Deformation Theory of Asymptotically Conical Coassociative 4-folds

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Abstract

Suppose that a coassociative 4-fold \( N \) in \( \mathbb{R}^7 \) is asymptotically conical to a cone \( C \) with rate \( \lambda < 1 \). If \( \lambda \in [-2, 1) \) is generic, we show that the moduli space of coassociative deformations of \( N \) which are also asymptotically conical to \( C \) with rate \( \lambda \) is a smooth manifold, and we calculate its dimension. If \( \lambda < -2 \) and generic, we show that the moduli space is locally homeomorphic to the kernel of a smooth map between smooth manifolds, and we give a lower bound for its expected dimension. We also derive a test for when \( N \) will be planar if \( \lambda < -2 \) and we discuss examples of asymptotically conical coassociative 4-folds.

1 Introduction

In this article we study coassociative 4-folds \( N \) in \( \mathbb{R}^7 \) which are asymptotically conical (AC). Our main result (cf. Theorem 6.4 & Corollary 6.5) is the following.

**Theorem 1.1** Let \( N \) be a coassociative 4-fold in \( \mathbb{R}^7 \) which is AC with rate \( \lambda < 1 \) to a cone \( C \). Let \( \mathcal{M}(N, \lambda) \) be the moduli space of coassociative deformations of \( N \) which are also AC with rate \( \lambda \) to \( C \).

(a) For generic \( \lambda \in [-2, 1) \) the deformation theory is unobstructed so \( \mathcal{M}(N, \lambda) \) is a smooth manifold near \( N \).

(b) For generic \( \lambda < -2 \), \( \mathcal{M}(N, \lambda) \) is locally homeomorphic to the kernel of a smooth map between smooth manifolds.

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In case (a), the dimension of $\mathcal{M}(N, \lambda)$ is equal to the dimension of the infinitesimal deformation space. We determine this dimension explicitly in Proposition 7.12 if additionally $\lambda \in [-2, 0)$, and we give upper and lower bounds for it otherwise in Proposition 7.13. In case (b), the map can be considered as a projection from the infinitesimal deformation space to the obstruction space and hence $\mathcal{M}(N, \lambda)$ is smooth if the latter space is zero. Moreover, we give a lower bound for the expected dimension of $\mathcal{M}(N, \lambda)$ for $\lambda < -2$ in Proposition 7.14.

These dimensions are given in terms of topological data from $N$ and $C$, and an analytic quantity determined by the cone $C$.

This article is motivated by the work of McLean [23, §4] on compact coassociative 4-folds and Marshall [21] on deformations of AC special Lagrangian (SL) submanifolds. The latter was in fact studied earlier by Pacini [24] using different methods. Asymptotically conical SL submanifolds are also discussed in the papers by Joyce [5]-[9] on SL submanifolds with conical singularities. Deformations of asymptotically cylindrical coassociative 4-folds are studied by Joyce and Salur in [11] and Salur in [25]. Other coassociative deformation theories have been studied by the author: coassociative 4-folds with conical singularities in [18] and compact coassociative 4-folds with boundary in [12] (with Kovalev).

We begin, in §2, by defining the submanifolds that we study here. Much of our work is analytic in nature, so to obtain many of the results we use weighted Sobolev spaces, which are a natural choice when studying AC submanifolds in this way. Thus, in §3, we define the weighted Banach spaces that we require.

In §4 we construct a deformation map which corresponds locally to the moduli space $\mathcal{M}(N, \lambda)$. We may therefore view the kernel of the linearisation of the deformation map at zero as the infinitesimal deformation space. We also define an associated map which is elliptic at zero since its derivative there acts as $d + d^*$ from pairs of self-dual 2-forms and 4-forms to 3-forms. This allows to prove that the kernel of the deformation map consists of smooth forms.

In §5 we discuss the Fredholm and index theory of the operator $d + d^*$: in particular, we describe a countable discrete set $D$, depending only on $C$, consisting of the rates $\lambda$ for which $d + d^*$ is not Fredholm. Our main result of the section (Theorem 5.10) identifies the obstruction space for our deformation theory.

In §6 we prove the deformation theory results which lead directly to Theorem 1.1. Section 7 then contains our aforementioned dimension calculations.

In §8 we construct two invariants of $N$ and hence derive a test for when $N$ will be planar. Finally, in §9 we discuss examples of AC coassociative 4-folds, including explicit examples which have rate $-3/2$. There are no known concrete examples of AC coassociative 4-folds with rate $\lambda < -2$, but such submanifolds
are essential for the desingularization theory of coassociative 4-folds with conical singularities, as discussed in [19].

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2 Asymptotically conical coassociative submanifolds of $\mathbb{R}^7$

We begin by defining coassociative 4-folds in $\mathbb{R}^7$, for which we introduce a distinguished 3-form on $\mathbb{R}^7$, following the notation in [10, Definition 11.1.1].

**Definition 2.1** Let $(x_1, \ldots, x_7)$ be coordinates on $\mathbb{R}^7$ and write $dx_{ij \ldots k}$ for the form $dx_i \wedge dx_j \wedge \ldots \wedge dx_k$. Define a 3-form $\varphi$ by:

$$\varphi = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}.$$ 

The 4-form $\ast \varphi$, where $\varphi$ and $\ast \varphi$ are related by the Hodge star, is given by:

$$\ast \varphi = dx_{4567} + dx_{2367} + dx_{2345} + dx_{1357} - dx_{1346} - dx_{1256} - dx_{1247}.$$ 

The subgroup of GL($7, \mathbb{R}$) preserving the 3-form $\varphi$ is $G_2$.

We can now characterise coassociative 4-folds in $\mathbb{R}^7$.

**Definition 2.2** A 4-dimensional submanifold $N$ of $\mathbb{R}^7$ is coassociative if and only if $\varphi|_N \equiv 0$ and $\ast \varphi|_N > 0$.

This is not the standard definition, which is formulated in terms of calibrated geometry, but is equivalent to it by [3, Proposition IV.4.5 & Theorem IV.4.6].

**Remark** The vanishing of $\varphi$ on a 4-fold $N$ in $\mathbb{R}^7$ forces $\ast \varphi$ to be nowhere vanishing on $N$. Thus, the condition $\ast \varphi|_N > 0$ amounts to a choice of orientation.

In order that we may study deformations of coassociative 4-folds, we need an important elementary result [23, Proposition 4.2].

**Proposition 2.3** Let $N$ be a coassociative 4-fold in $\mathbb{R}^7$. There is an isomorphism $j_N$ between the normal bundle $\nu(N)$ of $N$ in $\mathbb{R}^7$ and $\Lambda^2_+ T^* N$ given by $j_N : v \mapsto (v \cdot \varphi)|_N$.

**Note** Let $u \in T\mathbb{R}^7$. There exist unique normal and tangent vectors $v$ and $w$ on $N$ such that $u|_N = v + w$. Since $\varphi$ vanishes on the coassociative 4-fold $N$, $(w \cdot \varphi)|_N = 0$. Thus, $j_N(u) = j_N(v) \in \Lambda^2_+ T^* N$. So, $j_N : T\mathbb{R}^7|_N \to \Lambda^2_+ T^* N$. 

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Before defining AC submanifolds we clarify what we mean by a cone in $\mathbb{R}^n$.

**Definition 2.4** A cone $C \subseteq \mathbb{R}^n$ is a nonsingular submanifold, except perhaps at 0, satisfying $e^tC = C$ for all $t \in \mathbb{R}$. We call $\Sigma = C \cap S^{n-1}$ the link of $C$.

**Definition 2.5** Let $C$ be a closed cone in $\mathbb{R}^n$, let $\Sigma$ be the link of $C$ and let $N$ be a closed submanifold of $\mathbb{R}^n$. Then $N$ is asymptotically conical (AC) to $C$ (with rate $\lambda$) if there exist constants $\lambda < 1$ and $R > 1$, a compact subset $K$ of $N$, and a diffeomorphism $\Psi : (R, \infty) \times \Sigma \to N \setminus K$ such that

$$|\nabla^j (\Psi(r, \sigma) - \iota(r, \sigma))| = O(r^{\lambda - j}) \quad \text{for } j \in \mathbb{N} \text{ as } r \to \infty,$$

where $\iota : (0, \infty) \times \Sigma \to \mathbb{R}^n$ is the inclusion map given by $\iota(r, \sigma) = r\sigma$. Here $|.|$ is calculated using the conical metric $g_{\text{cone}} = dr^2 + r^2 g_\Sigma$ on $(0, \infty) \times \Sigma$, where $g_\Sigma$ is the round metric on $S^{n-1}$ restricted to $\Sigma$, and $\nabla$ is a combination of the Levi–Civita connection derived from $g_{\text{cone}}$ and the flat connection on $\mathbb{R}^n$, which acts as partial differentiation.

We also make the following definition.

**Definition 2.6** Let $(M, g)$ be a Riemannian $n$-manifold. Then $M$ is asymptotically conical (AC) (with rate $\lambda$) if there exist constants $\lambda < 1$ and $R > 1$, a compact $(n-1)$-dimensional Riemannian submanifold $(\Sigma, g_\Sigma)$ of $S^{n-1}$, compact $K \subseteq M$, and a diffeomorphism $\Psi : (R, \infty) \times \Sigma \to M \setminus K$ such that

$$|\nabla^j (\Psi^*(g) - g_{\text{cone}})| = O(r^{\lambda-1-j}) \quad \text{for } j \in \mathbb{N} \text{ as } r \to \infty,$$

where $(r, \sigma)$ are coordinates on $(0, \infty) \times \Sigma$, $g_{\text{cone}} = dr^2 + r^2 g_\Sigma$ on $(0, \infty) \times \Sigma$, $\nabla$ is the Levi–Civita connection of $g_{\text{cone}}$ and $|.|$ is calculated using $g_{\text{cone}}$.

Define $\iota : (0, \infty) \times \Sigma \to \mathbb{R}^n$ by $\iota(r, \sigma) = r\sigma$ and let $C = \text{Im } \iota$. Then $C$ is the asymptotic cone of $M$ and the components of $M_\infty = M \setminus K$ are the ends of $M$.

The condition $\lambda < 1$ in the definition above ensures that, by (2), the metric $g$ on $M$ converges to $g_{\text{cone}}$ at infinity.

**Note** By comparing (1) and (2), we see that if $N$ is a Riemannian submanifold of $\mathbb{R}^n$ which is AC to a cone $C$ with rate $\lambda$, it can be considered as an AC manifold with rate $\lambda$ and asymptotic cone $C$.

**Definition 2.7** Let $M$ be an AC manifold and use the notation of Definition 2.6. A radius function $\rho : M \to [1, \infty)$ on $M$ is a smooth function such that there exist positive constants $c_1 < 1$ and $c_2 > 1$ with $c_1 r < \Psi^*(\rho) < c_2 r$. 


If \( M \) is AC we may define a radius function \( \rho \) by setting \( \rho = 1 \) on \( K \), \( \rho(\Psi(r, \sigma)) = r \) for \( r > R + 1 \), and then extending \( \rho \) smoothly to our required function on \( M \).

We conclude the section with the following elementary result.

**Proposition 2.8** Suppose that \( N \) is a coassociative 4-fold in \( \mathbb{R}^7 \) which is AC with rate \( \lambda \) to a cone \( C \) in \( \mathbb{R}^7 \). Then \( C \) is coassociative.

**Proof:** By Definition 2.2, we have that \( \varphi|_{N} \equiv 0 \). We also have, using (1), that

\[
|\Psi^*(\varphi|_N) - \varphi|_C| = O(r^{\lambda - 1}) \quad \text{as } r \to \infty.
\]

Therefore, since \( \lambda - 1 < 0 \), \( \varphi|_C \to 0 \) as \( r \to \infty \). However, \( \varphi|_C \) must be independent of \( r \) since \( T_{r\sigma}C = T_{\sigma}C \) for all \( r > 0, \sigma \in \Sigma \). Hence \( \varphi|_C \equiv 0 \). \( \square \)

**Notes**

(a) Manifolds are taken to be nonsingular and submanifolds to be embedded, for convenience, unless stated otherwise.

(b) We use the convention that the natural numbers \( \mathbb{N} = \{0, 1, 2, \ldots\} \).

### 3 Weighted Banach spaces

We shall define *weighted* Banach spaces of forms on an AC manifold, following [1, §1]. We use the notation and definition of the usual ‘unweighted’ Banach spaces as in [10, §1.2]; that is, Sobolev and Hölder spaces are denoted by \( L^p_k \) and \( C^{k, a} \) respectively, where \( p \geq 1 \), \( k \in \mathbb{N} \) and \( a \in (0, 1) \). We also introduce the notation \( C^k_{\text{loc}} \) for the space of forms \( \xi \) such that \( f\xi \) lies in \( C^k \) for every smooth compactly supported function \( f \), and similarly define spaces \( L^p_k, \text{loc} \) and \( C^{k, a}_{\text{loc}} \).

For the whole of this section, we let \((M, g)\) be an AC \( n \)-manifold and \( \rho \) be a radius function on \( M \) as in Definition 2.7.

**Definition 3.1** Let \( p \geq 1 \), \( k \in \mathbb{N} \) and \( \mu \in \mathbb{R} \). The *weighted Sobolev space* \( L^p_{k, \mu}(\Lambda^m T^* M) \) of \( m \)-forms on \( M \) is the subspace of \( L^p_{k, \text{loc}}(\Lambda^m T^* M) \) such that

\[
\|\xi\|_{L^p_{k, \mu}} = \left( \sum_{j=0}^{k} \int_M |\rho^{j-\mu} \nabla^j \xi|^p \rho^{-n} dV_g \right)^{\frac{1}{p}}
\]

is finite. The normed vector space \( L^p_{k, \mu}(\Lambda^m T^* M) \) is a Banach space.
Note

\[ L^p(\Lambda^mT^*M) = L^p_{0,-\frac{\mu}{n}}(\Lambda^mT^*M). \]  

(3)

We now define dual weighted Sobolev space which shall be invaluable later.

**Definition 3.2** Use the notation from Definition 3.1. Let \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), let \( k, l \in \mathbb{N} \) and let \( \mu \in \mathbb{R} \). Define a pairing \( \langle \cdot , \cdot \rangle : L^p_{k, \mu}(\Lambda^mT^*M) \times L^q_{l, -n-\mu}(\Lambda^mT^*M) \to \mathbb{R} \) by

\[ \langle \xi, \eta \rangle = ||\xi \wedge \ast \eta||_{L^1}. \]

We shall occasionally refer to this as the dual pairing. For our purposes, we take the dual space of \( L^p_{k, \mu}(\Lambda^mT^*M) \) to be \( L^q_{l, -n-\mu}(\Lambda^mT^*M) \), with linear functionals represented by dual pairings.

**Definition 3.3** Let \( \mu \in \mathbb{R} \) and let \( k \in \mathbb{N} \). The weighted \( C^k \)-space \( C^k_{\mu}(\Lambda^mT^*M) \) of \( m \)-forms on \( M \) is the subspace of \( C^k_{\text{loc}}(\Lambda^mT^*M) \) such that the norm

\[ ||\xi||_{C^k_{\mu}} = \sum_{j=0}^{k} \sup_M |\rho^{-\mu} \nabla^j \xi| \]

is finite. We also define \( C^\infty_{\mu}(\Lambda^mT^*M) = \cap_{k \geq 0} C^k_{\mu}(\Lambda^mT^*M) \). Then \( C^k_{\mu}(\Lambda^mT^*M) \) is a Banach space but in general \( C^\infty_{\mu}(\Lambda^mT^*M) \) is not. Notice that we have a continuous embedding \( C^k_{\mu} \hookrightarrow C^l_{\nu} \) whenever \( k \geq l \) and \( \mu \leq \nu \).

We conclude this spate of definitions by defining weighted \( \text{Hölder} \) spaces. These shall not be required for the majority of the paper as weighted Sobolev and \( C^k \)-spaces will often suffice.

**Definition 3.4** Let \( a \in (0, 1) \), \( k \in \mathbb{N} \) and \( \mu \in \mathbb{R} \). Let \( d(x,y) \) be the geodesic distance between points \( x, y \in M \), let \( 0 < c_1 < c_2 < 1 \) be constant and let

\[ H = \{(x,y) \in M \times M : x \neq y, c_1 \rho(x) \leq \rho(y) \leq c_2 \rho(x) \text{ and} \]

\[ \text{there exists a geodesic in } M \text{ of length } d(x,y) \text{ from } x \text{ to } y \}. \]

A section \( s \) of a vector bundle \( V \) on \( M \), endowed with Euclidean metrics on its fibres and a connection preserving these metrics, is \( \text{Hölder continuous} \) (with exponent \( a \)) if

\[ [s]^a = \sup_{(x,y) \in H} \frac{|s(x) - s(y)|}{d(x,y)^a} < \infty. \]

We understand the quantity \( |s(x) - s(y)|_V \) as follows. Given \( (x,y) \in H \), there exists a geodesic \( \gamma \) of length \( d(x,y) \) connecting \( x \) and \( y \). Parallel translation
along γ using the connection on V identifies the fibres over x and y and the metrics on them. Thus, with this identification, |s(x) − s(y)|_V is well-defined.

The weighted Hölder space \( C^{k, a}_\mu(\Lambda^m T^* M) \) of m-forms ξ on M is the subspace of \( C^{k, a}_{\text{loc}}(\Lambda^m T^* M) \) such that the norm

\[
\|\xi\|_{C^{k, a}_\mu} = \|\xi\|_{C^k_\mu} + [\xi]^{k, a}_{\mu}
\]

is finite, where

\[
[\xi]^{k, a}_{\mu} = [\rho^{k+a-\mu} \nabla^k \xi]^a.
\]

Then \( C^{k, a}_\mu(\Lambda^m T^* M) \) is a Banach space. It is clear that we have an embedding \( C^{k, a}_\mu(\Lambda^m T^* M) \hookrightarrow C^{l, a}_\mu(\Lambda^m T^* M) \) whenever \( l \leq k \).

The set \( H \) in the definition above is introduced so that \( [\xi]^{k, a}_{\mu} \) is well-defined.

We shall need the analogue of the Sobolev Embedding Theorem for weighted spaces, which is adapted from [16, Lemma 7.2] and [1, Theorem 1.2].

**Theorem 3.5 (Weighted Sobolev Embedding Theorem)** Let \( p, q > 1 \), \( a \in (0, 1) \), \( \mu, \nu \in \mathbb{R} \) and \( k, l \in \mathbb{N} \).

(a) If \( k \geq l \), \( k - \frac{a}{p} \geq l - \frac{a}{q} \) and either \( p \leq q \) and \( \mu \leq \nu \), or \( p > q \) and \( \mu < \nu \), there is a continuous embedding \( L^p_{k, \mu}(\Lambda^m T^* M) \hookrightarrow L^q_{l, \nu}(\Lambda^m T^* M) \).

(b) If \( k - \frac{a}{p} \geq l + a \), there is a continuous embedding \( L^p_{k, \mu}(\Lambda^m T^* M) \hookrightarrow C^{l, a}_\mu(\Lambda^m T^* M) \).

We shall also require an Implicit Function Theorem for Banach spaces, which follows immediately from [14, Theorem 2.1].

**Theorem 3.6 (Implicit Function Theorem)** Let \( X \) and \( Y \) be Banach spaces and let \( U \subseteq X \) be an open neighbourhood of \( 0 \). Let \( \mathcal{F} : U \rightarrow Y \) be a \( C^k \)-map (\( k \geq 1 \)) such that \( \mathcal{F}(0) = 0 \). Suppose further that \( d\mathcal{F}|_0 : X \rightarrow Y \) is surjective with kernel \( K \) such that \( X = K \oplus A \) for some closed subspace \( A \) of \( X \). There exist open sets \( V \subseteq K \) and \( W \subseteq A \), both containing 0, with \( V \times W \subseteq U \), and a unique \( C^k \)-map \( \mathcal{G} : V \rightarrow W \) such that

\[
\mathcal{F}^{-1}(0) \cap (V \times W) = \{ (x, \mathcal{G}(x)) : x \in V \}
\]

in \( X = K \oplus A \).

### 4 The deformation map

For the rest of the paper, let \( N \subseteq \mathbb{R}^7 \) be a coassociative 4-fold which is asymptotically conical to a cone \( C \subseteq \mathbb{R}^7 \) with rate \( \lambda \), and use the notation of Definition
2.5 In particular, \((0, \infty) \times \Sigma \cong C\), where \(\Sigma = C \cap S^6\), with coordinates \((r, \sigma)\) on \((0, \infty) \times \Sigma\), and \((R, \infty) \times \Sigma \cong N \setminus K\), where \(K\) is a compact subset of \(N\) and \(R > 1\). Moreover, let \(\rho\) be a radius function on \(N\), as defined in Definition 2.5 and choose \(\Psi\) uniquely by imposing the condition that
\[
\Psi(r, \sigma) - \iota(r, \sigma) \in (T_r \sigma C)^\perp \quad \text{for all} \quad (r, \sigma) \in (R, \infty) \times \Sigma,
\]
which can be achieved by making \(R\) and \(K\) larger if necessary.

We shall endeavour to remind the reader of this notation when required.

4.1 Preliminaries

We wish to discuss deformations of \(N\); that is, coassociative submanifolds of \(\mathbb{R}^7\) which are ‘near’ to \(N\). We define this formally.

**Definition 4.1** The moduli space of deformations \(M(N, \lambda)\) is the set of coassociative 4-folds \(N' \subseteq \mathbb{R}^7\) which are AC to \(C\) with rate \(\lambda\) such that there exists a diffeomorphism \(h : \mathbb{R}^7 \to \mathbb{R}^7\), with \(h(N) = N'\), isotopic to the identity.

The first result we need is immediate from the proof of [13, Chapter IV, Theorem 9].

**Theorem 4.2** Let \(P\) be a closed submanifold of a Riemannian manifold \(M\). There exist an open subset \(V\) of the normal bundle \(\nu(P)\) of \(P\) in \(M\), containing the zero section, and an open set \(S\) in \(M\) containing \(P\), such that the exponential map \(\exp|_V : V \to S\) is a diffeomorphism.

**Note** The proof of this result relies entirely on the observation that \(\exp|_{\nu(P)}\) is a local isomorphism upon the zero section.

This information helps us prove a useful corollary.

**Corollary 4.3** Let \(P = \iota((R, \infty) \times \Sigma), Q = N \setminus K\) and define \(n_P : \nu(P) \to \mathbb{R}^7\) by \(n_P(\sigma, v) = v + \Psi(r, \sigma)\). There exist an open subset \(V\) of \(\nu(P)\), containing the zero section, and an open set \(S\) in \(\mathbb{R}^7\) containing \(Q\), such that \(n_P|_V : V \to S\) is a diffeomorphism. Moreover, \(V\) can be chosen to be an open neighbourhood of the zero section in \(C^1\).

**Proof:** Note that \(n_P\) takes the zero section of \(\nu(P)\) to \(Q\). By the definition of \(\Psi\) in Definition 2.5 \(n_P\) is a local isomorphism upon the zero section. Thus, by the note after Theorem 4.2 we have open sets \(V\) and \(S\) such that \(n_P|_V : V \to S\) is a diffeomorphism.
Since $\Psi - \iota$ is orthogonal to $(R, \infty) \times \Sigma$ by (4), it can be identified with a small section of the normal bundle. Hence $P$ lies in $S$ as long as $S$ grows with order $O(r)$ as $r \to \infty$. As we can form $S$ and $V$ in a translation equivariant way because we are working on a portion of the cone $C$, we can construct our sets with this growth rate at infinity and such that they do not collapse near $R$. Thus, we can ensure that $V$ is an open set in $C^1$. □

Recall that, by Proposition 2.8, $C$ is coassociative. Therefore, since $\Psi(r, \sigma) - \iota(r, \sigma)$ lies in $(T_r C)^\perp \cong \nu_r(C)$ for $r > R$ by (4), $\Psi - \iota$ can be identified with the graph of an element $\gamma_C$ of $\Lambda^2 T^* P$ by Proposition 2.3, using the notation of Corollary 4.3. Then

$$|\nabla_{\gamma_C} C| = O(r^{\lambda-j}) \quad \text{for } j \in \mathbb{N} \text{ as } r \to \infty,$$

(5) since $N$ is AC to $C$ with rate $\lambda$, where $\nabla_C$ and $|\cdot|$ are the Levi–Civita connection and modulus calculated using the conical metric. Thus, $\gamma_C$ lies in $C^\infty(\Lambda^2 T^* P)$. Moreover, we have a decomposition:

$$\mathbb{R}^7 = T_{\Psi(r, \sigma)} N \oplus \nu_r(C)$$

at $\Psi(r, \sigma)$. We can therefore identify $\nu_{\Psi(r, \sigma)}(N)$ with $\nu_r(C)$ and hence identify $\Lambda^2 T^* N$ and $\Lambda^2 T^* C$ near infinity. Formally, we have the following.

**Proposition 4.4** Use the notation of Corollary 4.3 and let $j_C$ and $j_N$ be the isomorphisms given by Proposition 2.3 applied to $C$ and $N$ respectively. There exists a diffeomorphism $\Upsilon : \nu(P) \to \nu(Q)$, with $\Upsilon(0) = 0$, and hence a diffeomorphism $\check{\Upsilon} : \Lambda^2 T^* P \to \Lambda^2 T^* Q$ given by $\check{\Upsilon} = j_N \circ \Upsilon \circ j_C^{-1}$.

**Proposition 4.5** Use the notation of Corollary 4.3 and Proposition 4.4. There exist an open set $U \subseteq \Lambda^2 T^* N$ containing the zero section and $W = (j_N \circ \Upsilon)(V)$, a tubular neighbourhood $T$ of $N$ in $\mathbb{R}^7$ containing $S$, and a diffeomorphism $\delta : U \to T$, affine on the fibres, that takes the zero section of $\Lambda^2 T^* N$ to $N$ and such that the following diagram commutes:

$$\begin{array}{ccc}
W & \xrightarrow{j_C^{-1}} & j_C(V) \\
\downarrow \delta & & \downarrow \delta \\
S & \xleftarrow{\nu_F} & V.
\end{array}$$

(6)

Moreover, we may choose $U$ to be a $C^1_1$-open neighbourhood of the zero section.

**Proof**: Define the diffeomorphism $\delta|_W : W \to S$ by (5). Interpolating smoothly over the compact set $K \subseteq N$, we extend $S$ to $T$, $W$ to $U$ and $\delta|_W$ to $\delta$ as required. Furthermore, by Corollary 4.3, $V$ can be chosen to be an open neighbourhood of the zero section in $C^1$. Hence, we can arrange the same for $U$. □
Proposition 4.7

Using the notation of Definitions 4.1 and 4.6, locally homeomorphic to the kernel of $F$ moduli space $M$. This leads immediately to our next result, which gives a local description of the

$\alpha$ can be identified with the graph of $\alpha$ and therefore with the graph of $\alpha$ for all $\alpha$. We use similar conventions to define subsets of the spaces discussed in 4.5. By Definition 2.2, $\text{Ker } F$ is coassociative. Note that $N$ of $\alpha$ given by Proposition 4.5 and $\Gamma$ by Definition 4.6.

We introduce the notation $C^k_\lambda(U) = \{ \alpha \in C^k_\lambda(\Lambda^2 T^* N) : \Gamma_{\alpha} \subseteq U \}$, where $U$ is given by Proposition 4.3 and $\Gamma_{\alpha}$ is the graph of $\alpha$. The fact that $U$ is a $C^1$-open set ensures that $C^k_\lambda(U)$ is an open subset of $C^k_\lambda(\Lambda^2 T^* N)$ for $k \geq 1$, since $\lambda < 1$. We use similar conventions to define subsets of the spaces discussed in 4.4, we showed that $\Psi$ where $K$ and $\alpha$ such that $\Psi$ is the deformation of $N$ corresponding to $\alpha$. We define $F : C^{1}_{\text{loc}}(U) \to C^{0}_{\text{loc}}(\Lambda^3 T^* N)$ by

$$F(\alpha) = f_{\alpha}(\varphi|_{N_{\alpha}}).$$

By Definition 2.2, $\text{Ker } F$ is the set of $\alpha \in C^{1}_{\text{loc}}(U)$ such that the deformation $N_{\alpha}$ of $N$ is coassociative. Note that $F$ is a nonlinear operator and that

$$dF|_{0}(\alpha) = d\alpha,$$

for all $\alpha \in C^{1}_{\text{loc}}(\Lambda^2 T^* N)$, by [23, p. 731] and our choice of $\delta$.

However, we only wish to consider smooth coassociative deformations $N_{\alpha}$ of $N$ which are asymptotically conical to $C$ with rate $\lambda$. If $N_{\alpha}$ is such a deformation, then $\alpha \in C^{\infty}(U)$ and there exists a diffeomorphism $\Psi_{\alpha} : (R, \infty) \times \Sigma \to N_{\alpha} \setminus K_{\alpha}$, where $K_{\alpha}$ is a compact subset of $N_{\alpha}$, as in Definition 2.3. We may define $\Psi_{\alpha}$ such that $\Psi_{\alpha}(r, \sigma) - \iota(r, \sigma)$ is orthogonal to $T_{r, \sigma}C$ for all $\sigma \in \Sigma$ and $r > R$.

Recall the notation of Corollary 4.3 and Proposition 4.4. Before Proposition 4.3, we showed that $\Psi - \iota$ can be identified with the graph of $\gamma_{C} \in C^{\infty}_\lambda(\Lambda^2 T^* P)$ and therefore with the graph of $\alpha_{C} = \tilde{\Psi}(\gamma_{C}) \in C^{\infty}_\lambda(\Lambda^2 T^* Q)$. Similarly, $\Psi_{\alpha} - \iota$ can be identified with the graph of $\alpha + \alpha_{C} \in C^{\infty}_\lambda(\Lambda^2 T^* Q)$. Hence, $\alpha \in C^{\infty}_\lambda(U)$.

We conclude that $N_{\alpha}$ is AC to $C$ with rate $\lambda$ if and only if $\alpha \in C^{\infty}_\lambda(U)$. This leads immediately to our next result, which gives a local description of the moduli space $M(N, \lambda)$ using the deformation map $F$.

Proposition 4.7 Using the notation of Definitions 4.1 and 4.6, $M(N, \lambda)$ is locally homeomorphic to the kernel of $F : C^{\infty}_\lambda(U) \to C^{\infty}(\Lambda^3 T^* N)$. 

4.2 The map $F$ and the associated map $G$

We introduce the notation $C^k_\lambda(U) = \{ \alpha \in C^k_\lambda(\Lambda^2 T^* N) : \Gamma_{\alpha} \subseteq U \}$, where $U$ is given by Proposition 4.3 and $\Gamma_{\alpha}$ is the graph of $\alpha$. The fact that $U$ is a $C^1$-open set ensures that $C^k_\lambda(U)$ is an open subset of $C^k_\lambda(\Lambda^2 T^* N)$ for $k \geq 1$, since $\lambda < 1$. We use similar conventions to define subsets of the spaces discussed in 4.4, we showed that $\Psi$ where $K$ and $\alpha$ such that $\Psi$ is the deformation of $N$ corresponding to $\alpha$. We define $F : C^{1}_{\text{loc}}(U) \to C^{0}_{\text{loc}}(\Lambda^3 T^* N)$ by

$$F(\alpha) = f_{\alpha}(\varphi|_{N_{\alpha}}).$$

By Definition 2.2, $\text{Ker } F$ is the set of $\alpha \in C^{1}_{\text{loc}}(U)$ such that the deformation $N_{\alpha}$ of $N$ is coassociative. Note that $F$ is a nonlinear operator and that

$$dF|_{0}(\alpha) = d\alpha,$$

for all $\alpha \in C^{1}_{\text{loc}}(\Lambda^2 T^* N)$, by [23, p. 731] and our choice of $\delta$.

However, we only wish to consider smooth coassociative deformations $N_{\alpha}$ of $N$ which are asymptotically conical to $C$ with rate $\lambda$. If $N_{\alpha}$ is such a deformation, then $\alpha \in C^{\infty}(U)$ and there exists a diffeomorphism $\Psi_{\alpha} : (R, \infty) \times \Sigma \to N_{\alpha} \setminus K_{\alpha}$, where $K_{\alpha}$ is a compact subset of $N_{\alpha}$, as in Definition 2.3. We may define $\Psi_{\alpha}$ such that $\Psi_{\alpha}(r, \sigma) - \iota(r, \sigma)$ is orthogonal to $T_{r, \sigma}C$ for all $\sigma \in \Sigma$ and $r > R$.

Recall the notation of Corollary 4.3 and Proposition 4.4. Before Proposition 4.3, we showed that $\Psi - \iota$ can be identified with the graph of $\gamma_{C} \in C^{\infty}_\lambda(\Lambda^2 T^* P)$ and therefore with the graph of $\alpha_{C} = \tilde{\Psi}(\gamma_{C}) \in C^{\infty}_\lambda(\Lambda^2 T^* Q)$. Similarly, $\Psi_{\alpha} - \iota$ can be identified with the graph of $\alpha + \alpha_{C} \in C^{\infty}_\lambda(\Lambda^2 T^* Q)$. Hence, $\alpha \in C^{\infty}_\lambda(U)$.

We conclude that $N_{\alpha}$ is AC to $C$ with rate $\lambda$ if and only if $\alpha \in C^{\infty}_\lambda(U)$. This leads immediately to our next result, which gives a local description of the moduli space $M(N, \lambda)$ using the deformation map $F$.

Proposition 4.7 Using the notation of Definitions 4.1 and 4.6, $M(N, \lambda)$ is locally homeomorphic to the kernel of $F : C^{\infty}_\lambda(U) \to C^{\infty}(\Lambda^3 T^* N)$. 

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We now prove that we can rewrite $F(\alpha)$ as a sum of $d\alpha$ and a term which is no worse than quadratic in $\alpha$ and $\nabla \alpha$. This result will be useful throughout the article, in particular we shall need it to derive regularity results in $4.3$.

**Proposition 4.8** Use the notation of Definitions $3.1$, $3.4$ and $4.6$. We can write

$$F(\alpha)(x) = d\alpha(x) + P_F(x, \alpha(x), \nabla \alpha(x))$$

(7)

for $x \in N$, where

$$P_F : \{(x, y, z) : (x, y) \in U, \ z \in T^*_x N \otimes \Lambda^2 T^*_x N\} \to \Lambda^3 T^* N$$

is a smooth map such that $P_F(x, y, z) \in \Lambda^3 T^*_x N$. Denote $P_F(x, \alpha(x), \nabla \alpha(x))$ by $P(\alpha)(x)$ for all $x \in N$.

Let $\mu < 1$. For each $n \in \mathbb{N}$, if $\alpha \in C^{n+1}_\mu(U)$ and $\|\alpha\|_{C^1}$ is sufficiently small, $P_F(\alpha) \in C^{n-2}_2(\Lambda^3 T^* N)$ and there exists a constant $c_n > 0$ such that

$$\|P_F(\alpha)\|_{C^{n+1}_2} \leq c_n \|\alpha\|^2_{C^{n+1}_2}.$$  

(8)

Let $p > 4$, $k \geq 1$, $l \in \mathbb{N}$ and $a \in (0, 1)$. If $\alpha \in L^p_{k+1, \mu}(U)$ or $\alpha \in C^{l+1, a}_\mu(U)$, with $\|\alpha\|_{C^1}$ sufficiently small, $P_F(\alpha) \in L^p_{k, 2\mu-2}(\Lambda^3 T^* N)$ or $C^{l, a}_{2\mu-2}(\Lambda^3 T^* N)$ and there exist constants $l_p, k > 0$ and $c_{l, a} > 0$ such that

$$\|P_F(\alpha)\|_{L^p_{k, 2\mu-2}} \leq l_{p, k} \|\alpha\|^2_{L^p_{k+1, \mu}} \quad \text{or} \quad \|P_F(\alpha)\|_{C^{l, a}_{2\mu-2}} \leq c_{l, a} \|\alpha\|^2_{C^{l+1, a}_\mu}.$$  

Remark: As $\mu < 1$, $2\mu - 2 < \mu - 1$, so $C^{n-2}_2 \hookrightarrow C^{n-1}_2$. Similar continuous embeddings occur for the weighted Sobolev and Hölder spaces. Furthermore, the conditions $p > 4$, $k \geq 1$ and $l \in \mathbb{N}$ ensure, by Definition $3.4$ and Theorem $3.5$ (b), that $C^{l+1, a}_\mu \hookrightarrow C^1_{l}$ and $L^p_{k+1, \mu} \hookrightarrow C^1_{l}$.

Proof: First, by the definition of $F$, $F(\alpha)(x)$ relates to the tangent space to the graph $\Gamma_\alpha$ of $\alpha$ at $\pi_\alpha(x) = (x, \alpha(x))$. Note that $T_{\pi_\alpha(x)} \Gamma_\alpha$ depends on both $\alpha(x)$ and $\nabla \alpha(x)$ and hence so must $F(\alpha)(x)$. We may then define $P_F$ by (7) such that it is a smooth function of its arguments as claimed.

We only prove the estimate (8) on $P_F$ as the results for weighted Sobolev and Hölder spaces can be deduced from the work presented here. Recall the notation at the start of this section and of Corollary $4.3$ and Proposition $4.4$. Let $\nabla_C$ denote the Levi–Civita connection of the conical metric on $C$. We argued after Corollary $4.3$ that we may identify the displacement $\Psi - \ell$ of $N$ from $C$, outside the compact subset $K$, with $\gamma_C \in C^{\infty}_0(\Lambda^2 C^* P)$. For each $\alpha \in C^1_{\text{loc}}(U)$, there exists a unique $\gamma \in C^1_{\text{loc}}(\Lambda^2 C^* P)$ such that $\alpha = \tilde{\gamma}(\gamma)$ on $N \setminus K$. Thus, define a function $F_C(\gamma + \gamma_C)$, for $\gamma \in C^1_{\text{loc}}(\Lambda^2 C^* P)$, on $(R, \infty) \times \Sigma$ by

$$F_C(\gamma + \gamma_C)(r, \sigma) = F(\alpha)(\Psi(r, \sigma)),$$

(9)
where $\alpha|_{N \setminus K} = \tilde{Y}(\gamma)$. Now define a smooth function $P_C$ by an equation analogous to (9):

$$F_C(\gamma + \gamma_C)(r, \sigma) = d(\gamma + \gamma_C)(r, \sigma) + P_C((r, \sigma), (\gamma + \gamma_C)(r, \sigma), \nabla_C(\gamma + \gamma_C)(r, \sigma)).$$

(10)

We notice that $F_C$ and $P_C$ are only dependent on the cone $C$ and, rather trivially, on $R$. Therefore, because of this fact and our choice of $\delta$ in Proposition 4.5, these functions have scale equivariance properties. We may therefore derive equations and inequalities on $\{R\} \times \Sigma$ and deduce the result on all of $(R, \infty) \times \Sigma$ by introducing an appropriate scaling factor of $r$.

Now, since the graph of $\alpha = 0$ corresponds to our coassociative 4-fold $N$, $F(\tilde{Y}(0)) = F(0) = 0$. So, taking $\gamma = 0$ in (9) gives:

$$F_C(\gamma_C) = d\gamma_C + P_C(\gamma_C) = 0,$$

(11)

adopting similar notation for $P_C(\gamma_C)$ as for $P_F(\alpha)$. From (7)-(11) we calculate:

$$P_F(\alpha)(\Psi(r, \sigma)) = d\gamma_C(r, \sigma) + P_C(\gamma + \gamma_C)(r, \sigma) = d\gamma_C(r, \sigma) + P_C(\gamma + \gamma_C)(r, \sigma) = (d\gamma_C(r, \sigma) + P_C(\gamma_C)(r, \sigma)).$$

(12)

Noting that $P_C$ is a function of three variables $x$, $y$ and $z$, we see that

$$P_C(\gamma + \gamma_C) - P_C(\gamma_C) = \int_0^1 \frac{d}{dt} P_C(t\gamma + \gamma_C) dt = \int_0^1 \gamma \cdot \frac{\partial P_C}{\partial y}(t\gamma + \gamma_C) + \nabla_C\gamma \cdot \frac{\partial P_C}{\partial z}(t\gamma + \gamma_C) dt.$$  

(13)

By Taylor’s Theorem,

$$P_C(\gamma + \gamma_C) = P_C(\gamma_C) + \gamma \cdot \frac{\partial P_C}{\partial y}(\gamma_C) + \nabla_C\gamma \cdot \frac{\partial P_C}{\partial z}(\gamma_C) + O(r^{-2}|\gamma|^2 + |\nabla_C\gamma|^2)$$  

(14)

when $r^{-1}|\gamma|$ and $|\nabla_C\gamma|$ are small.

Since $dF|_0(\alpha) = d\alpha$, as noted in Definition 4.6, $dF_C(\gamma + \gamma_C) = d\gamma$ and hence $dP_C|_{\gamma_C} = 0$. Thus, the first derivatives of $P_C$ with respect to $y$ and $z$ must vanish at $\gamma_C$ by (14). Therefore, given small $\epsilon > 0$, there exists a constant $A_0 > 0$ such that

$$\left| \frac{\partial P_C}{\partial y}(t\gamma + \gamma_C) \right| \leq A_0(r^{-2}|\gamma| + r^{-1}|\nabla_C\gamma|) \quad \text{and}$$

$$\left| \frac{\partial P_C}{\partial z}(t\gamma + \gamma_C) \right| \leq A_0(r^{-1}|\gamma| + |\nabla_C\gamma|)$$

(15)
for $t \in [0, 1]$, whenever

$$r^{-1}|\gamma|, r^{-1}|\gamma_C|, |\nabla C \gamma| \text{ and } |\nabla C \gamma_C| \leq \epsilon. \quad (17)$$

The factors of $r$ are determined by considering the scaling properties of $\gamma$ and $P_C$ and their derivatives under changes in $r$.

By (5), $r^{-1}|\gamma_C|$ and $|\nabla C \gamma_C|$ tend to zero as $r \to \infty$. We can thus ensure that (17) is satisfied by the $\gamma_C$ terms by making $R$ larger. Hence, (17) holds if $||\gamma||_{C^1_r} \leq \epsilon$. Therefore, putting estimates (15) and (16) in (13) and using (12),

$$|P_F(\alpha)(\Psi(r, \sigma))| = |P_C(\gamma + \gamma_C)(r, \sigma) - P_C(\gamma_C)(r, \sigma)| \leq A_0(r^{-1}|\gamma(r, \sigma)| + |\nabla C \gamma(r, \sigma)|)^2 \quad (18)$$

for all $r > R$, $\sigma \in \Sigma$, whenever $||\gamma||_{C^1_r} \leq \epsilon$.

Therefore, if $\alpha \in C^1_\mu(U)$, the corresponding $\gamma$ lies in $C^1_\mu(\Lambda^2 T^*P)$. Hence, by (18), $P_F(\alpha) \in C^0_\mu$ and there exists a constant $c_0$ such that

$$\sup_{N} |\rho^{2-2\mu} P_F(\alpha)| \leq c_0 \left( \frac{1}{\sup_{N} |\rho^{-\mu} \nabla x^i a_1|} \right)^2$$

whenever $||\alpha||_{C^1_r}$ is sufficiently small, where $\rho$ is a radius function on $N$. Thus, (8) holds for $n = 0$.

Similar calculations give analogous results to (18) for derivatives of $P_F$, from which we can deduce (8) for $n > 0$. We shall explain the method by considering the first derivative. If $\gamma \in C^2_{loc}(\Lambda^2 T^*P)$, we calculate from (13):

$$\nabla_C (P_C(\gamma + \gamma_C) - P_C(\gamma_C))$$

$$= \int_0^1 \nabla_C \left( \gamma \cdot \frac{\partial P_C}{\partial y}(t\gamma + \gamma_C) + \nabla_C \gamma \cdot \frac{\partial P_C}{\partial z}(t\gamma + \gamma_C) \right) dt$$

$$= \int_0^1 \nabla_C \gamma \cdot \frac{\partial P_C}{\partial y} + \gamma \cdot \left( \nabla_C (t\gamma + \gamma_C) \cdot \frac{\partial^2 P_C}{\partial y^2} + \nabla_C (t\gamma + \gamma_C) \cdot \frac{\partial^2 P_C}{\partial y \partial z} \right)$$

$$+ \nabla_C \gamma \cdot \frac{\partial P_C}{\partial z} + \nabla_C \gamma \cdot \left( \nabla_C (t\gamma + \gamma_C) \cdot \frac{\partial^2 P_C}{\partial z^2} + \nabla_C (t\gamma + \gamma_C) \cdot \frac{\partial^2 P_C}{\partial y \partial z} \right) dt.$$
since the second derivatives of $P_C$ are continuous functions defined on the closed bounded set given by $\|\gamma\|_{C^1} \leq \varepsilon$. We see that
\[
\left| \nabla (P_F(\alpha))(\Psi(r,\sigma)) \right| = \left| \nabla_C (P_C(\gamma + \gamma_C) - P_C(\gamma_C))(r,\sigma) \right|
\leq A_1 r \left( \sum_{i=0}^{2} r^{i-2} |\nabla_C \gamma(r,\sigma)| \right)^2
\]
whenever $\|\gamma\|_{C^1} \leq \varepsilon$. From this we can deduce the result (8) for $n = 1$.

In general we have the estimate
\[
\left| \nabla^j (P_F(\alpha))(\Psi(r,\sigma)) \right| \leq A_j r^j \left( \sum_{i=0}^{j+1} r^{i-1-j} |\nabla_C \gamma(r,\sigma)| \right)^2
\]
for some $A_j > 0$ whenever $\|\gamma\|_{C^1} \leq \varepsilon$. The result (8) for all $n \in \mathbb{N}$ follows. □

We now turn to the associated map $G$.

**Definition 4.9** Use the notation of Proposition 4.5 and Definition 4.6. Define
\[
G : C^1_{\text{loc}}(U) \times C^1_{\text{loc}}(\Lambda^4 T^*N) \to C^0_{\text{loc}}(\Lambda^3 T^*N)
\]
by:
\[
G(\alpha, \beta) = F(\alpha) + d^* \beta.
\]
By the observations made in Definition 4.6
\[
dG_{(0,0)}(\alpha, \beta) = d\alpha + d^* \beta
\]
for all $(\alpha, \beta) \in C^1_{\text{loc}}(\Lambda^4 T^*N \oplus \Lambda^4 T^*N)$. Thus $G$ is a nonlinear elliptic operator at $(0,0)$; that is, the linearisation of $G$ at $(0,0)$ is elliptic.

To complete this subsection, we relate the kernels of $F$ and $G$.

**Proposition 4.10** Use the notation of Definitions 3.1-3.4, 4.6 and 4.9. Let $p > 4$, $k \geq 1$, $l \in \mathbb{N}$, $a \in (0,1)$, $\mu < 0$ and $\nu \leq 0$. The kernels of $F$ and $G$ in $C^{l+1, a}$, $C^{l+1, a}_{\mu}$ and $L^p_{k+1, \nu}$ respectively are isomorphic.

*Proof:* Note that $\varphi$ is exact on any deformation $N_\alpha$ of $N$ because it is closed and vanishes on $N$. Thus, if $G(\alpha, \beta) = 0,$
\[
d(G(\alpha, \beta)) = d(F(\alpha)) + dd^* \beta = \Delta \beta = 0,
\]
since $F(\alpha)$ is exact. Recall we have a radius function $\rho$ on $N$. If $\beta$ decays with order $O(\rho^\mu)$, or $o(\rho^\mu)$, as $\rho \to \infty$, then $*\beta$ is a harmonic function on $N$ which tends to zero as $\rho \to \infty$. The Maximum Principle allows us to deduce that $*\beta = 0$ and conclude that $\beta = 0$, from which the proposition follows. □
4.3 Uniformly elliptic AC operators and regularity

We want to consider the regularity of solutions \( F(\alpha) = 0 \) near 0, but this equation is not elliptic. Therefore, we study \( G(\alpha, \beta) = 0 \) near \((0, 0)\), which is a non-linear elliptic equation. To do so, we must first discuss the regularity theory of certain linear elliptic operators, which we shall define, acting between weighted Banach spaces. We shall use the theory in [15] and [16], which centres around asymptotically cylindrical manifolds, so we begin with their definition.

**Definition 4.11** Recall Definition 3.1 and the notation introduced at the start of §4. In particular, we have a compact subset \( K \) of \( N \) such that \( N_\infty = N \setminus K \cong (R, \infty) \times \Sigma \) and a radius function \( \rho \) on \( N \). Let \( g \) be the metric on \( N \), considered as a Riemannian manifold. Define a metric \( \tilde{g} \) on \( N \) by \( \tilde{g} = \rho^{-2} g \). Further, define a coordinate \( t \) on \((T, \infty)\), where \( T = \log R \), by \( t = \log r \), let \( g_{\text{cyl}} = dt^2 + g_\Sigma \) be the cylindrical metric on \((T, \infty) \times \Sigma \) and let \( \nabla \) be the Levi–Civita connection of \( \tilde{g} \). We say that \((N, \tilde{g})\) is asymptotically cylindrical (ACyl).

For \( p \geq 1 \), \( k \in \mathbb{N} \) and \( \mu \in \mathbb{R} \) we define the Banach space \( \tilde{L}^p_{k, \mu}(\Lambda^m T^* N) \) to be the subspace of \( L^p_{k, \mu}(\Lambda^m T^* N) \) such that the following norm is finite:

\[
\|\xi\|_{\tilde{L}^p_{k, \mu}} = \left( \sum_{j=0}^k \int_N |\rho^{-\mu} \nabla^j \xi|_g^p dV_{\tilde{g}} \right)^{\frac{1}{p}}.
\]

**Definition 4.12** Use the notation from Definition 4.11. Suppose that \( \mathcal{P} : C^l_{\text{loc}}(\Lambda^m T^* N) \to C^0_{\text{loc}}(\Lambda^{m'} T^* N) \) and \( \mathcal{P}_\infty : C^l_{\text{loc}}(\Lambda^m T^* N_\infty) \to C^0_{\text{loc}}(\Lambda^{m'} T^* N_\infty) \) are linear differential operators of order \( l \). Suppose further that \( \mathcal{P}_\infty \) is invariant under the \( R^+ \)-action on \( N_\infty \cong (T, \infty) \times \Sigma \). We say that \( \mathcal{P} \) is cylindrical.

For \( \xi \in C^l_{\text{loc}}(\Lambda^m T^* N_\infty) \),

\[
\mathcal{P}_\xi = \sum_{i=0}^l \mathcal{P}_i \cdot \nabla^i \xi \quad \text{and} \quad \mathcal{P}_\infty \xi = \sum_{i=0}^l \mathcal{P}_{i, \infty} \cdot \nabla^i \xi,
\]

where \( \mathcal{P}_i \) and \( \mathcal{P}_{i, \infty} \) are tensors on \( N_\infty \) of type \((m+i, m')\) and \( \cdot \) means tensor product followed by contraction. If, for \( i = 0, \ldots, l \),

\[
|\nabla^j (\mathcal{P}_i - \mathcal{P}_{i, \infty})| \to 0 \quad \text{for } j \in \mathbb{N} \text{ as } t \to \infty,
\]

where \(|\cdot|\) is calculated using \( g_{\text{cyl}} \), we say that \( \mathcal{P} \) is asymptotically cylindrical (to \( \mathcal{P}_\infty \)). By [15, Theorem 3.7], \( \mathcal{P} \) is a continuous map from \( \tilde{L}^p_{k+l, \mu}(\Lambda^m T^* N) \) to \( \tilde{L}^p_{k, \mu}(\Lambda^{m'} T^* N) \) for all \( p > 1 \), \( k \in \mathbb{N} \) and \( \mu \in \mathbb{R} \).
Definition 4.13 Use the notation from Definitions 4.11 and 4.12 so we have an operator \( \mathcal{P} \) of order \( l \) acting between \( m \)- and \( m' \)-forms on \( N \). Let \( \nu \in \mathbb{R} \). We say that \( \mathcal{P} \) is asymptotically conical (AC) with rate \( \nu \) if the differential operator

\[
\mathcal{P}^\nu = \rho^{-m' + \nu} \rho^m
\]

is asymptotically cylindrical to a cylindrical map \( \mathcal{P}_\infty \), say. By [15, Proposition and Definition 4.4], \( \mathcal{P} : L^p_{k+1, \mu}(\Lambda^m T^* N) \to L^p_{k, \mu - \nu}(\Lambda^{m'} T^* N) \) is continuous for all \( p > 1, k \in \mathbb{N} \) and \( \mu \in \mathbb{R} \).

Note This definition can clearly be extended to linear differential operators acting between more general bundles of forms.

Notice that an AC operator with rate \( \nu \) reduces the growth rate of a form on the ends of \( N \) by \( \nu \). Examples of AC operators abound: \( d \) and \( d^* \) are first-order operators with rate 1 and the Laplacian is a second-order operator with rate 2.

Definition 4.14 Use the notation of Definition 4.13. We say that an AC operator \( \mathcal{P} \) is uniformly elliptic if it is elliptic and \( \mathcal{P}_\infty \) is elliptic.

Remark The definition of uniformly elliptic above implies uniform ellipticity in the sense of global bounds on the coefficients of the symbol.

The operators \( d + d^* \) and the Laplacian are uniformly elliptic AC operators.

We now turn to regularity results for smooth uniformly elliptic AC operators.

Theorem 4.15 Let \( V \) and \( W \) be bundles of forms on \( N \) and let \( \mathcal{P} \) be a smooth uniformly elliptic AC operator from \( V \) to \( W \) of order \( l \) and rate \( \nu \), in the sense of Definitions 4.13 and 4.14. Let \( p > 1, k \in \mathbb{N}, a \in (0, 1) \) and \( \mu \in \mathbb{R} \).

(a) Suppose that \( \mathcal{P} \xi = \eta \) holds for \( \xi \in L^1_{1, \text{loc}}(V) \) and \( \eta \in L^1_{0, \text{loc}}(W) \). If \( \xi \in L^p_{0, \mu}(V) \) and \( \eta \in L^p_{k, \mu - \nu}(W) \), then \( \xi \in L^p_{k+1, \mu}(V) \) and

\[
\|\xi\|_{L^p_{k+1, \mu}} \leq c \left( \|\eta\|_{L^p_{k, \mu - \nu}} + \|\xi\|_{L^p_{0, \mu}} \right)
\]

for some constant \( c > 0 \) independent of \( \xi \) and \( \eta \).

(b) Suppose that \( \mathcal{P} \xi = \eta \) holds for \( \xi \in C^1_{\text{loc}}(V) \) and \( \eta \in C^0_{\text{loc}}(W) \). If \( \xi \in C^k_{\mu}(V) \) and \( \eta \in C^{k+1, a}_{\mu - \nu}(W) \), then \( \xi \in C^{k+1, a}(V) \) and

\[
\|\xi\|_{C^{k+1, a}} \leq c' \left( \|\eta\|_{C^{k, a}_{\mu - \nu}} + \|\xi\|_{C^k_{\mu}} \right)
\]

for some constant \( c' > 0 \) independent of \( \xi \) and \( \eta \). Moreover, these estimates hold if the coefficients of \( \mathcal{P} \) only lie in \( C^k_{\text{loc}} \).
These results may be deduced from those given in [21 §6.1.1] or [22].

By taking $\eta = 0$ in Theorem 4.15(b), we have the following useful corollary.

**Corollary 4.16** In the notation of Theorem 4.15, if $\xi \in C^r_{\mu}(V)$ satisfies $\mathcal{P}\xi = 0$ then $\xi \in C^r_{\mu}(V)$.

We conclude by achieving the aim of this subsection.

**Proposition 4.17** Use the notation of Definition 4.9. Let $(\alpha, \beta) \in L^p_{k+1, \mu}(U) \times L^p_{k+1, \mu}(\Lambda^4 T^* N)$ for $p > 4$, $k \geq 2$ and $\mu < 1$. If $G(\alpha, \beta) = 0$ and $\|\alpha\|_{C^1}$ is sufficiently small, $(\alpha, \beta) \in C^\infty_{(U) \times C^\infty_{(\Lambda^4 T^* N)}$.

*Proof:* Notice first that $\alpha$ and $\beta$ lie in $C^2_{\mu}$ by Theorem 3.5 since $k + 1 - \frac{4}{p} > 2$.

As noted in the proof of Proposition 4.10, $G(\alpha, \beta) = 0$ implies that $\Delta \beta = 0$. Hence, by Corollary 4.16, $\beta \in C^\infty(\Lambda^4 T^* N)$.

For the following argument we find it useful to work with weighted Hölder spaces, defined in Definition 3.4. By Theorem 3.5, $\alpha \in C^{\mu, a}_0(U)$ with $a = 1 - 4/p \in (0, 1)$ since $p > 4$. Let $\pi_{\Lambda^2_*}$ be the projection from 2-forms to self-dual 2-forms on $N$. We know that $d^* \left( G(\alpha, \beta) \right) = d^* (d^* (F(\alpha))) = 0$ and also that

$$\tilde{F}(\alpha) = \pi_{\Lambda^2_*} \left( d^* (F(\alpha)) \right) = 0$$

is a nonlinear elliptic equation at 0, meaning that $d\tilde{F}|_0$ is an elliptic operator.

We can write $\tilde{F}$ as

$$\tilde{F}(\alpha)(x) = R_F (x, \alpha(x), \nabla \alpha(x)) \nabla^2 \alpha(x) + E_F (x, \alpha(x), \nabla \alpha(x)),$$

where $R_F$ and $E_F$ are smooth functions of their arguments, since $\tilde{F}(\alpha)$ is linear in $\nabla^2 \alpha$ with coefficients depending on $\alpha$ and $\nabla \alpha$. Define

$$S_\alpha(\gamma)(x) = R_F (x, \alpha(x), \nabla \alpha(x)) \nabla^2 \gamma(x)$$

for $\gamma \in C^{2}_{\text{loc}}(\Lambda^2_+ T^* N)$. Note that $S_\alpha$ is not the linearisation of $\tilde{F}$. Then $S_\alpha$ is a linear uniformly elliptic second order AC operator with rate 2, if $\|\alpha\|_{C^1}$ is sufficiently small, whose coefficients depend on $x$, $\alpha(x)$ and $\nabla \alpha(x)$. These coefficients therefore lie in $C^{\mu, a}_{\text{loc}}$.

Recall the notation and results of Proposition 4.8 and that $d^*$ is an AC operator with rate 1 in the sense of Definition 4.13. Thus, $d^* d\alpha + d^* (P_F(\alpha)) = 0$ and $d^* (P_F(\alpha)) \in C^{\kappa - \frac{2}{3}, a}_{2\mu - 3}(\Lambda^2 T^* N)$. Therefore,

$$S_\alpha(\alpha)(x) = -E_F (x, \alpha(x), \nabla \alpha(x)) \in C^{\kappa - \frac{2}{3}, a}_{2\mu - 3}(\Lambda^2 T^* N) \subseteq C^{\kappa - \frac{2}{3}, a}_{\mu - 2}(\Lambda^2 T^* N),$$

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since $\mu < 1$. However, $E_F(x, \alpha(x), \nabla\alpha(x))$ only depends on $\alpha$ and $\nabla\alpha$, and is at worst quadratic in these quantities by Proposition 4.8 so it must in fact lie in $C^{k-1}_{\mu-2}(\Lambda^2 T^* N)$ since we are given control on the decay of the first $k$ derivatives of $\alpha$ at infinity.

As $\alpha \in C^2_{\mu}(\Lambda^2 T^* N)$ and $S_\alpha(\alpha) \in C^{k-1}_{\mu-2}(\Lambda^2 T^* N)$, Theorem 4.13(b) implies that $\alpha \in C^{k+1}_{\mu}(\Lambda^2 T^* N)$. Therefore we have proved $\alpha \in C^{k+1}_{\mu}(\Lambda^2 T^* N)$ only knowing a priori that $\alpha \in C^k_{\mu}(\Lambda^2 T^* N)$. We proceed by induction to show that $\alpha \in C^l_{\mu}(\Lambda^2 T^* N)$ for all $l \geq 2$. The result follows from the elementary observation made at the end of Definition 3.4.

□

Taking $\beta = 0$ in Proposition 4.17 gives our main regularity result.

**Corollary 4.18** Use the notation of Definition 4.6. Let $\alpha \in L^p_{k+1, \mu}(U)$ for $p > 4$, $k \geq 2$ and $\mu < 1$. If $F(\alpha) = 0$ and $\|\alpha\|_{C^1_1}$ is sufficiently small, $\alpha \in C^\infty_{\mu}(U)$.

**5 The map $d + d^*$ and the exceptional set $D$**

We begin by defining the map of interest.

**Definition 5.1** Let $p > 4$, $k \geq 2$ and $\mu < 1$. Define the linear elliptic operator

$$ (d + d^*)_\mu : L^p_{k+1, \mu}(\Lambda^2 T^* N \oplus \Lambda^4 T^* N) \to L^p_{k, \mu-1}(\Lambda^3 T^* N) \quad (19) $$

by $(d + d^*)_\mu(\alpha, \beta) = d\alpha + d^*\beta$. Let $K(\mu) = \text{Ker}(d + d^*)_\mu$.

**Remark** The operator $(d + d^*)_\mu$ acts between the weighted Banach spaces claimed because it is AC with rate 1.

We now make an important observation.

**Lemma 5.2** Use the notation from Proposition 4.17 and Definitions 4.6 and 5.1. By making the open set $U$ smaller in $C^1_1$ if necessary,

$$ G : L^p_{k+1, \mu}(U) \times L^p_{k+1, \mu}(\Lambda^4 T^* N) \to L^p_{k, \mu}(\Lambda^3 T^* N). \quad (20) $$

Moreover, the linearisation of $(20)$ at $(0, 0)$ acts as $(d + d^*)_\mu$, as in $(19)$.

**Proof** By Proposition 4.8 and the fact that $d$ and $d^*$ are AC operators with rate 1 on $N$, we see that $G$ maps $(\alpha, \beta) \in L^p_{k+1, \mu}(U) \times L^p_{k+1, \mu}(\Lambda^4 T^* N)$ into $L^p_{k, \mu-1}(\Lambda^3 T^* N)$ if $\|\alpha\|_{C^1_1}$ is sufficiently small. This bound on the norm of $\alpha$ can be ensured by making the $C^1_1$-open set $U$ smaller. The description of the linearisation follows from the observations in Definition 4.9. □
It is clear from Propositions 4.7 and 4.10 that the kernel of the nonlinear map \( \mu = \lambda \), for \( \mu = \lambda \), is intimately linked with the moduli space \( \mathcal{M}(N, \lambda) \). Therefore, locally, one would expect the kernel \( \mathcal{K}(\lambda) \) of the linear map \( (d + d^*) \lambda \), given by \( \mathcal{K}(\lambda) \), to be related to \( \mathcal{M}(N, \lambda) \) as well by Lemma 5.2. In fact, we shall see that \( \mathcal{K}(\lambda) \) is directly connected with the infinitesimal deformations of \( N \).

5.1 Fredholm and index theory

We want to understand the Fredholm theory of \( (19) \) and so state a result adapted from \([16, Theorem 1.1 & Theorem 6.1]\).

**Theorem 5.3** Let \( V \) and \( W \) be bundles of forms over \( N \), let \( p > 1 \), let \( \mu, \nu \in \mathbb{R} \) and let \( k, l \in \mathbb{N} \). Let \( \mathcal{P} : L^p_{k+1, \mu}(V) \to L^p_{k, \mu-\nu}(W) \) be a uniformly elliptic AC operator of order \( l \) and rate \( \nu \) in the sense of Definitions 4.13-4.14. There exists a countable discrete set \( D(\mathcal{P}) \subseteq \mathbb{R} \), depending only on \( \mathcal{P} \) as in Definition 4.13, such that \( \mathcal{P} \) is Fredholm if and only if \( \mu \notin D(\mathcal{P}) \).

From this we know that, for each \( m \in \mathbb{N} \) with \( m \leq 4 \), there exists a countable discrete subset \( D(\Delta^m) \) of \( \mathbb{R} \) such that the Laplacian on \( \Lambda^m N \) is Fredholm if and only if \( \mu + 1 \notin D(\Delta^m) \). Thus, \((d + d^*)_\mu \) is Fredholm if \( \mu \notin (D(\Delta^2) \cup D(\Delta^4)) \). However, we can give an explicit description of the set \( D \) for which \( (19) \) is not Fredholm, following \([21, \S 6.1.2]\).

Recall the notation from Definitions 4.11-4.13 and consider the maps \( d \) and \( d^* \) acting on \( m \)-forms on \( N \). These are asymptotically conical with rate 1, so the related asymptotically cylindrical operators \( d^1 \) and \( (d^*)^1 \) are given by

\[
    d^1 = \rho^{-m} \, dp^m \quad \text{and} \quad (d^*)^1 = \rho^{-m+2} \, d^* \rho^m.
\]

Notice that \( \rho \) is asymptotic to \( r = e^t \), where the cylindrical coordinates \((t, \sigma)\) on the ends \( N_\infty \cong (T, \infty) \times \Sigma \) of \( N \) were introduced in Definition 4.11. Thus, we can take the cylindrical operator \( (d + d^*)_\infty \) associated to \( d + d^* \) to be

\[
    (d + d^*)_\infty = e^{-mt} (d + e^{2t} d^*) e^{mt}, \quad (21)
\]

acting on \( m \)-forms on the ends of \( N \).

Since \( N_\infty \cong (T, \infty) \times \Sigma \), an \( m \)-form \( \alpha \) on \( N_\infty \) can be written as

\[
    \alpha(t, \sigma) = \beta(t, \sigma) + dt \wedge \gamma(t, \sigma),
\]

where, for each fixed \( t \in (T, \infty) \), \( \beta(t, \sigma) \) and \( \gamma(t, \sigma) \) are \( m \)- and \((m-1)\)-forms on \( \Sigma \) respectively. Therefore, if \( \pi : (0, \infty) \times \Sigma \to \Sigma \) is the natural projection,
\[ \Lambda^m T^* N_\infty \cong \pi^*(\Lambda^m T^* \Sigma) \oplus \pi^*(\Lambda^m T^* \Sigma) \] Hence, \((d + d')_\infty\) maps sections of \(\pi^*(\Lambda^2 T^* \Sigma) \oplus \pi^*(\Lambda^2 T^* \Sigma)\) to sections of \(\pi^*(\Lambda^{odd} T^* \Sigma) \oplus \pi^*(\Lambda^{even} T^* \Sigma)\). Moreover, this action is given by:

\[
(d + d')_\infty \begin{pmatrix}
\alpha(t, \sigma) \\
\beta(t, \sigma) + \gamma(t, \sigma)
\end{pmatrix} = \begin{pmatrix}
d_\Sigma + d_\Sigma^* & -(\frac{\partial}{\partial t} + 3 - m) \\
\frac{\partial}{\partial t} + m & -(d_\Sigma + d_\Sigma^*)
\end{pmatrix} \begin{pmatrix}
\alpha(t, \sigma) \\
\beta(t, \sigma) + \gamma(t, \sigma)
\end{pmatrix}, \tag{22}
\]

where \(m\) denotes the operator which multiplies \(m\)-forms by a factor \(m\), and \(d_\Sigma\) and \(d_\Sigma^*\) are the exterior derivative and its formal adjoint on \(\Sigma\).

However, we wish only to consider elements of \(\Lambda^1 T^* \Sigma \oplus \Lambda^2 T^* \Sigma\) which correspond, via \(\pi^*\), to self-dual 2-forms on \(N_\infty\). Thus we define \(V_\Sigma \subseteq \Lambda^2 T^* \Sigma \oplus \Lambda^{odd} T^* \Sigma\) by

\[ V_\Sigma = \{ (\alpha, \ast_\Sigma \alpha + \beta) : \alpha \in \Lambda^2 T^* \Sigma, \beta \in \Lambda^3 T^* \Sigma \}, \tag{23} \]

where \(\ast_\Sigma\) is the Hodge star on \(\Sigma\). Then \(\pi^*(V_\Sigma) \cong \Lambda_\Sigma^2 T^* N_\infty \oplus \Lambda_\Sigma^3 T^* N_\infty\). We also want to project the image of \((d + d')_\infty\) to 3-forms on \(N_\infty\), so we let

\[ W_\Sigma = \{ (\beta, \alpha) : \alpha \in \Lambda^2 T^* \Sigma, \beta \in \Lambda^3 T^* \Sigma \}\]

and let \(\pi_{W_\Sigma}\) be the projection to \(W_\Sigma\). Note that \(\pi^*(W_\Sigma) \cong \Lambda^3 T^* N_\infty\).

For \(w \in C\), define a map \((d + d')_\infty(w) : C_{loc}(V_\Sigma \otimes C) \rightarrow C_{loc}(W_\Sigma \otimes C)\) by:

\[
(d + d')_\infty(w) \begin{pmatrix}
\alpha(\sigma) \\
\ast_\Sigma \alpha(\sigma) + \beta(\sigma)
\end{pmatrix} = \pi_{W_\Sigma} \circ \begin{pmatrix}
d_\Sigma + d_\Sigma^* & -(w + 3 - m) \\
w + m & -(d_\Sigma + d_\Sigma^*)
\end{pmatrix} \begin{pmatrix}
\alpha(\sigma) \\
\ast_\Sigma \alpha(\sigma) + \beta(\sigma)
\end{pmatrix}. \tag{24}
\]

Notice that we have formally substituted \(w\) for \(\frac{\partial}{\partial t}\) in \((22)\).

Let \(p > 4\) and \(k \geq 2\) as in \([16]\), noting that \(L^p_{k+1} \rightarrow C^2\) on \(\Sigma\) by the Sobolev Embedding Theorem. Define \(C \subseteq C\) as the set of \(w\) for which the map

\[ (d + d')_\infty(w) : L^p_{k+1}(V_\Sigma \otimes C) \rightarrow L^p_k(W_\Sigma \otimes C) \tag{25} \]

is not an isomorphism. By the proof of \([16\) Theorem 1.1], \(D = \{ \Re w : w \in C\}\). In fact, \(C \subseteq \mathbb{R}\) by \([21\ Lemma 6.1.13]\), which shows that the corresponding sets \(C(\Delta^m)\) are all real. Hence \(C = D\).

The symbol, hence the index \(\text{ind}_w\), of \((d + d')_\infty(w)\) is independent of \(w\). Furthermore, \((d + d')_\infty(w)\) is an isomorphism for generic values of \(w\) since \(D\) is countable and discrete. Therefore \(\text{ind}_w = 0\) for all \(w \in C\); that is,

\[ \dim \ker(d + d')_\infty(w) = \dim \text{coker}(d + d')_\infty(w), \]

where \(\text{coker}(d + d')_\infty(w)\) is the cokernel of \((d + d')_\infty(w)\).
so that \(25\) is not an isomorphism precisely when it is not injective.

The condition \((d + d^*)_\infty(w) = 0\), using \(24\) and elliptic regularity, corresponds to the existence of \(\alpha \in C^\infty(\Lambda^2 T^*\Sigma)\) and \(\beta \in C^\infty(\Lambda^3 T^*\Sigma)\) satisfying

\[
d_\Sigma \alpha = w\beta \quad \text{and} \quad d_\Sigma *_\Sigma \alpha + d^*_\Sigma \beta = (w + 2)\alpha.
\]

We first note that \(26\) implies that

\[
d_\Sigma d^*_\Sigma \beta = \Delta_\Sigma \beta = w(w + 2)\beta.
\]

(27)

Since eigenvalues of the Laplacian on \(\Sigma\) are positive, \(\beta = 0\) if \(w \in (-2, 0)\). If \(w = 0\) and we take \(\alpha = 0\), \(26\) forces \(\beta\) to be coclosed. As there are nontrivial coclosed 3-forms on \(\Sigma\), \((d + d^*)_\infty(0)\) is not injective and hence \(0 \in \mathcal{D}\). Suppose that \(w = -2\) lies in \(\mathcal{D}\). Then \(26\) gives \([\beta] = 0\) in \(H^3_{dR}(\Sigma)\). We know that \(\beta\) is harmonic from \(27\) so, by Hodge theory, \(\beta = 0\). Therefore \(-2 \in \mathcal{D}\) if and only if there exists a nonzero closed and coclosed 2-form on \(\Sigma\).

We state a proposition which follows from the work above.

**Proposition 5.4** Recall the definition of \(\Sigma\) from the start of §4 and denote the Hodge star, the exterior derivative and its formal adjoint on \(\Sigma\) by \(*_\Sigma\), \(d_\Sigma\) and \(d^*_\Sigma\) respectively. Let

\[
D(\mu) = \{ (\alpha, \beta) \in C^\infty(\Lambda^2 T^*\Sigma \oplus \Lambda^3 T^*\Sigma) : d_\Sigma \alpha = \mu \beta, \ d_\Sigma *_\Sigma \alpha + d^*_\Sigma \beta = (\mu + 2)\alpha \}. 
\]

The countable discrete set \(\mathcal{D}\) of \(\mu \in \mathbb{R}\) for which \((d + d^*)_\infty(\mu)\), given in \(19\), is not Fredholm is given by:

\[
\mathcal{D} = \{ \mu \in \mathbb{R} : D(\mu) \neq 0 \}.
\]

Moreover, \(-2 \in \mathcal{D}\) if and only if \(b^1(\Sigma) > 0\), and \(0 \in \mathcal{D}\).

A perhaps more illuminating way to characterise \(D(\mu)\) is by:

\[
(\alpha, \beta) \in D(\mu) \iff \xi = (r^{\mu+2}\alpha + r^{\mu+1} dr \wedge *_\Sigma \alpha, r^{\mu+3} dr \wedge \beta) \in C^\infty(\Lambda^2_T T^* C \oplus \Lambda^3 T^* C) \\
\text{is an } O(r^\mu) \text{ solution of } (d + d^*)_\infty(\xi) = 0 \text{ in } C^\infty(\Lambda^3 T^* C) \text{ as } r \to \infty.
\]

We now make a definition as in [16].

**Definition 5.5** Recall the map \((d + d^*)_\infty\) defined by \(22\), the bundle \(V_\Sigma\) given in \(23\) and the set \(\mathcal{D}\) given in Proposition \(5.4\). Let \(\mu \in \mathcal{D}\). We define \(d(\mu)\) to be the dimension of the vector space of solutions of \((d + d^*)_\infty(\xi) = 0\) of the form

\[
\xi(t, \sigma) = e^{\mu t} p(t, \sigma),
\]

where \(p(t, \sigma)\) is a polynomial in \(t \in (T, \infty)\) with coefficients in \(C^\infty(V_\Sigma \otimes \mathbb{C})\).

The next result is immediate from [16, Theorem 1.2].
Proposition 5.6 Use the notation of Proposition 5.4 and Definition 5.5. Let \( \lambda, \lambda' \notin \mathcal{D} \) with \( \lambda' \leq \lambda \). For any \( \mu \notin \mathcal{D} \) let \( i_\mu(d + d^*) \) denote the Fredholm index of the map \((19)\). Then

\[
i_\lambda(d + d^*) - i_{\lambda'}(d + d^*) = \sum_{\mu \in \mathcal{D} \cap (\lambda', \lambda)} d(\mu).
\]

Note A similar result holds for any uniformly elliptic AC operator on \( N \).

We make a key observation, which shall be used on a number of occasions later.

Proposition 5.7 Use the notation of Theorem 5.3. Let \( \lambda, \lambda' \in \mathbb{R} \) be such that \( \lambda' \leq \lambda \) and \( [\lambda', \lambda] \cap \mathcal{D}(\mathcal{P}) = \emptyset \). The kernels, and cokernels, of \( \mathcal{P} : L^p_{k+i, \mu}(V) \to L^p_{k, \mu-\nu}(W) \) when \( \mu = \lambda \) and \( \mu = \lambda' \) are equal.

Proof: Denote the dimensions of the kernel and cokernel of \( \mathcal{P} : L^p_{k+i, \mu}(V) \to L^p_{k, \mu-\nu}(W) \), for \( \mu \notin \mathcal{D}(\mathcal{P}) \), by \( k(\mu) \) and \( c(\mu) \) respectively. Notice that these dimensions are finite since \( \mathcal{P} \) is Fredholm if \( \mu \notin \mathcal{D}(\mathcal{P}) \). Since \( [\lambda', \lambda] \cap \mathcal{D}(\mathcal{P}) = \emptyset \),

\[
k(\lambda) - c(\lambda) = k(\lambda') - c(\lambda') \quad (28)
\]

We know that \( k(\lambda) \geq k(\lambda') \) because \( L^p_{k+1, \lambda'} \to L^p_{k+1, \lambda} \) by Theorem 3.5(a) as \( \lambda \geq \lambda' \). Similarly, since \( c(\mu) \) is equal to the dimension of the kernel of the formal adjoint operator acting on the dual Sobolev space with weight \( -4 - (\mu - \nu) \) (as noted in Definition 3.2), \( c(\lambda) \leq c(\lambda') \). Since the left-hand side of \((28)\) is non-negative and the right-hand side is non-positive, we conclude that both must be zero. The result follows from the fact that the kernel of \( \mathcal{P} \) in \( L^p_{k+1, \lambda'} \) is contained in the kernel of \( \mathcal{P} \) in \( L^p_{k+1, \lambda} \), and vice versa for the cokernels. \( \square \)

We conclude with an explicit description of the quantity \( d(\mu) \) for \( \mu \in \mathcal{D} \). This result, as can be seen from the proof, is similar to [5, Proposition 2.4].

Proposition 5.8 In the notation of Proposition 5.4 and Definition 5.5, \( d(\mu) = \dim D(\mu) \) for \( \mu \in \mathcal{D} \).

Proof: Use the notation of Proposition 5.4 and the work preceding it and Definition 5.5. Let \( p(t, \sigma) \) be a polynomial in \( t \in (T, \infty) \) of degree \( m \) written as

\[
p(t, \sigma) = \left( \sum_{j=0}^{m} p_j(\sigma) t^j, \sum_{j=0}^{m} (\ast \Sigma p_j(\sigma) + q_j(\sigma)) t^j \right).
\]
where $p_j \in C^\infty(\Lambda^2 T^* \Sigma)$ and $q_j \in C^\infty(\Lambda^3 T^* \Sigma)$ for $j = 0, \ldots, m$, with $p_m$ and $q_m$ not both zero, and let $\xi(t, \sigma) = e^{\mu t} p(t, \sigma)$ as in Definition 5.5. We want to find the dimension $d(\mu)$ of the space of $\xi$ such that $(d + d^*)_\infty \xi = 0$.

Using (22), $(d + d^*)_\infty \xi = 0$ is equivalent to

$$\sum_{j=0}^{m} j t^j (d_\Sigma p_j - \mu q_j) - \sum_{j=0}^{m} j t^j - 1 q_j = 0 \quad \text{and} \quad \tag{29}$$

$$\sum_{j=0}^{m} j t^j ((\mu + 2) p_j - d_\Sigma^* \Sigma p_j - d_\Sigma^* q_j) + \sum_{j=0}^{m} j t^j - 1 p_j = 0. \quad \tag{30}$$

Comparing coefficients of $t^m$ we deduce that $(p_m, q_m) \in D(\mu)$.

Suppose, for a contradiction, that $m \geq 1$. Comparing coefficients of $t^{m-1}$ in (29) and (30):

$$d_\Sigma p_{m-1} - \mu q_{m-1} = mq_m \quad \text{and} \quad d_\Sigma^* \Sigma p_{m-1} + d_\Sigma^* q_{m-1} - (\mu + 2) p_{m-1} = mp_m. \quad \tag{31}$$

We then compute using (31) and the fact that $(p_m, q_m) \in D(\mu)$:

$$m\langle p_m, p_m \rangle_{L^2} = \langle p_m, d_\Sigma^* \Sigma p_{m-1} + d_\Sigma^* q_{m-1} - (\mu + 2) p_{m-1} \rangle_{L^2} = \langle d_\Sigma^* \Sigma p_m - (\mu + 2)p_m, p_m \rangle_{L^2} + \langle d_\Sigma^* p_m, q_{m-1} \rangle_{L^2} = -\langle q_m, d_\Sigma p_{m-1} - \mu q_{m-1} \rangle_{L^2} = -m\langle q_m, q_m \rangle_{L^2}.$$  

Hence,

$$m\langle p_m \parallel p_m \parallel^2_{L^2} + \parallel q_m \parallel^2_{L^2} \rangle = 0$$

and so $p_m = q_m = 0$, our required contradiction.

Thus, the solutions $\xi$ must be of the from

$$\xi(t, \sigma) = e^{\mu t} p(t, \sigma) = e^{\mu t} (p_0(\sigma), *_{\Sigma} p_0(\sigma) + q_0(\sigma))$$

for $(p_0, q_0) \in D(\mu)$. The proposition follows. $\square$

### 5.2 The image of $d + d^*$

We remarked earlier upon the connection of the kernel of $(d + d^*)_\mu$ with the infinitesimal deformations of $N$. This suggests that the cokernel of $(d + d^*)_\mu$ is related to the obstruction theory for the deformation problem. When $(d + d^*)_\mu$
is Fredholm, the cokernel is isomorphic, via the dual pairing given in Definition 3.2, to the kernel of the adjoint map which we now define.

**Definition 5.9** Let $p > 4$, $k \geq 2$ and $\mu < 1$. Let $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $l \geq 2$. The adjoint map to (19) is given by

$$(d^*_+ + d)_{\mu} : L^q_{l+1, -3-\mu}(\Lambda^{3} T^* N) \to L^q_{l, -4-\mu}(\Lambda^2_+ T^* N \oplus \Lambda^{3} T^* N),$$

where $(d^*_+ + d)_{\mu}(\gamma) = d^*_+ \gamma + d\gamma$ and $d^*_+ = \frac{1}{2}(d^* + d^*)$. Let $C_+_{(\mu)} = \text{Ker}(d^*_+ + d)_{\mu}$.

It is straightforward to see, using integration by parts and the dual pairing, that $(d^*_+ + d)_{\mu}$ is the formal adjoint to $(d + d^*)_{\mu}$.

**Remark** The space $C_+_{(\mu)}$ is the kernel of an elliptic map so, for $\mu \notin \mathcal{D}$, it is a finite-dimensional space of smooth forms by Corollary 4.16. Thus, $C_+_{(\mu)}$ is independent of $l$ and we can choose $l \geq 2$ for use later.

The aim of this subsection is to prove the following.

**Theorem 5.10** Use the notation of Proposition 5.4 and Definition 5.9. Suppose further that $\mu \in (-\infty, 1) \setminus \mathcal{D}$ and let

$$C(\mu) = \{\gamma \in L^q_{l+1, -3-\mu}(\Lambda^{3} T^* N) : d\gamma = d^* \gamma = 0\}.$$  \hspace{1cm} (33)

There exist finite-dimensional subspaces $\tilde{C}(\mu)$ and $\mathcal{O}(N, \mu)$ of $L^p_{k, \mu-1}(\Lambda^{3} T^* N)$ such that

$$L^p_{k, \mu-1}(\Lambda^{3} T^* N) = \left(\text{d}(L^p_{k+1, \mu}(\Lambda^{2} T^* N)) + \text{d}^*(L^p_{k+1, \mu}(\Lambda^{3} T^* N))\right) \oplus \tilde{C}(\mu) \hspace{1cm} (34)$$

and

$$d(L^p_{k+1, \mu}(\Lambda^{2} T^* N)) = d(L^p_{k+1, \mu}(\Lambda^{2}_+ T^* N)) \oplus \mathcal{O}(N, \mu). \hspace{1cm} (35)$$

Moreover, $\tilde{C}(\mu) \cong C(\mu)$ and $\tilde{C}_+_{(\mu)} = \tilde{C}(\mu) \oplus \mathcal{O}(N, \mu) \cong C_+_{(\mu)}$ via the dual pairing. Furthermore:

(a) if $\mu < -2$, the sum in (34) is a direct sum;

(b) if $\mu \in [-2, 0)$, $\mathcal{O}(N, \mu) = 0$ and the sum in (34) is a direct sum;

(c) if $\mu \in [0, 1)$, $\mathcal{O}(N, \mu) = 0$ but the sum in (34) is not necessarily direct.

Before proving the theorem we make an elementary observation.

**Lemma 5.11** If $\mu < 0$, $d(L^p_{k+1, \mu}(\Lambda^{2} T^* N)) \cap d^*(L^p_{k+1, \mu}(\Lambda^{2} T^* N)) = \{0\}$.  \hspace{1cm} (32)
Proof: If $\beta \in L^p_{k+1,\mu}(\Lambda^4 T^* N)$ such that $d^* \beta$ is exact, then $dd^* \beta = 0$. Therefore $* \beta$ is a harmonic function which is $o(\rho^\mu)$ as $\rho \to \infty$. Applying the Maximum Principle, $* \beta = 0$.\hfill $\square$

Proof of Theorem 5.10(a). The key to proving this part of the theorem lies with comparing the image of $(d + d^*)$ with the image of the map

$$d + d^* : L^p_{k+1,\mu}(\Lambda^2 T^* N \oplus \Lambda^4 T^* N) \to L^p_{k,\mu-1}(\Lambda^3 T^* N)$$

given by $(d + d^*)(\alpha, \beta) = d\alpha + d^* \beta$. Clearly, there exists a finite-dimensional subspace $\tilde{C}(\mu)$ of $L^p_{k,\mu-1}(\Lambda^3 T^* N)$, which is isomorphic (via the dual pairing) to the annihilator of the image of (36), such that

$$L^p_{k,\mu-1}(\Lambda^3 T^* N) = \left( d(L^p_{k+1,\mu}(\Lambda^2 T^* N)) \oplus d^*(L^p_{k+1,\mu}(\Lambda^4 T^* N)) \right) \oplus \tilde{C}(\mu).$$

(The direct sum follows from Lemma 5.11 as $\mu < -2$.)

Now, the annihilator $\mathcal{A}(\mu)$ of the image of (36) is given by:

$$\mathcal{A}(\mu) = \{ \gamma \in L^p_{k+1,-3-\mu}(\Lambda^3 T^* N) : \langle \gamma, d\alpha + d^* \beta \rangle = 0 \}$$

for all $(\alpha, \beta) \in L^p_{k+1,\mu}(\Lambda^2 T^* N \oplus \Lambda^4 T^* N)$.

Therefore, using integration by parts (justified by the choice of weight for the dual Sobolev space), we deduce that $\mathcal{A}(\mu) = \mathcal{C}(\mu)$.

Using the Fredholmness of $(d + d^*)$, since $\mu \notin D$, there exists a finite-dimensional subspace $\tilde{C}_+(\mu)$ of $L^p_{k,\mu-1}(\Lambda^3 T^* N)$ such that

$$L^p_{k,\mu-1}(\Lambda^3 T^* N) = d(L^p_{k+1,\mu}(\Lambda_+^2 T^* N)) \oplus d^*(L^p_{k+1,\mu}(\Lambda^4 T^* N)) \oplus \tilde{C}_+(\mu).$$

Moreover, $\tilde{C}_+(\mu)$ is isomorphic to the annihilator of the image of (19), which is equal to $\mathcal{C}_+(\mu)$ as given in Definition 5.10 (again using integration by parts).

Equation (38) allows us to deduce that $d^*(L^p_{k+1,\mu}(\Lambda^3 T^* N))$ is closed. Hence, Lemma 5.11 allows us to deduce that

$$d(L^p_{k+1,\mu}(\Lambda_+^2 T^* N)) \cap d^*(L^p_{k+1,\mu}(\Lambda^4 T^* N)) = \{0\}.$$

Comparing (37) and (38) we see that there exists a finite-dimensional space $O(N, \mu)$ such that $\tilde{C}_+(\mu) = \tilde{C}(\mu) \oplus O(N, \mu)$, from which (a) follows.\hfill $\square$

We shall require some preliminary technical results before resuming our proof of Theorem 5.10. We consider the maps

$$d : L^p_{k+1,\mu}(\Lambda_+^2 T^* N) \to L^p_{k,\mu-1}(\Lambda^3 T^* N) \quad \text{and}$$

$$d : L^p_{k+1,\mu}(\Lambda^2 T^* N) \to L^p_{k,\mu-1}(\Lambda^3 T^* N).$$

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**Proposition 5.12** Let \( p > 4, k \geq 2 \) and \( \mu \in (-\infty, 1) \setminus \mathcal{D} \), where \( \mathcal{D} \) is given in Proposition 5.4. Let the annihilators of the images of (39) and (40) be \( \mathcal{A}_+(\mu) \) and \( \mathcal{A}(\mu) \) respectively. Then, if \( q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( l \geq 2 \),
\[
\mathcal{A}_+(\mu) = \{ \gamma \in L^q_{l+1,-3-\mu}(\Lambda^3 T^* N) : d^* \gamma \in L^q_{l,-4-\mu}(\Lambda^2 T^* N) \} \tag{41}
\]
and
\[
\mathcal{A}(\mu) = \{ \gamma \in L^q_{l+1,-3-\mu}(\Lambda^3 T^* N) : d^* \gamma = 0 \}. \tag{42}
\]
Furthermore, if \( \mu \in [-2, 1) \setminus \mathcal{D} \), \( \mathcal{A}_+(\mu) = \mathcal{A}(\mu) \).

**Proof:** The formulae (41) and (42) are easily deduced using the dual pairing and integration by parts. Clearly \( \mathcal{A}(\mu) \subseteq \mathcal{A}_+(\mu) \), so suppose \( \gamma \in \mathcal{A}_+(\mu) \) and \( \mu \geq -2 \). Notice that \( p > 4 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \) force \( q \in (1, \frac{1}{2}) \), and \( \mu \geq -2 \) implies that \( -4 - \mu \leq -2 \). Therefore, by Theorem 5.10(a), \( L^q_{l,-4-\mu} \hookrightarrow L^2_{0,-2} = L^2 \), recalling (3). Thus,
\[
\|d^* \gamma\|_{L^2_N}^2 = \int_N -d^* \gamma \wedge d^* \gamma = \int_N -d^* \gamma \wedge d^* \gamma = \int_N -d(\ast \gamma \wedge d\ast \gamma) = 0.
\]
The integration by parts is valid since \( \ast \gamma = o(\rho^{-3-\mu}) \) and \( d\ast \gamma = o(\rho^{-4-\mu}) \) as \( \rho \to \infty \), and \( -7 - 2\mu \leq -3 \). The proof is thus complete. \( \square \)

Using this proposition we can deduce the following invaluable result.

**Proposition 5.13** Recall the definition of \( \mathcal{D} \) in Proposition 5.4. For \( p > 4 \), \( k \geq 2 \) and \( \mu \in [-2, 1) \setminus \mathcal{D} \),
\[
d\left( L^p_{k+1,\mu}(\Lambda^2 T^* N) \right) = d\left( L^p_{k+1,\mu}(\Lambda^2 T^* N) \right) = d\left( L^p_{k+1,\mu}(\Lambda^2 T^* N) \right).
\]

**Proof:** By Proposition 5.12 the annihilators of the images of (39) and (40) are equal. We deduce that the closure of the ranges of (39) and (40) are equal. However, we know that \( \mu \notin \mathcal{D} \) so that (19) is Fredholm, which means that it has closed range. Therefore, (39) has closed range and hence
\[
d\left( L^p_{k+1,\mu}(\Lambda^2 T^* N) \right) = d\left( L^p_{k+1,\mu}(\Lambda^2 T^* N) \right).
\]
The left-hand side of this equation is contained in the image of (40), whereas the right-hand side contains the image of (40). The result follows. \( \square \)

**Proof of Theorem 5.17(b)-(c).** If \( \mu \in [-2, 1) \setminus \mathcal{D} \), (19) is Fredholm, so there exists a finite-dimensional subspace \( \mathcal{C}(\mu) \) of \( L^p_{k,\mu-1}(\Lambda^3 T^* N) \) such that
\[
L^p_{k,\mu-1}(\Lambda^3 T^* N) = \left( d\left( L^p_{k+1,\mu}(\Lambda^2 T^* N) \right) + d^* \left( L^p_{k+1,\mu}(\Lambda^3 T^* N) \right) \right) \oplus \mathcal{C}(\mu).
\]
The results now follow from Lemma 5.11 and Proposition 5.13. \( \square \)
5.3 The kernel of the adjoint map

We already remarked that studying the kernel of the adjoint map will help us to understand the obstructions to deformations of $N$. For this subsection we use the notation of Definition 5.9 and Proposition 5.12. Suppose further that $\mu \in [-2,1) \setminus D$, where $D$ is given in Proposition 5.4.

Let $\gamma \in A_+(\mu)$. By Proposition 5.12 $*\gamma \in L_{l+1,-3-\mu}(\Lambda^1 T^*N)$ and satisfies $d*\gamma = 0$. Recall that $\Psi : (R, \infty) \times \Sigma \cong N \setminus K$ and that $(r,\sigma)$ are the coordinates on $(R, \infty) \times \Sigma$. So, on $(R, \infty) \times \Sigma$,

$$\gamma(r,\sigma) = \chi(r,\sigma) + f(r,\sigma) \, dr$$

for a function $f$ and 1-form $\chi$ on $(R, \infty) \times \Sigma$, where $\chi$ has no $dr$ component. Write the exterior derivative on $(R, \infty) \times \Sigma$ in terms of the exterior derivative $d\Sigma$ on $\Sigma$ as:

$$d = d\Sigma + dr \wedge \partial \partial r$$

The equation $d*\gamma = 0$ then implies that $d\Sigma \chi = 0$ and $\partial \chi \partial r - d\Sigma f = 0$. (43)

Define a function $\zeta$ on $(R, \infty) \times \Sigma$ by

$$\zeta(r,\sigma) = -\int_r^\infty f(s,\sigma) \, ds.$$ 

This is well-defined since the modulus of $f$ is $o(r^{-3-\mu})$ as $r \to \infty$, where $-3-\mu \leq -1$ since $\mu \geq -2$. Noting that the modulus of $\chi$ with respect to $g_\Sigma$ is $o(r^{-2-\mu})$ as $r \to \infty$, with $-2 - \mu \leq 0$, we calculate using (43):

$$d\zeta(r,\sigma) = -\int_r^\infty d\Sigma f(s,\sigma) \, ds + f(r,\sigma) \, dr = -\int_r^\infty \frac{\partial \chi}{\partial r}(s,\sigma) \, ds + f(r,\sigma) \, dr$$

$$= \left[ -\chi(s,\sigma) \right]_r^\infty + f(r,\sigma) \, dr = \chi(r,\sigma) + f(r,\sigma) \, dr = *\gamma(\Psi(r,\sigma)).$$

If $\{R\} \times \Sigma$ has a tubular neighbourhood in $N$, which can be ensured by making $R$ larger if necessary, we can extend $\zeta$ smoothly to a function on $N$. Hence $\zeta \in L_{l+1,-2-\mu}(\Lambda^0 T^*N)$ with $d\zeta = *\gamma$ on $N \setminus K$. This leads us to the following.

**Proposition 5.14** Use the notation of Definition 5.9 and Propositions 5.4 and 5.12. Let $\gamma \in A_+(\mu)$ for $\mu \in [-2,1) \setminus D$. There exists $\zeta \in L_{l+2,-2-\mu}(\Lambda^0 T^*N)$ such that $*\gamma - d\zeta = \hat{\gamma}$ is a closed compactly supported 1-form. Moreover, the map $\gamma \mapsto [\hat{\gamma}]$ from $C_+(\mu) \subseteq A_+(\mu)$ to $H^1_{cs}(N)$ is injective.
Proof: Clearly our construction above ensures that \( \tilde{\gamma} \) is a closed 1-form which is zero outside of the compact subset \( K \) of \( N \). Thus \( [\xi] \in H^1_{cs}(N) \). Suppose that \( \gamma \in C^+ (\mu) \) and \([\tilde{\gamma}] = 0\). Then \( \xi = d\tilde{\zeta} \) for some function \( \tilde{\zeta} \) with compact support. Therefore

\[
0 = d^* \gamma = d^*(d\zeta + \tilde{\gamma}) = d^*d(\zeta + \tilde{\zeta}).
\]

Hence \( \zeta + \tilde{\zeta} \) is a harmonic function of order \( o(\rho^{-2-\mu}) \) as \( \rho \to \infty \). Since \( \mu \geq -2 \), the Maximum Principle forces \( \zeta + \tilde{\zeta} = 0 \) and the result follows. \( \square \)

It follows from Proposition \( 7.6 \) below that the map from \( C^+ (\mu) \) to \( H^1_{cs}(N) \) is an isomorphism when \( \mu \in [-2,0) \setminus D \).

6 The deformation theory

In this section we prove our main result, which is a local description of the moduli space \( \mathcal{M}(N, \lambda) \) of AC coassociative deformations of \( N \) with rate \( \lambda \). We remind the reader that we are using notation from the start of \( \S 4 \). We shall further assume that \( \lambda \not\in D \), where \( D \) is given by Proposition \( 5.4 \) and we choose some \( p > 4 \) and \( k \geq 2 \).

6.1 Deformations and obstructions

We begin by identifying the infinitesimal deformation space for our moduli space problem.

**Definition 6.1** The *infinitesimal deformation space* is

\[
\mathcal{I}(N, \lambda) = \{ \alpha \in L^p_{k+1, \lambda}(\Lambda^2 T^* N) : d\alpha = 0 \}.
\]

Note that \( \mathcal{I}(N, \lambda) \) is a subspace of \( \mathcal{K}(\lambda) \), given in Definition \( 5.1 \) and is thus finite-dimensional as \( \lambda \not\in D \). In fact, \( \mathcal{I}(N, \lambda) \cong \mathcal{K}(\lambda) \) if \( \lambda < 0 \) by Lemma \( 5.11 \).

We say that our deformation theory is *unobstructed* if \( \mathcal{M}(N, \lambda) \), given in Definition \( 4.1 \) is a smooth manifold near \( N \) of dimension \( \dim \mathcal{I}(N, \lambda) \).

By Proposition \( 4.7 \) and Corollary \( 4.18 \), \( \mathcal{M}(N, \lambda) \) is homeomorphic near \( N \) to the kernel of the map \( F \), given in Definition \( 4.6 \) near 0 in \( L^p_{k+1, \lambda}(U) \). Clearly \( \mathcal{I}(N, \lambda) \) is the tangent space to \( \text{Ker} F \) at 0, so it can be identified with the infinitesimal deformation space.

Furthermore, if there are no obstructions to the deformation theory of \( N \), then every infinitesimal deformation should extend to a genuine deformation and the moduli space should be a smooth manifold. This justifies our definition of an unobstructed deformation theory.
Theorem 5.10 identifies the obstruction space.

**Definition 6.2** The *obstruction space* is the finite-dimensional space

\[ O(N, \lambda) \cong \frac{d(L_{k+1}^p, \lambda(\Lambda^2 T^* N))}{d(L_{k+1}^p, \lambda(\Lambda^2 T^* N))} \]

given in (35). We observe that

\[ \dim O(N, \lambda) = \dim C_+ (\lambda) - \dim C(\lambda), \]

in the notation of Definition 5.9 and Theorem 5.10.

It shall become clear in the next subsection why we should think of \( O(N, \lambda) \) as the obstructions to our deformation theory. However, we notice by Theorem 5.10 that \( O(N, \lambda) = \{0\} \) for \( \lambda \in [-2, 1) \), which should mean that our deformation theory is unobstructed for these rates – we shall confirm that this is the case.

The key step in understanding the obstruction theory is contained in the next result, which studies the image of the deformation map \( F \).

**Proposition 6.3** Use the notation of Proposition 4.5 and Definition 4.6. Making the open set \( U \) smaller if necessary,

\[ F : L_{k+1}^p, \lambda(U) \rightarrow d(L_{k+1}^p, \lambda(\Lambda^2 T^* N)) \subseteq L_{k, \lambda-1}^p(\Lambda^3 T^* N). \]

**Proof:** It was noted in the proof of Proposition 4.10 that \( F(\alpha) \) is exact for \( \alpha \in L_{k+1}^p, \lambda(U) \). However, we need to know that we can choose a 2-form \( H(\alpha) \) lying in an appropriate weighted Sobolev space such that \( d(H(\alpha)) = F(\alpha) \).

Let \( u \) be the vector field given by dilations, which, in coordinates \((x_1, \ldots, x_7)\) on \( \mathbb{R}^7 \), is written:

\[ u = x_1 \frac{\partial}{\partial x_1} + \cdots + x_7 \frac{\partial}{\partial x_7}. \tag{44} \]

Then the Lie derivative of \( \varphi \) along \( u \) is:

\[ \mathcal{L}_u \varphi = d(u \cdot \varphi) = 3 \varphi. \tag{45} \]

Therefore, \( \psi = \frac{1}{3} u \cdot \varphi \) is a 2-form such that \( d\psi = \varphi \). Note that \( \psi|_C \equiv 0 \) since

\[ (u \cdot \varphi)|_C = u \cdot (\varphi|_C) = 0, \]

as \( u \in TC \) and \( C \) is coassociative by Proposition 2.8. Recall the definition of \( f_\alpha \) and \( N_\alpha \) in Definition 4.6 Define, for \( \alpha \in C^4_{\text{loc}}(U) \),

\[ H(\alpha) = f_\alpha^*(\psi|_{N_\alpha}) \]
so that

\[ F(\alpha) = f_{\alpha}^*(d\psi|_{\mathcal{N}_\alpha}) = d\left(f_{\alpha}^*(\psi|_{\mathcal{N}_\alpha})\right) = d(H(\alpha)). \]

Recall the diffeomorphism \( \Psi_\alpha : (R, \infty) \times \Sigma \to N_\alpha \setminus K_\alpha \), where \( K_\alpha \) is compact, introduced before Proposition 4.7, and the inclusion map \( \iota : (R, \infty) \times \Sigma \to C \) given by \( \iota(r, \sigma) = r\sigma \). The decay of \( H(\alpha) \) at infinity is determined by:

\[ \Psi_\alpha^*(\psi) = (\Psi_\alpha^* - \iota^*)(\psi) + \iota^*(\psi) = (\Psi_\alpha^* - \iota^*)(\psi) \]
since \( \psi|_C \equiv 0 \). For \( (r, \sigma) \in (R, \infty) \times \Sigma \),

\[ (\Psi_\alpha^* - \iota^*)(\psi)|_{(r, \sigma)} = \left(d\Psi_\alpha|_{(r, \sigma)}(\psi|_{\Psi_\alpha(r, \sigma)}) - dl_{(r, \sigma)}|_{\Psi_\alpha(r, \sigma)}(\psi|_{\Psi_\alpha(r, \sigma)}) \right) \]

\[ + dl_{(r, \sigma)}|_{\Psi_\alpha(r, \sigma)}(\psi|_{\Psi_\alpha(r, \sigma)} - |_{r}) \quad \text{(46)} \]

using the linearity of \( dl^* \) to derive the last term. Since \( |\psi| = O(r) \) and \( \Psi_\alpha \) satisfies (1) so that \( |d\Psi_\alpha^* - dl^*| = O(r^{\lambda-1}) \) as \( r \to \infty \), the expression in brackets in (46) is \( O(r^\lambda) \). The magnitude of the final term in (46) at infinity is determined by the behaviour of \( dl^* \), \( \nabla \psi \) and \( \Psi_\alpha - \iota \). Hence, as \( |dl^*| \) and \( |\nabla \psi| \) are \( O(1) \), using (11) again implies that this term is \( O(r^\lambda) \). We conclude that if \( \alpha \in L^p_{k+1, \lambda}(U) \) then \( H(\alpha) \in L^p_{k, \lambda}(\Lambda^2T^* N) \). Notice that \( H(\alpha) \) has one degree of differentiability less than one would expect since it depends on \( \alpha \) and \( \nabla \alpha \).

By Proposition 5.12 the annihilator \( A(\lambda) \) of the image of (10) for \( \mu = \lambda \) comprises of coclosed forms and lies in \( L^q_{l+1, -3-\lambda} \), where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( l \geq 2 \). For \( \gamma \in A(\lambda) \), recalling the dual pairing given in Definition 3.2

\[ \langle F(\alpha), \gamma \rangle = \langle d(H(\alpha)), \gamma \rangle = \langle H(\alpha), d^* \gamma \rangle = 0, \]

where the integration by parts is valid as \( H(\alpha) \in L^p_{k, \lambda} \) and \( \gamma \in L^q_{l+1, -3-\lambda} \). Thus, \( F(\alpha) \) must lie in the closure of the image of (10). \( \square \)

6.2 The moduli space \( \mathcal{M}(N, \lambda) \)

We now state and prove our main result.

**Theorem 6.4** Let \( N \) be a coassociative 4-fold in \( \mathbb{R}^7 \) which is asymptotically conical with rate \( \lambda \). Let \( p > 4, k \geq 2 \) and suppose that \( \lambda \notin \mathcal{D} \), where \( \mathcal{D} \) is given in Proposition 5.4. Use the notation of Definitions 4.1, 6.1, 6.2 and 6.2 and let \( \mathcal{B}(\lambda) = \{ \beta \in L^p_{k+1, \lambda}(\Lambda^4T^* N) : d^* \beta \in \overline{d(L^p_{k+1, \lambda}(\Lambda^2T^* N))} \subseteq L^p_{k, \lambda-1}(\Lambda^3T^* N) \} \).

There exist a manifold \( \hat{\mathcal{M}}(N, \lambda) \), which is an open neighbourhood of 0 in \( \mathcal{K}(\lambda) \), and a smooth map \( \pi : \hat{\mathcal{M}}(N, \lambda) \to \mathcal{O}(N, \lambda) \), with \( \pi(0) = 0 \), such that an open neighbourhood of 0 in \( \text{Ker} \pi \) is homeomorphic to \( \mathcal{M}(N, \lambda) \) near \( N \).
Moreover, if $\mathcal{O}(N, \lambda) = \{0\}$, $\mathcal{M}(N, \lambda)$ is a smooth manifold near $N$ with

$$\dim \mathcal{M}(N, \lambda) = \dim \mathcal{I}(N, \lambda) = \dim \mathcal{K}(\lambda) - \dim \mathcal{B}(\lambda);$$

that is, the deformation theory is unobstructed.

Note By Theorem 5.10, $\mathcal{B}(\lambda) = \{0\}$ if $\lambda < 0$ and, if $\lambda \geq 0$,

$$\mathcal{B}(\lambda) = \{\beta \in L^p_k(\Lambda^4 T^* N) : d^* \beta \in d(L^p_{k+1, \lambda}(\Lambda^2 T^* N))\}.$$

Moreover, $\mathcal{B}(\lambda)$ is finite-dimensional as it is isomorphic to a subspace of the harmonic functions in $L^p_k(\Lambda^4 T^* N)$.

Proof: Recall the open set $U$ given by Proposition 4.5. Make $U$ smaller if necessary so that Proposition 6.3 holds and that the $C^1$-norm of elements $\alpha \in L^p_{k+1, \lambda}(U)$ is small enough for Corollary 4.18 to apply. Define

\[
\begin{align*}
V &= L^p_{k+1, \lambda}(U) \times L^p_{k+1, \lambda}(\Lambda^4 T^* N) \\
X &= L^p_{k+1, \lambda}(\Lambda^4 T^* N) \oplus L^p_{k+1, \lambda}(\Lambda^4 T^* N) \\
Y &= \mathcal{O}(N, \lambda) \subseteq L^p_{k, \lambda-1}(\Lambda^3 T^* N) \quad \text{and} \\
Z &= d(L^p_{k+1, \lambda}(\Lambda^2 T^* N)) + d^*(L^p_{k+1, \lambda}(\Lambda^4 T^* N)) \subseteq L^p_{k, \lambda-1}(\Lambda^3 T^* N).
\end{align*}
\]

By Definition 3.1, $X$ is a Banach space and, as noted at the start of §4.2, $V$ is an open neighbourhood of zero in $X$. The obstruction space $Y$ is trivially a Banach space since it is finite-dimensional. Finally, $Z$ is a Banach space as it is a closed subset of a Banach space by the proof of Theorem 5.10. Define a smooth map $\mathcal{G}$ on the open subset $V \times Y$ of $X \times Y$ by

$$\mathcal{G}(\alpha, \beta, \gamma) = G(\alpha, \beta) + \gamma,$$

where $G$ is given in Definition 4.9. By Proposition 6.3, $\mathcal{G}$ maps into $Z$. Moreover, the linearisation of $\mathcal{G}$ at zero, $d\mathcal{G}_{|(0,0,0)} : X \times Y \to Z$, acts as

$$\mathcal{G}(\alpha, \beta, \gamma) \longmapsto d\alpha + d^* \beta + \gamma.$$

Therefore, $d\mathcal{G}_{|(0,0,0)}$ is surjective by Theorem 5.10. Furthermore, using the notation of Definition 5.1, we see that

$$\text{Ker } d\mathcal{G}_{|(0,0,0)} = \{(\alpha, \beta, \gamma) \in X \times Y : d\alpha + d^* \beta + \gamma = 0\}$$

$$\cong \text{Ker}(d + d^*) = \mathcal{K}(\lambda).$$
since, by definition of \( Y \), \((d + d^*) \lambda(X) \cap Y = \{0\}\). Finally note that there exists a closed subspace \( A \) of \( X \) such that \( X = K(\lambda) \oplus A \) as \( K(\lambda) \) is the kernel of a Fredholm operator.

We now apply the Implicit Function Theorem (Theorem 3.6) to \( G \). Thus, there exist open sets \( \hat{M}(N, \lambda) \subseteq K(\lambda) \), \( W_A \subseteq A \) and \( W_Y \subseteq Y \), each containing zero, with \( \hat{M}(N, \lambda) \times W_A \subseteq V \), and smooth maps \( W_A : \hat{M}(N, \lambda) \to W_A \) and \( W_Y : \hat{M}(N, \lambda) \to W_Y \) such that

\[
\text{Ker} \ G \cap (\hat{M}(N, \lambda) \times W_A \times W_Y) = \{ (\xi, W_A(\xi), W_Y(\xi)) \in K(\lambda) \oplus A \oplus Y : \xi \in \hat{M}(N, \lambda) \}.
\]

We take our map \( \pi = W_Y \). Thus, an open neighbourhood of zero in \( \text{Ker} \pi \) is homeomorphic to an open neighbourhood of zero in \( \text{Ker} \ G \subseteq V \).

If \( \lambda < 0 \), Proposition 4.10 and Theorem 5.10(a)-(b) imply that \( \text{Ker} \ G \subseteq V \) is isomorphic to \( \text{Ker} \ F \subseteq L^p_{k-1, \lambda}(U) \), \( \mathcal{I}(N, \lambda) \cong K(\lambda) \) and \( B(\lambda) = \{0\} \). Since \( \text{Ker} \ F \) consists of smooth forms by Corollary 4.18 and gives a local description of the moduli space by Proposition 4.17, the result follows for \( \lambda < 0 \).

Therefore, suppose \( \lambda \geq 0 \). Define a smooth map \( \pi_G \) on \( \text{Ker} \ G \) by \( \pi_G(\alpha, \beta) = \beta \). Notice that

\[
\text{Ker} \ \pi_G \cong \text{Ker} \ F = \{ \alpha \in L^p_{k+1, \lambda}(U) : F(\alpha) = 0 \}.
\]

Moreover, if \((\alpha, \beta) \in \text{Ker} \ G\),

\[
d^* \beta = -F(\alpha) \in d(L^p_{k+1, \lambda}(A^2 T^* N))
\]

by Proposition 6.3. Therefore, \( \pi_G : \text{Ker} \ G \to B(\lambda) \) and \( d\pi_G|_{(0,0)} : K(\lambda) \to B(\lambda) \).

Since \( d\pi_G|_{(0,0)} \) is clearly surjective, \( \pi_G \) is locally surjective. Moreover,\n
\[
\text{Ker} \ d\pi_G|_{(0,0)} \cong \{ \alpha \in L^p_{k+1, \lambda}(A^2 T^* N) : da = 0 \} = \mathcal{I}(N, \lambda) = \text{Ker} \ dF|_0.
\]

Hence, \( \text{Ker} \ F \) is locally diffeomorphic to \( \text{Ker} d\pi_G|_{(0,0)} \) and the result follows. \( \square \)

Theorem 5.10 gives us an immediate corollary to Theorem 6.4.

**Corollary 6.5** The deformation theory of AC coassociative 4-folds \( N \) in \( \mathbb{R}^7 \) with generic rate \( \lambda \in [-2, 1) \) is unobstructed.

**Remark** We have no guarantee that the deformation theory is unobstructed for generic rates \( \lambda < -2 \). However, we would expect the obstruction space to be zero for generic choices of \( N \).
7 The dimension of the moduli space $\mathcal{M}(N, \lambda)$

We shall continue to use the notation from the start of §4. We additionally recall the notation of Definitions 5.1 and 5.9 for $\mu \in \mathbb{R}$, we denote the kernel of $(d + d^*)_\mu$, given in (19), by $\mathcal{K}(\mu)$ and the kernel of $(d^*_\mu + d)_\mu$, given in (22), by $\mathcal{C}^0(\mu)$. We also remind the reader that $\mathcal{C}^0(\mu)$ is isomorphic to the cokernel of (19) via the pairing given in Definition 3.2 when $\mu \notin \mathcal{D}$, where $\mathcal{D}$ is given in Proposition 5.4. We shall split our study of the dimension of the moduli space into three ranges of rates $\lambda \notin \mathcal{D}$: $\lambda \in [-2, 0)$, $\lambda \in [0, 1)$ and $\lambda < -2$.

7.1 Topological calculations

The first proposition we state follows from standard results in algebraic topology if we consider our coassociative 4-fold $N$, which is AC to $C \cong (0, \infty) \times \Sigma$, as the interior of a manifold which has boundary $\Sigma$. This result is invaluable for our dimension calculations.

**Proposition 7.1** Recall the notation from the start of §4. Let the map $\phi_m : H^m_{cs}(N) \rightarrow H^m_{dR}(N)$ be defined by $\phi_m([\xi]) = [\xi]$. Let $r > R$ and let $\Psi_r : \Sigma \rightarrow N$ be the embedding given by $\Psi_r(\sigma) = \Psi(r, \sigma)$. Define $p_m : H^m_{dR}(N) \rightarrow H^m_{dR}(\Sigma)$ by $p_m([\xi]) = [\Psi_r^*\xi]$. Let $f \in C^\infty(N)$ be such that $f = 0$ on $K$ and $f = 1$ on $(R + 1, \infty) \times \Sigma$. If $\pi_\Sigma : (R, \infty) \times \Sigma \cong N \setminus K \rightarrow \Sigma$ is the projection map, define $\partial_m : H^m_{dR}(\Sigma) \rightarrow H^{m+1}_{cs}(N)$ by $\partial_m([\xi]) = [d(\pi_\Sigma^*f\xi)]$. Then the following sequence is exact:

$$
\cdots \rightarrow H^m_{cs}(N) \xrightarrow{\phi_m} H^m_{dR}(N) \xrightarrow{p_m} H^m_{dR}(\Sigma) \xrightarrow{\partial_m} H^{m+1}_{cs}(N) \rightarrow \cdots.
$$

(47)

**Remark** The fact that $H^0_{cs}(N) = H^1_{dR}(N) = \{0\}$ enables us to calculate the dimension of various spaces more easily using the long exact sequence (47).

We now identify a space of forms which we shall relate to the topology of $N$.

**Definition 7.2** Define $\mathcal{H}^m(N)$ by

$$
\mathcal{H}^m(N) = \{\xi \in L^2(\Lambda^m T^* N) : d\xi = d^*\xi = 0\}.
$$

(48)

Using (3), we notice that $L^2_{0,-2} = L^2$. Moreover, the elliptic regularity result Theorem 4.15(a) applied to the Laplacian can be improved as in [15, Proposition 5.3], using more theory of weighted Sobolev spaces than we wish to discuss here, to show that harmonic forms in $L^2_{0,-2}$, hence elements of $\mathcal{H}^m(N)$, lie in $L^2_{k,-2}$ for all $k \in \mathbb{N}$ and thus are smooth.
The moduli space of deformations for a compact coassociative 4-fold $P$, by [23, Theorem 4.5], is smooth of dimension $b^2_+ (P)$. This suggests that we need to define an analogue of $b^2_+ (N)$ for an AC coassociative 4-fold $N$.

**Definition 7.3** As noted in Definition 7.2, $H^2 (N)$ consists of smooth forms. The Hodge star maps $H^2 (N)$ into itself, so there is a splitting $H^2 (N) = \mathcal{H}^2_+ (N) \oplus H^2_- (N)$ where

$$\mathcal{H}^2_+ (N) = H^2 (N) \cap C^\infty (\Lambda^2_+ T^* N).$$

Define $J (N) = \phi^2 (H^2_{cs} (N))$, with $\phi$ as in Proposition 7.1. If $[\alpha], [\beta] \in J (N)$, there exist compactly supported closed 2-forms $\xi$ and $\eta$ such that $[\alpha] = \phi^2 ([\xi])$ and $[\beta] = \phi^2 ([\eta])$. We define a product on $J (N) \times J (N)$ by

$$[\alpha] \cup [\beta] = \int_N \xi \wedge \eta. \quad (49)$$

Suppose that $\xi'$ and $\eta'$ are also compactly supported with $[\alpha] = \phi^2 ([\xi'])$ and $[\beta] = \phi^2 ([\eta'])$. Then there exist 1-forms $\chi$ and $\zeta$ such that $\xi - \xi' = d\chi$ and $\eta - \eta' = d\zeta$. Therefore,

$$\int_N \xi' \wedge \eta' = \int_N (\xi - d\chi) \wedge (\eta - d\zeta) = \int_N \xi \wedge \eta - d\chi \wedge \eta - \xi' \wedge d\zeta = \int_N \xi \wedge \eta,$$

as both $\chi \wedge \eta$ and $\xi' \wedge \zeta$ have compact support. The product (49) on $J (N) \times J (N)$ is thus well-defined and is a symmetric topological product with a signature $(a, b)$. By [15, Example 0.15], $H^2 (N) \cong J (N)$ via $\alpha \mapsto [\alpha]$. We therefore define

$$b^2_+ (N) = \dim \mathcal{H}^2_+ (N).$$

Hence, $b^2_+ (N) = a$, which is a topological number.

We can now prove a useful fact about the kernel of $(d + d^*)_{\mu}$.

**Lemma 7.4** In the notation of Definitions 5.1 and 7.3,

$$\dim \mathcal{K}(\mu) \leq b^2_+ (N) \leq \dim \mathcal{K}(-2)$$

for all $\mu < -2$ such that $\mu \notin D$.

**Proof:** By Theorem 3.5(a), recalling that $p > 4$ and $k \geq 2$, $L^p_{k+1, \mu} \to L^0_{0, -2}$ and $L^p_{k+3, -2} \leftrightarrow L^p_{k+1, -2}$. The result follows from observations in Definitions 6.1 and 7.3. \qed

We want to make a similar statement about the cokernel of $(d + d^*)_{\mu}$. In fact, we can do a lot better, but we first need a result concerning functions on cones [5, Lemma 2.3].
Proposition 7.5 Recall the cone $C$ and its link $\Sigma$. Suppose that $f : C \to \mathbb{R}$ is a nonzero function such that

$$f(r, \sigma) = r^\mu f_S(\sigma)$$

for some function $f_S : \Sigma \to \mathbb{R}$ and $\mu \in \mathbb{R}$. Denoting the Laplacians on $C$ and $\Sigma$ by $\Delta_C$ and $\Delta_\Sigma$ respectively,

$$\Delta_C f(r, \sigma) = r^{\mu-2} (\Delta_\Sigma f_S(\sigma) - \mu(\mu+2)f_S(\sigma)).$$

Therefore, since $\Sigma$ is compact, $\mu(\mu+2) \geq 0$ and so there exist no nonzero homogeneous harmonic functions of order $O(r^\mu)$ on $C$ with $\mu \in (-2, 0)$.

Proposition 7.6 In the notation of Definition 5.9 and Theorem 5.10,

$$\dim C_+(\mu) = \dim C(\mu) = b^3(N)$$

for all $\mu \in (-2, 0) \setminus D$, and so they are independent of $\mu$ in this range.

Proof: Let $\mu \in (-2, 0) \setminus D$ and let $\gamma \in C_+(\mu) \subseteq L^q_{l+1, -3-\mu}(\Lambda^3 T^* N)$. Recall the closed compactly supported 1-form $\hat{\gamma} = *\gamma - d\zeta$ related to $\gamma$ as given by Proposition 5.14. Clearly $\Delta \zeta = -d^* \hat{\gamma}$ since $d^* \gamma = 0$. Hence, by Proposition 5.14 $d^* \hat{\gamma}$ lies in the image of the map

$$\Delta_{-2-\mu} = \Delta : L^q_{l+2, -2-\mu}(\Lambda^0 T^* N) \to L^q_{l, -4-\mu}(\Lambda^0 T^* N).$$

Therefore $d^* \hat{\gamma}$ is orthogonal, via the pairing given in Definition 3.2, to the kernel of the adjoint of $\Delta_{-2-\mu}$. Let $\nu \in (-2, 0) \setminus \{0\}$, Proposition 7.5 implies, as shown in [5] Definition 2.5 & Theorem 2.11, that there are no elements of $D(\Delta)$, defined in Theorem 5.3, between $-2-\mu$ and $-2-\nu$. Thus $\text{Coker } \Delta_{-2-\mu} = \text{Coker } \Delta_{-2-\nu}$. Obviously, $d^* \hat{\gamma}$ is then orthogonal to the kernel of the adjoint of $\Delta_{-2-\nu}$. Consequently, there exists $\zeta_\nu \in L^q_{l+2, -2-\nu}(\Lambda^0 T^* N)$ such that $\Delta_{-2-\nu} \zeta_\nu = -d^* \hat{\gamma}$ and hence $\zeta_\nu$ is harmonic. Moreover, $\zeta - \zeta_\nu = o(\rho^{-\min(\mu, \nu)-2})$ as $\rho \to \infty$, where $\rho$ is a radius function on $N$. Since $-\min(\mu, \nu) - 2 < 0$, we use the Maximum Principle to deduce that $\zeta - \zeta_\nu = 0$. Hence $*\gamma$ and $\gamma$ lie in $L^q_{l+1, -\nu-3}$ for any $\nu \in (-2, 0)$. The dimension of $C_+(\mu)$ is therefore constant for $\mu \in (-2, 0) \setminus D$.

We now note that, by [15] Example 0.15, $\mathcal{H}^3(N) \cong H^3_{dR}(N)$ via $\eta \mapsto [\eta]$, where $\mathcal{H}^3(N)$ is given by (15). By Theorem 3.3 (a), $L^q_{l+1, -3-\mu} \hookrightarrow L^q_{0, -2} = L^2$ for $\mu \geq -1$ and $L^q_{l+1, -2} \hookrightarrow L^q_{l+1, -3-\mu}$ if $\mu < -1$. By Theorem 5.10 $C_+(\mu) = C(\mu)$. Hence, $C_+(\mu) \subseteq \mathcal{H}^3(N)$ whenever $\mu \in [-1, 0) \setminus D$, and $\mathcal{H}^3(N) \subseteq C_+(\mu)$ if $\mu \in (-2, -1) \setminus D$. The result follows. \[\blacksquare\]
7.2 Rates $\lambda \in [-2, 0) \setminus D$

For convenience we introduce some extra notation.

**Definition 7.7** Since $D$ is discrete by Proposition 5.4, we may choose $\lambda_- < -2$ and $\lambda_+ \in (-2, 0)$ such that $[\lambda_-, \lambda_+] \cap D \subseteq \{-2\}$.

We are thus able to make the following proposition, which determines the dimension of $M(N, \lambda)$ in a special case.

**Proposition 7.8** Use the notation from Definitions 5.5 and 7.3. If $\beta^1(\Sigma) = 0$ and $\lambda \in [-2, 0) \setminus D$, 

$$\dim M(N, \lambda) = b^2_+ (N) + \sum_{\mu \in D \cap \{-2, \lambda\}} d(\mu).$$

**Proof:** Use the notation of Definition 7.7 and recall that $-2 \in D$ if and only if $\beta^1(\Sigma) > 0$ by Proposition 5.4. Since $-2 \notin D$, $[\lambda_-, \lambda_+] \cap D = \emptyset$. By Proposition 5.4, the dimensions of the kernel and cokernel of (19) are constant for $\mu \in [\lambda_-, \lambda_+]$. Therefore, by Proposition 7.4,

$$\dim K(\mu) = \dim K(-2) = b^2_+ (N) \quad \text{for } \mu \in [\lambda_-, \lambda_+].$$

Applying Propositions 5.6 and 7.6 completes the proof. □

The next proposition enables us to calculate the dimension of $M(N, \lambda)$ when $-2 \in D$. The proof is long, but the main idea is to identify the forms which are added to the kernel of (19) as the rate crosses $-2$. By the work in [16], these forms are precisely those which are asymptotic to homogeneous solutions $\xi \in C^\infty(\Lambda^2 T^* P + \Lambda^4 T^* C)$ of order $O(r^{-2})$ to $(d + d^*) \xi = 0$, as described after Proposition 5.3. In the course of the proof, we will need an elementary lemma.

**Lemma 7.9** If $P$ is a compact 4-dimensional Riemannian manifold,

$$d(C^\infty(\Lambda^2 T^* P)) = d(C^\infty(\Lambda^2 T^* P)).$$

**Proof:** Let $\theta \in C^\infty(\Lambda^2 T^* P)$. By Hodge theory, there exists $\psi \in C^\infty(\Lambda^1 T^* P)$ and $\omega \in C^\infty(\Lambda^2 T^* P)$, with $d\omega = 0$, such that $\theta = *d\psi + \omega$. (Here, $\omega$ is the sum of an exact form and a harmonic form.) Therefore, $d\theta = d*d\psi$. Letting $\Theta = *d\psi + d\psi \in C^\infty(\Lambda^2 T^* P)$, we have that $d\theta = d\Theta$. □

**Proposition 7.10** Use the notation of Definitions 7.3 and 7.4. If $\mu \in [\lambda_-, -2)$, 

$$\dim K(\mu) = \dim K(-2) = b^2_+ (N).$$

If $\nu \in (-2, \lambda_+]$, 

$$\dim K(\nu) = b^2_+ (N) - b^0(\Sigma) + b^1(\Sigma) + b^0(N) - b^1(N) + b^3(N).$$
Proof: Let \( \mu \in [\lambda_-, -2) \) and \( \nu \in (-2, \lambda_+] \). By Theorem 3.5(a), \( L^p_{k+1, \mu} \hookrightarrow L^p_{k+1, -2} \hookrightarrow L^p_{k+1, \nu} \), so \( K(\mu) \subseteq K(-2) \subseteq K(\nu) \).

Recall the map \((d + d^*)\infty\) defined by (22) and the bundle \( V_\Sigma \) given in (23). By Definition 5.5, \( d(-2) \) is the dimension of the vector space of solutions \( \xi(t, \sigma) \) to \((d + d^*)\infty\xi = 0\) of the form

\[
\xi(t, \sigma) = e^{-2t}p(t, \sigma),
\]

where \((t, \sigma) \in (T, \infty) \times \Sigma\) and \( p(t, \sigma) \) is a polynomial in \( t \) taking values in \( C^\infty(V_\Sigma \otimes \mathbb{C})\). Furthermore, by Proposition 5.8 and its proof, \( d(-2) = b_1(\Sigma) \) and \( \xi \) must be of the form

\[
\xi(t, \sigma) = e^{-2t}(\alpha(\sigma) + dt \wedge *_\Sigma \alpha(\sigma)),
\]

where \( \alpha \) is a closed and coclosed 2-form on \( \Sigma \) and \( *_\Sigma \) is the Hodge star on \( \Sigma \). Suppose \( \alpha \neq 0 \). We shall see that the forms \( \xi \) of this type correspond to forms on \( N \) which add to the kernel of (19) as the rate crosses \(-2\).

First, however, we must transform from the cylindrical picture to the conical one. Remember, in (21), we related \((d + d^*)\infty\) to the usual operators \( d \) and \( d^* \) acting on \( m \)-forms on \( N \setminus K \approx (R, \infty) \times \Sigma \). From this equation, since \( \xi \) is a 2-form, we notice that we need to consider \( \eta = e^{2t} \xi \) as the form on the cone related to \( \xi \). Changing coordinates \((t, \sigma)\) to conical coordinates \((r, \sigma)\), where \( r = e^t \), we may write

\[
\eta(r, \sigma) = \alpha(\sigma) + r^{-1}dr \wedge *_\Sigma \alpha(\sigma).
\]

We must now consider possible extensions of \( \eta \) to a form on \( N \). Recall that \( \Psi : (R, \infty) \times \Sigma \cong N \setminus K \) and that \( \rho \) is a radius function on \( N \) as in Definition 2.7. We also remind the reader of Proposition 4.4: that there is an isomorphism \( \tilde{\Upsilon} : \Lambda^2_T((R, \infty) \times \Sigma) \rightarrow \Lambda^2_T(N \setminus K) \).

By the work in [16] §5, as \([\lambda_-, \lambda_+] \cap D = \{-2\}, \gamma \in K(\nu) \) if and only if there exist \( \eta \) of the form (20), and \( \zeta \in K(\nu) \) with \( \zeta \) asymptotic to \( \tilde{\Upsilon}(\eta) \), such that \( \gamma - \zeta \in K(\mu) \). We can calculate:

\[
|\eta(r, \sigma)|_{g_{\text{cone}}} = O(r^{-2}) \quad \text{as } r \to \infty,
\]

where \( g_{\text{cone}} \) is the conical metric on \((R, \infty) \times \Sigma\) as given in Definition 2.6, since \( \alpha \) is a 2-form independent of \( r \). Thus, a form \( \zeta \) asymptotic to \( \tilde{\Upsilon}(\eta) \) must satisfy \( \zeta = O(\rho^{-2}) \) as \( \rho \to \infty \), so \( \zeta \notin K(-2) \). Hence, \( \gamma \notin K(-2) \).

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Use the notation of Proposition 7.1. We show that a form \( \zeta \) as in the previous paragraph will exist if and only if \([\alpha] \in p_2(H^2_\text{dir}(N))\). Necessity is clear, so we only consider sufficiency. Let \([\alpha] \in p_2(H^2_\text{dir}(N))\) with \(\alpha\) the unique harmonic representative in the class. Note that the map \(\pi_\Sigma : (R, \infty) \times \Sigma \cong N \setminus K \to \Sigma\) induces an isomorphism \(q_2 : H^2_\text{dir}(\Sigma) \to H^2_\text{dir}(N \setminus K)\). Now, \(\eta\), given by (47), is self-dual with respect to \(g_{\text{cone}}\), so

\[
\zeta' = \tilde{\Psi}(\eta) \in C^\infty(\Lambda^2_+ T^*(N \setminus K)).
\]

Moreover, since

\[
\eta = \alpha + d(*\Sigma \alpha \log \rho),
\]
we see that \([\zeta'] = [\eta] = [\alpha]\) in \(H^3_\text{dir}((R, \infty) \times \Sigma)\). Thus, \(q_2^{-1}([\zeta']) \in p_2(H^3_\text{dir}(N))\).

By the long exact sequence (47), the image of \(p_2\) is equal to the kernel of \(\partial_2\). Hence, if \(f \in C^\infty(N)\) such that \(f = 0\) on \(K\) and \(f = 1\) on \(\Psi((R + 1, \infty) \times \Sigma)\), \([d(f \zeta')] = 0\) in \(H^2_\text{dir}(N)\). Therefore, there exists a smooth, compactly supported, self-dual 2-form \(\chi\) on \(N\) such that \(d(f \zeta') = d\chi\) by Lemma 7.10.

We define \(\zeta = f \zeta' - \chi\), which is a closed self-dual 2-form on \(N\) asymptotic to \(\tilde{\Psi}(\eta)\), since it equals \(\tilde{\Psi}(\eta)\) outside the support of \(1 - f\) and the compactly supported form \(\chi\). We have therefore shown that the space of kernel forms added at rate \(-2\) is isomorphic to the image of \(p_2\). We can calculate the dimension of this image by (47):

\[
\dim p_2(H^2_\text{dir}(N)) = -b^0(\Sigma) + b^1(\Sigma) + b^0(N) - b^1(N) + b^3(N).
\]

By Proposition 5.7, there are no changes in \(K(\mu)\) for \(\mu \in [\lambda_-, -2)\) and the argument thus far shows that the kernel forms added at rate \(-2\) do not lie in \(K(-2)\). We conclude that \(K(\mu) = K(-2)\) for \(\mu \in [\lambda_-, -2)\), so \(\dim K(\mu) = \dim K(-2) = b^3_+(N)\) by Lemma 7.4. The latter part of the proposition follows from the formula and arguments above, since \(K(\nu)\) does not alter for \(\nu \in (-2, \lambda_+]\) by Proposition 5.7. \(\square\)

**Note** By (47), \(-b^0(\Sigma) + b^1(\Sigma) + b^0(N) - b^1(N) + b^3(N) = 0\) if \(b^1(\Sigma) = 0\).

Proposition 7.10 shows that the function \(k(\mu) = \dim K(\mu)\) is lower semi-continuous at \(-2\) and is continuous there if \(-2 \notin D\) by the note above. Our next result shows that \(c(\mu) = \dim C_+(\mu)\) is upper semi-continuous at \(-2\).

**Proposition 7.11** Use the notation of Definition 7.4. If \(\nu \in (-2, \lambda_+]\), then \(\dim C_+(\nu) = \dim C_+(-2) = b^3(N)\). If \(\mu \in [\lambda_-, -2)\),

\[
\dim C_+(\mu) = b^0(\Sigma) - b^0(N) + b^1(N).
\]
Proof: We know the index of (19) at rate \( \nu \) by Propositions 7.6 and 7.10:
\[
i_\nu(d + d^*) = b_\nu^2(N) - b^0(\Sigma) + b^1(\Sigma) + b^0(N) - b^1(N).
\]
By Proposition 5.6 recalling that \( d(-2) = b_1(\Sigma) \),
\[
i_\mu(d + d^*) = b_\mu^2(N) - b^0(\Sigma) + b^0(N) - b^1(N).
\]
Using Proposition 7.5 again we get the dimension of \( C_+^{(\mu)} \) claimed.

The only part left to do is to show that \( \dim C_+^{(\mu)} = b_3^3(N) \). Remember that if the rate for the kernel forms is \( \kappa \) then the ‘dual’ rate for the cokernel forms is \(-4 - (\kappa - 1) = -3 - \kappa \). Thus, by the work in [16], the forms \( \gamma \) which are added to the cokernel as the rate crosses \(-2 \) from above must be asymptotic to homogeneous forms on the cone of order \( O(r^{-1}) \). Therefore, \( \gamma \notin C(-2) \subseteq L_{l+1,-1}(\Lambda^3 T^* N) \) as required.

Note: The difference \( \dim C_+^{(\mu)} - \dim C_+^{(\nu)} \), as given in Proposition 7.11, is equal to \( \dim p_1(H_{dR}(N)) \) as defined in Proposition 7.1. We can prove an isomorphism between the image of \( p_1 \) and the cokernel forms added at \(-2 \) in a similar manner to the proof of Proposition 7.10, but we omit it.

Proposition 5.6 along with Propositions 7.6, 7.10 and 7.11 give us the dimension of \( K(\lambda) \), hence \( M(N, \lambda) \), for all \( \lambda \in [-2, 0) \setminus D \). This is because the dimension of the cokernel is constant for these choices of \( \lambda \), so the change of index of (19) equals the change in the dimension of \( K(\lambda) \).

Proposition 7.12 In the notation of Definitions 5.5 and 7.3, if \( \lambda \in [-2, 0) \setminus D \),
\[
\dim M(N, \lambda) = b_+^2(N) - b^0(\Sigma) + b^1(\Sigma) + b^0(N) - b^1(N) + b_3^3(N) + \sum_{\mu \in D \cap (-2, \lambda)} d(\mu).
\]

7.3 Rates \( \lambda \in [0, 1) \setminus D \)

We now discuss the case \( \lambda \geq 0 \) and begin by studying the point \( 0 \in D \). Recall that \( d(0) \), as given by Proposition 5.5, is equal to the dimension of
\[
D(0) = \{(\alpha, \beta) \in C(\Lambda^2 T^* \Sigma \oplus \Lambda^3 T^* \Sigma) : d_\Sigma \alpha = 0 \text{ and } d_\Sigma * \Sigma \alpha + d_\Sigma^* \beta = 2\alpha\},
\]
using the notation of Proposition 5.4.

It is clear, using integration by parts, that the equations which \((\alpha, \beta) \in D(0)\) satisfy are equivalent to
\[
d_\Sigma * \Sigma \alpha = 2\alpha \quad \text{and} \quad d_\Sigma^* \beta = 0.
\]
The latter equation corresponds to constant 3-forms on \( \Sigma \), so the solution set has dimension equal to \( b^0(\Sigma) \). If we define

\[
Z = \{ \alpha \in C^\infty(\Lambda^2 T^* \Sigma) : d_{\Sigma} \ast_{\Sigma} \alpha = 2\alpha \},
\]  

(51)

then \( d(0) = b^0(\Sigma) + \text{dim} \ Z \).

Suppose that \( \beta \in C^\infty(\Lambda^3 T^* \Sigma) \) satisfies \( d^*_{\Sigma} \beta = 0 \) and corresponds to a form on \( N \) which adds to the kernel of (19) at rate 0. There exists, by [16, §5], a 4-form \( \zeta \) on \( N \) asymptotic to \( r^3 dr \wedge \beta \) on \( N \setminus K \cong (R, \infty) \times \Sigma \), and a self-dual 2-form \( \xi \) of order \( o(1) \) as \( \rho \to \infty \) such that \( d\xi + d^*\zeta = 0 \), where \( \rho \) is a radius function on \( N \).

Since \( d^*\zeta \) is exact, \( \ast \zeta \) is a harmonic function which is asymptotic to a function \( c \), constant on each end of \( N \), as given in Definition 2.6. Applying [5, Theorem 7.10] gives a unique harmonic function \( f \) on \( N \) which converges to \( c \) with order \( O(\rho^\mu) \) for all \( \mu \in (-2, 0) \). The theorem cited is stated for an AC special Lagrangian submanifold \( L \), but only uses the fact that \( L \) is an AC manifold and hence is applicable here. Therefore, \( \ast \zeta - f = o(1) \) as \( \rho \to \infty \) and hence, by the Maximum Principle, \( \ast \zeta = f \).

We deduce that \( d^*\zeta \) and \( d\xi \) are \( O(\rho^{-3+\epsilon}) \) as \( \rho \to \infty \) for any \( \epsilon > 0 \) small, hence they lie in \( L^2 \). Integration by parts, now justified, shows that \( d^*\zeta = 0 \) and we conclude that \( \ast \zeta \) is constant on each component of \( N \). Hence, the piece of \( b^0(\Sigma) \) in \( d(0) \) that adds to the dimension of the kernel is equal to \( b^0(N) \). The other 3-forms \( \beta \) on \( \Sigma \) satisfying \( d^*_{\Sigma} \beta = 0 \) must correspond to cokernel forms, so \( b^0(\Sigma) - b^0(N) \) is subtracted from the dimension of the cokernel at 0.

For each end of \( N \), we can define self-dual 2-forms of order \( O(1) \) given by translations of it, written as \( \frac{\partial}{\partial x_j} \cdot \varphi \) for \( j = 1, \ldots, 7 \). If the end is a flat \( \mathbb{R}^4 \), we only get three such self-dual 2-forms from it. So, if \( k' \) is the number of ends which are not 4-planes,

\[
\text{dim} \ Z \geq 7k' + 3(b^0(\Sigma) - k') = 3b^0(\Sigma) + 4k'.
\]

Moreover, the translations of the components of \( N \) must correspond to kernel forms, since they are genuine deformations of \( N \). If a component of \( N \) is a 4-plane then there are only three nontrivial translations of it. Therefore, if \( k \) is the number of components of \( N \) which are not 4-planes, at least \( 3b^0(N) + 4k \) is added to the dimension of the kernel at rate 0 from \( \text{dim} \ Z \).

From this discussion, we can now state and prove the following inequalities for the dimension of \( \mathcal{M}(N, \lambda) \) for \( \lambda \in [0, 1) \setminus \mathcal{D} \).
Proposition 7.13 Use the notation from Proposition 5.6, Definition 7.3 and Theorem 6.4. If \( \lambda \in [0, 1) \setminus D \),

\[
\dim \mathcal{M}(N, \lambda) \leq \dim \mathcal{K}(0) + b^0(N) + \dim Z - \dim \mathcal{B}(\lambda) + \sum_{\mu \in D \cap (0, \lambda)} \mathcal{d}(\mu)
\]

where \( Z \) is defined in (51) and

\[
\dim \mathcal{K}(0) = b^2(N) - b^0(\Sigma) + b^1(\Sigma) + b^0(N) - b^1(N) + b^3(N) + \sum_{\mu \in D \cap (-2, 0)} \mathcal{d}(\mu).
\]

Moreover, if \( k \) is the number of components of \( N \) which are not a flat \( \mathbb{R}^4 \),

\[
\dim \mathcal{M}(N, \lambda) \geq \dim \mathcal{K}(0) + b^0(\Sigma) + 3b^0(N) + 4k - b^3(N) - \dim \mathcal{B}(\lambda) + \sum_{\mu \in D \cap (0, \lambda)} \mathcal{d}(\mu).
\]

Proof: By Theorem 6.4, \( \dim \mathcal{M}(N, \lambda) = \dim \mathcal{K}(\lambda) - \dim \mathcal{B}(\lambda) \). Using Propositions 5.6, 7.6 and 7.10 and the notation of Definition 7.7, we calculate:

\[
i_\lambda(d + d^*) = i_{\lambda^+}(d + d^*) + \sum_{\mu \in D \cap (-2, \lambda)} \mathcal{d}(\mu)
\]

\[
= b^2(N) - b^0(\Sigma) + b^1(\Sigma) + b^0(N) - b^1(N) + \sum_{\mu \in D \cap (-2, \lambda)} \mathcal{d}(\mu)
\]

\[
= \dim \mathcal{K}(0) - b^3(N) + \mathcal{d}(0) + \sum_{\mu \in D \cap (0, \lambda)} \mathcal{d}(\mu).
\]

(An argument using asymptotic behaviour of kernel forms as in the proof of Proposition 7.10 leads to lower semi-continuity of \( k(\mu) = \dim \mathcal{K}(\mu) \) at 0). From the discussion above, the part of \( \mathcal{d}(0) = b^0(\Sigma) + \dim Z \) that subtracts from the cokernel of (19) as the rate crosses 0 is greater than or equal to \( b^0(\Sigma) - b^0(N) \). Therefore, \( \dim \mathcal{C}_+(\lambda) \leq b^3(N) - b^0(\Sigma) + b^0(N) \) using Proposition 7.6 so

\[
\dim \mathcal{K}(\lambda) = i_\lambda(d + d^*) + \dim \mathcal{C}_+(\lambda)
\]

\[
\leq i_\lambda(d + d^*) + b^3(N) - b^0(\Sigma) + b^0(N)
\]

\[
= \dim \mathcal{K}(0) + b^0(N) + \dim Z + \sum_{\mu \in D \cap (0, \lambda)} \mathcal{d}(\mu).
\]

The first inequality follows.

Now, from the discussion above, at least \( b^0(N) + 3b^0(N) + 4k \) is added to the kernel of (19) from \( \mathcal{d}(0) \) as the rate crosses 0. Remember \( \dim \mathcal{C}_+(\lambda) \leq b^3(N) - b^0(\Sigma) + b^0(N) \). Therefore, at most this amount could be added to the
index of \(19\) from the sum of \(d(\mu)\) terms without adding to the kernel. Thus,
\[
\dim K(\lambda) \geq \dim K(0) + 4b^0(N) + 4k - b^3(N) + \sum_{\mu \in D \cap (0, \lambda)} d(\mu) - b^3(N) + b^0(\Sigma) - b^0(N) = \dim K(0) + b^0(\Sigma) + 3b^0(N) + 4k - b^3(N) + \sum_{\mu \in D \cap (0, \lambda)} d(\mu),
\]
from which we can easily deduce the second inequality. \(\square\)

7.4 Rates \(\lambda < -2, \lambda \notin D\)

By Theorem 6.4 and its proof, we can think of the map \(\pi\) as a projection from an open neighbourhood of 0 in the infinitesimal deformation space \(I(N, \lambda)\) to the obstruction space \(O(N, \lambda)\). Thus, the expected dimension of the moduli space is \(\dim I(N, \lambda) - \dim O(N, \lambda)\). We now find a lower bound for this dimension.

Proposition 7.14 Use the notation of Definitions 6.1, 6.2 and 7.3. The expected dimension of the moduli space satisfies
\[
\dim I(N, \lambda) - \dim O(N, \lambda) \geq b^2(N) - b^0(\Sigma) + b^0(N) - b^3(N) + b^3(N) + \sum_{\mu \in D \cap (\lambda, -2)} d(\mu).
\]

Proof: In Definition 6.2 we noted that
\[
\dim O(N, \lambda) = \dim C_+(\lambda) - \dim C(\lambda),
\]
where \(C(\lambda)\) is given in (33). Since \(L_{l+1, -1}^q \hookrightarrow L_{l+1, -3-\lambda}^q\) by Theorem 3.5(a) (as \(\lambda > -2\)), we deduce that \(\dim C(\lambda) \geq b^3(N)\) by Proposition 7.6. Therefore,
\[
\dim I(N, \lambda) - \dim O(N, \lambda) \geq \dim K(\lambda) - \dim C_+(\lambda) + b^3(N).
\]
Applying Propositions 7.9, 7.10 and 7.11 completes the proof. \(\square\)

Remark Any nonplanar coassociative 4-fold which is AC with rate \(\lambda\) has a 1-dimensional family of nontrivial AC coassociative deformations given by dilations. The obvious question is: where does this appear in the calculation of the dimension of the moduli space? The author believes that it should come from kernel forms of rate \(\mu\), where \(\mu\) is the greatest element of \(D \cap (-\infty, \lambda)\).
8 Invariants

Suppose for this section that $N$ is a coassociative 4-fold which is AC to a cone $C$ in $\mathbb{R}^7$ with rate $\lambda < -2$ and suppose, for convenience, that $N$ is connected. Recall the space $\mathcal{H}^2_+(N)$ and topological invariant $b_2^+(N)$ given in Definition 7.3.

Consider the deformation $N \mapsto e^t N$ for $t \in \mathbb{R}$. Clearly $e^t N$ is coassociative and AC to $C$ with rate $\lambda$ for all $t \in \mathbb{R}$. Let $u$ be the dilation vector field on $\mathbb{R}^7$ as given in (44). Define a self-dual 2-form $\alpha_u$ on $N$, as in the note after Proposition 2.3, by

$$\alpha_u = (u \cdot \varphi)|_{TN}.$$  \hfill (52)

The deformation corresponding to $t \alpha_u$, as given in Definition 4.6, is $e^t N$.

We calculate, using (45),

$$d\alpha_u = d(u \cdot \varphi)|_{TN} = 3\varphi|_{TN} = 0$$

since $N$ is coassociative. As $\lambda < -2$, $\alpha_u \in L^2(\Lambda^2 T^* N)$ and hence $t \alpha_u \in \mathcal{H}^2_+(N)$ for all $t \in \mathbb{R}$. So, there is a 1-dimensional subspace of $\mathcal{H}^2_+(N)$ whenever $e^t N \neq N$ for some $t \neq 0$; that is, when $N$ is not a cone. Consequently, $b_2^+(N) = 0$ forces $N \cong \mathbb{R}^4$, as $N$ is nonsingular.

This discussion leads to our first invariant.

**Definition 8.1** Let $N$ be an AC coassociative 4-fold in $\mathbb{R}^7$ with rate $\lambda < -2$, define $\alpha_u$ by (52) and let $X(N) = \|\alpha_u\|^2_{L^2}$. By the discussion above, this is a well-defined invariant of $N$.

We now consider, for $\Gamma$ a 2-cycle in $N$ and $D$ a 3-cycle in $\mathbb{R}^7$ such that $\partial D = \Gamma$, the integral $\int_D \varphi$.

Suppose $D'$ is another 3-cycle with $\partial D' = \Gamma$. Then $\partial(D - D') = 0$, so

$$\int_{D-D'} \varphi = [\varphi] \cdot [D - D'] = 0$$

since $[\varphi] \in H^3_{\text{dR}}(\mathbb{R}^7)$ is zero. Therefore $\int_D \varphi = \int_{D'} \varphi$; that is, the integral is independent of the choice of 3-cycle with boundary $\Gamma$.

Suppose instead that $\Gamma'$ is a 2-cycle in $N$ such that $\Gamma - \Gamma' = \partial E$ for some 3-cycle $E \subseteq N$. Let $D$ and $D'$ be 3-cycles such that $\partial D = \Gamma$ and $\partial D' = \Gamma'$. Then $\partial D = \partial(E + D') = \Gamma$ and thus

$$\int_D \varphi = \int_{E+D'} \varphi = \int_E \varphi + \int_{D'} \varphi = \int_{D'} \varphi,$$

since $E \subseteq N$ and $\varphi|_N = 0$. Hence, we have shown the integral only depends on the homology class $[\Gamma]$. 43
We can now define our second invariant for any coassociative 4-fold in $\mathbb{R}^7$.

**Definition 8.2** Let $N$ be a coassociative 4-fold in $\mathbb{R}^7$. Let $\Gamma$ be a 2-cycle in $N$ and let $D$ be a 3-cycle in $\mathbb{R}^7$ such that $\partial D = \Gamma$. Define $[Y(N)] \in H^3_{dR}(N)$ by:

$$[Y(N)] \cdot [\Gamma] = \int_D \varphi.$$

The work above shows that this is well-defined.

In the notation of Definition 8.2, we calculate, using (45) and (52),

$$3 \int_D \varphi = \int_D d(u \cdot \varphi) = \int_\Gamma u \cdot \varphi = \int_\Gamma \alpha_u \cdot [\Gamma].$$

Hence, $3[Y(N)] = [\alpha_u]$.

We remind the reader of Definitions 7.2 and 7.3, in particular that $J(N)$ is the image of $H^2_{cs}(N)$ in $H^2_{dR}(N)$. We showed that $\alpha_u \in H^2_{dR}(N)$ and we know that $H^2(N) \cong J(N)$ via $\gamma \mapsto [\gamma]$. Thus, $[Y(N)]$ and $[\alpha_u]$ lie in $J(N)$. Using the product (49), we deduce that

$$9[Y(N)]^2 = [\alpha_u] \cup [\alpha_u] = \int_N \alpha_u \wedge \alpha_u = \|\alpha_u\|^2_{L^2} = X(N).$$

We have used the fact that if $\alpha_u = \eta + d\xi$, for some compactly supported closed 2-form $\eta$ and $\xi \in C^\infty_{\lambda+1}(\Lambda^2 T^* N)$, then an integration by parts argument, valid since $\lambda < -2$, shows that

$$\int_N \alpha_u \wedge \alpha_u = \int_N \eta \wedge \eta.$$

We may thus derive a test which determines whether $N$ is a cone.

**Proposition 8.3** Let $N$ be a connected coassociative 4-fold in $\mathbb{R}^7$ which is AC with rate $\lambda < -2$. Use the notation of Definitions 7.3, 8.1 and 8.2. If $b^2_3(N) = 0$, $N$ is a cone and hence a linear $\mathbb{R}^4$ in $\mathbb{R}^7$ as $N$ is nonsingular. Moreover, $N$ is a cone if and only if $X(N) = 0$ or, equivalently, $[Y(N)]^2 = 0$.

We note, for interest, that a similar argument holds for a connected special Lagrangian $m$-fold $L$ in $\mathbb{C}^m$ ($m \geq 2$) which is AC to a cone $C \cong (0, \infty) \times \Sigma$ with rate $\lambda < -m/2$, in the following sense. Let $u$ be the dilation vector field on $\mathbb{C}^m$, and let $\omega_m$ and $\Omega_m$ be the symplectic and holomorphic volume forms on $\mathbb{C}^m$ respectively. Define

$$\beta_u = (u \cdot \omega_m)|_{TL} \quad \text{and} \quad \gamma_u = (u \cdot \Omega_m)|_{TL},$$
which are both zero if and only if $L$ is a cone. By the proof of [23, Theorem 3.6], $\beta_u$ and $\gamma_u$ are closed and $\gamma_u = *\beta_u$. Moreover, by [3] and Theorem 3.5(a), $\beta_u$ and $*\beta_u$ lie in $L^2$ as $\lambda < -m/2$.

By [13, Example 0.15], the closed and coclosed $(m-1)$-forms in $L^2$ uniquely represent the cohomology classes in $H^{m-1}_{dR}(L)$. Therefore, if $b^{m-1}(L) = 0$, $L$ is a cone and hence a linear $\mathbb{R}^m$ in $C^m$ as $L$ is nonsingular. Moreover, we may define the invariant $X(L) = \|\beta_u\|^2_{L^2}$ in an analogous way to $X(N)$ and see that $L$ is a cone if and only if $X(L) = 0$.

We define two invariants, $[Y(L)] \in H^1_{dR}(L)$ and $[Z(L)] \in H^{m-1}_{dR}(L)$ by

$$[Y(L)] \cdot [\Gamma] = \int_D \omega_m \quad \text{and} \quad [Z(L)] \cdot [\Gamma'] = \int_{D'} \text{Im} \Omega_m,$$

where $\Gamma$ is a 1-cycle in $L$, $D$ is a 2-cycle in $\mathbb{C}^m$ such that $\partial D = \Gamma$, $\Gamma'$ is an $(m-1)$-cycle in $L$ and $D'$ is an $m$-cycle in $\mathbb{C}^m$ such that $\partial D' = \Gamma'$. Analogues of these invariants on $\Sigma$ were introduced in [5, Definition 7.2].

Using similar calculations from our discussion of $[Y(N)]$, we can show that $2[Y(L)] = [\beta_u]$ and $m[Z(L)] = [\ast \beta_u]$. By [5, Proposition 7.3], $[Y(L)]$ lies in the image of the compactly supported cohomology for rates less than $-1$. Therefore, for $\lambda < -m/2 \leq -1$ (as $m \geq 2$), we calculate:

$$2m[Y(L)] \cup [Z(L)] = [\beta_u] \cup [\ast \beta_u] = \int_L \beta_u \wedge \ast \beta_u = \|\beta_u\|^2_{L^2} = X(L),$$

where the cup product is given by the pairing between $H^1_{cs}(L)$ and $H^{m}_{dR}(L)$, since $[\beta_u]$ can be represented by a compactly supported closed 1-form. Hence $X(L) = 0$ if and only if $[Y(L)] \cup [Z(L)] = 0$.

We write these observations as a proposition.

**Proposition 8.4** Let $L$ be a connected special Lagrangian $m$-fold in $\mathbb{C}^m$ ($m \geq 2$) which is AC with rate $\lambda < -m/2$. If $b^{m-1}(L) = 0$, $L$ is a cone and hence a linear $\mathbb{R}^m$ in $\mathbb{C}^m$ as $L$ is nonsingular. Moreover, in the notation above, $L$ is a cone if and only if $X(L) = 0$ or, equivalently, $[Y(L)] \cup [Z(L)] = 0$.

### 9 Examples

Before discussing examples we prove a general result which shows that if an AC coassociative 4-fold converges sufficiently quickly to a cone with symmetries, then it inherits those symmetries. The analogous statement is already known for AC special Lagrangian submanifolds.
Proposition 9.1 Let $N$ be a coassociative 4-fold which is asymptotically conical to a cone $C$ in $\mathbb{R}^7$ with rate $\lambda < -2$. Let $G$ be the identity component of the automorphisms of $C$, $\text{Aut}(C) \subseteq G_2$. Then $N$ is $G$-invariant.

Proof: Suppose for a contradiction that $N$ is not $G$-invariant. Then there exists a vector field $v$ in the Lie algebra $g$ of $G$ which preserves $C$ but not $N$. Since $v$ does not preserve $N$, it defines a nonzero closed self-dual 2-form $\alpha_v = (v \cdot \varphi)|_{TN}$ as in the note after Proposition 2.3. Moreover, $\alpha_v \in C^\infty_\lambda(\Lambda^2 T^*N)$ since the deformation of $N$ defined by $\alpha_v$, as in Definition 4.6, converges to $C$ at the same rate as $N$, recalling that $v$ preserves $C$.

Notice that, since $v \in g$, $G \subseteq G_2$ and $\varphi$ is invariant under $G_2$, the class of $\varphi$ in $H^3(\mathbb{R}^7, N)$, which is zero, is unchanged under flow along $v$. Let $u$ be the dilation vector field on $\mathbb{R}^7$ as given in (44). This leads us to define $\beta_v \in C^\infty(\Lambda^1 T^*N)$ by

$$\beta_v(x) = \frac{1}{3} \varphi(v, u, x)$$

for $x \in TN$. Thus, by (45), $d\beta_v = \alpha_v$ and, since $\alpha_v \in C^\infty_\lambda(\Lambda^2 T^*N)$ and $u = O(r)$ as $r \to \infty$, where $r$ is the radial distance in $\mathbb{R}^7$, $\beta_v \in C^\infty_{\lambda + 1}(\Lambda^1 T^*N)$.

Since $\alpha_v = d\beta_v$ is closed and self-dual, $d^*d\beta_v = 0$. If $\rho$ is a radius function on $N$ as in Definition 2.7, $d\beta_v = O(\rho^\lambda)$ and $\beta_v = O(\rho^{\lambda+1})$ as $\rho \to \infty$. As $L^2_{0, -2} = L^2$ by (4) and $\lambda < -2$, we see from Theorem 3.5(a) that $d\beta_v \in L^2$ and the following integration by parts argument is valid:

$$\|\alpha_v\|^2_{L^2} = \|d\beta_v\|^2_{L^2} = \langle d^*d\beta, \beta \rangle_{L^2} = 0.$$

Therefore, $\alpha_v = 0$, our required contradiction. □

9.1 SU(2)-invariant AC coassociative 4-folds

The coassociative 4-folds we discuss were constructed in [3, Theorem IV.3.2]. The construction involves the consideration of $\mathbb{R}^7$ as the imaginary part of the octonions, $\mathbb{O}$, and the definition of an action of SU(2) on $\text{Im} \mathbb{O}$ as unit elements in the quaternions, $\mathbb{H}$. We first define the action then state the result.

Definition 9.2 Write $\text{Im} \mathbb{O} \cong \text{Im} \mathbb{H} \oplus \mathbb{H}$. Define an action on $\text{Im} \mathbb{O}$ by

$$(x, y) \mapsto (qx\bar{q}, \bar{q}y)$$

for $q \in \mathbb{H}$ such that $|q| = 1$. This defines an action of SU(2) on $\text{Im} \mathbb{O} \cong \mathbb{R}^7$. 46
Proposition 9.3 Use the notation of Definition 9.2. Let \((e, f) \in \text{Im} \mathbb{H} \oplus \mathbb{H} \cong \text{Im} \mathcal{O}\) be such that \(|e| = |f| = 1\) and let \(c \in \mathbb{R}\). Then

\[ M_c = \{(s q e, r q f) : q \in \mathbb{H}, |q| = 1, \text{ and } s(4s^2 - 5r^2) = c \quad \text{for} \quad r \geq 0, \quad s \in \mathbb{R} \} \]

is an SU(2)-invariant coassociative 4-fold in \(\mathbb{R}^7 \cong \text{Im} \mathcal{O}\), where the action is given in Definition 9.2. Moreover, every coassociative 4-fold in \(\mathbb{R}^7\) invariant under this SU(2) action is of the form \(M_c\) for some choice of \(c, e, f\).

Suppose first that \(c = 0\). So, \(s = 0, s = \frac{\sqrt{5}}{2} r\) or \(s = -\frac{\sqrt{5}}{2} r\). Let

\[ M^0_c = \{(0, r q f) : r \geq 0\} \cong \mathbb{R}^4 \quad \text{and} \]

\[ M^{\pm}_c = \left\{ r \left( \pm \frac{\sqrt{5}}{2} q e q, q f \right) : r > 0 \right\} \cong (0, \infty) \times \mathcal{S}^3, \]

so that \(M_0 = M^0_0 \sqcup M^+_0 \sqcup M^-_0\). Thus, \(M_0\) is a SU(2)-invariant cone with three ends, two of which are diffeomorphic to cones on \(\mathcal{S}^3\) and one which is a flat \(\mathbb{R}^4\).

Suppose now \(c \neq 0\) and take \(c > 0\) without loss of generality. This forces \(s > 0\) and \(4s^2 - 5r^2 \neq 0\). It is then clear that \(M_c\) has two components, \(M^+_c\) and \(M^-_c\), corresponding to \(4s^2 - 5r^2 > 0\) and \(4s^2 - 5r^2 < 0\) respectively.

The component \(M^+_c\) is AC to the cone \(M^+_0\). We calculate the rate as follows. For large \(r\), \(s\) is approximately equal to \(\frac{\sqrt{5}}{2} r\) and hence \(4s^2 - 5r^2 = O(r^{-\frac{3}{2}})\). Therefore \(s = \frac{\sqrt{5}}{2} r + O(r^{-\frac{3}{2}})\) and thus \(M^+_c\) converges with rate \(-3/2\) to \(M^+_0\). For each \(r \neq 0\) we have an \(\mathcal{S}^3\) orbit in \(M^+_c\), but when \(r = 0\) there is an \(\mathcal{S}^2\) orbit. Therefore, topologically, \(M^+_c\) is an \(\mathbb{R}^2\) bundle over \(\mathcal{S}^2\). Hence \(H^3_{\text{dR}}(M^+_c) = \mathbb{R}\).

Suppose, for a contradiction, that \(b^2_2(M^+_c) = 1\), in the notation of Definition 9.3. Therefore, there exists a smooth, closed, self-dual 2-form in \(L^2 = L^2_{0, -2}\). The existence of this form, and the fact that the deformation theory of \(M^+_c\) is unobstructed by Corollary 6.5 means that there is a coassociative deformation \(\tilde{M}^+_c\) of \(M^+_c\) which is AC to \(M^+_0\) with rate less than \(-2\). However, \(M^+_0\) is SU(2)-invariant so, by Proposition 9.1, \(\tilde{M}^+_c\) must itself be invariant under SU(2). Proposition 9.3 describes the family of all coassociative 4-folds invariant under this action, so \(\tilde{M}^+_c\) must be AC with rate \(-3/2\), our required contradiction.

Hence, \(b^2_2(M^+_c) = 0\) and, since the SU(2) action has generic orbit \(\mathcal{S}^3\), we conclude that \(M^+_c\) is isomorphic to the bundle \(\mathcal{O}(-1)\) over \(\mathbb{C}\mathbb{P}^1\).

We now turn to \(M^-_c\). Here there are two ends, one of which has the same behaviour as the end of \(M^+_c\) and the other is where \(s \to 0\). For the latter case, we quickly see that \(s = O(r^{-4})\) and so \(M^-_c\) converges at rate \(-4\) to \(M^0_0\). As the case \(r = 0\) is excluded here, \(M^-_c\) is topologically \(\mathbb{R} \times \mathcal{S}^3\) and converges with rate \(-3/2\) to \(M^0_0\) and rate \(-4\) to \(M^0_0\) at its two ends. Hence \(H^3_{\text{dR}}(M^-_c) = 0\).
Consequently, for $c \neq 0$, $b_1^2(M_c) = 0$. Notice also that the link $\Sigma$ of the cone to which $M_c$ converges has $b_1(\Sigma) = 0$. Applying Corollary 6.3 and Proposition 7.12, the moduli space $\mathcal{M}(M_c, -3/2)$ is a smooth manifold with dimension

$$\sum_{\mu \in \mathcal{D} \cap (-\mathbf{2}, -3/2)} d(\mu),$$

using the notation of Proposition 5.4 and Definition 5.5.

**Note** Fox [2, Example 9.2 & Theorem 9.3] showed that the examples given in Proposition 9.3 are special cases of a surface bundle construction over a pseudoholomorphic curve in $S^6$ with null-torsion (in Proposition 9.3 the curve is a totally geodesic 2-sphere). The examples he produces are also coassociative 4-folds $N$ which are AC to some cone $C$ with rate $-3/2$, but the link $\Sigma$ of $C$ may now have $b_1(\Sigma) > 0$ and $b_2^1(N)$ may be non-zero.

### 9.2 2-ruled coassociative 4-folds

In [17], the author introduced the notion of 2-ruled 4-folds. A 4-dimensional submanifold $M$ in $\mathbb{R}^n$ is 2-ruled if it admits a fibration over a 2-fold $S$ such that each fibre is an affine 2-plane. As commented in [17, §3], if $S$ is compact then $M$ is AC with rate $\lambda \leq 0$ to its asymptotic cone. Therefore, 2-ruled 4-folds can be examples of AC submanifolds which are not cones. Unfortunately, the examples in [17, §5] of 2-ruled 4-folds are not coassociative, but are calibrated submanifolds of $\mathbb{R}^8$ known as Cayley 4-folds. However, the author has derived a system of ordinary differential equations whose solutions define 2-ruled coassociative 4-folds invariant under a $U(1)$ action. Though the solutions have not yet been found, it is clear that this system will provide AC examples.

The result [4, Theorem 3.5.1] gives a method of construction for coassociative 4-folds in $\mathbb{R}^7$ which are necessarily 2-ruled. However, the examples in [4] of coassociative 4-folds are not AC in the notion of Definition 2.5 but rather in a much weaker sense. In [2], Fox studies 2-ruled coassociative 4-folds, with a particular focus on cones. His works leads to a number of techniques for constructing 2-ruled coassociative 4-folds.

Recently the author ([20, Theorem 7.5]) has explicitly classified the 2-ruled coassociative cones and showed that the general cone can be constructed using a pseudoholomorphic curve in $S^6$ and holomorphic data on the curve. It is the hope of the author to use this result to produce nontrivial examples of AC 2-ruled coassociative 4-folds, using both the methods in [17] and other means.
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