An elementary solution to the Busemann-Petty problem

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Abstract

A unified analytic solution to the Busemann-Petty problem was recently found by Gardner, Koldobsky and Schlumprecht. We give an elementary proof of their formulas for the inverse Radon transform of the radial function $\rho_K$ of an origin-symmetric star body $K$.

Let $K$ and $L$ be two symmetric convex bodies in $\mathbb{R}^n$ such that for every hyperplane $H$ through the origin

$$\text{Vol}_{n-1}(K \cap H) \leq \text{Vol}_{n-1}(L \cap H);$$

does it follow that $\text{Vol}_n(K) \leq \text{Vol}_n(L)$? The answer to this question of Busemann and Petty [BP] is negative for $n \geq 5$ (Gardner [G1], Papadimitrakis [P]) and positive for smaller dimensions (Gardner [G2] for $n = 3$, Zhang [Z2] for $n = 4$). A unified solution was recently provided by Gardner, Koldobsky and Schlumprecht in [GKS], using Fourier transform. We give an elementary proof of their formulas for the inverse Radon transform of the radial function $\rho_K$. 

On $\mathbb{R}^n$, we denote the scalar product by $\langle \cdot, \cdot \rangle$ and the Euclidean norm by $| \cdot |$. We write $B^n$ for the unit ball and $S^{n-1}$ for the unit sphere, and $v_n$, $s_{n-1}$ denote their respective volumes. If $K \subset \mathbb{R}^n$ is a star body, its radial function $\rho_K$ is defined for every $x \in S^{n-1}$ by

$$\rho_K(x) = \sup \{ \lambda > 0; \lambda x \in K \}.$$ 

The connection between the Busemann-Petty problem and the spherical Radon transform $R$ is due to Lutvak [L]. Recall that $R$ acts on the space of continuous functions on $S^{n-1}$ by setting

$$Rf(\xi) = \int_{S^{n-1} \cap \xi^\perp} f(u) \, d\sigma_{n-2}(u)$$

for every $\xi \in S^{n-1}$; here $\sigma_{n-2}$ is the Haar measure of total mass $s_{n-2}$ on principal $n-2$ spheres. It follows from Lutvak [L], Zhang [Z2], that the Busemann-Petty problem has a positive answer in $\mathbb{R}^n$ if and only if every symmetric convex body $K$ in $\mathbb{R}^n$, with positive curvature and $C^\infty$ radial function, is such that $R^{-1}\rho_K$ is a non-negative function. In [GKS], the authors express $R^{-1}\rho_K$ in terms of

$$A_\xi(t) = \text{Vol}_{n-1}(K \cap (t\xi + \xi^\perp)), \quad \xi \in S^{n-1}$$
as follows:

**Theorem** Let $n \geq 3$. Let $K \subset \mathbb{R}^{n}$ be an origin-symmetric star body, with $C^\infty$ radial function $\rho_{K}$.

If $n$ is even, then

$$(-1)^{\frac{n}{2}} 2^{n} \pi^{n-2} \rho_{K} = R \left( \xi \mapsto A_{\xi}^{(n-2)}(0) \right).$$

If $n$ is odd, then

$$\frac{(-1)^{\frac{n+1}{2}} (2\pi)^{n-1}}{(n-2)!} \rho_{K} = R \left( \xi \mapsto \int_{0}^{\infty} t^{-n+1} \left( A_{\xi}(t) - \sum_{k=0}^{\frac{n-3}{2}} A_{\xi}^{(2k)}(0) \frac{t^{2k}}{(2k)!} \right) dt \right).$$

**Remark.** Let us recall why this solves the case $n = 4$ of the Busemann-Petty problem ([Z2], [GKS]). If $n = 4$, then $R^{-1} \rho_{K}(\xi) = -A''_{\xi}(0)/16\pi^{2}$. If $K$ is convex and symmetric, the latter is non-negative (by Brunn-Minkowski, the largest hyperplane section orthogonal to $\xi$ is indeed the one through the origin).

**Proof.** We first compute the Radon transform of $\xi \mapsto A_{\xi}(t)$, for any given $t \geq 0$. Let $e \in S^{n-1}$ and set $f(t) := R(\xi \mapsto A_{\xi}(t))(e)$. We identify $e^{\perp}$ and $\mathbb{R}^{n-1}$, and for $y \in \mathbb{R}^{n-1}$, we set $\phi(y) = \text{Vol}_{1}(K \cap (y + Re))$. Then

$$f(t) = \int_{S^{n-1}} \int_{x \in \mathbb{R}^{n}, \langle x, \xi \rangle = t} 1_{K}(x) d^{n-1}(x) d\sigma_{n-2}(\xi)$$

$$= \int_{S^{n-1}} \int_{y \in e^{\perp}, \langle y, \xi \rangle = t} \phi(y) d^{n-2}(y) d\sigma_{n-2}(\xi).$$

Considered as a function of $g$, the quantity

$$\int_{S^{n-1} \cap e^{\perp}} \int_{y \in e^{\perp}, \langle y, \xi \rangle = t} g(y) d^{n-2}(y) d\sigma_{n-2}(\xi)$$

(where $g$ is defined on $e^{\perp} \simeq \mathbb{R}^{n-1}$) is linear, continuous and rotation invariant. Hence there exists a measure $\mu_{t}$ on $\mathbb{R}^{+}$ such that for all $g$ the previous expression is equal to

$$\int_{\mathbb{R}^{+}} \left( \int_{S^{n-2}} g(ru) d\sigma_{n-2}(u) \right) d\mu_{t}(r).$$

Applying the definition of $\mu_{t}$ with the function $g = 1_{r B^{n-1}}$ yields

$$s_{n-2} \mu_{t}([0, r]) = \int_{S^{n-2}} \int_{\langle y, \xi \rangle = t} 1_{r B^{n-1}}(y) d^{n-2}(y) d\sigma_{n-2}(\xi)$$

$$= s_{n-2} \nu_{n-2} 1_{\{t \leq r\}}(r^{2} - t^{2})^{\frac{n-2}{2}}.$$

Consequently, $d\mu_{t}(r) = s_{n-3} r(r^{2} - t^{2})^{\frac{n-4}{2}} \Phi(r) dr$. Thus we have proved that

$$f(t) = s_{n-3} \int_{t}^{\infty} r(r^{2} - t^{2})^{\frac{n-4}{2}} \Phi(r) dr,$$
We conclude by exchanging the order of the Radon transform and the derivative. The first term is a polynomial in $t$.

Let

$$
F(t) = \int_0^\infty r(r^2 - t^2)^{\frac{n-4}{2}} \Phi(r) \, dr - t^{n-2} \int_0^1 u(u^2 - 1)^{\frac{n-4}{2}} \Phi(u) \, du.
$$

Then the quantity $\frac{F(t) - P(t)}{t^{n-1}}$ is equal to

$$
\int_t^\infty \left( \sum_{k=\frac{n-1}{2}}^{\infty} a_k (t^{-1} r)^{n-2-2k} \right) \Phi'(r) \, dr - \int_0^t \left( \sum_{k=0}^{\frac{n-1}{2}} a_k (t^{-1} r)^{n-2-2k} \right) \Phi'(r) \, dr
$$

$$
= \int_1^\infty \left( \sum_{k=\frac{n-1}{2}}^{\infty} a_k u^{n-2-2k} \right) \Phi'(tu) \, du - \int_0^1 \left( \sum_{k=0}^{\frac{n-1}{2}} a_k u^{n-2-2k} \right) \Phi'(tu) \, du.
$$

Notice that $\Phi$ is even, compactly supported and $C^\infty$ in some neighborhood of the origin. Our aim now is to relate $f(t)$ and $\Phi(0) = 2\rho_K(e) s_{n-2}$. The case $n = 4$ is very simple: $f(t) = 2\pi \int_0^\infty r \Phi(r) \, dr$, hence $f''(0) = -2\pi \Phi(0) = -16\pi^2 \rho_K(e)$. By exchanging the order of the Radon transform and the derivative, we conclude that $\rho_K$ is the Radon transform of $\xi \mapsto -A''_\xi(0)/16\pi^2$.

**If $n$ is even:**

$$
\frac{f(t)}{s_{n-3}} = \int_0^\infty r(r^2 - t^2)^{\frac{n-4}{2}} \Phi(r) \, dr - t^{n-2} \int_0^1 u(u^2 - 1)^{\frac{n-4}{2}} \Phi(u) \, du.
$$

The first term is a polynomial in $t$, of degree $n - 4$ and $\Phi$ is $C^\infty$ in some neighborhood of 0, thus

$$
f''(0) = -s_{n-3}(n-2)! \int_0^1 u(u^2 - 1)^{\frac{n-4}{2}} \Phi(0) \, du = (-1)^{\frac{n-2}{2}} 2^n \pi^{n-2} \rho_K(e).
$$

We conclude by exchanging the order of the Radon transform and the derivative.

**If $n$ is odd:** the basic principle is still very simple, but the technical details are slightly unpleasant. We shall begin by writing the proof as if $\Phi$ were $C^\infty$ on $\mathbb{R}$; but this is not true, because there are points of $e^\perp$ where our initial function $\phi$ is not differentiable, for example the points of the boundary of the projection of $K$ on $e^\perp$; we shall indicate afterwards the standard approximation argument that fixes this difficulty. Integrating by parts, we get

$$
F(t) := -\frac{n-2}{s_{n-3}} f(t) = \int_t^\infty (r^2 - t^2)^{\frac{n-2}{2}} \Phi'(r) \, dr.
$$

For $k \geq 0$, let $a_k = (-1)^k \left( \frac{n-2}{k} \right) = (-1)^k \frac{1}{k!} \prod_{j=0}^{k-1} \left( \frac{n-2}{2} - j \right)$. Notice that $\sum |a_k| < \infty$. Let

$$
P(t) = \sum_{k=0}^{\frac{n-3}{2}} a_k t^{2k} \int_0^\infty r^{n-2-2k} \Phi'(r) \, dr.
$$

Then the quantity $\frac{F(t) - P(t)}{t^{n-1}}$ is equal to

$$
\int_t^\infty \left( \sum_{k=\frac{n-1}{2}}^{\infty} a_k (t^{-1} r)^{n-2-2k} \right) \Phi'(r) \, dr - \int_0^t \left( \sum_{k=0}^{\frac{n-1}{2}} a_k (t^{-1} r)^{n-2-2k} \right) \Phi'(r) \, dr
$$

$$
= \int_1^\infty \left( \sum_{k=\frac{n-1}{2}}^{\infty} a_k u^{n-2-2k} \right) \Phi'(tu) \, du - \int_0^1 \left( \sum_{k=0}^{\frac{n-1}{2}} a_k u^{n-2-2k} \right) \Phi'(tu) \, du.
$$
By Fubini’s theorem and since \( \int_0^\infty \Phi'(tu) \, dt = -\Phi(0)/u \), we get
\[
\int_0^\infty \frac{F(t) - P(t)}{t^{n-1}} \, dt = \Phi(0) \left( \sum_{k=0}^{\infty} \frac{a_k}{n - 2 - 2k} \right) = c_n \rho_K(e),
\]
which is finite. Thus, \( P \) is the Taylor polynomial of \( F \) of order \( n - 3 \) at zero, and the above integral represents the action of the distribution \( t_+^{n+1} \) on \( F \). We obtain therefore
\[
\langle t_+^{n+1}, R(\xi \rightarrow A_\xi (t))(e) \rangle = -c_n \frac{s_{n-3}}{n-2} \rho_K(e).
\]
A soft manner to compute \( c_n \) is to replace \( \Phi \) by \( G(x) = e^{-x^2} \) in the previous computation. Once again, we end the proof by exchanging the order in which the Radon transform and the distribution \( t_+^{n+1} \) act (we shall give some explanation about this at the end).

We now explain how to deal with the fact that \( \Phi \) is not \( C^\infty \) everywhere. To every continuous and even function \( \Phi_1 \) on \( \mathbb{R} \), which is \( C^\infty \) in a neighborhood of 0 and supported on a fixed interval \([-R, R]\) containing the support of \( \Phi \), we associate the even function \( F_1 \) on \( \mathbb{R} \) defined for \( t \geq 0 \) by
\[
F_1(t) := -(n - 2) \int_t^\infty r(r^2 - t^2)^{\frac{n-4}{2}} \Phi_1(r) \, dr.
\]
Let \( Q(u) \) be the Taylor polynomial of degree \( n - 3 \) for \((1 - u^2)^{(n-4)/2}\) at the origin, and let \( P_1(t) := -(n - 2) \int_0^\infty r^{n-3}Q(t/r)\Phi_1(r) \, dr \) (of course, \( F_1 = F \) and \( P_1 = P \) when \( \Phi_1 = \Phi \)). One can get easily the following estimates (where \( C(n, R) \) or \( C(a, n, R) \) denote constants depending only upon \( n, R \) or \( a, n, R \)):

1. First, \( \|F_1\|_\infty \leq R^{n-2} \|\Phi_1\|_\infty \);
2. For every \( t \), we have \( |P_1(t)| \leq C(n, R) (1 + |t|^{n-3}) \|\Phi_1\|_\infty \);
3. Finally, when \( \Phi_1 \) vanishes on some neighborhood \((-a, a)\) of 0, one can see that \( |F_1(t) - P_1(t)| \leq C(a, n, R) t^{n-1} \|\Phi_1\|_\infty \) for \( 0 \leq t \leq 1 \).

These three estimates imply that the integral \( \int_0^\infty t^{-n+1}(F_1(t) - P_1(t)) \, dt \) converges to \( \int_0^\infty t^{-n+1}(F(t) - P(t)) \, dt \) when we let \( \Phi_1 \), equal to \( \Phi \) on a fixed interval \([-a, a]\) and supported on \([-R, R]\), tend uniformly to \( \Phi \).

Let us turn finally to the interchange of the actions of the Radon transform and the distribution \( t_+^{n+1} \) on the function \((\xi, t) \rightarrow A_\xi (t)\). It follows from our hypothesis that this function is \( C^\infty \) on \( S^{n-1} \times (-a, a) \) for some \( a > 0 \). Let us assume \( n = 5 \) for example. Since \( K \) is symmetric, we may write
\[
A_\xi(t) = f_0(\xi) + t^2 f_2(\xi) + t^4 g(\xi, t)
\]
where \( f_0, f_2 \) and \( g \) are continuous and bounded on \( S^{n-1} \) and \( S^{n-1} \times \mathbb{R} \) respectively. Since \( A_\xi \) vanishes for \( |t| > R \), we have \( g(\xi, t) = -t^{-4} f_0(\xi) - t^{-2} f_2(\xi) \) for \( t > R \), and
\[
\langle A_\xi, t_+^{n+4} \rangle = \int_0^R g(\xi, t) \, dt - \frac{R^{-3}}{3} f_0(\xi) - R^{-1} f_2(\xi),
\]
which shows that the interversion with the integral over $\xi \in S^{n-1}$ causes no trouble. □

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