PERFECT HYPERMOMENTUM FLUID:
VARIATIONAL THEORY
AND EQUATIONS OF MOTION

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Abstract

The variational theory of the perfect hypermomentum fluid is developed. The new type of the generalized Frenkel condition is considered. The Lagrangian density of such fluid is stated, and the equations of motion of the fluid and the Weyssenhoff-type evolution equation of the hypermomentum tensor are derived. The expressions of the matter currents of the fluid (the canonical energy-momentum 3-form, the metric stress-energy 4-form and the hypermomentum 3-form) are obtained. The Euler-type hydrodynamic equation of motion of the perfect hypermomentum fluid is derived. It is proved that the motion of the perfect fluid without hypermomentum in a metric-affine space coincides with the motion of this fluid in a Riemann space.

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1. INTRODUCTION

The Weyssenhoff–Raabe perfect spin fluid\textsuperscript{1,2} has a wide range of application in modern cosmology and astrophysics.\textsuperscript{3−7} The variational theory of this fluid is based on accounting the constraints in the Lagrangian density of the fluid with the help of Lagrange multipliers. Such theory was developed in case of a Riemann–Cartan space in Refs. 8 – 17 and in case of a metric-affine space in Refs. 13, 16. On the other variational methods of the perfect spin fluid in a Riemann–Cartan space see Refs. 18 and 19.

The Weyssenhoff–Raabe perfect spin fluid represents the particular case of the theory of the perfect fluid with intrinsic degrees of freedom. The natural generalization of the Weyssenhoff–Raabe perfect spin fluid is the perfect spin fluid with color charge, every particle of which is endowed with spin and intrinsic non-Abelian color charge.\textsuperscript{20−26} The variational theory of the fluid of such type was constructed in Refs. 21 – 23, 16, 25, 26. This model of the fluid can be used in nuclear physics for the generalization of the hydrodynamical description of adron multiple appearance\textsuperscript{27} (Landau model) and also in quark-gluon plasma physics and astrophysics.

The other significant generalization of the Weyssenhoff–Raabe perfect spin fluid is the perfect dilaton-spin fluid, every particle of which is endowed with intrinsic spin and dilatonic charge.\textsuperscript{28,29} Significance of matter with dilatonic charge is based on the fact that a low-energy effective string theory is reduced to the theory of interacting metric and dilatonic field.\textsuperscript{30}

The essential meaning of the fluid models discussed above consists in the fact that dynamics of this type of matter imposes the constraints on a metric and a connection of the space-time manifold and generates the Riemann–Cartan or the Weyl–Cartan geometrical structures of space-time.

The generalization of the spin and dilaton dynamical variables is the hypermomentum tensor $J^\alpha_\beta$ introduced in Ref. 31. It has the decomposition,

$$ J_{\alpha\beta} = S_{\alpha\beta} + \hat{J}_{(\alpha\beta)} + \frac{1}{4} g_{\alpha\beta} J, $$

(1.1)
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\[ S_{\alpha\beta} := J_{[\alpha\beta]}, \quad J := g^{\mu\nu} J_{\mu\nu}, \quad \hat{J}_{(\alpha\beta)} = J_{(\alpha\beta)} - \frac{1}{4} g_{\alpha\beta} J, \quad (1.2) \]

where \( S_{\alpha\beta} \) is the spin tensor, \( J \) is the dilatonic charge and \( \hat{J}_{(\alpha\beta)} \) is the intrinsic proper hypermomentum (shear) tensor.

The natural generalization of the perfect spin fluid and the perfect dilaton-spin fluid is the perfect fluid, every particle of which is endowed with intrinsic hypermomentum. This new hypothetical type of matter was announced in Refs. 32, 16 and named perfect fluid with intrinsic hypermomentum. The variational theory of such fluid was developed by various authors.\(^{33-43}\) In Refs. 36, 44 this type of matter was named hyperfluid. We shall name this type of fluid as perfect hypermomentum fluid.

The real existence of the hypermomentum fluid is the fact of fundamental meaning because as a source of gravitational field it generates the new type of space-time geometry, namely the geometry of a metric-affine space \((L_4, g)\) (see Ref. 44 and references therein). Nowadays, the metric-affine theory of gravitation arouses an interest in connection with the problem of the relation of the gravitation and the elementary particles physics\(^{45}\) and with the problem of renormalizability of the gravitational theory.\(^{46}\)

We shall develop the phenomenological macroscopic approach in the description of the perfect hypermomentum fluid. In this approach the physical variables which characterize the fluid (such as energy density, pressure, entropy, hypermomentum etc.) are considered to be smooth functions of space-time coordinates. This approach is compatible with the statistical continuous medium approach by means of the assumption that a fluid element represents a statistical subsystem with sufficiently large number of particles. The fluid element can be considered as a quasiclosed system with the properties that coincide with macroscopic properties of the fluid. The physical variables characterizing the fluid are regarded to be averaged over the fluid element.

The theory of the perfect fluid with intrinsic degrees of freedom being developed, the additional intrinsic degrees of freedom of a fluid element are described by the four vectors \( \vec{l}_p \) \((p = 1, 2, 3, 4)\), called directors, attached with each element of the fluid. Three of the
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directors \((p = 1, 2, 3)\) are space-like and the fourth one \((p = 4)\) is time-like.

In Riemann and Riemann–Cartan spaces a fluid element endowed with directors moves according to the Fermi transport that preserves the orthonormalization of the directors. In a metric-affine space, in which a metric and a connection are not compatible it is naturally consider the directors to be \textit{elastic}\textsuperscript{36–38} in the sense that they can undergo arbitrary deformations during the motion of the fluid. Nevertheless, in most theories it is accepted that the time-like director is collinear to the 4-velocity \(u^\alpha\) of the fluid element and orthogonality of space-like directors to a 4-velocity is maintained.

The distinction of the variational machinery consists in using the generalized Frenkel condition,\textsuperscript{32,33,36}

\[
J_{\alpha\beta} u^\beta = 0 , \quad J_{\alpha\beta} u^\alpha = 0 ,
\]

or the Frenkel condition in its standard classical form,\textsuperscript{37,39,40,42}

\[
S_{\alpha\beta} u^\beta = 0 ,
\]

where \(J_{\alpha\beta}\) and \(S_{\alpha\beta}\) are the specific (per particle) intrinsic hypermomentum tensor and the specific spin tensor of a fluid element, respectively. Another possibility is so called “unconstrained hyperfluid”\textsuperscript{41}, in which any type of Frenkel condition is absent.

In Ref. 41 it is mentioned that in case of the generalelized Frenkel condition (1.3) the dilatonic charge of a fluid element is expressed in terms of the shear tensor (see the decomposition (1.1)): \(J = (4/c^2)\hat{J}_{\alpha\beta} u^\alpha u^\beta\). In this case the hypermomentum fluid can not be of the pure dilatonic type with \(\hat{J}_{\alpha\beta} = 0\) and \(J \neq 0\). On the other hand, the Frenkel condition (1.4) leads to the unusual form of the evolution equation of the hypermomentum tensor,\textsuperscript{39,40} which does not demonstrate the Weyssenhoff-type dynamics. As to the unconstrained hyperfluid, in Ref. 41 it is stated that such type of fluid does not contain the Weyssenhoff spin fluid as a particular case. Therefore all three kinds of the approaches mentioned are not satisfactory from physical point of view.

In this paper we consider the new type of the generalized Frenkel condition, which allows to construct the hypermomentum perfect fluid theory with the dilaton-spin fluid and the
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Weyssenhoff spin fluid as particular cases. In our approach it is also essential that all four
directors are elastic. None of the orthogonality conditions of the four directors is maintained
during the motion of the fluid. Besides, the time-like director needs not to be collinear to
the 4-velocity of the fluid element. On the preliminary version of our results see Ref. 43.

Our paper is organized as follows.

In Sec. 2 the dynamical variables and constraints of the theory are discussed. In Sec. 3
the Lagrangian density of the perfect hypermomentum fluid is stated, and the equations of
motion of the fluid are derived. In Sec. 4 the Weyssenhoff-type evolution equation of the
hypermomentum tensor is stated. Then in Sec. 5 the expressions of the matter currents of
the hypermomentum fluid (the canonical energy-momentum 3-form, the metric stress-energy
4-form and the hypermomentum 3-form) are obtained. Sec. 6 is devoted to the derivation
of the Euler-type hydrodynamic equation of motion of the perfect hypermomentum fluid.
At last the peculiarities of the hypermomentum fluid motion are discussed in Sec. 7.

We use the exterior form variational method according to Trautman\textsuperscript{47,48}.

2. THE DYNAMICAL VARIABLES AND CONSTRAINTS

In the exterior form language the material frame of the directors turns into the coframe
of 1-forms \( l^p \) (\( p = 1, 2, 3, 4 \)), which have dual 3-forms \( l_q \), while the constraint

\[
l^p \wedge l_q = \delta^p_q \eta , \quad l^\alpha_l^\beta = \delta^\beta_\alpha, \quad (2.1)
\]

being fulfilled, where \( \eta \) is the volume 4-form and the component representations are intro-
duced,

\[
l^p = l^p_\alpha \theta^\alpha , \quad l_q = l^\beta_q \eta_\beta . \quad (2.2)
\]

Here \( \theta^\alpha \) is a 1-form basis and \( \eta_\beta \) is a 3-form defined as\textsuperscript{47}

\[
\eta_\beta = \bar{e}_\beta \eta = \ast \theta_\beta , \quad \theta^\alpha \wedge \eta_\beta = \delta^\alpha_\beta \eta , \quad (2.3)
\]
where $\mathcal{I}$ means the interior product, $\star$ is the Hodge dual operator and $\bar{e}_\beta$ is a basis vector, a coordinate system being nonholonomic in general.

Each fluid element possesses a 4-velocity vector $\bar{u} = u^\alpha \bar{e}_\alpha$ which is corresponded to a flow 3-form $u$ (Ref. 48), $u := \bar{u} \mathcal{I} \eta = u^\alpha \eta_\alpha$ and a velocity 1-form $\star u = u_\alpha \theta^\alpha = g(\bar{u},)$ with

$$\star u \wedge u = -c^2 \eta, \quad (2.4)$$

that means the usual condition $g(\bar{u}, \bar{u}) = -c^2$, where $g(,)$ is the metric tensor.

A fluid element moving, the fluid particles number and entropy conservation laws are fulfilled,

$$d(nu) = 0, \quad d(nsu) = 0, \quad (2.5)$$

where $n$ is the fluid particles concentration equal to the number of fluid particles per a volume unit, and $s$ is the the specific (per particle) entropy of the fluid in the rest frame of reference, respectively.

The measure of ability of a fluid element to perform the intrinsic motion is the quantity $\Omega_{p}^q$ which generalizes the fluid element “angular velocity” of the Weyssenhoff spin fluid theory. It has the form

$$\Omega_{p}^q \eta := u \wedge l_p^\alpha D l_p^\alpha, \quad (2.6)$$

where $\mathcal{D}$ is the exterior covariant differential with respect to a connection 1-form $\Gamma^\alpha_{\beta}$,

$$\mathcal{D} l_p^\alpha = dl_p^\alpha + \Gamma^\alpha_{\beta} l_p^\beta. \quad (2.7)$$

An element of the fluid with intrinsic hypermomentum possesses the additional “kinetic” energy 4-form,

$$E = \frac{1}{2} n J_{q}^{p} \Omega_{p}^{q} \eta = \frac{1}{2} n J_{q}^{p} u \wedge l_p^\alpha \mathcal{D} l_p^\alpha, \quad (2.8)$$

where $J_{q}^{p} := J^\alpha_{\beta} l_p^\alpha l_q^\beta$ is the specific intrinsic hypermomentum tensor representing the new dynamical quantity which generalizes the spin density of the Weyssenhoff fluid.
The hypermomentum tensor $J^p_q$ can be decomposed into irreducible parts,

$$J^p_q = \hat{J}^p_q + \frac{1}{4} \delta^p_q J , \quad J := J^p_p , \quad \hat{J}^p_p = 0 , \quad (2.9)$$

$$\hat{J}^p_q := S^p_q + \hat{J}^{(p)}_q , \quad S^p_q := J^{[p} q] , \quad \hat{J}^{(p)}_q = J^{(p)}_q - \frac{1}{4} \delta^p_q J . \quad (2.10)$$

Here $S^p_q$ is the specific spin tensor, $J$ is the specific dilatonic charge and $\hat{J}^{(p)}_q$ is the specific intrinsic proper hypermomentum (shear) tensor of a fluid element, respectively. We shall name the quantity $\hat{J}^p_q$ as the specific traceless hypermomentum tensor.

It is well-known that the spin tensor is spacelike in its nature that is the fact of fundamental physical meaning. This leads to the classical Frenkel condition, $S^\alpha_\beta u^\beta = 0$. We shall suppose here that the full traceless part of the hypermomentum tensor $\hat{J}^p_q$ (not only the spin tensor but also the tensor $\hat{J}^{(p)}_q$) has such property and therefore satisfies the generalized Frenkel conditions in the form,

$$\hat{J}^p_q u^p = 0 , \quad u^p := u^p_{\alpha p} , \quad (2.11)$$

$$\hat{J}^p_q u^q = 0 , \quad u^q := u^q_{\alpha q} , \quad (2.12)$$

which can be written in the following way,

$$\hat{J}^p_q l^p \wedge \ast u = 0 , \quad (2.13)$$

$$\hat{J}^p_q l^q \wedge u = 0 . \quad (2.14)$$

The Frenkel conditions (2.11), (2.12) are equivalent to the equality,

$$\Pi^r_l \Pi^l_q \hat{J}^r_t = \hat{J}^p_q , \quad \Pi^p_p := \delta^p_r + \frac{1}{c_2} u^p u_r . \quad (2.15)$$

Here $\Pi^p_p$ is the projection tensor, which separates the subspace orthogonal to the fluid velocity.

The internal energy density of the fluid $\varepsilon$ depends on the extensive (additive) thermodynamic parameters $n$, $s$, $J^p_q$ and obeys to the first thermodynamic principle,

$$d\varepsilon(n, s, J^p_q) = \frac{\varepsilon + p}{n} dn + n T ds + \frac{\partial \varepsilon}{\partial J^p_q} dJ^p_q , \quad (2.16)$$
where $p$ is the hydrodynamic fluid pressure and $T$ is the temperature.

We shall consider as independent variables the quantities $n$, $s$, $J^q$, $u$, $l^q$, $\theta^\sigma$, $\Gamma^\beta_\alpha$, the constraints (2.4), (2.5), (2.13), (2.14) being taken into account in the Lagrangian density by means of the Lagrange multipliers.

In what follows we need the variation,

$$
\delta \eta = \eta \frac{1}{2} g^{\alpha \beta} \delta g_{\alpha \beta} + \delta \theta^\sigma \wedge \eta_\sigma . \tag{2.17}
$$

As a result of the relation $\theta^\alpha \wedge u = u^\alpha \eta$ one has,

$$
\eta \delta u^\alpha = -\delta u \wedge \theta^\alpha + \delta \theta^\alpha \wedge u - u^\alpha \delta \eta . \tag{2.18}
$$

The relation $\ast u = g_{\alpha \beta} u^\alpha \theta^\beta$ yields the variation,

$$
\delta \ast u = g_{\alpha \beta} \theta^\alpha \delta u^\beta + u^\alpha \theta^\beta \delta g_{\alpha \beta} + u^\beta g_{\alpha \beta} \delta \theta^\sigma . \tag{2.19}
$$

As a result of the resolution of the constraints (2.1) and with the help of the relations (2.3), one can derive the variations,

$$
\eta \delta l^p_\alpha = -\delta \theta^\sigma \wedge \eta_\alpha l^p_\sigma + \delta l^p \wedge \eta_\alpha , \tag{2.20}
$$

$$
\eta \delta l^\alpha_p = \delta \theta^\alpha \wedge l_p - \delta l^q \wedge l^\alpha_q l_p . \tag{2.21}
$$

### 3. THE LAGRANGIAN DENSITY AND THE EQUATIONS OF MOTION OF THE FLUID

The perfect fluid Lagrangian density 4-form of the perfect hypermomentum fluid should be chosen as the remainder after subtraction the internal energy density of the fluid $\varepsilon$ from the “kinetic” energy (2.8) with regard to the constraints (2.4), (2.5), (2.13), (2.14) which should be introduced into the Lagrangian density by means of the Lagrange multipliers $\lambda$, $\varphi$, $\tau$, $\chi^q$, $\zeta_p$, respectively. As a result of the previous section the Lagrangian density 4-form has the form
\[ \mathcal{L}_m = L_m \eta = -\varepsilon(n, s, J^p_q)\eta + \frac{1}{2} n J^p_q u \wedge l^q \dot{D} l^p_\alpha + n u \wedge d\varphi + n \tau u \wedge ds + n \lambda (u \wedge u + c^2 \eta) + n \chi^q \dot{J}^p_q l_p \wedge \wedge u + n \zeta^p \dot{J}^p_q l^q \wedge u. \] (3.1)

The fluid motion equations and the evolution equation of the hypermomentum tensor are derived by the variation of (3.1) with respect to the independent variables \( n, s, J^{pq}, u, l^q \), and the Lagrange multipliers, the thermodynamic principle (2.16) being taken into account.

We shall consider the 1-form \( l^q \) as an independent variable and the 3-form \( l^p_q \) as a function of \( l^q \) by means of (2.1). As a result of such variational machinery one gets the constraints (2.4), (2.5), (2.13), (2.14) and the following variational equations,

\[
\begin{align*}
\delta n : & \quad (\varepsilon + p)\eta - \frac{1}{2} n J^p_q u \wedge l^q \dot{D} l^p_\alpha - n u \wedge d\varphi = 0, \quad (3.2) \\
\delta s : & \quad T \eta + u \wedge d\tau = 0, \quad (3.3) \\
\delta J^p_q : & \quad \frac{\partial \varepsilon}{\partial J^p_q} = \frac{1}{2} n \Omega^q_p - n (\chi^q u_q - \zeta^q u^q) + \frac{1}{4} n \delta^p_q (\chi^r u_r - \zeta^r u^r), \quad (3.4) \\
\delta u : & \quad d\varphi + \tau ds - 2 \lambda * u + \chi^q \dot{J}^p q \theta^\beta - \zeta^p \dot{J}^q u^q l^q + \frac{1}{2} J^p_q \delta^q l^p = 0, \quad (3.5) \\
\delta l^q : & \quad \frac{1}{2} \dot{J}^p \dot{J}^q l^p_\sigma - \chi^r \dot{J}^p q l_q l_p - \zeta^p \dot{J}^q q u = 0. \quad (3.6)
\end{align*}
\]

Here the “dot” notation for the tensor object \( \Phi \) is introduced,

\[ \dot{\Phi}^{\alpha \beta} := *(u \wedge D \Phi^{\alpha \beta}). \] (3.7)

Multiplying the equation (3.5) by \( u \) from the left externally and using (2.5) and (3.2), one derives the expression for the Lagrange multiplier \( \lambda \),

\[ 2 n c^2 \lambda = \varepsilon + p. \] (3.8)

As a consequence of the equation (3.2) and the constraints (2.4), (2.5), (2.13), (2.14) one can verify that the Lagrangian density 4-form (3.1) is proportional to the hydrodynamic fluid pressure, \( \mathcal{L}_m = p \eta \), which corresponds to Ref. 11.
4. THE EVOLUTION EQUATION OF THE HYPERMOMENTUM TENSOR

The variational equation (3.6) represents the evolution equation of the hypermomentum tensor. Multiplying the equation (3.6) by \( l^\beta \theta^\alpha \wedge \ldots \) from the left externally one gets,

\[
\frac{1}{2} J_\alpha^\beta - \chi^r J_r^\alpha u_\beta - \zeta_r J_r^\beta u_\alpha = 0 .
\]  

(4.1)

Contractions (4.1) with \( u_\alpha \) and then with \( u^\beta \) yield the expressions for the Lagrange multipliers,

\[
\zeta_r J_r^\beta = -\frac{1}{2c^2} J_\gamma^\beta u_\gamma ,
\]

(4.2)

\[
\chi^r J_r^\alpha = -\frac{1}{2c^2} J_\gamma^\alpha u_\gamma .
\]

(4.3)

After the substitution of (4.2) and (4.3) into (4.1) one gets the evolution equation of the hypermomentum tensor,

\[
\dot{J}_\alpha^\beta + \frac{1}{c^2} J_\gamma^\alpha u_\gamma u_\beta + \frac{1}{c^2} J_\beta^\gamma u_\gamma u_\alpha = 0 .
\]

(4.4)

This equation generalizes the evolution equation of the spin tensor in the Weyssenhoff fluid theory.

The equation (4.4) has the consequence,

\[
\dot{J}_\alpha^\beta u_\alpha u_\beta = 0 ,
\]

(4.5)

which permits to represent the evolution equation of the hypermomentum tensor (4.4) in the form,

\[
\Pi_\sigma^\alpha \Pi_\beta^\rho J_\sigma^\beta = 0 ,
\]

(4.6)

where the projection tensor \( \Pi_\sigma^\alpha \) has been defined in (2.15). The evolution equation of the hypermomentum tensor in the form (4.6) was derived in Ref. 42.

The contraction (4.4) on the indices \( \alpha \) and \( \beta \) gives with the help of (4.5) the dilatonic charge conservation law,

\[
\dot{J} = 0 .
\]

(4.7)
5. THE ENERGY-MOMENTUM TENSOR OF THE PERFECT HYPERMOMENTUM FLUID

With the help of the matter Lagrangian density (3.1) one can derive the external matter currents which are the sources of the gravitational field. In case of the perfect hypermomentum fluid the matter currents are the canonical energy-momentum 3-form $\Sigma_\sigma$, the metric stress-energy 4-form $\sigma^{\alpha\beta}$ and the hypermomentum 3-form $J^{\alpha\beta}$, which are determined as variational derivatives.

The variational derivative of the explicit form of the Lagrangian density (3.1) with respect to $\theta^\sigma$ yields the canonical energy-momentum 3-form,

$$
\Sigma_\sigma := \frac{\delta L_m}{\delta \theta^\sigma} = -\varepsilon \eta_\sigma + 2\lambda n u_\sigma u + 2c^2 \lambda n \eta_\sigma - n \chi^r_j (g_{\rho q} l_\rho^i u + u_\sigma l_\eta) + \frac{1}{2} n J^\rho_\rho \eta_\rho .
$$

(5.1)

Using the explicit form of the Lagrange multiplier (3.8), one gets,

$$
\Sigma_\sigma = p \eta_\sigma + \frac{1}{c^2} (\varepsilon + p) u_\sigma u + \frac{1}{c^2} n J^\rho_\rho \eta_\rho - n \chi^r_j (g_{\rho q} u + l_\rho^i u_\sigma) .
$$

(5.2)

On the basis of the evolution equation of the hypermomentum tensor (4.4) and with the help of (4.3) the expression (5.2) reads,

$$
\Sigma_\sigma = p \eta_\sigma + \frac{1}{c^2} (\varepsilon + p) u_\sigma u + \frac{1}{c^2} n g_{\alpha[\sigma} J^\alpha_{\beta]} u^\beta u .
$$

(5.3)

After some algebra one can get the other form of the canonical energy-momentum 3-form,

$$
\Sigma_\sigma = p \eta_\sigma + \frac{1}{c^2} (\varepsilon + p) u_\sigma u + \frac{1}{c^2} n S_{\alpha[\sigma} Q_{\beta]\gamma u^\gamma u^\alpha u ,
$$

(5.4)

where $Q_{\alpha\beta\gamma}$ are components of a nonmetricity 1-form,

$$
Q_{\alpha\beta} := -D g_{\alpha\beta} = Q_{\alpha\beta\gamma} \theta^\gamma .
$$

(5.5)

The metric stress-energy 4-form can be derived in the same way,

$$
\sigma^{\alpha\beta} := 2 \frac{\delta L_m}{\delta g_{\alpha\beta}} = T^{\alpha\beta} \eta ,
$$

$$
T^{\alpha\beta} = -\varepsilon g^{\alpha\beta} + 2n \lambda (u^\alpha u^\beta + c^2 g^{\alpha\beta}) - 2n \chi^r_j (g_{\rho q} l_\rho^i u + u_\sigma l_\eta) = pg^{\alpha\beta} + \frac{1}{c^2} (\varepsilon + p) u^\alpha u^\beta + \frac{1}{c^2} n J^{(\alpha\beta)}_{\gamma} u^\gamma .
$$

(5.6)
For the hypermomentum 3-form one finds
\[ \mathcal{J}^\alpha_\beta := -\frac{\delta \mathcal{L}_m}{\delta \mathcal{\Gamma}^\beta_\alpha} = \frac{1}{2} n J^\alpha_\beta u. \] (5.7)

Let us consider the special case of the perfect spin fluid with dilatonic charge, a fluid element of which does not possess the specific shear momentum tensor, \( \hat{J}^{(\rho q)} = 0 \), and is endowed only with the specific spin momentum tensor \( S^{pq} \) and the specific dilatonic charge \( J \). In this case the canonical energy-momentum 3-form (5.3) reads,
\[ \Sigma_\sigma = p \eta_\sigma + \frac{1}{c^2} (\varepsilon + p) u_\sigma u + \frac{1}{c^2} n g_{\alpha[\sigma} \dot{S}^\alpha_{\beta]} u^\beta u, \] (5.8)
where the specific energy density \( \varepsilon \) contains the energy density of the dilatonic interaction of the fluid. This expression coincides with the expression of the canonical energy-momentum 3-form of the perfect dilaton-spin fluid obtained in Ref. 28. If the dilatonic charge also vanishes, \( J = 0 \), then the expression (5.8) will describe the canonical energy-momentum 3-form of the Weyssenhoff perfect spin fluid in a metric-affine space.

6. The Hydrodynamic Equation of Motion of the Perfect Hypermomentum Fluid

As in case of the perfect dilaton-spin fluid the hydrodynamic Euler-type equation of motion of the perfect hypermomentum fluid can be derived as the consequence of the covariant energy-momentum quasi-conservation law. In a general metric-affine space the matter Lagrangian obeys the diffeomorphism invariance and the local \( GL(4, R) \)-gauge invariance that leads (when the equations of matter motion are fulfilled) to the corresponding Noether identities (see Ref. 44 and references therein):
\[ \mathcal{D} \Sigma_\sigma = (\bar{\epsilon}_\sigma | T^\alpha) \wedge \Sigma_\alpha - (\bar{\epsilon}_\alpha | R^\alpha_\beta) \wedge \mathcal{J}^\beta_\alpha - \frac{1}{2} (\bar{\epsilon}_\sigma | Q_{\alpha\beta}) \sigma^{\alpha\beta}, \] (6.1)
\[ \mathcal{D} \mathcal{J}^\alpha_\beta - \theta^\alpha \wedge \Sigma_\beta + \sigma^{\alpha\beta} = 0, \] (6.2)
where \( R^{\alpha\beta} \) is a curvature 2-form and \( T^\alpha \) is a torsion 2-form. It can be verified that the expressions of the canonical energy-momentum 3-form (5.3), the metric stress-energy 4-form
(5.6) and the hypermomentum 3-form (5.7) are compatible in the sense that they satisfy to the Noether identities (6.1) and (6.2).

Let us introduce a specific (per particle) dynamical momentum of a fluid element,

$$\pi_\sigma \eta := -\frac{1}{nc^2} * u \wedge \Sigma_\sigma , \tag{6.3}$$

$$\pi_\sigma = \frac{\varepsilon}{nc^2} u_\sigma - \frac{1}{c^2} S_{\sigma \rho \bar{\mu}} [\mathcal{D} u^\rho - \frac{1}{2c^2} \hat{J}_\sigma^\lambda u^\gamma \bar{u}] Q_{\lambda \gamma} . \tag{6.4}$$

Then the canonical energy-momentum 3-form (5.3) reads,

$$\Sigma_\sigma = p \eta_\sigma + n \left( \pi_\sigma + \frac{p}{nc^2} u_\sigma \right) u . \tag{6.5}$$

Let us substitute (6.5), (5.6) and (5.7) into (6.1) and take into account the fluid particles number conservation law (2.5) and the equality

$$\mathcal{D} \eta_\alpha = T^\beta \wedge \eta_{\alpha \beta} - \frac{1}{2} Q \wedge \eta_\alpha , \tag{6.6}$$

where according to Trautman\textsuperscript{47} 2-form fields $\eta_{\alpha \beta}$ are used,

$$\eta_{\alpha \beta} = \bar{e}_\beta \eta_\alpha = *(\theta_\alpha \wedge \theta_\beta) , \quad \theta^\sigma \wedge \eta_{\alpha \beta} = -2 \delta^\sigma_{[\alpha} \eta_{\beta]} , \tag{6.7}$$

and a Weyl 1-form $Q$ is introduced,

$$Q := g^{\alpha \beta} Q_{\alpha \beta} , \quad Q = Q_\alpha \theta^\alpha . \tag{6.8}$$

After some algebra one obtains the equation of motion of the perfect hypermomentum fluid in the form of the generalized hydrodynamic Euler-type equation,

$$u \wedge \mathcal{D} \left( \pi_\sigma + \frac{p}{nc^2} u_\sigma \right) = \frac{1}{n} \eta_\sigma \bar{e}_\sigma dp - (\bar{e}_\sigma ) T^\alpha \wedge \left( \pi_\alpha + \frac{p}{nc^2} u_\alpha \right) u$$

$$+ \frac{1}{2} (\bar{e}_\sigma ) R^\alpha_{\beta \gamma} \wedge J^\beta_\alpha u + \frac{1}{2} (\bar{e}_\sigma ) Q_{\alpha \beta} \left( \frac{\varepsilon + p}{nc^2} u^\alpha u^\beta + \frac{1}{c^2} (\bar{u} \mathcal{D} J^\alpha_{\gamma \mu} u^\beta u^\gamma ) \right) \eta . \tag{6.9}$$

The equation of the hypermomentum fluid motion (6.9) has the important consequence, which can be derived with the help of the procedure of the decomposition of the connection,\textsuperscript{44}

$$\Gamma^\alpha_{\beta} = \bar{\Gamma}^\alpha_{\beta} + \Delta^\alpha_{\beta} , \quad \Delta^\alpha_{\beta} = g^{\alpha \gamma} \left( \frac{1}{2} Q_{\gamma \beta} - \bar{e}_{[\gamma} Q_{\beta] \mu} \theta^\mu \right) , \tag{6.10}$$
where $\Gamma^{\alpha}_{\beta\gamma}$ denotes a connection 1-form of a Riemann–Cartan space $U_4$ with curvature, torsion and metric compatible with connection, and $\Delta^\alpha_{\beta\gamma}$ is so called a connection defect 1-form. The decomposition (6.10) of the connection induces corresponding decomposition of the curvature,

$$R^\alpha_{\beta\gamma} = \overline{\Gamma}^\alpha_{\beta\gamma} + \overline{\mathcal{D}} \Delta^\alpha_{\beta\gamma} + \Delta^\alpha_{\gamma\beta} = \overline{\Gamma}^\alpha_{\beta\gamma} + \frac{1}{4} \delta^\alpha_{\beta\gamma} \mathcal{R}^\tau_{\tau} + \mathcal{P}^\alpha_{\beta\gamma}, \quad \mathcal{P}^\alpha_{\beta\gamma} = \mathcal{P}[^{\alpha}_{\beta\gamma}] ,$$  

(6.11)

$$\mathcal{R}^\tau_{\tau} = \frac{1}{2} \mathcal{D} \mathcal{Q} = \frac{1}{2} (\dot{e}_\lambda \mathcal{D} Q_{\tau\lambda}) \theta^\lambda \wedge \theta^\tau + \frac{1}{2} Q_{\tau\tau} \mathcal{T}^\tau = \frac{1}{2} d\mathcal{Q} ,$$

(6.12)

where $\mathcal{D}$ is the exterior covariant differential with respect to the Riemann–Cartan connection 1-form $\overline{\Gamma}^\alpha_{\beta\gamma}$, and $\overline{\Gamma}^\alpha_{\beta\gamma}$ is the Riemann–Cartan curvature 2-form, $\mathcal{R}^\tau_{\tau}$ is the Weyl homothetic curvature 2-form. The corresponding decomposition of the specific dynamical momentum of a fluid element (6.4) reads,

$$\pi^\sigma = \frac{\varepsilon}{nc^2} u^\sigma + \frac{1}{c^2} \overline{\mathcal{D}} S^\rho_{\sigma\rho} u^\rho - \frac{1}{2c^2} (\overline{\dot{u}} \mathcal{D} \Delta^\rho_{\lambda}) \dot{J}^\rho_{\sigma\lambda} - \frac{1}{2c^2} (\overline{\dot{u}} \Delta^\lambda_{\rho}) \dot{J}^\rho_{\sigma\lambda} .$$

(6.13)

**Theorem 1.** If the equation of motion of the perfect hypermomentum fluid (6.9) is valid (the fluid particles number conservation law $d(nu) = 0$ being fulfilled), then the energy evolution equation along a streamline of the fluid reads,

$$\dot{\varepsilon} = \frac{\varepsilon + p}{n} \dot{n} ,$$

(6.14)

the motion of the perfect hypermomentum fluid is isentropic and the hypermomentum tensor evolution does not contribute to the energy change of a fluid element.

**Proof.** Let us evaluate the component of the equation (6.9) along the 4-velocity by contracting one with $u^\sigma$ and then use the decompositions (6.10), (6.13) and the Frenkel conditions (2.11), (2.12). The left-hand side of the equation obtained reads,

$$u^\sigma u^\wedge \mathcal{D} \left( \pi^\sigma + \frac{p}{n} \frac{\dot{u}}{c^2} \right) = (\overline{\dot{u}} \mathcal{D} \Delta^\rho_{\lambda}) \dot{J}^\rho_{\sigma\lambda} - \frac{1}{c^2} (\overline{\dot{u}} \Delta^\lambda_{\rho}) \dot{J}^\rho_{\sigma\lambda} .$$

(6.15)
\[
\frac{1}{n} \eta (\bar{u} \mid dp) + \frac{1}{2} (\bar{u} \mid Q^{\sigma \rho}) \left( \varepsilon + \frac{p}{nc^2} \right) u_\sigma u_\rho \eta
\]
\[
- \frac{1}{2c^2} (\bar{u} \mid Q^{\sigma \rho}) u_\sigma \hat{J}_{\rho \lambda} (\bar{u} \mid D u^\lambda) \eta - \frac{1}{2c^2} (\bar{u} \mid Q^{\alpha \sigma}) u_\sigma (\bar{u} \mid \Delta^{\beta \rho}) u_\rho \hat{J}_{\alpha \beta} \eta .
\] (6.16)

Equating (6.15) and (6.16) and taking into account the relation

\[
\Delta (\sigma \rho) = \frac{1}{2} Q_{\sigma \rho} ,
\] (6.17)

one obtains the equation,

\[
\bar{u} \mid d \left( \varepsilon + \frac{p}{n} \right) = \frac{1}{n} \bar{u} \mid dp .
\] (6.18)

As in case of the usual perfect fluid this equation means that along a streamline of the fluid the energy evolution equation takes the form (6.14). Comparing the equation (6.14) with the first thermodynamic principle (2.16), one can conclude that along a streamline of the fluid the conditions

\[
\dot{s} = 0 , \quad \frac{\partial \varepsilon}{\partial J_{pq}} \dot{J}_{pq} = 0
\] (6.19)

are valid. The first of these equalities means that the entropy conservation law is fulfilled along a streamline of the fluid and therefore that the motion of the perfect hypermomentum fluid is isoentropic. The second of these equalities means that the hypermomentum tensor evolution does not contribute to the energy change of the fluid element. As was to be proved.

**Remark.** The first equation (6.19) is the consequence of (2.5). The second equation (6.19) can be derived as the consequence of the fluid motion equation (3.4), the generalized Frenkel conditions (2.11), (2.12) and the evolution equation of the hypermomentum tensor (4.4). Therefore the conclusion (6.19) of the Theorem 1 means the consistency of the theory.

### 7. PECULIARITIES OF THE HYPERMOMENTUM FLUID MOTION

**Theorem 2.** When the traceless hypermomentum tensor vanishes, \( \hat{J}_{pq} = 0 \), the motion of the perfect hypermomentum fluid in a metric-affine space \((L_4, g)\) obeys the equation,
\[ u \wedge R \left( \frac{\varepsilon + p}{nc^2} u_\sigma \right) = \frac{1}{n} \eta \tilde{e}_\sigma | dp + \frac{1}{16} (\tilde{e}_\sigma | dQ) \wedge J u , \quad (7.1) \]

where \( R \) is the exterior covariant differential with respect to a Riemann (Levi–Civita) connection 1-form \( \tilde{\Gamma}_\alpha^\beta \).

**Proof.** Using the decomposition the nonmetricity 1-form on the Weyl’s piece and the traceless piece,

\[ Q_{\alpha\beta} = \tilde{Q}_{\alpha\beta} + \frac{1}{4} g_{\alpha\beta} Q , \quad g^{\alpha\beta} \tilde{Q}_{\alpha\beta} = 0 , \quad (7.2) \]

let us represent the connection 1-form of a metric-affine space \((L_4, g)\) as follows,

\[ \Gamma^\alpha_\beta = \tilde{\Gamma}^\alpha_\beta + W^\alpha_\beta + \tilde{\Delta}^\alpha_\beta , \quad \tilde{\Delta}^\alpha_\alpha = 0 , \quad (7.3) \]

where the connection defect 1-form corresponding to a connection 1-form of a Weyl space is introduced,

\[ \Delta^\alpha_\beta = \frac{1}{8} (\delta^\alpha_\beta Q + 2\theta^{\alpha[\beta} Q_{\gamma]}) . \quad (7.4) \]

In turn, the Riemann–Cartan connection 1-form can be decomposed as follows,

\[ \Gamma^\alpha_\beta = \tilde{\Gamma}^\alpha_\beta + \mathcal{K}^\alpha_\beta , \quad \mathcal{T}^\alpha =: \mathcal{K}^\alpha_\beta \wedge \theta^\beta , \quad (7.5) \]

\[ \mathcal{K}_{\alpha\beta} = 2\tilde{e}_[\alpha | T_{\beta]} - \frac{1}{2} \tilde{e}_\alpha \tilde{e}_\beta (T_\gamma \wedge \theta^\gamma) , \quad (7.6) \]

where \( \mathcal{K}^\alpha_\beta \) is a kontorsion 1-form.\(^{44}\) Taking into account the decompositions (7.2), (7.3), (7.5) and the expression for the specific dynamical momentum of a fluid element (6.4), the hydrodynamic equation of motion of the hypermomentum fluid (6.9) in case of \( \hat{P}^\alpha_q = 0 \) reads,

\[ u \wedge (\delta^\lambda_\sigma R \right. \frac{\varepsilon + p}{nc^2} u_\lambda) = \frac{1}{n} \eta \tilde{e}_\sigma | dp + \frac{1}{8} (\tilde{e}_\sigma | R^\lambda_\lambda) \wedge J u \]

\[-(\tilde{e}_\sigma | T^\lambda) \wedge \left( \frac{\varepsilon + p}{nc^2} \right) u_\lambda u - \frac{1}{8} (\tilde{e}_\sigma | Q) \left( \frac{\varepsilon + p}{n} \right) \eta + \frac{1}{2} (\tilde{e}_\sigma | \tilde{Q}_{\alpha\beta}) \left( \frac{\varepsilon + p}{nc^2} \right) u^\alpha u^\beta \eta . \quad (7.7) \]

By the direct calculations with the help of (6.10) and (7.6) one can check the equalities,

\[ u \wedge K^\lambda_\sigma u_\lambda = (\tilde{e}_\sigma | T^\lambda) \wedge u_\lambda u , \quad (7.8) \]

\[ u \wedge W^\lambda_\sigma u_\lambda = \frac{c^2}{8} (\tilde{e}_\sigma | Q) \eta , \quad (7.9) \]

\[ u \wedge \tilde{\Delta}^\lambda_\sigma u_\lambda = -\frac{1}{2} (\tilde{e}_\sigma | \tilde{Q}_{\alpha\beta}) u^\alpha u^\beta \eta . \quad (7.10) \]
On the basis of the equalities (7.8)–(7.10) and (6.12) the equation (7.7) yields the equation (7.1), as was to be proved.

**Corollary 1.** When the traceless hypermomentum tensor vanishes, the motion of the perfect hypermomentum fluid in a metric-affine space \((L_4, g)\) coincides with the motion of the perfect fluid with dilatonic charge in a Weyl–Cartan space \(Y_4\), the metric tensor, the torsion 2-form and the Weyl 1-form of which coincide with the metric tensor, the torsion 2-form and the Weyl 1-form of \((L_4, g)\), respectively.

**Proof.** These two statements are the consequences of the fact that in the equalities (7.8)–(7.10) torsion and traceless nonmetricity are arbitrary and can be put to be equal to zero. If only the traceless 1-form is equal to zero one gets a Weyl–Cartan space. If both the nonmetricity traceless 1-form and the torsion 2-form are equal to zero, then one gets a Weyl space. In these cases the equation of motion of the perfect hypermomentum fluid (7.1) in a metric-affine space \((L_4, g)\) coincides with the equations of motion of the perfect fluid with dilatonic charge in \(Y_4\) and \(W_4\), respectively, which are the particular cases (when \(S_{\alpha\beta} = 0\)) of the equation of motion of the perfect dilaton-spin fluid derived in Ref. 29.

**Corollary 3.** If all irreducible pieces of the hypermomentum tensor vanish, \(J_{\alpha\beta} = 0\), then the equation of motion of the perfect hypermomentum fluid in a metric-affine space \((L_4, g)\) coincides with the equation of motion of the usual perfect fluid in a Riemann space \(V_4\), the metric tensor of which coincides with the metric tensor of \((L_4, g)\).

**Proof.** If \(J = 0\) the equation (7.1) with the help of the Theorem 1 yields the equation of motion of the usual perfect fluid in a Riemann space,\(^{49}\)

\[
\frac{1}{c^2}(\varepsilon + p)\vec{u} \cdot \vec{D} u_\sigma = -\Pi_\sigma^\rho e_\rho dp ,
\]

(7.11)

where \(\Pi_\sigma^\rho\) is the projection tensor (2.15).
Corollary 4. The motion of the test particle with dilatonic charge in a metric-affine space $(L_4, g)$ coincides with the motion of this particle both in a Weyl-Cartan space $Y_4$ and in a Weyl space $W_4$; if the dilatonic charge of the particle vanishes, then the motion of this test particle in $(L_4, g)$ is realized along geodesics of a Riemann space $V_4$, the metric tensor of which coincides with the metric tensor of $(L_4, g)$.

Proof. The equation of motion of the test particle with dilatonic charge and with the mass $m_0 = \frac{\varepsilon}{(nc^2)} = \text{const}$ in a metric-affine space is obtained as the limiting case of the equation of motion (7.1) when the pressure $p$ vanishes. The statement of the Corollary 4 follows from the statements of the Corollaries 1–3.

The statements of the Corollaries 3 and 4 are the particular cases of the Theorem stated in Refs. 50 – 52 for the matter motion in a general metric-affine space.

Theorem. In $(L_4, g)$ the motion of matter without hypermomentum coincides with the motion in a Riemann space-time $V_4$, the metric tensor of which coincides with the metric tensor of $(L_4, g)$.

8. CONCLUSIONS

The essential feature of the constructed variational theory of the hypermomentum perfect fluid is the assumption that the frame realized by all four directors is elastic. The deformation of the directors during the motion of the fluid element, from one side, generates the space-time nonmetricity and, from the other side, allows nonmetricity of the space-time to be discovered. As the consequence of this fact the Lagrangian density (3.1) does not contain the term maintaining the orthogonality of the directors. The time-like director needs not to be collinear to the 4-velocity of the fluid element. The essential feature of our variational approach is the using the Frenkel conditions for the traceless hypermomentum tensor in the form (2.11) and (2.12), which do not coincide nor with their classical form, when the Frenkel condition is imposed on the spin tensor, $S_{\alpha\beta}u^\beta = 0$, nor with its generalized form, when the Frenkel condition is imposed on the full intrinsic hypermomentum tensor, $J_{\alpha\beta}u^\beta =$
\[ J_{\alpha\beta}u^\alpha = 0. \]

The expression of the energy-momentum tensor of the fluid (5.3) coincides with one obtained in the other approaches\textsuperscript{36,37}. But our approach does not contain the shortcomings, which are inherent in the previous theories of the hypermomentum perfect fluid. First of all, our variational theory contains the Weyssenhoff spin fluid as the particular case that is important from physical point of view. Then, the evolution equation of the hypermomentum tensor (4.4) demonstrates the Weyssenhoff-type dynamics. At last, the perfect hypermomentum fluid theory developed allows to describe as the special case the perfect spin fluid with dilatonic charge. It should be important to investigate the consequences of the employing the perfect fluid of such type as the gravitational field source in cosmological and astrophysical problems.

The perfect hypermomentum fluid model represents the medium with spin, shear momentum and dilatonic charge which generate the space-time metric-affine geometrical structure and interact with it. The influence of this geometry on fluid motion is described by the Euler-type hydrodynamic equation (6.9), the properties of which are established by Theorem 1 of Sec. 6. The main features of this motion are its isoentropic character and the fact that the hypermomentum tensor evolution does not contribute to the energy change of the fluid element.

The peculiarities of the hypermomentum fluid motion are discovered in Theorem 2 of Sec. 7. The important consequences of this Theorem mean that bodies and mediums without hypermomentum are not subjected to the influence of the possible nonmetricity of the space-time (in contrast to the generally accepted opinion) and therefore can not be the tools for the detection of the deviation of the space-time properties from the Riemann space structure.

Therefore for the investigation of the different manifestations of the possible space-time nonmetricity one needs to use the bodies and mediums endowed with the hypermomentum, i.e. particles and fluids with spin, dilatonic charge or with intrinsic hypermomentum.
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