Path generating transforms

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Abstract. We study combinatorial aspects of $q$-weighted, length-$L$ Forrester-Baxter paths, $P_{p,p',a,b,c}(L)$, where $p, p', a, b, c \in \mathbb{Z}_+$, $0 < p < p'$, $0 < a, b, c < p'$, $c = b \pm 1$, $L + a - b \equiv 0 \pmod{2}$, and $p$ and $p'$ are co-prime.

We obtain a bijection between $P_{p,p',a,b,c}(L)$ and partitions with certain prescribed hook differences. Thereby, we obtain a new description of the $q$-weights of $P_{p,p',a,b,c}(L)$. Using the new weights, and defining $s_0$ and $r_0$ to be the smallest non-negative integers for which $|ps_0 - p'r_0| = 1$, we restrict the discussion to $P_{s_0}^{p,p'}(L)$, and introduce two combinatorial transforms:

1: A Bailey-type transform $B$: $P_{s_0}^{p,p'}(L) \rightarrow P_{s_0+p}^{p,p'}(L')$, $L \leq L'$,

2: A duality-type transform $D$: $P_{s_0}^{p,p'}(L) \rightarrow P_{s_0+p}^{p,p'}(L)$.

We study the action of $B$ and $D$, as $q$-polynomial transforms on the $P_{s_0}^{p,p'}(L)$ generating functions, $\chi_{s_0}^{p,p'}(L)$. In the limit $L \rightarrow \infty$, $\chi_{s_0}^{p,p'}(L)$ reduces to the Virasoro characters, $\chi_{s_0}^{p,p'}$, of minimal conformal field theories $\mathcal{M}^{p,p'}$, or equivalently, to the one-point functions of regime-III Forrester-Baxter models.

As an application of the $B$ and $D$ transforms, we re-derive the constant-sign expressions for $\chi_{s_0}^{p,p'}$, first derived by Berkovich and McCoy.

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0. Introduction

0.1. Motivation. Many problems in exactly solvable models can be formulated as problems in combinatorics. In particular, the computation of \textit{one-point functions} in two dimensional lattice models \cite{7}, can be reduced to the evaluation of the generating functions of certain combinatorial objects known as \textit{one-dimensional configurations}. The purpose of this work is to study the combinatorics of a specific class of one-dimensional configurations called \textit{paths}. In particular, we wish to study the paths that originate in computing one-point functions in an infinite series of lattice models introduced by Forrester and Baxter \cite{19}.

Roughly speaking, we would like to show how a set of paths that belongs to a certain model in an infinite series, can be systematically obtained from a simpler model in the same series. The idea is to reduce the computation of one-point functions, in any model, recursively, down to a computation in the simplest possible model in the series. This latter computation can be trivial.

The above idea is not new. It originates, at the level of \textit{q}-series, in the Bailey transform \cite{6}, and in its extensions by Andrews \cite{4}. Combinatorially, it also appears at the level of \textit{q}-polynomials that count partition pairs \cite{14}. Finally, and closest to the spirit of this work, it appears in \cite{1, 11}, where certain infinite paths are \textit{q}-counted. In this work, we formulate the above idea for certain finite length paths, and thereby we are able to obtain their generating functions.

Our long term aim is to show that solutions in distinct models within a family, are not entirely independent, but are related to each other. Furthermore, if these solutions are understood combinatorially\footnote{In fact, we restrict our attention to paths of \textit{regime-III} Forrester-Baxter models. We refer the reader to \cite{19} for definitions of the various regimes. From now on, we use the words ‘Forrester-Baxter models’, with the above restriction in mind.} then one can obtain one from the other systematically, using combinatorial transforms. We anticipate that our approach will have applications to more general models, such as those based on the affine algebras \textit{sl}(n).

0.2. \(\mathcal{M}^{p,p'}\) and \(\chi^{p,p'}_{r,s}\). Consider the minimal conformal field theories, \(\mathcal{M}^{p,p'}\), of Belavin, Polyakov, and Zamolodchikov \cite{8} where \(p, p' \in \mathbb{Z}^+\) are co-prime and \(1 < p < p'\). \(\mathcal{M}^{p,p'}\) are also the critical limits of the exactly solvable Forrester-Baxter models \cite{19}.

The chiral half space of states \(\mathcal{M}^{p,p'}\) \cite{8} may be considered as the union of a certain set of highest weight Virasoro modules of central charge \(c = 1 - 6(p-p')^2/pp'\). Consider the characters, \(\chi^{p,p'}_{r,s}\), where \(p, p', r, s \in \mathbb{Z}^+\), \(0 < r < p\), \(0 < s < p'\), of the irreducible highest weight modules of the Virasoro algebra which comprise \(\mathcal{M}^{p,p'}\). In the language of conformal field theories, they are the simplest examples of \textit{conformal blocks on the torus} \cite{16}. In the language of lattice models, they are \textit{one-point functions} up to a normalisation constant.

\footnote{For example, by showing that what one is doing is nothing but counting certain objects that satisfy certain properties.}

\footnote{For an introduction to conformal field theories, see \cite{16}.}

\footnote{In fact, when we consider finite versions of these characters, our analysis naturally includes the cases \(\{p = 1, p'\}\), which correspond to the \textit{parafermionic} models \cite{16}.}

\footnote{For an introduction to exactly solvable lattice models, see \cite{7}.}
Following Baxter’s corner transfer matrix method [7], $\chi_{r,s}^{p,p'}$ can be interpreted as the generating functions of infinite length paths, $\mathcal{P}^{p,p'}_{r,s}$, that satisfy certain restrictions. These restrictions are encoded in the labels $p, p', r, s$ [19]. $\chi_{r,s}^{p,p'}$ can also be obtained as the generating functions of partitions with prescribed hook differences [5]. There is no known physical or algebraic motivation for this latter description.

0.3. $\mathcal{P}_{a,b,c}^{p,p'}(L)$. The infinite length paths, $\mathcal{P}_{a,b,c}^{p,p'}$, can be regarded as the $L \to \infty$ limit of length-$L$ paths $\mathcal{P}_{a,b,c}^{p,p'}(L)$, where $0 < a, b, c < p', c = b \pm 1$, $L + a - b \equiv 0 \pmod{2}$. Here, $r, s$ are related to $a, b, c$ by $r = \lfloor pc/p' \rfloor + (b - c + 1)/2$, $s = a$. In this paper, we work entirely at the level of $\mathcal{P}_{a,b,c}^{p,p'}(L)$, only taking the limit $L \to \infty$ at the end.

0.4. Purpose. The purpose of this work is three-fold:

1. We describe a bijection between the finite length paths $\mathcal{P}_{a,b,c}^{p,p'}(L)$, and partitions that satisfy prescribed hook difference conditions [8]. Thereby, we obtain a new description of the $q$-weights of $\mathcal{P}_{a,b,c}^{p,p'}(L)$.

2. We introduce two combinatorial transforms, $\mathcal{B}$ and $\mathcal{D}$, that act on finite length paths, and that can be used recursively to generate $\mathcal{P}_{s_0}^{p,p'}(L) \equiv \mathcal{P}_{s_0,s_0,r_0}^{p,p'}(L)$, for all allowed $p, p'$, from the combinatorially trivial $\mathcal{P}_{1,3}^{1,3}(L')$. Here, $s_0$ is such that $s_0$ and $r_0$ are the smallest non-negative integers for which $|ps_0 - p' r_0| = 1$, and $L \geq 0$ is even.

3. As an application, we compute the generating functions of $\mathcal{P}_{s_0}^{p,p'}(L)$, and reproduce the constant sign expressions for the characters $\chi_{r,s}^{p,p'}$, that were first obtained in [10].

0.5. Organisation. Although the bijection between $\mathcal{P}_{a,b,c}^{p,p'}(L)$ and the corresponding partitions with prescribed hook differences is logically the starting point of this work, we relegate its discussion and proof to an appendix. This is so that we don’t deviate from the main point of the paper, namely the combinatorics of the Forrester-Baxter paths and the $\mathcal{B}$ and $\mathcal{D}$ transforms.

In §1, we introduce the paths that we are interested in $q$-counting and their $q$-weights as originally defined in [19]. We define extra structures on the paths, namely, the bands and their parities, an alternative prescription for path weights that follows from the bijection described in Appendix A, the scoring and non-scoring vertices, and the striking sequence of a path.

In §2, we introduce the $\mathcal{B}$-transform that maps $\mathcal{P}_{s_0}^{p,p'}(L)$, into $\mathcal{P}_{s_0+r_0}^{p,p'+p}(L')$ for various $L'$, where $s_0$ and $r_0$ are as defined above. We also define the particle content of a path.

In §3, we introduce the $\mathcal{D}$-transform that maps $\mathcal{P}_{s_0}^{p,p'}(L)$ to $\mathcal{P}_{s_0}^{p+p',p}(L)$.

In §4, we digress to discuss the continued fraction description of the paths, and the related zones. We also discuss the $\mathbf{mn}$-system associated with $\mathcal{P}_{s_0}^{p,p'}(L)$, and introduce a matrix that generalises the Cartan matrix of the Lie algebra $\mathfrak{a}_t$, and that was first defined in [10]. We then describe what we mean by a sector $S(\vec{n}) \subset \mathcal{P}_{s_0}^{p,p'}(L)$, labelled by $\vec{n} \in \mathbb{N}^t$.

In §5, we compute the constant-sign generating functions of $\mathcal{P}_{s_0}^{p,p'}(L)$. In particular, we re-derive the constant-sign expressions of the generating functions, first obtained in [10]. Finally, we discuss the $L \to \infty$ limit of these generating functions.
1. Paths

1.1. Definitions. Let $p$ and $p'$ be positive coprime integers for which $0 < p < p'$. Then, given $a, b, c, L \in \mathbb{Z}$ such that $1 \leq a, b, c \leq p' - 1$, $b = c \pm 1$, $L \geq 0$, $L + a - b \equiv 0 \pmod{2}$, a path $h \in \mathcal{P}_{a,b,c}(L)$ is a sequence $h_0, h_1, h_2, \ldots, h_L$, of integers such that:

1. $1 \leq h_i \leq p' - 1$ for $0 \leq i \leq L$,
2. $h_{i+1} = h_i \pm 1$ for $0 \leq i < L$,
3. $a = h_0, b = h_L$.

Note that the values of $p$ and $c$ do not feature in the above restrictions. As described below, they specify how the elements of $\mathcal{P}_{a,b,c}(L)$ are weighted.

1.2. Heights, segments, and vertices. The integers $h_0, h_1, h_2, \ldots, h_L$ are readily depicted as a sequence of heights on a two-dimensional $L \times (p' - 1)$ grid. Adjacent heights are connected by line segments passing from $(i, h_i)$ to $(i + 1, h_{i+1})$ for $0 \leq i < L$.

Scanning the path from left to right, each of these line segments points either in the NE direction or in the SE direction. The following is a typical path in the set $\mathcal{P}_{5,3,4}^{11}(28)$:

![Typical path](image)

Figure 1.

Two adjacent line segments, one passing from $(i - 1, h_{i-1})$ to $(i, h_i)$, and the other from $(i, h_i)$ to $(i + 1, h_{i+1})$, define a vertex $v_i$. There are four types of vertices. They appear as follows:

![Types of vertices](image)

Figure 2.

They will be referred to as a straight-up vertex, a straight-down vertex, a peak-up vertex and a peak-down vertex respectively.

1.3. Forrester-Baxter weights. In $[19]$, each vertex $v_i$ is assigned a weight $ic_{FB}$, where the local weight function $c_{FB}(h_{i-1}, h_i, h_{i+1})$ is defined by:
\[
\begin{align*}
c_{FB}(h-1, h, h+1) &= \frac{1}{2}; \\
c_{FB}(h+1, h, h-1) &= \frac{1}{2}; \\
c_{FB}(h, h+1, h) &= -\left\lfloor \frac{h(p'-p)}{p'} \right\rfloor; \\
c_{FB}(h, h-1, h) &= \left\lfloor \frac{h(p'-p)}{p'} \right\rfloor,
\end{align*}
\]
where \(|n|\) is the integer part of \(n\). A path \(h\) is assigned a weight \(wt(h)\) given by:

\[
wt(h) = \sum_{i=1}^{L} ic_{FB}(h_{i-1}, h_{i}, h_{i+1}),
\]

where we take \(h_{L+1} = c\). In [19], the generating function of \(P_{a,b,c}^{p,p'}(L)\) is defined to be:\[ \phi_{a,b,c}^{p,p'}(L) = \sum_{h\in P_{a,b,c}^{p,p'}} q^{wt(h)}. \]

An expression for \(\phi_{a,b,c}^{p,p'}(L)\) was derived in [19, Theorem 2.3.1]. It turns out that there is a very convenient renormalisation \(\chi_{a,b,c}^{p,p'}(L)\) of \(\phi_{a,b,c}^{p,p'}(L)\) (which we give explicitly in Appendix A), from which the expression of [19] yields:

\[
\chi_{a,b,c}^{p,p'}(L) = \sum_{\lambda=-\infty}^{\infty} q^{\lambda^2 pp' + \lambda(p' r - pa)} \left[ \frac{L}{2} \right]_{q}^{L+a-b - p'\lambda}
\]

\[
- \sum_{\lambda=-\infty}^{\infty} q^{(\lambda p + r)(\lambda p' + a)} \left[ \frac{L}{2} \right]_{q}^{L+a-b - p'\lambda - a},
\]

where

\[
r = \lfloor pc/p' \rfloor + (b - c + 1)/2,
\]
and, as usual, the Gaussian polynomial \([ A \choose B ]_q\) is defined to be:

\[
[A \choose B]_q = \frac{\prod_{i=1}^{A} (1 - q^i)}{\prod_{i=1}^{B} (1 - q^i) \prod_{i=1}^{A-B} (1 - q^i)}
\]

for \(0 \leq B \leq A\), and \([ A \choose B ]_q = 0\) otherwise. In the limit \(L \to \infty\), we obtain

\[
\lim_{L \to \infty} \chi_{a,b,c}^{p,p'}(L) = \chi_{r,a}^{p,p'},
\]

where \(r\) is defined in [19] and

\(\hfill^6\text{In [19], this generating function is denoted by either } D_{L}(a,b,c) \text{ or } D_{L}^{(k)}(a,b,c), \text{ where } k = [c(p'-p)/p']\).
\[
\chi^{p,p'}_{r,s} = \frac{1}{(q)_\infty} \sum_{\lambda=-\infty}^{\infty} (q^{\lambda^2pp'+\lambda(p'r-ps)} - q^{(\lambda p+r)(\lambda p'+s)})
\]

is, up to a normalisation, the Rocha-Caridi expression \[21\] for the Virasoro character of central charge \(c = 1 - 6(p' - p)^2/pp'\) and conformal dimension \(\Delta^{p,p'}_{r,s} = ((p'r - ps)^2 - (p' - p)^2)/4pp'\). Therefore, \(\chi^{p,p'}_{a,b,c}(L)\) provides a finite analogue of the character \(\chi^{p,p'}_{r,a}\).

The expression obtained above for \(\chi^{p,p'}_{a,b,c}(L)\) is an alternating-sign \(q\)-polynomial. This expression is not combinatorial in the sense that we know that \(\chi^{p,p'}_{a,b,c}(L)\) is a generating function, and therefore all its non-vanishing coefficients are positive. We shall seek constant-sign expressions for \(\chi^{p,p'}_{a,b,c}(L)\), which in the limit \(L \to \infty\) will provide constant-sign expressions for the Virasoro characters \(\chi^{p,p'}_{r,a}\).

1.4. Bands and parities. In the path picture described above, there are \((p' - 1)\) heights. The regions between adjacent heights will be referred to as bands. There are \((p' - 2)\) bands. We assign a parity to each band: a band that lies between heights \(h\) and \((h+1)\) is even if \(\lfloor hp/p'\rfloor = \lfloor (h+1)p/p'\rfloor\), and odd otherwise. Scanning from below, the \(r\)th odd band lies between heights \(\lfloor rp'/p\rfloor\) and \(\lfloor rp'/p\rfloor + 1\). We will shade the odd bands more heavily than the even bands, as shown in Fig. 3, where \(p = 3\) and \(p' = 11\).

![Figure 3.](image)

Since \(\lfloor hp/p'\rfloor = 0\) for \(h = 1\) and \(\lfloor hp/p'\rfloor = p - 1\) for \(h = p' - 1\), we deduce that there are \((p - 1)\) odd bands and \((p' - p - 1)\) even bands. Furthermore, we may also readily deduce that if \(p' > 2p\) then the odd bands occur in isolation, with an even band on both sides. If \(p' < 2p\), the reverse is true. Finally, it is easily seen that the odd/even band structure is invariant under an up/down reflection.

A parity may now be assigned to each vertex of a path: it is the parity of the band in which the right edge lies.

\[\text{For physical reasons, the alternating-sign expressions are also called \textit{bosonic}. The constant-sign expressions are also called \textit{fermionic}. The study of constant-sign expressions for the Virasoro characters was initiated by the Stony Brook group. For further details, and original references, we refer the reader to [10] and references therein.}\]
1.5. Alternative prescription for path weights. The analysis of Appendix A shows that it is possible to assign a weight \( w_t(h) \) to each path \( h \) such that

\[
\chi_{a,b,c}^{p,p'}(L) = \sum_{h \in P_{a,b,c}^{p,p'}(L)} q^{w_t(h)}.
\]

Here, we describe how \( w_t(h) \) may be simply calculated using the path picture together with its band structure. First, we define new coordinates on the picture as follows:

\[
x = \frac{i - (h - a)}{2}, \quad y = \frac{i + (h - a)}{2}.
\]

Thus, the \( xy \)-coordinate system has its origin at the path’s initial point, and is slanted at \( 45^\circ \) to the original \( ih \)-coordinate system. Note that at each step in the path, either \( x \) or \( y \) is incremented and the other is constant. In this system, the path depicted in Fig. 3 has its first few coordinates at \((0,0), (0,1), (0,2), (0,3), (1,3), (1,4), (1,5), (1,6), (2,6), \ldots\) Now, for the \( i \)th vertex, we define \( c_i = c(h_{i-1}, h_i, h_{i+1}) \) according to the shape of the vertex and its parity.

| Vertex | \( c_i \) |
|--------|----------|
|        |         |
|        |         |
|        |         |
|        |         |
|        |         |
|        |         |
|        |         |

Here the unshaded band can be either an even or an odd band (or in the lowermost four cases, not a band in the model at all). Note that each vertex shape only contributes in one parity case. We shall refer to those four vertices, with assigned parity, for which, in general, the contribution is non-zero, as \emph{scoring} vertices. The other four vertices will be termed \emph{non-scoring}.

We now define

\[
wt(h) = \sum_{i=1}^{L} c_i.
\]

To illustrate this procedure, consider again the path \( h \) depicted in Fig. 3. The above table indicates that there are scoring vertices at \( i = 2, 7, 9, 12, 14, 16, 17, 19, 22, 23, 24, 26 \). This leads to

\[
wt(h) = 0 + 1 + 6 + 6 + 6 + 8 + 8 + 9 + 9 + 13 + 10 + 14 = 90.
\]
1.6. Striking sequence of a path. Consider each path $h$ as a sequence of straight lines, alternating in direction between NE and SE. Reading from the left, let the lengths of these lines be $w_1, w_2, w_3, \ldots, w_l$, for some $l$, so that each $w_i > 0$ and $w_1 + w_2 + \cdots + w_l = L(h)$, where $L(h)$ is the length of $h$. For each of these lines, the last vertex will be considered to be part of the line but the first will not. Then, the $i$th of these lines contains $w_i$ vertices, the first $w_i - 1$ of which are straight vertices. Then write $w_i = a_i + b_i$ so that $b_i$ is the number of scoring vertices in the $i$th line. The striking sequence of $h$ is then the array:

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_l \\ b_1 & b_2 & b_3 & \cdots & b_l \end{pmatrix}.$$ 

We define $m(h) = \sum_{i=1}^l a_i$, whereupon $\sum_{i=1}^l b_i = L(h) - m(h)$. We also define $\beta(h) = (b_1 + b_3 + \cdots) - (b_2 + b_4 + \cdots)$.

For example, for the path shown in Fig. 3, the striking sequence is:

$$\begin{pmatrix} 2 & 1 & 2 & 3 & 1 & 1 & 0 & 3 & 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 \end{pmatrix}.$$ 

In this case, $m(h) = 16$, and $\beta(h) = 0$.

**Lemma 1.1.** Let the path $h$ have the striking sequence $\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_l \\ b_1 & b_2 & b_3 & \cdots & b_l \end{pmatrix}$, with $w_i = a_i + b_i$ for $1 \leq i \leq l$. Then

$$wt(h) = \sum_{i=1}^l b_i(w_{i-1} + w_{i-3} + \cdots + w_{1+(i-1)\mod 2}).$$ 

**Proof:** First assume that the first $w_1$ segments of $h$ are in the NE direction. Then, for $i$ odd, the $i$th line is in the NE direction and its $x$-coordinate is $w_2 + w_4 + \cdots + w_{i-1}$. By the prescription of the previous section, and the definition of $b_i$, this line thus contributes $b_i(w_2 + w_4 + \cdots + w_{i-1})$ to the weight $wt(h)$ of $h$. Similarly, for $i$ even, the $i$th line is in the SE direction and contributes $b_i(w_1 + w_3 + \cdots + w_{i-1})$ to $wt(h)$. This proves the lemma if the first segments are in the NE direction. The reasoning is almost identical for the other case. \qed

1.7. Restricting the endpoints of paths. In the rest of this work, we restrict our attention to the set of paths $\mathcal{P}_{s_0}^{p,p'}(L) \equiv \mathcal{P}_{s_0,s_0+1}^{p,p'}(L)$, where $L \geq 0$ is even and $s_0$ is defined to be such that $s_0$ and $r_0$ are the smallest non-negative integers for which $|ps_0 - pr_0| = 1$. In the limit $L \to \infty$, the corresponding generating functions reduce to the Virasoro characters related to the models $\mathcal{M}^{p,p'}$, with smallest possible highest weights. The line $h = s_0$ in the path picture will be referred to as the ground-line.

Using the fact that $p$ and $p'$ are co-prime, it is straightforward to deduce that there do exist $s_0$ and $r_0$ satisfying $0 \leq r_0 \leq s_0 \leq p' - 1$ such that $|ps_0 - pr_0| = 1$. In fact, in the case $p = 1$ we immediately obtain $s_0 = 1$ and $r_0 = 0$, and in the case $p = p' - 1$ we immediately obtain $s_0 = 1$ and $r_0 = 1$. Otherwise, if $1 < p < p' - 1$, then necessarily $1 < s_0 < p' - 1$. Moreover, if $ps_0 - pr_0 = 1$, so that $ps_0/p' = r_0 + 1/p'$, the $h = s_0$ line is immediately below the $r_0$th odd band, and above an even band. On the other hand, if $ps_0 - pr_0 = -1$, so that $ps_0/p' = r_0 - 1/p'$, the $h = s_0$ line is immediately above the $r_0$th odd band, and
below an even band. We make use of this information to derive the following technical result for later convenience.

**Lemma 1.2.** For all $h^* \in \mathcal{P}_{s_0}(L)$, we have $\beta(h^*) = 0$.

**Proof:** We first define a flip transformation of a path. This consists of, exchanging two consecutive segments that form a down-peak for two that form an up-peak, or vice-versa. Thus a flip appears as follows:

Note that the segment preceding the two that are changed (which appears only if the peak is not the first vertex) and the segment succeeding the two that are changed, may each be either up or down, and each band may be even or odd. Consideration of all sixteen cases shows that if the two paths $h'$ and $h''$ differ by a flip transformation, then $\beta(h') = \beta(h'')$. In the case where the peak is the first vertex of the path, this holds if and only if the two bands are of opposite parity. This is always so for the $h = s_0$ case that we are dealing with here. (Note that if $h = 1$, this type of flip transformation is not valid and is not used.)

Now, consider the unique path $h(0)$ for which the first $L$ segments all lie in the $s_0$th band. We immediately obtain $\beta(h(0)) = 0$. Now, as is easily seen, any path $h^* \in \mathcal{P}_{p,p'}(L)$ can be obtained from $h(0)$ by a sequence of flip transformations. Therefore $\beta(h^*) = \beta(h(0)) = 0$, as required.

\[\square\]

2. The $B$-transform

In this section, we introduce the $B$-transform, the first of the two combinatorial transforms studied in this work.

The $B$-transform comprises three separate steps which we refer to as the $B_1$, $B_2$ and $B_3$-transforms. $B_1$ maps $\mathcal{P}_{s_0}(L)$ injectively to $\mathcal{P}_{s_0+k_0}(L')$. $B_2$ lengthens paths in a simple manner, in particular mapping $\mathcal{P}_{s_0}(L)$ to $\mathcal{P}_{s_0+k}(L')$ for $k \geq 0$. $B_3$ deforms, in a particular manner, a path within $\mathcal{P}_{s_0}(L)$. As we will see, taking the paths generated by the combined action of $B_1$ followed by $B_2$ as input to $B_3$, the paths generated by $B_3$ are naturally indexed by certain partitions $\lambda$ (See Appendix A.2 for a definition of a partition).

The $B$-transform comprising $B_1$ followed by $B_2$ followed by $B_3$, and involving the two parameters $k$ and $\lambda$, will be denoted $B(k, \lambda)$, and thus maps from $\mathcal{P}_{s_0}(L)$ to $\mathcal{P}_{s_0+k_0}(L')$. As we will show, this map: $(h, k, \lambda) \mapsto h'$ is actually a bijection.

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8Our $B$-transform is basically a generalisation of a transform introduced by Agarwal and Bressoud in [1, 11]. This, in turn, was motivated by the works of Bailey [6] and Burge [14]. The transform in [1, 11] acts on the infinite paths that pertain to the cases where $p = 2$ and $p' = 2k + 1$ for $k \in \mathbb{Z}_+$. In this work, we extend the transform of [1, 11], so that it acts on finite paths. We also generalise it to all co-prime $p, p'$. However, whereas the analysis of [1, 11] applies to $\mathcal{P}_{a,k,k+1}$, for all $a$, we restrict consideration to a single value of $a$ in the current paper.
2.1. The $B_1$-transform. The definition of the $B_1$-transform involves the band structure of $\mathcal{P}_{p,p'}(L)$. First note that the band structure of $\mathcal{P}_{p,p'}^{p'+p}(L')$ is easily obtained from that of $\mathcal{P}_{p,p'}(L)$. In the two cases, the number of odd bands is the same. Since the $r$th odd band for $\mathcal{P}_{p,p'}(L)$ has its lower edge at height $\lfloor rp'/p \rfloor$ and that of $\mathcal{P}_{p,p'}^{p'+p}(L')$ has its lower edge at $\lfloor r(p'+p)/p \rfloor = \lfloor rp'/p \rfloor + r$, we see that the distance between the odd bands has increased by exactly one, with the height of the lowermost having also increased by one. Note that, if $p > 1$, the starting point is on the upper or lower edge of the $r_0$th odd band both before and after the $B_1$-transform. If $p = 1$, the starting point remains at $h = 1$.

The image of the path is now obtained by examining the sequence of vertex types and inserting an extra one immediately prior to each scoring vertex. An extra straight-up vertex is inserted immediately prior to each odd straight-up vertex and each even peak-down vertex, and an extra straight-down vertex is inserted immediately prior to each odd straight-down vertex and each even peak-up vertex. In view of the odd bands having separated by one unit, and the change of the starting point, we see that the shapes and parities of the scoring vertices are naturally preserved under this transform.

For example, the following $\mathcal{P}_3^{3,8}(16)$ path (here $L = 16$ and we show $h_{L+1} = c = 4$):

![Diagram of $\mathcal{P}_3^{3,8}(16)$ path]

maps to this $\mathcal{P}_4^{4,11}(24)$ path:

![Diagram of $\mathcal{P}_4^{4,11}(24)$ path]

Note that the path obtained from a $B_1$-transform is such that there are no two consecutive scoring vertices, and that the first vertex is non-scoring.

**Lemma 2.1.** Let $h \in \mathcal{P}_{p,p'}^{r_0}(L)$ have striking sequence $\left( \begin{array}{c} a_1 \ b_1 \\ a_2 \ b_2 \\ \vdots \ b_3 \\ \vdots \ a_l \ b_l \end{array} \right)$, and let $h^{(0)} \in \mathcal{P}_{p,p'}^{r_0+p}(L^{(0)})$ be obtained from the action of the $B_1$-transform on $h$. Then
$h^{(0)}$ has striking sequence:

$$
\begin{pmatrix}
  a_1 + b_1 & a_2 + b_2 & a_3 + b_3 & \cdots & a_l + b_l \\
  b_1 & b_2 & b_3 & \cdots & b_l
\end{pmatrix},
$$

$m(h^{(0)}) = L$ and $L^{(0)} = 2L - m(h)$.

**Proof:** This follows directly from the definition of the striking sequences and the action of the $B_1$-transform.

**Lemma 2.2.** Let $h \in \mathcal{P}_{s_0}^{p,p'}(L)$ and $h^{(0)} \in \mathcal{P}_{s_0+r_0}^{p,p'+p}(L^{(0)})$ be the path obtained by the action of the $B_1$-transform on $h$. Then

$$
wt(h^{(0)}) = wt(h) + \frac{1}{4}(L - m)^2,
$$

where $m = m(h)$.

**Proof:** Let $h$ have striking sequence $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots, \frac{a_l}{b_l}\right)$, then Lemmas 2.1 and 1.3 show that

$$
wt(h^{(0)}) - wt(h) = (b_1 + b_3 + b_5 + \cdots)(b_2 + b_4 + b_6 + \cdots) = \frac{1}{4}((L - m)^2 - \beta(h)^2),
$$

the second equality resulting because $L - m = b_1 + b_2 + \cdots + b_l$ and $\beta(h) = (b_1 + b_3 + b_5 + \cdots) - (b_2 + b_4 + b_6 + \cdots)$. The lemma now follows on using Lemma 1.2.

**2.2. The $B_2$-transform.** Let $p' > 2p$ so that $h^{(0)} \in \mathcal{P}_{s_0}^{p,p'}(L')$ is a path for which there are no two neighbouring odd bands. By inserting a particle into $h^{(0)}$, we mean displacing $h^{(0)}$ two squares to the right and inserting two even edges. Since the path starts at height $s_0$ and the line $h = s_0$ borders both an even and an odd band (if $s_0 > 1$), this is possible in exactly one way. In this way, we obtain a path $h^{(1)}$ of length $L' + 2$. By repeating this process we may obtain a path $h^{(k)} \in \mathcal{P}_{s_0}^{p,p'}(L' + 2k)$ by successively inserting $k$ particles into $h^{(0)}$. We say that $h^{(k)}$ has been obtained by the action of a $B_2$-transform on $h^{(0)}$. When we need to show the dependence on $k$ explicitly, we refer to it as a $B_2(k)$-transform.

**Lemma 2.3.** Let $h \in \mathcal{P}_{s_0}^{p,p'}(L)$. Apply a $B_1$-transform to $h$ to obtain the path $h^{(0)} \in \mathcal{P}_{s_0+r_0}^{p,p'+p}(L^{(0)})$. Then obtain $h^{(k)} \in \mathcal{P}_{s_0+r_0}^{p,p'+p}(L^{(k)})$ by applying a $B_2(k)$-transform to $h^{(0)}$. If $m(k) = m(h^{(k)})$, then $L^{(k)} = L^{(0)} + 2k$, $m^{(k)} = m^{(0)}$ and

$$
wt(h^{(k)}) = wt(h) + \frac{1}{4}(L^{(k)} - m^{(k)})^2.
$$

**Proof:** That $L^{(k)} = L^{(0)} + 2k$ follows immediately from the definition of a $B_2$-transform. Lemma 2.2 yields:

$$
wt(h^{(0)}) = wt(h) + \frac{1}{4}(L - m(h))^2
$$

$$
= wt(h) + \frac{1}{4}\left(L^{(0)} - m(h^{(0)})\right)^2,
$$
the second equality following from Lemma 2.1. Let the striking sequence of $h^{(0)}$ be $(a_1 \ a_2 \ \cdots \ a_i \ b_1 \ b_2 \ \cdots \ b_i)$, whereupon that of $h^{(1)}$ is $\begin{pmatrix} 0 & 0 & a_1 & a_2 & \cdots & a_i \\ 1 & 1 & b_1 & b_2 & \cdots & b_i \end{pmatrix}$. Then, $m(h^{(1)}) = m(h^{(0)})$ and Lemma 1.1 shows that $wt(h^{(1)}) = wt(h^{(0)}) + L^{(0)} - m(h^{(0)}) + 1$. Repeated application then yields $m(h^{(k)}) = m(h^{(0)})$ and $wt(h^{(k)}) = wt(h^{(0)}) + k \left(L^{(0)} - m(h^{(0)})\right) + k^2$.

Then, using $L^{(k)} = L^{(0)} + 2k$, the lemma follows. \[ \square \]

2.3. Particle moves. In this section, we restrict attention to those $\mathcal{P}_{p,p'}^{\alpha} (L)$ for which $p' > 2p$, and specify six types of local deformations of a path. These deformations will be known as moves. In each of the six cases, a particular sequence of four segments of a path is changed to a different sequence, the remainder of the path being unchanged. The moves are as follows — the path portion to the left of the arrow is changed to that on the right:

Move. 1.

Move. 2.

Move. 3.

Move. 4.

Move. 5.

Move. 6.

Since $p' > 2p$, each odd band is straddled by a pair of even bands. Thus, there is no impediment to enacting moves 2 and 5 for paths $\mathcal{P}_{p,p'}^{\alpha} (L)$.

Note that moves 4–6 are inversions of moves 1–3. Also note that moves 2 and 3 (likewise moves 5 and 6) may be considered to be the same move since in the two cases, the same sequence of three edges is changed.

\[9\text{In the special cases } p = 1, 2, \text{ these moves are equivalent to those considered in } \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}. \text{ In the special cases } p' = p + 1, \text{ our results are equivalent to similar results obtained in } \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}. \]
Lemma 2.4. Let $h$ be a path for which four consecutive segments are as in one of diagrams on the left above. Let $\hat{h}$ be that obtained from $h$ by changing those segments according to the move. Then

$$\text{wt}(\hat{h}) = \text{wt}(h) + 1.$$  

Additionally, $\text{L}(\hat{h}) = \text{L}(h)$ and $m(\hat{h}) = m(h)$.

Proof: For each case, take the $xy$-coordinate of the leftmost point of this portion of a path to be $(x_0, y_0)$. Now consider the contribution to the weight of the three vertices in question before and after the move (although the vertex immediately before those considered may change, its contribution doesn’t). In each of the six cases, the contribution is $x_0 + y_0 + 1$ before the move and $x_0 + y_0 + 2$ afterwards. Thus the first statement holds. The final statement is obtained by inspecting all six moves. □

Now observe that for each of the moves specified above, the sequence of path segments before the move consists of an adjacent pair of scoring vertices followed by a non-scoring vertex. When $p' > 2p$ so that there are no two adjacent odd bands, all such combinations of vertices are present amongst the six moves. The specified move then consists of replacing such a combination with a non-scoring vertex followed by two scoring vertices. It is useful to interpret this as the pair of adjacent scoring vertices having moved rightward by one step. As anticipated above, we refer to a pair of adjacent scoring vertices as a particle. Thus each of the six moves here is a particle moving to the right by one step.

2.4. The $B_3$-transform. Since each of the moves described above moves a pair of scoring vertices to the right by one step, we see that a succession of such moves is possible until the pair is followed by another scoring vertex. If this itself is followed by yet another scoring vertex, we forbid further movement. However, if it is followed by a non-scoring vertex, further movement is allowed after considering the latter two of the three consecutive scoring vertices to be the particle (instead of the first two).

As above, let $h^{(k)}$ be a path resulting from a $B_2$-transform inserting $k$ particles into a path that itself is the image of a $B_1$ transform. We now consider moving these $k$ particles.

Lemma 2.5. There is a bijection between the paths obtained by moving the particles in $h^{(k)}$ and the partitions $\lambda$ with at most $k$ parts, none of which exceeds $m = m(h^{(k)})$. This bijection is such that if $h$ is the bijective image of a particular $\lambda$ then

$$\text{wt}(h) = \text{wt}(h^{(k)}) + \text{wt}(\lambda),$$

where $\text{wt}(\lambda) = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. Additionally, $\text{L}(h) = \text{L}(h^{(k)})$ and $m(h) = m(h^{(k)})$.

Proof: Since each particle moves by traversing a non-scoring vertex, and there are $m$ of these to the right of the rightmost particle in $h^{(k)}$, and there are no consecutive scoring vertices to its right, this particle can make $\lambda_1$ moves to the right, with $0 \leq \lambda_1 \leq m$. Similarly, the next rightmost particle can make $\lambda_2$ moves to the right with $0 \leq \lambda_2 \leq \lambda_1$. Here, the upper restriction arises because the two scoring vertices would then be adjacent to those of the first particle. Continuing in this way, we obtain that all possible final positions of the particles are indexed by
\( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) with \( m \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0 \), that is, by partitions of at most \( k \) parts with no part exceeding \( m \). Moreover, since by Lemma 2.4 the weight increases by one for each move, the weight increase after the sequence of moves specified by a particular \( \lambda \) is equal to \( wt(\lambda) \). The final statement also follows from Lemma 2.4.

We say that a path obtained by moving the particles in \( h^{(k)} \) has been obtained by the action of a \( B_3 \)-transform. If we wish to show the specific dependence on \( \lambda \), we refer to the transform as a \( B_3(\lambda) \)-transform.

Having defined \( B_1, B_2(k) \) for \( k \geq 0 \) and \( B_3(\lambda) \) for \( \lambda \) a partition with at most \( k \) parts, we now define a \( B(k, \lambda) \)-transform as the composition \( B(k, \lambda) = B_3(\lambda) \circ B_2(k) \circ B_1 \).

2.5. Particle content of a path. In this section let \( p' > 2p \) so that there are no two adjacent odd bands. Let \( h' \in \mathcal{P}^{p,p'}_{s_0}(L') \).

**Lemma 2.6.** There is a unique triple \((h, k, \lambda)\) where \( h \in \mathcal{P}^{p,p'-p}_{s_0\cdots r_0}(L) \) for some \( L \), such that the action of a \( B(k, \lambda) \)-transform on \( h \) results in \( h' \).

**Proof:** This is proved by reversing the constructions described in the previous sections. Locate the leftmost pair of consecutive scoring vertices in \( h' \), and move them leftward by reversing the particle moves, until they occupy the first two positions. Now ignoring these two vertices, do the same with the next leftmost pair of consecutive scoring vertices, moving them leftward until they occupy the third and fourth positions. Continue in this way until all consecutive scoring vertices occupy the leftmost positions of the path. Say there are \( 2k \) of them (that this number is even is implied by the starting point being between an odd and an even band), and denote this path \( h^{(k)} \). Clearly \( h' \) results from \( h^{(k)} \) by a \( B_3(\lambda) \)-transform for a particular \( \lambda \) with at most \( k \) parts.

Removing the first \( 2k \) segments of \( h^{(k)} \) produces a path \( h^{(0)} \in \mathcal{P}^{p,p'}_{s_0\cdots r_0}(L' - 2k) \) which has no two consecutive scoring vertices. Moreover, \( h^{(k)} \) arises by the action of a \( B_2(k) \)-transform on \( h^{(0)} \).

Finally, since \( h^{(0)} \) has by construction no pair of consecutive scoring vertices, and none at the first vertex, we may remove a non-scoring vertex before every scoring vertex to obtain a path \( h \in \mathcal{P}^{p,p'-p}_{s_0\cdots r_0}(L) \) for some \( L \), from which \( h^{(0)} \) arises by the action of a \( B_1 \)-transform. The lemma is then proved.

The value of \( k \) obtained above will be referred to as the particle content of \( h' \).

3. The \( D \)-transform

The \( D \)-transform\(^{10}\) is defined to act on each \( h \in \mathcal{P}^{p,p'}_{s_0}(L) \) to yield a path in \( \mathcal{P}^{p'-p,p'}_{s_0}(L) \). It is easily seen that the band structure of \( \mathcal{P}^{p'-p,p'}_{s_0}(L) \) is obtained from that of \( \mathcal{P}^{p,p'}_{s_0}(L) \) simply by replacing odd bands by even bands and vice-versa. The action on \( h \in \mathcal{P}^{p,p'}_{s_0}(L) \) yields a path \( \hat{h} \in \mathcal{P}^{p,p'}_{s_0\cdots r_0}(L) \) with exactly the same sequence of integer heights, i.e., \( \hat{h}_i = h_i \) for \( 0 \leq i \leq L \).

\(^{10}\)A similar duality-type transform appears in [4, 10].
Lemma 3.1. Let \( h \in \mathcal{P}_{s_0}^{p',p}(L) \) have striking sequence \( \left( a_1, b_1, a_2, b_2, \ldots, a_l, b_l \right) \), and let \( \hat{h} \in \mathcal{P}_{s_0}^{p'-p,p}(L) \) be obtained from the action of the \( D \)-transform on \( h \). Then \( \hat{h} \) has striking sequence:

\[
\begin{pmatrix}
    b_1 & b_2 & b_3 & \cdots & b_l \\
    a_1 & a_2 & a_3 & \cdots & a_l
\end{pmatrix}.
\]

Moreover, \( m(\hat{h}) = L(h) - m(h) \) and \( L(\hat{h}) = L(h) \).

Proof: This follows directly from the definition of the striking sequences after noting that the action of the \( D \)-transform exchanges odd bands for even bands and vice-versa. \( \square \)

Lemma 3.2. Let \( h \in \mathcal{P}_{s_0}^{p',p}(L) \), and \( \hat{h} \in \mathcal{P}_{s_0}^{p'-p,p}(L) \) be obtained by the action of a \( D \)-transform on \( h \). Then

\[
wt(h) + wt(\hat{h}) = \frac{1}{4} L^2.
\]

Proof: Let \( h \) have striking sequence \( \left( a_1, a_2, a_3, \ldots, a_l \right) \), and let \( w_i = a_i + b_i \) for \( 1 \leq i \leq l \). Then, using Lemmas 1.1 and 3.1, we obtain

\[
wt(h) \cdot wt(\hat{h}) = \sum_{i=1}^{l} b_i (w_{i-1} + w_{i-3} + \cdots + w_{1 + \text{mod} 2})
\]

\[
+ \sum_{i=1}^{l} a_i (w_{i-1} + w_{i-3} + \cdots + w_{1 + \text{mod} 2})
\]

\[
= \sum_{i=1}^{l} w_i (w_{i-1} + w_{i-3} + \cdots + w_{1 + \text{mod} 2})
\]

\[
= (w_1 + w_3 + w_5 + \cdots) \cdot (w_2 + w_4 + w_6 + \cdots) = L.
\]

The lemma then follows because \( (w_1 + w_3 + w_5 + \cdots) + (w_2 + w_4 + w_6 + \cdots) = L \) and \((w_1 + w_3 + w_5 + \cdots) - (w_2 + w_4 + w_6 + \cdots) = 0 \) because the start and endpoints of the paths have equal heights. \( \square \)

To obtain the particle content of a path \( h' \in \mathcal{P}_{s_0}^{p',p'}(L') \), where \( p' < 2p \), we first perform a \( D \)-transform on the path \( h' \) to obtain a path \( \hat{h}' \in \mathcal{P}_{s_0}^{p'-p,p}(L') \). The particle content of \( h' \) is defined to be equal to that of \( \hat{h}' \).

In what follows, we obtain the model \( \mathcal{P}_{s_0}^{p,p'}(L) \) by using a certain sequence of \( B \)- and \( D \)-transforms. This sequence is determined by the continued fraction expansion of \( p'/p \).

4. Continued fractions and the \( mn \)-system
4.1. Basic definitions. If \( p' \) and \( p \) are positive co-prime integers and
\[
\frac{p'}{p} = c_0 + \frac{1}{c_1 + \frac{1}{c_2 + \ddots + \frac{1}{c_{n-1} + \frac{1}{c_n}}}}
\]
with \( c_0 \geq 0 \), \( c_i \geq 1 \) for \( 0 < i < n \), and \( c_n \geq 2 \), then \((c_0, c_1, c_2, \ldots, c_n)\) is said to be the \textit{continued fraction} for \( p'/p \).

For later convenience, we now determine how the continued fractions of \( p'/p \) change under the \( B \) - and \( D \)-transforms.

**Lemma 4.1.** For positive co-prime integers \( p' \) and \( p \), let \( p'/p \) have continued fraction \((c_0, c_1, \ldots, c_n)\).

1. The continued fraction of \((p' + p)/p'\) is \((c_0 + 1, c_1, c_2, \ldots, c_n)\).
2. If \( p' > 2p \) then the continued fraction of \( p'/(p' - p) \) is \( (1, c_0 - 1, c_1, c_2, \ldots, c_n)\).

**Proof:** The first result follows immediately. So does the second after writing \( p'/p = 1 + 1/(p'/p - 1) \).

\[\square\]

4.2. Zones. For what follows, it is convenient to partition indices into \textit{zones}.

To this end, given \( p \) and \( p' \) with \( p'/p \) having continued fraction \((c_0, c_1, \ldots, c_n)\), define

\[
t_{\mu} = -1 - \delta_{\mu, n+1} + \sum_{i=0}^{\mu-1} c_i \quad (\mu = 0, 1, \ldots, n + 1).
\]

Then we say that the index \( j \) with \( 0 \leq j \leq t_{n+1} \) is in zone \( \mu \) if \( t_{\mu} < j \leq t_{\mu+1} \). Thus, there are \( n + 1 \) zones. Note that for \( 0 \leq \mu < n \), zone \( \mu \) contains \( c_{\mu} \) indices, and zone \( n \) contains \( c_n - 1 \) indices.

We define \( t = t_{n+1} \) and refer to it as the \textit{rank} of \( p'/p \). We also define the rank and number of zones of \( \mathcal{P}^{p,p'}_{a,b,c}(L) \) to be equal to the rank and number of zones of \( p'/p \) respectively. Then Lemma [4.1] shows that the rank is incremented under a \( B \)-transform while the number of zones is constant, and the rank is constant under a \( D \)-transform while the number of zones is incremented.

Let \( p' > 2p \) so that \( t_1 > 0 \). We now see that if we begin with \( \mathcal{P}^{1,3}_{a,b,c}(L') \) and apply a sequence of \( t - 1 \) \( B \)-transforms, with, for \( 1 \leq \mu \leq n \), the \((t - t_\mu)\)th \( B \)-transform immediately preceded by a \( D \)-transform, then we obtain elements of \( \bigcup_L \mathcal{P}^{p,p'}_{s_0}(L) \).

In what follows, we will use this sequence of transforms to obtain the paths that we wish to enumerate.

\[11\text{Although we make no use of this fact, it may be easily shown ([17] Lemma 3.9)] that with } s_0 \text{ and } r_0 \text{ defined to be the smallest non-negative co-prime integers such that } |ps_0 - p'r_0| = 1, \text{ then } s_0/r_0 \text{ has continued fraction } (c_0, c_1, \ldots, c_{n-1}) \text{ if } n > 0.\]

\[12\text{The } t_{n+1} \text{ defined here differs from that defined in [4].}\]

\[13\text{In fact, it is admissible to extend this sequence and consider it as passing from the trivial } \mathcal{P}^{1,3}_{a,b,c}(0) \text{ to } \bigcup_L \mathcal{P}^{p,p'}_{s_0}(L).\]
4.3. The *mn*-system. For each pair of positive co-prime integers \( p, p' \), we now define the associated *mn*-system\(^\text{14}\). Let \( p'/p \) have rank \( t \). The *mn*-system is then a set of \( t \) linear equations defining an interdependence between two \( t \)-dimensional vectors \( \hat{\mathbf{n}} = (n_1, n_2, \ldots, n_t) \) and \( \mathbf{m} = (m_0, m_1, \ldots, m_{t-1}) \). The equations are given by, for \( 1 \leq j \leq t \):

\[
\begin{align*}
    m_{j-1} - m_{j+1} &= m_j + 2n_j & \text{if } j = t_{\mu}, & \mu = 1, 2, \ldots, n; \quad (10) \\
    m_{j-1} + m_{j+1} &= 2m_j + 2n_j & \text{otherwise,} & \quad (11)
\end{align*}
\]

where we set \( m_t = m_{t+1} = 0 \).

Note that if each \( n_i \) is a non-negative integer then each \( m_j \) is an even non-negative integer. Also note that it is possible to eliminate \( m_j \) for \( 1 \leq j \leq t \) from the above equations to obtain:

\[
\sum_{i=1}^{t} l_in_i = \frac{m_0}{2},
\]

for certain positive integer values \( l_1, l_2, \ldots, l_t \). These values are referred to as *string lengths* in \([10]\) and feature prominently in the analysis there. We don’t require them in the present paper. In what follows, \( m_0 \) will be identified with the length \( L \) of a path.

As an example, consider the case where \( p = 9 \) and \( p' = 31 \). Here, the continued fraction of \( 31/9 \) is \((3, 2, 4)\), whereupon \( n = 2, t_1 = 2, t_2 = 4 \) and \( t = t_3 = 7 \). The *mn*-system of equations yields the following:

\[
\begin{align*}
    m_7 &= 0; \\
    m_6 &= 2n_7; \\
    m_5 &= 2n_6 + 4n_7; \\
    m_4 &= 2n_5 + 4n_6 + 6n_7; \\
    m_3 &= 2n_4 + 2n_5 + 6n_6 + 10n_7; \\
    m_2 &= 2n_3 + 4n_4 + 2n_5 + 8n_6 + 14n_7; \\
    m_1 &= 2n_2 + 2n_3 + 6n_4 + 4n_5 + 14n_6 + 24n_7; \\
    L &= m_0 = 2n_1 + 4n_2 + 2n_3 + 8n_4 + 6n_5 + 20n_6 + 34n_7.
\end{align*}
\]

In addition to the above \( t \) equations (not counting \( m_t = 0 \)), it will be useful to set \( m_{-1} = 0 \) and to use the appropriate \([10]\) or \([11]\) in the case \( j = 0 \) to obtain one further equation. Thence define the \( t \)-dimensional vector \( \mathbf{n} = (n_0, n_1, \ldots, n_{t-1}) \).

The significance of the *mn*-system will be revealed after we examine sequences of \( B \)- and \( D \)-transforms, and insert a number of particles at each stage.

4.4. The generalised Cartan matrix. The above expressions for the *mn*-system may be conveniently expressed in matrix form. The matrix so involved is a generalisation of the Cartan matrix of the Lie algebra of type A.

Given \( p \) and \( p' \), define \( t_\mu \) as in \([10]\) and let \( t = t_{n+1} \). Now let \( \mathbf{C} \) be the \( t \times t \) tri-diagonal matrix with entries \( C_{ij} \) for \( 0 \leq i, j \leq t-1 \) where, when the indices are in this range,

\footnote{The *mn*-systems defined here were first defined in \([10]\), where the interrelationship between \( \mathbf{m} \) and \( \mathbf{n} \) was derived by an altogether different method.}

\footnote{In this paper, the vectors \( \mathbf{m}, \mathbf{n} \) and \( \hat{\mathbf{n}} \) should be considered as column vectors. However, for typographical convenience, we shall express their components in row vector form.}
\[ C_{j,j-1} = -1, \quad C_{j,j} = 1, \quad C_{j,j+1} = 1, \quad \text{if } j = t_{\mu}, \quad \mu = 1, 2, \ldots, n; \]
\[ C_{j,j-1} = -1, \quad C_{j,j} = 2, \quad C_{j,j+1} = -1, \quad 0 \leq j < t \text{ otherwise.} \]

**Lemma 4.2.** For fixed \( p \) and \( p' \), let \( m \) and \( n \) satisfy the \( mn \)-system. Then:
\[
2n = -Cm.
\]

**Proof:** This follows immediately from the definition of \( C \) and equations (10) and (11) for \( 0 \leq j < t \).

**Corollary 4.3.** For fixed \( p \) and \( p' \), let \( m \) and \( n \) satisfy the \( mn \)-system. Then, on setting \( L = m_0 \):
\[
\sum_{i=1}^{t} m_i n_i = \begin{cases} 
-\frac{1}{2}m^TCm + \frac{1}{2}Lm_1 + \frac{1}{2}L^2 & : \text{if } t_1 = 0; \\
-\frac{1}{2}m^TCm - \frac{1}{2}Lm_1 + L^2 & : \text{if } t_1 > 0.
\end{cases}
\]

**Proof:** Using Lemma 4.3, \( \sum_{i=1}^{t} m_i n_i = -\frac{1}{2}m^TCm - m_0n_0 \). In the case \( t_1 = 0 \), expression (11) gives \( 2n_0 = -m_1 - m_0 \). In the case \( t_1 > 0 \), expression (11) gives \( 2n_0 = m_1 - 2m_0 \). The result then follows after substituting \( m_0 = L \).

### 4.5. Sectors

In this section, we use the sequence of \( B \)- and \( D \)-transforms detailed at the end of Section 4.2 to pass from \( \mathcal{P}_1^{1,3}(L) \) to \( \mathcal{P}_{p_0}^{p,p'}(L') \). First we prove two short lemmas.

**Lemma 4.4.** Let \( h \in \mathcal{P}_{p_0}^{p,p'}(L) \) and let \( h' \in \mathcal{P}_{p_0+p_0}^{p,p'}(L') \) arise through the action of a \( B(n',\lambda) \)-transform on \( h \). If \( m = m(h) \) and \( m' = m(h') \), then \( m - L \) and \( L' + m = 2m' + 2n' \).

**Proof:** By Lemma 2.4, the action of the \( B_1 \)-transform yields a path \( h^{(0)} \) of length \( 2L - m \), with \( m(h^{(0)}) = L \). Thereupon, using Lemma 2.3, the action of a \( B_2(n') \)-transform on \( h^{(0)} \) results in a path \( h^{(k)} \) of length \( 2L - m + 2n' \) for which \( m(h^{(k)}) = L \). Since a \( B_3(\lambda) \)-transform does not change either property, the lemma follows.

**Lemma 4.5.** Let \( h \in \mathcal{P}_{p_0}^{p,p'}(L) \) and let \( h' \in \mathcal{P}_{2p_0-r_0}^{p,p'}(L') \) arise through the action of a \( D \)-transform on \( h \) followed by a \( B(n',\lambda) \)-transform. If \( m = m(h) \) and \( m' = m(h') \), then \( m - L \) and \( L' - m = m' + 2n' \).

**Proof:** Let \( \hat{h} \in \mathcal{P}_{p_0}^{p'-p}(L) \) result from the action of the \( D \)-transform on \( h \). By Lemma 3.1, \( m(\hat{h}) = L - m \) and \( L(\hat{h}) = L \). Then, using Lemma 4.4, \( m' = L \) and \( L' + (L - m) = 2m' + 2n' \). The required expressions now follow.

We now define subsets \( \mathcal{S}_{p,p'}(\hat{n}) \) of \( \mathcal{P}_{p_0}^{p,p'}(L) \) indexed by \( \hat{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t \), which we refer to as sectors. For \( p' > 2p \), the sector \( \mathcal{S}_{p,p'}(\hat{n}) \) is obtained as follows. First let \( m = (m_0, m_1, \ldots, m_t) \) be obtained from the \( mn \)-system of \( p'/p \) and set \( m_t = 0 \). Now let \( \lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(t-1)}) \) be a sequence

\[ ^{16} \text{We take } 0 \in \mathbb{N}. \]
of partitions such that for each $1 \leq i < t$, $\lambda(i)$ has at most $n_i$ parts, none of which exceeds $m_i$.

We now create a sequence $(h^{(t)}, h^{(t-1)}, \ldots, h^{(1)})$ of paths as follows. First let $h^{(t)} \in P_{1,3}^{1,3}(2n_t)$. Note that this path is unique. Now for $1 \leq j < t$, obtain the path $h^{(j)}$ from $h^{(j+1)}$ as follows. If $j \neq t$ for all $\mu = 1, 2, \ldots, n$, obtain $h^{(j)}$ by applying a $B(n_j, \lambda(j))$-transform to $h^{(j+1)}$. In the case where $j = t$, for $\mu = 1, 2, \ldots, n$, apply a $D$-transform to $h^{(j+1)}$ immediately before applying a $B(n_j, \lambda(j))$-transform.

**Lemma 4.6.** Let $p' > 2p$ and let the sequence $(h^{(t)}, h^{(t-1)}, \ldots, h^{(1)})$ of paths be obtained as above. Then, for each $1 \leq j \leq t$, we have $m(h^{(j)}) = m_j$ and $L(h^{(j)}) = j - 1$.

**Proof:** The result follows immediately in the case $j = t$ since $L(h^{(t)}) = 2n_t = m_t$ and $m(h^{(t)}) = m_t - 1$ and $m(h^{(t)}) = 0$ by direct inspection of $h^{(t)}$. Now proceed by (decreasing) induction on $t$. Let $1 \leq k < t$ and assume that the result holds for $j = k + 1$, so that $m(h^{(k+1)}) = m_{k+1}$ and $L(h^{(k+1)}) = m_k$. If $k \neq t$ for all $\mu = 1, 2, \ldots, n$, Lemma 4.1 immediately shows that $m(h^{(k)}) = L(h^{(k+1)}) = m_k$ and $L(h^{(k)}) = 2m_k + 2n_k - m_{k+1}$, which, by (1), is $m_{k-1}$. The case $k = t$ for $\mu = 1, 2, \ldots, n$, follows similarly from Lemma 4.1 and (1). The required result therefore holds in the case $j = k$, whereupon the lemma follows by induction.

For $p' > 2p$, we now define the sector $S^{p,p'}(\hat{n})$ to be the set of all paths obtained as above for all $\lambda$ within the specified constraints. Lemma 4.1 then shows that for $h \in S^{p,p'}(\hat{n})$, the values $L(h)$ and $m(h)$ are constant and given by $m_0$ and $m_1$ respectively. In particular, $S^{p,p'}(\hat{n}) \subset P_{s_0}^{p,p'}(m_0)$. For $p' < 2p$, the sector $S^{p,p'}(\hat{n})$ is defined to be precisely the image of the sector $S^{p' - p,p'}(\hat{n}) \subset P_{s_0}^{p' - p,p'}(m_0)$ under the $D$-transform.

Now for $p' > 2p$, given $h \in P_{s_0}^{p,p'}(L)$ we can reverse the above construction (if $p' < 2p$ we simply apply a $D$-transform first). Let $h^{(t)} = h$ and define a sequence $(h^{(t)}, h^{(t-1)}, \ldots, h^{(1)})$ of paths, $\hat{n} = (n_1, n_2, \ldots, n_t)$ and a sequence $\lambda = (\lambda(1), \lambda(2), \ldots, \lambda(t-1))$ of partitions as follows. For $1 \leq j < t$, obtain $h^{(j+1)}$ from $h^{(j)}$ as follows. Lemma 4.1 shows that there is a unique triple $(h', k, \lambda)$ for which $h^{(j)}$ arises from the action of the $B(k, \lambda)$-transform on $h'$. Set $n_j = k$ and $\lambda(j) = \lambda$. If $j \neq t$ for $\mu = 1, 2, \ldots, n$, set $h^{(j+1)} = h'$. Otherwise, set $h^{(j+1)}$ to be the result of the $D$-transform acting on $h'$. In this way, we find that $h$ is present in one and only one sector, namely $h \in S^{p,p'}(\hat{n})$. Moreover, the sequence $\lambda$ of partitions is uniquely determined by $h$.

5. Generating functions and character formulae

5.1. Constant-sign generating functions for paths. In this section, we combine the techniques of the previous sections to calculate constant-sign generating functions for $P_{s_0}^{p,p'}(L)$. The first step is to determine such a function for all paths in a sector. Given co-prime $p, p'$, let $p'/p$ have rank $t$ and let $\hat{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t$. Now define the sector generating function to be:

$$S^{p,p'}(\hat{n}) = \sum_{h \in S^{p,p'}(\hat{n})} q^{\text{wt}(h)}.$$
Lemma 5.1. For $p' > 2p$, let $\hat{n} = (n_1, n_2, \ldots, n_t)$ and $m = (L, m_1, \ldots, m_t)$ satisfy the $mn$-system of $p'/p$. Then, if we set $\hat{n}' = (n_2, n_3, \ldots, n_t)$,

$$S^{p',p'}(\hat{n}) = q^{-(L-m_1)^2} \left[ \frac{m_1 + n_1}{n_1} \right]_q S^{p',p'}(\hat{n}').$$

Proof: Lemma 4.6 shows that for each $h \in S^{p',p'}(\hat{n})$, $L(h) = L$ and $m(h) = m_1$. Then, Lemma 2.6 shows that there is a bijection between $S^{p',p'}(\hat{n})$ and pairs $(h', \lambda)$ where $h' \in S^{p,p'-p}(\hat{n}')$ and $\lambda$ is a partition with at most $n_1$ parts, none of which exceeds $m_1$. This bijection is such that $h$ results from the action of the $B(n_1, \lambda)$-transform on $h'$. Moreover, Lemmas 2.3 and 2.5 show that $\text{wt}(h') = \text{wt}(h') + \frac{1}{2}(L-m_1)^2 + \text{wt}(\lambda)$. Therefore, if we let $\langle n_1, m_1 \rangle$ denote the set of partitions that have at most $n_1$ parts, none of which exceeds $m_1$,

$$S^{p',p'}(\hat{n}) = \sum_{h' \in S^{p',p'-p}(\hat{n}')} \sum_{\lambda \in \langle n_1, m_1 \rangle} q^{-(L-m_1)^2 + \text{wt}(\lambda) + \text{wt}(h')} \left( \sum_{\lambda \in \langle n_1, m_1 \rangle} q^{\text{wt}(\lambda)} \right) \left( \sum_{h' \in S^{p',p'-p}(\hat{n}')} q^{\text{wt}(h')} \right).$$

Since the generating function for all partitions $\lambda$ with at most $n_1$ parts, none of which exceeds $m_1$, is $\left[ \frac{m_1 + n_1}{n_1} \right]_q$ (see [3] for example), the lemma now follows. \qed

Lemma 5.2. Let $p'$ and $p$ be positive co-prime integers and let $t$ be the rank of $p'/p$. Let $\hat{n} = (n_1, n_2, \ldots, n_t)$ and $m = (L, m_1, \ldots, m_t)$ satisfy the $mn$-system of $p'/p$. If $p' > 2p$, then

$$S^{p',p'}(\hat{n}) = q^{+L(L-m_1)-\frac{1}{2} \sum_{j=1}^{t} m_j n_j} \prod_{j=1}^{t-1} \left[ \frac{m_j + n_j}{n_j} \right]_q,$$

and if $p' < 2p$, then

$$S^{p',p'}(\hat{n}) = q^{+Lm_1-\frac{1}{2} \sum_{j=1}^{t} m_j n_j} \prod_{j=1}^{t-1} \left[ \frac{m_j + n_j}{n_j} \right]_q.$$

Proof: We first prove that, for a given $t$, expression (14) follows from expression (13). So let $p' < 2p$, with $p'/p$ having rank $t$ and assume that (13) holds for rank $t$. In particular, it holds for the sector $S^{p'-p,p'}(n_1, n_2, \ldots, n_t)$. Notice that if $p' < 2p$ then the $mn$-system is the same for $p'/p$ as it is for $p'/p$. Using

$$\left[ \frac{m + n}{n} \right]_q = q^{mn} \left[ \frac{m + n}{n} \right]_{q^{-1}},$$

expression (13) may be rewritten

$$S^{p'-p,p'}(\hat{n}) = q^{+L(L-m_1)+\frac{1}{2} \sum_{j=1}^{t} m_j n_j} \prod_{j=1}^{t-1} \left[ \frac{m_j + n_j}{n_j} \right]_q^{-1}.$$
Since the action of a $D$-transform on $S^{p',p'}(\hat{n})$ yields exactly the set $S^{p',p'}(\hat{n})$, Lemma 3.2 gives:

$$S^{p',p'}(\hat{n}) = q^{4L^2 - \frac{1}{2}L(L-m_1) + \frac{1}{2} \sum_{j=1}^{k} m_j n_j} \prod_{j=1}^{k-1} \left[ \frac{m_j + n_j}{n_j} \right]_q,$$

whereupon (14) follows.

We now prove (13) for $p' > 2p$. In the case $p = 1$ and $p' = 3$, there is a unique path of length $L$, which comprises exactly $n_1 = L/2$ particles. Its weight is easily found to be $n_1^2$. Therefore $S^{1,3}(\hat{n}) = q^{4L^2}$. Since $p'/p = 3/1$ has continued fraction (3), then in this case $t = 1$ and hence $m_1 = 0$. Therefore, (13) holds for rank $t = 1$.

We now proceed by induction on the rank of $p'/p$. Assume that (13) and (14) hold for all $p'/p$ of rank $k - 1$, where $k > 1$. Now let $p'/p$ have rank $k$ so that if $p'/p$ has continued fraction $(c_0, c_1, \ldots, c_n)$ then $c_0 + c_1 + \ldots + c_n - 2 = k$. Note that $(p' - p)/p$ has continued fraction $(c_0 - 1, c_1, \ldots, c_n)$ and hence rank $k - 1$. Let $\hat{n}' = (n_2, n_3, \ldots, n_t)$. We consider separately the two cases $c_0 > 2$ and $c_0 = 2$ for which $t_1 > 1$ and $t_1 = 1$ respectively. In the $c_0 > 2$ case, the induction hypothesis gives:

$$S^{p,p'}(\hat{n}') = q^{4m_1(m_1-m_2) - \frac{1}{2} \sum_{j=2}^{k} m_j n_j} \prod_{j=2}^{k-1} \left[ \frac{m_j + n_j}{n_j} \right]_q.$$

Then Lemma 5.1 yields:

$$S^{p,p'}(\hat{n}) = q^{4L^2 - \frac{1}{2}L(L-m_1) + \frac{1}{2} \sum_{j=2}^{k} m_j n_j} \prod_{j=2}^{k-1} \left[ \frac{m_j + n_j}{n_j} \right]_q.$$

Expression (13) now follows for the case $t = k$, after noting that

$$(L - m_1)^2 = L(L-m_1) - m_1(L-m_1) = L(L-m_1) - m_1(m_1-m_2 + 2n_1),$$

where the second equality arises from (14) with $j = 1$.

In the $c_0 = 2$ case, $(p' - p)/p$ has continued fraction $(1, c_1, c_2, \ldots, c_n)$ and thus $2p < (p' - p)$. Then combining (14) at $t = k - 1$ with Corollary 5.1, results in:

$$S^{p,p'}(\hat{n}) = q^{4L^2 - \frac{1}{2}L(L-m_1) + \frac{1}{2} \sum_{j=2}^{k} m_j n_j} \prod_{j=2}^{k-1} \left[ \frac{m_j + n_j}{n_j} \right]_q.$$

In this case, expression (13) now follows for the case $t = k$, after noting that

$$(L - m_1)^2 = L(L-m_1) - m_1(L-m_1) = L(L-m_1) - m_1(m_2 + 2n_1),$$

where here, the second equality arises from (10) with $j = t_1 = 1$.

Thus, expressions (13) and (14) at $t = k$ follow from those at $t = k - 1$. The lemma then follows by induction.  \( \Box \)

We are now able to give a constant-sign generating function for all paths in $p_{0,0,0,s_0+1}(L)$ and thus provide a constant-sign expression for the finitised character $\chi_{p_{0,s_0}}(L)$.\(^{17}\)

\(^{17}\) We note that this expression combines the two expressions for $p' < 2p$, and $p' > 2p$ given in (10).
THEOREM 5.3. Let \( p \) and \( p' \) be positive co-prime integers and \( r_0 \) and \( s_0 \) the smallest positive integers such that \(|ps_0 - p'r_0| = 1\). Then for even \( L \geq 0 \):

\[
\lambda^{p,p'}_{s_0,r_0,s_0+1}(L) = \sum_m q^{\frac{1}{2}m^T C_m - \frac{1}{4}L^2} \prod_{j=1}^{t-1} \left[ m_j - \frac{1}{2}(Cm)_j \right]_q,
\]

where the summation is over vectors \( m = (m_1, m_2, \ldots, m_{t-1}) \) with \( m_j \in 2\mathbb{Z} \) and \( m_j \geq 0 \) for \( j = 1, 2, \ldots, t-1 \).

Proof: Since the two cases \( p' > 2p \) and \( p' < 2p \) correspond to \( t_1 > 1 \) and \( t_1 = 0 \) respectively, Corollary [13] shows that the exponents in (13) and (14) are each equal to \( \frac{1}{2}m^T C_m - \frac{1}{4}L^2 \). The current theorem therefore follows after summing over all sectors \((n_1, n_2, \ldots, n_t)\) which give \( m_0 = L \), and writing \( n_j = -\frac{1}{2}(Cm)_j \). That each \( m_j \) is even, follows inductively from (10) and (11).

5.2. Character formulae. In Theorem 5.3, we note that no explicit restrictions are given for the \( m_j \) (other than being even and non-negative). In fact, there do exist implicit restrictions, that arise because for

\[
\left[ \begin{array}{c} m_j + n_j \\ m_j \end{array} \right]_q
\]

\((n_j = -\frac{1}{2}(Cm)_j)\) to be non-zero requires \( n_j \geq 0 \). In order to more conveniently impose these restrictions and then to take the limit \( L \to \infty \), we now change to a further set of variables. This set will also enable a comparison with the results of [17] to be made. The variables will be \( \lambda_i^{(\mu)} \) for \( 0 \leq \mu \leq n \) and \( 1 \leq i \leq \delta_{\mu,n} \). It will turn out that we need only consider the cases for which each \( \lambda_i^{(\mu)} \) is actually a partition. It also turns out that we need only consider those cases for which certain \( \lambda_1^{(\mu)} \) and certain \( wt(\lambda_1^{(\mu)}) \) are bounded above by values that depend on \( L \) and the partitions \( \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(\mu-1)} \). These two bounds, if imposed, will be denoted \( \Lambda_0^{(\mu)} \) and \( w_0^{(\mu)} \) respectively.

For the purpose of having uniform expressions, set \((e_0,e_1,\ldots,e_{n-1},e_n) = (c_0,c_1,\ldots,c_{n-1},c_n-2)\). Now set \( m_0 = L \) and

\[
\lambda_i^{(\mu)} = \frac{1}{2}(m_{i\mu+i} - m_{i\mu+i+1}), \quad (1 \leq i < e_\mu, \quad 0 \leq \mu \leq n);
\]

\[
\lambda_{e_\mu}^{(\mu)} = \frac{1}{2}m_{e_\mu+1}, \quad (0 \leq \mu < n);
\]

\[
\lambda_{e_n}^{(n)} = \frac{1}{2}m_{n+1}.
\]

In addition, set:

\[
w_0 = L/2;
\]

\[
w_\mu = \lambda_{e_{\mu-1}}^{(\mu-1)}, \quad (1 \leq \mu \leq n);
\]

\[
\lambda_0^{(\mu)} = w_{\mu-1} - wt(\lambda^{(\mu-1)}), \quad (1 \leq \mu \leq n)
\]
Fraction

Proof: On using (21), we readily find that:

\( m_{\mu,i} = 2 \left( w_\mu - \sum_{j=1}^{i-1} \lambda_j^{(\mu)} \right), \quad (1 \leq i \leq e_\mu, \quad 0 \leq \mu \leq n); \)

\( n_{\mu,i} = \lambda_{i-1}^{(\mu)} - \lambda_i^{(\mu)}, \quad (1 + \delta_\mu, 0 \leq i \leq e_\mu, \quad 0 \leq \mu \leq n), \)

where the second expression arises on using Lemma 4.2. The restrictions \( n_j \geq 0 \) are thus equivalent to \( \lambda_1^{(0)} \geq \lambda_2^{(0)} \geq \ldots \geq \lambda_n^{(0)} \) and \( \lambda_0^{(\mu)} \geq \lambda_1^{(\mu)} \geq \ldots \geq \lambda_n^{(\mu)} \) for \( 1 \leq \mu \leq n \). That \( \lambda_{\mu,0}^{(\mu)} \geq 0 \) arises from (13) and (17). Thus \( \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n)} \) are indeed partitions with \( \lambda_0^{(\mu)} \geq \lambda_1^{(\mu)} \) for \( 1 \leq \mu \leq n \left( \lambda_1^{(0)} \right) \) is unbounded). That \( \lambda_0^{(\mu)} \geq 0 \) for \( 1 \leq \mu \leq n \) implies, via (20), that \( wt(\lambda^{(n)}) \leq w_\mu \) for \( 0 \leq \mu \leq n \). Additionally, we obtain \( \sum_{i=1}^{n} \lambda_i^{(n)} = 1 \sum_{i=1}^{n} \lambda_i^{(0)} = w_n \), so that \( wt(\lambda^{(n)}) = w_n \).

Thereupon, Theorem 5.3 may be rewritten as follows (cf. Theorem 3.8 of [17]):

**Theorem 5.4.** Let \( p \) and \( p' \) be positive co-prime integers and \( r_0 \) and \( s_0 \) the smallest non-negative integers such that \( |ps_0 - p'r_0| = 1 \). If \( p'/p \) has continued fraction \( (e_0, e_1, \ldots, e_{n-1}, e_n + 2) \), then for even \( L \geq 0 \):

\[
\chi^{p,p'}_{s_0,r_0,s_0+1}(L) = \sum q^{-n} \sum_{\lambda} \prod_{\mu=0}^{n} \lambda_{\mu,1}^{2-q} \prod_{i=1}^{e_\mu} \left[ 2 \left( w_\mu - \sum_{j=1}^{i} \lambda_j^{(\mu)} \right) + \lambda_{i-1}^{(\mu)} + \lambda_i^{(\mu)} \right],
\]

where the sum is over all sequences \( \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n)} \) of partitions for which, for \( 0 \leq \mu \leq n \), the partition \( \lambda^{(\mu)} = (\lambda_1^{(\mu)}, \lambda_2^{(\mu)}, \ldots, \lambda_n^{(\mu)}) \) satisfies \( \lambda_1^{(\mu)} \leq \lambda_0^{(\mu)} (0 < \mu) \) and \( wt(\lambda^{(\mu)}) \leq w_\mu \), where we define \( w_0 = L/2 \), \( w_\mu = \lambda_{\mu-1}^{(\mu)} \) for \( 1 \leq \mu \leq n \), and \( \lambda^{(n)} = w_{n-1} - wt(\lambda^{(n)}-1) \) for \( 1 \leq \mu \leq n \); and additionally also satisfy \( wt(\lambda^{(n)}) = w_n \). (In the above formula, \( \chi^{p,p'}_{s_0,r_0,s_0+1}(L) \).

**Proof:** On using (21), we readily find that:

\[
\frac{1}{4} \left( m^T C m - L^2 \right) = \sum_{\mu=0}^{n} \sum_{i=1}^{e_\mu} \lambda_{\mu,1}^{2-q}.
\]

The result then immediately follows from Theorem 5.3 using (21) and (22). \( \square \)

We are now able to take the \( L \to \infty \) limit. The following theorem deals with the \( p > 2 \) cases, for which \( n > 0 \) and \( e_1 > 0 \).

**Theorem 5.5.** Let \( p \) and \( p' \) be positive co-prime integers with \( p' > p > 2 \), and \( r_0 \) and \( s_0 \) the smallest non-negative integers such that \( |ps_0 - p'r_0| = 1 \). If \( p'/p \) has continued fraction \( (e_0, e_1, \ldots, e_{n-1}, e_n + 2) \), then:

\[
\chi^{p,p'}_{r_0,s_0}(L) = \sum q^{-n} \sum_{\lambda} \prod_{\mu=1}^{n} \prod_{i=1}^{e_\mu} \left[ 2 \left( w_\mu - \sum_{j=1}^{i-1} \lambda_j^{(\mu)} \right) + \lambda_{i-1}^{(\mu)} + \lambda_i^{(\mu)} \right],
\]

where
where the sum is over all sequences \( \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n)} \) of partitions for which, for \( 0 \leq \mu \leq n \), the partition \( \lambda^{(\mu)} = (\lambda^{(\mu)}_1, \lambda^{(\mu)}_2, \ldots, \lambda^{(\mu)}_n) \) satisfies \( \lambda^{(\mu)}_1 \leq \lambda^{(\mu)}_0 \) (1 < \( \mu \)) and \( wt(\lambda^{(\mu)}) \leq w_\mu \) (0 < \( \mu \)), where we define, \( w_\mu = \lambda^{(\mu-1)}_\mu \) for 1 \( \leq \mu \leq n \), and \( \lambda^{(\mu)}_0 = w_{\mu-1} - wt(\lambda^{(\mu-1)}) \) for 2 \( \leq \mu \leq n \); and additionally also satisfy \( wt(\lambda^{(n)}) = w_n \). (In the above formula, \( \lambda_{\mu,1} = \lambda^{(\mu)}_1 \).)

**Proof:** By (3),

\[
\chi^{p,p'}_{\eta_0,\eta_0} = \lim_{L \to \infty} \chi^{p,p'}_{\eta_0,\eta_0,\eta_0+1}(L),
\]

whereupon the result follows from Theorem 5.4 after noting that:

\[
\lim_{m \to \infty} \left[ \frac{m+n}{n} \right]_q = \lim_{m \to \infty} \left[ \frac{m+n}{m} \right]_q = \frac{1}{(q)_n}.
\]

\( \square \)

The case \( p = 2 \) has \( n = 1 \) and \( e_1 = 0 \). In this case, taking the \( L \to \infty \) limit of the expression given by Theorem 5.4, reproduces the summation expression for \( \chi^{2,2e_0+1}_{1,0} \) first given by Gordon [20] (see also [2, 8, 12, 1, 11]):

**Theorem 5.6.**

\[
\chi^{2,2e_0+1}_{1,0} = \sum_{\lambda} q^{\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{e_0-1}^2} \frac{(q)_{\lambda_1 - \lambda_2} \cdot (q)_{\lambda_1 - \lambda_3} \cdots (q)_{\lambda_{e_0-2} - \lambda_{e_0-1}}}{(q)_{\lambda_{e_0-1}}},
\]

where the sum is over all partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{e_0-1}) \).

**Proof:** With \( p = 2 \) and \( p' = 2e_0 + 1 \), the continued fraction of \( p'/p \) is \((e_0, 2)\), so that \( e_1 = 0 \). Then Theorem 5.4 requires a sum over all pairs of partitions \( \lambda^{(0)} \) and \( \lambda^{(1)} \) which have \( e_0 \) and \( e_1 \) parts respectively. Since it is required that \( wt(\lambda^{(1)}) = w_1 \) and \( w_1 = \lambda^{(0)}_0 \), it is necessary that \( \lambda^{(0)}_0 = 0 \). The only other constraint is \( wt(\lambda^{(0)}) \leq L/2 \). The result now follows exactly as in the proof of Theorem 5.5. \( \square \)

In the \( p = 1 \) case, the sum of Theorem 5.4 is over all partitions \( \lambda^{(0)} \) such that \( wt(\lambda^{(0)}) = L/2 \). Thus, as \( L \to \infty \), the exponent of \( q \) is unbounded and so \( \lim_{L \to \infty} \chi^{1,p'}_{1,1,2}(L) \) does not exist. This is to be expected since as \( L \to \infty \), every path in \( P^{1,p'}_{1,1,2}(L) \) has an unbounded number of scoring vertices.

**Appendix A. Bijection between paths and partitions**

In this Appendix, we describe a natural weight preserving bijection between the Forrester-Baxter paths \( P^{p,p'}_{a,b,c}(L) \) [11], and partitions with prescribed hook differences [5]. That such a bijection exists was anticipated in [5], where generating functions for the two are shown to be equal up to a normalisation.

**A.1. Path recurrence relations.** For \( 0 < h < p' \), define the functions \( r(h) = \lceil ph/p' \rceil \) and \( \tilde{r}(h) = \lfloor (p' - p)h/p' \rfloor \) and note that \( r(h) + \tilde{r}(h) = h - 1 \). From the definitions (1) and (2), it is easy to see that the path generating functions \( \phi^{p,p'}_{a,b,c}(L) \) satisfy the recurrences (19):

\[
\phi^{p,p'}_{a,b,b+1}(L) = q^{L(\tilde{r}(b+1))} \phi^{p,p'}_{a,b+1,b}(L-1) + q^{L/2} \phi^{p,p'}_{a,b-1,b}(L-1),
\]

\[
\phi^{p,p'}_{a,b,b-1}(L) = q^{L/2} \phi^{p,p'}_{a,b+1,b}(L-1) + q^{-L\tilde{r}(b-1)} \phi^{p,p'}_{a,b-1,b}(L-1);
\]
the boundary conditions:

\( (25) \quad \phi_{a,b-1,b}^{p,p'}(L) = 0 \) if \( b = 1 \),

\( (26) \quad \phi_{a,b+1,b}^{p,p'}(L) = 0 \) if \( b = p' - 1 \),

and initial conditions:

\( (27) \quad \phi_{a,b+1,0}^{p,p'}(0) = \phi_{a,0,0}^{p,p'}(0) = \delta_{a,b} \).

These five properties uniquely determine \( \phi_{a,b,c}^{p,p'}(L) \) in all cases. Now define

\( (28) \quad \chi_{a,b,b+1}^{p,p'}(L) = q^{-r(c)(a-b)\pm L)/2-(a-b)(a-c)/4} \phi_{a,b,b+1}^{p,p'}(L), \)

where \( c = b \pm 1 \). In translating the above recurrences to this new function, it is appropriate to treat the two cases of \( \hat{r}(b) = \hat{r}(b \pm 1) \) and \( \hat{r}(b) \neq \hat{r}(b \pm 1) \) separately. In the \( \hat{r}(b) = \hat{r}(b \pm 1) \) case, we find that (23) and (24) become

\( (29) \quad \chi_{a,b,b+1}^{p,p'}(L) = \chi_{a,b,b+1}^{p,p'}(L - 1) + q^{(a-b)/2} \phi_{a,b,b+1}^{p,p'}(L - 1), \)

and

\( (30) \quad \chi_{a,b,b}^{p,p'}(L) = \chi_{a,b,b}^{p,p'}(L - 1) + q^{(1+a-b)/2} \phi_{a,b,b}^{p,p'}(L - 1), \)

respectively. In the other case, where necessarily \( \hat{r}(b \pm 1) = \hat{r}(b) \), we have

\( (31) \quad \chi_{a,b,b+1}^{p,p'}(L) = \chi_{a,b,b+1}^{p,p'}(L - 1) + q^{(1+a-b)/2} \phi_{a,b,b+1}^{p,p'}(L - 1), \)

\( (32) \quad \chi_{a,b,b}^{p,p'}(L) = \chi_{a,b,b}^{p,p'}(L - 1) + q^{(1+a-b)/2} \phi_{a,b,b}^{p,p'}(L - 1). \)

The boundary and initial conditions are similar to those above. Namely:

\( (33) \quad \chi_{a,b-1,b}^{p,p'}(L) = 0 \) if \( b = 1 \),

\( (34) \quad \chi_{a,b+1,b}^{p,p'}(L) = 0 \) if \( b = p' - 1 \),

and

\( (35) \quad \chi_{a,b+1,0}^{p,p'}(0) = \chi_{a,0,0}^{p,p'}(0) = \delta_{a,b}. \)

**A.2. Partitions with prescribed hook differences.** A partition \( \mu = (\mu_1, \mu_2, \ldots, \mu_M) \) is a sequence of \( M \) integer parts \( \mu_1, \mu_2, \ldots, \mu_M \), satisfying \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_M > 0 \). The weight \( wt(\mu) \) of \( \mu \) is given by \( wt(\mu) = \sum_{i=1}^{M} \mu_i \). The partition \( \mu \) is often depicted by its Young diagram (also called Ferrars graph), \( F^\mu \) which comprises \( M \) left-adjusted rows, the \( i \)th row of which (reading down) consists of \( \mu_i \) cells. The coordinate \((i, j)\) of a cell is obtained by setting \( i \) and \( j \) to be respectively, the row and column (reading from the left) in which the cell resides. The \( k \)th diagonal of \( F^\mu \) comprises all those cells of \( F^\mu \) with coordinates \((i, j)\) which satisfy \( i - j = k \).

The partition \( \mu' \), conjugate to \( \mu \), is obtained by setting \( \mu'_j \) to be the number of cells in the \( j \)th column of \( F^\mu \). The hook difference at the cell with coordinate \((i, j)\) is then defined to be \( \mu_i - \mu'_j \). As an example, filling each cell of \( F^{(5,4,3,1)} \) with its hook difference, yields:

\[
\begin{array}{cccc}
1 & 2 & 2 & 3 & 4 \\
0 & 1 & 1 & 2 \\
-1 & 0 & 0 \\
-3 \\
\end{array}
\]
The bold entries are those on diagonal \(-1\). In what follows, we will be especially interested in the hook differences on certain diagonals.

Let \(K, i, N, M, \alpha, \beta\) be non-negative integers for which \(1 \leq i \leq K/2\), \(\alpha + \beta < K\) and \(\beta - i \leq N - M \leq K - \alpha - i\). In \(\chi\), \(D_{K,i}(N, M; \alpha, \beta)\) is defined to be the generating function for partitions \(\mu\) into at most \(M\) parts, each not exceeding \(N\) such that the hook differences on diagonal \(1 - \beta\) are at least \(\beta - i + 1\), and on diagonal \(\alpha - 1\) at most \(K - \alpha - 1\). In addition, if \(\alpha = 0\), the restriction that \(\mu_{N-L+1} > 0\) is also imposed; and if \(\beta = 0\), the restriction that \(\mu_1 > M - i\) is also imposed. It may be shown (see \(\chi\), page 346) that:

\begin{align}
D_{K,i}(N, M; \alpha, \beta) &= D_{K,i}(N, M-1; \alpha, \beta) + q^M D_{K,i}(N-1, M; \alpha+1, \beta-1), \\
D_{K,i}(M + K - i, M; 0, \beta) &= 0, \\
D_{K,i}(M - i, M; \alpha, 0) &= 0, \\
D_{K,i}(0, 0; \alpha, \beta) &= 1.
\end{align}

Now if we define

\[
\chi_{a,b,c}^{p,p'}(L) = D_{p,a} \left( \frac{L - a + b}{2}, \frac{L + a - b}{2}; p - r, r \right),
\]

with \(r = \lfloor pc/p' \rfloor + (b - c + 1)/2\). We find that \(\chi_{a,b,c}^{p,p'}(L)\) satisfies precisely the same recursion, boundary and initial conditions as \(\chi_{a,b,c}^{p,p'}(L)\). Since these conditions determine a function uniquely, we conclude that \(\tilde{\chi}_{a,b,c}^{p,p'}(L) = \chi_{a,b,c}^{p,p'}(L)\). That is, we obtain:

\begin{equation}
\chi_{a,b,c}^{p,p'}(L) = D_{p,a} \left( \frac{L - a + b}{2}, \frac{L + a - b}{2}; p - r, r \right).
\end{equation}

In \(\chi\), the above recurrences for \(D_{K,i}(N, M; \alpha, \beta)\) are solved to yield

\[
D_{K,i}(N, M; \alpha, \beta) = \sum_{\lambda} q^{\lambda(K\lambda - i)(\alpha + \beta) + K\beta\lambda} \left[ \frac{N + M}{M - K\lambda} \right]_q
- \sum_{\lambda} q^{\lambda(K\lambda + i)(\alpha + \beta) + K\beta\lambda + 2\lambda} \left[ \frac{N + M}{M - K\lambda - i} \right]_q.
\]

Using this and (38), we are led to the expression for \(\chi_{a,b,c}^{p,p'}(L)\) given in (3).

A.3. The bijection. In order to describe the bijection between the paths and the partitions described above, we recapitulate the derivation of (36) and (37) given in \(\chi\). Consider one of the partitions enumerated by \(D_{K,i}(N, M; \alpha, \beta)\). If the \(M\)th part is zero then the partition also appears in \(D_{K,i}(N, M - 1; \alpha, \beta)\). On the other hand, if the \(M\)th part is at least 1, then the partition obtained by decreasing each part by 1 appears in \(D_{K,i}(N - 1, M; \alpha + 1, \beta - 1)\) (the removal of the first part changes the diagonals on which the hook differences are considered — the changes in \(\alpha\) and \(\beta\) reflect this) Thus, (36) results.

Now, under the identification (38), expression (29) is equivalent to (36). The first term of (29) corresponds to a peak-down vertex at the \(L\)th position and the second term to a straight-up vertex at the \(L\)th position. This indicates that if \(\hat{r}(b) = \hat{r}(b + 1)\) (so that \(r(b) \neq r(b + 1)\) and hence the second edge of the vertex is
in an odd band), then a straight-up vertex corresponds to being able to remove the first column of length $M = (L + a - b)/2$ from the partition. On the other hand, if $\hat{r}(b) = \hat{r}(b + 1)$, then a peak-down vertex corresponds to leaving the partition unchanged, but reducing the constraining parameter $M$ by 1.

When $\hat{r}(b) \neq \hat{r}(b - 1)$, the analysis of (32) is similar. In this case, a peak-up vertex corresponds to being able to remove the first column of length $M = (L + a - b)/2$ from the partition, and a peak-down vertex corresponds to leaving the partition unchanged, but again reducing the constraining parameter $M$ by 1. We thus obtain the first and fourth entries in the first column of the following table.

Again consider one of the partitions enumerated by $D_{K,i}(N, M; \alpha, \beta)$. If the first part is less than $N$ then the partition also appears in $D_{K,i}(N - 1, M; \alpha, \beta)$. On the other hand, if the first part is exactly $N$, then the partition obtained by removing the first part appears in $D_{K,i}(N, M - 1; \alpha - 1, \beta + 1)$. Thus, (37) results.

Under the identification (38), expression (30) is equivalent to (37). The first and second terms of (30) correspond respectively to a straight-down vertex and a peak-up vertex at the $L$th position. Thus, when $\hat{r}(b) = \hat{r}(b - 1)$, a straight-down vertex corresponds to being able to remove the first part of length $N = (L - a + b)/2$ from the partition, and a peak-up vertex corresponds to leaving the partition unchanged, but reducing the constraining parameter $N$ by 1.

For $\hat{r}(b) \neq \hat{r}(b + 1)$, a similar analysis of (31) shows that a peak-down vertex corresponds to being able to remove the first part of length $N = (L - a + b)/2$ from the partition, and a straight-up vertex corresponds to reducing the parameter $N$ by 1.

By recursively applying the above rules, through a process of successive row and column removal, we eventually arrive at the empty partition corresponding to a path of length 0. By applying the procedure in reverse, traversing the path from left to right and keeping track of the values $N$ and $M$ as we proceed, we can build the required partition up from the empty partition. For this, the above description yields the moves given in Table 1.

| Vertex | Move | Vertex | Move |
|--------|------|--------|------|
| ![Add M column & increment N] | Add M column & increment N | ![Increment N] | Increment N |
| ![Add N row & increment M] | Add N row & increment M | ![Increment M] | Increment M |
| ![Increment N] | Increment N | ![Add M column & increment N] | Add M column & increment N |
| ![Increment M] | Increment M | ![Add N row & increment M] | Add N row & increment M |

Table 1.
Alongside each description, we give a line segment and an arrow which encapsulates the description. This provides a handier means of building the partition. In fact, it describes a construction of the partition’s profile (outline) as follows. Let the length zero path correspond to a single dot. Then, scanning the path from left to right, for each vertex, append the appropriate line segment to the profile so that the arrow points away from the initial dot. After the $L$th vertex has been considered, this profile is abutted into the axes in the 4th quadrant to produce the Young diagram of the required partition.

To illustrate this procedure, consider the following path in $P_{3,8}^{4,3,2}(15)$:

\begin{center}
\begin{tikzpicture}
\draw[very thick, ->] (0,0) -- (1,0);
\draw[very thick, ->] (1,0) -- (1,1);
\draw[very thick, ->] (1,1) -- (2,1);
\draw[very thick, ->] (2,1) -- (2,2);
\draw[very thick, ->] (2,2) -- (3,2);
\draw[very thick, ->] (3,2) -- (3,3);
\draw[very thick, ->] (3,3) -- (4,3);
\draw[very thick, ->] (4,3) -- (4,4);
\draw[very thick, ->] (4,4) -- (5,4);
\draw[very thick, ->] (5,4) -- (5,5);
\draw[very thick, ->] (5,5) -- (6,5);
\draw[very thick, ->] (6,5) -- (6,6);
\draw[very thick, ->] (6,6) -- (7,6);
\draw[very thick, ->] (7,6) -- (7,7);
\draw[very thick, ->] (7,7) -- (8,7);
\draw[very thick, ->] (8,7) -- (8,8);
\draw[very thick, ->] (8,8) -- (9,8);
\draw[very thick, ->] (9,8) -- (9,9);
\draw[very thick, ->] (9,9) -- (10,9);
\draw[very thick, ->] (10,9) -- (10,10);
\draw[very thick, ->] (10,10) -- (11,10);
\draw[very thick, ->] (11,10) -- (11,11);
\draw[very thick, ->] (11,11) -- (12,11);
\draw[very thick, ->] (12,11) -- (12,12);
\draw[very thick, ->] (12,12) -- (13,12);
\draw[very thick, ->] (13,12) -- (13,13);
\draw[very thick, ->] (13,13) -- (14,13);
\draw[very thick, ->] (14,13) -- (14,14);
\draw[very thick, ->] (14,14) -- (15,14);
\draw[very thick, ->] (15,14) -- (15,15);
\end{tikzpicture}
\end{center}

The above procedure then produces the following Young diagram (here the starting point is the unfilled circle).

\begin{center}
\begin{tikzpicture}
\draw[very thick, ->] (0,0) -- (1,0);
\draw[very thick, ->] (1,0) -- (1,1);
\draw[very thick, ->] (1,1) -- (2,1);
\draw[very thick, ->] (2,1) -- (2,2);
\draw[very thick, ->] (2,2) -- (3,2);
\draw[very thick, ->] (3,2) -- (3,3);
\draw[very thick, ->] (3,3) -- (4,3);
\draw[very thick, ->] (4,3) -- (4,4);
\draw[very thick, ->] (4,4) -- (5,4);
\draw[very thick, ->] (5,4) -- (5,5);
\draw[very thick, ->] (5,5) -- (6,5);
\draw[very thick, ->] (6,5) -- (6,6);
\draw[very thick, ->] (6,6) -- (7,6);
\draw[very thick, ->] (7,6) -- (7,7);
\draw[very thick, ->] (7,7) -- (8,7);
\draw[very thick, ->] (8,7) -- (8,8);
\draw[very thick, ->] (8,8) -- (9,8);
\draw[very thick, ->] (9,8) -- (9,9);
\draw[very thick, ->] (9,9) -- (10,9);
\draw[very thick, ->] (10,9) -- (10,10);
\draw[very thick, ->] (10,10) -- (11,10);
\draw[very thick, ->] (11,10) -- (11,11);
\draw[very thick, ->] (11,11) -- (12,11);
\draw[very thick, ->] (12,11) -- (12,12);
\draw[very thick, ->] (12,12) -- (13,12);
\draw[very thick, ->] (13,12) -- (13,13);
\draw[very thick, ->] (13,13) -- (14,13);
\draw[very thick, ->] (14,13) -- (14,14);
\draw[very thick, ->] (14,14) -- (15,14);
\draw[very thick, ->] (15,14) -- (15,15);
\end{tikzpicture}
\end{center}

Thus the required partition is $(6, 6, 6, 3, 2, 1, 1)$.

Using the above table, we see that the values of $M$ and $N$ at any vertex count the number of SE segments and NE segments respectively prior to that vertex. Thus in terms of the $xy$-coordinate system defined in Section 1.3, we simply have $x = M$ and $y = N$. The definition (8) then gives the weight (sum of parts) of the corresponding partition.

That the map from path to partition is in fact a bijection follows because given a partition and values of $a, b, c$ and $L$, we obtain $N = (L - a + b)/2$ and $M = (L + a - b)/2$ at the $L$th vertex. Then one of the two extremal line segments of the partition’s profile (extended to be of width $N$ and height $M$) corresponds to the $L$th vertex. The right edge of the vertex is determined by $b$ and $c$, whereupon Table 2 shows that only one left edge can occur. After removing the corresponding line segment from the partition’s profile, this process is recursively repeated to produce a unique complete path.

We note that in the case where $p' = p + 1$, which implies that all bands are odd, and where $a = b = 1$, this bijection reduces to that given in [18]. In the case $p = 2$, it reduces to that given in [13, 11, 11].
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