Alfvén polar oscillations of relativistic stars

Hajime Sotani1⋆ and Kostas D. Kokkotas1,2⋆

1 Theoretical Astrophysics, Eberhard-Karls University of Tübingen, 72076 Tübingen, Germany
2 Department of Physics, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece

Accepted 2009 February 12. Received 2009 February 11; in original form 2009 January 8

ABSTRACT
We study polar Alfvén oscillations of relativistic stars endowed with a strong global poloidal dipole magnetic field. Here, we focus only on the axisymmetric oscillations which are studied by numerically evolving the two-dimensional perturbation equations. Our study shows that the spectrum of the polar Alfvén oscillations is discrete in contrast to the spectrum of axial Alfvén oscillations which is continuous. We also show that the typical fluid modes, such as the $f$ and $p$ modes, are not significantly affected by the presence of the strong magnetic field.

Key words: MHD – relativity – stars: magnetic fields – stars: neutron – stars: oscillations – gamma-rays: theory.

1 INTRODUCTION
The soft gamma repeaters (SGRs) are the objects radiating sporadic X-ray and gamma-ray bursts, whose typical luminosities are around $10^{41}$ erg s$^{-1}$. Rarely, the SGRs were associated with the emission of stronger gamma-rays with peak luminosities of about $10^{44} \sim 10^{46}$ erg s$^{-1}$. These phenomena are typically referred as ‘giant flares’. Up to now, at least three giant flares have been detected in three different SGRs, which are: the SGR 0526–66 in 1979; the SGR 1900+14 in 1998 and the SGR 1806–20 in 2004. During these giant flares in SGRs, we observe an initial strong sharp burst followed by a decaying tail which may last for hundreds of seconds. The timing analysis of the decaying tail of the latest two giant flares revealed the existence of quasi-periodic oscillations (QPOs), whose specific frequencies are approximately 18, 26, 30, 92, 150, 625 and 1840 Hz for SGR 1900+14 in 1998 and 28, 53, 84 and 155 Hz for SGR 1900+14 (see Watts & Strohmayer 2006 for a review). The most promising mechanism to produce these QPOs is the magnetar models. Magnetars are thought to be neutron stars with the spectrum of the polar Alfvén oscillations has been studied in more detail in two recent papers by Colaiuda, Beyer & Stergioulas (2007). These attempts were partially successful, i.e. some of the observed frequencies are in good agreement with the frequencies of torsional oscillations. But, very soon it has been realized the difficulty to explain all the observed frequencies in SGRs by using only the torsional oscillations of the crust (magnetized or not). In an attempt to explain the QPOs without making use of the crust torsional oscillations, Levin (2006) using a toy model suggested that the torsional Alfvén oscillations could not form a discrete spectrum but instead they should form a continuum, where he strongly pointed out the importance for the crust-core coupling, soon later Glampedakis, Samuelson & Andersson (2006) came with a similar suggestion i.e. that the QPO spectrum may be explained by considering the torsional Alfvén oscillations of the magnetar’s core. About a year later, this suggestion has been verified with a more elaborate model by Levin (2007), while next year Sotani, Kokkotas & Stergioulas (2008a) working on a more realistic model showed that the spectrum of the torsional Alfvén oscillations of relativistic magnetars forms a continuum and that there exist two distinct families as the so called ‘lower’ and ‘upper’ QPOs. They also figure out that the observed QPO frequencies in SGRs (at least the lower observed frequencies) can be explained by using the frequencies of torsional Alfvén oscillations as well as the crust torsional oscillations.

This last result about the torsional Alfvén oscillations has been studied in more detail in two recent papers by Colaiuda, Beyer & Kokkotas (2009) and Cerda-Duran, Stergioulas & Font (2009). One could actually argue that the features of torsional Alfvén oscillations are quite well understood. However, the spectrum of the polar Alfvén oscillations of magnetars is quite unexplored. So far, there are only a

*E-mail: sotani@astro.auth.gr (HS); kostas.kokkotas@uni-tuebingen.de (KDK)
few studies of polar oscillations of the magnetized stars, where, for example, they consider only the fluid modes for Ap stars (Shibahashi & Takata 1993), the oscillations in a spherical shell for Newtonian stellar models with an incompressible fluid (Rincon & Rieutord 2003) and the oscillations in the crust region of Newtonian models (Lee 2007).

From the previous discussion, one can realize that the polar part of the spectrum of relativistic magnetized stars has not yet been studied in detail while it is still unknown whether spectrum of the polar Alfvén oscillations is discrete or continuous. Moreover, some of the higher QPO frequencies observed may be related to polar and not to axial Alfvén oscillations. Finally, polar oscillations are related to the emission of gravitational waves and the ultimate result will be the estimation of the amount of the energy released in the form of gravitational waves during the giant flares (Abbott et al. 2007).

In an attempt to shed some light to the various features of polar Alfvén oscillations, we performed two-dimensional numerical evolutions of the perturbation equation of relativistic magnetar models endowed with a global poloidal dipole magnetic field. It should be mentioned here that in this study we omit the effects due to the presence of a crust and we focus only on the axisymmetric polar oscillations of the core. The omission of the crust simplifies considerably the boundary conditions of the problem without affecting significantly the actual frequencies of the oscillation modes. As a future project, we can foresee the inclusion of the crust which induces additional families of shear and interfacial modes (e.g. Vavoulidis, Kokkotas & Stavridis 2008) which can be used to explain the observed QPO frequencies. In other words, this article is the first step towards a detailed study of the oscillation spectrum of magnetized relativistic stars.

This paper is structured as follows. In the next section, we describe the equilibrium configuration adopted in this paper. In Section 3, we drive the basic perturbation equations with the appropriate boundary conditions. The numerical results are shown in Section 4 and finally we give a conclusion in Section 5. Unless otherwise noted, we adopt units of $c = G = 1$, where $c$ and $G$ denote the speed of light and the gravitational constant, respectively, while the metric signature is $(-, +, +, +)$.

## 2 Equilibrium Configuration

We assume that the background models of the strongly magnetized stars that we will study are spherically symmetric and non-rotating. This choice is justified since all known magnetars rotate very slowly, i.e. periods of a few seconds while it is true that even for magnetars with strong magnetic fields such as $B \gtrsim 10^{15}$ G, the deformation due to the magnetic pressure is not significant (see e.g. Colaiuda et al. 2008; Haskell et al. 2008). 1 This is expected since even for the case of magnetars the magnetic energy is much smaller than the gravitational binding energy, i.e. $E_m/E_b \sim 10^{-4} (B/10^{15} \text{ G})^2$, where $E_m$ and $E_b$ are the magnetic and gravitational binding energies, respectively. In other words, we consider a static spherically symmetric neutron star, which is a solution of the Tolman–Oppenheimer–Volkov (TOV) equations. In this case, the metric is defined as

$$ds^2 = -e^{2\Lambda(r)}dr^2 + e^{2\Lambda(r)}d\theta^2 + r^2(d\phi^2 + \sin^2\theta d\phi^2),$$

where $e^{-2\Lambda} = 1 - 2m(r)/r$ and the fluid four-velocity is $u^\mu = (e^{-\Phi}, 0, 0, 0)$. On this stellar model, we superimpose a dipole magnetic field. We assume that the star consists of a perfect fluid and in addition we make the assumption of the "the ideal magnetohydrodynamics (MHD) approximation". This implies that a comoving observer cannot feel the presence of an electric field. Then the stress-energy tensor is given by

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu} + H^\mu H_\nu \left( u^\mu u^\nu + \frac{1}{2} g^{\mu\nu} \right) - H^{\mu \nu},$$

where $\epsilon$, $p$ and $H^{\mu\nu}$ are the energy density, the pressure and the normalized magnetic field defined as $H^{\mu\nu} = B^{\mu\nu}/\sqrt{4\pi}$, respectively. In particular, in this paper we adopt the pure poloidal dipole magnetic field, while the effect due to the toroidal magnetic fields will be left for future studies. The pure poloidal magnetic field is given as

$$(H_r, H_\theta, H_\phi) = \left[ \frac{\epsilon^2 a_1(r)}{\sqrt{4\pi}} \cos \theta, -\frac{\epsilon^{-2} \partial_r a_1(r)}{\sqrt{4\pi}} \sin \theta, 0 \right],$$

where $a_1(r)$ is a radial component of the electromagnetic four-potential, which is the same as in Sotani et al. (2007). This potential $a_1(r)$ is actually determined as the solution of the differential equation

$$\frac{d^2 a_1}{dr^2} + \frac{d}{dr}(\Phi - \Lambda) \frac{da_1}{dr} = -2e^{2\Lambda} a_1 = -4\pi \epsilon e^{2\Lambda} j_1(r),$$

where $j_1(r)$ is a radial component of the four-current given by $j_1(r) = f_0 r^2 (\epsilon + p)$ where $f_0$ is a constant. In order to determine the form of $a_1(r)$, we impose the regularity condition at the stellar centre, i.e. $a_1(r) = a_1 r^2 + O(r^3)$, while at the stellar surface $a_1(r)$ should be smoothly connected to the exterior vacuum solution.

1 In this paper, for simplicity, we will not take into account the deformation of the star due to the presence of the strong magnetic field. These types of deformations will potentially affect, quantitatively, the oscillation spectrum, although one should not expect any important qualitative effect.
3 PERTURBATION EQUATIONS

3.1 Axisymmetric polar perturbations

We restrict attention to axisymmetric polar-type perturbations with the relativistic Cowling approximation, that is, the metric perturbations are ignored, i.e. we set $\delta g_{\mu\nu} = 0$. Note that the polar type of perturbations are independent of the axial-type ones, which can be understood by recalling the nature of the axisymmetric perturbations (Lee 2007). Then, the Lagrangian displacement vector for axisymmetric polar-type perturbations can be written as

$$\xi^r = \frac{e^{-\Lambda}}{r^2} W(t, r, \theta),$$

$$\xi^\phi = -\frac{1}{r^2} V(t, r, \theta),$$

$$\xi^\theta = 0.$$  

Here, we should note that the angular dependence of Lagrangian displacement vector $\xi^r$ for polar parity can be, in principle, described via a spherical harmonic decomposition, i.e. $\xi^r \sim Y_{lm}, \xi^\phi \sim \partial_\theta Y_{lm}$ and $\xi^\theta \sim \partial_\phi Y_{lm}/\sin^2 \theta$. Then, the components of the perturbed four-velocity $\delta u^\mu$ in the Cowling approximation can be written as

$$\delta u^t = \delta u^\phi = 0,$$

$$\delta u^r = \frac{1}{r} e^{-\Phi-\Lambda} \partial_t W,$$

$$\delta u^\phi = -\frac{1}{r} e^{-\Phi} \partial_t V,$$

where $\partial_t$ denotes the partial derivative with respect to $t$. Additionally, the Lagrangian variation of the baryon number density, $\Delta n$, is defined as $\Delta n = -\nabla^i (\xi^i)$, where $\nabla^i$ and $\Delta$ denote, respectively, the covariant derivative in a three-dimensional spatial part of the metric and the Lagrangian variation. With the earlier definition for the Lagrangian displacement vector, the Lagrangian variation of the baryon number density takes the form

$$\frac{\Delta n}{n} = -\frac{e^{-\Lambda}}{r^2} \frac{\partial W}{\partial r} + \frac{1}{r^2} \left( \frac{\partial V}{\partial \theta} + \cot \theta V \right).$$

Furthermore, for adiabatic perturbations, with the help of the first law of thermodynamics, the perturbed density and pressure are given as

$$\delta \epsilon = (\epsilon + p) \frac{\Delta n}{n} - \frac{\partial \delta n}{\partial t} \xi^t,$$

$$\delta p = \gamma p \frac{\Delta n}{n} - \frac{\partial \delta p}{\partial r} \xi^r,$$

where $\gamma$ is the adiabatic constant defined as

$$\gamma = \left( \frac{\partial \ln p}{\partial \ln n} \right)_i = \left( \frac{\Delta p}{p} \right) \left( \frac{\Delta n}{n} \right)^{-1},$$

and we also use the following relation between the Lagrangian perturbation $\delta$ and Eulerian perturbation $\delta$,

$$\Delta f(t, r) \simeq \delta f(t, r) + \frac{\delta f}{\delta r} \xi^r.$$  

The above assumptions lead to the following form of the $r$ and $\theta$ components of the linearized equation of motion with vanishing shear modulus, i.e. the equation (35) in Sotani et al. (2007) with $\mu = 0$,

$$(\epsilon + p + H^2)e^{-\Phi} \delta u^r_{,r} = -(\delta \epsilon + \delta p) \Phi' e^{-2\Lambda} - e^{-2\Lambda} \delta p_{,r} + \left( \Lambda' + \frac{2}{r} \right) H' \delta H' + H' \delta H_{,r} + H' \delta H_{,\phi} + H^\phi \delta H_{,\phi} + H' \delta H_{,r} + \left(-2\Phi' e^{-2\Lambda} H_{,\phi} + H_{,\phi} - 2e^{-2\Lambda} H^\phi + \cot \theta H^r - e^{-2\Lambda} H_{,r} \right) \delta H^\phi + e^{-2\Lambda} H_{,r} \delta H_{,\phi},$$

$$(\epsilon + p + H^2) e^{-\Phi} \delta u^\phi_{,r} = -\frac{1}{r^2} \delta p_{,r} + \left( \Phi' + \Lambda' + \frac{4}{r} \right) H^r + H_{,r} - \frac{1}{r^2} H_{,r} \delta H' + H^\phi \delta H_{,\phi} + \frac{1}{r^2} H_{,r} \delta H_{,\phi},$$

where a prime (') denotes the derivative with respect to $r$, i.e. $' \equiv \partial_r$. On the other hand, the linearized induction equation (37) in Sotani et al. (2007) yields the following relations for the first time derivative of the components of $\delta H^\mu$,

$$\delta H^r_{,t} = e^{-\Phi} \left[ H' \delta u^r_{,\phi} + H^\phi \delta u^r_{,\phi} \right],$$

$$\delta H^\phi_{,t} = e^{-\Phi} \left[ H' \delta u^\phi_{,\phi} + H^r \delta u^\phi_{,\phi} \right].$$
\[ H_0 = -e^\phi \left\{ \left( \frac{\alpha'}{r} \right) H' + H_\phi \right\} \delta u' + \left( \cot \theta H' + H_\phi \right) \delta u_\theta + H^\phi \delta u'_{\phi} - H^\phi \delta u_{\phi}, \tag{19} \]

\[ H_\phi = -e^\phi \left\{ \left( \Phi' + \frac{2}{r} \right) H^\phi + H_\phi \right\} \delta u' + \left( \cot \theta H^\phi - \Phi' + H_\phi \right) \delta u_\theta + H^\phi \delta u'_\phi - H^\phi \delta u_{\phi}, \tag{20} \]

Substituting the previous relations into the linearized equations of motion (16) and (17), one can obtain the following coupled system of evolution equations for the two functions describing the fluid perturbations

\[
\begin{bmatrix}
A_{00} & A_{01} \\
A_{10} & A_{11}
\end{bmatrix}
\begin{bmatrix}
\delta u' \\
\delta u_\theta
\end{bmatrix}
= \begin{bmatrix}
F_{W} \\
F_{V}
\end{bmatrix},
\tag{22}
\]

where the coefficients \( A_{00}, A_{01}, A_{10} \) and \( A_{11} \) are

\[ A_{00} = (\epsilon + p + H^\phi H_\theta + H^\phi H_\phi) \frac{1}{r^2} e^{-2\phi - \Lambda}. \tag{23} \]

\[ A_{01} = \frac{1}{r^2} e^{-2\phi} H^\phi H_\phi. \tag{24} \]

\[ A_{10} = -\frac{1}{r^2} e^{-2\phi - \Lambda} H^\phi H_\phi. \tag{25} \]

\[ A_{11} = - (\epsilon + p + H^\phi H_\theta + H^\phi H_\phi) \frac{1}{r^2} e^{-2\phi}. \tag{26} \]

while the two expressions \( F_W \) and \( F_V \) are

\[ F_W = - \left( \delta \epsilon + \delta p \right) \Phi e^{-2\Lambda} - e^{-2\phi} \delta p_\phi + \left( \frac{\alpha'}{r} \right) H' \delta H' + H^\phi \left( \delta H'_\phi + \delta H'_\theta + \delta H'_\phi \right) + H^\phi \delta H'_\phi + H^\phi \delta H'_\phi \]

\[ + \left[ -2 \delta \Phi e^{-2\Lambda} H_\phi + H_\phi - 2 e^{-2\Lambda} H^\phi + \cot \theta H' - e^{-2\Lambda} H_\phi \right] \delta H^\phi - e^{-2\Lambda} H_\phi \delta H^\phi \]

\[ + \left[ -2 \delta \Phi e^{-2\Lambda} H_\phi + H_\phi - 2 e^{-2\Lambda} H^\phi + \cot \theta H' - e^{-2\Lambda} H_\phi \right] \delta H^\phi - e^{-2\Lambda} H_\phi \delta H^\phi. \tag{27} \]

\[ F_V = - \frac{1}{r^2} \delta p_\phi + \left[ \Phi' + \frac{4}{r^2} \right] H^\phi + H_\phi - \frac{1}{r^2} H^\phi \delta H^\phi + H^\phi \delta H^\phi + H^\phi \delta H^\phi \]

\[ + \left[ \Phi' + \frac{4}{r^2} \right] H^\phi + H_\phi - \frac{1}{r^2} H^\phi \delta H^\phi + H^\phi \delta H^\phi + H^\phi \delta H^\phi \]

\[ + \left[ \frac{1}{r^2} \left( H_\phi H_\phi - H_\phi H_\phi \right) - 2 \sin \theta \cos \theta H^\phi \right] \delta H^\phi - \frac{1}{r^2} H_\phi \delta H^\phi. \tag{28} \]

Note that the above system of evolution equations has been written for a general equilibrium magnetic field \( H^\phi(r, \theta, \phi) \), i.e. the form of the magnetic field has not yet been specified.

### 3.2 Dipole magnetic field

As we mentioned earlier, in this article we study the perturbations of dipole fields. In order to simplify the boundary condition at the stellar surface, we introduce two new functions \( w(t, r, \theta) \) and \( v(t, r, \theta) \) defined as

\[ w = \epsilon W \sin \theta, \tag{29} \]

\[ v = \epsilon V \sin \theta. \tag{30} \]

Note that in the new functions the trigonometric term, \( \sin \theta \), has been introduced in an attempt to improve the numerical stability of the code. Then, by restricting our attention only to axisymmetric dipole poloidal magnetic fields describe by equation (3), we get a simplified form of the linearized equation of motion (22), that is

\[ \begin{bmatrix}
A_{00} & A_{01} \\
A_{10} & A_{11}
\end{bmatrix}
\begin{bmatrix}
\delta u' \\
\delta u_\theta
\end{bmatrix}
= \epsilon \sin \theta \begin{bmatrix}
F_{W} \\
F_{V}
\end{bmatrix}, \tag{31}
\]

where the coefficients \( A_{00}, A_{01}, A_{10} \) and \( A_{11} \) become

\[ A_{00} = e^{-2\phi - \Lambda} \left[ \epsilon + p + \frac{1}{4\pi r^2} (e^{-\Lambda} a'_1 \sin \theta)^2 \right]. \tag{32} \]

\[ A_{01} = -\frac{a_1 a'_1}{2\pi r^2} e^{-2\phi - 2\Lambda} \sin \theta \cos \theta. \tag{33} \]
\[ A_{10} = \frac{a_0 a_1'}{2\pi r^2} e^{-2\Phi - \lambda} \sin \theta \cos \theta, \]  

\[ A_{11} = -e^{-\Phi} \left[ (\epsilon + p)r^2 + \frac{a_0^2}{\pi r^2} \cos^2 \theta \right] \]  

and the expressions \( F_w \) and \( F_v \) will be simplified to the following forms

\[ F_w = -(\delta \epsilon + \delta p) \Phi e^{-2\lambda} - e^{-2\lambda} \delta p + \frac{e^{-\lambda}}{\sqrt{4\pi}} \left\{ -4\pi j_1 \delta H^\theta + a_1' \left[ \left( \Phi' + \frac{2}{r} \right) e^{-2\lambda} \delta H^\theta + e^{-2\lambda} \delta H^\phi - \frac{1}{r^2} \delta H'' \right] \right\} \sin \theta, \]  

\[ F_v = -\delta p + \sqrt{4\pi} c_0^2 j_1 \delta H' \sin \theta \epsilon + \frac{e^{-\lambda} a_1'}{\sqrt{\pi}} \left[ \left( \Phi' + \frac{2}{r} \right) \delta H^\theta + \frac{1}{r^2} e^{2\lambda} \delta H'' \right] \cos \theta. \]  

Note that in order to derive the above equations we used the relation

\[ \delta \epsilon = -\frac{e^{-\lambda}}{\sqrt{4\pi r^2 \epsilon \sin \theta}} \left[ e^{-\lambda} a_1'(w \cos \theta + w', \sin \theta) + 2a_1(v \sin \theta - v', \cos \theta) \right], \]  

\[ \delta H^\theta = -\frac{e^{-\lambda}}{\sqrt{4\pi r^2 \epsilon \sin \theta}} \left\{ \left[ \left( \Phi' + \frac{2}{r} \right) + \frac{e'}{\epsilon} \right] a_1' - \frac{2a_1}{r^2} e^{2\lambda} + 4\pi c_0^2 j_1 \right\} e^{-\lambda} \epsilon \sin \theta \]  

\[ + 2v \left( a_1' - \frac{2a_1}{r^2} \right) \frac{a_1' \epsilon}{\epsilon} \cos \theta - e^{-\lambda} a_1' w' \sin \theta + 2a_1 v' \cos \theta \]  

with \( \delta H^\phi = 0. \)

Finally, the variations of pressure and density are described by

\[ \delta p = \frac{\gamma p}{\epsilon \sin \theta} (-e^{-\lambda} w' + v), \]  

\[ \delta \epsilon = \frac{\epsilon + p}{\epsilon \sin \theta} (-e^{-\lambda} w' + v) + \frac{pe'}{\epsilon^2 \sin \theta} e^{-\lambda} w. \]  

### 3.3 Boundary conditions

For axisymmetric polar perturbations, in general, the angular dependences of the functions \( w(t, r, \theta) \) and \( v(t, r, \theta) \) can be written as \( w \propto \sin \theta P_\ell(\cos \theta) \) and \( v \propto \sin \theta \partial_\theta P_\ell(\cos \theta) \), where \( P_\ell(\cos \theta) \) is the Legendre polynomial of the order of \( \ell \). Thus, the boundary conditions on the axis (\( \theta = 0 \)) will be

\[ w = 0 \text{ and } v = 0, \]  

while on the equatorial plane (\( \theta = \pi/2 \)) we set

\[ w = 0 \text{ and } v = 0 \text{ for odd } \ell, \]  

\[ w = 0 \text{ and } v = 0 \text{ for even } \ell. \]  

Finally, at the stellar centre we impose the regularity conditions for \( w \) and \( v \), i.e. \( w = 0 \) and \( v = 0 \) at \( r = 0 \). While the boundary condition at the stellar surface is that the Lagrangian perturbation of the pressure should be zero, i.e. \( \delta P = 0 \), which is equivalent to \( w = 0 \) and \( v = 0 \) as we will use a barotropic equation of state (EoS) and \( \epsilon = 0 \) at the stellar surface for the background.

### 4 NUMERICAL RESULTS

#### 4.1 Code tests

For our studies, we adopted a polytropic EoS defined as

\[ p = K n_b m_b \left( \frac{n}{n_0} \right)^\Gamma \text{ and } \epsilon = n m_b + \frac{p}{\Gamma - 1}, \]  

where \( m_b \) and \( n_0 \) are the baryon mass and normalized number density given by \( m_b = 1.66 \times 10^{24} \text{ g} \) and \( n_0 = 0.1 \text{ fm}^{-3} \), respectively. Specifically, we adopt the values of \( \Gamma \) and \( K \) as \( \Gamma = 2.46 \) and \( K = 0.00936 \), whose values are in good agreement with the tabulated data for...
the realistic EoS A (Pandharipande 1971). For this EoS, we focus only on the stellar model with mass $M = 1.4\, M_\odot$, radius $R = 10.35$ km and compactness $M/R = 0.2$.

For the time evolution of the perturbation equation (31), we use the iterated Crank–Nicholson scheme (Teukolsky 2000), which keeps second-order accuracy for both space and time. In order to eliminate the spurious higher frequencies, we also add a fourth-order Kreiss–Oliger dissipation into the evolution equations (Gustafsson, Kreiss & Oliger 1995), where the coefficient of this dissipation term, $\epsilon_D$, depends on the numerical grid size. This type of numerical viscosity has been used successfully in Sotani & Kokkotas (2004). Most of the numerical calculations in this paper were done in the two-dimensional region of $(r, \theta)$ with $50 \times 40$ grid points, where $\theta$ is in the range of $0 \leq \theta \leq \pi/2$. The values of $\theta = 0$ and $\pi/2$ correspond to the axis and the equatorial plane, respectively. It should be mentioned that the results were not significantly improved when a larger number of grid points have been used, i.e. $50 \times 80$ or $100 \times 40$. The numerical evolutions were stable for long enough to allow us to extract the specific frequencies for each spatial point by performing a fast Fourier transformation (FFT) of the time-varying perturbation functions. Since in this paper we evolve the perturbed models for about $T = 200$ ms, in the FFT we can expect that possible maximum error in the estimation of the frequency $\Delta f$ is around $\Delta f \sim 1/T \approx 5$ Hz.

Before studying in detail the magnetar oscillations, we performed a test of the two-dimensional evolution code on non-magnetized stellar models. In general for a spherically symmetric star which is not endowed with a magnetic field, the oscillations can be decomposed into spherical harmonics $Y_{\ell m}$ which are independent of the angular index $\ell$. Thus by using the appropriate boundary conditions we can determine the specific frequencies by transforming it into an eigenvalue problem. The idea here is to compare the frequencies determined by the eigenvalue methods to those derived from our evolution code in order to check the accuracy of the new method. For the solution of the eigenvalue problem, we use the same code as in Sotani & Kokkotas (2004). The regularity condition at stellar centre demands that the behaviour of $W$ and $V$ in the vicinity of stellar centre is of the form $W \sim Cr^{\ell+1} + \cdots$ and $V \sim -C r^{\ell} + \cdots$, where $C$ is some constant. Thus, we set the initial data for the time evolution in the form

$$w(r, \theta) = \epsilon r^{\ell+1} \sin \theta P_{\ell}(\cos \theta),$$

$$v(r, \theta) = -\frac{\epsilon}{\ell} r^\ell \sin \theta \partial_\theta P_{\ell}(\cos \theta).$$

On a non-rotating cold neutron star without magnetic fields, there exists only one family of fluid mode, i.e. the $p$ (pressure) modes. The $f$ mode is a fundamental mode of the pressure mode family, i.e. the $p_0$ mode. There exists only one such mode, while there are an infinite number of higher $p$ modes for every $\ell$. Traditionally, the $p$ modes are named as $p_1, p_2, \ldots$ in ascending order of those frequencies. Note that the subscript used in the notation of the $p$ modes shows in addition the number of radial nodes in the eigenfunction, while the eigenfunction of the $f$ mode has no nodes. In Table 1, we can compare the frequencies determined by the solution of the eigenvalue problem and those derived by solving the two-dimensional perturbation equations. We list results only for the $f, p_1$ and $p_2$ modes with harmonic indices $\ell = 2, 3$ and 4. Since the oscillations with lower frequencies (or with longer wavelength) are not so sensitive to the number of grid points, the lower frequencies is expected to be calculated with higher accuracy. Actually, we find that the $f$ mode with $\ell = 2$ shows a deviation from the values derived by the eigenvalue problem of the order of 2.6 per cent while the $f$ mode with $\ell = 4$ shows a deviation of about 3.7 per cent. In any case, the results derived with the time-evolution code show a maximum deviation which is always below 5 per cent even for the higher frequency modes. We consider that this accuracy is acceptable for the type of study that we perform here, i.e. to quantify the effect of the magnetic field on the fluid modes and to find out whether the polar Alfvén oscillations are discrete or continuous.

### 4.2 Polar oscillations of magnetars

As initial data for the evolution of the perturbation equations, we used the analytic forms for the perturbation functions $w(r, \theta)$ and $v(r, \theta)$ described by the equations (47) and (48). We set initial data for angular indices $\ell = 2, 3$ and 4 and imposed at the equatorial plane the boundary conditions described by the equations (44) and (45). Actually, the perturbations on an axisymmetric background can be decomposed into two classes that satisfy different conditions at the equatorial plane, i.e. one class satisfies the condition specified in equation (44) and the other one satisfies the condition given by equation (45). These conditions describe perturbations for even and odd values of the angular index $\ell$.

The various features of those oscillations are examined in two ways. The first involves checking of the FFT amplitude at various points inside the star. If the spectrum is continuous, the peaks in FFT will depend on the position, while if the peaks are independent of the observer’s position then they correspond to a discrete spectrum. The second check involves the study of the phase of oscillation for each peak frequency.

| $\ell$ | $f^{(E)}$ | $f^{(T)}$ | $p_{1}^{(E)}$ | $p_{1}^{(T)}$ | $p_{2}^{(E)}$ | $p_{2}^{(T)}$ |
|-------|----------|----------|--------------|--------------|--------------|--------------|
| 2     | 2.68     | 2.75     | 6.71         | 6.87         | 9.97         | 10.21        |
| 3     | 3.26     | 3.41     | 7.65         | 7.89         | 11.07        | 11.38        |
| 4     | 3.75     | 3.89     | 8.49         | 8.73         | 12.08        | 12.38        |

*Note.* Here, the superscripts (E) and (T) denote the frequencies found by solving the eigenvalue problem and those by evolving the two-dimensional perturbation equations, respectively.
Alfvén polar oscillations of magnetars

Figure 1. The FFT of the perturbation function $v(t, r, \theta)$ for a stellar model with $B = 1 \times 10^{16}$ G. The three lines in each figure correspond to three different angular positions, $\theta \sim 0, \pi/4$ and $\pi/2$ and the different figures correspond to the different radial positions, $r \sim 0, R/2$ and $R$. In all figures, we can see one peak for the Alfvén oscillations, which is corresponding to $2a_0$ mode (i.e. the fundamental $l = 2$ polar Alfvén mode) and another peak at higher frequencies corresponding to the frequency of the $f$ mode with $\ell = 2$ ($2f$).

Figure 2. The phase of the fundamental polar Alfvén mode with $\ell = 2$ ($2a_0$) for a magnetar with $B = 1 \times 10^{16}$ G. The left- and right-hand panels correspond to the perturbation functions $w(t, r, \theta)$ and $v(t, r, \theta)$, respectively. We observe that the phases for both functions $w$ and $v$ are almost constant except for some regions around $\theta \sim \pi/3$ for the function $w$ and a part of $r/R \sim 0.8$ for the function $v$. We identify that these two regions with somehow strange behaviour of the phase are related to the numerical loss of accuracy in the calculation of the effective amplitudes, where the eigenfunctions are almost zero because these are nodal points (see the left-hand panels in Figs 3 and 4). Thus, we can argue that the phase of the oscillation modes is independent of the position. With the above two tests, we conclude that the Alfvén oscillations of polar parity are described by a discrete spectrum, in contrast to the spectrum of the axial Alfvén oscillations. This feature seems to be very similar to the case of inertial modes of rotating stars. That is, the spectrum of the axial inertial modes seems to be continuous, at least in the slow-rotation approximation (e.g. Kojima 1998; Beyer & Kokkotas 1999), while the polar modes admit a discrete spectrum.

Figure 3. Effective amplitude of the eigenfunction for $w(t, r, \theta)$ for polar Alfvén oscillation modes. The three panels correspond to Alfvén oscillations with $\ell = 2$ (left-hand panel), $\ell = 3$ (centre) and $\ell = 4$ (right-hand panel).
Figure 4. Effective amplitude of the eigenfunction for $v(t, r, \theta)$ with the several Alfvén oscillation modes. The three panels are corresponding to the Alfvén oscillations with $\ell = 2$ (left-hand panel), with $\ell = 3$ (centre) and with $\ell = 4$ (right-hand panel).

Figure 5. The frequencies of the Alfvén modes as functions of magnetic field strength.

Different initial data sets, such as initial deformation for $\ell = 3$ and $4$, verify the previous results, i.e. the spectrum is discrete and the oscillation modes have constant phase. In Figs 3 and 4, we show the effective amplitudes of the eigenfunctions $u(r, \theta)$ and $v(r, \theta)$ with the corresponding peak frequencies in FFT, 300 Hz (for $\ell = 2$), 465 Hz (for $\ell = 3$) and 500 Hz (for $\ell = 4$), where we assumed that magnetic field strength is $B = 1 \times 10^{16}$ G.

In Fig. 5, we show the results of our examination for the dependence of the polar Alfvén modes on magnetic field strength. This dependence of the polar Alfvén oscillations on the magnetic field strength can be eventually used to explain the observed higher QPO frequencies such as 150, 625 and 1840 Hz in SGR 1806$-20$ or 155 Hz in SGR 1900$+14$. It should be noted that for weaker magnetic fields it is difficult to determine the frequencies of the Alfvén modes because the corresponding peaks in FFT become a few orders of magnitude smaller compared to those of the fluid modes. This observation suggests that the polar Alfvén oscillations will play no role on the dynamics of magnetars with weaker magnetic fields. In order to find the dependence of the polar Alfvén mode frequencies on moderate and weak magnetic field strengths, one may study the problem by using mode analysis i.e. by decomposing the perturbations into spherical harmonics $Y_{\ell m}$. This type of study has already been done for Newtonian stars by Lee (2007). Actually, since we find that the polar Alfvén modes form a discrete spectrum, one can safely study the problem by using modal decomposition. Additionally, Lee (2007) observed that for oscillations confined in the crust there exists a critical value for the magnetic field strength. That is, the frequencies of the magnetic modes become smaller as the magnetic field becomes weaker and then, depending on the stellar model, they seem to disappear after a critical value. We have observed a similar disappearance of the modes, but we cannot give a conclusive answer since this might be a result of the numerical method that we used. Specifically, as the magnetic field becomes weaker the energy on the Alfvén modes becomes minimal and the Fourier transform is dominated by the fluid modes, thus it becomes difficult if not impossible to extract the frequencies of the Alfvén modes. Thus, it is risky to conclude with time evolutions that there is a clear cut-off frequency. Instead, in order to answer this, mainly academic, question we plan to study the problem by using modal analysis of the perturbation equations. Concluding, the observation by Lee (2007) remains to be proved for the polar Alfvén modes in the framework of general relativity.

Finally, we examine how the fluid modes depend on the magnetic field strength. For a non-rotating magnetars with cold EoS, there exists only the family of pressure modes ($f$ and $p_i$ modes) in addition to the polar Alfvén modes mentioned earlier. However, these fluid modes are hardly affected in any way by the presence of the ultrastrong magnetic field unless its strength is larger than about $10^{17}$ G. This weak dependence on the magnetic field can be understood by realizing that restoring force for these fluid modes is the pressure and the magnetic pressure is typically very weak compared to the pressure of the fluid. In Fig. 6, we plot the frequencies of the $f$ and $p_1$ modes with harmonic indices $\ell = 2, 3$ and $4$ as functions of magnetic field strength. The marks correspond to frequencies of the modes of the magnetized stars while the horizontal dashed lines are the frequencies for the non-magnetized ones. From this figure, we can see that the effect of magnetic field on...
the frequencies of fluid modes can be traced only when the magnetic field becomes stronger than a few times $10^{16}$ G. Still the deviations between the frequencies for non-magnetized stars and those for magnetars with $B = 4 \times 10^{16}$ G are very small such as 1.1 per cent for $2f$, 1.3 per cent for $3f$ and 0.3 per cent for $4f$ modes while 0.4 per cent for $2p_1$, 0.3 per cent for $3p_1$ and 0.7 per cent for $4p_1$ modes. Since the frequencies of the polar Alfvén modes depend on the magnetic field, it is possible for ultrastrong magnetic fields to reach those of the $f$ modes.

5 CONCLUSION

In this paper, we study the polar-type oscillations of strongly magnetized neutron stars. As a first step, we derive the perturbation equations for an arbitrary magnetic field although for the numerical studies we adopted a stellar model with a global dipole magnetic field. In this study, we ignored the effects due to the presence of the crust, since these types of modes are confined mainly in the core of the magnetar. By using the two-dimensional time evolutions of the perturbation equations, together with the appropriate initial data sets, we estimate the Alfvén oscillation modes as well as the fluid modes. We used two techniques to examine the form of the oscillation spectrum, i.e. whether is continuous as in the case of axial perturbations or discrete. Actually, by examining the FFT amplitude at various points inside the star and phase of the corresponding frequencies, we conclude that the Alfvén oscillations with polar parity are described by a spectrum consisting only of discrete modes. There is a physical reason why the polar oscillations form a discrete spectrum while the axial ones a continuum. For the axial oscillations, the restoring force is only the magnetic one and around the magnetic pole the direction of the magnetic force becomes complicated, see also the discussion in Sotani et al. (2008a), Colaiuda et al. (2009) and Cerda-Duran et al. (2009). On the other hand, for polar oscillations the restoring force is not only the magnetic one but also the pressure. Thus in the absence of magnetic field these modes are still present while the axial ones reduce to a zero-frequency spectrum. As we mentioned earlier, this behaviour reminds the spectral properties observed the oscillations of rotating stars. There, the axial modes that had as restoring force only the Coriolis one were showing a continuous spectrum (see e.g. Kojima 1998; Beyer & Kokkotas 1999) while the polar ones were admitting a discrete spectrum.

Finally, the frequencies of the fundamental polar Alfvén oscillations for typical magnetars are of the order of a few 100 Hz and their dependence on the magnetic field can be used to explain some of the observed high QPO frequencies in SGRs.

This is the first study of its kind for axisymmetric polar Alfvén oscillations; it remains open to understand the form of the spectrum for non-axisymmetric perturbations while more complicated forms of the magnetic field geometry should be assumed. For example, the inclusion of a toroidal component as in Sotani, Colaiuda & Kokkotas (2008b) will affect the form of the oscillation spectrum. Finally, the inclusion of the crust enriches the spectrum (Vavoulidis et al. 2008) and the various new features have to be taken into account in the attempts to understand the structure of magnetars via their QPOs. Actually, the introduction of crust, although it makes the problem more complicated, might erase partially or completely the degeneracy in axial-mode spectrum.

ACKNOWLEDGMENTS

We thank Erich Gaertig for valuable comments. This work was supported via the Transregio 7 ‘Gravitational Wave Astronomy’ financed by the Deutsche Forschungsgemeinschaft DFG (German Research Foundation).

REFERENCES

Abbott B. et al., 2007, Phys. Rev. D, 76, 062003
Beyer H. R., Kokkotas K. D., 1999, MNRAS, 308, 745
Cerda-Duran P., Stergioulas N., Font J. A., 2009, preprint (astro-ph/0902.1472)
Colaiuda A., Ferrari V., Gualtieri L., Pons J. A., 2008, MNRAS, 385, 2080
Colaiuda A., Beyer H. R., Kokkotas K. D., 2009, preprint (astro-ph/0902.1401)
Duncan R. C., Thompson C., 1992, ApJ, 392, L9
Gaertig E., Kokkotas K. D., 2008, Phys. Rev. D, 78, 064063

© 2009 The Authors. Journal compilation © 2009 RAS, MNRAS 395, 1163–1172
