QUADRATIC TRANSFORMATION AND MATRIX BIOORTHOGONAL POLYNOMIALS: AN LU FACTORIZATION APPROACH

KIRAN KUMAR BEHERA

Abstract. The manuscript presents the LU approach to matrix biorthogonal polynomials when all the even ordered entries in the Gram matrix are zero. This arises in case of a quadratic transformation which is briefly discussed. Further, the main diagonal of the Gram matrix is a zero diagonal and we present the theory that follows from this fact. Precisely, we discuss the Christoffel transformation and matrix representations of the kernel polynomials, usually called the ABC Theorem. Finally, we provide an illustration of our results assuming the Gram matrix has Hankel symmetry.

1. Introduction

In recent years, the Gaussian (or LU) decomposition technique has been used as the basis for an alternative approach for instance, to the theories of generalized orthogonal polynomials, multiple orthogonal polynomials, integrable systems and multivariate orthogonal polynomials [1, 2, 4, 8]. We specially refer to [5] and references therein for a systematic theory of LU factorization in case of matrix biorthogonal polynomials. This approach brings the Gram matrix with respect to a bilinear functional to the center of analysis and usually the starting point is the LU decomposition of the Gram matrix.

A bilinear form on the ring \( \Pi_n[z] \) of polynomials in the variable \( z \), where \( \Pi_n \) is the ring of \( n \times n \) matrices, is the map

\[
\langle \cdot, \cdot \rangle : \Pi_n[z] \times \Pi_n[z] \mapsto \Pi_n,
\]
satisfying the following properties

(i) \( \langle C_1 \mathcal{P}(z) + C_2 \mathcal{Q}(z), \mathcal{R}(\omega) \rangle = C_1 \langle \mathcal{P}(z), \mathcal{R}(\omega) \rangle + C_2 \langle \mathcal{Q}(z), \mathcal{R}(\omega) \rangle \),

(ii) \( \langle \mathcal{P}(z), C_1 \mathcal{Q}(\omega) + C_2 \mathcal{R}(\omega) \rangle = \langle \mathcal{P}(z), \mathcal{Q}(\omega) \rangle C_1^T + \langle \mathcal{P}(z), \mathcal{R}(\omega) \rangle C_2^T \),

for any \( C_1, C_2 \in \Pi_n \) and \( \mathcal{P}(z), \mathcal{Q}(z), \mathcal{R}(z) \in \Pi_n[z] \). Further, if

\[ \mathcal{P}(z) = \sum_{k=0}^{m} p_k z^k, \quad \mathcal{Q}(z) = \sum_{l=0}^{n} q_l z^l, \quad p_k, q_k \in \Pi_n, \]

the bilinear form acts as

\[ \langle \mathcal{P}(z), \mathcal{Q}(\omega) \rangle = \sum_{k=0}^{m} \sum_{l=0}^{n} p_k m_{kl} q_l, \quad m_{kl} = \langle z^k I, \omega^l I \rangle, \]

Key words and phrases. LDU decomposition; quadratic transformation; matrix biorthogonal polynomials; Christoffel transform; Hankel symmetry.

This research is supported by the Dr. D. S. Kothari postdoctoral fellowship scheme of University Grants Commission (UGC), India.
where $I$ is the identity matrix in $\Pi_n$. This gives rise to the Gram matrix $M$ where

$$M = [m_{ij}] = \begin{pmatrix} m_{00} & m_{01} & \cdots \\ m_{10} & m_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad i, j \geq 0. \tag{1.1}$$

We also note that if $\mathcal{X}(z) := (I, zI, z^2I, \cdots)^T$, then $M$ has the alternative description $M = \langle \mathcal{X}(z), \mathcal{X}(\omega) \rangle$.

The rudiments of this approach are as follows. The bilinear form $\langle \cdot, \cdot \rangle$ is said to be quasi-definite whenever the associated Gram matrix has all its leading principal minors different from zero. In this case, the Gram matrix has a unique LU decomposition, but in fact, is written in the form $M = L_1^{-1} D L_2^{-T}$, where $D$ is a diagonal matrix, and $L_1$, $L_2$ are lower triangular matrices with $I$ as the diagonal entries.

Two vectors $P(z) := L_1 \mathcal{X}(z)$ and $Q(\omega) := L_2 \mathcal{X}(\omega)$ are defined so that if

$$P(z) = \begin{pmatrix} p_0(z) & p_1(z) & \cdots \end{pmatrix}^T, \quad Q(\omega) = \begin{pmatrix} q_0(\omega) & q_1(\omega) & \cdots \end{pmatrix}^T, \tag{1.2}$$

then $p_k(z)$ and $q_l(\omega)$ are matrix polynomials whose coefficients are easily determined from $L_1$ and $L_2$ respectively. This serves as the foundation to discuss various properties related to biorthogonality, spectral transformations like the Christoffel and Geronimus transformations and so on. We note down three such properties, or rather representations for future reference.

Suppose the underlying bilinear form is quasi-definite. Then we have the following representation [5]

$$p_n(z) = \theta_* \begin{pmatrix} \mathcal{M}^{[n]} \\ \vdots \\ \mathcal{M}_n \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ z^{n-1} \\ z^n \end{pmatrix}, \quad n = 0, 1, \cdots, \tag{1.3}$$

where the superscript $[n]$ denotes a truncated matrix and $\theta_*$ denotes the fact that $p_n(z)$ is, in fact, the Schur complement of $\mathcal{M}^{[n]}$ in the above matrix. A similar expression also exists for $q_n(\omega)$ and we will explain this in greater detail in Section [2]. Further, given the biorthogonal matrix polynomial sequences $\{p_n(z)\}$ and $\{q_n(\omega)\}$, the $j^{th}$ Christoffel-Darboux kernel polynomial is defined as [5]

$$K^{[j]}(z, \omega) = \sum_{k=0}^{j} q_k^T(\omega) d_k^{-1} p_k(z), \quad D = \text{diag} (d_{00}, d_{11}, \cdots). \tag{1.4}$$

These matrix kernel polynomials also possess the following representation, the so called ABC Theorem (named after Aitken, Berg and Collar) [20]

$$K^{[j]}(z, \omega) = (\mathcal{X}^{[j]}(\omega))^T [\mathcal{M}^{[j]}]^{-1} \mathcal{X}^{[j]}(z), \tag{1.5}$$

which is a direct consequence of (1.4).

Henceforward, for ease of reference we will call $m_{ij}$ the even (odd) ordered Gram entry if $i + j$ is even (odd). Our primary concern in the present manuscript is to discuss
the above factorization technique when all the even ordered Gram entries are zero matrices. Note that in such a case, the main diagonal of $M$ is the zero diagonal and hence the underlying bilinear form is no longer quasi-definite. This forces the main diagonal of the matrix $D$ in the ensuing $LDU$ decomposition to consist of zero entries and hence it has to be substituted appropriately. This will lead to a revision of the definition (1.4) of the matrix kernel polynomials and consequently the ABC Theorem (1.5). Among other things, the quasi-determinant representations (1.3) will also be no longer valid since for $p_0(z), M^{[0]}$ is the zero matrix. The precise goal of the present manuscript is to address the above concerns.

The layout of the manuscript is as follows. In rest of the present section, we briefly discuss the theory where Gram matrices with even ordered Gram entries equal to zero arise. In Section 2, we obtain the $LDU$ factorization of such Gram matrices, biorthogonality relations and representations of matrix biorthogonal polynomials. Section 3 demonstrates the Christoffel transformation of such Gram matrices. The associated matrix kernel polynomials followed by the appropriate form of the ABC Theorem are presented. Section 4 provides an illustration of the results in the special case of Hankel symmetry.

1.1. Unwrapping of measure. The quadratic transformation $\lambda \mapsto \lambda^2$ is perhaps the simplest case in the general theory of polynomial mappings. One of the direction in which this has been studied is the following [17]. Given $\{p_n(\lambda)\}$, a monic orthogonal polynomial sequence (MOPS), the problem is to find another MOPS $\{s_n(\lambda)\}$ such that $s_{2k}(\lambda) = p_k(t(\lambda)), k \geq 0$, where $t(\lambda)$ is a monic polynomial of degree 2. The sequence $\{s_n(\lambda)\}$ is then completed by defining $s_{2k+1}(\lambda) = (\lambda - a)p_k(t(\lambda))$. For some other approaches, we refer to [9, 10, 14, 15].

The connection of the transformation $\lambda \mapsto \lambda^2$ with moment theory in the context of present manuscript arises in what is known as unwrapping of measures. To begin with, given a sequence of $p \times p$ Hermitian matrices, $S_0, S_1, \cdots, S_{2n}$, the truncated Hamburger (TH) moment problem is to find a Hermitian matrix measure $\mu(u), -\infty < u < \infty$, such that

$$S_k = \int_{-\infty}^{\infty} u^k d\mu(u), \quad k = 0, 1, \cdots, 2n. \quad (1.6)$$

The Hamburger-Nevanlinna Theorem solves (1.6) as an interpolation problem in the class $\mathcal{N}_p$ of Nevanlinna functions and is presented below.

**Theorem A** ([3][11][16]). If $\mu(u), (-\infty < u < \infty)$ is a solution to the TH problem (1.6), then there exists $F(\lambda) \in \mathcal{N}_p$ such that

$$F(\lambda) = \int_{-\infty}^{\infty} \frac{1}{u - \lambda} d\mu(\lambda) \quad (1.7)$$
for which

\[
\lim_{\lambda \to \infty} \lambda^{2n+1} \left[ F(\lambda) + \frac{S_0}{\lambda} + \frac{S_1}{\lambda^2} + \cdots + \frac{S_{2n-1}}{\lambda^{2n}} \right] = -S_{2n}
\]  

(1.8)

uniformly in the sector \( \pi \varepsilon := \varepsilon < \arg \lambda < \pi - \varepsilon \), for some \( \varepsilon \in (0, \pi/2) \). Conversely, if (1.8) holds, at least for \( \lambda = iy \ (y \to +\infty) \), for some \( F(\lambda) \in N_p \), then \( F(\lambda) \) has the representation (1.7), where \( \mu(u) \) has \( 2n+1 \) moments \( S_0, S_1, \cdots, S_{2n} \).

If the relation (1.8) holds for all \( n = 0, 1, \cdots \), we have the asymptotic expansion

\[
F(\lambda) \sim -\frac{S_0}{\lambda} - \frac{S_1}{\lambda^2} - \cdots - \frac{S_{2n-1}}{\lambda^{2n}} - \cdots, \quad \lambda \in \pi \varepsilon.
\]

(1.9)

Hence, if we let \( \mathcal{H}(\lambda) = F(\lambda^2) \), then from (1.9), we have

\[
\mathcal{H}(\lambda) \sim -\frac{h_0}{\lambda} - \frac{h_1}{\lambda^2} - \cdots - \frac{h_{2n-1}}{\lambda^{2n}} - \cdots, \quad \lambda \in \pi \varepsilon,
\]

where \( h_{2k}=0, \ h_{2k+1} = S_k, \ k \geq 0 \), giving rise to the following moment matrix

\[
\mathcal{M} = \begin{pmatrix}
0 & m_{0,1} & 0 & m_{0,3} & \cdots \\
m_{1,0} & 0 & m_{1,2} & 0 & \cdots \\
0 & m_{2,1} & 0 & m_{2,3} & \cdots \\
m_{3,0} & 0 & m_{3,2} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}, \quad m_{i,j} = h_{i+j}, \quad i, j = 0, 1, \ldots.
\]

(1.10)

The motivation to introduce quadratic transformation in (1.9) comes from the fact that if we consider the problem (1.6) over the interval \([0, \infty)\), we have the truncated Stieltjes (TS) moment problem which is solved by \( \varphi(\lambda) \) in the Stieltjes class \( S_p \). A result like Theorem A exists [18] for the TS problem in which \( \varphi(\lambda) \) satisfies (1.8) and (1.6), but over the interval \([0, \infty)\). Further, it is well known that [12, 19, 21]

\[
\varphi(\lambda) = \int_0^\infty \frac{1}{u-\lambda} d\mu(u) \Longrightarrow \lambda \varphi(\lambda^2) = \int_{-\infty}^\infty \frac{\text{sign} u}{2} \frac{1}{u-\lambda} d\mu(u^2),
\]

thus reducing a Stieltjes moment problem to a Hamburger one. The quadratic transformation \( \lambda \mapsto \lambda^2 \) is the first part of this phenomenon of unwrapping of measures and provides the background for the problem under consideration.

2. The generic \( LDU \) decomposition

In this section we find the \( LDU \) decomposition of the Gram matrix \( \mathcal{M} \) (without assuming Hankel symmetry) given by (1.10), where \( L, D, U \) have the forms

\[
L = \begin{pmatrix}
I_2 & 0 & 0 & \cdots \\
L_{10} & I_2 & 0 & \cdots \\
L_{20} & L_{21} & I_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots 
\end{pmatrix}, \quad D = \begin{pmatrix}
I_2 & U_{01} & U_{02} & \cdots \\
0 & I_2 & U_{12} & \cdots \\
0 & 0 & I_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots 
\end{pmatrix}, \quad U = \begin{pmatrix}
I_2 & U_{01} & U_{02} & \cdots \\
0 & I_2 & U_{12} & \cdots \\
0 & 0 & I_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots 
\end{pmatrix}
\]

\[
M = \begin{pmatrix}
0 & m_{0,1} & 0 & m_{0,3} & \cdots \\
m_{1,0} & 0 & m_{1,2} & 0 & \cdots \\
0 & m_{2,1} & 0 & m_{2,3} & \cdots \\
m_{3,0} & 0 & m_{3,2} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}, \quad M = \begin{pmatrix}
\lambda & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots 
\end{pmatrix}
\]
In the second step, we determine the facts that and using the fact that \( l \).

\[
\mathcal{L}_{ij} = \begin{pmatrix} l_{2i,2j} & 0 \\ 0 & l_{2i+1,2j+1} \end{pmatrix}, \quad \mathcal{U}_{ij} = \begin{pmatrix} u_{2i,2j} & 0 \\ 0 & u_{2i+1,2j+1} \end{pmatrix},
\]

\[
\mathcal{D}_{ii} = \begin{pmatrix} 0 & d_{2i,2i+1} \\ d_{2i+1,2i} & 0 \end{pmatrix}, \quad \mathcal{I}_2 = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{I} \end{pmatrix}.
\]

(2.1)

with \( l_{i,j}, u_{i,j}, d_{i,i} \in \Pi_n \) and

\[
m_{i,2k} = h_{i,0}u_{0,2k} + h_{i,2}u_{2,2k} + \cdots + h_{i,2k}, \quad i = 1,3,\ldots, \\
m_{i,2k+1} = h_{i,1}u_{1,2k+1} + h_{i,3}u_{3,2k+1} + \cdots + h_{i,2k+1}, \quad i = 0,2,\ldots,
\]

for \( k = 0,1,2,\ldots \). Using an algorithmic approach, we determine \( h_{ij} \) and \( u_{ij} \) from (2.2).

In the second step, we determine \( l_{ij} \) and \( d_{ij} \) from \( h_{ij} \).

To proceed, let us work out the special cases \( m_{i,2} \) and \( m_{i,4} \). We will repeatedly use the facts that \( h_{ij} = 0 \) if \( j = i+1 \) for \( i = 1,3,5,\ldots \) or \( i+j \) is even or \( j \geq i+2 \). Since \( m_{i,2} = h_{i,0}u_{0,2} + h_{i,2} \), we have \( m_{1,2} = h_{1,0}u_{0,2} \) for \( i = 1 \) which implies \( u_{0,2} = h_{1,0}^{-1}m_{1,2} \).

Then,

\[
h_{i,2} = m_{i,2} - h_{i,0}h_{1,0}^{-1}m_{1,2} = \theta_*(\begin{pmatrix} h_{1,0} & m_{1,2} \\ h_{i,0} & m_{i,2} \end{pmatrix}) = \Theta_{i,2}^{(2)}, \quad \text{say.}
\]

Next, from \( m_{i,4} = h_{i,0}u_{0,4} + h_{i,2}u_{2,4} + h_{i,4} \), we substitute \( i = 1 \) to obtain (since \( h_{1,2} = h_{1,4} = 0 \)) \( u_{0,4} = h_{1,0}^{-1}m_{1,4} \). Observe that \( h_{i,2} \) has been obtained in the previous step for \( m_{i,2} \). So we put \( i = 3 \) to obtain

\[
h_{3,2}u_{2,4} = m_{3,4} - h_{3,0}h_{1,0}^{-1}m_{1,4} = \theta_*(\begin{pmatrix} h_{1,0} & m_{1,4} \\ h_{3,0} & m_{3,4} \end{pmatrix}) = \Theta_{3,4}^{(2)}, \quad \text{say.}
\]

Then, \( u_{2,4} = h_{3,2}^{-1}\Theta_{3,4}^{(2)} \). Finally,

\[
h_{i,4} = m_{i,4} - h_{i,0}u_{0,4} - h_{i,2}u_{2,4} = \theta_*(\begin{pmatrix} h_{1,0} & m_{1,4} \\ h_{i,0} & m_{i,4} \end{pmatrix}) - h_{i,2}h_{3,2}^{-1}\Theta_{3,4}^{(2)}.
\]

Then, proceeding as in the case of \( m_{1,2} \) we have

\[
h_{i,4} = \theta_*\left(\begin{pmatrix} h_{3,2} & \Theta_{3,4}^{(2)} \\ h_{i,2} & \Theta_{i,4}^{(2)} \end{pmatrix}\right) = \Theta_{i,4}^{(4)}, \quad \text{say, where} \Theta_{i,4}^{(2)} := \theta_*\left(\begin{pmatrix} h_{1,0} & m_{1,4} \\ h_{i,0} & m_{i,4} \end{pmatrix}\right).
\]
Now, it is matter of induction to prove the following.

**Theorem 2.1.** Define for \( i = 1, 3, 5, \ldots \)

\[
\theta_{i,2k}^{(2j)} = \theta_{i,2k} \begin{pmatrix} h_{2j-1,2j-2} & \theta_{2j-1,2k}^{(2j-2)} \\
 h_{i,2j-2} & \theta_{i,2k}^{(2j-2)} \end{pmatrix}, \quad j = 1, 2, \ldots, \quad k = 0, 1, \ldots ,
\]

where \( \theta_{i,2k}^{(0)} = m_{i,2k} \). Then

\[
h_{i,2l} = \theta_{i,2l}^{(2l)}, \quad u_{2l,2k} = h_{2l+1,2l}^{-1} \theta_{2l+1,2k}^{(2l)} \quad l = 0, 1, \ldots .
\]

**Proof.** Observe that we have actually proved the theorem for the cases \( j = 0, 1 \) and \( k, l = 0, 1, 2 \) in the preceding discussion. Hence, we assume that (2.3) holds for \( l = 0, 1, \ldots, j-1 \) and \( k = j \) and prove that the same holds for \( l = j \). For a simpler form, we note

\[
\theta_{i,2k}^{(2j)} = \theta_{i,2k}^{(2j-2)} - h_{i,2j-2} h_{2j-1,2j-2} \theta_{2j-1,2k}^{(2j-2)} - \theta_{i,2k}^{(0)} = m_{i,2k}.
\]

We first obtain \( u_{2j,2k} \) from the relation for \( m_{i,2k} \) in (2.2) by substituting \( i = 2j+1, j < k \). Recall that \( h_{2j+1,2j+2} = 0 \) and \( h_{2j+1,2j+1} = 0 \) for \( * - (2j+1) > 1 \). Then,

\[
h_{2j+1,2j} u_{2j,2k} = m_{2j+1,2k} - h_{2j+1,2j} h_{2j+1,2j+2} u_{2j,2k} - \cdots - h_{2j+1,2j+2} u_{2j,2k},
\]

\[
= \theta_{2j+1,2k}^{(2j)} - h_{2j+1,2j} h_{3,2} \theta_{2j+1,2k}^{(2j)} - \cdots - h_{2j+1,2j} u_{2j,2k},
\]

\[
= \cdots
\]

\[
= \theta_{2j+1,2k}^{(2j-2)} - h_{2j+1,2j} h_{2j-1,2j-2} \theta_{2j-1,2k}^{(2j-2)} = \theta_{2j+1,2k}^{(2j)}
\]

This gives \( u_{2j,2k} = h_{2j+1,2j} \theta_{2j+1,2k}^{(2j)} \), thereby proving the expression for \( u_{2l,2k} \) for \( l = j \). With \( u_{2j,2k} \) determined, we put \( k = j \) in the relation for \( m_{i,2k} \) from (2.2) to obtain similarly

\[
h_{i,2j} = \theta_{i,2j}^{(0)} - h_{i,2j} \theta_{1,2j}^{(0)} - h_{i,2} u_{2,2j} - h_{i,4} u_{4,2j} - \cdots - h_{i,2j} u_{2j,2j},
\]

\[
= \cdots
\]

\[
= \theta_{i,2j}^{(2j-2)} - h_{i,2j-2} h_{2j-1,2j-2} \theta_{2j-1,2j}^{(2j-2)} = \theta_{i,2j}^{(2j)}
\]

This proves the relation (2.3) for \( l = j \), thus completing the proof. \( \square \)

### 2.1. Biorthogonality relations.

Having decomposed the moment matrix \((1.10)\) which henceforth, we write in the form \( \mathcal{M} = \mathcal{L}_1^{-1} \mathcal{T} \mathcal{L}_2^{-T} \), let us define two block vectors with polynomial entries as

\[
P(z) = \mathcal{L}_1 \mathcal{X}(z), \quad Q(\omega) = \mathcal{L}_2 \mathcal{X}(\omega),
\]

(2.5)
where \( \mathcal{P}(z) \) and \( \mathcal{Q}(\omega) \) are as in (1.2). Explicitly for \( \mathcal{P}(z) \) we have

\[
\begin{pmatrix}
I & 0 & 0 & 0 & \cdots \\
0 & I & 0 & 0 & \cdots \\
L^{(1)}_{20} & 0 & I & 0 & \cdots \\
0 & L^{(1)}_{31} & 0 & I & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
I \\
zI \\
z^2I \\
z^3I \\
\vdots
\end{pmatrix}
= \begin{pmatrix}
L^{(1)}_{00} \\
L^{(1)}_{11}z \\
L^{(1)}_{20} + L^{(1)}_{22}z^2 \\
L^{(1)}_{31}z + L^{(1)}_{33}z^3 \\
\vdots
\end{pmatrix},
\]

which leads to the forms

\[
p_{2j}(z) = L^{(1)}_{2j,0} + L^{(1)}_{2j,2}z^2 + \cdots + z^{2j},
\]

\[
p_{2j+1}(z) = L^{(1)}_{2j+1,1}z + L^{(1)}_{2j+1,3}z^3 + \cdots + z^{2j+1}, \quad j = 0, 1, \ldots
\]

A similar expression for \( \mathcal{Q}(\omega) \) is

\[
q_{2j}(\omega) = L^{(2)}_{2j,0} + L^{(2)}_{2j,2}\omega^2 + \cdots + \omega^{2j},
\]

\[
q_{2j+1}(\omega) = L^{(2)}_{2j+1,1}\omega + L^{(2)}_{2j+1,3}\omega^3 + \cdots + \omega^{2j+1}, \quad j = 0, 1, \ldots
\]

We note that while \( p_{2j}(z) \) and \( q_{2j}(\omega) \) contain only even powers of \( z \) and \( \omega \) respectively, \( p_{2j+1}(z) \) and \( q_{2j+1}(\omega) \) have only the odd powers. This difference will be reflected throughout the manuscript, where we derive results separately for the indices \( 2j \) and \( 2j + 1 \). A fundamental reason for this, as will be observed, is that the matrix \( D \) is no longer a diagonal but a block diagonal matrix of \( 2 \times 2 \) blocks with matrix entries.

We begin with the following.

**Proposition 2.2.** The matrix polynomial sequences \( \{p_n(z)\}_{n=0}^\infty \) and \( \{q_n(\omega)\}_{n=0}^\infty \) satisfy the relations

\[
\langle p_{2j}, q_k \rangle = d_{2j,2j+1}d_{2j+1,k},
\]

\[
\langle p_{2j+1}, q_k \rangle = d_{2j+1,2j}d_{2j,k}, \quad j, k = 0, 1, 2, \ldots \tag{2.6}
\]

**Proof.** Using the properties of the bilinear form \( \langle \cdot, \cdot \rangle \) and the definition (2.5) we have

\[
\langle p_i, q_j \rangle = \langle \mathcal{P}_i(z), \mathcal{Q}_j(\omega) \rangle = \langle (L_1X(z))_i, (L_2X(\omega))_j \rangle
= \left( L_1\langle X(z), X(\omega) \rangle L_2^T \right)_{ij} = D_{ij}
\]

since \( M = L_1^{-1}DL_2^{-T} \). The relations (2.6) follow from \( D \) being a block diagonal matrix with the constituent blocks given by (2.1). \( \square \)

Similar relations hold for \( q_n(\omega) \) too

\[
\langle p_k(z), q_{2j}(\omega) \rangle = d_{2j+1,2j}d_{2j+1,k},
\]

\[
\langle p_k(z), q_{2j+1}(\omega) \rangle = d_{2j,2j+1}d_{2j,k}, \quad j, k = 0, 1, 2, \ldots \tag{2.7}
\]

Here, we emphasise the relations (2.6) and (2.7) show that the sequence of matrix polynomials \( \{p_i(z)\} \) and \( \{q_i(z)\} \) are not biorthogonal in the usual sense by which we
Another way to view this fact is to note that the above bilinear form will always involve only the even powers of $z$ leading to even ordered Gram entries which are zero by definition.

2.2. Quasideterminant representation. The theory of quasideterminants originated in attempts to define a determinant for matrices with entries over a non-commutative ring [13]. A quasideterminant is not actually an analogue of the commutative determinant, but rather of the ratio of the determinant of an $n \times n$ matrix to the determinant of an $(n - 1) \times (n - 1)$ sub-matrix. This provides the crucial link when we want to find the matrix analogues of the determinant representations of monic orthogonal polynomials with scalar coefficients.

Let $\mathcal{V}$ be a $2 \times 2$ matrix with entries $v_{ij}, i, j = 1, 2$, where $v_{ij}$ come from a non-commutative ring. The simplest example of quasideterminants of $\mathcal{V}$ are the following

$$|\mathcal{V}|_{11} = v_{11} - v_{12} \cdot v_{21}^{-1} \cdot v_{21}, \quad |\mathcal{V}|_{12} = v_{12} - v_{11} \cdot v_{21}^{-1} \cdot v_{22},$$

$$|\mathcal{V}|_{21} = v_{21} - v_{22} \cdot v_{12}^{-1} \cdot v_{11}, \quad |\mathcal{V}|_{22} = v_{22} - v_{21} \cdot v_{11}^{-1} \cdot v_{12},$$

which are also known as the Schur complements of the respective entries. We will use the form $|\mathcal{V}|_{22}$ to find the quasideterminant representations for $p_j(z)$ and $q_j(z)$ and use the notation $\theta_\ast$ to denote this fact, that is

$$\theta_\ast \mathcal{V} = \theta_\ast \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = v_{22} - v_{21} \cdot v_{11}^{-1} \cdot v_{12}.$$

The first step in obtaining the representations are the orthogonality relations

$$\langle p_{2j}(z), z^k \rangle = \begin{cases} 0, & k = 0, 1, \cdots, 2j; \\ d_{2j,2j+1}, & k = 2j + 1. \end{cases} \quad (2.8)$$

$$\langle p_{2j+1}(z), z^k \rangle = \begin{cases} 0, & k = 0, 1, \cdots, 2j - 1; \\ d_{2j-1,2j}, & k = 2j. \end{cases} \quad (2.9)$$

that follow easily from (2.6). We derive the quasideterminant representation for $p_{2j}(z)$ only. The others follow similarly.

**Theorem 2.3.** Consider the Gram matrix

$$\mathcal{M}_{\ast \ast}^{[2j]} = \begin{pmatrix} m_{01} & m_{03} & \cdots & m_{0,2j-1} \\ m_{21} & m_{23} & \cdots & m_{2,2j-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{2j-2,1} & m_{2j-2,3} & \cdots & m_{2j-2,2j-1} \end{pmatrix}.$$
Then, \( p_{2j}(z) \), \( j = 0, 1, \cdots \), has the quasideterminant representation

\[
p_{2j}(z) = \theta_{z} \begin{pmatrix}
    I \\
    z^2 I \\
    \vdots \\
    z^{2j-2} I \\
    z^{2j} I
\end{pmatrix},
\]

so that the orthogonality relations (2.8) give

\[
\mathcal{L}_{2j,0}^{(1)}(I, z^k) + \mathcal{L}_{2j,2}^{(1)}(z^2, z^k) + \mathcal{L}_{2j,4}^{(1)}(z^4, z^k) + \cdots + \langle z^j, z^k \rangle = 0,
\]

for \( k = 1, 3, \cdots, 2j - 1 \), which leads to the system of equations represented as

\[
\left( \mathcal{L}_{2j,0}^{(1)} \quad \mathcal{L}_{2j,2}^{(1)} \quad \cdots \quad \mathcal{L}_{2j,2j-2}^{(1)} \right) \mathcal{M}_{oe}^{[2j]} = - \left( m_{2j,1} \quad m_{2j,3} \quad \cdots \quad m_{2j,2j-1} \right).
\]

Since

\[
p_{2j}(z) = z^{2j} + \left( \mathcal{L}_{2j,0}^{(1)} \quad \mathcal{L}_{2j,2}^{(1)} \quad \cdots \quad \mathcal{L}_{2j,2j-2}^{(1)} \right) \begin{pmatrix}
    I \\
    : \\
    z^{2j-2} I
\end{pmatrix},
\]

we have the quasideterminant representation (2.10).

Consider another Gram matrix

\[
\mathcal{M}_{oe}^{[2j]} = \begin{pmatrix}
    m_{10} & m_{12} & \cdots & m_{1,2j-2} \\
    m_{30} & m_{32} & \cdots & m_{3,2j-2} \\
    \vdots & \vdots & \ddots & \vdots \\
    m_{2j-1,0} & m_{2j-1,2} & \cdots & m_{2j-1,2j-2}
\end{pmatrix}.
\]

Then, we have the following quasideterminant representations for \( j = 0, 1, \cdots \),

\[
p_{2j+1}(z) = \theta_{z} \begin{pmatrix}
    m_{10} & m_{12} & \cdots & m_{1,2j-2} \\
    m_{30} & m_{32} & \cdots & m_{3,2j-2} \\
    \vdots & \vdots & \ddots & \vdots \\
    m_{2j+1,0} & m_{2j+1,2} & \cdots & m_{2j+1,2j-2}
\end{pmatrix},
\]

Similarly, from (2.7) we have for \( j = 0, 1, \cdots \),

\[
q_{2j}(z) = \theta_{z} \begin{pmatrix}
    m_{1,2j} \\
    m_{3,2j} \\
    \vdots \\
    m_{2j-1,2j}
\end{pmatrix}.
\]

Proof. Since

\[
p_{2j}(z) = \mathcal{L}_{2j,0}^{(1)} + \mathcal{L}_{2j,2}^{(1)} z^2 + \mathcal{L}_{2j,4}^{(1)} z^4 + \cdots + z^{2j},
\]

we have the quasideterminant representation (2.10). \( \Box \)
In the remaining portion of the manuscript, we will be concerned more with the non-zero entries of all the matrices involved rather than the actual expressions of these entries. Our focus will be on structural relations between the various entities that we derive.

3. The Christoffel transformation

The underlying theme while discussing Christoffel transformation is that it should reflect the fact that \( p_{2j}(z) \) and \( p_{2j+1}(z) \) are different as polynomials. This is achieved by truncating after \( 2k - 1 \) rows and columns so that we obtain matrices of order \( 2k \times 2k \), thereby preserving the \( 2 \times 2 \) block structure.

If we observe, the analysis so far has its origins in the single most important fact that the main diagonal (and hence every alternate diagonal due to the symmetry associated with the quadratic transformation \( z \mapsto z^2 \)) of the Gram matrix is the zero diagonal. Hence, a desirable starting point could also have been the Gram matrix \( \hat{M} \) defined as

\[
\hat{M} = \Lambda M = \begin{pmatrix}
m_{10} & 0 & m_{12} & 0 & \cdots \\
0 & m_{21} & 0 & m_{23} & \cdots \\
m_{30} & 0 & m_{32} & 0 & \cdots \\
0 & m_{41} & 0 & m_{43} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad \Lambda = \begin{pmatrix}
0 & I & 0 & \cdots \\
0 & 0 & I & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

and then proceeding with the \( \hat{L}_1^{-1} \hat{D} \hat{L}_2^{-T} \) factorization. It turns out that \( \hat{L}_1 \) and \( \hat{L}_2 \) have the same structure as \( L_1 \) and \( L_2 \) while \( \hat{D} \) is a diagonal matrix. In fact given any Gram matrix \( \bar{G} \), the transformation \( \bar{G} \mapsto \hat{G} := \Lambda \bar{G} \) is known as the Christoffel transformation of the Gram matrix \( \bar{G} \) and is well studied [67]. We derive some of the existing results for \( z [5] \) in the present case of quadratic transformation \( z \mapsto z^2 \).

Since \( \hat{M} = \Lambda M \), we have

\[
\hat{L}_1^{-1} \hat{D} \hat{L}_2^{-T} = \Lambda L_1^{-1} D L_2^{-T} \implies \hat{D} \hat{L}_2^{-T} = \sigma D L_2^{-T}; \quad \sigma := \hat{L}_1 \Lambda L_1^{-1}.
\]

We call \( \sigma \) the connector which is also given by the expression \( \sigma = \hat{D} \hat{L}_2^{-T} L_2^{-T} D^{-1} \). A key role played by \( \sigma \), which somewhat explains the use of the term connector, is brought out in the following result.

**Proposition 3.1.** Given the matrix polynomial sequence \( \{p_n(z)\}_{n=0}^{\infty} \) associated with the Gram matrix \( M \), let \( \{\hat{p}_n(z)\}_{n=0}^{\infty} \) be associated with the Christoffel transformation \( \hat{M} \) of \( M \).
Then, we have
\[
\hat{p}_{2j} = \frac{1}{z}p_{2j+1}(z),
\]
\[
\hat{p}_{2j+1} = \frac{1}{z}(p_{2j+2}(z) - p_{2j+2}(0)p_{2j}(0)^{-1}p_{2j}(z)), \quad j = 0, 1, \ldots.
\]  
(3.1)

**Proof.** Since \(P(z) = L_1X(z)\), we have
\[
\sigma P(z) = \hat{D}\hat{L}_2^T\hat{L}_2^T\hat{D}^{-1}L_1X(z) = \hat{D}\hat{L}_2^T\hat{M}^{-1}\hat{L}_1X(z) = z\hat{P}(z).
\]
Further, comparing both the expressions for \(\sigma\), we conclude that \(\sigma\) has to be necessarily of the form
\[
\sigma = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & \cdots \\
\sigma_{10} & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & \sigma_{32} & 0 & 1 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{bmatrix}, \quad \sigma_{2j+1,2j} = \hat{d}_{2j+1,2j+1}d_{2j,2j+1}^{-1}, \quad j = 0, 1, \ldots
\]

Then from \(\sigma P(z) = z\hat{P}(z)\), we obtain
\[
p_{2j+2}(z) + \sigma_{2j+1,2j}p_{2j}(z) = z\hat{p}_{2j+1}(z), \quad j = 0, 1, \ldots.
\]
The value of \(\sigma_{2j+1,2j}\) is found by substituting \(z = 0\) in the above relation leading to (3.1).

We obtain in similar fashion
\[
Q(\omega) = L_2X(\omega) \implies Q^T(\omega)D^{-1} = \hat{Q}^T(\omega)\hat{D}^{-1}\sigma
\]  
(3.2)
leading to structural relations between \(q_k(z)\) and \(\hat{q}_l(z)\). However, because of the appearance of the expressions \(Q^T(\omega)\) and \(\hat{Q}^T(\omega), (3.2)\) serves as the motivation to define the kernel polynomials related to the Gram matrices \(M\) and \(\hat{M}\).

3.1. **Christoffel-Darboux kernels.** Let us first work out a special case that will serve as motivation for the definitions. Consider the following
\[
\begin{bmatrix}
\mathcal{L}^{(1)}_{00} & 0 & 0 & 0 & 0 \\
0 & \mathcal{L}^{(1)}_{11} & 0 & 0 & 0 \\
\mathcal{L}^{(1)}_{20} & 0 & \mathcal{L}^{(1)}_{22} & 0 & 0 \\
0 & \mathcal{L}^{(1)}_{31} & 0 & \mathcal{L}^{(1)}_{33} & 0 
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 
\end{bmatrix}
\begin{bmatrix}
I \\
zI \\
z^2I \\
z^3I 
\end{bmatrix}
= \begin{bmatrix}
p_0(z) \\
p_2(z) \\
0 \\
0
\end{bmatrix},
\]
which is to pick only \(p_{2j}(z)\). Denoting the right hand most vector as \(P_c^{[4]}(z)\) we define the matrix kernel polynomial \(\mathcal{K}^{[2]}(z, \omega)\) as
\[
\mathcal{K}^{[2]}(z, \omega) = [Q^{[4]}(\omega)]^T[D^{[4]}]^{-1}P_c^{[4]}(z) = \sum_{j=0}^{1} q_{2j+1}^{T}(\omega)d_{2j,2j+1}^{-1}p_{2j}(z).
\]
Hence, we give the following
Definition 3.2. Given the biorthogonal polynomial sequences \{\mathcal{P}_n(z)\}_{n=0}^{\infty} and \{\mathcal{Q}_n(z)\}_{n=0}^{\infty} that arise from a Gram matrix with even ordered moments as zero, the associated kernel polynomials are defined as

\[
\mathcal{K}^{[2n]}(z, \omega) = \sum_{j=0}^{n} q^T_{2j+1}(\omega) d^{-1}_{2j,2j+1} p_{2j}(z),
\]

\[
\mathcal{K}^{[2n+1]}(z, \omega) = \sum_{j=0}^{n} q^T_{2j}(\omega) d^{-1}_{2j+1,2j} p_{2j+1}(z), \quad n = 0, 1, \ldots.
\] (3.3)

Next the left hand side of the equality \(Q^T(\omega) D^{-1} = \mathcal{K}^{[2]}(\omega) \mathcal{K}^{[2]}(\omega)\) as obtained in (3.2) gives

\[
\begin{pmatrix}
q^T_0(\omega) & q^T_1(\omega) & q^T_2(\omega) & q^T_3(\omega)
\end{pmatrix}
\begin{pmatrix}
0 & d^{-1}_{10} & 0 & 0 \\
d^{-1}_{01} & 0 & 0 & 0 \\
0 & 0 & 0 & d^{-1}_{32} \\
0 & 0 & d^{-1}_{23} & 0
\end{pmatrix}
\begin{pmatrix}
p_0(z) \\
p_1(z) \\
p_2(z) \\
p_3(z)
\end{pmatrix}
\]

which is \(\mathcal{K}^{[2]}(z, \omega)\). Similarly, the right hand side of the equality gives

\[
\begin{pmatrix}
q^T_0(\omega) & q^T_1(\omega) & q^T_2(\omega) & q^T_3(\omega)
\end{pmatrix}
\begin{pmatrix}
\hat{d}^{-1}_{00} & 0 & 0 & 0 \\
0 & \hat{d}^{-1}_{11} & 0 & 0 \\
0 & 0 & \hat{d}^{-1}_{22} & 0 \\
0 & 0 & 0 & \hat{d}^{-1}_{33}
\end{pmatrix}
\begin{pmatrix}
0 & I & 0 & 0 \\
\sigma_{10} & 0 & I & 0 \\
0 & 0 & 0 & I \\
0 & \sigma_{32} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
p_0(z) \\
p_1(z) \\
p_2(z) \\
p_3(z)
\end{pmatrix}
\]

which gives

\[
z \sum_{j=0}^{1} q^T_{2j+1}(\omega) \hat{d}^{-1}_{2j+1,2j+1} p_{2j+1}(\omega) - q^T_3(\omega) \hat{d}^{-1}_{33} p_3(z).
\]

Hence we give the following

Definition 3.3. Given the Gram matrix \(\mathcal{M}\), let \(\hat{\mathcal{P}}_n(z)\) and \(\hat{\mathcal{Q}}_n(z)\) be the biorthogonal polynomial sequences associated with the Christoffel transformation \(\hat{\mathcal{M}}\) of \(\mathcal{M}\). Then, the associated kernel polynomials are defined as

\[
\mathcal{K}^{[2n]}(z, \omega) = \sum_{j=0}^{n} \hat{q}^T_{2j}(\omega) \hat{d}^{-1}_{2j,2j} \hat{p}_{2j}(z),
\]

\[
\mathcal{K}^{[2n+1]}(z, \omega) = \sum_{j=0}^{n} \hat{q}^T_{2j+1}(\omega) \hat{d}^{-1}_{2j+1,2j+1} \hat{p}_{2j+1}(z), \quad n = 0, 1, \ldots.
\] (3.4)

The preceding discussion immediately gives
Proposition 3.4. The kernel polynomials $K^{[j]}(z, \omega)$ and $\hat{K}^{[j]}(z, \omega)$ associated, respectively, with $M$ and $\hat{M}$ are related as

\[
K^{[2n]}(z, \omega) = z\hat{K}^{[2n+1]}(z, \omega) - \hat{q}_{2n+1}(\omega)\hat{\theta}_{2n+1,2n+1}p_{2n+2}(z), \\
K^{[2n+1]}(z, \omega) = z\hat{K}^{[2n]}(z, \omega), \quad n = 0, 1, \ldots.
\]

(3.5)

Proof. The first relation follows from extending the particular case above to matrices truncated after $(2n+1)$ rows and columns. The second relation follows on the same lines as the first one except for the fact that we replace $(p_0(z), 0, p_2(z), 0, \ldots, p_{2n}(z), 0)^T$ with the vector $(0, p_1(z), 0, p_3(z), \ldots, 0, p_{2n+1}(z))^T$. □

Next, we derive the ABC Theorems for both $M$ and $\hat{M}$. We will use the following three matrices

\[
\Theta^{[2n]} = \begin{pmatrix}
0 & 0 & \cdots & 0 & \mathcal{I} \\
\mathcal{I} & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & \mathcal{I} & 0 \\
\end{pmatrix}, \quad (\Theta^{[2n]})^{-1} = (\Theta^{[2n]})^T,
\]

\[
\Pi_e = \begin{pmatrix}
\vec{e}_0 & 0 & \vec{e}_2 & 0 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
\vec{e}_1 & 0 & \vec{e}_3 & 0 & \cdots \\
\end{pmatrix} \quad \text{and} \quad \Pi_o = \begin{pmatrix}
0 & \vec{e}_1 & 0 & \vec{e}_3 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
\end{pmatrix},
\]

where $\vec{e}_i = (0, \ldots, i, \ldots, 0), \quad i = 0, 1, \ldots$ are the canonical vectors and $0$ is the zero vector. The arrows indicate that the $(\vec{e}_i)$’s are written as columns in $\Pi_e$ and $\Pi_o$.

Theorem 3.5 (ABC Theorem). The matrix kernel polynomials associated with the Gram matrix $M$ have the representation

\[
k^{[2n]}(z, \omega) = (\mathcal{X}^{[2n]}(\omega))^T \Pi_o^{[2n]} (\mathcal{M}_e^{[2n]})^{-1} \Pi_e^{[2n]} \mathcal{X}^{[2n]}(z), \\
k^{[2n+1]}(z, \omega) = (\mathcal{X}^{[2n]}(\omega))^T \Pi_e^{[2n]} (\mathcal{M}_o^{[2n]})^{-1} \Pi_o^{[2n]} \mathcal{X}^{[2n]}(z),
\]

where $\mathcal{M}_e^{[2n]} = (\mathcal{L}_1^{[2n]})^{-1} \mathcal{D}_e^{[2n]} (\mathcal{L}_2^{[2n]})^{-T}$ and $\mathcal{M}_o^{[2n]} = (\mathcal{L}_1^{[2n]})^{-1} \mathcal{D}_o^{[2n]} (\mathcal{L}_2^{[2n]})^{-T}$ with

\[
\mathcal{D}_e^{[2n]} = \text{diag } (d_0, d_1, d_2, \ldots, d_{2n,2n+1}, d_{2n+1,2n+2}) \Theta^{[2n]}, \\
\mathcal{D}_o^{[2n]} = \Theta^{[2n]} \text{diag } (d_{0,0}, d_{0,1}, \ldots, d_{2n+1,2n+2} d_{2n,2n+1}).
\]

Proof. We prove the expression only for $K^{[2n]}(z, \omega)$. Along with the following relations

\[
\mathcal{L}_1^{[2n]} \Pi_e^{[2n]} \mathcal{X}^{[2n]}(z) = \begin{pmatrix} p_0(z) & 0 & \cdots & p_{2n}(z) & 0 \end{pmatrix}^T, \\
\mathcal{L}_2^{[2n]} \Pi_o^{[2n]} \mathcal{X}^{[2n]}(\omega) = \begin{pmatrix} 0 & q_1(\omega) & \cdots & 0 & q_{2n+1}(\omega) \end{pmatrix}^T,
\]

we note that the kernel polynomial $K^{[2n]}(z, \omega)$ contains terms involving $q_{2j+1}(\omega)$ and $p_{2j}(z)$. Hence, we need to make one more transformation that interchanges 0 and $q_{2j+1}(\omega)$, which we do by post-multiplying $\mathcal{L}_2^{[2n]} \Pi_o^{[2n]} \mathcal{X}^{[2n]}(\omega)$ with the matrix $\Theta^{[2n]}$. Since $K^{[2n]}(z, \omega) = [\mathcal{L}_2^{[2n]} \Pi_o^{[2n]} \mathcal{X}(\omega)]^T [\Theta^{[2n]}]^T (\mathcal{D}^{[2n]})^{-1} \mathcal{L}_1^{[2n]} \Pi_e^{[2n]} \mathcal{X}^{[2n]}(z)$, the expression for $K^{[2n]}(z, \omega)$ follows. □
4. Illustration: Hankel symmetry

We begin this section with the question: when does the Gram matrix \( M \) given by (1.10) have a factorization of the form \( M = \mathcal{L}\mathcal{D}\mathcal{L}^T \)? We consider the part of the matrix \( M \) above the diagonal and look for a map that transforms the condensed matrix \( \tilde{M} \) to the block structured matrix \( \mathcal{M} \), \( \mathcal{M} \leadsto \mathcal{M} \), as given below

\[
\begin{pmatrix}
m_{0,1} & m_{0,3} & \cdots \\
m_{1,2} & m_{1,4} & \cdots \\
\vdots & \vdots & \ddots \\
\end{pmatrix} \leadsto \begin{pmatrix}
0 & m_{01} & 0 & m_{03} & \cdots \\
m_{10} & 0 & m_{12} & 0 & \cdots \\
0 & m_{21} & 0 & m_{23} & \cdots \\
m_{30} & 0 & m_{32} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

If we impose Hankel symmetry so that \( m_{i,j} = m_{k,l} \) if \( i + j = k + l \), we may interpret the above map as each entry \( m_{i,j} \) being mapped to a block matrix on the right, which implies that we need to use the Kronecker product

\[\tilde{M} \otimes J_2, \quad \text{where} \quad J_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.\]

So, if \( \tilde{M} = \tilde{\mathcal{L}}\tilde{\mathcal{D}}\tilde{\mathcal{L}}^T \), then we have \( \tilde{M} \otimes J_2 = \tilde{\mathcal{L}}\tilde{\mathcal{D}}\tilde{\mathcal{L}}^T \otimes J_2 \). The use of Kronecker product along with the following properties

\[
(A \otimes B)(C \otimes D) = AC \otimes BD, \quad (A \otimes B)^T = A^T \otimes B^T, \quad (A \otimes B)^{-1} = A^{-1} \otimes B^{-1}
\]

simplifies the transition to the block structure to a great extent and we get

\[
(\mathcal{L} \otimes I)(\mathcal{D} \otimes J_2)(\mathcal{L}^T \otimes I) = (\mathcal{L} \mathcal{D} \otimes J_2)(\mathcal{L}^T \otimes I) = \mathcal{L}\mathcal{D}\mathcal{L}^T \otimes J_2.
\]

Consequently we obtain the following

\[
\tilde{M} = \tilde{\mathcal{L}}\tilde{\mathcal{D}}\tilde{\mathcal{L}}^T \implies M = (\tilde{\mathcal{L}} \otimes I)(\tilde{\mathcal{D}} \otimes J_2)(\tilde{\mathcal{L}}^T \otimes I) = \mathcal{L}^{-1}\mathcal{D}\mathcal{L}^{-T},
\]

which provides the relation between the respective factorizations. Further, with (4.1) if we let \( \mathcal{L}_1^{-1} = \tilde{\mathcal{L}} \otimes I \) and \( \mathcal{L}_2^{-T} = (\tilde{\mathcal{L}}^T \otimes I) \), then \( \mathcal{L}_1 = \mathcal{L}_2 = \tilde{\mathcal{L}}^{-1} \otimes I \). The polynomial sequence \( \{p_n(z)\}_{n=0}^{\infty} \) is generated from \( \mathcal{L}_1 \mathcal{X}(z) = (\tilde{\mathcal{L}}^{-1} \otimes I)\mathcal{X}(z) \) as

\[
\begin{pmatrix}
\tilde{\mathcal{L}}_{00} & 0 & 0 & \cdots \\
0 & \tilde{\mathcal{L}}_{00} & 0 & \cdots \\
\tilde{\mathcal{L}}_{10} & 0 & \tilde{\mathcal{L}}_{11} & \cdots \\
0 & \tilde{\mathcal{L}}_{10} & 0 & \tilde{\mathcal{L}}_{11} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
I \\
zI \\
z^2I \\
z^3I \\
\vdots
\end{pmatrix}
= \begin{pmatrix}
p_0(z) \\
p_1(z) \\
p_2(z) \\
p_3(z) \\
\vdots
\end{pmatrix},
\]

so that we have the forms

\[
p_{2j}(z) = \tilde{\mathcal{L}}_{j,j}z^{2j} + \tilde{\mathcal{L}}_{j,j-1}z^{2j-2} + \cdots + \tilde{\mathcal{L}}_{j,1}z^2 + \tilde{\mathcal{L}}_{j,0},
\]

\[
p_{2j+1}(z) = zp_{2j}(z), \quad j = 0, 1, \ldots.
\]
Hence, if we assume that the condensed matrix \( \tilde{M} \) is positive-definite, or in other words its entries are the moments coming from a determined matrix Hamburger moment problem, the factorization \( \tilde{M} = \tilde{L}\tilde{D}\tilde{U} \) exists and all of the above results follow through. We also note that the matrices \( M_{\text{co}}^{[2j]} \) and \( M_{\text{oe}}^{[2j]} \) used in the quasi-determinant representations of \( p_n(z) \) and \( q_n(z) \) are nothing but truncations of \( \tilde{M} \) and hence are invertible.

This also brings us to the following observation that because of the underlying Hankel symmetry, we have \( q_j(z) = p_j(z) \), \( j = 0, 1, \ldots \), where \( q_j(z) \) is obtained from \( \tilde{L}\tilde{X}(z) \). Further, if we denote \( \tilde{D} = \text{diag}(\tilde{d}_{0,0}, \tilde{d}_{1,1}, \ldots) \), then \( d_{2j,2j+1} = d_{2j+1,2j} = \tilde{d}_{jj}, j = 0, 1, \ldots \).

The biorthogonal relations (2.6) immediately yield
\[
\langle p_{2j}, p_k \rangle = d_{jj} \delta_{2j+1,k}, \quad \langle p_{2j+1}, p_k \rangle = d_{jj} \delta_{2j,k}, \quad j, k = 0, 1, 2, \ldots,
\]
which shows that \( \{p_n(z)\}_{n=0}^{\infty} \) is a polynomial sequence that is biorthogonal to itself with respect to the bilinear form \( \langle \cdot, \cdot \rangle \), which is, thus, not quasi-definite. In the scalar case, the relations (4.2) exist and such systems are called almost orthogonal, arising in works related to indefinite analogues of the Hamburger and Stieltjes moment problems. We refer the reader to [12, Sections 2, 3] for necessary references in this direction and a view of the unwrapping of measures via continued fractions. Further, some of the results in the present section that are reduced to special forms due to Hankel symmetry are also presented [12], though in the scalar case, where the approach is through the \( LU \) decomposition of the underlying Jacobi matrices.

We end with the special forms of the kernel polynomials and the ABC Theorem in case of Hankel symmetry. From (3.3), the kernel polynomials associated to Gram matrix \( M \) are given by
\[
K^{[2n]}(z, \omega) = \omega \sum_{j=0}^{n} p_{2j}^T(\omega)\tilde{d}_{jj}^{-1}p_{2j}(z), \quad n = 0, 1, \ldots,
\]
\[
K^{[2n+1]}(z, \omega) = z \sum_{j=0}^{n} p_{2j}^T(\omega)\tilde{d}_{jj}^{-1}p_{2j}(z), \quad n = 0, 1, \ldots,
\]

having representations as obtained in Theorem 3.5. We only note that \( M_{\text{co}}^{[2n]} \) and \( M_{\text{oe}}^{[2n]} \) are still not presented in the form of a \( LDU \) decomposition since \( D_{\text{co}}^{[2n]} \) and \( D_{\text{oe}}^{[2n]} \) are not diagonal. This is perhaps because the above forms (4.3), even in case of Hankel symmetry, appear as forward shifts in \( \omega \) and \( z \) respectively and needs to be investigated further.

References

[1] M. Adler and P. van Moerbeke, Group factorization, moment matrices, and Toda lattices, Internat. Math. Res. Notices 1997, no. 12, 555–572.

[2] M. Adler, P. van Moerbeke and P. Vanhaecke, Moment matrices and multi-component KP, with applications to random matrix theory, Comm. Math. Phys. 286 (2009), no. 1, 1–38.
[3] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, translated by N. Kemmer, Hafner Publishing Co., New York, 1965.

[4] C. Álvarez-Fernández, U. Fidalgo Prieto and M. Mañas, Multiple orthogonal polynomials of mixed type: Gauss-Borel factorization and the multi-component 2D Toda hierarchy, Adv. Math. 227 (2011), no. 4, 1451–1525.

[5] C. Álvarez-Fernández and M. Mañas, Chapter in Orthogonal polynomials: current trends and applications. Edited by Francisco Marcellán and Edmundo J. Huertas. SEMA SIMAI Springer Series, 22. Springer, Cham, [2021], ©2021. 327 pp.

[6] C. Álvarez-Fernández et al., Christoffel transformations for matrix orthogonal polynomials in the real line and the non-Abelian 2D Toda lattice hierarchy, Int. Math. Res. Not. IMRN 2017, no. 5, 1285–1341.

[7] C. Álvarez-Fernández and M. Mañas, On the Christoffel-Darboux formula for generalized matrix orthogonal polynomials, J. Math. Anal. Appl. 418 (2014), no. 1, 238–247.

[8] G. Ariznabarreta and M. Mañas, Multivariate orthogonal polynomials and integrable systems, Adv. Math. 302 (2016), 628–739.

[9] D. Bessis and P. Moussa, Orthogonality properties of iterated polynomial mappings, Comm. Math. Phys. 88 (1983), no. 4, 503–529.

[10] J. A. Charris and M. E. H. Ismail, Sieved orthogonal polynomials. VII. Generalized polynomial mappings, Trans. Amer. Math. Soc. 340 (1993), no. 1, 71–93.

[11] G. Chen and Y. Hu, The truncated Hamburger matrix moment problems in the nondegenerate and degenerate cases, and matrix continued fractions, Linear Algebra Appl. 277 (1998), no. 1-3, 199–236.

[12] M. Derevyagin, On the relation between Darboux transformations and polynomial mappings, J. Approx. Theory 172 (2013), 4–22.

[13] I. Gelfand et al., Quasideterminants, Adv. Math. 193 (2005), no. 1, 56–141.

[14] J. S. Geronimo and W. Van Assche, Orthogonal polynomials on several intervals via a polynomial mapping, Trans. Amer. Math. Soc. 308 (1988), no. 2, 559–581.

[15] M. E. H. Ismail, On sieved orthogonal polynomials. III. Orthogonality on several intervals, Trans. Amer. Math. Soc. 294 (1986), no. 1, 89–111.

[16] I. V. Kovalishina, Analytic theory of a class of interpolation problems, Izv. Akad. Nauk SSSR Ser. Mat. 47 (1983), no. 3, 455–497.

[17] F. Marcellán and J. Petronilho, Orthogonal polynomials and quadratic transformations, Portugal. Math. 56 (1999), no. 1, 81–113.

[18] F. J. Narcowich, R-operators. II. On the approximation of certain operator-valued analytic functions and the Hermitian moment problem, Indiana Univ. Math. J. 26 (1977), no. 3, 483–513.

[19] B. Simon, The classical moment problem as a self-adjoint finite difference operator, Adv. Math. 137 (1998), no. 1, 82–203.

[20] B. Simon, The Christoffel-Darboux kernel, in *Perspectives in partial differential equations, harmonic analysis and applications*, 295–335, Proc. Sympos. Pure Math., 79, Amer. Math. Soc., Providence, RI.

[21] H. S. Wall, *Analytic Theory of Continued Fractions*, D. Van Nostrand Company, Inc., New York, NY, 1948.

Department of Mathematics, Indian Institute of Science Bangalore-560012, Karnataka, India

Email address: kiranbehera@iisc.ac.in