He’s frequency–amplitude formulation for nonlinear oscillators using Jacobi elliptic functions

Alex Elías-Zúñiga¹, Luis Manuel Palacios-Pineda², Isaac H Jiménez-Cedeño³, Oscar Martínez-Romero¹ and Daniel Olvera Trejo¹

Abstract
In this work, the Duffing’s type analytical frequency–amplitude relationship for nonlinear oscillators is derived by using He’s formulation and Jacobi elliptic functions. Comparison of the numerical results obtained from the derived analytical expression using Jacobi elliptic functions with respect to the exact ones is performed by considering weak and strong Duffing’s nonlinear oscillators.

Keywords
Jacobian elliptic functions, Duffing-type oscillators, frequency-amplitude, He's formulation, fractal Fangzhu singular oscillator

Introduction
Many dynamic systems that arise in physics and engineering applications are known to be modeled by a homogeneous nonlinear differential equation whose closed-form solution is unknown. Then, numerical or approximate methods have to be used to predict its dynamic response behavior and determine the frequency–amplitude relationship needed to understand its qualitative and quantitative dynamic response.¹–¹⁸ However, in a few cases, the closed-form solution of some nonlinear oscillators is known. In fact, the cubic, the quadratic–cubic, and the cubic–quintic Duffing-type oscillators are those whose analytical closed form solutions are described exactly in terms of Jacobi elliptic functions.¹⁹–²² It is also known that when the rational or irrational functions describe the restoring forces of nonlinear oscillators, the original nonlinear differential equations can be written in an equivalent form using methods that transform the original equations into Duffing-type oscillators and thus, the approximate frequency–amplitude relationship can be determined by the analytical closed-form solutions of the corresponding Duffing-type oscillator.

However, to find a simple and accurate frequency–amplitude expression, Professor He has used an ancient Chinese mathematical algorithm in which the frequency–amplitude of various nonlinear oscillators was found.²³–²⁵

Therefore, the article aims to enhance the accuracy of He’s formulation using trial residuals based on Jacobi elliptic functions, which are a more general class of periodic functions that include the trigonometric functions as a

¹ Mechanical Engineering and Advanced Materials Department, School of Engineering and Science, Tecnologico de Monterrey, Ave. Eugenio Garza Sada 2501, Monterrey 64849, Mexico
² Tecnológico Nacional de México/Instituto Tecnológico de Pachuca, Pachuca, Mexico
³ Engineering Department, Safran Mexico, Chihuahua, Mexico

Corresponding author:
Alex Elías-Zúñiga, Tecnológico de Monterrey Avenida Eugenio Garza Sada 2501 Sur Monterrey, 64849 Mexico.
Email: aelias@tec.mx

Creative Commons CC BY: This article is distributed under the terms of the Creative Commons Attribution 4.0 License (https://creativecommons.org/licenses/by/4.0/) which permits any use, reproduction and distribution of the work without further permission provided the original work is attributed as specified on the SAGE and Open Access pages (https://us.sagepub.com/en-us/nam/open-access-at-sage).
particular case. To demonstrate the accuracy achieved by using Jacobi functions to find the approximate frequency–amplitude expression by He’s algorithm, two cases are examined. The first case focuses on determining the approximate frequency of the Fangzhu singular equation of motion, while the second case focuses on obtaining the frequency–amplitude equation for Duffing-type oscillators considering even and odd nonlinearities.

Preliminaries: He’s frequency–amplitude approach

In this section, we provide a brief summary of He’s formulation to derive frequency–amplitude expressions of nonlinear oscillators of the form

\[ \ddot{u} + f(u) = 0, \quad u(0) = A_0, \quad \dot{u}(0) = v_0 \]  

using the ancient Chinese algorithm\textsuperscript{23–25}

\[
\omega_{He}^2 = \frac{\omega_1^2 R_2(t) - \omega_2^2 R_2(t)}{R_2(t) - R_1(t)}
\]

where \( \omega_1 \) and \( \omega_2 \) are location points, \( \omega_i \) are arbitrary chosen frequencies with \( \omega_2 > \omega_1 \), and \( R_i(t) \) are trial residual functions defined as

\[
R_1(t) = -A \omega_1^2 \cos \omega_1 t + f(A \cos \omega_1 t)
\]

\[
R_2(t) = -A \omega_2^2 \cos \omega_2 t + f(A \cos \omega_2 t)
\]

and \( u_i(t) = A_i \cos \omega_i t, i = 1, 2 \). \( R_i(t) \) are assumed to be time-dependent and therefore, their average values can be computed from the following expression

\[
R_i = \frac{4}{T} \int_0^{T/4} R_i(t) w(t) \, dt
\]

where \( T \) is the motion period, and \( w(t) \) is a weighted function that can take any simple form.

We shall next modify He’s approach by introducing Jacobi elliptic functions instead of trigonometric ones.

Frequency–amplitude formulation based on elliptic functions

It is well known that most of the nonlinear oscillators can be expressed in the form

\[ \ddot{y} + f(y) = 0, \quad y(0) = 1, \quad \dot{y}(0) = v \]  

where \( A_0 \) is the initial amplitude of oscillation and \( f(u) \) are the conservative system restoring forces. If the following transformation \( y = u/A_0 \) is introduced, then equation (6) can be written as

\[
\ddot{y} + f(y) = 0, \quad y(0) = 1, \quad \dot{y}(0) = v
\]

Based on the ancient Chinese method, two trial functions are needed to find the approximate solution of equation (7) that are assumed to be \( y_1 = A_1 \text{cn}(\omega_1 t, k_1^2) \), and \( y_2 = A_2 \text{cn}(\omega_2 t, k_2^2) \) that satisfy the following Duffing differential equations

\[
\ddot{y}_1 + \omega_1^2 y_1 + \omega_1 y_1^3 = 0, \quad \text{and} \quad \ddot{y}_2 + \omega_2^2 y_2 + \omega_2 y_2^3 = 0
\]

where \( A_i, \omega_i, \) and \( k_i \) are trial amplitudes, frequencies, and modulus of the Jacobi elliptic functions \( \text{cn}(\omega_1 t, k_1^2) \) and \( \text{cn}(\omega_2 t, k_2^2) \) that need to be found. Thus, the residuals to find the approximate solution of equation (7) are...
\[ R_1(t) = f(y_1) - A_1(1 - 2k_i^2)\omega_i^2 cn(\omega_it, k_i^2) = 2A_1k_i^2\omega_i^2 cn^3(\omega_it, k_i^2) \quad (9) \]
\[ R_2(t) = f(y_2) - A_2(1 - 2k_i^2)\omega_i^2 cn(\omega_2t, k_i^2) = 2A_2k_i^2\omega_i^2 cn^3(\omega_2t, k_i^2) \quad (10) \]

Since the restoring forces of nonlinear oscillators can be expressed in equivalent form as polynomial expressions of the form

\[ f(y_i) = x_0y_i + x_1y_i^3 + x_2y_i^5 + x_3y_i^7 + x_4y_i^{11} \quad (11) \]

then, equations (9) and (10) can be written as

\[ R_1(t) = \frac{1}{64}(32A_1^2x_{22} + (64A_1x_0 + 48A_1^3x_1 + 40A_1^5x_2 + 35A_1^7x_3 + 32A_1(-2 + k_i^2)\omega_i^2)\cos(\phi_1) \]
\[ + 32A_1^3x_{222}\cos(2\phi_1) + (16A_1^3x_1 + 20A_1^5x_2 + 21A_1^7x_3 - 32A_1k_i^2\omega_i^2)\cos(3\phi_1) \]
\[ + A_1^4(4x_2 + 7A_1^3x_3)\cos(5\phi_1) + A_1^5x_3\cos(7\phi_1)) \quad (12) \]

\[ R_2(t) = \frac{1}{64}(32A_2^2x_{22} + (64A_2x_0 + 48A_2^3x_1 + 40A_2^5x_2 + 35A_2^7x_3 + 32A_2(-2 + k_i^2)\omega_i^2)\cos(\phi_2) \]
\[ + 32A_2^3x_{222}\cos(2\phi_2) + (16A_2^3x_1 + 20A_2^5x_2 + 21A_2^7x_3 - 32A_2k_i^2\omega_i^2)\cos(3\phi_2) \]
\[ + A_2^4(4x_2 + 7A_2^3x_3)\cos(5\phi_2) + A_2^5x_3\cos(7\phi_2)) \quad (13) \]

where the identities \( \cos \phi_i = cn(\omega_it, k_i^2) \) and \( \sin \phi_i = sn(\omega_it, k_i^2) \) have been used. Here \( \phi_i \) represents the Jacobi amplitude given by the expression

\[ \phi_i = am(\omega_it, k_i^2) = \int_{0}^{\omega_it} dn(\omega_it, k_i^2) d(\omega_it) \quad (14) \]

where \( dn(\omega_it, k_i^2) \) is a Jacobi elliptic function.

From He’s formulation, the residual equations (12) and (13) are found using the following relations

\[ \bar{R}_i = \frac{4}{T} \int_{0}^{T/4} R_i(t) dt \quad (15) \]

that yield after integration the expressions of \( \bar{R}_i \)

\[ \bar{R}_i = 1/(1680T\omega_i)(840A_1^2Tx_{22}x_0 + (6720A_1x_0 + 5040A_1^3x_1 + 4200A_1^5x_2 + 3675A_1^7x_3 \]
\[ - 6720A_1^2\omega_i^2 + 3360A_1k_i^2\omega_i^2)sn(T\omega_i/4, k_i^2)) + 1680A_1^2x_{222}sn(T\omega_i/2, k_i^2) \]
\[ + (560A_1^3x_1 + 700A_1^5x_2 + 735A_1^7x_3 - 1120A_1k_i^2\omega_i^2)sn(3T\omega_i/4, k_i^2) + (84A_1^5x_2 + 147A_1^7x_3)sn(5T\omega_i/4, k_i^2) \]
\[ + 15A_1^3x_3sn(7T\omega_i/4, k_i^2)) \quad (16) \]

Recalling that \( cn \) and \( sn \) are periodic Jacobi elliptic functions with period \( T = 4K(k_i^2)/\omega_i \), where \( K(k_i^2) \) are the complete elliptic integral of the first kind with modulus \( k_i^2 \), then equation (16) reduces to

\[ \bar{R}_i = (210A_1x_0 + 140A_1^3x_1 + 112A_1^5x_2 + 96A_1^7x_2 - 70(3A_1 - 2A_1k_i^2)\omega_i^2 + 105A_1^2x_{22}K(k_i^2))/(210K(k_i^2)) \quad (17) \]
Thus from equation (2), the approximate frequency–amplitude relationship of nonlinear oscillators modeled by an equation of the form

$$
\ddot{y} + \omega_0^2 y + \sum_{j=1}^{n} \omega_j^2 y_j = f(t),
$$

is given as

$$
\omega_{AE}^2 = (-2\omega_0^2 (105A_2x_2 + 70A_2^2x_1 + 56A_2^3x_2 + 48A_2^4x_3 + 35(-3A_2 + 2A_2^2k_2)\omega_0^2)K(k_1^2) \\
+ \left(2(105A_1x_2 + 70A_1^2x_1 + 56A_1^3x_2 + 48A_1^4x_3 + 35(-3A_1 + 2A_1k_1)\omega_0^2)\omega_0^2 \\
+ 105x_{22}(-2A_2\omega_1 + A_1\omega_2)(2A_2\omega_1 + A_1\omega_2)K(k_1^2)K(k_2^2)\right) \\
\left(-2(105A_2x_2 + 70A_2^2x_1 + 56A_2^3x_2 + 48A_2^4x_3 + 35(-3A_2 + 2A_2k_2)\omega_0^2)K(k_1^2) \\
+ 105x_{22}(-2A_2\omega_1 + A_1\omega_2)(2A_2\omega_1 + A_1\omega_2)K(k_1^2)K(k_2^2)\right) \\
+ 70(-3A_1 + 2A_1k_1)\omega_0^2 + 105(A_1 - A_2)(A_1 + A_2)x_{22}K(k_1^2)K(k_2^2) \right)
$$

In the next section, we shall next evaluate the accuracy of using Jacobi elliptic functions by examining the approximate frequency–amplitude solution of the Fangzhu singular oscillator, and the accuracy of equation (19) in providing the frequency–amplitude value of nonlinear Duffing-type oscillators considering weak and strong nonlinearities.

**Results**

The accuracy of the proposed approach is evaluated by examining two cases.

**Case (a): The Fangzhu singular oscillator**

First, the frequency–amplitude of the Fangzhu singular oscillator is determined. This singular oscillator arises during the mathematical modeling of the Fangzhu water harvester device considering the influence of its nanoscale surface morphology. Its equation of motion is given as

$$
\ddot{y} + \omega_0^2 y + \frac{Q}{y^a} = g(t) \text{ with } y(0) = A, \ y'(0) = 0
$$

where $y$ is the distance of the attracted molecule from its equilibrium position, $a$ is a positive fractional or integer number, $g(t)$ is a driving force, and $\omega_0$ represents a positive number related to the water harvester device surface morphology. Introducing the transformation $t = \tau/Q$ equation (20) becomes

$$
\frac{d^2 y}{d\tau^2} + \omega_0^2 y + \frac{1}{\tau^a} = G(\tau)
$$

Since we are interested in determining the frequency–amplitude relation of equation (21), it is assumed that $G(\tau) = 0$. Thus, equation (21) becomes

$$
\frac{d^2 y}{d\tau^2} + \omega_0^2 y + \frac{1}{\tau^a} = 0, \text{ with } y(0) = A, \ y'(0) = 0
$$

Notice that equation (22) does not have critical points. Furthermore, equation (22) has exact solution when $\omega = 0$ and $a = 1$ and a pseudo-period of the weak periodic solution given by $T = 2\sqrt{2 \pi A}$. However, the closed-form solution of equation (22) is unknown. Therefore, in order to determine the approximate frequency–
amplitude expression of the weak periodic solution of the Fangzhu equation, it is assumed that $y_i(t) = A_i \sin(\omega_i t + k_i^2)$, $i = 1, 2$. Thus, from equation (22) the following trial residuals are obtained

$$R_i(t) = A_i(-\omega_i^2 \sin(\omega_i t + k_i^2) + \sin(\omega_i t + k_i^2) + f(y_i(t)))$$  \hspace{1cm} (23)$$

where $f(y_i(t)) = y_i + 1/y_i$. Expanding equation (23) yields

$$R_i(t) = 1 + A_i \omega_i^2 \sin(\omega_i t + k_i^2) + \sin(\omega_i t + k_i^2) + A_i \omega_i^2 \sin(\omega_i t + k_i^2) + \sin(\omega_i t + k_i^2)$$  \hspace{1cm} (24)$$

If the Jacobi elliptic identities $\cos \phi_i = \cos(\omega_i t, k_i^2)$ and $\sin \phi_i = \sin(\omega_i t, k_i^2)$ are applied then, equation (24) becomes

$$R_i(t) = 1 + \cos \phi_i(A_i \cos + (k_i^2 - 1)A_i \omega_i^2 - A_i k_i^2 \omega_i^2 \cos \phi_i)$$  \hspace{1cm} (25)$$

The trial residual functions $R_i(t)$ are computed using equation (15). This yields, after considering the limits of integration $\tau \rightarrow T/4$ and $\tau \rightarrow 0$, and the fact that the period of the Jacobi elliptic functions $\sin(\omega_i t, k_i^2)$ and $\cos(\omega_i t, k_i^2)$ is $4K(k_i^2)/\omega_i$, the expression

$$\bar{R}_i = 1 + A_i \omega_i^2 \Big( A_i(3 + z) \omega_0^2 - A_i(3 - 2k_1^2 + z) \omega_i^2 \Big) \Gamma(1 + x/2)$$  \hspace{1cm} (26)$$

Using equation (2) gives the approximate frequency–amplitude relation

$$\omega_{AE} = \frac{1}{\omega_i} \left[ \left( \left( (3 + z) \omega_0^2 - (3 - 2k_1^2 + z) \omega_i^2 \right) \omega_i^2 K(k_i^2) + \left( \omega_i^2 \left( -(3 + z) \omega_0^2 + (3 - 2k_1^2 + z) \omega_i^2 \right) + \left( 4(\omega_i^2 - \omega_0^2) K(k_i^2) \Gamma((5 + z)/2) \right) \times \left( A_i \omega_i^2 \omega_0^2 (3 - 2k_1^2 + z) \omega_i^2 K(k_i^2) + (3 + z) \omega_0^2 - (3 - 2k_1^2 + z) \omega_i^2 K(k_i^2) \right) \right) \right]$$  \hspace{1cm} (27)$$

where $A_1 = A_2 = A$.  

Equation (27) illustrates that the frequency varies inversely with the oscillatory amplitude. This qualitative behavior of $\omega_{AE}$ as a function of the oscillation amplitude agrees with the results obtained by He et al.  

Furthermore, when $z = 1$ and $\omega_0 = 0$, equation (27) provides the frequency value of 1.2533 if $A = 1$, $\omega_1 = 1$, $\omega_2 = 2$, and $k_1 = k_2 = 0.5968i$, where $i = \sqrt{-1}$. This result agrees with the frequency value obtained from the exact solution derived by Gadella and Lara, Garcia and Gasull.  

If now $z = 8/53$ and $\omega_0 = 1$ with $k_1 = k_2 = 0.4254i$, using equation (27) gives $\omega_{AE} = 1.5196$ while the exact numerical frequency value of equation (22) is $\omega_{Exact} = 1.5167$. This implies a relative error value of 0.188%. For values of $0.01 \leq z \leq 1$ and $0 < \omega_0 \leq 100$ with $k_1 = k_2 = 0.4254i$, the relative error does not exceed the value of 5.8%, which corroborates the accuracy of our proposed approach.

On the other hand, it is important to point out that our solution procedure can be extended to study the fractal form of equation (22) given as

$$\frac{d^2y}{dt^2} + \omega^2 y + \frac{1}{y^2} = 0, \text{ with } y(0) = A, \ y'(0) = 0$$  \hspace{1cm} (28)$$

where $\gamma$ is a fractal parameter. Using the two-scale transform method $T_t = \tau^\gamma$, equation (28) becomes

$$\frac{d^2y}{dT_t^2} + \omega^2 y + \frac{1}{y^2} = 0, \text{ with } y(0) = A, \ y'(0) = 0$$  \hspace{1cm} (29)$$
for which the solution \( y(t) = Acn(\omega_1 t^7, k_1^7) \) holds.\(^{28}\)

**Case (b): Duffing-type oscillators**

The second case focuses on determining the frequency–amplitude relationship for Duffing-type cubic, cubic–quintic, cubic–quintic–heptic oscillators, and for the quadratic Duffing–Helmhotz oscillator, since these arise in many physical and engineering systems and their closed-form solutions are known.\(^{1,2}\) Furthermore, these Duffing-type oscillators can be used to transform nonlinear oscillators that have rational or irrational restoring forces into polynomial ones.\(^{32–40}\)

Table 1 illustrates the errors attained by considering different values of the nonlinear terms \( a_1, a_2, a_22, \) and \( a_3 \) in equation (18). For comparison purposes, Table 1 lists the frequency values obtained by using the approach introduced by Ren and Hu to derive the frequency–amplitude equation\(^{41}\)

\[
\omega_{RE}^2 = \omega_0 + \frac{2}{3} x_1 A^2 + \frac{8}{15} x_2 A^4 + \frac{16}{35} x_3 A^6 + \frac{\pi}{4} x_{22} A
\]

where the term related to the quadratic nonlinearity of equation (18) has been considered. In all cases examined here, the values of \( \omega_0 = 1, A = 1, A_1 = A_2 = 1, k_1^2 = k_2^2 = 0.06 \) with \( \omega_1 = 1, \) and \( \omega_2 = 2\omega_1 \) were selected to minimized the error. Notice from Table 1 that in general, the error attained from equation (19) is smaller than that of \( \omega_{RE} \). This reduced error is expected since the Duffing-type equations' exact solution is based on Jacobi elliptic functions. Table 1 shows the values of \( \omega_{AE} \) and \( \omega_{RE} \) are similar and compare well with respect to the exact

| \( x_{22} \) | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( \omega_{Exact} \) | \( \omega_{AE} \) | Relative error (%) | \( \omega_{RE} \) | Relative error (%) |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1.9099 | 1.8790 | 0.6799 | 1.8554 | 2.9378 |
| 10 | 10 | 10 | 10 | 5.2040 | 5.1586 | 0.8804 | 5.0423 | 3.2073 |
| 100 | 100 | 100 | 100 | 16.1641 | 16.0232 | 0.8793 | 15.6606 | 3.2152 |
| 1000 | 1000 | 1000 | 1000 | 51.0218 | 50.5773 | 0.8788 | 49.4322 | 3.2156 |
| 1 | 1 | 0 | 0 | 1.6043 | 1.6021 | 0.1349 | 1.5659 | 2.4536 |
| 10 | 10 | 0 | 0 | 4.0744 | 4.0366 | 0.9374 | 3.9396 | 3.4228 |
| 100 | 100 | 0 | 0 | 12.5214 | 12.3923 | 1.0420 | 12.0916 | 3.5548 |
| 1000 | 1000 | 0 | 0 | 39.4794 | 39.0681 | 1.0527 | 38.1191 | 3.5684 |
| 0 | 1 | 0 | 0 | 1.3177 | 1.3176 | 0.0121 | 1.2909 | 2.0745 |
| 0 | 10 | 0 | 0 | 2.8667 | 2.8259 | 1.4391 | 2.7688 | 3.5308 |
| 0 | 100 | 0 | 0 | 8.5335 | 8.3956 | 1.6435 | 8.2259 | 3.7395 |
| 0 | 1000 | 0 | 0 | 26.804 | 26.3721 | 1.6633 | 25.8392 | 3.7597 |
| 0 | 1 | 0 | 0 | 1.2647 | 1.2638 | 0.0708 | 1.2382 | 2.1343 |
| 0 | 10 | 0 | 0 | 2.5836 | 2.5685 | 0.5893 | 2.5166 | 2.6635 |
| 0 | 100 | 0 | 0 | 7.5425 | 7.5231 | 0.2579 | 7.3711 | 2.3253 |
| 0 | 1000 | 0 | 0 | 23.6406 | 23.5923 | 0.2045 | 23.1157 | 2.2708 |
| 0 | 0 | 1 | 0 | 1.2305 | 1.2320 | 0.1183 | 1.2071 | 1.9412 |
| 0 | 0 | 10 | 0 | 2.3879 | 2.4090 | 0.8759 | 2.3603 | 1.1680 |
| 0 | 0 | 100 | 0 | 6.8362 | 6.9757 | 2.0001 | 6.8347 | 0.0206 |
| 0 | 0 | 1000 | 0 | 21.3722 | 21.8456 | 2.1671 | 21.4043 | 0.1497 |
| 0 | 1 | 1 | 0 | 1.5235 | 1.5131 | 0.6447 | 1.4832 | 2.72015 |
| 0 | 1 | 0 | 0 | 1.6753 | 1.6636 | 0.7001 | 1.6300 | 2.7766 |
| 0 | 10 | 10 | 0 | 4.3059 | 4.2782 | 0.6477 | 4.1918 | 2.7233 |
| 0 | 100 | 100 | 100 | 13.2511 | 13.1780 | 0.5546 | 12.9118 | 2.6281 |
| 0 | 1000 | 1000 | 1000 | 41.7858 | 41.5600 | 0.5433 | 40.7203 | 2.6166 |
| 0 | 0 | 1 | 0 | 1.4450 | 1.4399 | 0.3541 | 1.4108 | 2.4235 |
| 0 | 0 | 10 | 0 | 3.3589 | 3.3703 | 0.3366 | 3.3022 | 1.7184 |
| 0 | 0 | 100 | 0 | 10.1327 | 10.2086 | 0.7439 | 10.0024 | 1.3028 |
| 0 | 0 | 1000 | 0 | 31.8817 | 32.1370 | 0.7943 | 31.4877 | 1.2513 |
frequency, \( \omega_{\text{Exact}} \), whose value can be obtained by numerical integration of equation (18). It is also seen from Table 1 that for increasing values of the nonlinear parameters \( \alpha \), the errors attained tend to decrease.

Conclusion

He’s formulation for getting the frequency–amplitude relationship for nonlinear oscillators provides accurate results when the two trial residuals are based on Jacobi elliptic functions. The simplicity of Professor He’s formulation can now be expanded to obtain analytical expressions for the frequency of nonlinear oscillators, since most of these can be written as a Duffing-type equation using transformation techniques. Furthermore, using the two-scale transform, our results can be applied to obtain approximate frequency–amplitude expressions for the fractal Fangzhu singular oscillator and for the fractal Duffing-type equations. Therefore, this paper provides evidence of the applicability of He’s formulation for all homogeneous single-degree-of-freedom nonlinear oscillators found in physics, aerospace, and engineering applications.

Authors’ contributions

AE-Z: Conceptualization, formal analysis, funding acquisition, investigation, project administration, writing—original draft, review and editing. LMP-P: Formal analysis, investigation, software, visualization, writing—review. IHJ-C: Formal analysis, investigation, software, visualization. OM-R: Formal analysis, investigation, software, visualization. DOT: Formal analysis, investigation, software, visualization.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: Tecnológico de Monterrey funded this research through the Research Group of Nanotechnology for Devices Design, and by the Consejo Nacional de Ciencia y Tecnología de México (Conacyt), Project Numbers 242269, 255837, 296176, and National Lab in Additive Manufacturing, 3D Digitizing and Computed Tomography (MADiT) LN299129.

ORCID iDs

Alex Elías-Zúñiga https://orcid.org/0000-0002-5661-2802
Luis Manuel Palacios-Pineda https://orcid.org/0000-0001-5297-2950
Daniel Olvera Trejo https://orcid.org/0000-0002-4385-6269

References

1. Hayashi C. Nonlinear oscillation in physical systems. Princeton: Princeton University Press, 1964.
2. Nayfeh AH and Mook DT. Non-linear oscillations. New York: John Wiley, 1973.
3. Sanders JA and Verhulst F. Averaging methods in nonlinear dynamical systems. New York: Springer-Verlag, 1985.
4. He JH. Homotopy perturbation technique, Comp Methods Appl Mech Eng 1999; 178: 257–262.
5. He JH. The homotopy perturbation method for nonlinear oscillators with discontinuities. Int J Nonlinear Mech 2007; 42: 335–341.
6. He JH. Some asymptotic methods for strongly nonlinear equations. Int J Nonlinear Mech 2007; 42: 335–341.
7. He JH. New interpretation of homotopy perturbation method. Int J Nonlinear Mech 2007; 42: 335–341.
8. He JH. Homotopy perturbation method with an auxiliary term. Int J Nonlinear Mech 2007; 42: 335–341.
9. He JH. Asymptotic methods for solitary solutions and compactons. Int J Nonlinear Mech 2007; 42: 335–341.
10. He JH. Homotopy perturbation method with two expanding parameters. Int J Nonlinear Mech 2007; 42: 335–341.
11. Elías-Zúñiga A, López de Lacalle LN, et al. Approximate solutions of delay differential equations with constant and variable coefficients by the enhanced multistage homotopy perturbation method. J Vibr Control 2015; 21: 932–946.
12. Compeán FI, Olvera D, Campa FJ, et al. Characterization and stability analysis of a multivariable milling tool by the enhanced multistage homotopy perturbation method. Proc IMechE Part B: J Engineering Manufacture 2012; 226: 230–241.
13. He JH. A tutorial review on fractal spacetime and fractional calculus. Int J Modern Phys C 2014; 25: 1450002.
14. He JH and Wu XH. Variational iteration method: new development and applications. Comp Math Appl 2007; 54: 881–894.
15. Zhou Y, Pang S, Chong G, et al. A variational iteration method integral transform technique for handling heat transfer problems. Therm Sci 2017; 21: S55–S61.
16. Meng ZJ, Zhou YM and Wang HQ. Local fractional variational iteration algorithm III for the diffusion model associated with non-differentiable heat transfer. *Therm Sci* 2016; 20: S781–S784.
17. He JH. Max-min approach to nonlinear oscillators. *Int J Nonlinear Sci Numer Simul* 2008; 9: 207–210.
18. He JH. Variational approach for nonlinear oscillators. *Chaos Solitons Fractals* 2007; 34: 1430–1439.
19. Beatty MF. Stability of a body supported by a simple vehicular shear suspension system. *Int J Non-Linear Mech* 1989; 24: 65–77.
20. Beatty MF. Finite amplitude vibrations of a body supported by simple shear springs. *J Appl Mech* 1984; 51: 361–366.
21. Elías-Zúñiga A. Exact solution of the quadratic mixed-parity Helmholtz Duffing oscillator. *Appl Math Comput* 2012; 218: 7590–7594.
22. Elías-Zúñiga A. Exact solution of the cubic-quintic Duffing oscillator. *Appl Math Model* 2013; 37: 2574-2579.
23. He JH. Ancient Chinese algorithm: the Ying Buzu Shu (method of surplus and deficiency) vs. Newton iteration method. *Handbook of elliptic integrals for engineers and physicists*.
24. He JH. Amplitude-frequency relationship for conservative nonlinear oscillators with odd nonlinearities. *Int J Appl Comput*.
25. He JH. An improved amplitude-frequency formulation for nonlinear oscillators. *Int J Nonlinear Sci Numer Simul* 2008; 9: 211–212.
26. Byrd PF and Friedman MD. *Handbook of elliptic integrals for engineers and physicists*. New York: Springer-Verlag, 1953.
27. He CH, He JH and Sedighi HM. Fangzhu: an ancient Chinese nanotechnology for water collection: history, mathematical insight, promises and challenges. *Math Methods Appl Sci* 2020; 2017; 3: 1557–1560.
28. Wang KL. Effect of Fangzhu’s nanoscale surface morphology on water collection. *Math Meth Appl Sci*. Epub ahead of print 26 June 2020. DOI: 10.1002/mma.6569.
29. He JH and El-Dib YO. Homotopy perturbation method for Fangzhu oscillator. *J Math Chem*. Epub ahead of print 16 September 2020. DOI: 10.1007/s10910-020-01167-6.
30. Gadella M and Lara LP. On the solutions of a nonlinear ‘pseudo’-oscillator equation. *Phys Scr* 2014; 89: 105205.
31. García JD and Gasull A. Weak periodic solutions of $\xi x^2 + 1 = 0$ and the Harmonic Balance Method. *Phys Conf Ser* 2017; 811: 012003.[1]
32. Elías-Zúñiga A and Martínez-Romero O. Energy method to obtain approximate solutions of strongly nonlinear oscillators. *Math Probl Eng* 2013; 2013: 620591.
33. Elías-Zúñiga A, Martínez-Romero O and Córdoba-Díaz RK. Approximate solution for the Duffing-harmonic oscillator by the enhanced cubication method. *Math Probl Eng* 2012; 2012: 618750.
34. Elías-Zúñiga A and Martínez-Romero O. Accurate solutions of conservative nonlinear oscillators by the enhanced cubication method. *Math Probl Eng* 2013; 2013: 842423.
35. Elías-Zúñiga A and Martínez-Romero O. Investigation of the equivalent representation form of damped strongly nonlinear oscillators by a nonlinear transformation approach. *J Appl Math* 2013; 2013: 245092.
36. Elías-Zúñiga A. “Quintication” method to obtain approximate analytical solutions of non-linear oscillators. *Appl Math Comput* 2014; 243: 849–855.
37. Elías-Zúñiga A, Palacios-Pineda LM, Olvera-Trejo D, et al. Lyapunov equivalent representation form of forced, damped, nonlinear, two degree-of-freedom systems. *Appl Sci* 2018; 8: 649.
38. Elías-Zúñiga A, Palacios-Pineda LM, Martínez-Romero O, et al. Equivalent representation form in the sense of Lyapunov, of nonlinear forced, damped second order differential equations. *Nonlinear Dyn* 2018; 92: 2143–2158.
39. Elías-Zúñiga A, Palacios-Pineda LM, Jiménez-Cedeño IH, et al. Equivalent power-form transformation for fractal Bratu’s equation. *Fractals* 2020; https://doi.org/10.1142/S0218348X21500195.
40. Elías-Zúñiga A, Palacios-Pineda LM, Jiménez-Cedeño IH, et al. Equivalent power-form representation of the fractal Toda oscillator. *Fractals* 2020; https://doi.org/10.1142/S0218348X21500341.
41. Anjum N and He JH. Laplace transform: making the variational iteration method easier. *Appl Math Lett* 2019; 92:134–138.