Systemic risk measures with markets volatility

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Abstract As systemic risk has become a hot topic in the financial markets, how to measure, allocate and regulate the systemic risk are becoming especially important. However, the financial markets are becoming more and more complicate, which makes the usual study of systemic risk to be restricted. In this paper, we will study the systemic risk measures on a special space $L^p(\cdot)$ where the variable exponent $p(\cdot)$ is no longer a given real number like the space $L^p$, but a random variable, which reflects the possible volatility of the financial markets. Finally, the dual representation for this new systemic risk measures will be studied. Our results show that every this new systemic risk measure can be decomposed into a convex certain function and a simple-systemic risk measure, which provides a new ideas for dealing with the systemic risk.

Keywords risk measure; systemic risk; variable exponent; dual representation;

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1 Introduction

The financial crisis has not only caused the public attention to systemic risk, but also highlighted the need to measure and manage systemic risk. The measurement of systemic risk involves two problems: the quantification of the systemic risk of the financial markets, and the allocation of that risk to individual institutions. This led to a focus on the research of systemic risk measures.

In the seminal paper, Artzner et al. (1997, 1999) firstly introduced the class of coherent risk measures. However, the traditional risk measures failed to capture sufficiently of the perilous systemic risk. Much of the recent research has focused on measuring the systemic risk. Systemic risk measures were introduced axiomatically in Chen et al. (2013). For more studies of systemic risk measures, see Tarashev et al. (2010), Acharya et al. (2012), Gauthier et al. (2012), Brunnermeier and Cheridito (2014), Armenti et al. (2015), Biagini et al. (2015), Feinstein et al. (2015) and the references therein.

However, the financial markets are becoming much more complicated that the common systemic risk measures may not describe the systemic risk availably. This arise the awareness of the urgent need for designing more appropriate systemic risk measures under a financial systems with greater uncertainty and volatility. The current volatility of systemic risk is reflected in the potentially conflicting views on the relationship between the structure of the financial network and the extent of financial contagion. In other words, it is the volatility of the financial markets. Taking this into consideration, we would like to emphasize that our
study of systemic risk measures will not focus on the common space of financial positions, but on a special space, the variable exponent Bochner-Lebesgue space, which is denoted by $L^{p(\cdot)}$. Under this space, the order $p(\cdot)$ is no longer a fixed positive number like $L^p$, but a measurable function. More concretely, the variable exponent $p(\cdot)$ reflects the uncertainty and volatility of the financial markets.

The variable exponent Lebesgue spaces appeared in the literature for the first time already in Orlicz (1931) and the variable exponent Bochner-Lebesgue space and was studied in detail by Cheng and Xu (2013). For more studies on variable exponent Lebesgue spaces, see Harjulehto et al. (2010), Kempka (2010), Diening et al. (2009), Hästö (2009), Kempka (2009), Xu (2009), Xu (2008), Almeida et al. (2008), Kováčik and Rákosník (1991), Musielak (1983), Nakano (1950) and the references therein.

The main focus of this paper is the study of systemic risk measures on variable exponent Bochner-Lebesgue space $L^{p(\cdot)}$. Traditional risk measurement strategies of financial systems always assumed that a risk measure is a map $\rho$ that evaluates the risk $f$ of financial positions directly. In the present paper, we make the measurement of systemic risk on $L^{p(\cdot)}$ into two steps. Firstly, we define a certain function $\phi$ that for any systemic risk $f \in L^{p(\cdot)}$, $\phi(f)$ is a function with a certain order, i.e. $\phi(f) \in L^p$, $p \in [1, +\infty]$; Secondly, we define a simple-systemic risk measure $\varrho : L^p \to \mathbb{R}$, which makes $\varrho(\phi(f))$ is a real number for any $f \in L^{p(\cdot)}$. Since each step is common for us, these two steps provide a novel approach to measure the systemic risk in uncertain markets. We will also show that each systemic risk measure $\rho$ on $L^{p(\cdot)}$ can be decomposed into a convex certain function $\phi$ and a simple-systemic risk measure $\varrho$, i.e. $\rho(f) = (\varrho \circ \phi)(f)$, $f \in L^{p(\cdot)}$. At last, the dual representation of systemic risk measures on $L^{p(\cdot)}$ is given.

The rest of the paper is organized as follows. In Section 2, we will briefly review the definition and the main properties of variable exponent Bochner-Lebesgue spaces. In Section 3, we will introduce the definition of systemic risk measures on variable exponent Bochner-Lebesgue spaces as well as the definition of convex certain function and simple-systemic risk measures. Section 4 is devoted to the new measurement of systemic risk on variable exponent Bochner-Lebesgue spaces, that is, each systemic risk measures on variable exponent Bochner-Lebesgue spaces can be decomposed into a convex certain function and a simple-systemic risk measure. Finally, in Section 5, we will study the dual representation of systemic risk measures.

2 Preliminaries

In this section, we will briefly introduce the definition and the main properties of variable exponent Bochner-Lebesgue spaces and the preliminaries.

From now on, let $(\Omega, \mathcal{F}, \mu)$ be a $\sigma$-finite complete measurable space, $E$ be a given reflexive Banach space with zero element $\theta$ and dual space $E^*$. Throughout this paper, we always assume $E$ is partially ordered by a given cone $K$, $E^*$ is partially ordered by $K_0$ where $K_0 := \{ f \in E : \langle X, f \rangle \geq 0 \text{ for any } X \in K \}$ is the positive dual cone of $K$. 


Remark 2.1. The partial order relation \( \geq_K \) is defined as follows, for any \( x, y \in E \)

\[
x \geq_K y \iff x - y \in K.
\]

Remark 2.2. The cone \( K \) is consisted of the ‘admissible’ price functionals. On the other hand, The cone \( K \) is also introduced to play the role of the solvency set of financial positions which denotes the way that a set of investors jointly interprets the common notion of the cost of financial positions.

We suppose that the numeraire asset is some interior point \( z \in \text{int}(K) \). The asset \( z \) is actually either a ‘reference cash stream’ according to Stoica (2006), or a ‘relatively secure cash stream’ according to Jaschke and Küchler (2001).

The Banach space valued Bochner-Lebesgue spaces with variable exponent were first introduced by Cheng and Xu (2013). Now, we will recall the definition and the related properties of this special space. We denote by \( S(\Omega, \mu) \) the set of all \( \mathcal{F} \)-measurable functions \( p(\cdot) : \Omega \to [1, \infty] \), which are called variable exponent on \( \Omega \). For a function \( p(\cdot) \in S(\Omega, \mu) \), we denote \( p'(\cdot) \in S(\Omega, \mu) \) by \( 1/p(y) + 1/p'(y) = 1 \).

The following definitions come from Cheng and Xu (2013).

Definition 2.1. A function \( f : \Omega \to E \) is strongly \( \mathcal{F} \)-measurable if there exists a sequence \( \{f_n\}_{n \geq 1} \) of \( \mu \)-simple functions converging to \( f \) \( \mu \)-almost everywhere.

Definition 2.2. The Bochner-Lebesgue space with variable exponent, which is denoted by \( L^{p(\cdot)}(\Omega, E) \), is the collection of all strongly \( \mathcal{F} \)-measurable functions \( f : \Omega \to E \) endowed with the norm

\[
\|f\|_{L^{p(\cdot)}(\Omega, E)} := \inf\{b > 0, \rho_{p(\cdot)}(f/b) \leq 1\}
\]

where

\[
\rho_{p(\cdot)}(f) := \int_{\Omega} \|f(y)\|^{p(y)}d\mu(y) \quad \text{and} \quad p(\cdot) \in S(\Omega, \mu).
\]

Remark 2.3. The variable exponent Bochner-Lebesgue space \( L^{p(\cdot)}(\Omega, E) \) was consisted of vector-valued measurable functions, which take values in a given Banach space \( E \). Moreover, if \( E \) is reflexive Banach space, then \( E^* \) is also reflexive. By Diestel and J.Uhl (1977), \( E^* \) has the Randon-Nikodym property. Under this condition, \( L^{p(\cdot)}(\Omega, E) \) is a reflexive Banach space itself. See Cheng and Xu (2013).

Remark 2.4. If \( E \) is a reflexive Banach space, then the dual of \( L^{p(\cdot)}(\Omega, E) \) is characterized by the mapping \( g \mapsto V_g \)

\[
L^{p(\cdot)}(\Omega, E^*) \rightarrow (L^{p(\cdot)}(\Omega, E))^* \quad \text{as follows}
\]

\[
\langle V_g, f \rangle = \int_{\Omega} \langle g, f \rangle d\mu, \quad \text{for any} \quad f \in L^{p(\cdot)}(\Omega, E).
\]

See Cheng and Xu (2013).
The variable exponent Lebesgue spaces appeared firstly in Orlicz (1931). Recent years, the studies of variable exponent Lebesgue spaces have attracted many attentions, especially the variable exponent Bochner-Lebesgue spaces, which was studied by Cheng and Xu (2013). In the space $L^{p(\cdot)}(\Omega, E)$, the order $p(\cdot)$ is no longer a fixed positive number, but a random variable, which belong to the set of $\mathcal{F}$-measurable functions.

From now on, we denote by $L^{p(\cdot)} := L^{p(\cdot)}(\Omega, E)$ the variable exponent Bochner-Lebesgue space. In this paper, we will use the space $L^{p(\cdot)}$ to describe the space of financial positions. This is based on two considerations. From the perspective of the markets, one is hard to evaluate a deterministic order $p$ due to the possible volatility of the markets. On the other hand, from the Bayesian statistical point, the order $p$ can be considered as a kind of parameter, and hence should be assumed to be a random variable.

3 The definition of systemic risk measures

During the most financial markets, systemic risk is defined as involving the risk of breakdown among institutions and other market participants in a chain-like fashion that has the potential to affect the entire financial system negatively. More concretely, the risk of 'domino effect' certainly seems central to the concept of systemic risk, as does the risk of some triggering event that causes the first domino to fall. In general, the systemic risk may refer to the potential for substantial volatility in asset prices, corporate liquidity, bankruptcies, and efficiency losses brought on by economic shocks.

Since the financial markets are becoming complicated, the systemic risk appears to be much more uncertain and volatile than before. Thus, we use the variable exponent Bochner-Lebesgue space $L^{p(\cdot)}$ to describe the systemic risk of the financial markets with uncertainty and volatility. However, measuring the systemic risk on $L^{p(\cdot)}$, in fact, is not straightforward. Instead of defining the systemic risk measures on $L^{p(\cdot)}$ directly, we would like to define two special functions first: the convex certain function and the simple-systemic risk measure. These two functions make the measurement of systemic risk into two steps: the certain function convert the uncertainty of systemic risk into certainty, then the simple-systemic risk measure quantify the risk which is simplified by the certain function.

Definition 3.1. A convex certain function is a function $\phi : E \to \mathbb{R}$ that satisfies $\phi(L^{p(\cdot)}) = L^p$ and the following properties,

A1 Monotonicity: for any $x, y \in E$, $x \geq_K y$ implies $\phi(x) \geq \phi(y)$;

A2 Convexity: for any $x, y \in E$ and $\lambda \in [0, 1]$, $\phi(\lambda x + (1 - \lambda)y) \leq \lambda\phi(x) + (1 - \lambda)\phi(y)$;

A3 Surjectivity: $\phi(E) = \mathbb{R}$.

Remark 3.1. The $E$ is a given Banach space which is defined in Section 2. And the order in A1 is the partial order under a cone $K$ which is defined by Remark 2.1, that is the Banach
space $E$ is partially ordered by a given cone $K$. Property A3 tells us that the function $\phi$ is a non-constant function with no lower bound.

Now, we consider a function $T_\phi : \mathbb{R} \to \mathbb{R}$ which is defined by

$$T_\phi(a) := \phi(az).$$

(3.1)

The next lemma provides a sufficient condition for $\phi$ as a function from $E$ to $\mathbb{R}$ to satisfy the requirement $\phi(L^{p(\cdot)}) = L^p$ from the definition of convex certain function.

**Lemma 3.1.** Suppose a certain function $\phi : E \to \mathbb{R}$ that satisfies the properties A1 – A2. If the following assumption holds, then $\phi(L^{p(\cdot)}) = L^p$.

(•) The function $T_\phi$ satisfies $\|T_\phi(Z)\|_p < \infty$ and $\|T_\phi^{-1}(Z)\|_p < \infty$ for any $Z \in L^p$.

**Proof.** For any $f \in L^{p(\cdot)}$, we consider a $Z$ which is defined by

$$Z(\omega) := \left\{ \begin{array}{ll} \max\{a|f(\omega) \leq_K az\}, & w \in A \\
\min\{a|az \leq_K f(\omega)\}, & w \in \Omega \setminus A \end{array} \right.$$ 

where $A = \{\omega \in \Omega \mid \phi(f(\omega)) \geq 0\}$. From the property of A1, we get

$$0 \leq \phi(f(\omega)) \leq \phi(Z(\omega)z) \text{ for any } w \in A$$

and

$$0 > \phi(f(\omega)) \geq \phi(Z(\omega)z) \text{ for any } w \in \Omega \setminus A.$$ 

This leads to

$$|\phi(f(\omega))| \leq |\phi(Z(\omega)z)| = |T_\phi(Z(\omega))| \text{ for any } w \in \Omega.$$ 

It follows that for any $f \in L^{p(\cdot)}$, there exists $Z \in L^p$ such that

$$\mathbb{E}[|\phi(f)|^p] \leq \mathbb{E}[|T_\phi(Z)|^p] \quad \text{for } p < \infty$$

and

$$\inf\{b \in \mathbb{R} \mid |\phi(f)| \leq b\} \leq \inf\{b \in \mathbb{R} \mid |T_\phi(Z)| \leq b\} \quad \text{for } p = \infty.$$ 

By the assumption (•), since $\|T_\phi(Z)\|_p < \infty$ for any $Z \in L^p$, we have

$$\|\phi(f)\|_p \leq \|T_\phi(Z)\|_p < \infty \text{ for any } f \in L^{p(\cdot)},$$

which means $\phi(L^{p(\cdot)}) \subseteq L^p$. For any $X \in L^p$, we can define $Y$ by

$$Y(\omega) := T_\phi^{-1}(X(\omega)) \text{ for all } \omega \in \Omega.$$ 

Since $\|T_\phi^{-1}(Z)\|_p < \infty$ for any $Z \in L^p$, it is not hard to check that $Y \in L^p$. Hence, there exists a vector $Yz \in L^{p(\cdot)}$ such that

$$\phi(Yz) = T_\phi(Y) = X,$$

which means that $L^p \subseteq \phi(L^{p(\cdot)})$ and we arrive at $\phi(L^{p(\cdot)}) = L^p$. \qed
Remark 3.2. Note that for any certain function $\phi$, which satisfies $A1 - A3$, $\|T_\phi(Z)\|_p < \infty$ and $\|T_\phi^{-1}(Z)\|_p < \infty$ for any $Z \in L^p$ are automatically satisfied.

In fact, the convex certain function $\phi$ is used to convert the uncertainty of systemic risk into certainty. Then, in order to measure the systemic risk on $L^{p(\cdot)}$, we still need a simple-systemic risk measure to quantify the risk which is simplified by the convex certain function.

Definition 3.2. A simple-systemic risk measure is a function $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ that satisfies the following properties,

B1 Monotonicity: for any $X, Y \in L^p$, $X \geq Y$ implies $\rho(X) \geq \rho(Y)$;

B2 Convexity: for any $X, Y \in L^p$ and $\lambda \in [0, 1]$, $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda) \rho(Y)$;

B3 Constancy: for any $a \in \mathbb{R}$, $\rho(a) = a$.

Remark 3.3. The properties B1 – B2 are very well known and have been studied in detail in the study of convex risk measures (see for instance, Föllmer and Schied 2002). The property B3 can be understood as a technical condition.

Furthermore, we consider a function $\rho : L^{p(\cdot)} \to \mathbb{R} \cup \{+\infty\}$ and denote $\rho_E$ the restriction of $\rho$ to $E$. Now, we will introduce the definition of systemic risk measures on variable exponent Bochner-Lebesgue space $L^{p(\cdot)}$ by axiomatic approach.

Definition 3.3. A systemic risk measure is a function $\rho : L^{p(\cdot)} \to \mathbb{R} \cup \{+\infty\}$ that satisfies $\rho_E(L^{p(\cdot)}) = L^p$ and the following properties,

C1 Monotonicity: for any $f, g \in L^{p(\cdot)}$, $f \geq_K g$ implies $\rho(f) \geq \rho(g)$;

C2 Preference consistency: If $\rho(f(\omega)) \leq \rho(g(\omega))$ for all $\omega \in \Omega$, then $\rho(f) \leq \rho(g)$;

C3 Convexity: for any $f, g \in L^{p(\cdot)}$ and $\lambda \in [0, 1]$, $\rho(\lambda f + (1 - \lambda)g) \leq \lambda \rho(f) + (1 - \lambda) \rho(g)$;

C4 Risk convexity: if $\rho(h(\omega)) = \lambda \rho(f(\omega)) + (1 - \lambda) \rho(g(\omega))$ for a given scalar $\lambda \in [0, 1]$ and for all $\omega \in \Omega$, then $\rho(h) \leq \lambda \rho(f) + (1 - \lambda) \rho(g)$;

C5 Surjectivity: $\rho(E) = \mathbb{R}$.

Remark 3.4. The properties C1 and C3 can be interpreted in the same way as in the definition of simple-systemic risk measures. The property C2 means that if the risk of economy $f(\omega) \in E$ is greater than the risk of economy $g(\omega) \in E$ for almost all $\omega \in \Omega$, then the risk of the random economy $f \in L^{p(\cdot)}$ should be greater than the risk of the random economy $g \in L^{p(\cdot)}$. The property C4 tells us that if the risk of the economy $h(\omega)$ is the convex combination of the risk of the economies $f(\omega)$ and $g(\omega)$ for all $\omega \in \Omega$, then the risk of the random economy $h \in L^{p(\cdot)}$ is at most the risk of the convex combination of the risks of the random economies $f, g \in L^{p(\cdot)}$. The condition $\rho_E(L^{p(\cdot)}) = L^p$ is a technical requirement.
The properties C5 and A3 of the corresponding functions are closely linked and we will need those properties for our decomposition of the measurement in the following section, i.e. \( \phi(E) = \mathbb{R} = \rho(E) \) is needed.

We will see in the following section that each systemic risk measure on \( L^p(\cdot) \) can be decomposed into a convex certain function \( \phi \) and a simple-systemic risk measure \( \varrho \). In other words, the following section will show that the measurement of systemic risk on \( L^p(\cdot) \) can be simplified into two steps.

4 How to measure the systemic risk on \( L^p(\cdot) \)

Since the main focus of this paper is to study the systemic risk measures on \( L^p(\cdot) \), the question of how to measure the systemic risk on \( L^p(\cdot) \) becomes especially critical.

In this section, we will provide a structural decomposition result which show that any systemic risk measure on \( L^p(\cdot) \) can be decomposed into a convex certain function and a simple-systemic risk measure. Furthermore, we also show that any convex certain function and simple-systemic risk measure can aggregate into a systemic risk measure on \( L^p(\cdot) \).

Theorem 4.1. A function \( \rho: L^p(\cdot) \to \mathbb{R} \cup \{+\infty\} \) is a systemic risk measure if and only if there exists a convex certain function \( \phi: E \to \mathbb{R} \) and a simple-systemic risk measure \( \varrho: L^p \to \mathbb{R} \cup \{+\infty\} \) such that \( \rho \) is the composition of \( \varrho \) and \( \phi \), i.e.

\[
\rho(f) = (\varrho \circ \phi)(f) \quad \text{for all} \quad f \in L^p(\cdot).
\] (4.1)

Proof. We first show the ’only if’ part. Suppose \( \rho \) is a systemic risk measure and define a function \( \phi \) by

\[
\phi(x) := \rho(x)
\] (4.2)

for any \( x \in E \). Since \( \rho \) satisfies the convexity C3, it follows

\[
\phi(\lambda x + (1 - \lambda)y) = \rho(\lambda x + (1 - \lambda)y) \leq \lambda \rho(x) + (1 - \lambda)\rho(y) = \lambda \phi(x) + (1 - \lambda)\phi(y)
\]

for any \( x, y \in E \) and \( \lambda \in [0, 1] \). Thus, \( \phi \) satisfies the convexity A2. Similarly, the monotonicity A1 of \( \phi \) can also be implied by the monotonicity C1 of \( \rho \). Since \( \rho \) satisfies the surjectivity C5, it follows immediately from [1,2] that \( \phi \) satisfies surjectivity A3. Moreover, by the definition of systemic risk measure, \( \rho_E(L^p(\cdot)) = L^p \), it follows again from [1,2] that \( \phi \) satisfies \( \phi(L^p(\cdot)) = L^p \). Thus, \( \phi \) is a convex certain function.

Next, we consider a function \( \varrho: \phi(L^p(\cdot)) \to \mathbb{R} \cup \{+\infty\} \), which is defined by

\[
\varrho(X) := \rho(f) \quad \text{where} \quad f \in L^p(\cdot) \text{ with } \phi(f) = X.
\] (4.3)

Now, we need to show that \( \varrho \) is well defined. Suppose \( f, g \in L^p(\cdot) \) with \( \phi(f) = \phi(g) \), we have

\[
\rho(f(\omega)) = \phi(f)(\omega) \geq \phi(g)(\omega) = \rho(g(\omega))
\]
and
\[ \rho(f(\omega)) = \phi(f)(\omega) \leq \phi(g)(\omega) = \rho(g(\omega)) \]
for all \( \omega \in \Omega \). Thus, by the property \( \text{C2} \) of \( \rho \), we have \( \rho(f) = \rho(g) \), which means \( g \) is well defined. Next, we want to show that the \( \varrho \) defined above is a simple-systemic risk measure. Suppose \( X, Y \in \phi(L^p(\cdot)) \) with \( X \geq Y \), there exists \( f, g \in L^p(\cdot) \) such that \( \phi(f) = X, \phi(g) = Y \). Then, we have
\[ \rho(f(\omega)) = \phi(f)(\omega) \geq \phi(g)(\omega) = \rho(g(\omega)) \]
for all \( \omega \in \Omega \). Thus, it follows again from the property \( \text{C2} \) of \( \rho \) that
\[ \varrho(X) = \rho(f) \geq \rho(g) = \varrho(Y) \]
which implies \( \varrho \) satisfies the monotonicity \( \text{B1} \). Let \( X, Y \in \phi(L^p(\cdot)) \) and \( \lambda \in [0, 1] \), we consider
\[ Z := \lambda X + (1 - \lambda)Y. \]
Suppose that \( f, g, h \in L^p(\cdot) \) such that
\[ \varrho(X) = \rho(f), \varrho(Y) = \rho(g), \varrho(Z) = \rho(h) \]
with \( \phi(f) = X, \phi(g) = Y \) and \( \phi(h) = Z \). Then
\[
\begin{align*}
\rho(h(\omega)) &= \phi(h)(\omega) \\
&= Z(\omega) \\
&= \lambda X(\omega) + (1 - \lambda) Y(\omega) \\
&= \lambda \phi(f)(\omega) + (1 - \lambda) \phi(g)(\omega) \\
&= \lambda \rho(f(\omega)) + (1 - \lambda) \rho(g(\omega))
\end{align*}
\]
for all \( \omega \in \Omega \). Thus, the property \( \text{C4} \) of \( \rho \) yields
\[ \varrho(Z) = \rho(h) \leq \lambda \rho(f) + (1 - \lambda) \rho(g) = \lambda \varrho(X) + (1 - \lambda) \varrho(Y), \]
which means \( \varrho \) satisfies the convexity \( \text{B2} \). From the property \( \text{A3} \) of \( \phi \) it follows that for any \( a \in \mathbb{R} \), there exists \( x \in E \) such that \( \phi(x) = a \). Then, we have \( \varrho(a) = \rho(x) \). Thus, \( \rho(x) = \phi(x) = a \) and this implies \( \varrho(a) = a \) for any \( a \in \mathbb{R} \), which means that \( \varrho \) satisfies the property \( \text{B3} \). Thus, \( \varrho \) is a simple-systemic risk measure and from (4.2) and (4.3), we have \( \rho = \varrho \circ \phi \).

Next, we will show the ‘ if ’ part. Suppose \( \phi \) is a convex certain function and \( \varrho \) is a simple-systemic risk measure. Furthermore, define \( \rho = \varrho \circ \phi \). Since \( \varrho \) and \( \phi \) are monotone and convex, it is not hard to check that \( \rho \) satisfies monotonicity \( \text{C1} \) and convexity \( \text{C3} \). Now, suppose \( f, g \in L^p(\cdot) \) which satisfies
\[ (\varrho \circ \phi)(f(\omega)) = \rho(f(\omega)) \geq \rho(g(\omega)) = (\varrho \circ \phi)(g(\omega)) \]
for all \( \omega \in \Omega \). Then, the property \( \text{B3} \) of \( \varrho \) implies
\[ \phi(f(\omega)) \geq \phi(g(\omega)) \]
for all \( \omega \in \Omega \), which means \( \phi(f) \geq \phi(g) \). Hence, by the property \( \text{B1} \) of \( \varrho \), we have
\[ \rho(f) = (\varrho \circ \phi)(f) \geq (\varrho \circ \phi)(g) = \rho(g), \]
which yields $\rho$ satisfies the property $C_2$. Next, we will show that $\rho$ satisfies the property $C_4$. To this end, we suppose $f, g, h \in L^p$ and $\lambda \in [0, 1]$ with

$$\rho(h(\omega)) = \lambda \rho(f(\omega)) + (1 - \lambda) \rho(g(\omega))$$

for all $\omega \in \Omega$. This means

$$(\phi \circ \phi)(h(\omega)) = \lambda (\phi \circ \phi)(f(\omega)) + (1 - \lambda) (\phi \circ \phi)(g(\omega))$$

for all $\omega \in \Omega$. Then, the property $B_3$ of $\phi$ implies

$$\phi(h(\omega)) = \lambda \phi(f(\omega)) + (1 - \lambda) \phi(g(\omega))$$

for all $\omega \in \Omega$, which yields $\phi(h) = \lambda \phi(f) + (1 - \lambda) \phi(g)$. Hence, by the property $B_2$ of $\phi$, we have

$$\rho(h) = (\phi \circ \phi)(h) \leq \lambda (\phi \circ \phi)(f) + (1 - \lambda) (\phi \circ \phi)(g) = \lambda \rho(f) + (1 - \lambda) \rho(g),$$

which means that $\rho$ satisfies the property $C_4$. Now, we only need to show that $\rho$ satisfies the property $C_5$. By the property $B_3$ of $\phi$ and the property $A_3$ of $\phi$, we have

$$\rho(E) = \phi(\phi(E)) = \phi(\mathbb{R}) = \mathbb{R},$$

which is just the property $C_5$ of $\rho$. Thus, the $\rho$ defined above is a systemic risk measure.

**Remark 4.1.** Theorem 4.1 not only provide a decomposition result for systemic risk measure on $L^p$, but also propose a idea to deal with the systemic risk on a market with uncertainty and volatility. More concretely, we first use the convex certain function $\phi$ to convert the uncertainty of systemic risk into certainty, then we quantify the simplified risk by the simple-systemic risk measure. This means that a regulator who deal with the measurement of this systemic risk can construct a reasonable systemic risk measure by choosing an appropriate certain function and an appropriate simple-systemic risk measure. The certain function should reflect his preferences towards the uncertainty and volatility of the financial markets.

In the following section, we will study the dual representation of the systemic risk measures on $L^p$ with the help of the acceptance sets of $\phi$ and $\phi$.

## 5 Dual representation

Before we study the dual representation of the systemic risk measures on $L^p$, the acceptance sets should be defined. Since every systemic risk measure $\rho$ can be decomposed into a convex certain function $\phi$ and a simple-systemic risk measure $\phi$, we only need to define the acceptance sets of $\phi$ and $\phi$, i.e.

$$\mathcal{A}_\phi := \{(c, X) \in \mathbb{R} \times L^p : \phi(X) \leq c\}$$

(5.1)
and

\[ A_\phi := \{(Y, f) \in L^p \times L^{p(\cdot)} : \phi(f) \leq Y\}. \quad (5.2) \]

We will see later on that these acceptance sets can be used to provide the dual representation of systemic risk measures on \( L^{p(\cdot)} \). The following properties are needed in the subsequent study.

**Definition 5.1.** Let \( M \) and \( N \) be two ordered linear spaces. A set \( A \subset M \times N \) satisfies \( f \)-monotonicity if \((m, n) \in A, q \in N \) and \( n \geq q \) imply \((m, q) \in A\). A set \( A \subset M \times N \) satisfies \( b \)-monotonicity if \((m, n) \in A, p \in M \) and \( p \geq m \) imply \((p, n) \in A\).

**Proposition 5.1.** Suppose \( \rho = \varrho \circ \phi \) is a systemic risk measure with convex certain function \( \phi : E \to \mathbb{R} \) and a simple-systemic risk measure \( \varrho : L^p \to \mathbb{R} \cup \{+\infty\} \). The corresponding acceptance sets \( A_{\varrho} \) and \( A_{\phi} \) are defined by (5.1) and (5.2). Then, \( A_{\varrho} \) and \( A_{\phi} \) are convex sets and they satisfy the \( f \)-monotonicity and \( b \)-monotonicity.

**Proof.** It is not hard to check the properties by the definition of \( \phi \) and \( \varrho \).

The next proposition provides the primal representation of systemic risk measures on \( L^{p(\cdot)} \) at the point of acceptance sets. This result will be used for the dual representation of the systemic risk measures on \( L^{p(\cdot)} \).

**Proposition 5.2.** Suppose \( \rho = \varrho \circ \phi \) is a systemic risk measure with convex certain function \( \phi : E \to \mathbb{R} \) and a simple-systemic risk measure \( \varrho : L^p \to \mathbb{R} \cup \{+\infty\} \). The corresponding acceptance sets \( A_{\varrho} \) and \( A_{\phi} \) are defined by (5.1) and (5.2). Then, for any \( f \in L^{p(\cdot)} \),

\[ \rho(f) = \inf \{ c \in \mathbb{R} : (c, X) \in A_{\varrho}, (X, f) \in A_{\phi} \} \quad (5.3) \]

where we set \( \inf \emptyset = +\infty \).

**Proof.** Since \( \rho = \varrho \circ \phi \), we have

\[ \rho(f) = \inf \{ c \in \mathbb{R} : (\varrho \circ \phi)(f) \leq c \}. \quad (5.4) \]

By the definition of \( A_{\varrho} \), we know that \( \varrho(X) = \inf \{ c \in \mathbb{R} : (c, X) \in A_{\varrho} \} \).

for all \( X \in L^p \). Then, from (5.4) and (5.5),

\[ \rho(f) = \inf \{ c \in \mathbb{R} : (c, \phi(f)) \in A_{\phi} \}. \]

It is not hard to check that

\[ \{ c \in \mathbb{R} : (c, \phi(f)) \in A_{\phi} \} = \{ c \in \mathbb{R} : (c, X) \in A_{\varrho}, (X, f) \in A_{\phi} \}. \]

Thus,

\[ \rho(f) = \inf \{ c \in \mathbb{R} : (c, X) \in A_{\varrho}, (X, f) \in A_{\phi} \}. \]

Now, with the help of Proposition 5.2, we will introduce the main result of this section: the dual representation of the systemic risk measures on \( L^{p(\cdot)} \).
**Theorem 5.1.** Suppose \( \rho = \varrho \circ \phi \) is a systemic risk measure characterized by a lower-\( \alpha \)-seminontinuity \( \text{simple-systemic risk measure} \ \rho \) and a \( \text{continue convex certain function} \ \phi \). Then, for any \( f \in L^p(\cdot), \ \rho(f) \) is of the following form

\[
\rho(f) = \sup_{(\hat{f}, \hat{Y}) \in P} \left\{ \langle \hat{f}, f \rangle - \alpha(\hat{Y}, \hat{f}) \right\} \tag{5.6}
\]

where \( \alpha : L^q \times (L^p(\cdot))^* \to \mathbb{R} \cup \{+\infty\} \) is defined by

\[
\alpha(\hat{Y}, \hat{f}) := \sup_{(c,Y) \in A_{\varrho}} \left\{ -c - \langle \hat{Y}, (Y - X) \rangle + \langle \hat{f}, g \rangle \right\}
\]

and

\[
P := \{ (\hat{Y}, \hat{f}) \in L^q \times (L^p(\cdot))^*, \alpha(\hat{Y}, \hat{f}) < \infty \}.
\]

**Proof.** By Proposition 5.2, we have

\[
\rho(f) = \inf_{(c,X) \in A_{\varrho}} \{ c + I_{A_{\varrho}}(c, X) \in A_{\phi} \}
\]

for any \( f \in L^p(\cdot) \). Furthermore, we can rewritten it by

\[
\rho(f) = \inf_{(c,X) \in \mathbb{R} \times L^q} \{ c + I_{A_{\varrho}}(c, X) + I_{A_{\phi}}(X, f) \} \tag{5.7}
\]

where the indicator function of a set \( A \in \mathcal{X} \times \mathcal{Y} \) is defined by

\[
I_A(x, y) := \begin{cases} 0, & (x, y) \in \mathcal{X} \times \mathcal{Y} \\ \infty, & \text{otherwise} \end{cases}
\]

From Proposition 5.1, we know that \( A_{\varrho} \) and \( A_{\phi} \) are convex sets. Thus,

\[
I'_{A_{\varrho}}(\hat{c}, \hat{X}) = \sup_{(\hat{c}, \hat{X}) \in A_{\varrho}} \{ \hat{c} \hat{X} + \langle \hat{X}, X \rangle \}, \ \hat{c} \in \mathbb{R}, \ \hat{X} \in L^q
\]

and

\[
I'_{A_{\phi}}(\hat{Y}, \hat{f}) = \sup_{(\hat{Y}, \hat{f}) \in A_{\phi}} \{ \hat{Y} \hat{f} + \langle \hat{f}, \hat{f} \rangle \}, \ \hat{Y} \in L^q, \ \hat{f} \in (L^p(\cdot))^*.
\]

On the other hand, since \( \varrho \) is lower-\( \alpha \)-seminontinue, it follows that \( A_{\varrho} \) is closed. Thus, by the duality theorem for conjugate functions, we have

\[
I_{A_{\varrho}}(c, X) = I''_{A_{\varrho}}(c, X) = \sup_{(\hat{c}, \hat{X}) \in \mathbb{R} \times L^q} \{ \hat{c} c + \langle \hat{X}, X \rangle - I'_{A_{\varrho}}(\hat{c}, \hat{X}) \}
\]

\[
= \sup_{(\hat{c}, \hat{X}) \in \mathbb{R} \times L^q} \left\{ \hat{c} c + \langle \hat{X}, X \rangle - \sup_{(\hat{c}, \hat{X}) \in A_{\varrho}} \{ \hat{c} c + \langle \hat{X}, X \rangle \} \right\}.
\]

Similarly, we have

\[
I_{A_{\phi}}(X, f) = I''_{A_{\phi}}(X, f) = \sup_{(\hat{Y}, \hat{f}) \in L^q \times (L^p(\cdot))^*} \{ \langle \hat{Y}, X \rangle + \langle \hat{f}, f \rangle - I'_{A_{\phi}}(\hat{Y}, \hat{f}) \}
\]

\[
= \sup_{(\hat{Y}, \hat{f}) \in L^q \times (L^p(\cdot))^*} \{ \langle \hat{Y}, X \rangle + \langle \hat{f}, f \rangle - \sup_{(\hat{Y}, \hat{f}) \in A_{\phi}} \{ \langle \hat{Y}, Y \rangle + \langle \hat{f}, f \rangle \} \}.
\]
Thus, we have
\[
\rho(f) = \inf_{(c, X) \in \mathbb{R} \times L^p} \left\{ c + I_{A_\phi}(c, X) + I_{A_\phi}(X, f) \right\}
\]
\[
= \inf_{(c, X) \in \mathbb{R} \times L^p} \sup_{(\hat{c}, \hat{X}) \in \mathbb{R} \times L^q} \left\{ c(1 + \hat{c}) + \langle \hat{X} + \hat{Y}, X \rangle + \langle \hat{f}, f \rangle - I'_{A_\phi}(\hat{c}, \hat{X}) - I'_{A_\phi}(\hat{Y}, \hat{f}) \right\}.
\]

By the Theorem 7 of Rockafellar (1974), because of the lower-semicontinuity of \(\phi\) and the continuity of \(\phi\), we can interchange the supremum and the infimum above, i.e.
\[
\rho(f) = \sup_{(\hat{c}, \hat{X}) \in \mathbb{R} \times L^q} \inf_{(c, X) \in \mathbb{R} \times L^p} \left\{ c(1 + \hat{c}) + \langle \hat{X} + \hat{Y}, X \rangle + \langle \hat{f}, f \rangle - I'_{A_\phi}(\hat{c}, \hat{X}) - I'_{A_\phi}(\hat{Y}, \hat{f}) \right\}.
\]

With \(\alpha(\hat{Y}, \hat{f})\) is defined by
\[
\alpha(\hat{Y}, \hat{f}) := \sup_{(\tau, X) \in A_\phi} \left\{ -\tau - \langle \hat{Y}, \hat{X} - \tau \rangle + \langle \hat{f}, \hat{f} \rangle \right\}
\]
and
\[
P := \left\{ (\hat{Y}, \hat{f}) \in L^q \times (L^p)\,^*, \alpha(\hat{Y}, \hat{f}) < \infty \right\},
\]
it immediately follows that
\[
\rho(f) = \sup_{(\hat{Y}, \hat{f}) \in P} \left\{ \langle \hat{f}, f \rangle - \alpha(\hat{Y}, \hat{f}) \right\}.
\]

**Remark 5.1.** Note that, the proof of Theorem 5.1 above utilized the primal representation of systemic risk measures in Proposition 5.2, which means that the acceptance sets \(A_\phi\) and \(A_\phi\) played a vital in the proof of dual representation of systemic risk measures. Thus, the dual representation of systemic risk measures \(\rho\) on \(L^p\) still dependent on the convex certain function \(\phi\) and a simple-systemic risk measure \(\phi\).

**References**

[1] Acharya, V., Pedersen, L., Philippon, T., M., R., Measuring systemic risk. CEPR Discussion Paper 8824, 2012. [http://www.cepr.org/pubs/dps/DP8824.asp](http://www.cepr.org/pubs/dps/DP8824.asp)

[2] Almeida, A., Hasanov, J., Samko, S., Maximal and potential operators in variable exponent Morrey spaces, Georgian Math. J., 15, 195-208, 2008.
[3] Armenti, Y., Crepey, S., Drapeau, S., Papapantoleon, A., Multivariate shortfall risk allocation and systemic risk, arXiv: 1507.05351, 2015.

[4] Artzner, P., Dellbaen, F., Eber, J.M., Heath, D., Thinking coherently, Risk, 10, 68-71, 1997.

[5] Artzner, P., Dellbaen, F., Eber, J.M., Heath, D., Coherent measures of risk, Math. Finance, 9(3), 203-228, 1999.

[6] Biagini, F., Fouque, J.P., Frittelli. M., A unified approach to systemic risk measures via acceptance sets, arXiv: 1503.06354, 2015.

[7] Brunnermeier, M.K., Cheridito, P., Measuring and allocating systemic risk, 2014. http://ssrn.com/abstract=2372472

[8] Chen, C., Iyengar, G., Moallemi, C. C., An axiomatic approach to systemic risk, Management Science, 59(6), 1373-1388, 2013.

[9] Cheng, C., Xu, J., Geometric properties of Banach space valued Bochner-Lebesgue spaces with variable exponent, Journal of Mathematical Inequalities, 7(3), 461-475, 2013.

[10] Diestel, J., Uhl, J. J., Jr, Vector Measures, Amer. Math. Soc., 1977.

[11] Diening, L., Hästö, P., Roudenko, S., Function spaces of variable smoothness and integrability, J. Funct. Anal., 256, 1731-1768, 2009.

[12] Feinstein, Z., Rudloff, B., Weber, S., Measures of systemic risk, arXiv: 1502.07961, 2015.

[13] Föllmer, H., Schied, A., Convex measures of risk and trading constrains. Finance Stoch., 6, 429-447, 2002.

[14] Gauthier, C., Lehar, M., Souissi, M., Macroprudential capital requirements and systemic risk, J. Financ Intermed, 21(4), 594-618, 2012.

[15] Harjulehto, P., Hästö, P., Lê, Út V., Nuortio, M., Overview of differential equations with nonstandard growth, Nonlinear Anal., 72, 4551-4574, 2010.

[16] Hästö, P., Local-to-global results in variable exponent spaces, Math. Res. Lett., 16, 263-278, 2009.

[17] Jaschke, S., Küchler, U., Coherent risk measures and good-deal bounds, Finance Stoch., 5, 181-200, 2001.

[18] Kempka, H., 2-microlocal Besov and Triebel-Lizorkin spaces of variable integrability, Rev. Mat. Complut, 22, 227-251, 2009.

[19] Kempka, H., Atomic, molecular and wavelet decomposition of generalized 2-microlocal Besov spaces, J. Funct. Spaces Appl., 8, 129-165, 2010.
[20] Kováčik, O., Rákosník, J., On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czech Math. J., 41, 592-618, 1991.

[21] Musielak, J., Orlicz Spaces and Modular Spaces, Springer Verlag, Berlin, 1983.

[22] Nakano, H., Modulared semi-ordered linear spaces, Maruzen, Tokyo, 1950.

[23] Orlicz, W., Über konjugierte Exponentenfolgen. Studia Math., 3, 200-211, 1931.

[24] Rockafellar, R., Conjugate duality and optimization. CBMS-NSF regional conference series in applied mathematics, Society for Industrial and Applied Mathematics, 1974.

[25] Stoica, G., Relevant coherent measures of risk, J. Math. Econom., 42, 794-806, 2006.

[26] Tarashev, N., Borio, C., Tsatsaronis, K., Attributing systemic risk to individual institutions. Working Paper No. 308. Bank for International Settlements, Basel, 2010. [http://www.bis.org/publ/work308.pdf](http://www.bis.org/publ/work308.pdf)

[27] Xu, J., Variable Besov and Triebel-Lizorkin spaces, Ann. Acad. Sci. Fenn. Math., 33, 511-522, 2008.

[28] Xu, J., An atomic decomposition of variable Besov and Triebel-Lizorkin spaces, Armenian J. Math., 2, 1-12, 2009.