Critical thickness of an optimum extended surface characterized by uniform heat transfer coefficient

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March 18, 2015

Abstract

We consider the heat transfer problem associated with a periodic array of extended surfaces (fins) subjected to convection heat transfer with a uniform heat transfer coefficient. Our analysis differs from the classical approach as (i) we consider two-dimensional heat conduction and (ii) the base of the fin is included in the heat transfer process. The problem is modeled as an arbitrary two-dimensional channel whose upper surface is flat and isothermal, while the lower surface has a periodic array of extensions/fins which are subjected to heat convection with a uniform heat transfer coefficient. Using the generalized Schwarz-Christoffel transformation the domain is mapped onto a straight channel where the heat conduction problem is solved using the boundary element method. The boundary element solution is subsequently used to pose a shape optimization problem, i.e. an inverse problem, where the objective function is the normalized Shape Factor and the variables of the optimization are the parameters of the Schwarz-Christoffel transformation. Numerical optimization suggests that the optimum fin is infinitely thin and that there exists a critical Biot number that characterizes whether the addition of the fin would result in an enhancement of heat transfer. The existence of a critical Biot number was investigated for the case of rectangular fins. It is concluded that a rectangular fin is effective if its thickness is less than $1.64 \frac{k}{h}$, where the $h$ is the heat transfer coefficient and $k$ is the thermal conductivity. This result is independent of both the thickness of the base and the length of the fin.

Keywords

Optimum fin design; shape Optimization; Inverse Design; Heat Convection; uniform heat transfer coefficient; generalized Schwarz-Christoffel transformation; Laplace equation.

1 Introduction

Assuming isothermal boundary conditions, any extension from a surface would result in a reduction in heat transfer rate. This statement can be deduced from geometrical inclusion theorems [1]. It implies that the heat transfer rate across a two-dimensional channel, bounded by isothermal surfaces, is higher than that across a similar channel whose surfaces are extended, i.e. by adding fins [2], assuming that the temperature difference between the surfaces remains the same. Hence, within the approximation of isothermal conditions and constant temperature difference, the addition of fins would simply reduce the heat transfer rate! Because isothermal conditions can be realized in the limit of infinite Biot number, the following

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question is raised: **When is an extended surface (fin) effective?** This question will be elucidated in this paper, where we show that the limit of infinite Biot number is singular \[3, 4\], and a critical thickness exists \[5\].

This paper considers the fundamental problem of finding the optimum shape of an extended surface/fin such that the heat transfer rate is maximized. In particular, we consider the inverse design problem associated with two-dimensional (2D) heat conduction in a finite 2D periodic channel/slab with a flat isothermal upper boundary and a periodic lower boundary which is subjected to convection with a constant heat transfer coefficient, i.e. convection heat transfer is only considered to the extent that it provides a boundary condition for the conduction problem. The heat flux is proportional to the temperature difference between the surface and the far field \[2\] i.e.

\[
k \frac{\partial T}{\partial n} = h (T_\infty - T_{\text{surface}}).
\]

Isothermal conditions can be realized in the limit of strong convection, i.e. large Biot number. Here, we should point out that isothermal conditions have been widely used in the heat transfer analysis of extended surfaces due to the very little information available on the coupling
between fin conduction and fluid convection, and the weak dependence of the heat transfer coefficient on the temperature difference between the base of the fin and its tip ([7]; [6], 4.36; [8], 4.5).

As we have mentioned, the optimization problem is an inverse design problem in the sense that the objective is to find the geometry that maximizes the heat transfer rate as opposed to the classical/direct problem, where the objective is to find the heat transfer rate associated with a given geometry. The objective function is the Shape Factor [21][9], i.e. the total heat transfer rate, and the variable of the optimization is the shape of the pipe which is parameterized through the parameters of the generalized Schwarz-Christoffel transformation. Hence, using Geometry Parametrization [10, 11, 12, 13], the Shape Optimization problem is posed as a nonlinear programming problem (constrained nonlinear optimization [14]), which is solved numerically [15].

For regular, symmetric, isothermal, doubly-periodic walls, the heat conduction problem in a semi-infinite domain was addressed by Fyrillas & Pozrikidis [17] using both boundary integral and asymptotic methods. For 2D periodic channels/slabs, the problem has been addressed by Brady & Pozrikidis [16] where the authors considered the heat conduction problem associated with irregular isothermal periodic surfaces, using the generalized Schwarz-Christoffel transformation developed by Davis [19], Floryan [20] and Floryan & Zemach [21]. It was concluded that for regular geometries the shape plays a more important role in the total transport rate rather than the total arc-length while, for self-similar irregularities, the height of the roughness is the significant factor. These conclusions lead naturally to the following question: Given the arc-length and the period of a periodic surface/curve, what is the geometry that maximizes the overall transport rate; This Shape Optimization problem was addressed by Leontiou, Kotsonis & Fyrillas [18].

From an engineering perspective, knowledge of a surface geometry that maximizes the transport rate offers opportunities for new designs that exhibit enhanced characteristics and properties. For example, the problem of transport across an uneven surface, described in the preceding paragraph, is relevant to a variety of engineering applications involving heat transfer across rough and irregular boundaries, such as the surface of a circuit board in microelectronics [22, 23, 24, 25]. In general, heat transfer in slab-like configurations is of interest to problems associated with Heat Transport from Extended Surfaces (Fins) [8] and inverted high conductivity fins/inserts [26]. Published work in these thematic areas [27, 28, 29, 30] suggests that there is potential for significant improvements if one considers a two-dimensional (2-D) heat conduction model as suggested by Aziz [31].

In Section §2 we address the heat transfer problem associated with a periodic array of periodic extensions/fins of uniform convection heat transfer coefficient. In Section §3 we address the shape optimization problems associated with the optimum shape of the fins such that the heat transfer rate is maximized. In particular, in Section §3.2 we investigate under what conditions a rectangular fin enhances the heat transfer rate.

2 Shape factor of a periodic array of extended surfaces (fins)

In this section we consider 2-D heat conduction in a finite slab. The geometry of the slab is periodic in the horizontal direction and bounded in the vertical direction by an isothermal ($T_0$) flat surface at the top, while the bottom surface is subjected to convection with a uniform convection heat transfer coefficient ($h$) and a constant far-field temperature $T_\infty$ [2]. The
bottom surface is not flat, rather a periodic array of extensions is present (extended surfaces, fins), in order to enhance the heat transfer rate [2, 8]. Continuity of the heat flux at the bottom surface implies that the heat conduction rate, \( k \cdot \nabla T = k \frac{\partial T}{\partial n} \), must be equal to the heat convection rate \( h(T - T_\infty) \), where \( k \) is the thermal conductivity.

We non-dimensionalize lengths with the distance between the fins (period \( L \)), and the temperature by subtracting \( T_0 \) and dividing by the temperature difference \( T_\infty - T_0 \). The dimensional analysis leads to the following definition for the Biot number, \( Bi = L \frac{h}{k} \). The domain and the dimensionless parameters associated with the problem are clearly indicated in Fig. 1.

![Figure 1](image_url)

Figure 1: Schematic representation of the problem in the physical domain. All variables are non-dimensional; lengths are non-dimensionalized with the distance between the periodic fins, i.e. period \( L \). The dimensionless thickness of the base of the fin is \( H_b \), and the length of the fin is \( H_f = H - H_b \). The non-dimensional temperatures are \( T = 0 \) at the top boundary and \( T_\infty = 1 \) at the far field.

## 2.1 Conformal transformation of the physical domain into a straight channel

To find the heat transfer rate of such a periodic slab we first transform it into a straight channel [32, 33, 18, 4, 35]. The relevant transformation, the generalized Schwarz-Christoffel transformation applicable for periodic channels, has been developed by Davis and Floryan [19, 20]

\[
\begin{align*}
  z(\hat{w}; \alpha) &= R \int_{\hat{w}_0}^{\hat{w}_N} \prod_{l=-\infty}^{l=\infty} \prod_{j=0}^{j=N} \left( \sinh \left[ \frac{w_N \pi}{2h} (\hat{\theta} - \hat{w}_j - l) \right] \right)^{\alpha_j} d\hat{\theta}, \\
  \end{align*}
\]

where the inner product identifies the number of elements \( N \) of the discretized lower boundary (fin), and the infinite outer product the periodic nature of the domain. We normalize lengths in the complex domain with \( w_N \), i.e. the period in the transformed domain. Hence, in the above transformation, \( \hat{w}_j \) are the normalized images of the \( z_j \) vertices, \( \alpha \) represents the \( N+1 \)-tuple \( \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_N \) which are equal to the turning angles multiplied by \( \pi \) (the angles are taken to be positive for a clockwise rotation, and \( \alpha_0 \) and \( \alpha_N \) are defined with respect to the \( x \)-axis), \( R \) is a complex constant, and \( \hat{h} \) is the normalized height of the channel in the transformed domain (without loss of generality we assume that \( h = H \)). For the configurations we will consider \( R \) is a real number and can be obtained by requiring that the
upper wall of the physical plane, i.e. the line $z = i \, H$, transforms to $w = i \, h$:

$$\text{Im} \left[ z [i \, \hat{h}; \alpha] \right] = H.$$  \hspace{1cm} (2)

In addition, in view of the geometry, we must have

$$\sum_{j=0}^{N} \alpha_j = 0.$$  \hspace{1cm} (3)

Given the domain, the parameters of the transformation (1) can be calculated by solving a system of non-linear equations [19, 20]. However, in this work, we pose an optimization problem where the lower boundary is the variable of the optimization. In particular, we look for the optimum shape of the extended surface such that the heat transport rate is maximized. We pose the problem with respect to the parameters $\alpha$’s of the Schwarz-Christoffel transformation. Essentially, we parameterize the lower boundary [10, 11, 12, 13] with respect to the parameters $\alpha$’s, which are the variables of the optimization procedure, while the objective function is the heat transfer rate (the Shape Factor). An expression for the Shape Factor is obtained in the following section using the boundary element method.

2.2 Shape Factor of an extended surface (fin)

In view of the conformal transformation and the boundary condition on the lower surface we can obtain the following expressions for the Shape Factor ($S$) [2] associated with an extended surface of unit span:

$$S = - \int_{0}^{1} \frac{\partial T}{\partial \hat{\eta}} [\hat{\xi}, \hat{\eta} = 0] \, d\hat{\xi} = Bi \int_{0}^{1} \left( 1 - T [\hat{\xi}, \hat{\eta} = 0] \right) \left| \frac{dz}{dw} \right|_{\hat{w} = \hat{\xi}} \, d\hat{\xi}. \hspace{1cm} (4)
$$

We also define the fin effectiveness ($\varepsilon_f$) as the ratio of the Shape Factor, as defined above, to the Shape Factor of the base without fins, i.e. $S_b = Bi/(1 + Bi \, H_b)$:

$$\varepsilon_f = \frac{S}{S_b}. \hspace{1cm} (5)$$

Hence, the definition of the Shape Factor and the fin effectiveness associated with a fin departs from the classical definitions [2], as it includes the area of the base not covered by the fin. Based on the above definition, an addition of an extended surface or fin would improve the heat transfer rate of the base, if its effectiveness is greater than one.

The temperature along the lower surface, which includes the fin, can be obtained by applying the boundary element method [30, 37, 31, 38, 39, 17, 5]. An appropriate Green’s function is that associated with a periodic array of sources of period 1 located along an insulated lower surface and a Dirichlet boundary condition along the top surface as described in [33, 34, 4, 35]:

$$G[\hat{\xi}' - \hat{\xi}, \hat{\eta}' = 0] = - \left( \hat{h} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \tanh \left[ 2 \pi m \hat{h} \right] \cos \left[ 2 \pi m (\hat{\xi}' - \hat{\xi}) \right] \right). \hspace{1cm} (6)$$
Figure 2: Mapping of the physical domain onto a straight channel using the generalized Schwarz-Christoffel transformation (equation 1). Lengths in the physical and complex domain are non-dimensionalized with $L$ and $w_N$, respectively.

The mathematical domain along with the boundary conditions is shown in Fig. 3. Applying the boundary element method and the boundary conditions we obtain a Fredholm Integral equations of the second kind for the temperature along the lower surface,

$$T[\hat{\xi}] = Bi \int_0^1 G [\hat{\xi}' - \hat{\xi}] \left| \frac{d\hat{z}}{d\hat{w}} \right|_{\hat{w}=\hat{\xi}} \left( T[\hat{\xi}] - 1 \right) \ d\hat{\xi}.$$  

The integral equation is solved numerically using the collocation boundary element method. It is important to note that in the above formulation the Generalized Schwarz-Christoffel transformation appears in the boundary element formulation similar to. Another complication is that unlike, the singularities of the transformation also appear in the boundary element formulation, which are addressed using Gauss-Jacobi
quadrature as explained in the following section, where we pose the Shape Optimization problem. The objective function is the fin effectiveness ($\varepsilon_f$), i.e. the normalized Shape Factor, and the variable of the optimization is the shape of the lower surface (fin); the latter is characterized through the parameters of the generalized Schwarz-Christoffel transformation.

3 Optimum extended surfaces

The formulation of the Shape Optimization problem follows along the same lines as the problems formulated by Fyrillas and Leontiou & Fyrillas [32, 33, 34, 18, 4, 35]. The objective function is the fin effectiveness (Eq. 5) and the constraints are dictated in view of the geometrical configuration (Fig. 2):

$$\text{maximize } \varepsilon_f[\alpha, \hat{w}_N]$$

subject to the constraints

$$\sum_{j=0}^{N-1} |z_{j+1} - z_j| = P$$
$$\sum_{j=0}^{N} \alpha_j = 0$$
$$\text{Im}[z_i] = y_i \geq (H - H_b)$$
$$\text{Re}[z_N] = x_N = 1$$
$$\text{Im}[z_N] = y_N = 0,$$

where the perimeter $P$, the height $H$ and the height of the base $H_b$ are assigned a priori. Note that the period is equal to one as it is used for non-dimensionalization, and the third constraint is an explicit equation to obtain the real constant $R$ (Eq. 2). The fourth constraint defines the length of the fin and hence, the thickness of the plate. The equality $H_f = H - H_b$ is achieved for sufficiently large perimeter $P$. 
Similar to our previous work \cite{32, 33, 34, 18, 1, 35} the integral \( \hat{z}_i \) (Eq. 1) is calculated using Gauss-Jacobi quadrature \cite{40}, and we choose the \( \hat{w}_j \)s to be equispaced between 0 and 1. The infinite product appearing in integral (1) can be truncated to a small value without affecting the accuracy due to the exponential decay of the hyperbolic sines \cite{20}. The numerical optimization has been performed using the NLPQL optimization code developed by Schittkowski \cite{15}.

In Fig. 4 we show numerical results of the fin effectiveness maximization problem (Eq. 8) associated with a slab of height \( H = 0.5 \) and \( H_b = 0.1 \), for different values of the perimeter (\( P \)) and Biot number (\( Bi \)). The results suggest that in all cases the optimum fin is infinitely thin, elongated in the vertical direction. It is very important to note that for a large Biot number the presence of the fin might not enhance the heat transfer rate from the base, i.e. the addition of a fin reduces the heat transfer rate. This can be justified as in the case of isothermal conditions the addition of a fin would result in a reduction of heat transfer (see Introduction, §1 first paragraph).

The above results/conclusions suggest that: (i) the optimum fin is infinitely thin and elongated, and (ii) there exists a critical Biot number which characterizes whether a fin is effective or not. Hence, in the next section we determine the critical Biot number of a rectangular fin.

3.1 Critical thickness characterizing the effectiveness of a fin

The results obtained through the Shape Optimization procedure, suggest that for a fixed perimeter the optimum fin is an infinitely thin extended surface. It is tempting to infer that this is an artifact of the uniform heat transfer coefficient; had conjugate heat transfer been considered, the optimization procedure would have led to a different result. Furthermore, an interesting result that has emerged from the optimization analysis is that there exists a critical Biot number, associated with a given fin geometry, that determines whether the addition of the fin would enhance the heat transfer rate. In this section, we elucidate this point by considering the heat transfer rate associated with particular geometries, i.e. we consider the effectiveness of rectangular fin \cite{2}.

We consider the classical configuration of a periodic array of rectangular fins attached to an infinite rectangular base Fig 5a. Similar to the previous section, we non-dimensionalize lengths with the period of the array (\( L \)), to obtain the dimensionless Biot number \( Bi = hL/K \). In Fig. 5b, we show a plot of the fin effectiveness Vs the Biot number for a rectangular fin of thickness \( H_t = 0.1 \), attached to base of thickness \( H_b = 0.1 \). The critical Biot number, i.e. the value of the Biot number where the effectiveness of the fin is equal to one, is \( Bi_{critical} \approx 16.4 \) and it is independent of the length of the fin \( H_f \). We have considered different sizes of rectangular fins and concluded that:

1. A rectangular fin is effective if \( Bi \ H_t \leq 1.64 \). Expressed in dimensional variables, \( fin \ thickness < 1.64 \ k/h \). This is independent of both the thickness of the base and the length of the fin.

2. The maximum effectiveness is realized at \( Bi = 0 \) and is equal to \( \varepsilon_f [Bi = 0] = 2H_f + 1 \).

The results have been verified through finite difference and finite element simulations.
Figure 4: Optimum surfaces/fins that maximize the heat transfer rate in a finite channel of height $H = 0.5$. The fins are assumed to be attached to a base whose minimum thickness is $H_f = H - H_b = 0.1$. In Fig. (a) we show results for $Bi = 1$ and two different perimeters: $P = 1.24$ (dashed curve, $\varepsilon_f = 1.0$); $P = 1.68$ (solid curve, $\varepsilon_f = 1.4$). In Fig. (b) we show results for $Bi = 10$ and two different perimeters: $P = 1.15$ (dashed curve, $\varepsilon_f = 0.5$); $P = 1.63$ (solid curve, $\varepsilon_f = 1.0$). In Fig. (c) we show results for $Bi = 100$ and two different perimeters: $P = 1.27$ (dashed curve, $\varepsilon_f = 0.5$); $P = 1.72$ (solid curve, $\varepsilon_f = 0.99$).
4 Conclusions

In this work we consider the heat transfer, shape optimization problem associated with a periodic array of extended surfaces subjected to convection with a uniform heat transfer coefficient. We address both the heat transfer problem and the Shape Optimization, inverse design problem, of finding the geometry that maximizes the heat transfer rate.

The problem is formulated as a two dimensional, arbitrary channel of unit length which is bounded from above by a flat isothermal surface and from below by periodic extensions/fins with a uniform heat transfer coefficient. Given a set of geometrical constraints that characterize the geometry of the fin and the base, the objective is to find the optimum shape of the fin such that the heat transfer rate is maximized. Within the approximation of uniform heat transfer coefficient, the optimization procedure suggests that the optimum fin is infinitely thin and long.

Furthermore, the optimization procedure has revealed a very interesting result. There is a critical Biot number that characterizes the fin effectiveness. For values of the Biot number less than the critical a fin enhances the heat transfer rate, while for higher values it attenuates the heat transfer rate, hence it is not effective. This is investigated further by considering the simplest case of a rectangular fin of uniform cross section. Numerical results have elucidated the fin effectiveness and are summarized as follows:

1. A rectangular fin is effective if $Bi H_t \leq 1.64$. Expressed in dimensional variables, fin thickness $< 1.64 \frac{k}{h}$. This is independent of both the thickness of the base and the length of the fin.

2. The maximum effectiveness is realized at $Bi = 0$ and is equal to $\varepsilon_f [Bi = 0] = 2H_f + 1$.

Figure 5: Fig (a): A periodic array of rectangular fins. Fig (b): Fin effectiveness ($\varepsilon_f$) Vs Biot number ($Bi$) for a rectangular fin of $H_b = H_t = 0.1$. The three curves correspond to three different lengths ($H_f$). The lower curve corresponds to $H_f = 0.01$ while the middle and top curves to $H_f = 0.1$ and $H_f = 1$, respectively.
Acknowledgment.
The work was funded by Porfyrios Chap Glass Ltd. The authors would like to thank Klaus Schittkowski for providing his NLPQL numerical optimization code [15].

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