JACOB’S LADDERS AND MULTIPLICATIVE ALGEBRA OF 
REVERSELY ITERATED INTEGRALS (ENERGIES) ON THE 
CRITICAL LINE

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Abstract. Certain completely logarithmic formula for a set of reverse ly iter-
ated integrals (energies) is proved in this paper. Namely, in this case we have 
that integral powers of \( \ln T \) are contained on input as well as on output of 
corresponding integrals (energies).

1. Introduction

Let us define the following matrix of reversely iterated segments (comp. [3], 
(2.3))

\[
\begin{vmatrix}
T, T + \ln^{-p} T \\
q^n, q
\end{vmatrix}_{p,q}, \quad p, q = 1, \ldots, k, \quad k \leq k_0,
\]

where

\[
\varphi_1\{[T, T + \ln^{-p} T]\} = [T, T + \ln^{-p} T],
\]

(1.2)

\[
T = T, T + \ln^{-p} T = T + \ln^{-p} T,
\]

(1.3) \( \Delta_p(T, \ln^{-p} T) = \bigcup_{q=1}^k [\tilde{T}, T + \ln^{-p} T], \quad p = 1, \ldots, k. \)

Properties of these sets are listed below (comp. [3], (2.5) – (2.7)): since

\[
\ln^{-p} T = o\left(\frac{T}{\ln T}\right), \quad p = 1, \ldots, k
\]

then

\[
||[T, T + \ln^{-p} T]| = T + \ln^{-p} T - T = o\left(\frac{T}{\ln T}\right), \quad q = 1, \ldots, k,
\]

(1.4)

\[
||[T + \ln^{-p} T, T]| \sim (1 - c)\pi(T); \quad \pi(T) \sim \frac{T}{\ln T}, \quad q = 2, \ldots, k.
\]

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Remark 1. Consequently, the asymptotic behavior of each of disconnected sets (1.3) is as follows: if \( T \to \infty \) then the components recede unboundedly each from other and all together are receding to infinity. Hence, each of the sets (1.3) behaves like an one-dimensional Friedmann-Hubble expanding universe.

Now we define the following correspondence

\[
\frac{q}{[T, T + \ln^{-p} T]} \to \int_{T}^{T + \ln^{-p} T} \prod_{r=0}^{q-1} \frac{1}{2} + i\varphi_r(t) \left| \frac{1}{2} + i\varphi_r(t) \right|^2 dt
\]

acting on the set of elements of the matrix (1.1). In this paper we obtain a canonical formula for these integrals.

2. Theorem on exclusivity of integer powers of \( \ln T \)

2.1. Let us remind that we have proved the following theorem (see [3], (2.1) – (2.7)): for every \( L_2 \)-orthogonal system

\[
\{f_n(t)\}_{n=1}^{\infty}, \quad t \in [0, 2l], \quad l = o \left( \frac{T}{\ln T} \right), \quad T \to \infty
\]

there is a continuum set of \( L_2 \)-orthogonal systems

\[
\{F_n(t; T, k, l)\}_{n=1}^{\infty} = \left\{f_n(\varphi_1^k(t) - T) \prod_{r=0}^{k-1} |\vec{Z}[\varphi_1^r(t)]|^{l}, \quad t \in [T, T + 2l], \right. \]

(2.1)

where

\[
\varphi_1^k[T, T + 2l] = [T, T + 2l], \quad k = 1, \ldots, k_0, \]

\[
[T, T + 2l] = [T, T + 2l], \quad T \to \infty,
\]

i.e. the following formula is valid

\[
\int_{T}^{T + 2l} f_m(\varphi_1^k(t) - T)f_n(\varphi_1^k(t) - T) \prod_{r=0}^{k-1} \vec{Z}^2[\varphi_1^r(t)] dt =
\]

(2.2)

\[
= \left\{ \begin{array}{ll}
0 & , \quad m \neq n, \\
A_n & , \quad m = n,
\end{array} \right.
\]

\[
A_n = \int_{0}^{2l} f_n^2(t) dt.
\]
Of course, we have that

\[\tilde{Z}^2(t) = \frac{d\varphi_1(t)}{dt} = \frac{Z^2(t)}{2\Phi_{\varphi_1}[\varphi(t)]} = \frac{\zeta(\frac{1}{2} + it)}{\omega(t)}, \quad \varphi_1(t) = \frac{1}{2} \varphi(t),\]

(2.3)

\[\omega(t) = \left(1 + O\left(\frac{\ln \ln t}{\ln t}\right)\right) \ln t.\]

and

\[Z(t) = e^{i\varphi(t)},\]

\[\varphi(t) = -\frac{t}{2} \ln \pi + \text{Im} \ln \Gamma\left(\frac{1}{4} + \frac{t}{2}\right),\]

Hence, for the classical Fourier’s orthogonal system

\[f_1(t) = 1\]

we have the following formula

\[\int_T^{T+2l} \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1(t)] \, dt = 2l,\]

i.e. (see (2.3),(2.5))

\[\int_T^{T+2l} \prod_{r=0}^{k-1} \frac{\zeta(\frac{1}{2} + i\varphi_1(t))}{\omega[\varphi_1(t)]} \, dt = 2l.\]

(2.6)
Next, let us remind that the disconnected set (see \[3\], (2.9))

\[ \Delta(T, k, l) = \bigcup_{r=0}^{k} [T, T + 2l] \]

has the following properties (see \[3\], (2.7), (2.10), (4.3))

\[ [T, T + 2l] \prec \bigcup_{r=0}^{k} [T, T + 2l] \prec \bigcup_{r=0}^{k} [T, T + 2l] \prec \ldots, \]

(2.7)

\[ \phi_r(t) \in [T, T + 2l], \ r = 0, 1, \ldots, k, \]

\[ \ln t \sim \ln T, \ \forall t \in (T, T + 2l), \ k = 1, \ldots, k_0. \]

Now we obtain from (2.6) by the mean-value theorem and (2.7) that

(2.8)

\[ \int_{T}^{T + 2l} \prod_{r=0}^{k-1} \left| \zeta \left( \frac{1}{2} + i \phi_r(t) \right) \right|^2 \ dt \sim 2l \ln^k T, \ T \to \infty. \]

Consequently, we obtain from (2.8) in the case

\[ 2l = \ln^{-p} T = o \left( \frac{T}{\ln T} \right), \ k = q \]

the following formula (comp. (1.7))

Theorem.

(2.9)

\[ \int_{T}^{q T + \ln^{-p} T} \prod_{r=0}^{q-1} \left| \zeta \left( \frac{1}{2} + i \phi_r(t) \right) \right|^2 \ dt \sim \ln^{q-p} T, \]

where

\[ \phi_r^0(t) : \ \phi_r^0(t) = t, \ \phi_r^1(t) = \phi_1(t), \ \phi_r^2(t) = \phi_1(\phi_1(t)), \ldots \]

(2.10)

\[ \phi_r^r(t) \in \left[ T, T + \ln^{-p} T \right], \ r = 0, 1, \ldots, q \]

(see (2.7)) and \( k_0 \in \mathbb{N} \) is an arbitrary and fixed number.

Remark 2. The formula (2.9) is the first completely logarithmic formula in the theory of the Riemann zeta-function in the following sense

\[ \int_{T}^{q T + \ln^{-p} T} \left( 2.9 \right) (1 + o(1)) \ln^{q-p} T, \ T \to \infty. \]

Namely, the integer powers of \( \ln T \) are contained on input as well as on the output of the integral (2.9). Of course, the formula (2.9) is not accessible by the current methods in the theory of the Riemann zeta-function.
3. Interpretations of the iterated integrals under study

3.1. We define the following planar figures

\[ S^k_{p,q}(T) = \begin{cases} 
(t, y) : t \in [\frac{q}{T}, T + \ln^{-p} T], \\
y \in \left[0, \prod_{r=0}^{q-1} |\zeta(\frac{1}{2} + i \varphi(t))| \right]^2, \\
p, q = 1, \ldots, k, T \to \infty.
\end{cases} \]  

(3.1)

Then

\[ \int_{\frac{T}{T}}^{T + \ln^{-p} T} \prod_{r=0}^{q-1} |\zeta(\frac{1}{2} + i \varphi(t))| ^2 dt = m\{S^k_{p,q}(T)\}. \]  

(3.2)

Remark 3. By (3.2) we have usual geometric interpretation of the integrals in the formula (3.2) as the measures of corresponding planar figures (3.1), and, of course, (see (2.9), (3.2))

\[ m\{S^k_{p,q}(T)\} \sim \ln^{q-p} T, \ T \to \infty. \]  

(3.3)

3.2. Let us consider an oscillating process (of any nature) described by the function

\[ f^k_{p,q}(t) = \begin{cases} 
\prod_{r=0}^{q-1} |\zeta(\frac{1}{2} + i \varphi(t))|, & t \in [\frac{q}{T}, T + \ln^{-p} T] \\
0, & \text{otherwise}
\end{cases} \]  

(3.4)

Then we have by means of the Plancherel’s \( L^2 \)-theory that

\[ \int_{0}^{\infty} \{f^k_{p,q}(t)\}^2 dt = \int_{\frac{T}{T}}^{T + \ln^{-p} T} \prod_{r=0}^{q-1} |\zeta(\frac{1}{2} + i \varphi(t))| ^2 dt = \int_{0}^{\infty} \{F^k_{p,q}(\omega)\}^2 d\omega, \]  

(3.5)

where

\[ F^k_{p,q}(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f^k_{p,q}(t) \cos(\omega t) dt \]  

is the Fourier’s cosine transformation of \( f^k_{p,q}(t) \) and

\[ \{F^k_{p,q}(\omega)\}^2 d\omega \]  

is the energy corresponding to the interval of frequencies

\[ [\omega, \omega + d\omega). \]

Remark 4. Now we have by (3.5) the following energetic interpretation

\[ \int_{\frac{T}{T}}^{T + \ln^{-p} T} \prod_{r=0}^{q-1} |\zeta(\frac{1}{2} + i \varphi(t))| ^2 dt = E^k_{p,q}(T), \]  

(3.6)

where \( E^k_{p,q}(T) \) is the total energy of the oscillating process (3.4) and (see (2.9), (3.6))

\[ E^k_{p,q}(T) \sim \ln^{q-p} T, \ T \to \infty. \]  

(3.7)
4. On generators of the main set of energies and constraints on behavior of $\zeta \left( \frac{1}{2} + it \right)$

Since (see (2.9))

$$\int_q^{T + \ln^{-p} T} \prod_{r=0}^{q-1} \left| \zeta \left( \frac{1}{2} + i\varphi_r^q(t) \right) \right|^2 dt \sim \ln^{q-p} T,$$

$$\int_T^{Q} \prod_{r=0}^{Q-1} \left| \zeta \left( \frac{1}{2} + i\varphi_r^Q(t) \right) \right|^2 dt \sim \ln^{Q-p} T$$

we obtain the following

**Corollary 1.** For every fixed $(P, Q): P \neq Q, \ P, Q = 1, \ldots, k$

we have that

$$\int_T^{T + \ln^{-p} T} \prod_{r=0}^{p-1} \left| \zeta \left( \frac{1}{2} + i\varphi_r^p(t) \right) \right|^2 dt \sim \begin{cases} \int_T^{Q} \prod_{r=0}^{Q-1} \left| \zeta \left( \frac{1}{2} + i\varphi_r^Q(t) \right) \right|^2 dt \Bigg|_{p \to P} \\
\int_T^{T + \ln^{-p} T} \prod_{r=0}^{P-1} \left| \zeta \left( \frac{1}{2} + i\varphi_r^P(t) \right) \right|^2 dt \Bigg|_{q \to Q} 
\end{cases},$$

$p, q = 1, \ldots, k, \ T \to \infty.$

**Remark 5.** We have by (4.1) that every fixed energy with $P \neq Q$ is the generator of all over main set of energies that correspond to $p, q = 1, \ldots, k$.

The energies in (4.1) correspond to the segments

$$[T, T + \ln^{-p} T], [T, T + \ln^{-p} T].$$

**Remark 6.** If $q \neq Q$ then we see that big distance (comp. (1.4), (1.5))

$$A \frac{T}{\ln T} \to \infty, \ T \to \infty$$

separates the segments (4.2).

**Remark 7.** Now, we will give an interpretation of the set of formulae (4.1) as a continuum set of constraints on behavior of the Riemann function

$$\zeta \left( \frac{1}{2} + it \right), \ t \to \infty.$$ 

In this direction we have that by the constraints (4.1) is expressed a high degree of inner binding of the set of values of the function $\zeta \left( \frac{1}{2} + it \right)$, although at big distances (see (4.3)).

Next, if we use the mean-value theorem in (4.1) then we obtain the following.
Corollary 2. There are numbers
\[ d_r^1 = \varphi_1^r(c_1) \in (Q_r^1, Q_r^1 + \ln^{-p} T), \ r = 0, 1, \ldots, q - 1, \]
\[ d_r^2 = \varphi_1^r(c_2) \in (Q_r^2, Q_r^2 + \ln^{-p} T), \ r = 0, 1, \ldots, Q - 1, \]
(comp. (2.10)), of course,
\[ d_0^1 = \varphi_1^0(c_1) = c_1, \ d_0^2 = \varphi_1^0(c_2) = c_2, \]
such that
\[ |[T, T + \ln^{-p} T] \prod_{r=0}^{q-1} \left( \zeta \left( \frac{1}{2} + id_r^1 \right) \right)^2 \sim \]
\[ \sim \left\{ \prod_{r=0}^{q-1} \left( \zeta \left( \frac{1}{2} + id_r^2 \right) \right)^2 \right\} , T \to \infty. \]

Remark 8. The formula (4.4) expresses one kind of continuum set of constraints that we have mentioned above. Namely, certain type of constraints corresponds to each formula that follows from (4.1).

5. LAW OF MULTIPLICATION OF ENERGIES

Next, we obtain from (4.1) the following

Corollary 3.
\[ \int_T^{T + \ln^{-p_1} T} Q_{q_1-1}^{r_1} \left| \zeta \left( \frac{1}{2} + i\varphi(t) \right) \right|^2 dt \times \]
\[ \times \int_T^{T + \ln^{-p_2} T} Q_{q_2-1}^{r_2} \left| \zeta \left( \frac{1}{2} + i\varphi(t) \right) \right|^2 dt \sim \]
\[ \sim \left\{ \prod_{r=0}^{p-1} \left( \zeta \left( \frac{1}{2} + i\varphi(t) \right) \right)^2 dt \right\} , T \to \infty, \]

where
\[ -k + 1 \leq q_1 + q_2 - (p_1 + p_2) \leq k - 1, \]
\[ p_1, q_1, p_2, q_2 = 1, \ldots, k. \]

Remark 9. We see that the main set of energies is not closed with respect to multiplication of the kind (5.1). Namely, if
\[ (p_1, q_1) = (1, k), \ (p_2, q_2) = (k, 1), \]
for example, then (see (5.2))
\[ q_1 + q_2 - (p_1 + p_2) = 2k - 2 > k - 1, \ k \geq 2. \]
However, in this case of constraints it follows from (5.1), (comp. (4.4)) that we may use the condition
\[-k_0 + 1 \leq q_1 + q_2 - (p_1 + p_2) \leq k_0 - 1\]
instead of (5.2).

6. Unit energies and non-local equivalences of them

Next, we obtain from (2.9) or (4.1) at \(p = q\) the following

Corollary 4.

(6.1) \[
\int_{T}^{T + \ln^{-p}T} \prod_{r=0}^{p} \left| \zeta \left( \frac{1}{2} + i \varphi_1(t) \right) \right|^2 dt \sim 1, \quad p = 1, \ldots, k, \quad T \to \infty,
\]
i. e. these integrals play a role of the asymptotic unit elements in the main set of energies.

Remark 10. Since (see [3], (5.6))

(6.2) \[
\frac{k}{T} - \frac{k-1}{T} \sim (1-c) \frac{T}{\ln T}, \quad T \to \infty, k = 1, \ldots, k_0,
\]
then we obtain from (6.1) the following non-local equivalences of the unit energies

(6.3) \[
\int_{T}^{T + \ln^{-1}T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim \int_{T}^{T + \ln^{-2}T} \prod_{r=0}^{2} \left| \zeta \left( \frac{1}{2} + i \varphi_1(t) \right) \right|^2 dt \sim \cdots \sim \int_{T}^{T + \ln^{-k}T} \prod_{r=0}^{k} \left| \zeta \left( \frac{1}{2} + i \varphi_1(t) \right) \right|^2 dt
\]
together with corresponding set of constraints (comp. (4.4)).

Remark 11. We see that the initial unit energy transmission (see (6.3))

(6.4) \[
\int_{T}^{T + \ln^{-1}T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim 1.
\]
corresponds to the translations on big distances (comp. (6.2))

\[\frac{1}{T} \to \frac{2}{T} \to \cdots \to \frac{k}{T}\]

Thus, the operation

\[
\int_{T}^{T + \ln^{-p}T} \prod_{r=0}^{p} \left| \zeta \left( \frac{1}{2} + i \varphi_1(t) \right) \right|^2 dt \sim 1
\]
asymptotically preserves the value of unit energy.
7. **Inverse Energies**

Next, we obtain from (5.1) in the case

\[(p_1, q_1) = (p, q), \ (p_2, q_2) = (q, p)\]

the following

**Corollary 5.**

\[
\begin{align*}
\int_T^{T+\ln^{-p} T} \prod_{r=0}^{q-1} \left| \zeta \left( \frac{1}{2} + i\varphi_1(t) \right) \right|^2 dt \times \\
\times \int_T^{T+\ln^{-q} T} \prod_{r=0}^{p-1} \left| \zeta \left( \frac{1}{2} + i\varphi_1(t) \right) \right|^2 dt \sim 1, \ p, q = 1, \ldots, k, \ T \to \infty,
\end{align*}
\]

i.e. the inverse energy to

\[
\int_T^{T+\ln^{-p} T} \prod_{r=0}^{q-1} \left| \zeta \left( \frac{1}{2} + i\varphi_1(t) \right) \right|^2 dt
\]

is the following one

\[
\int_T^{T+\ln^{-q} T} \prod_{r=0}^{p-1} \left| \zeta \left( \frac{1}{2} + i\varphi_1(t) \right) \right|^2 dt,
\]

and vice versa.

**Example.**

\[
\left\{ \int_T^{T+\ln^{-q} T} \prod_{r=0}^{256} \left| \zeta \left( \frac{1}{2} + i\varphi_1(t) \right) \right|^2 dt \right\}^{-1}
\]

\[
\sim \int_T^{T+\ln^{-257} T} \prod_{r=0}^{256} \left| \zeta \left( \frac{1}{2} + i\varphi_1(t) \right) \right|^2 dt, \ T \to \infty.
\]

**Remark 12.** We see that each energy in (7.1) is asymptotically balanced by another one (a kind of balance on a lever). Moreover, let us remind that there is a system of constraints corresponding to (7.1) (comp. (4.4)).

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