Dynamical fluctuations in temporal networks based on factorial moment approach

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Abstract. The factorial moments analyses are performed to study the scaling properties of dynamical fluctuations in temporal networks based on empirical data sets. The intermittent behavior is not evident between normalized factorial moment $F_q$ and bin number $M$ over the whole time range. But the strict power-law dependence of $F_q$ on $F_2$, $F_q \sim F_2^{\beta_q}$, indicates that the interaction has self-similarity structure in time interval and the fluctuations are not purely random but dynamical and correlated. We further find that $\beta_q$ has a well-fitted power-law relationship with $q - 1$. The exponents are very close to that for systems undergoing second-order phase transition in Ginzburg-Landau theory and describe the strength of intermittent behavior in the dynamical fluctuations.

1. Introduction

Interactions in complex systems are not static but change over time, which can be modelled in terms of temporal networks [1]. Temporal network consists of a set of contacts $(n_i, n_j, t)$, emphasizing on the time when node $i$ and $j$ have a connection. The addition of time dimension provides a new sight into the framework of complex network theory. In temporal networks, both structural properties and spreading dynamics crucially depend on the time-ordering of links.

The research of temporal networks has attracted great attention and it mainly focuses on two major aspects from the point view of time dimension. One is corresponding to the strategy of time aggregation especially when the topological characteristics are more relevant than the temporal properties. The topological structure of temporal network is achieved through aggregating contacts over a certain time interval and the temporal network is then represented as a series of snapshots of static graphs. Consequently, many existing concepts and tools of static graphs can be adopted to analyze temporal networks, since it is usually easier to analyze static networks. For example, the degree of a node $k_i(t)$ is described as the number of links that it has to other nodes within the time window $[t, t + \Delta t]$. The error and attack strategies in static networks have been applied to evaluate the temporal vulnerability [2], and so on. In order to understand the structure of temporal networks, it plays a crucial role to choose an optimal time interval $\Delta t$ to construct static graphs from temporal networks. Krings et al. studied the influences of time intervals when aggregating the mobile phone network over time [3]. Holme analyzed three ways of constructing static snapshots from temporal networks [4], but no candidate weighing out as a best choice. It is now still an open question on how to choose the time interval to represent temporal networks.
The other aspect is related to using dynamical processes to probe into the influence of time series on temporal network. We should take into account the time-ordering of each contact and the inter-event time between two consecutive contacts. The inter-event time distribution in temporal network follows a power-law, which is also called burstiness [5]. Although it is recognized that time-ordering and bursty characters have strong influences on the dynamical processes of temporal networks, numerous studies have appeared to arrive at contradictory results. Lambiotte et al. had stressed that time-ordering and burstiness of contacts were critical in spreading process, which led to slow down spreading [6]. In the work of Rocha et al, they concluded that temporal correlations accelerated outbreaks [7] in SI and SIR model. Miritello et al. demonstrated that bursts hindered propagation at large scales, but group conversations favored local rapid cascades [8].

Despite the promoting results in temporal networks, this field is still in its early stages about how temporal effect and topological structure interplay and hence affect the dynamical process. In this paper, based on empirical data sets, we will investigate the scaling properties of the dynamical fluctuations of contacts and nodes in temporal networks by means of the factorial moment approach. We are aiming at extracting the fundamental properties from the large amount of data and revealing the influences of time effects on temporal networks from a new perspective.

The rest of the paper is organized as follows. Section II briefly introduces the method of factorial moments. In Section III we give a brief description of the data sets and present the corresponding results, especially the scaling properties of fluctuations for contacts and nodes in the empirical data sets. Conclusions are offered in the final section.

2. Method of factorial moments

Temporal network consists of a sequence of contacts \((n_i, n_j, t)\), representing that node \(i\) and node \(j\) has a contact at time \(t\). The number of contacts characterizes the frequency that individuals are connected with each other and the number of nodes describe the activeness that individuals are involved. In this paper, factorial moment approach will be used to study the fluctuations of active contacts and nodes in temporal networks as well as the corresponding scaling properties among these fluctuations.

Factorial moments are originally introduced in nuclear physics to study the multiplicity fluctuation of hadrons produced during the high energy collisions[9, 10]. The factorial moment approach provides a general and sensitive method to characterize whether the fluctuations in multiplicity distributions arise from statistical contributions or dynamical interactions. Here we will employ the method to study the multiplicity of contacts and nodes in temporal networks as well as the corresponding scaling properties among these fluctuations.

Consider the time series of contacts (or nodes) \(y(t)\), where \(t\) is the time that contacts happen and \(t\) ranges from 0 to \(T\). We divide the whole time range \(T\) into \(M\) equal bins (the remainders are discarded). So the time interval in each bin is \(\Delta t = T/M\). Within each bin window \(m\) \((m = 1, 2, ..., M)\), denote the number of contacts (or nodes) as \(n_m\). Of course, \(n_m\) fluctuates for different bin windows. To measure the fluctuations and correlations, the \(q-th\) order factorial moment is introduced as,

\[
f_q = \frac{1}{M} \sum_{m=1}^{M} n_m(n_m - 1)...(n_m - q + 1) = \langle n_m(n_m - 1)...(n_m - q + 1) \rangle. \quad (1)
\]

In factorial moments, \(f_1 = \langle n_m \rangle\) is the mean number of contacts (or nodes) under a certain bin size, averaged over all the bins \(m\). Note that \(n_m\) must be greater than \(q\) \((n_m > q)\) in order to contribute to \(f_q\), and \(q\) is usually an integer. As \(M\) increases, \(\Delta t\) is decreased and the average
multiplicity \( \langle n_m \rangle \) in a bin decreases. This may lead to \( n_m < q \) which is not allowed. Thus high \( q \) corresponds to higher \( n_m \) in the bin under consideration, i.e., large fluctuations from \( \langle n_m \rangle \) [11].

Normalized factorial moments are more generally used,

\[
F_q = \frac{f_q}{f_1}.
\]

It was proved that \( F_q \) can filter out the statistical fluctuations [9, 10]. For uncorrelated statistical (Poissonian or Gaussian) distributions, \( F_q = 1 \) for all orders \( q \). The method of factorial moments has been applied to analyze different complex systems, such as multiplicity of produced hadrons [12], human electroencephalogram and gait series in biology [13, 14], financial price series [15], critical fluctuations in Bak-Sneppen model [16], spectra analysis of complex networks [17], to name a few. Specially it exhibits that the system has self-similarity structures when \( F_q \) has a power-law dependence on the bin size \( M \).

\[
F_q \propto M^{\alpha_q}, \quad \alpha_q > 0.
\]

This phenomenon is referred to as the intermittency. Intermittency basically means random deviations from smooth or regular behavior and is expected in a variety of statistical systems at the phase transition point of the second-order type.

If \( F_q \) does not have an evident scaling behavior with \( M \), the relationship between \( F_q \) and \( F_2 \) will be investigated,

\[
F_q \propto F_2^{\beta_q},
\]

This is more widely valid for intermittent systems even though Eq. (3) does not hold [18, 19].

### 3. Results and discussions

In this paper the factorial moments analyses are performed to uncover the scaling properties of the fluctuations in temporal networks based on the following two empirical data sets.

**Prostitution**: The data set consists of sexual contacts between sex buyers and sellers from a Brazilian web forum [20]. The time resolution is 1 day and the whole time range is \( T = 2232 \) days.

**Conference**: The data set was collected at a 3-day conference from face-to-face interactions between conference participants. A contact is recorded every 20-second intervals if two individuals are within range of 1.5m [21]. The whole time range is \( T = 212340 \) seconds.

We now divide the whole time range \( T \) into \( M \) bins and count the number of contacts and nodes in each bin window. Calculate \( f_q \) and \( F_q \) according to Eq. (1) and (2), respectively. It is noticed that \( f_q \) is averaged over all bins (known as the horizontal average).

Figure 1 presents the time series of node and contact activities for *Prostitution* data. It is observed that the same data set has similar temporal patterns. We next go into the details of these time series. We plot in Fig. 1 the log-log graph of \( F_q \) as a function of \( M \) for contacts (open circles) and nodes (filled circles). With \( M \) ranging from about 3 to 60 bins, it means that the time interval \( \Delta t \) extends approximately from 30 to 750 days. The increase of bin number \( M \) means that the fluctuations of arbitrary sizes can appear in the system, and consequently leads to the growth of factorial moment \( F_q \) with \( M \). In Fig. 1 we show that there is no power-law relationship between \( F_q \) and \( M \) of Eq. (3) over the whole range, but within some smaller ranges there is approximate linearity in each [22]. As stated in Ref. [15], for uncorrelated Poissonian or Gaussian distributions, \( F_q = 1 \) for all orders \( q \); whereas for correlated distributions, \( F_q \) should increase with the growth of bin number \( M \). Figure 1 suggests that the fluctuations of contacts and nodes in *Prostitution* data are not random Poisson distribution but have dynamical and
Figure 1. Log-log plot of factorial moments $F_q$ as a function of bin size $M$ for Prostitution data with $q$ varying from 2 to 6. (a) The fluctuations of contacts (open circles). (b) The fluctuations of nodes (filled circles). The symbols are the same below.

Figure 2. Log-log plot of $F_q$ as a function of $M$ for Conference data with the range of $q$ from 2 to 6. (a) for contacts; (b) for nodes.

correlated behaviors inside. The same phenomena have also been observed in Conference data in Fig. 2.

Further investigations have been performed on $F_q$ and $F_2$. We plot $F_q$ as a function of $F_2$ on the log-log scale for Prostitution and Conference data sets in Fig. 3 and 4, respectively. The figures present remarkable linearity over the entire range. Thus we have a strict scaling relationship between $F_q$ and $F_2$ for all $q$,

$$F_q \propto F_2^{\beta_q}.$$  \hspace{1cm} (5)

The result also implies that $\beta_q$ is independent of $M$, which suggests a common feature of scaling invariance in temporal networks.

Next we investigate the dependence of $\beta_q$ on $q$. The plot of $\beta_q$ as a function of $(q - 1)$ is presented on a log-log scale in Fig. 5 for Prostitution data and in Fig. 6 for Conference data. There is also a strict linear relationship between $\beta_q$ and $(q - 1)$ for all $q$. Now one has

$$\beta_q \propto (q - 1)^\gamma.$$  \hspace{1cm} (6)

The linear fits are also plotted in the figures. In Prostitution data, $\gamma$ are 1.44 for contacts and 0.629 for nodes. In Conference data $\gamma$ are 1.211 and 1.367 for contacts and nodes, respectively.
It is known that $\gamma$ is 1.304 for systems undergoing second-order phase transition in Ginzburg-Landau (GL) theory and $\gamma$ is independent of the details of the GL parameters [23, 24]. Except the exponent $\gamma$ of node fluctuation for *Prostitution* data in Fig. 4(b), the other three exponents are all very close to 1.3. If we recall Fig. 2 for the measure of intermittency, we obtain that the more linear $F_q$ and $M$ are, the more likely the exponents $\gamma$ are close to 1.3. The exponent $\gamma$ indicates the strength of intermittent behavior in the dynamical fluctuations of temporal networks.

### 4. Conclusions

The method of factorial moments are performed to study the scaling properties of dynamical fluctuations in temporal networks based on empirical data sets of *Prostitution* and *Conference*. The phenomena of intermittency $F_q \sim M^{\alpha q}$ over the whole range is not evident. But we have obtained a strict power-law relationship between $F_q$ and $F_2$, $F_q \sim F_2^{\beta q}$ for all orders $q$. The result implies that the fluctuations are not purely random, but have dynamical and correlated behaviors embedded in the system. We further find that $\beta_q$ scales with $q$ as $\beta_q \sim (q - 1)^\gamma$. Some of the exponents $\gamma$ are very close to that found in the system undergoing second phase-transition in GL theory. We suggest that the exponent $\gamma$ describes the strength of intermittent behavior in the dynamical fluctuations of temporal networks.

Still, there are some issues to be addressed. First, what is the driving mechanism(s) behind
Figure 5. Scaling properties between $\beta_q$ and $(q-1)$ for *Prostitution* data. The scaling exponents are about: (a) 1.341 for contacts; and (b) 1.104 for nodes.

Figure 6. Scaling properties between $\beta_q$ and $(q-1)$ for *Conference* data. The scaling exponents are about: (a) 0.992 for contacts; and (b) 1.345 for nodes.

these scaling properties of fluctuations in temporal networks? Second, why are some scaling exponents close to that of Ginzburg-Landau second-order phase transition? Are they belong to the same universal class? All these topics cannot be covered in this paper and will be discussed later.

The scaling invariances of fluctuations shed light on the temporal correlations of contact series and provide a new sight into understanding the influence of time in temporal networks.

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