A novel method to compute one loop 4 point functions

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Abstract. We present a part of an ongoing work initiated by Prof. Shimizu on a method to compute numerically two loop scalar integrals as a sum of two dimensional integrals of generalized scalar one loop four point functions. This implies to have a formula for the one loop scalar four point functions which is valid outside the physical domain, this is the object of the present work.

1. Introduction

This work has been started after a visit of Prof. Shimizu in our institute, he explained to us his idea about computing scalar two loop 3 and 4 point functions numerically as a sum of two dimensional integrals of generalized scalar one loop four point functions. He gave us his notes (a part in English and a part in Japanese!) and he trusted us to push forward this project. I (the author, J.Ph. G.) would like to take this opportunity to thank him for introducing me to the Japanese culture (despite the fact that I was not a diligent student!) and for his kindness, I really enjoyed to work with him.

More precisely, Prof. Shimizu showed that any scalar two loop three point function can be written as

$$I_{3\text{2-loop}} = \sum \int_0^1 d\xi \int_0^1 d\rho \tilde{I}_{4}^4(\xi,\rho)$$

and any scalar two loop four point function can be written as

$$I_{4\text{2-loop}} = \sum \int_0^1 d\xi \int_0^1 d\rho \tilde{I}_{5}^5(\xi,\rho)$$

where $\tilde{I}_4^4$ (respectively $\tilde{I}_5^5$) is the generalized scalar one loop four point function (resp. the generalized scalar one loop five point function). Here, a generalized scalar one loop $N$ point function differs from a scalar one loop $N$ point function by the integration volume over the Feynman parameters: it is not always the usual hyper-simplex, it can be an hyper-cube or an hyper-prism depending on the topology of the scalar two loop diagram.

As a one loop five point function can be expressed in term of one loop four point functions irrespective of the Feynman parameter phase space volume : $\tilde{I}_5^5 = \sum_i c_i \tilde{I}_4^4$, so one expects that
the scalar two loop three and four point functions can be calculated numerically as a sum of two
dimensional integrals of generalized one loop scalar four point functions. Besides the integration
volume issue, there is another problem to perform this program. Indeed, we cannot use the
usual four point function library (FF [1], LOOPTOOLS [2], D0C [3], OLO [4], ...) because
they return a result only if the kinematics defining the four point function is in the physical
domain. In our case, the kinematics will depend on the two extra integration variables $\xi$ and $\rho$,
and so there is no guarantee that it will belong to such a domain.

So the first step to start this program is to have a scalar four point function formula which
is valid outside the physical domain. This is the object of this work. Furthermore, there are
very few independent methods for the computation of scalar one loop four point functions in the
massive case proposed [5], [6], despite the fact that there are many numerical implementations of
these formulae. But for our purpose, the analytic continuation of ref. [5] with such a kinematics
remains puzzling and we have had to develop a completely novel method to compute one loop
scalar Feynman integrals in a more open way. We will focus here on the case of the four point
function but the method can be applied to two or three point functions too.

2. Definitions

\[ I_4^4 = \int_0^1 \prod_{i=1}^4 dx_i \delta(1 - \sum_{j=1}^4 x_j) \left( -\frac{1}{2} \tilde{X}^T S \tilde{X} - i \lambda \right)^{-2} \]  

(3)

where the $S$ matrix contains all the information on the kinematics and can be expressed in term
of the scalar product of the propagator 4-momenta and the internal masses:

\[ S_{i,j} = (q_i - q_j)^2 - m_i^2 - m_j^2 \]  

(4)

The raw vector $\tilde{X}$ contains the Feynman parameters as component:

\[ \tilde{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} \]

and the superscript $T$ stands for the matrix transpose.
Expressing $x_4$ in term of the other Feynman parameters to get rid of the $\delta$ constraint, we can write the scalar one loop four point function as:

$$I_4 = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \frac{1}{(D(X) - i\lambda)^2}$$  \hspace{1cm} (5)$$

with

$$D(X) = X^T A X + B^T X + C, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The components of the $3 \times 3$ matrix $A$, the three dimensional vector $B$ as the scalar $C$ can be expressed in term of the components of the $S$ matrix

$$A_{ij} = -\frac{1}{2} (S_{ij} - S_{4i} - S_{4j} + S_{44}) \quad i, j \neq 4 \hspace{1cm} (6)$$

$$B_j = S_{44} - S_{4j} \quad j \neq 4 \hspace{1cm} (7)$$

$$C = -\frac{1}{2} S_{44}$$  \hspace{1cm} (8)$$

It is important to remark that the matrix $A$ is proportional to a Gram matrix defined in term of the 4-vectors $r_i$ ($q_i = k + r_i$ where $k$ is the momentum running into the loop). More precisely, $A = 1/2 G^{(4)}$. The superscript 4 indicates that we single out the row and the column 4 of the $S$ matrix. If another number for a row and a column is singled out, $A$ will be proportional to another Gram matrix which has the same determinant as $G^{(4)}$. As the final result will depend on these Gram matrices only through their determinant, we will drop thereafter the superscript indicating which row and column have been singled out.

3. Description of the method

We will express the three dimensional integral over the simplex as a sum of three dimensional integrals on one eighth of $\mathbb{R}^3$. In order to achieve that, we will make an extensive use of the following identity$^1$:

$$\frac{1}{D^{\alpha+1}} = \frac{1}{\alpha} \Delta_n \left[ 2^{n-2\alpha} - \nabla^T \left( \frac{2X + A^{-1}B}{D^\alpha} \right) \right]$$  \hspace{1cm} (9)$$

where $n$ is the number of independent integration variables (here 3) and:

$$\Delta_n = B^T A^{-1} B - 4 C$$

The eq. (9) is interesting to use in the case where $\alpha = n/2$ because in such a case only the boundary term remains. But, in general, $\alpha \neq n/2$, for example for the four point function $\alpha = 1$ and $n = 3$. The idea is to introduce a new identity which enables to adjust the power of the denominator of eq. (5). For that, we introduce the following relation valid for $\mu > 1/\nu > 0$ and any complex number $D$:

$$\int_0^\infty \frac{d\xi}{(D + \xi^\nu)^\mu} = \frac{1}{\nu} B \left( \frac{1}{\nu}, \mu - \frac{1}{\nu} \right) \frac{1}{D^{\mu-1/\nu}}$$  \hspace{1cm} (10)$$

where $B(x, y)$ is the Euler beta function defined by

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}$$

$^1$ Integrating this equality over the simplex gives the usual relations between scalar one loop four point function in different space-time dimensions [7],[8]
A first step consists of an adjustment of the power of the denominator in the l.h.s of eq. (9) in such way that only the boundary term of eq. (9) remains. In the case of the four point function, cf. eq. (5) \( n = N - 1 \) is equal to 3. Imposing \( n - 2\alpha \) to vanish thus forces \( \alpha \) to be 3/2, hence a power \( \alpha + 1 = 5/2 \) in the l.h.s. of eq. (9). However in eq. (5), \( D \) is raised to the power 2, not 5/2. In order to shift the power of the denominator which arises naturally in eq. (5), by 1/2, we use identity (10) i.e. the power of \( D \) appearing in the denominator of the integrand of eq. (5). The representation of \( 1/D^2 \) thus obtained is substituted into eq. (5). This provides a representation of \( I_4^1 \) in whose integrand \( D \) is replaced by \( (D + \xi^\nu) \). Identity (9) is then applied to this new integrand considered as a function of the three integration variables \( x_1, x_2, x_3 \) whereas \( \xi \) is seen as a fixed parameter. In order that the first term of the thereby modified (9) be made vanishing, \( \mu \) is chosen to be 3/2 thus \( \nu \) shall be chosen equal to 2. Eq. (5) thus reads:

\[
I_4^1 = \int_0^\infty d\xi \frac{2}{B(2,1/2)} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \times \int_0^{1-x_1-x_2} dx_3 \frac{1}{D(x_1,x_2,x_3) + \xi^2 - i\lambda)^{3/2}} \tag{11}
\]

and a direct application of eq. (9), gives:

\[
I_4^1 = \frac{1}{3 B(2,1/2)} \int_0^\infty d\xi \frac{2}{\Delta_3/4 - i\lambda} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \times \int_0^{1-x_1-x_2} dx_3 \nabla^T \left( \frac{2X + A^{-1}B}{(D(x_1,x_2,x_3) + \xi^2 - i\lambda)^{3/2}} \right) \tag{12}
\]

with \( \Delta_3 = B^T A^{-1} B - 4C \).

Note that we have to be careful because we are working with complex numbers and it is not obvious that the integration over \( \xi \) in eq. (12) is well defined. More precisely, we can ask ourselves what happens if \( \text{Im}(\Delta_3) \rightarrow 0 \) and so the poles in \( \xi \) lie on the real axis. To answer such a question, we have to study what is the residue of the integrand of eq. (12) when \( \xi = \pm \sqrt{\Delta_3/4} \). The residue of the poles in \( \xi \), named \( R \), is given by :

\[
R = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \nabla^T \left( \frac{2X + A^{-1}B}{(D(X) + \Delta_3/4)^{3/2}} \right) \tag{13}
\]

If \( \text{Im}(\Delta_3) = 0 \), the imaginary part of the denominator is driven by \( \text{Im}(D(X)) \) which is of constant sign when the Feynman parameters travel across the simplex, that means that the function \( (D(X) + \Delta_3/4)^{3/2} \) never crosses its cut in this domain. So \( (2X + A^{-1}B)/(D(X) + \Delta_3/4)^{3/2} \) is a gentle function when \( X \) travels across the simplex. It is easy to show that, for such a function,

\[
\nabla^T \left( \frac{2X + A^{-1}B}{(D(X) + \Delta_3/4)^{3/2}} \right) = 0 \tag{14}
\]

So for \( \text{Im}(\Delta_3) = 0 \), \( R = 0 \) and the values \( \xi = \pm \sqrt{\Delta_3/4} \) are not poles for the integrand of eq. (12).

By computing explicitly the gradient in (12) and rearranging different terms, we end up with:

\[
I_4^1 = -\frac{2}{3} \frac{1}{B(2,1/2)} \int_0^\infty d\xi \frac{1}{\xi^2 - \Delta_3/4 - i\lambda} \sum_{i=1}^{4} \frac{\tilde{b}_i}{\det(G)} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{(\tilde{D}_i(x_1,x_2) + \xi^2 - i\lambda)^{3/2}} \tag{15}
\]

\(^2\) We skip the small imaginary part because this discussion is relevant only in the case of complex masses.
where we have introduced the following quantities:

\[ \Delta_3 = -2 \frac{\det(S)}{\det(G)} \]

\[ \bar{b}_i = \det(S) \sum_{j \in S} S^{-1}_{ij} \]

The new quadratic forms \( \mathcal{D}_i(x_1, x_2) \) are related to the original one by:

\[
\mathcal{D}_1(x_1, x_2) = D(0, x_1, x_2) \quad \mathcal{D}_2(x_1, x_2) = D(x_2, 0, x_1) \\
\mathcal{D}_3(x_1, x_2) = D(x_1, x_2, 0) \quad \mathcal{D}_4(x_1, x_2) = D(1 - x_1 - x_2, x_1, x_2)
\]

A second step consists of the application of the same method to the integration over the two left Feynman parameters \( x_1 \) and \( x_2 \) in (15): the new quadratic forms \( \mathcal{D}_i(x_1, x_2) \) can be written as:

\[
\mathcal{D}_i(x_1, x_2) = \bar{X}^T E_i \bar{X} + F_i^T \bar{X} + G_i
\]

where \( E_i \) is a \( 2 \times 2 \) matrix, \( F_i \) a two dimensional column vector and \( G_i \) a scalar. The two dimensional vector \( \bar{X} \) is defined by:

\[
\bar{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

In order that only the boundary term remains when applying eq. (9), we have to shift the power of the denominator and thus to introduce a new integral representation for it. After the second step, we end up with the integration on the left over Feynman parameter of a new quadratic form.

The third step, using the same method, will trade this remaining Feynman parameter integration with an integration over \( \sigma \) on half \( \mathbb{R} \), \( \sigma \) being the variable introduced to shift the power of the denominator in the last step.

We skip all the details and, after step 2 and 3, the final result is given by:

\[
I_4 = \frac{4}{3} \frac{1}{B(2,1/2)} \frac{1}{B(3/2, 1/2)} \frac{1}{B(1,1/2)} \sum_{i \in S} \sum_{j \in S \setminus \{i\}} \sum_{k \in S \setminus \{i,j\}} \frac{\bar{b}_i}{\det(G)} \frac{\bar{b}_j^{(ij)}}{\det(G^{(ij)})} \frac{\bar{b}_k^{(ij)}}{\det(G^{(ij)})} \int_0^\infty d\xi \int_0^\infty d\rho \int_0^\infty d\sigma \\
\times \frac{1}{\xi^2 - \Delta_3/4 - i \lambda} \frac{1}{\rho^2 - \Delta_2/4 - i \lambda} \\
\times \frac{1}{\xi^2 + \rho^2 + \sigma^2 - \Delta_1/4 - i \lambda} (D_{ijk} + \xi^2 + \rho^2 + \sigma^2 - i \lambda)^{1/2}
\]

Some explanations are needed in order to understand the notations of eq. (16). We introduce a set \( S = \{1, 2, 3, 4\} \) which contains the labels of the propagator of the four point functions. At the first step, we work with a \( 4 \times 4 \) \( S \) matrix from which we extracted a \( 3 \times 3 \) Gram matrix \( G \). At the second step, the quadratic forms \( \mathcal{D}_i(x_1, x_2) \) are the quadratic forms obtained by pinching one of the propagator of the initial diagram to which corresponds a \( 3 \times 3 \) \( S \) matrix named \( S^{(i)} \) and which is obtained from the initial \( S \) matrix by removing the row and the column \( i \). From this reduced \( S \) matrix, we can extract a reduce Gram matrix named \( G^{(i)} \). At the third step, we work with one dimensional quadratic forms which are the one obtained by pinching two propagators of the initial diagram, this leads to a \( 2 \times 2 \) \( S \) matrix named \( S^{\{i,j\}} \) obtained
by removing the rows and the columns $i$ and $j$. Again, from that matrix we can extract a one dimensional Gram matrix named $G^{(i,j)}$. At any step, we never performed a relabelling of the subscripts describing the left rows and columns of the $S$ matrix, this is for this reason that the subscripts $j$ and $k$ run over a subset of the set $S$.

At the end of the steps two and three, the following quantities have appeared which are defined as:

$$\tilde{D}_{ijk} = m_l^2 \quad \text{with} \quad l = S \setminus \{ i \ j \ k \}$$

and

$$\Delta_{ij}^1 = - \frac{2 \det (S^{(i,j)})}{\det (G^{(i,j)})}$$

$$\Delta_{i}^2 = \frac{2 \det (S^{(i,i)})}{\det (G^{(i,i)})}$$

4. Integration

It remains to perform the three integrations over $\xi$, $\rho$ and $\sigma$.

$$L_{ijk} = - \frac{\kappa}{4^3} \int_0^\infty d\xi \int_0^\infty d\rho \int_0^\infty d\sigma$$

$$\times \frac{1}{\xi^2 - \frac{2\lambda}{3} + i\lambda} \frac{1}{\xi^2 + \rho^2 - \frac{2\lambda}{3} - i\lambda}$$

$$\times \frac{1}{\xi^2 + \rho^2 + \sigma^2 - \frac{2\lambda}{3} - i\lambda}$$

$$\left( \tilde{D}_{ijk} + \xi^2 + \rho^2 + \sigma^2 - i\lambda \right)^{1/2}$$

with

$$\kappa = - \frac{256}{3} \frac{1}{B(2,1/2)} \frac{1}{B(3/2,1/2)} \frac{1}{B(1,1/2)}$$

To do that, we will prove the following intermediate result. Let us consider the following integral:

$$J = \int_0^\infty d\xi \frac{1}{\xi^2 + A + B} \frac{1}{(\xi^2 + B)^{1/2}}$$

where $A$ and $B$ are complex numbers. We have to distinguish two cases depending on $S_A$ and $S_B$: the sign of the imaginary part of $A$ (respectively $B$). Using Feynman parameter method, we can show that:

$$J = \int_0^1 dz \frac{1}{z^2 + (1 - z^2) A}$$

if $S_A = S_B$ and:

$$J = i \frac{1}{S_B} \int_0^\infty \frac{dz}{B z^2 - (1 + z^2) A} - \int_1^\infty \frac{dz}{B z^2 + (1 - z^2) A}$$

otherwise.

Let us sketch the way we proceed.

- For the $\sigma$ integration: this integration is of the type (17), so we use eq. (18) or (19) to trade the sigma integration with a $z$ integration.
• For the $\rho$ integration: this integration can be done using partial fraction decomposition and eq. (10).

• For the $\xi$ integration: again, this integration is of the type (17) and can be trade, using eq. (18) or (19), with a $v$ integration.

To go further, we have to distinguish the different cases, there are 8 cases to be considered. We present here, as example, only two cases.

4.1. Case $\text{Im}(\Delta_3) > 0$, $\text{Im}(\Delta_2) > 0$, $\text{Im}(\Delta_{ij}^i) > 0$, $\text{Im}(\tilde{D}_{ijk}) < 0$

This corresponds to the real mass case. After the integration over $\xi$, $\rho$ and $\sigma$, we end up with:

\[
L_{ijk} = \frac{1}{2} \int_0^1 \frac{dv}{v^2 P_{ijk} + R_{ij}} \left[ \int_0^1 \frac{dz}{z^2 (T + Q_i) + (1 - z^2) T} \right.
\]

\[
- \int_0^1 \frac{dz}{z^2 (v^2 P_{ijk} + R_{ij} + Q_i + T) + (1 - z^2) T} \right] \tag{20}
\]

where we have introduced the following quantities:

\[P_{ijk} = \tilde{D}_{ijk} + \frac{\Delta_{ij}^i}{4} \tag{21}\]
\[Q_i = \frac{\Delta_3}{4} - \frac{\Delta_2}{4} \tag{22}\]
\[R_{ij} = \frac{\Delta_2}{4} - \frac{\Delta_{ij}^i}{4} \tag{23}\]
\[T = -\frac{\Delta_3}{4} \tag{24}\]

Then, we made the following change of variables $x = \sqrt{z}$ and $y = \sqrt{v} z$ and exchange the order of the $y$ and $x$ integration. After these steps, the $x$ integration is easy to perform and setting $u^2 = y$, we get the following result:

\[
L_{ijk} = -\frac{1}{4} \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( u^2 P_{ijk} + R_{ij} + Q_i + T \right) \right.
\]

\[
- \ln \left( \frac{Q_i + T}{u^2 Q_i + T} \right) \right] \tag{25}\]

As an illustration, let us consider another case.

4.2. Case $\text{Im}(\Delta_3) < 0$, $\text{Im}(\Delta_2) > 0$, $\text{Im}(\Delta_{ij}^i) > 0$, $\text{Im}(\tilde{D}_{ijk}) < 0$

For this case, we will give no intermediate steps and the final results is given by:

\[
L_{ijk} = \frac{1}{4} \left\{ i \int_0^\infty \frac{du}{u^2 P_{ijk} Q_i + R_{ij} T} \left[ \ln \left( \frac{R_{ij} + Q_i}{u^2 (P_{ijk} + R_{ij} + Q_i) - T} \right) - \ln \left( \frac{Q_i}{u^2 Q_i - T} \right) \right] \right.
\]

\[
+ \int_1^\infty \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{R_{ij} + Q_i}{u^2 (P_{ijk} + R_{ij} + Q_i) + T} \right) - \ln \left( \frac{Q_i}{u^2 Q_i + T} \right) \right] \right.
\]

\[
+ \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{R_{ij} + Q_i}{u^2 P_{ijk} + R_{ij} + Q_i + T} \right) - \ln \left( \frac{Q_i}{Q_i + T} \right) \right] \} \tag{26}\]
The leftover $u$ integration is standard to perform and can be expressed in term of $\text{Li}_2$ and logarithm functions.

Nice features of the method are that for each cases (for instance eq. (25) or eq. (26)), the residue at the poles $u^2 = R_{ij} T/(P_{ijk} Q_i)$ or $u^2 = -R_{ij} T/(P_{ijk} Q_i)$ is zero by construction and also that for each integral, the logarithms never cross their cuts when $u$ travels along the path on the real axis defined by the bounds of the integration.

5. Conclusion
We presented a novel method to compute the scalar one loop four point functions, this method enables to compute four point functions with kinematics outside the physical domain in a very open way. In addition, the results are expressed in term of the determinant of the $\mathcal{S}$ matrix and reduced $\mathcal{S}$ matrices and also in term of the determinant of the Gram matrix and reduced Gram matrices. The drawback of this method is that it leads at the end to a formula which has more $\text{Li}_2$ than the other formula [5]. This method extends rather easily to the case where there are collinear/soft divergences. The results of all these formula have been coded into a FORTRAN 95 program and checked against the existing results [4], [9].

References
[1] van Oldenborgh G J 1991 Comput. Phys. Commun. 66 1–15
[2] Hahn T and Perez-Victoria M 1999 Comput. Phys. Commun. 118 153–165 (Preprint hep-ph/9807565)
[3] Nhung D T and Ninh L D 2009 Comput. Phys. Commun. 180 2258–2267 (Preprint 0902.0325)
[4] van Hameren A 2011 Comput. Phys. Commun. 182 2427–2438 (Preprint 1007.4716)
[5] ’t Hooft G and Veltman M J G 1979 Nucl. Phys. B153 365–401
[6] Denner A, Nierste U and Scharf R 1991 Nucl. Phys. B367 637–656
[7] Tarasov O V 1996 Phys. Rev. D54 6479–6490 (Preprint hep-th/9606018)
[8] Binoth T, Guillet J P and Heinrich G 2000 Nucl. Phys. B572 361–386 (Preprint hep-ph/9911342)
[9] Cullen G, Guillet J P, Heinrich G, Kleinschmidt T, Pilon E, Reiter T and Rodgers M 2011 Comput. Phys. Commun. 182 2276–2284 (Preprint 1101.5595)