Topics in 
Cubic Special Geometry

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Abstract

We reconsider the sub-leading quantum perturbative corrections to $\mathcal{N} = 2$ cubic special Kähler geometries. Imposing the invariance under axion-shifts, all such corrections (but the imaginary constant one) can be introduced or removed through suitable, lower unitriangular symplectic transformations, dubbed Peccei-Quinn (PQ) transformations.

Since PQ transformations do not belong to the $d = 4$ $U$-duality group $G_4$, in symmetric cases they generally have a non-trivial action on the unique quartic invariant polynomial $\mathcal{I}_4$ of the charge representation $\mathbf{R}$ of $G_4$. This leads to interesting phenomena in relation to theory of extremal black hole attractors; namely, the possibility to make transitions between different charge orbits of $\mathbf{R}$, with corresponding change of the supersymmetry properties of the supported attractor solutions. Furthermore, a suitable action of PQ transformations can also set $\mathcal{I}_4$ to zero, or vice versa it can generate a non-vanishing $\mathcal{I}_4$: this corresponds to transitions between “large” and “small” charge orbits, which we classify in some detail within the “special coordinates” symplectic frame.

Finally, after a brief account of the action of PQ transformations on the recently established correspondence between Cayley’s hyperdeterminant and elliptic curves, we derive an equivalent, alternative expression of $\mathcal{I}_4$, with relevant application to black hole entropy.
1 Introduction

Special Kähler geometry (SK) characterizes the scalar manifolds of Abelian vector multiplets in $\mathcal{N} = 2$ supergravity theory in $d = 4$ space-time dimensions (see e.g. [1, 2, 3, 4], and Refs. therein). Along the years, it has played a key role in various important developments in black hole (BH) physics.

Among these, the Attractor Mechanism [5] shed light on the dynamics of scalar fields coupled to BPS (Bogomol’ny-Prasad-Sommerfeld) and non-BPS extremal BHs. Through the introduction of an effective BH potential $V_{BH}$ [6], this mechanism describes the stabilization of the scalar fields in terms of the BH conserved charges in the near-horizon limit of the extremal BH background (see e.g. [7, 8, 9, 10, 11, 12], also for reviews and lists of Refs.).

Within theories with $\mathcal{N} = 2$ local supersymmetry emerging from Calabi-Yau compactifications of superstrings or $M$-theory, the Attractor Mechanism has played a key role in the study of connections with topological string partition functions [13] and relations with microstates counting (see for instance [9]), and also in the investigation of dynamical phenomena, such as wall crossing and split attractor flow (see e.g. [14], and Refs. therein).

In some seminal papers dating back to mid 90’s [5], the Attractor Mechanism was discovered by Ferrara, Kallosh and Strominger in $\mathcal{N} = 2$, $d = 4$ ungauged supergravity coupled to $n_V$ vector multiplets. This theory proved to be an especially relevant and rich framework for the study of the attractor dynamics of scalar flows coupled to extremal BHs.

An important arena in which many advances have been made along the years is provided by a particular yet broad class of SK geometries, namely the ones determined by an holomorphic prepotential function $F$ which is purely cubic in the complex scalar fields.
themselves:
\[ F_d \equiv \frac{1}{3!} d_{ijk} z^i z^j z^k. \] (1.1)

\( F_d \) defines the so-called \( d \)-SK geometries \[15, 16\]. These geometries naturally arise as the large volume limit of \( CY_3 \) compactifications of Type II(A) superstring theories, in which \( d_{ijk} \) is given by the triple intersection numbers of the \( CY_3 \) internal manifold itself (see Sec. 2.1.4 for further details, and list of Refs.).

Moreover, up to the so-called minimal coupling sequence (with quadratic prepotential) \[17\], all non-compact symmetric coset \( d \)-spaces, defined by a prepotential of the form \[16\]; \( G_4 \) is the \( d = 4 \) \( U \)-duality group, and \( H_4 \) is its maximal compact subgroup (with symmetric embedding). In symmetric SK geometries the Attractor Mechanism enjoys a noteworthy geometrical interpretation, related to the fascinating interplay among orbits of the charge irrepr. \( R \) of \( G_4 \) \[19, 20\], the solution of the Attractor Eqs. \[20\] and the related “moduli spaces” \[21\]. Through the Bekenstein-Hawking entropy (\( S \)) -area (\( A \)) formula \[22\]
\[ \frac{S}{\pi} = \frac{A}{4} = \sqrt{|I_4(Q)|} \], (1.2)

the entropy of the BH is given in terms of the unique invariant polynomial \( I_4 \) of the charge irrepr. \( R \) of \( G_4 \), which is quartic in charges \( Q \). It is also worth recalling that also the recently introduced first order approach to non-BPS scalar flows \[23\] has been completely solved in terms of geometrical quantities (\( U \)-duality invariants) in \[24\].

It is therefore natural to ask what is the role and the effect of sub-leading corrections to the \( \mathcal{N} = 2 \) purely cubic prepotential (1.1). As it is well known (see the recent discussion in \[25\], and Refs. therein), such corrections are of both quantum perturbative and non-perturbative nature, and not all of them are consistent with the Peccei-Quinn axion-shift symmetry \[26\], nor all of them actually affect the SK geometry of the scalar manifold itself (see e.g. \[27\]).

In this paper, extending on some previous results in \[15, 28, 29\], we further develop the study of those sub-leading corrections to \( d \)-SK geometries (1.1) which are consistent with the axion-shift symmetry and which do not affect the geometry of the vector multiplets’ scalar fields.\(^2\)

It is known \[15, 28\] that these sub-leading corrections can be included (or removed from) the \( \mathcal{N} = 2 \) symplectic sections by acting with suitable symplectic transformations, and this provides an effective shortcut to the process of solving the Attractor Eqs. (alias criticality conditions for \( V_{BH} \)) in the so corrected \( d \)-SK geometries. As we will find in the present investigation, such symplectic transformations have a group structure (we dub them Peccei-Quinn (PQ) symplectic transformations), but they do not belong to the suitable symplectic representation of \( G_4 \) itself.

At least for symmetric \( d \)-SK geometries, this leads to interesting consequences in the theory of charge orbits and “moduli spaces” of extremal BH attractor solutions. Indeed, the PQ transformations do not affect the geometry of the scalar manifold, neither

\(^1\)Here \( U \)-duality is referred to as the “continuous” limit (valid for large values of the charges) of the non-perturbative string theory symmetries introduced by Hull and Townsend in \[18\].

\(^2\)For a recent discussion of the unique (constant imaginary) term which is consistent with axion-shift and affects the geometry, see e.g. \[23\].
statification of the charge irrepr. space $\mathbb{R}$ into disjoint orbits, nor the structure of the corresponding “moduli spaces” of attractors\textsuperscript{3}, but they can change the value and the sign of $\mathcal{I}_4$, thus possibly switching from one charge orbits to another.

For instance, an extremal “small” BH configuration (with vanishing entropy according to formula (1.2)) within the $d$-SK geometry (1.1) can acquire, by introducing the quantum perturbative correction under consideration, a non-vanishing area of the event horizon, and thus a “large” nature (namely, a non-vanishing $\mathcal{I}_4$, and thus entropy, according to (1.2)). The opposite phenomenon can occur too, namely that “large” extremal BH configuration can become “small” for particular choices of the supporting charge vectors.

Another possible phenomenon is that the supersymmetry preserving features of the attractor configurations of $d$-SK geometry (1.1) can change in presence of those subleading corrections accounted for by PQ transformations. This is somewhat analogous to some phenomena observed in presence of the “$+i\lambda$” correction in the prepotential in [31].

By exploiting the PQ symplectic transformation, we will also study how the effective BH potential $V_{BH}$ gets modified in presence of the aforementioned corrections, and what is the fate of those charge configurations which support axion-free attractor solutions within the theory determined by (1.1). In general, the solutions of Attractor Eqs. for the corrected $d$-SK geometries can be obtained by considering the solutions in the purely cubic theory [29, 32], and by transforming the charges in such formulae with a suitable PQ transformation.

We will also briefly comment on the action of the PQ group on the roots of certain cubic elliptic curves, which have been recently connected [33] to the Cayley’s hyperdeterminant [34], namely to the (opposite of) $\mathcal{I}_4$ for the noteworthy triality-symmetric so-called $stu$ supergravity model [35]. This might lead to an interpretation of the PQ transformation within the intriguing “BH/qubit correspondence” [36].

Finally, we derive an alternative expression of $\mathcal{I}_4$ for symmetric $d$-SK geometries, and more in general for symmetric cubic geometries (such as the ones of some $\mathcal{N} > 2$-extended, $d = 4$ supergravities). This result allows for a consistent treatment of some expressions of the BH entropy available in the literature (see e.g. [32]). Furthermore, its further generalisation to the case of non-symmetric geometries (in which $\mathcal{I}_4$ is not generally related to the BH entropy) explicitly shows the contribution of the so-called $E$-tensor [16] introducing an explicit dependence on (some of the) scalar degrees of freedom.

The plan of the paper is as follows.

In Sec. 2.1 we analyse the PQ symplectic transformations within $\mathcal{N} = 2$, $d = 4$ SK geometry. More specifically, in Sec. 2.1.1 we recall the general structure of sub-leading terms in cubic prepotential, and their consistency with axion-shift symmetry. The PQ symplectic group is introduced in Sec. 2.1.2 and its relation to the $U$-duality group

\textsuperscript{3}In this respect, the general analysis and findings of the present paper explains the result obtained in Sec. 3 and App. A of [30], also providing a way to generalise them to generic BH charge configuration, and to a generic model with $\nu V$ vector multiplets.

Moreover, through the action of PQ symplectic group, also the results concerning non-perturbative instantonic corrections to the prepotential, obtained in Sec. 4 and App. B of [30], can be generalised to include the sub-leading quantum perturbative corrections under consideration. See treatment below for further comments.
clarified in Sec. 2.1.3. Moreover, Sec. 2.1.4 considers some aspects of stringy origin and topological interpretation of some generators of the PQ group.

Then, Sec. 2.2 applies this general formalism to relevant issues within the theory of extremal black hole attractors. Secs. 2.2.1 and 2.2.2 and is devoted to the study and classification (within symmetric cubic geometries) of the PQ group on the unique invariant polynomial $I_4$ of the charge representation $R$ of the $U$-duality group. At the end of Sec. 2.2.2 we briefly comment on the relevance of the PQ group for the attractor values of the scalars, i.e. for the non-degenerate critical points of the effective BH potential $V_{BH}$. The transformation properties of the latter are studied in Sec. 2.2.3 with an analysis of the possible axion-free supporting charge configurations.

Sec. 2.3 briefly analyses the “PQ-deformation” of the recently established intriguing relation between Cayley’s hyperdeterminant and elliptic curves.

Finally, in Sec. 3 an equivalent, alternative expression for $I_4$ is derived, by exploiting the identities characterising symmetric cubic special geometries, with relevant consequences on the matching of known expressions of the black hole entropy. In particular, the new expression $I_4$ allows one to relate its the scalar-dependence in non-symmetric geometries directly to the so-called $E$-tensor.

2 Peccei-Quinn Symplectic Transformations

2.1 General Theory

Let us consider $\mathcal{N} = 2$, $d = 4$ ungauged Maxwell-Einstein supergravity, whose vector multiplets’ scalar manifold is endowed with special Kähler (SK) geometry, based on an holomorphic prepotential function $F$, homogeneous of degree 2 in the contravariant symplectic sections $X^\Lambda$ (the reader is addressed e.g. to [1, 2, 3, 4] for a thorough introduction and list of Refs.).

2.1.1 Cubic Special Geometries and Axion-Shifts

We start and define the most general form of cubic prepotential as follows$^4$ ($d_{\Lambda\Sigma\Xi} = d_{(\Lambda\Sigma\Xi)} \in \mathbb{C}$):

$$
F \equiv \frac{1}{3!} d_{\Lambda\Sigma\Xi} \frac{X^\Lambda X^\Sigma X^\Xi}{X^0} = \\
= \frac{1}{3!} (\text{Red}_{ijk} + i \text{Im}d_{ijk}) \frac{X^i X^j X^k}{X^0} + \frac{1}{2} (\text{Red}_{0ij} + i \text{Im}d_{0ij}) X^i X^j + \\
+ \frac{1}{2} (\text{Red}_{00i} + i \text{Im}d_{00i}) X^i X^0 + \frac{1}{3!} (\text{Red}_{000} + i \text{Im}d_{000}) (X^0)^2. 
$$

$^4$Greek capital and Latin lowercase indices respectively run $0, 1, \ldots, n_V$ and $1, \ldots, n_V$ throughout. The naught index pertains to the graviphoton, while $n_V$ denotes the number of Abelian vector multiplets coupled to the supergravity one. Therefore, we work within the so-called symplectic basis of special coordinates (see e.g. [10, 2] and Refs. therein), which is manifestly covariant with respect to the $d = 5$ $U$-duality group $G_5$. 

4
By denoting the real and imaginary part of \( X^i \) respectively as \( X^i \equiv R^i + iI^i \), the corresponding Kähler potential reads

\[
K \equiv -\ln \left( i \left( X^\Lambda F_\Lambda - X^\Lambda F_\Lambda \right) \right) \\
= -\frac{4}{3} \text{Re} \delta_{ijk} I^i I^j I^k - \frac{2}{3} \text{Im} \delta_{ijk} R^i R^j R^k - 2i \text{Im} \delta_{ijk} I^i I^j I^k \\
-2i \text{Im} \delta_{0ij} R^i R^j - 2i \text{Im} \delta_{0ij} I^i I^0 - 2i \text{Im} \delta_{00i} R^i - \frac{2}{3} i \text{Im} \delta_{000}. \tag{2.2}
\]

Thus, the invariance of \( K \) under Peccei-Quinn (PQ) perturbative (continuous) axion-shift symmetry \[ R^i \to R^i + \alpha^i, \quad \alpha^i \in \mathbb{R} \tag{2.3} \]
yields

\[
\text{Im} \delta_{ijk} = \text{Im} \delta_{0ij} = \text{Im} \delta_{00i} = 0. \tag{2.4}
\]

The resulting axion-shift-invariant expression of \( K \) then simply reads

\[
K = -\frac{4}{3} \text{Re} \delta_{ijk} I^i I^j I^k - \frac{2}{3} i \text{Im} \delta_{000}, \tag{2.5}
\]

and the prepotential \( F \) given by \( \Delta \) can accordingly be split as

\[
F = \mathcal{F} + \mathfrak{F}, \tag{2.6}
\]

where

\[
\mathcal{F} \equiv \frac{1}{3!} \text{Re} \delta_{ijk} \frac{X^i X^j X^k}{X^0} + \frac{i}{3!} \text{Im} \delta_{000} (X^0)^2 \tag{2.7}
\]
is the part contributing to \( K \) given by \( \Delta \) and thus to the SK geometry, and

\[
\mathfrak{F} \equiv \frac{1}{2} \text{Re} \delta_{0ij} X^i X^j + \frac{1}{2} \text{Re} \delta_{00i} X^i X^0 + \frac{1}{3!} \text{Re} \delta_{000} (X^0)^2 \tag{2.8}
\]
is a quadratic form in \( X^\Lambda \), which does not contribute to \( K \). Thus, \( \mathcal{F} \) given by \( \Delta \) is the most general cubic prepotential which is consistent with the PQ axion-shift \( \Delta \) and which affects the geometry of the scalar manifold itself \cite{27}. Some issues within the SK geometry based on \( \mathcal{F} \) have been recently investigated in \cite{25} (see also \cite{28}).

On the other hand, \( \text{Re} \delta_{ijk} \) is usually denoted simply by the real symbol \( d_{ijk} \), and the holomorphic function

\[
\mathcal{F}_d \equiv \frac{1}{3!} d_{ijk} \frac{X^i X^j X^k}{X^0} \tag{2.9}
\]
is the prepotential of the so-called \( d \)-SK geometries\footnote{Regardless of the explicit form of \( d_{ijk} \), the corresponding special Kähler manifold has always at least \( n_V + 1 \) global isometries, namely an overall scaling and PQ axion-shifts (see Eq. \( \Delta \)), forming the group \( SO(1,1) \times \mathbb{R}^n \), which can be considered the “minimal \( G_4 \)” of \( d \)-SK geometries. Its relation to \( d = 5 \) uplift and further details can be found e.g. in \cite{37} (see also Refs. therein).} \cite{39,16}. This will be the most general framework we will be considering in the applications of Sec. 2.2
For later convenience, let us compute the derivatives of $\mathcal{F}$ with respect to the sections $X^\Lambda$:

$$\mathcal{F}_\Lambda \equiv D_\Lambda \mathcal{F} = \frac{\partial \mathcal{F}}{\partial X^\Lambda} = \begin{cases} \mathcal{F}_0 = \frac{1}{2} \text{Red}_{00}X^i + \frac{1}{3} \text{Red}_{000}X^0; \\ \mathcal{F}_i = \text{Red}_{0ij}X^j + \frac{1}{2} \text{Red}_{00}X^0. \end{cases} \quad (2.10)$$

It has been known (see e.g. [15, 28, 29]) that $\mathcal{F}$ can be introduced (or removed) in any $\mathcal{N} = 2$ prepotential by performing suitable symplectic transformations. More specifically, through the action of particular symplectic transformations one can introduce the effect of the sub-leading quantum perturbative terms (2.8) into the explicit expression of horizon values of attractors and into the corresponding value of BH entropy [28, 29].

A major part of the present investigation is devoted to a thorough analysis of this issue in full generality. In particular, we will focus on the effect of $\mathcal{F}$ on the BH entropy in the general framework of $d$-SKG, with leading cubic prepotential given by (2.9). This will naturally lead to the study of the effect of the so-called Peccei-Quinn transformations, namely particular symplectic transformations deeply related to $\mathcal{F}$, on the duality invariants and supersymmetry properties of extremal BH attractor solutions.

The results recently obtained in Sec. 3 of [30] provide an explicit example (with $n_V = 2$ and for a particular charge configuration) of some aspects of the general treatment given here. Indeed, the prepotential given by Eq. (3.7) of [30] is nothing but a particular case of the general structure (2.6)-(2.8).

2.1.2 The Peccei-Quinn Symplectic Group

Given an element

$$S \equiv \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \in GL(2n_V + 2, \mathbb{R}), \quad (2.11)$$

it belongs to the symplectic group $Sp(2n_V + 2, \mathbb{R}) \subsetneq GL(2n_V + 2, \mathbb{R})$ iff

$$S^T \Omega S = \Omega \iff S^{-1} = \Omega^{-1}S^T \Omega = -\Omega S^T \Omega, \quad (2.12)$$

where $\Omega$ is the $(2n_V + 2) \times (2n_V + 2)$ symplectic metric (the subscripts denote the dimensions of the square block components):

$$\Omega \equiv \begin{pmatrix} 0_{n_V+1} & I_{n_V+1} \\ -I_{n_V+1} & 0_{n_V+1} \end{pmatrix}. \quad (2.13)$$

7As shown in [35], the symplectic connection of SK geometry is flat.

8In this respect (and referring to the equation numbering of [30]), it is worth noting that the second of Eqs. (3.8) can be directly obtained from the general expression (2.9) for $d$-SKG geometry, because the sub-leading quantum perturbative terms appearing in Eq. (3.7) do not affect the Kähler potential and thus the metric.

9In all the following treatment, we work in the (semi)classical limit of large (continuous) charges, thus the field of definition of considered linear and symplectic groups is $\mathbb{R}$, and not $\mathbb{Z}$, as instead it would pertain to the quantum level.
The finite condition of symplecticity (2.12) translates on the square block components of \( S \) as follows:

\[
U^T V - W^T Z = I_{nV+1};
\]
\[
U^T W - W^T U = Z^T V - V^T Z = 0_{nV+1}.
\]

In general, the \( U \)-duality group \( G_4 \) of \( \mathcal{N} = 2, \ d = 4 \) supergravity is embedded into \( Sp(2nV + 2, \mathbb{R}) \) through its relevant (namely, smallest symplectic) (ir)repr. \( R \) (see e.g. \cite{2} and Refs. therein):

\[
G_4 \subseteq Sp(2nV + 2, \mathbb{R}).
\]

The vector of the fluxes of the two-form field strengths of the Abelian vector fields and of their duals

\[
Q \equiv (p^\Lambda, q_\Lambda)^T = (p^0, p^i, q_0, q_i)^T,
\]

as well as the vector of the holomorphic sections

\[
V \equiv (X^\Lambda, F_\Lambda)^T = (X^0, X^i, F_0, F_i)^T,
\]
sit in \( \mathbb{R} \), and thus they are \( Sp(2nV + 2, \mathbb{R}) \)-covariant, transforming under \( S \) as follows:

\[
Q' = S Q = \left( U^\Lambda_{\Sigma p} + Z^\Lambda_{\Sigma q_\Sigma} \right);
\]
\[
V' = S V = \left( U^\Lambda_{\Sigma X_\Sigma} + Z^\Lambda_{\Sigma F_\Sigma} \right).\]

Now, by recalling (2.10), it is immediate to realize that \( \mathcal{F}_\Lambda \) can be generated or removed by performing a suitable symplectic finite transformation on \( V \). Indeed, the identification

\[
\mathcal{F}_\Lambda \equiv F_\Lambda - V^\Lambda_{\Sigma} F_\Sigma = W_{\Lambda\Sigma} X_\Sigma = W_{\Lambda 0} X^0 + W_{\Lambda i} X^i
\]
defines, through Eq. (2.20), the components of the \((nV + 1) \times (nV + 1)\) sub-matrix \( W_{\Lambda\Sigma} \):

\[
W_{\Lambda\Sigma} = \begin{pmatrix}
W_{00} & W_{0j} \\
W_{i0} & W_{ij}
\end{pmatrix} = \frac{1}{3!} \begin{pmatrix}
2 \text{Red}_{000} & 3 \text{Red}_{00j} \\
3 \text{Red}_{00i} & 6 \text{Red}_{0ij}
\end{pmatrix} = \begin{pmatrix}
\varrho & c_j \\
c_i & \Theta_{ij}
\end{pmatrix} = W_{(\Lambda\Sigma)},
\]

which inherits the symmetry properties from the relevant components of the \( d_{\Lambda\Sigma\Xi} \) tensor. Note that we re-named the quantities for simplicity’s sake (\( \Theta_{ij} = \Theta_{(ij)} \)).

Thus, we are going to deal with particular symplectic transformations defined as follows:

1. In order to keep the contravariant symplectic sections \( X^\Lambda \) (and thus the coordinates of the scalar manifold) \textit{invariant} under the considered transformations, (2.20) imposes

\[
Z^\Lambda_{\Sigma} \equiv 0, \ U^\Lambda_{\Sigma} \equiv \delta^\Lambda_{\Sigma}.
\]
2. In order to generate or remove $\mathcal{F}_\Lambda$, as stated above one must define $W_{\Lambda \Sigma}$ as in Eq. (2.22), and furthermore Eq. (2.20) yields

$$V_{\Sigma} \equiv \delta_{\Sigma}^\Lambda.$$  (2.24)

The $(n_V + 1) \times (n_V + 1)$ matrices $U$, $Z$, $V$ and $W$ defined by Eqs. (2.22), (2.24) and (2.23) do satisfy the finite symplecticity condition (2.12), and we denote the corresponding symplectic matrix as

$$O \equiv \begin{pmatrix} I_{n_V + 1} & 0_{n_V + 1} \\ W & I_{n_V + 1} \end{pmatrix}.$$  (2.25)

It is easy to realize that $O$ given by (2.25) belongs to the $(n_V + 2)\cdot (n_V + 2)$-dimensional Abelian group

$$\mathcal{PQ}(2n_V + 2, \mathbb{R}) \equiv Sp(2n_V + 2, \mathbb{R}) \cap LUT(2n_V + 2, \mathbb{R}),$$  (2.26)

which we will henceforth refer to as the Peccei Quinn symplectic group. In (2.26) $LUT(2n_V + 2, \mathbb{R})$ is the $(n_V + 1)^2$-dimensional Abelian group of lower unitriangular $2 (n_V + 1) \times 2 (n_V + 1)$ real matrices, which are unipotent (see e.g. [10]). Correspondingly, the Peccei-Quinn (PQ) symplectic Lie algebra $pq(2n_V + 2, \mathbb{R})$ is given by

$$pq(2n_V + 2, \mathbb{R}) \equiv sp(2n_V + 2, \mathbb{R}) \cap lut(2n_V + 2, \mathbb{R}),$$  (2.27)

namely by the strictly lower triangular $2 (n_V + 1) \times 2 (n_V + 1)$ real matrices (which are nilpotent) with symmetric lower $(n_V + 1) \times (n_V + 1)$ block.

Matrices with structure as $O$ given by (2.25), and thus belonging to the group $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ defined above, appear also in other contexts. For instance, they are a particular case (with $A = I_{n_V + 1}$) of the quantum perturbative duality transformations in supersymmetric Yang-Mills theories coupled to supergravity (see e.g. [11], and Eq. (5.4) of [11]).

Let us give here some other explicit results, useful in the subsequent treatment. Eqs. (2.21, (2.22) and (2.24) imply

$$\mathcal{F}_\Lambda \equiv F'_{\Lambda} - F_{\Lambda},$$  (2.28)

Thus, within the framework under consideration, it follows that

$$F_{\Lambda} \equiv D_{\Lambda}F = \frac{\partial F}{\partial X_\Lambda} = \begin{cases} F_0 = -\frac{1}{3} \text{Re} d_{ijk} \frac{X_i X_j X_k}{(X_0)^2} + \frac{i}{3} \text{Im} d_{000} X^0; \\ F_i = \frac{1}{2} \text{Re} d_{ijk} \frac{X_j X_k}{X_0} ; \\ F'_{\Lambda} \equiv D_{\Lambda} \mathcal{F} + D_{\Lambda}F = D_{\Lambda}F = \frac{\partial F}{\partial X_\Lambda}, \end{cases}$$  (2.29)

where Eqs. (2.6) and (2.7) were used.
Moreover, by using \( \text{eq} \), the inverse of matrix \( O \) can be easily computed to be simply

\[
O^{-1} = \begin{pmatrix} I_{nV+1} & 0_{nV+1} \\ -\mathcal{W} & I_{nV+1} \end{pmatrix}.
\] (2.31)

Thus, by recalling Eqs. (2.19), (2.20), and the expressions (2.25) and (2.31) along with Eq. (2.22), one can write down the finite transformations of \( Q \) and \( V \) under the action of a generic element of \( \mathcal{P} \mathcal{Q} (2n_V + 2, \mathbb{R}) \) (the unwritten matrix components vanish throughout):

\[
Q' = OQ = \begin{pmatrix} p^0 \\ q_0 + q_i p^i + c_j p^j \\ q_i + c_i p^0 + \Theta_{ij} p^j \end{pmatrix} \Leftrightarrow Q = O^{-1} Q' = \begin{pmatrix} p^0 \\ q_0 - q_i p^i - c_j p^j \\ q_i - c_i p^0 - \Theta_{ij} p^j \end{pmatrix};
\]

(2.32)

\[
V' = OV = \begin{pmatrix} X^0 \\ X^i \\ F_0 + q_i X^0 + c_j X^j \\ F_i + c_i X^0 + \Theta_{ij} X^j \end{pmatrix} \Leftrightarrow V = O^{-1} V' = \begin{pmatrix} X^0 \\ X^i \\ F_0 - q_i X^0 - c_j X^j \\ F_i - c_i X^0 - \Theta_{ij} X^j \end{pmatrix}.
\]

(2.33)

### 2.1.3 Relation with \( U \)-Duality Transformations

In order to highlight some important features of the Peccei-Quinn transformations defined above, it is here convenient to briefly recall the properties of \( V \) and related quantities under the action of \( Sp (2n_V + 2, \mathbb{R}) \) (see e.g. \[42\] and Refs. therein).

The holomorphic sections \( V \) defined in (2.13) belong to the holomorphic (chiral) ring over the Kähler-Hodge bundle defined over the vector multiplets’ scalar manifold. Under a finite symplectic transformation \( S \in Sp (2n_V + 2, \mathbb{R}) \) defined by (2.11)-(2.15), \( V \) transform as

\[
V (z) \xrightarrow{S} SV' (z) = \exp [-f (z')] SV' (z').
\] (2.34)

\( z \) and \( z' \) collectively denote the scalar field parametrization (namely, the coordinate frame) before and after the application of \( S \). Thus, the action of \( S \) generally induces a (generally non-linear) coordinate transformation

\[
z \rightarrow z'.
\] (2.35)

Thus, the holomorphic superpotential \( W \equiv \langle Q, V (z) \rangle \equiv \Omega^T \Omega V (z) \) transforms as (recall \( \text{eq} \))

\[
W \xrightarrow{S} \exp [-f (z')] \langle Q', V' (z') \rangle \equiv \exp [-f (z')] W',
\] (2.36)

namely with an holomorphic overall factor \( \exp [-f (z')] \). The holomorphic function \( f (z') \) appearing in (2.34) and (2.36) is the gauge function of the Kähler transformation induced by \( S \) on the Kähler potential \( \mathcal{K} (z, \bar{z}) = - \ln [i \langle \nabla (z), V (z) \rangle] \) itself (recall Eq. (2.34)):

\[
\mathcal{K} (z, \bar{z}) \xrightarrow{S} - \ln \left[ i \langle \nabla (\bar{z}), V' (z') \rangle \right] + f (z') + \tilde{f} (\bar{z}) = \mathcal{K}' (z', \bar{z}) + f (z') + \tilde{f} (\bar{z}).
\] (2.37)
Eqs. (2.36) and (2.37) yield that the covariantly holomorphic sections $V(z, \overline{z}) \equiv \exp[K(z, \overline{z})/2] V(z)$, belonging to the Kähler-Hodge $U(1)$ bundle, transform under $S$ as follows (recall (2.34) and (2.37)):

$$V(z, \overline{z}) \xrightarrow{S} \exp[-i\text{Im}(f(z'))] SV'(z', \overline{z'}) ,$$

namely with an overall phase (Kähler-Hodge $U(1)$ factor) $\exp[-i\text{Im}(f(z'))]$. This in turn implies that the $N=2$ central charge $Z(z, \overline{z}) \equiv \langle Q, V(z, \overline{z}) \rangle$ transforms as

$$Z(z, \overline{z}) \xrightarrow{S} \exp[-i\text{Im}(f(z'))] Z'(z', \overline{z'}) .$$

A general consequence of Eqs. (2.34)-(2.39) is the following.

Under a transformation $S \in Sp(2(n_V + 2), \mathbb{R})$, $W(z)$ and $Z(z, \overline{z})$ are invariant iff $S$ does not induce any change in the coordinates of the scalar manifold. By looking at the conditions (2.14)-(2.15), it is immediate to realize that $O \in PQ(2n_V + 2, \mathbb{R})$ represented by (2.25) is actually the most general element of $Sp(2n_V + 2, \mathbb{R})$ that does not induce any transformation of coordinates on the scalar manifold, and thus leaves both $W$ and $Z$ (as well as the corresponding covariant derivatives $D_i W$ and $D_i Z$) invariant.

A direct consequence of this is that the effective BH potential [43]

$$V_{BH} \equiv |Z|^2 + g^5 (D_i Z) \overline{D_i Z}$$

is also invariant under $PQ(2n_V + 2, \mathbb{R})$:

$$V_{BH}(z, \overline{z}; Q) \xrightarrow{O} V_{BH}(z, \overline{z}; Q) .$$

For this reason, while $PQ(2n_V + 2, \mathbb{R})$ can be efficiently used to investigate the effects of $S$ given by (2.8) on the attractor points of $V_{BH}$ itself and on the BH entropy (through the study of the transformation properties of the quartic $G_4$-invariant $I_4$; see Sec. 2.2.1), its use in relation to $Z$, $D_i Z$ and $V_{BH}$ has some caveats, pointed out at the start of Sec. 2.2.3. The analysis of the latter Sec. relies on the results of [43] (see also [10] for a review, and Refs. therein) on the axion-free supporting charge configurations, and related supersymmetry properties, in $d$-SK geometries.

We are now going to show that

$$pq(2n, \mathbb{R}) \subseteq \frac{sp(2n_V + 2, \mathbb{R})}{g_4} ,$$

which thus implies, through exponential map:

$$PQ(2n, \mathbb{R}) \subseteq \frac{Sp(2n_V + 2, \mathbb{R})}{G_4} .$$

Namely, the PQ symplectic transformations lie in $Sp(2n_V + 2, \mathbb{R})$ outside of the $d = 4$ $U$-duality group $G_4$, whose Lie algebra is denoted by $g_4$ throughout. Thus, (2.27) and
can respectively be recast as
\[
pq (2nV + 2, \mathbb{R}) \equiv \frac{\text{sp} (2nV + 2, \mathbb{R})}{G_4},
\]
\[
\downarrow \exp
\]
\[
\mathcal{P}\mathcal{Q} (2nV + 2, \mathbb{R}) \equiv \frac{Sp (2nV + 2, \mathbb{R})}{G_4} \cap LUT (2nV + 2, \mathbb{R}),
\]
where \(\text{exp}\) denotes the exponential map.

Clearly, (2.42)-(2.44) hold whenever \(g_4\) is well defined, for instance in the \(N = 2\) models whose vector multiplets’ scalar manifold is a symmetric coset \(G_4/H_4\), with \(H_4\) being the maximal compact subgroup (with symmetric embedding) of \(G_4\) itself (see e.g. [16] and Refs. therein; see also [44] for a recent survey). Besides the \(\text{minimally coupled}\}\) \(\mathbb{CP}^n\) sequence with quadratic prepotential, these models are given by all symmetric \(d\)-\(SK\) geometries, whose prepotential is given by (2.49), with \(d_{ijk}\) satisfying the identity
\[
d_{r(pqd_{ij})k}\partial^{rkl} = \frac{4}{3}\delta_{[p}^{[r}\delta_{q]}^{l]}d_{qij]},
\]
which implies that \(d_{ijk}\) and its contravariant counterpart \(d^{ijk}\) are both \(G_5\)-invariant (scalar-independent) tensors (see Sec. 3 for further elucidation). Moreover, for all \(d\)-\(SK\) geometries a “minimal” \(G_4 \equiv SO (1, 1) \times \mathbb{R}^{nV}\) always exists (see Footnote 3).

Furthermore, for a symmetric \(d\)-\(SK\) geometry, the expression of the unique quartic invariant polynomial \(\mathcal{I}_4 (Q)\) of the symplectic repr. \(R\) of \(G_4\) reads (in the “special coordinates” sympletic basis [19]):
\[
\mathcal{I}_4 (Q) \equiv - (p^0)^2 q_0^2 - (p^i q_i)^2 - 2p^0 q_0 p^j q_i + 4q_0 \mathcal{I}_3 (p) - 4p^0 \mathcal{I}_3 (q) + 4 \{\mathcal{I}_3 (p), \mathcal{I}_3 (q)\},
\]
where
\[
\mathcal{I}_3 (p) \equiv \frac{1}{3!} d_{ijk} p^i p^j p^k; \quad \mathcal{I}_3 (q) \equiv \frac{1}{3!} d^{ijk} q_i q_j q_k; \quad \{\mathcal{I}_3 (p), \mathcal{I}_3 (q)\} \equiv \frac{\partial \mathcal{I}_3 (p)}{\partial p^0} \frac{\partial \mathcal{I}_3 (q)}{\partial q_i}.
\]

In \(d\)-\(SK\) geometries, the manifestly \((\mathfrak{g}_5 \oplus \mathfrak{so} (1, 1))\)-covariant form of the symplectic embedding of the infinitesimal transformation of the \(G_4\) is provided by the following 2 \((nV + 1) \times 2 (nV + 1)\) matrix \((i, j, k = 1, ..., nV)\) [37]:
\[
\mathcal{X} \equiv \begin{pmatrix}
3\lambda & b_j & 0 & 0^j \\
c^i & A^i_j + \lambda \delta^i_j & 0^i & d^{ijk} b_k \\
0 & 0_j & -3\lambda & -c^j \\
0_i & d_{ijk} c^k & -b_i & A^i_j - \lambda \delta^i_j
\end{pmatrix},
\]
where \(A^i_j\) is the electric-magnetic representation of the \(A_5\) algebra, \(\lambda\) is the \(\mathfrak{so} (1, 1)\) parameter, \(c^i\) are the parameters of the PQ axion-shift transformations \(t_{+2}\), and \(b_i\) are the parameters of the additional transformations \(t_{-2}\), not implementable on the vector potentials \(A^0, A^i\), which complete the algebra to \(g_4\) (subscripts denote weights w.r.t. \(\mathfrak{so} (1, 1)\)):
\[
g_4 = (\mathfrak{g}_5)_0 \oplus (\mathfrak{so} (1, 1))_0 \oplus t_{+2} \oplus t_{-2}.
\]
Thus, the matrix $X$ given by (2.48) realizes the Lie algebra $g_4$ of the $U$-duality group $G_4$ in its symplectic irrepr. $\mathbf{R}$, defining the embedding (2.16). By comparing the matrix $X$ given by (2.48) with the infinitesimal form of $O$ given by (2.25), namely with the strictly lower triangular matrix
\begin{equation}
O_{\text{inf}} = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 2 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 2
\end{pmatrix} \in \mathfrak{p}q (2n_V + 2, \mathbb{R}),
\end{equation}

one can conclude that results (2.42), and thus (2.43), hold.

### 2.1.4 Stringy Origin

It is here worth briefly commenting on the stringy origin of the components of the matrix $W_{\Lambda \Sigma}$ given by (2.22). For more details, and a list of Refs., we address the reader e.g. to the treatment of [28, 47, 48].

In Type IIA compactifications over Calabi-Yau threefolds ($CY_3$), it holds that
\begin{equation}
W_{0i} \equiv c_i = \frac{c_2}{24} = \frac{c_2 \cdot J_i}{24} = \frac{1}{24} \int_{CY_3} c_2 \wedge J_i,
\end{equation}

where $c_2$ is the second Chern class\(^\text{10}\) of $CY_3$, and $\{J_i\}_{i=1,\ldots,n_V}$ is a basis of $H^2(CY_3,\mathbb{R})$, the second cohomology group of $CY_3$.

Moreover, the coefficients of $F$ (as given by Eq. (2.7)) have the following stringy interpretation [28, 51, 52, 53]:
\begin{align}
\frac{1}{3!} \text{Red}_{ijk} &= C_{ijk}; \\
\frac{1}{3!} \text{Imd}_{000} &= -\frac{\zeta(3)}{(2\pi)^3} \chi,
\end{align}

where $C_{ijk}$ and $\chi$ respectively are the classical triple intersection numbers\(^\text{11}\) and Euler character of the $CY_3$, and $\zeta$ is the Riemann zeta function.

Notice that the other components of $W_{\Lambda \Sigma}$, namely $W_{00} \equiv \varrho$ and $W_{ij} \equiv \Theta_{ij}$, do not have an interpretation in terms of topological invariants of the internal manifold (see e.g. the discussion in [47], at least in the compactification framework under consideration. For this reason, they are usually disregarded in the stringy literature (see e.g. [28], in particular the discussion of Eq. (3.48) therein; see also [29]). However, it is worth pointing out that $W_{00}$ and $W_{ij}$ are important for fixing the integral basis for $V$ itself (see e.g. the discussion in [54, 55, 47]).

When setting $\varrho = \Theta_{ij} = 0$, the transformation (2.32) yields
\begin{equation}
\begin{pmatrix}
p^0 \\
p^i \\
q_0 \\
q_i
\end{pmatrix} \xrightarrow{\mathcal{O}^{-1}} \begin{pmatrix}
p'^0 \\
p'^i \\
q_0' - c_j p'^j \\
q_i' - c_i p'^0
\end{pmatrix},
\end{equation}

\(^{10}\)Note that, e.g., in presence of $R^2$-corrections, the second Chern class also contributes non-homogeneously to the BH entropy (see e.g. [49, 50]).

\(^{11}\)Actually, quantum (perturbative and non-perturbative) effects can also affect $\text{Red}_{ijk}$, i.e. (through Eq. (2.52)) the classical triple intersection numbers (see e.g. [28, 47], and Refs. therein).
which is a Witten theta-shift \cite{Witten} of electric charges via magnetic charges (in a generally axionful background).

Nevertheless, $\mathcal{W}_{00}$ and $\mathcal{W}_{ij}$ are perfectly consistent in a fully general supergravity analysis, and we will consider them non-vanishing throughout the applicative developments treated below.

In general, the term determined by $\text{Re} d_{ijk}$ in the general cubic prepotential (given by Eqs. (2.6)-(2.8)) is the leading one for large values of the scalar fields (moduli), and it defines the purely cubic prepotential (2.9) of the $d$-SK geometry of the complex structure (or Kähler structure) deformation moduli space of the large volume limit of the internal manifold $CY_3$ (in Type II compactifications). All other terms in Eqs. (2.6)-(2.8) define sub-leading contributions, which are of quantum perturbative nature, and consistent with the continuous PQ axion-shift symmetry (2.3). All such sub-leading terms, but the purely imaginary constant determined by $i \text{Im} d_{000}$ (and eventual renormalization of classical triple intersection numbers; see Footnote 6), can be taken into account by means of the group $\mathcal{P} \mathcal{Q} (2n_V + 2, \mathbb{R})$.

Non-perturbative effects (which can generally traced back to world-sheet instantons, i.e. to non-perturbative phenomena in the non-linear sigma model) usually exhibit exponential dependence on the moduli, and they are thus exponentially suppressed in the large volume limit (see e.g. \cite{28} and \cite{57, 58}). They break down the perturbative continuous PQ axion-shift symmetry (2.3) to its discrete form, namely \cite{17}

$$X^i \to X^i + 1.$$ \hspace{1cm} (2.55)

In some stringy framework, exponential terms (e.g. polylogarithmic functions) can arise also from quantum perturbative corrections (see e.g. the discussion in \cite{28} and \cite{57, 58}). The effect of non-perturbative, exponential corrections to cubic prepotentials on the spectrum and the stability of extremal BH attractors has been recently addressed in \cite{30}, whose findings confirm the general belief that non-perturbative correction lift the “flat” directions (if any) of the perturbative theory.\textsuperscript{12} At the level of the prepotential, this can be traced back to the fact that exponential corrections to the purely cubic holomorphic prepotential (2.9) $d$-SK geometries (of the kind given by Eq. (4.1) of \cite{30}) affect the geometry properties of the scalar manifold itself.

2.2 Application to Black Hole Attractors, Entropy and Supersymmetry

As pointed out in Sec. 2.1.3, the Peccei-Quinn symplectic group $\mathcal{P} \mathcal{Q} (2n_V + 2, \mathbb{R})$ is a proper subgroup of $\text{Sp}(2n_V + 2, \mathbb{R})_{G_4}$. The latter is the most general group acting linearly on the charges $Q$ which can change the value and possibly the sign of the unique quartic invariant $I_4(Q)$ of the symplectic (ir)repr. $R$ of $G_4$ itself.

In the following treatment, within the manifestly $G_3$-covariant “special coordinates” symplectic frame, we will analyse how $\mathcal{P} \mathcal{Q} (2n_V + 2, \mathbb{R})$ acts on $I_4(Q)$, on the non-degenerate critical points of the effective BH potential $V_{BH}$ (alias extremal BH attractors)\textsuperscript{12} Actually, also quantum perturbative corrections, such as the one given by the term $i \text{Im} d_{000}$ in (2.7) (with stringy origin given by (2.53)) can lift (some of the) “flat directions” of extremal BH attractor solutions \cite{59}.
[5], and on their supersymmetry properties. We will work within the $d$-SK geometries determined by the prepotential (2.9). When they involve the contravariant tensor $\delta^{ijk}$, the results on the transformation properties of $I_4$ generally hold only for $d$-SK geometries such that the coset $G_4/H_4$ is symmetric (see e.g. [16], and Refs. therein).

By suitably adapting its action, $\mathcal{P}\mathcal{Q}(2nV + 2, \mathbb{R})$ reveals to be a very effective tool to investigate the effect of the quantum perturbative sub-leading corrections (2.8) to the leading $d$-SK prepotential (2.9), some of which have a topological interpretation (see Sec. 2.1.4).

We anticipate that, under certain conditions on the ratio between the charges $Q$ and the parameters ($\rho, c_i, \Theta_{ij}$) of the finite PQ transformation $\mathcal{O}$ (given by Eq. (2.25) and (2.22)), the action of $\mathcal{P}\mathcal{Q}(2nV + 2, \mathbb{R})$ can give rise to a “transition” among the various orbits of $R$ of $G_4$, which in turn changes the supersymmetry-preserving features of the extremal BH attractor solutions.\(^{13}\)

2.2.1 Transformation of $I_4$

We start and apply the finite transformation\(^{14}\) $\mathcal{O}^{-1} \in \mathcal{P}\mathcal{Q}(2nV + 2, \mathbb{R})$ (given by (2.32)) to the $G_4$-invariant quartic polynomial $I_4(Q)$ given by (2.46)-(2.47). Thus, after some algebra, the following result is achieved:

$$\mathcal{P}\mathcal{Q}(2nV + 2, \mathbb{R}) \ni \mathcal{O}^{-1}: I_4(Q) \rightarrow I_4(\mathcal{O}^{-1}Q') = I_4(\mathcal{O}^{-1}Q') = I_4(Q') + I_4(Q', \rho, c_i, \Theta_{ij}),$$

\(^{13}\)Thus, our results should have interesting connections with the $d = 3$ timelike-reduced geodesic formalism and results of [60], whose thorough investigation we leave for further future study. For some developments in a $d = 4$ framework, see [61] (and also [7]).

\(^{14}\)We consider $\mathcal{O}^{-1}$ rather than $\mathcal{O}$ (a choice which is clearly immaterial at group level) because operationally (as discussed in [28]) one would like to include the effects of the sub-leading ($\rho, c_i, \Theta_{ij}$)-dependent terms in the prepotential (2.6)-(2.8) on the Bekenstein-Hawking BH entropy [22] by simply performing the computations within the purely cubic prepotential (2.9) (see e.g. the analysis of [43]) and then by applying the transformation $\mathcal{O}^{-1}$ on $Q$. Note that we will not deal here with the term $\frac{1}{3}Imd_{000}(X^0)^2$ in (2.7), which has been recently studied in [25].
where the quartic quantity $I_4$, describing the “PQ-deformation” of $I_4(Q)$, is given by the following expression\textsuperscript{15}:

$$I_4(Q; \varrho, c_i, \Theta_{ij}) = 2(p^0)^4 \left( \frac{1}{3} d^{ijk} c_i c_j c_k - \frac{1}{2} g^2 \right)$$

$+$ $2(p^0)^3 \left( q_0 - \varrho c_i p^i - d^{ijk} q_i c_j c_k + d^{ijk} c_i c_j \Theta_{klp} \right)$

$+$ $2(p^0)^2 \left( -2(c_i p^i)^2 + 2q_0 c_i p^i + \varrho p^i q_i - \varrho \Theta_{ij} p^i p^j - 2 d^{ijk} q_i c_j \Theta_{klp} \right)$

$+$ $2(p^0) \left( 2p^i q_i c_j p^j - 2c_i \Theta_{ij} p^i p^j p^k + q_0 \Theta_{ij} p^i p^j - \frac{1}{2} \varrho d^{ijk} p^i p^j p^k \right)$

$+$ $- (\Theta_{ij} p^i p^j)^2 + 2p^i q_i \Theta_{ij} p^j p^k - \frac{2}{3} c_i p^l d^{ijk} p^j p^k$

$$- 2 d^{ijk} d^{ilm} p^i p^j p^k q_i \Theta_{ms} p^s + d^{ijk} d^{ilm} p^i p^j p^k \Theta_{ls} \Theta_{mt} p^s p^t. \quad (2.57)$$

Note that the degree-4 homogeneity of $I_4$ in the charges is not spoiled, due to the linearity of the action of $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ on the charges themselves.

We now analyse various particular (both “large” and “small”) charge configurations, showing how the action of $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ can give rise to two types of phenomena, both corresponding to switching among different $\mathbb{R}$-orbits:

- change of sign of $I_4$:

  $$I_4(Q) \geq 0 \quad \mathcal{PQ} I_4(Q) + I_4(Q; \varrho, c_i, \Theta_{ij}) \leq 0, \quad (2.58)$$

  corresponding to a switch between different “large” $\mathbb{R}$-orbits \textsuperscript{20};

- generation of a non-vanishing $I_4$:

  $$I_4(Q) = 0 \quad \mathcal{PQ} I_4(Q) + I_4(Q; \varrho, c_i, \Theta_{ij}) \geq 0, \quad (2.59)$$

  or the other way around, generation of a vanishing $I_4$:

  $$I_4(Q) \geq 0 \quad \mathcal{PQ} I_4(Q) + I_4(Q; \varrho, c_i, \Theta_{ij}) = 0, \quad (2.60)$$

  both corresponding to a switch between a “large” and a “small” $\mathbb{R}$-orbit (usually named “charge orbit”).

\textsuperscript{15}Throughout the subsequent treatment, we omit the priming of the $O^{-1}$-transformed charges.
Some comments on the meaning of Eqs. (2.58)-(2.60) are in order.

- Firstly, let us recall that, through the Bekenstein-Hawking formula (1.2), “large” and “small” charge orbits respectively corresponds to $I_4 \neq 0$ and $I_4 = 0$; furthermore, “small” orbits split in light-like (3-charge), critical (2-charge) and doubly-critical (1-charge) ones [19, 62, 63, 64, 65].

Then, the general treatment of Sec. [2.3] implies that, in presence of $(q, c_i, \Theta_{ij})$-dependent sub-leading contributions (2.8) (recall the change of notation (2.22)) to the purely cubic prepotential (2.9) of $d$-SK geometry, the BH entropy $S$ becomes $(q, c_i, \Theta_{ij})$-dependent:

$$S = \frac{A}{4} = \sqrt{|I_4(q) + I_4(q; q, c_i, \Theta_{ij})|},$$  

(2.61) where $I_4(q; q, c_i, \Theta_{ij})$ is defined in (2.57). Consequently, depending on the relations between $I_4(q)$ and $I_4(q; q, c_i, \Theta_{ij})$, the phenomena (2.58)-(2.60) can occur, and the ones related to $c_i$ have, by virtue of (2.51), a clear topological interpretation within Type II CY$_3$-compactifications.

It should be remarked that the geometry of the symmetric coset $G_4/H_4$ is unaffected by the action of $Sp(2n_V + 2, \mathbb{R})$ (which just produces a change of coordinates; see Sec. [2.1.3]), and thus a fortiori by the action of its proper subgroup $PQ(2n_V + 2, \mathbb{R})$. Furthermore, by virtue of the treatment of Sec. [2.1.3], $PQ(2n_V + 2, \mathbb{R})$ does not act on the coordinates of the scalar manifolds, and thus does not induce any Kähler gauge transformation (2.37) on $K$, nor any holomorphic scaling (2.36) on $W$ (and $D_W$) and local phase transformation (2.39) on $Z$ (and $D_Z$) itself. Thus, the only effect of $PQ(2n_V + 2, \mathbb{R})$ on the BH effective potential $V_{BH}$ and its non-degenerate critical points (alias extremal BH attractors) [5] is a $(q, c_i, \Theta_{ij})$-dependent transformation of the charge vector $Q$, as given by (2.32). This fact will allow us to analyse the axion-free-supporting nature of the BH charge configurations in presence of non-vanishing parameters $q$, $c_i$ and $\Theta_{ij}$ by relying on the results of [14] (holding for generic (2.30)). The results recently obtained in Sec. 3 of [30] are an expected confirmation of all this reasoning.

By virtue of the transition from (1.2) to (2.61) through (2.56), $Sp(2n_V + 2, \mathbb{R})$ (and therefore its proper subgroup $PQ(2n_V + 2, \mathbb{R})$) does not affect the geometry of the scalar manifold, but it may affect the “magnitude” of the near-horizon space-time BH background, since its action may change the event horizon area $A$ of the extremal BH, and thus the (semi)classical Bekenstein-Hawking BH entropy $S$. The phenomena described by Eqs. (2.58)-(2.60) correspond to $(q, c_i, \Theta_{ij})$-dependent transformations moving from one charge orbit to another in the representation space $R$ of $G_4$.

The geometry and the classification of BH charge orbits (and related “moduli spaces”[14]) is not affected by $Sp(2n_V + 2, \mathbb{R})$ (and therefore by $PQ(2n_V + 2, \mathbb{R})$), but symplectic transformations can induce “transmutations” of the nature of the charge vector $Q \rightarrow Q([q, c_i, \Theta_{ij}])$, and thus of its supersymmetry preserving properties. As we will see in the case study considered in Sec. [2.2.2], in the case of $PQ(2n_V + 2, \mathbb{R})$ the actual occurrence of these phenomena depends on the very relations between $Q$ and the transformation parameters $(q, c_i, \Theta_{ij})$ themselves.

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16This has been recently confirmed by the analysis of the particular model of Sec. 3 of [30].
2.2.2 Analysis of “Large” and “Small” Configurations

The above treatment will be further clarified by the various examples which we are going to treat, generalising and systematically developing some points mentioned in [28]. We will make extensive use of formulæ (2.32) and (2.56)-(2.61).

1. “Large” \((p^0, q_0)\) (Kaluza-Klein) configuration. It supports non-BPS \(Z_H \neq 0\) (possibly axion-free [43]) attractors, and it is the supergravity analogue of \(D0-D6\) configuration in Type \(II\):

\[
Q \equiv (p^0, 0, q_0, 0)^T \Rightarrow I_4(Q) = - (p^0)^2 q_0^2 < 0. \tag{2.62}
\]

The action of \(\mathcal{P}Q(2n_V + 2, \mathbb{R})\) reads

\[
\begin{pmatrix}
  p^0 \\
  0 \\
  q_0 \\
  0
\end{pmatrix} \xrightarrow{\mathcal{O}^{-1}} \begin{pmatrix}
  p^0 \\
  0 \\
  q_0 - qp^0 \\
  -c_i p^0
\end{pmatrix}, \tag{2.63}
\]

and thus it generates \(c_i\)-dependent electric charges \(q_i\)’s, which in Type \(II\) compactifications corresponds to a stack of \(D2\) branes depending on the components of the second Chern class \(c_2\) of \(CY_3\) (recall Eq. (2.51)). The corresponding transformation of \(I_4\) reads

\[
- (p^0)^2 q_0^2 < 0 \xrightarrow{\mathcal{O}^{-1}} (p^0)^4 \left[ \frac{2}{3} d_{ijk} c_i c_j c_k - \left( \frac{q_0}{p^0} - \varrho \right)^2 \right] \gtrless 0. \tag{2.64}
\]

Thus, depending on whether

\[
\frac{2}{3} d_{ijk} c_i c_j c_k \gtrless \left( \frac{q_0}{p^0} - \varrho \right)^2, \tag{2.65}
\]

a “large” \((I_4 > 0):\)BPS or non-BPS \(Z_H = 0\), a “small” \((I_4 = 0):\)BPS or non-BPS), or a “large” non-BPS \(Z_H \neq 0\) \((I_4 < 0)\) BH charge configuration is generated by the action of \(\mathcal{P}Q(2n_V + 2, \mathbb{R})\). As anticipated in the above treatment, (2.65) shows that the relations among the components of \(Q\) and the parameters of the PQ symplectic transformation turn out to be crucial for the properties of the resulting charge configuration. The change of the axion-free-supporting nature of this configuration will be analysed in Sec. 2.2.3.

2. “Large” \((p^0, q_i)\) (“electric”) configuration. Depending on \(I_4(Q) \gtrsim 0\), it supports all kind of attractors (possibly axion-free [43]). It is the supergravity analogue of \(D2-D6\) configuration in Type \(II\):

\[
Q \equiv (p^0, 0, 0, q_i)^T \Rightarrow I_4(Q) = - \frac{2}{3} p^0 d_{ijk} q_i q_j q_k \gtrsim 0. \tag{2.66}
\]
The action of $\mathcal{P}Q(2n_V + 2, \mathbb{R})$ is
\[
\begin{pmatrix}
  p^0 \\
  0 \\
  0 \\
  q_i
\end{pmatrix}
\xrightarrow{\sigma^{-1}}
\begin{pmatrix}
  p^0 \\
  0 \\
  -g p^0 \\
  q_i - c_i p^0
\end{pmatrix},
\]
and thus it generates a $g$-dependent electric charge $q_0$. The corresponding transformation of $I_4$ reads
\[
-\frac{2}{3} p^0 d^{ijk} q_i q_j q_k \gtrless 0
\]
\[
\downarrow \mathcal{O}^{-1}
\]
\[
-\frac{2}{3} p^0 d^{ijk} q_i q_j q_k + 2 (p^0)^2 \left[ \left(\frac{1}{3} d^{ijk} c_i c_j c_k - \frac{1}{2} q^2 \right) \left(\frac{1}{2} - p^0 d^{ijk} q_i c_j c_k + d^{ijk} q_i q_j c_k \right) \right] \gtrless 0.
\]
Thus, depending on the sign (or on the vanishing) of the quantity in the last line of (2.68), the same comments made for configuration 1 hold in this case. The change of the axion-free-supporting nature of this configuration will be analysed in Sec. 2.2.3.

3. “Large” $(p^i, q_0)$ (“magnetic”) configuration. It is the “electric-magnetic dual” of the “electric” configuration 2. It is then interesting to compare the action of $\mathcal{P}Q(2n_V + 2, \mathbb{R})$ (which is asymmetric on magnetic and electric charges) on configurations 2 and 3. Depending on $I_4(Q) \gtrless 0$, this configuration supports all kind of attractors (possibly axion-free [43]). It is the supergravity analogue of $D0$-$D4$ configuration in Type II:
\[
Q \equiv (0, p^i, q_0, 0)^T \rightarrow I_4(Q) = \frac{2}{3} q_0 d_{ijk} p^i p^j p^k \gtrless 0.
\]
The action of $\mathcal{P}Q(2n_V + 2, \mathbb{R})$ is
\[
\begin{pmatrix}
  0 \\
  p^i \\
  q_0 \\
  0
\end{pmatrix}
\xrightarrow{\sigma^{-1}}
\begin{pmatrix}
  0 \\
  p^i \\
  q_0 - c_i p^i \\
  -\Theta_{ij} p^j
\end{pmatrix},
\]
and thus it generates $\Theta_{ij}$-dependent electric charges $q_i$’s. The corresponding transformation of $I_4$ reads
\[
\frac{2}{3} q_0 d_{ijk} p^i p^j p^k \gtrless 0
\]
\[
\downarrow \mathcal{O}^{-1}
\]
\[
\frac{2}{3} q_0 d_{ijk} p^i p^j p^k - \left(\Theta_{ij} p^j p^i\right)^2 - \frac{2}{3} c_i p^i d_{ijk} p^j p^k + d_{ijk} d_{ilm} p^j p^k \Theta_{is} \Theta_{mt} p^s p^t \gtrless 0.
\]
Thus, depending on the sign (or on the vanishing) of the quantity in the last line of (2.71), the same comments as made for above configurations hold. The change of the axion-free-supporting nature of this configuration will be analysed in Sec. 2.2.3.

Note that for $\Theta_{ij} = 0$, an example treated in [28] is recovered.
4. **“Small” lightlike (3-charge) \( q_i \) (“electric”) configuration.** This is the limit \( p^0 = 0 \) of configuration 2. In Type II, it corresponds to only \( D2 \) branes:

\[
Q \equiv (0, 0, 0, q_i)^T \Rightarrow \mathcal{I}_4 (Q) = 0,
\]

such that (recall definition (2.51))

\[
\mathcal{I}_3 (q) \neq 0,
\]

corresponding to a “large” BH in \( d = 5 \), with near-horizon geometry \( AdS_2 \times S^3 \) (see e.g. [43], and Refs. therein). Since there are no magnetic charges, \( \mathcal{P} \mathcal{Q} (2n \nu + 2, \mathbb{R}) \) is inactive on this configuration, which is thus left unchanged:

\[
\begin{pmatrix}
0 \\
0 \\
0 \\
q_i
\end{pmatrix}
\xrightarrow{\mathcal{O}^{-1}}
\begin{pmatrix}
0 \\
0 \\
0 \\
q_i
\end{pmatrix}
\]

(2.74)

5. **“Small” lightlike (3-charge) \( p^i \) (“magnetic”) configuration.** This is the limit \( q_0 = 0 \) of configuration 3. In Type II, it corresponds to only \( D4 \) branes:

\[
Q \equiv (0, p^i, 0, 0)^T \Rightarrow \mathcal{I}_4 (Q) = 0,
\]

such that (recall definition (2.51))

\[
\mathcal{I}_3 (p) \neq 0,
\]

corresponding to a “large” black string in \( d = 5 \), with near-horizon geometry \( AdS_3 \times S^2 \) (see e.g. [43], and Refs. therein). This configuration is the “electric-magnetic dual” of the “electric” configuration 4. However, differently from what happens for configuration 4, \( \mathcal{P} \mathcal{Q} (2n \nu + 2, \mathbb{R}) \) is active in this case (due to its asymmetric action on electric and magnetic charges):

\[
\begin{pmatrix}
0 \\
p^i \\
0 \\
0
\end{pmatrix}
\xrightarrow{\mathcal{O}^{-1}}
\begin{pmatrix}
0 \\
p^i \\
-c_{ij}p^j \\
-\Theta_{ij}p^j
\end{pmatrix}
\]

and it generates \( \Theta_{ij} \)-dependent electric charges \( q_i \)’s, as well as \( c_i \)-dependent electric charge \( q_0 \). In Type II compactifications, the latter corresponds to a stack of \( D0 \) branes depending on the components of the second Chern class \( c_2 \) of CY\(_3\) (recall Eq. (2.51)). The corresponding transformation of \( \mathcal{I}_4 \) reads

\[
0 \xrightarrow{\mathcal{O}^{-1}} (\Theta_{ij}p^j p^i)^2 - \frac{2}{3} c_{ij}p^j d_{ijk}p^j p^k + d_{ijk}d_{ilm}p^j p^k \Theta_{ls} \Theta_{mt} p^s p^t \geq 0.
\]

(2.78)

Thus, according to (2.78), a “large” (\( \mathcal{I}_4 > 0 \):BPS or non-BPS \( Z_H = 0 \)), a “small” (\( \mathcal{I}_4 = 0 \):BPS or non-BPS), or a “large” non-BPS \( Z_H \neq 0 \) (\( \mathcal{I}_4 < 0 \)) BH charge configuration can be generated. In case the quantity in (2.78) does not vanish, this is an example of phenomenon (2.59). Note that for \( \Theta_{ij} = 0 \), an example treated in [28] is recovered.
6. **“Small” critical (2-charge) \( q_i \) (“electric”) configuration.** This is the limit \( \mathcal{I}_3(q) = 0 \) of configuration 4. In Type II, it corresponds to only 2D branes:

\[
Q \equiv (0, 0, 0, q_i)^T \Rightarrow \mathcal{I}_4(Q) = 0,
\]

such that (recall definition (2.74))

\[
\begin{cases}
\mathcal{I}_3(q) = 0; \\
\partial \mathcal{I}_3(q) / \partial q_i \neq 0 \text{ for some } i,
\end{cases}
\]

(2.80)

corresponding to a “small” lightlike BH in \( d = 5 \). Since there are no magnetic charges, \( \mathcal{P}Q(2n_V + 2, \mathbb{R}) \) is inactive on this configuration, which is thus left unchanged (see Eq. (2.74)).

7. **“Small” critical (2-charge) \( p_i \) (“magnetic”) configuration.** This is the limit \( \mathcal{I}_3(q) = 0 \) of configuration 5. In Type II, it corresponds to only 2D branes:

\[
Q \equiv (0, p_i, 0, 0)^T \Rightarrow \mathcal{I}_4(Q) = 0,
\]

such that (recall definition (2.74))

\[
\begin{cases}
\mathcal{I}_3(p) = 0; \\
\partial \mathcal{I}_3(p) / \partial p_i \neq 0 \text{ for some } i,
\end{cases}
\]

(2.82)

corresponding to a “small” lightlike black string in \( d = 5 \). This configuration is the “electric-magnetic dual” of the “electric” configuration 6. However, differently from what happens for configuration 6, \( \mathcal{P}Q(2n_V + 2, \mathbb{R}) \) is active in this case, due to its asymmetric action on electric and magnetic charges. As given by Eq. (2.77), \( \Theta_{ij} \)-dependent electric charges \( q_i \)'s and \( c_i \)-dependent electric charge \( q_0 \) are generated. The corresponding transformation of \( \mathcal{I}_4 \) reads

\[
0 \overset{\sigma^{-1}}{\rightarrow} -\left( \Theta_{ij} p^i p^j \right)^2 + d_{ijk} d^{lm} p^i p^j \Theta_{ks} \Theta_{lt} p^s p^t \geq 0.
\]

(2.83)

Thus, according to (2.83), a “large” (\( \mathcal{I}_4 > 0 \): BPS or non-BPS \( Z_H = 0 \)), a “small” (\( \mathcal{I}_4 = 0 \): BPS or non-BPS), or a “large” non-BPS \( Z_H \neq 0 \) (\( \mathcal{I}_4 < 0 \)) BH charge configuration can be generated. In case the quantity in (2.83) does not vanish, this is an example of phenomenon (2.59).

8. **“Small” doubly-critical (1-charge) \( q_i \) (“electric”) configuration.** This is the limit \( \partial \mathcal{I}_3(q) / \partial q_i = 0 \) of configuration 6. In Type II, it corresponds to only 2D branes:

\[
Q \equiv (0, 0, 0, q_i)^T \Rightarrow \mathcal{I}_4(Q) = 0,
\]

such that (recall definition (2.74))

\[
\begin{cases}
\mathcal{I}_3(q) = 0; \\
\partial \mathcal{I}_3(q) / \partial q_i = 0 \forall i; \\
q_i \neq 0 \text{ for some } i,
\end{cases}
\]

(2.85)

corresponding to a “small” critical BH in \( d = 5 \). Since there are no magnetic charges, \( \mathcal{P}Q(2n_V + 2, \mathbb{R}) \) is inactive on this configuration, which is thus left unchanged (see Eq. (2.74)).
9. "Small" **doubly-critical** (1-charge) \( p^i \) ("magnetic") configuration. This is the limit \( \partial I_3 (p) / \partial p^i = 0 \) of configuration 7. In Type II, it corresponds to only D4 branes:

\[
Q \equiv (0, p^i, 0, 0)^T \Rightarrow I_4 (Q) = 0,
\]

such that (recall definition (2.47))

\[
\begin{cases}
I_3 (p) = 0; \\
\partial I_3 (p) / \partial p^i = 0 \forall i; \\
p^i \neq 0 \text{ for some } i,
\end{cases}
\]

(2.87)

(corresponding to a "small" critical black string in \( d = 5 \)). This configuration is the "electric-magnetic dual" of the "electric" configuration 8. However, differently from what happens for configuration 8, \( \mathcal{P}Q (2n_V + 2, \mathbb{R}) \) is active (see Eq. (2.77)) in this case, due to its asymmetric action on electric and magnetic charges. It generates \( \Theta_{ij} \)-dependent electric charges \( q_i \)'s and \( c_i \)-dependent electric charge \( q_0 \).

The corresponding transformation of \( I_4 \) reads

\[
0 \xrightarrow{\mathcal{O}^{-1}} - (\Theta_{ij} p^i p^j)^2 \leq 0.
\]

(2.88)

Thus, according to (2.83), a "small" \( (I_4 = 0) \) BPS or non-BPS, or a "large" non-BPS \( Z_H \neq 0 (I_4 < 0) \) BH charge configuration can be generated. In case the quantity in (2.88) is strictly negative, this is an example of phenomenon (2.59).

10. "Small" **doubly-critical** (1-charge) \( p^0 \) ("magnetic" Kaluza-Klein) configuration. This is the limit \( q_0 = 0 \) of configuration 1. In Type II, it corresponds to only D6 branes:

\[
Q \equiv (p^0, 0, 0, 0)^T \Rightarrow I_4 (Q) = 0,
\]

(2.89)

The action of \( \mathcal{P}Q (2n_V + 2, \mathbb{R}) \) reads

\[
\begin{pmatrix}
p^0 \\
0 \\
0 \\
0
\end{pmatrix} \xrightarrow{\mathcal{O}^{-1}} \begin{pmatrix}
p^0 \\
0 \\
-\rho p^0 \\
-c_i p^0
\end{pmatrix},
\]

(2.90)

and thus it generates \( \rho \)-dependent electric charge \( q_0 \) and \( c_i \)-dependent electric charges \( q_i \)'s. These latter in Type II compactifications corresponds to a stack of \( D2 \) branes depending on the components of the second Chern class \( c_2 \) of CY3 (recall Eq. (2.51)). The corresponding transformation of \( I_4 \) reads

\[
0 \xrightarrow{\mathcal{O}^{-1}} (p^0)^4 \left( \frac{2}{3} d^{ijk} c_i c_j c_k - \rho^2 \right) \gtrless 0.
\]

(2.91)

Thus, depending on whether

\[
\frac{2}{3} d^{ijk} c_i c_j c_k - \rho^2 \gtrless 0,
\]

(2.92)
a “large” ($\mathcal{I}_4 > 0$: BPS or non-BPS $Z_H = 0$), a “small” ($\mathcal{I}_4 = 0$: BPS or non-BPS), or a “large” non-BPS $Z_H \neq 0$ ($\mathcal{I}_4 < 0$) BH charge configuration is generated. In case the quantity in (2.88) is non-vanishing, this is an example of phenomenon (2.59). Note that for $\varrho = 0$, an example treated in [28] is recovered, namely:

$$\begin{cases}
0 \to 4 (p^0)^4 \mathcal{I}_3 (\mathbf{c}) \geq 0; \\
\mathcal{I}_3 (\mathbf{c}) \equiv \frac{1}{3!} \epsilon^{ijk} c_i c_j c_k.
\end{cases}$$

(2.93)

11. “Small” doubly-critical (1-charge) $q_0$ ("electric" Kaluza-Klein) configuration. This is the limit $p^0 = 0$ of configuration 1. In Type II, it corresponds to only $D0$ branes:

$$\mathcal{Q} \equiv (0, 0, q_0, 0)^T \Rightarrow \mathcal{I}_4 (\mathcal{Q}) = 0,$$

(2.94)

This configuration is the “electric-magnetic dual” of the “magnetic” configuration 10. Since there are no magnetic charges, $\mathcal{P}Q (2n_V + 2, \mathbb{R})$ is inactive on this configuration:

$$\begin{pmatrix}
0 \\
0 \\
q_0 \\
0
\end{pmatrix} \mathcal{O}^{-1}
\begin{pmatrix}
0 \\
0 \\
q_0 \\
0
\end{pmatrix}.$$

(2.95)

We conclude this Sec. with a comment on the attractor values of the scalars, i.e. on the non-degenerate critical points of the effective BH potential $V_{BH}$. In presence of the sub-leading quantum perturbative corrections (2.8), the expressions of such critical points can be obtained from the ones for the uncorrected (not necessarily cubic) SK geometry, by applying a suitable transformation of $\mathcal{P}Q (2n_V + 2, \mathbb{R})$ on the charges.

This fact has been known for some time [28, 29]. In the case in which the uncorrected geometry is a $d$-SK geometry with prepotential (2.4), this provides a generally more efficient approach to the computation of the attractor horizon (purely charge-dependent) values of the scalars. Namely, one has to start from the general expression of the extremal BH attractors for $d$-SK geometries [29, 32], and then apply the suitable transformation $\mathcal{O}^{-1}$ (2.32) of $\mathcal{P}Q (2n_V + 2, \mathbb{R})$ on the charges. As an example, in this way the results recently obtained in Sec. 3 and App. A of [30] can be recovered.

2.2.3 Transformation of $V_{BH}$

As mentioned above, $\mathcal{P}Q (2n_V + 2, \mathbb{R})$, when acting both on the charges $\mathcal{Q}$ and on the covariantly holomorphic symplectic sections $\mathcal{V}$, leaves $Z$ and $D_i Z$, and thus $V_{BH}$ given by (2.40), invariant.

Actually, in order to investigate the effect of the quantum perturbative sub-leading corrections (2.8) to any $\mathcal{N} = 2$ prepotential on $Z, D_i Z, V_{BH}, \partial_i V_{BH}, D_i \partial_j V_{BH}, D_i \partial_j \bar{V}_{BH}$ etc., one should act with $\mathcal{P}Q (2n_V + 2, \mathbb{R})$ only on charges. In order to show this, let us consider (without any loss of generality for our purposes) the $\mathcal{N} = 2$ central charge $Z \equiv \langle \mathcal{Q}, \mathcal{V} \rangle \equiv Q^T \Omega \mathcal{V}$. By recalling that $\mathfrak{F}$ can be introduced through the action of $\mathcal{O} \in$
\[ \mathcal{P}Q(2n_{\mathcal{V}} + 2, \mathbb{R}) \] on the sections, the expression of \( Z \) for any \( \mathcal{N} = 2 \) prepotential corrected with \( \mathfrak{T} \) is given by

\[ Z' \equiv Z(\mathcal{O}V(z, \bar{z}) ; Q) \equiv \langle Q, \mathcal{O}V \rangle \equiv Q^T \mathcal{O}V = Z(\mathcal{V}(z, \bar{z}) ; \mathcal{O}^{-1}Q), \] (2.96)

where in the second line the symplectic nature of \( \mathcal{O} \) has been exploited. Thus, the expression of \( Z \) for any \( \mathcal{N} = 2 \) prepotential corrected with \( \mathfrak{T} \) is nothing but the expression of \( Z \) computed for the uncorrected prepotential, with the charges transformed through \( \mathfrak{T} \) given by (2.25). The very same holds also for \( W, D_iW, D_iZ, V_{BH}, \partial_iV_{BH}, D_i\partial_jV_{BH}, D_i\partial_j\mathcal{O}V_{BH} \), and in general for all quantities depending on scalars and charges. In the case of the locus \( \partial_iV_{BH} = 0 \), this allows to easily compute the \( \mathfrak{T} \)-corrected attractors, once the ones for the uncorrected prepotential are known (see the discussion at the end of Sec. 2.2.2). In the case in which the uncorrected SK geometry is a cubic one, with prepotential (2.9), this reasoning provides a general alternative approach for the generalization (for all charge configurations in which the treatment of the purely cubic case is feasible \cite{29, 32, 43\}) of the computations recently performed in Sec. 3 and App. A of [30].

In light of the previous reasoning, the explicit expressions of \( Z, D_iZ \) and \( V_{BH} \) for an \( \mathfrak{T} \)-corrected \( d \)-SK geometry can be immediately obtained by applying the charge transformation \( \mathcal{O}^{-1} \) (given by (2.32)) to Eqs. (4.9), (4.10) and (2.13) of \cite{43\}, respectively.

Since it is crucial to our treatment, we here consider only the \( \mathfrak{T} \)-corrected expression of \( V_{BH} \) for \( d \)-SK geometries. As mentioned, the expression of \( V_{BH} \) for \( d \)-SK geometries is given by Eq. (2.13) of \cite{43\}, which we report here for ease of comparison:

\[ 2V_{BH}(z, \bar{z}; Q) = \left[ \nu (1 + 4g) + \frac{h^2}{36\nu} + \frac{3}{48\nu} g^{ij} h_i h_j \right] (p^0)^2 + \]
\[ + \left[ 4\nu g_{ij} + \frac{1}{4\nu} (h_i h_j + g_{mn} h_m h_n) \right] p^i p^j + \]
\[ + \nu \left[ q_0^2 + 2x^i q_0 q_i + \left( x^i x^j + \frac{1}{4} g^{ij} \right) q_i q_j \right] + \]
\[ + 2\nu g_i - \frac{h}{12\nu} h_i - \frac{1}{8\nu} g^{mn} h_m h_n \right] p^0 p^i + \]
\[ - \frac{1}{3\nu} \left[ -h p^0 q_0 + 3q_0 p^i h_i - (h x^i + \frac{3}{4} g^{ij} h_j) p^0 q_i \right] + \]
\[ + 3 (h_j x^i + \frac{1}{2} g^{mn} h_m h_j) q_i p^j \right]; \] (2.97)
where the following notation has been introduced (see e.g. [43] for further details):

\[
\begin{align*}
\left\{
\begin{array}{l}
z^i & \equiv x^i - i\lambda^i; \\
\nu & \equiv \frac{1}{3} d_{ijk} \lambda^j \lambda^k; \\
h_{ij} & \equiv d_{ijk} x^k; \\
d_{ij} & \equiv d_{ijk} \lambda^k; \\
g_{ij} & = -\frac{1}{4} \left( \frac{\partial d_{ij}}{\partial v} - \frac{\partial d_{ji}}{\partial v} \right); \\
g & \equiv g_{ij} x^i x^j.
\end{array}
\right.
\end{align*}
\]

(2.98)

It is worth recalling that (2.97) was recently re-obtained as the \( \text{Im}d_{000} = 0 \) limit of the more general quantum perturbative result of [25]. Consistently with the above reasoning, straightforward computations lead to the following expression of the \( \mathfrak{g} \)-corrected expression of \( V_{BH} \) for \( d \)-SK geometries:

\[
V_{BH}(z, \bar{z}, \mathcal{Q}) \xrightarrow{\mathcal{O}^{-1}} V_{BH}(z, \bar{z}, \mathcal{O}^{-1} \mathcal{Q}) = V_{BH}(z, \bar{z}, \mathcal{Q}) + \mathfrak{W}_{BH}(z, \bar{z}, \mathcal{Q}, q, c_i, \Theta_{ij}),
\]

(2.99)

where \( \mathfrak{W}_{BH} \) describes the “\( \mathfrak{PQ}\)-deformation” of \( V_{BH} \):

\[
2\mathfrak{W}_{BH}(z, \bar{z}, \mathcal{Q}, q, c_i, \Theta_{ij}) = \frac{1}{\nu} \left[ \begin{array}{c}
g'^2 (p^0)^2 + (c_i p^i)^2 - 2q_0 q^0 p^0 - 2q_0 c_i p^i + 2g^0 c_i p^i \\
+ 2x^i \left( -p_0^0 q_0 c_i - q_0 \Theta_{ij} p^j \\
+ (x^i x^j + \frac{1}{4} g^{ij}) \right) \begin{array}{c}
- q_i q_j + (p^0)^2 c_i c_j + p^0 c_i \Theta_{jk} p^k \\
- \Theta_{ik} p^k q_j + p^0 \Theta_{ik} p^k c_j + \Theta_{ik} p^k \Theta_{jl} p^l
\end{array}
\end{array} \right]
\]

\[
- \frac{1}{3\nu} \left[ \begin{array}{c}
h_p (g^0 + c_i p^i) \\
-3 (g^0 + c_j p^j)p^i h_i \\
+ (h x^j + \frac{3}{4} g^{ij} h_j) p^0 (c_i p^0 + \Theta_{ik} p^k) \\
-3 (h_j x^i + \frac{1}{2} g^{im} h_{mj}) (c_i p^0 + \Theta_{ik} p^k) p^j
\end{array} \right].
\]

(2.100)
Eqs. (2.99), (2.97) and (2.100), once specified for the particular $n_V = 2$ model treated in [30] (see Eq. (3.7) therein), allows one to easily recover Eq. (A.12) therein. Furthermore, by setting $p^0 = 0 = q_i$ (i.e. by considering the $D_0 - D_4$ configuration), Eq. (2.100) yields that the $\tilde{g}$-corrected $V_{BH}$ does not depend at all on $\varrho$; this fact generalizes the comment below Eq. (3.1) of [30].

Let us now consider the part of $V_{BH}$ (2.97) linear in the axions $x^i$. Eq. (2.97) yields

$$2 V_{BH}\big|_{\text{linear in } x^i} = \frac{2}{\nu} x^i q_0 q_i + 2 \nu g_i p^0 p^j - \frac{1}{2\nu} g^{ik} h_{kj} q_i q_j.$$

(2.101)

This implies that the BH charge configurations which support the axion-free solution $x^i = 0 \forall i$ at least as a particular solution of the axionic Attractor Eqs. $\partial V_{BH}/\partial x^i = 0$ are the following ones [43]:

$$\begin{cases} (p^0, q_0); \\
(p^0, q_i); \\
(p^i, q_0), \end{cases}$$

(2.102)

namely the “large” configurations 1, 2 and 3 treated in Sec. 2.2.2.

Through Eqs. (2.99), (2.97) and (2.100), the action of $P Q (2 n_V + 2, \mathbb{R})$ transforms (2.101) as follows:

$$2 \left[ V_{BH} + \mathcal{W}_{BH} \right]\big|_{\text{linear in } x^i} = \frac{2}{\nu} x^i q_0 q_i + 2 \nu g_i p^0 p^j - \frac{1}{2\nu} g^{ik} h_{kj} q_i q_j$$

$$+ 2 \nu x^i \left( -p^0 q_0 c_i - q_0 \Theta_{ij} p^j \right) - 2 \nu x^i \left( -p^0 q_i + q (p^0)^2 c_i + q p^0 \Theta_{ij} p^j \right)$$

$$+ c_j p^j q_i + p^0 c_i c_j p^j + c_j p^j \Theta_{ik} p^k$$

$$+ \frac{1}{2\nu} g^{im} h_{mj} \left( c_i p^0 + \Theta_{ikp^k} \right) p^j.$$

(2.103)

The rather intricate expression (2.103) implies that, in presence of the sub-leading quantum perturbative corrections (2.8), the configurations (2.102) do not support axion-free solutions any more, and that in general there are no axion-free-supporting BH charge configurations at all\(^{17}\), unless some extra assumptions are made. For instance, (2.103) yields the following axion-free-supporting conditions for the charge configurations (2.102):

$$2 \left[ V_{BH} + \mathcal{W}_{BH} \right]\big|_{\text{linear in } x^i, (p^0, q_0)} = \frac{2}{\nu} x^i c_i p^0 (-q_0 + q p^0) = 0 \iff \begin{cases} c_i = 0; \\
\text{and/or} \\
q_0 = q p^0; \end{cases}$$

(2.104)

$$2 \left[ V_{BH} + \mathcal{W}_{BH} \right]\big|_{\text{linear in } x^i, (p^0, q_i)} = \frac{2}{\nu} x^i q p^0 (-q_i + p^0 c_i) = 0 \iff \begin{cases} q = 0; \\
\text{and/or} \\
q_i = p^0 c_i; \end{cases}$$

(2.105)

\(^{17}\)This result is consistent with the analysis of the particular $n_V = 2$ model in $D_0$-$D_4$ configuration worked out in [30].
\[2 [V_{BH} + \mathfrak{W}_{BH}]_{\text{linear in } x^i(p^0,q^0)} = 2 \nu \left( -\delta^i_m q_0 + \delta^i_m c_k p^k + \frac{1}{4} g^{il} d_{klm} p^k \right) \Theta_{ij} p^j x^m = 0\]

\[
\begin{cases}
\Theta_{ij} = 0; \\
\text{and/or} \\
-\delta^i_m q_0 + \delta^i_m c_k p^k + \frac{1}{4} g^{il} d_{klm} p^k = 0.
\end{cases}
\] (2.106)

It is known [21] that in symmetric d-SK geometries, the “moduli space” of non-BPS \(Z_H \neq 0\) attractors is the scalar manifold of the \(d = 5\) uplifted theory. This can be easily seen in the \((p^0, q^0)\) configuration. Indeed, by setting \(x^i = 0 \forall i\), the effective BH potential (2.97) reads

\[2 V_{BH}(p^0,q^0,x^i=0) = \nu (p^0)^2 + \frac{1}{\nu} q_0^2,\] (2.107)

thus depending only on the Kaluza-Klein volume \(\nu\). The \(n_V\) real “rescaled” dilatons \(\lambda_i\) (2.108)

\[\hat{\lambda}^i \equiv \nu^{-\frac{1}{3}} \lambda^i,\] (2.109)

are “flat directions” of the critical value \(2.107\).

The action of \(PQ(2n_V + 2, \mathbb{R})\) may make the emergence of “moduli spaces” of attractors less manifest but, as stated above, does not change their geometrical structure. From (2.104), in \(\mathfrak{F}\)-corrected d-SK geometry the Kaluza-Klein charge configuration \((p^0, q^0)\) (with no further constraints) is axion-free-supporting for \(c_i = 0 \forall i\). In such a case, Eqs. (2.104) and (2.64) respectively yield

\[2 [V_{BH} + \mathfrak{W}_{BH}](p^0,q_0,x^i=0) = \nu (p^0)^2 + \frac{1}{\nu} (q_0 - \eta p^0)^2;\] (2.110)

\[- (p^0)^2 q_0^2 q^{-1} - (p^0)^2 (q_0 - \eta p^0)^2.\] (2.111)

Thus, the PQ-transformed BH charge configuration \((p^0, q_0)\) with \(c_i = 0 \forall i\) (and \(q_0 \neq \eta p^0\)) still supports non-BPS \(Z_H \neq 0\) (possibly axion-free) extremal BH attractors, whose “moduli space” is still manifest from (2.110). Note that the case \(q_0 = \eta p^0\) is troublesome, because it does not stabilize the Kaluza-Klein volume through the Attractor Mechanism.

### 2.3 Cayley’s Hyperdeterminant and Elliptic Curves

Recently, in [33], an intriguing relation between elliptic curves and the Cayley’s hyperdeterminant [34] was found.

More specifically, it was shown that if the cubic elliptic curve

\[y^2 = ax^3 + bx^2 + cx + d\] (2.112)

has a Mordell-Weil group containing a subgroup isomorphic to \(\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2\), then it can be transformed into the Cayley’s hyperdeterminant \(\text{Det}(\psi)\), which is nothing but
the (opposite of the) quartic scalar invariant built out of the unique rank-4 completely
symmetric primitive invariant tensor of the repr. \((2,2,2)\) of \([SL(2,\mathbb{R})]^3\), which in turn is
the \(U\)-duality group of the \(\mathcal{N} = 2, d = 4\) so-called \(stu\) model \([35]\):
\[
\mathcal{I}_{4,stu}(Q) = - \left(p^0\right)^2 q_0^2 - \left(p^1\right)^2 q_1^2 - \left(p^2\right)^2 q_2^2 - \left(p^3\right)^2 q_3^2
- 2p_0^0 q_0 q_1 - 2p_0^0 q_0 q_2 - 2p_0^0 q_0 q_3 + 2p_1^1 q_1 q_2 + 2p_1^1 q_1 q_3 + 2p_2^2 q_2 q_3
+ 4q_0^1 p_0^2 p_3^3 - 4p_0^0 q_1 q_2 q_3 = - \text{Det}(\psi).
\]
(2.113)

This expression can be obtained from the general one \((2.46)-(2.47)\), by specifying the \(stu\)
model data:
\[
d_{ijk} = 6\delta_{1(i}\delta_{2|j|}\delta_{3|k)}; \quad d^{ijk} = 6\delta_{l(i}\delta_{2|j|}\delta_{3|k)};
\]
(2.114)

consistent with \((2.43)\).

Under the aforementioned assumption on the Mordell-Weil group, the elliptic curve
\((2.112)\) can be factorised as \([33]\)
\[
y^2 = 4 \left(l - kr\right) \left(n - mx\right) \left(q - px\right),
\]
(2.115)

and through the positions (with \(u, v\) unknowns) \([33]\)
\[
y = uv^2 - ev + g; \quad x = v; \quad a = -4km; \quad b = 4kmrt + 4kpts + 4mprs; \quad c = -4rts \left(kt + mr + ps\right); \quad d = 4r^2 s^2 t^2;
\]
(2.116-2.121)

finally \((2.112)\) can be recast in the form
\[
u^2 v^2 + k^2 t^2 + m^2 r^2 + p^2 s^2 - 2ktuv - 2mruv - 2psuv - 2kmrt - 2kpts - 2mprs + 4kmvp + 4rstu = 0,
\]
(2.122)

which corresponds to the vanishing of \(\mathcal{I}_{4,stu}(Q)\) as given by \((2.113)\), under the (non-
unique) following mapping of the charge vector:
\[
Q \equiv \left(p^0, p^1, p^2, p^3, q_0, q_1, q_2, q_3\right)^T = (u, k, m, p, -v, t, r, s)^T.
\]
(2.123)

Interestingly, the two unknowns \(u\) and \(v\) corresponds to the magnetic \((D6)\) and electric
\((D0)\) Kaluza-Klein charges in the reduction \(d = 5 \to d = 4\).

Under the position \((2.123)\), the vanishing of \(\mathcal{I}_{4,stu}(Q)\), a necessary condition defining
the “small” orbits of the \((2,2,2)\) of \([SL(2,\mathbb{R})]^3\) \([66]\), can be recast in the form \((2.112)\), with
\[
y = p^0 q_0^2 + q_0 \left(p^1 q_1 + p^2 q_2 + p^3 q_3\right) + 2q_1 q_2 q_3; \quad x = -q_0; \quad a = -4p^1 p^2 p^3; \quad b = 4 \left(p^1 q_1 p^2 q_2 + p^1 q_1 p^3 q_3 + p^2 q_2 p^3 q_3\right); \quad c = -4q_1 q_2 q_3 \left(p^1 q_1 + p^2 q_2 + p^3 q_3\right); \quad d = 4q_1^2 q_2^2 q_3.
\]
(2.124-2.129)
In light of the treatment given in Secs. 2.1 and 2.2 it is worth pointing out that the above construction admits a “$PQ (8, \mathbb{R})$-deformation”.

The “$PQ (8, \mathbb{R})$-deformation” of the Cayley’s hyperdeterminant can be obtained from the general result (2.56)-(2.57) by using the $stu$ model data (2.114) (here $i, j = 1, 2, 3$):

$$I_{stu} (Q) + I_{stu} (\psi; g, c_i, \Theta_{ij}) = - (p^0)^2 (q_0 - gp^0 - c_ip^i)^2 - (p^0 p^j (q_0 - gp^0 - c_jp^j) (q_i - c_ip^0 - \Theta_{ij} p^j)$$

$$+ \sum_{i=1}^3 |\epsilon_{ijk}| p^j (q_j - c_jp^0 - \Theta_{ij} p^j) p^k (q_k - c_kp^0 - \Theta_{km} p^m))$$

$$+ 4 (q_0 - gp^0 - c_ip^i)^2 p^1 p^2 p^3 - 4p^0 \prod_{i=1}^3 (q_i - c_ip^0 - \Theta_{ij} p^j)$$

$$= - \text{Det} (\psi; g, c_i, \Theta_{ij}) .$$

(2.130)

The various terms (unknowns and coefficients) of the corresponding cubic elliptic curve (2.112) are given by the $PQ (8, \mathbb{R})$-transformed Eqs. (2.124)-(2.129), namely:

$$y = (q_0 - gp^0 - c_ip^i)^2 + (q_0 - gp^0 - c_ip^i) p^i (q_i - c_ip^0 - \Theta_{ij} p^j)$$

$$+ 2 \prod_{i} (q_i - c_ip^0 - \Theta_{ij} p^j) ;$$

(2.131)

$$x = - (q_0 - gp^0 - c_ip^i) ;$$

(2.132)

$$a = - 4p^0 p^2 p^3 ;$$

(2.133)

$$b = 2 \sum_{i} |\epsilon_{ijk}| p^j (q_j - c_jp^0 - \Theta_{ij} p^i) p^k (q_k - c_kp^0 - \Theta_{km} p^m) ;$$

(2.134)

$$c = - 2 \sum_{k} |\epsilon_{klm}| p^j (q_l - c_lp^0 - \Theta_{lm} p^m) p^m (q_m - c_mp^0 - \Theta_{mr} p^r) \prod_{i} (q_i - c_ip^0 - \Theta_{ij} p^j) ;$$

(2.135)

$$d = 4 \left[ \prod_{i} (q_i - c_ip^0 - \Theta_{ij} p^j) \right]^2 .$$

(2.136)

Clearly, the roots of the elliptic cubic curve (2.112) (with data (2.124)-(2.129)) are not the same as the roots of (2.112) (with data (2.131)-(2.136)). In general, the action of $PQ (8, \mathbb{R})$ amounts to a $(g, c_i, \Theta_{ij})$-redefinition of the vertices of the hypercube whose associate hyperdeterminant is Det$(\psi)$ given by (2.113) [34].

Let us further remark that in realistic superstring compactifications leading to the $stu$ model in the supergravity limit, the values of the parameters $c_i = \frac{c_2 V}{24}$ (where $c_2$ is the second Chern class; see Sec. 2.1.4) can be computed to read [37, 38]:

$$\text{Type IIA on K3 fibrations} : c_{2,1} = c_{2,3} = 24; \quad c_{2,2} = 92;$$

$$\text{Heterotic on } T^4 \times T^2 \text{ or } K3 \times T^2 : c_{2,1} = c_{2,3} = 0.$$  

(2.137)

In view of the recent progress within the fascinating BH/qubit correspondence [36], $PQ (8, \mathbb{R})$ may well have a role on the quantum information side; we leave the study of this interesting issue for future investigation.
3 An Alternative Expression for $I_4$

By refining and extending the analysis of \cite{25} and considering $d$-SK geometries based on the purely cubic holomorphic prepotential \cite{29}, we will now derive an alternative expression of the quartic invariant $I_4$ given by \cite{246-247}.

A crucial quantity in such developments is the so-called $E$-tensor. Such a rank-5 tensor was firstly introduced in \cite{16} (see also the treatment of \cite{46}), and it expresses the deviation of the considered geometry from being symmetric. Its definition reads (see e.g. \cite{10, 25} for a recent treatment, and Refs. therein):

$$E_{mijkl} = \frac{1}{3} D^m D_i C_{jkl}. \hspace{1cm} (3.1)$$

This definition can be elaborated further, by recalling the properties of the so-called $C$-tensor $C_{ijk}$. This is a rank-3 tensor with Kähler weights $(2, -2)$, defined as (see e.g. \cite{1, 69}):

$$C_{ijk} \equiv \langle D_i D_j V, D_k V \rangle = e^K (\partial_i N_{\Lambda \Sigma}) D_j X^\Lambda D_k X^\Sigma = e^K (\partial_i X^\Lambda) (\partial_j X^\Sigma) (\partial_k X^\Xi) \partial_\Xi F_\Lambda (X) \equiv e^K W_{ijk}, \hspace{1cm} \nabla W_{ijk} = 0, \hspace{1cm} (3.2)$$

where $N_{\Lambda \Sigma}$ is the $N = 2$, $d = 4$ kinetic vector matrix, and the second line holds only in “special coordinates”. $C_{ijk}$ is completely symmetric and covariantly holomorphic:

$$C_{ijk} = C_{(ijk)}; \hspace{1cm} \nabla_i C_{jkl} = 0. \hspace{1cm} (3.3)$$

By further steps, detailed in \cite{25}, the expression for $E_{mijkl}$ defined by \eqref{3.1} can thus be further elaborated as follows:

$$C_{p(ijk)q} g^{p\eta} g^{\eta r} C_{\eta r} = \frac{4}{3} C_{(ijk)l} g_{l\eta} + E_{(ijkl)}. \hspace{1cm} (3.4)$$

Formulae \eqref{3.1} and \eqref{3.4} hold for a generic SK geometry. By considering $d$-SK geometries based on the purely cubic holomorphic prepotential \cite{29} in the “special coordinates” symplectic basis, \eqref{3.4} can then be recast as

$$(X^0)^3 e^{3K} d_p(ijkl) g^{pr} g^{qs} d_{rst} = \frac{4}{3} X^0 e^K d_{(ijkl)t} + E_{(ijkl)}, \hspace{1cm} (3.5)$$

where $g^{ij}$ and $g_{ij}$ are defined in \cite{29} (see e.g. \cite{43} for further details).

Let us now introduce the “rescaled metric” \cite{70} \cite{43} and, for later convenience, its derivatives with respect to $\hat{\lambda}^i$ (the unique set of scalars on which it actually depends):

$$a_{ij} \equiv 4\nu^{2/3} g_{ij} = \left( \frac{1}{4} d_i d_j - \hat{d}_{ij} \right) \leftrightarrow a^{ij} = \frac{1}{4} \nu^{-2/3} g^{ij} = \frac{1}{2} \hat{\chi} \hat{\lambda}^i - \hat{d}^{ij}; \hspace{1cm} (3.6)$$

$$\frac{\partial a_{ij}}{\partial \hat{\chi}^k} = \frac{1}{2} \left( \hat{d}_{ik} \hat{d}_j + \hat{d}_{jk} \hat{d}_i \right) - d_{ij}; \hspace{1cm} (3.7)$$

$$\frac{\partial a_{ij}}{\partial \hat{\lambda}^k} = \frac{1}{2} \left( \delta_{ij} \hat{\lambda}^k + \delta_{ji} \hat{\lambda}^k \right) + \hat{d}^i \hat{d}^{jm} d_{klm}; \hspace{1cm} (3.8)$$

29
where $\hat{d}_{ij}, \hat{d}^{ij}$ and $\hat{d}_i$ are the “hatted” counterpart of the quantities defined in (2.98) (also recall the splitting $z^i \equiv x^i - i\lambda^i$ in the first line of (2.98), as well as (2.10) and (2.109)):

$$\hat{d}_{ij} \equiv d_{ijk} \hat{\lambda}^k; \quad \hat{d}_i \equiv d_{ijk} \hat{\lambda}^k; \quad (3.9)$$

$$\hat{d}^{ij} \hat{d}_{jk} \equiv \delta^i_k \Rightarrow \hat{d}^{im} \hat{d}_{jm} \equiv -\hat{d}^{ij} \hat{d}^{ml} d_{jkl}. \quad (3.10)$$

Thus, by fixing the Kähler gauge $X^0 \equiv 1$, after some algebra one achieves the following result:

$$d_{p(ij} d_{k)q} d^{pqv} = \frac{4}{3} \delta_v^i (d_{ijk}) + 2^{5/3} \nu^{5/3} E^v_{ijkl}; \quad (3.11)$$

where

$$d^{pqv} \equiv a^{pr} a^{qs} a^{vt} d_{rst}; \quad (3.12)$$

$$E^v_{ijkl} \equiv a^{vt} E_{ijkl}. \quad (3.13)$$

From (3.11), one can re-derive the explicit expression of $E_{ijkl}$ given by Eq. (4.21) of [25], implying that in any $d$-SK geometry $\nu^{5/3} E_{ijkl}$ depends only on the “rescaled $d = 4$ dilatons” $\hat{\lambda}^i$.

Let us now introduce the following $p^i$-dependent quantities, which are scalar-independent in any $d$-SK geometry:

$$d_{ij} \equiv d_{ijk} p^k; \quad d_{ij} \equiv \frac{\partial \mathcal{I}_3(p)}{\partial p^i \partial p^j}; \quad (3.14)$$

from which the following behaviors follow: $d_{ij} \sim [p]^2$ and $d_{ij} \sim [p]^{-2}$.

Thus, whenever $d_{ij}$ has maximal rank $n_V$, by contracting (3.11) with $p^k p^l q_{st} d^{pt}$, a little algebra leads to the result

$$-(p^i q_i)^2 + d_{ijkl} d^{lm} p^j p^k q_{lm} = \frac{1}{3} d_{ijkl} p^j p^k q_{lm} d^{lm} + 2^{5/3} \nu^{5/3} E^m_{ijkl} p^i p^j p^k q_{lm} d^{lm}. \quad (3.15)$$

By plugging (3.15) into the general expression of $\mathcal{I}_4$ given by (2.46)-(2.47), one obtains the following alternative expression:

$$\mathcal{I}_4 = -(p^0)^2 q_0^2 - 2 p^0 q_0 p^i q_i + \frac{1}{3} (2 q_0 + q_i q_j d^{ij}) d_{klm} p^k p^l p^m - \frac{2}{3} p^0 d^{ijk} q_i q_j q_k + 2^{5/3} \nu^{5/3} E^m_{ijkl} p^j p^l p^k q_{lm} d^{lm}. \quad (3.16)$$

which manifestly shows the contribution of the $E$-tensor as a source of dependence on $\nu$ and $\hat{\lambda}^i$’s for non-symmetric $d$-SK geometries, and more in general for all $d$-SK geometries in which the term $E^m_{ijkl} p^j p^l p^k q_{lm} d^{lm}$ does not vanish. Note that (3.16) is well defined whenever $d_{ij}$ (introduced in (3.11)) has maximal rank $n_V$.

Some comments on the alternative formula (3.16) for $\mathcal{I}_4$ are in order.

---

Footnotes:

15 Note that in $d$-SK geometries all geometrical quantities under consideration are real.

19 For homogeneous non-symmetric $d$-SK geometries, the expression of the $E$-tensor was explicitly computed in [71].

20 Attention should be paid not to confuse the scalar-independent quantities $d_{ij}$ and $d^{ij}$ defined by (3.14) with the $\lambda^i$-dependent quantities $d_{ij}$ and $d_{ij}$ defined in (2.98).
1. In symmetric d-SK geometries (see e.g. [46, 16], and Refs. therein) \( E_{ijkl} = 0 \), as a consequence of the covariant constancy of the Riemann tensor \( R_{ijkl} \) itself (see e.g. [25] for a recent treatment):

\[
D_m R_{ijkl} = 0. \tag{3.17}
\]

This implies, through Eq. (3.14):

\[
C_p (kl) C_{ij} n g^{pm} g^p q_{mn} = \frac{4}{3} g(l|m C_{ijkl}) \Leftrightarrow g^{pm} R_{(i|m)j(l|kl)} = - \frac{2}{3} g(l|m C_{ijkl}), \tag{3.18}
\]

whose specification in the manifestly \( G_5 \)-covariant “special coordinates” symplectic basis gives the identity (2.45), which is consistently the \( E_{ijkl} = 0 \) limit of (3.11).

By recalling definition (3.12), (2.45) (holding for symmetric d-SKG, and more in general in all cases in which \( E_{ijkl} = 0 \) globally) implies that \( d_{ijk} \) is a constant, scalar-independent tensor:

\[
\frac{\partial d_{ijk}}{\partial z^l} = 0. \tag{3.19}
\]

Furthermore, the \( E_{ijkl} = 0 \) limit of (3.16) yields

\[
I_4 = \frac{1}{3} \left( 2 q_0 + q_i q_j d^{ij} \right) d_{klm} p^k p^l p^m - \frac{2}{3} p^0 d_{ijk} q_i q_j q_k, \tag{3.20}
\]

which is a manifestly \( G_5 \)-invariant, alternative simple expression of \( I_4 \), in \( \mathcal{N} = 2 \) symmetric d-SK geometries, as well as in all \( d = 4 \mathcal{N} > 2 \)-extended supergravity theories whose scalar manifold is characterised by a symmetric cubic geometry\(^{21}\). In particular, for \( G_4 = E_7(-25) \) (\( \mathcal{N} = 2, d = 4 \mathcal{J}_3 \)-based “magic” supergravity) and \( G_4 = E_7(7) \) (\( \mathcal{N} = 8, d = 4 \mathcal{J}_3 \)-based maximal supergravity), (3.20) provides an equivalent expression of the Cartan-Cremmer-Julia [72, 73] unique quartic invariant of the fundamental irrepr. 56 of the exceptional Lie group \( E_7 \). It is also worth remarking that for symmetric d-SKG (and more in general in all cases in which \( E_{ijkl} = 0 \) globally) the expressions (2.46)-(2.47) and (3.20) actually are scalar-independent and thus purely charge-dependent, and therefore \( I_4 \) actually is the unique quartic invariant polynomial of the relevant symplectic (ir)repr. \( R \) of the \( d = 4 \mathcal{U} \)-duality group \( G_4 \).

2. The alternative expression (3.16) for \( I_4 \) is necessary to consistently match some known expressions of BH entropy with the formalism of d-SK geometries. Concerning this, the \( p_0 = 0 \) limit of (3.20) yields

\[
I_4 = \frac{1}{3} \left( 2 q_0 + q_i q_j d^{ij} \right) d_{klm} p^k p^l p^m, \tag{3.21}
\]

matching Eqs. (50)-(51) of [32]. Actually, since the treatment of [32] deals with generic (not necessarily symmetric, nor homogeneous) d-SK geometries, one should actually use the full formula (3.16). Consequently, the consistency of the results\(^{21}\) with the exception of \( \mathcal{N} = 4 \) “pure” and of \( \mathcal{N} = 5 \) supergravities, these also are all \( \mathcal{N} > 2 \)-extended theories which can be uplifted to \( d = 5 \) dimensions (see e.g. [44] for quick reference Tables, and Refs. therein).
(50)-(51) of [32] with the general formula (3.10) yields the following constraint on the on-shell expression of the $E$-tensor $(\text{at least for } p^0 = 0)$:

$$E^m_{ijkl}|_{\partial V_B H = 0} p^j p^k p^l q_m d^{in} = 0.$$  \hspace{1cm} (3.22)

It is known that the configuration $(p^i, q_0, q_i)$ does not support axion-free attractor solutions [43], thus (3.22) should be considered in an axionful background. However, the $E$-tensor is insensitive to the presence of non-vanishing axions, because it only depends on $\nu$ and $\hat{\lambda}$'s, as given by Eq. (4.21) of [25].

3. The observations at point 1 are no more generally true in non-symmetric $d$-SK geometries, and in all cases in which the $E$-tensor does not vanish globally\footnote{For some elaborations on this issue, see e.g. the recent treatment given in [25].}. In this case, $\mathcal{I}_4$ is no more an invariant of the $U$-duality group $G_4$ (whose transitive action on the scalar manifold is spoiled in the non-homogeneous case; see e.g. [16]). Concerning this, it is worth recalling that $G_4$ always contains (and for totally generic $d_{ijk}$'s, coincides with) the semi-direct product of PQ axion-shifts $\mathbb{R}^n$ and an overall rescaling $SO(1,1)$, namely (see e.g. [16]):

$$SO(1,1) \times_s \mathbb{R}^n \subset G_4.$$  \hspace{1cm} (3.23)

Within this framework, some analysis of the dependence on the scalar degrees of freedom can be made. First of all, one can easily verify that in $d$-SK geometries all relevant geometrical quantities considered above are independent of the $d = 4$ axions $x^i$, namely the real parts of the $d = 4$ complex scalars coordinatising the special Kähler vector multiplets' scalar manifolds of $\mathcal{N} = 2$, $d = 4$ supergravity. This can ultimately be traced back to the $d = 5$ origin of all $d$-SK geometries, which are the only SK geometries which can be uplifted to 5 space-time dimensions. Then, (3.6)-(3.12) and (3.6) yield that

$$\frac{\partial (\nu^{5/3} E^v_{ijkl})}{\partial \nu} = 0; \hspace{1cm} \frac{\partial E^v_{ijkl}}{\partial \hat{\lambda}^v} = \frac{3}{25} d_{p(ij)k}|d_{rst} \left[ \frac{1}{2} \left( \delta^p \hat{\lambda}^r + \delta^r \hat{\lambda}^p \right) + \hat{d}^{pm} \hat{d}^{rn} d_{emn} \right] a^{qs} a^{vt}$$

$$= \frac{3}{25} \left[ d_{v(ij)k} d_r v^q \hat{\lambda}^r + d_{p(ij)k} q_{d_{sel} \hat{\lambda}^p a^{qs} a^{vt}} + 2d_{p(ij)k} q_{d_r v^q \hat{d}^{pm} \hat{d}^{rn} d_{emn}} \right].$$  \hspace{1cm} (3.25)

The result (3.24) was derived in [25]. On the other hand, (3.25) expresses the way the $E$-tensor depends on $\hat{\lambda}$'s, encoding the non-symmetric nature of the corresponding $d$-SK geometry.

**Acknowledgments**

We would like to thank Sergio Ferrara for enlightening discussions.
The work of S. B. is supported by the ERC Advanced Grant no. 226455, “Supersymmetry, Quantum Gravity and Gauge Fields” (SUPERFIELDS).

The work of A. M. has been supported by an INFN visiting Theoretical Fellowship at SITP, Stanford University, CA, USA. Furthermore, he would like to thank Ms. Hanna Hacham for her nice and inspiring hospitality in Palo Alto, CA, USA.

The work of R. R. has been supported in part by Dipartimento di Scienze Fisiche of “Federico II” University and INFN-Sez. di Napoli. He would also like to thank INFN-LNF for kind hospitality and support.

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