CLUSTERING OF CONSECUTIVE NUMBERS IN PERMUTATIONS UNDER A MALLOWS DISTRIBUTION

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ABSTRACT. Let \( A_{l,k}^{(n)} \subset S_n \) denote the set of permutations of \([n]\) for which the set of \(l\) consecutive numbers \(\{k, k+1, \cdots, k+l-1\}\) appears in a set of consecutive positions. Under the uniformly random measure \(P_n\) on \(S_n\), one has \(P_n(A_{l,k}^{(n)}) \sim \frac{n}{\pi^{l-1}}\) as \(n \to \infty\). In this paper we consider the probability of the clustering of consecutive numbers under Mallows distributions \(P_q\), \(q > 0\). Because of a duality, it suffices to consider \(q \in (0,1)\). In particular, we show that for fixed \(q\), \(\lim_{l \to \infty} \lim_{n \to \infty} P_q(A_{l,k}^{(n)}) = (\prod_{i=1}^{\infty} (1-q^i))^2\), if \(\lim_{n \to \infty} \min(k_n, n-k_n) = \infty\), and that for \(q_n = 1 - \frac{c}{\pi^\alpha}\), with \(c > 0\) and \(\alpha \in (0,1)\), \(P_q(A_{l,k}^{(n)})\) is on the order \(\frac{1}{n^{\alpha (l-1)}}\), uniformly over all sequences \(\{k_n\}_{n=1}^{\infty}\). Thus, letting \(N_l^{(n)} = \sum_{k=1}^{n-l+1} 1_{A_{l,k}^{(n)}}\) denote the number of sets of \(l\) consecutive numbers appearing in sets of consecutive positions, we have

\[
\lim_{n \to \infty} E_{P_q^n} N_l^{(n)} = \begin{cases} 
\infty, & \text{if } l < \frac{1+\alpha}{\alpha} \\
0, & \text{if } l > \frac{1+\alpha}{\alpha}.
\end{cases}
\]

1. Introduction and Statement of Results

Let \(l \geq 2\) be an integer. Let \(P_n\) denote the uniform probability measure on the set \(S_n\) of permutations of \([n] := \{1, \cdots, n\}\), and denote a permutation \(\sigma \in S_n\) by \(\sigma = \sigma_1 \sigma_2 \cdots \sigma_n\). The set of \(l\) consecutive numbers \(\{k, k+1, \cdots, k+l-1\} \subset [n]\) appears in a set of consecutive positions in the permutation if there exists an \(m\) such that \(\{k, k+1, \cdots, k+l-1\} = \{\sigma_m, \sigma_{m+1}, \cdots, \sigma_{m+l-1}\}\). Let \(A_{l,k}^{(n)} \subset S_n\) denote the event that the set of \(l\) consecutive numbers \(\{k, k+1, \cdots, k+l-1\}\) appears in a set of consecutive

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positions. It is immediate that for any $1 \leq k, m \leq n - l + 1$, the probability that $\{k, k + 1, \ldots, k + l - 1\} = \{\sigma_m, \sigma_{m+1}, \ldots, \sigma_{m+l-1}\}$ is equal to $\frac{l!}{n!}$.

Thus,

\begin{equation}
(1.1) \quad P_n(A_{l;k}^{(n)}) = (n - l + 1) \frac{l!(n - l)!}{n!} \sim \frac{l!}{n^{l-1}}, \text{ as } n \to \infty, \text{ for } l \geq 2.
\end{equation}

Let $A_{l}^{(n)} = \bigcup_{k=1}^{n-l+1} A_{l;k}^{(n)}$ denote the event that there exists a set of $l$ consecutive numbers appearing in a set of consecutive positions, and let $N_{l}^{(n)} = \sum_{k=1}^{n-l+1} A_{l;k}^{(n)}$ denote the number of sets of $l$ consecutive numbers appearing in sets of consecutive positions. Then

\begin{equation}
(1.2) \quad E_n N_{l}^{(n)} = (n - l + 1)^2 \frac{l!(n - l)!}{n!} \sim \frac{l!}{n^{l-2}}, \text{ as } n \to \infty, \text{ for } l \geq 2.
\end{equation}

Using the inequality

\[
\sum_{k=1}^{n-k+1} P_n(A_{l;k}^{(n)}) - \sum_{1 \leq j < k \leq n-l+1} P_n(A_{l;j}^{(n)} \cap A_{l;k}^{(n)}) \leq P_n(A_{l}^{(n)}) \leq \sum_{k=1}^{n-k+1} P_n(A_{l;k}^{(n)}),
\]

along with the fact that for $j, k, m, r$, with $\{j, j+1, \ldots, j+l-1\} \cap \{k, k+1, \ldots, k+l-1\} = \emptyset$ and $\{m, m+1, \ldots, m+l-1\} \cap \{r, r+1, \ldots, r+l-1\} = \emptyset$, the probability that both $\{k, k+1, \ldots, k+l-1\} = \{\sigma_m, \sigma_{m+1}, \ldots, \sigma_{m+l-1}\}$ and $\{j, j+1, \ldots, j+l-1\} = \{\sigma_r, \sigma_{r+1}, \ldots, \sigma_{r+l-1}\}$ is equal to, $\frac{(l!)^2(n-2l)!}{n!}$, it is easy to show that

\begin{equation}
(1.3) \quad P_n(A_{l}^{(n)}) \sim \frac{l!}{n^{l-2}}, \text{ as } n \to \infty, \text{ for } l \geq 3.
\end{equation}

It follows from (1.2) (or from (1.3)) that for $l \geq 3$, the sequence $\{N_{l}^{(n)}\}_{n=1}^{\infty}$ converges to zero in probability. On the other hand, when $l = 2$, $\{N_{l}^{(n)}\}_{n=1}^{\infty}$ converges in distribution to a Poisson random variable with parameter 2.

This result goes back over 75 years; see [8], [5].

In this paper, we consider the above described asymptotic phenomenon concerning the clustering of consecutive numbers in the case that the random permutation is distributed according to a Mallows distribution. For each $q > 0$, the Mallows distribution with parameter $q$ is the probability measure
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$P^q_n$ on $S_n$ defined by

$$P^q_n(\sigma) = \frac{q^{\text{inv}(\sigma)}}{Z_n(q)}, \sigma \in S_n,$$

where $\text{inv}(\sigma)$ is the number of inversions in $\sigma$, and $Z_n(q)$ is the normalization constant, given by

$$Z_n(q) = \prod_{k=1}^n \frac{1 - q^k}{1 - q}.$$

Thus, for $q \in (0,1)$, the distribution favors permutations with few inversions, while for $q > 1$, the distribution favors permutations with many inversions. Of course, the case $q = 1$ yields the uniform distribution. Recall that the reverse of a permutation $\sigma = \sigma_1 \cdots \sigma_n$ is the permutation $\sigma^{\text{rev}} := \sigma_n \cdots \sigma_1$.

The Mallows distributions satisfy the following duality between $q > 1$ and $q \in (0,1)$:

$$P^q_n(\sigma) = P^\frac{1}{q}_n(\sigma^{\text{rev}}), \text{ for } q > 0, \sigma \in S_n \text{ and } n = 1, 2, \cdots.$$  

Since the set $A_{l,k}^{(n)}$ is invariant under reversal, for our study of clustering it suffices to consider the case that $q \in (0,1)$.

When $q \to 0$, the Mallows distribution $P^q_n$ converges weakly to the degenerate distribution on the identity permutation, and of course the identity permutation belongs to $A_{l,k}^{(n)}$ for all $k$ and $l$. Because the smaller $q$ is, the more the distribution favors permutations with few inversions, and as such, the smaller $q$ is, the more the distribution favors permutations which are close to the identity permutation, it seems intuitive that the smaller $q$ is, the more clustering there will be. However, whereas the structure of the Mallows distribution lends itself naturally to proving theorems concerning the inversion statistic [6], it is less transparent how to exploit that structure with regard to this clustering statistic. For example, the set $A_{l,k}^{(n)}$ is the disjoint union of the $n - l + 1$ sets $\{k, k + 1, \ldots, k + l - 1\} = \{\sigma_m, \sigma_{m+1}, \ldots, \sigma_{m+l-1}\}$, $m = 1, \cdots, n - l + 1$. In the case of the uniform distribution, these $n - l + 1$ sets all have the same probability. However, in the case of $P^q_n$, $q \in (0,1)$, we expect that for certain $m$, these sets will have probability less than what they have under the uniform distribution, and for
other \(m\) these sets will have probability greater than what they have under the uniform distribution.

For results concerning the behavior under a Mallows distribution of other permutation statistics—cycle counts and increasing subsequences, see [1], [2] and [3].

Before presenting our asymptotic results, we begin with the following duality with regard to \(k\), which will be useful, and whose short proof is given toward the end of this section.

**Proposition 1.**

(1.4) \[
P_q^n(A_{l;k}^{(n)}) = P_q^n(A_{l;n+2-k-l}^{(n)}), \quad k = 1, 2, \ldots, n - l + 1.
\]

We now consider asymptotic results in the case of fixed \(q \in (0, 1)\). It turns out that in this case, the probability of arbitrarily long clusters is bounded away from zero. Let

\[
C_q := \prod_{i=1}^{\infty} (1 - q^i).
\]

**Theorem 1.** Let \(q \in (0, 1)\).

i. \[
\lim_{l \to \infty} \lim_{n \to \infty} P_q^n(A_{l;k}^{(n)}) = \lim_{l \to \infty} \lim_{n \to \infty} P_q^n(A_{l;n+2-k-l}^{(n)}) = C_q \prod_{i=1}^{k-1} (1 - q^i), \quad k = 1, 2, \ldots;
\]

ii. \[
\lim_{l \to \infty} \lim_{n \to \infty} P_q^n(A_{l;k_n}^{(n)}) = C_q^2, \quad \text{if} \quad \lim_{n \to \infty} \min(k_n, n - k_n) = \infty.
\]

**Proposition 2.**

(1.5) \[
C_q \sim e^{-\frac{\pi^2}{6(1 - q)}}, \quad \text{as} \quad q \to 1.
\]

Theorem 1 and Proposition 2 yield the following corollary.

**Corollary 1.** If \(\lim_{n \to \infty} \min(k_n, n - k_n) = \infty\), then

(1.6) \[
\lim_{l \to \infty} \lim_{n \to \infty} P_q^n(A_{l;k_n}^{(n)}) \sim e^{-\frac{\pi^2}{3(1 - q)}}, \quad \text{as} \quad q \to 1.
\]
Remark. In particular, if $q_m = 1 - \frac{c}{\log m}$, then (1.6) gives
\[
\lim_{l \to \infty} \lim_{n \to \infty} P_n^m(A_{l,k_n}^{(n)}) \sim m^{-\frac{2}{3\pi}}, \quad \text{as } m \to \infty.
\]

We now consider asymptotic results in the case that $q_n = 1 - \frac{c}{n^\alpha}$ with $c > 0$ and $\alpha \in (0,1)$. We use the notation $a_n \lesssim b_n$ as $n \to \infty$ to indicate that $\lim \sup_{n \to \infty} a_n / b_n \leq 1$.

**Theorem 2.** Let $q_n = 1 - \frac{c}{n^\alpha}$, with $c > 0$ and $\alpha \in (0,1)$. Then
\[
\frac{(l - 1)!}{(2l)!} \frac{c^{l-1} l!}{n^{\alpha(l-1)}} \lesssim P_n^q(A_{l,k_n}^{(n)}) \lesssim \frac{1}{l} \frac{c^{l-1} l!}{n^{\alpha(l-1)}},
\]
for any choice of $\{k_n\}_{n=1}^\infty$, and the convergence is uniform over all $\{k_n\}_{n=1}^\infty$. If $k_n$ satisfies $\min(k_n, n-k_n) \to \infty$, then (1.7) holds with an improved upper bound:
\[
\frac{(l - 1)!}{(2l)!} \frac{c^{l-1} l!}{n^{\alpha(l-1)}} \lesssim P_n^q(A_{l,k_n}^{(n)}) \lesssim \left( \int_0^1 x^{l-1} e^{-(l-1)x} \, dx \right) \frac{c^{l-1} l!}{n^{\alpha(l-1)}}.
\]

Recall that $N_l^{(n)} = \sum_{k=1}^{n-l+1} 1_{A_{l,k}^{(n)}}$ denotes the number of sets of $l$ consecutive numbers appearing in sets of consecutive positions. Theorem 2 yields the following corollary.

**Corollary 2.** Let $q_n = 1 - \frac{c}{n^\alpha}$ with $c > 0$ and $\alpha \in (0,1)$, Then there exist constants $C_l^{(-)}, C_l^{(+) > 0}$ such that
\[
C_l^{(-)} n^{1-(l-1)\alpha} \leq E_n^q N_l^{(n)} \leq C_l^{(+) n^{1-(l-1)\alpha}}.
\]

In particular,
\[
\lim_{n \to \infty} E_n^q N_l^{(n)} = \begin{cases} 
\infty, & \text{if } l < \frac{1+\alpha}{\alpha}; \\
0, & \text{if } l > \frac{1+\alpha}{\alpha}.
\end{cases}
\]

**Remark 1.** For $\tau \in S_l$, let $A_{l,r,k}^{(n)} \subset A_{l,k}^{(n)}$ denote the event that the set of $l$ consecutive numbers $\{k, k+1, \ldots, k+l-1\} \subset [n]$ appears in a set of consecutive positions in the permutation and also that the relative positions
of these consecutive numbers correspond to the permutation \( \tau \). That is, 
\{k, k+1, \ldots, k+l-1\} = \{\sigma_m, \sigma_{m+1}, \ldots, \sigma_{m+l-1}\}, for some \( m \), and \( \sigma_{m+i-1} - (k-1) = \tau_i \), \( i = 1, \ldots, l \). Then \( A_{l,k}^{(n)} = \cup_{\tau \in S_l} A_{l,\tau,k}^{(n)} \). Small changes in the proof of Theorem 2, which we leave to the reader, show that (1.7) and (1.8) hold with \( P_{n,k}^{q_n}(A_{l,k}^{(n)}) \) replaced by \( P_{n,k}^{q_n}(A_{l,\tau,k}^{(n)}) \) and with \( l! \) deleted from the numerator in the upper and lower bounds, for all \( \tau \in S_l \). In particular, if \( \tau = id \), then \( A_{l,\tau,k}^{(n)} \) is the event that the numbers \{k, \ldots, k+l-1\} form an increasing run in the permutation, and if \( \tau \) satisfies \( \tau^{rev} = id \), then \( A_{l,\tau,k}^{(n)} \) is the event that the numbers \{k, \ldots, k+l-1\} form a decreasing run in the permutation.

**Remark 2.** Let \( K^{(-)}(l) = \frac{(l-1)!}{(2l)!} \) and \( K^{(+)}(l) = \int_0^1 x^{l-1}e^{-(l-1)x}dx \) denote the coefficients of \( \frac{e^{l-1}x}{n(l-1)} \) on the left and right hand sides respectively of (1.8). We have \( K^{(-)}(l) \sim \sqrt{\pi} l^{-\frac{3}{2}} 4^{-l} \). One can show that

\[
K^{(+)}(l) = \int_0^1 x^{l-1}e^{-(l-1)x}dx = \frac{(l-1)!}{(l-1)^l} \left(1 - e^{-(l-1)} \sum_{i=0}^{l-1} \frac{(l-1)^i}{i!}\right).
\]

Thus, \( K^{(+)}(l) \lesssim \frac{(l-1)!}{(l-1)^l} \sim \sqrt{2\pi} e^{-\frac{1}{2}}e^{-l} \). On the other hand, a rudimentary asymptotic analysis we performed on the interval \([\frac{l-1}{2}, -\frac{l-1}{2}]\) yields \( K^{(+)}(l) \gtrsim e^{\frac{1}{2}} l^{-\frac{3}{2}} e^{-l} \). We have \( K^{(-)}(2) = \frac{1}{12} \approx 0.083 \) and \( K^{(+)}(2) = 1 - \frac{3}{2} \approx 0.281 \).

Now we consider the case \( q = q_n = 1 - \frac{c}{n} \) and \( q = q_n = 1 - o\left(\frac{1}{n}\right) \).

**Theorem 3.** i. Let \( q_n = 1 - \frac{c}{n} \), with \( c > 0 \). Let \( k_n \sim dn \) with \( d \in (0, 1) \).

Then

\[
(1.9) \quad P_{n,k}^{q_n}(A_{l,k}^{(n)}) \lesssim \frac{1}{(1 - e^{-cd})^l} \left( \int_{e^{-cd}}^{1} y^{l-1}e^{(\log \frac{1 - e^{-cd}}{1 - e^{-c}})e^{-cd}(l-1)y} dy\right) e^{l-1} \frac{l!}{n(l-1)}.
\]

ii. Let \( q_n = 1 - o\left(\frac{1}{n}\right) < 1 \). Then for any choice of \( \{k_n\}_{n=1}^{\infty} \),

\[
(1.10) \quad P_{n,k}^{q_n}(A_{l,k}^{(n)}) \lesssim \frac{l!}{n^l}. \]

**Remark.** In part (i), we expect that the asymptotic behavior of \( P_{n,k}^{q_n}(A_{l,k}^{(n)}) \), when \( k_n \sim dn \), is in fact independent of \( d \in (0, 1) \). We note for all \( d \in
we have from (1.1) that \( P_n(q) \) (or for that matter, for any \( q < 1 \)), therefore matching up with (1.10). In the case of the uniform distribution \( (q = 1) \), we have from (1.1) that \( P_n^{(q)}(A_{l,k_n}) \sim \frac{n}{n-\tau} \), for any choice of \( k_n \). Since we expect \( P_n^{(q)}(A_{l,k_n}) \) to be decreasing in \( q \), we expect that the asymptotic inequality in (1.10) is an asymptotic equality. That is, asymptotically, we expect that the cluster event \( A_{l,k_n}^{(n)} \) cannot distinguish between \( P_n^{(1)} \) and \( P_n^{(q)} \), if \( q_n = 1 - o(\frac{1}{n}) \).

We can give a quick proof that \( P_n^{(q)}(A_{l,k_n}^{(n)}) \geq \frac{1}{2} \frac{n}{n-\tau} \), if \( q_n = 1 - o(\frac{1}{n}) \) (or for that matter, for any \( q < 1 \)). Note that \( \text{inv}(\sigma^{\text{rev}}) = \frac{1}{n}(n-1) - \text{inv}(\sigma) \). Let \( A_{l,k_n,+}^{(n)} \) (respectively, \( A_{l,k_n,-}^{(n)}, A_{l,k_n,0}^{(n)} \)) denote the subset of \( A_{l,k_n}^{(n)} \) containing those permutations \( \sigma \) satisfying \( \text{inv}(\sigma) > \frac{1}{n}(n-1) \) (respectively, \( \text{inv}(\sigma) < \frac{1}{n}(n-1), \text{inv}(\sigma) = \frac{1}{n}(n-1) \)). It is easy to show from the definition of the Mallows distribution that \( P_n^{(q)}(A_{l,k_n,+}^{(n)}) \geq P_n^{(1)}(A_{l,k_n,-}^{(n)}) \) and \( P_n^{(q)}(A_{l,k_n,0}^{(n)}) \geq P_n^{(1)}(A_{l,k_n,0}^{(n)}) \). Since \( A_{l,k_n}^{(n)} \) is invariant under reversal, it follows that \( P_n^{(1)}(A_{l,k_n,+}^{(n)}) + P_n(A_{l,k_n,0}^{(n)}) \geq \frac{1}{2} P_n^{(1)}(A_{l,k_n}^{(n)}) \sim \frac{1}{2} \frac{n}{n-\tau} \).

We now give the proof of Proposition 1.

**Proof of Proposition 1.** We defined above the reverse \( \sigma^{\text{rev}} \) of a permutation \( \sigma \in S_n \). The complement of \( \sigma \) is the permutation \( \sigma^{\text{com}} \) satisfying \( \sigma^{\text{com}} = n + 1 - \sigma_i, \ i = 1, \ldots, n \). Let \( \sigma^{\text{rev-com}} \) denote the permutation obtained by applying reversal and then complementation to \( \sigma \) (or equivalently, applying complementation and then reversal). Since \( \sigma^{\text{rev-com}} < \sigma^{\text{rev-com}} \) if and only \( \sigma_{n+1-j} < \sigma_{n+1-i} \), it follows that \( \sigma \) and \( \sigma^{\text{rev-com}} \) have the same number of inversions, and thus, from the definition of the Mallows distribution, \( P_n^{(r)}(\{\sigma\}) = P_n^{(r)}(\{\sigma^{\text{rev-com}}\}) \). Using this along with the fact that \( \sigma \in A_{l,k}^{(n)} \) if and only if \( \sigma^{\text{rev-com}} \in A_{l,n+2-k-1}^{(n)} \) proves (1.4).

We end this section by describing two different on-line constructions of a random permutation on \( S_n \) distributed according to the Mallows distribution
with parameter $q \in (0, 1)$. Both of these constructions will be utilized in our proofs.

For the first construction, let $\{X_j\}_{j=1}^n$ be independent random variables with $X_j$ distributed as a geometric random variable starting from zero with parameter $q$ and truncated at $j - 1$; that is,

\begin{equation}
(1.11) \quad P(X_j = m) = \frac{(1 - q)q^m}{1 - q^j}, \quad m = 0, \cdots, j - 1.
\end{equation}

Consider a horizontal line on which to place the numbers in $[n]$. We begin by placing down the number 1. Then inductively, if we have already placed down the numbers $1, 2, \cdots, j - 1$, the number $j$ gets placed down in the position for which there are $X_j$ numbers to its right. Thus, for example, for $n = 4$, if $X_2 = 1$, $X_3 = 2$ and $X_4 = 0$, then we obtain the permutation 3214. To see that this construction does indeed induce the Mallows distribution with parameter $q$, note that the number of inversions in the constructed permutation $\sigma$ is $\sum_{j=2}^n X_j$, and thus using (1.11),

$$P(X_j = x_j, j = 2, \cdots, n) = \frac{1}{Z_n(q)} q^{\sum_{j=2}^n x_j} = \frac{2^{\text{inv}(\sigma)}}{Z_n(q)}.
$$

In the second on-line construction, we actually construct a permutation of $N$ and then reduce it to a permutation in $S_n$. Consider the geometric distribution, starting from 1, with parameter $1 - q \in (0, 1)$: $p := \{p_n\}_{n=1}^{\infty}$ where $p_n = (1 - q)q^{n-1}$, $n = 1, 2, \cdots$. Take a countable sequence of independent samples from this distribution: $n_1, n_2, \cdots$. Let $\Pi_1 = n_1$ and then for $k \geq 2$, let $\Pi_k = \psi_k(n_k)$, where $\psi_k$ is the increasing bijection from $N$ to $N - \{\Pi_1, \cdots, \Pi_{k-1}\}$. Thus, the sequence of samples 7, 3, 4, 3, 7, 2, 5, $\cdots$ yields the permutation $\Pi$ of $N$ beginning with $\Pi_1 = 7, \Pi_2 = 3, \Pi_3 = 5, \Pi_4 = 4, \Pi_5 = 11, \Pi_6 = 2, \Pi_7 = 10$. Such a permutation of $N$ is called a $p$-shifted permutation. For fixed $n$, we define a permutation on $S_n$ by $\Pi_{i_1} \Pi_{i_2} \cdots \Pi_{i_n}$, where $i_1 < i_2 \cdots < i_n$ and $\{\Pi_{i_j}\}_{j=1}^{n} = [n]$. This random permutation has the Mallows distribution with parameter $q$; see [4], [7].
We prove Theorem 1 and Proposition 2 in section 2, Theorem 2 in section 3 and Theorem 3 in section 4. Much of the construction in the proof of Theorem 1 will be used also in the proofs of Theorems 2 and 3.

2. Proofs of Theorem 1 and Proposition 2

Proof of Theorem 1. We consider the first on-line construction, implemented via the sequence \( \{X_n\}_{n=1}^\infty \) of independent truncated geometric random variables with parameter \( 1 - q \). Although \( P_n^q \) denotes the Mallows distribution with parameter \( q \) on \( S_n \), we will also use this notation for probability when discussing events related to the random variables \( \{X_j\}_{j=1}^n \). Fix \( k \in \mathbb{N} \). We begin by considering the event, which we denote by \( B_l; k \), that after the first \( k + l - 1 \) positive integers have been placed down on the horizontal line, the set of \( l \) numbers \( \{k, k+1, \ldots, k+l-1\} \) appears in a set of \( l \) consecutive positions. Then \( B_l; k = \bigcup_{a=0}^{k-1} B_l; k; a \), where the events \( \{B_l; k; a\}_{a=0}^{k-1} \) are disjoint, with \( B_l; k; a \) being the event that the set of \( l \) numbers \( \{k, k+1, \ldots, k+l-1\} \) appears in a set of \( l \) consecutive positions and also that exactly \( a \) of the numbers in \([k-1]\) are to the right of this set. We calculate \( P_n^q(B_l; k; a) \).

Suppose that we have already placed down on the horizontal line the numbers in \([k-1]\). Their relative positions are irrelevant for our considerations. Now we use \( X_k \) to insert on the line the number \( k \). Suppose that \( X_k = a \), \( a \in \{0, \ldots, k-1\} \). Then the number \( k \) is inserted on the line in the position for which \( a \) of the numbers in \([k-1]\) are to its right. Now in order for \( k+1 \) to be placed in a position adjacent to \( k \), we need \( X_{k+1} \in \{a, a+1\} \).

(If \( X_{k+1} = a \), then \( k+1 \) will appear directly to the right of \( k \), while if \( X_{k+1} = a+1 \), then \( k+1 \) will appear directly to the left of \( k \).) If this occurs, then \( \{k, k+1\} \) are adjacent, and \( a \) of the numbers in \([k-1]\) are to the right of \( \{k, k+1\} \). Continuing in this vein, for \( i \in \{1, \ldots, l-2\} \), given that the numbers \( \{k, \ldots, k+i\} \) are adjacent to one another, and \( a \) of the numbers in \([k-1]\) appear to the right of \( \{k, \ldots, k+i\} \), then in order for \( k+i+1 \) to be placed so that \( \{k, k+1, \ldots, k+i+1\} \) are all adjacent to
one another (with \(a\) of the numbers in \([k-1]\) appearing to the right of these numbers), we need \(X_{k+i+1} \in \{a, \cdots, a+i+1\}\). We conclude then that

\[
P_n^q(B_l;k:a) = \prod_{j=0}^{l-1} P_n^q(X_{k+j} \in \{a, \cdots, a+j\})
\]

(2.1)

\[
P_n^q(B_l;k:a) = \prod_{j=0}^{l-1} P_n^q(X_{k+j} \in \{a, \cdots, a+j\}) = \prod_{j=0}^{l-1} \frac{(1-q) \sum_{i=0}^{j} q^{i+j}}{1-q^{k+j}} = q^\alpha \prod_{b=k}^{l} \frac{(1-q^b)}{(1-q^{b+l-1})}.
\]

We now consider the conditional probability, \(P_n^q(A_{l;k}^{(n)} | B_l;k:a)\), that is, the probability, given that \(B_l;k:a\) has occurred, that the numbers \(k, l, \cdots, n\) are inserted in such a way so as to preserve the mutual adjacency of the set \(\{k, \cdots, k+l-1\}\). We will obtain lower and upper bounds on this conditional probability. However, first we note that it is clear from the on-line construction that \(P_n^q(A_{l;k}^{(n)} | B_l;k:a)\) is decreasing in \(n\). Thus, since \(P_n^q(A_l;k:a)\) is independent of \(n\), it follows that \(P_n^q(A_{l;k}^{(n)})\) is decreasing in \(n\). Consequently \(\lim_{n \to \infty} P_n^q(A_{l;k}^{(n)})\) exists.

We now turn to a lower bound on \(P_n^q(A_{l;k}^{(n)} | B_l;k:a)\). Our lower bound will be the probability of the event that all of the remaining numbers are inserted to the right of the set \(\{k, \cdots, k+l-1\}\). This event is given by \(\cap_{j=0}^{n-k-l} \{X_{k+l+j} \leq a+j\}\). Thus, we have

\[
P_n^q(A_{l;k}^{(n)} | B_l;k:a) \geq P(\cap_{j=0}^{n-k-l} \{X_{k+l+j} \leq a+j\}) = \prod_{j=0}^{n-k-l} \frac{(1-q) \sum_{i=0}^{a+j} q^i}{1-q^{k+l+j}} = \]

(2.2)

\[
\prod_{j=0}^{n-k-l} \frac{1-q^{a+j+1}}{1-q^{k+l+j}} = \begin{cases} 
\prod_{b=a+n-k-l+2}^{l} \frac{(1-q^b)}{(1-q^{b+l-1})}, & \text{if } a \geq 2k + 2l - n - 1; \\
\prod_{b=a+1}^{a+n-k-l+1} \frac{(1-q^b)}{(1-q^b)}, & \text{if } a < 2k + 2l - n - 1.
\end{cases}
\]

Writing \(P_n^q(A_{l;k}^{(n)}) = \sum_{a=0}^{n} P_n^q(B_l;k:a)P(A_{l;k}^{(n)} | B_l;k:a)\), (2.1) and (2.2) yield for sufficiently large \(n\),

\[
P_n^q(A_{l;k}^{(n)}) \geq \prod_{b=k}^{l} \frac{(1-q^b)}{(1-q^{b+k-l+1})} \sum_{a=0}^{k-1} q^\alpha \prod_{b=a+n-k-l+2}^{l} \frac{(1-q^b)}{(1-q^{b+l-1})}.
\]

(2.3)
Letting \( n \to \infty \) in (2.3) and using the fact that the limit on the left hand side exists, we have

\[
\lim_{n \to \infty} P_q^n(A_{l;k}) \geq \frac{\prod_{b=k}^{l-1} (1 - q^b)}{\prod_{b=k}^{l-1} (1 - q^b)} \sum_{a=0}^{k-1} q^a \prod_{b=a+1}^{k+l-1} (1 - q^b).
\]

Now letting \( l \to \infty \) gives

\[
\lim_{l \to \infty} \lim_{n \to \infty} P_q^n(A_{l;k}) \geq C_q \prod_{i=1}^{\infty} (1 - q^i),
\]

where, we recall, \( C_q = \prod_{i=1}^{\infty} (1 - q^i) \).

We now turn to an upper bound on \( P_q^n(A_{l;k} | B_{l;k;a}) \). We begin with some preliminary facts. We have

\[
P(X_m \in \{ j + 1, \cdots, j + l - 1 \}) = \sum_{b=j+1}^{j+l-1} \frac{1 - q^b}{1 - q^m} = \frac{q^{j+1} - q^{j+l}}{1 - q^m}.
\]

If \( X \) is the corresponding non-truncated geometric random variable with parameter \( 1 - q \), then

\[
P(X \notin \{ j + 1, \cdots, j + l - 1 \}) = \sum_{b=j+1}^{j+l-1} (1 - q)^b = q^{j+1} - q^{j+l}.
\]

Thus, \( P(X \notin \{ j + 1, \cdots, j + l - 1 \}) > P(X_m \notin \{ j + 1, \cdots, j + l - 1 \}) \). Also note that \( P(X \notin \{ j + 1, \cdots, j + l - 1 \}) \) is increasing in \( j \).

Now we can describe the upper bound on \( P_q^n(A_{l;k} | B_{l;k;a}) \), the conditional probability given \( B_{l;k;a} \) that the numbers \( k + l, \cdots, n \) are inserted in such a way so as to preserve the mutual adjacency of the set \( \{k, \cdots, k + l - 1\} \). First the number \( k + l \) is inserted. The probability that its insertion preserves the mutual adjacency property of the set \( \{k, \cdots, k + l - 1\} \) is \( P(X_{k+l} \notin \{a + 1, \cdots, a + l - 1\}) \), and by the previous paragraph, this probability is less than \( P(X \notin \{a + 1, \cdots, a + l - 1\}) \), where \( X \) is the corresponding untruncated geometric random variable. If the insertion of \( k + l \) preserves the mutual adjacency, then either \( X_{k+l} \in \{0, \cdots, a\} \) or \( X_{k+l} \in \{a + l, \cdots, k + l - 1\} \). If \( X_{k+l} \in \{0, \cdots, a\} \), then in order for the mutually adjacency to be preserved when the number \( k + l + 1 \) is inserted, one needs
\[ P(X_{k+1+l} \notin \{a+2, \cdots, a+l\}) \], while if \( X_{k+l} \in \{a+l, \cdots, k+l-1\} \), then one needs \( P(X_{k+1+l} \notin \{a+1, \cdots, a+l-1\}) \). By the previous paragraph, either of these probabilities is less than \( P(X \notin \{a+2, \cdots, a+l\}) \). Thus, an upper bound for the conditional probability given \( B_a \) that the insertion of \( k+l \) and \( k+l+1 \) preserves the mutual adjacency is \( P(X \notin \{a+1, \cdots, a+l-1\})P(X \notin \{a+2, \cdots, a+l\}) \). Continuing in this vein, we conclude that

\[
P_n^q(A_{t;k}^{(n)} | B_{t;k};a) \leq \prod_{i=1}^{n-k-l+1} P_n^q(X \notin \{a+i, \cdots, a+i+l-2\}) = \prod_{i=1}^{n-k-l+1} \left( 1 - q^{a+i} + q^{a+i+l-1} \right).
\]

(2.7)

Using this upper bound, we obtain an upper bound on \( P_n^q(A_{t;k}^{(n)}) \). From (2.1) and (2.7), we have

\[
P_n^q(A_{t;k}^{(n)}) \leq \prod_{b=1}^{l} (1 - q^b) \frac{q^{a+l}}{1 - q^b} \prod_{a=0}^{k-1} \prod_{i=1}^{n-k-l+1} \left( 1 - q^{a+i} + q^{a+i+l-1} \right).
\]

(2.8)

Letting \( n \to \infty \) and using the fact that the limit on the left hand side exists, we have

\[
\lim_{n \to \infty} P_n^q(A_{t;k}^{(n)}) \leq \prod_{b=1}^{l} (1 - q^b) \frac{q^{a+l}}{1 - q^b} \prod_{a=0}^{k-1} \prod_{i=1}^{\infty} \left( 1 - q^{a+i} + q^{a+i+l-1} \right).
\]

(2.9)

\[
\lim_{l \to \infty} \lim_{n \to \infty} P_n^q(A_{t;k}^{(n)}) \leq C_0 \prod_{i=1}^{k-1} \frac{1}{1 - q^i}.
\]

Part (i) of the theorem follows from (2.5), (2.9) and Proposition 1.

Now consider part (ii) of the theorem, where \( k = k_n \). For a lower bound, letting \( \gamma_{n,l} = 2k_n + 2l - 2 - n \), (2.1) and (2.2) with \( k = k_n \) yield

\[
P_n^q(A_{t;k_n}^{(n)}) \geq \prod_{b=1}^{l} (1 - q^b) \left( \sum_{a=0}^{\min(k_n-1, \gamma_{n,l})} q^{a+l-1} \frac{q^{a+n-k_n-l+1} \prod_{b=k_n}^{a+n-k_n-l} (1 - q^b)}{\prod_{b=k_n}^{a+n-k_n-l+2} (1 - q^b)} \right) + \sum_{a=\min(k_n-1, \gamma_{n,l})+1}^{k_n-1} q^{a+l} \frac{q^{a+n-k_n-l+1} \prod_{b=a+n-k_n-l+2}^{a+n-k_n-l+2} (1 - q^b)}{\prod_{b=a+n-k_n-l+1}^{a+n-k_n-l+2} (1 - q^b)}.
\]
where we use the convention that sums of the form $\sum_{a=A_2}^{A_1} a$, with $A_2 < A_1$ are equal to zero. Letting $n \to \infty$ and using the assumption that $\lim_{n \to \infty} \min(k_n, n-k_n) = \infty$, it follows that

$$
\lim_{n \to \infty} \inf_{l \to \infty} P_n^q(A_{l:k_n}^{(n)}) \geq \prod_{b=1}^{l} (1 - q^b) \sum_{a=0}^{\infty} q^{al} \prod_{b=a+1}^{\infty} (1 - q^b).
$$

Now letting $l \to \infty$ gives

$$
(2.10) \quad \lim_{l \to \infty} \lim_{n \to \infty} \inf_{l \to \infty} P_n^q(A_{l:k_n}^{(n)}) \geq C_q^2.
$$

For an upper bound, letting $n \to \infty$ in (2.8) with $k = k_n$, and using the assumption that $\lim_{n \to \infty} \min(k_n, n-k_n) = \infty$, we have

$$
(2.11) \quad \lim_{n \to \infty} \sup_{l \to \infty} P_n^q(A_{l:k_n}^{(n)}) \leq \prod_{b=1}^{l} (1 - q^b) \sum_{a=0}^{\infty} q^{al} \prod_{i=1}^{\infty} (1 - q^{a+i} + q^{a+i+l-1}).
$$

Now letting $l \to \infty$ gives

$$
(2.12) \quad \lim_{l \to \infty} \lim_{n \to \infty} \sup_{l \to \infty} P_n^q(A_{l:k_n}^{(n)}) \leq C_q^2.
$$

Part (ii) of the theorem follows from (2.10) and (2.12).

**Proof of Proposition 2.** We have

$$
(2.13) \quad \log C_q = \log \prod_{b=1}^{\infty} (1 - q^b) = \sum_{b=1}^{\infty} \log(1 - q^b),
$$

and

$$
(2.14) \quad \int_{1}^{\infty} \log(1 - q^x) dx \leq \sum_{b=1}^{\infty} \log(1 - q^b) \leq \int_{2}^{\infty} \log(1 - q^x) dx.
$$

Making the change of variables $y = q^x$ gives

$$
(2.15) \quad \int_{a}^{\infty} \log(1 - q^x) dx = - \frac{1}{\log q} \int_{0}^{q^a} \log(1 - y) \frac{dy}{y} \sim \left( \int_{0}^{1} \log(1 - y) \frac{dy}{y} \right) \frac{1}{1 - q}, \quad \text{as} \quad q \to 1, \quad \text{for} \quad a > 0.
$$

However, $\int_{0}^{1} \log(1 - y) dy = - \int_{0}^{1} (\sum_{n=1}^{\infty} \frac{y^{n-1}}{n}) dy = - \sum_{n=1}^{\infty} \frac{1}{n^2} = - \frac{\pi^2}{6}$. Using this with (2.13)-(2.15), we obtain (1.5), proving the proposition.
3. Proof of Theorem 2

We will prove (1.7) and (1.8) in tandem. Note that the lower bounds in (1.7) and (1.8) are the same; only the upper bounds differ. Recall that (1.8) is stated to hold under the assumption \( \min(\frac{k_n}{n^\alpha}, \frac{n-k_n}{n^\alpha}) = \infty \), while (1.7) is stated to hold with no assumption on \( \{k_n\}_{n=1}^\infty \). Thus, we need to prove the common lower bound in (1.7) and (1.8), as well as the upper bound in (1.7), with no assumption on \( \{k_n\}_{n=1}^\infty \), while we need to prove the upper bound in (1.8) under the above noted assumption on \( \{k_n\}_{n=1}^\infty \). In fact, for our proofs, we will always need to assume that

\[
(3.1) \lim_{n \to \infty} \frac{k_n}{n^\alpha} = \infty.
\]

What allows us to make this assumption is Proposition 1. Thus, in the sequel we will always assume that (3.1) holds.

For the upper bound, we can follow the same construction used in the upper bound in Theorem 1. We start from (2.8) with \( k \) and \( q \) replaced by \( k_n \) and \( q_n \). We have

\[
(3.2) 1 - q_n^b = 1 - (1 - \frac{c}{n^\alpha})^b \sim \frac{bc}{n^\alpha}, \text{ for } b \in \mathbb{N},
\]

and

\[
(3.3) 1 - q_n^{k_n+i} = 1 - (1 - \frac{c}{n^\alpha})^{k_n+i} \geq 1 - e^{-\frac{ck_n+i}{n^\alpha}}.
\]

From (3.2) and (3.3) along with the assumption on \( q_n \) and the assumption (3.1) on \( k_n \), the term multiplying the summation in (2.8) satisfies

\[
(3.4) \frac{\prod_{k=1}^l (1 - q_n^k)}{\prod_{b=k_n}^{k_n+l-1} (1 - q_n^b)} \sim \frac{l!c^l}{n^{\alpha l}}.
\]

Using (3.2), the summation in (2.8) satisfies

\[
(3.5) \sum_{a=0}^{k_n-1} q_n^{a+l} \prod_{i=1}^{n-k_n-l+1} (1 - q_n^{a+i} + q_n^{a+i+l-1}) \sim \sum_{a=0}^{k_n-1} q_n^{a+l} \prod_{i=1}^{n-k_n-l+1} \left(1 - \frac{q_n^{a+i+l-1}(1-1)c}{n^\alpha}\right).
\]

We split up the continuation of the proof of the upper bound between the cases (1.7) and (1.8). First consider (1.7), where no assumption is made on
\( k_n \) (accept, of course, for (3.1)). In this case, from (2.8) with \( k_n \) and \( q_n \) in place of \( k \) and \( q \), along with (3.4) and (3.5), we have

\[
P_n^{kl}(A_{l,k_n}^{(n)}) \lesssim \frac{llc}{n^\alpha} \sum_{a=0}^{k_n-1} q_n^{al} \leq \frac{llc}{n^\alpha} \frac{1}{1-q_n^l} \sim \frac{1}{l} \frac{llc^{l-1}}{n^{\alpha(l-1)}},
\]

which is the upper bound in (1.7).

Now consider the case (1.8), where the assumption \( \min(\frac{k_n}{n}, \frac{n-k_n}{n}) = \infty \) is made. For this case, we need to analyze (3.5) more carefully. We analyze the product on the right hand side of (3.5). We write

(3.6) \[
\log \prod_{i=1}^{n-k_n-l+1} \left( 1 - \frac{q_n^{a+i(l-1)c}}{n^\alpha} \right) = \sum_{i=1}^{n-k_n-l+1} \log \left( 1 - \frac{q_n^{a+i(l-1)c}}{n^\alpha} \right).
\]

We have

(3.7) \[
\int_0^{n-k_n-l+1} \log \left( 1 - \frac{q_n^{a+x(l-1)c}}{n^\alpha} \right) dx \leq \sum_{i=1}^{n-k_n-l+1} \log \left( 1 - \frac{q_n^{a+i(l-1)c}}{n^\alpha} \right) \leq \int_1^{n-k_n-l+2} \log \left( 1 - \frac{q_n^{a+x(l-1)c}}{n^\alpha} \right) dx.
\]

Making the change of variables, \( y = q_n^a \), we have

(3.8) \[
\int_A^B \log \left( 1 - \frac{q_n^{a+x(l-1)c}}{n^\alpha} \right) dx = -\frac{1}{\log q_n} \int_{q_n^a}^{q_n^B} \log \left( 1 - \frac{q_n^{a+(l-1)c}y}{n^\alpha} \right) dy.
\]

From (3.8) and the assumptions on \( q_n \) and \( k_n \), both the left and the right hand sides of (3.7) are asymptotic to \( \frac{n^\alpha}{c} \int_0^1 \log \left( 1 - \frac{q_n^{a+(l-1)c}y}{n^\alpha} \right) dy \), which in turn is asymptotic to \(- (l-1) q_n^a\), uniformly over \( a \in \{0, \cdots, k_n-1\} \). Using this with (3.6) and (3.7) gives

(3.9) \[
\prod_{i=1}^{n-k_n-l+1} \left( 1 - \frac{q_n^{a+i(l-1)c}}{n^\alpha} \right) \sim e^{-(l-1)q_n^a}, \ \text{uniformly over} \ a \in \{0, \cdots, k_n-1\}.
\]

From (2.8) with \( k_n \) and \( q_n \) in place of \( k \) and \( q \), along with (3.4), (3.5) and (3.9), we obtain

(3.10) \[
P_n^{kl}(A_{l,k_n}^{(n)}) \lesssim \frac{llc}{n^\alpha} \sum_{a=0}^{k_n-1} q_n^{al} e^{-(l-1)q_n^a}.
\]
By the assumptions on $k_n$ and $q_n$, $\sum_{a=0}^{k_n-1} q_n^a e^{-(l-1)q_n^a}$ is asymptotic to $\int_0^{k_n} q_n^a e^{-(l-1)q_n^a} \, dx$. Making the change of variables $y = q_n^a$, this integral is equal to $-\frac{1}{\log q_n} \int_{(l-1)q_n^a}^{(l-1)q_n^{a+1}} y^{l-1} e^{-(l-1)y} \, dy$, which in turn is asymptotic to $\frac{n^a}{c} \int_0^1 y^{l-1} e^{-(l-1)y} \, dy$. Thus,

\begin{equation}
(3.11) \sum_{a=0}^{k_n-1} q_n^a e^{-(l-1)q_n^a} \sim \frac{n^a}{c} \int_0^1 y^{l-1} e^{-(l-1)y} \, dy.
\end{equation}

From (3.10) and (3.11), we conclude that

$$P_n^q(\mathcal{A}^{(n)}_{l;k,n}) \lesssim \left( \int_0^1 y^{l-1} e^{-(l-1)y} \, dy \right) \frac{c^{l-1}l!}{n^{\alpha(l-1)}},$$

which is the upper bound in (1.8).

We now turn to the lower bound. Our only assumption on $k_n$ is (3.1). The method used in the proof of Theorem 1 and in the proof of the upper bound here, via the first on-line construction of a Mallows distribution, is not precise enough to be of use in the proof of the lower bound here. For this, we utilize the second on-line construction for a permutation in $S_n$ with a Mallows distribution. As with the first construction, we will use $P_n^q$ for probabilities of events related to the second construction. With regard to the second construction, for $j \in \{0, \cdots, k-1\}$, let $C_{j;k,l}$ denote the event that exactly $j$ numbers from the set $\{1, \cdots, k-1\}$ appear in the permutation before any number from the set $\{k, \cdots, k+l-1\}$ appears. We calculate $P_n^q(C_{j;k,l})$ explicitly. For $a, b \in \mathbb{N}$, let $r_{a,b}$ denote the probability that in the second construction, the first number that appears from the set $\{1, \cdots, a+b\}$ comes from the set $\{1, \cdots, a\}$. Then

\begin{equation}
(3.12) r_{a,b} = \frac{\sum_{j=1}^a (1-q)q^{j-1}}{\sum_{j=1}^{a+b} (1-q)q^{j-1}} = \frac{1-q^a}{1-q^{a+b}}.
\end{equation}

For convenience, define $r_{0,b} = 0$. Then from the second on-line construction, it follows that

\begin{equation}
(3.13) P_n^q(C_{j;k,l}) = \left( \prod_{i=1}^j r_{k-i,l} \right) (1-r_{k-j-1,l}), \quad j = 0, \cdots, k-1.
\end{equation}
From (3.12) and (3.13), we have

\[
P_n^q(C_{j,k,l}) = \left( \prod_{i=1}^{j} \frac{1 - q^{k-i}}{1 - q^{k-1+i}} \right) \frac{q^{k-j-1} - q^{k-j-1+l}}{1 - q^{k-j-1+l}} = (1 - q^l)q^{k-1-j} \prod_{b=k-j}^{\min(k-j+l-1,k-1)} (1 - q^b) \prod_{l=k+1}^{\max(k-j+l,k)} (1 - q^b) \]

(3.14)

In order for the event \( A_{l,k}^{(n)} \) to occur, all \( l \) numbers must appear consecutively in the second on-line construction. Thus, given the event \( C_{j,k,l} \), in order for the event \( A_{l,k}^{(n)} \) to occur, all of the other \( l - 1 \) numbers in \( \{k, \ldots , k + l - 1\} \) must occur immediately after the appearance of the first number from this set. Given \( C_{j,k,l} \), after the appearance of the first number from \( \{k, \ldots , k + l - 1\} \), there are still \( k - 1 - j \) numbers from \( \{1, \ldots , k - 1\} \) that have not yet appeared, as well as a certain amount of numbers from \( \{k + l, \ldots , n\} \). Thus, a lower bound on \( P_n^q(A_{l,k}^{(n)} \mid C_{j,k,l}) \) is obtained by assuming that none of the numbers from \( \{k + l, \ldots , n\} \) have yet appeared. (Here it is appropriate to note that if we calculate an upper bound by assuming that all of the numbers from \( \{k + l, \ldots , n\} \) have already appeared, then the upper bound we arrive at is not as good as the upper bound in (1.8).)

In order to calculate explicitly this lower bound, for \( a, b, c \in \mathbb{N} \), let \( r_{a,b,c} \) denote the probability that the first number that appears from the set \( \{1, \ldots , a + b + c\} \) comes from the set \( \{1, \ldots , a\} \cup \{a + b + 1, \ldots , a + b + c\} \). Then

\[
r_{a,b,c} = \frac{\sum_{j=1}^{a} (1 - q)q^{j-1} + \sum_{j=a+b+1}^{a+b+c} (1 - q)q^{j-1}}{\sum_{j=1}^{a+b+c} (1 - q)q^{j-1}} = \frac{1 - q^a + q^{a+b} - q^{a+b+c}}{1 - q^{a+b+c}}.
\]

From the second construction, the lower bound on \( P_n^q(A_{l,k}^{(n)} \mid C_{j,k,l}) \), obtained by assuming that none of the numbers from \( \{k + l, \ldots , n\} \) have yet appeared,
is given by

\[(3.15)\]

\[ P_n(\{A_{ijk}^{(n)}(C_{jk,i}) \geq \prod_{i=1}^{l-1} (1 - r_{k-1-j,i,n-k-l+1}) = \prod_{i=1}^{l-1} q^{k-1-j} - q^{k-1-j+i} / 1 - q^{n-l-j+i} \]

\[ q^{(l-1)(k-1-j)} \prod_{b=n-l-j+1}^{l-1} (1 - q^b) \]

From (3.14) and (3.15), with \(k_n\) and \(q_n\) in place of \(k\) and \(q\), we have

\[(3.16)\]

\[ P_n(\{A_{ijk}^{(n)}(C_{jk,i}) \geq \sum_{j=0}^{k_n-1} P_n(\{A_{ijk}^{(n)}(C_{jk,i}) \geq \sum_{j=1}^{k_n-1} (1 - q_n^{k_n-j-1}) \prod_{b=k_n-j}^{l-1} (1 - q_n^{b}) \prod_{b=k_n-j}^{l-1} (1 - q_n^{b}) \prod_{b=n-l-j+1}^{l-1} (1 - q_n^{b}) \]

By the assumption on \(q_n\), the right hand side of (3.16) satisfies

\[(3.17)\]

\[ \sum_{j=1}^{k_n-1} (1 - q_n^{k_n-j-1}) \prod_{b=k_n-j}^{l-1} (1 - q_n^{b}) \prod_{b=k_n-j}^{l-1} (1 - q_n^{b}) \prod_{b=n-l-j+1}^{l-1} (1 - q_n^{b}) \geq \]

\[ \prod_{b=1}^{l} (1 - q_n^{b}) \sum_{j=1}^{k_n-1} q_n^{l(k_n-1-j)} \prod_{b=k_n-j}^{l-1} (1 - q_n^{b}) q_n^{l(k_n-1-j)} \prod_{b=n-l-j+1}^{l-1} (1 - q_n^{b}) \]

And

\[(3.18)\]

\[ \sum_{j=1}^{k_n-1} q_n^{l(k_n-1-j)} (1 - q_n^{k_n-1-j})^{l-1} \sim \int_0^{k_n-1-l} q_n^{l} (1 - q_n^{x})^{l-1} dx. \]

Making the change of variables \(y = q_n^{x}\), and using the assumption on \(q_n\) and the assumption on \(k_n\) in (3.1), we have

\[(3.19)\]

\[ \int_0^{k_n-1-l} q_n^{l} (1 - q_n^{x})^{l-1} dx = - \frac{1}{\log q_n} \int_0^{q_n^{k_n-1-l}} y^{l-1} (1 - y)^{l-1} dy \sim \]

\[ \frac{n^\alpha}{\Gamma(2l)} = \frac{n^\alpha}{\Gamma(2l)} \]

From (3.16)-(3.19), we conclude that

\[ P_n(\{A_{ijk}^{(n)}(C_{jk,i}) \geq \frac{(l-1)!^2}{(2l)! n^{\alpha(l-1)}} \]
which is the lower bound in (1.7) and (1.8).

For the upper and lower bounds in (1.7), the only assumption on $k_n$ was (3.1). It is clear from the proofs that if we fix $\alpha' \in (\alpha, 1)$ and let $k'_n = \lceil n^{\alpha'} \rceil$, then the upper and lower bounds in (1.7) are uniform over sequences $\{k_n\}_{n=1}^{\infty}$ satisfying $k_n \geq k'_n$. From this along with (1.4), it follows that the upper and lower bounds in (1.7) are in fact uniform over all sequences $\{k_n\}_{n=1}^{\infty}$. □

4. Proof of Theorem 3

Proof of part (i). We follow a slightly more precise version of the construction used in the upper bound in Theorems 1 and 2. From (2.6) it follows that $P_n^q(X_m \not\in \{j+1, \ldots, j+l-1\})$ is monotone increasing in $j$. Thus, the same argument leading up to (2.7) shows that

$$P_n^q(A^{(n)}_{l;k} | B_{l;k,a}) \leq \prod_{i=1}^{n-k-l+1} \left( 1 - \frac{q^{a+i} - q^{a+i+l-1}}{1 - q^{k+l+i-1}} \right).$$

We consider (4.1) with $k_n$ and $q_n$ in place of $k$ and $q$. From the assumption on $q_n$, we have

$$\prod_{i=1}^{n-k_n-l+1} \left( 1 - \frac{q_n^{a+i} - q_n^{a+i+l-1}}{1 - q_n^{k_n+l+i-1}} \right) \sim \prod_{i=1}^{n-k_n-l+1} \left( 1 - \frac{(l-1)cn^{-1}q_n^{a+i}}{1 - q_n^{k_n+l+i-1}} \right).$$

And

$$\log \prod_{i=1}^{n-k_n-l+1} \left( 1 - \frac{(l-1)cn^{-1}q_n^{a+i}}{1 - q_n^{k_n+l+i-1}} \right) \sim \sum_{i=1}^{n-k_n-l+1} \log \left( 1 - \frac{(l-1)cn^{-1}q_n^{a+i}}{1 - q_n^{k_n+l+i-1}} \right) \sim \int_0^{n-k_n-l+1} \log \left( 1 - \frac{(l-1)cn^{-1}q_n^{a+i}}{1 - q_n^{k_n+l+i-1}} \right) dx.$$

Making the change of variables $y = q_n^x$, using the assumptions on $k_n$ and $q_n$ and defining

$$\gamma(c, d) = \log \frac{1 - e^{-cd}}{1 - e^{-c}} < 0,$$
in order to simplify notation in the sequel, we have

\[ \int_{0}^{n-k_n-l+1} \log \left( 1 - \frac{(l-1)cn^{-1}q_n^{x+1}}{1-q_n^{k_n+l+x-1}} \right) dx = \]

\[- \frac{1}{\log q_n} \int_{q_n^{n-k_n-l+1}}^{1} \frac{\log \left( 1 - \frac{(l-1)cn^{-1}q_n^y}{1-q_n^{k_n+l+y-1}} \right)}{y} dy \leq \]

\[ \frac{1}{\log q_n} \int_{q_n^{n-k_n-l+1}}^{1} \frac{(l-1)cn^{-1}q_n^y}{1-q_n^{k_n+l+y-1}} dy \sim \]

\[- (l-1)q_n^a \int_{e^{-c(1-x)}}^{1} \frac{1}{1-e^{-cd}y} dy = \]

\[ (l-1)q_n^a e^{cd} \log \frac{1-e^{-cd}}{1-e^{-c}} = (l-1)q_n^a e^{cd} \gamma(c,d). \]

From (4.1)-(4.5), we conclude that

\[ P_n(A_{k_n}) \leq e^{(l-1)q_n^a e^{cd} \gamma(c,d)}. \]

Using (4.6) with (2.1), along with the assumptions on \( k_n \) and \( q_n \), gives

\[ P_n(A_{k_n}) \leq \sum_{a=0}^{k_n-1} q_n^a \prod_{b=k_n+l-1}^{l-1} (1-q_n^b) e^{(l-1)q_n^a e^{cd} \gamma(c,d)} \sim \]

\[ \frac{1}{n^{l-1-1}} \int_{0}^{dn} q_n^x e^{(l-1)q_n^x e^{cd} \gamma(c,d)} dx. \]

Making the change of variables \( y = q_n^x \) and using the assumption on \( q_n \), we obtain

\[ \int_{0}^{dn} q_n^x e^{(l-1)q_n^x e^{cd} \gamma(c,d)} dx = - \frac{1}{\log q_n} \int_{q_n^{dn}}^{1} y^{l-1} e^{(l-1)e^{cd} \gamma(c,d)} dy \sim \]

\[ \frac{n}{e} \int_{e^{-cd}}^{1} y^{l-1} e^{(l-1)e^{cd} \gamma(c,d)} dy. \]

From (4.7), (4.8) and (4.4), we arrive at (1.9), which completes the proof of part (i).

**Proof of part (ii).** We write \( q_n = 1 - \epsilon(n) \), where \( 0 < \epsilon(n) = o(\frac{1}{n}) \). We follow the proof of part (i) through the first three lines of (4.5), the only change
being that the term $cn^{-1}$ is replaced by $\epsilon(n)$. Starting from there, we have

$$
\int_{0}^{n-k_n-l+1} \log \left( 1 - \frac{(l-1)\epsilon(n) q_n^{a+x}}{1 - q_n^{k_n+l+x-1}} \right) \, dx \leq
$$

(4.9)

$$
\frac{1}{\log q_n} \int_{q_n^{-k_n-l+1}}^{1} \frac{(l-1)\epsilon(n) q_n^{a}}{1 - q_n^{k_n+l-1} y} dy =
$$

$$
\frac{(l-1)\epsilon(n) q_n^{a}}{\log q_n} q_n^{-(k_n+l-1)} \log \left( \frac{1 - q_n^{n}}{1 - q_n^{k_n+l-1}} \right).
$$

Since $\epsilon(n) = \sigma(n)$, we have $1 - q_n^{n} \sim n\epsilon(n) / 1 - q_n^{k_n+l-1} \sim k_n\epsilon(n)$, $q_n^{-(k_n+l-1)} \sim 1$ and $q_n^{a} \sim 1$, uniformly over $a \in \{0, \cdots, k_n - 1\}$. Using this with (4.9), we have

$$
\int_{0}^{n-k_n-l+1} \log \left( 1 - \frac{(l-1)\epsilon(n) q_n^{a+x}}{1 - q_n^{k_n+l+x-1}} \right) \, dx \lesssim (l-1) \log \frac{k_n}{n},
$$

(4.10)

uniformly over $a \in \{0, \cdots, k_n - 1\}$.

From (4.1)-(4.3) (with $cn^{-1}$ replaced by $\epsilon(n)$) and (4.10), we conclude that

(4.11)

$$
P_{n}(A^{(n)}_{i; k_n} | B_{i; k; a}) \lesssim \left( \frac{k_n}{n} \right)^{l-1}.
$$

Using (4.11) with (2.1), along with the assumption on $q_n$, we conclude that

$$
P_{n}(A^{(n)}_{i; k_n}) \lesssim \sum_{a=0}^{k_n-1} q_n^{a} \frac{k_n}{n} \frac{1}{\prod_{b=k_n}^{k_n+l-1} (1 - q_n^{b})} \left( \frac{k_n}{n} \right)^{l-1} \sim
$$

(4.12)

$$
l!\epsilon(n)^{l} \left( \frac{k_n}{n} \right)^{l-1} \sum_{a=0}^{k_n-1} q_n^{a} \left( \frac{k_n}{n} \right)^{l-1} 1 - q_n^{k_n+l} \sim
$$

$$
l! \frac{k_n \epsilon(n)}{n^{l-1} \epsilon(n)} = \frac{l!}{n^{l-1}}.
$$

□

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