Convexity and robustness of the Rényi entropy

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March 9, 2021

Abstract. We study convexity properties of Rényi entropy as function of $\alpha > 0$ on finite alphabets. We also describe robustness of the Rényi entropy on finite alphabets, and it turns out that the rate of respective convergence depends on initial alphabet. We establish convergence of the disturbed entropy when the initial distribution is uniform but the number of events increases to $\infty$ and prove that limit of Rényi entropy of binomial distribution is equal to Rényi entropy of Poisson distribution.

Keywords: Discrete distribution, Rényi entropy, Convexity.

Mathematics Subject Classification (2020): 60E05, 94A17.

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space supporting all distributions considered below. For any $N \geq 1$ introduce the family of discrete distributions $p = (p_1, p_2, \ldots, p_N)$ with probabilities $p_i \geq 0$, $1 \leq i \leq N$, $N \geq 1$, $p_1 + \ldots + p_N = 1$.

In the present paper we investigate some properties of the Rényi entropy, which was proposed by Rényi in [1],

$$H_\alpha(p) = \frac{1}{1-\alpha} \log \left( \sum_{k=1}^{N} p_k^\alpha \right), \quad \alpha > 0, \alpha \neq 1,$$

including its limit value as $\alpha \to 1$, i.e., the Shannon entropy

$$H(p) = -\sum_{k=1}^{N} p_k \log(p_k).$$

Due to this continuity, it is possible to put $H_1(p) = H(p)$. We consider the Rényi entropy as a functional of various parameters. The first approach is to fix the distribution and consider $H_\alpha(p)$ as the function of $\alpha > 0$. Some of the properties of $H_\alpha(p)$ as the function of $\alpha > 0$ are well known. In particular, it is known
that $H_\alpha(p)$ is continuous and non-increasing in $\alpha \in (0, \infty)$, $\lim_{\alpha \to 0^+} H_\alpha(p) = \log m$, where $m$ is the number of non-zero probabilities, and $\lim_{\alpha \to +\infty} H_\alpha(p) = -\log \max_k p_k$. However, for the reader’s convenience, we provide the short proofs of this and some other simple statements in the Appendix. One can see that these properties of the entropy itself and its first derivative are common for all finite distributions. Also, it is known that Rényi entropy is Schur concave as a function of distribution vector, that is

$$(p_i - p_j) \left( \frac{\partial H_\alpha(p)}{\partial p_i} - \frac{\partial H_\alpha(p)}{\partial p_j} \right) \leq 0, \ i \neq j.$$  

Some additional results such as lower bounds on the difference in Rényi entropy for distributions defined on countable alphabets could be found in [2]. Those results usually use Rényi divergence of order $\alpha$ of a distribution $P$ from a distribution $Q$

$$D_\alpha (P||Q) = \frac{1}{\alpha - 1} \log \left( \sum_{i=1}^{N} \frac{p_i^\alpha}{q_i^{\alpha-1}} \right),$$

which is very similar to Kullback-Leibler divergence. Some of Rényi divergences most important properties were reviewed and extended in [3]. Rényi divergence for most commonly used univariate continuous distributions could be found in [4]. Rényi entropy and divergence is widely used in majorization theory [5, 6], statistics [7, 8], information theory [2, 3, 9] and many other fields. Boundedness of Rényi entropy was shown in [10] for discrete log-concave distributions depending on it’s variance. There are other operational definitions of Rényi entropy given in [11], which are used in practice. However, in the present paper we restrict ourselves with standard Rényi entropy and go a step ahead in comparison with standard properties, namely, we investigate convexity of the Rényi entropy with the help of the second derivative. It turned out that from this point of view, the situation is much more interesting and uncertain in comparison with the behavior of the 1st derivative, and crucially depends on the distribution. One might say that all the standard guesses are wrong. Of course, the second derivative is continuous (evidently, it simply means that it is continuous at 1 because at all other points, the continuity is obvious), but then the surprises begin. If the second derivative starts with a positive value at zero, it can either remain positive or have inflection points, depending on the distribution. If it starts from the negative value, it can have the first infection point both before 1 and after 1, due to the distribution, too (point 1 is interesting as some crucial point for entropy, so, we compare the value of inflection points with it). The value of the second derivative at zero is bounded from below but unbounded from above. Some superposition of entropy is convex, and this fact simultaneously describes why other similar properties depend on distribution. Due to the over-complexity of some expressions, which defied analytical consideration, we propose several illustrations performed by numerical methods. We investigate robustness of the Rényi entropy w.r.t. the distribution, and it turns out that the
rate of respective convergence depends on initial distribution, too. Further, we establish convergence of the disturbed entropy when the initial distribution is uniform but the number of events increases to \( \infty \) and prove that limit of Rényi entropy of binomial distribution is equal to entropy of Poisson distribution. It was previously proved in [12] that Shannon entropy of binomial distribution is increasing to entropy of Poisson distribution. Our proof of this particular fact is simpler because uses only Lebesgue’s dominated convergence theorem. The paper is organized as follows. Section 2 is devoted to the convexity properties of the Rényi entropy, Section 3 describes robustness of the Rényi entropy, and Section 4 contains some auxiliary results.

2 Convexity of the Rényi entropy

To start, we consider the general properties of the 2nd derivative of the Rényi entropy.

2.1 The form and the continuity of the 2nd derivative

Let’s denote

\[ S_i(\alpha) = \sum_{k=1}^{N} p_k^\alpha \log^i p_k, i = 0, 1, 2, 3. \]

Denote also

\[ f(\alpha) = \log \left( \sum_{k=1}^{N} p_k^\alpha \right). \]

Obviously, function \( f \in C^\infty(\mathbb{R}^+) \), and its first derivatives equal

\[ f'(\alpha) = \frac{S_1(\alpha)}{S_0(\alpha)}, \quad f''(\alpha) = \frac{S_2(\alpha)S_0(\alpha) - S_1^2(\alpha)}{S_0^2(\alpha)}, \]

\[ f'''(\alpha) = \frac{S_3(\alpha)S_0^2(\alpha) - 3S_2(\alpha)S_1(\alpha)S_0(\alpha) + 2S_1^3(\alpha)}{S_0^3(\alpha)}. \]

In particular, if to consider the random variable \( \xi \) taking values \( \log p_k \) with probability \( p_k \), then

\[
\begin{align*}
 f'(1) &= E(\xi) < 0, \\
 f''(1) &= E(\xi^2) - (E(\xi))^2 > 0, \\
 f'''(1) &= E(\xi^3) - 3E(\xi^2)E(\xi) + 2(E(\xi))^3, \\
\end{align*}
\]

and the sign of \( f'''(1) \) is not clear (as we can see below, it can be both + and −).

Lemma 2.1. Let \( p_k \not= 0 \) for all \( 1 \leq k \leq N \). Then

(i) (a) The 2nd derivative \( H''_\alpha(p) \) equals

\[
H''_\alpha(p) = -\frac{1}{(1-\alpha)^3} \left( \sum_{k=1}^{N} ((1-\alpha)q_k'(\alpha) + 2q_k(\alpha)) \log \frac{q_k(\alpha)}{p_k} \right),
\]

where

\[ q_k(\alpha) = \frac{p_k^\alpha}{\sum_{k=1}^{N} p_k^\alpha}. \]
(b) The 2nd derivative $H''(p)$ can be also presented as
\[
H''(p) = -\frac{1}{3} f'''(\theta)
\] (2.3)
for some $0 < \theta < \alpha$.

(ii) The 2nd derivative $H''(p)$ is continuous on $\mathbb{R}^+$ if we put
\[
H''(1) = -\frac{1}{3} f'''(1) = -\frac{1}{3} (E(\xi^3) - 3E(\xi^2)E(\xi) + 2E(\xi)^3).
\]

Proof. Equality (2.2) is a result of direct calculations. Concerning equality (2.3), we can present $H(\alpha)$ as
\[
H(\alpha) = \frac{f(\alpha) - f(1)}{1 - \alpha},
\]
therefore, $-H(p)$ is a slope function for $f$. Taking successive derivatives, we get from standard Taylor formula that
\[
H' (p) = \frac{f'(\alpha)(1 - \alpha) + f(\alpha)}{(1 - \alpha)^2} = -\frac{1}{2} f''(\eta),
\]
and
\[
H''(p) = \frac{f''(\alpha)(1 - \alpha)^2 + 2f'(\alpha)}{(1 - \alpha) + 2f(\alpha)} = -\frac{1}{3} f'''(\theta),
\]
where $\eta, \theta \in (0, \alpha)$. If $\alpha \to 1$, then both $\eta$ and $\zeta$ tend to 1. Taking into account (2.1), we immediately get both equality (2.3) and statement (ii). \qed

2.2 Behavior of the 2nd derivative at the origin

Let us consider the starting point for the 2nd derivative, i.e., the behavior of $H''(p)$ at zero as a function of a distribution vector $p$. Analyzing (2.2), we see that $H''(p)$ as function of $\alpha$ is continuous in 0. Moreover,
\[
q_k(0) = 1/N, \quad q'_k(0) = \frac{\log p_k}{N} - \frac{\sum_{k=1}^{N} \log p_k}{N},
\]
so we can present $H''(p)$ as
\[
H''(p) = \sum_{k=1}^{N} \left( \frac{1}{N} \log p_k - \frac{1}{N^2} \sum_{i=1}^{N} \log p_i + \frac{2}{N} \right) \log \frac{1}{Np_k}
= \sum_{k=1}^{N} \left( \frac{1}{N} \log p_k - \frac{1}{N^2} \sum_{i=1}^{N} \log p_i + \frac{2}{N} \right) (\log N + \log p_k)
= 2 \log N + \frac{1}{N} \sum_{k=1}^{N} (\log p_k)^2 - \frac{1}{N^2} \left( \sum_{k=1}^{N} \log p_k \right)^2 + \frac{2}{N} \sum_{k=1}^{N} \log p_k.
\]
Now we are interested in the sign of $H''(\alpha)$. Give an example of distributions for which $H''(\alpha) > 0$ it is very simple, one of such examples is given at Figure 1. Concerning negative $H''(\alpha)$, it is also possible, however, at this moment we prefer to start with a more general result.

**Lemma 2.2.** If some probability vector is $p$ a point of local extremum of $H''(\alpha)$ then either $p = p(\text{uniform}) = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)$ or it contains two different probabilities.

**Proof.** Let us formulate the necessary conditions for $H''(\alpha, p)$ to have a local extremum at some point. Taking into account limitation $\sum_{k=1}^{N} p_k = 1$, these conditions have a form

$$
\begin{cases}
2 \log N + \frac{1}{N} \sum_{k=1}^{N} (\log p_k)^2 - \frac{1}{N^2} \left(\sum_{k=1}^{N} \log p_k\right)^2 + \frac{2}{N} \sum_{k=1}^{N} \log p_k \rightarrow \text{extr} \\
\sum_{k=1}^{N} p_k = 1.
\end{cases}
$$

We create a Lagrangian function

$$
L = \lambda_0 \left(2 \log N + \frac{1}{N} \sum_{k=1}^{N} (\log p_k)^2 - \frac{1}{N^2} \left(\sum_{k=1}^{N} \log p_k\right)^2 + \frac{2}{N} \sum_{k=1}^{N} \log p_k\right)
$$

$$
+ \lambda \left(\sum_{k=1}^{N} p_k - 1\right).
$$

If some $p$ is an extreme point then there exist $\lambda_0$ and $\lambda$ such that $\lambda_0^2 + \lambda^2 \neq 0$ and $\frac{\partial L}{\partial p_i}(p) = 0$ for all $1 \leq i \leq N$, i.e.,

$$
\frac{\partial L}{\partial p_i} = \lambda_0 \left(\frac{2}{Np_i} \log p_i - \frac{2}{N^2p_i} \left(\sum_{k=1}^{N} \log p_k\right) + \frac{2}{Np_i}\right) + \lambda = 0.
$$

If $\lambda_0 = 0$ then $\lambda = 0$. However, $\lambda_0^2 + \lambda^2 \neq 0$, therefore we can put $\lambda_0 = 1$. Then

$$
-\lambda p_i = \frac{2}{N} \log p_i - \frac{2}{N^2} \left(\sum_{k=1}^{N} \log p_k\right) + \frac{2}{N}.
$$

taking a sum of these equalities we get that $\lambda = -2$ whence

$$
p_i - \frac{1}{N} \log p_i = \frac{1}{N} - \frac{1}{N^2} \left(\sum_{k=1}^{N} \log p_k\right).
$$

So, if distribution vector $p$ is an extreme point then $p_1 - \frac{1}{N} \log p_1 = \ldots = p_N - \frac{1}{N} \log p_N$. Let’s have a look at continuous function $f(x) = x - \frac{1}{N} \log x$, $x \in (0, 1)$. Its derivative equals

$$
f'(x) = 1 - \frac{1}{Nx} = 0 \Leftrightarrow x = \frac{1}{N}, \quad \text{sign}(f'(x)) = \text{sign} \left( x - \frac{1}{N} \right),
$$

$$
5
$$
Then we will show that $p$ contains only two different probabilities.

Thus, if the vector of probabilities is a vector of local extremum of $\mathcal{H}''_0(p)$, then it contains no more than two different probabilities. Obviously, it can be $p = p(\text{uniform}) = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)$.

**Remark 2.3.** Note that $\mathcal{H}''_0(p(\text{uniform})) = 0$. Therefore, in order to find the distribution for which $\mathcal{H}''_0(p) < 0$ let us consider the distribution vector that contains only two different probabilities $p_0, q_0$ such that:

$$
\begin{align*}
\begin{cases}
p_0 - q_0 = \frac{1}{N} (\log p_0 - \log q_0), \\
k p_0 + (N-k)q_0 = 1,
\end{cases}
\end{align*}
$$

(2.5)

where $N, k \in \mathbb{N}$, $N > k$ and $p_0, q_0 \in (0, 1)$.

**Lemma 2.4.** Let $p$ be distribution vector satisfying (2.5). Then $\mathcal{H}''_0(p) < 0$.

**Proof.** First, we will show that $\mathcal{H}''_0(p)$ is non-positive. For that we rewrite $\mathcal{H}''_0(p)$ in terms of $p_0$ and $q_0$:

$$
\begin{align*}
\mathcal{H}''_0(p) &= 2 \log N + \frac{1}{N} (k(\log p_0)^2 + (N-k)(\log q_0)^2) - \frac{1}{N^2} (k \log p_0 + (N-k) \log q_0)^2 \\
&+ \frac{2}{N} (k \log p_0 + (N-k) \log q_0) \\
&= 2 \log N + \frac{k(N-k)}{N^2} ((\log p_0)^2 - 2 \log p_0 \log q_0 + (\log q_0)^2) \\
&+ \frac{2k}{N} (\log p_0 - \log q_0) + 2 \log q_0 = 2 \log N q_0 + k(N-k)(p_0 - q_0)^2 + 2k(p_0 - q_0).
\end{align*}
$$

We know that $k p_0 + (N-k)q_0 = 1$, whence $k = \frac{N q_0 - 1}{q_0 - p_0}$, and $N-k = \frac{1-N p_0}{q_0 - p_0}$. Then

$$
\begin{align*}
\mathcal{H}''_0(p) &= 2 \log N q_0 + (1-N q_0)(N p_0 - 1) + 2(1-N q_0) \\
&= 2 \log N q_0 + N(p_0 - q_0) + 1 - N^2 p_0 q_0 \\
&= \log(N q_0)^2 + \log \frac{p_0}{q_0} + 1 - N^2 p_0 q_0 = \log N^2 p_0 q_0 - N^2 p_0 q_0 + 1.
\end{align*}
$$

Note that $\log x - x + 1 < 0$ for $x > 0, x \neq 1$. We want to show that under conditions (2.5) $N^2 p_0 q_0$ can’t be equal to 1. Suppose that $N^2 p_0 q_0 = 1$. Then it follows from (2.5) that

$$
\frac{k}{N^2} + (N-k)q_0^2 = q_0.
$$

It means that $q_0$ and $p_0$ are algebraic numbers. Thus, their difference $p_0 - q_0$ is also algebraic. On the other hand, by Lindemann–Weierstrass theorem $\frac{1}{N} (\log p_0 - \log q_0)$ is transcendental number, which contradicts (2.5). So $N^2 p_0 q_0 \neq 1$ and $\mathcal{H}''_0(p) < 0$. 

\[\square\]
Theorem 2.5. For any \( n > 2 \) there exists \( N \geq n \) and a probability vector \( p = (p_1, \ldots, p_N) \) such that \( H'_0(p) < 0 \).

Proof. Consider the distribution vector \( p \) that satisfies conditions (2.3). From Lemma 2.4 we know that \( H'_0(p) < 0 \). Now we want to show that there exist arbitrarily large \( N \in \mathbb{N} \) and distribution vector \( p \) of length \( N \) that satisfy those conditions. For that we denote

\[
x = Np_0, \quad y = Nq_0, \quad r = \frac{k}{N} = \frac{y - 1}{y - x}.
\]

Then \( 0 < x < 1 < y \) and \( r < 1 \) and \( x - y = \log x - \log y \). Function \( x - \log x \) is decreasing on \((0, 1)\), is increasing on \((1, +\infty)\) and is equal to 1 at point 1. Let \( y = y(x) \) be implicit function defined by \( x - y = \log x - \log y \). By that we get 1-to-1 correspondence from \( x \in (0, 1) \) to \( y \in (1, +\infty) \). We also have function \( r(x) = \frac{y(x) - 1}{y(x) - x} \). If we find \( x' \in (0, 1) \) such that \( r' = r(x') \) is rational then we can pick \( N, k \in \mathbb{N} \) such that \( r' = \frac{k}{N} \) and get distribution vector \( p \) satisfying (2.3) with \( p_0 = \frac{k}{N} \), \( q_0 = \frac{y}{y} \). However, we won’t find such \( x' \), we will just show that they exist. To do that observe that \( y(x) \) is continuous function of \( x \) and so is function \( r(x) = \frac{y(x) - 1}{y(x) - x} \). What’s more,

\[
y(x) \to +\infty, \quad x \to 0 \quad \text{so} \quad r(x) \to 1, \quad x \to 0 + .
\]

Let’s fix \( x_0 \in (0, 1) \), \( r(x_0) < 1 \). Then for any \( r' \in (r(x_0), 1) \) there exists \( x' \in (0, x_0) \) such that \( r(x') = r' \). By taking \( r' \in \mathbb{Q} \) we get that there exists \( x' \) such that \( \frac{k}{N} < 1 \) and is rational. Finally, we want to show that \( N \) can be arbitrarily large. For that simply observe that \( \frac{k}{N} = r' \) so as \( r' \to 1- \) we get that \( N \to +\infty \).

\[ \square \]

Lemma 2.6. Let \( N \) be fixed. Then \( H''_0(p) \) as the function of vector \( p \) is bounded from below and is unbounded from above.

Proof. Recall that \( H''_0(p) = 0 \) on the uniform distribution and exclude this case from further consideration. In order to simplify the notations, we denote \( x_k = \log p_k \), and let

\[
S_N := N(H''_0(p) - 2 \log N) = \sum_{k=1}^{N} (x_k)^2 - \frac{1}{N} \left( \sum_{k=1}^{N} x_k \right)^2 + 2 \sum_{k=1}^{N} x_k.
\]

Note that there exists \( n \leq N - 1 \) such that

\[
x_1 < \log \frac{1}{N}, \ldots, x_n < \log \frac{1}{N}, \quad x_{n+1} \geq \log \frac{1}{N}, \ldots, x_N \geq \log \frac{1}{N}.
\]

Further, denote the rectangle \( A = [\log \frac{1}{N}; 0]^{N-n} \subset \mathbb{R}^{N-n} \), and let

\[
S_{N,1} = \sum_{k=1}^{n} x_k, \quad S_{N,2} = \sum_{k=n+1}^{N} x_k.
\]
Let's establish that $\mathcal{H}_0''(p)$ is bounded from below. In this connection, rewrite $S_N$ as

$$S_N = \sum_{k=1}^{n} x_k^2 + \sum_{k=n+1}^{N} x_k^2 - \frac{1}{N} ((S_{N,1})^2 + 2S_{N,1}S_{N,2} + (S_{N,2})^2) + 2S_{N,1} + 2S_{N,2}.$$ 

By Cauchy–Schwarz inequality we have

$$\left(\sum_{k=1}^{n} x_k^2\right)^2 \leq n \sum_{k=1}^{n} x_k^2, \quad \left(\sum_{k=n+1}^{N} x_k\right)^2 \leq (N-n) \sum_{k=n+1}^{N} x_k^2.$$ 

Therefore

$$S_N \geq \left(1 - \frac{n}{N}\right) \sum_{k=1}^{n} x_k^2 + \frac{n}{N} \sum_{k=n+1}^{N} x_k^2 - \frac{2}{N} S_{N,1}S_{N,2} + 2S_{N,1} + 2S_{N,2}$$

$$= \sum_{k=1}^{n} \left(\left(1 - \frac{n}{N}\right) x_k^2 + x_k \left(2 - \frac{2}{N} S_{N,2}\right)\right) + \frac{n}{N} \sum_{k=n+1}^{N} x_k^2 + 2S_{N,2}$$

$$= \frac{1}{N} \sum_{k=1}^{n} ((N-n) x_k^2 + 2x_k (N - S_{N,2})) + \frac{n}{N} \sum_{k=n+1}^{N} x_k^2 + 2S_{N,2}.$$ 

There exists $M > 0$ such that for every $n \leq N-1$ we have $|S_{N,2}| \leq M$ because $A$ is compact and $S_{N,2}$ is continuous on $A$. Obviously, $\frac{n}{N} \sum_{k=n+1}^{N} x_k^2 \geq 0$. Finally, for every $1 \leq k \leq n$ we have that $(N-n) x_k^2 + 2x_k (N - S_{N,2})$ is bounded from below by the value $-\frac{(N-S_{N,2})^2}{N-n} \geq -N^2 - M^2$. Resuming, we get that $S_N$ is bounded from below, and consequently $\mathcal{H}_0''(p)$ is bounded from below for fixed $N$.

Now we want to establish that $\mathcal{H}_0''(p)$ is not bounded from above. In this connection, let $\varepsilon > 0$, and let us consider the distribution of the form $p_1 = \varepsilon$, $p_2 = ... = p_N = \frac{1-\varepsilon}{N-1}$. Then we have

$$\mathcal{H}_0''(p) = 2\log N + \frac{1}{N} \sum_{k=1}^{N} \left(\log p_k\right)^2 - \frac{1}{N^2} \left(\sum_{k=1}^{N} \log p_k\right)^2 + \frac{2}{N} \sum_{k=1}^{N} \log p_k$$

$$= 2\log N + \frac{N-1}{N} \left(\log \frac{1-\varepsilon}{N-1}\right)^2 + \frac{1}{N} (\log \varepsilon)^2 - \frac{1}{N^2} \left(\frac{(N-1) \log \frac{1-\varepsilon}{N-1}}{N} + \log \varepsilon\right)^2$$

$$+ \frac{2(N-1)}{N} \log \frac{1-\varepsilon}{N-1} + \frac{2}{N} \log \varepsilon = \left(1 - \frac{1}{N^2}\right) \left(\frac{1}{N} - \frac{1}{N^2}\right) (\log \varepsilon)^2$$

$$+ \left(\frac{2}{N} - \frac{2(N-1)}{N^2} \log \frac{1-\varepsilon}{N-1}\right) \log \varepsilon + 2\log N + \left(\frac{N-1}{N} - \frac{(N-1)^2}{N^2}\right) \left(\log \frac{1-\varepsilon}{N-1}\right)^2$$

$$+ \frac{2(N-1)}{N} \log \frac{1-\varepsilon}{N-1} \to +\infty, \quad \varepsilon \to 0 + .$$
2.3 Superposition of entropy that is convex

Now we establish that the superposition of entropy with some decreasing function is convex. Namely, we shall consider function

\[ G_\beta(p) = -\mathcal{H}_{1+\frac{1}{\beta}}(p) = \beta \log \left( \sum_{k=1}^{N} p_k^{1+1/\beta} \right), \quad \beta > 0 \tag{2.6} \]

and prove its convexity. Since now we consider the tools that do not include differentiation, we can assume that some probabilities are zero. In order to provide convexity, we start with the following simple and known result whose proof is added for the reader’s convenience.

**Lemma 2.7.** For any measure space \((\mathcal{X}, \Sigma, \mu)\) and any measurable \(f \in L^p(\mathcal{X}, \Sigma, \mu)\) for some interval \(p \in [a, b]\), \(\|f\|_p = \|f\|_{L^p(\mathcal{X}, \Sigma, \mu)}\) is log-convex as a function of \(1/p\) on this interval.

**Proof.** For any \(p_2, p_1 > 0\) and \(\theta \in (0, 1)\), denote \(p_0 = \left(\frac{\theta}{p_1} + \frac{1-\theta}{p_2}\right)^{-1}\) and observe that

\[ \theta p_0/p_1 + (1-\theta)p_0/p_2 = 1. \]

Therefore, by the Hölder inequality

\[
\|f\|^p_p = \int_{\mathcal{X}} |f(x)|^{\theta p_1} \cdot |f(x)|^{(1-\theta)p_2} \mu(dx)
\leq \left( \int_{\mathcal{X}} |f(x)|^{p_1} \mu(dx) \right)^{\theta p_0/p_1} \left( \int_{\mathcal{X}} |f(x)|^{p_2} \mu(dx) \right)^{(1-\theta)p_0/p_2},
\]

whence

\[ \log \|f\|_p \leq \theta \log \|f\|_{p_1} + (1-\theta) \log \|f\|_{p_2}, \]

as required. \(\square\)

**Corollary 2.8.** For any probability vector \(p = (p_k, 1 \leq k \leq N)\), function \(G_\beta(p), \beta > 0\), is convex.

**Proof.** Follows from Lemma 2.7 by setting \(\mathcal{X} = \{1, \ldots, N\}, \mu(A) = \sum_{k \in A} p_k, f(k) = p_k, k \in \mathcal{X}. \square\)

**Remark 2.9.** It follows immediately from (2.6) that for the function

\[ G_\beta(p) = \beta \log \sum_{k=1}^{N} p_k^{1+1/\beta}, \quad \beta > 0 \]

then \(\mathcal{H}_\alpha(p) = G_{\frac{1}{\alpha-1}}(p)\). For \(\alpha > 1\), \(\frac{1}{\alpha-1}\) is convex. If it happened that there is such \(p\) that \(G_\beta(p)\) is non-decreasing on an interval then \(G_{\frac{1}{\alpha-1}}(p)\) be convex on
that interval and $H_{\alpha}(p)$ be convex, too. However,

$$G'_{\beta}(p) = \log \sum_{k=1}^{N} p_k^{1+1/\beta} - \frac{1}{\beta} \frac{\sum_{k=1}^{N} p_k^{1+1/\beta} \log p_k}{\sum_{k=1}^{N} p_k^{1+1/\beta}}$$

$$= - \sum_{k=1}^{N} p_k^{1+1/\beta} \log \frac{p_k^{1/\beta}}{\sum_{k=1}^{N} p_k^{1+1/\beta}} \leq 0.$$  

In some sense, this is a reason why we can not say something definite concerning the 2nd derivative of entropy either on the whole semiaxes or even in the interval $[1, +\infty)$.

2.4 Graphs of $H_{\alpha}(p)$ and it’s second derivative of several probability distributions

![Graphs of $H_{\alpha}(p)$ and $H''_{\alpha}(p)$](image)

Figure 1: Graph of $H_{\alpha}(p)$ and $H''_{\alpha}(p)$, where $p_1 = p_2 = 0.4, p_3 = 0.2$. Here $H_{\alpha}(p)$ is convex as a function of $\alpha > 0$. 

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Figure 2: Graph of $H_\alpha(p)$ and $H_\alpha''(p)$, where $p_1 = \ldots = p_{198} = \frac{1}{100}, p_{199} = p_{200} = \frac{101}{100}$. Dot is the point where $H_\alpha''(p) = 0$ and this point is $\alpha = 0.99422$.

Figure 3: Graph of $H_\alpha(p)$ and $H_\alpha''(p)$, where $p_1 = \ldots = p_{10} = 0.01, p_{11} = p_{12} = 0.15, p_{13} = p_{14} = 0.3$. Here second derivative becomes positive long before point 1 (at point 0.11318).

Figure 4: Graph of $H_\alpha(p)$ and $H_\alpha''(p)$, where $p_1 = \ldots = p_{10} = 0.08, p_{11} = 0.2$. Here second derivative becomes positive after point 1 (at point 2.9997).
Figure 5: Graph of $H_{\alpha}(p)$ and $H'_{\alpha}(p)$, where $p_1 = ... = p_{100} = 0.0001, p_{101} = ... = p_{200} = 0.0079, p_{201} = 0.2$. Here second derivative has two zeros.

Figure 6: Graph of $H_{\alpha}(p)$, where $p_1 = p_2 = 0.05, p_3$ is changing from 0 to 0.9 and $p_4 = 1 - p_1 - p_2 - p_3$. 
3 Robustness of the Rényi entropy

Now we study the asymptotic behavior of the Rényi entropy depending on the behavior of the involved probabilities. The first problem is the stability of the entropy w.r.t. involved probabilities and the rate of its convergence to the limit value when probabilities tend to their limit value with the fixed rate.

3.1 Rate of convergence of the disturbed entropy when the initial distribution is arbitrary but fixed

Let’s look at distributions that are “near” some fixed distribution \( p = (p_k, 1 \leq k \leq N) \) and construct the approximate distribution \( p(\epsilon) = (p_k(\epsilon), 1 \leq k \leq N) \) as follows. Now we can assume that some probabilities are zero, and we shall see that this assumption influences the rate of convergence of the Rényi entropy to the limit value. So, let \( 0 \leq N_1 < N \) be a number of zero probabilities, and for them we consider approximate values of the form \( p_k(\epsilon) = c_k \epsilon, 0 \leq c_k \leq 1, 1 \leq k \leq N_1. \) Further, let \( N_2 = N - N_1 \geq 1 \) be a number of non-zero probabilities, and for them we consider approximate values of the form \( p_k(\epsilon) = p_k + c_k \epsilon, |c_k| \leq 1, N_1 + 1 \leq k \leq N, \) where \( c_1 + ... + c_N = 0 \) and \( \epsilon \leq \min_{N_1+1 \leq k \leq N} p_k. \) Assume also that there exists \( k \leq N \) such that \( c_k \neq 0, \) otherwise \( H_\alpha(p) - H_\alpha(p(\epsilon)) = 0. \) So, we disturb intial probabilities linearly in \( \epsilon \) with different weights whose sum should necessarily be zero. These assumptions supply that \( 0 \leq p_k(\epsilon) \leq 1 \) and \( p_1(\epsilon) + ... + p_N(\epsilon) = 1. \) Now we want to find out how entropy of the disturbed distribution will differ from the initial entropy, depending on parameters \( \epsilon, N \) and \( \alpha. \) We start with \( \alpha = 1. \)

**Theorem 3.1.** Let number \( N \) and coefficients \( c_1, ..., c_N \) be fixed, and let \( \alpha = 1. \) We have three different situations:

(i) Let \( N_1 \geq 1 \) and there exists \( k \leq N_1 \) such that \( c_k \neq 0. \) Then

\[
H_1(p) - H_1(p(\epsilon)) \sim \epsilon \log \epsilon \sum_{k=1}^{N_1} c_k, \ \epsilon \to 0.
\]

(ii) Let for all \( k \leq N_1 \) \( c_k = 0 \) and \( \sum_{k=N_1+1}^{N} c_k \log p_k \neq 0. \) Then

\[
H_1(p) - H_1(p(\epsilon)) \sim \epsilon \sum_{k=N_1+1}^{N} c_k \log p_k, \ \epsilon \to 0.
\]

(iii) Let for all \( k \leq N_1 \) \( c_k = 0 \) and \( \sum_{k=N_1+1}^{N} c_k \log p_k = 0. \) Then

\[
H_1(p) - H_1(p(\epsilon)) \sim \frac{\epsilon^2}{2} \sum_{k=N_1+1}^{N} \frac{c_k^2}{p_k}, \ \epsilon \to 0.
\]
Proof. First of all, we will find asymptotic behavior of two auxiliary functions as \( \varepsilon \to 0 \). First, let \( 0 \leq c_k \leq 1 \). Then
\[
 c_k \varepsilon \log(c_k \varepsilon) = c_k \varepsilon \log \varepsilon + c_k \varepsilon \log c_k = c_k \varepsilon \log \varepsilon + o(\varepsilon \log \varepsilon), \; \varepsilon \to 0.
\]
Second, let \( p_k > 0, |c_k| \leq 1 \). Taking into account Taylor expansion of logarithm
\[
 \log(1 + x) = x - \frac{x^2}{2} + o(x^2), \; x \to 0,
\]
we can write:
\[
(p_k + c_k \varepsilon) \log(p_k + c_k \varepsilon) - p_k \log p_k = c_k \varepsilon \log p_k + (p_k + c_k \varepsilon) \log(1 + c_k p_k^{-1} \varepsilon)
\]
\[
= c_k \varepsilon \log p_k + (p_k + c_k \varepsilon) \left( c_k p_k^{-1} \varepsilon - \frac{1}{2} (c_k p_k^{-1} \varepsilon)^2 + o(\varepsilon^2) \right)
\]
\[
= \varepsilon (c_k \log p_k + c_k) + \varepsilon^2 \left( c_k^2 p_k^{-1} - \frac{1}{2} c_k^2 p_k^{-1} \varepsilon^2 \right) + o(\varepsilon^2), \; \varepsilon \to 0.
\]
In particular, we immediately get from (3.1) that
\[
(p_k + c_k \varepsilon) \log(p_k + c_k \varepsilon) - p_k \log p_k = o(\varepsilon \log \varepsilon), \; \varepsilon \to 0,
\]
and
\[
(p_k + c_k \varepsilon) \log(p_k + c_k \varepsilon) - p_k \log p_k = \varepsilon (c_k \log p_k + c_k) + o(\varepsilon), \; \varepsilon \to 0
\]
Now simply observe the following.

(i) \[
\lim_{\varepsilon \to 0} \frac{\mathcal{H}_1(p) - \mathcal{H}_1(p(\varepsilon))}{\varepsilon \log \varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon \log \varepsilon} \sum_{k=1}^{N_1} c_k \varepsilon \log c_k \varepsilon
\]
\[
+ \frac{1}{\varepsilon \log \varepsilon} \sum_{k=N_1+1}^{N} ((p_k + c_k \varepsilon) \log(p_k + c_k \varepsilon) - p_k \log p_k)
\]
\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon \log \varepsilon} \left( \sum_{k=1}^{N_1} (c_k \varepsilon \log \varepsilon + o(\varepsilon \log \varepsilon)) + \sum_{k=N_1+1}^{N} o(\varepsilon \log \varepsilon) \right)
\]
\[
= \sum_{k=1}^{N_1} c_k.
\]

(ii) Since for any \( k \leq N_1 \) we have that \( c_k = 0 \) and the total sum \( c_1 + ... + c_N = 0 \).
then \( c_{N_1+1} + \ldots + c_N = 0 \). Furthermore, in this case
\[
\lim_{\varepsilon \to 0} \frac{\mathcal{H}_1(p) - \mathcal{H}_1(p(\varepsilon))}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sum_{k=N_1+1}^{N} ((p_k + c_k \varepsilon) \log(p_k + c_k \varepsilon) - p_k \log p_k)
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sum_{k=N_1+1}^{N} (\varepsilon (c_k \log p_k + c_k) + o(\varepsilon))
= \sum_{k=N_1+1}^{N} (c_k \log p_k + c_k) = \sum_{k=N_1+1}^{N} c_k \log p_k.
\]

(iii) In this case we have the following relations:
\[
\lim_{\varepsilon \to 0} \frac{\mathcal{H}_1(p) - \mathcal{H}_1(p(\varepsilon))}{\varepsilon^2} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \sum_{k=N_1+1}^{N} ((p_k + c_k \varepsilon) \log(p_k + c_k \varepsilon) - p_k \log p_k)
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \sum_{k=N_1+1}^{N} (\varepsilon (c_k \log p_k + c_k) + \frac{c_k^2 \varepsilon^2}{2p_k} + o(\varepsilon^2))
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \sum_{k=N_1+1}^{N} \left( \frac{c_k^2 \varepsilon^2}{2p_k} + o(\varepsilon^2) \right) = \frac{1}{2} \sum_{k=N_1+1}^{N} \frac{c_k^2}{p_k}.
\]
Theorem is proved. \( \square \)

Now we proceed with \( \alpha < 1 \).

**Theorem 3.2.** Let number \( N \) and coefficients \( c_1, \ldots, c_N \) be fixed, and let \( \alpha < 1 \). Then we have three different situations:

(i) Let \( N_1 \geq 1 \) and there exists \( k \leq N_1 \) such that \( c_k \neq 0 \). Then
\[
\mathcal{H}_\alpha(p) - \mathcal{H}_\alpha(p(\varepsilon)) \sim \frac{\varepsilon^{\alpha}}{\alpha-1} \left( \sum_{k=1}^{N_1} c_k^\alpha \right) \left( \sum_{k=N_1+1}^{N} p_k^\alpha \right)^{-1}, \varepsilon \to 0.
\]

(ii) Let for all \( k \leq N_1 c_k = 0 \) and \( \sum_{k=N_1+1}^{N} c_k p_k^{\alpha-1} \neq 0 \). Then
\[
\mathcal{H}_\alpha(p) - \mathcal{H}_\alpha(p(\varepsilon)) \sim \frac{\alpha \varepsilon^\alpha}{\alpha-1} \left( \sum_{k=N_1+1}^{N} c_k p_k^{\alpha-1} \right) \left( \sum_{k=N_1+1}^{N} p_k^\alpha \right)^{-1}, \varepsilon \to 0.
\]

(iii) Let for all \( k \leq N_1 c_k = 0 \) and \( \sum_{k=N_1+1}^{N} c_k p_k^{\alpha-1} = 0 \). Then
\[
\mathcal{H}_\alpha(p) - \mathcal{H}_\alpha(p(\varepsilon)) \sim \frac{\alpha \varepsilon^{2\alpha}}{2} \left( \sum_{k=N_1+1}^{N} c_k^2 p_k^{2\alpha-2} \right) \left( \sum_{k=N_1+1}^{N} p_k^\alpha \right)^{-1}, \varepsilon \to 0.
\]
Proof. Similarly to proof of Theorem 3.1, we start with several asymptotic relations as \( \varepsilon \to 0 \). Namely, let \( p_k > 0 \), \( |c_k| \leq 1 \). Taking into account Taylor expansion of \( (1 + x)^\alpha \) that has a form

\[
(1 + x)^\alpha = 1 + \alpha x + o(x), \ x \to 0,
\]

we can write:

\[
\begin{align*}
\alpha c_k(p_k + c_k \varepsilon)^{\alpha - 1} &= \alpha c_k p_k^{\alpha - 1}(1 + c_k p_k^{-1} \varepsilon)^{\alpha - 1} \\
&= \alpha c_k p_k^{\alpha - 1}(1 + (\alpha - 1)c_k p_k^{-1} \varepsilon + o(\varepsilon)) \\
&= \alpha c_k p_k^{\alpha - 1} + \alpha(\alpha - 1)c_k^2 p_k^{\alpha - 2} \varepsilon + o(\varepsilon), \ \varepsilon \to 0.
\end{align*}
\]

As a consequence, we get the following asymptotic relations:

\[
\alpha c_k(p_k + c_k \varepsilon)^{\alpha - 1} = o(\varepsilon^{\alpha - 1}), \ \varepsilon \to 0,
\]

and

\[
\alpha c_k(p_k + c_k \varepsilon)^{\alpha - 1} = \alpha c_k p_k^{\alpha - 1} + o(1), \ \varepsilon \to 0
\]

(i) Applying L’Hospital’s rule, we get:

\[
\lim_{\varepsilon \to 0} \frac{H_\alpha(p) - H_\alpha(p(\varepsilon))}{\varepsilon^\alpha} = \lim_{\varepsilon \to 0} \frac{1}{(\alpha - 1)\varepsilon^\alpha} \log \left( \sum_{k=1}^{N_1} (c_k \varepsilon)^\alpha + \sum_{k=N_{1+1}}^{N} (p_k + c_k \varepsilon)^\alpha \right)
\]

\[
- \frac{1}{(\alpha - 1)\varepsilon^\alpha} \log \left( \sum_{k=1}^{N} p_k^\alpha \right) = \frac{1}{\alpha - 1} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\alpha - 1}
\]

\[
\times \frac{\sum_{k=1}^{N_1} \alpha c_k \alpha^{\alpha - 1} + \sum_{k=N_{1+1}}^{N} \alpha c_k(p_k + c_k \varepsilon)^{\alpha - 1}}{\sum_{k=1}^{N_1} (c_k \varepsilon)^\alpha + \sum_{k=N_{1+1}}^{N} (p_k + c_k \varepsilon)^\alpha}
\]

\[
= \frac{1}{\alpha - 1} \lim_{\varepsilon \to 0} \frac{\sum_{k=1}^{N_1} c_k^\alpha + \varepsilon^{1-\alpha} \sum_{k=N_{1+1}}^{N} o(\varepsilon^{\alpha - 1})}{\sum_{k=1}^{N_1} (c_k \varepsilon)^\alpha + \sum_{k=N_{1+1}}^{N} (p_k + c_k \varepsilon)^\alpha}
\]

\[
= \frac{1}{\alpha - 1} \left( \sum_{k=1}^{N_1} c_k^\alpha \right) \left( \sum_{k=N_{1+1}}^{N} p_k^\alpha \right)^{-1}.
\]

(ii) In this case we can transform the value under a limit as follows:

\[
\lim_{\varepsilon \to 0} \frac{H_\alpha(p) - H_\alpha(p(\varepsilon))}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{(\alpha - 1)\varepsilon} \left( \log \left( \sum_{k=N_{1+1}}^{N} (p_k + c_k \varepsilon)^\alpha \right) - \log \left( \sum_{k=1}^{N} p_k^\alpha \right) \right)
\]

\[
= \frac{1}{\alpha - 1} \lim_{\varepsilon \to 0} \frac{\sum_{k=N_{1+1}}^{N} \alpha c_k(p_k + c_k \varepsilon)^{\alpha - 1}}{\sum_{k=1}^{N} (p_k + c_k \varepsilon)^\alpha}
\]

\[
= \frac{1}{\alpha - 1} \lim_{\varepsilon \to 0} \frac{\sum_{k=N_{1+1}}^{N} (\alpha c_k p_k^{\alpha - 1} + o(1))}{\sum_{k=1}^{N} (p_k + c_k \varepsilon)^\alpha}
\]

\[
= \frac{\alpha}{\alpha - 1} \left( \sum_{k=N_{1+1}}^{N} c_k p_k^{\alpha - 1} \right) \left( \sum_{k=1}^{N} p_k^\alpha \right)^{-1}.
\]
(iii) Finally, in the 3rd case,

\[
\lim_{\varepsilon \to 0} \frac{\mathcal{H}_\alpha(p) - \mathcal{H}_\alpha(p(\varepsilon))}{\varepsilon^2} = \lim_{\varepsilon \to 0} \frac{1}{(\alpha - 1)\varepsilon^2} \left( \log \left( \sum_{k=N_1+1}^{N} (p_k + c_k \varepsilon)^\alpha \right) - \log \left( \sum_{k=1}^{N} p_k^\alpha \right) \right)
\]

\[
= \frac{1}{\alpha - 1} \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \sum_{k=N_1+1}^{N} \alpha c_k (p_k + c_k \varepsilon)^\alpha
\]

\[
= \frac{1}{\alpha - 1} \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \sum_{k=N_1+1}^{N} (\alpha \varepsilon \frac{c_k p_k^\alpha - 1}{\alpha} + \alpha (\alpha - 1) c_k^2 p_k^{2\alpha - 2} \varepsilon + o(\varepsilon))
\]

\[
= \frac{1}{\alpha - 1} \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \sum_{k=N_1+1}^{N} (\alpha (\alpha - 1) c_k^2 p_k^{2\alpha - 2} \varepsilon + o(\varepsilon))
\]

\[
= \frac{\alpha}{2} \left( \sum_{k=N_1+1}^{N} c_k^2 p_k^{2\alpha - 2} \right) \left( \sum_{k=N_1+1}^{N} p_k^\alpha \right)^{-1}.
\]

Theorem is proved. \(\square\)

Now we conclude with \(\alpha > 1\). In this case, five different asymptotics are possible.

**Theorem 3.3.** Let number \(N\) and coefficients \(c_1, \ldots, c_N\) be fixed, and let \(\alpha > 1\). Then five different situations are possible:

(i) Let \(\sum_{k=N_1+1}^{N} c_k p_k^{\alpha-1} \neq 0\). Then whatever \(N_1 \geq 0\) and \(\alpha > 1\) are equal, we have that

\[
\mathcal{H}_\alpha(p) - \mathcal{H}_\alpha(p(\varepsilon)) \sim \frac{\alpha \varepsilon}{\alpha - 1} \left( \sum_{k=N_1+1}^{N} c_k p_k^{\alpha-1} \right) \left( \sum_{k=N_1+1}^{N} p_k^\alpha \right)^{-1}, \varepsilon \to 0.
\]

(ii) Let \(\sum_{k=N_1+1}^{N} c_k p_k^{\alpha-1} = 0, N_1 \geq 1\), and there exists \(k \leq N_1\) such that \(c_k \neq 0\). Then for \(\alpha < 2\) it holds that

\[
\mathcal{H}_\alpha(p) - \mathcal{H}_\alpha(p(\varepsilon)) \sim \frac{\varepsilon^\alpha}{\alpha - 1} \left( \sum_{k=1}^{N_1} c_k^\alpha \right) \left( \sum_{k=N_1+1}^{N} p_k^\alpha \right)^{-1}, \varepsilon \to 0.
\]

(iii) Let \(\sum_{k=N_1+1}^{N} c_k p_k^{\alpha-1} = 0, N_1 \geq 0\) and for all \(k \leq N_1\) we have that \(c_k = 0\). Then for \(\alpha < 2\) it holds that

\[
\mathcal{H}_\alpha(p) - \mathcal{H}_\alpha(p(\varepsilon)) \sim \frac{\alpha \varepsilon^2}{2} \left( \sum_{k=N_1+1}^{N} c_k^2 p_k^{2\alpha - 2} \right) \left( \sum_{k=N_1+1}^{N} p_k^\alpha \right)^{-1}, \varepsilon \to 0.
\]
(iv) Let \( \sum_{k=N_1+1}^{N} c_k p_k^{\alpha-1} = 0, \ \alpha = 2. \) Then whatever \( N_1 \geq 0 \) and \( c_k \) for \( k \leq N_1 \) are equal, we have that

\[
H_\alpha(p) - H_\alpha(p(\varepsilon)) \sim \varepsilon^2 \left( \sum_{k=1}^{N} c_k^2 \right) \left( \sum_{k=N_1+1}^{N} p_k^2 \right)^{-1}, \ \varepsilon \to 0.
\]

(v) Let \( \sum_{k=N_1+1}^{N} c_k p_k^{\alpha-1} = 0, \ \alpha > 2. \) Then whatever \( N_1 \geq 0 \) and \( c_k \) for \( k \leq N_1 \) are equal, we have that

\[
H_\alpha(p) - H_\alpha(p(\varepsilon)) \sim \frac{\alpha \varepsilon^2}{2} \left( \sum_{k=N_1+1}^{N} c_k p_k^{\alpha-2} \right) \left( \sum_{k=N_1+1}^{N} p_k^\alpha \right)^{-1}, \ \varepsilon \to 0.
\]

**Proof.** As in the proof of Theorem 3.2, we shall use expansions (3.2) and (3.3). The main tool will be L’Hospital’s rule.

(i) Let \( \sum_{k=N_1+1}^{N} c_k p_k^{\alpha-1} \neq 0. \) Then whatever \( N_1 \geq 0 \) and \( \alpha > 1 \) are equal, we have the following relations:

\[
\lim_{\varepsilon \to 0} \frac{H_\alpha(p) - H_\alpha(p(\varepsilon))}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{(\alpha-1)\varepsilon} \log \left( \sum_{k=1}^{N} (c_k \varepsilon)^\alpha + \sum_{k=N_1+1}^{N} (p_k + c_k \varepsilon)^\alpha \right)
- \frac{1}{(\alpha-1)\varepsilon} \log \left( \sum_{k=1}^{N} p_k^\alpha \right) = \frac{1}{\alpha-1} \left( \sum_{k=1}^{N} c_k \varepsilon^{\alpha-1} \right) + \frac{1}{\alpha-1} \left( \sum_{k=N_1+1}^{N} (c_k \varepsilon)^{\alpha-1} + \sum_{k=N_1+1}^{N} (p_k + c_k \varepsilon)^{\alpha-1} \right)
= \alpha \left( \sum_{k=1}^{N} c_k p_k^{\alpha-1} \right) \left( \sum_{k=N_1+1}^{N} p_k^\alpha \right)^{-1}.
\]

(ii) Let \( \sum_{k=N_1+1}^{N} c_k p_k^{\alpha-1} = 0, \ \alpha > 1. \) Then and \( N_1 \geq 1 \) and there exists \( k \leq N_1 \) such that \( c_k \neq 0. \) Then for \( \alpha < 2 \) we have that

\[
\lim_{\varepsilon \to 0} \frac{H_\alpha(p) - H_\alpha(p(\varepsilon))}{\varepsilon^\alpha} = \lim_{\varepsilon \to 0} \frac{1}{(\alpha-1)\varepsilon^\alpha} \log \left( \sum_{k=1}^{N} (c_k \varepsilon)^\alpha + \sum_{k=N_1+1}^{N} (p_k + c_k \varepsilon)^\alpha \right)
- \frac{1}{(\alpha-1)\varepsilon^\alpha} \log \left( \sum_{k=1}^{N} p_k^\alpha \right) = \frac{1}{\alpha-1} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{\alpha-1}} \left( \sum_{k=1}^{N_1} c_k \varepsilon^{\alpha-1} + \sum_{k=N_1+1}^{N} (c_k \varepsilon)^{\alpha-1} \right)
= \frac{1}{\alpha-1} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{\alpha-1}} \left( \sum_{k=1}^{N_1} c_k \varepsilon^{\alpha-1} + \sum_{k=N_1+1}^{N} (p_k + c_k \varepsilon)^{\alpha-1} \right)
= \frac{1}{\alpha-1} \left( \sum_{k=1}^{N_1} c_k^\alpha \right) \left( \sum_{k=N_1+1}^{N} p_k^\alpha \right)^{-1}.
\]
(iii) Let $\sum_{k=N_1+1}^N c_k p_k^{\alpha - 1} = 0$, $N_1 \geq 0$ and for all $k \leq N_1$ we have that $c_k = 0$. Then for $\alpha < 2$ it holds that

$$
\lim_{\epsilon \to 0} \frac{H_\alpha(p) - H_\alpha(p(\epsilon))}{\epsilon^2} = \lim_{\epsilon \to 0} \frac{1}{(\alpha - 1)\epsilon^2} \left( \log \left( \sum_{k=N_1+1}^N (p_k + c_k \epsilon)^\alpha \right) - \log \left( \sum_{k=1}^N p_k^\alpha \right) \right)
$$

$$
= \frac{1}{\alpha - 1} \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \frac{\sum_{k=N_1+1}^N \alpha c_k (p_k + c_k \epsilon)^{\alpha - 1}}{\sum_{k=N_1+1}^N (p_k + c_k \epsilon)^\alpha}
$$

$$
= \frac{1}{\alpha - 1} \lim_{\epsilon \to 0} \frac{\sum_{k=N_1+1}^N (\alpha c_k p_k^{\alpha - 1} + \alpha (\alpha - 1) c_k^2 p_k^{\alpha - 2} \epsilon + o(\epsilon))}{2\epsilon \left( \sum_{k=N_1+1}^N (p_k + c_k \epsilon)^\alpha \right)}
$$

$$
= \frac{1}{\alpha - 1} \lim_{\epsilon \to 0} \frac{\sum_{k=N_1+1}^N (\alpha (\alpha - 1) c_k^2 p_k^{\alpha - 2} + o(\epsilon))}{2\epsilon \left( \sum_{k=N_1+1}^N (p_k + c_k \epsilon)^\alpha \right)}
$$

$$
= \frac{\alpha}{2} \left( \sum_{k=N_1+1}^N c_k p_k^{\alpha - 2} \right) \left( \sum_{k=N_1+1}^N p_k^\alpha \right)^{-1}.
$$

(iv) Obviously, in the case $\alpha = 2$ we have the simple value of the entropy:

$$
H_2(p) = -\log \left( \sum_{k=1}^N p_k^2 \right).
$$

Therefore, if $\sum_{k=N_1+1}^N c_k p_k^{\alpha - 1} = 0$, $\alpha = 2$, then, whatever $N_1 \geq 0$ and $c_k$ for $k \leq N_1$ are equal, we have that

$$
\lim_{\epsilon \to 0} \frac{H_2(p) - H_2(p(\epsilon))}{\epsilon^2} = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \log \left( \sum_{k=1}^{N_1} (c_k \epsilon)^2 + \sum_{k=N_1+1}^N (p_k + c_k \epsilon)^2 \right)
$$

$$
- \frac{1}{\epsilon^2} \log \left( \sum_{k=1}^{N_1} p_k^2 \right) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon}
$$

$$
\times \frac{\sum_{k=1}^{N_1} 2c_k^2 \epsilon + \sum_{k=N_1+1}^N 2c_k (p_k + c_k \epsilon)}{\sum_{k=1}^{N_1} (c_k \epsilon)^2 + \sum_{k=N_1+1}^N (p_k + c_k \epsilon)^2}
$$

$$
= \lim_{\epsilon \to 0} \frac{\sum_{k=1}^{N_1} c_k^2 + \sum_{k=N_1+1}^N c_k^2}{\sum_{k=1}^{N_1} (c_k \epsilon)^2 + \sum_{k=N_1+1}^N (p_k + c_k \epsilon)^2}
$$

$$
= \left( \sum_{k=1}^N c_k^2 \right) \left( \sum_{k=N_1+1}^N p_k^\alpha \right)^{-1}.
$$

(v) Let $\sum_{k=N_1+1}^N c_k p_k^{\alpha - 1} = 0$, $\alpha > 2$. Then whatever $N_1 \geq 0$ and $c_k$ for $k \leq N_1$
are equal, we have that

\[
\lim_{\varepsilon \to 0} \frac{\mathcal{H}_\alpha(p) - \mathcal{H}_\alpha(p(\varepsilon))}{\varepsilon^2} = \lim_{\varepsilon \to 0} \frac{1}{(\alpha - 1)\varepsilon^2} \log \left( \sum_{k=1}^{N_1} (c_k \varepsilon)^\alpha + \sum_{k=N_1+1}^{N} (p_k + c_k \varepsilon)^\alpha \right) - \frac{1}{(\alpha - 1)\varepsilon^2} \log \left( \sum_{k=1}^{N_1} p_k^\alpha \right) = \frac{1}{\alpha - 1} \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \left( \sum_{k=1}^{N_1} (c_k \varepsilon)^\alpha + \sum_{k=N_1+1}^{N} (p_k + c_k \varepsilon)^\alpha \right)
\]

\[
= \frac{1}{\alpha - 1} \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \left( \sum_{k=1}^{N_1} \alpha c_k^\alpha \varepsilon^{\alpha-1} + \sum_{k=N_1+1}^{N} \alpha (\alpha - 1) c_k^2 p_k^{\alpha-2} \varepsilon + o(\varepsilon) \right)
\]

Theorem is proved.
3.2 Convergence of the disturbed entropy when the initial distribution is uniform but the number of events increases to $\infty$

The second problem is to establish conditions of stability of the entropy of uniform distribution when the number of events tends to $\infty$. Let $N > 1$, $p_N(\text{uni}) = (\frac{1}{N}, \ldots, \frac{1}{N})$ be a vector of uniform $N$-dimensional distribution, $\varepsilon = \varepsilon(N) \leq \frac{1}{N}$, and $\{c_{kN}; N \geq 1, 1 \leq k \leq N\}$ be a family of fixed numbers (not totally zero) such that $|c_{kN}| \leq 1$ and $\sum_{k=1}^{N} c_{kN} = 0$. Note that for any $N \geq 1$ there are strictly positive numbers $c_{kN}$ for some $k$ and consider the disturbed distribution vector $p'_N = (\frac{1}{N} + c_{1N}\varepsilon, \ldots, \frac{1}{N} + c_{NN}\varepsilon)$.

**Theorem 3.4.** Let $\varepsilon(N) = o\left(\frac{1}{N}\right)$, $N \to \infty$. Then

$$\mathcal{H}_\alpha(p_N) - \mathcal{H}_\alpha(p'_N) \to 0, \; N \to \infty.$$  

**Proof.** We know that $N \varepsilon \to 0$, $N \to \infty$ and the family of numbers $\{c_{kn}; n \geq 1, 1 \leq k \leq n\}$ is bounded. Therefore the values

$$\sup_{n \geq 1, 1 \leq k \leq n} (1 + N c_{kn}\varepsilon) \to 1, \quad \inf_{n \geq 1, 1 \leq k \leq n} (1 + N c_{kn}\varepsilon) \to 1, \; N \to \infty$$

as the function of $N$, and for every $N \geq 1$ $\sup_{n \geq 1, 1 \leq k \leq n} (1 + N c_{kn}\varepsilon) \geq 1$. Recall that function $x \log x$ is increasing in $x \geq 1$ and $x \log x \leq 0$ for $0 < x < 1$. Moreover, Renyi entropy is maximal on the uniform distribution. As a consequence of all these observations and assumptions we get that

$$0 \leq \mathcal{H}_1(p_N) - \mathcal{H}_1(p'_N) = \frac{1}{N} \sum_{k=1}^{N} (1 + N c_{kn}\varepsilon) \log(1 + N c_{kn}\varepsilon)$$

$$\leq \frac{1}{N} \sum_{k=1}^{N} \sup_{n \geq 1, 1 \leq k \leq n} (1 + N c_{kn}\varepsilon) \log \sup_{n \geq 1, 1 \leq k \leq n} (1 + N c_{kn}\varepsilon)$$

$$= \sup_{n \geq 1, 1 \leq k \leq n} (1 + N c_{kn}\varepsilon) \log \sup_{n \geq 1, 1 \leq k \leq n} (1 + N c_{kn}\varepsilon) \to 0, \; N \to \infty.$$  

Let $\alpha > 1$. Then

$$0 \leq \mathcal{H}_\alpha(p_N) - \mathcal{H}_\alpha(p'_N) = \frac{1}{\alpha - 1} \log \left( \frac{1}{N} \sum_{k=1}^{N} (1 + N c_{kn}\varepsilon)^\alpha \right)$$

$$\leq \frac{1}{\alpha - 1} \log \left( \frac{1}{N} \sum_{k=1}^{N} \left( \sup_{n \geq 1, 1 \leq k \leq n} (1 + N c_{kn}\varepsilon) \right)^\alpha \right)$$

$$= \frac{\alpha}{\alpha - 1} \log \left( \sup_{n \geq 1, 1 \leq k \leq n} (1 + N c_{kn}\varepsilon) \right) \to 0, \; N \to \infty.$$
Similarly, for $0 < \alpha < 1$ we produce the transformations:

\[
0 \leq H_\alpha(p_N) - H_\alpha(p'_N) = \frac{1}{\alpha - 1} \log \left( \frac{1}{N} \sum_{k=1}^{N} (1 + Nc_k N \epsilon)^\alpha \right)
\]

\[
\leq \frac{1}{\alpha - 1} \log \left( \frac{1}{N} \sum_{k=1}^{N} \left( \inf_{n \geq 1, 1 \leq k \leq n} (1 + Nc_k \epsilon) \right)^\alpha \right)
\]

\[
= \frac{\alpha}{\alpha - 1} \log \left( \inf_{n \geq 1, 1 \leq k \leq n} (1 + Nc_k \epsilon) \right) \to 0, \quad N \to \infty,
\]

and the proof follows.

\[\square\]

### 3.3 Binomial and Poisson distribution

In this section we look at convergence of Rényi entropy of binomial distribution to Rényi entropy of Poisson distribution.

**Theorem 3.5.** Let $\lambda > 0$ be fixed. For any $\alpha > 0$

\[
\lim_{n \to \infty} H_\alpha \left( B \left( n, \frac{\lambda}{n} \right) \right) = H_\alpha(Poi(\lambda)).
\]

**Proof.** First, let $\alpha = 1$. We will find and regroup entropy of binomial and Poisson distribution.

\[
H_1 ( B (n, p)) = - \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} \log \left( \binom{n}{k} p^k (1 - p)^{n-k} \right)
\]

\[
= - \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} \log \left( \binom{n}{k} \right) - n (p \log p + (1 - p) \log(1 - p))
\]

\[
= - \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} \log(n! - \log k! - \log(n - k)!) + np \log n
\]

\[
- np \log np - n \log(1 - p) + np \log(1 - p) = np \log(1 - p) - n \log(1 - p)
\]

\[
- np \log np + np \log n - \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} \log(n! - \log k! - \log(n - k)!).
\]

\[
H_1(Poi(\lambda)) = - \sum_{k=0}^{\infty} e^{-\lambda \frac{k}{\lambda}!} \log \left( e^{-\lambda \frac{k}{\lambda}!} \right) = \lambda - \lambda \log \lambda + \sum_{k=0}^{\infty} e^{-\lambda \frac{k}{\lambda}!} \log k!
\]

We want to show componentwise convergence of entropies. For that let’s take $np = \lambda$ and observe that:

\[
np \log(1 - p) = \lambda \log(1 - p) \to \lambda \log 1 = 0, \quad n \to \infty, \quad p \to 0.
\]

\[22\]
\[-n \log(1 - p) = \log \left( 1 - \frac{\lambda}{n} \right)^{-n} \rightarrow \lambda, \; n \rightarrow \infty, \; p \rightarrow 0.\]

\[-np \log np = -\lambda \log \lambda.\]

\[np \log n - \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} \log n! - \log k! - \log(n - k)! \]
\[= \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} \log n - \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} \log n! - \log k! - \log(n - k)! \]
\[= \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} \left( \log n^k - \log n! + \log k! + \log(n - k)! \right) \]
\[= \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} \left( \log \frac{n^k(n - k)!}{n!} + \log k! \right) \]

It is well-known that \(\frac{\log(x)}{x} \leq 1, \; x > 0\). Using this fact, we get the following representation

\[\binom{n}{k} p^k (1 - p)^{n-k} \log \frac{n^k(n - k)!}{n!} = \frac{n!}{(n - k)!k!} \left( \frac{\lambda}{n} \right)^k \left( 1 - \frac{\lambda}{n} \right)^{n-k} \log \frac{n^k(n - k)!}{n!} \]
\[= \frac{\lambda^k}{k!} \left( 1 - \frac{\lambda}{n} \right)^{n-k} \frac{n!}{n^k(n - k)!} \log \frac{n^k(n - k)!}{n!} \]
\[\leq \frac{\lambda^k}{k!} \left( 1 - \frac{\lambda}{n} \right)^{n-k} \leq \frac{\lambda^k}{k!}.\]

For the second part of sum simply observe that:

\[\binom{n}{k} p^k (1 - p)^{n-k} \log k! = \frac{n!}{(n - k)!k!} \left( \frac{\lambda}{n} \right)^k \left( 1 - \frac{\lambda}{n} \right)^{n-k} \log k! \]
\[= \frac{\lambda^k}{k!} \log k! \left( 1 - \frac{\lambda}{n} \right)^{n-k} \frac{n!}{(n - k)!n^k} \]
\[\leq \frac{\lambda^k}{k!} \log k! \]
\[ \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (1 + \log k!) < \infty, \text{ thus, by Lebesgue's dominated convergence theorem:} \]

\[
\lim_{n \to \infty} \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} \left( \log \frac{n^k (n-k)!}{n!} + \log k! \right)
\]

\[
= \sum_{k=0}^{\infty} \lim_{n \to \infty} \binom{n}{k} p^k (1 - p)^{n-k} \left( \log \frac{n^k (n-k)!}{n!} + \log k! \right)
\]

\[
= \sum_{k=0}^{\infty} \lim_{n \to \infty} \frac{\lambda^k}{k!} (1 - \frac{\lambda}{n})^{n-k} \frac{n!}{(n-k)!n^k} \left( \log \frac{n^k (n-k)!}{n!} + \log k! \right)
\]

\[
= \sum_{k=0}^{\infty} e^{-\lambda \frac{\lambda^k}{k!}} \log k!
\]

Finally, we get that

\[
\lim_{n \to \infty} \mathcal{H}_1 \left( B \left( n, \frac{\lambda}{n} \right) \right) = \mathcal{H}_1(Poi(\lambda)).
\]

For \( \alpha \neq 1 \) we have:

\[
\mathcal{H}_\alpha(\text{binomial}) = \frac{1}{1 - \alpha} \log \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} \alpha,
\]

\[
\mathcal{H}_\alpha(\text{poisson}) = \frac{1}{1 - \alpha} \log \sum_{k=0}^{\infty} e^{-\lambda \frac{\lambda^k}{k!}} \alpha.
\]

Thus, to show that

\[
\lim_{n \to \infty} \mathcal{H}_\alpha \left( B \left( n, \frac{\lambda}{n} \right) \right) = \mathcal{H}_\alpha(Poi(\lambda)),
\]

it is enough to show convergence of sums which follows from Lebesgue’s dominated convergence theorem and

\[
\left( \binom{n}{k} p^k (1 - p)^{n-k} \right)^\alpha \leq \left( \frac{\lambda^k}{k!} \right)^\alpha, \sum_{k=0}^{\infty} \left( \frac{\lambda^k}{k!} \right)^\alpha < +\infty.
\]

\[ \square \]

### 4 Appendix

We let \( 0 \log 0 = 0 \) by continuity and prove several auxiliary results. Stating these three lemmas, we assume that \( p_i \geq 0, 1 \leq i \leq N \) are fixed.

**Lemma 4.1.** \( H_\alpha(p) \to H(p), \alpha \to 1. \)
Proof. Using L’Hospital’s rule, we get the following relations:

\[
\lim_{\alpha \to 1} H_\alpha(p) = \lim_{\alpha \to 1} \frac{1}{1-\alpha} \log \left( \sum_{k=1}^{N} p_k^\alpha \right) = \lim_{\alpha \to 1} \frac{1}{1-\alpha} \frac{\sum_{k=1}^{N} p_k^\alpha \log(p_k)}{\sum_{k=1}^{N} p_k^\alpha} \\
= - \sum_{k=1}^{N} p_k \log(p_k) = H(p).
\]

Let \( \mathcal{H}_1(p) := \mathcal{H}(p) \) (Shannon entropy), and so \( H_\alpha(p) \) is defined for all \( \alpha > 0 \) and is continuous in \( \alpha \).

**Lemma 4.2.** \( H_\alpha(p) \) is non-increasing in \( \alpha > 0 \).

Proof. Indeed,

\[
\frac{\partial H_\alpha(p)}{\partial \alpha} = \frac{1}{(1-\alpha)^2} \log \left( \sum_{i=1}^{N} p_i^\alpha \right) + \frac{1}{1-\alpha} \frac{\sum_{k=1}^{N} p_k^\alpha \log(p_k)}{\sum_{k=1}^{N} p_k^\alpha} \\
= \frac{1}{(1-\alpha)^2} \sum_{k=1}^{N} p_k^\alpha \left( \log \left( \sum_{i=1}^{N} p_i^\alpha \right) + \log \left( \frac{1}{p_k} \right) \right) \\
= \frac{-1}{(1-\alpha)^2} \sum_{k=1}^{N} p_k^\alpha \log \left( \frac{p_k^\alpha}{\sum_{i=1}^{N} p_i^\alpha} \right).
\]

Let \( q_k = \frac{p_k^\alpha}{\sum_{i=1}^{N} p_i^\alpha} \). Then

\[
\frac{\partial H_\alpha(p)}{\partial \alpha} = \frac{-1}{(1-\alpha)^2} \sum_{k=1}^{N} q_k \log \left( \frac{q_k}{p_k} \right) \leq 0.
\]

The fact that \( \mathcal{H}_\alpha(p) \leq \mathcal{H}_1(p) \leq \mathcal{H}_\beta(p) \), where \( 0 < \beta < 1 < \alpha \) follows from Lemma 4.1.

**Lemma 4.3.** \( H_\alpha(p) \) is less than or equal to \( \log N \) and it reaches maximum when distribution is uniform.

Proof. Let \( 1 \leq m \leq N \) be the number of non-zero probabilities. Then we have:

\[
\lim_{\alpha \to 0^+} H_\alpha(p) = \lim_{\alpha \to 0^+} \frac{1}{1-\alpha} \log \left( \sum_{k=1}^{N} p_k^\alpha \right) = \log m \leq \log N.
\]

So \( H_\alpha(p) \leq \log N \) due to Lemma 4.2. For the second part, put \( p_1 = \ldots = p_N = \frac{1}{N} \).

\[
\mathcal{H}_1(p) = - \sum_{k=1}^{N} \frac{1}{N} \log \left( \frac{1}{N} \right) = \log N.
\]
\[ H_\alpha(p) = \frac{1}{1 - \alpha} \log \left( \sum_{k=1}^{N} \frac{1}{N^\alpha} \right) = \frac{1}{1 - \alpha} \log \left( \frac{N}{N^\alpha} \right) = \log N. \]

\[ \square \]

**Remark 4.4.** Let \( 1 \leq m \leq N \) be the number of non-zero probabilities and without loss of generality let \( p_k < p_1 = \ldots = p_{N_1} \) for every \( N_1 + 1 \leq k \leq N \). Then we can also define:

\[ H_0(p) := \lim_{\alpha \to 0^+} H_\alpha(p) = \lim_{\alpha \to 0^+} \frac{1}{1 - \alpha} \log \left( \sum_{k=1}^{N} p_k^\alpha \right) = \log m. \]

\[ H_\infty(p) := \lim_{\alpha \to +\infty} H_\alpha(p) = \lim_{\alpha \to +\infty} \frac{1}{1 - \alpha} \log \left( \sum_{k=1}^{N} p_k^\alpha \right) = \]

\[ = \lim_{\alpha \to +\infty} \frac{\sum_{k=1}^{N} p_k^\alpha \log p_k}{\sum_{k=1}^{N} p_k^\alpha} \]

\[ = \lim_{\alpha \to +\infty} \frac{N_1 \log p_1 + \sum_{k=N_1+1}^{N} \left( \frac{p_k}{p_1} \right)^\alpha \log p_k}{N_1 + \sum_{k=N_1+1}^{N} \left( \frac{p_k}{p_1} \right)^\alpha} \]

\[ = - \log p_1. \]

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