ALGEBRAIC FUNCTIONAL EQUATIONS OF CLASS GROUPS

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Abstract. We generalize Iwasawa’s theorem on class group over $\mathbb{Z}_p$-extensions to all $\mathbb{Z}_d^p$-extensions.

1. Introduction

Let $k$ be a number field and let $K/k$ be an abelian extension. If $K/k$ is a finite extension, we define a map from the group of ideals of $k$ to that of $K$ by taking $a$ to the fractional ideal $a\mathcal{O}_K$. This map induces the homomorphism $c_{K/k} : \mathcal{C}_k \to \mathcal{C}_K$ of class groups which is called the capitulation map. Previous studies indicate that there is certain connection between the order of $\ker(c_{K/k})$ and the ramification of $K/k$.

A famous theorem of Suzuki [Suz91] that generalizes both Hilbert’s theorem 94 and the principal ideal theorem of class field theory says if $K/k$ is unramified, then the order of $\ker(c_{K/k})$ is divisible by the degree $[K : k]$. More results in this aspect can be found in [Gon07]. In contrast, Iwasawa [Iwa73] proves that if $K/k$ is a $\mathbb{Z}_p$-extension, so it is almost totally ramified at some place dividing $p$, then the the capitulation kernel $\hat{\mathfrak{X}}_K$ (see below) is pseudo-null. In this note, we generalize this result of Iwasawa to every $\mathbb{Z}_d^p$-extension of $k$, and then use it to establish a pseudo-isomorphism of Iwasawa modules that generalizes [Iwa73, Theorem 11].

From now on we let $K/k$ be a $\mathbb{Z}_d^p$-extension. For a number field $E \subset K$, let

$$\hat{\mathfrak{X}}_E = \bigcup_{E \subset E' \subset K} \ker(c_{E'/E})$$

where $E'/E$ runs through finite subextensions in $K/E$. Put

$$\hat{\mathfrak{X}}_K = \lim_{\leftarrow k \subset E \subset K} \hat{\mathfrak{X}}_E,$$

with the limit taken over the norm maps. Let $\sim$ denote pseudo-isomorphism.

Theorem 1. We have $\hat{\mathfrak{X}}_K \sim 0$.
Let $\mathfrak{X}_E$ denote the Sylow $p$-subgroup of the class group $\mathfrak{C}_E$ of $E$ so that $\mathfrak{X}_E$ is actually a subgroup of $\mathfrak{X}_E$. Denote
\[
\mathfrak{X}_K := \lim_{\longleftarrow} \mathfrak{X}_E.
\]
Consider the dual group $\mathfrak{M}_E := \text{Hom}(\mathfrak{X}_E, \mathbb{Q}_p/\mathbb{Z}_p)$. For $k \subset E \subset E' \subset K$, let $\mathfrak{M}_{E'} \rightarrow \mathfrak{M}_E$ be the homomorphism dual to the restriction of $c_{E'/E}$ to $\mathfrak{X}_E$. Denote
\[
\mathfrak{M}_K := \lim_{\longleftarrow} \mathfrak{M}_E.
\]
Write $\Gamma$ for the Galois group of $K/k$ and $\Lambda_\Gamma$ for the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$.

**Theorem 2.** The $\Lambda_\Gamma$-module $\mathfrak{M}_K$ is finitely generated.

Let
\[
\overset{\pounds}{\cdot} : \Lambda_\Gamma \longrightarrow \Lambda_\Gamma
\]
be the involution of $\mathbb{Z}_p$-algebra induced from $\Gamma \longrightarrow \Gamma$, $\gamma \mapsto \gamma^{-1}$. For a $\Lambda_\Gamma$-module $\mathfrak{D}$, let $\mathfrak{D}^{\overset{\pounds}{\cdot}}$ denote the module with $\mathfrak{D}$ as the underlying $\mathbb{Z}_p$-module while $\Gamma$ acts on $\mathfrak{D}$ via $\overset{\pounds}{\cdot} : \Gamma \longrightarrow \Gamma$.

**Theorem 3.** We have
\[
\mathfrak{M}_K \sim \mathfrak{X}_K^{\overset{\pounds}{\cdot}}.
\]

Suppose $k_0$ is a subfield of $k$ such that $K/k_0$ is an abelian extension having Galois group $\text{Gal}(K/k_0) = \Gamma \times \Theta$ with $\Theta$ finite of order prime to $p$. Then every pro-$p$ $\Lambda_\Gamma$-module $\mathfrak{D}$ can be written as
\[
\mathfrak{D} = \bigoplus_{\chi} \mathfrak{D}_{\chi},
\]
where $\chi$ runs through all $\mathbb{Q}_p^*$-valued characters of $\Theta$ and
\[
\mathfrak{D}_{\chi} = \{ x \in \mathfrak{D} \mid \sigma x = \chi(\sigma) \cdot x, \text{ for all } \sigma \in \Theta \}.
\]
It is clear that $(\mathfrak{D}^{\overset{\pounds}{\cdot}})_{\chi} = (\mathfrak{D}_{\chi^{-1}})^{\overset{\pounds}{\cdot}}$ and by Theorem 3
\[
\mathfrak{M}_K \sim (\mathfrak{X}_K)^{\overset{\pounds}{\cdot}}.
\]

We call the above pseudo-isomorphisms algebraic functional equations, because they induce the corresponding equalities of characteristic ideals. Such kind of functional equations, existing for both Tate-Shafarevich groups and dual Selmer groups of abelian varieties over $\mathbb{Z}_p$-extension of global fields \cite{LLTT18}, is crucial in proving the Iwasawa main conjecture for constant ordinary abelian varieties over global function field of characteristic $p$ \cite{LLTT16}. It is worthwhile to mention that over global function fields of characteristic $p$ our theorems hold trivially, because the global unit group $U_K$ has trivial $p$-primary part (cf. \S 3.3 and \S 3.3 especially the exact sequence (15), Lemma 3.1.1 and the proof of Theorem 3).

Theorem 1 is proved in \S 2.3 by following the path of Iwasawa \cite[Theorem 10]{Iwa73}, with the help of Proposition 2.2.3 which in turn is based on Monsky’s theorem \cite{Mon81} (see \S 2.1). Theorems 2, 3 are proved in \S 3.3 together they generalize Theorem 11 of \cite{Iwa73}.

We thank I. Longhi for helping us with the proof of Lemma 2.1.4.
1.1. Notation and preliminary remarks. Let $K/k$ be a $\mathbb{Z}_p^d$-extension with ramification locus $S$. Let $\mathcal{P}$ denote the set of places of $k$ dividing $p$.

Let $E$ stand for a finite intermediate extension of $K/k$ and denote $\Gamma_E = \text{Gal}(K/E)$. Put $\Gamma^{(n)} := \Gamma_p^n$, $k_n := K^{\Gamma^{(n)}}$, the $n$th layer of $K/k$. Let $\mathcal{I}_n$ denote the augmentation ideal of $\Lambda$ and put $\mathcal{I}_n := \ker(\Lambda \to \mathbb{Z}_p[\text{Gal}(k_n/k)])$ so that $\mathcal{I}_0 = \mathcal{I}_1$.

For simplicity, we set the following convention for functors on the category of finite intermediate extensions of $K/k$. If $\mathcal{O}$ is a contravariant (resp. covariant) functor, while the direct limits of $\mathcal{O}$ for $\mathcal{P}$ denote its value at $E$, then for all intermediate field $K$, we define $\mathcal{O}_K$ to be the inverse limit (resp. direct limit) of $\mathcal{O}_E$ for $E$ taken over all finite intermediate extensions of $K/k$. For example if $T$ denotes a set of places of $K$ then $T_E$ is the places of $E$ sitting over $T$. This defines $S_E$ and $\mathcal{P}_E$ as well as $S_K$ and $\mathcal{P}_K$. Write $\mathcal{O}_n$ for $\mathcal{O}_{k_n}$.

We use a capital letter in fraktur font (resp. Roman or calligraphy font) to denote a contravariant (resp. covariant) functor, while the direct limits of $\mathcal{X}_E$ and $\mathcal{C}_E$ taken over capitulation maps $c_{E/E'} : \mathcal{C}_E \to \mathcal{C}_{E'}$ will be denoted by $\mathcal{X}_K$ and $\mathcal{C}_K$ respectively. Then $\mathcal{M}_K$ and $\mathcal{X}_K$ are Pontryagin dual to each other [Kap50].

For a Galois group $G$ of number field extension, let $G_w$ and $G_w^0$ denote respectively the decomposition subgroup and the inertia subgroup at a place $w$.

For a topological group $A$, let $A^\vee$ denote $\text{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$, where $\mathbb{Q}_p/\mathbb{Z}_p$ is endowed with the discrete topology. In most cases, $A$ is pro-$p$ or $p$-primary and discrete so that $A^\vee$ is its Pontryagin dual. We also identify $A^\vee$ with $\text{H}^1(A, \mathbb{Q}_p/\mathbb{Z}_p)$ when $A$ is a compact group. In this note, functors in capital fraktur font always have compact values at every $E$. Thus, if $\mathcal{O} \to \mathcal{D}$ is a morphism with $\mathcal{O}_E \to \mathcal{D}_E$ surjective for every $E$, then $\mathcal{O}_K \to \mathcal{D}_K$ is also surjective. To see this, we apply the dual $\mathcal{D}^\vee_E \to \mathcal{O}^\vee_E$, which implies the injectivity of $\mathcal{D}^\vee_K \to \mathcal{O}^\vee_K$, and then obtain the desired surjectivity via the duality [Kap50].

Since $\mathcal{X}_K$ is torsion, it is annihilated by some non-zero $f \in \Lambda$. Then $\mathcal{M}_K$ is annihilated by $f^2$, whence torsion. To see this, we may assume that for every intermediate extension $E$ of $K/k$, the extension $K/E$ contains no nontrivial unramified intermediate extension. Then the restriction map $\mathcal{M}_E \to \mathcal{M}_E'$ is injective for $E \subset E' \subset K$, and hence $\mathcal{X}_K \to \mathcal{X}_E$ is surjective. This implies $f \cdot \mathcal{X}_E = 0$, so $f^2 \cdot \mathcal{M}_E = 0$ and $f^2 \cdot \mathcal{M}_K = 0$.

2. Towers of $\Gamma$ modules

2.1. Monsky’s Theorem. Endow $\mathfrak{m}_{p^\infty}$ with the discrete topology. Let $\hat{\Gamma}$ denote all continuous characters from $\Gamma$ to $\mathfrak{m}_{p^\infty}$. Every $\chi \in \hat{\Gamma}$ is of finite order, it factors through $\Gamma \to \Gamma/\Gamma^{(n)}$, for some $n$, and hence extends uniquely to a continuous $\mathbb{Z}_p$-algebra homomorphism $\chi : \Lambda \to \mathcal{O}$, where $\mathcal{O}$ is the ring of integers of $\mathbb{Q}_p$.

Definition 2.1.1. (1) Define the zero set of an element $\theta \in \Lambda$ to be

$$\Delta_\theta := \{\chi \in \hat{\Gamma} \mid \chi(\theta) = 0\}.$$
(2) A $\mathbb{Z}_p$-flat $Z$ of codimension $m$ is a subset of $\hat{\Gamma}$ consisting of solutions to the system of equations

$$\chi(\xi_j) = \zeta_j, \ j = 1, \ldots, m,$$

where $\xi_1, \ldots, \xi_m$ are elements of $\Gamma$, extendable to a $\mathbb{Z}_p$-basis, $\zeta_1, \ldots, \zeta_m \in \mathbb{Z}_p^\infty$.

(3) We say $\tau_1, \ldots, \tau_c$ is a set of tight $\mathbb{Z}_p$-generators of $\Gamma$ if each $\tau_i \notin \Gamma^{(1)}$ and the topological closure of $<\tau_i>$ and $<\tau_j>$ are distinct for $i \neq j$.

**Theorem 2.1.2 (Monsky, [Mon81, Theorem 2.2.6]).** The zero set $\Delta_f$ of a non-zero $f \in \Lambda$ is a proper subset of $\hat{\Gamma}$. There are $\mathbb{Z}_p$-flats $Z_1, \ldots, Z_l$ such that

$$\Delta_f = Z_1 \cup \cdots \cup Z_l.$$

Let us introduce more notation. For $n \geq 0$, $\sigma \in \Gamma$, $\sigma \neq id$, denote

$$\omega_{\sigma, n} := \sigma^n - 1,$$

which is regarded as an element of $\Lambda$. Put $\omega_{\sigma, -1} = 1$. For $m \geq n \geq -1$, write

$$\nu_{\sigma, n, m} := \omega_{\sigma, m} / \omega_{\sigma, n}$$

which is also an element of $\Lambda$. Because $\nu_{\sigma, r, r+1}$ is the $p^{r+1}$th cyclotomic polynomial in $\sigma$, it is irreducible in $\mathbb{Z}_p[\sigma]$. Hence in $\Lambda$, if $r \neq r'$, then $\nu_{\sigma, r, r+1}$ is relatively prime to $\nu_{\sigma, r', r'+1}$. Also, if $\sigma$ and $\sigma'$ are linearly independent over $\mathbb{Z}_p$, then $\nu_{\sigma, n, m}$ and $\nu_{\sigma', n, m'}$ are relatively prime in $\Lambda$.

Due to technical reason, we will need to deal with ideals other than $I_n$, especially when $d \geq 2$. We introduce the ideals

$$J_n := (\nu_{\tau_1, 0, n}, \ldots, \nu_{\tau_c, 0, n}) \subset \Lambda,$$

where $\tau_1, \ldots, \tau_c$ is a chosen set of tight $\mathbb{Z}_p$-generators of $\Gamma$. It is clear that for $m \geq n$, the inclusion $J_m \subset J_n$ holds. Since $\omega_{\tau, 0} \cdot \nu_{\tau, 0, n} = \omega_{\tau, n}$, the ideal $J_n = (\omega_{\tau_1, n}, \ldots, \omega_{\tau_c, n})$ is inside $J_n$. For a chosen $\mathbb{Z}_p$-basis $\sigma_1, \ldots, \sigma_d$ of $\Gamma$, we also need to consider ideals,

$$J_{\mathbf{u}} := (\nu_{\sigma_1, r_1, n_1}, \nu_{\sigma_2, r_2, n_2}, \ldots, \nu_{\sigma_d, r_d, n_d}) \subset \Lambda,$$

with $\mathbf{u} = (n_1, \ldots, n_d)$, $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{Z}^d$ such that $r_i \geq -1$ and $n_i > r_i$, for every $i$, the latter condition will be abbreviated as $\mathbf{u} > \mathbf{r}$.

For an ideal $I = (\theta_1, \ldots, \theta_l) \subset \Lambda$, denote

$$\Delta_I := \bigcap_{\theta \in I} \Delta_\theta = \bigcap_{i=1}^l \Delta_{\theta_i}.$$ 

Write $\Delta_{\mathbf{u}, \mathbf{r}}$ for $\Delta_{J_{\mathbf{u}}}$, if in $(\mathbf{u})$ each $\zeta_j$ is of order $p^{r_j}$, then

$$Z \subset \Delta(\nu_{\tau_1, r_1-1, 1}, \ldots, \nu_{\tau_m, r_m-1, 1}) = \Delta_{\nu_{\tau_1, r_1-1, 1}} \cap \cdots \cap \Delta_{\nu_{\tau_m, r_m-1, 1}}.$$ 

**(2)** If the order of $\chi \in \hat{\Gamma}$ does not exceed $p^n$ or $p^{n_1}$, for every $i = 1, \ldots, d$, then for all $x \in J_{\mathbf{u}, \mathbf{r}} + J_n$ the value $\chi(x)$ is divisible by the minimum of $p^n$ and $p^{n_1 - r_1}, \ldots, p^{n_d - r_d}$.

**Lemma 2.1.3.** The following statements hold true:

(a) If the order of $\chi \in \hat{\Gamma}$ does not exceed $p^n$ or $p^{n_i}$, for every $i = 1, \ldots, d$, then for all $x \in J_{\mathbf{u}, \mathbf{r}} + J_n$ the value $\chi(x)$ is divisible by the minimum of $p^n$ and $p^{n_1 - r_1}, \ldots, p^{n_d - r_d}$.
(b) For a fixed \(_{\Sigma}\), the intersection \( \bigcap_{\nu \geq \Sigma n > 0} (\mathcal{I}_{\Sigma n} + \mathcal{I}_n) = 0 \).

(c) Let \( x \in \Lambda_r \). Then \( x \in \mathcal{I}_{\Sigma n} \) if and only if \( \chi(x) = 0 \) for all \( \chi \in \Delta_{\Sigma n} \).

Proof. Let \( p^\alpha \) denote the order of \( \chi(\sigma) \). For \( m \geq \alpha \), we have \( \chi(\nu_{\sigma, r, m}) = 0 \), if \( \alpha > r \), while \( \chi(\nu_{\sigma, r, m}) = p^{m-r} \), if \( \alpha \leq r \). The assertion (a) follows.

To prove (b), let \( x \) be an element of the intersection in question. In view of Theorem 2.1.2, we have to show \( \chi(x) = 0 \) for all \( \chi \in \hat{\Gamma} \). Let \( \alpha \) denote the maximum of \( r_1, ..., r_d \). For an integer \( \beta > \alpha \), put \( \nu = (\beta, ..., \beta) \), \( n = \beta \). Then by (a), the value \( \chi(x) \) is divisible by \( p^{\beta-\alpha} \). Since this holds for all \( \beta \), we must have \( \chi(x) = 0 \).

We prove (c) by induction on \( d \). The statement trivially holds for \( d = 0 \). Suppose \( d > 0 \), let \( \Gamma' \subset \Gamma \) be the subgroup topologically generated by elements of the basis \( \sigma_1, ..., \sigma_d \) other than \( \sigma_1 \). If \( d = 1 \), then both \( \Gamma' \) and \( \hat{\Gamma} \) are the trivial group and \( \Lambda_{\Gamma'} = \mathbb{Z}_p \). In this case, put \( \mathcal{I}_{\Sigma, n'} = (0) \) and \( \mathcal{I}_{\Sigma', n'} = \hat{\Gamma} \). If \( d \geq 2 \), let \( r' = (r_2, ..., r_d) \) and \( n' = (n_2, ..., n_d) \). Put \( T := \sigma_1 - 1 \in \Lambda_{\Gamma} \). Then \( \nu_{\sigma_1, r_1, n_1} \) is a distinguished polynomial in \( T \) of degree \( \delta := p^{r_1} - [p^{r_1}] \), where \([\cdot]\) is the Gauss symbol. The natural map \( \hat{\Gamma} \to \Gamma' \), \( \chi \mapsto \hat{\chi} \), induces a surjection \( \rho : \Delta_{\Sigma, n} \to \Delta_{\Sigma', n'} \) such that the fibre \( \rho^{-1}(\hat{\chi}) \) for each \( \hat{\chi} \in \Delta_{\Sigma', n'} \) is of cardinality \( \delta \) and a character \( \chi \in \rho^{-1}(\hat{\chi}) \) is determined by the value \( \chi(\sigma_1) \), or equivalently, by \( \chi(T) \). By the Weierstrass division theorem [Bou72, VII, §3, Proposition 5], we may assume that
\[
x = \sum_{i=0}^{\delta-1} y_i \cdot T^i, \quad y_0, ..., y_{\delta-1} \in \Lambda_{\Gamma'}.
\]
Thus, for \( \chi \in \rho^{-1}(\hat{\chi}) \),
\[
0 = \chi(x) = \sum_{i=0}^{\delta-1} \hat{\chi}(y_i) \cdot \chi(T)^i.
\]
This means the polynomial \( \sum_{i=0}^{\delta-1} \hat{\chi}(y_i) \cdot T^i \) in \( X \) has \( \delta \) distinct roots, hence must be trivial. Therefore, each \( y_i \) is annihilated by characters in \( \Delta_{\Sigma', n'} \). The induction hypothesis implies \( y_i \in \mathcal{I}_{\Sigma, n} \) for every \( i \), so \( x \in \mathcal{I}_{\Sigma n} \).

Lemma 2.1.4. Suppose \( \mathcal{Y} \) is a finitely generated \( \Lambda_{\Gamma} \)-module. Then \( \bigcap_n \mathcal{I}_n \mathcal{Y} = 0 \) so that the natural maps \( \mathcal{Y} \to \mathcal{Y}/\mathcal{I}_n \mathcal{Y} \) induce the isomorphism
\[
\mathcal{Y} \xrightarrow{\sim} \hat{\mathcal{Y}} := \varprojlim_n \mathcal{Y}/\mathcal{I}_n \mathcal{Y}.
\]

Proof. The homomorphism \( \mathcal{Y} \to \hat{\mathcal{Y}} \) has dense image, and it is surjective because both \( \mathcal{Y} \) and \( \hat{\mathcal{Y}} \) are compact. It is sufficient to show \( \bigcap_n \mathcal{I}_n \mathcal{Y} = 0 \). Lemma 2.1.3 (b) says the assertion holds for \( \mathcal{Y} = \Lambda_{\Gamma} \). In general, we have a surjective homomorphism \( \phi : \bigoplus_{i=1}^l \Lambda_{\Gamma} \to \mathcal{Y} \), with \( \mathcal{I}_n \mathcal{Y} = \phi(\bigoplus_{i=1}^l \mathcal{I}_n) \). Suppose \( x \in \bigcap_n \mathcal{I}_n \mathcal{Y} \). Denote \( X = \phi^{-1}(x) \) which is a compact subset of \( \bigoplus_{i=1}^l \Lambda_{\Gamma} \). Since \( X \cap \bigoplus_{i=1}^l \mathcal{I}_n \mathcal{Y} \neq \emptyset \), for every \( n \), and \( \bigcap_n \bigoplus_{i=1}^l \mathcal{I}_n = 0 \), we must have \( 0 \in X \). Therefore, \( x = \phi(0) = 0 \). \( \square \)
2.2. The norm maps. Fix a basis $\sigma_1, \ldots, \sigma_d$ of $\Gamma$ and set

$$\nu_{n,m} := \prod_{i=1}^{d} \nu_{\sigma_i,n,m} \in \Lambda\Gamma.$$ 

Since $\mathcal{I}_n$ is generated by $\omega_{\sigma_i,n}$ and $\nu_{\sigma_i,n,m} \cdot \omega_{\sigma_i,n} = \omega_{\sigma_i,m}$, we have $\nu_{n,m} \cdot \mathcal{I}_n \subset \mathcal{I}_m$. For a $\Lambda\Gamma$-module $\mathcal{Y}$, the map sending $y \in \mathcal{Y}$ to $\nu_{n,m} \cdot y$ induces an endomorphism $\mathcal{Y}/\mathcal{I}_m \mathcal{Y} \overset{\nu_{n,m}}{\longrightarrow} \mathcal{Y}/\mathcal{I}_m \mathcal{Y}$, which turns out to be the norm map of $\Gamma^{(n)}/\Gamma^{(m)}$ acting on $\mathcal{Y}/\mathcal{I}_m \mathcal{Y}$, hence independent of the choice of the basis.

In general, if for each $n$, there associates a $\Lambda\Gamma$ submodule $\mathcal{I}_n \subset \mathcal{Y}$ containing $\mathcal{I}_n \mathcal{Y}$ such that $\nu_{n,m} \cdot \mathcal{I}_n \subset \mathcal{I}_m$ for $m \geq n$, we shall also denote the $\Lambda\Gamma$-map induced from $\nu_{n,m}$ as $\mathcal{Y}/\mathcal{I}_n \mathcal{Y} \overset{\nu_{n,m}}{\longrightarrow} \mathcal{Y}/\mathcal{I}_m \mathcal{Y}$.

Let $\mathcal{J}_n$ be defined by choosing a tight set $\tau_1, \ldots, \tau_c$ of generators of $\Gamma$.

Lemma 2.2.1. We have $\nu_{n,m} \cdot \mathcal{J}_n \subset \mathcal{J}_m$.

Proof. We have to show that $\nu_{n,m} \cdot \nu_{\tau_i,0,n} \in \mathcal{J}_m$. Since modulo $\mathcal{I}_m$, the image $\nu_{n,m} \cdot \nu_{\tau_i,0,n}$ is independent of the choice of the basis, we may assume that $\tau_i = \sigma_1$. Then $\nu_{n,m}(\nu_{\tau_i,0,n}) = \nu_{\sigma_1,0,m} \cdot \nu_{\sigma_2,n,m} \cdots \nu_{\sigma_d,n,m} \in \mathcal{J}_m$. □

Definition 2.2.2. For a $\Lambda\Gamma$-module $\mathcal{Y}$, define

$$\mathcal{Y}_{(n,m)} := \ker(\mathcal{Y}/\mathcal{I}_n \mathcal{Y} \overset{\nu_{n,m}}{\longrightarrow} \mathcal{Y}/\mathcal{I}_m \mathcal{Y}),$$

$$\mathcal{Y}_{(n)} := \bigcup_{m \geq n} \mathcal{Y}_{(n,m)},$$

and

$$\mathcal{Y} := \lim_{\leftarrow n} \mathcal{Y}_{(n)}.$$

Proposition 2.2.3. Suppose $\mathcal{Y}$ is a finitely generated $\Lambda\Gamma$-module. Then $\mathcal{Y} \sim 0$.

Proof. Consider an exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{Y} \longrightarrow \mathfrak{Z} \longrightarrow \mathcal{N} \longrightarrow 0,$$

with $\mathcal{M}$, $\mathcal{N}$ pseudo-null and $\mathfrak{Z}$ a direct sum $\bigoplus_{i=1}^{r} \Lambda\Gamma/(f_i)$ where $f_i$ is either 0 or a power of an irreducible element of $\Lambda\Gamma$. Since $\mathcal{M} \oplus \mathcal{N}$ is pseudo-null, there are relatively prime $g_1, g_2 \in \Lambda\Gamma$ annihilating both $\mathcal{M}$ and $\mathcal{N}$. These would lead to an exact sequence

$$0 \longrightarrow a \longrightarrow \mathcal{Y} \longrightarrow \mathfrak{Z} \longrightarrow b \longrightarrow 0,$$

with both $a$ and $b$ annihilated by $g_1^2$ and $g_2^2$, hence pseudo-null. Thus, it remains to show that $\mathcal{Y} = 0$ for $g_1 = \Lambda\Gamma/(f)$. Let $\bar{x} \in \mathcal{Y} \subset \mathcal{Y}$ (Lemma 2.1.4) and let $x \in \Lambda\Gamma$ be a lifting of $\bar{x}$. For each $n$, there is $m > n$ such that

$$\nu_{n,m} \cdot x \equiv a \cdot f \pmod{\mathcal{I}_m},$$

(3)
for some $a \in \Lambda_{\Gamma}$. Put $\omega := \omega_{r_{1},0} \cdots \cdots \omega_{r_{s},0}$, $y := \omega \cdot x$. Since $\omega \cdot J_{m} \subset J_{m}$,

$$\nu_{n,m} \cdot y \equiv \omega a \cdot f \pmod{J_{m}}.$$

(4)

Suppose $f = 0$. For every $\chi \in \Delta_{\rho \delta_{n}}$, we have $\chi(\nu_{n,m}) \not\equiv 0$, while $\chi(I_{n}) = 0$, so $\chi(y) = 0$. Then it follows that $y \in I_{n}$ (Lemma 2.1.3(c)). Since this holds for all $n$, Lemma 2.1.4 says $y = 0$, whence $x = 0$ as desired.

Suppose $f \not\equiv 0$ and let $Z_{1}, \ldots, Z_{t}$ be the $\mathbb{Z}_{p}$-flats in Theorem 2.1.2. For each $i$, let $\chi(\xi_{i}) = \zeta_{i}$ be one of the defining equations (1) of $Z_{i}$, so that (2) says $Z_{i} \subset \Delta_{\nu_{i}}$, if $\zeta_{i}$ is of order $p^{\nu}$ and $\epsilon_{i} := \nu_{\xi_{i}, \tau_{i-1}, \tau_{i}}$. Then we put $\epsilon := \epsilon_{1} \cdots \epsilon_{t}$ to have

$$\Delta_{f} \subset \Delta_{\nu_{i}}.$$

(5)

Denote $\nu_{n,m}^{(i)} := \nu_{\sigma_{i+1}, n,m} \cdots \nu_{\sigma_{d}, n,m}$, $-1 = (-1, \ldots, -1)$, and $m^{(i)} = (m^{(i)}_{1}, \ldots, m^{(d)})$, with $m^{(i)}_{j} = n$ for $j \leq i$; $m^{(i)}_{j} = m$, for $j > i$. We claim that if

$$\nu_{n,m}^{(i)} \cdot t_{i} \equiv b_{i} \cdot f \pmod{J_{-1,m(i)}}$$

(6)

holds for $t_{i}, b_{i} \in \Lambda_{\Gamma}$, then there is some $b_{i+1} \in \Lambda_{\Gamma}$ such that

$$\nu_{n,m}^{(i+1)} \cdot \epsilon t_{i} \equiv b_{i+1} \cdot f \pmod{J_{-1,m(i+1)}}.$$ 

Then, beginning with (1), for which $i = 0$, by repeatedly applying the above implication, we can deduce

$$\epsilon^{d-1} y \equiv b_{d} \cdot f \pmod{J_{n}},$$

which means $\omega \epsilon^{d-1} x \in I_{n} + (f)$. As this happens for every $n$, Lemma 2.1.4 says $\omega \epsilon^{d-1} x \in (f)$. The proposition is proved, if $f$ is relatively prime to $\omega \epsilon$, and it remains to treat the case where $f$ is a power of $\nu_{\sigma, \alpha-1, \alpha}$, $\alpha \geq 0$, $\sigma$ extendable to a $\mathbb{Z}_{p}$-basis of $\Gamma$, because every irreducible factor of $\omega \epsilon$ is of such type. We may also assume that $\sigma = \tau_{1}$. Because $\tau_{1}, \ldots, \tau_{c}$ form a tight generating set, we can write

$$\omega = \nu_{\sigma, \alpha-1, \alpha}^{\delta} \cdot \omega'$$

with $\delta = 0, 1$, and $\omega'$ relatively prime to $f$. Note that if $\delta = 1$, then $\alpha = 0$ and $\sigma_{1} = \tau_{j}$, for some $j$, we may assume that the basis $\sigma_{1}, \ldots, \sigma_{d}$ are taken from $\{\tau_{1}, \ldots, \tau_{c}\}$.

Put $z = \omega' x$, $\bar{a} = (a_{1}, \ldots, a_{d})$ with $a_{1} = \alpha$, $a_{j} = -1$ for $j \neq 1$. By (3),

$$\nu_{n,m} \cdot z \equiv \omega' a \cdot f \pmod{J_{\bar{a}, \bar{m}(0)}}.$$

(7)

Again, we claim that for $n > \alpha$, if

$$\nu_{n,m}^{(i)} \cdot s_{i} \equiv c_{i} \cdot f \pmod{J_{\bar{a}, \bar{m}(i)}}$$

holds for $s_{i}, c_{i} \in \Lambda_{\Gamma}$, then there is some $c_{i+1} \in \Lambda_{\Gamma}$ such that

$$\nu_{n,m}^{(i+1)} \cdot s_{i} \equiv c_{i+1} \cdot f \pmod{J_{\bar{a}, \bar{m}(i+1)}}.$$

Then we can deduce that $\omega' x \in J_{\bar{a}, \bar{m}(0)} + (f)$, for all $n > \alpha$, and hence $\omega' x \in (f)$. Therefore, $\bar{x} = 0$, since $\omega'$ is relatively prime to $f$.

The proofs of the claims rely on Lemma 2.1.3(c). Denote $\bar{u}^{(i)} := (u_{1}, \ldots, u_{d})$, with $u_{i+1} = n$ and $u_{j} = -1$ for $j \neq i + 1$. Let $\chi \in \Delta_{\bar{u}^{(i)}, \bar{m}(i)}$. The inclusion

$$\Delta_{\bar{u}^{(i)}, \bar{m}(i)} \subset \Delta_{\nu_{\sigma_{i+1}, n,m}(i), m(i)} \cap \Delta_{-1, m(i)}$$

(8)
leads to \( \chi(\nu_{\sigma,1,n,m}) = 0 = \chi(\mathscr{I}_{-1,m}(\iota)) \), so \((\mathfrak{I})\) yields \( \chi(b_i \cdot f) = 0 \). Now, if \( \chi(f) = 0 \), then \((\mathfrak{I})\) implies \( \chi(\epsilon) = 0 \); otherwise, we have \( \chi(b_i) = 0 \). Therefore, \( \chi(b_i) = 0 \) always holds. Lemma 2.1.3(c) says, for each \( \nu \), we may assume that for each \( \nu \), \( \nu \) lies in that all characters applied will be in \( \Delta \). Let \( \nu \), then \((\mathfrak{I})\) implies \( \nu \) if \( \nu \) never zero. Let \( \nu \), contained in \( \Delta \). Theorem 1. By replacing \( \nu \), by \( \nu \), we can write \( \nu \), with \( \nu \), in \( \mathfrak{I}^{(i)}_{\Delta_{-1,m}(1)} \). By \((\mathfrak{I})\) again,

\[
eb_l = \nu_{\sigma,1,n,m} \cdot b_{l+1} + b_{l+1},
\]

with \( b_{l+1} \in (\nu_{\sigma,1,-1,n}, \ldots, \nu_{\sigma,1,-1,n}, \nu_{\sigma,2,-1,m}, \ldots, \nu_{\sigma,2,-1,m}) \subset \mathfrak{I}^{(i)}_{-1,m}(1) \). By \((\mathfrak{I})\) again,

\[
\nu_{\sigma,1,n,m} \cdot (\nu_{\nu,n,m}^{(i+1)} \cdot \epsilon \cdot t_i - b_{l+1} \cdot f) \in \mathfrak{I}^{(i)}_{-1,m}(1).
\]

In view of Lemma 2.1.3 this proves the first claim, because \( \Delta^{(i)} \cap \Delta_{-1,m}(1) = \emptyset \), if \( \nu \in \Delta_{-1,m}(1) \), then \( \chi(\nu_{\sigma,1,n,m}) \neq 0 \) and consequently \( \chi(\nu_{\nu,n,m}^{(i+1)} \cdot t_i - b_{l+1} \cdot f) = 0 \).

The proof of the second claims is similar to the previous one, the basic difference lies in that all characters applied will be in \( \Delta^{(i)}_{\beta,\alpha,\beta} \) for some \( \beta > \alpha \), so that \( \chi(f) \) is never zero. Let \( \mathfrak{a}^{(i)} = (\mathfrak{a}^{(i)}, \ldots, \mathfrak{a}^{(i)}) \) be such that \( \mathfrak{a}^{(i)} = \mathfrak{a}^{(i)} \), for \( j \neq i + 1 \), \( \mathfrak{a}^{(i)} = n \).

Let \( \chi \in \Delta_{\beta,\alpha,\beta}^{(i)}, \ldots, \chi \in \Delta_{\beta,\alpha,\beta}^{(i)} \).

Since \( \chi(\nu_{\sigma,1,n,m}) = 0 = \chi(\mathfrak{I}^{(i)}_{\Delta_{\beta,\alpha,\beta}(1)}) \), the congruence \((\mathfrak{I})\) yields \( \chi(c_i \cdot f) = 0 \), hence \( \chi(c_i) = 0 \). Lemma 2.1.3(c) says \( c_i \in \mathfrak{I}^{(i)}_{\Delta_{\beta,\alpha,\beta}(1)} \), and we can write

\[
c_i = \nu_{\sigma,1,n,m} \cdot c_{i+1} + c_{i+1},
\]

with \( c_{i+1} \in (\nu_{\sigma,1,n}, \ldots, \nu_{\sigma,1,n}, \nu_{\sigma,2,n} + 1, \ldots, \nu_{\sigma,2,n} + 1) \subset \mathfrak{I}^{(i)}_{\Delta_{\beta,\alpha,\beta}(1)} \). Thus, by \((\mathfrak{I})\),

\[
\nu_{\sigma,1,n,m} \cdot (\nu_{\nu,n,m}^{(i+1)} \cdot s_i - c_{i+1} \cdot f) \in \mathfrak{I}^{(i)}_{\Delta_{\beta,\alpha,\beta}(1)}.
\]

But \( \chi(\nu_{\sigma,1,n,m}) \neq 0 \), for all \( \chi \in \Delta_{\beta,\alpha,\beta}^{(i)} \), it follows that \( \nu_{\nu,n,m}^{(i+1)} \cdot s_i - c_{i+1} \cdot f \) is contained in \( \mathfrak{I}^{(i)}_{\Delta_{\beta,\alpha,\beta}(1)} \).

\[
\text{Remark 2.2.4. If we replace } \mathfrak{I}, \mathfrak{I}, \mathfrak{J} \text{ in Definition 2.2.2 by smaller } \mathfrak{I}, \mathfrak{I}, \mathfrak{J} \text{ satisfying } \nu_{\alpha,\beta}(\mathfrak{I}) \subset \mathfrak{I} \text{, for } m > n \text{, and let } \mathfrak{J}_{(n)}, \mathfrak{J} \text{ be the resulting counterpart of } \mathfrak{J}_{(n)}, \mathfrak{J} \text{, then } \mathfrak{J} \subset \mathfrak{J} \subset \mathfrak{J} \text{. Therefore, } \mathfrak{J} \sim 0 \text{ as well. See the proof of Theorem 1.}
\]

2.3. The module \( \hat{\mathfrak{X}}_K \). Now we prove Theorem 1. By replacing \( k \) by \( k_n \), if necessary, we may assume that for each \( \nu \) in the ramification locus \( S \), there is some integer \( e \) such that

\[
\Gamma/\Gamma^0 \simeq \mathbb{Z}_p^e, \tag{9}
\]

and

\[
\Gamma = \bigcup_{\nu \in S} \Gamma^0. \tag{10}
\]

Let \( K_n \) denote the maximal unramified abelian \( p \)-extension over \( k_n \) so that by Class Field Theory, \( \mathfrak{X}_n = \text{Gal}(K_n/k_n) \). For \( m > n \), the norm map \( \mathfrak{X}_m \rightarrow \mathfrak{X}_n \) is compatible with the restriction of Galois action \( \text{Gal}(K_m/k_m) \rightarrow \text{Gal}(K_n/k_n) \), we have \( \mathfrak{X}_K = \text{Gal}(L/K) \), for \( L = \bigcup_n K_n \). Denote \( G := \text{Gal}(L/k) \). Then \( G/\mathfrak{X}_K = \Gamma \) and since \( L/K \) is unramified, at every place \( \nu \) of \( k \), we have \( G^0 \simeq \Gamma^0 \), a commutative
group.

Suppose \( S = \{ v_1, \ldots, v_s \} \). For each \( j \), choose a place \( u_j \) of \( L \) sitting over \( v_j \) and then choose a \( \mathbb{Z}_p \)-basis \( \tilde{\xi}_1^{(j)}, \ldots, \tilde{\xi}_d^{(j)} \) of \( G_{u_j}^0 \). By (10) and (11), we can choose these bases to have the union of their images under \( G \to \Gamma \) form a tight set \( \gamma := \{ \tau_1, \ldots, \tau_r \} \) of generators of \( \Gamma \). Then among \( \gamma \), we choose a \( \mathbb{Z}_p \)-basis \( \{ \sigma_1, \ldots, \sigma_d \} \) of \( \Gamma \). We lift each \( \sigma_i \) to some \( \tilde{\xi}_i^{(j)} \) and denote it by \( \tilde{\sigma}_i \). Every \( g \in G \) can be uniquely written as

\[
g = \tilde{\sigma}_1 \cdot \ldots \cdot \tilde{\sigma}_d \cdot x_g, \quad a_1, \ldots, a_d \in \mathbb{Z}_p, x_g \in \mathcal{X}_K.
\]

Let \( J_{\text{ram}} \) denote the \( \mathbb{Z}_p \)-submodule of \( \mathcal{X}_K \) generated by

\[
\{ x_g \mid g = \tilde{\gamma} \cdot \tilde{\xi}_1^{(j)} \cdot \tilde{\gamma}^{-1} \text{ for some } j = 1, \ldots, s, i = 1, \ldots, d, \tilde{\gamma} \in \Gamma \}.
\]

The commutator \( \tilde{\sigma}_i \cdot \tilde{\sigma}_j \cdot \tilde{\sigma}_i^{-1} \cdot \tilde{\sigma}_j^{-1} \in J_{\text{ram}} \), because \( g = \tilde{\sigma}_i \cdot \tilde{\sigma}_j \cdot \tilde{\sigma}_i^{-1} \) in (11) yields

\[
\tilde{\sigma}_i \cdot \tilde{\sigma}_j \cdot \tilde{\sigma}_i^{-1} = \tilde{\sigma}_j \cdot x_{i,j}, \quad x_{i,j} \in J_{\text{ram}}.
\]

The canonical action of \( \Gamma \) on \( \mathcal{X}_K \) coincides with the conjugation in \( G \), namely, if \( \gamma \in \Gamma \), \( x \in \mathcal{X}_K \), and \( \tilde{\gamma} \in G \) is a lifting of \( \gamma \), then

\[
\gamma x = \tilde{\gamma} \cdot x \cdot \tilde{\gamma}^{-1}.
\]

Hence \( \gamma^{-1}x \) is the same as the commutator \( \tilde{\gamma} \cdot x \cdot \tilde{\gamma}^{-1} \cdot x^{-1} \) in \( G \). Set

\[
J := \mathcal{I}_0 \cdot \mathcal{X}_K + J_{\text{ram}} \subset \mathcal{X}_K.
\]

Note that \( J \) is a \( \Gamma \)-module, since \( \Gamma \) acts trivially on \( \mathcal{X}_K/\mathcal{I}_0 \mathcal{X}_K \). Let \( \tilde{J} \subset G \) denote the closed subgroup generated by \( J \) and \( \{ \tilde{\sigma}_1, \ldots, \tilde{\sigma}_d \} \). Then for every place \( u \) of \( L \), the inertia subgroup \( G_{u}^0 \subset \tilde{J} \), because \( u = \gamma u_j \) for some \( j \), and hence \( \tilde{J} \) contains the \( \mathbb{Z}_p \)-basis \( \tilde{\gamma} \cdot \tilde{\xi}_1^{(j)} \cdot \tilde{\gamma}^{-1}, i = 1, \ldots, d_i \), of \( G_{u}^0 \). Thus, \( \tilde{J} \) is the closed subgroup of \( G \) topologically generated by all commutators and all inertia subgroups of \( G \). Therefore, the fixed field \( L^J \) is the maximal unramified abelian \( p \)-extension \( K_0 \) of \( k \), with \( \text{Gal}(K_0/k) = \mathcal{X}_0 \). By the uniqueness of the expression (11), we have

\[
\mathcal{X}_K/J \simeq G/\tilde{J} = \text{Gal}(K_0/k) = \mathcal{X}_0.
\]

Also, (10) implies \( K \cap K_0 = k \), hence \( \text{Gal}(KK_0/K) \simeq \text{Gal}(K_0/k) \). By (12),

\[
\text{Gal}(KK_0/K) = \mathcal{X}_K/J \simeq \mathcal{X}_0.
\]
To apply the above argument to \( k_n \), let \( G^{(n)} \) denote the pre-image of \( \Gamma^{(n)} \) under \( G \rightarrow \Gamma \) and set \( \tilde{\sigma}_{i,n} := \tilde{\sigma}^p_i \). Every element \( g \in G^{(n)} \) can be uniquely expressed as

\[
g = \tilde{\sigma}_{i,n}^1 \cdots \tilde{\sigma}_{d,n}^a \cdot x_g, \quad a_1, \ldots, a_d \in \mathbb{Z}_p, \quad x_g \in \mathfrak{X}_K.
\]

Write \( \tilde{\xi}^{(j)}_{s_1,n}, \ldots, \tilde{\xi}^{(j)}_{s_d,n} \) for \( (\tilde{\xi}^{(j)}_{s_1})^{p^n}, \ldots, (\tilde{\xi}^{(j)}_{s_d})^{p^n} \). They form a \( \mathbb{Z}_p \)-basis of the inertia subgroup of \( G^{(n)}_{\tilde{\xi}^{(j)}} \). Let \( J^{(n)}_{\text{ram}} \) denote the \( \mathbb{Z}_p \)-submodule of \( \mathfrak{X}_K \) generated by

\[
\{x_g \mid g = \tilde{\gamma} \cdot \tilde{\xi}^{(j)}_{s_1,n} \cdot \tilde{\gamma}^{-1}, \text{ for some } j = 1, \ldots, s, i = 1, \ldots, d_j, \tilde{\gamma} \in G\}.
\]

Set

\[
J_n := \mathcal{I}_n \cdot \mathfrak{X}_K + J^{(n)}_{\text{ram}} \subset \mathfrak{X}_K.
\]

Let \( \tilde{J}_n \subset G \) denote the closed subgroup generated by \( J_n \) and \( \{\tilde{\sigma}_{i,n}, \ldots, \tilde{\sigma}_{d,n}\} \).

**Lemma 2.3.1.** The fixed field of \( \tilde{J}_n \subset G^{(n)} \) is \( K_n \). We have

\[
\text{Gal}(KK_n/K) = \mathfrak{X}_K/J_n \simeq G^{(n)}/\tilde{J}_n = \text{Gal}(K_n/k_n) = \mathfrak{X}_n.
\]

Furthermore, \( J_n \) is a \( \Lambda \)-module, hence a \( \Lambda_\Gamma \)-module.

**Proof.** It remains to show that \( J_n \) is invariant under the action of \( \Gamma \), or equivalently \( K_n/k \) is a Galois extension. But this follows from the fact that \( k_n/k \) is Galois and \( K_n \) is the maximal unramified abelian \( p \)-extension over \( k_n \). \( \square \)

The next lemma follows \[Iwa73\, Theorem \,7\].

**Lemma 2.3.2.** If \( m \geq n \), then \( \nu_{n,m}(J_n) \subset J_m \) and that induces the commutative diagram

\[
\begin{array}{ccc}
\mathfrak{X}_K/J_n & \xrightarrow{\sim} & \mathfrak{X}_n \\
\downarrow{\nu_{n,m}} & & \downarrow{c_{km/kn}} \\
\mathfrak{X}_K/J_m & \xrightarrow{\sim} & \mathfrak{X}_m.
\end{array}
\]

**Proof.** Since \([10]\) says \( K/k \) contains no non-trivial unramified subextension, the restriction of Galois action \( \mathfrak{X} \rightarrow \mathfrak{X}_n \) is surjective for every \( n \). Let \( x_n \in \mathfrak{X}_n \) and let \( x \in \mathfrak{X}_K \) such that \( x|_{k_m} = x_n \). Denote \( x|_{k_m} := x_m \). Let \( \lfloor \cdot \rfloor_m \) denote the Artin map and let \( \mathfrak{L} \) be an ideal in \( \mathcal{O}_{k_m} \) with \( \lfloor \mathfrak{L} \rfloor_m = x_m \). If \( I = N_{k_m/k_n}(\mathfrak{L}) \), then \( x_n = \lfloor I \rfloor_n \), hence

\[
\nu_{n,m}(x|_{k_n}) = c_{km/k_n}(x_n) = [N_{k_m/k_n}(\mathfrak{L})]_m = \prod_{\gamma \in \text{Gal}(k_m/k_n)} \gamma[\mathfrak{L}]_m = \nu_{n,m}(x)|_{k_m}.
\]

In particular, if \( x \in J_n \), then the left-hand side is trivial, hence \( \nu_{n,m}(x) \in J_m \). \( \square \)

**Proof of Theorem 1** By Lemma 2.3.2, we can write for the capitulation kernel

\[
\dot{x}_n = \bigcup_{n} \ker(\mathfrak{X}_K/J_n \xrightarrow{\nu_{n,m}} \mathfrak{X}_K/J_n).
\]

This makes it possible to apply the technique developed in the previous sections. Because \( J \subset \mathfrak{X}_K \) is of finite index, for convenience, we replace \( \mathfrak{X}_K \) by \( J \). To be more precise, since \( \mathcal{I}_n \cdot \mathfrak{X}_K \subset J_n \), we have \( \mathcal{I}_n \cdot J \subset J_n \), put \( \dot{J}_{(n,m)} := \ker( J/J_n \xrightarrow{\nu_{n,m}} J/J_m) \), \( \dot{J}_{(n)} := \bigcup_{m \geq n} \dot{J}_{(n,m)} \), and \( \dot{J} := \lim_n \dot{J}_{(n)} \). Then it is sufficient to show \( \dot{J} \sim 0 \).
ALGEBRAIC FUNCTIONAL EQUATIONS

Take \( \mathfrak{Y} := J \), \( \mathfrak{J}_n := J_n \cap \mathfrak{J}_n \cdot \mathfrak{Y} \) and let \( \mathfrak{Y}_{(n)} := (\mathfrak{I}_n, \mathfrak{J}_n) \). The homomorphisms \( \mathfrak{Y}/\mathfrak{J}_n \rightarrow \mathfrak{Y}/\mathfrak{J}_n \cdot \mathfrak{Y} \), for all \( n \), yield a homomorphism \( \mathfrak{Y} \rightarrow \mathfrak{Y} \) fitting into the commutative diagram

\[
\begin{array}{ccc}
\mathfrak{Y} & \longrightarrow & \mathfrak{Y} \\
\downarrow & & \downarrow \\
\mathfrak{J} & \longrightarrow & J.
\end{array}
\]

Since the down-arrows are injective, Proposition 2.2.3 implies \( \mathfrak{Y} \sim 0 \).

To see the difference between \( \mathfrak{I}_n \) and \( J_n \), we first check that for \( x \in \mathfrak{X}_K \), the element \( \omega_{\sigma_1,n} \cdot x = \nu_{\sigma_1} \cdot \omega_{\sigma_1,0} \cdot x \in \mathfrak{J}_n \cdot J \), because \( \sigma_1 \in \mathfrak{g} \) and \( \omega_{\sigma_1,0} \cdot x \in \mathfrak{J}_0 \cdot \mathfrak{X}_K \subset J \).

This shows \( \mathfrak{J}_n \cdot \mathfrak{X}_K \subset \mathfrak{J}_n \cdot J \cap \mathfrak{J}_n = \mathfrak{J}_n \). Denote \( \tilde{\xi}_i := g \), \( \tilde{\xi}_i := g_n \) and let \( \tau \in \mathfrak{g} \) be the image of \( g \) under \( \mathcal{G} \). Let \( x = x_g \in \mathcal{J}_\text{ram} \) so that \( g = \tilde{g} \cdot x \) with \( \tilde{g} = \tilde{g}_1 \cdot \ldots \cdot \tilde{g}_d \). Then

\[
\tilde{g}_1 \cdot \ldots \cdot \tilde{g}_d \cdot x_g = g_n = g \cdot x = (\tilde{g} \cdot x) \cdot \nu_{\tau,0} x.
\]

Similarly, for \( g' = \tilde{g} \cdot x' \), with \( x = x_{g'} \in \mathcal{J}_\text{ram} \), and \( g' = g \cdot \gamma^{-1} \), then

\[
\tilde{g}_1 \cdot \ldots \cdot \tilde{g}_d \cdot x_{g'} = g_n = g' \cdot x' = (\tilde{g} \cdot x') \cdot \nu_{\tau,0} x' = \tilde{g} \cdot x' \cdot \nu_{\tau,0} x'.
\]

Since \( x_{g_n}, x_{g'} \in \mathcal{J}_\text{ram}(n) \), writing \( \mathfrak{X}_K \) additively, we deduce from the above formulae

\[
x_{g_n} - x_{g'} = \nu_{\tau,0,n}(x - x') \in \mathfrak{J}_n.
\]

Therefore, we have shown that the \( \mathbb{Z}_p \)-module \( J_n/\mathfrak{J}_n \) is generated by the classes of \( x_{g_n} \), with \( g = \tilde{\xi}_i \), \( j = 1, \ldots, s \), \( i = 1, \ldots, d_j \). Then we observe that \( x_{g_n} \) is trivial if the above \( \tau = \sigma_1 \). This shows the \( p \)-rank of \( J_n/\mathfrak{J}_n \) is at most \( c - d \), which equals 0 if \( d = 1 \). The exact sequence

\[
0 \longrightarrow J_n/\mathfrak{J}_n \longrightarrow \mathfrak{Y}/\mathfrak{J}_n \longrightarrow J/J_n \longrightarrow 0
\]

gives rise to the exact sequence

\[
a_n \longrightarrow (\mathfrak{Y}/\mathfrak{J}_n) \longrightarrow (J/J_n) \longrightarrow b_n,
\]

with \( p \)-ranks of both \( a_n \) and \( b_n \) bounded by \( c - d \). Consequently, the cokernel of the induced map \( \mathfrak{Y} \longrightarrow J \) is of \( p \)-ranks bounded by \( c - d \). Hence \( J \) is pseudo-null. \( \square \)

3. Cohomology groups of global units

The proof of Theorem 2 involves cohomology of unit groups. Denote the group of global units of \( E \) by \( U_E := \mathcal{O}_E^\times \). The cohomology groups \( H^i(\Gamma, U_K), i = 1, 2 \), has been studied in \( \text{Iwa83} \), \( \text{Yam84} \) for \( d = 1 \) case. We are going to show that for general \( d \), they are co-finitely generated \( \mathbb{Z}_p \)-modules. Put

\[
U_E := \mathbb{Q}_p/\mathbb{Z}_p \otimes U_E,
\]

and

\[
\mathfrak{M}_E := \text{Gal}(\mathcal{E}^{ab,p}/E),
\]

where \( \mathcal{E}^{ab,p} \) is the maximal pro-\( p \) abelian extension of \( E \), unramified outside \( p \).
3.1. **An exact sequence.** Let \( \text{Div}_E \) and \( P_E \) denote the groups of divisors and principal divisors of \( E \). In [Iwa83 Proposition 1], Iwasawa deduces the seven-term exact sequence:

\[
0 \longrightarrow H^1(G, U_E) \longrightarrow \text{Div}^G_E / P_k \overset{\alpha_{E/k}}\longrightarrow \mathcal{C}^G_E \overset{\beta_{E/k}}\longrightarrow H^2(G, U_E) \tag{14}
\]

with \( G := \text{Gal}(E/k) \) and \( b_E := \ker(H^2(G, E^*) \longrightarrow H^2(G, \text{Div}_E)) \).

The exact sequence identifies \( H^1(G, U_E) \) with the kernel of \( \alpha_{E/k} \). The restriction of \( \alpha_{E/k} \) to \( \mathcal{C}_k \subset \text{Div}^G_E / P_k \) is the capitulation homomorphism \( c_{E/k} \). Therefore, we have the exact sequence

\[
0 \longrightarrow \ker(c_{E/k}) \longrightarrow H^1(G, U_E) \longrightarrow (\text{Div}^G_E \cap P_k) / (\text{Div}_k \cap P_k) \longrightarrow 0. \tag{15}
\]

If \( v \in S \) and \( w \) is a place of \( E \) sitting over \( v \), then the sum

\[
\epsilon_v := \sum_{\gamma \in G/G_v} \gamma w
\]

is fixed by \( G \). Set \( g_{E/k,v} := |G_v^0| \). In \( \text{Div}_E \), we have \( g_{E/k,v} \cdot \epsilon_v = v \). The sequence

\[
0 \longrightarrow \text{Div}_k \longrightarrow \text{Div}^G_E \overset{l_{E/k}}\longrightarrow \bigoplus_{v \in S} (\mathbb{Z}_p / g_{E/k,v} \mathbb{Z}_p) \cdot \epsilon_v \longrightarrow 0, \tag{16}
\]

with \( l_{E/k} \) the projection onto the \( S_E \)-component, is exact. Combining (15) and (16), we obtain the exact sequence

\[
0 \longrightarrow \ker(c_{E/k}) \longrightarrow H^1(G, U_E) \overset{l_{E/k}}\longrightarrow \bigoplus_{v \in S} \mathbb{Z}_p / g_{E/k,v} \mathbb{Z}_p. \tag{17}
\]

By taking the injective limit of the above sequence, one can show that \( H^1(\Gamma, U_K) \) is co-finitely generated over \( \mathbb{Z}_p \). As for \( H^2(\Gamma, U_K) \), we have the following:

**Lemma 3.1.1.** The following statements are equivalent:

(a) The \( \Lambda_\Gamma \)-module \( M_K \) is finitely generated.

(b) The abelian group \( \mathcal{X}_K^\Gamma \) has finite \( p \)-rank.

(c) The abelian group \( H^2(\Gamma, U_K) \) has finite \( p \)-rank.

**Proof.** Since \( \mathcal{X}_K^\Gamma[p] \) is the Pontryagin dual of \( M_K / (p \Lambda_\Gamma + \mathcal{I}_\Gamma) M_K \), the equivalence between (a) and (b) follows from Nakayama’s Lemma.

Denote \( Q_E := \text{Im}(\alpha_{E/k}) \cap \mathcal{X}_E^\Gamma \). Then (14) induces the exact sequence

\[
Q_K \longrightarrow \mathcal{X}_K^\Gamma \longrightarrow H^2(\Gamma, U_K) \overset{\beta}{\longrightarrow} H^2(k, K^*) .
\]

Now, the \( p \)-rank of \( Q_K \) is \( \leq \) the \( p \)-rank of \( \mathcal{X}_k \) plus the cardinality of \( S \). Also, if \( v \not\in S \), then the composition of \( H^2(\Gamma, U_K) \overset{\beta}{\longrightarrow} H^2(k, K^*) \longrightarrow H^2(k_v, K_v^*) \) is the trivial homomorphism. Hence \( \text{Im}(\beta) \) is embedded into \( \prod_{v \in S} \mathbb{Q}_p / \mathbb{Z}_p \) by the local invariant maps and is of \( p \)-rank bounded by the cardinality of \( S \). Therefore, (b) and (c) are equivalent.

\( \square \)
3.2. **The structure of $U_K^\vee$.** Recall that $U_K$ denotes the group of global units of $K$ and $U_K^\vee = \text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p \otimes U_K, \mathbb{Q}_p/\mathbb{Z}_p)$.

Let $\mathcal{V}_E$ denote the group of $\mathcal{P}_E$-units of $E$. Kummer’s theory yields the commutative diagram

$$
\begin{array}{cccccc}
U_E \otimes \mathbb{Z}_p/\mathbb{Z}_p & \rightarrow & \mathcal{V}_E \otimes \mathbb{Z}_p/\mathbb{Z}_p & \rightarrow & E^* \otimes \mathbb{Z}_p/\mathbb{Z}_p \\
\uparrow & & \downarrow \cong & & \downarrow \\
H^1(\mathbb{M}_E, \mathfrak{m}_{p^n}) & \rightarrow & H^1(\mathbb{M}_E, \mathfrak{m}_{p^n}) & \rightarrow & H^1(k, \mathfrak{m}_{p^n}).
\end{array}
$$

Since $E^*/\mathcal{V}_E$ and $\mathcal{V}_E/U_E$ are torsion free, arrows in the upper row of the diagram are injective. Let $n$ goes to $\infty$. We obtain the commutative diagram

$$
\begin{array}{cccccc}
U_E^\vee & \rightarrow & \mathcal{V}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & H^1(\mathbb{M}_E, \mathfrak{m}_{p^n}) \\
\uparrow \iota_E & & \downarrow j_E & & \downarrow r_E \\
U_K^\vee & \rightarrow & \mathcal{V}_K \otimes \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & H^1(\mathbb{M}_K, \mathfrak{m}_{p^n}).
\end{array}
$$

**Lemma 3.2.1.** The restriction map $H^1(\mathbb{M}_E, \mathfrak{m}_{p^n}) \xrightarrow{r_E} H^1(\mathbb{M}_K, \mathfrak{m}_{p^n})^{\Gamma_E}$ has finite kernel and cokernel.

**Proof.** [Tan10, Lemma 3.2.1]. \hfill \Box

**Lemma 3.2.2.** The cokernel of $\mathcal{V}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow H^1(\mathbb{M}_E, \mathfrak{m}_{p^n})$ is finite.

**Proof.** We show that as $n \rightarrow \infty$, the cokernel of $\mathcal{V}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow H^1(\mathbb{M}_E, \mathfrak{m}_{p^n})$ remains bounded. By Kummer’s theory, an element of $H^1(\mathbb{M}_E, \mathfrak{m}_{p^n})$ is represented by an $f \in E^*$ modulo $(E^*)^{p^n}$ such that $p^n \mid \text{ord}_v(f)$ for every $v \notin \mathcal{P}_E$. Let $w_1, \ldots, w_r$ be a set of generators of $\mathcal{X}_E$ and let $p^m$ be the exponent of $\mathcal{X}_E$ so that each $w_i^{p^m} = (f_i)$ for some $f_i \in E^*$. If $n > m$, then modulo $(E^*)^{p^m}$, $f$ can be expressed as a product of powers of $f_1^{p^{n-m}}, \ldots, f_r^{p^{n-m}}$ together with elements of $\mathcal{V}_E$. Hence the cokernel in question has order bounded by $p^{mr}$. \hfill \Box

**Corollary 3.2.3.** The natural map $U_E \rightarrow U_K^{\Gamma_E}$ has finite kernel, its cokernel is cofinitely generated over $\mathbb{Z}_p$ of corank bounded by $|S_E|$.

**Proof.** The exact sequence

$$
U_E^\vee \xrightarrow{i_E} \mathcal{V}_E \xrightarrow{\delta_E} \prod_{v \in \mathcal{P}_E} \mathbb{Z}_v,
$$
where $\tilde{O}_E$ is defined by taking valuations at all $v \in P_E$ induces the commutative diagram of complexes

\[
\begin{array}{ccc}
\mathcal{U}_E' & \longrightarrow & \mathcal{V}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p \\
\alpha_E & \downarrow & \beta_E \\
\mathcal{U}_K' & \longrightarrow & \mathcal{V}_K \otimes \mathbb{Q}_p/\mathbb{Z}_p
\end{array}
\]

$\prod_{v \in P_E} \mathbb{Q}_p/\mathbb{Z}_p \cdot v$

$\prod_{w \in P_K} \mathbb{Q}_p/\mathbb{Z}_p \cdot w$.

Write $A_E := \ker(\tilde{O}_E)$, $B_E := \ker(\tilde{U}_K \circ \beta_E)$. Then $A_E/\mathcal{U}_E$ is finite. In view of the diagram (18) and Lemma 3.2.1, 3.2.2, we need to show that the cokernel of $A_E$ has corank bounded by $|S_E|$. If $w$ is a place of $K$ sitting over $v$ of $E$ with $v \notin S_E$, then $w$ is unramified under $K/k$, and hence the map

$$\mathbb{Q}_p/\mathbb{Z}_p \cdot v \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \cdot w$$

is injective. Therefore, we have the exact sequence

$$0 \longrightarrow A_E \longrightarrow B_E \longrightarrow \prod_{v \in S_E} \mathbb{Q}_p/\mathbb{Z}_p \cdot v.$$

\[\square\]

**Lemma 3.2.4.** Let $\Omega_K$ be a finitely generated $\Lambda_{\Gamma}$-module. If $\Omega_K$ is of rank $r$ over $\Lambda_{\Gamma}$, then $\Omega_K/I_n\Omega_K$ has $\mathbb{Z}_p$-rank

$$r_n = r \cdot p^{dn} + O(p^{(d-1)n}).$$

**Proof.** There exists an exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{r} \Lambda_{\Gamma} \longrightarrow \Omega_K \longrightarrow T \longrightarrow 0.$$

Since $T$ is torsion, [Tan14, Lemma 1.13] says $T/I_nT$ has $\mathbb{Z}_p$-rank $= O(p^{(d-1)n})$. Hence

$$r_n \leq r \cdot p^{dn} + O(p^{(d-1)n}).$$

To obtain an inequality in the other direction, we use an exact sequence

$$\Omega_K \longrightarrow \bigoplus_{i=1}^{r} \Lambda_{\Gamma} \longrightarrow T' \longrightarrow 0,$$

with $T'$ torsion. \[\square\]

Let $r_1$ and $r_2$ denote the number of real and complex places of $K$.

**Proposition 3.2.5.** The $\Lambda_{\Gamma}$-module $\mathcal{U}_K'$ is finitely generated, of rank $r_1 + r_2$.

**Proof.** Since $|S_{K_0}| = O(p^{(d-1)n})$, Corollary 3.2.3 says $\mathcal{U}_K'/\mathcal{I}_n\mathcal{U}_K'$, which is the Pontryagin dual of $\mathcal{U}_K^{(n)}$, has $\mathbb{Z}_p$-rank equals $(r_1 + r_2) \cdot p^{dn} + O(p^{(d-1)n})$. In particular, $\mathcal{U}_K'/p\Lambda_{\Gamma} + \mathcal{I}_\Gamma \mathcal{U}_K'$ is finite. Hence $\mathcal{U}_K'$ is finitely generated. By Lemma 3.2.4, the rank of $\mathcal{U}_K'$ equals $r_1 + r_2$. \[\square\]
3.3. The module $\mathfrak{M}_K$. Let $K'/k$ be an intermediate $\mathbb{Z}_p^{d-1}$-extension of $K/k$ with Galois group $\Gamma'$. From now on, the symbol $F$ will denote a finite intermediate extension of $K'/k$, while $E$ denotes a finite intermediate extension of $K/k$ such that $E \cap K' = F$. We shall fix a $\mathbb{Z}_p$-extension $k^{(\infty)}/k$ with $K = K'k^{(\infty)}$ and $K' \cap k^{(\infty)} = k$. Let $k^{(n)}$ denote the $n$th layer of $k^{(\infty)}/k$ so that $E = Fk^{(n)}$ for some $n$. Put $F^{(\infty)} := Fk^{(\infty)}$. Denote $\Psi = \text{Gal}(K/K')$ and let $\psi$ be a topological generator of $\Psi$.

For each $F$, define

\[ N_F := \{ x \in U_F \mid x^{p^n} \in \bigcap_{F \subset E \subset F^{(\infty)}} N_{E/F}(U_E), \text{ for some } n \}. \]

Direct computation shows

\[ H^2(F^{(\infty)}/F, U_{F^{(\infty)}}) = \mathbb{Q}_p/\mathbb{Z}_p \otimes (U_F/N_F). \]  \hspace{1cm} (19)

Proof of Theorem 2. In view of Lemma 3.1.1, the proposition is equivalent to the assertion that $H^2(\Gamma, U_K)$ has finite $p$-rank. For $d = 1$, the value of this $p$-rank is shown in [Iwa83, Yam84], the equality (19) also implies it is bounded by the rank of $U_k$. We prove the theorem by induction on $d$.

By (14), we have the exact sequence

\[ 0 \rightarrow \hat{\mathcal{C}}_{E/F} \rightarrow \mathcal{C}_E^\psi \rightarrow H^2(E/F, U_E), \]

where $\hat{\mathcal{C}}_{E/F}$ is generated by $\text{Div}_E^\psi$. Let $E$ goes to $K$ and take the direct limit to obtain the exact sequence

\[ 0 \rightarrow \hat{\mathcal{C}}_{K/K'} \rightarrow \mathcal{C}_K^\psi \rightarrow H^2(\Psi, U_K). \]

Since the $p$-part of $\mathcal{C}_K^\psi$ is Pontryagin dual to $\mathfrak{M}_K/\mathcal{J}_{\mathfrak{I}}\mathfrak{M}_K$, it remains to show both $\hat{\mathcal{C}}_{K/K'}^\psi$ and $H^2(\Psi, U_K)^\psi$ are of finite $p$-ranks.

The equality (19) says $H^2(\Psi, U_K)$ is a quotient of $U_{K'}$. Hence its Pontryagin dual, denote $\mathfrak{J}_{K'}$, is a $\Lambda_{\Gamma'}$ submodule of $U_{K'}^\psi$. Proposition 3.2.5 implies $\mathfrak{J}_{K'}$ is $\Lambda_{\Gamma'}$ finitely generated. Then $H^2(\Psi, U_K)^\psi$, being the Pontryagin dual of $\mathfrak{J}_{K'}/\mathcal{J}_{\mathfrak{I}}\mathfrak{J}_{K'}$, must have finite $p$-rank.
Let \( T \subset S \) be the subset consisting of places \( v \) with \( \Gamma_v^0 \) of rank \( d \). We choose \( K' \) such that \( K/K' \) is unramified outside \( T_K' \), which is a finite set. Apply \((16)\) to the case where \( k = F \) and then let \( E \) goes to \( K \) to obtain the exact sequence

\[
0 \rightarrow \bar{C}_{K'} \rightarrow \hat{C}_{K/K'} \rightarrow V \rightarrow 0,
\]

where \( \bar{C}_{K'} \) is the image of the capitulation homomorphism \( C_{K'} \rightarrow C_K \) and \( V \) is a quotient of \( \prod_{w \in T_K'} \mathbb{Q}_p/\mathbb{Z}_p \cdot w \). Obviously, the \( p \)-rank of \( V \) is finite. Now \( \bar{C}_{K'/K} \) is a \( \Lambda_{\Gamma'} \) submodule of \( M_{K'} \), whence finitely generated by the induction hypothesis. Consequently, \( \bar{C}_{K'/K} \), the Pontryagin dual of \( \bar{C}_{K'/K} \), must have finite \( p \)-rank. The above exact sequence implies the \( p \)-rank of \( \hat{C}_{K/K'} \) is finite. \( \square \)

For \( E \subset E' \subset K \), let \( r_{E'}^E : M_E \rightarrow M_{E'} \) and \( r_{E'}^E : M_{E'} \rightarrow M_E \) denote respectively the restriction and the corestriction. Extend them to homomorphisms

\[
r_{E'}^E : M_{E} \times X_E \rightarrow M_{E'} \times X_{E'}
\]

and

\[
r_{E'}^E : M_{E'} \times X_{E'} \rightarrow M_{E} \times X_E
\]

such that the restrictions to the second factors are respectively the capitulation and the norm. Consider the natural pairing \( \langle \cdot, \cdot \rangle_E : X_E \times M_E \rightarrow \mathbb{Q}_p/\mathbb{Z}_p \). Then

\[
\mathfrak{A} := \{ M_E, X_E, \langle \cdot, \cdot \rangle_E, r_{E'}^E, r_{E}^E \}
\]

form a \( T \)-system in the sense of \([LLTT18, \S 2.1]\). Write \( \hat{M}_E \) for \( \bigcup_E \ker(r_{E'}^E) \).

**Lemma 3.3.1.** We have \( \hat{M}_K = 0 \).

**Proof.** Let \( E_0/k \) be the maximal unramified subextension of \( K/k \). Then the restriction \( M_E \rightarrow M_{E'} \) is injective for \( E_0 \subset E \subset E' \subset K \). Therefore, \( M_E = 0 \), for \( E_0 \subset E \), and hence \( \hat{M}_K = 0 \). \( \square \)

**Proof of Theorem 3** Theorem 1 and Lemma 3.3.1 together imply the system \( \mathfrak{A} \) is pseudo-controlled. Then apply \([LLTT18, \text{Theorem 1}]\). \( \square \)

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