Circular Polarization of the CMB: A probe of the First stars

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The Cosmic Microwave Background (CMB) is linearly polarized. It is predicted that there is no significant intrinsic circular polarization (CP) in the standard cosmology. In this paper, we study the circular polarization of the CMB due to the Faraday conversion (FC), in particular FC due to the supernova remnants of the First stars, also called the Pop III stars. The mechanism of Faraday conversion channels a pre-existing linear polarization into circular polarization, in presence of magnetic field and scattering of photons with relativistic electrons. We derive an analytic form for the angular power spectrum of the CP of the CMB generated by the FC. We apply this result to the particular case of the FC triggered by explosions of the first stars and estimate the angular power spectrum, $C_l^{VV}$. We show that the amplitude of $l(l+1)C_l^{VV}/(2\pi) > 10^{-2} \mu K^2$ for $l > 100$, with the age of the Pop III SN remnant to be $10^4$ years and frequency of CMB observation as 30 GHz. We expect CP of the CMB to be a very promising probe of the yet unobserved first stars, primarily due to the expected high signal with no appreciable foreground, along with an unique frequency dependence.

I. INTRODUCTION

Observations of the Cosmic Microwave Background (CMB) are essential to modern cosmology. In particular, a precise measurement of the CMB polarization is one of the major goals for the ongoing and future CMB observations. In the standard cosmology, the CMB is predicted to be linearly polarized, because the linear polarization of the CMB is produced by the anisotropic Thomson scattering around the epoch of recombination [1, 2]. Theoretical studies of the CMB polarization predict that there exists 10% of the linear polarization [3–6]. Since the first detection of CMB polarization anisotropy by DASI [7], several observations have measured the angular power spectrum of the polarization and the cross-correlation (e.g., Ref. [8] for one of recent works). These observational results are consistent with the theoretical predictions. On the other hand, the circular polarization (CP, hereafter) of the CMB is usually assumed to be zero, because there is no generation mechanism at the epoch of recombination in the standard cosmology.

However, the CP of the CMB can be generated in the free-streaming regime after the epoch of recombination due to scattering of CMB photons with relativistic electrons or Compton scattering. An unpolarized or linearly polarized CMB undergoing Compton scattering with a quadrupole anisotropy in the scattering centers, can gain CP. There are additional mechanisms which can enhance the generation of CP such as the following.

The linear polarization of the CMB can be converted to CP with the presence of relativistic magnetized plasma [9, 10]. This conversion is formalized by the generalized Faraday rotation and known as the Faraday conversion. Cooray et al. have discussed the FC conversion with magnetic relativistic plasma in galaxy clusters and shown that the resultant degree of the CP reaches $10^{-9}$ at frequencies of 10 GHz for 10 $\mu$Gauss magnetic fields in a galaxy cluster [11].

The CP of the CMB can be generated by another mechanism. Mohammadi have investigated the generation of the CP of the CMB through their scattering with the cosmic neutrino background [12]. Giovannini has shown that the curvature perturbations can produce the CP with the presence of primordial magnetic fields around the last scattering surface [13, 14]. Sawyer has discussed the CP due to photon-photon interactions mediated by neutral hydrogen background [15]. In addition, some new physics effects can induce the CP of the CMB [16–18].

Recently Mainini et al. have performed the first attempt to detect the CP of the CMB since ’90 [19]. They have improved the upper limit on the degree of the CP, which is between $5 \times 10^{14}$ and $0.4 \times 10^{-4}$ at large angular scales (between $8^\circ$ and $24^\circ$). However, this limit is very far from $10^{-9}$ degree of the CP predicted in a cosmological context.

In this paper, we will evaluate the CP of the CMB via the Faraday conversion (FC, hereafter) in Supernova (SN) remnants of Population III (Pop III) stars. The formation of the first stars is a important milestone in the evolution of the structure formation. After photon decoupling, overdensity regions began to grow and collapsed to dark matter halos. Inside of dark matter minihalos, formation of luminous objects like stars was not solely driven by gravity. Sufficient amount of baryon gas inside of a dark matter halo must cool to eventually form stars. These first born stars are thought to be very massive and are termed as the First stars or alternatively Pop III stars. It is believed that Pop III stars formed in small halos ($10^6$–$10^8 M_\odot$) at $z \sim 20$-30 (see Ref. [20, 21] for a recent review). Although Pop III stars are key to early structure formations as the first luminous objects and the sources of cosmic reionization and cosmic metal pollution, no Pop III stars have been directly observed and there is some debate on their properties including mass range of Pop III stars In the isolation scenario, Pop III star mass is predicted to be massive, 100–500 $M_\odot$ [22–27].

It is known that Pop III stars with mass $> 10 M_\odot$ cause SNe at their death. The detection of SNe of Pop III stars
are expected as one of possible probes of first stars, because these SNe could be much brighter than their progenitors or host galaxies. In particular, Pop III stars with 140–260 M⊙ explode as pair-instability SNe, which is up to 100 times more energetic than Type Ia and Type II SNe [28]. Many works have been done to investigate the observability of Pop III SNe [29–37]. According to these works, SNe of Pop III stars could be found by the James Webb Space Telescope (JWST) or the Wide-Field Infrared Survey Telescope (WFIRST). Additionally, SN remnants of Pop III stars may be also detectable. Meiksin and Whalen have found that SN remnants of Pop III stars in 10⁷ M⊙ halos can produce observable radio signatures [38]. Oh et al. have shown that SN remnants of Pop III stars may induce additional CMB temperature anisotropy through the Sunyaev-Zel’dovich effect [39].

In this paper, adopting a simple analytic model for the evolution of SN remnants, we study the FC in a SN remnant of a Pop III star. Our aim is to evaluate the anisotropy of the CMB CP. Therefore, we calculate the power spectrum of the CMB CP.

Throughout this paper, we adopt a flat ΛCDM cosmology, with hcp = 0.7, Ωc = 0.23, and Ωb = 0.046.

II. CIRCULAR POLARIZATION DUE TO FARADAY CONVERSION

Due to the Thomson scattering with the presence of quadrupole temperature anisotropy, CMB becomes linearly polarized during the epoch of recombination. As the CMB propagates through the inter-galactic medium, the galaxy clusters, there are secondary effects which are imprinted on the CMB spectra and polarization properties. One of such secondary effect is the creation of the CP of CMB through the FC. Compton scattering with relativistic electrons in astrophysical objects under the presence of magnetic field can transform linear polarization (hereafter, LP) in CMB into CP through the mechanism of FC. In this section, after giving a brief review of the Stokes parameters, we formulate the generation of CP of the CMB due to the FC obtain the analytic form of the CP angular power spectrum.

A. Stokes parameters

First of all, we consider a monochromatic electro-magnetic (EM) wave propagating along ẑ. This EM wave is characterized by two mutually perpendicular electric field components on the x–y plane. At a given point in space, the amplitude of the electric field vectors pointing along x̂ and ẏ respectively are described by [41]

\[
E_x = E_x^0(t) \cos(\omega t - \phi_x(t)), \\
E_y = E_y^0(t) \cos(\omega t - \phi_y(t)).
\]

The extent of polarization is generally quantified in terms of the so-called Stokes parameters I, Q, U and V. These Stokes parameters for a monochromatic EM wave are defined as

\[
I = \langle E_x^0 \rangle^2 + \langle E_y^0 \rangle^2, \\
Q = \langle E_x^0 E_y^0 \rangle - \langle E_x^0 \rangle \langle E_y^0 \rangle, \\
U = 2 \langle E_x^0 \rangle \langle E_y^0 \rangle \cos(\phi_x - \phi_y), \\
V = 2 \langle E_x^0 E_y^0 \rangle \sin(\phi_x - \phi_y).
\]

However, in practice, we measure EM waves at frequency ω with bandwidth Δω. That is, measured EM waves can be expressed as a superposition of many waves around ω. For such EM waves, the Stokes parameters is obtained by the time averaging of the electric field components of the EM waves,

\[
I = \langle \langle E_x^0 \rangle^2 \rangle + \langle \langle E_y^0 \rangle^2 \rangle, \\
Q = \langle \langle E_x^0 \rangle^2 \rangle - \langle \langle E_y^0 \rangle^2 \rangle, \\
U = \langle 2 E_x^0 E_y^0 \cos(\phi_x - \phi_y) \rangle, \\
V = \langle 2 E_x^0 E_y^0 \sin(\phi_x - \phi_y) \rangle,
\]

where the bracket ⟨ ⟩ denote the time averaging with the time interval over which the measurement is performed.

As shown in above equations, the I parameter represents the intensity of the EM wave, the Q and U parameters are associated to LP of the EM wave and the V parameter quantifies the extent of CP. The parameters I and V are coordinate independent and dimensionless true observables. The parameters Q and U transform under a rotation of...
the coordinate system while $Q^2 + U^2$ is an invariant under the rotation of the axes. The sign of the parameter $V$ related to the rotating direction of electric field components on the $x$-$y$ plane. EM waves with $V > 0$ and $V < 0$ are called, respectively, right-handed and left-handed circular polarized waves. A linearly polarized wave is a combination of one left circularly and one right circularly polarized waves with equal amplitudes, that is, $V = 0$. A circularly polarized wave is created when the left and right circularly polarized components have unequal amplitudes. Thomson scattering of photons with non-relativistic electrons does not create CP. Therefore CMB coming from the last scattering surface is not circularly polarized. Relativistic electrons can, however, create CP through Compton scattering [3, 42]. In this case CP is created from an unpolarized or linearly polarized CMB, given there is a quadrapole anisotropy present at the Compton scattering centers.

In this paper, we will only consider the CP generation due to conversion of linear polarization into CP through the method of FC [9].

**B. Angular power spectrum of circular polarization**

For the analysis of the CMB anisotropy, it is useful to perform an angular decomposition of the anisotropic values in multipole space. Let $V(\hat{n})$ be the CP at a given direction $\hat{n}$ on the sky. Since the Stokes parameter $V$ is a dimensionless invariant quantity, it can be expanded in the basis of scalar spherical harmonics $Y_{lm}(\hat{n})$ in the following way,

$$V(\hat{n}) = \sum_{lm} V_{lm} Y_{lm}(\hat{n}).$$

Using the coefficients $V_{lm}$, we can write the angular power spectrum of the CMB circular polarization, $C_{VV}^l$, as

$$C_{VV}^l = \frac{1}{2l + 1} \sum_{m} V_{lm}^* V_{lm}.$$

Let us consider the CP of the CMB due to FC. The observed CP is given in terms of the Stokes $V$ parameter by

$$V(\hat{n}) = -2 \int_{r_\star}^{0} dr \ U(r, \vec{x}, \hat{n}) \alpha(r, \vec{x}, \hat{n}, \hat{b}),$$

where $r$ is the comoving distance, $*$ denotes the value at the last scattering surface and $U(\vec{x}, \hat{n})$ is the Stokes parameter at comoving space position $\vec{x}$ and observation direction $\hat{n}$ with $\vec{x} = r \hat{n}$. Here $\alpha(r, \vec{x}, \hat{n}, \hat{b})$ is the FC rate with the magnetic field direction $\hat{b}$, which we will discuss in more detail in the next section. As shown in Eq. (A5) of the appendix, $\alpha(r, \vec{x}, \hat{n}, \hat{b})$ can be decomposed as

$$\alpha(r, \vec{x}, \hat{n}, \hat{b}) = 2\pi \sqrt{\frac{32\pi}{15}} \int d^3k \ (2\pi)^3 \tilde{\alpha}(z, k) \left(2 Y_0^{0}(\hat{n}) + -2 Y_2^{0}(\hat{n})\right) \sum_{l} (-i)^l j_l(kr) \sum_{m=-l}^{m=l} Y_{lm}^*(\hat{k}) Y_{lm}(\hat{n}),$$

where $\pm 2Y_l^m(\hat{n})$ are spin-2 spherical harmonics, $j_l(x)$ is the spherical Bessel function, and, for simplicity, we assume that $\hat{b}$ is $(0, 0)$ in a polar coordinate system for the sky, $\hat{b} = (\theta, \phi)$.

Generally, the stokes parameters for a linear polarzation, $U$ and $Q$, can be expanded as [6],

$$(Q \pm iU)(r, \vec{x}, \hat{n}) = \int \frac{d^3k}{(2\pi)^3} \sum_{l} \sum_{m=-2}^{m=2} (E_{lm} \pm iB_{lm}) \pm2 G_{l}^m(\vec{x}, -\hat{n})$$

where $\pm2 G_{l}^m$ is a mode function for a spin-2 field,

$$\pm2 G_{l}^m(\vec{x}, \hat{n}) = (-i)^l \sqrt{\frac{4\pi}{2l+1}} \pm2 Y_l^m(\hat{n}) e^{ik\vec{x}}.$$

In Eq. (8), $E_{lm}$ and $B_{lm}$ are coefficients for so-called E- and B-mode polarizations [3–6]. For simplicity, we assume that B-mode polarization vanishes hereafter. In this assumption, the parameter $U$ is given in terms of $E_{lm}$ and $B_{lm}$ as

$$2iU(r, \vec{x}, \hat{n}) = \int \frac{d^3k}{(2\pi)^3} \sum_{l} \sum_{m=-2}^{m=2} E_{lm} (2G_{l}^m - -2 G_{l}^m).$$
With Eq. (7) and (10), we can decompose the Stokes parameter $V(\hat{n})$ in Eq. (6) in spherical harmonics as shown in Eq. (4). After a lengthy calculation, we finally obtain the angular power spectrum of $V$ as

$$C_{l}^{VV} \approx \frac{128}{15\pi} \sum_{l^\prime l} \sum_{l^\prime m^\prime = 2}^{m^\prime = 2} \sum_{m = -2}^{m} \int_{0}^{D_{\alpha}^{2}(r)P_{\alpha}(\frac{l^\prime}{r})P_{E_{\nu_{l}}}(k, r)j_{\nu_{l}}^{2}(kr)I_{r}^{2}, \quad (11)$$

where $P_{E_{\nu_{l}}}(k, r)$ is the power spectrum of E-mode polarization, $E_{\nu_{l}}$, at a comoving distance $r$, $D_{\alpha}^{2}(r)P_{\alpha}(k)$ is the power spectrum of the FC rate at $r$, and $I_{r}$ is given by Eq. (A12). The detailed derivation of Eq. (11) is in the appendix. Eq. (A12) tells us that, depending on the power spectrum, $D_{\alpha}^{2}P(k)$, the E-mode polarization is converted to the CP.

### III. FARADAY CONVERSION FROM POP III STARS

Within the $\Lambda$CDM framework of structure formation, dark matter minihalos formed around redshift of $z \sim 20-30$. Inside of the dark matter minihalos, Pop III stars were born. Several dynamical feedback processes along with entropy and angular-momentum properties of the pre-stellar gas determined the final mass of the Pop III stars, which is typically estimated to span between 60-300 $M_{\odot}$.

In this paper, we investigate the CP signal in the CMB coming from the explosions of the Pop III stars. A supernova generated due to a Pop III star explosion produces a large outburst of energy [36] and a shock wave. As the shock wave propagates through the ambient medium of the explosion, a high magnetic field and a large number of relativistic electrons are produced. Consequently, CMB photons passing through SN remnants of Pop III stars could be significantly affected by the mechanism of FC.

Adopting a simple analytic model of explosion of Pop III star, we estimates the FC induced by one SN remnant of Pop III stars in this section. In order to estimate the angular power spectra of CP, $C_{l}^{VV}$, obtained in the previous section, we need to calculate the FC rate at $r$, $P_{\alpha}$. Based on the halo model, we evaluate $P_{\alpha}$ due to SN remnants of Pop III stars.

#### A. Faraday Conversion due to a Pop III star explosion

Faraday conversion rate $\alpha$ in Eq. (6) is given by [9]

$$\alpha(z, \hat{z}, \hat{n}, \hat{b}) = \alpha_{0} \sin(\theta_{B}) \frac{2 + 2}{2 + 2},$$

$$\alpha_{0} = C_{\gamma} \frac{c^{2}}{n_{rel}\epsilon_{\min}(B_{\mathrm{mag}})^{2} \nu^{-2/3}}, \quad (12)$$

where $\theta_{B}$ is the angle between the direction of the line of sight, $\hat{n}$ and the magnetic field direction $\hat{b}$, $n_{rel}$ is the number density of relativistic electrons and $\gamma$ denote the power-law distribution of the relativistic electrons which described in terms of the Lorentz factor $\epsilon$ as $n_{rel}(\gamma) = n_{0}\epsilon^{-\gamma}$ between $\epsilon_{\min} < \epsilon < \epsilon_{\max}$. The parameter $C_{\gamma}$ in Eq. (12) is provided by Ref. [9],

$$C_{\gamma} = \left\{ \begin{array}{ll}
\frac{-2^{\gamma-1}}{\gamma-2} & (\frac{e}{2\pi m_{e}c})^{2} \log(\nu(z)/\nu_{L}) - 1, \\
\frac{e^{\gamma}}{2\pi m_{e}c}^{2} & \gamma \neq 2,
\end{array} \right. \quad \gamma = 2. \quad (13)$$

When a SN occurs, large amount of energy is injected into the surrounding gas. As a result, the shock is created and relativistic electrons and magnetic fields are generated inside the shocked gas (SN remnants). In order to estimate the FC rate in SN remnants, we evaluate the number density of relativistic electrons $n_{rel}$ and magnetic fields $B_{\mathrm{mag}}$, assuming the regime at which the shock front expands adiabatically. This regime is known as the blast wave regime and the dynamics of this regime is expressed in the Sedov similar solution which describes a point blast spherical explosion in an ambient medium [43]. In this solution, the radius of the shock $r_{s}$ is given by

$$r_{s} = 2.3 \text{ pc} \left( \frac{E_{\mathrm{SN}}}{10^{51} \text{erg}} \right)^{\frac{1}{2}} \left( \frac{\rho_{b}}{10^{-21} \text{g/cm}^{3}} \right)^{-\frac{1}{2}} \left( \frac{t_{age}}{100 \text{ yr}} \right)^{\frac{3}{2}}, \quad (14)$$

where $E_{\mathrm{SN}}$ is the energy of the SN explosion, $\rho_{b}$ is the baryon mass density of an ambient medium, the shock expands into, and $t_{age}$ is the time since the explosion.
The energy $E_{\text{SN}}$ generally depends on a mass of the exploded Pop III star. Since Pop III stars are predicted to be massive, 60-300 $M_\odot$, we assume that Pop III stars have 100 $M_\odot$ mass and explode as pair-instability SNe which is 100 times more powerful than typical type II SNe. Therefore, we adopt $E_{\text{SN}} = 10^{53}$ ergs in this paper.

The shock wave expands in the interior of a dark matter halo at first, although $r_s$ could become larger than the size of the halo as $t_{\text{age}}$ increases. Even in this scenario, we simply assume that the shock continues to expand following Eq. (14). Therefore, we set $\rho_b$ to

$$\rho_b = \begin{cases} \rho_b^{\text{halo}}, & r_s \leq R_{\text{vir}} \\ \rho_b, & r_s > R_{\text{vir}}, \end{cases} \tag{15}$$

where $R_{\text{vir}}$ is the virial radius of the halo and $\rho_b$ is the mean baryon mass density in the universe. To obtain the baryon mass density inside of a dark matter halo $\rho_b^\text{halo}$, we assume that the baryon mass distributes homogeneously inside the virial radius. Therefore, the baryon mass density in the halo is given by

$$\rho_b^\text{halo} = \frac{\Omega_b}{\Omega_b + \Omega_c} \frac{3M}{4\pi R_{\text{vir}}^3}. \tag{16}$$

The blast wave phase continues until the cooling of the SN remnant becomes effective. One of the important cooling mechanisms is the inverse Compton (IC) scattering. The cooling time, $t_{\text{IC}}$, for the IC scattering is independent of temperature and density,

$$t_{\text{IC}} = \frac{3m_e c}{4\sigma_T \rho_{\text{CMB}}} \approx 1.4 \times 10^7 \text{ yr} \left( \frac{1 + z}{20} \right)^{-4}, \tag{17}$$

where $\sigma_T$ is the Thomson cross-section of an electron and $\rho_{\text{CMB}}$ is the CMB energy density. Eq. (17) sets an upper limit on $t_{\text{age}} \lesssim 10^7$ yr, used for the estimation of $r_s$ in Eq. (14).

Let us evaluate the number density of relativistic electrons. Suppose that the fraction $f_{\text{rel}}$ of the explosion energy $E_{\text{SN}}$ gets converted into relativistic energy of electrons. For a hydrodynamic shock, the inner radius of the shocked regime (SN remnant) is estimated to be \cite{44}

$$r_p = r_s \left( \frac{\eta}{\eta - 1} \right)^{\frac{1}{3}}. \tag{18}$$

This equation is based on the conservation of mass in the shock enclosed regime, where the density rises by a factor of $\eta$. We adopt $\eta = 4$, based on the properties of monoatomic gas. We simply assume that the relativistic electrons are confined in the region between the radius $r_p$ and $r_s$, whose volume is $V_{\text{rem}} = 4\pi r_p^3 - 4\pi r_s^3/3$. Assuming the power-law distribution of the relativistic electrons as mentioned above, we can obtain the normalization $n_0$ of the distribution from

$$f_{\text{rel}} E_{\text{SN}} = V_{\text{rem}} \int_{\epsilon_{\text{min}}}^{\epsilon_{\text{max}}} n_0 m_e c^2 \epsilon^{1-\gamma}. \tag{19}$$

In our calculation, we use $\gamma = 2$, $\epsilon_{\text{min}} = 100$ and $\epsilon_{\text{max}} = 300$.

For magnetic fields, we also introduce the parameter $f_{\text{mag}}$ which denotes the fraction of the energy of a Pop III star explosion into magnetic field energy. The magnetic field amplitude $B_{\text{mag}}$ is obtained by solving the equation

$$\frac{B_{\text{mag}}^2}{8\pi} V_{\text{rem}} = f_{\text{mag}} E_{\text{SN}}. \tag{20}$$

Although the parameters $f_{\text{rel}}$ and $f_{\text{mag}}$ are theoretically uncertain, we set $f_{\text{rel}} = 0.1$ and $f_{\text{mag}} = 0.1$ in this paper.

### B. Power spectrum of the FC induced by Pop III stars

In the ΛCDM model, Pop III stars are predicted to have formed inside of dark matter halos, allowing us to adopt the halo model to calculate $P_s$. For simplicity, we assume that Pop III stars form in halos with the virial temperature $T_{\text{vir}} > 10^4$ K where atomic hydrogen cooling is effective. We also assume that one Pop III star is formed per halo and consequently, one halo hosts one Pop III SN remnant. We write the power spectrum for FC from Pop III star SN remnants in the following way,

$$D^2(\alpha) P_s(k) = P_{\text{mat}}(k) \left[ \int_{M_{\text{thr}}} dM \frac{dn}{dM} b(M, z) \tilde{\alpha}_0 \right]^2, \tag{21}$$
where \( \alpha_0 \) is the growth factor analogous as shown in Eq. (A14), \( P_{\text{mat}} \) is the linear matter power spectrum, \( dn/dM \) is the mass function \( \alpha_0 \) is the Fourier component of \( \alpha_0 \) for one SN remnant and \( b(M, z) \) is the linear bias. Here, considering the halo correlation term, we neglect the one-halo Poisson term, which quantifies the correlation within a given halo. In this work, we are interested in large-scale CP \( (l < 2000) \) which justifies to neglect the correlation contribution within a given halo.

In Eq. (21), \( \nu_{\text{thr}} \) is the threshold mass of the halos hosting the Pop III stars and corresponds to the virial mass with \( T_{\text{vir}} = 10^4 \, \text{K} \) [45].

\[
M_{\text{thr}} = 3.5 \times 10^7 h^{-1} \left( \frac{T_{\text{vir}}}{10^4 \, \text{K}} \right)^{3/2} \left( \frac{\Omega_m}{\Omega_m(z) 18\pi^2} \right)^{-2} \left( \frac{1 + z}{10} \right)^{-3/2},
\]

where \( \mu \) is the mean molecular weight and \( m_p \) is the proton mass. For a fully ionized plasma, we used, \( \mu = 0.6 \).

We choose the mass function, \( dn/dM \), to be alternately defined in terms of a multiplicity function \( f(\nu) \) as,

\[
\frac{dn}{dM} dM = \frac{\bar{\rho}}{M} f(\nu) d\nu.
\]

Here \( \bar{\rho} \) is the mean matter density in the universe and \( \nu \) is defined as \( \nu = (\delta_c/\sigma_M)^2 \) where \( \delta_c \) is the threshold overdensity of spherical collapses at redshift \( z \) and \( \sigma_M \) is the rms linear density fluctuations obtained with a top-hat filter of mass \( M \) at an initial time (see [40] and references therein).

For the function \( f(\nu) \), we adopt the function proposed by Sheth and Tormen [46]

\[
\nu f(\nu) = A(1 + \nu_{\text{thr}}^p \nu^p) \left( \frac{\nu_1}{2\pi} \right)^{\frac{1}{2}} e^{-\nu_{\text{thr}}^p / 2},
\]

where \( \nu_1 = a \nu \) with \( a = 0.7, p = 0.3 \) and \( A \) is the normalization constant determined by \( \int f(\nu) d\nu = 1 \). Following [40], linear bias, \( b_\nu \), in terms of \( \nu_1 \) is given by

\[
b(\nu) = 1 + \frac{\nu - 1}{\delta_c (1 + \nu_1^p)}.
\]

In Fig. 1, we plot the power spectrum of \( \alpha, D(z)^2 P_\alpha(k) \). Since we consider only the halo-halo correlation term, as shown in Eq. (21), the spectral shape is same as the one of the linear matter power spectrum. However, in general, the one halo Poisson term is non-negligible on small scales and taking into account the one-halo Poisson term is required. In this case, the shape of the power spectrum \( D_\alpha^2 P_\alpha \) would deviate from the linear matter power spectrum around \( k > 10^2 \, \text{Mpc}^{-1} \), which also corresponds to the typical scale of the SN remnants of Pop III stars.

As the redshift decreases, the power spectrum amplitude grows. This is simply due to the growth of the density fluctuations, as a result of which the resultant number density of halos becomes larger at lower redshifts. Fig. 1 also shows that the power spectrum depends on the age of SN remnants \( t_{\text{age}} \) and the observation frequency \( \nu \).

### IV. RESULTS FOR THE PREDICTED \( C_{lVV} \) FROM POP III STARS

We numerically calculate the angular power spectrum of the CMB CP due to the SN remnants of Pop III stars, substituting the power spectrum \( P_\alpha \) obtained in the previous section to Eq. (11).

Although Eq. (11) involves multiple summations of multipoles, we can reduce the calculation. Due to the property of the Wigner-3j symbols in Eq. (A12), the non-zero contributions come from the terms with \( l' + L \) is odd in Eq. (11). Therefore, non-vanishing \( I_{lm} \) require \( m' \neq 0 \) in Eq. (11). Under the assumption that the CMB is statistically isotropic, the angular correlation of multipole components \( V_{lm} \) defined in Eq. (4) is independent of \( m \). Therefore the calculation with only \( m = 0 \) in Eq. (11) is enough to obtain the angular power spectrum. Additionally, in order to reduce the computation, we focus on the contributions with \( m'' = 0 \).

In Fig. 2, we present the results of \( T_{lMB}^2 C_{lVV} \) in the units of \( (\mu K)^2 \). Here we assume that Pop III stars exist in high redshifts between \( z = 24 \) and 17 and we set \( \nu = 30 \, \text{GHz}, f_{\text{mag}} = 0.1 \) and \( f_{\text{mag}} = 0.1 \). Although \( P_\alpha \) has a peak at \( k \sim 10^{-2} \) corresponding to \( l \sim 100 \), the angular power spectrum of the CMB CP \( C_{lVV} \) peaks at higher \( l \sim 2000 \). This is because \( C_{lVV} \) is the convolution between the \( P_\alpha \) and the E-mode power spectrum. We also show the dependence on \( t_{\text{age}} \) in Fig. 2. As the SN remnants expand with \( t_{\text{age}} \) increase, the number density of relativistic electrons and magnetic field energy decrease. Accordingly the FC becomes ineffective in the case with large \( t_{\text{age}} \). For comparison, we also plot the angular power spectrum with \( \nu = 1 \, \text{GHz} \). The spectrum is strongly sensitive to the frequency of the CMB observation. Although the frequency dependence depends on the power-law index \( \gamma \) of the relativistic electron distributions, the amplitude of \( C_{lVV} \) is proportional to \( \nu^{-\gamma} \) for the case with \( \gamma = 2 \).
FIG. 1: Power spectra of Faraday conversion generated by the First stars. The solid lines correspond to $t_{\text{age}} = 10^4$ years and $\nu = 30$ GHz. The dotted lines correspond to $t_{\text{age}} = 10^6$ years and $\nu = 30$ GHz. Lastly, the dashed lines correspond to $t_{\text{age}} = 10^6$ years and $\nu = 1$ GHz. Here, $\nu$ is the frequency of the CMB photons observed today.

FIG. 2: Angular power spectra of circular polarization from Eq. (11) times $T_{\text{CMB}}^2$ in $(\mu \text{K})^2$. We have chosen a few cases for the age of the remnant of the Pop III explosion, with $10^4 < t_{\text{age}} < 10^6$ in years. The frequency of observation of the CMB has been chosen to be $\nu = 1$ and 30 GHz. Other parameters for this plot are described in Sec. (IV).
FIG. 3: Angular power spectra of circular polarization from Eq. (11) times $T^2_{\text{CMB}}$ in ($\mu$K)^2. We have chosen a few redshift slices and the age of the remnant of the Pop III explosion to be $t_{\text{age}} = 10^4$ in years. The frequency of observation of the CMB has been chosen to be $\nu = 30$ GHz for this plot. Other parameters for this plot are described in Sec. (IV).

In Fig. (3), we show the redshift dependence of $C_{l}^{VV}$. The figure tells us that $C_{l}^{VV}$ decreases with increasing redshift. This is primarily due to the grow of the number density of Pop III stars in lower redshifts. In this paper, although we do not take into account the redshift dependence of the Pop III star properties, their properties strongly depend on redshift. In particular, the metal pollution due to explosions of Pop III stars make mass of Pop III stars small and, finally, the abundance of Pop III stars are dominated by Pop II stars. These redshift evolutions induce the suppression in the efficiency of the FC in lower redshifts. Therefore, $C_{l}^{VV}$ in lower redshifts is expected to lower than our estimation. However, this is beyond the scope of this paper.

V. DISCUSSION AND SUMMARY

In this paper, we have studied the CP signals of the CMB as it propagates through the Pop III star explosions. In the standard cosmology, the CMB coming from the last scattering surface only has a linear polarization and is not circularly polarized. However, this existing linear polarization gets converted into CP through the process of the FC as it passes through magnetized plasma medium with relativistic electrons.

We have derived an analytic form for the angular power spectrum of the CMB CP. This analytic form is general to the CP due to the FC. The angular power spectrum is the convolution between the CMB E-mode and the FC rate power spectra.

Using this analytic form, we have estimated the CP of the CMB due to SN remnants of Pop III stars. In order to calculate the FC rate due to a SN remnant, we have adopted the Sedov similar solution for the evolution of the SN remnant. We obtain the power spectrum of the FC rate based on the halo model. The obtained CP strongly depends on the observation frequency. With increasing frequency, the signals of the CP fall off. Since the SN remnant evolution suppresses the number density of relativistic electrons and magnetic field amplitude, the signals of the CP also decrease with $t_{\text{age}}$ growing. The CP also depends on the energy of the explosion. The angular power spectrum $C_{l}^{VV}$ is roughly proportional to $E_{SN}^{(16+2\gamma)/10}$ with the spectral index $\gamma$ of the relativistic electron distribution. For $\gamma = 2$, we retrieve an approximate quadratic dependence of $C_{l}^{VV}$ on $E_{SN}$.

Our evaluation in this paper uses a very simple model for the explosion and ambient medium of the Pop III stars. We have assumed the direction of the magnetic field generated by their explosions, to be aligned to the $\hat{z}$ axis. In other words, we have assumed $B_{\text{mag}} = B_{z}$. If we relax this assumption, and consider a general direction of $B_{\text{mag}}$, we may use $B_{z} = B_{\text{mag}}/\sqrt{3}$ according to the equipartition over the all directions. The signal of CP due to only $B_{z}$ is
then suppressed by a factor of 1/9. In this case, there are additional contribution due $B_x$ and $B_y$, the components of magnetic field in the $\hat{x}$ and $\hat{y}$ directions. These contributions could be additive or subtractive and a precise estimate is difficult without a more detailed numerical modeling.

Throughout this paper, we have assumed that energy of each Pop III explosion is $E_{SN} = 10^{53}$ ergs. However, $E_{SN}$ depends on various properties of Pop III stars and is theoretically uncertain. Because $C_2^{VV}$ is proportional to the square of $E_{SN}$ in our case with $\gamma = 2$, a reduction in the energy of explosion by a factor of 10 suppresses the CP signals by a factor of 100. This change is significant. We will address detailed modeling of the energy distribution of such explosions in a future work.

In Ref. [11], the signal of CP of the CMB due to the propagation through galaxy clusters is predicted to be $< 10^{-9}$ ($\mu K$)$^2$, for $\nu > 10$GHz. Our predicted signals due to the Pop III stars are much higher for even $\nu \sim 30$GHz and could be observable through near future CMB experiments. This is because the number density of the host minihalos is higher than that of galaxy clusters. In addition, the averaged $B_{mag}$ in post explosion remnant environment for the Pop III stars is much higher than the average galaxy cluster magnetic field, which is around a few $\mu$G.

We propose that if the future CMB experiments are equipped with CP measuring instruments, the CMB observation can also be used as a probe of the Pop III stars. An unique frequency signature of the CP signal due to the FC and absence of any significant foreground makes CP signal to be a promising probe of the Pop III stars, which are yet unobserved.

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Appendix A: Derivation of $C_ℓ^{VV}$

In this appendix, we derive the angular power spectrum of the CP of the CMB, Eq. (11). As shown in Eq. (6), the observed Stokes parameter $V$ due to the FC is given by

$$V(\hat{n}) = -2 \int_{r_*}^{0} dr \ U(r, \vec{x}, \hat{n}) \alpha(r, \vec{x}, \hat{n}, \hat{b}).$$  \hspace{1cm} (A1)

According to Eq. (12), $\alpha$ is proportional to $(\sin \theta_B)^{(\gamma+2)/2}$ where $\theta_B$ is the angle between the line-of-sight direction $\hat{n}$ and the direction of magnetic fields $\hat{b}$. In order to express the $\theta_B$-dependence explicitly, we rewrite $\alpha$ as

$$\alpha(r, \vec{x}, \hat{n}, \hat{b}) = \alpha_0(r, \vec{x})(\sin \theta_B)^{(\gamma+2)/2}. \hspace{1cm} (A2)$$

For simplicity, we consider only $z$-direction component of magnetic fields with adopting $\gamma = 2$. In this case, the $\theta_B$-dependence can be written as

$$(\sin \theta_B)^{(\gamma+2)/2} = \sin^2 \theta = \frac{1}{2} \sqrt{\frac{32\pi}{15}} (2Y_2^0(\hat{n}) - 2Y_2^0(\hat{n})), \hspace{1cm} (A3)$$

where $\theta$ is a polar angle component of $\hat{n}$ in a spherical coordinate system $\hat{n} = (\theta, \phi)$ for the sky. We perform the Fourier decomposition of $\alpha_0(r, \vec{x})$,

$$\alpha_0(r, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \tilde{\alpha}(r, k) \exp(\vec{k} \cdot \vec{x}). \hspace{1cm} (A4)$$

Accordingly, $\alpha(r, \vec{x}, \hat{n}, \hat{b})$ is expressed as

$$\alpha(z, \vec{x}, \hat{n}, \hat{b}) = 2\pi \sqrt{\frac{32\pi}{15}} \int \frac{d^3k}{(2\pi)^3} \tilde{\alpha}(z, k) \left(2Y_2^0(\hat{n}) - 2Y_2^0(\hat{n})\right) \sum_{l} \left((-i)^l j_l(kr) \sum_{m=-l}^{m=l} Y_{lm}(\hat{k}) Y_{lm}(\hat{n})\right), \hspace{1cm} (A5)$$

where we use the Rayleigh expansion,

$$e^{ik \cdot \vec{x}} = \sum_{lm} 4\pi (-i)^l j_l(kr) Y_l^m(\hat{k}) Y_l^m(\hat{n}), \hspace{1cm} (A6)$$

with $\hat{k} = \vec{k}/k$.

As for the Stokes parameter $U$, we can write from Eq. (10)

$$U(r, \vec{x}, \hat{n}) = \frac{i}{2} \int \frac{d^3k}{(2\pi)^3} \sum_{l} \sum_{m=-2}^{m=2} E_{lm} \left(2G^m_l - 2G^m_l\right). \hspace{1cm} (A7)$$

Substituting Eqs. (A5) and (A7) provides

$$V(\hat{n}) = -8\pi^2 i \sqrt{\frac{32\pi}{15}} \sum_{l} \sum_{m=-2}^{m=2} \sum_{m'=-l'}^{m'=l'} \sum_{m''=-l''}^{m''=l''} \int_{r_*}^{0} dr \ \int \frac{d^3k}{(2\pi)^3} \ \int \frac{d^3k'}{(2\pi)^3}$$

$$\times (-i)^{l+l'+l''} \sqrt{\frac{4\pi}{2l'+1}} \tilde{\alpha}(r, k') E_{lm}(k, r) j_{l'}(k'r) j_{l''}(kr) Y_{lm'}(\hat{k}) Y_{lm''}^*(\hat{k})$$

$$\times \left(2Y_2^m(\hat{n}) - 2Y_2^m(\hat{n})\right) \left(2Y_2^0(\hat{n}) + 2Y_2^0(\hat{n})\right) Y_{lm'}(\hat{n}) Y_{lm''}^*(\hat{n}). \hspace{1cm} (A8)$$

Decomposing the Stokes parameter $V$ to spherical harmonics as shown Eq. (4), we obtain

$$V_{lm} = -8\pi^2 i \sqrt{\frac{32\pi}{15}} \sum_{l'} \sum_{m'=-l'}^{m'=l'} \sum_{m''=-l''}^{m''=l''} \int_{r_*}^{0} dr \ \int \frac{d^3k}{(2\pi)^3} \ \int \frac{d^3k'}{(2\pi)^3}$$

$$\times (-i)^{l+l'+l''} \sqrt{\frac{4\pi}{2l'+1}} \tilde{\alpha}(r, k') E_{lm}(k, r) j_{l'}(k'r) j_{l''}(kr) Y_{lm'}(\hat{k}) Y_{lm''}^*(\hat{k})$$

$$\times \int d^2\hat{n} \left(2Y_2^m(\hat{n}) - 2Y_2^m(\hat{n})\right) \left(2Y_2^0(\hat{n}) + 2Y_2^0(\hat{n})\right) Y_{lm'}(\hat{n}) Y_{lm''}^*(\hat{n}) Y_{lm}(\hat{n}). \hspace{1cm} (A9)$$
Since the product of two spherical harmonics can be given in terms of $3j$ symbols, the non-zero contributions come from the terms including $2Y_l^m(\hat{n}) - 2Y_{l}^{0}(\hat{n})$ and $-2Y_l^m(\hat{n})Y_2^0(\hat{n})$ which are represented as

$$2Y_l^m(\hat{n}) - 2Y_{l}^{0}(\hat{n}) = \sum_{l} \sqrt{\frac{5(2l+1)(2L+1)}{4\pi}} \binom{l}{m} \binom{L}{0} \binom{l}{2} \binom{L}{2} \binom{L}{m} \binom{L}{m} \binom{L}{0} \binom{L}{0} Y_l^m(\hat{n})$$

$$-2Y_l^m(\hat{n})Y_2^0(\hat{n}) = \sum_{l} \sqrt{\frac{5(2l+1)(2L+1)}{4\pi}} \binom{l}{m} \binom{L}{0} \binom{l}{2} \binom{L}{2} \binom{L}{m} \binom{L}{m} \binom{L}{0} \binom{L}{0} Y_l^m(\hat{n}).$$ 

(A10)

Integrating Eq. (A9) over $\hat{n}$ yields

$$V_{lm} = -8\pi^2 i \sqrt{\frac{32\pi}{15}} \sum_{l'} \sum_{l''} \sum_{m''} \int d\mathbf{r} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \alpha(r,k')E_{l'm'}(r,k')j_{l''}(kr)j_{l'''}(kr)Y_{l''m''}^*(\hat{k})Y_{l'''}m''''(\hat{k})I_{lm}.$$ 

(A11)

Here $I_{lm}$ represents the integration over $\hat{n}$,

$$I_{lm} = \int d^2 \hat{n} \left( 2Y^m_l(\hat{n}) - 2Y^0_l(\hat{n}) \right) Y^m_{l'}(\hat{n})Y^m_{l''}(\hat{n})Y^m_{l'''}(\hat{n})Y^m_{l''''}(\hat{n}).$$

(A12)

where, for non-zero $I_{lm}$, $m'' + m'''$ should be $m$. Note that in the above expression, $m' = 0$ is not allowed since it requires $l' + l = \text{even}$, which is opposed by the first difference term in Eq. (A12) that requires $l' + l$ to be odd in order to have a non-vanishing difference term.

Let us calculate the angular power spectrum of $V_{lm}$, plugging Eq. (A11) to Eq. (5),

$$C_l = \frac{64\pi^2}{15} \sum_{l' l'' l'''} \sum_{m'''} \sum_{M'''} \sum_{m''} \sum_{M''} \int d\mathbf{r} \int dr' \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \int \frac{d^3K}{(2\pi)^3} \int \frac{d^3K'}{(2\pi)^3} \alpha(r,k')\alpha(r',k'')E_{l'm'}(r,k)E_{l'M'}(K,k'')j_{l''}(kr)j_{l'''}(kr')Y^*_{l''m''}(\hat{k})Y^*_{l'''}m'''(\hat{k})Y^*_{l'M'M'}(k')I_{lm}(l')I_{lm}(l')$$

(A13)

Here, for convenience, we decompose $\alpha(r,k')$ and $E_{lm}(r,k)$ as

$$\alpha(r,k) = D_\alpha(r) \alpha(k), \quad E_{lm}(r,k) = D_E(r) E_{lm}(k),$$

where $D_\alpha$ and $D_E$ are the growth functions, and we define the power spectra of $\alpha$ and $E_{lm}$ as

$$\langle \alpha(k)\alpha^*(k') \rangle = (2\pi)^3\delta^3(k-k')P_{\alpha}(k), \quad \langle E_{lm}(k)E^*_{lm}(k') \rangle = (2\pi)^3\delta^3(k-k')P_{E}(k).$$

(A15)

Since we are interested in the scales corresponding to $l > 100$, we can apply the Limber approximation to Eq. (A13),

$$\int k^2dk P(k)j_l(kr)j_l(k'r) \approx \frac{\pi\delta(r-r')}{2r^2} P(k)|_{k=l/r}.$$ 

(A16)
Therefore, finally, we obtain the angular power spectrum of the Stokes parameter $V$ as

$$\begin{align*}
C_l &= \frac{128}{15} \sum_{l''} \sum_{m''=2} \sum_{m'''} \int dr \int \frac{k^2 dk}{(2l''+1)r^2} \tilde{D}_a^2(r) D_B^2(r) P_2 \left( \frac{l''}{r} \right) P_{E_{lm}}(k) j_{l''}(kr) I_{lm}. \\
\text{(A17)}
\end{align*}$$

In our final calculation of $C_{VV}^l$, we have simplified the evaluation of $I_{lm}$ by using Eq. (B2) and Eq. (B4) of Ref. [47] for wigner-3js with zero azimuthal numbers. For the wigner-3js with non-zero azimuthal numbers, we have used Eq. (A5c) of Ref. [48] wherever appropriate.