Scalar wormholes with nonminimal derivative coupling

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Received 16 November 2011, in final form 4 March 2012
Published 28 March 2012
Online at stacks.iop.org/CQG/29/085008

Abstract

We consider static spherically symmetric wormhole configurations in a gravitational theory of a scalar field with a potential $V(\phi)$ and nonminimal derivative coupling to the curvature described by the term $(\epsilon g_{\mu\nu} + \kappa G_{\mu\nu})\phi^{,\mu}\phi^{,\nu}$ in the action. We show that the flare-out conditions providing the geometry of a wormhole throat could be fulfilled both if $\epsilon = -1$ (phantom scalar) and $\epsilon = +1$ (ordinary scalar). Supposing additionally a traversability, we construct numerical solutions describing traversable wormholes in the model with arbitrary $\kappa$, $\epsilon = -1$ and $V(\phi) = 0$ (no potential). The traversability assumes that the wormhole possesses two asymptotically flat regions with corresponding Schwarzschild masses. We find that asymptotic masses of a wormhole with nonminimal derivative coupling could be positive and/or negative depending on $\kappa$. In particular, both masses are positive only provided $\kappa < \kappa_1 \leq 0$; otherwise, one or both wormhole masses are negative. In conclusion, we give qualitative arguments that a wormhole configuration with positive masses could be stable.

PACS numbers: 04.20.--q, 04.20.Jb, 04.50.Kd

1. Introduction

Nonminimal generalizations of general relativity imply a straightforward coupling between matter fields and the spacetime curvature and play an important role in modern theoretical physics. The well-known example of nonminimal theories could be represented by scalar–tensor theories of gravity with the action generally given as\textsuperscript{4}

$$S = \int d^4x \sqrt{-g} [F(\phi, R) + K(\phi, X) + V(\phi)] + S_m.$$  \hspace{1cm} (1)

\textsuperscript{4} Throughout this paper we use units such that $G = c = 1$. The metric signature is $(- + + +)$, and the conventions for curvature tensors are $R^\mu_{\nu\rho\kappa} = \Gamma^\mu_{\rho\nu,\kappa} - \cdots$ and $R_{\mu\nu} = R^a_{\mu\nu}$. 

0264-9381/12/085008+12$33.00 © 2012 IOP Publishing Ltd Printed in the UK & the USA
where \( \phi \) is the scalar field, \( X = \phi_{\mu} \phi^{\mu} \), \( R \) is the scalar curvature and \( S_{\mu} \) is an action of ordinary matter (not including the scalar field). Here the function \( F(\phi, R) \) provides a nonminimal coupling between the scalar field \( \phi \) and the curvature, \( K(\phi, X) \) represents a generalized kinetic term and \( V(\phi) \) is a scalar field potential. Note that the theory (1) includes a lot of extensively investigated models; among them are the \( f(R) \) gravity and the Gauss–Bonnet gravity, the K-essence scalar theory, models with quintessence, quintom, phantom, dilaton, tachyon and so on.

A further extension of scalar–tensor theories is represented by models with nonminimal couplings between derivatives of the scalar field and the curvature. In general, one could have various forms of such couplings. For instance, in the case of four derivatives one could have the terms \( \kappa_1 R \phi_{\mu} \phi^{\mu}, \kappa_2 R_{\mu\nu\rho\sigma} \phi^{\mu} \phi^{\nu} \phi^{\rho} \phi^{\sigma}, \kappa_3 R \phi, \kappa_4 R_{\mu\nu} \phi \phi^{\mu} \phi^{\nu}, \kappa_5 R_{\mu} \phi \phi^{\mu} \) and \( \kappa_6 \Box^2 \phi^2 \), where the coefficients \( \kappa_1, \ldots, \kappa_6 \) are coupling parameters with dimensions of length-squared. However, as was discussed in [2–5], using total divergencies and without loss of generality one can keep only the first two terms.

As was shown by Amendola [2], a theory with derivative couplings cannot be recast into the Einsteinian form by a conformal rescaling \( \tilde{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu} \). He also supposed that an effective cosmological constant and then the inflationary phase can be recovered without considering any effective potential if a nonminimal derivative coupling is introduced. Amendola himself [2] investigated a cosmological model with the only derivative coupling term \( \kappa_2 R_{\mu\nu} \phi^{\mu} \phi^{\nu} \) and presented some analytical inflationary solutions. A general model containing \( \kappa_1 R \phi_{\mu} \phi^{\mu} \) and \( \kappa_2 R_{\mu\nu} \phi^{\mu} \phi^{\nu} \) has been discussed by Capozziello et al [3, 4]. They showed that the de Sitter spacetime is an attractor solution in the model.

Note that generally field equations in the model with terms \( \kappa_1 R \phi_{\mu} \phi^{\mu} \) and \( \kappa_2 R_{\mu\nu} \phi^{\mu} \phi^{\nu} \) contain higher (third) derivatives of the metric and the scalar field. However, as was shown in our work [5], the order of field equations reduces to second one in the particular case when the kinetic term is only coupled to the Einstein tensor, i.e. \( \kappa G_{\mu\nu} \phi^{\mu} \phi^{\nu} \). In [5, 6], we studied in detail the exact cosmological scenarios with a nonminimal derivative coupling \( \kappa G_{\mu\nu} \phi^{\mu} \phi^{\nu} \), examining both the quintessence and the phantom cases with zero and constant potentials. It is worth noting that taking into account the nonminimal derivative coupling reveals new interesting features in a cosmological behavior. In general, we found [5, 6] that the universe has two quasi-de Sitter phases and transits from one to another without any fine-tuned potential, determined only by the coupling parameter. Further investigations of cosmological models with nonminimal derivative coupling have been continued in [7–16].

In this paper, we will study static spherically symmetric wormhole configurations in the scalar–tensor theory with the nonminimal derivative coupling. The wormholes supported by a minimally coupled scalar field are well known in the literature (see, for example, [23–25]). It is also known that they have a number of features unacceptable with the physical point of view. In particular, they have negative asymptotical Schwarzschild masses and are unstable [25–28]. We will show that the model with nonminimal derivative coupling provides...
scalar wormholes of a new type, possessing positive asymptotical masses, and give some arguments on their stability.

2. Model with nonminimal derivative coupling

2.1. Action and field equations

Let us consider a gravitational theory of a scalar field $\phi$ with nonminimal derivative coupling to the curvature which is described by the following action:

$$ S = \int d^4x \sqrt{-g} \left\{ \frac{R}{8\pi} - \frac{\epsilon g_{\mu\nu} + \kappa G_{\mu\nu}}{4\pi} \phi_{,\mu} \phi_{,\nu} - 2V(\phi) \right\}, \quad (2) $$

where $V(\phi)$ is a scalar field potential, $g_{\mu\nu}$ is a metric, $g = \det(g_{\mu\nu})$, $R$ is the scalar curvature, $G_{\mu\nu}$ is the Einstein tensor and $\kappa$ is the derivative coupling parameter with the dimension of length-squared.

Varying action (2) with respect to the metric $g_{\mu\nu}$ leads to the gravitational field equations

$$ G_{\mu\nu} = \frac{8\pi}{\epsilon} \left[ \epsilon T_{\mu\nu} + \kappa \Theta_{\mu\nu} - \delta_{\mu\nu} V(\phi) \right], \quad (3) $$

with

$$ T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2, \quad (4) $$

$$ \Theta_{\mu\nu} = -\frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi R + 2 \nabla_\alpha \phi \nabla_\mu \phi R_{\alpha \nu} + \nabla^\alpha \phi \nabla^\beta \phi R_{\alpha \mu \nu \beta} + \nabla_\mu \nabla^\alpha \phi \nabla_\nu \phi - \nabla_\mu \nabla_\nu \phi \Box \phi $$

$$ - \frac{1}{2} (\nabla \phi)^2 G_{\mu\nu} + g_{\mu\nu} \left[ -\frac{1}{2} \nabla^\alpha \nabla^\beta \phi \nabla_\alpha \phi \nabla_\beta \phi + \frac{1}{4} (\Box \phi)^2 - \nabla_\alpha \phi \nabla_\beta \phi R_{\alpha \beta} \right]. \quad (5) $$

Similarly, variation of action (2) with respect to $\phi$ provides the scalar field equation of motion:

$$ \left[ \frac{\epsilon g^{\mu\nu} + \kappa G^{\mu\nu}}{4\pi} \right] \nabla_\mu \nabla_\nu \phi = V_\phi, \quad (6) $$

where $V_\phi \equiv dV(\phi)/d\phi$.

Note that due to the Bianchi identity $G_\mu^{\mu} = 0$ the right-hand side of equation (3) should obey the relation

$$ \left[ \left( \frac{\epsilon T_\nu^{\nu} + \kappa \Theta_\nu^{\nu}}{4\pi} \right) - \delta_\nu^{\nu} V(\phi) \right]_\mu = 0. \quad (7) $$

One can check straightforwardly that the substitution of expressions (4) and (5) into (7) yields the equation of motion of scalar field (6). Therefore, relation (7) will take place provided equation (6) is fulfilled. In other words, equation (6) is a differential consequence of equation (3).

2.2. Field equations for a static spherically symmetric configuration

Consider a static spherically symmetric configuration in theory (2). In this case, the spacetime metric can be taken as follows:

$$ ds^2 = -A(u) \, dt^2 + A^{-1}(u) \, dr^2 + r^2(u) \, d\Omega^2, \quad (8) $$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2$ is the linear element of the unit sphere, and $A(u)$ and $r(u)$ are the functions of the radial coordinate $u$. Also, the scalar field $\phi$ depends only on $u$, so that $\phi = \phi(u)$.

Now the field equations (3) and (6) yield

$$ 2 \frac{r''}{r} + \frac{A'}{A} r' + \frac{r'^2}{r} - \frac{1}{A r^2} = -4\pi \epsilon \phi^2 - 8\pi A^{-1} V(\phi) $$

$$ + 8\pi \kappa \phi^2 \left( \frac{3A' r'}{2r} + \frac{A r'^2}{2r^2} + \frac{1}{2r^2} + \frac{A r''}{r} \right) + 16\pi \kappa \phi^3 \phi' A r', \quad (9a) $$

3
\[
\frac{A'}{A} + \frac{r''}{r} - \frac{1}{A r^2} = 4\pi \epsilon \phi'^2 - 8\pi A^{-1} V(\phi) + 8\pi \kappa \phi'^2 \left(-\frac{1}{2r^2} + \frac{3A'}{2r^2} + \frac{3A'}{2r} \right),
\]

\[
\frac{1}{2} \frac{A''}{A} + \frac{r''}{r} + \frac{A'}{Ar} = -4\pi \epsilon \phi'^2 - 8\pi A^{-1} V(\phi)
\]

\[
+ 8\pi \kappa \phi'^2 \left(\frac{A'}{r} + \frac{1}{2} \frac{A''}{r^2} + \frac{1}{4} \frac{A'}{A} + \frac{1}{4} \frac{A''}{A} \right) + 8\pi \kappa \phi \phi'' \left(\frac{A'}{r} + \frac{1}{2} \frac{A'}{A} \right),
\]

where the prime means d/dr, and equations (9a), (9b) and (9c) are \(\phi\), \(\phi'\) and \(\phi''\) components of equation (3), respectively. Equations (9) represent a system of four ordinary differential equations of second order for three functions \(r(u), A(u)\) and \(\phi(u)\). As was mentioned above, equation (9d) is a differential consequence of equations (9a)–(9c). It is also worth noting that equations (9a), (9c) and (9d) are of second order, while equation (9b) is a first-order differential constraint for \(r(u), A(u)\) and \(\phi(u)\).

Combining the above equations one can easily rewrite them into a more compact form:

\[
\frac{r''}{r} = -4\pi \epsilon \phi'^2 + 4\pi \kappa A \left(\phi'^2 \frac{r'}{r} \right)' + 4\pi \kappa \phi'^2 \frac{1}{r^2},
\]

\[
(A'r^2)' = -16\pi \epsilon r^2 V + 4\pi \kappa (A'A^2 \phi'^2)' + 8\pi \kappa \phi'^2 (AA'r' + A^2 r'^2 - A),
\]

\[
A(r^2)'' - A'' r^2 = 2 + 4\pi \kappa [\phi'^2 (2A'r' - AA'r^3)]' + 8\pi \kappa A\phi'^2,
\]

\[
\epsilon (Ar^2 \phi')' + \kappa [\phi'(AA'r' + A^2 r'^2 - A)]' = r^2 V_\phi.
\]

3. The wormhole solution

In this section, we will focus our attention on wormhole solutions of the field equations (9), or equivalently (10), obtained in the previous section. Note that equations (9) are a rather complicated system of nonlinear ordinary differential equations, and we do not know if it is possible to find any exact analytical solution to this system. Instead, we will construct wormhole solutions numerically, studying previously their asymptotical properties near and far from the wormhole throat.

To describe a traversable wormhole, metric (8) should possess a number of specific properties. In particular, (i) the radial coordinate \(u\) runs through the domain \((-\infty, +\infty)\); (ii) there exist two asymptotically flat regions \(R_{\pm}: u \to \pm \infty \) connected by the throat; (iii) \(r(u)\) has a global positive minimum at the wormhole throat \(u = u_0\); without loss of generality one can set \(u_0 = 0\), so that \(r_0 = \min[r(u)] = r(0)\) is the throat radius; (iv) \(A(u)\) is everywhere positive and regular, i.e. there are no event horizons and singularities in the spacetime. Taking into account necessary conditions for the minimum of function, we also obtain

\[
r_0' = 0, \quad r_0'' > 0,
\]

\[10\]
where the subscript ‘0’ means that values are calculated at the throat \( u = 0 \). Concerning wormholes, the above relations are known as the flare-out conditions\(^8\).

### 3.1. Initial condition analysis

Let us consider the field equations at the throat \( u = 0 \). By assuming \( r_0' = 0 \), equations (9a) and (9b) after a little algebra yield

\[
\begin{align*}
\frac{1}{r_0'} &= -\frac{4\pi (\epsilon A_0 \phi_0^2 - 2V_0)}{1 - 4\pi \kappa A_0 \phi_0^2}, \\
\frac{r_0''}{r_0'} &= \frac{4\pi \phi_0^2 (\epsilon - 8\pi \kappa V_0)}{(1 - 4\pi \kappa A_0 \phi_0^2)^2}.
\end{align*}
\]

To provide the flare-out conditions (11), the right-hand sides of equations (12) and (13) should be positive. This is possible if \( \phi_0^2 \neq 0 \) and the following inequalities take place:

\[
\begin{align*}
\epsilon A_0 \phi_0^2 - 2V_0 &< 0, \\
\epsilon - 8\pi \kappa V_0 &< 0.
\end{align*}
\]

For given \( \epsilon \) and \( \kappa \), these inequalities give restrictions for initial values of \( A_0 \phi_0^2 \) and \( V_0 = V(\phi_0) \) at the throat. Let us consider separately various cases.

(1) \( \kappa = 0 \). This is the case of a minimally coupled scalar field. Now (15) yields \( \epsilon < 0 \), i.e. \( \epsilon = -1 \), for any \( V_0 \). In turn, if \( \epsilon = -1 \) equation (14) is fulfilled provided \( V_0 > \frac{1}{4}A_0 \phi_0^2 \).

Thus, we have obtained a well-known result that a solution with the throat in general relativity with a minimally coupled scalar field is permitted only for phantom fields with negative kinetic energy (see, for example, [29]).

Furthermore, we will consider cases with nonminimal derivative coupling.

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\(^8\) Answer to referee’s remark. Let us construct an embedding function. With this aim we consider a spacetime section \( t = \) const, \( \theta = \pi/2 \):

\[
dz^2 = \frac{du^2}{A(u)} + r^2(u) \, d\phi^2.
\]

We assume that \( r(u) \) is decreasing if \( u \) runs from \(-\infty\) to \( u_0 \) and increasing if \( u \) runs from \( u_0 \) to \( \infty \), and \( r_0 = r(u_0) > 0 \) is the minimum of \( r(u) \). Let us define a new radial coordinate \( r_\pm = r(u) \) separately in the regions \( \mathcal{R}_- : u < u_0 \) and \( \mathcal{R}_+ : u > u_0 \). Since \( r(u) \) is monotonic in each region, we can, in principle, obtain \( u = u(r_\pm) \) and \( du = u'(r_\pm) \, dr_\pm = dr_\pm/r_\pm'(u) \, dr(u) \). Now we obtain

\[
dz^2 = \frac{dr_\pm^2}{A(u) r_\pm^2(u)} + r_\pm^2 \, d\phi^2,
\]

where \( u = u(r_\pm) \). To construct an embedding function, we consider the Euclidean space with the metric given in cylindrical coordinates:

\[
dz^2 = dz^2 + dr^2 + r^2 \, d\phi^2.
\]

The embedding function is defined as \( z = z(r) \). An induced metric on the cylindrical surface \( z = z(r) \) reads

\[
\begin{align*}
dr^2 &= (1 + z'^2(r)) \, dr^2 + r^2 \, d\phi^2. \\
Comparing the induced metric with the metric of the spacetime section in each separate region \( \mathcal{R}_\pm \), we find 1 + \( z'^2(r) = 1/A(u) r_\pm^2(u) \) or

\[
\begin{align*}
z'(r) &= \pm \left[ \frac{1}{A(u) r_\pm^2(u)} - 1 \right]^{1/2}.
\end{align*}
\]

Note that \( A(u) \) is everywhere positive and regular, and \( r'(u_0) = 0 \) because of the throat conditions. Therefore, \( z'(r) \) turns out to be infinite at \( r = r_0 \), or, equivalently, \( u = u_0 \). Graphically, this means that the tangent to the line \( z = z(r) \) becomes vertical. Finally, rotating the line \( z = z(r) \) around the axis \( z \) we can obtain the embedding diagram for wormhole geometry.
(2) $\epsilon = 1, \kappa > 0$. Constraints (14) and (15) give

(A) $A_0\phi_0^2 < \frac{1}{4\pi \kappa}$, \hspace{1cm} $V_0 > \frac{1}{8\pi \kappa}$,

(B) $A_0\phi_0^2 > \frac{1}{4\pi \kappa}$, \hspace{1cm} $V_0 < \frac{1}{8\pi \kappa} < 2A_0\phi_0^2$.

Thus, in the case $\kappa > 0$, $\epsilon = 1$ there are domains of initial values $A_0\phi_0^2$ and $V_0$ which provide the flare-out conditions (11). Stress also that the initial value $V_0$ should be necessarily positive at the throat.

(3) $\epsilon = 1, \kappa < 0$. It is easy to check that equations (14) and (15) are inconsistent in this case, and hence the flare-out conditions (11) are not fulfilled.

(4) $\epsilon = -1, \kappa > 0$. Equations (14) and (15) give

$$A_0\phi_0^2 < \frac{1}{4\pi \kappa}, \hspace{1cm} V_0 > -\frac{1}{2}A_0\phi_0^2. \hspace{5cm} (16)$$

(5) $\epsilon = -1, \kappa < 0$. Equations (14) and (15) give

$$A_0\phi_0^2 > 0, \hspace{1cm} -\frac{1}{2}A_0\phi_0^2 < V_0 < \frac{1}{8\pi |\kappa|}. \hspace{5cm} (17)$$

Thus, in case $\epsilon = -1$ there are domains of initial values $A_0\phi_0^2$ and $V_0$ which provide the flare-out conditions (11). It is worth noting that the value $V_0 = 0$ is admissible, and so one may expect to obtain a wormhole solution without a potential.

In the model with nonminimal derivative coupling there is a nontrivial case $\epsilon = 0$, when the free kinetic term is absent. Let us also consider this case.

(6) $\epsilon = 0, \kappa > 0$. Equations (14) and (15) give

$$A_0\phi_0^2 < \frac{1}{4\pi \kappa}, \hspace{1cm} V_0 > 0.$$

Thus, in this case there are domains of initial values $A_0\phi_0^2$ and $V_0$ which provide the flare-out conditions. As well as for $\epsilon = 1$ and $\kappa > 0$, the initial value $V_0$ should be necessarily positive at the throat.

(7) $\epsilon = 0, \kappa < 0$. Equations (14) and (15) are inconsistent.

Summarizing, we can conclude that the flare-out conditions (11) in the model with nonminimal derivative coupling can be fulfilled for various values of $\epsilon$ and $\kappa$. Respectively, the flare-out conditions provide an existence of solutions with the throat. It is especially worth noting that the throat in the model with nonminimal derivative coupling can exist not only if $\epsilon = 1$ (phantom case), but also if $\epsilon = 1$ (normal case) and $\epsilon = 0$ (no free kinetic term).

To finish the analysis of the field equations (9) at the throat, let us consider the metric function $A(r)$ and its first and second derivatives at $u = 0$. The value $A_0$ is a free parameter. Although $A'_0$ is also free, we assume, just for simplicity, $A'_0 = 0$. Note that in this case $A(u)$ has an extremum at the throat $u = 0$. Using equation (10c), we can find

$$A_0'' = -\frac{8\pi \kappa A_0\phi_0^2 + 2r_0^2 V_0}{r_0^4 - 1 - 4\pi \kappa A_0\phi_0^2}. \hspace{5cm} (18)$$

The sign of $A_0''$ determines a kind of the extremum of $A(u)$; it is a maximum if $A_0'' < 0$, and a minimum if $A_0'' > 0$. It is worth noting that, with the physical point of view, the maximum (minimum) of $A(r)$ corresponds to the maximum (minimum) of the gravitational potential. In turn, the gravitational force is equal to zero at extrema of the gravitational potential; moreover, in the vicinity of maximum (minimum) the gravitational force is repulsive (attractive). As a consequence, whether the throat is repulsive or attractive depends on the sign of $A_0''$.

As an example, let us consider the model with $\epsilon = -1$. By using relations (16) and (17), we can see that $A_0'' < 0$ if $\kappa > 0$ and $A_0'' > 0$ if $\kappa < 0$. Hence, the throat is repulsive in the first case and attractive in the second one.
3.2. Asymptotical analysis

While the throat is an essential feature of the wormhole geometry, its asymptotical properties could be varied for different models. Traversable wormholes are usually assumed possessing two asymptotically flat regions connected by the throat, and in this paper, we will look for wormhole solutions with an appropriate asymptotical behavior.

The spacetime with metric (8) has two asymptotically flat regions \( R_\pm \): \( u \to \pm \infty \) provided \( \lim_{u \to \pm \infty} |r(u)/|u|| = \delta_\pm \) and \( \lim_{u \to \pm \infty} A(u) = A_\pm \). Since a flat spacetime is necessarily empty, we also have to suppose that \( \lim_{u \to \pm \infty} \phi(u) = \phi_\pm \) and \( \lim_{u \to \pm \infty} V(\phi(u)) = V(\phi_\pm) = 0 \). Assume the following asymptotics at \( |u| \to \infty \):

\[
\begin{align*}
    r(u) & = \delta_\pm |u| \left[ 1 + \frac{\alpha_\pm}{|u|} + O(u^{-2}) \right], \\
    A(u) & = A_\pm \left[ 1 - \frac{\beta_\pm}{|u|} + O(u^{-2}) \right], \\
    \phi(u) & = \phi_\pm \left[ 1 - \frac{\gamma_\pm}{|u|} + O(u^{-2}) \right], \\
    V(\phi(u)) & = O(u^{-5}).
\end{align*}
\]

Substituting the above expressions into equation (9b) and collecting leading terms gives

\[
A_\pm = \delta_\pm^{-2}.
\]

Thus, the asymptotical form of metric (8) is

\[
ds^2 = -\delta_\pm^{-2} \left( 1 - \frac{\beta_\pm}{|u|} \right) dt^2 + \delta_\pm^2 \left( 1 - \frac{\beta_\pm}{|u|} \right)^{-1} dr^2 + \delta_\pm^2 u^2 \left( 1 + \frac{\alpha_\pm}{|u|} \right)^2 d\Omega^2.
\]

Introducing in the asymptotically flat regions \( R_\pm \) new radial coordinates

\[
\rho_\pm = \delta_\pm |u| \left[ 1 + \frac{\alpha_\pm}{|u|} \right]
\]

and neglecting terms \( O(\rho_\pm^{-2}) \), we obtain

\[
ds^2 = -\left( 1 - \frac{2m_\pm}{\rho_\pm} \right) \rho_\pm^2 dt^2 + \left( 1 - \frac{2m_\pm}{\rho_\pm} \right)^{-1} \rho_\pm^2 + \rho_\pm^2 d\Omega^2,
\]

where \( 2m_\pm = \delta_\pm \beta_\pm t_\pm = \delta_\pm^{-1} t_\pm \). This is nothing but two Schwarzschild asymptotics at \( u \to \pm \infty \) with masses \( m_\pm \). Taking into account that \( \delta_\pm |u| = \lim_{u \to \pm \infty} r(u) \) and \( \delta_\pm = \pm \lim_{u \to \pm \infty} r(u) \), we can find the following asymptotical formula:

\[
m_\pm = \lim_{u \to \pm \infty} [r(u)(1 - r^2(u)A(u))].
\]

3.3. The exact wormhole solution with \( \kappa = 0 \)

Let us discuss the particular case \( \kappa = 0 \) (no nonminimal derivative coupling). In this case, the system of field equations (10) reduces to well-known equations for a minimally coupled scalar field:

\[
\begin{align*}
    \frac{r''}{r} & = -4\pi \epsilon \phi^2, \\
    (A' r^2)' & = -16\pi r^2 V,
\end{align*}
\]
\[A(r^2)′′ - A′r^2 = 2,\]  
\[\epsilon (Ar^2φ')' = r^2V_φ.\]  
(27c)  
(27d)

Supposing additionally \(\epsilon = -1\) (phantom scalar) and \(V = 0\) (no potential term), one can find an exact wormhole solution to the system (27) (see [23, 24]). Adopting the result of [25], we can write down the solution as follows:

\[dx^2 = -e^{2\lambda(u)}\, dr^2 + e^{-2\lambda(u)} \left[ du^2 + (u^2 + u_0^2)\, d\Omega^2 \right],\]  
\[φ(u) = \left(\frac{m^2 + u_0^2}{4\pi m^2}\right)^{1/2} \lambda(u),\]  
(28)  
(29)

where \(\lambda(u) = (m/u_0) \arctan(u/u_0)\) and \(m\) and \(u_0\) are two free parameters. Taking into account the following asymptotical behavior:

\[e^{2\lambda(u)} = \exp \left(\frac{\pm \pi m}{u_0} \right) \left[ 1 - \frac{2m}{u} \right] + O(u^{-2})\]

in the limit \(u \to \pm \infty\), we may see that the spacetime with metric (28) possesses two asymptotically flat regions. These regions are connected by the throat whose radius corresponds to the minimum of the radius of a two-dimensional sphere, \(r^2(u) = e^{-2\lambda(u)}(u^2 + u_0^2)\). The minimum of \(r(u)\) is achieved at \(u = m\) and is equal to

\[r_0 = \exp \left(-\frac{m}{u_0} \arctan\frac{m}{u_0}\right) (u^2 + u_0^2)^{1/2}.\]

Asymptotical masses, corresponding to \(u \to \pm \infty\), are

\[m_\pm = \pm m \exp(\pm \pi m/2u_0).\]

It is worth noting that the masses \(m_\pm\) have both different values and different signs, and so wormholes supported by the minimally coupled scalar field inevitably have a negative mass in one of the asymptotical regions.

Note also that there is a particularly simple case \(m = 0\) when the static solutions (28) and (29) reduce to

\[dx^2 = -dr^2 + du^2 + (u^2 + u_0^2)\, d\Omega^2,\]  
\[φ(u) = (4\pi)^{-1/2} \arctan(u/u_0).\]  
(30)  
(31)

In this case both asymptotical masses are equal to zero, \(m_\pm = 0\), and so such wormhole is massless.

### 3.4. Numerical analysis

In this section, we present results of the numerical analysis of the field equations (9). Note that in order to realize a numerical analysis into practice one first needs to specify a form of the potential \(V(φ)\). The requirement of asymptotical flatness dictates \(\lim_{u \to \pm \infty} V(φ(u)) = V(φ_\pm) = 0\). The simplest choice obeying this asymptotical behavior corresponds to the zero potential, and hereafter we will assume \(V(φ) ≡ 0\). As the initial condition analysis has shown, the flare-out conditions with \(V_0 = 0\) are only fulfilled in the case of \(\epsilon = -1\).

The initial conditions for the system of second-order ordinary differential equations (9) read \(u = 0, \ r(0) = r_0, \ r'(0) = r'_0, \ A(0) = A_0, \ A'(0) = A'_0, \ φ(0) = φ_0\) and \(φ'(0) = φ'_0\). Without loss of generality, one can set \(A_0 = 1\) and \(φ_0 = 0\). Since \(u = 0\) is assumed to be a
wormhole throat, one obtains \( r_0' = 0 \). Now the throat’s radius \( r_0 \) given by equation (12) can be found as

\[
r_0 = \sqrt{\frac{1 - 4\pi \kappa \phi_0'^2}{4\pi \phi_0'^2}}.
\]

(32)

Then the only two free parameters determining the initial conditions remain: \( \Lambda_0' \) and \( \phi_0' \).

First, let us consider the case \( \Lambda_0' = 0 \). In figure 1, we represent numerical solutions for \( r(u) \), \( A(u) \) and \( \phi(u) \) for various values of \( \kappa \). Note that both \( r(u) \) and \( A(u) \) are even functions possessing an extremum at the throat \( u = 0 \); \( r(u) \) has a minimum due to the flare-out conditions, and, as was mentioned above, \( A(u) \) has a maximum if \( \kappa > 0 \) and a minimum if \( \kappa < 0 \). In the case of \( \kappa = 0 \) one obtains \( A(u) = 1 \); this coincides with the analytical result (30). The function \( \phi(u) \) is odd; it smoothly varies between two asymptotical values \( -\phi_+ \) and \( \phi_+ \), where \( \phi_+ = \lim_{u \to -\infty} \phi(u) \). The functions \( r(u) \) and \( A(u) \) given in figure 1 also possess a proper asymptotical behavior: \( \lim_{u \to \pm \infty} r(u) = \delta_\pm |u| \) and \( \lim_{u \to \pm \infty} A(u) = \delta_\pm^{-2} \). In the case of \( \Lambda_0' = 0 \), we have \( \delta_- = \delta_+ = \delta \), and the value of \( \delta \) depends generally on \( \kappa \), i.e. \( \delta = \delta(\kappa) \).

The asymptotical Schwarzschild masses \( m_\pm \) corresponding to the numerical solutions \( r(u) \) and \( A(u) \) are shown in figure 3. Because of the symmetry \( u \leftrightarrow -u \) we have \( m_- = m_+ = m \).

Moreover, \( m \) is positive if \( \kappa < 0 \), negative if \( \kappa > 0 \) and \( m = 0 \) for \( \kappa = 0 \). Note also that the wormhole solutions exist only for \( \kappa < \kappa_{\text{max}} \) and \( m \to -\infty \) if \( \kappa \to \kappa_{\text{max}} \).

The numerical solutions for \( r(u) \) and \( A(u) \) in the case of \( \Lambda_0' \neq 0 \) are shown in figure 2. It is worth noting that in this case both \( r(u) \) and \( A(u) \) have different asymptotics at \( u \to \pm \infty \): \( \lim_{u \to \pm \infty} r(u) = \delta_\pm |u| \) and \( \lim_{u \to \pm \infty} A(u) = \delta_\pm^{-2} \), where \( \delta_- \neq \delta_+ \). As a consequence, we obtain a wormhole configuration with two different asymptotical masses \( m_\pm \). The value of \( m_\pm \) depends on \( \kappa \); this dependence is shown in figure 3. Note that for \( \kappa < \kappa_1 \) both \( m_+ \) and \( m_- \) are positive, for \( \kappa > \kappa_2 \) both \( m_+ \) and \( m_- \) are negative, and for \( \kappa_1 < \kappa < \kappa_2 \) (in particular, for \( \kappa = 0 \)) the asymptotical masses \( m_\pm \) have opposite signs. Note also that \( \kappa < \kappa_{\text{max}} \) and \( m_\pm \to -\infty \) if \( \kappa \to \kappa_{\text{max}} \).

Note that a qualitative behavior of numerical solutions for \( r(u) \), \( A(u) \) and \( \phi(u) \) does not depend on the initial value \( \phi_0' \), and the above results were obtained for the specific choice \( \phi_0' = 0.1 \).
Figure 2. Graphs for $\delta^{-1} r(u)$, $\delta^2 A(u)$ and $\phi(u)$ are constructed for the initial values $A'_0 = 0.05$, $\phi'_0 = 0.1$ and $\kappa = -40, -20, -10, -5, -1, 0, 1, 2, 3$ (up–bottom for $r(u)$, bottom–up for $A(u)$, bottom–up for the right branch of $\phi(u)$); $\delta = \delta_+ = \lim_{u \to \infty} r(u)/u$ ($\delta_+ \neq \delta_-$). The bold lines correspond to $\kappa = 0$ (no nonminimal derivative coupling).

Figure 3. Asymptotical masses $m_{\pm}$ versus $\kappa$ for $A'_0 = 0$ and $A'_0 = 0.05$. Note that $m_{-} = m_{+}$ if $A'_0 = 0$. In the case of $A'_0 = 0.05$, the masses $m_{-}$ and $m_{+}$ are represented by the upper and lower graphs, respectively; $m_{-}(k_1) = m_{+}(k_2) = 0$.

4. Summary and discussion

We have constructed static spherically symmetric wormhole configurations in a gravitational theory of a scalar field with nonminimal derivative coupling to the curvature. Carrying out a local analysis of the flare-out conditions, we have shown that the solutions with a throat in theory (2) with a nonzero derivative coupling parameter $\kappa \neq 0$ could exist for all possible values of $\epsilon = -1, 0, +1$; in the case of $\epsilon = -1$, the value $V_0 = 0$ is admissible, while in the case of $\epsilon = +1$ and $\epsilon = 0$ the initial value $V_0$ should be necessarily positive at the throat. For comparison, it is worth noting that the solutions with a throat in the minimally coupling model with $\kappa = 0$ are forbidden for an ordinary scalar field with $\epsilon = +1$ [29].

Note that the flare-out condition does not guarantee by itself the existence of asymptotic regions, and so it is a necessary but not sufficient condition for a wormhole solution to exist. Assuming additionally an asymptotical flatness, we have found numerical solutions describing
traversable (Lorentzian) wormholes in the particular case of $V(\phi) \equiv 0$ (no potential) and $\epsilon = -1$.\(^9\)

The wormhole solutions constructed in the paper could be classified by their asymptotic behavior which, in turn, is determined by asymptotics of the metric functions $r(u)$ and $A(u)$ at $u \to \pm \infty$. If $A'_0 = 0$, then both $r(u)$ and $A(u)$ are even, and so a wormhole configuration is symmetrical relative to the throat $u = 0$. In this case, both asymptotical masses $m_{\pm}$ are equal, i.e. $m_- = m_+ = m$, and one has the following cases: (i) $m > 0$ if $\kappa < 0$; (ii) $m = 0$ if $\kappa = 0$; (iii) $m < 0$ if $\kappa > 0$. In the case of $A'_0 \neq 0$, a wormhole configuration has no symmetry relative to the throat; hence, asymptotical masses $m_{\pm}$ are different, i.e. $m_- \neq m_+$, and one obtains (i) $m_- > 0, m_+ > 0$ if $\kappa < \kappa_1 < 0$; (ii) $m_- \leq 0, m_+ > 0$ if $\kappa_1 < \kappa < \kappa_2$; (iii) $m_- < 0, m_+ \leq 0$ if $\kappa > \kappa_2 > 0$. Thus, depending on $\kappa$, a wormhole could possess positive and/or negative asymptotical Schwarzschild masses. It is worth emphasizing that both masses are positive only provided $\kappa < \kappa_1$ (for a symmetrical wormhole configuration one has $\kappa_1 = \kappa_2 = 0$), otherwise one or both wormhole masses are negative. For example, let us consider the case $\kappa = 0$, when the scalar field is minimally coupled to the curvature. In this case, well-known wormhole solutions have been analytically obtained by Ellis $[23]$ and Bronnikov $[24]$ (see the discussion in section 3). Such wormholes inevitably possess a negative Schwarzschild mass in one of the asymptotical regions, or, in the particular case of a symmetric wormhole configuration, both asymptotical masses are equal to zero. (Note that our numerical calculations given in figures 1–3 completely reproduce this particular analytical result.)

The stability of wormhole configurations is an important test of their possible viability. The stability of wormholes supported by phantom scalar fields was intensively investigated in the literature $[25–28]$, and the final resolution states that both static and nonstatic (see $[25]$) scalar wormholes are unstable. Although this result is technically complicated, there is a simple qualitative explanation of this instability. Actually, as was mentioned above, a scalar wormhole inevitably possesses a negative Schwarzschild mass in one of the asymptotical regions; for example, let it be $\mathcal{R}_- : u \to -\infty$. This means that the gravitational potential is decreasing and the gravitational force is repulsive far from the throat. Consider now a small scalar perturbation of the wormhole geometry localized near the throat. Such a perturbation shall play a role of a small bunch of energy density and, because of the repulsive character of the gravitational field, it shall be pushed to the infinity $\mathcal{R}_-$. Similarly, any scalar perturbations will propagate from the throat vicinity to infinity, and this indicates an instability of the throat.

In comparison with wormholes supported by a phantom scalar field minimally coupled to the curvature, the scalar wormholes with nonminimal derivative coupling obtained in this paper have a more general asymptotic behavior. Namely, depending on a value of the nonminimal derivative coupling parameter $\kappa$ one of the following qualitatively different cases is realized: (i) one of the asymptotic wormhole masses or both of them are negative and (ii) both asymptotic masses are positive. Taking into account the previous qualitative consideration, one can expect that a wormhole configuration will be unstable in the first case and stable in the second one. Actually, if both asymptotic masses are positive, then the gravitational potential is increasing and the gravitational field is attractive on both sides of the wormhole throat. In this case, all scalar perturbations should be localized in the vicinity of the throat, and this would provide a stability.

Of course, it is necessary to emphasize that the above consideration of wormhole stability has only a qualitative character. To answer finally the question—are scalar wormholes with

\(^9\) In the case of $\epsilon = +1$ or $\epsilon = 0$, one needs to consider a more general form of the potential $V(\phi)$, such that $V(0) = 0$ and $V(\phi_{\pm}) = 0$, where $\phi_{\pm} = \lim_{u \to \pm \infty} \phi(u)$. This case is much more complicated for numerical analysis, and we have not yet obtained any systematic results. For this reason, here we have restricted ourselves to the case of $V = 0$, $\epsilon = -1$.\)
nonminimal derivative coupling stable or not?—we need additional investigations, which are in progress.

Acknowledgments

The work was supported in part by the Russian Foundation for Basic Research grants no. 11-02-01162. SS appreciates Douglas Singleton and California State University Fresno for hospitality during the Fulbright scholarship visit.

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