Clifford algebra-parametrized octonions and generalizations

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Abstract

Introducing products between multivectors of \( \mathcal{C}l_{0,7} \) (the Clifford algebra over the metric vector space \( \mathbb{R}^{0,7} \)) and octonions, resulting in an octonion, and leading to the non-associative standard octonionic product in a particular case, we generalize the octonionic \( X \)-product, associated with the transformation rules for bosonic and fermionic fields on the tangent bundle over the 7-sphere \( S^7 \), and the \( XY \)-product. This generalization is accomplished in the \( u \)- and \( (u,v) \)-products, where \( u, v \in \mathcal{C}l_{0,7} \) are fixed, but arbitrary. Moreover, we extend these original products in order to encompass the most general — non-associative — products \( (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times \mathcal{C}l_{0,7} \rightarrow \mathbb{R} \oplus \mathbb{R}^{0,7} \), \( \mathcal{C}l_{0,7} \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \rightarrow \mathbb{R} \oplus \mathbb{R}^{0,7} \) and \( \mathcal{C}l_{0,7} \times \mathcal{C}l_{0,7} \rightarrow \mathbb{R} \oplus \mathbb{R}^{0,7} \). We also present the formalism necessary to construct Clifford algebra-parametrized octonions, which provides the structure to present the \( \mathbb{O}_{1,u} \) algebra. Finally we introduce a method to construct \( \mathbb{O} \)-algebras endowed with the \( (u,v) \)-product from \( \mathbb{O} \)-algebras endowed with the \( u \)-product. These algebras are called \( \mathbb{O} \)-like algebras and their octonionic units are parametrized by arbitrary Clifford multivectors. When \( u \) is restricted to the underlying paravector space \( \mathbb{R} \oplus \mathbb{R}^{0,7} \hookrightarrow \mathcal{C}l_{0,7} \) of the octonion algebra \( \mathbb{O} \), these algebras are shown to be isomorphic. The products between Clifford multivectors and octonions, leading to an octonion, are shown to share graded-associative, supersymmetric properties. We also investigate the generalization of Moufang identities, for each one of the products introduced.

Key words: Clifford algebras, octonions, graded-associative algebras.

MSC classification: 15A66, 17A35, 17C60, 81T60.

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1 Introduction

The $X$-product was originally introduced in order to correctly define the transformation rules for bosonic [vector] and fermionic [spinor] fields on the tangent bundle over the 7-sphere $S^7$. This product is closely related to the parallel transport of sections of the tangent bundle, at $X \in S^7$, i.e., $X \in \mathcal{O}$ such that $XX = X\bar{X} = 1$. The $X$-product is also shown to be twice the parallelizing torsion [2], given by the torsion tensor. This tensor is not constant due to the non-associativity of the octonion algebra $\mathcal{O}$, non-vanishing due to the non-commutativity of $\mathcal{O}$, and in particular, it is used to investigate the $S^7$ Kać-Moody algebra [11, 13]. The $X$-product has also been used to obtain triality maps and $G_2$ actions [4, 5], and it leads naturally to remarkable geometric and topological properties, for instance the Hopf fibrations $S^3 \cdots S^7 \to S^4$ and $S^8 \cdots S^{15} \to S^7$ [6, 7], and twistor formalism in ten dimensions [2]. The paramount importance of octonions in the search for unification is based, for instance, in the fact that by extending the division-algebra-valued superalgebras to octonions, in $D = 11$ an octonionic generalized Poincaré superalgebra can be constructed, the so-called octonionic $M$-algebra that describes the octonionic $M$-theory [5], where the octonionic super-2-brane and the octonionic super-5-brane sectors are shown to be equivalent [9]. Also, there are other vast generalizations and applications of the octonionic formalism [10, 11] such as the classification of quaternionic and octonionic spinors [12] and the pseudo-octonionic formalism [13].

This paper is intended to generalize the $X$- and $XY$-products, and to provide formal prerequisites to define new Clifford algebra-parametrized octonions associated with the $(1,u)$-product, besides extending the original Moufang identities to the new products to be defined, when it is possible. In this sense, we produce a copy of $\mathcal{O}$, but endowed with the $(1,u)$-product, instead of the standard octonionic product. Moreover we define and investigate the fundamental properties of the $(1,u)$-, $u$- and $(u,v)$-products, $u,v \in \mathcal{C}_0,7$, which naturally generalize the $X$- and $XY$-products, $X,Y \in \mathcal{O}$ such that $XX = X\bar{X} = 1 = YY = Y\bar{Y}$, i.e., $X,Y \in S^7$. Octonionic products between Clifford algebra multivectors that results in an octonion are also introduced, and the Moufang identities are shown to be generalizable for the $(1,u)$-product, while for other products it is shown not to be possible to extend the Moufang identities, even to new Moufang-like identities, obtained from the original ones using the automorphism and anti-automorphisms of Clifford algebras. This paper is organized as follows: Section 2 is devoted to present exterior and Clifford algebras and in Section 3 we review the fundamental properties of the octonionic algebra $\mathcal{O}$, defined in terms of the Clifford algebra $\mathcal{C}_0,7$ and its associated Clifford product that defines the octonionic product [14]. The octonionic product is chosen to be defined in terms of the Clifford product in order that the arena of the whole formalism to be the Clifford algebra. In Section 4 the $X$-product and the $XY$-product [7] are generalized, encompassing ‘octonionic products’ between arbitrary multivectors of $\mathcal{C}_0,7$ and octonions. After introducing products between Clifford multivectors and octonions, that results in an octonion, we illustrate the use of the formalism, giving some examples of useful computations. Section 5 is devoted to introduce octonionic, Clifford algebra-parametrized units, associated with the $(1,u)$-product and, using these Clifford algebraic-dependent octonionic units, we assert the main results generalizing the formalism presented in [7], this time encompassing the whole Clifford algebra $\mathcal{C}_0,7$ instead of simply its paravector subspace $\mathbb{R} \oplus \mathbb{R}^{0,7} \to \mathcal{C}_0,7$. Although the product between $\mathcal{C}_0,7$ and $\mathcal{O}$ (that results in an octonion) is not associative, it is shown to be graded-associative, if we apply the graded involution on elements of $\mathcal{C}_0,7$. Also, the generalized Moufang identities, for the $(1,u)$-product, are presented and discussed. In Section 6 an ‘octonionic
product' between multivectors of $\mathcal{Cl}_{0,7}$ is presented, which allows us to construct more general octonion algebras. It is shown that the Moufang identities are not valid for this product. Finally in Section 7 we construct $\mathbb{O}$-like algebras with the octonionic product structure based on the just defined generalized octonionic products.

2 Preliminaries

Let $V$ be a finite $n$-dimensional real vector space. We consider the tensor algebra $\bigoplus_{i=0}^{\infty} T^i(V)$ from which we restrict our attention to the space $\Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k(V)$ of multivectors over $V$. $\Lambda^k(V)$ denotes the space of the antisymmetric $k$-tensors, the $k$-forms. Given $\psi \in \Lambda(V)$, $\hat{\psi}$ denotes the reversion, an algebra anti-automorphism given by $\hat{\psi} = (-1)^{[k/2]}\psi$ ($[k]$ denotes the integer part of $k$). $\hat{\psi}$ denotes the main automorphism or graded involution, given by $\hat{\psi} = (-1)^k \psi$. The conjugation is defined as the reversion followed by the main automorphism. If $V$ is endowed with a non-degenerate, symmetric, bilinear map $g : V \times V \to \mathbb{R}$, it is possible to extend $g$ to $\Lambda(V)$. Given $\psi = u_1 \wedge \cdots \wedge u_k$ and $\phi = v_1 \wedge \cdots \wedge v_l$, one defines $g(\psi, \phi) = \det(g(u_i, v_j))$ if $k = l$ and $g(\psi, \phi) = 0$ if $k \neq l$. Finally, the projection of a multivector $\psi = \psi_0 + \psi_1 + \cdots + \psi_n$, $\psi_k \in \Lambda^k(V)$, on its $p$-vector part is given by $\langle \psi \rangle_p = \psi_p$, while projection in its $p$ and $q$ components is given by $\langle \psi \rangle_{p\oplus q} = \psi_p + \psi_q$. The Clifford product between $w \in V$ and $\psi \in \Lambda(V)$ is given by $w\psi = w \wedge \psi + w \cdot \psi$. The Grassmann algebra $(\Lambda(V), g)$ endowed with this product is denoted by $\mathcal{C}(V, g)$ or $\mathcal{C}_{p,q}$. The Clifford algebra associated with $V \simeq \mathbb{R}^{p,q}$, $p + q = n$.

3 Octonions

The octonion algebra $\mathbb{O}$ is defined as the paravector space $\mathbb{R} \oplus \mathbb{O}^{0,7}$ endowed with the product $\circ : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \to \mathbb{R} \oplus \mathbb{R}^{0,7}$, denominated octonionic standard product. The identity $\{ e_0 = 1 \} \times \{ e_0 = 1 \}$ in the underlying paravector space is an orthonormal basis $\{ e_0 \}_{a=1}^7$, in the Clifford algebra $\mathcal{C}_{0,7}$ associated with $\mathbb{O}$, generates the octonions. It is well known that the octonionic product can be constructed using the Clifford algebra $\mathcal{C}_{0,7}$ as

$$A \circ B = \langle AB(1 - \psi) \rangle_{0\oplus 1}, \quad A, B \in \mathbb{R} \oplus \mathbb{R}^{0,7},$$

where $\psi = e_1e_2e_3 + e_2e_3e_4 + e_3e_4e_5 + e_4e_5e_6 + e_5e_6e_7 + e_6e_7e_8 + e_7e_8e_9 \in \Lambda^3(\mathbb{R}^{0,7}) \hookrightarrow \mathcal{C}_{0,7}$ and the juxtaposition denotes the Clifford product $\mathcal{C}$. The idea of introducing the octonionic product from the Clifford one in this context is to present hereon our formalism using solely Clifford algebras. Indeed, as $\mathbb{O}$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}^{0,7}$ as a vector space, the octonionic product fundamentally takes two arbitrary elements of the paravector space $\mathbb{R} \oplus \mathbb{R}^{0,7}$ — which is itself endowed with the octonionic product — resulting in another element of the paravector space. But looking to octonions inside the Clifford algebra arena we can go beyond the paravector space and exploit the whole Clifford algebra space, which is the way we use to generalize the $X$- and $XY$-products.

It is now immediate, from Eq. 1, to verify the usual rules between basis elements under the octonionic product $\circ$:

$$e_a \circ e_b = e_{ab}^ce_c - \delta_{ab} \quad (a, b, c = 1, \ldots, 7),$$

where we denote $e_{ab}^c = 1$ for the cyclic permutations $(abc) = (124),(235),(346),(457),(561),(672)$ and $(713)$. Explicitly, the multiplication table is given by $[4]$

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$[4]$Octonions are usually defined by these rules, but here we point out the usefulness of considering these rules (Eq. 2) as derived from Eq. 1, using Clifford algebras.
All the relations above can be expressed as $e_a \circ e_{a+1} = e_{a+3} \mod 7$. Since we can consider the underlying vector space of $\mathbb{O}$ as being $\mathbb{R} \oplus \mathbb{R}^{0,7} \hookrightarrow \mathbb{C}l_{0,7}$, the Clifford conjugation of $X = x^0 + x^a e_a \in \mathbb{O}$ is given by $\bar{X} = x^0 - x^a e_a$.

4 The $u$-product and generalizations

Hereon we suppose that $u \in \mathbb{C}l_{0,7} \equiv \mathbb{C}l(\mathbb{R}^{0,7}, g)$. Given fixed but arbitrary $X,Y \in \mathbb{R} \oplus \mathbb{R}^{0,7}$ such that $X \bar{X} = \bar{X}X = 1 = YY = Y \bar{Y}$ ($X,Y \in S^7$), the $X$-product is defined \[\mathbb{I} \mathbb{II} \mathbb{III}\] by

$$A \circ_X B := (A \circ X) \circ (\bar{X} \circ B) = X \circ ((\bar{X} \circ A) \circ B) = (A \circ (B \circ X)) \circ \bar{X}. \quad (3)$$

The $XY$-product is defined as:

$$A \circ_{X,Y} B := (A \circ X) \circ (\bar{Y} \circ B). \quad (4)$$

In particular, the $(1,X)$-product is given by

$$A \circ_{1,X} B := A \circ (\bar{X} \circ B). \quad (5)$$

$X$ is the unit of $(1,X)$-product above, since $A \circ_{1,X} X = X \circ_{1,X} A = A \mathbb{I} \mathbb{II} \mathbb{III}$.

We could propose a natural generalization of the $X$-product, introducing the $u$-product as:

$$A \circ_u B := (Au) \circ (u^{-1} B). \quad (6)$$

But if the products $Au$ and $u^{-1} B$ are to be interpreted as Clifford products, the elements $u \in \mathbb{C}l_{0,7}$, such that $Au$ and $u^{-1} B$ are octonions, should be scalars. In this case $A \circ_u B \equiv A \circ B$ and there would be nothing new to investigate.

In order that Eq.(6) to make sense, all quantities between parenthesis must be octonions, and to avoid the trivial case (where $u$ must be a scalar), we have to define a product between octonions and Clifford multivectors that results in an octonion.

Then, for (homogeneous) multivectors $u = u_1 \ldots u_k \in \Lambda^k(\mathbb{R}^{0,7}) \hookrightarrow \mathbb{C}l_{0,7}$, where $\{u_p\}_{p=1}^k \subset \mathbb{R}^{0,7}$ ($k = 1, \ldots, 7$) and $A \in \mathbb{R} \oplus \mathbb{R}^{0,7}$, we define the product $\bullet_\mathbb{C}$ as

$$\bullet_\mathbb{C} : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times \Lambda^k(\mathbb{R}^{0,7}) \to \mathbb{R} \oplus \mathbb{R}^{0,7}$$

$$(A, u) \mapsto A \bullet_\mathbb{C} u = ((\cdots (A \circ u_1) \circ u_2) \circ \cdots) \circ u_{k-1}) \circ u_k. \quad (7)$$

Note that the product $\bullet_\mathbb{C}$ is defined in such a way that, after the octonionic product between $A$ and $u_1$ is firstly performed, the result is computed via the octonionic product, now, with $u_2$. This process lasts up to the last octonionic product with $u_k$. Parentheses in Eq.(7) emphasize the order of performance associated with the — non-associative — octonionic product, and the symbol $\bullet_\mathbb{C}$ remembers us the octonion $A$ enters in the left entry in the product in Eq.(7).
We also define the product \( \bullet \) as

\[
\bullet : \Lambda^k(\mathbb{R}^{0,7}) \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \rightarrow \mathbb{R} \oplus \mathbb{R}^{0,7}
\]

\[
(u, A) \rightarrow u \bullet A = u_1 \circ (\cdots \circ (u_{k-1} \circ (u_k \circ A)) \cdots).
\]  

(8)

Analogously, parentheses in Eq. (8) emphasize the order of performance associated with the octonionic product inside each parenthesis in Eq. (8), and the symbol \( \bullet \) remembers us the element \( A \) enters in the right entry in the product in Eq. (8).

**Remark 1:** It is clear that by extending the product in Eq. (7) to the scalars — elements of \( \Lambda^0(\mathbb{R}^{0,7}) \) — we have that \( A \bullet a = aA \), which denotes the trivial multiplication by scalars, where \( a \in \mathbb{R} = \Lambda^0(\mathbb{R}^{0,7}) \). Therefore now it is possible to extend by linearity the product \( \bullet \) to the whole exterior algebra \( \Lambda(\mathbb{R}^{0,7}) = \oplus_{a=0}^7 \Lambda^a(\mathbb{R}^{0,7}) \) in such a way that the extended products are now denoted by

\[
\bullet : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \rightarrow \mathbb{R} \oplus \mathbb{R}^{0,7}
\]

(9)

\[
\bullet : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times \Lambda(\mathbb{R}^{0,7}) \rightarrow \mathbb{R} \oplus \mathbb{R}^{0,7}
\]

(10)

**Remark 2:** In Remark 1 we have extended by linearity the products in Eqs. (7) and (8) from \( \Lambda^k(\mathbb{R}^{0,7}) \) to the whole exterior algebra \( \Lambda(\mathbb{R}^{0,7}) \) in such a way that now \( u \in \Lambda(\mathbb{R}^{0,7}) \). Therefore, when we restrict \( u \) to the paravector space, i.e., \( u \in \Lambda^0(\mathbb{R}^{0,7}) \oplus \Lambda^1(\mathbb{R}^{0,7}) = \mathbb{R} \oplus \mathbb{R}^{0,7} \), then \( u \) becomes an octonion, and all the products \( \bullet, \bullet, \circ \), and the usual octonionic product \( \circ \), are equivalent, since all these products take in this case two elements of the paravector space \( \mathbb{R} \oplus \mathbb{R}^{0,7} \) and maps them to another element of the paravector space, via the usual octonionic product defined by Eq. (11).

By abuse of notation we shall use hereon the symbol \( \bullet \) uniquely to denote both products \( \bullet \) and \( \bullet \cdot \), in Eq. (9) and Eq. (10), and each one of the above-mentioned products are to be clearly implicit, as there exists an octonion in the left or right entry of the product \( \bullet \).

Now, after the algebraic pre-requisites have been introduced, given \( u \in \Lambda(\mathbb{R}^{0,7}) \), the \( u \)-product is defined as

\[
\circ_u : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \rightarrow (\mathbb{R} \oplus \mathbb{R}^{0,7})
\]

\[
(A, B) \rightarrow A \circ_u B := (A \bullet u) \circ (u^{-1} \bullet B).
\]  

(11)

It is worthwhile to note that

\[
A \circ_u B = (A \circ (B \bullet u)) \bullet u^{-1} = u \bullet ((u^{-1} \bullet A) \circ B).
\]  

(12)

**Example 1:** Let us calculate the product \( e_1 \circ_u e_4 \), given \( u = e_2 e_7 \):

\[
e_1 \circ_u e_4 = [e_1 \bullet (e_2 e_7)] \circ [(e_2 e_7)^{-1} \bullet e_4]
\]

\[
= [(e_1 \circ e_2) \circ e_7] \circ [-e_2 \circ (e_7 \circ e_4)]
\]

\[
= [e_4 \circ e_7] \circ [-e_2 \circ e_5]
\]

\[
= -e_5 \circ (-e_3)
\]

\[
= -e_2.
\]  

(13)

We note that \( e_1 \circ e_4 = -e_2 \) too. We can prove, using Eq. (9) and the property \( \epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{jm} - \delta_{jm} \delta_{jl} + \epsilon_{ijm} \) that \( A \circ_u B = A \circ B \), whenever \( u \) is a homogeneous Clifford algebra element of unit norm. When \( u \) is a paravector — an element of \( \mathbb{R} \oplus \mathbb{R}^{0,7} \) — it is clear that the \( u \)-product is equivalent to the \( X \)-product.
In analogy to the $XY$-product and the $(1,X)$-product, just respectively defined by Eqs. (14) and (15), it is also possible to define another product, the $(1,u)$-product, as

$$\circ_{1,u} : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \rightarrow \mathbb{R} \oplus \mathbb{R}^{0,7}$$

$$\quad (A, B) \mapsto A \circ_{1,u} B := A \circ (u^{-1} \bullet B). \quad (14)$$

Finally, Eq. (14) can be generalized, given fixed $u, v \in \mathcal{C}_{0,7}$, as:

$$\circ_{u,v} : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \rightarrow \mathbb{R} \oplus \mathbb{R}^{0,7}$$

$$\quad (A, B) \mapsto A \circ_{u,v} B := (A \bullet u) \circ (v^{-1} \bullet B). \quad (15)$$

**Example 2:** Let us now calculate the product $e_1 \circ_{u,v} e_4$, where $u = e_4 e_6 e_7$ and $v = e_1 e_5$:

$$e_1 \circ_{u,v} e_4 = [e_1 \bullet (e_4 e_6 e_7)] \circ [(e_1 e_5)^{-1} \bullet e_4]$$

$$= [(e_1 \circ e_4) \circ e_6] \circ (e_7) \circ (-e_1 \circ (-e_7))$$

$$= (-e_7 \circ e_7) \circ (-e_3)$$

$$= -e_3 \quad (16)$$

### 5 $\mathbb{O}$-units associated with the $(1,u)$-product

We enunciate in what follows some results necessary to show that the octonion algebra $\mathbb{O}$ is isomorphic to the algebra $\mathbb{O}_{1,u} \equiv (\mathbb{R} \oplus \mathbb{R}^{0,7}, \circ_{1,u})$. Lemmata 1, 2, 3, 4 and 5 are demonstrated by performing all the possible combinations of $\{e_a\}_{a=1}^7$ and then extending this process by linearity to the whole $\mathcal{C}_{0,7}$. It is implicit that $u \in \mathcal{C}_{0,7}$ is not a scalar, since in this case there would be nothing to prove. In what follows, $A, B, C \in \mathbb{O}$. The next Lemma concerns the graded-associativity of the $\bullet$-product.

**Lemma 1:** The elements $u \in \Lambda^2(\mathbb{R}^{0,7}) \hookrightarrow \mathcal{C}_{0,7}$ satisfy the relation $(u \bullet A) \circ B = u \bullet (A \circ B)$, while the elements $u \in \Lambda^{2+1}(\mathbb{R}^{0,7}) \hookrightarrow \mathcal{C}_{0,7}$ satisfy the relation $(u \bullet A) \circ B = -u \bullet (A \circ B)$. These results can be expressed as

$$(u \bullet A) \circ B = \hat{u} \bullet (A \circ B)$$

Moreover, the relations

$$A \circ (B \bullet u) = (A \circ B) \bullet \hat{u}$$

hold for all $u \in \mathcal{C}_{0,7}$.

**Remark 3:** Since we have just asserted that $A \bullet B \equiv A \circ B$ when $A, B$ are octonions, the assertions of Lemma 1 are obviously equivalent to $(u \bullet A) \bullet B = \hat{u} \bullet (A \bullet B)$ and $A \bullet (B \bullet u) = (A \bullet B) \bullet \hat{u}$, from where it can be seen that we can correctly denominate this property as ‘graded-associativity’.

**Lemma 2:** The elements $u \in \mathcal{C}_{0,7}$ satisfy

$$(u \bullet A) \circ B = -(u \bullet B) \circ A$$

**Lemma 3:** The elements $u \in \mathcal{C}_{0,7} \setminus \Lambda^6(\mathbb{R}^{0,7})$ satisfy

$$u \bullet A = A \bullet \hat{u}$$

In the particular case where $u = e_a e_b e_c e_d e_f e_g \in \Lambda^6(\mathbb{R}^{0,7})$ and $A = e_h$, where none of the subindices equals each other, the identity $u \bullet A = -1 = A \bullet u$ holds.
**Lemma 4:** The elements $u \in C\ell_{0,7}$ satisfy the relation
\[
u^{-1} \bullet (u \bullet A) = A = (A \bullet u) \bullet u^{-1}
\] (17)

**Lemma 5:** Given $u \in C\ell_{0,7}$ it follows that
\[
u \bullet (A \bullet u) = (u \bullet A) \bullet u = u \bullet A \bullet u
\] (18)

The lemmata above allow to generalize the $\mathcal{O}$-product, also including ‘octonionic products’ between octonions and multivectors of $C\ell_{0,7}$. These lemmata provide the formal pre-requisite to construct the multiplicative table of the Clifford algebra-dependent units $\{E_A\}_{A=1}^7$, defined by
\[
E_1 = u \bullet e_1, \quad E_2 = u \bullet e_2, \quad E_3 = u \bullet e_3, \quad E_4 = \hat{u} \bullet e_4,
\]
\[
E_5 = u \bullet e_5, \quad E_6 = \hat{u} \bullet e_6 \text{ and } E_7 = \hat{u} \bullet e_1.
\] (19)

Doing the explicit verification on each one of the octonions defined by Eqs. (19), and subsequently extending by linearity in $C\ell_{0,7}$, it follows the

**Lemma 6:** The elements $E_a$ anticommute under the $(1, u)$-product, i.e,
\[
E_a \circ_{1,u} E_b = -E_b \circ_{1,u} E_a
\]

**Proof:** When $\{a, b\} = \{1, 2, 3, 5\}$, we see that
\[
E_a \circ_{1,u} E_b = (u \bullet e_a) \circ (u^{-1} \bullet (u \bullet e_b))
\]
\[
= (u \bullet e_a) \circ e_b, \text{ using Lemma 4}
\]
\[
= -(u \bullet e_b) \circ e_a, \text{ by Lemma 2}
\]
\[
= -(u \bullet e_b) \circ (u^{-1} \bullet (u \bullet e_a))
\]
\[
= -E_b \circ_{1,u} E_a.
\] (20)

The other cases, when $\{a, b\} = \{4, 6, 7\}$, are analogously demonstrated. \hfill \square

The multiplication table of $\{E_a\}$ is inherited from table (3). We express the table describing the $(1, u)$-product of the octonions $E_A$:

|     | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ | $E_7$ |
|-----|-------|-------|-------|-------|-------|-------|-------|
| $E_1$ | $-1$  | $E_4$ | $E_7$ | $-E_2$ | $E_6$ | $-E_5$ | $-E_3$ |
| $E_2$ | $-E_4$ | $-1$  | $E_5$ | $E_1$  | $-E_3$ | $E_7$  | $-E_6$ |
| $E_3$ | $-E_7$ | $-E_5$ | $-1$  | $E_6$  | $E_2$  | $-E_4$ | $E_1$  |
| $E_4$ | $E_2$  | $-E_1$ | $-E_6$ | $-1$  | $E_7$  | $E_4$  | $-E_5$ |
| $E_5$ | $-E_6$ | $E_3$  | $-E_2$ | $-E_7$ | $-1$  | $E_1$  | $E_4$  |
| $E_6$ | $E_5$  | $-E_7$ | $E_4$  | $-E_3$ | $-E_1$ | $-1$  | $E_2$  |
| $E_7$ | $E_3$  | $E_6$  | $-E_1$ | $E_5$  | $-E_4$ | $-E_2$ | $-1$   |

With this table we immediately see that $\{E_A\}_{A=1}^7$ are the octonion units associated with the $(1, u)$-product. We exhibit below the computations related to the first line of the table above. The other lines follow in an analogous way.

\[
E_1 \circ_{1,u} E_2 = (u \bullet e_1) \circ (u^{-1} \bullet (u \bullet e_2))
\]
\[
= (u \bullet e_1) \circ e_2, \text{ using Lemma 4}
\]
\[
= \hat{u} \bullet (e_1 \circ e_2), \text{ by Lemma 1}
\]
\[
= \hat{u} \bullet e_6
\]
\[
= E_6
\] (21)
We also have

\[
E_1 \circ_{1,u} E_3 = (u \cdot e_1) \circ (u^{-1} \cdot (u \cdot e_3)) = (u \cdot e_1) \circ e_3 \\
= \hat{u} \cdot (e_1 \circ e_3) = \hat{u} \cdot e_4 \\
= E_4, \quad (22)
\]

\[
E_1 \circ_{1,u} E_4 = -E_4 \circ_{1,u} E_1 = -(\hat{u} \cdot e_4) \circ (u^{-1} \cdot (u \cdot e_1)) \\
= -(\hat{u} \cdot e_4) \circ e_1 = -u \cdot (e_4 \circ e_1) = -u \cdot e_3 \\
= -E_3, \quad (23)
\]

\[
E_1 \circ_{1,u} E_5 = (u \cdot e_1) \circ (u^{-1} \cdot (u \cdot e_5)) \\
= (u \cdot e_1) \circ e_5 = \hat{u} \cdot (e_1 \circ e_5) = \hat{u} \cdot e_7 \\
= E_7, \quad (24)
\]

\[
E_1 \circ_{1,u} E_6 = -E_6 \circ_{1,u} E_1 = -(\hat{u} \cdot e_6) \circ (u^{-1} \cdot (u \cdot e_1)) \\
= -(\hat{u} \cdot e_6) \circ e_1 = -u \cdot (e_6 \circ e_1) = -u \cdot e_2 \\
= -E_2. \quad (25)
\]

**Remark 4:** The Moufang identities [4][10]

\[
(A \circ B) \circ (C \circ A) = A \circ (B \circ C) \circ A, \quad (26)
\]

\[
(A \circ B \circ A) \circ C = A \circ (B \circ (A \circ C)), \quad (27)
\]

\[
(A \circ B) \circ (C \circ A) = A \circ (C \circ B) \circ A, \quad (28)
\]

\[
C \circ (A \circ B \circ A) = ((C \circ A) \circ B) \circ A, \quad (29)
\]

$A, B, C \in \mathcal{O}$ can be immediately generalized for the $(1, u)$-product, using table [5], as

\[
(A \circ_{1,u} B) \circ_{1,u} (C \circ_{1,u} A) = A \circ_{1,u} (B \circ_{1,u} C) \circ_{1,u} A, \quad (30)
\]

\[
(A \circ_{1,u} B \circ_{1,u} A) \circ_{1,u} C = A \circ_{1,u} (B \circ_{1,u} (A \circ_{1,u} C)), \quad (31)
\]

\[
(A \circ_{1,u} B) \circ_{1,u} (C \circ_{1,u} A) = A \circ_{1,u} (C \circ_{1,u} B) \circ_{1,u} A, \quad (32)
\]

\[
C \circ_{1,u} (A \circ_{1,u} B \circ_{1,u} A) = ((C \circ_{1,u} A) \circ_{1,u} B) \circ_{1,u} A, \quad (33)
\]

In the case of the products

\[
\bullet_\iota : (\mathbb{R} \oplus \mathbb{R}^{0.7}) \times \Lambda(\mathbb{R}^{0.7}) \to \mathbb{R} \oplus \mathbb{R}^{0.7} \quad \text{and} \quad \bullet_j : \Lambda(\mathbb{R}^{0.7}) \times (\mathbb{R} \oplus \mathbb{R}^{0.7}) \to \mathbb{R} \oplus \mathbb{R}^{0.7}, \quad (34)
\]

it can be shown that uniquely using the Clifford conjugation and the graded involution it is not possible to get some kind of Moufang identities. The following counterexamples show why it is not possible.
Example 3: One of the Moufang identities for octonions is expressed as
\[(A \circ B) \circ (C \circ A) = A \circ (B \circ C) \circ A, \quad A, B, C \in \mathbb{O}.\] (35)

Suppose that an immediate generalization for such an identity is the expression
\[(u \bullet A) \bullet (B \bullet u) = u \bullet (A \bullet B) \bullet u, \quad u \in \mathcal{C}_{0,7},\] (36)
or \[(u \bullet A) \bullet (B \bullet u) = \hat{u} \bullet (A \bullet B) \bullet u, \quad \text{or} \quad (u \bullet A) \bullet (B \bullet u) = \hat{u} \bullet (A \bullet B) \bullet u, \quad \text{or the product given by Eq.} \ (36) \ 	ext{for any combination of the graded involution and/or Clifford conjugation acting on } u.\]

In order to the expressions become more clear, we denote Eq. (36) as
\[(u \bullet A) \circ (B \bullet u) = u \bullet (A \circ B) \bullet u, \quad u \in \mathcal{C}_{0,7},\] (37)
since the product \(\bullet\) between paravectors — elements of \(\mathbb{R} \oplus \mathbb{R}^{0,7}\) — is equal to the product \(\circ\). Let \(u = e_6e_7e_1e_3, \ A = e_2\) and \(B = e_5\). On the one hand,
\[\left(e_6e_7e_1e_3 \cdot e_2\right) \circ \left(e_5 \cdot e_6e_7e_1e_3\right) = -e_4,\] (38)
and
\[e_6e_7e_1e_3 \cdot \left(e_2 \circ e_5\right) \cdot e_6e_7e_1e_3 = -e_4.\] (39)

On the other hand, if we take \(u = e_1e_2e_3e_6, \ A = e_4\) and \(B = e_7\), we have that:
\[\left(e_1e_2e_3e_6 \cdot e_4\right) \circ \left(e_7 \cdot e_1e_2e_3e_6\right) = e_6,\] (40)
while
\[e_1e_2e_3e_6 \cdot \left(e_4 \circ e_7\right) \cdot e_1e_2e_3e_6 = -e_6.\] (41)

We then realize that for distinct elements \(u \in \Lambda^4(\mathbb{R}^{0,7}),\) we have \((u \bullet A) \circ (B \bullet u) = u \bullet (A \circ B) \bullet u,\) and \((u \bullet A) \circ (B \bullet u) = -u \bullet (A \circ B) \bullet u.\) These last two relations cannot be mutually satisfied by two elements in \(\mathcal{C}_{0,7}\) presenting the same degree. Analogous counterexamples can be shown for the product given by Eq. (36) with any combination of the graded involution and/or Clifford conjugation acting on \(u.\) It is also possible to show that the other Moufang identities given by Eqs. (27), (28), (29) cannot be generalized for the products \(\circ : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times \Lambda(\mathbb{R}^{0,7}) \rightarrow \mathbb{R} \oplus \mathbb{R}^{0,7}\) and \(\bullet : \Lambda(\mathbb{R}^{0,7}) \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \rightarrow \mathbb{R} \oplus \mathbb{R}^{0,7},\) uniquely using the Clifford conjugation and the graded involution.

6  Octonionic product between Clifford multivectors and generalizations

Given vectors \(\{u_p\}_{p=1}^k \subset \mathbb{R}^{0,7}\) and \(\{v_q\}_{q=1}^k \subset \mathbb{R}^{0,7},\) \(1 \leq k \leq 7,\) and elements \(u = u_1 \ldots u_k, \ v = v_1 \ldots v_k \in \mathcal{C}_{0,7},\) we define the “octonionic product” between elements of the Clifford algebra \(\mathcal{C}_{0,7}\) as
\[\odot : \mathcal{C}_{0,7} \times \mathcal{C}_{0,7} \rightarrow \mathbb{R} \oplus \mathbb{R}^{0,7},\]
\[(u, v) \mapsto u \odot v := u_1 \odot (u_2 \odot (\cdots \odot (u_k \bullet v) \cdots ))\] (42)

Parentheses in Eq. (42) emphasize the order of performance associated with the non-associative octonionic product. After calculating the product \(u_k \bullet v,\) the result is computed via the octonionic product with \(u_{k-1},\) and then successively until the last reminiscent result, which is computed with \(u_1\) through the usual octonionic product. It can be easily seen that, when we restrict elements of \(\mathcal{C}_{0,7}\) to the paravector space \(\mathbb{R} \oplus \mathbb{R}^{0,7},\) then \(A \odot B = A \odot B\), where \(A, B \in \mathbb{R} \oplus \mathbb{R}^{0,7}.\)
Example 4: Let us compute the product $e_1 e_2 \odot_\ell e_3 e_4$:

$$e_1 e_2 \odot_\ell e_3 e_4 = e_1 \odot (e_2 \bullet (e_3 e_4)) = e_1 \odot (e_5 \odot e_4) = e_1 \odot (-e_7) = e_3.$$  

(43)

We also define the product

$$\odot_J : \mathcal{C}_{\ell,0,7} \times \mathcal{C}_{\ell,0,7} \to \mathbb{R} \oplus \mathbb{R}^{0,7}$$

$$(u, v) \mapsto u \odot_J v := ((\cdots \odot (u \bullet v_1) \odot v_2) \cdots) \odot v_k$$  

(44)

It is immediate to see that $A \odot_J B = A \odot B$, whenever $A, B \in \mathbb{R} \oplus \mathbb{R}^{0,7}$, since we have just seen that $A \bullet B = A \odot B$, by Eqs. (?).

It is now immediate to see that the Moufang identities 26, 27, 28, 29 are not valid to the products $\odot_J$ and $\odot_\ell$.

Example 5: Let us see whether the Moufang identity given by Eq. 26 can be generalized to the $\odot_\ell$ product. On the one hand let us compute the product $e_7 e_3 \odot_\ell (e_5 e_4 \odot_\ell e_1 e_6) \odot_\ell e_7 e_3$:

$$e_7 e_3 \odot_\ell (e_5 e_4 \odot_\ell e_1 e_6) \odot_\ell e_7 e_3 = e_7 e_3 \odot_\ell (e_5 \odot (e_4 \bullet e_1 e_6)) \odot_\ell e_7 e_3$$

$$= e_7 e_3 \odot_\ell (e_5 \odot (e_2 \odot e_6)) \odot_\ell e_7 e_3$$

$$= e_7 e_3 \odot_\ell (e_5 \odot e_7) \odot_\ell e_7 e_3$$

$$= e_7 e_3 \odot_\ell e_4 \odot_\ell e_7 e_3$$

$$= (e_7 \odot e_6) \bullet e_7 e_3$$

$$= -e_2 \bullet e_7 e_3$$

$$= e_6 \odot e_3$$

$$= e_4.$$  

(45)

On the other hand we have:

$$(e_7 e_3 \odot_\ell e_5 e_4) \odot_\ell (e_1 e_6 \odot_\ell e_7 e_3) = (e_7 \odot (e_2 \odot e_4)) \odot (e_1 \odot (e_2 \odot e_3))$$

$$= (e_7 \odot e_1) \odot (e_1 \odot e_5)$$

$$= e_3 \odot e_6$$

$$= -e_4.$$  

(46)

Then a counterexample for the validity of the Moufang identity Eq. 26, for the $\odot_\ell$, was exhibited, and similarly it is easy to see that the other identities given by Eqs. 27, 28, 29 are not generalizable too.

Definition 11 allows us to see that the $(1, u)$-product can also be generalized in order to encompass elements $u \in \mathcal{C}_{\ell,0,7}$ in the first or in the second entry, as follows (by abuse of notation we denote distinct products $\odot_{1,u} : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \to \mathbb{R} \oplus \mathbb{R}^{0,7}$, $\odot_{1,u} : \mathcal{C}_{\ell,0,7} \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \to \mathbb{R} \oplus \mathbb{R}^{0,7}$, $\odot_{1,u} : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times \mathcal{C}_{\ell,0,7} \to \mathbb{R} \oplus \mathbb{R}^{0,7}$ and $\odot_{1,u} : \mathcal{C}_{\ell,0,7} \times \mathcal{C}_{\ell,0,7} \to \mathbb{R} \oplus \mathbb{R}^{0,7}$ by the same symbol $\odot_{1,u}$):

$$\odot_{1,u} : \mathcal{C}_{\ell,0,7} \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \to \mathbb{R} \oplus \mathbb{R}^{0,7}$$

$$(v, A) \mapsto v \odot_{1,u} A := v \bullet (u^{-1} \bullet A).$$  

(47)

Hereon we can opt to use $\odot_J$ or $\odot_\ell$ in the definitions below, and therefore we adopt the symbol $\odot$ to denote any one of them. Obviously if one of the two products $\odot_J$ and $\odot_\ell$ is to
be chosen, the respective choice must also be made in all the following definitions. Now we define

\[
\circ_{1,u} : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times C_{0,7} \to \mathbb{R} \oplus \mathbb{R}^{0,7} \\
(A, v) \mapsto A \circ_{1,u} v := A \circ (u^{-1} \odot v),
\]  

(48)

It is immediate that by Eq. (47) we have \( A \circ_{1,u} u = A \), so the element \( u \) is the right unit of the product defined in Eq. (48). Using Lemma 4 and Eq. (47) we can also prove that \( u \circ_{1,u} A = A \), so that \( u \) is also the left unit related to the product defined in Eq. (48), and we conclude that the unit associated with the \((1, u)\)-product is \( u \in C_{0,7} \). The last \( \circ_{1,u} \) extension, given \( z \in C_{0,7} \) fixed but arbitrary, is defined by

\[
\circ_{1,u} : C_{0,7} \times C_{0,7} \to \mathbb{R} \oplus \mathbb{R}^{0,7} \\
(v, z) \mapsto v \circ_{1,u} z := v \odot (u^{-1} \odot z),
\]  

(49)

Given \( z, t \in C_{0,7} \), the products given in Section 4 are immediately extended\(^2\) if we define:

\[
\circ_u : C_{0,7} \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \to \mathbb{R} \oplus \mathbb{R}^{0,7} \\
(v, A) \mapsto v \circ_u A := (v \odot u) \circ (u^{-1} \bullet A),
\]  

(50)

\[
\circ_u : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times C_{0,7} \to \mathbb{R} \oplus \mathbb{R}^{0,7} \\
(A, v) \mapsto A \circ_u v := (A \bullet u) \circ (u^{-1} \odot v),
\]  

(51)

\[
\circ_u : C_{0,7} \times C_{0,7} \to \mathbb{R} \oplus \mathbb{R}^{0,7} \\
(v, z) \mapsto v \circ_u z := (v \odot u) \circ (u^{-1} \odot z).
\]  

(52)

Finally the \((u, v)\)-product defined by Eq. (45) can also be extended as:

\[
\circ_{u,v} : C_{0,7} \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \to \mathbb{R} \oplus \mathbb{R}^{0,7} \\
(z, A) \mapsto z \circ_{u,v} A := (z \odot u) \circ (v^{-1} \bullet A),
\]  

(53)

\[
\circ_{u,v} : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times C_{0,7} \to \mathbb{R} \oplus \mathbb{R}^{0,7} \\
(A, z) \mapsto A \circ_{u,v} z := (A \bullet u) \circ (v^{-1} \odot z),
\]  

(54)

\[
\circ_{u,v} : C_{0,7} \times C_{0,7} \to \mathbb{R} \oplus \mathbb{R}^{0,7} \\
(t, z) \mapsto t \circ_{u,v} z := (t \odot u) \circ (v^{-1} \odot z).
\]  

(55)

7 Generalizations and equivalence between \( \mathbb{O} \)-like algebras

From table (5), we see that \( \mathbb{O} \cong \mathbb{O}_{1,u} \). Now it is shown how, from a copy of \( \mathbb{O}_u \), we can obtain for instance the algebra \( \mathbb{O}_{u,C} := (\mathbb{R} \oplus \mathbb{R}^{0,7}, \circ_{u,C}) \), \( u \in C_{0,7} \) and \( C \) is an octonion. The

\(^2\) We analogously denote the products below by the same symbols, by abuse of notation.
shown to be equivalent, since the former originates the latter from iterated products: immediate obtained for such new octonionic units. Also, the algebra-parametrized octonionic units associated with the natural formalism we use to generalize the process is accomplished by taking in the algebra \( O_u = (O, \circledast) \), products-\((1, v)\), i.e., given \( A, B \in \mathbb{R} \oplus \mathbb{R}^{0, 7} \), we calculate the product:

\[
A \circ u (v^{-1} \circ u B) = (A \circ u) \circ \{ u^{-1} \cdot [(v^{-1} \circ u) \circ (u^{-1} \cdot B)] \} = (A \circ u) \circ \{ u^{-1} \cdot [u \cdot ((u^{-1} \circ v^{-1}) \circ B)] \} = (A \circ u) \circ [(u^{-1} \circ v^{-1}) \circ B] = (A \circ u) \circ (C \circ B) = A \circ u_C B,
\]

where we have defined

\[
C = u^{-1} \circ v^{-1} \in \mathbb{R} \oplus \mathbb{R}^{0, 7}.
\]

The algebra \( O_u \) then originates \( O_{u_C} \).

In an analogous way the algebra \( O_u \) gives rise to the algebra \( O_{B \circ u} \), by the

**Theorem 1** \((A \circ u B) \circ u (B^{-1} \circ u C) = \pm A \circ u \circ B \circ u C, \quad \forall A, B, C \in \mathbb{R} \oplus \mathbb{R}^{0, 7}, u \in \mathcal{C} \ell_{0, 7}

**Proof:**

\[
(A \circ u B) \circ u (B^{-1} \circ u C) = \{ [(A \circ u) \circ (u^{-1} \cdot B)] \circ u \} \circ \{ u^{-1} \cdot [(B^{-1} \circ u) \circ (u^{-1} \cdot C)] \} = \{ [(A \circ (B \circ u)) \circ u^{-1}] \circ u \} \circ \{ u^{-1} \cdot [u \cdot ((u^{-1} \cdot B^{-1}) \circ C)] \} = [A \circ (B \circ u)] \circ [(u^{-1} \circ B^{-1}) \circ C] = \pm [A \circ (B \circ u)] \circ [(B \circ u)^{-1} \circ C] = \pm A \circ u_B C.
\]

If \( B \in \mathbb{R} \oplus \mathbb{R}^{0, 7} \) is such that \( BB = B \), i.e., \( B \in S^7 \), then Theorem 1 is enunciated as

\[
(A \circ u B) \circ u (B \circ u C) = \pm A \circ u_B u C, \quad \forall A, B, C \in \mathbb{R} \oplus \mathbb{R}^{0, 7}, u \in \mathcal{C} \ell_{0, 7}
\]

### 8 Concluding Remarks

We generalized the \( X \)- and \( XY \)-products, introducing the products \( \circ u : (\mathbb{R} \oplus \mathbb{R}^{0, 7}) \times (\mathbb{R} \oplus \mathbb{R}^{0, 7}) \to \mathbb{R} \oplus \mathbb{R}^{0, 7} \) and \( \circ u_v : (\mathbb{R} \oplus \mathbb{R}^{0, 7}) \times (\mathbb{R} \oplus \mathbb{R}^{0, 7}) \to \mathbb{R} \oplus \mathbb{R}^{0, 7} \), where \( u, v \in \mathcal{C} \ell_{0, 7} \) are chosen to be fixed but arbitrary. After the formal definitions we extended these products in order to encompass the products \( (\mathbb{R} \oplus \mathbb{R}^{0, 7}) \times \mathcal{C} \ell_{0, 7} \to \mathbb{R} \oplus \mathbb{R}^{0, 7}, \mathcal{C} \ell_{0, 7} \times (\mathbb{R} \oplus \mathbb{R}^{0, 7}) \to \mathbb{R} \oplus \mathbb{R}^{0, 7} \) and \( \mathcal{C} \ell_{0, 7} \times \mathcal{C} \ell_{0, 7} \to \mathbb{R} \oplus \mathbb{R}^{0, 7} \). We also furnish the mathematical requirements to introduce Clifford algebra-parametrized octonionic units associated with the \((1, u)\)-product and, consequently, to produce a copy of \( O \), but now endowed with the \((1, u)\)-product. The Moufang identities are immediately obtained for such new octonionic units. Also, the algebra \( O_u \) and \( O_{B \circ u} \) are shown to be equivalent, since the former originates the latter from iterated \( u \)-products. The products \( \odot : \mathcal{C} \ell_{0, 7} \times (\mathbb{R} \oplus \mathbb{R}^{0, 7}) \to \mathbb{R} \oplus \mathbb{R}^{0, 7} \) and \( \odot : (\mathbb{R} \oplus \mathbb{R}^{0, 7}) \times \mathcal{C} \ell_{0, 7} \to \mathbb{R} \oplus \mathbb{R}^{0, 7} \) are graded-associative, indicating the rise of a supersymmetric structure out of these products. Some more general algebras can be constructed from simpler ones, as the explicit construction given by Eq. (40). Although the products \( \odot : \Lambda(\mathbb{R}^{0, 7}) \times (\mathbb{R} \oplus \mathbb{R}^{0, 7}) \to \mathbb{R} \oplus \mathbb{R}^{0, 7} \) and \( \odot : (\mathbb{R} \oplus \mathbb{R}^{0, 7}) \times \Lambda(\mathbb{R}^{0, 7}) \to \mathbb{R} \oplus \mathbb{R}^{0, 7} \) are equivalent, in a sense, to the left and right actions presented in [1], this new approach allows us to generalize these actions in order to enclose all the products defined in this paper. Moreover, the present approach permits the graded-associativity shown in Lemma 1 to become transparent, and to use a simpler notation. Finally, by considering octonions inside the Clifford algebra arena we can go beyond the paravector space \( \mathbb{R} \oplus \mathbb{R}^{0, 7} \) and approach Clifford algebra \( \mathcal{C} \ell_{0, 7} \) space, which is the most natural formalism we use to generalize the \( X \) and \( XY \)-products. Explict applications of the present formalism in some physical theories are to be presented in a forthcoming paper [17].
Acknowledgements

The authors thank to Journal of Algebra Referee for elucidating and enlightening viewpoints, and also Dr. R. A. Mosna for useful discussions.

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