OPTIMAL CONTROL AND ZERO-SUM GAMES FOR MARKOV CHAINS OF MEAN-FIELD TYPE

SALAH EDDINE CHOUTRI AND BOUALEM DJEHICHE

Department of Mathematics, KTH Royal Institute of Technology
100 44, Stockholm, Sweden

HAMIDOU TEMBINE

Learning & Game Theory Laboratory, New York University
19 Washington Square North New York, NY 10011, USA

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Abstract. We establish existence of Markov chains of mean-field type with unbounded jump intensities by means of a fixed point argument using the total variation distance. We further show existence of nearly-optimal controls and, using a Markov chain backward SDE approach, we suggest conditions for existence of an optimal control and a saddle-point for respectively a control problem and a zero-sum differential game associated with payoff functionals of mean-field type, under dynamics driven by such Markov chains of mean-field type.

1. Introduction. A Markov chain of mean-field type (also known as nonlinear Markov chain) is a pure jump process with a discrete state space whose jump intensities further depend on the marginal law of the process. It is obtained as the limit of a system of pure jump processes with mean-field interaction, when the system size tends to infinity. The marginal law of the nonlinear process, obtained as a deterministic limit of the sequence of empirical distribution functions representing the states of the finite systems, satisfies a ‘nonlinear’ Fokker-Planck or masters equation called the McKean-Vlasov equation. In a sense, it represents the law of a typical trajectory in the underlying collection of interacting jump processes. In particular, optimal control and games based on the nonlinear process dynamics would give an insight into the effect of the design of control and game strategies for large system of interacting jump processes.

This class of processes is widely used for modeling purposes in chemistry, physics, biology and economics. Nicolis and Prigogine [24] were among the first authors to propose such a class of nonlinear processes as a mean-field model of a chemical reaction with spatial diffusion. It plays the same role as nonlinear diffusion processes play in the study of diffusion equations and more generally PDEs driven by nonlocal operators, with mean-field interaction (see Kolokoltsov [20], Jourdain et al. [19] and Sznitman [30] and the references therein). Mean-field models of the so-called first and second Schlögl processes [28] and the auto-catalytic process, which are widely

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* Corresponding author: H. Tembine.
used to model chemical reactions, provide interesting examples of Markov chains of
mean-field type with unbounded jump intensities, and have been studied in depth
in Dawson and Zheng [7], Feng and Zhang [17] and Feng [16]. These nonlinear
processes are obtained as limits of systems of birth and death processes with mean
field interaction. For application in the spread of epidemics see e.g. Djehiche and
Kaj [9], Djehiche and Schied [10] and Léonard [22]. For an account of existence
and uniqueness of such nonlinear jump processes with bounded jump intensities we
refer to Oelschläger [25]. See [23] for the case of unbounded jumps.

In the study of mean-field models, it is more or less decisive to make the right
choice of an adequate distance (among many others) on the set of probability mea-
sures which carries the topology of weak convergence. The total variation distance
is usually the natural one to use in the study of standard Markov chains and is easy
to manipulate. But, the fact that it does not necessarily guarantee finite moments
(except when the state-space is finite), it may not be suitable for mean-field models
when the mean-field interaction is given by e.g. the mean or the second moment,
whereas the Wasserstein distance is designed to guarantee finite moments.

Nonetheless, In this paper we formulate our findings using the total variation
distance only. We first give another proof of existence and uniqueness of Markov
chains of mean-field type using a fixed point argument. The proof is based on a
Girsanov-type change of measure and the Csiszár-Kullback-Pinsker inequality. As
we will see below, the full use of the total variation distance requires $L^2$-boundedness
of the Girsanov density, which is insured by imposing an extra regularity condition
of the intensity matrix of the Markov chain (see (A6) and (A7) below) compared
with what should be natural if the Wasserstein distance is used. Furthermore, we
consider optimal control and zero-sum games associated with payoff functionals of
mean-field type, when the nonlinear Markov chain is controlled through its jump
intensities. More precisely, we consider pure jump processes $x$ whose (eventually
unbounded) jump intensities at time $t$ depend on the whole path over the time
interval $[0,T]$ and also on the marginal law of $x(t)$, as long as they are predictable.
In a sense, this way of constructing a nonlinear jump process is a generalization of
the classical thinning procedure of a point process. A similar program for controlled
diffusion processes is performed in [8], with obvious overlap in the used methods
and techniques.

The main results on optimal control and zero-sum games are derived using tech-
niques involving Markov chain backward stochastic differential equations
(BSDE), where existence of an optimal control and a saddle-point strategy of the
game boil down to finding a minimizer and a min-max of an underlying Hamiltonian
$H$. Since the mean-field coupling through the marginal law of the controlled Markov
chain makes the Hamiltonian $H$, evaluated at time $t$, depend on the whole path of
the control process over the time interval $[0,t]$, we cannot follow the frequently used
procedure in standard optimal control and perform a deterministic minimization
of $H$ over the set of actions $U$ and then apply a Beneš-type progressively measur-
able selection theorem to produce an optimal control. We should rather take the
essential infimum of $H$ over the set $U$ of progressively measurable controls. This
nonlocal feature of the dependence of $H$ on the control does not seem covered by
the existing powerful measurable selection theorem. Therefore, our main results are
formulated by assuming existence of an essential minimum $u^* \in U$ of $H$ and use
suitable comparison results of Markov chain BSDEs to show that $u^*$ is in fact an
optimal control, simply because we don’t know any suitable measurable selection
2. Preliminaries. Let \( I = \{0, 1, 2, \ldots \} \) equipped with its discrete topology and \( \sigma \)-field and let \( \Omega := D([0, T], I) \) be the space of functions from \([0, T]\) to \( I \) that are right continuous with left limits at each \( t \in [0, T] \) and are left continuous at time \( T \). We endow \( \Omega \) with the Skorohod metric \( d_0 \) so that \((\Omega, d_0)\) is a complete separable metric (i.e. Polish) space. Given \( t \in [0, T] \) and \( \omega \in \Omega \), put \( x(t, \omega) = \omega(t) \) and denote by \( \mathcal{F}_t := \sigma(x(s), 0 \leq s \leq t), 0 \leq t \leq T \), the filtration generated by \( x \). Denote by \( \mathcal{F} \) the Borel \( \sigma \)-field over \( \Omega \). It is well known that \( \mathcal{F} \) coincides with \( \sigma(x(s), 0 \leq s \leq T) \). Set, for \( t \in [0, T] \), \( |x|_t := \sup_{0 \leq s \leq t} |x(s)| \) and \( \|a\|^2 := \sum_{i,j : j \neq i} |a_{ij}|^2 \) for \( a = (a_{ij}, i, j \in I, j \neq i) \).

To \( x \) we associate the indicator process \( I_i(t) = 1_{\{x(t) = i\}} \) whose value is 1 if the chain is in state \( i \) at time \( t \) and 0 otherwise, and the counting processes \( N_{ij}(t), i \neq j \), independent of \( x(0) \), such that

\[
N_{ij}(t) = \#\{\tau \in (0, t] : x(\tau^-) = i, x(\tau) = j\}, \quad N_{ij}(0) = 0,
\]

which count the number of jumps from state \( i \) into state \( j \) during the time interval \((0, t] \). Obviously, since \( x \) is right continuous with left limits, both \( I_i \) and \( N_{ij} \) are right continuous with left limits. Moreover, by the relationship

\[
x(t) = \sum_i i I_i(t), \quad I_i(t) = I_i(0) + \sum_{j : j \neq i} (N_{ji}(t) - N_{ij}(t)), \tag{2.1}
\]

the state process, the indicator processes, and the counting processes carry the same information which is represented by the natural filtration \( \mathbb{F}^0 := (\mathcal{F}_t^0, 0 \leq t \leq T) \) of \( x \).

We denote by \( \mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T} \) the completion of \( \mathbb{F}^0 \) with the \( P \)-null sets of \( \Omega \). Hereafter, a process from \([0, T] \times \Omega \) into a measurable space is said predictable (resp. progressively measurable) if it is predictable (resp. progressively measurable) w.r.t. the predictable \( \sigma \)-field on \([0, T] \times \Omega \) (resp. \( \mathbb{F} \)).

Note that (2.1) is equivalent to the following useful representation

\[
x(t) = x(0) + \sum_{i,j : i \neq j} (j - i)N_{ij}(t). \tag{2.2}
\]
Below, \(C\) denotes a generic positive constants which may change from line to line.

2.1. Markov chains. Let \(G = (g_{ij}, i, j \in I)\) be a \(Q\)-matrix, so that
\[
g_{ij} \geq 0, \quad i \neq j, \quad \sum_{j : j \neq i} g_{ij} = g_{ii} < +\infty.
\]
(2.3)

In view of e.g. Theorem 4.7.3 in [15], or Theorem 20.6 in [27] (for the finite state-space and time independent case), given the \(Q\)-matrix \(G\) and a probability measure \(\xi\) over \(I\), there exists a unique probability measure \(P\) on \((\Omega, F)\) under which the coordinate process \(x\) is a time-homogeneous Markov chain with intensity matrix \(G\) and starting distribution \(\xi\) i.e. such that \(P \circ x^{-1}(0) = \xi\). Equivalently, \(P\) solves the martingale problem for \(G\) with initial probability distribution \(\xi\) meaning that, for every \(f\) on \(I\), the process defined by
\[
M^f_t := f(x(t)) - f(x(0)) - \int_{[0,t]} Gf(x(s)) \, ds
\]
is a local martingale relative to \((\Omega, F, F_0)\), where
\[
Gf(i) := \sum_j g_{ij} f(j) = \sum_{j : j \neq i} g_{ij} (f(j) - f(i)), \quad i \in I,
\]
and
\[
Gf(x(s)) = \sum_{i,j \neq i} I_i(s) g_{ij} (f(j) - f(i)).
\]
(2.5)

By Lemma 21.13 in [27], the compensated processes associated with the counting processes \(N_{ij}\), defined by
\[
M_{ij}(t) = N_{ij}(t) - \int_{[0,t]} I_i(s^-) g_{ij} \, ds, \quad M_{ij}(0) = 0,
\]
are zero mean, square integrable and mutually orthogonal \(P\)-martingales whose predictable quadratic variations are
\[
\langle M_{ij} \rangle_t = \int_{[0,t]} I_i(s^-) g_{ij} \, ds.
\]
(2.7)

Moreover, at jump times \(t\), we have
\[
\Delta M_{ij}(t) = \Delta N_{ij}(t) = I_i(t^-) I_j(t).
\]
(2.8)

Thus, the optional variation of \(M\)
\[
[M](t) = \sum_{0 < s \leq t} |\Delta M(s)|^2 = \sum_{0 < s \leq t} \sum_{i,j : j \neq i} |\Delta M_{ij}(s)|^2
\]
is
\[
[M](t) = \sum_{0 < s \leq t} \sum_{i,j : j \neq i} I_i(s^-) I_j(s).
\]
(2.9)

We call \(M := \{M_{ij}, \ i \neq j\}\) the accompanying martingale of the counting process \(N := \{N_{ij}, \ i \neq j\}\) or of the Markov chain \(x\).

For a real-valued matrix \(m(t) := (m_{ij}(t), i, j \in I)\) indexed by \(I \times I\), we let
\[
\|m(t)\|_2^2 := \sum_{i,j : i \neq j} |m_{ij}(t)|^2 g_{ij} 1_{\{w(t^-) = i\}} < \infty.
\]
(2.10)
Let \((Z_{ij}, i \neq j)\) be a family of predictable processes and set
\[
\|Z(t)\|_g^2 := \sum_{i,j: i \neq j} Z_{ij}^2(t)I_i(t^-)g_{ij}, \quad 0 < t \leq T,
\]
where
\[
\sum_{0 < s \leq t} Z(s) \Delta M(s) := \sum_{0 < s \leq t} \sum_{i,j: i \neq j} Z_{ij}(s) \Delta M_{ij}(s).
\]

Consider the local martingale
\[
W(t) = \int_0^t Z(s)dM(s) := \sum_{i,j: i \neq j} \int_0^t Z_{ij}(s)dM_{ij}(s).
\]
Then, the optional variation of the local martingale \(W\) is
\[
[W](t) = \sum_{0 < s \leq t} |Z(s)\Delta M(s)|^2 = \sum_{0 < s \leq t} \sum_{i,j: i \neq j} |Z_{ij}(s)\Delta M_{ij}(s)|^2
\]
and its compensator is
\[
\langle W \rangle_t = \int_{[0,t]} \|Z(s)\|_g^2 ds.
\]
Provided that
\[
E \left[ \int_{(0,T]} \|Z(s)\|_g^2 ds \right] < \infty,
\]
\(W\) is a square-integrable martingale and its optional variation satisfies
\[
E[|W|(t)] = E \left[ \sum_{0 < s \leq t} |Z(s)\Delta M(s)|^2 \right] = E \left[ \int_{(0,t]} \|Z(s)\|^2_g ds \right].
\]
Moreover, the following Doob’s inequality holds:
\[
E \left[ \sup_{0 \leq t \leq T} \left| \int_0^t Z(s)dM(s) \right|^2 \right] \leq 4E \left[ \int_{(0,T]} \|Z(s)\|_g^2 ds \right].
\]
If \(\bar{Z}\) is another predictable process that satisfies (2.16), setting
\[
\langle Z(t), \bar{Z}(t) \rangle_g := \sum_{i,j: i \neq j} Z_{ij}(t)\bar{Z}_{ij}(t)I_i(t^-)g_{ij}, \quad 0 \leq t \leq T,
\]
and considering the martingale
\[
\bar{W}(t) = \int_0^t \bar{Z}(s)dM(s) := \sum_{i,j: i \neq j} \int_0^t \bar{Z}_{ij}(s)dM_{ij}(s),
\]
it is easy to see that
\[
E[|W,\bar{W}|(t)] = E \left[ \int_{(0,t]} \langle Z(s), \bar{Z}(s) \rangle_g ds \right].
\]
Since, the filtration \(\mathcal{F}\) generated by the chain \(x\) is the same as the filtration generated by the family of counting processes \(\{N_{ij}, i \neq j\}\), we state the following martingale representation theorem (see e.g. [2], Theorem T11 or [27], IV-21, Theorem 21.15).
Proposition 1 (Martingale representation theorem). If \( L \) is a (right-continuous) square-integrable \( \mathbb{F} \)-martingale, there exists a unique (\( dP \times g_{ij}I_i(s^-)ds \)-almost everywhere) family of predictable processes \( Z_{ij}, i \neq j \), satisfying

\[
E \left[ \int_{(0,T]} \|Z(s)\|^2ds \right] < +\infty, \tag{2.21}
\]

where

\[
\|Z(t)\|^2 := \sum_{i,j: i \neq j} Z_{ij}^2(t)I_i(t^-)g_{ij}, \quad 0 \leq t \leq T, \tag{2.22}
\]

such that

\[
\mathcal{L}_t = \mathcal{L}_0 + \int_0^t Z(s)dM(s), \quad 0 \leq t \leq T, \tag{2.23}
\]

where

\[
\int_0^t Z(s)dM(s) := \sum_{i,j: i \neq j} \int_0^t Z_{ij}(s)dM_{ij}(s).
\]

In particular, at jump times \( t \), we have

\[
\Delta \mathcal{L}_t = \sum_{i,j: i \neq j} Z_{ij}(t)\Delta M_{ij}(t) = \sum_{i,j: i \neq j} Z_{ij}(t)I_i(t^-)I_j(t).
\]

Next, we give an important application of Proposition 1 to the local martingale \( M^f \) given by (2.4) where an explicit form of the process \( Z \) can be displayed in terms of the function \( f \). At jump times \( t \), we have

\[
\Delta M^f_{ij} = M^f_{ij} - M^f_{ij} = \sum_{i} I_i(t^-) \sum_{j: j \neq i} I_j(t)(f(j) - f(i)) = \sum_{i,j: j \neq i} (f(j) - f(i))\Delta M_{ij}(t),
\]

since, by (2.8), at a jump time \( t \), \( I_i(t^-)I_j(t) = \Delta M_{ij}(t) \) and

\[
I_i(t^-)I_i(t) = I_{x(t^-) = i, x(t) = i} = 0.
\]

We may now define \( Z^f(t) = (Z^f_{ij}(t))_{ij} \) by

\[
Z^f_{ij}(t) := f(j) - f(i), \quad i, j \in I,
\]

to obtain

\[
M^f_{ij} = \int_{[0,t]} Z^f(s)dM(s) = \sum_{i,j: j \neq i} (f(j) - f(i))M_{ij}(t), \quad 0 \leq t \leq T. \tag{2.25}
\]

Provided that \( \sum_{i,j: i \neq j} (f(j) - f(i))^2g_{ij} < +\infty \), \( M^f \) is a square-integrable martingale.

For later use we need the following exponential estimate.

Lemma 2.1. Assume further that there exists \( \alpha > 0 \) such that the \( Q \)-matrix \( G = (g_{ij})_{ij} \) satisfies

\[
\sum_{i,j: i \neq j} e^{\alpha |j-i|}g_{ij} < +\infty. \tag{2.26}
\]

If there exists a constant \( \beta \geq \alpha \) such that

\[
E[e^{\beta x(0)}] < +\infty, \tag{2.27}
\]

then

\[
E[e^{\beta x(t)}] \leq \kappa_0, \tag{2.28}
\]
where
\[ \kappa_0 := E[e^{\alpha x(0)}]^{1/2} \exp \frac{T}{2} \sum_{i:j; i \neq j} (e^{\alpha j-i} - 1) g_{ij}. \]

In particular, for any \( q \geq 1 \), there exists a positive constant \( C_q \) which depends only on \( q, \alpha \) and \( \kappa_0 \) such that
\[ E[|x|^q_T] \leq C_q. \tag{2.29} \]

Proof. We note that by (2.3), using the same argument as [15], Theorem 6.4.1, the Markov chain \( x \) can be represented (in distribution) as the pure jump process \( \hat{x} \) defined by
\[ \hat{x}(t) = \zeta + \sum_{k \in \mathbb{Z}} k N^0_k \left( \int_0^t g_{\hat{x}(s)} \hat{x}(s) + k ds \right). \tag{2.30} \]
(by extending \( G = (g_{ij}, (i, j) \in I \times I) \) to \( (i, j) \in I \times \mathbb{Z} \) through setting \( g_{ij} = 0 \) for \( j < 0 \), where on a possibly larger probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\mathbb{P}})\) there exists a sequence \((N^0_k, k \in \mathbb{Z})\) of independent Poisson processes with intensity 1 and independent of \( \zeta \), such that \( \zeta \) and \( x(0) \) have the same distribution \( \xi(dy) \). Using the fact that \( t \to N^0_k(t) \) is a.s. increasing (indeed, by stationarity of the Poisson process we have for \( s \leq t \), \( \hat{\mathbb{P}}(N^0_k(t) - N^0_k(s) \geq 0) = \hat{\mathbb{P}}(N^0_k(t - s) \geq 0) = 1 \)) we have
\[ N^0_k \left( \int_0^t g_{\hat{x}(s)} \hat{x}(s) + k ds \right) = N^0_k \left( \int_0^t \sum_i 1_{\{\hat{x}(s) = i\}} g_{i+k} ds \right) \leq N^0_k \left( t \sum_i g_{i+k} \right), \]
applying the Cauchy-Scharz inequality we obtain
\[ E[e^{\alpha x^T}] = \hat{E}[e^{\alpha \hat{x}}] \leq \left( \hat{E}[e^{\alpha \zeta}] \right)^{1/2} \left( \hat{E} \left[ \exp \alpha \sum_{k \in \mathbb{Z}} |k| N^0_k \left( \sum_i g_{i+k} T \right) \right] \right)^{1/2}, \]
where \( \hat{E} \) denotes the expectation w.r.t. \( \hat{\mathbb{P}} \). Using the explicit form of the moment generating function of each of the independent Poisson processes, we obtain
\[ \hat{E} \left[ \exp \left( \alpha \sum_{k \in \mathbb{Z}} |k| N^0_k \left( T \sum_i g_{i+k} \right) \right) \right] = \exp \left( T \sum_{k \in \mathbb{Z}} (e^{\alpha |k|} - 1) \sum_i g_{i+k} \right) \]
\[ = \exp \left( T \sum_{i,j; i \neq j} (e^{\alpha j-i} - 1) g_{ij} \right). \]
Since \( \zeta \) and \( x(0) \) have the same distribution, we finally obtain
\[ E[e^{\alpha x^T}] \leq \left( E[e^{\alpha x(0)}] \right)^{1/2} \exp \frac{T}{2} \sum_{i:j; i \neq j} (e^{\alpha j-i} - 1) g_{ij} = \kappa_0. \tag{2.31} \]

2.2. Probability measures on \( I \). Let \( \mathcal{P}(I) \) denote the set of probability measures on \( I \). For \( \mu, \nu \in \mathcal{P}(I) \), the total variation distance is defined by the formula
\[ d(\mu, \nu) = 2 \sup_{A \subset I} |\mu(A) - \nu(A)| = \sum_{i \in I} |\mu(\{i\}) - \nu(\{i\})|. \tag{2.32} \]
Furthermore, let \( \mathcal{P}(\Omega) \) be the set of probability measures \( P \) on \( \Omega \) and \( \mathcal{P}_2(\Omega) \) be the subset of probability measures \( P \) on \( \Omega \) such that
\[ \|P\|_2 := \int_{\Omega} |w|^2 P(dw) = E[[x]^2] < +\infty, \]
Indeed, we have

\[ d(P, Q) := 2 \sup_{A \in \mathcal{F}} |P(A) - Q(A)|. \]  

(2.33)

Similarly, on the filtration \( \mathcal{F} \), we define the total variation metric between two probability measures \( P \) and \( Q \) as

\[ D_t(P, Q) := 2 \sup_{A \in \mathcal{F}_t} |P(A) - Q(A)|, \quad 0 \leq t \leq T. \]  

(2.34)

It satisfies

\[ D_s(P, Q) \leq D_t(P, Q), \quad 0 \leq s \leq t. \]  

(2.35)

For \( P, Q \in \mathcal{P}(\Omega) \) with time marginals \( P_t := P \circ x^{-1}(t) \) and \( Q_t := Q \circ x^{-1}(t) \), the total variation distance between \( P_t \) and \( Q_t \) satisfies

\[ d(P_t, Q_t) \leq D_t(P, Q), \quad 0 \leq t \leq T. \]  

(2.36)

Indeed, we have

\[ d(P_t, Q_t) := 2 \sup_{B \in \mathcal{F}} |P_t(B) - Q_t(B)| = 2 \sup_{B \in \mathcal{F}} |P(x^{-1}(t)(B)) - Q(x^{-1}(t)(B))| \]

\[ \leq 2 \sup_{A \in \mathcal{F}_t} |P(A) - Q(A)| = D_t(P, Q). \]

Endowed with the total variation metric \( D_T \), \( \mathcal{P}(\Omega) \) is a complete metric space. Moreover, \( D_T \) carries out the usual topology of weak convergence. But, \((D_T, \mathcal{P}_2(\Omega))\) may not be complete simply because the total variation metric does not guarantee existence of finite moments. This makes this distance less suitable for the study of models of mean-field type where the mean-field interaction is of the type \( E[x(t)] \) or \( E[\varphi(x(t))] \) when \( \varphi \) is a Lipschitz function. Nevertheless, as we show it below, the following subset of \( \mathcal{P}_2(\Omega) \)

\[ \mathcal{D}_{p, \kappa} := \{ Q \in \mathcal{P}_2(\Omega), dQ = XdP, X \text{ is } \mathcal{F}_T\text{-measurable and } E[X^p] \leq \kappa \}, \]

where \( p > 1 \) and \( \kappa > 0 \) are given constants, which fits with our framework, turns out a complete metric space when endowed with the total variation norm \( D_T \). Using this space requires a higher degree of smoothness on the intensity matrix of the Markov chain we impose below.

3. Jump processes of mean-field type. In this section we prove existence of a unique probability measure \( \tilde{P} \) on \((\Omega, \mathcal{F})\) under which the coordinate process \( x \) is a jump process with intensities \( \lambda_{ij}(t, x, \tilde{P} \circ x^{-1}(t)) \), \( i, j \in I \), where we allow the jump intensities at time \( t \) depend on the whole path \( x \) over the time interval \( [0, T] \) and also on the marginal law of \( x(t) \), as long as the intensities are predictable. Because of the dependence of its jump intensities on the marginal law, we call it jump process of mean-field type. If the intensities are deterministic functions of \( t \) and the marginal law of \( x(t) \) i.e. they are of the form \( \lambda_{ij}(t, \tilde{P} \circ x^{-1}(t)) \), \( i, j \in I \), we call \( x \) a Markov chain of mean-field type or simply an nonlinear Markov chain.

The probability measure \( \tilde{P} \) is constructed as follows. We start with the probability measure \( P \) which solves the martingale problem associated with \( G = (g_{ij}) \), introduced in Section (2.1), making the coordinate process \( x \) a time-homogeneous Markov chain. Then, using a Girsanov-type change of measure in terms of a Doléans-Dade exponential martingale for jump processes which involves the intensities \( \lambda_{ij} \) and \( g_{ij} \), we obtain our probability measure \( \tilde{P} \). It is also possible to choose \( G \) time-dependent. But, it is easier to deal with time-independent intensities.
Let $\lambda$ be a measurable process from $[0,T] \times I \times I \times \Omega \times \mathcal{P}(I)$ into $(0, +\infty)$ such that

(A1) For every $Q \in \mathcal{P}_2(\Omega)$, the process $(\lambda_{ij}(t,x,Q \circ x^{-1}(t)))_t$ is predictable.

(A2) There exists a positive constant $c_1$ such that for every $(t,i,j) \in [0,T] \times I \times I; i \neq j, w \in \Omega$ and $\mu \in \mathcal{P}(I)$,

$$\lambda_{ij}(t,w,\mu) \geq c_1 > 0.$$  

(A3) For $p = 1, 2$, and for every $t \in [0,T]$, $w \in \Omega$ and $\mu \in \mathcal{P}_2(I)$,

$$\sum_{i,j \neq i} |j-i|^p \lambda_{ij}(t,w,\mu) \leq C(1 + |w|^p_2 + \int |y|^p_2 \mu(dy)).$$

(A4) The $Q$-matrix $(g_{ij})$ and the probability measure $\xi$ on $I$ satisfy

$$\sum_{i,j \neq i} |j-i|^2 g_{ij} < \infty, \quad ||\xi||^2 := \int |y|^2 \xi(dy) < \infty.$$  

(A5) For $p = 1, 2$, and for every $t \in [0,T]$, $w, \tilde{w} \in \Omega$ and $\mu, \nu \in \mathcal{P}(I)$,

$$\sum_{i,j \neq i} |j-i|^p |\lambda_{ij}(t,w,\mu) - \lambda_{ij}(t,\tilde{w},\nu)| \leq C(|w-\tilde{w}|_p + d^p(\mu,\nu)).$$

**Remark 1.**

1. Assumption (A4) is needed to guarantee that the chain has finite second moment.

2. Since $|j-i| \geq 1$, we obtain from (A5) the following Lipschitz property of the intensity matrix

$$||\lambda(t,w,\mu) - \lambda(t,\tilde{w},\nu)|| \leq C(|w-\tilde{w}|_4 + d(\mu,\nu)).$$

**Example 3.1. A mean-field Schlögl model.** In the mean-field version of the Schlögl model (cf. [7], [17] and [16]) the intensities are

$$\lambda_{ij}(w,\mu) := \begin{cases} 
\nu_{ij} & \text{if } j \neq i + 1, \\
\nu_{i+1} + ||\mu||_1 & \text{if } j = i + 1,
\end{cases}$$

(3.1)

where $||\mu||_1 = \int |y| \mu(dy)$ is the first moment of the probability measure $\mu$ on $I$ and $(\nu_{ij})$ is a $Q$-matrix satisfying $\inf_{i \in I} \nu_{i+1} > 0$ and there exists $N_0 \geq 1$ such that $\nu_{ij} = 0$, for $|i-j| \geq N_0$. The martingale problem formulation states that, for every $f$ on $I$, the process defined by

$$M^f_t := f(x(t)) - f(x(0)) - \int_{[0,t]} (\tilde{G}(s)f)(x(s)) ds$$

is a local martingale relative to $(\Omega, \mathcal{F}, \mathbb{F})$, where

$$\tilde{G}(s)f(i) = \sum_{j \neq i} \nu_{ij}(f(j) - f(i)) + \sum_j jP(x(s) = j)(f(i+1) - f(i)).$$

(3.2)

Let $P$ be the probability under which $x$ is a time-homogeneous Markov chain such that $P \circ x^{-1}(0) = \xi$ and with time-independent $Q$-matrix $(g_{ij})_{ij}$ satisfying (2.3) and (2.26).

Assume further that

$$c_2 := \inf_{i,j \neq j} g_{ij} > 0.$$  

(3.3)

This condition is needed below to obtain estimates involving the density of Girsanov-Doléans-Dade type change of measure between two probability measures under
which the chain has jump intensities are \( \lambda_{ij} \) and \( g_{ij} \), respectively. This amounts to only taking into account nonzero jump intensities.

To ease notation we set, for \((t, i, j) \in [0, T] \times I \times I \) and \( Q \in \mathcal{P}(\Omega) \)

\[
\lambda^Q_{ij}(t) := \lambda_{ij}(t, x, Q \circ x^{-1}(t)), \quad \lambda^Q_0(t) := -\sum_{j \neq i} \lambda_{ij}^Q(t). \quad (3.4)
\]

Let \( P^Q \) be the measure on \((\Omega, \mathcal{F})\) defined by

\[
dP^Q := L^Q(T)dP,
\]

where

\[
L^Q(T) := \prod_{i,j \neq j} \exp \left\{ \int_{[0,T]} \ln \frac{\lambda^Q_{ij}(t)}{g_{ij}} dN_{ij}(t) - \int_0^T (\lambda^Q_{ij}(t) - g_{ij}) I_i(t) dt \right\},
\]

is the Doleans-Dade exponential. It is the solution of the following linear stochastic integral equation

\[
L^Q(t) = 1 + \int_{[0,t]} L^Q(s^-) \sum_{i,j \neq j} I_i(s^-) \ell^Q_{ij}(s) dM_{ij}(s),
\]

where

\[
\ell^Q_{ij}(s) = \left\{ \begin{array}{ll}
\lambda^Q_{ij}(s)/g_{ij} - 1 & \text{if } i \neq j, \\
0 & \text{if } i = j,
\end{array} \right.
\]

and \((M_{ij})_{ij}\) is the \(P\)-martingale given in (2.6).

If \( L^Q \) is a \( P \)-martingale, then by Girsanov theorem, \( P^Q \) is a probability measure on \((\Omega, \mathcal{F})\) under which the coordinate process \( x \) is a jump process with intensity matrix \( \lambda^Q := (\lambda^Q_{ij}(t))_{i,j} \) and starting distribution \( P^Q \circ x^{-1}(0) = \xi \). In particular, the compensated processes associated with the counting processes \( N_{ij} \) defined by

\[
M^Q_{ij}(t) := N_{ij}(t) - \int_{[0,t]} I_i(s^-) \lambda^Q_{ij}(s) ds,
\]

are zero mean, square integrable and mutually orthogonal \( P^Q \)-martingales whose predictable quadratic variations are

\[
\langle M^Q_{ij} \rangle_t = \int_{[0,t]} I_i(s^-) \lambda^Q_{ij}(s) ds.
\]

Using (3.8), we may write \( M^Q_{ij} \) in terms of \( M_{ij} \) as follows.

\[
M^Q_{ij}(t) = M_{ij}(t) - \int_{[0,t]} \ell^Q_{ij}(s) I_i(s^-) g_{ij} ds.
\]

Now, since \( L^Q \) is a positive \( P \)-local martingale, it is a supermartingale. Thus, \( E[L^Q(T)] \leq 1 \). In order to show that is a \( P \)-martingale, we need to show that \( E[L^Q(T)] = 1 \). We note that the imposed conditions (A1)-(A4) on the intensity matrix \( \lambda^Q \) do not fit with the assumptions displayed in the literature ranging from [2], Theorem T11, to [29], Theorem 2.4, to guarantee that \( L^Q \) is a \( P \)-martingale.

To show that \( L^Q \) is a \( P \)-martingale we will use the following apriori estimate.

**Lemma 3.2.** Let \( Q \in \mathcal{P}_2(\Omega) \) and assume \( \lambda^Q \) and \( \xi \) satisfy (A1)-(A4). If \( P^Q \), given by (3.5), is a probability measure on \((\Omega, \mathcal{F})\), then

\[
\|P^Q\|_2^2 = E_{P^Q}[\|x\|_T^2] \leq C e^{C_T}(1 + \|\xi\|_2^2 + \|Q\|_2^2) < +\infty.
\]

(3.12)
In particular, 
\[ \|P\|_2^2 = E\|x\|_T^2 \leq C e^{CT}(1 + \|\xi\|_2^2) < +\infty. \] (3.13)

Proof. Since, by Girsanov theorem, under \(P^Q\), \(x\) has jump intensity \(\lambda_{ij}(t, x, Q \circ x^{-1}(t))\), applying a similar formula as (2.4) to \(f(x) = x\), where instead of \(G\) we use the matrix \(\lambda^Q\), we obtain
\[ x(t) = x(0) + \int_{[0,t]} \lambda^Q(s)x(s) \, ds + M^x(t), \] (3.14)
where
\[ M^x(t) = \int_{[0,t]} Z^x(s)dM(s) = \sum_{i,j} (j - i)M_{ij}(t), \] (3.15)
with
\[ \|Z^x(s)\|_{\lambda^Q}^2 = \sum_{i,j} (j - i)^2I_i(s^-)\lambda^Q_{ij}(s), \]
and
\[ \lambda^Q(s)x(s) = \sum_{i,j; j \neq i} (j - i)I_i(s^-)\lambda^Q_{ij}(s), \]
which, in view of (A3), satisfy
\[ |\lambda^Q(s)x(s)|^2 + \|Z^x(s)\|_{\lambda^Q}^2 \leq C(1 + |x|^2 + \int w^2 Q(dw)), \] 0 \leq s \leq T. (3.16)

Therefore, applying the Cauchy-Schwarz inequality together with (2.18) to (3.14) we obtain
\[ E_{P^Q}\|x\|_T^2 \leq C E_{P^Q} \left[ |x(0)|^2 + \int_{[0,T]} (|\lambda^Q(s)x(s)|^2 + \|Z^x(s)\|_{\lambda^Q}^2) \, ds \right], \]
and by (3.16) we get
\[ E_{P^Q}\|x\|_T^2 \leq C \left(1 + \|\xi\|^2 + \int_{[0,T]} \left( E_{P^Q}\|x\|_T^2 + \int w^2 Q(dw) \right) \, ds \right). \] (3.17)
Using \(\int w^2 Q(dw) \leq \int |w|^2 Q(dw) \leq \|Q\|_2^2\) and applying Gronwall’s inequality we finally get (3.12).

Proposition 2. Let \(Q \in \mathcal{P}_2(\Omega)\) and assume \(\lambda^Q\) and \(\xi\) satisfy (A1)-(A4). Then, \(L^Q\) is a \(P\)-martingale.

Proof. The proof is inspired by the proof of Proposition (A.1) in [12]. As mentioned above, it suffices to prove that \(E[L^Q(T)] = 1\). For \(n \geq 0\), let \(\lambda^n\) be the predictable intensity matrix given by \(\lambda^n_{ij}(t) := \lambda^Q_{ij}(t)1_{\{\omega, |x(\omega)|_{i-1} \leq n\}}\) and let \(L^n\) be the associated Doolean-Dade exponential and \(P^n\) the positive measure defined by \(dP^n = L^n(T) dP\). Noting that, for \(i, j \in I, i \neq j, |i - j| \geq 1\), by (A3), we have
\[ \lambda_{ij}(t, w, \mu) \leq C(1 + |w|_I + \int |y|\mu(dy)). \]
Thus, for every \(n \geq 1\), \(\lambda^n_{ij}(t) \leq C(1 + n + \|Q\|_2)\), i.e. \(\lambda^n_{ij}\) is bounded. In view of [2], Theorem T11, \(L^n\) is a \(P\)-martingale. In particular, \(E[L^n(T)] = 1\) and \(P^n\) is a probability measure. By (3.13), \(|x|_T < \infty, P\text{-a.s.}\). Therefore, on the set \(\{\omega, |x(\omega)|_{T} \leq n_0\}\), for all \(n \geq n_0\), \(L^n(T, \omega) = L^Q(T, \omega)\). This in turn yields that \(L^n(T) \rightarrow L^Q(T), P\text{-a.s., as } n \rightarrow +\infty\). Now, if \((L^n(T))_{n \geq 1}\) is uniformly integrable,
the $P$-a.s. convergence implies $L^1(P)$-convergence of $L^n(T)$ to $L^Q(T)$, yielding $E[L^Q(T)] = 1$. It remains to show that $(L^n(T))_{n \geq 1}$ is uniformly integrable:

$$
\lim_{a \to \infty} \sup_{n \geq 1} \int_{\{L^n(T) > a\}} L^n(T) \, dP = 0.
$$

For $m \geq 1$, set $\theta_m = \inf\{t \leq T, |x|_t \geq m\}$ if the set is nonempty and $\theta_m = T + 1$ if it is empty. Denoting by $E^n$ the expectation w.r.t. $P^n$, we have

$$
\int_{\{\theta_m \leq T\}} L^n(T) \, dP = P^n(\theta_m \leq T) = P^n(|x|_T \geq m) 
\leq E^n[|x|_T] / m \leq C/m,
$$

where, by (3.12), $C$ does not depend on $n$.

Let $\eta > 0$. Choose $m_0 \geq 1$ such that $C/m_0 < \eta$. We have, for all $n \geq m_0$,

$$
L^n(T \wedge \theta_{m_0}) = L^n(T \wedge \theta_{m_0}).
$$

This entails that, as $a \to \infty$,

$$
\sup_{n \geq 1} \int_{\{L^n(T \wedge \theta_{m_0}) > a\}} L^n(T \wedge \theta_{m_0}) \, dP = \max_{n \leq m_0} \int_{\{L^n(T \wedge \theta_{m_0}) > a\}} L^n(T \wedge \theta_{m_0}) \, dP \to 0.
$$

So there exists $a_0 > 0$ such that whenever $a > a_0$,

$$
\max_{n \leq m_0} \int_{\{L^n(T \wedge \theta_{m_0}) > a\}} L^n(T \wedge \theta_{m_0}) \, dP < \eta.
$$

We have

$$
\sup_{n \geq 1} \int_{\{L^n(T) > a\}} L^n(T) \, dP \leq \sup_{n \geq 1} \int_{\{L^n(T) > a, \theta_{m_0} \leq T\}} L^n(T) \, dP
+ \sup_{n \geq 1} \int_{\{L^n(T) > a, \theta_{m_0} > T\}} L^n(T) \, dP
\leq \sup_{n \geq 1} \int_{\{\theta_{m_0} \leq T\}} L^n(T) \, dP + \sup_{n \geq 1} \int_{\{L^n(T \wedge \theta_{m_0}) > a\}} L^n(T \wedge \theta_{m_0}) \, dP
\leq C/m_0 + \eta < 2\eta,
$$

in view of (3.18) and (3.19). This finishes the proof since $\eta$ is arbitrary.

Next, we will show that there is $\hat{Q}$ such that $P^\hat{Q} = \hat{Q}$, i.e., $\hat{Q}$ is a fixed point. It is the probability measure under which the coordinate process is a jump process of mean-field type.

**Lemma 3.3.** Let $p > 1$ and $\kappa > 0$ be given constants. The set $\mathbb{D}_{p,\kappa}$ defined by

$$
\mathbb{D}_{p,\kappa} := \{Q \in \mathcal{P}(\Omega), dQ = XdP, X \text{ is } \mathcal{F}_T \text{-measurable and } E[X^p] \leq \kappa\},
$$

endowed with total variation norm $D_T$ is a complete metric space.

**Proof.** Let $(Q_n)_{n \geq 0}$ be a Cauchy sequence in $(\mathbb{D}_{p,\kappa}, D_T)$. Thus, for any $n \geq 0$,

$$
dQ^n = X_n dP \text{ with } E[X^n] \leq \kappa.
$$

Since $(\mathcal{P}(\Omega), D_T)$ is a complete metric space and $\mathbb{D}_{p,\kappa} \subset \mathcal{P}(\Omega)$, there exists a probability $Q \in \mathcal{P}(\Omega)$ such that $D_T(Q_n, Q) \to 0$ as $n \to \infty$. Next, since $p > 1$, there exists a subsequence, which we still denote by $(X_n)_{n \geq 0}$, and an $\mathcal{F}_T$-measurable random variable $X$ such that $X_n \to X$ weakly in $L^p(\Omega, \mathcal{F}_T, P)$, as $n \to \infty$. But, for any $A \in \mathcal{F}_T$

$$
\lim_{n} Q^n(A) = \lim_{n} E[X_n 1_A] = E[X 1_A] = Q(A).
$$
This entails that the probability \( Q \) has a density w.r.t. \( P \) which is given by \( X \). On the other hand, by the semi-continuity of the norm w.r.t. the weak topology, we obtain \( E[X^p] \leq \lim\inf_n E[X^n_p] \leq \kappa \). Finally, by Hölder’s inequality, we have

\[
E_Q[|x|^2_T] = E[|x|^2_T X] \leq (E[X^p])^{\frac{2}{p}} (E[|x|^{2q}_T])^{\frac{1}{q}} < +\infty
\]

for \( q \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), since, by (2.26), \( |x|_T \) satisfies (2.29). Hence, \( Q \) belongs to \( \mathbb{D}_{p,\kappa} \) and \( (\mathbb{D}_{p,\kappa}, D_T) \) is a complete metric space.

**Theorem 3.4.** Assume (A1)-(A5) and consider the metric space \( (\mathcal{P}_2(\Omega), D_T) \).

Then, the map

\[
\Phi : \mathcal{P}_2(\Omega) \to \mathcal{P}_2(\Omega) \quad Q \mapsto \Phi(Q) := P^Q, \quad dP^Q := L^Q(T)dP, \quad P^Q \circ x^{-1}(0) = \xi,
\]

is well defined.

Assume further that the intensities \( \lambda_{ij} \) satisfy the following condition.

(A6) For every \( t \in [0, T] \), \( w \in \Omega \) and \( \mu \in \mathcal{P}_1(I) \),

\[
\sum_{i,j : j \neq i} \lambda_{ij}^3(t, w, \mu) \leq C(1 + |w|_t + \int |y|d\mu(dy)).
\]

(A7) For \( \alpha := \frac{2CT}{c_2} \), where \( C \) is the constant in (A6), such that

\[
\sum_{i,j : j \neq i} e^{\alpha|i-j|}g_{ij} < \infty, \quad \int e^{\alpha y}d\xi(dy) < \infty.
\]

Then, the probability density \( L^Q(T) \) is bounded in \( L^2(P) \). Moreover, there exists a constant \( \kappa > 0 \) such that \( \Phi(\mathbb{D}_{2,\kappa}) \subset \mathbb{D}_{2,\kappa} \). Furthermore, \( \Phi \) admits a fixed point in \( \mathbb{D}_{2,\kappa} \).

If \( \hat{Q} \) denotes such a fixed point, it satisfies

\[
\|\hat{Q}\|_2^2 \leq Ce^{2CT}(1 + \|\xi\|_2^2) < \infty.
\]

**Proof.** Let \( Q \in \mathcal{P}_2(\Omega) \). Then, by (3.12),

\[
\|\Phi(Q)\|_2^2 = E_{\Phi(Q)}[|x|^2_T] \leq Ce^{CT}(1 + \|\xi\|_2^2 + \|Q\|_2^2) < +\infty,
\]

which implies that \( \Phi(Q) \in \mathcal{P}_2(\Omega) \), since \( \|\xi\|_2^2 < +\infty \) by (A4).

Next, we show that \( E[(L^Q(T))^2] \leq \kappa \) for some constant \( \kappa > 0 \). Let \( h_{ij}(t) := \lambda_{ij}^q(t)/g_{ij} \) and define

\[
\varphi_T := \prod_{i,j \neq j} \exp \left\{ \int_{[0,T]} \ln h_{ij}^2(t) dN_{ij}(t) - \int_0^T \left( \frac{h_{ij}^4(t)}{2} - \frac{1}{2} \right) g_{ij}I_i(t)dt \right\}
\]

and

\[
\psi_T := \prod_{i,j \neq j} \exp \left\{ \int_0^T \left( \frac{h_{ij}^4(t)}{2} - \frac{1}{2} - 2(h_{ij}(t) - 1) \right) g_{ij}I_i(t)dt \right\}.
\]

We have

\[
(L^Q(T))^2 = \varphi_T \psi_T.
\]
The choice of \( \varphi_T \) is made so that the process \( \varphi_t^2 \) is a Doléans-Dade positive supermartingale satisfying \( E[\varphi_T^2] \leq 1 \). Furthermore, by (2.3), (3.3) and (A6), we have

\[
\psi_T \leq \prod_{i,j \neq j} \exp \left\{ \int_0^T \left( \frac{(\lambda_i^Q(t))^4}{2g_{ij}} + \frac{3}{2} g_{ij} \right) dt \right\}
\]

\[
\leq \exp \left\{ \frac{CT}{2} \left( 1 + |x_T| + \|Q\|_1 + \frac{3T}{2} \sum_{i,j} g_{ij} \right) \right\}.
\]

Therefore,

\[
E[\psi_T^2]^{1/2} \leq \mu_T E \left[ e^{\frac{CT}{2}|x_T|} \right]^{1/2},
\]

where

\[
\mu_T := \exp \left\{ \frac{T}{2} \left[ \frac{C}{c^2} (1 + \|Q\|_1) + \sum_{i,j \neq j} g_{ij} \right] \right\}.
\]

Since, by Hölder inequality, we have

\[
E[(L^Q(T))^2] \leq E[\varphi_T^2]^{1/2} E[\psi_T^2]^{1/2} \leq E[\psi_T^2]^{1/2},
\]

It follows from (3.21) that

\[
E[(L^Q(T))^2] \leq \mu_T E \left[ e^{\frac{CT}{2}|x_T|} \right]^{1/2}.
\]

In view of (2.31), we obtain

\[
E[(L^Q(T))^2] \leq \mu_T \kappa_0 := \kappa.
\]

Next, we show the contraction property of the map \( \Phi \) on \( D_{2,\kappa} \). To this end, given \( Q, \tilde{Q} \in P_2(\Omega) \), we use an estimate of the total variation distance \( D_T(\Phi(Q), \Phi(\tilde{Q})) \) in terms of the relative entropy \( H(\Phi(Q) \| \Phi(\tilde{Q})) \) between \( \Phi(Q) \) and \( \Phi(\tilde{Q}) \) given by the celebrated Csiszár-Kullback-Pinsker inequality:

\[
D_T^2(\Phi(Q), \Phi(\tilde{Q})) \leq 2H(\Phi(Q) \| \Phi(\tilde{Q}))
\]

where,

\[
H(\Phi(Q) \| \Phi(\tilde{Q})) = E_{\Phi(Q)} \left[ \log \frac{L^Q(T)}{L^{\tilde{Q}}(T)} \right].
\]

In view of (3.6), we have

\[
\log \frac{L^Q(T)}{L^{\tilde{Q}}(T)} = \sum_{i,j \neq j} \int_{[0,T]} \ln \frac{\lambda_i^Q(t)}{\lambda_i^{\tilde{Q}}(t)} dN_{ij}(t) - \int_0^T (\lambda_i^Q(t) - \lambda_i^{\tilde{Q}}(t)) I_i(t) dt.
\]

Taking expectation w.r.t. \( \Phi(Q) \), using (3.9), we obtain

\[
H(\Phi(Q) \| \Phi(\tilde{Q})) = \sum_{i,j \neq j} E_{\Phi(Q)} \left[ \int_0^T \left( (\tau(\lambda_i^Q(t)) - \tau(\lambda_i^{\tilde{Q}}(t))) - (\lambda_i^Q(t) - \lambda_i^{\tilde{Q}}(t)) \log \lambda_i^{\tilde{Q}}(t) \right) I_i(t) dt \right],
\]

where \( \tau(x) := x \log x - x + 1, x > 0 \) is a convex function. We note that the r.h.s. of this last equality is non-negative, since, by convexity, we have

\[
\tau(x) \geq \tau(y) + (x - y)\tau'(y),
\]
where $\tau'(y) = \log y$. Using Taylor expansion we get
\[
\tau(x) = \tau(y) + (x - y)\tau'(y) + \frac{1}{2}(x - y)^2\tau''(z),
\]
for some $z \in \{tx + (1-t)y : t \in (0,1)\}$, where $\tau''(z) = 1/z$. Taking $x, y$ such that $x, y \geq c_1 > 0$, as in (A2), we obtain
\[
\tau(x) \leq \tau(y) + (x - y)\tau'(y) + \frac{1}{2c_1}(x - y)^2.
\]  
(3.24)

Applying (3.24) to the entropy (3.23), we obtain
\[
H(\Phi(Q)|\Phi(\tilde{Q})) \leq \frac{1}{2c_1} \sum_{i,j: i \neq j} E_{\Phi(Q)} \left[ \int_0^T (\lambda_{ij}^Q(t) - \lambda_{ij}^{\tilde{Q}}(t))^2 dt \right].
\]
Combining this inequality with (3.22), we obtain
\[
D_2^T(\Phi(Q), \Phi(\tilde{Q})) \leq \frac{1}{c_1} \sum_{i,j: i \neq j} E_{\Phi(Q)} \left[ \int_0^T (\lambda_{ij}^Q(t) - \lambda_{ij}^{\tilde{Q}}(t))^2 dt \right].
\]  
(3.25)

We may use (A5) to obtain
\[
\sum_{i,j: i \neq j} E_{\Phi(Q)} \left[ (\lambda_{ij}^Q(t) - \lambda_{ij}^{\tilde{Q}}(t))^2 \right] \leq Cd^2(Q_t, \tilde{Q}_t) \leq CD_2^T(Q, \tilde{Q}).
\]

Therefore,
\[
D_2^T(\Phi(Q), \Phi(\tilde{Q})) \leq 2H(\Phi(Q)|\Phi(\tilde{Q})) \leq \frac{C}{c_1} \int_0^T D_2^T(Q, \tilde{Q}) dt.
\]

Iterating this inequality, we obtain, for every $N > 0$,
\[
D_2^T(\Phi^N(Q), \Phi^N(\tilde{Q})) \leq CN \int_0^T \frac{(T-t)^{N-1}}{(N-1)!} D_2^T(Q, \tilde{Q}) dt \leq \frac{C^NT^N}{N!} D_2^T(Q, \tilde{Q}),
\]
where $\Phi^N$ denotes the $N$-fold composition of the map $\Phi$. Hence, for $N$ large enough, $\Phi^N$ is a contraction which implies that $\Phi$ admits a unique fixed point.

Finally, using (3.17) with $\Phi(\tilde{Q}) = \tilde{Q}$, noting that $\int w_2^2 \tilde{Q}(dw) \leq \int |w|^2 E_{\tilde{Q}}[|x|^2]$, we get
\[
E_{\tilde{Q}}[|x|^2] \leq C \left(1 + \|\xi\|_2^2 + 2 \int_{[0,T]} E_{\tilde{Q}}[|x|^2] ds\right).
\]

Applying Gronwall’s inequality we obtain the estimate
\[
\|\tilde{Q}\|_2^2 \leq Ce^{2CT}(1 + \|\xi\|_2^2) < \infty.
\]

\[\square\]

**Remark 2.** The mean-field Schlögl model (3.1) satisfies (A6) and (A7).

**Corollary 1.** The mapping $t \mapsto P \circ x^{-1}(t)$ is continuous. More precisely, we have
\[
d(P_t, P_s) \leq C(1 + \|P\|_2)(t-s), \quad 0 \leq s \leq t \leq T.
\]  
(3.26)

**Proof.** The inequality (3.26) follows by applying the above estimates to the martingale (2.4) with $f(x) = I_{\{x\in A\}}$, $A \subset I$, where we use the matrix $\lambda$ instead of $G$. \[\square\]
3.1. Markov chain BSDEs. An important consequence of Proposition 1 are solutions \((Y, Z)\) of Markov chain backward stochastic differential equations (BSDEs) defined on \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{P})\) by

\[-dY(t) = f(t, \omega, Y(t^-), Z(t))dt - Z(t)dM(t), \quad Y(T) = \zeta,\]

where \(\zeta \in L^2(\mathcal{F}_T, \mathbb{R})\) and \(f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{I \times I} \rightarrow \mathbb{R}\) is an integrable and \(\mathbb{F}\)-predictable function.

It is easily seen that if \((Y, Z)\) solves (3.27) then it admits the following representation:

\[Y(t) = \mathbb{E}\left[\zeta + \int_t^T f(s, \omega, Y(s^-), Z(s)) ds \bigg| \mathcal{F}_t\right], \quad t \in [0, T].\]

Moreover, \(t \mapsto Y(t)\) is right-continuous with left limits. Therefore, \(Y(t^-) = Y(t)\) a.e. Hence, we may write

\[Y(t) = \mathbb{E}\left[\zeta + \int_t^T f(s, \omega, Y(s), Z(s)) ds \bigg| \mathcal{F}_t\right], \quad t \in [0, T].\]

Existence and uniqueness results of solutions of Markov chain BSDEs (3.27) based on the martingale representation theorem (\(L^2\)-theory) have been recently studied in a series of papers by Cohen and Elliott (see e.g. [3] and the references therein). Their approach essentially adapts the method for solving Brownian motion driven BSDEs established first in [26]. Recently, Confortola et al. [5] derived existence and uniqueness results for more general classes of BSDEs driven by marked point processes under only \(L^1\)-integrability conditions. In this paper we use the \(L^2\)-theory as we want to use the martingale representation theorem in our optimal control problem.

Below, we establish existence of an optimal control and a saddle-point for the zero-sum game using some properties of the following class of BSDEs.

\[-dY(t) = \phi(t, x, Z(t))dt - Z(t)dM(t), \quad Y(T) = \zeta,\]

\((M_{ij})_{ij}\) is the \(\mathcal{P}\)-martingale given in (2.6) and the driver \(\phi\) is essentially of the form

\[\phi(t, x, p) := f(t, x) + \langle \ell(t, x), p \rangle_g,\]

where \(p := (p_{ij}, i, j \in I)\) is a real-valued matrix indexed by \(I \times I\) and \(\ell = (\ell_{ij}, i, j \in I)\) is given by

\[\ell_{ij}(s, x) := \begin{cases} \lambda_{ij}(s, x)/g_{ij} - 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}\]

where the predictable process \(\lambda(t, x) = (\lambda_{ij}(t, x), i, j \in I)\) is the intensity matrix of the chain \(x\) under a probability measure \(\widetilde{P}\) on \((\Omega, \mathcal{F})\) given by a similar formula as (3.5)-(3.6). In particular, as in (3.11), the processes

\[\widetilde{M}_{ij}(t) = M_{ij}(t) - \int_{[0,t]} \ell_{ij}(s, x) I_i(s^-) g_{ij} ds\]

are zero mean, square integrable and mutually orthogonal \(\widetilde{P}\)-martingales whose predictable quadratic variations are

\[\langle \widetilde{M}_{ij} \rangle_t = \int_{[0,t]} I_i(s^-) \lambda_{ij}(s, x) ds.\]

Moreover, \(\phi\) satisfies a ‘stochastic’ Lipschitz condition. More precisely, we make the following assumptions on the driver \(\phi\) and the terminal value \(\xi\).
(H1) P-a.s., for all $(t, \omega) \in [0, T] \times \Omega$, $z_1 = (z_{1j})$, $z_2 = (z_{2j})$, $x_{ij}, z_{ij} \in \mathbb{R}$,

$$\phi(t, \omega, z_1) - \phi(t, \omega, z_2) \leq a(t)\|z_1 - z_2\|_g.$$

(see Notation 2.10) where $(a(t))$ is a nonnegative and progressively measurable process which belongs to $L^2([0, T] \times \Omega, dt \times dP$).

(H2) $\phi(t, \omega, 0)$ is bounded.

(H3) $\zeta$ is an $\mathcal{F}_T$-measurable and bounded random variable.

First we establish a comparison result for solutions of the BSDE (3.28). This result is a key argument in the proof of existence and uniqueness of solutions of our BSDE.

**Proposition 3.** Let, for $i = 1, 2$, $(Y^i, Z^i)$ be the solutions of the BSDE (3.28) associated with $(\phi^i, \zeta^i)$ respectively, where $(\phi^i, \zeta^i)$ satisfy (H1) to (H3). Assume that

(H4) $\zeta^1 \geq \zeta^2$, P-a.s.;

(H5) for any $(t, \omega) \in [0, T] \times \Omega$, $\phi^1(t, \omega, Z^2(t)) \geq \phi^2(t, \omega, Z^2(t))$ P-a.s.

then, $Y^1 \geq Y^2$ on $[0, T]$ P-a.s.

**Proof.** Set

$$\hat{Y} := Y^1 - Y^2, \quad \hat{Z} := Z^1 - Z^2, \quad \hat{\zeta} := \zeta^1 - \zeta^2.$$

Using (H4) and (H5), we obtain

$$\begin{align*}
\hat{Y}_t = \hat{\zeta} + \int_t^T (\phi^1(s, x, Z^1(s)) - \phi^1(s, x, Z^2(s)))ds \\
+ \int_t^T (\phi^1(s, x, Z^1(s)) - \phi^2(s, x, Z^2(s)))ds - \int_t^T \hat{Z}(s)dM(s) \\
\geq \int_t^T (\phi^1(s, x, Z^1(s)) - \phi^1(s, x, Z^2(s)))ds - \int_t^T \hat{Z}(s)dM(s) \quad \text{P-a.s.}
\end{align*}$$

Since

$$\phi^1(s, x, Z^1(s)) - \phi^1(s, x, Z^2(s)) = \langle \ell(s, x), \hat{Z}(s) \rangle_g,$$

by (3.31)

$$\int_0^t \hat{Z}(s)d\hat{M}(s) = \int_0^t \hat{Z}(s)dM(s) - \int_0^t \langle \ell(s, x), \hat{Z}(s) \rangle_g ds$$

is a zero-mean $\hat{P}$-martingale. Since $\hat{P}$ is absolutely continuous w.r.t. $P$, we also have

$$\hat{Y}_t \geq -\int_t^T \hat{Z}(s)d\hat{M}(s) \quad \text{P}-\text{a.s.}$$

Hence, taking conditional expectation w.r.t. $\mathcal{F}_t$, we obtain

$$\hat{Y}_t \geq -\mathbb{E} \left[ \int_t^T \hat{Z}(s)d\hat{M}(s) | \mathcal{F}_t \right] = 0 \quad \text{P and } \hat{P}\text{-a.s.}$$

\[\Box\]

**Theorem 3.5.** Let $(\phi, \zeta)$ satisfies the assumptions (H1) to (H3). Then, the BSDE (3.28) associated with $(\phi, \zeta)$ admits a solution $(Y, Z)$ consisting of an adapted process $Y$ which is right-continuous with left limits and a predictable process $Z$ which satisfy

$$E \left[ \sup_{t \in [0, T]} |Y(t)|^2 + \int_{[0, T]} \|Z(s)\|_g^2 ds \right] < +\infty.$$

This solution is unique up to indistinguishability for $Y$ and for $Z$ equality $dP \times g_{ij}I(s^-)ds$-almost everywhere.
Using Proposition 3, the proof of the theorem is similar to that of the Brownian motion driven BSDEs derived in [18], Theorem I-3, using an approximation scheme by an increasing sequence of standard Markov chain BSDEs for which existence, uniqueness and comparison results are similar to that of the Brownian motion driven BSDEs derived in [26] and [13], along with the properties (2.17) and (2.18) related to the martingale \( W \) displayed in (2.13) together with Itô’s formula for semimartingales driven by counting processes. We omit the details.

4. **Optimal control of jump processes of mean-field type.** In this section we perform a detailed study of the control problem using the total variation distance as a carrier of the topology of weak convergence.

Let \( (U, \delta) \) be a compact metric space with its Borel field \( B(U) \) and \( U \) the set of \( \mathbb{F} \)-progressively measurable processes \( u = (u(t), 0 \leq t \leq T) \) with values in \( U \). We call \( U \) the set of admissible controls. In this section we consider a control problem of the jump process of mean-field type introduced above, where the control enters the jump intensities.

For \( u \in U \), let \( P^u \) be the probability measure on \( (\Omega, \mathcal{F}) \) under which the coordinate process \( x \) is a jump process with intensities

\[
\lambda^u_{ij}(t) := \lambda_{ij}(t, x, P^u \circ x^{-1}(t), u(t)), \quad i, j \in I, \quad 0 \leq t \leq T, \tag{4.1}
\]

satisfying the following assumptions similar to (A1)-(A5).

(B1) For any \( u \in U, i, j \in I \), the process \( (\lambda_{ij}(t, x, P^u \circ x^{-1}(t), u(t)))_t \) is predictable.

(B2) There exists a positive constants \( c_1 \) such that for every \( (t, i, j) \in [0, T] \times I \times I; i \neq j, w \in \Omega, u \in U \) and \( \mu \in \mathcal{P}(I) \)

\[
\lambda_{ij}(t, w, \mu, u) \geq c_1 > 0.
\]

(B3) For \( p = 1, 2 \) and for every \( t \in [0, T], w \in \Omega, u \in U \) and \( \mu \in \mathcal{P}_2(I) \)

\[
\sum_{i,j : j \neq i} |j - i|^p \lambda_{ij}(t, w, \mu, u) \leq C(1 + |w|^p_t + \int |y|^p \mu(dy)).
\]

(B4) For \( p = 1, 2 \) and for every \( t \in [0, T], w, \tilde{w} \in \Omega \) and \( \mu, \nu \in \mathcal{P}(I) \)

\[
\sum_{i,j : j \neq i} |j - i|^p |\lambda_{ij}(t, w, \mu, u) - \lambda_{ij}(t, \tilde{w}, \nu, v)| \leq C(|w - \tilde{w}|^p_t + dP(\mu, \nu) + \delta^p(u, v)).
\]

(B5) For every \( t \in [0, T], w \in \Omega, u \in U \) and \( \mu \in \mathcal{P}_2(I) \)

\[
\sum_{i,j : j \neq i} \lambda^u_{ij}(t, w, u, \mu) \leq C(1 + |w|_t + \int |y| \mu(dy)).
\]

(B6) Condition (A7) holds.

Existence of \( P^u \) such that \( P^u \circ x^{-1}(0) = \xi \) is derived as a fixed point of \( \Phi^u \) defined in the same way as in Theorem 3.4 except that the intensities \( \lambda_{ij}() \) further depend on \( u \), which does not rise any major issues.

Let \( P \) be the probability measure on \( (\Omega, \mathcal{F}) \) under which \( x \) is a time-homogeneous Markov chain such that \( P \circ x^{-1}(0) = \xi \) and with \( Q \)-matrix \( (q_{ij})_{ij} \) satisfying (2.3), (2.26) and (3.3). We have

\[
dP^u := L^u(T)dP, \tag{4.2}
\]

where, for \( 0 \leq t \leq T \),
\[ L^u(t) := \prod_{i,j} \exp \left\{ \int_{(0,t]} \ln \frac{\lambda^u_{ij}(s)}{g_{ij}} dN_{ij}(s) - \int_0^t (\lambda^u_{ij}(s) - g_{ij}) I_i(s) ds \right\}, \]  

which satisfies

\[ L^u(t) = 1 + \int_{(0,t]} L^u(s^-) \sum_{i,j:i \neq j} I_i(s^-) \ell^u_{ij}(s) dM_{ij}(s), \]  

where \( \ell^u_{ij}(s) := \ell_{ij}(s,x,P^u \circ x^{-1}(s),u(s)) \) is given by the formula

\[ \ell^u_{ij}(s) = \left\{ \begin{array}{ll} \frac{\lambda^u_{ij}(s)}{g_{ij}} - 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{array} \right. \]  

and \( (M_{ij})_{ij} \) is the \( P \)-martingale given in (2.6). Moreover, in a similar way as in (3.11), the accompanying martingale \( M^u = (M^u_{ij})_{ij} \) satisfies

\[ M^u_{ij}(t) = M_{ij}(t) - \int_{(0,t]} \ell^u_{ij}(s) I_i(s^-) g_{ij} ds. \]  

The condition (B5) corresponds to (A6) of Theorem 3.4 and together with (B6) is imposed to guarantee that \( L^u \) is \( L^2(P) \)-bounded.

We first derive continuity of the map \( u \mapsto P^u \) and then state the optimal control problem we want to solve.

Let \( E^u \) denote the expectation w.r.t. \( P^u \). By (3.20), we have, for every \( u \in U \),

\[ \|P^u\|^2_2 = E^u[|x|^2_T] \leq Ce^{CT} (1 + \|\xi\|^2_2) < \infty. \]  

We further have the following estimate of the total variation between \( P^u \) and \( P^v \).

**Lemma 4.1.** For every \( u, v \in U \), it holds that

\[ D^2_T(P^u, P^v) \leq C \sup_{0 \leq t \leq T} E^u[\delta^2(u(t), v(t))]. \]  

In particular, the function \( u \mapsto P^u \) from \( U \) into \( P_2(\Omega) \) is Lipschitz continuous: for every \( u, v \in U \),

\[ D_T(P^u, P^v) \leq C \delta(u, v). \]  

Moreover,

\[ K_T := \sup_{u \in U} \|P^u\|_2 \leq C < \infty, \]  

for some constant \( C > 0 \) that depends only on \( T \) and \( \xi \).

**Proof.** A similar estimate as (3.25) yields

\[ D^2_T(P^u, P^v) \leq \frac{1}{c_1} \sum_{i,j:i \neq j} E^u \left[ \int_0^T (\lambda^u_{ij}(t) - \lambda^v_{ij}(t))^2 dt \right]. \]  

Using (B4), we obtain

\[ D^2_T(P^u, P^v) \leq \frac{C}{c_1} E^u \left[ \int_0^T d^2(P^u(t), P^v(t)) + \delta^2(u(t), v(t)) dt \right]. \]

By (2.36) and Gronwall inequality we finally obtain

\[ D^2_T(P^u, P^v) \leq C \sup_{0 \leq t \leq T} E^u[\delta^2(u(t), v(t))]. \]

Inequality (4.9) follows from (4.8) by letting \( u(t) = u \in U \) and \( v(t) = v \in U \). It remains to show (4.10). But, this follows from (4.7) and the continuity of the function \( u \mapsto P^u \) from the compact set \( U \) into \( P_2(\Omega) \). \( \square \)
In the rest of the paper, we will assume that \( x(0) \) is a given deterministic point in \( I \) and \( \mathcal{F}_0 \) is the trivial \( \sigma \)-algebra.

Let \( f \) be a measurable function from \( [0, T] \times \Omega \times \mathcal{P}_2(I) \times U \) into \( \mathbb{R} \) and \( h \) be a measurable function from \( I \times \mathcal{P}_2(I) \) into \( \mathbb{R} \) such that

\[
\text{(B7)} \quad \text{For any } u \in U \text{ and } Q \in \mathcal{P}_2(\Omega), \text{ the process } (f(t, x, Q \circ x^{-1}(t), u(t)))_t \text{ is progressively measurable. Moreover, } h(x(T), Q \circ x^{-1}(T)) \text{ is } \mathcal{F}_T\text{-measurable.}
\]

\[
\text{(B8)} \quad \text{For every } t \in [0, T], \ w \in \Omega, \ u, v \in U \text{ and } \mu, \nu \in \mathcal{P}_2(I), \quad |\phi(t, w, \mu, u) - \phi(t, w, \nu, v)| \leq C(d(\mu, \nu) + \delta(u, v)),
\]

for \( \phi \in \{f, h\} \).

\[
\text{(B9)} \quad f \text{ and } h \text{ are uniformly bounded.}
\]

The cost functional \( J(u), u \in U \) associated with the controlled the jump process through the intensities \( \lambda_{ij}(t, u(t)) \) is

\[
J(u) := E^u \left[ \int_0^T f(t, x, P^u \circ x^{-1}(t), u(t)) \, dt + h(x(T), P^u \circ x^{-1}(T)) \right], \quad (4.12)
\]

where \( f \) and \( h \) satisfy (B7), (B8) and (B9) above.

Any \( \tilde{u} \in U \) satisfying

\[
J(\tilde{u}) = \min_{u \in U} J(u) \quad (4.13)
\]

is called optimal control. The corresponding optimal dynamics is given by the probability measure \( \tilde{P} \) on \( (\Omega, \mathcal{F}) \) defined by

\[
d\tilde{P} = L^{\tilde{u}}(T) \, dP, \quad (4.14)
\]

where \( L^{\tilde{u}} \) is given by the same expression as (4.3) and under which the coordinate process \( x \) is a jump process with intensities

\[
\lambda_{ij}^{\tilde{u}}(t) := \lambda_{ij}(t, x, P^{\tilde{u}} \circ x^{-1}(t), \tilde{u}(t)), \quad i, j \in I, \ 0 \leq t \leq T.
\]

We want to prove existence of such an optimal control and characterize the optimal cost functional \( J(\tilde{u}) \) in terms of a BSDE.

For \( (t, w, \mu, u) \in [0, T] \times \Omega \times \mathcal{P}_2(I) \times U \) and a matrix \( p = (p_{ij}) \) on \( I \times I \) with real-valued entries, we introduce the Hamiltonian associated with the optimal control problem (4.12)

\[
H(t, w, \mu, p, u) := f(t, w, \mu, u) + \langle \ell(t, w, \mu, u), p \rangle_g, \quad (4.15)
\]

where

\[
\langle \ell(t, w, \mu, u), p \rangle_g := \sum_{i,j: i \neq j} p_{ij} \ell_{ij}(t, w, \mu, u) g_{ij} \mathbf{1}_{\{w(t) = i\}}.
\]

Recalling that \( \ell_{ij}(t, w, \mu, u) g_{ij} = \lambda_{ij}(t, w, \mu, u) - g_{ij} \) for \( i \neq j \), we have

\[
\langle \ell(t, w, \mu) - \ell(t, w, \nu, v), p \rangle_g = \sum_{i,j: i \neq j} p_{ij} \left( \lambda_{ij}(t, w, \mu) - \lambda_{ij}(t, w, \nu) \right) g_{ij} \mathbf{1}_{\{w(t) = i\}}.
\]

Using (B4) we obtain

\[
|\lambda_{ij}(t, w, \mu, u) - \lambda_{ij}(t, w, \nu, v)| \leq C(d(\mu, \nu) + \delta(u, v)), \quad j \neq i.
\]

Therefore, provided that \( \|p(t)\|_g < +\infty \) (see the notation (2.10)),

\[
|\langle \ell(t, w, \mu, u) - \ell(t, w, \nu, v), p \rangle_g| \leq \frac{1}{c^2} \|p(t)\|_g \|\lambda(t, w, \mu, u) - \lambda(t, w, \nu, v)\|_g \leq C\|p(t)\|_g (d(\mu, \nu) + \delta(u, v))(\sum_{i,j: i \neq j} g_{ij})^{1/2},
\]
Therefore, that is

Moreover, noting that for $i \neq j$, $|i - j| \geq 1$, we may use (B3) to obtain

$$
\ell_{ij}(t, w, \mu, u) \leq 1 + \frac{\lambda_{ij}(t, w, \mu, u)}{g_{ij}} \leq 1 + \frac{C}{c_2}(1 + |w|_t + \int |y|\mu(dy)),
$$

that is

$$
\ell_{ij}(t, w, \mu, u) \leq C(1 + |w|_t + \int |y|\mu(dy)), \quad i, j \in I, \ i \neq j.
$$

Therefore,

$$
|H(t, w, \mu, p, u) - H(t, w, \mu, p', u)| \leq C \left(1 + |w|_t + \int |y|\mu(dy)\right) \| (p - p')(t)\|_p.
$$

(4.17)

Next, we show that the cost functional $J(u), u \in U$, can be expressed by means of solutions of a linear BSDE.

**Proposition 4.** For every $u \in U$, the BSDE

$$
\begin{align*}
- dY^u(t) & = H(t, x, P^u \circ x^{-1}(t), Z^u(t), u(t))dt - Z^u(t)\ dM(t), \\
Y^u(T) & = h(x(T), P^u \circ x^{-1}(T)),
\end{align*}
$$

(4.18)

admits a solution $(Y, Z)$ which consists of an $\mathbb{F}$-adapted process $Y$ which is right-continuous with left limits and a predictable process $Z$ which satisfy

$$
E \left[ |Y^u|_T^2 + \int_0^T \| Z^u(s) \|_g^2 ds \right] < +\infty.
$$

(4.19)

This solution is unique up to indistinguishability for $Y$ and equality $dP \times g_{ij} I_i(s^-)ds$-almost everywhere for $Z$.

Moreover, $Y^u_0 = J(u)$.

**Proof.** Since the Hamiltonian $H(t, w, \mu, p, u)$ is linear in $p$, by Theorem 3.5, existence and uniqueness of solutions of the BSDE (4.18) satisfying (4.19) follows from (4.17), the boundedness of $h(x(T), P^u \circ x^{-1}(T))$ and the boundedness of $H(t, x, P^u \circ x^{-1}(t), 0, u(t))$ which follows from (B9).

It remains to show that $Y^u_0 = J(u)$. Indeed, in terms of the $(\mathbb{F}, P^u)$-martingale

$$
M^u_s(t) = M_{ij}(t) - \int_{[0,t]} \ell_{ij}^u(s)I_i(s^-)g_{ij}ds,
$$

the process $(Y^u, Z^u)$ satisfies, for $0 \leq t \leq T$,

$$
Y^u(t) = h(x(T), P^u \circ x^{-1}(T)) + \int_t^T f(s, x, P^u \circ x^{-1}(s), u(s))ds - \int_t^T Z^u(s)dM^u(s).
$$

Therefore, $P^u$-a.s.,

$$
Y^u(t) = E^u \left[ \int_t^T f(s, x, P^u \circ x^{-1}(s), u(s))ds + h(x(T), P^u \circ x^{-1}(T)) \bigg| \mathcal{F}_t \right].
$$

In particular,

$$
Y^u(0) = E^u \left[ \int_0^T f(s, x, P^u \circ x^{-1}(s), u(s))ds + h(x(T), P^u \circ x^{-1}(T)) \right] = J(u),
$$

where $\sum_{i, j \neq j} g_{ij} < +\infty$, by (2.3).

Hence, in view of (B4), the Hamiltonian $H$ satisfies

$$
|H(t, w, \mu, p, u) - H(t, w, \nu, p, v)| \leq C(1 + \|p(t)\|_g)(d(\mu, \nu) + \delta(u, v)).
$$

(4.16)

where

$$
\sum_{i, j \neq j} g_{ij} < +\infty,
$$

by (2.3).
since $\mathcal{F}_0$ is the trivial $\sigma$-algebra.

4.1. **Existence of an optimal control.** In the remaining part of this section we want to find $\hat{u} \in \mathcal{U}$ such that $\hat{u} = \arg\min_{u \in \mathcal{U}} J(u)$. A way to find such an optimal control is to proceed as in Proposition 4 and introduce a linear BSDE whose solution $Y^*$ satisfies $Y^*_0 = \inf_{u \in \mathcal{U}} J(u)$. Then, by comparison (cf. Proposition 3), the problem can be reduced to minimizing the corresponding Hamiltonian and the terminal value $h$ w.r.t. the control $u$.

As explained in the introduction, since the marginal law $P^u \circ x^{-1}(t)$ of $x(t)$ under $P^u$ depends on the whole path of $u$ over $[0,t]$ and not only on $u(t)$, we should minimize the Hamiltonian $H(t,x,P^u \circ x^{-1}(t),z,u(t))$ w.r.t. the whole set $\mathcal{U}$ of admissible stochastic controls. Therefore, we should take the essential infimum of the Hamiltonian over $\mathcal{U}$, instead of the minimum over $\mathcal{U}$. Therefore, for the associated BSDE to make sense, we should show that it exists and is progressively measurable. This is shown in the next proposition.

Let $\mathbb{L}$ denote the $\sigma$-algebra of progressively measurable sets on $[0,T] \times \Omega$. For $z \in \mathbb{R}^{I \times I}$, the set of real-valued $I \times I$-matrix, set

$$H(t,x,z,u) := H(t,x,P^u \circ x^{-1}(t),z,u(t)). \quad (4.20)$$

Since $H$ is linear in $z$ and a progressively measurable process, it is an $\mathbb{L} \times \mathcal{B}(\mathbb{R}^{I \times I})$-random variable.

**Proposition 5.** There exists an $\mathbb{L}$-measurable process $H^*$ such that for every $z \in \mathbb{R}^{I \times I}$,

$$H^*(t,x,z) = \text{ess inf}_{u \in \mathcal{U}} H(t,x,z,u), \quad dP \times dt \text{-a.s.} \quad (4.21)$$

Moreover, $H^*$ is Lipschitz continuous in $z$: For every $z,z' \in \mathbb{R}^{I \times I}$,

$$|H^*(t,x,z) - H^*(t,x,z')| \leq C(1 + |x|_t + \sup_{u \in \mathcal{U}} \|P^u\|_2)\|(z-z')(t)\|_g, \quad (4.22)$$

and there exists a predictable process $\hat{\ell}(t,x,z,\bar{z}) = (\hat{\ell}_{ij}(t,x,z,\bar{z}))_{ij}$ satisfying

$$\hat{\ell}_{ij}(t,x,z,\bar{z}) > -1, i \neq j,$$

such that

$$H^*(t,x,y,z) = \text{ess inf}_{u \in \mathcal{U}} f(t,x,u) + \langle \hat{\ell}(t,x,z,0),z \rangle_g, \quad (4.23)$$

and for every $z,\bar{z} \in \mathbb{R}^{I \times I}$

$$H^*(t,x,y,z) - H^*(t,x,y,\bar{z}) = \langle \hat{\ell}(t,x,z,\bar{z}),z-\bar{z} \rangle_g, \quad (4.24)$$

if $\theta$ is an $\mathbb{L}$-measurable process with values in $\mathbb{R}^{I \times I}$, then

$$H^*(t,x,\theta_t) = \text{ess inf}_{u \in \mathcal{U}} H(t,x,\theta_t,u), \quad dP \times dt \text{-a.e.} \quad (4.25)$$

**Proof.** The proof of (4.21) and (4.25) is similar to the one of Propositions 4.4 and 4.6 in [8]. We give it in the appendix for the sake of completeness. We prove the inequality (4.22). We have

$$|H^*(t,x,z) - H^*(t,x,z')|$$

$$= \left| \text{ess inf}_{u \in \mathcal{U}} H(t,x,P^u \circ x_t^{-1},z,u) - \text{ess inf}_{u \in \mathcal{U}} H(t,x,P^u \circ x_t^{-1},z',u) \right|$$

$$\leq \text{ess sup}_{u \in \mathcal{U}} \left| H(t,x,P^u \circ x_t^{-1},z,u) - H(t,x,P^u \circ x_t^{-1},z',u) \right|$$

$$\leq C(1 + |x|_t + \sup_{u \in \mathcal{U}} \|P^u\|_2)\|(z-z')(t)\|_g.$$
by (4.17), where by the continuity of \( u \rightarrow P^u \), \( K_T := \sup_{u \in U} \|P^u\|_2 \) is finite.

Next, we will show that there exists a predictable process \( \hat{\ell}(t, x, z, \bar{z}) = (\hat{\ell}_{ij}(t, x, z, \bar{z}))_{ij} \) satisfying \( \hat{\ell}_{ij}(t, x, z, \bar{z}) > -1, i \neq j \), such that (4.23) holds.

We claim that, for every \( z, \bar{z} \in \mathbb{R}^T, \), (4.24) holds (cf. [4] pp. 482-483), where \( \hat{\ell}(t, x, z, \bar{z}) = (\hat{\ell}_{ij}(t, x, z, \bar{z}))_{ij} \) is given by

\[
\hat{\ell}(t, x, z, \bar{z}) = \alpha(t, x, z, \bar{z})\hat{\ell}(t, x, z, \bar{z}) + (1 - \alpha(t, x, z, \bar{z}))\hat{\ell}(t, x, z, \bar{z}), \tag{4.26}
\]

with

\[
\alpha(t, x, z, \bar{z}) = \frac{H^*(t, x, y, z) - H^*(t, x, y, \bar{z}) - \langle \hat{\ell}(t, x, z, \bar{z}) - \hat{\ell}(t, x, z, \bar{z}), z - \bar{z} \rangle_g}{\langle \hat{\ell}(t, x, z, \bar{z}) - \hat{\ell}(t, x, z, \bar{z}), z - \bar{z} \rangle_g} \in [0, 1], \tag{4.27}
\]

where, \( \hat{\ell} = (\hat{\ell}_{ij})_{ij} \) and \( \hat{\ell} = (\hat{\ell}_{ij})_{ij} \) satisfy

\[
\langle \hat{\ell}(t, x, z, \bar{z}), z - \bar{z} \rangle_g \leq H^*(t, x, y, z) - H^*(t, x, y, \bar{z}) \leq \langle \hat{\ell}(t, x, z, \bar{z}), z - \bar{z} \rangle_g, \tag{4.28}
\]

with, for \( i \neq j \),

\[
\hat{\ell}_{ij}(t, x, z, \bar{z}) := \inf\sup_{u \in U} \ell_{ij}^u(t) \mathbf{1}_{\{z_{ij} > \bar{z}_{ij}\}} + \sup\inf_{u \in U} \ell_{ij}^u(t) \mathbf{1}_{\{z_{ij} \leq \bar{z}_{ij}\}},
\]

\[
\hat{\ell}_{ij}(t, x, z, \bar{z}) := \inf\sup_{u \in U} \ell_{ij}^u(t) \mathbf{1}_{\{z_{ij} > \bar{z}_{ij}\}} + \sup\inf_{u \in U} \ell_{ij}^u(t) \mathbf{1}_{\{z_{ij} \leq \bar{z}_{ij}\}}.
\]

Indeed, we have

\[
H^*(t, x, y, z) - H^*(t, x, y, \bar{z}) \geq \inf\sup_{u \in U} \ell_{ij}^u(t) (z - \bar{z})_g \geq (\inf\sup_{u \in U} \ell_{ij}^u(t), (z - \bar{z})^+)_g - (\inf\sup_{u \in U} \ell_{ij}^u(t), (z - \bar{z})^-)_g = \langle \hat{\ell}(t, x, z, \bar{z}), z - \bar{z} \rangle_g,
\]

where, \( (z - \bar{z})^\pm := ((z_{ij} - \bar{z}_{ij})^\pm)_{ij} \), with \( \rho^+ = \max(\rho, 0) \) and \( \rho^- = \max(-\rho, 0) \), \( \rho \in \mathbb{R} \). By symmetry, we also have

\[
H^*(t, x, y, z) - H^*(t, x, y, \bar{z}) \leq \langle \hat{\ell}(t, x, z, \bar{z}), z - \bar{z} \rangle_g.
\]

Combining these two inequalities and choosing \( \hat{\ell} \) as in (4.26), we obtain (4.24).

Moreover, since for every \( u \in U \), \( i \neq j \), \( \ell_{ij}^u = \frac{\lambda_{ij}^u}{\gamma_{ij}} - 1 \), the intensity processes defined by

\[
\lambda_{ij}(t, x, z, \bar{z}) := \inf\sup_{u \in U} \lambda_{ij}^u(t) \mathbf{1}_{\{z_{ij} > \bar{z}_{ij}\}} + \sup\inf_{u \in U} \lambda_{ij}^u(t) \mathbf{1}_{\{z_{ij} \leq \bar{z}_{ij}\}},
\]

\[
\bar{\lambda}_{ij}(t, x, z, \bar{z}) := \inf\sup_{u \in U} \lambda_{ij}^u(t) \mathbf{1}_{\{z_{ij} > \bar{z}_{ij}\}} + \sup\inf_{u \in U} \lambda_{ij}^u(t) \mathbf{1}_{\{z_{ij} \leq \bar{z}_{ij}\}},
\]

are related to \( \hat{\ell}_{ij} \) and \( \bar{\ell}_{ij} \) by the formula

\[
\lambda_{ij}(t, x, z, \bar{z}) = (1 + \hat{\ell}_{ij}(t, x, z, \bar{z}))g_{ij}, \quad \bar{\lambda}_{ij}(t, x, z, \bar{z}) := (1 + \bar{\ell}_{ij}(t, x, z, \bar{z}))g_{ij}. \tag{4.29}
\]

Thus, \( \hat{\ell}_{ij}(t, x, z, \bar{z}) \) and \( \bar{\ell}_{ij}(t, x, z, \bar{z}) \) are both strictly larger than \(-1 \). From (4.29), it follows that the intensity process \( \lambda(t, x, z, \bar{z}) \) associated to \( \hat{\ell}(t, x, z, \bar{z}) \) through the formula \( \hat{\lambda}_{ij}(t, x, z, \bar{z}) := (1 + \hat{\ell}_{ij}(t, x, z, \bar{z}))g_{ij}, i \neq j \), reads

\[
\hat{\lambda}_{ij}(t, x, z, \bar{z}) = \alpha(t, x, z, \bar{z})\hat{\lambda}_{ij}(t, x, z, \bar{z}) + (1 - \alpha(t, x, z, \bar{z}))\lambda_{ij}(t, x, z, \bar{z}),
\]

for which the Girsanov theorem holds (Proposition 2 ). \( \square \)

Now, define the \( \mathcal{F}_T \)-measurable random variable

\[
h^*(x) := \inf\sup_{u \in U} h(x(T), P^u \circ x^{-1}(T)), \tag{4.30}
\]
and from $H^*(t, x, y, 0) = \text{ess inf}_{u \in \mathcal{U}} f(t, x, u)$, we obtain (3.29). Therefore, by Theorem 3.5, the BSDE

$$
\begin{cases}
-dY^*(t) = H^*(t, x, Z^*(t)) dt - Z^*(t) dM(t), & 0 \leq t < T, \\
Y^*(T) = h^*(x),
\end{cases}
$$

admits a unique solution $(Y^*, Z^*)$ which satisfies

$$
E \left[ \sup_{t \in [0, T]} |Y^*(t)|^2 + \int_{(0, T]} \|Z^*(s)\|^2 ds \right] < +\infty.
$$

We have the following comparison result.

**Proposition 6** (Comparison result). For every $t \in [0, T]$, it holds that

$$
Y^*(t) \leq Y^u(t), \quad P\text{-a.s.}, \quad u \in \mathcal{U}.
$$

**Proof.** We have

$$
Y^*(t) - Y^u(t) = h^*(x) - h(x(T), P^u \circ x^{-1}(T)) - \int_t^T (Z^*(s) - Z^u(s)) dM(s)
$$

$$
+ \int_t^T (H^*(s, x, Z^*(s)) - H(s, x, P^u \circ x^{-1}(s), Z^*(s), u(s))) ds
$$

$$
+ \int_t^T (H(s, x, P^u \circ x^{-1}(s), Z^*(s), u(s)) - H(s, x, P^u \circ x^{-1}(s), Z^u(s), u(s))) ds.
$$

Using the definition of $H^*$ and $h^*$ and noting that

$$
H(s, x, P^u \circ x^{-1}(s), Z^*(s), u(s)) - H(s, x, P^u \circ x^{-1}(s), Z^u(s), u(s)) = \langle \ell(s, u(s)), Z^*(s) - Z^u(s) \rangle_g,
$$

we have

$$
Y^*(t) - Y^u(t) \leq \int_t^T \langle \ell(s, u(s)), Z^*(s) - Z^u(s) \rangle_g ds - \int_t^T \langle \ell(s, u(s)), Z^*(s) - Z^u(s) \rangle_g ds
$$

$$
- \int_t^T (Z^*(s) - Z^u(s)) dM^u(s),
$$

where $M^u$ is the $P^u$-martingale defined in (4.6). Taking the $P^u$-conditional expectation w.r.t. $\mathcal{F}_t$, we obtain $Y^*(t) \leq Y^u(t)$, $\forall u \in \mathcal{U}$. $\square$

**Proposition 7** ($\varepsilon$-optimality). Assume that for any $\varepsilon > 0$ there exists $u^\varepsilon \in \mathcal{U}$ such that $P$-a.s.,

$$
\begin{cases}
H^*(t, x, Z^*(t)) \geq H(t, x, Z^*(t), P^u \circ x^{-1}(t), u^\varepsilon(t)) - \varepsilon, & 0 \leq t < T, \\
h^*(x) \geq h(x(T), P^u \circ x^{-1}(T)) - \varepsilon.
\end{cases}
$$

Then,

$$
Y^*(t) = \text{ess inf}_{u \in \mathcal{U}} Y^u(t), \quad 0 \leq t \leq T.
$$

**Proof.** Let $(Y^\varepsilon, Z^\varepsilon)$ be the solution the following BSDE

$$
Y^\varepsilon(t) = h(x(T), P^u \circ x^{-1}(T)) + \int_t^T H(s, x, Z^\varepsilon(s), P^u \circ x^{-1}(s), u^\varepsilon(s)) ds
$$

$$
- \int_t^T Z^\varepsilon(s) dM(s).
$$

We have

$$
Y^*(t) - Y^\varepsilon(t) = h^*(x) - h(x(T), P^u \circ x^{-1}(T)) - \int_t^T (Z^*(s) - Z^\varepsilon(s)) dM(s)
$$

$$
+ \int_t^T \{H^*(s, x, Z^*(s)) - H(s, x, P^u \circ x^{-1}(s), Z^*(s), u^\varepsilon(s))\} ds
$$

$$
+ \int_t^T \{H(t, x, P^u \circ x^{-1}(s), Z^\varepsilon(s), u^\varepsilon(s)) - H(s, x, P^u \circ x^{-1}(s), Z^\varepsilon(s), u^\varepsilon(s))\} ds.
$$

Since

$$
H^*(s, x, Z^*(s)) - H(s, x, P^u \circ x^{-1}(s), Z^*(s), u^\varepsilon(s)) \geq -\varepsilon,
$$
and $h^*(x) - h(x(T), P^u \circ x^{-1}(T)) \geq -\varepsilon$, applying a similar argument as in the proof of Proposition 6, we obtain $Y^*(t) \geq Y^u(t) - \varepsilon(T + 1)$. Therefore, for every $0 \leq t \leq T$, $Y^*(t) = \text{ess inf}_{u \in \mathcal{U}} Y^u(t)$.

In next theorem, we characterize the set of optimal controls associated with (4.13) under the dynamics $P^u$.

**Theorem 4.2** (Existence of optimal control). If there exists $u^* \in \mathcal{U}$ such that

$$H^*(t, x, Z^*(t)) = H(t, x, P^{u^*} \circ x^{-1}(t), Z^*(t), u^*(t)), \quad 0 \leq t < T,$$

and

$$h^*(x) = h(x(T), P^{u^*} \circ x^{-1}(T)).$$

Then,

$$Y^*(t) = Y^{u^*}(t) = \text{ess inf}_{u \in \mathcal{U}} Y^u(t), \quad 0 \leq t \leq T. \quad (4.37)$$

In particular, $Y^*_0 = \text{inf}_{u \in \mathcal{U}} J(u) = J(u^*)$.

**Proof.** By comparison, the conditions (4.35) and (4.36) imply that $Y^* = Y^{u^*}$. Due to (4.34), we arrive at (4.37). \qed

**Remark 3.** If the marginal law $P^u \circ x^{-1}$ of $x_s$ is a function of $(x, u(s))$ only and does not depend on the whole path of $u$ over $[0, s]$, it suffices to take the minimum of $H$ and $h$ over the compact set of controls $\mathcal{U}$, instead of taking the essential infimum over $\mathcal{U}$. An optimal control over $[0, T]$ can be obtained by pasting the minima of $H$ and $h$ as follows. By Beneš selection theorem [1], there exist two measurable functions $u_1^*$ from $[0, T] \times \Omega \times \mathbb{R}^I$ into $U$ and $u_2^*$ from $I$ into $U$ such that

$$H^*(t, x, z) := \inf_{u \in \mathcal{U}} H(t, x, P^u \circ x^{-1}(t), z, u) = H(t, x, P^{u_1^*} \circ x^{-1}(t), z, u_1^*(t, x, z))$$

and

$$h^*(x) := \inf_{u \in \mathcal{U}} h(x(T), P^u \circ x^{-1}(T)) = h(x(T), P^{u_2^*} \circ x^{-1}(T)).$$

Thus, the progressively measurable function $u^*$ defined by

$$\tilde{u}(t, x, z) := \begin{cases} u_1^*(t, x, z), & t < T, \\ u_2^*(x(T)), & t = T, \end{cases}$$

satisfies

$$H^*(t, x, z) = H(t, x, P^{\tilde{u}} \circ x^{-1}(t), z, \tilde{u}) \quad \text{and} \quad h^*(x) = h(x(T), P^{\tilde{u}} \circ x^{-1}(T)). \quad \Box$$

We end this section by providing an example where an optimal control exists.

**Example 4.3.** Assume that the set of $L^2$-bounded densities $\{L^u(T), u \in \mathcal{U}\}$ is weakly sequentially compact for the topology $\sigma(L^1, L^\infty)$. Consider a cost functional of the form

$$J(u) = E^u \left[ \int_0^T f(t, x, E^u[\alpha(x(t))], u(t))dt + h(x(T), E^u[\beta(x(T))] \right],$$

where $\alpha, \beta, f, h$ are bounded functions and $(y, a) \in \mathbb{R} \times U \mapsto f(\cdot, \cdot, y, a)$ and $y \in \mathbb{R} \mapsto h(\cdot, y)$ are continuous. Then, an optimal control exists. Indeed, let $(u_n)_{n \geq 0}$ be a sequence in $\mathcal{U}$ such that

$$\inf_{u \in \mathcal{U}} J(u) = \lim_{n \to \infty} J(u_n).$$
By weak compactness of the set of densities \( \{ L^u(T), u \in \mathcal{U} \} \), there exist \( u^* \in \mathcal{U} \) and a subsequence \( (L^{u_{n_k}}(T))_{k \geq 0} \) which converges weakly to \( L^{u^*}(T) \). Since \( \alpha, \beta \) are bounded we have, for any \( t \leq T \),

\[
\lim_{k \to \infty} E^{u_{n_k}}[\alpha(x(t))] = \lim_{k \to \infty} E[L^{u_{n_k}}(T)\alpha(x(t))] = E[L^{u^*}(T)\alpha(x(t))] = E^{u^*}[\alpha(x(t))],
\]

\[
\lim_{k \to \infty} E^{u_{n_k}}[\beta(x(T))] = \lim_{k \to \infty} E[L^{u_{n_k}}(T)\beta(x(T))] = E[L^{u^*}(T)\beta(x(T))] = E^{u^*}[\beta(x(T))].
\]

Using the boundedness and continuity of \( f \) and \( h \), by the dominated convergence theorem, we obtain

\[
\lim_{k \to \infty} \int_0^T f(s, x, E^{u_{n_k}}[\alpha(x(t))]) ds + h(x(T), E^{u_{n_k}}[\beta(x(T))]) = \int_0^T f(s, x, E^{u^*}[\alpha(x(t))]) ds + h(x(T), E^{u^*}[\beta(x(T))])
\]

in \( L^p \), for any \( p \geq 1 \). Thus, the weak convergence of \( (L^{u_{n_k}}(T))_{k \geq 0} \), we have that

\[
\lim_{k \to \infty} J(u_{n_k}) = J(u^*)
\]

which implies that \( J(u^*) = \inf_{u \in \mathcal{U}} J(u) \), i.e. \( u^* \) is optimal for \( J \).

If the set of intensities \( \lambda^u \) is mean-field free i.e. \( \lambda^u(t) := \lambda(t, x, u(t)) \) and for each \((t, w) \in [0, T] \times \Omega, \lambda(t, y, \mathcal{U}) \) is convex (Roxin’s condition) i.e. for every \( u_1, u_2 \in \mathcal{U} \), and \( a \in [0, 1] \), there exists an admissible control \( u \in \mathcal{U} \) such that

\[
\lambda(t, w, u(t, w)) = a \lambda(t, w, u_1(t, w)) + (1 - a) \lambda(t, w, u_2(t, w)),
\]

then, mimicking the proof of Theorems 3 and 4 in [1], the set of densities \( \{ L^u(T), u \in \mathcal{U} \} \) is convex and weakly sequentially closed. Being \( L^2 \)-bounded, it is weakly sequentially compact. The proof of these results relies on the measurable selection theorem, which does not seem extend to intensities \( \lambda^u \) of mean-field type which, at each time \( t \), depend on the whole path of \( u \) over \([0, t]\).

4.2. Existence of nearly-optimal controls. The sufficient condition (4.35)-(4.36) is quite hard to verify in concrete situations. This makes Theorem 4.2 less useful for showing existence of optimal controls. Nevertheless, near-optimal controls enjoy many useful and desirable properties that optimal controls do not have. Thanks to Ekeland’s variational principle [11], that we will use below, under very mild conditions on the control set \( \mathcal{U} \) and the payoff functional \( J \), near-optimal controls always exist while optimal controls may not exist or are difficult to establish.

We introduce the Ekeland metric \( d_E \) on the space \( \mathcal{U} \) of admissible controls defined as follows. For \( u, v \in \mathcal{U} \),

\[
d_E(u, v) := \hat{P}\{ (\omega, t) \in \Omega \times [0, T], \delta(u(t)(\omega), v(t)(\omega)) > 0 \},
\]

where \( \hat{P} \) is the product measure of \( P \) and the Lebesgue measure on \([0, T]\).

In our proof of existence of near-optimal controls, we use \( L^2 \)-boundedness of the Girsanov density \( L^u \), which, in view of Theorem 3.4, is satisfied under assumptions (B6) and (B7). We have

**Lemma 4.4.** (i) \( d_E \) is a distance. Moreover, \((\mathcal{U}, d_E)\) is a complete metric space.

(ii) Let \((u^n)\) and \( u \) be in \( \mathcal{U} \). If \( d_E(u^n, u) \to 0 \) then \( E[\int_0^T \delta^2(u^n(t), u(t)) dt] \to 0 \).

**Proof.** For a proof of (i), see [14]. The proof of completeness of \((\mathcal{U}, d_E)\) needs only completeness of the metric space \((U, \delta)\).

(ii) Let \((u^n)\) and \( u \) be in \( \mathcal{U} \). Then, by definition of the distance \( d_E \), since \( d_E(u^n, u) \to 0 \) then \( \delta(u^n(t), u(t)) \) converges to 0, \( dP \times dt \)-a.e. Now, since the set \( U \) is compact, the sequence \( \delta(u^n, u) \) is bounded. Thus, by dominated convergence, we have \( E[\int_0^T \delta^2(u^n(t), u(t)) dt] \to 0 \).
Proposition 8. Assume (B1) to (B9) hold and let \((u^n)_n\) and \(u\) be in \(\mathcal{U}\). If \(d_E(u^n, u) \to 0\) then \(D^2_T(P^n, P^u) \to 0\). Moreover, for every \(t \in [0, T]\), \(L^{u^n}(t)\) converges to \(L^u(t)\) in \(L^1(P)\).

**Proof.** In view of Lemma (4.4), we have \(\mathbb{E}[\int_0^T \delta^2(u(t), u^n(t))dt] \to 0\). Therefore, the sequence \((\int_0^T \delta^2(u(t), u^n(t))dt)_n\) converges in probability w.r.t. \(P\) to 0 and by compactness of \(\mathcal{U}\), it is uniformly bounded. On the other hand, since \(L^u(T)\) is integrable then the sequence \((L^u(T) \int_0^T \delta^2(u(t), u^n(t))dt)_n\) converges also in probability (w.r.t. \(P\)) to 0. Next, by the uniform boundedness of \((\int_0^T \delta^2(u(t), u^n(t))dt)_n\), the sequence \((L^u(T) \int_0^T \delta^2(u(t), u^n(t))dt)_n\) is uniformly integrable. Finally, since
\[
\mathbb{E}[\int_0^T \delta^2(u(t), u^n(t))dt] = \mathbb{E}[L^u(T) \int_0^T \delta^2(u(t), u^n(t))dt],
\]

it follows that \(\mathbb{E}[\int_0^T \delta^2(u(t), u^n(t))dt] \to 0\) as \(n \to +\infty\). To conclude it is enough to use the inequality (4.8).

To prove that \(L^{u^n}(t)\) converges to \(L^u(t)\) in \(L^1(P)\), using the fact that \((L^{u^n}(t))_n\) is \(L^2(P)\)-bounded (hence, uniformly integrable) it suffices to show that \(L^{u^n}(t)\) converges to \(L^u(t)\) in probability w.r.t. \(P\), as \(n \to +\infty\). Using the following relationship between the Hellinger distance w.r.t. \(P\), and the Total variation distance \(D_1(P^{u^n}, P^u)\):
\[
E \left[ \left( \sqrt{L^{u^n}(t)} - \sqrt{L^u(t)} \right)^2 \right] := D^2_1(P^{u^n}, P^u) \leq 2D_1(P^{u^n}, P^u),
\]
and that \(D_1(P^{u^n}, P^u)\) tends to 0, we obtain \(\sqrt{L^{u^n}(t)}\) converges to \(\sqrt{L^u(t)}\) in probability (P), as \(n \to +\infty\). This in turn yields that \(L^{u^n}(t)\) converges to \(L^u(t)\) in probability (P), as \(n \to +\infty\).

\(\square\)

Proposition 9 (Existence of near optimal control). For any \(\varepsilon > 0\), there exists a control \(u^\varepsilon \in \mathcal{U}\) such that
\[
J(u^\varepsilon) \leq \inf_{u \in \mathcal{U}} J(u) + \varepsilon. \tag{4.39}
\]

\(u^\varepsilon\) is called near or \(\varepsilon\)-optimal for the payoff functional \(J\).

**Proof:** The result follows from Ekeland’s variational principle, provided that we prove that the payoff function \(J\), as a mapping from the complete metric space \((\mathcal{U}, d_E)\) to \(\mathbb{R}\), is lower bounded and lower-semicontinuous. Since \(f\) and \(h\) are assumed uniformly bounded, \(J\) is obviously bounded. We now show continuity of \(J\): \(J(u^n)\) converges to \(J(u)\) when \(d_E(u^n, u) \to 0\).

Integrating by parts, we obtain
\[
J(u) = E[\int_0^T L^u(t) f(t, x, P^u \circ x^{-1}(t), u(t))dt + L^u(T) h(x(T), P^u \circ x^{-1}(T))].
\]

Using the inequality
\[
|L^{u^n}(t) f(t, x, u^n) - L^u(t) f(t, x, u^n)| \leq |L^{u^n}(t) - L^u(t)| |f(t, x, u^n)| + L^u(t) |f(t, x, u^n) - f(t, x, u)|,
\]
where, we set \(f(t, x, u) := f(t, x, P^u \circ x^{-1}(t), u(t))\), and (B8) together with the boundedness of \(f\), by Proposition 8, \(E[\int_0^T L^{u^n}(t) f(t, x, u^n)dt] \to E[\int_0^T L^u(t) f(t, x, u)dt]\), as \(d_E(u^n, u) \to 0\). A similar argument yields convergence of
\[ E[L^{u}(T)h(x(T), P^{u} \circ x^{-1}(T))] \] to \[ E[L^{u}(T)h(x(T), P^{u} \circ x^{-1}(T))] \] when \( d_{E}(u^{n}, u) \) tends to 0.

5. **The two-players zero-sum game problem.** In this section we consider a two-players zero-sum game. Let \( \mathcal{U} \) (resp. \( \mathcal{V} \)) be the set of admissible \( U \)-valued (resp. \( V \)-valued) control strategies for the first (resp. second) player, where \( (U, \delta_{1}) \) and \( (V, \delta_{2}) \) are compact metric spaces.

For \((u, v), (\bar{u}, \bar{v}) \in U \times V\), we set
\[
\delta((u, v), (\bar{u}, \bar{v})) := \delta_{1}(u, \bar{u}) + \delta_{2}(v, \bar{v}).
\]

The distance \( \delta \) defines a metric on the compact space \( U \times V \).

Let \( P \) be the probability measure on \((\Omega, \mathcal{F})\) under which \( x \) is a time-homogeneous Markov chain such that \( P \circ x^{-1}(0) = \xi \) and with \( Q \)-matrix \((g_{ij})_{ij}\) satisfying \((2.3)\), \((2.26)\) and \((3.3)\).

For \((u, v) \in \mathcal{U} \times \mathcal{V}\), let \( P^{u,v} \) be the measure on \((\Omega, \mathcal{F})\) defined by
\[
dP^{u,v} := L^{u,v}(T)dP,
\]
where
\[
L^{u,v}(t) := \prod_{i,j} \exp \left\{ \int_{(0,t]} \ln \frac{\lambda^{u,v}_{ij}(s)}{g_{ij}} dN_{ij}(s) - \int_{0}^{t} (\lambda^{u,v}_{ij}(s) - g_{ij})I(s)ds \right\},
\]
\[
\lambda^{u,v}_{ij}(t) := \lambda_{ij}(t, x, P^{u,v} \circ x^{-1}(t), u(t), v(t)), \quad i, j \in I, \quad 0 \leq t \leq T,
\]
satisfying the following assumptions.

(C1) For any \((u, v) \in \mathcal{U} \times \mathcal{V}, i, j \in I\), the process \((\lambda_{ij}(t, x, P^{u,v} \circ x^{-1}(t), u(t), v(t)))_{t} \) is predictable.

(C2) There exists a positive constants \( c_{1} \) such that for every \((t, i, j) \in [0, T] \times I \times I; i \neq j, w \in \Omega, u \in U, v \in V \) and \( \mu \in \mathcal{P}(I) \)
\[
\lambda_{ij}(t, w, u, v) \geq c_{1} > 0.
\]

(C3) For \( p = 1, 2 \) and for every \( t \in [0, T], w \in \Omega, u \in U, v \in V \) and \( \mu \in \mathcal{P}_{2}(I) \),
\[
\sum_{i,j \neq i} |j - i|^{p} \lambda_{ij}(t, w, u, v) \leq C(1 + |w|_{t}^{p} + \int |y|^{p} \mu(dy)).
\]

(C4) For \( p = 1, 2 \) and for every \( t \in [0, T], w, \bar{w} \in \Omega, (u, v), (\bar{u}, \bar{v}) \in U \times V \) and \( \mu, \nu \in \mathcal{P}(I) \),
\[
\sum_{i,j \neq i} |j - i|^{p} |\lambda_{ij}(t, w, u, v) - \lambda_{ij}(t, \bar{w}, \bar{u}, \bar{v})| \leq C(|w - \bar{w}|_{t}^{p} + d^{p}(\mu, \nu) + \delta^{p}((u, v), (\bar{u}, \bar{v}))).
\]

(C5) For every \( t \in [0, T], w \in \Omega, u \in U, v \in V \) and \( \mu \in \mathcal{P}_{1}(I) \),
\[
\sum_{i,j \neq i} \lambda^{4}_{ij}(t, w, u, v) \leq C(1 + |w|_{t} + \int |y| \mu(dy)).
\]

(C6) Condition \((A7)\) holds.

By Proposition 2, these assumptions guarantee that \( P^{u,v} \) is a probability measure on \((\Omega, \mathcal{F})\) under which the coordinate process \( x \) is a chain with intensity matrix \( \lambda^{u,v} \). Let \( E^{u,v} \) denote the expectation w.r.t. \( P^{u,v} \).

Let \( f \) be a measurable function from \([0, T] \times \Omega \times \mathcal{P}_{2}(I) \times U \times V \) into \( \mathbb{R} \) and \( h \) be a measurable function from \( I \times \mathcal{P}_{2}(I) \) into \( \mathbb{R} \) such that
(C7) For every \( t \in [0, T] \), \( w \in \Omega \), \((u, v), (\hat{u}, \hat{v}) \in U \times V \) and \( \mu, \nu \in \mathcal{P}_2(I) \),
\[
|\phi(t, w, \mu, u, v) - \phi(t, w, \nu, \hat{u}, \hat{v})| \leq C(d(\mu, \nu) + \delta((u, v), (\hat{u}, \hat{v}))),
\]
for \( \phi \in \{f, h\} \).

(C8) For every \( t \in [0, T] \), \( w \in \Omega \), \((u, v) \in U \times V \) and \( \mu \in \mathcal{P}_2(I) \),
\[
|f(t, w, \mu, u, v)| \leq C(1 + |w|_t + \int |g(\mu(dy))|).
\]

(C9) \( f \) and \( h \) are uniformly bounded.

The performance functional \( J(u, v), (u, v) \in U \times V \), associated with the controlled Markov chain is
\[
J(u, v) := E^{u, v} [\int_0^T f(t, x, P^{u,v} \circ x^{-1}(t), u(t), v(t))dt + h(x(T), P^{u,v} \circ x^{-1}(T))]. \tag{5.5}
\]

The zero-sum game we consider is between two players, where the first player (with control \( u \)) wants to minimize the payoff \( (5.5) \), while the second player (with control \( v \)) wants to maximize it. The zero-sum game boils down to showing existence of a saddle-point for the game i.e. to show existence of a pair \((\hat{u}, \hat{v})\) of strategies such that
\[
J(\hat{u}, \hat{v}) \leq J(\hat{u}, v) \leq J(\hat{u}, \hat{v}) \leq J(u, \hat{v}), \tag{5.6}
\]
for each \((u, v) \in U \times V \).

The corresponding optimal dynamics is given by the probability measure \( \hat{P} \) on \((\Omega, \mathcal{F})\) defined by
\[
d\hat{P} = L^{\hat{u}, \hat{v}}(T)dP, \tag{5.7}
\]
under which the chain has intensity \( \lambda^{\hat{u}, \hat{v}} \).

For \((t, w, \mu, u, v) \in [0, T] \times \Omega \times \mathcal{P}_2(I) \times U \times V \) and matrices \( p = (p_{ij}) \) with real-valued entries, we introduce the Hamiltonian associated with the optimal control problem \( (5.5) \)
\[
H(t, w, \mu, p, u, v) := f(t, w, \mu, u, v) + \langle f(t, w, \mu, u, v), p \rangle_g, \tag{5.8}
\]
where we recall that \( \ell_{ij}(t, w, \mu, u, v)g_{ij} = \lambda_{ij}(t, w, \mu, u, v) - g_{ij} \) for \( i \neq j \).

In a similar way as for \((4.16)\) and \((4.17)\), whenever \( \|p\|_g(t) \) and \( \|p'(t)\|_g \) are finite, the Hamiltonian \( H \) satisfies
\[
|H(t, w, \mu, p, u, v) - H(t, w, \nu, \hat{u}, \hat{v})| \leq C(1 + \|p(t)\|_g(d(\mu, \nu) + \delta((u, \hat{u}), (v, \hat{v})))), \tag{5.9}
\]
and
\[
|H(t, w, \mu, p, u, v) - H(t, w, \mu, p', u, v)| \leq C(1 + |w|_t + \int |g(\mu(dy))| \|p - p'(t)\|_g). \tag{5.10}
\]

Next, let \( z \in R^{I \times I} \) and set
- \( \mathcal{H}(t, x, z) := \text{ess sup}_{v \in V} \text{ess inf}_{u \in U} H(t, x, z, u, v) \),
- \( \bar{\mathcal{H}}(t, x, z) := \text{ess inf}_{v \in V} \text{ess sup}_{u \in U} H(t, x, z, u, v) \),
- \( h(x) := \text{ess sup}_{v \in V} \text{ess inf}_{u \in U} h(x(T), P^{u,v} \circ x^{-1}(T)) \),
- \( \bar{h}(x) := \text{ess inf}_{v \in V} \text{ess sup}_{u \in U} h(x(T), P^{u,v} \circ x^{-1}(T)) \),
- \((Y, Z)\) the solution of the BSDE associated with \((\mathcal{H}, h)\) and \((\bar{Y}, \bar{Z})\) the solution of the BSDE associated with \((\bar{\mathcal{H}}, \bar{h})\).
Following a similar proof as the one leading to Proposition 5, \( H(t,x,p) \) and \( \overline{H}(t,x,p) \) are Lipschitz continuous in \( p \) with the Lipschitz constant \( C(1 + |x|_t + \sup_{(u,v) \in U \times V} \|P_{u,v}\|_2) \). Furthermore, the driver of the BSDE associated with \((Y,Z)\) reads

\[
H(t,x,y,z) := \text{ess sup}_{u \in U} \text{ess inf}_{v \in V} \{ f(t,x,u,v) + \langle \ell^{u,v}(t), z \rangle_g \},
\]  

(5.11)

and the one associated with \((\overline{Y}, \overline{Z})\) is

\[
\overline{H}(t,x,y,z) := \text{ess inf}_{u \in U} \text{ess sup}_{v \in V} \{ f(t,x,u,v) + \langle \ell^{u,v}(t), z \rangle_g \}.
\]  

(5.12)

To \( H \) we associate \( \hat{\beta}(t,x,z,\bar{z}) = (\hat{\beta}_{ij}(t,x,z,\bar{z}))_{ij} \) given by

\[
\hat{\beta}(t,x,z,\bar{z}) = \theta_\beta(t,x,z,\bar{z})\beta(t,x,z,\bar{z}) + (1 - \theta_\beta(t,x,z,\bar{z}))\overline{\beta}(t,x,z,\bar{z}),
\]  

(5.13)

with

\[
\theta_\beta(t,x,z,\bar{z}) = \frac{H(t,x,y,z) - H(t,x,y,\bar{z}) - (\beta(t,x,z,\bar{z}), z - \bar{z})_g}{(\beta(t,x,z,\bar{z}) - \overline{\beta}(t,x,z,\bar{z}), z - \bar{z})_g} \in [0,1],
\]  

(5.14)

where \( \overline{\beta} = (\overline{\beta}_{ij})_{ij} \) and \( \beta = (\beta_{ij})_{ij} \) read, for \( i \neq j \),

\[
\overline{\beta}_{ij}(t,x,z,\bar{z}) := \text{ess sup}_{v \in V} \text{sup}_{u \in U} \ell^{u,v}_{ij}(t)1_{\{z_{ij} > \bar{z}_{ij}\}} + \text{ess inf}_{v \in V} \text{inf}_{u \in U} \ell^{u,v}_{ij}(t)1_{\{z_{ij} \leq \bar{z}_{ij}\}},
\]

\[
\beta_{ij}(t,x,z,\bar{z}) := \text{ess sup}_{v \in V} \text{sup}_{u \in U} \ell^{u,v}_{ij}(t)1_{\{z_{ij} > \bar{z}_{ij}\}} + \text{ess inf}_{v \in V} \text{inf}_{u \in U} \ell^{u,v}_{ij}(t)1_{\{z_{ij} \leq \bar{z}_{ij}\}}.
\]

To \( \overline{H} \) we associate \( \overline{\theta}(t,x,z,\bar{z}) = (\overline{\theta}_{ij}(t,x,z,\bar{z}))_{ij} \) given by

\[
\overline{\theta}(t,x,z,\bar{z}) = \theta_\theta(t,x,z,\bar{z})\overline{\beta}(t,x,z,\bar{z}) + (1 - \theta_\theta(t,x,z,\bar{z}))\theta(t,x,z,\bar{z}),
\]  

(5.15)

with

\[
\theta_\theta(t,x,z,\bar{z}) = \frac{\overline{H}(t,x,y,z) - \overline{H}(t,x,y,\bar{z}) - (\overline{\beta}(t,x,z,\bar{z}), z - \bar{z})_g}{(\overline{\beta}(t,x,z,\bar{z}) - \theta(t,x,z,\bar{z}), z - \bar{z})_g} \in [0,1],
\]  

(5.16)

where \( \theta = (\theta_{ij})_{ij} \) and \( \overline{\theta} = (\overline{\theta}_{ij})_{ij} \) are given by \((i \neq j)\)

\[
\overline{\theta}_{ij}(t,x,z,\bar{z}) := \text{ess sup}_{v \in V} \text{sup}_{u \in U} \ell^{u,v}_{ij}(t)1_{\{z_{ij} \geq \bar{z}_{ij}\}} + \text{ess inf}_{v \in V} \text{inf}_{u \in U} \ell^{u,v}_{ij}(t)1_{\{z_{ij} < \bar{z}_{ij}\}},
\]

\[
\theta_{ij}(t,x,z,\bar{z}) := \text{ess sup}_{v \in V} \text{sup}_{u \in U} \ell^{u,v}_{ij}(t)1_{\{z_{ij} \geq \bar{z}_{ij}\}} + \text{ess inf}_{v \in V} \text{inf}_{u \in U} \ell^{u,v}_{ij}(t)1_{\{z_{ij} < \bar{z}_{ij}\}}.
\]

We omit the proof of the next lemma as it is similar to the proof of (4.24).

**Lemma 5.1.** \( H \) and \( \overline{H} \) are balanced: For every \( z, \bar{z} \in \mathbb{R}^{t \times I} \),

\[
\begin{align*}
H(t,x,y,z) - H(t,x,y,\bar{z}) &= (\hat{\beta}(t,x,z,\bar{z}), z - \bar{z})_g, \\
\overline{H}(t,x,y,z) - \overline{H}(t,x,y,\bar{z}) &= (\overline{\theta}(t,x,z,\bar{z}), z - \bar{z})_g,
\end{align*}
\]  

(5.17)

where \( \hat{\beta} \) and \( \overline{\theta} \) are given by (5.13) and (5.15).

Then, as previously, there exists a unique solution \((Y,Z)\) (resp. \((\overline{Y}, \overline{Z})\)) to the BSDE associated with \((H, h)\) (resp. \((\overline{H}, \overline{h})\)).

**Definition 5.2** (Isaacs’ condition): We say that the Isaacs’ condition holds for the game if

\[
\begin{align*}
\begin{cases}
H(t,x,z) = \overline{H}(t,x,z), & 0 \leq t \leq T, \\
\overline{h}(x) = \overline{h}(x).
\end{cases}
\end{align*}
\]

Applying the comparison theorem for BSDEs, we obtain the following
Proposition 10. For every $t \in [0,T]$, it holds that $Y_{t} = \overline{Y}(t) := Y(t)$, $P$-a.s.. Moreover, if the Issac’s condition holds, then

$$Y(t) = \overline{Y}(t) := Y(t), \quad P\text{-a.s.,} \quad 0 \leq t \leq T. \tag{5.18}$$

In the next theorem, we formulate conditions for which the zero-sum game has a value. For $(u, v) \in \mathcal{U} \times \mathcal{V}$, let $(Y^{u,v}, Z^{u,v})$ be the solution of the BSDE

$$\begin{cases}
- \dot{Y}^{u,v}(t) = H(t, x, P^{u,v} \circ x^{-1}(t), Z^{u,v}(t), u(t), v(t))dt - Z^{u,v}(t)dM(t), \\
Y^{u,v}(T) = h(x(T), P^{u,v} \circ x^{-1}(T)).
\end{cases} \tag{5.19}$$

Theorem 5.3 (Existence of a value for the game). Assume that, for every $0 \leq t < T$,

$$H(t, x, Z(t)) = \overline{H}(t, x, Z(t)).$$

If there exists $(\hat{u}, \hat{v}) \in \mathcal{U} \times \mathcal{V}$ such that, for every $0 \leq t < T$,

$$H(t, x, Z(t)) = \text{ess inf}_{u \in \mathcal{U}} H(t, x, Z(t), u, \hat{v}) = \text{ess sup}_{v \in \mathcal{V}} H(t, x, Z(t), \hat{u}, v), \tag{5.20}$$

and

$$h(x) = \overline{h}(x) = \text{ess inf}_{u \in \mathcal{U}} h(x(T), P^{u,\hat{v}} \circ x^{-1}(T)) = \text{ess sup}_{v \in \mathcal{V}} h(x(T), P^{\hat{u},v} \circ x^{-1}(T)). \tag{5.21}$$

Then,

$$Y(t) = \text{ess inf}_{u \in \mathcal{U}} \text{ess sup}_{v \in \mathcal{V}} Y^{u,v}(t) = \text{ess sup}_{v \in \mathcal{V}} \text{ess inf}_{u \in \mathcal{U}} Y^{u,v}(t), \quad 0 \leq t \leq T. \tag{5.22}$$

Moreover, the pair $(\hat{u}, \hat{v})$ is a saddle-point for the game.

Proof. Let $(u, v) \in \mathcal{U} \times \mathcal{V}$ and $(\bar{Y}^{u}, \bar{Z}^{u})$ and $(\bar{Y}^{v}, \bar{Z}^{v})$ be the solution of the following BSDE

$$\begin{cases}
- \dot{\bar{Y}}^{u}(t) = \text{ess sup}_{v \in \mathcal{V}} H(t, x, \bar{Z}^{u}(t), u, v)dt - \bar{Z}^{u}(t)dM(t), \quad 0 \leq t < T, \\
\bar{Y}^{u}(T) = \text{ess sup}_{v \in \mathcal{V}} h(x(T), P^{u,v} \circ x^{-1}(T)),
\end{cases} \tag{5.23}$$

$$\begin{cases}
- \dot{\bar{Y}}^{v}(t) = \text{ess inf}_{u \in \mathcal{U}} H(t, x, \bar{Z}^{v}(t), u, v)dt - \bar{Z}^{v}(t)dM(t), \quad 0 \leq t < T, \\
\bar{Y}^{v}(T) = \text{ess inf}_{u \in \mathcal{U}} h(x(T), P^{u,v} \circ x^{-1}(T)).
\end{cases} \tag{5.24}$$

By uniqueness of the solutions of the BSDEs, we have

$$\bar{Y}^{u}(t) = \text{ess sup}_{v \in \mathcal{V}} Y^{u,v}(t), \quad \bar{Y}^{v}(t) = \text{ess inf}_{u \in \mathcal{U}} Y^{u,v}(t), \tag{5.25}$$

and, by comparison, we have

$$\bar{Y}^{u}(t) \geq Y(t) \geq \text{ess sup}_{v \in \mathcal{V}} \bar{Y}^{v}(t).$$

Therefore,

$$\text{ess inf}_{u \in \mathcal{U}} \bar{Y}^{u}(t) \geq Y(t) \geq \text{ess sup}_{v \in \mathcal{V}} \bar{Y}^{v}(t).$$

But, by (5.20) and (5.21), in view of the uniqueness of the solutions of the BSDEs we have $\bar{Y}^{v}(t) = Y(t) = \bar{Y}^{v}(t)$.

Therefore,

$$\bar{Y}^{u}(t) = \text{ess inf}_{u \in \mathcal{U}} \bar{Y}^{u}(t) = Y(t) = \bar{Y}^{v}(t) = \text{ess sup}_{v \in \mathcal{V}} \bar{Y}^{v}(t) = Y^{\hat{u},\hat{v}}(t).$$
Using (5.25), we obtain
\[
Y(t) = Y\tilde{u},\tilde{v}(t) = \text{ess sup}_{v \in V} Y\tilde{u},v(t) = \text{ess inf}_{u \in U} Yu,\tilde{v}(t).
\]
Therefore,
\[
Y\tilde{u},v(t) \leq Y\tilde{u},\tilde{v}(t) \leq Y\tilde{u},v(t).
\]
Thus, \(Y\tilde{u},\tilde{v}(0)\) is the value of the game and \((\tilde{u}, \tilde{v})\) is a saddle-point. \(\square\)

**Remark 4.**
1. As mentioned in Remark (3), if the marginal law \(P^{u,v} \circ x^{-1}(s)\) of \(x(s)\) is a function of \((u(s), v(s))\) only and does not depend on the whole path of \((u, v)\) over \([0, s]\), it suffices to take the minimum and the maximum resp. of \(H\) and \(h\) over the compact set \(U\) and \(V\), resp., instead of taking the essential infimum over \(U\) and the essential maximum over \(V\). By the measurable selection theorem (see e.g. [1]), a saddle-point over \([0, T]\) can be obtained by pasting the saddle-points of \(H\) and \(h\).
2. It is possible to characterize the optimal controls \(\hat{u}\) and the equilibrium points \((\hat{u}, \hat{v})\) in terms of a stochastic maximum principle. This approach is discussed in [6].

6. Appendix.

**Proof of (4.21) in proposition 5.** For \(n \geq 0\) let \(z_n \in Q^{I \times I}\), the \(I \times I\)-matrix with rational entries. Then, since \((t, \omega) \mapsto H(t, \omega, z_n, u)\) is \(L\)-measurable, its essential infimum w.r.t. \(u \in U\) is well defined i.e. there exists an \(L\)-measurable r.v. \(H^n\) such that
\[
H^n(t, x, z_n) = \text{ess inf}_{u \in U} H(t, x, z_n, u), \quad dP \times dt\text{-a.s.} \quad (6.1)
\]
Moreover, there exists a set \(J_n\) of \(U\) such that \((t, \omega) \mapsto \inf_{u \in J_n} H(t, \omega, z_n, u)\) is \(L\)-measurable and
\[
H^n(t, x, z_n) = \inf_{u \in J_n} H(t, x, z_n, u), \quad dP \times dt\text{-a.s.} \quad (6.2)
\]
Next, set \(N = \bigcup_{n \geq 0} N_n\), where
\[
N_n := \{(t, \omega) : H^n(t, \omega) \neq \inf_{u \in J_n} H(t, \omega, z_n, u)\}.
\]
Then, \(dP \times dt(N) = 0\).

We define \(H^*\) as follows: For \((t, \omega) \in N^c\) (the complement of \(N\)),
\[
H^*(t, x, z) = \begin{cases} 
\inf_{u \in J_n} H(t, x, z_n, u) & \text{if } z = z_n \in Q^{I \times I}, \\
\lim_{z_n \to z} \inf_{u \in J_n} H(t, x, z_n, u) & \text{otherwise}.
\end{cases} \quad (6.3)
\]
The last limit exists due to the fact that, for \(n \neq m\), we have
\[
\left| \inf_{u \in J_m} H(t, x, z_m, u) \right| - \inf_{u \in J_n} H(t, x, z_n, u) |H^*(t, x, z_m) - H^*(t, x, z_n)|
\leq \text{ess sup}_{u \in U} |H(t, x, P^u \circ x^{-1}, z_n, u) - H(t, x, P^u \circ x^{-1}, z_m, u)|
\leq C(1 + |x| + \sup_{u \in U} \|P^u\|2)(|z_n - z_m|)(t)||g.
\]
We now show that, for every \(z \in R^{I \times I}\),
\[
H^*(t, x, z) = \text{ess inf}_{u \in U} H(t, x, z, u), \quad dP \times dt\text{-a.s.} \quad (6.4)
\]
If \(z \in Q^{I \times I}\), the equality follows from the definitions (6.1) and (6.3). Assume \(z \notin Q^{I \times I}\) and let \(z_n \in Q^{I \times I}\) such that \(z_n \to z\). Further, let \(\varphi(t, x)\) be a progressively
measurable process such that $\varphi(t, x) \leq H(t, x, z, u)$ for all $u \in U$. Thus, for every
\[ \eta > 0 \] there exists $n_0 \geq 0$ such
\[ \varphi(t, x) \leq H(t, x, z_n, u) + \eta, \quad n \geq n_0, \ u \in U. \]
Therefore, $\varphi(t, x) \leq \star H^*(t, x, z_n) + \eta, \ n \geq n_0$. Letting $n \to \infty$, we obtain $\varphi(t, x) \leq \star H^*(t, x, z) + \eta$. Sending $\eta$ to 0, we finally get $\varphi(t, x) \leq \star H^*(t, x, z)$, i.e.
\[ \text{ess inf}_{u \in U} H(t, x, z, u) \leq \star H^*(t, x, z), \quad dP \times dt\text{-a.s.} \]
On the other hand, in view of (6.3) and the linearity of $H$ in $z$, we have $\star H^*(t, x, z) \leq H(t, x, z, u), \ u \in U$. Thus,
\[ \star H^*(t, x, z) \leq \text{ess inf}_{u \in U} H(t, x, z, u). \]
This finishes the proof of (6.4).

**Proof of (4.25)** in proposition 5. Noting that for any $z \in \mathbb{R}^{I \times I}$ and $u \in U$, $\star H^*(t, x, z) \leq H(t, x, z, u)$, we have
\[ \star H^*(t, x, \theta_t) \leq H(t, x, \theta_t, u), \quad u \in U. \]
Next, let $\Phi$ be an $L$-measurable process such that $\Phi(t, \omega) \leq H(t, x, \theta_t, u)$ for any $u \in U$. Assume first that $\theta$ is uniformly bounded. Then there exists a sequence of $L$-processes $\theta^n_{n \geq 0}$ such that for any $n \geq 0$, $\theta^n$ takes its values in $Q^{I \times I}$, is piecewise constant and satisfies $\|\theta^n - \theta\|_\infty := \sup_{(t, \omega)} \|\theta^n_t(\omega) - \theta_t(\omega)\|_g \to 0$ as $n \to \infty$. Furthermore, in view of the conditions (B4) and (B8), we have
\[ |H(t, x, \theta_t, u) - H(t, x, \theta^n_t, u)| \leq C(1 + \|x\|_t + \|P^n\|_2)\|\theta^n - \theta\|_\infty. \]
Now, let $\epsilon > 0$ and $n_0$ such that for any $n \geq n_0$, $\|\theta^n - \theta\|_\infty \leq \epsilon$. Then, for $n \geq n_0$ and $u \in U$ we have
\[ \Phi(t, \omega) \leq H(t, x, \theta^n_t, u) + \epsilon C(1 + \|x\|_t + \|P^n\|_2), \]
which implies that
\[ 1_{B^k_t} \Phi(t, \omega) \leq 1_{B^k_t} \{ H(t, x, z^n_k, u) + \epsilon C(1 + \|x\|_t + \|P^n\|_2) \}, \]
where $B^k_t$ is a subset of $[0, T] \times \Omega$ in which $\theta^n$ is constant and equals to $z^n_k \in Q^{I \times I}$. Therefore
\[ 1_{B^k_t} \Phi(t, \omega) \leq 1_{B^k_t} \{ \text{ess inf}_{u \in J^k_t} H(t, x, z^n_k, u) + \epsilon C(1 + \|x\|_t + \|P^n\|_2) \}
\leq 1_{B^k_t} \{ \star H^*(t, x, z^n_k) + \epsilon C(1 + \|x\|_t + \|P^n\|_2) \}
\leq 1_{B^k_t} \{ \star H^*(t, x, \theta^n_t) + \epsilon C(1 + \|x\|_t + \|P^n\|_2) \}, \]
where $J^k_t$ is the countable subset of $U$ defined in (6.2) and associated with $z^n_k$. Summing over $k$, we obtain
\[ \Phi(t, \omega) \leq \star H^*(t, x, \theta_t) + 2\epsilon C(1 + \|x\|_t + \|P^n\|_2), \quad (6.5) \]
since $H^*$ is stochastic Lipschitz w.r.t. $z$ (see (4.22)). Thus,
\[ |H^*(t, x, \theta_t) - \star H^*(t, x, \theta^n_t)| \leq \epsilon C(1 + \|x\|_t + \|P^n\|_2) \]
for $n \geq n_0$. Send now $\epsilon$ to 0 in (6.5) to obtain that $\Phi(t, \omega) \leq \star H^*(t, x, \theta_t)$ which means
\[ \star H^*(t, x, \theta_t) = \text{ess inf}_{u \in U} H(t, x, \theta_t, u), \quad dP \times dt\text{-a.e.} \]
If \( \theta \) is not bounded, we can find a sequence of bounded \( \mathbb{L} \)-processes \( (\bar{\theta}^n)_{n \geq 0} \) such that \( \bar{\theta}^n \to \theta \) as \( n \to \infty \), \( dP \times dt \)-a.e.

Therefore, we have

\[
H^*(t, x, \bar{\theta}^n(t)) = \text{ess inf}_{u \in U} H(t, x, \bar{\theta}^n(t), u), \quad dP \times dt \text{-a.e.} \tag{6.6}
\]

But, the stochastic Lipschitz property of \( H^* \) and the linearity of \( H \) w.r.t. \( z \) imply that, as \( n \to \infty \),

\[
H^*(t, x, \theta^n(t)) \to H^*(t, x, \theta(t)), \quad \text{ess inf}_{u \in U} H(t, x, \bar{\theta}^n(t), u) \to \text{ess inf}_{u \in U} H(t, x, \bar{\theta}, u).
\]

We then obtain the desired result by taking the limit in (6.6).

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REFERENCES

[1] V. E. Beneš, Existence of optimal stochastic control laws, \textit{SIAM J. Control}, 9 (1971), 446–472.
[2] P. Brémaud, \textit{Point Processes and Queues: Martingale Dynamics}, Springer-Verlag, New York-Berlin, 1981.
[3] S. N. Cohen and R. J. Elliott, Existence, uniqueness and comparisons for BSDEs in general spaces, \textit{Annals of Probability}, 40 (2012), 2264–2297.
[4] S. N. Cohen and R. J. Elliott, \textit{Stochastic Calculus and Applications}, Second edition. Probability and its Applications. Springer, Cham, 2015.
[5] F. Confortola, M. Fuhrman and J. Jacod, Backward stochastic differential equations driven by a marked point process: an elementary approach, with an application to optimal control, Preprint, \texttt{arXiv:1407.0876 [math.PR]}, 2014.
[6] S. E. Choutri and H. Tembine, A stochastic maximum principle for markov chains of mean-field type, \textit{Games}, 9 (2018), Paper No. 84, 21 pp, \texttt{https://doi.org/10.3390/g9040084}.
[7] D. Dawson and X. Zheng, Law of large numbers and central limit theorem for unbounded jump mean-field models, \textit{Advances in Applied Mathematics}, 12 (1991), 293–326.
[8] B. Djehiche and S. Hamadène, Optimal control and zero-sum stochastic differential game problems of mean-field type, \textit{Appl Math Optim.}, 2018, \texttt{https://doi.org/10.1007/s00245-018-9525-6}.
[9] B. Djehiche and I. Kaj, The rate function for some measure-valued jump processes, \textit{The Annals of Probability}, 23 (1995), 1414–1438.
[10] B. Djehiche and A. Schied, Large deviations for hierarchical systems of interacting jump processes, \textit{Journal of Theoretical Probability}, 11 (1998), 1–24.
[11] I. Ekeland, \textit{On the variational principle}, \textit{J. Math. Anal. Appl.}, 47 (1974), 324–353.
[12] N. El Karoui and S. Hamadène, BSDEs and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations, \textit{Stochastic Processes and their Application}, 107 (2003), 145–169.
[13] N. El Karoui, S. Peng and M.-C. Quenez, Backward stochastic differential equations in finance, \textit{Mathematical Finance}, 7 (1997), 1–71.
[14] R. J. Elliott and M. Kohlmann, The variational principle and stochastic optimal control, \textit{Stochastics}, 3 (1980), 229–241.
[15] S. N. Ethier and T. G. Kurtz, \textit{Markov Processes: Characterization and Convergence}, John Wiley & Sons, Inc., New York, 1986.
[16] S. Feng, Large deviations for empirical process of mean-field interacting particle system with unbounded jumps, \textit{The Annals of Probability}, 22 (1994), 2122–2151.
[17] S. Feng and X. Zheng, Solutions of a class of nonlinear master equations, \textit{Stochastic processes and their applications}, 43 (1992), 65–84.
[18] S. Hamadène and J. P. Lepeltier, Backward equations, stochastic control and zero-sum stochastic differential games, \textit{Stochastics Stochastics Rep.}, 54 (1995), 221–231.
[19] B. Jourdain, S. Méléard and W. Woyczynski, Nonlinear SDEs driven by Lévy processes and related PDEs, *ALEA Lat. Am. J. Probab. Math. Stat.*, 4 (2008), 1–29.

[20] V. N. Kolokoltsov, *Nonlinear Markov Processes and Kinetic Equations*, Cambridge University Press, Cambridge, 2010.

[21] V. N. Kolokoltsov, Nonlinear Markov games on a finite state space (mean-field and binary interactions), *International Journal of Statistics and Probability*, 1 (2012), 77.

[22] C. Léonard, Some epidemic systems are long range interacting particle systems, Stochastic processes in epidemic systems (eds. J.P. Gabriel et al.), Lecture Notes in Biomathematics, volume 86, 1990, Springer.

[23] C. Léonard, Large deviations for long range interacting particle systems with jumps, *Annales de l'IHP Probabilités et Statistiques*, 31 (1995), 289–323.

[24] G. Nicolis and I. Prigogine, *Self Organization in Non-Equilibrium Systems*, New York-London-Sydney, 1977.

[25] K. Oelschläger, A martingale approach to the law of large numbers for weakly interacting stochastic processes, *The Annals of Probability*, (1984), 458–479.

[26] E. Pardoux and S. Peng, Adapted Solution of a Backward Stochastic Differential Equation, *Systems and Control Letters*, 14 (1990), 55–61.

[27] L. C. G. Rogers and D. Williams, *Diffusions, Markov Processes and Martingales-Volume 2: Itô Calculus*, Cambridge University Press, Cambridge, 2000.

[28] F. Schlögl, Chemical reaction models for non-equilibrium phase transitions, *Zeitschrift für Physik*, 253 (1972), 147–161.

[29] A. Sokol and N. R. Hansen, Exponential martingales and changes of measure for counting processes, *Stochastic Analysis and Applications*, 33 (2015), 823–843.

[30] A.-S. Sznitman, *Topics in propagation of chaos, Ecole d’Été de Probabilités de Saint-Flour XIX 1989, 1964* (1991), 165–251.

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E-mail address: tembine@nyu.edu
E-mail address: boualem@kth.se
E-mail address: choutri@kth.se