Observability of fractional order differential impulsive multi control problem with fractional integral nonlocal initial condition
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Abstract :
In this paper ,the observability of fractional differential impulsive multi control abstract problem with fractional integral nonlocal initial condition have been studied as abstract Cauchy problem for using Banach fixed point which defined on space of presented problem which is piecewise continuous space and proprieties of the initial observable condition for their problem.

1.Introduction:
The observability of linear and linear impulsive abstract problems which defined in infinite dimensional continues or piecewise continuous appearing in many researches [5][3].The observability depended on nonlinear part and many methods of certain fixed point theorems depened on their problems. The impulsive fractional order abstract control problems with general nonlocal initial condition have been appeared in limited classes with different approach such as, [4],[6],[11],[8],[9].

Consider the following impulsive multi control fractional differential abstract problem with fractional integral nonlocal initial condition:

\[
D^αx(t) = A(t)x(t) + f_1\left(t, x(t), \int_0^t h(t, s, x(s))ds\right) + \sum_{i=1}^2 \int_0^{\tau_i} g_i(t, s, x(s))\,ds + \int_0^{\tau_i} h(t, s, x(s))\,ds + \sum_{i=1}^2 B_iu_i(t), \quad t \neq t_k
\]

\[
u_1(t) = -Kx(t), \quad u_2(t) \leq v(t) + a(t)\int_0^{\tau_i} (t - \tau)^{\alpha - 1} u_2(t)\,d\tau,
\]

\[
\Delta x(t_k) = x(t_k^+) - x(t_k^-),
\]

\[
x(0) + \int_0^\infty x(\tau)\,d\tau = x_0,
\]

\[
y(t) = C(t)x(t) + C^*(t).
\]

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where \( t \in J = [0,T], k = 1,2,\ldots,m, \quad 0 < \alpha \leq 1 \), \( D^\alpha \) is the caputo fractional derivative. \( v(t), a(t) \) are nonnegative functions. Assume a bounded operator \( A(t): X \to X \) (\( X \) Banach space ), \( f_1, f_2: J \times X \times X \to X \), \( h: t \times s \times X \to X \), 0≤ s ≤ t ≤ T , 0 =t_0 < t_1 < \cdots < t_m < t_{m+1} = T ,  
\[ \Delta x(t_k) = x(t_k^+) - x(t_k^-), \quad x(t_k^+), \quad x(t_k^-), \]
denoted the left and the right limit of \( x \) at \( t_k \), respectively \( g: PC([0,T]; X) \to X \) is a given function. \( y(.) \) is referred to as the output which is belong to Banach space \( Y \). \( C: X \to Y, \quad C^*: Y \to X \) is a bounded linear operator.

Our aim to study and present the observeability of fractional differential impulsive multi control abstract problem with fractional integral nonlocal initial condition (1-5) with necessary and sufficient conditions that which guaranty the problem initial observable.

2. Preliminaries:
The following definitions and results are need it later on for investigate the initial observable for problem(1-3).

**Definition(2.1), [7]:**
The Riemann- Liounille fractional integral of a function \( f \) with order \( \alpha > 0 \), is
\[ \mathcal{I}_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0, \alpha > 0 \]. (4)

**Definition(2.2), [2]:**
The Caputa fractional derivative of a function \( f \) with order \( \alpha > 0 \), where \( n-1 < \alpha \leq n \),and \( n \in N \), is defined by:
\[ D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s) ds, \quad t > 0, \alpha > 0 \]. (5)

Where \( f \) is absolutely continuous derivative up to \( n-1 \). If \( 0 < \alpha \leq 1 \) then
\[ D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds. \]

**Lemma (2.3), [7]:**
Let \( \alpha > 0 \) and \( f \) be a suitable function. Then we have
\[ I_{0+}^\alpha D_{0+}^\alpha f(t) = f(t) - f(0), \text{where} 0 < \alpha \leq 1 \]. (6)

**Definition (2.4), [10]:**
The function \( x(.) \in PC([0,T]; X) \) is a mild solution of abstract problem (1-5) which is equivalent to
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\[ x(t) = \begin{cases} 
  x_0 - I^\beta g(x) + \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \int_0^t [A(s) - K] x(s) + \\
  f_1(s, x(s), \int_0^s h(t, \tau, x(\tau)) d\tau) f_2(s, x(s), D^\alpha x(s)) + B_1 u_1(s) ds, & t \in [0, t_1] \\
  x_0 - I^\beta g(x) + \sum_{k=1}^m \delta_k (y(t_i)) + \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \int_0^t [A(s) - K] x(s) + \\
  f_1(s, x(s), \int_0^s h(t, \tau, x(\tau)) d\tau) f_2(s, x(s), D^\alpha x(s)) + B_1 u_1(s) ds, & t \in (t_k, t_{k+1}] 
\end{cases} \]

If satisfies the integral \((7)\).

\textbf{Lemma (2.1.1), [10]:}

A Mittage-Leffler function \(E_{\alpha,\beta}(A t^\alpha)\) satisfies the following
1. \(E_{\alpha,1}(A t^\alpha) \leq K_{E_{\alpha,1}} \|e^{A t}\|, \alpha > 1, K_{E_{\alpha,1}} > 1\), such that
2. \(E_{\alpha,\alpha}(A t^\alpha) \leq K_{E_{\alpha,\alpha}} \|e^{A t}\|, \alpha > 1, K_{E_{\alpha,\alpha}} > 1\), where \(A \in \mathbb{R}^{n \times n}\).

\textbf{Lemma (2.1.2), [10]:}

Let \(\alpha > 0\) \(v(t)\) is a nonnegative function locally integrable on \([0, T]\) and \(a(t)\) a nonnegative, nondecreasing continuous function defined on \([0, T]\), \(a(t) < M\) and suppose \(z(t)\) is nonnegative and locally integrable on \([0, T]\) with \(z(t) \leq v(t) + a(t) \int_0^t (t - \tau)^{\alpha-1} z(\tau) d\tau\). If \(v(t)\) is a nondecreasing function on \([0, T]\), we have \(z(t) \leq v(t) E_{\alpha}(\Gamma(\alpha)a(t)t^\alpha)\).

\textbf{Hypothesis :}

Let \(B_r = \{x \in X : \|x\| \leq r\}\) is a neighborhood of zero, \(t \in [0, T]\).

\(h1): \|A(t)\|_{B(X)} \leq M\) where \(A : J \to B(X)\) is bounded linear operator and \(B_r \subseteq \bar{M}, \|K\| \leq \bar{M}, \bar{M}, M > 0\).

\(h2): \int f_1 f_2 : J \times X \times X \to X\) is continuous and there exist constant \(N_1 > 0\) and \(N_2 > 0\) such that
\[ \|f_1(t, x, u) - f_1(t, y, v)\| \leq N_1 [\|x - y\| + \|u - v\|], \quad x, y, u, v \in B_r \]
\[ \|f_2(t, x, u) - f_2(t, y, v)\| \leq N_1 [\|x - y\| + \|u - v\|], \quad x, y, u, v \in B_r \]
\[ = \max_{t \in J} \|f_2(t, 0, 0)\| \]
\[ \|f_2(t, x, u)\| \leq K f_2(t) \Omega_{f_2}(\|x\| + \|u\|) \leq K f_2(t) \Omega_{f_2}(r + \frac{\tau^{\gamma-1}}{\Gamma(\gamma)} \sup_{t \in J} \|u(t)\|) \]

\(h3): A\) continuous function \(h : \Delta \times X \to X\) with fixed constants \(H_1 > 0\) and \(H_2 > 0\) satisfy
\[ \|h(t, s, x_1) - h(t, s, x_2)\| \leq H_1 \|x_1 - x_2\|, \quad x_1, x_2 \in B_r\] and \(H_2 = \max_{t \in J} \|h(t, 0, 0)\| \)
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(h4): \( \|I_x^k (x_1) - I_x^k (x_2)\| \leq \ell_1 \|x_1 - x_2\| \) and \( \|I_x^k (x)\| < \ell_2 \) for each \( x_1, x_2, x \in X \) and \( k=1, \ldots, m, \ell_1, \ell_2 > 0 \).

(h5):
\[
\| \beta_\alpha \| g(t_2) - I_\beta \| g(t_1) \| \leq \frac{1}{[\Gamma (\beta + 1)]} \| g(t_2) - g(t_1) \| \leq \frac{\ell_2}{[\Gamma (\beta + 1)]} \| x_1 - x_2 \|
\]

for \( x_1, x_2 \in \mathcal{PC}([0, T]: \mathbb{X}) \).

From the following system:

\[
\begin{align*}
D\alpha x(t) &= A(t)x(t) + f_1 (t, x(t), \int_0^t h(t, s, x(s))ds) f_2 (t, x(t), D\alpha x(t)) + \sum_{i=1}^{2} B_i u_i(t), t \neq t_k \\
u_i(t) &= -Kx(t), \quad u_2(t) \leq v(t) + a(t) \int_0^t (t - \tau)^{-\alpha-1} u_2(\tau)d\tau, \\
\Delta x(t_k) &= x(t_k^+) - x(t_k^-), \\
x(0) + I_\beta g(x) &= x_0, \\
y(t) &= C(t)x(t) + C^*(t).
\end{align*}
\]

C: \( X \rightarrow Y \), \( C^*: Y \rightarrow X \) is a bounded linear operator, the homogenous part is:

\[
\begin{align*}
x(t) &= \begin{cases} \\
\left[ x_0 - I_\beta g(x) \right] + \frac{1}{\Gamma (\alpha)} \int_0^t (t - s)^{\alpha-1} \left[ (A(s) - K)x(s) + B_2 u_2(s) \right]ds, & t \in [0, t_1] \\
\left[ x_0 - I_\beta g(x) \right] + \sum_{i=1}^{2} I_i \left( x(t_i^-) \right) + \frac{1}{\Gamma (\alpha)} \int_0^t (t - s)^{\alpha-1} \left[ (A(s) - K)x(s) + B_2 u_2(s) \right]ds, & t \in (t_k, t_{k+1}], k = 1, 2, \ldots, m.
\end{cases}
\end{align*}
\]

(8) and

\[
y(t) = \begin{cases} \\
C(t) \left[ x_0 - I_\beta g(x) \right] C^*(t) + \frac{C(t)}{\Gamma (\alpha)} \int_0^t (t - s)^{\alpha-1} \left[ (A(s) - K)x(s) + B_2 u_2(s) \right]ds C^*(t), & t \in [0, t_1] \\
C \sum_{i=1}^{2} I_i \left( x(t_i^-) \right) + \frac{C}{\Gamma (\alpha)} \int_0^t (t - s)^{\alpha-1} \left[ (A(s) - K)x(s) + B_2 u_2(s) \right]ds, & t \in (t_k, t_{k+1}], k = 1, 2, \ldots, m.
\end{cases}
\]

(9) Let \( \Omega = \mathcal{PC}(J; \mathbb{Y}) \), now assume the operator \( H: X \rightarrow Y \) as

\[
H \left[ x_0 - g(x(t)) \right] = \begin{cases} \\
C(t) \left[ x_0 - I_\beta g(x) \right] C^*(t) + \frac{C(t)}{\Gamma (\alpha)} \int_0^t (t - s)^{\alpha-1} \left[ (A(s) - K)x(s) + B_2 u_2(s) \right]ds C^*(t), & t \in [0, t_1] \\
C \sum_{i=1}^{2} I_i \left( x(t_i^-) \right) C^*(t) + \frac{C}{\Gamma (\alpha)} \int_0^t (t - s)^{\alpha-1} \left[ (A(s) - K)x(s) + B_2 u_2(s) \right]ds C^*(t), & t \in (t_k, t_{k+1}], k = 1, 2, \ldots, m.
\end{cases}
\]

(10) As the same proving of results in [4], we can prove the following,
Remarks (3.1):
1. The system in (8) is initial observable if kernel={0}.
2. The system in (8) is continuously initially observable if
\[
\|H[x_0 - I^\beta g(x(t))]| = \|x_0 - I^\beta g(x(t))\|.
\]
3. If a system in (8) is initially observable which implies the map H is injective but not surjective.
4. When system in (8) is continuous initially observable implies that \( H^{-1}: Y \rightarrow X \) exists and bounded that is there exists \( \tilde{R} > 0 \) Such that \( \|H^{-1}v\| \leq \tilde{R}\|v\| \) for all \( v \in Y \).

As the same proving of result in [1], we can prove the following:

Lemma (3.2):
The system in (8) is continuously initially observable on \([0,T]\) if and only if the system
\[
x(t) = \begin{cases} 
  x_0 - I^\beta g(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B_2 u_2(s) \,ds, & t \in [0,t_1] \\
  x_0 - g(x) + \sum_{k=1}^k l_i(x(t_k)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \,B_2 u_2(s) \,ds, & t \in (t_k, t_{k+1}], k = 1,2,\ldots.
\end{cases}
\]
is exactly controllable on \([0,T]\).

Concluding remarks (3.3):
If the linear part equation
\[
x(t) = \begin{cases} 
  x_0 - I^\beta g(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B_2 u_2(s) \,ds, & t \in [0,t_1] \\
  x_0 - I^\beta g(x) + \sum_{k=1}^k l_i(x(t_k)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \,B_2 u_2(s) \,ds, & t \in (t_k, t_{k+1}], k = 1,2,\ldots,
\end{cases}
\]
is exactly controllable on \([0,T]\) then by remark (3.1), is continuously initially observable on \([0,T]\), thus remark (3.1) (4) is also satisfied.
Since the system in (8) is continuously initially observable so that the initial state \( x_0 - I^\beta g(x) \) of the system (8) can be obtained as follow:

\[
H^{-1}y(t) =\begin{cases} 
  H^{-1} \left[ C(t) \left[ x_0 - I^\beta g(x) \right] C^*(t) + \frac{C(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ A(s) - K \right] x(s) \,ds \right] C^*(t), & t \in [0,t] \\
  H^{-1} \left[ C(t) \left[ x_0 - I^\beta g(x) \right] C^*(t) + C(t) \sum_{i=1}^m l_i(x(t_k)) C^*(t) \right] \left[ A(s) - K \right] x(s) \,ds \right] C^*(t), & t \in (t_k, t_{k+1}], k = 1,2,\ldots,m
\end{cases}
\]
From (10), we have that
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\[
H^{-1}y(t) = \begin{cases} 
[x_0 - I^\beta g(x)] + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} \left[ A(s) - K \right] x(s) \, ds, & t \in [0,t_1] \\
[x_0 - I^\beta g(x)] + \sum_{i=1}^{k} I_i \left( x(t_i) \right) + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} \left[ A(s) - K \right] x(s) \, ds, & t \in (t_k, t_{k+1}], k = 1,2,\ldots,m 
\end{cases}
\]

(11) From (11) the equation (8) become

\[
x(t) = H^{-1}y(t) = \begin{cases} 
[x_0 - I^\beta g(x)] + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} \left[ A(s) - K \right] x(s) \, ds, & t \in [0,t_1] \\
[x_0 - I^\beta g(x)] + \sum_{i=1}^{k} I_i \left( x(t_i) \right) + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} \left[ A(s) - K \right] x(s) \, ds, \quad t \in (t_k, t_{k+1}], k = 1,2,\ldots,m 
\end{cases}
\]

(12) In the following formulation, we generalize of concluding remark (3.3).

4. The problem formulation:

Consider the the fractional differential impulsive multi control abstract problem with fractional integral nonlocal initial conditions (1-5) and let the output \( y_1(t) = C(t)x(t)C^*(t) \). now substitutes (7) in \( y_1(t) \)

\[
y_1(t) = C(t)[x_0 - I^\beta g(x)]C^*(t) + \frac{c(t)}{\Gamma(a)} \int_0^t (t-s)^{a-1} \left[ A(s) - K \right] x(s) \, ds \]

(13) For \( u_1,u_2 \in L^2(J,U) \), to calculate the finite time observer, need to construct the initial state implicitly function \( x(.) \) for arbitrary control function \( u_1,u_2 \in L^2(J,U) \), the nonlocal initial state \( x_0 - I^\beta g(x) \) of the problem (13) can be obtain by:

\[
C(t)[x_0 - I^\beta g(x)]C^*(t) + \frac{c(t)}{\Gamma(a)} \int_0^t (t-s)^{a-1} \left[ A(s) - K \right] x(s) \, ds = y_1(t) - \frac{c(t)}{\Gamma(a)} 
\]

(14) Now from \( H \) is invertible operator, then,
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\[
\begin{align*}
& H^{-1} \left( C(t)[x_0 - I^\alpha g(x)]C'(t) + \frac{c(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ A(s) - K]x(s)ds \right] C'(t) \right) = H^{-1} \left( y_1(t) - \frac{c(t)}{\Gamma(\alpha)} \right) \\
& \left[ \int_0^t (t-s)^{\alpha-1} \left[ f_1(s,x(s), \int_0^t h(t,\tau, x(\tau)) d\tau, f_2(s,x(s),D^\alpha x(s)) + B u(s) \right] ds \right] C'(t), t \in [0,t_1] \\
& H^{-1} \left( C(t)[x_0 - I^\alpha g(x)]C'(t) + C(t) \sum_{i=1}^k l_i(x(t_i))C'(t) + \frac{c(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ A(s) - K]x(s)ds \right] C'(t) \right) \\
& = H^{-1} \left( y_1(t) - \frac{c(t)}{\Gamma(\alpha)} \right) \left[ \int_0^t (t-s)^{\alpha-1} \left[ f_1(s,x(s), \int_0^t h(t,\tau, x(\tau)) d\tau, f_2(s,x(s),D^\alpha x(s)) + B u(s) \right] ds \right] C'(t), t \in [0,t_1] \\
& \left( 15 \right) \text{From equations (10) and (15) and Substituting in (7), we get:}
\end{align*}
\]

\[x_0 - g(x) =
\]

\[H^{-1} \left( y_1(t) - \frac{c(t)}{\Gamma(\alpha)} \right) \left[ \int_0^t (t-s)^{\alpha-1} \left[ f_1(s,x(s), \int_0^t h(t,\tau, x(\tau)) d\tau, f_2(s,x(s),D^\alpha x(s)) + B u(s) \right] ds \right] C'(t), t \in [0,t_1] \\
\left( 16 \right) \text{Remark ( 4.1):}
\]

The equation in (16) is a finite time observer which provide the mild solution \( x(.) \in PC([0,T]; X) \) to have a fixed point for all control functions \( u_1, u_2 \in L^2([0,T]; U) \).

3. Main results:

Consider the abstract control problem (1-5) and consider their mild solution (7) with hypothesis(h1-h5) and we needs the following adopted in the main result:

(g1) Let \( X \rightarrow Y \), \( C^*: Y \rightarrow X \) is bounded Linear operators , there exist \( L_1, L_2 > 0 \) such that

\[\|C(t)x C^*(t)\|_Y \leq L_1 L_2 \|x\|_Y, x \in X.\]

(g2) If \( T \in R^+ \) where \( R^+ \) is the set of positive numbers.

(g3) \( \delta < 1 \)

\[\hat{R} + \frac{\hat{R} \hat{L}_K}{\Gamma(\alpha)} \left( N_2 \hat{N}_1 \left( r + \frac{\Gamma(\alpha-\alpha)}{\Gamma(2-\alpha)} \sup_{0 \leq s \leq T} \| y(s) \| \right) + N_2 \right) + N_1 \hat{N}_2 \left( r + t_1(H_1 r + H_4) \right) + K_1 K_2 K_{E_{\alpha-1}} e^{\Gamma(\alpha) (t_1 r + t_1^\alpha)} \left( M + \hat{M} \hat{M} \right) < r, t \in [0, t_1].\]
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Lemma (5.1):
Consider the abstract control problem (1-5) and let \( \tilde{M} = PC([0,T];B_\alpha) \) be a nonempty subset \( Z = PC([0,T];X) \) where \( X \) a complex Banach space with norm \( \| x \|_{pc} = \sup_{t \in [0,T]} \| x(t) \|, t \in [0,T] \). Then \( M \) is a closed set.

Proof:
Let \( w^n \in \tilde{M} \) be a sequence, \( w^n \to w \) as \( n \to \infty \), where \( w^n \) is a continuous sequence functions at \( t = t_k, \ k = 1,2,\ldots,m \) and only left continuous at \( t = t_k \) and only right limit \( x(t_k^+) \) exists with \( u_1, u_2 \in L^2(J,U) \) these sequence is pointwise convergent to \( w \), now we need to prove \( w \in \tilde{M} \), so our aim that \( w \in Z \) and \( \| w(t) \| \leq r \), with \( u_1, u_2 \in L^2(J,U) \). Now to prove that \( w \in Z \), since \( w^n \in \tilde{M} \) pointwise converges, thus sequence \( w^n \) is uniformly convergence to \( w \), hence \( w \in Z \). Now to show that \( \| w(t) \| \leq r \), from above sequence \( w^n \) is uniformly convergent to \( w \) and \( \| w^n - w \|_{pc} = \sup_{t \in [0,T]} \| w^n(t) - w(t) \| \) in a complex Banach space \( Z \) then \( \sup_{t \in [0,T]} \| w^n(t) - w(t) \| \to 0 \), as \( n \to \infty \) for all \( 0 \leq t \leq T \), we have that

\[
\| w(t) \|_{pc} = \left\| \lim_{n \to \infty} w^n(t) \right\|_{pc} = \lim_{n \to \infty} \| w^n(t) \|_{pc} \leq \lim_{n \to \infty} r.
\]

Therefore \( \tilde{M} \) is a closed subset of \( Z \).

Lemma (5.2):
Assume that the hypotheses (h1-h3) and (h5) holds. From (13) we defined \( y_1(t) \) and \( y_2(t) \) as a nonlinear observations such that

\[
y_1(t) = \begin{cases}
C(t)[x_3 - I^\alpha g(x_3)]C^*(t) + \frac{C(t)}{\Gamma(\alpha)} \left[ \int_0^t (t-s)^{\alpha-1} [A(s) - K]x_1(s) + f_3(s, x(s), \int_0^s h(t, x(t))dt) f_2(s, x(s), D^\alpha x(s)) + B_2 u_2(s) \right] ds \] C^*(t), \quad t \in [0,t_k) \\
C(t)[x_3 - g(x_3)]C^*(t) + C(t) \sum_{i=1}^k I_i (x_i(t_i)) C^*(t) + \frac{C(t)}{\Gamma(\alpha)} \left[ \int_0^t (t-s)^{\alpha-1} [A(s) - K]x_1(s) + f_3(s, x(s), \int_0^s h(t, x(t))dt) f_2(s, x(s), D^\alpha x(s)) + B_2 u_2(s) \right] ds \] C^*(t), \quad t \in (t_k, t_{k+1}).
\end{cases}
\]
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\[
y_2(t) = \begin{cases}
C(t)[x_0 - I^\alpha g(x_2)]C^*(t) + \frac{c(t)}{\Gamma(\alpha)} \left[ \int_0^t (t-s)^{\alpha-1} [A(s) - K]x_2(s) \right]
+ f_1(s, x(s), \int_0^t h(t, \tau, x(\tau))d\tau) f_2(s, x(s), D^\alpha x(s)) + B_2 u_2(s) ds \right] C^*(t), t \in [0, t_1] \\
C(t)[x_0 - g(x_2)]C^*(t) + C(t) \sum_{1}^{k} I_i (x_2(t_i)) C^*(t) + \frac{c(t)}{\Gamma(\alpha)} \left[ \int_0^t (t-s)^{\alpha-1} [A(s) - K]x_2 \right]
+ f_1(s, x(s), \int_0^t h(t, \tau, x(\tau))d\tau) f_2(s, x(s), D^\alpha x(s)) + B_2 u_2(s) ds \right] C^*(t), t \in (t_k, t_{k+1}]
\end{cases}
\]

If \( \bar{R}_3 = L_1 L_2 \left( r + (r + \frac{\tau^\beta}{\Gamma(\beta+1)} r + \| g(0) \| ) + Lm \ell_2 + \frac{L_{L_2}}{\Gamma(\alpha)} \frac{\tau^\alpha}{\Gamma(\alpha+1)} (M + \tilde{M} \tilde{M}) r \right) + N_2 \tilde{N}_1 (r + \frac{\tau^\alpha}{r(2-\alpha)} \sup_{0 \leq s \leq T} \| y(s) \|) + N_1 \tilde{N}_2 (r + T (H_1 r + H_1)) + K_1 K_{\tilde{R}_1} \tilde{R}_1 \tau^\alpha \end{equation}

Then

1. \( \| y(t) \| \leq \left\{ \begin{array}{ll}
\bar{R}_3, & t \in [0, t_1] \\
\bar{R}_4, & t \in (t_k, t_{k+1}], k = 1, 2, \ldots, m \end{array} \right. \)

2. \( \| y_1(t) - y_2(t) \| \leq \left\{ \begin{array}{ll}
\tilde{\ell}_3, & t \in [0, t_1] \\
\tilde{\ell}_4, & t \in (t_k, t_{k+1}], k = 1, 2, \ldots, m \end{array} \right. \)

Proof:

\[
\| y(t) \|_Y = \left\| C(t)[x_0 - I^\alpha g(x)]C^*(t) + \frac{c(t)}{\Gamma(\alpha)} \left[ \int_0^t (t-s)^{\alpha-1} [A(s) - K]x(s) \right]
+ f_1(s, x(s), \int_0^t h(t, \tau, x(\tau))d\tau) f_2(s, x(s), D^\alpha x(s)) + B_2 u_2(s) ds \right] C^*(t), t \in [0, t_1] \\
\| y(t) \|_Y \leq \left\| C(t)[x_0 - I^\alpha g(x)]C^*(t) \|_Y + \frac{1}{\Gamma(\alpha)} \left\| C(t) \left[ \int_0^t (t-s)^{\alpha-1} [A(s) - K]x(s) \right] \|_Y \right. \right.
\]

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\[
\begin{aligned}
&+f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) \\
&+B_1 \left[ v(t) + a(t) \int_0^{\tau} (t-\tau)^{\alpha-1} u_2(\tau)d\tau \right] ds \right] C^*(t) \bigg\|_y, \\
\|y(t)\|_y \leq L_1 L_2 \|x_0\| + \|f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) + B_2 \ v(t) \ E_q(\Gamma(\alpha) a(t)t^\alpha) \bigg\|_y, \\
\|y(t)\|_y \leq L_1 L_2 \|x_0\| + \|f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) + B_2 \ v(t) \ E_q(\Gamma(\alpha) a(t)t^\alpha) \bigg\|_y, \\
\|y(t)\|_y \leq L_1 L_2 \|x_0\| + \|f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) + B_2 \ v(t) \ E_q(\Gamma(\alpha) a(t)t^\alpha) \bigg\|_y, \\
\|y(t)\|_y \leq L_1 L_2 \|x_0\| + \|f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) + B_2 \ v(t) \ E_q(\Gamma(\alpha) a(t)t^\alpha) \bigg\|_y, \\
\|y(t)\|_y \leq L_1 L_2 \|x_0\| + \|f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) + B_2 \ v(t) \ E_q(\Gamma(\alpha) a(t)t^\alpha) \bigg\|_y, \\
&+ \bigg\| f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) + f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) - f_1\left(s,0,0\right) f_2(s,0,0) \bigg\| \\
&+ \|B_2\| \|v(t)\| \|E_q(\Gamma(\alpha) a(t)t^\alpha)\| \bigg\| \\
\leq L_1 L_2 \|x_0\| + \|f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) + f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) - f_1\left(s,0,0\right) f_2(s,0,0) \bigg\| \\
&+ \bigg\| f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) + f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) - f_1\left(s,0,0\right) f_2(s,0,0) \bigg\| \\
&+ \|B_2\| \|v(t)\| \|E_q(\Gamma(\alpha) a(t)t^\alpha)\| \bigg\| \\
&+ \|f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) + f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) - f_1\left(s,0,0\right) f_2(s,0,0) \bigg\| \\
&+ \|K_1 \ | \ K_2 E_{\alpha,\delta} e^{\Gamma(\alpha) a(t)t^\alpha} \bigg\| \\
\|y(t)\|_y \leq L_1 L_2 \|x_0\| + \|f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) + f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) - f_1\left(s,0,0\right) f_2(s,0,0) \bigg\| \\
&+ \|K_1 \ | \ K_2 E_{\alpha,\delta} e^{\Gamma(\alpha) a(t)t^\alpha} \bigg\| \\
&+ \|f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) + f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) - f_1\left(s,0,0\right) f_2(s,0,0) \bigg\| \\
&+ \|K_1 \ | \ K_2 E_{\alpha,\delta} e^{\Gamma(\alpha) a(t)t^\alpha} \bigg\| \\
\|y(t)\|_y \leq L_1 L_2 \|x_0\| + \|f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) + f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) - f_1\left(s,0,0\right) f_2(s,0,0) \bigg\| \\
&+ \|K_1 \ | \ K_2 E_{\alpha,\delta} e^{\Gamma(\alpha) a(t)t^\alpha} \bigg\| \\
&+ \|f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) + f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) - f_1\left(s,0,0\right) f_2(s,0,0) \bigg\| \\
&+ \|K_1 \ | \ K_2 E_{\alpha,\delta} e^{\Gamma(\alpha) a(t)t^\alpha} \bigg\| \\
\|y(t)\|_y \leq L_1 L_2 \|x_0\| + \|f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) + f_1\left(s,x(s), \int_0^{\tau} h(t,\tau, x(\tau))d\tau\right) f_2(s,x(s), D^\alpha x(s)) - f_1\left(s,0,0\right) f_2(s,0,0) \bigg\|
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\[
\begin{align*}
\leq L_1L_2(r + (r + \frac{\tau^\beta_G}{\Gamma(\beta + 1)})r + \|g(0)\|) + \frac{L_1L_2}{\Gamma(\alpha)} \tau^\alpha \left( M + \tilde{M}M \right) r \\
+ N_2\tilde{N}_1 \left( r + \frac{T^{\alpha-\alpha}}{\Gamma(2-\alpha)} \sup_{s \leq T} \|y(s)\| \right) + N_2 + N_1\tilde{N}_2 \left( r + T(H_1r + H_1) \right) + K_1K_2K_{\alpha_3} \xi_{\alpha_3} e^{\Gamma(\alpha)(T + T^\alpha)}
\end{align*}
\]

Now, \( \|y(t)\|_{L^2} \leq \tilde{K}_3 \), where

\[
\tilde{K}_3 = L_1L_2(r + (r + \frac{\tau^\beta_G}{\Gamma(\beta + 1)})r + \|g(0)\|) + \frac{L_1L_2}{\Gamma(\alpha)} \tau^\alpha \left( M + \tilde{M}M \right) r + N_2\tilde{N}_1 \left( r + \frac{T^{\alpha-\alpha}}{\Gamma(2-\alpha)} \sup_{s \leq T} \|y(s)\| \right) + N_2 + N_1\tilde{N}_2 \left( r + T(H_1r + H_1) \right) + K_1K_2K_{\alpha_3} \xi_{\alpha_3} e^{\Gamma(\alpha)(T + T^\alpha)}
\]

For \( \in (t_k, t_{k+1}) \), \( k = 1, 2, \ldots, m \), \( \|y(t)\|_{L^2} \)

\[
\|y(t)\|_{L^2} = \left\| C(t) \left[ x_0 - I^\beta g(x) \right] C^*(t) + C(t) \sum_{l=1}^k l_1(t(t_l))C^*(t) + \frac{C(t)}{\Gamma(\alpha)} \int_0^t (t-s)_{\alpha-1} \left[ A(s) - B_1K \right] x(s) + f_1 \left( s, x(s), \int_0^t h(t, \tau, x(\tau)) d\tau \right) f_2(s, x(s), D^\alpha x(s)) + B_2 \left[ v(t) + a(t) \int_0^\tau (t-\tau)_{\alpha-1} u_2(\tau) d\tau \right] \right\| C^*(t)
\]

\[
\leq \left\| C(t) \left[ x_0 - I^\beta g(x) \right] C^*(t) \right\| + \left\| C(t) \sum_{l=1}^k l_1(t(t_l))C^*(t) \right\|
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)_{\alpha-1} \left[ A(s) - B_1K \right] x(s) + f_1 \left( s, x(s), \int_0^t h(t, \tau, x(\tau)) d\tau \right) f_2(s, x(s), D^\alpha x(s)) + B_2 \left[ v(t) + a(t) \int_0^\tau (t-\tau)_{\alpha-1} u_2(\tau) d\tau \right] \right\| ,
\]

condition (g1), yield

\[
\|y(t)\|_{L^2} \leq L_1L_2 \left\| x_0 - I^\beta g(x) \right\| + L_1L_2 \left\| \sum_{l=1}^k l_1(t(t_l)) \right\|
\]

\[
+ \frac{L_1L_2}{\Gamma(\alpha)} \left\| \int_0^t (t-s)_{\alpha-1} \left[ A(s) - B_1K \right] x(s) + f_1 \left( s, x(s), \int_0^t h(t, \tau, x(\tau)) d\tau \right) f_2(s, x(s), D^\alpha x(s)) + B_2 \left[ v(t) + a(t) \int_0^\tau (t-\tau)_{\alpha-1} u_2(\tau) d\tau \right] \right\|
\]

conditions (h1-h5) and first part of proving, we get

\[
\|y(t)\|_{L^2} \leq L_1L_2 \left( r + (r + \frac{\tau^\beta_G}{\Gamma(\beta + 1)})r + \|g(0)\| \right) + L_1L_2 m^2
\]

\[
+ \frac{L_1L_2}{\Gamma(\alpha)} \left( \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1)} \right) \left( M + \tilde{M}M \right) r \\
+ N_2\tilde{N}_1 \left( r + \frac{T^{\alpha-\alpha}}{\Gamma(2-\alpha)} \sup_{s \leq T} \|y(s)\| \right) + N_2 + N_1\tilde{N}_2 \left( r + T(H_1r + H_1) \right) + K_1K_2K_{\alpha_3} \xi_{\alpha_3} e^{\Gamma(\alpha)(T + T^\alpha)},
\]

hence,

\[
\|y(t)\|_{L^2} \leq \tilde{K}_4.
\]
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where

\[ \dot{X}_1 = L_1L_2(r + (r + (r + (r + g(0)))) + L_1L_2 + \frac{L_1L_2}{\Gamma(\alpha+1)}(M + \tilde{M}) r + N_2 \tilde{N}_1(r + \sup_{0\leq \tau \leq T} \|y(\tau)\|) + N_1 \tilde{N}_2(r + T(H_1 r + H_1)) + K_1 K_2 K_{\alpha-1} e^{T(\alpha-1)} T^\alpha) \]

For \( t \in [0, t_1] \). To satisfy Lipshtiz property,

\[ \|y_1(t) - y_2(t)\| \leq \|C(t)[x_0 - l^{\beta} g(x_1)]C'(t) + \frac{C(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ [A(s) - B_1 K]x_1(s) + B_2 \left( v(t) + a(t) \int_0^t (t-s)^{\alpha-1} u_2(s) ds \right) \right] ds \]

\[ \leq \|C(t)\| \|C'(t)\| \|l^{\beta} g(x_1) - l^{\beta} g(x_2)\| + \frac{\|C(t)\| \|C'(t)\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ [A(s) - B_1 K]x_1(s) - x_2(s) \right] ds \]

\[ \leq L_1L_2 \frac{\tau^{\beta} G}{\Gamma(\beta+1)} \|x_1(s) - x_2(s)\| + \frac{L_1L_2}{\Gamma(\alpha)} \frac{\tau^\alpha}{\Gamma(\alpha+1)} \left( M + \tilde{M} \right) [x_1(s) - x_2(s)] \]

\[ \leq L_1L_2 \frac{\tau^{\beta} G}{\Gamma(\beta+1)} \|x_1(s) - x_2(s)\| + \frac{L_1L_2}{\Gamma(\alpha)} \frac{\tau^\alpha}{\Gamma(\alpha+1)} \left( M + \tilde{M} \right) [x_1(s) - x_2(s)] \]
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\[ \| y_1(t) - y_2(t) \|_Y \leq \left[ L_1 L_2 \frac{r^\alpha}{\Gamma(\alpha+1)} \left[ (M + \tilde{M}) + N_2 T_1 (1 + \frac{r^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{t \in [0,T]} \| x_2(s) \|) \right] \right] \| x_1(s) - x_2(s) \| \]

Hence, \( \| y_1(t) - y_2(t) \|_Y \leq \beta_3 \| x_1(t) - x_2(t) \| \). For \( t \in [0,t_1] \)

For \( \in (t_k, t_{k+1}], k = 1,2,\ldots,m \). To satisfy Lipshtiz property, \( \| y_1(t) - y_2(t) \|_Y \leq \| C(t) \|_Y x_0 - I^\beta g(x_1) \| C^*(t) + C(t) \sum_{i=1}^{k} I_i (x_1(t_i)) C^*(t) + \frac{C(t)}{r(\alpha)} \| \int_0^T (t-s)^{\alpha-1} \]

\[ \left[ A(s) - B_1(s) \right] x_1(s) + f_1 \left( s, x_1(s), \int_0^T h(t, \tau, x_1(\tau)) d\tau \right) f_2 (s, x_1(s), D^\alpha x_1(s)) \]

\[ + B_2 \left[ v(t) + a(t) \int_0^T (t-\tau)^{\alpha-1} u_2(\tau) d\tau \right] \right] ds \| C^*(t) - C(t) \|_Y x_0 - I^\beta g(x_2) \| C^*(t) \]

\[ - C(t) \sum_{i=1}^{k} I_i (x_2(t_i)) C^*(t) \frac{C(t)}{r(\alpha)} \| \int_0^T (t-s)^{\alpha-1} \]

\[ \left[ A(s) - B_1(s) \right] x_2(s) + f_1 \left( s, x_2(s), \int_0^T h(t, \tau, x_2(\tau)) d\tau \right) f_2 (s, x_2(s), D^\alpha x_2(s)) \]

\[ - B_2 \left[ v(t) + a(t) \int_0^T (t-\tau)^{\alpha-1} u_2(\tau) d\tau \right] \right] ds \| C^*(t) \| \leq L_1 L_2 \| I^\beta g(x_2) - I^\beta g(x_1) \| + L_1 L_2 m_\beta \| x_1(s) - x_2(s) \| + \frac{r^\alpha}{\Gamma(\alpha+1)} L_1 L_2 \| x_1(s) - x_2(s) \| + \frac{r^\alpha}{\Gamma(\alpha+1)} \left[ f_1 \left( s, x_1(s), \int_0^T h(t, \tau, x_1(\tau)) d\tau \right) f_2 (s, x_1(s), D^\alpha x_1(s)) - f_1 \left( s, x_2(s), \int_0^T h(t, \tau, x_2(\tau)) d\tau \right) f_2 (s, x_2(s), D^\alpha x_2(s)) \right] ds \]

Conditions (g1), (h1), (h2) and (h5), obtain,

\[ \| y_1(t) - y_2(t) \|_Y \leq \beta_4 \| x_1(t) - x_2(t) \| \] for \( t \in (t_k, t_{k+1}], k = 1,2,\ldots,m \).

**Theorem (5.3):**

Assume that the hypotheses (h1-h5) and conditions g3(i),(ii) are satisfied. Then the impulsive multi control fractional differential abstract problem with fractional integral nonlocal initial condition(1-5) has a unique fixed point \( x(\cdot) \in PC([0,T]:X) \) for all control function \( u_1(\cdot), u_2(\cdot) \in L^2([0,T]:U) \).

**Proof:**

Define the nonlinear map: \( \Phi: \tilde{M} = PC([0,T]:B_r) \rightarrow \mathbb{R} = PC([0,T]:X) \) as follows:

\( \Phi(x(t)) = \frac{C(t)}{r(\alpha)} \int_0^T (t-s)^{\alpha-1} \left[ f_1 \left( s, x_1(s), \int_0^T h(t, \tau, x_1(\tau)) d\tau \right) f_2 (s, x_1(s), D^\alpha x_1(s)) + B_2 \left[ v(t) + a(t) \int_0^T (t-\tau)^{\alpha-1} u_2(\tau) d\tau \right] \right] ds \| C^*(t) \sum_{i=1}^{k} I_i (y(t_i)) \right] (t) \in (t_k, t_{k+1}] \)
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\[(\varphi x)(t) = \begin{cases} 
H^{-1}[y_1(t) - \left[ \left( \frac{c(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(s, x_1(s), \int_0^t h(t, \tau, x_1(\tau))d\tau \right) f_2(s, x_1(s), D^\alpha x_1(s), 
\quad + B_2 \left[ v(t) + a(t) \int_0^{\tau} (t-\tau)^{\alpha-1} u_2(\tau)d\tau \right] ds \right] C^\alpha(t) + \frac{1}{(\Gamma(\alpha))^\gamma} \int_0^t (t-s)^{-\alpha} 
\left[ [A(s) - B_1 K] x(s) + f_1(s, x_1(s), \int_0^{\tau} h(t, \tau, x_1(\tau))d\tau \right) f_2(s, x_1(s), D^\alpha x_1(s), 
\quad + B_2 \left[ v(t) + a(t) \int_0^{\tau} (t-\tau)^{\alpha-1} u_2(\tau)d\tau \right] ds, & t \in [0, t_1] 
\right. 
\left. H^{-1}[y_1(t) - \left[ \left( \frac{c(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(s, x_1(s), \int_0^t h(t, \tau, x_1(\tau))d\tau \right) f_2(s, x_1(s), D^\alpha x_1(s), 
\quad + B_2 \left[ v(t) + a(t) \int_0^{\tau} (t-\tau)^{\alpha-1} u_2(\tau)d\tau \right] ds \right] C^\alpha(t) + \sum_{i=1}^{\gamma} I_i \left( y(t_i) \right) + \frac{1}{(\Gamma(\alpha))^\gamma} \int_0^t (t-s)^{-\alpha} 
\left[ [A(s) - B_1 K] x(s) + f_1(s, x_1(s), \int_0^{\tau} h(t, \tau, x_1(\tau))d\tau \right) f_2(s, x_1(s), D^\alpha x_1(s), 
\quad + B_2 \left[ v(t) + a(t) \int_0^{\tau} (t-\tau)^{\alpha-1} u_2(\tau)d\tau \right] ds, & t \in (t_k, t_{k+1}] \end{cases} \right.
\]

(17)
for all control function \(u_1(., .), u_2(., .) \in L^2([0, T], U).\)

Our interest to prove \(\varphi x\) has a fixed point. So we need to do the following steps:

Step1: \(M = PC([0, T], E)\) is a closed subset of \(Z = PC([0, T], X)\).
Step2: \(\varphi M \subseteq M\) with \(u_1(., .), u_2(., .) \in L^2([0, T], U).\)
Step3: \(\varphi\) is a contraction on \(M\) for \(u_1(., .), u_2(., .) \in L^2([0, T], U).\) From Lemma (5.2).

step (1) have been satisfied. For proving step (2), we need lemma (5.2), now let \(x \in M\)
1. \(\varphi x \in Z\) for \(u_1(., .), u_2(., .) \in L^2([0, T], U).\)
2. \(\|\varphi x(t)\| \leq r\), for \(u_1(., .), u_2(., .) \in L^2([0, T], U).\) From (17), it is clear (1) satisfied.
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To prove (2), thus

\[ \left\| H^{-1}[y_1(t)] - \left[ \frac{c(t)}{r_{(a)}} \int_0^t (t-s)^{a-1} \left[ f_1(s, x_1 x_1(s), \int_0^t h(t, \tau, x_1(\tau))d\tau \right] f_2(s, x_1(s)) D^a x_1(s) + B_2 \left[ v(t) + a(t) \int_0^t (t-\tau)^{a-1} u_2(\tau)d\tau \right] ds \right] \right\| C^a(t) + \frac{1}{r_{(a)}} \int_0^t (t-s)^{a-1} \left[ f_1(s, x_1 x_1(s), \int_0^t h(t, \tau, x_1(\tau))d\tau \right] f_2(s, x_1(s)) D^a x_1(s) + B_2 \left[ v(t) + a(t) \int_0^t (t-\tau)^{a-1} u_2(\tau)d\tau \right] ds, \quad t \in [0, t_1] \]

\[ \left\| H^{-1}[y_1(t)] - \left[ \frac{c(t)}{r_{(a)}} \int_0^t (t-s)^{a-1} \left[ f_1(s, x_1 x_1(s), \int_0^t h(t, \tau, x_1(\tau))d\tau \right] f_2(s, x_1(s)) D^a x_1(s) + B_2 \left[ v(t) + a(t) \int_0^t (t-\tau)^{a-1} u_2(\tau)d\tau \right] ds \right] \right\| C^a(t) + \frac{1}{r_{(a)}} \int_0^t (t-s)^{a-1} \left[ f_1(s, x_1 x_1(s), \int_0^t h(t, \tau, x_1(\tau))d\tau \right] f_2(s, x_1(s)) D^a x_1(s) + B_2 \left[ v(t) + a(t) \int_0^t (t-\tau)^{a-1} u_2(\tau)d\tau \right] ds, \quad t \in (t_k, t_{k+1}] \]

From (18) and boundedness of $H^{-1}$ which given from remark (3.1)(4), yield

\[ \left\| y_1(t) \right\| \leq \left\{ \frac{c(t) + c'(t)}{r_{(a)}} \right\} \left\| f_1(s, x_1 x_1(s), \int_0^t h(t, \tau, x_1(\tau))d\tau \right\| f_2(s, x_1(s)) \]

\[ + B_2 \left[ v(t) + a(t) \int_0^t (t-\tau)^{a-1} u_2(\tau)d\tau \right] ds \]

\[ \left\| A(s) - B_1 K \right\| ||x(s)|| + \left\| f_1(s, x_1 x_1(s), \int_0^t h(t, \tau, x_1(\tau))d\tau \right\| f_2(s, x_1(s), D^a x_1(s) \]

\[ + B_2 \left[ v(t) + a(t) \int_0^t (t-\tau)^{a-1} u_2(\tau)d\tau \right] ds, \quad t \in [0, t_1] \]

\[ \left\| y_1(\tau) \right\| \leq \left\{ \frac{c(t) + c'(t)}{r_{(a)}} \right\} \left\| f_1(s, x_1 x_1(s), \int_0^t h(t, \tau, x_1(\tau))d\tau \right\| f_2(s, x_1(s)) \]

\[ + B_2 \left[ v(t) + a(t) \int_0^t (t-\tau)^{a-1} u_2(\tau)d\tau \right] ds \]

\[ \left\| A(s) - B_1 K \right\| ||x(s)|| + \left\| f_1(s, x_1 x_1(s), \int_0^t h(t, \tau, x_1(\tau))d\tau \right\| f_2(s, x_1(s), D^a x_1(s) \]

\[ + B_2 \left[ v(t) + a(t) \int_0^t (t-\tau)^{a-1} u_2(\tau)d\tau \right] ds, \quad t \in (t_k, t_{k+1}] \]
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\[
\begin{align*}
&\leq \bar{K}R_3 + \left[ \frac{2R_3}{\Gamma(\alpha)} \frac{t_{k+1}^{\alpha-\alpha}}{\Gamma(\alpha+1)} \left( N_2 \bar{N} \left( r + \frac{t_{k+1}^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0 \leq s \leq \tau} \| y(s) \| \right) + N_2 \right) + N_1 \bar{N} \left( r + t_1 (H_1 r + \\
&+ K_1 K_2 K_{\xi} \left( \frac{t_k^{\alpha-\alpha}}{\Gamma(\alpha+1)} \right) M + \bar{M} \right) \right],

&\quad t \in [0, t_1].

&\leq \bar{K}R_4 + \left[ \frac{2R_4}{\Gamma(\alpha)} \frac{t_{k+1}^{\alpha-\alpha}}{\Gamma(\alpha+1)} \left( N_2 \bar{N} \left( r + \frac{t_{k+1}^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0 \leq s \leq \tau} \| y(s) \| \right) + N_2 \right) + N_1 \bar{N} \left( r + t_{k+1} (H_1 r \\
&+ K_1 K_2 K_{\xi} \left( \frac{t_{k+1}^{\alpha-\alpha}}{\Gamma(\alpha+1)} \right) M + \bar{M} \right) \right],

&\quad t \in [t_k, t_{k+1}].
\end{align*}
\]

condition (g3)(i), given that,
\[\|(\phi x)(t)\| \leq r, \text{ for } t \in [0, t_1] \text{ and } t \in (t_k, t_k), k = 1, ..., m\]

To satisfy step (3).
\[\|(\phi x_1)(t) - \phi x_2)(t)\| \leq \]

\[
\begin{align*}
&H^{-1}[y_1(t) - \\
&\quad \left[ \frac{c(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ f_1 \left( s, x_1(s), \int_0^t h(t, \tau, x_1(\tau))d\tau \right) f_2 \left( s, x_1(s), D^{\alpha} x_1(s) \right) + \\
&+ B_2 \left[ v(t) + a(t) \int_0^t (t-\tau)^{\alpha-1} u_2(\tau)d\tau \right] \right] dsC^*(t) \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ [A(s) - B_1 K] x_1(s) + \\
&f_1 \left( s, x_1(s), \int_0^t h(t, \tau, x_1(\tau))d\tau \right) f_2 \left( s, x_1(s), D^{\alpha} x_1(s) \right) + B_2 \left[ v(t) + a(t) \int_0^t (t-\tau)^{\alpha-1} u_2(\tau)d\tau \right] \right] ds \\
&- H^{-1}[y_2(t) - \\
&\quad \left[ \frac{c(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ f_1 \left( s, x_2(s), \int_0^t h(t, \tau, x_2(\tau))d\tau \right) f_2 \left( s, x_2(s), D^{\alpha} x_2(s) \right) + \\
&+ B_2 \left[ v(t) + a(t) \int_0^t (t-\tau)^{\alpha-1} u_2(\tau)d\tau \right] \right] dsC^*(t) \\
&- \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ [A(s) - B_1 K] x_2(s) + \\
&f_1 \left( s, x_2(s), \int_0^t h(t, \tau, x_2(\tau))d\tau \right) f_2 \left( s, x_2(s), D^{\alpha} x_2(s) \right) + B_2 \left[ v(t) + a(t) \int_0^t (t-\tau)^{\alpha-1} u_2(\tau)d\tau \right] \right] ds \right], \quad t \in [0, t_1]
\end{align*}
\]

\[\|(\phi x_1)(t) - \phi x_2)(t)\| \leq \]
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\[
\begin{align*}
\|H^{-1}[y_1(t) - \left[ &\frac{c(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ f_1 \left( s, x_1(s), \int_0^t h(t, \tau, x_1(\tau)) \, d\tau \right) f_2 \left( s, x_1(s), D^\alpha x_1(s) \right) + \\
& B_2 \left[ v(t) + a(t) \int_0^t (t-\tau)^{\alpha-1} u_2(\tau) \, d\tau \right] \right] \right] ds \| - \\
& \sum_{k=1}^m I_1 \left( x_2(t_k) \right) + \\
& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ [A(s) - B_1 K] x_2(s) + \\
f_1 \left( s, x_2(s), \int_0^t h(t, \tau, x_1(\tau)) \, d\tau \right) f_2 \left( s, x_2(s), D^\alpha x_1(s) \right) \right] ds \\
& + B_2 \left[ v(t) + a(t) \int_0^t (t-\tau)^{\alpha-1} u_2(\tau) \, d\tau \right] \right] ds - H^{-1}[y_2(t) - \\
& \left[ \frac{c(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ f_1 \left( s, x_2(s), \int_0^t h(t, \tau, x_1(\tau)) \, d\tau \right) f_2 \left( s, x_2(s), D^\alpha x_1(s) \right) \right] ds \right] ds - H^{-1}[y_2(t) - \\
& \left[ \frac{c(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ f_1 \left( s, x_2(s), \int_0^t h(t, \tau, x_1(\tau)) \, d\tau \right) f_2 \left( s, x_2(s), D^\alpha x_1(s) \right) \right] ds \right] ds \\
& + B_2 \left[ v(t) + a(t) \int_0^t (t-\tau)^{\alpha-1} u_2(\tau) \, d\tau \right] \right] ds \right] ds, \quad t \in (t_k, t_{k+1}], k = 1, 2, \ldots m .
\end{align*}
\]

Then

\[
\| (\varphi x_1(t) - \varphi x_2(t)) \| \leq \| H^{-1} \| \| y_1(t) - y_2(t) \| + \frac{\| H^{-1} \| \| c(t) \| \| c'(t) \|}{\Gamma(\alpha)} \\
\left[ f_1 \left( s, x_1(s), \int_0^t h(t, \tau, x_1(\tau)) \, d\tau \right) f_2 \left( s, x_1(s), D^\alpha x_1(s) \right) + \\
\left[ f_1 \left( s, x_2(s), \int_0^t h(t, \tau, x_1(\tau)) \, d\tau \right) f_2 \left( s, x_2(s), D^\alpha x_1(s) \right) \right] ds, \left[ f_0^t (t-s)^{\alpha-1} \right] \right] ds, \quad t \in [0, t_1] \\
\| (\varphi x_1(t) - \varphi x_2(t)) \| \leq \| H^{-1} \| \| y_1(t) - y_2(t) \| 
\]
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From Lemma (5.2), we obtain:

\[
\|\varphi x_1(t) - \varphi x_2(t)\| \leq \bar{K} \ell_3 \|x_1(s) - x_2(s)\| + \frac{R \ell_1 \ell_2}{\Gamma(\alpha)} \left[ \left( M + \bar{M} \bar{M} \right) + N_2 \bar{N}_1 \left( 1 + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0 \leq s \leq t} \|x_2(s)\| \right) \right] \\
+ N_1 \left[ TH_1 \|x_1(s) - x_2(s)\| K_{f_2}(t) \Omega_{f_2}(r + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0 \leq s \leq t} \|x_2(s)\|) \right] \\
+ \frac{R \ell_1 \ell_2}{\Gamma(\alpha)} \left[ \left( M + \bar{M} \bar{M} \right) + N_2 \bar{N}_1 \left( 1 + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0 \leq s \leq t} \|x_2(s)\| \right) \right] \\
+ N_1 \left[ TH_1 \|x_1(s) - x_2(s)\| K_{f_2}(t) \Omega_{f_2}(r + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0 \leq s \leq t} \|x_2(s)\|) \right],
\]

\(t \in [0, t_1].\)

Thus,

\[
\|\varphi x_1(t) - \varphi x_2(t)\| \leq \bar{K} \ell_3 + \frac{R \ell_1 \ell_2}{\Gamma(\alpha)} \left[ \left( M + \bar{M} \bar{M} \right) + N_2 \bar{N}_1 \left( 1 + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0 \leq s \leq t} \|x_2(s)\| \right) \right] \\
\|x_1(s) - x_2(s)\| < \delta \|x_1(s) - x_2(s)\|, t \in [0, t_1] \\
\|\varphi x_1(t) - \varphi x_2(t)\| \leq \left[ \bar{K} \ell_3 + \frac{\tau^{\alpha}}{\Gamma(\alpha)} + 1 \right] \frac{R \ell_1 \ell_2}{\Gamma(\alpha)} \left[ \left( M + \bar{M} \bar{M} \right) + N_2 \bar{N}_1 \left( 1 + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0 \leq s \leq t} \|x_2(s)\| \right) \right] \\
+ L_1 L_2 m \ell_2 + \left[ \frac{\tau^{\alpha}}{\Gamma(\alpha)} \left( M + \bar{M} \bar{M} \right) + N_2 \bar{N}_1 \left( 1 + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0 \leq s \leq t} \|x_2(s)\| \right) \right] \left( T_{H_1} K_{f_2}(t) \Omega_{f_2}(r + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0 \leq s \leq t} \|x_2(s)\|) \right), t \in (t_k, t_{k+1}], k = 1, 2, ... m
\]
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From condition (g3)(ii). Hence \[ \| (\varphi x_1)(t) - \varphi x_2)(t) \| \leq \delta \| x_1(s) - x_2(s) \| \]
Therefore, \( \varphi(x)(\cdot) \) is contraction. Thus \( \varphi x = x \) and we had
\[ C(t) (\varphi x)(t) C^*(t) = y(t) \], hence
\[ C(t) x(t) C^*(t) = y(t) \].

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