Dynamics of a Hybrid Circuit System With Lossless Transmission Line

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ABSTRACT In this paper, a hybrid circuit system composed of a Chua’s circuit with lossless transmission line and a \( RC \) oscillating circuit is considered. First, the equations describing the hybrid circuit system are reduced to a set of hybrid equations made up of a delayed neutral-type differential equation and a differential equation. By constructing the appropriate Lyapunov function, a novel condition of global stability is derived, which depends on the lower bound of \( f'(x) \). Using the length of transmission line as the bifurcation parameter, the conditions of local stability and Hopf bifurcation are obtained. Finally, the realization of our proposed circuit is given and two examples are given to verify the obtained theoretical analysis by using Matlab and Multisim14.0.

INDEX TERMS Lossless transmission line, hybrid circuit system, Hopf bifurcation, differential equation, stability.

I. INTRODUCTION
Chua’s circuit is one of the significant nonlinear circuit systems, which was proposed by Leon O. Chua in 1983. As shown in Fig. 2, Chua’s circuit is a simplest third-order autonomous circuit composed of linear resistance, capacitance, inductance and nonlinear “Chua’s diode”. It is well known that Chua’s circuit has rich dynamic behaviors such as multi-stability, periodic solution, quasi periodic solution, chaos and so on. These kinds of dynamic behaviors have wide applications in many areas, i.e., image encryption [1]–[3], secure communication [4]–[6], wave filter [7] and so on. Therefore, many scholars have conducted research on this field by using circuit experiment, computer simulation, theoretical analysis and other methods [8]–[26].

In practical circuit systems, lossless transmission line (LTL) is widely used because it can transfer information or energy from one place to another effectively. There exists a time delay in circuit when using lossless transmission line because of the propagation time through the transmission line. Thus, the circuit system with LTL can be considered as a delayed partial differential equation, which is an infinite dimensional system. This means that the circuit system with LTL has more complicated dynamic behaviors than simple circuit system. In recent years, many scholars have paid more attention on the dynamics of circuit system with LTL [19]–[26]. In [19], Sharkovsky et al. first proposed time-delayed Chua’s circuit (TDCC) by replacing the parallel LC resonator in traditional Chua’s circuit with a lossless transmission line and they found that chaos exists in TDCC. In [20], the existence of periodic limit cycles and chaotic attractors in TDCC has been proved by spectral approach. In [22], the authors have found the period-doubling bifurcation phenomenon of synchronous chaotic oscillators with time-delayed variation under LTL coupling. In [23], the authors have proved the existence of multiple amplitude phase-locked periodic solutions in a class circuit system with lossless transmission line. In [25], the author discussed the stability of TDCC and the existence of Hopf bifurcation phenomenon.

Oscillating circuit is a kind of circuit that can produce oscillating current whose size and direction change with period. There are two main types of oscillation circuits: \( RC \) oscillating circuit and \( LC \) oscillating circuit. The circuit diagrams of \( RC \) oscillating circuit and \( LC \) oscillating circuit are shown in Figure 1. From Figure 1 (a), it can be seen
that $RC$ oscillating circuit is composed of a resistor and a capacitor. Because of its simple structure, $RC$ oscillating circuit is widely used in many fields such as measurement, automatic control, wireless communication and telecontrol. In wireless communication, $RC$ oscillating circuit is coupled with functional circuit to realize system frequency regulation. For example, the high-frequency sinusoidal signals are often used as a carrier wave to modulate the audio signals for long-distance transmission. It is well known that TDCC is widely used in secure communication because it can produce chaos. To adjust the frequency of chaotic signal and ensure the stable transmission of chaotic signal, engineers always couple an $RC$ oscillating circuit on TDCC. Therefore, it is of great practical significance to study this kind of coupled circuit composed of TDCC and $RC$ oscillating circuit. However, few works study the dynamics of hybrid circuit system composed of TDCC and $RC$ oscillating circuit.

Inspired by the observation, in this paper, we investigate a hybrid circuit system as shown in Figure 2. From Figure 2, we can see the hybrid circuit system is composed of a TDCC and a $RC$ oscillating circuit. The main contribution of this paper is as follows:

1) A hybrid circuit system composed of a Chua’s circuit with lossless transmission line and a $RC$ oscillating circuit is proposed. The reliability and efficiency of our proposed circuit is better than the circuit proposed in [25] because the $RC$ oscillating circuit in the hybrid circuit in the proposed circuit can be used to realize the circuit power regulation. As is known to all, power regulation is important to the reliability and efficiency of the circuit system as well as to the quality of the output waveform.

2) The equations describing the hybrid circuit system are reduced to a set of hybrid equations made up of a delayed neutral-type differential equation and a differential equation. By constructing the appropriate Lyapunov function, the conditions of global stability are derived. Using the length of LTL as the bifurcation parameter, the conditions of local stability and the Hopf bifurcation are obtained.

The structure of this paper is as follows: In Section 2, circuit analysis of hybrid Circuit system is given. In Section 3, the global stability condition of the system is obtained by using Lyapunov function. In Section 4, the existence condition of Hopf bifurcation is obtained. In Section 5, simulations are given to verify the correctness of the obtained theorem.

II. CIRCUIT ANALYSIS

In this section, we give the circuit analysis of the proposed hybrid circuit system described in Figure 2. First we study the circuit 1, which is a hybrid circuit system composed of a Chua’s circuit and $RC$ oscillating circuit. The main contribution of this paper is to study the dynamics of hybrid circuit system composed of TDCC and $RC$ oscillating circuit. Therefore, it is of great practical significance to study this kind of coupled circuit composed of TDCC and $RC$ oscillating circuit.
Replacing $t$ by $t - (l/s)$ in (2.5), one can obtain
\[ \kappa_1 (l - at) = -\sigma_1 (at - l). \]  
(2.6)

Defining $u_1 (t)$ and $h = 2l/a$, we have
\[ u_1 (t) = \sigma_1 (at + l), \]  
(2.7)
\[ u_1(t - h) = \kappa_1 (at - l). \]  
(2.8)

By (2.3) and (2.6)-(2.8), one has
\[ v_1(l, t) = \kappa_1 (l - at) + \sigma_1 (at + l) = u_1(t) - u_1(t - h), \]  
(2.9)
\[ i_1 (l, t) = \frac{1}{Z} \left( \kappa_1 (l - at) - \sigma_1 (at + l) \right) \]  
\[ = -\frac{1}{Z} (u_1(t) + u_1(t - h)). \]  
(2.10)

Substituting (2.9) and (2.10) into (2.2), we can get the differential equation of Circuit 1:
\[ -\frac{1}{Z} (u_1(t) + u_1(t - h)) \]  
\[ = f_1 \left( \frac{R_1 + Z}{Z} u_1(t) + \frac{R_1 - Z}{Z} u_1(t - h) - E \right) \]  
\[ + C_1 \frac{d}{dt} \left( \frac{R_1 + Z}{Z} u_1(t) + \frac{R_1 - Z}{Z} u_1(t - h) \right) \]  
\[ + \frac{1}{R} \left( \frac{R_1 + Z}{Z} u_1(t) + \frac{R_1 - Z}{Z} u_1(t - h) - V_2 \right). \]  
(2.11)

Now, let us consider Circuit 2, according to Kirchhoff’s Current Laws, one can obtain
\[ i_2 = i c_2 + f_2 (V_2) - (I_1 - I_2), \]  
(2.12)
where $i_2 = \frac{V_2}{R_2}$ is the current through resistor $R_2$, $i c_2 = \frac{V_2}{RC}$ is the current through capacitor $C_2$, $I_1 - I_2 = \frac{1}{R} \left( \frac{R_1 + Z}{Z} u_1(t) + \frac{R_1 - Z}{Z} u_1(t - h) - 2V_2 \right)$ is the current flowing from the Circuit 2, $f_2 (\cdot)$ is the current expression of the $V_2$. By (2.11), (2.12), the hybrid circuit system can be described as follows:
\[
\begin{cases}
-\frac{1}{Z} (u_1(t) + u_1(t - h)) \\
= f_1 \left( \frac{R_1 + Z}{Z} u_1(t) + \frac{R_1 - Z}{Z} u_1(t - h) - E \right) \\
+ C_1 \frac{d}{dt} \left( \frac{R_1 + Z}{Z} u_1(t) + \frac{R_1 - Z}{Z} u_1(t - h) \right) \\
+ \frac{1}{R} \left( \frac{R_1 + Z}{Z} u_1(t) + \frac{R_1 - Z}{Z} u_1(t - h) - V_2 \right), \\
\end{cases}
\]  
(2.13)
\[
\begin{cases}
V_2 \frac{R_2}{R_2} = f_2 (V_2) \\
= -\frac{1}{Z} \left( \frac{R_1 + Z}{Z} u_1(t) + \frac{R_1 - Z}{Z} u_1(t - h) - 2V_2 \right) \\
+ C_2 \frac{dV_2}{dt}.
\end{cases}
\]

Letting $x_2 (t) = \frac{R_1 + Z}{Z} u_1(t), u_1(t) = \frac{Z}{Z + R_1} x_2 (t), u_1(t - h) = \frac{Z}{Z + R_1} x_2 (t - h), q = \frac{R_1 - Z}{Z + R_1} x_1(t) = V_2$, we have
\[
\begin{cases}
\frac{dx_1(t)}{dt} = a_1 x_1(t) + a_2 (x_2(t) + q x_2(t - h)) \\
- a_3 \beta (x_1(t)), \\
\frac{d}{dt} (x_2(t) + q x_2(t - h)) = a_4 x_1(t) - a_4 (x_2(t)) \end{cases}
\]  
(2.14)
where $a_1 = \frac{R - 2R_1}{RC_2 C_1}, a_2 = \frac{1}{RC_2}, a_3 = \frac{1}{C_2}, a_4 = \frac{1}{RC_1}, a_5 = \frac{1}{(Z + R_1) C_1}, a_6 = \frac{1}{C_1}$.

**Remark 1:** The RC oscillating circuit in the hybrid circuit is used to realize the circuit power regulation. As is known to all, power regulation is important to the reliability and efficiency of the circuit system as well as to the quality of the output waveform. Thus, the reliability and efficiency of our proposed circuit is better than the circuit proposed in [25].

**III. GLOBAL STABILITY ANALYSIS**
To obtain the main result, we assume $|q| < 1, E = 0$ and $f_1 (0) = 0$, then we can calculate that the equilibrium of (2.14) is zero.

Let
\[
M = \max \left\{ \frac{a_2 - a_5 (3 + q) + 2\beta - a_4 (1 + 2q)}{2a_6 (1 + q)}, \frac{2a_1 + (a_2 + a_4) (1 + q)}{2a_3}, \frac{a_2 q - a_5 (1 + q + q^2) - a_4 q (1 + 2q)}{2a_6 (1 + q)} \right\},
\]
and $m = \inf_{|x| \geq 0} \frac{f_0 (x)}{x}$, one has

**Theorem 1:** If $f_1 (0) = 0$ and $m > M > 0$, the zero equilibrium of (2.14) is globally stable.

**Proof:** Consider the following Lyapunov function
\[
V (x_1(t), x_2(t)) = \frac{D_{x_2} x_2^2}{2} + \frac{x_1^2 (t)}{2} + \beta \int_{-h}^{t} e^{\alpha t} x_2^2 (t + \theta) d\theta,
\]  
(3.1)
where $D_{x_2} = x_2 (t) + q x_2 (t - h)$ and $\alpha, \beta$ are positive number. Taking the derivative of (3.1) yields
\[
\dot{V} = x_1 (t) (a_1 x_1 (t) + a_2 D_{x_2} (t) - a_3 f_2 (x_1(t))) + D_{x_2} (a_4 x_1 (t) - a_5 (x_2 (t) + x_2 (t - h)) - a_4 D_{x_2} \\
- a_6 f_1 (D_{x_2})) - a_\beta e^{-at} \int_{-h}^{t} e^{\alpha t} x_2^2 (s) ds + a_\beta e^{-at} \left( e^{\alpha t} x_2^2 (t) - e^{\alpha (t-h)} x_2^2 (t - h) \right).
\]  
(3.2)
Let $s = t + \theta$, according to (3.1), one can obtain
\[
\beta \int_{-h}^{0} e^{\beta t} x_2^2 (t + \theta) = \beta e^{-\alpha t} \int_{t-h}^{t} e^{\alpha t} x_2^2 (t + \theta)
= V (x_1 (t), x_2 (t)) - \frac{D x_2^2}{2} - \frac{x_1^2}{2}.
\] (3.3)

Then, (3.2) can be written as
\[
\dot{V} = -\alpha V + (a_1 + \frac{\alpha}{2}) x_1^2 (t) + (-a_5 + \beta - a_4 + \frac{\alpha}{2}) x_2^2 (t)
+ \left( -a_5 q^2 - a_4 q^2 + \frac{\alpha q^2}{2} - \beta e^{-ah} \right) x_2^2 (t - h)
+ (a_2 + a_4) x_1 (t) x_2 (t) + q (a_2 + a_4) x_1 (t) x_2 (t - h)
+ (-a_5 (1 + q) - 2a_4q + \alpha q) x_2 (t) x_2 (t - h)
- a_6 D x_2 f_1 (D x_2) - a_3 x_1 (t) x_1 (t).
\] (3.4)

Hence, we can obtain
\[
\dot{V} = -\alpha V + (a_1 + \frac{\alpha}{2}) x_1^2 (t) + (-a_5 + \beta - a_4 + \frac{\alpha}{2}) x_2^2 (t)
+ \left( -a_5 q^2 - a_4 q^2 + \frac{\alpha q^2}{2} - \beta e^{-ah} \right) x_2^2 (t - h)
+ (a_2 + a_4) x_1 (t) x_2 (t) + q (a_2 + a_4) x_1 (t) x_2 (t - h)
+ (-a_5 (1 + q) - 2a_4q + \alpha q) x_2 (t) x_2 (t - h)
- a_6 D x_2 f_1 (D x_2) - a_3 x_1 (t) x_1 (t).
\] (3.5)

As
\[
\inf_{|x| \geq 0} \frac{f_1 (x)}{x} = m > M > 0,
\] (3.6)

let $\epsilon$ satisfy $m > \epsilon > M > 0$, one can obtain
\[
f_2 (x_1 (t)) x_1 (t) = -x_1^2 (t) \left( \frac{f_2 (x_1 (t))}{x_1 (t)} - \epsilon \right),
-D x_2 f_1 (D x_2) = -\left( D x_2 \right)^2 \left( \frac{f_1 (D x_2)}{D x_2} - \epsilon \right).
\] (3.7)

Then
\[
\dot{V} = -\alpha V - a_6 \left( D x_2 \right)^2 \left( \frac{f_1 (D x_2)}{D x_2} - \epsilon \right)
- a_3 x_1^2 (t) \left( \frac{f_2 (x_1 (t))}{x_1 (t)} - \epsilon \right)
+ \left( a_1 + \frac{\alpha}{2} \right) x_1^2 (t)
+ \left( -a_5 + \beta - a_4 + \frac{\alpha}{2} \right) x_2^2 (t)
+ \left( -a_5 q^2 - a_4 q^2 + \frac{\alpha q^2}{2} - \beta e^{-ah} \right) x_2^2 (t - h)
+ (a_2 + a_4) x_1 (t) x_2 (t)
+ q (a_2 + a_4) x_1 (t) x_2 (t - h)
+ (-a_5 (1 + q) - 2a_4q + \alpha q) x_2 (t) x_2 (t - h)
- a_6 \epsilon (D x_2)^2 - a_3 x_1^2 (t).
\] (3.8)

Let $x_1 (t) = u_1$, $x_2 (t) = u_2$ and $x_2 (t - h) = \nu$, one can obtain
\[
P_a (u_1, u_2, \nu) = \left( a_1 + \frac{\alpha}{2} - a_3 \epsilon \right) u_1^2
+ \left( -a_5 + \beta - a_4 + \frac{\alpha}{2} - a_6 \epsilon \right) u_2^2
+ \left( -a_5 q^2 - a_4 q^2 + \frac{\alpha q^2}{2} - \beta e^{-ah} - a_6 q^2 \epsilon \right) \nu^2
+ (a_2 + a_4) u_1 u_2 + q (a_2 + a_4) u_1 \nu
+ (-a_5 (1 + q) - 2a_4q + \alpha q - 2a_6 q \epsilon) u_2 \nu.
\] (3.9)

Assuming $a = 0$, $\epsilon < \frac{-a_5 (1 + q) - 2a_4q + \alpha q}{2a_6 q}$ and using fundamental inequality $ab \leq \frac{a^2 + b^2}{2}$, one have
\[
P_0 (u_1, u_2, \nu) \leq \left( a_1 - a_3 \epsilon + \frac{a_2 + a_4}{2} \right) u_1^2
+ + \left( \frac{a_2 + a_4}{2} - \frac{a_5 (3 + q)}{2} + \beta \right) a_4 (1 + q) - a_6 \epsilon (1 + q) u_2^2
+ \left( q (a_2 + a_4) - a_5 (1 + q + 2 \epsilon) \right) u_1 u_2
+ \left( a_6 \epsilon q (1 + q) - a_5 \epsilon q (1 + q) \right) \nu^2.
\] (3.10)

As $\epsilon > \frac{2a_1 + (a_2 + a_4) (1 + q)}{2a_6 q} > \frac{2a_2 - a_5 (3 + q) + 2 \beta - a_4 (1 + 2q)}{2a_6 q (1 + q)}$ and
$\epsilon > \frac{a_2 q - a_5 (1 + q + 2 \epsilon) - a_4 (1 + 2 \epsilon)}{2a_6 (1 + q)}$, one has $P_0$ is negative.

By continuity, it follows that $P_a$ is negative when $\alpha > 0$. Hence, we have
\[
\dot{V} \leq -\alpha V - a_6 \left( D x_2 \right)^2 \left( \frac{f_1 (D x_2)}{D x_2} - \epsilon \right)
- a_3 x_1^2 (t) \left( \frac{f_2 (x_1 (t))}{x_1 (t)} - \epsilon \right).
\] (3.11)

As $\frac{f_2 (x_1 (t))}{x_1 (t)} > \epsilon$ and $\frac{f_1 (D x_2)}{D x_2} > \epsilon$, we can conclude $\dot{V} \leq -\alpha V \leq 0$, which means $V$ is nonincreasing and bounded below. By the theorem 3.1 of [28]–[33], we can obtain that the zero solution of the system (2.14) is globally asymptotically stable. The proof is complete.

**Remark 2:** From theorem 1, we know that the condition of global stability does not depend on the equilibrium, but depends on the lower bound of $\frac{f_1 (x)}{x}$.

**IV. BIFURCATION ANALYSIS**

In this section, we give the bifurcation analysis of (2.14). For convenience, we define $f_1 = m_i x + n_i x^3$. Linearizing (2.14), one can obtain:
\[
\begin{align*}
\frac{dx_1 (t)}{dt} &= (a_1 - a_1) x_1 (t) + a_2 (x_2 (t) + qx_2 (t - h)), \\
\frac{dx_2 (t) + qx_2 (t - h)}{dt} &= a_4 x_1 (t) \\
\frac{dx_2 (t) + qx_2 (t - h)}{dt} &= -a_5 (x_2 (t) + x_2 (t - h)) \\
- (a_4 + a_2) (x_2 (t) + qx_2 (t - h)) &= 0.
\end{align*}
\] (4.1)
where \( \alpha_1 = a_3m_1, \alpha_2 = a_5m_2 \). Then characteristic equation of (4.1) is
\[
F(\lambda) = \det \begin{bmatrix}
1 & 0 \\
0 & 1 + qe^{-\lambda h}
\end{bmatrix}
- \begin{bmatrix}
a_1 - a_1 & a_2 + qe^{-\lambda h} \\
a_4 - a_5 - a_5e^{-\lambda h} & (a_4 + a_2) (1 + qe^{-\lambda h})
\end{bmatrix}
= 0.
\]
(4.2)

Then, one can obtain
\[
F(\lambda) = \lambda^2 + c_1\lambda + c_2 + \left(q\lambda^2 + c_3\lambda + c_4 \right) e^{-\lambda h} = 0,
\]
(4.3)

where
\[
c_1 = -a_1 + a_4 + a_5 + a_1 + a_2,
\]
\[
c_2 = -a_1a_5 - a_2a_4 - a_1a_4 - a_1a_2 + a_5a_1 + a_4a_1 + a_1a_2,
\]
\[
c_3 = a_5 + q(-a_1 + a_4 + a_1 + a_2),
\]
\[
c_4 = a_5a_1 - a_1a_5 + q(-a_2a_4 - a_1a_4 - a_1a_2 + a_4a_1 + a_1a_2).
\]

Suppose (4.3) has a pair of pure imaginary root \( \lambda = \pm j\omega (>0) \), then separating the real and imaginary parts, one can obtain
\[
\begin{align*}
(c_4 - q\omega^2) \cos \omega h + c_3\omega \sin \omega h &= \omega^2 - c_2, \\
c_3\omega \cos \omega h - (c_4 - q\omega^2) \sin \omega h &= -c_1\omega.
\end{align*}
\]
(4.4)

By simple calculation, we have
\[
\cos \omega h = \frac{(\omega^2 - c_2)(c_4 - q\omega^2) - c_1c_4\omega^2}{(c_4 - q\omega^2)^2 + c_3^2\omega^2}.
\]
(4.6)

By (4.4) and (4.5), we have
\[
\omega^4 + d_1\omega^2 + d_2 = 0,
\]
(4.7)

where \( d_1 = \frac{c_2^2 - q(2c_2 + 1)\omega^2}{q^2 - 1} \), \( d_2 = \frac{c_2^2 - c_3^2}{q^2 - 1} \). Denote \( \omega^2 = z^2 \), \( z^2 + d_1z + d_2 = 0 \).

The roots of (4.8) are \( z_{1,2} = \frac{-d_1 \pm \sqrt{d_1^2 - 4d_2}}{2} \).

**Lemma 1:** (A1) If \( d_1 > 0, d_2 < 0 \), the equation of (4.12) has only one positive roots; (A2) If \( d_1 < 0, d_2 > 0 \), the equation of (4.8) has two positive roots.

Without loss of generality, we assume (4.7) has two positive roots \( z_1 \) and \( z_2 \). Then, we have \( \omega_1 = \sqrt{z_1} \) and \( \omega_2 = \sqrt{z_2} \).

By (4.6), one has
\[
h_k^0 = \frac{1}{\omega_k} \arccos \left( \frac{(\omega_k^2 - c_2)(c_4 - q\omega_k^2) - c_1c_4\omega_k^2}{(c_4 - q\omega_k^2)^2 + c_3^2\omega_k^2} \right) + 2\pi.
\]
(4.9)

Then, we define \( h_0 = \min_{k=1,2} \{ h_k^0 \} \).

Letting \( h = 0 \), (4.3) can be rewritten as
\[
(1 + q)\lambda^2 + (c_1 + c_3)\lambda + c_2 + c_4 = 0.
\]
(4.10)

By simple calculation, one has
\[
\lambda_{1,2} = \frac{-(c_1 + c_3) \pm \sqrt{(c_1 + c_3)^2 - 4(1 + q)(c_2 + c_4)}}{2(1 + q)}.
\]

It is easy to obtain if \( c_1 + c_3 > 0, c_2 + c_4 > 0 \) and \( \Delta = (c_1 + c_3)^2 - 4(1 + q)(c_2 + c_4) \geq 0 \), all roots of (4.10) have negative real part when \( h = 0 \).

Denote \( g(z_k) = z_k^2 + d_1z_k + d_2 \).

**Lemma 2:** Suppose that \( z_k = \omega_k^2, k = 1, 2 \) and \( g' (z_k) \neq 0 \). Hence, we can obtain
\[
\frac{d\left(\text{Re} \lambda(h)\right)}{dh} \Bigg|_{h=h_k^0} \neq 0.
\]

**Proof:** Differentiating the (4.3) with respect to \( h \), one has
\[
\frac{d\lambda}{dh} \Bigg|_{h=h_k^0} = 2\lambda + c_1 + (2q\lambda + c_3) e^{-\lambda h} - h_0 e^{-\lambda h_0} \left( q\lambda^2 + c_3\lambda + c_4 \right)
\]
\[
= \frac{\lambda}{\lambda} \left( q\lambda^2 + c_3\lambda + c_4 \right) e^{-\lambda h_0} - \frac{h_0}{\lambda}.
\]
(4.11)

By (4.4) and (4.5), one can obtain
\[
\left[ \frac{d\left(\text{Re} \lambda (h)\right)}{dh} \right]^{-1} \Bigg|_{h=h_k^0} = \left( \frac{(2\lambda + c_1) e^{\lambda h} + (2q\lambda + c_3)}{\lambda(q\lambda^2 + c_3\lambda + c_4)} \right)_{h=h_k^0}
\]
\[
= \frac{1}{K} \left( c_1\omega_k^2 + 2c_2\omega_k^4 + 2qc_4\omega_k^2 - 2q^2\omega_k^4 - c_3^2\omega_k^2 \right)
\]
\[
= \frac{1}{K} \left( -2q^2 - 1 \right) z_k^2 - \left( c_3 - 2qc_4 + 2c_2 - c_1^2 \right) z_k
\]
\[
= -\frac{z_k}{K} \left( q^2 - 1 \right) g' (z_k),
\]

where \( K = c_2^2 \omega_k^4 + \omega_k^2 (c_4 - q\omega_k^2)^2 > 0, g'(z_k) \neq 0 \). Therefore, we can conclude
\[
\text{sign} \left\{ \frac{d\left(\text{Re} \lambda (h)\right)}{dh} \right\} \Bigg|_{h=h_k^0} = \text{sign} \left\{ \frac{d\left(\text{Re} \lambda (h)\right)}{dh} \right\}^{-1} \Bigg|_{h=h_k^0}
\]
\[
= \text{sign} \left\{ -z_k \left( q^2 - 1 \right) g' (z_k) \right\} \neq 0.
\]

The lemma 2 is proved.

**Lemma 3:** Consider the exponential polynomial \([33]–[34] \)
\[
P(\lambda, e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_m}) = \lambda^n + P_1^{(0)} \lambda^{n-1} + \ldots + P_{n-1}^{(0)} \lambda + P_n^{(0)}
\]
\[
+ \left[ P_1^{(1)} \lambda^{n-1} + \ldots + P_{n-1}^{(1)} \lambda + P_n^{(1)} \right] e^{-\lambda \tau_1}
\]
\[
+ \ldots + \left[ P_1^{(m)} \lambda^{n-1} + \ldots + P_{n-1}^{(m)} \lambda + P_n^{(m)} \right] e^{-\lambda \tau_m}
\]
(4.12)

where \( \tau_i \geq 0 (i = 1, 2, \ldots, m) \) and \( P_j^{(i)} (j = 1, 2, \ldots, n) \) are constants. As \( (\tau_1, \tau_2, \ldots, \tau_m) \) vary, the sum of the order of the zeros of \( P(\lambda, e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_m}) \) on the open right half plane can change only if a zero appears on or crosses the imaginary axis.

**Theorem 2:** Following Lemma 1, 2 and 3, if one of (A1), (A2) and \( c_1 + c_3 > 0, c_2 + c_4 > 0 \) holds, we can find the
system (2.14) is asymptotically stable for all \( h \in [0, h_0) \); System (2.14) undergoes Hopf bifurcation at the zero equilibrium point when \( h = h_0 \).

Remark 3: From theorem 2, it is found that \( h \) has a major impact on the dynamics of hybrid circuit. As \( h=2l/a \), we find that the dynamic behaviors of the system (2.7) can be controlled by adjusting the length of transmission line.

V. NUMERICAL EXAMPLES

In this section, we use Multisim14.0 and Matlab to verify the theoretical analysis. Multisim14.0 is a circuit simulation software. By using Multisim14.0, the circuit implementation of Figure 2 is given in Figure 3. We choose \( f_i = m_i x + n_i x^3 \) \((i = 1, 2)\), where \( m_i = 1, n_i = 1/3 \).

Example 1: This example verifies the Theorem 1. Let \( R = 1, R_1 = 2, R_2 = 2, C_1 = 0.5, C_2 = 1, Z = 1.5, \alpha = 10 \), we can find \( |q| = 0.1429 < 1, M = 0.2143 < \varepsilon < 0.3571 < m = 1 \) and \( \inf_{|x| \geq H} \frac{f(x)}{x} > 1 > M > 0 \), according to Theorem 1, we know the zero equilibrium point of system (2.14) with \( E = 0 \) is globally exponentially stable, which is shown in Figure 4 and 5. Figure 4 is the results of Matlab and Figure 5 is the results of Multisim14.0.

Example 2: This example verifies the Theorem 2. Let \( R = 1.5, R_1 = 0.1, R_2 = 0.45, C_1 = 0.2, C_2 = 0.8, Z = 2, m_i = 1, n_i = 1/3, q = -0.9048 \), we have \( |q| < 1, d_1 > 0, d_2 < 0, c_1 + c_3 > 0, c_2 + c_4 > 0 \) and \( \Delta = (c_1 + c_3)^2 - 4(1 + q)(c_2 + c_4) > 0 \). By calculation, we can obtain \( \omega_0 = 0.1325 \) and \( h_0 = 7.5794 \). Let \( h = 7.2 < h_0 \), by theorem 2, we know the system (2.14) is globally exponentially stable at zero point.
Asymptotically stable near the zero equilibrium point which is shown in Figure 6 and 7. Figure 6 is the results of Matlab and Figure 7 is the results of Multisim 14.0. Let $h = 7.8 > h_0$, by theorem 2, we know the system (2.14) undergoes a Hopf bifurcation which is shown in Figure 8 and 9. Figure 7 is the results of Matlab and Figure 8 is the results of Multisim 14.0.

VI. CONCLUSIONS

Recently, with the progress of electronic technology, nonlinear electronic devices are widely used in the hybrid circuit system, which leads to a large number of nonlinear phenomena in hybrid circuit system such as oscillation, quasi-oscillation, chaos and so on. It is found that these dynamic behaviors generated by hybrid circuit system have many applications such as image encryption [1]–[3], secure communication [4]–[6], wave filter [7] and so on. In this paper, we investigate the dynamics of a hybrid circuit system composed of a Chua’s circuit with lossless transmission line and a non-linear oscillating circuit. First, the equation describing hybrid circuit system is reduced to a mixture equation set made up of a delayed neutral-type differential equation and a differential equation. By constructing the appropriate Lyapunov function, a novel condition of global stability are derived. Using the length of transmission line $l$ as the bifurcation parameter, the conditions of local stability and the Hopf bifurcation are obtained. Finally, the realization of our proposed circuit is given and two examples are given to verify the obtained theoretical analysis by using Matlab and Multisim 14.0.

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