Knudsen Type Group and Boltzmann Type Equation for Time in $\mathbb{R}$

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Abstract We consider a certain Boltzmann type equation on a bounded physical and a bounded velocity space under the presence of both, reflective as well as diffusive boundary conditions. We provide conditions on the shape of the physical space and on the relation between the reflective and the diffusive part in the boundary conditions such that the associated Knudsen type semigroup is reversible. Furthermore, we provide conditions under which there exists a unique global solution to the Boltzmann type equation for time $t \geq 0$ or for time $t \in \mathbb{R}$.

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1 Introduction

Boltzmann type equations provide a mathematical description of a rarefied gas in a vessel, in the present paper with reflective and diffusive boundary conditions.

In the fundamental paper [9] O. E. Lanford demonstrated that the limit of a time reversible microscopic $N$-particle system may establish a time irreversible kinetic equation. In particular, supposing $N \to \infty$ and, for the particle size, $\varepsilon \to 0$ such that $N \cdot \varepsilon^{d-1}$ is bounded, the Boltzmann equation has been derived. This result of O. E. Lanford holds on some finite time interval, assuming that at initial time zero the particles are independently distributed.

In recent investigations M. Pulvirenti et al [12, 13] have shown that under appropriate additional hypotheses even initially correlated $N$-particle systems can converge to the Boltzmann equation. A detailed analysis of initial configurations and how they contribute to the convergence of $N$-particle systems to the Boltzmann equation is due to T. Bodineau et al [1]. How initial conditions on the $N$-particle systems affect the proof of irreversibility of the solution to the Boltzmann equation is a major topic in this research. In particular, Boltzmann’s $H$-theorem on non-increasing entropy is important in [1].

Boltzmann’s $H$-theorem is a crucial tool in order to prove irreversibility of solutions to the Boltzmann equation. As for example in [1], the proof of a corresponding $H$-theorem is possible in a number of situations, in particular when the physical space $\Omega$ is $\mathbb{R}^d$ and the velocity space $V$ is $\{v \in \mathbb{R}^d : |v| > 0\}$. On the other hand, for bounded $\Omega$ or $V$, certain types of boundary conditions may not be compatible to known proofs of the $H$-theorem. For example, the proof in [14], Theorem 1 of Section 1.1.2 cannot be applied or modified in order to cope with diffusive boundary conditions as used in the present paper.
In Caprino et al [2] it has been shown that for bounded $\Omega$ and $V = \{ v \in \mathbb{R}^d : |v| > v_0 > 0 \}$, and appropriate initial configurations, a certain N-particle system converges to the unique stationary solution of a Boltzmann equation satisfying certain diffusive boundary conditions. Stationarity is the only well-established form of time reversibility relative to Boltzmann equations. Boundedness from below and above of the stationary solution to a Boltzmann equation as in [2] has been proved in [10].

In the present paper we consider a Boltzmann type equation on a bounded physical space $\Omega$ and a bounded velocity space $V$. The boundary conditions include both, a reflective and a diffusive part. The first problem we are interested in is how reversibility of the corresponding Knudsen type transport semigroup is related to the shape of the boundary and the boundary conditions. For example, pure reflection of particles on the physical boundary $\partial \Omega$ inducing reversibility is, up to the modulus of the velocity, described by well-established theory of mathematical billiards in dimension $d = 2$ and $d = 3$. However, more realistic is the presence of both, reflective and diffusive boundary conditions. A detailed spectral analysis of the Knudsen type semigroup is developed in Section 4 in order to approach reversibility.

The second problem we study in this paper is existence and uniqueness of global solutions to Boltzmann type equations, first for time $t \geq 0$. This includes a sufficient condition on the initial probability density of the particles at time $t = 0$. The results of Section 4 on the associated Knudsen type transport semigroup are then applied to prove existence and uniqueness of solutions to Boltzmann type equations for time $t \in \mathbb{R}$, see Theorem 5.8.

1.1 Main results

For a certain class of physical spaces which includes certain convex polygons in dimension $d = 2$ or a certain class of polyhedrons in $d = 3$ let us consider the Knudsen type semigroup $S(t), t \geq 0$. Formally it is introduced for time $t \geq 0$ by the solution to the initial boundary value problem

$$\left( \frac{d}{dt} + v \cdot \nabla_r \right) (S(t)p_0)(r,v) = 0 \quad \text{on} \quad (r,v,t) \in \Omega \times V \times [0,\infty),$$

$S(0)p_0 = p_0$, and the boundary conditions

$$(S(t)p_0)(r,v) = \omega (S(t)p_0)(r, R_r(v)) + (1 - \omega)J(r,v)(S(\cdot)p_0)M(r,v), \quad t > 0,$$

for all $(r,v) \in \partial^{(1)} \Omega \times V$ with $v \cdot n(r) \leq 0$. Here $n(r)$ is the outer normal at $r$ belonging to the boundary part $\partial^{(1)} \Omega$ obtained from $\partial \Omega$ by removing all vertices and edges (in $d = 3$) and “$\circ$” denotes the inner product in $\mathbb{R}^d$. Furthermore, $R_r(v) := v - 2v \cdot n(r) \cdot n(r)$ for $(r,v) \in \partial^{(1)} \Omega \times V$ indicates reflection of the velocity $v$ at the boundary point $r$,

$$J(r,v)(S(\cdot)p_0) = \int_{v \cdot n(r) \geq 0} v \cdot n(r) S(t)p_0(r,v) \, dv,$$

and $M$ is a certain positive continuous function on $\{(r,v) : r \in \partial^{(1)} \Omega, \ v \in V, \ v \cdot n(r) \leq 0\}$, bounded from below and above, that quantifies the diffusive part of the boundary conditions. The constant $\omega \in (0,1)$ controls the relation between reflective and diffusive boundary conditions. The following spectral property is the first major result.

Proposition 1 (Corollary 4.4 (c) below) There exists $m_1 > -\infty$ such that for $t \geq 0$, the resolvent set of $S(t)$ contains the set $\{ \lambda = e^{i\mu} : \Re \mu < m_1 \} \cup \{ \lambda = e^{i\mu} : \Re \mu > 0 \}$. 

2
Proposition 1 is the crucial technical part of the paper since it includes explicit solutions to certain Banach space valued differential equations, for which there is no known theory. Under additional conditions on the shape of ∂Ω, we have proved the following.

**Theorem 2** (Theorem 4.3 below) There exists $k_0 \in \mathbb{N}$ such that for $2^{-\frac{1}{k_0}} < \omega < 1$, the semigroup $S(t), t \geq 0$, extends to a strongly continuous group in $L^1(\Omega \times V)$ which we will denote by $S(t), t \in \mathbb{R}$.

Concerning Boltzmann type equations for time $t \geq 0$, the following result on unique existence of global solutions to their integrated (mild) versions

$$p(r, v, t) = S(t) p_0(r, v) + \lambda \int_0^t S(t-s) Q(p, p) (r, v, s) \, ds.$$  \hspace{1cm} (1.1)

has been proved under reasonable assumptions on the collision operator $Q$. Here $\lambda > 0$ is a certain constant controlling the existence and uniqueness of preliminary local solutions.

**Theorem 3** (Theorem 5.6 below) Let $p_0 \in L^1(\Omega \times V)$ with $\|p_0\|_{L^1(\Omega \times V)} = 1$ and suppose that there are constants $0 < c \leq C < \infty$ with $c \leq p_0 \leq C$ a.e. on $\Omega \times V$. Then there exists a unique solution $p$ to (1.1) on $\Omega \times V \times [0, \infty)$ with $p(\cdot, \cdot, 0) = p_0$. The solution $p \equiv p(p_0)$ to (1.1) has the following properties.

1. The map $[0, \infty) \ni t \mapsto p(\cdot, \cdot, t) \in L^1(\Omega \times V)$ is continuous with respect to the topology in $L^1(\Omega \times V)$.

2. We have $\|p(\cdot, \cdot, t)\|_{L^1(\Omega \times V)} = 1, t \geq 0$.

3. For every $t \geq 0$, there exists a constant $c_t > 0$ such that $p(\cdot, \cdot, t) \geq c_t$ a.e. on $\Omega \times V$.

4. The norm $\|p(\cdot, \cdot, t)\|_{L^\infty(\Omega \times V)}$ is exponentially bounded.

It also has been proved that a similar statement holds for the differential form

$$\frac{d}{dt} p(\cdot, \cdot, t) = Ap(\cdot, \cdot, t) + \lambda Q(p, p)(\cdot, \cdot, t), \quad t \geq 0,$$  \hspace{1cm} (1.2)

with $p(\cdot, \cdot, 0) = p_0 \in D(A)$, cf. Corollary 5.7 below. Here, $(A, D(A))$ denotes the infinitesimal operator of the strongly continuous semigroup $S(t), t \geq 0, \text{ in } L^1(\Omega \times V)$ and $d/dt$ is a derivative in $L^1(\Omega \times V)$. At $t = 0$ it is the right derivative. This solution coincides with the solution to the equation (1.1) if we suppose in Theorem 1 $p(\cdot, \cdot, 0) = p_0 \in D(A)$.

The extension of Theorem 2 and its differential version (1.2) to time in $\mathbb{R}$ is based on Theorem 1.

**Theorem 4** (Theorem 5.8 below) Suppose the conditions of Theorem 3 are satisfied.

(a) Let $p_0 \in L^1(\Omega \times V)$ with $\|p_0\|_{L^1(\Omega \times V)} = 1$ and suppose that there are constants $0 < c \leq C < \infty$ with $c \leq p_0 \leq C$ a.e. on $\Omega \times V$. Then for every $\tau \leq 0$ there exists a unique solution $p$ to

$$p(r, v, t) = S(t) p_0(r, v) + \lambda \int_{\tau}^{t} S(t-s) Q(p, p) (r, v, s) \, ds.$$  \hspace{1cm} (1.3)
on \( \Omega \times V \times [\tau, \infty) \) with \( p(\cdot, \cdot, 0) = p_0 \). We have properties (1)-(3) of Theorem 3 for \( t \geq \tau \) as well as

\[
\|p(\cdot, \cdot, t)\|_{L^\infty(\Omega \times V)} \leq a \cdot \exp \left\{ \lambda b \cdot (t - \tau) \right\}, \quad t \in [\tau, \infty),
\]

where the constants \( a, b > 0 \) do not depend on \( \tau \).

(b) Let \( p_0 \in D(A) \) with \( \|p_0\|_{L^1(\Omega \times V)} = 1 \) and suppose that there are constants \( 0 < c \leq C < \infty \) with \( c \leq p_0 \leq C \) a.e. on \( \Omega \times V \). Then for every \( \tau \leq 0 \) there is a unique solution \( p(\cdot, \cdot, t) \in D(A), t \in [\tau, \infty) \), to the equation

\[
\frac{d}{dt} p(\cdot, \cdot, t) = Ap(\cdot, \cdot, t) + \lambda Q(p, p)(\cdot, \cdot, t)
\]

with \( p(\cdot, \cdot, 0) = p_0 \). Here, \( d/dt \) is a derivative in \( L^1(\Omega \times V) \). At \( t = \tau \) it is the right derivative. This solution coincides with the solution to the equation (1.3) if we suppose there \( p(\cdot, \cdot, 0) = p_0 \in D(A) \). We have properties (1)-(3) of Theorem 3 for \( t \geq \tau \) as well as (1.4).

2 Preliminaries

Let \( d \in \{2, 3\} \). For a set \( \Gamma \subseteq \mathbb{R}^d \) let \( \overline{\Gamma} \) denote its closure in \( \mathbb{R}^d \). Let \( \Omega \subseteq \mathbb{R}^d \) be a bounded domain, called the physical space. Let us suppose that \( \partial \Omega = \bigcup_{i=1}^{n_\Omega} \overline{\Gamma}_i \) where the \( \Gamma_i \) are smooth \((d-1)\)-dimensional manifolds such that \((\overline{\Gamma}_i \setminus \Gamma_i) \cap \Gamma_j = \emptyset, i \neq j \subseteq \{1, \ldots, n_\Omega\} \). Let us furthermore assume that for \( i \neq j \) the intersection \( \overline{\Gamma}_i \cap \overline{\Gamma}_j \) either is the empty set or a smooth closed \((d-2)\)-dimensional manifold which, by definition, for \( d = 2 \) is just a single point in \( \mathbb{R}^2 \).

In case of \( d = 3 \), for any two neighboring \( \Gamma_i \) and \( \Gamma_j \) let us call \( x \in \overline{\Gamma}_i \cap \overline{\Gamma}_j \) an end point if for any open ball \( B_x \) in \( \mathbb{R}^d \) with center \( x \) the set \( \partial B_x \cap (\overline{\Gamma}_i \cap \overline{\Gamma}_j) \) consists of at most one point. For the boundary \( \partial \Omega \) let us also assume that there is \( \xi \in (0, \pi) \) such that for any two neighboring \( \Gamma_i \) and \( \Gamma_j \) and any \( x \in \overline{\Gamma}_i \cap \overline{\Gamma}_j \), not being an end point if \( d = 3 \), we have the following. For the angle \( \xi(i, j; x) \) between the rays \( R_i \) and \( R_j \) from \( x \) tangential to \( \Gamma_i \) and \( \Gamma_j \) respectively, and orthogonal to \( \overline{\Gamma}_i \cap \overline{\Gamma}_j \) if \( d = 3 \), it holds that

\[
\xi < \xi(i, j; x) < 2\pi - \xi.
\]

The latter guarantees compatibility with Assumption A in the proof of Theorem 2.1 in \cite{2}, as we will discuss in Remark 3 below.

Let \( V := \{v \in \mathbb{R}^d : 0 < v_{\min} < |v| < v_{\max} < \infty\} \) be the velocity space and let \( \lambda > 0 \). Denote by \( n(r) \) the outer normal at

\[
r \in \partial^{(i)} \Omega := \bigcup_{i=1}^{n_\Omega} \overline{\Gamma}_i,
\]

indicate the inner product in \( \mathbb{R}^d \) by “\( \cdot \)”, and let \( R_v(v) := v - 2v \cdot n(r) \cdot n(r) \) for \( (r, v) \in \partial \Omega \times V \).

For \((r, v, t) \in \Omega \times V \times [0, \infty)\), consider the Boltzmann type equation

\[
\frac{d}{dt} p(r, v, t) = -v \cdot \nabla_r p(r, v, t) + \lambda Q(p, p)(r, v, t)
\]
with boundary conditions
\[
p(r, v, t) = \omega p(r, R_v(v), t) + (1 - \omega)J(r, t)(p)M(r, v), \quad r \in \partial(1)\Omega, \ v \circ n(r) \leq 0,
\] (2.3)
for some \(\omega \in [0, 1] \) and initial condition \(p(0, r, v) := p_0(r, v)\). Consider also its integrated (mild) version
\[
p(r, v, t) = S(t) p_0(r, v) + \lambda \int_0^t S(t - s) Q(p, p) (r, v, s) \, ds.
\] (2.4)

The following global conditions (i)-(viii) on the terms in (2.2)-(2.4) and the shape of \(\Omega\) will be in force throughout the sections of the paper, as it appears in the preliminary paragraphs of the sections.

(i) For all \(t \geq 0\) and all \(r \in \partial(1)\Omega\), the function \(J\) is given by
\[
J(r, t)(p) = \int_{\text{von}(r) \geq 0} v \circ n(r) p(r, v, t) \, dv.
\]
(ii) The real function \(M\) on \(\{(r, v) : r \in \partial(1)\Omega, \ v \in V, \ v \circ n(r) \leq 0\}\) has positive lower and upper bounds \(M_{\min}\) and \(M_{\max}\), is continuous on every \(\Gamma_i, \ i \in \{1, \ldots, n_\partial\}\), and satisfies
\[
\int_{\text{von}(r) \leq 0} |v \circ n(r)| M(r, v) \, dv = 1, \quad r \in \partial(1)\Omega.
\]
(iii) \(S(t), \ t \geq 0\), is formally given by the solutions to the initial boundary value problems
\[
\left( \frac{d}{dt} + v \circ \nabla_r \right) (S(t)p_0)(r, v) = 0 \quad \text{on} \quad (r, v, t) \in \Omega \times V \times [0, \infty), \quad (2.5)
\]
\(S(0)p_0 = p_0\), and
\[
(S(t)p_0)(r, v) = \omega (S(t)p_0)(r, R_v(v)) + (1 - \omega)J(r, t)(S(\cdot)p_0)M(r, v), \quad t > 0,
\]
for all \((r, v) \in \partial(1)\Omega \times V\) with \(v \circ n(r) \leq 0\). We call \(S(t), \ t \geq 0\), \textit{Knudsen type semigroup}. If \(\omega = 0\) we indicate this in the notation by \(S_0(t), \ t \geq 0\). We follow [2] and call \(S_0(t), \ t \geq 0\), just \textit{Knudsen semigroup}.

(iv) Denoting by \(\chi\) the indicator function and setting \(p := 0\) as well as \(q := 0\) on \(\Omega \times (\mathbb{R}^d \setminus V) \times [0, \infty)\), the collision operator \(Q\) is given by
\[
Q(p, q)(r, v, t) = \frac{1}{2} \int_\Omega \int_V \int_{S_{d-1}^+} B(v, v_1, e) h_r(r, y) \chi_{\{(v^*, v^*_1) \in V \times V\}} \times \frac{\partial}{\partial e} \left( p(r, v^*, t)q(y, v^*_1, t) - p(r, v, t)q(y, v_1, t) \right) \, de \, dv_1 \, dy.
\]
Here \(S_{d-1}^+\) is the unit sphere. Moreover, \(S_{d-1}^+ = S_{d-1}^+(v-v_1) := \{e \in S_{d-1} : e \circ (v-v_1) > 0\}\), \(v^* := v - e \circ (v-v_1)e\), \(v^*_1 := v_1 + e \circ (v-v_1)e\) for \(e \in S_{d-1}^+\) as well as \(v, v_1 \in V\), and \(de\) refers to the normalized Lebesgue measure on \(S_{d-1}^+\).
(v) The collision kernel $B$ is assumed to be non-negative, bounded, and continuous on $V \times V \times S^{d-1}$, symmetric in $v$ and $v_1$, and satisfying $B(v^*, v_1^*, e) = B(v, v_1, e)$ for all $v, v_1 \in V$ and $e \in S^{d-1}_+$ for which $(v^*, v_1^*) \in V \times V$.

(vi) $h_\gamma$ is a continuous function on $\overline{\Omega} \times \overline{\Omega}$ which is non-negative and symmetric, and vanishes for $|r - y| \geq \gamma > 0$.

In order to prepare condition (vii), let us take a closer look at the collision operator. Let

$$\sigma := \frac{v - v_1 - 2e \circ (v - v_1)e}{|v - v_1 - 2e \circ (v - v_1)e|}$$

and

$$\tilde{B}(v, v_1, \sigma) := B(v, v_1, e) \cdot \chi_{\{(v^*, v_1^*) \in V \times V\}}(v, v_1, e).$$

We observe that, for given $v, v_1 \in \mathbb{R}^d$ with $v \neq v_1$, there is a bijective function $S^{d-1}_+(v - v_1) \ni e \mapsto \sigma \in S^{d-1} \setminus \{(v - v_1)/|v - v_1|\}$ with Jacobian determinant of its inverse

$$\left|\frac{de}{d\sigma}\right| = (2\sin(\alpha/2))^{2-d}, \quad d = 2, 3,$$

where $\alpha \equiv \alpha(v; v_1, \sigma) \in (0, \pi]$ is the angle between $\sigma$ and $v - v_1$ and $d\sigma$ refers to the normalized Lebesgue measure on $S^{d-1}$. We shall therefore treat the term $|de/d\sigma|$ as $|de/d\sigma|(v; v_1, \sigma)$.

Let us fix $v \in \mathbb{R}^d$ and $\sigma \in S^{d-1}$, and introduce the maps

$$\varphi(w) \equiv \varphi(v; w, \sigma) := -v + 2w + \frac{|v - w|^2}{\sigma \circ (v - w)} \sigma$$

as well as

$$\psi(w) \equiv \psi(v; w, \sigma) := \varphi(w) + v - w = w + \frac{|v - w|^2}{\sigma \circ (v - w)} \sigma$$

where $w \in \{u \in \mathbb{R}^d : \sigma \circ (v - u) > 0\}$. We note that $\varphi$ is injective. In particular, it holds that $v_1 = \varphi(v_1^*)$ with Jacobian determinant

$$\left|\frac{dv_1}{dv_1^*}\right| = \frac{2^{d-1}|v - v_1^*|^2}{(\sigma \circ (v - v_1^*))^2}, \quad d = 2, 3.$$

Moreover, we observe $v^* = \psi(v_1^*) \equiv \psi(v; v_1^*, \sigma)$. Introduce

$$\tilde{B}_J(v, v_1, \sigma) := \tilde{B}(v, v_1, \sigma) \left|\frac{de}{d\sigma}\right|(v, v_1, \sigma).$$

For the next calculation, we take into consideration that, for given $\sigma \in S^{d-1}$ and $v \in V$, it holds that

$$\left\{v_1^* := \frac{v + v_1}{2} - \frac{|v - v_1|}{2} \sigma : v_1 \in \mathbb{R}^d\right\} = \{v_1^* \in \mathbb{R}^d : (v - v_1^*) \circ \sigma > 0\}.$$

Furthermore, we note that, for fixed $\sigma \in S^{d-1}$ and $v \in V$, $\{\varphi(v_1^*) : v_1 \in V\}$ is not necessarily a subset of $V$. Keeping in mind the definitions of $\tilde{B}$ and $\tilde{B}_J$ we extend $\tilde{B}_J$ from $V \times V \times S^{d-1}$ to $V \times \mathbb{R}^d \times S^{d-1}$ by zero. It turns out that with

$$p_\gamma(r, v_1, t) := \int_{y \in \Omega} p(y, v_1, t) h_\gamma(r, y) \, dy$$
it holds that
\[
\int_V \int_{S^d_+} B(v, v_1, e) \chi_{\{(v^*, v_1^*) \in V \times V\}} p(r, v^*, t) p_\gamma(r, v_1^*, t) \, dv_1 \\
= \int_{S^d_+} \int_{\{v_1^* \in V\}} \left( \frac{2^{d-1} B_J(v, \varphi(v_1^*), \sigma) |v - v_1^*|^2}{(\sigma \circ (v - v_1^*))^2} \right) p(r, \psi(v_1^*), t) p_\gamma(r, v_1^*, t) \, dv \, d\sigma \\
= \int_V \int_{S^d_+} \left( \frac{2^{d-1} B_J(v, \varphi(v; v_1, \sigma), \sigma) |v - v_1|^2}{(\sigma \circ (v - v_1))^2} \right) p(r, \psi(v; v_1, \sigma), t) \, d\sigma \, p_\gamma(r, v_1, t) \, dv_1.
\]

We obtain the following \textit{Carleman type representation} of the collision operator
\[
Q(p, p)(r, v, t) = \int_V \int_{S^d_+} \left( \frac{2^{d-1} B_J(v, \varphi(v; v_1, \sigma), \sigma)}{\cos^2 \alpha(v; v_1, \sigma)} \right) p(r, \psi(v; v_1, \sigma), t) \, d\sigma \, p_\gamma(r, v_1, t) \, dv_1 \\
- \int_V \int_{S^d_+} B_J(v, v_1, \sigma) p(r, v, t) \, d\sigma \, p_\gamma(r, v_1, t) \, dv_1.
\]

**Remark 1** This representation of the collision operator can alternatively be obtained by first adjusting Lemma 7 of [3] to the particular form of the items in (iv),(v),(vi). In a second step one has to substitute the inner integral with respect to the canonical map (using the velocity symbols of [4]) \( S^d_+(v - v_1^*) \mapsto \{v' : (v' - v) \circ (v_1^* - v) = 0\} \).

We add a technical condition on the collision kernel. It is motivated by the subsequent Example 2.1 and Remark 2

\[(vii)\]
\[
\sup_{\sigma \in S^d_+ (v - v_1), v, v_1 \in V} \left| \frac{2^{d-1} B_J(v, \varphi(v; v_1, \sigma), \sigma)}{\cos^2 \alpha(v; v_1, \sigma)} \right| =: b < \infty
\]

**Example 2.1** Consider
\[
\tilde{B}(v, v_1, \sigma) := \frac{\sin^D \alpha}{C + |v - v_1|^\beta} \cdot A(|v_1 - v_1|) a(\sin \alpha) \cdot \chi_{\{(v^*, v_1^*) \in V \times V\}}(v, v_1, \alpha)
\]
where \([0, \pi) \ni \alpha := \arccos(\sigma \circ (v - v_1)/|v - v_1|), \beta \geq 0, C > 0, D \geq 2, a \) is a bounded non-negative real function on \([0, 1]\), and \(A\) is a bounded non-negative real function on \([0, \infty)\). This collision kernel satisfies (vii) by the following.

Let us recall \(|de/d\sigma| = (2 \sin(\alpha/2))^{2-d}\). We observe that under the map \(v_1^* \mapsto \varphi(v_1^*) = v_1\) the angle \(\alpha/2 = \arccos(\sigma \circ (v - v_1^*)/|v - v_1^*|)\) turns into \(\alpha\). Hence \(|v - \varphi(v_1)| = |v - v_1|/\cos \alpha\) if \(\alpha \in [0, \pi/2)\). We obtain
\[
\tilde{B}_J(v, \varphi(v_1), \sigma) = \frac{2^{D+2-d} \sin^{(D+2-d)} \alpha \cdot \cos^D \alpha}{C + (|v - v_1|/\cos \alpha)^\beta} \times
\]
\[
\times A(|v - v_1|/\cos \alpha) a(\sin 2\alpha) \cdot \chi_{\{(v^*, v_1^*) \in V \times V\}}(v, v_1, 2\alpha)
\]
whenever \(\sigma \in S^d_+(v - v_1)\) or equivalently \(\alpha \in [0, \pi/2)\). By \(D \geq 2\) we have (vii).
Remark 2 Let us look at the left-hand side of (2.6) under the common assumption that
\[ B(v, v_1, e) = B(v, v_1, e_1) \quad \text{where} \quad e_1 := \frac{v - v_1 - e \circ (v - v_1) e}{|v - v_1 - e \circ (v - v_1)|} = \frac{v - v_1^*}{|v - v_1^*|} \] (2.8)
for all \( v, v_1 \in V, \ e \in S^d_{+}(v - v_1) \). Replacing \( e \) by \( e_1 \) in Example 2.1 the angle \( \alpha \) turns into \( \pi - \alpha \), and \( (v^*, v_1^*) \) turns into \( (v_1^*, v^*) \). Thus condition (2.8) is satisfied in Example 2.1. Consider now the range of \( (v^*, v_1^*) \) when keeping \( v, v_1 \in V \) fixed but leaving \( \sigma \in S^d_{+}(v - v_1) \) variable. It becomes evident that under condition (2.8), for all non-negative measurable \( f, g : S^d_{+}(v - v_1) \mapsto \mathbb{R} \) and all \( (v, v_1) \in V \times V \) with \( (v^*, v_1^*) \in V \times V \) it holds that
\[ \int_{S^d_{+}} B(v, v_1, e)f(v^*)g(v_1^*) de = \int_{S^d_{+}} B(v, v_1, e)g(v^*)f(v_1^*) de . \] (2.9)
In particular, in (2.6) the symbols \( p \) and \( p_\gamma \) can be interchanged.

Let us turn to the last of the global conditions, namely condition (viii) below. For this, recall the structure of \( \partial \Omega \) from the beginning of this section. For \( (y, v) \in \Omega \times V \) introduce
\[ T_\Omega \equiv T_\Omega(y, v) := \inf \{ s > 0 : y - sv \notin \Omega \} \]
and
\[ y^- \equiv y^-(y, v) := y - T_\Omega(y, v)v . \] (2.10)
For all \( (y, v) \in \bigcup_{i=1}^{m_\sigma} \Gamma_i \times V = \partial^{(1)}\Omega \times V \) with \( v \circ n(y) \leq 0 \) as well as \( y^-(y, R_y(v)) \in \partial^{(1)}\Omega \), let
\[ \sigma(y, v) := (y^-(y, R_y(v)), R_y(v)) . \]
We note that there exist \( m_\sigma \equiv m_\sigma(\sigma) \in \mathbb{N} \) and mutually disjoint sets \( G_i \subseteq \{(r, v) : r \in \partial^{(1)}\Omega, \ v \in V, \ v \circ n(r) \leq 0\} \), \( i \in \{1, \ldots, m_\sigma\} \), satisfying
\[ \bigcup_{i=1}^{m_\sigma} G_i = \{(r, v) : r \in \partial^{(1)}\Omega, \ v \in V, \ v \circ n(r) \leq 0\} \]
such that the following holds. For each \( i \in \{1, \ldots, m_\sigma\} \) there exists \( j \in \{1, \ldots, m_\sigma\} \) such that \( \sigma \) maps \( G_i \) bijectively and continuously to \( G_j \).

In this way we understand \( \sigma \) as a map defined for \( (y, v) \in \partial^{(1)}\Omega \times V \) with \( v \circ n(y) \leq 0 \). For its \( k \)-fold composition \( \sigma^k(y, v) = (y^{(k)}(y, v), v^{(k)}(y, v)) \equiv (y^{(k)}, v^{(k)}) \), \( k \in \mathbb{N} \), we suppose also that by iteration \( y^{(j)} \in \partial^{(1)}\Omega, \ j \in \{1, \ldots, k\} \), and we indicate this by the phrase “for a.e. \( y \in \partial^{(1)}\Omega \)”.

Set \( (y^{(0)}, v^{(0)}) := (y, v) \). An important role will play the following condition on the shape of \( \Omega \).

(viii) There exist \( k^{(0)} \in \mathbb{N} \) and \( \sigma_{min} > 0 \) such that for a.e. \( (y, v) \in \partial^{(1)}\Omega \times V \) with \( v \circ n(y) \leq 0 \) and all \( j \in \mathbb{Z}_+ \) we have
\[ \sigma_{min} \leq |y^{(j)} - y^{(j+1)}| + |y^{(j+1)} - y^{(j+2)}| + \ldots + |y^{(j+k^{(0)}-1)} - y^{(j+k^{(0)})}| . \]
Condition (viii) is primarily used in the proof of the subsequent Lemma 3.1. This lemma is fundamental for the whole paper. Since condition (viii) appears again in quite another but more complex context in Subsection 4.1 we postpone the discussion of its verifiability to Remark 10 below.
3 Basic Properties of the Knudsen Type Semigroup

Suppose the global conditions (i)-(viii). In the paper we will work with spaces of the form $L^1(E)$. It will always be clear from the context what we understand by the Borel $\sigma$-algebra $B$ over $E$ and by the Lebesgue measure on $(E, B)$. More precisely, let $L^1(E)$ denote the space of all (equivalence classes of) measurable functions on $E$ which are absolutely integrable with respect to the Lebesgue measure on $E$. Moreover, it will also always be clear from the context of the actual section, subsection, statement, etc. whether we are concerned with a space of complex valued functions, or if it is sufficient to deal with a space of real valued functions. By the normalization condition of (ii) and the boundary conditions in (iii), $S(t)$ maps $L^1(\Omega \times V)$ linearly to $L^1(\Omega \times V)$ with operator norm one.

3.1 The Knudsen Type Semigroup Preserves a.e. Boundedness

This subsection is entirely devoted to Lemma 3.1 below. In its proof we will apply [2], Theorem 2.1. It says that the Knudsen semigroup $S_0(t)$, $t \geq 0$, in $L^1(\Omega \times V)$ given by (2.5) for $\omega = 0$, initial condition $S_0(0)p_0 = p_0 \in L^1(\Omega \times V)$, and boundary conditions

$$(S_0(t)p_0)(r, v) = J(r, t)(S_0(\cdot)p_0)M(r, v), \quad t > 0,$$

for all $(r, v) \in \partial \Omega \times V$ with $v \circ \eta(r) \leq 0$, admits a unique non-negative stationary element $\eta_0 \in L^1(\Omega \times V)$ with $\|\eta_0\|_{L^1(\Omega \times V)} = 1$. In other words, it holds that $S_0(t)\eta_0 = \eta_0$ for all $t \geq 0$. Moreover, for any $\eta > 0$ there exists $T_0(\eta) > 0$ such that, for any $t \geq T_0(\eta)$ and for any probability density $f$ on $\Omega \times V$, we have $\|S_0(t)f - \eta_0\|_{L^1(\Omega \times V)} \leq \eta$.

The following remarks explain in which sense the setup of [2], Theorem 2.1 and its proof, is compatible with our framework.

Remark 3 Recall the hypotheses on $\partial \Omega$ from the beginning of Section 2. In particular, recall the notion of $\partial^{(1)}\Omega = \bigcup_{i=1}^{n_\Omega} \Gamma_i$.

Theorem 2.1 of [2] is formulated and proved for a domain $\Omega$ with “sufficiently smooth” boundary. The proof can be rewritten for our hypotheses on the boundary $\partial\Omega$ by replacing “$\partial\Omega$” there with $\partial^{(1)}\Omega$. Furthermore, if a formula or statement in the proof of Theorem 2.1 in [2] is formulated for “all” points belonging to $\partial\Omega$, it is now valid for a.e. all such points. In addition, the Assumptions A and B within the proof of Theorem 2.1 of [2] are formulated using the notation of the ongoing proof. This implies that the points $y, y', y_1 \in \partial\Omega$ appearing in Assumption A are arranged in a way that the straight connections between the points $y$ and $y_1$ and between the points $y'$ and $y_1$, excluding $y, y', y_1$, lie entirely in $\Omega$. For our purposes, Assumption A has to be reformulated as follows.

There exists an $\varepsilon > 0$ such that for all $y, y' \in \partial^{(1)}\Omega$, there is a $y_1 \in \partial^{(1)}\Omega$ with

$$\min(|y_1 - y|, |y_1 - y'| \geq \varepsilon \text{ and},$$

$$\text{(A.4)} \quad \min(|y_1 - y|, |y_1 - y'| \geq \varepsilon \text{ and},$$

$$\text{(A.5)} \quad \text{denoting } e(y, y_1) := (y_1 - y)/|y_1 - y| \text{ and similarly } e(y', y_1), \text{ it holds that}$$

$$|e(y', y_1) \circ n(y')| \geq \varepsilon, \quad |e(y, y_1) \circ n(y_1)| \geq \varepsilon, \quad |e(y, y_1) \circ n(y)| \geq \varepsilon.$$
In this sense, the Assumptions A and B formulated of the proof of [2], Theorem 2.1, are satisfied in our paper by (2.1) and (ii).

**Remark 4** There is just one minor change necessary in the proof of [2], Theorem 2.1, to show the following. There is a non-negative element \( \eta \in L^1(\Omega \times V) \) with \( \|\eta\|_{L^1(\Omega \times V)} = 1 \) which is stationary under the Knudsen type semigroup \( S(t) \), \( t \geq 0 \), i.e.

\[
S(t)\eta = \eta, \quad \text{for all } t \geq 0.
\]

Furthermore, for any \( \eta > 0 \) there exists \( T(\eta) > 0 \) such that, for any \( t \geq T(\eta) \) and for any probability density \( f \), we have

\[
\|S(t)f - \eta\|_{L^1(\Omega \times V)} \leq \eta.
\]

This change is the subsequent one. Using our notation, the two sentences after (A.8) in [2] have to be updated as follows. The velocities \( u = (y - y_1)/t^* \) and \(-u\) allow us to go both directions from the boundary points \( y_1 \) and respectively \( y \) with exit probability densities uniformly bounded from below. Indeed, according to (A.5) and (A.6), we have for \( u \) and \( y_1 \)

\[
\omega \left| \frac{u \circ n(y_1)|\eta(y_1, R_y(u), \cdot)|}{J(y_1, \cdot)} \right| + (1 - \omega) |u \circ n(y_1)|M(y_1, u)
\geq (1 - \omega)\|u\| \left| n(y_1) \right| M(y_1, u) \geq (1 - \omega)v_{\min} \varepsilon_{\min}.
\]

By means of this modification of the proof of [2], Theorem 2.1, we have verified the existence of a non-negative function \( \eta \in L^1(\Omega \times V) \) with \( \|\eta\|_{L^1(\Omega \times V)} = 1 \) and (3.1), (3.2).

**Lemma 3.1** Let \( \omega \in (0, 1) \) and \( p_0 \in L^\infty(\Omega \times V) \). There are finite real numbers \( p_{0, \text{min}} \) and \( p_{0, \text{max}} \) such that

\[
p_{0, \text{min}} \leq S(t) p_0 \leq p_{0, \text{max}} \quad \text{a.e. on } \Omega \times V
\]

for all \( t \geq 0 \). In particular, if \( p_0 \geq 0 \) and \( \|1/p_0\|_{L^\infty(\Omega \times V)} < \infty \) then we may suppose \( p_{0, \text{min}} > 0 \).

Proof. **Step 1** Without loss of generality, we may suppose that \( \eta : \Omega \times V \to [0, \infty] \) is defined everywhere on \( \Omega \times V \) such that for every \( (r, v) \in \Omega \times V \) there are \( a \equiv a(r, v) < 0 \) and \( b \equiv b(r, v) > 0 \) with \( r + av \in \partial \Omega \) and \( r + bv \in \partial \Omega \), \( \{r + cv : c \in (a, b)\} \subset \Omega \), and \( \eta(r + cv, v) = \eta(r, v) \equiv \eta(r, v), c \in (a, b) \). In addition, we even may suppose that \( \eta(r + av, v) = \eta(r + bv, v) = \eta(r, v) \).

For such a boundary point \( y = r + bv \) we recall the notation \( y^-(y, v) = r + av \), see (2.10).

Let \( L^1_w \) be the space of all equivalence classes of measurable functions \( f \) defined on \( (y, v) \in \partial \Omega \times V \) with \( v \circ n(y) \leq 0 \) such that

\[
\|f\|_{L^1_w} := \int_{y \in \partial(1) \Omega} \int_{v \circ n(y) \leq 0} |v \circ n(y)| f(y, v) \, dv \, dy < \infty.
\]

Furthermore, the map \( Sf(y, v) := f(y^-(y, R_y(v)), R_y(v)) = f(\sigma(y, v)), (y, v) \in \partial \Omega \times V \) with \( v \circ n(y) \leq 0 \), is by

\[
\|Sf\|_{L^1_w} = \int_{y \in \partial(1) \Omega} \int_{v \circ n(y) \leq 0} |v \circ n(y)| \left| f(y^-(y, R_y(v)), R_y(v)) \right| \, dv \, dy
\]
We obtain
\[
\int_{v \in \partial^{(1)} \Omega} \int_{t_{\min}}^{t_{\max}} y \circ n(y) \left| f \left( y^-(y, v), R_y(-\alpha e), R_y(-\alpha e) \right) \right| \, \rho(y) \, dv \, dy
\]

Furthermore, we note that
\[
\left| f \left( (v-y)^-, v \right) \right| \leq L_y \left| v \right| (v-y)^- \| f \|_{L_w}
\]

a linear operator \( S : L^1_w \to L^1_w \) with operator norm one. For this calculation we have used \( e \circ n(y) = R_y(-e) \circ n(y) \) as well as \( de = dR_y(-e) \) in order to obtain the fourth line from the third.

**Step 2** Next we aim to demonstrate that \( \overline{f} \in L^1_w \). For this we denote \((\partial \Omega)_y := \{ r \in \partial^{(1)} \Omega : r + \alpha(y-r) : \alpha \in (0,1) \} \subset \Omega \), \( y \in \Omega \), and \((\Omega)_r := \{ y \in \Omega : r + \alpha(y-r) : \alpha \in (0,1) \} \subset \Omega \), \( r \in \partial \Omega \). Furthermore, we introduce
\[
\rho_d(r) = \int_{y \in (\Omega)_r} |y-r|^{1-d} \cdot \frac{n(r) \circ (r-y)}{|y-r|} \, dy, \quad r \in \partial^{(1)} \Omega,
\]

and observe that, by the piecewise smoothness of \( \partial \Omega \), there is a constant \( c_d > 0 \) only depending on \( \Omega \) such that \( c_d \leq \rho_d(r) < \infty \) for all \( r \in \partial \Omega \). Set \( C_{v,d} := (v_{\max} - v_{\min})/d \) as well as \( C_d := (1 - \omega)M_{\min} \cdot C_{v,d} \), and denote by \( L \) the Lebesgue measure on \((S^{d-1}, B(S^{d-1}))\).

We will write \( v = \alpha e \) where \( \alpha \in (v_{\min}, v_{\max}) \) and \( e \in S^{d-1} \). Here we mention that \( y^-(y, \alpha e) \in \partial \Omega \) is independent of \( \alpha \in (v_{\min}, v_{\max}) \) and therefore may appear as \( y^-(y, \cdot e) \). We obtain
\[
1 = \int_{\Omega} \int_{V} \overline{f}(y, v) \, dv \, dy = \int_{\Omega} \int_{V} \overline{f}(y^-(y, v), v) \, dv \, dy
\]

Furthermore, we note that
\[
J(y, \cdot)(\overline{f}) = \int_{v \circ n(y) \geq 0} v \circ n(y) \overline{f}(y, v) \, dv
\]
Introducing $k$ all In other words, we have

These boundary conditions on $y$ that is independent of $y$ for all boundary conditions on $\Omega$. In this step we apply the results of Steps 1 and 2. According to (iii) we have the

**Step 3** In this step we apply the results of Steps 1 and 2. According to (iii) we have the boundary conditions on $\bar{g}$

$$
\bar{g}(y,v) = (1 - \omega) J(y,\cdot)(\bar{g}) \cdot M(y,v), \quad y \in \partial \Omega, \quad v \circ n(y) \leq 0.
$$

These boundary conditions on $\bar{g}$ can be rewritten as

$$
\bar{g}(y,v) = (1 - \omega) M(y,v) J(y,\cdot)(\bar{g}) + \omega \bar{g}(y,R_y(v))
= (1 - \omega) M(y,v) J(y,\cdot)(\bar{g}) + \omega \bar{g}(y,R_y(v),R_y(v))
= (1 - \omega) M(y,v) J(y,\cdot)(\bar{g}) + \omega (S\bar{g})(y,v), \quad y \in \partial \Omega, \quad v \circ n(y) \leq 0.
$$

Together with the just shown $\bar{g} \in L_w^1$ and (3.4) the latter says that, among other things, that $MJ(\bar{g}) \in L_w^1$. Therefore

$$
\bar{g} = (1 - \omega) \sum_{k=0}^\infty \omega^k S^k(MJ(\bar{g}))
= (1 - \omega) MJ(\bar{g}) + (1 - \omega) \sum_{k=0}^\infty \omega^{k+1} S^{k+1}(MJ(\bar{g}))
$$

a.e. on $\{(y,v) \in \partial \Omega \times V : v \circ n(y) \leq 0\}$, where the infinite sums converge in $L_w^1$.

**Step 4** The next two steps are devoted to upper bounds of $J(y,\cdot)(\bar{g})$. In the present one we still construct upper bounds depending on $y \in \partial \Omega$. In Step 5 below we will derive a bound that is independent of $y$ and use it to show $||\bar{g}||_{L^\infty(\Omega \times V)} < \infty$.

Taking into consideration that according to (iii), there exist constants $M_{\min}, M_{\max} \in (0,\infty)$ such that $M_{\min} \leq M(y,v) \leq M_{\max}$, it turns out that $S^k M$ is uniformly bounded for all $k \in \mathbb{Z}_+$. Moreover extending $J(\bar{g})$ to all $(y,v) \in \partial \Omega \times V$ with $v \circ n(y) \leq 0$ by $J(\bar{g})(y,v) := J(y,\cdot)(\bar{g})$, from (3.4) and $MJ(\bar{g}) \in L_w^1$ we may conclude that $S^k J(\bar{g}) \in L_w^1$ for all $k \in \mathbb{Z}_+$. It follows now from (3.9) and (3.3) that

$$
J(y,\cdot)(\bar{g}) = (1 - \omega) \int_{v \circ n(y) \leq 0} |v \circ n(y)| \sum_{k=0}^\infty \omega^k S^{k+1}(MJ(\bar{g}))(y,v) \, dv, \quad y \in \partial \Omega.
$$

Introducing

$$
\delta(k,y,\omega) := (1 - \omega) \int_{w \circ n(y) \leq 0} |w \circ n(y)| S^k M(y,w) \, dw, \quad k \in \mathbb{N},
$$

(3.11)
we obtain from (3.10) that

\[
(1 - \omega) \int_{w(y) \leq 0} |w \circ n(y)| S(MJ(\theta))(y, w) \, dw \\
= (1 - \omega) \frac{\delta(1, y, \omega)}{1 - \delta(1, y, \omega)} \cdot \int_{v(y) \leq 0} |v \circ n(y)| \sum_{k=1}^{\infty} \omega^k S^{k+1}(M)J(\theta))(y, v) \, dv, \quad y \in \partial^{(1)} \Omega.
\]

Inserting this in (3.10) gives

\[
J(y, \cdot)(\theta) = \frac{1 - \omega}{1 - \delta(1, y, \omega)} \int_{v(y) \leq 0} |v \circ n(y)| \sum_{k=1}^{\infty} \omega^k S^{k+1}(M)(\theta))(y, v) \, dv.
\]  

(3.12)

Letting \( k_0 \) be the integer introduced in condition (viii) and iterating the calculations from (3.10) to (3.12) \( k_0 \) times we verify

\[
\theta(y) \equiv \theta(k_0, y, \omega) := \sum_{k=1}^{k_0} \omega^{k-1} \delta(k, y, \omega) \leq 1
\]

as well as

\[
J(y, \cdot)(\theta) = \frac{1 - \omega}{1 - \theta(y)} \int_{v(y) \leq 0} |v \circ n(y)| \sum_{k=k_0}^{\infty} \omega^k S^{k+1}(M)(\theta))(y, v) \, dv, \quad y \in \partial^{(1)} \Omega.
\]  

(3.13)

Similarly we also obtain \( \theta(k_0, y, \omega) + \omega^{k_0} \delta(k_0+1, y, \omega) = \theta(k_0+1, y, \omega) \leq 1 \). By \( M \geq M_{\text{min}} > 0 \) we get \( S^{k_0+1}M \geq M_{\text{min}} > 0 \) and therefore from (3.11)

\[
(1 - \omega)M_{\text{min}} \int_{w(y) \leq 0} |w \circ n(y)| \, dw \leq \delta(k_0 + 1, y, \omega), \quad y \in \partial^{(1)} \Omega.
\]

Thus

\[
\theta(y) \equiv \theta(k_0, y, \omega) \leq 1 - \omega^{k_0}(1 - \omega)M_{\text{min}} \int_{w(y) \leq 0} |w \circ n(y)| \, dw, \quad y \in \partial^{(1)} \Omega,
\]  

(3.14)

where we observe that the right-hand side does not depend on \( y \) and is smaller than one. In other words, (3.14) implies the existence of \( \kappa < 1 \) such that

\[
\theta(y) \equiv \theta(k_0, y, \omega) \leq \kappa < 1, \quad y \in \partial^{(1)} \Omega.
\]

Furthermore according to (ii), there exists \( 0 < M_{\text{max}} < \infty \) such that \( M(y, v) \leq M_{\text{max}} \) and hence \( S^{k+1}M(y, v) = M(\sigma^{k+1}(y, v)) \leq M_{\text{max}} \) for \( (y, v) \in \partial^{(1)} \Omega \times V \) with \( v \circ n(y) \leq 0 \).

From (3.13) it follows now that

\[
J(y, \cdot)(\theta) \leq \frac{1 - \omega}{1 - \kappa} \sum_{k=k_0}^{\infty} \omega^k \int_{v(y) \leq 0} |v \circ n(y)| S^{k+1}M(y, v) \cdot S^{k+1}J(\theta))(y, v) \, dv \\
\leq \frac{1 - \omega}{1 - \kappa} M_{\text{max}} \sum_{k=k_0}^{\infty} \omega^k \int_{v(y) \leq 0} |v \circ n(y)| S^{k+1}J(\theta)(y, v) \, dv \quad y \in \partial^{(1)} \Omega.
\]  

(3.15)
Step 5 In this step we show \( \|g\|_{L^\infty(\Omega \times V)} < \infty \). Recall the notation from the end of Section 2. Let \((y, v) \in \partial^{(1)} \Omega \times V\) with \(v \circ n(y) \leq 0\). For \(e := -v/|v| \in S_{+}^{d-1}(n(y))\), let \(r := y - (y, e)\), i.e. \(e = (y - r)/|y - r|\). In the remainder of this step we suppose \(r \in \partial^{(1)} \Omega\) and iteratively also \(y^{(j)} \in \partial^{(1)} \Omega\), \(j \in \mathbb{N}\), and we indicate this in the text by the phrase “for a.e. \(y \in \partial^{(1)} \Omega\)”.

Furthermore, denote \(e_k := -v^{(k)}/|v^{(k)}|\), \(k \in \mathbb{N}\). For a.e. \(y \in \partial^{(1)} \Omega\) introduce the distance

\[
|y - y^{(k)}|_k := \sum_{j=1}^{k} |y^{(j)} - y^{(j-1)}|, \quad k \in \mathbb{N}.
\]

Note that

\[
\frac{dy^{(k)}}{dr} = \frac{dy^{(k)}}{de_1} / \frac{dr}{de_1} = \frac{dy^{(k)}}{de_1} / \frac{dr}{de} = \frac{|y - y^{(k)}|_k e \circ n(r)|}{|y - r|^{d-1} e_k \circ n(y^{(k)})|}, \quad k \in \mathbb{N},
\]

where

\[
\frac{dr}{de_1} = \frac{dr}{de} \cdot \frac{de}{de_1} = \frac{dr}{de}
\]

by symmetry of \(e\) and \(-e_1\) about \(n(y)\), and

\[
\frac{dy^{(k)}}{de_1} = \frac{|y - y^{(k)}|_k}{|e_k \circ n(y^{(k)})|}
\]

is usually motivated by means of a ray from \(y\) in direction of \(e_1\) as follows. The ray passes through \(\Omega\) until it hits \(y^{(1)} \in \partial^{(1)} \Omega\). Then \(\Omega\) is reflected about the straight line \((d = 2)\) or plane \((d = 3)\) orthogonal to \(n(y^{(1)})\) and containing \(y^{(1)}\). In this way the ray passes through \(k - 1\) more consecutively reflected copies of \(\Omega\).

With these preparations in mind we obtain for a.e. \(y \in \partial^{(1)} \Omega\) and \(k \in \mathbb{N}\)

\[
\int_{v \circ n(y) \leq 0} |v \circ n(y)| S^k J(y)(y, v) \, dv = \int_{v \circ n(y) \leq 0} |v \circ n(y)| J(y^{(k)}, v^{(k)}) \, dv
\]

= \(\int_{S_{+}^{d-1}} \int_{v_{\text{max}}}^{v_{\text{min}}} \alpha^d e \circ n(y) J(y^{(k)}, \cdot)(\tilde{g}) \, d\alpha \, de\)

= \(C_{v, d+1} \int_{S_{+}^{d-1}} e \circ n(y) J(y^{(k)}, \cdot)(\tilde{g}) \, de\)

= \(C_{v, d+1} \int_{r \in \partial \Omega} e \circ n(y) \cdot |e \circ n(r)| J(y^{(k)}, \cdot)(\tilde{g}) \, dr\)

= \(C_{v, d+1} \int_{r \in \partial \Omega} e \circ n(y) \cdot |e \circ n(r)| \int_{w \circ n(y^{(k)}) \leq 0} |w \circ n(y^{(k)})| \tilde{g}(y^{(k)}, w) \, dw \, dr\)

= \(C_{v, d+1} \int_{y^{(k)} \in \partial \Omega} e \circ n(y) \cdot |e \circ n(y^{(k)})| \int_{w \circ n(y^{(k)}) \leq 0} |w \circ n(y^{(k)})| \tilde{g}(y^{(k)}, w) \, dw \, dy^{(k)}\).

where we have applied (3.6) in the second last line. By condition (viii) we have for all \(k \geq k_0\)

\[
\frac{e \circ n(y) \cdot |e_k \circ n(y^{(k)})|}{|y - y^{(k)}|^{d-1}_k} \leq \sigma_{\text{min}}^{-d}
\]
for a.e. \((y, v) \in \partial^{(1)} \Omega \times V\) with \(v \circ n(y) \leq 0\). Thus, for a.e. \(y \in \partial^{(1)} \Omega\),

\[
\int_{\text{von}(y) \leq 0} |v \circ n(y)| S^k J(\overline{g})(y, v) \, dv \\
\leq C_{v, d+1} \sigma_{\min}^{-1-d} \int_{y(k) \in \partial \Omega} \int_{\text{von}(y(k)) \leq 0} |w \circ n(y^{(k)})| |\overline{g}(y^{(k)}, w)| \, dw \, dy^{(k)} \\
= C_{v, d+1} \sigma_{\min}^{-1-d} \|\overline{g}\|_{L^1}, \quad k \geq k_0.
\]

(3.16)

It follows now from (3.7), (3.15), and (3.16) that, for a.e. \((y, v) \in \partial^{(1)} \Omega\),

\[
J(y, \cdot)(\overline{g}) \leq C_{v, d+1} \sigma_{\min}^{-1-d} M_{\max} \frac{\omega_{\max}}{1 - \kappa} \cdot \|\overline{g}\|_{L^1} =: C_J < \infty.
\]

Noting that this implies \(S^k(MJ(\overline{g}))(y, v) \leq M_{\max} C_J\) for a.e. \((y, v) \in \partial^{(1)} \Omega \times V\) with \(v \circ n(y) \leq 0\) and \(k \in \mathbb{Z}_+\), we may now conclude from (3.9) and the first paragraph of Step 1 that

\[
\|\overline{g}\|_{L^\infty(\Omega \times V)} \leq M_{\max} C_J < \infty.
\]

**Step 6** Let us demonstrate \(\|1/\overline{g}\|_{L^\infty(\Omega \times V)} < \infty\). For \(1 > \varepsilon > 0\) let \(C_\Omega(y, \varepsilon)\) denote the open cone in \(\mathbb{R}^d\) with vertex \(y \in \partial^{(1)} \Omega\) given by \(\{x \in \mathbb{R}^d : ((y - x)/|x - y|) \circ n(y) > \varepsilon\}\). Furthermore, for \(y \in \partial^{(1)} \Omega\) introduce

\[
(\partial \Omega)_{y, \varepsilon} := \{r \in (\partial \Omega)_y \cap C_\Omega(y, \varepsilon) : ((r - y)/|r - y|) \circ n(r) > \varepsilon\}.
\]

Using the notation \(e := (y - r)/|y - r|\), for every \(1 > \varepsilon > 0\) it holds that

\[
\inf_{y \in \partial^{(1)} \Omega, r \in (\partial \Omega)_{y, \varepsilon}} \left\{ \frac{e \circ n(y) \cdot |e \circ n(r)|}{|r|^{d-1}} \right\} =: c_\varepsilon > 0.
\]

As already explained in the beginning of Step 1 of this proof we have \(\overline{g}(y^-, y, v) = \overline{g}(y, v)\) for \(v \circ n(y) \geq 0\) and \(y \in \partial^{(1)} \Omega\). Using the boundary conditions (3.8) we obtain now

\[
J(y, \cdot)(\overline{g}) = \int_{\text{von}(y) \geq 0} v \circ n(y) \cdot \overline{g}(y, v) \, dv \\
= \int_{\text{von}(y) \geq 0} v \circ n(y) \cdot \overline{g}(y^-(y, v), v) \, dv \\
\geq (1 - \omega) M_{\min} \int_{\text{von}(y) \geq 0} v \circ n(y) \cdot J(y^-(y, v), \cdot)(\overline{g}) \, dv \\
= (1 - \omega) M_{\min} \int_{S^{d-1}(n(y))} \int_{v_{\min}}^{v_{\max}} \alpha^d e \circ n(y) \cdot J(y^-(y, \alpha e), \cdot)(\overline{g}) \, d\alpha \, de \\
= (1 - \omega) M_{\min} C_{v, d} \int_{r \in (\partial \Omega)_y} \frac{|(r - y) \circ n(y)| \cdot (r - y) \circ n(r)}{|y - r|^{d+1}} J(r, \cdot)(\overline{g}) \, dr \\
\geq c_\varepsilon (1 - \omega) M_{\min} C_{v, d} \int_{r \in (\partial \Omega)_{y, \varepsilon}} J(r, \cdot)(\overline{g}) \, dr, \quad y \in \partial^{(1)} \Omega.
\]

(3.17)
Assume that there exist \( y \in \partial \Omega \) and a sequence \( y_k \in \partial^{(1)} \Omega \), \( k \in \mathbb{N} \), with \( y_k \xrightarrow{k \to \infty} y \) as well as \( J(y_k, \cdot)(\overline{g}) \xrightarrow{k \to \infty} 0 \). From (3.17) it follows that for every \( \varepsilon > 0 \) and \( \delta > 0 \) there is a \( k \in \mathbb{N} \) such that

\[
\delta \geq \int_{r \in (\partial \Omega) y_k, \varepsilon} J(r, \cdot)(\overline{g}) \, dr.
\]

Thus, \( J(r, \cdot)(\overline{g}) = 0 \) for a.e. \( r \in (\partial \Omega) y \). Plugging this in the left-hand side of (3.17), and iterating the last conclusion, we may even state that \( J(r, \cdot)(\overline{g}) = 0 \) for a.e. \( r \in \partial^{(1)} \Omega \).

Recalling (3.6) and introducing \( r^{-}(r, v) \) as \( y^{-}(y, v) \) in (2.10), it turns out that

\[
0 = \int_{r \in \partial^{(1)} \Omega} J(r, \cdot)(\overline{g}) \, dr = \int_{r \in \partial^{(1)} \Omega} \int_{|v| < r} |v \circ n(r)| \overline{g}(r, v) \, dv \, dr
\]

\[
= \int_{v \in V} \int_{r \in \partial^{(1)} \Omega} |v \circ n(r)| \overline{g}(r, v) \, dv \, dr
\]

\[
= \int_{v \in V} \int_{r \in \partial^{(1)} \Omega} \frac{|v \circ n(r)|}{r - r^{-}(r, v)} \int_{0}^{r - r^{-}(r, v)} \overline{g}(r + \alpha v/|v|, v) \, d\alpha \, dr \, dv
\]

\[
= \int_{V} \int_{\Omega} \frac{1}{|r^{-}(r, v) - r^{-}(r, v)|} \overline{g}(r, v) \, dr \, dv.
\]

In other words, the above assumption would lead to \( \overline{g} = 0 \) a.e. on \( \Omega \times V \). This proves the existence of a lower bound \( c_J > 0 \) on \( J(r, \cdot)(\overline{g}) \), uniformly for a.e. \( r \in \partial^{(1)} \Omega \).

This yields \( S^k(MJ(\overline{g}))(y, v) \geq M_{\min} c_J > 0 \) for a.e. \( (y, v) \in \partial^{(1)} \Omega \times V \) with \( v \circ n(y) \leq 0 \) and \( k \in \mathbb{Z}_+ \). It follows now from (3.3) and the first paragraph of Step 1 that

\[
\|\overline{g}\|_{L^\infty(\Omega \times V)} \geq M_{\min} c_J > 0,
\]

i.e. \( 1/\|\overline{g}\|_{L^\infty(\Omega \times V)} < \infty \).

**Step 7** Let \( p_0 \in L^\infty(\Omega \times V) \). By the result of Step 6 there exists \( a > 0 \) with \( -a \overline{g} \leq p_0 \leq a \overline{g} \) a.e. on \((\Omega \times V)\). Furthermore, in case of \( p_0 \geq 0 \) and \( 1/p_0 \) \( L^\infty(\Omega \times V) \) < \( \infty \), there is \( b > 0 \) such that \( b \overline{g} \leq p_0 \) a.e. on \((\Omega \times V)\), see Step 5. This implies

\[
-a \|\overline{g}\|_{L^\infty(\Omega \times V)} \leq -a \overline{g} \leq S(t)p_0 \leq a \overline{g} \leq a \|\overline{g}\|_{L^\infty(\Omega \times V)} \quad \text{a.e. on } (\Omega \times V) \text{ for all } t \geq 0
\]

and, if \( p_0 \geq 0 \) and \( 1/p_0 \|_{L^\infty(\Omega \times V)} \) < \( \infty \), also

\[
0 < b \cdot \text{ess inf } \overline{g}(y, w) \leq b \overline{g} \leq S(t)p_0 \quad \text{a.e. on } (\Omega \times V) \text{ for all } t \geq 0.
\]

The lemma follows.

\[\square\]

### 3.2 Construction of the Knudsen Type Semigroup

Let us now introduce the concept of rays and paths. Let \((r, v) \in \Omega \times V\) or \((r, v) \in \partial^{(1)} \Omega \times V\) with \( v \circ n(r) \geq 0 \). The map \([\tau_0, \tau_1] \ni \tau \mapsto r - \tau v \) with \( 0 \leq \tau_0 < \tau_1 \leq \infty \) is called a ray. We mention that the time \( T_\Omega \equiv T_\Omega(r, v) = \inf \{ s > 0 : r - sv \notin \Omega \} \) can be interpreted as the first exit time from \( \Omega \) of the ray \([0, \infty) \ni \tau \mapsto r - \tau v \).

Suppose we are given \( t \geq 0 \) and \((r, v) \equiv (r_0, v_0) \in \Omega \times V\). If \( T_\Omega(r, v) < t \) and \( r_1 := r - T_\Omega(r, v)v \in \partial^{(1)} \Omega \) then from the point \( r_1 \) we simultaneously follow all rays \([0, \infty) \ni \tau \mapsto \)
\( r_1 - \tau v_1 \) for which \( v_1 \circ n(r_1) \geq 0 \) until \( T_{\Omega}(r, v) + \tau = t \leq T_{\Omega}(r, v) + T_{\Omega}(r_1, v_1) \) or these rays exit from \( \Omega \) for the first time at some \( r_2 \in \partial \Omega \).

We start over, simultaneously from all \( r_2 \in \partial^{(1)} \Omega \), along all rays \( [0, \infty) \ni \tau \mapsto r_2 - \tau v_2 \) for which \( v_2 \circ n(r_2) \geq 0 \), and continue in this manner until \( \sum_{l=0}^{m-1} T_{\Omega}(r_l, v_l) < t \) but \( \sum_{l=0}^{m} T_{\Omega}(r_l, v_l) \geq t \) for some \( m \in \mathbb{Z}_+ \). Here we use the convention \( \sum_{l=0}^{-1} = 0 \). We note that \( m \) depends on \( r_0, v_0, v_1, \ldots, v_{m-1} \).

Any ray \( [0, \infty) \ni \tau \mapsto r_m - \tau v_m \) takes at time \( \tau = t - \sum_{l=0}^{m-1} T_{\Omega}(r_l, v_l) \) the value of some point \( r_e \in \overline{\Omega} \). In this way, we have constructed sequences or rays for which we now consider only the restrictions to \([0, T_{\Omega}(r_0, v_0))\), \([0, T_{\Omega}(r_1, v_1))\), \ldots, \([0, T_{\Omega}(r_{m-1}, v_{m-1}))\), \([0, t - \sum_{l=0}^{m-1} T_{\Omega}(r_l, v_l)]\).

To prepare the next definition we re-parametrize the rays to maps over consecutive intervals \( \tau \in [0, T_{\Omega}(r_0, v_0))\), \( \tau \in [T_{\Omega}(r_0, v_0), T_{\Omega}(r_0, v_0) + T_{\Omega}(r_1, v_1))\), \ldots, \( \tau \in [\sum_{l=0}^{m-1} T_{\Omega}(r_l, v_l), t)\). It is important to understand that \( \tau \) stands for reversed time from \( t \) to zero, i.e. at time \( t \) we have \( \tau = 0 \) and at zero time we have \( \tau = t \).

**Definition 3.2** Let \( t > 0 \) and \((r, v) \in \Omega \times V\).

(a) A path \( \pi \) with time range \([0, t]\) pinned in at time \( t \) in \((r, v)\) is a finite collection of re-parametrized rays of the form

\[
\left[ \sum_{l=0}^{k-1} T_{\Omega}(r_l, v_l), \sum_{l=0}^{k} T_{\Omega}(r_l, v_l) \right] \ni \tau \mapsto r_k - \left( \tau - \sum_{l=0}^{k-1} T_{\Omega}(r_l, v_l) \right) v_k , \tag{3.18}
\]

\( k = 0, \ldots, m-1 \), such that \( \sum_{l=0}^{m-1} T_{\Omega}(r_l, v_l) < t \) as well as \( \sum_{l=0}^{m} T_{\Omega}(r_l, v_l) \geq t \) and

\[
\left[ \sum_{l=0}^{m-1} T_{\Omega}(r_l, v_l), t \right] \ni \tau \mapsto r_m - \left( \tau - \sum_{l=0}^{m-1} T_{\Omega}(r_l, v_l) \right) v_m , \tag{3.19}
\]

\( m \in \mathbb{Z}_+ \), where we use the convention \( \sum_{l=0}^{-1} = 0 \). Here, we suppose

\[ v_k \circ n(r_k) \geq 0 , \quad r_k = r_{k-1} - T_{\Omega}(r_{k-1}, v_{k-1})v_{k-1} \in \partial^{(1)} \Omega , \quad k = 1, \ldots, m, \]

and \( r_0 := r \) as well as \( v_0 := v \). Furthermore, \( v_e := v_m \) and we suppose

\[ r_e := r_m - \left( t - \sum_{l=0}^{m-1} T_{\Omega}(r_l, v_l) \right) v_m \in \Omega . \]

(b) For fixed \( r, v, t \) as above, let \( \pi(r, v, t) \) be the set of all paths \( \pi \) with time range \([0, t]\) pinned at \( t \) in \((r, v)\).

**Remark 5** Let \( p_0 \) be a probability density everywhere defined on \( \Omega \times V \). A more intuitive explanation of the term path \( \pi \) with time range \([0, t]\) pinned at time \( t \) in \((r, v)\) is to follow \( S(t)p_0 \) along a path \( \pi \) backward in time. This is from time \( t \) to time \( 0 \), with the understanding that starting with \( S(t)p_0(r, v) \), after \( \tau \in [0, t] \) units backward in time we have arrived at some \( S(t - \tau)p_0(r', v') \). In particular, we keep in mind the boundary conditions of (iii), together with (i). In this sense, a path \( \pi \) consists of \( m + 1 \) re-parametrized rays along each of which \( S(t - \cdot)p_0 \) is constant. These are

- if \( T_{\Omega}(r_0, v_0) \equiv T_{\Omega}(r, v) < t \), i.e. \( m \geq 1 \), the ray \( r_0 - \tau v_0 \) from \( r = r_0 \in \Omega \) to \( r_1 \in \partial^{(1)} \Omega \) with velocity \( v_0 = v \), i.e. over the time range \( \tau \in [0, T_{\Omega}(r_0, v_0)) \),

- if \( \sum_{l=0}^{k} T_{\Omega}(r_l, v_l) < t \), the re-parametrized ray \( r_k - \left( \tau - \sum_{l=0}^{k-1} T_{\Omega}(r_l, v_l) \right) v_k \) from \( r_k \in \partial^{(1)} \Omega \) to \( r_{k+1} \in \partial^{(1)} \Omega \) with velocity \( v_k \), and therefore over the time range of (3.18), \( k = 1, \ldots, m - 1 \), if \( m \geq 1 \),

17
the proofs of Lemma 3.3 and Theorem 4.5.

Below we will frequently refer to Remark 5 and the following Remark 6, for example in the proofs of Lemma 3.3 and Theorem 4.5.
Remark 6 The subsequent explicit representation of the semigroup $S(t)$, $t \geq 0$, is a consequence of (3.20). Let $p_0$ be a probability density everywhere defined on $\Omega \times V$ and let $(r, v, t) \in \Omega \times V \times [0, \infty)$. We have

$$S(t)p_0(r, v) = \chi_{\{0\}}(m)p_0(r_e, v_0)$$

$$+ \chi_{\{1\}}(m) \left( \omega p_0(r_e, R_{r_1}(v_0)) + (1 - \omega)M(r_1, v_0) \int_{v_1 \in (r_1, \infty)} v_1 \circ n(r_1)p_0(r_e, v_1) dv_1 \right)$$

$$+ \chi_{\{2\}}(m) \left( \omega \left[ \omega p_0(r_e, R_{r_2}(R_{r_1}(v_0))) \right. \right.$$

$$\left. + (1 - \omega)M(r_2, R_{r_1}(v_0)) \int_{v_2 \in (r_2, \infty)} v_2 \circ n(r_2)p_0(r_e, v_2) dv_2 \right)$$

$$\left. + (1 - \omega)M(r_1, v_0) \int_{v_1 \in (r_1, \infty)} v_1 \circ n(r_1) \times \right.$$  

$$\left[ \omega p_0(r_e, R_{r_2}(v_1)) + (1 - \omega)M(r_2, v_1) \int_{v_2 \in (r_2, \infty)} v_2 \circ n(r_2)p_0(r_e, v_2) dv_2 \right] dv_1 \right)$$

$$+ \chi_{\{3\}}(m) \text{-term} + \ldots, \quad (r, v, t) \in \Omega \times V \times [0, \infty), \quad (3.22)$$

where we recall that, for fixed $(r, v, t) \in \Omega \times V \times [0, \infty)$, the number $m$ is a function of $v_1, v_2, \ldots, v_e$, cf. Definition 3.2 and Remark 5.

Lemma 3.3 The semigroup $S(t)$, $t \geq 0$, is strongly continuous in $L^1(\Omega \times V)$.

Proof. Step 1 Let $p_0$ be a probability density everywhere defined on $\Omega \times V$ and let $(r, v, t) \in \Omega \times V \times [0, \infty)$. Let us use the notation of Remarks 5 and 6 and let us continue from (3.20). For $\varepsilon > 0$ set $\Omega_\varepsilon := \{ x \in \Omega : |y - x| > \varepsilon \text{ for all } y \in \partial \Omega \}$. Now, choose $\varepsilon > 0$ such that $\Omega_{\frac{\varepsilon}{2}} \neq \emptyset$.

In fact, if $t < \varepsilon/v_{\max}$ then for all $r \in \Omega_\varepsilon$ we have $T_{\Omega}(r_0, v_0) = T_{\Omega}(r, v) > t$. Consequently $m = 0$ and $(r, v) = (r_e + tv_e, v_e) = (r_e + tv, v)$. Furthermore, (3.20) implies for sufficiently small $\varepsilon > 0$

$$S(t)p_0(r, v) = p_0(r_e, v_e) = p_0(r - tv, v), \quad r \in \Omega_\varepsilon, \quad v \in V, \quad t < \varepsilon/v_{\max}. \quad (3.23)$$

We also observe that $\bar{p}_0 = p_0$ a.e. on $\Omega \times V$ implies $S(t)\bar{p}_0 = S(t)p_0$ for all $t \geq 0$.

Step 2 Let $p_0 \in L^\infty(\Omega \times V)$ and let $\varphi \in C_b(\Omega \times V)$ be uniformly continuous and non-negative on $\Omega \times V$. Introduce $a(t) := \sup_{v \in V} \sup_{r \in \Omega_\varepsilon} |\varphi(r - tv, v) - \varphi(r, v)|$. Choosing $t = \varepsilon/(2v_{\max})$ and recalling (3.23) we verify

$$\left| \int_{\Omega} \int_V (S(t)p_0 - p_0) \varphi \, dv \, dr \right|$$

$$\leq \left| \int_{\Omega} \int_V (p_0(r - tv, v) - p_0(r, v)) \varphi(r, v) \, dv \, dr \right| + \int_{\Omega \setminus \Omega_\varepsilon} \int_V (S(t)p_0 + p_0) \varphi \, dv \, dr$$

$$\leq \left| \int_{\Omega_\varepsilon} \int_V (p_0(r - tv, v)\varphi(r - tv, v) - p_0(r, v)\varphi(r, v)) \, dv \, dr \right|$$

$$+ a(t) \int_{\Omega_\varepsilon} \int_V p_0(r - tv, v) \, dv \, dr + \int_{\Omega \setminus \Omega_\varepsilon} \int_V (S(t)p_0 + p_0) \varphi \, dv \, dr$$

19
\[
\leq \|\varphi\|_V \left( \int_{\Omega \setminus \Omega_2} p_0 \, dv \, dr + a(t) + \|\varphi\|_V + \int_{\Omega \setminus \Omega_2} S(t)p_0 \, dv \, dr + \|\varphi\|_V + \int_V p_0 \, dv \, dr \right) \\
\leq 2\|\varphi\|_V \left( \int_{\Omega \setminus \Omega_2} p_0 \, dv \, dr + a(t) + \|\varphi\|_V \left( 1 - \int_{\Omega_2} \int_V p_0(r - tv) \, dv \, dr \right) \right) \\
\leq 3\|\varphi\|_V \left( \int_{\Omega \setminus \Omega_2} p_0 \, dv \, dr + a(t) \xrightarrow{t \to 0} 0 \right). \tag{3.24}
\]

According to Lemma 3.1 and \( p_0 \in L^\infty(\Omega \times V) \) there is some \( b \in (0, \infty) \) with \( \|S(t)p_0 - p_0\|_{L^\infty(\Omega \times V)} < b \) for all \( t \geq 0 \). By suitable approximation of test functions in \( L^1(\Omega \times V) \) we verify \( \int_{\Omega} \int_V (S(t)p_0 - p_0) \varphi \, dv \, dr \xrightarrow{t \to 0} 0 \) for all \( \varphi \in L^1(\Omega \times V) \). In particular,
\[
\int_{\Omega} \int_V (S(t)p_0 - p_0) \varphi \, dv \, dr \xrightarrow{t \to 0} 0, \quad \varphi \in L^\infty(\Omega \times V). \tag{3.25}
\]

**Step 3** Now let \( p_0 \in L^1(\Omega \times V) \) and \( \varphi \in L^\infty(\Omega \times V) \). Let us recall from the beginning of Section 2 that for every \( t \geq 0 \), \( S(t) \) maps \( L^1(\Omega \times V) \) into \( L^1(\Omega \times V) \) with operator norm one. By (3.25) we have for \( \varphi \in L^\infty(\Omega \times V) \), \( N \in \mathbb{N} \), and \( p_{0,N} := p_0 - (p_0 \wedge N) \)
\[
\left| \limsup_{t \to 0} \int_{\Omega} \int_V (S(t)p_0 - p_0) \varphi \, dv \, dr \right| \\
= \left| \lim_{t \to 0} \int_{\Omega} \int_V (S(t)(p_0 \wedge N) - (p_0 \wedge N)) \varphi \, dv \, dr \\
+ \limsup_{t \to 0} \int_{\Omega} \int_V (S(t)p_{0,N} - p_{0,N}) \varphi \, dv \, dr \right| \\
= \left| \limsup_{t \to 0} \int_{\Omega} \int_V (S(t)p_{0,N} - p_{0,N}) \varphi \, dv \, dr \right| \leq 2\|p_{0,N}\|_{L^1(\Omega \times V)} \|\varphi\|_{L^\infty(\Omega \times V)}.
\]

Since the right-hand side can be made arbitrarily small we have (3.25) for all \( p_0 \in L^1(\Omega \times V) \). In other words, we have \( S(t)p_0 \xrightarrow{t \to 0} p_0 \) weakly in \( L^1(\Omega \times V) \). It follows now from [11], Theorem 1.4 of Chapter 2, or [3], Theorem 1.6 of Chapter 1, that \( S(t)p_0 \xrightarrow{t \to 0} p_0 \) strongly in \( L^1(\Omega \times V) \).

## 4 Spectral Properties of the Knudsen Type Semigroup and Group

Let us suppose that the global conditions (i)-(viii) are satisfied. For \( f \in L^1(\partial \Omega \times V) \) recall the definition
\[
J(r, \cdot) (f) = \int_{\omega \cap (r) \geq 0} w \circ n(r) f(r, w) \, dw, \quad r \in \partial^{(1)} \Omega,
\]
where, as in Section 3, the notation \( J(r, \cdot) (f) \) indicates that \( f \) does not depend on \( t \). Recall also the boundary conditions
\[
f(r, v) = \omega f(r, R_r(v)) + (1 - \omega) M(r, v) J(r, \cdot) (f) \tag{4.1}
\]
Furthermore, let $L^1_u$ be the space of all equivalence classes of measurable functions defined on $\{(r, w) \in \partial(1) \Omega \times V : w \circ n(r) \geq 0\}$ such that
\[
\|f\|_{L^1_u} := \int_{r \in \partial(1) \Omega} \int_{w \circ n(r) \geq 0} w \circ n(r)|f(r, w)| \, dw \, dr < \infty.
\]

Obviously the restriction of $f \in L^1_u$ to $\{(r, w) \in \partial(1) \Omega \times V : w \circ n(r) \geq 0\}$ belongs to $L^1_u$. In addition every $f \in L^1_u$ can uniquely be extended to an element of $L^1_b$ by the boundary conditions (4.1). Let us introduce a map $U$ defined on $L^1_u$ by
\[
Uf(r, w) := \omega f(r^-, R_{r^-}(w)) + (1 - \omega) M(r^-, w) J(r^-, \cdot)(f)
\]
where $(r, w) \in \partial \Omega \times V$ such that $(r^-, w) \equiv (r^-(r, w), w) \in \partial(1) \Omega \times V$ and $w \circ n(r^-) \leq 0$. Let $\mu \in \mathbb{C}$. For $f \in L^1_u$, $g \in L^1(\Omega \times V)$ and $(r^-, w) \in \partial(1) \Omega \times V$ with $w \circ n(r^-) \leq 0$ introduce
\[
a(\mu)f(r, w) := e^{-T_\Omega(r, w)\mu} Uf(r, w)
\]
as well as
\[
b(\mu)g(r, w) := \int_0^{T_\Omega(r, w)} e^{-\beta \mu} g(r - \beta w, w) \, d\beta.
\]

### 4.1 Spectral Properties on the boundary $\partial \Omega$

Let $(A, D(A))$ denote the infinitesimal operator of the strongly continuous semigroup $S(t)$, $t \geq 0$, in $L^1(\Omega \times V)$, cf. also Lemma 3.3.

**Lemma 4.1** (a) The operator $U$ is a linear operator $L^1_u \to L^1_u$ with operator norm one. If for $f \in L^1_u$ we have $f \geq 0$ or $f \leq 0$, then $\|Uf\|_{L^1_u} = \|f\|_{L^1_u}$.

(b) Let $\mu \in \mathbb{C}$ and $g \in L^1(\Omega \times V)$. There is a unique $f \in D(A)$ such that $\mu f - A f = g$ if and only if there is a unique $\tilde{f} \in L^1_u$ such that
\[
\tilde{f} = a(\mu)\tilde{f} + b(\mu) g.
\]

**Proof.** Step 1 Let us verify part (a). Observe that by the boundary conditions (4.1) for $f \in L^1_b$ and $(r, w) \in \partial(1) \Omega \times V$ with $w \circ n(r) \geq 0$ we have
\[
Uf(r, w) = f(r^-(r, w), w), \quad (r, w) \in \partial(1) \Omega \times V \quad \text{with} \quad w \circ n(r) \geq 0.
\]

Furthermore, recall that the restriction of $f \in L^1_b$ to $\{(r, w) \in \partial(1) \Omega \times V : w \circ n(r) \geq 0\}$ belongs to $L^1_u$ and that $f \in L^1_u$ can uniquely be extended to an element of $L^1_b$. For $f \in L^1_b$,
it holds that
\[
\int_{r \in \partial^{(1)}\Omega} \int_{w \in \{n(r) \geq 0\}} w \circ n(r) |f(r^{-}(r, w), w)| \, dw \, dr
\]
\[
= \int_{r \in \partial^{(1)}\Omega} \int_{w \in \{n(r) \leq 0\}} |w \circ n(r)||f(r^{-}, w)| \cdot \left( \frac{w \circ n(r)}{|w \circ n(r)|} \cdot \frac{dr^{-}}{dr^{-}} \right) \, dw \, dr^{-}
\]
\[
= \int_{y \in \partial^{(1)}\Omega} \int_{v \in \{n(y) \geq 0\}} v \circ n(y)|f(y, -v)| \, dv \, dy
\]
\[
\leq \int_{y \in \partial^{(1)}\Omega} \int_{v \in \{n(y) \geq 0\}} v \circ n(y)|f(y, v)| \, dv \, dy
\]
where the last \(\leq\) sign holds because of (4.1) and (ii). We mention also that the equality sign holds whenever \(f\) is non-negative or non-positive.

**Step 2** We demonstrate the first part of (b). Let \(g \in L^{1}(\Omega \times V)\) and \(\mu \in \mathbb{C}\). Let us assume that there is a unique \(f \in D(A)\) with \(\mu f - A f = g\). Recalling condition (iii) this equation takes the form
\[
\mu f(y, w) + w \circ \nabla y f(y, w) = g(y, w), \quad (y, w) \in \Omega \times V.
\]
(4.5)
Here, \(w \circ \nabla y f\) is a directional derivative with respect to the norm in \(L^{1}(\Omega \times V)\). It follows that
\[
f(r - \gamma w, w) = e^{\gamma \mu} \left( f(r, w) - \int_{0}^{\gamma} e^{-\beta \mu} g(r - \beta w, w) \, d\beta \right), \quad \gamma \in [0, T_{\Omega}(r, w)],
\]
(4.6)
a.e. on \(\{(r, w) \in \partial^{(1)}\Omega \times V : w \circ n(r) \geq 0\}\). Conversely, applying the differentiation \(w \circ \nabla y\) in \(L^{1}(\Omega \times V)\) to (4.6) we obtain (4.3). In addition, as a consequence of (4.6), for a.e. \(\{(r, w) \in \partial^{(1)}\Omega \times V : w \circ n(r) \geq 0\}\) and \(r^{-} \equiv r^{-}(r, w) = r - T_{\Omega}(r, w)\)
\[
f(r, w) = e^{-T_{\Omega}(r,w)\mu} f(r^{-}, w) + \int_{0}^{T_{\Omega}(r,w)} e^{-\beta \mu} g(r - \beta w, w) \, d\beta.
\]
As a consequence of (iii) and Lemma 3.3 the restriction \(\tilde{f}\) of \(f \in D(A)\) to \(\partial \Omega \times V\) belongs to \(L^{1}_{b}\). Recalling (4.4), the last equation coincides therefore a.e. on \(\{(r, w) \in \partial^{(1)}\Omega \times V : w \circ n(r) \geq 0\}\) with
\[
f(r, w) = e^{-T_{\Omega}(r,w)\mu} U f(r, w) + \int_{0}^{T_{\Omega}(r,w)} e^{-\beta \mu} g(r - \beta w, w) \, d\beta.
\]
Since the restriction \(\tilde{f}\) of \(f \in D(A)\) to \(\{(r, w) \in \partial^{(1)}\Omega \times V : w \circ n(r) \geq 0\}\) is also the restriction of \(\tilde{f} \in L^{1}_{b}\) to \(\{(r, w) \in \partial^{(1)}\Omega \times V : w \circ n(r) \geq 0\}\) we have \(\tilde{f} \in L^{1}_{b}\). Thus the last equality coincides with (4.3).

**Step 3** We show the second part of (b). Let us assume that there is a unique \(\tilde{f} \in L^{1}_{b}\) such that we have (4.3). As mentioned above \(\tilde{f} \in L^{1}_{u}\) can uniquely be extended to some \(\tilde{f} \in L^{1}_{b}\). Keeping in mind relation (4.4) from (4.3) it follows that
\[
\tilde{f}(r, w) = \tilde{f}(r, w) = e^{-T_{\Omega}(r,w)\mu} f(r^{-}, w) + \int_{0}^{T_{\Omega}(r,w)} e^{-\beta \mu} g(r - \beta w, w) \, d\beta.
\]
for a.e. \((r, w) \in \partial^{(1)} \Omega \times V\) with \(w \circ n(r) \geq 0\). From here we define a function \(\hat{f}\) by

\[
\hat{f}(r - \gamma w, w) := e^{\gamma \mu} \left( \hat{f}(r, w) - \int_0^\gamma e^{-\beta \mu} g(r - \beta w, w) \, d\beta \right), \quad \gamma \in [0, T_\Omega(r, w)],
\]

(4.7)

for a.e. \((r, w) \in \partial^{(1)} \Omega \times V\) with \(w \circ n(r) \geq 0\). We observe

\[
\left\|\hat{f}\right\|_{L^1(\Omega \times V)} = \int_V \left\|\hat{f}(\cdot, w)\right\|_{L^1(\Omega)} \, dw
\]

\[
= \int_V \int_{\{r \in \partial^{(1)} \Omega : w \circ n(r) \geq 0\}} w \circ n(r) \int_0^{T_\Omega(r, w)} \left| \hat{f}(r - \beta w, |w|, w) \right| \, d\beta \, dr \, dw
\]

\[
= \int_{\partial^{(1)} \Omega} \int w \circ n(r) \int_0^{T_\Omega(r, w)} \left| \hat{f}(r - \beta w, |w|, w) \right| \, d\beta \, dw \, dr
\]

\[
= \int_{\partial^{(1)} \Omega} \int w \circ n(r) \int_0^{T_\Omega(r, w)} \left| \hat{f}(r - \beta w, w) \right| \, d\beta \, dw \, dr,
\]

(4.8)

no matter whether \(\left\|\hat{f}\right\|_{L^1(\Omega \times V)}\) is finite or not. Now we substitute \(\hat{f}(r - \beta w, w)\) by (4.7) and estimate

\[
\left\|\hat{f}\right\|_{L^1(\Omega \times V)} \leq \int_{\partial^{(1)} \Omega} \int w \circ n(r) \int_0^{T_\Omega(r, w)} \left| e^{\beta \mu} \left| \hat{f}(r, w) \right| \right| d\beta \, dw \, dr
\]

\[
+ \int_{\partial^{(1)} \Omega} \int w \circ n(r) \int_0^{T_\Omega(r, w)} \int_0^\beta e^{(\beta - \alpha) \mu} |g(r - \alpha w, w)| \, d\alpha \, d\beta \, dw \, dr
\]

\[
\leq \int_0^{T_\Omega(r, w)} \left| e^{\beta \mu} \right| \, d\beta \cdot \left\|\hat{f}\right\|_{L^1_C} + \varepsilon^{2 \text{diam}(\Omega)} |\Re(\mu)|/|\mu| \cdot (\text{diam}(\Omega))/2 \times
\]

\[
\times \int_{\partial^{(1)} \Omega} \int w \circ n(r) \int_0^{T_\Omega(r, w)} |g(r - \alpha w, w)| \, d\alpha \, dw \, dr
\]

\[
\leq \frac{1}{|\Re(\mu)|} \cdot \left\|\hat{f}\right\|_{L^1_C} + \varepsilon^{2 \text{diam}(\Omega)} |\Re(\mu)|/|\mu| \cdot (\text{diam}(\Omega))/2 \cdot \left\|g\right\|_{L^1(\Omega \times V)} < \infty
\]

where in the step from the second last to the last line we have applied (4.8) once again. This says \(\hat{f} \in L^1(\Omega \times V)\).

Our task is now to show that \(\hat{f}\) belongs to \(D(A)\) and that we have \(\mu \hat{f} - A \hat{f} = g\). Choosing \(\gamma = 0\) in (4.7) we deduce that \(\hat{f}\) coincides a.e. with the restriction of \(\hat{f}\) to \(\{(r, w) \in \partial \Omega \times V : w \circ n(r) \geq 0\}\). In other words, we have (4.6) with \(\hat{f}\) instead of \(f\). Its equivalence to (4.5) has already been mentioned in Step 2 of this proof. \(\square\)

In order to solve (4.3) for \(f \in L^1_u\) let us examine the operator \(id - a(\mu)\). As a consequence of Lemma 4.1 (b) it is sufficient to focus on \(\mu \in \mathbb{C}\) with \(\Re(\mu) \leq 0\). For \((r, w) \in \partial^{(1)} \Omega \times V\) with \(w \circ n(r) \geq 0\) introduce

\[
A(\mu)f(r, w) := \omega e^{-T_\Omega(r, w)\mu} f(\gamma^r, R^r_w(w))
\]

(4.9)

and

\[
B(\mu)f(r, w) := (1 - \omega)e^{-T_\Omega(r, w)\mu} M(r^r_w, w)J(r^r_\gamma, \cdot)(f)
\]

23
which, because of Lemma \ref{lem:4.1} (a), are well-defined on \( f \in L^1_w \). Relation \eqref{eq:1.2}, yields the decomposition
\[
a(\mu)f(r, w) = A(\mu)f(r, w) + B(\mu)f(r, w).
\] (4.10)

By Lemma \ref{lem:4.1} (a), \( a(\mu), A(\mu), \) and \( B(\mu) \) are bounded linear operators on \( L^1_w \). Next we are concerned with the bijectivity of \( id - A(\mu) \) on \( L^1_w \).

For this we need some preparations. For \((r, w) \in \partial^{(1)} \Omega \times V \) with \( w \circ n(r) \geq 0 \) let us introduce \( \mathbb{T}^0(r, w) := (r, w) \),
\[
\mathbb{T}(r, w) := (r^-(r, w), R_{r^-}(w)),
\]
and the abbreviations \( l_k \equiv l_k(r, w) := T_\Omega \left( \mathbb{T}^k(r, w) \right) \) as well as \((r_k, w_k) \equiv (r_k(r, w), w_k(r, w)) := \mathbb{T}^k(r, w), \) \( k \in \mathbb{Z}_+ \). Noting that
\[
\mathbb{T}^{-1}(r, w) = (r^-(r, -R_r(w)), R_v(w))
\]
for all \((r, w) \in \partial^{(1)} \Omega \times V \) with \( w \circ n(r) \geq 0 \) we also abbreviate \( l_k \equiv l_k(r, w) := T_\Omega \left( \mathbb{T}^k(r, w) \right) \) and \((r_k, w_k) \equiv (r_k(r, w), w_k(r, w)) := \mathbb{T}^k(r, w), \) \( -k \in \mathbb{N} \). We emphasize that the notation just introduced is adjusted to the remainder of this section. It is not compatible to Definition \ref{def:3.2} and Remarks \ref{rem:5} as well as \ref{rem:6} but will not cause any ambiguity.

**Remark 7** Let us take advantage of a well known fact from the ergodic theory of \( d \)-dimensional mathematical billiards, \( d = 2, 3 \). For Lebesgue-a.e. \((r, w) \in \partial^{(1)} \Omega \times V \) with \( w \circ n(r) \geq 0 \) there is a \( \tau(r, w) \leq \text{diam}(\Omega) \) such that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} l_k = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} l_{-k} = \tau(r, w). 
\] (4.11)

Relation \eqref{eq:4.11} holds even Lebesgue-a.e. on \((r, v) \in \partial^{(1)} \Omega \times S^{d-1} : v \circ n(r) \geq 0 \} \). One may recover these results by using the excellent source \[3\] in the following way. First recall the Ergodic Theorem of Birkhoff-Khinchin, Theorem II.1.1 together with the Corollary II.1.4. Then recall from \[3\], (IV.2.3) and the formula above (IV.2.3), that the billiard map \( T \) is measure preserving with respect to some probability measure \( \nu \) on \((r, v) \in \partial^{(1)} \Omega \times S^{d-1} : v \circ n(r) \geq 0 \} \), where \( S^{d-1} \) denotes the unit sphere. In fact, the Radon-Nikodym derivative of \( \nu \) with respect to the Lebesgue measure \( \lambda \) is
\[
\frac{d\nu}{d\lambda} = c_\nu \cdot n(r) \circ v 
\] (4.12)
where \( c_\nu > 0 \) is a normalizing constant. Now it remains to mention that \( \mathbb{T} \) leaves the modulus of the velocity \(|w|\) invariant an that, assuming for a moment that \( S^{d-1} \subset V \), the map \( \mathbb{T} \) is for \(|w| = 1 \) identical with the billiard map \( T \) of \[3\], up to orientation.

Introduce
\[
d := \{(r, v) \in \partial^{(1)} \Omega \times S^{d-1} : v \circ n(r) \geq 0, \text{ and we have } \text{(4.11)}\}.
\]
Remark 8 In order to motivate a technical condition in Lemma 4.12 Proposition 4.3 and Corollary 4.4 below, let us take a look at subsets of \( \{(r, w) \in \partial^{(1)} \Omega \times V : w \circ n(r) \geq 0\} \) on which the limits (4.11) are strictly positive. In particular, \( \tau(r, v) > 0 \) for Lebesgue-a.e. \( (r, v) \in d \) and also \( \tau(r, w) > 0 \) for Lebesgue-a.e. \( (r, w) \in \partial^{(1)} \Omega \times V \) with \( w \circ n(r) \geq 0 \) will now be derived from

\[
\int_d \lim_{n \to \infty} \frac{1}{n} \# \{0 \leq j < n : T^j(r, v) \in A\} \nu(d(r, v)) = \nu(A) \quad (4.13)
\]

for all Borel subsets \( A \) of \( d \), see [3] below Theorem II.1.1. For this we may consider \( l_k \equiv l_k(r, v) \) as a function \( l_k(r, v) \equiv l(T^k(r, v)) \), \( k \in \mathbb{N} \), \( (r, v) \in d \). Indeed, let us assume that \( B := \{(r, v) \in d : \tau(r, v) = 0\} \) has positive Lebesgue measure or equivalently \( \nu(B) = 0 \). With \( \varepsilon > 0 \), \( f_\varepsilon(r, v) := \chi_{[\varepsilon, \infty)}(l(T(r, v))) \), \( (r, v) \in d \), and \( A_\varepsilon := \{(r, v) \in d : l(T(r, v)) \geq \varepsilon\} \) the following chain of inclusions holds.

\[
B = \{(r, v) \in d : \tau(r, v) = 0\} \subseteq \left\{(r, v) \in d : \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f_\varepsilon(T^k(r, v)) = 0\right\} =: B_\varepsilon. \quad (4.14)
\]

Now replace \( A \) in (4.13) by \( A_\varepsilon \) and let \( \varepsilon \to 0 \). Then the right-hand side of (4.13) tends to one by the definition of \( A_\varepsilon \). However, the integral on the left-hand side can be decomposed into the two parts \( \int_{B_\varepsilon} \) and \( \int_{d \setminus B_\varepsilon} \). We observe \( \lim_{\varepsilon \to 0} \int_{B_\varepsilon} = 0 \) by (4.14) and lim sup_{\varepsilon \to 0} \int_{d \setminus B_\varepsilon} \leq 1 - \nu(B) \) by construction and \( \nu(B_\varepsilon) \geq \nu(B) \). The assumption \( \nu(B) > 0 \) implies now that lmsup of the left-hand side of (4.13) is bounded by \( 1 - \nu(B) < 1 \). Thus the above assumption cannot hold, i. e. \( \tau(r, v) > 0 \) for Lebesgue-a.e. \( (r, v) \in d \) and also \( \tau(r, w) > 0 \) for Lebesgue-a.e. \( (r, w) \in \partial^{(1)} \Omega \times V \) with \( w \circ n(r) \geq 0 \).

In other words, in Remark 8 it is demonstrated that with

\[
D := \{(r, w) \in \partial \Omega \times V : w = |w| \cdot v, \ (r, v) \in d, \ \tau(r, w) > 0\}
\]

the Lebesgue measure of \( \{(r, w) \in \partial^{(1)} \Omega \times V : w \circ n(r) \geq 0\} \setminus D \) is zero. In particular we stress that if \( w = |w| \cdot v \in V, \ (r, v) \in d \), and \( \tau(r, v) > 0 \) then we have \( |w| \cdot \tau(r, w) = \tau(r, v) \).

Remark 9 Let us introduce a probability measure \( N \) on \( \{(r, w) \in \partial^{(1)} \Omega \times V : w \circ n(r) \geq 0\} \) by its Radon-Nikodym density with respect to the Lebesgue measure. For this, let \( \Lambda \) denote the Lebesgue measure on \( \{(r, v) \in \partial^{(1)} \Omega \times S^{d-1} : v \circ n(r) \geq 0\} \) and let \( \Lambda \) be the Lebesgue measure on \( \{(r, w) \in \partial^{(1)} \Omega \times V : w \circ n(r) \geq 0\} \). Set

\[
\frac{dN}{d\Lambda}(r, w) := c_N \cdot |w| \frac{d\nu}{d\Lambda}(r, w/|w|) = c_N \cdot w \circ n(r) \quad (4.15)
\]

where \( c_N > 0 \) is a normalizing constant, cf. (4.12). Let us recall the following facts. The billiard map \( T \) of [3] preserves the measure \( \nu \) of [3], Section IV.2. Assuming for a moment that \( S^{d-1} \subset V \), the map \( T \) coincides for fixed modulus of the velocity \( |w| = 1 \) with \( T \), up to orientation. In addition, \( T \) leaves the modulus of the velocity \( |w| \) invariant. As a consequence we conclude that the map \( T \) preserves the measure \( N \). Furthermore, the definition (4.13) shows the identity of \( L^1(D, N) \) with \( L^1_u \), up to zero-sets; we indicate this by \( L^1(D, N) \simeq L^1_u \).
Recall the definition of $\tau(r, w) \cdot |w|$ in (4.11) and introduce

$$c_\tau := \underset{(r, w) \in D}{\text{ess inf}} \tau(r, w) \cdot |w|$$

as well as

$$C_\tau \equiv C_{\tau, k_0} := \underset{(r, w) \in D, k \geq k_0}{\text{ess sup}} \frac{k \cdot \tau(r, w) - \sum_{i=1}^{k} l_i(r, w)}{k \cdot \tau(r, w)}, \quad k_0 \in \mathbb{N}.$$

**Remark 10** Below we will suppose $C_\tau \equiv C_{\tau, k_0} < 1$. To motivate this assumption let $\Omega$ be a convex polygon in dimension $d = 2$ or a convex polyhedron in dimension $d = 3$. Suppose that the inner angles between two neighboring edges if $d = 2$ or faces if $d = 3$ are at least $\pi/2$.

Consider points $(r, v) \in \partial^{(1)} \Omega \times S^{d-1}$ with $v \circ n(r) > 0$ and $(r_k, v_k) = T_k(r, v)$ such that $r_k$ is neither a vertex nor an edge for every $k \in \mathbb{Z}$. For $d = 2$ denote by $s_{\text{min}}$ the length of the shortest edge of $\partial \Omega$. For $d = 3$ denote by $s_{\text{min}}$ the infimum over all distances between two points $y_1$ and $y_2$ belonging to two faces of $\partial \Omega$ having no edge in common. It follows that

$$c_\tau \geq \frac{1}{3} s_{\text{min}}.$$ 

As above, let $\text{diam}(\Omega)$ denote the diameter of $\Omega$. Then for any $j \in \mathbb{Z}$ it holds that

$$s_{\text{min}} \leq \left\{ \begin{array}{cl} |r_j - r_{j+1}| + |r_{j+1} - r_{j+2}| & \text{if } d = 2 \\ |r_j - r_{j+1}| + |r_{j+1} - r_{j+2}| + |r_{j+2} - r_{j+3}| & \text{if } d = 3 \end{array} \right..$$

This is obvious for $d = 2$. For $d = 3$ it follows from the fact that there cannot be more than three consecutive reflections on faces having a vertex in common. Now we may choose $k_0 = 2$ if $d = 2$ and $k_0 = 3$ if $d = 3$ to obtain

$$C_\tau \leq \frac{(2d - 1) \cdot \text{diam}(\Omega) - s_{\text{min}}}{(2d - 1) \cdot \text{diam}(\Omega)} < 1.$$ 

Introduce

$$m := \frac{\log \omega \cdot v_{\text{max}}}{c_\tau} \quad \text{if } c_\tau > 0 \text{ and } m := -\infty \text{ if } c_\tau = 0.$$ 

Throughout this section, let us use the convention $\# \sum_{i=0}^{-1} = 0$.

**Lemma 4.2** Let $\omega \in (0, 1)$. (a) There exists an at most countable subset $\mathcal{M}$ of $[m, 0]$ such that we have the following. For

$$\mu \in \mathcal{M} := \{ \lambda \in \mathbb{C} : \Re \lambda \in (-\infty, 0] \setminus \mathcal{M} \}$$

and any given a.e. bounded measurable $\psi$ there exists an a.e. unique measurable $\varphi$, both defined on $\{(r, w) \in \partial^{(1)} \Omega \times V : w \circ n(r) \geq 0\}$, such that

$$\varphi(r, w) - \omega e^{-T_\Omega(r, w) \mu} \varphi(r, T_\Omega^{-1}(w)) = \psi(r, w) \quad (4.16)$$

for a.e. $(r, w) \in \partial^{(1)} \Omega \times V$ with $w \circ n(r) \geq 0$.

(b) Suppose $(1 - C_{\tau, k_0}) c_\tau > 0$ for some $k_0 \in \mathbb{N}$. Then for

$$\mu \in \mathcal{M}_m := \{ \mu \in \mathbb{C} : \Re \mu < \frac{m}{1 - C_\tau} \}$$

the operator $id - A(\mu)$ is bijective in $L^1_u$. 

26
Proof. Step 1 We point out two equivalent representations of \((4.16)\), namely \((4.17)\) and \((4.19)\) below. Relation \((4.16)\) is equivalent to

\[
\varphi(r, w) - \exp \left\{ -n \cdot \left( \frac{\mu}{n} \sum_{k=0}^{n-1} l_k - \log \omega \right) \right\} \varphi(r_n, w_n) = \sum_{k=0}^{n-1} \exp \left\{ -k \cdot \left( \frac{\mu}{k} \sum_{i=0}^{k-1} l_i - \log \omega \right) \right\} \psi(r_k, w_k), \quad n \in \mathbb{N}. \quad (4.17)
\]

Recalling the definitions above Remark 7 and replacing \((r, w)\) with \((r, -R_r(w))\), \(R_r(w)\), and hence \((r^-, R^-_r(w))\) with \((r, w)\), equation \((4.16)\) turns into

\[
\varphi(r, w) = e^{T_\omega(r^-, R^-_r(w)), R_r(w)} \varphi(r^-, R^-_r(w), R_r(w)) - e^{T_\omega(r^-, R^-_r(w)), R_r(w)} \psi(r^-, R^-_r(w), R_r(w)). \quad (4.18)
\]

Similarly to \((4.17)\) we obtain from \((4.18)\)

\[
\varphi(r, w) - \exp \left\{ n \cdot \left( \frac{\mu}{n} \sum_{k=1}^{n} l_{-k} - \log \omega \right) \right\} \varphi(r_{-n}, w_{-n}) = - \sum_{k=1}^{n} \exp \left\{ k \cdot \left( \frac{\mu}{k} \sum_{i=1}^{k} l_{-i} - \log \omega \right) \right\} \psi(r_{-k}, w_{-k}), \quad n \in \mathbb{N}. \quad (4.19)
\]

Step 2 In this step, we construct the set \(\mathbb{M}\). Recall the hypothesis \(\omega \in (0, 1)\). Let

\[
\mathbb{D}_\mu^- := \{(r, w) \in \mathbb{D} : \Re \mu \cdot \tau(r, w) < \log \omega\},
\]

and

\[
\mathbb{D}_\mu^+ := \{(r, w) \in \mathbb{D} : \Re \mu \cdot \tau(r, w) > \log \omega\}, \quad \Re \mu \leq 0.
\]

Furthermore introduce

\[
\mathbb{D} \setminus (\mathbb{D}_\mu^- \cup \mathbb{D}_\mu^+) = \left\{ (r, w) \in \mathbb{D} : \Re \mu = \frac{\log \omega}{\tau(r, w)} \right\} =: \mathbb{D}_\mu^0, \quad \Re \mu \leq 0.
\]

We observe that for \(\omega \in (0, 1)\) and \(a \in (-\infty, 0)\), the set \(\mathbb{D}_{a+b}^-\) is independent of \(b \in \mathbb{R}\). Since, for fixed \(\omega \in (0, 1)\), there are at most countably many \(a \in (-\infty, 0)\) such that \(\mathbb{D}_{a+b}^-\) has positive Lebesgue measure the set

\[
\mathbb{M} := \{ \mu \in \mathbb{C} : \Re \mu \leq 0, \ \mathbb{D}^-_\mu \text{ has zero Lebesgue measure} \} \quad (4.20)
\]

coincides with \(\{ \mu \in \mathbb{C} : \Re \mu \leq 0 \}\) except for a union of at most countably many vertical lines of the form \(\{ \mu \in \mathbb{C} : \Re \mu \in \mathbb{M} \}\) where \(\mathbb{M}\) is an at most countable set of non-positive real numbers. We mention that for \(\mu \in \mathbb{M}\) we have Lebesgue-a.e. on \(\{(r, w) \in \partial^{(1)} \Omega \times V : w \circ n(r) \geq 0 \}\) the alternative that either \((r, w) \in \mathbb{D}_\mu^0\) or \((r, w) \in \mathbb{D}_\mu^-\). Furthermore if \(c_\tau > 0\), we have \(\Re \mu \cdot c_\tau < \log \omega \cdot v_{\max}\) if and only if \(\Re \mu \leq m\) and therefore

\[
\mathbb{D}_\mu^+ = \emptyset \quad \text{and} \quad \mathbb{D}_\mu^- = \emptyset, \quad \text{if } c_\tau > 0 \text{ and } \Re \mu < m. \quad (4.21)
\]
In other words, we have \( M \subset [m, 0] \) and \( M = \{ \mu \in \mathbb{C} : \Re \mu \in (-\infty, 0) \setminus M \} \), no matter if \( c_r > 0 \) or \( c_r = 0 \).

**Step 3** Next we will be concerned with the existence/non-existence of the limits as \( n \to \infty \) in (4.17) and (4.19). In this step we restrict the analysis to \( (r, w) \in D^< \cup D^> \). It holds that

\[
\varphi_1^<(r, w) := -\sum_{k=1}^{\infty} \exp \left\{ k \cdot \left( \frac{\mu}{k} \sum_{i=1}^{k} l_{-i}(r, w) - \log \omega \right) \right\} 
\]

\[
= -e^{\mu l_{-1}(r, w) - \log \omega} \sum_{k=0}^{\infty} \exp \left\{ k \cdot \left( \frac{\mu}{k} \sum_{i=1}^{k} l_{-i}(r, -1, w - 1) - \log \omega \right) \right\} 
\]

\[
= e^{\mu l_{-1}(r, w) - \log \omega} \cdot (-1 + \varphi_1^<(r, -1, w - 1)) ,
\]

(4.22)

no matter whether the sum \( \varphi_1^<(r, w) \) converges or not. We obtain immediately that the sum \( \varphi_1^<(r, w) \) converges if and only if \( \varphi_1^<(r_k, w_k) \) converges for every \( k \in \mathbb{Z} \).

Iterating (4.22) we find

\[
\varphi_1^<(r, w) = -\sum_{k=1}^{n} \exp \left\{ k \cdot \left( \frac{\mu}{k} \sum_{i=1}^{k} l_{-i}(r, w) - \log \omega \right) \right\} 
\]

\[+ \exp \left\{ \sum_{i=1}^{n} (\mu \cdot l_{-i}(r, w) - \log \omega) \right\} \varphi_1^<(r_{-n}, w_{-n}), \quad n \in \mathbb{N}.
\]

(4.23)

Recalling that \( (r, w) \in D^< \cup D^> \), we observe

\[
\sum_{i=1}^{\infty} (\Re \mu \cdot l_{-i}(r, w) - \log \omega) = \text{sign} (\Re \mu \cdot \tau(r, w) - \log \omega) \cdot \infty .
\]

(4.24)

In the remainder of this step we show the equivalence of the following.

(1) The sum \( \varphi_1^<(r, w) \) given in the first line of (4.22) converges.

(2) All \( \varphi_1^<(r_k, w_k), k \in \mathbb{Z} \), converge.

(3) We have \( \Re \mu \cdot \tau(r, w) < \log \omega \), i.e. \( (r, w) \in D^< \).

The equivalence of (1) and (2) has been noted above. Let us proceed from (1) to (3) and back to (1). If the sum \( \varphi_1^<(r, w) \) converges then the first term on the right-hand side of (4.23) converges to \( \varphi_1^<(r, w) \) as \( n \to \infty \). Since \( \varphi_1^<(r_{n}, w_{n}) \) cannot tend to zero as \( n \to \infty \) by (4.22), from (4.23) and (4.24) we may now conclude the following.

If \( \varphi_1^<(r, w) \) converges then

\[
\lim_{n \to \infty} \sum_{i=1}^{n} (\Re \mu \cdot l_{-i}(r, w) - \log \omega) = -\infty ,
\]

(4.25)

By (4.24) we have (3). In this case, there is a \( k_1 \in \mathbb{N} \) such that for \( j > k_1 \)

\[
\Re \mu \cdot \sum_{i=1}^{j} l_{-i}(r, w) - j \log \omega < \frac{j}{2} (\Re \mu \cdot \tau(r, w) - \log \omega) .
\]
Letting \( k, l > k_1 \) and using the first line of (4.22) we obtain
\[
\left| \sum_{j=1}^{k} \exp \left\{ j \cdot \left( \frac{\mu}{j} \sum_{i=1}^{j} l_{-i}(r, w) - \log \omega \right) \right\} - \sum_{j=1}^{l} \exp \left\{ j \cdot \left( \frac{\mu}{j} \sum_{i=1}^{j} l_{-i}(r, w) - \log \omega \right) \right\} \right|
\leq \sum_{j=k \land l+1}^{\infty} \exp \left\{ j \cdot \left( \Re \mu \cdot \sum_{i=1}^{j} l_{-i}(r, w) - \log \omega \right) \right\}
\leq \sum_{j=k \land l+1}^{\infty} \left( \frac{j}{2} (\Re \mu \cdot \tau(r, w) - \log \omega) \right) \xrightarrow{k,l \to \infty} 0,
\]
(4.26)
i.e. the sum \( \varphi_1^>(r, w) \) converges provided that \( \Re \mu \cdot \tau(r, w) < \log \omega \). We may now state the equivalence of the properties (1)-(3).

Introducing
\[
\varphi_1^>(r, w) := \sum_{k=0}^{\infty} \exp \left\{ -k \cdot \left( \frac{\mu}{k} \sum_{i=0}^{k-1} l_i(r, w) - \log \omega \right) \right\}
\]
the equivalence of the following properties (4)-(6) can be proved in a similar way.

(4) The sum \( \varphi_1^>(r, w) \) converges.

(5) All \( \varphi_1^>(r_k, w_k) \), \( k \in \mathbb{Z} \), converge.

(6) We have \( \Re \mu \cdot \tau(r, w) > \log \omega \), i.e. \( (r, w) \in D_\mu^\geq \).

**Step 4** Let us prove part (a). Without mentioning this again in the present step, we take advantage of the equivalences (1)-(3) and (4)-(6) of Step 3. From (4.17) and (4.19) we obtain necessarily
\[
\varphi(r, w) = \sum_{k=0}^{\infty} \exp \left\{ -k \cdot \left( \frac{\mu}{k} \sum_{i=0}^{k-1} l_i(r, w) - \log \omega \right) \right\} \psi(r_k, w_k),
\]
(4.27)
for a.e. \( (r, w) \in D_\mu^\geq \), as well as
\[
\varphi(r, w) = -\sum_{k=1}^{\infty} \exp \left\{ k \cdot \left( \frac{\mu}{k} \sum_{i=1}^{k} l_{-i}(r, w) - \log \omega \right) \right\} \psi(r_{-k}, w_{-k}),
\]
(4.28)
for a.e. \( (r, w) \in D_\mu^\leq \) whenever \( \mu \in \mathbb{M} \). Checking back with (4.16) we verify that (4.27), (4.28) is the unique solution to (4.16).

**Step 5** In order to verify part (b), let us also recall the definition of \( (r_{-k}, w_{-k}) \), \( k \in \mathbb{N} \), above Remark 7 and the definition of \( \tau(r, w) \) in (4.11). Review thoroughly Remark 9. The bijection \( T \) and hence also \( T^{-1} \), preserve the measure \( N \) given by (4.15).

Note that for \( \Re \mu < m \) it holds that \( D = D^\leq \) by (4.21). By the construction of \( m \) and \( M_m \) in Lemma 142 \( \Re \mu \cdot (1 - C_\tau) c_\tau / v_{\text{max}} - \log \omega \) is for \( \mu \in M_m \) and negative on \( D_\).
By means of the above preparations it follows as in (4.26) that for \( \psi \in L^1_u \simeq L^1(D, N) \) and \( \mu \in M_m \) we have
\[
\left| \sum_{k=n_0}^{n} \exp \left\{ k \cdot \left( \frac{\mu}{k} \sum_{i=1}^{k} l_{-i} - \log \omega \right) \right\} \psi(r_{-k}, w_{-k}) \right| \\
\leq \sum_{k=n_0}^{n} \exp \left\{ k \cdot \left( \frac{\Re \mu}{k} \sum_{i=1}^{k} l_{-i} - \log \omega \right) \right\} \left| \psi(r_{-k}, w_{-k}) \right| \\
= \sum_{k=n_0}^{n} \exp \left\{ k \cdot \Re \mu \left( \frac{1}{k} \sum_{i=1}^{k} l_{-i} - \tau(r, w) \right) \right\} \cdot e^{k \cdot (\Re \mu - \tau(r, w) - \log \omega)} \left| \psi(r_{-k}, w_{-k}) \right| \\
\leq \sum_{k=n_0}^{n} e^{-k \Re \mu C \tau(r, w)} \cdot e^{k \cdot (\Re \mu - \tau(r, w) - \log \omega)} \left| \psi(r_{-k}, w_{-k}) \right| \\
\leq \sum_{k=n_0}^{n} e^{k \cdot (\Re \mu - (1 - C) c / \nu_{\max} - \log \omega)} \left| \psi(r_{-k}, w_{-k}) \right|, \quad k_0 \leq n_0 \leq n, \ n \in \mathbb{N}.
\] We obtain
\[
\int_D \left| \sum_{k=n_0}^{n} \exp \left\{ k \cdot \left( \frac{\mu}{k} \sum_{i=1}^{k} l_{-i} - \log \omega \right) \right\} \psi(r_{-k}, w_{-k}) \right| dN \\
\leq \int_D \sum_{k=n_0}^{n} e^{k \cdot (\Re \mu - (1 - C) c / \nu_{\max} - \log \omega)} \left| \psi(r_{-k}, w_{-k}) \right| dN \\
= \sum_{k=n_0}^{n} e^{k \cdot (\Re \mu - (1 - C) c / \nu_{\max} - \log \omega)} \int_D \left| \psi(r, w) \right| dN \\
\leq e^{n_0 \cdot (\Re \mu - (1 - C) c / \nu_{\max} - \log \omega)} \int_D \left| \psi(r, w) \right| dN \\
\to 0 \quad \text{as } n_0, n \to \infty \text{ where } n_0 \leq n,
\] the third line because of the fact that \( \tau \) preserves the measure \( N \). This together with the Lebesgue-a.e. result (4.28) shows that the term on the right-hand side of (4.19) converges to (4.28) in \( L^1_u \simeq L^1(D, N) \). In particular, for \( \varphi \in L^1_u \) and \( \mu \in M_m \) we get
\[
\exp \left\{ n \cdot \left( \frac{\mu}{n} \sum_{k=1}^{n} l_{-k} - \log \omega \right) \right\} \varphi(r_{-n}, w_{-n}) \xrightarrow{n \to \infty} 0 \quad \text{in } L^1_u.
\]

Considering (4.16) and the equivalent equation (4.19) for \( \varphi, \psi \in L^1_u \) we obtain necessarily (4.28) where the infinite sum converges in \( L^1_u \). On the other hand, we may check representation (4.28) of \( \varphi \) back with (4.16), now as an equation in \( L^1_u \). Recalling that \( A(\mu) \) is a bounded linear operator in \( L^1_u \) we establish bijectivity of \( id - A(\mu) \) in \( L^1_u \) for \( \mu \in M_m \). \( \square \)

Let
\[
W_k(r, w) := \begin{cases} 
\exp \left\{ -k \cdot \left( \frac{\mu}{k} \sum_{i=0}^{k-1} l_i(r, w) - \log \omega \right) \right\} & \text{if } (r, w) \in D^\mu_/> \\
- \exp \left\{ (k + 1) \cdot \left( \frac{\mu}{k+1} \sum_{i=1}^{k+1} l_{-i}(r, w) - \log \omega \right) \right\} & \text{if } (r, w) \in D^<_\mu, \quad k \in \mathbb{Z}_+.
\end{cases}
\]
and, for definiteness, $W_k(r, w) := 0$ if $(r, w) \in \{(r, w) \in \partial^{(1)} \Omega \times V : w \circ n(r) \geq 0\} \setminus (D_\mu \cup D_\mu)$, $k \in \mathbb{Z}$. In addition, let

$$(p_k, u_k) \equiv (p_k(r, w), u_k(r, w))$$

$$:= \begin{cases} (r_k(r, w), w_k(r, w)) & \text{if } (r, w) \in D_\mu^> \\ (r_{-k-1}(r, w), w_{-k-1}(r, w)) & \text{if } (r, w) \in D_\mu^< \end{cases}, \quad k \in \mathbb{Z}_+.$$
converges in $L^1_u$. It defines a bounded linear operators $X_\mu$ in $L^1_u$.

(b) Suppose

$$2^{-\frac{1}{k_0}} < \omega < 1. \quad (4.32)$$

Then there exists $m_1 \leq m$ such that for

$$\mu \in \mathbb{M}_{m_1} := \{ \lambda \in \mathbb{C} : \Re \lambda < m_1 \}$$

we have $\|X_\mu\|_{B(L^1_u, L^1_u)} < 1$. For $\mu \in \mathbb{M}_{m_1}$ and $\psi \in L^1_u$, the equation

$$\varphi - (\text{id} - A(\mu))^{-1}B(\mu)\varphi = \psi$$

has the unique solution

$$\varphi = \sum_{n=0}^{\infty} X_\mu^n \psi \quad (4.33)$$

where the infinite sum converges in $L^1_u$. Moreover, the operator $\text{id} - (\text{id} - A(\mu))^{-1}B(\mu)$ is bijective in $L^1_u$.

Proof. Step 1 We show (a). Let $\mu \in \mathbb{M}_m$ and recall the definitions of $A(\mu)$ and $B(\mu)$,

$$A(\mu)\psi(r, w) = \omega e^{-T_0(r, w)\mu} \psi(r_1, w_1)$$

and, taking into consideration the identity $(r, w) = (r_1, R_{r_1}(w_1))$, 

$$B(\mu)\psi(r, w) = (1 - \omega) e^{-T_0(r, w)\mu} M(r_1, R_{r_1}(w_1)) J(r_1, \cdot)(\psi), \quad \psi \in L^1_u.$$ 

As a consequence of [4.2] and Lemma 4.1 (a), the map $\varphi \mapsto M(r^{-}, w)J(r^{-}, \cdot)(\varphi)$ is a bounded linear operator $L^1_u \mapsto L^1_u$. Thus, also $B(\mu)$ is a bounded linear operator $L^1_u \mapsto L^1_u$. Furthermore, by Lemma 4.2 (b), $(\text{id} - A(\mu))^{-1}$ and therefore also $\text{id} - (\text{id} - A(\mu))^{-1}B(\mu)$ are bounded linear operators $L^1_u \mapsto L^1_u$.

Now recall that for $c_\tau > 0$ and $\mu \in \mathbb{M}_m$ we have $D_{\mu}^+ = \emptyset$ and $D_{\mu}^- = \emptyset$ by (4.21). Using representation (4.28) for $(\text{id} - A(\mu))^{-1}$ and the above definition of $B(\mu)$ it turns out that

$$X_\mu = (\text{id} - A(\mu))^{-1}B(\mu)$$

where the sum (4.31) converges in $L^1_u$.

Step 2 We verify (b). Similar to Step 5 of the proof of Lemma 4.2 for $\psi \in L^1_u$ we obtain

$$| (\text{id} - A(\mu))^{-1}B(\mu) \psi | \leq \sum_{k=1}^{\infty} \exp \left\{ \frac{\mu}{k} \sum_{i=1}^{k} l_{-i} - \log \omega \right\} \left( B(\mu) \psi \right)(r_{-k}, w_{-k})$$

$$\leq \frac{(1 - \omega)}{\omega} \sum_{k=1}^{k_0} \exp \left\{ (k - 1) \cdot \left( \frac{\Re \mu}{k - 1} \sum_{i=1}^{k-1} l_{-i} - \log \omega \right) \right\} \times$$

$$\times M \left( r_{1-k}, R_{r_1}(w_{1-k}) \right) J(r_{1-k}, \cdot)(|\psi|)$$

32
\[ + \frac{(1 - \omega)}{\omega} \sum_{k=k_0+1}^{\infty} \exp \left\{ (k - 1) \cdot \left( \frac{\Re \mu}{k - 1} \sum_{i=1}^{k-1} l_{-i} - \log \omega \right) \right\} \times \]

\[ \times M \left( r_{1-k}, R_{r_{1-k}}(w_{1-k}) \right) J(r_{1-k}, \cdot)(|\psi|) \]

\[ \leq \frac{(1 - \omega)}{\omega} \sum_{k=1}^{k_0} \omega^{-(k-1)} M \left( r_{1-k}, R_{r_{1-k}}(w_{1-k}) \right) J(r_{1-k}, \cdot)(|\psi|) \]

\[ + \frac{(1 - \omega)}{\omega} \sum_{k=k_0+1}^{\infty} e^{(k-1) \cdot (\Re \mu \cdot (1-C_T) c_T/v_{\max} - \log \omega)} M \left( r_{1-k}, R_{r_{1-k}}(w_{1-k}) \right) J(r_{1-k}, \cdot)(|\psi|) . \]

Since \( T^{-1} \) preserves the measure \( N \) we have

\[ \int_{\mathcal{D}} M \left( r_{1-k}, R_{r_{1-k}}(w_{1-k}) \right) J(r_{1-k}, \cdot)(|\psi|) dN = \int_{\mathcal{D}} M (r, R_r(w)) J(r, \cdot)(|\psi|) dN \]

\[ = \int_{\mathcal{D}} |\psi(r, w)| dN , \quad \psi \in L^1_u , \]

where we have applied the definition of \( N \) in (4.15) and (ii) in the last line. Therefore

\[ \int_{\mathcal{D}} |(id - A(\mu))^{-1}B(\mu)\psi| dN \]

\[ \leq \frac{(1 - \omega)}{\omega} \sum_{k=1}^{k_0} \omega^{-(k-1)} \int_{\mathcal{D}} |\psi(r, w)| dN \]

\[ + \frac{(1 - \omega)}{\omega} \sum_{k=k_0+1}^{\infty} e^{(k-1) \cdot (\Re \mu \cdot (1-C_T) c_T/v_{\max} - \log \omega)} \int_{\mathcal{D}} |\psi(r, w)| dN \]

\[ = \frac{(1 - \omega)}{\omega} \left( \frac{\omega^{-k_0} - 1}{\omega^{-1} - 1} + \frac{e^{k_0 \cdot (\Re \mu \cdot (1-C_T) c_T/v_{\max} - \log \omega)}}{1 - e^{\Re \mu \cdot (1-C_T) c_T/v_{\max} - \log \omega}} \right) \int_{\mathcal{D}} |\psi(r, w)| dN . \]

Noting that for \( \omega \in (0, 1) \)

\[ \frac{(1 - \omega)}{\omega} \cdot \frac{\omega^{-k_0} - 1}{\omega^{-1} - 1} = \omega^{-k_0} - 1 < 1 \]

is equivalent to (4.32) we may now claim that there is \( m_1 \leq m \) such that for \( \mu \in \mathbb{M}_{m_1} = \{ \mu \in \mathbb{C} : \Re \mu < m_1 \} \)

\[ \| X_\mu \|_{B(L^1_u, L^1_v)} = \|(id - A(\mu))^{-1}B(\mu)\|_{B(L^1_u, L^1_v)} < 1 . \]

Since \( id - A(\mu) \) is bijective in \( L^1_u \) by Lemma 4.2 (b), bijectivity of \( id - (id - A(\mu))^{-1}B(\mu) \) in \( L^1_u \) means that for any \( \psi \in L^1_u \) there is a unique \( \varphi \in L^1_v \) such that

\[ (id - A(\mu))\psi(r, w) = (id - A(\mu))\varphi(r, w) - B(\mu)\varphi(r, w) \quad (4.34) \]

for a.e. \( (r, w) \in \partial^{(1)}\Omega \times V \) with \( w \circ n(r) \geq 0 \). We have necessarily obtained (4.33). On the other hand, checking back representation (4.33) of \( \varphi \) with (4.34) we verify bijectivity of \( id - (id - A(\mu))^{-1}B(\mu) \) in \( L^1_u \).

\[ \square \]
4.2 Spectral Properties on the state space $\Omega$ and the Knudsen Type Group

Let us recall that $(A, D(A))$ denotes the infinitesimal operator of the strongly continuous semigroup $S(t)$, $t \geq 0$, in $L^1(\Omega \times V)$, cf. also Lemma 4.3 and Lemma 4.4 (b). Note the difference to the operator $A(\mu)$ given by (4.9).

**Corollary 4.4** Let $c_\tau$ and $C_\tau$ be given as in Lemma 4.2. Suppose

$$(1 - C_{\tau, k_0}) c_\tau > 0 \quad \text{for some } k_0 \in \mathbb{N}$$

and $2^{-\frac{1}{k_0}} < \omega < 1$. Let $m_1$ and $\mathbb{M}_{m_1}$ be as introduced in Proposition 4.3.

(a) For $\mu \in \mathbb{M}_{m_1}$ the operator $id - A(\mu) - B(\mu) = id - a(\mu)$ is a bijection in $L^1_u$.
(b) For $\mu \in \mathbb{M}_{m_1}$ and $g \in L^1(\Omega \times V)$ the equation $\mu f - Af = g$ has a unique solution $f \in D(A)$.
(c) Denoting by $\sigma(A)$ the spectrum of $A$ and by $\sigma(S(t))$ the spectrum of $S(t)$ we have the spectral mapping relation

$$\sigma(S(t)) \setminus \{0\} = \{e^{t\mu} : \mu \in \sigma(A)\}, \quad t > 0.$$ 

Furthermore, for $t \geq 0$, the resolvent set $\rho(S(t))$ of $S(t)$ contains the set $\{\lambda = e^{t\mu} : \mu \in \mathbb{M}_{m_1}\}$ $\cup \{\lambda = e^{t\mu} : \mathfrak{R}\mu > 0\}$.

**Proof.** Step 1 Part (a) is an immediate consequence of (4.30), Lemma 4.2 (b), and Proposition 4.3 (b). Furthermore, from (4.8) we obtain

$$\int_{r \in \partial(1) \Omega} \int_{w \circ n(r) \geq 0} w \circ n(r) \int_0^{T_{1}(r, w)} |e^{-\beta}||g(r - \beta w, w)| d\beta dw dr \leq e^{\text{diam}(\Omega)|\mathfrak{R}(\mu)|} \int_V \|g(\cdot, w)\|_{L^1(\Omega)} dw = e^{\text{diam}(\Omega)|\mathfrak{R}(\mu)|} \|g\|_{L^1(\Omega \times V)} < \infty,$$

i. e.

$$b(\mu)g \in L^1_u \quad \text{if } g \in L^1(\Omega \times V).$$

For part (b) it is now sufficient to note that the equation $\mu f - Af = g$ is equivalent to (4.3), see Lemma 4.4 (b).

Step 2 It remains to verify part (c). In this step we prepare the proof of the spectral mapping relation $\sigma(S(t)) \setminus \{0\} = \{\lambda = e^{t\mu} : \mu \in \sigma(A)\}, \quad t > 0$. The actual proof will be carried out in Steps 3 and 4 below. Let us keep on using the symbols and terminology of [I]. In particular, let $\sigma(\cdot)$ and $A\sigma(\cdot)$ denote the spectrum and approximate point spectrum of an operator. According to [I], Theorems 3.6 and 3.7 of Chapter IV, we have to demonstrate that for any $\lambda \neq 0$ belonging to the approximate point spectrum $A\sigma(S(t))$ of $S(t)$ we have $\lambda \in \{e^{t\mu} : \mu \in A\sigma(A)\}, \quad t > 0$.

Now, let $t > 0$ and $e^{t\mu} \in A\sigma(S(t))$. By [I], Lemma 1.9 of Chapter IV, there exists a sequence $f_n \in L^1(\Omega \times V)$ with $\|f_n\|_{L^1(\Omega \times V)} = 1$, $n \in \mathbb{N}$, and

$$\lim_{n \to \infty} \|S(t)f_n - e^{t\mu}f_n\|_{L^1(\Omega \times V)} = 0.$$ \quad (4.35)
The continuous Markov process \( X_t, t \geq 0 \), introduced after (3.20) can be modified in a way that its trajectories never reach \( \partial \Omega \setminus \partial^{(1)} \Omega \). The trajectories move uniformly with constant velocity from time zero until the first time they hit \( \partial^{(1)} \Omega \). Then they move again uniformly with constant velocity until the second time they hit \( \partial^{(1)} \Omega \). By condition (viii) the hitting times do not accumulate.

Until (4.36) below, suppose we are given a sequence \( b_n \in L^1(\Omega \times V) \) with \( \|b_n\|_{L^1(\Omega \times V)} = 1 \), and \( b_n \geq 0 \), \( n \in \mathbb{N} \). Let \( P_{b_n} \) denote the probability measure over the \( \sigma \)-algebra \( \mathcal{F} \) generated by the cylindrical sets of the trajectories of \( X_t, t \geq 0 \), for which the image measure of \( P_{b_n} \) under the map \( \{X_t, t \geq 0\} \mapsto X_0 \) is \( b_n \) times Lebesgue measure on \( \Omega \times \overline{V} \). Let \( E_{b_n} \) stand for the expectation with respect to \( P_{b_n}, n \in \mathbb{N} \). This notation allows the interpretation that \( E_{b_n} \) is the expectation relative to the process \( X_t, t \geq 0 \), started with the initial probability density \( b_n \).

Furthermore, by \( 0 < v_{\text{min}} \leq v \), cf. Subsection 2.1, there is a constant \( c > 0 \) such that \( |X_{t_2} - X_{t_1}| \leq c|t_2 - t_1| \) for all \( 0 \leq t_1 < t_2 \) and all trajectories of \( X_t, t \geq 0 \). This means that \( E_{b_n}[|X_{t_2} - X_{t_1}|^2] \leq c|t_2 - t_1|^2 \). According to [7], Subsections 9.1, 9.2 and particularly Theorem 2 of Subsection 9.2, there is a subsequence \( P_{b_{n_k}}, k \in \mathbb{N} \), converging weakly to some probability measure \( P \) on \( \mathcal{F} \). Let \( \nu \) denote the image measure of \( P \) under the map \( \{X_t, t \geq 0\} \mapsto X_0 \). Theorem 2 of Subsection 9.2 of [7] says in particular that for all \( t > 0 \) and all continuous maps \( g : C([0, t]; \Omega \times \overline{V}) \mapsto \mathbb{R} \) the following holds.

\[
\lim_{k \to \infty} \int g(\{X_u, u \in [0, t]\}) \, dP_{b_{n_k}} = \int g(\{X_u, u \in [0, t]\}) \, dP.
\]

Specified to \( g(h) := \psi(h(0)) \), \( h \in C([0, t]; \Omega \times \overline{V}), \psi \in C(\Omega \times \overline{V}) \), it holds that

\[
\lim_{k \to \infty} \int \psi(r, w) b_{n_k}(r, w) \, dr \, dw = \int \psi(r, w) \, \nu(dr \times dw).
\]

This gives rise to denote by \( E_{\nu} \) the expectation with respect to \( P \). Here we have the interpretation that \( E_{\nu} \) is the expectation relative to the process \( X_t, t \geq 0 \), started with the initial probability measure \( \nu \). Thus, for all continuous \( g : C([0, t]; \Omega \times \overline{V}) \mapsto \mathbb{R} \)

\[
\lim_{k \to \infty} \int_{\Omega \times \overline{V}} E_{b_{n_k}}[g(\{X_u, u \in [0, t]\})] \, d\nu = \int_{\Omega \times \overline{V}} E_{\nu}[g(\{X_u, u \in [0, t]\})].
\]

Let us now return to the sequence \( f_n \in L^1(\Omega \times V) \) with \( \|f_n\|_{L^1(\Omega \times V)} = 1 \), \( n \in \mathbb{N} \), introduced in the beginning of this step. Then (4.37) and (4.36) hold for \( b_{n_k} \) replaced with \( f_{n_k}^+ / \|f_{n_k}^+\|_{L^1(\Omega \times V)} \) or \( f_{n_k}^- / \|f_{n_k}^-\|_{L^1(\Omega \times V)} \), where \( f_{n_k}^+ := f_{n_k} \vee 0 \) as well as \( f_{n_k}^- := (-f_{n_k}) \vee 0 \), and \( \nu \) replaced with the respective non-negative Borel probability measures on \( \Omega \times \overline{V}, \nu^+ \) or \( \nu^- \). Choosing another subsequence if necessary, we may suppose that the limits

\[
l^+ := \lim_{k \to \infty} \|f_{n_k}^+\|_{L^1(\Omega \times V)} \quad \text{and} \quad l^- := \lim_{k \to \infty} \|f_{n_k}^-\|_{L^1(\Omega \times V)} = 1 - l^+
\]

exist. It follows now from (4.37) and (4.36) that with \( \mu := l^+ \nu^+ - l^- \nu^- \) we have

\[
\lim_{k \to \infty} \int \psi(r, w) f_{n_k}(r, w) \, dr \, dw = \int \psi(r, w) \mu(dr \times dw)
\]

and

\[
\lim_{k \to \infty} \int_{\Omega \times \overline{V}} E_{f_{n_k}}[g(\{X_u, u \in [0, t]\})] \, d\nu = \int_{\Omega \times \overline{V}} E_{\nu}[g(\{X_u, u \in [0, t]\})].
\]
for \( g \) and \( \psi \) as above. Here the expectations \( E_{f_n} \) and \( E_u \) come with the interpretation as weighted differences of the expectations relative to the process \( X_t \), \( t \geq 0 \), started with the respective initial probability densities and initial probability measures.

Below we shall consider two cases separately, namely that \( \mu \) is the zero measure, or alternatively that \( \mu \) is not the zero measure, i.e. a certain signed measure with total variation one.

**Step 3** Assume in this step that \( \mu \) is the zero measure. Recall Steps 1 and 2 of the proof of Lemma 3.3 and the notation introduced there. In particular, for \( \varphi \in C_b(\Omega \times V) \), uniformly continuous on \( \Omega \times V \) recall \( a(t) \equiv a_\varphi(t) := \sup_{r \in V} \sup_{v \in \Omega} |\varphi(r - tv, v) - \varphi(r, v)| \). We obtain as in (3.24)

\[
\left| \int_{\Omega} \int_V (S(u)f_{n_k} - f_{n_k}) \varphi \, dv \, dr \right| \leq 3\|\varphi\| \int_{\Omega \setminus \Omega_{2e}} \int_V |f_{n_k}| \, dv \, dr + a(u), \quad k \in \mathbb{N}, \ u \in (0, t].
\]

Denoting by \( |\mu| \) the variation of \( \mu \) it follows that

\[
\limsup_{k \to \infty} \left| \int_{\Omega} \int_V (S(u)f_{n_k} - f_{n_k}) \varphi \, dv \, dr \right| \leq \limsup_{k \to \infty} 3\|\varphi\| \int_{\Omega \setminus \Omega_{2e}} \int_V |f_{n_k}| \, dv \, dr + a(u) = 3\|\varphi\| \cdot |\mu| (\Omega \setminus \Omega_{2e}) + a(u) \quad \to 0,
\]

For the equality sign in the second last line, recall (4.38). Similarly, we get

\[
\limsup_{k \to \infty} \left| \int_{\Omega} \int_V (S(s + u)f_{n_k} - S(s)f_{n_k}) \varphi \, dv \, dr \right| \leq a_\varphi(u) \quad \to 0
\]

for all \( 0 < u < t, \ 0 < s < t \) with \( s + u < t \). The last two relations imply that for any set \( \Phi \) of test functions containing all \( \varphi \in C_b(\Omega \times V) \) having a common modulus of continuity and a common bound on the norm in \( C_b(\Omega \times V) \), the family

\[
[0, t] \ni u \mapsto \int_{\Omega} \int_V (S(u)f_{n_k}) \varphi_k \, dv \, dr, \quad \varphi_k \in \Phi, \ k \in \mathbb{N},
\]

is equicontinuous and equibounded. Keeping in mind that every such set \( \Phi \) is totally bounded in \( C_b(\Omega \times V) \) and that there is an increasing sequence of such sets \( \Phi \) whose union is dense in \( C_b(\Omega \times V) \) we may select a set \( \Phi \) and a sequence \( \varphi_k \in \Phi \) such that \( \int_{\Omega} \int_V f_{n_k} \varphi_k \, dv \, dr > \frac{1}{2} \).

It follows now precisely as in [1], proof of Lemma 3.9, part (a) \( \Rightarrow \) (b), that there is an \( m \in \mathbb{Z} \) such that \( \mu + \frac{2\pi}{\bar{t}} \in A\sigma(A) \) provided that \( \lambda = e^{t\mu} = e^{t(\mu + 2\pi \bar{t})} \in A\sigma(S(t)) \). As explained in the first paragraph of Step 2, this implies the spectral mapping relation \( \sigma(S(t)) \setminus \{0\} = \{e^{t\mu} : \mu \in \sigma(A)\}, \ t > 0 \).

**Step 4** Assume now that \( \mu \) is not the zero measure. For \( m \in \mathbb{Z} \) and \( \psi \in C(\bar{\Omega} \times V) \) equations (3.31) and (4.39) imply the existence of the limit

\[
\left| \int_{[0, t] \times \bar{\Omega} \times V} \frac{e^{-u(2\pi i \bar{t})}}{\bar{t}} e^{-u\mu} S(u)f_{n_k}(r, w) \cdot \psi(r, w) \, dw \, dr \, du \right|
\]

\[
= \left| \int_{\bar{\Omega} \times V} E_{f_{n_k}} \left( \int_0^t e^{-u(2\pi i \bar{t})} e^{-u\mu} \cdot \psi(X_u) \, du \right) \, dw \right|
\]

\[
\to \left| \int_{\bar{\Omega} \times V} E_{\mu} \left( \int_0^t e^{-u(2\pi i \bar{t})} e^{-u\mu} \cdot \psi(X_u) \, du \right) \mu(dr \times dw) \right|. \quad (4.40)
\]
By the Stone-Weierstraß Theorem, the (complex) $C([0,t] \times \Omega \times V)$ coincides with the closed linear span of

$$\left\{ e^{-u(\frac{2\pi m}{t})} \cdot \psi : m \in \mathbb{Z}, \psi \in C(\Omega \times V) \right\} .$$

Since $\mu$ is not the zero measure in (4.40), there exist a particular $m \in \mathbb{Z}$ and a particular $\psi \in C(\Omega \times V)$ with $\|\psi\| = 1$ such that

$$\lim_{k \to \infty} \left| \int_{[0,t] \times \Omega \times V} e^{-u(\frac{2\pi m}{t})} e^{-u} S(u) f_n(r, w) \cdot \psi(r, w) \, dw \, dr \, du \right| > 0. \quad (4.41)$$

As a consequence there is $c > 0$ and $k_0 \in \mathbb{N}$ such that for all $k > k_0$ we have

$$\left\| \int_0^t e^{-u(\frac{2\pi m}{t})} e^{-u} S(u) f_n(r, w) \, du \right\|_{L^1(\Omega \times V)} \geq \left| \int_{[0,t] \times \Omega \times V} e^{-u(\frac{2\pi m}{t})} e^{-u} S(u) f_n(r, w) \cdot \psi(r, w) \, dw \, dr \, du \right| > c. \quad (4.42)$$

On the other hand, for $m \in \mathbb{Z}$ chosen in (4.41), it holds that

$$f_n - e^{-tu} S(t) f_n = f_n - e^{-t(\mu + \frac{2\pi m}{t})} S(t) f_n = \left( \left( \mu + \frac{2\pi m \cdot i}{t} \right) \cdot id - A \right) \cdot \int_0^t e^{-u(\frac{2\pi m}{t})} \left( e^{-u} S(u) f_n \right) \, du, \quad k \in \mathbb{N}, \quad (4.43)$$

cf. [4], Lemma 1.9 of Chapter II. Introduce

$$\hat{f}_k := \int_0^t e^{-u(\frac{2\pi m}{t})} \left( e^{-u} S(u) f_n \right) \, du, \quad k \in \mathbb{N}.$$ 

Relation (4.42) says that $\lim \inf_{k \to \infty} \| \hat{f}_k \|_{L^1(\Omega \times V)} > c > 0$. Imposing now (4.35) on the left-hand side of (4.43) we verify $\lim_{k \to \infty} ((\mu + \frac{2\pi m}{t}) \hat{f}_k - A \hat{f}_k) = 0$ in $L^1(\Omega \times V)$ and, moreover with $F_k := \hat{f}_k / \| \hat{f}_k \|_{L^1(\Omega \times V)}$ for sufficiently large $k \in \mathbb{N},$

$$\lim_{k \to \infty} \left\| \left( \mu + \frac{2\pi m \cdot i}{t} \right) F_k - AF_k \right\|_{L^1(\Omega \times V)} = 0.$$ 

Also in the case when $\mu$ is not the zero measure, we have demonstrated that $\mu + \frac{2\pi m}{t} \in A \sigma(A)$ provided that $\lambda = e^{i\mu} = e^{i(\mu + \frac{2\pi m}{t})} \in A \sigma(S(t))$ where $m \in \mathbb{Z}$ is the number chosen in (4.41). In other words, together with [4], Theorems 3.6 and 3.7 of Chapter IV, we have proved the spectral mapping relation

$$\sigma(S(t)) \setminus \{0\} = \{ e^{i\mu} : \mu \in \sigma(A) \}, \quad t > 0. \quad (4.44)$$

**Step 5** For the remainder of part (c) note that $\rho(S(t))$ contains the set $\{ \lambda = e^{i\mu} : \mu \in \mathbb{M}_{m_1} \}$ because of part (b) of this corollary and the spectral mapping relation (4.44).
Recall also from Lemma 3.3 and the construction in (iii) that $S(t)$, $t \geq 0$, is a strongly continuous semigroup with operator norm $\|S(t)\| = 1$ in $B(L^1(\Omega \times V), L^1(\Omega \times V))$ for all $t \geq 0$. Consequently, $\rho(S(t))$ contains the set $\{\lambda = e^{i\mu} : \Im \mu > 0\}$ for all $t \geq 0$ by the Hille-Yosida Theorem and again the spectral mapping relation (4.44).

Let us conclude this section with a result concerning the reversibility of the semigroup $S(t)$, $t \geq 0$.

**Theorem 4.5** Let $c_r$ and $C_r$ be given as in Lemma 4.2. Suppose

$$(1 - C_{r,k_0}) c_r > 0 \quad \text{for some } k_0 \in \mathbb{N}$$

and (4.32), i.e.

$$2^{-\frac{1}{k_0}} < \omega < 1.$$  

Then $S(t)$, $t \geq 0$, extends to a strongly continuous group in $L^1(\Omega \times V)$ which we will denote by $S(t)$, $t \in \mathbb{R}$.

Proof. As an immediate consequence of Lemma 3.3 and the classical result [11], Theorem 6.5 of Chapter I, we have to show that $\lambda = 0$ does not belong to the spectrum of $S(t_0)$ for some $t_0 > 0$.

**Step 1** Assuming the contrary it follows from Corollary 1.4 (c) that $\lambda = 0$ is an isolated point in the spectrum of $S(t)$ for all $t > 0$. By [4], Proposition 1.10 in Chapter IV, $\lambda = 0$ belongs to the approximate spectrum of $S(t)$, $t > 0$, in the context of [4], Chapter IV. This means by [4], Lemma 1.9 of Chapter IV, that for all $t > 0$ there is a sequence $f_n \in L^1(\Omega \times V)$ with

$$\|f_n\|_{L^1(\Omega \times V)} = 1, \ n \in \mathbb{N}, \ \text{and} \ \|S(t)f_n\|_{L^1(\Omega \times V)} \xrightarrow{n \to \infty} 0. \quad (4.45)$$

**Step 2** It is now our turn to demonstrate that (4.45) cannot hold. For this let $0 < t < k_0(1 - C_{r,k_0}) c_r/v_{\max}$ which implies by the definitions of $c_r$ and $C_r$

$$0 < t < k_0(1 - C_{r,k_0}) c_r/v_{\max} < k_0(1 - C_{r,k_0}) \tau(r, w) \leq \sum_{i=1}^{k_0} I_i(r, w), \quad (r, w) \in D.$$  

Figuratively, this means that particles realizing the density flow $S(u)$, $0 \leq u < t$, hit the boundary $\partial \Omega$ maximally $k_0$ times. Mathematically, we have the explicit representation of $S(t)$ in (3.22), see Remark 6. Letting now in (3.22) $f_0 \in L^1(\Omega \times V)$ and $t$ be fixed as above, representation (3.22) holds for a.e. $(r, v) \in \Omega \times V$.

Recall the terminology of Definition 3.2. It follows from (3.22) and the above choice of $t$ that

$$S(t)f_0(r, v) = \chi_{\{0\}}(m)f_0(r_e, v_0) + \omega \chi_{\{1\}}(m)f_0(r_e, R_{r_1}(v_0))$$

$$+ \ldots + \omega^{k_0} \chi_{\{k_0\}}(m)f_0(r_e, R_{r_{k_0}}( \ldots R_{r_1}(v_0) \ldots ))$$

$$+ \mathbf{R}(f_0; r, v, t) \quad \text{a.e. on } (r, v) \in \Omega \times V \quad (4.46)$$

where the term $\mathbf{R}(f_0; r, v, t)$ contains all the remaining items of the right-hand side of (3.22) not appearing in the first two lines of (4.46). Introduce also

$$\mathbf{Q}(f_0; r, v, t) := \chi_{\{0\}}(m)f_0(r_e, v_0) + \omega \chi_{\{1\}}(m)f_0(r_e, R_{r_1}(v_0))$$

$$+ \ldots + \omega^{k_0} \chi_{\{k_0\}}(m)f_0(r_e, R_{r_{k_0}}( \ldots R_{r_1}(v_0) \ldots ))$$

38
where we recall that all terms which contribute to $R(f_0; \cdot, \cdot, t)$ as well as $Q(f_0; \cdot, \cdot, t)$, except for $\omega$, are functions of $(r, v) \in \Omega \times V$. We have

$$\|R(f_0; \cdot, \cdot, t)\|_{L^1(\Omega \times V)} \leq \|R(|f_0|; \cdot, \cdot, t)\|_{L^1(\Omega \times V)}$$

$$= \|S(t)|f_0|\|_{L^1(\Omega \times V)} - \|Q(|f_0|; \cdot, \cdot, t)\|_{L^1(\Omega \times V)}$$

(4.47)

since, replacing in (4.46) $f_0$ with $|f_0|$, all terms there are non-negative. In the context of Definition 3.2 we keep in mind the following. For $(r, v), (\bar{r}, \bar{v}) \in \Omega \times V$ with $(r, v) \neq (\bar{r}, \bar{v})$ let us follow the two paths $\pi$ with time range $[0, t]$ pinned at time $t$ in $(r, v)$ and $(\bar{r}, \bar{v})$ generated by deterministic reflections at the boundary $\partial \Omega$. Supposing that these paths do not terminate in an edge or vertex of $\partial \Omega$, we observe that they do not intersect both, the space as well as the velocity variable, at any time between $t$ and 0. Consequently, there is an one-to-one map a.e. on $\Omega \times V$ between the start point $(r, v) \equiv (r_0, v_0)$ at time $t$ and the end point

$$\begin{cases} 
(r_e, v_0) & \text{if no reflection between } t \text{ and } 0 \\
(r_e, R_{r_1}(\ldots R_{r_k}(v_0) \ldots)) & \text{if } k \text{ reflections between } t \text{ and } 0,
\end{cases} \quad k \in \{1, \ldots, k_0\},$$

(4.49)

at time 0 of those deterministic paths. For use in (4.49) below, we observe also that the volume element $|v|^{-1} \, dr \, dv$ is preserved along those paths. Therefore,

$$\|Q(|f_0|; \cdot, \cdot, t)\|_{L^1(\Omega \times V)} = \|Q(f_0; \cdot, \cdot, t)\|_{L^1(\Omega \times V)}$$

which together with (4.47) gives

$$\|R(f_0; \cdot, \cdot, t)\|_{L^1(\Omega \times V)} + \|Q(f_0; \cdot, \cdot, t)\|_{L^1(\Omega \times V)}$$

$$\leq \|S(t)|f_0|\|_{L^1(\Omega \times V)} = \|f_0\|_{L^1(\Omega \times V)} = \|f_0\|_{L^1(\Omega \times V)}.$$ 

(4.48)

It follows now from (4.46) and (4.48) and the just mentioned preservation of the volume element $|v|^{-1} \, dr \, dv$ that

$$\|S(t)f_0\|_{L^1(\Omega \times V)} \geq 2 \|Q(f_0; \cdot, \cdot, t)\|_{L^1(\Omega \times V)} - \|f_0\|_{L^1(\Omega \times V)}$$

$$\geq 2 \omega^{k_0} \chi_{\{0\}}(m)f_0(r_e, v_0) + \chi_{\{1\}}(m)f_0(r_e, R_{r_1}(v_0))$$

$$+ \ldots + \chi_{\{k_0\}}(m)f_0(r_e, R_{r_{k_0}}(\ldots R_{r_1}(v_0) \ldots)) - \|f_0\|_{L^1(\Omega \times V)}$$

$$\geq (2\omega^{k_0} - 1)\|f_0\|_{L^1(\Omega \times V)}$$

(4.49)

where we note that by hypothesis (4.32) we have $2\omega^{k_0} - 1 > 0$. Thus, (4.49) contradicts (4.45).

As a continuation of Lemma 3.1, we are now interested in boundedness from below and above along the group $S(t)$, $t \in \mathbb{R}$.

Corollary 4.6 Let the conditions of Theorem 4.5 be satisfied. For $f_0 \in L^\infty(\Omega \times V)$ there are finite real numbers $p_{0,\min}$ and $p_{0,\max}$ such that

$$p_{0,\min} \leq S(t)p_0 \leq p_{0,\max} \quad \text{a.e. on } \Omega \times V$$

for all $t \in \mathbb{R}$. In particular, if $p_0 \geq 0$ and $\|1/p_0\|_{L^\infty(\Omega \times V)} < \infty$ then we may suppose $p_{0,\min} > 0$.
Proof. The crucial property is that there exist real numbers $0 < q_t < g_u < \infty$ such that for the unique probability density $\gamma$ invariant with respect to $S(t)$, $t \in \mathbb{R}$, it holds that $q_t \leq \gamma \leq g_u$ a.e. on $\Omega \times V$, see Steps 5 and 6 of the proof of Lemma 3.1. The following observation shows that for

$$p_0 \in L^1(\Omega \times V) \quad \text{with} \quad p_0 \geq 0 \quad \text{and} \quad t < 0 \quad \text{we have} \quad q := S(t)p_0 \geq 0. \quad (4.50)$$

Assuming this was not the case, for $q^+ := q \vee 0$ and $q^- := (-q) \vee 0$, without loss of generality, we could suppose $\|q^+\|_{L^1(\Omega \times V)} > 0$ and $\|q^-\|_{L^1(\Omega \times V)} > 0$ by the strong continuity of $S(u)$, $t < u < 0$, in $L^1(\Omega \times V)$. Furthermore, we would obtain

$$\|p_0\|_{L^1(\Omega \times V)} = \|S(-t)q\|_{L^1(\Omega \times V)} \leq \|S(-t)q^+\|_{L^1(\Omega \times V)} + \|S(-t)q^-\|_{L^1(\Omega \times V)} = \|q^+\|_{L^1(\Omega \times V)} + \|q^-\|_{L^1(\Omega \times V)} = \|q\|_{L^1(\Omega \times V)} = \|S(t)p_0\|_{L^1(\Omega \times V)} \quad (4.51)$$

because $S(-t)$ has operator norm one in $B(L^1(\Omega \times V), L^1(\Omega \times V))$. However, comparing (4.51) with $\|S(-t)p_0\|_{L^1(\Omega \times V)} = \|p_0\|_{L^1(\Omega \times V)}$, which holds because of $p_0 \geq 0$, we could conclude

$$\|p_0\|_{L^1(\Omega \times V)} = \|q^+\|_{L^1(\Omega \times V)} + \|q^-\|_{L^1(\Omega \times V)}$$

which is impossible since $S(-t)q^+ \geq 0$ as well as $S(-t)q^- \leq 0$ and $\|S(-t)q^+\|_{L^1(\Omega \times V)} = \|q^+\|_{L^1(\Omega \times V)} > 0$ as well as $\|S(-t)q^-\|_{L^1(\Omega \times V)} = \|q^-\|_{L^1(\Omega \times V)} > 0$. We have verified (4.50). Now, the statement follows as in Step 7 of the proof of Lemma 3.1.

\[\square\]

5 \ Solutions to the Boltzmann Type Equation

We are interested in global solutions to the Boltzmann type equations (2.2) and (2.4) for $t \geq 0$ and $t \in \mathbb{R}$. It is beneficial to construct local solutions in a first step. In particular, the proof of the existence global solutions on $t \in \mathbb{R}$ uses crucially the Knudsen type group $S(t)$, $t \in \mathbb{R}$. The initial values at time zero are probability densities on $\Omega \times V$ being bounded from above and from below. Also in this section we shall suppose the global conditions (i)-(viii).

5.1 Construction of local Solutions to the Boltzmann Type Equation

By the normalization condition in (ii), $S(t)$, $t \geq 0$, given in (iii) maps $L^1(\Omega \times V)$ linearly to $L^1(\Omega \times V)$ with operator norm one. We observe furthermore that by the definitions in (iv)-(vi), for fixed $t \geq 0$, $Q$ maps $(p(\cdot, \cdot, t), q(\cdot, \cdot, t)) \in L^1(\Omega \times V) \times L^1(\Omega \times V)$ to $L^1(\Omega \times V)$ such that

$$\int_{\Omega} \int_{V} Q(p(\cdot, \cdot, t), q(\cdot, \cdot, t))(r, v) \, dv \, dr = 0 \quad (5.1)$$

and

$$\|Q(p(\cdot, \cdot, t), q(\cdot, \cdot, t))\|_{L^1(\Omega \times V)} \leq 2 \|h_r\| \|B\| \|p(\cdot, \cdot, t)\|_{L^1(\Omega \times V)} \|q(\cdot, \cdot, t)\|_{L^1(\Omega \times V)} \|L^1(\Omega \times V)\| \quad (5.2)$$

where $\| \cdot \|$ denotes the sup-norm.
Remark 12 We are mainly concerned with solutions \( p \in (L^1(\Omega \times V))^{[0,T]} \) to (2.4). Since for any \( p_0 \in L^1(\Omega \times V) \) we have (2.3) for \( S(t)p_0 \) and all \( t > 0 \) by (iii), the equation (2.4) shows that all \( p(\cdot, \cdot, t), t > 0, \) satisfy the boundary conditions (2.3).

Let \( T > 0 \) and let \( (L^1(\Omega \times V))^{[0,T]} \) be the space of all measurable real functions \( f(r,v,t), (r,v) \in \Omega \times V, t \in [0,T], \) such that \( F_f(t) := f(\cdot, \cdot, t) \in L^1(\Omega \times V) \) for all \( t \in [0,T] \) and \( F_f \in C([0,T]; L^1(\Omega \times V)). \) With the norm
\[
\|f\|_{1,T} := \sup_{t \in [0,T]} \|f(\cdot, \cdot, t)\|_{L^1(\Omega \times V)}
\]
\((L^1(\Omega \times V))^{[0,T]} \) is a Banach space.

Let \( p_0 \in L^1(\Omega \times V) \) and let \( p, q \in (L^1(\Omega \times V))^{[0,T]} \). For \((r,v,t) \in \Omega \times V \times [0,T] \) we set
\[
\Psi(p_0,p)(r,v,t) := S(t)p_0(r,v) + \lambda \int_0^t S(t-s) Q(p,p)(r,v,s) \, ds \tag{5.3}
\]
where the integral converges in \( L^1(\Omega \times V) \). We have \( \Psi(p_0,p) \in (L^1(\Omega \times V))^{[0,T]} \) and according to (5.2)
\[
\|\Psi(p_0,p)\|_{1,T} \leq \|p_0\|_{L^1(\Omega \times V)} + \lambda T \|Q(p,p)\|_{1,T} \\
\leq \|p_0\|_{L^1(\Omega \times V)} + 2\lambda T \|h_\gamma\|_1 \|B\| \|p\|_{1,T}. \tag{5.4}
\]
Moreover, if \( \lambda \leq 1/(16T \|h_\gamma\|_1 \|B\|) \) and \( \|p_0\|_{L^1(\Omega \times V)} \leq 3/2 \) then (5.4) leads to
\[
\|p\|_{1,T} \leq 2 \quad \text{implies} \quad \|\Psi(p_0,p)\|_{1,T} \leq 2. \tag{5.5}
\]
Using symmetry and bilinearity of \( Q(\cdot, \cdot) \), for \( p_0, p'_0 \in L^1(\Omega \times V) \) and \( q, p_1, p_2 \in (L^1(\Omega \times V))^{[0,T]} \) we obtain from (5.2)
\[
\|\Psi(p_0,p_1) - \Psi(p'_0,p_2)\|_{1,T} \\
\leq \max_{t \in [0,T]} \|S(t)(p_0 - p'_0)\|_{L^1(\Omega \times V)} + \lambda T \|Q(p_1 + p_2, p_1 - p_2)\|_{1,T} \\
\leq \|p_0 - p'_0\|_{L^1(\Omega \times V)} + 2\lambda T \|h_\gamma\|_1 \|B\| \|p_1 + p_2\|_{1,T} \|p_1 - p_2\|_{1,T}. \tag{5.6}
\]
It follows from (ii) and (iii) that \( \frac{d}{dt} \int_\Omega \int_V S(t)p_0(r,v) \, dr \, dv = 0, \ t \geq 0. \) Relation (5.1) gives now
\[
\frac{d}{dt} \int_\Omega \int_V \Psi(p_0,p)(r,v,t) \, dr \, dv = 0, \ t \in [0,T]. \tag{5.7}
\]
Let \( \mathbb{1} \) denote the function constant to one on \( \Omega \times V \). Furthermore, for \( t \in [0, \infty) \) introduce
\[
c_{t,\text{min}}^\mathbb{1} := \left( \sup_\tau \|1/(S(\tau) \mathbb{1})\|_{L^\infty(\Omega \times V)} \right)^{-1} \quad \text{and} \quad c_{t,\text{max}}^\mathbb{1} := \sup_\tau \|S(\tau) \mathbb{1}\|_{L^\infty(\Omega \times V)} \tag{5.8}
\]
where, for \( 0 \leq t < \infty \) the supremum is taken over \([0,t]\) and for \( t = \infty \) the supremum is taken over \([0,\infty)\). Lemma 3.11 says \( 0 < c_{t,\text{min}}^\mathbb{1} \leq c_{t,\text{max}}^\mathbb{1} < \infty, \ t \in [0,\infty]. \) By (iv)-(vi) it holds for non-negative \( p, q \in (L^1(\Omega \times V))^{[0,T]} \) that
\[
Q(p,q)(r,v,t) \geq -\|h_\gamma\|_1 \|B\| \cdot \frac{1}{2} (\|p\|_{1,T} q(r,v,t) + \|q\|_{1,T} p(r,v,t)), \tag{5.9}
\]
Let us assume
\[ \hat{p}_{t, \text{max}} := \text{ess sup}\{p(r', v', t') : r' \in \Omega, \ v' \in V, \ t' \in [0, t]\} < \infty \]
and \( \|p\|_{1, T} \leq 1 \). Relation (5.9) together with definition (5.3) and Lemma 3.1 applied to \( p_0 \) as well as \( p(\cdot, \cdot, s), s \in [0, t] \), imply that \( 0 \leq \|S(\tau - s)p(\cdot, \cdot, s)\| \leq \|S(\tau - s)(\hat{p}_{t, \text{max}} \cdot I)\| \leq \hat{p}_{t, \text{max}} c_{t, \text{max}} I \) for all \( 0 \leq s \leq \tau \leq t \) and hence

\[ \Psi(p_0, p)(r, v, t) \geq p_{0, \text{min}} - \lambda t \|h_{\gamma}\| \|B\| \cdot \hat{p}_{t, \text{max}} c_{t, \text{max}}, \quad (5.10) \]

\( r \in \Omega, \ v \in V, \ t \in [0, T] \). For the sake of clarity of the subsequent analysis we stress the different definitions of \( p_{0, \text{min}} \) as well as \( p_{0, \text{max}} \) where \( p_0 \in L^1(\Omega \times V) \), cf. Lemma 3.1 and \( \hat{p}_{t, \text{max}} \) where \( p \in (L^1(\Omega \times V))^{[0, T]} \), cf. above (5.10).

Let \( \mathcal{N} \) denote the set of all non-negative \( p_0 \in L^1(\Omega \times V) \) with \( \|1/p_0\|_{L^\infty(\Omega \times V)} < \infty \), \( \|p_0\|_{L^\infty(\Omega \times V)} < \infty \), and \( \|p_0\|_{L^1(\Omega \times V)} = 1 \). For all \( p_0 \in \mathcal{N} \), we may and do assume (5.3) with \( p_{0, \text{min}} > 0 \) and \( p_{0, \text{max}} < \infty \).

Furthermore for \( p_0 \in \mathcal{N} \), let \( \mathcal{M} \equiv \mathcal{M}(p_0) \) be the set of all \( p \in (L^1(\Omega \times V))^{[0, T]} \) such that \( \|p(\cdot, \cdot, t)\|_{L^1(\Omega \times V)} = 1 \) and \( \frac{1}{2}p_{0, \text{min}} \leq p(\cdot, \cdot, t) \leq p_{0, \text{max}} + \frac{1}{2}p_{0, \text{min}} \) a.e. on \( \Omega \times V, \ t \in [0, T] \). We note that \( \mathcal{M} \) may depend on the choice of \( p_{0, \text{min}} \) and \( p_{0, \text{max}} \).

**Lemma 5.1** Fix \( T > 0 \) as well as \( c \geq 1 \), and let \( b > 0 \) be the number defined in condition (vii).

(a) Let \( p_0 \in \mathcal{N} \) and let \( p, q \in (L^1(\Omega \times V))^{[0, T]} \) be non-negative with \( \|p\|_{1, T} \leq 1 \). Furthermore, let \( 0 \leq \beta < p_{0, \text{min}} \). If \( 0 < \lambda \leq (p_{0, \text{min}} - \beta)/(T \|h_{\gamma}\| \|B\| \cdot \hat{p}_{T, \text{max}} c_{T, \text{max}}) \) then

\[ \|\Psi(p_0, p)(\cdot, \cdot, t)\|_{L^1(\Omega \times V)} = 1 \]

and

\[ \beta \leq \Psi(p_0, p)(\cdot, \cdot, t) \leq p_{0, \text{max}} + \lambda T \|h_{\gamma}\| \|B\| \cdot b \cdot \hat{p}_{T, \text{max}} c_{T, \text{max}}, \quad t \in [0, T]. \]

(b) Let \( p_0 \in \mathcal{N} \) and let \( p, q \in (L^1(\Omega \times V))^{[0, T]} \) be non-negative with \( \|p\|_{1, T} \leq 1 \). Furthermore, let \( \beta = \frac{1}{2}p_{0, \text{min}}, \quad \hat{p}_{T, \text{max}} \leq c \|q\|_{1, T} \cdot (p_{0, \text{max}} + \frac{1}{2}p_{0, \text{min}}), \) and \( \hat{p}_{T, \text{max}} \leq p_{0, \text{max}} + \frac{1}{2}p_{0, \text{min}} \). If

\( 0 < \lambda \leq \frac{1}{2}p_{0, \text{min}}/(T \|h_{\gamma}\| \|B\| (\|p_1\|_{1, T} + \|p_2\|_{1, T})) \) then we have

\[ \frac{1}{2}p_{0, \text{min}} \leq \Psi(p_0, p)(\cdot, \cdot, t) \leq p_{0, \text{max}} + \frac{1}{2}p_{0, \text{min}}, \quad t \in [0, T]. \]

(c) Let \( p_0 \in L^1(\Omega \times V) \) and \( p_1, p_2, q \in (L^1(\Omega \times V))^{[0, T]} \) and \( \delta \in (0, 1) \). For \( 0 < \lambda \leq \delta/(2T \|h_{\gamma}\| \|B\| (\|p_1\|_{1, T} + \|p_2\|_{1, T})) \) we have

\[ \|\Psi(p_0, p_1) - \Psi(p_0, p_2)\|_{1, T} \leq \delta \|p_1 - p_2\|_{1, T}. \quad (5.11) \]

(d) In particular, let \( p_0 \in \mathcal{N}, \ p_1, p_2 \in \mathcal{M}, \ q \in (L^1(\Omega \times V))^{[0, T]} \) be non-negative, and \( \delta \in (0, 1) \). For

\[ 0 < \lambda \leq \frac{3\delta}{8}p_{0, \text{min}}/(T \|h_{\gamma}\| \|B\| \cdot (p_{0, \text{max}} + \frac{1}{2}p_{0, \text{min}}) c_{T, \text{max}}), \]

we have \( \Psi(p_0, p_1), \Psi(p_0, p_2) \in \mathcal{M} \) and (5.11).
Proof. Recalling \( \|p\|_{1,T} \leq 1 \) and looking at representation (2.7) we find
\[
\Psi(p_0, p)(\cdot, \cdot, t) \leq p_{0, \max} + \lambda \|h_\gamma\| b \cdot \int_0^t S(t-s)(\hat{p}_{T, \max} \cdot I) \, ds, \quad t \in [0, T].
\]
Parts (a) and (b) of the Lemma are now a consequence of (5.3), (5.7), (5.10), and (2.7) together with (vii). For part (b) we note that \( \|B\| \leq b \), cf. Section 2. Parts (c) and (d) follow from (5.6), and parts (a) and (b). For part (d) we note that \( c_{T, \max}^1 \geq 1 \).

Let \( T > 0 \). Let us iteratively construct a solution \( p \equiv p(p_0) \) to (2.4) restricted to \((r, v, t) \in \Omega \times V \times [0, T]\) By (i)-(iii) we have then the boundary conditions (2.3). Set
\[
p^{(0)}(\cdot, \cdot, t) := p_0, \quad p^{(n)}(\cdot, \cdot, t) := \Psi \left( p_0, p^{(n-1)} \right)(\cdot, \cdot, t), \quad t \in [0, T], n \in \mathbb{N}.
\]
We note that with \( d(q_1, q_2) := \|q_1 - q_2\|_{1,T}, q_1, q_2 \in \mathcal{M} \), the pair \((\mathcal{M}, d)\) is a complete metric space. Furthermore, Lemma 5.1 (a) and (b) imply that, for \( \lambda \) as in the first part of Lemma 5.1 (b), \( q \in \mathcal{M} \equiv \mathcal{M}(p_0) \) yields \( \Psi(p_0, q) \in \mathcal{M} \). An immediate consequence of Lemma 5.1 (d) and the Banach fixed point theorem is now part (b) of the following proposition. The first part of (a) is a consequence of Lemma 5.1 (c), (5.5), and again the Banach fixed point theorem. The continuity statement in the second part of (a) follows from (5.6).

**Proposition 5.2** Fix \( T > 0 \) and let \( b > 0 \) be the number defined in condition (vii).

(a) Fix \( 0 < \lambda \leq 1/(16T \|h_\gamma\| \|B\|) \). Let \( p_0 \in L^1(\Omega \times V) \) with \( \|p_0\|_{L^1(\Omega \times V)} \leq 3/2 \). There is a unique element \( p \equiv p(p_0) \in (L^1(\Omega \times V))^{[0,T]} \) such that
\[
p = \Psi(p_0, p).
\]
The map \( \{q_0 \in L^1(\Omega \times V) : \|q_0\|_{L^1(\Omega \times V)} \leq 3/2 \} \ni p_0 \mapsto p(p_0) \in (L^1(\Omega \times V))^{[0,T]} \) is continuous.

(b) Assume \( p_0 \in \mathcal{N} \) and \( 0 < \lambda < \frac{3}{2}p_{0, \min}/(T\|h_\gamma\|b \cdot (p_{0, \max} + \frac{1}{2}p_{0, \min}) c_{T, \max}^1) \). There is a unique element \( p \equiv p(p_0) \in \mathcal{M} \) such that
\[
p = \Psi(p_0, p).
\]
In other words, for \( p_0 \in \mathcal{N} \) the equation (2.4), restricted to \((r, v, t) \in \Omega \times V \times [0, T]\), has a unique solution \( p \equiv p(p_0) \). This solution satisfies
\[
\frac{1}{2}p_{0, \min} \leq p(p_0)(\cdot, \cdot, t) \leq p_{0, \max} + \frac{1}{2}p_{0, \min} \quad \text{a.e. on } \Omega \times V
\]
and \( \|p(p_0)(\cdot, \cdot, t)\|_{L^1(\Omega \times V)} = 1, t \in [0, T] \). Furthermore,
\[
\lim_{n \to \infty} \|p^{(n)} - p(p_0)\|_{1,T} = 0.
\]

### 5.2 Global Solutions to the Boltzmann Type Equation for \( t \geq 0 \) and \( t \in \mathbb{R} \)

Lemma 5.3 and Theorem 1.5 will now be applied to investigate the existence of solutions to the integrated (mild) version of the Boltzmann equation (2.4). As above, let \((A, D(A))\) denote the infinitesimal operator of the strongly continuous semigroup \( S(t), t \geq 0, \) in \( L^1(\Omega \times V) \). Obviously, \((A, D(A))\) is also the infinitesimal operator of the strongly continuous group \( S(t), t \in \mathbb{R}, \) whenever the hypotheses of Theorem 1.5 are satisfied.
Proposition 5.3 (a) Let $p_0 \in L^1(\Omega \times V)$. There exists $T_{\text{max}} \equiv T_{\text{max}}(p_0) \in (0, \infty]$ such that the following holds. The equation (2.4) with $p(\cdot, \cdot, 0) = p_0$ has a unique solution $p(\cdot, \cdot, t) \in L^1(\Omega \times V)$, $t \in [0, T_{\text{max}})$, which is continuous in $t \in [0, T_{\text{max}})$ with respect to the topology in $L^1(\Omega \times V)$. Moreover, if $T_{\text{max}} < \infty$ then $\lim_{t \to T_{\text{max}}} \|p(\cdot, \cdot, t)\|_{L^1(\Omega \times V)} = \infty$.

(b) Let $p_0 \in D(A)$. The solution $p(\cdot, \cdot, t)$, $t \in [0, T_{\text{max}})$, to the equation (2.4) given in part (a) satisfies

$$\frac{d}{dt} p(\cdot, \cdot, t) = Ap(\cdot, \cdot, t) + \lambda Q(p, p)(\cdot, \cdot, t), \quad t \in [0, T_{\text{max}}),$$

with $p(\cdot, \cdot, t) \in D(A)$. Here, $d/dt$ is a derivative in $L^1(\Omega \times V)$. At $t = 0$ it is the right derivative.

(c) Let $p_0 \in L^1(\Omega \times V)$. For the solution $p(\cdot, \cdot, t)$, $t \in [0, T_{\text{max}})$, to the equation (2.4) given in part (a) it holds that

$$\int_\Omega \int_V p_0(r, v) dv \, dr = \int_\Omega \int_V p(r, v, t) dv \, dr, \quad t \in [0, T_{\text{max}}).$$

Proof. **Step 1** We prove part (a). We recall that the equation (2.4) is the integrated (mild) version of (5.12) with boundary conditions included in $p(\cdot, \cdot, t) \in D(A)$. For $q_1, q_2 \in L^1(\Omega \times V)$ we have

$$\|Q(q_1, q_1) - Q(q_2, q_2)\|_{L^1(\Omega \times V)} \leq \|Q(q_2, q_1 - q_2, q_1 - q_2)\|_{L^1(\Omega \times V)} \leq \|B\| \|h_\gamma\| \|q_1 + q_2\|_{L^1(\Omega \times V)} \|q_1 - q_2\|_{L^1(\Omega \times V)},$$

i.e. $L^1(\Omega \times V) \ni q \mapsto Q(q, q) \in L^1(\Omega \times V)$ is locally Lipschitz continuous with constant $2C\|B\| \|h_\gamma\|$ on $\{q \in L^1(\Omega \times V) : \|q\|_{L^1(\Omega \times V)} \leq C\}$ for any $C > 0$. Keeping the previous lemma in mind, [11] Theorem 1.4 of Chapter 6, says now that there is a unique solution $p$ to (2.4) with $p(\cdot, \cdot, 0) = p_0$ on some time interval $t \in [0, T_{\text{max}})$ such that

$$T_{\text{max}} < \infty \quad \text{implies} \quad \lim_{t \to T_{\text{max}}} \|p(\cdot, \cdot, t)\|_{L^1(\Omega \times V)} = \infty.$$

Furthermore, the just quoted source says also that $p(\cdot, \cdot, t)$ is continuous on $t \in [0, T_{\text{max}})$ with respect to the topology in $L^1(\Omega \times V)$.

**Step 2** We prove part (b). For $h \in L^1(\Omega \times V)$ we have

$$\|Q(q + h, q + h) - Q(q, q) - 2Q(q, h)\|_{L^1(\Omega \times V)} = \|Q(h, h)\|_{L^1(\Omega \times V)} / \|h\|_{L^1(\Omega \times V)}$$

which converges to zero as $\|h\|_{L^1(\Omega \times V)} \to 0$ by (5.2). Therefore the map $L^1(\Omega \times V) \ni q \mapsto Q(q, q) \in L^1(\Omega \times V)$ is Fréchet differentiable. The Fréchet derivative at $q \in L^1(\Omega \times V)$ has the representation

$$\nabla Q(q, q)(\cdot) = 2Q(\cdot, q).$$

Next we remind of the notation $B \equiv B(L^1(\Omega \times V), L^1(\Omega \times V))$ of the space of all bounded linear operators $L^1(\Omega \times V) \to L^1(\Omega \times V)$ endowed with the operator norm. We observe that

$$\|\nabla Q(p, p)(\cdot) - \nabla Q(q, q)(\cdot)\|_B = 2\|Q(\cdot, p) - Q(\cdot, q)\|_B \sup_{\|h\|_{L^1(\Omega \times V)} = 1} \|Q(h, p - q)\|_{L^1(\Omega \times V)}.$$
converges to zero as \( \|p - q\|_{L^1(\Omega \times V)} \to 0 \) by (5.2). Thus the Fréchet derivative \( \nabla Q \) as a map \( L^1(\Omega \times V) \to B(L^1(\Omega \times V), L^1(\Omega \times V)) \) is continuous on \( L^1(\Omega \times V) \) is continuous. Now Theorem 1.5 of Chapter 6, says that if \( p_0 \in D(A) \) then the solution \( p(\cdot, \cdot, t) \) to (2.4) satisfies \( p(\cdot, \cdot, t) \in D(A) \) as well as (6.12).

**Step 3** Part (c) is an immediate consequence of (5.7).

We mention that, even if \( p(\cdot, \cdot, 0) = p_0 \) is positively bounded from below, it is not yet clear whether or not \( p(\cdot, \cdot, t) \) is a.e. non-negative for all \( t \in [0, T_{\text{max}}) \). We will address this problem in Lemma 5.5. Together with Proposition 5.3 this will lead to global solutions of (2.4), i.e., to solutions for \( t \in [0, \infty) \). See Theorem 5.6 and Corollary 5.7.

Let \( p_0 \) be a probability density and let \( 0 < T < T_{\text{max}}(p_0) \). Denote by \( p \) the solution to (2.4) on \( \Omega \times V \times [0, T] \) with \( p(\cdot, \cdot, 0) = p_0 \) given in Lemma 5.3 (a). Note that in Proposition 5.3 (a) it has not been shown that \( p \) is non-negative. However, the continuity with respect to \( L^1(\Omega \times V) \) of the map \([0, T] \ni t \mapsto p(\cdot, \cdot, t)\), stated in Proposition 5.3 (a), implies that

\[
\|p\|_{1,T} \leq C_T \quad \text{for some} \ 0 < C_T < \infty.
\]  

Let us recall the notation of the introductory part to the present Section 3. In the present subsection we shall use the decomposition \( Q(p, p) = Q^+(p, p) - Q^-(p, p) \) of the collision operator specified by

\[
Q^+(p, p)(r, v, t) = \int_{V} \int_{S_+} B(v, v_1, e) p(r, v^*, t) p_\gamma(r, v_1^*, t) \chi_{\{(v^*, v_1^*)\in V \times V_1\}} dv_1 \quad (5.14)
\]

in the sense of an element in \( L^1(\Omega \times V) \). The expression

\[
\hat{B}(v, v_1) := \int_{S_+} B(v, v_1, e) \cdot \chi_{\{(v^*, v_1^*)\in V \times V_1\}} dv_1 \quad (5.15)
\]

is by condition (v) well-defined for \((v, v_1) \in V \times V \). Keeping in mind (vi) and the definition of \( p_\gamma \) in the introductory part to the present Section 3 the term

\[
\hat{B}_p(r, v, t) := \lambda \int_{V} \hat{B}(v, v_1) p_\gamma(r, v_1, t) dv_1
\]

is well-defined and bounded on \((r, v, t) \in \Omega \times V \times [0, T] \). Here

\[
\left| \hat{B}_p(r, v, t) \right| \leq \lambda h_\gamma \|B\| \|p(\cdot, \cdot, t)\|_{L^1(\Omega \times V)} \leq C_T \lambda h_\gamma \|B\|, \quad (5.13)
\]

cf. also (5.13). In fact, we have \( Q^-(p, p)(\cdot, \cdot, t) = p(\cdot, \cdot, t) \hat{B}_p(\cdot, \cdot, t) \in L^1(\Omega \times V) \). Moreover, by (vi), the map \( \Omega \ni r \mapsto p_\gamma(r, \cdot, t) \) is bounded and uniformly continuous with respect to the topology of \( L^1(V) \) for any \( t \in [0, T] \). Thus by (v), \( \hat{B}_p(\cdot, \cdot, t) \) is bounded and continuous on \( \Omega \times V \) for any \( t \in [0, T] \).

**Remark 13** Let \( t \in [0, T] \). According to (2.4) and Proposition 5.3 (a) \([0, t] \ni s \mapsto S(t - s) Q(p, p)(\cdot, \cdot, s) \) is Bochner integrable, i.e.

\[
\int_0^t \|S(t - s) Q(p, p)(\cdot, \cdot, s)\|_{L^1(\Omega \times V)} ds < \infty.
\]
By \((5.13)\) we have \(\int_0^t \|p(\cdot, s)\|_{L^1(\Omega \times V)} \, ds < \infty\). Together with \((5.15)\), \(\|S(t-s)\|_B = 1\) as the operator norm in \(B(L^1(\Omega \times V), L^1(\Omega \times V))\), and \(Q^-(p, s)(\cdot, s) = p(\cdot, s)\hat{B}_p(\cdot, s)\), \(s \in [0, t]\), this leads to
\[
\int_0^t \|S(t-s)Q^-(p, s)(\cdot, s)\|_{L^1(\Omega \times V)} \, ds < \infty
\]
as well as
\[
\int_0^t \|S(t-s)Q^+(p, s)(\cdot, s)\|_{L^1(\Omega \times V)} \, ds < \infty.
\]
In other words, \(S(t-\cdot)Q^-(p, p) \in L^1([0, t]; L^1(\Omega \times V))\) as well as \(S(t-\cdot)Q^+(p, p) \in L^1([0, t]; L^1(\Omega \times V))\). According to [8], Appendix C, both integrals
\[
\int_0^t S(t-s)Q^-(p, p)(\cdot, s) \, ds \quad \text{as well as} \quad \int_0^t S(t-s)Q^+(p, p)(\cdot, s) \, ds
\]
can be evaluated a.e. on \(\Omega \times V\) as ordinary Lebesgue integrals. Replacing the bounds, these integrals can also be evaluated over arbitrary Borel subsets of \([0, t]\) instead of over the whole interval \([0, t]\).

**Remark 14** From Remark [13] and \((2.4)\) we obtain for a.e. \((r, v) \in \Omega \times V\) and all \(0 \leq \tau \leq T_{\Omega}(r, v) \wedge t\)
\[
p(r - \tau v, v, t - \tau) - p(r, v, t)
\]
\[
= -\int_0^\tau \lambda Q(p, p)(r - sv, v, t - s) \, ds
\]
\[
= \int_0^\tau \lambda Q^-(p, p)(r - sv, v, t - s) \, ds - \int_0^\tau \lambda Q^+(p, p)(r - sv, v, t - s) \, ds \quad (5.16)
\]
which includes \(\int_0^\tau |\lambda Q^-(p, p)(r - sv, v, t - s)| \, ds < \infty\) as well as \(\int_0^\tau |\lambda Q^+(p, p)(r - sv, v, t - s)| \, ds < \infty\). By approximation on \(\tau \in \{ta : a \in [0, 1]\}\) is rational) along rays \(\tau \mapsto r - \tau v\) relation \((5.16)\) holds also for a.e. \((r, v) \in \partial \Omega \times V\) with \(v \circ n(r) \geq 0\).

For the next lemma we stress that \(\hat{B}_p\) is well-defined and bounded on \(\overline{\Omega} \times V \times [0, T]\) for any \(0 < T < T_{\max}\), cf. \((5.15)\). We remind of the notation \(\text{diam}(\Omega) = \sup\{|r_1 - r_2| : r_1, r_2 \in \Omega\}\) and denote \(T_m := \text{diam}(\Omega)/v_{\min}\).

**Lemma 5.4** Let \(p_0 \in L^1(\Omega \times V)\) and \(0 < T < T_{\max}\). Let \(p(\cdot, s), s \in [0, T]\) be the solution to \((2.4)\) with \(p(\cdot, 0) = p_0\) given in Proposition 5.3 (a). Then for all \(t \in [0, T]\), all \((r, v) \in \Omega \times V\) as well as all \((r, v) \in \partial^{(1)} \Omega \times V\) such that \(v \circ n(r) \geq 0\), and all \(0 \leq \tau \leq T_{\Omega}(r, v) \wedge t\) we have
\[
0 < \exp \left\{ -\lambda(T_m \wedge T) \|h_\tau\|B \right\}
\]
\[
\leq \psi(r, v, t; \tau) := \exp \left\{ \int_0^\tau \hat{B}_p(r - sv, v, t - s) \, ds \right\}
\]
\[
\leq \exp \left\{ \lambda(T_m \wedge T) \|h_\tau\|B \right\} < \infty.
\]
Suppose \((5.10)\) for some \(r, v, t, \tau\) as above. Then
\[
p(r - \tau v, v, t - \tau)
\]
\[
= \psi(r, v, t; \tau) \left( -\int_0^\tau \frac{\lambda Q^+(p, p)(r - sv, v, t - s)}{\psi(r, v, t; s)} \, ds + p(r, v, t) \right). \quad (5.17)
\]
Proof. As already noted below (5.15), \( \hat{b}_p(\cdot, \cdot, t) \) is bounded and continuous on \( \Omega \times V \) for any \( t \in [0, T] \). In particular, \( [0, T_\Omega \wedge t] \ni \tau \mapsto \hat{b}_p(r - \tau v, t) \) is continuous for all \( (r, v) \in \Omega \times V \) or \( (r, v) \in \partial^{(1)} \Omega \times V \) with \( v \circ n(r) \geq 0 \). We note also that by (5.15) we have
\[
0 < \exp \{ -\lambda(T_m \wedge T) \| h_\gamma \| \| B \| \} \leq \inf \psi
\]
as well as
\[
\sup \psi \leq \exp \{ \lambda(T_m \wedge T) \| h_\gamma \| \| B \| \} < \infty
\]
where both, the infimum as well as the supremum, are taken over the set \( \{ (r, v, t, \tau) : (r, v) \in \Omega \times V, t \in [0, T], \tau \in [0, T_\Omega(r, v) \wedge t] \} \).

For the rest of the proof, let \( (r, v) \in \Omega \times V \) or \( (r, v) \in \partial^{(1)} \Omega \times V \) with \( v \circ n(r) \geq 0 \) such that we have (5.16). By the just mentioned boundedness of \( \psi \), for
\[
\varphi(\tau, t) := p(r, v, t + \tau) \psi(r, v, t + \tau; \tau)
\]
\[
= p(r, v, t + \tau) \exp \left\{ \int_0^\tau \hat{b}_p(r - sv, v, t + \tau - s) \, ds \right\}, \quad \tau \in [0, T_\Omega \wedge t], \quad (5.18)
\]
the map \( [0, T_\Omega \wedge t] \ni \tau \mapsto \varphi(\tau, t - \tau), \tau \in [0, T_\Omega \wedge t] \), is well-defined whenever (5.16). From (5.18) we get
\[
\varphi(\tau, t - \tau) - \varphi(0, t) = p(r, v, t) \left( \exp \left\{ \int_0^\tau \hat{b}_p(r - sv, v, t - s) \, ds \right\} - 1 \right)
\]
\[
= \int_0^\tau \hat{b}_p(r - sv, v, t - s) p(r, v, t) \exp \left\{ \int_0^s \hat{b}_p(r - uv, v, t - u) \, du \right\} \, ds
\]
\[
= \int_0^\tau \hat{b}_p(r - sv, v, t - s) \varphi(s, t - s) \, ds . \quad (5.19)
\]
Recalling Remarks 13 and 14 we note that
\[
f(\tau, t) := \psi(r, v, t + \tau; \tau) \left( - \int_0^\tau \frac{\lambda Q(p, p)(r - sv, v, t + \tau - s)}{\psi(r, v, t + \tau; s)} \, ds + p(r, v, t + \tau) \right) \quad (5.20)
\]
\( \tau \in [0, T_\Omega \wedge t] \), is well-defined for \( (r, v) \) as chosen for this proof. Moreover we observe that by (5.18)
\[
f(\tau, t) = - \psi(r, v, t + \tau; \tau) \int_0^\tau \frac{\lambda Q(p, p)(r - sv, v, t + \tau - s)}{\psi(r, v, t + \tau; s)} \, ds + \varphi(\tau, t) . \quad (5.21)
\]
Using (5.18)-(5.21) the following is a straightforward calculation,
\[
f(\tau, t - \tau) - f(0, t) = - \psi(r, v, t; \tau) \int_0^\tau \frac{\lambda Q(p, p)(r - sv, v, t - s)}{\psi(r, v, t; s)} \, ds
\]
\[
+ \varphi(\tau, t - \tau) - \varphi(0, t)
\]
\[
= - \exp \left\{ \int_0^\tau \hat{b}_p(r - uv, v, t - u) \, du \right\} \int_0^\tau \frac{\lambda Q(p, p)(r - sv, v, t - s)}{\psi(r, v, t; s)} \, ds
\]
\[
+ \int_0^\tau \hat{b}_p(r - sv, v, t - s) \varphi(s, t - s) \, ds
\]
Let us look at (5.22) as an equation for \([0,T](5.22)\) under the initial condition obtained from (5.20) replacing there \(t\) by \(\tau\) of Proposition 5.3 (a). If there exist constants \(\lambda > c\), (5.23), (5.24) is false.

Let \(\text{Lemma 5.5} \rightleftharpoons (5.17)\) .

Proof. Without loss of generality suppose \(\lambda > c\) such that there is \(t < \tau\) for some \(\lambda > c\). Let \(\textbf{v} = \textbf{0}\) and \(\textbf{v} = \textbf{0}\) a.e. on \(\Omega \times V\). \(\Omega \times V\) and assume now that there is \(t \in [T, T_{\max})\) such that

\[
\text{leb}\left(\{(r, v) \in \Omega \times V : p(r, v, s) \leq 0\} \right) = 0 \quad \text{for} \quad 0 \leq s < t
\]

and

\[
\text{leb}\left(\{(r, v) \in \Omega \times V : p(r, v, s) \leq 0\} \right) > 0 \quad \text{for} \quad s = t \quad \text{or} \quad t < s < s_1
\]

for some \(t < s_1 < T_{\max}\). Note that the opposite to this assumption is \(p(\cdot, \cdot, s) > 0\) a.e. on \(\Omega \times V\) for every \(s \in [0, T_{\max})\). Therefore it is our objective to show that the assumption \((5.23), (5.24)\) is false.

Let us introduce the times

\[
t_k \equiv t_{\Omega,k}(r_0, v_0, \ldots, v_k) = t \land \sum_{l=0}^k T_\Omega(r_l, v_l), \quad k \in \mathbb{Z}_+,
\]

a slight modification of the times \(t_k\) used in (3.20). Let \(t \in [0, T_{\max})\). For a.e. \((r, v) \in (r_0, v_0) \in \Omega \times V\) and a.e. \((r, v) \equiv (r_0, v_0) \in \partial(1)\Omega \times V\) with \(v \circ n(r) \geq 0\) an iteration of (5.11)
leads to
\[
\begin{align*}
p(r, v, t) &= \left( \int_0^{t_0} \frac{\lambda Q^+(p, p)(r_0 - sv_0, v_0, t - s)}{\psi(r_0, v_0, t; s)} \, ds \right. \\
& \quad + \frac{\omega \chi_{[1, \infty)}(m)p(r_1, R_{r_1}(v_0), t - \tilde{t}_0) + \chi_{[0)}(m)p_0(r_e, v_0)}{\psi(r_0, v_0, t; \tilde{t}_0)} \\
& \left. + \sum_{k=1}^{\infty} (1 - \omega)M(r_1, v_0) \int_{v_1 \circ n(r_1) \geq 0} (v_1 \circ n(r_1)) \ldots \times \\
& \quad \times (1 - \omega)M(r_k, v_k-1) \int_{v_k \circ n(r_k) \geq 0} (v_k \circ n(r_k)) \times \\
& \quad \left( \chi_{[k, \infty)}(m) \int_{\tilde{t}_k-1}^{\tilde{t}_k} \frac{\lambda Q^+(p, p)(r_k - (s - \tilde{t}_{k-1})v_k, v_k, t - s)}{\psi(r_k, v_k, t - \tilde{t}_{k-1}; s - \tilde{t}_{k-1})} \, ds \\
& \quad + \frac{\omega \chi_{[k+1, \infty)}(m)p(r_{k+1}, R_{r_{k+1}}(v_k), t - \tilde{t}_{k+1}) + \chi_{[k)}(m)p_0(r_e, v_k)}{\psi(r_k, v_k, t - \tilde{t}_{k-1}; t_k - \tilde{t}_{k-1})} \right) \, dv_k \ldots \, dv_1 \right) \\
& \quad \frac{(5.25)}{}
\end{align*}
\]

where we note that the structure of this sum is a slight reordering of the structure of the sums \((3.20)\) and \((3.22)\). By the assumption \((5.23)\), \((5.24)\) all summands are non-negative. Thus, the infinite sum converges since the iteration of \((5.17)\) shows that any partial sum is bounded by the non-negative \(p(r, v, t)\). Recall also \((3.20)\).

The infinite sum on the right-hand side of \((5.25)\) is not smaller than the term obtained by replacing there \(\int_{v_k \circ n(r_k) \geq 0} \) with \(\int_{v_k \circ n(r_k) \geq \varepsilon} \) for some arbitrary \(\varepsilon \in (0, 1)\), for all \(k \in \mathbb{N}\). The term obtained covers a subset of paths in \(\pi(r, v, t)\) consisting of boundedly many, say at most \(k_{\varepsilon}\) rays. For an explanation, observe that the velocities \(v_0, v_1, \ldots, v_e\) are bounded which makes, because of \(v_k \circ n(r_k) \geq \varepsilon\), the length \(T_{\Omega}(r_k, v_k)\) of the time interval along a single ray of a path bounded from below by some positive time.

In other words, replacing on the right-hand side of \((5.25)\) \(\int_{v_k \circ n(r_k) \geq 0} \) by \(\int_{v_k \circ n(r_k) \geq \varepsilon} \) the resulting term consists of products of at most \(k_{\varepsilon}\) factors. Let us also mention that \((5.23)\) implies non-negativity of all terms \(Q^+\) in \((5.25)\). According to \(\sup \psi < \infty\) by Lemma \(5.4\), \(M(r_{k+1}, v_k) \geq M_{\text{min}} > 0\) by (ii), \(1 - \omega > 0\), and the hypothesis \(p_0 > c > 0\), the right-hand side and hence the left-hand side of \((5.25)\) is bounded from below by some positive constant. The latter and \((5.17)\) imply that \((5.24)\) cannot hold.

As already mentioned, this means \(p(\cdot, \cdot, t) > 0\) a.e. on \(\Omega \times V\) for every \(t \in [0, T_{\text{max}}]\). Now we repeat the reasoning of the last paragraph for arbitrary \(t \in [0, T_{\text{max}}]\). We obtain that the left-hand side of \((5.25)\) is bounded from below by some positive constant \(c_t\) for every \(t \in [0, T_{\text{max}}]\).

Let \(b > 0\) be the number defined in condition (vii). Furthermore, let us remind of the notations \(c_{\text{max}} = \sup_{t \geq 0} \|S(t)\|_{L^\infty(\Omega \times V)}\), which is finite by Lemma \(3.1\) and \(p_{0, \text{max}} \equiv p_{0, \text{max}}(p_0)\), cf. Lemma \(3.1\). We may now state the following result on the existence of global solutions to the equation \((2.4)\).

**Theorem 5.6** Let \(p_0 \in L^1(\Omega \times V)\) with \(\|p_0\|_{L^1(\Omega \times V)} = 1\) and suppose that there are constants \(0 < c \leq C < \infty\) with \(c \leq p_0 \leq C\) a.e. on \(\Omega \times V\). Then there exists a unique solution \(p\) to \((2.4)\) on \(\Omega \times V \times [0, \infty)\) with \(p(\cdot, \cdot, 0) = p_0\). The solution \(p \equiv p(p_0)\) to \((2.4)\) has the following properties.
(1) The map \([0, \infty) \ni t \mapsto p(\cdot, \cdot, t) \in L^1(\Omega \times V)\) is continuous with respect to the topology in \(L^1(\Omega \times V)\).

(2) We have \(\|p(\cdot, \cdot, t)\|_{L^1(\Omega \times V)} = 1\), \(t \geq 0\).

(3) For every \(t \geq 0\), there exists a constant \(c_t > 0\) such that \(p(\cdot, \cdot, t) \geq c_t \) a.e. on \(\Omega \times V\).

(4) We have

\[
\|p(\cdot, \cdot, t)\|_{L^\infty(\Omega \times V)} \leq p_{0,\max} \cdot \exp \left\{ \lambda \|h_\gamma\| b c_{\infty,\max}^I \cdot t \right\}, \quad t \in [0, \infty).
\]

Proof. **Step 1** The properties (1)-(3) are an immediate consequence of Proposition 5.3, parts (a) as well as (c), and Lemma 5.5. In particular we obtain \(T_{\max} = \infty\).

Let us verify (4). It follows from (2.4), (2.7), and property (3) of this theorem that

\[
\|p(\cdot, \cdot, t)\|_{L^\infty(\Omega \times V)} \leq p_{0,\max} + \lambda \|h_\gamma\| b \int_0^t S(t - s)(\|p(\cdot, \cdot, s)\|_{L^\infty(\Omega \times V)} \cdot 1) \, ds
\]

\[
\leq p_{0,\max} + \lambda \|h_\gamma\| b c_{\infty,\max}^I \int_0^t \|p(\cdot, \cdot, s)\|_{L^\infty(\Omega \times V)} \, ds, \quad t \geq 0,
\]

where we have not yet excluded that both sides are infinity.

Recalling that \(c_{T_{\max}}^I \leq c_{\infty,\max}^I < \infty\) we may use Proposition 5.2 (b) to claim that there is some \(T \equiv T(\lambda, p_0) > 0\) such that \(\|p(\cdot, \cdot, t)\|_{L^\infty(\Omega \times V)}\) is bounded on \(t \in [0, T]\). This allows to apply Grönwall’s inequality to (5.26) in order to obtain (4) for \(t \in [0, T]\).

**Step 2** We keep in mind the particular form of (4) on every time interval \(t \in [0, T_1]\) on which the left-hand side and hence the right-hand side of (5.26) is finite. In order to apply Grönwall’s inequality on \(t \geq 0\) it is sufficient to show that there is no \(t_0 > 0\) such that

\[
\limsup_{t \uparrow t_0} \|p(\cdot, \cdot, t)\|_{L^\infty(\Omega \times V)} < \infty \quad \text{and} \quad \|p(\cdot, \cdot, t_0)\|_{L^\infty(\Omega \times V)} = \infty.
\]

This is what we are concerned with in the remainder of the proof. We therefore assume that there was a \(t_0 > 0\) with (5.27). It is our aim to lead this assumption to a contradiction.

Let \(n \in \mathbb{N}\) and \(B_n := \{(r, v) : p(r, v, t_0) > n\}\). By the assumption we have \(\text{leb}(B_n) > 0\). Recalling Step 1 of the proof of Lemma 5.3 let us choose \(\varepsilon_n > 0\) such that \(\text{leb}(B_n \cap \Omega_{\varepsilon_n}) > 0\) and \(t_0 \geq \varepsilon_n/(2v_{\max})\). For \(s := \varepsilon_n/(2v_{\max})\) we have

\[
r - \tau v \in \Omega \quad \text{for all } \tau \in [0, s], \ r \in \Omega_{\varepsilon_n}, \ v \in V.
\]

This implies \(\{r - \tau v : (r, v) \in (B_n \cap \Omega_{\varepsilon_n}) \times V\} \subset \Omega\) and

\[
\text{leb} \left( \{r - \tau v : (r, v) \in (B_n \cap \Omega_{\varepsilon_n}) \times V\} \right) \geq c_n, \quad \tau \in [0, s],
\]

for some \(c_n > 0, n \in \mathbb{N}\). Choosing now \(n > \limsup_{t \uparrow t_0} \|p(\cdot, \cdot, t)\|_{L^\infty(\Omega \times V)}\) relations (5.27) and (5.28) contradict (5.17).

We have verified that there is no \(t_0 > 0\) with (5.27). Applying now Grönwall’s inequality to (5.26) we obtain (4) for \(t \geq 0\). \(\Box\)

Recall that \((A, D(A))\) denotes the infinitesimal operator of the strongly continuous semigroup \(S(t), t \geq 0\), in \(L^1(\Omega \times V)\). Together with Proposition 5.3 (b) we obtain the following.
Corollary 5.7 Let $p_0 \in D(A)$ with $\|p_0\|_{L^1(\Omega \times V)} = 1$ and suppose that there are constants $0 < c \leq C < \infty$ with $c \leq p_0 \leq C$ a.e. on $\Omega \times V$. Then there is a unique solution $p(\cdot, \cdot, t) \in D(A), t \in [0, \infty)$, to the equation

$$\frac{d}{dt} p(\cdot, \cdot, t) = Ap(\cdot, \cdot, t) + \lambda Q(p, p)(\cdot, \cdot, t)$$

with $p(\cdot, \cdot, 0) = p_0$. Here, $d/dt$ is a derivative in $L^1(\Omega \times V)$. At $t = 0$ it is the right derivative. This solution coincides with the solution to the equation (2.4) with $p(\cdot, \cdot, 0) = p_0 \in D(A)$ of Theorem 5.6 and has therefore the properties (1)-(4) of Theorem 5.6.

Let us recall Theorem 4.5 and Corollary 4.6. The main result of the paper is the following.

Theorem 5.8 Suppose that the conditions of Theorem 4.5 are satisfied.

(a) Let $p_0 \in L^1(\Omega \times V)$ with $\|p_0\|_{L^1(\Omega \times V)} = 1$ and suppose that there are constants $0 < c \leq C < \infty$ with $c \leq p_0 \leq C$ a.e. on $\Omega \times V$. Then for every $\tau \leq 0$ there exists a unique solution $p$ to

$$p(r, v, t) = S(t)p_0(r, v) + \lambda \int_\tau^t S(t - s)Q(p, p)(r, v, s)\, ds.$$  \hspace{1cm} (5.29)

on $\Omega \times V \times [\tau, \infty)$ with $p(\cdot, \cdot, 0) = p_0$. We have properties (1)-(3) of Theorem 5.6 for $t \geq \tau$ as well as

$$\|p(\cdot, \cdot, t)\|_{L^\infty(\Omega \times V)} \leq p_{0, \max} \cdot \exp \{\lambda h_\gamma \|b_{\infty, \max} \cdot (t - \tau)\}, \quad t \in [\tau, \infty).$$  \hspace{1cm} (5.30)

(b) Let $p_0 \in D(A)$ with $\|p_0\|_{L^1(\Omega \times V)} = 1$ and suppose that there are constants $0 < c \leq C < \infty$ with $c \leq p_0 \leq C$ a.e. on $\Omega \times V$. Then for every $\tau \leq 0$ there is a unique solution $p(\cdot, \cdot, t) \in D(A), t \in [\tau, \infty)$, to the equation

$$\frac{d}{dt} p(\cdot, \cdot, t) = Ap(\cdot, \cdot, t) + \lambda Q(p, p)(\cdot, \cdot, t)$$

with $p(\cdot, \cdot, 0) = p_0$. Here, $d/dt$ is a derivative in $L^1(\Omega \times V)$. At $t = \tau$ it is the right derivative. This solution coincides with the solution to the equation (5.29) if there $p(\cdot, \cdot, 0) = p_0 \in D(A)$. We have properties (1)-(3) of Theorem 5.6 for $t \geq \tau$ as well as (5.30).

Proof. For part (a) we refer to 4.5, Corollary 4.6, and Theorem 5.6 and in particular to Step 1 of its proof. For part (b) there is just to demonstrate that $p_0 \in D(A)$ implies $p(\cdot, \cdot, \tau) \in D(A)$. However, analyzing the equation

$$\frac{d}{dt} q(\cdot, \cdot, t) = -Aq(\cdot, \cdot, t) - \lambda Q(q, q)(\cdot, \cdot, t), \quad t \geq 0, \quad q(\cdot, \cdot, 0) = p_0$$

as in the proof of Proposition 5.3, Step 2, and Corollary 5.4, $p(\cdot, \cdot, \tau) \in D(A)$ follows from [11], Theorem 1.5 of Chapter 6, [11]. Here we note that by Theorem 4.5, $(-A, D(A))$ is the generator of the strongly continuous semigroup $S(-u), u \geq 0.$ \hfill $\Box$
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