Achievable spectral radii of symplectic Perron-Frobenius matrices

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Abstract

A pseudo-Anosov surface automorphism $\phi$ has associated to it an algebraic unit $\lambda_\phi$ called the dilatation of $\phi$. It is known that in many cases $\lambda_\phi$ appears as the spectral radius of a Perron-Frobenius matrix preserving a symplectic form $L$. We investigate what algebraic units could potentially appear as dilatations by first showing that every algebraic unit $\lambda$ appears as an eigenvalue for some integral symplectic matrix. We then show that if $\lambda$ is real and the greatest in modulus of its algebraic conjugates and their inverses, then $\lambda^n$ is the spectral radius of an integral Perron-Frobenius matrix preserving a prescribed symplectic form $L$. An immediate application of this is that for $\lambda$ as above, $\log (\lambda^n)$ is the topological entropy of a subshift of finite type.

1 Introduction

We recall that a self-homeomorphism $\phi$ of a surface $F$ with $\chi(F) < 0$ is called pseudo-Anosov if it leaves invariant a pair of transverse, singular, measured foliations $\mathcal{F}^s$, $\mathcal{F}^u$ called the stable and unstable foliations, respectively. Associated to such a map is an algebraic unit $\lambda_\phi$ called the dilatation of $\phi$ which measures how the map stretches $\mathcal{F}^s$ and shrinks $\mathcal{F}^u$. The dilatation encodes a variety of dynamical properties, for example the topological entropy of $\phi$ is $\log(\lambda_\phi)$. Recently there has been a great deal of interest in the dilatations of pseudo-Anosov automorphisms, including a recent paper of Farb, Leininger, and Margalit which explores connections between low dilatation pseudo-Anosovs and 3-manifolds (see [6]). More generally, the question of which dilatations can be realized by some pseudo-Anosov has received attention (see for example [9] and [12]).

There are a number of ways to find the dilatation $\lambda_\phi$ of a pseudo-Anosov $\phi$. By taking suitable branched coverings, $\lambda_\phi$ can be made to appear as the largest root of an integral symplectic matrix. In fact, in [14] Penner describes a symplectic pairing which is preserved by the action of $\phi$ by an integral Perron-Frobenius matrix. This matrix encodes the action of $\phi$ on a train track $\tau$ which carries it, and the dilatation appears as the spectral radius of
the matrix (for more on train tracks and pseudo-Anosovs, see \[1\], \[14\], and \[8\]). Different train tracks and different pseudo-Anosovs will have different symplectic pairings associated to them. The pairing in general may have degeneracies, but in large classes of examples the pairing is non-degenerate (and in fact a symplectic form).

The motivation for this paper came from thinking about what algebraic units appear as spectral radii of integral symplectic Perron-Frobenius matrices, and hence could potentially appear as dilatations of pseudo-Anosov automorphisms. Additionally, we want to be able to construct these matrices to preserve a prescribed symplectic form.

Let \( \lambda \in \mathbb{R} \) be an algebraic unit, that is, \( \lambda \) is the root of a polynomial which is irreducible over the integers and of the form \( p(t) = t^g + a_g t^{g-1} + \ldots + a_2 t \pm 1 \). If also \( |\lambda| > 1 \), \( \lambda \) has algebraic multiplicity 1, and for all other roots \( \omega \) of \( p(t) \) we have \( |\lambda + \lambda^{-1}| > |\omega + \omega^{-1}| \) we will say \( \lambda \) is a Perron unit. From \( p(t) \), we can form a self-reciprocal (or palindromic) polynomial \( q(t) = t^g p(t) p(t^{-1}) \) for which \( \lambda \) and \( \lambda^{-1} \) are both roots. If \( \lambda \) is a Perron unit, then it is the unique largest root of \( q(t) \).

We want to find Perron units which appear as the spectral radius of a symplectic Perron-Frobenius matrix. In particular, we will prove:

**Main Theorem.** Let \( \lambda \) be a Perron unit, and let \( L \) be any integral symplectic form.

Then for some \( n \in \mathbb{N} \), \( \lambda^n \) is the spectral radius of an integral Perron-Frobenius matrix which preserves the symplectic form \( L \).

The proof is constructive enough that it is possible to find a matrix for \( \lambda \) with the assistance of a computer.

The rest of this paper is divided into three parts. In the first part, we give a canonical form for integral symplectic matrices so that it is easy to construct a matrix preserving a given symplectic form and having a given self-reciprocal polynomial as its characteristic polynomial. In the second part, we show how to conjugate a power of these matrices to be Perron-Frobenius. In particular, we prove:

**Theorem.** Let \( M \) be an integral matrix with a unique, real eigenvalue of largest modulus greater than 1. Suppose also that this eigenvalue has algebraic multiplicity 1, and that \( M \) preserves a symplectic form \( L \).

Then \( \exists n \in \mathbb{N} \) and \( B \in \text{GL}(2g) \) such that \( B^{-1} M^n B \) is an integral, Perron-Frobenius matrix. Furthermore, \( B^{-1} M^n B \) will also preserve \( L \).

In the final section, we give an immediate application of some of these results to subshifts of finite type. Given an integral Perron-Frobenius matrix, it is always possible to build a larger Perron-Frobenius matrix whose entries are all either 0 or 1. This new matrix will have the same spectral radius as the original one, so the results above show that every Perron unit appears as the spectral radius of a such a matrix. In fact, up to multiplication by \( t^k \),
the characteristic polynomial of the new matrix is the same as the one it was built from. We include this discussion both as a simple application and because it may also be useful in studying pseudo-Anosovs.

Although the motivation for this paper was to study pseudo-Anosov maps, there are applications of these results outside the study of surface automorphisms. See for example [11]. To the author’s knowledge these results are unknown, though some may seem like basic facts.

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2 A Canonical Form for Self-Reciprocal Polynomials

In this section, we establish a canonical form for integral matrices with self-reciprocal characteristic polynomial. These matrices preserve a symplectic form which is standard in the sense that it arises naturally from the study of surface automorphisms.

A polynomial $p(t)$ over the integers is *self-reciprocal* if its coefficients are palindromic, i.e, $p(t)$ has the form

$$p(t) = 1 - a_2t - a_3t^2 - ... - a_{g+1}t^g - a_g t^{g+1} - ... - a_2 t^{2g-1} + t^{2g}$$  \hspace{1cm} (1)

Let $Sp(2g)$ be the symplectic group over $\mathbb{R}^{2g}$. Up to change of basis, we may represent any non-degenerate, skew-symmetric bilinear form by either

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & & & \\ 0 & & & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

or

$$K = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$
Where $I$ represents the $g \times g$ identity matrix. We specify $J$ because it is the symplectic form we usually think of when considering the action of a surface automorphism on the first homology group of the surface. We include $K$ because it is easier to work with in obtaining the results of this section.

We now define two standard forms for a matrix which has the self-reciprocal polynomial $p(t)$ as its characteristic polynomial. We will also show that each preserves one of the standard symplectic forms above. The first canonical form, denoted $A$ below, preserves $J$ (that is, $A^T J A = J$).

$$A = \begin{pmatrix} 0 & \cdots & 0 & -1 \\ 0 & a_2 & 0 & a_3 & \cdots & 0 & a_g & 1 & a_{g+1} \\ 1 & 0 & \ddots & \vdots \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 1 & 0 & a_g \end{pmatrix}$$

By performing the change of basis which carries $J$ to $K$, we obtain a second canonical form, denoted $B$, which preserves $K$.

$$B = \begin{pmatrix} 0 & \cdots & \cdots & -1 \\ 1 & \cdots & \cdots & a_2 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 1 & a_2 & a_3 & \cdots & a_{g+1} \\ 0 & \cdots & \ddots & \ddots & \ddots \end{pmatrix}$$

The proofs of this section could be considered tedious, and the uninterested reader should have no problems skipping to section 3 after first reading theorem 3.

**Lemma 1.** $A$ preserves the symplectic form $J$ and $B$ preserves the symplectic form $K$.

**Proof.** It suffices to show that $B$ preserves $K$. Let $\{e_1, \ldots, e_{2g}\}$ denote the standard basis vectors for $\mathbb{R}^{2g}$. We note that the action of $B$ on $e_i$ is:
\[ Be_i = e_{i+1} \text{ if } 1 \leq i \leq g \]
\[ Be_i = a_{i-g+1}e_{g+1} + e_{i+1} \text{ if } g + 1 \leq i \leq 2g - 1 \]
\[ Be_{2g} = -e_1 + \sum_{i=2}^{g+1} a_i e_i \]

We now show that if \( \langle \cdot, \cdot \rangle \) is the bilinear form coming from \( K \), \( \langle Be_i, Be_k \rangle = \langle e_i, e_k \rangle \).

Since this is all computational, we will do only a few cases here. A key observation to simplify calculations is that for \( 1 \leq i \leq g \) we have \( \langle e_i, e_k \rangle \neq 0 \) if and only if \( k = g + i \). In particular, \( \langle e_i, e_{g+1} \rangle \neq 0 \) if and only if \( i = 1 \).

First we will let \( 1 \leq i \leq g \). Then:

\[
\langle Be_i, Be_k \rangle = \langle e_{i+1}, Be_k \rangle = \begin{cases} 
\langle e_{i+1}, e_{k+1} \rangle & \text{if } 1 \leq k \leq g \\
\langle e_{i+1}, a_{k-g+1}e_{g+1} \rangle + \langle e_{i+1}, e_{k+1} \rangle & \text{if } g + 1 \leq k \leq 2g - 1 \\
\langle e_{i+1}, -e_1 \rangle + \langle e_{i+1}, \sum_{j=2}^{g+1} a_j e_j \rangle & \text{if } k = 2g 
\end{cases}
\]

But checking our form \( K \), we see that

\[
\langle Be_i, Be_k \rangle = \begin{cases} 
0 & \text{if } 1 \leq k \leq g \\
0 + 1 & \text{if } k = g + i \text{ and } g + 1 \leq k \leq 2g - 1 \\
0 + 0 & \text{if } k \neq g + i \text{ and } g + 1 \leq k \leq 2g - 1 \\
1 + 0 & \text{if } i = g \text{ and } k = 2g \\
0 + 0 & \text{if } i \neq g \text{ and } k = 2g 
\end{cases}
\]

A slightly more complicated case occurs if we let \( g + 1 \leq i \leq 2g - 1 \) and \( k = 2g \). Then:

\[
\langle Be_i, Be_k \rangle = a_{i-g+1} \langle e_{g+1}, Be_{2g} \rangle + \langle e_{i+1}, Be_{2g} \rangle
= a_{i-g+1} + 0 + 0 + \sum_{j=2}^{g+1} a_j \langle e_{i+1}, e_j \rangle
= a_{i-g+1} - a_{i-g+1}
= 0
\]

The other cases are not more difficult than the two above.
Now we will show that $A$ and $B$ both have characteristic polynomials of form (1).

**Lemma 2.** The characteristic polynomials of $A$ and $B$ are both 
\[ p(t) = 1 - a_2 t - a_3 t^2 - ... - a_{g+1} t^g - a_g t^{g+1} - ... - a_2 t^{2g-1} + t^{2g}. \]

**Proof.** As with the proof of lemma 1, we prove our result for $B$ and the result immediately follows for $A$.

Let $B_0 = B - tI$, and let $B_{k+1}$ be the matrix obtained from $B_k$ by blocking off the first row and first column. Then the $(0, 2g - k)$ minor of $B_k$ is 1 for $0 \leq k < g$. Thus we see that
\[
\det (B - tI) = 1 + a_2 (-t) + (-a_3) (-t)^2 + ... + (-1)^g a_g (-t)^{g-1} + (-t)^g \det B_g
\]
(2)

Where $B_g$ has form:
\[
B_g = \begin{pmatrix}
a_2 - t & a_3 & \cdots & a_{g+1} \\
1 & -t & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & -t
\end{pmatrix}
\]

Let $D_g = B_g$ and for $l \geq g$ let $D_{l-1}$ be the matrix obtained from $D_l$ by blocking off the last row and last column. Then for $g \geq l > 2$, the $(0, l)$ minor of $D_l$ is 1. Thus we have:

\[
\det B_g = (-1)^{g+1} a_{g+1} + ... + (-t)^i (-1)^{g+1-i} a_{g+1-i} + ... + (-t)^{g-3} (-1)^4 a_4 + (-t)^{g-2} \det D_2
\]
\[
= (-1)^{g+1} a_{g+1} + ... + (-1)^{g+1} t^i + ... + (-1)^{g+1} t^{g-3} + (-t)^{g-2} \det D_2
\]
(3)

Notice that in the equation above that if $g$ is even, then every coefficient is negative. If $g$ is odd, every coefficient is positive. Now,

\[
\det D_2 = \det \begin{pmatrix}
a_2 - t & a_3 \\
1 & -t
\end{pmatrix} = t^2 - a_2 t - a_3
\]
(4)

Now by substituting (4) into (3) into (2), we obtain our result.

Putting lemmas 1 and 2 together, we have the following theorem:

**Theorem 3.** Every algebraic unit is an eigenvalue of some symplectic matrix.

**Proof.** Let $\lambda$ be an algebraic unit with minimum polynomial $q(t) = 1 + b_2 t + b_3 t^2 + ... + b_g t^{g-1} + t^g$. Then $t^g q(t) q(t^{-1})$ is a self-reciprocal polynomial. Applying lemmas 1 and 2 we obtain our result.
3 Changing Basis to be Perron-Frobenius

We say a real matrix $M$ is *Perron-Frobenius* if it has all nonnegative entries and $M^k$ has strictly positive entries for some $k \in \mathbb{N}$. Such matrices have important applications in dynamical systems, graph theory, and in studying pseudo-Anosov surface automorphisms. A key result about such matrices was proved in the early 20th century:

**Perron-Frobenius Theorem.** Let $M$ be Perron-Frobenius. Then $M$ has a unique eigenvalue of largest modulus $\lambda$. Furthermore, $\lambda$ is real, positive, and has an associated real eigenvector with all positive entries.

The eigenvalue $\lambda$ is called the *spectral radius* or *growth rate* of $M$. The main purpose of this section is to find integral matrices which can be conjugated to be Perron-Frobenius. We’d also like to do this in a way which preserves a fixed symplectic form (for example, the symplectic form $J$ from section 2). In particular, we prove the following:

**Theorem 4.** Let $M \in \text{Sp}(2g, \mathbb{Z}, L)$ such that $M$ has a unique, real eigenvalue of largest modulus greater than 1. Suppose also that this eigenvalue has algebraic multiplicity 1. Then $\exists n \in \mathbb{N}$ and $B \in \text{GL}(2g)$ such that $B^{-1}M^nB$ is a Perron-Frobenius matrix in $\text{Sp}(2g, \mathbb{Z}, L)$.

Here we denote by $\text{Sp}(2g, \mathbb{Z}, L)$ the group of $2g \times 2g$ integer matrices which preserve a fixed symplectic form $L$. When we do not care to fix a particular symplectic form, we will use the notation $\text{Sp}(2g)$ to mean the group of symplectic linear transformations on $\mathbb{R}^{2g}$.

We also obtain a similar result for integral, nonsingular matrices (see corollary 12). Given a matrix $M$ with a unique real eigenvalue of largest modulus greater than 1, we will denote this eigenvalue $\lambda_M$ and its associated eigenvector $v_M$. We will refer to $\lambda_M$ and $v_M$ as the dominating eigenvalue and dominating eigenvector, respectively.

The idea behind the proof will be to find an integral basis $\{b_1, ..., b_{2g}\}$ for $\mathbb{R}^{2g}$ such that $v_M$ is contained in the cone determined by $b_1, ..., b_{2g}$. We also need that if $W$ is the co-dimension 1 invariant subspace of $M$ such that $v_M \notin W$, then $b_1, ..., b_{2g}$ all lie on the same side of $W$ as $v_M$. To make the notion of side precise, denote by $W^+$ as the set of all vectors in $\mathbb{R}^{2g}$ that can be written as $av_M + w$ where $a \in \mathbb{R}^+$ and $w \in W$.

**Lemma 5.** Let $M$ be a matrix with a dominating real eigenvalue $\lambda_M$ and associated real eigenvector $v_M$. Say $\{b_1, ..., b_{2g}\}$ is a basis for $\mathbb{R}^{2g}$ such that $b_1, ..., b_{2g} \in W^+$ and $v_M$ is contained in the interior of the cone determined by $b_1, ..., b_{2g}$.

Then for some $n \in \mathbb{N}$, $M^n$ has all positive entries after changing to the basis above.

**Proof.** Since we can replace $M$ by $M^2$ if necessary, we may assume $\lambda_M$ is positive. Let
\( \lambda_2, \ldots, \lambda_n \) be the other eigenvalues of \( M \) and let \( v_M, v_2, \ldots, v_{2g} \) be a Jordan basis for \( M \) (i.e., a basis in which the linear transformation represented by \( M \) is in Jordan canonical form). Note that \( v_2, \ldots, v_{2g} \) span \( W \).

Consider a Jordan block associated to some eigenvalue \( \lambda_i \) of \( M \):

\[
J_i = \begin{pmatrix}
\lambda_i & 1 \\
& \lambda_i \\
& & \ddots \\
& & & 1 \\
& & & & \lambda_i
\end{pmatrix}
\]

The definition of matrix multiplication guarantees that each entry of \( J_i^k \) will be a polynomial in \( \lambda_i \). Each diagonal entry will equal \( \lambda_i^k \) and every other entry of \( J_i^k \) will have degree strictly less than \( k \). Thus we see that if \( v_j \) is a Jordan basis vector corresponding to the eigenvalue \( \lambda_i \) we get \( \frac{J_i^k v_j}{\lambda_i^k} \to 0 \) as \( k \to \infty \), which implies:

\[
\text{Corollary 6. Let } M \text{ as in lemma } \square \text{ and } v \in W^+. \text{ Then the distance between } \frac{M^k v}{||M^k v||} \text{ and } \frac{v_M}{||v_M||} \text{ approaches } 0 \text{ as } k \to \infty.
\]

Our goal is now to construct a matrix \( B \in \text{Sp} (2g, \mathbb{Z}, L) \) such that the columns of \( B \) form a basis satisfying the hypotheses of lemma \( \square \). The idea will be to construct a set of symplectic basis vectors which define a very narrow cone, and then apply a slightly perturbed symplectic isometry of \( S^{2g-1} \) to move that cone into the correct position.

A symplectic linear transformation \( \tau \) is a (symplectic) transvection if \( \tau \neq 1 \), \( \tau \) is the identity map on a codimension 1 subspace \( U \), and \( \tau v - v \in U \) for all \( v \in \mathbb{R}^{2g} \). Geometrically,
tranvection is a shear fixing the hyperplane $U$. A symplectic transvection preserving the symplectic form $J$ can be written

$$\tau_{u,a}v = v + aJ(v, u)u$$

for some scalar $a$ and vector $u \in \mathbb{R}^{2g}$. Note that the fixed subspace is $< u >^\perp$ and that it contains $u$. $\text{Sp}(2g)$ is generated by transvections (see [10]). If we wish to preserve a symplectic form $L$ different from $J$, simply replace $J$ with $L$ in the formula.

Let $u \in \mathbb{R}^{2g}$ be the vector $(-1, 1, ..., -1, 1)$ and set $a = 1$. Let $e_1, ..., e_{2g}$ be the standard basis for $\mathbb{R}^{2g}$. Notice $J(e_i, u) = 1$, so $\tau_{u,1}e_i = e_i + u$. Thus, in matrix form:

$$\tau_{u,1} = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

Composing this with transvections $\tau_{e_k,2}$ with $k$ even, we get the symplectic matrix

$$A = \begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

This matrix preserves the symplectic form $J$, and is also Perron-Frobenius. In fact, we can find such a matrix for any integral symplectic form:

**Lemma 7.** There is a Perron-Frobenius matrix in $\text{Sp}(2g, \mathbb{Z}, L)$ for any integral symplectic form $L$.

*Proof.* Non-degeneracy of $L$ guarantees that there is $u \in \mathbb{Q}^{2g}$ such that $L(e_i, u) = 1$ for every basis vector $e_i$. Let $w = (1, 1, ..., 1) \in \mathbb{Q}^{2g}$, and notice that $L(u, w) = -2g$. Then $\tau_{u,a}e_i = e_i + au$ for for a very large we have that $\tau_{u,a}e_i$ is close to $cu$ for some $c \in \mathbb{N}$. Now by continuity, $L(\tau_{u,a}e_i, w) = l < 0$ and for $b \in \mathbb{N}$ we have $\tau_{w,-b}\tau_{u,a}e_i = \tau_{u,a}e_i - blw$. Thus for $b$ large enough, $\tau_{w,-b}\tau_{u,a}e_i$ is a rational vector with positive entries for all $i$. This transformation has Perron-Frobenius matrix representation. If it is not integral, we can adjust the values of $a$ and $b$ to clear denominators. \qed

Let $U(g)$ denote the group of unitary linear transformations of $\mathbb{C}^g$. Equivalently, we can
think of the unitary group as a group of matrices: 
\[ U(g) = \{ M | M \in GL(g, \mathbb{C}), M^* M = I \} \]
where \( M^* \) denotes the conjugate transpose of \( M \).

We identify \( U(g) \) with a subgroup of \( GL(2g, \mathbb{R}) \) as follows: Let \( M \in U(g) \). Replace every entry \( m = re^{i\theta} \in \mathbb{C} \) in \( M \) by the scaled 2 \( \times \) 2 rotation matrix \( R = \begin{pmatrix} r \cos(\theta) & -r \sin(\theta) \\ r \sin(\theta) & r \cos(\theta) \end{pmatrix} \).

We now can consider \( U(g) \) as a group of real matrices acting on \( \mathbb{R}^{2g} \). Notice that if \( m \mapsto R \), then \( \bar{m} \mapsto R^T \). Thus, if \( M = [m_{i,j}] \in U(g) \) is identified with \( N = [R_{i,j}] \), we have \( M^* M = [\bar{m}_{i,j}]^T [m_{i,j}] \mapsto [R^T_{i,j}]^T [R_{i,j}] = N^T N = I \). Hence with this identification \( U(g) \) is a subgroup of the real orthogonal group \( O(2g) \) (in fact it is a subgroup of \( SO(2g) \)).

Notice that the symplectic form \( J \) gets identified with the complex matrix
\[
\begin{pmatrix}
-i & \\
& \ddots & -i \\
& & \ddots & -i
\end{pmatrix}
\]
which is in the center of \( U(g) \). Then if \( M \in U(g) \) we have \( M^* J M = J \), and thus \( U(g) \) is a subgroup of \( Sp(2g) \). Below is a more powerful result which is proved in [13] as lemma 2.17.

**Lemma 8.** \( Sp(2g) \cap O(2g) = U(g) \)

We also need the following fact:

**Lemma 9.** The unitary group \( U(g) \) acts transitively on \( S^{2g-1} \subseteq \mathbb{R}^{2g} \).

**Proof.** The \( S^{2g-1} \) sphere can be thought of as all vectors in \( \mathbb{C}^g \) having unit length. Let \( v \in S^{2g-1} \) and \( \{ e_1, \ldots, e_g \} \) be the standard basis for \( \mathbb{C}^g \). Using the Gram-Schmidt process, we can extend \( v \) to an orthonormal basis \( \{ v, v_2, \ldots, v_g \} \) for \( \mathbb{C}^g \). Then the change of basis matrix is in \( U(g) \) and sends \( e_1 \) to \( v \). \( \square \)

At one point during the proof of our main theorem, it will become important to know that \( Sp(2g, \mathbb{Q}) \) is dense in \( Sp(2g) \). This follows quickly from the Borel Density Theorem, but we include an elementary proof.

**Lemma 10.** \( Sp(2g, \mathbb{Q}) \) is dense in \( Sp(2g) \).

**Proof.** Let \( M' \in Sp(2g, \mathbb{R}, J) \). Perturb the entries of \( M' \) by a small amount to obtain a matrix \( M \) with rational entries. We will systematically modify the columns \( a_1, b_1, \ldots, a_g, b_g \) of \( M \) to form a new \( M \) which preserves \( J \) and still differs from \( M' \) by a small amount. Here for convinence we let \( <,> \) denote the symplectic form given by \( J \).

We iterate the following procedure for each pair of columns \( a_i, b_i \), starting with \( a_1, b_1 \). First, say \( <a_i, b_i> = 1 + \eta_i \) where \( \eta_i \) is a small, rational number (its magnitude depends on the
size of the perturbation of $M'$. Replace $a_i$ with $\frac{a_i}{1+\eta_k}$, so that now $<a_i,b_j>\geq 1$. Now we modify each pair of columns $a_j,b_j$ with $j > i$. Set $\epsilon_{i,j} = <a_i,a_j>$ and $\delta_{i,j} = <b_i,a_j>$. Replace $a_j$ with $a_j - \epsilon_{i,j}b_i - \delta_{i,j}a_i$, so that now $<a_i,a_j>\geq <b_i,a_j> \geq 0$. Note that $\epsilon_{i,j}$ and $\delta_{i,j}$ are also small rational numbers. Now modify $b_j$ by a similar procedure, so that $<a_i,b_j>\geq <b_i,b_j> = 0$.

Now repeat the procedure with the columns $a_{i+1},b_{i+1}$. After modifying every column we obtain a new $M$ which is in $\text{Sp}(2g,\mathbb{Q},J)$. Furthermore, since at each stage the modifications to the columns are small, $M$ is still close to $M'$.

We’re now ready to prove theorem 4. Throughout we will use the notation that if $v \in \mathbb{R}^{2g}\backslash \{0\}$ then $\hat{v}$ denotes the normalization $v/\|v\| \in S^{2g-1}$. If $M$ is a matrix with no zero columns, then $\hat{M}$ will denote the matrix obtained by normalizing each of the columns.

**proof of theorem 4** Let $M \in \text{Sp}(2g,\mathbb{Z},L)$ with dominating real eigenvalue $\lambda$ and associated eigenvector $v_M$. Let $W$ be the co-dimension 1 invariant subspace of $M$ with $v_M \notin W$, and $W^+$ the component of $\mathbb{R}^{2g}\backslash W$ containing $v_M$. Set $\epsilon$ to be the minimal distance in $S^{2g-1}$ from $\hat{v}_M$ to $W \cap S^{2g-1}$. Then by lemma 7 and corollary 8 there exists $n \in \mathbb{N}$ and $A \in \text{Sp}(2g,\mathbb{Z},L)$ such that $A$ is Perron-Frobenius and the convex hull $H$ of the columns of $\hat{A}^n$ has diameter less than $\epsilon$ (here we take $H \subseteq S^{2g-1}$ and measure distance in $S^{2g-1}$).

Let $\nu$ be in the interior of $H$. Since $U(g)$ acts transitively on $S^{2g-1}$ (lemma 9), there is $S \in U(g)$ such that $S\nu = v_M$. As a real linear transformation, $S$ is orthogonal and hence $\text{diam}(H) = \text{diam}(S(H))$. Thus the columns of $S\hat{A}^n$ are contained in $W^+$. $U(g)$ is a subgroup of $\text{Sp}(2g)$ (lemma 8), so $S \in \text{Sp}(2g)$. Furthermore, by lemma 10 we may perturb $S$ slightly so that now $S \in \text{Sp}(2g,\mathbb{Q},L)$. Set $B' = S\hat{A}^n$, note $B' \in \text{Sp}(2g,\mathbb{Q},L)$. Scale $B'$ by an integer $\alpha$ so that $B = \alpha B'$ is a nonsingular, integral matrix.

Set $d = \text{det} B$. Then $B^{-1} = \frac{1}{d}C$, where $C$ is the adjugate of $B$. In particular, $C$ is integral.

Consider the projection map $\text{SL}(2g,\mathbb{Z}) \rightarrow \text{SL}(2g,\mathbb{Z}/d\mathbb{Z})$. Since $\text{SL}(2g,\mathbb{Z}/d\mathbb{Z})$ is finite, for some $m \in \mathbb{N}$ we have $M^m$ in the kernel of this map. Hence, we can write $M^m = I + d\Lambda$ for some integral matrix $\Lambda$. Putting this together, we have:

$$B^{-1}M^mB = \frac{1}{d}C(I + d\Lambda)B$$

$$= I + C\Lambda B$$

In particular, $B^{-1}M^mB$ is integral. By construction, the columns of $B$ give a basis satisfying the conditions of lemma 5, so for large enough $k \in \mathbb{N}$ we have $B^{-1}M^{mk}B$ is Perron-Frobenius and integral. Furthermore $B^{-1}M^{mk}B$ is symplectic since $B$ is a scaled symplectic matrix.

\qed
Using theorems 3 and 4, we can prove our main result, which we restate here:

**Theorem 11.** Let \( \lambda \) be a Perron unit, and let \( L \) be any integral symplectic form.

Then for some \( n \in \mathbb{N} \), \( \lambda^n \) is the spectral radius of an integral Perron-Frobenius matrix which preserves the symplectic form \( L \).

**Proof.** Using the canonical form of section 2, we can build a matrix \( M \in \text{Sp}(2g, \mathbb{Z}, J) \) with \( \lambda \) its spectral radius. For some \( B' \in \text{GL}(2g, \mathbb{Q}) \) we have \( (B')^T J B' = L \). Scale \( B' \) by an integer \( \alpha \) so that \( B = \alpha B' \) is integral. Now proceeding with the argument at the end of the proof for theorem 4 we get that \( B^{-1} M^r B \in \text{Sp}(2g, \mathbb{Z}, L) \). Now we can apply theorm 4 to obtain our result.

We end this section by noting that if the matrix \( M \) is not symplectic, we can modify the hypotheses slightly to achieve a result similar to theorem 4. The proof uses similar ideas, but is actually significantly easier.

**Corollary 12.** Let \( M \) be an integral, nonsingular matrix with a unique, real eigenvalue of largest modulus greater than 1. Suppose also that this eigenvalue has algebraic multiplicity 1.

Then \( \exists n \in \mathbb{N} \) such that \( M^n \) is conjugate to an integral Perron-Frobenius matrix.

**Proof.** Let \( \delta = \det M \), and pick a \( B' \in \text{SL}(r, \mathbb{Q}) \) such that the columns of \( B' \) satisfy the conditions of lemma 5. Choose \( \alpha \in \mathbb{Z} \) such that \( B = \alpha B' \) has integer entries and \( \delta \) divides every entry of \( \bar{B} \). Assuming we also chose \( \alpha \) to be large, we may set \( B = \bar{B} + I \) and the columns of \( \bar{B} \) will still satisfy lemma 5.

Consider \( d = \det B \). Calculating the determinant by cofactor expansion, we see that \( d = (\text{sum of terms divisible by } \delta) + 1 \). In particular, \( \delta \) is relatively prime to \( d \), so \( M \) has a projection to \( \text{GL}(r, \mathbb{Z}/d\mathbb{Z}) \). We now raise \( M \) to a power \( m \) so that \( M^m = I + d\Lambda \) and proceed with the argument of theorem 4.

\[ \square \]

4 Subshifts of Finite Type

We will now apply the previous two sections to symbolic dynamics, in particular to subshifts of finite type.

Let \( M \) be an \( n \times n \) matrix of 0’s and 1’s. Let \( A_n = \{1, 2, ..., n\} \), and form \( \Sigma_n = A_n \times \mathbb{Z} \). We can think of \( \Sigma_n \) as the set of all bi-infinite sequences in symbols from \( A_n \), and we endow it
with the product topology. Now we form a subset $\Lambda_M \subseteq \Sigma^n$ by saying $(s_i) \in \Lambda_M$ if the $s_i, s_{i+1}$ entry of $M$ is equal to 1 for all $i$. We can think of the $i, j$ entry of $M$ as telling us whether it is possible to transition from state $i$ to state $j$. Now let $\sigma$ be the automorphism of $\Lambda_M$ obtained by shifting every sequence one place to the left. The dynamical system $(\Lambda_M, \sigma)$ is called a subshift of finite type, and can be thought of as a zero-dimensional dynamical system. These dynamical systems have relatively easy to understand dynamics and are often used to model more complicated systems (for example, pseudo-Anosov automorphisms).

Let $M = [m_{i,j}]$ be a square matrix with nonnegative, integer entries. We form a directed graph $G$ from $M$ as follows. $G$ has one vertex for each row of $M$. Then connect the $i$-th vertex to the $j$-th vertex by $m_{i,j}$ edges, each directed from vertex $i$ to vertex $j$. We call $M$ the transition matrix for $G$. If $M$ is Perron-Frobenius, then the graph $G$ will be strongly connected and the $i, j$-th entry of $M^k$ represents the number of paths of length $k$ from vertex $i$ to vertex $j$. The spectral radius $\lambda$ of $M$ can be interpreted as the growth rate of the number of paths of length $k$ in $G$, i.e. \[ \lim_{k \to \infty} \frac{M^k}{\lambda^k} = P \neq 0. \]

We now show how to go from an integral Perron-Frobenius matrix $M$ to another matrix with the same spectral radius whose entries are all 0 or 1. This construction can also be found in [9]. Given a directed graph $G$ with Perron-Frobenius transition matrix $M$, label the edges of $G$ as $e_1, ..., e_n$ and the vertices $v_1, ..., v_m$. From $G$, we form a directed graph $H$ as follows: the vertex set $w_1, ..., w_n$ of $H$ is in 1-1 correspondence with the edge set of $G$ ($w_i \leftrightarrow e_i$). If the edge $e_i$ terminates at the vertex from which $e_j$ emanates, then we place an edge in $H$ from $w_i$ to $w_j$. Let $N$ be the transition matrix of $H$. Note that by construction, every entry of $N$ is either a 0 or a 1.

A subgraph of a graph $G$ is a cycle if it is connected and every vertex has in and out valence 1. If $M$ is a transition matrix for $G$, it is possible to reformulate the calculation of the characteristic polynomial $p(t) = \det(tI - M)$ in terms of cycles in $G$ (see [3]):

**Lemma 13.** Let $G$ be a graph with transition matrix $M$. Denote by $C_i$ the collection of all subgraphs which have $i$ vertices and are the disjoint union of cycles. For $C \in C_i$, denote by $\#(C)$ the number of cycles in $C$. Then the characteristic polynomial $p(t) = \det(tI - M)$ is

\[ p(t) = t^m + \sum_{i=1}^{m} c_i t^{m-i} \]

where $m$ is the number of vertices in $G$ and

\[ c_i = \sum_{C \in C_i} (-1)^{\#(C)} \]

Using this formula, we can prove that the characteristic polynomial of $N$ (as above) has a nice form, and in particular that the spectral radius of $N$ is the same as the spectral radius
Theorem 14. Let $M$ be the transition matrix for a graph with $m$ vertices and let $N$ be an $n \times n$ matrix of 0’s and 1’s built from $M$ by the construction above.

Then if $p(t) = \det (tI - M)$ is the characteristic polynomial of $M$, the characteristic polynomial of $N$ is $q(t) = t^{n-m}p(t)$.

Proof. Let $G$ be the graph associated to $M$, and $H$ the graph associated with $N$. Order the vertices of $G$, and for each vertex $v$ fix a lexicographic order of $(in-edge, out-edge)$ pairs of edges incident to $v$. Let $D_i$ be the collection of subgraphs of $H$ which can be written as a union of disjoint cycles with $i$ total vertices. For $D \in D_i$, there is a canonical projection of $D$ to a collection of paths in $G$ (using the fact that vertices in $H$ come from edges in $G$). Let $D^*_i$ be the subset of $D_i$ containing those disjoint unions of cycles in $H$ which do not project to a disjoint union of cycles in $G$. We will show that there is a bijection between elements of $D^*_i$ having an odd number of components and elements of $D_i$ having an even number of components.

Let $D \in D^*_i$ and say $D$ has an odd number of components. Call $C$ its projection to a collection of paths in $G$. Since $C$ is not a disjoint union of cycles, there must be vertices of $G$ that are either visited by two different paths in $C$ and/or are visited twice by the same path. Choose $v$ to be the minimal such vertex in the ordering of vertices of $G$, and note that $v$ must have in-valence and out-valence both of at least 2. Choose two in/out-edge pairs, $(e, f)$ and $(e', f')$, such that each pair occurs in some path in $C$ and so that they are minimal among such pairs in the ordering of edges incident to $v$. Note that $D$ contains vertices in $H$ corresponding to $e, e', f, f'$ and must contain edges from $e$ to $f$ and from $e'$ to $f'$. Build $D' \in D^*_i$ by letting $D'$ have the same vertex collection as $D$, but instead of containing edges from $e$ to $f$ and from $e'$ to $f'$ it contains edges from $e$ to $f'$ and $e'$ to $f$ (call this operation an edge swap).

If the pairs $(e, f)$ and $(e', f')$ are both part of the same cycle in $D$, then $D'$ will have one more component than $D$. If they are part of two different cycles, then $D'$ will have one less component. In either case, $D'$ has an even number of components and we have constructed a well-defined map from elements of $D^*_i$ having odd components to elements having even components. Note also that the projection $C'$ of $D'$ still visits $v$ twice, and contains in/out-edge pairs $(e, f')$ and $(e', f)$. Thus we can define the inverse of this map in exactly the same way, and hence we have a bijection.

Because of the bijection we built above, we see that disjoint unions of cycles in $D^*_i$ cancel out when $q(t)$ when it is computed using lemma [13]. Elements of $D_i \setminus D^*_i$ are in bijective correspondence with cycles in $C_i$, so we get our conclusion.

Finally, we have:
Theorem 15. Let $\lambda$ be a Perron unit. Then there is $k \in \mathbb{N}$ such that $\log(\lambda^k)$ is the topological entropy of some subshift of finite type.

This follows directly from theorems 4, 3, and comments of Fathi, Laudenbach, and Poéaru on subshifts of finite type (see [7]).

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