Asymptotics of the Airy-kernel determinant

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Abstract. The authors use Riemann-Hilbert methods to compute the constant that arises in the asymptotic behavior of the Airy-kernel determinant of random matrix theory.
1 Introduction

Let $K_s$ be the trace-class operator with kernel
\[ K_s(t,u) = \frac{\text{Ai}(t)\text{Ai}'(u) - \text{Ai}(u)\text{Ai}'(t)}{t-u} \]  
(see [31]) acting on $L^2(-s, \infty)$. Here $\text{Ai}(x)$ is the Airy function (see, e.g., [1]). In this paper we are concerned with the behavior of $\det(I - K_s)$ as $s \to +\infty$. Our main result is the following.

Theorem 1. The large-$s$ asymptotic behavior of the Fredholm determinant $\det(I - K_s)$ is given by the formula
\[
\ln \det(I - K_s) = -\frac{s^3}{12} - \frac{1}{8} \ln s + \chi + O(s^{-3/2}),
\]
(2) where
\[
\chi = \frac{1}{24} \ln 2 + \zeta'(-1),
\]
(3) and $\zeta(s)$ is the Riemann zeta-function.

The Airy-kernel determinant $\det(I - K_s)$ is the edge scaling limit for the largest eigenvalue of a random $n \times n$ Hermitian matrix $H$ from the Gaussian Unitary Ensemble (GUE) (see [29, 31]) as $n \to \infty$: More precisely, if $\lambda_1(H) \geq \lambda_2(H) \geq \cdots \geq \lambda_n(H)$ denote the eigenvalues of $H$, then
\[
\det(I - K_s) = \lim_{n \to \infty} \text{Prob}\{H \in \text{GUE} : (\lambda_1(H) - \sqrt{2n})2^{1/2}n^{1/6} \leq -s\}
\]
(4) (See [21, 31], and also [8] for some history of [1]).

This determinant also describes the distribution of the longest increasing subsequence of random permutations [3, 25]. Namely, let $\pi = i_1i_2\cdots i_n$ be a permutation in the group $S_n$ of permutations of $1, 2, \ldots, n$. Then a subsequence $i_{k_1}, i_{k_2}, \ldots, i_{k_r}$, $k_1 < k_2 < \cdots < k_r$, of $\pi$ is called an increasing subsequence of length $r$ if $i_{k_1} < i_{k_2} < \cdots < i_{k_r}$. Let $l_n(\pi)$ denote the length of a longest increasing subsequence of $\pi$ and let $S_n$ have the uniform probability distribution. Then $l_n(\pi)$ is a random variable, and [3]
\[
\det(I - K_s) = \lim_{n \to \infty} \text{Prob}\{\pi \in S_n : (l_n(\pi) - 2\sqrt{n})n^{-1/6} \leq -s\}
\]
(5)

The distribution $F_{TW}(x) \equiv \det(I - K_{-x})$, known as the Tracy-Widom distribution, admits the following integral representation [31]:
\[
F_{TW}(x) = \exp\left\{-\int_{-x}^{\infty} (y-x)u^2(y)dy\right\},
\]
(6)
where \( u(y) \) is the (global) Hastings-McLeod solution of the Painlevé II equation
\[
u''(y) = yu(y) + 2u^3(y),
\] specified by the following asymptotic condition:
\[
u(y) \sim Ai(y) \quad \text{as} \quad y \to +\infty.
\] The behavior of \( u(y) \) as \( y \to -\infty \) is given by the relation \[22]\:
\[
u(y) = \sqrt{-\frac{y}{2}} \left( 1 + \frac{1}{8y^3} + O(y^{-6}) \right), \quad y \to -\infty,
\]
from which one learns that as \( s \to +\infty \),
\[- \int_{-s}^{\infty} (s + y)u^2(y)dy + \frac{s^3}{12} + \frac{1}{8} \ln s = as + b + o(1)
\]
for some constants \( a, b \). The content of Theorem 1 is that \( a = 0 \) and \( b = \chi \) as in \[3\]. The value \[3\] of the constant \( \chi \) was conjectured by Tracy and Widom in \[31\] on the basis of the numerical evaluation of the l.h.s. of \[10\] as \( s \to +\infty \) and by taking into account the Dyson formula for a similar constant in the asymptotics of the so-called sine-kernel determinant \[31\]. The sine-kernel determinant describes the gap probability for GUE in the bulk scaling limit as \( n \to \infty \) \[29\].

Dyson’s conjecture for the constant in the asymptotics of the sine-kernel determinant was proved rigorously in independent work by Ehrhardt \[19\] and one of the authors \[26\], and a third proof was given later in \[17\]. The two latter works use a Riemann-Hilbert-problem approach. The proof in \[26\] relies on a priori information from \[33\], whereas the proof in \[17\] is self-contained. The proof of Theorem 1 in this paper follows the method in \[17\].

As discussed in \[17\], the key difficulty in evaluating constants such as \( \chi \) in \[2\] in the asymptotic expansion of the determinants, is that in the course of the analysis one most naturally obtains expressions only for the logarithmic derivative with respect to some auxiliary parameter, say \( \alpha \), in the problem, and not the determinant itself. After evaluation of these expressions asymptotically, the constant of integration remains undetermined. In \[17\] and \[26\], this difficulty is overcome by utilizing a scaling limit of finite-\( n \) random matrices together with universality in the sense of random matrix theory (see, e.g., \[15\]), in a way that is inspired by, but different from, Dyson \[18\]. We proceed as follows.

Consider the scaled Laguerre polynomials \( p_k(x) \) defined for some integer \( n \) by the orthogonality relation
\[
\int_0^\infty e^{-4nx} p_k(x)p_m(x)dx = \delta_{k,m}, \quad k, m = 0, 1 \ldots,
\]
The polynomial \( p_k(x) = \alpha_k x^k + \cdots \) is of degree \( k \) and is related to the standard Laguerre polynomial \( L_k^{(0)}(x) \) (see \[30\]) as follows:
\[
p_k(x) = 2\sqrt{n}L_k^{(0)}(4nx)
\]
with leading coefficient
\[ \kappa_k = (-1)^k \frac{2\sqrt{n}}{k!} (4n)^k. \] (12)

The scaling here is chosen so that the asymptotic density of zeros of the polynomial \( p_n(x) \) (with index \( n \)) is supported on the interval \((0, 1)\) (as opposed to \((0, 4n)\) for \( L_n^{(0)}(x) \)). See [30, 16] and below.

In the unitary random matrix ensemble defined by the Laguerre weight, the distribution function of the eigenvalues is given by the expression:
\[
dP(x_0, \ldots, x_{n-1}) = \frac{1}{C_n n!} \prod_{0 \leq i < j \leq n-1} (x_i - x_j)^2 \prod_{j=0}^{n-1} e^{-4x_jn} \, dx_j,
\] (13)
where the normalization constant
\[
C_n = \frac{1}{n!} \int_0^\infty \cdots \int_0^\infty \prod_{0 \leq i < j \leq n-1} (x_i - x_j)^2 \prod_{j=0}^{n-1} e^{-4x_jn} \, dx_j.
\] (14)

By a well known identity (e.g. [30, 7]), the r.h.s. of the above expression gives
\[
C_n = \prod_{k=0}^{n-1} \kappa_k^{-2} = (4n)^{-n^2} \prod_{k=0}^{n-1} k!^2,
\] (15)
where (12) was used.

For \( \alpha \geq 0 \), the probability \( D_n(\alpha) \) that the interval \((\alpha, \infty)\) has no eigenvalues is given by
\[
D_n(\alpha) = \frac{1}{C_n n!} \int_0^\alpha \cdots \int_0^\alpha \prod_{0 \leq i < j \leq n-1} (x_i - x_j)^2 \prod_{j=0}^{n-1} e^{-4x_jn} \, dx_j.
\] (16)

By standard arguments (cf. [7, 29]), this quantity can be written as the Fredholm determinant of an integral operator on \( L_2(0, \infty) \) in the following way:
\[
D_n(\alpha) = \det(I - K_n \chi_{(\alpha, \infty)}), \quad K_n(x, y) = \frac{1}{4} \frac{\omega_n(x)\omega_{n-1}(y) - \omega_n(y)\omega_{n-1}(x)}{y - x},
\] (17)
where
\[
\omega_k(x) = e^{-2nx}p_k(x), \quad k = 0, 1, \ldots,
\] (18)
and \( \chi_{(\alpha, \infty)} \) is the characteristic function of the interval \((\alpha, \infty)\).
If \( x = 1 + 1/(2n) + u/(2n)^{2/3} \) with \( u \) fixed, then as \( n \to \infty \), we obtain from classical results on the asymptotics of the Laguerre polynomials (see [32][30]):

\[
\omega_n \left( 1 + \frac{1}{2n} + \frac{u}{(2n)^{2/3}} \right) = \omega_n \left( \frac{1}{4n} (4n + 2 + 2(2n)^{1/3}u) \right) = (-1)^n \frac{2\sqrt{n}}{(2n)^{1/3}} \{ \text{Ai} (u) + O(n^{-2/3}) \};
\]

\[
\omega_{n-1} \left( 1 + \frac{1}{2n} + \frac{u}{(2n)^{2/3}} \right) = \omega_{n-1} \left( \frac{1}{4n} \left[ 4(n-1) + 2 + 2(2[n-1])^{1/3} \right] \left( u + \frac{2}{(2n)^{1/3}} \right) + O(n^{-2/3}) \right),
\]

where \( \text{Ai} (x) \) is the standard Airy function. Let

\[
K_{\text{airy}}(u, v) = \frac{\text{Ai} (u) \text{Ai}'(v) - \text{Ai} (v) \text{Ai}'(u)}{u - v}.
\]

Set

\[
u^{(n)} = 1 + \frac{u}{(2n)^{2/3}} + \frac{1}{2n}, \quad v^{(n)} = 1 + \frac{v}{(2n)^{2/3}} + \frac{1}{2n}.
\]

It follows from (19) that for any fixed \( u, v \) we have

\[
\lim_{n \to \infty} \frac{1}{(2n)^{2/3}} K_n \left( u^{(n)}, v^{(n)} \right) = K_{\text{airy}}(u, v).
\]

In fact, this asymptotics is uniform for \( u, v \geq L_0 \), where \( L_0 \) is an arbitrary constant. Indeed, for any \( L_0 \) there exists \( C = C(L_0) > 0 \), \( c = c(L_0) > 0 \) such that

\[
\left| \partial_u^j \partial_v^k \left( \frac{1}{(2n)^{2/3}} K_n \left( u^{(n)}, v^{(n)} \right) - K_{\text{airy}}(u, v) \right) \right| \leq C e^{-cu} e^{-cv} \frac{1}{n^{2/3}}, \quad j, k = 0, 1.
\]

This estimate can be proved in a same manner as estimate (3.8) in [8]. In [8] the authors use global estimates for orthogonal polynomials on \( \mathbb{R} \) taken from [16]: Here the relevant global estimates can be obtained from [32].

As in [8], estimate (20) immediately implies that for any fixed \( s \in \mathbb{R} \),

\[
\lim_{n \to \infty} D_n \left( 1 - \frac{s}{(2n)^{2/3}} \right) = \det \left( I - K_{\text{airy}} \chi_{(-s, \infty)} \right).
\]

Below we obtain the asymptotics of the determinant \( \det \left( I - K_{\text{airy}} \chi_{(-s, \infty)} \right) \equiv \det \left( I - K_s \right) \) as \( s \to +\infty \). In order to do this, we analyze the asymptotics of (17) for all \( \alpha \) from \( \alpha \) close to zero to \( \alpha = 1 - s/(2n)^{2/3} \). Note that the determinant (17) has the structure of so-called integrable determinants [23]. Therefore, it is not surprising that there exists a differential identity for \( \frac{d}{da} \ln D_n (\alpha) \) in terms of the solution of a related Riemann-Hilbert problem. Solving the Riemann-Hilbert problem asymptotically as \( n \to \infty \), we find the asymptotics of this
logarithmic derivative uniform for \( \alpha \in [0, 1-s_0/(2n)^{2/3}] \), \((2n)^{2/3} > s_0\) for some (large) \( s_0 > 0 \). Integrating these asymptotics from \( \alpha \) close to zero to \( \alpha = 1 - s/(2n)^{2/3} \), \( s_0 < s < (2n)^{2/3} \) we obtain the asymptotics of \( D_n(1 - s/(2n)^{2/3}) \) provided we know the asymptotics of \( D_n(\alpha) \) for \( \alpha \) close to zero. The latter, however, is readily obtained from the series expansion of the multiple integral formula for \( D_n(\alpha) \) (see (22,27) below). More precisely, the “inner workings” of the method in this paper (cf. also (133) in [17]) can be seen from formula (161) below, which is obtained by integrating the derivative \((d/d\alpha') \ln D_n(\alpha')\) from \( \alpha' = \alpha_0 \) to \( \alpha' = \alpha \). The key fact is that the estimate on the derivative is uniform for \( 0 \leq \alpha' \leq 1 - s/(2n)^{2/3}, \ s > s_0 \) (see (152,153)): This leads to the error estimate \( O(1/(n(1-\alpha)^{3/2})) \) in (161). Using (27), we can then let \( \alpha_0 \to 0 \): The singularities on the l.h.s. and the r.h.s. of (161) cancel out, and we are left with (162). Using (162), we immediately obtain Theorem 1. Note that in our calculations formula (3) for \( \chi \) does not arise from an evaluation of \( D_n(\alpha_0) \) as \( n \to \infty \) for some fixed \( \alpha_0 \). Rather it arises, somewhat paradoxically, from the behavior of \( D_n(\alpha_0) \) as \( \alpha_0 \to 0 \) with \( n \) fixed as given in (27).

In Section 2 the series expansion of \( D_n(\alpha) \) for \( n \) fixed and \( \alpha \to 0 \) is derived, as indicated above. In Section 3 we obtain an asymptotic \((n \to \infty)\) solution of the Riemann-Hilbert problem related to (17). Moreover, in Section 3, a differential identity for \( d/d\alpha \ln D_n(\alpha) \) is obtained in terms of the matrix elements (and their first derivatives) of the solution to the Riemann-Hilbert problem at the point \( \alpha \). An alternative derivation of this identity, which is closer to the spirit of integrable systems and \( \tau \)-functions (see, e.g., [5, 9, 23]), is given in the Appendix. The identity is then evaluated asymptotically in Section 4 using asymptotics found in Section 3. In Section 5 the identity is integrated, and the results of Section 2 are then used to complete the proof.

**Remark.** Universality allows for considerable freedom in the choice of the approximating ensemble in the above method. We choose to consider the Laguerre ensemble, although we could have considered, for example, GUE itself: for GUE, however, the analysis turns out to be algebraically more complicated. (For example, in the GUE case there will be two endpoints instead of one endpoint at \( x = 1 \), see (12) et seq.) In choosing the approximating ensemble, it is essential that the various constants that arise can be evaluated explicitly as in (27) and also in formula (17) in [17]. In both cases we see that ultimately the formula for the desired constant arises from classical formulae for the Legendre polynomials.

In physics, and also in mathematical physics, universality is often viewed as a passive statement that certain systems "behave in a similar fashion". The thrust of this paper, going back to Dyson [18], is that universality can be used as an active analytical tool to obtain estimates for asymptotic problems of mathematical and physical interest.

**Addendum.** We draw the attention of the reader to the work [2] of Baik, Buckingham, and DiFranco, in which the authors give a different proof of (3) together with related results for GOE and GSE. The paper [2] appeared after our paper was written and refereed.
2 Expansion of $D_n(\alpha)$ as $\alpha \to 0$.

In this section we derive a series expansion for $D_n(\alpha)$ as $\alpha \to 0$. Changing the variables $x_j = (\alpha/2)(t_j + 1)$ and expanding the exponent in (16), we obtain for fixed $n$:

$$D_n(\alpha) = \frac{1}{C_n n!} \left( \frac{\alpha}{2} \right)^{n(n-1)+n} \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{0 \leq i < j \leq n-1} (t_i - t_j)^2 \prod_{j=0}^{n-1} (1 - 2\alpha n(t_j + 1) + O(\alpha^2)) dt_j = \frac{1}{C_n n!} \left( \frac{\alpha}{2} \right)^n A_n (1 + O_n(\alpha)), \quad (22)$$

where

$$A_n = \frac{1}{n!} \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{0 \leq i < j \leq n-1} (t_i - t_j)^2 \prod_{j=0}^{n-1} dt_j \quad (23)$$

can be expressed in terms of the product of the leading coefficients (cf. (14,15)) of the Legendre polynomials:

$$A_n = \prod_{k=0}^{n-1} \frac{2^{2k} (k!)^4}{[(2k)!]^2} \cdot \frac{2}{2k + 1} \quad (24)$$

The asymptotics of $A_n$ as $n \to \infty$ (used first by Widom in [33], and then in [17]) are given by the expression

$$\ln A_n = -n^2 \ln 2 + n \ln(2\pi) - \frac{1}{4} \ln n + \frac{1}{12} \ln 2 + 3\zeta'(-1) + \tilde{\delta}_n, \quad n \to \infty. \quad (25)$$

where $\zeta'(x)$ is the derivative of Riemann’s zeta-function, and $\tilde{\delta}_n \to 0$ as $n \to \infty$. The zeta-function originates from the expansion of the product of factorials.

The asymptotics of $C_n$ (15) have a similar form,

$$\ln C_n = -\left( \frac{3}{2} + \ln 4 \right) n^2 + n \ln(2\pi) - \frac{1}{6} \ln n + 2\zeta'(-1) + \delta_n, \quad \delta_n \to 0, \quad n \to \infty. \quad (26)$$

Substituting the asymptotics (25,26) into (22), we obtain for $\alpha > 0$:

$$\ln D_n(\alpha) = \left( \frac{3}{2} + \ln \alpha \right) n^2 - \frac{1}{12} \ln \frac{n}{2} + \zeta'(-1) + \delta_n + O_n(\alpha), \quad (27)$$

where $\delta_n$ depends on $n$ only and $\delta_n \to 0$ as $n \to \infty$. Note for later application (see proof of Lemma 2) that the error term $O_n(\alpha)$ is analytic in $\alpha$, in particular, $(d/d\alpha)O_n(\alpha) = O_n(1)$. We shall use formula (27) in the last section.

Caveat: $O_n(\alpha) \to 0$ as $\alpha \to 0$, $n$ fixed: no claim is made about $O_n(\alpha)$ as $n \to \infty$. 

7
3 Differential identity and the Riemann-Hilbert problem

3.1 Initial transformations

In what follows, unless explicitly stated otherwise, we will always assume $0 < \alpha < 1$. At certain points in the text, however, we will also consider $\alpha$ in a small neighborhood $D_{\varepsilon_0}(0)$ of $\alpha = 0$ (see the discussion in the end of Section 3.1.)

The multiple integral (16) can be written as (cf. (14,15)):

$$D_n(\alpha) = \frac{1}{C_n} \frac{1}{\prod_{j=0}^{n-1} \theta_j^{-2}},$$ (28)

where $\theta_j$ are the leading coefficients of the polynomials $q_j(x) = \theta_j x^j + \cdots$ satisfying

$$\int_0^\alpha q_k(x)q_m(x)e^{-4nx}dx = \delta_{km}, \quad k, m = 0, 1, \ldots$$ (29)

It is convenient to write this orthogonality relation in the form

$$\int_0^\alpha q_j(x)x^k e^{-4nx}dx = \frac{\delta_{jk}}{\theta_j}, \quad k = 0, 1, \ldots, j = 0, 1, 2, \ldots$$ (30)

Note, in particular, that

$$\int_0^\alpha q_j(x)\frac{\partial}{\partial \alpha} q_j(x)e^{-4nx}dx =$$

$$\int_0^\alpha q_j(x) \left( \frac{d\theta_j}{d\alpha} x^j + \text{polynomial of degree less than } j \right) e^{-4nx}dx = \frac{1}{\theta_j} \frac{d\theta_j}{d\alpha}. \quad (31)$$

Using relation (31), we obtain

$$\frac{d}{d\alpha} \ln D_n(\alpha) = \frac{d}{d\alpha} \ln \prod_{j=0}^{n-1} \theta_j^{-2} = -2 \sum_{j=0}^{n-1} \frac{d\theta_j}{\theta_j} d\alpha = -2 \sum_{j=0}^{n-1} \int_0^\alpha q_j(x) \frac{\partial}{\partial \alpha} q_j(x)e^{-4nx}dx =$$

$$- \int_0^\alpha \frac{\partial}{\partial \alpha} \left( \sum_{j=0}^{n-1} q_j^2(x) \right) e^{-4nx}dx = -\frac{d}{d\alpha} \left( \int_0^\alpha \sum_{j=0}^{n-1} q_j^2(x)e^{-4nx}dx \right) + \sum_{j=0}^{n-1} q_j^2(\alpha)e^{-4n\alpha}. \quad (32)$$

By (29) with $k = m = j$, the last integral (inside the brackets) in (32) equals $n$ and hence vanishes upon differentiation.

Applying the Christoffel-Darboux formula,

$$\sum_{j=0}^{n-1} q_j^2(x) = \frac{\theta_{n-1}}{\theta_n} (q_n(x)q_{n-1}(x) - q_n(x)q'_{n-1}(x)),$$ (33)
to the last sum in (32), we obtain
\[
\frac{d}{d\alpha} \ln D_n(\alpha) = \frac{\theta_{n-1}}{\theta_n} e^{-4n\alpha} (q_n'(\alpha)q_{n-1}(\alpha) - q_n(\alpha)q_{n-1}'(\alpha)).
\] (34)

Here and below the prime denotes differentiation w.r.t. the argument \(x\).

Formula (34) shows that \(\frac{d}{d\alpha} \ln D_n(\alpha)\) depends only on \(q_n, q_{n-1}\). This property is crucial for the analysis below.

As noted in [20], orthogonal polynomials can be represented in terms of a solution to an associated Riemann-Hilbert problem. In the present case, the relevant Riemann-Hilbert problem is formulated as follows: Find a \(2 \times 2\) matrix-valued function \(V(z)\) satisfying the conditions:

(a) \(V(z)\) is analytic for \(z \in \mathbb{C} \setminus [0, \alpha]\).

(b) Let \(x \in (0, \alpha)\). \(V(z)\) has \(L_2\) boundary values \(V_+(x)\) as \(z\) approaches \(x\) from above, and \(V_-(x)\), from below. They are related by the jump condition
\[
V_+(x) = V_-(x) \begin{pmatrix} 1 & e^{-4nx} \\ 0 & 1 \end{pmatrix}, \quad x \in (0, \alpha).
\] (35)

(c) \(V(z)\) has the following asymptotic behavior as \(z \to \infty\):
\[
V(z) = \left( I + O\left( \frac{1}{z}\right) \right) z^{n\sigma_3}, \quad \text{where} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (36)

This Riemann-Hilbert problem (RHP) has a unique solution for any \(n, \alpha > 0\), and, in particular, \(V_{11}(z) = q_n(z)/\theta_n\) and \(V_{21}(z) = -2\pi i \theta_{n-1} q_{n-1}(z)\). Therefore we can rewrite the differential identity (34) in terms of \(V(z)\) in the form
\[
\frac{d}{d\alpha} \ln D_n(\alpha) = \frac{e^{-4n\alpha}}{2\pi i} \left( V_{11}(\alpha)V_{21}'(\alpha) - V_{11}'(\alpha)V_{21}(\alpha) \right).
\] (37)

In this section our task is to solve the RHP for \(V(z)\) asymptotically (in other words, to find asymptotics of the polynomials \(q_k(z)\)) as \(n \to \infty\). The results will then be used in Section 4.2 to evaluate the r.h.s. of (37).

Following the steepest descent method for RH problems as described in [16, 7], we first of all need to find a so-called \(g\)-function: In the present situation this reduces to finding a function analytic outside the interval \((-\infty, \alpha)\) and continuous up to the boundary with the properties:

(a) \(g(z) = \ln(z) + O(1/z)\) as \(z \to \infty\);

\[\footnote{An alternative derivation of this identity is presented in the Appendix.}\]
(b) There exists a constant $l$ such that the boundary values $g_\pm(x) = \lim_{\varepsilon \to 0} g(x \pm i\varepsilon)$ of $g(z)$ are related as follows:

$$g_+(x) + g_-(x) - 4x - l = 0, \quad x \in (0, \alpha); \quad (38)$$

(c) On $(0, \alpha)$, $g_+(x) - g_-(x)$ is purely imaginary, and $i(d/dx)(g_+(x) - g_-(x)) > 0$;

(d) $e^{g_+(x)-g_-(x)} = 1$ on $(-\infty, 0)$.

A standard computation shows that if such a function $g(z)$ exists than it is unique.

Formally, the derivative $g'(z)$ of $g(z)$ must have the properties:

(a') $g'(z) = 1/z + O(1/z^2)$ as $z \to \infty$;

(b') $g'_+(x) + g'_-(x) = 4$ for $x \in (0, \alpha)$.

It is easy to verify that the following function satisfies these conditions:

$$g'(z) = 2 + \frac{1 + \alpha - 2z}{\sqrt{z(z - \alpha)}}. \quad (39)$$

(In fact, $g'(z)$ is the unique function with $L^p$ boundary values $g'_\pm$ satisfying (a') and (b') for any $1 < p < 2$.) In (39), the branch is chosen so that $\sqrt{z(z - \alpha)}$ is analytic in the complement of $(0, \alpha)$ and positive for $z > \alpha$.

Therefore,

$$g(z) = \int_\alpha^z g'(t)dt + C,$$

where the constant $C$ is determined from the condition that $g(z) - \ln(z) = O(1/z)$ as $z \to \infty$. This gives

$$g(z) = 2z - \alpha + \ln \frac{\alpha}{4} + \int_\alpha^z \frac{1 + \alpha - 2t}{\sqrt{t(t - \alpha)}}dt, \quad (40)$$

and it is easy to verify that $g(z)$ indeed satisfies (a)–(d). From (38,40) we now see that

$$l = -2\alpha + 2 \ln \frac{\alpha}{4}. \quad (41)$$

We need to analyze the RHP for $V(z)$ asymptotically as $n \to \infty$ uniformly for $0 < \alpha < 1 - s_0/(2n)^{2/3}$ where $s_0$ is a fixed (large) number. The steepest descent method continues with the following steps (see [16, 7]):

1) the RHP for $V$ is conjugated by $e^{ng(z)\sigma_3}$;

2) Note that as the contour for the RHP is $(0, \alpha)$, the extra condition (4.14) for $g(z)$ in [16] is redundant in the present situation.
2) the contour \((0, \alpha)\) is split into lenses;

3) matching parametrices for the solution to the RHP are constructed (i) away from the end-points 0 and \(\alpha\), (ii) in neighborhoods of 0 and \(\alpha\), respectively.

By means of these steps, the RHP reduces as \(n \to \infty\) to a small norm problem which can be solved by a Neuman series.

All these steps go through in the standard way except for the construction of the parametrix in a neighborhood of \(\alpha\). As we see from [16, 7] the method requires that in a neighborhood \(|z - \alpha| \leq \varepsilon\), \(\varepsilon\) small and fixed,

\[
(g_+ - g_-(z)) = (z - \alpha)^\beta(c + O(z - \alpha)),
\]

for some \(c \neq 0\) and some exponent \(\beta > 0\). (In [16], \(\beta = 3/2\).) In our case for \(0 < z < \alpha\),

\[
(g_+ - g_-(z)) = 2 \int_0^z \frac{1 + \alpha - 2t}{\sqrt{t(t - \alpha)}} dt = \frac{4}{\sqrt{\alpha}}(z - \alpha)^{1/2}(1 - \alpha + O(z - \alpha)).
\]

For any fixed \(0 < \alpha < 1\) we see that \((g_+ - g_-)(z)\) satisfies (42). As \(\alpha \to 1\), we have to make the neighborhood \(|z - \alpha| < \varepsilon\) smaller and smaller. The constant \(c\) in (42) depends then on \(\alpha\), but that, in itself, is not an insurmountable problem. The real problem is that, unlike the situation in [26], the parametrix away from the points 0, \(\alpha\) (see [16, 7]) contains certain terms of the form \((z/(z - \alpha))^{1/4}\) evaluated on \(\{z : |z - \alpha| = \varepsilon\}\), and as a result is not uniformly bounded when \(1 - \alpha\), and hence \(\varepsilon\), approach zero. At the same time, there is not enough decay in the other relevant quantities to compensate for this. The problem can be circumvented, however, by introducing a transformation of the \(z\)-plane that “regularizes” the RHP in a neighborhood of \(z = \alpha\). Namely, set

\[
\lambda = \frac{1 - \alpha}{\alpha} \frac{z}{1 - z}, \quad z \neq 1.
\]
This fractional-linear transformation maps the interval $[0, \alpha]$ onto $[0, 1]$, the point $z = 1$ is mapped to infinity, and infinity is mapped to $\lambda = -(1 - \alpha)/\alpha$. The inverse transform is

$$z = \frac{\alpha \lambda}{1 - \alpha + \alpha \lambda}, \quad \lambda \neq -\frac{1 - \alpha}{\alpha}. \quad (45)$$

Thus $z(\lambda)$ is analytic from $\mathbb{C} \setminus \{-(1 - \alpha)/\alpha\}$ into $\mathbb{C}$, taking the complement of $[0, 1]$ onto $\mathbb{C} \setminus [0, \alpha]$.

The fact that in our case we could not obtain an estimate of the form (42) uniformly as $\alpha \uparrow 1$ originates in the vanishing of the numerator in the integral for $g_+ - g_-$ in (43) at the point $t = (1 + \alpha)/2 \in (\alpha, 1)$. Under the transformation $z \to \lambda$ the point $(1 + \alpha)/2$ is mapped to $\lambda = 1 + \alpha^{-1}$. This point is at a positive distance from the contour $0 < \lambda < 1$ for $\alpha \in (0, 1)$. This means that we will be able to construct a parametrix for the solution of the RHP in the $\lambda$ variable in a fixed neighborhood about $\lambda = 1$. On the other hand, the point $\lambda = -(1 - \alpha)/\alpha$ (the image of $z$-infinity) now approaches the contour as $\alpha \uparrow 1$, and we will need to contract the neighborhood of $\lambda = 0$ so that this point remains outside. We shall see, however, that this neighborhood presents no problem, as the relevant terms of the jump matrix for the final $R$-RHP (see (70) and the argument after (103) below) decay sufficiently fast on the boundary of the neighborhood.

For any $\lambda \in \mathbb{C} \setminus ([0, 1] \cup \{-(1 - \alpha)/\alpha\})$ set

$$U(\lambda) \equiv V(z(\lambda)), \quad (46)$$

where $z(\lambda) = \alpha \lambda/(1 - \alpha + \alpha \lambda)$ as in (45).

Then we obtain the following Riemann-Hilbert problem for $U(\lambda)$:

(a) $U(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus ([0, 1] \cup \{-(1 - \alpha)/\alpha\})$.

(b) Let $\lambda \in (0, 1)$. $U$ has $L_2$ boundary values $U_+(\lambda)$ as $\lambda$ approaches the real axis from above, and $U_-(\lambda)$, from below. They are related by the jump condition

$$U_+(\lambda) = U_-(\lambda) \begin{pmatrix} 1 & e^{-4n\pi(\lambda)} \\ 0 & 1 \end{pmatrix}, \quad \lambda \in (0, 1). \quad (47)$$

(c) $U(\lambda)$ has the following asymptotic behavior as $\lambda \to -\frac{1 - \alpha}{\alpha}$ ($z \to \infty$):

$$U(\lambda) = \left[ I + O\left(\frac{1}{z(\lambda)}\right) \right] z(\lambda)^{n\sigma_3}. \quad (48)$$

We transfer $g(z)$ to the $\lambda$-plane by defining

$$\tilde{g}(\lambda) \equiv g(z(\lambda)), \quad \text{for} \quad \lambda \in \mathbb{C} \setminus \left[ -\frac{1 - \alpha}{\alpha}, 1 \right]. \quad (49)$$
Necessarily, \( \hat{g}(\lambda) \) is analytic on its domain. We obtain
\[
\hat{g}(\lambda) = 2\alpha(\alpha - 1) \frac{1 - \lambda}{1 - \alpha + \alpha \lambda} + \alpha + \ln \frac{\alpha}{4} + (1 - \alpha)^{3/2} \int_{1}^{\lambda} \frac{1 + \alpha(1 - t)}{(1 - \alpha(1 - t))^{3/2}} \frac{dt}{\sqrt{t(t - 1)}}. \tag{50}
\]

Note that \( \hat{g}(\lambda + 0) - \hat{g}(\lambda - 0) = 2\pi i \) on \((-\alpha^{-1} - 1, 0)\) as this interval is the image of the half-axis \((-\infty, 0)\) in the \(z\)-variable, where it is easy to conclude (cf. (52) below) that \( g_+(z) - g_-(z) = 2\pi i \). This jump in the \(\lambda\)-variable is also easy to obtain directly from (50).

Let
\[
h(\lambda) = \frac{2(1 - \alpha)^{3/2}}{e^{i\pi/2}} \int_{1}^{\lambda} \frac{1 + \alpha(1 - t)}{(1 - \alpha(1 - t))^{3/2}} \frac{dt}{\sqrt{t(1 - t)}} \tag{51}
\]

which is analytic in \( \mathbb{C} \setminus ((-\infty, 0) \cup (1, \infty)) \). Here we choose the branch so that \( \sqrt{t(1 - t)} \) is analytic in \( \mathbb{C} \setminus ((-\infty, 0) \cup (1, \infty)) \) and positive for \( t \in (-1, 1) \). The function \( h(\lambda) \) is the analytic continuation of \( \hat{g}(\lambda + 0) - \hat{g}(\lambda - 0) \) off the interval \((0, 1)\).

Note that
\[
h(0) = \frac{2}{e^{i\pi/2}} \int_{1}^{0} \frac{1 + \alpha - 2\alpha x}{\sqrt{x(1 - x)}} dx = 2\pi i. \tag{52}
\]

Now transform the RHP for \( U \) as follows:
\[
\tilde{T}(\lambda) = e^{-n\sigma_3/2} U(\lambda) e^{-n(\hat{g}(\lambda) - l/2)\sigma_3}, \quad \lambda \in \mathbb{C} \setminus \left( [0, 1] \cup \left\{ -\frac{1 - \alpha}{\alpha} \right\} \right). \tag{53}
\]

We easily obtain then that \( \tilde{T}(\lambda) \) satisfies:

(a) \( \tilde{T}(\lambda) \) is analytic for \( \lambda \in \mathbb{C} \setminus [0, 1] \).

(b) For \( \lambda \in (0, 1) \) the boundary values of \( \tilde{T}(\lambda) \) are related by the jump condition
\[
\tilde{T}_+ (\lambda) = \tilde{T}_- (\lambda) \left( \begin{array}{cc} e^{-nh(\lambda)} & 1 \\ 0 & e^{nh(\lambda)} \end{array} \right), \quad \lambda \in (0, 1). \tag{54}
\]

(c) \( \tilde{T}(\lambda) \) has the following asymptotic behavior as \( \lambda \to -\frac{1 - \alpha}{\alpha} \):
\[
\tilde{T}(\lambda) = I + O \left( \lambda + \frac{1 - \alpha}{\alpha} \right). \tag{55}
\]

Note that the problem is now normalized to \( I \) at \( \lambda = -\frac{1 - \alpha}{\alpha} \).

Since \( \det \tilde{T}(\lambda) = 1 \) and \( \tilde{T}(\lambda) \) is analytic at infinity, it follows that \( \tilde{T}(\infty) \) is invertible. The function \( T(\lambda) \) defined by
\[
T(\lambda) = \tilde{T}(\infty)^{-1} \tilde{T}(\lambda) \tag{56}
\]
is the solution to the same Riemann-Hilbert problem as $\tilde{T}(\lambda)$, with the $(c)$ condition replaced by

$$T(\lambda) = I + O(1/\lambda), \quad \lambda \to \infty. \quad (57)$$

Clearly,

$$\tilde{T}(\lambda) = T^{-1} \left( -\frac{1 - \alpha}{\alpha} \right) T(\lambda). \quad (58)$$

We now show that the RHP for $T$ is solvable for all $0 \leq \alpha < 1$. For $0 < \alpha < 1$ the existence of such a $T(\lambda)$ follows simply by pushing forward $V(z)$, the solution of the RHP (35, 36) for the polynomials orthogonal on $(0, \alpha)$ with the weight $e^{-4nz}$: the existence of $V(z)$ itself follows from the basic results of [20, 12]. So we are reduced to showing that $T(\lambda)$ exists in the case $\alpha = 0$ when the mapping $V(z) \to T(\lambda)$ breaks down. For $\alpha = 0$, $h(\lambda) = 4 \ln(\sqrt{\lambda} + i\sqrt{1 - \lambda})$, $0 < \lambda < 1$. If $(\lambda - 1)^{1/2}$ (resp., $\lambda^{1/2}$) denotes the branch which is analytic in $\mathbb{C} \setminus [-\infty, 1]$ (resp., $\mathbb{C} \setminus [-\infty, 0]$), then in particular $(\lambda - 1)^{1/2} = -((\lambda - 1)^{1/2} = i\sqrt{1 - \lambda}$, $0 < \lambda < 1$, and we find

$$e^{nh(\lambda)} = \left( \frac{(\lambda - 1)^{1/2} + \lambda^{1/2}}{(\lambda - 1)^{1/2} + \lambda^{1/2}} \right)^{2n}. \quad (59)$$

Thus if $r(\lambda) = ((\lambda - 1)^{1/2} + \lambda^{1/2})/2$, then

$$\begin{pmatrix} e^{-nh(\lambda)} & 1 \\ 0 & e^{nh(\lambda)} \end{pmatrix} = \begin{pmatrix} (r_-/r_+)^{2n} & 1 \\ 0 & (r_+/r_-)^{2n} \end{pmatrix}, \quad 0 < \lambda < 1. \quad (60)$$

Setting $Z(\lambda) = T(\lambda)r(\lambda)^{2n\sigma_3}$, we see that $Z(\lambda)$ solves the RHP:

(a) $Z(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus [0, 1]$.

(b) For $\lambda \in (0, 1)$ the boundary values of $Z(\lambda)$ are related by the jump condition

$$Z_+(\lambda) = Z_-(\lambda) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \lambda \in (0, 1). \quad (61)$$

(c) $Z(\lambda)$ has the following asymptotic behavior as $\lambda \to \infty$:

$$Z(\lambda) = (I + O(1/\lambda))\lambda^{n\sigma_3}. \quad (62)$$

This is the standard RHP for polynomials orthogonal on $(0, 1)$ with the unit weight. Therefore the desired solution $T(\lambda)$ exists for $\alpha = 0$ as well. This completes the proof of solvability of the RHP for $T(\lambda)$ for all $0 \leq \alpha < 1$. The above proof of solvability for all $n$ is included only for completeness (cf. the last remark at the end of Section 3.5).

As is standard in applications of the steepest descent method, we now deform the RHP as follows. Let $\Sigma = \cup_{j=1}^3 \Sigma_j$ be the oriented contour as in Figure 2. Define a matrix-valued
Figure 2: Contour for the $S$-Riemann-Hilbert problem and the circular neighborhoods $U_{1,0}$ of the points 1, 0. These neighborhoods will be introduced below in connection with the construction of parametrices.

function $S(\lambda)$ on $\mathbb{C} \setminus \Sigma$ by the expressions:

$$S(\lambda) = \begin{cases} T(\lambda), & \text{for } \lambda \text{ outside the lens}, \\ T(\lambda) \begin{pmatrix} 1 \\ -e^{-nh(\lambda)} & 0 \\ 0 & 1 \end{pmatrix}, & \text{for } \lambda \text{ in the upper part of the lens}, \\ T(\lambda) \begin{pmatrix} 1 \\ e^{nh(\lambda)} & 0 \\ 0 & 1 \end{pmatrix}, & \text{for } \lambda \text{ in the lower part of the lens}. \end{cases} \tag{63}$$

It is easy to verify that $S(\lambda)$ solves the following RHP:

(a) $S(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus \Sigma$, where $\Sigma = \bigcup_{j=1}^{3} \Sigma_j$.

(b) The boundary values of $S(\lambda)$ are related by the jump condition

$$S_+(\lambda) = S_-(\lambda) \begin{pmatrix} 1 \\ e^{\mp nh(\lambda)} & 0 \\ 0 & 1 \end{pmatrix}, \quad \lambda \in (\Sigma_1 \cup \Sigma_3) \setminus \{0,1\}, \tag{64}$$

where the plus sign in the exponent is on $\Sigma_3$, and the minus sign, on $\Sigma_1$.

$$S_+(\lambda) = S_-(\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \lambda \in \Sigma_2 \equiv (-1,1).$$

(c) $S(\lambda) = I + O(1/\lambda)$ as $\lambda \to \infty$.

For a fixed $0 < \varepsilon < 1/4$, consider the circular neighborhood $U_1$ of radius $\varepsilon$ at the point $\lambda = 1$. Consider also the neighborhood $U_0$ of $\lambda = 0$ of radius $\varepsilon_3(1 - \alpha)$ for a fixed $1/2 > \varepsilon_3 > 0$. Note that $U_0$ contracts with growing $n$ for $\alpha = 1 - s_0/(2n)^{2/3}$. The point $-(1 - \alpha)/\alpha$ lies outside $U_0$ for all $\alpha \in (0,1)$. 

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In $U_0$, we can expand the integrand in (51) in powers of $t$ and $t/(1 - \alpha)$:

\[
\begin{align*}
 h(\lambda) &= h(0) + \frac{2}{e^{\pi/2} \sqrt{1 - \alpha}} \int_0^\lambda (1 + \alpha - \alpha t) \left( 1 - \frac{2\alpha t}{1 - \alpha} + O\left( \frac{t^2}{(1 - \alpha)^2} \right) \right) \times \\
 &\quad \left( 1 + t/2 + O(t^2) \right) \frac{dt}{\sqrt{t}} \\
 &= h(0) + \frac{4\sqrt{\lambda}}{e^{\pi/2} \sqrt{1 - \alpha}} \left( 1 + \alpha + \frac{1 - 6\alpha - 3\alpha^2}{6(1 - \alpha)} \lambda + O\left( \frac{\lambda^2}{(1 - \alpha)^2} \right) \right),
\end{align*}
\]  
(65)

uniformly in $\alpha$, and where $h(0) = 2\pi i$ (see (52)). It is the presence of $\sqrt{1 - \alpha}$ in the denominator that will allow us to construct a solution to the RHP using a contracting neighborhood $U_0$ as $\alpha$ approaches 1.

We shall now show that the the jump matrices for $S(\lambda)$ on $\Sigma_1 \cup \Sigma_3 \setminus (U_1 \cup U_0)$ are uniformly exponentially close to the identity (see (69) below) as $n(1 - \alpha)^{3/2} \to \infty$.

To estimate the real part of $h(\lambda)$ outside of the neighborhoods $U_0$ and $U_1$, we now describe the form of the lens more precisely. First, we assume that the contour $\Sigma_1$ is the mirror image of $\Sigma_3$, i.e.

\[
\Sigma_3 = \overline{\Sigma_1}.
\]

Therefore, we only need to describe the structure of the contour $\Sigma_1$. We assume that for $0 \leq \Re \lambda \leq 1/2$ the contour $\Sigma_1$ lies above the straight line originating at zero, and making a positive angle $\gamma_0$ with the real axis (see Figure 3). The value of the angle $\gamma_0$ will be specified later on. Similarly, the part of the contour between the vertical line $\Re \lambda = 1/2$ and the boundary of the neighborhood $U_1$ lies above the line $\Im \lambda = (1 - \Re \lambda) \tan \gamma_1$ where, again, the value of the angle $\gamma_1 < \gamma_0$ will be specified later on. Note that the contour $\Sigma$ has a well-defined limit as $\alpha \downarrow 0$.

Let $\lambda_0$ (resp., $\lambda_1$) be the point of intersection of the contour $\Sigma_1$ and the boundary of the disc $U_0$ (resp., $U_1$) (see again Figure 3). Let $\Re \lambda = \mu$, $\Im \lambda = u$. Thus, $\lambda = \mu + iu$, and on $\Sigma_1$,
\(\Re\lambda_0 \leq \mu \leq \Re\lambda_1\). Fix some small \(\varepsilon_2 > 0\). Suppose that \(\Sigma_1\) and \(\Sigma_3\) are so close to the real axis that
\[
\frac{|u|}{\mu} < \varepsilon_2, \quad \frac{|u|}{1 - \mu} < \varepsilon_2, \quad \Re\lambda_0 \leq \mu \leq \Re\lambda_1.
\]  
(66)

In particular, this implies that \(\tan \gamma_0 < \varepsilon_2\) and \(\tan \gamma_1 < \varepsilon_2\). Furthermore, as \(1 + \alpha - \alpha \mu > 1\), we have
\[
\frac{|u|}{1 + \alpha - \alpha \mu} < |u|,
\]
and, as \(1 - \alpha + \alpha \mu > \alpha \mu\),
\[
\frac{\alpha |u|}{1 - \alpha + \alpha \mu} < \frac{|u|}{\mu}.
\]

The above inequalities allow us to perform the following estimate on \(h(\lambda)\) for \(\lambda = \mu + iu\) in \((\Sigma_1 \cup \Sigma_3) \setminus \{U_1 \cup U_0\}\). Using (51), we obtain
\[
h(\lambda) = h(\mu) + \frac{2(1 - \alpha)^{3/2}}{e^{\pi/2}} \int_{\mu}^{\mu + iu} \frac{d(\mu + iv)}{1 + \alpha - \alpha \mu - \alpha iv\sqrt{1 - \alpha}} = \tag{67}
\]
\[
h(\mu) + \frac{2(1 - \alpha)^{3/2}}{\sqrt{1 - \alpha}} \frac{1 + \alpha - \alpha \mu}{(1 - \alpha + \alpha \mu)^2} \int_{0}^{u} \left(1 - \frac{i\alpha v}{1 + \alpha - \alpha \mu}\right) \left(1 + \frac{i\alpha v}{1 - \alpha + \alpha \mu}\right)^{-2} \times
\]
\[
\left(1 + \frac{i\alpha v}{\mu}\right)^{-1/2} \left(1 - \frac{i\alpha v}{1 - \mu}\right)^{-1/2} dv = h(\mu) + \frac{2(1 - \alpha)^{3/2}}{\sqrt{1 - \alpha} \mu(1 - \alpha + \alpha \mu)^2} \frac{1 + \alpha - \alpha \mu}{(1 - \alpha + \alpha \mu)^2} u [1 + O(\varepsilon_2)],
\]
where the constant in the error term is uniform for \(0 \leq \alpha < 1\).

The fraction \(u/(1 - \alpha + \alpha \mu)^2\) in the last equation of (67) can be estimated for some \(\varepsilon_4 > 0\) as
\[
\frac{|u|}{(1 - \alpha + \alpha \mu)^2} > \varepsilon \sin \gamma_1 > \varepsilon_4, \quad \text{for} \quad \frac{1}{2} \leq \mu \leq \Re\lambda_1,
\]
\[
\frac{|u|}{(1 - \alpha + \alpha \mu)^2} > \frac{\mu \tan \gamma_0}{(1 - \alpha + \alpha \mu)^2} = \frac{\tan \gamma_0}{\mu(\alpha + (1 - \alpha)/\mu)^2} > \tag{68}
\]
\[
\frac{\tan \gamma_0}{(1 + \varepsilon_2/(\varepsilon_3 \sin \gamma_0))^2} \geq \varepsilon_4, \quad \text{for} \quad \Re\lambda_0 \leq \mu \leq \frac{1}{2},
\]
where \(\varepsilon_4\) depends only on \(\varepsilon\) and \(\varepsilon_i, i = 2, 3, \gamma_0, \gamma_1\), which in turn depend only on \(\varepsilon, \varepsilon_2, \varepsilon_3\).

Since \(\Re h(\mu) = 0\), we obtain from (68) as \(n \to \infty\) for sufficiently small \(\varepsilon_2 > 0\):
\[
|e^{-nh(\lambda)}| = O(e^{-\rho c}), \quad \lambda \in \Sigma_1 \setminus \{U_0 \cup U_1\},
\]
\[
|e^{nh(\lambda)}| = O(e^{-\rho c}), \quad \lambda \in \Sigma_3 \setminus \{U_0 \cup U_1\}
\]
(69)

uniformly for \(\alpha \in [0, 1 - s_0/(2n)^{3/2}]\) for some (large) \(s_0 > 0\) and all \(n > s_0^{3/2}/2\), for some \(c = c(\varepsilon, \varepsilon_2, \varepsilon_3) > 0\), where
\[
\rho = n|1 - \alpha|^{3/2}.
\]
So except for the jump on the interval $(0,1)$ and the jumps inside $U_1, U_0$, the jumps of $S(\lambda)$ are indeed exponentially close to the identity as $\rho \to \infty$.

For later purposes, we shall need the series expansion of $h(\lambda)$ at $\lambda = 0, 1$. We have:

$$h(\lambda) = 2\pi i + \frac{4\sqrt{\lambda}}{e^{i\pi/2}\sqrt{1-\alpha}} \left( 1 + \alpha + \frac{1 - 6\alpha - 3\alpha^2}{6(1-\alpha)} \lambda + O\left( \frac{\lambda^2}{(1-\alpha)^2} \right) \right), \quad \lambda \to 0; \quad (70)$$

$$h(\lambda) = 4(1-\alpha)^{3/2} \sqrt{u} \left( 1 - (\alpha + 1/6)u + (\alpha^2 + 3\alpha/10 + 3/40)u^2 + O(u^3) \right), \quad \lambda = 1 + u, \quad u \to 0. \quad (71)$$

In (70) the cut of the root lies to the left of $\lambda = 0$, and $-\pi < \arg \lambda < \pi$, whereas in (71) the cut lies to the right of $\lambda = 1$, and $0 < \arg u < 2\pi$.

Note the crucial fact that, as follows from (70), (71), the quantity $n|h(\lambda)|$ (resp., $n|h(\lambda) - 2\pi i|$) is uniformly large on the boundary $\partial U_1$ (resp., $\partial U_0$) for some (large) $s_0 > 0$ for all $\alpha \in (0,1-s_0/(2n)^{2/3}]$, if $(2n)^{2/3} > s_0$. Indeed, it is of order $s_0^{3/2}$ for $\lambda$ on $\partial U_1$ (resp., of order $n$ for $\lambda$ on $\partial U_0$). This will allow us to obtain the desired asymptotic solution of the Riemann-Hilbert problem.

For technical reasons (see the end of the section 4.2. below and also proof of Corollary 2 in [17]), we need to control the solution of the RHP for all $\alpha \in D_{\varepsilon_0}(0) \cup [\varepsilon_0, 1-s_0/(2n)^{2/3}]$, where $D_{\varepsilon_0}$ denotes the disc of radius $\varepsilon_0$ about zero in the complex $\alpha$-plane with $\varepsilon_0$ small. For all $\alpha \in D_{\varepsilon_0}(0)$ we use the fixed contour $\Sigma = \Sigma_{\alpha=0}$ in Figure 3 corresponding to $\alpha = 0$. By the preceding calculation we see that $|\Re h(\lambda; \alpha = 0)| \geq c_0 > 0$ for all $\lambda \in (\Sigma_1 \cup \Sigma_3) \setminus (U_0 \cup U_1)$. Thus

$$|e^{-nh(\lambda; \alpha = 0)}| \leq e^{-nc_0}, \quad \lambda \in (\Sigma_1 \cup \Sigma_3) \setminus (U_0 \cup U_1). \quad (72)$$

Hence, by continuity, we must have

$$|e^{-nh(\lambda, \alpha)}| \leq e^{-nc_0} \quad (73)$$

for all $\lambda \in (\Sigma_1 \cup \Sigma_3) \setminus (U_0 \cup U_1)$ and all $\alpha \in D_{\varepsilon_0}(0), 0 < c_0 < c_0, \varepsilon_0$ sufficiently small.

We now begin the construction of parametrices which give, in their respective regions, the leading contribution to the asymptotics for the RHP.

### 3.2 Parametrix in $\mathbb{C} \setminus (U_1 \cup U_0)$

First, because of the exponential convergence described above, we expect the following model problem to play a role in constructing a parametrix for the solution of the RHP as $n \to \infty$:

(a) $N(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus [0,1]$,

(b) $N_+(\lambda) = N_-(\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \lambda \in (0,1), \quad (74)$
The solution \( N(\lambda) \) can be found in the standard way by first transforming \( N(\lambda) \) with a \( 2 \times 2 \) unitary transformation to the form for which the jump matrix is diagonal and then solving the two resulting scalar Riemann-Hilbert problems (cf. [7]). We obtain
\[
N(\lambda) = \frac{1}{2} \begin{pmatrix}
  m + m^{-1} & -i(m - m^{-1}) \\
  i(m - m^{-1}) & m + m^{-1}
\end{pmatrix},
\]
where \( m(\lambda) \) is analytic outside \([0, 1]\) and \( m(\lambda) \to +1 \) as \( \lambda \to \infty \). Note that \( \det N(\lambda) = 1 \) and that \( N(\lambda) \) is the unique \( L^p \) solution of the RHP for any \( 1 < p < 4 \).

### 3.3 Parametrix at \( \lambda = 1 \)

Now let us construct a parametrix in \( U_1 \). We look for an analytic matrix-valued function \( P_1(\lambda) \) in \( U_1 \) which has the same jump relation as \( S(\lambda) \) on \( \Sigma \cap U_1 \) and instead of a condition at infinity satisfies the matching condition on the boundary
\[
P_1(\lambda)N^{-1}(\lambda) = I + O(1/\rho), \quad \lambda \in \partial U_1, \quad \rho = n|1 - \alpha|^{3/2},
\]
uniformly in \( \lambda \) and \( \alpha \) as \( \rho \to \infty \).

Define:
\[
\phi(\lambda) = \begin{cases}
  e^{i\pi h(\lambda)/2}, & \text{for } \Im \lambda > 0, \\
  h(\lambda)/2, & \text{for } \Im \lambda < 0.
\end{cases}
\]

This function is analytic in \( U_1 \) outside \((1 - \varepsilon, 1]\).

We look for \( P_1(z) \) in the form:
\[
P_1(\lambda) = E_n(\lambda)\hat{P}(\lambda)e^{n\phi(\lambda)\sigma_3},
\]
where \( E_n(\lambda) \) is analytic and invertible (\( \det E_n \neq 0 \)) in a neighborhood of \( U_1 \), and therefore does not affect the jump and analyticity conditions for \( \hat{P}(\lambda)e^{n\phi(\lambda)\sigma_3} \).

As \( P_1(\lambda) \) is required to satisfy the jump relations (64) for \( S \), it is easy to verify that \( \hat{P}(\lambda) = E_n(\lambda)^{-1}P_1(\lambda)e^{-n\phi(\lambda)\sigma_3} \) satisfies jump conditions with \textit{constant} jump matrices:
\[
\hat{P}_+(\lambda) = \hat{P}_-(\lambda) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \lambda \in (\Sigma_1 \cup \Sigma_3) \cap U_1 \setminus \{1\},
\]
\[
\hat{P}_+(\lambda) = \hat{P}_-(\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \lambda \in \Sigma_2 \cap U_1.
\]

Now introduce a mapping of \( U_1 \) onto a new \( \zeta \)-plane
\[
\zeta = n^2\phi(\lambda)^2 = 4n^2(1-\alpha)^3u(1-(2\alpha+1/3)u+(3\alpha^2+14\alpha/15+8/45)u^2+O(u^3)), \quad \lambda = 1+u,
\]
where \( n = 2\) and \( \alpha \geq 1/2 \).
Figure 4: Contour of the Riemann-Hilbert problem for $\Psi(\zeta)$ (the case of $U_1$).

where we used (71). The expansion at $\lambda = 1$ is uniform for $\alpha$ in a bounded set.

Choosing a sufficiently small $\varepsilon > 0$, we see that $\zeta(\lambda)$ is analytic and one-to-one in the neighborhood $U_1$.

Note that if $\alpha \in [0, 1 - s_0/(2n)^{2/3}]$ then $|\zeta| = O(\rho^2)$ uniformly large, if $s_0$ is large, on the boundary $\partial U_1$ and in $\alpha$. This is a crucial fact in the present work. When $\alpha = 1 - s_0/(2n)^{2/3}$, we have $\rho = s_0^{3/2}/2$.

Let us now choose the exact form of the contours in $U_1$ so that their images under the mapping $\zeta(\lambda)$ are straight lines (see Figure 4). Set

$$\hat{P}(\lambda) = \Psi(\zeta),$$

(82)

So the jump matrices for $\Psi(\zeta)$ are the same as for $\hat{P}(\lambda)$ (they are shown in Figure 4). A matrix $\Psi(\zeta)$ satisfying these jump conditions was constructed in [27] in terms of Bessel functions, namely:

1) region I

$$\Psi(\zeta) = \frac{1}{2} \begin{pmatrix} H_0^{(1)}(e^{-i\pi/2} \zeta^{1/2}) & H_0^{(2)}(e^{-i\pi/2} \zeta^{1/2}) \\ \pi \zeta^{1/2} \left(H_0^{(1)}\right)'(e^{-i\pi/2} \zeta^{1/2}) & \pi \zeta^{1/2} \left(H_0^{(2)}\right)'(e^{-i\pi/2} \zeta^{1/2}) \end{pmatrix},$$

(83)

2) region II

$$\Psi(\zeta) = \frac{1}{2} \begin{pmatrix} H_0^{(2)}(e^{i\pi/2} \zeta^{1/2}) & -H_0^{(1)}(e^{i\pi/2} \zeta^{1/2}) \\ -\pi \zeta^{1/2} \left(H_0^{(2)}\right)'(e^{i\pi/2} \zeta^{1/2}) & \pi \zeta^{1/2} \left(H_0^{(1)}\right)'(e^{i\pi/2} \zeta^{1/2}) \end{pmatrix},$$

(84)
3) region III

\[ \Psi(\zeta) = \begin{pmatrix} I_0(\zeta^{1/2}) \\
\pi i \zeta^{1/2} I'_{0}(\zeta^{1/2}) \\
-\zeta^{1/2} K_0'(\zeta^{1/2}) \end{pmatrix}, \tag{85} \]

where \(-\pi < \arg(\zeta) < \pi\).

Here the square root \(\sqrt{\zeta}\) has the cut on \((-\infty, 0)\). Hence, \(\sqrt{\zeta} = -n\phi(\lambda)\) for \(-\pi < \arg(\zeta) < \pi\).

The large-\(\zeta\) asymptotics of Bessel functions give (here we choose \(s_0\), depending only on \(\varepsilon\), sufficiently large):

\[
\Psi(\zeta) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\
-\frac{1}{2}\sqrt{\pi} \\
\frac{\zeta}{2} \end{array} \right) + O(\zeta^{-3/2}) \tag{86} \]

uniformly on the boundary \(\partial U_1\).

Thus

\[
P_1(\lambda) = E_n(\lambda) \Psi(\lambda) e^{n\phi(\lambda)\sigma_3}, \tag{87} \]

where the function \(E_n(\lambda)\) is found from the matching condition to be

\[
E_n(\lambda) = \frac{1}{\sqrt{2}} N(\lambda) \left( \begin{array}{cc} 1 & -i \\
-1 & 1 \end{array} \right) (\pi \sqrt{\zeta})^{\sigma_3/2}. \tag{88} \]

Now to complete the construction of the parametrix it only remains to show that \(E_n(\lambda)\) is an analytic function in \(U_1\) (clearly, \(\det E_n(\lambda) \neq 0\)). First, we show that it has no jump on the real \(\zeta\)-axis. This is easy to verify using the jump condition for \(N(\lambda)\) and the identity \(\zeta = \zeta e^{-2\pi i}\) on the negative half axis. Moreover, a simple calculation shows that \(E_n(\lambda)\) has no pole at \(\lambda = 1\). Thus, \(E_n(\lambda)\) is analytic in \(U_1\), and the parametrix in \(U_1\) is given by the equations (79, 82, 83, 84, 85, 88) for \(\alpha \in [0, 1)\).

Below we shall need the first three terms in the matching condition for \(P_1\). Using (86), we obtain

\[
P_1(\lambda) N^{-1}(\lambda) = I + \Delta_1(\lambda) + \Delta_2(\lambda) + O\left(\frac{1}{\rho^3}\right), \quad \lambda \in \partial U_1. \tag{89} \]

Here

\[
\Delta_1(\lambda) = \frac{1}{8\sqrt{\zeta}} N(\lambda) \left( \begin{array}{cc} -1 & -2i \\
-2i & 1 \end{array} \right) N(\lambda)^{-1} = \frac{1}{16\sqrt{\zeta}} \left( \begin{array}{cc} -3m^2 + m^{-2} & -i(3m^2 + m^{-2}) \\
-i(3m^2 + m^{-2}) & 3m^2 - m^{-2} \end{array} \right), \tag{90} \]

\[
\Delta_2(\lambda) = \frac{3}{2\sqrt{\zeta}} N(\lambda) \left( \begin{array}{cc} -1 & 4i \\
-4i & -1 \end{array} \right) N(\lambda)^{-1} = \frac{3}{2\sqrt{\zeta}} \left( \begin{array}{cc} -1 & 4i \\
-4i & -1 \end{array} \right), \tag{90} \]

where \(m(\lambda)\) is defined in (76). Note that both \(\Delta_1(\lambda)\) and \(\Delta_2(\lambda)\) are meromorphic functions in \(U_1\) with a simple pole at \(\lambda = 1\).
Recall that we use the contour \( \Sigma = \Sigma_{\alpha=0} \) for all \( \alpha \in D_{\varepsilon_0}(0) \), \( \varepsilon_0 \) small. For such \( \alpha \), the map \( \lambda \to \zeta \) maps \( U_1 \) (consisting of the three regions separated by \( \Sigma \)) onto a set, region to region, where the lines separating each region are now no longer straight but lie in small cones about the original ones. The opening angles of the cones are proportional to \( |\Im \alpha| \). Using the same definition for \( \Psi \) as in (83–85) for each of the new regions I, II, III, we find again that (86) is valid, and that \( P_1(\lambda)N^{-1}(\lambda) \) has the same expansion (89) as in the case \( 0 \leq \alpha < 1 \). Note that the values of \( \varepsilon_0 \) and \( s_0 \) can be changed (now and below) if necessary.

### 3.4 Parametrix at \( \lambda = 0 \)

The construction of the parametrix in \( U_0 \) is similar. Recall, however, that the radius of \( U_0 \) is \( \varepsilon_3(1-\alpha) \), so it decreases as \( \alpha \to 1 \), i.e. as the pole of \( h(\lambda) \) approaches the point \( \lambda = 0 \). We shall see that this neighborhood produces asymptotics for the RHP in inverse powers of \( n \).

We look for an analytic matrix-valued function \( P_0(z) \) in the neighborhood \( U_0 \) which satisfies the same jump conditions as \( S(\lambda) \) on \( \Sigma \cap U_0 \), and satisfies the matching condition

\[
P_0(\lambda)N^{-1}(\lambda) = I + (1 - \alpha)^{-1/2}O(1/n)
\]

uniformly in \( \lambda \) on the boundary \( \partial U_0 \) as \( n \to \infty \).

Below we define functions in \( U_0 \) which play the same role as \( \phi, E_n, \) and \( \hat{P} \) in \( U_1 \). We use the same notation for these quantities as before. Namely, let

\[
\phi(\lambda) = \begin{cases} 
  e^{i\pi (h(\lambda) - 2\pi i)/2}, & \text{for } \Im \lambda > 0, \\
  (h(\lambda) - 2\pi i)/2, & \text{for } \Im \lambda < 0.
\end{cases}
\]

This function is analytic in \( U_0 \) outside \([0, \varepsilon_3(1-\alpha)]\).

As above, we look for the parametrix \( P_0(\lambda) \) in the form:

\[
P_0(\lambda) = E_n(\lambda)\hat{P}(\lambda)e^{n\phi(\lambda)\sigma_3},
\]

We obtain that

\[
\hat{P}_+(\lambda) = \hat{P}_-(\lambda) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \lambda \in ((\Sigma_1 \cup \Sigma_3) \cap U_0) \setminus \{0\},
\]

\[
\hat{P}_+(\lambda) = \hat{P}_-(\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \lambda \in \Sigma_2 \cap U_0.
\]

We choose the following mapping of \( U_0 \) onto a \( \zeta \)-plane

\[
\zeta = n^2\phi(\lambda)^2 = e^{-i\pi}4n^2(1+\alpha)^2 \frac{\lambda}{1-\alpha} \frac{2}{1-\alpha} \left(1 + \frac{1}{3(1-\alpha^2)} \lambda + O \left( \frac{\lambda^2}{(1-\alpha)^2} \right) \right),
\]

where we used (70).
Choosing a sufficiently small $\varepsilon_3 > 0$, we see that $\zeta(\lambda)$ is analytic and one-to-one in the neighborhood $U_0$.

Let us also choose the exact form of the contours in $U_0$ so that their images under the mapping $\zeta(\lambda)$ are direct lines. In the $\zeta$-plane the contour and the jump matrices for $\hat{P}(\lambda)$ are the same as in Figure 4 with the only difference that all directions are reversed (pointing away from $\zeta = 0$). It is easily seen that the function

$$\hat{P}(\lambda) = \sigma_3 \Psi(\zeta) \sigma_3,$$

where $\Psi(\zeta)$ is given by (83-85) satisfies the jump conditions in this case.

Finally, we calculate $E_n$ and obtain

$$P_0(\lambda) = E_n(\lambda) \sigma_3 \Psi(\zeta(\lambda)) \sigma_3 e^{n\phi(\lambda)} \sigma_3,$$

where

$$E_n(\lambda) = \frac{1}{\sqrt{2}} N(\lambda) \begin{pmatrix} 1 & i \ i & 1 \end{pmatrix} (\pi \sqrt{\zeta})^{\sigma_3/2}$$

(98)

(the analyticity of $E_n(\lambda)$ in $U_0$ is verified as above).

Then we see immediately from (86,95) that

$$P_0(\lambda) N^{-1}(\lambda) = E_n(\lambda) \sigma_3 \Psi(\zeta) \sigma_3 e^{n\phi(\lambda)} \sigma_3 N^{-1}(\lambda) = I + \frac{1}{\sqrt{\lambda}} O \left( \frac{1}{\sqrt{\zeta}} \right) = I + \frac{1}{\sqrt{1 - \alpha}} O \left( \frac{1}{n} \right)$$

(99)

uniformly in $\lambda \in \partial U_0$ and $\alpha \in [0, 1 - s_0/(2n)^{2/3}]$. Of course, the bound in (99) blows up if $\alpha \to 1$ too rapidly: for $0 \leq \alpha < 1 - s_0/(2n)^{2/3}$, we see that the error term is $O(n^{-2/3})$.

Thus the construction of the parametrix in $U_0$ is now complete.

Using the expansion of $\Psi(\zeta)$, we can extend (99) to a full asymptotic series in inverse powers of $n$. Substituting (86) into (99), we obtain in particular:

$$P_0(\lambda) N^{-1}(\lambda) = I + \Delta_1(\lambda) + \Delta_2(\lambda) + \frac{1}{\sqrt{1 - \alpha}} O \left( \frac{1}{n^3} \right),$$

(100)

where

$$\Delta_1(\lambda) = \frac{1}{8 \sqrt{\zeta}} N(\lambda) \begin{pmatrix} -1 & 2i \ 2i & 1 \end{pmatrix} N(\lambda)^{-1} = \frac{1}{16 \sqrt{\zeta}} \begin{pmatrix} m^2 - 3m^{-2} & i(m^2 + 3m^{-2}) \ i(m^2 + 3m^{-2}) & -m^2 + 3m^{-2} \end{pmatrix},$$

$$\Delta_2(\lambda) = \frac{3}{2 i \sqrt{\zeta}} N(\lambda) \begin{pmatrix} -1 & -4i \ 4i & -1 \end{pmatrix} N(\lambda)^{-1} = \frac{3}{2 i \sqrt{\zeta}} \begin{pmatrix} -1 & -4i \ -4i & -1 \end{pmatrix}.$$

(101)

As above, note that $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ are meromorphic functions in $U_0$ with a simple pole at $\lambda = 0$.

For sufficiently small $\varepsilon_0$, the estimate (100) extends uniformly for $\alpha \in D_{\varepsilon_0}(0) \cup [0, 1 - s_0/(2n)^{2/3}]$ for all $n > s_0^{3/2}/2$, and $\lambda \in \partial U_0$ as in Section 3.3.
3.5 Final transformation of the problem

Now construction of the parametrices is complete, and we are ready for the last transformation of the Riemann-Hilbert problem. Let

\[
R(\lambda) = \begin{cases} 
S(\lambda)N^{-1}(\lambda), & \lambda \in \mathbb{C} \setminus (U_0 \cup U_1 \cup \Sigma), \\
S(\lambda)P_0^{-1}(\lambda), & \lambda \in U_0 \setminus \Sigma, \\
S(\lambda)P_1^{-1}(\lambda), & \lambda \in U_1 \setminus \Sigma.
\end{cases} \tag{102}
\]

It is easy to see that this function has jumps only on \( \partial U_1 \), \( \partial U_0 \), and parts of \( \Sigma_1 \), \( \Sigma_3 \) lying outside the neighborhoods \( U_0 \), \( U_1 \) (we denote these parts \( \Sigma_{\text{out}} \)). The contour is shown in Figure 5. Outside this contour, \( R(\lambda) \) is analytic. Besides, \( R(\lambda) = I + O(1/\lambda) \) as \( \lambda \to \infty \).

The jumps are as follows:

\[
R_+(\lambda) = R_-(\lambda)N(\lambda) \begin{pmatrix} 1 & 0 \\ e^{+\text{nh}(\lambda)} & 1 \end{pmatrix} N(\lambda)^{-1}, \quad \lambda \in \Sigma_{\text{out}}^1 \cup \Sigma_{\text{out}}^3,
\]

where the "-" sign in the exponent is taken on \( \Sigma_{\text{out}}^1 \), and "+", on \( \Sigma_{\text{out}}^3 \), \( R_+(\lambda) = R_-(\lambda)P_0(\lambda)N(\lambda)^{-1}, \quad \lambda \in \partial U_0 \setminus \{\text{intersection points}\}, \)

\[
R_+(\lambda) = R_-(\lambda)P_1(\lambda)N(\lambda)^{-1}, \quad \lambda \in \partial U_1 \setminus \{\text{intersection points}\}.
\]

The jump matrix on \( \Sigma_{\text{out}} \) can be uniformly estimated (both in \( \lambda \) and \( \alpha \in [0, 1 - s_0/(2n)^{2/3}] \) as \( I + O(\exp(-c\rho)) \)), where \( c \) is a positive constant. In view of the estimates (69), this is obviously true outside a fixed neighborhood of \( \lambda = 0 \), say when \( |\lambda| \geq 1/2 \). However, since the parametrix \( N(\lambda) \) is of order \( 1/\lambda^{1/4} \) for \( \lambda \) close to zero, and the contour approaches \( \lambda = 0 \) as \( \alpha \to 1 \), we need a more detailed analysis for \( |\lambda| \leq 1/2 \). In that case, we use (67) to write for all \( \alpha \in [0, 1) \) (in what follows the same symbols \( C \) and \( c \) stand for various positive constants independent of \( \alpha, \lambda, \) and \( n \)):

\[
\left| \frac{1}{\sqrt{\lambda}} e^{-n\text{kh}(\lambda)} \right| < \frac{C}{\sqrt{\mu}} \exp \left[ -cn \frac{\sqrt{\mu/(1-\alpha)}}{(1 + \alpha \mu/(1-\alpha))^2} \right] = \frac{C}{\sqrt{1-\alpha}} \frac{1}{t} \exp \left[ -cn \frac{t}{(1 + \alpha t^2)^2} \right] \equiv f(t), \tag{104}
\]
where \( t = \sqrt{\mu/(1 - \alpha)} \), \( \mu = \Re \lambda \). We need to find the maximum value of \( f(t) \) in the interval

\[
t_1 \equiv \sqrt{(\varepsilon_3/\varepsilon_2) \sin \gamma_0} \leq t \leq \frac{1}{\sqrt{2(1 - \alpha)}} \equiv t_2
\]

for all \( \alpha \in [0, 1 - s_0/(2n)^{2/3}] \). For this purpose, it is convenient to consider the following two cases separately.

1) \( \alpha t^2 \leq 1 \). Then \( 1 + \alpha t^2 \leq 2 \), and we have

\[
f(t) < \frac{C}{t \sqrt{1 - \alpha}} \exp[-cnt] \equiv f_1(t).
\]

The derivative \( f_1'(t) < 0 \) for \( t > 0 \), which implies

\[
\max_{t \in [t_1, 1/\sqrt{\alpha}]} f(t) < f_1(t_1) < C e^{n^{1/3} e^{-cn}} \leq Ce^{-cn}.
\]

If \( 1/\sqrt{\alpha} > t_2 \) this is all we need. Otherwise consider

2) \( \alpha t^2 > 1 \). Then \( 1 + \alpha t^2 < 2\alpha t^2 \), and we have

\[
f(t) < \frac{C}{t \sqrt{1 - \alpha}} \exp[-cnt/t^3] \equiv f_2(t).
\]

The only maximum of \( f_2(t) \) is at the point \( t_c = (3cn)^{1/3} \). Now choose sufficiently large \( s_0 > 0 \) (depending on \( \varepsilon_2, \varepsilon_3 \)). Then

\[
t_2 = \frac{1}{\sqrt{2(1 - \alpha)}} < cn^{1/3} s_0^{1/2} < t_c.
\]

Therefore

\[
\max_{t \in [1/\sqrt{\alpha}, t_2]} f(t) < f_2(t_2) < Ce^{-cn(1-\alpha)^{3/2}} = Ce^{-c\rho}.
\]

Combining (106,108), we finally obtain that the jump matrix on \((\Sigma_1 \cup \Sigma_3) \setminus (U_0 \cup U_1)\) is the identity up to an error of order

\[
\left| \frac{1}{\sqrt{\lambda}} e^{-n \Re h(\lambda)} \right| < Ce^{-c\rho}
\]

for all \( \alpha \in [0, 1 - s_0/(2n)^{2/3}] \), \( 2n > s_0^{3/2} \).

This estimate can be readily extended to complex \( \alpha \in \mathcal{D}_{\varepsilon_0}(0) \). The jump matrices on \( \partial U_{0,1} \) admit the uniform expansions given by (100,89).

A consequence of the above considerations is the following result:
Lemma 1 Let $\rho = n|1 - \alpha|^{3/2}$, $\alpha \in D_{s_0}(0) \cup [0, 1 - s_0/(2n)^{2/3}]$, $U = U_0 \cup U_1$, $\tilde{\Sigma} = \Sigma^{\text{out}} \cup \partial U$. Also let $\tilde{U}_1$ be the circle centered at $\lambda = 1$ of radius $\varepsilon/2$. Then, for sufficiently small $\varepsilon, \varepsilon_j, j = 0, 2, 3$ ($\varepsilon_j, j = 0, 2, 3$ are the $\varepsilon$-parameters introduced above in the definition of the contour $\tilde{\Sigma}$), there exists $s_0 > 0$ such that for all $\alpha \in D_{s_0}(0) \cup [0, 1 - s_0/(2n)^{2/3}]$, and $n > s_0^{3/2}/2$, a (unique) solution $R(\lambda)$ of the R-RH problem exists. Moreover, the function $R(\lambda)$ admits the following asymptotic expansion, which (and the derivative of which) is uniform for $\alpha \in D_{s_0}(0) \cup [0, 1 - s_0/(2n)^{2/3}]$ and all $\lambda \in \tilde{U}_1$, as $\rho \to \infty$:

\[
R(\lambda) = I + R_1(\lambda) + R_2(\lambda) + \cdots + R_k(\lambda) + R_{r(k+1)}(\lambda),
\]

\[
R_{r(k+1)}(\lambda) = O(\rho^{-k-1}), \quad \frac{d}{d\lambda} R_{r(k+1)}(\lambda) = O(\rho^{-k-1}),
\]

$k = 1, 2, \ldots$. The functions $R_j(\lambda) = O(\rho^{-j})$ are constructed by induction as follows:

\[
R_1(\lambda) = \frac{1}{2\pi i} \int_{\partial U} \Delta_1(s) \frac{ds}{s - \lambda}, \quad R_2(\lambda) = \frac{1}{2\pi i} \int_{\partial U} (R_1(s)\Delta_1(s) + \Delta_2(s)) \frac{ds}{s - \lambda},
\]

\[
\ldots, \quad R_k(\lambda) = \frac{1}{2\pi i} \int_{\partial U} \sum_{j=1}^{k} R_{k-j}(-s)\Delta_j(s) \frac{ds}{s - \lambda}, \quad R_0 \equiv I.\]

**Remark.** The uniformity means that for sufficiently small $\varepsilon, \varepsilon_j, j = 0, 2, 3$, there exist positive constants $s_0$, $c_1$, and $c_2$ independent of $\alpha, n, \lambda$ such that

\[
|R_{r(k+1)}| \leq \frac{c_1}{\rho^{k+1}}, \quad \left|\frac{d}{d\lambda} R_{r(k+1)}\right| \leq \frac{c_2}{\rho^{k+1}},
\]

$\forall \lambda \in \tilde{U}_1, \forall \alpha \in D_{s_0}(0) \cup [0, 1 - s_0/(2n)^{2/3}], \forall n > s_0^{3/2}/2$.

We also note that, $\rho > s_0^{3/2}/2, \forall \alpha \in D_{s_0}(0) \cup [0, 1 - s_0/(2n)^{2/3}], n > s_0^{3/2}/2$.

**Proof of Lemma 1.** We shall follow a similar line of argument to the one which was used to prove similar statement in [17] (Lemma 1). For simplicity, as in [17], we will only prove the expansion (110) in the case $k = 2$, which is all that is needed for the problem at hand. We shall also adopt the notation:

\[
R_{r(3)}(\lambda) \equiv R_r(\lambda).
\]

Besides, as before, the symbol $c$ will stand for various positive constants independent of $\alpha$, $\lambda$, and $n$.

Write the jump condition for $R(\lambda)$ in the form

\[
R_{0+} + R_{1+} + R_{2+} + R_{r+} = (R_{0-} + R_{1-} + R_{2-} + R_{r-})(I + \Delta_1 + \Delta_2 + \Delta_r).
\]

Here $\Delta_1$ and $\Delta_2$ are given by (10190) on $\partial U_0, \partial U_1$, respectively, and we set $\Delta_1 = \Delta_2 = 0$ on the rest of the contour. A direct analysis of the expressions (10190) shows that $\Delta_k =$
\(O((n^{-k}|1-\alpha|^{-1/2})\) on \(\partial U_0\), and \(\Delta_k = O(\rho^{-k})\) on \(\partial U_1\). Similarly, \(\Delta_r = O(1/\rho^3)\) on \(\partial U_1\) (this error term arises from the Bessel asymptotics), \(\Delta_r = O(|1-\alpha|^{1/4}/\rho^3)\) on \(\partial U_0\), and, by \([109]\), \(\Delta_r = O(e^{-c\rho})\) on \(\tilde{\Sigma} \setminus \partial U\).

We now show that we can define \(R_1\) and \(R_2\) so that they are of order \(1/\rho\) and \(1/\rho^2\), respectively. We then show that the remainder \(R_r\) is of order \(1/\rho^3\). Set

\[R_0 = I.\]

We define \(R_j\) by collecting in (114) the terms that we want to be of the same order. First,

\[R_1 + (\lambda) = R_1 - (\lambda) + \Delta_1(\lambda), \quad \lambda \in \tilde{\Sigma}.\]  

(115)

We are looking for a function \(R_1(\lambda)\), which is analytic outside \(\tilde{\Sigma}\), satisfying \(R_1(\lambda) = O(1/\lambda), \lambda \to \infty\), and the above jump condition. The solution to this RH-problem is given by the Sokhotsky-Plemelj formula,

\[R_1(\lambda) = C(\Delta_1),\]  

(116)

where

\[C(f) = \frac{1}{2\pi i} \int_{\tilde{\Sigma}} f(s) \frac{ds}{s-\lambda}\]

is the Cauchy operator on \(\tilde{\Sigma}\). The condition \(\Delta_1(\lambda) = O(1/\rho), \lambda \in \tilde{\Sigma}, \rho \to \infty\) (uniform in \(\alpha\)), implies that there exist \(c, \delta, s_0 > 0\) such that

\[|R_1(\lambda)| \leq c/\rho, \quad n \geq \frac{s_0^{3/2}}{2}\]  

(117)

uniformly in \(\alpha \in D_{s_0}(0) \cup [0, 1 - s_0/(2n)^{2/3}]\) and \(\lambda\) satisfying \(\text{dist}(\lambda, \tilde{\Sigma}) \geq \delta\). Actually, this estimate is uniform for all \(\lambda \in \mathbb{C} \setminus \tilde{\Sigma}\) up to \(\tilde{\Sigma}\). Indeed, since

\[R_1(\lambda) = \frac{1}{2\pi i} \int_{\partial U} \Delta_1(s) \frac{ds}{s-\lambda},\]  

(118)

for \(\lambda\) outside a fixed neighborhood of zero, this is seen by shifting the contour to a fixed distance from the point \(\lambda\). Inside that neighborhood, the distance of the shift will depend on \(\alpha\). Namely, the distance is \(\varepsilon'|1-\alpha|\) for a fixed (sufficiently small) \(\varepsilon' > 0\). Then

\[|C(\Delta_1)| \leq \max |\Delta_1| \frac{1}{|1-\alpha|} + \frac{c}{\rho} \leq \frac{c}{n|1-\alpha|^{3/2}} + \frac{c}{\rho} = \frac{c}{\rho},\]  

(119)

on and close to \(\partial U_0\). Here we used the estimate \(\Delta_1 = O(n^{-1}\lambda^{-1/2})\), so that in the neighborhood of the circle \(\partial U_0\) the inequality

\[\max |\Delta_1| \leq \frac{c}{n|1-\alpha|^{1/2}}\]
holds. It should be observed that, by the same deformation of the contour of integration in (118), one obtains the analytic continuations of both the functions $R_{1+}(\lambda)$ and $R_{1-}(\lambda)$ in the neighborhood of the contour $\partial U$ and hence in the neighborhood of $\tilde{\Sigma}$ (we note that on the part $\Sigma^{\text{out}}$ of the contour $\tilde{\Sigma}$ $R_1(\lambda)$ has no jump). Moreover, the estimate (117) is preserved under this analytic continuation.

Now define $R_2(\lambda)$ by the jump condition
\begin{equation}
R_{2+}(\lambda) = R_{2-}(\lambda) + R_{1-}(\lambda)\Delta_1(\lambda) + \Delta_2(\lambda), \quad \lambda \in \tilde{\Sigma},
\end{equation}
(120)
together with the requirement of analyticity for $\lambda \in \mathbb{C} \setminus \tilde{\Sigma}$, and the condition $R_2(\lambda) = O(1/\lambda)$ for $\lambda \to \infty$. The solution to this RHP is
\begin{equation}
R_2(\lambda) = C(R_1 - \Delta_1 + \Delta_2), \quad \lambda \in \mathbb{C} \setminus \tilde{\Sigma}.
\end{equation}
(121)
Using (117) and the estimates for $\Delta_2$, we obtain in the same way as for $R_1$,
\begin{equation}
|R_2(\lambda)| \leq c/\rho^2, \quad \lambda \in \mathbb{C} \setminus \tilde{\Sigma}, \quad n \geq \frac{s_0^{3/2}}{2}
\end{equation}
(122) with the same uniformity and analyticity properties in $\alpha$ and $\lambda$.

Now from (114,115,120) we obtain
\begin{equation}
R_{r+}(\lambda) = R_{r-}(\lambda) + M(\lambda) + R_{r-}(\lambda)\Delta(\lambda), \quad \lambda \in \tilde{\Sigma},
\end{equation}
(123)
where
\begin{equation}
M \equiv R_{2-}\Delta_1 + (R_{1-} + R_{2-})\Delta_2 + (I + R_{1-} + R_{2-})\Delta_r, \quad \Delta \equiv \Delta_1 + \Delta_2 + \Delta_r.
\end{equation}

**Remark.** In the terminology of [13], equation (123) is an inhomogeneous RH-problem of type 2.

Since $R_r = R - I - R_1 - R_2$, the matrix function $R_r(\lambda)$ is analytic outside $\tilde{\Sigma}$ and satisfies the condition $R_r(\lambda) = O(1/\lambda)$ as $\lambda \to \infty$. Therefore
\begin{equation}
R_r(\lambda) = C(M) + C(R_r - \Delta), \quad \lambda \in \mathbb{C} \setminus \tilde{\Sigma}.
\end{equation}
(124)

Hence
\begin{equation}
R_{r-}(\lambda) = C_-(M) + C_-(R_r - \Delta), \quad \lambda \in \tilde{\Sigma},
\end{equation}
(125)
where $C_-(f) = \lim_{\lambda' \to \lambda} C(f)$, as $\lambda'$ approaches a point $\lambda \in \tilde{\Sigma}$ from the $-$ side of $\tilde{\Sigma}$. Now defining the operator
\begin{equation}
C_\Delta(f) \equiv C_-(f\Delta),
\end{equation}
we represent (125) in the form
\begin{equation}
(I - C_\Delta)(R_{r-}) = C_-(M).
\end{equation}
(126)
By virtue of the estimates (100), (89), and (109) we have that

$$||\Delta||_{L^2(\tilde{\Sigma})\cap L^\infty(\tilde{\Sigma})} \leq \frac{c}{\rho},$$

(127)

for all $\alpha \in D_{c\delta}(0) \cup [0, 1 - s_0/(2n)^{2/3}]$ and $n > s_0^{3/2}/2$.

The Cauchy operator $C_-$ is bounded in the space $L^2(\tilde{\Sigma})$ (see, e.g., [28]), and by a standard scaling argument (the Cauchy operator is homogeneous of degree 0), its norm is bounded by a constant independent of $\alpha$. This together with the $L^\infty$ part of the estimate (127) implies that the operator norm $||C_\Delta||_{L^2} = O(1/\rho)$, and hence $I - C_\Delta$ is invertible by a Neumann series for $s_0$ (and, therefore, $\rho$) sufficiently large. Thus (126) gives

$$R_{r-} = (I - C_\Delta)^{-1}(C_-(M)),$$

(128)

and this proves the solvability of the $R$-RH problem for all $\alpha \in D_{c\delta}(0) \cup [0, 1 - s_0/(2n)^{2/3}]$ and $n > s_0^{3/2}/2$. Moreover, using the $L^2$ part of the estimate (127), we conclude that $||C_-(M)||_{L^2(\tilde{\Sigma})} = O(\rho^{-3})$. Together with (128) this yields the uniform estimate

$$||R_{r-}\ll_{L^2(\tilde{\Sigma})} \leq \frac{c}{\rho^3},$$

(129)

$$\forall \alpha \in D_{c\delta}(0) \cup [0, 1 - s_0/(2n)^{2/3}], \ n > s_0^{3/2}/2.$$

The solution $R(\lambda)$ of the $R$-RH problem is given by the integral representation

$$R(\lambda) = I + R_1(\lambda) + R_2(\lambda) + C(M) + C(R_{r-}\Delta)(\lambda),$$

(130)

$$\lambda \in \mathbb{C} \setminus \tilde{\Sigma}.$$

**Remark.** Let $\Omega_k, \ k = 1, 2, 3, 4$ denote the connected components of the set $\mathbb{C} \setminus \tilde{\Sigma}$. Then, using again the possibility of the contour deformation when solving the integral equation (126), and taking into account the triviality of the jump matrix monodromy at each node point of the contour $\tilde{\Sigma}$, we conclude that the restriction $R|_{\Omega_k}(\lambda)$ is continuous in $\overline{\Omega}_k$ for each $k$ (see e.g. [4]). This means that equation (130) defines the solution of the $R$-RH problem in the classical, point-wise continuous, sense.

Combining the inequality (129) with equation (130), we can complete the proof of the lemma. Indeed, assuming that $\lambda \in \tilde{U}_1$, we immediately obtain the estimate

$$|C(M)(\lambda)| \leq \frac{c}{\rho^3}, \ n > s_0^{3/2}/2,$$

(131)

for the fourth term in the r.h.s. of (130), and the estimate

$$|C(R_{r-}\Delta)(\lambda)| \leq c||R_{r-}\ll_{L^2(\tilde{\Sigma})}|\Delta||_{L^2(\tilde{\Sigma})} \leq \frac{c}{\rho^3},$$

(132)

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for the fifth term. Both the estimates are uniform in \( \alpha \in \mathcal{D}_{\varepsilon_0}(0) \cup [0, 1 - s_0/(2n)^{2/3}] \). Together they yield the estimate

\[
|R_r(\lambda)| \leq \frac{c}{\rho^3}, \quad n > s_0^{3/2}/2,
\]

(133)

uniformly in \( \alpha \in \mathcal{D}_{\varepsilon_0}(0) \cup [0, 1 - s_0/(2n)^{2/3}] \) and \( \lambda \) lying in \( \widetilde{U}_1 \). This establishes part of the estimate (110) for the error term. The estimate for the derivative follows immediately from (124). This completes the proof of the lemma (in the case \( k = 2 \)). □

**Remark.** (Cf. Remark 2 in [17].) Part of the assertion of Lemma 1 is that the solution of the \( R \)-RH problem, and hence of the original \( T \)-RH problem, exists and is unique for all \( \alpha \in \mathcal{D}_{\varepsilon_0}(0) \cup [0, 1) \) and all \( n > 0 \) for some (possibly smaller) \( \varepsilon_0 > 0 \). Indeed, by the discussion following (58), the \( T \)-RH problem, and hence the \( R \)-RH problem, is solvable for all \( \alpha \in [0, 1), n > 0 \). Since, by the previous remark, the solution of the \( R \)-RH problem is continuous up to the contour, the problem is easily seen to be solvable for \( \alpha \in \mathcal{D}_{\varepsilon_0}(0), 0 < n \leq s_0^{3/2}/2 \) for some \( \varepsilon'_0 > 0 \) by continuity of the jump matrix at \( \alpha = 0 \). By Lemma 1, the \( R \)-RH problem is solvable for all \( \alpha \in \mathcal{D}_{\varepsilon_0}(0), n > s_0^{3/2}/2 \). Thus the \( R \)-RH problem, and hence the \( T \)-RH problem, is solvable for all \( n > 0 \) on \( \mathcal{D}_{\varepsilon''_0}(0) \cup [0, 1) \), where \( \varepsilon'' = \min\{\varepsilon_0, \varepsilon'_0\} \).

## 4 Evaluation of the differential identity

### 4.1 Exact transformations

We start with the differential identity (37). Note that since \( V(z) \) is related to \( U(\lambda) \) by the expression (46)

\[
U(\lambda) = V(z(\lambda)), \quad z = \frac{\alpha \lambda}{1 - \alpha + \alpha \lambda},
\]

we have

\[
\frac{d\lambda}{dz} \bigg|_{z=\alpha} = \frac{1}{\alpha(1 - \alpha)},
\]

(134)

and (37) can be rewritten in terms of \( U(\lambda) \) as follows

\[
\frac{d}{d\alpha} \ln D_n(\alpha) = \frac{e^{-4n\alpha}}{2\pi i \alpha(1 - \alpha)}(U_{11}(1)U'_{21}(1) - U'_{11}(1)U_{21}(1)).
\]

(135)

Note that the derivatives in (135) are taken w.r.t. \( \lambda \).
By (53, 58), the matrix elements of $U(\lambda)$ can be expressed in terms of $T(\lambda)$ as follows:

$$U_{11}(\lambda) = \left[ T^{-1} \left( -\frac{1-\alpha}{\alpha} \right) T(\lambda) \right]_{11} e^{n\hat{g}(\lambda)},$$

$$U_{21}(\lambda) = \left[ T^{-1} \left( -\frac{1-\alpha}{\alpha} \right) T(\lambda) \right]_{21} e^{-n\hat{g}(\lambda)}.$$

(136)

Furthermore, for $\lambda$ outside the lens in $U_1$

$$T(\lambda) = S(\lambda), \quad S(\lambda) = R(\lambda)P_1(\lambda).$$

(137)

Note also that by (87)

$$S_{j1} = (R(\lambda)E_\alpha(\lambda)\Psi(\zeta))_{j1}e^{n\phi(\lambda)}, \quad j = 1, 2,$$

and, as follows from the definitions of the functions $\phi$, $h$, and the properties of $g(z)$,

$$\phi(\lambda) + \hat{g}(\lambda) = \mp \frac{1}{2} h + \hat{g} = \pm \frac{\hat{g}_+ - \hat{g}_-}{2} + \hat{g}_\pm = \frac{\hat{g}_+ + \hat{g}_-}{2} = 2z(\lambda) + l/2,$$

where $\hat{g}_\pm(\lambda)$ stand for the analytic continuation of these functions. Here the upper sign corresponds to $\Im \lambda > 0$, and the lower, to $\Im \lambda < 0$.

Hence, (135) finally gives

$$\frac{d}{d\alpha} \ln D_n(\alpha) = \frac{1}{2\pi i\alpha(1-\alpha)}((RE\Psi)_{11}(1)(RE\Psi)_{21}'(1) - (RE\Psi)'_{11}(1)(RE\Psi)_{21}(1)),$$

(138)

where we used the fact that $\det T^{-1}(-(1-\alpha)/\alpha) = 1$. In (138), the derivative at $\lambda = 1$ is taken along a path in $U_1$ outside the lens. In the next subsection we use the solution of the Riemann-Hilbert problem for $R(\lambda)$ (found in Section 3) to construct the asymptotics of the r.h.s. of (138).

### 4.2 Asymptotics

Consecutive asymptotic terms in the expansion of the logarithmic derivative (138) are generated by consecutive terms in (110):

$$R(\lambda) = I + R_1(\lambda) + R_2(\lambda) + \cdots.$$ 

Thus, setting $R = I$ in (138) gives the main asymptotic term of $\frac{d}{d\alpha} \ln D_n(\alpha)$:

$$\frac{1}{2\pi i\alpha(1-\alpha)}((E\Psi)_{11}(1)(E\Psi)'_{21}(1) - (E\Psi)'_{11}(1)(E\Psi)_{21}(1)),$$

(139)

Using (88) and (85), we obtain

$$(E\Psi)_{11}(\zeta) = \mu_+(\lambda), \quad (E\Psi)_{21}(\zeta) = -i\mu_-(\lambda),$$

(140)
where
\[ \mu_{\pm}(\lambda) = \sqrt{\frac{\pi}{2}} e^{1/4} (m^{-1}(\lambda) I_0(\sqrt{\lambda}) \pm m(\lambda) I_0'(\sqrt{\lambda})). \]  
(141)

Using the expansion of Bessel functions as \( \zeta \to 0 \) (i.e. \( \lambda \to 1 \)), we obtain
\[ \mu_{\pm}(1) \equiv M = \sqrt{\pi n (1-\alpha)^{3/4}}, \quad \mu_{\pm}'(1) \equiv a \pm b, \]
\[ a = M \left[ n^2 (1-\alpha)^3 - \frac{\alpha}{2} + \frac{1}{6} \right], \quad b = M n (1-\alpha)^{3/2}. \]  
(142)

Substituting these values into (139), we find the main asymptotic term
\[ \frac{d}{d\alpha} \ln D_n(\alpha) \sim \frac{n^2}{\alpha} (1-\alpha)^2. \]  
(143)

To obtain the next term, we need to compute first
\[ R_1(1) = \frac{1}{2\pi i} \int_{\partial U} \frac{\Delta_1(\lambda)}{\lambda - 1} d\lambda, \quad R_1'(1) = \frac{1}{2\pi i} \int_{\partial U} \frac{\Delta_1(\lambda)}{(\lambda - 1)^2} d\lambda. \]  
(144)

We now examine \( \Delta_{1,2} \) in the neighborhoods of the points \( \lambda = 0, 1 \). Using (95) and expanding the matrix elements of \( N(\lambda) \), we obtain from (101):
\[ \Delta_1(\lambda) = \frac{C_1}{\lambda} + \frac{\sqrt{1-\alpha}}{32n(1+\alpha)} \begin{pmatrix} F_0(\alpha) - 5/2 & i(F_0(\alpha) + 7/2) \\ i(F_0(\alpha) + 7/2) & -F_0(\alpha) + 5/2 \end{pmatrix} + O(\lambda), \]
\[ F_0(\alpha) = \frac{1 - 6\alpha - 3\alpha^2}{6(1-\alpha^2)}, \quad C_1 = \frac{\sqrt{1-\alpha}}{32n(1+\alpha)} \begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix}, \quad \lambda \in U_0. \]  
(145)

For \( \Delta_2(\lambda) \), we obtain similarly:
\[ \Delta_2(\lambda) = \frac{3(1-\alpha)(1+O(\lambda))}{2^{9} n^2 (1+\alpha)^2 \lambda} \begin{pmatrix} 1 & 4i \\ -4i & 1 \end{pmatrix}, \quad \lambda \in U_0. \]  
(146)

In \( U_1 \), a similar calculation based on (91) and (81) gives (\( \lambda = 1 + u \))
\[ \Delta_1(\lambda) = \frac{A_1}{u} + \frac{1}{32n(1-\alpha)^{3/2}} \begin{pmatrix} -5/2 + \alpha + 1/6 & -i(7/2 + \alpha + 1/6) \\ -i(7/2 + \alpha + 1/6) & -(-5/2 + \alpha + 1/6) \end{pmatrix} + \]
\[ \frac{u}{32n(1-\alpha)^{3/2}} \begin{pmatrix} 3/2 - (5/2)(\alpha + 1/6) + F_1(\alpha) & -i(-3/2 + (7/2)(\alpha + 1/6) + F_1(\alpha)) \\ -i(-3/2 + (7/2)(\alpha + 1/6) + F_1(\alpha)) & -(3/2 - (5/2)(\alpha + 1/6) + F_1(\alpha)) \end{pmatrix} + \]
\[ O(u^2), \quad 1 + u = \lambda, \quad \lambda \in U_1, \quad A_1 = \frac{1}{32n(1-\alpha)^{3/2}} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \]
\[ F_1(\alpha) = \frac{\alpha}{6} - \frac{31}{4 \cdot 45}, \quad \Delta_2(\lambda) = \frac{3(1 + (2\alpha + 1/3)u + O(u^2))}{2^{9} n^2 (1-\alpha)^3 u} \begin{pmatrix} -1 & 4i \\ -4i & -1 \end{pmatrix}, \quad \lambda \in U_1. \]  
(147)
Now the expressions for $R(1)$ and $R'(1)$ are obtained from the above results and \((144)\) by a straightforward residue calculation:

$$ R_1(1) = \begin{pmatrix} \delta \\ \eta \end{pmatrix}, \quad \delta = \frac{1}{32n(1-\alpha)^{3/2}} \left[ \frac{5}{2} - \alpha - 1/6 - \frac{(1-\alpha)^2}{1+\alpha} \right], $$

$$ \eta = \frac{i}{32n(1-\alpha)^{3/2}} \left[ \frac{7}{2} + \alpha + 1/6 - \frac{(1-\alpha)^2}{1+\alpha} \right], $$

$$ R'_1(1) = \begin{pmatrix} \sigma \\ \tau \end{pmatrix}, \quad \sigma = \frac{1}{32n(1-\alpha)^{3/2}} \left[ -3/2 + (5/2)(\alpha + 1/6) - F_1(\alpha) + \frac{(1-\alpha)^2}{1+\alpha} \right], $$

$$ \tau = \frac{i}{32n(1-\alpha)^{3/2}} \left[ -3/2 + (7/2)(\alpha + 1/6) + F_1(\alpha) + \frac{(1-\alpha)^2}{1+\alpha} \right]. $$

Note that the contours $\partial U_{0,1}$ are traversed in the negative direction.

We shall be using the following notation for the expansion terms of the logarithmic derivative \((138)\). We denote $R_k \cdot R_m$ ($R_0 \equiv I$) the term given by

$$ \frac{1}{2\pi i(1-\alpha)^2} ((R_k E \Psi)_{11}(1)(R_m E \Psi)'_{21}(1) + (R_m E \Psi)_{11}(1)(R_k E \Psi)'_{21}(1) $$

$$ -(R_k E \Psi)'_{11}(1)(R_m E \Psi)_{21}(1) - (R_m E \Psi)'_{11}(1)(R_k E \Psi)_{21}(1)). $$

For example, the main term \((139)\) is $I \cdot I$. We can now evaluate the next ($R_1 \cdot I$) term in the expansion. It is written as follows:

$$ \frac{d}{d\alpha} \ln D_n(\alpha) - \frac{n^2}{\alpha}(1-\alpha)^2 \sim \frac{1}{2\pi i(1-\alpha)^2} \left\{ R_1(1) \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\} M(-i\mu'_-(1)), $$

$$ M \left\{ R_1(\lambda) \begin{pmatrix} \mu_+(\lambda) \\ -i\mu_-(\lambda) \end{pmatrix} \right\}'_1 $$

$$ M \left\{ R_1(1) \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\} \frac{M^2(\tau + i\sigma)}{\pi i(1-\alpha)^2} = \frac{\alpha}{4(1-\alpha^2)}, $$

where we first simplified the expression substituting the above symbolic representation of $R_1$ in terms of $\delta, \eta, \sigma, \tau$, and used their numerical values only at the last step.

It turns out that the two terms in the asymptotics just obtained is all we need (up to the error term). The following lemma is the main result of this section:

**Lemma 2** There exists $s_0 > 0$ such that the expansion

$$ \frac{d}{d\alpha} \ln D_n(\alpha) = \frac{n^2}{\alpha}(1-\alpha)^2 + \frac{\alpha}{4(1-\alpha^2)} + r(n, \alpha), $$

$$ r(n, \alpha) = \frac{1}{1-\alpha} O \left( \frac{1}{\rho} \right), \quad \rho = n|1-\alpha|^{3/2}, $$

(152) (153)
holds uniformly in $\alpha \in (0, 1 - s_0/(2n)^{2/3}]$ for all $n > s_0^{3/2}/2$.

**Proof.** It only remains to prove the expression for the error term. We consider the expansion of $R(\lambda)$ up to the third term: $R = I + R_1 + R_2 + R_r$. Since $R_k = O(\rho^{-k})$ and, according to (142), $\mu'_\pm(1)\mu_\pm(1) = O(\rho^3)$, it is not difficult to deduce from (138) (cf. (151)) that the contribution of the terms $R_r \cdot R_1, R_2 \cdot R_2$ and higher are of order $(\alpha(1-\alpha))^{-1}O(\rho^{-1})$. Thus we shall need to consider in detail only the following 4 terms: $R_1 \cdot R_1, R_2 \cdot I, R_2 \cdot R_1, R_r \cdot I$.

For the $R_1 \cdot R_1$ term, which we denote $L_{11}$, we obtain after a calculation similar to (151):

$$L_{11} = -\frac{n^2}{\alpha}(1-\alpha)^2(\delta^2 + \eta^2) = \frac{1}{2^8\alpha(1-\alpha^2)}(\alpha + 2/3)(2 + 5\alpha - \alpha^2).$$  

(154)

For further analysis, we need to calculate $R_2(1)$. It is given by the formula:

$$R_2(1) = \frac{1}{2\pi i} \int_{\partial U} \frac{R_1(\lambda)\Delta_1(\lambda) + \Delta_2(\lambda)}{\lambda - 1} d\lambda.$$  

(155)

The solution of the Riemann-Hilbert problem for $R_1$ inside $U_{1,0}$ is given by the expression (which we write on the boundary)

$$R_1(\lambda) = \frac{A_1}{\lambda - 1} + \frac{C_1}{\lambda} - \Delta_1(\lambda), \quad \lambda \in \partial U,$$  

(156)

where $A_1, C_1$ are defined in (147,145). Note that outside $U_{1,0}$ the solution is

$$R_1(\lambda) = \frac{A_1}{\lambda - 1} + \frac{C_1}{\lambda}.$$  

It is easily seen that the jump, analyticity conditions, and the condition at infinity of the Riemann-Hilbert problem for $R_1(\lambda)$ are satisfied, and therefore, by uniqueness, this is the solution.

The expansions for $\Delta_{1,2}$ obtained above and the formulas (156,155) give, by a residue calculation, the final expression for $R_2(1)$:

$$R_2(1) = \begin{pmatrix} \gamma & -\beta \\ \beta & \gamma \end{pmatrix}, \quad \gamma = \frac{-1}{2^9n^2(1-\alpha)^3} \left[ (3\alpha - 1) \left( 1 - \frac{(1 - \alpha)^2}{3(1 + \alpha)} \right) + 3 - \frac{(1 - \alpha)^2}{1 + \alpha} \right],$$  

(157)

where the expression for $\beta$ is omitted as it is not needed below.

To compute the “$R_2 \cdot I$” term (which we denote $L_{20}$) note first that the contribution of the terms in that expression involving $R'_2(1)$ is of order $(\alpha(1-\alpha))^{-1}O(\rho^{-1})$ and we need not calculate them. The remainder gives a nontrivial contribution, and we obtain:

$$L_{20} = \frac{2bM\gamma}{\pi\alpha(1-\alpha)} + \frac{1}{\alpha(1-\alpha)}O \left( \frac{1}{n(1-\alpha)^{3/2}} \right).$$  

(158)
The expression for $\gamma$ tells us that this is equal to $-L_{11}$ up to the error term. Thus, we conclude that the contributions of $R_2 \cdot I$ and $R_1 \cdot R_1$ terms cancel each other.

The analysis of the $R_2 \cdot R_1$ term is now easy to carry out, and we find that this term is of order $(\alpha(1 - \alpha))^{-1}O(\rho^{-2})$.

For any matrix elements of $R_2(1)$ (we only know they are of order $O(\rho^{-3})$), we obtain that the $R_2 \cdot I$ term is of order $(\alpha(1 - \alpha))^{-1}O(\rho^{-1})$.

Thus, in view of uniformity of the error term in the expansion of $R(\lambda)$, the lemma is proven but with the remainder

$$r(n, \alpha) = \frac{1}{\alpha(1 - \alpha)} O \left( \frac{1}{\rho} \right).$$

(159)

We now show that $\alpha$ in the denominator here can be omitted. First, we notice that $r(n, \alpha) = O_n(1)$ as $\alpha \to 0$ and $n$ is fixed: this follows immediately after substitution of the expansion (27) into the l.h.s. of (152). However, we need an estimate which is uniform in $n$. To obtain such an estimate, we use the extensions of our expressions for complex $\alpha$ discussed above.

As follows from (152,27), $r(n, \alpha)$ is an analytic function of $\alpha$ in $D_{\varepsilon_0}(0)$. Thus

$$r(n, \alpha) = \frac{1}{2\pi i} \int_{\partial D_{\varepsilon_0/2}(0)} r(n, \tilde{\alpha}) \frac{d\tilde{\alpha}}{\tilde{\alpha} - \alpha}, \quad |\alpha| < \varepsilon_0/4.$$  

(160)

Since by (159), $r(n, \tilde{\alpha})$ is uniformly bounded on $\partial D_{\varepsilon_0/2}(0)$, it follows that $r(n, \alpha)$ is uniformly bounded by $O(1/\rho)$ for all $\alpha \in D_{\varepsilon_0/4}(0)$, and all $n > s_0^{3/2}/2$. Lemma 2 is proven. □

5 Proof of Theorem 1

Integrating the differential identity (152) from $\alpha_0$ (close to zero from above) to any $\alpha_0 < \alpha \leq 1 - s_0/(2n)^{2/3}$, we obtain:

$$\ln D_n(\alpha) - \ln D_n(\alpha_0) = n^2 \left( \ln \frac{\alpha}{\alpha_0} - 2(\alpha - \alpha_0) + \frac{\alpha^2 - \alpha_0^2}{2} \right) - \frac{1}{8} \ln \frac{1 - \alpha^2}{1 - \alpha_0^2} + O \left( \frac{1}{n(1 - \alpha)^{3/2}} \right)$$

(161)

for all $n > s_0^{3/2}/2$. Note from (153) that the term $O(1/n(1 - \alpha)^{3/2})$ does not depend on $\alpha_0$. Substituting for $\ln D_n(\alpha_0)$ the expansion (27) and taking the limit $\alpha_0 \to 0$, we obtain for any $0 < \alpha \leq 1 - s_0/(2n)^{2/3}$,

$$\ln D_n(\alpha) = n^2 \left( \frac{3}{2} + \ln \alpha - 2\alpha + \frac{\alpha^2}{2} \right) - \frac{1}{12} \ln n - \frac{1}{8} \ln(1 - \alpha^2) +$$

$$\frac{1}{12} \ln 2 + \zeta'(-1) + O \left( \frac{1}{n(1 - \alpha)^{3/2}} \right) + \delta_n.$$  

(162)
Fix any $s > s_0$ and, for $n$ sufficiently large, set $\alpha = 1 - s/(2n)^{2/3}$. Now take the limit $n \to \infty$. As $n \to \infty$, the r.h.s. of (162) becomes

$$-\frac{s^3}{12} - \frac{1}{8} \ln s + \frac{1}{24} \ln 2 + \zeta'(-1) + O(s^{-3/2}).$$

(163)

On the other hand, as $s$ is any fixed number $s > s_0$, the l.h.s. of (162) converges to $\ln \det(I - K_s)$ by (21). □

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6 Appendix

Here we present an alternative derivation of the identity (37). Let

$$\phi(x) = \frac{1}{2} \omega_{n-1}(x), \quad \psi(x) = \frac{1}{2} \omega_n(x).$$

(164)

The determinant (17) is written then as follows:

$$D_n(\alpha) = \det \left( I - \frac{\phi(x)\psi(y) - \phi(y)\psi(x)}{x - y} \chi(\alpha, \infty) \right).$$

(165)

The operator $K(x, y) = (\phi(x)\psi(y) - \phi(y)\psi(x))/(x - y)$ is of integrable type, and hence (see, e.g., [23, 6, 9]) $D_n(\alpha)$ is related to the following Riemann-Hilbert problem for a $2 \times 2$ matrix-valued function $Y(z)$ (Figure 6):

(a) $Y(z)$ is analytic for $z \in \mathbb{C} \setminus [\alpha, \infty)$.

(b) Let $x \in (\alpha, \infty)$. $Y(z)$ has $L_2$ boundary values $Y_+(x)$ as $z$ approaches $x$ from above, and $Y_-(x)$, from below. They are related by the jump condition

$$Y_+(x) = Y_-(x) \nu_Y(x), \quad \nu_Y(x) = \begin{pmatrix} 1 + 2\pi i \phi(x)\psi(x) & -2\pi i \psi(x)^2 \\ 2\pi i \phi(x)^2 & 1 - 2\pi i \phi(x)\psi(x) \end{pmatrix}, \quad x \in (\alpha, \infty).$$

(166)
(c) $Y(z)$ has the following asymptotic behavior at infinity:

$$Y(z) = I + O\left(\frac{1}{z}\right), \quad \text{as } z \to \infty. \quad (167)$$

As in [5, 24], it is possible to reduce the RHP for $Y$ to an equivalent RHP with an “elementary”, in fact constant, jump matrix (see (174) below). Note first that for any functions $\tilde{\psi}(x), \tilde{\phi}(x)$ such that $\psi(x)\tilde{\phi}(x) - \phi(x)\tilde{\psi}(x) = 1$, we have

$$v_Y(x) = A(x) \begin{pmatrix} 1 & -2\pi i \\ 0 & 1 \end{pmatrix} A^{-1}(x), \quad A(x) = \begin{pmatrix} \psi(x) & \tilde{\psi}(x) \\ \phi(x) & \tilde{\phi}(x) \end{pmatrix}, \quad (168)$$

Note that the condition on $\tilde{\psi}(x), \tilde{\phi}(x)$ is equivalent to the following one:

$$\det A(x) = 1.$$

Let

$$\Phi(z) = \begin{pmatrix} \psi(z) & e^{2nz} \int_0^\infty \frac{\psi(\xi)}{\xi-z} e^{-2n\xi}d\xi \\ \phi(z) & e^{2nz} \int_0^\infty \frac{\phi(\xi)}{\xi-z} e^{-2n\xi}d\xi \end{pmatrix}. \quad (169)$$

The function $\Phi(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}_+$. Using the orthogonality property of the polynomials $p_n(x), p_{n-1}(x)$ with respect to the weight $e^{-4nx}$, we see that $\Phi(z)$ solves the following RHP on $\mathbb{R}_+$:

(a) $\Phi(z)$ is analytic for $z \in \mathbb{C} \setminus [0, \infty)$.

(b) For $x \in (0, \infty)$ the $L_2$ boundary values $\Phi_+(x)$ and $\Phi_-(x)$ are related by the jump condition

$$\Phi_+(x) = \Phi_-(x) \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}, \quad x \in (0, \infty). \quad (170)$$

(c) $\Phi(z)$ has the following asymptotic behavior as $z \to \infty$:

$$\Phi(z) = \left(I + O\left(\frac{1}{z}\right)\right) \left(\frac{\gamma_n}{2} e^{-2nz} z^n\right)^{\sigma_3}. \quad (171)$$
By standard arguments, see [7], det \( \Phi(z) = 1 \). Hence, we see that for \( x > 0 \), we can take
\[
A(x) = \Phi_+(x). 
\]
(172)

The decomposition (168) suggests the following transformation of the Riemann-Hilbert problem. Let
\[
X(z) = Y(z)\Phi(z),
\]
(173)

It is easy to verify that \( X(z) \) satisfies the following problem:

(a) \( X(z) \) is analytic for \( z \in \mathbb{C} \setminus [0, \alpha] \).

(b) For \( x \in (0, \alpha) \) the \( L_2 \) boundary values \( X_+(x) \) and \( X_-(x) \) are related by the jump condition
\[
X_+(x) = X_-(x) \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}, \quad x \in (0, \alpha).
\]
(174)

(c) \( X(z) \) has the following asymptotic behavior as \( z \to \infty \):
\[
X(z) = \left( I + O \left( \frac{1}{z} \right) \right) \left( \frac{\kappa_n}{2} e^{-2nz} \sigma_3 \right),
\]
(175)

Thus \( X(z) \) satisfies the same RHP as \( \Phi(z) \), but now on the interval \((0, \alpha)\).

The transformation
\[
V(z) = \left( \sqrt{2\pi i \kappa_n} \right)^{-\sigma_3} X(z)e^{2nz\sigma_3} (2\pi i)^{\sigma_3/2}
\]
(176)
converts the RHP to the RHP for \( V(z) \) of Section 3.

We now turn to the derivation of the identity for \( D_n(\alpha) \). Write the determinant (165)
in the form
\[
D_n(\alpha) = \det(I - K),
\]
where \( K \) is an integral operator acting on functions \( f(x) \) from \( L^2(\alpha, \infty) \) as follows:
\[
(Kf)(x) = \int_{\alpha}^{\infty} K(x, y)f(y)dy, \quad K(x, y) = \frac{\phi(x)\psi(y) - \phi(y)\psi(x)}{x - y}.
\]
The logarithmic derivative of \( D_n(\alpha) \) w.r.t. \( \alpha \) has the form
\[
\frac{d}{d\alpha} \ln D_n(\alpha) = -\text{tr} \left( (I - K)^{-1} \frac{dK}{d\alpha} \right) = ((I - K)^{-1}K)(\alpha, \alpha) = ((I - K)^{-1}(K - I + I))(\alpha, \alpha) = R(\alpha, \alpha),
\]
(177)
where \( R(x, y) \) is the kernel of the operator \((I - K)^{-1} - I\). As noted above, the kernel \( K(x, y) \) has the structure of an “integrable” kernel. A consequence of this fact is the identity

\[
R(x, y) = \frac{-F_1(x)F_2(y) + F_2(x)F_1(y)}{x - y},
\]

where the \( F_j(z) \) are expressed in terms of the solution of the Riemann-Hilbert problem for \( Y(z) \) as follows:

\[
F_j(z) = Y_{+,j1} + Y_{+,j2}\phi, \quad j = 1, 2.
\]

Comparing this with the definition (173) of \( X(z) \) we see that

\[
F_j(z) = X_{j1}(z), \quad j = 1, 2.
\]

Substituting then \( R(\alpha, \alpha) = \lim_{x \to \alpha} R(x, \alpha) \) into (177), we obtain:

\[
\frac{d}{d\alpha} \ln D_n(\alpha) = X_{11}(\alpha)X'_{21}(\alpha) - X'_{11}(\alpha)X_{21}(\alpha),
\]

which expresses the logarithmic derivative of \( D_n(\alpha) \) in terms of the solution of the Riemann-Hilbert problem for \( X(z) \). Now the function \( X(z) \) is related to \( V(z) \) by the expression (176). In particular,

\[
X_{11}(z) = \frac{\kappa_0}{2} e^{-2nz}V_{11}(z), \quad X_{21}(z) = \frac{1}{\pi i \kappa_n} e^{-2nz}V_{21}(z).
\]

Calculating the derivatives of these quantities at \( z = \alpha \) and substituting into (181), we finally obtain (37).

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