ON PRODUCTS OF $\mathfrak{sl}_n$ CHARACTERS AND SUPPORT CONTAINMENT

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Abstract. Let $\lambda$, $\mu$, $\nu$, and $\rho$ be dominant weights of $\mathfrak{sl}_n$ satisfying $\lambda + \mu = \nu + \rho$. Let $V_\lambda$ denote the highest weight module corresponding to $\lambda$. Lam, Postnikov, Pylyavskyy conjectured a sufficient condition for $V_\lambda \otimes V_\mu$ to be contained in $V_\nu \otimes V_\rho$ as $\mathfrak{sl}_n$-modules. In this note we prove a weaker version of the conjecture. Namely we prove that under the conjectured conditions every irreducible $\mathfrak{sl}_n$-module which appears in the decomposition of $V_\lambda \otimes V_\mu$ does appear in the decomposition of $V_\nu \otimes V_\rho$.

1. Introduction

Let $\lambda$, $\mu$ be two dominant weights of $\mathfrak{sl}_n$. Recall that the weight lattice in this case is $\mathbb{Z}^n/(1, \ldots, 1)$. Thus dominant weights can be viewed as partitions with $n$-th part equal to zero. Equivalently, dominant weights can be associated with Young diagrams with $n - 1$ rows. Let $V_\lambda$ denote the highest weight module corresponding to $\lambda$. Recall that as $\lambda$ runs through all dominant weights, the $V_\lambda$-s constitute the set of irreducible $\mathfrak{sl}_n$-modules. Since $\mathfrak{sl}_n$ is semisimple, the tensor product $V_\lambda \otimes V_\mu = \bigoplus \nu c^{\nu}_{\lambda, \mu} V_\nu$ decomposes into a direct sum of $V_\nu$-s. The coefficients $c^{\nu}_{\lambda, \mu}$ which appear in this decomposition are the celebrated Littlewood-Richardson coefficients. Let $\chi_\lambda$ denote the polynomial character of irreducible representation $V_\lambda$. Then $\chi_\lambda = s_\lambda(x_1, \ldots, x_n, 0, \ldots)$ is the evaluation of Schur function $s_\lambda$ modulo the relation $x_1 \cdots x_n = 1$. For the background on representation theory of $\mathfrak{sl}_n$ and Schur functions see [Hum], [Sta]. Schur functions form a basis for the ring $\Lambda$ of symmetric functions with $c^{\nu}_{\lambda, \mu}$ as structure constants. Note that under the substitution $x_{n+1} = \cdots = 0$ the Schur functions $s_\lambda$ with $\lambda$ having more than $n$ parts vanish. This causes a subtle difference between multiplication of $\chi_\lambda$-s and multiplication of Schur functions: some terms appearing in the latter vanish in the former.

One can ask when $V_\lambda \otimes V_\mu$ is contained in $V_\nu \otimes V_\rho$ as an $\mathfrak{sl}_n$-module. Of course one way to answer this question is just to say that for all $\kappa$ one should have $c^{\kappa}_{\lambda, \mu} \leq c^{\kappa}_{\nu, \rho}$. However, one might hope to find simple sufficient and/or necessary conditions for the containment to hold. An obviously related question is when the differences of the form $s_\nu s_\rho - s_\lambda s_\mu$ are Schur-nonnegative. Some conjectures and results of this form have appeared in the literature, see [BM, FFLP, LLT, LP, LPP, Oko, RS].

The first thing to note is that the highest weight appearing in $V_\lambda \otimes V_\mu$ is $\lambda + \mu$. Thus, in order for $V_\lambda \otimes V_\mu$ to be a submodule of $V_\nu \otimes V_\rho$ we need to have $\lambda + \mu \leq \nu + \rho$ in dominance order. It is natural to investigate what happens if we restrict our
attention to the case when equality holds, i.e. $\lambda + \mu = \nu + \rho$. For this situation, Lam, Postnikov and Pylyavskyy made a conjecture concerning a sufficient condition for $V_\lambda \otimes V_\mu$ to be a submodule of $V_\nu \otimes V_\rho$, or equivalently for $\chi_\nu \chi_\rho - \chi_\lambda \chi_\mu$ to be $\chi$-nonnegative.

Let $\alpha_{ij} = e_i - e_j$ be the roots of the type $A$ root system. Call a polytope *alcoved* if its faces belong to hyperplanes given by the equations $\langle \alpha_{ij}, \tau \rangle = m$, where $\langle , \rangle$ is the standard inner product and $m \in \mathbb{Z}$. Alcoved polytopes are studied in [LPo]. Given two weights $\lambda, \mu$ one can consider the minimal alcoved polytope $P_{\lambda,\mu}$ containing $\lambda$ and $\mu$. $P_{\lambda,\mu}$ is always a parallelepiped in which $\lambda$ and $\mu$ are a pair of opposite vertices. An example for $sl_3$ is shown in Figure 1. The weights $\tau$ inside $P_{\lambda,\mu}$ can be characterized by the following condition: for all $1 \leq i, j \leq n$, the number $\tau_i - \tau_j$ lies weakly between $\lambda_i - \lambda_j$ and $\mu_i - \mu_j$. Let $\nu$ and $\rho$ be another pair of weights.

**Conjecture 1.** [LPP2] If $\lambda + \mu = \nu + \rho$ and $\nu, \rho \in P_{\lambda,\mu}$, then $\chi_\nu \chi_\rho - \chi_\lambda \chi_\mu$ is $\chi$-nonnegative.

**Example 2.** It is easy to see in Figure 1 that points $\rho = (11,7,0)$, $\nu = (5,2,0)$ lie inside marked $P_{\lambda,\mu}$ with $\lambda = (12,7,0)$, $\mu = (4,2,0)$. In this case

$$
\chi_\nu \chi_\rho - \chi_\lambda \chi_\mu = \chi_{(13,12,0)} + \chi_{(6,4,0)} + \chi_{(7,6,0)} + \chi_{(8,8,0)} + \chi_{(7,3,0)} + \chi_{(8,5,0)} + \chi_{(9,7,0)}
$$

$$
+ \chi_{(10,9,0)} + \chi_{(11,11,0)} + \chi_{(8,2,0)} + \chi_{(9,4,0)} + \chi_{(10,6,0)} + \chi_{(11,8,0)} + \chi_{(12,10,0)}.
$$

We prove the following weaker statement.

**Theorem 3.** If $\lambda + \mu = \nu + \rho$ and $\nu, \rho \in P_{\lambda,\mu}$, then every $\chi_\kappa$ occurring in $\chi_\lambda \chi_\mu$ with a non-zero coefficient does also occur in $\chi_\nu \chi_\rho$ with a non-zero coefficient.

The paper goes as follows. In Section 2 we review the theory of Horn-Klyachko inequalities. We prove Lemma 6 which plays a key role later. In Section 3 we review Rhoades-Skandera theory of Temperley-Lieb immanants. In Section 4 we apply the theory of Temperley-Lieb immanants to prove Lemma 11. Finally we combine Lemma 6 and Lemma 11 to obtain proof of Theorem 3.

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2. Horn-Klyachko inequalities

For a finite set $I = \{i_1 > \cdots > i_r\}$ of positive integers, define the corresponding partition $\lambda(I)$ by

$$\lambda(I) = (i_1 - r, i_2 - (r - 1), \ldots, i_r - 1).$$

**Definition 4.** Define $T_r^n$ to be the set of triples $(I, J, K)$ of subsets of $\{1, \ldots, n\}$ of the same cardinality $r$ such that the Littlewood-Richardson coefficient $c_{\lambda(I), \lambda(J)}^{\lambda(K)}$ is positive. A **Horn-Klyachko inequality** for a triple of partitions $\alpha, \beta, \gamma$ has the form

$$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j$$

for a triple $(I, J, K)$ in $T_r^n$ and some $r < n$.

The following fact was proved in [K] [KT], see also [HM] for a survey:

**Theorem 5.** For a triple of partitions $\alpha, \beta, \gamma$ of length $n$, the Littlewood-Richardson coefficient $c_{\alpha, \beta}^{\gamma}$ is positive if and only if $\sum_{i=1}^n \gamma_i = \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i$ and Horn-Klyachko inequalities for $\alpha, \beta, \gamma$ are valid for all $(I, J, K) \in T_r^n$ and all $r < n$.

Let partitions $\lambda, \mu, \nu, \rho$ with at most $n$ parts satisfy the conditions of Conjecture [K] and $\gamma$ be a partition such that $c_{\lambda\mu} > 0$. Consider a triple

$$(I = (i_1, \ldots, i_r), J = (j_1, \ldots, j_r), K = (k_1, \ldots, k_r))$$

in $T_r^n$. Given permutations $\{l_1, \ldots, l_r\}$ of $I$ and $\{m_1, \ldots, m_r\}$ of $J$, switch $l_p$ and $m_p$ in some of the pairs $(l_p, m_p)$. This operation yields $2^r$ possible pairs $(I', J')$.

**Lemma 6.** Assume there exist permutations $\{l_1, \ldots, l_r\}$ of $I$ and $\{m_1, \ldots, m_r\}$ of $J$ such that all possible triples $(I', J', K)$ are in $T_r^n$. Then the Horn-Klyachko inequality corresponding to the triple $(I, J, K)$ holds for $\nu, \rho, \gamma$.

**Proof.** Since $\nu, \rho \in P_{\lambda, \mu}$, for $i, j \geq 1$ both $\nu_i - \nu_j$ and $\rho_i - \rho_j$ are between $\lambda_i - \lambda_j$ and $\mu_i - \mu_j$, which implies

$$|(\nu_i - \nu_j) - (\rho_i - \rho_j)| \leq |(\lambda_i - \lambda_j) - (\mu_i - \mu_j)|.$$

Rearranging terms, we obtain

$$|(\nu_i + \rho_j) - (\nu_j + \rho_i)| \leq |(\lambda_i + \mu_j) - (\lambda_j + \mu_i)|.$$

This inequality combined with the equality $(\nu_i + \rho_j) + (\nu_j + \rho_i) = (\lambda_i + \mu_j) + (\lambda_j + \mu_i)$ following from $\lambda + \mu = \nu + \rho$, shows that $\nu_i + \rho_j$ and $\nu_j + \rho_i$ are between $\lambda_i + \mu_j$ and $\lambda_j + \mu_i$. We use the fact that for all $i, j \geq 1$ we have

$$\nu_i + \rho_j \geq \min\{\lambda_i + \mu_j, \lambda_j + \mu_i\}.$$

For every $p \in \{1, \ldots, r\}$, choose $(l'_p, m'_p)$ to be a permutation of $(l_p, m_p)$ such that $\lambda_{l'_p} + \mu_{m'_p} = \min\{\lambda_p + \mu_{m_p}, \lambda_{m_p} + \mu_p\}$, and let $I' = \{l'_1, \ldots, l'_r\}$, $J' = \{m'_1, \ldots, m'_r\}$ be the corresponding subsets of $\{1, \ldots, n\}$. By the assumption of the lemma, $c_{\lambda\mu} > 0$ and $(I', J', K)$ is in $T_r^n$. Therefore, by Theorem 5, the Horn-Klyachko inequality for $\lambda, \mu, \gamma$ and the triple $(I', J', K)$ holds:

$$\sum_{p=1}^r \lambda_{l'_p} + \sum_{p=1}^r \mu_{m'_p} \geq \sum_{k \in K} \gamma_k.$$
Observe that
\[
\sum_{i \in I} \nu_i + \sum_{j \in J} \rho_j = \sum_{p=1}^{r} \nu_{p} + \sum_{p=1}^{r} \rho_{m_{p}} = \sum_{p=1}^{r} (\nu_{p} + \rho_{m_{p}}) \geq \\
\sum_{p=1}^{r} \min \{\lambda_{l_{p}} + \mu_{m_{p}}, \lambda_{m_{p}} + \mu_{l_{p}}\} = \sum_{p=1}^{r} (\lambda_{l_{p}} + \mu_{m_{p}}) \geq \sum_{k \in K} \gamma_{k}.
\]
Therefore, the Horn-Klyachko inequality for \(\nu, \rho, \gamma\) and the triple \((I, J, K)\) holds. \(\square\)

3. Temperley-Lieb immanants

In this section we review the theory of Temperley-Lieb immanants developed by Rhoades and Skandera. We limit ourselves to discussing Theorem 9 and Theorem 7 of which we make use in this paper. For detailed exposition of the beautiful results of Rhoades and Skandera we refer reader to the original papers [RS], [RS2]. One can also find a (more detailed than here) review in [LPP].

The symmetric functions \(h_k = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}\) are called the homogeneous symmetric functions. For background on them, see [Sta]. Given two sets \(V = (v_1 \geq v_2 \cdots \geq v_n \geq 0)\) and \(U = (u_1 \geq u_2 \cdots \geq u_n \geq 0)\) one can construct the generalized Jacobi-Trudi matrix \(X_{V,U} = (h_{v_i-u_j})_{i,j=1}^{n}\). For example, for \(V = (4,3,3,2)\) and \(U = (3,2,1,0)\) we get

\[
X_{V,U} = \begin{bmatrix}
h_1 & h_2 & h_3 & h_4 \\
1 & h_1 & h_2 & h_3 \\
0 & 1 & h_1 & h_2
\end{bmatrix}
\]

Note that for the operation \(\lambda = \lambda(I) = (i_1 - r, \ldots, i_r - 1)\) defined in Section 2 we have the Jacobi-Trudi identity \(s_{\lambda(I)} = \det X_{I,\{r,\ldots,2,1\}}\). (See [Sta]).

The Temperley-Lieb algebra \(TL_n(\xi)\) is the \(\mathbb{C}[\xi]\)-algebra generated by \(t_1, \ldots, t_{n-1}\) subject to the relations \(t_i^2 = \xi t_i, t_it_jt_i = t_j\) if \(|i - j| = 1\), and \(t_it_j = t_jt_i\) if \(|i - j| \geq 2\). The dimension of \(TL_n(\xi)\) equals the \(n\)-th Catalan number \(C_n = \frac{1}{n+1}\binom{2n}{n}\). A 321-avoiding permutation is a permutation \(w \in S_n\) that has no reduced decomposition of the form \(w = \cdots s_is Js_i\cdots\) with \(|i - j| = 1\). (These permutations are also called fully-commutative.) A natural basis of the Temperley-Lieb algebra is \(\{t_w \mid w\text{ is a 321-avoiding permutation in } S_n\}\), where \(t_w := t_{i_1} \cdots t_{i_r}\), for a reduced decomposition \(w = s_{i_1} \cdots s_{i_r}\).

For any permutation \(v \in S_n\) and a 321-avoiding permutation \(w \in S_n\), let \(f_w(v)\) be the coefficient of the basis element \(t_w \in TL_n(2)\) in the basis expansion of \((t_{i_1} - 1) \cdots (t_{i_r} - 1) \in TL_n(2)\), where \(v = s_{i_1} \cdots s_{i_r}\) is a reduced decomposition. Rhoades and Skandera [RS2] defined the Temperley-Lieb immanant \(\text{Imm}^{TL}_w(x)\) of an \(n \times n\) matrix \(X = (x_{ij})\) by

\[
\text{Imm}^{TL}_w(X) := \sum_{v \in S_n} f_w(v) x_{1,v(1)} \cdots x_{n,v(n)}.
\]

**Theorem 7.** Rhoades-Skandera [RS2] Proposition 2.3, Proposition 3.2 Temperley-Lieb immanants of generalized Jacobi-Trudi matrices are Schur-nonnegative.

**Remark 8.** In [RS2] two stronger statements (Proposition 2.3 and Proposition 3.2) are proved, from which Theorem 7 follows in a straightforward way.
A product of generators (decomposition) \( t_i \cdots t_i \) in the Temperley-Lieb algebra \( TL_n \) can be graphically presented by a Temperley-Lieb diagram with \( n \) non-crossing strands connecting the vertices 1, \ldots, 2n, possibly with some internal loops. The left endpoints are assumed to be labeled 1, \ldots, \( n \) from top to bottom and the right endpoints are assumed to be labeled 2n, \ldots, \( n+1 \) from top to bottom.

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
1 & t_1 & t_2 & t_{n-1}
\end{array}
\]

The map that sends \( t_w \) to the non-crossing matching given by its Temperley-Lieb diagram is a bijection between basis elements \( t_w \) of \( TL_n \), where \( w \) is 321-avoiding, and non-crossing matchings on the vertex set \([2n]\).

Following [RS2], for a subset \( S \subseteq [2n] \), let us say that a Temperley-Lieb diagram (or the associated element in \( TL_n \)) is \( S \)-compatible if each strand of the diagram has one endpoint in \( S \) and the other endpoint in its complement \([2n] \setminus S\). Coloring vertices in \( S \) black and the remaining vertices white, a basis element \( t_w \) is \( S \)-compatible if and only if each edge in the associated matching has two vertices of different color. Let \( \Theta(S) \) denote the set of all 321-avoiding permutations \( w \in S_n \) such that \( t_w \) is \( S \)-compatible. An example for \( n = 5 \), \( S = \{3, 6, 7, 8, 10\} \) is shown in the figure below, where all possible compatible non-crossing matchings are presented.

For two subsets \( I, J \subseteq [n] \) of the same cardinality, let \( \Delta_{I,J}(X) \) denote the minor of an \( n \times n \) matrix \( X \) in the row set \( I \) and the column set \( J \). Let \( I^\wedge := \{2n+1-i \mid i \in I\} \).

**Theorem 9.** Rhoades-Skandera [RS2 Proposition 4.4], cf. Skandera [Sk2] For two subsets \( I, J \subseteq [n] \) of the same cardinality and \( S = J \cup (I^\wedge) \), we have

\[
\Delta_{I,J}(X) \cdot \Delta_{I^\wedge,J^\wedge}(X) = \sum_{w \in \Theta(S)} \text{Imm}^T_w(X).
\]

**Example 10.** Take \( I = \{1, 2\} \), \( J = \{1, 3\} \), and

\[
X = \begin{bmatrix}
x_{11} & x_{12} & x_{13} & x_{14} \\
x_{21} & x_{22} & x_{23} & x_{24} \\
x_{31} & x_{32} & x_{33} & x_{34} \\
x_{41} & x_{42} & x_{43} & x_{44}
\end{bmatrix}
\]

Then \( S = \{1, 3, 5, 6\} \), and the elements of \( \Theta(S) \) are shown in the figure below. In this case Theorem 9 yields the decomposition

\[
t_1 t_2 t_1 t_3 = \xi t_1 t_3
\]
Let $T_n$.

Lemma 11. In the setup of Lemma 6, there exist permutations $\{l_1, \ldots, l_r\}$ and $\{m_1, \ldots, m_r\}$ of $I$ and $J$ respectively such that all possible triples $(I', J', K)$ are in $T_n$.

Proof. Let $X_{V,U}$ be the generalized Jacobi-Trudi matrix for column set $U = (r, r, \ldots, 1, r - 1, \ldots, 1)$, and row set $V = I \cup J$ in some chosen non-increasing arrangement. Let $\#I$ and $\#J$ denote the sets of numbers of the rows of $I$ and $J$ in the chosen non-increasing arrangement of $I \cup J$. Since $(I, J, K) \in T_n$, we have $\lambda(I) \lambda(J) > 0$. Hence $s_{\lambda(I) \lambda(J)}$ is present in the decomposition of $s_{\lambda(I)} s_{\lambda(J)}$, which by Jacobi-Trudi identity equals to the product $\Delta_{\#I, \{2r,2r-2,\ldots,2\}} \Delta_{\#J,\{2r-1,2r-3,\ldots,1\}}$ of complementary minors of $X_{V,U}$. This product, in turns, by Theorem 9 equals to $\sum_{w \in \Theta(S)} \Imm^TL_w(X_{V,U})$, where $S = \#J \cup \{4r, 4r - 2, \ldots, 2r + 2\}$ is the subset of the vertices $1, \ldots, 4r$ of the Temperley-Lieb diagram which are colored black.

Since $s_{\lambda(K)}$ is in the Schur function decomposition of $\sum_{w \in \Theta(S)} \Imm^TL_w(X_{V,U})$, it is present in the Schur function decomposition of one of the immanants $\Imm^TL_w(X_{V,U})$ for some $321$-avoiding permutation $w \in \Theta(S)$. For this $321$-avoiding permutation $w$, the basis element $t_w$ and the corresponding non-crossing matching $M_w$ of the Temperley-Lieb diagram with columns $V$ and $U$ are $S$-compatible. Therefore, all edges of $M_w$ have endpoints of different color in the Temperley-Lieb diagram on vertices $\{1, 2, \ldots, 4r\}$ where $S$ is colored black and $\{4r\}/S$ colored white.

We proceed now to construct the needed permutations $\{l_1, \ldots, l_r\}$ of $I$ and $\{m_1, \ldots, m_r\}$ of $J$ based on $S$ and $M_w$. We go along $V$ from top to bottom (see Figure 2(i)) and label vertices in $I$ that are connected to vertices in $J$ by edges in $M_w$ (suppose that there are $k$ such vertices in $I$) with variables $l_1, \ldots, l_k$ as we meet them. We also label the vertex in $J$ connected to $l_i$ ($i \leq k$) by $m_i$.

Next, we remove the vertices $l_1, \ldots, l_k, m_1, \ldots, m_k$ from $V$ and call the remaining set $V'$. We also go along $U$ and discard every pair of vertices in $U$ connected by an edge in $M_w$, and call the remaining set $U'$. We go along $V'$ from top to bottom and label the white vertices that we meet by $l_{k+1}, \ldots, l_r$, and the black vertices we meet by $m_{k+1}, \ldots, m_r$ from top to bottom. For $f \geq 1$, we also label the vertices in $U'$ connected by edges in $M_w$ to $l_{k+f}$ by $p_{k+f}$, and those connected to $m_{k+f}$ by $q_{k+f}$. (See Figure 2(ii)). Note that every vertex in $V$ between adjacent vertices of $V'$ is connected by an edge in $M_w$ to another vertex between the same vertices.

4. PROOF OF THE MAIN THEOREM
of $V^-$ because $M_w$ is a non-crossing, and the same is true about $U$. Therefore, in building $V^-$ and $U^-$ we discarded segments of even lengths from $V$ and $U$.

Claim. For $f \geq 1$, vertices $l_{k+f}$ and $q_{k+f}$ are white and odd-numbered in the Temperley-Lieb diagram for $S$ and $M_w$; vertices $p_{k+f}$ and $m_{k+f}$ are black and even-numbered. Also, $l_{k+f+1} > m_{k+f} > l_{k+f}$ and $p_{k+f+1} < q_{k+f} < p_{k+f}$. (See Figure 2(ii))

Proof. Since we discarded segments of even lengths from $U$ to obtain $U^-$ and the colors in $U$ were alternating from top to bottom beginning with the black even vertex $4r$, the colors in $U^-$ are also alternating from top to bottom beginning with a black even vertex. Therefore, vertices in $U^-$ from top to bottom are $p_{k+1} > q_{k+1} > p_{k+2} > \ldots > p_r > q_r$, where $p_{k+f}$ is black and $q_{k+f}$ is white for $f \geq 1$. Because the restriction of the matching $M_w$ to $U^- \cup V^-$ is non-crossing, the inequalities $p_{k+1} > q_{k+1} > p_{k+2} > \ldots > p_r > q_r$ for $U^-$ imply that $l_{k+1} < m_{k+1} < l_{k+2} < \ldots < l_r < m_r$ for $V^-$. The colors in $V^-$ alternate and have a white odd vertex at the top because the colors in $U^-$ alternate with a black even vertex at the top. Therefore, $l_{k+f}$ is white and $m_{k+f}$ is black for $f \geq 0$. The statements about being odd/even now follow from the fact that we discarded segments of even lengths from $U$ and $V$ to obtain $U^-$ and $V^-$. We now build a new coloring $S'$ of $U \cup V$ based on the transpositions $(l_p, m_p)$ that may have occurred in going from $I, J$ to $I', J'$. We only allow ourselves to recolor both elements in a pair $\{2m, 2m-1\} \in \{2r+1, \ldots, 4r\}$ of vertices in the second column of Temperley-Lieb diagram for $S'_0 = \#J' \cup \{4r, 4r-2, \ldots, 2r+2\}$, because the columns $4r+1-2m$ and $4r+2-2m$ of $X_{U,V}$ are identical and hence such a recoloring produces the same pair of complementary minors $\Delta_{\#J',4r-1,2r+1}$, $\Delta_{\#J',4r-3,2r+3}$ of $X_{V,U}$ as $S'_0$ does, and therefore by Jacobi-Trudi identity the product of these complementary minors is $s_{\lambda(I')} s_{\lambda(J')}$. Rule of recoloring. For every pair $l_{k+f}$ and $m_{k+f}$ ($f \geq 1$) that exchanged colors in transition from $I, J$ to $I', J'$, recolor the pairs $(p_{k+f}, p_{k+f} - 1), (p_{k+f} - 2, p_{k+f} - 3), \ldots, (q_{k+f} + 1, q_{k+f})$. The recoloring is permissible because the vertex $p_{k+f}$ is even by the Claim. (See Figure 2(iii))

![Figure 2](image-url)

**Figure 2.**

*Why the rule produces a coloring compatible with $M_w$. The vertices between $p_{k+f}$ and $q_{k+f}$ either all changed color or all stayed the same, so an edge in $M_w$ that connected two vertices in $U$ between $p_{k+f}$ and $q_{k+f}$ now has its endpoints*
changed or not changed simultaneously, so they are of different color in the new coloring.

A pair \((l_{k+f}, m_{k+f})\) changes color simultaneously with the pair \((p_{k+f}, q_{k+f})\), so \(l_{k+f}\) and \(p_{k+f}\), and \(m_{k+f}\) and \(q_{k+f}\) change or do not change their color simultaneously, so the endpoints of the edges between \(U^-\) and \(V^-\) remain colored differently in the new coloring.

A pair of vertices \((l_p, m_p)\) in \(V\) connected by an edge in \(M_w\) changes color simultaneously when the corresponding transposition occurs, so the endpoints of such an edge remain colored differently. Finally, a pair of vertices in \(U\) between \(q_{k+f}\) and \(p_{k+f+1}\) connected by an edge in \(M_w\) never changes color, so such an edge has its endpoints colored differently in the new coloring. We considered all possibilities for an edge in \(M_w\) relative to \(U^-\) and \(V^-\) in a non-crossing matching, so \(M_w\) is compatible with the new coloring.

We already noticed that the new coloring produces the product of complementary minors of \(X_{V,U}\) equal to \(s_{\lambda(I')}s_{\lambda(J')}\). The fact that the new coloring is compatible with \(M_w\) implies that the immanant \(\text{Imm}^TL_w(X_{V,U})\) is present in the decomposition \(s_{\lambda(I')}s_{\lambda(J')} = \sum_{w \in T(S)} \text{Imm}^TL_w(X_{V,U})\). Since \(s_{\lambda(K)}\) is in the decomposition of \(\text{Imm}^TL_w(X_{V,U})\) which is Schur-nonnegative by Theorem \(\Box\) \(s_{\lambda(K)}\) is present in the Schur function decomposition of \(s_{\lambda(I')}s_{\lambda(J')}\). Therefore \(c_{\lambda(\lambda(I'))\lambda(J')} > 0\) and \((I', J', K) \in T^n\) for all \(I', J'\) that can be obtained by transposing pairs \((l_p, m_p)\) in \(I_J\).

We are ready to prove Theorem \(\Box\)

**Proof.** From Lemma \(\Box\) and Lemma \(\Box\) it follows that whenever the Horn-Klyachko inequality for triple \((I, J, K)\) holds for \(\lambda, \mu, \gamma\), it also holds for \(\nu, \rho, \gamma\). Thus all possible \(\gamma\)'s for which all needed Horn-Klyachko inequalities hold or, equivalently, \(c_{\lambda,\mu} > 0\), also have the property that \(c_{\gamma,\rho} > 0\). \(\Box\)

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