Families index theory for Overlap lattice Dirac operator. I.

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Abstract

The index bundle of the Overlap lattice Dirac operator over the orbit space of lattice gauge fields is introduced and studied. Obstructions to the vanishing of gauge anomalies in the Overlap formulation of lattice chiral gauge theory have a natural description in this context. Our main result is a formula for the topological charge (integrated Chern character) of the index bundle over even-dimensional spheres in the orbit space. It reduces under suitable conditions to the topological charge of the usual (continuum) index bundle in the classical continuum limit (this is announced and sketched here; the details will be given in a forthcoming paper). Thus we see that topology of the index bundle of the Dirac operator over the gauge field orbit space can be captured in a finite-dimensional lattice setting.

1 Introduction

The index bundle of the Dirac operator over the orbit space of gauge fields was studied many years ago by Atiyah and Singer \[1,2\]. Obstructions to the vanishing

\[\text{1 Detailed treatments of Dirac operator index theory can be found in [2].}\]
of chiral gauge anomalies have a natural description in this context [1, 3]. More recently, the overlap formalism [4], developed by Narayanan and Neuberger as a way to formulate chiral gauge theories on the lattice, has provided the ingredients for a lattice version of Dirac operator index theory when the base manifold (spacetime) is an even-dimensional torus. A lattice version of the index arose there as the fermionic topological charge of the lattice gauge field, and can be expressed as the index of the Overlap lattice Dirac operator, introduced in [5]. The fact that the overlap formulation successfully reproduces the global gauge anomaly and obstructions to the vanishing of local gauge anomalies [3, 7, 8, 10] indicates that it should also lead to a lattice version of families index theory for the Dirac operator. The purpose of the present paper is to show that this is indeed the case.

We develop a families index theory for the Overlap lattice Dirac operator coupled to SU(N) lattice gauge fields on the 4-torus. (The specialisation to 4 dimensions is for simplicity and physical relevance; everything generalises straightforwardly to arbitrary even-dimensional torus [12].) After recalling the lattice setup in §2, we show in §3 that an index bundle over the orbit space of lattice gauge fields can be defined in a natural way. In §4 we derive our main result: a formula ("lattice families index theorem") for the topological charge (integrated Chern character) of this index bundle over a generic even-dimensional sphere in the orbit space of lattice gauge fields (Theorem 1, eq. (4.1) below). It contains an integer-valued part which has no continuum analogue; in §5 we outline the argument in the forthcoming paper [12] showing that the rest of the expression reduces to the topological charge of the usual (continuum) index bundle in the classical continuum limit (Theorem 2).

When the group \( G \) of gauge transformations is restricted to \( G_0 = \{ \phi \in G \mid \phi(x_0) = 1 \} \) (where \( x_0 \) is some basepoint in the 4-torus) the orbit space \( \mathcal{U}/G_0 \) is a smooth manifold (since \( G_0 \) acts freely on the space \( \mathcal{U} \) of lattice gauge fields). In this case the above-mentioned part of the families index formula which has no continuum analogue vanishes in the classical continuum limit, as we will see in §5. The above-mentioned

\[^2\]A first step in this direction was made earlier in [11].
classical continuum limit result then implies that the finite-dimensional lattice setting is capturing topology of the continuum index bundle over $\mathcal{A}/\mathcal{G}_0$ when the lattice is sufficiently fine.

In §6 we discuss how obstructions to the vanishing of gauge anomalies in the overlap formulation of lattice chiral gauge theories have natural descriptions in this context. We conclude in §7 by discussing some implications and possible generalisations of our results, and some related results to be shown in subsequent papers. The background material in §2-3 is intended to make this paper accessible to nonspecialists in lattice gauge theory.

This is the first in a series of papers. In the follow-up paper [12], we treat the general case of arbitrary even-dimensional torus and prove the classical continuum limit result announced and outlined in §5, as well as a result mentioned in the concluding remark 2. In [13], we apply the results of the present paper to demonstrate an interplay between topological features of the space of SU(N) lattice gauge fields on $T^4$ and the existence question for $\mathcal{G}_0$ gauge fixings on the lattice which do not have the Gribov problem. We find that certain obstructions to the existence of such gauge fixings in the continuum are absent on the lattice, and that instead the topological sectors (specified by the fermionic topological charge) contain noncontractible spheres of various even dimensions when $N \geq 3$, and noncontractible circles in the $N = 2$ case. Further applications of the lattice families index theory developed here to the global gauge anomaly on the lattice and related issues will be given in [14].

2 Lattice setup

We take the spacetime manifold to be the Euclidean 4-torus $T^4$ and the gauge group to be SU(N). A hypercubic lattice on $T^4$ determines the space $\mathcal{C}$ of lattice spinor fields, the space $\mathcal{U}$ of lattice gauge fields, and the group $\mathcal{G}$ of lattice gauge transformations. $\mathcal{C}$ is the finite-dimensional complex vector space consisting of the spinor-valued functions on the lattice sites, i.e. the functions on the lattice sites with values in $\mathbb{C}^4 \otimes \mathbb{C}^N$. 
$C^4$ and $C^N$ are the spin and SU(N) representation spaces; for simplicity we are specialising to the fundamental representation of SU(N). It has the inner product

$$\langle \psi_1, \psi_2 \rangle = a^4 \sum_x \psi_1(x)^* \psi_2(x) \quad (2.1)$$

($a$=lattice spacing). $U$ consists of the SU(N)-valued functions on the links of the lattice. It has the finite-dimensional manifold structure

$$U \cong SU(N) \times SU(N) \times \cdots \times SU(N) \quad (2.2)$$

(one copy for each link). The group $G$ consists of the maps $\phi : \{\text{lattice sites}\} \to SU(N)$. It acts on $C$ and $U$ by

$$\phi \cdot \psi(x) = \phi(x) \psi(x) \quad (2.3)$$

$$(\phi \cdot U)_{\mu}(x) = \phi(x) U_{\mu}(x) \phi(x + ae_\mu)^{-1} \quad (2.4)$$

($e_\mu=$unit vector in the positive $\mu$-direction). Continuum spinor fields, gauge fields and gauge transformations have lattice transcripts, defined in a natural way: In the case of spinor field $\psi(x)$ or gauge transformation $\phi(x)$ we restrict $x$ to the lattice sites to get elements in $C$ or $G$, respectively. In the case of gauge field $A_{\mu}(x)$ the lattice transcript $U_{\mu}(x)$ is the parallel transport from $x + ae_\mu$ to $x$. Then (2.3)--(2.4) are the lattice transcripts of the continuum gauge transformations. For a given continuum gauge field $A$, the classical continuum limit of a quantity depending on the lattice gauge field $U$ (e.g., the index of a lattice Dirac operator) is the limit of infinitely many subdivisions of the hyper-cubic lattice (i.e. lattice spacing $a \to 0$) with $U$ being the lattice transcript of $A$. The classical continuum limit of quantities depending on families of lattice gauge fields is defined analogously.

The covariant derivative $\partial_{A\mu} = \partial_\mu + A_\mu$ can be approximated on the lattice by the covariant finite difference operators $\frac{1}{a} \nabla_\mu^\pm$, $\frac{1}{a} \nabla_\mu$ where

$$\nabla_\mu^+ \psi(x) = U_\mu(x) \psi(x + ae_\mu) - \psi(x) \quad (2.5)$$

$$\nabla_\mu^- \psi(x) = \psi(x) - U_\mu(x - ae_\mu)^{-1} \psi(x - ae_\mu) \quad (2.6)$$

and $\nabla_\mu = \frac{1}{2}(\nabla_\mu^+ + \nabla_\mu^-)$. Note that $(\nabla_\mu^\pm)^* = -\nabla_\mu^\mp$, so $\nabla_\mu^* = -\nabla_\mu$. 

4
3 Lattice version of Dirac operator index theory

The Dirac operator $\mathcal{D}^A = \gamma^\mu \partial^A_\mu$ has the naive lattice approximation

$$\hat{\nabla} = \gamma^\mu \frac{1}{a} \partial^A_\mu$$

(3.1)

However, the index theory for this operator is trivial: $\text{index } \hat{\nabla}^U = 0 \ \forall U \in \mathcal{U}$. This well-known fact is a simple consequence of the chiral symmetry

$$\gamma_5 \hat{\nabla} + \hat{\nabla} \gamma_5 = 0$$

(3.2)

and the finite-dimensionality of $\mathcal{C}$. A nontrivial lattice version of the Dirac operator index can instead be constructed by noting that index $\mathcal{D}^A$ is minus the spectral flow of $H^A_m = \gamma_5 (\mathcal{D}^A - m)$ as $m$ increases from $m < 0$ to $m > 0$. (We are following the physics convention where the $\gamma^\mu$’s are hermitian; then $\mathcal{D}^A$ is antihermitian and $H^A_m$ is hermitian.) A lattice analogue of $H^A_m$ is

$$H^U_m = \gamma_5 (D^U_w - m)$$

(3.3)

where

$$D^U_w = \hat{\nabla}^U + a \frac{r}{2} \Delta^U$$

(3.4)

is the Wilson–Dirac operator [15]. The Wilson term $a \frac{r}{2} \Delta$, where $\Delta = \frac{1}{a^2} (\nabla^\mu)^* \nabla^\mu = \frac{1}{a^2} \sum_\mu \nabla^-_\mu - \nabla^+_\mu$ is the lattice Laplace operator and $r > 0$ the Wilson parameter, breaks the chiral symmetry and thereby allows for a nontrivial index theory. It is known that when $m < 0$ the hermitian operator $H^U_m$ has symmetric spectrum and no zero-modes [4]. Hence the spectral flow as $m$ increases from any $m_1 < 0$ to some $m_2 > 0$ is equal to half the spectral asymmetry of $H^U_m$ at $m = m_2$, i.e. $\frac{1}{2} \text{Tr}(\epsilon^U)$, where

$$\epsilon^U = \frac{H^U_m}{|H^U_m|}$$

(3.5)

This suggests defining the lattice version of the index by

$$\text{index } \mathcal{D}^A \rightarrow - \frac{1}{2} \text{Tr}(\epsilon^U)$$

(3.6)
for some suitable value of the parameter $m$ in (3.3) (which we have suppressed in the notation in (3.3)–(3.6)). In fact the right-hand side of (3.6), with

$$m = \frac{rm_0}{a}, \quad 0 < m_0 < 2$$

(3.7)

is precisely the definition of the topological charge of $U$ which arose in the overlap formulation of chiral gauge theories on the lattice [4]. In [16] this was shown to reduce to index $\mathcal{D}^A$ in the classical continuum limit for any choice of $m$ satisfying (3.7) above.

It is also these choices of $m$ which give the correct classical continuum limit for the families index theory that we develop in the following.

For this definition of the lattice version of the index we must exclude from $U$ the lattice gauge fields $U$ for which $H^U$ has zero-modes, so that $\epsilon^U$ is well-defined. This determines a decomposition of $U$ into topological sectors labelled by the lattice index. It has been shown in [17] (see also [18]) that $H^U$ has no zero-modes when each $U(p)$ is sufficiently close to the identity, where $U(p)$ is the product of the $U_\mu(x)$’s around a plaquette $p$ in the lattice. (We refer to [18] for the currently best bound on $||1 - U(p)||$ which ensures this.) Consequently, the absence of zero-modes for $H^U$ is guaranteed close to the classical continuum limit, since when $U$ is the lattice transcript of a continuum field $A$, and $p$ is the plaquette specified by a lattice site $x$ and directions $\mu$ and $\nu$, we have $1 - U(p) = a^2 F_{\mu\nu}(x) + O(a^3)$. (See [16] for a more detailed discussion of this point.) From now on, we take $m$ as in (3.7) and assume that a choice of $m_0 \in (0, 2)$ has been made, and $U$ denotes the space of lattice gauge fields with the $U$’s for which $H^U$ has zero-modes excluded.

The lattice version (3.6) of index $\mathcal{D}^A$ can be expressed as the index of the Overlap lattice Dirac operator $\mathcal{D}$, given by

$$D = \frac{1}{a}(1 + \gamma_5 \epsilon)$$

(3.8)

with $\epsilon$ as in (3.5). The nullspace of this operator is invariant under $\gamma_5$ and therefore has a chiral decomposition

$$\ker D^U = (\ker D^U)_+ \oplus (\ker D^U)_-$$

(3.9)
The index can then be defined as

\[
\text{index } D^U = \dim(\ker D^U)_+ - \dim(\ker D^U)_- \tag{3.10}
\]

and coincides with the above lattice version of index \(\eta^A\):

\[
\text{index } D^U = -\frac{1}{2}\text{Tr}(\epsilon^U) \tag{3.11}
\]

While this was obtained directly in [5], it can also be seen using the fact that (3.8) satisfies [13]

\[
\gamma_5 D + D \gamma_5 = aD\gamma_5D \quad \text{(GW relation)} \tag{3.12}
\]

\[
D^* = \gamma_5 D \gamma_5 \quad \text{(\(\gamma_5\)-hermiticity)} \tag{3.13}
\]

The first relation was originally studied by Ginsparg and Wilson [20], and later rediscovered by Hasenfratz and collaborators [21], who noted that the nullspace of a solution \(D\) is invariant under \(\gamma_5\) (since if \(D\psi = 0\) then \(D\gamma_5\psi = (aD\gamma_5D - \gamma_5D)\psi = 0\)). Furthermore, they showed that solutions of (3.12)–(3.13) satisfy the following index formula (see also [22]):

\[
\text{index } D^U = -\frac{a}{2}\text{Tr}(\gamma_5 D^U) \tag{3.14}
\]

Substituting (3.8) in (3.14) leads to (3.11).

Having reviewed the previously developed lattice index theory for the Dirac operator we now proceed to develop a lattice version of the families index theory. Rather than the index (3.10), the central object here is the index bundle, formally given, in analogy with the continuum case, by

\[\text{“index } D = (\ker D)_+ - (\ker D)_-\] \tag{3.15}\]

where

\[
(\ker D)_\pm = \{(\ker D^U)_\pm \}_{U \in \mathcal{U}} \tag{3.16}
\]

If \((\ker D)_+\) and \((\ker D)_-\) were vector bundles then (3.15) would be a well-defined element in \(K(\mathcal{U})\), the K-theory of \(\mathcal{U}\). However, as in the continuum setting, the
dimensions of \((\ker D^U)_+\) and \((\ker D^U)_-\) can jump as \(U\) varies, even though their difference, index \(D^U\), remains constant. In the continuum some trickery is required to deal with this aspect so as to make the index bundle into a well-defined element in the K-theory (see, e.g., [2]). But in the lattice setting this aspect can be dealt with in a simple way, by exploiting the finite-dimensionality of \(C\), as follows. Besides the usual chiral decomposition,

\[
C = C_+ \oplus C_- \quad (\gamma_5 = \pm 1 \text{ on } C_\pm), \tag{3.17}
\]

there is another decomposition determined by \(e^U\) (which played a central role in the overlap formalism [4], see also [23, 24]):

\[
C = \hat{C}_+^U \oplus \hat{C}_-^U \quad (-e^U = \pm 1 \text{ on } \hat{C}_\pm^U) \tag{3.18}
\]

By (3.8),

\[
-\epsilon^U = \gamma_5 (1 - aD^U) \tag{3.19}
\]

which looks formally like a gauge field-dependent lattice deformation of \(\gamma_5\), hence the \(\pm\) convention in (3.18). It follows from (3.19) that

\[
(\ker D^U)_\pm \subset \hat{C}_\pm^U \cap C_\pm \tag{3.20}
\]

In light of this we can define \(V^U\) to be the orthogonal complement of \((\ker D^U)_+\) in \(\hat{C}_+^U\), and \(W^U\) the orthogonal complement of \((\ker D^U)_-\) in \(C_-\), i.e.

\[
\hat{C}_+^U = (\ker D^U)_+ \oplus V^U \tag{3.21}
\]

\[
C_- = (\ker D^U)_- \oplus W^U. \tag{3.22}
\]

From (3.8) (or (3.12) and (3.19)) we get

\[
\gamma_5 D^U = D^U e^U \tag{3.23}
\]

which implies that \(D^U\) maps \(\hat{C}_\pm^U\) to \(C_\mp\).

\textit{Lemma 1.} The map \(D^U : \hat{C}_+^U \to C_-\) restricts to an isomorphism \(D^U : V^U \cong W^U\).
Proof. It suffices to show that the orthogonal complement of the image of $D^U$ on $\mathcal{C}$ is precisely $\ker D^U$. But this is a simple consequence of the $\gamma_5$-hermiticity property (3.13) and the invariance of $\ker D^U$ under $\gamma_5$. ■

Now set
\begin{equation}
V = \{V^U\}_{U \in \mathcal{U}}, \quad W = \{W^U\}_{U \in \mathcal{U}} \tag{3.24}
\end{equation}

These are not vector bundles in general, since $\dim V^U$ and $\dim W^U$ can jump as $U$ varies, but we will formally treat them as vector bundles in the following. By Lemma 1, $V$ and $W$ are isomorphic. Using this and (3.21)–(3.22), formal K-theoretic manipulations give
\begin{align*}
\text{index } D &= (\ker D)_+ - (\ker D)_- \\
&\cong (\ker D)_+ \oplus V - (\ker D)_- \oplus V \\
&\cong (\ker D)_+ \oplus V - (\ker D)_- \oplus W \\
&= \widehat{\mathcal{C}}_+ - \mathcal{C}_- \tag{3.25}
\end{align*}

where
\begin{equation}
\widehat{\mathcal{C}}_+ = \{\widehat{\mathcal{C}}^U\}_{U \in \mathcal{U}} \tag{3.26}
\end{equation}

and $\mathcal{C}_-$ in (3.25) is to be interpreted as the trivial vector bundle over $\mathcal{U}$ with fibre $\mathcal{C}_-$. Unlike the initial expression, the final expression (3.25) for index $D$ is a well-defined element in $K(\mathcal{U})$ due to the following:

Lemma 2. $\widehat{\mathcal{C}}_+$ is a smooth vector bundle over $\mathcal{U}$.

Proof. Since $\dim V^U = \dim W^U$ we have
\begin{equation}
\text{index } D^U = \dim \widehat{\mathcal{C}}^U_+ - \dim \mathcal{C}_- \tag{3.27}
\end{equation}

hence $\dim \widehat{\mathcal{C}}^U_+$ is locally constant under variations of $U$. Furthermore, $\widehat{\mathcal{C}}_+ = \mathcal{P}(\mathcal{C})$ where $\mathcal{C}$ is to be interpreted as the trivial vector bundle over $\mathcal{U}$ with fibre $\mathcal{C}$ and $\mathcal{P} : \mathcal{C} \to \mathcal{C}$ is the smooth vector bundle map defined on the fibres by
\begin{equation}
P^U = \frac{1}{2}(1 - e^U) \tag{3.28}
\end{equation}
Then by a standard mathematical result (see, e.g., Prop. 1.3.2.(ii) of [25]), \( \hat{C}_+ \) is a smooth vector bundle over \( U \).

We therefore take (3.25) as the definition of the index bundle of \( D \). Due to gauge covariance it descends to an element in the K-theory of the orbit space, i.e.

\[
\text{index } D \in K(U/G) \quad (3.29)
\]

Due to the triviality of \( C_- \) in (3.25), the topology of the index bundle is determined solely by \( \hat{C}_+ \). In particular, for the nonzero degree parts of the Chern character we have

\[
\text{ch}_n(C_-) = 0 \quad \text{and} \quad 3\text{ch}_n(\text{index } D) = \text{ch}_n(\hat{C}_+ - C_-) = \text{ch}_n(\hat{C}_+) \quad (n \geq 1) \quad (3.30)
\]

Thus the topological charge of the index bundle over general \( 2n \)-dimensional spheres in the orbit space, which is obtained by integrating the Chern character, is the same as for the vector bundle \( \hat{C}_+ \) for \( n \geq 1 \). We remark that, since \( \hat{C}_+ \) is a unitary vector bundle (with unitary structure determined by the inner product (2.1)), its topological charge over odd-dimensional spheres always vanishes. So it is only the even-dimensional case that is of interest.

4 Topological charge of the index bundle over \( 2n \)-spheres in the orbit space

Generically, a \( 2n \)-sphere in \( U/G \) can be presented as a \( 2n \)-dimensional family \( U(\theta,t) \) of lattice gauge fields, parameterised by the \( 2n \)-ball \( B^{2n} \) with boundary \( S^{2n-1} \) (\( \theta \in S^{2n-1} \) and \( t \) is the radial coordinate in \( B^{2n} \)), with all the boundary points \( U(\theta,1) \) being gauge equivalent, i.e. \( U(\theta,1) = \phi_\theta \cdot U \) \((U \equiv U(0,1))\) for some family \( \{\phi_\theta\}_{\theta \in S^{2n-1}} \) of lattice gauge transformations. Set \( P^{(\theta,t)} = P^{U(\theta,t)} \) (given by (3.28)). The main result of this paper is the following formula, which can be regarded as a “lattice families index theorem”:

\[\text{index } D = \dim \hat{C}^U_+ - \dim C_- = \text{index } D^U.\]

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3The zero degree part of the Chern character is just \( \text{ch}_0(\text{index } D) = \dim \hat{C}^U_+ - \dim C_- = \text{index } D^U \).
Theorem 1. The topological charge of \( \hat{C}_+ \) and therefore of the index bundle – over the above \( 2n \)-sphere in \( \mathcal{U}/\mathcal{G} \) (\( n \geq 1 \)) is

\[
Q_{2n} = \frac{1}{(2\pi i)^n} \left( \frac{1}{n!} \int_{B^{2n}} \text{Tr} \left[ P^{(\theta,t)} (dP^{(\theta,t)})^{2n} \right] - \frac{(1)^n (n-1)!}{2 (2n-1)!} \int_{S^{2n-1}} \text{Tr} \left[ \mathcal{U}^{(\phi^{-1}d\phi)}^{2n-1} \right] \right) + 2 \sum_x \text{deg}(\phi(x))
\]

(4.1)

where \( \text{Tr} \) is the trace for linear operators on \( \mathcal{C} \), \( d = d_\theta + d_t \) is the exterior derivative on \( B^{2n} \) and \( \text{deg}(\phi(x)) \) is the degree of the map \( S^{2n-1} \to \text{SU}(N), \theta \mapsto \phi_\theta(x) \).

Remark. The last term \( 2 \sum_x \text{deg}(\phi(x)) \) vanishes in the \( n = 1 \) case (since \( \text{SU}(N) \) contains no noncontractible circles), but can be nonvanishing for \( 2 \leq n \leq N \). This even integer-valued term has no continuum analogue. The rest of (4.1) reduces to the topological charge of the continuum Dirac index bundle in the classical continuum limit, cf. §5. We will also see there that when \( \mathcal{G} \) is replaced by the constrained group of gauge transformations \( \mathcal{G}_0 \), for which the orbit space is a smooth manifold, the term \( 2 \sum_x \text{deg}(\phi(x)) \) gives no contribution to the classical continuum limit.

Proof of Theorem 1. The restriction of the vector bundle \( \hat{C}_+ \) to the \( 2n \)-sphere in \( \mathcal{U}/\mathcal{G} \) has the following characterisation. Set \( \hat{C}_+^{(\theta,t)} := \hat{C}_U^{(\theta,t)} \) and define the vector bundle

\[
E := \{ \hat{C}_+^{(\theta,t)} \} \}_{(\theta,t) \in B^{2n}}
\]

(4.2)

over \( B^{2n} \). The action of the gauge transformation \( \phi_\theta \) on \( \mathcal{C} \) restricts to an isomorphism

\[
\phi_\theta : \hat{C}_+^U \xrightarrow{\sim} \hat{C}_+^{(\theta,1)}
\]

(4.3)

This determines an equivalence relation \( \sim \) identifying each \( \hat{C}_+^{(\theta,1)} \) with \( \hat{C}_+^U \). The bundle \( \hat{C}_+ \) over the \( 2n \)-sphere in \( \mathcal{U}/\mathcal{G} \) is then given by \( E/\sim \). Topologically, this is equivalent to the bundle over \( S^{2n} \) constructed as follows. Let \( \tilde{B}^{2n} \) denote another copy of the \( 2n \)-ball, with coordinates \( (\theta,s) \), and let \( \tilde{E} \) denote the trivial bundle over \( \tilde{B}^{2n} \) with fibre \( \hat{C}_+^U \). Then \( E \) and \( \tilde{E} \) can be glued together along the common boundary \( S^{2n-1} \) of \( B^{2n} \) and \( \tilde{B}^{2n} \) via (4.3) to get a bundle \( \mathcal{E} \) over \( S^{2n} = B^{2n} \cup_{S^{2n-1}} \tilde{B}^{2n} \) with the same topology as the restriction of \( \hat{C}_+ \) to the \( 2n \)-sphere in \( \mathcal{U}/\mathcal{G} \).
To get a formula for the topological charge we introduce a covariant derivative (connection) in $\mathcal{E}$. With $d = d_\theta + d_t$,

$$\nabla^{(\theta,t)} = P^{(\theta,t)} \circ d \circ P^{(\theta,t)}$$

(4.4)
is a connection in the bundle $E$ over $B^{2n}$, and, with $d = d_\theta + d_s$,

$$\tilde{\nabla}^{(\theta,s)} = P^U \circ (d + f(s)\phi^{-1}_\theta d_\theta \phi_\theta) \circ P^U$$

(4.5)
is a connection in the bundle $\tilde{E}$ over $\tilde{B}^{2n}$, where $f$ is some cutoff function with $f(s) = 1$ and $f(s) = 0$ in neighborhoods of $s = 1$ and $s = 0$, respectively. A simple calculation shows that

$$\nabla^{(\theta,1)} = \phi_\theta \circ \tilde{\nabla}^{(\theta,1)} \circ \phi^{-1}_\theta$$

(4.6)

Consequently, $\nabla$ and $\tilde{\nabla}$ fit together to give a connection in $\mathcal{E}$. Hence the topological charge of this bundle is

$$Q_{2n} = \frac{1}{(2\pi i)^n n!} \left[ \int_{B^{2n}} \text{Tr} (F_\nabla)^n - \int_{\tilde{B}^{2n}} \text{Tr} (F_{\tilde{\nabla}})^n \right]$$

(4.7)

where $F_\nabla$ and $F_{\tilde{\nabla}}$ are the curvatures of $\nabla$ and $\tilde{\nabla}$, respectively.

Using the fact that for general connection $\nabla$ we have $\nabla \circ \nabla = F_\nabla$ (i.e. wedge multiplication by $F_\nabla$), simple calculations in the present case give

$$F_\nabla^{(\theta,t)} = P^{(\theta,t)} dP^{(\theta,t)} dP^{(\theta,t)}$$

(4.8)

$$\quad (F_\nabla)^n = P(dP)^{2n}$$

(4.9)
and

$$F_{\tilde{\nabla}}^{(\theta,s)} = \left[ f'(s) ds \wedge \phi^{-1}_\theta d_\theta \phi_\theta + f(s)(f(s) - 1)(\phi^{-1}_\theta d_\theta \phi_\theta)^2 \right] P^U$$

(4.10)

$$\quad (F_{\tilde{\nabla}})^n = nf'(s)(f(s)(f(s) - 1))^{n-1} ds \wedge (\phi^{-1}_\theta d_\theta \phi_\theta)^{2n-1} P^U$$

(4.11)

where we have used the fact that $\phi^{-1}_\theta d_\theta \phi_\theta$ maps $\tilde{C}_+^U$ to itself, i.e. $[\phi^{-1}_\theta d_\theta \phi_\theta, P^U] = 0$. After substituting these in (4.7), the radial parameter in the second integral can be integrated out to obtain

$$Q_{2n} = \frac{1}{(2\pi i)^n n!} \left( \int_{B^{2n}} \text{Tr} [P(dP)^{2n}] + \chi(n) \int_{S^{2n-1}} \text{Tr} [P^U (\phi^{-1}_\theta d_\theta \phi_\theta)^{2n-1}] \right)$$

(4.12)
where
\[
\chi(n) = -n \int_0^1 f'(s)(f(s)(f(s) - 1))^{n-1} ds = (-1)^n \frac{n!(n-1)!}{(2n-1)!}
\] (4.13)

To get the second equality in (4.13), note that the integral depends only on the values of \(f(s)\) at \(s=0\) and \(s=1\); to evaluate it we can therefore replace \(f(s)\) by \(s\).

The first term in (4.12) is the first term in (4.1). Substituting \(P^U = \frac{1}{2}(1 - \epsilon^U)\) the second term in (4.12) becomes
\[
\frac{\chi(n)}{2(2\pi i)^n n!} \left( \int_{S^{2n-1}} \text{Tr} \left[ (\phi^{-1}_\theta d_\theta \phi_\theta)^{2n-1} \right] - \int_{S^{2n-1}} \text{Tr} \left[ \epsilon^U (\phi^{-1}_\theta d_\theta \phi_\theta)^{2n-1} \right] \right). \tag{4.14}
\]
The second term here is the second term in (4.1). The first term here can be rewritten as
\[
\frac{\chi(n)}{2(2\pi i)^n n!} \int_{S^{2n-1}} \sum_x 4 \text{tr} \left[ (\phi_\theta(x)^{-1} d_\theta \phi_\theta(x))^{2n-1} \right] \tag{4.15}
\]
where \(\text{tr}\) is the trace for the fundamental representation of SU(N) and the factor 4 comes from the trivial trace over spinor indices. To show that this coincides with the last term in (4.1), thereby completing the proof of the Theorem, it remains to show that
\[
\frac{\chi(n)}{(2\pi i)^n n!} \int_{S^{2n-1}} \text{tr} \left[ (\phi_\theta(x)^{-1} d_\theta \phi_\theta(x))^{2n-1} \right] = \text{deg}(\phi(x)) \tag{4.16}
\]
To get this, note that
\[
\text{deg}(\phi(x)) = \frac{-1}{(2\pi i)^n n!} \int_{\tilde{B}^{2n}} \text{tr} \tilde{F}^n \tag{4.17}
\]
where \(\tilde{F}\) is the curvature of the gauge field \(\tilde{A}(\theta, s) = f(s)\phi^{-1}_\theta(x) d_\theta \phi_\theta(x)\) on \(\tilde{B}^{2n}\), since this is gauge-equivalent to zero at the boundary \(S^{2n-1}\) of \(\tilde{B}^{2n}\) via the gauge transformation \(\theta \mapsto \phi^{-1}_\theta(x)\). After integrating out the radial parameter in \(\tilde{B}^{2n}\), (4.17) reduces to the left-hand side of (4.16). This completes the proof of Theorem 1.

\[4\text{Note that } \tilde{F}^n \text{ is given by (4.11) without the } P^U \text{ and with } \phi_\theta \mapsto \phi_\theta(x).\]
5 Classical continuum limit

In this section we consider the situation where the $2n$-sphere $\{U^{(\theta,t)}\}$ in $U/G$ is the lattice transcript of a generic $2n$-sphere $\{A^{(\theta,t)}\}$ in the orbit space $A/G$ of smooth continuum SU(N) gauge fields on the 4-torus. I.e. $A^{(\theta,t)} = \phi_\theta \cdot A$ ($A \equiv A^{(0,1)}$) where $\{\phi_\theta : T^4 \to SU(N) , \theta \in S^{2n-1}\}$ corresponds to a smooth map $\Phi : S^{2n-1} \times T^4 \to SU(N)$ via $\phi_\theta(x) = \Phi(\theta, x)$. In this case it is well-known that the topological charge of the continuum Dirac index bundle over the $2n$-sphere in $A/G$ equals the degree $\deg(\Phi)$ of the map $\Phi$. (This follows, e.g., from the result in [1].) We announce and outline the argument for the following result, which will be shown in [2]:

**Theorem 2.** (i) The first two terms in the lattice families index formula (4.1) reduce to the topological charge of the continuum Dirac index bundle over the $2n$-sphere in $A/G$ in the classical continuum limit, i.e.

$$
\lim_{a \to 0} \frac{1}{(2\pi)^n} \left( \frac{1}{n!} \int_{B^{2n}} \text{Tr} \left[ P(dP)^{2n} \right] - \frac{(-1)^n}{2} \frac{(n-1)!}{(2n-1)!} \int_{S^{2n-1}} \text{Tr} \left[ \epsilon U(\phi^{-1}_\theta d_\theta \phi_\theta)^{2n-1} \right] \right) = \deg(\Phi)
$$

(5.1)

(ii) When the gauge transformations are constrained to belong to $G_0 := \{ \phi \in G | \phi(x_0) = 1\}$, where $x_0$ is an arbitrary basepoint in $T^4$, the remaining term $2 \sum_x \deg(\phi(x))$ in the lattice families index formula (4.1) vanishes.

**Corollary.** The topological charge of the index bundle of the Overlap lattice Dirac operator over the lattice transcript of a generic $2n$-sphere in the continuum orbit space $A/G_0$ coincides with the continuum topological charge $\deg(\Phi)$ when the lattice is sufficiently fine.

Theorem 2 (i) and (ii) together imply that the topological charge of the lattice index bundle reduces to that of the continuum index bundle over the $2n$-sphere in $A/G_0$ in the $a \to 0$ limit (classical continuum limit). Since the lattice and continuum topological charges are both integers, the Corollary follows. We remark that the gauge field orbit space specified by the constrained gauge transformations is a smooth manifold, since $G_0$ (unlike $G$) acts freely on the space of gauge fields. (For this
reason the gauge transformations were constrained to belong to $G_0$ in the study of the continuum Dirac index bundle in [1].

In the following we outline the proof of Theorem 2 (the details will be given in [12]). We begin by considering the term $2 \sum_x \text{deg}(\phi(x))$ in the lattice families index formula (1.1) in the present case where the $2n$-sphere in the lattice orbit space is the lattice transcript of the $2n$-sphere in the continuum orbit space. I.e. $\phi_\theta$ is the lattice transcript of a $S^{2n-1}$-family of continuum gauge transformations, also denoted $\phi_\theta$, and $U$ is the lattice transcript of $A$. Since $\phi_\theta(x)$ depends smoothly on $x$, and $T^4$ is connected, the degree $\text{deg}(\phi(x))$ of the map $S^{2n-1} \rightarrow \text{SU}(N), \theta \mapsto \phi_\theta(x)$ is independent of $x$. Denoting this by $\text{deg}(\phi)$, we therefore have $2 \sum_x \text{deg}(\phi(x)) = 2N \text{deg}(\phi)$ where $N$ = the number of lattice sites. Thus this term diverges in the continuum limit if $\text{deg}(\phi) \neq 0$, but vanishes when $\text{deg}(\phi) = 0$. The latter occurs when the gauge transformations belong to $G_0$, since in this case $\phi_\theta(x_0) = 1$ for all $\theta \in S^{2n-1}$, hence $\text{deg}(\phi) = \text{deg}(\phi(x_0)) = 0$. This shows Part (ii) of the Theorem.

The proof of Part (i) of the Theorem is based on the following technical result, which we announce here and prove in [12]:

\[
\lim_{a \rightarrow 0} \int_{B^{2n}} \text{Tr} \left[ (dP)^{2n} P \right] = \frac{1}{(2\pi i)^2} \int_{B^{2n} \times T^4} \text{tr} \left[ dt A^{(\theta,t)}(x) dx^\mu \wedge (d_\theta A^{(\theta,t)}(x) dx^\nu)^{n-1} \wedge F^{(\theta,t)}(x) \right]
\]

(5.2)

and

\[
\lim_{a \rightarrow 0} \int_{S^{2n-1}} \text{Tr} \left[ (\phi^{-1}_\theta d_\theta \phi_\theta)^{2n-1} \epsilon U \right] = \frac{1}{2(2\pi i)^2} \int_{S^{2n-1} \times T^4} \text{tr} \left[ (\phi^{-1}_\theta(x) d_\theta \phi_\theta(x))^{2n-1} \wedge F(x)^2 \right]
\]

(5.3)

where $F^{(\theta,t)}$ and $F$ are the curvatures of $A^{(\theta,t)}$ and $A$ (= $A^{(0,1)}$), respectively. In the $n = 1$ case this has essentially already been derived in a related context in [24, 10]. The derivation for general $n$ is essentially a generalisation of the one given there. Now, substituting these into the left-hand side of (5.4) gives

\[
\frac{1}{(2\pi i)^{n+2}} \left[ \frac{1}{n!} \int_{B^{2n} \times T^4} \text{tr} \left[ dt A^{(\theta,t)}(x) dx^\mu \wedge (d_\theta A^{(\theta,t)}(x) dx^\nu)^{n-1} \wedge F^{(\theta,t)}(x) \right] \right. - \left. \frac{(-1)^n n!}{4} \int_{S^{2n-1} \times T^4} \text{tr} \left[ (\phi^{-1}_\theta(x) d_\theta \phi_\theta(x))^{2n-1} \wedge F(x)^2 \right] \right]
\]

(5.4)

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On the other hand, the degree of $\Phi$ can be expressed as

$$
\text{deg}(\Phi) = \frac{1}{(2\pi i)^{n+2}(n+2)!} \left[ \int_{B^{2n} \times T^4} \text{tr} \mathcal{F}^{n+2} - \int_{\tilde{B}^{2n} \times T^4} \text{tr} \tilde{\mathcal{F}}^{n+2} \right]
$$

(5.5)

where $\mathcal{F}, \tilde{\mathcal{F}}$ are the curvatures of the gauge fields

$$
\mathcal{A}(\theta,t,x) = A^{(\theta,t)}(x) \, dx^\mu, \quad \tilde{\mathcal{A}}(\theta,s,x) = A_\mu(x) \, dx^\mu + f(s)\phi^{-1}(x) \, d_\theta \phi\theta(x)
$$

(5.6)

on $B^{2n} \times T^4$ and $\tilde{B}^{2n} \times T^4$, respectively, since these are related at the common boundary $S^{2n-1} \times T^4$ by the gauge transformation $\Phi : S^{2n-1} \times T^4 \to \text{SU}(N)$. From (5.6) we calculate

$$
\mathcal{F}^{n+2} = (n+1)(n+2) d_t A^{(\theta,t)}(x) \, dx^\mu \wedge (d_\theta A^{(\theta,t)}(x) \, dx^\nu)^{n-1} \wedge F^{(\theta,t)}(x)
$$

(5.7)

$$
\tilde{\mathcal{F}}^{n+2} = (n+1)(n+2) \frac{n}{2} f'(s)(f(s)(f(s)-1))^{n-1} F(x)^2 \wedge ds \wedge (\phi^{-1}(x) \, d_\theta \phi\theta(x))^{2n-1}
$$

(5.8)

After substituting these in (5.5) and integrating out the radial parameter in $\tilde{B}^{2n}$ in the second integral, (5.4) is obtained, thereby establishing (5.1).

6 Relation to gauge anomalies in lattice chiral gauge theory

In the continuum, the chiral fermion determinant is not really a function of the gauge field, but rather a section in the $\text{U}(1)$ determinant line bundle associated with the index bundle of the Dirac operator over the space of gauge fields (see, e.g., [1, 3] for the details of this and the following remarks). To get the determinant as a function of the gauge field, a choice of trivialisation of the determinant line bundle must be made. Different choices of trivialisations correspond to different (gauge field-dependent) complex phase choices for the determinant. By gauge covariance, the determinant line bundle descends to a line bundle over the orbit space of gauge fields, and the trivialisations which lead to a gauge-invariant chiral fermion determinant are precisely those which descend to trivialisations of the line bundle over the orbit space. Therefore, the obstructions to getting a gauge-invariant chiral fermion determinant are precisely the obstructions to trivialising the determinant line bundle over the
orbit space. The primary obstructions are the obstructions to trivialising the line bundle over the 2-spheres in the orbit space (cf. the discussion in the final section of [3]). Since the Chern character of the determinant line bundle coincides with the first Chern character $\text{ch}_1(\text{index } \mathcal{O})$ of the index bundle (i.e. the degree 2 part of the total character $\text{ch}(\text{index } \mathcal{O})$), these obstructions are the topological charges of the index bundle over the 2-spheres in the orbit space. For the generic 2-spheres considered in the preceding section, which correspond to maps $\Phi : S^1 \times T^4 \rightarrow SU(N)$, the topological charge coincides with the winding number obstruction studied in [3] (i.e. the winding number of the phase of $\det(\mathcal{D}_{\theta} \cdot A)$ as $\theta$ goes around $S^1$).

The situation for the lattice version of the chiral fermion determinant arising in the overlap formulation [4] is completely analogous. It is again a section in a $U(1)$ determinant line bundle [4, 7] (see also [10] for further discussion). The line bundle is $\Lambda_{\text{max}}^+ \hat{\mathcal{C}}_+ \otimes (\Lambda_{\text{max}}^- \mathcal{C}_-)^*$. But this is precisely the determinant line bundle associated with the index bundle (3.25) of the Overlap lattice Dirac operator that we have introduced in §3. Thus the primary obstructions to the existence of a gauge-invariant phase choice for the overlap chiral fermion determinant are the topological charges of the lattice index bundle over the 2-spheres in the orbit space of lattice gauge fields. For generic 2-spheres, these coincide with the winding number obstructions for the overlap determinant studied in [10]. Indeed, the formula for the topological charge $Q_2$, given by the $2n = 2$ case our families index formula (1.1), coincides with the winding number formula Eq. (3.11) of [10].

When the gauge group is $SU(N)$ with $N \geq 3$ there are always nonvanishing obstructions of the primary type discussed above. This is because of the existence of topologically nontrivial maps $\Phi : S^1 \times T^4 \rightarrow SU(N)$ for all $N \geq 3$. Hence the determinant line bundle over the orbit space cannot be trivialised in these cases and no

\footnote{In fact the lattice version of the chiral fermion determinant in the overlap formulation equals the determinant of $D : \hat{\mathcal{C}}_+ \rightarrow \mathcal{C}_-$ where $D$ is the overlap Dirac operator [26, 10].}

\footnote{The topological structure of $SU(N)$ is $S^3 \times S^5 \times \cdots \times S^{2N-1}$ modulo a finite set of equivalence relations. Thus when $N \geq 3$ these is always an $S^5$ factor and topologically nontrivial maps $\Phi : S^1 \times T^4 \rightarrow SU(N)$ can be constructed from maps $S^1 \times T^4 \rightarrow S^5$ with nonvanishing degree.}
gauge-invariant phase choice for the chiral fermion determinant (with fermion in the fundamental representation) exists. The same is true in the lattice setting, at least when the lattice is sufficiently fine, due to the classical continuum limit result of [10] (or equivalently, the $2n=2$ case of Theorem 2 of the present paper).

On the other hand, for gauge group SU(2) there are no topologically nontrivial maps $\Phi : S^1 \times T^4 \to SU(2)$. Hence the primary obstructions all vanish in the continuum setting. There might still be nonvanishing primary obstructions in the lattice setting though; all we can say for certain at present is that they vanish in the classical continuum limit. However, there is another obstruction, namely Witten’s global gauge anomaly [27]. The presence of this obstruction in the SU(2) theory in the lattice setting has been demonstrated both numerically [6, 8] and analytically (in the classical continuum limit) in [9]. This obstruction also has a natural description in the context of families index theory for the Dirac operator: There is a canonical trivialisation of the U(1) determinant line bundle over 1-dimensional balls (i.e. line segments) in the space of gauge fields; it specifies a real line bundle which descends to a real line bundle over circles in the orbit space with structure group $O(1) \cong \mathbb{Z}_2$. (The circles come from line segments with gauge equivalent end points.) The global gauge anomaly is the obstruction to trivialising this bundle. In the continuum setting this description is implicit in [27]. This viewpoint on the global anomaly, and related issues, will be discussed in detail in the lattice setting in [14].

7 Concluding remarks

(1) We have shown that the index bundle of the Overlap lattice Dirac operator over the orbit space of SU(N) lattice gauge fields can be constructed in a natural way, and have derived a formula (“lattice families index theorem”) for its topological charge over generic $2n$-spheres in the orbit space. An unanticipated feature is the presence

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7The nonvanishing of Witten’s global anomaly in the continuum in the SU(2) case is related to the fact that the space of maps $T^4 \to SU(2)$ has two connected components, i.e. there are maps (gauge transformations) which cannot be continuously deformed to the trivial map (see [24]).
(when $2 \leq n \leq N$) of an integer-valued term which has no continuum analogue and which generally diverges in the classical continuum limit. However, when the gauge transformations are constrained to belong to $G_0 = \{ \phi \in G \mid \phi(x_0) = 1 \}$ (as was also done in [1] to make the orbit space a smooth manifold), this term vanishes for the $2n$-spheres in $U/G_0$ which are lattice transcripts of $2n$-spheres in the continuum orbit space $\mathcal{A}/G_0$. The rest of the formula reduces in the classical continuum limit to the topological charge of the continuum index bundle over $2n$-sphere in the continuum orbit space. (This was announced and outlined here; the key technical part of the argument will be given in [12].) Thus we have seen how topology of the continuum Dirac index bundle over $\mathcal{A}/G_0$ can be captured in a finite-dimensional lattice setting.

An implication of this is that $2n$-spheres in $U/G_0$ which arise as lattice transcripts of noncontractible $2n$-spheres in $\mathcal{A}/G_0$ (i.e. those with $\text{deg}(\Phi) \neq 0$) are again noncontractible. (For if this were not the case, the lattice index bundle over the $2n$-sphere would be trivialisable and hence have vanishing topological charge, in contradiction with the classical continuum limit result, Theorem 2.) All this provides further evidence that the orbit space of lattice gauge fields is a good finite-dimensional model for the orbit space of continuum SU(N) gauge fields on an even-dimensional torus, the topology of which is of considerable mathematical interest and potential physical relevance. We emphasize that, a priori, it is not at all clear that this should be the case. The situation is complicated by the fact that the measure-zero subspace of lattice gauge fields $U$ for which $H^U$ has zero-modes needs to be excised from $U$, and it is difficult to say anything at all about what the resulting topological sectors of $U$ look like. Although the lattice orbit space seems to be reproducing the topology of the continuum one, the situation before modding out by the gauge transformations is quite different: In the continuum, the topological sectors are just affine $\infty$-dimensional vector spaces with no nontrivial topology; however, in [13] we will show that the topological sectors of $U$ do have nontrivial topology no matter how fine the lattice is:

Using the results of the present paper we will show that they contain noncontractible

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8Earlier evidence for this came from the demonstrations that the lattice theory reproduces the global gauge anomaly and obstructions to the vanishing of local gauge anomalies [4, 5, 8, 9, 10].
spheres of various dimensions. These are not simply due to the presence of noncontractible spheres in SU(N) but directly reflect the topological structure of \( U \) resulting from excising the \( U \)'s for which \( H^U \) has zero-modes. (E.g. in the SU(2) case we find that there are noncontractible circles.) This is closely connected with the existence question for \( G_0 \) gauge-fixings in the lattice theory, which we also discuss in [13].

(2) The topological charge of the continuum Dirac index bundle over a \( 2n \)-sphere in the orbit space can be expressed as the index of a Dirac operator in \( 2n+4 \) dimensions, or \( 2n+2m \) dimensions when the dimension of the spacetime is \( 2m \) (see, e.g., [3]). There is an analogous result in the present lattice setting. It has already been given in the \( n=1 \) case in [11]; the generalisation to arbitrary \( n \) is straightforward and will be given in [12].

(3) The construction of the lattice index bundle, and the derivation of the families index formula (4.1), go through for general “Overlap-type” lattice Dirac operators of the form \( D^U = \frac{1}{a}(1 + \gamma_5 e^U) \) where \( e^U : \mathcal{C} \rightarrow \mathcal{C} \) is any operator depending smoothly and gauge-covariantly on \( U \) and with the properties \((e^U)^2 = 1 \) and \((e^U)^* = e^U \). Such \( D \) are precisely the solutions to the GW relation (3.12) with the \( \gamma_5 \) hermiticity property (3.13). Besides the Overlap Dirac operator (3.8), another solution is the lattice Dirac operator resulting from the perfect action approach of Ref. [28], although this is given via recursion relations and no closed form expression is known. Other solutions have been presented in [29]. However, at present we can only say with certainty that the classical continuum limit result, Theorem 2, holds for the case where \( D \) is the Overlap Dirac operator.

(4) From a mathematical viewpoint, an obvious question is how to generalise the constructions and results of this paper to spacetime manifolds other than the tori. The problem is to find suitable generalisations of the “naive” lattice Dirac operator \( \nabla^U \) in (3.1) and the lattice Laplace operator \( \Delta^U \) in (3.4); these can then be fed into the formulae (3.3)–(3.8) to get the discrete Dirac operator \( D^U \) and the constructions and derivation of the families index formula go through as before, cf. Remark 3 above. (A suitable range for the parameter \( m \) would also need to be determined.)
Given a polyhedral cell decomposition of a general spacetime manifold, a discrete Laplace operator $\Delta$ is easy to construct (e.g. along the lines of [30]). Constructing the discrete $\nabla$ seems less easy though. For this one needs to find a way to incorporate the spin structure into the discrete setting. In the tori case the spin structure is particularly simple and easily incorporated into the discrete setting with hyper-cubic cell decomposition, cf. (3.1). How to do this in the general case is less obvious. This is a problem for future work.

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