JORDAN DERIVATIONS AND LIE DERIVATIONS ON PATH ALGEBRAS

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ABSTRACT. Without the faithful assumption, we prove that every Jordan derivation on a class of path algebras of quivers without oriented cycles is a derivation and that every Lie derivation on such kinds of algebras is of the standard form.

1. Introduction

Let $R$ be a commutative ring with identity, $A$ be a unital algebra over $R$ and $Z(A)$ be the center of $A$. We set $a \circ b = ab + ba$ and $[a,b] = ab - ba$ for all $a, b \in A$. Recall that an $R$-linear mapping $\Theta$ from $A$ into itself is called a derivation if

$$\Theta(ab) = \Theta(a)b + a\Theta(b)$$

for all $a, b \in A$, a Jordan derivation if

$$\Theta(a \circ b) = \Theta(a) \circ b + a \circ \Theta(b)$$

for all $a \in A$, and a Lie derivation if

$$\Theta([a,b]) = [\Theta(a), b] + [a, \Theta(b)]$$

for all $a, b \in A$. Moreover, in the 2-torsion free case the definition of a Jordan derivation is equivalent to $\Theta(a^2) = \Theta(a)a + a\Theta(a)$ for all $a \in A$. We say a Lie derivation $\Theta$ is standard if it can be expressed as

$$\Theta = D + \Phi,$$  (♠)

where $D$ is an ordinary derivation of $A$ and $\Phi$ is a linear mapping from $A$ into the center $Z(A)$ of $A$.

Jordan derivations and Lie derivations of associative algebras play significant roles in various mathematical areas, in particular in matrix theory, in ring theory and in the theory of operator algebras. There are two common problems in this context:

(a) whether Jordan derivations are derivations;
(b) whether Lie derivations are of the standard form (♠).

Let $A, B$ be two unital algebras over the commutative ring $R$, $M$ be a faithful $(A, B)$-bimodule and $N$ be a $(B, A)$-bimodule. We denote the full matrix algebra
of all $n \times n$ matrices over $\mathcal{R}$, the triangular algebra consisting of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$, and the generalized matrix algebra consisting of $\mathcal{A}, \mathcal{B}, \mathcal{M}$ and $\mathcal{N}$ by
\[
M_{n \times n}(\mathcal{R}), \quad \mathcal{T}_{\mathcal{R}} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} \quad \text{and} \quad \mathcal{G}_{\mathcal{R}} = \begin{bmatrix} A & M \\ \mathcal{N} & B \end{bmatrix}
\]
respectively. In [12] Jacobson and Rickart proved that every Jordan derivation on $M_{n \times n}(\mathcal{R})$ is a derivation. Zhang and Yu [20] obtained that every Jordan derivation on $\mathcal{T}_{\mathcal{R}}$ is a derivation. Xiao and Wei [17] extended this result to the higher case and obtained that any Jordan higher derivation on a triangular algebra is a higher derivation. Alaminos et al [2] showed that every Lie derivation on the full matrix algebra $M_{n \times n}(\mathcal{F})$ of all $n \times n$ matrices over a field $\mathcal{F}$ of characteristic zero has the standard form (♠). Cheung [10] considered Lie derivations of triangular algebras and gave a sufficient and necessary condition which enables every Lie derivations of $\mathcal{T}_{\mathcal{R}}$ to be standard (♠). Yu and Zhang extended Cheung’s work to the nonlinear case [19]. Benkovic investigated the structure of Jordan derivations and Lie derivations from $\mathcal{T}_{\mathcal{R}}$ into its bimodule in [4] and [6]. More recently, the current authors [15, 16] described the structures of Jordan and Lie derivations on the generalized matrix algebra $\mathcal{G}_{\mathcal{R}}$. It was shown that under certain conditions, every Jordan derivation on $\mathcal{G}_{\mathcal{R}}$ can be decomposed as the sum of a derivation and an anti-derivation. We also proved that under some mild assumptions, every Lie derivation on $\mathcal{G}_{\mathcal{R}}$ has the standard form (♠). In [7] Benkovič studied generalized Jordan derivations and generalized Lie derivations of the triangular algebra $\mathcal{T}_{\mathcal{R}}$ and investigated whether they also have the nice properties which are similar to those of Jordan derivations and Lie derivations.

We need to point out that most of existing works related to Jordan derivations and Lie derivations of matrix algebras heavily depend on the faithful condition. For instance, the $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ in $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{G}_{\mathcal{R}}$ is always assumed to be faithful. There is a commonly consensus in the study of related topics. It seems that the assumption concerning the faithfulness is a very natural condition and that without it, one hardly get any useful results. To the best of our knowledge there are no any articles treating Jordan derivations and Lie derivations of matrix algebras without the faithful assumption except for [10]. When studying some harder problems, it sometimes occurs that even the assumption concerning faithfulness is too weak and then the problem has to be approached via some stronger assumptions, for example loyalty ($a \mathcal{M} b = 0$ implies that $a = 0$ or $b = 0$), which was already used when studying the Lie isomorphisms on triangular algebras. Therefore we propose a challenging question:

**Question 1.1.** Without the faithful assumption, what can we say about the Jordan derivations and Lie derivations of matrix algebras?

Path algebras of quivers come up naturally in the study of tensor algebras of bimodules over semisimple algebras. It is well known that any finite dimensional basic $\mathcal{K}$-algebra is given by a quiver with relations when $\mathcal{K}$ is an algebraically closed field. In [11], Guo and Li studied the Lie algebra of differential operators on a path algebra $\mathcal{K}\Gamma$ and related this Lie algebra to the algebraic and combinatorial properties of the path algebra $\mathcal{K}\Gamma$. The main purpose of this paper is to study Jordan derivations and Lie derivations of a class of path algebras of quivers without oriented cycles, which can be viewed as one-point extensions. The distinguished feature of our work is that the faithful assumption is removed. We prove that every
Jordan derivation on a class of path algebras of quivers without oriented cycles is a derivation and that every Lie derivation on such kinds of algebras is of the standard form \( \langle \rangle \).

2. Path algebras and triangular algebras

Let us give a quick review of path algebras of quivers and triangular algebras. For more details, we refer the reader to \([3]\).

2.1. Path algebras. A quiver \( \Gamma \) is an oriented graph. Let us denote the set of vertices by \( \Gamma_0 \) and denote the set of arrows between vertices by \( \Gamma_1 \). Throughout this paper, we always assume that \( \Gamma_0 \) and \( \Gamma_1 \) are both finite sets. In this case, we say that the quiver \( \Gamma = (\Gamma_0, \Gamma_1) \) is finite. If \( \alpha \) is an arrow from the vertex \( i \) to the vertex \( j \), then we write \( s(\alpha) = i \) and \( e(\alpha) = j \). A vertex \( i \) is called a sink if there is no arrow \( \alpha \) such that \( s(\alpha) = i \) and is called a source if there is no arrow \( \alpha \) such that \( e(\alpha) = i \). A nontrivial path in \( \Gamma \) is an ordered sequence of arrows \( p = \alpha_n \cdots \alpha_1 \) such that \( e(\alpha_m) = s(\alpha_{m+1}) \) for each \( 1 \leq m < n \). Define \( s(p) = s(\alpha_1) \) and \( e(p) = e(\alpha_n) \). A trivial path is the symbol \( e_i \) for each \( i \in \Gamma_0 \). In this case, we set \( s(e_i) = e(e_i) = i \).

A nontrivial path \( p \) is called an oriented cycle if \( s(p) = e(p) \). Denote the set of all paths by \( \mathcal{P} \).

Let \( K \) be a field and \( \Gamma \) be a quiver. Then the path algebra \( K\Gamma \) is the \( K \)-algebra generated by the paths in \( \Gamma \) and the product of two paths \( x = \alpha_n \cdots \alpha_1 \) and \( y = \beta_t \cdots \beta_1 \) is defined by

\[
xy = \begin{cases} 
\alpha_n \cdots \alpha_1 \beta_t \cdots \beta_1, & e(y) = s(x) \\
0, & \text{otherwise}
\end{cases}
\]

Clearly, \( K\Gamma \) is an associative algebra with the identity \( 1 = \sum_{i \in \Gamma_0} e_i \), where \( e_i (i \in \Gamma_0) \) are pairwise orthogonal primitive idempotents of \( K\Gamma \).

A relation \( \sigma \) on a quiver \( \Gamma \) over a field \( K \) is a \( K \)-linear combination of paths \( \sigma = \sum_{i=1}^{m} k_i p_i \), where \( k_i \in K \) and \( e(p_1) = \cdots = e(p_n) \). Define \( s(p_1) = \cdots = s(p_n) \). Moreover, the number of arrows in each path is assumed to be at least 2. Let \( \rho \) be a set of relations on \( \Gamma \) over \( K \). The pair \( (\Gamma, \rho) \) is called a quiver with relations over \( K \). Denote by \( \langle \rho \rangle \) the ideal of \( K\Gamma \) generated by the set of relations \( \rho \). The \( K \)-algebra \( K(\Gamma, \rho) = K\Gamma / \langle \rho \rangle \) is always associated with \( (\Gamma, \rho) \). For arbitrary element \( x \in K\Gamma \), write by \( \mathcal{P} \) the corresponding element in \( K(\Gamma, \rho) \). It is well known that every basic finite dimensional algebra over an algebraically closed field \( K \) is isomorphic to some \( K(\Gamma, \rho) \).

**Example 2.1.** Let \( K \) be a field and \( \Gamma \) be the following quiver

\[
\begin{array}{c}
1 \overset{\alpha_1}{\longrightarrow} 2 \overset{\alpha_2}{\longrightarrow} 3 \cdots \rightarrow \alpha_{n-1} \overset{\alpha_n}{\longrightarrow} n
\end{array}
\]

Then \( K\Gamma \) is isomorphic to the upper triangular matrix algebra \( T_n(K) \).

**Example 2.2.** Let \( K \) be a field and \( \Gamma \) be the following quiver

\[
\begin{array}{c}
1 \overset{\alpha}{\longrightarrow} 2 \overset{\beta}{\longrightarrow} 3 \overset{\gamma}{\longrightarrow} 4 \overset{\varepsilon}{\longrightarrow} 5 \overset{\eta}{\longrightarrow} 6
\end{array}
\]

Then \( K\Gamma \) is isomorphic to the upper triangular matrix algebra \( T_6(K) \).
Then a basis of the path algebra $\mathcal{K}\Gamma$ is 

$$\{e_1, e_2, e_3, e_4, e_5, \alpha, \beta, \gamma, \varepsilon, \eta, \varepsilon\beta\alpha, \varepsilon\gamma\alpha, \varepsilon\beta\gamma\varepsilon, \varepsilon\gamma\beta\varepsilon, \varepsilon\gamma\beta\gamma\varepsilon\}. $$

If $\rho = \{\varepsilon\beta\}$, then a basis of $\mathcal{K}(\Gamma, \rho)$ is 

$$\{e_1, e_2, e_3, e_4, e_5, \alpha, \beta, \gamma, \varepsilon, \eta, \beta\alpha, \gamma\alpha, \beta\gamma\alpha, \varepsilon\beta\alpha, \varepsilon\gamma\alpha, \varepsilon\beta\gamma, \varepsilon\gamma\beta, \varepsilon\gamma\beta\gamma\varepsilon\}. $$

2.2. **Triangular algebras and one-point extension.** Let us begin with the definition of triangular algebras over a field $\mathcal{K}$. Let $\mathcal{K}$ be a field and $\mathcal{A}$ and $\mathcal{B}$ two $\mathcal{K}$-algebras. Let $\mathcal{A}\mathcal{M}\mathcal{B}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule. Note that $\mathcal{M}$ need not to be faithful neither as left $\mathcal{A}$-module nor as right $\mathcal{B}$-module here. Then one can define 

$$\mathcal{T}_K = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \mid a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M} \right\}$$

to be an associative algebra under matrix-like addition and matrix-like multiplication. The center of $\mathcal{T}_K$ is 

$$\mathcal{Z}(\mathcal{T}_K) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a \in \mathcal{Z}(\mathcal{A}), b \in \mathcal{Z}(\mathcal{B}), am = mb, \forall m \in \mathcal{M} \right\}. $$

In particular, if $\mathcal{B}$ is equal to the field $\mathcal{K}$, then the triangular algebra $\mathcal{T}_K = [\mathcal{A} \mathcal{M} \mathcal{B}]$ is called a *one-point extension* of $\mathcal{A}$ by the bimodule $\mathcal{A}\mathcal{M}\mathcal{B}$. This terminology comes up in connection with path algebras. For convenience, we explain the reason here. Let $\Lambda = \mathcal{K}(\Gamma, \rho)$ be a finite dimensional path algebra of the quiver $(\Gamma, \rho)$ with relations. Let $i$ be a source in $\Gamma$ and $\mathbf{e}_i$ the corresponding idempotent in $\Lambda$. Note that $\Gamma$ is a quiver without oriented cycles. Clearly, there are no nontrivial paths ending in $i$. This implies that $\mathbf{e}_i\mathbf{e}_i\Lambda\mathbf{e}_i \simeq \mathcal{K}$ and $\mathbf{e}_i\Lambda(1 - \mathbf{e}_i) = 0$. Therefore 

$$\Lambda \simeq \begin{bmatrix} (1 - \mathbf{e}_i)\Lambda(1 - \mathbf{e}_i) & (1 - \mathbf{e}_i)\Lambda e_1 \\ 0 & \mathcal{K} \end{bmatrix}. $$

Let us denote by $(\Gamma', \rho')$ the quiver obtained by removing the vertex $i$ and the relations starting at $i$ and write $\Lambda' = \mathcal{K}(\Gamma', \rho')$. Then $(1 - \mathbf{e}_i)\Lambda(1 - \mathbf{e}_i) \simeq \Lambda'$. Thus $\Lambda$ can be constructed from $\Lambda'$ by adding one point $i$, together with arrows and relations which start at $i$. 

It is worth noting that $(1 - \mathbf{e}_i)\Lambda\mathbf{e}_i$ is not faithful as a left $\Lambda'$-module in general. We illustrate an example here.

**Example 2.3.** Let $\mathcal{K}$ be a field and $\Lambda = \mathcal{K}\Gamma$ a path algebra, where $\Gamma$ is the following quiver 

```
1 ----> 2 ----> 3 ----> 4
\alpha \beta \gamma
```

Obviously, vertex 1 is a source in $\Gamma$. Then 

$$\Lambda \simeq \begin{bmatrix} (1 - e_1)\Lambda(1 - e_1) & (1 - e_1)\Lambda e_1 \\ 0 & \mathcal{K} \end{bmatrix}. $$

It is easy to check that $(1 - e_1)\Lambda e_1$ is not faithful as a left $(1 - e_1)\Lambda(1 - e_1)$ module. 

On the other hand, let us take $e = e_1 + e_2$. It is easy to check that $(1 - e)\Lambda e = 0$. Then the algebra $\mathcal{K}\Gamma$ can also be viewed as a triangular algebra as follows: 

$$\Lambda \simeq \begin{bmatrix} e\Lambda e & e\Lambda(1 - e) \\ 0 & (1 - e)\Lambda(1 - e) \end{bmatrix}. $$

Clearly, $e\Lambda(1 - e)$ is not faithful as left $e\Lambda e$-module and as right $(1 - e)\Lambda(1 - e)$-module.
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3. Jordan Derivations on Path Algebras

Let us first recall some indispensable descriptions concerning derivations and Jordan derivations of the triangular algebra $T_K$.

Lemma 3.1. [10] Lemma 5] An $K$-linear mapping $\Theta$ from $T_K = [A \oplus M \oplus B]$ into itself is a derivation if and only if it has the form

$$\Theta\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) & am_0 - m_0b + \tau_2(m) \\ 0 & \mu_4(b) \end{bmatrix}, \quad \forall \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in T_K,$$

where $m_0 \in M$ and

$$\delta_1 : A \longrightarrow A, \quad \tau_2 : M \longrightarrow M, \quad \mu_4 : B \longrightarrow B$$

are all $K$-linear mappings satisfying the following conditions

1. $\delta_1$ is a derivation of $A$ and $\mu_4$ is a derivation of $B$.
2. $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$ and $\tau_2(mb) = \tau_2(m)b + m\mu_4(b)$.

Lemma 3.2. [11] Lemma 3.2] Let $K$ be a field with $\text{Char}K \neq 2$. An $K$-linear mapping $\Theta$ from $T_K = [A \oplus M \oplus B]$ into itself is a Jordan derivation if and only if it has the form

$$\Theta\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) & am_0 - m_0b + \tau_2(m) \\ 0 & \mu_4(b) \end{bmatrix}, \quad \forall \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in T_K,$$

where $m_0 \in M$ and

$$\delta_1 : A \longrightarrow A, \quad \tau_2 : M \longrightarrow M, \quad \mu_4 : B \longrightarrow B$$

are all $K$-linear mappings satisfying conditions

1. $\delta_1$ is a Jordan derivation of $A$ and $\mu_4$ is a Jordan derivation of $B$.
2. $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$ and $\tau_2(mb) = \tau_2(m)b + m\mu_4(b)$.

It should be remarked that if $\Gamma$ is a quiver without oriented cycles, then the path algebra $K(\Gamma, \rho)$ can be viewed as a one-point extension algebra. We now give the form of Jordan derivations in this background.

Lemma 3.3. Let $T_K = [A \oplus M \oplus B]$ be a one-point extension of $A$ by the bimodule $AM_K$ with $M \neq 0$ and $\Theta$ be a Jordan derivation of $T_K$. Then $\Theta$ has the form

$$\Theta\left(\begin{bmatrix} a & m & k \\ 0 & 0 & k \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) & am_0 - m_0k + \tau_2(m) \\ 0 & 0 \end{bmatrix}, \quad \forall \begin{bmatrix} a & m & k \\ 0 & 0 & k \end{bmatrix} \in T_K,$$

where $m_0 \in M$ and both $\delta_1 : A \longrightarrow A$ and $\tau_2 : M \longrightarrow M$ are $K$-linear mappings satisfying conditions

1. $\delta_1$ is a Jordan derivation of $A$.
2. $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$.

Proof. Clearly, by Lemma 3.2 we only need to show $\mu_4(k) = 0$ for all $k \in K$. In fact, it follows from $\tau_2$ being $K$-linear that $\tau_2(mk) = \tau_2(m)k$ for all $m \in M$ and $k \in K$. Then the condition (2) of Lemma 3.2 implies that $m\mu_4(k) = 0$ for all $k \in K$ and $m \in M$. Note that $M$ is a nonzero $K$-vector space. This forces $\mu_4(k) = 0$ for all $k \in K$. \hfill \Box

Now we are in a position to state the main result of this section.
Theorem 3.4. Let $K$ be a field with $\text{Char} K \neq 2$ and $\Lambda = K(\Gamma, \rho)$ a finite dimensional path algebra over $K$ of the quiver $(\Gamma, \rho)$ with relations. Then every Jordan derivation on $\Lambda$ is a derivation.

Proof. If the quiver only contains one vertex, then $\Lambda \simeq K$. Since $K$ is a field, every Jordan derivation on $K$ is a derivation.

Now suppose that the number of vertices in $\Gamma$ is not less than 2. Note that $\Gamma$ is a quiver without oriented cycles. Take a source $i$ in $\Gamma$ and let $\bar{e}_i$ be the corresponding idempotent in $\Lambda$. Then we have

$$\Lambda \simeq \begin{bmatrix} (1 - \bar{e}_i)\Lambda(1 - \bar{e}_i) & (1 - \bar{e}_i)\Lambda\bar{e}_i \\ 0 & K \end{bmatrix}.$$

Let us denote $(1 - \bar{e}_i)\Lambda(1 - \bar{e}_i)$ by $\Lambda'$. We assert that each Jordan derivation on $\Lambda$ is a derivation if and only if each Jordan derivation on $\Lambda'$ is a derivation. In fact, the assertion holds true when $(1 - \bar{e}_i)\Lambda\bar{e}_i = 0$. If $(1 - \bar{e}_i)\Lambda\bar{e}_i \neq 0$, then the assertion follows Lemma 3.1 and Lemma 3.3. Then it is sufficient to determine whether every Jordan derivation on $\Lambda'$ is a derivation. Note that $\Lambda' \simeq K(\Gamma', \rho')$, where $(\Gamma', \rho')$ is obtained by removing the vertex $i$ from $\Gamma$. We continuously repeat this process and ultimately arrive at the algebra $K$ after finite times, since $\Gamma_0$ is a finite set. Clearly, every Jordan derivation on $K$ is a derivation. This completes the proof. □

Corollary 3.5. Let $K$ be a field with $\text{Char} K \neq 2$ and $\Lambda = K(\Gamma, \rho)$ a finite dimensional path algebra of the quiver $(\Gamma, \rho)$ with relations. Suppose that $\Theta$ is a derivation of $\Lambda$ with $\Theta(x) \in Z(\Lambda)$ for all $x \in \Lambda$. Then $\Theta = 0$.

Proof. Let $i$ be a source in $\Gamma$. Then $\Lambda = K(\Gamma, \rho) \simeq \begin{bmatrix} \Lambda' & (1 - \bar{e}_i)\Lambda\bar{e}_i \\ 0 & K \end{bmatrix}$. Let $\Theta$ be a derivation of $\Lambda$. By Lemma 3.3 and Theorem 3.4 it follows that $\Theta$ has the form

$$\Theta \left( \begin{bmatrix} a & m \\ 0 & k \end{bmatrix} \right) = \begin{bmatrix} \delta_1(a) & am_0 - m_0k + \tau_2(m) \\ 0 & 0 \end{bmatrix}, \quad \forall \begin{bmatrix} a & m \\ 0 & k \end{bmatrix} \in \Lambda,$$

where $m_0 \in (1 - \bar{e}_i)\Lambda\bar{e}_i$ and $\delta_1 : \Lambda' \to \Lambda'$, $\tau_2 : (1 - \bar{e}_i)\Lambda\bar{e}_i \to (1 - \bar{e}_i)\Lambda\bar{e}_i$ are all $K$-linear mappings satisfying conditions

1. $\delta_1$ is a derivation of $\Lambda'$.
2. $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$.

Assume that $\Theta(x) \in Z(\Lambda)$ for all $x \in \Lambda$. Then

$$\Theta \left( \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \right) = \begin{bmatrix} \delta_1(a) & 0 \\ 0 & 0 \end{bmatrix}, \quad \forall \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in \Lambda,$$

where $\delta_1(x) \in Z(\Lambda')$ for all $x \in \Lambda'$. This implies that $\Theta \neq 0$ if and only if $\delta_1 \neq 0$. Since $\Gamma_0$ is a finite set, repeat this process finite times we obtain that $\Theta \neq 0$ if and only if some derivation on $K$ is nonzero. However, every derivation on $K$ is zero. This forces $\Theta$ to be zero. □

Moreover, the proof of Theorem 3.4 implies that one-point extension preserves the property of every Jordan derivation being a derivation.

Corollary 3.6. Let $K$ be a field and $A$ a finite dimensional $K$-algebra with every Jordan derivation being a derivation. Let $\Lambda$ be a one-point extension of $A$ by the bimodule $AM_K$. Then every Jordan derivation of $\Lambda$ is a derivation.
At the end of this section, let us characterize Jordan generalized derivations and generalized Jordan derivations of \( \Lambda \). Recall that a linear map \( f : \Lambda \to \Lambda \) is called a **Jordan generalized derivation** if there exists a linear map \( d : \Lambda \to \Lambda \) such that
\[
f(x \circ y) = f(x) \circ y + x \circ d(y)
\]
for all \( x, y \in \Lambda \), where \( d \) is called an associated linear map of \( f \). A linear map \( f : \Lambda \to \Lambda \) is called a **generalized Jordan derivation** if there exists a linear map \( d : \Lambda \to \Lambda \) such that
\[
f(x \circ y) = f(x)y + f(y)x + xd(y) + yd(x)
\]
for all \( x, y \in \Lambda \). A linear map \( f : \Lambda \to \Lambda \) is called a **generalized derivation** if there exists a linear map \( d : \Lambda \to \Lambda \) such that
\[
f(xy) = f(x)y + xd(y)
\]
for all \( x, y \in \Lambda \).

**Proposition 3.7.** Let \( K \) be a field with \( \text{Char} K \neq 2 \) and \( \Lambda = K(\Gamma, \rho) \) a finite dimensional path algebra of the quiver \( (\Gamma, \rho) \) with relations. Then

1. Every Jordan generalized derivation of \( \Lambda \) is a generalized derivation.
2. Every generalized Jordan derivation of \( \Lambda \) is a generalized derivation.

**Proof.** (1) Let \( f \) be a Jordan generalized derivation on \( \Lambda \). Firstly, we claim that \( f(1) \in Z(\Lambda) \). In fact, since \( \Gamma \) has no oriented cycles, let us assume that \( i \) is a source in \( \Gamma_0 \). Then
\[
\Lambda = K(\Gamma, \rho) \cong \begin{bmatrix} \Lambda' & (1 - \tau_i)\Lambda \tau_i \\ 0 & K \end{bmatrix}.
\]
It follows from [14] Lemma 2.4 that
\[
f(1) = \begin{bmatrix} (1 - \tau_i)f(1)(1 - \tau_i) & 0 \\ 0 & \tau_if(1)\tau_i \end{bmatrix}
\]
and \([x, y], f(1)\] = 0 for all \( x, y \in \Lambda \). Note that \( K \) is a field and then clearly \( \tau_if(1)\tau_i \in Z(K) = K \). On the other hand, taking \( x = \begin{bmatrix} \tau_i & 0 \\ 0 & \tau_i \end{bmatrix} \) and \( y = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \), where \( m \in (1 - \tau_i)\Lambda \tau_i \), in \([x, y], f(1)\] = 0 leads to
\[
(1 - \tau_i)f(1)(1 - \tau_i)m = m\tau_if(1)\tau_i.
\]
This implies that \( f(1) \in Z(\Lambda) \) if and only if \((1 - \tau_i)f(1)(1 - \tau_i) \in Z(\Lambda') \). Repeat this process finite times, we arrive at the algebra \( K \). Then the commutativity of \( K \) gives \( f(1) \in Z(\Lambda) \). Hence we have from [14] Theorem 2.3 that for all \( x \in \Lambda \),
\[
f(x) = f(1)x + d(x),
\]
where \( d \) is a Jordan derivation of \( \Lambda \). By Theorem 3.4, \( d \) is a derivation of \( \Lambda \). It follows from [14] Proposition 2.1 that \( f \) is a generalized derivation of \( \Lambda \).

(2) Let \( f \) be a generalized Jordan derivation of \( \Lambda \). Then we have from [7] Lemma 4.1 that for all \( x \in \Lambda \), \( f(x) = f(1)x + d(x) \), where \( d \) is a Jordan derivation of \( \Lambda \). By Then \( d \) is a derivation of \( \Lambda \) Theorem 3.4. It follows from [14] Proposition 2.1 that \( f \) is a generalized derivation of \( \Lambda \).

**Remark 3.8.** Lemma 4.1 of [7] and Lemma 2.4 of [14] were obtained without the faithful assumption of bimodule \( A_M B \).
4. Lie derivations on path algebras

In [10], Cheung characterized Lie derivations of triangular algebras. The general form of Lie derivations was described and a sufficient and necessary condition which enables every Lie derivation to be standard was given. The current authors [15] studied the structure of Lie derivations of generalized matrix algebras and also provided certain sufficient condition such that every Lie derivation being of the standard form (♠). In this section we will investigate Lie derivations on a class of path algebras without the faithful assumption.

Lemma 4.1. [10] Proposition 4] A Lie derivation \( \Theta \) from \( T_K = [A M B] \) into itself is of the form

\[
\Theta \left( \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \right) = \begin{bmatrix} \delta_1(a) + \delta_4(b) & am_0 - m_0b + \tau_2(m) \\ 0 & \mu_1(a) + \mu_4(b) \end{bmatrix}, \quad \forall \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in T_K,
\]

where \( m_0 \in M \) and

\[
\delta_1 : A \rightarrow A, \quad \delta_4 : B \rightarrow Z(A), \quad \tau_2 : M \rightarrow M, \quad \mu_1 : A \rightarrow Z(B), \quad \mu_4 : B \rightarrow B
\]

are all \( K \)-linear mappings satisfying the following conditions

1. \( \delta_1 \) is a Lie derivation of \( A \), \( \mu_1([a,a']) = 0 \) for all \( a, a' \in A \) and \( \tau_2(am) = a\tau_2(m) + \delta_1(a)m - m\mu_1(a) \) for all \( a \in A, m \in M \);
2. \( \mu_4 \) is a Lie derivation of \( B \), \( \delta_4([b,b']) = 0 \) for all \( b, b' \in B \) and \( \tau_2(mb) = \tau_2(m)b + \mu_4(b)m \) for all \( b \in B, m \in M \).

Furthermore, a Lie derivation \( \Theta \) on \( T_K \) is of the standard form (♠) if and only if there exist linear mappings \( l_A : A \rightarrow Z(A) \) and \( l_B : B \rightarrow Z(B) \) satisfying

3. \( \delta = \delta_1 - l_A \) is a derivation on \( A \), \( l_A([a,a']) = 0 \) for all \( a, a' \in A \) and \( l_A(a)m = m\mu_1(a) \) for all \( a \in A, m \in M \);
4. \( \mu = \mu_1 - l_B \) is a derivation on \( B \), \( l_B([b,b']) = 0 \) for all \( b, b' \in B \) and \( ml_B(b) = \mu_4(b)m \) for all \( b \in B, m \in M \).

Let \( R \) be a commutative ring with identity and \( A \) be an \( R \)-algebra. Then \( W(A) \) defined in [10] is the smallest subalgebra of \( A \) satisfying the following conditions:

1. \( [x,y] \in W(A) \) for all \( x, y \in A \);
2. Suppose that \( x \in A \) and \( f(t) \) is a polynomial in \( R[t] \). If \( f'(x) = 0 \), then \( f(x) \in W(A) \);
3. Suppose that \( x \in A \) and \( f(t) \) is a polynomial in \( R[t] \). If \( f(x) \in W(A) \) and \( f'(x) \) is invertible, then \( x \in W(A) \);
4. \( W(A) \) contains all the idempotents in \( A \);
5. \( W(A) \) contains all the elements of the form \( x^p \), where \( x \in A \) and \( p \geq 0 \) is the characteristic of \( A \);
6. \( \{x \in W(A) \mid x^{-1} \in W(A)\} \subseteq W(A) \);
7. If \( x \in A \) is invertible with \( x^n \in W(A) \) for some positive integer \( n \), then \( nx \in W(A) \).

Then Cheung in [10] gives a sufficient condition for a Lie derivation of \( T_K \) being of the standard form.

Lemma 4.2. [10] Theorem 11] Let \( T_K = [A M B] \) be a triangular algebra. Then every Lie derivation of \( T_K \) is of the standard form (♠) if the following conditions are satisfied:

1. Every Lie derivation of \( A \) is of standard form and \( A = W(A) \);
(2) Every Lie derivation of $B$ is of standard form and $B = W(B)$.

Since a field $K$ is an algebra over itself, obviously $K = W(K)$. Surprisingly, for a finite dimensional path algebra $\Lambda = K(\Gamma, \rho)$, the equality $W(\Lambda) = \Lambda$ also holds.

**Lemma 4.3.** Let $K$ be a field and $\Gamma$ a finite quiver without oriented cycles. Let $\Lambda = K(\Gamma, \rho)$ be the path algebra of $\Gamma$ with relations. Then $W(\Lambda) = \Lambda$.

**Proof.** If $\Gamma$ only contains one vertex, then $\Lambda \simeq K$. In this case we get $W(\Lambda) = \Lambda$.

Now suppose that the number of vertices in $\Gamma$ is not less than 2. It follows from the condition (4) in the definition of $W(\Lambda)$ that all the trivial paths are contained in $W(\Lambda)$. On the other hand, for an arbitrary nontrivial path $p = \alpha_n \cdots \alpha_1$, we have $p = [p, s(p)]$, which is due to the fact $ps(p) = p$ and $s(p)p = 0$. Then the condition (1) of the definition of $W(\Lambda)$ implies that $p \in W(\Lambda)$. Therefore all paths in $(\Gamma, \rho)$ are contained in $W(\Lambda)$. Hence $\Lambda = W(\Lambda)$. □

Now we are ready to give the main result of this section.

**Theorem 4.4.** Let $\Lambda = K(\Gamma, \rho)$ a finite dimensional path algebra of the quiver $(\Gamma, \rho)$ with relations and $\Theta$ be a Lie derivation from $\Lambda$ into itself. Then $\Theta$ is of the standard form (♠). Moreover, the standard decomposition is unique.

**Proof.** If $\Gamma$ only contains one vertex, then $\Lambda \simeq K$. Clearly, every Lie derivation on $K$ is of the standard form (♠) and $W(K) = K$.

Now assume that the number of vertices in $\Gamma$ is not less than 2. Take a source $i$ in $\Gamma$ and let $e_i$ be the corresponding idempotent in $\Lambda$. Then $\Lambda \simeq \left[ (1 - e_i) \Lambda (1 - e_i) \quad (1 - e_i) \Lambda e_i \right]$. Let us denote $(1 - e_i) \Lambda (1 - e_i)$ by $\Lambda'$. In view of Lemma 4.3, we have $\Lambda' = W(\Lambda')$.

By Lemma 4.2, we obtain that each derivation on $\Lambda$ is of the standard form if and only if each Lie derivation on $\Lambda'$ is of the standard form. Repeating this process continuously, we ultimately conclude that each Lie derivation on $\Lambda$ is of the standard form if and only if each Lie derivation on $K$ is of the standard form. This is due to the fact that $\Gamma$ is a finite quiver. Therefore every Lie derivation on $\Lambda$ is of the standard form (♠).

Let us see the uniqueness of the standard decomposition. Suppose that $\Theta = D + \Phi = D' + \Phi'$. Then $D - D' = \Phi' - \Phi$. Clearly, the image of $\Phi' - \Phi$ is in $Z(\Lambda)$. This shows that $(D - D')(x) \in Z(\Lambda)$ for all $x \in \Lambda$. It follows from Corollary 3.5 that $D - D' = 0$. Consequently, $D = D'$ and $\Phi = \Phi'$.

**Corollary 4.5.** Let $\Theta_{\text{Lied}}$ be a Lie derivation of $\Lambda = K(\Gamma, \rho)$. Then there exists a derivation $D$ of $\Lambda$ such that $\Theta_{\text{Lied}}(x) = D(x)$ for all $x = \sum_i k_i p_i \in \Lambda$, where $p_i$ are non-trivial paths in $\Gamma$.

**Proof.** Let us denote by $\Gamma'$ the quiver obtained via adding a vertex to $\Gamma$ with no arrows starting and ending at it. Then

$$\Lambda' = K(\Gamma', \rho) \simeq \left[ \begin{array}{cc} \Lambda & 0 \\ 0 & K \end{array} \right].$$
Let \( \Theta_{\text{Lied}} \) be a Lie derivation on \( \Lambda \) and let \( f \) be a Lie derivation on \( K \). It follows from Lemma 4.4 that 
\[
\Theta'_{\text{Lied}} \left( \begin{bmatrix} x & 0 \\ 0 & b \end{bmatrix} \right) = \begin{bmatrix} \Theta_{\text{Lied}}(x) & 0 \\ 0 & f(b) \end{bmatrix}, \quad \forall \begin{bmatrix} x & 0 \\ 0 & b \end{bmatrix} \in \Lambda'
\]
is a Lie derivation on \( \Lambda' \). We have from Theorem 4.3 that \( \Theta'_{\text{Lied}} \) is of standard form. Then by Lemma 4.1 there exists linear map \( \phi : \Lambda \to Z(\Lambda) \) such that \( \Theta_{\text{Lied}} - \phi \Lambda \) is a derivation and \( \phi(\{x, y\}) = 0 \) for all \( x, y \in \Lambda \). For a nontrivial path \( p \in \Gamma \), it is easy to check that \( \overline{p} = [\overline{p}, s(p)] \). Thus \( \phi(\overline{p}) = 0 \). This implies that for arbitrary 
\[
x = \sum_i k_i \overline{p}_i \in \Lambda,
\]
where \( p_i \) is a non-trivial path in \( \Gamma \), \( \phi(\overline{p}) = 0 \). Define \( D \) to be \( \Theta_{\text{Lied}} - \phi \Lambda \). Then \( D(x) = \Theta_{\text{Lied}}(x) - \phi(\overline{x}) = \Theta_{\text{Lied}}(x) \). \( \square \)

**Lemma 4.6.** \([11]\) Theorem 2.5] A linear mapping \( \Theta \) from a path algebra \( K\Gamma \) into itself is a derivation if and only if \( \Theta \) satisfies
\[
(1) \quad \Theta(e_i) = \sum_{q \in \mathcal{P}, x(q) \neq e(q)} k^q_i q;
\]
\[
(2) \quad \Theta(p) = \sum_{q \in \mathcal{P}, x(q) \neq e(q)} k^p_i q + \sum_{q \in \mathcal{P}, x(q) = e(p)} k^p_i q + \sum_{q \in \mathcal{P}, x(q) \neq e(q)} k^p_i pq.
\]
The coefficients \( k^q_i \) are subject to certain conditions.

**Lemma 4.7.** Let \( \Gamma \) be a connected quiver without oriented cycles and \( K\Gamma \) be the path algebra of \( \Gamma \). If \( \Theta \) is a Lie derivation of \( K\Gamma \) with the standard decomposition \( \Theta = D + \Phi \), then we have
\[
\Phi(e_i) = k_i
\]
for an arbitrary trivial path \( e_i \in \Gamma \), where \( k_i \in K \).

**Proof.** Suppose that
\[
\Phi(e_i) = \sum_j k_j e_j + \sum_{p \in \mathcal{P}, s(p) \neq e(p)} k_p p.
\]
Clearly, for every trivial path \( e_t \),
\[
eq t)
\]
On the other hand,
\[
\Phi(e_i)e_t = k_t e_t + \sum_{p \in \mathcal{P}, s(p) \neq e(p), t(p) = t} k_p p. \tag{4.3}
\]
Note that \( \Phi(x) \in Z(\Lambda) \) for all \( x \in \Lambda \). Combining (4.2) with (4.3) leads to
\[
\sum_{p \in \mathcal{P}, s(p) \neq e(p), t(p) = t} k_p p = \sum_{p \in \mathcal{P}, s(p) \neq e(p), t(p) = t} k_p p.
\]
This implies that for all nontrivial path \( p \) with \( s(p) = t \) or \( e(p) = t \), the coefficients \( k_p = 0 \). Note that \( e_t \) is arbitrary. Thus the coefficients of all nontrivial paths is zero. So (4.1) becomes to
\[
\Phi(e_i) = \sum_j k_j e_j.
\]

\[1\] See \([11]\) Theorem 2.5 for details.
Now assume that \( \alpha \) is an arrow in \( \Gamma \) with \( s(\alpha) = x \) and \( e(\alpha) = y \). It is easy to check that \( \alpha \Phi(e_i) = k_x \alpha \) and \( \Phi(e_i) \alpha = k_y \alpha \). It follows from \( \Phi(\Lambda) \in Z(\Lambda) \) that \( k_x = k_y \). Note that \( \Gamma \) is a connected quiver. Then the coefficients of all trivial paths in \( \Phi(e_i) \) are the same. That is, \( \Phi(e_i) = k_i \sum_j e_j = k_i. \)

We can characterize Lie derivations of a finite dimensional path algebra now.

**Theorem 4.8.** Let \( \Gamma \) a quiver without oriented cycles and \( K \Gamma \) be a finite dimensional path algebra of \( \Gamma \). Then a linear mapping \( \Theta \) from \( K \Gamma \) into itself is a Lie derivation if and only if \( \Theta \) satisfies the following conditions

1. \( \Theta(e_i) = k_i + \sum_{q \in \beta, s(q) \neq e(q)} k_q^e q; \)
2. For each nontrivial path \( p \),

\[
\Theta(p) = \sum_{q \in \beta, s(q) \neq e(q)} k_q^{\beta(p)} qp + \sum_{q \in \beta, s(q) = e(p)} k_q^p q + \sum_{q \in \beta, s(q) \neq e(q)} k_q^{\beta(p)} pq.
\]

The coefficients \( k_q^p \) here are subject to certain conditions the same as in \[11\] Theorem 2.5.

**Proof.** It follows from Theorem \[14\] Lemma \[16\] and Lemma \[17\] easily. \( \square \)

**Example 4.9.** Let \( \Gamma \) be a quiver as follows.

\[
\begin{array}{c}
\cdot \\vdash \alpha \\vdash \beta \\vdash \cdot \\
1 \quad 2 \quad 3
\end{array}
\]

Let \( \Theta \) be a linear mapping from \( K \Gamma \) into itself defined by \( \Theta(e_1) = k_1 - \alpha \), \( \Theta(e_2) = k_2 + \alpha + \beta \), \( \Theta(e_3) = k_3 - \beta \) and \( \Theta(\alpha) = \Theta(\beta) = 0 \), where \( k_i \in K \) for \( i = 1, 2, 3 \). If there exists some \( i \in \{1, 2, 3\} \) such that \( k_i \neq 0 \), then \( \Theta \) is a Lie derivation of \( K \Gamma \) but not a derivation.

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