The structure of Schmidt subspaces of Hankel operators: a short proof

by

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Abstract. We give a short proof of the main result of a previous paper of ours: every Schmidt subspace of a Hankel operator is the image of a model space by an isometric multiplier. This class of subspaces is closely related to nearly $S^*$-invariant subspaces, and our proof uses Hitt’s theorem on the structure of such subspaces. We also give a formula for the action of a Hankel operator on its Schmidt subspace.

1. Introduction and main result

1.1. Hankel operators. We denote by $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ the open unit disk in the complex plane, and by $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ the unit circle. Let

$$\mathcal{H}^2 = \left\{ \sum_{k=1}^{\infty} \hat{f}(k) z^k : \sum_{k=0}^{\infty} |\hat{f}(k)|^2 < \infty \right\}$$

be the standard Hardy space of functions analytic in the unit disk (see e.g. [6] for the background); here $\hat{f}(\cdot)$ are the Fourier coefficients of $f$. As is customary, we identify elements of $\mathcal{H}^2$ with their boundary values on the unit circle.

Further, let $P$ be the orthogonal projection onto $\mathcal{H}^2$ in $L^2(\mathbb{T})$, i.e.

$$P : \sum_{k=-\infty}^{\infty} \hat{f}(k) z^k \mapsto \sum_{k=0}^{\infty} \hat{f}(k) z^k, \quad f \in L^2(\mathbb{T}).$$

For a symbol $u \in \text{BMOA}(\mathbb{T})$ (see e.g. [6] Chapter X] for the definition of this function space) we define the Hankel operator $H_u$ acting on $\mathcal{H}^2$ by

(1.1) \quad $H_u f = P(u \cdot \hat{f}), \quad f \in \mathcal{H}^2$.  

2020 Mathematics Subject Classification: 47B35, 30H10.
Key words and phrases: Hankel operators, Hardy space, model spaces, nearly invariant subspaces.

Received 17 July 2019; revised 14 January 2020.
Published online 27 May 2020.
Thus, $H_u$ is an anti-linear operator. By the Nehari–Fefferman theorem (see e.g. [3] Section 1.1) condition $u \in \text{BMOA}(\mathbb{T})$ ensures that $H_u$ is bounded. Denoting by $(\cdot, \cdot)$ the standard inner product in $\mathcal{H}^2$, for all $n, m \geq 0$ we have

$$(H_u z^n, z^m) = (P(u \overline{z^n}), z^m) = (u \overline{z^n}, z^m) = (u, z^{m+n}) = \hat{u}(n + m).$$

Thus, $H_u$ is the anti-linear realisation of the infinite Hankel matrix

$$\{\hat{u}(n + m)\}_{n,m \geq 0}$$

in the Hardy class $\mathcal{H}^2$. In Section 1.5 we recall the relation of $H_u$ to a linear realisation of the Hankel matrix in $\mathcal{H}^2$. Observe that $H_u^2$ is a bounded linear self-adjoint and non-negative operator on $\mathcal{H}^2$.

Our first aim is to describe the Schmidt subspaces

$$E_{H_u}(s) := \text{Ker}(H_u^2 - s^2 I), \quad s > 0,$$

as a class of subspaces in $\mathcal{H}^2$. Since $H_u$ commutes with $H_u^2$, we see that $E_{H_u}(s)$ is an invariant subspace for $H_u$ (this is one of the advantages of working with the anti-linear realisation $H_u$). Our second aim is to give the formula for the action of $H_u$ on this subspace.

### 1.2. Model spaces and isometric multipliers.

Recall that a non-constant bounded analytic function $\theta$ on $\mathbb{D}$ is called inner if its non-tangential boundary values on the unit disk satisfy $|\theta(z)| = 1$ for a.e. $z \in \mathbb{T}$. For an inner function $\theta$ the model space $K_\theta$ (see e.g. [7] Chapter 3 for the background) is defined as

$$K_\theta = \mathcal{H}^2 \cap (\theta \mathcal{H}^2)^\perp.$$

A convenient equivalent description of $K_\theta$ is

$$(1.2) \quad h \in K_\theta \iff h \in \mathcal{H}^2 \text{ and } z \overline{\theta} h \in \mathcal{H}^2.$$

Observe that for $h \in K_\theta$, the combination $z \overline{\theta} h$ is again in $K_\theta$.

As usual, we denote by $Sf(z) = zf(z)$ the shift operator in $\mathcal{H}^2$ and by $S^*$ the adjoint of $S$ in $\mathcal{H}^2$. Recall that the significance of model spaces stems from Beurling’s theorem which implies that a proper subspace of $\mathcal{H}^2$ is invariant under $S^*$ if and only if it is a model space.

If $p$ is an analytic function in the unit disk, we will say that $p$ is an isometric multiplier on $K_\theta$ if for every $f \in K_\theta$ we have $pf \in \mathcal{H}^2$ and $\|pf\| = \|f\|$. In this case we denote

$$pK_\theta := \{pf : f \in K_\theta\}.$$

We note that for a subspace $pK_\theta \subset \mathcal{H}^2$, the choice of the parameters $p$ and $\theta$ in this representation is not unique. One can multiply $p$ and $\theta$ by arbitrary unimodular constants and one can also perform Frostman shifts on $pK_\theta$; see Section 2.1 for the details.
1.3. Main result and discussion

**Theorem 1.1 (3).** Let $H_u$ be a bounded Hankel operator $\{1.1\}$ in $\mathcal{H}^2$. Every non-trivial Schmidt subspace $E_{H_u}(s)$, $s > 0$, is of the form $pK_\theta$, where $\theta$ is an inner function and $p$ is an isometric multiplier on $K_\theta$. Moreover, there exists a unimodular constant $e^{i\varphi}$ such that the action of $H_u$ on this subspace is given by

$$H_u(p) = se^{i\varphi}p\bar{\theta}h,$$ \hspace{1cm} h \in K_\theta.

**Remarks.** 1. By (1.2), the combination $z\theta h$ in (1.3) is in $\mathcal{H}^2$; in fact, it is in $K_\theta$.

2. The constant $e^{i\varphi}$ depends on the choice of the parameters $p, \theta$ in the representation $E_{H_u}(s) = pK_\theta$. In particular, by making a suitable choice of unimodular constants in the definitions of $p$ and $\theta$, one can achieve $e^{i\varphi} = 1$ in (1.3).

3. For an inner function $\theta$, consider the Hankel operator $H_{S^*\theta}$. It is not difficult to see that

$$\text{Ran } H_{S^*\theta} = E_{H_{S^*\theta}}(1) = K_\theta$$

and the action of $H_{S^*\theta}$ on $K_\theta$ is given by the anti-linear involution

$$H_{S^*\theta}h = \bar{z}\theta h, \quad h \in K_\theta.$$ Comparing with Theorem 1.1, we see that such Hankel operators can be regarded as the “simplest” ones from the point of view of our analysis: they have only one non-trivial Schmidt subspace and we can choose $p = 1$. Here and in what follows, $1$ is the function identically equal to $1$ in $\mathcal{H}^2$.

4. Denoting by $M_p$ the operator of multiplication by $p$, we can rewrite formula (1.3) as

$$H_uM_p = se^{i\varphi}M_pH_{S^*\theta} \quad \text{on } K_\theta.$$ Thus, up to the multiplicative constant $se^{i\varphi}$, the operator $M_p$ intertwines the action of $H_u$ on the Schmidt subspace with the action of the anti-linear involution $H_{S^*\theta}$ on $K_\theta$.

5. In [3], formula (1.3) was discussed only in the case $\theta(0) = 0$. This case is important because the condition $\theta(0) = 0$ is equivalent to $1 \in K_\theta$ and thus to $p \in pK_\theta$. In fact, in this case for every $h \in K_\theta$ we have

$$\text{(1.4) } (ph, \overline{p(0)p}) = p(0)(ph, p) = p(0)(h, 1) = p(0)h(0) = (ph, 1),$$

and therefore $\overline{p(0)p}$ is the orthogonal projection of $1$ onto the subspace $pK_\theta$.

6. There seems to be a close analogy between Theorem 1.1 and the structure of Toeplitz eigenspaces. Let $v \in L^\infty(\mathbb{T})$ and let $T_v$ be the Toeplitz operator with symbol $v$:

$$T_vf = P(v \cdot f).$$
Then (see [4]) all eigenspaces of \( T_v \) have the form \( pK_\theta \). In fact, Toeplitz operators and operators of the form \( H^2_u \) satisfy similar commutation relations; see Remark 2.2 below.

7. A natural question is whether any subspace of the form \( pK_\theta \) (where \( \theta \) is an inner function and \( p \) is an isometric multiplier on \( K_\theta \)) can appear as a Schmidt subspace of a bounded Hankel operator. Let \( p = p_ip_o \) be the inner-outer factorisation of \( p \); thus, a subspace \( pK_\theta \) is characterised by the triple \( \theta, p_i, p_o \). In [3, Section 6] it is shown that any pair \( \theta, p_i \) of inner functions can appear in the description of a Schmidt subspace of a bounded Hankel operator. We do not know what the class of admissible outer factors \( p_o \) is; we regard this as an interesting open problem.

1.4. Nearly \( S^* \)-invariant subspaces and Hitt’s theorem. To set the scene, we recall that, by a direct calculation, for \( f \in H^2 \) and for \( w \in \mathbb{D} \) one has

\[
S^*(I - wS^*)^{-1}f(z) = \frac{f(z) - f(w)}{z - w}.
\]

In particular, if \( M \subset H^2 \) is an \( S^* \)-invariant subspace (i.e. if \( M \) is either a model space or \( M = H^2 \)), then we have the implication

\[
f \in M \implies \frac{f(z) - f(w)}{z - w} \in M.
\]

(1.5)

Let \( M \subset H^2 \) be a closed linear subspace. We denote by \( Z(M) \subset \mathbb{D} \) the set of common zeros of elements of \( M \), i.e.

\[
Z(M) = \{ w \in \mathbb{D} : f(w) = 0 \ \forall f \in M \}.
\]

The subspace \( M \) is called nearly \( S^* \)-invariant if for some \( w \in \mathbb{D} \setminus Z(M) \) one has the implication (compare with (1.5))

\[
f \in M, \ f(w) = 0 \implies \frac{f(z)}{z - w} \in M.
\]

(1.6)

This definition was originally suggested by D. Hitt [5] with \( w = 0 \), in which case it becomes

\[
f \in M, \ f(0) = 0 \implies S^*f \in M.
\]

(1.7)

A simple argument presented later in [11, Proposition 5.1] shows that if (1.6) holds for some \( w \in \mathbb{D} \setminus Z(M) \), then it holds for all such \( w \), and so the notion of nearly \( S^* \)-invariance is independent of the choice of the base point \( w \).

The fundamental result giving the structural description of nearly \( S^* \)-invariant subspaces is the following theorem by D. Hitt [5] (see also [9, 11] for alternative proofs).

**Theorem 1.2 ([5]).** Let \( M \subset H^2 \) be a non-trivial nearly \( S^* \)-invariant subspace. Then \( M = pN \), where \( N \subset H^2 \) is an \( S^* \)-invariant subspace and \( p \) is an isometric multiplier on \( N \).
By Beurling’s theorem, \( N \) is either a model space or \( N = \mathcal{H}^2 \); in the second case \( p \) must be an inner function.

A converse of Hitt’s theorem is obvious: if \( p \) is an isometric multiplier on an \( S^* \)-invariant subspace \( N \), and \( p(w) \neq 0 \), then \( M = pN \) satisfies (1.6).

Hitt’s theorem seems to be closely related to Theorem 1.1. However, in [3] the authors were unable to use Hitt’s theorem directly (even though its key ideas were used in the proof). The reason for this is that, on the one hand, due to some operator algebra (see (2.5) below) it was important to use (1.6) with respect to the base point \( w = 0 \); on the other hand, it may happen that \( 0 \in \mathcal{Z}(E_{H_u}(s)) \) (see e.g. [3, Section 6]).

This obstacle is overcome in the present paper through the use of a Möbius map. More precisely, our plan of the proof is as follows. At the first step we consider the case \( 0 \notin \mathcal{Z}(E_{H_u}(s)) \). We prove that in this case \( E_{H_u}(s) \) satisfies (1.7). Thus, \( E_{H_u}(s) \) is nearly \( S^* \)-invariant and so we use Hitt’s theorem to obtain the representation \( E_{H_u}(s) = pK_\theta \). Some additional algebra yields formula (1.3) for the action of \( H_u \).

At the second step we consider the case \( 0 \in \mathcal{Z}(E_{H_u}(s)) \). We choose a point \( \alpha \in \mathbb{D} \setminus \mathcal{Z}(E_{H_u}(s)) \) and fix a Möbius map \( \mu \) sending \( \alpha \) to \( 0 \) and consider the associated unitary operator \( U_\mu \) on \( \mathcal{H}^2 \). It is easy to check that \( U_\mu E_{H_u}(s) \) is a Schmidt subspace of another bounded Hankel operator \( H_w \) and that \( 0 \notin \mathcal{Z}(E_{H_w}(s)) \). This reduces the problem to the one considered at the first step of the proof.

Our proof is informed by the intuition coming from [3], and in fact we reproduce some simple elements of the argument of [3].

1.5. Linear Hankel operators. Here we rewrite Theorem 1.1 in terms of linear (rather than anti-linear) Hankel operators on the Hardy space. We follow [3, Appendix] almost verbatim.

Let \( J \) be the linear involution in \( L^2(\mathbb{T}) \),

\[
Jf(z) = f(\bar{z}), \quad z \in \mathbb{T},
\]

and let \( \mathcal{C} \) be the anti-linear involution in \( \mathcal{H}^2 \),

\[
\mathcal{C}f(z) = \overline{f(\bar{z})}, \quad z \in \mathbb{T}.
\]

For a symbol \( u \in \text{BMOA}(\mathbb{T}) \), define the linear Hankel operator \( G_u \) in \( \mathcal{H}^2 \) by

\[
G_u f = P(u \cdot Jf), \quad f \in \mathcal{H}^2.
\]

We have \( G_u = H_u \mathcal{C} \) and \( G_u^* = \mathcal{C} H_u \), and so from Theorem 1.1 we obtain

**Theorem 1.3.** Let \( s \) be a singular value of \( G_u \). Then there exists an inner function \( \theta \) and an isometric multiplier \( p \) on \( K_\theta \) such that

\[
\text{Ker}(G_u^* G_u - s^2 I) = \mathcal{C}(pK_\theta), \quad \text{Ker}(G_u G_u^* - s^2 I) = pK_\theta.
\]
The action
\[ G_u : \text{Ker}(G_u^*G_u - s^2 I) \to \text{Ker}(G_uG_u^* - s^2 I) \]
is given by
\[ G_uC(pf) = sp\theta f, \quad f \in K_{\theta}. \]

2. The case \( 0 \notin \mathcal{Z}(E_{H_u}(s)) \)

2.1. Frostman shifts. Let \( \theta \) be an inner function; then (see e.g. [2, Theorem 10]) for any \( |\alpha| < 1 \) one has
\[ K_{\theta} = g_\alpha K_{\theta_\alpha}, \]
where
\[ (2.1) \quad \theta_\alpha(z) = \frac{\alpha - \theta(z)}{1 - \overline{\alpha}\theta(z)}, \quad g_\alpha(z) = \frac{1 - \alpha\theta(z)}{\sqrt{1 - |\alpha|^2}}, \]
and \( g_\alpha \) is an isometric multiplier on \( K_{\theta_\alpha} \). It follows that if \( p \) is an isometric multiplier on \( K_{\theta} \), then
\[ (2.2) \quad pK_{\theta} = pg_\alpha K_{\theta_\alpha}, \]
where \( pg_\alpha \) is an isometric multiplier on \( K_{\theta_\alpha} \). Conversely, if
\[ pK_{\theta} = \tilde{p}K_{\tilde{\theta}}, \]
where \( p \) is an isometric multiplier on \( K_{\theta} \) and \( \tilde{p} \) is an isometric multiplier on \( K_{\tilde{\theta}} \), then, again by [2, Theorem 10],
\[ \tilde{p} = c_1 pg_\alpha, \quad \tilde{\theta} = c_2 \theta_\alpha, \]
where \( \alpha \in \mathbb{D} \) and \( c_1, c_2 \) are unimodular complex numbers.

2.2. Some algebra of model spaces

Lemma 2.1. Let \( \theta \) be an inner function in the unit disk. Then
\[ S^*(K_{\theta} \cap \mathbb{1}^\perp) = K_{\theta} \cap (S^*\theta)^\perp. \]

Proof. It suffices to prove the identity
\[ K_{\theta} \cap \mathbb{1}^\perp = S(K_{\theta} \cap (S^*\theta)^\perp). \]
Let \( h \in K_{\theta} \cap \mathbb{1}^\perp \). Write \( h = Sg \) with \( g = S^*h \in K_{\theta} \). Then
\[ (g, S^*\theta) = (Sg, \theta) = (h, \theta) = 0, \]
and so \( g \in K_{\theta} \cap (S^*\theta)^\perp \). Conversely, let \( g \in K_{\theta} \cap (S^*\theta)^\perp \). Write any \( f \in \mathcal{H}^2 \) as \( f = c\mathbb{1} + Sv \) with \( v \in \mathcal{H}^2 \); then
\[ (Sg, \theta f) = \overline{c}(Sg, \theta) + (Sg, \theta Sv) = \overline{c}(g, S^*\theta) + (g, \theta v) = 0, \]
and so \( Sg \in K_{\theta} \). Clearly, \( Sg \perp \mathbb{1} \), and so \( Sg \in K_{\theta} \cap \mathbb{1}^\perp. \)
2.3. Some identities for \( H_u \). Hankel operators \( H_u \) satisfy the key identity
\[
S^*H_u = H_u S;
\]
in fact, this identity characterises the class of all Hankel operators. Recalling that
\[
SS^* = I - (\cdot, 1) 1,
\]
from (2.3) and from \( u = H_u 1 \) one obtains
\[
S^*H_u^2 S = H_u^2 - (\cdot, u) u.
\]
Multiplying (2.4) by \( S^* \) on the right and rearranging, we arrive at
\[
S^*H_u^2 - H_u^2 S^* = (\cdot, 1) S^*H_u u - (\cdot, Su) u.
\]
This relation is key to checking the definition (1.7) of nearly \( S^* \)-invariance. Finally, it is straightforward to check that \( H_u \) satisfies
\[
(S^*H_u^2 f, g) = (H_u g, f), \quad f, g \in \mathcal{H}^2.
\]

Remark 2.2. Observe that the Toeplitz operators \( T_v \) on \( \mathcal{H}^2 \) satisfy the commutation relation
\[
S^* T_v S = T_v;
\]
formula (2.4) can be viewed as a rank one perturbation of this relation.

2.4. Proof of the representation \( E_{H_u}(s) = pK_\theta \) in the case \( 0 \notin \mathcal{Z}(E_{H_u}(s)) \). Here we assume that \( 0 \notin \mathcal{Z}(E_{H_u}(s)) \) and prove the first part of Theorem 1.1

(1) For \( f \in E_{H_u}(s) \) and \( g \in E_{H_u}(s) \) we have
\[
(S^*H_u^2 f, g) - (H_u^2 S^* f, g) = s^2(S^* f, g) - (S^* f, H_u^2 g)
\]
and therefore, by (2.5),
\[
(f, Su)(u, g) = 0.
\]
By assumption, there exists an element \( h \in E_{H_u}(s) \) with \( (h, 1) \neq 0 \). Take \( g = H_u h; \) then, using (2.6), we get
\[
(u, g) = (H_u 1, g) = (H_u g, 1) = (H_u^2 h, 1) = s^2(h, 1) \neq 0,
\]
and so \( (f, Su) = (S^* f, u) = 0. \) Now applying (2.5) to \( f \), we find
\[
(H_u^2 - s^2 I)S^* f = 0,
\]
i.e. \( S^* f \in E_{H_u}(s) \). Putting this together, we see that we have checked the inclusion
\[
S^*(E_{H_u}(s) \cap 1^\perp) \subset E_{H_u}(s) \cap u^\perp.
\]

(2) By Hitt’s theorem, \( E_{H_u}(s) = pN \), where \( p \) is an isometric multiplier on \( N \) and \( N \) is either a model space or \( N = \mathcal{H}^2 \); we need to eliminate the
second possibility. Suppose \( N = \mathcal{H}^2 \). Since by assumption \( 0 \notin \mathcal{Z}(E_{H_u}(s)) \), we see that \( p(0) \neq 0 \). Then
\[
p\mathcal{H}^2 \cap \mathbb{1}^\perp = zp\mathcal{H}^2.
\]
It follows that
\[
S^* (p\mathcal{H}^2 \cap \mathbb{1}^\perp) = p\mathcal{H}^2.
\]
Comparing with (2.7), we conclude that \( u \perp E_{H_u}(s) \). Then for any \( h \in E_{H_u}(s) \),
\[
s^2(h, \mathbb{1}) = (H^2_u h, \mathbb{1}) = (H_u \mathbb{1}, H_u h) = (u, H_u h) = 0,
\]
and so \( E_{H_u}(s) \perp \mathbb{1} \), contrary to our assumption.

2.5. Proof of (1.3) in the case \( 0 \notin \mathcal{Z}(E_{H_u}(s)) \). Here we assume \( 0 \notin \mathcal{Z}(E_{H_u}(s)) \) and prove formula (1.3) for the action of \( H_u \) on \( E_{H_u}(s) = pK_\theta \).

(1) Let us first assume that \( \theta(0) = 0 \); then \( \mathbb{1} \in K_\theta \), \( p \in pK_\theta \) and by (1.4) the element \( \overline{p(0)}p \) is the orthogonal projection of \( \mathbb{1} \) onto \( pK_\theta \). Next, let \( u_\mathbb{1} \) be the orthogonal projection of \( u \) onto \( E_{H_u}(s) \). Since \( H_u \) commutes with \( H^2_u \), and therefore with the orthogonal projection onto \( E_{H_u}(s) \), we see that
\[
(2.8) \quad u_\mathbb{1} = H_u(\overline{p(0)}p) = p(0)H_up.
\]
Further, by (2.7), we have
\[
(2.9) \quad S^* (pK_\theta \cap p^\perp) \subset pK_\theta \cap u_\mathbb{1}^\perp.
\]
Lemma 2.1 implies
\[
S^* (pK_\theta \cap p^\perp) = S^* (p(K_\theta \cap 1^\perp)) = pS^* (K_\theta \cap 1^\perp)
\]
\[
= p(K_\theta \cap (S^* \theta)^\perp) = pK_\theta \cap (pS^* \theta)^\perp.
\]
Comparing this with (2.9), we obtain
\[
(2.10) \quad u_\mathbb{1} = cpS^* \theta
\]
with some constant \( c \). Putting this together with (2.8), we get
\[
(2.11) \quad H_up = \frac{c}{p(0)}pS^* \theta.
\]
In order to evaluate \( c \), let us compute the norms on both sides of (2.10):
\[
\|u_\mathbb{1}\|^2 = \|p(0)H_up\|^2 = |p(0)|^2(H^2_up, p) = s^2|p(0)|^2(\mathbb{1}, \mathbb{1}) = s^2|p(0)|^2,
\]
\[
\|cpS^* \theta\| = |c|\|pS^* \theta\| = |c|\|S^* \theta\| = |c|\|\mathcal{Z}\theta\| = |c|.
\]
It follows that \( |c| = s|p(0)| \). Substituting this into (2.11), we obtain
\[
H_up = se^{i\varphi}pz\theta
\]
with some unimodular complex number \( e^{i\varphi} \), which is exactly the required formula (1.3) for \( h = \mathbb{1} \). From this we easily get (1.3) for a general \( h \in K_\theta \):
\[
H_u(ph) = P(\overline{hup}) = P(hP(u\mathbb{1})) = P(\overline{hH_up}) = se^{i\varphi}P(\overline{hpz}\theta) = se^{i\varphi}\overline{hpz}\theta.
\]
(2) Now let $E_{H_u}(s) = pK_\theta$, with $\theta(0) \neq 0$. Choose $\alpha = \theta(0)$ and write

$$pK_\theta = pg_\alpha K_{\theta\alpha}$$

as in (2.2). Since $\theta_\alpha(0) = 0$, by the previous step of the proof we have

$$H_u(pg_\alpha h) = se^{i\phi}zg_\alpha \overline{h}, \quad h \in K_{\theta\alpha}.$$ 

Directly from the definitions (2.1) one has $g_\alpha = -\theta g_\alpha$, and so, denoting $g_\alpha h = v \in K_\theta$, we obtain

$$H_u(pv) = -se^{i\phi}z\theta v, \quad v \in K_\theta,$$

as required. \[\blacksquare\]

3. The case $0 \in \mathcal{Z}(E_{H_u}(s))$

3.1. Conformal maps. For $\alpha \in \mathbb{D}$, let $\mu : \mathbb{D} \to \mathbb{D}$ be the conformal map

$$\mu(z) = \frac{\alpha - z}{1 - \overline{\alpha}z},$$

and consider the corresponding unitary operator on the Hardy class,

$$U_\mu f(z) = \frac{\sqrt{1 - |\alpha|^2}}{1 - \overline{\alpha}z} f(\mu(z)), \quad z \in \mathbb{D}.$$ 

Observe that $\mu$ is an involution, $\mu \circ \mu = \text{id}$, and $U_\mu^2 = I$.

**Lemma 3.1.** Let $U_\mu$ be as defined above, and let $u \in \text{BMOA}(\mathbb{T})$. Then

$$U_\mu H_u U_\mu = H_w, \quad \text{where} \quad w = -S^*((Su) \circ \mu).$$

**Proof.** Computing the Jacobian of the change of variables $e^{it} \mapsto \mu(e^{it})$ on the unit circle, for $h_1, h_2 \in \mathcal{H}^2$ we get

$$(H_u h_1, h_2) = (u, h_1 h_2) = \left(u \circ \mu, (h_1 \circ \mu)(h_2 \circ \mu) \frac{1 - |\alpha|^2}{|1 - \overline{\alpha}z|^2}\right).$$

Writing for $|z| = 1$,

$$\frac{1 - |\alpha|^2}{|1 - \overline{\alpha}z|^2} = -\left(\frac{1 - |\alpha|^2}{1 - \overline{\alpha}z}\right)^2 \overline{z}\mu(z),$$

we get

$$(H_u h_1, h_2) = -\overline{z}\mu(z) u \circ \mu, (U_\mu h_1)(U_\mu h_2)) = (w, (U_\mu h_1)(U_\mu h_2))$$

$$= (H_w U_\mu h_1, U_\mu h_2) = (U_\mu H_w U_\mu h_1, h_2). \quad \blacksquare$$

**Lemma 3.2.** Let $\theta$ be an inner function and let $p$ be an isometric multiplier on $K_\theta$. Then $U_\mu(pK_\theta) = (p \circ \mu)K_{\theta\circ\mu}$, and $p \circ \mu$ is an isometric multiplier on $K_{\theta\circ\mu}$. 
Proof. Clearly, $U_{\mu}(pK_{\theta}) = (p \circ \mu)U_{\mu}(K_{\theta})$. Also, $U_{\mu}(\theta\mathcal{H}^2) = (\theta \circ \mu)\mathcal{H}^2$ and so $U_{\mu}K_{\theta} = K_{\theta \circ \mu}$. ■

3.2. Proof of Theorem 1.1 in the case $0 \in \mathcal{Z}(E_{H_u}(s))$. Let us choose $\alpha \in \mathbb{D} \setminus \mathcal{Z}(E_{H_u}(s))$. Consider the conformal map $\mu$ and the unitary operator $U_{\mu}$ corresponding to this point $\alpha$. By the choice of $\alpha$, we have $0 \notin \mathcal{Z}(U_{\mu}E_{H_u}(s))$. Moreover, by Lemma 3.1, the subspace $U_{\mu}E_{H_u}(s)$ is a Schmidt subspace of a Hankel operator $H_w$, $U_{\mu}E_{H_u}(s) = \text{Ker}(U_{\mu}H_u^2U_{\mu} - s^2I) = E_{H_w}(s)$.

Thus, by the already proven case of Theorem 1.1 applied to $H_w$, we deduce that

$$E_{H_w}(s) = pK_{\theta},$$

where $p$ is an isometric multiplier on $K_{\theta}$, and that $H_w$ acts on $E_{H_w}(s)$ according to (1.3):

$$H_w(ph) = se^{i\varphi}p\bar{z}\theta\bar{h}, \quad h \in K_{\theta}.$$  (3.1)

By Lemma 3.2 we obtain

$$E_{H_u}(s) = U_{\mu}E_{H_u}(s) = U_{\mu}(pK_{\theta}) = (p \circ \mu)K_{\theta \circ \mu},$$

which proves the first part of the theorem. It remains to check formula (1.3) for the action of $H_u$.

Denote $U_{\mu}h = v \in K_{\theta \circ \mu}$. Apply $U_{\mu}$ on both sides of (3.1). For the left hand side, we have

$$U_{\mu}H_w(ph) = U_{\mu}H_w(pU_{\mu}v) = U_{\mu}H_wU_{\mu}((p \circ \mu)v) = H_u((p \circ \mu)v).$$

For the right hand side, we have

$$U_{\mu}(se^{i\varphi}p\bar{z}\theta\bar{h}) = se^{i\varphi}(p \circ \mu)(\theta \circ \mu)U_{\mu}(\bar{z}\theta\bar{h}).$$

By the definition of $U_{\mu}$,

$$U_{\mu}(\bar{z}\theta\bar{h}) = \sqrt{1 - |\alpha|^2} \frac{\mu(z)h(\mu(z))}{1 - \alpha z} = \sqrt{1 - |\alpha|^2} \frac{\alpha - \bar{z}}{1 - \alpha z} \frac{1}{\bar{h}(\mu(z))}$$

$$= -\bar{z} \sqrt{1 - |\alpha|^2} \frac{1 - \bar{\alpha}z}{1 - \alpha \bar{z}} \bar{h}(\mu(z)) = -\bar{z} U_{\mu}\bar{h}.$$  (1.3)

Putting this together, we obtain

$$H_u((p \circ \mu)v) = -se^{i\varphi}(p \circ \mu)\bar{z}(\theta \circ \mu)\bar{v},$$

for all $v \in K_{\theta \circ \mu}$. This is the required formula (1.3).

Acknowledgments. The authors are grateful to V. Kapustin for useful discussions and to the anonymous referees for helpful suggestions.
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