Magnetoelastic effect in dipolar clusters

Paula Mellado,1 Andres Concha,1 and Sergio Rica1

1School of Engineering and Applied Sciences, Universidad Adolfo Ibáñez, Santiago, Chile

We combine the anisotropy of magnetic interactions and the point symmetry of finite solids in the study of dipolar clusters as new basic units for multiferroics metamaterials. The Hamiltonian of magnetic dipoles with an easy axis at the vertices of polygons and polyhedra, maps exactly into a Hamiltonian with symmetric and antisymmetric exchange couplings. The last one gives rise to a Dzyaloshinskii-Moriya contribution responsible for the magnetic modes of the systems and their symmetry groups, which coincide with those of a particle in a crystal field with spin-orbit interaction. We find that the clusters carry spin current and that they manifest the magnetoelastic effect. We expect our results to pave the way for the rational design of magnetoelastic devices at room temperature.

Introduction. The conciliation of crystal symmetry and magnetic phenomena has been a key element in the understanding of matter and the quest for new materials. Magnetic degrees of freedom coupled to physical symmetry preclude the realization of magnetic phases, that may manifest as ferromagnetic, ferrimagnetic, and superconductor materials among others [1–3]. The magnetic degrees of freedom coupled to physical symmetry preclude the realization of magnetic phases, that may manifest as ferromagnetic, ferrimagnetic, and superconductor materials among others [1–3]. The magnetic degrees of freedom coupled to physical symmetry preclude the realization of magnetic phases, that may manifest as ferromagnetic, ferrimagnetic, and superconductor materials among others [1–3]. The magnetic degrees of freedom coupled to physical symmetry preclude the realization of magnetic phases, that may manifest as ferromagnetic, ferrimagnetic, and superconductor materials among others [1–3]. The magnetic degrees of freedom coupled to physical symmetry preclude the realization of magnetic phases, that may manifest as ferromagnetic, ferrimagnetic, and superconductor materials among others [1–3]. The magnetic degrees of freedom coupled to physical symmetry preclude the realization of magnetic phases, that may manifest as ferromagnetic, ferrimagnetic, and superconductor materials among others [1–3]. The magnetic degrees of freedom coupled to physical symmetry preclude the realization of magnetic phases, that may manifest as ferromagnetic, ferrimagnetic, and superconductor materials among others [1–3]. The magnetic degrees of freedom coupled to physical symmetry preclude the realization of magnetic phases, that may manifest as ferromagnetic, ferrimagnetic, and superconductor materials among others [1–3]. The magnetic degrees of freedom coupled to physical symmetry preclude the realization of magnetic phases, that may manifest as ferromagnetic, ferrimagnetic, and superconductor materials among others [1–3]. The magnetic degrees of freedom coupled to physical symmetry preclude the realization of magnetic phases, that may manifest as ferromagnetic, ferrimagnetic, and superconductor materials among others [1–3].

Main results. In this article, we find new simple systems/mechanisms that may realize the ME effect at room temperature by means of magnetic modes induced by the interplay of dipolar interactions and geometric constraints. More precisely, we study the ME effect in dipolar systems realized by regular polygons and polyhedra decorated with easy axis magnetic dipoles at their vertices. The dipolar Hamiltonian is mapped exactly into a symmetric and antisymmetric contribution, where the antisymmetric part takes the form of a Dzyaloshinskii-Moriya (DM) interaction. Energy minimization of the dipolar energy yields the lowest energy magnetic configurations of regular n-sided polygons and several platonic solids. We demonstrate that the magnetic states realize the ME effect and posses multipolar moments, spin current and ME polarization. Exact diagonalization of the interaction matrix of the ground state sectors yields double degenerate spectra. These degeneracies do not match the dimensions of the pertinent point groups. For example, in the tetrahedral cluster with lowest magnetic energy mode shown in Fig. 1, the dimensions of the irreducible representations (irreps) of the tetrahedral point group $T$ are 1,1,1 and 3. The two non-trivial subgroups of $T$ compatible with the magnetic configuration, $D_2$ and $C_2$, have only one dimensional irreps each. The degeneracy of the eigenvalues are tied to the symmetry and not to the specific form of the Hamiltonian, thus we have map the ground state (GS) sector in each case, into an effective Hamiltonian $\hat{H}_f$ where collinear dipoles are coupled via Ising-like interactions. The spectrum of $\hat{H}_f$ yields doublets and more important, $\hat{H}_f$ enlightens the symmetries of the GS of the dipolar clusters. We found that the symmetry group of regular polygons with a number of vertices $n = 2(2s + 1)$ ($s > 0$ and integer) sides is the double chiral dihedral point group $D_n$. The generators of $D_n$ allowed to determine $Q$ that in this case is diagonal, symmetric and has two independent coefficients. For regular polygons with $n = 4s$, the symmetry group corresponds to the double point group $D_{2d}$ which yields symmetric diagonal $Q$ with one independent coefficient. The outcome for regular polyhedra is related to regular polygons. Indeed the symmetry group of the cube and the octahedron is $D_{2d}$ and thus in both cases $Q$ is diagonal with a single independent coefficient. For the tetrahedron the symmetry group is $D_4$ and $Q$ has two
independent matrix elements along the diagonal. Double

groups are subset of $SU(2)$ and arise in systems with half-

integer angular momentum and spin orbit interaction.

Magnetoelectric effect. The ME effect can be intro-
duced via an expansion of the free energy in terms of $H$
and $E$, namely $F(E, H) = F_0 - \frac{\mu_0}{2\pi} E_i E_j - \frac{\mu_0}{8\pi} H_i H_j - Q_{ij} E_i E_j + \ldots$ where $\epsilon_{ij}, \mu_{ij}$ are, respectively, the dielectric and the magnetic permeability. Derivative of $F$ in $H$
gives $M$ and derivative of $F$ in $E$ gives $P$, there-
fo in the linear ME effect $P = QH$ and $M = QE$ in
proper units. The ME tensor changes sign upon $r \to -r$
or $t \to -t$, so that a linear ME requires a simultaneous
violation of $I$ and $T$ symmetries. To describe ME effect
in terms of observable order parameters a common ap-

proach is to associate the shape of $Q$ to ME moments
that arise from the magnetic multipolar expansion $[8]$.

Expanding the magnetization energy in an inhomoge-
neous magnetic field, $H$, in powers of the field grad-
ents at some reference point: $\mathcal{H}_{\text{int}} = - \int (m(r) \cdot H(0) - x_i m_j \partial_i H_j(0)) \, d^3 r - \ldots = -M \cdot H(0) - a(\nabla \cdot H)|_{r=0} - t \cdot (\nabla \times H)|_{r=0} - q_{ij} (\partial_i H_j + \partial_j H_i)|_{r=0} - \ldots$, one identifies directly: $a = \frac{1}{2} \int r \cdot m(r) \, d^3 r$ as a monopolar moment; $t = \frac{1}{2} \int r \times m(r) \, d^3 r$, as a toroidal moment dual to the anti-
symmetric part of the tensor $\partial_i H_j$; and, a traceless sym-
metric tensor $q_{ij} = \frac{1}{2} \int (x_i m_j + x_j m_i - \frac{2}{3} \delta_{ij} r \cdot m(r)) d^3 r$ that describes the quadrupole magnetic moment of the system.

A microscopic mechanism connecting the electric
dipole with the spin operator is the spin-orbit interaction
that transfers anisotropy from the real space into the spin
space. The ME effect and the spin current $\mathcal{J}_s \propto s_i \times s_j$ are directly related in non-collinear spin structures as for
instance the spiral state. In magnets, $\mathcal{J}_s$ is associated with the spin rigidity and it is induced between two spins with generic non-parallel configurations. In Ref. [19] it has been shown that $\mathcal{J}_s$ in noncollinear magnets leads to the electric polarization $P \propto e_{ij} \times \mathcal{J}_s$, where $e_{ij}$ is the director vector joining spins $s_i$ and $s_j$.

The model. The dipolar classical hamiltonian for the
systems, in units of Joule $[J]$, reads

$$\mathcal{H}_{\text{dip}} = \gamma \sum_{i \neq k=1}^{N} \frac{m_i \cdot m_k - 3(m_i \cdot \hat{e}_{ik})(m_k \cdot \hat{e}_{ik})}{|r_i - r_k|^3},$$

here $\hat{e}_{ik} = (r_i - r_k)/|r_i - r_k|$, and $\gamma = \frac{\mu_0 m_0^2}{8\pi}$ has units of $[\text{Nm}^2]$ and contains the physical parameters involved in the energy such that $\mu_0$, the magnetic permeability, and $m_0$, the intensity of the magnetic moments with units $[\text{m}^2 \text{A}]$. From now on we normalize all distances by the cluster side length $L$, that is $\hat{x}_i = x_i/L$. Dipoles magnetic
moments are normalized by $m_i = m_0 \hat{m}_i$, have unit
vector: $\hat{m}_i = (\sin \alpha_i \cos \varphi_i, \sin \alpha_i \sin \varphi_i, \cos \alpha_i)$, and are located at the vertices $\hat{x}_i$ of regular polygons or platonic solids. They rotate in an easy plane described in terms of a polar angle $\alpha_i$ chosen respect to the $\hat{z}$ axis, and a fixed azimuthal angle $\varphi_i$ that accounts for the projection in the $\hat{x} - \hat{y}$ plane of the vector joining the site $i$ with the centroid of the cluster.

It is straightforward to show that the dipolar energy of our dipolar clusters is separable into symmetric and antisymmetric exchange contributions. Indeed for odd polygons,

$$E^{(\text{odd})} = \gamma \sum_{k=1}^{N} \frac{1}{\Delta_k} \left[ \sum_{l=-s}^{s} (\hat{m}_i \cdot \hat{m}_{i+k}) + \frac{3}{2} \tan \left( \frac{\pi}{n} k \right) \sum_{l=-s}^{s} \left( \hat{m}_i \times \hat{m}_{i+k} \right) \cdot \hat{z} \right],$$

with $\Delta_k = \sin \left( \frac{\pi}{n} k \right)/\sin(\pi/n)$ and $s$ is related to the number of vertices via $n = 2s + 1$. For polygons with even number of vertices the dipolar energy reads,

$$E^{(\text{even})} = \gamma \sum_{k=1}^{N} \frac{1}{\Delta_k} \left[ \sum_{l=1}^{n} \left( \hat{m}_i \cdot \hat{m}_{i+k} \right) + \frac{3}{2} \tan \left( \frac{\pi}{n} k \right) \sum_{l=1}^{n} \left( \hat{m}_i \times \hat{m}_{i+k} \right) \cdot \hat{z} \right] + \gamma \frac{n/2}{\Delta_k^{n/2}} \left[ -2\hat{m}_i \cdot \hat{m}_{i+n/2} + 3\hat{m}_i \cdot \hat{m}_{i+n/2} \right].$$

The first term in Eq. (2) and Eq. (3) is a symmetric exchange interaction between all dipoles. The second term is an antisymmetric exchange, $h_{\text{DM}} = J_{\text{DM}} \cdot (\hat{m}_i \times \hat{m}_{i+k})$, with $J_{\text{DM}} = \frac{3 \mu_0}{8\pi L^2 \Delta_k} \tan \left( \frac{\pi}{n} k \right) \hat{z}$ (in units of $[\text{Nm}^2]$) also known as the Dzyaloshinskii-Moriya interaction [20]. Dipolar energy for even polygons has two addi-
tional exchange contributions between dipoles located at opposite vertices in the cluster. These terms are written separately from the main sum because in the limit: $\lim_{k \to n/2} \tan \left( \frac{\pi}{n} k \right) \to \infty$, together with $(\hat{m}_i \times \hat{m}_{i+k}) \to 0$. More important, they compel for opposite dipoles to be in a collinear configuration. The spin orbit interaction

$\text{FIG. 1. Minimum energy magnetic configurations of pen-
tagonal, hexagonal, tetrahedral, cubic and octahedral dipolar}

clusters. The angle of rotation $\alpha$ and the easy plane of rotation
are shown.
shown here, is also manifested in the energy of polyhedral clusters, as we show in the supplemental information [21].

Classical ground states. For an even regular polygon the ground state can be computed directly from Eq. (3) and it is \( \alpha_k = (-1)^{k+1} \frac{\pi}{2} \). Fig. 1 shows the resulting antiferromagnetic mode for the hexagonal cluster. For odd polygons, the ground state configuration of Eq. (2) is satisfied by polar angles
\[
(\alpha_0, \alpha_{-1}, \alpha_{-2}, \ldots, \alpha_N, \alpha_{N+1}, \ldots, \alpha_{N+1-N/2})
\]
that satisfy \( \alpha_{-k} = -\alpha_k \) by symmetry and may be computed numerically (See supplemental information [21]).

Moreover we considered the dipolar hamiltonian of three platonic solids: the tetrahedron, the cube, and the octahedron. Energy minimization of Eq. (1) resulted in the lowest energy magnetic configurations shown in Fig. 1. For the tetrahedron the GS polar angles for all dipoles yield \( \alpha_k^{(t)} = \pi/2 \). The cube is such that \( \alpha_k^{(c)} = (-1)^{k+1} \arctan(1/\sqrt{2}) \). In the octahedron, collinear dipoles have equal \( \alpha \) and at all faces the sum of \( \alpha \) yields \( \pi \). For all polygons and polyhedra the net magnetization along \( \hat{z} \), \( m_z = \sum \cos(\alpha_k) = 0 \).

Magnetoelectric moments. The moments of finite clusters plays a crucial role toward the implementation of ME effect in two and three dimensional natural or tailored made lattices [22]. The GS of the dipolar clusters studied here are odd under \( \mathcal{I} \) and \( \mathcal{T} \), and therefore a non zero ME response is expected. The ME responses for all clusters are summarized in Table I and Table II. In Table II the first three rows show ME moments \( t = \sum \hat{x}_k \times \hat{m}_k \), \( q_{a\beta} = \sum_k (\hat{x}_k \cdot \hat{m}_k^a + \hat{x}_k \cdot \hat{m}_k^\beta - \frac{1}{3} \delta_{a\beta} \hat{x}_k \cdot \hat{m}_k) \) and \( a = \sum \hat{x}_k \cdot \hat{m}_k \). Fourth and fifth rows show the spin current \( \mathcal{J}_s \) and ME polarization \( \mathcal{P} \), where \( a, t \) and \( q \) are given in units of \( [mL] \), \( \mathcal{J}_s \) in units of \( \gamma/L^2 = [\text{Joule-m}] \) and \( \mathcal{P} \) in units of \( \gamma/L \) (Numerical values can be found in the supplemental information [21]). We found that most polygons with \( n = 2s + 1 \) vertices realize a toroidal moment in the \( x-y \) plane and quadrupolar moment. \( \mathcal{J}_s \) and \( \mathcal{P} \) are also manifested in all odd polygons. Even polygons have spin current along the \( \hat{z} \) axis, \( \mathcal{J}_z = (n-1) \sin \left( \frac{\pi}{n} \right) \) and polarization \( \mathcal{P} = \sin \left( \frac{\pi}{n} \right) (\cos \left( \frac{\pi}{n} \right), \sin \left( \frac{\pi}{n} \right), 0) \) in the \( x-y \) plane. The square has quadrupolar moment \( q_{x^2-y^2} = 4\sqrt{2} \) (in units of \( m_0L \)) but aside from it, \( q_{a\beta}, a \) and \( t \) cancel out in all even polygons. The ME response in these cases is not due to multipole moments from the second order terms in the series expansion of \( \mathcal{H}_{int} \). Indeed, the antiferromagnetic ground state configuration of hexagonal and octagonal clusters resemble a magnetic hexapole and octupole respectively.

In three dimensional clusters, there is not eulerian trail, and therefore \( \mathcal{J}_s \) and \( \mathcal{P} \) are computed using the hamiltonian path, a trail that visits each site once. Polyhedra, except the cube, have toroidal moment along the \( \hat{y} \) axis, like the odd polygons. The tetrahedral cluster has \( t_y = -1 \) in dimensionless units, spin current and ME polarization along the \( \hat{z} \) and \( \hat{x} \) axes respectively. The cube has no moments but \( \mathcal{J}_s \) and \( \mathcal{P} \) in the \( x-y \) plane. The octahedral cluster has monopolar, quadrupolar and toroidal moments in the \( x-y \) plane, spin current with components along all axes and \( \mathcal{P} = 0 \).

Ground state sector. The GS sector of a cluster \( \mathcal{C} \), has hamiltonian \( \mathcal{H}_{GS}^{(C)} = \sum_{j,k} \hat{m}_j \hat{m}_k^{(C)} ) \hat{m}_k \). Coefficients, \( \mathcal{A}_{j,k}^{(C)} \), are equal to the dipolar energy between dipoles \( j \) and \( k \) in the ground state of cluster \( \mathcal{C} \), further \( \mathcal{A}_{j,k}^{(C)} \) is symmetric and has no diagonal elements. We gain insight into the symmetries of even polygons and polyhedra by solving the spectrum of the interaction matrix \( \mathcal{A}_{j,k}^{(C)} \). Indeed diagonalization of such a matrix for all clusters, yields eigenvalues with even degeneracies which do not correspond to the dimensions of the irreps of the corresponding point groups. We address this issue by building an effective Hamiltonian \( \mathcal{H}_{eff} \) where matrix elements consist of link variables that represent Ising interactions among collinear dipoles. As an example, consider the hexagonal cluster in Fig. 1. Exact diagonalization of the interaction matrix \( \mathcal{A}_{j,k}^{(h)} \) yields a spectrum with multiplicities \( \{1, 2, 2, 1\} \), where the GS has degeneracy 2. The point group of the hexagon is the dihedral group \( D_h \) with \( h = 12 \) symmetry elements. Of the 12 symmetries, the magnetic configuration preserves two 3-fold rotations respect to the principal axis (\( \hat{z} \)), and two 2-fold rotations respect to axes perpendicular to \( \hat{z} \). Take now any pair of collinear dipoles (there are three of them in the hexagon), say the \( (p, q) \) pair, and associate an Ising variable (matrix element) \( u_{p,q} \) according to the following rules: if \( \hat{m}_p \cdot \hat{m}_q = 1 \), \( u_{p,q} = 1 = -u_{q,p} \); if \( \hat{m}_p \cdot \hat{m}_q = -1 \), \( u_{p,q} = 1 = u_{q,p} \) provided \( p \) and \( q \) point into the cluster and \( u_{p,q} = -1 = u_{q,p} \) provided \( p \) and \( q \) point out of the cluster, in all other cases \( u_{p,q} = 0 \). Applying this procedure to the hexagon yields a \( 6 \times 6 \) hermitic matrix consisting of three \( \sigma_y \) Pauli matrices along its diagonal, namely \( \mathcal{H}_{eff}^{(h)} = (\sigma_y, \sigma_y, \sigma_y, \sigma_y, \sigma_y, \sigma_y) \). The spectrum of \( \mathcal{H}_{eff}^{(h)} \) is \( \{1, -1, -1, -1, 1, 1\} \), and therefore it preserves the degeneracies of the eigenvalues of \( \mathcal{A}_{j,k}^{(h)} \). \( \mathcal{H}_{eff} \) reduces \( \mathcal{H}_{GS} \) to its diagonal form and, more important, it produces a repre-
For instance, if the operation which flips the Ising variables and fixes the problem.

The number of independent matrix elements of $Q$ corresponds to the number of times that the scalar irreps, $\Gamma_1$, is contained in the decomposition of the direct product $\Gamma_E \otimes \Gamma_M$. For $D_n$ point groups, the number of times that $\Gamma_1$ is contained is equal to 2, while for $D_{2d}$, $\Gamma_1$ is contained once. Therefore in the first case the number of independent matrix elements of $Q$ is two, while in the second case is one. Applying the generators of symmetries of the respective point groups to $Q$, ($Q$ transforms according to the rules of an axial vector) it is straightforward to determine the positions of those coefficients in each case. For the tetrahedron, cube and octahedron the symmetry groups are $D_4$, $D_{2d}$, $D_{2d}$ respectively. Table II shows the ME tensor in all cases [21].

| $C$ | $H_{f}^{(C)}$ | SG | $Q$ |
|-----|---------------|----|-----|
| Square | $(\hat{\sigma}_x, -\hat{\sigma}_y)$ | $D_{2d}$ | $(Q_{11}, -Q_{11})$ |
| Hexagon | $(\hat{\sigma}_y, \hat{\sigma}_x, \hat{\sigma}_y)$ | $D_6$ | $(Q_{11}, Q_{22})$ |
| Tetrahedron | $(\hat{\sigma}_y, \hat{\sigma}_x)$ | $D_4$ | $(Q_{11}, Q_{11}, Q_{22})$ |
| Cube | $(\hat{\sigma}_x, -\hat{\sigma}_x, \hat{\sigma}_x, -\hat{\sigma}_x)$ | $D_{2d}$ | $(Q_{11}, -Q_{11}, 0)$ |
| Octahedron | $(\hat{\sigma}_x, -\hat{\sigma}_x, \hat{\sigma}_x, -\hat{\sigma}_x)$ | $D_{2d}$ | $(Q_{11}, -Q_{11}, 0)$ |

Acknowledgments. This work was supported in part by Fondecyt under Grant No. 11121397 (PM), and by

Double groups and $Q$. The program implemented on the hexagon, was applied to all even polygons and polyhedra examined in this paper. In polygons with $n = 4s$ vertices, the symmetry group corresponds to the double dihedral group $D_{2d}$, for polygons with $n = 2(2s + 1)$ vertices it is the dihedral group $D_n$. $D_n$ and $\tilde{D}_{2d}$ differ in that the first is chiral while the second is not. This has an impact in the shape of $Q$. Indeed, $Q$ is a second rank axial tensor which connects a polar vector with an axial vector. For a system, whose symmetries are determined by a point group $G$, a polar vector $\mathbf{E}$ and an axial vector $\mathbf{M}$ transform according to irreps $\Gamma_E$ and $\Gamma_M$ of $G$.

Conclusions. We have shown that magnetic dipoles at the sites of two and three dimensional clusters, some of them motif of crystallographic space groups, are active for ME effect, carry spin current and in several cases manifest antisymmetric and symmetric ME moments. We found that the symmetries of the GS sector of these clusters are realized by double point groups, extensions of ordinary point groups that accommodate states with half-integer angular momentum, and consequently possess even dimensional representations. The dipolar hamiltonian in these systems exposes a spin-orbit coupling that manifests in a Dzyaloshinskii-Moriya interaction which explains the onset of double group symmetries. The origin of half-integer angular momentum associated to even dimensional irreps can be explained in terms of the spin current in these clusters. Indeed, in two dimensions a magnetic flux $\Phi$ can be defined from the spin current across the cluster. For the case of even polygons with circumradius $\rho = \frac{L}{2 \sin(\pi/m)}$, this flux becomes:

$$\Phi = \frac{\rho_0}{m_0} \mathbf{J}_z = \frac{(n-1) \sin \left( \frac{\pi}{m} \right)}{8 \pi \sin \left( \frac{\pi}{2} \right)}$$

in units of $[\text{m}^2 \text{m} / \text{L}]$. $\Phi$ is proportional to a magnetic charge $g$, $2\pi \Phi = g$, which in the large limit, $g \rightarrow \frac{(n-1)}{2}$. In even clusters $g$ takes half-integer values in units of $[\text{m}^2 / \text{L}^2]$ and it is responsible of a change of the net angular momentum of our clusters from integer to half-integer values [24]. For exciting magnetic clusters a viable experimental route could involve optical probes, as has been shown in [25] for a collection of molecules on a neutral substrate.
Fondecyt under Grant No. 1181382 (SR). P.M. acknowledges support from the Simons Foundation.

[1] L. Savary and L. Balents, Reports on Progress in Physics 80, 016502 (2016).
[2] P. F. S. Rosa, J. Kang, Y. Luo, N. Wakeham, E. D. Bauer, F. Ronning, Z. Fisk, R. M. Fernandes, and J. D. Thompson, Proceedings of the National Academy of Sciences 114, 5384 (2017). ISSN 0027-8424.
[3] S. F. Edwards and P. W. Anderson, Journal of Physics F: Metal Physics 5, 965 (1975).
[4] P. Mellado, A. Concha, and L. Mahadevan, Physical review letters 109, 257203 (2012).
[5] I. E. Dzyaloshinskii, Soviet Physics JETP 10, 628 (1960).
[6] M. Date, J. Kanamori, and M. Tachiki, Journal of the Physical Society of Japan 16, 2589 (1961).
[7] L. D. Landau, J. Bell, M. Kearsley, L. Pitaevskii, E. Lifshitz, and J. Sykes, Electrodynamics of continuous media, vol. 8 (elsevier, 2013).
[8] N. A. Spaldin, M. Fiebig, and M. Mostovoy, Journal of Physics: Condensed Matter 20, 434203 (2008).
[9] D. N. Astrov, J. Exptl. Theoret. Phys. (U.S.S.R.) 38, 984 (1960).
[10] D. Astrov, Sov. Phys. JETP 13, 729 (1961).
[11] I. Dzyaloshinskii, J. Phys. Chem. Solids 4, 241 (1958).
[12] K. Siratori, K. Kohn, and E. Kita, Acta Phys. Pol. A 81, 431 (1992).
[13] T. Kimura, G. Lawes, T. Goto, Y. Tokura, and A. Ramirez, Physical Review B 71, 224425 (2005).
[14] T. Kimura, Annu. Rev. Condens. Matter Phys. 3, 93 (2012).
[15] M. Matsubara, S. Manz, M. Mochizuki, T. Kubacka, A. Iyama, N. Aliouane, T. Kimura, S. L. Johnson, D. Meier, and M. Fiebig, Science 348, 1112 (2015).
[16] T. Kimura, Annu. Rev. Mater. Res. 37, 387 (2007).
[17] Y. Kitagawa, Y. Hiraoka, T. Honda, T. Ishikura, H. Nakamura, and T. Kimura, Nature Materials 9, 797 (2010).
[18] M. Mostovoy, Physical Review Letters 96, 067601 (2006).
[19] H. Katsura, N. Nagaosa, and A. V. Balatsky, Physical review letters 95, 057205 (2005).
[20] T. Moriya, Physical review 120, 91 (1960).
[21] See supplemental material for details.
[22] K. T. Delaney, M. Mostovoy, and N. A. Spaldin, Phys. Rev. Lett. 102, 157203 (2009).
[23] J. Shi, P. Zhang, D. Xiao, and Q. Niu, Physical review letters 96, 076604 (2006).
[24] P. Mellado, O. Petrova, and O. Tchernyshyov, Physical Review B 91, 041103 (2015).
[25] M. Wei, M. Niu, P. Bi, X. Hao, S. Ren, S. Xie, and W. Qin, Advanced Optical Materials 5, 1700644 (2017).