SHORTCUT GRAPHS AND GROUPS

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Abstract. We introduce shortcut graphs and groups. Shortcut graphs are graphs in which cycles cannot embed without metric distortion. Shortcut groups are groups which act properly and cocompactly on shortcut graphs. These notions unify a surprisingly broad family of graphs and groups of interest in geometric group theory and metric graph theory, including: the 1-skeletons of systolic and quadric complexes (in particular finitely presented C(6) and C(4)-T(4) small cancellation groups), 1-skeletons of finite dimensional CAT(0) cube complexes, hyperbolic graphs, standard Cayley graphs of finitely generated Coxeter groups and the standard Cayley graph of the Baumslag-Solitar group BS(1, 2). Most of these examples satisfy a strong form of the shortcut property.

The shortcut properties also have important geometric group theoretic consequences. We show that shortcut groups are finitely presented and have exponential isoperimetric and isodiametric functions. We show that groups satisfying the strong form of the shortcut property have polynomial isoperimetric and isodiametric functions.

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1. Introduction

A main current in geometric group theory is the study of groups acting on spaces satisfying various kinds of combinatorial nonpositive curvature properties. These spaces are typically associated with graphs having nice metric properties. For example, the 1-skeletons of CAT(0) cube complexes [23, 8, 7], systolic complexes [7] and quadric complexes [13], all of which arose independently in the geometric group theory and metric graph theory literature [10, 17, 12, 13, 2, 20, 19, 25, 3]. In this paper we introduce a metric property that captures an aspect of nonpositive curvature that is shared by these graphs and a surprisingly large number of other graphs of importance in group theory and metric graph theory.

A shortcut graph is a graph $\Gamma$ for which there is a bound on the lengths of its isometrically embedded cycles. A strongly shortcut graph $\Gamma$ has a bound on the lengths of its $K$-bilipschitz cycles, for some fixed $K > 1$. A group is (strongly) shortcut if it acts properly and cocompactly on a (strongly) shortcut graph.

Initial motivation for the study of shortcut graphs and groups arose from systolic and quadric complexes. Chepoi characterized systolic complexes as those flag simplicial complexes whose 1-skeletons contain isometrically embedded cycles only of length three (i.e. their only isometrically embedded cycles are triangles) [7]. We similarly characterized quadric complexes as those square-flag square complexes whose 1-skeletons contain isometrically embedded cycles only of length four [13]. Hence the 1-skeletons of systolic and quadric complexes are shortcut. In particular, systolic and quadric groups, and thus finitely presented $C(6)$ and $C(4)\cdot T(4)$ small cancellation groups, are shortcut [26, 13]. As we will see, many prominent classes of graphs and groups satisfy the shortcut property.

1.1. Summary of results. The following theorems summarized our main results.

**Theorem A** (Corollary 4.4). Shortcut groups are finitely presented.

**Theorem B** (Theorem 4.5, Theorem 4.9). Shortcut graphs and groups have exponential isoperimetric and isodiametric functions. Strongly shortcut graphs and groups have polynomial isoperimetric and isodiametric functions. Consequently, shortcut groups have decidable word problem.

**Theorem C** (Theorem 6.2, Corollary 6.6, Theorem 6.18, Theorem 6.12, Theorem 6.22, Theorem 6.25). The following classes of graphs are strongly shortcut.

- Hyperbolic graphs
- 1-skelettons of finite dimensional CAT(0) cube complexes
- 1-skelettons of systolic complexes
- 1-skelettons of quadric complexes
- Standard Cayley graphs of finitely generated Coxeter groups
- All Cayley graphs of $\mathbb{Z}$ and $\mathbb{Z}^2$

In particular, hyperbolic groups, cocompactly cubulated groups, systolic groups, quadric groups, $C(6)$ small cancellation groups, $C(4)\cdot T(4)$ small cancellation groups and Coxeter groups are all strongly shortcut.

**Theorem D** (Theorem 7.14, Theorem 7.18). The Baumslag-Solitar group $BS(1, 2)$ is shortcut but not strongly shortcut. Moreover, $BS(1, 2)$ has a Cayley graph which is shortcut and a Cayley graph which is not shortcut.
1.2. Structure of the paper. In Section 2 we present the main definitions of the paper. In Section 3 we prove basic properties of shortcut graphs and groups. In Section 4 we construct disk diagrams for shortcut graphs and study their filling invariants. In Section 5 we prove that products of (strongly) shortcut graphs are (strongly) shortcut and that finite graphs of (strongly) shortcut groups with finite edge groups are (strongly) shortcut. In Section 6 we prove that various classes of graphs and groups are strongly shortcut. In Section 7 we study the shortcut property in Cayley graphs of BS(1, 2).

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2. Definitions

A graph $\Gamma$ is a 1-dimensional polyhedral complex whose edges are isometric to $[0, 1] \subset \mathbb{R}$. In this way each graph is equipped with both the structure of a cellular complex with edges and vertices and the structure of a geodesic metric giving a distance between any pair of its points. A combinatorial map of graphs is a continuous map $\Gamma_1 \rightarrow \Gamma_2$ in which each vertex of $\Gamma_1$ maps onto a vertex of $\Gamma_2$ and each closed edge of $\Gamma_1$ maps onto a vertex or closed edge of $\Gamma_2$. A combinatorial map is degenerate if some closed edge maps onto a vertex. Otherwise it is nondegenerate.

2.1. Isometric and almost isometric cycles. Let $\Gamma$ be a graph. A combinatorial path in $\Gamma$ is a nondegenerate combinatorial map $P \rightarrow \Gamma$ from a graph $P$ that is homeomorphic to a compact interval of $\mathbb{R}$. A cycle $C$ is a graph homeomorphic to a circle. A cycle in $\Gamma$ is a nondegenerate combinatorial map $C \rightarrow \Gamma$ from a cycle $C$. The length of a path or cycle, denoted $|P|$ or $|C|$, is the number of its edges.

A cycle $f: C \rightarrow \Gamma$ is isometric if it is an isometric embedding. Corollary 3.2 below shows that $f$ is isometric if and only if

$$d_\Gamma(f(p), f(q)) \geq \frac{|C|}{2}$$

for every antipodal pair of points $p, q \in C$. With this in mind we give the following definition. A cycle $f: C \rightarrow \Gamma$ is $\xi$-almost isometric, for $\xi \in (0, 1)$, if

$$d_\Gamma(f(p), f(q)) \geq \xi \frac{|C|}{2}$$

for every antipodal pair of points $p, q \in C$. One may imagine that if $f$ is not isometric then there is a “shortcut” in $\Gamma$ between a pair of its antipodes and if $f$ is not $\xi$-almost isometric then there is such a “shortcut” which reduces the distance by a constant factor.

2.2. Shortcut graphs and groups. A connected simplicial graph $\Gamma$ is shortcut if, for some $\theta \in \mathbb{N}$, every isometric cycle $C \rightarrow \Gamma$ has length $|C| \leq \theta$. A connected simplicial graph $\Gamma$ is strongly shortcut if, for some $\theta \in \mathbb{N}$ and some $\xi \in (0, 1)$, every $\xi$-almost isometric cycle $C \rightarrow \Gamma$ has length $|C| \leq \theta$.

Remark 2.1. It follows immediately from the definitions that if $\Gamma'$ is an isometrically embedded subgraph of a (strongly) shortcut graph $\Gamma$ then $\Gamma'$ is also (strongly) shortcut.
Proposition 3.5 below says that $\Gamma$ is strongly shortcut if and only if there is a $K > 1$ and a bound on the lengths of the $K$-bilipschitz cycles of $\Gamma$. Since a 1-bilipschitz map is the same thing as an isometric embedding, we can thus define both properties together as follows: $\Gamma$ is shortcut if there is a $K \geq 1$ and a bound on the lengths of the $K$-bilipschitz cycles of $\Gamma$; if this $K$ can be chosen strictly greater than 1 then $\Gamma$ is strongly shortcut.

A group $G$ is (strongly) shortcut if it acts properly and cocompactly on a (strongly) shortcut graph.

3. Basic properties

In this section we prove some basic properties of (strongly) shortcut graphs and groups. In particular, we prove that the strong shortcut property is equivalent to the existence of a $K > 1$ for which there is a bound on the lengths of the $K$-bilipschitz cycles. This characterization is particularly interesting in light of a characterization of Hume and MacKay of a hyperbolic graph as a graph for which there is a bound on the lengths of its 18-bilipschitz cycles [16]. We also show that a (strongly) shortcut group acts freely and cocompactly on a (strongly) shortcut graph.

Proposition 3.1. Let $\Gamma$ be a graph and let $\bar{\xi} \in (0, 1]$. A cycle $f : C \to \Gamma$ satisfies

$$d_\Gamma(f(p), f(q)) \geq \bar{\xi}\frac{|C|}{2}$$

for every antipodal pair of points $p, q \in C$ if and only if

$$d_\Gamma(f(p), f(q)) \geq d_C(p, q) - (1 - \bar{\xi})\frac{|C|}{2}$$

for every pair of points $p, q \in C$.

Proof. The “if” part follows by applying the inequality to each antipodal pair of $p$ and $q$. To prove the “only if” part, let $p, q \in C$. Let $p'$ be the antipode of $p$. Then, since $f$ is 1-Lipschitz, we have

$$d_\Gamma(f(p), f(p')) \leq d_\Gamma(f(p), f(q)) + d_\Gamma(f(q), f(p'))$$

$$\leq d_\Gamma(f(p), f(q)) + d_C(q, p')$$

$$= d_\Gamma(f(p), f(q)) + d_C(p, p') - d_C(p, q)$$

$$= d_\Gamma(f(p), f(q)) + \frac{|C|}{2} - d_C(p, q)$$

but $\bar{\xi}\frac{|C|}{2} \leq d_\Gamma(f(p), f(p'))$ and so we have $d_\Gamma(f(p), f(q)) \geq d_C(p, q) - (1 - \bar{\xi})\frac{|C|}{2}$. \hfill \Box

Corollary 3.2. Let $\Gamma$ be a graph and let $f : C \to \Gamma$ be a cycle. Then $f$ is isometric if and only if

$$d_\Gamma(f(p), f(q)) \geq \frac{|C|}{2}$$

for every antipodal pair of points $p, q \in C$.

Proof. This follows from the fact that $f$ is 1-Lipschitz and by applying Proposition 3.1 with $\bar{\xi} = 1$. \hfill \Box
Proposition 3.3. Let $\Gamma$ be a graph and let $f: C \to \Gamma$ be a cycle in $\Gamma$ of length $|C| \geq 4$. If $f$ is not isometric then
\[ d_\Gamma(f(u), f(v)) < d_C(u, v) \]
for some pair of vertices $u, v \in C^0$ with $d_C(u, v) \geq \left\lfloor \frac{|C|}{2} \right\rfloor - 1$. If $f$ is not $\xi$-almost isometric, for some $\xi \in (0, 1)$, then
\[ d_\Gamma(f(u), f(v)) < \xi d_C(u, v) \]
for some pair of vertices $u, v \in C^0$ with $d_C(u, v) \geq \left\lfloor \frac{|C|}{2} \right\rfloor - 1$.

Proof. Let $\bar{\xi} \in (0, 1]$. Suppose
\[ d_\Gamma(f(p), f(q)) < \bar{\xi} \frac{|C|}{2} \]
for some pair of antipodal points $p, q \in C$. If $\bar{\xi} < 1$ then this is equivalent to $f$ not being $\bar{\xi}$-almost isometric and, otherwise, this is equivalent to $f$ not being isometric.

Let $C : [0, d] \to \Gamma$ be a geodesic from $f(p)$ to $f(q)$ where $d = d_\Gamma(f(p), f(q))$. If $C^{-1}(\Gamma^0) = \emptyset$ then $f(p)$ and $f(q)$ are contained in the interior of some common edge $e$ of $\Gamma$. Then the edges $e_1$ and $e_2$ of $C$ with $p \in e_1$ and $q \in e_2$ map onto $e$. Then there are endpoints $u \in e_1$ and $v \in e_2$ such that $f(u) = f(v)$ and $d_C(p, u) + d_C(q, v) \leq 1$.

So we have $d_\Gamma(f(u), f(v)) = 0$ and $d_C(u, v) \geq d_C(p, q) - d_C(p, u) - d_C(q, v) \geq \left\lfloor \frac{|C|}{2} \right\rfloor - 1$.

Assume now that $p$ and $q$ do not map to the same edge in $\Gamma$. Let $\alpha^{-1}(\Gamma^0) = \{x_1, x_2, \ldots, x_k\}$ with $0 \leq x_1 < x_2 < \cdots < x_k \leq d$. Then $\alpha|[0, x_1]$ and $\alpha|[x_k, d]$ factor through $f$ so their images contain vertices $u, v \in C^0$ with $d_C(p, u) < 1$ and $d_C(q, v) < 1$ such that $f(u) = \alpha(x_1)$ and $f(v) = \alpha(x_k)$. Moreover
\[
d_\Gamma(f(u), f(v)) = x_k - x_1 = d - x_1 - (d - x_k) = d - d_C(p, u) - d_C(q, v) < \bar{\xi} d_C(p, q) - d_C(p, u) - d_C(q, v) \leq \bar{\xi} (d_C(p, q) - d_C(p, u) - d_C(q, v)) \leq \bar{\xi} d_C(u, v)\]
and $d_C(u, v) \geq d_C(p, q) - d_C(p, u) - d_C(q, v) > \left\lfloor \frac{|C|}{2} \right\rfloor - 2$. Since $|C|$ and $d_C(u, v)$ are integers we obtain $d_C(u, v) \geq \left\lfloor \frac{|C|}{2} \right\rfloor - 1$. \qed

Remark 3.4. In fact, if $f: C \to \Gamma$ is not isometric then we can improve the $u$ and $v$ obtained from Proposition 3.3 so that $d_C(u, v) \geq \left\lfloor \frac{|C|}{2} \right\rfloor$. This does not hold for $f$ not $\xi$-almost isometric. (Consider the quotient map from $C$ oriented and of even length that identifies two antipodal edges of $C$ in an orientation reversing way.) In order to present a unified proof without additional case analysis, we give the weaker statement in Proposition 3.3.

Proposition 3.5. Let $\Gamma$ be a graph. Then $\Gamma$ is strongly shortcut if and only if there exists $K > 1$ such that there is a bound on the length of $K$-bilipschitz cycles of $\Gamma$. 
Proof. If a cycle $f: C \to \Gamma$ is not $\xi$-almost isometric then, for some pair of antipodal points $p, q \in C$,

$$\xi d_C(p, q) = \frac{\xi |C|}{2} > d_\Gamma(f(p), f(q))$$

and so $f$ is not $\frac{1}{\xi}$-bilipschitz. This proves the “only if” part of the proposition.

To prove the “if” part of the proposition, suppose $\theta$ bounds the length of the $K$-bilipschitz cycles of $\Gamma$ where $K > 1$. Let $1 - \frac{(K-1)^3}{15K^2(K+1)} < \xi < 1$. We will show that there is a bound on the lengths of the $\xi$-almost isometric cycles of $\Gamma$. Let $f: C \to \Gamma$ be a $\xi$-almost isometric cycle of $\Gamma$. We will define a sequence of paths $(P_i)$, a sequence of cycles $(C_i)_i$, a sequence of finite graphs $(\Gamma_i)_i$ and sequences of combinatorial maps as in the following commuting diagram.

Where it makes sense, we will use the same notation to refer to points and subspaces as we do to refer to their images under maps. We begin with $\Gamma_0 = C_0 = C$ and $f_0 = f$. Suppose we inductively have $C_i \hookrightarrow \Gamma_i \xrightarrow{f_i} \Gamma$. If the composition of these maps is $K$-bilipschitz then we terminate the sequence with $n = i$. Otherwise, let $u_i, v_i \in C_i$ be a furthest pair of vertices in $C_i$ for which $d_\Gamma(f_i(u_i), f_i(v_i)) < \frac{1}{K} d_C(u_i, v_i)$. Let $Q_i$ be a geodesic segment of $C_i$ between $u_i$ and $v_i$. Let $P_i$ be the closure of the complement of $Q_i$ and let $P_i^\circ$ be the interior of $P_i$. Let $R_i \to \Gamma$ be a geodesic from $f_i(u_i)$ to $f_i(v_i)$. We obtain $f_{i+1}: \Gamma_{i+1} \to \Gamma$ from $f_i$ and $R_i \to \Gamma$ by identifying the endpoints of $R_i$ with $\{u_i, v_i\}$. Let $C_{i+1} = P_i \cup R_i$ in $\Gamma_{i+1}$. The sequence always terminates since $|C_i|$ is strictly decreasing.

Our goal is to show that $\frac{|C_n|}{|C_0|}$ is uniformly bounded away from zero. Thus we will show that if we have arbitrarily long $\xi$-almost isometric cycles then we must also have arbitrarily long $K$-bilipschitz cycles.

For each $i$, the interior $P_i^\circ$ embeds in both $C_i$ and $C_{i+1}$. Let $P_{0,j}^\circ$ be the limit of the diagram

in the category of topological spaces and continuous maps. Concretely, we have $P_{0,0}^\circ = C_0$ and $P_{0,1}^\circ = P_0^\circ$ and $P_{0,j}^\circ = P_{0,j-1}^\circ \cap P_{j-1}^\circ$ where the intersection is taken
in $C_{j-1}$. Thus we have the following commutative diagram of embeddings.

$$
\begin{array}{cccccccc}
P_0^o & \hookrightarrow & P_1^o & \hookrightarrow & P_2^o & \ldots & \hookrightarrow & P_{n-1}^o & \hookrightarrow & P_n^o \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_0 & \hookrightarrow & C_1 & \hookrightarrow & C_2 & \ldots & \hookrightarrow & C_{n-2} & \hookrightarrow & C_{n-1} & \hookrightarrow & C_n
\end{array}
$$

We can think of $P_{0,i}^o$ as original points of $C$ that are not replaced until at least step $j$ of the construction of the $C_i$, where the $i$th step of the construction refers to the operation of replacing $Q_i \to \Gamma$ with $R_i \to \Gamma$ in order to obtain $C_{i+1} \to \Gamma$ from $C_i \to \Gamma$.

Fix $j$ and suppose $Q_i \subset P_{0,i}^o$ for all $i < j$, where $P_{0,i}^o$ is viewed as a subspace of $C_i$ via the inclusion $P_{0,i}^o \hookrightarrow C_i$. So, for $i < j$, we have lifts $Q_i \hookrightarrow P_{0,i}^o$ as in the diagram

$$
Q_0 \quad Q_1 \quad Q_2 \quad \ldots \quad Q_{j-2} \quad Q_{j-1}
$$

and we have disjoint unions $P_{0,i}^o = Q_i \cup P_{0,i+1}^o$. So, for $i < j$, we may think of the $Q_i$ as subspaces of $C$. The $Q_i$ are disjoint in $C$ and $C \setminus P_{0,j}^o = \bigcup_{i=0}^{j-1} Q_i$. Since $C_j$ is obtained from $C$ by replacing $Q_i$ with $R_i$, for each $i < j$, we see that the $R_i$, with $i < j$, embed disjointly in $C_j$ and the complement in $C_j$ of $P_{0,j}^o$ is $\bigcup_{i=0}^{j-1} R_i$. Since $f$ is $\xi$-almost isometric we have

$$|Q_i| - (1 - \xi)|C|/2 \leq |R_i| < 1/K|Q_i|$$

for all $i < j$, by Proposition 3.1. Hence, if $i < j$ then we have the following inequality.

$$(*) \quad K|R_i| < |Q_i| < (1 - \xi)\left(\frac{K}{K-1}\right)|C|/2$$

Moreover, we can find a pair of points $p, q$ in the closure of $C \setminus \bigcup_{i=1}^{j-1} Q_i$ at distance $d_C(p, q) \geq |C|/2 - K(1-\xi)|C|/4(K-1)$. Let $S_1$ and $S_2$ be the two segments of $C$ between $p$ and $q$. If $I_1 = \{i < j : Q_i \subset S_1\}$ then

$$d_T(f(p), f(q)) \leq |S_1| - \sum_{i \in I_1} |Q_i| + \sum_{i \in I_1} |R_i|$$

$$< |S_1| - \sum_{i \in I_1} |Q_i| + \frac{1}{K} \sum_{i \in I_1} |Q_i|$$

$$= |S_1| - \frac{K-1}{K} \sum_{i \in I_1} |Q_i|$$
and the same holds for $S_2$ and so, by Proposition 3.1,

$$|C| - \frac{K - 1}{K} \sum_{i < j} |Q_i|$$

$$\geq 2d(f(p), f(q))$$

$$\geq 2dC(p, q) - (1 - \xi)|C|$$

$$\geq |C| - \frac{K(1 - \xi)}{2(K - 1)}|C| - (1 - \xi)|C|$$

which gives us the following inequality.

$$\sum_{i < j} |Q_i| \leq (1 - \xi) \left( \frac{K(3K - 2)}{(K - 1)^2} \right) \frac{|C|}{2}$$

We will now prove that $Q_i \subset P_{0,i}$ for $1 \leq i < n$. For the sake of finding a contradiction, suppose $j \geq 1$ is the least integer with $Q_j \not\subset P_{0,i}$. As above we view the $R_i$ with $i < j$ as disjoint segments of $C_j$ with $C_j \setminus \bigcup_{i=0}^{j-1} R_i = P_{0,j}$. It is possible that, for some $i < j$, we have $Q_j \cap R_i \neq \emptyset$ in $C_j$ but $R_i \not\subset Q_j$. This may happen for at most two $R_i$ since such $R_i$ must contain an endpoint of $Q_j$. Let $Q^{-}_j$ be obtained from $Q_j$ by subtracting the interiors of any such $R_i$ and let $Q^{+}_j \to C_j$ extend $Q_j \to C_j$ so as to include full copies of any such $R_i$. Let $Q^{-} \subset C$ be obtained from $Q^{-}_j \subset C_j$ by replacing any $R_i \subset Q^{-}_j$, where $i < j$, with $Q_i \subset C$. Let $Q^{+} \to C$ be obtained from $Q^{+}_j \to C_j$ by replacing any $R_i \to C_j$, where $i < j$, with $Q_i \to C$. Let $R^{+} \to \Gamma$ be obtained from $Q^{+}_j \to \Gamma$ by replacing $Q_j \to \Gamma$ with $R_j \to \Gamma$. Then $R^{+} \to \Gamma$ and the composition $Q^{+} \to C \xrightarrow{f} \Gamma$ have the same endpoints in $\Gamma$ and we have

$$|R^{+}| = |R_j| + |Q^{+}_j \setminus Q_j|$$

$$< \frac{1}{K} |Q_j| + |Q^{+}_j \setminus Q_j|$$

$$= \frac{1}{K} (|Q^{-}_j| + |Q_j \setminus Q^{-}_j|) + |Q^{+}_j \setminus Q_j|$$

$$\leq \frac{1}{K} |Q^{-}_j| + |Q_j \setminus Q^{-}_j| + |Q^{+}_j \setminus Q_j|$$

$$= \frac{1}{K} |Q^{-}_j| + |Q^{+}_j \setminus Q^{-}_j|$$

$$\leq \frac{1}{K} |Q^{-}_j| + \frac{1}{K} |Q^{+} \setminus Q^{-}|$$

$$= \frac{1}{K} |Q^{+}|$$

where the final inequality follows from the fact that $Q^{+}_j \setminus Q^{-}_j$ consists of up to two copies of segments $R_i$ which are replaced with corresponding $Q_i$ in $Q^{+} \setminus Q^{-}$. By assumption, $Q_j$ nontrivially intersects at least one $R_i$, with $i < j$. Let $m$ be minimal such that $Q_m$ nontrivially intersects $R_m$. Since $Q_j$ is not equal to this $R_m$ we see that $|Q^{+}| > |Q_m|$. Hence, if $Q^{+} \to C_m$ were an isometric embedding then this would contradict the choice of $v_m$ and $v_m$. So, $Q^{+} \to C_m$ is not an isometric
embedding and so $|Q^+| > \frac{|C_m|}{2}$. But then

$$
|Q^+| > \frac{|C_m|}{2} \geq \frac{|C|}{2} - \sum_{k < m} |Q_k| \geq \frac{|C|}{2} - (1 - \xi) \left( \frac{K(3K - 2)}{(K - 1)^2} \right) \frac{|C|}{2}
$$

by (~†~) while

$$
|Q_0| < (1 - \xi) \left( \frac{K}{(K - 1)} \right) \frac{|C|}{2}
$$

by (~∗~). So $|Q^+| \leq |Q_0|$ would imply

$$
1 - (1 - \xi) \left( \frac{K(3K - 2)}{(K - 1)^2} \right) < (1 - \xi) \left( \frac{K}{(K - 1)} \right)
$$

which after some manipulation gives $\xi < 1 - \frac{(K - 1)^2}{K(4K - 3)}$ which one can show contradicts our choice of $\xi > 1 - \frac{(K - 1)^3}{13K^2(K + 1)}$. Hence $|Q^+| > |Q_0|$ so if $Q^+ \rightarrow C$ were an isometric embedding then this would contradict the choice of $u_0$ and $v_0$. So, $Q^+ \rightarrow C$ is not an isometric embedding and so $|Q^+| > \frac{|C|}{2}$. On the other hand

$$
|Q^+| \leq |Q^-| + 2 \max_{i < j} |Q_i|
$$

$$
\leq |Q_j^-| + \sum_{i < j} (|Q_i| - |R_i|) + 2 \max_{i < j} |Q_i|
$$

$$
\leq \frac{|C|}{2} + \sum_{i < j} (|Q_i| - |R_i|) + 2 \max_{i < j} |Q_i|
$$

$$
\leq \frac{|C|}{2} + 2 \sum_{i < j} (|Q_i| - |R_i|) + 2 \max_{i < j} |Q_i|
$$

$$
\leq \frac{|C|}{2} + 4 \sum_{i < j} |Q_i|
$$

$$
\leq \frac{|C|}{2} + (1 - \xi) \left( \frac{4K(3K - 2)}{(K - 1)^2} \right) \frac{|C|}{2}
$$

where the last inequality follows from (~†~). We have

$$
1 - \xi < \frac{(K - 1)^3}{13K^2(K + 1)} < \frac{(K - 1)^2}{12K(K + 1)} = \frac{(K - 1)^2}{4K(3K + 3)} < \frac{(K - 1)^2}{4K(3K - 2)}
$$

and so $|Q^+| < |C|$ so $Q^+$ embeds in $C$ and the endpoints $u, v$ of $Q^+$ in $C$ are at distance

$$
d_C(u, v) \geq \frac{|C|}{2} - (1 - \xi) \left( \frac{4K(3K - 2)}{(K - 1)^2} \right) \frac{|C|}{2}.
$$

But we also have

$$
d_f(f(u), f(v)) \leq |R^+| \leq \frac{1}{K} |Q^+| \leq \frac{1}{K} \left( \frac{|C|}{2} + (1 - \xi) \left( \frac{4K(3K - 2)}{(K - 1)^2} \right) \frac{|C|}{2} \right)
$$
which, by Proposition 3.1, implies
\[
\frac{1}{K} \left( \frac{|C|}{2} + (1 - \xi) \left( \frac{4K(3K - 2)}{(K - 1)^2} \right) \frac{|C|}{2} \right) \\
\geq \frac{|C|}{2} - (1 - \xi) \left( \frac{4K(3K - 2)}{(K - 1)^2} \right) \frac{|C|}{2} - (1 - \xi) \frac{|C|}{2}
\]
which is equivalent to
\[
1 - \xi \geq \frac{(K - 1)^3}{K(13K^2 + 2K - 7)}.
\]
But
\[
1 - \xi < \frac{(K - 1)^3}{13K^2(K + 1)} = \frac{(K - 1)^3}{K(13K^2 + 13K)} < \frac{(K - 1)^3}{K(13K^2 + 2K - 7)}
\]
so we have a contradiction. Therefore we have proved that \( Q_i \subset P_{0,i} \) for \( 1 < i < n \).

Then the \( Q_i \) are all pairwise disjoint in \( C \) and \( C \setminus (\bigcap_i P_i) = \bigcup_i Q_i \) so \( C_n \) is obtained from \( C \) by replacing \( Q_i \subset C \) with \( R_i \), for each \( i \). Then since \( f_n \) is \( K \)-bilipschitz and by \((\dagger)\) and \((\ddagger)\), we have
\[
\theta \geq |C_n| \geq |C| - \sum_i |Q_i| \geq |C| - (1 - \xi) \left( \frac{K(3K - 2)}{(K - 1)^2} \right) \frac{|C|}{2} \\
> |C| - \left( \frac{(K - 1)^2}{4K(3K - 2)} \right) \left( \frac{K(3K - 2)}{(K - 1)^2} \right) \frac{|C|}{2} \\
= \frac{7}{8} |C|
\]
So \( |C| < \frac{8}{7} \theta \) and we see that \( \frac{8}{7} \theta \) bounds the lengths of \( \xi \)-almost isometric cycles of \( \Gamma \).

\[\square\]

**Proposition 3.6.** Let \( \Gamma \) be a (strongly) shortcut graph. Then the graph obtained from \( \Gamma \) by subdividing each edge is (strongly) shortcut.

**Proof.** Let \( \Gamma' \) be the barycentric subdivision of \( \Gamma \). Then \( \Gamma' \) is isometric to \( \Gamma \) after scaling the metric by a factor of 2. Since isometric cycles are embedded, they have no backtracks and so every isometric cycle of \( \Gamma' \) is the subdivision of an isometric cycle of \( \Gamma \). Hence, if \( \theta \) bounds the lengths of the isometric cycles of \( \Gamma \) then \( 2\theta \) bounds the lengths of the isometric cycles of \( \Gamma' \). So if \( \Gamma \) is shortcut then \( \Gamma' \) is shortcut. Similarly, if there is a bound on the lengths of the \( K \)-bilipschitz cycles of \( \Gamma \) then there is a bound on the lengths of the \( K \)-bilipschitz cycles of \( \Gamma' \). So, by Proposition 3.5, if \( \Gamma \) is strongly shortcut then \( \Gamma' \) is strongly shortcut. \[\square\]

**Proposition 3.7.** Let \( G \) be a (strongly) shortcut group. Then \( G \) acts freely and cocompactly on a (strongly) shortcut graph.

**Proof.** Let \( G \) act properly and cocompactly on a (strongly) shortcut graph \( \Gamma \). If \( \Gamma \) has a single vertex then \( G \) is finite and so acts freely on any Cayley graph of \( G \) which is strongly shortcut because it is connected and finite. So we may assume that \( \Gamma \) has more than one vertex.

By Proposition 3.6, we may assume that \( G \) acts on \( \Gamma \) without edge inversions. Let \( \pi : G \times \Gamma^0 \to \Gamma^0 \) be the projection onto the second factor. Define a graph \( \tilde{\Gamma} \) on the vertex set \( G \times \Gamma^0 \) where \( \tilde{\Gamma} \) has an edge joining \((g,v)\) and \((g',v')\) for each edge joining \( v \) and \( v' \). Then the diagonal action \( G \subset G \times \Gamma^0 \) given by
\[
g \cdot (g',v) = (gg',gv)
\]
extends to \( \hat{\Gamma} \) and the projection \( \pi \) extends to a \( G \)-equivariant nondegenerate combinatorial map \( \pi: \hat{\Gamma} \to \Gamma \). That \( G \) acts on \( \Gamma \) without edge inversions rules out nontrivial fixed points of midpoints of edges of \( \hat{\Gamma} \) and so the action of \( G \) on \( \hat{\Gamma} \) is free. Let \( \{v_1, v_2, \ldots, v_k\} \) be a set of orbit representatives of \( G \cap \Gamma^0 \). Let \( \hat{\Gamma} \) be the induced subgraph of \( \hat{\Gamma} \) on \( \bigcup_{g \in G} \{ (g, gv_i) \} \). We will prove that \( \hat{\Gamma} \) is (strongly) shortcut and that the action of \( G \) on \( \hat{\Gamma} \) is cocompact.

Let \( \hat{\pi}: \hat{\Gamma} \to \Gamma \) be the restriction of \( \pi \) to \( \hat{\Gamma} \). Since \( G \) acts properly on \( \Gamma \), the preimage under \( \hat{\pi} \) of a vertex of \( \Gamma \) is finite and so \( \hat{\Gamma} \) is locally finite. Also, the vertex set of \( \hat{\Gamma} \) is the union of finitely many orbits \( \bigcup_{i=1}^k G \cdot (1, v_i) \) so the action of \( G \) on \( \hat{\Gamma} \) is cocompact. It remains to prove that \( \hat{\Gamma} \) is (strongly) shortcut.

For a vertex \( v \in \Gamma^0 \), we have \( v = gv_i \) for some \( i \) and so \( \hat{\pi}(g, gv_i) = v \). Moreover, since \( \hat{\Gamma} \) is an induced subgraph of \( \hat{\Gamma} \), for each pair of vertices \( (g, u), (h, v) \in \Gamma^0 \), we see that \( \hat{\pi} \) induces a bijection between the set of edges between \( (g, u) \) and \( (h, v) \) and the set of edges between \( u \) and \( v \). This implies that for any \( (g, u), (h, v) \in \Gamma^0 \), we can lift any path of nonzero length \( \alpha: P \to \Gamma \) between \( u \) and \( v \) to a path \( \hat{\alpha}: P \to \hat{\Gamma} \) from \( (g, u) \) to \( (h, v) \). The lift is not unique since, if the sequence of vertices visited by \( \alpha \) is \( (u = u_0, u_1, u_2, \ldots, u_k = v) \) then, for \( 0 < i < k \), the lift of \( u_i \) in \( \hat{\alpha} \) may be any \( (g, u_i) \in \hat{\pi}^{-1}(u_i) \).

If \( \Gamma \) is strongly shortcut then let \( \theta \geq 3 \) bound the lengths of the \( \hat{\xi} \)-almost isometric cycles of \( \Gamma \). Otherwise, let \( \theta \geq 3 \) bound the lengths of the isometric cycles of \( \Gamma \) and set \( \hat{\xi} = 1 \). Let \( \hat{\xi} = 1 + \frac{\xi}{2} \) and let \( f: C \to \hat{\Gamma} \) be a \((\xi\text{-almost})\) isometric cycle of length \(|C| > \theta \). By Proposition 3.3,

\[
d_{\Gamma}(\hat{\pi} \circ f(u), \hat{\pi} \circ f(v)) < \hat{\xi}d_{C}(u, v)
\]

for some \( u, v \in C^0 \) with \( d_{C}(u, v) \geq \left\lfloor \frac{|C|}{2} \right\rfloor - 1 \). If \( d_{\Gamma}(\hat{\pi} \circ f(u), \hat{\pi} \circ f(v)) > 0 \) then let \( \alpha: P \to \Gamma \) be a geodesic from \( \hat{\pi} \circ f(u) \) to \( \hat{\pi} \circ f(v) \). Otherwise, let \( \alpha: P \to \Gamma \) be a path of length 2 from \( \hat{\pi} \circ f(u) \) to \( \hat{\pi} \circ f(v) = \hat{\pi} \circ f(u) \). This is always possible since \( \Gamma \) is a connected graph on more than one vertex. By the previous paragraph, we may lift \( \alpha \) to a path \( \hat{\alpha}: P \to \hat{\Gamma} \) from \( f(u) \) to \( f(v) \). So we see that either

\[
d_{\Gamma}(f(u), f(v)) < \hat{\xi}d_{C}(u, v)
\]

or

\[
d_{\Gamma}(f(u), f(v)) \leq 2
\]

and so, by Proposition 3.1, one of

(§)

\[
d_{C}(u, v) - (1 - \hat{\xi})\frac{|C|}{2} < \hat{\xi}d_{C}(u, v)
\]

or

(¶)

\[
d_{C}(u, v) - (1 - \hat{\xi})\frac{|C|}{2} \leq 2
\]

must hold. Since \( d_{C}(u, v) \geq \frac{|C|}{2} - \frac{3}{2} \) we see that (¶) gives the bound \(|C| \leq \frac{7}{\hat{\xi}} \). On the other hand, (§) is equivalent to

\[
(1 - \hat{\xi})d_{C}(u, v) < (1 - \hat{\xi})\frac{|C|}{2}
\]

and so gives

\[
(\hat{\xi} - \frac{|C|}{2}) < (1 - \hat{\xi})\frac{3}{2}
\]
which is impossible if $\xi = 1$ and otherwise gives the bound $|C| \leq \frac{3(1-\xi)}{\xi-\xi}$. \qed

4. Filling properties and disk diagrams

In this section we study disk diagrams and the isoperimetric and isodiametric functions of (strongly) shortcut graphs and groups. Let $\Gamma$ be a graph and let $\theta \in \mathbb{N}$. For the purposes of the current discussion, a cycle $C$ is always based and oriented. Hence two cycles $f_1, f_2 : C \to \Gamma$ may be distinct even if $f_1 = f_2 \circ \psi$ for some $\psi \in \text{Aut}(C)$. Let

$$S_\theta = \{f : C \to \Gamma : |C| \leq \theta\}$$

be the set of cycles in $\Gamma$ of length less than or equal to $\theta$. The $\theta$-filling $F_\theta(\Gamma)$ is the 2-complex whose 1-skeleton is $\Gamma$ and whose 2-skeleton has a unique 2-cell with attaching map $f : C \to \Gamma$ for each $f \in S_\theta$. If a group $G$ acts on $\Gamma$ then $G$ acts on $S_\theta$ by $g \cdot f = \varphi_g \circ f$ where $\varphi_g \in \text{Aut}(\Gamma)$ is the automorphism by which $g$ acts on $\Gamma$. Thus the action of $G$ on $\Gamma$ extends to an action on $F_\theta(\Gamma)$ such that an element $g \in G$ stabilizes a 2-cell $F$ if and only if $g$ stabilizes $F$ pointwise.

For $\theta, N \in \mathbb{N}$ with $3 \leq \theta \leq N$ and $\xi \in (0, 1)$ consider the following property.

(\text{\textbf{1}}) Every cycle $C \to \Gamma$ with $\theta < |C| \leq N$ is not $\xi$-almost isometric.

\textbf{Remark 4.1.} If $\Gamma$ is shortcut then $\Gamma$ satisfies (\text{\textbf{1}}) for $\theta$ bounding the lengths of isometric cycles of $\Gamma$, for any $N \geq \theta$ and for $\xi \in \left(\frac{N-2}{N}, 1\right)$. Of course, if $\Gamma$ is strongly shortcut, then it satisfies (\text{\textbf{1}}) for a fixed $\xi$ not depending on $N$.

\textbf{Construction 4.2.} Let $\Gamma$ be a graph satisfying (\text{\textbf{1}}). Given a cycle $f : C \to \Gamma$ of length $|C| \leq N$ we will inductively construct a disk diagram $D_f \to F_\theta(\Gamma)$ for $f$.

If $|C| \leq \theta$ then $D_f \to F_\theta(\Gamma)$ is just a single 2-cell mapping to the 2-cell of $F_\theta(\Gamma)$ whose attaching map is isomorphic to $f$. Otherwise $f$ is not $\xi$-almost isometric and so, by Proposition 3.3,

$$d_\Gamma(f(u), f(v)) < \xi d_C(u, v)$$

for some pair of vertices $u, v \in C^0$ with $d_C(u, v) \geq \left\lfloor \frac{|C|}{\theta} \right\rfloor - 1$. Let $P$ and $Q$ be the two segments of $C$ joining $u$ and $v$. Let $g : R \to \Gamma$ be a geodesic path from $f(u)$ to $f(v)$ in $\Gamma$ and note that $|R| < \xi \min\{|P|, |Q|\}$. Glue $f$ and $g$ together along $u \sim g^{-1}(f(u))$ and $v \sim g^{-1}(f(v))$ to obtain a combinatorial map $h : (C \sqcup R)/\sim \to \Gamma$ with $h|_{P \cup R}$ and $h|_{Q \cup R}$ cycles of length

$$|P| + |R| < |P| + \xi|Q| < |C|$$

and

$$|Q| + |R| < |Q| + \xi|P| < |C|$$

and so, by induction we have disk diagrams $D_{h|_{P \cup R}} \to F_\theta(\Gamma)$ and $D_{h|_{Q \cup R}} \to F_\theta(\Gamma)$ for $h|_{P \cup R}$ and $h|_{Q \cup R}$. Gluing $D_{h|_{P \cup R}} \to F_\theta(\Gamma)$ and $D_{h|_{Q \cup R}} \to F_\theta(\Gamma)$ together along $R$ we obtain a disk diagram $D_f \to F_\theta(\Gamma)$ for $f$.

4.1. Simple connectedness.

\textbf{Theorem 4.3.} Let $\Gamma$ be a shortcut graph and let $\theta \geq 3$ bound the lengths of the isometric cycles of $\Gamma$. Then $F_\theta(\Gamma)$ is simply connected.

\textit{Proof.} Let $f : C \to \Gamma$ be a cycle in $\Gamma$. Then, by Remark 4.1 $\Gamma$ satisfies (\text{\textbf{1}}) for $\theta$ as given, for $N = |C|$ and for $\xi = \frac{N-1}{N}$. Hence, we may apply Construction 4.2 to obtain a disk diagram for $f$. \qed
Corollary 4.4. Let $G$ be a (strongly) shortcut group. Then there is a compact 2-complex $X$ with $G = \pi_1(X)$ such that the universal cover $\tilde{X}$ of $X$ has (strongly) shortcut 1-skeleton. In particular, the group $G$ is finitely presented.

Proof. By Proposition 3.7, there is a free and cocompact action of $G$ on a (strongly) shortcut graph $\Gamma$. In particular, the graph $\Gamma$ is shortcut so let $\theta \geq 3$ bound the lengths of the isometric cycles of $\Gamma$ and let
\[ \mathcal{C} = \{ f : (\tilde{C}, v) \to \Gamma : |\tilde{C}| \leq \theta \} \]
be the set of all based oriented cycles of length at most $\theta$ in $\Gamma$. Then $G$ acts on $\mathcal{C}$ so $G$ acts on the 2-complex $X$ obtained from $\Gamma$ by gluing in a 2-cell along each $f \in \mathcal{C}$.

The 2-complex $\tilde{X}$ is a supercomplex of the $\theta$-filling $F_\theta(\Gamma)$ and $X^1 = F_\theta(\Gamma)^1 = \Gamma$ so, by Theorem 4.3, the 2-complex $\tilde{X}$ is simply connected. The $G$-action on $\tilde{X}$ is free since if some $g \in G$ stabilizes a 2-cell then it must fix its boundary so, by the freeness of the action on $X^1 = \Gamma$, we have $g = 1$. \hfill \square

4.2. Isoperimetric function.

Theorem 4.5. Let $\Gamma$ be a graph. If $\Gamma$ is shortcut then, for $\theta$ large enough, the Dehn function $\Delta$ of the filling $F_\theta(\Gamma)$ satisfies $\Delta(n) \leq 2^n$. If $\Gamma$ is strongly shortcut for $\xi \in (0, 1)$ and $r > \frac{1}{1-\log_2(1+\xi)}$ then, for $\theta$ large enough, the Dehn function $\Delta$ of the filling $F_\theta(\Gamma)$ satisfies $\Delta(n) \leq n^\theta$.

Proof. Suppose $\Gamma$ is shortcut and let $\theta \geq 3$ bound the lengths of the isometric cycles of $\Gamma$. Let $\Delta : \mathbb{N} \to \mathbb{N}$ be the Dehn function of $F_\theta(\Gamma)$. We will prove, by induction on $n$ that $\Delta(n) \leq 2^n$. If $n \leq \theta$ then this clearly holds since any cycle of length at most $\theta$ bounds a 2-cell in $F_\theta(\Gamma)$. Let $f : C \to \Gamma$ be a cycle of length $n > \theta$. Applying Construction 4.2 to $f$ with $N = n$ and $\xi = \frac{n-1}{n}$ we see that $f$ bounds a disk diagram $D_f$ which is the union of two disk diagrams of boundary length less than $n$. Hence $f$ bounds a disk of area at most $2\Delta(n-1)$. By induction
\[ 2\Delta(n-1) \leq 2 \cdot 2^{n-1} = 2^n \]
and so we have $\Delta(n) \leq 2^n$.

Suppose $\Gamma$ is strongly shortcut. Choose $L \in \mathbb{N}$ with $L > 3$. Let $\theta \geq \frac{L}{1-\xi}$ bound the lengths of the $\xi$-almost isometric cycles of $\Gamma$. We will prove that the Dehn function of $F_\theta(\Gamma)$ satisfies $\Delta(n) \leq n^{\log_b(2)}$ for $b = \frac{2L}{(L-3)\xi + L + 3}$. Note that $b > 1$ and that $b$ tends to $\frac{2}{1+\xi}$ as $L$ goes to infinity so that $\log_b(2)$ tends to $\frac{\log_2(2)}{\log_2(\frac{2}{1+\xi})} = \frac{1}{1-\log_2(1+\xi)}$. So if $r > \frac{1}{1-\log_2(1+\xi)}$ we may choose $L$ large enough that $r > \log_b(2)$. Going forward we assume that we have such a choice of $L$.

The argument proceeds as in the shortcut case but in the inductive step $f$ bounds a disk diagram which is the union of two disk diagrams of boundary length strictly less than
\[ \xi \frac{n^2}{2} + \left\lceil \frac{n}{2} \right\rceil + 1 \leq \xi \frac{n^2}{2} + \frac{n}{2} + 1 = \frac{1}{2} \left( \xi + 1 + \frac{3}{n} \right)n \leq \frac{1}{2} \left( \xi + 1 + \frac{3}{\theta} \right)n \]
so, since $\theta \geq \frac{L}{1-\xi}$ we have a disk diagram for $f$ of area at most
\[ 2\Delta \left( \left\lceil \frac{1}{2} \left( \xi + 1 + \frac{3(1-\xi)}{L} \right)n \right\rceil \right) = 2\Delta \left( \left\lceil \frac{1}{2L} \left( (L-3)\xi + L + 3 \right)n \right\rceil \right) = 2\Delta \left( \left\lceil \frac{1}{b}n \right\rceil \right) \]
and so by induction we have
\[
2\Delta\left(\left\lfloor \frac{1}{b}n \right\rfloor \right) \leq 2\left(\left\lfloor \frac{1}{b}n \right\rfloor \right)^{\log_b(2)} \leq 2\left(\frac{1}{b}n \right)^{\log_b(2)} = 2n^{\log_b(2)} = n^{\log_b(2)}
\]
and so we have that
\[
\Delta(n) \leq n^{\log_b(2)} < n^r.
\]
\[\Box\]

**Corollary 4.6.** Let \( G \) be a group. If \( G \) is shortcut then it has an exponential isoperimetric function. If \( G \) is strongly shortcut then it has a polynomial isoperimetric function.

**Corollary 4.7.** Let \( G \) be a shortcut group. Then \( G \) has a decidable word problem.

**Corollary 4.8.** Let \( \Gamma \) be a graph that is strongly shortcut for some \( \xi \in (0, \sqrt{2} - 1) \). Then \( \Gamma \) is hyperbolic.

**Proof.** We have \( \frac{1}{1-\log_b(1+\xi)} < \frac{1}{1-\log_b(\sqrt{2})} = 2 \) so we can choose \( r < 2 \) such that \( r > \frac{1}{1-\log_b(1+\xi)} \). Then by Theorem 4.5, for \( \theta \) large enough, the Dehn function \( \Delta \) of the filling \( F_\theta(\Gamma) \) satisfies \( \Delta(n) \leq n^r \) so is subquadratic. Then, by the isoperimetric gap [10, 22, 4], \( \Gamma \) has linear isoperimetric function and so is hyperbolic. \[\Box\]

### 4.3. Isodiametric function.

**Theorem 4.9.** Let \( \Gamma \) be a graph. If \( \Gamma \) is shortcut then, for \( \theta \) large enough, the filling \( F_\theta(\Gamma) \) has an exponential isodiametric function. If \( \Gamma \) is strongly shortcut then, for \( \theta \) large enough, the filling \( F_\theta(\Gamma) \) has a polynomial isodiametric function.

**Proof.** For a cycle \( f: C \to \Gamma \) let \( \text{diam}(f) \) denote the minimum diameter of a disk diagram for \( f \). Observe that in Construction 4.2 \( \text{diam}(f) \leq \text{diam}(h|_{P\cup R}) + \text{diam}(h|_{Q\cup R}) \). Indeed we may glue together minimal diameter disk diagrams of \( h|_{P\cup R} \) and \( h|_{Q\cup R} \) along \( R \) to obtain a disk diagram for \( f \). Using this observation, the proof follows virtually identically to that of Theorem 4.5. \[\Box\]

**Corollary 4.10.** Let \( G \) be a group. If \( G \) is shortcut then it has an exponential isodiametric function. If \( G \) is strongly shortcut then it has a polynomial isodiametric function.

### 5. Combinations

In this section we show that (strongly) shortcut graphs and groups are closed under products and that a finite graph of (strongly) shortcut groups with finite edge groups is (strongly) shortcut.

#### 5.1. Products.

Let \( \Gamma_1 \) and \( \Gamma_2 \) be simplicial graphs. The product graph \( \Gamma_1 \times \Gamma_2 \) of \( \Gamma_1 \) and \( \Gamma_2 \) is the 1-skeleton of the CW complex product of \( \Gamma_1 \) and \( \Gamma_2 \). The vertex set of \( \Gamma_1 \times \Gamma_2 \) is \( \Gamma_1^0 \times \Gamma_2^0 \) and the edges of \( \Gamma_1 \times \Gamma_2 \) are given by \( (u_1, u_2) \sim (v_1, v_2) \) whenever

\[
u_1 = v_1 \text{ and } u_2 \sim v_2
\]

or

\[
u_1 \sim v_1 \text{ and } u_2 = v_2
\]

where \( \sim \) is the edge relation.

**Theorem 5.1.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be (strongly) shortcut graphs. Then \( \Gamma_1 \times \Gamma_2 \) is (strongly) shortcut.
Proof. Let \( \Gamma_1 \) and \( \Gamma_2 \) be (strongly) shortcut and let \( \theta \) bound the lengths of the \((\xi\text{-almost})\) isometric cycles of the \( \Gamma_i \). Let \( \Gamma = \Gamma_1 \times \Gamma_2 \) and let \( f : C \to \Gamma \) be a cycle of length \( |C| \geq 2\theta \). We combine the shortcut and strongly shortcut cases as follows. If the \( \Gamma_i \) are strongly shortcut then we have \( \theta \) and \( \xi \) as given. Otherwise, by Remark 4.1, the \( \Gamma_i \) satisfy (1) for \( \theta \) as given, for \( N = |C| \) and for some \( \xi \) depending on \( N \). We will show that for some antipodal pair of points \( p, q \in C \) we have \( d_T(f(p), f(q)) < \left( \frac{1+\xi}{2} \right) \frac{|C|}{2} \).

Each edge of \( C \) projects nondegenerately onto exactly one of \( \Gamma_1 \) or \( \Gamma_2 \). Call those edges that project nondegenerately onto \( \Gamma_1 \) horizontal edges and those that project nondegenerately onto \( \Gamma_2 \) vertical edges. Without loss of generality the number of horizontal edges is greater than or equal to the number of vertical edges. Let \( f_1 : C_1 \to \Gamma_1 \) be the cycle obtained from \( C \to \Gamma_1 \) by contracting the vertical edges of \( C \). Then \( N \geq |C_1| \geq \frac{|C|}{2} \geq \theta \) so we have \( d_T(f_1(p_1), f_1(q_1)) < \xi d_{C_1}(p_1, q_1) \) for some antipodal pair of points \( p_1, q_1 \in C_1 \). Let \( p', q' \in C \) map to \( p_1 \) and \( q_1 \) under the contraction map \( C \to C_1 \). We may choose \( p' \) and \( q' \) so that they are not contained in the interior of any vertical edge. Let \( \ell \) be the number of vertical edges in a geodesic segment of \( C \) between \( p' \) and \( q' \). Then \( d_C(p', q') = \frac{|C_1|}{2} + \ell \) while \( d_T(f(p'), f(q')) < \xi \frac{|C_1|}{2} + \ell \). Let \( P \subset C \) be a geodesic segment of length \( \frac{|C|}{2} \) containing \( p' \) and \( q' \) and let \( p \) and \( q \) be the endpoints of \( P \) with \( p \) nearest to \( p' \) and \( q \) nearest to \( q' \). Then \( d_C(p, q) = \frac{|C|}{2} \) while

\[
d_T(f(p), f(q)) \leq d_T(f(p), f(p')) + d_T(f(p'), f(q')) + d_T(f(q'), f(q))
\leq d_C(p, p') + d_T(f(p'), f(q')) + d_C(q', q)
= d_C(p, q) - d_C(p', q') + d_T(f(p'), f(q'))
< \frac{|C|}{2} - \left( \frac{|C_1|}{2} + \ell \right) + \xi \frac{|C_1|}{2} + \ell
= \frac{|C|}{2} - (1 - \xi) \frac{|C_1|}{2}
= \left( \frac{1+\xi}{2} \right) \frac{|C|}{2}
\]
and so \( d_T(f(p), f(q)) < \left( \frac{1+\xi}{2} \right) \frac{|C|}{2} \). \( \square \)

**Corollary 5.2.** Let \( G_1 \) and \( G_2 \) be (strongly) shortcut groups. Then \( G_1 \times G_2 \) is (strongly) shortcut.

5.2. **Trees of shortcut graphs.** Let \( T \) be a tree. An **arc decomposition** of \( T \) is a partition of the set of edges of \( T \) such that the following conditions hold.

1. The union of the edges in each part is isomorphic to a path, which we refer to as an **arc**.
2. The interior vertices of each arc all have degree two.

The endpoints of the arcs of an arc decomposition are called **nodes**. Every tree comes equipped with a default arc decomposition whose arcs are simply its edges. A **tree of graphs with discrete edge graphs** is a surjective (possibly degenerate) combinatorial map \( \Gamma \to T \) from a graph \( \Gamma \) to a tree \( T \) that is equipped with an arc decomposition such that the following conditions hold.

1. The preimage of each node \( v \) is a connected subgraph \( \Gamma_v \) called the **vertex graph** at \( v \) of \( \Gamma \to T \).
Proof. Let \( e \) and \( C \) be as in Lemma 5.4 and let \( P \) be the closure of a component of \( C \setminus f^{-1}(w) \). We need to show that \( |P| \leq L + 1 \). The initial and terminal edges \( e_1 \) and \( e_2 \) of \( P \) map to the same edge of \( T \) and so either \( |P| \leq L + 1 \) or \( |P| \geq |C| - L + 1 \). But \( |C| - L + 1 \geq \frac{2|C|}{3} \) while, by our choice of \( w \), we have \( |P| \leq \frac{2|C|}{3} \). Hence \( |P| \leq L + 1 \) as required.

A cycle with degeneracies is \( \xi \)-almost isometric if

\[
\delta_T(f(p), f(q)) \geq \xi \frac{|C|}{2}
\]

for any antipodal pair of points \( p, q \in C \).
Lemma 5.6. Let $\Gamma$ be strongly shortcut with $\theta$ bounding the lengths of the $\xi$-almost isometric cycles of $\Gamma$. Then there exist $\xi' \in (0,1)$ and $\theta' \in \mathbb{N}$ depending only on $\xi$ and $\theta$ such that $\theta'$ bounds the lengths of the $\xi'$-almost isometric cycles with degeneracies of $\Gamma$.

Proof. Let $\theta$ bound the lengths of the $\xi$-almost isometrically embedded cycles of $\Gamma$. Let $f : C \to \Gamma$ be a cycle with degeneracies. Define $S \subset C$ as the union of all edges of $C$ that map to vertices under $f$. We may assume that $S \neq C$ since, otherwise $f$ is the constant map and so satisfies the conclusion of the lemma for any $\xi'$. Let $(P_i)_{i=1}^t$ be the sequence of components of $S$ in the order they are visited in some traversal of $C$. We begin by showing, for $\theta'' = \max \{ \theta, \frac{4}{1-\xi} \}$ and $\xi'' = \frac{1+\xi}{2}$, that if $|C| > \theta''$ and $|P_i|$ is even for each $i$ then there exist antipodal points $p,q \in C$ such that $d_\Gamma(f(p), f(q)) < \xi'' \frac{|C|}{2}$. Call the condition that the $|P_i|$ are even the parity condition. Later we will use this result to prove the statement of the lemma, for $\theta' \geq \max \{ \theta'', \frac{2(2+\xi'')}{1-\xi} \}$ and $\xi' = \frac{1+\xi''}{2}$, with no assumption on the parities of the $|P_i|$.

Assume $f$ satisfies the parity condition and $|C| > \theta''$. We obtain from $f$ a cycle without degeneracies $f' : C \to \Gamma$ by setting $f'|_{C \setminus S} = f|_{C \setminus S}$ and mapping each edge of $P_i$ onto $f(e_i)$ where $e_i$ is the edge that follows $P_i$ in some fixed orientation of $C$. This is possible since $f$ satisfies the parity condition. Thus $f'$ folds $P_i$ onto $f(e_i)$ in a zig-zag fashion. Then for any point $p \in C$, we have $d_\Gamma(f(p), f'(p)) \leq 1$. If $|C| > \theta''$ then there is a pair of antipodal points $p,q \in C$ such that $d_\Gamma(f'(p), f'(q)) \leq \xi'' \frac{|C|}{2}$. Hence

$$d_\Gamma(f(p), f(q)) < \xi'' \frac{|C|}{2} + 2 = \left( \xi + \frac{4}{|C|} \right) \frac{|C|}{2} \leq \left( \xi + \frac{4}{\theta''} \right) \frac{|C|}{2} \leq \xi'' \frac{|C|}{2}$$

as required.

We now consider a general $f : C \to \Gamma$ that does not necessarily satisfy the parity condition. Assume that $|C| > \theta'$. Let $i_0 < i_1 < \cdots < i_{m-1}$ be the set of indices for which $|P_i|$ is not even and assume $|C| > \theta'$. Obtain a cycle $f' : C'' \to \Gamma$ from $f$ by contracting an edge in each $P_{i_j}$ with $j$ odd and expanding a vertex $v$ to an edge $e \to f(v)$ in each $P_{i_j}$ with $j$ even. Then $f'$ satisfies the parity condition and we have $|C| \leq |C''| \leq |C| + 1$. There is a relation $R \subset C \times C''$ with $pRq$ if and only if one of the following holds.

1. $p$ was obtained directly from $q$
2. $p$ is contained in an edge that was contracted to $q$
3. $p$ is a vertex which was expanded to an edge that contains $q$

By the alternating nature of the expansions and contractions we see that if $pRq'$ and $qRq''$ then $d_{C'}(p,q) \geq d_{C'}(p',q') - 1$. By the previous paragraph, we have a pair of antipodal points $p', q' \in C'$ such that $d_{C'}(f'(p'), f'(q')) < \xi'' \frac{|C'|}{2}$. Take any $p'', q'' \in C$ satisfying $p''Rq'$ and $q''Rq''$. Then $f(p'') = f'(p')$ and $f(q'') = f'(q')$ and so $d_\Gamma(f(p''), f(q'')) < \xi'' \frac{|C'|}{2} \leq \xi'' \frac{|C|+1}{2}$ and yet $d_{C'}(p'', q'') \geq \frac{|C|}{2} - 1$. Hence, as $f$ is 1-Lipschitz, for some antipodal pair of points $p,q \in C$ we have

$$d_\Gamma(f(p), f(q)) < \xi'' \frac{|C|+1}{2} + 1 = \left( \xi'' + \frac{\xi''+2}{|C|} \right) \frac{|C|}{2} < \left( \xi'' + \frac{\xi''+2}{\theta''} \right) \frac{|C|}{2} \leq \xi'' \frac{|C|}{2}$$

as required. \qed
Theorem 5.7. Let φ: Γ → T be a tree of graphs with discrete edge graphs satisfying the following conditions.

1. The vertex graphs Γ_v are uniformly (strongly) shortcut in the sense that there exists θ ≥ 3 (and ξ ∈ (0,1)) such that θ bounds the lengths of the (ξ-almost) isometric cycles of every vertex graph.

2. The arcs of T all have some common length M such that, for every attaching map Γ_v,Φ of φ, the diameter of Γ_v,Φ(Γ^Φ) is at most M.

Then Γ is (strongly) shortcut.

Proof. We will first consider the case where the vertex graphs are shortcut. Let f: C → Γ be an isometric cycle. If f maps entirely into a single vertex graph then |C| ≤ θ by hypothesis. So, suppose the image of f contains some edge in the lift P of an arc P of T. Then, since f is injective, it must traverse all of P and, by consideration of φ ∘ f, it must also traverse some other lift ¯P of P in the opposite direction. Let Q and Q' be the segments of C which map isomorphically to P and ¯P. Let u ∈ Q and u' ∈ Q' be endpoints of Q and Q' mapping to the same vertex graph. Then dT(u, f(u')) ≤ M and so dC(u, u') ≤ M whereas Q and Q' each have length M. Hence the geodesic segment R of C between u and u' is disjoint from the interiors of Q and Q'. The same goes for the geodesic segment R' between the other pair of endpoints of Q and Q'. Then C is covered by the segments Q, Q', R and R', each of which has length at most M. Hence the lengths of the isometric cycles of Γ are bounded by max{θ, 4M}.

The case where the vertex graphs are strongly shortcut requires a more delicate argument relying on the preceding lemmas. By Lemma 5.6 we can replace θ and ξ so that θ bounds the lengths of the ξ-almost isometric cycles with degeneracies of all the vertex graphs. Let ξ' = \( \frac{ξ + 2}{3} \) and let θ' = max{θ, \( \frac{24M+6}{ξ-1} \)}. Let f: C → Γ be a ξ'-almost isometric cycle. We will prove that |C| ≤ θ'. Since \( \frac{24M+6}{ξ-1} \geq 4 \) we may assume that |C| ≥ 4.

If the image of f is contained entirely in a single vertex graph then |C| ≤ θ ≤ θ' since ξ' ≥ ξ. So let us assume that f is not confined to a single vertex graph. Then φ ∘ f maps some pair of distinct edges e and e' of C onto a common edge of some arc P of T. Then f maps e and e' onto a pair of edges in the same relative position in lifts of P. Let ¯e and ¯e' be the images of e and e' under f. Since P has length M and the attaching maps of φ have diameter bounded by M, we have dT(¯e, ¯e') ≤ 2M where p_e and p_e' are the midpoints of ¯e and ¯e'. Then by Proposition 3.1 dC(p_e, p_e') ≤ 2M + (1 - ξ')\( \frac{|C|}{2} \) where p_e and p_e' are the midpoints of e and e'. Let L = 2M + (1 - ξ')\( \frac{|C|}{2} \). If L ≥ \( \frac{|C|}{3} + 1 \) then we have

\[
0 \leq 2M + \left(1 - \frac{ξ + 2}{3}\right)\frac{|C|}{2} - \frac{|C|}{3} - 1
= 2M - 1 + \left(\frac{1}{2} - \frac{ξ + 2}{6} - \frac{1}{3}\right)|C|
= 2M - 1 + \left(\frac{ξ + 1}{6}\right)|C|
\]

and so |C| ≤ \( \frac{6(2M-1)}{ξ+1} \) ≤ \( \frac{24M+6}{ξ-1} \) ≤ θ'. So we may assume that L < \( \frac{|C|}{3} + 1 \) and can apply Lemma 5.5 to φ ∘ f and L to obtain a vertex w ∈ T^0 such that any segment Q ⊂ C whose interior is disjoint from (φ ∘ f)^{-1}(w) has length |Q| ≤ L + 1. Let v ∈ (φ ∘ f)^{-1}(w) be a vertex.
Suppose \( w \) is an interior vertex of an arc \( P \) of \( T \). Let \( p \) be the antipode of \( v \) and let \( p' \) be a point of \((\varphi \circ f)^{-1}(w)\) that is nearest to \( p \). Then \( d_{C}(p, p') \leq \frac{L+1}{2} \) and so \( d_{C}(p', v) \geq \frac{|C|}{2} - \frac{L+1}{2} \). So, since arcs have length \( M \) and the images of attaching maps of \( \varphi \) have diameter at most \( M \), we have

\[
2M \geq d_{\Gamma}(f(p'), f(v))
\]

\[
\geq d_{C}(p', v) - (1 - \xi') \frac{|C|}{2}
\]

\[
\geq \frac{|C|}{2} - \frac{L+1}{2} - (1 - \xi') \frac{|C|}{2}
\]

\[
= \xi' \frac{|C|}{2} - \frac{L+1}{2}
\]

where the second inequality holds by Proposition 3.1. So, recalling that \( L = 2M + (1 - \xi') \frac{|C|}{2} \), we have

\[
2M \geq \xi' \frac{|C|}{2} - M - (1 - \xi') \frac{|C|}{4} - \frac{1}{2}
\]

which gives \( |C| \leq \frac{4M+2}{4\xi - 1} = \frac{4M+2}{4\xi - 1} < \frac{24M+6}{4\xi} < \theta' \).

Suppose \( w \) is a node of \( T \). Then \((\varphi \circ f)^{-1}(w) = f^{-1}(\Gamma_{w})\). Let \((P_{i})_{i}\) be the components of \( f^{-1}(\Gamma_{w}) \) and let \((Q_{j})_{j}\) be the closures of the components of \( C \setminus f^{-1}(\Gamma_{w}) \). Then \(|Q_{j}| \leq L + 1\) for each \( j \) and \( f \) maps each \( P_{i} \) into \( \Gamma_{w} \) and maps each \( Q_{j} \) into the closure of the complement of \( \Gamma_{w} \). We will define a cycle with degeneracies \( f' : C \rightarrow \Gamma_{w} \) that agrees with \( f \) on the \( P_{i} \) and that maps each \( Q_{j} \) onto a geodesic of \( \Gamma_{w} \). To see that this is possible, we need only to show that the endpoints of each \( Q_{j} \) map to a distance of at most \( |Q_{j}| \) in \( \Gamma_{w} \). The endpoints of \( Q_{j} \) map to a distance of at most \( M \) since \( M \) bounds the diameters of the attaching maps of \( \varphi \). So we need only consider the case where \( |Q_{j}| < M \). But then \( Q_{j} \) is not long enough for \( f|Q_{j} \) to traverse the lift of an arc of \( T \) since the arcs have length \( M \). Hence the endpoints of \( Q_{j} \) map to the same vertex of \( \Gamma_{w} \). So we are able to define \( f' : C \rightarrow \Gamma_{w} \). Then, for a point \( p \in C \), we have \( d_{\Gamma}(f(p), f'(p)) \leq M + \frac{L+1}{2} \).

So if \( p, q \in C \) are any antipodal pair then

\[
d_{\Gamma_{w}}(f'(p), f'(q)) \geq d_{\Gamma}(f'(p), f'(q))
\]

\[
\geq d_{\Gamma}(f(p), f(q)) - 2M - L - 1
\]

\[
\geq \xi' \frac{|C|}{2} - 2M - L - 1
\]

and so \( d_{\Gamma_{w}}(f'(p), f'(q)) \geq (\xi' - \frac{2(2M+L+1)}{|C|}) \frac{|C|}{2} \). So as long as \( \xi' - \frac{2(2M+L+1)}{|C|} \geq \xi \), then \( f' \) is \( \xi \)-almost isometric and thus we have \( |C| \leq \theta \). If \( \xi' - \frac{2(2M+L+1)}{|C|} < \xi \), then we have

\[
0 < \xi - \frac{\xi + 2}{3} + \frac{2(2M + 2M + (1 - \xi + 2) \frac{|C|}{2}) + 1}{|C|}
\]

\[
= \frac{2\xi - 2}{3} + \frac{8M + 2}{|C|} + \left(1 - \frac{\xi + 2}{3} \right) + \frac{2}{|C|}
\]

\[
= \frac{\xi - 1}{3} + \frac{8M + 2}{|C|}
\]

and so \( |C| < \frac{24M+6}{4\xi} \leq \theta' \). \(\Box\)
Corollary 5.8. Let $\mathcal{G}$ be a finite graph of (strongly) shortcut groups with finite edge groups. Then the fundamental group of $\mathcal{G}$ is (strongly) shortcut.

Proof. Let $\Gamma$ be the underlying graph of $\mathcal{G}$. We construct a graph of spaces $\mathcal{H}$ on $\Gamma$ such that the fundamental group functor sends $\mathcal{H}$ to $\mathcal{G}$. See Scott and Wall for this viewpoint on graphs of groups [24]. By Corollary 4.4, we can choose the vertex spaces so that their universal covers have (strongly) shortcut 1-skeleton. The 1-skeleton $\tilde{\Gamma}$ of the universal cover of $\mathcal{H}$ has the structure $\tilde{\Gamma} \to T$ of a tree of graphs $\tilde{\Gamma} \to T$ where $T$ is the Bass-Serre tree of $\mathcal{G}$. The fundamental group $\pi_1(\mathcal{G})$ acts freely and cocompactly on $\Gamma$. For $M$ large enough, subdividing each edge of $T$ into an arc of length $M$ results in a tree of graphs that satisfies the conditions of Theorem 5.7.

Corollary 5.9. Amalgamations and HNN extensions of (strongly) shortcut groups over finite subgroups are (strongly) shortcut.

Note that BS(1, 2) is an HNN extension of $\mathbb{Z}$ but is not strongly shortcut. Hence, we see that the condition that the edge groups be finite is essential in the strong shortcut case.

6. Examples

In this section we prove that hyperbolic graphs, 1-skeletons of CAT(0) cube complexes, the standard Cayley graphs of finitely generated Coxeter groups and all Cayley graphs of $\mathbb{Z}$ and $\mathbb{Z}^2$ are strongly shortcut. In particular, hyperbolic groups, cocompactly cubulated groups and finitely generated Coxeter groups are strongly shortcut.

6.1. Hyperbolic graphs. The thinness of geodesic bigons in a hyperbolic graph immediately implies the shortcut property. In this section we will prove that hyperbolic graphs are in fact strongly shortcut. To do so we will make use of the following proposition whose proof is given in Bridson and Haefliger [5].

Proposition 6.1 (Specialization of Proposition 1.6 of Part III of Bridson and Haefliger [5]). Let $\Gamma$ be a $\delta$-hyperbolic graph. Let $f : P \to \Gamma$ be a 1-Lipschitz map to $\Gamma$ from a compact interval $P \subset \mathbb{R}$. If $Q \subset \Gamma$ is the image of a geodesic joining the endpoints of $f$, then

$$d_\Gamma(x, f(P)) \leq \delta \max \{0, \log_2 |P|\} + 1$$

for every $x \in Q$.

Theorem 6.2. Let $\Gamma$ be a hyperbolic graph. Then $\Gamma$ is strongly shortcut.

Proof. Let $\delta \geq 1$ be a hyperbolicity constant for $\Gamma$. Suppose $f : C \to \Gamma$ is a $\frac{3}{4}$-almost isometric cycle of length $|C| \geq 2$. Let $y, y' \in C$ be a pair of antipodal points and let $P_1 \subset C$ and $P_2 \subset C$ be the two segments of $C$ between $y$ and $y'$. Let $Q$ be the image of a geodesic in $\Gamma$ from $f(y)$ to $f(y')$ and let $x$ be the midpoint of $Q$. Then, by Proposition 6.1, there are points $p_1 \in P_1$ and $p_2 \in P_2$ such that $f(p_1)$ and $f(p_2)$ are each at distance at most $\delta \log_2 \frac{|C|}{2} + 1$ from $x$ in $\Gamma$. Then, since $f$ is $\frac{3}{4}$-almost
isometric, we have \(|Q| \geq \frac{3|C|}{8}\) and
\[
\frac{3|C|}{16} \leq \frac{1}{2}|Q| = d_\Gamma(f(y), x) \\
\leq d_\Gamma(f(y), f(p_1)) + d_\Gamma(f(p_1), x) \\
\leq d_C(y, p_1) + \delta \log_2 \frac{|C|}{2} + 1
\]
and so \(d_C(y, p_1) \geq \frac{3|C|}{16} - \delta \log_2 \frac{|C|}{2} - 1\). By the same argument we have the same lower bound for \(d_C(y, p_2)\) and \(d_C(y', p_1)\) and \(d_C(y', p_2)\). Hence
\[
d_C(p_1, p_2) \geq \frac{3|C|}{8} - 2\delta \log_2 \frac{|C|}{2} - 2
\]
and so, by Proposition 3.1, we have
\[
d_\Gamma(f(p_1), f(p_2)) \geq \frac{3|C|}{8} - 2\delta \log_2 \frac{|C|}{2} - 2 - \left(1 - \frac{3}{4}\right) \frac{|C|}{2} = \frac{|C|}{4} - 2\delta \log_2 \frac{|C|}{2} - 2
\]
but \(f(p_1)\) and \(f(p_2)\) are both within a distance of \(\delta \log_2 \frac{|C|}{2} + 1\) to \(x\) and so
\[
d_\Gamma(f(p_1), f(p_2)) \leq 2\delta \log_2 \frac{|C|}{2} + 2.
\]
Hence we have
\[
|C| \leq 16\left(\delta \log_2 \frac{|C|}{2} + 1\right)
\]
which bounds the length \(|C|\) of \(f\).

\[\square\]

**Corollary 6.3.** Hyperbolic groups are strongly shortcut.

6.2. CAT(0) cube complexes. In this section we will prove that the 1-skeleton of a finite-dimensional CAT(0) cube complex is strongly shortcut. The proof rests on a theorem about edge colorings of cycles.

Let \(C\) be a cycle. An edge coloring of \(C\) is a function \(\alpha: C^{(1)} \to W\) from the set \(C^{(1)}\) of edges of \(C\) to some set \(W\) of colors. A cycle \(C\) along with an edge coloring \(\alpha: C^{(1)} \to W\) is a wall cycle if \(\alpha\) is surjective and, for each \(w \in W\), the number \(|\alpha^{-1}(w)|\) of edges of color \(w\) is even. In this case we may refer to the elements of \(W\) as walls.

Let \((C, \alpha)\) be a wall cycle. A combinatorial segment \(P \subset C\) crosses a wall \(w \in W\) if the number of edges of \(P\) colored \(w\) is odd. A combinatorial segment \(P \subset C\) begins and ends with a wall \(w \in W\) if the initial and terminal edges of \(P\) map to \(w\) under \(\alpha\). Two distinct walls \(w_1, w_2 \in W\) cross if for some combinatorial segment \(P \subset C\), we have that \(P\) begins and ends with one of the two walls and \(P\) crosses the other of the two walls. The dimension \(d\) of a wall cycle \((C, \alpha)\) is defined as \(d = \max\{1, n\}\) where \(n\) is the size of the largest set \(S \subseteq W\) of pairwise crossing walls. The wall crossing distance \(d_\alpha(u, v)\) between a pair of vertices \(u, v \in C^0\) is defined as the number of walls crossed by a segment \(P \subset C\) from \(u\) to \(v\). Note that the choice of segment \(P\) does not matter since each wall appears an even number of times along \(C\).

**Proposition 6.4.** Let \(X\) be a CAT(0) cube complex. Let \(W\) be the set of hyperplanes of \(X\) and let \(\beta: X^{(1)} \to W\) map each edge \(e\) of \(X\) to the hyperplane that \(e\) crosses. Then for any cycle \(f: C \to X^1\), the coloring \((C, \alpha)\) is a wall cycle, where \(\alpha(e) = \beta(f(e))\) for \(e \in C^{(1)}\). Moreover, two crossing walls of \((C, \alpha)\) must cross in \(X\) and so the dimension of \(X\) is at least the dimension of \((C, \alpha)\). Lastly, the wall crossing distance on \((C, \alpha)\) satisfies \(d_\alpha(u, v) = d_{X^1}(f(u), f(v))\).
Proof: That \((C, \alpha)\) is a wall cycle is a consequence of the fact that hyperplanes of a \(\text{CAT}(0)\) complex are two-sided. That two crossing walls of \((C, \alpha)\) must cross in \(X\) is a consequence of the fact that hyperplanes are connected and two-sided. The dimension of a \(\text{CAT}(0)\) cube complex is equal to the size of the largest set of its pairwise crossing hyperplanes. Finally, the combinatorial distance between two vertices of a \(\text{CAT}(0)\) cube complex is equal to the number of hyperplanes separating them. \(\square\)

In light of Proposition 6.4, the following theorem implies that the 1-skeleta of \(d\)-dimensional \(\text{CAT}(0)\) cube complexes are strongly shortcut.

**Theorem 6.5.** Let \((C, \alpha)\) be a \(d\)-dimensional wall cycle. If \(d_\alpha(u, v) \geq \frac{(5d-1)}{5d} |C|\) for all antipodal pairs of vertices \(u, v \in C^0\) then \(|C| \leq \frac{50d^2}{5d-1}\).

**Corollary 6.6.** The 1-skeleta of finite dimensional \(\text{CAT}(0)\) cube complexes are strongly shortcut.

**Corollary 6.7.** Cocompactly cubulated groups are strongly shortcut.

A group is cocompactly cubulated if it acts properly and cocompactly on a \(\text{CAT}(0)\) cube complex. Such groups include right-angled Artin groups whose standard Cayley graphs are 1-skeleta of \(\text{CAT}(0)\) cube complexes [6] but not all Artin groups are cocompactly cubulated [11, 15]. It is thus natural to ask if all Artin groups are (strongly) shortcut.

The proof of Theorem 6.5 relies on several lemmas and on the following theorem of Turan.

**Theorem 6.8** (Turan’s Theorem). Let \(\Gamma\) be a simplicial graph on \(n\) vertices. If every complete subgraph of \(\Gamma\) has at most \(d \in \mathbb{N}\) vertices then \(\Gamma\) has at most \((\frac{4}{d})^{n^2/2}\) edges.

Several proofs of Turan’s Theorem are given in Aigner and Ziegler [1].

**Lemma 6.9.** Let \((C, \alpha)\) be a wall cycle and suppose that for some \(\xi \in (0, 1)\) we have \(d_\alpha(u, v) \geq \frac{\xi |C|}{2}\) for every antipodal pair \(u, v \in C^0\). Let \(W' = \{w \in W : |\alpha^{-1}(w)| = 2\}\). Then \(|W \setminus W'| \leq \frac{1-\xi}{\xi} |W|\).

Proof. Partition \(C^{(1)}\) into two sets \(S\) and \(T\) such that \(|(\alpha|_S)^{-1}(w)| = |(\alpha|_T)^{-1}(w)|\) for each \(w \in W\). This is always possible since \(|\alpha^{-1}(w)|\) is even for each \(w \in W\). Viewing the elements of \(S\) as colored by \(W\), every color appears in \(S\) and those colors in \(W \setminus W'\) appear at least twice. Hence \(|W| \leq |S| - |W \setminus W'| = \frac{|C|}{2} + |W'| - |W|\). Let \(u\) and \(v\) be an antipodal pair of vertices. Then we have

\[
\xi \frac{|C|}{2} \leq d_\alpha(u, v) \leq |W| \leq \frac{|C|}{2} + |W'| - |W|
\]

and so \(|W'| \geq |W| - (1 - \xi) \frac{|C|}{2} \geq |W| - \frac{1-\xi}{\xi} |W| = \frac{2\xi - 1}{\xi} |W|\). Hence we have \(|W \setminus W'| = |W| - |W'| \leq (1 - \frac{2\xi - 1}{\xi}) |W| = \frac{1-\xi}{\xi} |W|\). \(\square\)

Let \((C, \alpha)\) be a wall cycle and let \(w\) be a wall of \((C, \alpha)\). Let \(X_w \subset C\) denote the set of all midpoints of edges colored \(w\) and let \(\text{diam} X_w\) denote the diameter of \(X_w\) as a metric subspace of \((C, d_C)\). For a pair of vertices \(u, v \in C^0\) we say that \(w\) contributes to \(\{u, v\}\) if a geodesic segment from \(u\) to \(v\) crosses \(w\). Hence \(d_\alpha(u, v)\) is equal to the number of walls contributing to \(\{u, v\}\).
Lemma 6.10. Let \((C, \alpha)\) be a wall cycle and let \(w \in W\) be a wall such that the number of edges colored \(w\) is exactly 2. Then \(w\) contributes to \(\{u, v\}\) for exactly \(\diam X_w\) antipodal pairs of vertices \(u, v \in C^0\).

Proof. Let \(P \subset C\) be a segment beginning and ending with \(w\) of length \(|P| = \diam X_w + 1\). Then \(w\) contributes to an antipodal pair \(\{u, v\}\) if and only if one of \(u\) or \(v\) is an interior vertex of \(P\) and there are exactly \(|P| - 1 = \diam X_w\) such pairs.

Lemma 6.11. Let \((C, \alpha)\) be a wall cycle and suppose that, for some \(\xi \in (0, 1)\), we have \(d_\alpha(u, v) \geq \xi \frac{|C|}{2}\) for all antipodal pairs of vertices \(u, v \in C^0\). Let \(w \in W\) be a wall. Then \(w\) crosses at least \(\diam X_w - 1 - (1 - \xi) \frac{|C|}{2}\) walls.

Proof. Consider first the case where \(\diam X_w = \frac{|C|}{2}\). Then there exist a pair of antipodal edges \(e, e'\) colored \(w\). Let \(u \in e\) and \(u' \in e'\) be a pair of antipodal vertices and let \(P \subset C\) be a segment with endpoints \(u\) and \(u'\). Note that \(P\) contains exactly one of \(e\) or \(e'\). Without loss of generality \(P\) contains \(e\). Then, since \(d_\alpha(u, u') \geq \xi \frac{|C|}{2}\) and \(P\) must cross at least \(\xi \frac{|C|}{2} - 1\) walls aside from \(w\). Then the same must hold for \(P \cup e'\) and so \(w\) crosses at least \(\xi \frac{|C|}{2} - 1 = \diam X_w - 1 - (1 - \xi) \frac{|C|}{2}\) walls.

Consider now the case \(\diam X_w < \frac{|C|}{2}\). We have a geodesic segment \(P \subset C\) beginning and ending with \(w\) such that \(|P| = \diam X_w + 1\). Let \(u\) and \(v\) be the endpoints of \(P\), let \(u'\) be the antipode of \(u\) and let \(Q\) be the geodesic segment containing \(P\) and having endpoints \(u\) and \(u'\). Then we have
\[
\xi \frac{|C|}{2} \leq d_\alpha(u, u') \leq d_\alpha(u, v) + d_\alpha(v, u') \\
\leq d_\alpha(u, v) + d_C(v, u') \\
= d_\alpha(u, v) + \frac{|C|}{2} - (\diam X_w + 1)
\]
and so we have \(d_\alpha(u, v) \geq \diam X_w + 1 - (1 - \xi) \frac{|C|}{2}\). But \(w\) crosses at least \(d_\alpha(u, v) - 1\) walls and so we are done.

Proof of Theorem 6.5. Let \(\xi = \left(\frac{3d-1}{5d}\right)\). For each vertex pair \(\{u, v\}\) and each wall \(w \in W\), let \(\I_w^{\{u, v\}}\) be defined as follows.
\[
\I_w^{\{u, v\}} = \begin{cases} 
1 & \text{if } w \text{ contributes to } \{u, v\} \\
0 & \text{otherwise}
\end{cases}
\]
Let \(W' \subseteq W\) be the set of walls which color exactly two edges of \(C\). We have
\[
d_\alpha(u, v) = \sum_{w \in W'} \I_w^{\{u, v\}}
\]
and, by Lemma 6.10, for \(w \in W'\) we have
\[
\diam X_w = \sum_{\{u, v\} \in A} \I_w^{\{u, v\}}
\]
where \(A\) is the set of antipodal pairs of vertices. Let \(\Gamma\) be the simplicial graph with vertex set \(W\) and where two walls are joined by an edge if they cross. Then we
have

$$|Γ^{(1)}| \geq \frac{1}{2} \sum_{w ∈ W'} (\deg(w))$$

$$≥ \frac{1}{2} \sum_{w ∈ W'} (\sum_{\{u,v\} ∈ A} \mathbb{I}_w^{(u,v)}) - \frac{1}{2} |W'| - \frac{1}{2} |W'|(1 - ξ) \frac{|C|}{2}$$

$$= \frac{1}{2} \sum_{\{u,v\} ∈ A} \left(\sum_{w ∈ W'} \mathbb{I}_w^{(u,v)}\right) - \frac{1}{2} \sum_{w ∈ W' \backslash W'} \left(\sum_{\{u,v\} ∈ A} \mathbb{I}_w^{(u,v)}\right) - \frac{1}{2} |W'| - \frac{1}{2} |W'|(1 - ξ) \frac{|C|}{2}$$

$$= \frac{1}{2} \sqrt{ξ} \left(\frac{|C|}{2}\right)^2 - \frac{1}{2} |W'\backslash W'| - \frac{1}{2} |W'| - \frac{1}{2} |W'|(1 - ξ) \frac{|C|}{2}$$

$$≥ \frac{1}{2} \sqrt{ξ |W|^2 - \frac{1}{2} (1 - ξ^2) |W'\backslash W'|} \cdot \frac{|C|}{2} - \frac{1}{2} |W'| - \frac{1}{2} |W'|(1 - ξ) \frac{|C|}{2}$$

$$= (\xi - 1 - ξ \frac{1 - ξ}{ξ^2} - \frac{1}{|W'|}) \frac{|W|^2}{2}$$

$$= (\xi + 1 - \frac{1}{ξ^2} - \frac{2}{ξ |C|}) \frac{|W|^2}{2}$$

where the second inequality holds by Lemma 6.11 and the second to last inequality holds by Lemma 6.9. We now verify that $4x - 3 ≤ x + 1 - \frac{1}{ξ^2}$ for $x ∈ [\frac{3}{5}, 1]$, noting that it suffices to check the inequality for $x = \frac{3}{5}$ and $x = 1$ since $x \mapsto x + 1 - \frac{1}{x^2}$ is a concave function. Then, since $ξ = \frac{5d - 1}{5d} ∈ [\frac{3}{5}, 1]$ we have

$$|Γ^{(1)}| ≥ \left(4ξ - 3 - \frac{2}{ξ |C|}\right) \frac{|W|^2}{2} = \left(4(5d - 1) \frac{5d}{5d - 1} - 3 - \frac{2}{|C|} \cdot \frac{5d}{5d - 1}\right) \frac{|W|^2}{2}$$

and, since $(C, α)$ is $d$-dimensional, we have

$$\frac{4(5d - 1)}{5d} - 3 - \frac{2}{|C|} \cdot \frac{5d}{5d - 1} ≤ \frac{d - 1}{d}$$

by Turan’s Theorem (Theorem 6.8). After some rearranging and cancellation this inequality becomes $|C| ≤ \frac{5d^2}{5d - 1}$. 

6.3. Cayley graphs of Coxeter groups. In this section we use the cubulation of Coxeter groups of Niblo and Reeves [21] and our result on $\text{CAT}(0)$ cube complexes to prove that Coxeter groups are strongly shortcut.

Let $Γ$ be a simplicial graph on the vertex set $\{v_1, v_2, \ldots, v_n\}$ with every edge labeled by an integer at least 2. If $Γ$ has an edge $e$ from $v_i$ to $v_j$ then let $m_{ij} = m_{ji}$ denote the label of $e$. The Coxeter group $C_Γ$ defined by $Γ$ is given by the following
presentation
\[ \langle v_1, v_2, \ldots, v_n \mid v_k^2 = 1 \text{ for all } k \text{ and } (v_1v_j)^m = 1 \text{ for all edges } \{v_i, v_j\} \in \Gamma^{(1)} \rangle \]

For a Coxeter group \( C_\Gamma \), Niblo and Reeves [21] construct a finite dimensional CAT(0) cube complex into whose 1-skeleton the Cayley graph \( \text{Cay}(C_\Gamma, \Gamma^0) \) isometrically embeds. Hence, since the 1-skeletons of CAT(0) cube complexes are strongly shortcut, we have the following theorem.

**Theorem 6.12.** Coxeter groups are strongly shortcut.

### 6.4. Systolic and quadric complexes

In this section we will prove that the 1-skeletons of systolic and quadric complexes are strongly shortcut. To do so we will rely on Corollary 6.6, the characterizations of disk diagrams in systolic and quadric complexes and a theorem about transforming 2-dimensional systolic complexes into quadric complexes.

A **bridged graph** is a connected graph whose isometric cycles all have length three [25]. Bandelt characterized **hereditary modular graphs** as those connected graphs whose isometric cycles all have length four [3]. Chepoi characterized **systolic complexes** as the flag simplicial completions of bridged graphs [7] and we will use this as a definition here. The present author characterized **quadric complexes** as those square complexes obtained from hereditary modular graphs by gluing in a square along each embedded 4-cycle [13] and we will use this as a definition here.

We require the following lemmas concerning disk diagrams in systolic and quadric complexes.

**Lemma 6.13** (Chepoi [7, Theorem 8.1, Claim 1]). Every cycle in a systolic complex has a systolic disk diagram.

**Lemma 6.14** ([13, Lemma 1.6]). Every cycle in a quadric complex has a CAT(0) square complex disk diagram.

The present author and Osajda proved the following theorem [14], which we also need.

**Theorem 6.15** ([14, Theorem 3.2]). Let \( \Gamma \) be the 1-skeleton of a 2-dimensional systolic complex. Let \( v \in \Gamma^0 \) be a vertex. Let \( \Gamma' \) be the graph obtained from \( \Gamma \) by deleting every edge whose endpoints are equidistant to \( v \). Then \( \Gamma' \) is the 1-skeleton of a quadric complex.

**Lemma 6.16.** Let \( D \subset \mathbb{R}^2 \) be a systolic disk diagram. Let \( v \in \partial D \) be a vertex on the boundary of \( D \). Let \( \Gamma \) be the plane graph obtained from the 1-skeleton \( D^1 \) by deleting every edge whose endpoints are equidistant to \( v \). Then the square complex \( D_- \subset \mathbb{R}^2 \) obtained from \( \Gamma \) by including any planar region bounded by an embedded 4-cycle of \( \Gamma \) is a CAT(0) square complex disk diagram.

**Proof.** By Theorem 6.15, so long as every embedded 4-cycle of \( \Gamma \) bounds a planar region of \( \mathbb{R}^2 \), the square complex \( D_- \) is quadric. But quadric complexes are simply connected, by Lemma 6.14 and planar quadric complexes are CAT(0) so proving that every embedded 4-cycle \( C \) of \( \Gamma \) bounds a planar region of \( \mathbb{R}^2 \) will suffice to prove the lemma. But such a \( C \) is an embedded 4-cycle of \( D^1 \) which is a bridged graph so some antipodal pair of vertices of \( C \) are joined by an edge \( e \) in \( D^1 \) making \( C \cup \{e\} \) the union of two 3-cycles \( C_1 \) and \( C_2 \) with \( C_1 \cap C_2 = \{e\} \). Then, since \( D \) is a flag simplicial complex, the 3-cycles \( C_1 \) and \( C_2 \) bound planar regions and thus so does \( C \). \( \square \)
Let $D \subset \mathbb{R}^2$ be a simplicial disk diagram whose boundary $\partial D$ is an embedded cycle. Let $v \in \partial D$ be a vertex and let $E$ be the set of edges of $\partial D$ whose endpoints are equidistant to $v$ in $D^1$. Let $D_+ \subset \mathbb{R}^2$ be the disk diagram obtained from $D$ by adding a new triangle $T_v$ to $D$ for each $e \in E$ by identifying $e$ with an edge of $T_v$. Notice that $D^1$ is convex in $D^1_+$. Thus no boundary edge of $D_+$ has endpoints that are equidistant to $v$ in $D^1_+$ and if $D$ is systolic then so is $D_+$.

**Lemma 6.17.** Let $D \subset \mathbb{R}^2$ be a systolic disk diagram whose boundary $\partial D$ is a $K$-bilipschitz embedded cycle $\partial D$ of $D^1$. Let $v \in \partial D$ be a vertex and let $E$ be the set of edges of $\partial D$ whose endpoints are equidistant to $v$ in $D^1$. Then at most $|E| \leq \frac{K-1}{K} \cdot |\partial D| + 1$.

Consequently, if $|\partial D| \geq \frac{2K(2K+3)}{K-1}$ then the boundary $\partial D_+$ of the disk diagram $D_+ \subset \mathbb{R}^2$ obtained from $D$ as above is an embedded $\frac{3-2K}{K^2}$-almost isometric cycle.

**Proof.** One of the two embedded paths in $\partial D$ from $v$ to its antipode $\bar{v}$ contains at least $\left\lceil \frac{|E|-1}{2} \right\rceil$ edges of $E$. But then $d_{\partial D}(v, \bar{v}) \leq \frac{|\partial D|}{2} - \left\lceil \frac{|E|-1}{2} \right\rceil \leq \frac{|\partial D|}{2} - \frac{|E|-1}{2}$, which by $K$-bilipschitz embeddedness implies $\frac{1}{K} \cdot \frac{|\partial D|}{2} \leq \frac{|\partial D|}{2} - \frac{|E|-1}{2}$. After rearranging we obtain $|E| \leq \frac{K-1}{K} \cdot |\partial D| + 1$, as required.

Now let $D_+ \subset \mathbb{R}^2$ be obtained from $D$ as above. Let $p, \bar{p} \in \partial D_+$ be antipodal. There exist vertices $u, v \in D^1 \cap \partial D_+$ with $d_{\partial D_+}(u, p) \leq 1$ and $d_{\partial D_+}(v, \bar{p}) \leq 1$. Then $d_{\partial D_+}(u, v) \geq \frac{|\partial D_+|}{2} - 2$ but $d_{\partial D_+}(u, v) \leq d_{\partial D}(u, v) + |E| \leq d_{\partial D}(u, v) + \frac{K-1}{K} \cdot |\partial D_+| + 1$ so $d_{\partial D}(u, v) \geq \frac{|\partial D_+|}{2} - 2 - \frac{K-1}{K} \cdot |\partial D_+| - 1 = \frac{2-K}{2K} \cdot |\partial D_+| - 3$. Hence by $K$-bilipschitz embeddedness of $\partial D$ in $D^1$, by convexity of $D^1$ in $D^1_+$ and by $|\partial D_+| \geq |\partial D| \geq \frac{2K(2K+3)}{K-1}$, we have

$$d_{D^1_+}(p, \bar{p}) \geq d_{D^1_+}(u, v) - 2 = d_{D^1}(u, v) - 2 \geq \frac{1}{K} d_{\partial D}(u, v) - 2 \geq \frac{2-K}{2K} \cdot |\partial D_+| - 3 = \frac{2-K}{2K} \cdot \frac{K(2K+3)}{K^2} \cdot \frac{|\partial D_+|}{2} \geq \frac{3-2K}{K^2} \cdot |\partial D_+|$$

as required.

**Theorem 6.18.** The 1-skeletons of systolic and quadric complexes are strongly shortcut.

**Proof.** Let $X$ be a systolic or quadric complex. For the sake of finding a contradiction, assume that $X^1$ is not strongly shortcut. Then there exists a sequence $(f_n: C_n \to X^1)_{n \in \mathbb{N}}$ of cycles such that $f_n$ is an $\frac{n+1}{n}$-bilipschitz embedding and $|C_n| \geq n$. We will use the $f_n$ to construct a sequence of spaces $(D_n)_n$ such that
(1) $D_n \subset \mathbb{R}^2$ is a CAT(0) square complex disk diagram, and

(2) $\partial D_n$ is an embedded $\xi_n$-almost isometric cycle in $D_n^1$,

where $|\partial D_n| \to \infty$ and $\xi_n \to 1$ as $n \to \infty$. This will suffice since, by taking the wedge of the $D_n$, we contradict Corollary 6.6.

In the case where $X$ is quadric, let $f_n: D_n \to X$ be the CAT(0) square complex disk diagram for $f_n$ guaranteed by Lemma 6.14. Then, since $f_n = f_n|_{\partial D_n}$ is an $\frac{n+1}{n}$-bilipschitz embedding and $f_n|_{D_n^1}$ is 1-Lipschitz, the inclusion $\partial D_n \to D_n^1$ is $\frac{n+1}{n}$-bilipschitz and so is an $\frac{n}{n+1}$-almost isometric embedding of a cycle of length $|\partial D_n| = |C_n| \geq n$.

In the case where $X$ is systolic, we will need a slightly more sophisticated argument. Pick a subsequence $(f_{n_k}: C_{n_k} \to X^1)_{k \in \mathbb{N}}$ such that $f_{n_k}$ is $K_k$-bilipschitz with $K_k = \frac{k+1}{k}$ and $|C_{n_k}| \geq \frac{2K_k(2K_k+3)}{K_k-1}$. Let $f_{n_k}: D_{n_k} \to X$ be the systolic disk diagram for $f_n$ guaranteed by Lemma 6.13. Then $D_{n_k}$ satisfies the conditions of Lemma 6.17 with $K = K_k$, thus we obtain a systolic disk diagram $D_{n_k,+} \subset \mathbb{R}^2$ from $D_{n_k}$, as above, whose boundary $\partial D_{n_k,+}$ is an embedded $\frac{3-2K_k}{K_k}$-almost isometric cycle of length $|\partial D_{n_k,+}| \geq |\partial D_{n_k}| = |C_{n_k}| \geq \frac{2K_k(2K_k+3)}{K_k-1}$.

Moreover, for some vertex $v_k \in \partial D_{n_k,+}$ no edge $e$ of $\partial D_{n_k,+}$ has endpoints that are equidistant to $v_k$ in $D_{n_k,+}$.

Let $D_{n_k,-} \subset \mathbb{R}^2$ be the CAT(0) square complex disk diagram obtained from $D_{n_k,+}$ as in the statement of Lemma 6.16 with $v$ set to $v_k$. Then $\partial D_{n_k,-} = \partial D_{n_k,+}$ and $D_{n_k,-} \subset D_{n_k,+}$ so $\partial D_{n_k,-}$ is an embedded $\frac{3-2K_k}{K_k}$-almost isometric cycle of length $|\partial D_{n_k,-}| \geq \frac{2K_k(2K_k+3)}{K_k-1}$ in $D_{n_k,-}$. But $K_k \to 1$ so $\frac{3-2K_k}{K_k} \to 1$ and $\frac{2K_k(2K_k+3)}{K_k-1} \to \infty$ as $k \to \infty$, thus the $(D_{n_k,-})_k$ are as required.

Corollary 6.19. Systolic and quadric groups are strongly shortcut.

Wise proved that finitely presented $C(6)$ small cancellation groups are systolic [26] and the present author proved that finitely presented $C(4)$-$T(4)$ small cancellation groups are quadric [13] so we have the following corollary.

Corollary 6.20. Finitely presented $C(6)$ and $C(4)$-$T(4)$ small cancellation groups are strongly shortcut.

6.5. Cayley graphs of $\mathbb{Z}$ and $\mathbb{Z}^2$. We have shown that the 1-skeletons of CAT(0) cube complexes are strongly shortcut. In particular, the standard Cayley graphs of the finitely generated free abelian groups are strongly shortcut. In this section we will strengthen this result for $\mathbb{Z}$ and $\mathbb{Z}^2$ by showing that all of their Cayley graphs are strongly shortcut.

Lemma 6.21. Let $\Gamma$ be a graph and suppose there is a continuous $(K, M)$-quasi-isometric embedding $\iota: \Gamma \to \mathbb{R}^2$. Let $\xi \in (0, 1)$ and let $f: C \to \Gamma$ be a $\xi$-almost isometric cycle. Suppose the image of $\iota \circ f$ is contained in the $N$-neighborhood of a line $L \subset \mathbb{R}$. Then $|C| \leq \frac{2K}{\xi}(M + 2N)$.  

Proof. By continuity, for some pair of antipodal points $p, q \in C$, the points $\iota \circ f(p)$ and $\iota \circ f(q)$ project perpendicularly to the same point of $L$. Then

$$\frac{1}{K}d_{\Gamma}(f(p), f(q)) - M \leq d_{\mathbb{R}}(\iota \circ f(p), \iota \circ f(q)) \leq 2N$$
and so we have

\[ \frac{|C|}{2} \leq d_T(f(p), f(q)) \leq K(M + 2N) \]

and so we have \(|C| \leq \frac{2K}{\xi}(M + 2N)\). \qed

Since the inclusion map \(Z \hookrightarrow \mathbb{R} \times \{0\}\) extends to a continuous quasi-isometric embedding from any Cayley graph of \(Z\), we obtain as a corollary of Lemma 6.21 the following theorem.

**Theorem 6.22.** Every Cayley graph of \(Z\) is strongly shortcut.

The Cayley graphs of \(Z\) are all quasi-isometric to \(\mathbb{R}\) and so are hyperbolic. Thus Theorem 6.22 also follows from Theorem 6.2. In the remainder of this section we will prove the strong shortcut property for Cayley graphs of \(Z^2\) where we cannot rely on hyperbolicity. In fact, we cannot even rely on the quasi-isometry type of \(Z^2\) as the following example makes clear.

**Example 6.23.** Let \(\Gamma\) be the standard Cayley graph of \(Z^2\). For each \(n \in \mathbb{N}\), let \(A_n\) be the induced subgraph on \(\{0, 1, \ldots, n\}^2\), let \(P_n\) be the induced subgraph on \(\{0, 1, \ldots, n\} \times \{0\}\) and let \(Q_n\) be the induced subgraph on \(\{0\} \times \{0, 1, \ldots, n\}\). Then \(C_n = P_n \cup Q_n \cup (P_n + (0, n)) \cup (Q_n + (n, 0))\) is the embedded cycle that “bounds” \(A_n\). Note that the \(A_n' = A_n + \left(\frac{n(n+1)}{2}, \frac{n(n+1)}{2}\right)\) are disjoint. Let \(C_n' = C_n + \left(\frac{n(n+1)}{2}, \frac{n(n+1)}{2}\right)\) and let \(\Gamma'\) be the graph obtained from \(\Gamma\) by subdividing the edges of each \(A_n' \setminus C_n'\). Then the \(C_n'\) are isometrically embedded in \(\Gamma'\). Thus \(\Gamma'\) is not shortcut and yet \(\Gamma'\) is quasi-isometric to \(\Gamma\) and thus to \(\mathbb{R}^2\).

Let \(S\) be a generating set of \(Z^2\), let \(\Gamma\) be the Cayley graph of \((\mathbb{Z}^2, S)\) and let \(\iota: \Gamma \to \mathbb{R}^2\) be the \((K, M)\)-quasi-isometry obtained by extending the inclusion map \(Z^2 \to \mathbb{R}^2\) to \(\Gamma\) in such a way that the restriction of \(\iota\) to each edge is a constant-speed geodesic.

**Lemma 6.24.** Let \(f: C \to \Gamma\) be a \(\xi\)-almost isometric embedding. For some constants \(A\) and \(B\) depending only on \(S\), there is a line in \(\mathbb{R}^2\) whose \((|1 - \xi|A|C| + B)\)-neighborhood contains the image of \(\iota \circ f\).

**Proof.** For \(x \in \mathbb{R}^2\) let \(|x|\) denote the standard Euclidean norm of \(x\). Let \(t \in S\) achieve \(|t| = \max\{|s| : s \in S\}\). Let \(V\) be the 1-dimensional vector subspace of \(\mathbb{R}^2\) generated by \(t\). For \(s \in S\) let \(s_t\) be the perpendicular projection of \(s\) onto \(V\) and let \(\alpha = \max\{|s_t| : s \in S \setminus \{\pm t\}\}\). Then \(\alpha < |t|\).

By continuity, some pair of antipodal points \(p, q \in C\) satisfy \(\iota \circ f(p) - \iota \circ f(q) \in V\). Pick \(u \in \mathbb{Z}^2\) such that \(|u - \iota \circ f(p)| \leq 1\). Then for some \(r \in \mathbb{Z}\), we have \(|u + rt - \iota \circ f(q)| \leq 1 + |t|\). Then \(d_T(u, f(p)) \leq K(M+1)\) and \(d_T(u + rt, f(q)) \leq K(M+1+|t|)\).

Hence we have

\[
\frac{|C|}{2} \leq d_T(f(p), f(q)) \leq d_T(f(p), u) + d_T(u, u + rt) + d_T(u + rt, f(q)) \\
\leq |r| + K(2M + 2 + |t|)
\]

and so \(|r| \geq \frac{|C|}{2} - K(2M + 2 + |t|)\). Let \(P \subset C\) be a segment with endpoints \(p\) and \(q\). Each edge of \(\Gamma\) is labeled by a generator in \(S\). Pull these labels back to \(C\) under \(f\). Let \(T\) be the union of all the \(t\)-labeled edges of \(C\) and let \(\ell\) be the total
length of the segments of $T \cap P$. Consider the projection of the path $\iota \circ f|_P$ onto the line $\iota \circ f(p) + V$. It has arclength at most $\ell |t| + \left(\frac{|C|}{2} - \ell\right)\alpha$. But the endpoints of $\iota \circ f|_P$ are $\iota \circ f(p)$ and $\iota \circ f(q)$, which are of distance at least $(|r| - 1)|t| - 2$ apart and so $(|r| - 1)|t| - 2 \leq \ell |t| + \left(\frac{|C|}{2} - \ell\right)\alpha$. Combining this inequality with $|r| \geq \frac{|C|}{2} - K(2M + 2 + |t|)$ we have

$$\left(\frac{|C|}{2} - K(2M + 2 + |t|) - 1\right)|t| - 2 \leq \ell |t| + \left(\frac{|C|}{2} - \ell\right)\alpha$$

which, after some manipulation gives

$$\ell \geq \left(\frac{|C|}{2} - \frac{K(2M + 2 + |t|)|t| + |t| + 2}{|t| - \alpha}\right)\frac{|C|}{2}$$

and so we have the following:

$$|P \setminus T| = \frac{|C|}{2} - \ell \leq \left((1 - \frac{\xi}{\alpha})|t| - \frac{|C|}{2} + \frac{K(2M + 2 + |t|)|t| + |t| + 2}{|t| - \alpha}\right)\frac{|C|}{2}$$

But then the projection of $\iota \circ \alpha|_P$ to $V^\perp$ must have length at most

$$\left(\frac{(1 - \frac{\xi}{\alpha})|t|^2}{|t| - \alpha}\right)\frac{|C|}{2} + \frac{K(2M + 2 + |t|)|t|^2 + |t|^2 + 2|t|}{|t| - \alpha}$$

and so the image of $\iota \circ f|_P$ must be contained in a neighborhood of radius

$$\left(\frac{(1 - \frac{\xi}{\alpha})|t|^2}{2(|t| - \alpha)}\right)\frac{|C|}{2} + \frac{K(2M + 2 + |t|)|t|^2 + |t|^2 + 2|t|}{2(|t| - \alpha)}$$

about the line $\iota \circ f(p) + V$. Then the lemma holds with $A = \frac{|t|^2}{4(|t| - \alpha)}$ and $B = \frac{K(2M + 2 + |t|)|t|^2 + |t|^2 + 2|t|}{2(|t| - \alpha)}$.

**Theorem 6.25.** Every Cayley graph of $\mathbb{Z}^2$ is strongly shortcut.

**Proof.** Let $f : C \to \Gamma$ be a $\xi$-almost isometric cycle. By Lemma 6.24 we have a line $L \subset \mathbb{R}^2$ whose $(1 - \xi)A|C| + B)$-neighborhood contains the image of $\iota \circ f$. So, by Lemma 6.21, we have

$$|C| \leq \frac{2K}{\xi} (M + 2(1 - \xi)A|C| + 2B)$$

and so

$$\left(1 - \frac{4K}{\xi}(1 - \xi)A\right)|C| \leq \frac{2K}{\xi} (M + 2B)$$

which gives us a bound on the length of $|C|$ assuming we have $1 - \frac{4K}{\xi}(1 - \xi)A > 0$. But this condition is equivalent to $\xi > \frac{4K}{\xi}(1 - \xi)A$. Hence, for $\xi \in \left(\frac{4K}{\xi}(1 - \xi)A, 1\right)$, there is a bound on the length of the $\xi$-almost isometric cycles of $\Gamma$. \hfill \square

7. THE BAUMSLAG-SOLITARY GROUP BS(1,2)

The Baumslag-Solitar group BS(1,2) is defined by the following presentation.

$$\langle a, t \mid tat^{-1} = a^2 \rangle$$

In this section we will show that the standard Cayley graph of $G = BS(1,2)$ is shortcut but that adding the generator $\tau = t^2$ results in a Cayley graph $Cay(G, \langle a, t, \tau \rangle)$.
which is not shortcut. Hence we see that there exists a shortcut group with exponential Dehn function [9] and that the shortcut property for a Cayley graph is not invariant under a change of generating set. We also see that there exists a shortcut group which is not strongly shortcut, since strongly shortcut groups have polynomial isoperimetric function, by Corollary 4.6.

Let $\Gamma$ be the Cayley graph of $BS(1,2)$ with generating set $\{a, t\}$. Since $BS(1,2)$ is an HNN extension it has a Bass-Serre tree $T$. Every vertex of $T$ has two outgoing edges labeled $t$ and one incoming edge labeled $t$.

**Lemma 7.1.** Every element of $BS(1,2)$ can be written uniquely in the form $t^m a^kt^n$ where $m, k, n \in \mathbb{Z}$ and $k$ is even only if $k = m = 0$.

**Proof.** Given any word representing an element of $BS(1,2)$ in the standard generators, we may commute positive powers of $t$ to the right and negative powers of $t$ to the left using the relations $t^m a^k = a^{2^m k} t^n$ and $a^k t^{-n} = t^{-n} a^{2^k}$, with $n \geq 0$, to obtain a representative of the form $t^m a^k t^n$. Then we may apply the relation $a^k = t^n a^{k/2^m} t^{-n}$ if $k$ is a nonzero integer multiple of $2^n$, with $n \geq 0$, to obtain a representative of the form $t^m a^k t^n$ where $k$ is even if and only if $k = m = 0$.

To see that this form is unique, let $t^m a^k t^n = t^m a^k t^n'$. By consideration of the Bass-Serre tree $T$ we must have $m + n = m' + n'$. Without loss of generality $m \geq m'$ and so we have

$$a^k t^n = t^{m-m'} a^k t^n = a^{m-m'} k^{-m-m'} t^n = a^{2m-m'} k t^n'$$

and so, as the base group embeds in an HNN extension, we have $k' = 2m-m' k$.

So, in the case where $k = m = 0$, we have $k' = 0$ and so $m' = 0$, which implies $(m', k', n') = (m, k, n)$. If $k \neq 0$ then $k' \neq 0$ and so $k$ and $k'$ are both odd integers. Hence $2m-m' = 1$, which again implies $(m', k', n') = (m, k, n)$. \qed

It follows from Lemma 7.1 that we have a one-to-one correspondence

$$\varphi : G \to \mathbb{Z} \left[ \frac{1}{2} \right] \times \mathbb{Z}$$

$$t^m a^k t^n \mapsto (2^m k, m+n)$$

with inverse

$$\varphi^{-1} : \mathbb{Z} \left[ \frac{1}{2} \right] \times \mathbb{Z} \to G$$

$$(r, z) \mapsto \begin{cases} t^{\nu(r)} a^{r/2^{\nu(r)}} t^{z-\nu(r)} & \text{if } r \neq 0 \\ t^z & \text{if } r = 0 \end{cases}$$

where $\mathbb{Z} \left[ \frac{1}{2} \right]$ is the set of dyadic rationals and $\nu(r)$ is defined as follows.

$$\nu(r) = \max \left\{ m \in \mathbb{Z} : r \text{ is an integer multiple of } 2^m \right\}$$

The **height** of a point $(r, z)$ is $z$. We use $\mu(g)$ to denote the height of $\varphi(g)$ for $g \in BS(1,2)$. For a word $w$ in a generating of $BS(1,2)$ we define the height $\mu(w)$ of $w$ as $\mu(g)$ for the element $g \in BS(1,2)$ that is represented by the word $w$.

Pushing forward the group operation to $\mathbb{Z} \left[ \frac{1}{2} \right] \times \mathbb{Z}$ gives the following operation.

$$(r, z) \cdot (r', z') = (r + 2^r r', z + z')$$
Pushing forward the Cayley graph structure gives the following edges.
\[(r, z) \xrightarrow{z} (r + 2z, z)\]
\[(r, z) \xrightarrow{1} (r, z + 1)\]

The Bass-Serre tree \(T\) may be identified with the quotient of this graph which identifies \((r, z)\) and \((r', z)\) if \(r - r'\) is an integer multiple of \(2z\). This identification preserves height so we may refer to the height \(\mu(v)\) of a vertex \(v\) of \(T\).

**Lemma 7.2.** Let \(u\) and \(v\) be distinct vertices of \(T\) with \(\mu(u) \geq \mu(v)\) and let \(u^{(-1)} \xrightarrow{1} u\) be the unique incoming edge of \(v\) from \(P \subset T\). Then, since \(u = v\) is an embedded path and each vertex of \(P\) is outgoing edge and that height increases when traversing the outgoing edges and decreases when traversing the incoming edges. Let \(T'\) be a finite subtree of \(T\) and suppose, for the sake of finding a contradiction, that \(v\) and \(v'\) are distinct minimal height vertices of \(T'\). Let \(P\) be the unique embedded path from \(v\) to \(v'\) in \(T\). Then \(P \subset T'\) and so, by minimality of the height of \(v\), the first edge of \(P\) is outgoing from \(v\). Then, since \(P\) is an embedded path and each vertex of \(T\) has at most one incoming edge, every subsequent edge of \(P\) is also directed away from \(v\) and towards \(v'\). Thus \(v'\) has greater height than \(v\), a contradiction.

**Lemma 7.3.** Let \(u, v \in T\) with \(\mu(u) - \mu(v) = h > 0\) and let
\[u^{(-h)} \xrightarrow{1} u^{(-h+1)} \xrightarrow{1} \ldots \xrightarrow{1} u^{(-1)} \xrightarrow{1} u\]
be the unique directed path of length \(h\) ending at \(u\) in \(T\). If \((u = u_0, u_1, \ldots, u_k = v)\) is the sequence of vertices of a path from \(u\) to \(v\) in \(T\) then \(u_i = u^{(i-1)}\) for some \(i \in \{1, 2, \ldots, k\}\).

**Proof.** The proof is by induction on \(h\). By Lemma 7.2, we have \(u_i = u^{(i-1)}\) for some \(i \in \{1, 2, \ldots, k\}\). This proves the base case \(h = 1\). For \(h > 1\) the lemma follows from the induction hypothesis applied to \((u_i, u_{i+1}, \ldots, u_k = v)\).

**Lemma 7.4.** Every finite subtree of \(T\) has a unique vertex of minimal height.

**Proof.** This follows from the fact that each vertex of \(T\) has two outgoing edges and one incoming edge and that height increases when traversing the outgoing edges and decreases when traversing the incoming edges. Let \(T'\) be a finite subtree and suppose, for the sake of finding a contradiction, that \(v\) and \(v'\) are distinct minimal height vertices of \(T'\). Let \(P\) be the unique embedded path from \(v\) to \(v'\) in \(T\). Then \(P \subset T'\) and so, by minimality of the height of \(v\), the first edge of \(P\) is outgoing from \(v\). Then, since \(P\) is an embedded path and each vertex of \(T\) has at most one incoming edge, every subsequent edge of \(P\) is also directed away from \(v\) and towards \(v'\). Thus \(v'\) has greater height than \(v\), a contradiction.

### 7.1. Geodesics in \(BS(1, 2)\)

We now prove some lemmas about geodesics in the Cayley graph \(\Gamma\). We will describe paths in \(\Gamma\) using words in the letters \(\{a\pm 1, t\pm 1\}\). We use the notation \(w_1 \equiv w_2\) to mean that the words \(w_1\) and \(w_2\) represent the same element of \(BS(1, 2)\). Of course if \(w_1 \equiv w_2\) then paths described by \(w_1\) and \(w_2\) and starting at the same initial vertex must have the same final vertex.
Remark 7.5. A path in $\Gamma$ may be projected to a path in $T$. A backtrack in the projection corresponds to a subword of the form $ta^k t^{-1}$ or $t^{-1} a^{2k} t$. The initial and terminal edges of a path described by $twt^{-1}$ project to the same edge in $T$ if and only if $w \equiv a^k$. The initial and terminal edges of a path described by $t^{-1} wt$ project to the same edge in $T$ if and only if $w \equiv a^{2k}$.

Lemma 7.6. Words of the following forms do not describe geodesic paths.

1. $ta^\pm 1 t^{-1}$
2. $t^{-1} a^k$ and $a^k t$ with $|k| \geq 2$
3. $a^\pm t^{-1} a^{-\varepsilon}$ with $\varepsilon = \pm 1$.
4. $t^{-1} w_1 t w_2 t^{-1}$ with $w_1 \equiv a^{k_1}$ and $w_2 \equiv a^{k_2}$
5. $w \equiv t^h$ with $w \not\equiv t^h$.
6. $w_1 w_2$ with $w_1 \mu(w_1) < 0$

Proof. The following equivalences prove nongeodesicity for (1), (2), (3) and (4).

1. $t a^\pm 1 t^{-1} \equiv a^{\pm 2}$
2. $t^{-1} a^k \equiv a^{\pm 1} t^{-1} a^{k+2}$
3. $a^\varepsilon t^{-1} a^{-\varepsilon} \equiv t^{-1} a^\varepsilon$
4. $t^{-1} w_1 t w_2 t^{-1} \equiv t^{-1} a^{k_1} a^{k_2} \equiv t^{-1} a^{2k_2} a^{k_1} \equiv t^{-1} t w_2 t^{-1} w_1 \equiv w_2 t^{-1} w_1$

(5) Suppose $w$ is geodesic with $w \equiv t^h$. Note that $\mu(w) = h$ so $w$ must contain at least $|h|$ instances of $t^\varepsilon$ where $\varepsilon$ is the sign of $h$. Hence, as $|w| \leq |t^h| = h$, we see that $w$ cannot contain any instance of $a^{\pm 1}$. But $w$ may not contain any backtracks either and so $w = t^h$.

(6) Suppose $w = w_1 w_2$ is geodesic with $w \equiv a^k$ and $\mu(w_1) < 0$. We view $w$ as a path $P \to \Gamma$. We pull back labels and directions from $\Gamma$ so that the edges of $P$ are directed and labeled with the generators $a$ and $t$. In this way, each subpath of $P$ is labeled by a word in $a$, $t$ and their inverses. By Lemma 7.4, there is a unique vertex $v$ of minimal height of the projection of $w$ to the Bass-Serre tree $T$ and $\mu(v) \leq \mu(w_1) < 0 = \mu(1) = \mu(w)$. Then $w$ must contain a subpath labeled $t^{-1} a^\varepsilon t$ with the $a^\varepsilon$ part mapping to $v$ under the projection to $T$. Then (2) implies that $|k| = 1$. So $t^{-1} a^\varepsilon t = t^{-1} a^{\pm 1} t$ corresponds to a nonbacktracking path in $T$. Since the projection of $w$ to $T$ is a closed path and $v$ is a cutpoint of $T$, it follows that $w$ contains another subpath labeled $t^{-1} a^{\pm 1} t$ such that $a^{\pm 1}$ maps to $v$ under the projection to $T$. Then $w$ contains a subword labeled $t^{-1} a^{\varepsilon_1} t w t^{-1} a^{\varepsilon_2} t$ with $\varepsilon_i \in \{\pm 1\}$ and $w' \equiv a^k$. But, by (4), we know that $t^{-1} a^{\varepsilon_1} t w' t^{-1}$ is not geodesic, which is a contradiction.

Lemma 7.7. If $\mu(w) = 0$ and every prefix $w'$ of $w$ has $\mu(w') \geq 0$ then $w \equiv a^k$ for some $k$.

Proof. Viewing $w$ as a path in $\mathbb{Z} \left[\frac{1}{2}\right] \times \mathbb{Z}$ starting at $(0,0)$, we see that at each step the first coordinate changes by a positive power of 2. Hence the endpoint of the path is $(k,0)$ for some integer $k$.

A word $w$ is ascending if it contains only positive powers of $t$ and descending if it contains only negative powers of $t$.

Lemma 7.8. Let $w$ be a geodesic word with $w \equiv t^{-h} a^k$ where $h \geq 0$. Then no prefix $w'$ of $w$ satisfies $\mu(w') < -h$ and we have $w = xy$ where $x$ is ascending and $y$ is descending.
Proof. Let $w = w_1 w_2$ where $w_1$ is the smallest prefix of $w$ with $\mu(w_1) = -h$. View $w_1$ as a path $f: P \to T$ and let $\tilde{f}: P \to T$ be the projection of this path to $T$. Then $\tilde{f}$ starts at the vertex of $T$ corresponding to the coset $\langle a \rangle$ and $\tilde{f}$ ends at a vertex of height $-h$ and $\tilde{f}$ does not reach the height $-h$ until its final step. Thus, by Lemma 7.3, the final vertex of $\tilde{f}$ is the vertex of $T$ corresponding to the coset $t^{-h} \langle a \rangle$. Thus $w_1 \equiv t^{-h} a^\ell$, for some $\ell \in \mathbb{Z}$, and so $w_2 \equiv w_1^{-1} w \equiv a^{-\ell} t^h t^{-h} a^k \equiv a^{k-\ell}$. If $w$ had a prefix $w'$ with $\mu(w') < -h$ then $w'$ would have to be longer than $w_1$ and so we would have $w' = w_1 w_2^\prime$. Then $w_2^\prime$ would be a prefix of $w_2$ of height $\mu(w_2') = \mu(w') - \mu(w_1) < 0$, which by Lemma 7.6 would contradict the geodesicity of $w$.

We now prove that $w = xy$ such that $x$ is ascending and $y$ is descending. If $w$ has no such decomposition then $w$ has a subword of the form $t^{-1} w' t$. An innermost such subword has the form $t^{-1} a^k t$. By Lemma 7.6(2), we have $|k| \leq 1$. So, since $t^{-1} t$ is not geodesic, we have $w = w_1 t^{-1} a^k t w_2$ with $\varepsilon = \pm 1$. We have $\mu(w_1 t^{-1} a^\varepsilon) \geq -h$ and so $\mu(w_1 t^{-1} a^\varepsilon t) > -h$. So $\mu(w_2) > 0$ and the shortest prefix of $w_2$ of negative height has the form $w_2' t^{-1}$ with $\mu(w_2') = 0$. Then, by Lemma 7.7, we have $w_2' \equiv a^k$ for some $k$. But then

$$w_1 t^{-1} a^\varepsilon t w_2' t^{-1} \equiv w_1 t^{-1} w_2' t^{-1} a^\varepsilon \equiv w_1 w_2' t^{-1} a^\varepsilon$$

and so $w_1 t^{-1} a^\varepsilon t w_2' t^{-1}$ is a nongeodesic subword of $w$, which is a contradiction. □

Lemma 7.9. Let $h \geq 1$, let $k \geq 2^h$ and let $\varepsilon = \pm 1$. Let $w$ be a geodesic word with $w \equiv t^{-h} a^\varepsilon k$. Then the first letter of $w$ is not $t^{-1}$.

Proof. Suppose the first letter of $w$ is $t^{-1}$. Then, by Lemma 7.8, we see that $w$ is descending. Hence

$$w = t^{-\ell_1} a^{k_1} t^{-\ell_2} a^{k_2} \cdots t^{-\ell_m} a^{k_m}$$

with $\sum_i \ell_i = h$ and $\ell_i > 0$ for all $i$. So we have

$$w \equiv t^{-L} \prod_{i \neq j} a^{k_i} a^{2^{L_1} k_{i_1} + 2^{L_2} k_{i_2} + \cdots + 2^{L_m} k_{i_m}}$$

where $L_j = \sum_{i \neq j} \ell_i$ and so $\varepsilon k = 2^{L_1} k_{1_1} + 2^{L_2} k_{2_2} + \cdots + 2^{L_m} k_{m_m}$. But, by Lemma 7.6(2), we have $|k_i| \leq 1$ for all $i$ and so

$$|k| \leq 2^{L_1} + 2^{L_2} + \cdots + 2^{L_m}$$

with

$$0 = L_m < L_{m-1} < \cdots < L_1 < h$$

which implies $|k| \leq \sum_{j=0}^{h-1} 2^j = 2^h - 1$, a contradiction. □

Lemma 7.10. Let $h \geq 1$, let $0 \leq k \leq 2^h$ and let $\varepsilon = \pm 1$. Let $w$ be a geodesic word with $w \equiv t^{-h} a^\varepsilon k$. Then $w$ is descending and every prefix $w'$ of $w$ satisfies $w' \equiv t^{-h'} a^{\varepsilon k'}$ where $0 \leq h' \leq h$ and $0 \leq k' \leq 2^h$.

Proof. Since there is an automorphism of BS(1,2) fixing $t$ and sending $a$ to $a^{-1}$, we may assume that $\varepsilon = 1$. The proof that $w$ is descending is by induction on the length of $w$. If $|w| = 1$ then $w = t^{-1}$ and so satisfies the required conditions. Assume now that $|w| > 1$. Consider the path $f: P \to T$ followed by $w$ in the Bass-Serre tree $T$. Let $v_1$ and $v_2$ be the initial and final vertices of this path. The shortest path in $T$ from $v_1$ to $v_2$ is labeled $t^{-h}$. By Lemma 7.6(6), the path $f$ may not traverse an edge below $v_2$. Hence, any instance of $t$ in $w$ corresponds to an edge of $T$ which is ascended by $f$ and later descended. That is, the instance of $t$ is
the first letter of a subword $tw^t-1$ of $w$ with $w' \equiv a^{k'}$ for some $k'$. Then, if $w$ has an instance of $t$ then, by Lemma 7.6(4), it must occur to the left of any negative power of $t$. So $w = a^n w'$ for some $n \in \{-1, 0, 1\}$ and some word $w'$. But then

$$w' \equiv t^{-1} a^{-\eta} w \equiv t^{-1} a^{-\eta} t^{-b} a^k \equiv t^{-(h+1)} a^{-2b \eta + k}$$

and $|−2^b \eta + k| \leq 2^h + 2^h = 2^{h+1}$ and so, by induction, $w'$ must have a prefix of the form $t^{-1}$ or $a^{\varepsilon} t^{-1}$, where $\varepsilon'$ is the sign of $−2^b \eta + k$. But then $w$ must contain a subword $tt^{-1}$ or $ta^{\varepsilon'} t^{-1}$, which are not geodesic. So we see that $w$ is descending.

It remains to show that every prefix $w'$ of $w$ satisfies the condition $(\ast)$ that $w' \equiv t^{-b'} a^{k'}$ where $0 \leq h' \leq h$ and $0 \leq k' \leq 2^h$. That $w' \equiv t^{-b'} a^{k'}$ with $0 \leq h' \leq h$ holds because $w$ is descending and $f$ does not descend below $v_2$ in $T$. Assume for the sake of finding a contradiction that $w$ does not satisfy $(\ast)$. Note that $w' \equiv t^{-b'} a^{k'}$ satisfies $0 \leq k'$ (respectively $k' \leq 2^h$) if and only if $w't^{-1} \equiv t^{-b'} a^{k'} t^{-1} \equiv t^{-(h'+1)} a^{2^h k'}$ satisfies $0 \leq 2k'$ (respectively $2k' \leq 2^h+1$). So the shortest prefix of $w$ that violates $(\ast)$ has the form $w'a^{\varepsilon}$ and the shortest prefix of $w$ of length at least $|w'|$ that satisfies $(\ast)$ has the form $w'a^{\varepsilon}w''a^{-\varepsilon}$, where for some $h', h'' \geq 0$, either $\varepsilon = -1$ and $w' \equiv t^{-k'}$ and $w'a^{\varepsilon}w''a^{-\varepsilon} \equiv t^{-(h'+h'')} a^{2^h k'}$ or $\varepsilon = 1$ and $w' \equiv t^{-k'} a^{k'}$ and $w'a^{\varepsilon}w''a^{-\varepsilon} \equiv t^{-(h'+h'')} a^{2^h k'}$. We will prove that $a^{\varepsilon}w''a^{-\varepsilon} \equiv t^{-h''}$, which contradicts geodesicity of $w$ by Lemma 7.6(5). In the case $\varepsilon = -1$, we have

$$a^{\varepsilon}w''a^{-\varepsilon} \equiv (w')^{-1} w' a^{\varepsilon} w'' a^{-\varepsilon} \equiv t^{h'} t^{-(h'+h'')} \equiv t^{-h''}$$

and, in the case $\varepsilon = 1$, we have

$$a^{\varepsilon}w''a^{-\varepsilon} \equiv (w')^{-1} w' a^{\varepsilon} w'' a^{-\varepsilon} \equiv a^{-2^h t^{h'} t^{-(h'+h'')} a^{2^h k'}} a^{2^h k'} \equiv a^{-2^h t^{h''} a^{2^h k'}} \equiv t^{-h''}$$

so we have our contradiction in either case. \hfill \square

Lemma 7.11. Let $h \geq 1$, let $0 \leq k \leq 2^h$ and let $\varepsilon = \pm 1$. The following statements describe precisely which initial letters a geodesic word $w \equiv t^{-h} a^{k}$ may have.

1. If $k < \left(\frac{2}{3}\right)2^h$ then any geodesic word $w \equiv t^{-h} a^{k}$ has the form $w = t^{-1} w'$.
2. If $\left(\frac{2}{3}\right)2^h < k < \left(\frac{5}{3}\right)2^h$ then any geodesic word $w \equiv t^{-h} a^{k}$ has the form $w = t^{-1} w'$ or $w = a^{\varepsilon} w''$ and there exist geodesics of both forms.
3. If $k > \left(\frac{5}{3}\right)2^h$ then any geodesic word $w \equiv t^{-h} a^{k}$ has the form $w = a^{\varepsilon} w'$.

Proof. The proof is by induction on $h$. If $h = 1$ then $k \in \{0, 1, 2\}$. If $k = 0$ then $k < \frac{1}{3} = \left(\frac{2}{3}\right)2^h$ and the only geodesic word $w \equiv t^{-h} a^{k}$ is $t^{-1}$. If $k = 1$ then $k < \frac{4}{3} = \left(\frac{2}{3}\right)2^h$ and the only geodesic word $w \equiv t^{-h} a^{k}$ is $t^{-1} a^{\varepsilon}$. If $k = 2$ then $k > \frac{4}{3} = \left(\frac{2}{3}\right)2^h$ and the only geodesic word $w \equiv t^{-h} a^{k}$ is $a^{\varepsilon} t^{-1}$. So, in all cases, the lemma holds for $h = 1$.

If $h = 2$ then $k \in \{0, 1, 2, 3, 4\}$. By Lemma 7.10, we need only consider descending words whose prefixes are equivalent to $t^{-h'} a^{k'}$.
for \( h' \in \{0,1,2\} \) and \( k' \in \{0,1,\ldots,2^h\} \). We may also exclude words with backtracks and, by Lemma \( 7.6(2) \), those containing \( t^{-1}a^{\ell} \) with \( |\ell| \geq 2 \) and, by Lemma \( 7.6(3) \), those containing \( a^\ell t^{-1}a^{-\varepsilon} \). Then the list of all possible geodesics is

\[
\begin{align*}
t^{-2}a^{\varepsilon} & \\
t^{-1}a^{\varepsilon}t^{-1}a^{\varepsilon} & \equiv t^{-2}a^{\varepsilon(2+\ell)} \\
a^{\varepsilon}t^{-2}a^{-\varepsilon} & \equiv t^{-2}a^{\varepsilon(4-\ell)}
\end{align*}
\]

with \( \ell \in \{0,1\} \). Only one pair of the words in this list, namely \( a^\varepsilon t^{-2}a^{-\varepsilon} \) and \( t^{-1}a^{\ell}t^{-1}a^{-\varepsilon} \), are equivalent and they have the same length. Hence the list is exactly the list of all geodesics equivalent to \( t^{-2}a^{\varepsilon} \) with \( k \in \{0,1,2,3,4\} \). Now, if \( k < (\frac{2}{3})2^h \) then \( k \in \{0,1,2\} \). All geodesics equivalent to \( t^{-2}a^{\varepsilon} \) with \( k \in \{0,1,2\} \) are of the first two forms in the list which have initial letter \( t^{-1} \). The only \( k \) with \( (\frac{2}{3})2^h < k < (\frac{3}{5})2^h \) is \( k = 3 \) and \( t^{-2}a^{-3} \) is equivalent, with \( \ell \) set to \( 1 \), to both the second form and the third form, which have initial letter \( t^{-1} \) and \( a^\varepsilon \). Finally, if \( k > (\frac{2}{3})2^h \) then \( k = 4 \) is equivalent only to the last form in the list with \( \ell = 0 \) and this form has initial letter \( a^\varepsilon \). So we see that the lemma holds for \( h = 2 \).

Going forward we assume that \( h > 2 \).

Suppose \( k < (\frac{2}{3})2^h \). To show that a geodesic \( w \equiv t^{-h}a^{\varepsilon}k \) has initial letter \( t^{-1} \) it suffices, by Lemma \( 7.10 \), to rule out the possibility that \( w \) has the form \( a^\varepsilon t^{-1}w' \). If that were the case then, by Lemma \( 7.10 \), the first letter of \( w' \) would be either \( t^{-1} \) or \( a^{-\varepsilon} \) and we would have \( w' \equiv ta^{-\varepsilon}t^{-h}a^{\varepsilon}k \equiv t^{-(h-1)}a^{-\varepsilon}(2^h-k) \). If \( 2^h-k > 2^{h-1} \) then, by Lemma \( 7.9 \), the first letter of \( w' \) is not \( t^{-1} \) and so would have to be \( a^{-\varepsilon} \). But \( a^\varepsilon t^{-1}a^{-\varepsilon} \) is not geodesic, by Lemma \( 7.6(3) \). So we have \( 2^{h-1} \geq 2^h-k > (\frac{2}{3})2^{h-1} \). Then, applying \( (2) \) and \( (3) \) inductively to \( h-1 \) and \( 2^h-k \) and \( -\varepsilon \), we see that \( w' \) is equivalent to a geodesic of the form \( a^{-\varepsilon}w'' \). But then \( a^\varepsilon t^{-1}a^{-\varepsilon}w'' \) is geodesic and this cannot be by Lemma \( 7.6(3) \).

Suppose \( k > (\frac{2}{3})2^h \). Let \( w \) be a geodesic with \( w \equiv t^{-h}a^{\varepsilon}k \). Suppose the initial letter of \( w \) is not \( a^\varepsilon \). Then, by Lemma \( 7.10 \), the first letter of \( w \) is \( t^{-1} \) and the second letter of \( w \) is either \( t^{-1} \) or \( a^\varepsilon \). By Lemma \( 7.9 \), the second letter of \( w' \) cannot be \( t^{-1} \) and so we have \( w = t^{-1}a^\varepsilon w' \) for some \( w' \). Hence \( w' \equiv a^{-\varepsilon}t^{-h}a^{\varepsilon}k \equiv t^{-(h-1)}a^{-\varepsilon}(k-2^{h-1}) \) with \( k-2^{h-1} \leq 2^{h-1} \) and \( k-2^{h-1} > (\frac{2}{3})2^{h-1} = (\frac{2}{3})2^{h-1} \). So, inductively applying \( (2) \) and \( (3) \), we see that \( w' \) can be replaced by a geodesic of the form \( a^\varepsilon w'' \). But then \( t^{-1}a^\varepsilon a^\varepsilon w'' \) is a geodesic, which is a contradiction.

Suppose \( (\frac{2}{3})2^h < k < (\frac{5}{6})2^h \). That any geodesic \( w \equiv t^{-h}a^{\varepsilon}k \) has the form \( w = t^{-1}w' \) or \( w = a^\varepsilon w'' \) follows from Lemma \( 7.10 \). Consider first the case where we have a geodesic of the form \( w = t^{-1}w' \). Then, by Lemma \( 7.10 \), the initial letter of \( w' \) is either \( t^{-1} \) or \( a^\varepsilon \). But \( w' \equiv tw \equiv t^{-(h-1)}a^{\varepsilon}k \) with \( k > (\frac{2}{3})2^h > 2^{h-1} \) and so, by Lemma \( 7.9 \), the initial letter of \( w' \) is \( a^\varepsilon \). So \( w' = a^\varepsilon w' \) with \( u' = a^{-\varepsilon}w' = t^{-(h-1)}a^{-\varepsilon}(k-2^{h-1}) \) and \( k-2^{h-1} < (\frac{5}{6})2^{h-1} = (\frac{2}{3})2^{h-1} \). So, by induction, we have \( u' = t^{-1}x' \) with \( x' \equiv t^{-(h-2)}a^{-\varepsilon}(k-2^{h-1}) \) and \( k-2^{h-1} > (\frac{2}{3})2^{h-1} - 2^{h-1} = (\frac{2}{3})2^{h-2} \). Then, either by induction if \( k \leq 2^{h-2} \) or otherwise by Lemma \( 7.9 \), we have that \( x' \equiv t^{-1}a^\varepsilon y' \). Thus we have a geodesic \( t^{-1}a^\varepsilon t^{-1}a^\varepsilon y' = t^{-h}a^{\varepsilon}k \). But \( a^\varepsilon t^{-2}a^{-\varepsilon} \) has the same length as \( t^{-1}a^\varepsilon t^{-1}a^\varepsilon \) and \( a^\varepsilon t^{-2}a^{-\varepsilon} \) is not geodesic, so \( a^\varepsilon t^{-2}a^{-\varepsilon}y' \) is a geodesic with \( a^\varepsilon t^{-2}a^{-\varepsilon}y' = t^{-h}a^{\varepsilon}k \). Now, consider the case where we have a geodesic of the form \( w = a^\varepsilon w'' \). By Lemma \( 7.10 \), the next two letters of \( w'' \) are either \( t^2 \) or \( t^{-1}a^{-\varepsilon} \) but \( a^\varepsilon t^{-1}a^{-\varepsilon} \) is not geodesic, by Lemma \( 7.6(3) \), so we must have \( w = a^\varepsilon t^{-2}x'' \).
Lemma 7.13. If \( C \) is an isometrically embedded cycle of length \( |C| \), then there exists a word \( w = a^\varepsilon w_1a^{\varepsilon^2}w_2 \) such that \( w \) is an isometrically embedded cycle of length \( |C| > 5 \) and \( \varepsilon = \pm 1 \).

Proof. By Lemma 7.8, we have \( w = wa^\ell y \). Since \( y \) is descending, we have \( y \equiv t^{-h}a^\varepsilon m \) for some \( m \in \mathbb{Z} \). Since the first letter of \( y \) is \( t^{-1} \), by Lemma 7.9, we have \( |m| \leq 2^h \). Then, by Lemma 7.11, we have \( |m| < \left( \frac{5}{3} \right)^2 \). Now \( a^\varepsilon y \) is a subword of \( w \) and \( a^\varepsilon y \equiv a^\varepsilon t^{-h}a^\varepsilon m \equiv t^{-h}a^\varepsilon (m+2^h) \) with \( m + 2^h > 2^h - \left( \frac{5}{3} \right)^2 \). Then, either \( m \geq 0 \) so that \( m + 2^h \geq 2^h > \left( \frac{5}{3} \right)^2 \) or \( m < 0 \) so that \( 2^h > m + 2^h > 0 \) and, by Lemma 7.11, we have \( m + 2^h > \left( \frac{5}{3} \right)^2 \). Hence we have \( -\left( \frac{1}{2} \right)^2 m < \left( \frac{5}{3} \right)^2 \). Applying the exact same arguments to the subwords \( x^{-1}a^{-\varepsilon x^{-1}} \) of \( w^{-1} = y^{-1}a^\ell x^{-1} \), we see that \( x^{-1} \equiv t^{-h}a^{-\varepsilon m} \) with \( -(\frac{1}{2})^2 m < n < \left( \frac{5}{3} \right)^2 \). Hence

\[
\varepsilon \equiv a^\varepsilon t^h a t^{-h} a^\varepsilon m \equiv a^\varepsilon (n + 2^h) \|
\]

and so \( k = \varepsilon (n + 2^h) \) which gives

\[
-\left( \frac{2}{3} \right)^2 2^h + 2^h |\ell| < \varepsilon k < \left( \frac{5}{3} \right)^2 2^h + 2^h |\ell|
\]

and so, as \( |\ell| \geq 2 \) we see that \( \varepsilon k > 0 \). Then \( k \) has the same sign as \( \ell \) so \( \varepsilon k = |k| \) and we have

\[
-\left( \frac{2}{3} \right)^2 2^h < |k| - 2^h |\ell| < \left( \frac{5}{3} \right)^2 2^h
\]

as required. \( \square \)

7.2. Isometric cycles in \( BS(1,2) \). Let \( f : C \to \Gamma \) be a cycle in \( \Gamma \). The edges of \( C \) are naturally directed and labeled by the generators \( a \) and \( b \) by pulling back from \( \Gamma \). In this way, each path \( P \to C \) is labeled by a word in \( a, b \) and their inverses.

A traversal of a cycle \( C \) is an immersed path \( P \to C \) such that \( |P| = |C| \). A traversal of \( C \) is equivalent to a choice of basepoint and orientation of \( C \). Note that if \( f : C \to \Gamma \) is a cycle in \( \Gamma \) then the label of any traversal of \( C \) determines \( f \) up to translation by elements of \( BS(1,2) \).

Lemma 7.13. Let \( f : C \to \Gamma \) be an isometrically embedded cycle of length \( |C| > 5 \). Then some traversal of \( C \) is labeled by a word of the form

\[
w = a^\varepsilon_1 w_1 a^\varepsilon_2 w_2
\]
with \(\varepsilon_1, \varepsilon_2 \in \{\pm 1\}\) such that \(w_i\) satisfies the following properties for each \(i\).

1. \(w_i\) is geodesic with initial letter \(t\) and terminal letter \(t^{-1}\).
2. \(w_i \equiv a^{2k_i}\) with \(|k_1 + k_2| \leq 1\).

**Proof.** Let \(f : C \to \Gamma\) be an isometrically embedded cycle. Let \(f\) be the projection of \(f\) to the Bass-Serre tree \(T\). By Lemma 7.4, there is a unique vertex \(v\) of minimal height of \(f(C)\). Since the preimage of \(v\) in \(\Gamma\) has no embedded cycles, we cannot have \(f(C) = \{v\}\). Then \(C\) must contain a subpath \(t^{-1}a^kt\) with the \(a^k\) part mapping to \(v\) under \(f\). Since \(f\) is an isometric embedding and \(|C| > 5\), any subpath of \(f\) of length two is geodesic. Then Lemma 7.6(2) implies that \(|k| = 1\). So \(t^{-1}a^kt = t^{-1}a^\pm 1t\) corresponds to a nonbacktracking path in \(T\). It follows, since \(f\) is a closed path and \(v\) is the endpoint of \(T\), that \(C\) contains another subpath labeled \(t^{-1}a^\pm 1t\) such that \(f\) sends \(a^\pm 1\) to \(v\). So some traversal of \(C\) is labeled by a word

\[
w = a^{\varepsilon_1}tu_1t^{-1}a^{\varepsilon_2}tu_2t^{-1}
\]

with \(\varepsilon_1, \varepsilon_2 \in \{\pm 1\}\), such that \(u_i \equiv a^{k_i}\) for some \(k_1, k_2 \in \mathbb{Z}\). Let \(w_i = tu_1t^{-1}\). Then \(w_i \equiv a^{2k_i}\) so we have

\[
\varepsilon_1 + 2k_1 + \varepsilon_2 + 2k_2 = 0
\]

since \(w\) is trivial in \(G\), and this implies \(|k_1 + k_2| \leq 1\). Also we have

\[
a^{\varepsilon_i}w_i a^{\varepsilon_{i+1}} \equiv w_i a^{\varepsilon_i}a^{\varepsilon_{i+1}}
\]

which shows that \(a^{\varepsilon_i}w_i a^{\varepsilon_{i+1}}\) has the same length and endpoints as \(w_i a^{\varepsilon_i}a^{\varepsilon_{i+1}}\). But \(w_i a^{\varepsilon_i}a^{\varepsilon_{i+1}}\) cannot be geodesic since it either backtracks or contains \(t^{-1}a^k\) with \(|k| = 2\) and so \(a^{\varepsilon_i}w_i a^{\varepsilon_{i+1}}\) is not geodesic either. Hence, since \(f\) is an isometric embedding, the complementary path \(w_{i+1}\) is geodesic. \(\square\)

**Theorem 7.14.** The standard Cayley graph of of BS(1, 2) is shortcut.

**Proof.** Let \(\Gamma\) be the standard Cayley graph of BS(1, 2). We will show that there are no isometrically embedded cycles \(f : C \to \Gamma\) of length \(|C| > 5\). For the sake of deriving a contradiction, suppose \(f\) is such a cycle. Then, by Lemma 7.13, some traversal of \(C\) is labeled

\[
w = a^{\varepsilon_1}w_1 a^{\varepsilon_2}w_2
\]

such that \(w_i\) is geodesic with initial letter \(t\) and terminal letter \(t^{-1}\) and \(w_i \equiv a^{2k_i}\) with \(|k_1 + k_2| \leq 1\). Then, by 7.12, we have

\[
w_i = x_i a^{\ell_i} y_i
\]

where \(x_i\) is ascending and has initial and terminal letter \(t\), where \(y_i\) is descending and has initial and terminal letter \(t^{-1}\), where \(\ell_i\) has the same sign as \(k_i\) and where

\[
-\left(\frac{2}{3}\right)2^{h_i} < |k_i| - 2^{h_i}|\ell_i| < \left(\frac{5}{3}\right)^2 2^{h_i}
\]

with \(0 \leq h_i = \mu(x_i) = -\mu(y_i)\). Since \(x_i\) has terminal letter \(t\) and \(y_i\) has initial letter \(t^{-1}\), we have \(h_i \geq 1\) and, by Lemma 7.6(1), we have \(|\ell_i| \geq 2\).

We may assume that \(h_1 \leq h_2\) since otherwise we may replace \(w\) with a cyclic permutation. We must have \(|k_i| \geq 1\) since otherwise \(w_1 = 1\) which does not start with \(t\). Hence, as \(|k_1 + k_2| \leq 1\) we must have that \(k_1\) and \(k_2\) have opposite signs. Since there is an automorphism of BS(1, 2) fixing \(t\) and mapping \(a \mapsto a^{-1}\) we may assume that \(k_1 > 0\) and \(k_2 < 0\). Then \(\ell_1 > 0\) and \(\ell_2 < 0\).

Let \(p_1 \in C\) be the midpoint of the subpath \(a^{\ell_1}\) of \(w\). Abusing notation we write the two segments of \(C\) between \(p_1\) and \(p_2\) as \(a^{\frac{1}{2}\ell_1} y_1 a^{\varepsilon_2} x_2 a^{\frac{1}{2}\ell_2}\) and \(a^{\frac{1}{2}\ell_2} y_2 a^{\varepsilon_1} x_1 a^{\frac{1}{2}\ell_1}\).
which may be thought of as combinatorial paths in the barycentric subdivision of \( \Gamma \). Since \( f \) is an isometric embedding, one of these two paths must be geodesic in \( \Gamma \). We have
\[
y_1a^2x_2a^{-1} = y_1a^2a^{-2h_2} x_2 \equiv a^{-2h_2-h_1} y_1a^2x_2
\]
and so
\[
a^{-1}y_2a^3x_1 \equiv y_2a^{-2h_2} a^3x_1 \equiv y_2a^3x_1a^{-2h_2-h_1}
\]
So, as \( h \), \( |\ell| \), and \( a \) are integers, we obtain \( 2^{h_2-h_1} \leq |\ell| \). Since \( h \) is an integer, this inequality may only hold if \( 2^{h_2-h_1} \geq 1 \). Therefore, we have \( 2^{h_2-h_1} \geq 1 \). But this is a contradiction because we established above that \( h_1 \geq 1 \).

7.3. A Cayley graph of BS(1,2) that is not shortcut. We now turn our attention to a different generating set of BS(1,2). Let \( \Gamma \) be the Cayley graph of BS(1,2) with the generating set \( \{a,t,\tau\} \) where \( \tau = t^2 \).

**Lemma 7.15.** Let \( k \geq 1 \) and let \( 0 \leq z_{\text{max}} \leq k \). Suppose
\[
\sum_{z=-m}^{z_{\text{max}}} \alpha_z 2^z = 2^k \pm 1
\]
where \( \alpha_z \in \mathbb{Z} \) and \( m \geq 0 \).

- If \( z_{\text{max}} = 0 \) then \( \sum_z |\alpha_z| \geq 2^{k-z_{\text{max}}} - 1 \)
- If \( z_{\text{max}} = 1 \) then \( \sum_z |\alpha_z| \geq 2^{k-z_{\text{max}}} \)
- If \( z_{\text{max}} \geq 2 \) then \( \sum_z |\alpha_z| \geq 2^{k-z_{\text{max}}} + 1 \)
Proof. If \( z_{\text{max}} = 0 \) then

\[
2^k - 1 \leq 2^k + 1 = \left| \sum_{z = -m}^{z_{\text{max}}} \alpha_z 2^z \right| \leq \sum_{z = -m}^{z_{\text{max}}} |\alpha_z| 2^z_{\text{max}} = \sum_{z = -m}^{z_{\text{max}}} |\alpha_z|
\]

and so we have \( \sum_{z} |\alpha_z| \geq 2^{k - z_{\text{max}}} - 1 \).

Suppose \( z_{\text{max}} = 1 \) and \( \sum_{z} |\alpha_z| < 2^{k - z_{\text{max}}} \). Then \( \sum_{z} |\alpha_z| \leq 2^{k - z_{\text{max}}} - 1 \) and so

\[
\left| \sum_{z = -m}^{z_{\text{max}}} \alpha_z 2^z \right| \leq \sum_{z = -m}^{z_{\text{max}}} |\alpha_z| 2^z_{\text{max}} \leq (2^{k - z_{\text{max}}} - 1) \cdot 2^z_{\text{max}} = 2^k - 2
\]

which is a contradiction. So we have \( \sum_{z} |\alpha_z| \geq 2^{k - z_{\text{max}}} \).

Now, suppose \( z_{\text{max}} \geq 2 \). Among all \( m \geq 0 \) and \( (\alpha_z)_z \) that satisfy

\[
\sum_{z = -m}^{z_{\text{max}}} \alpha_z 2^z = 2^k + 1
\]

choose an \( m \geq 0 \) and \( (\alpha_z)_z \) that minimizes \( \sum_{z} |\alpha_z| \). We will show that \( \sum_{z} |\alpha_z| \geq 2^{k - z_{\text{max}}} + 1 \). We claim that for \( z < z_{\text{max}} \), we have \( |\alpha_z| \leq 1 \). Indeed, if \( |\alpha_z| \geq 2 \), then we can replace \( \alpha_z \) with \( \alpha_z - \varepsilon 2 \) and \( \alpha_z + 1 \) with \( \alpha_z + 1 + \varepsilon \), where \( \varepsilon \) is the sign of \( \alpha_z \). This reduces \( \sum_{z} |\alpha_z| \) while preserving \( \sum_{z} \alpha_z 2^z \) and so contradicts minimality of \( m \) and \( (\alpha_z)_z \). Since \( 2^k + 1 \) is not even, there must be some \( \alpha_z \neq 0 \) with \( z \leq 0 \).

So if \( \alpha_{z_{\text{max}}} \geq 2^{k - z_{\text{max}}} \), then \( \sum_{z} |\alpha_z| \geq 2^{k - z_{\text{max}}} + 1 \). So we may assume that \( \alpha_{z_{\text{max}}} \leq 2^{k - z_{\text{max}}} \). Say \( \alpha_{z_{\text{max}}} = 2^{k - z_{\text{max}}} - \ell \) with \( \ell \geq 1 \). Then \( 2^k + 1 - \alpha_{z_{\text{max}}} \cdot 2^z_{\text{max}} \geq \ell 2^z_{\text{max}} - 1 \) and so \( \sum_{z = -m}^{z_{\text{max}} - 1} \alpha_z 2^z \geq \ell 2^z_{\text{max}} - 1 \). But, since \( \alpha_z \leq 1 \) for \( z < z_{\text{max}} \), we have

\[
\ell 2^z_{\text{max}} - 1 \leq \sum_{z = -m}^{z_{\text{max}} - 1} \alpha_z 2^z \leq \sum_{z = -m}^{z_{\text{max}} - 1} 2^z = 2^z_{\text{max}} - 2^{-m} \leq 2^z_{\text{max}}
\]

and so \( (\ell - 1)4 \leq (\ell - 1)2^z_{\text{max}} \leq 1 \) which implies \( \ell = 1 \). Thus if \( \alpha_{z'} < 1 \) for some \( z' \) with \( 0 \leq z' < z_{\text{max}} \), then we would have

\[
2^z_{\text{max}} - 1 \leq \sum_{z = -m}^{z_{\text{max}} - 1} \alpha_z 2^z
\]

\[
= \sum_{z = -m}^{z_{\text{max}} - 1} \alpha_z 2^z + \sum_{z = 0}^{z_{\text{max}} - 1} \alpha_z 2^z_{z'} + \sum_{z = 0}^{z_{\text{max}} - 1} \alpha_z 2^{z'}
\]

\[
\leq \sum_{z = -m}^{z_{\text{max}} - 1} 2^{z} + \sum_{z = 0}^{z_{\text{max}} - 1} 2^{z'}
\]

\[
= 1 - 2^{-m} + 2^z_{\text{max}} - 1 - 2^{z'} < 2^z_{\text{max}} - 1
\]

a contradiction. Hence \( \alpha_z = 1 \) for \( 0 \leq z < z_{\text{max}} \) so that \( \sum_{z} |\alpha_z| \geq 2^{k - z_{\text{max}}} - 1 + z_{\text{max}} \geq 2^{k - z_{\text{max}}} + 1 \) as required.

\[ \square \]

Lemma 7.16. Let \( k \) and \( \ell \) be nonnegative integers with \( \ell \leq k \) and \( k \geq 2 \). Then the word

\[ w = \tau^{\ell} a^{r \cdot a^{-k - 1} a^{k - \ell}} \]

describes a geodesic in \( \Gamma \).

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Proof. Consider the bijection \(\varphi : \text{BS}(1, 2) \to \mathbb{Z}[\frac{1}{2}] \times \mathbb{Z}\) described near the beginning of Section 7. Then, under this bijection, \(w\) describes a path from \((0, 2(k-\ell))\) to \((4^k \pm 1, 2(k - \ell))\). Indeed, following the description of the directed labeled edges we have the following:

\[
(0, 2(k - \ell)) \xrightarrow{\tau^{k = 2^t \ell}} (0, 2k)
\]
\[
\xrightarrow{a} (2^{2k}, 2k) = (4^k, 2k)
\]
\[
\xrightarrow{\tau^{k = t - 2k}} (4^k, 0)
\]
\[
\xrightarrow{a^\pm 1} (4^k \pm 1, 0)
\]
\[
\xrightarrow{\tau^{k - t = 2(k - t)}} (4^k \pm 1, 2(k - \ell))
\]

Consider any path \((r_j, z_j)_{j=0}^m\) from \((0, 2(k-\ell))\) to \((4^k \pm 1, 2(k - \ell))\) following edges of the Cayley graph. It will suffice to show that \(m \geq 2k + 2\), since \(2k + 2 = |w|\). For each \(j\), either

\[
(r_{j+1}, z_{j+1}) - (r_j, z_j) = (0, \pm 1)
\]

which we call a horizontal step or

\[
(r_{j+1}, z_{j+1}) - (r_j, z_j) = (0, \pm 2)
\]

which we call a vertical step. Since \(4^k \pm 1\) is not a power of 2, there must be at least two horizontal steps in \((r_j, z_j)_j\). Moreover, since \(4^k \pm 1\) is not even, we must have \(z_j \leq 0\) for some \(j\). Let \(z_{\text{max}} = \max_j z_j\). By joining consecutive \((r_j, z_j)\) by horizontal or vertical segments in \(\mathbb{R}^2 \supset \mathbb{Z}[\frac{1}{2}] \times \mathbb{Z}\) we extend \((r_j, z_j)_{j=0}^m\) to a piecewise affine map \(f : P \to \mathbb{R}^2\) where \(|P| = m\). The projection \(P \to \mathbb{R}\) of this map onto the second component captures the vertical behaviour of \(f\). We collapse any edges of \(P\) that map to points under \(P \to \mathbb{R}\) to obtain a map \(\bar{f} : P \to \mathbb{R}\) where \(|\bar{P}|\) is equal to the number of vertical steps of \((r_j, z_j)_j\). The map \(\bar{f}\) is 2-Lipschitz and \(0, 2(k - \ell), z_{\text{max}} \in \bar{f}(\bar{P})\) with \(z_{\text{max}} \geq 2(k - \ell)\) and \(0 \leq 2(k - \ell)\). Then

\[
\left| \bar{f}^{-1}([k - \ell, \infty)) \right| \geq 2 \cdot \frac{z_{\text{max}} - 2(k - \ell)}{2}
\]

and

\[
\left| \bar{f}^{-1}(]-\infty, k - \ell[) \right| \geq 2 \cdot \frac{2(k - \ell)}{2}
\]

where \(|\bar{f}^{-1}(I)|\) is the sum of the lengths of all maximal segments of \(f^{-1}(I)\). Hence \(|\bar{P}| \geq z_{\text{max}}\) and so \((r_j, z_j)_j\) takes at least \(z_{\text{max}}\) vertical steps. Then, since there must also be at least two horizontal steps, we see that if \(z_{\text{max}} \geq 2k\) then \(m \geq 2k + 2\). So we may assume that \(z_{\text{max}} < 2k\).

We split into three cases: \(z_{\text{max}} = 0\), \(z_{\text{max}} = 1\) and \(z_{\text{max}} \geq 2\). If \(z_{\text{max}} = 0\) then, by Lemma 7.15, there are at least \(2^{2k} - 1\) horizontal steps. So it suffices to show that \(2^{2k} - 1 \geq 2k + 2\) but, since \(k \geq 2\), this follows from the fact that \(2^x \geq x + 3\) for all \(x \geq 4\).

If \(z_{\text{max}} = 1\) then, by Lemma 7.15, there are at least \(2^{2k-1}\) horizontal steps. There will also be at least \(z_{\text{max}} = 1\) vertical step. But a closed path cannot contain a single vertical step so there are at least 2 vertical steps. So it suffices to show that
$2^{k-1} + 2 \geq 2k + 2$ but, since $k \geq 2$, this follows from the fact that $2^x \geq x + 1$ for all $x \geq 1$.

If $z_{\text{max}} \geq 2$ then, by Lemma 7.15, there are at least $2^{k-z_{\text{max}}} + 1$ horizontal steps. There are also at least $z_{\text{max}}$ vertical steps. So it suffices to show that $2^{k-z_{\text{max}}} + 1 + z_{\text{max}} \geq 2k + 2$ but, since $z_{\text{max}} < 2$, this follows from the fact that $2^x \geq x + 1$ for all $x \geq 1$.

\begin{lemma}

The word

$$w = a\tau^k a\tau^{-k} a^{-1} \tau^k a^{-1} \tau^{-k}$$

describes an isometric cycle in $\Gamma$ for all $k \geq 1$.

\begin{proof}
The word $w$ has length $4k+4$. After possible inversion and/or the application of the automorphism of $BS(1, 2)$ fixing $t$ and sending $a \mapsto a^{-1}$, the cyclic subwords of $w$ of length $2k+2$ are all of the form in Lemma 7.16 and so are geodesic. Hence, by Proposition 3.3, the word $w$ describes an isometric cycle in $\Gamma$.
\end{proof}

\end{lemma}

Then we have the following theorem and we see that the shortcut property for a Cayley graph is not invariant under a change of generating set.

\begin{theorem}

Let $\Gamma$ be the Cayley graph of $BS(1, 2)$ with generating set $\{a, t, \tau\}$ where $\tau = t^2$. Then $\Gamma$ is not shortcut.

\end{theorem}

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