V.M. Red’kov

Ricci Coefficients in Covariant Dirac Equation, Symmetry Aspects and Newman-Penrose Approach

September 6, 2011

The paper investigates how the Ricci rotation coefficients act in the Dirac equation in presence of external gravitational fields described in terms of Riemannian space-time geometry. It is shown that only 8 different combinations of the Ricci coefficients $\gamma_{abc}(x)$ are involved in the Dirac equation. They are combined in two 4-vectors $B_a(x)$ and $C_a(x)$ under local Lorentz group which has status of the gauge symmetry group. In all orthogonal coordinates one of these vectors, ”pseudovector” $C_a(x)$, vanishes identically. The gauge transformation laws of the two vectors are found explicitly. Connection of these $B_a(x)$ and $A_a(x)$ with the known Newman-Penrose coefficients is established. General study of gauge symmetry aspects in Newman-Penrose formalism is performed. Decomposition of the Ricci object, ”tensor” $\gamma_{abc}(x)$, into two ”spinors” $\gamma(x)$ and $\bar{\gamma}(x)$ is done. At this Ricci rotation coefficients are divided into two groups: 12 complex functions $\gamma(x) = (\gamma^\alpha_{\beta\rho\sigma})$ and 12 conjugated to them $\bar{\gamma}(x) = (\gamma^\beta_{\alpha\rho\sigma})$. Components of spinor $\bar{\gamma}(x)$ coincide with 12 spin coefficients by Newman-Penrose $\kappa, \pi, \epsilon, \rho, \lambda, \alpha, \sigma, \beta, \tau, \nu, \gamma$. For listing these it is used a special letter-notation $L_i, N_i, M_i, \bar{M}_i$. The formulas for gauge transformations of spin coefficients under local Lorentz group are derived. There are given two solutions to the gauge problem: one in the compact form of transformation laws for spinors $\gamma(x)$ and $\bar{\gamma}(x)$, and another as detailed elaboration of the latter in terms of 12 spin coefficients.

1 Dirac equation and Ricci coefficients

The known Dirac equation on the background of a curved space-time involves the Ricci rotation coefficients [1], the non-linear objects of a curved space-time geometry, and it has the form [2-4] ($A_a(x)$ stands for an external electromagnetic field, $c = 1, \hbar = 1$)

*redkov@dragon.bas-net.by
\[
\left\{ \gamma^\alpha \left[ i \left( \frac{\partial}{\partial x^\alpha} + \Gamma_\alpha \right) - e A_\alpha \right] - m \right\} \Psi = 0 ,
\]
\[
\gamma^\alpha(x) = \gamma^b e^\alpha_{(b)}(x) , \quad \Gamma_\alpha(x) = \frac{1}{2} \sigma^{ab} e^\beta_{(a)} e^\gamma_{(b)\beta\alpha} , \quad \sigma^{ab} = \frac{1}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a) ,
\] (1)
or
\[
\left\{ \gamma^c \left[ i (e^\alpha_{(c)} \partial_\alpha + \frac{1}{2} \sigma^{ab} \gamma_{abc}) - e A_c \right] - m \right\} \Psi = 0
\] (2)
where \( \gamma_{abc}(x) \) are the Ricci rotation symbols [1]
\[
\gamma_{bac}(x) = - \gamma_{abc}(x) = - e_{(b)\beta\alpha} e^\beta_{(a)} e^\alpha_{(c)}
\] (3)
and \( A_a(x) = e^\alpha_{(a)}(x) A_\alpha(x) \) designates tetrad (vierbein) components of the electromagnetic field 4-vector. Now, with the use of the known formula for product of three Dirac matrices [5]
\[
\gamma^c \gamma^a \gamma^b = \gamma^c g^{ab} - \gamma^a g^{cb} + \gamma^b g^{ac} + i \gamma^5 \epsilon^{cabk} \gamma_k , \quad \gamma^5 = -i \gamma^0 \gamma^1 \gamma^2 \gamma^3 , \quad \epsilon^{0123} = +1
\]
we can easily produce
\[
\gamma^c \sigma^{ab} = \frac{1}{2} (g^{ca} \gamma^b - g^{cb} \gamma^a + i \gamma^5 \epsilon^{cabk} \gamma_k) .
\] (4)
Taking in mind eq. (4), eq. (2) can be transformed into
\[
\left\{ \gamma^k \left[ i (e^\alpha_{(k)} \partial_\alpha + \frac{1}{2} e^\alpha_{(k)} e^\gamma_{(k)\alpha} \gamma_{abc}) - e A_c \right] - m \right\} \Psi = 0 .
\] (5)

With the notation
\[
B_k(x) = \frac{1}{2} e^\alpha_{(k)} ;_\alpha(x) , \quad C_k(x) = \frac{1}{4} e^{abc} \gamma_{abc}(x)
\] (6)
the Dirac equation (1) will get the form
\[
\left\{ \gamma^k \left[ i (e^\alpha_{(k)} \partial_\alpha + B_k - i \gamma^5 C_k) - e A_a \right] - m \right\} \Psi = 0 .
\] (7)

This form of the Dirac equation is remarkable in some aspects. The first one is that the vector field \( C_k(x) \) involved in eq. (7) vanishes identically in all orthogonal coordinates and their accompanying tetrads. Therefore, the Dirac equation will take on the simpler form
\[
\left\{ \gamma^k \left[ i (e^\alpha_{(k)} \partial_\alpha + B_k) - e A_a \right] - m \right\} \Psi = 0 .
\] (8)

Let us prove this property of the vector \( C_k(x) \). By definition, \( \gamma_{abc} = - \gamma_{bac} \); for the following is is useful to introduce a quantity antisymmetric with respect to \( bc \):
\[
\lambda_{abc}(x) = \gamma_{abc}(x) - \gamma_{acb}(x) .
\]
For $\lambda_{abc}(x)$ there exists representation in terms of ordinary derivatives:

$$\lambda_{abc} = \gamma_{abc} - \gamma_{acb} = (e(\alpha)_{\alpha;\beta} - e(\alpha)_{\beta;\alpha})e(\alpha)_{(c)}e(\beta)_{(b)}$$

$$= (\partial_{\beta}e(\alpha)_{\alpha} - \Gamma_{\alpha\beta}^{\rho}e(\alpha)_{\rho} - \partial_{\alpha}e(\alpha)_{\beta} + \Gamma_{\beta\alpha}^{\rho}e(\alpha)_{\rho})e(\alpha)_{(c)}e(\beta)_{(b)}$$

$$= (\partial_{\beta}e(\alpha)_{\alpha} - \partial_{\alpha}e(\alpha)_{\beta})e(\alpha)_{(c)}e(\beta)_{(b)} .$$

Also the identity

$$\gamma_{abc} = \frac{1}{2}(\lambda_{abc} + \lambda_{bca} - \lambda_{cab}) = \frac{1}{2}(\gamma_{abc} - \gamma_{acb} + \gamma_{bca} - \gamma_{bac} - \gamma_{cab} + \gamma_{cbb})$$ (9)

holds. With the use of eqs. (9), for components of $C_k(x)$ (6) it follows

$$C_0(x) = e^{abc}_{0} \gamma_{abc}(x) = (\lambda_{123} + \lambda_{231} + \lambda_{312}) ,$$

$$C_1(x) = e^{abc}_{1} \gamma_{abc}(x) = -(\lambda_{203} + \lambda_{302} + \lambda_{023}) ,$$

$$C_2(x) = e^{abc}_{2} \gamma_{abc}(x) = (\lambda_{301} + \lambda_{013} + \lambda_{130}) ,$$

$$C_3(x) = e^{abc}_{3} \gamma_{abc}(x) = -(\lambda_{012} + \lambda_{120} + \lambda_{201}) .$$ (10)

Let us consider these relations (10) in a space-time with a diagonal metric tensor:

$$dS^2 = h^2_0(x) (dx^0)^2 - h^2_i(x) (dx^i)^2 ,$$

and its accompanying tetrad

$$e(\alpha)_{\alpha}(x) = \begin{pmatrix} h_0 & 0 & 0 & 0 \\ 0 & h_1 & 0 & 0 \\ 0 & 0 & h_2 & 0 \\ 0 & 0 & 0 & h_3 \end{pmatrix} .$$ (11)

Taking into account (11) and (4), for eq. (10) we can easily obtain the form

$$C_0(x) = [\partial_{2}e(1)_{3} - \partial_{3}e(1)_{2}]e(2)_{3}e^3_{(3)} + [\partial_{3}e(2)_{1} - \partial_{1}e(2)_{3}]e^3_{(2)} + [\partial_{1}e(3)_{2} - \partial_{2}e(3)_{1}]e^1_{(1)} ,$$

$$C_1(x) = -[\partial_{3}e(2)_{0} - \partial_{0}e(2)_{3}]e^3_{(3)} - [\partial_{0}e(3)_{2} - \partial_{2}e(3)_{0}]e^0_{(2)} - [\partial_{2}e(0)_{3} - \partial_{3}e(0)_{2}]e^2_{(2)} ,$$

$$C_2(x) = [\partial_{0}e(3)_{1} - \partial_{1}e(3)_{0}]e^0_{(1)}e^1_{(1)} + [\partial_{1}e(0)_{3} - \partial_{3}e(0)_{1}]e^1_{(3)} + [\partial_{3}e(1)_{0} - \partial_{0}e(1)_{3}]e^3_{(3)} ,$$

$$C_3(x) = -[\partial_{1}e(0)_{2} - \partial_{2}e(0)_{1}]e^1_{(2)}e^2_{(2)} - [\partial_{2}e(1)_{0} - \partial_{0}e(1)_{2}]e^2_{(0)}e^0_{(0)} - [\partial_{0}e(2)_{1} - \partial_{1}e(2)_{0}]e^0_{(0)}e^1_{(1)} .$$

It should be noted that these relations involve only non-diagonal elements of the tetrad matrix $e(\alpha)_{\beta}(x)$, therefore all quantities $C_k(x)$ vanish identically. So, always in any space-time models characterized by (11) the above vector combination of the Ricci coefficient is zero:

$$C_k(x) = \frac{1}{4} \epsilon^{abc}_{k} \gamma_{abc}(x) \equiv 0 .$$ (12)
We are to give attention to separation of these 8 relevant constituents from \( \gamma_{abc} \). To this end, as the first step, one decomposes \( \gamma_{abc}(x) \) into the sum

\[
\gamma_{abc} = [\gamma_{abc} - A \epsilon_{abc}^n C_n(x)] + A \epsilon_{abc}^n C_n(x) = \Delta_{[abc]}(x) + A \epsilon_{abc}^n C_n(x) \tag{13}
\]

where \( A \) is to be chosen later. Indeed, let us require

\[
\epsilon^{abc}_m \Delta_{[abc]}(x) = 0 \quad \text{or} \quad 4 C_m(x) - A (\epsilon^{abc}_m \epsilon_{abc}^n) C_n(x) = 0.
\]

From this, with relation \( \epsilon^{abc}_m \epsilon_{abc}^n = -6 \delta^m_n \), it follows

\[
(4 + 6 A) C_m = 0 \quad \text{or} \quad A = -\frac{2}{3}.
\]

With the use of the following notation for 3-rank tensor dual to a vector \( C_n \):

\[
C_{[abc]}(x) = -\frac{2}{3} \epsilon_{abc}^n C_n(x) \tag{14}
\]

the expansion (13) looks as

\[
\gamma_{abc}(x) = \Delta_{[abc]}(x) + C_{[abc]}(x) . \tag{15}
\]

The latter provides us with the decomposition of the \( \gamma_{abc}(x) \) into the sum of \( \Delta_{[abc]}(x) \), orthogonal to the Levi-Civita symbol

\[
\epsilon^{abc}_m \Delta_{[abc]}(x) = 0 , \tag{16}
\]

and the \( C_{[abc]}(x) \) non-orthogonal to the Levi-Civita symbol

\[
\epsilon^{abc}_m C_{[abc]}(x) = \epsilon^{abc}_m \left[ -\frac{2}{3} \epsilon_{abc}^n C_n(x) \right] = +4 C_m(x) = \epsilon^{abc}_m \gamma_{abc}(x) . \tag{17}
\]

Further, taking in mind the known formula

\[
\epsilon^{klm}_n \epsilon_{abc}^n = (-1) \begin{pmatrix} \delta^k_a & \delta^k_b & \delta^k_c \\ \delta^l_a & \delta^l_b & \delta^l_c \\ \delta^m_a & \delta^m_b & \delta^m_c \end{pmatrix}
\]

one produces the following form of \( C_{[abc]}(x) \) in terms of the Ricci coefficients:

\[
C_{[abc]}(x) = -\frac{2}{3} \epsilon_{abc}^n C_n(x) = -\frac{2}{3} \epsilon_{abc}^n \frac{1}{4} \epsilon^{klm}_n \gamma_{klm}(x) = \frac{1}{3} (\gamma_{abc} + \gamma_{bca} + \gamma_{cab}) .
\]

In addition, for \( \Delta_{abc} \) one gets to

\[
\Delta_{[abc]}(x) = \gamma_{abc} - C_{[abc]} = \frac{2}{3} \gamma_{abc} + \frac{1}{3} (\gamma_{acb}(x) - \gamma_{bca}(x)) . \tag{18}
\]
Take notice that one cannot obtain from \( C_{abc}(x) \), by means of simplification over any pair of indices, a non-zero vector. In other words, this tensor is ir reducible. But such a trick is possible with the \( \Delta_{[abc]}(x) \):

\[
\Delta_{[abc]}(x) = [\Delta_{[abc]}(x) - \alpha (g_{ac}B_b(x) - g_{bc}B_a(x))] + \alpha (g_{ac}B_b(x) - g_{bc}B_a(x)) = E_{[abc]}(x) + B_{[abc]}(x),
\]

where

\[
B_b(x) = \gamma_{kb}^k(x) = -\gamma_b^k, \quad B_{[abc]}(x) = \alpha (g_{ac}B_b(x) - g_{bc}B_a(x)), \quad E_{[abc]} = \Delta_{[abc]}(x) - B_{[abc]}(x).
\]

The choice \( \alpha = +1/3 \) insures the properties

\[
B_{[kk]}^k(x) = B_b(x), \quad B_{[bk]}^k(x) = -B_b(x),
\]

\[
E_{[kk]}^k(x) = 0, \quad E_{[bk]}^k(x) = 0.
\]

Besides, the quantity \( E_{[abc]}(x) \) is orthogonal to the Levi-Civita symbol:

\[
\epsilon_{abc} E_{[abc]}(x) = 0.
\]

So, we get to the result we need: the Ricci object can be composed as the sum

\[
\gamma_{abc}(x) = C_{[abc]}(x) + B_{[abc]}(x) + E_{[abc]}(x)
\]

where the tensors \( C_{[abc]}(x) \) and \( B_{[abc]}(x) \) are the combinations which are relevant as we concern the Dirac equation in any curved space-time model. Besides, in any orthogonal coordinate system the tensor \( C_{[abc]}(x) \) vanishes identically.

2 Gauge properties of \( B_a \) and \( C_a \)

Now we are going to consider how the above two vector fields \( B_a(x) \) and \( C_a(x) \) behave with respect to any local tetrad Lorentz transformation \([1]\). For more generality we will take account of proper as well as improper Lorentz matrices

\[
\epsilon_{(a)}^{\beta}(x) = L_a^b(x) \epsilon_{(b)}^{\beta}(x), \quad L_a^b(x) = L_a^b(k(x), k^*(x)).
\]

Starting from definition of \( B_a(x) \), one can readily produce

\[
B'_a(x) = \nabla_{\beta} \left[ \epsilon_{(a)}^{\beta}(x) \right] = \nabla_{\beta} \left[ L_a^b(x) \epsilon_{(b)}^{\beta}(x) \right] = L_a^b(x) \left( \nabla_{\beta} \epsilon_{(b)}^{\beta}(x) \right) + \frac{\partial L_a^b(x)}{\partial x^\beta} \epsilon_{(b)}^{\beta}
\]

from where it follows the transformation law for \( B_a(x) \) we need:

\[
B'_a(x) = L_a^b(x) B_b(x) + \frac{\partial L_a^b(x)}{\partial x^\beta} \epsilon_{(b)}^{\beta}.
\]
Analogously let us analyze the case of \( C_a(x) \). Now, it will be convenient to go from the known formulas for gauge transformation of the Ricci coefficients \([1]\)

\[
\gamma'_{abc}(x) = L_a^{k}(x)L_b^{l}(x)L_c^{n}(x) \gamma_{klm}(x) + L_a^{k}(x)g_{kl} \frac{\partial L_b^{l}(x)}{\partial x^\mu}L_c^{n}(x)e_{(n)}^{\mu}(x). \tag{25}
\]

Instead of \( L_a^{k}(x)g_{kl} \) we will write \( L_{al}(x) \) and so on; with this notation the orthogonality condition for Lorentz matrices will take the form \( L_{ab} = L_{ba}^{-1} \). Multiplying eq. (25) by \( \frac{1}{4}\epsilon^{abc}_{\phantom{abc}d} \), we get to

\[
C'_{d}(x) = \frac{1}{4} \epsilon^{abc}_{\phantom{abc}d} L_a^{k}(x)L_b^{l}(x)L_c^{n}(x) \gamma_{klm}(x)
+ \frac{1}{4} \epsilon^{abc}_{\phantom{abc}d} L_{al}(x) \frac{\partial L_b^{l}(x)}{\partial x^\mu}L_c^{n}(x)e_{(n)}^{\mu}(x). \tag{26}
\]

Now, taking into account the known formula

\[
\epsilon^{abcd} L_a^{k}L_b^{l}L_c^{n}L_d^{m} = + \det [L_s^{t}] \epsilon^{klmn},
\]

we will have

\[
\epsilon^{abc}_{\phantom{abc}d} L_a^{k}L_b^{l}L_c^{n} = + \det [L_s^{t}] \epsilon^{klmn} L_{dm},
\]

with the use of which in eq. (26) we get to the required gauge law \( (\det (L_s^{t}) = \det L) \):

\[
C'_{d}(x) = \det L(x) L_{d}^{m}(x) C_{m}(x)
+ \frac{1}{4} \epsilon^{abc}_{\phantom{abc}d} \left[ L_{al}(x) \left( \frac{\partial(L^{-1})^{l}_{b}(x)}{\partial x^\mu} \right) \right] L_{c}^{n}(x)e_{(n)}^{\mu}(x). \tag{27}
\]

3 Connection with the Newman-Penrose spin formalism

Now we are going to dwell upon the structure of the Dirac equation \([4]\) in a detailed component-based form\(^1\). As a first step let us write down the Dirac equation in the 2-spinor (splitted) form (the conventional notation for spinors based on dotted and undotted indices is used)

\[
\sigma^a \left[ i (e_\beta^{(a)} \partial_\beta + B_a + iC_a) - eA_a \right] \xi = m \eta,
\]

\[
\sigma^a \left[ i (e_\beta^{(a)} \partial_\beta + B_a - iC_a) - eA_a \right] \eta = m \xi. \tag{28}
\]

From this, allowing for the explicit form of the Pauli two-by-two matrices, and introducing a special letter designation according to

\[
B_{(0)} + B_{(3)} = \hat{B}_0, \quad B_{(0)} - B_{(3)} = \hat{B}_1,
\]

\[
B_{(1)} - iB_{(2)} = \hat{B}_2, \quad B_{(1)} + iB_{(2)} = \hat{B}_3,
\]

\[
C_{(0)} + C_{(3)} = \hat{C}_0, \quad C_{(0)} - C_{(3)} = \hat{C}_1,
\]

\(^1\)In the widely used method of spin coefficients by Newman-Penrose \([4]\) just a such approach is exploited; we consider the above Dirac equation \([4]\) in that spin-coefficient language and then compare it with the form based on the use of the vectors \( B_a(x) \) and \( C_a(x) \) \([5]\).
Also, one is to employ the letter notation for elements of the matrices $\sigma^\beta(x)$ and $\tilde{\sigma}^\beta(x)$:

$$\begin{align*}
\sigma^\beta(x) &= \begin{pmatrix}
e^{\beta}_0 + e^{\beta}_3 & e^{\beta}_1 - ie^{\beta}_2 \\
e^{\beta}_1 + ie^{\beta}_2 & e^{\beta}_0 - e^{\beta}_3
\end{pmatrix} = \sqrt{2} \begin{pmatrix}l^\beta(x) & m^\beta(x) \\
\bar{m}^\beta(x) & \bar{n}^\beta(x)\end{pmatrix}, \\
\tilde{\sigma}^\beta(x) &= \begin{pmatrix}e^{\beta}_0 - e^{\beta}_3 & -e^{\beta}_1 + ie^{\beta}_2 \\
-e^{\beta}_1 - ie^{\beta}_2 & e^{\beta}_0 + e^{\beta}_3
\end{pmatrix} = \sqrt{2} \begin{pmatrix}n^\beta(x) & -\bar{m}^\beta(x) \\
-m^\beta(x) & l^\beta(x)\end{pmatrix},
\end{align*}$$

and further

$$\sigma^\beta \tilde{\sigma}_{\beta;\alpha} = 2 \begin{pmatrix}n^\beta \bar{m}_{\beta;\alpha} - m^\beta \bar{n}_{\beta;\alpha} & n^\beta m_{\beta;\alpha} - m^\beta n_{\beta;\alpha} \\
m^\beta \bar{l}_{\beta;\alpha} + l^\beta \bar{m}_{\beta;\alpha} & -\bar{m}^\beta m_{\beta;\alpha} + l^\beta n_{\beta;\alpha}\end{pmatrix},$$

for eqs. (28) we get the form

$$\begin{align*}
\sigma^\alpha \left[ i (e^\beta_{(a)\beta} + B_a + iC_a) - eA_a \right] \xi = \left(\begin{array}{l}i(e^\beta_0 \partial_\beta + \hat{B}_0 + i\hat{C}_0) - e\hat{A}_0 \\
i(e^\beta_3 \partial_\beta + \hat{B}_3 + i\hat{C}_3) - e\hat{A}_3 \end{array}\right) \xi = m\eta,
\end{align*}$$

The forms obtained reflect explicitly that only 8 of the 24 Ricci coefficients are involved in the Dirac equation: $\hat{B}_0, \hat{B}_1, \hat{B}_2, \hat{B}_3$ and $\hat{C}_0, \hat{C}_1, \hat{C}_2, \hat{C}_3$.

Now we will go into the notation accepted in the Newman-Penrose approach [4]. First, for describing the connection $B_{\alpha}(x)$ in spinor basis one is to introduce the following designation

$$B_{\alpha}(x) = \begin{pmatrix}0 & 0 \\
0 & \Sigma_{\alpha}\end{pmatrix} = \frac{1}{8} \begin{pmatrix}\sigma^\beta \tilde{\sigma}_{\beta;\alpha} - \tilde{\sigma}^\beta \sigma_{\beta;\alpha} & 0 \\
0 & \sigma^\beta \tilde{\sigma}_{\beta;\alpha} - \tilde{\sigma}^\beta \sigma_{\beta;\alpha}\end{pmatrix},$$

where

$$\sigma_{\beta}(x) = \sigma^a e_{(a)\beta}(x), \quad \tilde{\sigma}_{\beta}(x) = \tilde{\sigma}^a e_{(a)\beta}(x),$$

$$\sigma_{\beta;\alpha}(x) = \sigma^a e_{(a)\beta;\alpha}, \quad \tilde{\sigma}_{\beta;\alpha} = \tilde{\sigma}^a e_{(a)\beta;\alpha}.$$

Also, one is to employ the letter notation for elements of the matrices $\sigma^\beta(x)$ and $\tilde{\sigma}^\beta(x)$:
\[ \sigma_{\beta\alpha} \sigma^\beta = 2 \left( \begin{array}{cc} n_{\beta\alpha} l^3 - m_{\beta\alpha} \bar{m}^\beta & n_{\beta\alpha} m^\beta - m_{\beta\alpha} n^\beta \\ -m_{\beta\alpha} l^3 + l_{\beta\alpha} \bar{m}^\beta & -m_{\beta\alpha} m^\beta + l_{\beta\alpha} n^\beta \end{array} \right). \]

So, the expression for connection \( \Sigma_{\alpha}(x) \) is

\[ 2\Sigma_{\alpha} = \left( \begin{array}{cc} -l^3 n_{\beta\alpha} - m^\beta \bar{m}_{\beta\alpha} & 2 n^3 m_{\beta\alpha} \\ 2 l^3 \bar{m}_{\beta\alpha} & l^3 n_{\beta\alpha} + m^3 \bar{m}_{\beta\alpha} \end{array} \right). \] (32)

In getting (32) one must allow for that arbitrary generally covariant scalar products

\[ l^3(x), n^3(x), m^3(x), \bar{m}^3(x) \]

are generally covariant invariants, therefore the identities of the form

\[ n^3 l_{\beta} = \text{inv} \implies n^3_{\beta\alpha} l_{\beta} + n^3 l_{\beta\alpha} = 0, \]
\[ n^3 m_{\beta} = \text{inv} \implies n^3_{\beta\alpha} m_{\beta} + n^3 m_{\beta\alpha} = 0, \]
\[ n^3 \bar{m}_{\beta} = \text{inv} \implies n^3_{\beta\alpha} \bar{m}_{\beta} + n^3 \bar{m}_{\beta\alpha} = 0 \]

and so on hold. In the same manner we find the expression for \( \bar{\Sigma}_{\alpha}(x) \)-connection:

\[ \bar{\sigma}_{\beta\alpha} \bar{\sigma}^\beta = 2 \left( \begin{array}{cc} \bar{m}^\beta & n_{\beta\alpha} - m_{\beta\alpha} \\ n_{\beta\alpha} & l_{\beta\alpha} \end{array} \right) \]

\[ = 2 \left( \begin{array}{cc} l^3 n_{\beta\alpha} - m^\beta \bar{m}_{\beta\alpha} & -l^3 m_{\beta\alpha} + m^3 l_{\beta\alpha} \\ \bar{m}^\beta n_{\beta\alpha} - n^3 \bar{m}_{\beta\alpha} & -\bar{m}^\beta m_{\beta\alpha} + n^3 l_{\beta\alpha} \end{array} \right), \]

\[ \bar{\sigma}_{\beta\alpha} \bar{\sigma}^\beta = 2 \left( \begin{array}{cc} l_{\beta\alpha} m_{\beta\alpha} & n_{\beta\alpha} - m_{\beta\alpha} \\ \bar{m}_{\beta\alpha} n_{\beta\alpha} & \bar{m}_{\beta\alpha} m_{\beta\alpha} \end{array} \right) \]

and finally

\[ 2 \bar{\Sigma}_{\alpha} = \left( \begin{array}{cc} l^3 n_{\beta\alpha} + \bar{m}^\beta m_{\beta\alpha} & -2 l^3 m_{\beta\alpha} \\ -2 n^3 m_{\beta\alpha} & -l^3 n_{\beta\alpha} - \bar{m}^3 m_{\beta\alpha} \end{array} \right). \] (33)

To write down the Dirac equation

\[ i \sigma^\alpha(x) \left( \partial_\alpha + \Sigma_{\alpha}(x) \right) \xi(x) = m \eta(x), \]
\[ i \bar{\sigma}^\alpha(x) \left( \partial_\alpha + \bar{\Sigma}_{\alpha}(x) \right) \eta(x) = m \xi(x), \]

in the Newman-Penrose approach we are to develop expressions for involved differential operators in corresponding notation. To this end we obtain

\[ i \sigma^\alpha(x) \left( \partial_\alpha + \Sigma_{\alpha}(x) \right) = i \sqrt{2} \left[ \left( \begin{array}{cc} l^a \partial_\alpha & m^a \partial_\alpha \\ \bar{m}^a \partial_\alpha & n^a \partial_\alpha \end{array} \right) \right] \]

\[ + \left( \begin{array}{c} -\frac{1}{2} (l^3 n_{\beta\alpha} + m^3 \bar{m}_{\beta\alpha}) \partial_\alpha + m^a \eta \bar{m}_{\beta\alpha} + \frac{1}{2} (l^3 n_{\beta\alpha} + m^3 \bar{m}_{\beta\alpha}) m^\alpha + l^a n^3 m_{\beta\alpha} \\ -\frac{1}{2} (l^3 n_{\beta\alpha} + m^3 \bar{m}_{\beta\alpha}) \bar{m}^\alpha + n^3 l^3 \bar{m}_{\beta\alpha} + \frac{1}{2} (l^3 n_{\beta\alpha} + m^3 \bar{m}_{\beta\alpha}) n^\alpha + \bar{m}^a n^3 m_{\beta\alpha} \end{array} \right). \]
Let us introduce conventions:

\[ l^\alpha \partial_\alpha = \nabla_l, \quad n^\alpha \partial_\alpha = \nabla_n, \]
\[ m^\alpha \partial_\alpha = \nabla_m, \quad \bar{m}^\alpha \partial_\alpha = \nabla_{\bar{m}}; \]

besides it will be convenient to omit pairs of mute indices:

\[ \bar{m}_\beta \alpha l^\beta m^\alpha = \bar{m}lm = -il\bar{mn}, \]
\[ -\frac{1}{2}(n_\beta \alpha l^\beta \bar{m}_\beta \alpha m^\beta)l^\alpha = -\frac{1}{2}(nl + \bar{mn})l \]

and so on. Thus, the above equation will look as

\[ i\sigma^\alpha(x)(\partial_\alpha + \Sigma_\alpha(x)) = \left( \begin{array}{ccc}
\nabla_l - \frac{1}{2}(nl + \bar{mn})l + \bar{m}lm & \nabla_m + \frac{1}{2}(nl + \bar{mn})m + mnl & \\
\nabla_{\bar{m}} - \frac{1}{2}(nl + \bar{mn})\bar{m} + \bar{m}m & \nabla_n + \frac{1}{2}(nl + \bar{mn})n + m\bar{m} & \\
\end{array} \right). \quad (34) \]

In the same manner we find

\[ i\bar{\sigma}^\alpha(x)(\partial_\alpha + \bar{\Sigma}_\alpha(x)) = \left( \begin{array}{ccc}
\nabla_n + \frac{1}{2}(nl + \bar{mn})n + \bar{m}mn & -\nabla_m + \frac{1}{2}(nl + \bar{mn})m - mnl & \\
\n\bar{\nabla}_{\bar{m}} - \frac{1}{2}(nl + \bar{mn})\bar{m} - \bar{m}m & \bar{\nabla}_l + m\bar{m}l - \frac{1}{2}(nl + \bar{mn})l & \\
\end{array} \right). \quad (35) \]

Let us define 12 complex combinations of the Ricci coefficients:

\[ \begin{align*}
(l\bar{m})l &= L_1, & (l\bar{m})m &= M_1, \\
(-nm)l &= L_2, & (-nm)m &= M_2, \\
\frac{1}{2}(ln + \bar{mn})l &= L_3, & \frac{1}{2}(ln + \bar{mn})m &= M_3, \\
(l\bar{m})\bar{m} &= \bar{M}_1, & (l\bar{m})n &= N_1, \\
(-nm)\bar{m} &= \bar{M}_2, & (-nm)n &= N_2, \\
\frac{1}{2}(ln + \bar{mn})\bar{m} &= \bar{M}_3, & \frac{1}{2}(ln + mn)n &= N_3.
\end{align*} \quad (36) \]

These \((L_i, N_i, M_i, \bar{M}_i)\) are related straightforwardly with the so-called Newman-Penrose coefficients [4]

\[(k, \pi, \epsilon; \rho, \lambda, \alpha; \sigma, \mu, \beta; \tau, \nu, \gamma)\]

according to

\[ L_1 = k^*, \quad L_2 = \pi^*, \quad L_3 = \epsilon^*, \quad N_1 = \tau^*, \quad N_2 = \nu^*, \quad N_3 = \gamma^*, \quad M_1 = \rho^*, \quad M_2 = \lambda^*, \quad M_3 = \alpha^*, \quad \bar{M}_1 = \sigma^*, \quad \bar{M}_2 = \mu^*, \quad \bar{M}_3 = \beta^* \cdot \]

In this notation, the above differential operators read as

\[ i\sigma^\alpha(x)(\partial_\alpha + \Sigma_\alpha(x)) = i\sqrt{2} \left( \begin{array}{cc}
(\nabla_l + L_3 - M_1) & (\nabla_m + L_2 - M_3) \\
(\nabla_{\bar{m}} + M_3 - N_1) & (\nabla_n + M_2 - N_3) \\
\end{array} \right), \quad (37) \]

\[ i\bar{\sigma}^\alpha(x)(\partial_\alpha + \bar{\Sigma}_\alpha(x)) = i\sqrt{2} \left( \begin{array}{cc}
(\nabla_n + \bar{M}_3^* - \bar{N}_1^*) & -\nabla_m + \bar{M}_2^* - \bar{N}_3^* \\
-\nabla_{\bar{m}} + L_2 - M_3^* & (\nabla_l + L_3^* - \bar{M}_1^*) \\
\end{array} \right). \quad (38) \]
It should be noted that in eqs. (37) and (38) only the following 8 spin coefficients and their conjugates

\[
L_2, L_3, \ M_2, \ M_3, \ N_1, \ N_3, \ M_1, \ M_3,
\]

\[
L_2^*, L_3^*, \ \bar{M}_2, \ \bar{M}_3, \ \bar{N}_1^*, \ \bar{N}_3^*, \ \bar{M}_1^*, \ \bar{M}_3^*,
\]

are involved, and what is more, these quantities enter the equations only in combinations

\[
L_3 - M_1, \ L_2 - M_3, \ \bar{M}_3 - N_1, \\
\bar{M}_2 - N_3, \ (L_3 - M_1)^*, \ (L_2 - M_3)^*,
\]

\[
(M_3 - N_1)^*, \ (M_2 - N_3)^*.
\]

In other words, this means that only 8 real-valued combinations of the Ricci object enter the Dirac equation.

It remains to establish how these spin-coefficients-based parameters refer to the above vectors \( C_a(x) \) and \( B_a(x) \). By straightforward comparison we derive

\[
\hat{B}_0 + i\hat{C}_0 = \sqrt{2} (L_3 - M_1),
\]

\[
\hat{B}_0 - i\hat{C}_0 = \sqrt{2} (L_3^* - M_1^*),
\]

\[
\hat{B}_1 + i\hat{C}_1 = \sqrt{2} (M_2 - N_3),
\]

\[
\hat{B}_1 - i\hat{C}_1 = \sqrt{2} (M_2^* - N_3^*),
\]

\[
\hat{B}_2 + i\hat{C}_2 = \sqrt{2} (L_2 - M_3),
\]

\[
\hat{B}_2 - i\hat{C}_2 = \sqrt{2} (M_3^* - N_1^*),
\]

\[
\hat{B}_3 + i\hat{C}_3 = \sqrt{2} (M_3 - N_1),
\]

\[
\hat{B}_3 - i\hat{C}_3 = \sqrt{2} (L_2^* - M_3^*),
\]

(39)

from here it follows

\[
\hat{B}_0 = 2^{-1/2}(L_3 - M_1 + L_3^* - M_1^*),
\]

\[
\hat{B}_1 = 2^{-1/2}(\bar{M}_2 - N_3 + \bar{M}_2^* - N_3^*),
\]

\[
\hat{B}_2 = 2^{-1/2}(L_2 - M_3 + \bar{M}_3^* - N_1^*),
\]

\[
\hat{B}_3 = 2^{-1/2}(M_3 - N_1 + L_2^* - M_3^*),
\]

\[
i\hat{C}_0 = 2^{-1/2}(L_3 - M_1 - L_3^* + M_1^*),
\]

\[
i\hat{C}_1 = 2^{-1/2}(\bar{M}_2 - N_3 - \bar{M}_2^* + N_3^*),
\]

\[
i\hat{C}_2 = 2^{-1/2}(L_2 - M_3 - \bar{M}_3^* + N_1^*),
\]

\[
i\hat{C}_3 = 2^{-1/2}(M_3 - N_1 - L_2^* + M_3^*).
\]

(40)

As it should be expected, the \( \hat{B}_0, \hat{B}_1 \) and \( \hat{C}_0, \hat{C}_1 \) are real-valued, but \( \hat{B}_3 = \hat{B}_2^* \), \( \hat{C}_3 = \hat{C}_2^* \).
4 Newman-Penrose coefficients in spinor approach

To present time, the method of spin coefficients proposed by Newman and Penrose has become prevalent in studying gravitational fields and particle fields on curved space-time background (see, for example, the textbook by Rindler and Penrose [4] or surveys by Frolov [6], Alekseev and Khlebnikov [7]). The point we are going to elaborate below is that the spin coefficients provide us with gauge non-invariant characteristics of a gravitational field. So the question of their gauge properties is of special physical meaning, important conceptually and relevant to ordinary day-to-day technical work with gravitational fields.

Let us begin with special definition of spin coefficients in spinor approach. Starting from ordinary (non-isotropic) tetrad and conventional Ricci rotation coefficients

\[ e_\beta^\gamma(x), \quad \gamma_{abc}(x) = -e_\alpha^\beta(x)\cdot\epsilon_{\mu} \epsilon^\nu \epsilon^\rho \cdot e_\gamma(\nu) ; \]

instead of \( \gamma_{abc}(x) \) one can introduce two complex-valued (say "spinor") objects \( \gamma(x) \) and \( \tilde{\gamma}(x) \):

\[ \gamma(x) = \left( \sigma^a \sigma^b \otimes \sigma^c \right) \gamma_{abc}(x) , \quad \tilde{\gamma}(x) = \left( \sigma^a \sigma^b \otimes \sigma^c \right) \gamma_{abc}(x) ; \]

(41)

\( \gamma(x) \) and \( \tilde{\gamma}(x) \) stand for objects with four 2-spinor indices. Inversely, initial Ricci coefficients can be reconstructed as follows

\[ \gamma_{klm}(x) = A \left[ \text{Tr}(\sigma_k \sigma_l \otimes \text{Tr}(\sigma_n) \gamma(x) + A \left[ \text{Tr}(\sigma_k \sigma_l \otimes \text{Tr}(\sigma_n) \tilde{\gamma}(x) . \right. \right. \right. \right.

(42)

It readily can be found \( A = 1/16 \) by substituting expressions (41) for \( \gamma(x) \) and \( \tilde{\gamma}(x) \) into (42):

\[ \gamma_{klm}(x) = A \text{Tr} \left( \sigma_k \sigma_l \sigma_a \sigma_b \right) \text{Tr} \left( \sigma_n \sigma_c \right) \gamma_{abc}(x) \]

\[ + A \text{Tr} \left( \sigma_k \sigma_l \sigma_n \sigma_c \right) \text{Tr} \left( \sigma_a \sigma_b \right) \gamma_{abc}(x) \]

and taking into account the known formulas for Pauli matrix traces

\[ \text{Tr} \left( \sigma_k \sigma_l \sigma_a \sigma_b \right) = 2(g_{kl}g_{ab} - g_{ka}g_{lb} + g_{kb}g_{la} - i\epsilon_{klab}) , \]

\[ \text{Tr} \left( \sigma_k \sigma_l \sigma_n \sigma_c \right) = 2(g_{kl}g_{ab} - g_{ka}g_{lb} + g_{kb}g_{la} + i\epsilon_{klab}) , \]

\[ \text{Tr}(\sigma_n \sigma_c) = 2g_{nc} , \quad \text{Tr}(\sigma_a \sigma_c) = 2g_{nc} . \]

Further, with the use of Ricci coefficient definition, spinors \( \gamma(x) \) and \( \tilde{\gamma}(x) \) will read

\[ \gamma(x) = -\sigma_{\alpha;\beta}(x) \sigma^\alpha(x) \otimes \sigma^\beta(x) , \quad \tilde{\gamma}(x) = -\sigma_{\alpha;\beta}(x) \tilde{\sigma}^\alpha(x) \otimes \tilde{\sigma}^\beta(x) . \]

(43)

For the following it will be more convenient instead of \( \gamma(x) \), \( \tilde{\gamma}(x) \) to use slightly modified spinors \( \Gamma(x) \) and \( \bar{\Gamma}(x) \):

\[ \Gamma(x) = (\epsilon \otimes I \otimes I \otimes I) \gamma(x) , \quad \bar{\Gamma}(x) = (I \otimes \epsilon \otimes \epsilon \otimes \epsilon) \tilde{\gamma}(x) , \]

where \( \epsilon = -i \sigma^2 \). These spinors are symmetrical over two first indices:

\[ \Gamma(x) = [ \gamma(\alpha;\beta)\rho\sigma(x) ] , \quad \bar{\Gamma}(x) = [ \tilde{\gamma}(\alpha;\beta)\rho\sigma(x) ] . \]
Explicit spinor-indices structure of all other quantities involved looks as follows

$$\sigma^a = \sigma^a_{\alpha\beta}, \quad \bar{\sigma}^a = \bar{\sigma}^{\alpha\beta},$$

$$\gamma(x) = (\gamma^a_{\beta\mu}(x)) = [ (\sigma^a_{\alpha\nu} \sigma^b_{\nu\beta} - \sigma^c_{\nu\beta} \gamma_{abc}(x) )],$$

$$\bar{\gamma}(x) = (\bar{\gamma}^a_{\beta\mu}(x)) = [ (\sigma^b_{\alpha\nu} \sigma^c_{\nu\beta} + \sigma^c_{\nu\beta} \gamma_{abc}(x) )],$$

$$\sigma^\nu(x) = (\epsilon^{\nu\alpha\beta}_a(x)) = [ \sigma^a_{\alpha\beta} \bar{e}^{\nu\alpha\beta}_a(x) ],$$

$$\bar{\sigma}^\nu(x) = (\bar{\epsilon}^{\nu\alpha\beta}_a(x)) = [ \bar{\sigma}^{\alpha\beta} e^{\nu\alpha\beta}_a(x) ].$$

Now, with the use of special letter-notation (31) for $\Gamma(x)$ and $\bar{\Gamma}(x)$ one gets

$$\Gamma = - \left( \begin{array}{cc} 2l_{\alpha\beta} \bar{m}^\alpha & l_{\alpha\beta} n^\alpha + m_{\alpha\beta} \bar{m}^\alpha \\ l_{\alpha\beta} n^\alpha + m_{\alpha\beta} \bar{m}^\alpha & -2n_{\alpha\beta} \bar{m}^\alpha \end{array} \right) \otimes \left( \begin{array}{cc} l^\beta & m^\beta \\ \bar{m}^\beta & n^\beta \end{array} \right),$$

$$\bar{\Gamma} = - \left( \begin{array}{cc} 2l_{\alpha\beta} m^\alpha & l_{\alpha\beta} n^\alpha + \bar{m}_{\alpha\beta} m^\alpha \\ l_{\alpha\beta} n^\alpha + \bar{m}_{\alpha\beta} m^\alpha & -2n_{\alpha\beta} m^\alpha \end{array} \right) \otimes \left( \begin{array}{cc} l^\beta & \bar{m}^\beta \\ m^\beta & n^\beta \end{array} \right).$$

It should be noted $(\Gamma(x))^* = \bar{\Gamma}(x)$, where the symbol $*$ stands for complex conjugation. Therefore one may consider transformation law for the $\Gamma(x)$ only. Besides, you may omit indices $(\alpha, \beta)$ for short – then you arrive at

$$\Gamma(x) = -2 \left( \begin{array}{cc} \bar{m} & \frac{1}{2}(l n + m \bar{m}) \\ \frac{1}{2}(l n + m \bar{m}) & -n \bar{m} \end{array} \right) \otimes \left( \begin{array}{cc} l \\ \bar{m} \\ m \\ n \end{array} \right). \quad (44)$$

All 12 independent components of the $\Gamma(x)$ (the first matrix in (44) is symmetrical one) may be listed in the manner we like: with the help of 12 different letter symbols – see (36). The quantities $(L_i, N_i, M_i, \bar{M}_i)$ may be straightforwardly connected with the so-called Newman-Penrose spin coefficients $(k, \pi, \epsilon; \rho, \lambda, \alpha; \mu, \beta; \tau, \nu, \gamma)$:

$$L_1 = c \ k^* \quad L_2 = c \ \pi^* \quad L_3 = c \ \epsilon^*$$

$$N_1 = c \ \tau^* \quad N_2 = c \ \nu^* \quad N_3 = c \ \gamma^*$$

$$M_1 = c \ \rho^* \quad M_2 = c \ \lambda^* \quad M_3 = c \ \alpha^*$$

$$\bar{M}_1 = c \ \sigma^* \quad \bar{M}_2 = c \ \mu^* \quad \bar{M}_3 = c \ \beta^*$$

where $c = 2^{3/2}$.

5 Gauge transformation

Now the task is to consider the problem of general 6-parametric gauge transformations for spin coefficients $(L_i, N_i, M_i, \bar{M}_i)$ under local Lorentz group. Let us start with the known gauge
law for the ordinary Ricci object $\gamma_{abc}(x)$:

$$\gamma'_{abc}(x) = L_a^k(x) L_b^l(x) L_c^n(x) \gamma_{klm}(x)$$

$$+ L_a^k(x) g_{kl} \left[ \frac{\partial}{\partial x^m} L_b^l(x) \right] L_c^n(x) \epsilon_{m}(x).$$

As the $4 \times 4$ Lorentz matrix $L_a^b(x)$ depends upon both $k(x)$ and conjugate $k^*(x)$, the second term in the formula contains both $(\partial/\partial x^\mu) k_a$ and $(\partial/\partial x^\mu) k^*_a$.

It will be seen, simplification we arrive at in spinor basis $\gamma(x)$ and $\gamma(x)$ is that each of spinors $\gamma(x)$ and $\gamma(x)$ transforms independently within itself, besides terms $(\partial/\partial x^\mu) k_a$ and $(\partial/\partial x^\mu) k^*_a$ enter its own transformation law. The task is to find these two formulas.

Let us proceed from spinors $\gamma(x)$ and $\gamma(x)$ in a primed tetrad $e^\alpha_{(a)}(x) = L_a^b(x) e^\alpha_{(b)}(x)$:

$$\gamma'(x) = -\sigma^\alpha_{\alpha;\beta} \sigma^\alpha \otimes \sigma^\beta, \quad \gamma'(x) = -\sigma^\alpha_{\alpha;\beta} \sigma^\alpha \otimes \sigma^\beta. \quad (45)$$

Taking the the known relations [5]

$$\bar{\sigma}^\alpha(x) = B(k(x)) \bar{\sigma}(x) B(k^*(x)), \quad \sigma^\alpha(x) = B(k^*(x)) \sigma(x) B(k(x)),$$

we get to

$$\gamma'(x) = - \left[ \frac{\partial B(k)}{\partial x^\mu} (\bar{\sigma}^\nu \sigma^\nu) B(k) + B(k) \bar{\sigma}_{\nu \mu} \sigma^\nu B(k) \right]$$

$$+ B(k) \bar{\sigma}_\nu \frac{\partial B(k^*)}{\partial x^\mu} B(k^*) \sigma^\nu B(k^*) \] \otimes B(k) \sigma^\mu B(k^*),$$

$$\gamma'(x) = - \left[ \frac{\partial B(k^*)}{\partial x^\mu} (\sigma^\nu \bar{\sigma}_\nu) B(k^*) + B(k^*) \sigma_{\nu \mu} \bar{\sigma}^\nu B(k^*) \right]$$

$$+ B(k^*) \sigma_\nu \frac{\partial B(k)}{\partial x^\mu} B(k) \bar{\sigma}^\nu B(k^*) \] \otimes B(k) \bar{\sigma}^\mu B(k^*).$$

In both formulas third terms vanish because of two identities

$$\bar{\sigma}_\nu \left[ \frac{\partial B(k^*)}{\partial x^\mu} B(k^*) \right] \sigma^\nu \equiv 0, \quad \sigma_\nu \left[ \frac{\partial B(k)}{\partial x^\mu} B(k) \right] \bar{\sigma}^\nu \equiv 0$$

hold. So that we arrive at

$$\gamma'(x) = (B(k) \otimes \tilde{B}(k) \otimes B(k^*) \otimes \tilde{B}(k^*)) \gamma(x) + 4 \left( B(k) \frac{\partial B(k)}{\partial x^\mu} \right) \otimes B(k^*) \sigma^\mu B(k),$$

$$\gamma'(x) = [(B(k^*) \otimes \tilde{B}(k^*) \otimes B(k) \otimes \tilde{B}(k^*)) \gamma(x) + 4 \left( B(k^*) \frac{\partial B(k^*)}{\partial x^\mu} \right) \otimes B(k) \bar{\sigma}^\mu B(k)].$$

From this, with the use of

$$\epsilon \ B(k) = \tilde{B}(k) \epsilon, \quad (\epsilon_{\mu \nu} = -i \sigma^2; \epsilon_{\mu \nu} = -i \sigma^2),$$

13
gauge formulas for $\Gamma(x)$ and $\bar{\Gamma}(x)$ follow

$$\Gamma'(x) = (\tilde{B}(k) \otimes \tilde{B}(k) \otimes B(k^*) \otimes \tilde{B}(k)) \Gamma(x) + 4\epsilon(B(k)\frac{\partial B(k)}{\partial x^\mu}) \otimes (B(k^*)\sigma^\mu B(k)),$$

$$\bar{\Gamma}'(x) = (B(k^*) \otimes B(k^*) \otimes \tilde{B}(k) \otimes B(k^*)) \bar{\Gamma}(x) + 4(B(k^*)\frac{\partial B(k^*)}{\partial x^\mu}) \epsilon \otimes (\epsilon B(k)\sigma^\mu B(k^*)) \epsilon. \quad (46)$$

In more detailed form the law (46) means

$$L'_i = bb^*F_i - dd^*G_i - db^*H_i - d^*b\Delta_i \quad (48)$$

$$M'_i = -cb^*F_i - ad^*G_i + ab^*H_i + cd^*\Delta_i$$

$$N'_i = cc^*F_i + aa^*G_i - ad^*H_i - a^*c\Delta_i \quad (49)$$

where

$$F_1 = [(b^2L_1 + d^2L_2 - 2bdL_3) - 2l^\mu(b\partial_\mu d - d\partial_\mu b)],$$

$$G_1 = [(b^2N_1 + d^2N_2 - 2bdN_3) - 2n^\mu(b\partial_\mu d - d\partial_\mu b)],$$

$$H_1 = [(b^2M_1 + d^2M_2 - 2bdM_3) - 2m^\mu(b\partial_\mu d - d\partial_\mu b)],$$

$$\Delta_1 = [(b^2M_1 + d^2M_2 - 2bdM_3) - 2m^\mu(b\partial_\mu d - d\partial_\mu b)];$$

$$F_2 = [(c^2L_1 + a^2L_2 - 2acL_3) - 2l^\mu(c\partial_\mu a - a\partial_\mu c)],$$

$$G_2 = [(c^2N_1 + a^2N_2 - 2acN_3) - 2n^\mu(c\partial_\mu a - a\partial_\mu c)],$$

$$H_2 = [(c^2M_1 + a^2M_2 - 2acM_3) - 2m^\mu(c\partial_\mu a - a\partial_\mu c)],$$

$$\Delta_2 = [(c^2M_1 + a^2M_2 - 2acM_3) - 2m^\mu(c\partial_\mu a - a\partial_\mu c)];$$

$$F_3 = [-bcL_1 - adL_2 + (ab + cd)L_3 - 2l^\mu(a\partial_\mu b - c\partial_\mu c)],$$

$$G_3 = [-bcN_1 - adN_2 + (ab + cd)N_3 - 2n^\mu(a\partial_\mu b - c\partial_\mu c)],$$

$$H_3 = [-bcM_1 - adM_2 + (ab + cd)M_3 - 2m^\mu(a\partial_\mu b - c\partial_\mu c)],$$

$$\Delta_3 = [-bcM_1 - adM_2 + (ab + cd)M_3 - 2m^\mu(a\partial_\mu b - c\partial_\mu c)];$$

The quantities $(a, b, c, d)$ are elements of spinor $(2 \times 2)$ – transformation $B(k(x)) \in SL(2,C)$ corresponding to the local Lorentz matrix:

$$e^\mu_{(b)}(x) = L_b^a(k(x), k^*(x)), \quad B(k(x)) = \begin{pmatrix} a(x) & c(x) \\ d(x) & b(x) \end{pmatrix}. \quad (50)$$
Example, spin coefficients of spherical tetrad

Here one illustrative example will be considered: let us find explicit spin coefficients in spherical tetrad of Minkowski space by means of a direct gauge transformation of zero spin coefficients in Cartesian tetrad. When starting spin coefficient vanish identically, in the above formulas (48) and (49) are present only second non-uniform terms.

For elements \((a, b, c, d)\) of spinor matrix corresponding to transition from Cartesian tetrad to spherical one, we have

\[
a = \cos \frac{\theta}{2} e^{i\phi/2}, \quad c = \sin \frac{\theta}{2} e^{-i\phi/2}, \quad d = -\sin \frac{\theta}{2} e^{i\phi/2}, \quad b = \cos \frac{\theta}{2} e^{-i\phi/2}. \tag{51}
\]

These parameters are functions of variables \((\theta, \phi)\), therefore in (48) and (49) it is convenient index \(\mu\) to refer to \(x^\mu = (t, r, \theta, \phi)\). We have

\[
\sigma^\mu(x) = (\partial x^\mu/\partial x^i)\sigma^i(x), \quad \sigma^j(x) = \sigma^a e^j_a \tag{52}
\]

where \(x^\mu\) and \(x^i\) stand for spherical and Cartesian coordinates respectively, \(e^j_a(\theta, \phi) = \delta^j_a\) is a Cartesian tetrad defined in Cartesian coordinates. Correspondingly, \(\sigma^\mu(x)\) is

\[
\sigma^\mu(x) = \begin{pmatrix}
l^\mu(x) & m^\mu(x) \\
m^\mu(x) & \bar{m}^\mu(x)
\end{pmatrix} = \frac{\partial x^\mu}{\partial x^a} \sigma^a. \tag{53}
\]

From (53), with the relations

\[
\frac{\partial r}{\partial x} = \sin \theta \cos \phi, \quad \frac{\partial r}{\partial y} = \sin \theta \sin \phi, \quad \frac{\partial r}{\partial z} = \cos \theta,
\]

\[
\frac{\partial \theta}{\partial x} = \frac{\cos \theta \cos \phi}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos \theta \sin \phi}{r}, \quad \frac{\partial \theta}{\partial z} = -\frac{\sin \theta}{r},
\]

\[
\frac{\partial \phi}{\partial x} = -\frac{\sin \phi \sin \theta}{r}, \quad \frac{\partial \phi}{\partial y} = \frac{\cos \phi \sin \theta}{r}, \quad \frac{\partial \phi}{\partial z} = 0
\]

it follows \((l^\mu, n^\mu, m^\mu, \bar{m}^\mu)\):

\[
l^\mu = \begin{pmatrix}
1 \\
\cos \theta \\
-\frac{1}{r} \sin \theta \\
0
\end{pmatrix}, \quad m^\mu = \begin{pmatrix}
0 \\
\sin \theta e^{-i\phi} \\
\frac{1}{r} \cos \theta e^{-i\phi} \\
-\frac{i}{r} e^{-i\phi} \sin \theta
\end{pmatrix}, \quad
\]

\[
n^\mu = \begin{pmatrix}
1 \\
-\cos \theta \\
\frac{1}{r} \sin \theta \\
0
\end{pmatrix}, \quad \bar{m}^\mu = \begin{pmatrix}
0 \\
\sin \theta e^{+i\phi} \\
\frac{1}{r} \cos \theta e^{+i\phi} \\
+\frac{i}{r} e^{+i\phi} \sin \theta
\end{pmatrix}.
\]
In addition

\[
\frac{\partial b}{\partial \theta} - \frac{\partial d}{\partial \theta} = -\frac{1}{2}, \quad \frac{\partial a}{\partial \theta} - \frac{\partial c}{\partial \theta} = -\frac{1}{2},
\]
\[
\frac{\partial d}{\partial \phi} - \frac{\partial b}{\partial \phi} = -\frac{i}{2} \sin \theta, \quad \frac{\partial a}{\partial \phi} - \frac{\partial c}{\partial \phi} = +\frac{i}{2} \sin \theta,
\]
\[
\frac{\partial b}{\partial \theta} - \frac{\partial a}{\partial \theta} = 0, \quad \frac{\partial d}{\partial \phi} - \frac{\partial c}{\partial \phi} = -\frac{i}{2} \sin \theta.
\]

With the use of which we get to the \( F_i, G_i, H_i, \Delta_i \):

\[
F_1 = -\frac{1}{r} \sin \theta, \quad F_2 = -\frac{1}{r} \sin \theta, \quad G_1 = +\frac{1}{r} \sin \theta, \quad G_2 = +\frac{1}{r} \sin \theta,
\]
\[
H_1 = \frac{1}{r} (+1 + \cos \theta) e^{-i\phi}, \quad H_2 = \frac{1}{r} (-1 + \cos \theta) e^{-i\phi},
\]
\[
\Delta_1 = \frac{1}{r} (-1 + \cos \theta) e^{+i\phi}, \quad \Delta_2 = \frac{1}{r} (+1 + \cos \theta) e^{+i\phi},
\]
\[
F_3 = 0, \quad G_3 = 0, \quad H_3 = +\frac{1}{r} \cot \theta e^{-i\phi}, \quad \Delta_3 = -\frac{1}{r} \cot \theta e^{+i\phi}.
\]

Now, from (48) and (49) we can readily calculate spherical spin coefficients:

\[
L_i = 0, \quad N_i = 0,
\]
\[
M_1 = \frac{2}{r}, \quad M_2 = 0, \quad M_3 = \frac{1}{r} \cot \theta,
\]
\[
\bar{M}_1 = 0, \quad \bar{M}_2 = \frac{2}{r}, \quad \bar{M}_3 = \frac{1}{r} \cot \theta.
\]

These quantities are needed in working with spherical coordinates and tetrad. It is seen that a special gauge transformation is responsible for their explicit form.

7 Conclusions

It is shown that only 8 different combinations of the Ricci coefficients are involved in the Dirac equation on a curved space-time background. In other words, \( S = 1/2 \) particle effectively observes only ‘a third’ of geometric characteristics of any space-time model, and by no means responds to remaining 16 ones. These eight ones may be collected in two 4-vectors \( B_a(x) \) and \( C_a(x) \) under local Lorentz group which has status of the gauge symmetry group. In all orthogonal coordinates one of these vectors, ”pseudovector” \( C_a(x) \), vanishes identically. The gauge transformation laws of vectors \( B_a(x) \) and \( C_a(x) \) are found explicitly. The Ricci rotation coefficients, being exploited in the generally covariant linear Dirac equation, assume their very strict and subtle gauge symmetry properties. Connection of these \( B_a(x) \) and \( C_a(x) \) with the known Newman-Penrose coefficients is established. Insight into the Ricci object in terms of the two vector fields \( B_\alpha, C_\alpha \) seems deeper and simpler than the spin coefficients method.
General study of gauge symmetry in Newman-Penrose formalism is done. To this end, decomposition of the tensor $\gamma_{abc}(x)$ into two spinors $\gamma(x)$ and $\bar{\gamma}(x)$ is performed. At this Ricci rotation coefficients are divided into two groups: 12 complex functions $\gamma(x) = (\gamma^\alpha_{\beta\rho\sigma})$ and 12 conjugated to them $\bar{\gamma}(x) = (\gamma_{\alpha}^{\beta\rho\sigma})$. Components of spinor $\gamma(x)$ coincide with 12 spin coefficients by Newman-Penrose $\kappa, \pi, \epsilon, \rho, \lambda, \alpha, \sigma, \beta, \tau, \nu, \gamma$. For listing these it is used a special letter-notation $L_i, N_i, M_i, \bar{M}_i (i = 1, 2, 3)$.

The formulas for gauge transformations of spin coefficients under local Lorentz group are derived on the base of spinor approach. In contrast to a generally accepted treatment of gauge symmetry in Newman-Penrose formalism with only 2-parametric Lorentz matrices the formulas obtained are applicable for any general 6-parametric transformation. There are given two solutions to the gauge problem: one in the compact form of transformation laws for spinors $\gamma(x)$ and $\bar{\gamma}(x)$, and another as detailed description of the latter in terms of spin coefficients.

References

[1] Ivanitskaja O.S. *Extended Lorentz transformations and their applications*. (Nauka i technika, Minsk, 1969) (in russian); *Lorentzian basis and gravitational effects in Einstein's theory of gravity*. (Nauka i technika, Minsk, 1976) (in russian).

[2] Mitskevich N.V. *Physical fields in general relativity*. (Nauka, Moskow, 1969) (in russian).

[3] Gorbatsievich A.K. *Quantum mechanics in general relativity. Basic principles and elementary applications*. (Nauka i technika, Minsk, 1985) (in russian).

[4] Penrose R., Rindler W. *Spinors and space-time. Vol. 1 and Vol. 2*. (Nauka, Moskow, 1987) (in russian).

[5] Fedorov F.I. *The Lorentz group*. (Nauka, Moskow, 1979) (in russian).

[6] Frolov V.P. *Newman-Penrose method in general relativity*. Trudy FIAN. 96, 72-180 (1977) (in russian).

[7] Alekseev G.A., Khlebnikov V.I. *Newman-Penrose formalism and its application in general relativity*. Fiz. Elem. Chast. Atom. Yadra. 9, 790-870 (1978) (in russian).