Rankin-Selberg methods for closed string amplitudes

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Abstract. After integrating over supermoduli and vertex operator positions, scattering amplitudes in superstring theory at genus $h \leq 3$ are reduced to an integral of a Siegel modular function of degree $h$ on a fundamental domain of the Siegel upper half plane. A direct computation is in general unwieldy, but becomes feasible if the integrand can be expressed as a sum over images under a suitable subgroup of the Siegel modular group; if so, the integration domain can be extended to a simpler domain at the expense of keeping a single term in each orbit – a technique known as the Rankin-Selberg method. Motivated by applications to BPS-saturated amplitudes, Angelantonj, Florakis and I have applied this technique to one-loop modular integrals where the integrand is the product of a Siegel-Narain theta function times a weakly, almost holomorphic modular form. I survey our main results, and take some steps in extending this method to genus greater than one.

1. Introduction

According to the basic postulate of superstring theory, scattering amplitudes of $n$ external states at $h$-th order in perturbation theory are given by correlation functions of $n$ vertex operators, integrated over the moduli space $\mathcal{M}_{h,n}$ of $n$-punctured super-Riemann surfaces $\Sigma$ of genus $h$ [1] [2]. After integrating over the fermionic moduli and the locations of the punctures (with due care to the subtle topology of supermoduli space [3] [4]), such amplitudes reduce to an integral over the moduli space $\mathcal{M}_{h,0}$ of ordinary Riemann surfaces without marked point. The latter is a quotient of the Teichmüller space $\mathcal{T}_h$ by the mapping class group $\Gamma_h$. For $1 \leq h \leq 3$, $\mathcal{T}_h$ is isomorphic (via the period map, $\Sigma \mapsto \Omega \equiv \Omega_1 + i\Omega_2$, away from suitable divisors) to the degree $h$ Siegel upper-half plane $\mathcal{H}_h$, while $\Gamma_h$ is identified with the Siegel modular group $Sp(h,\mathbb{Z})$, acting on $\mathcal{H}_h$ by fractional linear transformations. Thus, scattering amplitudes at genus $h$ are ultimately written as modular integrals

\begin{equation}
A_h = \text{R.N.} \int_{\mathcal{F}_h} d\mu_h F_h(\Omega), \quad d\mu_h = |\Omega_2|^{-\frac{h+1}{2}} d\Omega_1 d\Omega_2
\end{equation}

where $\mathcal{F}_h = \Gamma_h \backslash \mathcal{H}_h$ is a fundamental domain of the action of $\Gamma_h$ on $\mathcal{H}_h$, $F(\Omega)$ is a function on $\mathcal{H}_h$ invariant under $\Gamma_h$, $|\Omega_2| = \det|\Im(\Omega)| > 0$, $d\mu_h$ is the standard invariant measure on $\mathcal{H}_h$, and R.N. denotes a suitable infrared renormalization.
prescription. For \( h > 3 \), the Teichmüller space \( T_h \) embeds as a subvariety of codimension one or greater in the Siegel upper-half plane \( H_h \), and \( d\mu_h \) is replaced by a suitable measure with support on \( F_h \) — a complication which we shall not confront.

In general, integrals of the type (1.1) are untractable (except, perhaps, numerically), due to the complicated nature of the modular function \( F_h \), but also to the unwieldy shape of any fundamental domain: for \( h = 2 \), it takes no less than 25 inequalities to define \( F_2 \) [5]. Yet, the computation of such integrals is an unavoidable step in extracting any phenomenological prediction of superstring theory, and in investigating some of its structural properties such as invariance under dualities. The daunting task of integrating (1.1), however, can be considerably simplified if the integrand can be written as a sum over images (or Poincaré series) of a given function \( f_h(\Omega) \),

\[
F_h(\Omega) = \sum_{\gamma \in \Gamma_h, \infty \setminus \Gamma_h} f_h(\gamma \cdot \Omega)
\]

where \( f_h(\gamma \cdot \Omega) = f_h(\gamma \cdot \Omega) \) and \( f_h(\Omega) \) is invariant under a subgroup \( \Gamma_{h, \infty} \subset \Gamma_h \). If the sum is absolutely convergent, exchanging it with the integral extends the integration domain to a larger fundamental domain \( F_{h, \infty} \), while restricting the sum to a single coset,

\[
\mathcal{A}_h = R.N. \int_{F_{h, \infty}} d\mu_h f_h(\Omega).
\]

This ‘unfolding trick’, at the heart of the Rankin-Selberg method in number theory, is expedient if both \( F_{h, \infty} \) and \( f_h \) are simpler than \( F_h \) and \( F_h \).

A special class of amplitudes where this method is advantageous arises when the integrand \( F_h(\Omega) \) factorizes as \( \Phi(\Omega) \times I_{d+k,d,h} \), where \( \Phi(\Omega) \) is a (non-holomorphic, in general) Siegel modular form of weight \(-k/2\) and \( I_{d+k,d,h} \) is the Siegel-Narain theta series

\[
I_{d+k,d,h}(G, B, Y; \Omega) = |\Omega_2|^{d/2} \sum_{p^\alpha \in \Lambda} e^{-\pi \Omega_{2,\alpha,\beta} M_2(p^\alpha, p^\beta) + \pi i \Omega_{1,\alpha,\beta} (p^\alpha, p^\beta)}
\]

where the sum runs over \( h \)-tuples of vectors \( p^\alpha, \alpha = 1 \ldots h \) in an even self-dual lattice of signature \( (d+k, d) \) (hence \( k \in 8\mathbb{Z} \)) with quadratic form \( (\cdot, \cdot) \in 2\mathbb{Z} \) equipped with positive definite quadratic form \( M_2(\cdot, \cdot) \). The latter is parametrized by the Grassmannian

\[
G_{d+k,d} = [SO(d+k) \times SO(d)] \setminus SO(d+k, d),
\]

which can be coordinatized by a real positive definite symmetric matrix \( G_{ij} \), a real antisymmetric matrix \( B_{ij} \) and a real rectangular matrix \( Y_{i,a} \) \( (i, j = 1 \ldots d, a = 1 \ldots k) \). This type of theta series arises in any string vacua which involve a \( d \)-dimensional torus with constant metric \( G_{ij} \) and Kalb-Ramond field \( B_{ij} \), equipped with a \( U(1)^k \) flat connection with holonomies \( Y_{i,a} \). Importantly, \( I_{d+k,d,h} \) is invariant under \( \Gamma_h \times O(d+k, d, \mathbb{Z}) \), where the last factor is the automorphism group of the lattice \( \Lambda \) (also known as T-duality group), acting by right-multiplication on the coset \( \{1\} \).

In such cases, the integral

\[
\mathcal{A}_h = R.N. \int_{F_h} d\mu_h I_{d+k,d,h}(G, B, Y; \Omega) \Phi(\Omega).
\]
can be computed by expressing $\Gamma_{d+k,d,h}$ as a sum of Poincaré series under $\Gamma_h$, and applying the unfolding trick to each one of them. This ‘lattice unfolding technique’ has been the method of choice for one-loop amplitudes in the physics literature [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], and has been very useful in extracting asymptotic expansions at particular boundary components of $G_{d+k,d}$. However, its main drawback is that the various terms in the Poincaré series decomposition are not invariant under $O(d + k, d, \mathbb{Z})$, even though the sum is. As a result, the result of the unfolding trick is not manifestly invariant under $O(d + k, d, \mathbb{Z})$.

Another option, advocated in [18] and further developed in [19, 20] (see [21] for a complementary survey), is to represent the other factor $\Phi(\Omega)$ in the integrand (or the full integrand, in the absence of any Siegel-Narain theta series) as a Poincaré series, and use it to unfold the integration domain. This technique is of course limited by our ability to find absolutely convergent Poincaré series representations for (in general, non-holomorphic) Siegel modular forms. For genus one, it turns out that any almost, weakly holomorphic modular form of negative weight under $\Gamma_1 = SL(2, \mathbb{Z})$ (or congruence subgroups thereof) can be represented as a linear combination of certain absolutely convergent Poincaré series, first introduced by Niebur [22] and Hejhal [23] and revived in recent mathematical work [24, 25, 26, 27]. Almost, weakly holomorphic integrands $\Phi(\Omega)$ are non-generic, but do occur for certain classes of ‘BPS-saturated’ amplitudes, which play a central role for determining threshold corrections to gauge couplings and for testing non-perturbative dualities (see e.g. [28] for a review). As we shall see, the unfolding trick produces a sum over lattice vectors with fixed integer norm, manifestly invariant under T-duality. Physically, it can be interpreted as a sum of field-theory type amplitudes, with BPS states running in the loop. In particular, it exposes the singularities of the amplitude, originating from BPS states becoming massless. The price to pay is that the behavior at the boundary components is obscured, although it can be recovered in some cases, showing agreement with – and uncovering hidden structure in – the result of the usual lattice unfolding technique.

A third, and most radical option, is to represent 1 as a Poincaré series, and use it to unfold the integral. Indeed, it is well-known that 1 is a residue of non-holomorphic Siegel-Eisenstein series, the simplest conceivable example of Poincaré series. This trick, which we refer to as the Rankin-Selberg-Zagier method, is very useful to evaluate integrals of the type (1.6) with $\Phi = 1$ (hence $k = 0$). It expresses the result, for any genus, as a Langlands-Eisenstein series of $SO(d, d, \mathbb{Z})$, verifying a conjecture put forward in [29].

We begin our survey of various applications of the Rankin-Selberg method (or unfolding trick) to closed string amplitudes in [2] by computing one-loop integrals of symmetric lattice partition functions by means of a non-holomorphic Eisenstein series insertion. In [23] we move on to more general modular integrals of non-symmetric lattice partition functions against harmonic elliptic genus, which we compute by representing the latter as linear combination of Niebur-Poincaré series. In the last section [3] we take steps towards extending the Rankin-Selberg-Zagier method to higher genus.

Acknowledgements: I wish to thank C. Angelantonj and I. Florakis for a very enjoyable collaboration on the results reported in §2-3, K. Bringmann and D. Zagier for valuable advice during the course of this project, and the organizers of the String Math 2013 conference for their kind invitation to speak.
2. One-loop modular integrals with trivial elliptic genus

We start with the simplest case of one-loop modular integrals of the form \( \langle 1,6 \rangle \) with \( \Phi = 1 \) (hence \( k = 0 \)). Such integrals were computed by the ‘lattice unfolding technique’ in \([6,7]\) for \( d = 1 \), \([8]\) for \( d = 2 \), \([16]\) for \( d \geq 3 \), and a conjectural relation to constrained Epstein series was put forward in \([29]\). In this section, we shall calculate them instead by inserting by hand a non-holomorphic Eisenstein series in the integrand, computing the integral by the unfolding trick and taking a suitable residue at the end. We use the standard notations \( \tau = \tau_1 + i\tau_2 \), \( q = e^{2\pi i\tau} \) for the period \( \Omega_{11} \) and modulus of an elliptic curve, and write \( \Gamma = \Gamma_1 = SL(2, \mathbb{Z}), \mathcal{H}_1 = \mathcal{H} \), etc.

2.1. Non-holomorphic Eisenstein series. The non-holomorphic Eisenstein series for \( \Gamma = SL(2, \mathbb{Z}) \) is defined by the sum over images

\[
E^*(s; \tau) = \zeta^*(2s) \sum_{\gamma \in \Gamma_1} \tau_2^s |\gamma| = \frac{1}{2} \zeta^*(2s) \sum_{(c,d)=1} \frac{\tau_2^s}{|c\tau + d|^{2s}},
\]

where \( \Gamma_1 \) is the subgroup of upper-triangular matrices in \( \Gamma \), and

\[
\zeta^*(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s) = \zeta(1 - s)
\]

is the completed Riemann zeta function. The sum converges absolutely for \( \Re(s) > 1 \), and has a meromorphic continuation to all \( s \). The normalization in \((2.1)\) ensures that \( E^*(s; \tau) \) invariant under \( s \mapsto 1 - s \), and has only simple poles at \( s = 0 \) and \( s = 1 \). The key point for our purposes is that the residue at \( s = 1 \) is constant – in agreement with the fact that \( E^*(s; \tau) \) is an eigenmode of the Laplacian \( \Delta_{\mathcal{H}} \) on \( \mathcal{H} \), with vanishing eigenvalue at \( s = 0 \) or \( 1 \).

\[
[\Delta_{\mathcal{H}} - \frac{1}{2} s(s - 1)] E^*(s; \tau) = 0,
\]

\( \Delta_{\mathcal{H}} = 2\tau_2^2 \partial_{\tau} \partial_{\bar{\tau}} \). More precisely, the Laurent expansion at \( s = 1 \) is given by the first Kronecker limit formula,

\[
E^*(s; \tau) = \frac{1}{2(s - 1)} + \frac{1}{2} \left( \gamma_E - \log(4\pi \tau_2 |\eta(\tau)|^4) \right) + O(s - 1),
\]

where \( \eta = \sqrt{1/24} \prod_{n \geq 1} (1 - q^n) \) is the Dedekind eta function and \( \gamma_E \) is Euler’s constant. All these statements are easy consequences of the Fourier series representation of \( E^*(s; \tau) \), or Chowla-Selberg formula,

\[
E^*(s; \tau) = \zeta^*(2s) \tau_2^s + \zeta^*(2s - 1) \tau_2^{1-s} + 2 \sum_{N \not\equiv 0} |N|^{s-\frac{1}{2}} \sigma_{2s}(N) \tau_2^{1/2} K_{s-\frac{1}{2}}(2\pi |N| \tau_2) e^{2\pi i N \tau_1},
\]

where \( \sigma_t(N) = \sum_{d|N} d^t \) is the divisor function and \( K_{t}(z) \) is the modified Bessel function of the second kind. Below we shall denote the first line of \((2.5)\), which dominates the behavior at \( \tau_2 \to \infty \), by \( E^*_0(s; \tau) \).

For any modular function \( F(\tau) \) of rapid decay (such as the modulus square \( |\psi|^2 \) of a holomorphic cusp form), we consider the modular integral (also known as the Rankin-Selberg transform)

\[
\mathcal{R}^*(F; s) = \int_\mathcal{F} d\mu E^*(s; \tau) F(\tau).
\]
For $\Re(s) > 1$, the sum over cosets in (2.1) can be exchanged with the integral, so that the integration domain is extended to the strip
\begin{equation}
S = \Gamma_{\infty} \setminus \Gamma = \{ \tau_2 > 0, -\frac{1}{2} < \tau_1 \leq \frac{1}{2} \},
\end{equation}
at the expense of retaining the contribution of the unit coset only,
\begin{equation}
\mathcal{R}^*(F; s) = \zeta^*(2s) \int_{\tilde{S}} d\mu \tau_2^s F(\tau) = \zeta^*(2s) \int_0^\infty d\tau_2 \tau_2^{s-n-2} F_0(\tau_2) .
\end{equation}
The last equality expresses $\mathcal{R}^*(F; s)$ as a Mellin transform of the zero-th Fourier coefficient $F_0(\tau_2) = \int_{-1/2}^{1/2} d\tau_1 F(\tau)$. At the same time, $\mathcal{R}^*(F; s)$ inherits the meromorphicity and invariance under $s \mapsto 1 - s$ satisfied by $E^*$. In the case where $F = |\psi|^2$, $\mathcal{R}^*(F; s)$ is proportional to the $L$-series $\sum |a_n|^2 n^{-s}$, whose analyticity and functional equation are thereby determined. This is one of the main uses of the Rankin-Selberg method in number theory [30].

More importantly for our purposes, the fact that the residue of $E^*(s; \tau)$ at $s = 1$ is constant implies that the residue of $\mathcal{R}^*(F; s)$ at $s = 1$ is proportional to the modular integral of $F$,
\begin{equation}
\text{Res}_{s=1} \mathcal{R}^*(F; s) = \frac{1}{\pi} \int_{\tilde{F}} d\mu F.
\end{equation}
Unfortunately, this statement only holds for functions $F$ of rapid decay, which rules out the interesting case $F = \Gamma_{d,d,1}$. In the next section, following [31] we discuss how the unfolding trick can nevertheless be used after proper regularization.

2.2. Rankin-Selberg-Zagier method. Let us now consider a modular function $F$ with polynomial growth at the cusp,
\begin{equation}
F(\tau) \sim \varphi(\tau_2), \quad \varphi(\tau_2) = \sum_\alpha c_\alpha \tau_2^\alpha .
\end{equation}
In order to regulate infrared divergences in the integral (2.6), we truncate the integration domain to $F_T = F \cap \{ \tau_2 > T \}$ and define the renormalized integral as
\begin{equation}
\text{R.N.} \int_{F_T} d\mu F(\tau) = \lim_{T \to \infty} \left[ \int_{F_T} d\mu F(\tau) - \hat{\varphi}(T) \right],
\end{equation}
where $\hat{\varphi}(T)$ is the anti-derivative of $\varphi(\tau_2)$,
\begin{equation}
\hat{\varphi}(T) = \sum_{\alpha \neq 1} c_\alpha \frac{T^{\alpha-1}}{\alpha-1} + c_1 \log T .
\end{equation}
On the other hand, we define the Rankin-Selberg transform of $F$ as the Mellin transform of the zero-th Fourier coefficient, minus its leading behavior,
\begin{equation}
\mathcal{R}^*(F; s) = \zeta^*(2s) \int_0^\infty d\tau_2 \tau_2^{s-2} (F_0 - \varphi) .
\end{equation}
The renormalized integral (2.11) is then related to the Rankin-Selberg transform (2.13) via a generalization of (2.9) [31]
\begin{equation}
\text{R.N.} \int_{F_T} d\mu F(\tau) = 2 \text{Res}_{s=1} \left[ \mathcal{R}^*(F; s) + \zeta^*(2s) h_T(s) + \zeta^*(2s-1) h_T(1-s) \right] - \hat{\varphi}(T) .
\end{equation}
where \( h_T(s) \) is the meromorphic function of \( s \) defined by

\[
(2.15) \quad h_T(s) = \int_0^T \frac{d\tau_2 \varphi(\tau_2)\tau_2^{s-2}}{\sum_{\alpha} c_\alpha \frac{T^{\alpha+s-1}}{\alpha+s-1}}.
\]

Note that the right-hand side of (2.14) is by construction independent of the infrared cut-off \( T \). The derivation of (2.14) is based on the generalized unfolding trick for modular integrals on the truncated fundamental domain,

\[
(2.16) \quad \int_{\mathcal{F}_T} d\mu F \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} f|_\gamma = \int_{\mathcal{S}_T} d\mu F f - \int_{\mathcal{F}_T} d\mu F \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} f|_\gamma,
\]

where \( \mathcal{S}_T \) is the truncated strip \( \{\tau_2 < T, -1/2 < \tau_1 < 1/2\} \). Applying this observation to the regulated integral \( R^*_T(F; s) \equiv \int_{\mathcal{F}_T} d\mu F E^*(s; \tau) \) and reorganizing terms gives \[ Eq. (27)\]

\[
(2.17) \quad R^*(F; s) = R^*_T(F; s) + \int_{\mathcal{F}_T} d\mu (F E^*(s; \tau) - \varphi E_0^*(s; \tau_2)) - \zeta^*(2s) h_T(s) - \zeta^*(2s - 1) h_T(1 - s),
\]

from which (2.14) follows. Another consequence of (2.17) is that the Rankin-Selberg transform \( R^*(F; s) \) has a meromorphic continuation in \( s \), invariant under \( s \mapsto 1 - s \), and analytic away from \( s = 0, 1, \alpha_i, 1 - \alpha_i \). A particularly pleasant feature of the renormalization prescription (2.11) is that the Rankin-Selberg transform \( R^*(F; s) \) coincides with the renormalized integral

\[
(2.18) \quad R^*(F; s) = R.N. \int_{\mathcal{F}} d\mu F(\tau) E^*(s; \tau).
\]

Moreover, if \( F \) is constant, \( R^*(F; s) \) vanishes and therefore \( R.N. \int_{\mathcal{F}} d\mu E^*(s; \tau) = 0 \).

### 2.3. Constrained Epstein series

The Rankin-Selberg-Zagier method discussed in the previous subsection applies immediately to modular integrals of Siegel-Narain theta series for even self-dual lattices of signature \((d, d)\),

\[
(2.19) \quad \Gamma_{d,d}(G, B; \tau) = \tau_2^{d/2} \sum_{(m_i, n^i) \in \mathbb{Z}^{2d}} e^{-\pi T_2 \mathcal{M}^2(m_i, n^i) + 2\pi i T_2 m_i n^i},
\]

where \( \mathcal{M}^2 \) is the positive definite quadratic form

\[
(2.20) \quad \mathcal{M}^2(m_i, n^i) = (m_i + B_{ik} n^k) G^{ij}(m_j + B_{jl} n^l) + n^i G_{ij} n^j
\]

and \( (G_{ij}, B_{ij}) = (G_{ji}, -B_{ji}) \) parametrize the Grassmannian \( G_{d,d} \). Eq. (2.19) defines a modular function \( F \) on \( \mathcal{H} \) of polynomial growth characterized by

\[
(2.21) \quad \varphi(\tau_2) = \tau_2^d, \quad h_T(s) = \frac{T^{s+d/2-1}}{s + d/2 - 1}, \quad \varphi(\tau_2) = \begin{cases} \tau_2^{d-1} / (\frac{d}{2} - 1) & \text{if } d \neq 2 \\ \log \tau_2 & \text{if } d = 2 \end{cases}.
\]
Its Rankin-Selberg transform is

\begin{equation}
\mathcal{R}^*(\Gamma_{d,d}; s) = \zeta^*(2s) \int_0^\infty \frac{d\tau_2}{\tau_2} \sum_{(m_i, n_i) \in \mathbb{Z}^{2d}\setminus(0,0)} e^{-\pi \tau_2 \mathcal{M}^2(m_i, n_i)}
\end{equation}

\begin{equation}
= \zeta^*(2s) \frac{\Gamma(s + \frac{d}{2} - 1)}{\pi^{\frac{d}{2}-1}} \mathcal{E}_V^d(s + \frac{d}{2} - 1; G, B) \equiv \mathcal{E}_V^d(s + \frac{d}{2} - 1; G, B)
\end{equation}

where \(\mathcal{E}_V^d\) is the constrained Epstein series, absolutely convergent for \(\Re(s) > d\) \[29\]

\begin{equation}
\mathcal{E}_V^d(s; G, B) = \sum_{(m_i, n_i) \in \mathbb{Z}^{2d}\setminus(0,0)} [\mathcal{M}^2(m_i, n_i)]^{-s},
\end{equation}

and \(\mathcal{E}_V^d(s; G, B)\) is its ‘completion’. The results of \[22\] show that \(\mathcal{E}_V^d(s)\) admits a meromorphic continuation in \(s\), invariant under \(s \mapsto d - 1 - s\), with simple poles at \(s = 0, \frac{d}{2} - 1, \frac{d}{2}, d - 1\) (or double poles at \(s = 0\) and \(s = 1\) if \(d = 2\)). For \(d \neq 2\), the residues at \(s = \frac{d}{2}\) or \(s = \frac{d}{2} - 1\) produce the modular integral of interest:

\begin{equation}
\text{R.N.} \int_{\mathcal{F}} d\mu \Gamma_{d,d} = \zeta^*(2) \frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}} \text{Res}_{s=\frac{d}{2}} \mathcal{E}_V^d(s; G, B) = \frac{\Gamma\left(\frac{d}{2} - 1\right)}{\pi^{\frac{d}{2}-1}} \mathcal{E}_V^d\left(\frac{d}{2} - 1; G, B\right),
\end{equation}

rigorously proving a conjecture in \[29\]. Physically, the integral \[22\] computes (among other things) the one-loop contribution to \(R^4\) couplings in type II strings compactified on a torus \(T^d\) \[32\] \[16\]. The right-hand side is interpreted as a sum of one-loop contributions from particles of momentum \(m_i\) and winding \(n_i\) along the torus, with mass \(\mathcal{M}^2(m_i, n_i)\), satisfying the BPS constraint \(m_i n_i = 0\). It is manifestly invariant under the T-duality group \(O(d, d, \mathbb{Z})\), under which \(m_i, n_i\) transform in the vector (defining) representation. For \(s \to 1\) the \(s\)-dependent generalization \[22\] can be thought of as the dimensionally regularized amplitude, i.e. the amplitude in \(D = 10 - (d + 2s - 2)\) non-compact dimensions. The case \(s = 2\) also computes \(D^4 R^4\) couplings at one-loop in type II string theory on \(T^d\) \[36\]. Mathematically, \[22\] is recognized as a Langlands-Eisenstein series of \(O(d, d)\) with infinitesimal character \(\rho - 2s\lambda_1\) (where \(\rho\) is the Weyl vector and \(\lambda_1\) the weight of the vector representation) \[37\] \[38\]. The residue of this Langlands-Eisenstein series at \(s = \frac{d}{2}\) yields the minimal theta series associated to the minimal representation of \(O(d, \hat{d})\) \[39\].

For \(d = 2\), the Grassmannian \(G_{2,2}\) reduces to a product of two upper half planes \(\mathcal{H}_T \times \mathcal{H}_U\), where \(T\) and \(U\) are the Kähler modulus and complex structure moduli, respectively, while the T-duality group decomposes into \(SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \cong \sigma_{T,U}\), where \(\sigma_{T,U}\) is an involution exchanging \(T\) and \(U\). The BPS constraint \(m_1 n_1 + m_2 n_2 = 0\) can be solved explicitly, allowing to rewrite the constrained Epstein series as a product of two non-holomorphic Eisenstein series \[18\],

\begin{equation}
\mathcal{E}_V^d(s; T, U) = 2 E^*(s; T) E^*(s; U).
\end{equation}
Extracting the residue at \( s = 0 \) or 1 by means of the Kronecker limit formula (2.4) leads to

\[
\int_F I_{2,2}(T, U; \tau) \, d\mu = -\log \left( T_2 U_2 |\eta(T) \eta(U)|^4 \right) + \text{cte},
\]

which agrees with [8], up to a renormalization scheme-dependent additive constant. The reader familiar with [8] may appreciate the elegance of the Rankin-Selberg-Zagier method compared with the lattice unfolding method.

3. One-loop modular integrals with harmonic elliptic genus

While the integrand \( \Phi \) in type II one-loop string amplitudes is of polynomial growth at the cusp \( \tau \to i \infty \), this is not the case for heterotic strings, due to the tachyon in the spectrum before imposing the GSO projection. Instead, \( \Phi \) is a modular form of negative modular weight \( w = -k/2 \) with a first order pole at the cusp. The Rankin-Selberg-Zagier method described in §2.2 is not directly applicable, however the unfolding trick could still be used provided \( \Phi \) had a uniformly convergent Poincaré series representation

\[
\Phi = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} f |_{w, \gamma}
\]

with suitable \( \Gamma_\infty \)-invariant seed \( f (\tau) \). \( f \) should grow as \( 1/\eta^{2w} \) as \( \tau \to 0 \) if (3.1) is to represent a modular form with a \( \kappa \)-th order pole at the cusp, but uniform convergence requires \( f (\tau) \ll \tau_2 \frac{1 - \frac{w}{2}}{2s - w} \) as \( \tau_2 \to \infty \). The naive choice \( f (\tau) = 1/\eta^{2w} \) is fine for weight \( w > 2 \) but fails for \( w \leq 2 \).

3.1. Selberg-Poincaré and Niebur-Poincaré series. A first, natural option for regulating the sum is to insert a non-holomorphic convergence factor à la Hecke-Kronecker, i.e. choose a seed \( f (\tau) = \frac{\tau - w}{2s - w} \). The resulting Selberg-Poincaré series

\[
E(s, \kappa, w; \tau) \equiv \frac{1}{2} \sum_{(c,d)=1} \frac{(ct + d)^{-w} \tau_2^{s - \frac{w}{2}}}{|ct + d|^{2s - w}} e^{-2\pi i k \frac{c + d}{cd}}
\]

converges absolutely for \( \Re(s) > 1 \), but analytic continuation to the desired value \( s = \frac{w}{2} \) is non-trivial, as it depends on deep properties of Kloosterman sums is tricky, and in general non-holomorphic [40, 41]. Another undesirable feature of the Selberg-Poincaré series (3.2) is that it is not an eigenmode of the weight \( w \) Laplacian on \( \mathcal{H} \), rather

\[
\Delta_{\mathcal{H}, w} + \frac{1}{2} (s - \frac{w}{2})(1 - \frac{w}{2} - s) \left[ E(s, \kappa, w) = 2\pi \kappa \left( s - \frac{w}{2} \right) E(s + 1, \kappa, w) \right],
\]

so that the analytic continuation to \( s = \frac{w}{2} \) is not guaranteed to yield a holomorphic result, nor even harmonic [42].

A much more convenient choice, which does not require analytic continuation, is the Niebur-Poincaré series

\[
\mathcal{F}(s, \kappa, w; \tau) = \frac{1}{2} \sum_{(c,d)=1} (ct + d)^{-w} \frac{\tau_2^{s - \frac{w}{2}}}{|ct + d|^{2s - w}} e^{-2\pi i k \Re(\frac{c + d}{cd})}
\]

Here \( \Delta_{\mathcal{H}, w} = 2D_{w-2}D_w \) where \( D_{w}, D_w \) are defined in (3.8). Notice the change of convention compared to [19].

The relation between the Poincaré series (3.2) and (3.4) can be found in [19 App. B].
first introduced by Niebur [22] (for weight zero) and Hejhal [23] and revived in recent mathematical work on Mock modular forms [24, 25, 26, 27]. The seed $f(\tau) = M_{s,w}(\kappa\tau_2) e^{-2\pi i \kappa \tau_2}$, where

$$M_{s,w}(y) = \frac{|4\pi y|^{-\frac{w}{2}}}{\tau} M_{\frac{w}{2} \Omega(s), s - \frac{1}{2}} (4\pi y)$$

is proportional to the Whittaker function $M_{\lambda,\mu}(z)$, is uniquely determined by the requirements that $F(s, \kappa, w; \tau)$ be an eigenmode of the Laplacian $H$

$$\left[\Delta_{H,w} + \frac{i}{2} (s - \frac{w}{2})(1 - \frac{w}{2} - s)\right] F(s, \kappa, w; \tau) = 0,$$

and that $f(\tau)$ has the desired growth at $\tau_2 \to \infty$ and $\tau_2 \to 0$,

$$f(\tau) \sim \tau_2^{-\infty} \frac{\Gamma(2s)}{\Gamma(s + \frac{w}{2})} q^{-\kappa}, \quad f(\tau) \sim \tau_2^{0} |4\pi \kappa \tau_2|^s \frac{\pi}{\tau} e^{-2\pi i \kappa \tau_2}.$$  

The last property ensures that $F(s, \kappa, w)$ converges absolutely for $\Re(s) > 1$, while a more detailed argument based on the Fourier expansion of $F(s, \kappa, w)$ (which can be found in [27, 19]) shows that $F(s, \kappa, w)$ is holomorphic for $\Re(s) > \frac{1}{2}$ [43, 44]. Besides being an eigenmode of $\Delta_{H,w}$, $F(s, \kappa, w)$ also transforms in a simple way under the lowering, raising and Hecke operators $D, \bar{D}, H_m$ defined by

$$D_w = \frac{i}{\pi} \left( \frac{\partial}{\partial \tau} - \frac{iw}{2\tau_2} \right), \quad \bar{D}_w = -i\pi \tau_2^{2} \partial_{\bar{\tau}},$$

$$\langle H_m \cdot \Phi \rangle(\tau) = \sum_{s,\kappa,\mu,\omega} \sum_{b \equiv s, \kappa, \omega \mod B} d^{-w} \Phi \left( \frac{\alpha \tau + b}{d} \right),$$

namely [19]

$$D_w \cdot F(s, \kappa, w; \tau) = 2\kappa (s + \frac{w}{2}) F(s, \kappa, w + 2; \tau),$$

$$\bar{D}_w \cdot F(s, \kappa, w; \tau) = \frac{1}{8\kappa} (s - \frac{w}{2}) F(s, \kappa, w - 2; \tau),$$

$$H_m \cdot F(s, \kappa, w; \tau) = \sum_{d \mid (\kappa, m)} d^{1-w} F(s, \kappa m/d^2, w; \tau).$$

The decisive advantage of Niebur’s Poincaré series over Selberg’s, however, is that the value $s = 1 - \frac{w}{2}$, degenerate with the value $s = \frac{w}{2}$ under the Laplacian (3.6), lies in the convergence domain $\Re(s) > 1$ (except for $w = 0$, which requires a more careful treatment). Eigenmodes of the Laplacian (3.6) with $s = \frac{w}{2}$ or equivalently $s = 1 - \frac{w}{2}$ are known as weak harmonic Maass forms (WHMS), and have a Fourier expansion near $\tau = \infty$ of the form

$$\Phi = \sum_{m=\kappa} \frac{(4\pi \tau_2)^{1-w}}{w-1} \tilde{b}_0 + \sum_{m=1} \frac{m^{w-1}}{\Gamma(1-w, 4\pi m \tau_2)} q^{-m}.$$  

Weak holomorphic modular forms are a special case of WHMS, where the negative frequency coefficients $\tilde{b}_m$ vanish. Mock modular forms are defined as the analytic part $\Phi^+ = \sum_{m=1}^{\infty} a_m q^m$ of a WHMS. Acting on any WHMS $\Phi$ of weight $w$ with the lowering operator $D$ produces the complex conjugate of a holomorphic modular form $\Psi = \sum_{m \geq 1} b_m q^m$ of weight $2 - w$ (the shadow) while the iterated raising operator $D^{1-w}$ produces a weakly holomorphic modular form

\[\text{Here we restrict to the case where the shadow is regular at } \tau = \infty, \text{ see } [44] \text{ for the general expansion.}\]
\[ \Xi = \sum_{m=-\infty}^{\infty} m^{1-w} a_m q^m \] of weight \( 2 - w \) (the "ghost") such that \( \Phi^+ \) is an Eichler integral of \( \Xi \). In the case of \( F(1 - \frac{w}{2}, \kappa, w) \), the shadow is the usual Poincaré series \[ \Psi \propto \mathcal{P}(-\kappa, 2 - w) = \sum q^{\ell_1 | \kappa, 2 - w} \], while the ghost is the Niebur-Poincaré series \[ \Xi \propto F(1 - \frac{w}{2}, \kappa, 2 - w) \]. In particular, \( \Psi \) is a cusp form of weight \( 2 - w \), so must vanish for \( w = 0, -2, -4, -6, -8, -12 \). Indeed, for these values, \( F(1 - \frac{w}{2}, \kappa, w) \) is an ordinary weak holomorphic modular form, e.g.

\[ (3.12) \quad F(1, 1, 0) = J + 24 \,, \quad F(2, 1, -2) = 3! \frac{E_4 E_6}{\Delta} \,, \quad F(7, 1, -12) = 13! / \Delta \, , \ldots \]

where \( E_4, E_6 \) are the usual Eisenstein series of weight 4, 6 under \( SL(2, \mathbb{Z}) \), \( \Delta = \eta^{24} \) is the modular discriminant and \( J = \frac{E_4^3}{E_6} - 744 = 1/q + \mathcal{O}(q) \) is the usual Hauptmodul. In contrast, for \( w = -10 \), \( F(1, 1, -10) \) is a genuine WHMS, with irrational positive frequency Fourier coefficients and non-trivial shadow, proportional to \( \Delta \) [45].

It is worth noting that for \( s = 1 - \frac{w}{2} \), the seed of the Niebur-Poincaré series simplifies to

\[ (3.13) \quad f(\tau) = \Gamma(2 - w) \left( q^{-\kappa} - q^{\kappa} \sum_{\ell=0}^{w} \frac{(4\pi \kappa \tau_2)^\ell}{\ell!} \right) , \]

which plainly shows the improved ultraviolet behavior compared to the naive choice \( f(\tau) \sim q^{-\kappa} \).

Using the Niebur-Poincaré series \( F(s, \kappa, w) \) with \( s = 1 - \frac{w}{2} \), we can now represent any weakly holomorphic modular form \( \Phi \) of weight \( w \leq 0 \) as a linear combination of Niebur-Poincaré series\(^5\)

\[ (3.14) \quad \Phi = \frac{1}{\Gamma(2 - w)} \sum_{-\kappa \leq m \leq 0} a_m F(1 - \frac{w}{2}, m, w; \tau) + a_0' \delta_{w, 0} \]

where the coefficients are read off from the polar part \( \Phi = \sum_{-\kappa \leq m \leq 0} a_m q^m + \mathcal{O}(1) \) at the cusp \( \tau \to \infty \). Indeed, the difference between the left and right-hand sides of (3.14) is a harmonic Maass form of negative weight which is exponentially suppressed at the cusp (for suitable choice of \( a_0' \) if \( w = 0 \)), hence vanishes [24]. In particular, while each term in (3.14) may be a WHMF with non-trivial shadow, the shadow cancels in the linear combination (3.14). More generally, using the fact that almost, weakly holomorphic modular forms of weight \( w < 0 \) are linear combinations of iterated derivatives \( D^n \Phi_{w-2n} \) of weakly holomorphic modular forms of weight \( w - 2n \), along with (3.10), we can similarly represent any almost, weakly holomorphic modular form of weight \( w < 0 \) as a linear combination

\[ (3.15) \quad \Phi = \sum_{p=0}^{n} \sum_{m=1}^{\kappa} a_p(m) F(1 - \frac{w}{2} + p, m, w; \tau) + a_0' \delta_{w, 0} \]

where \( n \) is the depth (i.e. the maximal power of \( \hat{E}_2 \)). As an example relevant for the computation of threshold corrections to gauge couplings in heterotic string theory

\[ ^5 \text{Similarly, modular forms under congruence subgroups of } SL(2, \mathbb{Z}) \text{ can be represented as linear combinations of Niebur-Poincaré series attached to all cusps, see [20] for the example of the Hecke congruence subgroup } \Gamma_0(N). \]

\[ ^6 \text{This is not the case for almost, weakly holomorphic modular forms of weight } w > 0, (\hat{E}_2)^n J \text{ being a counter-example. Such cases can be treated by considering } s \text{-derivatives of } F(s, \kappa, w) \text{ at } s = w/2 \text{ [46].} \]
compactified on $K_3 \times T^2$, we quote
\begin{equation}
\frac{\hat{E}_2 E_4 E_6 - E_3^2}{\Delta} = F(2, 1, 0) - 6 F(1, 1, 0) + 864 .
\end{equation}

3.2. One-loop BPS state sums. Using the representation \[3.15\], any one-loop modular integral of the form \[1.6\] is reduced to a linear combination of modular integrals of Niebur-Poincaré series against lattice-partition functions,
\begin{equation}
I_{d+k,d}(G, B, Y; s, \kappa) = \text{R.N.} \int_{\mathcal{F}} d\mu \Gamma_{d+k,d}(G, B, Y; \tau) F(s, \kappa, -\frac{k}{2}, \tau),
\end{equation}
where the Siegel-Narain theta series is given by \[47\]
\begin{equation}
\Gamma_{d+k,d}(G, B, Y; \tau) = \frac{4}{\tau_2} \sum_{(m_i, n_i, q_i) \in \Lambda} \frac{1}{q_1^{dL} q_2^{p_R}}.
\end{equation}

Here, $m_i, n_i$ run over integers while $q_i$ takes values in an even self-dual Euclidean lattice $\Lambda_E$ of dimension $k$ (hence $k$ must be a multiple of 8, and $\Lambda = \mathbb{Z}^{d,d} \oplus \Lambda_E$), $p_L^2 - p_R^2 = 4(m_i n_i^2 + \frac{1}{2}(q_i^a)^2)$ and $p_L^2 + p_R^2 = M^2(m_i n_i, q_i^a)$ is a positive definite quadratic form on $\Lambda$ parametrized by the Grassmanian $G_{d+k,d}$, coordinatized by $(G_{ij}, B_{ij}, Y^a_i)$. It is worth noting that the lattice partition function satisfies the differential equation \[29\]
\begin{equation}
[\Delta_{G_{d+k,d}} - 2 \Delta_{H_{-k/2}} + \frac{1}{2} d(d + k - 2) - \frac{k}{2}] \Gamma_{d+k,d} = 0.
\end{equation}
where $\Delta_{G_{d+k,d}}$ is the Laplace-Beltrami operator on $G_{d+k,d}$.

Due to the exponential growth near the cusp, the integral must be regulated by truncating the fundamental domain to $\mathcal{F}$ and taking the limit $\mathcal{F} \to \infty$, as in \[2.11\]. The integral over $\mathcal{F}$ can be computed using the generalized unfolding trick \[2.10\]. Carrying out these steps, one finds that the modular integral \[3.17\] away from the loci in $G_{d+k,d}$ where one of the lattice vectors becomes null ($p_R^2 = 0$) can be written as an infinite sum \[19, 24\]
\begin{equation}
I_{d+k,d}(s, \kappa) = \sum_{\text{BPS}} \int_0^\infty \frac{d\tau_2}{\tau_2} \tau_2^{d/2} \mathcal{M}_{s, -\frac{k}{2}}(-\kappa \tau_2) e^{-\pi \tau_2 (p_L^2 + p_R^2)/2}
\end{equation}
\begin{equation}
= (4\pi \kappa)^{\frac{1}{2}} \Gamma(s + 2d+k) - 1
\end{equation}
\begin{equation}
\times \sum_{\text{BPS}} 2F1 \left( s - \frac{k}{2}, s + \frac{2d+k}{4} - 1; 2s; \frac{4s}{p_L^2} \right) \left( \frac{p_L^2}{4\kappa} \right)^{1-s - \frac{2d+k}{4}},
\end{equation}
where the sum runs over $(m_i, n_i, q_i^a) \in \Lambda$ subject to the quadratic ‘BPS’ constraint
\begin{equation}
p_L^2 - p_R^2 = 4(m_i n_i^2 + \frac{1}{2}(q_i^a)^2) = 4\kappa.
\end{equation}
The unfolding method shows that the sum converges absolutely for $\Re(s) > \frac{2d+k}{4}$ (away from afore-mentioned loci) and has a meromorphic continuation to $\Re(s) > 1$, with a simple pole at $s = \frac{2d+k}{4}$ \[24\]. Thus $I_{d+k,d}(s, \kappa)$ defines a function on the Grassmanian $G_{d+k,d}$, manifestly invariant under the automorphism group $O(d+k, d, \mathbb{Z})$ of the lattice $\Lambda$, and eigenmode of $\Delta_{G_{d+k,d}}$, as a consequence of \[3.19\] and \[3.20\],
\begin{equation}
[\Delta_{G_{d+k,d}} + \frac{1}{16} (2d+k - 4s)(2d+k + 4s - 4)] \Gamma_{d+k,d}(s, \kappa) = 0.
\end{equation}
\footnote{For the values $s = 1 - \frac{2d+k}{4} + n$ relevant for the expansion \[3.14\], the summand in \[3.20\] can be rewritten in terms of elementary functions \[19\].}
At the value \( s = \frac{2d+k}{4} \), the eigenvalue vanishes but the renormalized integral (3.17) must be defined by subtracting the pole. The finite remainder \( \tilde{I}_{d+k,d}(\frac{2d+k}{4},\kappa) \) is then a quasi-harmonic function on \( G_{d+k,d} \), mapped to a constant function by the Laplacian \( \Delta_{G_{d+k,d}} \).

Physically, (3.20) is interpreted as a sum of field-theoretical one-loop amplitudes, with BPS particles of mass \( p_L^2 + p_R^2 - 4\kappa \) propagating in the loop. The singularities on the loci where a lattice vector becomes null, \( p_R^2 = 0 \), originate from one of these particles becomes massless. The manifestly T-duality invariant BPS sum (3.22) should be contrasted from the result obtained in [10, 11] by the lattice unfolding method. It is worth noting that the result (3.20) generalizes easily to modular integrals of Niebur-Poincaré series times lattice partition functions with momentum insertions [19, §3.3].

### 3.3. Fourier-Jacobi expansion

While the BPS state sum (3.20) is manifestly invariant under T-duality and exhibits singularities from massless states in a transparent fashion, it is in general non-trivial to extract the asymptotic expansion at a particular boundary component of the Narain moduli space \( O(d+k,d) \), i.e. at infinity in a particular Weyl chamber. Of course, this asymptotic expansion is precisely what is provided by the lattice unfolding method. In this section, we shall explain how to extract it from the BPS sum (3.20), in the special case \( d = 2, k = 0, \kappa = 1 \) [46]. The generalization to \( \kappa \neq 1 \) is straightforward, but the extension to arbitrary lattices is an open problem.

For two-dimensional lattices with \( \kappa = 1 \), the BPS constraint (3.21) implies that the integer matrix

\[
\gamma = \begin{pmatrix} m_1 & -m_2 \\ n_2 & n_1 \end{pmatrix} = \begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m'_1 & -m'_2 \\ n'_2 & n'_1 \end{pmatrix}
\]

is an element of \( SL(2,\mathbb{Z}) \). As written above, such matrices can be decomposed into products of an upper triangular matrix with \( M \in \mathbb{Z} \) and coset representatives of \( \Gamma_{\infty} \setminus SL(2,\mathbb{Z}) \). After Poisson resummation over the integer \( M \), the first line of (3.20) can be rewritten as

\[
I_{2,2}(T,U;s,1) = \sum_{M \in \mathbb{Z}} \sum_{\gamma \in \Gamma_{\infty}} \int_0^{\infty} \frac{d\tau_2}{\tau_2} M_{s,0}(-\kappa\tau_2) \sqrt{\frac{T_2U_2}{\tau_2}} \exp \left[ -\pi \tau_2 \left( \frac{T_2}{U_2} + \frac{U_2}{T_2} \right) - \frac{\pi M^2 T_2 U_2}{\tau_2} + 2\pi i M (T_1 - U_1) \right] |_\gamma
\]

where the slash operator \( |_\gamma \) now acts by replacing \( U \mapsto \frac{m'_1U - m'_2}{n'_1U + n'_2} \). Thus, the right-hand side is a sum of Poincaré series in \( U \), with \( T \)-dependent coefficients. Evaluating the integral over \( \tau_2 \) in the chamber where \( T_2 \) is larger than all \( U_2 |_{\gamma} \), we find

\[
I_{2,2}(T,U;s,1) = 2^{2s} \sqrt{4\pi \Gamma(s - \frac{1}{2})} T_2^{1-s} \frac{E^*(s;U)}{\pi^{-s} \Gamma(s)} + 4 \sum_{M > 0} \sqrt{\frac{T_2}{M}} K_{s-\frac{1}{2}}(2\pi MT_2) \left[ e^{2\pi i MT_1} F(s, M, 0; U) + \text{c.c} \right].
\]
This provides the asymptotic expansion of $I_{2,2}(T,U; s, 1)$ near the dimension-one boundary component $T \to \infty$ keeping $U$ fixed and arbitrary.

For $s = 1$, based on (3.22) we expect $\tilde{I}_{2,2}(T,U;1,1)$ to be a quasi-harmonic modular form in $(T,U)$. Indeed, one may use $K_{1/2}(x) = \sqrt{2\pi} e^{-x}$, (3.12), (3.10) and (2.4) to obtain

$$I_{2,2}(T,U;1,1) = -24 \log(4\pi T_2 U_2 |\eta(U)|^4 |\eta(T)|^4)$$

$$- 8\pi T_2 + 2 \sum_{N>0} \frac{1}{N} \left[ q_T^N H_N \cdot J(U) + \text{c.c.} \right]$$

The second line is recognized as the real part of the logarithm of Borcherds’ infinite product [48, Eq. 7.1]

$$\log \left[ q_T (J(T) - J(U)) \right] = - \sum_{N>0} \frac{1}{N} q_T^N H_N^{(U)} \cdot J(U).$$

Combining (3.26) and (3.27), we arrive at the well-known result [10] (up to an additive constant)

$$R.N. \int_{\mathcal{F}} d\mu \Gamma_{2,2}(T,U) (J(\tau)+24) = - \log |J(T) - J(U)|^4 - 24 \log \left[ T_2 U_2 |\eta(T) \eta(U)|^4 \right].$$

For $s = n + 1$ with $n$ integer, one can similarly use the properties

$$2 (-2N)^n \sqrt{NT_2} K_{n+\frac{1}{2}}(2\pi NT_2) e^{2\pi i NT_1} = D_T^n q_T^N$$

$$(2\kappa)^n n! F(n+1, \kappa, 0; U) = D_U^n F(n+1, \kappa, -2n; U)$$

$$\pi^{n+1} E^*(n+1; U) = (2\pi)^n D_U^n E(n+1, 0, -2n; U),$$

to express $I_{2,2}(n+1, 1)$ as

$$I_{2,2}(n+1, 1) = 4 \Re \left[ \frac{(-D_T D_U)^n}{n!} f_n(T,U) \right],$$

where $f_n(T,U)$ is a generalized prepotential, which is a linear combination harmonic Maass form of weight $-2n$ in $U$, with coefficients which are holomorphic in $T$.

$$f_n(T,U) = 2 (2\pi)^{2n+1} E(n+1, 0, -2n; U) + \sum_{M>0} \frac{2}{(2M)^{2n+1}} q_T^M H_M^{(U)} \cdot F(n+1, 1, -2n; U).$$

Holomorphicity in $U$ may be restored by replacing $E(n+1, 0, -2n; U)$ and $F(n+1, 1, -2n; U)$ by their analytic parts, proportional to the Eichler integrals of the usual holomorphic Eisenstein series $E_{2n+2}(U)$ and Poincaré series $F(n+1, 1, 2n+2; U)$. The resulting generalized non-holomorphic prepotential $\tilde{f}_n(T,U)$ will no longer be covariant under T-duality, but rather transform as an Eichler integral, picking up additional polynomials of degree $2n$ in $(T,U)$ under $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \ltimes \sigma_{T,U}$. For $n = 1$, $f_1(T,U)$ describes the one-loop correction to the prepotential in $\mathcal{N} = 2$ heterotic string vacua, and was indeed observed to transform by period integrals in the prescient paper [49]. Generalized prepotentials with $n = 2$ also arose in the study of $F^4$ corrections in $D = 8$ heterotic string vacua [11, 14, 50], and were introduced for general $n$ in [15, 51]. Our approach gives a straightforward derivation of their modular properties.
4. Higher-loop modular integrals

In this last section, we tackle the case of higher-loop modular integrals of the form \((1.4)\), which was one of our main motivations for developing the Rankin-Selberg technique. Unfortunately, Siegel-Poincaré series of degree \(h \geq 2\) are terra incognita in the mathematical literature, and we shall content ourselves with modular integrals of a symmetric lattice partition function and trivial elliptic genus,

\[
A_h(G, B) = \text{R.N.} \int_{\mathcal{F}_h} d\mu_h \Gamma_{d,d,h}(G, B; \Omega) .
\]

Our aim will to compute \((4.1)\) using the same strategy as in \([2]\) by inserting a non-holomorphic Eisenstein series in the integral, applying the unfolding trick and extracting a suitable residue.

4.1. Non-holomorphic Eisenstein series. Recall that the Siegel upper half plane of degree \(h\),

\[
\mathcal{H}_h = \{ \Omega = \Omega_1 + i\Omega_2 \in \mathbb{C}^{h \times h}, \Omega = \bar{\Omega}^t; \Omega_2 > 0 \}
\]

admits a transitive action \(\Omega \mapsto (A\Omega + B)(C\Omega + D)^{-1}\) of the Siegel modular group \(\Gamma_h\),

\[
\Gamma_h = Sp(h, \mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{Z}^{2h \times 2h}, \begin{array}{l} AB^t = BA^t, CD^t = DC^t \\ AD^t - BC^t = 1 \end{array} \right\} .
\]

The completed non-holomorphic Eisenstein series of weight 0 under \(\Gamma_h\) is defined by \([52, 53, 54]\),

\[
E_h^*(s; \Omega) = \mathcal{N}_h(h) \sum_{\gamma \in \Gamma_{\infty}} |\Omega_2|^s_0 \gamma, \quad \mathcal{N}_h(h) = \zeta^*(2s) \prod_{j=1}^{[h/2]} \zeta^*(4s - 2j)
\]

where \(|\Omega_2| = \det \Omega_2\) and \(|w\gamma|\) denotes the Petersson slash operator \(F[w\gamma(\Omega)] = [\det(C\Omega + D)]^{-w} F[(A\Omega + B)(C\Omega + D)^{-1}]\) and \(\Gamma_{\infty}\) is the subgroup of \(\Gamma_h\) of matrices with \(C = 0\). Equivalently,

\[
E_h^*(s; \Omega) = \mathcal{N}_h(h) \sum_{(C, D) \in SL(h, \mathbb{Z})/\mathbb{Z}^{h, 2h}} \left[ \frac{|\Omega_2|}{|C\Omega + D|^2} \right]^s
\]

where the sum runs over pairs of coprime symmetric integer matrices \((C, D)\), modulo a common left multiplication by \(GL(h, \mathbb{Z})\). The sum converges absolutely for \(\Re(s) > \frac{h+1}{2}\), has a meromorphic continuation to the \(s\)-plane, and is an eigenmode of the Laplace-Beltrami operator on \(\mathcal{H}_h\),

\[
\Delta_{\mathcal{H}_h} E_h^*(s; \Omega) = \frac{1}{2}h(s(2s - h - 1) E_h^*(s; \Omega) .
\]

With the choice of normalization in \((4.4)\), \(E_h^*(s; \Omega)\) is invariant under \(s \mapsto \frac{h+1}{2} - s\), with poles at most at \(s = j/4\) with \(0 \leq j \leq 2h + 2\) \([53]\).

The Fourier expansion with respect to \(\Gamma_{\infty}\) takes the form

\[
E_h^*(s; \Omega) = \sum_{\sum \not\in \mathbb{Z}^{h, h}} E_h^*(T; s; \Omega_2) e^{2\pi i T[\Omega_{\infty}]}\]

\[\text{for } h = 1, E_1^*(s; \Omega) \text{ reduces to the Eisenstein series } (2.1) \text{ for } SL(2, \mathbb{Z}) = Sp(1, \mathbb{Z}).\]
where the sum runs over half-integer symmetric $h \times h$ matrices $T$ (i.e. such that $2T$ is integer with even diagonal entries). The zero-th Fourier mode is given by [53]

$$E_h^*(T = 0; \Omega_2, s) = \sum_{r=0}^{h} \zeta^*(2s - r) \prod_{i=1}^{r} \zeta^*(4s - r - i) \prod_{i=r+1}^{[h/2]} \zeta^*(4s - 2i) \times |\Omega_2|^{s - \frac{r+1}{h}(2s - \frac{r+1}{2})} E^{r;SL(h,\mathbb{Z})}_{\Lambda^r,V} \left(2s - \frac{r+1}{2}; \hat{\Omega}_2\right)$$

where $\hat{\Omega}_2 = \Omega_2/|\Omega_2|^{1/h}$ and $E^{r;SL(h,\mathbb{Z})}_{\Lambda^r,V}$ is the completed Langlands-Eisenstein series of $SL(h,\mathbb{Z})$ with infinitesimal character $\rho - 2s\lambda_r$, where $\lambda_r$ is the weight associated to the $r$-fold antisymmetric product of the defining representation,

$$E^{r;SL(h,\mathbb{Z})}_{\Lambda^r,V}(s; \hat{\Theta}) = \prod_{j=1}^{r-1} \zeta^*(2s - j) \sum_{Q \in \mathbb{Z}^h \times \mathbb{Z}/GL(r,\mathbb{Z})} |\det(Q^t\hat{\Theta}Q)|^{-s} ,$$

with the understanding that $E^{r;SL(h,\mathbb{Z})}_{\Lambda^r,V} = 1$. In (4.10) the sum runs over primitive integer $h \times r$ matrices $Q$ modulo right action of $GL(r,\mathbb{Z})$. It satisfies the functional equation

$$E^{r;SL(h,\mathbb{Z})}_{\Lambda^r,V}(s; \hat{\Theta}) = E^{r;SL(h,\mathbb{Z})}_{\Lambda^r,V}(s - h; \hat{\Theta}) .$$

Most importantly, $E_h^*(s; \Omega)$ has a simple pole with residue $r_h$ at $s = \frac{h+1}{2}$ (and consequently a simple pole with at $s = 0$ with residue $-r_h$) where

$$r_h = -\text{Res}_{s=0} N_h(s) = \frac{1}{2} \prod_{j=1}^{[h/2]} \zeta^*(2j + 1) ,$$

which can be read off from the terms with $r = 0$ in (4.11).

### 4.2. Rankin-Selberg method

For a non-holomorphic modular form $F(\Omega)$ of weight 0 and of rapid decay at the cusp, the modular integral

$$\mathcal{R}_h^*(F; s) = \int_{\mathcal{F}_h} d\mu_h E_h^*(s; \Omega) F(\Omega)$$

over a fundamental domain $\mathcal{F}_h$ of the Siegel upper half plane is convergent whenever $\Re(s) > h + 1$, and can be computed by the unfolding trick: the sum over $\Gamma_{\infty} \backslash \Gamma$ is traded for an integral over the ‘generalized strip’

$$\mathcal{S}_h = \Gamma_{\infty} \backslash \Gamma_h = GL(h,\mathbb{Z}) \backslash (\mathcal{P}_h \times [-\frac{1}{2}, \frac{1}{2}]^{h(h+1)/2}) ,$$

where $\mathcal{P}_h = GL(h,\mathbb{R})/SO(h) = \mathbb{R}^+ \times SL(h,\mathbb{R})/SO(h)$ is the space of positive definite symmetric real matrices. Integrating along $\Omega_1$ replaces $F(\Omega)$ by its zeroth Fourier coefficient $F_0(\Omega_2) = \int_0^1 d\Omega_1 F(\Omega)$, leading to

$$\mathcal{R}_h^*(F; s) = N_h(s) \int_{GL(h,\mathbb{Z}) \backslash \mathcal{P}_h} \frac{d\Omega_2}{|\Omega_2|^{h+1-s}} F_0(\Omega_2) .$$

The integration domain $GL(h,\mathbb{Z}) \backslash \mathcal{P}_h$ is the product of a semi-infinite line $\mathbb{R}^+$, associated to the determinant $|\Omega_2|$, times a fundamental domain for the action of $SL(h,\mathbb{Z})$ on the space of positive definite symmetric real matrices of determinant one, e.g. the one constructed by Minkowski [55].
The Rankin-Selberg transform, defined by (4.13), inherits the analytic properties of $E_1^s(s; \Omega)$, in particular it is meromorphic in $s$ with a simple pole at $s = 0, \frac{h+1}{2}$ and satisfies the functional equation

\begin{equation}
R_h^s(F; s) = R_h^s(F; \frac{h+1}{2} - s).
\end{equation}

Since the residue of $E_1^s(s; \Omega)$ is a constant (4.12), the modular integral of $F$ over $\mathcal{F}_h$ is proportional to the residue of $R_h^s(F; s)$ at the same point,

\begin{equation}
\int_{\mathcal{F}_h} d\mu F = \frac{1}{r_h} \text{Res} \left._{s=\frac{h+1}{2}} R_h^s(F; s) \right|_{s=\frac{h+1}{2}}
\end{equation}

4.3. Higher-loop BPS state sums. The Rankin-Selberg method described in the previous subsection is, unfortunately, not directly applicable to the modular integral (4.1), since the Siegel-Narain theta series (1.4) is not of rapid decay at $\Omega_2 \to \infty$. This is best seen after expliciting (1.4) as

\begin{equation}
\Gamma_{d,d,h}(G, B; \Omega) = \Omega_2^{d/2} \sum_{(m_\alpha, n^\beta) \in \mathbb{Z}^{2d}} e^{-\pi Tr(M^2 \Omega_2) + 2\pi i m_\alpha n^\beta \Omega_{1,\alpha \beta}}
\end{equation}

where

\begin{equation}
M^{2,\alpha \beta} = (m_\alpha + B_{jk} n^{k \alpha}) G^{ij} (m_\beta + B_{jk} n^{k \beta}) + n^{i \alpha} G_{ij} n^{j \beta}
\end{equation}

is the Gram matrix of the positive definite quadratic form (2.20) on $h$-tuples of vectors $(m_i, n^i)^\alpha$ in $\mathbb{Z}^{2d}$. The $h$-tuple contributes to the zero-th Fourier coefficient $F_0(\Omega_2)$ of $\Gamma_{d,d,h}$ whenever $m^{i \alpha} n^{j \beta} = 0$ for all $\alpha, \beta$, i.e. when the $h$ vectors $(m_i, n^i)^\alpha$ span an isotropic subspace of $\mathbb{R}^{d \times d}$. The contribution is exponentially suppressed as $\Omega_2 \to \infty$ unless the Gram matrix $M^{2,\alpha \beta}$ has vanishing determinant, i.e. when the $h$ vectors $(m_i, n^i)^\alpha$ are linearly dependent. Since the dimension of the maximal isotropic subspace of $\mathbb{R}^{d \times d}$ is $d$, this is always the case if $d < h$. As in the genus one case (4.13), it is natural to extend the definition of the Rankin-Selberg transform (4.15) by subtracting the non-decaying part of $F_0(\Omega_2)$. For the case at hand, $F = \Gamma_{d,d,h}(g, B; \Omega)$, one obtains

\begin{equation}
R_h^s(\Gamma_{d,d,h}; s) = N_h(s) \int_{GL(h, \mathbb{Z})/\mathcal{P}_h} \frac{d\Omega_2}{|\Omega_2|^{\frac{h+1}{2}}} \sum_{(m_\alpha, n^\beta) \in \mathbb{Z}^{2d} \times h, m^{i \alpha n^j \beta} = 0} e^{-\pi Tr(M^2 \Omega_2)}
\end{equation}

which is an empty sum if $h < d$. By the unfolding trick again, this can be written as an integral over the full space of positive definite symmetric matrices $\mathcal{P}_h$, at the expense of restricting the sum to $GL(h, \mathbb{Z})$ orbits,

\begin{equation}
R_h^s(\Gamma_{d,d,h}; s) = N_h(s) \int_{\mathcal{P}_h} \frac{d\Omega_2}{|\Omega_2|^{\frac{h+1}{2}}} \sum_{(m_\alpha, n^\beta) \in \mathbb{Z}^{2d} \times h / GL(h, \mathbb{Z}), m^{i \alpha n^j \beta} = 0} e^{-\pi Tr(M^2 \Omega_2)}
\end{equation}

This integral can be carried out using 

\begin{equation}
\int_{\mathcal{P}_h} d\Omega_2 |\Omega_2|^\frac{h+1}{2} e^{-\pi Tr(Q \Omega_2)} = \Gamma_h(\delta) |Q|^{-\delta}
\end{equation}
where the right-hand side is fixed by invariance under $SL(h, \mathbb{R})$ and dimensional analysis, up to a multiplicative factor given by
\begin{equation}
\Gamma_h(s) = \prod_{k=0}^{h-1} \pi^{k/2} \Gamma(s - \frac{k}{2}).
\end{equation}

Using (4.22), we find that the regularized Rankin-Selberg transform of $\Gamma_{d,d,h}$ is given by the 'higher genus BPS sum'
\begin{equation}
\mathcal{R}_h^\star(\Gamma_{d,d,h}; s) = N_h(s) \Gamma_h \left( s - \frac{h+1-d}{2} \right) \sum_{\text{BPS}} [\det(M^2)]^{-s + \frac{h+1-d}{2}}
\end{equation}
where
\begin{equation}
\sum_{\text{BPS}} = \sum_{\{m_\alpha, n_\beta\} \in \mathbb{Z}^2} m_\alpha \beta = 0, \text{rk}(m_\alpha, n_\beta) \geq h.
\end{equation}

For $d > h$, this is recognized as a Langlands-Eisenstein series of $SO(d, d, \mathbb{Z})$ with infinitesimal parameter $\rho - 2s', \lambda_h$, with $s' = s - \frac{h+1-d}{2}$, attached to the representation $\Lambda^h V$ where $V$ is the vector representation,
\begin{equation}
\mathcal{R}_h^\star(\Gamma_{d,d,h}; s) \propto \mathcal{E}^{r,SO(d,d,\mathbb{Z})}_h(s - \frac{h+1-d}{2}; G, B) \quad (d > h)
\end{equation}

For $d = h$, the representation $\Lambda^h V$ decomposes into a sum of two irreps with weight $2\lambda_S$ and $2\lambda_C$ where $\lambda_S, \lambda_C$ are the weights associated to the two inequivalent spinor representations. Consequently, we expect
\begin{equation}
\mathcal{R}_h^\star(\Gamma_{d,d,h}; s) \propto \mathcal{E}^{r,SO(h,h,\mathbb{Z})}_h(2s-1; G, B) + \mathcal{E}^{r,SO(h,h,\mathbb{Z})}_C(2s-1; G, B) \quad (d = h)
\end{equation}
These identifications are consistent with the fact that $\mathcal{R}_h^\star(\Gamma_{d,d,h}; s)$ is an eigenmode of the Laplace-Beltrami operator on $G_{d,d}$ with eigenvalue
\begin{equation}
\left[ \Delta_{G_{d,d}} - \frac{h}{4}(2s-d)(2s+d-h+1) \right] \mathcal{R}_h^\star(\Gamma_{d,d,h}; s) = 0,
\end{equation}
as follows from (4.17) and the generalization of (3.19) to genus $h$ [29],
\begin{equation}
\left[ \Delta_{G_{d,d}} - \Delta_{H_h} + \frac{dh(d-h-1)}{4} \right] \Gamma_{d,d,h} = 0.
\end{equation}
The proportionality factor in (4.22) will be determined for $d = 2$ and $d = 3$ in [43.57] and [43.58] below.

4.4. Higher loop string and field theory amplitudes. By a similar reasoning as in §2.2, the modular integral (4.1) should be proportional to the residue of the regularized Rankin-Selberg transform $\mathcal{R}_h^\star(\Gamma_{d,d,h}; s)$, up to a renormalization scheme-dependent subtraction $\delta$,
\begin{equation}
\text{Res}_{s = \frac{h+1}{2}} \mathcal{R}_h^\star(\Gamma_{d,d,h}; s) = \frac{1}{\rho_h} \text{R.N.} \int_{\mathcal{F}_h} d\mu \Gamma_{d,d,h} + \delta.
\end{equation}
Unfortunately, we have not yet been able to imitate the method of [31] to compute the subtraction $\delta$. Using invariance under $s \mapsto \frac{h+1}{2} - s$ and assuming that the
It is interesting to observe that the contribution of the terms with zero winding, \( n^\alpha = 0 \), is exactly of the form expected for a subintegral two-loop field theory integral can be written as

\[
\mathcal{A}_h = \delta + \int_{GL(2;\mathbb{Z})/\mathcal{P}_h} \frac{d\Omega_2}{\Omega_2^{h+1}} \sum_{m_i, n_i \in \mathbb{Z}^{d \times h}, \text{Rk}(m_i, n_i) \geq h} e^{-\pi \text{Tr}(M^2 \Omega_2)}.
\]

It is interesting to observe that the contribution of the terms with zero winding, \( n^\alpha = 0 \), is exactly of the form expected for a subintegral two-loop field theory integral. For \( h = 2 \), it was indeed noted in the context of \( D^4R^4 \) couplings in eleven-dimensional supergravity \[57\] that the three Schwinger parameters \( L_1, L_2, L_3 \) could be mapped by a variable change

\[
V = (L_1 L_2 + L_2 L_3 + L_1 L_3)^{-1/2}, \quad \tau_1 = \frac{L_2}{L_2 + L_3}, \quad \tau_2 = \frac{1}{V(L_2 + L_3)},
\]

to \( \mathbb{R}_+^4 \times \mathcal{F}_{\Gamma_0(2)} \), where \( \mathcal{F}_{\Gamma_0(2)} \) is the fundamental domain of the action of the Hecke subgroup \( \Gamma_0(2) \) on the Poincaré upper half plane parametrized by \( \tau \). Using the invariance of the integrand under a larger group \( SL(2, \mathbb{Z}) \), it was shown that the unregulated two-loop field theory integral can be written as

\[
\mathcal{A}_2^{F.T.} = \int_{GL(2;\mathbb{Z})/\mathcal{P}_2} \frac{d\Omega_2}{\Omega_2^{3/2}} \sum_{m_i, n_i \in \mathbb{Z}^{d \times h}} e^{-\pi \text{Tr}(m_i^\alpha \Omega_2 \alpha \beta G_{ij})}
\]

with

\[
\Omega_2 = \begin{pmatrix} L_1 + L_2 & L_2 \\ L_2 & L_2 + L_3 \end{pmatrix} = \frac{1}{V \tau_2} \begin{pmatrix} |\tau_1|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix}. \tag{4.34}
\]

This indeed matches the zero-winding contribution to \[4.31\], with the understanding that \( \delta \) incorporates contributions with \( \text{Rk}(m_i^\alpha) < 2 \), which are responsible for infrared (and, in field theory, ultraviolet) divergences. It would be interesting to perform a similar matching for the \( D^6R_4 \) couplings in type II on \( T^d \), proportional to the integral \[4.1\] at 3 loops \[58\, 59\].

### 4.5. Lattice unfolding method.

While we are not able to compute the subtraction \( \delta \) yet, in this section and the next we shall compute the modular integral using the lattice unfolding method, and comparing with the Rankin-Selberg transform in cases where both results are available in closed form. For this purpose, we shall use the Lagrangian representation of the lattice partition function, where modular invariance is manifest,

\[
\Gamma_{d,d,h}(G, B; \Omega) = V_d^h \sum_{(M, N) \in \mathbb{Z}^{2d \times d}} \exp \left( -\pi G_{ij} (M_\alpha^i - \Omega_{\alpha \beta} N^{i \beta}) [\Omega_2^{-1}]^{\alpha \gamma} (M_\gamma^j - \tilde{\Omega}_{\gamma \delta} N^{j \delta}) + 2\pi i B_{ij} M_\alpha^i N^{i \alpha} \right).
\]

where \( V_d = \sqrt{\text{det} G_{ij}} \). This expression follows from \[4.13\] by Poisson resummation on \( m_i^\alpha \), and is manifestly invariant under \( Sp(h, \mathbb{Z}) \) action on \( \Omega \), with \( (M_\alpha^i, N^{i \alpha}) \) transforming in the defining representation of \( Sp(h, \mathbb{Z}) \) for any \( i = 1 \ldots d \). Orbits under \( Sp(h, \mathbb{Z}) \) are classified (in part) by the rank of the \( d \times 2h \) matrix \( (M_\alpha^i, N^{i \alpha}) \) and by the \( d \times d \) antisymmetric matrix \( m^{ij} = M_\alpha^i N^{i \alpha} \).
4.5.1. **Zero orbit.** The term with \( M_{\alpha}^i = N^{i\alpha} = 0 \) is invariant under the action of \( Sp(h, \mathbb{Z}) \). Its integral over \( F_h \) is proportional to the volume \( \mathcal{V}_h \) of the fundamental domain,

\[
\mathcal{A}^{(0)}_h = \mathcal{V}_h V^h_d.
\]

where \( \mathcal{V}_h \) is given by [60]

\[
\mathcal{V}_h = \int_{\mathcal{F}_h} d\mu_h = 2 \prod_{j=1}^h \zeta^*(2j),
\]

so that \( \mathcal{V}_1 = \pi/3, \mathcal{V}_2 = \pi^3/270, \mathcal{V}_h = \zeta^*(h) \mathcal{V}_{h-1} \) whenever \( h \geq 2 \).

4.5.2. **Rank one orbit.** If \( (M_{\alpha}^i, N^{i\alpha}) \) has rank one, it can be mapped by \( Sp(h, \mathbb{Z}) \) to an orbit representative with \( M_{\alpha}^i = 0 \) unless \( \alpha = h \), and \( N^{i\alpha} = 0 \) for all \( i, \alpha \). The stabilizer of such an element is the Fourier-Jacobi subgroup spanned by

\[
\Gamma_j = \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \cdot \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \mu^t & \kappa \\ 0 & \mu & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.
\]

where \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) runs over elements of \( Sp(h-1, \mathbb{Z}) \), \( \lambda, \mu \in \mathbb{Z}^{h-1}, \kappa \in \mathbb{Z} \). Decomposing the period matrix

\[
\Omega = \begin{pmatrix} \rho_1 & \rho_1 u_2 - u_1 \\ u_2^t \rho_1 - u_1^t & \sigma_1 \end{pmatrix} + 1 \begin{pmatrix} \rho_2 & \rho_2 u_2 \\ u_2^t \rho_2 - u_1^t + u_1^t u_2 \end{pmatrix},
\]

where \( t \in \mathbb{R}^+, \rho = \rho_1 + i \rho_2 \in \mathcal{H}_{h-1}, u_1, u_2 \in \mathbb{R}^{h-1}, \sigma_1 \in \mathbb{R} \), the measure on \( \mathcal{H}_h \) can be written as

\[
\frac{d\Omega d\bar{\Omega}}{|\Omega|^h} = \frac{d\rho d\bar{\rho}}{|\rho|^h} \frac{dt}{t^{h+1}} du_1 du_2 d\sigma_1.
\]

At the cost of restricting to an orbit representative of the above form, the integration domain \( \mathcal{F}_h \) can therefore be unfolded unto \( \Gamma_j \setminus \mathcal{H}_h = \mathbb{R}^+ \times \mathcal{F}_{h-1} \times T^{2h-1} \), where \( T^{2h-1} \) is a twisted torus parametrized by \( u_1, u_2, \sigma_1 \). Denoting \( M_h^1 = m^t \), we find

\[
\mathcal{A}^{(1)}_h = \frac{1}{2} \sum_{m^t \neq 0} V^h_d \int_{\mathcal{F}_{h-1}} d\mu_{h-1} \int_{T^{2h-1}} du_2 d\sigma_1 \int_0^\infty \frac{dt}{t^{h+1}} e^{-\pi m^t g_i m^t / t}
\]

\[
= \frac{1}{2} \mathcal{V}_{h-1} V^h_d \int_{\mathcal{F}_{h-1}} d\mu_{h-1} \int_{T^{2h-1}} du_2 d\sigma_1 \int_0^\infty \frac{dt}{t^{h+1}} e^{-\pi m^t g_i m^t / t}
\]

\[
= \frac{1}{2} \mathcal{V}_{h-1} V^h \int_{\mathcal{F}_{h-1}} d\mu_{h-1} \int_{T^{2h-1}} du_2 d\sigma_1 \int_0^\infty \frac{dt}{t^{h+1}} e^{-\pi m^t g_i m^t / t}
\]

where \( \mathcal{E}_{V,s}^{SL(d)}(\hat{G}) \) is the completed Epstein zeta series in the vector representation, evaluated at \( \hat{G} = G_{i\bar{i}} / |\det(G)|^{2/d} \).

4.5.3. **Rank \( h \) orbit with \( N_{\alpha}^i = 0 \).** If \( N_{\alpha}^i = 0 \) and \( M^{i\alpha} \) is a generic matrix of rank \( h \), then the stabilizer of \( (M^{i\alpha}, 0) \) is the subgroup of matrices with \( A = D = 1, C = 0 \). The integral can be unfolded unto the generalized strip \([1,13] \), and after a trivial
integration over \( \Omega_1 \), produces

\[
\mathcal{A}_h^{(h)} = V_d h \int_{GL(h,\mathbb{Z})/P_h} \frac{d\Omega_2}{|\Omega_2|^{h+1}} \sum_{M^i_k \in \mathbb{H}^{h \times d} / GL(h,\mathbb{Z})} \exp \left( -\pi G_{ij} M^i_k [\Omega_2^{-1}]^\alpha_\beta M^i_k \right)
\]

\[
= V_d h \int_{P_h} \frac{d\Omega_2}{|\Omega_2|^{h+1}} \sum_{Rk(M^i_k) = h} \exp \left( -\pi G_{ij} M^i_k [\Omega_2^{-1}]^\alpha_\beta M^i_k \right)
\]

(4.42)

Ignoring for a moment the constraint \( \text{Rk}(M^i_k) = h \) and performing a Poisson re-summation \( M^i_k \rightarrow m^i_\alpha \), we observe that the first line matches (for genus 2, and presumably genus 3 as well) the field theory amplitude (4.33). The integral over \( P_h \) can be computed using (4.22), and yields

\[
(4.43) \quad \mathcal{A}_h^{(h)} = V_d h \Gamma_h(\frac{h+1}{2}) \sum_{M^i_k \in \mathbb{H}^{h \times d} / GL(h,\mathbb{Z})} \left[ \text{det} M^i_k G_{ij} M^i_k \right] \frac{h+1}{2} \]

For \( d = h \), the sum over \( M^i_k \) can be further evaluated using (61)

\[
\sum_{M \in M_0(\mathbb{Z})/GL(h,\mathbb{Z})} |M|^{-s} = \zeta(s)\zeta(s-1)\ldots\zeta(s-h+1)
\]

Defining \( \zeta_h^*(s) = \prod_{k=0}^{h-1} \zeta^*(s-k) \), we find

\[
(4.45) \quad \mathcal{A}_h^{(h)} = V_d^{-1} \zeta_h^*(h+1).
\]

4.5.4. Orbits with \( m^{ij} \neq 0 \). The orbits above all had \( m^{ij} = M^i_\alpha N^{j_\alpha} \), hence led to contributions independent of the B-field \( B_{ij} \). For \( d \geq 2h \), the generic orbit with \( m^{ij} \neq 0 \) breaks \( Sp(h,\mathbb{Z}) \) entirely, hence can be unfolded on the full Siegel upper-half plane \( \mathcal{H}_h \). The integrals over \( \Omega_{1\alpha\beta} \) are Gaussian, while the integral over \( \Omega_{2\alpha\beta} \) can be expressed in terms of the matrix Bessel function of (56). There are also contributions from orbits which leave part of \( Sp(h,\mathbb{Z}) \) unbroken. We shall not attempt to classify these orbits in full generality, instead we focus on some simple cases where the full integral is within reach.

4.6. Some simple cases.

4.6.1. \( d = 1 \), any \( h \). For \( d = 1 \), the rank 0 and 1 are the only possible orbits. Using (4.36) and (4.44), we arrive at

\[
(4.46) \quad \mathcal{A}_h^{d=1} = V_h R^h + \zeta^*(h) V_{h-1} R^{-h} = V_h (R^h + R^{-h}) ,
\]

in accordance with T-duality. Conversely, T-duality can be used to prove the recursion formula \( V_h = \zeta^*(h) V_{h-1} \) hence (4.37). For \( h = 1 \) the same result follows from the Rankin-Selberg transform (18)

\[
(4.47) \quad \mathcal{R}_1^2(\Gamma_{1,1,1};s) = 2\zeta^*(2s)\zeta^*(2s-1)(R^{1-2s} + R^{2s-1}) .
\]

The Rankin-Selberg transform vanishes for \( h > 1 \), and (4.46) should originate entirely from the subtraction \( \delta \) in (4.30).
4.6.2. $d = h = 2$. We now consider the genus-two amplitude on $T^2$. By an $Sp(2, \mathbb{Z})$ rotation one can choose
\begin{equation}
(M^{\alpha}_i, N^{\alpha}) = \begin{pmatrix}
0 & p & 0 \\
j_1 & j_2 & j_3 \\
q & 0 & 0 \\
\end{pmatrix}.
\end{equation}
If $p \neq 0$, the choice of the first vector $(M^{\alpha}_i, N^{\alpha})$ breaks $Sp(2, \mathbb{Z})$ to the Fourier-Jacobi group $\Gamma_f = SL(2, \mathbb{Z}) \times \mathbb{Z}^2 \ltimes \mathbb{Z}$. If $q = 0$ one can set $j_3 = 0$ by means of an $SL(2, \mathbb{Z})$ transformation. If $j_1 = 0$, $(M^{\alpha}_i, N^{\alpha})$ has rank 1 case so (4.41) applies. If $j_1 \neq 0$, $M^{\alpha}_i$ has rank 2 and $N^{\alpha}$ vanishes so (4.42) applies instead. Including the zero orbit, we find that the contributions with $m^{ij} = pq = 0$ sum up to
\begin{equation}
A_{2}^{\text{deg}} = \zeta^*(2)\zeta^*(4)T_2^2 + \zeta^*(2)E_1^*(2; U) + \zeta^*(2)\zeta^*(3)T_2^{-1}.
\end{equation}
If $pq \neq 0$, such that we can choose $0 \leq j_1, j_2, j_3 < |q|$ by means of a $\mathbb{Z}^2 \ltimes \mathbb{Z}$ transformation. Using the parametrization (4.39), the integration domain then unfolds onto $\mathbb{R}^+ \times F_1(\rho) \times \mathbb{R}^3(u_1, u_2, \sigma_1)$. The integral over $u_1, \sigma_1$ and $u_2$ (performed in this order) is Gaussian, with a saddle point at
\begin{equation}
(4.50)\quad u_1 = \frac{j_1}{q}, \quad u_2 = -\frac{j_3}{q}, \quad \sigma_1 = \frac{pU_1}{q|U|^2} + \frac{j_1j_3 - j_2q + j_3^2\rho_1}{q^2}
\end{equation}
leading to
\begin{equation}
(4.51)\quad A_2^{\text{n.d.}} = \sum_{(p,q)\neq (0,0)} |q|^3 \int_{F_1} \frac{d\rho d\sigma}{\rho_2^2} \int_0^\infty dt \frac{\sqrt{U_2}}{t^3 |q|^3 |T_2^{3/2}| U} \varepsilon_{\frac{\alpha^2\rho_2}{\rho_2}} - \frac{x^2 \rho_2}{\rho_2^2} + 2\pi i pq T_1
\end{equation}
where the factor $|q|^3$ in front counts the number of $j_i$’s such that $0 \leq j_1, j_2, j_3 < |q|$. In total, we find
\begin{equation}
(4.52)\quad A_2^{d=2} = \zeta^*(2) [E_1^*(2; T) + E_1^*(2; U)],
\end{equation}
which we recognize as the spinor and conjugate spinor Epstein series of $SO(2, 2)$ with $s = 2$, as conjectured in [29]. In the decompactification limit, setting $T = iR_1R_2, U = R_1/R_2$ and taking $R_2 \to \infty$, we see that $A_2^{d=2}$ grows as $R_2^2$ times $A_2^{d=1}(R_1)$, as it should.

Alternatively, we can compute the Rankin-Selberg transform (1.32), and extract the residue at $s = 3/2$. Denoting $(m^{i}_1, n^{i}_1) = (m^{i}_1, n^{i})$ and $(m^{i}_2, n^{i,2}) = (\tilde{m}^{i}, \tilde{n}^{i})$, the quadratic constraints in the BPS sum (4.25) read
\begin{equation}
(4.53)\quad m_1 n^1 + m_2 n^2 = 0, \quad \tilde{m}_1 \tilde{n}^1 + \tilde{m}_2 \tilde{n}^2 = 0, \quad m_1 \tilde{n}^1 + m_2 \tilde{n}^2 + \tilde{m}_1 n^1 + \tilde{m}_2 n^2 = 0.
\end{equation}
The first two constraints can be solved as in [18] §3.2,
\begin{equation}
(4.54)\quad \begin{pmatrix}
m_1 \\
\tilde{m}_1 \\
m_2 \\
\tilde{m}_2 \\
n^1 \\
\tilde{n}^1 \\
n^2 \\
\tilde{n}^2
\end{pmatrix} = \begin{pmatrix}
ck_1 & ck_2 & -dk_2 & dk_1 \\
\tilde{ck}_1 & \tilde{ck}_2 & -\tilde{dk}_2 & \tilde{dk}_1
\end{pmatrix}
\end{equation}
The third constraint requires $(c\tilde{d} - d\tilde{c})(k_2 \tilde{k}_1 - k_1 \tilde{k}_2) = 0$, while the condition $\text{Rk}(m^{i}_1, n^{i,2}) \geq 2$ requires one of the two factors in this product to be non-vanishing. There are therefore two possible branches:
\begin{align}
(4.55)\quad i) \quad & (\tilde{c}, \tilde{d}) = (c, d) \neq (0, 0), \quad \gcd(c, d) = 1, \quad k_2 \tilde{k}_1 - k_1 \tilde{k}_2 \neq 0 \\
ii) \quad & (\tilde{k}_1, \tilde{k}_2) = (k_1, k_2) \neq (0, 0), \quad \gcd(k_1, k_2) = 1, \quad c\tilde{d} - d\tilde{c} \neq 0
\end{align}
In either case,

\begin{align}
  i) \det(M^2) &= (k_2 \tilde{k}_1 - k_1 \tilde{k}_2)^2 \frac{|c + dT|^4}{4T_2^2} \\
  ii) \det(M^2) &= (cd - de)^2 \frac{|k_1 + k_2 U|^4}{4U_2^2}
\end{align}

(4.56)

In the first branch, the sum over \(c, d\) produces the Eisenstein series \(E_1^*(2s - 1; T) / \zeta^*(4s - 2)\), while the sum over matrices \(M = \begin{pmatrix} k_1 & k_2 \\ \tilde{k}_1 & \tilde{k}_2 \end{pmatrix} \) modulo \(GL(2, \mathbb{Z})\) can be computed using (4.44). In total we find

(4.57) \( \mathcal{R}_\beta^d (\Gamma_{2,2}; s) = \zeta^*(2s) \zeta^*(2s - 1) \zeta^*(2s - 2) [E_1^*(2s - 1; T) + E_1^*(2s - 1; U)] \),

in agreement with (4.27). The residue at \(s = 3/2\) reproduces (4.52), as it should.

4.6.3. \(d = h = 3\). We have not attempted to compute the three-loop integral \(A_3^{d=3}\) directly, nor the Rankin-Selberg transform \(\mathcal{R}_\beta^d (\Gamma_{3,3}; s)\), however Eq. (4.27) and the known properties of the Eisenstein series \(E_{SO(3,3)}^* = E_{SL(4)}^*\) and \(E_{C}^{*,SO(3,3)} = E_{A,V}^{*,SL(4)}\) suggest that the Rankin-Selberg transform should be given by

(4.58) \( \mathcal{R}_\beta^d (\Gamma_{3,3}; s) = \zeta^*(2s) \zeta^*(2s - 1) \zeta^*(2s - 2) \zeta^*(2s - 3) \times \left[ E_{SO(3,3)}^* (2s - 1) + E_{C}^{*,SO(3,3)} (2s - 1) \right] \).

The residue at \(s = 2\) produces

(4.59) \( A_3^{d=3} = \zeta^*(2) \zeta^*(4) \left( E_{SO(3,3)}^* (3) + E_{C}^{*,SO(3,3)} (3) \right) \),

in accordance with the conjecture in [29]. The results for the three-loop amplitude on \(T^2\) and \(S^1\) follow by decompactification,

(4.60) \( A_3^{d=2} = \zeta^*(2) \zeta^*(4) (E_1(3; T) + E_1(3; U)) \)
\( A_3^{d=1} = 2 \zeta^*(2) \zeta^*(4) \zeta^*(6) (R^3 + 1/R^3) \)

The \(d = 1\) result agrees with (4.46), it would be useful to obtain the \(d = 2\) case from the subtraction \(\delta\) in (4.30).

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