Exact spectra of conformal supersymmetric nonlinear sigma models in two dimensions

N. Read\textsuperscript{1} and H. Saleur\textsuperscript{2,3}

\textsuperscript{1} Department of Physics, Yale University, P.O. Box 208120, New Haven, CT 06520-8120
\textsuperscript{2} Department of Physics, University of Southern California, Los Angeles, CA 90089-0484
\textsuperscript{3} CIT-USC Center for Theoretical Physics, USC, Los Angeles CA 90089-2535

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We study two-dimensional nonlinear sigma models in which the target spaces are the coset supermanifolds \( U(n + m|n)/U(1) \times U(n + m - 1|n) \cong \mathbb{CP}^{n+m-1|n} \) (projective superspaces) and \( \text{OSp}(2n + m|2n)/\text{OSp}(2n + m - 1|2n) \cong S^{2n+m-1|2n} \) (superspheres). These quantum field theories live in Hilbert spaces with indefinite inner products. These theories possess non-trivial conformally-invariant renormalization-group fixed points, or in some cases, lines of fixed points. Some of the conformal fixed-point theories can also be obtained within Landau-Ginzburg theories. We obtain the complete spectra (with multiplicities) of exact conformal weights of fermions with quenched disorder \([1–8]\), and recent generalizations \([9–22]\), as well as other statistical mechanics models, and also to the spin quantum Hall transition in noninteracting fermions with quenched disorder.

I. INTRODUCTION

In this paper we consider two-dimensional (2D) nonlinear sigma models, that have \( \mathbb{Z}_2 \)-graded Lie algebras (superalgebras) as internal symmetries. Such models have arisen in various contexts: the integer quantum Hall (IQH) effect \([1\, 8]\) and recent generalizations \([9\, 10]\), as well as other problems of fermions with quenched disorder \([11\, 12]\); percolation, polymers, and other statistical mechanics problems (Refs. \([23\, 24]\) and this paper); and strings in anti-de Sitter space (reviewed in Ref. \([30]\), and see Refs. \([31\, 32]\)). In the models on which we concentrate here, the target spaces are particular Riemannian supersymmetric spaces \([1\, 8]\), in cases in which the underlying manifolds are compact. These spaces are \( U(n + m|n)/U(1) \times U(n + m - 1|n) \cong \mathbb{CP}^{n+m-1|n} \) (projective superspaces) and \( \text{OSp}(2n + m|2n)/\text{OSp}(2n + m - 1|2n) \cong S^{2n+m-1|2n} \) (superspheres). In most cases (in a suitable range for \( m \), and for \( n \) sufficiently large), the beta function for the coupling in the nonlinear sigma model is nonzero, and there is a single non-trivial renormalization-group (RG) fixed-point theory for each model. We make use of lattice models that we argue have transitions in the same respective universality classes to obtain the exact complete spectra of energy and momentum eigenvalues (i.e. conformal weights) of states and the corresponding local operators, with their multiplicities, in the conformal field theories (CFTs) that describe the RG fixed points. In two cases, the beta function in the sigma model is identically zero nonperturbatively, and we find the exact spectrum at one special point on each of these two lines of fixed-point theories. In some cases, there are also supersymmetric Landau-Ginzburg theories \([29]\) that have a transition in the same universality class.

In more detail, the field theories and their critical points that we discuss are as follows. For the case of \( \mathbb{CP}^{n+m-1|n} \), the fields can be represented by complex components \( z^a (a = 1, \ldots, n + m) \), \( \zeta_a (\alpha = 1, \ldots, n) \), where \( z^a \) is commuting, \( \zeta_a \) is anticommuting. In these coordinates, at each point in spacetime, the solutions to the constraint \( z^1_a + \zeta^1_\alpha = 1 \) (we use the conjugation \( \dagger \) that obeys \( (\eta \xi)^\dagger = \xi \eta \) for any \( \eta, \xi \), modulo \( U(1) \) phase transformations \( z^a \rightarrow e^{i B} z^a, \zeta^\alpha \rightarrow e^{i B} \zeta^\alpha \), parametrize \( \mathbb{CP}^{n+m-1|n} \). The Lagrangian density in two-dimensional Euclidean spacetime is

\[
\mathcal{L} = \frac{1}{2 g_s^2} \left[ (\partial \mu - i a_\mu) z^a_\dagger (\partial \mu + i a_\mu) z^a + (\partial \mu - i a_\mu) \zeta^\dagger_\alpha (\partial \mu + i a_\mu) \zeta^\alpha \right] + i \frac{\theta}{2 \pi} ( \partial_\mu a_\nu - \partial_\nu a_\mu ),
\]

where \( a_\mu (\mu = 1, 2) \) stands for the combination

\[
a_\mu = \frac{i}{2} [ z^a_\dagger \partial_\mu z^a + \zeta^\dagger_\alpha \partial_\mu \zeta^\alpha - (\partial z^a_\dagger) z^a - (\partial \zeta^\dagger_\alpha) \zeta^\alpha ],
\]

and we work on a rectangle with periodic boundary conditions on \( z^a, \zeta^\alpha \). The fields are subject to the constraint, and under the \( U(1) \) gauge invariance, \( a_\mu \) transforms as a gauge potential; a gauge must be fixed in any calculation. This set-up is similar to the nonsupersymmetric
\( \mathbb{CP}^{m-1} \) model in Refs. [33,34]. The coupling constants are \( g_\beta^2 \), the usual sigma model coupling (there is only one such coupling, because the target supermanifold is a supersymmetric space, and hence the metric on the target space is unique up to a constant factor), and \( \theta \), the coefficient of the topological term, so \( \theta \) is defined modulo \( 2\pi \).

First we note a well-known important point about the supersymmetric models: the physics is the same for all \( n \), in the following sense. For example, in the present model, correlation functions of operators that are local functions (possibly including derivatives) of components \( a \leq n_1 + m \), \( \alpha \leq n_1 \), exist for \( n = n_1 \), say, and are equal to those of the same operators for any other value \( n > n_1 \), due to cancellation of the “unused” even and odd index values. This can be seen in perturbation theory because the “unused” index values appear only in summations in closed loops, and their contributions cancel, but is also true nonperturbatively (it can be shown in the lattice constructions we discuss below). In particular, the renormalization group (RG) flow of the coupling \( g_\beta^2 \) is the same as for \( \mathbb{CP}^{m-1} \), namely (we will not be precise about the normalization of \( g_\beta^2 \))

\[
\frac{dg_\beta^2}{dl} = \beta(g_\beta^2) = mg_\beta^4 + O(g_\beta^6) \tag{1.3}
\]

where \( l = \ln L \), the logarithm of the length scale at which the coupling is defined [see e.g. Ref. [33], eq. (3.4)]. (The beta function for \( \theta \) is zero in perturbation theory, and that for \( g_\beta^2 \) is independent of \( \theta \).) For \( m > 0 \), if the coupling is weak at short length scales, then it flows to larger values at larger length scales. For \( \theta \neq \pi \) (mod 2\( \pi \)), the coupling becomes large, the U(n + m|n) symmetry is restored, and the theory is not perturbative. However, a transition is expected at \( \theta = \pi \) (mod 2\( \pi \)). For \( m > 2 \), this transition is believed to be first order, while it is second order for \( m \leq 2 \) [24]. In the latter case, the system with \( \theta = \pi \) flows to a conformally-invariant fixed-point theory. At the fixed point, a change in \( \theta \) is a relevant perturbation that makes the theory massive. Our exact results describe these critical theories for \( m \leq 2 \).

For \( m = 2 \), the RG flow is the same as for the familiar O(3) sigma model. For \( n = 0 \), the \( \theta = \pi \) critical theory is the SU(2) level 1 Wess-Zumino-Witten (WZW) model [37]. For \( n > 0 \), a supersymmetric extension of this familiar system is obtained. For \( m = 1 \), we argue that the critical theory is related to the critical CFT of percolation.

For \( m = 0 \), the beta function is exactly zero (see Sec. II A), and the sigma model exhibits a line of fixed points, with continuously-varying scaling dimensions for general \( n > 1 \). We argue that these dimensions do not depend on \( \theta \). We find an exact description of the spectrum in one theory with the same symmetry, and argue that it represents one special value of the coupling \( g_\beta^2 \). This theory describes “dense polymers” (there are also variant dense polymer theories with OSP(2n|2n) supersymmetry). These points will be discussed further in Sec. I A, an important role is played by the fact that for \( n = 1 \), this theory is just non-interacting massless scalar fermions (with central charge \( c = -2 \)). This case has similarities with the sigma model with a supergroup manifold SL(n|n)/GL(1) as target space in Ref. [33,34] (for a careful discussion of the target manifold, see Ref. [27]).

The other nonlinear sigma model has target space \( S^{2(n+m-1)/2} \), it is a supersymmetric extension of the usual O(m)-vector model. As coordinates, we use a real scalar field,

\[
\phi \equiv (\phi_1, \ldots, \phi_{2n+m}, \psi_1, \ldots, \psi_{2n}),
\]

(the 2\( n + m \) components \( \phi_a \) are commuting, the remaining \( 2n \) components \( \psi_a \) are anticommuting), and

\[
\phi \cdot \phi' = \sum_a \phi_a \phi_a' + \sum_{\alpha \beta} J_{\alpha \beta} \psi_a \psi_{\beta'} \tag{1.4}
\]

is the osp(2\( n + m | 2n \))-invariant bilinear form \( [J_{\alpha \beta}] \) is a nondegenerate symplectic form, for which we can take the standard form, consisting of diagonal blocks \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

In the nonlinear sigma model, the fields are restricted to obey the constraint

\[
\phi \cdot \phi = 1, \tag{1.5}
\]

which defines the unit supersphere, \( S^{2(n+m-1)/2} \). The Lagrangian density of the sigma model is

\[
\mathcal{L} = \frac{1}{2g_\beta^2} \partial_\mu \phi \cdot \partial_\mu \phi. \tag{1.6}
\]

The perturbative beta function for the coupling \( g_\beta^2 \) in this model is (again, it is independent of \( n \) to all orders)

\[
\beta(g_\beta^2) = (m - 2)g_\beta^4 + O(g_\beta^6). \tag{1.7}
\]

For \( m > 2 \), the RG flow is towards strong coupling at large length scales, and it is expected that the symmetry is restored, and no transition occurs (for \( n = 0 \), \( m \geq 2 \), a broken symmetry phase is not allowed in 2D, and no phase with power-law correlations is known in this model for \( n = 0 \), \( m > 2 \)). For \( m < 2 \), zero coupling is an attractive fixed point, and one then expects a transition to separate this regime from strong coupling. Our exact results describe this critical point. In the case \( m = 1 \), \( n = 0 \), it is best to use a lattice version of the model, as will be described in Sec. I and then this is simply the 2D Ising phase transition. For \( m = 0 \), this critical point is used to describe dilute polymers (self-avoiding walks).

For \( m = 2 \), the theory is more interesting, as the perturbative beta function for \( g_\beta^2 \) vanishes to all orders, similar to the case of the \( \mathbb{CP}^{n-1/2} \) model. For the action
as given, we again expect that the beta function vanishes exactly, and there is a line of fixed point theories with continuously-varying scaling dimensions. However, the lattice version of the model, discussed in Sec. I, is the usual XY model for \( n = 0 \), and a transition occurs due to vortex excitations \([38]\), which are not accounted for in the action \([14]\). The supersymmetric extension should have excitations that correspond to vortices, that can appear as perturbations in the action, and which become relevant above a critical value of the (renormalized) coupling \( g_2^2 \). Then, as in the Kosterlitz-Thouless (KT) theory of the 2D XY model \([38]\), there will be a low temperature (small \( g_2^2 \)) phase with power-law correlations with continuously-varying exponents, and a high-temperature (large \( g_2^2 \)) phase that is massive, separated from each other by a critical point. It is for this critical theory that we obtain the exact spectrum.

The critical CFTs in the \( S^{2n+m-1/2n} \) models can also be reached starting from Landau-Ginzburg or \( \phi^4 \) theories, in which the field \( \phi \), unconstrained this time, has Lagrangian density

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial_\mu \phi + \frac{1}{2} r \phi \cdot \phi + \frac{1}{8} \lambda (\phi \cdot \phi)^2.
\]  

\( (1.8) \)

Such a supersymmetric approach to dilute polymers (the \( m = 0 \) case) was proposed by Parisi and Sourlas \([29]\) (this works in any spacetime dimension). In these theories, a transition can occur when the renormalized mass-squared \( r \) changes sign, again only if \( m \leq 2 \). At the critical point, \( \lambda \) flows away from zero in any dimension less than four, and reaches a nontrivial fixed point. These critical points should be the same CFTs as in the nonlinear sigma models of the same symmetry. The “low temperature” \( (r < 0) \) regime of these models should correspond to the weakly-coupled regime of the \( S^{2n+m-1/2n} \) sigma models.

We now turn to the lattice models we use, which originated as descriptions of percolation, polymers, and some other problems in statistical mechanics. The standard approach to percolation, polymers (self-avoiding walks) and related 2D problems leads to a CFT description of the critical phenomena involving a single scalar field with a background charge, and has been well-studied. It is less widely appreciated that there are representations of the original lattice models using supersymmetry \([29]\). The cancellation between fermion and boson states in loops (percolation cluster boundaries, or polymer loops) in graphical expansions can be used to weight these loops by 1 (for percolation) or 0 (for polymers), in place of the usual \( Q \to 1 \) limit in percolation (based on the \( Q \)-state Potts model) or \( m \to 0 \) limit in polymers (based on the \( O(m) \)-vector model). In the cases of percolation \([14]\) and of dense polymers, we will show that the supersymmetric model is a vertex model (or in an anisotropic limit, a quantum spin chain) and there is a corresponding nonlinear sigma model, the \( \mathbb{C}P^{n+m-1/n} \) model at \( m = 1 \) (percolation), 0 (dense polymers). For dilute polymers, the corresponding nonlinear sigma model is the \( S^{2n+m-1/2n} \) model at \( m = 0 \). The consequence of the existence of supersymmetric models for these problems is that there should be versions of the CFT descriptions that exhibit the same global supersymmetries.

Our results for the percolation case apply directly to the critical theory of the spin quantum Hall (SQH) transition in noninteracting fermions with quenched disorder, for which some exact exponents have been found previously by a mapping of a lattice model onto percolation \([13]\). While much has been learned about percolation and polymers without the use of the supersymmetric models, in the random fermion problems such as the SQH and IQH transitions the supersymmetry is virtually all we have to guide us in seeking the critical theories. Therefore, a fuller understanding of CFTs with the appropriate type of supersymmetry should aid our understanding of the latter problems. Our results constitute another step towards the understanding of previously unknown CFTs that have superalgebras as symmetries, and central charge 0, \(-2\) or other values. An important part of our results is that these theories are not rational CFTs, even though all the conformal weights are rational numbers, because the spectra do not consist of a finite number of representations of a chiral algebra of integer spin fields. In particular, the chiral algebras do not contain affine Lie superalgebras, nor do the theories appear to be related to such superalgebras in any simple way. While the presence of symmetries acting locally on the degrees of freedom does imply the existence of local currents, these do not generate an affine Lie superalgebra because they are not purely right- or left-moving (i.e., not chiral—a fact that will manifest itself in logarithms in their correlators), and so cannot be part of any chiral algebra. We anticipate that such features may be common in the presently unknown CFTs for the random fermion problems.

We note that there is other previous work on applications of sigma models with supergroup manifolds as the target space, in which the CFTs are also not current algebras, though closely related to them. These theories are related to other random fermion problems, the so-called chiral ensembles \([23,28]\). There is a proposal to apply one of these theories, using the work in Refs. \([31,32]\), to the IQH transition \([24]\). Certain other random fermion problems can be tackled fruitfully by current-algebra methods \([32,46]\).

We now describe our method, in brief overview. We first discuss vertex models with supersymmetry, and their relation to nonlinear sigma models and Landau-Ginzburg theories. Special cases of these models describe some properties of percolation and polymers. The models can be defined with periodic boundary conditions on, say, a rectangle with sides \( l_1, l_2 \). They can be viewed as 1D quantum systems in the usual way, say by constructing a transfer matrix \( T = e^{-H} \), and then \( Z = \text{STr} e^{-Hl_{12}} \) is
the partition function, which (by construction) is $Z = 1$ for percolation and dilute polymers, and $Z = 0$ for dense polymers. The supertrace, denoted $\text{STr}$, is defined as $\text{STr} (\cdots) = \text{Tr} (\cdots (-1)^F \cdots)$, where $F = 0, 1$ is the grading of states; in many models, $F$ is the fermion number. The partition function $Z$ is 1 or 0 in these special cases because of cancellations between states associated with the same energy eigenvalue (i.e. eigenvalue of $H$). To obtain the energy eigenvalues, we will calculate $Z_{\text{mod}} = \text{Tr} e^{-H \beta}$. This means modifying the boundary condition in the 2 (imaginary time) direction so as to insert an additional factor $(-1)^F$. For the original periodic boundary conditions, all fields are periodic around either fundamental cycle of the torus. In the modified partition function, the boundary condition is modified (twisted) such that odd (fermionic) fields are antiperiodic along the 2 (time) direction, periodic along the 1 (space) direction, while even (bosonic) fields remain periodic. By varying the aspect ratio of the system, the energies and momenta, and their multiplicities, can be read off from the modified partition function. [We note that to incorporate supersymmetry as an internal symmetry, it is convenient to use a space of states that includes some that have negative norm (more pedantically, it is the norm-squared that is negative). This is connected with the violation of the spin-statistics theorem in the continuum field theories. This allows $H$ to be Hermitian with respect to the indefinite inner product, but nonetheless because of the negative norms, viewed as a matrix $H$ is not diagonalizable. It can be brought to Jordan normal form, and then the diagonal elements of $e^{-H \beta}$ are of the form $e^{-E \beta / 2}$ with multiplicities given by the sum of the sizes of the Jordan blocks that contain $E$. These Jordan blocks are related to the fact that these CFTs are logarithmic. We will not obtain any information about the Jordan block structure, and will simply think of energy eigenvalues $E$ with multiplicities.]

By evaluating the transfer matrix products, the vertex models become loop models, that is sums over configurations of nonintersecting loops, with various weights. In the supersymmetry point of view, each loop has states in the defining representation of the superalgebra running around the loop, and the state on a loop is chosen independently of those on other loops. In particular, for periodic boundary conditions each loop, whether or not it winds the cycles of the torus, is given a weight which is the supertrace (denoted $\text{str}$) of 1 in the defining representation of the superalgebra, which is set up to be $\text{str} 1 = m$ in general, and thus 1 (for percolation), 0 (for polymers). The effect of the modified boundary condition is that a loop that winds along the 2-direction an odd number of times loses the cancellation of even and odd states, and is given a factor which is the trace (not the supertrace) of 1 in the defining representation, while loops that wrap an even number of times are given unchanged weights.

The loop models can in turn be represented using a scalar field with a background charge, with certain boundary condition effects (see Ref. and references therein). We will show that these can be modified to include the differing factor for each loop that winds the 2 direction an odd number of times. After performing a Poisson resummation, we obtain the modified partition function in the continuum limit, which gives the spectrum of each CFT. For low-lying levels, the multiplicities are dimensions of representations of the superalgebra, and this should continue to hold for higher levels, where the multiplicities are positive integers.

In Section IV we describe explicitly the supersymmetric vertex models we use, show how they can be mapped onto loop models (special cases of which describe percolation and polymers), and discuss general aspects of the critical phenomena further. In Sec. II we give the construction of the modified partition function for a general loop model. In Sec. V this is applied to the sigma (or Landau-Ginzburg) models that correspond to percolation, dilute and dense polymers, and also some other models. Appendix A gives details of the derivation of the modified partition function. Appendix B contains results on finite size systems with free boundary condition in the space direction. These give the spectrum of the boundary CFT for percolation, whereas the rest of the paper is about the bulk CFTs. For this case, we show that the states enumerated by the modified partition function are complete. Some partial results on completeness of states in finite size with periodic boundary conditions in the space direction are also given.

II. SUPERSYMMETRIC MODELS

In this Section, we describe the supersymmetric lattice models for percolation and polymers. We also explain the relation to the field theories introduced in Sec. I and give some further field-theoretic arguments.

A. Supersymmetric vertex models and CP$^{n+m-1}$|n sigma models

A supersymmetric representation of the standard model of bond percolation in 2D was constructed in Ref. Some salient points are reviewed here (some further details can be found in Appendix B). We also describe the mapping to the corresponding CP$^{n+m-1}$|n sigma models, and use in understanding the physics of the lattice models.

The standard model of 2D percolation is bond percolation on the square lattice. Clusters can be represented alternatively by their boundaries, which are configurations of space-filling, non-intersecting (and non-self-intersecting) loops on edges of the square lattice (the loops are allowed to touch, without intersecting, at the
vertices). The latter square lattice is the medial graph of that on which the bonds are placed in percolation. This mapping has been pictured, and the weights given, many times (see e.g. \cite{49,50,53}), so we will omit the details, except for the important point that there is a global factor of 1 for each loop, in addition to local weights that are independent of the global properties. The model can be formulated on a simply-connected portion of the plane, or on a cylinder, or on a torus, that is with periodic boundary conditions in two directions. The global factor per loop applies to loops that wind around the nontrivial cycles of the cylinder or torus, as well as to those that are homotopic to a point. Because percolation (or Potts) clusters admit a two-coloring with opposite colors on the two sides of any segment of a loop, the loops that wind around the system must do so in pairs, and because they do not intersect, all the loops that wind the system must be homotopic to one another. The partition function is the sum over all distinct configurations of such loops, with equal weight, such that \( Z = 1 \).

A popular representation for the percolating clusters uses a \( Q \)-state Potts model on the underlying graph. This yields a loop model with the same local weights but a global factor \( \sqrt{Q} \) per loop \( \mathcal{F} \), so percolation corresponds to the limit \( Q \to 1 \) (when the model is formulated on the torus, there is a slight difference between the loop model and the Potts model \( \mathcal{F} \)). Affleck \cite{51} showed that the loop model can be reproduced using a representation of the transfer matrix in a space with \( m \) states per site, with \( Q = m^2 \). The transfer matrix has SU(\( m \)) symmetry.

We now describe the supersymmetric vertex model, which is closely related to that in Ref. \cite{54}, and a special case of which represents percolation \cite{14}. The transfer matrix acts in a space that is a \( \mathbb{Z}_2 \)-graded tensor product (denoted \( \otimes \)) of \( \mathbb{Z}_2 \)-graded vector spaces at each site; these spaces have graded dimension \( n+m|m \), with \( n+m \geq 0 \) (we will also refer to the total dimension of spaces, which here would be \( 2n+m \)). This space is represented using boson and fermion oscillators, with constraints. We consider sites labeled \( i = 0, \ldots, 2l_i - 1 \), with periodic boundary condition. For \( i \) even we have boson operators \( b_i^\dagger, b_i^\alpha, [b_i^\dagger, b_j^\alpha] = \delta_{ij}\delta_{\alpha\beta} \) (\( a = b = 1, \ldots, n+m \), and fermion operators \( f_i^\dagger, f_i^\alpha, \{f_i^\dagger, f_j^\beta\} = \delta_{ij}\delta_{\alpha\beta} \) (\( \alpha = 1, \ldots, n \)). For \( i \) odd we have similarly boson operators \( \tilde{b}_i^\dagger, \tilde{b}_i^\alpha, [\tilde{b}_i^\dagger, \tilde{b}_j^\alpha] = \delta_{ij}\delta_{\alpha\beta} \) (\( a = b = 1, \ldots, n+m \), and fermion operators \( \tilde{f}_i^\dagger, \tilde{f}_i^\alpha, \{\tilde{f}_i^\dagger, \tilde{f}_j^\beta\} = -\delta_{ij}\delta_{\alpha\beta} \) (\( \alpha = 1, \ldots, n \)). Notice the minus sign in the anticommutator; since our convention is that the \( \tilde{\cdot} \) stands for the adjoint, this minus sign implies that the norms of any two states that are mapped onto each other by the action of a single \( \tilde{T}_{ia} \) or \( \tilde{T}_{i\dagger a} \) have opposite signs, and the “Hilbert” space has an indefinite inner product.

The supersymmetry generators are the bilinear forms \( b_i^\dagger b_j^\alpha \), \( f_i^\dagger f_j^\alpha \), \( b_i^\dagger \tilde{f}_j^\alpha \), \( \tilde{b}_i^\dagger \tilde{f}_j^\alpha \), \( f_i^\dagger \tilde{f}_j^\alpha \) for \( i \) even, and correspondingly \( -\tilde{b}_i^\dagger b_j^\alpha, \tilde{T}_{i\dagger a}^\dagger T_{ia}, \tilde{T}_{i\dagger a}^\dagger \tilde{T}_{ia}, \tilde{f}_i^\dagger f_j^\alpha, f_i^\dagger \tilde{f}_j^\alpha \) for \( i \) odd, which for each \( i \) have the same (anti-)commutators as those for \( i \) even. Under the transformations generated by these operators, \( b_i^\dagger, f_i^\dagger \) (\( i \) even) transform as the fundamental (defining) representation \( \mathcal{V} \) of \( gl(n+m|m) \), \( \tilde{T}_{i\dagger a}^\dagger, \tilde{T}_{ia}^\dagger \) (\( i \) odd) as the dual fundamental \( V^\ast \) (which differs from the conjugate of the fundamental, due to the negative norms). We always work in the subspace of states that obey the constraints

\[
\begin{align*}
\tilde{b}_i^\dagger b_i^\alpha + f_i^\dagger f_i^\alpha &= 1 \quad (i \text{ even}), \\
\tilde{b}_i^\dagger b_i^\alpha - \tilde{T}_{i\dagger a}^\dagger T_{ia} &= 1 \quad (i \text{ odd})
\end{align*}
\]

(we use the summation convention for repeated indices of types \( a \) or \( \alpha \)). These specify that there is just one particle, either a boson or a fermion, at each site, and so we have the graded tensor product of alternating irreducible representations \( V, V^\ast \) as desired. In the spaces \( V^\ast \) on the odd sites, the odd states (those with fermion number \( -\tilde{T}_{i\dagger a}^\dagger T_{ia} \) equal to one) have negative norm.

A note on the superalgebras is in order (these remarks are well-known, and can also be found in e.g. Ref. \cite{52}). The above bilinears generate the superalgebra \( gl(n+m|m) \) in the defining representation \( V \); they span the Hermitian matrices on \( V \). One generator, denoted \( \mathcal{E} \), and given by \( \mathcal{E} = \tilde{b}_i^\dagger b_i^\alpha + f_i^\dagger f_i^\alpha \) (or the identity matrix) in the defining representation, commutes with the others. The generators which correspond to the Hermitian matrices with vanishing supertrace (in \( V \)), form a subsuperalgebra, denoted \( sl(n+m|m) \). For \( m \neq 0 \), \( \mathcal{E} \) has nonzero supertrace, and is eliminated by restricting to \( sl(n+m|m) \). In these cases, \( sl(n+m|m) \) is a simple superalgebra. But for \( m = 0 \), \( \mathcal{E} \) has supertrace 0, and generates an ideal in \( sl(n+m|m) \). The quotient superalgebra, denoted \( psl(n|m) \), can be obtained by taking the quotient of \( sl(n+m|m) \) by the ideal. For \( n > 1 \), \( psl(n|m) \) is simple. The spaces \( V, V^\ast \) are, strictly speaking, not representations of \( psl(n|m) \) because \( \mathcal{E} \) is nonzero in these spaces (\( \mathcal{E} = +1 \) in \( V \), \( -1 \) in \( V^\ast \)), but the tensor products of equal numbers of factors \( V, V^\ast \), as we use in our models, are. We also note that we use lower case letters for the superalgebras (which are always over \( \mathbb{C} \)), while upper case letters such as \( U(n+m|m) \) denote a corresponding supergroup, with the real form of the underlying Lie group specified—in this example, it is \( U(n+m) \times U(n) \). The real form is determined by the unitary transformations with respect to the inner products in the spaces in our models.

The invariant transfer matrix is constructed as follows. First we note that for any two sites \( i \) (even), \( j \) (odd), the combinations

\[
\begin{align*}
b_i^\dagger b_j^\alpha + f_i^\dagger f_j^\alpha, \quad \tilde{b}_j^\dagger b_i^\alpha + \tilde{f}_j^\dagger f_i^\alpha
\end{align*}
\]

are invariant under \( gl(n+m|m) \), thanks to our use of the dual \( V^\ast \) of \( V \). Then the transfer matrix acting on sites \( i, i+1 \) with \( i \) even is
\[ T_i = (1 - p_A) + p_A (b_{i+1}^\dagger d_{i+1} + f_{i+1}^\dagger f_{i+1}) \]
\[ \times (b_i^\dagger b_{i+1, a} + f_i^\dagger f_{i+1, a}) \]  
while for \( i \) odd
\[ T_i = p_B + (1 - p_B) (b_{i+1}^\dagger b_{i+1, a} + f_{i+1}^\dagger f_{i+1, a}) \]
\[ \times (b_i^\dagger b^{\dagger}_{i+1, a} + f_i^\dagger f^{\dagger}_{i+1, a}). \] (2.4)

The complete transfer matrix is
\[ T = T_1 T_3 \cdots T_{2l_1 - 1} T_0 T_2 \cdots T_{2l_1 - 2}, \] (2.6)
and \( i = 2l_1 \) is interpreted as \( i = 0 \). Then \( Z = \text{STr} T^2 \).

We usually assume \( 0 \leq p_S \leq 1 \), where \( S = A, B \). By taking either of the two terms in \( T_i \) for each vertex in the graph, the expansion in space-filling loops mentioned above is obtained \([3]\), with factors \( p_S, 1 - p_S \) for each vertex, and a factor \( (n + m) - m = \text{str} = m \) for each loop (in all the models considered in this paper, we will consider only \( m, n \) integer, with \( n + m, n \geq 0 \)). The latter is equal to the supertrace in the fundamental representation (denoted \( \text{str} \)), of 1, because the evaluation of contributions for each loop can be viewed in terms of states in \( V \) flowing around the loop. This holds true for topologically nontrivial loops as well as for loops homotopic to a point. The transfer matrix \( T \) on a chain with periodic boundary conditions, as considered here, is translationally invariant (under translation by two sites), as well as supersymmetric. We note that these constructions give representations of the Temperley-Lieb algebra (at certain parameter values determined only by \( m \) in the various graded vector spaces, as discussed further in App. \( \mathbb{3} \)). The special case \( m = 1 \) gives percolation \((Z = 1 \text{ for } m = 1)\). This version has the advantage of being mathematically well-defined, because there is a mathematically-meaningful, nontrivial transfer matrix at \( m = 1 \), unlike the case of the \( SU(m) \) model \([\mathbb{2}]\) which requires that a poorly-defined limit \( m \rightarrow 1 \) be taken in order to obtain any quantity of interest. For \( m = 1 \), the parameters \( p_A \) and \( p_B \) are the probabilities of links of the square lattice being occupied by bonds in the percolation problem, with \( A \) as the horizontal, and \( B \) as the vertical links. For all \( m \), the system is isotropic when \( p_A = p_B \), while the transition occurs when \( p_A = 1 - p_B \).

In the anisotropic (time- or 2-continuum) limit \( p_A \rightarrow 0 \) with \( p_A/(1 - p_B) \) fixed, we can write \( T \approx e^{-p_A/(1 - p_B)} H \), where the Hamiltonian \( H \) is
\[
H = -\epsilon \sum_{i} (b_{i+1}^\dagger b_{i+1} + f_{i+1}^\dagger f_{i+1})(b_i^\dagger b_{i+1, a} + f_i^\dagger f_{i+1, a}) \]
\[ - \epsilon^{-1} \sum_{i} (b_i^\dagger b_{i+1, a} + f_i^\dagger f_{i+1, a})(b_{i+1}^\dagger b_{i+1, a} + f_{i+1}^\dagger f_{i+1, a}) \]
\[ + (\epsilon + \epsilon^{-1}) l_1, \] (2.7)
and \( \epsilon = \sqrt{p_A/(1 - p_B)} \). This Hamiltonian, like the transfer matrix above, is invariant under \( \text{sl}(n + m|n) \) (or \( \text{psl}(n|n) \) for \( m = 0 \)) for all parameter values \( p_A, p_B \). This Hamiltonian can be viewed as that of an antiferromagnetic superspin chain, built from the generators listed above times those on the neighboring site, contracted using the invariant bilinear form on the superalgebra, up to constant terms. When \( \epsilon \neq 1 \), the two types of terms in \( H \) have different coefficients, and this perturbation ("staggering the coupling") is a relevant perturbation that takes the system off the transition point (see e.g. Ref. \([\mathbb{4}]\)). For \( m = 1 \), the ground state energy of \( H \) is zero, which follows from an argument similar to that in Ref. \([\mathbb{5}]\), or from the facts that the partition function is 1, and that there are no energies less than zero (which is less obvious). The preceding family of supersymmetric vertex models, or the corresponding superspin chain Hamiltonians, are the \( \text{sl}(n + m|n) \)-invariant models for which we find exact results in the following.

For such vertex models or spin chains, there is a corresponding continuum quantum field theory, which is a nonlinear sigma model with target space the symmetry supergroup [here \( U(n + m|n) \)], modulo the isoropy supergroup of the highest weight state (see e.g. Refs. \([\mathbb{6}, \mathbb{7}]\) for related, but non-supersymmetric, examples, and Refs. \([\mathbb{2}], \mathbb{8}\) for supersymmetric examples in random fermion problems). For the present cases we obtain \( U(n + m|n)/U(1) \times U(n + m - 1|n) \cong \mathbb{C} \mathbb{P}^{n+m-1|n} \), a supersymmetric version of complex projective space. This mapping can be controlled by generalizing the spin chain model by replacing the fundamental by a "higher spin" (larger) irreducible representation (irrep), represented by the size of the highest weight \( S \), \( 2S \) integer (with \( S = 1/2 \) for the fundamental), and its dual on alternate sites. For an appropriate family of such irreps, such that the isoropy supergroup is the same, the sigma model can be obtained with a bare coupling constant (at the scale of the lattice spacing) \( \epsilon_0^2 \sim 1/S \), and the mapping is exact at large \( S \). In the construction above, this can be done by replacing 1 on the right hand side of Eqs. \((\mathbb{2}), (\mathbb{3})\) by the integer \( 2S \), and using the same Hamiltonian \((\mathbb{2}), (\mathbb{3})\), up to an additive constant (the transfer matrix of the vertex model would need to be modified). The action of the nonlinear sigma model obtained this way is as in Eq. \((\mathbb{1})\), where the bare value of \( \theta \) is \( \theta = 2\pi S + O(\epsilon - 1) \); note that \( \theta \) can be varied by staggering the couplings \([\mathbb{8}]\). In fact, as \( \epsilon \) varies from 0 to \( \infty \), \( \theta \) varies from 0 to 4\( \pi S \), and so passes through \( \pi \) \((\text{mod } 2\pi)\) \(2S\) times; for \( 2S \) odd, one such point occurs at \( \epsilon = 1 \) for symmetry reasons.

A great deal of insight into the vertex models or spin chains of interest, with \( S = 1/2 \), can now be gained by analyzing the sigma models, and vice versa. When the beta function Eq. \((\mathbb{1})\) for \( \epsilon_0^2 \) is positive at weak coupling, it is expected that the same fixed point CFT is reached in the universal large length-scale behavior of the spin chain models for all odd values of \( 2S \), when \( \epsilon = 1\) \((\theta = \pi)\) (more generally, also at all the \( 2S \) critical values of \( \epsilon \), for
all $S$). In particular, for $m = 1$, this CFT, to which the $\text{CP}^{n-1|n}$ sigma model at $\theta = \pi$ flows, is related to the critical CFT of percolation. We should point out that problems like percolation are defined geometrically, and that it is possible to define many scaling dimensions, or correlation functions, for geometric or topological properties of clusters. On the other hand, we are dealing here with concrete models, and for these the spectra of states and corresponding operators in CFT are precisely defined, and may not include all possible operators that yield geometric properties of clusters. The model we are using emphasizes the properties of the loops, which are the boundaries (hulls) of percolation clusters \cite{13}.

As argued in Sec. \ref{sec:11} for $m = 2$, the RG flow in the sigma model is the same as for the familiar $O(3)$ sigma model. For $n = 0$, the spin chains are the familiar Heisenberg spin chains, and for $2S$ odd, the critical theory is the $SU(2)$ level 1 Wess-Zumino-Witten (WZW) model \cite{37}. For $n > 0$, a supersymmetric extension of this familiar system is obtained. We will return to results for this model in Sec. \ref{sec:V} below.

Also, the case $m = 0$ (and $S = 1/2$) can serve as a model for dense polymers. The general set-up for polymers is discussed more fully in Section \ref{sec:11}. Here we just specify that dense polymers are obtained when one or more long polymer loops are forced into contact. Models of dense polymers involve space-filling non-intersecting loops, with a global factor $m = 0$ for each loop. One such model, for loops on the square lattice, is obtained from the $\text{psl}(n|n)$ invariant vertex model. The structure of this model implies that it contains information on exponents for only even numbers of polymers winding around the torus. The partition function of this model vanishes, as long as the boundary conditions are $\text{psl}(n|n)$-invariant, as they are here. The case $n = 1$ is a free-fermion model \cite{38}.

Define fermion operators $F^i_\mu = f^i_\mu b_1^\dagger$ (i even), $\bar{F}^i = b_1 f^i_\mu$ (i odd); the space of states is just a Fock space, without constraints, in these variables. These operators flip the superspins, and are related directly to the odd generators of the superalgebra. The Hamiltonian in terms of $F$, $\bar{F}$ is noninteracting, and can easily be solved exactly; indeed, for $m = 0$, $n = 1$, this can be done for all $S$ and $\epsilon$. The result is free massless scalar complex, or “symplectic”, fermions \cite{23,17,53}, with Lagrangian density in the continuum limit ($\eta$, $\eta^\dagger$ are anticommuting complex fields):

$$\mathcal{L} = \frac{1}{2g_2^2} \partial_\mu \eta^\dagger \partial^\mu \eta, \quad (2.8)$$

where (for each value of $\epsilon$) $g_2^2$ is precisely proportional to $1/2S$. By construction, the free fermions are the Goldstone modes (spin waves) of the spontaneously-broken supersymmetry of the antiferromagnetic chain. The odd elements of the superalgebra act on $\eta$ as $\eta \rightarrow \eta + \xi$, $\xi$ a complex constant, so no other terms with zero or two derivatives are possible in the action. The spin-wave theory of the antiferromagnet is exact in this case.

In the case of general $n$, the naive continuum (or large $S$) limit in this $m = 0$ case is the sigma model with target space $\text{CP}^{n-1|n}$. In this model, the perturbative beta function vanishes identically. This can be seen either from direct calculations, which have been done to at least four-loop order \cite{38}, or from an argument similar to that in Ref. \cite{11}: for $n = 1$, the sigma model reduces to the massless free fermion theory \cite{28} (the sigma model target space for $n = 1$ has dimension 0/1 over the complex numbers), and further the theta term becomes trivial in this case. Thus, for all sigma model couplings $g_2^2 > 0$, the theory for $n = 1$ is non-interacting. The free-fermion theory is conformal, with Virasoro central charge $c = -2$; $\theta$ is clearly a redundant perturbation in this case, as it does not appear in the action (a similar argument appeared in Ref. \cite{15}). By the same argument as before, the conformal invariance with $c = -2$ should hold for all $n$, and also for all $g_2^2$ and $\theta$, though the action is no longer non-interacting in general. Thus the beta function is also identically zero non-perturbatively. In general, the scaling dimensions will vary with the coupling $g_2^2$, so changing $g_2^2$ is an exactly marginal perturbation, though for $n = 1$ the coupling can be scaled away, so there is no dependence on the coupling in the exponents related to those multiplets of operators that survive at $n = 1$. Hence for $n = 1$, the exactly-marginal perturbation that changes $g_2^2$ is redundant. The dense-polymer problem (the $S = 1/2$ case) apparently corresponds to only one special value of $g_2^2$. We also expect that, at the special value of $g_2^2$, $\theta$ remains redundant for all $n$, because the vertex model remains critical when staggering is present, as we know from the $n = 1$ case, and in the modified Coulomb gas derived below, this constrains all the parameters, so staggering does not change any scaling dimensions even for $n > 1$ (whereas we cannot change $g_2^2$ within the vertex model at fixed $S = 1/2$, and this leaves open the possibility that scaling dimensions, other than those of operators that occur for $n = 1$, change with this coupling). It is likely that $\theta$ is redundant for all $g_2^2$ and all $n$.

Finally, the cases $m < 0$ of the $\text{sl}(n + m|n)$ (or percolation-type) models may also be of physical interest. For sufficiently weak coupling, the flow of $g_2^2$ is towards zero at large length scales, for all $\theta$. There could be a transition at some finite $g_2^2$. When $n + m = 1$, the action consists only of interacting scalar fermions, and there can be no theta term. Then using similar arguments as in the dense polymer case, the critical theory will be independent of $\theta$ for all $n \geq 1 - m$. The exact results presented below may describe these transition points, at least for $m = -2$, $-1$. 7
### B. Polymers and $S^{2n+m-1|2n}$ sigma models

Polymers are modeled as self-avoiding and non-intersecting random walks. There is a standard method in which this is represented using a scalar field theory with $m$ component scalar fields, and the limit $m \to 0$ in which closed loops disappear yields the statistics of polymers \[5,4\]. Such a model is $O(m)$-invariant. Parisi and Sourlas introduced a supersymmetric version, for which we gave the Lagrangian already in Eq. (1.8). The critical point of that theory for $m = 0$ represents “dilute” polymers.

The lattice model usually used in obtaining exact results for 2D polymers is a strongly-coupled version of the field theory, obtained by truncating the high-temperature expansion of a lattice model \[55\]. A supersymmetric version of this model can be constructed with $OSp(2n + m|2n)$ supersymmetry, as follows: We use the honeycomb graph, which can be constructed by tiling the torus with regular hexagons. The partition function is

$$Z = \int \prod_i d[\phi_i] e^{-J \sum_{<ij>} \phi_i \cdot \phi_j / T}, \quad (2.11)$$

where $d[\phi_i]$ is the normalized invariant measure on the supersphere, $J$ is a constant, and $T$ is temperature. This partition function can be shown to be independent of $n$, with no approximation. The high-temperature (strong-coupling) expansion of this model is obtained by expanding the exponential in powers of $J/T$. It can be shown to be independent of $n$ to all orders. In general the expansion involves diagrams with any number of lines (with associated factor $J/T$ per line) on each link of the graph, with a condition that an even number of lines end at each vertex. Each line represents the vector representation propagating along the links. We truncate the expansion of the exponential to the first two terms \[55\], and then the graphical expansion is the same as in the model above, with $K \propto J/T$. Either of these constructions can be used as the supersymmetric version of the $O(m)$ model. \[For m = 1, n = 0, this model accurately represents the (untruncated) Ising model on the honeycomb graph, with $K = \tanh J/T$.\]

Finally, either model can be represented by a transfer matrix on the tensor product of complex $2n + m + 1|2n$-dimensional spaces $C \oplus V'$ (where $V'$ has been complexified), one for each vertical link in a horizontal row of the graph; the states represent either the empty (singlet) or the occupied (vector) state. The partition function is then the supertrace of this transfer matrix to the power $l_2$. One then expects the critical theory of any of these models to be the same as in those discussed in the introduction. We remark that the Lie superalgebra-invariant approach to polymers is different from the $N = 2$ superconformal approach proposed in \[57\], although relations between the two are not excluded.

Apart from the $m = 0$ polymer case, the $OSp(2n + m|2n)$-invariant models \[2.9\] with $m \neq 0$ may also be of interest. The cases $m > 2$ show no critical phenomena, but the cases $m = 1, 2$ exhibit the standard Ising and KT transitions for $n = 0$. Again, for $n > 0$ we obtain supersymmetric extensions which have transitions at which the CFTs include the exponents and correlators of the usual theories. In spite of having the same supersymmetry, these models do not have the same representation structure as the random-bond Ising model \[10\]. In Sec. \[IVD\] below, we also discuss some results for the case $m = -2$.

In the $O(m)$ models \[2.9\], there is also a large $K$, “low $T$,” regime. For all $K$ in the range $K_c < K < \infty$, the system flows to the same fixed point CFT (which varies with $m$). For $m = 2$, this coincides with the “critical” branch, discussed above, but is a distinct CFT for $m < 2$ \[54\]. For $m = 0$, this yields a model of dense polymers, which in this case has no restriction on the numbers of loops that can wind the two directions on the torus \[58\]. We will consider this briefly below. Naively, or based on the Landau-Ginzburg theory \[13\], the low $T$ region of...
the $O(m)$ or $\text{OSp}(2n + m|2n)$-invariant models would be expected to be a broken symmetry regime of the model \cite{1}, described by the weakly coupled $S^{2n+m-1|2n}$ sigma model, see Eq. (1.8). The perturbative beta function (1.7) for the coupling $g^2$ in this model is such that for $m < 2$ zero coupling is an attractive fixed point at large distances (which tends to confirms the existence of a transition in this range of $m$). However, this noninteracting behavior is at odds with the conformally-invariant behavior of the $O(m)$ models, which map to loop models, that we study here, so apparently the latter do not describe this regime of the $S^{2n+m-1|2n}$ sigma model. Instead, the low $T$ $O(m)/$loop models are more closely related to the $sl(n + m|n)$-invariant models, as we will see. This distinction is most likely due to the truncation of the high-$T$ expansions, as discussed above, which does not change the universality class in the high-$T$ and critical regions, which are the same as in the continuum theory (1.8), but could change the low-$T$ behavior. In terms of the physics of polymers, what we call dense polymers is the $m = 0$ model in which dense loops are strictly non-intersecting, and the resulting CFT has $c = -2$. If we adopt instead de Gennes’ model of weak repulsions, as in Lagrangian \cite{3}, then intersections are suppressed, but allowed, and it seems that this “dense” phase is attracted to the non-interacting fixed point at $g^2 = 0$ in the $S^{2n-1|2n}$ sigma model, a theory of free scalar bosons and fermions with central charge $c = -1$. (This distinction of the two models relies on the topology of loops in 2D, so may not hold in higher dimensions.) We expect that the introduction of a small amplitude for allowing crossings in the former dense polymer model produces a crossover to the latter fixed point—the apparent violation of Zamolodchikov’s $c$-theorem being possible, because the theories in question are not unitary.

In the Introduction we have already briefly discussed the model with partition function (2.11) for $m = 2$. For $n = 0$, it is the XY model, and the transition is understood in terms of vortices \cite{8}, which are characterized by integers, since the fundamental homotopy group $\pi_1(S^1) = \mathbb{Z}$. On the $S^{2n+1|2n}$, for $n > 0$, $\pi_1(S^{2n+1|2n}) = \pi_1(S^{2n+1}) = 0$, and there are no topologically stable vortices in the continuum $S^{2n+1|2n}$ sigma model. But we still believe that the partition function of the lattice model on the torus with periodic boundary conditions is the same for all $n$, and similarly for the continuum model (1.4) in which the fields are assumed continuous, so there are no vortices. At weak coupling, we can attempt a semiclassical calculation within the continuum sigma model (1.6). For $n = 0$, one expands around the saddle point where $\phi$ is constant, and a set of saddles where $\phi$ winds in one or both directions on the torus. [In fact, such partition functions for winding numbers $M'$, $M$ appear in a different context in eq. (1.3) below.] For $n > 0$, the natural saddle points that correspond to winding configurations for $n = 0$ are those where the field winds (as a function of position going around a cycle on the torus) around a great circle of the sphere, which break the symmetry down to $\text{osp}(2n|2n)$. These saddle points are unstable, but the multiplicity of the unstable modes is $2n|2n$, and so the negative eigenvalue cancels in the formal one-loop fluctuation determinant. Summing over such saddle points and working to one-loop order, we can reproduce the partition function of the continuum $S^{1|0}$ model. Thus it is very plausible that these configurations are associated with the winding modes for $n = 0$, which correspond in CFT to operators that insert vortices.

C. Modifying boundary conditions

We outlined in the Introduction how we would modify the boundary condition in the 2 direction so as to evaluate $Z_{\text{mod}} = \text{Tr} e^{-H_{12}}$ instead of $Z = \text{STr} e^{-H_{12}}$. Note that this modified boundary condition breaks the supersymmetry. More generally, we will evaluate $Z_{\text{mod}} = \text{Tr} e^{P \tau n_1 - H_{12}}$, where $P$ is the momentum, and $\tau_R$ is a free real parameter. The insertion of the translation by $\tau_R \hat{l}_1$ in the 1 direction makes the torus non-rectangular.

It is not difficult to see that the effect of the modified boundary condition is that, for each loop that winds in the 2 direction (crosses the 1 axis), which would previously have picked up the factor $\text{str}1$, now picks up $\text{str} (-1)^{n_2 \hat{l}}$, where $n_2$ is the number of times it cuts the 1-axis, and $f = 0$ or 1 represents the grading of states in the fundamental: for the sl$(n + m|n)$- (respectively, os$p(2n + m|2n)$-) invariant models, $f = 0$ for the first $n + m$ (resp., $2n + m$) basis states in $V$ (resp., $V'$), $f = 1$ for the remaining $n$ (resp., $2n$). Hence, loops that wind the 2 direction an even number of times are given the factor $m$ as usual, while those that wind an odd number of times are given $2n + m$ (resp., $4n + m$).

In discussing the spectrum, we will often refer to operators as well as to states. The models we are studying are all compact, that is there is a finite number of states per site in the chain on which the transfer matrix acts (in the naive continuum field theories, the target spaces are all compact, or effectively so as excursions to large field values are suppressed). In such systems, one expects that all operators that are strictly local in terms of these variables (i.e., operators whose correlators with local functions of these variables are single-valued) correspond to states in the CFT, so the spectrum of one is the spectrum of the other. There may also be meaningful operators that are not local, such as twist fields or disorder operators in the Ising model. These are not expected to correspond to states. Examples in the models we study include the Potts spin operator, for which there is a nonlocal construction in the supersymmetric vertex model \cite{3}, operators studied by Duplantier \cite{9} and by
The $n = 1$ case of percolation is related to the $n = 1$ case of the model for the spin QH transition \([13]\). In general, the naive continuum theory for the spin QH effect is the sigma model with target space $\text{OSp}(2n|2n)/U(n|n)$, where the real form of the even part of osp this time is $\text{SO}^*(2n) \times \text{Sp}(2n)$ (class C in Ref. \([14]\)). This target space is noncompact for $n > 1$, but compact and the same as $\mathbb{CP}^{n-1}$ for $n = 1$. In the noncompact cases, states lie in infinite-dimensional representations, and the spectrum is expected to be continuous. On the other hand, the natural local operators (at least, those that are local functions expected to be continuous) are typically associated to them, which will be summed over. The heights are real “height” variables to each plaquette of the lattice. The loops are here viewed as having two possible orientations (represented by an arrow on the loop) assigned to them, which will be summed over. The heights associated to a given configuration of directed loops are defined, up to addition of a constant, so that on crossing a loop from left to right (looking along the direction of the arrow on the loop), the height $\varphi$ changes by $\varphi_0$, a constant. Locally, the heights are well-defined, because of the absence of any sources or sinks for the arrows on the loops. On the torus, loops that wind around the two cycles can lead to heights that are not single valued, but are single-valued modulo $\varphi_0$. Suppose the height changes by $\delta_1 \varphi = M \varphi_0$ in the 1 direction, and by $\delta_2 \varphi = M' \varphi_0$ in the 2 direction (where 1 and 2 refer to a choice of fundamental cycles on the torus, which is not necessarily rectangular now). Then, because the loops are non-intersecting, it can be shown \([15]\) that this corresponds to at least $|M \land M'|$ distinct loops, each of which wind the 2 direction $|M/(M \land M')|$ times, and the 1 direction $|M'/(M \land M')|$ times (disregarding the orientation of each loop), plus any number of loops that are homotopic to a point. Here $m \land n$ denotes the highest common factor of two integers, with $m \land 0 = m$ for all $m \geq 0$.

Now each loop that is homotopic to a point is given a global factor $m = 2 \cos \varphi_0 e_0$ by the sum over the two possible orientations, and the same factor is desired for each homotopically non-trivial loop in the unmodified partition function. For our modified partition function, this can be simply modified to $2 \cosh \alpha$ for each loop that winds the 2 direction an odd number of times. The arguments of Ref. \([15]\) can now be easily modified to show that, in the continuum limit taken at appropriate values for the local weight factors (discussed in Sec. II), the modified partition function can be written as

\[ Z_{\text{mod}}[g, \varphi_0, e_0, \alpha] = \sum_{M', M \in \mathbb{Z}} Z_{M', M}(gf^2) w(M, d; f, e_0, \alpha), \tag{3.1} \]

\[ w(M, d; f, e_0, \alpha) = \cos(2\pi df e_0) \delta_{M/d \equiv 0} + \cosh(\alpha) \delta_{M/d \equiv 1}. \tag{3.2} \]

where $d = M' \land M$, $f = \varphi_0/(2\pi)$, and in the Kronecker $\delta$ the congruences are modulo 2. The notation is somewhat redundant, since the parameters $g$, $f$, $e_0$ enter only in the combinations $gf^2$ and $f e_0$, but it is conventional and we will keep it. The continuum scalar field partition function is given by

\[ Z_{M', M}(g) = Z_0(g) \exp \left\{ -\pi g [M'^2 + M^2 (\tau_R^2 + \tau_I^2)] - 2 M M' \tau_R \tau_I / \tau_1 \right\}, \tag{3.3} \]

and

\[ Z_0(g) = \frac{\sqrt{q}}{\tau_1^{1/2} \eta(q) \eta(q^2)}. \tag{3.4} \]

$\tau = \tau_R + i\tau_I = \omega_2/\omega_1$ is the complex aspect ratio (modular parameter) of the torus ($\tau_I = l_2/l_1 > 0$), $q = e^{2\pi i \tau}$, and

\[ \eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{3.5} \]

is the Dedekind function. We note that the Virasoro central charge of the continuum CFT is
that in Ref. [48])

as it is simply a generating function for the spectrum.

 charged with the coupling $g$ for the scalar field depends on the
details of the local weights in general. The expression (3.1)
is clearly not modular invariant. However, the underlying
unmodified partition function, in which the weight is al-
ways the cosine factor, is modular invariant by construc-
tion, because of the sum over all winding numbers that it
contains. $Z_{\text{mod}}$ does not have to be modular invariant,
as it is simply a generating function for the spectrum.

Next we perform a Poisson resummation on the $M$
variable. For each $M \neq 0$, this has to be done by group-
ing the terms into sums of $Z_{M',M}(gf^2)$ over $M'$ a multi-
ple of some positive integer (possibly 1). As in Ref. [48],
the general structure of the result is

$$Z_{\text{mod}}[g, 2\pi f, e_0, \alpha] =$$

$$= \frac{1}{\eta(q)\eta(\bar{q})} \left\{ \sum_P q^{\Delta_{e_0+P,f,0}(g)} \overline{\sum_{q^{\Delta_{e_0+P,f,0}(g)}}} + \sum_{M>0, N>0, P:} \Lambda_{\text{mod}}(M, N; f, e_0, \alpha)$$

$$\times q^{\Delta_{P/(N,f),M}(g)} \overline{\sum_{q^{\Delta_{P/(N,f),M}(g)}}} \right\}. \quad (3.7)$$

The summations are over the integers, except for restric-
tions shown explicitly; note that, for any integers $N, M, N|M$ means $M$ is divisible by $N$. Also

$$\Delta_{e,m}(g) = \frac{1}{4} \left( \frac{e}{\sqrt{g}} + m\sqrt{g} \right)^2, \quad (3.8)$$

for any values of the electric charge $e$ and the magnetic
charge $m$. The coefficients $\Lambda_{\text{mod}}(M, N; f, e_0, \alpha)$ are de-

proved in Appendix A, with the result (more compact than that in Ref. [48])

$$\Lambda_{\text{mod}}(M, N; f, e_0, \alpha) =$$

$$\frac{2}{\mu(d)} \sum_{d>0, d|M} \phi(M/d) w(M, d; f, e_0, \alpha). \quad (3.9)$$

In this expression, $\mu$ and $\phi$ are the Möbius and Euler
functions, respectively, the definitions of which are re-
called in App. A. The unmodified partition function $Z$

is given by the same expressions but with $\Lambda$ in place of
$\Lambda_{\text{mod}}$; $\Lambda$ differs from $\Lambda_{\text{mod}}$ by having $\cos 2\pi df e_0$ in place of
$w(M, d; f, e_0, \alpha)$, irrespective of whether $M/d$ is even
or odd.

In terms of CFT, the right- and left-moving conformal
weights of the states are found by expanding $Z_{\text{mod}}$ in
powers of $q$ and $\overline{q}$, and comparing with the form

$$Z_{\text{mod}} = (q\overline{q})^{-c/24} \text{Tr} q^{L_0} \overline{q}^{\overline{L}_0}; \quad (3.10)$$

the right- (resp., left-) moving conformal weight $h (\overline{h})$ is
the eigenvalue of $L_0 (\overline{L}_0)$. Alternatively, in terms of the
Hamiltonian $H = 2\pi (L_0 + \overline{L}_0)/l_1$ and momentum $P = 2\pi (L_0 - \overline{L}_0)/l_1$, the energy and momentum eigenvalues
(measured from the ground state values) are $E = 2\pi (h + \overline{h})/l_1$, $P = 2\pi (h - \overline{h})/l_1$. The momentum is related to
the conformal spin $s = h - \overline{h}$, which is always an integer.

For the percolation-type, sl($\alpha + m|n$)-invariant models,
$M'$ and $M$ are both even, hence so is $M' \wedge M$. Then we
can move a factor of two into $\varphi_0$ and $\alpha$, so now $f = \varphi_0/\pi$, and the modified partition function is given by

$$Z_{\text{mod}}[g, 2\varphi_0 = 2\pi f, e_0, 2\alpha]$$

as given in Eqs. (3.7), (3.8), (3.9). We are now ready to find the spectra of the various
models discussed in Section II.

IV. RESULTS

In this Section we apply the formulas of the preceding
Section to the models in Section II, by recalling the val-
ues of the parameters that enter in the continuum limit as in Ref. [48]. We begin with the percolation (CP$^{[n]}$)
case as it has direct application to the spin QH transition of
recent interest. We will refer to “states” or to “op-
erators” interchangeably, where the operators are those
that correspond to the states. This does not rule out the
possible interest of operators that do not correspond to
states, but they would be nonlocal with respect to the
variables used in the construction of our models.

A. CP$^{[n]}$ model at $\theta = \pi$, and percolation

To obtain the model of percolation hulls, or the CP$^{[n]}$
sigma model at $\theta = \pi$, we have $m = 2\cos(\pi e_0/2) = 1$
(we use $e_0 = \pm 2/3$), $f = 1/2$, $2\cos \alpha = 2n + 1$, and
we use the result that in the continuum theory, $g = 8/3$
[48], and hence $c = 0$. By the remarks at the end of
Sec. II, we must calculate $Z_{\text{mod}}[8/3, \pi, 2/3, 2\alpha]$. First we point out some conformal weights that appear in the spectrum. If we ignore the contributions from expanding the products in the Dedekind functions, which we view loosely as giving “descendants”, then the scalar (momentum or conformal spin zero) states are either of
purely electric type, with dimensions

$$h = h = \Delta_{\varphi_0 + 2P, 0} - \frac{1}{24} \left( \frac{3P + 1)^2 - 1}{24} \right), \quad (4.1)$$

or of purely magnetic type, with dimensions

$$h = \overline{h} = \Delta_{0, M/2} - \frac{1}{24} \left( \frac{4M^2 - 1}{24} \right). \quad (4.2)$$

The latter correspond to the $M$-hull operators ($M > 0$
[48], while the former are thermal [12], and the sequences
overlap infinitely often. There are many other values at
both zero and nonzero conformal spin. For \( M > 0 \), the general right-moving weight is given by
\[
h = \frac{(3P/N + 2M)^2 - 1}{24},
\] (4.3)

(plus positive integers in the case of descendants), and the conformal spin is \( s = PM/N \) (plus integers). This \( h \) corresponds to position \((-P/N, M)\) in the Kac table of \( c = 0 \) Virasoro representations. Since \( N \) can be arbitrarily large for some \( M \) (as long as \( N|M \) and \( N \wedge P = 1 \)), many nondegenerate Virasoro representations occur in the spectrum. These include some with \( h \) in the range \(-1/24 < h < 0\), however, even when \( h \) or \( \bar{h} \) is negative, \( h + \bar{h} \) (or the excitation energy) is positive. The multiplicities of these states are nonzero, since the lowest weight \( h \) at which a given \( N \) occurs is when \( N = M \) and \( P \) is close to \(-2M^2/3\) (such a state has conformal spin of order \(-2N^2/3\) for large \( N \)). Then the multiplicity is just \( \Lambda_{\text{mod}}(N, N) \), which goes as \( \sim (2 \cosh 2N\alpha)/N \) for large \( \alpha \) (or \( n \)). Thus there are infinitely many equivalence classes of conformal weights under congruence modulo 1.

This implies, that even though the conformal weights are all rational numbers, the theory is not a rational CFT, since there can be no chiral algebra (containing only chiral fields of integer spin) under which the states fall into a finite number of representations.

Next we give some results for the total multiplicities (including descendants) of low-lying levels. The ground state \( h = \bar{h} = 0 \) is non-degenerate. The next-lowest levels are the one-hull operators, with \( h = \bar{h} = 1/8 \), which occur in the two series mentioned above. The total multiplicity with which these occur in \( Z_{\text{mod}} \) is
\[
D_1 = 1 + \Lambda_{\text{mod}}(1, 1) = 1 + 2 \cosh 2\alpha
= (2n + 1)^2/2 - 1 = D_{\text{adj}},
\] (4.4)

the dimension of the adjoint representation of \( \text{sl}(n+1|n) \), as expected \[1\]. It is significant that this number is positive, even, and can be identified as (in general, a sum of) dimensions of representations of \( \text{sl}(n+1|n) \) (it must be even, since except for the ground state all states must cancel in pairs when \( Z \) is calculated). If the CFT underlying this system contained right- and left-moving affine Lie superalgebra of \( \text{sl}(n+1|n) \) at some level, then the multiplicities of primary fields would have to be products, or in the simplest case squares, of dimensions of representations of \( \text{sl}(n+1|n) \). Taking into account the left-right symmetry of the system, that is obviously not the case.

The next levels include the two-hull operators, with \( h = \bar{h} = 5/8 \). These occur for \( M = 0, P = 1, \) and \( M = 2, P = 0, N = 1 \). The total multiplicity is
\[
D_2 = 1 + \Lambda_{\text{mod}}(2, 1) = 1 + \cos \pi e_0 + \cosh 4\alpha
= (2n + 1)^4/2 - 2(2n + 1)^2 + 3/2.
\] (4.5)

For \( n = 1 \), this is \( D_2 = 24 \). This may possibly be a sum of adjoints (each of dimension 8) and an indecomposable representation of \( \text{sl}(2|1) \) of dimension 8 (discussed further in Appendix \ref{appendix}). The singlet operator in the indecomposable can appear as a relevant “thermal” (non-symmetry breaking) perturbation of the critical theory.

Among non-scalar operators, the most interesting are the spin-1 “currents” \( h = 1, \bar{h} = 0 \). They are obtained from (i) \( M = 0 \), as a descendant of the identity, (ii) \( M = 1, P = 1, N = 1 \). The total multiplicity is \( 1 + \Lambda_{\text{mod}}(1, 1) = D_1 = D_{\text{adj}} \). Thus they are precisely the currents of the global \( \text{sl}(n+1|n) \) supersymmetry, which must exist by Noether’s theorem, but which are apparently not chiral since if they were they would generate an affine Lie superalgebra. Hence, they must be logarithmic operators \[17\]: even though \( \bar{h} = 0 \), these operators have nontrivial action on the left-moving sector.

There is also a large multiplet of fields (one of them being the stress energy tensor) with \( h = 2, \bar{h} = 0 \): the total multiplicity is \( D_1 + D_2 \), which may be interpreted as descendants of the currents, together with a “primary” multiplet of dimension \( D_2 \). The latter is presumably again an indecomposable representation, since the stress tensor, which is invariant under \( \text{sl}(n+1|n) \), must be paired with other fields so that \( Z \) can be 1 \[18\].

Finally, we mention the marginal operators of dimension \( (h, \bar{h}) = (1, 1) \). The total multiplicity turns out to be \( 2D_1 \). It is likely that these actually form two adjoints, and that there is no invariant marginal operator. For \( n > 0 \), this is much less than would exist if there were right- and left-moving chiral currents of weight 1, that would generate an affine Lie superalgebra; in that case there would be at least \( (D_{\text{adj}})^2 \) weight \((1, 1)\) states. Since there are not, we again conclude that the currents are not chiral.

We should comment here that the weights \((1/3, 1/3)\) do not appear in the spectrum. These were proposed \[13\] as the weights of a local spin-zero symmetry-breaking perturbation that had been studied numerically \[11\].

Problems with this proposal were pointed out previously \[14, 63\]. The present results show that no such local operator exists in the model in question, and the operator in the superspin chain found in Ref. \[13\] must have some other conformal weights in the long-distance limit. The only spin-zero operators close by are the one- and two-hull operators identified above. It was argued that the requisite operator appears as the leading term in the operator product of the one-hull operator with itself. Since the one-hull operator has even parity \[13\], it can appear in the product and now seems to be the best candidate. This is consistent with the fact that the operator in Ref. \[13\] is not a singlet, but part of a multiplet, which however does not contain a singlet—just like the one-hull operator in the adjoint representation. This would predict that the exponent \( \mu \) of Ref. \[1\] (see also Ref. \[13\]) should be \( \mu = 2/(2 - h - \bar{h}) = 8/7 \approx 1.14, \) within \( \pm 25\% \).
theory, or at special values of the coupling.) There are also other relevant spin-zero ($\hbar = \overline{h} < 1$) operators.

At nonzero spin, there are $8n^2$ currents with $h = 1$, $\overline{h} = 0$. The currents have multiplicity $D_2$, which equals the dimension $D_{\text{adj}}$ of the adjoint representation of osp$(2n|2\ell)$, as appropriate for the currents. The adjoint representation consists of the super-antisymmetric second-rank tensors, that are antisymmetric in the even-even part, and so on. At weights $(h, \overline{h}) = (2, 0)$, there is a total of $D_1 + D_2$ states. Even though the subset of $D_2$ states look formally like descendants of the currents, we prefer to interpret them as transforming in the supersymmetric representation, with the singlet as the stress tensor $\Theta$. This assignment would again be expected from the Landau-Ginzburg point of view, where the stress tensor has the form $\partial \phi \cdot \partial \phi + \ldots$, and so is part of the supersymmetric representation. This is clear at $\lambda = 0$, and should be maintained at finite $\lambda$, and hence in the critical theory. However, the currents might be expected to have Virasoro descendants at level 1, so our interpretation may have some problems in CFT, if not in LG theory; this issue is presumably connected with the logarithmic operators that are part of the currents.

Finally, there are marginal operators at $(1, 1)$, with multiplicity $2D_2$. For $n > 0$, this is again fewer than in current algebra. This multiplet is most likely two adjoints, as in percolation, so that there are no invariant marginal perturbations.

C. CP$^{n-1|n}$ model and dense polymers

For dense polymers, we first discuss the model with psl$(n|n)$ supersymmetry, then the osp$(2n|2\ell)$-invariant model, which contains additional scaling dimensions. For the former, the modified partition function we require is $Z_{\text{mod}}[g, 2\pi f, c_0, 2\alpha]$, with $g = 2$, $f = 1/2$, $c_0 = \pm 1$, $2\cosh \alpha = 2n$. The central charge is $c = -2$.

For $n = 1$, the expression for $Z_{\text{mod}}$ simplifies drastically. This is the case in which the lattice model is essentially a psl$(1|1)$ spin chain, which is a free fermion system that can be solved directly, so we expect to obtain the modified partition function of symplectic fermions $\Theta$. The expression Eq. (3.1) for $Z_{\text{mod}}$ becomes for the percolation-type models

$$Z_{\text{mod}}[g, 2\phi_0 = 2\pi f, c_0, 2\alpha] =$$

$$\sum_{M', M \in \mathbb{Z}} Z_{M', M}(gf^2)w(M, d; f, c_0, 2\alpha),$$

$$w(M, d; f, c_0, 2\alpha) = \cos(2\pi df c_0) \delta_{M/d = 0} + \cosh(2\alpha) \delta_{M/d = 1},$$

where again $d = M' \wedge M$. Inserting the above values for dense polymers and $n = 1$, the Poisson resummation and use of the Jacobi identity give

of the numerical value, 1.45.

B. Critical $S^{2n-1|2\ell}$ model and polymers

A similar analysis can be done for the dilute polymer model, that is the critical point in the $S^{2n-1|2\ell}$ sigma model. In this case, the parameters are $m = 2\cos(\pi e_0) = 0$ (we use $e_0 = 1/2$), $f = 1/2$, $2\cosh \alpha = 4n$, $g = 3/2$. The modified partition function is $Z_{\text{mod}}[3/2, \pi, 1/2, \alpha]$. Many aspects of the results are similar to the CP$^{n|n}$ case, already discussed: The theory is irrational, since it contains infinitely many conformal weights that do not differ by integers. The weights are

$$h = \frac{(8P/N + 3M)^2 - 4}{96}$$

plus non-negative integers for $M > 0$, and $h = [(4P + 1)^2 - 1]/24$ plus non-negative integers for $M = 0$. The spin zero operators include some with the former weights with $P = 0$ (the $M$-leg operators, $M > 0$ [63]), and the latter weights which are those of thermal operators [44, 5].

Some of the low-lying states are as follows. The lowest excited states are at $h = \overline{h} = 5/96$, the 1-leg operators. Their multiplicity is $D_1 = \Lambda_{\text{mod}}(1, 1) = 4n$. This is the dimension of the defining (“vector”) representation of osp$(2n|2\ell)$, as expected (and see a similar result in Ref. [58]). As in the discussion of percolation, this speaks against identifying the CFT as containing an affine osp$(2n|2\ell)$ Lie superalgebra. The next lowest spin-zero states are the thermal, 2-leg operators, with $h = \overline{h} = 1/3$, and multiplicity $D_2 = 1 + \Lambda_{\text{mod}}(2, 1) = 8n^2$. It makes sense to identify this representation of osp$(2n|2\ell)$ with the supersymmetric second-rank tensors built from the graded tensor product of the vector representation with itself, which has the correct dimension (“supersymmetric” here means they are symmetric in the even-even, and antisymmetric in the odd-odd components). It is indecomposable and contains a singlet. (For $n = 1$, we can use the isomorphism osp$(2|2) \cong sl(2|1)$, and this indecomposable representation is the same as the 8-dimensional one mentioned in the last Subsection.) In a Landau-Ginzburg view of the critical theory as an RG fixed point in the theory $\overline{\text{L}}$, this singlet operator would be $\phi \cdot \phi$, which at $\lambda = 0$ clearly lies in a multiplet of similar super-symmetric tensors, and this should be maintained at finite $\lambda$ (and hence in the critical theory), because the degeneracy of the multiplet is needed to maintain the partition function $Z = 1$ throughout the RG flow. (Similar arguments regarding the multiplicities of energy levels at each momentum in a finite size system can be given for the sl$(n + m|n)$-invariant models, using the sigma model at weak coupling. Of course, such arguments cannot predict coincidences of energies that occur in the critical
\[
Z_{\text{mod}}[2, \pi, 1, 0] = \frac{1}{\eta(0)\eta(\frac{\pi}{2})} \sum_{m \in \mathbb{Z}, c \in \mathbb{Z} + 1/2} q^{(2c+m)^2/8} \eta^{(2c-m)^2/8} = 4 \left| q^{1/12} \prod_{n=1}^{\infty} (1 + q^n)^2 \right|^2 = \det(-D_{\text{AP}}),
\]

where \(D_{\text{AP}}\) is the Laplacian on the torus, with periodic boundary condition along the 1 cycle, and antiperiodic along the 2 cycle. Thus this is indeed the free fermion partition function. This conformal field theory contains the \(\text{psl}(1|1)\) affine Lie superalgebra in its chiral algebra (as remarked in a similar context in Ref. [57]); the right-moving currents are \(\partial \eta, \partial \eta^*\).

For \(n > 1\), there are in general no cancellations, and the situation is similar to the percolation and dilute polymer cases. The values for the conformal weights, neglecting descendants, are

\[
h = \frac{(2P/N + M)^2 - 1}{8},
\]

for \(M > 0\), and \(h = \frac{(2P + 1)^2 - 1}{8}\) for \(M = 0\). In particular, for spin zero, these give only the values \(h = \overline{h} = (M^2 - 1)/8\) \((M > 0)\) which are the dimensions of the 2M-leg operators [53]. These are expected as our model contains only even numbers of polymer loops winding on the torus, and in our model local operators always conserve the number of polymer lines mod 2. All conformal weights are non-negative.

The ground state is degenerate, and its degeneracy is \(D_1 = 2 + \Lambda_{\text{mod}}(1,1) = 4n^2\). This was expected, as a natural generalization of the four-fold degeneracy of the symplectic fermion ground state for \(n = 1\) [33], as in both cases the states form the indecomposable adjoint representation of \(\text{gl}(n|n)\) (isomorphic to \(V \otimes V^*\)). The singlet is the identity operator, and the multiplet represents the 2-leg operators. Notice that this multiplet cannot be decomposed into representations of right- and left-moving \(\text{psl}(n|n)\) algebras, even for \(n = 1\). The next spin-zero states occur at \(h = \overline{h} = 3/8\), the 4-leg operators. The multiplicity for general \(n\) is \(D_2 = \Lambda_{\text{mod}}(2,1) = 8n^2(n^2 - 1)\), so they are absent for \(n = 1\). At nonzero spin, we find at \((h, \overline{h}) = (1,0)\) a total of \(2D_1 + D_2 = 8n^4\) states. These are more than simply the \(4n^2 - 2\) currents associated with \(\text{psl}(n|n)\) symmetry. At \((2,0)\) we find a multiplicity \(3D_1 + D_2 = 4n^2(2n^2 + 1)\). The marginal operators, \(h = 1, \overline{h} = 1\), have total multiplicity \(16n^2(4n^4 - 3n^2 + 2)/3\) (which is integer for all \(n\)). (All these multiplicities agree with those for free fermions at \(n = 1\).) For \(n = 1\), the only invariant marginal operator is redundant, corresponding to the fact that in the free fermion theory, a change in coupling \(g_\sigma^2\) can be scaled away and has no effect. As explained in Sec. [14], if our theory describes a special point on the \(\text{CP}^{n-1}|n\) sigma model line of CFTs, an exactly marginal (but not redundant) operator would be expected for \(n > 1\). There is nothing to rule out the possibility that the invariant operator of the \(n = 1\) case becomes such an operator for \(n > 1\), and the identification is presumably correct.

An alternative model of dense polymers is the low \(T\) regime of the \(\text{osp}(2n|2n)\)-invariant polymer-type model [23]. The continuum limit of this model is the same as for the \(\text{psl}(n|n)\) model, except that the restriction in the latter to even numbers of loops \(M'\), \(M\) winding the torus is dropped, which means that we use \(f = 1/4, \alpha (\text{not } 2\alpha)\), and \(2\cosh \alpha = 4n\) for \(\text{osp}(2n|2n)\) symmetry. Then the modified partition function we need is \(Z_{\text{mod}}[2, \pi/2, \pm 1, \alpha]\). As expected, in this theory, \(M\)-leg operators [57] with all \(M > 0\) occur, in particular \(M = 1\), with conformal weight \(h = \overline{h} = -3/32\). The latter has multiplicity \(4n\), the dimension of the vector representation. The identity and 2-leg operators at \(h = \overline{h} = 0\) have multiplicity \(8n^2\) in this case, which we presume is the indecomposable supersymmetric representation, with the identity as the singlet. The structure of higher levels is generally similar to the \(\text{psl}(n|n)\) dense polymer model: one can check that all conformal weights in the spectrum of the latter still occur, but there are many additional weights, and the multiplicities appear to reflect the \(\text{osp}(2n|2n)\) symmetry, as in the examples just given.

D. Other supersymmetric models

Here we briefly consider some other models that arise by taking \(m \neq 0, 1\) in the \(\text{sl}(n + m|n)\)-invariant models, or \(m \neq 0\) in the \(\text{osp}(2n + m|2n)\)-invariant models.

One such model is obtained at \(m = 2\) in the supersymmetric vertex model, or the \(\text{CP}^{n+1|n}\) sigma model at \(\theta = \pi\). For \(n = 0\), this is simply the \(\text{SU}(2)\)-invariant point of the six-vertex model, or spin-1/2 Heisenberg spin-chain, without staggering of the couplings. The critical theory is the \(\text{SU}(2)\) WZW model at level 1. For \(n > 0\), operators with the same conformal weights as those in the latter theory will appear, and possibly others. We study this using our modified partition function, with parameter values \(m = 2, c_0 = 0, f = 1/2, 2\cosh \alpha = 2n + 2, g\) takes the self-dual value \(g = 4\). The central charge \(c = 1\), and the conformal weights are

\[
h = \Delta = \frac{(P/N + M)^2}{4},
\]

plus positive integers. For \(n = 0\), one checks (using Möbius inversion again on \(G\) in Eq. (10) that our formulas reduce to the standard Coulomb gas partition function; in this case \(\Lambda_{\text{mod}}(M, N) = 2\delta_{N,1}\). However, for \(n > 0\), all values \(N > 1\) show up, and the structure is quite similar to the \(\text{CP}^{n|n}\) case: the theory is not rational for any choice of chiral algebra.
Some low-lying conformal weights are \((1/4, 1/4)\), which has multiplicity \(2+\Lambda_{\text{mod}}(1,1) = (2n+2)^2\). This is correct for \(n = 0\), and for general \(n\) is the square of the dimension of the fundamental representation. However, since the fundamental of \(\mathfrak{sl}(n+2|n)\) is not self-dual, the WZW model for this symmetry algebra should have twice this multiplicity for these states (thus this WZW model does not go continuously to the limit \(n = 0\)). This is larger (by 1) than the dimension of the adjoint, which would be the answer expected from the weak-coupling sigma model analysis alluded to in Sec. [5][3] (since the partition function \(Z > 1\) in the present case, such isolated singlets are possible above the ground state), but it is the minimal result that agrees with the \(n = 0\) case. The currents, with weights \((1,0)\), have multiplicity \(1+\Lambda_{\text{mod}}(1,1) = (2n+2)^2 - 1 = D_{\text{adj}}\) again in this case. The next lowest spin-zero states are at \((1,1)\) and have total multiplicity \(3+\Lambda_{\text{mod}}(2,1) + \Lambda_{\text{mod}}(1,1) = (2n+2)^2/2 + 1\). For \(n = 0\), this is \(9 = (D_{\text{adj}})^2\), but for \(n > 0\) is again less than \((D_{\text{adj}})^2\). Even though for \(n = 0\) this theory is precisely a WZW model, for \(n > 0\) it is not, and the operators in it that correspond to those for \(n = 0\) must have operator products that involve those not present at \(n = 0\), and that violate the current algebra structure.

It may be useful here to say a few words more formally about the relation of the theories at different points in this paper. The relevant set-up has been described already in Ref. [32][4], so we can ad hoc arguments at various points in this paper. The relevant set-up has been described already in Ref. [32][4], so we can be brief. The point is that a suitable element \(Q = \sum_i Q_i\), a sum of positive odd roots \(Q_i = \sum_{a=1}^n f_{ia}^a b_i^{a+\alpha} (i \text{ even and the corresponding } \mathfrak{sl}(n+2|n) \text{ generators for } i \text{ odd})\) in the superalgebra, obeys \(Q^2 = 0\), and generates a correspondence of the even and odd states that we would like to cancel. In the space of states, the chomology space \(\text{Ker} Q / \text{Im} Q\) (where \(\text{Ker}\) is the kernel, \(\text{Im}\) is the image, of a map) can be shown explicitly to be isomorphic to the \(n = 0\) Hilbert space. More generally, one can relate \(n\) to \(0 \leq n' < n\) in this way, and similar relations hold for other \(m\) and for osp algebras. The algebras of operators on these spaces are similarly related by taking the chomology of the algebra in the larger \(n\) theory (where \(Q\) acts on operators by graded commutation), both in finite size and in the continuum CFTs. In particular, note that the chiral algebra of the chomology can be larger than the chomology (which is chiral) of the chiral algebra. In the \(\mathbb{CP}^{n+1|n}\) case just discussed, the chiral algebra of the chomology is the SU(2) affine Lie algebra, but this is not contained in the chiral algebra of the \(\mathfrak{sl}(n+2|n)\) invariant theory. The chiral algebra of the latter may be more like a W-algebra, as in the theories in Ref. [32][4]. Similar arguments apply to the cancellation of integrations over even and odd variables in the integrals that we have also used earlier.

In the osp\((2n + m|2n)\)-invariant models, the cases \(m = 1\) and \(m = 2\) give supersymmetric extensions of the Ising and KT transitions, respectively. Similar phenomena as for the SU(2) model just discussed are expected here; they are not be described by osp\((2n+m|2n)\) affine Lie superalgebras. In particular for \(m = 1\), \(n = 0\), the Ising transition is related to a free Majorana fermion field. The loop gas description reproduces the correct modular-invariant partition function on the torus for the Ising model \([15]\). For the critical point in the \(S^{2n|2n}\) sigma model with \(n > 0\), one might imagine that the theory becomes \(2n+1\) free Majoranas, and \(n\) beta-gamma (bosonic ghost) pairs, which is a representation of osp\((2n + 1|2n)\) current algebra at level 1 (using the normalization for the level that is standard for \(O(n)\) current algebra) \([30]\). However, that is not what occurs in our model, and the theory is interacting, not free.

The other case \(m = 2\) in the osp\((2n + m|2n)\)-invariant, polymer-type models is interesting. The partition function we find is of similar structure as the others. This \(S^{2n+1|2n}\) sigma model possesses a line of fixed points with continuously varying scaling dimensions (like the \(\mathbb{CP}^{n-1|n}\) model, though here \(c = 1\)), and the KT point, at which we find the spectrum, is a special point where all the dimensions become rational. The multiplicities are quite similar to other cases. For example, the spin zero states include the vector representation of multiplicity \(4n + 2\) at \(h = \tilde{h} = 1/16\), and a traceless supersymmetric tensor, of multiplicity \((4n + 2)^2/2\), at \(h = \tilde{h} = 1/4\).

The states at \(h = 1, \tilde{h} = 0\) have multiplicity \((4n + 2)^2/2 - 1, \text{ the dimension of the adjoint, so they can be identified with the currents. There are } (4n + 2)^4/4 \text{ marginal operators at } h = \tilde{h} = 1. \text{ This coincides with } (D_{\text{adj}} + 1)^2 \text{ in this case, so is actually larger then } (D_{\text{adj}})^2. \text{ We expect that this set includes fourth-rank supersymmetric tensors, and others, and there should be just two invariant marginal operators. One of the perturbations represents a change in the coupling } g_{\sigma}^2 \text{ in the } S^{2n+1|2n} \text{ sigma model, which in the absence of the other perturbation is exactly marginal. By perturbing the action by both these operators, it should be possible to reproduce the Kosterlitz RG flow diagram near the KT fixed point \([8]\).}

The case \(m = -2\) in the critical \(O(m)\) \((S^{2n-3|2n})\) models is also amusing. Here the Landau-Ginzburg theory \([8]\) predicts that, for \(n = 1\), the theory is free massless symplectic fermions \([29]\) and \(c = -2\). For \(n = 1\), we do indeed find the same critical modified partition function as in \(\text{eq. } (4.9)\). For \(n > 1\), we find that the set of conformal weights \((h, \tilde{h})\) is the same as in the \(\mathfrak{psl}(n|n)\) \((\mathbb{CP}^{n-1|n})\) dense polymer model, but the multiplicities are different, and reflect the dimensions of representations of osp\((2n - 2|2n)\). For example, at \(h = \tilde{h} = 0\), there are \(4n\) operators, which we interpret as two singlets plus the vector. One invariant operator is the identity, the other is the relevant perturbation that corresponds to \(\phi \cdot \phi\) in the Landau-Ginzburg picture, which for \(n = 1\) is
just the fermion mass term. At (1, 0), we find $8n^2 - 4n + 2$ operators, which we interpret as the $8n^2 - 8n + 3$ currents in the adjoint, plus the derivatives of the (0, 0) fields other than the identity. At the next-lowest spin-zero weights, $h = \frac{3}{2}$, we find $8n(n-1)$ states, the dimension of the traceless supersymmetric second-rank tensor of $\text{osp}(2n - 2|2n)$. In the Ginzburg-Landau picture, the trace of the multiplet of supersymmetric second-rank tensors in $\phi$ is the invariant $\phi \cdot \phi$ identified at $h = \frac{3}{2} = 0$; this is allowed by symmetry requirements. It is clear that the theory for $n > 1$ is not free scalar fields.

V. CONCLUSION

To conclude, we have found a rich structure in the conformal field theories of a number of critical points that arise in nonlinear sigma models with target spaces $\text{CP}^{n+m-1|n}$ and $S^{2+n+m-1|2n}$, and are related to models in statistical mechanics such as percolation and polymers (critical and low $T$). For $m$ an integer in the range $-2 \leq m \leq 2$, these give 14 different series of theories, of which we explicitly mentioned results for 8 (two further cases with $m = -2$ lie at $c = -\infty$, so should be excluded). All the theories contain only rational conformal weights, yet except in some exceptional cases at particular $n$, for $n > 0$ there are infinitely many fractional conformal weights, and there can be no chiral algebra that will organize them into a finite number of representations, so the theories are not rational. None of the theories with $n > 0$, except the $\text{psl}(1|1)$ model, contains the affine Lie superalgebra associated with its global supersymmetry in its chiral algebra, as shown by the absence of commuting left and right actions of the global symmetry on the multiplets of states, and other reasons. On the other hand, the large multiplicities of the states point to an enlarged global symmetry algebra of nonlocal charges, which is at least as large as the commutant of the Temperley-Lieb algebra in the lattice models (or some analogue in the osp-invariant cases).

There are many interesting open problems. It would be of great interest to find the full symmetry structure of the operators, the chiral algebra, and the operator product expansions of the fields. With this information, it may be possible to extend the results to the $n > 1$ spin quantum Hall transition, and even to the integer quantum Hall transition and other noncompact supersymmetric nonlinear sigma models in general.

ACKNOWLEDGMENTS

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APPENDIX A: RESUMMATION OF MODIFIED PARTITION FUNCTION

In this Appendix, we give the derivation of the result (3.7), (3.8), (3.9) from Eq. (3.3).

First, in the $M = 0$ terms, $M'/M \wedge M$ is always even, and as in Ref. [48] these terms can be resummed to give

$$\frac{1}{\eta(q)\eta(q')}\sum_{p} q^{\Delta_{\phi_{0}^{+}}+p/1,0}(q) q^{\Delta_{\phi_{0}^{+}}+p/1,0}(q').$$

(A1)

In the $M \neq 0$ terms, the terms $\pm|M|$ give the same result, and we write them in terms of $M > 0$, with a factor of 2. Then for each $M > 0$, the sum can be written

$$Z_{\pm M} = 2 \sum_{d>|d|d|d|d|d>0} w(M,d;f,e,\alpha) \sum_{M':M'\wedge M=d} Z_{M',M}(gf^2).$$

(A2)

We need to find the sums

$$f(d) = \sum_{M':M'\wedge M=d} Z_{M',M}(gf^2)$$

(A3)

(in the following, $d$, $d'$, etc. will always stand for positive divisors of $M$, i.e. $d|M$). On the other hand, sums over $M'$ of the following form can be Poisson-resummed:

$$g(d) = \sum_{M':d|M'} Z_{M',M}(gf^2)$$

(A4)

Clearly, we have

$$g(d) = \sum_{d':d|d} f(d').$$

(A5)

Then one of the Möbius inversion formulas (pp. 234–239 in Ref. [47]) can be used to obtain

$$f(d) = \sum_{d':d|d'} \mu(d'/d)g(d').$$

(A6)

The Möbius function $\mu$ is defined by $\mu(n) = (-1)^r$, if $n$ is an integer that is a product $n = \prod_{i=1}^{r} p_i$ of $r$ distinct primes, $\mu(1) = 1$, and $\mu(x) = 0$ otherwise or if $x$ is not an integer. The two central properties of $\mu(x)$ that lead to the Möbius inversion formulas are [48]

$$\sum_{d':d|d',d'|d'} \mu(d'/d') = \delta_{d,d'},$$

(A7)

which we use here, and
\[
\sum_{d':d|d',d'|d''} \mu(d''/d') = \delta_{d,d''}. \tag{A8}
\]
These can be proved by noticing that for \(d \neq d''\), there are finitely many terms, and they cancel in pairs. Hence we have
\[
Z_{\pm M} = \sum_{d'} G(M, d'; f, e_0, \alpha) \frac{1}{\eta(q)\eta(q^2)} \times \sum_{P'} q^{\Delta_{P'/\langle\langle d', M, f \rangle\rangle} \eta^2_{P'/\langle 2 \rangle, M, f \rangle} N(d', P; M, M, F), (g},\tag{A9}
\]
where
\[
G(M, d'; f, e_0, \alpha) = 2 \sum_d w(M, d; f, e_0, \alpha) \mu(d'/d)/d'.
\tag{A10}\]

We now wish to group together the terms in which \(P\) and \(d\) have common factors. For each term in the sum, let \(d'' = P' \land d\), and \(P' = Pd'\), \(d' = N d^2\), and we can use \(N > 0\) as a summation variable in place of \(d''\). Then
\[
Z_{\pm M} = \sum_{d'} G(M, d'; f, e_0, \alpha) \frac{1}{\eta(q)\eta(q^2)} \times \sum_{N>0; P; N=d, P; N=1} q^{\Delta_{P'/\langle\langle d', M, f \rangle\rangle} \eta^2_{P'/\langle 2 \rangle, M, f \rangle} N(d', P; M, M, F), (g},\tag{A11}
\]
where
\[
\Lambda_{\text{mod}}(M, N; f, e_0, \alpha) =
2 \sum_{d,d':N|d,d'\neq d'} w(M, d; f, e_0, \alpha) \mu(d'/d)/d'.
\tag{A12}\]
Finally, the sum over \(d'\) can be performed:
\[
\sum_{d':N|d'} \mu(d'/d')/d' = M^{-1} \sum_{t=0}^{M}(M/N) t \mu(M/(dt))
= c_{M/d}(M/N)/M,\tag{A13}
\]
where \(t\) is an integer, and \(c_n(m)\) is Ramanujan’s sum (Ref. [7], Thm. 271). By Thm. 272 of Ref. [7], and some elementary manipulation, we obtain
\[
\sum_{d':N|d'} \frac{\mu(d'/d')}{d'} = \frac{\mu(\frac{N}{N_{\text{gcd}}}) \phi(M/d)}{M \phi(\frac{N}{N_{\text{gcd}}})).\tag{A14}\]
and Eq. (1.9) follows. Here \(\phi(n)\) is Euler’s totient function, which is defined for positive integers \(n\) as the number of integers \(m\) such that \(1 \leq m \leq n\) and \(n \land m = 1\), and is given explicitly by
\[
\phi(n) = n \prod_{p|p|n} (1 - 1/p), \tag{A15}
\]
where the product is over primes \(p\). This completes the derivation of Eq. (3.7).

**APPENDIX B: FINITE SIZE AND COMPLETENESS OF STATES**

In this Appendix we supplement the main text by presenting results for one case, the percolation model, in a finite size system with free, rather than periodic, boundary conditions. In this model, we can verify that our set of states is complete, that is, the number of states of the supersymmetric chain coincides with the number of states counted by the modified partition function. The exponents found are the boundary hull exponents, and their descendants. The continuum limit for free boundary conditions in other cases can be found similarly. We also sketch a partial proof for completeness of the states for periodic boundary conditions.

We consider the system with free boundary conditions. We write the operators \(T_i\) defined in Sec. II A as
\[
T_i = (1 - p_A) + p_A e_i, \quad (i \text{ even})
T_i = p_B + (1 - p_B)e_i, \quad (i \text{ odd}).\tag{B1}
\]
Then \(e_i, i = 0, \ldots, 2l - 2\), satisfy the Temperley-Lieb algebra relations
\[
e_i^2 = m e_i,
[e_i, e_{i \pm 1}] = e_i, \quad j \neq i, j \pm 1\tag{B2}\]
(in fact, these are satisfied by \(e_i\) for all \(i = 0, \ldots, 2l - 1\). One can also introduce a trace \(\text{Tr}_m\), which in the supersymmetric construction is \(\text{Tr}_n = \text{Str}\), and which obeys \(m \text{Tr}_m e_i = \text{Tr}_m W (i = 0, \ldots, 2l - 2\)), where \(W\) is any string of \(e_i\)s with \(j = 0, \ldots, i - 1\). The relations (B2), together with the normalization for the trace \(\text{Tr}_m = m^{2l}\), are sufficient to evaluate the trace \(\text{Tr}_m\) of any product of \(e_i\), with \(i = 0, \ldots, 2l - 2\), independent of the vector space (or graded vector space) on which the Temperley-Lieb algebra is represented. Here we will consider only \(m = 1\).

We are mainly interested in the spectrum of the transfer matrix obtained by multiplying the elementary vertex matrices along a given row:
\[
T = T_1T_3 \cdots T_{2l-3}T_2 \cdots T_{2l-2}.\tag{B3}
\]
To obtain this spectrum, we consider the usual trace \(\text{Tr}\) of the transfer matrix \(T\) raised to the power \(t_2\), that is, the modified partition function of the model on an annulus. This modified partition function can be expanded graphically. The result is a gas of dense loops with weights \(p_S\)
or $1 - p_S$ ($S = A, B$) for every point where loops come together, and a global weight factor $\text{tr}(-1)^F = 1$ for contractible loops. The weight of non-contractible loops (necessarily in the time direction, since the system has free boundary conditions in the space direction) is $2n+1$. It would be $n + 1 - n = 1$ if the partition function were defined as the supertrace $\text{STr}$ (or $\text{Tr}_m$), leading to the partition function $Z = 1$. Note that for free boundary conditions, a loop can wind the time direction at most once.

The modified partition function with weight $2n+1$ per non-contractible loop can be computed using a mapping onto the 6-vertex model. To do so, we observe that the loop model partition function is the same as the well-known reformulation of the $Q$ state Potts model (with $Q = 1$) partition function using the polygon decompositions of the surrounding (or medial) lattice $R$. The mapping onto the 6-vertex model then follows from orienting the loops, and gluing them at the interaction vertices. The resulting 6-vertex model has anisotropy parameter $\Delta = \frac{1}{2}$. The loop orientation can be thought of as an isospin $S^z = \pm 1/2$, and the insertion of a term $k^{2S^z}$ in the trace finally produces the right weight for non-contractible loops provided

$$ (2)_k = 2n + 1, \quad (B4) $$

where as usual we have set $(x)_k = (k^x - k^{-x})/(k - k^{-1})$. In the following we also use the parameter $\alpha = \ln k$, such that $(2)_k = 2 \cosh \alpha$.

The 6-vertex model has a Hilbert space $\mathcal{H}_{6\text{-vertex}}$ which is entirely unrelated to the one of the supersymmetric model; in particular its dimension is $2^{2l_1}$! The point however is that the 6-vertex model is solvable, so all of the transfer matrix eigenvalues can be obtained: these eigenvalues will also be the ones of the initial supersymmetric problem, up to multiplicities which we now elucidate.

To proceed, we recall that the 6-vertex model transfer matrix enjoys a crucial property: it has a quantum group $\text{sl}(2)_t$ isospin symmetry with $t = e^{i\pi/3}$, $[T_{6\text{-vertex}}, \text{sl}(2)_t] = 0$. Since the theory of representations of $\text{sl}(2)_t$ for $t$ a root of unity is a bit intricate, let us for a while assume the vertex model is at some generic coupling, with $t$ not a root of unity. In that case, the quantum group symmetry results in a splitting of $\mathcal{H}_{6\text{-vertex}}$ into irreducible representations of $\text{sl}(2)_t$ according to

$$ \mathcal{H}_{6\text{-vertex}} = \sum_{j=0}^{l_1} \left[ \left( \begin{array}{c} 2l_1 \\ l_1 - j \end{array} \right) - \left( \begin{array}{c} 2l_1 \\ l_1 - j - 1 \end{array} \right) \right] \rho_j, \quad (B5) $$

where $j$ is the $\text{sl}(2)_t$ isospin, the $\rho_j$ are $\text{sl}(2)_t$ representations, and the coefficients are binomial coefficients. Then there are

$$ \left( \begin{array}{c} 2l_1 \\ l_1 - j \end{array} \right) - \left( \begin{array}{c} 2l_1 \\ l_1 - j - 1 \end{array} \right), \quad (B6) $$

representations of isospin $j$, each coming with its own eigenvalue. The number $[B10]$ is nothing but the dimension of the irreducible representation $\Pi_j$ of the commutant algebra of $\text{sl}(2)_t$, the Temperley-Lieb algebra.

The generating function of the eigenvalues of the transfer matrix in this representation $Z_j = \sum \lambda_j^k$ (that is, the generating function of the eigenvalues at isospin $j$) can be obtained by taking the generating function in the sector $S^z = j$ and subtracting the generating function in the sector of isospin $S^z = j + 1$: $Z_j = Z_{S^z=j} - Z_{S^z=j+1}$. Commutation with the $\text{sl}(2)_t$ algebra ensures coincidence of the eigenvalues that have to be subtracted, even in finite size.

The generating function of the 6-vertex model eigenvalues reads then

$$ Z_{6\text{-vertex}} = \sum_{j=0}^{l_1} (2j + 1) Z_j. \quad (B7) $$

To obtain what will be the generating function of the $\text{sl}(n+1|n)$ model eigenvalues after the 6-vertex model parameters are adjusted so $t = e^{i\pi/3}$, we need to give a weight $2n+1$ per non-contractible loop, that is introduce the term $k^{2S^z}$ in the trace. Since the 6-vertex transfer matrix commutes with $\text{sl}(2)_t$, this simply gives a multiplicity factor which is the $k$-deformation of the one in $[B7]$:

$$ Z_{\text{mod}} = \text{Tr} T^{l_2} = \sum_{j=0}^{l_1} (2j + 1)_k Z_j. \quad (B8) $$

The decomposition of the $\text{sl}(n+1|n)$ Hilbert space in terms of irreducible representations of the Temperley-Lieb algebra also follows:

$$ \mathcal{H}_{\text{susy}} = \sum_{j=0}^{l_1} (2j + 1)_k \Pi_j. \quad (B9) $$

[For $n = 1$, the first few multiplicities are $(1)_k = 1$, $(3)_k = 8$, $(5)_k = 55, \ldots$, and all are Fibonacci numbers.] The total cardinality reads then

$$ \text{Dim} \mathcal{H}_{\text{susy}} = \sum_{j=0}^{l_1} (2j + 1)_k \left[ \left( \begin{array}{c} 2l_1 \\ l_1 - j \end{array} \right) - \left( \begin{array}{c} 2l_1 \\ l_1 - j - 1 \end{array} \right) \right]. \quad (B10) $$

For future purposes, we note that the cardinality can also be written

$$ \text{Dim} \mathcal{H}_{\text{susy}} = \sum_{j=1}^{l_1} \left( \begin{array}{c} 2l_1 \\ l_1 - j \end{array} \right) \left[ (2j + 1)_k - (2j - 1)_k \right] + \left( \begin{array}{c} 2l_1 \\ l_1 \end{array} \right) .$$
\[
= 2 \sum_{j=1}^{l_1} \left( \frac{2l_1}{l_1 - j} \right) \cosh 2j \alpha \left( \frac{2l_1}{l_1} \right) \\
= (e^\alpha + e^{-\alpha})^{2l_1} = (2n + 1)^{2l_1}.
\] (B11)

We now let \( t = e^{i\pi/3} \). None of the previous results change, except that \( \mathcal{H}_{6-vertex} \) splits into indecomposable representations of \( \text{sl}(2)_{\lambda} \) instead of irreducible ones. Similarly, the representations of the Temperley-Lieb algebra have to be handled more carefully. However, the arguments using the counting of states still hold by continuity, and so do those that lead to the partition functions in the continuum limit.

The conformally-invariant limit for the \( \text{sl}(n + 1/n) \) model is obtained in the isotropic case when \( p_S = 1/2 \). Up to a non-universal bulk free energy term (which is trivial here, because the unmodified partition function is \( Z \)), the expression of \( Z_j \) in the conformal limit follows from the computations in \[79\]. In the limit where \( t \to e^{i\pi/3} \), one finds

\[
Z_j \to \frac{q^{j(2j-1)/3} - q^{j(j+1)(2j+3)/3}}{P(q)}.
\] (B12)

Here, \( P(q) = \prod_{n=1}^{\infty} (1 - q^n) \), and \( q = e^{-\pi i z/l_1} \). The generating function for the spectrum of the \( \text{sl}(n + 1/n) \) model with free boundary conditions finally follows:

\[
Z_{\text{mod}} = \sum_{j=0}^{\infty} (2j+1)_k \frac{q^{j(2j-1)/3} - q^{j(j+1)(2j+3)/3}}{P(q)}.
\] (B13)

The unmodified partition function is reproduced meanwhile by taking

\[
Z = \sum_{j=0}^{l_1} (2j+1)_k Z_j.
\] (B14)

as \( t \to e^{i\pi/3} \). That it is equal to 1 in that limit is not obvious from this formula alone: to prove it, one needs to invoke the additional cancellations that arise from representation theory of \( \text{sl}(2) \), when \( t = e^{i\pi/3} \). These cancellations completely mimic the ones of the continuum limit, which give here

\[
Z = \frac{1}{P(q)} (1 - q - q^2 + q^5 + q^7 - q^{12} + \ldots),
\] (B15)

which is equal to one by Euler’s pentagonal number theorem.

Of course, the fact that \( Z = 1 \) follows much more directly from the fact that it coincides with the partition function of the supersymmetric model with periodic boundary conditions in the time direction.

In conclusion, we see that the eigenvalues of the transfer matrix based on \( \text{sl}(n + 1/n) \) are the same as the eigenvalues of the 6-vertex transfer matrix. The only things that differ are multiplicities, none of which are zero, i.e. any eigenvalue of one transfer matrix is also an eigenvalue of the other.

The operator content of the \( \text{sl}(n + 1/n) \) model with free boundary conditions is therefore made only of \( j \) hull operators with surface dimensions

\[
h_j = \frac{j(2j-1)}{3}
\] (B16)

and multiplicities (neglecting the effect of descendants this time)

\[
D_j = (2j + 1)_k.
\] (B17)

These multiplicities are much larger than dimensions of irreducible representations of \( \text{sl}(n + 1/n) \); the commutant of the Temperley-Lieb algebra in the supersymmetric representation used here is much larger than (the products of generators of) the global supersymmetry \( \text{sl}(n + 1/n) \) \[74\]. The values of \( D_j \) can however be understood by a simple argument: calling \( D_1 = D_{\text{adj}} = (2n + 1)^2 - 1 \) the dimension of the adjoint, one finds that

\[
D_j = D_{\text{adj}} - D_{j-1} - D_{j-2}.
\]

This is easily interpreted graphically: with a pair of non contracted lines we associate the adjoint. Multiplying \( 2(j-1) \) non-contracted lines by the adjoint gives either \( 2j \) non contracted lines, or \( 2(j-1) \) if one of the added lines gets contracted with the ones existing before, or \( 2(j-2) \) if the two added lines get contracted.

Here, it may be useful to comment briefly on the decomposition of products of the type \((V \otimes V^*)^j\) in the case \( n = 1 \). Following notations of \[74\], one has for instance

\[
V \otimes V^* = \Pi(0,0) + \Pi(0,1),
\]

\[
(V \otimes V^*)^2 = \Pi(0,0) \oplus \Pi(0,2) \oplus 4\Pi(0,1) \oplus \Pi(1/2, 3/2)
\]

\[
\oplus \Pi(-1/2, 3/2) \oplus \text{[Indecomp.],}
\]

where in particular \( \Pi(0,1) \) is the adjoint, and \( \text{[Indecomp.]} \) stands for an eight-dimensional indecomposable block. Recall that the representations \( \Pi(b,j) \) are generically typical, with dimension \( 8j \) and vanishing supertrace, while \( \Pi(\pm j,j) \) are atypical, with dimension \( 4j + 1 \), and supertrace equal to ±1. We observe in the previous examples that the “true” singlet appears only once: it can be proved that this is a general feature (invariant vectors, that are annihilated by all the generators, appear in the indecomposable blocks in general, but they result from the action of some generators on other vectors inside the block). This is compatible (though not necessarily equivalent) with having \( Z = 1 \). With our normalizations for the transfer matrix, the eigenvalue associated with the true singlet is \( \lambda = 1 \).

A similar approach could presumably be used to study periodic boundary conditions. Commutation with the
quantum group \( \text{sl}(2)_t \) would then be replaced by the existence of commutative diagrams as discussed in \([2]\). We are not going to discuss this in detail here, but would like to indicate how the counting of states works in that case.

The modified partition function of the \( \text{sl}(n+1|n) \) model with periodic boundary condition in the space direction, \( Z_{\text{mod}} \), will be obtained by combining partition functions of the 6-vertex model with a given value of the isospin \( S^z = M \), and a twist angle \( \phi \) (we use the notations of \([72]\)). This corresponds in the Coulomb gas at coupling \( g^2 = g'/4 = 2/3 \), to \( m = S^z = M, e = \phi/2\pi + P \), where \( P \) is an arbitrary integer. The 6-vertex partition function in the continuum limit is

\[
Z_{S^z}^{\phi/2\pi} = \frac{1}{\eta(q)\eta(\bar{q})} \sum_P q^{\Delta_{\phi/2\pi + P,S^z}(g/4)} \sum_{\mu} \prod_{\alpha} \frac{1}{\eta(q)\eta(\bar{q})}.
\]

The partition function of the supersymmetric model for a given value of \( M \) reads meanwhile, for \( M \neq 0 \),

\[
Z_{\pm M} = \sum_{d'} G(M, d'; 1/2, e_0, 2\alpha) \frac{1}{\eta(q)\eta(\bar{q})} \sum_{\mu} q^{\Delta_{\phi/2\pi + P,M}(g/4)} \prod_{\alpha} \frac{1}{\eta(q)\eta(\bar{q})},
\]

(B20)

The sum over \( P' \) can be decomposed into a sum of \( d' \) terms for all the congruences modulo \( d' \); it follows that

\[
Z_{\pm M} = \sum_{d'} G(M, d'; 1/2, e_0, 2\alpha) \sum_{k=0}^{d'-1} Z_{M/k}^{d'}. \quad \text{(B21)}
\]

It is very likely that the same expression holds in the finite case as well. If we assume this is true, we can check that the right number of states for the supersymmetric model is obtained. Indeed, each \( Z_{S^z}^{\phi/2\pi} \) being a partition function of the 6-vertex model at isospin \( S^z \) accounts for \( \left( \frac{2l_1}{l_1 - S^z} \right) \) states, independently of the value of the twist angle \( \phi \). Hence the number of states contributing to \( Z_{\pm M} \) is

\[
\left( \frac{2l_1}{l_1 - M} \right) \sum_{d'} d' G(M, d'; 1/2, e_0, 2\alpha).
\]

Now,

\[
\sum_{d'} d' G(M, d'; 1/2, e_0, 2\alpha) = 2 \sum_{d', d} w(M, d; 1/2, e_0, 2\alpha) \mu(d'/d). \quad \text{(B22)}
\]

Recall that \( d|M, d'|M \). Then \( \sum_{d'} \mu(d'/d) = \delta_{d,M} \) from Eq. \([A7]\), and the right hand side reduces to \( 2 \cosh 2M \alpha \). It follows that the number of states contributing to \( Z_{\pm M} \) is \( 2 \cosh 2M \alpha \left( \frac{2l_1}{l_1 - M} \right) \), for \( M \neq 0 \). The case \( M = 0 \) is trivial: the corresponding partition function of the supersymmetric model has the form \([A1]\), and is reproduced in the 6-vertex model by taking \( S^z = 0 \) and \( \phi/2\pi = e_0 \), with a corresponding number of states equal to \( \left( \frac{2l_1}{l_1} \right) \).

For a system of size \( 2l_1 \), the allowed values of the isospin run between 0 and \( l_1 \), so using \([B11]\) we find the right size for \( \text{Dim} \mathcal{H}_{\text{susy}} \).

Notice that in finite size, there will be numerous coincidences of levels ensured by the commutative diagrams of \([2]\). These should guarantee that subtractions of partition functions are possible in the finite size case as well, like in the case of free boundary conditions.

We should point out that the arguments given in this Appendix generalize to other \( m \) and \( n \) values, including the range \( m > 2 \) where the models are gapped, not conformal. This includes the cases with \( n = 0 \), the \( \text{SU}(m) \) spin chains with alternating \( n \) (or \( V \)) and \( \mathfrak{g} \) (or \( V^* \)) representations. For \( \epsilon = 1 \), the eigenvalues of these models can be found from the Bethe ansatz on the corresponding 6-vertex model \([72]\), and the multiplicities can now be found as here.
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