SUMMATION FORMULAS FOR GJMS-OPERATORS AND Q-CURVATURES ON THE MÖBIUS SPHERE

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Abstract. For the Möbius sphere $S^{q,p}$, we confirm the recursive formulas for GJMS-operators and $Q$-curvatures formulated by the first author in “On conformally covariant powers of the Laplacian” (http://arxiv.org/abs/0905.3992).

October 26, 2009

Contents

1. Introduction and statement of results 1
2. Proof of Theorem 1.1 6
3. Proof of Theorem 1.2 15
Appendix. Proofs of some auxiliary results 18
References 20

1. INTRODUCTION AND STATEMENT OF RESULTS

In [J09b], one of the authors formulated conjectural recursive formulas for GJMS-operators and Branson’s $Q$-curvatures. In the present article, we prove these conjectures for the space $S^{q,p} = S^q \times S^p$ with the signature $(q,p)$-metric $g_{S^q} - g_{S^p}$ given by the round metrics on the factors. Let $n = q + p$.

A GJMS-operator $P_{2N}$, $N \geq 1$, is a rule which associates to any pseudo-Riemannian manifold $(M, g)$ a differential operator of the form

$$P_{2N}(g) = \Delta_g^N + \text{lower order terms},$$

where $\Delta_g = -\delta_g d$ is the Laplace–Beltrami operator of $g$. The operators $P_{2N}(g)$ are given by universal polynomial formulas in terms of the metric $g$, its inverse, the curvature tensor of $g$, and its covariant derivatives. Moreover, the operators $P_{2N}(g)$ are conformally covariant in the sense that

$$e^{(\frac{n}{2} + N)\varphi} \left(P_{2N}(e^{2\varphi} g)(u)\right) = P_{2N}(g) \left(e^{\left(\frac{n}{2} - N\right)\varphi} u\right) \quad (1.1)$$

2010 Mathematics Subject Classification. Primary 53B20, 53B30; Secondary 05A19, 33C20, 53A30.

The work of the first author was supported by SFB 647 “Raum-Zeit-Materie” of DFG. The work of the second author was partially supported by the Austrian Science Foundation FWF, grants Z130-N13 and S9607-N13, the latter in the framework of the National Research Network “Analytic Combinatorics and Probabilistic Number Theory.”
for all $\varphi \in C^\infty(M)$ and $u \in C^\infty(M)$. The operators $P_{2N}$ were derived in [GJMS92] from the $N^{th}$ powers of the Laplace–Beltrami operator of the Fefferman–Graham ambient metric [FG07]. On manifolds of even dimension $n$, this construction is obstructed at $2N = n$, i.e., it only yields a finite sequence $P_2(g), \ldots, P_n(g)$ of operators for which (1.1) is valid for all metrics. However, the obstructions should not be attributed to the method of construction. In fact, Graham [G92] proved that on manifolds of dimension 4, there is no conformally covariant cube of the Laplacian. More generally, Gover and Hirachi [GH04] proved that on manifolds of even dimension $n$, there is no conformally covariant power of the Laplacian the order of which exceeds $n$. Nevertheless, for even $n$ and locally conformally flat metrics, the obstructions vanish, and the construction of [GJMS92] still yields an infinite sequence of conformally covariant operators. In the present work, we will be concerned with such a case. For odd $n$, the operators $P_{2N}$ exist to all even orders $2N \geq 2$.

The GJMS-operators are not uniquely determined by the requirement of their conformal covariance. It remains a challenge to understand their structure, and to characterize them among other conformally covariant operators with leading part a power of the Laplacian.

Explicit formulas for $P_2$ and $P_4$ are well-known. These operators coincide with the Yamabe-operator

$$\Delta - \left(\frac{n}{2} - 1\right)J, \quad J = \frac{\text{scal}}{2(n-1)},$$

and the Paneitz-operator

$$\Delta^2 + \delta ((n-2)Jg - 4P) d + \left(\frac{n}{2} - 2\right) \left(\frac{n}{2}J^2 - 2P^2 - \Delta J\right).$$

(1.3)

Here

$$P(g) = \frac{1}{n-2} (\text{Ric}(g) - J(g)g)$$

denotes the Schouten tensor. In (1.3), it is regarded as an endomorphism of $T^*M$.

However, starting with $N = 3$, the structure of $P_{2N}$ is much less understood. The recent theory in [J09a] introduced new ideas for unveiling the structure of high order GJMS-operators. One of the central findings in [J09a] was a recursive structure among a sequence of differential operators which are closely related to the GJMS-operators and their constant terms. This construction motivated the definition of the so-called $Q$-curvature polynomials. Finally, a thorough investigation of their recursive structure led in [J09b] to the formulation of a conjectural recursive formula for all GJMS-operators for locally conformally flat metrics.

For sufficiently small values of $N$, it is possible to confirm these formulas. In fact, one special case of these formulas is the recursive description

$$P_6^0 = \left[2(P_2P_4 + P_4P_2) - 3P_2^3\right]^0 - 48\delta(P^2d)$$

(1.4)

of the non-constant part of the GJMS-operator $P_6$ for locally conformally flat metrics. Similarly, the non-constant part of the Paneitz operator $P_4$ can be written in the form

$$P_4^0 = (P_2^2)^0 - 4\delta(Pd).$$

(1.5)
For detailed proofs of (1.4) and (1.5) we refer to [J09a].

We proceed with the formulation of the two main conjectures of [J09b]. For this purpose, we need to introduce some notation. A sequence $I = (I_1, \ldots, I_r)$ of integers $I_j \geq 1$ will be regarded as a composition of the sum $|I| = I_1 + I_2 + \cdots + I_r$. As usual, a composition of a positive integer $m$ is a representation of $m$ as a sum $m = m_1 + m_2 + \cdots + m_r$ of positive integers $m_1, m_2, \ldots, m_r$, where two representations which contain the same summands but differ in the order of the summands are regarded as different. $|I|$ will be called the size of $I$. For any composition $I$, we set

$$P_{2I} = P_{2I_1} \circ \cdots \circ P_{2I_r}. \quad (1.6)$$

For $N \geq 1$, let

$$M_{2N} = \sum_{|I|=N} m_I P_{2I} \quad (1.7)$$

with the multiplicities $m_I$ defined by

$$m_I = (-1)^{|I|}|I|! (|I| - 1)! \prod_{j=1}^{r} \frac{1}{I_j! (I_j - 1)!} \prod_{j=1}^{r-1} \frac{1}{I_j + I_{j+1}}. \quad (1.8)$$

Here, an empty product has to be interpreted as 1. In particular, $m_{(N)} = 1$ for all $N \geq 1$, and $m_I = m_{I^{-1}}$, where $I^{-1}$ is the reverse composition of $I = (I_1, I_2, \ldots, I_r)$, that is, $I^{-1} = (I_r, I_{r-1}, \ldots, I_1)$.

The first conjecture of [J09b] states that, for locally conformally flat manifolds of dimension $n \geq 3$, we have

$$M_{2N} = -c_N \delta(\mathcal{P}^{N-1}d), \quad (1.9)$$

where

$$c_N = 2^{N-1} N! (N - 1)!. \quad (1.10)$$

Eq. (1.9) is a summation formula for the non-constant part of the operator $M_{2N}$. We note that the sum (1.7) contains $2^{N-1}$ terms. Eq. (1.9) asserts that, in this sum defining the order $2N$ operator $M_{2N}$, huge cancellations take place. Since $m_{(N)} = 1$, Eq. (1.9) can be regarded as a formula for $P_{2N}^{\sigma}$ in terms of a linear combination of the second order operator $\delta(\mathcal{P}^{N-1}d)$ and compositions of lower order GJMS-operators. In that sense, it is a recursive formula for $P_{2N}^{\sigma}$. Eqs. (1.4) and (1.5) are the first two special cases.

In the present paper, we confirm (1.9) for $S^{q,p}$, by proving the following summation formula.

**Theorem 1.1.** On $S^{q,p}$, we have

$$\mathcal{M}_{4N} = (2N)! (2N-1)! \left(\frac{1}{2} - B^2 - C^2\right), \quad N \geq 1 \quad (1.11)$$

and

$$\mathcal{M}_{4N+2} = (2N+1)! (2N)! (-B^2 + C^2), \quad N \geq 0. \quad (1.12)$$
Here
\[B^2 = -\Delta g + \left(\frac{q - 1}{2}\right)^2 \quad \text{and} \quad C^2 = -\Delta g + \left(\frac{p - 1}{2}\right)^2.\] (1.13)

Note that Theorem 1.1 also determines the constant terms of all \(M_{2N}\).

In order to see that Theorem 1.1 confirms (1.9), we observe that Theorem 1.1 implies
\[M_{4N}^0 = (2N)! (2N - 1)! (\Delta g + \Delta g), \quad N \geq 1,\]
\[M_{4N+2}^0 = (2N + 1)! (2N)! (\Delta g - \Delta g), \quad N \geq 0.\]

But using
\[P = \frac{1}{2} \begin{pmatrix} 1_s & 0 \\ 0 & -1_s \end{pmatrix},\] (1.14)
these identities can be written in the form
\[M_{4N}^0 = -c_{2N} \delta(P^{2N-1} d),\]
\[M_{4N+2}^0 = -c_{2N+1} \delta(P^{2N} d),\]
where \(c_{2N}\) and \(c_{2N+1}\) are defined in (1.10).

The second conjecture of [J09b] concerns the constant terms of GJMS-operators. These give rise to Branson’s \(Q\)-curvatures. More precisely, for even \(n\) and \(2N < n\), the equation
\[P_{2N}(1) = (-1)^N \left(\frac{n}{2} - N\right) Q_{2N}\]
defines a scalar curvature quantity \(Q_{2N}\). The quantities \(Q_2, \ldots, Q_{n-2}\) are called the subcritical \(Q\)-curvatures. The critical \(Q\)-curvature \(Q_n\) can be defined through the subcritical ones by continuation in dimension. For the details we refer to [B95] and [J09a].

Explicit formulas for \(Q_2\) and \(Q_4\) follow from (1.2) and (1.3):
\[Q_2 = J \quad \text{and} \quad Q_4 = \frac{n}{2} j^2 - 2|P|^2 - \Delta J.\]

For \(N \geq 3\), explicit formulas for \(Q_{2N}\) are substantially more complicated. Therefore, it is of some interest to establish recursive formulas for \(Q\)-curvatures. It is the main feature of the second conjecture to propose such a formula. In order to motivate its formulation, we start with the description of two special cases. First of all, we rewrite \(Q_4\) in the form
\[Q_4 = -P_2(Q_2) - Q_2^2 + 2! 2^3 v_4,\] (1.15)
where
\[v_4 = \frac{1}{4} \text{tr}(\wedge^2 P).\]

Next, for locally conformally flat metrics, we have (see [J09a])
\[Q_6 = \left[-2P_2(Q_4) + 2P_4(Q_2) - 3P_2^2(Q_2)\right] - 6 \left[Q_4 + P_2(Q_2)\right] Q_2 - 2! 3! 2^5 v_6,\] (1.16)
where
\[v_6 = -\frac{1}{8} \text{tr}(\wedge^3 P).\]
More conceptually, the coefficients $v_4$ and $v_6$ are Taylor coefficients of the function

$$v(r) = \det \left(1 - \frac{r^2}{4} P\right) = 1 + v_2 r^2 + v_4 r^4 + v_6 r^6 + \cdots,$$  

(1.17)

which describes the volume form of a Poincaré–Einstein metric associated to the locally conformally flat metric $g$ (see [G00], [FG07]). Eq. (1.17) implies

$$v_{2k} = \left(-\frac{1}{2}\right)^k \text{tr}(\wedge^k P).$$

Now, one can prove that the formulas (1.15) and (1.16) are equivalent to the respective identities

$$Q_4 + P_2(Q_2) = 2! 2^4 w_4$$  

(1.18)

and

$$-Q_6 - 2P_2(Q_4) + 2P_4(Q_2) - 3P_2^2(Q_2) = 2! 3! 2^6 w_6,$$  

(1.19)

where the quantities $w_4$ and $w_6$ are Taylor coefficients of

$$w(r) = \sqrt{v(r)},$$

i.e.,

$$(1 + r^2 w_2 + r^4 w_4 + r^6 w_6 + \cdots)^2 = v(r).$$

More explicitly, we have

$$2w_2 = v_2,$$

$$2w_4 = \frac{1}{4}(4v_4 - v_2^2),$$

$$2w_6 = \frac{1}{8}(8v_6 - 8v_2v_4 + v_2^3).$$

In these terms, the second conjecture states that

$$\sum_{|I|=N} m_I \frac{P_{2I}(1)}{n - I_{\text{last}}} = N! (N-1)! w_{2N}$$  

(1.20)

for locally conformally flat metrics and $N \geq 1$. Here, $I_{\text{last}}$ denotes the last entry of the composition $I$. For $I = (J, \frac{n}{2})$, the quotient in (1.20) is to be interpreted as

$$(1)^{\frac{n}{2}} P_{2J}(Q_n).$$

Again, the sum in (1.20) contains $2^{N-1}$ terms. By separating the contribution for $I = (N)$, Eq. (1.20) can be regarded as a formula for $Q_{2N}$. It expresses $Q_{2N}$ in terms of lower order $Q$-curvatures, lower order GJMS-operators and $w_{2N}$. Eqs. (1.18) and (1.19) are the first two special cases.

We confirm (1.20) for $S^q,p$ by proving the following summation formula.

**Theorem 1.2.** On $S^q,p$, we have

$$\sum_{|I|=N} m_I \frac{P_{2I}(1)}{n - I_{\text{last}}} = N! (N-1)! \sum_{M=0}^{N} (-1)^M \left(\frac{\frac{q}{2}}{M}\right) \left(\frac{\frac{p}{2}}{N-M}\right)$$  

(1.21)
for all $N \geq 1$.

In order to see that Theorem 1.2 confirms (1.20), we note that (1.14) and (1.17) imply
\[ v(r) = (1 - r^2/4)^q(1 + r^2/4)^p. \]
Hence
\[ w(r) = (1 - r^2/4)^{q/2}(1 + r^2/4)^{p/2}. \]
It follows that the right-hand side of (1.21) coincides with
\[ N! (N - 1)! 2^{2N} w_{2N}. \]

The remaining part of the paper is organized as follows. In Section 2, we prove Theorem 1.1. Section 3 contains the proof of a summation formula which contains Theorem 1.2 as a special case. The proofs of some technical results which are used in the course of these proofs are collected in an appendix.

\section{Proof of Theorem 1.1}

We start with a description of the GJMS-operators on $S^{q,p}$.

\textbf{Theorem 2.1} ([B95], Theorem 6.2). On $S^{q,p}$, the GJMS-operators are given by the product formulas
\[ P_{4N} = \prod_{j=1}^{N} (B + C + (2j - 1))(B - C - (2j - 1))(B + C - (2j - 1))(B - C + (2j - 1)) \] (2.1)
and
\[ P_{4N+2} = (-B^2 + C^2) \prod_{j=1}^{N} (B + C + 2j)(B - C - 2j)(B + C - 2j)(B - C + 2j). \] (2.2)

Here, the operators $B$ and $C$ are defined as the positive square roots of the non-negative operators $B^2$ and $C^2$.

The reader should note that the formulas (2.1) and (2.2) are equivalent to the product representations
\[ P_{4N} = \prod_{j=1}^{N} ((B^2 - C^2)^2 - 2(2j - 1)^2(B^2 + C^2) + (2j - 1)^4) \] (2.3)
and
\[ P_{4N+2} = (-B^2 + C^2) \prod_{j=1}^{N} ((B^2 - C^2)^2 - 2(2j)^2(B^2 + C^2) + (2j)^4). \] (2.4)

An advantage of the latter formulas is that they do not require to leave the framework of differential operators.

The formulas (2.1) and (2.2) can be restated uniformly as
\[ P_{2N} = 2^{2N} \left( (C + B + 1 - N)/2 \right)_N \left( (C - B + 1 - N)/2 \right)_N, \] (2.5)
where \((\alpha)_m\) is the usual Pochhammer symbol defined by \((\alpha)_m = \alpha(\alpha+1) \cdots (\alpha+m-1)\) for \(m \geq 1\), and \((\alpha)_0 = 1\). In terms of the variables

\[ X = C + B \quad \text{and} \quad Y = C - B, \]

the product formula (2.5) is equivalent to

\[ P_{2N} = 2^{2N} \left( (X + 1 - N)/2 \right)_N \left( (Y + 1 - N)/2 \right)_N. \]

In the sequel, we shall regard \(P_{2N}\) as this polynomial in the variables \(X\) and \(Y\).

In the proofs of Theorems 1.1 and 1.2, the basis \(\left( ((X+1-A)/2)_A \right)_{A=0,1,...}\) of the linear space of all polynomials in \(X\) will play an essential role. (A glance at (2.7) may suggest why this could be the case.) In the following lemma, we compute the structure coefficients with respect to multiplication of this basis of polynomials. They will enter the inductive proofs of Lemma 2.2 and of Theorems 2.2 and 3.1.

**Lemma 2.1.** For all non-negative integers \(A\) and \(B\), we have

\[ ((X + 1 - A)/2)_A \left( (X + 1 - B)/2 \right)_B \]

\[ = \sum_{j=0}^{[(A+B)/2]} (-1)^j \frac{(-A/2)_j (-B/2)_j (-A+B)/2)_j}{j!} \left( (X + 1 - A - B + 2j)/2 \right)_{A+B-2j}. \]

(2.8)

**Remark 2.1.** The sum on the right-hand side of (2.8) terminates latest at \(j = [(A+B)/2]\). If \(A\) or \(B\) should be even, this happens already at \(j = A/2\), respectively at \(j = B/2\).

**Proof.** The reader should recall the standard hypergeometric notation

\[ _pF_q \left[ a_1, \ldots, a_p; b_1, \ldots, b_q; z \right] = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{m! (b_1)_m \cdots (b_q)_m} z^m \]

where \((a_1)_m\), etc. are again Pochhammer symbols. In terms of this notation, the right-hand side of (2.8) reads

\[ ((X + 1 - A - B)/2)_{A+B} _3F_2 \left[ -A/2, -B/2; 1 \right] \left( (X - A - B)/2, (1 - X - A - B)/2 \right). \]

The \(_3F_2\)-series can be evaluated by means of the Pfaff–Saalschütz summation (cf. [S66, (2.3.1.3); Appendix (III.2)])

\[ _3F_2 \left[ a, b, -n; c, 1 + a + b - c - n \right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}, \]

where \(n\) is a non-negative integer. If we apply the formula, then we obtain the left-hand side of (2.8) after little simplification. \(\square\)

**Lemma 2.2.** For all positive integers \(a < N\), the partial sum

\[ S(N, a) = \sum_{J: |J| = a = N} m_{(J,a)} P_{2J} \]

(2.9)
satisfies
\[
S(N, a) = \binom{N-1}{a-1} \sum_{k=0}^{\lfloor(N-a)/2 \rfloor} \sum_{l=0}^{\lfloor(N-a)/2 \rfloor} (-1)^{N+k+l+a} 2^{2N-2k-2l-2a} \\
\cdot (X + 1 - N + a + 2k)/2)_{N-a-2k} \left((Y + 1 - N + a + 2l)/2\right)_{N-a-2l} \\
\cdot (-N + a)_{2k} (-N + a + b)_{2l} (-N/2)_k (-N/2)_l 4F3 \left[ \begin{array}{c} -\frac{1}{2}, -k, -l, \frac{1}{2} - \frac{N}{2} \end{array} \right| \frac{a}{2} - \frac{N}{2}, \frac{b}{2} - \frac{N}{2}, \frac{1}{2} + \frac{a}{2} - \frac{N}{2}, \frac{1}{2}; 1 \right]. \quad (2.10)
\]

Proof. We prove the claim by induction on \( N \), the case \( N = 2 \) being straightforward to check. Suppose that we have already established (2.10) up to \( N - 1 \). If \( a + |J| = N \), we write \( m_{(a, J)} \) as
\[
m_{(a, J)} = \begin{cases} \\
- \frac{N! (N - 1)!}{(N - a)! (N - a - 1)! N \cdot a! (a - 1)!} m_{(N-a)}, & \text{if } J = (N - a), \\
- \frac{N! (N - 1)!}{(N - a)! (N - a - 1)! (a + b) a! (a - 1)!} m_{(b, K)}, & \text{if } J = (b, K), \\
\end{cases}
\]
and, if \( J = (b, K) \), we write \( P_{2J} = P_{2b}P_{2K} \). Then the left-hand side of (2.10) becomes
\[
- \frac{N! (N - 1)!}{N \cdot a! (a - 1)! (N - a)! (N - a - 1)!} P_{2(N-a)} \\
- \sum_{b=1}^{N-a-1} \sum_{K: b+|K| = N-a} \frac{N! (N - 1)!}{(a + b) a! (a - 1)! (N - a)! (N - a - 1)!} m_{(b, K)} P_{2b}P_{2K}. \quad (2.11)
\]
If we now use the induction hypothesis, then this sum simplifies to
\[
- \sum_{b=1}^{N-a} \frac{N! (N - 1)!}{(a + b) a! (a - 1)! (N - a)! (N - a - 1)!} 2^{2b} ((X + 1 - b)/2)_b ((Y + 1 - b)/2)_b \\
\cdot \binom{N-a-b}{b-1} \sum_{k=0}^{\lfloor(N-a-b)/2 \rfloor} \sum_{l=0}^{\lfloor(N-a-b)/2 \rfloor} (-1)^{N+k+l+a+b} 2^{2N-2k-2l-2a-2b} \\
\cdot (X + 1 - N + a + b + 2k)/2)_{N-a-b-2k} \left((Y + 1 - N + a + b + 2l)/2\right)_{N-a-b-2l} \\
\cdot (-N + a + b)_{2k} (-N + a + b + 2l)_{2l} (-N + a + b)_{2l} ((-N + a)/2)_k ((-N + a)/2)_l \\
\cdot 4F3 \left[ \begin{array}{c} a \end{array} \right| \frac{a}{2} - \frac{N}{2}, \frac{b}{2} - \frac{N}{2}, \frac{1}{2} + \frac{a}{2} - \frac{N}{2}, \frac{1}{2}; 1 \right]. \quad (2.12)
\]
Here, the term containing \( P_{2(N-a)} \), appearing in (2.11), arises as the summand for \( b = N - a \), since this forces \( k \) and \( l \) to be zero.
Now, Lemma [2.1] implies the expansions
\[
((X + 1 - b)/2)_b ((X + 1 - N + a + b + 2k)/2)_{N-a-b-2k}
\]
\[
\sum_{j_1=0}^{\lfloor (N-a-2k)/2 \rfloor} (-1)^{j_1} \frac{(b/2)_{j_1} (- (N - a - b - 2k)/2)_{j_1} (-(N - a - 2k)/2)_{j_1}}{j_1!} \\
\cdot \left( (X + 1 - N + a + 2k + 2j_1)/2 \right)_{N+a-2k-2j_1}.
\]

and

\[
\sum_{j_2=0}^{\lfloor (N-a-2l)/2 \rfloor} (-1)^{j_2} \frac{(b/2)_{j_2} (- (N - a - b - 2l)/2)_{j_2} (-(N - a - 2l)/2)_{j_2}}{j_2!} \\
\cdot \left( (Y + 1 - N + a + 2l + 2j_2)/2 \right)_{N+a-2l-2j_2}.
\]

We use these in (2.12) and, in addition, perform the index transformation \(s_1 = k + j_1\) and \(s_2 = l + j_2\). Thus, we arrive at the expression

\[
\sum_{s_1=0}^{\lfloor (N-a)/2 \rfloor} \sum_{s_2=0}^{\lfloor (N-a)/2 \rfloor} \left( (X + 1 - N + a + 2s_1)/2 \right)_{N-a-2s_1} \left( (Y + 1 - N + a + 2s_2)/2 \right)_{N-a-2s_2} \\
\cdot \frac{N!(N-1)!}{(a+b)! (a-1)! (N-a)! (N-a-1)!} \left( \frac{N-a-1}{b-1} \right) (-1)^{N+s_1+s_2+a+b} 2^{N-2a} \\
\cdot \sum_{k=0}^{\lfloor (N-a-b)/2 \rfloor} \sum_{l=0}^{\lfloor (N-a-b)/2 \rfloor} \frac{(-b/2)_{s_1-k} (-b/2)_{s_2-l} ((-N + a)/2)_{s_1} ((-N + a)/2)_{s_2}}{k! l! (s_1-k)! (s_2-l)!} \\
\cdot \left( (-N + a + b + 1)/2 \right)_{s_1} \left( (-N + a + b + 1)/2 \right)_{s_2} \frac{(_4 F_3)}{2} \left[ \frac{a - \frac{1}{2} - k, -l, \frac{1}{2} + b - \frac{N}{2}, \frac{1}{2} + a + \frac{b}{2} - \frac{N}{2}; 1}{a} \right].
\]

In this expression, we now concentrate on the terms involving the summation index \(k\) only:

\[
\sum_{k=0}^{s_1} \frac{(-b/2)_{s_1-k} (- (N - a - b - 1)/2)_{s_1-k}}{k! (s_1-k)!},
\]

where \(s\) stands for the summation index of the \(_4 F_3\)-series in (2.19). Because of the term \((-k)_s\) in the numerator of the summand, we may start the summation at \(k = s\) (instead of at \(k = 0\)). Hence, if we write this sum in hypergeometric notation, we obtain

\[
\frac{(-b/2)_{s_1-s} (- (N - a - b - 1)/2)_{s_1-s}}{s!} \frac{(_2 F_1)}{s} \left[ \frac{-s_1 + s, \frac{1}{2} + a - \frac{N}{2} + b + s}{1 + \frac{1}{2} - s_1 + s} ; 1 \right].
\]
This $2F_1$-series can be evaluated by means of the Chu–Vandermonde summation formula (see \cite{S66} (1.7.7), Appendix (III.4)), so that the sum in (2.14) equals
\[
(-s_1)_s ((N - a - b - 1)/2)_s ((1 + a - N)/2)_s.
\]

Clearly, an analogous computation can be done for the sum over $l$ in (2.13). Altogether, we see that (2.13) simplifies to
\[
- \sum_{s_1=0}^{[(N-a)/2]} \sum_{s_2=0}^{[(N-a)/2]} \left( (X+1-N+a+2s_1)/2 \right)_{N-a-2s_1} \left( (Y+1-N+a+2s_2)/2 \right)_{N-a-2s_2}
\]
\[
\cdot \sum_{b=1}^{N-a} (-1)^{N+s_1+s_2+a+b} 2^{N-2s_1-2s_2-2a} N! (N-1)! \frac{(N-a-1)!}{(a+b)! (a-1)! (N-a)! (b-1)!}
\]
\[
\cdot \left( (-N+a)_2s_1 (-N+a)_2s_2 \right)_{s_1-s} \left( (-N+a+b+2s)/2 \right)_{s_2-s} \frac{s! s_2!}{s! (N-a)! (N-a-b-2s)!}.
\]

(2.15)

Next we concentrate on the terms involving the summation index $b$ only:
\[
\frac{1}{(N-a-2s-1)!} \sum_{b=1}^{N-a} (-1)^b \frac{(N-a-2s-1)}{a+b} \left( (-N+a+b+2s)/2 \right)_{s_1-s} \left( (-N+a+b+2s)/2 \right)_{s_2-s}.
\]

By Lemma A.2 with $X = -a - 1$ and $M = N-a-2s-1$, this is equal to
\[
\frac{(-1)^{N-a-2s-1}}{(-N+2s)_{N-a-2s}} ((-N+2s)/2)_{s_1-s} ((-N+2s)/2)_{s_2-s}
\]
\[
+ \chi(s_1 = s_2 = (N-a)/2) \cdot 2^{-N+a+2s}
\]
\[
= - \frac{a!}{(N-2s)!} \frac{(-N/2)_{s_1} (-N/2)_{s_2}}{(-N/2)_s^2} + \chi(s_1 = s_2 = (N-a)/2) \cdot 2^{-N+a+2s},
\]
where $\chi(S) = 1$ if $S$ is true and $\chi(S) = 0$ otherwise. If we substitute this in (2.15), then we obtain

$$
\sum_{s_1=0}^{[(N-a)/2]} \sum_{s_2=0}^{[(N-a)/2]} \left( (X+1-N+a+2s_1)/2 \right)_{N-a-2s_1} \left( (Y+1-N+a+2s_2)/2 \right)_{N-a-2s_2} 
\times (-1)^{N+s_1+s_2+a} 2^{N-2s_1-2s_2-2a} \frac{(N-1)!}{(a-1)! (N-a)!} 
\times \left( -N+a \right)_{2s_1} \left( -N+a \right)_{2s_2} \frac{(N-a/2)_{s_1} (N-a/2)_{s_2}}{s_1! s_2!} 
\times \sum_{s=0}^{s_1} \frac{(-1/2)_s (-s_1)_s (-s_2)_s N! (N-a-2s)!}{s! (N-2s)! (-N/2)_s^2 (N-a)!} 
\times \left( -N-a + 2s \right) \left( -N/2 \right)_s \left( N-a-2s \right)! 
\times \frac{N! (N-1)!}{a! (a-1)! ((N-a)/2)!^2} 
\times \sum_{s=0}^{(N-a)/2} \frac{2^{-N+a+2s} (-1/2)_s \left( -N-a/2 \right)_s^2 \left( N-a-2s \right)!}{s!} 
\times \chi(N-a \text{ is even}) \cdot \frac{N! (N-1)!}{a! (a-1)! ((N-a)/2)!^2} 
\times \sum_{s=0}^{(N-a)/2} \frac{2^{-N+a+2s} \left( -N-a/2 \right)_s^2 \left( N-a-2s \right)!}{s!} 
\times \chi(N-a \text{ is even}) \cdot \frac{N! (N-1)!}{a! (a-1)! ((N-a)/2)!^2} \frac{2^{-N+a} \left( N-a \right)!_{2F1} \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \right]}{\frac{a}{2} - \frac{N}{2} + \frac{1}{2} }.
$$

Here, the first sum is, upon rewriting, exactly equal to the right-hand side of (2.10) (except that $s_1$ and $s_2$ took over the role of $k$ and $l$). On the other hand, if we write the second sum in hypergeometric notation, we obtain

$$
\chi(N-a \text{ is even}) \cdot \frac{N! (N-1)!}{a! (a-1)! ((N-a)/2)!^2} 
\times \sum_{s=0}^{(N-a)/2} \frac{2^{-N+a+2s} \left( -N-a/2 \right)_s^2 \left( N-a-2s \right)!}{s!} 
\times \chi(N-a \text{ is even}) \cdot \frac{N! (N-1)!}{a! (a-1)! ((N-a)/2)!^2} \frac{2^{-N+a} \left( N-a \right)!_{2F1} \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \right]}{\frac{a}{2} - \frac{N}{2} + \frac{1}{2} }.
$$

The $2F1$-series can be evaluated by the Chu–Vandermonde summation formula, so that the above expression becomes

$$
\chi(N-a \text{ is even}) \cdot \frac{N! (N-1)!}{a! (a-1)! ((N-a)/2)!^2} 2^{-N+a} \left( N-a \right)!_{2F1} \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \right] /
\frac{\left( (a-N+2)/2 \right)_{(N-a)/2} \left( (a-N+1)/2 \right)_{(N-a)/2}}{((a-N+2)/2)_{(N-a)/2}}.
$$

which vanishes because of the term $((a-N+2)/2)_{(N-a)/2}$. This concludes the induction step.

\textbf{Theorem 2.2.} For all positive integers $N$, we have

$$
\sum_{|I|=N} m_I P_{2I} = \begin{cases} 
N! (N-1)! XY, & \text{if } N \text{ is odd}, \\
\frac{1}{2} N! (N-1)! (1 - X^2 - Y^2), & \text{if } N \text{ is even}, 
\end{cases}
$$

where $P_{2N}$ is defined in (2.1).
Proof. By the definition of $S(N, a)$ in (2.9), we have

$$
\sum_{|l|=N} m_l P_{2l} = P_{2N} + \sum_{a=1}^{N-1} S(N, a) P_{2a}.
$$

(2.17)

If, in this equation, we substitute the expressions for $P_{2a}$ and $S(N, a)$ given by (2.7) and (2.10), then we see that the left-hand side of (2.16) is equal to

$$
\sum_{a=1}^{N} \left( (X + 1 - a)/2 \right)_a \left( (Y + 1 - a)/2 \right)_a \left( N - 1 \right)_{a-1} \cdot
\sum_{k=0}^{[(N-a)/2]} \sum_{l=0}^{[(N-a)/2]} (-1)^{N+k+l+a} 2^{N-2-2k-2l} \cdot
\frac{((X + 1 - N + a + 2k)/2)_{N-a-2k} \left( (Y + 1 - N + a + 2l)/2 \right)_{N-a-2l}}{k! l!} \cdot
\frac{(-N + a)_{2k} (-N + a)_{2l} (-N/2)_k (-N/2)_l}{4F3 \left[ -\frac{1}{2}, -k, l, \frac{1}{2} - \frac{N}{2} ; 1 \right]}
$$

(2.18)

Here, the term $P_{2N}$, appearing on the left-hand side of (2.16), arises as the summand for $a = N$, since this forces $k$ and $l$ to be zero.

Lemma 2.1 implies the expansions

$$
\left( (X + 1 - a)/2 \right)_a \left( (X + 1 - N + a + 2k)/2 \right)_{N-a-2k} = \sum_{j_1=0}^{[(N-2k)/2]} (-1)^{j_1} (a/2)_{j_1} \left(- (N - a - 2k)/2\right)_{j_1} \left(- (N - 2k)/2\right)_{j_1} \cdot
$$

and

$$
\left( (Y + 1 - a)/2 \right)_a \left( (Y + 1 - N + a + 2l)/2 \right)_{N-a-2l} = \sum_{j_2=0}^{[(N-2l)/2]} (-1)^{j_2} (a/2)_{j_2} \left(- (N - a - 2l)/2\right)_{j_2} \left(- (N - 2l)/2\right)_{j_2} \cdot
$$

We use these in (2.18) and, in addition, perform the index transformation $s_1 = k + j_1$ and $s_2 = l + j_2$. Thus, the left-hand side in (2.16) can be written in the form

$$
\sum_{s_1=0}^{[N/2]} \sum_{s_2=0}^{[N/2]} \sum_{a=1}^{N} (-1)^{N+s_1+s_2+a} 2^{N-2-2k-2l} \left( N - 1 \right)_{a-1} \cdot
\left( (X + 1 - N + 2s_1)/2 \right)_{N-2s_1} \left( (Y + 1 - N + 2s_2)/2 \right)_{N-2s_2}
$$
Observe only: \(k\)-term (2.19). Clearly, an analogous computation can be done for the sum over \(m\), so that the sum in (2.20) equals

\[
\sum_{k=0}^{[(N-a)/2]} 2^{-2k} \frac{(-a/2)_{s_1-k}(-N+a)_{2k}(-k)_s}{k!(s_1-k)!} (-N-a/2)_k
\]

where \(s\) stands for the summation index of the \(4F_3\)-series in (2.19). Because of the term \((-k)_s\) in the numerator of the summand, we may start the summation at \(k = s\) (instead of at \(k = 0\)). Hence, if we write this sum in hypergeometric notation, we obtain

\[
\frac{(-a/2)_{s_1-s}(-(N-a-1)/2)_s(-s)_s}{s!} _2F_1\left[-s_1+s,\frac{1}{2}-\frac{N}{2}+\frac{a}{2}+s; 1\right].
\]

This \(2F_1\)-series can be evaluated by means of the Chu–Vandermonde summation formula, so that the sum in (2.20) equals

\[
\frac{(-s_1)_s(-(N-a-1)/2)_s ((1-N)/2)_{s_2}}{s_1!((1-N)/2)_s}.
\]

Clearly, an analogous computation can be done for the sum over \(l\) in (2.19). Altogether, we see that (2.19) simplifies to

\[
\sum_{s_1=0}^{[N/2]} \sum_{s_2=0}^{[N/2]} \sum_{a=1}^{N} (-1)^{N+s_1+s_2+a} 2^{2N} \binom{N-1}{a-1}
\]

\[
\cdot (-N/2)_{s_1}((1-N)/2)_{s_1}(-N/2)_{s_2}((1-N)/2)_{s_2}
\]

\[
\cdot \binom{X+1-N+2s_1}{N-2s_1} \binom{Y+1-N+2s_2}{N-2s_2}
\]

\[
\cdot (-N-a/2)_{s_1}(-(N-a)/2)_{s_2} \frac{(-1/2-s_1,-s_2,1/2+\frac{a}{2}-\frac{N}{2}; 1)}{s_1!s_2!} _4F_3\left[-\frac{1}{2},-s_1,-s_2,\frac{1}{2}+\frac{a}{2}-\frac{N}{2}; 1\right].
\]

Next we concentrate on the terms involving the summation index \(a\) only:

\[
\sum_{a=1}^{N} (-1)^a \binom{N-1}{a-1} (-N-a/2)_{s_1}(-(N-a)/2)_{s_2}(-(N-a-1)/2)_{s_2} (-N-a/2)_{s_1}
\]

\[
= \frac{(N-1)!}{2^{2s}(N-2s-1)!} \sum_{a=1}^{N-2s} (-1)^a \binom{N-2s-1}{a-1}
\]

\[
\cdot (-N-a-2s/2)_{s_1-s}(-(N-a-2s)/2)_{s_2-s}
\]
\[
\begin{align*}
&= -\frac{(N-1)!}{2^{2s}(N-2s-1)!} \sum_{a=0}^{N-2s-1} (-1)^a \binom{N-2s-1}{a} \cdot ((a-N+2s+1)/2)_{s_1-s} ((a-N+2s+1)/2)_{s_2-s}, \\
\end{align*}
\]

where, in abuse of notation, we wrote again \(s\) for the summation index of the \(\,\_\!\!\_\!\!\_\!\!\,F_3\)-series in (2.21). By Lemma A.1 this sum vanishes if the degree of
\[
((a-N+2s+1)/2)_{s_1-s} ((a-N+2s+1)/2)_{s_2-s}
\]
as a polynomial in \(a\) should be less than \(N-2s-1\). In fact, this is almost always the case since \(s_1-s+s_2-s \leq 2\lfloor N/2 \rfloor - 2s \leq N-2s\). More precisely, the sum in (2.22) is non-zero only if:

1. \(N\) is odd and \(s_1 = s_2 = (N-1)/2\);
2. \(N\) is even and \(s_1 = s_2 = N/2\) or \(\{s_1, s_2\} = \{N/2, (N-2)/2\}\).

We now discuss these cases separately.

1. Let \(N\) be odd. Then, as we observed above, in the sum (2.21) only the terms corresponding to \(s_1 = s_2 = (N-1)/2\) survive. Using Lemma A.1 we see that (2.21) reduces to
\[
-4 (-N)^{N-1} \left(\frac{X}{2}\right) \left(\frac{Y}{2}\right) \times \sum_{a=0}^{N-1/2} (-1)^{N-2s_2}2^{-N+2s+1} \frac{(N-1)!}{2^{2s}((N-1)/2)!} \frac{(-1/2)_s ((1-N)/2)_s}{s!(-N/2)_s} = 2^{1-N}XY \frac{N^2 (N-1)!}{((N-1)/2)!^2} \, _2F_1\left[-\frac{1}{2}, \frac{1}{2} - \frac{N}{2}; 1\right].
\]
The \(\,\_\!\!\_\!\!\_\!\!\_\!\!\,F_1\)-series can be evaluated by means of the Chu–Vandermonde summation formula. After some simplification, we obtain \(N! (N-1)! XY\), in accordance with the claim in (2.10).

2. Let \(N\) be even. As we observed above, there are now three terms in (2.21) which survive: the terms for \(s_1 = s_2 = N/2\), for \((s_1, s_2) = (N/2, (N-2)/2)\), and for \((s_1, s_2) = ((N-2)/2, N/2)\).

We begin with the term for \(s_1 = N/2\) and \(s_2 = (N-2)/2\). By Lemma A.1 this term equals
\[
-4 (-N)_N (-N)_{N-2} \left((Y-1)/2\right)_2 \times \sum_{a=0}^{(N-2)/2} (-1)^{N-2s_2}2^{-N+2s+1} \frac{(N-1)!}{2^{2s}(N/2)!((N-2)/2)!} \frac{(-1/2)_s ((2-N)/2)_s}{s!((1-N)/2)_s} = -2^{-N}Y^2 (N^2 (N-1)!){(N/2)!((N-2)/2)!} \, _2F_1\left[-\frac{1}{2}, \frac{1}{2} - \frac{N}{2}; 1\right].
\]
Again, the \(\,\_\!\!\_\!\!\_\!\!\,F_1\)-series can be evaluated by means of the Chu–Vandermonde summation formula. After some simplification, we obtain \(-\frac{1}{4}N! (N-1)! (Y^2 - 1)\). Clearly, an
analogous computation yields that the term for \( s_1 = (N - 2)/2 \) and \( s_2 = N/2 \) equals \(-\frac{1}{2} N! (N - 1)! (X^2 - 1)\). Finally, we consider the term in (2.21) corresponding to \( s_1 = s_2 = N/2 \). Here, in order to apply Lemma A.1 we need to compute the coefficient \( c_{N-2s-1} \) in the expansion

\[
((a - N + 2s + 1)/2)^{(N-2s)/2} ((a - N + 2s + 1)/2)^{(N-2s)/2} = \sum_{k=0}^{N-2s} c_k \binom{a}{k}.
\]

We have \( c_{N-2s} = 2^{-N+2s}(N-2s)! \). Comparison of coefficients of \( a^{-N-2s-1} \) on both sides of (2.23) then yields that

\[
c_{N-2s-1} = 2^{-N+2s} \left( -2 \sum_{\ell=1}^{(N-2s)/2} (2\ell - 1) + \sum_{\ell=1}^{(N-2s-1)/2} \ell \right) \\
= 2^{-N+2s} \left( -2 \left( \frac{N - 2s}{2} \right)^2 + \frac{(N - 2s)(N - 2s - 1)}{2} \right) \\
= -2^{-N+2s-1}(N - 2s).
\]

Hence, if we use this, together with Lemma A.1 we obtain that the term corresponding to \( s_1 = s_2 = N/2 \) in (2.21) equals

\[
\frac{(N-1)!}{(N/2)!^2} \left( N \; _2F_1 \left[ \begin{array}{c} \frac{1}{2}, -\frac{N}{2} \\ \frac{N}{2} \end{array} \middle| 1 \right] + \frac{N}{N - 1} \; _2F_1 \left[ \begin{array}{c} \frac{1}{2}, 1 - \frac{N}{2} \\ \frac{N}{2} \end{array} \middle| 1 \right] \right).
\]

Both \( _2F_1 \)-series can be evaluated by means of the Chu–Vandermonde summation formula. As it turns out, the first one sums to zero. If the result is simplified, one obtains \(-\frac{1}{2} N! (N - 1)! \) eventually.

Putting our results together, we obtain

\[
\frac{1}{2} N! (N - 1)! (-X^2 + 1 - Y^2 + 1 - 1) = \frac{1}{2} N! (N - 1)! (1 - X^2 - Y^2)
\]

for the left-hand side of (2.16), which is in accordance with the right-hand side. \(\Box\)

3. Proof of Theorem 1.2

As in Section 2 we regard all GJMS-operators as polynomials in the variables \( X \) and \( Y \) (see (2.7)). In particular, \( P_{2N} \) is divisible by the monomial \((X + 1 - N)\).

**Theorem 3.1.** For all positive integers \( N \), we have

\[
\sum_{a=1}^{N} \sum_{|J|=N-a} m_{(j,a)} \frac{1}{X + 1 - a} P_{2j} P_{2a}
\]
\[ = N!(N-1)! \sum_{k=0}^{N} (-1)^k \binom{(X - Y + 1)/2}{k} \binom{(X + Y + 1)/2}{N - k}. \quad (3.1) \]

**Proof.** By the definition of \( S(N, a) \) in (2.9), we have

\[
\sum_{a=1}^{N} \sum_{|J|=N-a} m_{(J,a)} \frac{1}{X + 1 - a} P_{2J} P_{2a} = \frac{1}{X + 1 - N} P_{2N} + \sum_{a=1}^{N-1} \frac{1}{X + 1 - a} S(N, a) P_{2a}.
\]

The reader should note that the only difference to (2.17) are the factors \( 1/(X + 1 - N) \) in front of \( P_{2N} \) and the factor \( 1/(X + 1 - a) \) in front of the summand of the sum over \( a \). We may therefore proceed as in the proof of Theorem 2.2, as long as the summation over \( a \) does not come into play. More precisely, by comparing with (2.21), we see that the left-hand side of (3.1) simplifies to

\[
\sum_{s_1=0}^{[N/2]} \sum_{s_2=0}^{[N/2]} \sum_{a=1}^{N} (-1)^{N+s_1+s_2+a} 2^{2N} \binom{N-1}{a-1} \\
\cdot (-N/2)_{s_1} ((1-N)/2)_{s_1} (-N/2)_{s_2} ((1-N)/2)_{s_2} \\
\cdot \frac{1}{X + 1 - a} (X + 1 - N + 2s_1)/2 \binom{(X + 1 - N + 2s_2)/2}{N - 2s_1} \\
\cdot (-N-a/2)_{s_1} (-N-a/2)_{s_2} \text{ }_{4}F_{3} \left[ \frac{-1/2, -s_1, -s_2, 1/2 + a/2 - N/2}{-N/2, 1/2 - N/2, a/2 - N/2}; 1 \right]. \quad (3.2)
\]

As in the proof of Theorem 2.2, we now concentrate on the terms involving the summation index \( a \) only:

\[
\sum_{a=1}^{N} (-1)^{a} \frac{1}{X + 1 - a} \binom{N-1}{a-1} \frac{(-N-a/2)_{s_1} (-N-a/2)_{s_2} (-N-a-1/2)_{s_2}}{(-(N-a)/2)_{s}} \\
\left[ \frac{(N-1)!}{2^{2s} (N-2s-1)!} \sum_{a=1}^{N-2s} (-1)^{a} \frac{1}{X + 1 - a} \binom{N-2s-1}{a-1} \right] \\
\cdot (-N-a-2s/2)_{s_{1-s}} (-N-a-2s/2)_{s_{2-s}} \\
\left[ \frac{(N-1)!}{2^{2s} (N-2s-1)!} \sum_{a=0}^{N-2s-1} (-1)^{a} \frac{1}{X - a} \binom{N-2s-1}{a} \right] \\
\cdot ((a-N+2s+1)/2)_{s_{1-s}} ((a-N+2s+1)/2)_{s_{2-s}},
\]

where, in abuse of notation, we wrote again \( s \) for the summation index of the \( _4F_3 \)-series in (3.2). By Lemma A.2 with \( M = N - 2s - 1 \), this is equal to

\[
\frac{(-1)^{N-2s} (N-1)!}{2^{2s}(X-N+2s+1)_{N-2s}} ((X + 1 - N + 2s)/2)_{s_{1-s}} ((X + 1 - N + 2s)/2)_{s_{2-s}} \\
\cdot \chi(s_1 = s_2 = N/2) \cdot 2^{-N}(N-1)!. \quad (3.3)
\]
We substitute this in (3.2). As before in the proof of Lemma 2.2 the second term in (3.3) does not contribute anything since the (remaining) sum over \( s \) vanishes. Therefore, substitution of (3.3) in (3.2) leads to the expression

\[
\sum_{s_1=0}^{\lfloor N/2 \rfloor} \sum_{s_2=0}^{\lfloor N/2 \rfloor} (-1)^{s_1+s_2} N^{-2s} \left( (X + 1 - N + 2s) / 2 \right)_{N-s \cdot s_1} \left( (X + 1 - N + 2s) / 2 \right)_{s_2-s} (s_1 - s)! (X - N + 2s + 1)_{N-s} \cdot \left( (Y + 1 - N + 2s_2) / 2 \right)_{N-s_2} \cdot \frac{(N - 1)!}{s_2! s!} (N - 1)^{s \cdot s_2 - s} \cdot \frac{1}{s_2! s!}.
\]

We write the sum over \( s_1 \) in hypergeometric notation to obtain

\[
\sum_{s_2=0}^{\lfloor N/2 \rfloor} (-1)^{s_2} 2^N (N/2)_{s_2} ((1 - N) / 2)_{s_2} \cdot \left( (X + 1 - N + 2s) / 2 \right)_{s_2-s} \left( (Y + 1 - N + 2s) / 2 \right)_{s_2-s} \frac{(N - 1)!}{s_2! s!} \cdot \frac{1}{s_2! s!}.
\]

The \( 2F_1 \)-series can be evaluated by means of the Chu–Vandermonde summation formula. After some simplification, we arrive at

\[
\sum_{s_2=0}^{\lfloor N/2 \rfloor} (-1)^{s_2} 2^N (N/2)_{s_2} ((1 - N) / 2)_{s_2} \cdot \left( (X + 1 - N + 2s) / 2 \right)_{s_2-s} \left( (Y + 1 - N + 2s) / 2 \right)_{s_2-s} \frac{(N - 1)!}{s_2! s!} \cdot \frac{1}{s_2! s!}.
\]

After having evaluated the \( 2F_1 \)-series by means of the Chu–Vandermonde summation formula, we are left with

\[
\sum_{s_2=0}^{\lfloor N/2 \rfloor} (-1)^{s_2} 2^N \frac{(N - 1)!}{s_2!} (N/2)_{s_2} ((1 - N) / 2)_{s_2} \cdot \left( (X + 2 - N) / 2 \right)_{s_2} \left( (Y + 1 - N + 2s_2) / 2 \right)_{s_2-s} \frac{(N - 1)!}{s_2! s!} \cdot \frac{1}{s_2! s!}.
\]

\[
= 2^N (N - 1)! \left( (Y + 1 - N) / 2 \right)_{N/2} 2F_1 \left[ \frac{X/2 - Y}{2}; 1 \right].
\]
By the transformation formula in Lemma A.3 this is equal to
\[(N-1)!((X+Y-2N+3)/2)N_{2} F_{1}\left[\frac{-X/2 + Y/2 - \frac{1}{2} - N}{2}; -1\right].\]
which agrees with the right-hand side of (3.1). This finishes the proof. \(\square\)

Now we are finally in the position to establish Theorem 1.2.

Proof of Theorem 1.2. Using \(n = q + p\), we find
\[X(1) = (C + B)(1) = \frac{n}{2} - 1 \quad \text{and} \quad Y(1) = (C - B)(1) = \frac{p - q}{2}.\]
Thus, we can write the constant term of \(P_{2N}\) in the form
\[P_{2N}\left(\frac{n}{2} - 1, \frac{p - q}{2}\right).\]
Now we calculate
\[
\sum_{|I|=N} m_{I} \frac{1}{n^{2} - I_{\text{last}}} P_{2I}(1)
= \sum_{a=1}^{N} \sum_{|J|=N-a} m_{(J,a)} \frac{1}{n^{2} - a} P_{2J}\left(\frac{n}{2} - 1, \frac{p - q}{2}\right) P_{2a}\left(\frac{n}{2} - 1, \frac{p - q}{2}\right)
= N!(N-1)! \sum_{k=0}^{N} (-1)^{k} \left(\frac{a}{k}\right) \left(\frac{b}{N-k}\right).
\]
In the last step we have applied Theorem 3.1. \(\square\)

APPENDIX. PROOFS OF SOME AUXILIARY RESULTS

Here we prove three technical lemmas which were used in Sections 2 and 3.

Lemma A.1. Let \(p(x)\) be a polynomial in \(x\), and suppose that
\[p(x) = \sum_{k \geq 0} c_{k} \binom{x}{k},\]
for (uniquely determined) coefficients \(c_{k}\). Furthermore, let \(M\) be an integer. Then
\[\sum_{a=0}^{M} (-1)^{a} \binom{M}{a} p(a) = (-1)^{M} c_{M}.\]

Proof. This is a classical fact from finite difference calculus. For the convenience of the reader, we give the simple proof. We calculate
\[
\sum_{a=0}^{M} (-1)^{a} \binom{M}{a} p(a) = \sum_{a=0}^{M} (-1)^{a} \binom{M}{a} \sum_{k \geq 0} c_{k} \binom{a}{k}
\]
= \sum_{k \geq 0} c_k \binom{M}{k} \sum_{a=0}^{M} (-1)^a \binom{M-k}{a-k}
= \sum_{k \geq 0} c_k \binom{M}{k} (-1)^k \delta_{M,k} = (-1)^M c_M.

The proof is complete. □

**Lemma A.2.** Let $p(x)$ be a polynomial in $x$, and suppose that
\[ p(x) = \sum_{k \geq 0} c_k \binom{x}{k} \]
for (uniquely determined) $c_k$. Furthermore, let $M$ be an integer. If the degree of $p(x)$ is at most $M$, then
\[ \sum_{b=0}^{M} \frac{(-1)^b}{X-b} \binom{M}{b} p(b) = (-1)^M \frac{M!}{(X-M)_{M+1}} p(X). \]
If the degree of $p(x)$ equals $M + 1$, then
\[ \sum_{b=0}^{M} \frac{(-1)^b}{X-b} \binom{M}{b} p(b) = (-1)^M \frac{M!}{(X-M)_{M+1}} p(X) + \frac{c_{M+1}}{M+1}. \]

**Proof.** It suffices to verify the claim for a basis of the vector space of polynomials in $x$ of degree at most $M + 1$. We choose \{\binom{x}{k}\}_{0 \leq k \leq M+1} as such a basis. If $k \leq M$, we have
\[ \sum_{b=0}^{M} \frac{(-1)^b}{X-b} \binom{M}{b} \binom{x}{k} = \frac{(-1)^k}{X-k} \binom{M}{k} \binom{1}{k} \binom{X}{k} = \frac{(-1)^k}{X-k} \binom{M}{k} \binom{X-k+1}{M-k} = \frac{(-1)^M}{k!} \frac{M!}{(X-M)_{M+1}} \binom{X}{k}, \]
while if $k = M + 1$ we obtain 0. Here, we used the Chu–Vandermonde summation formula to evaluate the $\binom{X}{k}$-series. All this is in agreement with our claims. □

**Lemma A.3.** For all non-negative integers $N$, we have
\[ _3F_2 \left[ a, \frac{1}{2} - \frac{N}{2}, -\frac{N}{2}; 1 \right] = 2^{-N} (e - a)_N \frac{1}{(e)_N} _2F_1 \left[ 1 - a - e - N, -N; 1 + a - e - N \right]; \]
(A.1)

**Proof.** In [KR03, (3.14)],
\[ _4F_3 \left[ a, \frac{1}{2} + a + b, e, 1 - N - e; 1 \right] = \frac{(a + e)_N}{(e)_N} _3F_2 \left[ 2a + b + a + e, a + b, -N; 2a + 2b, a + e; 1 \right]. \]
we let $b$ tend to infinity. As a result, we obtain the transformation formula

\[
\binom{3}{2} F_2 \left[ \frac{a, \frac{1}{2} - \frac{N}{2}, -\frac{N}{2}}{e, 1 - N - e}; 1 \right] = \frac{(a + e)_N}{(e)_N} \binom{2}{1} F_1 \left[ \frac{2 a, -N, 1}{a + e}; \frac{1}{2} \right].
\]

If we now apply the transformation formula (see \cite[(1.7.1.3)]{S66})

\[
\binom{2}{1} F_1 \left[ \frac{A, B}{C}; z \right] = (1 - z)^{-B} \binom{2}{1} F_1 \left[ \frac{C - A, B}{C}; -\frac{z}{1 - z} \right]
\]

to the $2F_1$-series on the right-hand side, then we obtain

\[
2^{-N} \frac{(a + e)_N}{(e)_N} \binom{2}{1} F_1 \left[ \frac{e - a, -N}{a + e}; -1 \right].
\]

Finally, if we reverse the order of summands in this $2F_1$-series (that is, if we denote the summation index in the $2F_1$-series by $s$, then we replace $s$ by $N - s$ and rewrite the result again in hypergeometric notation), then we arrive at the right-hand side of (A.1).

\[\square\]

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\end{itemize}

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