Another look at rational torsion of modular Jacobians

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We study the rational torsion subgroup of the modular Jacobian $J_0(N)$ for $N$ a square-free integer. We give a proof of a result of Ohta on a generalization of Ogg’s conjecture: For a prime number $p 
mid 6N$, the $p$-primary part of the rational torsion subgroup equals that of the cuspidal subgroup. Whereas previous proofs of this result used explicit computations of the cardinalities of these groups, we instead use their structure as modules for the Hecke algebra.

1. Introduction

Let $N$ be a square-free integer and let $J_0(N)$ be the Jacobian of the modular curve $X_0(N)$. In the case where $N$ is prime, Ogg (1) computed the order of the cuspidal subgroup [the subgroup of $J_0(N)$ generated by linear equivalence classes of differences of cusps] and conjectured that the cuspidal subgroup is the whole rational torsion subgroup (2). Mazur’s (3) proof of Ogg’s conjecture was one of the principal results of ref. 3.

Ogg’s computation of the order of the cuspidal subgroup has been generalized to other modular curves by Kubert and Lang (see ref. 4 for an example). The order of the cuspidal subgroup for $X_0(N)$ with square-free $N$ has been computed by Takagi (5) using Kubert–Lang theory.

A number of authors have considered the generalization of Ogg’s conjecture to $J_0(N)$, where $N$ is a positive integer that is not necessarily prime. Since the cuspidal subgroup of $J_0(N)$ may not consist entirely of rational points, the generalization states that the rational torsion subgroup of $J_0(N)$ is contained in the cuspidal subgroup of this Jacobian. See Lorenzini (6), Conrad et al. (7), Ohta (8, 9), Yoo (10), and Ren (11) for details. In particular, Ohta (9) has proved the generalization of Ogg’s conjecture to square-free $N$.

That is, he proved that the $p$-primary part of $J_0(N)(\mathbb{Q})_{\text{tor}}$ and of the cuspidal subgroup are equal for $p \geq 5$ (and $p = 3$ if $3 \nmid N$). The computation of the order of the class group by Takagi was a key input in Ohta’s proof.

In this article, we give a proof that, for a prime $p 
mid 6N$, the $p$-primary parts of $J_0(N)(\mathbb{Q})_{\text{tor}}$ and of the cuspidal subgroup are equal. Our proof is by “pure thought”—we do not compute the order of either group but instead analyze their structure as modules for the Hecke algebra.

Because we do not consider the $p$-primary parts for primes $p$ dividing $6N$, our results do not completely recover Ohta’s. We have not investigated the possibility of adapting our method to handle these primes. See the PhD thesis of Lui (12) for some computational evidence for the generalized Ogg conjecture for 2- and 3-primary parts.

1.1. Structure of the Proof. Fix a prime $p 
mid 6N$. Let $C$ denote the $p$-primary part of the cuspidal subgroup and $T$ denote the $p$-primary part of the rational torsion subgroup $J_0(N)(\mathbb{Q})_{\text{tor}}$. Our proof of the following theorem is based on the structure of $C$ and $T$ as Hecke modules. See Section 2.1 for precise definitions of the Hecke operators we use.

Theorem 1.1. The rational torsion subgroup $T$ coincides with its subgroup $C$.

Let $\mathbb{T}$ be the Hecke algebra acting on $J_0(N)$ and let $I \subseteq \mathbb{T}$ be the Eisenstein ideal (see Section 2.3 for the precise definitions). The following result is the essential contribution of this article:

Theorem 1.2. The annihilators of $T$ and $C$ as $\mathbb{T}$-modules satisfy

$$\text{Ann}_\mathbb{T}(C) \subseteq I \subseteq \text{Ann}_\mathbb{T}(T).$$

Because the annihilator of $T$ is contained in the annihilator of its submodule $C$, the two inclusions of Theorem 1.2 imply the equalities

$$\text{Ann}_\mathbb{T}(C) = I = \text{Ann}_\mathbb{T}(T).$$

[1.3]
We deduce Theorem 1.1 from [1,3] at the end of Section 2.5, using an argument of Mazur to show that the Pontryagin dual of $T$ is cyclic as a $T$-module.

**Corollary 1.4.** The index of $I$ in $\mathbb{T}$ is the order of $C = T$.

**Proof.** Indeed, the Pontryagin dual of $T$ is isomorphic to $\mathbb{T}/I$ because it is a cyclic $\mathbb{T}$-module by Theorem 2.5 and has annihilator $I$ by [1,3]. □

We prove the inclusions $I \subseteq \text{Ann}_T(T)$ and $\text{Ann}_T(C) \subseteq I$ separately, by independent arguments. The Eichler–Shimura relation implies $T$ is annihilated by $T_{q} - q - 1$ for almost all primes $q$ (Lemma 2.4). So, to prove the inclusion $I \subseteq \text{Ann}_T(T)$, it is enough to show that elements of this type generate $I$. That is the content of the following theorem, which we prove in Section 3 using Galois representations:

**Theorem 1.5.** Let $S$ be a finite set of prime numbers that contains all primes dividing $6Np$. The Eisenstein ideal $I$ is generated by the set of all $T_{q} - q - 1$ with $q \notin S$.

Let $\hat{T}$ be the Hecke algebra acting on the full space of modular forms (not just cusp forms) of weight 2 and level $\Gamma_0(N)$. Let $\hat{I} \subseteq \hat{T}$ be the Eisenstein ideal and $J \subseteq \hat{T}$ be the ideal generated by the set of all $T_{q} - q - 1$ for all $q \notin S$. Let

$$\alpha : \hat{T}/J \to \hat{T}/I$$

be the quotient map arising from the inclusion $J \subseteq \hat{I}$. The content of Theorem 1.5 is that $J = \hat{I}$ or, in other words, that $\alpha$ is an isomorphism. To give the flavor of the proof of Theorem 1.5, we now explain informally why $\hat{T}/J$ and $\hat{T}/I$ have the same minimal primes: Think of $\text{Spec}(\hat{T}/J)$ as the set of normalized Eisenstein eigenforms and $\text{Spec}(\hat{T}/J)$ as the set of those normalized eigenforms whose associated Galois pseudorepresentation is $\epsilon + 1$, where $\epsilon : \Gamma_0 \to \mathbb{Z}_p^\times$ is the $p$-adic cyclotomic character. Because of oldforms, the map sending a normalized eigenform to its Galois pseudorepresentation is not necessarily injective. However, for each $\ell | N$, the $U_\ell$ eigenvalue singles out a root of the characteristic polynomial of Frobenius at $\ell$, and the map sending a normalized eigenform to its pseudorepresentation plus this additional data is injective. We show that the normalized Eisenstein eigenforms fill out all the possibilities for roots of the characteristic polynomial of Frobenius at $\ell$ with pseudorepresentation $\epsilon + 1$. This shows that $\text{Spec}(\hat{T}/J) \subseteq \text{Spec}(\hat{T}/I)$ is an equality. In Section 3, we give a more precise version of this argument to establish that $\alpha$ is an isomorphism.

Our proof of the inclusion $\text{Ann}_T(C) \subseteq I$ is given in Section 5. It is based on an interpretation, due to Stevens (13), of $C$ in terms of the lattice of Eisenstein series that have integral periods. We use an integrality result about modular units to show that this lattice is contained in the lattice of Eisenstein series with integral $q$-expansion and thereby exhibits a subquotient of $C$ that has annihilator $I$.

**Remark 1.6.** An alternate proof of the inclusion $\text{Ann}_T(C) \subseteq I$ will be given in forthcoming work of Jordan, Ribet, and Scholl (14). This proof uses $p$-adic Hodge theory to show that the Hecke algebra $\hat{T}$ is $p$-saturated in the endomorphism ring of the generalized Jacobian of $X_0(N)$ relative to the cusps and then proceeds along the lines of the proof of ref. 15, equation 1.7.

**Remark 1.7.** In Section 4, we prove an integrality result about modular units (Proposition 4.4) using arithmetic geometry techniques. This integrality also follows from an explicit description of the modular units given by Takagi (5) using Kubert–Lang theory. We avoid invoking Takagi’s result both in accordance with our desire for a pure thought proof and to show that our method can be used to give alternate proofs of some of his results (Corollary 6.1).

## 2. Preliminaries

In this section we set up notation for modular forms and Hecke algebras, recall a duality result, and employ arguments from ref. 3 used to reduce the proof of Theorem 1.1 to Theorem 1.2.

### 2.1. Modular Forms

Let $M_2(N)$ denote the $\mathbb{C}$-vector space of modular forms of weight 2 and level $\Gamma_0(N)$ and let $S_2(N)$ and $E_2(N)$ be the subspaces of $M_2(N)$ consisting of cusp forms and Eisenstein series, respectively. For a subring $R \subseteq \mathbb{C}$, let $M_2(N, R) \subseteq M_2(N)$ denote the $R$-module of modular forms whose $q$-expansion (at the cusp $\infty$) is in $R[q]$. Let $S_2(N, R) = S_2(N) \cap M_2(N, R)$ and $E_2(N, R) = E_2(N) \cap M_2(N, R)$.

The dimension of $E_2(N)$ is $2^r - 1$, where $r$ is the number of prime divisors of $N$ (see ref. 16, theorem 4.6.2, p. 133, for example). For each $d | N$ with $d > 1$, let $E_d$ be the element of $E_2(N)$ such that for prime numbers $\ell$,

$$a_\ell(E_d) = \begin{cases} 1 & \text{if } \ell \mid d \\ \ell & \text{if } \ell \nmid d \end{cases}$$

if $\ell \mid N$, and $a_\ell(E_d) = 1 + \ell$ if $\ell \nmid N$. These eigenforms form a basis of $E_2(N)$. Since each $E_d$ is in $E_2(N, \mathbb{Z}_p)$, we infer that $E_2(N, \mathbb{Z}_p)$ is a free $\mathbb{Z}_p$-module of rank $2^r - 1$ (although not necessarily with \{ $E_d$ \} as basis).

### 2.2. Hecke Operators

The spaces of modular forms just introduced are equipped with actions of the classical Hecke operators $T_n$ for $n \geq 1$. These operators arise from correspondences on modular curves (see, for example, ref. 17, chap. 7). As such, they also act on geometric objects such as $J_0(N)$ and the divisor group of cusps, but there is some ambiguity as to how a correspondence acts (using either the “Picard” or the “Albanese” functoriality—see the discussion in ref. 18, pp. 443–444). A summary of our conventions is as follows:

- The endomorphism of $J_0(N)$ denoted $T_\ell$ here is the same as the one in ref. 18, p. 444 defined using Picard functoriality. It satisfies $T_\ell = \xi_\ell^*\xi_\ell$,
- The action of $T_\ell$ on the cusps is the dual of the “standard” action (for example, our action of $T_\ell$ on the cusps is the same as the action of $^* T_\ell$ on the cusps described in ref. 19, p. 15).

To avoid any ambiguity, we now spell this out in more detail.

We use the following notation for modular curves: $Y_0(N)^{an}$ is the quotient of the upper half-plane by $\Gamma_0(N)$, thought of as a complex analytic manifold, and $Y_0(N)$ is the smooth model of $Y_0(N)^{an}$ over $\mathbb{Z}_p$ (recall that $p \nmid N$, so such a smooth model exists). We use analogous notations $X_0(N)^{an}$ and $X_0(N)$ for the closed modular curve. Let $\Gamma_0(N) = X_0(N) - Y_0(N)$ denote the set of cusps. Let $J_0(N)$ denote the Jacobian variety of $X_0(N)$.

For a prime number $\ell$, let $t_\ell = (\frac{1}{\ell}, \frac{1}{\ell}, \frac{1}{\ell}) \in GL_2(\mathbb{Q})^+$ and consider the subgroup $\Gamma_0(N; t_\ell) = \Gamma_0(N) \cap t_\ell^{-1}\Gamma_0(N)t_\ell$. Let
Let \( Y_0(N; \ell)^{an} \) be the quotient of the upper half-plane by \( \Gamma_0(N; \ell) \). There is a correspondence

\[
Y_0(N; \ell)^{an} \to \pi \to Y_0(N; \ell)^{an},
\]

where \( \pi \) is the quotient map and \( t_\ell \) is the map induced by \( x \mapsto t_\ell x \). This correspondence extends uniquely to give a correspondence on \( Y_0(N) \) that preserves the cusps; we use the same names \( \pi \) and \( t_\ell \) for the maps in this correspondence. Define the element \( T_\ell \in \text{End}(J_0(N)) \) by

\[
T_\ell = (t_\ell)_* \pi^*.
\]

The same formula defines an endomorphism \( T_\ell \) of the group \( \text{Div}^0(C_0(N)) \) of degree-zero divisors on the cusps. The map

\[
\text{Div}^0(C_0(N)) \to J_0(N)
\]

sending a divisor to its class is equivariant for these actions of \( T_\ell \).

By identifying \( M_2(N) \) with the space of differential forms on \( Y_0(N)^{an} \), the formula \( T_\ell = (t_\ell)_{*} \pi^{*} \) gives an action of \( T_\ell \) on \( M_2(N) \). This action preserves the subspaces \( S_2(N) \) and \( E_2(N) \) and preserves the \( R \)-submodule \( M_2(N, R) \) if \( R \) is a \( \mathbb{Z}[1/N] \)-algebra.

### 2.3. Hecke Algebras
Let \( \mathcal{T} \subset \text{End}(M_2(N)) \) be the \( \mathbb{Z}(p) \)-algebra generated by the Hecke operators \( T_\ell \) for all primes \( \ell \geq 1 \). Let \( I \subset \mathcal{T} \) be the annihilator of the space of Eisenstein series. Let \( \mathcal{T} \) be the image of \( \mathcal{T} \) in \( \text{End}(S_2(N)) \) and let \( I \subset \mathcal{T} \) be the image of \( I \). As is customary, for a prime divisor \( \ell \) of \( N \), we denote the operator \( T_\ell \) by \( U_\ell \).

### 2.4. Duality
The following duality is well known. The duality 2.4. Duality.

\[
\text{Duality.}
\]

The pairing

\[
\text{Let } G_0(N)^{an} = \pi \to Y_0(N)^{an},
\]

is a perfect pairing of free \( \mathbb{Z}(p) \)-modules of finite rank.

2. Let \( X \subset M \) be a \( \mathcal{T} \)-submodule. The pairing 2.2 induces an isomorphism

\[
\text{Hom}(M/X, \mathbb{Z}(p)) \cong \text{Ann}_p(X).
\]

Moreover, if \( M/X \) is torsion-free, then 2.2 induces an isomorphism

\[
\text{Hom}(X, \mathbb{Z}(p)) \cong \mathcal{T}/\text{Ann}_p(X).
\]

### 2.5. The Eichler– Shimura Relation
The following lemma, which is well known, shows that many of the elements of \( I \) annihilate \( T \). In Section 3, we will prove the inclusion \( I \subset \text{Ann}_p(T) \) by showing that these elements generate \( I \).

#### Lemma 2.4
For every prime \( q \not\in S \), the group \( T \) is annihilated by \( T_q - q - 1 \).

**Proof**
Let \( q \) be a prime that is not in \( S \). The Eichler– Shimura relation states that \( T_q = F_{q} + V_q \) on \( J_0(N) / \mathbb{Z}(q) \) (ref. 3, p. 89), where \( F_q \) is the Frobenius endomorphism of the group scheme \( J_0(N) / \mathbb{Z}(q) \) and \( V_q \) is the Verschiebung. Since \( F_q \) is the identity on \( J_0(N)(\mathbb{F}_q) \), this implies that \( T_q = 1 + q \) on \( J_0(N)(\mathbb{F}_q) \). But, since \( q \not\mid 2N \), the reduction modulo \( q \) makes \( T \) an injection \( T \to J_0(N)(\mathbb{F}_q) \) of \( T \)-modules (see ref. 22, appendix, for example), so \( T_q = 1 + q \) on \( T \) as well.

### 2.6. Cyclicity of the Dual of \( T \)
**Theorem 1.1** follows from **Theorem 1.2** together with the following mild generalization of a theorem from Mazur (ref. 3, corollary II.14.8, p. 199), which appears in the work of Ohta (ref. 9, proposition 3.5.4).

**Theorem 2.5 (Mazur, Ohta)** The Pontryagin dual of \( T \) is cyclic as a \( \mathcal{T} \)-module.

**Proof**
It is equivalent to show that, for all maximal ideals of \( \mathcal{T} \), the \( m \)-torsion subgroup of \( T \), which we denote by \( T[m] \), has dimension at most 1 as a \( \mathcal{T}/m \)-vector space. Since, as was noted in the proof of **Lemma 2.4**, the reduction map \( T \to J_0(N)(\mathbb{F}_p) \) is injective, \( T[m] \) is contained in the \( m \)-torsion subgroup of \( J_0(N)[p] \). Furthermore, as in ref. 3, proposition II.14.7, p. 119, the Cartier–Serre isomorphism induces an injective homomorphism

\[
J_0(N)[p](\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p \to H^0(X_0(N) / \mathbb{Z}(p), \Omega^1).
\]
3. Galois Representations and the Annihilator of T

In this section, we prove Theorem 1.5 and the inclusion 
\( \text{Ann}_\mathbb{T}(T) \supseteq I \) of Theorem 1.2. Fix a finite set \( S \) of prime numbers containing all primes dividing \( 6Np \).

3.1. Galois Representations. Let \( J \subset \hat{\mathbb{T}} \) be the ideal generated by \( T_q - a - 1 \) for all \( q \notin S \).

Lemma 3.1. The ideal \( J \) contains the following elements:

a) \( T_q - a - 1 \) for each prime \( q \nmid N \),

b) \( (U_\ell - 1)(U_\ell - \ell) \) for each prime \( \ell \mid N \), and

c) \( \prod_{q \mid N} (U_q - 1) \), where the product is over the set of prime divisors of \( N \).

We prove Lemma 3.1 using properties of Galois representations associated to eigenforms. Before continuing with the proof, we review the required properties. To this end, fix a maximal ideal \( m \subset \hat{\mathbb{T}} \) and assume that \( m \supseteq J \).

For each minimal prime ideal \( p \subset m \), there is a corresponding (not necessarily cuspidal) eigenform \( f_p \) with coefficients in \( \mathcal{O}_p = \hat{\mathbb{T}}_m/p \) (which is a finite extension of \( \mathbb{Z}_p \)), defined by \( \phi_n(f_p) = T_n \mod p \). Let \( K_p \) be the field of fractions of \( \mathcal{O}_p \). To each \( f_p \), there is an associated two-dimensional semisimple Galois representation \( \rho_p \) with coefficients in \( K_p \), constructed by Shimura (see ref. 17). See ref. 24, theorem 3.1 for a list the properties of \( \rho_p \), which have been established by the work of many mathematicians; we will rewrite these properties in terms of a representation \( \rho_m \) that we now define.

Since \( \mathbb{T}_m \) is reduced (25), there is an injective homomorphism

\[
\mathbb{T}_m \hookrightarrow \mathbb{T}_m \otimes \mathbb{Q} = \prod_{p \mid m} K_p.
\]

The product of the Galois representations \( \rho_p \) is a continuous representation

\[
\rho_m : G_Q \rightarrow \text{GL}_2(\mathbb{T}_m \otimes \mathbb{Q}),
\]

with the following properties:

1) \( \rho_m \) is unramified outside \( Np \); 
2) \( \det(\rho_m) = \kappa_{\text{cyc}} \), the \( p \)-adic cyclotomic character; and

3) \( \text{tr}(\rho_m)(F_q) = T_q \) for each prime \( q \nmid Np \), where \( F_q \) is an arithmetic Frobenius.

Since \( m \supseteq J \), the Chebotarev density theorem implies that \( \text{tr}(\rho_m)(\sigma) \equiv \kappa_{\text{cyc}}(\sigma) + 1 \pmod{m} \) for every \( \sigma \in G_Q \). This implies that \( m \) is ordinary in the sense that \( T_n \) is invertible modulo \( m \). Indeed, a theorem of Fontaine (ref. 26, theorem 2.6) implies that the restrictions of \( \text{tr}(\rho_m) \mod m \) to the inertia group at \( p \) is the sum of two nontrivial characters in the nonordinary case. Let \( u_p \in \mathbb{T}_m \) denote the unit root of \( x^2 - T_p x + p \), which exists by Hensel’s lemma.

The restrictions of \( \rho_m \) to decomposition groups at primes dividing \( Np \) can be described as follows:

4) For every prime \( \ell \nmid N \) and every choice of arithmetic Frobenius \( F_{\ell} \in G_Q \), the trace of \( \rho_m(F_{\ell}) \) is given by

\[
\text{tr}(\rho_m)(F_{\ell}) = U_{\ell} + \ell U_{\ell}^{-1}.
\]

5) For every \( \sigma \) in a decomposition group at \( p \), the trace of \( \rho_m(\sigma) \) is given by

\[
\text{tr}(\rho_m)(\sigma) = \kappa_{\text{cyc}}(\sigma) \lambda^{-1}(\sigma) + \lambda(\sigma),
\]

where \( \lambda \) is the unramified character sending Frobenius to \( u_p \).

In the proof of Lemma 3.1 that follows, we refer to these five properties of \( \rho_m \) as Property 1 through Property 5.

Proof of Lemma 3.1. It is enough to show the containment after completion at all maximal ideals \( m \subset \hat{\mathbb{T}} \). We can and do assume \( m \supseteq J \) because the statement is clear if \( J_m = \mathbb{T}_m \).

The Chebotarev density implies that the image of \( \text{tr}(\rho_m) \) is contained in \( \mathbb{T}_m \) and that \( J_m \) is the ideal generated by

\[
\text{tr}(\rho_m)(\sigma) - \kappa_{\text{cyc}}(\sigma) - 1
\]

for all \( \sigma \in G_Q \). It follows from Property 3 that \( T_q - q - 1 \in J_m \) for all \( q \nmid N \).

To prove Lemma 3.1, part a, it remains to show that \( T_p - p - 1 \in J_m \) or, equivalently, that \( u_p - 1 \in J_m \). Let \( \sigma \) be an element of the inertia group at \( p \) such that \( \kappa_{\text{cyc}}(\sigma) \not\equiv 1 \pmod{p} \). Then Property 5 implies

\[
\text{tr}(\rho_m)(F_{\ell}) - \ell - 1 = U_{\ell} + \ell U_{\ell}^{-1} - 1 \in J_m,
\]

\[
\text{tr}(\rho_m)(\sigma F_{\ell}) - \kappa_{\text{cyc}}(\sigma) \kappa_{\text{cyc}}(F_{\ell}) \not\equiv 1 \pmod{J_m}.
\]

Subtracting these equations, we find that

\[
(\kappa_{\text{cyc}}(\sigma) - 1) \kappa_{\text{cyc}}(F_{\ell}) \not\equiv 0 \pmod{J_m}.
\]

By the assumption on \( \sigma \), the element \( (\kappa_{\text{cyc}}(\sigma) - 1) \kappa_{\text{cyc}}(F_{\ell}) \in \mathbb{Z}_p \) is a unit, so \( u_p - 1 \in J_m \).

Hence \( T_p - p - 1 \in J_m \), completing the proof of Lemma 3.1, part a.

For Lemma 3.1, part b, note that by Property 4

\[
\text{tr}(\rho_m)(F_{\ell}) - \ell - 1 = U_{\ell} + \ell U_{\ell}^{-1} - 1 \in J_m.
\]

Since \( (U_{\ell} - 1)(U_{\ell} - \ell) = U_{\ell}(U_{\ell} + \ell U_{\ell}^{-1} - 1) \), this implies \( (U_{\ell} - 1)(U_{\ell} - \ell) \in J_m \).

For Lemma 3.1, part c, let \( x = \prod_{\ell \mid N} (U_{\ell} - 1) \). We will show that \( x = 0 \) in \( \mathbb{T}_m \) by showing that \( x \) maps to zero in each factor \( K_p \) under the injective map \( \mathbb{T}_m \rightarrow \mathbb{T}_m \otimes \mathbb{Q} \). First suppose that \( f_p \) is an Eisenstein series. The only eigenforms in \( E_2(N) \) are the forms \( E_d \) defined in Section 2.1; hence, \( f_p = E_d \) for some \( d \mid N \) with \( d > 1 \). In particular, \( u_p(f_p) = 1 \) for every prime \( \ell \mid d \), so \( x \) maps to zero in
that factor. Next suppose that $f_p$ is a cuspform. Then $a_{\ell}(f_p) = 1$ for some $\ell \mid N$ by the following lemma of Ribet and Yoo. Hence $x = 0$ in $\mathbb{T}_m$.

**Lemma 3.3 (Ribet, Yoo).** Let $M$ be a square-free integer and let $p \nmid 6M$. Suppose that $f$ is a newform of weight $2$ and level $M$ such that $a_{\ell}(f) \equiv q + 1 \pmod{p}$ for all $q \nmid Mp$. Then $a_{\ell}(f) = 1$ for some $\ell \mid M$.

**Proof.** See ref. 27.

3.2. Proof that $J = \bar{I}$. We now prove *Theorem 1.5*, which is the claim that $J = \bar{I}$, by showing that the elements of $J$ listed in Lemma 3.1 generate $I$.

**Proof of Theorem 1.5.** We will show that the natural surjection $\alpha : \mathbb{T}/J \rightarrow \bar{\mathbb{T}}/\bar{I}$ is an isomorphism. This will imply that the elements $T_q - q - 1$ for $q \nmid S$ generate $J$ and hence also generate $I$.

Let $N = \ell_1 \cdots \ell_r$, be the prime factorization of $N$ and let $R = \mathbb{Z}_p[x_1, \ldots, x_r]$. By Lemma 3.1, part a, the $\mathbb{Z}_p$-algebra homomorphism

$$s : R \rightarrow \mathbb{T}/J, \quad x_i \mapsto U_i - 1$$

is surjective. Let $\mathcal{I} \subset R$ be the ideal generated by $x_1x_2 \cdots x_r$ and $x_i(x_i + 1 - \ell_i)$ for $i = 1, \ldots, r$. The map $s$ induces a surjection

$$\bar{s} : R/\mathcal{I} \rightarrow \bar{\mathbb{T}}/\bar{I} \text{ because } s(\mathcal{I}) = 0 \text{ by Lemma 3.1, parts } b \text{ and } c.$$

We claim that $R/\mathcal{I}$ is a free $\mathbb{Z}_p$-module of rank $2^r - 1$, which is the same as the rank of $\bar{\mathbb{T}}/\bar{I}$ by Lemma 2.3. Indeed, consider the ideal $\mathcal{I}' \subset \mathcal{I}$ generated by $x_i(x_i + 1 - \ell_i)$ for $i = 1, \ldots, r$. Then $R/\mathcal{I}'$ is freely generated by the $2^r$ monomials of degree at most $1$ in each $x_i$. The ideal generated by $x_1x_2 \cdots x_r$ in $R/\mathcal{I}'$ is equal to its $\mathbb{Z}_p$-span, which is a direct summand of $R/\mathcal{I}$ as $\mathbb{Z}_p$-modules. Since $\mathcal{I} = \mathcal{I}' + x_1x_2 \cdots x_rR$, this implies that $R/\mathcal{I}$ is free of rank $2^r - 1$.

The composition

$$\alpha \circ \bar{s} : R/\mathcal{I} \rightarrow \bar{\mathbb{T}}/\bar{I}$$

is a surjective homomorphism of free $\mathbb{Z}_p$-modules of the same finite rank and is therefore an isomorphism. Since $s$ is surjective, this implies that $\alpha$ is an isomorphism, which is the content of *Theorem 1.5*.

Since $\text{Ann}_\mathbb{Q}(T) \supseteq J$ by Lemma 2.4, *Theorem 1.5* implies the inclusion $\text{Ann}_\mathbb{Q}(T) \supseteq \bar{I}$, which is one of the two inclusions of *Theorem 1.2*.

4. The Period Lattice of Eisenstein Series

In this section, we introduce the period lattice in the space of Eisenstein series. This lattice is closely related to the cuspidal subgroup of $J_0(N)$ and to modular units by the work of Stevens (13). We show that the $q$-expansions of the elements of the period lattice have coefficients in $\mathbb{Z}_p$ by first proving an analogous integrality property for $q$-expansions of modular units. Recall from Section 2.2 the notation $Y_0(N)_{an} \subset X_0(N)_{an}$ for the analytic modular curves, $Y_0(N) \subset X_0(N)$ for their $\mathbb{Z}_p$-models, and $C_0(N) = X_0(N) - Y_0(N)$ for the set of cusps. If $R$ is a commutative $\mathbb{Z}_p$-algebra, let $Y_0(N)_R$ and $X_0(N)_R$ denote the base changes of $Y_0(N)$ and $X_0(N)$ to $R$.

4.1. The Period Lattice. For each Eisenstein series $f \in E_2(N)$, consider the period map $\int f : H_1(Y_0(N)_{an}, \mathbb{Z}) \rightarrow \mathbb{C}$, defined by

$$\int f : \gamma \mapsto \int_{\gamma} f(z) \; dz.$$

The period lattice $\mathcal{E}$ is the $\mathbb{Z}_p$-module consisting of those Eisenstein series $f$ for which the image of $\int f$ is contained in $\mathbb{Z}_p$.

Let $U = (\mathcal{O}_{Y_0(N)_{an}}/\mathbb{C}^\times) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p$ be the $\mathbb{Z}_p$-module of (analytic) modular units modulo scalars. We will refer to elements of $U$ as scalar classes of modular units; note that the divisor of a modular unit depends only on its scalar class. Let $C = \mathbb{D}^0(C_0(N)) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p$ be the $\mathbb{Z}_p$-module of degree-zero divisors with cuspidal support. By definition,

$$C = \bar{C}/\text{div}(U).$$

Stevens establishes the isomorphism

$$\mathcal{D} : U \rightarrow \mathcal{E}, \quad \mathcal{D}(u) \log(q) = d \log(u),$$

where $d \log(u) = \log(2\pi i \mathcal{D}(u))$ for all $u \in U$ (ref. 13, p. 521), where $\text{Res} : \mathcal{E} \rightarrow \bar{C}$ is the residue map defined by

$$\text{Res}(f) = \sum_{x \in C_0(N)} \text{Res}_x(f(z) \; dz)[x].$$

We will use this isomorphism, together with an integrality result for modular units, to study the integrality of $q$-expansions of elements of the period lattice.

4.2. $\mathbb{Z}_p$-Integrality of Modular Units. We show that every (analytic) modular unit of level $\Gamma_0(N)$ can be written as a complex constant times a $\mathbb{Z}_p$-integral modular unit—that is, a unit in the ring of regular functions of the smooth model of the modular curve over $\mathbb{Z}_p$. Then we show that $\mathbb{Z}_p$-integral modular units have $q$-expansions in $\mathbb{Z}_p$ using the algebraic description of the infinite cusps.

The following lemmas provide the reduction from analytic modular units to integral modular units in two steps: first from analytic to rational and then from rational to integral. The second lemma is established using a similar method to that used in the proof of the $q$-expansion principle.

**Lemma 4.2.** The homomorphism

$$\mathcal{O}_{Y_0(N)_{an}}^\times \rightarrow \mathcal{O}_{Y_0(N)_{an}}^\times/\mathbb{C}^\times$$

sending a unit on $Y_0(N)_{an}$ to the scalar class of its associated analytic modular unit is surjective.

**Proof.** Since the divisor map $\text{div} : \mathcal{O}_{Y_0(N)_{an}}^\times/\mathbb{C}^\times \rightarrow \text{Div}^0(C_0(N))$ is injective, *Lemma 4.2* states that for each analytic unit $u_{an} \in \mathcal{O}_{Y_0(N)_{an}}$, there is an algebraic unit $u \in \mathcal{O}_{Y_0(N)}$ with the same divisor.

Let $u_{an} \in \mathcal{O}_{Y_0(N)_{an}}$ and let $D = \text{div}(u_{an})$ in $\bar{C}$, and consider the divisor class $D \in \text{Pic}(X_0(N)_{an})$ of $D$. Then $D$ is in the kernel of the composite map

$$\text{Pic}(X_0(N)_{an}) \rightarrow \text{Pic}(X_0(N)_\mathbb{C}) \rightarrow \text{Pic}(X_0(N)_{an}).$$

The first map is injective by ref. 28, tag 0CC5, and the second map is injective (in fact, an isomorphism) by the comparison between algebraic and geometric geometry (see ref. 29, theorem 4.4, p. 329). This injectivity implies that $D = 0$ in $\text{Pic}(X_0(N)_{an})$, so $D = \text{div}(u)$ for some rational function $u$ on $X_0(N)_{an}$. Since $D$ is supported on the cusps, $u \in \mathcal{O}_{Y_0(N)_{an}}$ as desired.

**Lemma 4.3.** Base change induces a surjection

$$\mathcal{O}_{Y_0(N)}^\times \rightarrow \mathcal{O}_{Y_0(N)_\mathbb{Q}}^\times/\mathbb{Q}^\times.$$
**Proof.** Let \( u \in \mathcal{O}^X_{Y_0(N)_{\mathbb{Q}}}, \) thought of as a rational function on the arithmetic surface \( Y_0(N) \) with zeros and poles along the special fiber \( Y_0(N)_{\mathbb{F}_p} \). Choose a closed point \( x \in Y_0(N)_{\mathbb{F}_p} \) with residue field \( k \), a finite extension of \( \mathbb{F}_p \), and identify the local ring at \( x \) with \( W(k)[T] \), which is possible because \( Y_0(N) \) is smooth. Consider the pullback \( \tilde{u} \) of \( u \) to \( W(k)[T] \). Since \( u \) is a unit on \( Y_0(N)_{\mathbb{Q}} \), the constant term \( \tilde{u}(0) \) of \( \tilde{u} \) is nonzero, so we can write \( \tilde{u}(0) = p^n \alpha \) for some \( n \in \mathbb{Z} \) and \( \alpha \in W(k)^\times \). We claim that \( p^{-n}u \) is a unit in \( \mathcal{O}_{Y_0(N)} \). Indeed, the divisor of \( p^{-n}u \) is supported on \( Y_0(N)_{\mathbb{F}_p} \), by assumption, and, since \( Y_0(N) \) is normal, it must be a union of irreducible codimension 1 subvarieties. Since \( Y_0(N) \) is smooth, \( Y_0(N)_{\mathbb{F}_p} \) is itself irreducible, so the divisor of \( p^{-n}u \) is either empty or all of \( Y_0(N)_{\mathbb{F}_p} \). But, by the definition of \( n \), \( x \) is not in the divisor of \( p^{-n}u \), so the divisor is empty and \( p^{-n}u \) is a unit. \( \Box \)

**Proposition 4.4.** The \( q \)-expansion of every element of \( \mathcal{O}^X_{Y_0(N)} \), thought of as an analytic modular unit, is in \( \mathbb{Z}(q)[q^{-1}]^\times \).

**Proof.** The \( q \)-expansion of a modular unit is given by pulling back to the infinity cusp. The infinity cusp extends to the integral model as a morphism

\[ \infty : \text{Spec}(\mathbb{Z}[q][q^{-1}]) \to Y_0(N) \]

of schemes (defined using the Tate elliptic curve). Hence there is a commutative diagram

\[ \begin{array}{ccc} \mathcal{O}^X_{Y_0(N)} & \to & \mathbb{Z}(q)[q^{-1}]^\times \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y_0(N),\mathbb{Q}}^X & \to & \mathbb{Z}(q)[q^{-1}]^\times, \end{array} \]

where the horizontal arrows are the \( q \)-expansions (that is, pullback along \( \infty \)). This commutativity implies that the \( q \)-expansion of every element in the image of \( \mathcal{O}^X_{Y_0(N)} \to \mathcal{O}_{Y_0(N),\mathbb{Q}}^X \) belongs to \( \mathbb{Z}(q)[q^{-1}]^\times \). \( \Box \)

### 4.3. \( q \)-Expansions of Elements of the Period Lattice.

We note the effect of the isomorphism \( \mathcal{D} \) from (4.1) on \( q \)-expansions. Let \( u \in \mathcal{O}^X_{Y_0(N),\mathbb{Q}} \) and let \( u(q) \) be the \( q \)-expansion of \( u \) and \( \mathcal{D}(u)(q) \) be the \( q \)-expansion of \( \mathcal{D}(u) \). The chain rule and the fact that \( dq = 2\pi i dq \) imply

\[ \mathcal{D}(u)(q) = \frac{u'(q)}{u(q)}, \]

where \( u'(q) = \frac{d}{dq} u(q) \).

**Theorem 4.6.** For each \( f \in \mathcal{E} \), the \( q \)-expansion of \( f \) has coefficients in \( \mathbb{Z}(q) \). In other words, \( \mathcal{E} \subseteq E_2(N, \mathbb{Z}(q)) \).

**Proof.** Let \( S \subseteq \mathcal{O}^X_{Y_0(N),\mathbb{Q}} \) be the image of the map \( \mathcal{O}^X_{Y_0(N)} \to \mathcal{O}_{Y_0(N),\mathbb{Q}}^X \) sending an integral modular unit to the associated analytic modular unit. By Lemmas 4.2 and 4.3, the composition \( S \to \mathcal{O}_{Y_0(N),\mathbb{Q}}^X \to \mathcal{O}_{Y_0(N),\mathbb{Q}}^X / \mathbb{C}^X \) is surjective. Then, by Steven's theorem (Eq. 4.1), \( \mathcal{D}(S) \) spans \( \mathcal{E}, \) so it is enough to show that \( \mathcal{D}(S) \subseteq E_2(N, \mathbb{Z}(q)) \).

Let \( u \in S \) and write \( \mathcal{D}(u) = u(q) + \sum_{n \in \mathbb{Z}} a_n u^n \). Since \( u \) is in \( \mathcal{O}_{Y_0(N),\mathbb{Q}}^X \), the \( q \)-expansion of \( \mathcal{D}(u) \) is in \( \mathbb{Z}(q)[q^{-1}]^\times \) and \( \mathcal{D}(u) \in E_2(N, \mathbb{Z}(q)) \). \( \Box \)

### 5. The Annihilator of the Cuspidal Subgroup

We established the inclusion \( \text{Ann}_T(T) \supseteq I \) in Section 3. To complete the proof of Theorem 1.2 we need to show that \( C \) is large in the sense that \( \text{Ann}_T(C) \) is contained in \( I \). The following result implies this needed fact:

**Proposition 5.1.** There is a subquotient of \( C \) whose Pontryagin dual is a free \( \mathbb{T}/I \)-module of rank 1.

**Proof.** For this proof, let \( M = M_2(N, \mathbb{Z}(p)) \), \( S = S_2(N, \mathbb{Z}(p)) \), and \( E = E_2(N, \mathbb{Z}(p)) \). Steven's isomorphism \( \mathcal{D} : U \sim \mathcal{E} \) (Eq. 4.1) and the inclusion \( \mathcal{E} \subseteq E \) of Theorem 4.6 induce a natural surjection

\[ C \cong \tilde{C} / \text{Res}(\mathcal{E}) \to \tilde{C} / \text{Res}(E). \]

On the other hand, the residue theorem gives an exact sequence

\[ 0 \to S \to M \xrightarrow{\text{Res}} \tilde{C}, \]

so the residue map induces an injective homomorphism

\[ \frac{M}{S + E} \xrightarrow{\text{Res}} \tilde{C} / \text{Res}(E). \]

Note that the residue map is Hecke equivariant with respect to the action of Hecke operators on \( \tilde{C} \) by Picard functoriality (see ref. 19, p. 36, for example), which induces our chosen action of \( \mathbb{T} \) on \( C \). Hence \( C \) has a subquotient that is isomorphic to \( \mathbb{F}_p M / \mathbb{T} \mathbb{F}_p \).

Let \( X = \frac{M}{S + E} \) and \( \mathcal{I} \) remain to show that the Pontryagin dual \( X^\vee \) of \( X \) is isomorphic to \( \mathbb{T} / I \).

Consider the exact sequence

\[ 0 \to S \to M \xrightarrow{\text{Res}} \tilde{C} \]

By Lemma 2.1, statement 2, there are duality isomorphisms

\[ \text{Hom}_{\mathbb{Z}(p)}(S, \mathbb{Z}(p)) \cong \mathbb{T}, \quad \text{Hom}_{\mathbb{Z}(p)}(M / E, \mathbb{Z}(p)) \cong \mathbb{I}. \]

Since \( M / E \) is torsion-free, \( \text{Ext}^1_{\mathbb{Z}(p)}(M / E, \mathbb{Z}(p)) = 0. \) Because \( X \) is finite, there are isomorphisms \( \text{Hom}_{\mathbb{Z}(p)}(X, \mathbb{Z}(p)) = 0 \) and \( \text{Ext}^1_{\mathbb{Z}(p)}(X, \mathbb{Z}(p)) \cong X^\vee \). Applying \( \text{Hom}_{\mathbb{Z}(p)}(\cdot, \mathbb{Z}(p)) \) to \( [5.2] \), we obtain an exact sequence

\[ 0 \to \tilde{I} \to \mathbb{T} \xrightarrow{X^\vee} 0, \]

so \( X^\vee \cong \mathbb{T} / I \). \( \Box \)

The proposition implies that \( C \) has a subquotient whose annihilator is \( I \). Since any element of \( \mathbb{T} \) that annihilates \( C \) will annihilate this subquotient, it follows that \( \text{Ann}_T(C) \subseteq I \) and this completes the proof of Theorem 1.2. By Corollary 1.4, \( C \) has the same cardinality as its subquotient \( X \). We record some implications of this as a corollary:

**Corollary 5.3.**

1. The period lattice \( E \) coincides with the \( q \)-expansion lattice \( E_2(N, \mathbb{Z}(q)) \).
2. There is an isomorphism of \( \mathbb{T} \)-modules

\[ \frac{M}{S + E} \to C \]

sending \( f \in M \) to the class of \( \text{Res}(f) \) in \( C \).
3) The Pontryagin dual of $C$ is a cyclic $\mathbb{T}/I$-module generated by the homomorphism $\lambda : C \to \mathbb{Q}_p/\mathbb{Z}_p$ defined as follows: Let $x \in C$, choose $f \in M$ such that $\text{Res}(f) \in \mathbb{C}$ represents the class of $x$, and write $f = ae + bg$ for $a, b \in \mathbb{Q}_p$, $e, g \in E$, and $g \in S$; then define $\lambda(x) = ba_1(g) \mod \mathbb{Z}_p$.

**Proof.** Continue with the notation as in the proof of Proposition 5.1. Recall that $X = \mathcal{M}/\mathcal{E}$ is Pontryagin dual to $\mathbb{T}/I$, so there is an equality of cardinalities $|X| = |\mathbb{T}/I|$. The module $X$ is a quotient of $\mathcal{C}/\mathcal{E}$ as follows:

$$X \xrightarrow{\text{Res}} \mathcal{C}/\mathcal{E} \xrightarrow{\mathcal{C}/\mathcal{Res}(E)} \mathcal{C}/\mathcal{Res}(E) \xrightarrow{\sim} C. \quad [5.4]$$

Moreover, $|C| = |\mathbb{T}/I|$ by Corollary 1.4. The maps in [5.4] imply a string of inequalities of cardinalities

$$|\mathbb{T}/I| = |X| \leq |\mathcal{C}/\mathcal{Res}(E)| \leq |\mathcal{C}/\mathcal{Res}(E)| = |C| = |\mathbb{T}/I|. \quad [5.4]$$

Since the beginning and the end of the string are equal, all the inequalities are in fact equalities. This implies that all the maps in [5.4] are isomorphisms.

The fact that the surjection $\mathcal{C}/\mathcal{Res}(E) \to \mathcal{C}/\mathcal{Res}(E)$ is an isomorphism implies $\mathcal{Res}(E) = \mathcal{Res}(E)$. Since $\mathcal{Res}$ is injective on $\mathcal{E}$, this shows that $E = E'$, proving statement 1. The map described in statement 2 is the composition of the isomorphisms in [5.4].

The homomorphism $\lambda$ defined in statement 3 is the composition of the isomorphism defined in statement 2 with the map $\lambda' : X \to \mathbb{Q}_p/\mathbb{Z}_p$ defined by $\lambda'(ae + bg) = ba_1(g) \mod \mathbb{Z}_p$. By statement 2, it is enough to show that $\lambda'$ generates the dual of $X$. First note that $\lambda'$ is well-defined since $\mathcal{M}/\mathcal{E} = \mathcal{S}/[\mathcal{S}/\mathcal{P}] \oplus \mathcal{F}[\mathcal{P}]$ as $\mathbb{Q}$-vector spaces. The element of $\text{Ext}^1_{\mathbb{Z}_p}(X, \mathbb{Z}_p)$ corresponding to $\lambda'$ is class of the extension

$$0 \to \mathbb{Z}_p \to \mathcal{Z}_p \to X \to 0,$$

where $\mathcal{Z}_p = \{(x, \alpha) \in \mathbb{X} \times \mathbb{Q}_p | \lambda'(x) \equiv \alpha \mod \mathbb{Z}_p\}$. On the other hand, the proof of Proposition 5.1 shows that $\text{Ext}^1_{\mathbb{Z}_p}(X, \mathbb{Z}_p)$ is a free $\mathbb{T}/I$-module generated by the class of the extension

$$0 \to \mathbb{Z}_p \to \mathcal{Z}_p \to X \to 0,$$

where $\mathcal{Z} = \mathcal{M}/\mathcal{E} \oplus \mathbb{Z}_p$. There is a map

$$Z \to \mathcal{Z}', \quad (f, \alpha) \to (f, \alpha + ba_1(g)),$$

where $f = ae + bg \in M$ with $e \in E$, $g \in S$, $a, b \in \mathbb{Q}_p$, and $\alpha \in \mathbb{Z}_p$. A simple computation shows that this map is an isomorphism of extensions. Hence the class of the extension given by $\mathcal{Z}'$ generates $\text{Ext}^1_{\mathbb{Z}_p}(X, \mathbb{Z}_p)$ and $\lambda'$ generates the dual of $X$. \hfill \Box

### 6. Complement: Explicit Bases of Eisenstein Series and Modular Units

We give explicit bases of $\mathcal{E}$ and $U$ as $\mathbb{Z}(p)$-modules. For $d > 1$, let $\eta_d \in U$ be the scalar class of $\left(\frac{d}{\eta(z)}\right)^{12N}$, where $\eta(z)$ is the Dedekind eta function. Let $f_d = D(h_4) \in \mathcal{E}$, and note that

$$f_d(z) = 12N \left(E_2(dz) - E_2(z)\right),$$

where $E_2(z)$ is the (nonholomorphic) normalized Eisenstein series of weight 2 and level 1.

**Corollary 6.1.**

1. The set $\{f_d : d > 1, d \mid N\}$ is a basis for $E_2(N, \mathbb{Z}(p))$ as a $\mathbb{Z}(p)$-module.
2. The period lattice $\mathcal{E}$ coincides with the $q$-expansion lattice $E_2(N, \mathbb{Z}(p))$.
3. The set $\{h_d : d > 1, d \mid N\}$ is a basis for $U$ as a $\mathbb{Z}(p)$-module.

**Proof.** Note that statement 2 has already been proved in Corollary 5.3, statement 1; we give an alternate proof here. Since each $f_d$ is in $\mathcal{E}$, statement 1 implies that $E_2(N, \mathbb{Z}(p)) \subseteq \mathcal{E}$. Together with Theorem 4.6, this inclusion implies statement 2. Since $f_d = D(h_4)$ and $D$ is an isomorphism, statements 1 and 2 imply statement 3.

It remains to prove statement 1. Let $E' \subseteq E_2(N, \mathbb{Z}(p))$ be the $\mathbb{Z}(p)$ submodule generated by $\{f_d : d > 1, d \mid N\}$. Since the set $\{f_d : d > 1, d \mid N\}$ is a basis for $E_2(N, \mathbb{Q})$ (see ref. 16, theorem 4.6.2, p. 133, for example), the index of $E'$ in $E_2(N, \mathbb{Z}(p))$ is finite. To show this index is 1, it is enough to show that the restriction map

$$\text{Hom}_{\mathbb{Z}(p)}(E_2(N, \mathbb{Z}(p)), \mathbb{Z}(p)) \to \text{Hom}_{\mathbb{Z}(p)}(E', \mathbb{Z}(p)) \quad [6.2]$$

is surjective.

For any divisor $d \mid N$, define a functional $l_d : E_2(N, \mathbb{Z}(p)) \to \mathbb{Z}(p)$ by

$$l_d(f) = \sum_{t \mid d} \mu(d/t) \sigma_{d/t} a_1(f), \quad [6.3]$$

where $\mu$ is the Möbius function and $\sigma$ is the divisor-sum function. In terms of the duality pairing of Lemma 2.1, $l_d$ corresponds to the Hecke operator

$$t_d := \prod_{\ell \mid N, \ell \text{ prime}} (U_\ell - \ell - 1),$$

in that $l_d(f) = a_1(t_d f)$. Now let $d, s \mid N$ with $d, s > 1$. An elementary computation, either directly using [6.3] or by computing the action of $t_d$ on $f$, shows

$$l_d(f_s) = \begin{cases} -12Nd & s = d \\ 0 & s \neq d. \end{cases}$$

Since $-12Nd \in \mathbb{Z}(p)$, the elements $l_d$ generate $\text{Hom}_{\mathbb{Z}(p)}(E', \mathbb{Z}(p))$ and the map $l_d$ is surjective. \hfill \Box

**Corollary 6.1.** statement 3 was first proved by Takagi (ref. 5, theorem 4.1) using Kubert–Lang theory.

**Data, Materials, and Software Availability.** There are no data underlying this work.

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