INSTANTON IN THE NONPERTURBATIVE QCD VACUUM

N.O. Agasian\textsuperscript{a}, S.M. Fedorov\textsuperscript{b}

Institute of Theoretical and Experimental Physics,
117218, Moscow, B.Cheremushkinskaya 25, Russia

Abstract. The influence of nonperturbative fields on instantons in quantum chromodynamics is studied. Effective action for instanton is derived in bilocal approximation and it is demonstrated that stochastic background gluon fields are responsible for IR stabilization of instantons. It is shown that instanton size in QCD is of order of 0.25 fm. Comparison of obtained instanton size distribution with lattice data is made.

1. Instantons were introduced in 1975 by Polyakov and coauthors \cite{1}. These topologically nontrivial field configurations are essential for the solution of some problems of quantum chromodynamics. Instantons allow to explain anomalous breaking of $U(1)_A$ symmetry and the $\eta'$ mass \cite{2,3}, spontaneous chiral symmetry breaking (SCSB) \cite{4}. Taking into account instantons is of crucial importance for many phenomena of QCD (see \cite{5} and references therein).

At the same time, there is a number of serious problems in instanton physics. The first is IR inflation of instanton, i.e. divergence of integrals over instanton size $\rho$ at big $\rho$. Second, quasiclassical instanton anti-instanton vacuum lacks confinement.

The most popular model of instantons is the model of "instanton liquid", which was phenomenologically formulated by Shuryak \cite{6}. It states that average distance between pseudoparticles is $R \sim 1$ fm and their average size is $\bar{\rho} \sim 1/3$ fm. Thus, $\bar{\rho}/\bar{R} \sim 1/3$ and vacuum consists of well separated, and therefore not very much deformed, instantons and anti-instantons. However, the mechanism for the suppression of large-size instantons in the ensemble of topologically non-trivial fields is still not understood.

On the other hand, QCD vacuum contains not only quasiclassical instantons, but other nonperturbative fields as well. In this talk we will demonstrate that instanton can be stabilized in nonperturbative vacuum and exist as a stable topologically nontrivial field configuration against the background of stochastic nonperturbative fields, which are responsible for confinement, and will find quantitatively it's size. In this way, we will follow the analysis performed in \cite{7}.

2. Standard euclidian action of gluodynamics has the form

$$S[A] = \frac{1}{2g_0^2} \int d^4x \text{tr}(F_{\mu\nu}^2[A]) = \frac{1}{4} \int d^4x F_{\mu\nu}^a[A] F_{\mu\nu}^a[A], \quad (1)$$

where $F_{\mu\nu}[A] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu,A_\nu]$ is the strength of gluon field and we use the Hermitian matrix form for gauge fields $A_\mu(x) = g_0 A_\mu^a(x)t^a/2$ and $\text{tr} t^a t^b = \delta^{ab}/2$. We decompose $A_\mu$ as $A_\mu = A_\mu^{\text{inst}} + B_\mu + a_\mu$, where $A_\mu^{\text{inst}}$ is an

\textsuperscript{a}e-mail: agasian@heron.itep.ru

\textsuperscript{b}e-mail: fedorov@heron.itep.ru
instanton-like field configuration with a unit topological charge \( Q_T[A_{\text{inst}}] = 1 \); \( a_\mu \) is quantum field and \( B_\mu \) is nonperturbative background field (with zero topological charge), which can be parametrized by gauge invariant nonlocal vacuum averages of gluon field strength.

In general case effective action for instanton in NP vacuum takes the form

\[
Z = e^{-S_{\text{eff}}[A_{\text{inst}}]} = \int [Da_\mu] \left< e^{-S[A_{\text{inst}} + B + a]} \right>, \tag{2}
\]

where \( \langle \ldots \rangle \) implies averaging over background field \( B_\mu \).

Integration over \( a_\mu \) and \( B_\mu \) corresponds to averaging over fields that are responsible for the physics at different scales. Integration over \( a_\mu \) takes into account perturbative gluons and describes phenomena at small distances, while averaging over \( B_\mu \) (formally interaction with gluon condensate) accounts for phenomena at scales of confinement radius. Therefore averaging factorizes \( Z = \langle Z_1 \rangle \langle Z_2 \rangle \) (see [3] for details) and effective action appears to be sum of two terms, "perturbative" and "nonperturbative". Perturbative fluctuations were considered in [8,9], and it was shown that in NP vacuum standard perturbation theory for instantons changes, which results in "freezing" of effective coupling constant. The perturbative part of effective instanton action in stochastic vacuum \( S_{\text{eff}}[A_{\text{inst}}] \) was shown to be

\[
S_{\text{eff}}^P(\rho) = \frac{b}{2} \ln \frac{1/\rho^2 + m^2_\ast}{\Lambda^2} \tag{3}
\]

Here \( m_\ast \simeq 0.75 m_{0^{++}} \sim 1 \text{GeV} \), where \( 0^{++} \) is the lightest glueball.

Thus, we have for effective instanton action

\[
S_{\text{eff}}[A_{\text{inst}}] = S_{\text{eff}}^P[A_{\text{inst}}] + S_{\text{eff}}^{NP}[A_{\text{inst}}] \tag{4}
\]

\[
S_{\text{eff}}^{NP}[A_{\text{inst}}] = - \ln \langle Z_2(B) \rangle = - \ln \left< \exp \{-S[A_{\text{inst}} + B] + S[A_{\text{inst}}]\} \right> \tag{5}
\]

3. We consider effect of NP fields on instanton, i.e. we evaluate \( \langle Z_2 \rangle \). In this work we make use of the method of vacuum correlators, introduced in works of Dosch and Simonov [10]. NP vacuum of QCD is described in terms of gauge invariant vacuum averages of gluon fields (correlators)

\[
\Delta_{\mu_1 \nu_1 \ldots \mu_n \nu_n} = \langle \text{tr} G_{\mu_1 \nu_1}(x_1) \Phi(x_1, x_2) G_{\mu_2 \nu_2}(x_2) \ldots G_{\mu_n \nu_n}(x_n) \Phi(x_n, x_1) \rangle,
\]

where \( G_{\mu \nu} \) is gluon field strength, and \( \Phi(x, y) = \text{Pexp} \left( i \int_{y}^{x} B_\mu dz_\mu \right) \) is a parallel transporter, which ensures gauge invariance. In many cases bilocal approximation appears to be sufficient for qualitative and quantitative description.

\[\text{In operator product expansion method and in QCD sum rules nonperturbative field is characterized by a set of local gluon condensates } \langle G^2 \rangle, \langle G^3 \rangle, \ldots \]
of various physical phenomena in QCD. Moreover, there are indications that corrections due to higher correlators are small [11]. Tensor structure of bilocal correlator follows from antisymmetry in Lorentz indices. It is parametrized by two functions $D(z)$ and $D(z)$:

$$
\langle g^2 G^a_{\mu\nu}(x, x_0) G^b_{\rho\sigma}(y, x_0) \rangle = \langle G^2 \rangle \frac{\delta^{ab}}{N_c^2 - 1} \times
$$

$$
\times \left\{ \frac{D(z)}{12} \delta_{\mu\rho} \delta_{\nu\sigma} + \frac{D(z)}{6} (n_\mu n_\rho \delta_{\nu\sigma} + n_\nu n_\sigma \delta_{\mu\rho}) - (\mu \leftrightarrow \nu) \right\},
$$

(6)

where $G_{\mu\nu}(x, x_0) = \Phi(x, x_0) \Phi(x, x_0)$, $n_\mu = z_\mu / |z| = (x-y)_\mu / |x-y|$ is the unit vector, $\langle G^2 \rangle \equiv \langle g^2 G^a_{\mu\nu} G^a_{\mu\nu} \rangle$ and, as it follows from normalization, $D(0) + \overline{D}(0) = 1$.

Bilocal correlator was measured on the lattice (see [12] and references therein), and functions $D(z)$ and $\overline{D}(z)$ were found to be exponentially decreasing

$$
D(z) = A_0 \exp(-z/T_g),
$$

$$
\overline{D}(z) = A_1 z \exp(-z/T_g) / T_g,
$$

where $T_g$ is the gluonic correlation length. Besides, according to lattice measurements $A_1 \ll A_0 \sim A_0 / 10$. Lattice data are presented in Table 1. $SU(3)$ full stands for chromodynamics with 4 quarks, while $SU(2)$ and $SU(3)$ quenched mean pure $SU(2)$ and $SU(3)$ gluodynamics, respectively.

| $\langle G^2 \rangle$, GeV$^4$ | $T_g$, fm |
|----------------|----------|
| $SU(2)$ quenched | 13 | 0.16 |
| $SU(3)$ quenched | 5.92 | 0.22 |
| $SU(3)$ full | 0.87 | 0.34 |

To evaluate $S_{NP}^{\text{eff}}$ we use the cluster expansion:

$$
\langle \exp(x) \rangle = \exp \left( \langle x \rangle + \frac{\langle x^2 \rangle - \langle x \rangle^2}{2!} + \ldots \right)
$$

(7)

In bilocal approximation we find $S_{NP}^{\text{eff}} = S_{\text{dia}} + \frac{1}{2} S_{\text{dia}}^2 + S_{\text{para}} + S_1 + S_2$, where

$$
S_{\text{dia}} = -\frac{1}{2g^2} \int d^4 x \left\langle \text{tr} \left( [A_\mu, B_\nu] - [A_\nu, B_\mu] \right)^2 \right\rangle
$$

(8)

$$
S_{\text{para}} = -\frac{1}{2g^4} \int d^4 x d^4 y \left\langle \text{tr} \left( F_{\mu\nu}(x) G_{\mu\nu}(x) \right) \text{tr} \left( F_{\rho\sigma}(y) G_{\rho\sigma}(y) \right) \right\rangle
$$

(9)

$$
S_1 = \frac{2}{g^4} \int d^4 x d^4 y \left\langle \text{tr} \left( F_{\mu\nu}[A_\mu, B_\nu] \right) \text{tr} \left( F_{\rho\sigma}[A_\rho, B_\sigma] \right) \right\rangle
$$

(10)

$$
S_2 = \frac{2i}{g^4} \int d^4 x d^4 y \left\langle \text{tr} \left( F_{\mu\nu} G_{\mu\nu} \right) \text{tr} \left( F_{\rho\sigma}[A_\rho, B_\sigma] \right) \right\rangle
$$

(11)
We use notations $S_{\text{dia}}$ (diamagnetic) and $S_{\text{para}}$ (paramagnetic) for contributions (8) and (9) into interaction of instanton with background field. Next, $S_{\text{eff}}^{\text{NP}}$ can be expressed through bilocal correlator (see [7] for details), for instance

$$S_{\text{dia}} = \langle G^2 \rangle \frac{N_c}{N_c^2 - 1} \int d^4 x \int_0^1 \alpha d\alpha \int_0^1 \beta d\beta x^2 (A^a_\mu(x))^2 \times$$

$$\times [D((\alpha - \beta)x) + 2D((\alpha - \beta)x)]$$ (13)

4. We use standard form for instanton field configuration

$$A^{\text{inst}}_\mu = 2t^b R^{b\beta}_\mu \eta_{\mu\nu} \frac{(x - x_0)_\nu}{(x - x_0)^2} f \left( \frac{(x - x_0)^2}{\rho^2} \right),$$

where matrix $R^{b\beta}$ ensures embedding of instanton into $SU(N_c)$ group, $b = 1, 2, \ldots N_c^2 - 1; \beta = 1, 2, 3, \eta^a_{\mu\nu}$ are ‘t Hooft symbols. In singular gauge profile function $f(z)$ satisfies boundary conditions $f(0) = 1, f(\infty) = 0$ and the classical solution has the form $f(z) = 1/(1 + z^2)$. Of course, real instanton profile in NP vacuum is different. The problem of asymptotic behavior of instanton solution far from the center $|x| \gg \rho_c$ was studied in detail in Refs. [13–15].

Our numerical analysis shows that the value of $\rho_c$ is almost not affected by the asymptotic of classical instanton solution provided that condensate $\langle G^2 \rangle$ and correlation length $T_g$ have reasonable values.

Numerical calculations show that $S_{\text{dia}}(\rho)$ is a growing function of $\rho$, and it ensured IR stabilization of an instanton. Numerical results for instanton size distribution $dn/d^4zd\rho \sim \exp(-S_{\text{eff}})$ and corresponding lattice data [16] are presented in Fig. 1. All graphs are normalized to the commonly accepted instanton density $1 \text{ fm}^{-4}$. Different lattice groups roughly agree on instantons size within a factor of two, e.g. $\bar{\rho} = 0.3 \ldots 0.6 \text{ fm}$ for $SU(3)$ gluodynamics. There is no agreement at all concerning the density $N/V$. As a tendency, lattice studies give higher density and larger instantons than phenomenologically assumed.

Using our model we find for $\langle G^2 \rangle = 5.92 \text{ GeV}^4$ and $T_g = 0.22 \text{ fm}$ that $\rho_c \approx 0.15 \text{ fm}$, which is less than phenomenological ($\approx 0.3 \text{ fm}$) and lattice results (in full QCD we find $\rho_c \approx 0.25 \text{ fm}$). However, we can present physical arguments to explain these deviations. Lattice calculations include cooling procedure, during which some lattice configurations of gluon field are discarded. This procedure can result in a change in gluon condensate $\langle G^2 \rangle$, and thus instanton size distribution is calculated at a value of gluon condensate $\langle G^2 \rangle_{\text{cool}}$ which is smaller than physical value $\langle G^2 \rangle$. Therefore, lattice data for average instanton size $\bar{\rho}$ should be compared with our calculations for $\rho_c$, performed at smaller values of $\langle G^2 \rangle$. We show dependence of $\rho_c$ on $\langle G^2 \rangle$ for several
values of $T_g$ in Fig. 3. One can see that increase of $\langle G^2 \rangle$ results in decrease of instanton size, and that effect is a result of nonlocal "diamagnetic" interaction of instanton with NP fields.

We did not go beyond bilocal approximation in this work. As mentioned above, this approximation is good enough not only for qualitative, but also for quantitative description of some phenomena in nonperturbative QCD. In the problem under consideration there are two small parameters. These are $1/g^2(\rho_c) \sim 0.15 \ldots 0.25$ and $1/N_c$, and there powers grow in each term of cluster expansion. Moreover, we made an estimate for the sum of leading terms in cluster expansion [17], and found that IR stabilization stays intact ($\rho_c$ appears to be a little smaller). Thus, proposed model describes physics of single instanton stabilization in NP vacuum, not only qualitatively, but also quantitatively with rather good accuracy.

![Figure 1: Instanton density $dn/d^4zd\rho$ and lattice data [14]. $SU(3)$ full (solid line), $SU(2)$ quenched (dotted line) and $SU(3)$ quenched (dashed line)](image)

Figure 1: Instanton density $dn/d^4zd\rho$ and lattice data [14]. $SU(3)$ full (solid line), $SU(2)$ quenched (dotted line) and $SU(3)$ quenched (dashed line)

![Figure 2: Instanton size as a function of gluon condensate ($N_c = 3$, $N_f = 4$) at $T_g = 0.2$ fm (dotted line), $T_g = 0.3$ fm (dashed line), $T_g = 0.34$ fm (solid line)](image)

Figure 2: Instanton size as a function of gluon condensate ($N_c = 3$, $N_f = 4$) at $T_g = 0.2$ fm (dotted line), $T_g = 0.3$ fm (dashed line), $T_g = 0.34$ fm (solid line)
Acknowledgments

We are grateful to Yu.A. Simonov for helpful discussions and comments. The financial support of RFFI grant 00-02-17836 and INTAS grant CALL 2000 N 110 is gratefully acknowledged.

References

[1] A. M. Polyakov, *Phys. Lett.* **B59** (1975) 79; A. A. Belavin, A. M. Polyakov, A. S. Shvarts and Y. S. Tyupkin, *Phys. Lett.* **B59** (1975) 85.

[2] G. ’t Hooft, *Phys. Rev. Lett.* **37** (1976) 8.

[3] E. Witten, *Nucl. Phys.* **B149** (1979) 285; G. Veneziano, *Nucl. Phys.* **B159** (1979) 213.

[4] D. Diakonov and V. Y. Petrov, *Nucl. Phys.* **B272** (1986) 457.

[5] T. Schafer and E. V. Shuryak, *Rev. Mod. Phys.* **70** (1998) 323 [hep-ph/9610451].

[6] E. V. Shuryak, *Nucl. Phys.* **B203** (1982) 93.

[7] N. O. Agasian and S. M. Fedorov, [hep-ph/0111208].

[8] N. O. Agasian and Yu. A. Simonov, *Mod. Phys. Lett.* **A10** (1995) 1755.

[9] N. O. Agasian, *Phys. Atom. Nucl.* **59** (1996) 297.

[10] H. G. Dosch, *Phys. Lett.* **B190** (1987) 177; H. G. Dosch and Yu. A. Simonov, *Phys. Lett.* **B205** (1988) 339; Yu. A. Simonov, *Nucl. Phys.* **B307** (1988) 512.

[11] A. Di Giacomo, H. G. Dosch, V. I. Shevchenko and Yu. A. Simonov, [hep-ph/0007223].

[12] A. Di Giacomo, [hep-lat/0012013].

[13] D. Diakonov and V. Y. Petrov, *Nucl. Phys.* **B245** (1984) 259.

[14] A. B. Migdal, N. O. Agasian and S. B. Khokhlachev, *JETP Lett.* **41** (1985) 497; N. O. Agasian and S. B. Khokhlachev, *Sov. J. Nucl. Phys.* **55** (1992) 628, 633; N. O. Agasian, [hep-ph/9803252, hep-ph/9904227].

[15] A. E. Dorokhov, S. V. Esaiiehgan, A. E. Maximov and S. V. Mikhailov, *Eur. Phys. J.* **C13** (2000) 331 [hep-ph/9903450].

[16] A. Hasenfratz and C. Nieter, *Phys. Lett.* **B439** (1998) 366 [hep-lat/9806026].

[17] N.O. Agasian and S.M. Fedorov, ITEP-PH-6/2001