Free dihedral actions on Abelian varieties

Bruno Aguiló Vidal

Abstract
We give a simple construction for hyperelliptic varieties defined as the quotient of a complex torus by the action of a dihedral group that contains no translations and fixes no points. This generalizes a construction given by Catanese and Demleitner for $D_4$ in dimension three.

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Introduction

A Generalized Hyperelliptic Manifold $X$ is defined to be a quotient $X = T/G$ of a complex torus $T$ by the free action of a finite group $G$ which contains no translations. We say that $X$ is a Generalized Hyperelliptic Variety if moreover the torus $T$ is projective, i.e., it is an Abelian variety $A$.

Uchida and Yoshihara showed that the only non Abelian group that gives such an action in dimension three is the dihedral group $D_4$ of order 8 \[3\]. Later, Catanese and Demleitner gave a simple and explicit construction for that action \[2\] and completed the characterization of three-dimensional hyperelliptic manifolds \[1\].

The purpose of this note is to generalize Catanese and Demleitner’s construction to bigger dihedral groups acting in higher dimension. Specifically, for every $n \in \mathbb{N}$ we give a Generalized Hyperelliptic Variety of dimension $2n + 1$ defined by the action of the dihedral group $D_{4n}$ of order $8n$ acting on a family of Abelian varieties, from which the construction by Catanese and Demleitner remains as the particular case for $n = 1$.

We end with a simple corollary that explains how this allows us to create Generalized Hyperelliptic Varieties using any dihedral group.

The construction
Let $E, E'$ be any two elliptic curves, 

$$E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau), \quad E' = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau').$$
Now, for \( n \in \mathbb{N} \) set \( A' := E^{2n} \times E' \), \( A := A'/\langle w \rangle \), where \( w := (1/2, 1/2, ..., 1/2, 0) \).

**Theorem.** The Abelian Variety \( A \) admits a free action with no translations of the dihedral group \( D_{4n} \) of order \( 8n \).

**Proof.** First, let us recall that for \( k \in \mathbb{N} \), the dihedral group of order \( 2k \) is defined as

\[
D_k := \langle r, s \mid r^k = 1, s^2 = 1, (rs)^2 = 1 \rangle.
\]

Now, set, for \( z := (z_1, z_2, ..., z_{2n}, z_{2n+1}) \in A' \):

\[
r(z) := (-z_{2n}, z_1, z_2, ..., z_{2n-1}, z_{2n+1} + \frac{1}{4n})
\]

\[
= R(z) + (0, ..., 0, \frac{1}{4n}),
\]

\[
s(z) := (-z_{2n} + b_1, -z_{2n-1} + b_2, ..., -z_1 + b_{2n}, -z_{2n+1})
\]

\[
= S(z) + (b_1, b_2, ..., b_{2n}, 0),
\]

where, for \( i = 1, ..., n, b_{2i-1} := 1/2 + \tau/2 \) and \( b_{2i} := \tau/2 \).

**Step 1.** It is easy to verify that \( r \) and \( R \) have order exactly \( 4n \) on \( A' \), and that \( R(w) = w \), so that \( r \) descends to an automorphism of \( A \) of order exactly \( 4n \). Moreover, any power \( r^j, 0 < j < 4n \), acts freely on \( A \) since the \((2n+1)\)-th coordinate of \( r^j(z) \) equals \( z_{2n+1} + \frac{1}{4n} \), and clearly none of this powers is a translation.

**Step 2.** \( s^2(z) = z + w \), since for \( i = 1, ..., 2n, b_i - b_{2n+1-i} = 1/2 \); moreover, \( S(w) = w \), hence \( s \) descends to an automorphism of \( A \) of order exactly \( 2 \).

**Step 3.** We have

\[
rs(z) = r(-z_{2n} + b_1, -z_{2n-1} + b_2, ..., -z_1 + b_{2n}, -z_{2n+1})
\]

\[
= (z_1 - b_{2n}, -z_{2n} + b_1, ..., -z_2 + b_{2n-1}, -z_{2n+1} + \frac{1}{4n}).
\]

hence

\[
(rs)^2(z) = (z_1 - 2b_{2n}, z_2 + b_1 - b_{2n-1}, ..., z_i + b_{i-1} - b_{2n-(i-1)}, ..., z_{2n} + b_{2n-1} - b_1, z_{2n+1})
\]

\[
= z,
\]

and we have an action of \( D_{4n} \) on \( A \), since the orders of \( r \), \( s \) and \( rs \) are precisely \( 4n \), \( 2 \) and \( 2 \).

**Step 4.** We claim that also the symmetries in \( D_{4n} \) act freely on \( A \) and are not translations. Since there are exactly two conjugacy classes of symmetries,
those of $s$ and $rs$, it suffices to observe that these two transformations are not translations. In the next step we show that they both act freely.

**Step 5.** It is rather immediate that $rs$ acts freely, since $rs(z) = z$ in $A$ is equivalent to

$$(−b_{2n}, −z_{2n} − z_2 + b_1, ..., −z_2 − z_{2n} + b_{2n−1}, −2z_{2n+1} + \frac{1}{2n})$$

being a multiple of $w$ in $A'$, but this is absurd since $2w = 0$ and $−b_{2n} = \tau/2 \neq 0, 1/2$.

On the other hand, $s$ acts freely in $A$ because $s(z) = z$ is equivalent to

$$(−z_{2n} − z_1 + b_1, −z_{2n−1} − z_2 + b_2, ..., −z_1 − z_{2n} + b_{2n}, −2z_{2n+1})$$

being a multiple of $w$ in $A'$, but the first and $2n$-th coordinate of multiples of $w$ are equal, while here the difference between them is $1/2 \neq 0$.

\[\square\]

**Using any dihedral group**

Notice that, although the previous construction is somewhat restrictive because it works with very specific dihedral groups, since it is true that $D_n \subseteq D_{nk}$ for all $n, k \in \mathbb{N}$, we have the following corollary:

**Corollary.** For all $n \in \mathbb{N}$, there exists a free action of the dihedral group $D_n$ of order $2n$ on some Abelian variety of dimension $\frac{mcm(4,n)}{2} + 1$ that contains no translations.

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**References**

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