Application of quantum Pinsker inequality to quantum communications

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Abstract—Back in the 1960s, based on Wiener’s thought, Shikao Ikehara (first student of N.Wiener) encouraged the progress of Hisaharu Umegaki’s research from a pure mathematical aspect in order to further develop the research on mathematical methods of quantum information at Tokyo Institute of Technology. Then, in the 1970s, based on the results accomplished by Umegaki Group, Ikehara instructed the author to develop and spread quantum information science as the global information science. While Umegaki Group’s results have been evaluated as major achievements in pure mathematics, their contributions to current quantum information science have not been fully discussed. This paper gives a survey of my talk in the memorial seminar on Ikehara, in which Ikehara and Umegaki Group’s contributions to design theory of quantum communication have been introduced with specific examples such as quantum relative entropy and quantum Pinsker inequality.

I. INTRODUCTION

In the real world, we have no performance evaluation measures for communication system with operational meanings of the global information transmission and processing other than various signal detection criteria established by Wiener, and Shannon entropy by Shannon. The same is true even where the physical system for implementation is generalized into quantum system or relativistic system, which means that modern communication theory is the most successful field among other scientific theories in human history.

Quantum information science originates from quantum communication theory, based on which, quantum communication such as quantum key distribution and quantum symmetric key cipher has been developed. These are formulated based on and as quantum versions of statistical signal detection theory and Shannon’s information transmission theory. The former has a beautiful form as a design theory for detection and estimation techniques of signals transmitted in a quantum state established by Helstrom [1], Holevo [2] and Yuen [3]. The theory of Shannon information transmitted in a quantum state was started by Stratonovich, Holevo, et al and studied by Yuen [4] and Hirota [5] from the viewpoint of quantum state control as well as by Jozsa Group [6] and Masashi Ban of Tamagawa University Group [7] in the context of accessible information.

On the other hand, in mathematics, theories may be developed without regard for the operability of information handled by humans or with a focus on physical phenomena of a specific device. Shannon’s entropy clearly defines very common signals processed in human social activities and it is applied to the communication system with the operational meanings for the relevant information processing. Quantum information theory as mathematics can be modeled on it. However, unlike Shannon information, quantum entropy lacks versatility regard for operational meaning on information transmission. It is defined in consideration of mathematical form or application to physics. Therefore, no new information scientific technology is expected from simple generalization of the concepts of Wiener and Shannon. Only when its contribution to Wiener-Shannon system is proved, the mathematics is deemed to have contributed to the global information science.

It was Holevo who discovered a liaison between quantum entropy and Shannon information. In the study of accessible information (an application of Shannon’s mutual information) to quantum system, its upper bound is now called Holevo bound, which is given in the form of quantum entropy, and its measure is called Holevo information. Holevo derived the upper bound in flow of the result of Stratonovich on Nth extended system in relation to Shannon mutual information in quantum system (see Reference 8). Prior to that, Tamagawa University Group had published a paper providing specific examples of maximization of accessible information and superadditivity in Nth extended systems. Jozsa Group proved that the maximum amount of accessible information reaches Holevo information in the limit of pure quantum state system without external noise. Jozsa presented the results at the 3rd International Conference on Quantum Communication, Measurement (QCM 1996). While Holevo had shifted to research on quantum stochastic process, I recommended him that he should generalize Jozsa’s works. In a very short while, Holevo and Tamagawa University Group discussed the proof method in the general model of external noise system with Kitaev, and only after a week later, Holevo showed us the proposed proof on a white board at Tamagawa University.
We conveyed the results to Jozsa and Holevo submitted his proof to IEEE’s Transaction on Information Theory. Jozsa Group also started their consideration and the proof was completed by Shumacher-Westmoreland and submitted to the Physical Review. Although published at different times, these are now called Holevo-Shumacher-Westmoreland theorem as the formula of discrete channel capacity of classical-quantum composite system [9, 10].

Additionally, Holevo and Tamagawa University Group gave a guide on formula for Gaussian channel capacity and for reliability function in 1996–2000 [11, 12]. As a result, Holevo information, which is expressed in the form of quantum entropy and serves as a parameter of the extreme point of Shannon system, has greatly contributed to the real world.

On the other hand, Umegaki and his group developed as mathematics in a quantum entropy form without considering the operational meaning like Shannon information. This paper will clarify Umegaki’s contribution on the application of quantum information theory to real world issue.

II. PROGRESS IN QUANTUM ENTROPY

A. Approach of Umegaki and Holevo

Let us here denote the most fundamental formula in Shannon’s communication theory in the following.

\[ H(X) = -\sum_x P(x) \log P(x) \]  

\[ H(X|Y) = -\sum_y \sum_x P(y|x)P(x|y) \log P(x|y) \]  

\[ I(X,Y) = H(X) - H(X|Y) = H(Y) - H(Y|X) = H(X) + H(Y) - H(X,Y) \]

This is called von Neumann entropy. Research on a quantum version of relative entropy, which is regarded as a mathematical generalization of Shannon’s entropy theory, was begun for the first time in the world by Hisaharu Umegaki at Tokyo Institute of Technology. This was driven by the motivation of Shikao Ikehara to recommend the succession of Wiener thought [14,15]. Umegaki, for the first time in the world, defined the following quantum relative entropy on the von Neumann algebra in 1962 and formulated its various features [16-18].

**Definition - 1(Umegaki)**

\[ D_q(\rho||\sigma) = Tr\{\rho[\log \rho - \log \sigma]\} \]  

\[ \text{supp}(\rho) \subseteq \text{supp}(\sigma) \]

The receiving system performs a quantum measurement on the quantum system, and becomes a model for determining the classical parameter \( \{ x \} \) as a message. This is a problem of Accessible information (Shannon mutual information of classical-quantum composite systems).

\[ I_{\text{acc}} = \max_{\Pi} I(X_c,Y_q) \]

where \( \Pi \) is detection operator or positive operator valued measure (POVM). That is, a message \( x \) is mapped to a quantum density operator which corresponds to a concrete signal. The quantum density operator of the set is described as follows:

\[ \rho_{\epsilon} = \sum_x p(x)\rho_{\epsilon x}^q \]

Thus, from a mathematical definition point of view, the relative entropy looks like “General”. On the other hand, von Neumann defined entropy for quantum systems in response to the development of quantum statistical mechanics. The entropy of a quantum system with the quantum density operator: \( \rho_{X_q} \in \mathcal{D}(H_2) \) is

\[ S(X_q) = S(\rho_{X_q}) = -Tr\{\rho_{X_q} \log \rho_{X_q}\} \]

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**Definition - 2(Holevo)**

\[ \chi(\epsilon) = S(\rho_{\epsilon}) = \sum_x p(x)S(\rho_{\epsilon x}^q) \]
This is called the Holevo information. As stated in the introduction, the Holevo-Shumacher-Westmoreland theorem guarantees that the limit of Shannon information transmitted in a quantum system is the maximum value of Holevo information. This fact shows that quantum entropy theory contributes to the Shannon theory. 

As mentioned above, Umegaki developed with a foundation of mathematical statistics, and Holevo developed quantum communication theory while faithfully inheriting Shannon’s world.

B. More progress

Although quantum entropy can be developed by mathematical formalism, there is no guarantee that it will have the operational meaning applicable in the real world like Shannon theory. Wiener criticized simple mathematical formalism, claiming that unless a research directly transforms the technical system in the real world through mathematical generalization, it is not an authentic mathematical study. This is called Wiener’s criteria for mathematical generalization research. This paper discusses whether quantum entropy theory satisfies Wiener’s criteria.

Based on Umegaki’s quantum relative entropy, we can formally replace Shannon’s formula with its quantum version. It was carried out faithfully under the development of quantum entropy theory as mathematics by collaborating with both Japanese and foreign researchers such as Accardi, Belavkin, Ohya, and Petz [19].

Let us construct the Shannon’s mutual information by quantum entropy form. When the quantum density operator is \( \rho_{X,Y} \in D(H_X \otimes H_Y) \), we have

\[
S(X,Y) = -Tr\{\rho_{X,Y} \log \rho_{X,Y} \}
\]

Therefore, the quantum mutual information can be defined as follows.

\[
I_q(X,Y) = S(X) + S(Y) - S(X,Y)
\]

However, since the quantum entropy is not information in the meaning of a message, the above expression does not have an operational meaning of a general communication system. On the other hand, mathematically, the Holevo information can be expressed in this context. Let us assume that the quantum density operator as a set of classical-quantum composite systems is given by

\[
\rho_{X,Y} = \sum_x p(x)|x><x| \otimes \rho_{Y|X}^x
\]

So we have

\[
S(X,Y) = H(X) + \sum_x p(x)S(Y|X)
\]

From the above, also we have

\[
\chi(\epsilon) = I_{q\epsilon}(X,Y)
\]

In this way, the quantum mutual information contains formally the Holevo information, but the operational meaning is completely different, and only the Holevo information has significance for the Shannon system that has a great impact on the real world. On the other hand, from a mathematical point of view, the quantum mutual information can be expressed in terms of quantum relative entropy. That is,

\[
I_q(X,Y) = D_q(\rho_{X,Y} || \rho_{X} \otimes \rho_{Y})
\]

Furthermore, the Holevo information is also described by quantum relative entropy as follows:

\[
\chi(\epsilon) = \sum_x p(x) D_q(\rho_{X,Y}^x || \rho_{X,Y}^x)
\]

Thus, from the mathematical point of view, Umegaki’s quantum relative entropy is the most fundamental notion. Based on this, quantum entropy theory has advanced rapidly in the 21st century as a mathematical study applying quantum statistical physics [20]. In the next section, we focus on applicability of the abstract mathematics to real communication systems based on Pinsker inequalities that give linkages to statistics and signal detection theory.

III. Quantum Pinsker Inequality

Relative entropy is essentially the distance between two probability distributions in statistics. Thus, it is natural to consider the relationship with various mathematical distances. In general, the distance between two probability distributions is called a statistical distance or Kolmogorov distance, and is defined as follows.

\[
||P(x) - Q(x)||_c = \sum_x |P(x) - Q(x)|
\]

Such a concept of distance is often discussed in the language of distinguishability, and it is a source of great misunderstanding when one applies such mathematics to another problem. Here, it is discussed as a distance. The most important inequality in distance relations in statistics is the following Pinsker inequality shown by Pinsker in 1964.

**Theorem 1 (Pinsker)**

\[
D_c(P(x)||Q(x)) \geq \frac{1}{2 \ln 2} ||P(x) - Q(x)||^2_c
\]

By utilizing this, generalization to the mutual information for the composite system becomes possible as follows:

**Theorem 2**

\[
I(X,Y) \geq \frac{2}{\ln 2} \Delta_c^2
\]

\[
\Delta_c = \frac{1}{2} ||P(x,y) - P(x)P(y)||_c
\]
The trace distance is bounded by Holevo Information as follows:

$$\Delta_q = Tr\{\Pi^{opt}(\rho - \sigma)\} = \frac{1}{2}||\rho - \sigma||_q$$ (23)

where $\Pi^{opt}$ is detection operator or a positive operator valued measure (POVM). The relationship between relative entropy and statistical distance shifts to the relationship between quantum relative entropy and trace distance. It was expressed by the cooperation among Hiai, Ohya, and Tsukada as follows [21].

**Theorem - 3 (Quantum Pinsker Inequality)**

$$D_q(\rho||\sigma) \geq \frac{1}{2 \ln 2}||\rho - \sigma||_q^2$$ (24)

This can be further generalized to a quantum composite system. Let us assume quantum density operators in composite system as follows: $\rho_{X_0Y_0} \in D(H_X \otimes H_Y)$. $\Delta_q = 1/2||\rho_{X_0Y_0} - \rho_{X_0} \otimes \rho_{Y_0}||_q$ then we have the relation between quantum mutual information $I_q(X_q; Y_q)$ and trace distance.

**Theorem - 4**

$$I_q(X_q; Y_q) \geq \frac{1}{2 \ln 2}||\rho_{X_0Y_0} - \rho_{X_0} \otimes \rho_{Y_0}||_q^2 = \frac{1}{2} \Delta_q^2$$ (25)

At this stage, quantum entropy theory does not play an important role in the Wiener-Shannon systems, and does not contribute as a design theory for real communication technologies. But in the next section, we will show remarkable result for applications to real world issue.

**IV. Upper Bound Theory of Guessing Probability in QKD**

In the quantum entropy theory, only the Holevo information contributes to the Wiener and Shannon systems related to information communication systems, and it opened up the real world of optical quantum communication systems. On the other hand, in the context of Umegaki’s research, the Holevo information can be formally expressed as a special example of quantum mutual information from Eq(16). Here let us denote the trace distance as follows:

$$\Delta_q = \frac{1}{2}||\sum_x p(x)|x < x|_{X_0} \otimes \rho_{Y_0}^x - \sum_x p(x)|x > x|_{X_0} \otimes \rho_{Y_0}^x||_q$$ (26)

Here we can show the following important theorem [22] based on Eqs(17, 18, 25, 26):

**Theorem - 5**

The trace distance is bounded by Holevo Information as follows:

$$\chi(\epsilon) \geq \frac{2}{\ln 2} \Delta_q^2$$ (27)

Even at this stage, the trace distance of the two quantum density operators in the above show only the characteristics of the quantum system, and the relationship with the evaluation of the technical operation in the Wiener-Shannon system is not clear. That is, the contribution to the real system is not visible. In order to show that the theory of quantum entropy contributes to the Wiener-Shannon’s communication theory, it is necessary to show that the trace distance defined before observation contributes directly to traditional performance evaluation measures in Wiener-Shannon system.

Before entering the main topic, we discuss with regard to the trace distance in the theory of quantum key distribution (QKD), because there is a theory that is misunderstood. It is supposed in QKD theory that there are quantum density operators formed by real protocols and quantum density operator formed by ideal protocols. They introduced Helstrom’s quantum signal detection theory as a model to discriminate between these two quantum density operators and show the following average error probability or detection probability from the Helstrom formula.

$$P_e = \frac{1}{2} [1 - \Delta_q(\rho^{AE}_{\lambda}; \rho_{AE}')]$$ (28)

$$P_d = \frac{1}{2} [1 + \Delta_q(\rho^{AE}_{\lambda}; \rho_{AE}')]$$ (29)

where $\Delta_q = 1/2||\rho_{AE}^{\lambda} - \rho_{AE}'||_q$ is the trace distance. In the first place, there is no physical communication system that transmits and receives real and ideal quantum density operators, so this model cannot be a tool for discussing the security of QKD. In addition, $\Delta_q$ is a parameter and cannot have a meaning as probability by itself. So the trace distance does not contribute to security evaluation at this stage.

However, in 2009, Yuen provided a significant inequality. That is, when the attacker accesses the number of signals $M = 2^K$ with key sequence length $|K|$ for real protocols in the context of the security of QKD, for the statistical distance $\Delta_c$ of the probability distributions after quantum measurement, the upper bound of the guessing probability is given as follows [23,24,25].

**Theorem - 6 (Yuen)**

$$P(K)_{guess} \leq \frac{1}{M} + \Delta_c$$ (30)

$$\Delta_c = \frac{1}{2}||P(x, y) - P(x)P(y)||_c$$

The author’s group were able to conclude from the discussion with Yuen that the above relationship could be applied to the level of trace distance before making specific observations. The final expression is as follows [22,26].
Theorem 7
Let the trace distance of the quantum density operators between an actual protocol and the ideal one be:
\[
\Delta_q = \max_{\Lambda} \text{Tr}\Lambda(\sum_k p(k)p_E^k - \rho_K \otimes \rho_E) \quad (31)
\]
\[k \in M, \quad \Lambda : \text{POVM}\]
Then the average guessing probability for real QKD signals is
\[
\frac{1}{M} \leq P(K)_{\text{guess}} \leq \frac{1}{M} + \Delta_q \quad (32)
\]
where \(\chi(\epsilon) \geq \frac{\epsilon}{\ln 2} \Delta_q^2\) from the theorem 5.

The above equations can be obtained from the relationship between the statistical distance and the trace distance, but another direct proof is shown in the appendix for the convenience of the readers.

From the results of Theorem 5 and Theorem 7, theory of Umegaki and his group contributes to the design theory of actual communication technologies in a sense different from the Holevo-Shumacher-Westmoreland theorem via the Holevo information.

V. CONCLUSION
Quantum communication theory based on the basic concept of Wiener and Shannon has already contributed to the real optical communication system in a concrete manner [27, 28, 29]. I believe that this success originates from Ikehara’s human resource development activities for mathematical basic research of quantum information, which were passed down to the later generations as part of Umegaki’s extensive mathematical research. I also believe that philosophies of Ikehara and Umegaki greatly influenced researchers of the world who contributed to the development of today’s quantum information science via the researchers trained at the international conference [30-39] that I established under the direction of Ikehara.

Finally, I add a short remark. As Wiener pointed out, in order to create new concepts in the future, the researchers of information science must lead quantum information science. In other words, a mathematical research is expected to be carried out in consideration of real communication system from the perspective of information science, not quantum statistical physics. We have excellent researchers who have approached from information science and mathematics in Japanese community. Thus I appreciate the great contributions of the Japanese researchers listed below:
In the entropy theoretical approaches, Masaki Sohma [40], Masahito Hayashi [41], Keiji Matsumoto [42, 43] and Tomohiro Ogawa and Hiroshi Nagaoka [44]. In the signal detection approaches, Akio Fujiwara [45], Masashi Ban [46], Masao Osaki [47], Kenji Nakahira [48], Kentaro Kato [49], Tsuyoshi Usuda [50], Jun Suzuki [51]. In the coding theory approach, Mitsuhiro Hamada [52].

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Then, the average guessing probability is given by

\[
P(K)_{\text{guess}} = \max_{i=1}^{M} \sum_{i=1}^{M} p(y_i)p(x_i|y_i)
\]

\[
= \max_{i=1}^{M} \sum_{i=1}^{M} p(i)p(y_i|x_i)
\]

\[
= P_d
\]

(39)

The lower bound of the detection probability is simply \(1/M\) as the pure guessing in the signal detection theory. It can be given by \(\Delta_q = 0\) which means that the real case is equal to the ideal case [22,26]. That is, there is no correlation between key sequence \(K\) and observation data \(E\) of Eve. So one can denote associated with Shannon theory in the perfect case as follows:

\[
P(K|E) = P(K), \quad \text{or} \quad H(K|E) = H(K)
\]

(40)