CONSTRUCTING EMBEDDINGS AND ISOMORPHISMS OF
FINITE ABSTRACT SEMIGROPS

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\textbf{Abstract.} We present a search algorithm for constructing embeddings and
deciding isomorphisms of semigroups, working with their multiplication tables.
The algorithm is used for enumerating diagram semigroups up to isomorphism
and for finding minimal degree representations.

1. INTRODUCTION

Semigroups are abstract algebraic structures with a single associative binary
operation, often called multiplication. Deciding whether or not a semigroup $S$
embeds into another semigroup $T$ ($T$ contains a copy of $S$), or whether they are
isomorphic ($S$ and $T$ are essentially the same) is a non-trivial task, even if they
are explicitly represented as multiplication tables. For instance, one reason they
are non-trivial is that algebraic structure isomorphism can be reduced to graph-
isomorphism [24].

Since the only requirement for semigroup multiplication is associativity, semi-
group elements come in a great variety: numbers, functions, matrices, relations,
diagrams, just to name a few. Embedding a given semigroup into a semigroup of
another kind is a way of constructing different representations of the semigroup. A
basic result of semigroup theory is that any semigroup of order $n$ can be embed-
ded into a transformation semigroup of degree $n + 1$ (the semigroup analogue of
Cayley’s Theorem for groups, e.g. [18], Chapter 1). In practice, we would like to
have a transformation representation of lower degree. Since transformation semi-
groups are finite state automata (without specifying initial and accepting states),
the embedding problem is equivalent to the question: \textit{What is the minimal number
of states needed for a computation?} Current software packages for transformation
semigroups lack the functionality of reducing the degree. Here we give a customized
backtrack search algorithm that hugely improves the current computational capa-
bilities in semigroup theory, so we can answer some open problems with a little
computation.

Similarly, for the question \textit{What is computable with $n$ states?}, we need a practical
algorithm to find isomorphic semigroups. Using the algorithm described here, we
extended results of [3, 28] and enumerated all 4-state finite computations up to
isomorphism [9]. The enumeration was also extended for more general diagram
semigroups [11].

2. Notation

\textit{Semigroups and Homomorphisms.} A semigroup is simply a set $S$ together with an
associative binary operation $S \times S \rightarrow S$. Given two semigroups $(S, \cdot)$ and $(T, \star)$,
a \textit{homomorphism} is a function $\varphi : S \rightarrow T$, such that $\varphi(u \cdot v) = \varphi(u) \star \varphi(v)$ for all
An embedding is an injective homomorphism, denoted by $S \hookrightarrow T$. The image of an embedding of $S$ into $T$ is an isomorphic copy of $S$ inside $T$.

**Multiplication Tables.** Let $S$ be a finite semigroup such that $|S| = n$. We fix an order on the semigroup elements $s_1, \ldots, s_n$, so we can refer to the elements by their indices. Then the multiplication table, or Cayley table of $S$ is a matrix $S_{n \times n}$ with entries from $\{1, \ldots, n\}$, such that $S_{i,j} = k$ if $s_i s_j = s_k$, and we can simply write $ij = k$. The $i$th row is $S_{i,\ast}$ and the $j$th column vector is $S_{\ast,j}$.

**Transformation Semigroups.** A transformation is a function $f : X \to X$ from a set to itself. If $|X| = n \in \mathbb{N}$, then the set is denoted by positive integers $\{1, \ldots, n\}$ and a transformation $t$ is denoted by simply listing the images of the points: $[t(1), t(2), \ldots, t(n)]$. A transformation semigroup $(X, S)$ of degree $n$ is a collection $S$ of transformations of an $n$-element set, closed under function composition. The semigroup of all transformations of $n$ points is the full transformation semigroup $T_n$. The group consisting of all permutations (bijective transformations) of degree $n$ is the symmetric group $S_n$. The cyclic (one-generated) group of order $n$ is denoted by $\mathbb{Z}_n$. The dihedral group, the symmetry group of a regular $n$-gon, containing $2n$ elements, is denoted by $D_n$.

Transformations $s, t \in T_n$ are conjugate if there exists a permutation $g \in S_n$ such that $t = g^{-1}sg$. In other words, we get $t$ if we relabel the points of $s$ according to the permutation $g$.

## 3. Algorithm for Embedding Multiplication Tables

Given semigroups $S$ and $T$ such that $|S| \leq |T|$, we would like to know if we can construct an embedding $S \hookrightarrow T$, and if so, actually construct one. Let $m = |S|$ and $n = |T|$, so we need a map $\varphi : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that $\varphi(i) \varphi(j) = \varphi(ij)$, $\varphi$ is a homomorphism, and $\varphi(i) = \varphi(j)$ implies $i = j$, so $\varphi$ is injective.

**Example 3.1.** For permutation group $\mathbb{Z}_2 = \{(\cdot), (1, 2)\}$ and transformation semigroup $T_2 = \{[1, 1], [1, 2], [2, 1], [2, 2]\}$, the maps $1 \mapsto 2$, $2 \mapsto 3$ give an embedding (encoding the elements by integers corresponding to their lexicographic ordering). The isomorphic copy of the source semigroup is indicated by boldface inside the target semigroup.

| $\mathbb{Z}_2$ | $T_2$ |
|----------------|-------|
| 1 2            | 1 2 3 4 |
| 1 2            | 1 1 4 4 |
| 2 1            | 2 1 3 4 |
| 1 2 3 4        |            |

For finding embeddings, the brute-force algorithm goes through all possible maps by checking all possible assignments of the $m$ elements of $S$ to the $n$ elements of $T$. There are $\frac{n!}{(n-m)!}$ such maps. By gradually exploiting more information about $\varphi$, $S$ and $T$ we can construct increasingly more efficient algorithms.

If some elements violate the homomorphism property, then regardless where the other elements are sent, we cannot have a homomorphism. In other words, a partial non-solution cannot be extended to a solution. Therefore the classical backtrack method [20] can be applied to this problem. For constructing the embedding $\varphi : S \hookrightarrow T$, let $p$ be a partial solution represented by a sequence of integers, such that the $i$th element is $\varphi(i)$. So, $p = (\varphi(1), \varphi(2), \ldots, \varphi(l))$ for some $l \leq |S|$. Being
a homomorphism requires that if \( ij = k \) (in the multiplication table of \( S \)), then \( \varphi(i)\varphi(j) = \varphi(k) \), (in the multiplication table of \( T \)). However, \( k, \varphi(k) \) may not be in the partial solution yet. If a product \( ij = k \) is not in the domain of the partial map \( \varphi \), then the product \( \varphi(i)\varphi(k) \), should also be undefined, i.e. not being in the sequence \( p \). Now assume that \( p \) could be a homomorphism. Then we extend its sequence by choosing a new \( \varphi(l+1) \) from the remaining elements of \( T \) and check whether the homomorphism property is true for products containing \( l+1 \). We also have to check for previous undefined entries, since they may evaluate to \( l+1 \) and thus become defined. If the above conditions are true, then we can continue extending \( p \) by \( l+2 \). If not, then according to backtrack, we choose another \( \varphi(l+1) \), or if there is no such candidate, then going back to \( l \) and looking for alternative \( \varphi(l) \), and so on.

We can further reduce the number of choices available for the search algorithm by using precomputed information about the semigroup elements. This is done by giving equivalence relations on \( S \) and \( T \) such that elements of a class in \( S \) may only be mapped by an embedding to an element of a single class of \( T \). In other words, we classify elements of \( S \) and \( T \) by properties that are invariant under embeddings. Therefore, when trying to extend a partial solution, we need to look for candidates only in the corresponding class. We call this algorithm the partitioned backtrack search.

For abstract semigroups, one such invariant is the semigroup generalization of the order of the element.

**Definition 3.2** (Index-period). For an element \( a \) of finite semigroup \( S \), the index-period is the pair \((m, r)\) of the smallest values \( m \geq 1, r \geq 1 \) such that \( a^{m+r} = a^m \).

**Definition 3.3.** Let \( S \) be a finite semigroup. Then the equivalence \( \cong_e \) is defined by
\[
s \cong_e t \iff s \text{ and } t \text{ have the same index-period values.}
\]

How might partitioning by \( \cong_e \) improve the search? In a two-fold way: we can detect easily if embedding is not possible, and we can reduce the search space.

Prior to the search, we check whether for each \( A \in S/\cong_e \) the corresponding class \( B \in T/\cong_e \) (with same index-period as \( A \)) satisfies \(|A| \leq |B|\). If these conditions are not fulfilled, then the embedding is not possible.

Assuming that the conditions are satisfied, let’s denote the class in \( T/\cong_e \) corresponding to the class \( A \in S/\cong_e \) by \( \varphi(A) \). Then the size of the search space is given by
\[
\prod_{A \in S/\cong_e} \frac{|\varphi(A)|!}{(|\varphi(A)| - |A|)!},
\]
showing that the efficiency depends on the number of the classes and their cardinalities.

**Example 3.4.** If \(|S| = 5\) and \(|T| = 10\) the search space size is \( \frac{10!}{(10-5)!} = 30240 \). Assuming that we can partition \( S \) into \( \cong_e \)-classes of size 2, 3 and \( T \) into corresponding classes of size 3, 5 (and a class of size 2 with index-period not appearing in \( S \)), then the search space size is reduced to \( \frac{3!}{(3-2)!} \cdot \frac{5!}{(5-3)!} = 360 \).
4. Deciding Isomorphism of Multiplication Tables

Isomorphism is a special case of an embedding, thus we can define more invariants and a finer partitioning of elements by using a stronger equivalence relation. These invariant properties can be any statistics of a multiplication table that do not take into account any information of the actual ordering of the elements in the table. For instance, the number of elements in a table that have \( k \) occurrences in total is a usable invariant, while the number of occurrences of the \( k \)-th element is not. Therefore, frequency distributions are the prime candidates for invariant properties.

A frequency distribution takes a multiset and enumerates its distinct elements paired with the number of occurrences of the elements. For instance, the frequency distribution of a row vector \([7, 2, 1, 2, 5, 2, 7]\) is \([[1, 1], [2, 3], [5, 2], [7, 2]]\), meaning that 1 appears once, 2 appears three times and 5 and 7 twice. However, we cannot retain the element information as it depends on the sorting of the semigroup elements. We keep only the sorted frequency values \([1, 2, 2, 3]\). Sorting is crucial here to decide whether two distributions are the same or not. For example, the vector \([2, 4, 2, 4]\) has the same frequency distribution as \([1, 3, 3, 1]\) and \([2, 2, 4, 4]\), but \([2, 4, 4, 4]\) has a different one. We can also take the frequency distribution of frequency values, since it is derived from data containing no information on the ordering of the elements.

We can define invariant properties both on the element and on the table level. In addition to the index-period values, the element level invariants are:

- **frequency**: the number of occurrences of the element in the table,
- **diagonal frequency**: the number of occurrences of the element in the diagonal of the table,
- **row frequencies**: the number of occurrences of the element \( i \) in its row \( S_{i,*} \),
- **column frequencies**: the number of occurrences of the element \( i \) in its column \( S_{*,i} \).

We put these invariants together in a single aggregated data structure called element profile. With this, similar to \( \cong_e \), we can define an equivalence relation for constructing isomorphisms.

**Definition 4.1.** Let \( S \) be a finite semigroup. Then the equivalence \( \cong_i \) is defined by

\[
\text{if } s \cong_i t \iff s \text{ and } t \text{ have the same element profile.}
\]

The table level invariants can be used to decide the possibility of the isomorphism. They are:

- **frequency distribution elements**: the numbers of occurrences of elements in the multiplication table,
- **column and row frequencies**: the frequency distribution of column and row frequencies,
- **diagonal frequencies**: number of occurrences of diagonal elements,
- **idempotent frequencies**: frequency distributions can also be calculated for the set of idempotents, giving a very strong semigroup invariant,
- **element profiles**: the set of element profiles and their frequency distributions,
How strong are these invariants? Some of them are sensitive enough to tell apart groups, despite their very special nature (multiplication tables are Latin squares, all principal ideals are the same, there is only one idempotent).

Example 4.2. Let $S$ be the semigroup generated by transformations $[2, 1, 1], [2, 3, 2]$, and $[3, 1, 3]$. Using only index-period values to classify its 15 elements, the size of the search space to find isomorphisms to another representation of $S$ is 2903040. Classifying by element profiles defined above reduces to search space size to 768.

5. Applications

5.1. Automorphism Groups of Semigroups. Backtrack search methods can systematically produce all solutions. In case of isomorphism, the set of solutions corresponds to the automorphism group of the semigroup. For verification purposes, we recomputed the results of [1] (not shown here), using our method on small semigroups [6]. We also used this technique for finding isomorphism classes and determining the automorphism groups of all transformation semigroups of degree 4 [9], see Table 1.

5.2. Diagram Semigroup embeddings. From the point of view of theoretical computer science and automata theory, transformation semigroups are the most important representations. However, they are just one special case of diagram semigroups, where semigroup elements can be composed by adjoining diagrams. Several of these are studied as diagram algebras and in representation theory: $\mathcal{PB}_n$ partitioned binary relations [23], $B_n$ binary relations [26], $\mathcal{PT}_n$ partial transformation semigroup [14], $\mathcal{P}_n$ (bi)partition monoid [17, 19, 22], $\mathcal{B}_n$ Brauer monoid [4], $S_n$ symmetric group [5, 7], $\mathcal{T}_n$ full transformation semigroup [14], $I_n$ symmetric inverse monoid [21], $I^*_n$ dual symmetric inverse monoid [15], and $\mathcal{T}_L n$ Temperley-Lieb monoid [16]. For the partial order of diagram semigroups of the same degree see Figure 1.

Various diagram representations can be considered as different computational paradigms. The interesting question is how much ‘bigger’ a less capable computational device should be in order to realize a more powerful one. For instance, to emulate computations of equivalence classes with transformations, for $n = 1$ we have to double the degree $\mathcal{P}_1 \hookrightarrow \mathcal{T}_2$, and for $n = 2$ we need at least 5 states:
Table 1. Automorphism groups of all the 132069775 nonempty transformation semigroups of degree 4 up to conjugation. The last column is the SmallGroup library identification [2].

| #conjugacy classes | automorphism group | (order,index) |
|---------------------|--------------------|---------------|
| 131286736           | 1                  | (1,1)         |
| 748946              | \( \mathbb{Z}_2 \) | (2,1)         |
| 29138               | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) | (4,2)         |
| 1296                | \( D_4 \)          | (8,3)         |
| 1144                | \( S_3 \)          | (6,1)         |
| 969                 | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) | (8,5)         |
| 717                 | \( \mathbb{Z}_3 \) | (3,1)         |
| 296                 | \( D_6 \)          | (12,4)        |
| 182                 | \( \mathbb{Z}_2 \times D_4 \) | (16,11)       |
| 58                  | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3 \) | (24,14)       |
| 48                  | \( S_3 \times S_3 \) | (36,10)       |
| 47                  | \( S_4 \)          | (24,12)       |
| 44                  | \( \mathbb{Z}_2 \times S_4 \) | (48,48)       |
| 29                  | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) | (16,14)       |
| 28                  | \( D_4 \times S_3 \) | (48,38)       |
| 22                  | \( \mathbb{Z}_2 \times S_3 \times S_3 \) | (72,46)       |
| 12                  | \( (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_2 \) | (32,27)       |
| 8                   | \( S_4 \times S_3 \) | (144,183)     |
| 8                   | \( (S_3 \times S_3) \times \mathbb{Z}_2 \) | (72,40)       |
| 8                   | \( D_4 \times S_4 \) | (192,1472)    |
| 8                   | \( \mathbb{Z}_2 \times S_3 \times S_4 \) | (288,1028)    |
| 6                   | \( \mathbb{Z}_2 \times (S_3 \times S_3) \times \mathbb{Z}_2 \) | (144,186)     |
| 6                   | \( \mathbb{Z}_2 \times S_2 \times S_4 \) | (96,226)      |
| 4                   | \( (S_3 \times S_3) \times \mathbb{Z}_2 \times S_3 \) | (432,741)     |
| 4                   | \( (\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_2 \) | (18,4)        |
| 4                   | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times D_4 \) | (32,46)       |
| 2                   | \( (S_3 \times S_3) \times \mathbb{Z}_2 \times S_4 \) | (1728,47847)  |
| 2                   | \( \mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_2 \) | (36,13)       |
| 2                   | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3 \) | (48,51)       |
| 1                   | \( \mathbb{Z}_2 \times D_4 \times S_3 \) | (96,209)      |

\( \mathbb{P}_2 \hookrightarrow \mathcal{T}_5 \). Similar results include: \( \mathfrak{B}_1 \cong \mathcal{T}_1 \), \( \mathfrak{B}_2 \hookrightarrow \mathcal{T}_3 \), TL1 \( \cong \mathcal{T}_1 \), TL2 \( \hookrightarrow \mathcal{T}_2 \), TL3 \( \hookrightarrow \mathcal{T}_4 \), \( \mathfrak{P}_1 \hookrightarrow \mathfrak{B}_2 \), \( \mathfrak{B}_3 \hookrightarrow B_3 \), \( \mathcal{I}_4 \hookrightarrow B_3 \), \( \mathcal{T}_2 \hookrightarrow \mathfrak{B}_3 \) but \( \mathcal{T}_3 \) does not embed into \( \mathfrak{B}_6 \).

**Definition 5.1.** For a semigroup \( S \) the minimal \( D \)-diagram representation degree is \( \mu_D(S) = \min \{ n \mid S \hookrightarrow D_n \} \) where \( D \in \{ \mathbb{P}B, B, \mathbb{P}T, T, S, \mathcal{I}, \mathcal{I}^*, P, \mathfrak{B}, \mathcal{T} \} \).

**Example 5.2.** We can find minimal degree diagram representations of a matrix semigroup over \( \mathbb{Z} \) by using its multiplication table. Let

\[
S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\},
\]

and using the ordering of the element matrices...
Table 2. Number of embeddings of full transformation semigroups. Embedding the trivial monoid is equivalent to finding idempotent elements \((e^2 = e)\) up to conjugation in the target semigroup.

| \(\mathcal{T}_m \xrightarrow{k} \mathcal{T}_n\) | \(n = 1\) | \(n = 2\) | \(n = 3\) | \(n = 4\) | \(n = 5\) | \(n = 6\) |
|---|---|---|---|---|---|---|
| \(m = 1\) | 1 | 2 | 3 | 5 | 7 | 11 |
| \(m = 2\) | 0 | 1 | 3 | 12 | 35 | 110 |
| \(m = 3\) | 0 | 0 | 1 | 4 | 17 | 64 |
| \(m = 4\) | 0 | 0 | 0 | 1 | 2 | 6 |
| \(m = 5\) | 0 | 0 | 0 | 0 | 1 | |

be its multiplication table. Then \(\mu_T(S) = \mu_P(S) = 3\), so we can have transformation semigroup representation of \(S\) using 3 points, but in this particular case we cannot reduce the degree by switching to the more general partition monoid representation. Also, we need two more points to represent \(S\) as Brauer and Temperley-Lieb monoid: \(\mu_B(S) = \mu_{TL}(S) = 5\).

Beyond the question whether embedding is possible, we are also interested in the number of different embeddings, up to conjugation and up to automorphism. We use the notation \(S \xrightarrow{k} T\) to indicate that there are \(k\) distinct copies of \(S\) inside \(T\). Table 2 summarizes \(k\)-embeddings for low degree transformation semigroups.

There are 282 non-empty transformation semigroups on 3 points up to conjugation. For degree 4, this number is 132069775 [9]. How many isomorphic copies of the degree 3 transformation semigroups can we find inside \(T_4\)? Counting the embeddings up to conjugation we find only 2347 subsemigroups of \(T_4\) isomorphic to some subsemigroup of \(T_3\); so most degree 4 transformation semigroups are ‘new’. Calculating the same number for \(T_5\) yields 18236; a modest increase compared to the still unknown, but expected-to-be gigantic, number of degree 5 transformation semigroups.

5.3. Embeddings into 2-generated subsemigroups. Here we will answer a couple of open questions stated in [8] using the embedding algorithm. There the primary interest is in the embeddability into a 2-generated semigroup, denoted by \(S \xrightarrow{2\text{-gen}} T\). Being \(n\)-generated means that the semigroup can be generated by \(n\) elements. It is easy to prove that \(\mathcal{B}_n \xrightarrow{2\text{-gen}} \mathcal{B}_{n+2}\), but there is no 2-generated subsemigroup of \(\mathcal{B}_{n+1}\) where we can embed \(\mathcal{B}_n\). The same was conjectured for the partition monoid, but contrary to the expectation we found that \(\mathcal{P}_2 \xrightarrow{2\text{-gen}} \mathcal{P}_3\), in 3 different ways.

To obtain this result, we enumerated conjugacy class representatives of subsemigroups of the target semigroup that are 2-generated, then filtered them for the
property of being at least as big as the source semigroup. Finally, we used our embedding search to check each of the candidate 2-generated subsemigroups.

6. Conclusion

We presented customized backtrack search algorithms for constructing semigroup embeddings and isomorphisms. In the SubSemi package [10] developed for the GAP computer algebra system [15] we implemented the above algorithms. Despite being inefficient for large semigroups due to the multiplication table representation, they proved to be immensely useful in practical computations. They have been used extensively in several computational semigroup theory projects [9,11,12], where they have provided a huge amount of interesting raw data for mathematical research.

To our best knowledge, the SubSemi package is the first software tool capable of calculating general semigroup embeddings. For the isomorphism calculation, we can compare our method to SmallestMultiplicationTable function in the SEMIGROUPS package [25] (based on the SmallestImageSet function in GAP [15] and depending on the GRAPE package [27]). This function calculates the smallest multiplication table in lexicographic ordering for the semigroup (by calculating a group action orbit of the symmetric group), which can be used for isomorphism testing. Its complexity depends on the size of the semigroup since we have to act on the table by the corresponding symmetric group, while our method does not require group actions and it uses more information about the semigroup structure, thus it is more scalable. In practice, roughly speaking, with SubSemi we can calculate embeddings and isomorphisms as long as the multiplication table of the target semigroup fits into the memory (e.g. embeddings into $T_6$ with $6^6 = 46656$ elements).

With these algorithmic advances, we can further advance our theoretical knowledge of semigroups and finite state automata, which in turn will help us to design more efficient algorithms.

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