Entanglement dynamics under local Lindblad evolution

Sandeep K. Goyal[†] and Sibasish Ghosh[∗]

The Institute of Mathematical Sciences, CIT campus, Chennai 600 113, India

The phenomenon of entanglement sudden death (ESD) in finite dimensional composite open systems is described here for both bi-partite as well as multipartite cases, where individual subsystems undergo Lindblad type heat bath evolution. ESD is found to be generic for non-zero temperature of the bath. At $T=0$, one-sided action of the heat bath on pure entangled states of two qubits does not show ESD.

I. INTRODUCTION

Entanglement is considered to be the most useful resource in Quantum Information Theory [1]: it is essential for quantum teleportation, superdense coding, communication complexity problem, one-way computation etc. Not only that creating entangled state is a non-trivial task, to store or transmit entangled states in an error-free manner is also difficult, if not impossible, due to the very fragile character of quantum system—every quantum system has a high possibility to interact with its environment, and thereby, the system will, in general, get entangled with its environment. This will give rise to the phenomenon of decoherence [2].

In general, the purity of any initial state of the quantum system goes down with time in the presence of decoherence. This decoherence time depends on the system as well as on the character of the interaction of the system with its environment. So, due to the monogamy property of entanglement, the initial entanglement (if any) of a bi-partite or multipartite quantum system will, in general, decay (to zero) when each individual system undergoes a decoherence procedure. What can be said about the associated rate of the above-mentioned decay in entanglement? How does one compare the rate of decoherence of the individual subsystems and the rate of decay in initial entanglement among the subsystems? In this connection, Yu and Eberly [3] described a phenomenon called “entanglement sudden death (ESD)” in which the entanglement decay rate is shown to be exponentially larger than the rate of decoherence. This happens whenever the individual qubits of a two-qubit system undergo evolution under local heat bath action at zero temperature.

ESD was shown for a certain class of two-qubit states which were initially entangled at zero temperature ($T=0$). It was then generalized for $T \neq 0$ for “$X$” states [4]. In this paper, we want to find out the largest set of two-qubit states each of which undergoes ESD due to Markovian local heat bath dynamics. If the initial bath state is the thermal state, at finite temperature ($T \neq 0$) it is shown that all two-qubit states show ESD. The same is true also where initial states of the bath is a squeezed thermal state. On the other hand, under one sided Markovian heat bath evolution (i.e, only one of the two qubits is undergoing Markovian heat bath evolution), one does not see any ESD at $T = 0$ for any two-qubit pure state if the initial state of the bath is the vacuum state. However, we will find ESD if the initial state of the bath is the squeezed vacuum. This feature does not hold for one-sided quantum non-demolition (QND) evolution [10] with squeezed thermal state as the initial state of the bath. In this case, no two-qubit pure entangled state show ESD.

In deriving all this, we have used the factorization rule for concurrence [7]. We have also described the ESD phenomena for states of $d \otimes 2$ systems as well as those of $n$-qubits by using the above factorization rule.

The structure of the paper is as follows: in section II we discuss the ESD in two-qubit system with the bath acting only on one qubit. In section III, we generalize the factorization law for entanglement decay [7] to $d \otimes 2$ dimensional systems. In section IV, we use the result of section III and show ESD in $n$-qubit systems. In section V, we show ESD when the bath is in squeezed thermal state initially. In section VI, we consider the QND-type evolution of the individual qubits. Finally in section VII we will derive the sufficient condition for ESD in all dimensions. Section VIII contains our conclusion and discussions on some open problems.

II. TWO-QUBIT UNDER LOCAL BATH

We study a 2-qubit system initially prepared in an entangled state where one of them (say the first one) is interacting with a reservoir at temperature $T$. If the initial state of the bath in contact of the system under consideration be the thermal state, then the dynamics of a single-qubit density matrix $\rho$ describing the system (under the Born-Markov-rotating wave approximation) is given by:

$$
\frac{d\rho}{dt} = \frac{(N+1)\gamma}{2} [2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-] \\
+ \frac{(N)\gamma}{2} [2\sigma_+\rho\sigma_- - \sigma_-\sigma_+\rho - \rho\sigma_-\sigma_+] 
$$

where $N$ is the mean occupation number of the reservoir, $\gamma$ is the spontaneous decay rate of the qubits, $\sigma_+ = |1\rangle\langle 0|$ and $\sigma_- = |0\rangle\langle 1|$. Here we are ignoring the unitary part of the evolution which is irrelevant for our purpose [3] [12].

[†] Electronic address: goyal@imsc.res.in
[∗] Electronic address: sibasish@imsc.res.in
We can rewrite eq. (1) as:
\[ \dot{\rho} = \Lambda[\rho] \Rightarrow \dot{\rho}_{ij} = \sum_{kl} L_{ij,kl} \rho_{kl} \] (2)

where \( L \) is the matrix representation for \( \Lambda \) (called Lindblad operator [13]). The solution for eq. (2) is:
\[ \rho(t)_{ij} = \sum_{kl} V_{ij,kl} \rho(0)_{kl}. \] (3)

Here \( V = \exp(Lt) \) is a completely positive map as \( \rho(t) \) is a valid density matrix of the qubit at any time \( t \). Our aim is to find entanglement in the evolved 2-qubit state after applying the map \( V \) on one qubit. For that purpose we will use the factorization law for entanglement decay [7] which states that the concurrence of a two qubit pure state \( |\chi\rangle \) with one qubit being subject to an arbitrary channel \( V \) will satisfy the equation \( C(\rho_{AB}) = C(|\chi\rangle) C((I \otimes V)(|\phi^+\rangle|\phi^+\rangle)) \) where \( \rho_{AB} = (I \otimes V)(|\chi\rangle\langle\chi|) \). To show entanglement sudden death (ESD) in any state [3], it is sufficient to show that the state \( |\phi^+\rangle \) evolves to a separable state under the action of given single qubit operation \( V \). We can calculate the \( V \) matrix for our case:
\[ V = \begin{bmatrix} \frac{1+\cot^2}{2} + x \cot(\theta) & 0 & 0 & y_2 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ y_1 & 0 & 0 & 1+\cot^2 - x \cot(\theta) \end{bmatrix} \] (4)

where \( y_1 = 2x \cot(\theta)N, \ y_2 = 2x \cot(\theta)(1+N), \ \tan(\theta) = \left( \frac{2x(1+2N)}{x^2-1} \right), \ x = \exp\left[-\frac{1}{2}\gamma(1+2N)t\right]. \) When we apply this \( V \) on one side of \( |\phi^+\rangle \) we will get the mixed state:
\[ M(t) = \begin{bmatrix} \frac{1+\cot^2}{2} + x \cot(\theta) & 0 & 0 & x \\ 0 & y_2 & 0 & 0 \\ 0 & 0 & y_1 & 0 \\ x & 0 & 0 & 1+\cot^2 - x \cot(\theta) \end{bmatrix} \] (5)

For the matrix \( M(t) \) to represent a separable state, the partial transpose of \( M(t) \) should be a positive semi-definite matrix [8, 9]. If \( \sinh(\gamma(1+2N)t) \geq 2 \sqrt{\frac{N(N+1)}{(1+2N)^2}} \), the matrix \( M(t) \) is positive semi-definite under partial transposition and hence is separable. This shows that the operator \( V \) acting on one-qubit evolves the maximally entangled state \( |\phi^+\rangle \) into a separable state. The factorization law for entanglement decay implies that all the two qubit pure entangled states show ESD.

Now consider the zero temperature case \( (T = 0) \). For \( T = 0 \) the mean occupation number \( N \) is zero and hence \( y_1 \) is also zero. The matrix \( M \) matrix will never be positive under partial transposition and hence will always represent an entangled state. The direct implication of this result is that there is no pure two qubit entangled state that will show ESD at \( T = 0 \).

For \( T \neq 0 \), every two-qubit pure state shows ESD and since mixed states are convex combinations of pure states, every two-qubit mixed state will show ESD. However, at \( T = 0 \) some mixed states do show ESD even though no pure state does.

In section [IV] we will extend our result to the \( n \)-qubit case. But before that we will derive the generalized factorization law for entanglement decay, i.e., the factorization law for \( d \otimes 2 \) systems in the next section.

### III. FACTORIZATION LAW FOR ENTANGLEMENT DECAY FOR \( d \otimes 2 \) SYSTEMS

In this section, we consider the effect of any single-qubit trace-preserving map \( \$ \) on side \( B \) of any bipartite density matrix \( \rho_{AB} \), where \( \text{dim} \mathcal{H}_A = d \). In ref. [17] one can see some discussion on the factorization law for entanglement decay for \( d \otimes d \) systems.

To start with, we consider first its effect on any pure state \( |\chi\rangle_{AB} \) having Schmidt form \( |\chi\rangle_{AB} = \sqrt{\rho}|00\rangle_{AB} + \sqrt{1-\rho}|11\rangle_{AB} \). Here \( |0\rangle_B, |1\rangle_B \) are the eigenvectors of \( \sigma_z \) for the subsystem \( B \) and \( |0\rangle_A, |1\rangle_A \) are taken from the standard orthonormal basis \( \{|0\rangle_A, |1\rangle_A, \ldots, |d-1\rangle_A\} \) of \( \mathcal{H}_A \) with \( 0 < p < 1 \).

As \( (I_A \otimes \$)(|\chi\rangle_{AB}) \) is a two-qubit density matrix, one can find out its concurrence \( C((I_A \otimes \$)(|\chi\rangle_{AB})). \) So, according to fig. [4], we have
\[ C((I_A \otimes \$)(|\chi\rangle_{AB}))) = C(N : (|\phi^+\rangle_{DE}(|\chi\rangle_{AD} \otimes \rho_{EC}(\$))|\phi^+\rangle_{DE}) , \] (6)
where $\rho_{EC} = (I_E \otimes \mathbb{S})\left((\phi^+)_{EC}(\phi^+)\right)$, $\mathcal{N}: Z \equiv Z/(\text{Tr}Z)$ and $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is a two qubit maximally entangled state.

We now perform the POVM $\{M = \sqrt{p}|0\rangle_1|0\rangle + \sqrt{1-p}|1\rangle_1, \text{Id}_{d \times d} - M\}$ on side $A$ of the $d \otimes 2$ density matrix $\rho_{AC}(\mathbb{S})$. So in the case when $M$ is clicked, the normalized output state is given by

$$\mathcal{N}: (M \otimes I_C)\rho_{AC}(\mathbb{S})(M \otimes I_C) : = \sigma_{AC}, \quad (7)$$

Note here that $\rho_{AC}(\mathbb{S})$ is same as $\rho_{EC}(\mathbb{S})$ with just $E$ replaced by $A$. Along the line of (7), it will follow immediately that

$$\mathcal{N}: \left<\phi^+|DE \left(\left<\chi\right|_{AD} N \otimes \rho_{EC}(\mathbb{S}) \left|\phi^+\right>\right) \right|_{DE} : = \sigma_{AC}. \quad (8)$$

Thus we see that $(I_A \otimes \mathbb{S})\left|(\chi)_{AB}\left(\chi\right)\right\rangle = \sigma_{AC}$. So we have

$$\left|(I_A \otimes \mathbb{S})\left|(\chi)_{AB}\left(\chi\right)\right\rangle\right\rangle = \mathcal{N}: (M' \otimes I_C)\{(I_A \otimes \mathbb{S})\left(|\phi^+\rangle_{AB}|\phi^+\rangle\right)\}(M' \otimes I_C) : \equiv \sigma_{AC} \quad (say), \quad (9)$$

where $M' = U_A\mathcal{M}U_A^\dagger = U_A(\mathbb{S}\sqrt{p}|0\rangle_1|0\rangle + \mathbb{S}\sqrt{1-p}|1\rangle_1)U_A^\dagger$ and $|\phi^+\rangle_{AB} = (U_A \otimes U_B)|\phi^+\rangle_{AB}$. One can then write

$$\sigma_{AC} = \mathcal{N}: (U_A \otimes I_B)[\left(M \otimes I_C\right)\{(I_A \otimes \mathbb{S})\left(|\chi\rangle_{AB}|\chi\rangle\right)\}(U_A^\dagger \otimes I_B) : = \mathcal{N}: (U_A \otimes I_B)\tau_{AC}(U_A^\dagger \otimes I_B) : \quad (10)$$

Therefore, $\mathcal{C}(\sigma_{AC}) = \mathcal{C}(\tau_{AC})$.

Now

$$\tau_{AC} \tau_{AC} = \tau_{AC}(r_y \otimes r_y)\tau_{AC}^*(r_y \otimes r_y)$$

$$(MVA \otimes IC)\rho_{AC}(\mathbb{S})(V_A^T M \otimes IC)(r_y \otimes r_y)$$

$$\times \left(MVA^* \otimes IC\right)\rho_{AC}(\mathbb{S})^*(V_A^T M \otimes IC)(r_y \otimes r_y), \quad (11)$$

which will, in turn, show that $\det(\tau_{AC} \tau_{AC} - \lambda I_{d \times 4}) = \det(\eta_{AC}(M\sigma_y \otimes \sigma_y)\eta_{AC}^*(M\sigma_y \otimes \sigma_y) - \lambda I_{d \times 4})$ with $\eta_{AC} = (V_A \otimes IC)\rho_{AC}(\mathbb{S})(V_A^T \otimes IC)$.

Thus the concurrence of $\tau_{AC}$ is same as that of the two-qubit state $\eta_{EC}$. So $\mathcal{C}((I_A \otimes \mathbb{S})\left(|\chi\rangle_{AB}\left(\chi\right)\right) = \mathcal{C}(\eta_{EC})$. Now, for the state $\eta_{EC}$, we have already seen that the factorization rule holds good for concurrence. Therefore,

$$\mathcal{C}((I_A \otimes \mathbb{S})\left(|\chi\rangle_{AB}\left(\chi\right)\right) = \mathcal{C}(\eta_{AC}), \quad (12)$$

Next we consider the case when $|\chi\rangle_{AB}$ has the following Schmidt decomposition:

$$|\chi\rangle_{AB} = (U_A \otimes U_B)(\sqrt{p}|00\rangle_{AB} + \sqrt{1-p}|11\rangle_{AB}) = (U_A \otimes U_B)|\chi_{00}\rangle_{AB} \quad (13)$$

where $U_A$ is a $d \times d$ unitary matrix while $U_B$ is a $2 \times 2$ unitary matrix.

Let $\rho_{AB}$ be a mixed state on $\mathbb{C}^d \otimes \mathbb{C}^2$. Let us consider an arbitrary ensemble representation for $\rho_{AB}$:

$$\rho_{AB} = \sum_{j=1}^{N} p_j |\phi_j\rangle_{AB}\langle\phi_j|, \quad (14)$$

$E_F(\rho)$ being the entanglement of formation. Now

$$E_F(|\phi_j\rangle_{AB}\langle\phi_j|) = f(\mathcal{C}(|\phi_j\rangle_{AB}\langle\phi_j|)), \quad (15)$$
where $f(x)$ is a monotonic function of $x \in [0, 1]$. Here

$$E_F(\mathbb{I}_A \otimes \mathbb{S}(\rho_{AB}))$$

$$= E_F \left( \sum_{j=1}^{N} p_j (\mathbb{I}_A \otimes \mathbb{S})(\phi_j)_{AB} \langle \phi_j | \right)$$

$$\leq \sum_{j=1}^{N} p_j E_F( (\mathbb{I}_A \otimes \mathbb{S})(\phi_j)_{AB} \langle \phi_j |)$$

$$= \sum_{j=1}^{N} p_j f( (\mathbb{S}(\phi_j)_{AB} \langle \phi_j |))$$

$$= \sum_{j=1}^{N} p_j f( C( (\mathbb{I}_A \otimes \mathbb{S})(\phi_j)_{AB} \langle \phi_j |)) C( (\phi_j)_{AB} \langle \phi_j |)$$

$$= \sum_{j=1}^{N} p_j f( C( (\mathbb{I}_A \otimes \mathbb{S})(\phi_j)_{AB} \langle \phi_j |)) C( (\phi_j)_{AB} \langle \phi_j |)$$

$$\left( 18 \right)$$

Nevertheless, when $C( (\mathbb{I}_A \otimes \mathbb{S})(\phi_j)_{AB} \langle \phi_j |)$ becomes zero $f( C( (\mathbb{I}_A \otimes \mathbb{S})(\phi_j)_{AB} \langle \phi_j |))$ becomes zero. So, in that case, $E_F( (\mathbb{I}_A \otimes \mathbb{S})(\rho_{AB})) = 0$.

**IV. ESD in $n$-QUBIT SYSTEM**

In this section we will be dealing with the evolution of entanglement of an $n$-qubit system. We will prove that at finite temperature, all the $n$-qubit states show ESD under the map $V$ – given in eq $(4)$ acting on each qubit locally. To make the proof less cumbersome we will do it for the three-qubit case only. Let the state of the three-qubit system be $|\psi\rangle_{ABC}$ evolving under the bath $V \otimes V \otimes V$. The final state is:

$$\rho = (V \otimes V \otimes V)(|\psi\rangle_{ABC} \langle \psi|)$$

$$= (\mathbb{I} \otimes \mathbb{I} \otimes V)(V \otimes \mathbb{I} \otimes \mathbb{I})(V \otimes V \otimes \mathbb{I}) (|\psi\rangle_{ABC} \langle \psi|)$$

$$= (\mathbb{I} \otimes \mathbb{I} \otimes V)(V \otimes V \otimes \mathbb{I})\rho_{A:BC}$$

$$= (\mathbb{I} \otimes \mathbb{I} \otimes V)\rho_{A:BC}$$

$$\left( 19 \right)$$

where $\rho_{A:BC} = (V \otimes \mathbb{I} \otimes \mathbb{I})(|\psi\rangle_{ABC} \langle \psi|)$ and $\rho_{A:BC} = (\mathbb{I} \otimes V \otimes \mathbb{I})\rho_{A:BC}$. We have seen in section $11$ that $V$ evolves $|\phi^+\rangle$ to a separable state in some time $t$, and so the factorization law for entanglement decay for $2 \otimes d$ system implies that the state $\rho_{A:BC}$ will be separable in the partition $A : BC$ at time $t$. As this time $t$ is the maximum time any state can take to lose entire entanglement by the action of channel $V$ on one subsystem, we can see that at the same time $t$ the reduced state $\rho_{BC} \equiv \text{Tr}[\rho_{A:BC}]$ becomes separable in the partition $B : C$. Hence we get the full separability in the partition $A : B : C$. So all the three-qubit pure states show ESD. Mixed states are the convex sum of pure states. If all the pure states show ESD, all the mixed states will also show ESD. This result can be generalized to any the $n$-qubit case and we will get full $n$ separability at time $t$.

**V. SQUEEZED THERMAL BATH**

In this section we will consider the case of an $n$-qubit system $S$ interacting with a squeezed thermal bath acting locally on each of the $n$ individual qubits. The evolution of the reduced density matrix of the system $S$ in the interaction picture has the form $[10, 12]$:

$$\frac{d}{dt}\rho^s(t) =$$

$$\gamma_0 (N + 1) \left( \sigma_+ \rho^s(t) \sigma_+ - \frac{1}{2} \sigma_+ \sigma_- \rho^s(t) - \frac{1}{2} \rho^s(t) \sigma_+ \sigma_- \right)$$

$$+ \gamma_0 N \left( \sigma_+ \rho^s(t) \sigma_- - \frac{1}{2} \sigma_+ \sigma_- \rho^s(t) - \frac{1}{2} \rho^s(t) \sigma_- \sigma_+ \right)$$

$$- \gamma_0 M \sigma_+ \rho^s(t) \sigma_+ - \gamma_0 M^* \sigma_- \rho^s(t) \sigma_- \left( 20 \right)$$

Here $\gamma_0$ is the spontaneous emission rate given by $\gamma_0 = 4\omega^2|d|^2/3\hbar e^3$, and $\sigma_+$, $\sigma_-$ are the standard raising and lowering operators, respectively given by $\sigma_+ = |1\rangle\langle 0|$, $\sigma_- = |0\rangle\langle 1|$ and $2N + 1 = \cosh(2\omega)/(2N_{th} + 1)$. $M = -\frac{1}{2}\sinh(2\omega)e^{i\omega}2N_{th} + 1$, $N_{th} = \frac{2\hbar^2}{e^2} - 1$. Here $N_{th}$ is Planck’s distribution giving the number of thermal photons at the frequency $\omega$, $r$ and $\phi$ are squeezing parameters, $d$ is the transition matrix elements of the dipole operator and $c$ is the speed of light in vacuum. We have neglected the Hamiltonian evolution part in eq $(20)$ as we did in the evolution equation $(1)$. Moreover, eq $(20)$ is in a Lindblad form and hence it corresponds to a completely positive map $V$ $[11, 12]$.

From eqn $(20)$ we can find our $V_{sq}$ matrix which can be written as:

$$V_{sq} = \begin{bmatrix} \alpha & 0 & 0 & \beta \\ 0 & ye^{-i\omega t} & ze^{-i\omega t} & 0 \\ 0 & ze^{i\omega t} & ye^{i\omega t} & 0 \\ \mu & 0 & 0 & \nu \end{bmatrix},$$

$$\left( 21 \right)$$

where

$$\alpha = \frac{N(1 + x^2) + x^2}{2N + 1}$$

$$\beta = \frac{N(1 - x^2)}{2N + 1}$$

$$\mu = \frac{(N + 1)(1 - x^2)}{2N + 1}$$

$$\nu = \frac{N(1 + x^2) + 1}{2N + 1}$$

$$x^2 = \exp[-\gamma_0(2N + 1)t]$$

$$y = \cosh \left( \frac{\gamma_0 at}{2} \right)$$

$$z = \sinh \left( \frac{\gamma_0 at}{2} \right)$$

$$a = \sinh(2\omega)(2N_{th} + 1)$$

The state corresponding to this channel $V_{sq}$ can be given by Choi-Jamilowski isomorphism $[14, 15]$. The corre-
sponding state is:

\[
(\mathbb{I} \otimes V_{sq})(|\phi^+\rangle\langle\phi^+|) = M_{sq}
\]

\[
= \begin{bmatrix}
\alpha & 0 & 0 & ye^{-i\omega t} \\
0 & \beta & ze^{-i\omega t} & 0 \\
0 & ze^{i\omega t} & \mu & 0 \\
ye^{i\omega t} & 0 & 0 & \nu
\end{bmatrix}
\]

This matrix \(M_{sq}\) is a positive semi-definite matrix. The positivity under partial transposition, i.e,

\[
\alpha \nu - |\epsilon|^2 \geq 0, \quad \beta \mu - \eta^2 \geq 0, \quad (23)
\]
equivalently

\[
\frac{N(N + 1)}{2N + 1} 4 \cosh^2 \left( \frac{\gamma_0 (2N + 1) t}{2} \right) - \sinh^2 \left( \frac{\gamma_0 t}{2} \right) \geq 0,
\]

\[
\frac{N(N + 1)}{2N + 1} 4 \sinh^2 \left( \frac{\gamma_0 (2N + 1) t}{2} \right) - \cosh^2 \left( \frac{\gamma_0 t}{2} \right) \geq 0.
\]

ensures separability. It is enough to show that equation \((23)\) is true in order to show that the state \(M_{sq}\) is separable as \(\cosh^2 \left( \frac{\gamma_0 (2N + 1) t}{2} \right) \geq \sinh^2 \left( \frac{\gamma_0 t}{2} \right)\) and \(\sinh^2 \left( \frac{\gamma_0 t}{2} \right) \leq \cosh^2 \left( \frac{\gamma_0 t}{2} \right)\). At \(t = 0\) this condition \((23)\) is violated. At very large \(t\), this condition is true if \(2N + 1 > a\), i.e, \(2N + 1 > \sinh(2r)(2N_{th} + 1)\). Now \(\sinh(2r) \leq \cosh(2r)\) which implies \(2N + 1 > \sinh(2r)(2N_{th} + 1)\), since \(2N + 1 = \cosh(2r)(2N_{th} + 1)\). This shows that there exists a finite time \(t\) at which \(M_{sq}\) will turn from an entangled state to a separable state. This shows that one sided operation of \(V_{sq}\) on \(|\phi^+\rangle\langle\phi^+|\) evolves it into a separable state. Now using the factorization law for entangled states for \(d \otimes 2\) case (as was done in section \VI\), we can say that all \(n\)-qubit states shows ESD.

When \(r = 0\), i.e, when there is no squeezing then \(\sinh(2r) = 0\) and \(2N + 1\) is always greater than \(\sinh(2r)(2N_{th} + 1)\). This shows that when there is squeezing, the system takes more time to lose the entanglement. So instead of choosing the thermal bath one can choose squeezed thermal bath and delay the ESD process.

At \(T = 0\), ESD does not occur for a pure two-qubit entangled state if one considers the effect of \(V\) on one of the qubits where the initial state of the bath is a vacuum and the evolution is governed by equation \((1)\). But for the case of squeezed bath, at \(T = 0\), the mean occupation number \(N\) is not zero and hence all the pure state show ESD. This implies that at zero temperature one can reduce the time of dissipation of the entanglement by switching on squeezing. Conversely, switching on squeezing at \(T \neq 0\) delays ESD.

VI. QUBIT IN QUANTUM NON DEMOLITION (QND) INTERACTION WITH BATH

In this section we will study the evolution of a system of qubits in QND interaction with bath. QND open quantum systems are those systems in which the system Hamiltonian commutes with the interaction Hamiltonian. In this section we will be considering the evolution which Banerjee et al. has considered in \[10\]. We can write the CP map \(V_{QND}\) and the \(M_{QND}\) following equation \((10)\) of reference \[10\]:

\[
V_{QND} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{-i\omega t}e^{-(\gamma(t))^2\gamma(t)} & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
M_{QND} = \begin{pmatrix}
1 & 0 & 0 & e^{-i\omega t}e^{-(\gamma(t))^2\gamma(t)} \\
0 & 0 & 0 & 0 \\
e^{-i\omega t}e^{-(\gamma(t))^2\gamma(t)} & 0 & 0 & 1
\end{pmatrix},
\]

where \(\omega\) is the natural frequency of the system and \(\gamma(t)\) is the time dependent spontaneous decay parameter (see \[10\]). We can see that the matrix \(M_{QND}\) is not positive under partial transposition and hence under the one-sided action of \(V_{QND}\), \(|\phi^+\rangle\langle\phi^+|\) will not become separable. Therefore this map will not show ESD.

VII. SUFFICIENT CONDITION FOR ESD IN ANY FINITE DIMENSIONAL SYSTEM

In this section we will derive the sufficient condition for ESD for any multipartite system where the dimension of each subsystem is \(d\). For that purpose we will consider the state-channel duality for \(d \otimes d\) system. Consider a map \(\$\) acting on the \(B\) subsystem, where the state of the system \(AB\) is \(|\chi\rangle_{AB} = \sum_{i=1}^{d} \sqrt{p_i}|i\rangle\). The action of the channel on the state \(|\chi\rangle_{AB}\) can be written as:

\[
(\mathbb{I}_A \otimes \$)(|\chi\rangle_{AB}|\chi\rangle) = \mathbb{N} : \langle\Phi^+_{DE}(|\chi\rangle_{AD}(\chi) \otimes \rho_{EC}(\$))|\phi^+\rangle_{DE}) =
\]

\[
\mathbb{N} : (\mathbb{M}_A \otimes \mathbb{I}_C)\rho_{AC}(\mathbb{M}_A \otimes \mathbb{I}_C)^\dagger : \Rightarrow E((|I_A \otimes \$\rangle \langle I_A|)_{AB}(\chi)) = E(F \mathbb{N} : (\mathbb{M}_A \otimes \mathbb{I}_C)\rho_{AC}(\mathbb{M}_A \otimes \mathbb{I}_C)^\dagger :).
\]

where \(\mathbb{M}_A = \sum_{i} p_{i}|i\rangle\langle i|\) and \(\rho_{AC} = (\mathbb{I} \otimes \$)(|\phi^+\rangle\langle\phi^+|)\). If \(\$\) turns out to be an entanglement breaking channel \[10\] (equivalently, if \((\mathbb{I}_A \otimes \$)(|\phi^+\rangle_{AB}\langle\phi^+|)\) is separable), then \(\mathbb{N} : (\mathbb{M}_A \otimes \mathbb{I}_C)\rho_{AC}(\mathbb{M}_A \otimes \mathbb{I}_C)^\dagger : \) is also separable and hence, \(E((|I_A \otimes \$\rangle \langle I_A|)_{AB}(\chi)) = 0\). Thus separability of \((|I_A \otimes \$\rangle \langle I_A|)_{AB}(\phi^+\langle \phi^+|)\), for a given \$, is sufficient for ESD.

Now, based on the discussion in section \[11\] we can claim that if we have a channel \$ which shows ESD in
$d \otimes d$ systems, then the same operation, acting locally on each subsystem, will show ESD in multipartite system as well.

**VIII. CONCLUSION**

In this paper we have shown that if the initial bath state is the thermal state, at finite temperature ($T \neq 0$) all two-qubit states as well as multiqubit states show ESD. The same result has been shown for squeezed thermal initial states of the bath. We have found that under the one-sided action of the Markovian heat bath evolution, one does not see any ESD at $T = 0$ in two-qubit pure state where the initial state of the bath is the vacuum state. This result is in sharp contrast with the case when the initial state of the bath is squeezed vacuum in which all two-qubit states show ESD. It has been shown that squeezing delays ESD in the finite temperature case but speeds it up at zero temperature. We have found that for the action of one-sided QND evolution with squeezed thermal state as the initial state of the bath, no two-qubit pure entangled state show ESD. Finally we have derived the sufficient conditions for the ESD for any multipartite system of identical (but distinguishable) particles. There are articles where one can find ESD in the systems evolving under non-Markovian evolution [18].

After having the full understanding of ESD in Lindblad type of evolution, the following problems become prominent: estimating the exact time to ESD for mixed entangled states; the factorization law for concurrence or any other useful measure of entanglement for two-qubit mixed entangled state; a general scheme for controlling ESD; a complete analysis of ESD in multipartite states and the quasi-covariance of dynamical evolution. One can have a full understanding of ESD only after all the problems have been solved.

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[1] N.A. Nielsen, I.L. Chuang, *Quantum Computation and Quantum Information*, (Cambridge University Press, Cambridge, 2002).
[2] W.H. Zurek, Phys. Today, 44, 36 (1991).
[3] T.Yu and J.H. Eberly, Phys. Rev. Lett. 93, 140404 (2004).
[4] A. Al-Qasimi and D.F.V. James, Phys. Rev. A 77, 012117 (2008).
[5] M. Ali, G. Alber, and A.R.P. Rau, J. Phys. B: At. Mol. Opt. Phys. 42, 025501 (2009).
[6] E. Andersson, J.D. Cresser, and M.J.W. Hall, J. Mod. Opt. 54, 15 (2007).
[7] T. Konrad, F. Melo, M. Tiersch, C. Kasztelan, A. Aragao and A. Buchleitner, Nature Physics 4, 99 (2008).
[8] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[9] M. Horodecki, P. Horodecki, R. Horodecki, Phys. Lett. A 223, 1, (1996).
[10] S. Banerjee and R. Srikant, Eur. Phys. J. D 46, 335-344 (2008).
[11] M.O. Scully, M.S. Zubairy, *Quantum optics* (Cambridge University Press, Cambridge, 1997).
[12] H.P. Breuer, F. Petruccione, *The theory of open Quantum Systems* (Oxford University Press, 2002).
[13] G. Lindblad, commun. Math. Phys. 48, 119-130, (1976).
[14] A. Jamiołkowski, Rep. Math. Phys. 3, 275, (1972).
[15] M. Choi, Linear Algebra and Its Applications, 10, 285-290, (1975).
[16] M. Horodecki, PW. Shor, M. Ruskai, Rev. Math. Phys. 15, 629, (2003).
[17] Z. Li, S. Fei, Z. D. Wang, and W. M. Liu, PRA 79, 024303, (2009).
[18] M. Ikram, F. Li, M.S. Zubairy, Phys. Rev. A 75, 062336, (2007).