HERONIAN MEAN DERIVATIVE-BASED SIMPSON’S-TYPE SCHEME FOR RIEmann-STIELTJES INTEGRAL

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Abstract

In this paper, a new heronian mean derivative-based quadrature scheme of Simpson’s 1/3-type is proposed for the approximation of the Riemann-Stieltjes integral (RS-integral). Theorems are proved related to the basic form, composite form, local and global errors of the new scheme for the RS-integral. The reduction of the new proposed scheme is verified using g(t) = t for Riemann integral. The theoretical results of the new proposed scheme have been proved by experimental work using programming in MATLAB against existing schemes. The order of accuracy, computational cost and average CPU time (in seconds) of the new proposed scheme are determined. The results obtained show the effectiveness of the proposed scheme compared to the existing schemes.

Keywords: Quadrature rule, Riemann-Stieltjes, Simpson’s 1/3 rule, Composite form, Local error, Global error, Cost-effectiveness, Time-efficiency, Heronian Mean.

I. Introduction

Mathematical models are often nonlinear, as the physical phenomenon in nature is not always simple and linear. For example nonlinear equations [XIX],[XX] and their solvers [VII],[XXII], etc. The approximate calculation of a definite integral is the fundamental problem in numerical integration and this approximate value is known as the area under the curve used in engineering applications. Indefinite integral I(f) = \int_a^b f(x)dx, the analytic solution is not always available for integrand f(x) = e^{x^2} or sin x^2. Such definite integrals can also be evaluated with numerical methods.
numerical methods of integration is known as quadrature schemes. A lot of work has been done in the literature to improve the approximations of quadrature rules for the Riemann integral, and some studies have pointed to the approximation of the RS-integral. The Riemann-Stieltjes RS-integral is a modification of the Riemann integral. Suppose \( f(x) \) is bounded function in \([a, b]\), and another function \( \alpha(x) \) which is monotonically increasing in \([a, b]\), the RS-integral is defined [IV] as:

\[
RS(f(x); \alpha; a, b) = \int_{a}^{b} f(x)\,d\alpha(x)
\]  

(1)

where \( f(x) \) is integrand and \( \alpha(x) \) is an integrator.

RS-integrals are applied in several fields of mathematics. For instance, Statistics and theory of Probability, Complex analysis, Functional analysis, the theory of the Operator, and other fields. As mentioned above, quadrature approximations for the Riemann integral are extremely available in literature as in [II], [XVIII]. Further, it is used for integral equations in [XXI], and utilized in reluctance motors [X]. In contrast, only some works in the past used numerical approximation for the RS-integrals. Initially, Trapezoid-type approximation with Hadamard inequality can be used in [XI] for the RS-integral. Further, applied the relative convexity concept in [XII] in which some important inequalities were developed for the approximation of RS-integral using the midpoint and Simpson rules. Authors in [XXIII] introduced a new family of closed Newton-Cotes quadrature schemes with Midpoint derivative for the Riemann integral.

There are so many approaches used to improve Newton-Cotes formulas numerically. Authors in [XIV],[XV],[XVI],[XVII] derived derivative-based closed Newton-Cotes quadrature schemes using different means such as geometric, harmonic, heronian and centroidal at the evaluation of function derivative. Similarly, in [V],[VI] cubature schemes using derivatives were proposed.

The first time in [XXV], the midpoint derivative-based trapezoid scheme for the RS-integral was presented. Afterward in [XXIV], authors proposed the composite form of trapezoidal rule for the RS-integral without numerical experiments. Recently, Memon et al. [VIII],[IX] proposed efficient derivative-based and derivative-free quadrature schemes for the RS-integral with numerical verifications.

In this research work, a new heronian mean derivative-based Simpson’s 1/3-type scheme is proposed for the approximation of the RS-integral. The theorems in basic and composite forms and related error analysis are described. The present research verified the validity of the new scheme by numerical experiments from literature to show cost efficiency, time efficiency and rapid convergence.

II. Some Existing Schemes for the RS-Integral

Some existing schemes: T [XI], ZT [XXV], MZT [VIII], can be described for the RS-integral in basic form in (2)-(4) as:

\[
T \approx \left( \frac{1}{b-a} \int_{a}^{b} g(t)\,dt - g(a) \right) f(a) + \left( g(b) - \frac{1}{b-a} \int_{a}^{b} g(t)\,dt \right) f(b)
\]  

(2)

\[
ZT \approx \left( \frac{1}{b-a} \int_{a}^{b} g(t)\,dt - g(a) \right) f(a) + \left( g(b) - \frac{1}{b-a} \int_{a}^{b} g(t)\,dt \right) f(b) + \left( \int_{a}^{b} \int_{a}^{b} g(x)\,dx\,dt - \frac{b-a}{2} \int_{a}^{b} g(t)\,dt \right) f''(c_{ZT})
\]  

(3)

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where, \( c_{ZT} = \frac{(-2b^2+a^2-ab)\int_a^b g(t)dt + 6b \int_a^b g(t)dxdt - 6 \int_a^b f'(t)g(t)dxdt}{6 \int_a^b f(t)\int_a^b g(x)dxdt - 3(b-a) \int_a^b g(t)dt} \).

\[
MZT \approx \left( \frac{-b^2}{b-a} \right) \frac{rb}{a^2} g(t)dt - g(a) \int f(a) + \left( g(b) - \frac{1}{b-a} \frac{rb}{a^2} g(t)dt \right) f''(c_{MZT})
\]

\[
+ \left( \int_a^b f(t)g(x)dxdt - \frac{b-a}{2} \int_a^b g(t)dt \right) f''(c_{MZT})
\]

where, \( c_{MZT} = \frac{(-2b^2+a^2-ab)\int_a^b g(t)dt + 6b \int_a^b g(t)dxdt - 6 \int_a^b f'(t)g(t)dxdt}{6 \int_a^b f(t)\int_a^b g(x)dxdt - 3(b-a) \int_a^b g(t)dt} \).

The composite forms of the CT, ZCT and MZT schemes are described in (5)-(7) as:

\[
CT \approx \left[ \frac{n}{b-a} \int_a^b x^k g(t)dt - g(a) \right] f(a) + \frac{n}{b-a} \sum_{k=1}^{n} \left[ \int_a^b x^k g(t)dt - \int_a^b g(t)dt \right] f(b)
\]

\[
ZCT \approx \left[ \frac{n}{b-a} \int_a^b x^k g(t)dt - g(a) \right] f(a) + \frac{n}{b-a} \sum_{k=1}^{n} \left[ \int_a^b x^k g(t)dt - \int_a^b g(t)dt \right] f(b) + \sum_{k=1}^{n} \int_a^b x^k g(t)dxdt - \frac{n}{b-a} \int_a^b x^k g(t)dt \right] f''(c_{ZT, k})
\]

\[
\int_a^b f(t)dt \approx MZCT = \left[ \frac{n}{b-a} \int_a^b x^k g(t)dt - g(a) \right] f(a) + \frac{n}{b-a} \sum_{k=1}^{n} \left[ \int_a^b x^k g(t)dt - \int_a^b g(t)dt \right] f(b) + \sum_{k=1}^{n} \int_a^b x^k g(t)dxdt - \frac{n}{b-a} \int_a^b x^k g(t)dt \right] f''(c_{ZT, k})
\]

Where,

\[
c_{ZT, k} = \frac{(-2x_k^2+x_k x_{k-1}-x_{k-1} x_k) \int_a^b x^k g(t)dt + 6b \int_a^b x^k g(t)dxdt - 6 \int_a^b g(t)dxdt - 6 \int_a^b y^k g(t)dxdt}{6 \int_a^b f(t)\int_a^b g(x)dxdt - \frac{3}{n} \int_a^b x^k g(t)dxdt}
\]

\[
c_{MZT, k} = \frac{(-2x_k^2+x_k x_{k-1}+x_{k-1} x_k) \int_a^b x^k g(t)dt + 6b \int_a^b x^k g(t)dxdt - 6 \int_a^b g(t)dxdt - 6 \int_a^b y^k g(t)dxdt}{6 \int_a^b f(t)\int_a^b g(x)dxdt - \frac{3}{n} \int_a^b x^k g(t)dxdt}
\]

III. Proposed Heronian Mean Derivative-Based Simpson’s 1/3-type Scheme for the Riemann-Stieltjes Integral

We base the proposed heronian mean derivative-based Simpson’s 1/3-type schemes for the RS-integral approximation on the well-known closed Newton-Cotes heronian mean derivative-based quadrature rule – the Simpson’s 1/3 (HeMS13) rule – which is defined in basic form in (8).

\[
\int_a^b f(x)dx \approx HeMS13 = \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{(b-a)^5}{2880} \int_a^b f(x)dxdt - \frac{(b-a)^5}{17280} f^{(5)}(\xi)
\]

The Simpson’s 1/3 rule for the Riemann integral as in (8) provides an exact answer for all polynomials whose degree is four or less, so its precision is 4.

Based on (8), the proposed scheme, i.e. HeMS13 for the RS-integral, in basic form is derived in Theorem 1.

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Theorem 1. Let \( f(t) \) and \( g(t) \) be continuous on \([a, b]\) and \( g(t) \) be increasing there. Then the proposed heronian mean derivative-based Simpson’s 1/3 scheme for the RS-integral can be described as:

\[
\int_a^b f(t)g(t) \approx \text{HeMS13} = \left( \frac{4}{(b-a)^2} \int_a^b f(\int_a^t g(x)dx)dt \right) f(a)
- \frac{1}{b-a} \int_a^b g(t)dt - g(a)
\]

\[
+ \left( \frac{4}{b-a} \int_a^b g(t)dt - \frac{8}{(b-a)^2} \int_a^b \int_a^t g(x)dx dt \right) f \left( \frac{a+b}{2} \right)
\]

\[
+ \left( g(b) - \frac{3}{b-a} \int_a^b g(t)dt + \frac{4}{(b-a)^2} \int_a^b \int_a^t g(x)dx dt \right) f(b)
\]

\[
+ \left( \frac{96}{(b-a)^2} \left( 3 \int_a^b x^3 dx - 10ab - 7a^2 \right) \int_a^b \int_a^t g(x)dx dt \right) f(4) \left( \frac{a+\sqrt{ab}+b}{3} \right), \tag{9}
\]

Proof of Theorem 1.

To derive the heronian mean derivative-based Simpson’s 1/3 scheme for the RS-integral, we search numbers: \( a_0, b_0, c_0, d_0 \) so that:

\[
\int_a^b f(t)g(t) \approx a_0f(a) + b_0f \left( \frac{a+b}{2} \right) + c_0f(b) + d_0f \left( \frac{a+\sqrt{ab}+b}{3} \right) \tag{10}
\]

is exact for \( f(t) = 1, t, t^2, t^3, t^4. \) That is,

\[
\int_a^b 1dg = a_0 + b_0 + c_0
\]

\[
\int_a^b tdg = a_0a + b_0 \left( \frac{a+b}{2} \right) + c_0b
\]

\[
\int_a^b t^2dg = a_0a^2 + b_0 \left( \frac{a+b}{2} \right)^2 + c_0b^2
\]

\[
\int_a^b t^3dg = a_0a^3 + b_0 \left( \frac{a+b}{2} \right)^3 + c_0b^3
\]

\[
\int_a^b t^4dg = a_0a^4 + b_0 \left( \frac{a+b}{2} \right)^4 + c_0b^4 + 24d_0
\]

By using integration by parts of the RS-integral, as in [XXV], we have the following system of equations (11)-(15):

\[
a_0 + b_0 + c_0 = g(b) - g(a) \tag{11}
\]

\[
a_0a + b_0 \left( \frac{a+b}{2} \right) + c_0b = bg(b) - ag(a) - f_a^b \int_a^t g(t)dt \tag{12}
\]

\[
a_0a^2 + b_0 \left( \frac{a+b}{2} \right)^2 + c_0b^2 = b^2g(b) - a^2g(a) - 2b \int_a^b \int_a^t g(t)dt +
\]

\[
2f_a^b \int_a^t g(x)dx dt \tag{13}
\]

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\[a_0a^3 + b_0\left(\frac{a + b}{2}\right)^3 + c_0b^3 = b^3g(b) - a^3g(a) - 3b^2 \int_a^b g(t)dt + 6b \int_a^b \int_a^t g(x)dxdt - 6 \int_a^b \int_a^y g(x)dxdt \quad (14)\]

\[a_0a^4 + b_0\left(\frac{a + b}{2}\right)^4 + c_0b^4 + 24d_0 = b^4g(b) - a^4g(a) - 4b^3 \int_a^b g(t)dt + 12b^2 \int_a^b \int_a^t g(x)dxdt - 24b \int_a^b \int_a^y g(x)dxdt - 24 \int_a^b \int_a^t \int_a^y g(x)dxdydt \quad (15)\]

The system of linear equations \((11)-(15)\) can be described with the coefficient matrix as:

\[
M = \begin{bmatrix}
1 & 1 & 1 & 0 \\
\frac{a + b}{2} & a & b & 0 \\
\frac{(a + b)^2}{2} & a^2 & b^2 & 0 \\
\frac{(a + b)^3}{2} & a^3 & b^3 & 0 \\
\frac{(a + b)^4}{2} & a^4 & b^4 & 24
\end{bmatrix}
\]

The reduced row echelon form of \(M\) is:

\[
M \approx M_R = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

As in \(M_R\), \(\text{rank}(M) = 4\). In order to check the linearly independent rows, take \(a = -1\) and \(b = 1\) in matrix \(M\) then

\[
M = \begin{bmatrix}
1 & 1 & 1 & 0 \\
-1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
-1 & 0 & 1 & 0 \\
1 & 0 & 1 & 24
\end{bmatrix}
\]

This shows that the first three and fifth are linearly independent rows whereas the fourth row is linearly dependent. To find the coefficients \(a_0, b_0, c_0\) and \(d_0\), we solve equations \((11), (12), (13)\) and \((15)\) simultaneously, to have:

\[a_0 = \frac{4}{(b-a)^2} \int_a^b \int_a^t g(x)dxdt - \frac{1}{b-a} \int_a^b g(t)dt - g(a),\]

\[b_0 = \frac{4}{b-a} \int_a^b g(t)dt - \frac{8}{(b-a)^2} \int_a^t \int_a^y g(x)dxdt,\]

\[c_0 = g(b) - \frac{3}{b-a} \int_a^b g(t)dt + \frac{4}{(b-a)^2} \int_a^t \int_a^y g(x)dxdt,\]

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\[ d_0 = -\frac{(a-b)^2(3a+5b)}{96} \int_a^b g(t)dt + \frac{(17b^2-10ab-7a^2)}{48} \int_a^b \int_a^t g(x)dxdt \]
\[-b \int_a^b \int_a^t g(x)dxdy dt + \int_a^b \int_a^t \int_a^y g(x) dxdy dz dt \]

Putting the values of coefficients \( a_0, b_0, c_0 \) and \( d_0 \) in (10), we have:

\[
\int_a^b f(t)dg \approx HeMS13 = \left( \frac{4}{(b-a)^2} \int_a^b \int_a^t g(x)dxdt \right) f(a) + \frac{4}{b-a} \int_a^b g(t)dt - \frac{8}{(b-a)^2} \int_a^b \int_a^t g(x)dxdt \right) f\left( \frac{a+b}{2} \right) + \frac{4}{b-a} \int_a^b g(t)dt + \frac{4}{(b-a)^2} \int_a^b \int_a^t g(x)dxdt \right) f(b) + \frac{(b-a)^2(3a+5b)}{96} \int_a^b g(t)dt + \frac{17b^2-10ab-7a^2}{48} \int_a^b \int_a^t g(x)dxdt \right) f(4) \left( \frac{a+b}{3} \right)
\]

which is the same as (9), and the precision of this scheme is 4.

The local error term of the proposed heronian mean derivative-based Simpson’s 1/3 scheme is described in Theorem 2.

**Theorem 2.** Let \( f(t) \) and \( g(t) \) be continuous on [a, b] and \( g(t) \) be increasing there. Then the proposed HeMS13 scheme for the RS-integral with local error term can be defined as:

\[
\int_a^b f(x)dg = HeMS13 + R_{HeMS13}[f]
\]

with HeMS13 as in (9), and error term \( R_{HeMS13}[f] \) as:

\[
R_{HeMS13}[f] = \left( \frac{(a-b)^2(3a+5b)}{96} \int_a^b g(t)dt + \frac{b(2a+2b-5a^2)}{6} \int_a^b \int_a^t g(x)dxdt \right) f(5)(\xi)g(\eta),
\]

where \( \xi, \eta \in (a, b) \).

**Proof of Theorem 2.**

Since the precision of the proposed HeMS13 scheme is 4, we take \( f(t) = \frac{t^5}{5!} \) to find the leading error term defined as:

\[
R_{HeMS13}[f] = \frac{1}{5!} \int_a^b t^5 dg - HeMS13(t^5, g; a, b)
\]

We learn, from [XXV], that:

\[
\int_a^b t^5 dg = \frac{1}{120} (b^5 - a^5) g(b) - \frac{b^4}{24} g(t)dt + \frac{b^3}{6} \int_a^t g(x)dxdt
\]

\[
-b^2 \int_a^b \int_a^t \int_a^y g(x)dx dy dt + b \int_a^b \int_a^t \int_a^y g(x) dx dy dz dt
\]

\[
- \int_a^b \int_a^t \int_a^y \int_a^z g(x) dx dy dz dt
\]

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By Theorem 1 and scheme (9), we have:

\[
\text{HeMS13}(t^5; g; a, b) = \left( \frac{4}{(b-a)^2} \int_a^b g(x) dx - \frac{1}{(b-a)^2} \int_a^b g(t) dt - g(a) \right) \frac{a^5}{5!} \\
+ \left( \frac{4}{(b-a)^2} \int_a^b g(t) dt - \frac{8}{(b-a)^2} \int_a^b g(x) dx \frac{(a+b)^5}{252} \right) \\
+ \left( g(b) - \frac{3}{b-a} \int_a^b g(t) dt + \frac{4}{(b-a)^2} \int_a^b g(x) dx \frac{b^5}{252} \right) \\
+ \left( \frac{(a-b)^2(3a+5b)}{96} \int_a^b g(t) dt + \frac{17b^2-10ab-7a^2}{48} \int_a^b g(x) dx \frac{b^5}{252} \right) \\
+ \left( -b \int_a^b \int_a^t g(t) dt + b \frac{2b-8\sqrt{ab} - a}{3} \int_a^b \int_a^t g(x) dx \frac{b^5}{252} \right)
\]

We use (18) and (19) in (17) to get:

\[
R_{\text{HeMS13}} \int f = \left( \frac{(a-b)^2(9a^2+32ab+50a\sqrt{ab}+5b\sqrt{ab}b^2)}{2880} \int_a^b g(t) dt \right) \\
+ (a-b)(5a^2+24ab+14a\sqrt{ab}+34b\sqrt{ab}-5b^2) \int_a^b \int_a^t f(x) dx \frac{b^5}{252} \\
- \frac{b(2a+2\sqrt{ab}b)}{6} \int_a^b \int_a^t \int_a^y g(x) dx dy dz dt + \frac{2b-8\sqrt{ab} - a}{3} \int_a^b \int_a^t \int_a^y g(x) dx dy dz dt \\
- \int_a^b \int_a^t \int_a^y \int_a^z f(x) dx dy dz dw dt \int(5)g(\eta).
\]

This is the required local error term of the proposed HeMS13 scheme for RS-integrals.

**Theorem 3.** When \( g(t) = t \), the proposed HeMS13 scheme (9) with the error term (16) for the RS-integrals is reduced to the corresponding HeMS13 scheme [XVI], i.e. (8) for the Riemann integrals.

**Proof of Theorem 3.**

By Theorem 2, we have:

\[
\int_a^b f(t) g(t) = \int_a^b f(t) dt = \left( \frac{4}{(b-a)^2} \int_a^b \int_a^t x dx dt - \frac{1}{(b-a)^2} \int_a^b \int_a^t t dt - g(a) \right) f(a) \\
+ \left( \frac{4}{(b-a)^2} \int_a^b \int_a^t t dt - \frac{8}{(b-a)^2} \int_a^b \int_a^t x dx dt \right) f \left( \frac{a+b}{2} \right) \\
+ \left( g(b) - \frac{3}{b-a} \int_a^b \int_a^t t dt + \frac{4}{(b-a)^2} \int_a^b \int_a^t x dx dt \right) f(b) \\
+ \left( \frac{(a-b)^2(3a+5b)}{96} \int_a^b \int_a^t x dx dt + \frac{17b^2-10ab-7a^2}{48} \int_a^b \int_a^t x dx dt \right) f(4) \frac{(a+b+\sqrt{ab}b)}{3} \\
+ \left( -b \int_a^b \int_a^t \int_a^y x dx dy dt + \int_a^b \int_a^t \int_a^y x dx dy dz dt \right) f(4) \frac{(a+b+\sqrt{ab}b)}{3} \\
+ \left( \frac{(a-b)^2(9a^2+32ab+50a\sqrt{ab}+5b\sqrt{ab}b^2)}{2880} \int_a^b \int_a^t t dt \right) \\
- \frac{b(2a+2\sqrt{ab}b)}{6} \int_a^b \int_a^t \int_a^y g(x) dx dy dz dt + \frac{2b-8\sqrt{ab} - a}{3} \int_a^b \int_a^t \int_a^y g(x) dx dy dz dt \\
- \int_a^b \int_a^t \int_a^y \int_a^z f(x) dx dy dz dw dt \int(5)g(\eta).
\]

It is obvious to get:

\[
\int_a^b t dt = \frac{b^2-a^2}{2}, \\
\int_a^b \int_a^t x dx dt = \frac{b^3}{6} - \frac{a^2b}{2} + \frac{a^3}{3}.
\]
\[ \int_a^b f(x) dy \]
\[
\begin{align*}
&+ \left[ g(x_1) - \frac{3}{b-a} \int_a^x g(t) \, dt + \frac{4}{(b-a)^2} \int_a^x \int_a^t g(x) \, dx \, dt \right] f(x_1) \\
&+ \left( \frac{(b-a)^2}{96} - \frac{(3x_1+5x_2)}{96} \right) \int_a^x g(t) \, dt + \frac{17x_2^2 - 10x_1x_2 - 7x_2^2}{48} \int_a^x \int_a^t g(x) \, dx \, dt \\
&+ \left[ \frac{4}{b-a} \int_{x_1}^{x_2} \int_{x_1}^{x_2} g(x) \, dx \, dy \right] f(x_1) \\
&+ \left[ g(x_2) - \frac{3}{b-a} \int_{x_1}^{x_2} g(t) \, dt + \frac{4}{(b-a)^2} \int_{x_1}^{x_2} \int_{x_1}^{x_2} g(x) \, dx \, dt \right] f(x_2) \\
&+ \left( \frac{(b-a)^2}{96} - \frac{(3x_1+5x_2)}{96} \right) \int_{x_1}^{x_2} g(t) \, dt + \frac{17x_2^2 - 10x_1x_2 - 7x_2^2}{48} \int_{x_1}^{x_2} \int_{x_1}^{x_2} g(x) \, dx \, dt \\
&+ \ldots + \left[ \frac{4}{(b-a)^2} \int_{x_{k-1}}^{x_k} \int_{x_{k-1}}^{x_k} g(x) \, dx \, dy \right] f\left( \frac{x_{k-1}+x_k}{2} \right) \\
&+ \left[ g(x_k) - \frac{3}{b-a} \int_{x_{k-1}}^{x_k} g(t) \, dt + \frac{4}{(b-a)^2} \int_{x_{k-1}}^{x_k} \int_{x_{k-1}}^{x_k} g(x) \, dx \, dt \right] f(x_k) \\
&+ \left( \frac{(b-a)^2}{96} - \frac{(3x_{k-1}+5x_k)}{96} \right) \int_{x_{k-1}}^{x_k} g(t) \, dt + \frac{17x_k^2 - 10x_{k-1}x_k - 7x_k^2}{48} \int_{x_{k-1}}^{x_k} \int_{x_{k-1}}^{x_k} g(x) \, dx \, dt \\
&+ \ldots \\
&+ \int_{x_{k-1}}^{x_k} \int_{x_{k-1}}^{x_k} \int_{x_{k-1}}^{x_k} g(x) \, dx \, dy \, dz \\
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\end{align*}
\]
\[ + \ldots + \left[ \frac{4}{(b-a)^2} \int_{x_{n-1}}^{b} \int_{x_{n-1}}^{t} g(x) dx dt - \frac{1}{b-a} \int_{x_{n-1}}^{b} g(t) dt - g(x_{n-1}) \right] f(x_{n-1}) \\
+ \left[ \frac{4}{(b-a)^2} \int_{x_{n-1}}^{b} g(t) dt - \frac{b}{(b-a)^2} \int_{x_{n-1}}^{x_{n-1}^t} g(x) dx dt \right] f \left( \frac{x_{n-1} + b}{2} \right) \\
+ \left[ g(b) - \frac{3}{b-a} \int_{x_{n-1}}^{b} g(t) dt + \frac{4}{(b-a)^2} \int_{x_{n-1}}^{x_{n-1}^t} g(x) dx dt \right] f(b) \\
+ \left( \frac{b(a)}{3} (3x_{n-1} + 5b) \right) \int_{x_{n-1}}^{x_{n-1}^t} g(x) dx dt + \frac{17b^2 - 10x_{n-1} b - 7x_{n-1}^t}{48} \int_{x_{n-1}}^{x_{n-1}^t} g(x) dx dt \\
- \frac{b}{x_{n-1}^t} \int_{x_{n-1}}^{x_{n-1}^t} g(x) dx dy dt + \frac{b}{x_{n-1}^t} \int_{x_{n-1}^t}^{x_{n-1}} g(x) dx dy dz dt \right] f \left( \frac{x_{n-1} + \sqrt{x_{n-1}^t + b}}{2} \right) \\
= \left[ \frac{4n^2}{(b-a)^2} \int_{x_{1}^a}^{x_{1}} g(x) dx dt - \frac{n}{b-a} \left[ \int_{x_{1}^a}^{x_{1}} g(t) dt - g(a) \right] f(a) \right] f(x_1) \\
+ \left[ \frac{4n^2}{(b-a)^2} \int_{x_{1}^a}^{x_{1}} g(t) dt - \frac{8n^2}{(b-a)^2} \int_{x_{1}^a}^{x_{1}} g(x) dx dt \right] f \left( \frac{x_1 + x_2}{2} \right) \\
+ \left[ \frac{4n^2}{(b-a)^2} \int_{x_{1}^a}^{x_{1}} g(t) dt - \frac{8n^2}{(b-a)^2} \int_{x_{1}^a}^{x_{1}} g(x) dx dt \right] f \left( \frac{x_1 + x_2}{2} \right) \\
+ \left[ \frac{4n^2}{(b-a)^2} \int_{x_{1}^a}^{x_{1}} g(t) dt - \frac{8n^2}{(b-a)^2} \int_{x_{1}^a}^{x_{1}} g(x) dx dt \right] f \left( \frac{x_{n-1} + x_k}{2} \right) \\
+ \left[ \frac{4n^2}{(b-a)^2} \int_{x_{1}^a}^{x_{1}} g(t) dt - \frac{n}{b-a} \left( 3 \int_{x_{1}^a}^{x_{1}} g(t) dt + \int_{x_{1}^a}^{x_1} g(x) dx dt \right) \right] f(x_1) \\
+ \left[ \frac{4n^2}{(b-a)^2} \int_{x_{1}^a}^{x_{1}} g(t) dt - \frac{n}{b-a} \left( 3 \int_{x_{1}^a}^{x_{1}} g(t) dt + \int_{x_{1}^a}^{x_1} g(x) dx dt \right) \right] f(x_2) + \ldots \right] \int_{x_{1}^a}^{x_{1}} g(t) dt - \frac{n}{b-a} \left( 3 \int_{x_{1}^a}^{x_{1}} g(t) dt + \int_{x_{1}^a}^{x_{1}} g(x) dx dt \right) - \frac{n}{b-a} \left( 3 \int_{x_{1}^a}^{x_{1}} g(t) dt + \int_{x_{1}^a}^{x_1} g(x) dx dt \right) \right] f(x_{n-1}) \\
+ \left[ g(b) - \frac{3n}{b-a} \int_{x_{n-1}}^{b} g(t) dt + \frac{4n^2}{(b-a)^2} \int_{x_{n-1}}^{b} g(x) dx dt \right] f(b) \\
+ \left( \frac{h^2(3x_{n-1} + 5b) b^6}{49} \int_{x_{n-1}}^{x_{n-1}^t} g(t) dt + \frac{17b^2 - 10x_{n-1} b - 7x_{n-1}^t}{48} \int_{x_{n-1}}^{x_{n-1}^t} g(x) dx dt \\
- \frac{b}{x_{n-1}^t} \int_{x_{n-1}}^{x_{n-1}^t} g(x) dx dy dt + \frac{b}{x_{n-1}^t} \int_{x_{n-1}^t}^{x_{n-1}} g(x) dx dy dz dt \right] f \left( \frac{x_{n-1} + \sqrt{x_{n-1}^t + b}}{2} \right) \\
= \left[ \frac{4n^2}{(b-a)^2} \int_{x_{x_{k-1}}^{x_{k-1}}} g(x) dx dt - \frac{n}{b-a} \left[ \int_{x_{x_{k-1}}^{x_{k-1}}} g(t) dt - g(a) \right] f(a) \right] f(x_k) \\
+ \left[ \frac{4n^2}{(b-a)^2} \int_{x_{x_{k-1}}^{x_{k-1}}} g(t) dt - \frac{2n}{b-a} \int_{x_{x_{k-1}}^{x_{k-1}}} g(x) dx dt \right] f \left( \frac{x_{k-1} + x_k}{2} \right) \\
+ \left[ \frac{4n^2}{(b-a)^2} \int_{x_{x_{k-1}}^{x_{k-1}}} g(t) dt - \frac{2n}{b-a} \int_{x_{x_{k-1}}^{x_{k-1}}} g(x) dx dt \right] f \left( \frac{x_{k-1} + x_k}{2} \right) \\
+ \left[ \frac{4n^2}{(b-a)^2} \int_{x_{x_{k-1}}^{x_{k-1}}} g(t) dt - \frac{2n}{b-a} \int_{x_{x_{k-1}}^{x_{k-1}}} g(x) dx dt \right] f \left( \frac{x_{k-1} + x_k}{2} \right) \\
+ \left[ \frac{4n^2}{(b-a)^2} \int_{x_{x_{k-1}}^{x_{k-1}}} g(t) dt - \frac{2n}{b-a} \int_{x_{x_{k-1}}^{x_{k-1}}} g(x) dx dt \right] f \left( \frac{x_{k-1} + x_k}{2} \right) \right] f(x_k) \\
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\[ \int_{a}^{b} f(x) \, dx = \sum_{k} \left( \frac{b^2}{2} \left( x_k + x_{k+1} \right) + \frac{17x_k^2 - 10x_kx_{k+1} - 7x_k^2}{48} \right) f(x_k) \]

So the proposed composite heronian mean derivative-based Simpson’s 1/3 scheme for the RS-integral is proved.

It is noted that the global error terms of the proposed HeMCS13 scheme cannot be defined in classical form.

**IV. Results and Discussion**

Some authors [XI],[XII],[XXV], in previous studies did not perform experimental works for their theoretical works on quadrature schemes for RS-integral. In this study, experimental works have been performed on quadrature schemes for RS-integral which confirm the validity of theoretical results. Three numerical problems have been solved for each scheme taken from [VIII], [IX], [III], [XIII], [I] etc, which were determined using MATLAB software. All the results are noted in Intel (R) Core (TM) Laptop with RAM 8.00GB and a processing speed of 1.00GHz-1.61GHz.

Double-precision arithmetic is used for numerical results.

**Example 1.** \( \int_{0.5}^{1.5} \sin 5 \, dx \) (cos x) = 0.227676016130689

**Example 2.** \( \int_{2}^{6} \sin x \, dx (x^3) = -59.655908136641912 \)

**Example 3.** \( \int_{1}^{6} e^x \, dx \) sin x = 187.4269314248657

The absolute error and computational order of accuracy (COC) formulae are taken from [IX]. In Table 1, the absolute error drops have been compared for the proposed HeMCS13 scheme and other existing schemes; CT, ZCT and MZCT under similar conditions, and it is observed that the proposed HeMCS13 scheme has the smallest error for all examples. When the number of strips is increased, it is observed from Figs. 1-3 through the line plots of decreasing error distributions that errors in the proposed scheme reduce rapidly in the comparison of other schemes.

Using the COC formula, the observed COC have been determined for the used methods, and are listed in Tables 2-4 for Examples 1-3, respectively versus several strips. The numbers in Tables 2-4 verify the theoretical accuracy of the discussed methods, including the proposed HeMCS13 scheme for the RS-integral. The order of accuracy of the proposed HeMCS13 scheme is 4 which is the same as the MZCT scheme, but the error reduction is rapid for the former. The CT scheme shows the order of accuracy of 2, whereas for the ZCT scheme, due to the issues and mistakes highlighted in [VIII], the order oscillates and doesn’t converge to 4.

In Table 5, the total evaluations required per strip are summarized for the discussed methods, which are necessary to compute the computational costs. In Table 6, we list the total computational cost and the average CPU usage in seconds for the three integrals mentioned in Examples 1-3 using CT, ZCT, MZCT and HeMCS13 schemes. It is observed from numerical results that the proposed scheme took less cost to achieve the error 10^{-5} as compared to existing schemes for all test problems, and the similar

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performance is obvious from Table 6 regarding the smaller average CPU time to achieve the error $10^{-5}$ for the proposed method against others for Examples 1-3.

**Table 1:** Absolute error comparison by HeMCS13 and other schemes for Examples 1-3.

| Quadrature variants | Example 1 (m=20) | Example 2 (m=100) | Example 3 (m=20) |
|---------------------|------------------|-------------------|------------------|
| CT                  | 1.1862E-03       | 4.9713E-04        | 3.9042E-02       |
| ZCT                 | 1.6698E-03       | 5.5959E-04        | 3.9042E-02       |
| MZCT                | 1.8552E-06       | 1.2428E-09        | 2.4399E-06       |
| HeMCS13             | 1.5471E-07       | 8.3806E-11        | 3.8454E-07       |
Table 2: Comparison of COC in all methods for Example 1

| Number of strips (m) | CT      | ZCT    | MZCT   | HeMCS13 |
|---------------------|---------|--------|--------|---------|
| 1                   | NA      | NA     | NA     | NA      |
| 2                   | 2.5728  | 3.3788 | 5.0724 | 4.1456  |
| 4                   | 2.0420  | 4.3106 | 4.1474 | 4.0337  |
| 8                   | 2.0091  | -0.8154| 4.0348 | 4.0083  |
| 16                  | 2.0022  | 2.1713 | 4.0086 | 4.0021  |
| 32                  | 2.0006  | 3.2118 | 4.0021 | 4.0005  |

Table 3: Comparison of COC in all methods for Example 2

| Number of strips (m) | CT      | ZCT    | MZCT   | HeMCS13 |
|---------------------|---------|--------|--------|---------|
| 5                   | NA      | NA     | NA     | NA      |
| 10                  | 2.0018  | 1.9243 | 4.0025 | 4.0264  |
| 20                  | 2.0004  | 2.4487 | 4.0007 | 4.0066  |
| 40                  | 2.0001  | 2.3985 | 4.0001 | 4.0017  |
| 80                  | 2.0001  | 2.3985 | 4.0000 | 4.0004  |

Table 4: Comparison of COC in all methods for Example 3

| Number of strips (m) | CT      | ZCT    | MZCT   | HeMCS13 |
|---------------------|---------|--------|--------|---------|
| 1                   | NA      | NA     | NA     | NA      |
| 2                   | 1.9319  | 1.9319 | 3.8984 | 4.0187  |
| 4                   | 1.9844  | 1.9844 | 3.9761 | 4.0049  |
| 8                   | 1.9962  | 1.9962 | 3.9941 | 4.0012  |
| 16                  | 1.9990  | 1.9990 | 3.9985 | 4.0003  |
| 32                  | 1.9998  | 1.9998 | 3.9985 | 4.0000  |

Table 5: Computational cost in quadrature variants form strips.

| Quadrature Variants | Total evaluations |
|---------------------|-------------------|
| CT                  | 2m+3 [XI]         |
| ZCT                 | 5m+3 [XI]         |
| MZCT                | 5m+3 [XI]         |
| Proposed HeMCS13    | 7m+3              |

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Table 6: Computational cost and CPU time comparison to achieve at most 1E-05 absolute error in quadrature variants for Examples 1-3.

| Quadrature Variants | Computational cost | CPU time (in seconds) |
|---------------------|--------------------|-----------------------|
|                     | Example 1 | Example 2 | Example 3 | Example 1 | Example 2 | Example 3 |
| CT                  | 439       | 1415      | 2503      | 68.04     | 12.82     | 432.30     |
| ZCT                 | 1043      | 3503      | 6253      | 552.60    | 153.98    | 6469.38    |
| MZCT                | 73        | 78        | 78        | 28.01     | 5.26      | 27.35      |
| HeMCS13             | 59        | 45        | 66        | 26.65     | 5.99      | 27.82      |

V. Conclusion

A new heronian mean derivative-based quadrature scheme of Simpson’s 1/3 type was proposed for efficient approximations of the RS-integral and extended for higher strips in a composite sense. The theorems regarding the local and global error terms were proved. Three numerical problems were tested from the literature to discuss the performance of the proposed scheme against few other existing schemes. The error drops, observed orders of accuracy and computations performance in terms of evaluations and CPU usage show the dominance of the proposed scheme over other discussed schemes for the evaluation for the RS-integral numerically.

Conflict of Interest:

No conflict of interest regarding this article

References

I. Bartle, R.G. and Bartle, R.G., The elements of real analysis, (Vol. 2). John Wiley & Sons, 1964.
II. Bhatti AA, Chandio MS, Memon RA and Shaikh MM, A Modified Algorithm for Reduction of Error in Combined Numerical Integration, Sindh University Research Journal-SURJ (Science Series) 51.4, (2019): 745-750.
III. Burden, R.L., Faires, J.D., Numerical Analysis, Brooks/Cole, Boston, Mass, USA, 9th edition, 2011.
IV. Dragomir, S.S., and Abelman S., Approximating the Riemann-Stieltjes integral of smooth integrands and of bounded variation integrators, Journal of Inequalities and Applications, 2013.1 (2013), 154.
V. Malik K., Shaikh, M. M., Chandio, M. S. and Shaikh, A. W. Error analysis of closed Newton-Cotes cubature schemes for double integrals. : J. Mech. Cont. & Math. Sci., 15 (11): 95-107, 2020.

Kashif Memon et al
VI. Malik K., Shaikh, M. M., Chandio, M. S. and Shaikh, A. W. Some new and efficient derivative-based schemes for numerical cubature. J. Mech. Cont. & Math. Sci., 15 (10): 67-78, 2020.

VII. Mastoi, Adnan Ali, Muhammad Mujtaba Shaikh, and Abdul Wasim Shaikh, A new third-order derivative-based iterative method for nonlinear equations. J. Mech. Cont. & Math. Sci., 15 (10): 110-123, 2020.

VIII. Memon K, Shaikh MM, Chandio MS and Shaikh AW, A Modified Derivative-Based Scheme for the Riemann-Stieltjes Integral, Sindh University Research Journal-SURJ (Science Series) 52.1, (2020): 37-40.

IX. Memon, A. A., Shaikh, M. M., Bukhari, S. S. H., & Ro, J. S. Look-up Data Tables-Based Modeling of Switched Reluctance Machine and Experimental Validation of the Static Torque with Statistical Analysis. Journal of Magnetics, 25(2): 233-244, 2020.

X. Mercer, P.R., Hadamard's inequality and Trapezoid rules for the Riemann-Stieltjes integral, Journal of Mathematica Analysis and Applications, 344 (2008), 921-926.

XI. Mercer, P.R., Relative convexity and quadrature rules for the Riemann-Stieltjes integral, Journal of Mathematica inequality, 6 (2012), 65-68.

XII. Protter, M.H. and Morrey, C.B., A First Course in Real Analysis. Springer, New York, NY, 1977.

XIV. Ramachandran Thiagarajan, Udayakumar.D, Parimala .R, Geometric mean derivative-based closed Newton-Cotes quadrature, International Journal of Pure & Engineering Mathematics, 4, 107-116, April 2016.

XV. Ramachandran Thiagarajan, Udayakumar.D, Parimala .R, Harmonic mean derivative-based closed Newton-Cotes quadrature, IOSR-Journal of Mathematics, 12, 36-41, May-June 2016.

XVI. Ramachandran Thiagarajan, Udayakumar.D, Parimala .R, Heronian mean derivative-based closed Newton cotes quadrature, International Journal of Mathematical Archive, 7, 53-58, July 2016.

XVII. Ramachandran Thiagarajan, Parimala .R, Centroidal mean derivative-based closed Newton cotes quadrature, International Journal of Science and Research, 5, 338-343, August 2016.

XVIII. Shaikh, MM., MS Chandio and AS Soomro, A Modified Four-point Closed Mid-point Derivative Based Quadrature Rule for Numerical Integration, Sindh University Research Journal-SURJ (Science Series), 48.2, 2016.

XIX. Shaikh, Muhammad Mujtaba, Shafiq-ur-Rehman Massan, and Asim Imdad Wagan. "A new explicit approximation to Colebrook’s friction factor in rough pipes under highly turbulent cases." International Journal of Heat and Mass Transfer 88 (2015): 538-543.

XX. Shaikh, Muhammad Mujtaba, Shafiq-ur-Rehman Massan, and Asim Imdad Wagan. "A sixteen decimal places' accurate Darcy friction factor database using non-linear Colebrook's equation with a million nodes: A way forward to the soft computing techniques." Data in brief 27 (2019): 104733.

Kashif Memon et al
XXI. Shaikh, Muhammad Mujtaba. "Analysis of Polynomial Collocation and Uniformly Spaced Quadrature Methods for Second Kind Linear Fredholm Integral Equations—A Comparison." Turkish Journal of Analysis and Number Theory 7.4 (2019): 91-97.

XXII. Umar, Sehrish, Muhammad Mujtaba Shaikh, and Abdul Wasim Shaikh. A new quadrature-based iterative method for scalar nonlinear equations. J. Mech. Cont. & Math. Sci., 15 (10): 79-93, 2020.

XXIII. Zhao, W., and H. Li, Midpoint Derivative-Based Closed Newton-Cotes Quadrature, Abstract And Applied Analysis, Article ID 492507, (2013).

XXIV. Zhao, W., Z. Zhang, and Z. Ye, Composite Trapezoid rule for the Riemann-Stieltjes Integral and its Richardson Extrapolation Formula, Italian Journal of Pure and Applied Mathematics, 35 (2015), 311-318.

XXV. Zhao, W., Z. Zhang, and Z. Ye, Midpoint Derivative-Based Trapezoid Rule for the Riemann-Stieltjes Integral, Italian Journal of Pure and Applied Mathematics, 33, (2014), 369-376.