Resource Theory of Special Antiunitary Asymmetry

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We propose the resource theory of a special antiunitary asymmetry in quantum theory. The notion of antiunitary asymmetry, in particular, PT-asymmetry is different from the usual resource theory for asymmetry about unitary representation of a symmetry group, as the PT operator is an antiunitary operator with P being any self-inverse unitary and T being the time-reversal operations. Here, we introduce the PT-symmetric states, PT-covariant operations and PT-asymmetry measures. For single qubit system, we find duality relations between the PT-asymmetry measures and the coherence. Moreover, for two-qubit states we prove the duality relations between the PT-asymmetry measures and entanglement measure such as the concurrence. This gives a resource theoretic interpretation to the concurrence which is lacking till today. Thus, the PT-asymmetry measure and entanglement can be viewed as two sides of an underlying resource. Finally, the PT-symmetric dynamics is discussed and some open questions are addressed.

Introduction.– Quantum resource theories [1, 2] have played a pivotal role in the development and quantitative understanding of various physical phenomena in quantum physics and quantum information theory. A resource theory consists of two basic elements: free operations and free states. Any operation (or state) is dubbed as a resource if it falls out of the set of free operations (or the set of free states). The most significant resource theory is entanglement [3], which is a basic resource for various quantum information processing protocols, such as the superdense coding [4], teleportation [5] and remote state preparation [6]. The other notable examples include the resource theories of thermodynamics [7], asymmetry [8–14], coherence [15–22] and steering [23]. The main advantages of having a resource theory for some physical quantity are the succinct understanding of various physical processes and operational quantification of the relevant resources at ones disposal.

The Hermiticity is one fundamental requirement of quantum mechanics for the Hamiltonian of a quantum system, which guarantees that the energies are real and the total probability of the quantum state is conserved during the evolution of the system. However, it has been proved that a broad class of non-Hermitian Hamiltonian with PT-symmetry can also have real spectra and probability conservation by redefined inner product [24–29], where P denotes the parity operator and T denotes the time reversal operator. This implies that PT-symmetric theory constitutes a complex generalization of conventional quantum mechanics [26]. Moreover, in the system with PT-symmetric non-Hermitian Hamiltonian, a number of interesting phenomena and applications appear in both classical and quantum regimes, such as undirectional invisibility [30–32], non-Hermitian Bloch oscillation [33, 34], perfect laser absorbers [35–37], ultrafast quantum state transformation [38], quantum state discrimination with single-shot measurement [39] and the potential violation of the no-signalling principle [40, 41]. However, most research focus on the PT-symmetric Hamiltonian, never consider the quantum state with PT-symmetry. Thus, the following questions arise: how to define a PT-symmetric quantum state, what is the physical meaning of PT-symmetric states and how to define measures of PT-asymmetry.

Recently, the quantification of time reversal asymmetry [42] and CPT asymmetry [43, 44] have been considered in antiunitary and unitary representations, respectively. However, there still remains a question in which representation to choose those relevant operations [45]. In this work, we use the framework of quantum resource theory to quantify PT-asymmetry and investigate the relationship between PT-asymmetry measures, quantum coherence and entanglement. Note that here P is a self-inverse unitary operator (need not be parity operator) and T is time-reversal operator. PT operator can be realized as a special kind of antiunitary operator [46, 47], which is in contrast with the resource theory of asymmetry on the unitary representation of a symmetric group [8–14]. Thus, the resource of PT-asymmetry will be a special kind of resource theory of antiunitary asymmetry. Though we cannot tensor the antilinear operator with the identity operator consistently, because antilinear operators are nonlocal, nevertheless they have been used to measure entanglement of a given bipartite state [48–53]. And there is a famous entanglement measure—the concurrence [49], which is indeed constructed from antilinear operators. It is quite satisfying that the resource theory of antiunitary asymmetry provides a unified view of two fundamental resources of quantum world such as the coherence and entanglement. For single qubit, we reveal a duality relation between the PT-asymmetry measure and the coherence. For two-qubit pure states, we prove duality relations between the PT-asymmetry measures and entanglement measure such as the concurrence. Amazingly, we find that the pure bipartite state is maximally entangled if and only

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if it is $PT$-symmetric. Therefore, entanglement is a special $PT$-symmetry in some sense. Furthermore, as $K = \ast$ is an unphysical operator, that is $K$ cannot be realized in a physical system, then it is hard to calculate the $PT$-asymmetry measure. However, we show that via the embedding quantum simulator [54–59], the $PT$ asymmetry measure can be calculated efficiently.

$PT$-symmetric state. – Consider the self-inverse unitary operator $P$ and time reversal operator $T$, where $P$ and $T$ satisfy the following condition:(1) $P = P^T$, $P^2 = I$, (2) $T = U^*K$, where $U$ is a unitary operator with $U = U^T$ and $K = \ast$ is the complex conjugation and (3) $\{P, T\} = 0$. Note that any antiunitary operator $\Theta$ with $\Theta = \Theta^T = \Theta^{-1}$ can be written in the form $VK$, where $V$ is a unitary operator with $V = V^T$ and $K$ is the complex conjugation with respect to a given basis. Such antiunitary operator is called conjugation and plays an important role in quantum information theory [50, 60]. It is easy to see that such conjugation is equivalent to the $PT$ operator defined above. Thus, the resource theory of antiunitary asymmetry considered in this work is a special kind of antiunitary asymmetry resource theory and may indicate the way towards formulating the general resource theory of antiunitary asymmetry.

Throughout this paper, we assume that self-inverse unitary operator and time reversal operator always satisfy these conditions. Given a quantum state $\rho$, once we apply the operations $P$ and $T$, the final state will be $PT\rho PT$ (Since $PT\rho PT = PUK\rho KU^*P = PU\rho U^*P$ is a quantum state). If the initial state is equal to the final state, that is $[\rho, PT] = 0$, then we call the state $\rho$ is $PT$-symmetric state. If the initial state is not equal to the final state, that is $[\rho, PT] \neq 0$, then we call the state $\rho$ is $PT$-asymmetric. Moreover, we denote the set of all $PT$-symmetric states by $\emph{Sym}(P, T)$.

$PT$-covariant operation. – To characterize the quantum operation which transform the $PT$-symmetric states to the $PT$-symmetric states, we distinguish quantum operations with and without subselection. Any quantum operation $\Phi$ can be described using a set of Kraus operators $\{K_\mu\}$ with $\Phi(\cdot) = \sum\lambda K_\lambda(\cdot)K_\lambda^\dagger$. The operation $\Phi : \mathcal{D}(H) \rightarrow \mathcal{D}(H)$ is called $PT$-covariant if $\Phi(PT\rho PT) = PT\Phi(\rho)PT$, that is $[\Phi, PT] = 0$. Such operations are denoted by $\emph{Phi}_{\emph{PT}CO}$. (Of course, we can also consider the $PT$-covariant operation with different $PT$, that is $\Phi(P_1T_1(\cdot)P_1T_1) = P_2T_2\Phi^\dagger(\cdot)P_2T_2$.) Besides this, we also need to consider the quantum operations with subselection. Thus, a quantum operation $\Phi$ is called selective $PT$-covariant if the Kraus operators $\{K_\mu\}$ of $\Phi$ satisfy $K_\mu(PT\rho PT)K_\mu^\dagger = PTK_\mu(\cdot)K_\mu^\dagger PT$ for any $\mu$.

The measure of $PT$-asymmetry for a state. – When the state is $PT$-asymmetric, that is it breaks the $PT$-symmetry, we want to quantify how much the $PT$-symmetry is broken by the given state. Thus, we need to introduce the $PT$ asymmetry measure, like the entanglement measure [61, 62], asymmetry measure [11, 14] and coherence measure [15, 16]. Now, we list the conditions that any function $\Gamma$ from a state to a real number needs to satisfy in order to be a proper $PT$-asymmetry measure.

For any proper $PT$-asymmetry measure $\Gamma$, it needs to satisfy the following conditions:

(C1) $\Gamma(\rho, PT) = 0$ iff $[\rho, PT] = 0$.

(C2) Monotone under $PT$-covariant operations $\Phi_{PTCO}$, that is $\Gamma(\Phi_{PTCO}(\rho), PT) \leq \Gamma(\rho, PT)$.

(C2') Monotone under selective $PT$-covariant operations: $\sum\mu p_\mu \Gamma(\rho_\mu, PT) \leq \Gamma(\rho, PT)$, where $K_\mu(PT(\cdot)PT)K_\mu^\dagger = PTK_\mu(\cdot)K_\mu^\dagger PT$ and $\rho_\mu = K_\mu^\dagger \rho K_\mu$ with $p_\mu = \text{Tr}(K_\mu^\dagger \rho K_\mu)$.

(C3) Convexity: $\Gamma(\sum n p_n \rho_n, PT) \leq \sum n p_n \Gamma(\rho_n, PT)$, where $\{\rho_n\}$ is a set of states and $p_n \geq 0$ with $\sum n p_n = 1$.

We now give several $PT$-asymmetry measures via the relative entropic, the skew information and the fidelity measures.

Relative entropy of $PT$-asymmetry. – The quantum relative entropy for states $\rho$ and $\sigma$ is defined as $S(\rho || \sigma) := \text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \sigma)$. The relative entropy of $PT$-asymmetry measure $\Gamma_r$ is defined as

$$
\Gamma_r(\rho, PT) = \min_{\sigma \in \emph{Sym}(P, T)} S(\rho || \sigma).
$$

First, we get a closed form expression of $\Gamma_r$ to avoid the minimization and it is given by

$$
\Gamma_r(\rho, PT) = S(\rho || PT^r) = S(\rho^T) - S(\rho),
$$

where $\rho^T = 1/2(\rho + PT\rho PT)$ is $PT$-symmetric and $\Gamma_r$ fulfills the conditions (C1), (C2), (C2') and (C3) as a proper $PT$-asymmetry measure (See Appendix A). Then, for any state $\rho$, $\Gamma_r(\rho, PT) = S(\rho^T) - S(\rho) \leq 1$, as $S(\sum \rho_i \rho_{i'}) \leq \sum \rho_i S(\rho_i) + H(\{\rho_i\})$ [63] and $(S(\rho^T) + S(\rho^T)) = S(\rho)$. Since $S(\sum \rho_i \rho_{i'}) = \sum \rho_i S(\rho_i) + H(\{\rho_i\})$ is equivalent to that $\rho_i$ have orthogonal supports [63], then $\Gamma_r(\rho, PT) = 1$ if $\rho \perp PTP$.

Skew information of $PT$-asymmetry. – Let us define the skew information of $PT$-asymmetry $\Gamma_s$ as

$$
\Gamma_s(\rho, PT) := 1/2 \text{Tr} \left( [\rho^{1/2}, PT^{1/2}]^2 \right)
= 1 - \text{Tr} \left( \rho^{1/2}PT^r \rho^{1/2}PT \right),
$$

where $[\cdot, \cdot]$ denote the commutator and $[\rho^{1/2}, PT^{1/2}]^2 = \rho + PT\rho PT - 2\rho^{1/2}PT\rho^{1/2}PT$. Note that, in the definition of Wigner-Yanase-Dyson skew information $I(\rho, O) = -1/2 \text{Tr} \left( [\rho^{1/2}, O] \right)$, the operator $O$ is required to be an observable [64], that is $O$ must be a Hermitian, however $PT$ is not a linear operator, thus $PT$ is not an observable. Therefore, we cannot use the properties of skew information to state that
\[ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] and time reversal operator \( T = * \)

\( \Gamma_s \) satisfy the conditions (C1), (C2), (C2') and (C3). However, \( \Gamma_s \) still fulfills these conditions (See Appendix B). Obviously, for any state \( \rho \), \( \Gamma_s(\rho, PT) \leq 1 \) and the equality holds iff \( \rho = PT \rho PT \).

The above two quantities \( \Gamma_F \) and \( \Gamma_s \) are proper PT-asymmetry measures (An example is presented in Fig.1). Of course, there may be other possible \( PT \)-asymmetry measure, like the \( \rho \)-asymmetry measure induced by the trace norm, the Hilbert Schmidt norm and so on. Here, we introduce another interesting \( PT \)-asymmetry measure defined by the fidelity.

**Fidelity measure of \( PT \)-asymmetry.** – Let us consider the \( PT \)-asymmetry measure defined as

\[
\Gamma_F(\rho, PT) = 1 - F(\rho, PT) = 1 - \text{Tr} \left( \sqrt{\sqrt{\rho} PT \rho PT \sqrt{\rho}} \right),
\]

which fulfills the conditions (C1), (C2), (C2') and (C3) (See Appendix C).

Based on the proof in Proposition 7 (See Appendix B), we have \( \Gamma_F \leq \Gamma_s \). Following Ref.[50], \( \Gamma_F(\rho, PT) \) can be written as

\[
\Gamma_F(\rho, PT) = \min_k \sum_k p_k \Gamma_F(\psi_k, PT),
\]

where the minimum is taken over all the decomposition of \( \rho = \sum_k p_k |\psi_k \rangle \langle \psi_k| \). Furthermore, the optimal decomposition can be found in Ref.[50].

**Duality of \( PT \)-Asymmetry, Coherence and Entanglement.** – Given a self-inverse unitary operator \( \mathcal{P} \) and a time reversal operator \( \mathcal{T} \), for any pure state \( \rho = |\psi \rangle \langle \psi| \), we have

\[
\Gamma_s(\psi, PT) = 1 - |\langle \psi | PT | \psi \rangle|^2,
\]

and

\[
\Gamma_F(\psi, PT) = 1 - |\langle \psi | PT | \psi \rangle|. \tag{7}
\]

This can be interpreted as follows. Imagine that we have two copies of a pure state \( |\psi \rangle \), and one is rotated in space by a unitary operator \( \mathcal{P} \) and the other is transformed under time reversal operator \( \mathcal{T} \). The final states will be \( \mathcal{P} |\psi \rangle \) and \( \mathcal{T} |\psi \rangle \), and we want to know whether these final states coincide or not. If they coincide, then this means that the effect of the parity operator and the time reversal operator leaves the state \( |\psi \rangle \) invariant, and we say \( |\psi \rangle \) has \( PT \)-symmetry. Otherwise, the state \( |\psi \rangle \) breaks the \( PT \)-symmetry.

Moreover, the spectrum of \( \rho^{PT} \) with \( \rho = |\psi \rangle \langle \psi| \) is \( \{ 1/2, -1/2 \} \) and \( |\langle \psi | PT | \psi \rangle|^2 \). Thus,

\[
\Gamma_s(\psi, PT) = H \left( \frac{1}{2} - \frac{1}{2} |\langle \psi | PT | \psi \rangle| \right)
\]

where \( H(p) = -p \log(p) - (1-p) \log(1-p) \) is the Shannon entropy for the probability distribution \( \{ p, 1-p \} \).

Let us consider the simplest case: a single qubit system. We take \( \mathcal{P} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( \mathcal{T} = * \). Then for pure qubit state \( |\psi \rangle = (\psi_1, \psi_2)^t \), where \( t \) denotes the transpose,

\[
\Gamma_s(\psi, PT) = 1 - |\langle \psi | PT | \psi \rangle|^2 = 1 - |\psi_1 \psi_2 + \psi_2 \psi_1|^2 = 1 - 4|\psi_1|^2 |\psi_2|^2,
\]

and

\[
\Gamma_F(\psi, PT) = H \left( \frac{1}{2} - 2|\psi_1|^2 |\psi_2|^2 \right).
\]

Thus, a pure state \( |\psi \rangle \) is \( PT \)-symmetric iff \( |\psi_1| = |\psi_2| = 1/\sqrt{2} \). This suggests that there should be a connection between the quantum coherence [15] and the \( PT \)-asymmetry measure. In fact, for a single qubit system we have duality relations between the \( l_1 \)-norm of coherence and the \( PT \)-asymmetry measures as given by

\[
\Gamma_s(\psi, PT) + C_{l_1}(\psi)^2 = 1,
\]

\[
\Gamma_F(\psi, PT) + C_{l_1}(\psi) = 1, \tag{9}
\]

where \( C_{l_1}(\psi) = \sum_{i \neq j} |\rho_{ij}| = 2|\psi_1||\psi_2| \) is the \( l_1 \)-norm of coherence for a single qubit. Therefore, a maximally pure coherent state is actually a \( PT \)-symmetric state.

However, in two-qubit system, we have two different ways to consider the \( PT \)-asymmetry. On the one hand, we can construct the \( PT \)-operators on 2-qubit system using \( \mathcal{P} \), \( \mathcal{T} \) operators on single qubit systems like \( PT_1 \mathcal{T}_1 \otimes PT_2 \mathcal{T}_2 \). On the other hand, we can construct \( \mathcal{P} \), \( \mathcal{T} \) operators on 2-qubit system which cannot be constructed from single qubit systems, and this may be connected with entanglement closely.

For two-qubit pure state \( |\Psi \rangle \) the famous entanglement monotone – the concurrence [49] is defined as

\[
C(\Psi) = |\langle \Psi | \sigma_y \otimes \sigma_y K |\Psi \rangle| \tag{10}
\]

Now, we prove duality relations between the \( PT \)-asymmetry measures and the concurrence. Using the definitions of \( \Gamma_s \), \( \Gamma_F \) and \( C(\psi) \) for any pure two qubit state, we have the following theorem.

**Theorem 1.** Given a two-qubit system with self-inverse unitary operator \( \mathcal{P} = \sigma_y \otimes \sigma_y \) and time reversal operator \( \mathcal{T} = * \), for pure bipartite states \( |\Psi \rangle \) we have

\[
\Gamma_s(\Psi, PT) + C(\Psi)^2 = 1, \tag{11}
\]

\[
\Gamma_F(\Psi, PT) + C(\Psi) = 1, \tag{12}
\]
where $H(p) = -p \log(p) - (1-p) \log(1-p)$ is the Shannon entropy for the probability distribution \{p, 1-p\} and $C(\Psi)$ is the concurrence for pure state $\Psi$.

For any two-qubit mixed states $\rho$, the equalities may not hold. However, we still have the following inequality:

\[
\Gamma_\tau(\rho, PT) + C(\rho)^2 \leq 1,
\]

\[
\Gamma_F(\rho, PT) + C(\rho) \leq 1,
\]

\[
\Gamma_r(\rho, PT) \leq H\left(\frac{1}{2} - \frac{1}{2} C(\rho)\right),
\]

where $C(\rho) = \min \sum_k p_k C(\Psi_k)$ and the minimum is taken over all the pure states decomposition of $\rho = \sum_k p_k |\Psi_k\rangle \langle \Psi_k| \ [49, 50]$.

In fact, we can prove the following

\[
\Gamma_F(\rho, PT) + CoA(\rho) = 1,
\]

where the concurrence of assistance $CoA(\rho) = \max \sum_k p_k C(\Psi_k)$ and the maximum is taken over all the pure states decomposition of $\rho = \sum_k p_k |\Psi_k\rangle \langle \Psi_k| \ [65, 66]$.

The proof of this theorem is presented in the Appendix D.

Since the concurrence quantifies the entanglement of a pure bipartite state, thus the above proposition shows that a pure state has more $PT$-symmetry with $P = \sigma_y \otimes \sigma_y$ and $T = *$ if and only if this state is more entangled, i.e., the pure state $\Psi$ is a $PT$-symmetric state iff $\Psi$ is maximally entangled. Therefore, entanglement is a special kind of $PT$-symmetry in some sense. Our formalism, the resource theory of antimatter asymmetry, in fact, provides a unified view of two fundamental resources such as the quantum coherence and the entanglement.

To generalize these notions, we consider the relationship between $PT$-asymmetry and entanglement in multi-qubit system. In $N$-qubit system with $N \geq 2$, there exists a systematic procedure to define entanglement monotone for pure states via three operational building blocks [52, 55]: $K = *$, $\sigma_y$ and $g_{ij} \sigma_y \sigma_j$, where $g_{ij} = diag \{-1, 1, 0, 0\}$, $\sigma_0 = I$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$ ad $\sigma_3 = \sigma_z$. If $N$ is even, then the simplest entanglement monotone is $\langle \Psi| \sigma_y^{\otimes N} K |\Psi\rangle$ and if $N$ is odd, $\sum_{ij} g_{ij} \langle \Psi| \sigma_i \otimes \sigma_y^{\otimes N-1} K |\Psi\rangle$ $|\Psi| \sigma_j \otimes \sigma_y^{\otimes N-1} K |\Psi\rangle$ is the simplest entanglement monotone [52, 55]. With this construction, for $N=2$, we have the entanglement monotone-the concurrence: $C(\Psi) = |\langle \Psi| \sigma_y \otimes \sigma_y K |\Psi\rangle|$. For $N=3$, we have the 3-tangle [51] defined as $\tau_3(\Psi) := |\sum_{ij} g_{ij} \langle \Psi| \sigma_i \otimes \sigma_y^{\otimes 2} K |\Psi\rangle|$. Thus, we have defined entanglement monotone $\tau_N$ for any $N$-qubit system and it is easy to see that for the even and odd cases we have the following relations:

(i) if $N = 2k$, then

\[
\Gamma_r(\Psi, P_2 T) + \tau_N(\Psi)^2 = 1.
\]

(ii) if $N = 2k + 1$, then

\[
\tau_N(\Psi) = | - \Gamma_s(\Psi, P_0 T) + \Gamma_s(\Psi, P_2 T) + \Gamma_s(\Psi, P_3 T) - 1 |,
\]

where $P_2 = \sigma_y^{\otimes N}$, $T = *$ and $H(p) = -p \log(p) - (1-p) \log(1-p)$ is the Shannon entropy for the probability distribution \{p, 1-p\}.

Since $K = *$ is an unphysical operator, one may think that we need to perform full tomography to calculate $\langle \Psi| \sigma_y^{\otimes N} K |\Psi\rangle$ [54, 55]. However, based on the embedding quantum simulator (EQS), such quantity can be calculated efficiently [54, 55]. This technique of embedding quantum simulator [54–59] is described as follows: define a mapping $M : C^d \rightarrow \mathbb{R}^{2d}$ as

\[
|\Psi\rangle = \begin{pmatrix} \psi_1 \ \psi_2 \ \psi_3 \ \psi_4 \ \cdots \ \psi_{2d} \end{pmatrix} \rightarrow |\tilde{\Psi}\rangle = \begin{pmatrix} \psi_1 \ \psi_2 \ \psi_3 \ \psi_4 \ \cdots \ \psi_{2d} \end{pmatrix}
\]

The reverse mapping is given by $|\Psi\rangle = M |\tilde{\Psi}\rangle$, with $M = (1, i) \otimes I_d$ and $K |\Psi\rangle = M (\sigma_z \otimes I_d) |\Psi\rangle$. Thus, one has

\[
\langle \Psi| \sigma_y^{\otimes N} K |\Psi\rangle = \langle \tilde{\Psi}| M^\dagger P U M (\sigma_z \otimes I_d) |\Psi\rangle.
\]

By the embedding quantum simulator, the quantity $\langle \Psi| \sigma_y^{\otimes N} K |\Psi\rangle$ can be calculated efficiently, which means the $PT$-asymmetry measures such as $\Gamma_r, \Gamma_s, \Gamma_F$ can be calculated efficiently (see Fig. 2).

![FIG. 2. A quantum network for estimation of $PT$-asymmetry. The probability of finding the control qubit (the top line) in state $|0\rangle$: $p_0$ depends on the $PT$ asymmetry of $|\Psi\rangle$ [67, 68], that is $p_0 = 1 + |\langle \Psi| \sigma_y^{\otimes N} K |\Psi\rangle|^2/2$, where $T = U K$.](image-url)
the notion of $\mathcal{PT}$-symmetric states, $\mathcal{PT}$-covariant operations and $\mathcal{PT}$-asymmetry measures. We give several interesting $\mathcal{PT}$-asymmetry measures which are induced from different distance measures. Most importantly, we have proved the duality relations between the $\mathcal{PT}$-asymmetry measures, coherence and concurrence. This also gives new interpretations to quantum coherence and entanglement, which are special $\mathcal{PT}$-symmetries in some sense, thus unifying two fundamental resources of quantum world. Furthermore, we have argued that via the embedding quantum simulator, the $\mathcal{PT}$-asymmetry measures can be calculated efficiently. Finally, we have discussed the $\mathcal{PT}$-symmetric dynamics and proposed several open problems. Our findings will open up new ways of thinking about quantum coherence and entanglement from another resource theoretic point of view, i.e., the $\mathcal{PT}$-asymmetry measures. The $\mathcal{PT}$-asymmetry is a just special kind of antiunitary asymmetry resource theory and may pave the way to a general antiunitary asymmetry resource theory.

ACKNOWLEDGMENTS

K. B. thanks Yaobo Zhang for useful discussion. This work is supported by the Natural Science Foundations of China (Grants No:11171301, No: 10771191 and No: 11571307) and the Doctoral Programs Foundation of the Ministry of Education of China (Grant No. J20130061).
Proof. Due to the spectral decomposition theorem, $Q$ and $S$ can be written as $Q = \sum \lambda_i |\psi_i\rangle \langle \psi_i|$ and $S = \sum \mu_j |\phi_j\rangle \langle \phi_j|$, respectively. Thus, we have

\[
\text{Tr} (QPTSPT) = \sum_{ij} \lambda_i \mu_j \text{Tr} (|\psi_i\rangle \langle \psi_i| UPU \phi_j \langle \phi_j| KU \dagger P) = \sum_{ij} \lambda_i \mu_j |\langle \psi_i| PU |\phi_j^*\rangle|^2,
\]

where the first equality comes from the fact that $T = UK$, where $U$ is a unitary operator with $U = U^\dagger$ and $K = *$. Similarly, we have

\[
\text{Tr} (PTQPTS) = \sum_{ij} \lambda_i \mu_j |\langle \phi_j| PU |\psi_i^*\rangle|^2.
\]

Hence, to prove $\text{Tr} (QPTSPT) = \text{Tr} (PTQPTS)$, we only need to prove that for any two pure states $|\psi\rangle$ and $|\phi\rangle$

\[
|\langle \psi| V |\phi^*\rangle| = |\langle \phi| V |\psi^*\rangle|.
\]

where $V = PU$ is a unitary operator. Moreover, since $P = TPT = UKPUK = UP^\dagger U^\dagger$, then $P^\dagger U^\dagger V = U^\dagger P = U^\dagger P^\dagger$, which implies that $U^\dagger P^\dagger = PU$. That is, $V^\dagger V = V$. Therefore, $|\langle \psi| V |\phi^*\rangle| = |\langle \phi^*| V^\dagger |\psi\rangle| = |\langle \phi| V |\psi^*\rangle|$. \hfill \qed

Proposition 3. Given the self-inverse unitary operator $P$ and time reversal operator $T$, let $\Gamma_r(\rho, PT) = \min_{\sigma} \mathbb{S}(\rho || \sigma)$, then we have

\[
\Gamma_r(\rho, PT) = S(\rho || \rho^{PT}) = S(\rho^{PT}) - S(\rho),
\]

where $\rho^{PT} = \frac{1}{2}(\rho + PT \rho PT)$ is $PT$-symmetric.

Proof. Since it involves the complex conjugation $K$, taking trace may be complicated. As for any linear operator $A$, $KAK = A^*$ and $\text{Tr} (A^*) \neq \text{Tr} (A)$ in general, thus we need be more careful to deal with taking trace here. However, due to Lemma 2, for any two Hermitian operators $Q$ and $S$, we have

\[
\text{Tr} (QPTSPT) = \text{Tr} (PTQPTS).
\]

Then we follow the approach in [10, 11] to complete the proof. Due to the fact that $S(\rho || |\sigma|) \geq 0$ and $S(\rho || |\sigma|) = 0$ iff $\sigma = \rho$, thus

\[
\text{Tr} (\rho \log \sigma) \leq \text{Tr} (\rho \log \rho) \leq \frac{1}{2} \text{Tr} (\rho \log \rho) + \frac{1}{2} \text{Tr} (\rho \log \rho). \quad (A5)
\]

where the equality holds iff $\sigma = \rho$, which implies that

\[
\text{max}_\sigma \text{Tr} (\rho \log \sigma) = \text{Tr} (\rho \log \rho).
\]

First, for any $PT$-symmetric state $\sigma$, we have

\[
\text{Tr} (PT \rho PT \log \sigma) = \text{Tr} (\rho PT \log \sigma) = \text{Tr} (\rho \log \sigma),
\]

where the first equality comes from the equation (A4) and the second equality comes from the fact that $\sigma$ is $PT$-symmetric. Hence, we have

\[
\text{Tr} (\rho \log \sigma) \leq \frac{1}{2} \text{Tr} (\rho \log \sigma) + \frac{1}{2} \text{Tr} (\rho \log \sigma), \quad (A6)
\]

Appendix A: Relative entropy of $PT$-asymmetry

To prove the properties of $\Gamma_r$, we need the following lemma, which is not trivial, as $PT$ is antilinear operator.

Lemma 2. Given the self-inverse unitary operator $P$ and time reversal operator $T$, for any two Hermitian operators $Q$ and $S$, we have

\[
\text{Tr} (QPTSPT) = \text{Tr} (PTQPTS). \quad (A1)
\]
where \( \rho^{PT} = \frac{1}{2}(\rho + PT\rho PT) \). Thus, for the \( PT \)-symmetric state \( \rho^{PT} \), \( S(\rho|\rho^{PT}) = \text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \rho^{PT}) = \text{Tr}(\rho \log \rho) - \text{Tr}(\rho^{PT} \log \rho^{PT}) = S(\rho^{PT}) - S(\rho) \).

Next, we show that \( \min_{\sigma \in \text{Sym}(P,T)} S(\rho|\sigma) = S(\rho|\rho^{PT}) \), where the third equality comes from (A6) and the fourth equality comes from the inequality (A5). Therefore, the proof of the proposition is completed.

**Proposition 4.** Given the self-inverse unitary operator \( P \) and time reversal operator \( T \), then \( \Gamma_r \) satisfies the conditions (C1), (C2), (C2') and (C3), so it is a proper \( PT \)-asymmetry measure.

**Proof.** Since \( S(\rho|\sigma) = 0 \) iff \( \rho = \sigma \), thus \( \Gamma_r \) satisfy (C1). Besides, as the relative entropy is contracted under quantum operations [69] and jointly convex [70], then \( \Gamma_r \) satisfy the conditions (C2) and (C3). Moreover, we use the techniques in Ref. [11, 15] to prove that \( \Gamma_r \) satisfy the condition (C2').

Take a special self-inverse unitary operator \( P_0 = I \) and a time reversal operator \( T_0 = * \), then it is easy to see that there exist a set of orthonormal pure states \( \{ |\mu\rangle \} \) with \( |\mu\rangle |\mu\rangle \in \text{Sym}(P_0,T_0) \). For any selective \( PT \)-covariant operation \( \Phi \) with \( K_{\mu}(P(T\rho T)) K_{\mu}^\dagger = PTK_{\mu}(\rho K_{\mu}^\dagger)PT \) for any \( \mu \), it is easy to verify that the quantum operations \( \Phi \) with Kraus operators \( K_{\mu} = |\mu\rangle \langle \mu | \) is selective \( PT \)-covariant with respect to \( (P,T) \) and \( (P_0 \otimes P,T_0 \otimes T) \), that is, \( K_{\mu}PTK_{\mu}^\dagger = PTK_{\mu}PTK_{\mu}^\dagger \) for any \( \mu \). As we have proved that \( \Gamma_r \) satisfies the condition (C2), which implies that \( \Gamma_r(\sum_{\mu} p_\mu |\mu\rangle \langle \mu | \otimes \rho_\mu, P_0 T_0 \otimes PT) \leq \Gamma_r(\rho, PT) \). Moreover, \( \Gamma_r(\sum_{\mu} p_\mu |\mu\rangle \langle \mu | \otimes \rho_\mu, P_0 T_0 \otimes PT) \)

where the first equality comes from the Proposition 3 and the second equality comes from the following fact:

\[
S(\sum_i p_i |\langle i| \otimes \rho_i) = \sum_i p_i S(\rho_i) + H(\{ p_i \}),
\]

where \( \{ | i \rangle \} \) is a set of orthonormal pure states and \( H(\{ p_i \}) = \sum_i -p_i \log p_i \) is the Shannon entropy of the probability distribution \( \{ p_i \} \). Therefore, \( \Gamma_r \) satisfies the condition (C2'), that is, \( \Gamma_r \) is a proper \( PT \)-asymmetry measure.

**Lemma 5.** The measure \( \Gamma_r \) is additive, that is

\[
\Gamma_r(\rho \otimes \sigma, PT \otimes P'T') = \Gamma_r(\rho, PT) + \Gamma_r(\sigma, P'T')
\]

**Proof.** This comes directly from the representation (A3) of \( \Gamma_r \).

**Lemma 6.** The relative entropy of \( PT \)-asymmetry is asymptotically continuous, i.e., for \( \| \rho - \sigma \|_1 \leq \epsilon \leq 1/\epsilon \), then

\[
|\Gamma_r(\rho, PT) - \Gamma_r(\sigma, PT)| \leq 2\epsilon \log d + 2\eta(\epsilon),
\]

where \( d \) is the dimension of the system and \( \eta(\epsilon) = -\epsilon \log \epsilon \).

**Proof.** Due to \( \| \rho - \sigma \|_1 \leq \epsilon \), we have \( \| \rho^{PT} - \sigma^{PT} \|_1 \leq \epsilon \). Then by Fannes’ inequality [71] and (A3), we get the desired result.

**Appendix B: Skew information of \( PT \)-asymmetry**

**Proposition 7.** Given the self-inverse unitary operator \( P \) and time reversal operator \( T \), then \( \Gamma_s \) satisfies the conditions (C1), (C2), (C2') and (C3), so it is a proper \( PT \)-asymmetry measure.

**Proof.** If \( [\rho, PT] = 0 \), then obviously \( \Gamma_s(\rho) = 0 \). So we only need to prove the inverse direction. Since \( PT \rho^{1/2}PT = PUK_0^{1/2}U^\dagger \rho U U^\dagger K_0^{1/2}U = PU(\rho^{1/2}U)U^\dagger \rho U U^\dagger K_0^{1/2}U \) is positive and \( (PT \rho^{1/2}PT)^2 = \rho PT \rho PT \), then \( PT \rho^{1/2}PT \) is the square root of the quantum state \( PT \rho PT \). Besides, \( \text{Tr}(\rho^{1/2}PT \rho^{1/2}PT) = \text{Tr}(\rho^{1/2}PT \rho^{1/2}PT) = F(\rho, PT) \), where for any two states \( \rho_1 \) and \( \rho_2 \), \( F(\rho_1, \rho_2) := \text{Tr}(\rho_1^{1/2} \rho_2^{1/2}) \). That is, \( \Gamma_s(\rho, PT) \geq 1 - F(\rho, PT) \rho PT \). As \( \Gamma_s(\rho, PT) = 0 \), then \( F(\rho, PT) \rho PT = 1 \) which means \( \rho = PT \rho PT \). Thus \( \Gamma_s \) satisfy the condition (C1).

Since \( PT \rho^{1/2}PT \) is the square root of the quantum state \( PT \rho PT \), then \( \Gamma_s(\rho, PT) = 1 - \text{Tr}(\rho^{1/2}(PT \rho PT)^{1/2}) \). The convexity of \( \Gamma_s \):

\[
\Gamma_s(p\rho + (1-p)\sigma, PT) \leq p\Gamma_s(\rho, PT) + (1-p)\Gamma_s(\sigma, PT)
\]

comes from the following famous result [72]:

\[
\text{Tr}((p\rho_1 + (1-p)\rho_2)^{\alpha} (p\sigma_1 + (1-p)\sigma_2)^{1-\alpha}) \geq p \text{Tr}(\rho_1^{\alpha} \sigma_1^{1-\alpha}) + (1-p) \text{Tr}(\rho_2^{\alpha} \sigma_2^{1-\alpha})
\]

where \( \rho_1, \rho_2, \sigma_1, \sigma_2 \) are quantum states and \( p, \alpha \in [0,1] \).
Besides, for the quantity $D_\alpha(\rho_1, \rho_2) = \text{Tr} \left( \rho_1^\alpha \rho_2^{1-\alpha} \right)$ with $\alpha \in (0, 1)$, it holds that [73]:

$$D_\alpha(\rho_1, \rho_2) \leq D_\alpha(\Phi(\rho_1), \Phi(\rho_2))$$  \hspace{1cm} (B1)

where $\Phi$ is a quantum operation. Take $\alpha = 1/2$, then we will find that $\Gamma_\alpha$ satisfies the condition (C2).

Finally, similar to proof in Proposition 2, we take a special self-inverse unitary operator $P_0 = I$ and time reversal operator $T_0 = *$ with a set of orthonormal pure states $\{ | \mu \rangle \}_\mu$, $| \mu \rangle | \mu \rangle \in \text{Sym}(P_0, T_0)$. For any selective $\mathcal{PT}$-covariant operation $\Phi$ with $K_\mu(\mathcal{PT} (\cdot) \mathcal{PT}) K_\mu^\dagger = \mathcal{PT} K_\mu (\cdot) K_\mu^\dagger \mathcal{PT}$ for any $\mu$, $\mu$, it is easy to verify that the quantum operations $\tilde{\Phi}$ with Kraus operators $\tilde{K}_\mu = | \mu \rangle K_\mu$ is selective $\mathcal{PT}$-covariant with respect to $(P, T)$ and $(P_0 \otimes P, T_0 \otimes T)$, thus $\tilde{\Phi}(\rho) = \sum_\mu p_\mu | \mu \rangle | \mu \rangle \otimes \rho_\mu$, where $\rho_\mu = K_\mu \rho K_\mu^\dagger / p_\mu$ with $p_\mu = \text{Tr}(K_\mu \rho K_\mu^\dagger)$. As we have proved that $\Gamma_\alpha$ satisfies the condition (C2), which implies that $\Gamma_\alpha(\sum_\mu p_\mu | \mu \rangle | \mu \rangle \otimes \rho_\mu, P_0 T_0 \otimes \mathcal{PT}) \leq \Gamma_\alpha(\rho, \mathcal{PT})$. Moreover, it is easy to verify that

$$\Gamma_\alpha(\sum_\mu p_\mu | \mu \rangle | \mu \rangle \otimes \rho_\mu, P_0 T_0 \otimes \mathcal{PT}) = \sum_\mu p_\mu \Gamma_\alpha(\rho_\mu, \mathcal{PT}).$$  \hspace{1cm} (B2)

Hence, the condition (C2') holds for $\Gamma_\alpha$.

Appendix C: Geometric measure of $\mathcal{PT}$ asymmetry

Proposition 8. Given the self-inverse unitary operator $P$ and time reversal operator $T$, then $\Gamma_F$ satisfy the conditions (C1), (C2), (C2') and (C3), so it is a proper $\mathcal{PT}$-asymmetry measure.

Proof. (C1) is obvious, since $F(\rho, \mathcal{PT} \rho \mathcal{PT}) = 1$ iff $\rho = \mathcal{PT} \rho \mathcal{PT}$. The convexity (C3) of $\Gamma_F$ comes from the joint concavity of fidelity (See [74] and the reference therein). As fidelity is non-decreasing under CPTP maps (See [74] and the reference therein), thus $\Gamma_F$ satisfies the condition (C2). Moreover, using a similar method as the proof in $\Gamma_\alpha$ and $\Gamma_\alpha$, we can prove the condition (C2'). Take a special self-inverse unitary operator $P_0 = I$ and time reversal operator $T_0 = *$ with a set of orthonormal pure states $\{ | \mu \rangle \}_\mu$, $| \mu \rangle | \mu \rangle \in \text{Sym}(P_0, T_0)$. For any selective $\mathcal{PT}$-covariant operation $\Phi$ with $K_\mu(\mathcal{PT} (\cdot) \mathcal{PT}) K_\mu^\dagger = \mathcal{PT} K_\mu (\cdot) K_\mu^\dagger \mathcal{PT}$ for any $\mu$, it is easy to verify that the quantum operations $\tilde{\Phi}$ with Kraus operators $\tilde{K}_\mu = | \mu \rangle K_\mu$ is selective $\mathcal{PT}$-covariant with respect to $(P, T)$ and $(P_0 \otimes P, T_0 \otimes T)$, thus $\tilde{\Phi}(\rho) = \sum_\mu p_\mu | \mu \rangle | \mu \rangle \otimes \rho_\mu$, where $\rho_\mu = K_\mu \rho K_\mu^\dagger / p_\mu$ with $p_\mu = \text{Tr}(K_\mu \rho K_\mu^\dagger)$. As we have proved that $\Gamma_\alpha$ satisfies the condition (C2), which implies that $\Gamma_\alpha(\sum_\mu p_\mu | \mu \rangle | \mu \rangle \otimes \rho_\mu, P_0 T_0 \otimes \mathcal{PT}) \leq \Gamma_\alpha(\rho, \mathcal{PT})$. Moreover, it is easy to verify that

$$\Gamma_F(\sum_\mu p_\mu | \mu \rangle | \mu \rangle \otimes \rho_\mu, P_0 T_0 \otimes \mathcal{PT}) = \sum_\mu p_\mu \Gamma_F(\rho_\mu, \mathcal{PT}).$$  \hspace{1cm} (C1)

Hence, the condition (C2') holds for $\Gamma_F$.

Lemma 9. The $\mathcal{PT}$ asymmetry measure $\Gamma_F$ is continuous, that is

$$| \Gamma_F(\rho, \mathcal{PT}) - \Gamma_F(\sigma, \mathcal{PT}) | \leq 2 \sqrt{\| \rho - \sigma \|_1}.$$  \hspace{1cm} (C2)

Proof.

$$| \Gamma_F(\rho, \mathcal{PT}) - \Gamma_F(\sigma, \mathcal{PT}) | = \| F(\rho, \mathcal{PT} \rho \mathcal{PT}) - F(\sigma, \mathcal{PT} \sigma \mathcal{PT}) \|$$

$$\leq \| F(\rho, \mathcal{PT} \rho \mathcal{PT}) - F(\rho, \mathcal{PT} \mathcal{PT}) \| + | F(\rho, \mathcal{PT} \sigma \mathcal{PT}) - F(\sigma, \mathcal{PT} \sigma \mathcal{PT}) |$$

$$\leq \sqrt{1 - F(\mathcal{PT} \rho \mathcal{PT}, \mathcal{PT} \sigma \mathcal{PT})^2} + \sqrt{1 - F(\rho, \sigma)^2}$$

$$= 2 \sqrt{1 - F(\rho, \sigma)^2} \leq 2 \sqrt{2} \sqrt{1 - F(\rho, \sigma)}$$

$$\leq 2 \sqrt{\| \rho - \sigma \|_1}.$$  \hspace{1cm} (C3)

where the second inequality comes from [75] and the last inequality comes from the Fuchs-van de Graaf inequality [76].

Appendix D: Duality of $\mathcal{PT}$-Asymmetry and Entanglement

Theorem 10. Given a two-qubit system with the self-inverse unitary operator $P = \sigma_x \otimes \sigma_x$ and time reversal operator $T = *$, for pure bipartite states $| \Psi \rangle$ we have

$$\Gamma_\alpha(\Psi, \mathcal{PT}) + C(| \Psi \rangle) = 1,$$  \hspace{1cm} (D1)

$$\Gamma_F(\Psi, \mathcal{PT}) + C(| \Psi \rangle) = 1,$$  \hspace{1cm} (D2)

and

$$\Gamma(\Psi, \mathcal{PT}) = H \left( \frac{1}{2} - \frac{1}{2} C(| \Psi \rangle) \right)$$  \hspace{1cm} (D3)

where $H(\rho) = -p \log(p) - (1 - p) \log(1 - p)$ is Shannon entropy for the probability distribution $\{ p, 1 - p \}$ and $C(| \Psi \rangle)$ is the concurrence for pure state $| \Psi \rangle$.

For any two-qubit states $\rho$, the equalities may not hold. However, we still have the following inequality:

$$\Gamma_\alpha(\rho, \mathcal{PT}) + C(| \rho \rangle) \leq 1,$$  \hspace{1cm} (D4)

$$\Gamma_F(\rho, \mathcal{PT}) + C(| \rho \rangle) \leq 1,$$  \hspace{1cm} (D5)

$$\Gamma(\rho, \mathcal{PT}) \leq H \left( \frac{1}{2} - \frac{1}{2} C(| \rho \rangle) \right),$$  \hspace{1cm} (D6)

where $C(| \rho \rangle) = \min \sum_k p_k C(k)$ and the minimum is taken over all the pure states decomposition of $\rho = \sum_k p_k | \Psi_k \rangle | \Psi_k \rangle$ [49, 50].
In fact,
\[ \Gamma_F(\rho, PT) + CoA(\rho) = 1 \] (D7)
where the concurrence of assistance \( CoA(\rho) = \max \sum_k p_k C(\Psi_k) \) and the maximum is taken over all the pure states decomposition of \( \rho \).

Proof. The equations (D1), (D2), (D3) come directly from (6), (8) and (10) when \( \sigma = \sigma_y \otimes \sigma_y \) and \( T = * \). Thus we only need to verify that \( \sigma = \sigma_y \otimes \sigma_y \) and \( T = * \) satisfy these three condition. Obviously, \( \sigma_y \otimes \sigma_y = (\sigma_y \otimes \sigma_y)\dagger \) and \( (\sigma_y \otimes \sigma_y)^2 = I = T^2 \). Furthermore, as \( K(\sigma_y \otimes \sigma_y) = -\sigma_y, \) then \( K(\sigma_y \otimes \sigma_y) = \sigma_y \otimes \sigma_y \).

Due to the convexity of \( \Gamma_\sigma \), and \( C(\rho) \), for any pure state decomposition of \( \rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k| \), we have \( \Gamma(\rho, PT) \leq \sum_k p_k \Gamma_s(\Psi_k, PT)\) and \( C(\rho) \leq \sum_k p_k C(\Psi_k) \). Based on the convexity of \( f(x) = x^2 \), we have \( C(\rho)^2 \leq \sum_k p_k \rho C(\Psi_k)^2 \). Therefore, due to the equality (D1), we get the inequality (D4).

Similarly, (D5) comes directly from the convexity of \( \Gamma_F \) and \( C(\rho) \). And (D6) also comes from the convexity of \( \Gamma_\sigma \), \( C(\rho) \) and the concavity of Shannon entropy \( H \), as for any pure states decomposition of \( \rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k| \),
\[ \Gamma_r(\rho, PT) \leq \sum_k p_k \Gamma_r(\Psi_k, PT) = \sum_k p_k H(\frac{1}{2} - \frac{1}{2} C(\Psi_k)) \leq H(\frac{1}{2} - \frac{1}{2} C(\rho)) \]
Finally, (D7) comes directly from the definition of concurrence of assistance (CoA) and Eq.5.

Appendix E: PT-symmetric dynamics

Consider the unitary operator \( V \) which is \( PT \)-covariant, that is
\[ PT V \rho \dagger PT = V \rho PT V \dagger, \text{ for any } \rho, \] (E1)
then
\[ VPT = e^{i \theta} PT \] (E2)

Besides, the unitary operator \( V \) is called \( PT \)-invariant unitary if \( [V, PT] = 0 \). If \( [V, PT] = 0 \), then \( [V^\dagger, PT] = 0 \). Note that, if the Hamiltonian \( H \) satisfying \( \{ H, PT \} = 0 \), then the unitaries \( e^{iHt} \) satisfy \( [e^{iHt}, PT] = 0 \).

Similar to entanglement [77] and coherence [78], we can also consider the state transformation under \( PT \)-covariant operation.

**Proposition 11.** Pure state \( |\psi\rangle \) can be transformed to \( |\phi\rangle \) under selective \( PT \)-covariant operations if and only if \( \langle \phi | PT |\psi\rangle \leq \langle \phi | PT |\phi\rangle \).

**Proof.** The "only if" part is obvious, we only need to prove the "if" part. Define \( H_{PT} = \{ |\psi\rangle \in \mathcal{H} : \langle \phi | PT |\psi\rangle = \langle \phi | PT |\phi\rangle \} \), then \( H_{PT} \) is a real Hilbert space with \( \dim H_{PT} = \dim \mathcal{H} \) [50, 60]. Thus the basis of \( H_{PT} \) is also a basis of \( \mathcal{H} \), which is called \( PT \)-invariant basis. Here we use \( \{ |i\rangle_{O}\}_{i=1}^{d} \) and \( \{ |i\rangle_{N}\}_{i=1}^{d} \) to denote the initial given basis of \( \mathcal{H} \) and the \( PT \)-invariant basis, \( |\psi\rangle \) and \( |\phi\rangle \) denote the representation of state \( |\psi\rangle \) in the basis \( \{ |i\rangle_{O}\}_{i=1}^{d} \) and \( \{ |i\rangle_{N}\}_{i=1}^{d} \), respectively. Obviously, there exists a unitary operator \( U \) such that \( |\psi\rangle_N = U |\psi\rangle_O \) for any \( \psi \). Besides, the \( PT \) operator in the new basis \( \{ |i\rangle_N\}_{i=1}^{d} \) is equivalent to the complex conjugation with respect to this basis, denoted by \( K_N \). Then \( PT |\psi\rangle_O = K_N |\psi\rangle_N \) and \( \langle \psi | PT |\psi\rangle = \langle \psi | PT |\psi\rangle_O = \langle K_N |\psi\rangle_N = \langle |U^\dagger K_N U |\psi\rangle_O \) which implies that \( PT = U^\dagger K_N U \). Thus \( \langle \phi | PT |\psi\rangle \leq \langle \phi | PT |\phi\rangle \) is equivalent to \( \langle \phi | K_N |\psi\rangle_N \leq \langle \phi | K_N |\phi\rangle_N \). Due to Ref.[42], there exists a CPTP map \( \Phi \) with Kraus operators \( \{ K_\mu \} \) such that \( [K_\mu, K_N] = 0 \) and \( \Phi(|\psi\rangle_N) = |\phi\rangle_N |\phi\rangle_N \). Therefore, in the initial given basis \( \{ |i\rangle_O\}_{i=1}^{d} \), the quantum operation \( \Phi \) with \( K_\mu = U^\dagger K_N U \) satisfy the conditions \( [K_\mu, PT] = 0 \) and \( \Phi(|\psi\rangle) = |\phi\rangle |\phi\rangle \).

The \( PT \)-asymmetry measure introduced here is connected with the time reversal symmetry monotone in [42] up to a unitary. However, it is hard to find the unitary \( U \) such that \( \Gamma(\rho, PT) = \Gamma(U \rho U^\dagger, K_N) \) where \( K_N \) is the complex conjugation with the \( PT \)-invariant basis. And such a unitary, especially a global unitary on the composite system, may ruin the nonlocality of \( PT \)-symmetry.

Except the \( PT \)-covariant operations, we can also consider the operations which map the \( PT \)-symmetric state to \( PT \)-symmetric state, that is \( \Phi(Sym(\mathcal{P}, T)) \subset Sym(\mathcal{P}, T) \). Such operations are called \( PT \)-preserving operations. Obviously, all \( PT \)-covariant operations are \( PT \)-preserving operations. Moreover, for any operations \( \Phi = \sum_\mu K_\mu (\cdot) K_\mu^\dagger \) with \( K_\mu(Sym(\mathcal{P}, T))K_\mu^\dagger \subset Sym(\mathcal{P}, T) \), we call such operation selective \( PT \)-preserving operations, which is similar to the definition of incoherent operations [15]. We can weaken the conditions (C2) and (C2') to the following conditions:

(C2a) Monotone under \( PT \)-preserving operations \( \Phi_{PT_{pr}} \), that is \( \Gamma(\Phi_{PT_{pr}}(\rho, PT)) \leq \Gamma(\rho, PT) \).

(C2'a) Monotone under selective \( PT \)-preserving operations, that is \( \sum_\mu K_\mu(\rho, PT) \leq \Gamma(\rho, PT) \), where \( K_\mu(Sym(\mathcal{P}, T))K_\mu^\dagger \subset Sym(\mathcal{P}, T) \) and \( \rho_{\mu} = K_\mu \rho K_\mu^\dagger \) with \( \rho_{\mu} = Tr(K_{\mu} \rho K_{\mu}) \).

Therefore, we can also consider the \( PT \)-asymmetry monotone which satisfy the conditions (C1), (C2a), (C2a') and (C3), which will be explored in a future work.