Abstract

For many years, $^4$He typified Bose-Einstein superfluids, but recent advances in dilute ultra-cold alkali-metal gases have provided new neutral superfluids that are particularly tractable because the system is dilute. This chapter starts with a brief review of the physics of superfluid $^4$He, followed by the basic ideas of Bose-Einstein condensation (BEC), first for an ideal Bose gas and then considering the effect of interparticle interactions, including time-dependent phenomena. Extensions to more exotic condensates include magnetic dipolar gases, mixtures of two components, and spinor condensates that require a focused infrared laser for trapping of all the various hyperfine magnetic states in a particular hyperfine $F$ manifold of $m_F$ states. With an applied rotation, the trapped BECs nucleate quantized vortices. Recent theory and experiment have shown that laser coupling fields can mimic the effect of rotation. The resulting synthetic gauge fields have produced vortices in a nonrotating condensate.

Keywords: dilute quantum gas, Bose-Einstein condensation, quantized vortices, exotic condensates, synthetic gauge potentials
**Contents**

1 **Introduction to Bose superfluids: brief review of superfluid \(^4\)He**
   - 1.1 two-fluid model .................................................. 3
   - 1.2 quantized circulation ............................................. 4
   - 1.3 quasiparticles and the Landau critical velocity .......... 5

2 **Three-dimensional ideal Bose gas**
   - 2.1 qualitative picture of Bose-Einstein condensation in a uniform system .............................................. 5
   - 2.2 quantitative description of ideal Bose gas in an external potential \(V_{tr}\) ......................................................... 6
   - 2.3 ideal Bose gas in three-dimensional box with periodic boundary conditions ........................................... 7
   - 2.4 ideal Bose gas in three-dimensional harmonic trap .......................... 8
   - 2.5 ideal Bose gas in harmonic traps with reduced dimensions .......................... 9

3 **Energies and length scales for interacting Bose gas**
   - 3.1 interaction energy for a uniform gas .......................... 11
   - 3.2 healing length for a uniform gas ............................. 11

4 **Gross-Pitaevskii picture for a trapped Bose gas**
   - 4.1 static behavior .................................................... 12
   - 4.2 time-dependent GP equation ....................................... 14
   - 4.3 Bogoliubov spectrum: linearized hydrodynamic equations for a uniform Bose gas ........................................ 16
   - 4.4 linearized hydrodynamic equations for collective modes of a stationary condensate ................................ 18
   - 4.5 effect of attractive interactions .................................... 19

5 **Selected applications and comparison with experiments**
   - 5.1 free expansion for different trap aspect ratios ............ 20
   - 5.2 measurement of the condensate fraction ..................... 22
   - 5.3 low-lying collective oscillation modes ...................... 24

6 **Dipolar condensates**

2
1. Introduction to Bose superfluids: brief review of superfluid $^4$He

Superfluid $^4$He has played a central role in understanding the physics of Bose-Einstein condensation. It has the unusual property of remaining liquid to $T = 0$ K under atmospheric pressure. This behavior reflects its weak interatomic potential and the relatively large zero-point energy (because of the small mass). The common helium isotope $^4$He has two electrons, two protons and two neutrons and thus acts like a boson under exchange of two such atoms. Below a characteristic temperature $T_\lambda \approx 2.17$ K, it undergoes a phase transition from a normal weakly viscous fluid to a superfluid with remarkable properties (for a review of this fascinating system, see [1]).

1.1. two-fluid model

Superfluid $^4$He has remarkable hydrodynamic properties. It can flow through fine channels with no pressure drop, which suggests that it has zero viscosity, yet a direct measurement of the viscosity through torque on a rotating cylinder yields a value comparable with that for the normal phase above $T_\lambda$. This seemingly contradictory behavior was explained phenomenologically by a two-fluid model, in which the low-temperature phase has two interpenetrating components: (1) a superfluid with zero viscosity and irrotational flow velocity $v_s$ with $\nabla \times v_s = 0$ and (2) a normal (viscous) fluid with general velocity $v_n$. Each component has its own temperature-dependent mass density $\rho_s$ and $\rho_n$, with $\rho_s + \rho_n = \rho$ (the total mass density). The
superfluid component has \( \rho_s = 0 \) at \( T_\lambda \) and \( \rho_s = \rho \) at \( T = 0 \) K. The normal density \( \rho_n \) has been measured by the viscous drag on a set of oscillating disks that respond only to the normal component; the superfluid density is then given by \( \rho_s = \rho - \rho_n \).

Superfluid \(^4\)He can exhibit persistent currents in a multiply connected geometry like a torus or in a porous random medium such as a compacted powder. If the container and fluid rotate above \( T_\lambda \) and are then cooled below \( T_\lambda \) the superfluid will continue to rotate at essentially the original angular velocity \( \Omega \). If the container is slowly brought to rest, in many cases, the superfluid continues to rotate (in contrast to the behavior of a classical viscous fluid that would eventually come to rest). Such persistent currents typify both superfluid \(^4\)He and superconductors (where the charge makes the detection of such electrical currents relatively easy). There is a critical velocity \( v_c \) for superfluid flow that is set by the energy of the lowest excited state; \( v_c \) gives an upper limit to the magnitude of persistent currents in confined geometries. For a channel of lateral dimension \( \sim d \), the typical observed critical velocity is of order \( v_c \sim \hbar/Md \), where \( M \) is the atomic mass.

1.2. quantized circulation

For low flow velocity, liquid \(^4\)He is effectively incompressible with \( \nabla \cdot \mathbf{v}_s = 0 \). Since \( \mathbf{v}_s \) is also irrotational, it can be described with a velocity potential \( \Phi \) that satisfies Laplace’s equation, with \( \mathbf{v}_s = \nabla \Phi \). For a quantum fluid, we can be more specific, with \( \Phi = (\hbar/M)S \), where \( M \) is the atomic mass and \( S \) is the phase of a macroscopic one-body wave function characterizing the quantum-mechanical condensed state of superfluid \(^4\)He at \( T = 0 \) K. The irrotational condition on \( \mathbf{v}_s \) apparently suggests that the superfluid would not rotate at \( T = 0 \) K, but experiments showed that the rotating superfluid had a parabolic meniscus independent of temperature (like that of a classical viscous fluid). In 1949, Onsager proposed without explanation that the superfluid circulation \( \kappa = \oint \mathbf{dl} \cdot \mathbf{v}_s \) was quantized in units of \( \hbar/M \). Subsequently Feynman [2] suggested that rotating superfluid \(^4\)He has a uniform array of quantized vortex lines with singular vorticity \( \kappa \) at the center of each line[1] The velocity potential for a single vortex line is proportional to the azimuthal angle \( \phi \), which ensures that the quantum-mechanical wave function is single-

\[ ^1 \text{Donnelly} \text{ gives good account of this intriguing early history in Sec. 2.3.} \]
valued and reproduces Onsager’s quantized circulation. Feynman also chose the areal vortex density \( n_v \) to mimic solid-body rotation \( \mathbf{v}_{sb} = \Omega \times \mathbf{r} \), yielding \( n_v = M\Omega/\pi\hbar \), where \( \Omega \) is the external angular velocity.

1.3. quasiparticles and the Landau critical velocity

Landau \([4, 5]\) introduced a quasiparticle model to describe the normal component (for more accessible accounts, see \([1, 6]\)). Specifically the elementary excitations act like phonons at long wavelengths with a dispersion relation \( \omega_k = sk \), where \( s \approx 240 \text{ m/s} \) is the speed of compressional sound. For shorter wavelengths comparable with the interparticle spacing in liquid \(^4\text{He} \), the dispersion relation has a local minimum associated with what are called rotons

\[
\omega_k \approx \frac{\Delta}{\hbar} + \hbar (k - k_0)^2 \frac{2M^*}{M},
\]

where \( \Delta/k_B \approx 8.7 \text{ K} \) is the roton gap, \( k_0 \approx 1.9 \times 10^{10} \text{ m}^{-1} \), and \( M^* \approx 0.16M \) for \(^4\text{He} \). This model of independent phonons and rotons allows a straightforward calculation of the temperature-dependent normal-fluid density \( \rho_n \). More importantly, it also predicts a critical velocity as follows. Imagine a macroscopic object moving through the superfluid at \( T = 0 \text{ K} \). Conservation of energy and momentum indicates that this object cannot lose energy by creating a quasiparticle unless its speed exceeds a critical velocity \( v_c \) given by the minimum value of \( \omega_k/k \):

\[
v_c = \frac{\omega_k}{k} \bigg|_{\text{min}}.
\]

For superfluid \(^4\text{He} \), rotons determine this value as \( v_c \approx \Delta/hk_0 \approx 50 \text{ m/s} \).

2. Three-dimensional ideal Bose gas

This section provides a brief review of the ideal Bose gas, both the familiar uniform case and for a gas confined in a harmonic trap as in most experiments with ultracold atomic vapors (good general references are \([7, 8, 9]\)).

2.1. qualitative picture of Bose-Einstein condensation in a uniform system

Consider a uniform ideal gas with number density \( n \) and volume per particle \( n^{-1} \). One important characteristic length is the interparticle spacing \( \sim n^{-1/3} \). Suppose that the system is in thermal equilibrium at temperature \( T \). If \( M \) is the particle mass, the mean thermal momentum is \( p_T \sim \sqrt{Mk_B T} \).
The de Broglie relation then gives the thermal wavelength \( \lambda_T \sim \hbar/\sqrt{Mk_BT} \), which provides another characteristic length for this uniform ideal gas.

In the classical limit \( \hbar \to 0 \) or at high temperature, the thermal wavelength becomes small compared to the interparticle spacing \( n^{-1/3} \), which means that the effect of quantum diffraction and interference is negligible. This limit is equivalent to ray optics for light when the wavelength is very small compared to the typical dimension.

As \( T \) falls, however, the thermal wavelength eventually becomes comparable with \( n^{-1/3} \), and quantum interference between nearby particles becomes crucial. It is convenient to introduce the dimensionless phase-space density \( n\lambda_T^3 \), which is small in the classical limit. When \( n\lambda_T^3 \) is of order unity, however, the ideal gas changes character. An ideal gas of fermions undergoes a crossover from the classical ideal gas to a degenerate Fermi gas at the Fermi temperature \( T_F \) given by the qualitative relation \( n\lambda_T^3 \sim 1 \). Equivalently, \( k_BT_F \sim \hbar^2n^{2/3}/M \), is the usual expression in condensed-matter physics. For \( n \sim 10^{28} \text{ m}^{-3} \) and \( M \) equal to the electron mass, the Fermi temperature \( T_F \) for electrons in a metal is of order \( 10^4 \) K, whereas liquid \(^3\)He has similar number density but particle mass \( \sim 10^4 \) times larger, yielding \( T_F \sim 1 \) K.

In contrast, an ideal gas of bosons in three dimensions undergoes a sharp phase transition at a temperature \( T_c \) given by the similar criterion: \( n\lambda_T^3 \sim 1 \), or equivalently \( k_BT_c \sim \hbar^2n^{2/3}/M \), similar to that for a Fermi gas. For liquid \(^4\)He, the number density and mass are comparable to those for \(^3\)He, so that the transition temperature is of order 1 K. Nevertheless, the physics of the Bose-Einstein condensation is quite different from the onset of Fermi degeneracy, and we now explore the specific cases of a Bose gas in a box and in a general harmonic trap.

### 2.2. Quantitative Description of Ideal Bose Gas in an External Potential \( V_{tr} \)

Consider an external trap with potential \( V_{tr} \), and an associated complete set of quantum-mechanical single-particle states with energies \( \epsilon_j \). Assume an ideal Bose gas in equilibrium at temperature \( T \) and chemical potential \( \mu \) in this external potential. The mean occupation number of the state \( j \) is

\[
n_j = \frac{1}{\exp[\beta(\epsilon_j - \mu)] - 1} \equiv f(\epsilon_j), \tag{3}
\]

where \( \beta^{-1} = k_BT \), and \( f(\epsilon) = \{\exp[\beta(\epsilon-\mu)]-1\}^{-1} \) is the usual Bose-Einstein distribution function. The mean total particle number and mean total energy
are given by

\[ N(T, \mu) = \sum_j f(\epsilon_j), \quad (4) \]

\[ E(T, \mu) = \sum_j \epsilon_j f(\epsilon_j). \quad (5) \]

Formally, one can invert Eq. (4) to express the chemical potential as \( \mu(T, N) \), thus giving the energy in terms of the more familiar variables \( T, N \).

Since we shall deal with many states, it is convenient to introduce the density of states \( g(\epsilon) = \sum_j \delta(\epsilon - \epsilon_j) \) where the sum is over all distinct single-particle states (some of which may be degenerate). This applies for any particular trap potential. Equation (4) can then be rewritten as

\[ N(T, \mu) = \int d\epsilon g(\epsilon) f(\epsilon). \quad (6) \]

In the classical limit for fixed \( N \), the chemical potential is large and negative [since \( n_j \ll 1 \) in Eq. (3)], but as the temperature decreases, the chemical potential increases and eventually approaches the lowest single-particle energy \( \epsilon_0 \). This equality \( \mu(T_c, N) = \epsilon_0 \) defines the critical temperature for the onset of Bose-Einstein condensation (BEC). For \( T < T_c \), the chemical potential cannot increase, because the distribution function \( f(\epsilon_0) \) would become singular. In contrast, Eq. (6) continues to decrease for \( T < T_c \). The explanation is that a macroscopic number of particles starts to occupy the lowest single-particle state with occupation number \( N_0(T) \). The temperature dependence of \( N_0(T) \) follows from conservation of particles \( N = N_0(T) + N'(T) \), where

\[ N'(T) = \int_{\epsilon_0}^{\infty} d\epsilon \frac{g(\epsilon)}{\exp[\beta(\epsilon - \epsilon_0)] - 1} \quad (7) \]

defines the total number of particles not in the condensate. Note that this integral is finite only if \( g(\epsilon_0) \) vanishes; otherwise \( N'(T) \) diverges and BEC cannot occur.

2.3. ideal Bose gas in three-dimensional box with periodic boundary conditions

Imagine a cubical box of length \( L \) on a side. Plane waves have the familiar normalized wave function \( \psi(r) = V^{-1/2} \exp(ik \cdot r) \) with \( V = L^3 \) and single-particle energy \( \epsilon_k = \hbar^2 k^2 / 2M \). The states obey periodic boundary conditions...
for the specific wave vectors $k = (2\pi/L) (n_x, n_y, n_z)$, where $n_j$ is any integer. The lowest single-particle state is uniform with $k = 0$ and $\epsilon_0 = 0$. The corresponding density of states is readily found to be

$$g(\epsilon) = \frac{V}{4\pi^2} \left( \frac{2M}{\hbar^2} \right)^{3/2} \epsilon^{1/2},$$

which vanishes at $\epsilon = 0$. The condition for the onset of BEC yields the familiar condition for the phase-space density $n\lambda^3_{T_c} = \zeta(3/2) \approx 2.612$ at $T_c$.

Below $T_c$, the fraction of particles in the excited states (those with finite momentum) is given by

$$\frac{N'(T)}{N} = \left( \frac{T}{T_c} \right)^{3/2}.$$  

Conservation of total particles then yields the fraction of condensed particles in the zero-momentum single-particle ground state

$$\frac{N_0(T)}{N} = 1 - \left( \frac{T}{T_c} \right)^{3/2},$$

which increases continuously from 0 at $T_c$ to 1 at $T = 0$ K.

### 2.4. ideal Bose gas in three-dimensional harmonic trap

Typical experiments on dilute alkali-metal gases rely on magnetic traps that vary quadratically with the distance from the origin (see Chap. 1). Hence we consider a general three-dimensional harmonic trap potential

$$V_{tr}(r) = \frac{1}{2}M \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right),$$

with the familiar single-particle energies

$$\epsilon_{n_xn_yn_z} = \hbar (n_x\omega_x + n_y\omega_y + n_z\omega_z) + \epsilon_0,$$

where $n_x, n_y, n_z$ are non-negative integers and $\epsilon_0 = \frac{1}{2}\hbar(\omega_x + \omega_y + \omega_z)$ is the zero-point energy in this harmonic trap. The ground-state wave function $\psi_0(x, y, z)$ is a product of three Gaussians with characteristic dimensions $d_j = \sqrt{\hbar/M\omega_j}$, where $j = x, y, z$. If the sums in the density of states are approximated by integrals (which holds for $\epsilon \gg \epsilon_0$), the resulting density of states becomes

$$g(\epsilon) = \frac{\epsilon^2}{2\hbar^3\omega_0^3},$$

8
where $\omega_0 = (\omega_x \omega_y \omega_z)^{1/3}$ is the geometric-mean frequency.

The onset of BEC in this harmonic trap occurs for $\mu(T_c, N) = \epsilon_0$, with the transition temperature

$$k_B T_c = [\zeta(3)]^{-1/3} \hbar \omega_0 N^{1/3} \approx 0.94 \hbar \omega_0 N^{1/3},$$

(14)

where $\zeta(3) \approx 1.212$. Below $T_c$, the number of particles in the excited states decreases like $N'(T)/N = (T/T_c)^3$. The remaining particles occupy the ground state so

$$\frac{N_0(T)}{N} = 1 - \left(\frac{T}{T_c}\right)^3.$$

(15)

The condensate fraction grows from 0 at $T_c$ to 1 at $T = 0$ K as in the case of a uniform Bose gas, but with a temperature dependence different from that in Eq. (9). The single-particle ground-state wave function is that of a harmonic oscillator, $\psi_0(x, y, z) \propto \prod_j \exp(-x_j^2/2d_j^2)$. Note that the nontrivial trapping potential $V_{tr}$ introduces a new characteristic oscillator length $d_0 = \sqrt{\hbar/M \omega_0}$ that does not appear for the uniform ideal Bose gas. For a noninteracting gas (or small numbers of atoms) the zero-temperature condensate density has a Gaussian profile $n_0(r) = N|\psi_0(r)|^2$. For an interacting gas the size of the condensate is larger than the harmonic oscillator length as shown in Sec. 4.1, but Eq. (15) remains a good approximation since it is the Bose-Einstein statistics that determine $T_c$, not the interactions.

Images of the temperature-dependent condensate provided clear evidence for the formation of a BEC in dilute cold alkali-metal gases (see, for example, [7, 8, 9]). Typical atomic traps have $d_0 \sim$ a few $\mu$m, and $N \sim 10^6$, leading the transition temperatures of order 100-1000 nK, depending on the atomic mass.

2.5. ideal Bose gas in harmonic traps with reduced dimensions

The shape of the confining potentials for ultracold atoms (created by applied magnetic fields, laser light or other external fields) can be engineered to be highly asymmetric so that certain degrees of freedom are frozen out. Strong confinement in two directions, $\omega_z \ll \omega_x \approx \omega_y$, creates a long thin tube of atoms and such one-dimensional systems have some fascinating properties. For example in a Tonks-Giradeau gas (see Chap. 4) bosonic atoms act like fermions (in some but not all ways). This behavior arises when the confinement is so tight that atoms cannot pass each other, and this one-dimensional system of impenetrable bosons resembles beads on a string, or cars in a traffic
jam—atoms that cannot be at the same position along the system look as if they obey the Pauli exclusion principle.

Conversely for very tight confinement in one direction, $\omega_z \gg \omega_x \simeq \omega_y$ in Eq. (11), the cloud of atoms has a oblate (pancake) shape; when the temperature is low enough $k_B T \ll \hbar \omega_z$ the atoms move only in the radial plane giving a two-dimensional gas, e.g. for $\omega_z/2\pi = 2$ kHz this requires $T \ll 100$ nK which is readily achievable. Two-dimensional behavior has been investigated in films of superfluid helium but an important difference is that the helium system corresponds to a square well (with a flat potential between the walls) whereas in ultracold gases the atoms are usually in a harmonic potential—these two situations have different densities of the states [as we have seen for the three-dimensional cases in Eq. (8) and Eq. (13)]. A two-dimensional gas in a uniform potential is a special case: the density of states is independent of energy so the integral that gives $N'$ diverges in Eq. (7)—this means that BEC does not occur in this case. Nevertheless interesting phenomena occur in a two-dimensional gas in the quantum degenerate regime such as the Berezinskii-Kosterlitz-Thouless transition, when the density of atoms per unit area is sufficiently low, and the physics related to this is discussed in Chap. 4. Other trapping geometries have also been demonstrated, one of the most interesting being the creation of ring-shaped clouds where there can be persistent flow around the loop, as for superfluid helium in similar multiply connected systems. Another case is that of rapid rotation about the axis of a cylindrically symmetrical potential which is described in more detail in Sec. 9; centripetal acceleration causes the gas cloud to bulge outward along its equator so that it can become flattened into the 2-D regime, i.e. effectively the tightness of trap is reduced in radial direction in the rotating frame, as shown in Eq. (60).

3. Energies and length scales for interacting Bose gas

The inclusion of a two-body interaction potential $U(r - r')$ leads to several new features that play a crucial role in the description of realistic dilute cold Bose gases. Typically, the interactions have a short range much less that the interparticle spacing, and, at low temperature, only s-wave scattering is important. To incorporate these restrictions, it is convenient to introduce a simple model, approximating the isotropic interparticle potential as $U(r) = g\delta^{(3)}(r)$, where $g$ has the dimension of energy times volume. Standard scattering theory for two particles with identical masses $M$ yields
the identification

\[ g = \frac{4\pi\hbar^2 a}{M}, \tag{16} \]

where \( a \) is the s-wave scattering length that relates the phase of the scattered wave to that of incident wave. For the commonly used bosonic alkali-metal atoms (\(^{23}\)Na, \(^{87}\)Rb), the scattering length is positive and a few nm (a common trapped state of \(^7\)Li is unusual in having a large negative scattering length that leads to very different physics; see Sec. 4.5). Typically, the dimensionless parameter \( na^3 \) that characterizes the “diluteness” of the gas is generally small. In special cases, however, the scattering length is purposely enhanced with an applied magnetic field, as occurs in a “Fano-Feshbach resonance” (see Chap. 1).

### 3.1. interaction energy for a uniform gas

In an interacting gas with \( N \) particles, the \( i \)th particle at \( \mathbf{r}_i \) experiences an effective mean-field potential

\[ V(\mathbf{r}_i) = g \sum_{j \neq i} \delta^{(3)}(\mathbf{r}_i - \mathbf{r}_j) \approx gn(\mathbf{r}_i), \tag{17} \]

where the sum is over all other particles, and the last expression omits small corrections of order \( 1/N \). This effective potential is just the one-body Hartree interaction with all the other particles \( V_H(\mathbf{r}) = gn(\mathbf{r}) \), familiar from atomic and condensed-matter physics.

It takes an energy \( E(N+1) - E(N) = gn \) to add one additional condensate particle to a uniform interacting dilute condensed Bose gas in a box. This quantity is just the chemical potential \( \mu = (\partial E/\partial N)_V \) so that here \( \mu = gn \). For a uniform dilute gas in a box of volume \( V \), the thermodynamic definition of \( \mu \) implies the total energy \( E(V,N) = \frac{1}{2}gN^2/V \). Furthermore, the pressure is \( p = - (\partial E/\partial V)_N = \frac{1}{2}gn^2 \).

### 3.2. healing length for a uniform gas

In general, the kinetic energy operator is \( \mathcal{T} = -\hbar^2 \nabla^2/2M \). Since \( \nabla^2 \) has the dimension of an inverse squared length, comparison of the kinetic energy \( \mathcal{T} \) and the Hartree energy \( V_H = gn \) leads to a characteristic squared length

\[ \xi^2 = \frac{\hbar^2}{2Mgn} = \frac{1}{8\pi na^3}. \tag{18} \]
where the second form makes use of Eq. (16). If a uniform Bose gas is perturbed locally (by an impurity or a rigid external boundary, for example), \( \xi \) is the length over which the gas heals back to its equilibrium density \( n \). In the dilute limit \( na^3 \ll 1 \), the healing length \( \xi \) is large compared to the interparticle spacing \( n^{-1/3} \), confirming the mean-field character of the Hartree picture of the interaction.

4. Gross-Pitaevskii picture for a trapped Bose gas

At zero temperature, a uniform interacting Bose gas in a box has two microscopic length scales: the scattering length \( a \) and the interparticle spacing \( n^{-1/3} \) (or equivalently, the healing length \( \xi \)). In addition, an interacting trapped Bose gas has a length associated with the oscillator length \( d_0 \), which leads to many interesting new effects. Although the original papers of Gross [11] and Pitaevskii [12] focused on a uniform gas, Baym and Pethick [13] extended the treatment to a harmonic trap soon after the creation of a BEC in 1995 [10].

4.1. static behavior

Gross and Pitaevskii independently started from the time-independent Schrödinger equation for the condensate wave function in an ideal gas and added the nonlinear Hartree potential, leading to a nonlinear Schrödinger equation. With the addition of the trap potential, the Gross-Pitaevskii (GP) equation becomes

\[
\left( -\frac{\hbar^2 \nabla^2}{2M} + V_{\text{tr}} + g|\Psi|^2 \right) \Psi = \mu \Psi, \tag{19}
\]

where \( \mu \) is the chemical potential and \( \Psi \) is the condensate wave function. Strictly, \( \Psi \) is normalized to the condensate number \( N_0 \), but a dilute Bose gas at zero temperature has \( N_0 \approx N \) so that the condensate normalization is generally taken as \( \int dV |\Psi|^2 \approx N \).

For a deeper understanding of this time-independent GP equation, it is helpful to start from the corresponding GP energy functional

\[
E_{\text{GP}}[\Psi] = \int dV \left( \frac{\hbar^2 |\nabla \Psi|^2}{2M} + V_{\text{tr}}|\Psi|^2 + \frac{1}{2}g|\Psi|^4 \right) \tag{20}
\]
containing the kinetic energy, the trap energy (proportional to the condensate density), and the interaction energy (proportional to the condensate density squared). Variation of $E_{GP}[\Psi] - \mu N$ with respect to $\Psi^*$ at fixed normalization with the chemical potential as a Lagrange multiplier reproduces the time-independent GP equation, Eq. (19); but note that although this looks like a nonlinear eigenvalue problem, the right-hand side of Eq. (19) contains the Lagrange multiplier $\mu$, rather than the energy per particle.

For a harmonic potential as in Eq. (11), it is convenient to use the mean oscillator length $d_0$ as the length scale and the mean oscillator frequency $\omega_0$ times $\hbar$ as the energy scale. The resulting dimensionless form of the energy functional in Eq. (20) has a simple structure, with the first two terms of order unity (since they involve only the harmonic oscillator), but the interaction term contains a new dimensionless parameter $Na/d_0$ that characterizes the importance of the interaction energy relative to the other two terms. If this interaction parameter is small, then the BEC adjusts to balance the kinetic energy and the confining trap, like an ideal gas, but the situation is very different if this parameter is large.

Note that the ratio $a/d_0$ is typically $\sim 10^{-3}$, but the total number of atoms is of order $10^5 - 10^6$, so that this interaction parameter is generally large for most experiments. In this case, the repulsive interactions act to expand the condensate to a typical mean dimension $R_0$ that is considerably larger than $d_0$ (typically $R_0/d_0 \sim 10$ for $Na/d_0 \sim 10^3$). Since this expanded condensate has a small density gradient, the first term of Eq. (20) becomes small, and the remaining terms give the approximate Thomas-Fermi (TF) energy functional [13]

$$ E_{TF}[\Psi] = \int dV \left( V_{tr} |\Psi|^2 + \frac{1}{2} g |\Psi|^4 \right), $$

which involves only $|\Psi|^2$ and $|\Psi|^4$. Variation of this TF energy functional with respect to $|\Psi|^2$ immediately yields the TF approximation

$$ V_{tr}(r) + g|\Psi(r)|^2 = \mu, $$

which also follows by omitting the kinetic energy in Eq. (19). Solution of this algebraic equation gives the celebrated TF condensate density profile

$$ n(r) = \frac{\mu - V_{tr}(r)}{g} \theta[\mu - V_{tr}(r)], $$
where $\theta(x)$ is the unit positive step function. For the general three-dimensional harmonic trap in Eq. (11), this TF density is an inverted parabola

$$n(r) = n(0) \left(1 - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} - \frac{z^2}{R_z^2}\right)$$

(24)

where the right-hand side is positive and zero otherwise. Here, $n(0) = \mu/g$ is the central density, and $R_j^2 = 2\mu/M\omega_j^2$ are the squared condensate radii in the three orthogonal directions.

The normalization condition on the density $\int dV n(r) \approx N$ yields $N = 8\pi n(0) R_0^3/15$, where $R_0^3 = R_x R_y R_z$ depends on the chemical potential $\mu$. Some analysis yields the important dimensionless relation

$$\frac{R_0^5}{d_0^5} = 15 \frac{Na}{d_0},$$

(25)

which is large in the present TF limit. Similarly, the TF chemical potential becomes

$$\mu_{TF} = \frac{1}{2} M \omega_0^2 R_0^2 = \frac{1}{2} \hbar \omega_0 \frac{R_0^2}{d_0^2},$$

(26)

showing that $\mu_{TF} \gg \hbar \omega_0$ in the TF limit. Note that $\mu_{TF}$ is proportional to $N^{2/5}$, hence the thermodynamic relation $\mu = \partial E/\partial N$ yields the TF energy for the trapped condensate $E_{TF} = \frac{5}{7} \mu_{TF} N$.

It is natural to use the central density $n(0)$ to define the healing length in a nonuniform condensate, and Eq. (18) shows that $\xi^2 = d_0^4/R_0^2$; equivalently,

$$\frac{\xi}{d_0} = \frac{d_0}{R_0}.$$

(27)

The right-hand side is small in the TF limit, so that $\xi$ is also small compared to the mean oscillator length $d_0$. This clear separation of TF length scales $\xi \ll d_0 \ll R_0$ is frequently valuable in understanding the physics of the trapped TF condensate.

4.2. time-dependent GP equation

Equation (19) has the intuitive time-dependent generalization (known as the time-dependent GP equation)

$$i\hbar \frac{\partial \Psi(r, t)}{\partial t} = \left[-\frac{\hbar^2 \nabla^2}{2M} + V_{tr}(r) + g|\Psi(r, t)|^2\right] \Psi(r, t),$$

(28)
where $\Psi$ now depends on $t$ as well as on $r$. Comparison with Eq. (19) shows that a stationary solution has the time dependence $\exp(-i\mu t/\hbar)$. To understand this time-dependence, note that the condensate wave function $\Psi(r, t)$ is a matrix element of the Heisenberg field operator $\psi(r, t) = e^{iHt/\hbar}\psi(r, 0)e^{-iHt/\hbar}$.

Since the operator $\psi$ removes one particle, the states on the right and left have $N$ and $N-1$ particles, respectively. Their energy difference $E_N - E_{N-1}$ is the chemical potential $\mu = \partial E/\partial N$.

This nonlinear field equation can be recast in an intuitive hydrodynamic form by writing the condensate wave function as

$$\Psi(r, t) = |\Psi(r, t)|\exp[iS(r, t)] \tag{29}$$

in terms of the magnitude $|\Psi|$ and the phase $S$. The condensate (particle) density is simply $n(r, t) = |\Psi(r, t)|^2$, and the usual one-body definition of the particle current density for the Schrödinger equation shows that $j = |\Psi|^2\hbar\nabla S/M$. The hydrodynamic definition $j = n\mathbf{v}$ then gives the local superfluid velocity as

$$\mathbf{v}(r, t) = \frac{\hbar}{M}\nabla S(r, t) = \nabla \Phi(r, t), \tag{30}$$

where $\Phi = \hbar S/M$ is the velocity potential for this irrotational flow. The relation between the velocity $\mathbf{v}$ and the phase $S$ implies that the circulation $\kappa = \oint_C \mathbf{v} \cdot d\mathbf{l}$ around any closed path $C$ in the fluid is quantized in units of $\hbar/M$ [2]

$$\kappa = \frac{\hbar}{M} \oint_C \mathbf{v} \cdot d\mathbf{l} = \frac{\hbar}{M} \Delta S_C, \tag{31}$$

because $\Delta S_C$ must be an integral multiple of $2\pi$ to ensure that the condensate wave function $\Psi$ is single valued.

Substitute Eq. (29) into the time-dependent GP equation (28). The imaginary part gives the expected conservation of particles

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0. \tag{32}$$

In contrast the real part gives a generalized Bernoulli equation

$$\frac{1}{2}M\mathbf{v}^2 + V_{tr} - \frac{\hbar^2}{2M\sqrt{n}}\nabla^2\sqrt{n} + gn + M\frac{\partial \Phi}{\partial t} = 0. \tag{33}$$
Here, the explicitly quantum-mechanical term involving $\hbar^2$ is called the quantum pressure. It is usually small for low-lying collective modes of a trapped condensate in the TF limit, since the density is slowly varying. It can be important for finite-wavelength disturbances, however, as shown below for the Bogoliubov excitation spectrum (see, for example, Sec. 7.2 of [9]).

4.3. Bogoliubov spectrum: linearized hydrodynamic equations for a uniform Bose gas

As a first application of these hydrodynamic equations, consider a uniform stationary Bose gas in a box with periodic boundary conditions (so that $V_{tr}$ acts solely to fix the allowed plane waves, but otherwise plays no role). Assume small perturbations around the equilibrium number density $n_0$, writing $n \approx n_0 + n'$, where $n'$ is small. Similarly the velocity potential becomes $\Phi_0 + \Phi'$. The linearized equation of continuity has the simple form

$$\frac{\partial n'}{\partial t} + n_0 \nabla^2 \Phi' = 0$$  \hspace{1cm} (34)

because $v' = \nabla \Phi'$.

To zero order, the Bernoulli equation yields $gn_0 + M \partial \Phi_0 / \partial t = 0$. Since $\Phi_0 = \hbar S_0 / M$ and $gn_0$ is simply the equilibrium chemical potential $\mu_0$ for a uniform Bose gas, this result reproduces the time-dependent phase of the equilibrium condensate wave function $\Psi \propto \exp(-i \mu_0 t / \hbar)$, as seen by comparing Eqs (19) and (28).

In this example all wavenumbers are relevant, and it is essential to retain the quantum pressure in the Bernoulli equation. Thus, it is necessary to write $\sqrt{n} \approx \sqrt{n_0} + \frac{1}{2} n'/\sqrt{n_0}$. The first-order contribution yields the additional linear equation

$$-\frac{\hbar^2 \nabla^2 n'}{4Mn_0} + gn' + M \frac{\partial \Phi'}{\partial t} = 0.$$  \hspace{1cm} (35)

Assume plane-wave solutions $\propto \exp(i (k \cdot r - \omega t))$. The two linearized equations become

$$-\omega n' - n_0 k^2 \Phi' = 0,$$  \hspace{1cm} (36)

$$\left( \frac{\hbar^2 k^2}{4Mn_0} + g \right) n' - i \omega M \Phi' = 0.$$  \hspace{1cm} (37)

Solutions exist only if the determinant of coefficients vanishes, which leads to the celebrated Bogoliubov spectrum $\omega_k$ for a dilute Bose gas [14] (here
written in terms of the excitation energy $\epsilon_k = \hbar \omega_k$)

$$\epsilon_k^2 = \frac{g n_0 \hbar^2 k^2}{M} + \left( \frac{\hbar^2 k^2}{2M} \right)^2 = \frac{g n_0 \hbar^2 k^2}{M} + \left( \epsilon_0^k \right)^2,$$

where $\epsilon_0^k = \hbar^2 k^2 / 2M$ is the free-particle excitation spectrum. For a summary of Bogoliubov’s original derivation based on quantized field amplitudes, see Sec. 7.2 of [9].

This dispersion relation $\omega_k = \epsilon_k / \hbar$ has two distinct regions depending on the wave number $k$. At long wavelengths (small $k$), the spectrum is linear with $\epsilon_k \approx \hbar s k$, where $s = \sqrt{g n_0 / M}$ is the speed of sound. Note that $g$ must be positive for this uniform Bose gas, which means that the effective interparticle interaction must be repulsive. In contrast, at short wavelengths (large $k$), the spectrum is quadratic, with $\epsilon_k \approx \epsilon_0^k + g n_0$, shifted up from the free-particle value by the chemical potential $g n_0$ that reflects the presence of the background medium.

![Figure 1: Filled circles show measured excitation spectrum for dilute trapped BEC, with solid curve the Bogoliubov spectrum. For comparison dashed curve shows the free-particle quadratic spectrum. Reprinted with permission of the authors [15] and the American Physical Society.](image)

The crossover between these two regimes occurs at $k_{cr}^2 \approx 2M n_0 g / \hbar^2 = 1 / \xi^2$, where $\xi = \hbar / \sqrt{2M n_0 g}$ is the healing length for the uniform gas [see Eq. (18)]. If there were no interactions ($g \to 0$), then $k_{cr} \to 0$, and the
spectrum reduces to that of a free particle for all \( k \). Figure 1 shows the measured Bogoliubov excitation spectrum with no free parameters (the free-particle spectrum is the dashed line).

It is interesting to consider the Landau critical velocity \( v_c \) for this dilute interacting Bose gas. Since the Bogoliubov spectrum is linear at small \( k \) and changes to quadratic at large \( k \), its derivative \( d\omega_k/dk \) (the group velocity) is non-negative. Hence the minimum of \( \epsilon_k/\hbar k \) occurs at long wavelengths (in contrast to the case for superfluid \(^4\)He) and is simply the speed of sound

\[
v_c \approx s = \sqrt{\frac{n_0 g}{M}}.
\]

Note that the repulsive interactions \((g > 0)\) are essential to give a nonzero critical velocity. For an ideal gas, the spectrum is quadratic for all \( k \) and the corresponding critical velocity vanishes. Thus an ideal Bose gas at zero temperature indeed has a Bose-Einstein condensate, but it is not a superfluid in the conventional sense because its critical velocity vanishes.

4.4. linearized hydrodynamic equations for collective modes of a stationary condensate

A different and very important application of the hydrodynamic equations is to study the small low-lying oscillations of an initially static condensate with density \( n_0(\mathbf{r}) \). In this case, the equation of continuity (32) becomes \( \partial n'/\partial t + \nabla \cdot (n_0 \mathbf{v}') = 0 \), where \( n' \) and \( \mathbf{v}' \) are small perturbations. Similarly, the linearized form of the Bernoulli equation (33) yields \( gn' + M \partial \Phi' / \partial t = 0 \), where \( \nabla \Phi' = \mathbf{v}' \).

The time derivative of the first of these equations and the gradient of the second readily yield the generalized wave equation

\[
\frac{\partial^2 n'}{\partial t^2} = \nabla \cdot \left( \frac{n_0(\mathbf{r}) g}{M} \nabla n' \right),
\]

which explicitly incorporates the equilibrium condensate density \( n_0(\mathbf{r}) \) as an inhomogeneous effective potential. Note that the quantity \( g n_0(\mathbf{r})/M \) can be identified as the local squared speed of sound \( s^2(\mathbf{r}) \). Hence Eq. (40) can equivalently be written

\[
\frac{\partial^2 n'}{\partial t^2} = \nabla \cdot \left[ s^2(\mathbf{r}) \nabla n' \right],
\]

which is important in experimental studies of collective modes of trapped condensates, as described in Sec. 5.3.
4.5. effect of attractive interactions

For liquid $^4$He, the interatomic potential is overall repulsive because of the strong repulsive core and the weak attractive van der Waals attraction. This is why it remains a liquid down to zero temperature and can only be solidified by going to high pressure (many atmospheres) at low temperature.

The situation is very different for the alkali metals like Li, Na, K, and Rb, because each atom is much more polarizable than He. As a result, the interatomic potential typically has many bound states, and the sign of the s-wave scattering length depends critically on the relative position of the last bound state \[9\]. For example, $^7$Li atoms have a negative scattering length $a \approx -1.46$ nm equivalent to an attractive interaction.

As noted in connection with Eq. (39), a uniform dilute Bose gas with attractive interactions has an imaginary speed of sound and is thus unstable. With a trapped condensate, however, the kinetic trapping energy already provides a positive energy that can counteract the negative attractive interaction energy. For a quantitative analysis, consider a spherical condensate with oscillator length $d_0$ and use a trial Gaussian wave function

$$ \Psi(r) \propto \exp \left( -\frac{r^2}{2\beta^2 d_0^2} \right), $$

where $\beta$ is a variational parameter (note that the condensate radius is $\beta d_0$). The ground-state energy (20) is easily evaluated to give

$$ E_g(\beta) = \frac{1}{2} N \hbar \omega_0 \left[ \frac{3}{2} \left( \frac{1}{\beta^2} + \beta^2 \right) + \sqrt{\frac{2}{\pi}} \frac{N a}{d_0} \frac{1}{\beta^3} \right], $$

and $\beta < 1$ ($\beta > 1$) for attractive (repulsive) interactions. Here, the three terms represent the kinetic energy, the trap energy, and the interaction energy, respectively. This treatment also gives a quantitative justification of the approximation made in going from Eq. (20) to (21).

It is clear from inspection that $E_g(\beta)$ becomes large for $\beta \to \infty$ because of the spatial confinement of the harmonic trap. The detailed behavior for small $\beta$ depends crucially on the sign and magnitude of the interaction parameter $Na/d_0$. For a noninteracting gas with $a = 0$, the minimum of Eq. (43) occurs for $\beta = 1$, which is just the familiar result for an oscillator. Baym and Pethick \[13\] used this variational approach to study the expansion of the condensate for strong repulsive interactions. The situation is very different.
for an attractive interaction, no matter how weak, because the (now negative) cubic term dominates for \( \beta \to 0 \), and \( E_g(\beta) \) diverges to \(-\infty\). Hence any attractive system is globally unstable with respect to a collapse. For small and moderate values of \( N|a|/d_0 \), however, \( E_g(\beta) \) does retain a local minimum, and the system then remains locally stable. A straightforward analysis shows that the minimum disappears at the critical value \( N_c|a|/d_0 \approx 0.671 \) and that the critical condensate radius is reduced by a factor \( \beta_c = 5^{-1/4} \approx 0.669 \). For comparison, a numerical study of the GP equation \[16\] yields the value \( N_c|a|/d_0 \approx 0.575 \), which differs from the variational estimate by \( \approx 17\% \).

The attractive interaction in \(^7\)Li produces just this behavior. For a trap size of order 3 \( \mu \)m, the critical stability limit is \( N_c \sim 1250 \) atoms \[17\]. In the experiments, atoms from the surrounding gas continue to augment the condensate until it reaches the critical value and then collapses. This process repeats in a cyclic fashion.

5. Selected applications and comparison with experiments

The GP formalism, either in the form of the time-dependent equation (28) or the pair of coupled hydrodynamic equations (32) and (33), is remarkably rich and has many applications. Here we consider a few of the most important ones.

5.1. free expansion for different trap aspect ratios

Much of the experimental data relating to Bose-Einstein condensation in trapped gases has been obtained from images of the density distribution of an atomic cloud after the trap has been switched off. Indeed such time-of-flight expansion of an ultracold gas released from a harmonic trap provided some of the first evidence of Bose Einstein condensation \[10\]. In a series of observations taken a fixed time after release the measured size of the expanded cloud decreases abruptly as the temperature is reduced through the phase transition. A second characteristic sign of the transition is observed by recording the aspect ratio of the atom cloud (the ratio \( R_z/R_\perp \) of axial and radial widths) as it expands from an asymmetric trap; for a sufficiently long period of free expansion (where the cloud in large compared to its initial size) a thermal cloud has an aspect ratio of unity (circular shape) while a BEC has an elliptical distribution (as shown in Fig. 2). In the absence of interactions one can explain the difference above as follows. The asymptotic density profile of an ideal gas released from a potential reflects the initial
momentum distribution of the cloud in the trap (when the distance \( pt/M \) travelled by the atoms of momentum \( p \) and mass \( M \) in time \( t \) is much greater than the initial size of the cloud). A cloud of atoms that is far from the quantum regime has the same velocity distribution (Maxwell-Boltzmann) in all directions characterized by the temperature, and hence expands isotropically. On the other hand, atoms in a BEC occupy one quantum state and are described by a single wave function. The momentum distribution is the Fourier transform of the spatial wave function and has an aspect ratio of \( \sqrt{\omega_x/\omega_z} \) when viewed along the \( y \)-direction. The anisotropy of the confining potential is thus reflected in the aspect ratio of the expanding BEC, with the aspect ratio being inverted with respect to its initial value in trap, i.e. the cloud expands most rapidly in along the axis where the initial confinement is tightest, as expected from the Heisenberg uncertainty principle. However interactions must be taken into account for a complete description of the
expansion of a condensate. They give rise to a mean-field interaction energy which is converted to kinetic energy of motion during the release. For larger atom numbers this interaction term, proportional to the density, dominates the expansion of the condensate (as explained in Sec. 4.1). Gradients in the density drive the initial acceleration of the atoms so the cloud expands more rapidly in the most tightly confined direction. This causes the aspect ratio of the cloud to invert during expansion as in the non-interacting case, but when interactions are dominant the calculation requires solving the set of hydrodynamic equations given in Eq. (32) and (33), with the external potential \( V_{tr} \) set to zero at the release time (see also [13]). This approach based on mean-field theory predicted asymptotic aspect ratios for Bose Einstein condensates that agree with the experimental results to within a few percent, and this was one of the first validations of the mean-field description of interactions in dilute atomic vapors.

5.2. measurement of the condensate fraction

Superfluid \(^4\text{He}\) generally forms a macroscopic sample, which facilitates the study of the superfluid properties like the normal and superfluid densities and persistent currents. In contrast, it has proved extremely difficult to measure reliably the low-temperature condensate fraction for bulk uniform superfluid \(^4\text{He}\). Conceptually, the true many-body ground-state wave function \( \Psi_0 \) can be expanded in the complete set of states \( |\Phi_i\rangle \) for an ideal Bose gas in a box. The strong repulsive interactions for realistic superfluid \(^4\text{He}\) lead to a linear combination of many ideal-gas states with high energies in addition to the noninteracting ground state \( \Phi_0 \) with all the particles in the zero-momentum condensate. Correspondingly, the fraction of \(^4\text{He}\) atoms in the zero-momentum condensed state (the overlap \( |\langle \Phi_0 | \Psi_0 \rangle|^2 \)) is significantly reduced, leading to an estimated condensate fraction \( N_0 / N \sim 0.1 \), in contrast to the value 1 for an ideal uniform Bose gas.

Measurements of \( N_0 / N \) for superfluid \(^4\text{He}\) have relied on quasi-elastic scattering of high-energy neutrons. In principle, the single-particle momentum distribution function of bulk superfluid liquid helium has a delta function at \( k = 0 \), with a weight proportional to \( N_0 / N \). Nevertheless, the experiments have proved challenging with considerable uncertainty. Sokol [13] summarizes the experimental conclusions, confirming the theoretical estimates of \( N_0 / N \sim 0.1 \), and Sec. 8.5 of Ref. [8] gives an abbreviated account. The situation is very different for a dilute ultracold gas. Detection of the condensate fraction is reasonably straightforward and an obvious characteristic
feature of the onset BEC in a trap is the sudden appearance of the narrow condensate rising out of the broad thermal cloud as \( T \) falls below the actual transition temperature \([10]\). (Convincing direct measurements of the superfluid properties of quantum degenerate gases followed a few years after the first observation of BEC.)

![Figure 3: Scaled condensate-fraction ratio \( N_0(T)/N \) compared with that for an ideal gas in a general harmonic trap. Solid line is from Eq. (15) with \( T_0(N) \) the transition temperature given in Eq. (14). Measured data fit a similar curve but with a reduced transition temperature 0.94 \( T_0(N) \) (dashed curve). The inset shows the reduction in the number of atoms \( N \) as the system cools. (Note that \( N \) was relatively small in this early experiment.) Reprinted with permission of the authors [19] and the American Physical Society.]

In the original experiment by the JILA group [19], and in many subsequent experiments, the atomic cloud was allowed to expand for a definite time and then imaged by recording the shadow of the cloud arising from absorption of a pulse of resonant laser light. In general, the projected image has both the broad thermal background and the narrow condensate (for \( T < T_c \)). The ratio of the areas under these curves yields the condensate fraction \( N_0(T)/N \). Figure 3 shows the measured values which extend down to \( \sim 0.5T_c \). Since the gas is very dilute the condensate fraction closely resembles that for an ideal trapped gas. Corrections from finite \( N \) and many-body interactions are estimated to be at most a few percent. In recent experiments [20] the condensation fraction has been measured in a system where the scattering length can be tuned to be close to zero so approximating very closely to an ideal gas; this improves the ability to observe condensates with
small number of atoms and clarifies the interpretation of data like that shown in Fig. 3.

5.3. low-lying collective oscillation modes

Ever since the first observation of BEC in a dilute gas the study of collective oscillation modes has been crucial to our understanding of these mesoscopic systems. For typical experimental conditions these excitations dominate the low-frequency response of a weakly-interacting gas to an external perturbation. Initial experiments performed at temperatures well below \( T_c \) tested the mean field theory based on the Gross Pitaevskii equation. The very good agreement between theory and experiment with no free fit parameters provided strong evidence for the validity of this model. To model the system we return to the linearized hydrodynamic equation Eq. (41). For a uniform system the solutions to Eq. (41) correspond to sound waves propagating with the speed given in Eq. (39). In a trapped system the wavelength of the phonons must be considered in relation to the dimensions of the cloud. For wavelengths significantly less than all spatial dimensions of the condensate the solutions are sound waves propagating at the local sound velocity. Whereas for wavelengths comparable to the size of the condensate, the boundary conditions produce standing sound waves at discrete frequencies which are referred to as the collective oscillation modes of the system. Thus the collective oscillation frequencies of a BEC in a harmonic trap do not depend on interactions (number of atoms) even though of the speed of sound \( s \) does, i.e. \( \omega = sk \) where the wavenumber \( k \) is the same for different numbers of atoms in the same trap (in the TF regime). This arises because the speed of sound is \( s \approx (n_0g/M)^{1/2} \) (see Eq. (40)), and the wavelength of the lowest energy excitation is limited by the extent of the cloud in a given direction \( R_i \) [determined by \( \mu = n_0g = M\omega_i^2R_i^2/2 \) from Eq. (26) in the Thomas-Fermi regime]. Hence the lowest mode has a frequency of \( s/R_i \approx \omega_i \), which is the same as for the non-interacting case.

Collective oscillation modes are classified according to the symmetries of the trapping potential. In a spherical potential both angular momentum and its axial component are conserved quantities and solutions have the form \( n' = P_{\ell}^{(2n_r)}(r)Y_{\ell m}(\theta, \phi) \) where \( \ell \) is the orbital angular momentum quantum number, \( m \) is the magnetic quantum number, \( P_{\ell}^{(2n_r)}(r) \) are polynomials of degree \( 2n_r \) where \( n_r \) is the number of nodes of the radial wave function, and \( Y_{\ell m}(\theta, \phi) \) are spherical harmonics. The corresponding frequencies for a spherical harmonic potential are \( \omega(n_r, \ell) = \omega_0(2n_r^2 + 2n_r\ell + 3n_r + \ell)^{1/2} \), where
\( \omega_0 \) is the oscillation frequency of atoms in the trap \[8\]. For \( n_r = 0, \ell = 1 \) the motion is a simple dipole oscillation, or sloshing back and forth of the cloud; in a harmonic trap this centre-of-mass motion of the many-body system is decoupled from its internal degrees of freedom. Thus the dipole modes have frequencies exactly equal to those of the oscillations of individual atoms in the trap which facilitates calibration of experiments. The modes with orbital angular momentum quantum number \( \ell = 2 \) (and \( n_r = 0 \)) are quadrupole modes and in a spherical potential the oscillation frequency is \( \sqrt{2} \omega_0 \) for all five values of \( m \). The mode with \( n_r = 1, \ell = 0 \) is known as the “breathing mode” in which there is expansion and contraction along the radial direction changing the volume of the cloud but not its shape and this mode has a frequency of \( \sqrt{5} \omega_0 \). Experiments are rarely performed in a spherical trap but in many cases the potential does have a cylindrical symmetry about the \( z \)-axis \( (\omega_x = \omega_y = \omega_\perp) \) so that \( m \) remains a good quantum number (but not \( \ell \)), and two of the three dipole modes remain degenerate. In an axisymmetric potential the \( m = 0 \) modes with \( \ell = 0 \) and \( \ell = 2 \) are mixed to give modes for which the oscillations along the axial and radial directions are in phase (higher frequency) and out of phase (lower frequency), and the \( |m| = 2 \) modes remain degenerate (as do the \( |m| = 1 \) modes). The frequency spectrum of the modes and their shape are shown in Fig. 4 for a time averaged orbiting potential (TOP) trap with \( \omega_z/\omega_r = \sqrt{8} \) (see Sec. 12.2 of \[8\] for more general references).

Some of the first experiments on BEC measured the frequencies of the low-lying \( m = 0 \) mode and an \( m = 2 \) quadrupole mode and found good agreement with hydrodynamic theory when the number of atoms in the condensate is sufficiently large. After this verification, considerable subsequent theoretical and experimental effort was concentrated on the so-called scissors modes with \( |m| = 1 \) as described in Sec. 9.1. The scissors mode differs from the other collective oscillation modes because it involves rotational motion about an axis (lying in the \( xy \) plane) as illustrated in Fig. 4. Thus it is sensitive to the irrotational flow associated with superfluids.

Interactions give rise to a nonlinear term in the Schrödinger equation, Eq. (19), and this nonlinearity leads to a coupling between the collective excitations, e.g. second harmonic generation was observed when the trapping potential was adjusted so that the frequency of a high-lying mode was close to the second harmonic of the low-lying \( m = 0 \) mode \[21\]. When the BEC was driven at the frequency of the lower mode there was a large transfer of energy to the mode at twice the frequency.
Figure 4: The low energy collective oscillation mode spectrum of a trapped BEC in a TOP trap where $\omega_z/\omega_r = \sqrt{8}$: the three dipole modes are two degenerate modes at $\omega_r$ and one at $\omega_z$; the six quadrupole modes are the $m = 0$ low-lying and high-lying modes, the doubly degenerate $|m| = 2$ modes and the $|m| = 1$ or scissors modes. Figure drawn by E. Nugent, Oxford.
The first experiments on collective oscillations were performed with single component Bose gases in three-dimensional harmonic trapping potentials, which is by far the simplest situation to analyze. More recently there has been work on two-component gases, Fermi gases and in complex and low dimensional trapping potentials. The collective oscillation frequencies of a BEC do not depend on the strength of the interactions (number of atoms) in a harmonic trap for reasons explained above, but they do in a ring trap where the lowest-order collective modes are quantized flows around the ring [22]. A very special theoretical case is a two-dimensional interacting quantum gas confined in a harmonic potential which has a breathing mode at the universal frequency of precisely $2\omega_\perp$, twice the radial oscillation frequency, because of underlying symmetry of quantum mechanical Hamiltonian [23]. A wealth of new physics beyond mean field theory occurs in these systems and collective oscillations are an important way of uncovering new effects.

6. Dipolar condensates

It is helpful to start with a brief review of electric and magnetic dipoles and their interactions (Ref. [24] provides a good general review of these intriguing systems). In SI units, an elementary electric dipole moment $p$ has dimensions of charge-length (namely C m) with the electrostatic potential

$$\Phi(r) = \frac{p \cdot r}{4\pi\epsilon_0 r^3},$$  \hspace{1cm} (44)

where $\epsilon_0$ is the permittivity of the vacuum. A combination of the dipole electrostatic field $E(r) = -\nabla\Phi(r)$ and the energy $U = -p \cdot E(r)$ of an external dipole in this field yields the interaction energy $U_{dd}$ of two dipoles $p_1$ and $p_2$

$$U_{dd} = \frac{p_1 \cdot p_2 - 3(\hat{r}_{12} \cdot p_1)(\hat{r}_{12} \cdot p_2)}{4\pi\epsilon_0 r_{12}^3},$$  \hspace{1cm} (45)

where $r_{12} = r_1 - r_2$ is the vector joining the two dipoles. The most notable features are the rather complicated angular dependence and the inverse-cube dependence on the separation (a long-range interaction, in contrast to the usual short-range interaction $g\delta(r)$ discussed in Sec. 3).

To simplify this interaction energy, it is common to orient the dipoles with an extra applied electric field along $\hat{z}$, in which case Eq. (45) has the simpler form

$$U_{dd} = \frac{C_{dd}}{4\pi} \frac{1 - 3\cos^2 \theta}{r_{12}^3},$$  \hspace{1cm} (46)
where \( C_{dd} = \frac{p^2}{\epsilon_0} \) (assuming identical atomic dipoles) and \( \theta \) is the angle between the vector \( \mathbf{r}_{12} \) and the \( \hat{z} \) axis. Note the strong angular dependence that is proportional to the (negative of the) Legendre polynomial \( P_2(\cos \theta) = (3 \cos^2 \theta - 1)/2 \): two parallel dipoles aligned along the \( \hat{z} \) axis (head-to-tail) have negative (attractive) energy, whereas two parallel dipoles side by side have a positive (repulsive) energy. It is easy to see that the dimensional coupling constant \( C_{dd} \) has dimensions of energy-volume, exactly the same as the contact interaction constant \( g \) from Sec. 3.

A similar but considerably more intricate argument yields the identical expression \( (46) \) for the energy of oriented magnetic dipoles \( \mathbf{m} \), where \( \mathbf{m} \) has dimensions of current-area \((\text{A m}^2)\). The only difference is the definition of the magnetic-dipole dimensional coupling constant \( C_{dd} = \mu_0 m^2 \), where \( \mu_0 \) is the permeability of the vacuum. Note that, by definition, \( \epsilon_0 \mu_0 = c^{-2} \). Compared to an elementary electric dipole moment \( p \), an elementary magnetic moment \( m \) has the same dimension as \( cp \). Furthermore, a typical valence electron in an atom has orbital speed of order \( \alpha c \), where \( \alpha = e^2 / (4\pi \epsilon_0 \hbar c) \approx 1/137 \) is the fine-structure constant. The atomic magnetic moment arises from electronic motion with velocity \( v \), leading to the typical order-of-magnitude estimate \( m \sim vp \), with \( v/c \sim \alpha \). Thus the interaction energy of magnetic dipoles is smaller by a factor of \( \alpha^2 \sim 10^{-4} \) compared to that for electric dipoles \[24\].

One approach to parametrize the dipole-dipole interaction is to introduce a corresponding dipolar length \( a_{dd} = C_{dd}M/(12\pi \hbar^2) \). For many purposes, the \( s \)-wave scattering length \( a \) provides the relevant comparison, with the dimensionless ratio \( \epsilon_{dd} = a_{dd}/a \). For \(^{87}\text{Rb} \), this ratio \( a_{dd}/a \approx 0.007 \) is small so that dipolar effects are frequently negligible (but see Sec. 8 for an example where the dipolar energy is crucial).

Equivalently, this same dimensionless ratio becomes \( \epsilon_{dd} = C_{dd}/3g \)[24], where \( g \) is the contact coupling constant. At present, all experimental studies of dipolar effects have relied on the particular atomic species \(^{52}\text{Cr} \) that has a large intrinsic magnetic dipole moment \( m \approx 6\mu_B \), where \( \mu_B = e\hbar/2m_e \) is the Bohr magneton. Even in this most favorable case of \(^{52}\text{Cr} \), the dimensionless ratio is only \( \epsilon_{dd} \approx 0.16 \). The Stuttgart group has created a BEC of \(^{52}\text{Cr} \)[25], which has allowed many detailed experimental studies. More recently, Lev and colleagues have made a BEC of \(^{164}\text{Dy} \), which has an even larger atomic magnetic moment \( \mu \approx 10 \mu_B \)[26].

Recently, powerful laser techniques have created bosonic heteronuclear polar molecules such as \(^{40}\text{K}^{87}\text{Rb} \)[27, 28] that have large electric dipole moments with \( \epsilon_{dd} \approx 20 \)[24]. The phase-space density required for a quantum
degenerate gas of polar molecules has not yet been achieved, but strenuous efforts are currently being made in several experiments, and this is an extremely promising area for the future work. Such systems will qualitatively alter the study of ultracold quantum matter and offer many new possibilities. Reference [29] provides a recent review of such cold dipolar gases.

In the context of the Gross-Pitaevskii description, the addition of the long-range dipole-dipole interaction potential significantly affects the total interaction energy [the third term in Eq. (20)]. Specifically, the two-particle potential now becomes \( g\delta(r - r') + U_{dd}(r - r') \), leading to the more complicated contribution to the GP energy functional

\[
E_{\text{int}}[\Psi] = \frac{1}{2} \int dV \Psi^*(r) \Psi^*(r') \left[ g\delta(r - r') + U_{dd}(r - r') \right] \Psi(r') \Psi(r). \tag{47}
\]

Here, the first term leads to the local Hartree potential \( V_H = g|\Psi(r)|^2 \), but the second dipole term is intrinsically nonlocal and couples the total condensate densities at the two different positions. This second term significantly complicates the GP equation, and it is convenient to introduce an additional dipole-dipole interaction potential

\[
V_{dd}(r, t) = \int dV' U_{dd}(r - r') |\Psi(r', t)|^2, \tag{48}
\]

that acts in addition to the Hartree potential \( V_H = g|\Psi|^2 \). The resulting time-dependent GP equation

\[
i\hbar \frac{\partial \Psi}{\partial t} = \left( -\frac{\hbar^2 \nabla^2}{2M} + V_{tr} + g|\Psi|^2 + V_{dd} \right) \Psi. \tag{49}
\]

now contains the nonlocal integral contribution \( V_{dd} \). For the stationary case, the left-hand side of (49) simply reduces to \( i\mu \). Based on the form of Eq. (46), a cigar-shaped pure dipolar condensate should collapse, whereas a disk-shaped pure dipolar condensate should be stable. The repulsive contact interaction acts to counteract the collapse, and variational studies have confirmed these qualitative ideas, but the details become rather complicated [24]. Assume an axisymmetric trap with a particular aspect ratio (this depends on the trap frequencies \( \omega_z \) and \( \omega_{\perp} \)). Reference [30] used the BEC of \( ^{52}\text{Cr} \) to study the behavior of the condensate for various aspect ratios of the external trap. Most importantly, they also used a Feshbach resonance to control the \( s \)-wave scattering length and confirmed
the theoretical picture. For example, using a Feshbach resonance to tune the scattering length to zero, they demonstrated the stability of a pancake-shaped pure dipolar gas. Subsequently, Ref. [31] investigated the collapse dynamics of the dipolar condensate when the $s$-wave scattering length is too small to support the system. As might be expected from the $P_2(\cos \theta)$ form of the oriented dipole-dipole interaction Eq. (46), Fig. 5 shows that the collapse dynamics leads to a $d$-wave symmetric implosion (the system is axisymmetric around the horizontal axis).

7. Mixtures

This section and the next consider multicomponent BECs, but there are two very different types. The first is a mixture of different hyperfine states that are confined in a magnetic trap. The second is a “spinor condensate” that involves all the magnetic sublevels of a single hyperfine state. Since many of the sublevels cannot be magnetically trapped, this system requires an optical trap that can confine all the magnetic sublevels.

For definiteness, we consider the particular species $^{87}\text{Rb}$ that has been used in many of the experiments with mixtures. The nucleus has spin $3/2$ and the single $s$-state valence electron with spin $1/2$ leads to two hyperfine states. The lower one has $F = 1$ and the higher one has $F = 2$, separated
by an energy with an equivalent frequency of 6.8 GHz.

In a magnetic trap, among the $F = 1$ manifold, only the state $|1, -1\rangle$ can be trapped (using the standard notation $|F, m_F\rangle$) whereas the $F = 2$ manifold has two states $|2, 1\rangle$ and $|2, 2\rangle$ that can be trapped. Usually experiments focus on the pair $|1\rangle \equiv |1, -1\rangle$ and $|2\rangle \equiv |2, 1\rangle$ since they have essentially the same magnetic moments and thus behave similarly in a magnetic field (which means that they experience similar trap potentials).

### 7.1. interacting two-component mixtures

The simplest case of a mixture is two dilute Bose gases with order parameters $\Psi_1$ and $\Psi_2$ and coupling constants $g_{11}$ and $g_{22}$ for self-interaction and $g_{12} = g_{21}$ for mutual interaction. The principal new feature is that the GP equation for (say) $\Psi_1$ now includes an additional Hartree term $V_{H12}(r) = g_{12}|\Psi_2(r)|^2$ for the mutual interaction with species 2

$$i\hbar \frac{\partial \Psi_1(r, t)}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2M} + V_{tr1}(r) + g_{11}|\Psi_1(r, t)|^2 + g_{12}|\Psi_2(r, t)|^2 \right] \Psi_1(r, t),$$

and similarly for the GP equation for $\Psi_2$.

Suppose for simplicity that the two condensates are uniform. It is not difficult to see that the conditions for miscibility are $g_{11} > 0$, $g_{22} > 0$, and $g_{11}g_{22} > g_{12}^2$. The first and second conditions ensure that each pure condensate is stable against collapse (as is familiar for a single-component condensate). In contrast, the third condition ensures that the overlapping mixture is stable against phase separation. If this latter condition is violated, then the uniform two-component system will phase separate, and the long-wavelength collective modes of the coupled uniform system will have imaginary frequencies.

The situation in a trap is somewhat different, since the harmonic trap acts to confine the two components. Unless the self-interaction constants are essentially equal, one component will form a denser core, surrounded by (and partially overlapping with) the other less-dense component that experiences the inward pressure of the trap and outward pressure of the inner core (analogous to density stratification in an inhomogeneous self-gravitating body).

As explained below, external electromagnetic fields allow a convenient transformation between these hyperfine species, for example a sudden transition from $|1\rangle$ to $|2\rangle$. Reference [32] modeled this behavior with a single-component time-dependent GP equation with a time-dependent scattering
length $a(t)$ that changed from $a_{11}$ to $a_{22}$ discontinuously at $t = 0$. After a variable time-delay, the trap was turned off, allowing the condensate to expand. Initially, the condensate shrunk, followed by compressional oscillations. This behavior is understandable because $a_{11} > a_{22}$ for these particular states. Thus the initial density profile for $|1\rangle$ could no longer sustain the equilibrium radius once the species transformed to $|2\rangle$, experiencing radial shrinkage and radial oscillations. A fit to the observations indicated that $a_{11}/a_{22} \approx 1.06$.

### 7.2. electromagnetic coupling between two hyperfine states

The previous model in (50) assumes that the two components interact only through the density (the mean-field Hartree terms) that contain no phase information. Specifically, each condensate has the representation $\Psi_j(r) = |\Psi_j(r)| \exp[iS_j(r)]$. Each order parameter is single-valued when $r$ executes a closed path at any given instant, so that the phase is $2\pi$-periodic for any such closed path. As noted in connection with superfluid $^4$He, the circulation for each component is therefore quantized in units of $\hbar/M$. Technically, each complex one-component order parameter has $U(1)$ symmetry. Far below the superfluid transition temperature, the quantized circulation “explains” the existence of persistent currents because any fluctuation that can change the circulation is very improbable, reflecting the topological charge associated with the $2\pi$ periodicity.

The situation is very different when external electromagnetic fields couple the two components, leading to off-diagonal terms in the combined GP equations. The resulting total order parameter no longer has two separate complex scalar functions [each with $U(1)$ symmetry]; instead it now has two components with intrinsic coupling, qualitatively changing its character into a single $SU(2)$ structure. For elementary discussions of this essential model system, see, for example, [33, Chaps. 7-11] and [34, Sec. 4.4]. The dynamics of this spin-$\frac{1}{2}$ becomes similar to that familiar from nuclear magnetic resonance, obeying what are here called the “optical Bloch equations.” In the present case, the transformation from $|1\rangle$ to $|2\rangle$ is analogous to a $\pi$ pulse that rotates a spin up into a spin down. If the pulse is twice as long, the resulting $2\pi$ rotation reproduces the initial $|1\rangle$ state apart from an overall phase. Experiments [32] verify the periodic transfer between the two states as the length of the applied pulse varies (each hyperfine state has a slightly distinct resonant frequency, so that each can be imaged separately).
This electromagnetic coupling has the remarkable feature of allowing a
direct measurement of the relative phase between the two components, as
becomes clear from the dynamical equations of the coupled system
\[ i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \approx \begin{pmatrix} T + V_{tr1} + V_{H1} + V_{H12} & \frac{1}{2} \hbar \Omega(t) \exp(i\omega_{rf}t) \\ \frac{1}{2} \hbar \Omega(t) \exp(-i\omega_{rf}t) & T + V_{tr2} + V_{H2} + V_{H12} \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \]
(52)
where \( T \) denotes the kinetic energy, \( \omega_{rf} \) is the frequency separating the two
hyperfine states, and \( \Omega(t) \) reflects the two-photon electromagnetic coupling
between the two hyperfine states. It is known as the “Rabi” frequency and
depends on the strength of the electromagnetic coupling field (the time de-
dependence here arises because it can be turned on and off, but more general
situations can and do arise). Transforming to the rotating frame eliminates
the explicit rf time dependence.

The experiment starts with population in \(|1\rangle\) and applies a \( \pi/2 \) pulse that
creates a linear superposition of the two states, then waits a variable time and
applies a second \( \pi/2 \) pulse. The trap is turned off, expanding the condensate.
Resulting images of either component show oscillations depending on the
length of the variable delay, which provide a measure of the relative phase.

This and other experiments indicate that the electromagnetic coupling
has the remarkable property of changing the topology of this two-component
system. When the coupling is off, they are simply separate \( U(1) \) complex
scalar order parameters, each with its own phase angle and correspond-
ing quantized circulation (each has the topology of a cylinder). When the cou-
pling is on, they instead form a single coupled \( SU(2) \) system with two angles
on the polar sphere and no topological quantization (the topology is now that
of a sphere). The JILA group studied this system in considerable detail, and
its topology proved essential in the first experimental creation of a quantized
vortex in a dilute BEC \[36\], as discussed in Sec. 9.

8. Spinor condensates

The previous section focused on mixtures that usually involve two dif-
ferent \( F \) hyperfine manifolds in a magnetic trap. In this situation, only the
weak-field seeking states are relevant, since the other states no longer remain
confined. In contrast, a spinor condensate involves a single hyperfine mani-
fold, but it uses an optical trap that retains all the \( m_F \) magnetic states (the
simplest and usual case is \( F = 1 \) so that the \( m_F \) levels are \(+1, 0, \) and \(-1\)).
See Chap. 1 for an introductory discussion of these traps.
8.1. spinor condensates: special case of $F = 1$

The possibility of trapping all three magnetic sublevels in an optical trap rapidly led to the study of spin-1 Bose-Einstein condensates (as well as more general cases) \[38, 39\]. In contrast to the previous mixture of distinct species, the rotational invariance of the interactions between two spin-1 atoms now leads to special restrictions. Here, the macroscopic order parameter is a three-component vector (written as a transpose for convenience)

$$\Psi^T = (\Psi_1, \Psi_0, \Psi_{-1})$$

(53)

involving the three $m_F$ states. Only $s$-wave scattering is relevant in the present low-energy limit, and the interaction between the atoms has the familiar contact form

$$V_{\text{int}}(r_1 - r_2) = \delta(r_1 - r_2) \sum_F \frac{4\pi\hbar^2 a_F}{M} \mathcal{P}_F,$$

(54)

where $F$ is the magnitude of the total hyperfine spin of the two atoms $F = F_1 + F_2$ and $\mathcal{P}_F$ is the projection operator onto the appropriate value of the total $F$.

For bosons each with hyperfine spin $F_j = 1$, the only allowed total values are $F = 0$ and $F = 2$, with two scattering lengths $a_0$ and $a_2$. The interaction potential can then be rewritten in the equivalent form

$$V_{\text{int}}(r_1 - r_2) = \delta(r_1 - r_2) \left( g_0 + g_2 \mathbf{F}_1 \cdot \mathbf{F}_2 \right),$$

(55)

where the effective interaction constants are given by

$$g_0 = \frac{4\pi\hbar^2}{M} \frac{2a_2 + a_0}{3}, \quad g_2 = \frac{4\pi\hbar^2}{M} \frac{a_2 - a_0}{3}.$$  \hspace{1cm} (56)

In practice, the two scattering lengths are roughly comparable, so that $|g_2| \ll g_0$. Specifically, $g_2$ is small and positive for $^{23}\text{Na}$, but it is small and negative for $^{87}\text{Rb}$, which has crucial practical consequences for the distinct behavior of these two atomic species.

Ho \[38\] introduced an effective energy functional, writing the spinor order parameter in the form $\Psi_\alpha(r) = \sqrt{n(r)} \zeta_\alpha(r)$, where the spinor index runs over the values $1, 0, -1$, $n(r)$ is the common density for all three components, and $\zeta_\alpha(r)$ is a normalized spinor with $\zeta^\dagger \cdot \zeta = 1$. The ground state follows
by minimizing the energy with fixed total particle number (enforced with a
chemical potential), leading to an effective energy functional

$$K = E - \mu N$$

where

$$K = \int dV \left[ \frac{\hbar^2}{2M} \left| \nabla \sqrt{n} \right|^2 + \frac{\hbar^2}{2M} \left| \nabla \zeta \right|^2 - n[\mu - V_t(r)] + \frac{n^2}{2} \left( g_0 + g_2 \langle F \rangle^2 \right) \right]$$

Here the trap is taken as independent of the hyperfine state, and $\mu$ determines
the total number of atoms.

Apart from the gradient terms, the ground-state spinor $\zeta_\alpha$ follows by
minimizing the spin-dependent part of the energy $\frac{1}{2}n^2g_2\langle F \rangle^2$, where $\langle F \rangle = \sum_{\alpha\beta} \zeta_\alpha^\dagger F_{\alpha\beta} \zeta_\beta$ is the expectation value of the appropriate spin-1 matrices [9, Sec. 12.2.1]. If $g_2$ is positive (as for $^{23}$Na), then the minimum occurs for $|\langle F \rangle| = 0$. Such states are called “polar” and they are those obtained by spa-
tial rotations of the hyperfine state $|m_F = 0\rangle$. If $g_2$ is negative (as for $^{87}$Rb),
then the minimum occurs by maximizing the average spin with $|\langle F \rangle| = 1$. These states are called “ferromagnetic” and they are those obtained by spa-
tial rotation of the hyperfine state $|m_F = 1\rangle$.

8.2. experimental studies of spinor condensates

Soon after the optical trapping of a BEC of $^{23}$Na atoms, the MIT group
made a detailed experimental study of these fascinating spin-1 spin or sys-
tems, where the spin-dependent interaction constant $g_2$ in (56) is small and
positive, favoring polar configurations with $\langle F \rangle = 0$. For definiteness, we
focus on the formation of ground-state spin domains in an external magnetic
field [40]. The resulting spin structures can be either miscible or immiscible
depending on the applied magnetic field and on which of the $m_F$ components
are occupied. They imaged the spin domains by time-of-flight expansion fol-
lowed by a Stern-Gerlach separation of the various spin components in an
inhomogeneous magnetic field. Reference [41] comprehensively reviews this
and related experiments on the polar spinor condensate $^{23}$Na.

Recently, Stamper-Kurn’s group at Berkeley has studied the very different
situation in $^{87}$Rb, where the relevant spin-dependent coupling constant $g_2
in (56)$ is small and negative, favoring ferromagnetic spinor structures with $|\langle F \rangle| = 1$ [42]. Here, the relevant scattering length is not the usual s-wave
$a$, but the much smaller spin-dependent quantity $\Delta a = (a_2 - a_0)/3$ that
follows from Eq. (56). Consequently, the usually small dipolar length $a_{dd}$
now becomes crucial. Specifically, the dimensionless ratio $|a_{dd}/\Delta a| \approx 0.4$ is
no longer small, which leads to many intriguing new phenomena involving the dipolar energy (effectively a dipolar quantum fluid [42]).

Figure 6: Spontaneous dissolution of helical textures in a quantum degenerate $^{87}\text{Rb}$ spinor Bose gas. A magnetic field gradient prepares transversely magnetized (b) uniform or (a), (c) helical magnetic textures. The transverse magnetization after a variable time of free evolution is imaged in the $xz$ plane, with orientation indicated by the hue and amplitude brightness. (b) Uniform texture remains homogeneous for long evolution times, whereas (c) with initial pitch $\lambda = 60 \, \mu\text{m}$ dissolves over $\sim 200$ ms, leading to a sharply modulated texture. Reprinted with permission of the authors [42] and the American Physical Society.

The experiment involved an asymmetric triaxial trap with a thin elongated elliptical two-dimensional TF condensate that initially had a helical spin texture with a variable pitch (periodicity length $\lambda$ between 50 and 150 $\mu\text{m}$). This helical spin texture evolved for a variable time, and the vector magnetization was then measured nondestructively. For comparison, they also prepared similar uniform samples with much larger pitch ($\lambda \gg R_z$). Figure 6 shows the evolution of these initial textures (a), including both (b) uniform and (c) helical structures. For (c), note the small-scale structure of the magnetization of order 10 $\mu\text{m}$, much smaller than the initial helical pitch. Several experimental tests associate this short-range modulation with the dipolar interaction.
9. Rotating Bose gases and quantized vortices

The superfluid properties of liquid helium at low temperatures (below the lambda-point) were apparent soon after liquefaction was achieved at the beginning of the 20th Century, in particular the absence of viscosity that allows the liquid to leak through small channels. The properties of superfluid helium were explained in an intuitive phenomenological way by the two-fluid model (of Landau and others) outlined in Sec. 1.1; although there is a connection with Bose-Einstein condensation, it has not been central to understanding the superfluid helium as can be seen by comparing the physics discussed in Sec. 1.3 with that in 4.3.

The study of BEC in dilute atomic gases developed in a very different way. The first experimental breakthroughs confirmed the prediction of Bose and Einstein and found that the critical temperature was that predicted by statistical mechanics (within a few percent). About four years after the first BEC was created, experiments started to demonstrate the striking superfluid properties that distinguish it from a classical fluid. This section mainly concerns those methods that use rotation to probe the quantum properties. For completeness, however, we mention the work on the flow of the quantum fluid around a strongly repulsive potential created by a focused blue-detuned laser beam. Displacing this laser beam back and forth acts like a ‘macroscopic object’ moving inside the cloud of atoms; it was found that there is a critical velocity [defined in Eq. (39)] below which there is no observable dissipation of energy.

9.1. the scissors mode

Some of the first evidence for the superfluidity of dilute atomic gases in the BEC regime came from the quantitative measurements of the so-called scissors mode which derives its name from nuclear physics where there is an excitation of the protons and neutrons in deformed nuclei (as explained below) [43]. In the case of BEC the cloud of ultracold atoms is non-spherical because of asymmetry in the trapping potential, i.e. the three frequencies in Eq. (11) are not all the same (which is almost always true in practice), so that the density contours are deformed into ellipses. When the potential is suddenly rotated through a small angle about a symmetry axis, the deformed density distribution finds itself offset from its equilibrium position [44] and therefore starts rotating to and fro through a small angle without changing shape corresponding to the modes labelled $|m = 1|$ in Fig. 4. Superfluids
behave in a distinctive way in rotating systems because of their irrotation flow ($\nabla \times \mathbf{v} = 0$ as discussed in Sec. 1.1); this leads to a reduced moment of inertia as compared to the classical value for a ‘solid-body’ with the same density distribution.

In trapped three-dimensional ultracold gases the superfluid and BEC fraction are considered to be the same (the condensate fraction is small in helium, as discussed in Sec. 5.2). Two-dimensional systems, however, can be completely superfluid even when there is no BEC (see Sec. 9.2), and there is great interest in investigating the relationship between superfluidity and BEC.

As mentioned above the scissors mode experiments with ultracold gases were suggested [43] based on an analogy with groundbreaking work on superfluidity of nuclear matter. The scissors mode in strongly deformed nuclei was first suggested in a model where the protons and neutrons are assumed to be two distinct deformed quantum fluids, e.g. two ellipsoidal distributions with a common center. These deformed distributions rotate relative to each other around a common axis (two-rotor model) like the blades of a pair of scissors about the pivot.

9.2. the nucleation of vortices

Perhaps the most obvious way to create a vortex in a fluid is rotating the container as in experiments with superfluid helium, or using laser beams moving with a circular motion to stir the system like a spoon in a cup of tea. However the first vortex created in a BEC of ultracold Rb atoms was achieved at JILA by an ingenious scheme which directly controlled the phase of wave function, in a manner far removed from any previous work (on helium) [45]. That experiment used a mixture of ultracold atoms in two different magnetic states as described in Sec. 7. By driving a two-photon transition between the states (with a combination of microwave and RF radiation), a ring of atoms was created which had a phase winding of $2\pi$ corresponding to a vortex state, like a persistent flow with one unit of angular momentum (and further work on persistent flow is discussed below). In this scheme the center of the ring was filled with atoms in a different Zeeman sub-state, but these atoms were selectively removed to leave a cloud of atoms containing a more conventional vortex (and much of the rich physics of vortices and other topological structures in multi-component systems remains to be explored).

In principle this scheme is very flexible, but much of the subsequent work has used experimental methods in which mechanical rotation or stirring imparts angular moment to the condensate. Analogous to the “rotating bucket”
experiment in superfluid $^4$He, ultracold atoms have been trapped in a rotating potential whose contours of constant energy are elliptical (at a given instant of time). However once a BEC has formed and established long-range phase coherence there is significant hysteresis: the condensate can remain in a (metastable) state without vortices even when the confining potential rotates sufficiently fast that vortex states have lower energy. The nucleation of vortices requires a way to overcome the energy barrier. This has been achieved by driving the system at a frequency that resonantly excites a quadrupole mode so that deformation builds up until the cloud of atoms is sufficiently perturbed that vortices can enter; vortices form in the low-density outer reaches of the cloud and then move towards the center.

The characteristic core size in the healing length $\xi$ given in Eq. (18). The wave function for a vortex has the form $\Psi(r, \phi) = f(r/\xi)e^{i\phi}$, and a variational solution of the GP equation gives the approximate amplitude as $f(x) = x/\sqrt{x^2 + 2}$ which provides a good fit at both large and small $x = r/\xi$ (when compared to more accurate numerical solutions).

![Figure 7: Images of expanded Bose-Einstein condensates of $^{87}$Rb atoms showing (a) small vortex array for slow rotation, and (b) large vortex array for rapid rotation. Note the highly regular triangular form. (c) An array of vortices that has not settled down to a regular lattice as in the previous images; stirring or shaking such a system creates a tangle of vortex lines corresponding to a turbulent quantum fluid. Images (a) and (b) reprinted with permission of the authors [46] and the American Physical Society. Image (c) provided by R. Williams and S. Al-Assam [47].](image)

Rather than starting with a static BEC and trying to break up the established phase, an alternative strategy for nucleating vortices is to spin up a thermal cloud of trapped atoms and then cool them below the critical temperature in the rotating frame. Arranging preferentially to remove atoms
with less than average angular momentum during evaporation increases the rotation rate as the temperature decreases. This puts the system directly into the equilibrium state for the given rotation rate, i.e. the quantum fluid has the number of vortices corresponding to the rotation rate of an equivalent classical fluid, see Fig. 7 (as outlined in Sec. 1.2).

Generally the vortices are too small to be observed in a trapped BEC and images such as Fig. 7 are taken after a time-of-flight expansion. Indeed for pancake-shaped condensates the vortices expand proportionally more than the overall size of the cloud to give extremely high contrast images. By observing along the axis of rotation the cores of the vortices appear as dips in the density of the cloud of atoms (in images recorded by absorption of resonant laser light).

Recently another method has been developed that is closely related to the original phase imprinting scheme [47]. A Bose-Einstein condensate is loaded into a deep 2-D optical lattice to give an array small condensates (containing hundreds of atoms at each site); the phases of these separated groups of atoms can evolve independently so that when the optical lattice potential is rotated, about an axis perpendicular to the plane of the lattice, phase differences build up corresponding to the velocity field in a rotating system [see Eq. (30)]. Thus when the lattice depth is reduced adiabatically to zero the small clouds join together with relative phases that give vortices throughout the system (with no hysteresis). Vortices rotating in the same direction repel each other and they arrange themselves into a lowest energy configuration that is a regular triangular array (or Abrikosov lattice similar to that of vortices in superconductors), assuming that there is some damping in the system.

In some situations, the regular arrangement of vortices aligned along the rotation axis as in Figs. 7 can be disrupted to create a turbulent system. In superfluid \(^4\)He, the much smaller healing length and larger sample size allows for many vortices and quantum turbulence in the evolution of a tangle of vortices has been studied. Another fascinating type of system is ring vortices that resemble smoke rings in the condensate; these have been created in experiments with ultracold atoms by producing a sharp phase discontinuity in the BEC that evolves into a ring vortex (albeit with less contrast than for vortices whose cores go right through the cloud). In addition to studying the properties of individual vortices, these systems allow the study of the collective properties of “vortex matter”. Interesting effects are predicted to occur when the number of vortices becomes very large, corresponding to very
fast rotation, as described in the following section.

In addition to the work on BEC, vortices have been used to investigate superfluidity in quantum degenerate Fermi gases [48], which has connections with other systems such as superconductors, neutron stars etc. Cold atomic vapors have the important characteristic that the strength of interactions between atoms can be precisely controlled over an enormous range (for both fermions, bosons and mixtures) thus making these systems a fantastic test bed for theoretical ideas. The experiment of Zwierlein et al. at MIT [48] used the fermionic isotope of lithium ($^6$Li). Adjusting the magnetic field experienced by the atoms to increase the strength of interactions (near a Fano-Feshbach resonance where the scattering length is resonantly enhanced) causes pairs of atoms to bind together to form long-range molecules, i.e. molecules in a very high-lying vibrational level of the molecular potential where the internuclear spacing is orders of magnitude greater than that of a Li$_2$ molecule in its ground vibrational level. These molecules are composite bosons and hence at low temperature they are in a BEC. A vortex lattice was created in this BEC by imparting angular momentum to the system by rotation, as in experiments with bosonic atoms. The binding energy was decreased (by changing the magnetic field) so that the ultracold molecules became weakly bound pairs of fermions closely analogous to the Cooper pairs (of electrons) in superconductors. This allowed a detailed study of the crossover from a BEC of molecules to a Bardeen-Cooper-Schrieffer superfluid of loosely bound pairs, and this particular experiment showed that superfluidity as evidenced by the existence of vortices persisted in this BEC-BCS cross-over. The ability to scan the system from one regime to another is a truly remarkable feature of ultracold vapors.

The first observation of a vortex in an ultracold gas was a challenging experiment but nowadays vortices have been observed in a great variety of ways; even in non-rotating systems where vortices can (sometimes) arise spontaneously after a rapid quench, i.e. sudden cooling through the phase transition, or by merging Bose-Einstein condensed clouds that have different (random) phases. Another method that has be successfully used to nucleate vortices spontaneously, without rotation or stirring, is by merging of multiple trapped BECs [49]. In this experiment three BECs were formed that were separated from one another and then the potential barriers between them were lowered so that they merged together. The independent, uncorrelated BECs had different (random) phases and sometimes there was sufficient phase winding for vortices to form.
The physics of the persistent flow of a superfluid around a ring and a quantum vortex are closely linked. For a ring-shaped cloud of BEC the atoms are at a fixed radius and therefore the condition on circulation in Eq. (31) corresponds exactly to quantisation of angular moment in units of $\hbar$. Note, however, that vortices with more than one unit of circulation are energetically unstable, e.g. a vortex with two units rapidly breaks up into two vortices. This has been used to measure angular momentum of flow around a ring formed by the combination of a repulsive barrier (from a laser beam with blue frequency detuning) passing through the center of a trapping potential; slowly reducing the height of the central barrier to zero puts the atoms into a harmonic trap where the ring-shaped cloud rapidly separates in many vortices of unit circulation which can be counted. Vortices with circulation greater than unity can be energetically stable in anharmonic potentials, e.g. when there is a positive quartic ($r^4$) term in addition to harmonic confinement ($r^2$). Such traps are of interest in the pursuit of the quantum states that arise at very fast rotation rates as described below.

9.3. the use of rotation for direct quantum simulation

The rotation of neutral atoms can be considered as a way of simulating the effect of a magnetic field $\mathbf{B}$ on charged particles. A simple illustration of this analogy can be seen by comparing the form of the Lorentz force on a charged particle in a magnetic field, $q(\mathbf{v} \times \mathbf{B})$, and the Coriolis force on a neutral particle of mass $M$ in the rotating frame, $2M(\mathbf{v} \times \Omega)$, suggesting a mapping

$$qB = 2M\Omega.$$ (58)

where we generally take $\Omega = \Omega \hat{z}$. More rigorously one can consider the form of the Hamiltonian of a single neutral particle in a 3D harmonic trapping potential in the rotating frame,

$$H_\Omega = H_0 - \Omega \cdot \mathbf{L}$$

$$= \frac{1}{2M} [p - M(\Omega \times r)]^2 + \frac{1}{2}M(\omega^2 - \Omega^2)(x^2 + y^2) + \frac{1}{2}M\omega^2 z^2.$$ (60)

It is apparent that the kinetic energy term is formally equivalent to the usual gauge-invariant term $(p - A)^2/2M$, with $\mathbf{B} = \nabla \times \mathbf{A}$, that arises for a charged particle in a magnetic field. (We take the effective charge as 1
so \( \mathbf{A} \) has the dimension of momentum \( \text{[kg \cdot m/s]} \). Note that the centripetal acceleration which arises from rotation reduces the radial harmonic trapping frequency, \( \omega_{\perp} \rightarrow (\omega_{\perp}^2 - \Omega^2)^{1/2} \). Thus in experimental conditions where the rotation frequency \( \Omega \) tends to the radial trapping frequency \( \omega_{\perp} \) radial trapping becomes very weak and the atomic cloud expands to be a very thin pancake-shape. Under these conditions the temperature of the quantum degenerate gas is so low that \( k_B T \ll \hbar \omega_z \) and the motion in this direction is frozen out. This makes the Hamiltonian of Eq. (60) almost equivalent to that of a two-dimensional electron gas under a magnetic field (but with some residual harmonic potential in the radial direction). Interactions have not been included in this description but typically the chemical potential \( \mu = g_n(0) \) is small in this regime because of the cloud has low density, \( n(0) \), when it has spread out radially. The close analogy between the physics of rotating neutral atoms and electrons under a magnetic field has led to considerable interest in the possibility of achieving strongly-correlated quantum Hall states in a rapidly rotating atomic gas (for a recent review see \[51\]).

The use of rotation for simulating a gauge field with ultracold atoms has a significant drawback, namely exotic strongly-correlated states are predicted to occur in a Bose gas rotating very close to the centrifugal limit, \( \Omega \simeq \omega_{\perp} \), at which rotating atoms are expelled from the trap. There are considerable technical challenges associated with rotating close to the centrifugal limit, e.g. the harmonic trap must be very axisymmetric (\( \omega_x \) very close to \( \omega_y \)), and any static perturbations in the lab frame lead to substantial heating. These issues have prevented fractional quantum Hall effect physics from being achieved in a rotating Bose gas, though impressive attempts have been made \[52\]. In order to circumvent the pragmatic problems associated with rotation there has been a development of alternative methods for creating synthetic gauge fields for ultracold atoms \[50\], as described in the following section.

10. Synthetic (artificial) gauge fields—vortices without rotation

This particular vector potential \( \mathbf{A} = M\Omega \times \mathbf{r} \) in Eq. (60) yields an effective magnetic field \( \mathbf{B}_{\text{eff}} = \nabla \times \mathbf{A} = 2M\Omega \) that is uniform and the corresponding vector potential is analogous to what is called symmetric gauge. (The crucial requirement for a nonzero magnetic field is that the vector direction of \( \mathbf{A} \) differs from that of the intrinsic spatial dependence: for symmetric gauge, \( \mathbf{A} \) is along \( \mathbf{\hat{\phi}} \) and its magnitude depends on the distance \( r \).) Other choices yield the same magnetic field if they are related through a gauge transformation.
\( A \to A' = A + \nabla \Lambda \), where \( \Lambda \) is a scalar gauge function. Under such a gauge transformation, the quantum-mechanical state vector undergoes a phase change \( \psi \to \psi' = \exp(i\Lambda/\hbar)\psi \) \([34, \text{p. 200}]\). For example, another common choice takes \( A' = -2M\Omega y \hat{x} \) (Landau gauge) where the constant coefficient is chosen so that the two vector potentials yield the same effective uniform magnetic field \( \nabla \times A' = 2M\Omega \).

The line integral of \( A \) plays an essential role; the special choice of a closed contour \( C \) gives the familiar gauge-invariant result \( \oint_C A \cdot dl = \int dS \cdot \nabla \times A \), which is just the magnetic flux enclosed by the contour. In the present case of unit fictitious charge, the flux quantum is \( \hbar \), and the enclosed flux is simply \( \hbar \times N_v \), where \( N_v \) is the number of flux quanta enclosed by \( C \). Equivalently, the enclosed flux is \( \hbar \) times the net phase change around the closed contour (namely \( \hbar \times 2\pi N_v \)). In particular, if a quantum-mechanical state yields a net phase change when it executes a closed path \( C \), it experiences a gauge field \( A \) (either from real magnetism or from an artificial/synthetic gauge field).

This observation leads to the important concept of geometric phase, often called Berry’s phase \([53, 34, \text{Sec. 10.2}]\). If a nondegenerate quantum eigenstate \(|\chi_1(r)\rangle\) depends on \( r \) along a contour \( C \), then there is a net geometric phase change on once encircling the contour \( C \)

\[
\gamma_C = \hbar^{-1} \oint_C dl \cdot A(r),
\]

(61)

where the integrand

\[
A(r) \equiv i\hbar \langle \chi_1(r)|\nabla \chi_1(r) \rangle
\]

(62)

is a synthetic or artificial vector potential that acts just like any familiar applied gauge field \([50, \text{Sec. I.4}]\). The line integral \( \gamma_C \) is real and can be interpreted as the Berry’s phase \([53]\) acquired by an atom after adiabatic transport around the closed loop \( C \) in coordinate space. This quantity is called geometric phase because it depends only on the path and not on the speed (assumed slow).

To make a connection with cold atoms, it is helpful to summarize a simple toy model used in \([50, \text{Sec. I}]\). Specifically, consider a two-state atom with bare states \(|g\rangle \) and \(|e\rangle \), and a one-body Hamiltonian \( p^2/2M + V_{tr} \) that is diagonal in this basis. Add a coupling operator \( U \) that reflects, for example, electromagnetic coupling between the two bare states. Assume the general hermitian traceless form

\[
U = \frac{\hbar \Omega}{2} \begin{pmatrix}
\cos \theta & e^{-i\phi} \sin \theta \\
e^{i\phi} \sin \theta & -\cos \theta
\end{pmatrix},
\]

(63)
where $\Omega$ is a generalized Rabi frequency [compare Eq. (52)], $\theta$ is a mixing angle and $\phi$ determines the phase of the off-diagonal coupling terms. The $2 \times 2$ matrix is the product $\hat{\mathbf{n}} \cdot \sigma$ where the unit vector $\hat{\mathbf{n}}$ is characterized by the spherical polar angles $\theta, \phi$, and $\sigma = \sum_{j=1}^{3} \hat{x}_j \sigma_j$ represents the Pauli matrices. All three parameters $\Omega, \theta,$ and $\phi$ can depend on position $\mathbf{r}$.

The operator $U$ has two eigenvectors

$$|\chi_1\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix}, \quad |\chi_2\rangle = \begin{pmatrix} -e^{-i\phi} \sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}$$

(64)

with eigenvalues $\hbar \Omega/2$ and $-\hbar \Omega/2$, respectively; these eigenvectors are usually known as dressed states and are the eigenstates of the atoms in the radiation field. They can serve as the basis for an expansion of the full state vector of the particle:

$$|\Psi(\mathbf{r}, t)\rangle = \sum_{j=1}^{2} \psi_j(\mathbf{r}, t) |\chi_j(\mathbf{r})\rangle.$$  

(65)

Consider the action of the momentum operator $\mathbf{p} = -i\hbar \nabla$ on the state $|\Psi(\mathbf{r}, t)\rangle$. Since the state vector (65) is a product of two position-dependent functions, this operation produces $\mathbf{p}|\Psi\rangle = \sum_{j=1}^{2} [(\mathbf{p}\psi_j)|\chi_j\rangle + \psi_j \mathbf{p}|\chi_j\rangle]$. A little manipulation yields

$$\mathbf{p}|\Psi\rangle = \sum_{j,k=1}^{2} (\mathbf{p}\delta_{jk} - A_{jk}) \psi_k |\chi_j\rangle$$

(66)

with $A_{jk} = i\hbar \langle \chi_j | \nabla | \chi_k \rangle$ a $2 \times 2$ vector matrix in the dressed-state indices, and we have inserted the $2 \times 2$ completeness relation to obtain the second term in Eq. (66).

If the system is prepared with the particle in state $|\chi_1\rangle$, and the particle moves sufficiently slowly, it will adiabatically remain in this state. The projection of the kinetic-energy term $(\mathbf{p}^2/2M) |\Psi\rangle$ onto the state $|\chi_1\rangle$ yields the effective Schrödinger equation for the amplitude $\psi_1$ describing the center-of-mass motion in the first internal state

$$i\hbar \frac{\partial \psi_1}{\partial t} = \left[ \frac{(\mathbf{p} - A)^2}{2M} + V_{tr} + \frac{\hbar \Omega}{2} + W \right] \psi_1,$$

(67)

where the induced vector potential $\mathbf{A}$ is given in Eq. (62) and $W = |A_{12}|^2/2M$ is an induced scalar potential arising from the elimination of the second
atomic state $|\chi_2\rangle$. Note that this synthetic gauge field $A$ means that cold neutral atoms can serve to study magnetic effects usually associated with electromagnetic charge.

In the present case, the synthetic vector potential has the form $A(r) = \frac{1}{2}\hbar(\cos \theta - 1)\nabla \phi$, and the effective magnetic field becomes $B(r) = \nabla \times A = \frac{1}{2}\hbar \nabla (\cos \theta) \times \nabla \phi$. For a nonzero effective magnetic field, $\nabla \theta$ and $\nabla \phi$ must not be collinear.

Among the many proposals for experimental implementation of these ideas [50, Sec. II], we here focus on one scheme [54] that has proved practical in creating vortices with a synthetic gauge potential and no externally applied rotation [55]. Two slightly detuned laser beams counter-propagate along $\hat{x}$ with a difference wave vector $q\hat{x}$. Together these beams induce Raman transitions, and the corresponding Rabi frequency has a spatial dependence $\propto e^{iqx}$ that appears as the phase $\phi = qx$ of the coupling matrix $U$ in Eq. (63). In addition, a uniform magnetic Zeeman field along $\hat{y}$ provides detuning away from the resonant coupling and determines the mixing angle $\theta$. Detailed analysis in momentum space for this two-level system [54] shows that the minimum in the dispersion relation shifts away from $k_x = 0$ to a finite value with an approximate dispersion relation $(\hbar k_x - A_x)^2/2M$. (Reference [50, Sec. II.D] gives the corresponding real-space analysis.) This shift yields what is effectively a synthetic vector potential $A_x$ that depends on $q$ and on the Zeeman field. $A_x$ would be uniform in the absence of additional perturbations. In the presence of a time-dependent $A_x(t)$, however, the derivative $-\partial A_x/\partial t$ induces an effective electric field $E_x$. Reference [56] ramps a uniform $A$ from an initial value to a final value and measures the resulting pulsed electric field, obtaining values for both the canonical momentum $p$ and the mechanical momentum $Mv = p - A$.

Application of a variable magnetic field gradient along $\hat{y}$ supplies the necessary spatial variation $A_x(y)$ like the Landau gauge, and the combination of $\phi(x)$ and $\theta(y)$ together produce an effectively uniform rotation (synthetic magnetic field) over a reasonably broad spatial range. In practice the actual experiment involves all three $|m_F\rangle = 1, 0, -1$ magnetic substates, and the theory requires numerical analysis. Nevertheless, Ref. [55] was able to produce vortices without rotation, as shown in Fig. 8. One puzzle is that the vortices do not form a regular array, in contrast to general theoretical predictions, but this may arise from insufficient time to reach equilibrium.

This field is developing rapidly, especially in connection with trapped atoms in optical lattices (see Chaps. 1 and 5 for general discussions). Refer-
Figure 8: Vortex formation for increasing applied magnetic field gradient with $N = 1.4 \times 10^5$ atoms. The six images show the $|m_F = 0$ component, and increasing magnetic field gradient is equivalent to increasing synthetic gauge potential. Reprinted with permission of I. B. Spielman.

Reference [50] reviews various proposals for such synthetic gauge fields. Recently, Bloch’s group in Munich has studied a two-dimensional $^{87}$Rb BEC in a square lattice, producing strong effective magnetic fields with staggered (alternating striped) flux of order 1/2 flux quantum per plaquette [57]. This achievement is a significant step toward realizing highly correlated states that are analogs of fractional quantum-Hall states [51].

The situation becomes much more interesting if two or more degenerate states contribute simultaneously to the Berry’s phase [50, Sec. III]. Instead of the previous example of two well-separated states $|g\rangle$ and $|e\rangle$, consider a set of $q$ degenerate ground states $|\chi_j\rangle$. If the system starts in a linear combination of these states, then the adiabatic motion of the particle will remain within this degenerate set, leading to the following matrix generalization of the synthetic gauge field in Eq. (62):

$$A_{jk} = i\hbar \langle \chi_j(r) | \nabla | \chi_k(r) \rangle,$$

where the effective vector potential is now a $q \times q$ matrix in the internal space, as well as a vector in three-dimensional coordinate space. In general,
these matrices do not commute, which leads to non-Abelian gauge fields \[58\] that are qualitatively different from the Abelian case with \( q = 1 \).

One unusual aspect of these non-Abelian matrix gauge fields appears in the effective magnetic field \( B \). By definition, the velocity is the combination \( v = (p - A)/M \). Its Heisenberg equation of motion \( i\hbar \dot{v} = [v, \frac{1}{2} M v^2] \) leads to the expected form \( M \dot{v} = \frac{1}{2} (v \times B - B \times v) \), where the symmetrized structure arises because \( v \) and \( B \) need not commute. Here, the effective magnetic field \( B \) is given by the generalized curvature

\[
B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}, \quad \text{where} \quad F_{jk} = \partial_j A_k - \partial_k A_j - \frac{i}{\hbar} [A_j, A_k]. \tag{69}
\]

Even if these gauge fields are spatial constants, the effective magnetic field does not, in general, vanish because of the term \( \frac{1}{2} \epsilon_{ijk} [A_j, A_k] = (A \times A)_i \). This situation is dramatically different from that of the more familiar Abelian gauge field, where the magnetic field vanishes for a spatially constant \( A \).

Reference \[59\] analyzes a particular generalization of the experiments by the Spielman group involving two degenerate ground states obtained by balancing the linear and quadratic Zeeman effect along with the applied field gradient.

Another remarkable feature of these non-Abelian gauge fields is the possibility of generating an effective spin-orbit coupling for trapped neutral atoms \[59, 60, 61\]. In certain situations, the matrix vector potential \( A \) can lie in the \( xy \) plane and reduce to a linear combination of the \( 2 \times 2 \) Pauli matrices \( \sigma_x \) and \( \sigma_y \). Depending on the vector coefficients, the term \( (p \cdot A + A \cdot p)/2M \) in the Hamiltonian can produce a spin-orbit coupling involving products of momentum components and Pauli matrices. Similar coupling terms \( p \times \sigma \cdot \hat{z} = p_x \sigma_y - p_y \sigma_x \) and \( p_y \sigma_y - p_x \sigma_x \) are familiar in the condensed-matter fields of semiconductor spintronics and topological insulators, where they are known as Rashba and Dresselhaus spin-orbit coupling \[62\]. In this way, it may be possible to use cold atoms to explore some of the properties of these rather complicated semiconductors, although current experiments include only a single Pauli spin matrix and thus provide only an Abelian coupling \[61\]. There are also proposals for \( 3 \times 3 \) spin-1 generalizations of the Pauli matrices for spin-\( \frac{1}{2} \). Reference \[50\], Sec. III.D and Sec. V] summarizes this fascinating field and its connection with the rapidly emerging field of spintronics.
Acknowledgments

A. F. thanks G. Juzeliūnas and I. Spielman for very helpful discussions about synthetic gauge fields. Part of this work was completed at the Kavli Institute for Theoretical Physics, University of California Santa Barbara (National Science Foundation Grant No. PHY05-51164), and A. F. is grateful for the warm hospitality. C. F. thanks E. Nugent and R. Williams for their contributions.

References

[1] D. R. Tilley and J. Tilley, Superfluidity and Superconductivity (Institute of Physics, Bristol, England, 1990), third edition.

[2] R. P. Feynman, Application of quantum mechanics to liquid helium, in Progress in Low Temperature Physics, vol. 1, C. J. Gorter, editor (North-Holland, Amsterdam, 1955), p. 17.

[3] R. J. Donnelly, Quantized vortices in He II (Cambridge University Press, Cambridge, 1991).

[4] L. D. Landau, The theory of superfluidity of helium II, J. Phys. (U.S.S.R.) 5, 71 (1941).

[5] L. D. Landau, On the theory of superfluidity of helium II, J. Phys. (U.S.S.R.) 11, 91 (1947).

[6] E. M. Lifshitz and L. P. Pitaevskii, Statistical Physics (Pergamon Press, Oxford, 1980), part 2, chap. III.

[7] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Theory of Bose-Einstein condensation in the alkali gases, Rev. Mod. Phys. 71, 463 (1999).

[8] L. Pitaevskii and S. Stringari, Bose-Einstein Condensation (Oxford Science Publications, Oxford, 2003).

[9] C. J. Pethick and H. Smith, Bose-Einstein Condensation in Dilute Gases (Cambridge University Press, Cambridge, 2008), second edition.
[10] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, *Observation of Bose-Einstein condensation in a dilute atomic vapor*, Science **269**, 198 (1995).

[11] E. P. Gross, *Structure of a quantized vortex in boson systems*, Nuovo Cimento **20**, 454 (1961).

[12] L. P. Pitaevskii, *Vortex lines in an imperfect Bose gas*, Zh. Eksp. Teor. Fiz. **40**, 646 (1961) [Sov. Phys.—JETP **13**, 451 (1961)].

[13] G. Baym and C. J. Pethick, *Ground-state properties of magnetically trapped Bose-condensed rubidium gas*, Phys. Rev. Lett. **76**, 6 (1996).

[14] N. N. Bogoliubov, *On the theory of superfluidity*, J. Phys. (U.S.S.R.) **11**, 23 (1947).

[15] R. Ozeri, N. Katz, J. Steinhauer, and N. Davidson, *Bulk Bogoliubov excitations in a Bose-Einstein condensate*, Rev. Mod. Phys. **77**, 187 (2005).

[16] P. A. Ruprecht, M. J. Holland, K. Burnett, and M. Edwards, *Time-dependent solution of the nonlinear Schrödinger equation for Bose-condensed trapped neutral atoms*, Phys. Rev. A **51**, 4704 (1995).

[17] C. A. Sackett, J. M. Gerton, M. Welling, and R. G. Hulet, *Measurements of collective collapse in a Bose-Einstein condensate with attractive interactions*, Phys. Rev. Lett. **82**, 876 (1999).

[18] P. E. Sokol, *Bose-Einstein condensation in liquid helium*, in *Bose-Einstein Condensation* (Cambridge University Press, Cambridge, 1995), ed. A. Griffin, D. W. Snoke, and S. Stringari, Chap. 4.

[19] J. R. Ensher, D. S. Jin, M. R. Matthews, C. E. Wieman, and E. A. Cornell, *Bose-Einstein condensation in a dilute gas: measurement of energy and ground-state occupation*, Phys. Rev. Lett. **77**, 4984 (1996).

[20] N. Tammuz, R. P. Smith, R. Campbell, S. Beattie, S. Moulder, J. Dalibard, and Zoran Hadzibabic. *Can a Bose Gas Be Saturated?* Phys. Rev. Lett. **106**, 230401 (2011).
[21] G. Hechenblaikner, O. Maragò, E. Hodby, J. Arlt, S. Hopkins, and C. Foot, *Observation of harmonic generation and nonlinear coupling in the collective dynamics of a Bose-Einstein condensate*, Phys. Rev. Lett. **85**, 692 (2000).

[22] E. Nugent, D. McPeake, and J. McCann, *Superfluid toroidal currents in atomic condensates*, Phys. Rev. A **68**, 063606 (2003).

[23] L. P. Pitaevskii, and A. Rosch, *Breathing modes and hidden symmetry of trapped atoms in two dimensions*, Phys. Rev. A **55**, R853 (1997).

[24] T. Lahaye, C. Menotti, L. Santos, M Lewenstein, and T. Pfau, *The physics of dipolar bosonic quantum gases*, Rep. Prog. Phys. **72**, 126401 (2009).

[25] A. Griesmaier, J. Werner, S. Hensler, J. Stuhler, and T. Pfau, *Bose-Einstein condensation of chromium*, Phys. Rev. Lett. **94**, 160401 (2005).

[26] M Lu, N. Q. Burdick, S. H. Youn, and B. L. Lev, *Strongly dipolar Bose-Einstein condensate of dysprosium*, Phys. Rev. Lett. **107**, 190401 (2011).

[27] S. Ospelkaus, A. Pe’er, K.-K. Ni, J. J. Zirbel, B. Neyenhuis, S. Kotochigova, P. S. Julienne, J. Ye, and D. S. Jin, *Efficient state transfer in an ultracold dense gas of heteronuclear molecules*, Nature Phys. **4**, 622 (2008).

[28] K.-K. Ni, S. Ospelkaus, M. G. H. de Miranda, A. Pe’er, B. Neyenhuis, J. J. Zirbel, S. Kotochigova, P. S. Julienne, D. S. Jin, and J. Ye, *A high phase-space-density gas of polar molecules*, Science **322**, 231 (2008).

[29] L. D. Carr, D. DeMille, R. V. Krems, and J. Ye, *Cold and ultracold molecules: science, technology and applications*, New J. Phys. **11**, 055049 (2009).

[30] T. Koch, T. Lahaye, J. Metz, B. Fröhlich, A. Griesmaier, and T. Pfau, *Stabilization of a purely dipolar quantum gas against collapse*, Nature Phys. **4**, 218 (2008).

[31] T. Lahaye, J. Metz, B. Fröhlich, T. Koch, M. Meister, A. Griesmaier, T. Pfau, H. Saito, Y. Kawaguchi, and M. Ueda, *d-Wave collapse and explosion of a dipolar Bose-Einstein condensate*, Phys. Rev. Lett. **101**, 080401 (2008).
[32] M. R. Matthews, D. S. Hall, D. S. Jin, J. R. Ensher, C. E. Wieman, E. A. Cornell, F. Dalfovo, C. Minniti, and S. Stringari, *Dynamical response of a Bose-Einstein condensate to a discontinuous change in internal state*, Phys. Rev. Lett. **81**, 243 (1998).

[33] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics Quantum Mechanics* (Addison-Wesley, Reading, MA, 1965), Vol. III.

[34] D. J. Griffiths, *Introduction to Quantum Mechanics* (Pearson Prentice Hall, Upper Saddle River, NJ 2005), second edition.

[35] D. S. Hall, M. R. Matthews, C. E. Wieman, and E. A. Cornell, *Measurement of relative phase in two-component Bose-Einstein condensates*, Phys. Rev. Lett. **81**, 1543 (1998).

[36] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, and E. A. Cornell *Vortices in a Bose-Einstein condensate*, Phys. Rev. Lett. **83**, 2498 (1999).

[37] D. M. Stamper-Kurn, M. R. Andrews, A. P. Chikkatur, S. Inouye, H.-J. Miesner, J. Stenger, and W. Ketterle, *Optical confinement of a Bose-Einstein condensate*, Phys. Rev. Lett. **80**, 2027 (1998).

[38] T.-L. Ho, *Spinor Bose condensates in optical traps*, Phys. Rev. Lett. **81**, 742 (1998).

[39] T. Ohmi and K. Machida, *Bose-Einstein condensation with internal degrees of freedom in alkali atom gases*, J. Phys. Soc. Jpn. **67**, 1822 (1998).

[40] J. Stenger, S. Inouye, D. M. Stamper-Kurn, H.-J. Miesner, A. P. Chikkatur, and W. Ketterle, *Spin domains in ground-state Bose-Einstein condensates*, Nature **396**, 345 (1998).

[41] D. M. Stamper-Kurn and W. Ketterle, *Spinor condensates and light scattering from Bose-Einstein condensates*, in *Coherent Atomic Matter Waves*, Proceedings of the Les Houches Summer School, Session LXXII, 1999, edited by R. Kaiser, C. Westbrook, and F. David (Springer, New York, 2001), p. 137. This review is available as e-print: cond-mat 0005001.
[42] M. Vengalattore, S. R. Leslie, J. Guzman, and D. M. Stamper-Kurn, *Spontaneously modulated spin textures in a dipolar spinor Bose-Einstein condensate*, Phys. Rev. Lett. **100**, 170403 (2008).

[43] D. Guery-Odelin, and S. Stringari *Scissors mode and superfluidity of a trapped Bose-Einstein condensed gas*, Phys. Rev. Lett. **83**, 4452 (1999).

[44] O. Maragò, S. Hopkins, J. Arlt, E. Hodby, G. Hechenblaikner, and C. Foot, *Observation of the scissors mode and evidence for superfluidity of a trapped Bose-Einstein condensed gas*, Phys. Rev. Lett. **84**, 2056 (2000).

[45] B. P. Anderson, P. C. Haljan, C. E. Wieman, and E. A. Cornell, *Vortex Precession in Bose-Einstein Condensates: Observations with Filled and Empty Cores*, Phys. Rev. Lett. **85**, 2857 (2000).

[46] I. Coddington, P. C. Haljan, P. Engels, V. Schweikhard, S. Tung, and E. A. Cornell, *Experimental studies of equilibrium vortex properties in a Bose-condensed gas*, Phys. Rev. A **70**, 063607 (2004).

[47] R. A. Williams, S. Al-Assam, and C. J. Foot, *Observation of Vortex Nucleation in a Rotating Two-Dimensional Lattice of Bose-Einstein Condensates*, Phys. Rev. Lett. **104**, 050404 (2010).

[48] M. W. Zwierlein, J. R. Abo-Shaeer, A. Schirotzek, C. H. Schunck, and W. Ketterle. *Vortices and superfluidity in a strongly interacting Fermi gas* Nature **435**, 1047 (2005).

[49] D. R. Scherer, C. N. Weiler, T. W. Neely, and B. P. Anderson. *Vortex formation by merging of multiple trapped Bose-Einstein condensates* Phys. Rev. Lett. **98**, 110402 (2007).

[50] J. Dalibard, F. Gerbier, G. Juzeliūnas, and P. Öhberg, *Artificial gauge potentials for neutral atoms*, Rev. Mod. Phys. **83**, 1523 (2011).

[51] N. R. Cooper, *Rapidly Rotating Atomic Gases*, Advances in Phys. **57**, 539 (2008).

[52] V. Schweikhard, I. Coddington, P. Engels, V. Mogendorff, and E. A. Cornell, *Rapidly Rotating Bose-Einstein Condensates in and near the Lowest Landau Level*, Phys. Rev. Lett. **92**, 040404 (2004).
[53] M. V. Berry, *Quantal phase factors accompanying adiabatic changes*, Proc. R. Soc. Lond. A 392, 45 (1984).

[54] I. B. Spielman, *Raman processes and effective gauge potentials*, Phys. Rev. A 79, 063613 (2009).

[55] Y.-J. Lin, R. L. Compton, K. Jiménez-García, J. V. Porto, and I. B. Spielman, *Synthetic magnetic fields for ultracold neutral atoms*, Nature 462, 628 (2009).

[56] Y.-J. Lin, R. L. Compton, K. Jiménez-García, W. D. Phillips, J. V. Porto, and I. B. Spielman, *A synthetic electric force acting on neutral atoms*, Nature Physics 7, 531 (2011), doi:10.1038/nphys1954.

[57] M. Aidelsburger, M. Atala, S. Nascimbène, S. Trotzky, Y.-A. Chen, and I. Bloch, *Experimental realization of strong effective magnetic fields in an optical lattice*, Phys. Rev. Lett. 107, 255301 (2011).

[58] F. Wilczek and A. Zee, *Appearance of gauge structures in simple dynamical systems*, Phys. Rev. Lett. 52, 2111 (1984).

[59] T.-L. Ho and S. Zhang, *Bose-Einstein condensates with spin-orbit interaction*, Phys. Rev. Lett. 107, 150403 (2011).

[60] C. Wang, C. Gao, C.-M. Jian, and H. Zhai, *Spin-orbit coupled spinor Bose-Einstein condensates*, Phys. Rev. Lett. 105, 160403 (2010).

[61] Y.-J. Lin, K. Jiménez-García, and I. B. Spielman, *Spin-orbit-coupled Bose-Einstein condensates*, Nature 471, 83 (2011).

[62] J. Schliemann, D. Loss, and R. M. Westervelt, *Zitterbewegung of electrons and holes in III-V semiconductor quantum wells*, Phys. Rev. B 73, 085323 (2006).