Centralizers of Commuting Elements in Compact Lie Groups

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Abstract

The moduli space for a flat G-bundle over the two-torus is completely determined by its holonomy representation. When G is compact, connected, and simply connected, we show that the moduli space is homeomorphic to a product of two tori mod the action of the Weyl group, or equivalently to the conjugacy classes of commuting pairs of elements in G. Since the component group for a non-simply connected group is given by some finite dimensional subgroup in the centralizer of an n-tuple, we use diagram automorphisms of the extended Dynkin diagram to prove properties of centralizers of pairs of elements in G.

Keywords: Moduli space; Lie groups; Representation theory; Characteristic classes; Centralizers

Introduction

Classifying the moduli space of gauge equivalence classes of flat connections on a principal G-bundle over a compact Riemann surface Σ of genus g is of interest from various perspectives. For example, Atiyah-Bott [1] proved that this moduli space is equivalent to the finite dimensional representation space (Hom(π₁(Σ),G))/G by constructing a symplectic structure on the moduli space by symplectic reduction from the infinite-dimensional sympletic manifold of all connections. Every nontrivial triple on a flat principal bundles G. Witten [3] proved that the number of extra quantum vacuum states for a flat principal G-bundle over a spatial 3-torus T³ is the topological invariant called the Witten Index which is equal to g, the dual Coxeter number of the Lie group G.

Our primary motivation comes from Borel-Friedman-Morgan [4]. Given G as a compact, connected, semi-simple Lie group, they proved that principal G-bundles x with flat connections over a maximal two torus T² are classified up to restricted gauge equivalence by classifying commuting pairs of elements in the simply connected covering G of G that commute up to the center. The first invariant is the nontrivial characteristic class [w]∈H²(T, π₁(G))=π₁(G) with a subgroup of the center CG we fix a topological type of the bundle by w(z)=c∈CG. Since the characteristic class is completely defined by the holonomy representation ρ:π₁(T)=Z×Z→G where the images of ρ commute then for any lifts ˜x, ˜y∈G, we have [˜x, ˜y]=[w]=c. Elements with this property are called c-pairs or “almost commuting”.

Definition 1.1: A pair of elements x, y∈G commutes if [x, y]=1. A c-pair in the simply connected covering G of G is a pair of elements (x, y) where x, y∈G such that [x, y]=1 and [˜x, ˜y]=c∈CG.

To understand why a flat bundle is determined by its holonomy representation note the following. Let G be a compact, connected and not necessarily simply connected Lie group and π:G→G the universal covering map. Certainly the choice of a lift ˜x is unique up to an element in Ker(π)≡π₁(G) which is identified as a subgroup of the center of the simply connected covering. Extending this for a c-pair: for k∈Ker(π), [˜x, ˜y]=[k ˜x, ˜y]=c because k∈Ker(π) commutes with every element in G and is also invariant under the choice of x, y. We may define conjugation by ˜g∈G to be ˜g[˜x, ˜y]=g[˜x, ˜y]g⁻¹=[g ˜x, g ˜y]=g ˜x ˜y g⁻¹. This lift is independent of the choice of c∈CG and thus our c-pair is well-defined.

For completeness, we recall some definitions found in [5] on Dynkin diagrams and root/coroot systems that we will use throughout the paper. Let Φ be a reduced irreducible root system for a compact connected Lie group G, and let Δ={a₁,...,aₙ} be a choice of simple roots for G. Let d be the highest root of Φ with respect to Δ. Set ˜a = −d and let 3 = Δ ∪ {˜a} be the extended set of simple roots. Then 3 is the set of coroots a⁻¹ inverse to each root a∈Δ. If we define A to be the unique alcove containing the origin in the positive Weyl chamber associated to A then there is a bijection between the walls of A and 3. Therefore 3 is the set of nodes for the extended Dynkin diagram D(3).

For each element eCG the differential w∈W of the action of the center on the alcove is a linear map normalizing 3∋a and the action of w on the nodes of D(3) is a diagram automorphism. Given a maximal torus T⊂G, denote Lie(T)=h and the exponential map identifies T with h/Q where where Q=a=∑aₙ is the lattice associated to the coroots dual to a choice of simple roots a∈Δ for G. Denote the affine Weyl group by Wₐ. The alcove is defined over the maximal torus TCG as A=½Wₐ(Φ)∋h where Wₐ(Φ) acts simply transitively on the set of alcoves in the vector space V; thus there is an induced action of the center CG on the alcove A.

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Component Group of the Centralizer of Commuting Pairs

The work in [4] gave an explicit characterization of the moduli space of c-pairs in terms of the extended coroot diagram of a simply connected group G and the action of the Weyl group on that diagram. This beneficial relationship between the root/coroot system and holonomy plays an essential role. Let G be a compact connected Lie group. When G is disconnected, there is the appearance of a c-"1-chain" coming from a component group which may be a finite group of a certain order. A group G is reductive if any representation is a direct sum of irreducible representations.s Notice that when a group G is compact, it is equivalent to being reductive. We will use the following theorem by Borel.

Theorem 2.1 (Theorem 5): Let G be a compact, connected and simply connected Lie group. Then Z(x) is connected.

The following demonstrates the relationship between the conjugacy classes for commuting pairs (x, y) and flat G-bundles over T^x:

Proposition 2.2: Assume that G is a compact, connected and simply connected Lie group. For any maximal torus S, we have that 
\[\{\text{Hom}(\pi_1(T^x, x), G)/G\} \times \text{S}(x)/\text{W}\} \text{ is a homeomorphism.}\]

Proof: Fix generators \( (y_1, y_2) \) for \( \pi_1(T^x, x) \). Notice that we have a representation \( \rho: \pi_1(T^x) \to G \) where \( \rho(y_1) = x, \rho(y_2) = y \) and that these images define the commutator in G. In fact, the representation determines the commutator in the following sense. Let T be the maximal torus in G. Then for some \( g \in G \), \( g \gamma' \in T \) and \( g \gamma'' \in T \) since every element in G can be conjugated into the maximal torus. We want to show that both \( x, y \in T \). To do this, define conjugation by \( g \in G \) for the pair \( (x, y) \) by \( g(x, y) = (g(x), g(y)) = (x', y') \) where \( x', y' \in T \times T \) and \( x', y' \in T \times T \). The fact that G is simply connected implies that Z(x) is connected (2.1). Thus we may restrict to the connected component of the identity \( Z(x) \). Since \( x \in Z(x) \) we must show that \( TZ(x) \) because this would imply that both \( x, y \in T \). By definition of the representation, the image \( [x, y] = 1 \) so \( y \in T \) which implies we may project \( y \) to an element \( \gamma \in \text{W} \). If we conjugate the pair \( (x, y) \) by \( \gamma \), then \( (\gamma_x, \gamma_y) \in G \). Thus we have shown that \( (\gamma_x, \gamma_y) \) conjugates elements from \( T \times T \) to \( T \times T \) and \( \gamma \in W \).

Conversely, W acts by simultaneous conjugation on \( S \times S \) so that \( g\gamma W = g\gamma W \) when \( g \gamma W \in G \). Thus we have its action on the pair \( \gamma \times (S\times S) \) by \( (g\gamma) \times (S\times S) \) for \( (g\gamma, h) \in g\gamma W \). Define the commutator by \( [t, h] = [g\gamma, h] \). Since elements in S commute, if \( y, y' \) generate \( \pi_1(T^x) \) and \( \rho(y, y') = [g\gamma, h] \), then the holonomy determines the commutator and vice versa. Thus we have defined conjugacy of pairs by sending a pair homeomorphically to \( S \times S \) because the representation modulo conjugacy by G yields a commutator.

Given a commuting n-tuple \( \pi = (x_1, \ldots, x_n) \), the next corollary follows immediately because the fundamental group of the centralizer \( Z(x_i) \) is trivial, and for \( i \) a simply connected group, the component group \( \pi \cdot \pi Z(x_i) \) is contained in the fundamental group of the semisimple subgroup \( D \cdot Z(x_i) \).

Corollary 2.3: When G is a group of type \( A_n, C_n \) every commuting n-tuple can be conjugated into the maximal torus T in G so that the moduli space has the form \( M_{\pi}(T)/\pi \).

The corollary can also be seen directly as follows. If \( \pi = (x_1, \ldots, x_n) \) is a commuting n-tuple such that \( [x_i, x_j] = c \in G \) and \( [x_i, x_j] = 1, i = 2, 3, \ldots, n \), choosing \( x_i \) in the alcove over the torus implies that \( x_i \) projects to a Weyl element and therefore conjugates back into the maximal torus; every other element has trivial commutator and thus can be conjugated to the maximal torus. This also works when \( (x_i, x_j) \) for \( 1 \leq i \leq j \leq n \) is an arbitrary n-tuple because the lifts \( [x_i, x_j] = c \in G \) and for the cases of type \( A_n, C_n \) the center is generated by one cyclic element. Hence only one pair in the n-tuple determines what happens to the other elements in the n-tuple.

Corollary 2.4: Let \( \pi = (x_1, \ldots, x_n) \) be a commuting n-tuple and S a maximal torus in G. If G is simply connected then the component group \( \pi \cdot Z(S) \) is a subgroup of \( Z(n^2) \), where \( n \leq 6 \) and corresponds to the coroot integer for \( \pi \cdot S \) which is associated to the node in the extended Dynkin diagram \( D(G) \).

By [4], Lemma 3.1.5: for \( x_i \in Z(x_1, \ldots, x_n), \) \( \text{Stab}_{\pi \cdot Z(x_1, \ldots, x_n)}(x_i) \) is a subgroup of \( \pi \cdot Z(x_1, \ldots, x_n) \). If \( n = 6 \) is reductive, then \( \pi \cdot Z(x_1, \ldots, x_n) \) is a subgroup of the fundamental group \( \pi \cdot Z(x_1, \ldots, x_n) \), which in turn is a finite subgroup of the centralizer \( C \cdot Z(x_1, \ldots, x_n) \). If the fundamental group of the centralizer \( Z(x_1, \ldots, x_n) \) trivial, then \( \pi \cdot Z(x_1, \ldots, x_n) \) is a torus \( T^x \) and hence \( \pi \cdot Z(x_1, \ldots, x_n) \) is trivial. So suppose that \( \pi \cdot D \subset \pi \cdot Z(x_1, \ldots, x_n) \). Even if G is not simply connected but still connected, choosing \( x_i \) to lie in the connected component \( Z(x_i) \) of the centralizer of \( Z(x_i) \) will yield the same result.

Proposition 2.5: Let G be simply connected and let \( \Delta = \{a_1, \ldots, a_n\} \) be a choice of simple roots. Let \( \Delta_1 = \{a_1, a_2, \ldots, a_k\} \) for \( n \leq k \) be a choice of simple roots for \( Z(x_1, \ldots, x_n) \) and let \( h(\pi) \) be the real linear span of the cosets dual to the roots in \( \Delta_1 \). Then there is an exact sequence \( 1 \rightarrow Q(\pi) \rightarrow Q(\pi) \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 1 \).

Proof: By definition of the fundamental group, \( n = \gcd(g_1, g_2, \ldots, g_n) \) knowing that all the coroot integers for both the classical and exceptional groups are less than or equal to six, \( n \leq 6 \). Dividing each of the coroot integers in any group G by n, we may define a new integer \( g_1 = g_1/n \), \( g_2 = g_2/n \) for \( r > k \). By definition, this element will have order \( n \) in the central subgroup \( Q(\pi) \cap Q(\pi) \) and thus is a generator for the cokernel. Hence \( Q(\pi) \cap Q(\pi) \equiv \mathbb{Z}/n\mathbb{Z} \).

Proposition 2.6: For an arbitrary compact, connected simple group G and for a commuting n-tuple \( \pi = (x_1, \ldots, x_n) \), the component group of the centralizer of the n-tuple can be defined in terms of the roots as:

\[\pi \cdot Z(\pi) = \frac{\text{Stab}_{\pi \cdot Z(\pi)}(x_i)}{\pi(\pi(\pi))} \]

Proof: If G is not simply connected, then under complexification \( T = h/L \), where \( Q \subset L \subset P \). If \( \text{Stab}_{\pi}(\pi) \) is a subset of roots which annihilate x then the action of \( \pi \cdot Z(\pi) \) is defined with respect to this smaller subset of roots. If x corresponds to some node \( a_1 \) in the extended Dynkin diagram such that \( a_{x}^{1} \neq 1 \) then \( \Phi(\pi) = \{a_1 \in \Delta | k = 1 \} \). Since \( Q \subset L \subset P \) we have the nesting of tori \( h/P \subset h/L \subset L/Q \). Thus for the lift \( h/\pi \rightarrow h/Q \) sending \( x \leftrightarrow \tilde{x} \) its kernel consists of all the roots in L not in \( Q \) i.e. \( L/Q \). Therefore the roots which annihilate x are the same as those annihilating \( \tilde{x} \).

Let \( W(\pi(\pi)) \) be a subgroup in \( W_\pi \) defined by a subroot system when viewed as characters which annihilate x. The faithful action of \( W_\pi \) on \( h/Q \) yields a split exact sequence \( 1 \rightarrow \pi(\pi) \rightarrow W_\pi \rightarrow W_\pi/W_\pi \) and since the kernel is central, \( W_\pi = h/Q \times \mathbb{W} \) is a direct product because the action of the Weyl group is trivial on the center. Restrict the Weyl group to \( L : W_\pi = h/L \times \mathbb{W} \). The torus action of \( L/Q \) on \( h/Q \) provides the...
and hence $\Delta(\cdot) = b$. Therefore, we may define $Stab_{\mathfrak{n}(\mathfrak{g})}(x) = Stab_{\mathfrak{n}(\mathfrak{g})}(x)$ in the sense that the roots which annihilate $x$ can be used to define a subset $S \subseteq L/Q^+$, where $S = \pi(Z(x, \ldots, x))$. This allows for a component group larger than the fundamental group and therefore it is not necessarily cyclic. Since $S \subseteq \mathfrak{g}G$ it induces a well-defined cyclic permutation on the vertices in the alcove and its fixed space $b^\ast$ may be something other than the barycenter. $L$ is defined as follows. For $Z(x)$ the vector space is $b$ and its coroot lattice is the entire $Q$. Because we are considering commuting elements, we choose $x_i \in Z(x_i)$ to lie in $b(x_i)L$, where $b(x_i)$ is the vector space associated to $DZ(x_i)$. Because $Z(x_i)$ is not necessarily connected, the associated lattice is $Q^+ \subseteq L_i \subseteq P_i$ By induction, the element $y_i \in b(x_i,x_i,x_i)L_i$ and $Q^+ \subseteq L_i \subseteq P_i \subseteq \mathfrak{g}$ so that $L = L_i$ as the associated lattice to the centralizer of the prior $n-1$ elements. From the definition of these lattices, when they are quotiented out by the coroot lattice, they will either be a cyclic subgroup of the center whose order divides the order of the center or will be the entire center. Thus we have the above conclusion since $S \subseteq L/Q^+ \subseteq P/Q^+$.

**Properties of Centralizers**

In general, the component group of the centralizer of an ordered $n$-tuple is some subquotient of the Weyl group and lies in the connected component $Z(x, \ldots, x)$. Let $\Delta = \{a_1, a_2, \ldots, a_n\}$ be the set of extended simple roots for a Lie group $G$. Any closed subset of the extended simple roots for $G$ gives a subdiagram of the extended diagram. We are interested in the subset of roots $\Delta$ that annihilate the $n$-tuple. In particular, the simple root system for the centralizer $Z(x)$ for any $x \in G$ is defined as $\Delta = \{a_1, a_2, \ldots, a_n\}$, which has an associated Weyl group $W(\Phi(x))$. Let $T = \langle x_1, \ldots, x_n \rangle$ be a commuting $n$-tuple. Any element $x = \pi(Z_T(x))$ can be represented by $g \in Z(T)$ since $g$ normalizes $Z(T)$ and therefore via conjugation defines a map $\pi(x) = \pi(Z(x))$ defining the path components of $G$. Let $S_\Lambda$ be the maximal torus in the centralizer $Z(x, \ldots, x)$ generated by the roots in $\Lambda$ given by its Lie algebra $s = \sum_{a \in \Lambda} \ker(a)$. Since $G$ is reductive, there is a standard decomposition given by $G = (CG)^\times DG$, where $(CG)^\times$ is a central torus in the semisimple subgroup $DG$ of $G$ and $= (CG)^\times \cap DG$ is a finite subgroup of the center of $DG$. Thus we get a decomposition of the centralizer into $Z(S) = Z_1 \times \ldots \times Z_n$, where $Z_i$ is the obstruction to a lift which will lie in the center of $Z(x)$ which means that given the equivalence classes above, the obstruction will lie in $F$. The action of $F$ must be nontrivial in order to get the semidirect product by $F$.

**Example 3.1:** Let $G = SU(2)$. There is a nontrivial central action $c \in Z_2$, on the above over $A_1$ given by switching the two vertices, leaving the barycenter as the only fixed point. As described above, we may consider $SU(2) = SU(2)$ where the noncentral trivial action acts “diagonally” on each $SU(2)$ component, switching the vertices of the alcove over each copy of $A_1$. We denote by $A \times A$, the join of the alcovess over each $A_i$. In this case, the join of the two 1-simplices is a 2-simplex given as a square with the barycenter $b = \{0, b, b\}$ as the only fixed point under the central action. The join can be thought of as the Minkowski sum of two simplices: $S_1 + S_2 = \{x_1 + x_2 | x_1, x_2 \in S_1, x_2 \in S_2\}$.

In order to determine the component group of the centralizer of an $n$-tuple in a non-simply connected group we note that the finite principal subgroup contained in the center of each centralizer $Z(x, \ldots, x)$, for some $k$, at some point becomes the component group and therefore defines the singularities in the moduli space. For the classical groups, $Z(x)$ will be a product of type $A_n$, $B_n$, or $D_n$, and for the exceptional groups, $Z(x, \ldots, x)$ will be of type $A_n$, $D_n$. Therefore, it suffices to consider the diagonal action of the fundamental group $\pi_1(DZ(x))$ of groups of these types. Since the fundamental group is a subgroup of the center of the simply connected covering, for type $B_n$ we only consider the $\mathbb{Z}/2\mathbb{Z}$ action on the alcove given by flipping the two vertices; the action of any higher order central cyclic group is trivial.

**Example 3.2:** Consider a group of type $B_n$. Select $\tilde{x}$ to correspond to the vertex in the 3-simplex associated to the trivial node $\alpha_i^\vee$. The centralizer $Z(x)$ is the set of those elements which annihilate $x$. In the Lie algebra, these elements are precisely the generators for the maximal torus $Lie(S) = \mathfrak{z}$ in the Lie algebra $LDZ(x)$. When we remove the node $\alpha_i$ from the diagram, we are left within 3 nodes, each orthogonal to each other. Thus we get $LDZ(x) = SU(2) \times SU(2)$, and $Z = \{x, y, z\}$ is $\mathbb{Z}$. By [6], if a root system has a node with torsion prime $p$ then there exists a diagonal element $\epsilon p Z(x)$ of order $p$. For $B_n, p = 2$ and hence the finite diagonal subgroup is $\Delta = \{1, -1\}$ of order 2 in the center $CDZ(x) = \mathbb{Z}_2 \times \mathbb{Z}_2$. $\Delta$ acts on $LDZ(x)$ by flipping the two vertices corresponding to each copy of $SU(2)$. We write $LDZ(x) = DZ(x) / \delta = \mathbb{Z}_2 / \mathbb{Z}_2$.

In order to obtain a commuting triple, the choice for $y$ must come out of $DZ(x)$ since $\pi(Z(x))$ normalizes $Z(x)$. If $\tilde{y}$ is chosen so that it does not lie in the fixed point space under the diagonal action, the component group $\mathfrak{z} Z(x, y)$ is trivial. However, if $\tilde{y}$ lies somewhere in the fixed space under the action of the center, there will be a nontrivial component group $\pi(Z(x, y))$. The alcove for $SU(2) = T \times \mathbb{Z}^2$ is a 1-simplex. Thus the 3-simplex determined by the join of three 1-simplices is a cube and is $DZ(x)$. If $\tilde{y}$ is in the interior of the cube, then since $\mathbb{Z}_2$ stabilizes $y$, and we have $Z(x, y) = T \times \mathbb{Z}^2$. The maximal torus in $Z(x, y)$ is $\mathbb{Z}_2$. Select $x \in \mathbb{Z}_2$ because we require $[y, z] = 1$ but $[\tilde{y}, \tilde{z}] = \epsilon \in \mathbb{Z}_2$. Therefore, $Z(x, y) = \mathbb{Z}_2$.

Notice that the centralizer of a commuting triple will depend on the choices for each of the elements $x, y, z$. If instead, we select $\tilde{x}$ to correspond to the node $\alpha_1$ of the Lie algebra of the semisimple part of the centralizer is $LDZ(x)/SU(4)$. The real dimension of the maximal torus is zero thus $Z(x) = DZ(x)$. The diagonal element must be in the center $C SU(4) = \{ad : \omega \rightarrow \omega - 1\}$. In this case, $\Delta$ is simply the trivial action (multiplication by $z$).

**Definition 3.3:** Define the rank $rk(x_1, \ldots, x_n)$ of an $n$-tuple to be the rank of $Z(x_1, \ldots, x_n)$. An $n$-tuple has rank zero if and only if $Z(x_1, \ldots, x_n)$ is a finite group. A $c$-pair $(x, y)$ is in normal form with respect to the maximal torus $T$ in the alcove $A$ if $x \in T$ is the image under the exponential map of $\tilde{x} \in \mathfrak{h}$ and $y \in N(T)$ projects to $w_1 \in W$. Note $w \in W$ is the differential action of $ceCG$ that, as a group of affine isometries of the Lie algebra $t$ of the maximal torus $T$ normalizes the alcove $A$.

**Example 4.3:** Consider $G = SU(3)$ and $x \in T$, the maximal torus of $G$. Then the centralizer $Z(x) = T$. In the Lie algebra of $G$, select $\tilde{x}$ to be regular (in the interior of the alcove) so that $Z(x)$ is a connected, abelian, reductive subgroup $\{H \in \mathfrak{h} : [H, x] = 0\}$. By choosing $\tilde{y}$ to be regular, $Z(x, y) = Z(x) \cap Z(y) = ceCG$ and hence $(x, y)$ is a $c$-pair of rank zero.

**Remark 0.1:** For $ceCG$, denote by $\mathfrak{h}$ the torus in $T$ fixed under the action of the center. The choice of roots $\Delta(a) = \{a \in \mathfrak{a} | [a, \mathfrak{g}] = 0\}$ where $c = \exp(\lambda)$ for $x_\lambda^\vee \alpha_a \neq 0$ defining the fixed subtorus $S \subseteq T$ is independent of the choice of lift $\lambda$, let $x_\lambda^\vee \alpha_a \neq 0$ be such that $exp(\lambda) = e^x$. Then under exp $\rightarrow T$, the kernel of this map is an integral lattice defined with respect to $T$. Namely, $\ker(exp) = \mathfrak{g}$, but for $\lambda - \lambda \in \ker(\mathfrak{h})$ this implies that $\gamma \rightarrow \gamma \in \mathbb{Z}$ and hence $\Delta(\gamma) = \gamma(\delta)$ if and only if
follows directly from looking at the coroot integers for all the extended given by $A_{\tau} \cdot x_{\tau} \cdot T_{\tau}^{\pm 1}$ its centralizer is $Z(A_\tau) = \{[B]\cdot A_\tau : [B]\cdot A_\tau = [B]\cdot A_\tau\}$.

This implies that $[AB, t] = [BA, s]$. But since $st = t\in T, AB = BA$, Therefore, $Z(A_\tau, t) = Z(A_\tau) \cdot T_{\tau}^{\pm 1}$ which is connected and thus Proposition 3.5 applies. The conclusion follows because the components in the almost direct product are simply connected.

Proposition 3.8: Given $G \times G$, where $G_1, G_2$ are subgroup of $G$, $F \subseteq CG_1$ and $F \subseteq CG_2$, and $F \subseteq (DG_1 \times DG_2)$. Then for $[a, b] \in G \times G$,

\[
\{1\} \rightarrow F \rightarrow Z([a, b]) \rightarrow Z_{G_1}(a) \times Z_{G_2}(b) \rightarrow F.
\]

Proof. Consider the map $G \times G \rightarrow G \times F \times G / F / F$. The kernel is $\ker = \{[c, d] : c \in \text{Def}F\}$. Thus we have the injective map $\{1\} \rightarrow G \times G \rightarrow G / F / F$. By definition of the centralizer of an element in $G \times G$,

\[
Z([a, b]) = \{[c, d] : [ac, bd] = [ca, db]\}
\]

\[
= \{\{c, d\} : [c, d] = f(h, d) = f^{-1}(f, f) = f_{f_{f_{f_{f}}}}(, [c, d]) = 1\}
\]

This demonstrates that the coker of $\pi$ is $F$ and that $\pi$ is not surjective. Therefore, $\pi^{-1}(Z([a, b])) = Z([a, b])$. Note also that by the definition of the centralizer of $[a, b] \in G \times G$, that the generalized Stiefel-Whitney class $\{w_2, (a, c) - w_2(b, d)\}$ F. Hence $w_2 : HF(T) \rightarrow Z$ defines an obstruction.

Corollary 3.9: Following proposition 3.8, if $G = T$ for some torus and $G_1$ is of type $A_\tau$, then:

\[
F \rightarrow Z_{\tau}(A) \rightarrow Z_{\tau}(A) / Z_{\tau}(F) \rightarrow F \rightarrow \{1\}.
\]

Proof. Given a sequence $\{1\} \rightarrow T^{\tau} \rightarrow T^s \times T^s \times T^s \rightarrow F$ inside $Z_{\tau}(A) \times [s, B] \rightarrow [s, B] A \rightarrow [s, B]$ and in $Z_{\tau}(A) / Z_{\tau}(F)$ the $\{B : [A, B] = [B, A]\}$ which implies that $Z_{\tau}(A) / Z_{\tau}(F)$ and $AB = BA\xi$ for $\xi$ F. Thus they are equal up to an element in the finite group. Therefore we have $Z_{\tau}(A) / Z_{\tau}(F) = \pi^{-1}(Z_{\tau}(A) / Z_{\tau}(F)) / AB = BA\xi$. Suppose that $[A] \in A / F$ and consider its lift $A_\tau \rightarrow A$.

\[
F \rightarrow Z_{\tau}(A) \rightarrow Z_{\tau}(A) / Z_{\tau}(F) \rightarrow F \rightarrow \{1\}
\]

because the kernel is $\ker \pi F = \phi$ and from what we have already deduced, $AB = BA\xi$ for $\xi$ F. Hence:

\[
Z_{\tau}(A) / Z_{\tau}(F) = \pi^{-1}(Z_{\tau}(A) / Z_{\tau}(F)) / F.
\]

We used the simply connected component as follows. We consider $[B]$ such that exists $a$ with $AB = BA$ then multiplication of the equivalence classes is $[s, B][s, A] = [s, B] A = [s, B] A$. Since $\pi_1(\tilde{G}) = \phi$ when lift to the universal covering we can say that for $[t, A] \in \tilde{G} = T^s \times T^s \tilde{G}$ then $Z_{\tau}(A) / Z_{\tau}(F) / T^s \times T^s (A)$ and more importantly that $Z_{\tau}(A) / Z_{\tau}(F)$ is connected.

Corollary 3.10: Consider a subgroup in $G$ of the form $A_{\tau} A_{\tau} A_{\tau}$, then the centralizer of the element $[a, b] \in A_{\tau} A_{\tau} A_{\tau}$ for $r + k + n + 1$, is $Z_{\tau}(a, b) = Z_{\tau}(A_{\tau}) / F$, where $F = CDZ_{\tau} \times CDZ_{\tau}$. Then the centralizer of an element $[a, b] E A_{\tau} A_{\tau} A_{\tau}$ is of the form:

\[
Z_{\tau}(a, b) = \frac{Z_{\tau}(a) \times Z_{\tau}(b)}{F}.
\]
where \( F' = CDZ_{\theta_1} \cap CDZ_{\theta_2} \supseteq F \).

It does not necessarily follow that \( \pi Z(x_1, \ldots, x_n) \cap \pi DZ(x_i) \) because \( DZ(x_i) \) is not necessarily connected. The fact that for \( G \) of type \( D_n \), \( \pi D \supseteq C \) \( D \supseteq \mathbb{Z} \times \mathbb{Z} \) and that the characteristic class for a principal \( G \)-bundle over \( T^n \) lies in \( H^2(T^n, \pi_1(G)) \equiv \mathbb{Z} \times \mathbb{Z} \) means that there is a possibility that the component group of the centralizer of an \( n \)-tuple inside a group of type \( D_n \) will not be finite cyclic.

**Conclusion**

We have shown that for an arbitrary compact, connected simple group \( G \) and a commuting \( n \)-tuple, that the component group of the centralizer can be defined in terms of the roots and therefore, we may use diagram automorphisms to define the moduli space of commuting elements. We have also seen that the centralizer of a commuting \( n \)-tuple is determined by the order and choice of elements. For example, we can generate a commuting \( n \)-tuples of rank zero of arbitrary length by finding a nontrivial triple, say, and then adding arbitrarily many elements from the torus, thereby not altering the centralizer.

**References**

1. Atiyah M, Bott R (1982) The Yangs-Mills equations over Riemann surfaces. Phil Trans Roy Soc London A 308: 523-615.
2. Kac V, Smilga A (2000) Vacuum structure in supersymmetric Yang-Mills theories with any gauge group, The many faces of the superworld, World Sci Publ, River Edge, NJ, pp: 185-234.
3. Witten E (1998) Toroidal compactification without vector structure. J High Energy Phys 9802 1998, no. 02, 006.
4. Borel A, Friedman RD, Morgan JW (2002) Almost commuting elements of compact Lie groups. Mem Amer Math Soc 157: 747, x+136.
5. Bourbaki N (1981) Groupes et Algebres de Lie. Chap. 4,5, et 6, Masson, Paris.
6. Steinberg R (1975) Torsion in reductive groups. Advances in Math 15: 63-s92.
7. Molinari B (2000) Gauge spinors and string duality. Nuclear Physics B 577: 439-460.