Weak localization corrections to the thermal conductivity in s-wave superconductors

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We study the thermal conductivity in disordered s-wave superconductors. Expanding on previous works for normal metals, we develop a formalism that tackles particle diffusion as well as the weak localization (WL) and weak anti-localization (WAL) effects. Using a Green’s functions diagrammatic technique, which takes into account the superconducting nature of the system by working in Nambu space, we identify the system’s low-energy modes, the diffuson and the Cooperon. The time scales that characterize the diffusive regime are energy dependent; this is in contrast with the the normal state, where the relevant time scale is the mean free time $\tau_e$, independent of energy. The energy dependence introduces a novel energy scale $\varepsilon_*$, which in disordered superconductors ($\tau_e \Delta \ll 1$, with $\Delta$ the gap) is given by $\varepsilon_* = \sqrt{\Delta/\tau_e}$. From the diffusive behavior of the low-energy modes, we obtain the WL correction to the thermal conductivity. We give explicitly expressions in two dimensions. We determine the regimes in which the correction depends explicitly on $\varepsilon_*$ and propose an optimal regime to verify our results in an experiment.

I. INTRODUCTION

The study of quantum effects in the transport properties of disordered conductors has a long history. For thermal conductivity in normal metals, a fundamental question was whether such corrections obey the Wiedemann-Franz (WF) law relating the electrical conductivity $\sigma$ to thermal conductivity $K$ [1]. For non-interacting electrons, the WF law is expected to hold with the inclusion of quantum corrections in the weak localization regime but the numerical coefficient known as the Lorentz number $L_0 = K/\sigma T$, with $T$ the temperature, is reduced when approaching the Anderson localization transition [2]. Away from the transition, deviations have been calculated due to electron-electron interactions [3]. Mesoscopic fluctuations can also lead to violations of the WF law [4]. In the superconducting state the dc electrical resistance vanishes and hence there is no WF law; in fact, approaching the critical temperature from the normal state, superconducting fluctuations lead to a divergent electrical conductivity, whereas they only constitute a finite correction to $K$ [5]. Sufficiently far below the critical temperature, fluctuations are negligible and the leading order expression for the thermal conductivity of a BCS superconductor has been obtained in the early work of Ref. [6]. Further extensions to this result include the effects of electron-phonon scattering [7, 8], strong coupling [9], and paramagnetic impurities [10]. However, to the best of our knowledge, the question of the fate of the weak localization correction to the thermal conductivity in the superconducting state has so far only been addressed for SNS junctions [11] and not for the bulk.

In this paper, we analytically calculate the weak localization correction to the thermal conductivity in s-wave superconductors, including weak anti-localization in a system with spin-orbit scattering. To that end, we extend the formalism used to study diffusion in normal metals, see e.g. Ref. [12], so that it can be used for superconductors as well. Technically, we work with matrix Green’s functions in Nambu space. In the next section, we introduce the model for disordered superconductors to establish our notation. In Sec. III we study diffusion in disordered superconductors in depth by generalizing the ladder approximation. We focus on the A-type diffusons and Cooperons [13], since in a time reversal invariant system, the D-type diffusons do not contribute to thermal transport [9]. In contrast to the normal state, the diffusion constant in the superconducting state depends on energy (measured from the Fermi energy). This energy dependence manifests itself in the condition defining the diffusive regime in the time domain, which is now not simply given by the requirement of time being long compared to the impurity scattering time $\tau_e$. We find that the corresponding time scale in the superconducting state is different for energies below or above an energy scale $\varepsilon_*$ which is a function of the superconducting gap $\Delta$ and the scattering time; for disordered superconductors with $\tau_e \Delta \ll 1$, we find $\varepsilon_* = \sqrt{\Delta/\tau_e}$.

In Sec. IV we make use of the results of the preceding section to calculate the thermal conductivity from the Kubo formula. We recover previous results [9] for the Drude-Boltzmann contribution to the thermal conductivity, which, because of the opening of the superconducting gap, is suppressed as temperature is reduced. As the diffusion constant is energy dependent, we have to specify whether the phase-coherence length or the phase-coherence time is constant in a material in order to evaluate the weak localization correction. We obtain results for both scenarios; in general, the WL correction is temperature dependent. Interestingly, the suppression of the WL correction with decreasing temperature is generally stronger than that of the Drude-Boltzmann term. Of possible experimental interest is the temperature of order $T_\Delta = \frac{\Delta}{k_B}$, defined by $k_B T_\Delta = \Delta(T)$. On one hand, this temperature is sufficiently high that the strong (exponential) suppression
of the (Drude-Boltzmann) thermal conductivity has not yet taken place. On the other hand, for disordered superconductors, this temperature is low enough that most of the weak localization correction is already suppressed. This temperature is therefore optimal in order to observe the deviation of the WL correction in the superconducting state from its normal-state value, as we predict the thermal conductivity to be larger than expected from its value just above \( T_c \). We summarize our findings in Sec. V. A number of details can be found in Appendices A to E.

II. MODEL

The (mean field) Hamiltonian for a superconductor with s-wave pairing can be expressed in the Bogoliubov-de Gennes (BdG) form as [14]

\[
H = \sum_{k} \Psi_k \hat{H}_{\text{BdG}}(k) \Psi_k, \tag{1}
\]

with the Nambu vector

\[
\Psi_k = \begin{pmatrix} c_{k\uparrow} \\ \epsilon_i c_{-k\downarrow} \end{pmatrix}, \tag{2}
\]

where \( c_{k\sigma} \) and \( c_{k\sigma} \) are creation and annihilation operators for electrons with momentum \( k \) and spin \( \sigma \), respectively. The BdG Hamiltonian is given by

\[
\hat{H}_{\text{BdG}}(k) = \epsilon_k \tau_3 - \Delta \tau_1, \tag{3}
\]

where the hat denotes matrices in the Nambu space. Here, \( \epsilon_k = k^2/2m - \mu \), \( m \) is the electron mass, \( \mu = k_F^2/2m \) the Fermi energy with \( k_F \) the Fermi momentum, and \( \tau_i \) the Pauli matrices in Nambu space (we omit hats on these matrices for notational simplicity). For later use, we introduce the basis \( \{|e\}, |h\rangle \} \) in Nambu space, where the states \( |e\rangle \) and \( |h\rangle \) stand for electron and hole respectively. The Bogoliubov-de Gennes Hamiltonian (3) includes the non-interacting electron and hole Hamiltonians in its diagonal terms as well as the pairing term, given by the superconducting order parameter \( \Delta \), in its off-diagonal terms. The retarded and advanced Green's functions are then solutions of

\[
(E - \hat{H}_{\text{BdG}} \pm i0^+) \hat{G}_{E}^{R,A} = 1. \tag{4}
\]

We distinguish the four different elements of the matrix Green's function as follows

\[
\hat{G}_{E}^{R,A} = \begin{pmatrix} G_{E}^{R,A} & F_{E}^{R,A} \\ \bar{F}_{E}^{R,A} & \bar{G}_{E}^{R,A} \end{pmatrix}. \tag{5}
\]

The diagonal terms—that is, the electron and hole Green's functions—describe electron and hole propagation, respectively. The off-diagonal terms, known as anomalous Green's functions, account for particle-hole conversion, i.e., Andreev reflection.

III. PARTICLE DIFFUSION AND WEAK LOCALIZATION

In this section, we study the propagation of particles in disordered conventional superconductors in the weak disorder limit \( k_F l_c \gg 1 \). In this limit, localization affects the transport coefficients, but Anderson localization [15] does not yet take place. Throughout this section, we expand to the superconducting state the diagrammatic treatment of particle propagation in a normal metal presented in chapter 4 of Ref. [12]. The main technical change involves modifying the Feynman diagrams to include all the four components of the superconducting Green's function defined in Eq. (5) [16], see Fig. 1. We define the quantum

![Fig. 1. Feynman diagrams for the time ordered Green's functions in a normal metal and a superconductor. Time evolution occurs from left to right. The four components of the Green's function for a superconductor are distinguished by the arrows at the ends. The matrix formulation in the Nambu formalism is represented by an arrow-less line.](image-url)
The Drude-Boltzmann contribution \( \hat{P}_{0,\omega}(r, r') \) is given by

\[
\hat{P}_{0,\omega}(r, r') = \hat{G}_E^R(r, r') \otimes G^A_{E}(r', r)^T. \tag{10}
\]

The disorder-averaged superconducting retarded Green’s function can be explicitly calculated in momentum space, where it is given by [10]

\[
\overline{G}_E^R(k) = \frac{\bar{E} \tau_0 + \epsilon_k \tau_3 - \Delta \tau_1}{\bar{E}^2 - \epsilon_k^2 - \Delta^2}. \tag{11}
\]

with

\[
\bar{E} = E \left[ 1 + i \frac{1}{2\tau_e \sqrt{E^2 - \Delta^2}} \right], \tag{12}
\]

and

\[
\bar{\Delta} = \Delta \left[ 1 + i \frac{1}{2\tau_e \sqrt{E^2 - \Delta^2}} \right]. \tag{13}
\]

The Fourier transform of Eq. (11) into real space can then be calculated in the limit \( \mu \gg \epsilon, \Delta \), with \( \epsilon = \sqrt{E^2 - \Delta^2} \), by linearizing the spectrum around \( k_F \). We provide the explicit result in the two dimensional case, since it will be of particular interest for the weak localization correction. In the limit \( k_F R \gg 1 \) with \( R = r' - r \), we have (\( E > 0 \))

\[
\overline{G}_E^R(r, r') = \frac{m}{\sqrt{2\pi k_F R}} e^{i \frac{E}{k_F R}} e^{-\frac{m}{\pi} \left[ \frac{E}{e} \cos \left( k_F R + \frac{3\pi}{4} \right) \tau_0 - \sin \left( k_F R + \frac{3\pi}{4} \right) \tau_3 - \frac{\Delta \delta}{\epsilon} \cos \left( k_F R + \frac{3\pi}{4} \right) \tau_1 \right]}.
\]

The advanced Green’s function in real space can be arrived at using Eq. (8). Having obtained the disorder-averaged superconducting Green’s functions in real space, \( \hat{P}_{0,\omega}(r, r') \) can be found from Eq. (10) In the next section we use \( \hat{P}_{0,\omega}(r, r') \) to calculate the diffusion.
B. Diffusion in disordered superconductors: The diffuson

The diffuson \( \tilde{P}_{d,\omega}(r, r') \) is the classical probability of propagation from \( r \) to \( r' \) accounting for all paths including at least one scattering event. Summation over these paths is performed in the ladder approximation, as sketched in Fig. 2 giving the equation

\[
\tilde{P}_{d,\omega}(r, r') = \int d^d r_1 \int d^d r_2 \tilde{P}_{0,\omega}(r, r_1) \tilde{G}(r_1, r_2) \tilde{P}_{0,\omega}(r_2, r').
\]

The Drude-Boltzmann factors account for the trajectory before the first scattering event and after the last one, at \( r_1 \) and \( r_2 \) respectively. The structure factor \( \tilde{G}(r_1, r_2) \) includes all scattering events. In our formalism, it is a 4x4 matrix defined self-consistently by

\[
\tilde{G}(r_1, r_2) = \gamma_e \tilde{U}_0(\delta(d)(r_1 - r_2) + \int d^d r'' \tilde{P}_{0,\omega}(r_1, r'') \tilde{G}(r'', r_2)),
\]

see the bottom half of Fig. 2. The Drude-Boltzmann contribution decays on a length scale of the order of the mean free path \( l_c \), cf. Eq. (14). Here we are interested in the diffusive regime, where the length scale \( \lambda \) over which the structure factor varies is much longer than the mean free path, \( \lambda \gg l_c \). We can then approximate \( \tilde{G}(r_1, r_2) \approx \tilde{G}(r, r') \). In this limit, Eq. (15) can be approximately rewritten as

\[
\tilde{P}_{d,\omega}(r, r') = \langle \tilde{P}_0 \rangle_r \tilde{G}(r, r') \langle \tilde{P}_0 \rangle_r,
\]

with \( \langle \tilde{P}_0 \rangle_r \equiv \langle \tilde{P}_{0,\omega=0} \rangle_r \) and

\[
\langle \tilde{P}_{0,\omega} \rangle_r \equiv \int d^d r' \tilde{P}_{0,\omega}(r, r').
\]

Diffusion takes place at sufficiently long times beyond the scale \( \tau_{\min} \) so that terms of the order \( (\omega \tau_{\min})^2 \) and \( (\omega \tau_{\min})(l_c/\lambda)^2 \) can be neglected in comparison to those of order \( (\omega \tau_{\min}) \) and \( (l_c/\lambda)^2 \), respectively. For the diffusion, in a normal metal the scale \( \tau_{\min} \) is simply given by \( \tau_e = l_c/v_F \). Analogously, for a superconductor, we obtain the scattering time

\[
\tau_s = \frac{l_c}{v_g}, \quad v_g = v_F \frac{\epsilon}{|E|}
\]

with \( v_g \) the group velocity of the quasiparticles. However, we find that diffusion only sets in after the longer time

\[
\tau_{\min} = \max \left\{ \tau_s, \frac{\Delta}{e^2/\epsilon} \right\}.
\]

The second scale \( \Delta/e^2 \) appears in order that the diffusive modes [first two entries of Eq. (25) below] are decoupled from the massive modes (last two entries). The definition in Eq. (20) reduces to \( \tau_{\min} = \tau_s \) in the normal state, while in the superconducting one we find

\[
\tau_{\min} = \begin{cases} \frac{\Delta}{\epsilon}, & E < E_s, \\ \tau_s, & E > E_s. \end{cases}
\]

where \( E_s \) is defined as the energy at which \( \tau_s = \Delta/e^2 \). The magnitude of \( E_s \) is sensitive to the disorder strength in the superconductor. Writing \( E_s = \Delta + \epsilon_s \), we obtain

\[
\epsilon_s \simeq \begin{cases} \sqrt{\frac{\Delta}{\epsilon_s}}, & \tau_s \Delta \ll 1, \\ \frac{1}{2\Delta\tau_s}, & \tau_s \Delta \gg 1. \end{cases}
\]

where the condition \( \tau_s \Delta \ll 1 \) identifies the dirty regime, in which \( \epsilon_s \gg \Delta \), and \( \tau_s \Delta \gg 1 \) the clean case, where \( \epsilon_s \ll \Delta \). We will discuss in Sec. IV the relevance of this and other energy scales to the thermal conductivity.

To obtain the diffusion equation for the structure factor \( \tilde{G}(r) \), we expand the latter up to second order in \( r'' - r_1 \) around \( r'' = r_1 \) in the left hand side of Eq. (16). That equation can then be cast in the form

\[
\tilde{M}(r) \tilde{G}(r, r') = \gamma_e \delta(d)(r' - r),
\]

with the matrix operator

\[
\tilde{M}(r) = \tilde{U}_0^{-1} - \gamma_e \langle \tilde{P}_0 \rangle_r - \frac{\gamma_e}{2d} (\tilde{v}^2 \tilde{P}_0)_{r} \nabla_r^2,
\]

We have again neglected terms of order \( (\omega \tau_{\min})(l_c/\lambda)^2 \) by evaluating \( \langle \tilde{v}^2 \tilde{P}_0 \rangle_r \) at \( \omega = 0 \). We denote by removing the \( \omega \) subscript the Green’s function in real space [we remind that in the diffusive regime we only need to keep terms of order \( (\omega \tau_{\min})^0 \) and \( (\omega \tau_{\min})^1 \)]. Using the relation \( \langle \tilde{v}^2 \tilde{P}_0 \rangle_r = 2l_c^2 \langle \tilde{P}_0 \rangle_r \) and the definition of the potential matrix \( \tilde{U}_0 \) [see the text after Eq. (6)], the matrix operator \( \tilde{M}(r) \) is obtained straightforwardly.

We wish to study the structure of \( \tilde{M}(r) \) to understand the diffusive modes of \( \tilde{G}(r, r') \). It is convenient to introduce the states \( |a_\pm \rangle = \frac{1}{\sqrt{2}}(|e, e\rangle \pm |h, h\rangle) \) and \( |b_\pm \rangle = \frac{1}{\sqrt{2}}(|e, h\rangle \pm |h, e\rangle) \). We then work in the basis \( B = \{|a_-, \rangle, \cos(\theta) |a_+ \rangle \pm \sin(\theta) |b_+ \rangle, \cos(\theta) |b_- \rangle \pm \sin(\theta) |b_+ \rangle, \} \), where \( \Delta/E = \tan(\theta) \). In this basis, the structure of \( \tilde{M}(r) \) simplifies and the behavior of the diffusive modes can be singled out. Indeed, we find in the diffusive regime the result

\[
\tilde{M}(r) = \text{diag} \left\{ \tau_s D, \frac{\Delta}{e^2/\epsilon}, \right\}
\]

with \( D = -i\omega - D_s \nabla_r^2 \). Here \( D_s \) is the superconducting diffusion constant

\[
D_s = \frac{\nu_g l_c}{d}
\]

which, similarly to the scattering rate above, is energy dependent. On the other hand, the mean free path, proportional to \( \sqrt{D_s \tau_s} \), remains independent of energy and
equal to that in the normal state. These findings are in agreement with those in Ref. [20].

Equation (25) shows that in the diffusive limit \( \tilde{M}_\omega (r) \) is a diagonal matrix with two diffusive and two fast modes. We will neglect the fast modes and focus on the diffusive ones. To this end, we define \( M_\omega (r) \) as the 2 \( \times \) 2 matrix obtained by projecting \( \tilde{M}_\omega (r) \) into the subspace spanned by \( \{ a_- \cos (\theta ) | a_+ \sin (\theta ) \} \). According to [Eq. (23)], the structure factor \( \tilde{G}_\omega (r, r') \) in this subspace satisfies the equation

\[
\left[ \begin{array}{cc}
1 & 0 \\
0 & e^{i \hat P (r') - i \hat P (r)} D
\end{array} \right] \tilde{G}_\omega (r, r') = \gamma \delta^{(d)} (r' - r), \tag{27}
\]

where the terms in square brackets are the matrix \( \tilde{M}_\omega (r) \).

We can rewrite Eq. (17) using \( \tilde{G}_\omega (r, r') \) as

\[
\hat P_{d,\omega} (r, r') = \hat P_c \tilde{G}_\omega (r, r') \hat P_T,
\tag{28}
\]

where \( \hat P_c \) is the matrix with dimension 4 \( \times \) 2 that encompasses the first two columns of \( \hat P \omega \). This matrix has the useful property \( \gamma \hat P_c \hat P_T = 1 \). The diffusion thus found is a rank two matrix that takes the form \( \hat P_{d,\omega} (r, r') = \text{diag} \{ \hat P_{d,\omega} (r, r')_{1,1}, \hat P_{d,\omega} (r, r')_{2,2}, 0, 0 \} \) in the previously introduced basis \( B \); it has the useful property \( \gamma \hat P_c \hat P_T = 1 \).

\[
\hat P_{d,\omega} (r, r') = \text{diag} \{ \hat P_{d,\omega} (r, r')_{1,1}, \hat P_{d,\omega} (r, r')_{2,2} \} \text{ follows a diffusion equation given by}
\]

\[
\frac{v_F}{2 \pi \rho_0 \nu_g} \left( \begin{array}{cc}
1 & 0 \\
0 & e^{i \hat P (r') - i \hat P (r)} D
\end{array} \right) \hat P_{d,\omega} (r, r') = \delta^{(d)} (r' - r). \tag{29}
\]

The result resembles the diffusion in the normal metal, but in the superconducting state the diffusion constant and the scattering time depend on the group velocity \( v_g \) which is no longer equal to the Fermi velocity [Eq. (29)].

After applying the temporal Fourier transform, we obtain a direct relation between the probabilities of diffusion in the superconducting and normal states

\[
\hat P_{c,\omega} (r, r') = \int d^d r_1 \int d^d r_2 \left( \frac{G_{E,\omega} (r, r_1)}{G_E (r_1, r_1)^T} \right) \hat P_{c,\omega} (r_1, r_2) \left( \frac{G_{E,\omega} (r_2, r')}{G_E (r_2, r_2)^T} \right), \tag{32}
\]

\[
\hat G_{c,\omega} (r_1, r_2) = \gamma \hat U_c \left[ \delta^{(d)} (r_1 - r_2) + \int d^d r'' \left( \frac{G_{E,\omega} (r_1, r'')}{G_E (r_1, r_1)^T} \right) \hat G_{c,\omega} (r'', r_2) \right]. \tag{33}
\]

Since the disorder-averaged Green’s functions decay exponentially in real space [cf. Eq. (14)], it can already be seen above that the Cooperon is exponentially suppressed in \( |r' - r|/l_c \). We can simplify Eq. (33) by not-
Here, we define $R$ (in Eq. (32) which, neglecting terms of order $(\omega \tau_{\text{min}})^{2}$, is then given by $\Gamma_{\omega}(r, r') = \hat{\Gamma}_{\omega}(r, r')$ in Eq. (32) when we have assumed $\omega \tau_{\text{min}} \ll 1$, affects the return probability $P_{d, \omega}(r, r')$. Based on that condition, the diffusive behavior of the system breaks down when considering very short timescales. On the other hand, on long time scales diffusion is limited by the phase-coherence time $\tau_{\phi}$ (which can be related to the phase-coherence length via $L_{\phi} = \sqrt{D_{\phi} \tau_{\phi}}$). The return probability at zero frequency is then given by

$$\hat{P}_{d}(r, r') = \int_{0}^{\tau_{\phi}} dt \hat{P}_{d}(r, r; t),$$

(39)

where $\hat{P}_{d}(r, r'; t)$ is given in Eq. (30). Solving the diffusion equation (31) in $d$-dimensional free space we obtain

$$\hat{P}_{d}(r, r'; t)_{1,1} = \frac{2 \pi \rho_{0} v_{F}}{v_{F}} \left( \frac{1}{(4 \pi D_{\tau} t)^{d/2}} e^{-R^{2}/(4 D_{\tau} t)} \right),$$

(40)

where, as mentioned above, we focus for later use on the element $\hat{P}_{d}(r, r'; t)_{1,1} = \langle a_{-} | \hat{P}_{d}(r, r'; t) | a_{-} \rangle$. Inserting this result into Eq. (39) and performing the integral yields the return probability at zero frequency

$$\hat{P}_{d}(r, r')_{1,1} = \frac{4 \pi \rho_{0} D}{(4 \pi)^{d/2} D_{\tau}^{1-d/2}} \left( \frac{1}{\sqrt{\tau_{\min}} - \sqrt{\tau_{\phi}}} \right) \left( \frac{1}{\sqrt{\tau_{\min}} - \frac{\sqrt{\tau_{\phi}}}{\tau}} \right),$$

(41)

We remind that, unlike in the normal state where $\tau_{\min} = \tau_{c}$, in the superconducting state $\tau_{\min}$ of Eq. (20) is an energy-dependent quantity. The dependence is qualitatively different in the two regimes separated by the energy $E_{\ast}$ [see Eq. (22)], and the energy $E_{\ast}$ itself takes different values in the clean and dirty regimes.

IV. THERMAL CONDUCTIVITY

In this section, we connect the results of the previous section concerning particle propagation to the thermal conductivity which is a physical observable. We obtain quantum corrections to the known results for the Drude-Boltzmann contribution [3]. In particular, we derive explicit results for the weak localization correction to the thermal conductivity in two dimensions. Interestingly, in the superconducting state this correction displays a temperature dependence that differs from that in the normal...
with \( \tau \) time reversal symmetry it can be shown that the

D-ε\( T \) state (or its simple extension to be discussed below). Different regimes arise depending on the relations between temperature \( T \), order parameter \( \Delta(T) \), and the energy scale \( \varepsilon \approx E_c - \Delta \).

Our starting point is Kubo’s formula for the thermal conductivity \( K(T) \) \cite{20}; it can be written in terms of a product of Green’s functions \[ K = \frac{1}{4\pi k_B T^2 m^2} \int_\Delta dE \frac{E^2}{\cosh^2 \left( \frac{E}{2k_B T} \right)} I, \] (42)

where \( k_B \) is the Boltzmann constant and we take \( x \) as the direction of the temperature gradient (and hence of heat propagation in an isotropic material, to which we restrict our attention). Its diagrammatic representation can be seen in Fig. 4.

Using the above expression, we rewrite \( I = I_A - I_D \) as the difference between two integrals with

\begin{align*}
I_A &= \text{Re} \int \frac{d^4k}{(2\pi)^d} \frac{d^4k'}{(2\pi)^d} x_z x'_z \frac{1}{2} \text{Tr} \left[ \tau_3 \text{Im} \tilde{G}_E^R(k, k') \tau_3 \text{Im} \tilde{G}_E^R(k', k) \right], \\
I_D &= \text{Re} \int \frac{d^4k}{(2\pi)^d} \frac{d^4k'}{(2\pi)^d} x_z x'_z \frac{1}{2} \text{Tr} \left[ \tau_3 \text{Im} \tilde{G}_E^R(k, k') \tau_3 \text{Im} \tilde{G}_E^R(k', k) \right].
\end{align*}

(44)

In the regime \( \mu \gg \varepsilon \Delta \) discussed after Eq. (18), we can approximate \( k_z \approx k_F \mu _z \), where \( \mu \) is the unit vector on the Fermi surface, and similarly for \( k'_z \). Therefore, only the relative angle between the two momenta \( k, k' \) matters. Indeed, we discuss below the dependence of the disorder-averaged product of Green’s functions on the relative orientation of \( k \) and \( k' \).

Once this dependence is known, the integrals \( I_A \) and \( I_D \) can then be related to the so called A-type and D-type diffusive modes, respectively \cite{13}. The A-type modes contribution \( I_A \) is proportional to the probability of diffusion \( \tilde{P}_\omega (r, r') \) studied in Sec. III and the D-type modes one to \( \tilde{P}^D(r, r') = \tilde{G}_E^R(r, r') \otimes \tilde{G}_E^R(r', r) \). In systems with time reversal symmetry it can be shown that the D-type modes do not contribute to the thermal conductivity, i.e., \( I_D = 0 \) \cite{9}. With \( I = I_A \) we can calculate the thermal conductivity using the results from the previous section. We only briefly sketch how to use those results here; more details on how to relate transport coefficients to the propagation probability can be found in Ref. [12].

As done for the total probability of diffusion, we divide the different contributions to the thermal conductivity into \( K_0, K_d \) and \( K_c \) and define as such the integrals \( I_0, I_d \) and \( I_c \). These integrals are related to the quantities \( P_0(r, r')_{1,1}, P_d(r, r')_{1,1}, \) and \( P_c(r, r')_{1,1} \) defined in the previous section; here we use the identity \( \tilde{P}(r, r')_{1,1} = \text{Tr} \left[ \tau_3 \tilde{G}_E^R(r, r') \tau_3 \tilde{G}_E^R(r', r) \right] / 2 \), where we dropped the subscript \( \omega = 0 \) to simplify the notation and we remind that the term on the left hand side is defined as \( \tilde{P}(r, r')_{1,1} = (a_- | \tilde{P}(r, r') | a_-) \).

The Drude-Boltzmann integral \( I_0 \) represents propagation in a disordered medium without any scattering event taking place. In the absence of scattering, the initial and final momenta of the Green’s functions are the same, with \( \tilde{G}_E^R(k, k') = \tilde{G}_E^R(k', k) \otimes \tilde{G}_E^R(k', k) \). Then the angular integration in momentum space is equivalent to taking the product of momenta out of the integral as \( k_F^2 / d \), and the relation between \( I_0 \) and \( \tilde{P}_0(r, r')_{1,1} \) can be obtained by going into real space, using the Fourier transform for a translational invariant system

\[ \int \frac{d^4k}{(2\pi)^d} \tilde{G}_E^R(k) \otimes \tilde{G}_E^R(k)^T = \int d^d r \tilde{P}_0(r, r'). \] (46)

The Drude-Boltzmann integral is then

\[ I_0 = \frac{k_F^2}{d} \text{Re} \int d^d r \tilde{P}_0(r, r')_{1,1}, \] (47)

and using Eq. (18) see also Eq. (B6) we obtain \( I = k_F^2 / d \gamma_c \). Inserting the result into Eq. (42) we obtain the Drude-Boltzmann contribution to the thermal conductivity

\[ K_0 = \frac{D p_0}{2 k_B T^2} \int_\Delta dE \frac{E^2}{\cosh^2 \left( \frac{E}{2 k_B T} \right)} , \] (48)

where \( D = v_F l_c / d \) is the diffusion constant in the normal state. This formula agrees with previous calculations \cite{9}. It is equivalent to the result in the normal state with the sole difference that only states with energy \( E > \Delta \) contribute. The absence of states below the gap is reflected in the lower limit of the integral and leads to the exponential suppression of \( K_0 \) at temperatures \( k_B T \ll \Delta \).

For the diffusion integral \( I_d \), we find simply \( I_d = 0 \). This result is valid for isotropic scattering by impurities: the initial and final momenta of the Green’s functions \( (k, k') \) respectively have uncorrelated directions after a large number of scattering events, which leads to the vanishing of the angular integration in Eq. (44). Anisotropic scattering would result in the substitution of the scattering time \( \tau_c \) with the transport time in Eq. [18] [12].
(\(\tau_e\) enters that equation via the mean free path in the diffusion constant).

Similar considerations to those above make it possible to relate \(I_c\) to \(\hat{P}_c(r, r')_{1,1}\). The Cooperon accounts for an enhanced probability of a particle to return to its initial point; therefore, its initial and final momenta will be approximately opposite to each other. The integrand of \(I_c\) is then sharply peaked around \(\mathbf{k} = -\mathbf{k}'\), and can be approximated to be proportional to \(\delta^{(2)}(\mathbf{k} + \mathbf{k}')\) \(^{23}\). We again take the product of momenta out of the integral, and going over to real space yields

\[
I_c = -\frac{k_F^2}{\pi} \text{Re} \int d^4r' \hat{P}_c(r, r')_{1,1,1,1}. \tag{49}
\]

After substituting Eq. (37) into the above expression we have

\[
I_c = -\frac{k_F^2}{\pi} \text{Re} \hat{P}_d(r, r)_{1,1} \int d^4R \hat{f}(R)_{1,1}. \tag{50}
\]

Using the expressions for \(\hat{f}(R)_{1,1}\) given in Eq. (38) we find that for all dimensions

\[
\int d^4R \hat{f}(R)_{1,1} = \frac{\tau_e}{\pi \rho_0}, \tag{51}
\]

and inserting these results into Eq. (42) we arrive at

\[
K_c = -\frac{D}{4\pi^2 k_B T^2 \rho_0} \int_0^\infty dE \frac{E^2}{\cosh^2 \left( \frac{E}{3k_B T} \right)} \hat{P}_d(r, r)_{1,1,1,1}, \tag{52}
\]

where the return probability \(\hat{P}_d(r, r)_{1,1,1,1}\), given by Eq. (41), is a function of energy \(E\). This energy dependence leads to a temperature dependence of \(K_c\) which we study in the following for two dimensions. We note that it is crucial to retain this energy dependence. Neglecting the energy dependence of the return probability, we would find the incorrect result \(K_c/K_0 = -P_d(r, r)_{1,1,1,1}/2\pi^2 \rho_0^2\) and the temperature dependence of \(K_c\) would simply follow from the one in the normal state.

A. Regimes for the WL correction to the thermal conductivity

As remarked above, the dependence of the return probability \(P_d(r, r)_{1,1,1,1}\) on energy makes it possible for the WL correction \(K_c\) to the thermal conductivity to have a temperature dependence that differs from that of the main (Drude-Boltzmann) contribution \(K_0\). Here we explore when such a deviation takes place and under which conditions it could be observable. To this aim, let us introduce the temperature \(T_\Delta\) defined by \(k_B T_\Delta = \Delta(T_\Delta)\); for our purposes, the temperature dependence of the gap on temperature is approximately captured by the interpolation formula \(^{26}\)

\[
\Delta(T) \approx 1.76 k_B T_c \tanh \left( 1.74 \sqrt{\frac{T_c}{T} - 1} \right), \tag{53}
\]

with \(T_c\) the critical temperature of the superconductor. From this expression we find \(T_\Delta \approx 0.9 T_c\). Clearly, both \(K_0\) and \(K_c\) are exponentially suppressed in the low-temperature regime \(T \ll T_\Delta\), see Eqs. (48) and (52), making their accurate measurement challenging. Therefore, the high-temperature regime \(T_\Delta \lesssim T < T_c\) is most relevant in order to observe the effects of weak-localization. For completeness, we consider both regimes below (details of the calculations are presented in Appendix D).

A second relevant temperature scale, denoted by \(T_s\), can be defined via the equation \(k_B T_s = \varepsilon_s(T_s)\), where \(\varepsilon_s\) depends on temperature through the gap \(\Delta(T)\), see Eq. (22). For dirty superconductors, \(\tau_e \Delta(0) \ll 1\), we have \(T_s \approx T_c\); while for clean ones, \(\tau_e \Delta(0) \gg 1\), we find \(T_s \ll T_c\), indicating that qualitatively different behaviors can be expected in the two cases. Finally, with regard to the effect of phase coherence on \(K_c\), we consider two possibilities, namely an energy independent coherence time \(\tau_0\) or an energy independent coherence length \(L_\phi = \sqrt{D \tau_0} \). These two possibilities are equivalent in the normal state, but in the superconducting one they are not, due to the energy dependence of the diffusion constant \(D\), Eq. (26).

1. High-temperature regime

In the high-temperature regime \(T \gtrsim T_\Delta\), in order to find the leading contributions to the heat conductivity, we approximate \(k_B T \gg \Delta\). Then the WL correction in this regime does not depend on the gap \(\Delta\). Moreover, for a superconductor in the clean limit, since \(\varepsilon_s \ll \Delta\), the relative correction coincides with the one in the normal state \(^{27}\),

\[
\frac{K_c}{K_0} = \frac{-1}{\pi k_F l_c} \left\{ \ln \left( \frac{\tau_e}{\tau} \right) + 2 \ln \left( \frac{k_B T}{\varepsilon_s l_c} \right) \right\} \quad (\tau_0 \text{ fixed}), \quad (L_\phi \text{ fixed}). \tag{54}
\]

The same expressions hold for a dirty superconductor sufficiently close to \(T_c\), so that \(\varepsilon_s \ll k_B T\), but since, as mentioned above, \(T_s \sim T_c\), this result has very limited applicability. More interestingly, there exists an intermediate regime, \(T_\Delta \lesssim T \lesssim T_s\), in which the WL correction depends on the ratio \(k_B T/\varepsilon_s\),

\[
\frac{K_c}{K_0} = \frac{-1}{\pi k_F l_c} \left\{ \ln \left( \frac{\tau_e}{\tau} \right) + 2 \ln \left( \frac{k_B T}{\varepsilon_s l_c} \right) \right\} \quad (\tau_0 \text{ fixed}), \quad (\tau_0 \text{ fixed}). \tag{55}
\]

Note that, in the high-temperature regime, the temperature dependence of the WL correction is insensitive to the assumption of energy-independent dephasing time vs. length; this can be traced back to the fact that at the relevant energy scale (given by temperature), we have for the group velocity \(v_g = v_F\), see Eq. (19).
2. Low-temperature regime

In the low temperature regime $T \ll T_\Delta$ we have $k_B T \ll \Delta$, which results in the exponential suppression of both $K_0$ and $K_c$ discussed above. Their ratio, however, is not exponentially suppressed. Indeed, the WL correction for a dirty superconductor is given by

$$\frac{K_c}{K_0} = \frac{-1}{\pi k_F l_c} \left[ \ln \left( \frac{\tau_\phi}{\tau_e} \right) + \ln \left( \frac{\Delta k_B T}{\epsilon^2 \alpha} \right) \right] \quad (\tau_\phi \text{ fixed}),$$

and

$$\frac{K_c}{K_0} = \frac{-1}{\pi k_F l_c} \left[ 2 \ln \left( \frac{\tau_\phi}{\tau_e} \right) + \ln \left( \frac{\Delta^3 k_B T}{\epsilon^2 \alpha^3} \right) \right] \quad (L_\phi \text{ fixed}).$$

In both cases, at the cross-over temperature $T_\Delta$ the correction agrees with that found in the high-temperature regime. However, the temperature dependence is now sensitive to the assumption of energy-independent dephasing time/length.

For a clean superconductor in the regime $T_s < T < T_\Delta$, the normalized WL correction is

$$\frac{K_c}{K_0} = \frac{-1}{\pi k_F l_c} \left[ \ln \left( \frac{\tau_\phi}{\tau_e} \right) + \frac{1}{2} \ln \left( \frac{\Delta k_B T}{\epsilon^2 \alpha} \right) \right] \quad (\tau_\phi \text{ fixed}),$$

and

$$\frac{K_c}{K_0} = \frac{-1}{\pi k_F l_c} \left[ 2 \ln \left( \frac{\tau_\phi}{\tau_e} \right) + \ln \left( \frac{\Delta^3 k_B T}{\epsilon^2 \alpha^3} \right) \right] \quad (L_\phi \text{ fixed}).$$

We note that, according to Eqs. (54) and (57), for $T > T_s$ and assuming energy-independent dephasing length, the WL correction in the clean case coincides with that in the normal state. This finding resembles that for the WL correction to the heat conductance of superconductor/normal/superconductor junctions with short (shorter than dephasing length) normal part in the absence of phase gradient and gap differences [11]. In that case, the latter two assumptions ensure that the transmission probability of quasiparticles excitations through the junction is independent of energy. Similarly here, the assumptions of energy-independent dephasing length and sufficiently high temperature ensure that the return probability of Eq. (41) is energy-independent over the relevant energy range. For a clean superconductor there exists also a regime where $k_B T \ll \varepsilon_s$, where this energy independence does not hold. This regime is calculated in Appendix D, but we do not discuss it here further as it has a limited validity at temperatures where the thermal conductivity is strongly suppressed [28].

V. SUMMARY AND DISCUSSION

In this work, we have calculated the weak localization correction to the thermal conductivity in conventional disordered superconductors. As our starting point, we have studied diffusion with the help of a general formalism based on semiclassical Green’s functions and their corresponding matrix expressions in Nambu space, see Sec. III. The formalism can be straightforwardly expanded to tackle systems with different symmetries; as an example, in Appendix E we investigate diffusion in the presence of weak spin-orbit scattering.

![Diagram](image_url)
For both clean ($\tau_c \Delta >> 1$) and dirty ($\tau_c \Delta \ll 1$) superconductors, we have considered the high ($T> T_\Delta$) and low ($T<T_\Delta$) temperature regimes, as summarized in Fig. 5 for two dimensions. We highlight the regime $T_\Delta < T < T_c \approx T_s$, which exists only in dirty superconductors, as the most interesting for the experimental verification of our results. In this temperature range, the thermal conductivity is not yet exponentially suppressed but, at the same time, most of the decrease in the magnitude of the WL correction has taken place, see Fig. 5(a) and Eq. (23). An interesting question for future research is the generalization of the approach presented here to calculate transport properties in disordered $d$-wave superconductors $^{30-32}$, for which the weak localization effect $^{33}$, becomes the relative momentum squared $q^2$, and $\hat{M}_\omega(q)$ can be inverted to obtain

$$\hat{\Pi}_\omega(q) = \gamma_e \hat{M}_\omega(q)^{-1}. \quad (B1)$$

After calculating the inverse of $\hat{M}_\omega(q)$ explicitly, we can simplify it in the diffusive regime discussed in Sec. III B and $\hat{\Pi}_\omega(q)$ is reduced to a rank two matrix whose non-zero elements correspond to $\hat{\Gamma}_\omega(q) = \gamma_e M_\omega(q)^{-1}$, given in the basis $B_2 = \{|a_-\rangle, \cos(\theta)|a_+\rangle + \sin(\theta)|b_+\rangle\}$ by

$$\hat{\Gamma}_\omega(q) = \gamma_e \left( \hat{D}_{\alpha q^2-i\omega} ^0 1 \hat{D}_{\alpha q^2-i\omega} ^{-1} \right). \quad (B2)$$

The diffuson, given by

$$\hat{\Pi}_{\omega}(q) = \hat{\Pi}_{\omega}(q) \hat{\Pi}_{\omega}(q) \hat{P}_{\omega}(q), \quad (B3)$$

can be approximated in the limit of small relative momentum $q$ and relative frequency $\omega$ as

$$\hat{\Pi}_{\omega}(q) = \hat{\Pi}_{\omega}(0) \hat{\Pi}_{\omega}(q) \hat{P}_{\omega}(0), \quad (B4)$$

where $\hat{P}_{\omega}(q = 0) = \langle \hat{P}_0 \rangle_r$, given in the original Nambu basis $[\text{defined after Eq. (9)}]$ by

$$\langle \hat{\Pi}_0 \rangle_r = \frac{1}{2\gamma_e \epsilon^2} \times \left( \begin{array}{cccc} 2E^2 - \Delta^2 & -\Delta E & -\Delta E & \Delta^2 \\ -\Delta E & \Delta^2 & \Delta^2 & -\Delta E \\ -\Delta E & \Delta^2 & -\Delta E & -\Delta E \\ \Delta^2 & -\Delta E & -\Delta E & 2E^2 - \Delta^2 \end{array} \right), \quad (B5)$$

and in its eigenbasis $\hat{B}$ by

$$\langle \hat{\Pi}_0 \rangle_r = \frac{1}{\gamma_e} \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & E^2 + \Delta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right). \quad (B6)$$

The diffuson can then be calculated by direct matrix multiplication. We obtain

$$\hat{\Pi}_d(q) = \frac{1}{\tau_s D_{\alpha q^2-i\omega} \langle \hat{P}_0 \rangle_r}, \quad (B7)$$

Let us now consider the diffusion equation followed by $\hat{P}_{d,\omega}(r, r')_{1,1} \equiv \langle a_-| \hat{P}_{d,\omega}(r, r')|a_-\rangle$, given by the first element of Eq. (29)

$$\frac{v_F}{2\pi \rho_0 v_g} (\nabla^2_r - i\omega) \hat{P}_{d,\omega}(r, r')_{1,1} = \delta(d)(r' - r). \quad (A6)$$

After spatial integration, we find the normalization factor $A = v_F / 2\pi \rho_0 v_g$.

**Appendix B: Superconducting diffuson in momentum space**

To work in momentum space, we start by taking the Fourier transform Eq. (23). The Laplace operator $\nabla^2_r$ becomes the relative momentum squared $q^2$, and $\hat{M}_\omega(q)$ can be inverted to obtain

$$\hat{\Pi}_\omega(q) = \gamma_e \hat{M}_\omega(q)^{-1}. \quad (B1)$$

After calculating the inverse of $\hat{M}_\omega(q)$ explicitly, we can simplify it in the diffusive regime discussed in Sec. III B and $\hat{\Pi}_\omega(q)$ is reduced to a rank two matrix whose non-zero elements correspond to $\hat{\Gamma}_\omega(q) = \gamma_e M_\omega(q)^{-1}$, given in the basis $B_2 = \{|a_-\rangle, \cos(\theta)|a_+\rangle + \sin(\theta)|b_+\rangle\}$ by

$$\hat{\Gamma}_\omega(q) = \gamma_e \left( \hat{D}_{\alpha q^2-i\omega} ^0 1 \hat{D}_{\alpha q^2-i\omega} ^{-1} \right). \quad (B2)$$

The diffuson, given by

$$\hat{\Pi}_{\omega}(q) = \hat{\Pi}_{\omega}(q) \hat{\Pi}_{\omega}(q) \hat{P}_{\omega}(q), \quad (B3)$$

can be approximated in the limit of small relative momentum $q$ and relative frequency $\omega$ as

$$\hat{\Pi}_{\omega}(q) = \hat{\Pi}_{\omega}(0) \hat{\Pi}_{\omega}(q) \hat{P}_{\omega}(0), \quad (B4)$$

where $\hat{P}_{\omega}(q = 0) = \langle \hat{P}_0 \rangle_r$, given in the original Nambu basis $[\text{defined after Eq. (9)}]$ by

$$\langle \hat{\Pi}_0 \rangle_r = \frac{1}{2\gamma_e \epsilon^2} \times \left( \begin{array}{cccc} 2E^2 - \Delta^2 & -\Delta E & -\Delta E & \Delta^2 \\ -\Delta E & \Delta^2 & \Delta^2 & -\Delta E \\ -\Delta E & \Delta^2 & -\Delta E & -\Delta E \\ \Delta^2 & -\Delta E & -\Delta E & 2E^2 - \Delta^2 \end{array} \right), \quad (B5)$$

and in its eigenbasis $\hat{B}$ by

$$\langle \hat{\Pi}_0 \rangle_r = \frac{1}{\gamma_e} \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & E^2 + \Delta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right). \quad (B6)$$

The diffuson can then be calculated by direct matrix multiplication. We obtain

$$\hat{\Pi}_d(q) = \frac{1}{\tau_s D_{\alpha q^2-i\omega} \langle \hat{P}_0 \rangle_r}, \quad (B7)$$

Let us now consider the diffusion equation followed by $\hat{P}_{d,\omega}(r, r')_{1,1} \equiv \langle a_-| \hat{P}_{d,\omega}(r, r')|a_-\rangle$, given by the first element of Eq. (29)

$$\frac{v_F}{2\pi \rho_0 v_g} (\nabla^2_r - i\omega) \hat{P}_{d,\omega}(r, r')_{1,1} = \delta(d)(r' - r). \quad (A6)$$

After spatial integration, we find the normalization factor $A = v_F / 2\pi \rho_0 v_g$.
which corresponds to a rank two matrix that can be written as

\[
\hat{P}_{d,\omega}(q) = \frac{2\pi\rho v_a}{v_F} \left( \frac{D, q^2 - \omega}{D, q^2 - \omega} \frac{E^2 + \Delta^2}{\epsilon^2 + \Delta^2} \right). \tag{B8}
\]

in the basis \( \tilde{B}_2 = \{|a_-, \cos(\theta)|a_+\rangle, \sin(\theta)|b_+\rangle\} \). This is again equivalent to the result obtained by solving Eq. (29) after performing a Fourier transform into momentum space.

**Appendix C: Superconducting Cooperon**

Here we work out explicitly the relation between Cooperon and diffusion in the superconducting state. Since we are interested in the diffusive regime, the matrix \( \Gamma_{c,\omega}(r, r) = \Gamma_{d,\omega}(r, r) \) can be simplified as a 2×2 matrix that follows Eq. (27) in the subspace spanned by \( \tilde{B}_2 = \{|a_-, \cos(\theta)|a_+\rangle, \sin(\theta)|b_+\rangle\} \). As done in Eq. (28), we rewrite Eq. (34) as

\[
\hat{P}_{c,\omega}(r, r') = \hat{F}_\omega(R)\tilde{\Gamma}_{\omega}(r, r)\tilde{F}^T(R), \tag{C1}
\]

where \( \hat{F}_\omega(R) \) is defined, similarly to \( \hat{P}_\omega \), as the matrix containing the first two columns of \( \hat{F}(R) \) in the \( B \) basis. By substituting the expression for \( \hat{F}_\omega(r, r) \) as a function of \( \hat{P}_{d,\omega}(r, r) \) obtained from Eq. (28) we find

\[
\hat{P}_{c,\omega}(r, r') = \hat{A}(R)\hat{P}_{d,\omega}(r, r)\hat{A}(R)^T, \tag{C2}
\]

where \( A(R) = \gamma_1^2\tilde{F}_\omega(R)\tilde{F}^T \). (C3)

The matrix \( \hat{A}(R) \) is, like \( \hat{P}_{d,\omega}(r, r) \), a rank two matrix whose only non-zero terms exist in the subspace spanned by the basis \( \tilde{B}_2 = \{|a_-, \cos(\theta)|a_+\rangle, \sin(\theta)|b_+\rangle\} \). The Cooperon \( \hat{P}_{c,\omega}(r, r') \) will therefore also share this property, and we can work with [Eq. (C2)] in the \( \tilde{B}_2 \) basis subspace to ensure the invertibility of all terms involved and simplify the calculation. We write this as

\[
\hat{P}_{c,\omega}(r, r') = \hat{A}(R)\hat{P}_{d,\omega}(r, r)\hat{A}(R)^T, \tag{C4}
\]

where use of the sans serif fonts denotes the projection into the 2×2 subspace. We deduce from Eq. (29) that \( \hat{P}_{d,\omega}(r, r) \) is diagonal and proportional to the matrix \( \text{diag}[1, e^2/(E^2 + \Delta^2)] \) in the \( \tilde{B}_2 \) basis. We can then write in this basis

\[
\hat{P}_{c,\omega}(r, r') = \hat{P}_{d,\omega}(r, r)\hat{f}(R) \tag{C5}
\]

where

\[
\hat{f}(R) = \left( \begin{array}{cc}
1 & 0 \\
0 & e^{2 + \Delta^2}
\end{array} \right) \hat{A}(R) \left( \begin{array}{cc}
1 & 0 \\
0 & E^2 + \Delta^2
\end{array} \right) \hat{A}(R)^T. \tag{C6}
\]

The equation in the full 4×4 space can be obtained by expanding every matrix into the full \( B \) basis by filling in zeros in all the other elements of the matrix to obtain

\[
\hat{P}_{c,\omega}(r, r') = \hat{P}_{d,\omega}(r, r)\hat{f}(R). \tag{C7}
\]

In the two dimensional case, we have, in the \( \tilde{B}_2 \) basis

\[
\hat{f}(R) = e^{\frac{-R}{2k_F}} \frac{1}{\pi k_F R} \left( \frac{1}{\sqrt{E^2 + \Delta^2}} \frac{\cos^2(k_h R - \frac{\pi}{4}) + \cos^2(k_e R - \frac{\pi}{4})}{\cos^2(k_h R - \frac{\pi}{4}) - \cos^2(k_e R - \frac{\pi}{4})} \right), \tag{C8}
\]

where \( k_e = k_F + \frac{\epsilon}{v_F} \) and \( k_h = k_F - \frac{\epsilon}{v_F} \). We note that in contrast to \( \hat{P}_{d,\omega}(r, r') \), this matrix is not diagonal in the \( B \) basis; that is, we have not fully separated the two low-energy modes. However, we work in the limit \( \mu \gg \epsilon \), where the small difference in the frequency of oscillation between electrons and holes is negligible. The fast oscillations average out when integrating over a length long compared to the Fermi wavelength but small compared to the mean free path, so that we can obtain an approximate formula for \( \hat{f}(R) \) by replacing \( \cos^2(k_h R - \pi/4) \approx \cos^2(k_h R - \pi/4) \approx 1/2 \). In this approximation the proportionality factor between the Cooperon and the return probability \( \hat{P}_{d,\omega}(r, r) \), see Eq. (37), is the same as in the normal state.

**Appendix D: Evaluation of the weak localization correction**

The energy-dependent return probability \( \hat{P}_d(r, r)_{1,1} \), given in Eq. (41), has different behaviors below and above \( E_c \), see the definition of \( \tau_{\text{min}} \) in Eq. (21). Accordingly, the energy integral for the WL correction to the thermal conductivity, Eq. (52), is split into two parts,

\[
K_c = -\frac{1}{8\pi^2 k_B T^2} (I_1 + I_2), \tag{D1}
\]

which in two dimensions are explicitly

\[
I_1 = \int_{\Delta}^{E_c} \frac{dE}{E} \frac{E^2}{\cosh^2 \left( \frac{E}{2k_B T} \right)} \ln \left[ \frac{\tau_\phi (E^2 - \Delta^2)}{\Delta} \right] \tag{D2}
\]
and
\[ I_2 = \int_{E_*}^{\infty} dE \frac{E^2}{\cosh^2 (E/2k_BT)} \ln \left( \frac{\tau_0 \sqrt{E^2 - \Delta^2}}{\tau_e E} \right). \] (D3)

Below we consider two situations: energy-independent phase-coherence time \( \tau_0 \), and energy-independent phase-coherence length \( L_\phi = \sqrt{D \tau_0} \). These two scenarios are equivalent in the normal state, but yield different results in the superconducting one. We note that, strictly speaking, the lower integration limit of \( I_1 \) is not \( \Delta \) but, in the diffusion approximation, the quantity \( \Delta_* \) defined by requiring that, for the left hand side of Eq. (39) to be non-zero, \( \tau_0 \gg \tau_{\text{m}} \). For energy-independent phase time, under the usual assumption that \( \tau_0 \gg \tau_e \) (needed for the general applicability of the diffusive approximation [3]), we find for \( \Delta_* \), the equation \( \Delta_*^2 = (1/\tau_0 + \Delta) \); thus, for \( \tau_0 \gg 1/\Delta \), we have \( \Delta_* \approx \Delta \), an approximation that is valid for temperature not too close to absolute zero, \( k_BT \gg 1/\tau_0 \) (at lower temperatures, the WL correction is, with logarithmic accuracy, absent, since the modes with energy between \( \Delta \) and \( \Delta_* \) are not diffusive). The same approximation is valid in the case of energy-independent phase length (assumed to be long compared to the mean free path \( l_\epsilon \)) under the condition \( L_\phi \gg \xi \), with \( \xi = \sqrt{l_\epsilon \xi_\Delta} \), where \( \xi_\Delta = v_F/\Delta \) is the BCS coherence length for a clean superconductor.

1. Energy-independent \( \tau_0 \)

It is convenient to rewrite \( I_1 + I_2 = I_n + I_{\varepsilon_*} + I_3 \) with
\[ I_n = \int_{\Delta}^{\infty} dE \frac{E^2}{\cosh^2 (E/2k_BT)} \ln \left( \frac{\tau_0 \sqrt{E^2 - \Delta^2}}{\tau_e E} \right), \] (D4)
\[ I_{\varepsilon_*} = \int_{\Delta}^{E_*} dE \frac{E^2}{\cosh^2 (E/2k_BT)} \ln \left( \frac{E \sqrt{E^2 - \Delta^2}}{E_* \sqrt{E_*^2 - \Delta^2}} \right), \] (D5)
\[ I_3 = \int_{\Delta}^{\infty} dE \frac{E^2}{\cosh^2 (E/2k_BT)} \ln \left( \frac{\sqrt{E^2 - \Delta^2}}{E} \right), \] (D6)
where we have used the identity
\[ \tau_e = \frac{\Delta}{E_* (E_*^2 - \Delta^2)^{1/2}}. \] (D7)
which follows from the definition of \( E_* \), see Eq. (22).

The integral in Eq. (D4) is defined such that its contribution to the relative correction to the thermal conductivity \( K_f/K_0 \) coincides with that in the normal state, see Eq. (54). The other two integrals are then responsible for the temperature-dependent deviations from the normal state expression. We compute \( I_3 \) and \( I_{\varepsilon_*} \) for different temperature regimes with logarithmic accuracy; note that only \( I_{\varepsilon_*} \) depends on the disorder strength. We first consider the low-temperature regime \( T \ll T_\Delta \) for both the dirty and the clean case, and later the high-temperature regime \( T \gg T_\Delta \).

a. Low-temperature regime

In the low-temperature regime, since we have \( k_BT \ll \Delta \) the hyperbolic cosine can then be approximated as \( 1/\cosh^2 (E/2k_BT) \approx 4e^{-E/k_BT} \). Introducing the dimensionless integration variable \( \alpha = (E - \Delta)/k_BT \) and keeping only the leading term in the small parameter \( k_BT/\Delta \), we find
\[ I_3 \approx \frac{C}{2} \int_0^{\infty} d\alpha \ e^{-\alpha} \ln \left( \frac{2k_BT}{\Delta} \alpha \right) = \frac{C}{2} \int_0^{2e^{-\gamma_e} \Delta/k_BT} d\alpha \ e^{-\gamma_e} \frac{2e^{-\gamma_e} \Delta/k_BT}{\varepsilon_*^2} \] (D8)
with \( C = 4k_BT \Delta^2 e^{-\Delta/k_BT} \) and \( \gamma_E \approx 0.5772 \ldots \) the Euler-Mascheroni constant.

For the integral \( I_{\varepsilon_*} \) we can proceed with the same approximation for the hyperbolic cosine and the same change of integration variable to get
\[ I_{\varepsilon_*} \approx \frac{C}{2} \int_0^{\varepsilon_*} d\alpha \ e^{-\alpha} \ln \frac{2\Delta^3 \alpha}{(k_BT)^3 \alpha_*^2} = \frac{C}{2} \ln \left( \frac{2e^{-\gamma_e} \Delta^3 k_BT}{\varepsilon_*^4} \right) \] (D10)
The sum of Eqs. (D8) and (D10) leads to the last term in the top line of Eq. (59).

In the clean case, since \( \alpha_* k_BT/\Delta \ll 1 \), the integral simplifies to
\[ I_{\varepsilon_*} \approx \frac{C}{2} \int_0^{\varepsilon_*} d\alpha \ e^{-\alpha} \ln \frac{\alpha}{\alpha_*}. \] (D11)
At very low temperatures such that \( k_BT \ll \varepsilon_* \), we can extend the upper integration limit to infinity and thus find a logarithmic contribution of the form \( I_{\varepsilon_*} \approx C \ln (\varepsilon_* / k_BT/\varepsilon_*)^2/2 \); we also note here that for this contribution to be present the condition \( \tau_0 \gg 1/\Delta \) mentioned above is not sufficient, and a more stringent one \( \tau_0 \gg \tau_e^2 \xi_\Delta \), obtained from demanding \( E_* \gg \Delta_* \), is needed. At intermediate temperatures \( \varepsilon_* \ll k_BT \ll k_BT_\Delta \), on the other hand, there is no logarithmic contribution from \( I_{\varepsilon_*} \) and hence the last term in the top line of Eq. (57) is determined solely by Eq. (D8).

b. High-temperature regime

In the high-temperature regime \( T \gg T_\Delta \), we can approximate \( k_BT \gg \Delta \). The integral \( I_3 \sim k_BT^2 \Delta \) has then no logarithmic parameter dependence and can be neglected in comparison to \( I_n \sim (k_BT)^3 \ln (\tau_0/\tau_e) \). For \( I_{\varepsilon_*} \), we must again consider the various regimes separately. However, for \( k_BT \) large compared to \( E_* \) (which is always true in the clean case at high temperatures, while
it would require $T$ in the narrow range between $T_s$ and $T_c$ for the dirty case), we can approximate the hyperbolic cosine with unity; then $I_{\ast}$ becomes independent of temperature and displays no logarithmic parameter dependence; thus, as $I_1$ above, $I_{\ast}$ can be neglected in comparison to $I_n$ and we arrive at the result in the top line of Eq. \[ 14 \].

We are left with the dirty case in the regime $T_D \lesssim T \lesssim T_s$. Then $\Delta$ is small compared to both $E_s$ and the typical energy $E \sim T$, so that we can write

$$ I_{\ast} \approx \int_{\Delta}^{E_s} dE \frac{E^2}{\cosh^2 \left( \frac{E}{2k_B T} \right)} 2 \ln \left( \frac{E}{E_s} \right) \tag{D12} $$

which, with logarithmic accuracy, is

$$ I_{\ast} = I_{K_0} 2 \ln \left( \frac{k_B T}{\varepsilon_s} \right) \tag{D13} $$

with

$$ I_{K_0} = \int_{\Delta}^{\infty} dE \frac{E^2}{\cosh^2 \left( \frac{E}{2k_B T} \right)} \tag{D14} $$

Since we can also write $I_n = I_{K_0} \ln (\tau_n/\tau_c)$, the sum $I_n + I_{\ast}$ leads to the top line in Eq. \[ 9 \].

### 2. Energy-independent $L_\phi$

In the previous subsection, we assumed the phase-coherence time to be independent of energy. Since the group velocity $v_g$ [Eq. \[ 19 \]] in a superconductor and hence the diffusion constant $D_s$ [Eq. \[ 26 \]] are energy dependent, such a choice for the phase-coherence time leads to an energy-dependent phase-coherence length. As an alternative scenario, we consider here a constant phase-coherence length, expressed in terms of the dephasing time and diffusion constant as $L_\phi = \sqrt{D_s \tau_\phi}$. This choice now leads to an energy dependent phase-coherence time $\tau_\phi = L_\phi^2 / k_e v_g$. We substitute this expression for $\tau_\phi$ together with $\tau_c = l_c / v_F$ in Eq. \[ D2 \] and Eq. \[ D3 \] to rewrite the integrals in terms of length rather than time scales. We obtain $I_1 + I_2 = I_n + I_{\ast}$, with

$$ I_n = \int_{\Delta}^{\infty} dE \frac{E^2}{\cosh^2 \left( \frac{E}{2k_B T} \right)} \ln \left( \frac{L_\phi^2}{l_c^2} \right) \tag{D15} $$

and $I_{\ast}$ as defined in Eq. \[ D5 \]. The expressions for the different regimes can then be easily obtained using the results for $I_{\ast}$, in the preceding part of the appendix. Here we only note that the condition for the presence of the $I_{\ast}$, contribution in the clean case for the lowest temperature regime $T \ll T_s$ [see discussion after Eq. \[ D11 \]] can be written as $L_\phi \gg l_c$.

### Appendix E: Weak anti-localization: Spin-orbit scattering

In this appendix, we study weak anti-localization \[ 33 \] in the presence of spin-orbit scattering in disordered s-wave superconductors. To properly account for spin, we now define the Nambu vector as [cf. Eq. \[ 2 \]]

$$ \Psi_k = \begin{pmatrix} c_{k+} \cr c_{k+}^\dagger \cr c_{k-} \cr c_{k-}^\dagger \end{pmatrix} \tag{E1} $$

The full Nambu space is then the product between the space spanned by $\{ |c \rangle, |h \rangle \}$ (the basis used in the main text) and the spin space spanned by $\{ |\uparrow \rangle, |\downarrow \rangle \}$; the Pauli matrices $\tau_1$ and $\sigma_1$ act respectively on these two subspaces. The spin-orbit scattering can be expressed as an additional term in the Hamiltonian in the form \[ 12 \]

$$ \hat{H}^{so}_{\alpha\alpha'}(k, k') = iV^{so} \sigma_{\alpha\alpha'} \cdot (u_k \times u_{k'}) \otimes \tau_3, \tag{E2} $$

where $V^{so}$ is the strength of the spin-orbit scattering potential, $u_k = k/k$, the components of the operator $\sigma$ are the Pauli matrices $\{ \sigma_x, \sigma_y, \sigma_z \}$ and $\sigma_{\alpha\alpha'} = \langle \alpha' | \sigma | \alpha \rangle$ with $\alpha, \alpha' \in \{ \uparrow, \downarrow \}$. The full disorder potential now takes the form $V_{\alpha\alpha'}(k, k') = V_{\alpha\alpha'}(k, k') \otimes \tau_3$ with

$$ V_{\alpha\alpha'}(k, k') = V_0 \delta_{\alpha\alpha'} + iV^{so} \sigma_{\alpha\alpha'} \cdot (u_k \times u_{k'}). \tag{E3} $$

This leads to a new disorder parameter $\gamma_{tot} = \langle |V_{\alpha\alpha'}(k, k')|^2 \rangle_{k'} = \gamma_e + \gamma_{so}$, with $\gamma_{so} = 1/2\pi \rho_0 \tau_{so}$, $\tau_{so} = l_{so} / v_F$ and where $\gamma_e$ has been defined at the end of Sec. \[ II \].

The disorder-averaged superconducting Green’s function can be generalized to the full Nambu space as

$$ \overline{G_{R,A}} = \begin{pmatrix} G_{E,R} & F_{E,R} \cr F_{R,A} & G_{E,R} \end{pmatrix} \otimes \sigma_0, \tag{E4} $$

and the diffusion and the Cooperon can be calculated following a procedure similar to the one used in Sec. \[ III \]. We define

$$ \bar{p}^{\text{so}}_{d,\omega}(r, r') = \langle \hat{P}_0^{\text{so}}(r) \hat{F}_\omega^{\text{so}}(r, r') \hat{F}_{\omega'}^{\text{so}}(r') \rangle, \tag{E5} $$

$$ \bar{p}^{\text{so}}_{c,\omega}(r, r') = \langle \hat{P}_0^{\text{so}}(R) \hat{F}_\omega^{\text{so}}(r, r') \hat{F}_{\omega'}^{\text{so}}(R) \rangle, \tag{E6} $$

which generalize Eq. \[ 17 \] and Eq. \[ 34 \] respectively. We use lower-case $p$s to emphasize that not all elements of these matrices correspond to diffusions and Cooperons, as we will later see. The terms that do not take collisions into account, i.e. $\langle \hat{P}_0^{\text{so}}(r) \rangle$ and $\hat{F}^{\text{so}}(R)$, are related to those in the absence of spin-orbit scattering by $\langle \hat{P}_0^{\text{so}}(r) \rangle = \langle \hat{P}_0 \rangle \otimes \sigma_0 \otimes \sigma_0$ and $\hat{F}^{\text{so}}(R) = \hat{F}(R) \otimes \sigma_0 \otimes \sigma_0$; here $\langle \hat{P}_0 \rangle$ and $\hat{F}(R)$ are as those defined in Eq. \[ 18 \] and Eq. \[ 35 \] respectively, but with $\gamma_{tot}$ replacing $\gamma_e$. 
The equations followed by the structure factors are now given by

\[ \tilde{M}_\omega^{so}(r)\tilde{\Gamma}_\omega^{so}(r, r') = \gamma c \delta^{(d)}(r' - r) \tag{E7} \]

\[ \tilde{M}_\omega^{so}(r)\tilde{\Gamma}_\omega^{so}(r, r') = \gamma c \delta^{(d)}(r' - r). \tag{E8} \]

The diffusion matrices \( \tilde{M}_\omega^{so}(r) \) and \( \tilde{M}_\omega^{so}(r) \) are each defined by an equation similar to Eq. (24) but substituting \( \langle \bar{P}_0 \rangle_r \) by \( \langle \hat{P}_0 \rangle_r \) and \( U_v \) by the potential matrices \( \hat{U}_v^{so} \) and \( \hat{U}_v^{so} \). The potential matrices are no longer equivalent for the diffusion and the Cooperon due to the different spin and momenta relations between the retarded and advanced Green’s functions in the two cases. They are given by

\[ \hat{U}_v^{so} = \hat{U}_v \otimes \hat{u}_v^{so}, \tag{E9} \]

\[ \hat{U}_c^{so} = \hat{U}_v \otimes \hat{u}_c^{so}, \tag{E10} \]

with the (normal metal \cite{12}) matrices \( \hat{u}_v^{so} \) and \( \hat{u}_c^{so} \) given in the basis \( \{ |\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle \} \) by

\[ \hat{u}_v^{so} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{\gamma_v}{3\gamma_c} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \tag{E11} \]

\[ \hat{u}_c^{so} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{\gamma_v}{3\gamma_c} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{E12} \]

Here, each element \( \langle \gamma | \hat{u}_v^{so} | \alpha \rangle \) and \( \langle \gamma | \hat{u}_c^{so} | \alpha \rangle \) with \( \alpha, \beta, \gamma, \delta \in \{ \uparrow, \downarrow \} \), relates the spins of the Green’s functions before and after interacting with an impurity, as depicted in Fig. 6. After obtaining \( \hat{\Gamma}_\omega^{so}(r, r') \) and \( \hat{\Gamma}_\omega^{so}(r, r') \) from Eq. (E7) and Eq. (E8), the matrices \( \hat{p}_{d,\omega}^{so}(r, r') \) and \( \hat{p}_{c,\omega}^{so}(r, r') \) can be calculated using Eq. (E5) and Eq. (E6).

Not all terms in \( \hat{p}_{d,\omega}^{so}(r, r') \) and \( \hat{p}_{c,\omega}^{so}(r, r') \) represent diffusons or Cooperons. The trajectories represented by the retarded and advanced Green’s functions in the diffusion and the Cooperon are not independent and their spin configurations are related. The diffusion, for instance, is composed by a time reversed pair of trajectories; this implies that \( \alpha = \beta \) and \( \gamma = \delta \). We can obtain the diffusion by summing over the final spin constraint while taking this constraint into account. In this way we recover a 4x4 matrix in Nambu space, similar to \( \hat{P}_{d,\omega} \) of Sec. III, where each element now accounts for the probability of propagation with and without spin flip. The diffusion for a particle with initial spin \( \alpha \) is given by

\[ \langle i, j | \hat{P}_{d,\omega}^{so}(r, r') | i, j \rangle = \sum_{\beta} \langle i, j \beta \gamma | \hat{p}_{d,\omega}^{so}(r, r') | i, j \alpha \gamma \rangle, \tag{E13} \]

where \( |i\rangle, |j\rangle \in \{ |e\rangle, |h\rangle \} \) and \( |i, j\rangle \in \{ |e\rangle \otimes |a\rangle, |h\rangle \otimes |\alpha\rangle \} \) with \( \alpha \in \{ \uparrow, \downarrow \} \) and \( \alpha \neq \alpha \). The Cooperon also accounts for the probability of propagation with and without spin flip; however, the conditions on the spins are different since the advanced Green’s function (lower line in Fig. 6) now covers the trajectory in the opposite direction. It is now necessary that \( \alpha = \delta \) and \( \gamma = \beta \), and the Cooperon contribution for a particle with initial spin \( \alpha \) is given by

\[ \langle i, j | \hat{P}_{c,\omega}^{so}(r, r') | i, j \rangle = \sum_{\beta} \langle i, j \beta | \hat{p}_{c,\omega}^{so}(r, r') | i, j \beta \rangle. \tag{E14} \]

Direct calculation (cf. Ref. \cite{12}) shows that spin-orbit scattering does not affect the diffusion, \( \hat{P}_{d,\omega}^{so}(r, r') = \hat{P}_{d,\omega}(r, r') \), while the Cooperon is now qualitatively different, with

\[ \hat{P}_{c,\omega}^{so}(r, r') = -\frac{1}{2} \hat{P}_{c,\omega}(r, r'). \tag{E15} \]

As a consequence, in the presence of spin-orbit scattering the quantum correction to the thermal conductivity is

\[ \frac{K_c^{so}}{K_0^{so}} = -\frac{1}{2} \frac{K_c}{K_0}, \tag{E16} \]

where \( K_c/K_0 \) is the correction calculated in Sec. IV. This correction, known as weak-anti-localization (WAL) effect, increases the total thermal conductivity and is due to destructive interference between self-crossing paths.

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