The $\pi^0 - \eta - \eta'$ mixing in a generalized multi-quark interaction scheme.

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Isospin symmetry breaking effects related to the $\pi^0 - \eta$ and $\pi^0 - \eta'$ mixing angles are considered within a recently derived multi-quark interaction Lagrangian which accounts for all possible spin zero non-derivative type of vertices relevant at the scale of spontaneous breaking of chiral symmetry in 4D, including a subset of interactions which break the chiral symmetry explicitly. We work in the strange non-strange basis where the interactions of the $\pi^0$ with the $\eta$ and $\eta'$ mesons can be linearized in the mixing angles. We obtain a reduction of 40\% in the ratio of the mixing angles $\eta - \eta'$ when the explicit symmetry breaking terms are present. A certain scaling behavior is identified which allows to separate the isospin breaking effects in the meson spectrum from the dynamics leading to the generation of the quark masses. The results are compared with the values extracted from various approaches in the literature.

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The strong isospin symmetry is considered to be a very good approximation in the empirical description of a large bulk of strong interaction processes. This is related to the hierarchy in which breaking of the chiral symmetry $SU(3)_L \times SU(3)_R$ by different current quark masses occurs, down to $SU(2)_L \times U(1)_Y$ flavor symmetry if $m_u, m_d \ll m_s$. In the case of the pseudoscalar mesons it is accurate at the order of the ratio of the light and strange current quark masses $(m_u - m_d)/m_s$\(^\[1,2\]) and explains partly the small meson mass differences within charged isospin multiplets. A further source of isospin breaking is due to the electromagnetic interactions, which are expected to be suppressed at the scale of strong interactions.

A detailed quantitative analysis however requires isospin breaking corrections to be taken into account in a series of low energy phenomena, such as: the description of mass splittings of mesons, and Dashen’s theorem\(^3\); sum rules for quark condensates\(^2,4\); kaon decays\(^5\); $\pi - \pi$\(^6,7\) and $\pi - K$ scattering\(^8,9\) in relation to mesonic atoms\(^10,11\).

Strong isospin breaking effects become particularly relevant if a certain process depends crucially on the differences of the light quark masses. If in addition the electromagnetic interactions are a subleading effect, these processes provide for ideal tools in a quantitative analysis of quark mass ratios. In the latter category are the $\eta, \eta' \to 3\pi$ decays, the $\pi^0 - \eta$ and $\pi^0 - \eta'$ mixings, as well as the $\rho - \omega$ mixing in the vector channels.

Isospin breaking associated with the $\eta^0 - \eta - \eta'$ system has long been known to play a role in the Standard Model prediction of the CP violation related ratio $(\epsilon')_CP$\(^12,13\) representing a substantial correction to the QCD penguin contributions\(^14\). It affects the value of the $K^0 \to \pi^0\pi^0$ transition through the dominant QCD $Q_5$ penguin operator, which is one of the sources of uncertainties in the determination of $(\epsilon')_CP$\(^16\), for a recent review see\(^17\).

In chiral perturbation theory (ChPT)\(^18,19,2\), the $\pi^0 - \eta$ mixing angle occurs already at order $p^2$ and was first evaluated to order $p^4$ in the context of $K_{l3}$ form factors in\(^19\).

In the analysis of $\eta - \eta'$ mixing of\(^21\) the $U(1)_A$ anomaly is described by the gluon transition matrix element $<0|\bar{G}G|\eta'>$ and the quark flavor basis has been used with the decay constants following the pattern of particle state mixing in that basis. It has been shown that this approach leads to results consistent with many observables related to $\eta - \eta'$ mixing. In\(^21,22\) it has been extended to include the mixing to the neutral pion.

In the present work we address the $\pi^0 - \eta - \eta'$ mixings resulting from a recently proposed Lagrangian\(^23,24\). In this effective Lagrangian approach built from all spin 0 and non-derivative multi-quark interactions relevant at the scale of spontaneous chiral symmetry breaking the complete set of interactions which break explicitly the chiral symmetry was included for the first time. This Lagrangian represents a generalization of the original Nambu-Jona-Lasinio\(^25,26\) extended to the realistic three flavor and color case with $U(1)_A$ breaking six-quark ‘t Hooft interactions\(^27,41\) and an appropriate set of eight-quark interactions\(^42\). The last ones complete the number of vertices which are important in four dimensions for dynamical $SU(3)_L \times SU(3)_R$ chiral symmetry breaking\(^43,44\). The Lagrangian considers all interactions relevant at the same order in large $N_c$ counting as the $U(1)_A$ anomaly term.

In the isospin limit this generalized Lagrangian has been shown to accurately describe the low-lying spectrum of the pseudoscalar and scalar mesons\(^23\), together with a good description of the radiative two photon decays of the pseudoscalars and the strong two body decays of the scalars\(^24\). We take the Lagrangian in this limit to define the unperturbed system. Isospin breaking is introduced by allowing small variations in the up and down current quark masses and solving the gap equations.

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to extract the pertinent constituent quark masses and condensates to be used in the meson mass expressions.

In the following we indicate only the terms of the bosonized Lagrangian which are relevant for the present study, the terms related with the $\pi^0, \eta, \eta'$ masses

\[ L_{\text{mass}} = \frac{1}{2} h^{(2)}_{ab} \phi_a \phi_b + \frac{N_c I_0}{4\pi^2} \phi^4_a \]

\[ - \frac{N_c I_1}{24\pi^4} \left\{ \left( \phi_a^2 (2M_u^2 - M_s^2 - M_i^2) + \phi_b^2 (2M_s^2 - M_u^2 - M_i^2) + \phi_c^2 (2M_t^2 - M_u^2 - M_s^2) \right) \right\} \]  

and refer to [23], [24] for the full Lagrangian and derivation. The study, the terms related with the singlet, and the indices $h$ metric and antisymmetric structure constants. The cubic equations

\[ \kappa \phi_c (2M_u^2 - M_s^2 - M_i^2) + \phi_b c \left( 2M_s^2 - M_u^2 - M_i^2 \right) + \phi_c c \left( 2M_t^2 - M_u^2 - M_s^2 \right) \} \right\} \]  

are totally defined in terms of $h_a$, $\mu_a$ and the parameters $G, k, k_2, g_j, \{ i = 1...8 \}$ of the model. Here the indices $\{ a, b \} = 0, 3, 8$ label the flavor indices of the corresponding subset of diagonal Gell-Mann matrices and the singlet, and the $d_{abc}, f_{abc}$ are the conventional symmetric and antisymmetric structure constants. The $h_a$ ($a = 0, 3, 8), after a convenient redefinition to the flavor indices $i = u, d, s$ [43] satisfy the following system of cubic equations

\[ \Delta_i + \frac{k}{4} t_{ijk} h_j h_k + \frac{h_i}{2} (2G + g_1 h^2 + g_4) h_i + \frac{g_2}{2} h_i^3 \]

\[ + \frac{h_i}{4} \left[ 3g_3 h_i^2 + g_2 h_i^2 + 2(g_2 + g_6) \mu_i h_i + 4g_7 \mu_i \right] \]

\[ + \kappa t_{ijk} \mu_j h_k = 0, \]  

where $\Delta_i = M_i - m_i$; $t_{ijk}$ is a totally symmetric quantity, whose nonzero components are $t_{uuds} = 1$; there is no summation over the open index $i$ but we sum over the dummy indices, e.g. $h_i^2 = h_u^2 + h_d^2 + h_s^2$, $\mu_i h_i = \mu_u h_u + \mu_d h_d + \mu_s h_s$. The equations (4) must be solved self-consistently with the gap equations

\[ h_i + \frac{N_c}{6\pi^2} M_i \left[ 3I_0 - (3M_i^2 - M^2) I_1 \right] = 0. \]

and setting $\mu_i = m_i$, using a freedom associated with the Kaplan-Manohar ambiguity [46], [19], [23], [24]. Here $N_c = 3$ is the number of colors, and $M^2 = M_u^2 + M_s^2 + M_t^2$. The quark one-loop integrals $I_i$ ($i = 0, 1$) are the arithmetic average values $I_i = \frac{1}{4} [J_i (M_u^2) + J_i (M_s^2) + J_i (M_t^2)]$ [50, 52] where

\[ J_i (m^2) = \int_0^\infty \frac{dt}{t^2} \rho (t\Lambda^2) e^{-t m^2}, \]

with the Pauli-Villars [53] regularization kernel with two subtractions [54]

\[ \rho (t\Lambda^2) = 1 - (1 + t\Lambda^2) \exp (-t\Lambda^2). \]

The components of the fields $\phi$ are related as

\[ \phi_u = \phi_3 + \sqrt{2} \phi_0 + \phi_8 = \phi_3 + \eta_{ns}, \]

\[ \phi_d = -\phi_3 + \sqrt{2} \phi_0 + \phi_8 = -\phi_3 + \eta_{ns}, \]

\[ \phi_s = \sqrt{\frac{2}{3}} \phi_0 - 2 \phi_8 = \sqrt{2} \eta_s. \]

We also introduce the $\eta_{ns}$ and $\eta_s$ which stand for the flavor components of the physical $\eta, \eta'$ states in the non-strange and strange basis. In addition to the flavor mixing in the $\eta, \eta'$ channels the isospin breaking induces a coupling between the $\pi^0$ and these states,

\[ \pi^0 = \phi_3 + \epsilon \eta + \epsilon' \eta'. \]

To get the physical $\pi^0$, $\eta$ and $\eta'$ mesons we proceed as in [22]. Since $\phi_3$ couples weakly to the $\eta_{ns}$ and $\eta_s$ states (decoupling in the isospin limit) while the $\eta - \eta'$ mixing is strong, it is appropriate to use isoscalar $\eta_{ns}, \eta_s$ and isovector $\phi_3$ combinations as a starting point for a unitary transformation to the physical meson states $\pi^0, \eta, \eta'$. In this case the corresponding unitary matrix $U$ can be linearized in the $\pi^0 - \eta$ and $\pi^0 - \eta'$ mixing angles $\epsilon_1, \epsilon_2 \sim \mathcal{O} (\epsilon), \epsilon \ll 1$. To be specific [22],

\[ \begin{pmatrix} \eta^0 \\ \eta' \end{pmatrix} = U (\epsilon_1, \epsilon_2, \psi) \begin{pmatrix} \phi_3 \\ \eta_{ns} \end{pmatrix}, \]  

where

\[ U = \begin{pmatrix} 1 & \epsilon_1 + \epsilon_2 \cos \psi & -\epsilon_2 \sin \psi \\ -\epsilon_2 \cos \psi & \cos \psi & -\sin \psi \\ -\epsilon_1 \sin \psi & \sin \psi & \cos \psi \end{pmatrix} \]

In particular, in eq. (9) $\epsilon = \epsilon_2 + \epsilon_1 \cos \psi, \epsilon' = \epsilon_1 \sin \psi$.

The $\eta - \eta'$ mixing angle $\psi$ in the isospin limit is obtained from the mixing angle conventions summarized in the Appendix B of [53]. We have the following different possibilities of relating the physical states $(X, \bar{X})$ with the states of the strange non-strange basis

\[ \begin{pmatrix} X \\ \bar{X} \end{pmatrix} = R_\psi \begin{pmatrix} X_{ns} \\ X_s \end{pmatrix} = R_\psi \begin{pmatrix} -X_s \\ X_{ns} \end{pmatrix}, \]  

(12)
TABLE I: The pseudoscalar masses and weak decay constants (all in MeV) in the isospin limit used as input (marked with *) for different sets of the model. Parameter sets (a),(b) contain explicit symmetry breaking interactions (see Table III) and allow for a fit of the scalar masses and strong decays as well, $m_\sigma = 550$ MeV, $m_\kappa = 850$ MeV, $m_{a_0} = m_{f_0} = 980$ MeV [24]; set (c) does not. Set (a) corresponds to an octet-singlet mixing angle in the scalar sector of $\theta_S = 27.5^\circ$, set (b) to $\theta_S = 25^\circ$.

| Sets | $m_\pi$ | $m_K$ | $m_\eta$ | $m_{\eta'}$ | $f_\pi$ | $f_K$ |
|------|--------|--------|----------|------------|--------|------|
| a    | 138    | 494    | 547      | 958        | 92     | 113  |
| b    | 138    | 494    | 547      | 958        | 92     | 113  |
| c    | 138    | 494    | 475      | 958        | 92     | 115.7 |

TABLE II: Parameter sets of the model: $m_0$, $m_s$, and $\Lambda$ are given in MeV. The couplings have the following units: $[G] = \text{GeV}^{-2}$, $[\kappa] = \text{GeV}^{-5}$, $[g_1] = [g_2] = \text{GeV}^{-8}$. We also show here the values of constituent quark masses $M$ and $M_s$ in MeV. See also caption of Table I.

| Sets | $m_0$ | $m_s$ | $M$  | $M_s$ | $\Lambda$ | $G$  | $-\kappa$ | $g_1$ | $g_2$ |
|------|-------|-------|------|-------|------------|------|------------|-------|-------|
| a    | 4.0   | 100   | 373  | 544   | 828        | 10.48| 122.       | 328   | 173   |
| b    | 4.0   | 100   | 372  | 542   | 829        | 9.83 | 118.5      | 3305  | -158  |
| c    | 6.1   | 190   | 375  | 569   | 836        | 9.79 | 138.2      | 2500  | -100  |

TABLE III: Explicit symmetry breaking interaction couplings. The couplings have the following units: $[\kappa_2] = \text{GeV}^{-3}$, $[g_4] = [g_5] = [g_6] = [g_7] = [g_8] = \text{GeV}^{-4}$. See also caption of Table I.

| Sets | $\kappa_2$ | $-g_3$ | $g_4$ | $g_5$ | $-g_6$ | $-g_7$ | $g_8$ |
|------|------------|--------|-------|-------|--------|--------|-------|
| a    | 6.17       | 6497   | 1235  | 213   | 1642   | 13.3   | -64   |
| b    | 5.61       | 6472   | 702   | 210   | 1668   | 100    | -38   |
| c    | 0          | 0      | 0     | 0     | 0      | 0      | 0     |

TABLE IV: The mixing angles in the $\eta - \eta'$ system and isospin limit, and related weak decay constants for the sets discussed and in comparison with different approaches.

| Sets | $\theta_\rho^\prime$ | $\theta_\rho$ | $\theta_8$ | $\frac{f_\pi}{f_K}$ | $\frac{f_\rho}{f_K}$ |
|------|-----------------------|---------------|-------------|----------------------|---------------------|
| a    | -12                   | -1.42         | -21.37      | 1.172                | 1.318               |
| b    | -15                   | -4.42         | -24.37      | 1.172                | 1.322               |
| [20] phen. | -13.3             | -6.8        | -19.4       | 1.10                 | 1.19                |
| [20] phen. | -15.4             | -9.2        | -21.2       | 1.17                 | 1.26                |
| [58] CHPT | -10.5             | -1.5        | -20.0       | 1.24                 | 1.31                |
| [59] CHPT | -                   | -4.0        | -20.5       | 1.10                 | 1.28                |
| [60] sum rules | -               | -15.6       | -10.8       | 1.39                 | 1.39                |

TABLE V: Isospin breaking parameter $\delta = \frac{m_d - m_u}{m_d + m_u}$, current and constituent quark masses $m_u$, $m_d$, $M_u$, $M_d$ in MeV and $\pi^0 - \eta$, $\pi^0 - \eta'$ mixing angles $\epsilon$ and $\epsilon'$. Set (a') corresponds to scenario (i), see main text.

| Sets | $\delta$ | $m_u$ | $m_d$ | $M_u$ | $M_d$ | $m_{\pi^0}$ | $\epsilon$ | $\epsilon'$ | $\frac{\epsilon}{\epsilon'}$ |
|------|----------|-------|-------|-------|-------|--------------|-------------|-------------|-----------------------------|
| a    | 0.175    | 3.3   | 4.7   | 370.0 | 377.0 | 137.5        | 0.017       | 0.0031      | 5.47                        |
| a    | 0.20     | 3.2   | 4.8   | 369.4 | 377.4 | 137.3        | 0.017       | 0.0031      | 5.47                        |
| a    | 0.25     | 3.0   | 5.0   | 368.4 | 378.4 | 137.0        | 0.021       | 0.0039      | 5.47                        |
| a'   | 0.55     | 1.8   | 6.2   | 370.0 | 377.0 | 137.5        | 0.015       | 0.0027      | 5.45                        |
| b    | 0.175    | 3.3   | 4.7   | 368.6 | 375.6 | 137.5        | 0.0157      | 0.0026      | 6.08                        |
| b    | 0.20     | 3.2   | 4.8   | 368.1 | 376.1 | 137.3        | 0.0180      | 0.00296     | 6.08                        |
| b    | 0.25     | 3.0   | 5.0   | 367.1 | 377.1 | 137.0        | 0.0225      | 0.0037      | 6.08                        |
| c    | 0.175    | 5.0   | 7.2   | 373.7 | 377.1 | 137.8        | 0.0107      | 0.00123     | 8.75                        |
| c    | 0.20     | 4.9   | 7.3   | 373.4 | 377.3 | 137.8        | 0.0123      | 0.00140     | 8.75                        |
| c    | 0.25     | 4.6   | 7.6   | 372.9 | 377.8 | 137.6        | 0.0153      | 0.00175     | 8.75                        |
where the orthogonal $2 \times 2$ matrix $R_{\psi}$ is

$$R_{\psi} = \begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix}, \quad (13)$$

or with the states of the singlet-octet basis

$$\begin{pmatrix} \tilde{X} \\ X \end{pmatrix} = R_{\theta} \begin{pmatrix} X_{\eta} \\ X_{0} \end{pmatrix}. \quad (14)$$

Here $\theta$, being a solution of the equation $\tan 2\theta = x$, is the principal value of $\arctan x$, i.e. belongs to the interval $-(\pi/4) \leq \theta \leq (\pi/4)$. The angle $\psi$ is related with $\theta$ by the equation $\psi = \theta + \theta_{id}$, where $\theta_{id} (\theta_{id} + \psi_{id} = \pi/2)$ is determined by the equations $\sin \theta_{id} = \sqrt{2}/3$, $\cos \theta_{id} = 1/\sqrt{3}$, therefore $\psi = \psi_{id} + \arctan \sqrt{2} = \psi + 54.74^\circ$. It means that $\psi$ is restricted to the range $97.4^\circ \leq \psi \leq 99.74^\circ$. If the value of $\psi$ leaves the range, we must resort to the angle $\psi_2 = \theta - \psi_{id}$, taking values in the interval $-80.26^\circ \leq \psi_2 \leq 9.74^\circ$. These two angles correspond to two alternative phase conventions for a strange $\bar{s}s$-component. As a result of the following numerical calculations, in the case of the pseudoscalars the identification of the physical states is $X = \eta, \tilde{X} = \eta'$. Numerically the effects of isospin breaking are obtained by lifting the degeneracy in the up and down current quark mass values while keeping unchanged the complete parameter set $G, \kappa, \kappa_2, g_j \{j = 1...8\}$ as well as $m_s, M_s$ obtained from the fits in the isospin limit, see Tables I-III. This leads to a change in the constituent quark masses and condensates of the light quarks, which must be re-evaluated to the amount of isospin breaking, set e.g. by the ratio $\delta = \frac{m_u-m_d}{m_u+m_d}$, by solving four coupled equations, the three gap equations [5], linked to the SPA equations [4], and $\delta$, self-consistently. The value of the mixing angle in the $\eta - \eta'$ sector is taken to be the unperturbed one (given in Table IV), which is compatible with the state mixing scheme in the $\phi_3, \eta_{ns}, \eta_{s}$ basis.

We consider two scenarios: (i) the gap equations are solved by fixing the light current quark masses in the interaction terms to their unperturbed values (isospin limit), denoted as $m_0$, thus allowing variations in the up and down current masses only in the leading in $N_c$ quark mass terms, i.e. in $\Delta_i$ in eq. [4]; (ii) variations are taken to occur in all terms with current quark mass insertions. In the latter case the generation of constituent quark masses and condensates is understood to be fully non-perturbative in all flavors, as opposed to case (i) where the different constituent $M_u, M_d$ quark masses originate only from a perturbation in the LO current quark mass term. We arguably favor case (ii) to be more adequate for determining the physical constituent masses and condensates, assigning their origin to the non-perturbative character of strongly interacting systems.

Concerning the meson mass spectrum, the interaction terms in eq. (3) involving the $u, d$ current quark masses are treated perturbatively, as the isospin breaking induced in these NLO explicit chiral symmetry breaking multi-quark interactions may obtain sizeable corrections from electromagnetic interactions, not considered in the present work. It is noteworthy that treating the meson spectrum perturbatively in the $u, d$ current quark masses, one obtains that for both cases (i) and (ii) all matrix elements except the ones linking the $\phi_3 - \eta_s$ and $\phi_3 - \eta_{ns}$ states remain practically unchanged when compared to each other and to the isospin limit; and that furthermore the mixing $\phi_3 - \eta_s$ and $\phi_3 - \eta_{ns}$ can be made virtually the same for cases (i) and (ii) by rescaling the ratio $\delta$ such that the constituent quark masses match in both cases. The resulting mixing angles $\epsilon, \epsilon'$ will then be the same, and the difference in considerations (i), (ii) has only an impact on the values of the current quark masses.

Thanks to this approximate scaling behavior it is possible to separate the isospin breaking effects in the meson spectrum from the dynamics leading to the generation of the quark masses. One can thus have a hint which of the scenarios (i) or (ii) describes more appropriately the generation of masses, by comparing the light current quark masses emerging in both with empirical data.

We present first the results for the isospin limit, from which the couplings are taken over to implement isospin breaking. We consider the cases in which explicit symmetry breaking terms are present in the interaction Lagrangian, sets (a,b) in the Tables, and compare with the parameter set (c) in which explicit symmetry breaking occurs only through the LO current quark mass term. Table I indicates the mass spectra of the low lying pseudoscalar (and scalar meson nonets, see caption, for sets (a,b)) used in the fit of parameters. Table II shows the current and constituent quark masses in the isospin limit and the model parameters which do not break explicitly the chiral symmetry, the 4-quark coupling $G$, the 6-quark t Hooft determinant coupling $\kappa$, and two 8-quark cou-
plings $g_1, g_2$, of which $g_1$ is OZI-violating. In set (c) we put $g_1 = 2500$ without loss of generality, since a change in $g_1$ can be counterbalanced by a change in $G$ leaving all other parameters and observables unchanged, except for the low lying $\sigma$-meson mass, see e.g. $56$, $57$, which does not affect the pseudoscalar characteristics considered here. Table III displays the couplings related with the explicit symmetry breaking interactions. In Table IV further properties of the $\eta - \eta'$ system are presented, the angle $\theta$ in the singlet-octet basis and the related angles $\theta_0, \theta_8$, as well as the weak decays $f_0, f_8$, obtained following the methods of $58$, and compared with the results of other approaches $20$ and references therein, $58$, $59$, $60$. This comparison shows that set (a) yields results quite close to the chiral perturbation analysis of $60$ and follows the general trend of the other references in the table, with exception of the sum rules approach of $60$. Our values for $\theta_0, \theta_8$ are smaller, respectively larger than the ones presented in $20$, which is probably due to the different way in which the $U(1)_A$ anomaly is treated. One further notices that an increase in $\theta$ leads mainly to an increase in the $\theta_0$ angle, comparing sets (a) and (b), which have similar theoretical input. Having shown that the parameters obtained in the isospin limit lead to a good description of empirical data, we proceed to show the results of isospin breaking in Table V. We use the same three values of $\delta$ in each parameter set (a,b,c). The amount by which the current quark masses differ, related to $\delta$, leads to a slight reduction in the $\pi^0$ mass, which turns out to be at most 1 MeV. The larger observed empirical value for the difference $m_{\pi^+} - m_{\pi^0} \sim 4.5$ MeV is mainly of electromagnetic origin, not considered here. The main observation is that the ratio $\gamma$ remains almost constant within each set and that the sets (a,b) with explicit chiral symmetry breaking terms yield a ratio which is reduced by $\sim 40\%$ compared to set (c). The larger value of $\delta$ leads in sets (a,b) to current quark masses of the light quarks which are closer to the presently quoted values $m_u = 2.15(15)$ MeV, $m_d = 4.70(20)$ MeV, $64$. The case denoted as set (a') corresponds to solving the gap equations with the parameter set (a) keeping the NLO current quark mass insertions in (4) fixed to the value $m_0$. As mentioned above, a rescaling of $\delta$ leads to the same output for constituent masses and mixing angles. However the large value $\delta$ needed for that requires a much too large splitting in the $m_u, m_d$ values, as compared to empirical values.

Table VI collects values obtained in the literature, within different phenomenological approaches, as well as in experiments. Comparison of the different values has to be done with care, for a very careful discussion see $22$. In the experimental value $63$, the mixing $\pi^0 - \eta'$ has not been taken into account. In the ChPT result $58$ the $\eta'$ is considered as a background field.

We conclude that the explicit symmetry breaking interactions of the generalized NJL Lagrangian considered are relevant to reduce the ratio $\gamma$, bringing it closer to the phenomenological values. We obtain values for the $\epsilon$ mixing angle which lie within the results discussed in the literature. Unfortunately the value for $\epsilon'$ is much less discussed. We obtain $\epsilon$ and $\epsilon'$ reasonably close to the ones indicated in $21$, $22$ for current quark mass values in good agreement with the presently quoted average values. The corresponding sets (a,b) are the ones which also yield the best fits to other empirical data within the model variants.

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[27] G. 't Hooft, Phys. Rev. D 14, 3432 (1976).
[28] G. 't Hooft, Phys. Rev. D 18, 2199 (1978).
[29] V. Bernard, R. L. Jaffe, U.-G. Meißen, Phys. Lett. B 198, 92 (1987).
[30] V. Bernard, R. L. Jaffe, U.-G. Meißen, Nucl. Phys. B 308, 753 (1988).
[31] H. Reinhardt and R. Alkofer, Phys. Lett. B 207, 482 (1988).
[32] S. Klimt, M. Lutz, U. Vogl, W. Weise, Nucl. Phys. A 516, 429 (1990).
[33] U. Vogl, M. Lutz, S. Klimt, W. Weise, Nucl. Phys. A 516, 469 (1990).
[34] U. Vogl, W. Weise, Progr. Part. Nucl. Phys. 27, 195 (1991).
[35] M. Takizawa, K. Tsushima, Y. Kohyama, K. Kubodera, Nucl. Phys. A 507, 611 (1990).
[36] S. P. Klevansky, Rev. Mod. Phys. 64, 649 (1992).
[37] T. Hatsuda, T. Kunihiro, Phys. Rep. 247, 221 (1994).
[38] V. Bernard, A. H. Blin, B. Hiller, U.-G. Meißen, M. C. Ruivo, Phys. Lett. B 305, 163 (1993).
[39] V. Dmitrasinovic, Nucl. Phys. A 686, 379 (2001).
[40] M. C. Birse, T. D. Cohen, J. A. McGovern, Phys. Lett. B 388, 137 (1996).
[41] K. Naito, M. Oka, M. Takizawa, T. Umekawa, Progr. Theor. Phys. 109, 969 (2003).
[42] A. A. Osipov, B. Hiller, J. da Providência, Phys. Lett. B 634, 48 (2006).
[43] A. A. Andrianov, V. A. Andrianov, Theor. Math. Phys. 94, 3 (1993).
[44] A. A. Andrianov, V. A. Andrianov, Int. J. of Mod. Phys. A 8, 1981 (1993).
[45] A. A. Osipov, B. Hiller, Phys. Lett. B 271, 188 (2002).
[46] D. B. Kaplan, A. V. Manohar, Phys. Rev. Lett. 56, 2004 (1986).
[47] H. Leutwyler, Nucl. Phys. B 337, 108 (1990).
[48] J. F. Donoghue, D. Wyler, Phys. Rev. D 45, 892 (1992).
[49] H. Leutwyler, Phys. Lett. B 374, 163 (1996).
[50] A. A. Osipov, B. Hiller, Phys. Lett. B 515, 458 (2001).
[51] A. A. Osipov, B. Hiller, Phys. Rev. D 63, 084009 (2001).
[52] A. A. Osipov, B. Hiller, Phys. Rev. D 64, 087701 (2001).
[53] W. Pauli, F. Villars, Rev. Mod. Phys. 21, 434 (1949).
[54] V. Bernard, A. H. Blin, B. Hiller, Yuri P. Ivanov, A. A. Osipov, U.-G. Meissner, Annals of Phys. 249, 499 (1996), e-Print: [hep-ph/9506309].
[55] A. A. Osipov, H. Hansen, B. Hiller, Nucl. Phys. A 745, 81 (2004).
[56] A. A. Osipov, B. Hiller, A. H. Blin, J. da Providência, Annals of Phys. 322, 2021 (2007).
[57] B. Hiller, J. Moreira, A. A. Osipov, A. H. Blin, Phys Rev. D 81, 116005 (2010).
[58] J. L. Goity, A. M. Bernstein, B. R. Holstein, Phys Rev. D 66, 076014 (2002).
[59] R. Kaiser, H. Leutwyler, Europ. Phys. J. C 17, 623 (2000). H. Leutwyler, Nucl. Phys. Proc. Suppl. 64, 223 (1998).
[60] F. de Fazio, M. R. Pennington, JHEP 0007, 051 (2000).
[61] S.A. Coon, B.H.J. McKellar, M. D. scadron, Phys.Rev. D 34, 2784 (1986).
[62] J. Z. Bai et. al. [BES Collaboration], Phys. Rev. D 70 012006 (2004).
[63] W. B. Tippens et. al., Phys. rev. D 63 052001 (2001).
[64] J. Beringer et al., Particle Data Group, Phys. Rev. D 86, 010001 (2012).