SOME REMARKS ON HOMOGENEOUS KÄHLER MANIFOLDS

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Abstract. In this paper we provide a positive answer to a conjecture due to A. J. Di Scala, A. Loi, H. Hishi (see [3, Conjecture 1]) claiming that a simply-connected homogeneous Kähler manifold $M$ endowed with an integral Kähler form $\mu_0 \omega$, admits a holomorphic isometric immersion in the complex projective space, for a suitable $\mu_0 > 0$. This result has two corollaries which extend to homogeneous Kähler manifolds the results obtained by the authors in [8] and in [12] for homogeneous bounded domains.

1. Introduction and statement of the main results

The paper consists of three results: Theorem 1.1, Theorem 1.2 and Theorem 1.3. Our first result answer positively to a conjecture due to A. J. Di Scala, A. Loi, H. Hishi (see [3, Conjecture 1]), namely:

Theorem 1.1. Let $(M, \omega)$ be a simply-connected homogeneous Kähler manifold such that its associated Kähler form $\omega$ is integral. Then there exists a constant $\mu_0 > 0$ such that $\mu_0 \omega$ is projectively induced.

Before stating our second result (Theorem 1.2), we recall the definitions of a Berezin quantization and diastasis function. Let $(M, \omega)$ be a symplectic manifold and let $\{\cdot, \cdot\}$ be the associated Poisson bracket. A Berezin quantization on $M$ is given by a family of associative algebras $\mathcal{A}_h \subset C^\infty(M)$ such that

- $h \in E \subset \mathbb{R}^+$ and $\inf E = 0$
- exists a subalgebra $\mathcal{A} \subset (\oplus_{h \in E} \mathcal{A}_h, \ast)$, such that for an arbitrary element $f = f(h) \in \mathcal{A}$, where $f(h) \in \mathcal{A}_h$, there exists a limit $\lim_{h \to 0} f(h) = \varphi(f) \in C^\infty(M)$,
hold the following correspondence principle: for \( f, g \in \mathcal{A} \)
\[ \varphi(f \ast g) = \varphi(f) \varphi(g), \quad \varphi \left( \hbar^{-1}(f \ast g - g \ast f) \right) = i\{\varphi(f), \varphi(g)\}, \]

for any pair of points \( x_1, x_2 \in \Omega \) there exists \( f \in \mathcal{A} \) such that \( \varphi(f)(x_1) \neq \varphi(f)(x_2) \)

Let \((M, \omega)\) be a real analytic Kähler manifold. Let \( U \subset M \) be a neighborhood of a point \( p \in M \) and let \( \psi : U \to \mathbb{R} \) be a Kähler potential for \( \omega \). The potential \( \psi \) can be analytically extended to a sesquilinear function \( \psi(p, \overline{q}) \), defined in a neighborhood of the diagonal of \( U \times U \), such that \( \psi(q, \overline{q}) = \psi(q) \). Assume (by shrinking \( U \)) that the analytic extension \( \psi \) is defined on \( U \times U \). The Calabi’s diastasis function \( D : U \times U \to \mathbb{R} \) is given by:
\[ D(p, q) = \psi(p, p) + \psi(q, q) + \psi(p, q) + \psi(q, p). \]

Denoted by \( D_p(q) := D(p, q) \) the diastasis centered in a point \( p \), one can see that \( D_p : U \to \mathbb{R} \) is a Kähler potential around \( p \).

Our second result extends to any homogeneous Kähler manifolds the results obtained by the authors [8] for homogeneous bounded domains.

**Theorem 1.2.** Let \((M, \omega)\) be a homogeneous Kähler manifold. Then the following are equivalent:

(a) \( M \) is contractible.
(b) \((M, \omega)\) admits a global Kähler potential.
(c) \((M, \omega)\) admits a global diastasis \( D : M \times M \to \mathbb{R} \).
(d) \((M, \omega)\) admits a Berezin quantization.

Also to state the third result (Theorem 1.3) we need some definitions. The diastatic entropy has been defined by the second author in [12] (see also [13]) following the ideas developed in [9] and [10]. Assume that \( \omega \) admits a globally defined diastasis function \( D_p : M \to \mathbb{R} \) (centered at \( p \)). The diastatic entropy at \( p \) is defined as
\[ \text{Ent} (M, \omega)(p) = \min \left\{ c > 0 \mid \int_M e^{-c D_p} \frac{\omega^n}{n!} < \infty \right\}. \]

The definition does not depend on the point \( p \) chosen (see [13] Proposition 2.2)).

Assume that \( M \) is simply-connected and assume that there exists a line bundle \( L \to M \) such that \( c_1(L) = [\omega] \) (i.e. \( \omega \) is integral). Let \( h \) be an Hermitian metric on \( L \) such that \( \text{Ric}(h) = \omega \) and consider the Hilbert space of global holomorphic sections of \( L^\lambda = \otimes^\lambda L \) given by
\[ \mathcal{H}_{\lambda, h} = \left\{ s \in \text{Hol}(L) \mid \|s\|^2 = \int_M h^\lambda(s, s) \frac{\omega^n}{n!} < \infty \right\}, \]

where
\[ h^\lambda(s, s) = \int_M h^\lambda(s, s) \frac{\omega^n}{n!} < \infty \].
with the scalar product induced by $\| \cdot \|$. Let \( \{ s_j \}_{j=0,\ldots,N} \), \( N \leq \infty \), be an orthonormal basis for \( H_{\lambda,h} \). The \( \varepsilon \)-function is given by

\[
\varepsilon_\lambda(x) = \sum_{j=0}^{N} h^\lambda(s_j(x), s_j(x)).
\]

This definition depends only on the Kähler form \( \omega \). Indeed since \( M \) is simply-connected, there exists (up to isomorphism) a unique \( L \to M \) such that \( c_1(L) = [\omega] \), and it is easy to see that the definition does not depend on the orthonormal basis chosen or on the Hermitian metric \( h \) (see e.g [8] or [14, 15] for details). Under the assumption that \( \varepsilon_\lambda \) is a (strictly) positive function, one can consider the coherent states map \( f : M \to \mathbb{CP}^N \) defined by

\[
f(x) = [s_0(x), \ldots, s_j(x), \ldots].
\]

The fundamental link between the coherent states map and the \( \varepsilon \)-function is expressed by the following equation (see [16] for a proof)

\[
f^* \omega_{FS} = \lambda \omega + \frac{i}{2} \partial \bar{\partial} \varepsilon_\lambda,
\]

where \( \omega_{FS} \) is the Fubini–Study form on \( \mathbb{CP}^N \).

**Definition.** We say that \( \lambda \omega \) is a balanced metric if and only if the \( \varepsilon_\lambda \) is a positive constant.

We can now state our third and last result which extends to any homogeneous Kähler manifold, the result obtained by the second author [12, Theorem 2], about the link between diastatic entropy and balanced condition on homogeneous bounded domains.

**Theorem 1.3.** Let \( (M, \omega) \) be a contractible homogeneous Kähler manifold. Then \( \lambda \omega \) is balanced if and only if

\[
\text{Ent}(M, \omega) < \lambda.
\]

2. **Proof the main results**

We start with the following two lemmata, which provide a necessary and sufficient condition on the non-triviality of the Hilbert space \( \mathcal{H}_{\lambda,h} \).

**Lemma 2.1** (Rosemberg–Vergne [17]). Let \( (M, \omega) \) be a simply-connected homogeneous Kähler manifold with \( \omega \) integral. Then there exists \( \lambda_0 > 0 \) such that \( \mathcal{H}_{\lambda,h} \neq \{0\} \) if and only if \( \lambda > \lambda_0 \) and \( \lambda \omega \) is integral.

**Lemma 2.2.** Let \( (M, \omega) \) be a simply-connected homogeneous Kähler manifold. Then \( \mathcal{H}_{\lambda,h} \neq \{0\} \) if and only if \( \lambda \omega \) is a balanced metric.

\(^1\)The authors thanks Hishi Hideyuki for reporting this result.
Proof. Let $F \in \text{Aut}(M) \cap \text{Isom}(M, \omega)$ be a holomorphic isometry and let $\tilde{F}$ its lift to $L$ (which exists since $M$ is simply-connected). Note that, if $\{s_0, \ldots, s_N\}, \ N \leq \infty,$ is an orthonormal basis for $\mathcal{H}_{\lambda, h}$, then $\{\tilde{F}^{-1}(s_0(F(x))), \ldots, \tilde{F}^{-1}(s_N(F(x)))\}$ is an orthonormal basis for $\mathcal{H}_{\lambda, \tilde{F}^* h}$. Therefore

$$
\epsilon_{\lambda}(x) = \sum_{j=0}^{N} \tilde{F}^* h^\lambda \left( \tilde{F}^{-1}(s_j(F(x))), \tilde{F}^{-1}(s_j(F(x))) \right)
= \sum_{j=0}^{N} h^\lambda (s_j(F(x)), s_j(F(x))) = \epsilon_{\lambda}(F(x)).
$$

Since $\text{Aut}(M) \cap \text{Isom}(M, \omega)$ acts transitively on $M$, $\epsilon_{\lambda}$ is forced to be constant. \hfill \Box

Proof of Theorem 1.1. By Lemma 2.1 there exists $\lambda > \lambda_0$, such that the Hilbert space $\mathcal{H}_{\lambda, h} \neq \{0\}$. By Lemma 2.2, $\epsilon_{\lambda}$ is a positive constant, so the coherent states map $f$ given by (4) is well defined. Moreover, by (5), we have that $f^* \omega_{FS} = \lambda \omega$, i.e. $\lambda \omega$ is projectively induced for all $\lambda > \lambda_0$. The conclusion follows by setting $\mu_0 > \lambda_0$. \hfill \Box

The main ingredients for the proof of Theorem 1.2 are the following two lemmata. The first one is the celebrated theorem of Dorfmeister and Nakajima which provides a positive answer to the so called Fundamental Conjecture formulated by Vinberg and Gindikin. The second one is a reformulation due to Engliš of the Berezin quantization result in terms of the $\epsilon$-function and Calabi’s diastasis function.

Lemma 2.3 (Dorfmeister–Nakajima [4]). A homogeneous Kähler manifold $(M, \omega)$ is the total space of a holomorphic fiber bundle over a homogeneous bounded domain $\Omega$ in which the fiber $\mathcal{F} = E \times \mathbb{C}$ is (with the induced Kähler metric) the Kähler product of a flat homogeneous Kähler manifold $E$ and a compact simply-connected homogeneous Kähler manifold $C$.

Lemma 2.4 (Engliš [5]). Let $\Omega \subset \mathbb{C}^n$ be a complex domain equipped with a real analytic Kähler form $\omega$. Then, $(\Omega, \omega)$ admits a Berezin quantization if the following two conditions are satisfied:

1. the function $\epsilon_{\lambda}(x)$ is a positive constant (i.e. $\lambda \omega$ is balanced) for all sufficiently large $\lambda$;
2. the function $e^{-D(x,y)}$ is globally defined on $\Omega \times \Omega$, $e^{-D(x,y)} \leq 1$ and $e^{-D(x,y)} = 1$, if and only if $x = y$.

Proof of Theorem 1.2 (a) $\Rightarrow$ (b). By Lemma 2.3 since a homogeneous bounded domain is contractible, $M$ is a complex product $\Omega \times \mathcal{F}$, where $\mathcal{F} = E \times \mathbb{C}$ is (with the induced metric) a Kähler product of a flat Kähler manifold $E$ and a compact simply-connected homogeneous Kähler manifold $C$. On the other hand $E$ is Kähler
flat and therefore $E = \mathbb{C}^k \times T$ where $T$ is a product of flat complex tori. Hence $M$ is a complex product $\Omega \times \mathbb{C}^k \times T \times \mathbb{C}$. Since we assumed $M$ contractible, the compact factor $T \times \mathbb{C}$ has dimension zero and $M = \Omega \times \mathbb{C}^k$. It is well-known that $\Omega$ is biholomorphic to a Siegel domain (see, [6] for a proof), therefore $\Omega \times \mathbb{C}^k$ is pseudoconvex and a classical result of Hormander (see [2]) asserting that the equation $\bar{\partial} u = f$ with $f$ $\bar{\partial}$-closed form has a global solution on pseudoconvex domains, assures us the existence of a global potential $\psi$ for $\omega$ (see also [11, 12], and the proof of Theorem 4 in [3] for an explicit construction of the potential $\psi$).

(b) $\Rightarrow$ (c). Let $\psi : M \to \mathbb{R}$ be a global Kähler potential for $(M, \omega)$. By Lemma 2.3, $M = \Omega \times \mathbb{C}^k \times T \times \mathbb{C}$. The compact factor $T \times \mathbb{C}$ is a Kähler submanifold of $M$, therefore the existence of a global Kähler potential on $M$ implies that $\dim(T \times \mathbb{C}) = 0$. So $M = \Omega \times \mathbb{C}^k$.

Consider the Hilbert space $\mathcal{H}_{\lambda, h}$ defined in (2). Since $\Omega \times \mathbb{C}^k$ is contractible the line bundle $L \sim M \times \mathbb{C}$. So, the holomorphic section $s \in \mathcal{H}_{\lambda, h}$ can be viewed as a holomorphic function $s : M \to \mathbb{C}$. As Hermitian metric $h$ over $L$ we can take the one defined by $h(s, s) = e^{-\psi} |s|^2$. Hence $\mathcal{H}_{\lambda, h}$ can be identified with the weighted Hilbert space $\mathcal{H}_{\lambda \psi}$ (see [7]), of square integrable holomorphic functions on $M = \Omega \times \mathbb{C}^k$, with weight $e^{-\lambda \psi}$, namely

$$\mathcal{H}_{\lambda \psi} = \left\{ s \in \text{Hol}(M) \mid \int_M e^{-\lambda \psi} |s|^2 \omega^n \frac{n!}{n!} < \infty \right\}. \quad (7)$$

Assume $\lambda > \lambda_0$ with $\lambda_0$ given by Lemma 2.1, so that $\mathcal{H}_{\lambda \psi} \neq \{0\}$. Let $\{s_j\}$ be an orthonormal basis for $\mathcal{H}_{\lambda \psi}$, then the reproducing kernel is given by

$$K_{\lambda \psi}(z, \bar{w}) = \sum_{j=0}^{\infty} s_j(z) \overline{s_j(w)}. \quad (8)$$

Then, the $\varepsilon$-function (defined in (3)) reads as:

$$\varepsilon_{\lambda}(z) = e^{-\lambda \psi(z)} K_{\lambda \psi}(z, \bar{z}). \quad (9)$$

Let $\psi(z, \bar{w})$ be the analytic continuation of the Kähler potential $\psi$. By Lemma 2.2 there exists a constant $C$ such that

$$\varepsilon_{\lambda}(z, \bar{w}) = e^{-\lambda \psi(z, \bar{w})} K_{\lambda \psi}(z, \bar{w}) = C > 0. \quad (10)$$

Hence $K_{\lambda \psi}(z, \bar{w})$ never vanish. Therefore, for any fixed point $z_0$, the function

$$\Phi(z, \bar{w}) = \frac{K_{\lambda \psi}(z, \bar{w}) K_{\lambda \psi}(z_0, \bar{z}_0)}{K_{\lambda \psi}(z, \bar{z}_0) K_{\lambda \psi}(z_0, \bar{w})} \quad (10)$$

is well defined. Note that

$$D_{z_0}(z) = \frac{1}{\lambda} \log \Phi(z, \bar{z}) \quad (11)$$
is the diastasis centered in \( z_0 \) associated to \( \omega \) and that \( D_{z_0} \) is defined on whole \( M \). Since we can repeat this argument for any \( z_0 \in M \), we conclude that the diastasis \( D : M \times M \to \mathbb{R} \) is globally defined.

(c) \( \Rightarrow \) (d). Arguing as in “(b) \( \Rightarrow \) (c)”, the existence of a global diastasis implies that \( M \) is a complex product \( \Omega \times \mathbb{C}^k \). Therefore, as in [11] \( \mathcal{H}_{\lambda, h} \cong \mathcal{H}_{\lambda D_{z_0}} = \{ s \in \text{Hol}(M) \mid \int_M e^{-\lambda D_{z_0} x^2} |s|^2 \omega^n \omega^n! < \infty \} \). Assume \( \lambda > \lambda_0 \) with \( \lambda_0 \) given by Lemma 2.1 and consider the coherent states map \( f \) given by \( [4] \). By Lemma 2.2 \( \varepsilon_\lambda \) is a positive constant and by \( [5] \) we conclude that \( f^* \omega_{FS} = \lambda \omega \).

By [8, Example 6], the Calabi’s diastasis function \( D_{FS} \) associated to \( \omega_{FS} \) is such that \( e^{-D_{FS}} \) is globally defined on \( \mathbb{C}P^N \times \mathbb{C}P^N \). Since the diastasis \( D \) is globally defined on \( M \), by the hereditary property of the diastasis function (see [1, Proposition 6]) we get that, for all \( x, y \in M \),

\[
  e^{-D_{FS}}(f(x), f(y)) = e^{-\lambda D(x, y)} = \left( e^{-D(x, y)} \right)^\lambda
\]

is globally defined on \( M \times M \). Since, by [8, Example 6], \( e^{-D_{FS}(p, q)} \leq 1 \) for all \( p, q \in \mathbb{C}P^N \) it follows that \( e^{-D(x, y)} \leq 1 \) for all \( x, y \in M \). By Lemma 2.3 it remains to show that \( e^{-D(x, y)} = 1 \) iff \( x = y \). By (12) and by the fact that \( e^{-D_{FS}(p, q)} = 1 \) iff \( p = q \) (again by [8, Example 6]) this is equivalent to the injectivity of the coherent states map \( f \). This follows by [3, Theorem 3], which asserts that a Kähler immersion of a homogeneous Kähler manifold into a finite or infinite dimensional complex projective space is one to one.

(d) \( \Rightarrow \) (a). By the very definition of Berezin quantization there exists a global potential for \((M, \omega)\). By Lemma 2.3 we deduce, as above, that \( M \) is a complex product \( \Omega \times \mathbb{C}^k \), where \( \Omega \) is a bounded homogeneous domain which is contractible. □

**Proof of Theorem 1.3** By (c) in Theorem 1.2 the diastasis \( D : M \times M \to \mathbb{R} \) is globally defined. Assume that \( \lambda \omega \) is balanced i.e. that \( \varepsilon_\lambda \) is a positive constant. Since the \( \varepsilon_\lambda \) does not depend on the Kähler potential, by (9) we have

\[
  \varepsilon_\lambda(z, \overline{w}) = e^{-\lambda D_{z_0}(z, \overline{w})} K_{\lambda D_{z_0}}(z, \overline{w}) = C
\]

where \( D_{z_0}(z, \overline{w}) \) denote the analytic continuation of \( D_{z_0} \) and \( K_{\lambda D_{z_0}} \) is the reproducing kernel for \( \mathcal{H}_{\lambda D_{z_0}} \). By (10), with \( \overline{w} = \overline{z}_0 \), we get \( D_{z_0}(z, \overline{z}_0) = \frac{1}{\lambda} \log \Phi(z, \overline{z}_0) = 0 \). Hence

\[
  C = e^{-\lambda D_{z_0}(z, \overline{z}_0)} K_{\lambda D_{z_0}}(z, \overline{z}_0) = K_{\lambda D_{z_0}}(z, \overline{z}_0) \in \mathcal{H}_{\lambda D_{z_0}}.
\]

Thus \( \mathcal{H}_{\lambda D_{z_0}} \) contains the constant functions and

\[
  \int_M e^{-\lambda D_{z_0}} \omega^n \frac{\omega^n!}{n!} < \infty.
\]
Therefore, by definition of diastatic entropy,

$$\text{Ent}(M, \omega)(z_0) < \lambda < \infty.$$  

On the other hand, if for some $z_0$, $\text{Ent}(M, \omega)(z_0) < \lambda$, then $\mathcal{H}_{AD_{z_0}} \neq \{0\}$ and by Lemma 2.2 we conclude that $\lambda \omega$ is balanced. \(\square\)

Remark 2.5. From the previous proof, we see that if $M$ is simply-connected, then $\text{Ent}(M, \omega)(z_0) = \lambda_0$ for any $z_0 \in M$, where $\lambda_0$ is the positive constant defined in Lemma 2.1.

References

[1] E. Calabi, *Isometric Imbeddings of Complex Manifolds*, Ann. of Math. 58 (1953), 1-23.
[2] S. C. Chen, M. C. Shaw, *Partial differential equations in several complex variables*, AMS/IP Studies in Advanced Mathematics, 19. American Mathematical Society, Providence, RI; International Press, Boston, MA, (2001), xii+380 pp.
[3] A. J. Di Scala, A. Loi, H. Hishi, *Kähler immersions of homogeneous Kähler manifolds into complex space forms*, to appear in Asian Journal of Mathematics.
[4] J. Dorfmeister, K. Nakajima, *The fundamental conjecture for homogeneous Kähler manifolds*, Acta Math. 161 (1988), no. 1-2, 23-70.
[5] M. Engliš *Berezin Quantization and Reproducing Kernels on Complex Domains*, Trans. Amer. Math. Soc. vol. 348 (1996), 411-479.
[6] S. G. Gindikin, I. I. Piatetskii-Shapiro, E. B., *Classification and canonical realization of complex bounded homogeneous domains*, Trans. Moscow Math. Soc. 12 (1963), 404-437.
[7] A. Loi, R. Mossa, F. Zuddas, *The log-term of the disc bundle over a homogeneous Hodge manifold*, arXiv:1402.2089.
[8] A. Loi, R. Mossa, *Berezin quantization of homogeneous bounded domains*, Geom. Dedicata 161 (2012), 119-128.
[9] A. Loi, R. Mossa, *The diastatic exponential of a symmetric space*, Math. Z. 268 (2011), 3-4, 1057-1068.
[10] R. Mossa, *The volume entropy of local Hermitian symmetric space of noncompact type*, Differential Geom. Appl. 31 (2013), no. 5, 594-601.
[11] R. Mossa *A bounded homogeneous domain and a projective manifold are not relatives*, Riv. Mat. Univ. Parma 4 (2013), no. 1, 55-59.
[12] R. Mossa, *A note on diastatic entropy and balanced metrics*, J. Geom. Phys. 86 (2014), 492-496.
[13] R. Mossa, *Upper and lower bounds for the first eigenvalue and the volume entropy of non-compact Kähler manifold*, arXiv:1211.2705 [math.DG].
[14] R. Mossa, *Balanced metrics on homogeneous vector bundles*, Int. J. Geom. Methods Mod. Phys. 8 (2011), no. 7, 1433-1438.
[15] A. Loi, R. Mossa, *Uniqueness of balanced metrics on complex vector bundles*, J. Geom. Phys. 61 (2011), no. 1, 312-316.
[16] J. Rawnsley, *Coherent states and Kähler manifolds*, Quart. J. Math. Oxford (2), n.28 (1977), 403-415.
[17] J. Rosenberg, M. Vergne, *Harmonically induced representations of solvable Lie groups* J. Funct. Anal., 62 (1985), pp. 8-37.
[18] A. Loi, M. Zedda, \textit{Kähler-Einstein submanifolds of the infinite dimensional projective space}, Math. Ann. 350 (2011), 145-154.