TIME PARALLEL GRAVITATIONAL COLLAPSE SIMULATION

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Abstract. This article demonstrates the applicability of the parallel-in-time method Parareal to the numerical solution of the Einstein gravity equations for the spherical collapse of a massless scalar field. To account for the shrinking of the spatial domain in time, a tailored load balancing scheme is proposed and compared to load balancing based on number of time steps alone. The performance of Parareal is studied for both the sub-critical and black hole case; our experiments show that Parareal generates substantial speedup and, in the super-critical regime, can reproduce Choptuik’s black hole mass scaling law.

1. Introduction

Einstein’s field equations of general relativity (GR) consist of ten coupled, non-linear, hyperbolic-elliptic partial differential equations (PDEs). Because all forms of energy couple to gravity, there is an enormous dynamic range of spatiotemporal scales in GR. Hence, usually only the application of advanced numerical methods can provide solutions and in numerical relativity [2, 4] extensive use of high-performance computing (HPC) is made [27, 33].

Today, almost all HPC architectures are massively parallel systems connecting large numbers of compute nodes by a high-speed interconnect. In numerical simulations, the power of these systems can only be harnessed by algorithms that feature a high degree of concurrency; every algorithm with strong serial dependencies can only provide inferior performance on massively parallel computers. For the solution of PDEs, parallelization strategies have been developed mainly for spatial solvers. However, in light of the
rapid increase in the number of cores in supercomputers, methods that offer additional concurrency along
the temporal axis have recently begun to receive more attention.

The idea of parallelization-in-time was introduced in 1964 [35]. In the 1980s and 1990s, time and spacetime multigrid methods were studied [23–25]. More recently, the now widely used time parallel method Parareal was proposed [32]. Other recently introduced parallel-in-time methods are PFASST [12, 34], RIDC [10], or MGRIT [13]. A historical overview is offered in [18].

Given the demonstrated potential of parallel-in-time integration methods for large-scale parallel simulations [42, 44], these methods could be beneficial for the numerical relativity community. However, their application is not straightforward and often it is unclear a priori if good performance can be achieved. In this article, we therefore investigate the principal applicability of the time parallel Parareal method to solving Einstein’s equations describing spherical, gravitational collapse of a massless scalar field. The system is also referred to as an Einstein-Klein-Gordon system because it is equivalent to a Klein-Gordon equation expressed in the context of GR, i.e. on a back-reacting, curved geometry. It defines a basic gravitational field theory and is of interest therefore not only in numerical relativity but also in, e.g., quantum gravity [26, 30, 46]. A summary of numerically derived results is given in [22]; the work by Choptuik [8] brought forward novel, physical results and is of particular interest here because we will show that Parareal correctly reproduces the expected mass scaling law.

Mathematical theory shows that Parareal performs well for diffusive problems with constant coefficients [20]. For diffusive problems with space- and time-dependent coefficients [43], coefficients changing in time [3], or jumping coefficients on complex geometries [31], numerical experiments suggest that Parareal continues to converge quickly. In contrast, even for simple constant-coefficient hyperbolic problems, theory shows that Parareal can be expected to converge too slowly to provide meaningful speedup [20] although special cases exist where reasonable convergence is achievable [16]. Generally, for problems with imaginary eigenvalues and in particular hyperbolic PDEs, some form of stabilization is required for Parareal to provide speedup [7, 19, 41]. Surprisingly, this seems not to be the case for the equations describing gravitational collapse; we demonstrate that plain Parareal can achieve significant speedup. A detailed analytical investigation of why this is the case would definitely be of interest but is left out for future work. One reason could be that we solve in characteristic coordinates for which the discretization is aligned with the directions of propagation [17, 29].

The article is structured as follows: In Section 2 we define the system of Einstein field equations that we solve using Parareal. In addition, we give details on the numerical approach and discuss the interplay between Parareal and the particular structure of the spatial mesh. In Section 3 we discuss the Parareal method. Then, in Section 4 numerical results are presented. Finally, in Section 5 we conclude with a summary and discussion.

2. Equations

2.1. Gravitational collapse. The Einstein field equations in Planck units normalized to $4\pi G/c^4 = 1$ are

\[ G_{\mu\nu} = 2T_{\mu\nu}, \]  

(2.1.1)

where $\mu, \nu \in \{0, 1, 2, 3\}$ index time (via 0) and space (via 1, 2, and 3).\(^1\) Once the non-gravitational matter content is specified by a definition of the energy-momentum tensor $T_{\mu\nu}$, possibly along with equations of state that together satisfy the continuity equations $\nabla^\mu T_{\mu\nu} = 0$, Equation (2.1.1) defines a set of ten partial differential equations for ten unknown metric tensor field components $g_{\mu\nu}$.\(^2\) In all generality, the equations are coupled, non-linear, and hyperbolic-elliptic in nature. Six of the ten equations are hyperbolic evolution equations, while the remaining four are elliptic constraints on the initial data; they represent the freedom to choose spacetime coordinates. For the matter content, we consider a minimally coupled massless scalar field $\phi$ with energy-momentum tensor

\[ T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi. \]  

(2.1.2)

\(^1\)We omit the addition of the cosmological constant term $\Lambda g_{\mu\nu}$ on the left-hand side in Equation (2.1.1) because observations suggest $0 < \Lambda \ll 1$ (see, e.g., [28]); the term’s impact on black hole formation as studied here can be neglected.

\(^2\)We use the Einstein summation convention.
For the metric tensor field $g_{\mu\nu}$ in spherical symmetry it is natural to introduce a parametrization in terms of Schwarzschild coordinates $(t, r)$. Here, $t$ is the time coordinate of a stationary observer at infinite radius $r$, which measures the size of spheres centered at $r = 0$. In [8] the resulting Einstein field equations are analyzed numerically. In particular, adaptive mesh refinement [5] is used to resolve the black hole formation physics. In [21] the same investigation is carried out in double null or characteristic coordinates $(\tau, \rho)$ without mesh refinement (see, however, [39, 45]). Finally, in [30] the effect of quantum gravity modifications on the collapse is studied in adjusted characteristic coordinates. Here we use characteristic coordinates $(\tau, \rho)$ as well but exclude quantum gravity modifications. Also, for simplicity, we will refer to $\tau$ as a time coordinate and to $\rho$ as a space coordinate.

Making the \textit{ansatz}

\begin{equation}
(2.1.3) \quad g_{\mu\nu} dx^\mu dx^\nu = -2\partial_\tau r H d\tau d\rho + r^2 (d\theta^2 + \sin(\theta) d\phi^2)
\end{equation}

for the metric tensor field and using an auxiliary field $h$ for the spacetime geometry along with an auxiliary field $\Phi$ for the matter content, the complete field equations are

\begin{equation}
(2.1.4) \quad \partial_\tau r = \frac{h}{2}, \quad \partial_\tau \Phi = \frac{(H - h)(\Phi - \rho)}{2r},
\end{equation}

for $r$ and $\Phi$, and

\begin{equation}
(2.1.5) \quad \partial_\rho \phi = \frac{\partial_\rho r}{r} (\Phi - \phi), \quad \partial_\rho H = \frac{\partial_\rho r}{r} (\Phi - \rho)^2, \quad \partial_\rho h = \frac{\partial_\rho r}{r} (H - h),
\end{equation}

for $\phi$, $H$, and $h$ (see [21]). Overall the system can be seen as a wave equation for the massless scalar field $\phi$ on a back-reacting, curved geometry. Boundary conditions at $(\tau, \rho = 0)$ are $r \equiv 0$ and regularity of $\Phi$, $H$, and $h$, which implies $\Phi = \phi$ and $H = h$ at the boundary [11, 29]. Consistent initial data at $(\tau = 0, \rho)$ are

\begin{equation}
(2.1.6) \quad r = \frac{\rho}{2}, \quad \Phi = (1 + \rho \partial_\rho) \phi,
\end{equation}

where we choose for $\phi$ the Gaussian wave packet

\begin{equation}
(2.1.7) \quad \phi(0, \rho) = \phi_0 \frac{\rho^3}{1 + \rho^3} \exp \left(-\frac{[\rho - \rho_0]}{\delta_0} \right)^2.
\end{equation}

The initial scalar field configuration is thus characterized by an amplitude $\phi_0$, mean position $\rho_0$, and width $\delta_0$. Depending on the value of these parameters, the solution of Equations (2.1.4) and (2.1.5) can describe a bounce of the wave packet or black hole formation near the boundary at $r = 0$. A black hole appears when the outward null expansion

\begin{equation}
(2.1.8) \quad \Theta^+ = \frac{1}{\tau} \sqrt{\frac{2h}{H}},
\end{equation}

which measures the relative rate of change of a cross-sectional area element of a congruence of out-going null curves, approaches zero [36]. The black hole mass is

\begin{equation}
(2.1.9) \quad M = \frac{r}{2},
\end{equation}

evaluated at the point $(\tau^+, \rho^+)$ toward which $\Theta^+$ vanishes.

2.2. \textbf{Numerical solution.} The numerical grid is depicted in Figure 1a. It is parametrized by the characteristic coordinates $\tau$ and $\rho$, which are used for numerical integration; $\tau$ is used as coordinate representing time and $\rho$ as coordinate representing space. Integration thus takes place on a right triangle with initial data defined along the lower right-hand leg. Clearly, the spatial domain becomes smaller as the solution is advanced in $\tau$. Note that the domain is not exactly a right triangle because at the upper-most corner a small sub-triangle is missing. This "buffer" zone of extent $\lambda$ is needed for the spatial part of the numerical stencil to fit. The computational domain thus consists of all points $(\tau, \rho) \in [0, L - \lambda] \times [0, L]$ with $L = 80$, $\lambda = 0.625$, and $\rho \geq \tau$.

For the time parallel method Parareal (see Section 3), two serial time integration methods are required for the solution of the equations in (2.1.4). Here, we choose the explicit first-order Euler method on a coarse spacetime grid and the second-order Lax-Wendroff Richtmyer two-step method on a fine spacetime grid. For Parareal to be efficient, the cost of the coarse method has to be small compared to the cost of the fine
one: by choosing a simple first-order method on a coarse mesh for \( \mathcal{C} \) and a higher-order method on a finer mesh for \( \mathcal{F} \), we obtain a good coarse-to-fine ratio (see Section 3.4). In general, choosing a more accurate fine method would improve the coarse-to-fine ratio but might necessitate more iterations for Parareal to converge. For optimal speedup, a good balance between the difference in accuracy and difference in cost between \( \mathcal{C} \) and \( \mathcal{F} \) has to be found.

For the integration in space of the equations in (2.1.5) we use a second-order Runge-Kutta method \([29]\). Snapshots of scalar field evolution resulting from the chosen fine grid discretization are shown in Figure 1b, where \( \phi \) evolves along constant lines of \( \rho \) until a bounce occurs at \( r = 0 \). The figure also shows how the size of the domain decreases during the evolution: for \( \tau = 0 \) the left boundary is at \( \rho = 0 \) while for \( \tau = 20 \) it is at \( \rho = 20 \).

2.3. Mass scaling. In practice, the simulation terminates when a black hole forms because \( H \) grows without bound in this case (see \([11]\) for details). Figure 2a provides a simplified illustration of a black hole region (dotted portion) and shows where the simulation comes to a halt (dashed line). Thus, to determine the black hole mass \( M \), we record minimal expansion values via the scalar \( (r\Theta^+)_{\text{mi}} = \min_\rho \{r\Theta^+\} \) derived from Equation (2.1.8). The last such recorded minimal value before the termination of the simulation defines a characteristic coordinate \( (\tau^+, \rho^+) \) (see again Figure 2a), which we can use to define an \( r \) and \( M \) via Equation (2.1.9). The scalar \( (r\Theta^+)_{\text{mi}} \) approaches 0 when \( (\tau, \rho) \) nears \( (\tau^+, \rho^+) \), as is shown in the lower portion of Figure 2b.

Based on numerical experiments, Choptuik presents, among other things, a relation between the amplitude \( \phi_0 \) of the Gaussian in Equation (2.1.7) and the black hole mass \( M \) \([8]\). He shows that there is a critical value \( \phi_0^* \) such that for \( \phi_0 < \phi_0^* \) there is a bounce (sub-critical case), while for \( \phi_0 > \phi_0^* \) there is a black hole (super-critical case). Based thereon, he demonstrates that the black hole mass scales with \( \phi_0 - \phi_0^* > 0 \) according to the law \( M \propto (\phi_0 - \phi_0^*)^\gamma \) with \( \gamma \) being a positive constant of the same value for various initial data profiles. We demonstrate that Parareal can correctly capture this black hole mass scaling law although our coarse level Euler method alone cannot. Also, Parareal requires less wall-clock time than \( \mathcal{F} \), which can be beneficial for the investigation of the high-accuracy demanding critical solution \([8, 22]\) that requires the simulation of numerous black holes \([21]\). This analysis however is omitted in this article and left for future work.
3. PARAREAL

3.1. Algorithm. Parareal [32] is a method for the solution of initial value problems

\[ \partial_\tau u(\tau) = f(\tau, u(\tau)), \quad u(0) = u_0, \quad 0 \leq \tau \leq T. \]

Here, as is outlined in the previous section, \( f \) comes from discretizing Equations (2.1.4) and (2.1.5), and \( T = L - \lambda \) marks the end time. Parareal starts with a decomposition of the time domain into \( N_{pr} \) temporal subintervals (TSs) defined in terms of times \( \tau^p \) such that

\[ [\tau^1, \tau^2] \cup \ldots \cup [\tau^{N_{pr}-1}, \tau^{N_{pr}}] = [0, L - \lambda]. \]

Now denote by \( \mathcal{F} \) some serial time integration method of high accuracy and cost (in our case this is the second-order Lax-Wendroff Richtmyer two-step method), and by \( \mathcal{E} \) a cheap and possibly much less accurate method (in our case this is the explicit first-order Euler method). Instead of running the fine method subinterval by subinterval serially in time, Parareal performs the iteration

\[ u^{p+1}_{i+1} = \mathcal{E} \left( u^p_{i+1} \right) - \mathcal{E} \left( u^p_{i} \right) + \mathcal{F} \left( u^p_{i} \right), \]

where super-scripts index time or process number \( p \in \{1, \ldots, N_{pr} \} \) and sub-scripts iterations \( i \in \{1, \ldots, N_{it} \} \). The advantage is that the expensive computation of the fine method can be performed in parallel over all TSs at once. Here, we assume that the number of TSs is equal to the number \( N_{pr} \) of cores (or processes) used for the time direction. Good speedup can be obtained if \( \mathcal{E} \) is fast in comparison to \( \mathcal{F} \) but still accurate enough for Parareal to converge rapidly. See Section 3.4 for a more detailed discussion of Parareal’s speedup.

In Section 2.2 we hinted at the interchangeability of the characteristic coordinates \( \tau \) and \( \rho \) for the numerical integration. Therefore, theoretically, Parareal could also be used for the spatial integration to simultaneously parallelize both time and space. However, such an interweaving of two Parareal iterations is not discussed in this article; it is put aside for future work.

3.2. Spatial coarsening in Parareal. In order to make \( \mathcal{E} \) cheaper and improve speedup, we not only use a less accurate time stepper for \( \mathcal{E} \) but also employ a coarsened spatial discretization with a reduced number of degrees-of-freedom. Therefore, we need a spatial interpolation \( I \) and restriction \( R \) operator. In this case (see, e.g., [14]), the Parareal algorithm is given by

\[ u^{p+1}_{i+1} = I \mathcal{E} \left( Ru^p_{i+1} \right) - I \mathcal{E} \left( Ru^p_{i} \right) + \mathcal{F} \left( u^p_{i} \right). \]
3.3. Implementation. We have implemented two different realizations of Parareal. In a “standard” version \( \mathcal{P}_{st} \) (see Listing 3a), the Parareal correction is computed on each TS up to a uniformly prescribed iteration number. In contrast, in the “modified” implementation \( \mathcal{P}_{mo} \) (see Listing 3b), Parareal corrections are only performed on TSs where the solution may not yet have converged. Because Parareal always converges at a rate of at least one TS per iteration, we only iterate on a TS if its assigned MPI rank is greater than or equal to the current Parareal iteration number (see line 8 in Listing 3b). Otherwise, no further iterations are needed and performed, and the process remains idle. As the iteration progresses, more and more processes enter an idle state, which reduces communication and, as is shown below, runtime and energy consumption. In an implementation to be realized in future work, the criterion for convergence used here will be replaced by a check for some residual tolerance.

Another difference between the standard and modified implementation is that in the former, after each time parallel fine evolution, a copy of the fine grid solution has to be created (see line 10 in Listing 3a). In the modified Listing 3b this copying is circumvented by the use of two alternating indices “\( j \)” and “\( k \)” in lines 9 and 10, respectively. The iteration number determines their value which, in turn, determines the fine grid solution buffer that is used to send or receive data by means of the corresponding MPI routines (see lines 14 and 22 in Listing 3b). The two implementations also have slightly different requirements in terms of storage. As can be seen in line 15 in Listing 3a, in \( \mathcal{P}_{st} \) on the first TS or, equivalently, for the first MPI rank, the fine grid solution has to be assigned initial data at the beginning of each iteration. This requires one additional buffer to be held in storage. Other than that both implementations need one coarse grid solution buffer and three fine grid buffers for each TS.

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(a) The standard Parareal implementation \( \mathcal{P}_{st} \).

(b) The modified Parareal implementation \( \mathcal{P}_{mo} \).

\begin{verbatim}
1 if p > 1 then
2    Coarse(co; τ^1 → τ^p)
3    Interp(co → fi [0])
4 if p < N_p then
5    Coarse(co; τ^p → τ^p+1)
6    Interp(co → fi [2])
7 for i = 1 : N_s do
8    if p < N_p then
9        Fine(fi [0]; τ^p → τ^p+1)
10       fi [1] = fi [0]
11      fi [1] := fi [2]
12    if p > 1 then
13       MPIRecv (fi [0]; p ← p − 1)
14 else
15      Init(fi [0])
16      Restrict(fi [0] → co)
17 if p < N_p then
18    Coarse(co; τ^p → τ^p+1)
19    Interp(co → fi [2])
20    fi [1] := fi [2]
21 if p < N_p then
22    MPISend (fi [1]; p ⇒ p + 1)
\end{verbatim}

\begin{verbatim}
1 if p > 1 then
2    Coarse(co; τ^1 → τ^p)
3    Interp(co → fi [0])
4 if p < N_p then
5    Coarse(co; τ^p → τ^p+1)
6    Interp(co → fi [2])
7 for i = 1 : N_s do
8    if p := i then
9      j = (i+1) % 2
10     k = i % 2
11 if p < N_p then
12    Fine(fi [j]; τ^p → τ^p+1)
13 if p > i then
14    MPIRecv (fi [k]; p ← p − 1)
15    fi [j] := fi [2]
16    Restrict(fi [k] → co)
17 if p < N_p then
18    Coarse(co; τ^p → τ^p+1)
19    Interp(co → fi [2])
20    fi [j] := fi [2]
21 if p < N_p then
22    MPISend (fi [j]; p ⇒ p + 1)
\end{verbatim}

Figure 3. Pseudo code for the standard and modified Parareal implementation. The variable “\( co \)” is a coarse grid solution buffer and “\( fi \)” an array of three fine grid solution buffers.

\footnote{We also tested barycentric interpolation \([6, 15]\) but found the performance in terms of runtimes and speedup (see Sections 3.4 and 4) to be inferior.}
3.4. **Speedup.** We denote by $R_{co}$ the coarse and by $R_{fi}$ the fine time stepper’s runtime. Recalling that $N_{it}$ denotes the number of iterations required for Parareal to converge given $N_{pr}$ processes, Parareal’s theoretically achievable speedup is

\[
S = \left(1 + \frac{N_{it}}{N_{pr}}\right) \frac{R_{co}}{R_{fi}} + \frac{N_{it}}{N_{pr}} \frac{R_{co}}{R_{fi}} \leq \min \left\{ \frac{N_{pr}}{N_{it}}, \frac{R_{fi}}{R_{co}} \right\},
\]

as is discussed, e.g., in [34]. The estimate is valid only for the ideal case, where runtimes across subintervals are perfectly balanced. In the presence of load imbalances in time however, *i.e.* differences in the runtimes of $C$ and $F$ across TSs, maximum speedup is reduced [31]. Because the spatial domain we consider is shrinking in time, a tailored decomposition of the time axis has to be used to provide well balanced computational load, as is discussed in the next section.

3.5. **Load balancing.** Because we integrate over a triangular computational spacetime domain (see Figure 1a), a straight forward, uniform partitioning of the time axis results in *unevenly* distributed computational load in time. The first load balancing (LB) strategy, which henceforth we will refer to as LB$_{un}$, is based on this straight forward, basic decomposition of the time axis. It assigns to each TS the same number of time steps without regard for their computational cost. Because of the shrinking domain, TSs at later times carry fewer spatial degrees-of-freedom so that the per-process runtimes $R_{co}$ and $R_{fi}$ of the coarse and fine time stepper are larger for the earlier TSs than for the later ones. Figure 4a shows how this partition leads to an uneven distribution of computational load in time because the portion extending across the “early-middle” TS $[e, m]$ covers a larger area and thus a larger number of grid points than the portion over the “middle-late” TS $[m, l]$.

Figure 4 suggests that early in time TSs should have a shorter extent in time than later ones. Thus, in the second strategy, which in the following we will refer to as LB$_{ev}$, we also consider the *cost* of time steps in order to balance the runtime $R_{co} + R_{fi}$ over all processes $p$. We use a decomposition of the time axis in TSs such that the sum of the total coarse and fine runtime is *evenly* distributed over all TSs, *i.e.* such that $R_{co} + R_{fi} = N_{pr} (R_{co}^p + R_{fi}^p)$ for any process $p$. This is done by a bisection approach, making use of the fact the we use explicit rather than implicit time integrators (cf. the discussion in [31]), and thus that the cost of a time step $\tau \rightarrow \tau + \Delta \tau$ is directly proportional to the number of spatial degrees-of-freedom at $\tau$. Therefore, the total spacetime domain is first divided into two parts of roughly equal number of grid points as is sketched in Figure 4b. Then, each part is divided again and again until the required number of TSs is reached. Note that this limits the possible numbers of TSs to powers of 2.
Figure 5 shows Vampir\(^4\) traces for simulations featuring LB\(_{\text{un}}\) (Figure 5a) and LB\(_{\text{ev}}\) (Figure 5b). The horizontal axes correspond to runtime, while the vertical axes depict MPI rank numbers from 1 (lower) to 8 (upper). In each case, three Parareal iterations are performed. Green regions indicate the coarse and fine integrators carrying out work. Time spent in MPI receives (including waiting time) is shown in red. We observe how LB\(_{\text{un}}\) leads to load imbalance and incurs significant wait times in processes handling later TS. In contrast, the processes’ idle times (shown in red) in MPI receives are almost invisible in the case of LB\(_{\text{ev}}\). Elimination of wait times leads to a significant reduction in runtime and increase in speedup, as will be shown in Section 4.

![Vampir trace for LB\(_{\text{un}}\). The Parareal runtime is \(R_{\text{pa}} = 7.964\) (s).](image1)

![Vampir trace for LB\(_{\text{ev}}\). The Parareal runtime is \(R_{\text{pa}} = 5.436\) (s).](image2)

**Figure 5.** Vampir traces for the implementation \(\mathcal{P}_{\text{mo}}\) with \((N_{\text{pr}}, N_{\text{it}}) = (8, 3)\) for two different load balancing strategies.

### 3.6. Power consumption.

The simulations for this article were performed on the Cray XC40 supercomputer Piz Dora\(^5\) at the Swiss National Supercomputing Centre (CSCS) in Lugano, Switzerland. It features 1,256 compute nodes, which all hold two 12-core Intel Xeon E5-2690v3 CPUs. This results in a total of 30,144 compute cores and a peak performance of 1.254 PFlops; it occupies position 56 in the Top500 November, 2014 list.\(^6\) On Piz Dora, we used the GNU compiler collection\(^7\) version 4.9.2 and the runtimes we provide do not include the cost of I/O operations.

The formula for power consumption on Piz Dora including AC-DC conversion is\(^8\)

\[
P = \left[ \frac{E}{R_{\text{pa}}} + N_{\text{no}} \left( \frac{4,400}{192} + \frac{100}{4} \right) \right] \frac{100}{95} (\text{W}),
\]

where the power \(P\), energy \(E\), and Parareal runtime \(R_{\text{pa}}\) depend on \((N_{\text{pr}}, N_{\text{it}})\). The first term defines the power consumption of the compute nodes, of which there are \(N_{\text{no}} = \text{ceil}(N_{\text{pr}}/24)\) because each Piz Dora node is comprised of 24 processors. As for the second term, 4,400 (W) is the demand of the blowers per rack, which hold about 192 nodes. Finally, 100 (W) is the power consumption of the Aries\(^9\) chip in each blade; each blade contains four nodes. The factor 100/95 is for the AC-DC conversion. A simple model for energy overhead of Parareal is discussed in [3].

Figure 6a shows Parareal runtime measurements for LB\(_{\text{ev}}\), polynomial interpolation of order 5, and for the standard implementation (i.e. \(\mathcal{P}_{\text{st}}\)) as well as the modified one (i.e. \(\mathcal{P}_{\text{mo}}\)). The values shown are averages over eight simulations to reduce the impact of noise; the relative standard deviations are smaller.

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\(^4\)https://www.vampir.eu/

\(^5\)http://user.cscs.ch/computing_systems/piz_dora/

\(^6\)http://www.top500.org/list/2014/11

\(^7\)https://gcc.gnu.org

\(^8\)Based on personal communication with Gilles Fourestey from the Swiss National Supercomputing Center (CSCS) in Lugano, Switzerland and with Andrea Arteaga from the Swiss Federal Institute of Technology Zurich (ETHZ), Switzerland.

\(^9\)http://www.arieselec.com/
than 0.7%. The plot shows that the modified implementation can slightly outperform the standard one in terms of runtime for large iteration numbers.

Figure 6b shows the power consumption of Parareal for the same configurations as in Figure 6a. Again, we created statistical ensembles of size equal to 8 for each setting to determine averages; here, the relative standard deviations are smaller than 5%. Interestingly, there is a noticeable difference between the two versions in power uptake, despite the small differences in runtime. This probably is related to the fact that in the modified implementation, as \( N_{\text{it}} \) increases, the number of nodes that are carrying out work decreases (see Section 3.3). As the iteration continues, more and more cores are becoming idle and could in principle be powered off. These idle cores apparently need less power, either because the system is actually reducing, e.g., their frequency or simply because no memory accesses etc. are required. In the remainder of this article we use \( P_{\text{mo}} \). Since Parareal does not converge at all for third-order interpolation until \( i = N_{\text{it}} \) (see Section 4.1), we use fifth-order interpolation.

4. Results

For the results presented here we use a coarse grid resolution of \( (\Delta \tau)_{\text{co}} = (\Delta \rho)_{\text{co}} = \Delta_{\text{co}} = L/2,048 \approx 0.039 \) and a fine grid resolution of \( \Delta_{\text{f}} = \Delta_{\text{co}}/8 \approx 0.005 \). We have also determined a reference solution to approximately measure the serial fine stepper’s discretization error. For this we have used again the serial fine time stepper but with a step size of \( \Delta_{\text{re}} = \Delta_{\text{f}}/4 \approx 0.001 \).

4.1. Sub-critical. First we consider the sub-critical case, where no black holes form. Figure 7 shows for \( N_{\text{pr}} = 512 \) and two different sets of initial data parameters the relative defect

\[
D_{[i]} = \frac{\| r_{[i]} - r_{\text{f}} \|_2}{\| r_{\text{f}} \|_2},
\]

which measures the difference between the Parareal solution \( r_{[i]} \) after \( i \) iterations and the serial fine solution \( r_{\text{f}} \) as a function of the characteristic coordinate \( \tau \).

In Figure 7a we use the initial data parameters \( (\phi_0, \rho_0, \delta_0) = (0.035, 20, 1) \), which results in an “early” bounce of the wave packet at about \( \tau = 20 \). For the simulations in Figure 7b, the values are \( (\phi_0, \rho_0, \delta_0) = (0.01, 75, 1) \), which leads to a “late” bounce at about \( \tau = 75 \). Defects are plotted for \( N_{\text{it}} \in \{1, 2, 3, 4\} \) along with the serial fine solution’s estimated discretization error \( \| r_{\text{re}} - r_{\text{f}} \|_2/\| r_{\text{f}} \|_2 \) labeled “Fine”. Parareal converges in two iterations: for \( N_{\text{it}} = 2 \), the defect is below the discretization error for all \( \tau \). In fact, without the bounce region near \( \tau = 20 \), only one iteration would be required for convergence. For the late bounce scenario in Figure 7b, we also observe that the rate of convergence at the final time \( \tau = L - \lambda \) gives
an indication for the convergence at all $\tau$. In the following we thus focus on convergence at the final time. Convergence for the other evolved field $\Phi$ is not shown but was found to be at least as good as for $r$.

![Graphs showing defect of Parareal over time for different scenarios.](image)

**Figure 7.** Defect in $r$ between Parareal and fine method over time for fixed $N_{pr} = 512$.

Figures 8a and 8b illustrate the defect of Parareal at the end of the simulation at $\tau = L - \lambda$ for various values of $N_{pr}$ with third-order interpolation (left) and fifth-order interpolation (right). For third-order interpolation, Parareal does not converge at all. The configuration stalls at a defect of about $10^{-2}$ until the iteration count equals $N_{pr}$. There, Parareal converges by definition but cannot provide any speedup. In contrast, Parareal shows good convergence behavior for fifth-order interpolation. For $N_{pr}$ less than 64, the defect of Parareal falls below the approximate discretization error of the fine method after a single iteration. Otherwise, for $N_{pr} \geq 64$ up to $N_{pr} = 512$, two iterations are required. The resulting speedups with correspondingly adjusted values for $N_{it}$ are shown in Figure 8c for both load balancing strategies (see the discussion in Section 3.5). In addition, the projected speedup according to Equation (3.4.1) is shown. Up to $N_{pr} = 64$, for the advanced load balancing, speedup closely mirrors the theoretical curve while the basic load balancing performs significantly worse. For $N_{pr} \geq 64$, measured speedups fall short of the ideal values, peak at $N_{pr} = 256$, and then start to decrease.

Although the load balancing strategy $LB_{ev}$ results in significantly better speedup than the basic approach $LB_{un}$, the peak value provided by both schemes is essentially the same. This is because for increasingly large numbers of cores, the computational load per TS eventually becomes small and imbalances in computational load insignificant. Instead, runtime is dominated by overhead from, e.g., communication in time. The communication load is independent of the chosen load balancing and depends solely on the number of TSs; for every TS one message has to be sent and received once per iteration (save for the first and last TS). Therefore, it can be expected that ultimately both approaches to load balancing lead to comparable peak values. Below we demonstrate that the saturation in speedup is related to a significant increase in time spent in MPI routines; eventually, communication cost starts to dominate over the computational cost left on each time slice and the time parallelization saturates just as spatial parallelization does.

Figure 9 illustrates the reason behind the drop-off in speedup beyond $N_{pr} = 256$. First, define

\[
R_{p_{st}}^p = R_{p_{co}}^p + R_{p_{fi}}^p + \sum_{st} R_{p_{st}}^p,
\]

where $R_{p_{st}}^p$ denotes runtime spent in *stages* that are different from coarse and fine integration on the TS assigned to process $p$. For now, we consider only overhead from sending and receiving data as well as from interpolation, other overheads are not further analyzed here. Next, we introduce the *total* overhead on a
Figure 8. Parareal’s performance for the subcritical case in terms of convergence for polynomial interpolation orders 3 and 5, and speedup.

TS as the sum of all stage-runtimes

\[ O_{\text{to}}^p = \sum_{\text{st}} R_{\text{st}}^p, \]

which is also the runtime spent neither in the coarse nor fine integrator for a given \( p \). The average overhead is now defined as the geometric mean value of \( O_{\text{to}}^p \) over all TSs, which is

\[ O_{\text{av}} = \left( \prod_{p=1}^{N_{\text{pr}}} O_{\text{to}}^p \right)^{1/N_{\text{pr}}}. \]

Finally, we define the relative overhead for individual stages on a TS as

\[ O_{\text{st}}^p = \frac{R_{\text{st}}^p}{R_{\text{pa}}^p}, \]

where \( R_{\text{pa}}^p \) is the runtime of Parareal at processor \( p \). Ideally, as is assumed for the derivation of the speedup model given in Equation (3.4.1), \( R_{\text{co}}^p \) and \( R_{\text{fi}}^p \) are the dominant costs. In this case, \( R_{\text{co}}^p + R_{\text{fi}}^p \approx R_{\text{pa}}^p \) so that according to Equation (4.1.2) we have \( O_{\text{to}}^p \approx 0 \) and therefore \( O_{\text{av}} \approx 0 \) by definition. However, as can be seen in Figure 9a, \( O_{\text{av}} \) is small only for small values of \( N_{\text{pr}} \). For \( N_{\text{pr}} \geq 32 \) it increases rapidly, which indicates that the overhead from communication and other sources starts to play a more dominant role when \( N_{\text{pr}} \) is increased.

Figure 9b shows the relative overhead from Equation (4.1.5) for \( N_{\text{pr}} \in \{32, 512\} \) and \( p \in \{1, \ldots, N_{\text{pr}}\} \) for the three different stages \( \text{st} \in \{\text{Interpolation, Send, Receive}\} \); “Send” and “Receive” are referring to the corresponding MPI routines. There is a significant increase in relative overhead in all three stages as the number of cores grows, causing the eventual drop-off in speedup for increasing \( N_{\text{pr}} \).

4.2. Super-critical. We consider now the more complex case in which a black hole forms at some time during the simulation. The goal is to compute the black hole’s position via Equation (2.1.8) so that its mass can be determined from Equation (2.1.9) (see Section 2.3). Because the characteristic coordinates \((\tau, \rho)\) do not allow us to continue the simulation past the black hole formation event, we need a way to keep the simulation from terminating when \( \Theta^+ \) approaches 0 (see Figure 2b).

To avoid the need to adaptively modify the decomposition of the time domain, we carry out the super-critical case study using initial data parameter values near \((\phi_0, \rho_0, \delta_0) = (0.01, 75.1, 1)\), which we have also used for the results in Figure 7b. With these parameters and in particular for \( \phi_0 \geq 0.01 \), for all investigated partitions of the time axis with \( N_{\text{pr}} \leq 256 \), the black hole generated by the fine time integrator forms in the last TS unless \( \phi_0 \) becomes too large (\( \rho_0 \) and \( \delta_0 \) are fix). Thus, Parareal can be used over all TSs except
for the last one, where only the fine method is executed to compute the black hole’s position. The C++ implementation uses a try-throw-catch approach to prevent complete termination of the simulation; if the radicand in the definition of $\Theta^+$ in Equation (2.1.8) fails to be non-negative, an exception is thrown such that the Parareal iteration can continue. As the Parareal iteration converges and better and better starting values are provided for $\mathcal{F}$ on the last TS, the accuracy of the computed black hole position improves. A more general implementation aiming at production runs would need to allow for black hole formation in TSs before the last one but this is left for future work. In this article, the focus lies on investigating the principal applicability of Parareal to the simulation of gravitational collapse.

Figure 10 depicts the Choptuik scaling that results from solutions computed with Parareal for $N_{pr} = 256$ after the first three iterations. Table 1 lists the generated values of $\phi_0^*$ and $\gamma$ (see Section 2.3), and errors compared to the value provided by the fine integrator, which agrees with the result in [21]. As can be seen in Figure 10a, the coarse integrator $C$ alone cannot adequately resolve black holes with $\phi_0 - \phi_0^* \lesssim 10^{-9}$ (they are too small for $C$ to be “visible”) and its $\gamma$ is wrong by about 20%. This means that the coarse method is too “coarse” in the sense that, on its own, it cannot correctly capture the physics underlying the

*Figure 9. Overhead from communication and other sources increases with $N_{pr}$, which leads to Parareal’s speedup decay.*

*Figure 10. Parareal’s performance for the super-critical case.*
investigated problem. Nonetheless, Parareal is not only capable of generating the correct black hole physics but can do so after only one iteration.

| $\phi_0^*$          | Value | Error (%) | $\gamma$ | Value | Error (%) |
|---------------------|-------|-----------|----------|-------|-----------|
| Coarse              | 0.01057748 | 7.25 $\cdot 10^{-1}$ | 0.458    | 20.21 |
| $N_{it} = 1$        | 0.01055915 | 5.51 $\cdot 10^{-1}$ | 0.377    | 1.05  |
| $N_{it} = 2$        | 0.01050240 | 1.01 $\cdot 10^{-2}$ | 0.370    | 2.89  |
| $N_{it} = 3$        | 0.01050135 | 9.52 $\cdot 10^{-5}$ | 0.381    | 0     |
| Fine                | 0.01050134 | 0           | 0.381    | 0     |

Table 1. Approximate values and relative errors for the critical amplitude $\phi_0^*$ and resulting straight line slope $\gamma$.

Figure 10b visualizes the speedup achieved in the super-critical case including the theoretical estimate according to Equation (3.4.1). The numbers of iterations required for Parareal to converge are derived from an analysis just like the one plotted in Figure 8 for the sub-critical case and basically the values are identical. Up to 64 processes, good speedup close to the theoretical bound is observed. For larger core numbers however, speedup is reaching a plateau and performance is no longer increasing. As in the sub-critical case, as $N_{it}$ increases, the computing times per TS eventually become too small and Parareal’s runtime becomes dominated by, e.g., communication (see Figure 9). Even though the temporal parallelization eventually saturates, substantial acceleration of almost a factor of 30 using 128 cores in time is possible, corresponding to a parallel efficiency of about 23%.

5. Conclusion

The article assesses the performance of the parallel-in-time integration method Parareal for the numerical simulation of gravitational collapse of a massless scalar field in spherical symmetry. It gives an overview of the dynamics and physics described by the corresponding Einstein field equations and presents the employed numerical methods to solve them. Because the system is formulated and solved in characteristic coordinates, the computational spacetime domain is triangular so that later time steps carry fewer spatial degrees-of-freedom. A strategy for balancing computational cost per subinterval instead of just number of steps is presented and its benefits are demonstrated by traces using the Vampir tool. Numerical experiments are presented for both the sub- and super-critical case. Parareal converges rapidly for both and, for the latter, correctly reproduces Choptuik’s mass scaling law after only one iteration despite the fact that the used coarse integrator alone generates a strongly flawed mass scaling law. This underlines the capability of Parareal to quickly correct a coarse method that does not resolve the dynamics of the problem. The here presented results illustrate that Parareal and presumably other parallel-in-time methods as well can be used to improve utilization of parallel computers for numerical studies of black hole formation.

Multiple directions for future research emerge from the presented results. Evaluating performance gains for computing the critical solution [8, 22] would be valuable. Next, complexer collapse scenarios such as in the Einstein-Yang-Mills system [9], axial symmetry [37], or binary black hole spacetimes [38] could be addressed. An extended implementation of Parareal could utilize a more sophisticated convergence criterion, a more flexible black hole detection, and parallelism in space via, e.g., again Parareal. The latter would be possible because the integration along the characteristic we took to represent space is for the solution of initial value problems just like in the temporal direction. Another topic of interest is that of adaptive mesh refinement [1]: how it can be used efficiently in connection with Parareal or other time parallel methods seems to be an open problem. As discussed in the introduction, a mathematical analysis of the convergence behavior of Parareal for Einstein’s equations would be of great interest as well, particularly since the good performance is unexpected in view of the negative theoretical results for simple hyperbolic problems. Finally, incorporating a parallel-in-time integration method into a software library widely used
for black hole or other numerical relativity simulations would be the ideal way to make this new approach available to a large group of domain scientists.\textsuperscript{10}

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\section*{References}

\begin{enumerate}
\item Private communication with Jonathan Thornburg from the Indiana University in Bloomington (IN), United States of America.
\item M. Alcubierre. \textit{Introduction to 3+1 Numerical Relativity}, volume 140 of \textit{International Series of Monographs on Physics}. Oxford University Press, Oxford, UK, 1\textsuperscript{st} edition, Jun 2008. ISBN 9780199205677.
\item A. Arteaga, D. Ruprecht, and R. Krause. A stencil-based implementation of Parareal in the C++ domain specific embedded language STELLA. \textit{Applied Mathematics and Computation}, 2015. DOI 10.1016/j.amc.2014.12.055.
\item T. W. Baumgarte and S. L. Shapiro. \textit{Numerical Relativity. Solving Einstein’s Equations on the Computer}. Cambridge University Press, Cambridge, UK, 1\textsuperscript{st} edition, Aug 2010. ISBN 9780521514071.
\item M. J. Berger and J. Oliker. Adaptive mesh refinement for hyperbolic partial differential equations. \textit{Journal of Computational Physics}, 53(3):484–512, Mar 1984. DOI 10.1016/0021-9991(84)90073-1. URL http://www.sciencedirect.com/science/article/pii/0021999184900731.
\item J.-P. Berrut and L. N. Trefethen. Barycentric Lagrange Interpolation. \textit{SIAM Review}, 46(3):501–517, Sep 2004. DOI 10.1137/S0036144502417715.
\item F. Chen, J. S. Hesthaven, and X. Zhu. On the Use of Reduced Basis Methods to Accelerate and Stabilize the Parareal Method. In A. Quarteroni and G. Rozza, editors, \textit{Reduced Order Methods for Modeling and Computational Reduction}, volume 9 of \textit{MS&A - Modeling, Simulation and Applications}, pages 187–214. Springer International Publishing, 2014. DOI 10.1007/978-3-319-02090-7_7.
\item M. W. Choptuik. Universality and scaling in gravitational collapse of a massless scalar field. \textit{Physical Review Letters}, 70(1):9–12, Jan 1993. DOI 10.1103/PhysRevLett.70.9.
\item M. W. Choptuik, E. W. Hirschmann, and R. L. Marsa. New critical behavior in Einstein-Yang-Mills collapse. \textit{Physical Review D}, 60(12):124011, Nov 1999. DOI 10.1103/PhysRevD.60.124011. URL http://arxiv.org/abs/gr-qc/9903081.
\item A. J. Christlieb, C. B. Macdonald, and B. W. Ong. Parallel high-order integrators. \textit{SIAM Journal on Scientific Computing}, 32(2):818–835, 2010. DOI 10.1137/09075740X.
\item D. Christodoulou. Bounded variation solutions of the spherically symmetric Einstein-scalar field equations. \textit{Communications on Pure and Applied Mathematics}, 46(8):1131–1220, 1993. DOI 10.1002/cpa.3160460803.
\item M. Emmett and M. L. Minion. Toward an Efficient Parallel in Time Method for Partial Differential Equations. \textit{Communications in Applied Mathematics and Computational Science}, 7:105–132, 2012. DOI 10.2140/camcos.2012.7.105.
\end{enumerate}

\textsuperscript{10}A copy of the library Lib4PrM for the Parareal method can be obtained by cloning the Git repository https://scm.ti-edu.ch/repogit/lib4prm.
R. D. Falgout, S. Friedhoff, T. V. Kolev, S. P. MacLachlan, and J. B. Schroder. Parallel time integration with multigrid. *SIAM Journal on Scientific Computing*, 36:C635–C661, 2014. DOI 10.1137/130944230.

P. F. Fischer, F. Hecht, and Y. Maday. A parareal in time semi-implicit approximation of the Navier-Stokes equations. In R. Kornhuber and et al., editors, *Domain Decomposition Methods in Science and Engineering*, volume 40 of *Lecture Notes in Computational Science and Engineering*, pages 433–440, Berlin, 2005. Springer. DOI 10.1007/3-540-26825-1_44.

M. S. Floater and K. Hormann. Barycentric Rational Interpolation with no Poles and High Rates of Approximation. *Numerische Mathematik*, 107(2):315–331, Aug 2007. DOI 10.1007/s00211-007-0093-y.

M. J. Gander. Analysis of the Parareal Algorithm Applied to Hyperbolic Problems using Characteristics. *Bol. Soc. Esp. Mat. Apl.*, 42:21–35, 2008.

M. J. Gander. Analysis of the Parareal algorithm applied to hyperbolic problems using characteristics. *Boletín de la Sociedad Española de Matemática Aplicada*, (42):21–35, Mar 2008. URL http://www.sema.org.es/ojs/index.php?journal=sema&page=article&op=view&path%5B%5D=19.

M. J. Gander. 50 years of Time Parallel Time Integration. In *Multiple Shooting and Time Domain Decomposition*. Springer, 2015. URL http://www.unige.ch/~gander/Preprints/50YearsTimeParallel.pdf.

M. J. Gander and M. Petcu. Analysis of a Krylov Subspace Enhanced Parareal Algorithm for Linear Problem. *ESAIM: Proc.*, 25:114–129, 2008. DOI 10.1051/proc:082508.

M. J. Gander and S. Vandewalle. Analysis of the Parareal Time-Parallel Time-Integration Method. *SIAM Journal on Scientific Computing*, 29(2):556–578, 2007. DOI 10.1137/05064607X.

D. Garfinkle. Choptuik scaling in null coordinates. *Physical Review D*, 51(10):5558–5561, May 1994. DOI 10.1103/PhysRevD.51.5558. URL http://arxiv.org/abs/gr-qc/9412008.

C. Gundlach and J. M. Martín-García. Critical phenomena in gravitational collapse. *Living Reviews in Relativity*, 10(5), Dec 2007. DOI 10.12942/lrr-2007-5. URL http://arxiv.org/abs/0711.4620.

W. Hackbusch. Parabolic multi-grid methods. *Computing Methods in Applied Sciences and Engineering, VI*, pages 189–197, 1984. URL http://dl.acm.org/citation.cfm?id=4673.4714.

G. Horton. The time-parallel multigrid method. *Communications in Applied Numerical Methods*, 8 (9):585–595, 1992. DOI 10.1002/cnm.1630080906.

G. Horton, S. Vandewalle, and P. Worley. An Algorithm with Polylog Parallel Complexity for Solving Parabolic Partial Differential Equations. *SIAM Journal on Scientific Computing*, 16(3):531–541, 1995. DOI 10.1137/0916034.

V. Husain. Critical Behavior in Quantum Gravitational Collapse. *Advanced Science Letters*, 2(2):214–220(7), Jun 2009. DOI 10.1166/asl.2009.1028. URL http://arxiv.org/abs/0808.0949.

L. E. Kidder, M. A. Scheel, S. A. Teukolsky, E. D. Carlson, and G. B. Cook. Black hole evolution by spectral methods. *Physical Review D*, 62(8):084032, Sep 2000. DOI 10.1103/PhysRevD.62.084032. URL http://arxiv.org/abs/gr-qc/0005056.

E. Komatsu, J. Dunkley, M. R. Nolta, C. L. Bennett, B. Gold, G. Hinshaw, N. Jarosik, D. Larson, M. Limon, L. Paige, D. N. Spergel, M. Halpern, R. S. Hill, A. Kogut, S. S. Meyer, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright. Five-Year Wilkinson Microwave Anisotropy Probe Observations: Cosmological Interpretation. *The Astrophysical Journal Supplement Series*, 180(2):330, Feb 2009. DOI 10.1088/0067-0049/180/2/330. URL http://arxiv.org/abs/0803.0547.

A. Kreienbuehl. Quantum Cosmology, Polymer Matter, and Modified Collapse. PhD thesis, University of New Brunswick, Fredericton Campus, Department of Mathematics and Statistics, Aug 2011.

A. Kreienbuehl, V. Husain, and S. S. Seahra. Modified general relativity as a model for quantum gravitational collapse. *Classical and Quantum Gravity*, 29(9):095008, May 2012. DOI 10.1088/0264-9381/29/9/095008. URL http://arxiv.org/abs/1011.2381.

A. Kreienbuehl, A. Naegel, D. Ruprecht, R. Speck, G. Wittum, and R. Krause. Numerical simulation of skin transport using Parareal. *Computing and Visualization in Science*, 17:99–108, 2015. URL http://dx.doi.org/10.1007/s00791-015-0246-y.
[32] J.-L. Lions, Y. Maday, and G. Turinici. A "parareal" in time discretization of PDE’s. *Comptes Rendus de l’Académie des Sciences - Series I - Mathematics*, 332:661–668, 2001. DOI 10.1016/S0764-4442(00)01793-6.

[33] F. Loeffler, J. Faber, E. Bentivegna, T. Bode, P. Diener, R. Haas, I. Hinder, B. C. Mundim, C. D. Ott, E. Schnetter, G. Allen, M. Campanelli, and P. Laguna. The Einstein Toolkit: a community computational infrastructure for relativistic astrophysics. *Classical and Quantum Gravity*, 29(11):115001, May 2012. DOI 10.1088/0264-9381/29/11/115001. URL http://arxiv.org/abs/1111.3344.

[34] M. L. Minion. A Hybrid Parareal Spectral Deferred Corrections Method. *Communications in Applied Mathematics and Computational Science*, 5(2):265–301, 2010. DOI 10.2140/camcos.2010.5.265.

[35] J. Nievérgelt. Parallel methods for integrating ordinary differential equations. *Commun. ACM*, 7(12):731–733, 1964. DOI 10.1145/355588.365137.

[36] E. Poisson. *A Relativist’s Toolkit. The Mathematics of Black-Hole Mechanics*. Cambridge University Press, Cambridge, UK, 2004.

[37] F. Pretorius. *Numerical Simulations of Gravitational Collapse*. PhD thesis, The University of British Columbia, 2002.

[38] F. Pretorius. Evolution of Binary Black-Hole Spacetimes. *Physical Review Letters*, 95(12):121101, Sep 2005. DOI 10.1103/PhysRevLett.95.121101. URL http://arxiv.org/abs/gr-qc/0507014.

[39] F. Pretorius and L. Lehner. Adaptive mesh refinement for characteristic codes. *Journal of Computational Physics*, 198(1):10–34, 2004. DOI 10.1016/j.jcp.2004.01.001. URL http://arxiv.org/abs/gr-qc/0302003.

[40] D. Ruprecht. Convergence of Parareal with spatial coarsening. *PAMM*, 14(1):1031–1034, 2014. ISSN 1617-7061. DOI 10.1002/pamm.201410490.

[41] D. Ruprecht and R. Krause. Explicit parallel-in-time integration of a linear acoustic-advection system. *Computers & Fluids*, 59(0):72–83, 2012. DOI 10.1016/j.compfluid.2012.02.015.

[42] D. Ruprecht, R. Speck, M. Emmett, M. Bolten, and R. Krause. Poster: Extreme-scale space-time parallelism. In *Proceedings of the 2013 Conference on High Performance Computing Networking, Storage and Analysis Companion*, SC ’13 Companion, 2013. URL http://sc13.supercomputing.org/sites/default/files/PostersArchive/tech_posters/post148s2-file3.pdf.

[43] D. Ruprecht, R. Speck, and R. Krause. Parareal for diffusion problems with space- and time-dependent coefficients. In *Domain Decomposition Methods in Science and Engineering XXII*, volume 104 of *Lecture Notes in Computational Science and Engineering*, pages 3–10. Springer International Publishing Switzerland, 2015. DOI 10.1007/978-3-319-18827-0_1.

[44] R. Speck, D. Ruprecht, R. Krause, M. Emmett, M. L. Minion, M. Winkel, and P. Gibbon. A massively space-time parallel N-body solver. In *Proceedings of the International Conference on High Performance Computing, Networking, Storage and Analysis*, SC ’12, pages 92:1–92:11, Los Alamitos, CA, USA, 2012. IEEE Computer Society Press. DOI 10.1109/SC.2012.6.

[45] J. Thornburg. Adaptive mesh refinement for characteristic grids. *General Relativity and Gravitation*, 43(5):1211–1251, May 2011. DOI 10.1007/s10714-010-1096-z. URL http://arxiv.org/abs/0909.0036.

[46] J. Ziprick and G. Kunstatter. Dynamical singularity resolution in spherically symmetric black hole formation. *Physical Review D*, 80(2):024032, Jul 2009. DOI 10.1103/PhysRevD.80.024032. URL http://arxiv.org/abs/0902.3224.