Convolution morphisms and Kottwitz conjecture

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Abstract

We define etale cohomology of the moduli spaces of mixed characteristic local shtukas so that it gives smooth representations including the case where the relevant elements of the Kottwitz set are both non-basic. Then we relate the etale cohomology of different moduli spaces of mixed characteristic local shtukas using convolution morphisms, duality morphisms and twist morphisms. As an application, we show the Kottwitz conjecture in some new cases including the cases for all inner forms of $GL_3$ and minuscule cocharacters. We study also some non-minuscule cases and show that the Kottwitz conjecture is true for any inner form of $GL_2$ and any cocharacter if the Langlands parameter is cuspidal. On the other hand, we show that the Kottwitz conjecture does not hold as it is in non-minuscule cases if the Langlands parameter is not cuspidal. Further, we show that a generalization of the Harris–Viehmann conjecture for the moduli spaces of mixed characteristic local shtukas does not hold in Hodge–Newton irreducible cases.

Introduction

The Kottwitz conjecture says that etale cohomology of Rapoport–Zink spaces or more generally local Shimura varieties realize the local Langlands correspondence (cf. [Rap95, Conjecture 5.1], [RV14, Conjecture 7.4]). In [SW20], Scholze constructs local Shimura varieties as special cases of moduli spaces of mixed characteristic local shtukas. The Kottwitz conjecture makes sense also for the moduli spaces of mixed characteristic local shtukas. A weak version of the conjecture is studied by Hansen–Kaletha–Weinstein in [HKW22]. In the weak version, we ignore the action of the Weil groups and have an equality up to representations which have trace 0 on regular elliptic elements.

Let $p$ be a prime number. Let $G$ be a connected reductive group over a $p$-adic number field $F$. For $b, b' \in G(\bar{F})$ and a system $\mu_* = (\mu_1, \ldots, \mu_m)$ of cocharacters of $G$, we define a moduli space $\text{Sht}^{\mu_*}_{b,b'}$ of mixed characteristic local shtukas. See §2 for the precise definition.

In this paper, we introduce convolution morphisms, duality morphisms and twist morphisms between moduli spaces of mixed characteristic local shtukas. The convolution morphism is related to a convolution morphism on affine Grassmannians. Using these morphisms and the convolution products in the geometric Satake equivalence for $B^+_{un}$-Grassmannians, we relate the etale cohomology of different moduli spaces of mixed characteristic local shtukas. More concretely, we show the following:

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2010 Mathematics Subject Classification. Primary: 11F70; Secondary: 14G35.
Theorem 1 (Corollary 5.2). Assume that $G$ is quasi-split and take a Borel pair $T \subset B$ of $G$. Let $\mu_\bullet = (\mu_1, \ldots, \mu_m)$ be a system of dominant cocharacters of $T$ and $b_0, b_m \in G(\bar{F})$. Let $E$ be a finite extension of $F$ containing the fields of definition of $\mu_i$ for $1 \leq i \leq m$. We have

$$
\sum_{(b_i)_{1 \leq i \leq m-1} \in \check{I}^{\mu_\bullet}_{b_0, b_m}} H_* \left( \prod_{i=1}^{m-1} G_{b_i}(F), \bigotimes_{1 \leq i \leq m} H^*_c(Sht_{b_{i-1}, b_i}) \otimes \delta_{bi} \right) = \sum_{\lambda \in X^*(T)^+/G} V_{\mu_\bullet}^\lambda \otimes H_c^*(Sht_{b_0, b_m})
$$

as virtual representations of $G_{b_0}(F) \times G_{b_m}(F) \times W_E$, where $I^{\mu_\bullet}_{b_0, b_m}$ is a finite set defined in §5.

We note that even if $b_0$ and $b_m$ are basic, non-basic elements appear in $I^{\mu_\bullet}_{b_0, b_m}$ and there are contributions from cohomology of non-basic moduli spaces of local shtukas. For a derived category version of the above statement, see Proposition 5.1.

As an application of Theorem 1 (or its derived category version) together with duality morphisms and twist morphisms, we show new cases of the Kottwitz conjecture for the moduli spaces of mixed characteristic local shtukas. In particular, we show the following:

Theorem 2 (Corollary 7.6). Let $G$ be an inner form of $GL_3$ over $F$. Let $(G, b, \mu)$ be a local shtuka datum such that $\mu$ is minuscule and $b$ is basic. Let $\varphi: W_F \to \mathbb{L}GL_3$ be a discrete local $L$-parameter. Let $\pi$ and $\pi_b$ be the irreducible smooth representations of $G(F)$ and $G_b(F)$ corresponding to $\varphi$ via the local Langlands correspondence. Then we have

$$\mathcal{H}^* \left( R\text{Hom}_{G(F)} \left( R\Gamma_c(Sht_{1,b}^{\mu}), \pi \right) \right) \simeq \pi_b \boxtimes (r_\mu \circ \varphi)$$

as representations of $G_b(F) \times W_F$.

It is remarkable that the proof of Theorem 2 requires moduli spaces of local shtukas for non-minuscule cocharacters, even though the statement involves only minuscule cocharacters: Using a derived category version of Theorem 1, we can calculate a sum of cohomology of moduli spaces of local shtukas for a minuscule cocharacter and a non-minuscule cocharacter. Then we separate them into each term using the duality isomorphism. We also note that it is essential to introduce convolution morphisms for moduli spaces of mixed characteristic local shtukas with multiple legs in §3 because we use it in the proof of a compatibility result, Proposition 6.2, which plays an important role in the proof of Theorem 2.

Theorem 1 is useful also for studying non-minuscule cases. We give inductive formulas that enable us to calculate the cohomology of moduli spaces of local shtukas for inner forms of $GL_2$. We can summarize the results in §4 as the following theorem:

Theorem 3. Let $G$ be an inner form of $GL_2$ over $F$. Let $(G, b, \mu)$ be a local shtuka datum. Let $\rho$ be a discrete series representation of $G_b(F)$. We put

$$H_c^*(Sht_{1,b}^{\mu})[\rho] = \sum_{i,j \in \mathbb{Z}} (-1)^{i+j} \text{Ext}^i_{G_b(F)} \left( R^j\Gamma_c(Sht_{1,b}^{\mu}), \rho \right).$$

Then we can calculate $H_c^*(Sht_{1,b}^{\mu})[\rho]$ by inductive formulas. In particular, when $b$ is basic, we see the following:
(1) The Kottwitz conjecture for $\text{Sht}_{1,b}^\mu$ holds if the $L$-parameter is cuspidal or $G$ is not quasi-split.

(2) The Kottwitz conjecture for $\text{Sht}_{1,b}^\mu$ does not hold in general if the $L$-parameter is not cuspidal and $G$ is quasi-split.

We note that in the first statement of Theorem 3, $b$ can be non-basic and $\mu$ can be non-minuscule. Even if we are interested only in $H^*_c(\text{Sht}_{1,b}^\mu)[\rho]$ for a basic $b$, the inductive formulas for the calculation of $H^*_c(\text{Sht}_{1,b}^\mu)[\rho]$ involve moduli spaces of local shtukas for non-basic elements. Therefore it is important to study non-basic cases at the same time.

We note that Theorem 3 is compatible with the result in [HKW22], since the error term involves only representations which have trace 0 on regular elliptic elements. We remark also that this error term supports the expectation [Par16, Remark 4.6] in the geometrization of the local Langlands correspondence is true.

Further, we see that the Harris–Viemann conjecture for the moduli spaces of mixed characteristic local shtukas does not hold as it is in Example 8.10 and Remark 8.11. We note that Harris–Viemann conjecture for the moduli spaces of mixed characteristic local shtukas is proved in [GI16] and [Han21a] under the Hodge–Newton reducibility condition. On the other hand, the Hodge–Newton reducibility condition is not satisfied in Example 8.10.

In §1, we collect results on relative homology and the geometric Satake correspondence. In §2, we give a definition of a moduli space of mixed characteristic local shtukas. The definition which we give here is slightly different from that in [SW20]. Our definition is suitable to construct convolution morphisms between moduli spaces of mixed characteristic local shtukas in §3. In §3, we construct a twist morphism between moduli spaces of mixed characteristic local shtukas, which has an origin in the twist of a vector bundle by a line bundle. In §5, we discuss a relation between cohomology of different moduli spaces of mixed characteristic local shtukas using convolution morphisms. In §6, we construct a duality morphism, which has an origin in the dual of a vector bundle. In §7, we give an application to the Kottwitz conjecture. In §8, we give some inductive formulas on cohomology and discuss more about the Kottwitz conjecture in non-minuscule cases.

After we put a former version of this paper on arXiv, a preprint [Han21b] by Hansen appeared, where a cohomology version of Theorem 2 is proved for cuspidal local $L$-parameters of $\text{GL}_n$ using a result in [ALB21]. A merit of Theorem 2 is that it works for discrete local $L$-parameters.

Acknowledgements

The author is grateful to Peter Scholze for answering questions. He would like to thank David Hansen for helpful comments. He thank also Teruhisa Koshikawa for helpful comments and various discussions. This work was supported by JSPS KAKENHI Grant Number 18H01109.

Notation

For a field $F$, let $\Gamma_F$ denote the absolute Galois group of $F$. For a non-archimedean local field $F$, let $\bar{F}$ denote the completion of the maximal unramified extension of $F$. For an
1 Sheaves in ℓ-adic coefficients

1.1 Relative homology

Let \( p \) be a prime number. Let \( \Lambda \) be a solid \( \hat{\mathbb{Z}}^p \)-algebra. For a small \( v \)-stack \( X \), we define \( D_{\square}(X, \Lambda) \) as \([FS21, Definition VII.1.17]\). There is a symmetric monoidal structure \(- \otimes^\Lambda -\) on \( D_{\square}(X, \Lambda) \) constructed by \([FS21, Proposition VII.2.2]\). In the sequel, we simply write \( \otimes^\Lambda \) for \( \square \otimes^\Lambda \).

For a morphism \( f: X \to Y \) of small \( v \)-stacks, let \( f^\#\) be a left adjoint to \( f^*: D_{\square}(Y, \Lambda) \to D_{\square}(X, \Lambda) \) constructed by \([FS21, Proposition VII.3.1]\).

The following lemma is already known (cf. the proof of \([FS21, Proposition VII.6.3]\)).

Lemma 1.1. Let \( f: X \to Y \) be a quasi-compact, quasi-separated morphism of small \( v \)-stacks. Assume that \( \Lambda = \lim_{\leftarrow n \in I} \mathbb{Z}/n\mathbb{Z} \), where \( I \) is a filtered set of positive integers which are prime to \( p \). Then we have

\[
f_2^\# \Lambda \simeq \lim_{\leftarrow n \in I} f_2^\#(\mathbb{Z}/n\mathbb{Z}).
\]

Proof. We recall a proof. We may assume that \( Y \) is a spatial diamond. Then \( \Lambda \) is a pseudo-coherent object on \( X \) by the assumption on \( f \). Since \( f_2^\# \) preserves pseudo-coherent objects, \( f_2^\# \Lambda \) is also a pseudo-coherent object. Since each cohomology sheaf of \( f_2^\# \Lambda \) is a finitely presented solid sheaf, we have

\[
f_2^\# \Lambda \simeq \lim_{\leftarrow n \in I} (f_2^\# \otimes^\Lambda \mathbb{Z}/n\mathbb{Z}) \simeq \lim_{\leftarrow n \in I} f_2^\#(\mathbb{Z}/n\mathbb{Z})
\]

by \([FS21, Theorem VII.1.3, Proposition VII.3.1]\).

Lemma 1.2. Let \( f: X \to Y \) be a morphism of small \( v \)-stacks. Let \( \mathcal{F} \) be a solid \( \hat{\mathbb{Z}}^p \)-sheaf on \( X \). Let \( \{U_i\}_{i \in I} \) be a filtered direct system of quasi-compact open substacks of \( X \) such that \( X = \bigcup_{i \in I} U_i \). Let \( f_i \) and \( \mathcal{F}_i \) be the restriction to \( U_i \) of \( f \) and \( \mathcal{F} \) for \( i \in I \). Then we have

\[
f_2^\# \mathcal{F} \simeq \lim_{i \in I} f_i^\# \mathcal{F}_i.
\]

Proof. Let \( j_i: U_i \to X \) be the inclusion for \( i \in I \). Since \( f_2^\# \) commutes with a direct limit, it suffices to show \( \mathcal{F} \simeq \lim_{i \in I} j_i^\# \mathcal{F}_i \). By the projection formula, we may assume that \( \mathcal{F} = \hat{\mathbb{Z}}^p \). For any solid \( \hat{\mathbb{Z}}^p \)-sheaf \( \mathcal{G} \) on \( X \), we have

\[
\text{Hom}(\lim_{i \in I} j_i^\#, \hat{\mathbb{Z}}^p, \mathcal{G}) \simeq \lim_{i \in I} \text{Hom}(j_i^\#, \hat{\mathbb{Z}}^p, \mathcal{G}) \simeq \lim_{i \in I} \mathcal{G}(U_i) \simeq \mathcal{G}(X) \simeq \text{Hom}(\hat{\mathbb{Z}}^p, \mathcal{G}).
\]

Hence we obtain the claim.
Lemma 1.3. Let $F$ be a non-archimedean field with residue characteristic $p$. Let $d$ be a positive integer.

(1) Let

$$f : (\text{Spa} (\mathcal{O}_F[[x_1^{1/p^n}, \ldots, x_d^{1/p^n}]] \times_{\text{Spa} (\mathcal{O}_F)} \text{Spa} (F)) \to \text{Spa} (F)^\circ$$

be the natural morphism. Then we have $f_* \Lambda \simeq \Lambda$. Further, the geometric Frobenius morphism $x_i \mapsto x_i^p$ induces the multiplication by $p^d$ on $f_* \Lambda$.

(2) Let

$$f : (A_{d, F})^\circ \to \text{Spa} (F)^\circ$$

be the natural morphism. Then we have $f_* \Lambda \simeq \Lambda$. Further, the geometric Frobenius morphism $x_i \mapsto x_i^p$ induces the multiplication by $p^d$ on $f_* \Lambda$.

Proof. We show the first claim of (1). We may assume that $\Lambda = \widehat{\mathbb{Z}}^p$ and $F$ is algebraically closed of characteristic $p$. We write $\text{Spa} (\mathcal{O}_F[[x_1^{1/p^n}, \ldots, x_d^{1/p^n}]] \times_{\text{Spa} (\mathcal{O}_F)} \text{Spa} (F)$ as a union of affinoids isomorphic to $\text{Spa} (F(x_1^{1/p^n}, \ldots, x_d^{1/p^n}))$. By Lemma 1.2, it is reduced to show that $g_* \widehat{\mathbb{Z}}^p \simeq \widehat{\mathbb{Z}}^p$ for

$$g : \text{Spa} (F(x_1^{1/p^n}, \ldots, x_d^{1/p^n})) \to \text{Spa} (F).$$

By Lemma 1.1 and [FS21, Proposition VII.5.2], the claim follows from that $g_!(\mathbb{Z}/n\mathbb{Z}) \simeq (\mathbb{Z}/n\mathbb{Z})(-d)[-2d]$ for any integer $n$ prime to $p$. The claim on the geometric Frobenius morphism follows from the case for $g_!(\mathbb{Z}/n\mathbb{Z})$.

We can show the claim (2) similarly. \qed

Let $\ell$ be a prime number different from $p$.

Lemma 1.4. Let $G$ be a locally pro-$p$ group. Let $\mathcal{H}(G)$ be the Hecke algebra of $G$ with coefficients in $\Lambda$. Let $f : X \to Y$ be a morphism of small $v$-stacks which is a $G$-torsor. For a pro-$p$ open subgroup $K$ of $G$, let $f_K : X/K \to Y$ be the morphism induced by $f$. Let $g : Y \to Z$ be a morphism of small $v$-stacks. The morphisms $f_K$ and $(g \circ f_K)_*$ induce

$$\lim_K (g \circ f_K)_* f_K : D_{\boxplus}(Y, \Lambda) \to D_{\boxplus}(Z, \mathcal{H}(G))$$

(1) For $A \in D_{\boxplus}(Y, \Lambda)$, we have

$$(\lim_K f_K_\sharp f_K^* A) \otimes^L_{\mathcal{H}(G)} \Lambda \simeq A.$$

(2) Assume that $A \in D_{\boxplus}(Y, \Lambda)$ is obtained from $V \in D^b(G, \Lambda)$. Then we have

$$(\lim_K (g \circ f_K)_* (A) \otimes^L_{\mathcal{H}(G)} V) \otimes^L_{\mathcal{H}(G)} \Lambda \simeq g_* A.$$

Proof. (1) We have

$$(\lim_K f_K_\sharp f_K^* A) \otimes^L_{\mathcal{H}(G)} \Lambda \simeq ((\lim_K f_K_\sharp A) \otimes^L_{\mathcal{H}(G)} A) \otimes^L_{\mathcal{H}(G)} \Lambda \simeq ((\lim_K f_K_\sharp A) \otimes^L_{\mathcal{H}(G)} \Lambda) \otimes^L_{\mathcal{H}(G)} A.$$
Hence it suffices to show that the natural morphism \((\lim_{\rightarrow} f_{K,\natural} A) \otimes^L_{H(G)} \Lambda \to \Lambda\) is an isomorphism. We can check this \(v\)-locally on \(Y\) by [FS21 Proposition VII.3.1 (iii)]. Hence the claim follows.

(2) The morphism \(g_{\natural}\) induces

\[
g_{\natural} : D_{\square}(Y, H(G)) \to D_{\square}(Z, H(G)).
\]

By [FS21 Proposition VII.3.1 (i)], we have

\[
(\lim_{\rightarrow} f_{K,\natural} A) \otimes^L_{H(G)} \Lambda \cong g_{\natural} \left( (\lim_{\rightarrow} f_{K,\natural} A) \otimes^L_{H(G)} \Lambda \right)
\]

Combined with (1), it remains to show

\[
\lim_{\rightarrow} f_{K,\natural} A \cong \lim_{\rightarrow} f_{K,\natural} f_{K}^* A.
\]

We can check that the morphism

\[
\lim_{\rightarrow} f_{K,\natural} A \to \lim_{\rightarrow} f_{K,\natural} f_{K}^* A
\]

induced by \(V \to V^K \hookrightarrow f_{K}^* A\) is an isomorphism.

Let \(\Lambda\) be a \(\mathbb{Z}_{\ell}\)-algebra. For an Artin \(v\)-stack \(X\), let \(D_{\text{lis}}(X, \Lambda)\) be the category defined in [FS21 Definition VII.6.1].

**Lemma 1.5.** Let \(f : X \to Y\) be an \(\ell\)-cohomologically smooth morphism.

(1) We have \(f_{\natural}(D_{\text{lis}}(X, \Lambda)) \subset D_{\text{lis}}(Y, \Lambda)\).

(2) For \(A \in D_{\square}(Y, \Lambda)\), we have \((f^* A)^{\text{lis}} \cong f^* (A^{\text{lis}})\).

**Proof.** The claim (1) follows from [FS21 Definition VII.6.1]. For \(B \in D_{\square}(Y, \Lambda)\), we have

\[
\text{Hom}(B, (f^* A)^{\text{lis}}) \cong \text{Hom}(B, f^* A) \cong \text{Hom}(f_{\natural}(B), A)
\]

\[
\cong \text{Hom}(f_{\natural}(B), A^{\text{lis}}) \cong \text{Hom}(B, f^* (A^{\text{lis}})),
\]

where we use (1) at the third isomorphism. Hence the claim (2) follows.

**Lemma 1.6.** Let

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow g' & & \downarrow g \\
X & \xrightarrow{f} & Y
\end{array}
\]

be a cartesian diagram of small \(v\)-stack. Assume that \(g\) is \(\ell\)-cohomologically smooth. Then we have

\[
g^* Rf_{\text{lis}*} A \cong Rf'_{\text{lis}*} g'^* A
\]

for \(A \in D_{\text{lis}}(X, \Lambda)\).
Proof. This follows from [FS21, Proposition VII.2.4] and Lemma 1.5. □

Lemma 1.7. Let $f: X \to Y$ be an $\ell$-cohomologically smooth morphism. Let $A, B \in D_{\text{lis}}(Y, \Lambda)$. Then we have $f^* R\mathcal{H}om_{\text{lis}}(A, B) \cong R\mathcal{H}om_{\text{lis}}(f^* A, f^* B)$.

Proof. This follows from [FS21, Proposition VII.2.4] and Lemma 1.5. □

Lemma 1.8. Let $f: X \to Y$ be a morphism. Let $A \in D_{\text{lis}}(X, \Lambda)$ and $B \in D_{\text{lis}}(Y, \Lambda)$.

1. We have $R\mathcal{H}om_{\text{lis}}(B, Rf_{\text{lis}}^*(A)) \cong Rf_{\text{lis}}^* R\mathcal{H}om_{\text{lis}}(f^* B, A)$.

2. If $f$ is $\ell$-cohomologically smooth, then we have

$$R\mathcal{H}om_{\text{lis}}(f_\sharp(A), B) \cong Rf_{\text{lis}}^* R\mathcal{H}om_{\text{lis}}(A, f^* B).$$

Proof. [1] For $C \in D_{\text{lis}}(Y, \Lambda)$, we can check

$$R\text{Hom}(C, R\mathcal{H}om_{\text{lis}}(B, Rf_{\text{lis}}^*(A))) \cong R\text{Hom}(C, Rf_{\text{lis}}^* R\mathcal{H}om_{\text{lis}}(f^* B, A))$$

by adjoint. The claim [2] is proved similarly. □

For an $\ell$-cohomologically smooth morphism $f: X \to Y$, we put

$$D_f = (\varprojlim_n Rf^!(\mathbb{Z}/\ell^n\mathbb{Z})) \otimes_{\mathbb{Z}_\ell} \Lambda$$

and

$$f_\sharp(A) = f_\sharp(A \otimes_{D_f} D_f^{-1})$$

for $A \in D_{\text{lis}}(X, \Lambda)$. For an $\ell$-cohomologically smooth morphism $f: X \to *$, we write $D_X$ for $D_f$. For $f: X \to *$ and $A \in D_{\text{lis}}(X, \Lambda)$, we put $R\Gamma_\sharp(X, A) = f_\sharp(A)$. For $f: X \to \text{Spa} C$ and $A \in D_{\text{lis}}(X, \Lambda)$ where $C$ is an algebraically closed non-archimedean field of characteristic $p$, we put $R\Gamma_\sharp,C(X, A) = f_\sharp(A)$.

1.2 Geometric Satake equivalence

We recall the geometric Satake equivalence for $B^+_{\text{dR}}$-Grassmannians by Fargues–Scholze (cf. [FS21 VI, IX]).

Let $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$. Let $F$ be a finite extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$ with the residue field $\mathbb{F}_q$. For an algebraic field extension $k$ of $\mathbb{F}_q$, let Perf$_k$ denote the category of perfectoid spaces over $k$ with $v$-topology in the sense of [Sch17 §8].

Let $G$ be a connected reductive group over $F$. We define $v$-sheaves $LG$ and $L^+G$ over $\text{Spd} \mathbb{Q}_p$ by sending $S = \text{Spa}(R, R^\circ) \in \text{Perf}_{\mathbb{F}_q}$ with an untilt $S^\circ = \text{Spa}(R^\circ, R^\circ_{\text{dR}})$ to $B_{\text{dR}}(R^\circ)$ and $B^+_{\text{dR}}(R^\circ)$, where $B_{\text{dR}}(R^\circ)$ and $B^+_{\text{dR}}(R^\circ)$ are defined as in [Far16, Definition 1.32]. We put $\text{Gr}_G = LG/L^+G$ and

$$\mathcal{H}ck_G = [L^+G \setminus LG/L^+G].$$

For $A_1, A_2 \in D_{\mathfrak{m}}(\mathcal{H}ck_G, \Lambda)$, let $A_1 \ast A_2$ denote the convolution product of $A_1$ and $A_2$. Let $Q$ be a finite quotient of $W_F$ such that the action of $W_F$ on $\widehat{G}$ factors through $Q$. Let

$$S': \text{Rep}_\Lambda(\widehat{G} \times Q) \to D_{\mathfrak{m}}(\mathcal{H}ck_G, \Lambda)$$
denote the functor that gives the geometric Satake equivalence (cf. [FS21, IX.2]). This functor is symmetric monoidal functor by the construction (cf. [FS21, Proposition VI.10.2]).

For $V_1, V_2 \in \text{Rep}_A(\hat{G} \rtimes Q)$, let
\[ c_{V_1, V_2} : S'(V_1) \star S'(V_2) \simeq S'(V_2) \star S'(V_1) \]
be the commutativity constraint uniquely characterized by
\[ S'(V_1) \star S'(V_2) \xrightarrow{c_{V_1, V_2}} S'(V_2) \star S'(V_1) \]
\[ S'(V_1 \otimes V_2) \xrightarrow{S'(\sigma_{V_1, V_2})} S'(V_2 \otimes V_1), \]
where $\sigma_{V_1, V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1$ is the isomorphism switching $V_1$ and $V_2$.

Assume that $\mu \in X_s(T)^\dagger$. Let $Q_\mu \subset Q$ be the stabilizer of $\mu$. Let $r_{G, \mu}'$ be the highest weight $\mu$ irreducible representation of $\hat{G} \rtimes Q_\mu$. We put
\[ r_{G, \mu} = \text{Ind}_{\hat{G} \rtimes Q_\mu}^{\hat{G} \times Q} r_{G, \mu}'. \]
We simply write $r_\mu$ for $r_{G, \mu}$ if there is no confusion. We write $V_\mu$ for the representation space of $r_\mu$. We put $IC_\mu' = S'(V_\mu)$. We use the same notation $IC_\mu'$ for the pullback of $IC_\mu'$ to other spaces.

2 Moduli of local shtukas

Let $S = \text{Spa}(R, R^+) \in \text{Perf}_{\hat{F}}$. We put $W_{\bar{O}_F}(R^+) = W(R^+) \otimes_{W(F_q)} \bar{O}_F$. Take an topological nilpotent unit $\varpi_R$ in $R$. Let $\mathcal{Y}_{(0, \omega)}(S)$ be the adic space defined by the condition $p \neq 0$ and $[\varpi_R] \neq 0$ in $\text{Spa}(W_{\bar{O}_F}(R^+), W_{\bar{O}_F}(R^+))$. Then $\mathcal{Y}_{(0, \omega)}(S)$ has an action of the $q$-th power Frobenius element $\varphi_S$ induced by the $q$-th power map on $R$.

The quotient
\[ X_S = \mathcal{Y}_{(0, \omega)}(S)/\varphi_S \]
is called the relative Fargues–Fontaine curve for $S$ (cf. [SW20, Definition 15.2.6]). The construction glues together to give $X_S$ for any $S \in \text{Perf}_{\hat{F}}$.

We define a continuous map
\[ \kappa_S : \mathcal{Y}_{(0, \omega)}(S) \to (0, \infty) \]
by
\[ \kappa_S(x) = \frac{\log[|\varpi_R|_x]}{\log[|p|_x]} \]
where $\tilde{x}$ is the maximal generalization of $x \in \mathcal{Y}_{(0, \omega)}(S)$ and $| \cdot |_x$ denotes the valuation corresponding to $\tilde{x}$. For an interval $I$ in $(0, \infty)$, let $Y_I(S)$ denote the interior of $\kappa_S^{-1}(I)$.

For $S \in \text{Perf}_{\hat{F}}$, we put $\mathcal{B}(S) = \mathcal{O}(\mathcal{Y}_{(0, \omega)}(S))$. Then $\mathcal{B}$ is a $v$-sheaf by [FS21, Proposition II.2.1].

Let $G$ be a connected reductive group over $F$. Let $b \in G(\hat{F})$. We define an algebraic group $G_b$ over $F$ by
\[ G_b(R) = \{ g \in G(\hat{F} \otimes F R) \mid g(b \sigma \otimes 1) = (b \sigma \otimes 1)g \} \]
for any $F$-algebra $R$. We define a $G$-bundle $\mathcal{E}_{b,X_S}$ on $X_S$ by

$$(G_F \times_{\text{Spa}(F)} \mathcal{Y}(0,\infty)(S))/((b\sigma) \times \varphi_S)$$

If $b' = g^{-1}b\sigma(g)$ for $b,b',g \in G(\tilde{F})$, then the left multiplication by $g^{-1}$ induces an isomorphism

$$t_g : \mathcal{E}_{b,X_S} \to \mathcal{E}_{b',X_S}.$$  \hspace{1cm} (2.1)

We define a sheaf $\tilde{J}_b$ on $\text{Perf}_{\mathcal{Y}}$ by

$$\tilde{J}_b(S) = \text{Aut}(\mathcal{E}_{b,X_S})$$

for $S \in \text{Perf}_{\mathcal{Y}}$. In the sequel, we simply write $\mathcal{E}_b$ for $\mathcal{E}_{b,X_S}$ if there is no confusion. We define $\tilde{J}_b^{>0}$ as in [FS21, III.5]. Then we have $\tilde{J}_b = \tilde{J}_b^{>0} \times G_b(F)$ by [FS21, Proposition III.5.1]. If $b$ is basic, we have $\tilde{J}_b = G_b(F)$.

Let $b,b' \in G(\tilde{F})$. Let $\mu_1, \ldots, \mu_m$ be cocharacters of $G$. We put $\mu_\bullet = (\mu_1, \ldots, \mu_m)$. For $1 \leq i \leq m$, let $E_i$ be the field of definition of $\mu_i$.

**Definition 2.1.** We define the presheaf $\text{Sht}^\bullet_{G,b,b'}$ by sending $S = \text{Spa}(R,R^+)$ in $\text{Perf}_{\mathcal{Y}}$ to the isomorphism classes of the following objects;

- an untilt $S_i^\bullet$ of $S$ over $\tilde{E}_i$ for $1 \leq i \leq m$,
- a $G$-torsor $\mathcal{P}$ on $\mathcal{Y}(0,\infty)(S)$ with an isomorphism

$$\varphi_\mathcal{P} : (\varphi_S^*\mathcal{P})|_{\mathcal{Y}(0,\infty)(S)} \cup_{i=1}^m S_i^\bullet \simeq \mathcal{P}|_{\mathcal{Y}(0,\infty)(S)} \cup_{i=1}^m S_i^\bullet$$

which is meromorphic along the Cartier divisor $\bigcup_{i=1}^m S_i^\bullet \subset \mathcal{Y}(0,\infty)(S)$ and the relative position of $\varphi_S^*\mathcal{P}$ and $\mathcal{P}$ at $S_i^\bullet$ is bounded by $\sum_{j|S_j^\bullet = S_i^\bullet} \mu_j$ at all geometric rank $1$ points for all $1 \leq i \leq m$,

- an isomorphism

$$t_{[r,\infty]} : \mathcal{P}|_{\mathcal{Y}(r,\infty)(S)} \simeq G \times \mathcal{Y}(r,\infty)(S)$$

for large enough $r$ under which $\varphi_\mathcal{P}$ is identified with $b \times \varphi_S$ and an isomorphism

$$t_{(0,r']} : \mathcal{P}|_{\mathcal{Y}(0,r')(S)} \simeq G \times \mathcal{Y}(0,r')(S)$$

for small enough $r'$ under which $\varphi_\mathcal{P}$ is identified with $b' \times \varphi_S$.

If there is no confusion, we simply write $\text{Sht}^\bullet_{G,b,b'}$ for $\text{Sht}^\bullet_{G,b,b'}$. If $\mu_\bullet = (\mu)$, we simply write $\text{Sht}^\bullet_{G,b,b'}$ for $\text{Sht}^\bullet_{G,b,b'}$. We use similar abbreviations also for other spaces.

We define the right action of $\tilde{J}_b \times \tilde{J}_b'$ on $\text{Sht}^\bullet_{G,b,b'}$ by

$$(t_{[r,\infty]}, t_{(0,r')}) \mapsto (g^{-1} \circ t_{[r,\infty]}, g'^{-1} \circ t_{(0,r')})$$

for $(g,g') \in \tilde{J}_b \times \tilde{J}_b'$.

We define $G^\text{tw}_{G,\text{Spd} E_1 \times \cdots \times \text{Spd} E_m, \leq \mu\bullet}$ as in [SW20, Definition 23.4.1]. It is a spacial diamond by [SW20, Proposition 23.4.2]. We have a morphism

$$\pi^\mu_{G,b,b'} : \text{Sht}^\bullet_{G,b,b'} \to G^\text{tw}_{G,\text{Spd} E_1 \times \cdots \times \text{Spd} E_m, \leq \mu\bullet}$$
defined by forgetting \(\ell(0,r)\). The morphism \(\pi^{r}_{G,b,b'}\) is a \(\tilde{J}_b\)-torsor over a locally spatial subdiamond of \(\text{Gr}^{1\mu'}_{G,\text{Spd} \tilde{E}_1 \times \cdots \times \text{Spd} \tilde{E}_m, \leq \mu}\) by \([\text{Sch}17\text{, Proposition 11.20}]\). Hence, \(\text{Sht}^{\mu}_{G,b,b'}\) is a diamond by \([\text{Sch}17\text{, Proposition 11.6}]\) and \([\text{Far16\text{, 2.5, 2.6.2}}]\).

We have a natural inversing morphism
\[
\text{Sht}^{\mu}_{G,b,b'} \rightarrow \text{Sht}^{\mu-1}_{G,b',b}
\] (2.2)
compatible with the action of \(\tilde{J}_b \times \tilde{J}_{b'}\).

Let \(B(G)\) be the set of \(\sigma\)-conjugacy classes in \(G(\bar{F})\). We write \(B(G)\) for the set of the basic elements in \(B(G)\). Let \(\mu\) be a cocharacter of \(G\). We define \(B(G,\mu)\) as in \([\text{Kot97\text{, 6.2}}]\).

Assume that \(G\) is quasi-split. We fix subgroups \(A \subset T \subset B\) of \(G\) where \(A\) is a maximal split torus, \(T\) is a maximal torus and \(B\) is a Borel subgroup. We write \(X_\sigma(A)^+\) and \(X_\sigma(T)^+\) for the dominant cocharacter of \(A\) and \(T\). For \(b \in G(\bar{F})\), we define \(\nu_b \in X_\sigma(A)^+\) as in \([\text{Far16\text{, 2.2.2}}]\) using the slope morphism constructed in \([\text{Kot85\text{, 4.2}}]\).

Let \(B(G,\mu,[b])\) be the set of acceptable neutral elements in \(B(G)\) for \((\mu,[b])\) (cf. \([\text{GI16\text{, Definition 4.3}}]\)).

Lemma 2.2. Assume that \(b\) is basic. The map
\[
G(\bar{F}) \rightarrow G(\bar{F}) = G_b(\bar{F}); \ g \mapsto gb^{-1}
\]
induces bijections \(B(G) \rightarrow B(G_b), B(G)_{\text{bas}} \rightarrow B(G_b)_{\text{bas}}\) and \(B(G,\mu,[b]) \rightarrow B(G_b,\mu)\).

Proof. The claim follows from the equality
\[
(g'g\sigma(g')^{-1})b^{-1} = g'(gb^{-1})(b\sigma(g')b^{-1})^{-1}
\]
for \(g, g' \in G(\bar{F})\). \(\square\)

Proposition 2.3. Assume that \(b'\) is basic. We have a natural isomorphism
\[
\text{Sht}^{\mu}_{G,b,b'} \overset{\sim}{\longrightarrow} \text{Sht}^{\mu}_{G,b',b}^{-1,1}
\]
which is compatible with the action of \(\tilde{J}_b \times \tilde{J}_{b'}\).

Proof. We can view \(\text{Sht}^{\mu}_{G,b,b'}\) as a moduli space of modifications of \(G\)-torsors on a Fargues–Fontaine curve. The category of \(G\)-torsor is equivalent to the category of \(G_b\)-torsor on a Fargues–Fontaine curve as explained in the proof of \([\text{SW20\text{, Corollary 23.2.3}}]\). The claim follows from this equivalence. \(\square\)

Remark 2.4. Assume that \(b, b'\) are basic and \(m = 1\). Then a weak version of Kottwitz conjecture for \(\text{Sht}^{\mu}_{G,b,b'}\) holds by \([\text{HKW22\text{, Theorem 1.0.4}}]\), Lemma 2.2 and Proposition 2.3.

Remark 2.5. Assume that \(b, b'\) are basic and \(m = 1\). Under the isomorphism in Proposition 2.3, the inversing morphism (2.2) is identified with the Faltings–Fargues isomorphism proved in \([\text{SW20\text{, Corollary 23.2.3}}]\).

Lemma 2.6. Assume that \(b'\) is basic. If \(\text{Sht}^{\mu}_{G,b,b'}\) is not empty, then we have \([b] \in B(G,\mu,[b'])\).
Proof. By Proposition 2.3 we may assume that \( b' = 1 \) dropping the assumption that \( G \) is quasi-split. Then the claim follows from [CS17, Proposition 3.5.3].

We define a Weil descent datum of \( \tilde{J}_b \) by

\[
\tilde{J}_b \to \tilde{J}_{q(b)} = \sigma^*(\tilde{J}_b); \quad f \mapsto t_b \circ f \circ t_b^{-1},
\]
where \( t_b \) is defined in (2.1). Let \( \rho_G \) denote the half-sum of the positive roots of \( G \) with respect to \( T \) and \( B \). We put \( N_b = (2\rho_G, \nu_b) \).

Lemma 2.7. Let \( \Lambda \) be a solid \( \mathbb{Z}^p \)-algebra. Let \( f: \tilde{J}_b^{>0} \to * \) be the structure morphism. Then we have an isomorphism \( f_*(\Lambda) \simeq \Lambda(N_b) \) compatible with the actions of \( W_F \).

Proof. This is proved in the same way as [GI16, Lemma 4.17] using Lemma 1.3 and the definition of the Weil descent datum.

Let \( \delta_b: G_b(F) \to \Lambda^\times \) be the character obtained by the action of \( G_b(F) \) on \( D_f \), where \( f: \tilde{J}_b^{>0} \to * \).

3 Cohomology of moduli of local shtukas

Let \( \mu_* = (\mu_1, \ldots, \mu_m) \in (X_*(T))^m \). Let \( E \) be the field of definition of \( \mu_* \). The space \( \text{Sht}^{\mu_*}_{G,b,b'} \) is the moduli space of \((S^\flat, \mathcal{E}_b \to \mathcal{E}_{b'})\), where \( S^\flat \) is an untilt over \( \tilde{E} \) and \( \mathcal{E}_b \to \mathcal{E}_{b'} \) is a modification bounded by \( \mu_* \) along the Cartier divisor defined by \( S^\flat \).

Let \( \mathbb{C}^p_b \) denote the tilt of \( \mathbb{C}^p \). The untily \( \mathbb{C}^p_b \) of \( \mathbb{C}^p \) determine a morphism \( \text{Spa} \mathbb{C}^p_b \to \text{Spd} \mathbb{Q}_p \). For the arithmetic Frobenius element \( \sigma_E \in \text{Gal}(F^ur/E) \), we take \( m \) such that \( \sigma_E|_{F^ur} = \sigma^m \) and define a Weil descent datum of \( \text{Sht}^{\mu_*}_{G,b,b'} \) by

\[
\text{Sht}^{\mu_*}_{G,b,b'} \to \text{Sht}^{\mu_*}_{G,\sigma^m(b),\sigma^m(b')}, \quad \sigma^*(\text{Sht}^{\mu_*}_{G,b,b'});
\]

\[
(S^\flat, \mathcal{E}_b \overset{f}{\to} \mathcal{E}_{b'}) \mapsto (S^\flat, \mathcal{E}_{\sigma^m(b)} \overset{t_{m,b}^{-1}}{\to} \mathcal{E}_b \overset{f}{\to} \mathcal{E}_{\sigma^m(b')} \overset{t_{m,b}}{\to} \mathcal{E}_{\sigma^m(b')})
\]

where we put \( t_{m,b} = t_{\sigma^m(b)} \circ \cdots \circ t_b: \mathcal{E}_b \to \mathcal{E}_{\sigma^m(b)} \) using (2.1).

We have fiber products

\[
\begin{array}{cccccc}
\tilde{M}^{\tilde{b}}_b & \xrightarrow{\tilde{j}_b} & \tilde{M}_b & \xrightarrow{i_b} & * \\
\dagger_b & \downarrow & \dagger_b & \downarrow & \\
M^{\tilde{b}}_b & \xrightarrow{j_b} & M_b & \xrightarrow{i_b} & [*/G_b(F)] \\
\dagger_b & \downarrow & \dagger_b & \downarrow & \\
\text{Bun}^{\tilde{b}}_G & \xrightarrow{j_b} & \text{Bun}_G & \xrightarrow{i_b} & \text{Bun}^{\tilde{b}}_G
\end{array}
\]

and morphisms \( q_b: M_b \to [*/G_b(F)] \) and \( \tilde{q}_b: \tilde{M}_b \to * \) as [FS21 V.3]. Here \( i_b \) and \( \tilde{i}_b \) are sections of \( q_b \) and \( \tilde{q}_b \) respectively explained in [FS21 Proposition V.3.6]. Then \( i^b, j^b \) and \( \pi_b \) factor through

\[
\begin{array}{c}
i^b: \text{Bun}^b_G \to \text{Bun}^{\tilde{b}}_G \\
j^b: \text{Bun}^{\tilde{b}}_G \to \text{Bun}^b_G \\
\pi'_b: M_b \to \text{Bun}^{\tilde{b}}_G
\end{array}
\]

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Lemma 3.1. The functors $h_b$ and $R_{h_b}$ are quasi-inverses of the equivalence

$$h_b^*: D_{lis}(Bun^b, \Lambda) \to D_{lis}(\Lambda/[*/G_b(F)], \Lambda)$$

of categories.

Proof. Since $h_b^*$ is an equivalence of categories by [FS21 Proposition VII.7.1], its left adjoint $h_b$ and right adjoint $h_b$ give quasi-inverses.

We define $\tilde{i}_b!: D_{lis}(\Lambda, \Lambda) \to D_{lis}(\widehat{M}_b, \Lambda)$ and $\tilde{i}_b^a: D_{lis}(\widehat{M}_b, \Lambda) \to D_{lis}(\Lambda, \Lambda)$ by

$$\tilde{i}_b = \text{cone}(\tilde{j}_{b,3}j_b \to \text{id}) \circ \eta_b^a, \quad \tilde{i}_b^a = R\tilde{j}_{b,\text{lis}^a} \circ \text{fib}(\text{id} \to R\tilde{j}_{b,\text{lis}^a}j_b^a).$$

Then $\tilde{i}_b^a$ is a left adjoint of $\tilde{i}_b!$. We define $i_b!: D_{lis}(\Lambda/[*/G_b(F)], \Lambda) \to D_{lis}(\widehat{M}_b, \Lambda)$ and $i_b^a: D_{lis}(\widehat{M}_b, \Lambda) \to D_{lis}(\Lambda/[*/G_b(F)], \Lambda)$ by

$$i_b! = \text{cone}(j_{b,3}j_b \to \text{id}) \circ \eta_b^a, \quad i_b^a = Rj_{b,\text{lis}^a} \circ \text{fib}(\text{id} \to Rj_{b,\text{lis}^a}j_b^a).$$

Then $i_b^a$ is a left adjoint of $i_b!$. Further we define $i_b^b: D_{lis}(Bun^b, \Lambda) \to D_{lis}(\text{Bun}^{\leq b}, \Lambda)$ and $i_b^{b!}: D_{lis}(\text{Bun}^{\leq b}, \Lambda) \to D_{lis}(Bun^b, \Lambda)$ by

$$i_b^b = \pi_b \circ i_b! \circ \pi_b^b, \quad i_b^{b!} = Rh_b \circ i_b^a \circ \pi_b^{b!}.$$

Then $i_b^{b!}$ is a left adjoint of $i_b^b!$. We define $i_b^b: D_{lis}(Bun^b, \Lambda) \to D_{lis}(Bun_G^b, \Lambda)$ and $i_b^{b!}: D_{lis}(Bun_G^b, \Lambda) \to D_{lis}(Bun^b, \Lambda)$ by

$$i_b^b = j_{b}^{\leq b} \circ i_b^b, \quad i_b^{b!} = i_b^{b!} \circ j_{b}^{\leq b}.$$
Proof. Let $A \in D_{lis}({\mathcal{M}}_b, \Lambda)$. By Lemma \ref{lem:4.2}, there is $A_1 \in D_{lis}({\mathcal{M}}_b^\times, \Lambda)$ and $A_2 \in D_{lis}([*/G_b(F)], \Lambda)$ such that $j_{b,s}A_1 \rightarrow A \rightarrow i_{b,t}A_2 \rightarrow$ is a distinguished triangle. By taking $j_{b,s}$ and $j_{b,t}$, we have $A_1 \cong j_{b,s}A$ and $A_2 \cong i_{b,t}A$. Hence we obtain the first isomorphism in \ref{lem:2}. The second isomorphism in \ref{lem:2} follows from the first one by taking the right adjoint. The other claims are proved in the same way using Lemma \ref{lem:3.2}.

\begin{lem}
For $A \in D_{lis}({\mathcal{M}}_b, \Lambda)$ and $B \in D_{lis}([*/G_b(F)], \Lambda)$, we have an isomorphism $i_{b,1}(i_b^*(A) \otimes^L B) \cong A \otimes^L i_{b,1}(B)$.
\end{lem}

Proof. We have

$$A \otimes^L i_{b,1}(B) \cong \text{cone}(j_{b,s}j_b^*(A \otimes^L q_b^*B) \rightarrow A \otimes^L q_b^*B) \cong i_{b,1}(i_b^*(A \otimes^L B)),$$

where we use Lemma \ref{lem:3.3} at the second isomorphism.

\begin{lem}
We have $\text{fib}(D_{q_b} \rightarrow Rj_{b,lis}j_b^*D_{q_b}) \cong R(i_{b,lis})\Lambda$.
\end{lem}

Proof. By the change of coefficient and the inverse limit, we may assume that $\Lambda$ is torsion. Then we have $\text{fib}(D_{q_b} \rightarrow Rj_{b,lis}j_b^*D_{q_b}) \cong R(i_{b,lis})i_b^\prime\Lambda \cong R(i_{b,lis})\Lambda$.

\begin{lem}
We have $\mathbb{D} \circ i_{b,1} = i_{b,1} \circ \mathbb{D}$ and $i_b^\prime \circ \mathbb{D} = \mathbb{D} \circ i_b^\prime$.
\end{lem}

Proof. Let $A \in D_{lis}({\mathcal{M}}_b, \Lambda)$. We have

$$(i_b^\prime \circ \mathbb{D})(A) = i_b^\prime R\text{Hom}_{lis}(A, D_{q_b}) \cong Rq_{b,lis}(R\text{Hom}_{lis}(A, \text{fib}(D_{q_b} \rightarrow Rj_{b,lis}j_b^*D_{q_b}))).$$

By Lemma \ref{lem:3.5} this is isomorphic to

$$Rq_{b,lis}(R\text{Hom}_{lis}(A, i_{b,1}\Lambda)) \cong Rq_{b,lis}(Ri_{b,lis}R\text{Hom}_{lis}(i_b^*A, \Lambda)) \cong R\text{Hom}_{lis}(i_b^*\Lambda, \Lambda) = (\mathbb{D} \circ i_b^*)(A).$$

Hence we have $i_b^\prime \circ \mathbb{D} = \mathbb{D} \circ i_b^\prime$. Another claim follows from this by adjoint.

The following lemma is already known (cf. \cite{FS21}, IX.3).

\begin{lem}
We have $\mathbb{D} \circ j_b^\prime = j_b^\prime \circ \mathbb{D}$ and $i_{b,1} \circ \mathbb{D} = \mathbb{D} \circ i_{b,1}$.
\end{lem}

Proof. Let $A \in D_{lis}(\text{Bun}_G, \Lambda)$. We have

$$h_b^*(i_{b,1} \circ \mathbb{D})(A) \cong i_b^\prime \pi_b^*R\text{Hom}_{lis}(A, D_{Bun_G}) \cong i_b^\prime R\text{Hom}_{lis}(\bar{\pi}_b^*A, \pi_b^*D_{Bun_G}) \cong i_b^\prime \mathbb{D}((\pi_b^*A) \otimes D_{\bar{\pi}_b}) \cong \mathbb{D}(i_b^\prime \pi_b^*A) \otimes i_b D_{\bar{\pi}_b}^{-1},$$

where we use Lemma \ref{lem:3.4} at the second isomorphism and Lemma \ref{lem:3.6} at the fourth isomorphism. On the other hand we have

$$h_b^*(i_{b,1} \circ \mathbb{D})(A) \cong h_b^*R\text{Hom}_{lis}(i_{b,1}^*A, D_{Bun_G}) \cong R\text{Hom}_{lis}(h_b^*i_{b,1}^*A, h_b D_{Bun_G}) \cong \mathbb{D}(i_b^\prime \pi_b^*A) \otimes D_{h_b^{-1}},$$

where we use Lemma \ref{lem:3.4} at the second isomorphism. Hence $i_{b,1} \circ \mathbb{D} = \mathbb{D} \circ i_{b,1}$ follows from \cite{Sch17}, Proposition 23.12. Another claim follows from this by adjoint.

\begin{lem}
We have $\tilde{i}_b \cong \tilde{i}_b^\prime(\text{fib}(id \rightarrow Rj_{b,lis}j_b^*))$, $i_b^\prime \cong i_b^\prime(\text{fib}(id \rightarrow Rj_{b,lis}j_b^*))$ and $i_{b,1} \cong i_{b,1}^\prime(\text{fib}(id \rightarrow Rj_{b,lis}j_b^*))$.
\end{lem}

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Proof. For \( A \in D_{lis}(\mathcal{M}_b, A) \) and \( B \in D_{lis}([*/G_b(F)], \Lambda) \), we have
\[
\text{Hom}(i_{b!}(B), A) \cong \text{Hom}(i_{b!}(B), \text{fib}(A \to Rj_{b!\text{lis}}\ast j_b^!A)) \cong \text{Hom}(B, i_b^!(\text{fib}(A \to Rj_{b!\text{lis}}\ast j_b^!A)))
\]
by Lemma 3.2. Hence we obtain the second claim. Other claims are proved similarly. \( \square \)

**Lemma 3.9.** We have \( \mathcal{M}_b \cong i_{b!}^! \mathcal{M}_b \cong i_b^! \mathcal{M}_b \) and \( i_b^! \mathcal{M}_b \cong \mathcal{M}_b \).

**Proof.** We can check these using Lemma 3.8. \( \square \)

**Lemma 3.10.** We have \( \mathcal{M}_b \cong R\mathcal{M}_b \text{lis}, i_b^! \cong \mathcal{M}_b \text{lis}^\ast, \) and \( i_b^! \mathcal{M}_b \cong \mathcal{M}_b \text{lis}^\ast. \)

**Proof.** By Lemma 3.3, Lemma 3.8, and Lemma 3.9, we have
\[
i_b^! \mathcal{M}_b \text{lis}^\ast \cong \mathcal{M}_b \text{lis} \mathcal{M}_b \text{lis}^\ast, \mathcal{M}_b \text{lis} \mathcal{M}_b \text{lis}^\ast \cong \mathcal{M}_b \text{lis} \mathcal{M}_b \text{lis}^\ast \mathcal{M}_b \text{lis}^\ast.
\]
Hence \( i_b^! \cong i_b^! \mathcal{M}_b \text{lis} \text{lis}^\ast \) follows from Lemma 3.2 using Lemma 1.6. Other claims are proved similarly. \( \square \)

For a compact open subgroup \( K \) of \( G_b(F) \), we consider the fiber products

\[
\begin{array}{ccc}
\text{Sht}^\mu_{G,b,K,b',C_p} & \xrightarrow{f_K} & \text{Hck}^\mu_{b'} \times \text{Spa} C_p \\
\downarrow h_K & & \downarrow \iota_{b'} \\
[*/K] & \xrightarrow{h_K} & \text{Bun}^b_G \xrightarrow{\iota^b} \text{Bun}_G
\end{array}
\]

where \( h_K \) and \( \iota_{b'} \) are the compositions
\[
[*/K] \xrightarrow{h_K,G_b(F)} [*/G_b(F)] \xrightarrow{h_b} \text{Bun}^b_G, \\
\text{Spa} C_p \xrightarrow{\iota_{b'}} \text{Bun}^b_{G'} \times \text{Div}^m_X \xrightarrow{p_{1,\bigotimes}} \text{Bun}_G \times \text{Div}^m_X
\]
of the natural morphisms. Let \( p_{1,\bigotimes} \): \( \text{Hck}^\mu_{b'} \rightarrow \text{Hck}^\mu_{G'} \times \text{Div}^m_X \rightarrow \text{Bun}_G \times \text{Div}^m_X \).

We put
\[
f_K,\Lambda = p_{1,\bigotimes}^* i_{b!}^! h_K,\Lambda.
\]

**Remark 3.11.** If \( b \) is basic, \( f_K \) is etale, in particular \( \ell \)-cohomologically smooth. In this case, the above definition of \( f_K,\Lambda \) coincides with the general definition before.

We put
\[
R\Gamma_c(\text{Sht}^\mu_{G,b,K,b'}) = f_{\iota_{b'}!}( (f_K,\Lambda) \otimes \Lambda^1 \text{IC}_{\mu}^b).
\]

We can view
\[
R\Gamma_c(\text{Sht}^\mu_{G,b,K,b'}) \cong \iota_{b!}^* T_{\mu}^b (i_{b!} h_K,\Lambda)
\]
as an object of \( D(G_b(F) \times W_E) \) by [FS21 Corollary IX.2.3]. For a compact open subgroup \( K' \) of \( G_b(F) \), we define \( R\Gamma_c(\text{Sht}^\mu_{G,b,K',b'}) \) in the symmetric way. Since \( \text{IC}_{\mu}^b \) and \( \text{IC}_{-\mu}^b \) correspond under the natural isomorphism \( \text{Sht}^\mu_{G,b,b'} \cong \text{Sht}^\mu_{G,b',b} \), we have
\[
R\Gamma_c(\text{Sht}^\mu_{G,b,K',b'}) \cong \iota_{b'}^* T_{-\mu} (i_{b'}^! h_{K',\Lambda}).
\]
**Remark 3.12.** If $b$ is basic, $\Gamma_c(\text{Sh}_G, b)$ is identified with $(f_u \circ f_K)_!(\text{IC}_M)$. We define $\Gamma_c(\text{Sh}_G, b)$ as above since we do not have a good definition of

$$f_K!: D_{\text{lis}}(\text{Sh}_G, b, b', \mathbb{C}_p), \Lambda) \to D_{\text{lis}}(\text{Heck}_b, \Lambda)$$

for a general $b$.

We put

$$\Gamma_c(\text{Sh}_G, b) = \lim_{K \subseteq G_b(F)} \Gamma_c(\text{Sh}_G, b)$$

**Lemma 3.13.** We have $q_{b,!*} \circ i_{b,!*} = \text{id}$.

**Proof.** Let $B \in D_{\text{lis}}([*/G_b(F)], \Lambda)$. Then we have $i_{b,!*}(B) \cong \text{cone}(j_{b,!*} j_! \Lambda \to \Lambda) \otimes \Lambda^b q_b^* B$. Hence we have

$$(q_{b,!*} \circ i_{b,!*})(B) \cong q_{b,!*} \left( (j_{b,!*} j_! \Lambda \to \Lambda) \otimes D_{q_b}^{-1} \right) \otimes \Lambda^b B.$$  

It remains to show $q_{b,!*} \left( (j_{b,!*} j_! \Lambda \to \Lambda) \otimes D_{q_b}^{-1} \right) \cong \Lambda$. It suffices to show this after taking a pullback via $\text{Spa} \mathbb{C}_p \to [*/G_b(F)]$ since the induced actions of $G_b(F)$ on the both sides are trivial. Let $j_U: U \to M_{b, c_p}$ be a quasicompact open neighborhood of $\tilde{i}_b(\text{Spa} \mathbb{C}_p)$. We have

$$\tilde{i}_b (\text{cone}(j_{b,!*} j_! \Lambda \to \Lambda) \otimes \Lambda^{-1}) \cong \left( q_{b,!*} \circ j_U \right) \left( \text{cone}(\tilde{i}_b, \Lambda) \otimes D_{q_b}^{-1} \right).$$

Then the question is reduced to the torsion case by Lemma [11] since $q_b \circ j_U$ is quasi-compact, separated by [FS21 Proposition V.3.5]. In the torsion case, the claim follows from [FS21 Proposition VII.5.2] and $\text{cone}(j_{b,!*} j_! \Lambda \to \Lambda) \cong i_{b,!*} (\Lambda)$.

**Lemma 3.14.** For $A \in D_{\text{lis}}(M_b, \Lambda)$ and $B \in D_{\text{lis}}([*/G_b(F)], \Lambda)$, we have an isomorphism $q_{b,!*}(A \otimes \Lambda^{b,!*} i_{b,!*} B) \cong i_{b,!*} (A \otimes \Lambda^{b,!*} i_{b,!*} B)$.

**Proof.** We have

$$A \otimes \Lambda^{b,!*} i_{b,!*} B \cong \text{cone}(j_{b,!*} j_! A \to A) \otimes \Lambda^{b,!*} q_{b,!*} B$$

$$\cong \text{cone}(j_{b,!*} j_! (A \otimes D_{q_b}) \to A \otimes D_{q_b}) \otimes D_{q_b}^{-1} \otimes \Lambda^{b,!*} q_{b,!*} B$$

$$\cong (i_{b,!*} i_{b,!*} (A \otimes D_{q_b})) \otimes D_{q_b}^{-1} \otimes \Lambda^{b,!*} q_{b,!*} B,$$

where we use Lemma [3.3][2] at the last isomorphism. Hence we have

$$q_{b,!*}(A \otimes \Lambda^{b,!*} i_{b,!*} B) \cong q_{b,!*} ((i_{b,!*} i_{b,!*} (A \otimes D_{q_b})) \otimes D_{q_b}^{-1} \otimes \Lambda^{b,!*} B$$

$$\cong q_{b,!*} (i_{b,!*} i_{b,!*} (A \otimes D_{q_b})) \otimes \Lambda^{b,!*} B \cong i_{b,!*} (A \otimes D_{q_b}) \otimes \Lambda^{b,!*} B,$$

where we use Lemma [3.13] at the last isomorphism.

**Lemma 3.15.** Let $A \in D_{\text{lis}}(\text{Bun}_G, \Lambda)$ and $B \in D_{\text{lis}}(\text{Bun}_G, \Lambda)$. Then we have

$$\Gamma_c(\text{Bun}_G, A \otimes \Lambda^{b,!*} i_{b,!*} B) \cong \Gamma_c([*/G_b(F)], h_{b,!*} (i_{b,!*} A \otimes \Lambda^{b,!*} B) \otimes \Lambda^{b,!*} i_{b,!*} D_{q_b})$$

for a general $b$. 

**Proof.** We have

$$A \otimes \Lambda^{b,!*} i_{b,!*} B \cong \text{cone}(j_{b,!*} j_! A \to A) \otimes \Lambda^{b,!*} q_{b,!*} B$$

$$\cong \text{cone}(j_{b,!*} j_! (A \otimes D_{q_b}) \to A \otimes D_{q_b}) \otimes D_{q_b}^{-1} \otimes \Lambda^{b,!*} q_{b,!*} B$$

$$\cong (i_{b,!*} i_{b,!*} (A \otimes D_{q_b})) \otimes D_{q_b}^{-1} \otimes \Lambda^{b,!*} q_{b,!*} B,$$

where we use Lemma [3.3][2] at the last isomorphism. Hence we have

$$q_{b,!*} (A \otimes \Lambda^{b,!*} i_{b,!*} B) \cong q_{b,!*} ((i_{b,!*} i_{b,!*} (A \otimes D_{q_b})) \otimes D_{q_b}^{-1} \otimes \Lambda^{b,!*} B$$

$$\cong q_{b,!*} (i_{b,!*} i_{b,!*} (A \otimes D_{q_b})) \otimes \Lambda^{b,!*} B \cong i_{b,!*} (A \otimes D_{q_b}) \otimes \Lambda^{b,!*} B,$$

where we use Lemma [3.13] at the last isomorphism.
Proof. We have
\[ \Gamma^! (\text{Bun}_G, A \otimes \mathcal{O} i^! B) \cong \Gamma^! (\mathcal{M}_b, \pi_b^* A \otimes \mathcal{O} i_b^! h_b^* B) \]
\[ \cong \Gamma^! ([* / G_b (F)], i_b^* (\pi_b^* A \otimes D_{q_b}) \otimes \nabla h_b^* B) \]
\[ \cong \Gamma^! ([* / G_b (F)], h_b^* (i_b^* A \otimes \nabla B) \otimes \nabla i_b^! D_{q_b}) , \]
where we use Lemma \[3.14\] at the second isomorphism.

\[ \Box \]

**Lemma 3.16.** We have a natural isomorphism
\[ \lim_{K \subset G_b (F)} \Gamma_c (\text{Sh}_{G_b, b, K, K'}) \cong \lim_{K' \subset G'_b (F)} \Gamma_c (\text{Sh}_{G_b, b', K, K'}). \]

Proof. It suffices to show that
\[ \Gamma_c (\text{Sh}_{G_b, b, K, K'}) \cong \Gamma_c (\text{Sh}_{G_b, b', K, K'}) \]
for enough small \( K \) and \( K' \). We consider the following diagram:

\[ \begin{array}{ccc}
[\text{Spa} C_p / G_b (F)] & \xrightarrow{h_{K,b}} & [\text{Spa} C_p / K] \\
\text{Bun}_{G, C_p}^b & \xrightarrow{i^b} & \text{Bun}_{G, C_p}^b \\
\downarrow h_b & & \downarrow h_b
\end{array} \quad \begin{array}{ccc}
[\text{Spa} C_p / K'] & \xrightarrow{h_{K',b'}} & [\text{Spa} C_p / G'_b (F)] \\
\text{Bun}_{G, C_p}^b & \xrightarrow{p_1} & \text{Hck}^\mu_{C_p} \\
\downarrow h_{b'} & & \downarrow h_{b'}
\end{array} \]

We have
\[ \Gamma^! (\text{Hck}^\mu_{C_p}, p_1^! h_b^! h_{K!}! \Lambda \otimes \nabla p_2^! h_{K'!}! \Lambda \otimes \nabla \mathcal{O} \mu^* ) \]
\[ \cong \Gamma^! (B_{G, C_p}, T_{\mu^*} (i_b^! h_{K!}! \Lambda \otimes \nabla i_b^! h_{K'!}! \Lambda) \]
\[ \cong \Gamma^! ([\text{Spa} C_p / G'_b (F)], h_{b'}^! (i_b^* T_{\mu^*} (i_b^! h_{K!}! \Lambda \otimes \nabla h_{K'!}! \Lambda) \otimes s_{q_b} D_{q_b}) , \]
where we use Lemma \[3.15\] at the last isomorphism. We have
\[ h_{b'}^! h_{K'!}! \Lambda \otimes s_{q_b} D_{q_b} \cong h_{b'}^! h_{K'!}! ((h_{K'!}! \Lambda) \otimes D_{q_b}^{-1}) \otimes s_{q_b} D_{q_b} \]
\[ \cong h_{K'!}! h_{K'!}^* (D_{h_{q_b}^{-1}} \otimes s_{q_b} D_{q_b}) , \]
where we use Lemma \[3.1\] at the last isomorphism. Since \( h_{b'} \) and \( q_b \) are cohomologically smooth of the same dimension, \( D_{h_{q_b}^{-1}} \otimes s_{q_b} D_{q_b} \) is etale locally trivial. Hence we may assume that \( K' \) is small enough so that \( h_{K'!}! (D_{h_{q_b}^{-1}} \otimes s_{q_b} D_{q_b}) \cong \Lambda \). Then \[3.2\] is isomorphic to
\[ \Gamma^! ([\text{Spa} C_p / G'_b (F)], h_{b'}^! h_{K'!}^* T_{\mu^*} (i_b^! h_{K!}! \Lambda \otimes \nabla h_{K'!}! \Lambda) \cong \Gamma_c (\text{Sh}_{G_b, b, K, K'}) , \]
Since \[3.1\] is symmetric with respect to \( (b, K) \) and \( (b', K') \), the claim follows.

\[ \Box \]

**Proposition 3.17.** We have \( \Gamma_c (\text{Sh}_{G_b, b, K, K'}) \cong \Gamma_c (\text{Sh}_{G_b, b', K, K'}) \).

Proof. This follows from Lemma \[3.16\].

\[ \Box \]

**Proposition 3.18.** (1) If \( K \) is pro-p, then \( \Gamma_c (\text{Sh}_{G_b, K, b'}) \) is a compact object in \( D (G_b (F), \Lambda) \).
(2) For \( i \in \mathbb{Z} \), \( H^i_{\bullet}(\text{Sh}_{G,b,K,b}^\bullet) \) is finitely generated smooth \( G_{\bar{b}}(F) \)-representation.

(3) If \( \rho \) is an admissible representation of \( G_{\bar{b}} \) over \( \Lambda \), then \( R \text{Hom}_{G_{\bar{b}}}(R\Gamma_c(\text{Sh}_{G,b,K,b}^\bullet), \rho) \) is a perfect complex of \( \Lambda \)-modules.

(4) If \( \Lambda = \overline{\mathbb{Q}}_\ell \) and \( \rho \) is a finite length representation of \( G_{\bar{b}}(F) \) over \( \overline{\mathbb{Q}}_\ell \), then

\[
\lim_{K \subset G_{\bar{b}}(F)} R^i \text{Hom}_{G_{\bar{b}}(F)}(R\Gamma_c(\text{Sh}_{G,b,K,b}^\bullet), \rho)
\]

is finite length representation of \( G_{\bar{b}}(F) \) for \( i \in \mathbb{Z} \).

Proof. We have

\[
\lim_{K \subset G_{\bar{b}}(F)} R \text{Hom}_{G_{\bar{b}}}(R\Gamma_c(\text{Sh}_{G,b,K,b}^\bullet), \rho) \cong \bigoplus_{i,j} R^i \Gamma_c(\text{Sh}_{G,b,K,b}^\bullet)[\rho].
\]

Then the claims are proved in the same way as [FS21, IX.3] using Lemma B.7.

We put

\[
H^i_{\bullet}(\text{Sh}_{G,b,K,b}^\bullet) = \sum_{i \in \mathbb{Z}} (-1)^j R^j \Gamma_c(\text{Sh}_{G,b,K,b}^\bullet).
\]

4 Convolution morphism and twist morphism

4.1 Convolution morphism

Let \( \Delta_{m, \text{Spd } F} \) denote the diagonal subspace of \( (\text{Spd } F)^m \). For \( 1 \leq i < j \leq m \), let \( \text{pr}_{i,j} : (\text{Spd } F)^m \to (\text{Spd } F)^2 \) denote the projection to the \( (i, j) \)-component. We put

\[
U_m = (\text{Spd } F)^m \setminus \bigcup_{1 \leq i < j \leq m} \text{pr}_{i,j}^{-1}\left( \bigcup_{n \in \mathbb{Z}\setminus\{0\}} (\varphi \times 1)^n(\Delta_2, \text{Spd } F) \right).
\]

This is an open subspace of \( (\text{Spd } F)^m \) which contains \( \Delta_{m, \text{Spd } F} \).

Let \( b_0, \ldots, b_m \in G(\bar{F}) \) and \( \mu_\bullet = (\mu_1, \ldots, \mu_m) \) where \( \mu_i \in X_*(T) \) for \( 1 \leq i \leq m \). We put

\[
\text{Sh}_{G,b_0,b_m}^\mu U_m = \text{Sh}_{G,b_0,b_m}^\mu \times (\text{Spd } F)^m U_m.
\]

We define the convolution morphism

\[
m_{\mu_\bullet, U_m} : (\text{Sh}_{G,b_0,b_1}^{\mu_1} \times \cdots \times \text{Sh}_{G,b_{m-1},b_m}^{\mu_m}) \times (\text{Spd } F)^m U_m \to \text{Sh}_{G,b_0,b_m}^{\mu_\bullet} U_m
\]

over \( \text{Spd } E_1 \times \cdots \times \text{Spd } E_m \) as follows. Let \( S = \text{Spa}(R, R^+) \in \text{Perf}_{\text{f}} \) and

\[
(S^\Delta_1, P_i, \varphi_{P_i}, t_{(0,r)}, t_{[r', \infty]}, i)c_{1 \leq i \leq m}
\]

be objects giving an \( S \)-valued point of

\[
(\text{Sh}_{G,b_0,b_1}^{\mu_1} \times \cdots \times \text{Sh}_{G,b_{m-1},b_m}^{\mu_m}) \times (\text{Spd } F)^m U_m.
\]

Define \( P \) by gluing \( P_{1, y_{(0,r)}(S)} \) and \( P_m y_{(r', \infty)}(S) \) by the following modifications:
• Modifications occur only at $\bigcup_{i=1}^m \bigcup_{n \geq 0} \varphi^{-n}(S^4_i)$.

• Take $1 \leq i_0 \leq m$. Put

$$I_{i_0} = \{1 \leq i \leq m \mid S^d_i = S^d_{i_0}\}.$$ 

Define the modification at $S^d_{i_0}$ by the composite of the modifications at $S^d_i$ given by $\varphi_p$ for all $i \in I_{i_0}$. For $n > 0$, the modification at $\varphi^{-n}(S^d_{i_0})$ is given by the pullback under $\varphi^n$ of the modification at $S^d_{i_0}$.

Then $\mathcal{P}$ is naturally equipped with an isomorphism

$$\varphi_p : (\varphi^*_S \mathcal{P})|_{S \times \text{Spa} F^m \setminus \bigcup_{i=1}^m S^4_i} \simeq \mathcal{P}|_{S \times \text{Spa} F^m \setminus \bigcup_{i=1}^m S^4_i}.$$

Further, we have isomorphisms

$$\mathcal{P}|_{Y_{(0,r)}(S)} = \mathcal{P}|_{Y_{(0,r)}(S)} \xrightarrow{\iota_{(0,r),1}} G \times Y_{(0,r)}(S),$$

$$\mathcal{P}|_{Y_{(r',\infty)}(S)} = \mathcal{P}|_{Y_{(r',\infty)}(S)} \xrightarrow{\iota_{(r',\infty),m}} G \times Y_{(r',\infty)}(S).$$

These gives an $S$-valued point of $\text{Sht}^\mu_{G,b_0,b_m,U_m}$. Thus we obtain $m_{\mu*} : \text{Gr}_{G,\text{Spd} E_1 \times \cdots \times \text{Spd} E_m, \leq \mu*} \rightarrow \text{Gr}_{G,\text{Spd} E_1 \times \cdots \times \text{Spd} E_m, \leq \mu*}$ as in [SW20, Definition 20.4.4]. Then we have a convolution morphism

$$m_{\mu*} : \text{Gr}_{G,\text{Spd} E_1 \times \cdots \times \text{Spd} E_m, \leq \mu*} \rightarrow \text{Gr}_{G,\text{Spd} E_1 \times \cdots \times \text{Spd} E_m, \leq \mu*}$$

by [SW20, Proposition 20.4.5]. Note that

$$\text{Gr}_{G,\text{Spd} E_1 \times \cdots \times \text{Spd} E_m, \leq \mu*} \times (\text{Spd} F)^m U_m \simeq \text{Gr}_{G,\text{Spd} E_1 \times \cdots \times \text{Spd} E_m, \leq \mu*} \times (\text{Spd} F)^m U_m.$$ 

Then we have a morphism

$$\text{Sht}^\mu_{G,b_0,b_1} \times \cdots \times \text{Sht}^\mu_{G,b_m-1,b_m} \rightarrow \text{Gr}_{G,\text{Spd} E_1 \times \cdots \times \text{Spd} E_m, \leq \mu*}$$

by looking at a modification at each $S^4_i$. Then we have the commutative diagram

$$\begin{array}{ccc}
(Sht^\mu_{G,b_0,b_1} \times \cdots \times Sht^\mu_{G,b_m-1,b_m}) \times (Spd F)^m U_m & \xrightarrow{m_{\mu*} U_m} & Sht^\mu_{G,b_0,b_m,U_m} \\
\downarrow & & \downarrow \\
\widetilde{Gr}_{G,Spd E_1 \times \cdots \times Spd E_m, \leq \mu*} \times (Spd F)^m U_m & \rightarrow & \text{Gr}_{G,\text{Spd} E_1 \times \cdots \times \text{Spd} E_m, \leq \mu*} \times (\text{Spd} F)^m U_m
\end{array}$$

where the bottom morphism is induced by $m_{\mu*}$.

### 4.2 Twist morphism

Let $Z^0$ be the identity component of the center of $G$. Let $a, a' \in Z^0(\bar{F})$ and $\lambda \in \mathcal{X}_s(Z^0)$. Let $E$ be a finite extension of $F$ in $\mathbb{C}_p$ containing the fields of definition of $\mu$ and $\lambda$. We define the morphism

$$i^{\mu,\lambda}_{b,b',a,a'} : \text{Sht}^\mu_{G,b,b',\text{Spd} E} \times \text{Spd} E \rightarrow \text{Sht}^\lambda_{Z^0,a,a',\text{Spd} E} \xrightarrow{i^{\mu,\lambda}_{Z^0,a,a',\text{Spd} E}} \text{Sht}^\mu_{G,b,b',a,a'} \text{Spd} E.$$
as follows. Let \((S^x, \mathcal{E}_b \to \mathcal{E}_b')\) and \((S^z, \mathcal{E}_a \to \mathcal{E}_a')\) be modifications defining points in \(\text{Sht}_{G,b/b'}^\mu\) and \(\text{Sht}_{Z^0,a,a'}^\mu\). Then the diagonal arrow in the diagram

\[
\begin{array}{ccc}
\mathcal{E}_b \times Z^0 \mathcal{E}_a' & \longrightarrow & \mathcal{E}_b' \times Z^0 \mathcal{E}_a' \\
\uparrow & & \downarrow \\
\mathcal{E}_b \times Z^0 \mathcal{E}_a & \longrightarrow & \mathcal{E}_b' \times Z^0 \mathcal{E}_a
\end{array}
\]

defines the image of

\[
((S^x, \mathcal{E}_b \to \mathcal{E}_b'), (S^z, \mathcal{E}_a \to \mathcal{E}_a'))
\]

under \(t_{b,b',a,a'}^{\mu,\lambda}\) in \(\text{Sht}_{G,b/b'Z^0,a,a'}^\mu\). Note that we have equalities \(G_b(F) = G_{ab}(F)\) and \(G_{b'}(F) = G_{a'b'}(F)\).

**Proposition 4.1.** We have

\[
(R\Gamma_c(\text{Sht}_{G,b/b'}^\mu) \otimes^L R\Gamma_c(\text{Sht}_{Z^0,a,a'}^\lambda)) \otimes_{Z^0(F)} \overline{\mathbb{T}}_\ell \simeq R\Gamma_c(\text{Sht}_{G,b/b',ab,a'}^{\mu,\lambda})
\]

in the derived category of representations of \(G_b(F) \times G_{b'}(F) \times W_E\).

**Proof.** This follows from Lemma [14] and that \(t_{b,b',a,a'}^{\mu,\lambda}\) is a \(Z^0(F)\)-torsor. \(\square\)

## 5 Formula on cohomology

Let \(b_0, \ldots, b_m \in G(\overline{F})\) and \(\mu_1, \ldots, \mu_m \in X_s(T)^+\). Let \(E\) be a finite extension of \(F\) in \(\mathbb{C}_p\) containing \(E_i\) for \(1 \leq i \leq m\). Let

\[
m_{b_\bullet, \mu, \bullet} : \text{Sht}_{b_0,b_1,\text{Spd }\bar{E}}^{\mu_1} \times \text{Spd }\bar{E} \cdots \times \text{Spd }\bar{E} \text{Sht}_{b_{m-1},b_m,\text{Spd }\bar{E}}^{\mu_m} \to \text{Sht}_{b_0,b_m,\text{Spd }\bar{E}}^{[\mu,\bullet]}
\]

by the pullback of the convolution morphism \(m_{b_\bullet, \mu, \bullet, \bar{U}_m}\) defined in §4 under the morphism

\[
\text{Spd }\bar{E} = \Delta_m,\text{Spd }\bar{E} \hookrightarrow (\text{Spd }\bar{E})^m \longrightarrow \text{Spd }\bar{E}_1 \times \cdots \times \text{Spd }\bar{E}_m.
\]

The morphism \(m_{b_\bullet, \mu, \bullet}\) coincides with the morphism defined by the composition of modifications. This induces

\[
\overline{m}_{b_\bullet, \mu, \bullet} : (\text{Sht}_{b_0,b_1,\text{Spd }\bar{E}}^{\mu_1} \times \text{Spd }\bar{E} \cdots \times \text{Spd }\bar{E} \text{Sht}_{b_{m-1},b_m,\text{Spd }\bar{E}}^{\mu_m})/(\tilde{J}_{b_1} \times \cdots \times \tilde{J}_{b_{m-1}}) \to \text{Sht}_{b_0,b_m,\text{Spd }\bar{E}}^{[\mu,\bullet]},
\]

where \(\tilde{J}_{b_i}\) for \(1 \leq i \leq m - 1\) acts diagonally on the factor

\[
\text{Sht}_{b_{i-1},b_i,\text{Spd }\bar{E}}^{\mu_i} \times \text{Spd }\bar{E} \text{Sht}_{b_i,b_{i+1},\text{Spd }\bar{E}}^{\mu_{i+1}}
\]

and trivially on the other factors.

Let

\[
\tilde{\text{Gr}}_{G,\text{Spd }\bar{E}, \leq \mu, \bullet} \xrightarrow{m_{\mu, \bullet}} \text{Gr}_{G,\text{Spd }\bar{E}, \leq [\mu, \bullet]}
\]

be the pullback of

\[
m_{\mu, \bullet} : \tilde{\text{Gr}}_{G,\text{Spd }E_1 \times \cdots \times \text{Spd }E_m, \leq \mu, \bullet} \longrightarrow \text{Gr}_{G,\text{Spd }E_1 \times \cdots \times \text{Spd }E_m, \leq [\mu, \bullet]}
\]
under
\[ \text{Spd } \tilde{E} = \Delta_{m, \text{Spd } E} \hookrightarrow (\text{Spd } \tilde{E})^m \hookrightarrow \text{Spd } E_1 \times \cdots \times \text{Spd } E_m. \]

We define \( m_{\mu^\bullet, b_0, b_m} : \text{Sh}_E^{\mu^\bullet} \rightarrow \text{Sh}_E^{\mu^\bullet} \) by the fiber product
\[
\begin{array}{ccc}
\text{Sh}_E^{\mu^\bullet} & \xrightarrow{m_{\mu^\bullet, b_0, b_m}} & \text{Sh}_E^{\mu^\bullet} \\
\downarrow & & \downarrow \\
\text{Gr}_G, \text{Spd } E, \leq \mu^\bullet & \xrightarrow{m_{\mu^\bullet}} & \text{Gr}_G, \text{Spd } E, \leq \mu^\bullet
\end{array}
\]

Then \( \text{Sh}_E^{\mu^\bullet} \) is a moduli space of modifications
\[
\mathcal{E}_{b_0} \xrightarrow{f_1} \mathcal{E}_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} \mathcal{E}_{m-1} \xrightarrow{f_m} \mathcal{E}_{b_m}
\]
at \( S^i \) such that \( f_i \) is bounded by \( \mu_i \) for \( 1 \leq i \leq m \). We define a subspace \( \text{Sh}_E^{b_0, b_m, \mu^\bullet} \subset \text{Sh}_E^{\mu^\bullet} \) as a moduli space of modifications
\[
\mathcal{E}_{b_0} \xrightarrow{f_1} \mathcal{E}_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} \mathcal{E}_{m-1} \xrightarrow{f_m} \mathcal{E}_{b_m}
\]
at \( S^i \) such that \( f_i \) is bounded by \( \mu_i \) for \( 1 \leq i \leq m \) and \( \mathcal{E}_i \) is isomorphic to \( \mathcal{E}_b \) geometric fiberwisely for \( 1 \leq i \leq m-1 \).

We put
\[
I_{b_0, b_m}^{\mu^\bullet} = \{(b_1, \ldots, b_{m-1}) \in B(G)^{m-1} \mid \text{Sh}_E^{\mu_i} \neq \emptyset \text{ for } 1 \leq i \leq m\}.
\]

We take \( \mu_{m+1} \) such that \( b_m \in B(G, \mu_{m+1}, [1]) \). Then \( I_{b_0, b_m}^{\mu^\bullet} \) is a finite set, since it is contained in \( \prod_{1 \leq i \leq m-1} B(G, \Sigma_{j=i+1}^{m+1} \mu_j, [1]) \) by Lemma 2.6. For \( \lambda \in X_+(T) / \Gamma_E \), we put
\[
V^{\mu^\lambda}_\mu = \text{Hom}_{\text{L}(G)}(V_\lambda, \bigotimes_{1 \leq i \leq m} V_{\mu_i}).
\]

For \( (b_i)_{1 \leq i \leq m-1} \in I_{b_0, b_m}^{\mu^\bullet} \), we put \( N_{b_\bullet} = \sum_{1 \leq i \leq m-1} N_{b_i} \). We write \( \text{Gr}_G^{(\infty)} \) for the inverse image of \( \text{Gr}_G, \text{Spd } E, \leq \mu \) under \( L \text{Spd } E \rightarrow \text{Gr}_G, \text{Spd } E \).

**Proposition 5.1.** The sum
\[
\sum_{\lambda \in X_+(T) / \Gamma_E} V^{\mu^\lambda}_\mu \otimes^L R \Gamma_c(\text{Sh}_E^{\mu^\lambda})
\]
is decomposed into
\[
\left( \bigotimes_{1 \leq i \leq m} R \Gamma_c(\text{Sh}_E^{\mu^\lambda}) \otimes^L \bigotimes_{1 \leq i \leq m-1} \delta_{b_i} \right) \otimes^L \prod_{i=1}^{m-1} G_{b_i}(F)^{\Lambda[2N_{b_\bullet}]}
\]
for \( (b_i)_{1 \leq i \leq m-1} \in I_{b_0, b_m}^{\mu^\bullet} \) by distinguished triangles in the derived category of representations of \( G_{b_0}(F) \times G_{b_m}(F) \times W_E \).
Proof. Let \( IC_{\mu} \) be the external twisted product of \( IC_{\mu_1}, \ldots, IC_{\mu_m} \) on \( \widetilde{Gr}_{\text{Spd}, E} \leq \mu \). By the construction of convolution product \([ FS21 \text{ VI.8]}\) in geometric Satake equivalence and \([ FS21 \text{ Proposition VII.4.3]}, \) we have

\[
(m_{\mu})_\ast IC_{\mu} = \sum_{\lambda \in \mathcal{X}(T) \vdash T} V^\lambda \otimes^L IC_\lambda.
\]

Hence the sum

\[
\sum_{\lambda \in \mathcal{X}(T) \vdash T} V^\lambda \otimes^L R \Gamma_c(Sht^\lambda_{b_0,b_m})
\]

is isomorphic to \( R \Gamma_c(Sht^\mu_{b_0,b_m}; IC_{\mu}) \).

We put \( \mu = (\mu_1, \ldots, \mu_m) \). Let \( \{[h_{i,j-1}]\}_{1 \leq j \leq n} \) be the image of the projection \( I_{i,j} \to B(G) \) to the \((m-1)\)-th component. It suffices to show that \( R \Gamma_c(Sht^\mu_{b_0,b_m}; IC_{\mu}) \) is decomposed into

\[
(R \Gamma_c(Sht^\mu_{b_0,b_{m-1}}) \otimes^L R \Gamma_c(Sht^\mu_{b_{m-1},b_m}) \otimes^L \delta_{h_{i,j}}(F)) \otimes^L \Gamma_c(Sht^\mu_{b_0,b_m}) \Lambda[2N_{b_{m-1}}]
\]

for \( 1 \leq j \leq n \).

Let \( K \subset G_{b_0}(F) \) be enough small compact open subgroup. Then

\[
R \Gamma_c(Sht^\mu_{b_0,K,b_m}; IC_{\mu}) \cong t_{b_0}^* i_{b_0}^* T_{\mu} i_{b_0}! (h_K; \Lambda) \cong t_{b_0}^* i_{b_0}^* T_{\mu} i_{b_0}! (h_K; \Lambda)
\]

is decomposed into

\[
t_{b_0}^* i_{b_0}^* T_{\mu} i_{b_0}! (h_K; \Lambda)
\]

for \( 1 \leq j \leq n \) by Lemma 3.3. This is isomorphic to

\[
t_{b_0}^* i_{b_0}^* T_{\mu} i_{b_0}! (h_K; \Lambda) \cong \left( \lim_{K' \subset G_{b_{m-1}}(F)} h_{K', \Lambda} \right) \otimes^L R \Gamma_c(Sht^\mu_{b_0,K,b_{m-1}}) \otimes^L \Gamma_c(Sht^\mu_{b_0,b_m}) \Lambda[2N_{b_{m-1}}]
\]

by Lemma 2.7. By Lemma 1.4,

\[
\left( \lim_{K' \subset G_{b_{m-1}}(F)} h_{K', \Lambda} \right) \otimes^L R \Gamma_c(Sht^\mu_{b_0,K,b_{m-1}}) \otimes^L \Gamma_c(Sht^\mu_{b_0,b_m}) \Lambda[2N_{b_{m-1}}]
\]

is isomorphic to

\[
(R \Gamma_c(Sht^\mu_{b_0,b_{m-1}}) \otimes^L R \Gamma_c(Sht^\mu_{b_{m-1},b_m}) \otimes^L \delta_{h_{i,j}}(F)) \otimes^L \Gamma_c(Sht^\mu_{b_0,b_m}) \Lambda[2N_{b_{m-1}}]
\]

since \( t_{b_0}^* i_{b_0}^* T_{\mu} i_{b_0}! \), commutes with direct limits, tensors and changes of coefficients. Therefore we obtain the claim.

Corollary 5.2. We have

\[
\sum_{(b_i)_{1 \leq i \leq m-1} \in \Gamma_{b_{m-1},b_m}^\ast} H_S \left( \prod_{i=1}^{m-1} G_{b_i}(F), \bigotimes_{1 \leq i \leq m} H_c^\ast(Sht^\mu_{b_{i-1},b_i}) \otimes^L \delta_{b_i} \right) = \sum_{\lambda \in \mathcal{X}(T) \vdash T} V^\lambda \otimes^L H_c^\ast(Sht^\lambda_{b_0,b_m})
\]

as virtual representations of \( G_{b_0}(F) \times G_{b_m}(F) \times W_E \).
We have the commutative diagram

\[ \theta_{b,b'} : \text{Sht}_{G,b,b'}^\mu \to \text{Sht}_{G,(b'),(b)}^\mu \]

by sending \( f : \mathcal{E}_b \to \mathcal{E}_{b'} \) to \( \iota(f)^{-1} : \mathcal{E}_{(b')} \to \mathcal{E}_{(b)} \). The above isomorphism is compatible with actions of \( \mathcal{J}_b \times \mathcal{J}_{b'} \) and \( \mathcal{J}_{(b')} \times \mathcal{J}_{(b)} \) under the isomorphism

\[ \mathcal{J}_b \times \mathcal{J}_{b'} \to \mathcal{J}_{(b')} \times \mathcal{J}_{(b)} ; (g,g') \mapsto (\iota(g'),\iota(g)). \]

Then \( \theta_{b,(b)} \) is an involution on \( \text{Sht}_{G,b,b'}^\mu \). On the other hand, \( \theta \) induces a morphism \( \theta : \text{Hck}_{G} \to \text{Hck}_{G} \). Let \( E \) be the field of definition of \( \mu \). We have a natural morphism

\[ \pi_{b,b'}^\mu : \text{Sht}_{G,b,b'}^\mu \to \text{Hck}_{G,\text{Spd } E}. \]

We have the commutative diagram

\[
\begin{array}{ccc}
\text{Sht}_{G,b,b'}^\mu & \xrightarrow{\theta_{b,b'}} & \text{Sht}_{G,(b'),(b)}^\mu \\
\pi_{b,b'}^\mu | & | & \pi_{(b'),(b)}^\mu \\
\text{Hck}_{G,\text{Spd } E} & \xrightarrow{\theta} & \text{Hck}_{G,\text{Spd } E} \\
\end{array}
\]

We have \( \mathcal{S}'(r_\mu \circ \text{ad}(\tilde{\rho}(-1))) \cong \theta^* \text{IC}^\mu_\mu \) by \[\text{FS21} \text{ Proposition VI.12.1}\]. Hence \( \tilde{\rho}(-1) : r_\mu \circ \text{ad}(\tilde{\rho}(-1)) \to r_\mu \) induces \( M_\mu : \theta^* \text{IC}^\mu_\mu \to \text{IC}^\mu_{(b')} \). Hence we obtain the isomorphism

\[ \Gamma_c(\text{Sht}_{G,(b'),(b)}^\mu) \to \Gamma_c(\text{Sht}_{G,b,b'}^\mu) \]

induced by \( \theta_{b,b'} \).
Lemma 6.1. The isomorphism
\[
R\Gamma_c(\text{Sh}_{G,\iota(b)}^\mu) \to R\Gamma_c(\text{Sh}_{G,b}^\mu)
\]
is compatible with actions of \(\tilde{J}_b \times \tilde{J}_{b'}\) and \(\tilde{J}_{\iota(b')} \times \tilde{J}_{\iota(b)}\) under the isomorphism
\[
\tilde{J}_b \times \tilde{J}_{b'} \to \tilde{J}_{\iota(b')} \times \tilde{J}_{\iota(b)}; \ (g, g') \mapsto (\iota(g'), \iota(g)).
\]

Proof. This follows from the definition. \(\square\)

Further, we have an involution
\[
\theta_b: \text{Sh}_{G,b,1}^\mu \times \text{Sh}_{G,1,\iota(b)}^\mu \to \text{Sh}_{G,b,1}^\mu \times \text{Sh}_{G,1,\iota(b)}^\mu; \ (x, x') \mapsto (\theta_{\iota,\iota(b)}(x'), \theta_{b,1}(x)).
\]

We have a decomposition
\[
V_\mu \otimes V_\mu = \text{Sym}^2 V_\mu \oplus \bigwedge^2 V_\mu.
\]

Let
\[
\Psi_{b,\mu}: \left(R\Gamma_c(\text{Sh}_{b,1}^\mu) \otimes R\Gamma_c(\text{Sh}_{\iota(b)}^\mu)\right) \otimes_{G(F)} \Lambda \to \sum_{\lambda \in \text{X}_*(T)^+/T} V_\mu^\lambda \otimes R\Gamma_c(\text{Sh}_{b,\iota(b)}^\lambda)
\]
be the morphism given by Proposition 5.1. Let \(s_{b,\mu}\) be the involution on the source of \(\Psi_{b,\mu}\) induced by \(\theta_b\) and the multiplication by \((-1)^{(2\mu, \mu)}\). On the other hand, let \(t_{b,\mu}\) be the involution on the target of \(\Psi_{b,\mu}\) induced by the permutation \(\sigma_{V_\mu, V_\mu}\) on \(V_\mu \otimes V_\mu\) and \(\theta_{b,\iota(b)}: \text{Sh}_{b,\iota(b)}^\lambda \to \text{Sh}_{b,\iota(b)}^\lambda\).

Proposition 6.2. The morphism \(\Psi_{b,\mu}\) is compatible with the involutions \(s_{b,\mu}\) and \(t_{b,\mu}\).

Proof. By the characterization of the commutativity constraint, the equality
\[
\text{IC}_\mu^\prime \ast \text{IC}_\mu^\prime = \sum_{\lambda \in \text{X}_*(T)^+/T} V_\mu^\lambda \otimes \text{IC}_\lambda^\prime
\]
is compatible with the involutions \(c_{\text{IC}_\mu, V_\mu}\) and \(c_{V_\mu, \text{IC}_\mu}\). Hence the target of \(\Psi_{b,\mu}\) is equal to \(H^*_c(\text{Sh}_{b,\iota(b)}^\mu, \text{IC}_\mu^\prime \ast \text{IC}_\mu^\prime)\) with the involution given by \(c_{\text{IC}_\mu, V_\mu}\) and \(\theta_{b,\iota(b)}\). Let \(\sigma_{2, X}: (\text{Div}_X^\mu)^2 \to (\text{Div}_X^\mu)^2\) and \(\sigma_{2, G}: \text{Hck}^\{1,2\}_G \to \text{Hck}^\{1,2\}_G\) be the permutation of two Cartier divisors. Let \(\text{IC}_\mu^\prime \ast \text{IC}_\mu^\prime\) be the fusion product on \(\text{Hck}_G^\{1,2\}\). Here we use the notation at the beginning of [FS21] VI.9. Then we have the morphism
\[
\tilde{c}_{\text{IC}_\mu, V_\mu}: \sigma_{2, G}^*(\text{IC}_\mu^\prime \ast \text{IC}_\mu^\prime) \to \text{IC}_\mu^\prime \ast \text{IC}_\mu^\prime
\]
extending \(c_{\text{IC}_\mu, V_\mu}\).

The morphism \(\theta\) induces \(\theta^{(1),(2)}: \text{Hck}^\{1,2\}^\{1,(1),(2)\} \to \text{Hck}^\{1,2\}^\{1,(1),(2)\}\) switching two Cartier divisors. Here we use the notation in the proof of [FS21], Proposition VI.9.4. Then we have a morphism
\[
S_{\mu, \mu}: \theta^{(1),(2)*}(\text{IC}_\mu^\prime \boxtimes \text{IC}_\mu^\prime) \to \text{IC}_\mu^\prime \otimes \text{IC}_\mu^\prime
\]

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induced by \( M_\mu \) and switching two factors of IC'\( _\mu \). The morphism \( \theta \) induces \( \theta^{(1,2)} : \mathcal{H}ck_G^{(1,2)} \to \mathcal{H}ck_G^{(1,2)} \) switching two Cartier divisors. Then we have

\[
S'_{\mu,\mu} = \frac{m}(S_{\mu,\mu}) : \theta^{(1,2)}(\text{IC}'_\mu * \text{IC}'_\mu) \to \text{IC}'_\mu * \text{IC}'_\mu.
\]

Since \( \theta^{(1,2)} \circ \sigma_{2,G} \) is the automorphism of \( \mathcal{H}ck_G^{(1,2)} \) over (\text{Div}_X)^2 induced by \( \theta \), we also have \( \sigma_{2,G} : (\text{IC}'_\mu * \text{IC}'_\mu) \to (\text{IC}'_\mu * \text{IC}'_\mu) \) defined in the same way as \( M_\mu \).

Then it suffices to show that

\[
\theta^{(1,2)}(\text{IC}'_\mu * \text{IC}'_\mu) \xrightarrow{\sigma_{2,G}(M_{\mu,\mu})} \sigma_{2,G}(\text{IC}'_\mu * \text{IC}'_\mu) \xrightarrow{\tilde{c}_{\nu,\mu}} \text{IC}'_\mu * \text{IC}'_\mu
\]

and \( S'_{\mu,\mu} \) are equal. It suffices to check this on \( \mathcal{H}ck_G^{(1,2)} \times (\text{Div}_X)^{1,2} \times (\text{Div}_X)^{1,2}:\{1\},\{2\} \) by [FS21] Proposition VI.9.3. This follows from the constructions of \( \tilde{c}_{\nu,\mu}, S'_{\mu,\mu} \) and \( M_{\mu,\mu} \).

\[\square\]

7 Kottwitz conjecture

**Definition 7.1.** Let \( \varphi : W_\ell \to LG \) be an \( \ell \)-adic local L-parameter for \( G \) (cf. [Ima20, Definition 1.14]). We put

\[S_{\varphi} = \{ g \in \widehat{G}(\mathbb{Q}_\ell) \mid g \varphi g^{-1} = \varphi \} .\]

We say that \( \varphi \) is discrete if \( S_{\varphi}/Z(\widehat{G})^{\text{Tr}} \) is finite (cf. [Far10, Definition 4.1]).

Let \( b, b' \in \text{GL}_n(F) \) such that \([b] \in B(G, \mu, [b'])\). We put

\[H^{*}(\text{Sht}_{b,b'}^\mu)[\pi] = \sum_{i,j \in \mathbb{Z}} (-1)^{i+j} \text{Ext}_{G_\ell(F)}^{i,j}(R^j \Gamma_c(\text{Sht}_{b,b'}^\mu), \pi)\]

for an irreducible smooth representation \( \pi \) of \( G_b(F) \).

The following is a version of Kottwitz conjecture for moduli spaces of mixed characteristic local shtukas in GL\(_n\)-case (cf. [RV14, Conjecture 7.4]):

**Conjecture 7.2.** Assume that \( b,b' \) are basic. Let \( \varphi : W_\ell \to LG \) be a discrete local L-parameter. Let \( \pi_b \) and \( \pi_{b'} \) be the irreducible smooth representations of \( G_b(F) \) and \( G_{b'}(F) \) corresponding to \( \varphi \) via the local Langlands correspondence. Then we have

\[H^{*}(\text{Sht}_{b,b'}^\mu)[\pi_b] = \pi_{b'} \boxtimes (r_\mu \circ \varphi)\]

in Groth\( (G_{b'}(F) \times W_\ell) \).

For an object \( \mathcal{C} \) in a derived category, we put \( \mathcal{H}^{*}(\mathcal{C}) = \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^{i}(\mathcal{C}) \). The following conjecture is motivated by [Dat07, Théorème A].

**Conjecture 7.3.** Assume that \( b,b' \) are basic. Let \( \varphi : W_\ell \to LG \) be a discrete local L-parameter. Let \( \pi_b \) and \( \pi_{b'} \) be the irreducible smooth representations of \( G_b(F) \) and \( G_{b'}(F) \) corresponding to \( \varphi \) via the local Langlands correspondence. Then we have

\[\mathcal{H}^{*}(R \text{Hom}_{G_b(F)}(R \Gamma_c(\text{Sht}_{b,b'}^\mu), \pi_b)) \simeq \pi_{b'} \boxtimes (r_\mu \circ \varphi)\]

as representations of \( G_{b'}(F) \times W_\ell \).
Lemma 7.4. Assume that \( b \) is basic. Let \( \pi_b \) and \( \pi_{\iota(b)} \) be the irreducible smooth representations of \( G_b(F) \) and \( G_{\iota(b)}(F) \) corresponding via the local Jacquet–Langlands correspondence. Then the pullback of \( \pi_{\iota(b)} \) under the isomorphism \( \iota: G_b(F) \to G_{\iota(b)}(F) \) is isomorphic to \( \pi_b^\ast \).

Proof. By [Pra19, Corollary 1], we may assume that \( \iota(g) = t \cdot g^{-1} \). If \( b = 1 \), the claim follows from a theorem of Gelfand and Kazhdan (cf. [BZ76, 7.3. Theorem]). If regular elements \( g \in GL_n(F) \) and \( g' \in G_b(F) \) have the same reduced characteristic polynomial, then \( \iota(g) \in GL_n(F) \) and \( \iota(g') \in G_{\iota(b)}(F) \) are regular and have the same reduced characteristic polynomial. Hence the claim follows from the case where \( b = 1 \) and the characterization of the local Jacquet–Langlands correspondence.

We put \( \kappa(b) = v_F(\det(b)) \). For \( m_1, \ldots, m_n \in \mathbb{Z} \), let \( (m_1, \ldots, m_n) \) denote the cocharacter of \( GL_n \) or its standard Levi subgroup defined by \( z \mapsto \text{diag}(z^{m_1}, \ldots, z^{m_n}) \).

Theorem 7.5. Conjecture 7.3 is true in the following cases:

(1) \( \kappa(b) \equiv \kappa(b') \mod n \) and
\[
\mu = \frac{\kappa(b) - \kappa(b')}{n}(1, \ldots, 1).
\]

(2) \( \kappa(b) \equiv 0, 1, \kappa(b) \equiv \kappa(b') + 1 \mod n \) and
\[
\mu = \frac{\kappa(b) - \kappa(b') - 1}{n}(1, \ldots, 1) + (1, 0, \ldots, 0).
\]

(3) \( \kappa(b) \equiv 0, -1, \kappa(b) \equiv \kappa(b') - 1 \mod n \) and
\[
\mu = \frac{\kappa(b) - \kappa(b') + 1}{n}(1, \ldots, 1) + (0, \ldots, 0, -1).
\]

(4) \( \kappa(b) \equiv 1, \kappa(b') \equiv -1 \mod n \) and
\[
\mu = \frac{\kappa(b) - \kappa(b') - 2}{n}(1, \ldots, 1) + \begin{cases} (2, 0, \ldots, 0), \\ (1, 1, 0, \ldots, 0). \end{cases}
\]

(5) \( \kappa(b) \equiv -1, \kappa(b') \equiv 1 \mod n \) and
\[
\mu = \frac{\kappa(b) - \kappa(b') + 2}{n}(1, \ldots, 1) + \begin{cases} (0, \ldots, 0, -2), \\ (0, \ldots, 0, -1, -1). \end{cases}
\]

Proof. By the inversing isomorphism (2.2), the claims (3) and (5) are reduced to the claims (2) and (4). By Proposition 4.1, we may assume that \( \kappa(b) = \kappa(b') = 0 \) in (1) \( \kappa(b) = 0, -1, \kappa(b) = \kappa(b') + 1 \) in (2) and \( \kappa(b) = -1, \kappa(b') = 1 \) in (4). Further, we may assume that \( \kappa(b) = 0 \) in (2) by Lemma 6.1 and Lemma 7.4. Then the claim (1) is trivial. The claim (2) follows from the proof of [Dat07, Théorème A] taking care the degree in [Dat07, Théorème 4.1.2].
We show the claim. We may assume that \( b' = \iota(b) \). We put
\[
\mu_1 = (1, 0, \ldots, 0), \quad \mu_2 = (2, 0, \ldots, 0), \quad \mu_{1,1} = (1, 1, 0, \ldots, 0).
\]

Note that we have \( \{\mu_1, \mu_2\} = \{[1]\} \). Let \( \pi_1 \) be the irreducible smooth representations of \( GL_n(F) \) corresponding to \( \varphi \) via the local Langlands correspondence. By Proposition 5.1, Lemma 5.3, the claim (2) and [Dat07, Corollaire 4.2.1], we have
\[
(V_{\mu_2}^2)^* \otimes \mathcal{H}^* \left( R \text{Hom}_{G(b)}(F) \left( R\Gamma_c(Sht^{\mu_2}_{(b),b}), \pi(b) \right) \right)
\]
\[
+ (V_{\mu_1,1}^2)^* \otimes \mathcal{H}^* \left( R \text{Hom}_{G(b)}(F) \left( R\Gamma_c(Sht_{(b),1}^{\mu_1}), \pi_1(b) \right) \right)
\]
\[
\simeq \mathcal{H}^* \left( R \text{Hom}_{G(b)}(F) \left( R\Gamma_c(Sht_{(b),1}^{\mu_1}) \otimes R\Gamma_c(Sht_{(b),1}^{\mu_1}) \otimes \otimes_{G(b)}(F) \otimes_{\ell}, \pi(b) \right) \right)
\]
\[
\simeq \mathcal{H}^* \left( R \text{Hom}_{G(b)}(F) \left( R\Gamma_c(Sht_{(b),1}^{\mu_1}), \mathcal{H}^* \left( R \text{Hom}_{G(b)}(F) \left( R\Gamma_c(Sht_{(b),1}^{\mu_1}), \pi_1(b) \right) \right) \right) \right)
\]
\[
\simeq \mathcal{H}^* \left( R \text{Hom}_{G(b)}(F) \left( R\Gamma_c(Sht_{(b),1}^{\mu_1}), \pi_1 \boxplus \varphi \right) \right)
\]
\[
\simeq \pi_b \boxplus (\varphi \boxplus \varphi) \simeq \pi_b \boxplus \left( (\mu_2 \circ \varphi) \oplus (\mu_{1,1} \circ \varphi) \right)
\]

Using Proposition 5.2 we can separate the above equality to obtain the claim. \( \square \)

**Corollary 7.6.** Conjecture 7.3 is true if \( n \leq 3 \) and \( \mu \) is minuscule.

**Proof.** All the cases are contained in Theorem 7.3 \( \square \)

### 8 Inductive formula

For a smooth representation \( \pi \) of \( G(F) \) and the unipotent radical \( N \) of a parabolic subgroup of \( G \), let \( \pi_N \) denote the Jacquet module of \( \pi \) with respect to \( N \).

Assume that \( G = GL_2 \). Let \( T \) be the diagonal torus and \( B \) be the upper triangle Borel subgroup of \( GL_2 \). Let \( N \) be the unipotent radical of \( B \), and \( N^{\text{op}} \) be the unipotent radical of the opposite Borel subgroup \( B^{\text{op}} \). Let \( \delta_B : T(F) \to \mathbb{T}_\ell^\times \) be the modulus character with respect to \( B \). For \( b = \begin{pmatrix} \varpi_m & 0 \\ 0 & \varpi^l \end{pmatrix} \) with \( m < l \), let \( \delta_b : G_b(F) \to \mathbb{T}_\ell^\times \) be the character determined by \( \delta_B \) and the natural isomorphism \( G_b(F) \cong T(F) \).

**Lemma 8.1.** Let \( m \in \mathbb{Z} \). We put
\[
b = \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^m \end{pmatrix}, \quad b' = \begin{pmatrix} \varpi^{m-1} & 0 \\ 0 & \varpi^m \end{pmatrix}.
\]

Let \( \pi \) be an admissible representation of \( G(F) \). Then we have
\[
R^* \text{Hom}_{G_b(F)} \left( R^* \Gamma_c(Sht_{b,b'}^{(1,0)}), \pi \right) = -R^* \text{Hom}_{T(F)} \left( R^* \Gamma_c(Sht_{T,b,b'}^{(1,0)}), \pi_{N^{\text{op}}} \right) \left( \frac{1}{2} \right) \]

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Proof. By [Cas82, A.11 Proposition, A.12 Theorem], [GI16, Theorem 4.25] (cf. [Han21a]) and [Ren10, III.2.7 Théorème, VI.9.6 Proposition], we have

$$R^* \text{Hom}_{G(F)} \left( R^* \Gamma_c(\text{Sh}_{b,b'}^{(1,0)}), \pi \right) = R^* \text{Hom}_{G(F)} \left( \pi^*, R^* \Gamma_c(\text{Sh}_{b,b'}^{(1,0)})^* \right)$$

$$= R^* \text{Hom}_{T(F)} \left( (\pi^*)_N, -R^* \Gamma_c(\text{Sh}_{T,b,b'}^{(1,0)})^* \right) \left( \frac{1}{2} \right)$$

$$= -R^* \text{Hom}_{T(F)} \left( (\pi^*)_N, R^* \Gamma_c(\text{Sh}_{T,b,b'}^{(1,0)})^* \right) \left( \frac{1}{2} \right)$$

$$= -R^* \text{Hom}_{T(F)} \left( R^* \Gamma_c(\text{Sh}_{T,b,b'}^{(1,0)}), \pi_{N^w} \right) \left( \frac{1}{2} \right).$$

\[\square\]

**Proposition 8.2.** Let $\chi_1, \chi_2 : F^\times \to \overline{\mathbb{Q}}_l^\times$ be characters. Let $\varphi_{\chi_i} : \text{W}_F \to \overline{\mathbb{Q}}_l^\times$ be the character corresponding to $\chi_i$. We put $\rho = \chi_1 \boxtimes \chi_2$ as representations of $T(F)$. Let $m \geq 0$ and $m/2 \leq l \geq 0$. We put

$$b = \left( \begin{array}{cc} \varpi^l & 0 \\ 0 & \varpi^{m-l} \end{array} \right), \quad b_1 = \left( \begin{array}{cc} \varpi^{l-1} & 0 \\ 0 & \varpi^{m-l} \end{array} \right), \quad b_2 = \left( \begin{array}{cc} \varpi^{l-1} & 0 \\ 0 & \varpi^{m-1-l} \end{array} \right).$$

(1) Assume $m \neq 2l$. We put

$$b'_1 = \left( \begin{array}{cc} \varpi^l & 0 \\ 0 & \varpi^{m-l-1} \end{array} \right).$$

If $l = 0$, then we have

$$H_c^* (\text{Sh}_{b_1}^{(m,0)}) [\rho] = (-1)^m (\text{Ind}_{B(F)}^{G(F)} \rho) \boxtimes \varphi_{\chi_2}^m \left( \frac{m}{2} \right).$$

If $l \geq 1$, then we have

$$H_c^* (\text{Sh}_{b_1}^{(m,0)}) [\rho] = -H_c^* (\text{Sh}_{b_1}^{(m-1,0)}) [\rho] \boxtimes \varphi_{\chi_1} \left( -\frac{1}{2} \right) - H_c^* (\text{Sh}_{b_1}^{(m-2,0)}) [\rho] \boxtimes \varphi_{\chi_1} \boxtimes \varphi_{\chi_2}$$

$$\left\{ \begin{array}{ll} -H_c^* (\text{Sh}_{b_1}^{(m-1,0)}) [\text{Ind}_{B(F)}^{G(F)} \rho] \boxtimes \varphi_{\chi_2} \left( \frac{1}{2} \right) & \text{if } m = 2l + 1 \\
-H_c^* (\text{Sh}_{b_1}^{(m-1,0)}) [\rho] \boxtimes \varphi_{\chi_2} \left( \frac{1}{2} \right) & \text{if } m \geq 2l + 2. \end{array} \right.$$

(2) Assume $m = 2l$. If $l = 0$, then we have

$$H_c^* (\text{Sh}_{b_1}^{(0,0)}) [\text{Ind}_{B(F)}^{G(F)} \rho] = (\text{Ind}_{B(F)}^{G(F)} \rho) \boxtimes 1.$$

If $l \geq 1$, then we have

$$H_c^* (\text{Sh}_{b_1}^{(m,0)}) [\text{Ind}_{B(F)}^{G(F)} \rho] = -H_c^* (\text{Sh}_{b_1}^{(m-1,0)}) [\rho] \boxtimes \varphi_{\chi_1} \left( -\frac{1}{2} \right)$$

$$- H_c^* (\text{Sh}_{b_1}^{(m-1,0)}) [\rho^w \otimes \delta_{B}^{-1}] \boxtimes \varphi_{\chi_2} \left( \frac{1}{2} \right)$$

$$- H_c^* (\text{Sh}_{b_1}^{(m-2,0)}) [\text{Ind}_{B(F)}^{G(F)} \rho] \boxtimes \varphi_{\chi_1} \boxtimes \varphi_{\chi_2}.$$
Proof. First we show the claim (1) If $l = 0$, we have

$$R^* \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b,1}^{(m,0)}), \rho \right) = R^* \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b,1}^{(0,-m)}), \rho \right)$$

$$= (-1)^m R^* \text{Hom}_{G_b(F)} \left( \text{Ind}_{B(F)}^{G(F)} R^* \Gamma_c(\text{Sh}_{T,1}^{(0,-m)}), \frac{m}{2} \otimes \delta_{b_1}^{-1}, \rho \right)$$

$$= (-1)^m \text{Ind}_{B(F)}^{G(F)} \left( R^* \text{Hom}_{G_b(F)} \left( R^* \Gamma_c(\text{Sh}_{T,1}^{(0,-m)}), \frac{m}{2} \otimes \delta_{b_1}^{-1}, \rho \right) \otimes \delta_{b_1}^{-1} \right)$$

$$= (-1)^m \text{Ind}_{B(F)}^{G(F)} \left( R^* \text{Hom}_{G_b(F)} \left( R^* \Gamma_c(\text{Sh}_{T,1}^{(0,m)}), \frac{m}{2}, \rho \otimes \delta_b \right) \otimes \delta_B^{-1} \right)$$

$$= (-1)^m (\text{Ind}_{B(F)}^{G(F)} \rho) \boxtimes \varphi_{x_2}^{m} \left( \frac{m}{2} \right),$$

where we use $\text{Sh}_{1,b}^{(m-1,1)} = \emptyset$ and [GI16 Theorem 4.25] at the second equality. We assume that $l \geq 1$. By Proposition 5.1 and Lemma 5.3 the sum

$$R^* \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b,1}^{(m,0)}), \rho \right) + R^* \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b,1}^{(m-1,1)}), \rho \right)$$

is equal to the sum

$$R^* \text{Hom}_{G_{b_1}(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(m-1,0)}), R \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b,b_1}^{(1,0)}), \rho \right) \otimes \delta_{b_1}^{-1} \right)$$

$$+ R^* \text{Hom}_{G_{b_1}(F)} \left( R\Gamma_c(\text{Sh}_{b_{1}',1}^{(m-1,0)}), R \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b_{1}',1}^{(1,0)}), \rho \right) \otimes \delta_{b_{1}'}^{-1} \right).$$

Since the fiber of the natural morphism $\text{Sh}_{b,b_1}^{(1,0)} \to \text{Sh}_{T,b,b_1}^{(1,0)}$ is isomorphic to $\mathbb{B}^{m = m + 1 - 2l}$, we have

$$R^* \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b,b_1}^{(1,0)}), \rho \right) = - (\rho \otimes \delta_B) \boxtimes \varphi_{x_1} \left( - \frac{1}{2} \right).$$

Further, we have

$$R^* \text{Hom}_{G_{b_1}(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(m-1,0)}), R \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b,b_1}^{(1,0)}), \rho \right) \otimes \delta_{b_1}^{-1} \right)$$

$$= - R^* \text{Hom}_{G_{b_1}(F)} \left( R\Gamma_c(\text{Sh}_{b_{1},1}^{(m-1,0)}), \rho \right) \boxtimes \varphi_{x_1} \left( - \frac{1}{2} \right).$$

If $m = 2l + 1$, we have

$$R^* \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b,b_1}^{(1,0)}), \rho \right) = - \left( \text{Ind}_{B(F)}^{G(F)} \rho \right) \boxtimes \varphi_{x_2} \left( \frac{1}{2} \right)$$

by the claim in the case where $l = 0$.

If $m \geq 2l + 2$, since the fiber of the natural morphism $\text{Sh}_{b,b_1}^{(1,0)} \to \text{Sh}_{T,b,b_1}^{(0,1)}$ is isomorphic to $\mathbb{B}^{m = m - 2l}$, we have

$$R^* \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b,b_1}^{(1,0)}), \rho \right) = - (\rho \otimes \delta_B) \boxtimes \varphi_{x_2} \left( \frac{1}{2} \right).$$
Therefore

\[
R^* \text{Hom}_{G_a(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(m,0)}), \rho \right)
\]

\[
= R^* \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(m-1,0)}), R \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(1,0)}), \rho \otimes \delta_{b_1} \right) \right)
\]

\[
+ R^* \text{Hom}_{G_{b_1}'(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(m-1,0)}), R \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(1,0)}), \rho \otimes \delta_{b_1} \right) \right)
\]

\[
- R^* \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(m-1,1)}), \rho \right)
\]

\[
= -H_c^*(\text{Sh}_{b_1,1}^{(m-1,0)})[\rho] \otimes \varphi_{\chi_1} \left(-\frac{1}{2}\right) - H_c^*(\text{Sh}_{b_1,1}^{(m-2,0)})[\rho] \otimes \varphi_{\chi_1} \otimes \varphi_{\chi_2} - H_c^*(\text{Sh}_{b_1,1}^{(m-1,0)})[\rho] \otimes \varphi_{\chi_2} \left(\frac{1}{2}\right) \quad \text{if } m = 2l + 1,
\]

\[
-H_c^*(\text{Sh}_{b_1,1}^{(m-1,0)})[\rho] \otimes \varphi_{\chi_2} \left(\frac{1}{2}\right) \quad \text{if } m \geq 2l + 2.
\]

Next we show the claim [2]. The claim is trivial if \( l = 0 \). Assume that \( l > 0 \). We put \( \pi = \text{Ind}_{B(F)}^G(\rho) \) and

\[
b'_1 = \begin{pmatrix} 0 & \varpi^{l-1} \\ \varpi^l & 0 \end{pmatrix}.
\]

By Proposition 5.1 and Lemma 5.3, the sum

\[
R^* \text{Hom}_{G(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(m,0)}), \pi \right) + R^* \text{Hom}_{G(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(m-1,1)}), \pi \right)
\]

is equal to the sum

\[
R^* \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(m-1,0)}), R \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(1,0)}), \pi \right) \otimes \delta_{b_1} \right)
\]

\[
+ R^* \text{Hom}_{G_{b_1}'(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(m-1,0)}), R \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(1,0)}), \pi \right) \otimes \delta_{b_1} \right)
\]

We have

\[
R^* \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(1,0)}), \pi \right) = 0
\]

by [Dat07, Théorème A].

By Lemma 5.1 and the geometric lemma (cf. [Ren10, VI.5.1 Théorème]), we have

\[
R^* \text{Hom}_{G_{b_1}(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(m-1,0)}), R \text{Hom}_{G_b(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(1,0)}), \pi \right) \otimes \delta_{b_1} \right)
\]

\[
= -R^* \text{Hom}_{G_{b_1}(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(m-1,0)}), R \text{Hom}_{T(F)} \left( R \Gamma_c(\text{Sh}_{T,b_1}^{(1,0)}), \pi_{N_{b_1}} \right) \left(\frac{1}{2}\right) \otimes \delta_{b_1} \right)
\]

\[
= -R^* \text{Hom}_{G_{b_1}(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(m-1,0)}), R \text{Hom}_{T(F)} \left( R \Gamma_c(\text{Sh}_{T,b_1}^{(1,0)}), (\rho \otimes \delta_B) + \rho_w \right) \left(\frac{1}{2}\right) \otimes \delta_{b_1} \right)
\]

\[
= -R^* \text{Hom}_{G_{b_1}(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(m-1,0)}), \rho \right) \otimes \varphi_{\chi_1} \left(-\frac{1}{2}\right)
\]

\[
- R^* \text{Hom}_{G_{b_1}(F)} \left( R\Gamma_c(\text{Sh}_{b_1,1}^{(m-1,0)}), \rho_w \otimes \delta_{b_1} \right) \otimes \varphi_{\chi_2} \left(\frac{1}{2}\right).
\]

Hence

\[
H_c^*(\text{Sh}_{b_1,1}^{(m,0)})[\pi] = -H_c^*(\text{Sh}_{b_1,1}^{(m-1,1)})[\pi] - H_c^*(\text{Sh}_{b_1,1}^{(m-1,0)})[\rho] \otimes \varphi_{\chi_1} \left(-\frac{1}{2}\right)
\]

\[
- H_c^*(\text{Sh}_{b_1,1}^{(m-1,0)})[\rho_w \otimes \delta_{b_1}] \otimes \varphi_{\chi_2} \left(\frac{1}{2}\right).
\]
Therefore we obtain the claim. 

By Proposition 8.2, we can calculate $H^*_c(Sht_{b,1}^{(m,0)})[\rho]$ and $H^*_c(Sht_{b,1}^{(m,0)})[\text{Ind}^G_B(F) \rho]$ in Proposition 8.2 inductively. We do not pursue the explicit formula here, but record the following corollary.

**Corollary 8.3.** The $GL_2(F)$-representations $H^*_c(Sht_{b,1}^{(m,0)})[\rho]$ and $H^*_c(Sht_{b,1}^{(m,0)})[\text{Ind}^G_B(F) \rho]$ in Proposition 8.2 are linear combinations of proper parabolic inductions.

**Proof.** This follows from Proposition 8.2 by induction.

**Proposition 8.4.** We put

$$b_1 = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix}$$

and $b_m = b_1^m$ for $m \in \mathbb{Z}$. For an odd integer $m$, we put

$$b'_m = \begin{pmatrix} \sigma^{m-1} & 0 \\ 0 & \sigma^{-m+1} \end{pmatrix}.$$ 

Assume that $m \geq 2$. If $m$ is odd or $\varphi$ is cuspidal, we have

$$H^*_c(Sht_{b_m,1}^{(m,0)})[\pi_{b_m}] = H^*_c(Sht_{b_m,1}^{(m-1,0)})[\pi_{b_m-1}] \otimes \varphi - H^*_c(Sht_{b_m,1}^{(m-2,0)})[\pi_{b_m-2}] \otimes (r_{(1,1)} \circ \varphi).$$

If $m$ is even and $\varphi$ is not cuspidal, we have

$$H^*_c(Sht_{b_m,1}^{(m,0)})[\pi_{b_m}] = H^*_c(Sht_{b_m,1}^{(m-1,0)})[\pi_{b_m-1}] \otimes \varphi - H^*_c(Sht_{b_m,1}^{(m-2,0)})[\pi_{b_m-2}] \otimes (r_{(1,1)} \circ \varphi) - H^*_c(Sht_{b_m,1}^{(m-1,0)})[\chi \otimes \chi] \otimes \varphi \left( -\frac{1}{2} \right)$$

where $\chi$ is a character of $F^*$ such that $\pi_{b_m} \simeq St_\chi$.

**Proof.** Assume that $m$ is odd. By Proposition 5.1 and Lemma 5.3, the sum

$$R^* \text{Hom}_{GL_2(F)} \left( R\Gamma_c(Sht_{b_m,1}^{(m,0)}), \pi_{b_m} \right) + R^* \text{Hom}_{GL_2(F)} \left( R\Gamma_c(Sht_{b_m,1}^{(m-1,1)}), \pi_{b_m} \right)$$

is equal to

$$R^* \text{Hom}_{G_{b-1}(F)} \left( R\Gamma_c(Sht_{b_m-1,1}^{(m-1,0)}), R\text{Hom}_{GL_2(F)} \left( R\Gamma_c(Sht_{b_m,1}^{(1,0)}), \pi_{b_m} \right) \right).$$

Hence the claim follows from Corollary 7.46.

Assume that $m$ is even. By Proposition 5.1 and Lemma 5.3, the sum

$$R^* \text{Hom}_{GL_2(F)} \left( R\Gamma_c(Sht_{b_m,1}^{(m,0)}), \pi_{b_m} \right) + R^* \text{Hom}_{GL_2(F)} \left( R\Gamma_c(Sht_{b_m,1}^{(m-1,1)}), \pi_{b_m} \right)$$

is equal to the sum

$$R^* \text{Hom}_{G_{b-1}(F)} \left( R\Gamma_c(Sht_{b_m-1,1}^{(m-1,0)}), R\text{Hom}_{GL_2(F)} \left( R\Gamma_c(Sht_{b_m,1}^{(1,0)}), \pi_{b_m} \right) \right) + R^* \text{Hom}_{G_{b-1}(F)} \left( R\Gamma_c(Sht_{b_m,1}^{(m-1,0)}), R\text{Hom}_{GL_2(F)} \left( R\Gamma_c(Sht_{b_m,1}^{(1,0)}), \pi_{b_m} \right) \otimes \delta_B^1 \right).$$

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Hence, by Corollary 7.6, it suffices to show that
\[
R \text{Hom}_{GL_2(F)} \left( R \Gamma_c (\text{Sh}_{b_m, b_{m-1}}^{(1,0)}), \pi_{b_m} \right) = \begin{cases} 
0 & \text{if } \varphi \text{ is cuspidal,} \\
-((\chi \boxtimes \chi) \otimes \delta_B) \otimes \varphi \chi ( -\frac{1}{2}) & \text{if } \varphi \text{ is not cuspidal.}
\end{cases}
\]

By Lemma 8.1, we have
\[
R^\bullet \text{Hom}_{GL_2(F)} \left( R \Gamma_c (\text{Sh}_{b_m, b_{m-1}}^{(1,0)}), \pi_{b_m} \right) = -R^\bullet \text{Hom}_{T(F)} \left( R^\bullet \Gamma_c (\text{Sh}_{T, b_m, b_{m-1}}^{(1,0)}), (\pi_{b_m})_{N^\varphi} \right) \left( \frac{1}{2} \right).
\]

Hence the claim follows from \((\text{St}_{\chi})_{N^\varphi} \simeq (\chi \boxtimes \chi) \otimes \delta_B\).

**Proposition 8.5.** We put
\[
b_1 = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}
\]
and \(b_m = b_1^m\) for \(m \in \mathbb{Z}\). For \(m \geq 1\), we have
\[
R^\bullet \text{Hom}_{G_{b_m}(F)} \left( R \Gamma_c (\text{Sh}_{b_m, b_{m-1}}^{(m+1,0)}), \pi_{b_m} \right)
= R^\bullet \text{Hom}_{GL_2(F)} \left( R \Gamma_c (\text{Sh}_{b_1, b_{-1}}^{(1,0)}), R \text{Hom}_{G_{b_m}(F)} \left( R \Gamma_c (\text{Sh}_{b_{m-1}, b_{-1}}^{(m,0)}), \pi_{b_m} \right) \right) - R^\bullet \text{Hom}_{G_{b_{m-2}}(F)} \left( R \Gamma_c (\text{Sh}_{b_{m-2}, b_{-1}}^{(m-1,0)}), \pi_{b_{m-2}} \right) \otimes (r_{(1,1)} \circ \varphi).
\]

**Proof.** This follows from Proposition 5.1 and Lemma 5.3.

**Theorem 8.6.** Assume that \(n = 2\). Then Conjecture 7.2 is true if \(\kappa(b)\) is odd or \(\varphi\) is cuspidal.

**Proof.** We put
\[
b_1 = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}
\]
and to show the claim, we may assume that \(\mu = (m, 0)\) for some \(m \geq 0\) and \(b = 1\) or \(b_1\) by twisting.

Assume that \(\varphi\) is cuspidal. If \(b = 1\), we can show the claim by induction using Proposition 8.4. If \(b = b_1\), we can show the claim by induction using Proposition 8.5 and the case for \(b = 1\).

It remains to treat the case where \(\varphi\) is not cuspidal and \(b = b_1\). First, we can show that
\[
H^\bullet_c (\text{Sh}_{b_1, b_{-1}}^{(m,0)}), [\pi_{b_1}] - \pi_1 \boxtimes (r_{(m,0)} \circ \varphi)
\]
is a linear combination of proper parabolic inductions as representations of \(GL_2(F)\) using Corollary 8.3 and Proposition 8.4. Hence, the claim follows from Proposition 8.5 and [Dat07, Théorème A].

On the other hand, the following example shows that Conjecture 7.2 is not true if \(\mu\) is not minuscule and \(\varphi\) is not cuspidal.
Example 8.7. Let \( \mu = (2, 0) \) and \( b \) be a basic element such that \( \kappa(b) = 2 \). Assume that \( \varphi \) is not cuspidal and take a character \( \chi \) of \( F^\times \) such that \( \pi_1 \simeq \St_\chi \). Then we have

\[
R^i \Hom_{G_b(F)}(R\Gamma_c(\Sh_{b,1}^\mu), \pi_b) = \pi_1 \boxtimes (r_\mu \circ \varphi) + \left( \Ind_{B(F)}^{\GL_2(F)}(\chi \boxtimes \chi) \right) \boxtimes (r_{(1,1)} \circ \varphi)
\]

by Proposition 8.2 and Proposition 8.4.

Remark 8.8. Example 8.7 is compatible with the main theorem of [HKW22], since the representation \( \Ind_{B(F)}^{\GL_2(F)}(\chi \boxtimes \chi) \) has trace 0 on regular elliptic elements.

Example 8.10. Let \( \chi_1, \chi_2 : F^\times \to \overline{\mathbb{Q}}_\ell^\times \) be characters. Let \( \varphi_{\chi_1} : W_F \to \overline{\mathbb{Q}}_\ell^\times \) be the character corresponding to \( \chi_1 \). We put \( b = \left( \begin{array}{cc} \varpi & 0 \\ 0 & \varpi^2 \end{array} \right) \) and \( \mu = (3, 0) \). We put \( \rho = \chi_1 \boxtimes \chi_2 \) as representations of \( T(F) \). Then we have

\[
H^\bullet(\Sh_{G,b,1}^\mu)[\rho] = - (\Ind_{B(F)}^{G(F)}(\rho) \boxtimes \varphi_{x_1} \otimes \varphi_{x_2}^2 \left( \frac{1}{2} \right)).
\]

Proof. We put

\[
b_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & \varpi^2 \end{array} \right), \quad b_1' = \left( \begin{array}{cc} \varpi & 0 \\ 0 & \varpi \end{array} \right), \quad b_2 = \left( \begin{array}{cc} 1 & 0 \\ 0 & \varpi \end{array} \right).
\]

By Proposition 8.2, we have

\[
H^\bullet_c(\Sh_{b_1}^{(3,0)})[\rho] = - H^\bullet_c(\Sh_{b_1}^{(2,0)})[\rho] \otimes \varphi_{x_1} \left( \frac{1}{2} \right) - H^\bullet_c(\Sh_{b_2}^{(1,0)})[\rho] \otimes \varphi_{x_1} \otimes \varphi_{x_2} - H^\bullet_c(\Sh_{b_2}^{(2,0)})[\Ind_{B(F)}^{G(F)}(\rho)] \otimes \varphi_{x_2} \left( \frac{1}{2} \right)
\]

\[
- H^\bullet_c(\Sh_{b_1}^{(2,0)})(\Ind_{B(F)}^{G(F)}(\rho)) \boxtimes \varphi_{x_2} \left( \frac{1}{2} \right) = - \Ind_{B(F)}^{G(F)}(\rho) \boxtimes \varphi_{x_1} \otimes \varphi_{x_2}^2 \left( \frac{1}{2} \right) + (\Ind_{B(F)}^{G(F)}(\rho)) \boxtimes \varphi_{x_1} \otimes \varphi_{x_2} \left( \frac{1}{2} \right) + H^\bullet_c(\Sh_{b_2}^{(1,0)})([\rho] \otimes \varphi_{x_1} \otimes \varphi_{x_2}) + H^\bullet_c(\Sh_{b_2}^{(1,0)})(\rho \otimes \delta_B^{-1}) \otimes \varphi_{x_2}^2 \left( \frac{1}{2} \right) + H^\bullet_c(\Sh_{b_1}^{(1,0)}(\rho \otimes \delta_B^{-1})) \otimes \varphi_{x_1} \otimes \varphi_{x_2} \left( \frac{1}{2} \right)
\]

\[
- \Ind_{B(F)}^{G(F)}(\rho \otimes \delta_B^{-1}) \otimes \varphi_{x_1} \otimes \varphi_{x_2} \left( \frac{1}{2} \right).
\]

Therefore we obtain the claim.

Remark 8.11. We use notation in Example 8.10. We define \( I_{b,\mu,T} \) in the same way as [RV14, (31)]. Then we have \( I_{b,\mu,T} = \emptyset \). Therefore Example 8.10 shows that the non-minuscule generalization of [RV14, Conjecture 8.5] does not hold as it is. We note that \(([b], \mu)\) is not Hodge–Newton reducible (cf. [RV14, Definition 4.28]).

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