SUPERCYCLICITY OF THE LEFT AND RIGHT MULTIPLICATION OPERATORS ON BANACH IDEAL OF OPERATORS

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Abstract. Let $X$ be a Banach space with $\dim X > 1$ such that $X^*$, its dual, is separable and $\mathcal{B}(X)$ the algebra of bounded linear operators on $X$. In this paper, we study the passage of property of being supercyclic from an operator $T \in \mathcal{B}(X)$ to the left and right multiplication induced by $T$ on separable admissible Banach ideal of $\mathcal{B}(X)$. We give a sufficient condition for the tensor product $T \otimes R$ of two operators to be supercyclic. As a consequence, we give another equivalent conditions for the Supercyclicity Criterion.

1. Introduction and Preliminary

Throughout the paper, we denote by $X$ a Banach space with $\dim X > 1$ such that $X^*$, its dual, is separable, $\mathcal{B}(X)$ denote the algebra of bounded linear operators on $X$ and $\mathcal{K}(X)$ denote the algebra of compact operators on $X$. From [7, Proposition 2.8], if $X^*$ is separable, then $X$ is separable. Let $T \in \mathcal{B}(X)$, the orbit of a vector $x \in X$ under $T$ is the set

$$\text{Orb}(T, x) := \{T^n x : n \in \mathbb{N}\}.$$ 

An operator $T \in \mathcal{B}(X)$ is said to be hypercyclic if there is some vector $x \in X$ such that $\text{Orb}(T, x)$ is dense in $X$; such a vector $x$ is called a hypercyclic vector for $T$. Similarly, $T \in \mathcal{B}(X)$ is said to be supercyclic if there is some vector $x \in X$ such that

$$\mathcal{C}\text{Orb}(T, x) := \{\alpha T^n x : \alpha \in \mathbb{C}, n \in \mathbb{N}\}$$

is dense in $X$; such a vector $x$ is called supercyclic vector for $T$. From [15], an operator $T$ on $X$ is supercyclic if and only if for each pair $(U, V)$ of nonempty open subsets of $X$ there exist $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$ such that

$$\alpha T^n(U) \cap V \neq \emptyset.$$ 

There is a characterization of hypercyclicity called hypercyclicity criterion. It provides several sufficient conditions that ensure hypercyclicity. This criterion has been established by Carol Kitai in 1982 [17]. It is improved by several authors afterwards as Gethner and Shapiro [9] in 1987 and Juan Bes [3] in 1998. Recall that $T \in \mathcal{B}(X)$ satisfy the hypercyclicity criterion, if there exist two dense subsets $Y$ and $Z$ in $X$, a strictly increasing sequence $(n_k)_{k \geq 0}$ of positive integers and mappings $S_{n_k} : Z \to X$ such that:

(i) $T^{n_k} x \to 0$ for any $x \in Y$;
(ii) $(S_{n_k} y)_k$ converges to $0$ for any $y \in Z$;
(iii) $(T^{n_k} S_{n_k} y)_k$ converges to $y$ for any $y \in Z$.

In the same way, Salas in 1999 gave a characterization of supercyclic bilateral backward weighted shifts via the Supercyclicity Criterion, that is a sufficient condition for supercyclicity [23]. We say $T \in \mathcal{B}(X)$ satisfy the supercyclicity criterion, if there exist two dense subsets $D_1$ and $D_2$ in $X$, a sequence $(n_k)_{k \geq 0}$ of positive integers, and also there exist a mappings $S_{n_k} : D_2 \to X$ such that:

(i) $\|T^{n_k} x\| \cdot \|S_{n_k} y\| \to 0$ for every $x \in D_1$ and $y \in D_2$;
(ii) $T^{n_k} S_{n_k} y \to y$ for every $y \in D_2$.

2000 Mathematics Subject Classification. Primary 47A80, 47A53; secondary 47A10, 47A11.

Key words and phrases. Hypercyclicity, Supercyclicity, left multiplication, right multiplication, tensor product, Banach ideal of operators.
For a more general overview of hypercyclicity, supercyclicity and related properties in linear dynamics, we refer to [11, 12, 13, 15, 19, 24].

The left multiplication operator $L_T : B(X) \rightarrow B(X)$ induced by a fixed bounded linear operator $T \in B(X)$ is defined as $L_T(A) = TA$ and the right multiplication operator $R_T : B(X) \rightarrow B(X)$ induced by a fixed bounded linear operator $T \in B(X)$ is defined as $R_T(A) = AT$ where $A \in B(X)$. Recall [12] that $(J, \| \cdot \|_J)$ is said to be a Banach ideal of $B(X)$ if:

(i) $J \subset B(X)$ is a linear subspace;

(ii) The norm $\| \cdot \|_J$ is complete in $J$ and $\| S \| \leq \| S \|_J$ for all $S \in J$;

(iii) $\forall S \in J, \forall A, B \in B(X), \forall SAB \in J$ and $\| SAB \|_J \leq \| A \| \| S \|_J \| B \| J$;

(iv) The rank one operators $x \otimes x^* \in J$ and $\| x \otimes x^* \|_J = \| x \| \| x^* \|$ for all $x \in X$ and $x^* \in X^*$.

A rank one operator defined on $X$ as $(x \otimes x^*)(z) = (x^*, z)x = x^*(z)x$ for all $x \in X$, $x^* \in X^*$ and any $z \in X$. The space of finite rank operators $F(X)$ is defined as the linear span of the rank one operators, that is $F(X) = \{ \sum_{i=1}^{m} x_i \otimes x_i^*, \; x_i \in X, \; x_i^* \in X^*, \; m \geq 1 \}$. A Banach ideal $J$ of operators is said to be admissible if $F(X)$ is dense in $J$ with respect to the norm $\| \cdot \|_J$.

Note that $L_T : J \rightarrow J$ and $R_T : J \rightarrow J$ are well-defined and we have for all $A \in J$:

$$\| L_T(A) \|_J = \| TA \|_J \leq \| T \| \| A \|_J \; ; \; \| R_T(A) \|_J = \| AT \|_J \leq \| T \| \| A \|_J.$$ 

The study of hypercyclicity in operator algebras was initiated by Kit Chan in 1999 [5], who showed that hypercyclicity can occur on the operator algebra $B(H)$ with strong operator topology, when $H$ is a separable Hilbert space. Subsequently his idea was used by several authors, see for examples [4, 6, 11, 20, 24]. Recently, Gilmore et al in [10, 11], investigate and study the hypercyclicity properties of the commutator maps $L_T - R_T$ and the generalised Derivations $L_A - R_B$ built from the basic multiplications acting on separable Banach ideals of operators.

The motivation of this study is that Bonet et al. [4] use tensor product techniques developed in [18] to characterized the hypercyclicity of the left and the right multiplication operators on an admissible Banach ideal of operators.

Let $E$ and $F$ be normed linear spaces. Recall [22] that the projective tensor norm on $E \otimes F$ is the function $\Pi : E \otimes F \rightarrow [0, +\infty]$ defined for all $z \in E \otimes F$

$$\Pi(z) := \inf \{ \sum_{j=1}^{n} \| x_j \| \| y_j \| : z = \sum_{j=1}^{n} x_j \otimes y_j \}.$$ 

For elementary tensors $z = x \otimes y$ we just have $\Pi(z) = \| x \| \| y \|$, with this topology the space is denoted by $E \otimes_\pi F$, and its completion by $E \tilde{\otimes}_\pi F$. For a more general overview of the projective tensor norm and its related properties we refer to [10, 22].

The purpose of this paper is to characterize the supercyclicity of the left and the right multiplication on a separable admissible Banach ideal of operators and give a sufficient condition for the tensor product $T \tilde{\otimes} R$ of two operators to be supercyclic and some equivalent conditions for the Supercyclicity Criterion.

In section 2, we study the passage of property of being supercyclic from $T \in B(X)$ to the left and the right multiplication induced by $T$ on a separable admissible Banach ideal $J$ of $B(X)$. So, we prove that:

(i) $T$ satisfies the supercyclicity criterion on $X$ if and only if $L_T$ is supercyclic on $(J, \| \cdot \|_J)$.

(ii) $T^*$ satisfies the supercyclicity criterion on $X^*$ if and only if $R_T$ is supercyclic on $(J, \| \cdot \|_J)$.

In section 3, we give a sufficient condition for the tensor product $T \tilde{\otimes} R$ of two operators to be supercyclic. As a consequence, we give some equivalent conditions for the Supercyclicity Criterion.

2. SUPERCYCLICITY OF THE LEFT AND RIGHT MULTIPLICATION ON BANACH IDEAL OF OPERATORS.

We begin this section with the following lemma which will be used in sequel.
Lemma 2.1. Let $X$ be Banach space, $X^*$, its dual, is separable, $(J, \| \cdot \|_J)$ an admissible Banach ideal of $\mathcal{B}(X)$. If $D$ and $\Phi$ are a countable dense subsets of $X$ and $X^*$, respectively. Then the set
\[
\mathcal{X} := \text{span}\{ x \otimes \phi / x \in D, \phi \in \Phi \}
\]
is a countable dense subset of $J$ with respect to $\| \cdot \|_J$-topology.

Proof. Let $T \in J$. If $\epsilon > 0$ is arbitrary, then there is a finite rank operator $F$ such that $\| T - F \|_J < \frac{\epsilon}{2}$. Let $F = \sum_{i=1}^{N} \alpha_i a_i \otimes \varphi_i$, where $a_i \in X$, $\varphi_i \in X^*$ and $\alpha_i \in \mathbb{C}$ for $i = 1, 2, \ldots, N$. For every $i \in \{1, 2, \ldots, N\}$, there exist some $\phi_i \in \Phi$ such that $\| \varphi_i - \phi_i \| < \frac{\epsilon}{4N|\alpha_i||\phi_i|}$ and there exist $x_i \in D$ such that $\| a_i - x_i \| < \frac{\epsilon}{4N|\alpha_i||\phi_i|}$. Therefore
\[
\| F - \sum_{i=1}^{N} \alpha_i x_i \otimes \phi_i \|_J = \| \sum_{i=1}^{N} \alpha_i a_i \otimes \varphi_i - \sum_{i=1}^{N} \alpha_i x_i \otimes \phi_i \|_J
\]
\[
= \| \sum_{i=1}^{N} \alpha_i (a_i \otimes \varphi_i - x_i \otimes \phi_i) \|_J
\]
\[
\leq \sum_{i=1}^{N} |\alpha_i| \| a_i \otimes \varphi_i - x_i \otimes \phi_i \|_J
\]
\[
= \sum_{i=1}^{N} |\alpha_i| \| a_i \otimes (\varphi_i - \phi_i) + (a_i - x_i) \otimes \phi_i \|_J
\]
\[
\leq \sum_{i=1}^{N} |\alpha_i| (\| a_i \otimes \varphi_i - \phi_i \| + \| (a_i - x_i) \otimes \phi_i \|_J)
\]
\[
= \sum_{i=1}^{N} |\alpha_i| (\| a_i \otimes \phi_i \| + \| a_i - x_i \| \| \phi_i \|)
\]
\[
< \frac{\epsilon}{4} + \frac{\epsilon}{4}
\]
\[
= \frac{\epsilon}{2}
\]

Hence
\[
\| T - \sum_{i=1}^{N} \alpha_i x_i \otimes \phi_i \|_J = \| T - F + F - \sum_{i=1}^{N} \alpha_i x_i \otimes \phi_i \|_J
\]
\[
\leq \| T - F \|_J + \| F - \sum_{i=1}^{N} \alpha_i x_i \otimes \phi_i \|_J
\]
\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2}
\]
\[
= \frac{\epsilon}{2}
\]

Thus $\mathcal{X}$ is a countable dense subset of $J$ with respect to $\| \cdot \|_J$-topology. \hfill \square

In the setting of Banach ideals, J. Bonet et al \cite{4} characterised the hypercyclicity of the left and the right multipliers using tensor techniques developed in \cite{18}. For a separable admissible Banach ideal $J$ of $\mathcal{B}(X)$, they showed that
(i) $T$ satisfies the hypercyclicity criterion on $X$ if and only if $L_T$ is hypercyclic on $(J, \| \cdot \|_J)$.
(ii) $T^*$ satisfies the hypercyclicity criterion on $X^*$ if and only if $R_T$ is hypercyclic on $(J, \| \cdot \|_J)$. 

In the following, we prove that this results hold for supercyclically.

**Theorem 2.2.** Let $X$ be Banach space, with $\dim X > 1$ such that $X^*$, its dual, is separable and $T \in \mathcal{B}(X)$. Then $T$ satisfy the supercyclicity criterion on $X$ if and only $L_T$ is supercyclic on $(J, \|\cdot\|_J)$.

**Proof.** $\Rightarrow$) Assume that $T$ satisfy the supercyclicity criterion on $X$, then there exist a strictly increasing sequence $(n_k)_{k}$ of positive integers, dense subsets $D_1, D_2$ of $X$ and maps $S_{n_k} : D_2 \rightarrow X$ such that for all $x \in D_1$ and $y \in D_2$

a) $\|T^{n_k}x\|_{S_{n_k}y} \rightarrow 0$, as $k \rightarrow +\infty$;

b) $T^{n_k}S_{n_k}y \rightarrow y$, as $k \rightarrow +\infty$.

Let $\Phi$ be dense subset of $X^*$ and consider the sets $X_0 = \text{span}\{x \otimes \varphi/x \in D_1, \varphi \in \Phi\}$ and $Y_0 = \text{span}\{y \otimes \phi/y \in D_2, \phi \in \Phi\}$ and the maps $Q_{n_k} : Y_0 \rightarrow J$ define by

$$Q_{n_k}(\sum_{j=1}^{N_k} \beta_j y_j \otimes \phi_j) = \sum_{j=1}^{N_k} \beta_j (S_{n_k} y_j \otimes \phi_j).$$

By Lemma 2.1, $X_0$ and $Y_0$ are subsets of $J$ which are $\|\cdot\|_J$-dense in $J$. Let $A = \sum_{i=1}^{N_1} \alpha_i x_i \otimes \varphi_i \in X_0$ and $B = \sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j \in Y_0$, then

$$\|(L_T)^{n_k} A\|_J \|Q_{n_k} B\|_J = \|(L_T)^{n_k} (\sum_{i=1}^{N_1} \alpha_i x_i \otimes \varphi_i)\|_J \|Q_{n_k} (\sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j)\|_J$$

$$= \sum_{i=1}^{N_1} |\alpha_i| \|(L_T)^{n_k} x_i \otimes \varphi_i\|_J \sum_{j=1}^{N_2} |\beta_j| \|(S_{n_k} y_j \otimes \phi_j)\|_J$$

$$\leq \sum_{i=1}^{N_1} |\alpha_i| \|(L_T)^{n_k} x_i \otimes \varphi_i\|_J \sum_{j=1}^{N_2} |\beta_j| \|(S_{n_k} y_j \otimes \phi_j)\|_J$$

Using the assumption a). We show that $\|(L_T)^{n_k} A\|_J \|Q_{n_k} B\|_J \rightarrow 0$, as $k \rightarrow +\infty$. In the other hand we have:

$$L_T^{n_k} Q_{n_k}(B) = L_T^{n_k} Q_{n_k}(\sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j)$$

$$= \sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j$$

by using the assumption b). Hence $L_T$ satisfies the supercyclicity criterion on $(J, \|\cdot\|_J)$. Thus $L_T$ is supercyclic on $(J, \|\cdot\|_J)$. 


$\Leftarrow$ Suppose that $LT$ is supercyclic on $(J, \| \cdot \|_J)$. Assume that $x_1, x_2 \in X$ are linearly independent and define

$$\varphi : J \rightarrow X \mathbin{\bigoplus} X,$$

$$R \mapsto Rx_1 \mathbin{\oplus} Rx_2.$$  

Then $\varphi$ is surjective. Indeed, let $y_1, y_2 \in X$. By using the Hahn-Banach theorem, there exist $x_1^*, x_2^* \in X^*$ such that $x_1^*(x_1) = x_2^*(x_2) = 1$ and $x_1^*(x_2) = x_2^*(x_1) = 0$. Let $R = y_1 \mathbin{\otimes} x_1^* + y_2 \mathbin{\otimes} x_2^* \in J$ thus $\varphi(R) = Rx_1 \mathbin{\oplus} Rx_2 = y_1 \mathbin{\oplus} y_2$. For $A \in J$, we have

$$\begin{align*}
(\varphi \circ LT)A &= \varphi(TA) \\
&= (TA)x_1 \mathbin{\oplus} (TA)x_2 \\
&= T(Ax_1) \mathbin{\oplus} T(Ax_2) \\
&= (T \mathbin{\oplus} T)(Ax_1 \mathbin{\oplus} Ax_2) \\
&= (T \mathbin{\oplus} T) \circ \varphi(A) \\
&= (T \mathbin{\oplus} T) \circ \varphi(A).
\end{align*}$$  

Therefore, $\varphi \circ LT = (T \mathbin{\oplus} T) \circ \varphi$. Thus $T \mathbin{\oplus} T$ is supercyclic on $X \mathbin{\bigoplus} X$. Hence, by \cite{2} Lemma 3.1] $T$ satisfies the supercyclicity criterion. □

We have the following corollary.

**Corollary 2.3.** Let $X$ be Banach space, with $\dim X > 1$ such that $X^*$, its dual, is separable and $T \in B(X)$. Then the following are equivalent:

(i) $T$ satisfies the supercyclicity criterion on $X$.

(ii) $LT$ is supercyclic on $(K(X), \| \cdot \|)$ endowed with the norm operator topology.

(iii) $LT$ is supercyclic on $B(X)$ in the strong operator topology.

*Proof.* (i) $\Leftrightarrow$ (ii) Consequence of Theorem \cite{2}. Since $K(X)$ is an admissible Banach ideal of $B(X)$.

(i) $\Rightarrow$ (iii) Suppose that $T$ satisfy the supercyclicity criterion on $X$. Let $U$ and $V$ be two non-empty open subsets of $B(X)$ in the strong operator topology. Since $K(X)$ is dense in $B(X)$ with the strong operator topology \cite{3} Corollary 3], there exist $A_1, A_2 \in K(X)$ such that $A_1 \in U$ and $A_2 \in V$. Thus we can find $x_1, x_2 \in X \setminus \{0\}$ and $\epsilon_1, \epsilon_2 > 0$ such that

$$\{ A \in B(X) : \| (A - A_1)x_1 \| < \epsilon_1 \} \subset U$$

and

$$\{ A \in B(X) : \| (A - A_2)x_2 \| < \epsilon_2 \} \subset U.$$  

Let

$$U_i = \{ A \in K(X) : \| A - A_i \| < \frac{\epsilon_i}{\| x_i \|} \}.$$  

$U_1$ is a non-empty open subset of $K(X)$ with the norm operator topology. By Theorem \cite{2} with $J = K(X)$, $LT$ is supercyclic on $K(X)$, so there is some $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$ such that

$$\alpha(L_T)^n U_1 \cap U_2 \neq \emptyset.$$  

Hence, it follows that $\alpha(L_T)^n U \cap V \neq \emptyset$. Thus, $LT$ is supercyclic on $B(X)$ in the strong operator topology.

(i) $\Rightarrow$ (iii) By the same technique as in the proof of Theorem \cite{2}. □

**Theorem 2.4.** Let $X$ be Banach space, with $\dim X > 1$ such that $X^*$, its dual, is separable and $T \in B(X)$. Then $T^*$ satisfies the supercyclicity criterion on $X^*$ if and only if $RT$ is supercyclic on $(J, \| \cdot \|_J)$.

*Proof.* $\Rightarrow$ Assume that $T^*$ satisfy the supercyclicity criterion on $X^*$, then there exist a strictly increasing sequence $(n_k)_k$ of positive integers, dense subsets $\Phi_1, \Phi_2$ of $X^*$ and maps $M_{n_k} : \Phi_2 \rightarrow X^*$ such that for all $\varphi \in \Phi_1$ and $\phi \in \Phi_2$

a) $\| (T^*)^{n_k} \varphi \| M_{n_k} \phi \rightarrow 0$, as $k \rightarrow +\infty$;

b) $(T^*)^{n_k} M_{n_k} \phi \rightarrow \phi$, as $k \rightarrow +\infty$.  


Let $D$ be dense subset of $X$ and consider the sets $\Phi_0 = \text{span}\{x \otimes \varphi/x \in D, \varphi \in \Phi_1\}$ and $\Psi_0 = \text{span}\{y \otimes \phi/x \in D, \phi \in \Phi_2\}$ and the maps $N_{n_k} : \Psi_0 \rightarrow J$ define by

$$N_{n_k}(\sum_{j=1}^{N} \beta_j y_j \otimes \phi_j) = \sum_{j=1}^{N} \beta_j y_j \otimes M_{n_k} \phi_j.$$ 

By Lemma 2.1, $\Phi_0$ and $\Psi_0$ are subsets of $J$ which are $\|\cdot\|_J$-dense in $J$. Let $A = \sum_{i=1}^{N_1} \alpha_i x_i \otimes \varphi_i \in \Phi_0$ and $B = \sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j \in \Psi_0$, then

$$\|(R_T)^{n_k} A\|_J \|N_{n_k} B\|_J = \|(R_T)^{n_k} (\sum_{i=1}^{N_1} \alpha_i x_i \otimes \varphi_i)\|_J \|N_{n_k} (\sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j)\|_J$$

$$\leq \left(\sum_{i=1}^{N_1} \|\alpha_i\| \|x_i\| \|(T^*)^{n_k} \varphi_i\|\right) \left(\sum_{j=1}^{N_2} \|\beta_j\| \|y_j\| \|M_{n_k} \phi_j\|\right)$$

$$\leq \sum_{i \leq N_1, j \leq N_2} \|\alpha_i\| \|\beta_j\| \|(T^*)^{n_k} \varphi_i\| \|M_{n_k} \phi_j\| \|x_i\| \|y_j\|.$$ 

Using the assumption a). We show that $\|(R_T)^{n_k} A\|_J \|N_{n_k} B\|_J \rightarrow 0$, as $k \rightarrow +\infty$. In the other hand we have

$$(R_T)^{n_k} N_{n_k}(B) = (R_T)^{n_k} N_{n_k}(\sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j)$$

$$= (R_T)^{n_k} (\sum_{j=1}^{N_2} \beta_j y_j \otimes M_{n_k} \phi_j)$$

$$= \sum_{j=1}^{N_2} \beta_j y_j \otimes (T^*)^{n_k} M_{n_k} \phi_j$$

$$\rightarrow \sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j = B \text{ as } k \rightarrow +\infty,$$

by using the assumption b). Hence $R_T$ satisfies the supercyclicity criterion on $(J, \|\cdot\|_J)$. Thus $R_T$ is supercyclic on $(J, \|\cdot\|_J)$.

$\Leftarrow$ Suppose that $R_T$ is supercyclic on $(J, \|\cdot\|_J)$. Let $x_1^*, x_2^* \in X^*$ are linearly independent and define

$$\phi : J \rightarrow X^* \oplus X^*$$

$$R \mapsto R^* x_1^* \oplus R^* x_2^*.$$ 

Then $\phi$ is surjective, indeed, let $y_1^*, y_2^* \in X^*$, we take $x_1^*, x_2^* \in X^*$ such that $x_i^*(x_j) = \delta_{i,j}$ and put $R = x_1 \otimes y_1^* + x_2 \otimes y_2^*$, then $\phi(R) = (R^* x_1^*, R^* x_2^*) = (x_1^* \circ R, x_2^* \circ R) = (y_1^*, y_2^*)$. For $A \in J$,
we have

\[ (\phi \circ R_T)A = \phi(AT) = (AT)^*x_1^* \oplus (AT)^*x_2^* = T^*A^*x_1^* \oplus T^*A^*x_2^* = (T^* \oplus T^*)(A^*x_1^* \oplus A^*x_2^*) = ((T^* \oplus T^*) \circ \phi)A \]

Therefore, \( \phi \circ R_T = (T^* \oplus T^*) \circ \phi \). Thus \( T^* \oplus T^* \) is supercyclic on \( X^* \oplus X^* \). Hence, by [2, Lemma 3.1] \( T^* \) satisfies the supercyclicity criterion on \( X^* \). \(\)

We have the following corollary.

**Corollary 2.5.** Let \( X \) be Banach space, with \( \dim X > 1 \) such that \( X^* \), its dual, is separable and \( T \in \mathcal{B}(X) \). Then the following are equivalent:

(i) \( T^* \) satisfies the supercyclicity criterion on \( X^* \).

(ii) \( R_T \) is supercyclic on \( (\mathcal{K}(X), \| \|) \) endowed with the norm operator topology.

(iii) \( R_T \) is supercyclic on \( \mathcal{B}(X) \) in the strong operator topology.

**Proof.** (i) \(\Leftrightarrow\) (ii) Consequence of Theorem [2,4] Since \( \mathcal{K}(X) \) is an admissible Banach ideal of \( \mathcal{B}(X) \).

(i) \(\Rightarrow\) (iii) Suppose that \( T^* \) satisfies the supercyclicity criterion on \( X^* \). Let \( U \) and \( V \) be two non-empty open subsets of \( \mathcal{B}(X) \) in the strong operator topology. Since \( \mathcal{K}(X) \) is dense in \( \mathcal{B}(X) \) with the strong operator topology [2, Corollary 3], there exist \( A_1, A_2 \in \mathcal{K}(X) \) such that \( A_1 \in U \) and \( A_2 \in V \). Thus we can find \( x_1, x_2 \in X \setminus \{0\} \) and \( \epsilon_1, \epsilon_2 > 0 \) such that

\[ \{ A \in \mathcal{B}(X) : \| (A - A_1)x_1 \| < \epsilon_1 \} \subset U \]

and

\[ \{ A \in \mathcal{B}(X) : \| (A - A_2)x_2 \| < \epsilon_2 \} \subset U. \]

Let

\[ U_i = \{ A \in \mathcal{K}(X) : \| A - A_i \| < \frac{\epsilon_i}{\| x_i \|} \}. \]

\( U_i \) is a non-empty open subset of \( \mathcal{K}(X) \) with the norm operator topology. By Theorem [2,4] with \( J = \mathcal{K}(X) \), \( R_T \) is supercyclic on \( (\mathcal{K}(X), \| \|) \), so there is some \( \alpha \in \mathbb{C} \) and \( n \in \mathbb{N} \) such that

\[ \alpha (R_T)^n U_1 \cap U_2 \neq \emptyset. \]

Hence, it follows that \( \alpha (R_T)^n U \cap V \neq \emptyset \). Thus \( R_T \) is supercyclic on \( \mathcal{B}(X) \) in the strong operator topology.

(i) \(\Rightarrow\) (iii) By the same technique as in the proof of Theorem [2,4] \(\)

3. **Stability of supercyclicity tensor product.**

In [18] the authors gave a sufficient condition for the tensor product \( T \hat{\otimes} R \) of two operators to be hypercyclic. We inspired from this results, we give a sufficient condition to the tensor product \( T \hat{\otimes} R \) of two operators to be supercyclic.

**Definition 3.1.** Let \( X \) be Banach space. An operator \( T \in \mathcal{B}(X) \) is said to satisfy the Tensor Supercyclicity Criterion (TSC) if there exists dense subsets \( D_1, D_2 \subset X \), an increasing sequence \( (n_k)_k \) of positive integers, \( (\lambda_{n_k})_{k \in \mathbb{N}} \subset \mathbb{C} \setminus \{0\} \) and a sequence of mappings \( S_{n_k} : D_2 \rightarrow X \) such that:

(i) \( (\lambda_{n_k}T^{n_k}x)_{k \in \mathbb{N}} \) is bounded for all \( x \in D_1 \);

(ii) \( (\frac{\lambda_{n_k}}{x^{n_k}}S_{n_k}y)_{k \in \mathbb{N}} \) is bounded for all \( y \in D_2 \);

(iii) \( (T^{n_k}S_{n_k}y)_{k \in \mathbb{N}} \rightarrow y \) for all \( y \in D_2 \).

**Example 3.2.**

1. Clearly, a sequences of operators satisfying the Supercyclicity Criterion satisfy Tensor Supercyclicity Criterion.

2. The identity map on \( X \) satisfy the Tensor Supercyclicity Criterion.
(3) Any isometry on a Banach space satisfies the Tensor Supercontinuity Criterion with respect to the sequence of all positive integers.

**Theorem 3.3.** Let $E$ and $F$ be separable Banach spaces. If $T_1 \in \mathcal{B}(E)$ satisfies the Supercontinuity Criterion and $T_2 \in \mathcal{B}(F)$ satisfies the Tensor Supercontinuity Criterion, then
\[
T_1 \otimes \pi T_2 : E \otimes \pi F \rightarrow E \otimes \pi F
\]
satisfies the Supercontinuity Criterion. Accordingly, it is supercyclic.

**Proof.** Let $X_1, X_1 \subset E$, $Y_1, Y_2 \subset F$ be dense subspaces, $(\lambda^1_{nk})_{k \in \mathbb{N}}, (\lambda^2_{nk})_{k \in \mathbb{N}} \subset \mathbb{C}\{0\}$ and $S^1_{nk} : X^2 \rightarrow E$, $S^2_{nk} : Y^2 \rightarrow F$, $k \in \mathbb{N}$, linear maps satisfying the conditions of Supercontinuity Criterion and Tensor Supercontinuity Criterion for $T_1 \in \mathcal{B}(E)$ and $T_2 \in \mathcal{B}(F)$, respectively. We will see that $Z_1 := X_1 \otimes Y_1$, $Z_2 := X_2 \otimes Y_2$, $(\lambda_{nk} := \lambda^1_{nk} \lambda^2_{nk})_{k \in \mathbb{N}}$ and the maps $S_{nk} := S^1_{nk} \otimes S^2_{nk} : Z_2 \rightarrow E \otimes F$, are such that conditions of the Supercontinuity Criterion are satisfied for the operator $T := T_1 \otimes T_2 : E \otimes \pi F \rightarrow E \otimes \pi F$.

Indeed, if we compute on elementary tensors, we obtain for every $x_1 \in X_1$, $x_2 \in Y_1$, $y_1 \in X_2$ and $y_2 \in Y_2$:
\[
\lim_{k \to +\infty} \Pi(\lambda_{nk} T^{nk})(x_1 \otimes x_2) = \lim_{k \to +\infty} \Pi(\lambda^1_{nk} \lambda^2_{nk} (T_1^{nk} \otimes T_2^{nk})(x_1 \otimes x_2)) = \lim_{k \to +\infty} \Pi((\lambda^1_{nk} T_1^{nk} \otimes \lambda^2_{nk} T_2^{nk})(x_1 \otimes x_2)) = \lim_{k \to +\infty} \|\lambda^1_{nk} T_1^{nk} x_1\| \cdot \|\lambda^2_{nk} T_2^{nk} x_2\| = 0,
\]
since the first sequence tends to 0 and the second one is bounded. Analogously
\[
\lim_{k \to +\infty} \Pi\left(\frac{1}{\lambda_{nk}} S_{nk}(y_1 \otimes y_2)\right) = \lim_{k \to +\infty} \Pi\left(\frac{1}{\lambda^1_{nk} \lambda^2_{nk}} (S^1_{nk} \otimes S^2_{nk})(y_1 \otimes y_2)\right) = \lim_{k \to +\infty} \Pi\left(\frac{1}{\lambda^1_{nk}} S^1_{nk} \otimes \frac{1}{\lambda^2_{nk}} S^2_{nk}(y_1 \otimes y_2)\right) = \lim_{k \to +\infty} \left\|\frac{1}{\lambda^1_{nk}} S^1_{nk} y_1\right\| \cdot \left\|\frac{1}{\lambda^2_{nk}} S^2_{nk} y_2\right\| = 0,
\]
since the first sequence tends to 0 and the second one is bounded. Finally,
\[
\lim_{k \to +\infty} \Pi\left[(T^{nk} S_{nk})(y_1 \otimes y_2) - (y_1 \otimes y_2)\right] = \lim_{k \to +\infty} \Pi\left((T_1^{nk} \otimes T_2^{nk})(S^1_{nk} \otimes S^2_{nk})(y_1 \otimes y_2) - (y_1 \otimes T_2^{nk} S^2_{nk} y_2) + (y_1 \otimes T_2^{nk} S^2_{nk} y_2) - (y_1 \otimes y_2)\right) = \lim_{k \to +\infty} \Pi\left(\left((T_1^{nk} S^1_{nk} y_1 - y_1) \otimes T_2^{nk} S^2_{nk} y_2 + y_1 \otimes (T_2^{nk} S^2_{nk} y_2 - y_2)\right)\right) \leq \lim_{k \to +\infty} \{\Pi[\left(T_1^{nk} S^1_{nk} y_1 - y_1\right) \otimes T_2^{nk} S^2_{nk} y_2] + \Pi[y_1 \otimes (T_2^{nk} S^2_{nk} y_2 - y_2)]\} = \lim_{k \to +\infty} \left\{\left\|T_1^{nk} S^1_{nk} y_1 - y_1\right\| \left\|T_2^{nk} S^2_{nk} y_2\right\| + \left\|y_1\right\| \left\|T_2^{nk} S^2_{nk} y_2 - y_2\right\|\right\} = 0,
\]
which completes the proof by taking linear combinations of elementary tensors. $\square$

In the following Proposition, we show the connection between supercyclicity of tensor products and supercyclicity of direct sums, and yields another equivalent formulation in the context of tensor products of the supercyclicity criterion.

**Proposition 3.4.** Let $E, F$ be a separable Banach spaces with $\dim F \geq 2$ and $T \in \mathcal{B}(E)$. The following are equivalent:

(i) $T$ satisfies the supercyclicity criterion.

(ii) $T \circ \pi : E \otimes \pi F \rightarrow E \otimes \pi F$ is supercyclic for the projective tensor norm $\pi$. 
Proof. (i) ⇒ (ii) Is a consequence of Theorem 3.3 by taking $T_2 = I$.

(iii) ⇒ (i) See [2] Lemma 3.1.

(ii) ⇒ (iii) Since $\dim F \geq 2$, let $x_1^*, x_2^* \in F^*$ and consider the following commutative diagram:

\[
\begin{array}{ccc}
E \otimes F & \xrightarrow{T \otimes I} & E \otimes F \\
\downarrow \varphi & & \downarrow \varphi \\
E \oplus E & \xrightarrow{T \oplus I} & E \oplus E
\end{array}
\]

where $\varphi(\sum_{i \leq N} e_i \otimes x_i) := (\sum_{i \leq N} \langle x_i, x_1^* \rangle e_i, \sum_{i \leq N} \langle x_i, x_2^* \rangle e_i)$. $\varphi$ is surjective. Indeed, Let $e_1, e_2 \in E$, we take $x_1, x_2 \in F$ such that $x_i^*(x_j) = \delta_{i,j}$, so we have $\varphi(e_1 \otimes x_1 + e_2 \otimes x_2) = (e_1, e_2)$. Let $u = \sum_{i \leq N} e_i \otimes x_i \in E \otimes F$, then

\[
(T \oplus T \circ \varphi)(u) = (T \oplus T)(\sum_{i \leq N} \langle x_i, x_1^* \rangle e_i, \sum_{i \leq N} \langle x_i, x_2^* \rangle e_i)
\]

\[
= (T \oplus T)(\sum_{i \leq N} \langle x_i, x_1^* \rangle T e_i, \sum_{i \leq N} \langle x_i, x_2^* \rangle T e_i)
\]

\[
= (\sum_{i \leq N} \langle x_i, x_1^* \rangle T e_i, \sum_{i \leq N} \langle x_i, x_2^* \rangle T e_i)
\]

\[
= (\sum_{i \leq N} (T e_i \otimes x_1^*)x_i, \sum_{i \leq N} (T e_i \otimes x_2^*)x_i)
\]

\[
= (\sum_{i \leq N} (T \otimes I)(e_i \otimes x_1^*)x_i, \sum_{i \leq N} (T \otimes I)(e_i \otimes x_2^*)x_i)
\]

\[
= \varphi(\sum_{i \leq N} e_i \otimes x_i)
\]

\[
= \varphi(\sum_{i \leq N} e_i \otimes x_i).
\]

Thus, $T \oplus T$ is supercyclic on $E \oplus E$. □

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