Quasi-integrability in supersymmetric sine-Gordon models

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Abstract – The deformed supersymmetric sine-Gordon model, obtained through known deformation of the corresponding potential, is found to be quasi-integrable, like its non-supersymmetric counterpart, which was observed earlier. The system expectedly possesses finite number of conserved quantities, leaving out an infinite number of non-conserved anomalous charges. The quasi-integrability of this supersymmetric model heavily rely on the boundary conditions of the potential, otherwise rendered to be completely non-integrable. Moreover, interesting additional algebraic structures appear, absent in the non-supersymmetric counterparts.

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Introduction. – The integrability of finite-dimensional dynamical systems is related to the fact that the system possesses as many constants of motion as its number of degrees of freedom and it is described by the Liouville-Arnold theorem. There is no such unique definition to describe integrability for dynamical systems having infinite degrees of freedom or governed by partial differential equations (PDEs). The notion of integrability is described via various ways, for example, bi-Hamiltonian theory, method of inverse scattering transformation, Lax pair formalism etc. The integrable PDEs appearing in field theory are best studied via Lax pair method or zero curvature equation. These are deemed integrable if they contain infinitely many conserved quantities, which are responsible for the stability of corresponding soliton solutions [1]. In particular, these constants of motion uniquely define the dynamics of the system, rendering the latter to be completely solvable. The sine-Gordon (SG) model, in one space and one time (1 + 1) dimensions, is one such system that further incorporates semi-classical solitonic solutions, which are physically realizable, representing high degree of symmetry. The latter property, in turn, corresponds to the infinitely many conserved quantities. Such solitonic solutions of integrable models correspond to the zero-curvature condition [2,3], containing the connections that constitute the Lax pair, which linearize the corresponding non-linear system.

Real physical systems do not possess infinite degrees of freedom, and thus, a corresponding field-theoretical model cannot be integrable in principle. However, such systems do possess solitonic states, which are very similar to those of integrable models like the SG. Therefore, the study of physical continuous systems as slightly deformed integrable models is of conceptual interest. Recently, it was shown that the SG model can be deformed as an approximate system, leading to a finite number of conserved quantities [4,5]. An almost flat connection, leading to an anomalous zero-curvature condition was obtained, rendering the system quasi-integrable. It was numerically shown [4,6] that the system behaves almost like the integrable SG model for single and non-interacting multi-soliton states, but differs considerably in the presence of scattering. This hints at integrable models being asymptotic limits of physical systems, which can be modeled by their quasi-integrable counterparts.

On the other hand, the SG model admits a supersymmetric extension, embedded in the $N = 2$ superspace [7]. This model also is integrable [8,9] and can consistently be quantized [10,11]. Their integrability, through linearization by introducing proper Lax pair, leads to the super SG (SSG) equation as a consistency condition. Such a model contains fermionic components of the Majorana variety, which are of great interest in current material physics. Later it was shown that the SSG admits two independent ways of linearization, corresponding to two possible constructions of the Lax pair [12]. However, these two representations were found to be gauge-equivalent. The model was further shown to incorporate multi-soliton solutions [13]. As the quasi-integrable models are closer to physical systems, a corresponding deformation of SSG model can potentially lead to the realization of the same as a limiting case. This is further important from the
pure algebraic viewpoint, as the generalization to the superspace can expectedly induce richer mathematical structure.

In the present work, we attempt to construct a quasi-integrable SSG (QISSG) model. It is found that the generalization to the superspace indeed extends the corresponding group structure beyond the usual \(sl(2)\) loop algebra of the SG model [4,5]. However, the quasi-integrability, obtained through gauge transformation of the anomalou curvature condition, is retained and a finite number of conserved charges appear, with the asymptotic correspondence to the SSG generalized to the “non-commutative boundary”. In the following, we briefly introduce the nomenclature of SSG model, as well as that of quasi-integrability, followed by our generalization of the latter to the SSG model. We summarize the results at the end.

Basic concepts. –

Super sine-Gordon model. The sine-Gordon (SG) model [1], defined through the dynamical equation,

\[
\partial_x \partial_- \phi(x,t) = \sin \phi(x,t),
\]

(1)
is an integrable one, containing infinitely many conserved quantities. This system is realized in \((1+1)\) dimensions, is an integrable one, containing infinitely many conserved quantities of space-time, making \(\Phi(x,t)\) quasi-integrable, followed by our generalization of the latter to the SSG model. We summarize the results at the end.

The linearization of the SSG model was achieved [9], by identifying the corresponding Lax pair, which are now \(3\times3\) matrices in the superspace. Further, they were found to implement integrability through the zero-curvature condition. It was later shown [12] that the linear representation is not unique, and two such forms, related to each other by a super gauge transformation and to the super Bäcklund transformation of the equation.

Quasi-integrable models. Certain non-integrable theories share physical results similar to the integrable ones. Namely, solitonic solutions of certain respective models, belonging to this distinct class, have similar properties [4]. The same is observed in \((2+1)\) dimensions, for the Ward-modified chiral model and baby Skyrme model having many potentials [5]. Therefore, it has recently been attempted to identify such non-integrable models as quasi-integrable ones [4,15], in order to realize such models as parametric generalization of the integrable counterparts. Such generalization is complemented by anomaly functions \(P_n\), defined through [5],

\[
\frac{dQ_n}{dt} = P_n, \quad n \in \mathbb{Z},
\]

wherein the quantities \(Q_n\) are conserved for vanishing \(P_n\), a limit that leads to the corresponding integrable model. When one considers \(P_n\) for two soliton configurations in these theories, the corresponding anomaly has the intriguing property \(\int_{-\infty}^{\infty} \, dt \, P_n = 0\), which implies that the charges are asymptotically conserved: \(Q_n(t = \infty) = Q_n(t = +\infty)\). If the model possesses breather-like field configurations we obtain \(Q_n(t) = Q_n(t + T)\). On the other hand, for different configurations, like multi-soliton solutions, the anomalies \(P_n\) are found to be non-vanishing, but they display interesting boundary properties, having topological nature [5].

As an concrete example, a deformation of the SG potential is proposed as a quasi-integrable system, leading to a quasi zero-curvature condition: \(F_{+-} \neq 0\). A suitable gauge transformation then leads to quasi-conservation equations, with only a few anomaly functions \((P_n)\) vanishing. We generalize the construction of quasi-integrable SG model to the supersymmetric one, or a quasi-integrable supersymmetric sine-Gordon (QISSG) model, in the following.

The supersymmetric model. – To begin with, we consider the \(N = 2\) supersymmetric SG model [9], that yields the Lax pair,

\[
A_+ = \frac{i}{2\sqrt{2\lambda}} \frac{dV(\Phi)}{d\Phi} F_1 + \frac{1}{2\sqrt{2\lambda}} V(\Phi) B_1, \quad A_- = \frac{1}{\sqrt{2\lambda}} B_{-1} - \frac{i}{2} D_- \phi F_0.
\]

Here, the \(3 \times 3\) matrices have the forms

\[
F_1 = 2\sqrt{\lambda} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},
\]

\[
B_1 = \begin{pmatrix} \phi(x,t) \\ \psi_1(x,t) \\ \phi_1(x,t) \end{pmatrix},
\]

\[
B_{-1} = \begin{pmatrix} \psi_2(x,t) \\ \phi_2(x,t) \\ \psi_3(x,t) \end{pmatrix},
\]

\[
F_0 = \begin{pmatrix} \phi(x,t) \\ \psi_1(x,t) \\ \phi_1(x,t) \end{pmatrix}.
\]
The deformation is done by introducing a small parameter \( \lambda \) for the un-deformed potential. However, this is not the case for quasi-integrable deformation of the same. The Euler function \( E(\Phi) \) vanishes on-shell (eq. (2)), and therefore can be identified that, \( A_\pm \rightarrow B_\pm = G A_\pm G^{-1} - D_\pm GG^{-1} \),

\[
G = \exp \left[ \sum_{n=1}^{\infty} \zeta_n F_n \right].
\]

The choice of the transformation generator \( G \) utilizes the \( sl(2) \) algebraic structure, thereby making this transformation essentially a rotation in the space of the super-\( sl(2) \) loop algebra. The form of \( G \) enables the utilization of the Baker-Campbell-Hausdorff (BCH) formula,

\[
e^{X}Y_\pm e^{-X} = Y_\pm + [X,Y_\pm] + \frac{1}{2!}[X,[X,Y_\pm]] + \cdots,
\]

to evaluate the first term on the LHS of eq. (12), where it can be identified that,

\[
X = \sum_{n=1}^{\infty} \zeta_n F_n \quad \text{and} \quad Y_\pm = A_\pm.
\]

The rotated Lax component, \( B_\pm \), is evaluated to begin with, as a choice, which is not unique. This is what is expected from the notion of gauge-invariance. However, the choice of representation for the particular gauge group can result in analytical simplicity. In the present case, the additional presence of supersymmetry makes this choice effectfully crucial, as it prohibits the matrix \( B_{-1} \), in eq. (5), from being a semi-simple element of the \( sl(2) \) algebra in eq. (11). It can owe to the fermionic domain of the supersymmetric structure, incorporating its non-commutative nature in an indirect manner. Therefore, from the representation chosen in eqs. (4) and (5), we choose to evaluate \( B_\pm \) first, unlike the case of the non-supersymmetric system of ref. [5], where the negative component was chosen. In any case, the components will have the form

\[
B_{\pm} \equiv e^{X}Y_\pm e^{-X} - \sum_{n} D_\pm \zeta_n F_n.
\]

The first four non-trivial terms of the BCH expansion are of the form

\[
[X, Y_\pm] = \alpha(\Phi)\chi_{2m} B_{2m+1} + \beta(\Phi) \left\{ \chi_{2m} F_{2m+1} + \chi_{2m-1} F_{2m} \right\},
\]

\[
\frac{1}{2!}[X,[X,Y_\pm]] = \alpha(\Phi)\chi_{2m} \left\{ \chi_{2p} F_{2(m+p)+1} + \chi_{2p-1} F_{2(m+p)} \right\} + \beta(\Phi) \left\{ \chi_{2m} \chi_{2p} B_{2(m+p)+1} - \chi_{2m-1} \chi_{2p-1} B_{2(m+p)-1} \right\},
\]

leading to the \( sl(2) \) algebra,

\[
[\tau_+, \tau_-] = 2\tau_3, \quad [\tau_3, \tau_\pm] = \pm \tau_\pm.
\]

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\[
\begin{align*}
\frac{1}{3!} [X, [X, [X, Y_+]]] &= \\
\frac{2}{3} \left[ \alpha(\Phi) \zeta_{2m} \left\{ \zeta_{2p} \zeta_{2q} B_2(m+p+q) + 1 \right\} \right. \\
&- \zeta_{2p-1} \zeta_{2q-1} B_2(m+p+q-1) \left\} + \beta(\Phi) \zeta_{2m} \zeta_{2p} \right. \\
&\times \left\{ \zeta_{2q} F_2(m+p+q) + 1 + \zeta_{2q-1} F_2(m+p+q-1) \right\} \\
&- \beta(\Phi) \zeta_{2m-1} \zeta_{2p-1} \left\{ \zeta_{2q} F_2(m+p+q) - 1 \right\} \\
&+ \zeta_{2q-1} F_2(m+p+q-1) \right\}; \\
\frac{1}{4!} [X, [X, [X, Y_+]]] &= \\
\frac{1}{3} \left[ \alpha(\Phi) \zeta_{2m} \zeta_{2p} \left\{ \zeta_{2q} \zeta_{2r} F_2(m+p+q+r) + 1 \right\} \right. \\
+ \zeta_{2r-1} F_2(m+p+q+r) + \zeta_{2q-1} \left\{ \zeta_{2r} F_2(m+p+q+r) - 1 \right\} \\
&+ \zeta_{2r-1} F_2(m+p+q+r-1) \right\} \\
+ \beta(\Phi) \zeta_{2m} \zeta_{2p} \left\{ \zeta_{2q} \zeta_{2r} B_2(m+p+q+r) + 1 \right\} \\
&\left. - \zeta_{2q-1} \zeta_{2r-1} B_2(m+p+q+r-1) \right\} \\
&- \beta(\Phi) \zeta_{2m-1} \zeta_{2p-1} \left\{ \zeta_{2q} \zeta_{2r} B_2(m+p+q+r) - 1 \right\} \\
&\left. - \zeta_{2q-1} \zeta_{2r-1} B_2(m+p+q+r-3) \right\}; \\
\end{align*}
\] (15)

where

\[
\alpha(\Phi) = \frac{i}{\sqrt{2\lambda}} \frac{dV(\Phi)}{d\Phi}, \quad \beta(\Phi) = \frac{1}{\sqrt{2\lambda}} V(\Phi), \quad \text{and} \quad (m, p, q, r) = 1, 2, 3, \ldots.
\]

The gauge-fixing is implemented through the condition that the coefficients of \( F_n \), for all \( n \) in eq. (14), should vanish. This leads to the following consistency conditions:

\[
\begin{align*}
\text{for } F_1: \quad &D_+ \zeta_1 = \frac{1}{2} \alpha(\Phi), \\
\text{for } F_2: \quad &D_+ \zeta_2 = \beta(\Phi) \zeta_1, \\
\text{for } F_3: \quad &D_+ \zeta_3 = \beta(\Phi) \zeta_2, \\
\text{for } F_4: \quad &D_+ \zeta_4 = \beta(\Phi) \zeta_3 - \frac{2}{3} \zeta_4^2 + \alpha(\Phi) \zeta_2, \\
\end{align*}
\]

and so on. These relate the parameters with the variables of the system. The above equations have to be integrated to obtain \( \zeta_\alpha \)'s, including \( \alpha(\Phi) \) and \( \beta(\Phi) \) which, in general, can require non-trivial boundary conditions, both in \( \zeta \)-parameter space and \( \Phi \)-field space. Such behavior can be attributed to the supersymmetric nature of the present system, as discussed before. As a result, the gauge-fixed field component has the form

\[
B_+ = \sum_{n=0}^{\infty} b^{(2n+1)}_+ B_{2n+1}, \quad \text{(16)}
\]

with various coefficients \( b^{(2n+1)}_+ \):

\[
\begin{align*}
&b_+^{(1)} = \frac{1}{2} \beta(\Phi), \\
&b_+^{(3)} = \alpha(\Phi) \zeta_2 - \beta(\Phi) \zeta_4^2, \\
&b_+^{(5)} = \alpha(\Phi) \left\{ \zeta_2 - \frac{2}{3} \zeta_4^2 \right\} + \beta(\Phi) \left\{ \zeta_2^2 - 2 \zeta_3 \zeta_1 + \frac{1}{3} \zeta_4^4 \right\},
\end{align*}
\]

\[
\ldots
\]

\[
\ldots
\]

\[
\ldots
\]

\[
\ldots
\]

(17)

It is to be noted that, in evaluating \( B_+ \), the equation of motion has not been used, i.e., the Euler function \( E(\Phi, \zeta) \), corresponding to the deformed potential eq. (16), has not been set equal to zero. Therefore, the above results are off-shell. To determine the exact form of the coefficients \( b^{(2n+1)}_+ \), the terms \( \zeta_\alpha \) are to be evaluated, for which the super-differential equations are obtained as consistency conditions. The necessary boundary conditions for the same is provided by the equation of motion, which will be used in evaluating the other Lax component \( B_- \). For this purpose, we adopt the definitions

\[
\begin{align*}
A_- &= \frac{\kappa}{2} B_{-1} + \frac{\gamma(\Phi)}{2} F_0, \\
\kappa &= \sqrt{\frac{2}{\lambda}}, \quad \gamma(\Phi) = -i D_- \Phi. \quad \text{(18)}
\end{align*}
\]

As in case of \( B_+ \), the first three non-trivial BCH contributions have the form,

\[
\begin{align*}
[X, Y_-] &= \kappa \left\{ \zeta_{2m} \bar{F}_{2m+1} + \zeta_{2m-1} \bar{F}_{2m} \right\} - \gamma(\Phi) \zeta_{2m-1} B_{2m-1}, \\
\frac{1}{2!} [X, [X, Y_-]] &= \kappa \left\{ \zeta_{2m} \left\{ \zeta_{2p} B_{m+p}^2 + 2 \zeta_{2p-1} F_2(m+p) \right\} - \zeta_{2m-1} \left\{ \zeta_{2p} B_{m+p}^2 \right\} \right\} \\
&\left. + \zeta_{2m} \left\{ 2 \zeta_{2p} F_2(m+p) - \zeta_{2p-1} B_{m+p}^2 \right\} \right\} \\
&\left. - \gamma(\Phi) \zeta_{2m-1} \left\{ \zeta_{2p} F_2(m+p) - \zeta_{2p-1} F_2(m+p) \right\} \right\}, \\
\frac{1}{3!} [X, [X, [X, Y_-]]] &= \frac{2}{3} \kappa \left\{ \zeta_{2m} \left\{ \zeta_{2p} \left( \zeta_{2q} F_2(m+p+q) + 1 \right) \right\} + \zeta_{2q-1} \left\{ \zeta_{2p} F_2(m+p+q) - 1 \right\} \right\} \\
&\left. + \zeta_{2p-1} \left\{ \zeta_{2q} F_2(m+p+q-1) - 4 \zeta_{2q-1} F_2(m+p+q-3) \right\} \right\} \\
&\left. + \zeta_{2p-1} \left\{ 4 \zeta_{2q} \left( \zeta_{2q} F_2(m+p+q) - \zeta_{2q-1} F_2(m+p+q-1) \right) \right\} \right\} \\
&\left. - \zeta_{2p-1} \left\{ \zeta_{2q} F_2(m+p+q-1) - \zeta_{2q-1} F_2(m+p+q-3) \right\} \right\} \\
&\left. - \gamma(\Phi) \zeta_{2m-1} \left\{ \zeta_{2p} \zeta_{2q} B_{2m+p+q+1} \right\} \right\}, \\
&\left. - \gamma(\Phi) \zeta_{2m-1} \left\{ \zeta_{2p} \zeta_{2q} F_2(m+p+q+1) \right\} \right\}, \quad \text{(19)}
\end{align*}
\]

wherein the following additional matrices appear:

\[
\begin{align*}
F_2 &= \lambda^m F_0, \\
F_0 &= \frac{1}{2\lambda} \left[ F_1, B_{-1} \right] = 2 \begin{pmatrix}
0 & -i & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\end{align*}
\]

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Due to the action of the same pair of operators, larger subspace $\tilde{F}_2$ generated by $B$-terms \((5,6)\). This is an alternate signature of the present representation of eq. (5). Equations (21), (22) further ensure the uniqueness of the solution to a twisted loop algebra, of the form obtained in ref. \([16]\). To see this, one notes the following partially close algebra:

\[
\begin{align*}
[0, \tilde{F}_0] & = 2B_{-1}, & F_0, \tilde{F}_1 \rightarrow -4\tilde{F}_1, \\
F_0, \tilde{F}_1 & = 2\tilde{F}_0, & \tilde{F}_1, \tilde{F}_1 & = -4\tilde{F}_1.
\end{align*}
\]

(20)

From the above algebra, it is easy to identify the rotations, in the space of $\lbrack B_{-1}, \tilde{F}_0, 0 \rbrack$,

\[
\begin{align*}
B_{-1} & \rightarrow \tilde{F}_1 \rightarrow F_1 \rightarrow B_{-1}, \\
B_{-1} & \rightarrow \tilde{F}_0 \rightarrow F_0 \rightarrow B_{-1}.
\end{align*}
\]

(22)

generated by $F_{0,1}$ independently. However, rotation in a larger subspace $\lbrack B_{-1}, \tilde{F}_0, 0, \tilde{F}_0, 0 \rbrack$, is also identified as,

\[
\begin{align*}
B_{-1} & \rightarrow \tilde{F}_0 \rightarrow F_0 \rightarrow \tilde{F}_0 \rightarrow F_0 \rightarrow B_{-1}, \\
B_{-1} & \rightarrow \tilde{F}_1 \rightarrow F_1 \rightarrow \tilde{F}_1 \rightarrow F_1 \rightarrow B_{-1};
\end{align*}
\]

(23)

due to the action of same pair of operators $F_{0,1}$, however, now acting successively. Such extension to the matrix vector space can be considered as quasi-integrable generalization of the emergent twisted loop algebra. These additional algebraic structures further ensure the uniqueness of the SSG model, as inherent supersymmetry does not allow $B_{-1}$ to be a part of the $sl(2)$ algebra, in the present representation of eq. (5). Equations (21), (22) and (23) are unique to the present supersymmetric quasi-integrable model, absent in the non-supersymmetric counterparts \(([5,6])\). This is an alternate signature of the fermionic sector non-commutativity, following the apparent non-locality of parameters $\zeta_n$ and charges $\tilde{b}_{n+1}^{(2)}$, as discussed before. The final expression for the gauge field $B_{-}$ has the form

\[
B_{-} \equiv \sum_{n=0}^{\infty} \left( b_{n}^{(0)} B_{n}^{(0)} + b_{n+1}^{(2)} B_{n+1} \right)
\]

\[ + f(n) F_{n} + f(n) \tilde{F}_{n} + \tilde{f}(n) \tilde{F}_{n}, \]

(24)

with non-zero coefficients,

\[
\begin{align*}
b_{-1}^{(0)} & = \frac{1}{2} \kappa, & b_{-1}^{(1)} & = -3 \kappa \zeta_1, & b_{-1}^{(2)} & = 2 \kappa \zeta_2, \cdots; \\
b_{-1}^{(1)} & = -2 \gamma(\Phi) \zeta_1, & b_{-1}^{(3)} & = -2 \gamma(\Phi) \left( \zeta_1 - \frac{2}{3} \zeta_3 \right), \\
b_{-1}^{(2)} & = -2 \gamma(\Phi) \left( \zeta_2 + \frac{2}{3} \zeta_3 \zeta_1 \right) + \cdots; \\
f_{-1}^{(0)} & = -2 \gamma(\Phi), & f_{-1}^{(1)} & = -2 \gamma(\Phi), \\
f_{-1}^{(2)} & = -2 \gamma(\Phi) \zeta_2, & f_{-1}^{(3)} & = -2 \gamma(\Phi) \zeta_3, \\
f_{-1}^{(3)} & = -2 \gamma(\Phi) \zeta_4, & f_{-1}^{(4)} & = -2 \gamma(\Phi) \zeta_5, \\
& \cdots & \cdots
\end{align*}
\]

(25)

In evaluating the exact expressions for the above coefficients, the equation of motion, $(dV/d\Phi) + D_+ D_- \phi = 0$, has been utilized. This fact, as will be seen, enables us to determine the original coefficients $\zeta_n$ of the gauge transformation, for the integrable part of the system.

The quasi-integrability. The anomaly $X$ vanishes for integrable models. In the present non-integrable one, due to its presence, the anomalous coefficients $f(n), f(n)$ and $\tilde{f}(n)$ survive. It is observed that these coefficients vanish for $X = 0$. This allows for the definition of the quasi-continuity expressions

\[
\Gamma^{2n+1} := D_+ b_{-}^{(2n+1)} - D_- b_{-}^{(2n+1)}, \quad n = 0, 1, 2, \cdots
\]

(26)

from eqs. (16) and (24). The first of the above expressions vanishes for $\zeta_1 = \frac{1}{\sqrt{2 \lambda}} D_+ \Phi$, through the equation of motion. This is exactly the supersymmetric generalization of the result in \([5]\). Further, the first of the charge-densities,

\[
\Sigma^{(2n+1)} := \frac{1}{2} \left( b_{n+1}^{(2n+1)} - b_{n-1}^{(2n+1)} \right), \quad n = 0, 1, 2, \cdots
\]

(27)

leads to the total charge,

\[
Q^{(1)} := \int_{x, \theta} \Sigma^{(1)}
\]

\[ = \frac{1}{4 \sqrt{2} \lambda} \int_{x, \theta} V(\Phi)
\]

\[ - \frac{1}{2 \sqrt{2} \lambda} \int_{x, \theta} D_- (\Phi D_- \Phi),
\]

(28)

as $D^2 = 0$. The first term on the RHS is expected to yield a constant value, as $V(\Phi, \varepsilon)$ is an infinitesimal deformation away from the periodic potential $V(\Phi)$. The second term
is expected to vanish at the boundary of the superspace, as $\Phi$ is too. This invariably leads us to the conserved charge of the quasi-integrable model, yielding,

$$\frac{dQ^{(1)}}{dt} = 0.$$  \hfill (29)

This is not true, in general, for the higher-order charges of the model,

$$\frac{dQ^{(2n+1)}}{dt} = P_{(2n+1)} \neq 0, \quad n = 1, 2, 3.$$ \hfill (30)

This makes the system quasi-integrable. It is easy to see that, for $\varepsilon = 0$, $X = 0$, which leads to $P_{(2n+1)} = 0$, $n = 1, 2, 3$, and the SG model is recovered.

**Summary.** It has been shown that quasi-integrability can be attained in supersymmetric integrable models also, by taking the super SG model as an explicit example. It is to be mentioned here that the exact periodicity of the undeformed potential plays a crucial role in this formalism. Further, the extended structure of the superspace introduces extended sub-algebras (twisted loop algebras), forming closed vector spaces. A further unique aspect, that of non-locality of parameters, is observed. They yield a few conserved charges, as in the case of quasi-integrable non-supersymmetric models, with the others remaining non-conserved.

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