Local hidden variable modelling, classicality, quantum separability and the original Bell inequality

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Received 23 March 2010, in final form 26 August 2010
Published 16 December 2010
Online at stacks.iop.org/JPhysA/44/035305

Abstract

We introduce a general condition sufficient for the validity of the original Bell inequality (1964) in a local hidden variable (LHV) frame. This condition can be checked experimentally and incorporates only as a particular case the assumption on perfect correlations or anticorrelations usually argued for this inequality in the literature. Specifying this general condition for a quantum bipartite case, we introduce the whole class of bipartite quantum states, separable and nonseparable, that (i) admit an LHV description under any bipartite measurements with two settings per site; (ii) do not necessarily exhibit perfect correlations and may even have a negative correlation function if the same quantum observable is measured at both sites, but (iii) satisfy the ‘perfect correlation’ version of the original Bell inequality for any three bounded quantum observables $A_1, A_2 = B_1, B_2$ at sites ‘$A$’ and ‘$B$’, respectively. Analysing the validity of this general LHV condition under classical and quantum correlation scenarios with the same physical context, we stress that, unlike the Clauser–Horne–Shimony–Holt inequality, the original Bell inequality distinguishes between classicality and quantum separability.

PACS numbers: 03.65.Ta, 03.65.Ud, 03.67.–a

1. Introduction

Analysing in 1964 a possibility of a local hidden variable (LHV) description of bipartite\(^1\) quantum measurements on two-qubits, Bell introduced \([1]\) the LHV constraint on correlations, usually now referred to as the original Bell inequality. Both of Bell’s proofs \([1, 2]\) of this LHV inequality are essentially built up on two additional assumptions—a dichotomic character of Alice’s and Bob’s measurements plus the perfect correlation or anticorrelation of Alice’s and Bob’s outcomes for a definite pair of their local settings. Specifically, the latter assumption is usually abbreviated as the assumption on perfect correlations or anticorrelations.

\(^1\) In quantum information, two parties (observers) are usually named as Alice and Bob.
Bell’s proofs are now reproduced in any textbook on quantum information and there exists the widespread misconception that, in any LHV case, the original Bell inequality cannot hold without the additional assumptions used by Bell [1, 2], specifically, without the assumption on perfect correlations or anticorrelations—as it has been, for example, claimed by Simon [3] and Zukowski [4].

However, Bell’s additional assumptions are only sufficient but not necessary for the validity of the original Bell inequality in an LHV frame. Based on operator methods, we, for example, proved in [5–8] that, in either of the bipartite cases, classical or quantum, the original Bell inequality holds for Alice’s and Bob’s real-valued outcomes in \([-1, 1]\) of any spectral type, continuous or discrete, not necessarily dichotomic, and that there exist bipartite quantum states \(\rho\) on a Hilbert space \(\mathcal{H} \otimes \mathcal{H}\) that do not necessarily exhibit perfect correlations and may even have [8] a negative correlation if the same quantum observable is measured at both sites but satisfies the ‘perfect correlation’ (minus sign) version of the original Bell inequality:

\[
|\text{tr}[\rho(A_1 \otimes B_1)] - \text{tr}[\rho(A_1 \otimes B_2)]| \leq 1 - \text{tr}[\rho(B_1 \otimes B_2)],
\]

for any three bounded quantum observables \(A_1, A_2 = B_1, B_2\) on \(\mathcal{H}\), measured at the sites of Alice and Bob, respectively, and with eigenvalues in \([-1, 1]\).

Note that if bipartite measurements, with two settings per site, are performed on any of bipartite quantum states considered in [5–8], then these bipartite quantum measurements admit an LHV description. For a nonseparable quantum state, this fact follows from theorem 1 in [9] and the existence (see equation (A5) in [8]) for these bipartite measurements of a joint probability measure returning all observed joint distributions as marginals.

Our results in [5–8] clearly indicate that, in an LHV frame, the original Bell inequality holds for outcomes of any spectral type, not necessarily dichotomic, and under an additional assumption which is more general than the assumption on perfect correlations or anticorrelations and ensures the existence, in a quantum LHV case, of bipartite quantum states never violating the original Bell inequality (1).

The aim of this paper is to find such a general LHV condition and to specify the validity of this general condition under bipartite correlation scenarios on classical and quantum particles. The paper is organized as follows.

In section 2, for an arbitrary bipartite correlation scenario, with two settings per site and real-valued outcomes of any spectral type, discrete or continuous, we introduce a new condition sufficient for the validity of the original Bell inequality in an LHV frame. We prove that, in an LHV model of any type, this new LHV condition is more general than the assumption on perfect correlations or anticorrelations and incorporates the latter only as a particular case. We show that, in a dichotomic case, the new general LHV condition can be tested experimentally.

In section 3, we stress that, under a bipartite correlation experiment, performed on classical particles and described by bounded classical observables \(A_1, A_2 = B_1, B_2\) at the sites of Alice and Bob, respectively, the minus sign version of the new LHV condition and, therefore, the ‘perfect correlation’ version of the original Bell inequality are fulfilled for any initial state of classical particles and any type of Alice’s and Bob’s classical measurements—ideal (necessarily exhibiting perfect correlations if the same classical observable is measured at both sites) or non-ideal (not necessarily exhibiting perfect correlations).

2 For this misleading opinion, see, for example, the corresponding paragraph of the Wikipedia article on Bell’s theorem.

3 Throughout this paper, the term ‘classical’ is meant in its physical sense.

4 For correlations of an arbitrary type, not necessarily quantum, the original Bell inequality, in its ‘perfect correlation/anticorrelation’ version has the form (7).
In section 4, for a quantum correlation scenario, we specify the new general LHV condition in quantum terms. This allows us to introduce a new class of bipartite quantum states that admit an LHV description under any bipartite quantum measurements with two settings per site and satisfy the 'perfect correlation' version of the original Bell inequality for any bounded quantum observables $A_1$, $A_2 = B_1$, $B_2$ at the sites of Alice and Bob, respectively. These quantum states do not necessarily exhibit perfect correlations and may even have a negative correlation function whenever the same quantum observable is measured at both sites. We stress that an arbitrary separable quantum state does not need to belong to this new state class and that all bipartite quantum states specified by us earlier in [5–8] are included into this new state class only as a particular subclass.

In section 5, we summarize the main results of this paper and discuss their significance for the statistical analysis of correlation experiments on classical and quantum particles. We stress that these results rigorously disprove the widespread misconceptions existing in the literature since 1964, in particular, the recent claims of Simon [3] and Zukowski [4] on the relation between the original Bell inequality, perfect correlations, classicality and quantum separability.

2. General bipartite case

Consider the probabilistic description\(^5\) of a bipartite correlation scenario (‘gedanken’ experiment), specified at Alice’s and Bob’s sites by measurement settings $a_i$, $b_k$, $i, k = 1, 2$, and real-valued outcomes $\lambda_1, \lambda_2 \in [-1, 1]$, respectively. This correlation scenario is described by four bipartite joint measurements $(a_i, b_k)$ with joint probability distributions $P(a_i, b_k)$.

For a joint measurement $(a_i, b_k)$, denote by

$$\langle \lambda_n \rangle_{(a_i, b_k)} := \int \lambda_n P(a_i, b_k)(d\lambda_1 \times d\lambda_2), \quad n = 1, 2,$$

(2)

the mean values of outcomes at Alice’s ($n = 1$) and Bob’s ($n = 2$) sites and by

$$\langle \lambda_1 \lambda_2 \rangle_{(a_i, b_k)} := \int \lambda_1 \lambda_2 P(a_i, b_k)(d\lambda_1 \times d\lambda_2),$$

(3)

the expected value of the product of their outcomes. In quantum information, this product expectation is usually referred to as a correlation function or correlation, for short.

If, under a joint measurement $(a_i, b_k)$, Alice’s and Bob’s outcomes are perfectly correlated or anticorrelated in the sense that either the event $\{\lambda_1 = \lambda_2\}$ or the event $\{\lambda_1 = -\lambda_2\}$ is observed with certainty, then

$$P(a_i, b_k)(\{\lambda_1 = \lambda_2\}) = \int_{\lambda_1 = \lambda_2} P(a_i, b_k)(d\lambda_1 \times d\lambda_2) = 1$$

$$P(a_i, b_k)(\{\lambda_1 = -\lambda_2\}) = \int_{\lambda_1 = -\lambda_2} P(a_i, b_k)(d\lambda_1 \times d\lambda_2) = 1,$$

(4)

respectively. For two-valued outcomes $\lambda_1, \lambda_2 = \pm 1$, this perfect correlation/anticorrelation condition is equivalently represented by the restriction on the correlation function: $\langle \lambda_1 \lambda_2 \rangle_{(a_i, b_k)} = \pm 1$, introduced originally by Bell [1, 2].

The following theorem introduces a new condition sufficient for the validity of the original Bell inequality in an arbitrary LHV model. As it is further shown below, this new LHV condition is more general than the assumption on perfect correlations or anticorrelations and incorporates the latter only as a particular case.

\(^5\) For the general framework on the probabilistic description of a multipartite correlation scenario, see [9].
Theorem 1. Let a 2 × 2-setting bipartite correlation scenario, with outcomes λ₁, λ₂ ∈ [−1, 1] of an arbitrary spectral type, discrete or continuous, admit an LHV model for correlation functions, that is, each ⟨λ₁λ₂⟩(a₁b₁), i, k = 1, 2, admits representation

\[ \langle λ₁λ₂ \rangle^{(a₁b₁)} = \int_{Ω} f_1^{(a₁)}(ω)f_2^{(b₁)}(ω)v(ω)dω \] (5)

in terms of some variables ω ∈ Ω, a probability distribution v of these variables and real-valued functions \( f_1^{(a₁)}(ω), \ f_2^{(b₁)}(ω) \in [-1, 1] \). If, in this LHV model, the condition

\[ \int_{Ω} f_2^{(b₂)}(ω)(f_2^{(b₁)}(ω) ± f_1^{(a₂)}(ω))v(ω)dω \geq 0 \] (6)

is fulfilled in its minus sign or plus sign form, then the original Bell inequality:

\[ |\langle λ₁λ₂ \rangle^{(a₁b₁)} − ⟨λ₁λ₂⟩^{(a₂b₂)}| ≤ 1 ± |⟨λ₁λ₂⟩^{(a₂b₂)}| \] (7)

holds in its minus sign (‘perfect correlation’) or plus sign (‘perfect anticorrelation’) version, respectively.

Proof. In view of representation (5),

\[ ⟨λ₁λ₂⟩^{(a₁b₁)} − ⟨λ₁λ₂⟩^{(a₂b₂)} = \int_{Ω} f_1^{(a₁)}(ω)\left(f_2^{(b₁)}(ω) − f_2^{(b₂)}(ω)\right)v(ω)dω. \] (8)

From relation (8), the number inequality |x − y| \( ≤ 1 − xy, \ \forall x, y \in [-1, 1] \), and condition (6) it follows that

\[ |⟨λ₁λ₂⟩^{(a₁b₁)} − ⟨λ₁λ₂⟩^{(a₂b₂)}| ≤ \int_{Ω} |f_2^{(b₁)}(ω) − f_2^{(b₂)}(ω)|v(ω)dω \]
\[ ≤ \int_{Ω} (1 − f_2^{(b₂)}(ω)f_2^{(b₁)}(ω))v(ω)dω \]
\[ ≤ 1 ± |⟨λ₁λ₂⟩^{(a₂b₂)}|, \] (9)

where the minus (or plus) sign form of condition (6) corresponds to the minus (or plus) sign version of relation (9). This proves the statement.

At the end of this section, we show that, in a dichotomic case, the LHV condition (6) can be tested experimentally. Note that Bell’s assumption ⟨λ₁λ₂⟩^{(a₂b₁)} = ±1 also refers only to a dichotomic case.

According to the terminology introduced by Fine [10], a correlation LHV model is referred to as deterministic if the values of functions \( f_1^{(a₁)}, f_2^{(b₁)}, i, k = 1, 2, \) coincide with Alice’s and Bob’s outcomes under their corresponding measurements and stochastic, otherwise. If, in addition, functions \( f_1^{(a₁)}, f_2^{(b₁)}, i, k = 1, 2, \) are conditioned by any extra relation, then we refer [9] to such a correlation LHV model as conditional.

Therefore, otherwise expressed, theorem 1 reads that if a 2 × 2-setting correlation scenario, with outcomes λ₁, λ₂ ∈ [−1, 1] of any spectral type, admits a conditional correlation LHV model (5), (6), then the original Bell inequality (7) holds.

The original Bell inequality (7) represents an example of a conditional Bell-type inequality².

We stress that the LHV condition (6) does not, in general, imply any restriction on a value of the correlation ⟨λ₁λ₂⟩^{(a₂b₁)}. This, in particular, means that, in contrast to the claims of Simon and Zukowski in [3, 4], for the ‘perfect correlation’ (minus sign) form of inequality (7) to hold in an LHV frame, the expectation ⟨λ₁λ₂⟩^{(a₂b₁)} does not need to be even positive (see example 3 in section 4.1).

² For the definition of a Bell-type inequality, conditional or unconditional, see [11].
Condition (6) is, in particular, fulfilled if
\[ f_1^{(a_2)}(\omega) = \pm f_2^{(b_1)}(\omega), \] (10)
\[ \nu\text{-almost everywhere}^7 \text{ (a.e.) on } \Omega. \] Since, in an arbitrary LHV model, the values of functions \( f_1^{(a_2)}, f_2^{(b_1)} \) do not need to coincide with Alice’s and Bob’s outcomes under their measurements specified by settings \( a_2 \) and \( b_1 \), respectively, relation (10) does not, in general, mean the perfect correlation or anticorrelation of Alice’s and Bob’s outcomes under the joint measurement \((a_2, b_1)\).

Below we prove (propositions 1–3) that, for any spectral type of outcomes and in an LHV model of any type, the LHV condition (6) is more general than the assumption on perfect correlations or anticorrelations and incorporates the latter assumption only as a particular case.

**Proposition 1.** Let a \( 2 \times 2 \)-setting bipartite correlation scenario, with outcomes \( \lambda_1, \lambda_2 = \pm 1 \), admit a correlation LHV model (5) and, under the joint measurement \((a_2, b_1)\), Alice’s and Bob’s outcomes be perfectly correlated or anticorrelated\(^8\):
\[ \langle \lambda_1 \lambda_2 \rangle^{(a_2, b_1)} = \pm 1. \] (11)
Then this LHV model is conditioned by relation (6) and, therefore, by theorem 1, the original Bell inequality (7) holds in its ‘perfect correlation’ (minus sign) or ‘perfect anticorrelation’ (plus sign) form, respectively.

**Proof.** In view of equations (5) and (11),
\[ \int f_1^{(a_2)}(\omega)f_2^{(b_1)}(\omega)\nu(d\omega) = \pm 1, \] (12)
where \( f_1^{(a_2)}(\omega), f_2^{(b_1)}(\omega) \in [-1, 1] \). For functions with values in \([-1, 1]\), the plus sign or the minus sign version of equation (12) correspondingly implies that
\[ f_1^{(a_2)}(\omega)f_2^{(b_1)}(\omega) = 1 \iff f_1^{(a_2)}(\omega) = f_2^{(b_1)}(\omega) \in \{-1, 1\}, \nu\text{-a.e.} \]
\[ f_1^{(a_2)}(\omega)f_2^{(b_1)}(\omega) = -1 \iff f_1^{(a_2)}(\omega) = -f_2^{(b_1)}(\omega) \in \{-1, 1\}, \nu\text{-a.e.} \] (13)
These relations mean the validity of condition (10) and, therefore, condition (6). \( \square \)

If Alice’s and Bob’s outcomes take any values in \([-1, 1]\), possibly not discrete, then the assumption on perfect correlations or anticorrelations under the joint measurement \((a_2, b_1)\) is mathematically expressed by condition (4) but not by Bell’s restriction (11). For this general case, we have the following statement (see also proposition 2).

**Proposition 2.** Let, for a \( 2 \times 2 \)-setting bipartite scenario, with outcomes \( \lambda_1, \lambda_2 \in [-1, 1] \) of an arbitrary spectral type, the averages
\[ \langle \lambda_1^m \lambda_2^n \rangle^{(a_i, b_k)} = m + n \leq 2, \quad m, n = 0, 1, 2, \quad i, k = 1, 2, \] (14)
admit an LHV model
\[ \langle \lambda_1^m \lambda_2^n \rangle^{(a_i, b_k)} = \int_{\Omega} \left( f_1^{(a_i)}(\omega) \right)^m \left( f_2^{(b_k)}(\omega) \right)^n \nu(d\omega) \] (15)

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7 This terminology means that relation (10) can be violated only on a subset \( \Omega' \subset \Omega \) of zero measure \( \nu(\Omega') = 0 \).

8 For outcomes \( \pm 1 \), relations (4) and (11) are equivalent.
where functions $f_{1}^{(a)}(ω)$, $f_{2}^{(b)}(ω) ∈ [−1, 1]$. If, under the joint measurement $(a_2, b_1)$, Alice’s and Bob’s outcomes are perfectly correlated or anticorrelated, that is
\[ P^{(a_2, b_1)}(λ_1 = λ_2) = \int_{λ_1 = λ_2} P^{(a_1, b_1)}(dλ_1 × dλ_2) = 1 \]
or
\[ P^{(a_2, b_1)}(λ_1 = −λ_2) = \int_{λ_1 = −λ_2} P^{(a_1, b_1)}(dλ_1 × dλ_2) = 1, \]
respectively, then the LHV model (15) is conditioned by condition (6) and, therefore, by theorem 1, the original Bell inequality (7) holds in its ‘perfect correlation’ (minus sign) or ‘perfect anticorrelation’ (plus sign) version, respectively.

**Proof.** Due to representation (15) and the assumption on perfect correlations (the first line of equation (16)), we have
\[
0 \leq \int_{Ω} (f_{1}^{(a_2)}(ω) − f_{2}^{(b_1)}(ω))^2 ν(𝑑ω) \\
= \int (λ_1 − λ_2)^2 P^{(a_2, b_1)}(dλ_1 × dλ_2) \\
= \int_{λ_1 ≠ λ_2} (λ_1 − λ_2)^2 P^{(a_2, b_1)}(dλ_1 × dλ_2) \\
\leq 4 \int_{λ_1 ≠ λ_2} P^{(a_2, b_1)}(dλ_1 × dλ_2) \\
= 0.
\]
The similar relations (but with the plus sign in the first three lines and event $|λ_1 ≠ −λ_2|$ in the third and the fourth lines) hold in the case of perfect anticorrelations. These relations imply the validity of condition (10), hence, condition (6). □

Suppose now that a $2 \times 2$-setting correlation experiment admits an LHV model for joint probability distributions—in the sense that each joint distribution $P^{(a_i, b_k)}$, $i, k = 1, 2$, admits the representation
\[ P^{(a_i, b_k)}(dλ_1 × dλ_2) = \int_{Ω} P_1^{(a_i)}(dλ_1|ω) P_2^{(b_k)}(dλ_2|ω) ν(𝑑ω), \quad i, k = 1, 2, \]
in terms of some variables $ω ∈ Ω$ and conditional probability distributions $P_1^{(a_i)}(·|ω)$, $P_2^{(b_k)}(·|ω)$, defined ν-a.e. on Ω. In this LHV model, expectations $⟨λ_1 λ_2⟩^{(a_i, b_k)}$, $i, k = 1, 2$, admit representation (5) with functions
\[
f_1^{(a_i)}(ω) := \int λ_1 P_1^{(a_i)}(dλ_1|ω), \quad f_2^{(b_k)}(ω) := \int λ_2 P_2^{(b_k)}(dλ_2|ω), \quad i, k = 1, 2, \]
so that condition (6) takes the form
\[ \int λ_1 (λ_2 ≠ λ_1') μ(dλ_1' × dλ_2 × dλ_2') \geq 0, \]
where
\[ μ(dλ_1' × dλ_2 × dλ_2') = \int_{Ω} P_1^{(a_2)}(dλ_1'|ω) P_2^{(b_1)}(dλ_2|ω) P_2^{(b_2)}(dλ_2'|ω) ν(𝑑ω). \]

Note that, for outcomes of any spectral type, the existence of an LHV model (5) for correlation functions does not need to imply the existence of an LHV model (18) for joint probability distributions.

In view of equations (18)–(21), we have the following corollary of theorem 1.
Corollary 1. Let a $2 \times 2$-setting bipartite correlation scenario, with outcomes $\lambda_1, \lambda_2 \in [-1, 1]$ of any spectral type, admit a conditional LHV model (18), (20) for joint probability distributions. Then the original Bell inequality (7) holds.

Let us now show that, as it is the case for correlation LHV models, discussed above in propositions 1, 2, for an LHV model for joint probability distributions, condition (20) incorporates the assumption on perfect correlations or anticorrelations only as a particular case.

Proposition 3. Let a $2 \times 2$-setting bipartite correlation scenario, with outcomes $\lambda_1, \lambda_2 \in \Lambda \subseteq [-1, 1]$ of an arbitrary spectral type, admit an LHV model (18) for joint probability distributions. If, under the joint measurement $(a_2, b_1)$, Alice’s and Bob’s outcomes are perfectly correlated or anticorrelated, i.e. assumption (16) is fulfilled, then this LHV model is conditioned by the minus sign or plus sign version of condition (20), respectively, and, therefore, by corollary 1, the original Bell inequality (7) holds.

Proof. From equations (20) and (21) and the assumption on perfect correlations (the first line of equation (16)) it follows that

$$0 \leq \left| \int \lambda_2 (\lambda_2 - \lambda_1) \mu(d\lambda_1 \times d\lambda_2 \times d\lambda'_2) \right|$$

$$\leq \int |\lambda_1' - \lambda_2| \mu(d\lambda_1' \times d\lambda_2 \times \Lambda)$$

$$= \int_{\lambda_1' \neq \lambda_2} |\lambda_1' - \lambda_2| p(a_2, b_1)(d\lambda_1' \times d\lambda_2)$$

$$\leq 2 \int_{\lambda_1' \neq \lambda_2} p(a_2, b_1)(d\lambda_1' \times d\lambda_2)$$

$$= 0.$$  

These relations imply the validity of the minus sign version of condition (20). The similar relations (but with the plus sign in the first three lines and event $\{\lambda_1' \neq -\lambda_2\}$ in the third and fourth lines) hold in the case of perfect anticorrelations. This proves the statement. □

From propositions 1–3 it follows that, in an LHV model of any type, the assumption on perfect correlations or anticorrelations implies the validity of the general LHV condition (6). The converse of this statement is not, however, true.

In order to show this, let us consider a dichotomic bipartite case\(^9\) with outcomes $\pm 1$. Since $(\lambda_1')^2 = 1$, we have the following expression for the left-hand side of, for example, the minus sign form of condition (20):

$$\int \lambda_2 (\lambda_2 - \lambda_1') \mu(d\lambda_1' \times d\lambda_2 \times d\lambda'_2)$$

$$= \int \lambda_1' \lambda_2 (\lambda_1' \lambda_2 - 1) \mu(d\lambda_1' \times d\lambda_2 \times d\lambda'_2)$$

$$= 2\mu(\{\lambda_1' \lambda_2 = -1\}) - 4\mu(\{\lambda_1' \lambda_2 = -1, \lambda_1' \lambda'_2 = 1\}).$$  

\(^9\) In a dichotomic case, the existence of an LHV model for correlation functions implies the existence of an LHV model for joint probability distributions, see theorem 3 in section 4.1 of [9].
Taking into account that
\[ \mu((\lambda'_1\lambda_2 = -1, \lambda'_1\lambda'_2 = 1)) \leq \mu((\lambda'_1\lambda'_2 = -1)) \]
\[ \mu((\lambda'_1\lambda_2 = -1, \lambda'_1\lambda'_2 = 1)) \leq \mu((\lambda'_1\lambda'_2 = 1)) \]
\[ \mu((\lambda'_1\lambda_2 = -1, \lambda'_1\lambda'_2 = 1)) \leq \sqrt{\mu((\lambda'_1\lambda'_2 = -1))\mu((\lambda'_1\lambda'_2 = 1))} \]  
(24)
and substituting these relations into equation (23), we derive
\[ \int \lambda'_2(\lambda_2 - \lambda'_1)\mu(d\lambda'_1 \times d\lambda_2 \times d\lambda'_2) \]
\[ \geq 2\mu((\lambda'_1\lambda_2 = -1)) - 4\mu((\lambda'_1\lambda'_2 = 1)) \]  
(25)
and
\[ \int \lambda'_2(\lambda_2 - \lambda'_1)\mu(d\lambda'_1 \times d\lambda_2 \times d\lambda'_2) \]
\[ \geq 2\sqrt{\mu((\lambda'_1\lambda_2 = -1))} \cdot (\sqrt{\mu((\lambda'_1\lambda'_2 = -1))} - 2\sqrt{\mu((\lambda'_1\lambda'_2 = 1))}) \]  
(26)
Therefore, if either of the conditions
\[ \mu((\lambda'_1\lambda_2 = -1)) \geq 2\mu((\lambda'_1\lambda'_2 = 1)) \quad \text{or} \quad \mu((\lambda'_1\lambda_2 = -1)) = 0 \]  
(27)
is fulfilled, then the minus sign version of the general LHV condition (20) holds. Since from equation (21) it follows that
\[ \mu((\lambda'_1\lambda_2 = -1)) \equiv P^{(a_2,b_1)}((\lambda'_1\lambda_2 = -1)) \]
\[ \mu((\lambda'_1\lambda'_2 = 1)) \equiv P^{(a_2,b_1)}((\lambda'_1\lambda'_2 = 1)) \]  
(28)
in terms of joint probabilities, conditions (27) read
\[ P^{(a_2,b_1)}((\lambda'_1\lambda_2 = 1)) + 2P^{(a_2,b_1)}((\lambda'_1\lambda'_2 = 1)) \leq 1, \]
\[ P^{(a_2,b_1)}((\lambda'_1\lambda'_2 = 1)) = 1, \]  
(29)
respectively.

Thus, if, under the joint measurements \( (a_2, b_1) \) and \( (a_2, b_2) \), the probabilities of event \( \{\lambda_1\lambda_2 = 1\} \) satisfy either of the conditions in (29), then the minus sign form of the LHV condition (20) is fulfilled so that, according to corollary 1, the ‘perfect correlation’ version
\[ |\langle \lambda'_1\lambda_2 \rangle^{(a_2,b_1)} - \langle \lambda'_1\lambda_2 \rangle^{(a_2,b_2)}| \leq 1 - |\langle \lambda'_1\lambda_2 \rangle^{(a_2,b_2)}| \]  
(30)
of the original Bell inequality (7) holds.

Quite similarly, if, under the joint measurements \( (a_2, b_1) \) and \( (a_2, b_2) \), the probabilities of event \( \{\lambda_1\lambda_2 = -1\} \) satisfy either of the conditions
\[ P^{(a_2,b_1)}((\lambda'_1\lambda_2 = -1)) + 2P^{(a_2,b_2)}((\lambda'_1\lambda_2 = -1)) \leq 1, \]
\[ P^{(a_2,b_1)}((\lambda'_1\lambda_2 = -1)) = 1, \]  
(31)
then the ‘perfect anticorrelation’ version
\[ |\langle \lambda'_1\lambda_2 \rangle^{(a_2,b_1)} - \langle \lambda'_1\lambda_2 \rangle^{(a_2,b_2)}| \leq 1 + |\langle \lambda'_1\lambda_2 \rangle^{(a_2,b_2)}| \]  
(32)
of the original Bell inequality (7) holds.

We stress that all conditions in (29), (31) can be tested experimentally and that only the second condition in (29) and second condition in (31) mean, correspondingly, perfect correlations and perfect anticorrelations under the joint measurement \( (a_2, b_1) \).

In the following sections, we introduce classical and quantum correlation scenarios where the general LHV condition (6) and, therefore, the original Bell inequality (7) are fulfilled for the whole range of measurement settings.
3. Classical bipartite case

As an application of our results derived in section 2, consider the probabilistic description of a $2 \times 2$-setting bipartite correlation scenario, where every joint measurement $(a_i, b_k), i, k = 1, 2,$ is performed on the same, identically prepared pair of classical particles, each observed at one of the sites and not interacting with each other during a joint measurement. Let also a measurement device of each party not affect a measurement device and particle observed at another site, and the results of each joint measurement not in any way disturb results of other joint measurements. Due to this physical setting, the considered classical correlation scenario is local in the Einstein–Podolsky–Rosen (EPR) [12] sense\(^ {10} \).

In a classical EPR local bipartite case\(^ {11} \), a state of a bipartite classical system before measurements is represented by a probability distribution $\pi$ of some system’s variables $\theta \in \Theta$ such that, for any joint measurement $(a_i, b_k)$ performed on this classical system in a state $\pi$, the distribution $P_{\pi, (a_i, b_k)}$ has the factorizable form

$$P_{\pi, (a_i, b_k)}(d\lambda_1 \times d\lambda_2) = \int_{\Theta} P_{\pi, i}(d\lambda_1|\theta)P_{\pi, k}(d\lambda_2|\theta)\pi(\theta)(d\theta),$$

reducing to an image (37) of distribution $\pi$ if Alice’s and Bob’s classical measurements are ideal, i.e. describe a measured system property without an error. Due to the EPR locality of the considered classical correlation experiment, each of conditional distributions $P_{\pi, i}(\cdot|\theta)$, $P_{\pi, k}(\cdot|\theta)$ depends only on a setting of the corresponding measurement at the corresponding site.

Substituting representation (33) into equations (2) and (3), we obtain the following expressions for averages of Alice’s $(n = 1)$ and Bob’s $(n = 2)$ outcomes:

$$\langle \lambda_1 \rangle_{\pi}^{(a_i, b_k)} = \langle \lambda_1 \rangle_{\pi}^{(a_i, b_k)} = \int_{\Theta} A_i(\theta)\pi(\theta)(d\theta) := \langle \lambda_1 \rangle_{\pi}^{(A_i)} \text{, } i = 1, 2,$$

$$\langle \lambda_2 \rangle_{\pi}^{(a_i, b_k)} = \langle \lambda_2 \rangle_{\pi}^{(a_i, b_k)} = \int_{\Theta} B_k(\theta)\pi(\theta)(d\theta) := \langle \lambda_2 \rangle_{\pi}^{(B_k)} \text{, } k = 1, 2,$$

and the product expectations

$$\langle \lambda_1 \lambda_2 \rangle_{\pi}^{(a_i, b_k)} = \int_{\Theta} A_i(\theta)B_k(\theta)\pi(\theta)(d\theta) := \langle \lambda_1 \lambda_2 \rangle_{\pi}^{(A_i, B_k)} \text{, } i, k = 1, 2, \text{ (35)}$$

in terms of classical observables

$$A_i(\theta) := \langle \lambda_1 \rangle_{\pi}^{(a_i)} = \int_{\Theta} \lambda_1 P_{\pi, i}(d\lambda_1|\theta) \in [-1, 1],$$

$$B_k(\theta) := \langle \lambda_2 \rangle_{\pi}^{(b_k)} = \int_{\Theta} \lambda_2 P_{\pi, k}(d\lambda_2|\theta) \in [-1, 1].$$

If classical measurements, specified by settings $a_i$ and $b_k$, are ideal, then values of observables $A_i$ and $B_k$ coincide with Alice’s and Bob’s outcomes under these measurements, while the joint distribution $P_{\pi, \text{ideal}}$ has the form

$$P_{\pi, \text{ideal}}(D_1 \times D_2) = \pi\left(A_i^{-1}(D_1) \cap B_k^{-1}(D_2)\right). \quad \forall D_1, D_2 \subseteq [-1, 1]. \text{ (37)}$$

Here, $A_i^{-1}(D) := \{\theta \in \Theta \mid A_i(\theta) \in D\}$ denotes the preimage of a subset $D \subseteq [-1, 1]$ under the mapping $A : \Theta \to [-1, 1]$. Similarly, for the notation $B_k^{-1}(D)$.

If classical measurements $a_i$ and $b_k$ are non-ideal\(^ {12} \), then observables $A_i$ and $B_k$ describe these measurements only on average, in the sense that values $A_i(\theta)$, $B_k(\theta)$ of these observables

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\(^{10}\) For the mathematical formulation of the EPR locality, see section 3 in [9] and also section 4 in [13].

\(^{11}\) See section 3.1 in [9]

\(^{12}\) A non-ideal classical measurement is called randomized.
do not constitute Alice’s and Bob’s outcomes but only their averages \( \langle \lambda_1 | \theta \rangle_{\pi}^{(a_1)} \), \( \langle \lambda_2 | \theta \rangle_{\pi}^{(b_1)} \) for a certain initial \( \theta \in \Theta \).

Thus, the statistical context of our notation \( \langle \lambda_1 \lambda_2 \rangle_{\pi}^{(A_1 B_1)} \) is, in general, different from the context of the notation \( E_\pi (A_1 B_1) \) used in conventional probability theory for the expected value of the product of random variables.

From representations (33) and (35) it follows that any EPR local \( 2 \times 2 \)-setting correlation scenario on a classical state \( \pi \) admits the LHV model (18) for joint probability distributions and, hence, the LHV model (5) for correlation functions. Therefore, if classical observables (36) satisfy condition (6), then by theorem 1 the original Bell inequality (7) holds and, in view of notation (35), reads

\[
|\langle \lambda_1 \lambda_2 \rangle_{\pi}^{(A_1 B_1)} - \langle \lambda_1 \lambda_2 \rangle_{\pi}^{(A_1 B_2)}| \leq 1 + |\langle \lambda_1 \lambda_2 \rangle_{\pi}^{(A_2 B_2)}|.
\]  

(38)

If classical observables corresponding to settings \( a_2 \) and \( b_1 \) coincide: \( A_2 = B_1 \), then the minus sign form of condition (6) holds. Therefore, an arbitrary classical state \( \pi \) satisfies the ‘perfect correlation’ version

\[
|\langle \lambda_1 \lambda_2 \rangle_{\pi}^{(A_1 B_1)} - \langle \lambda_1 \lambda_2 \rangle_{\pi}^{(A_1 B_2)}| \leq 1 - |\langle \lambda_1 \lambda_2 \rangle_{\pi}^{(B_2 B_2)}|
\]  

(39)

of the original Bell inequality (7) for any three classical observables \( A_1, A_2 = B_1, B_2 \), measured, possibly on average, at the sites of Alice and Bob and having values in \([-1, 1]\).

If a joint classical measurement \((a_2, b_1)\) on a state \( \pi \) is ideal and the corresponding classical observables coincide: \( A_2 = B_1 \), then, due to equation (37), the outcome event \( \{\lambda_1 = \lambda_2\} \) is observed with certainty

\[
P_{\pi, \text{ideal}}^{(B_1 B_1)}(\{\lambda_1 = \lambda_2\}) := P_{\pi, \text{ideal}}^{(a_2 b_1)}(\{\lambda_1 = \lambda_2\})|_{a_2 = a_1} = 1.
\]  

(40)

Hence, under an ideal joint classical measurement of the same classical observable at both sites, Alice’s and Bob’s outcomes are necessarily perfectly correlated for any initial classical state \( \pi \).

However, if classical measurements of Alice and Bob on a state \( \pi \) are non-ideal, then a classical state \( \pi \) does not need to exhibit perfect correlations whenever the same classical observable is on average measured at both sites.

Thus, the ‘perfect correlation’ version (39) of the original Bell inequality is fulfilled for any classical state \( \pi \) and any three classical observables \( A_1, A_2 = B_1, B_2 \) measured ideally or non-ideally at the sites of Alice and Bob but the condition on perfect correlations necessarily holds only in the case of ideal classical measurements of the observable \( B_1 \) at both sites.

Consider now a possible physical context of a correlation scenario on a classical state \( \pi \) modelled at Alice’s and Bob’s sites by only three classical observables \( A_1, A_2 = B_1, B_2 \). Let Alice/Bob joint measurements be performed by identical measurement devices with settings \( a_1, a_2 = b_1, b_2 \) and upon a pair of classical particles, that are indistinguishable, identically prepared and do not interact with each other during measurements. In this case, due to the physical indistinguishability of sites and measurements of Alice and Bob specified by the same setting \( a_2 = b_1 \), these measurements are described by the same conditional averages \( \langle \lambda_1 | \theta \rangle_{\pi}^{(a_1 b_1)} = \langle \lambda_2 | \theta \rangle_{\pi}^{(b_1 b_1)} \) for any initial \( \theta \in \Theta \), and, therefore, in view of relations (36) should be on average modelled by the same classical observable \( A_2 = B_1 \) at both sites.

4. Quantum bipartite case

Consider now the probabilistic description of a quantum EPR local \( 2 \times 2 \)-setting correlation scenario, where any of the joint measurements \((a_i, b_k)\), \( i, k = 1, 2 \), is performed on the same identically prepared pair of quantum particles and the physical context of this quantum
scenario is similar to that of a classical EPR local scenario discussed in section 3—only with substitution of the term ‘classical particles’ by ‘quantum particles’.

In a quantum EPR local bipartite case, a state of a bipartite quantum system before measurements is represented by a density operator $\rho$ on a complex separable Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, possibly infinitely dimensional, and, for any joint measurement $(a_i, b_k)$ performed on this quantum system in a state $\rho$, the joint probability distribution $P^{(a_i, b_k)}_\rho$ takes the form

$$P^{(a_i, b_k)}_\rho(\mathrm{d}\lambda_1 \times \mathrm{d}\lambda_2) = \text{tr}\left[ \rho \left( M_1^{(a_i)}(\mathrm{d}\lambda_1) \otimes M_2^{(b_k)}(\mathrm{d}\lambda_2) \right) \right],$$

where $M_1^{(a_i)}$, $M_2^{(b_k)}$ are positive operator-valued (POV) measures, representing on $\mathcal{H}_1$ and $\mathcal{H}_2$ the corresponding quantum measurements of Alice and Bob. Due to the EPR locality of the considered quantum correlation experiment, each of these POV measures depends only on a setting of the corresponding measurement at the corresponding site.

Substituting equation (41) into equations (2) and (3), we get the following expressions for averages of Alice’s $(n = 1)$ and Bob’s $(n = 2)$ outcomes:

$$\langle \lambda_1 \rangle^{(a_i, b_k)}_\rho = \langle \lambda_1 \rangle^{(a_i, b_k)}_\rho = \text{tr}[\rho \{ A_i \otimes I_{\mathcal{H}_2} \}] := \langle \lambda_1 \rangle^{(A_i)}_\rho, \quad i = 1, 2,$n

$$\langle \lambda_2 \rangle^{(a_i, b_k)}_\rho = \langle \lambda_2 \rangle^{(a_i, b_k)}_\rho = \text{tr}[\rho \{ I_{\mathcal{H}_1} \otimes B_k \}] := \langle \lambda_2 \rangle^{(B_k)}_\rho, \quad k = 1, 2,$

and the product expectations

$$\langle \lambda_1 \lambda_2 \rangle^{(a_i, b_k)}_\rho = \int \lambda_1 \lambda_2 \text{tr}[\rho \{ M_1^{(a_i)}(\mathrm{d}\lambda_1) \otimes M_2^{(b_k)}(\mathrm{d}\lambda_2) \}]$$

$$= \text{tr}[\rho \{ A_i \otimes B_k \}] := \langle \lambda_1 \lambda_2 \rangle^{(A_i, B_k)}_\rho, \quad i, k = 1, 2,$

in terms of quantum observables

$$A_i := \int \lambda_1 M_1^{(a_i)}(\mathrm{d}\lambda_1), \quad B_k := \int \lambda_2 M_2^{(b_k)}(\mathrm{d}\lambda_2)$$

with eigenvalues in $[-1, 1]$, on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

If Alice’s and Bob’s quantum measurements specified by settings $a_i$ and $b_k$ are ideal, then eigenvalues of observables $A_i$ and $B_k$ coincide with Alice’s and Bob’s outcomes under these measurements while the POV measures $M_1^{(a_i)}$, $M_2^{(b_k)}$ describing these measurements are given by projection-valued measures $P^{(a_i)}_A$, $P^{(b_k)}_B$, uniquely corresponding to quantum observables $A_i$, $B_k$ due to the spectral theorem. If quantum measurements $a_i$ and $b_k$ are non-ideal, then observables (44) describe these quantum measurements only on average—in the sense of representations (42), (43).

For Alice/Bob joint measurements on a bipartite quantum state $\rho$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, we consider the possibility of a conditional LHV simulation specified in section 2.

We recall [6, 7] that, for any state $\rho$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, there exist self-adjoint trace class operators $T_\uparrow$ on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$ and $T_\downarrow$ on $\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$, not necessarily positive, such that

$$\text{tr}^{(2)}_{\mathcal{H}_2}[T_\uparrow] = \text{tr}^{(3)}_{\mathcal{H}_2}[T_\uparrow] = \rho, \quad \text{tr}^{(1)}_{\mathcal{H}_1}[T_\downarrow] = \text{tr}^{(2)}_{\mathcal{H}_2}[T_\downarrow] = \rho.$$  

(45)

Here, the indices of $T$ point to a direction of extension of a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, and the notation $\text{tr}^{(m)}[\cdot]$ means the partial trace over the elements of a Hilbert space $\mathcal{H}_m, m = 1, 2$, standing in the $m$th place of tensor products. In [6, 7], we refer to any of these dilations as a source operator for a bipartite state $\rho$. For each source operator, $\text{tr}[T] = 1$.

We introduce also the following new notion. We call a bounded operator $Z \neq 0$ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ as tensor-positive and denote it by $Z \otimes > 0$ if the scalar product

$$(\psi_1 \otimes \cdots \otimes \psi_N, Z \psi_1 \otimes \cdots \otimes \psi_N) \geq 0,$$

(46)

13 See section 3.1 in [9].

14 Due to this, an ideal quantum measurement is often referred to as projective.

15 In a quantum case, a non-ideal measurement is otherwise referred to as generalized.
for any vectors $\psi_i \in \mathcal{H}_i, \cdots, \psi_N \in \mathcal{H}_N$. Any positive operator is, of course, tensor-positive but the converse is not true. Due to equation (46) and the spectral decomposition of a positive operator on a complex separable Hilbert space, the relation

$$\text{tr}[Z [W_1 \otimes \cdots \otimes W_N]] \geq 0$$

(47)

holds for any tensor-positive $Z \otimes 0$ and any non-negative bounded operators $W_n \geq 0$, $n = 1, \ldots, N$, each defined on the corresponding complex separable Hilbert space $\mathcal{H}_n$.

The following statement (proved in the appendix) specifies the general LHV condition (6) in quantum terms.

**Theorem 2.** Let a quantum $2 \times 2$-setting bipartite correlation scenario, with outcomes $\lambda_1, \lambda_2 \in \Lambda \subseteq [-1, 1]$ of any spectral type, be specified by equations (41)–(44) and performed on a quantum state $\rho$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ that has a tensor-positive source operator $R_\rho \otimes 0$ on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$. Then this quantum correlation experiment admits an LHV model (18) for joint probability distributions. If, in addition,

$$\text{tr}[\sigma_{R_\rho} [B_1 \otimes B_2]] \equiv \text{tr}[\rho[A_2 \otimes B_2]] \geq 0$$

(48)

where

$$\sigma_{R_\rho} := \text{tr}_{\mathcal{H}_1}^3 \{ R_\rho \}$$

(49)

is a density operator on $\mathcal{H}_2 \otimes \mathcal{H}_2$, then this quantum correlation experiment admits the conditional LHV model (18), (20) and, therefore, by corollary 1 satisfies the original Bell inequality

$$\left| \langle \lambda_1 \lambda_2 \rangle \rho \{ A_1, B_2 \} - \langle \lambda_1 \lambda_2 \rangle \rho \{ A_1, B_1 \} \right| \leq 1 \equiv \langle \lambda_1 \lambda_2 \rangle \rho \{ A_2, B_2 \},$$

(50)

in its ‘perfect correlation’ (minus sign) or ‘perfect anticorrelation’ (plus sign) form, respectively. In quantum terms, this inequality reads

$$|\text{tr}[\rho[A_1 \otimes B_1]] - \text{tr}[\rho[A_1 \otimes B_2]]| \leq 1 \equiv |\text{tr}[\rho[A_2 \otimes B_2]]|.$$  

(51)

A similar statement holds for a state $\rho$ that has a tensor-positive source operator $R_\rho \otimes 0$ on $\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$.

We stress that condition (53) introduced in [5] for a separable quantum case and condition (42) in [7] represent particular cases of the quantum LHV condition (48).

From theorem 2 it follows that, for a state $\rho$ on $\mathcal{H} \otimes \mathcal{H}$ with a tensor-positive source operator $R_\rho \otimes 0$ and three *given* quantum observables $A_1, A_2 = B_1, B_2$ on $\mathcal{H}$, the original Bell inequality

$$|\text{tr}[\rho[A_1 \otimes B_1]] - \text{tr}[\rho[A_1 \otimes B_2]]| \leq 1 \equiv |\text{tr}[\rho[B_1 \otimes B_2]]|.$$  

(52)

in its minus sign or plus sign version, holds if

$$\text{tr}[\sigma_{R_\rho} [B_1 \otimes B_2]] \equiv \text{tr}[\rho[B_1 \otimes B_2]] \geq 0,$$

(53)

respectively. We stress that condition (53), which is the quantum version of the general LHV condition (6), does not mean the perfect correlation or anticorrelation of Alice’s and Bob’s outcomes if the observable $B_1$ is measured at both sites.

The following statement specifies the property of a bipartite quantum state ensuring the validity of the ‘perfect correlation’ (minus sign) version of the original Bell inequality (52) for any three quantum observables $A_1, A_2 = B_1, B_2$, measured, possibly on average, at the sites of Alice and Bob, respectively.

16 On a complex separable Hilbert space, every positive operator is self-adjoint.
Theorem 3. If, for a quantum state $\rho$ on $\mathcal{H} \otimes \mathcal{H}$, there exists a tensor-positive source operator $R \otimes 0$ on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ such that

$$\text{tr}_{\mathcal{H}}^{(k)}[R] = \rho, \quad k = 1, 2, 3,$$  

(54)

then this state $\rho$ satisfies the ‘perfect correlation’ version of the original Bell inequality:

$$\left| \langle \lambda_1 \lambda_2 \rangle^\rho_{(A_1, B_1)} - \langle \lambda_1 \lambda_2 \rangle^\rho_{(A_2, B_2)} \right| \leq 1 - \left( \langle \lambda_1 \lambda_2 \rangle^\rho_{(A_1, B_2)} \right),$$  

(55)

for any three bounded quantum observables $A_1$, $A_2 = B_1$, $B_2$ on $\mathcal{H}$, with eigenvalues in $[-1, 1]$ of an arbitrary spectral type, discrete or continuous. In quantum terms, this inequality reads

$$|\text{tr}[\rho(A_1 \otimes B_1)] - \text{tr}[\rho(A_1 \otimes B_2)]| \leq 1 - \text{tr}[\rho[B_1 \otimes B_2]].$$  

(56)

Proof. For a source operator $R \otimes 0$ specified by property (54), the reduced operator $\sigma_R$, given by (49), is equal to $\rho$. Therefore, the minus sign version of condition (48) takes the form

$$\text{tr}[\rho[B_1 \otimes B_2]] - \text{tr}[\rho[A_2 \otimes B_2]] \geq 0,$$  

(57)

which is always true if $A_2 = B_1$. By theorem 2, this proves the statement. \[\square\]

Due to theorems 2 and 3, every bipartite state $\rho$ with the state property (54) (i) admits an LHV description under any bipartite quantum measurements, ideal or non-ideal, with two settings per site; (ii) does not need to exhibit perfect correlations (may even have a negative correlation function $\langle \lambda_1 \lambda_2 \rangle^\rho_{(B_1, B_2)}$) if the same quantum observable is measured at both sites—but satisfies the ‘perfect correlation’ version (56) of the original Bell inequality for any three quantum observables $A_1$, $A_2 = B_1$, $B_2$, measured ideally or non-ideally at the sites of Alice and Bob, respectively, and having eigenvalues in $[-1, 1]$.

Note that an arbitrary separable quantum state does not need to have the state property (54) and, therefore, to satisfy inequality (56) for arbitrary $A_1$, $A_2 = B_1$, $B_2$, and that the class of bipartite quantum states specified by property (54) includes separable and nonseparable states introduced by us earlier in [5–7] only as a particular subclass.

We stress that the physical context of a quantum correlation scenario described by three quantum observables $A_1$, $A_2 = B_1$, $B_2$ is quite similar to the physical context of a classical correlation scenario described by three classical observables $A_1$, $A_2 = B_1$, $B_2$ and discussed in section 3—only with substitution of the term ‘classical particles’ by ‘quantum particles’.

Namely, let Alice/Bob joint measurements be performed by identical measurement devices with settings $a_1$, $a_2 = b_1$, $b_2$ and upon a pair of quantum particles that are indistinguishable, identically prepared and do not interact with each other during measurements. Then, due to the physical indistinguishability of sites and measurements of Alice and Bob specified by the same setting $a_2 = b_1$, these quantum measurements are described by the same operator averages $\int \lambda_1 \mathbf{M}_{1}^{(a_1 = b_1)}(d\lambda_1) = \int \lambda_2 \mathbf{M}_{2}^{(b_1)}(d\lambda_2)$ and, therefore, in view of relations (44), should be on average modelled by the same quantum observable $A_2 = B_1$ at both sites.

4.1. Examples

In this section, we present examples of bipartite quantum states, separable and nonseparable, specified by theorem 3. Satisfying inequality (56) for arbitrary three quantum observables $A_1$, $A_2 = B_1$, $B_2$, with eigenvalues in $[-1, 1]$, these bipartite quantum states do not need to exhibit perfect correlations and may even have a negative correlation function $\langle \lambda_1 \lambda_2 \rangle^\rho_{(B_1, B_2)}$ whenever the same quantum observable is measured at both sites.

(1) Every state $\rho$ on $\mathcal{H} \otimes \mathcal{H}$, reduced from a symmetric density operator $\sigma$ on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$.
(2) As it is proved by theorem 3 in [7], every Werner state [14]:

\[ W_d(\Phi) = \frac{1 + \Phi P_d^{(\Phi)}}{2} r_d^{(\Phi)} + \frac{1 - \Phi P_d^{(\Phi)}}{2} r_d^{(\Phi)}, \quad \Phi \in [-1, 1], \]  

on \( C^d \otimes C^d \) for \( d \geq 3 \), separable (\( \Phi \in [0, 1] \)) or nonseparable (\( \Phi \in [-1, 0) \)), and every separable Werner state \( W_2(\Phi), \Phi \in [0, 1] \) on \( C^2 \otimes C^2 \).

Here, \( P_d^{(\Phi)} \) are the orthogonal projections onto the symmetric (plus sign) and antisymmetric (minus sign) subspaces of \( C^d \otimes C^d \) with dimensions \( r_d^{(\pm)} = \text{tr}[P_d^{(\pm)}] = \frac{d(d+1)}{2}, \) respectively.

(3) As it is proved by theorem 2 in [8], each of the noisy states on \( C^d \otimes C^d \):

\[ \eta_\psi(\beta) = \beta |\psi\rangle \langle \psi| + (1 - \beta) \frac{I_{C^d \otimes C^d}}{d^2}, \quad \beta \in \left[ 0, \frac{1}{2\gamma_{\psi} + 1} \right], \]  

\[ \gamma_\psi := d \|\text{tr}_{C^d}(|\psi\rangle \langle \psi|)\| = d \|\text{tr}_{C^d}(|\psi\rangle \langle \psi|)\| \geq 1, \]

corresponding to a pure state \( |\psi\rangle \langle \psi| \). Here, \( \|\cdot\| \) means the operator norm and we take into account that, for any pure state on \( C^d \otimes C^d \), the eigenvalues (hence, the operator norms) of the reduced states \( \text{tr}_{C^d}(|\psi\rangle \langle \psi|) \) and \( \text{tr}_{C^d}(|\psi\rangle \langle \psi|) \) on \( C^d \) coincide.

In particular, the separable noisy singlet\(^{[7]}\):

\[ \eta_{\psi_2}(\beta) = \beta |\psi_2\rangle \langle \psi_2| + (1 - \beta) \frac{I_{C^2 \otimes C^2}}{4}, \quad \beta \in (0, \frac{1}{2}], \]

\[ \psi_2 = \frac{1}{\sqrt{2}} (e_1 \otimes e_2 - e_2 \otimes e_1), \]

for which the above parameter \( \gamma_{\psi_2} = 1 \). This noisy state constitutes the two-qubit Werner state \( W_2((1 - 3\beta)/2) \).

Note that whenever the spin observable \( \sigma_n \) along an arbitrary direction \( n \) in \( \mathbb{R}^3 \) is measured in the state \( \eta_{\psi_2}(\beta) \) at both sites, the correlation function is negative:

\[ \langle \lambda_1 \lambda_2 \rangle_{\eta_{\psi_2}(\beta)}(\sigma_n \otimes \sigma_n) = \text{tr}[\eta_{\psi_2}(\beta) (\sigma_n \otimes \sigma_n)] = -\beta < 0. \]  

This rules out perfect correlations. Furthermore, if Alice’s and Bob’s spin measurements on the state \( \eta_{\psi_2}(\beta) \) are projective, then, given an outcome, say of Alice, the conditional probability that Bob observes a different outcome is equal to \((1 + \beta)/2\) while the conditional probability that Bob observes the same outcome is \((1 - \beta)/2\).

5. Conclusion

In this paper, we introduce a \textit{new condition} sufficient for the validity of the original Bell inequality in an LHV frame. This LHV condition is more general than the assumption on perfect correlations or anticorrelations and incorporates the latter assumption only as a particular case. For dichotomic bipartite measurements, the new general LHV condition can be tested experimentally.

Specified for a quantum bipartite case, the new general LHV condition reduces to the form that does not necessarily imply any correlation between Alice’s and Bob’s outcomes if the same quantum observable is measured at both sites and leads to the existence of the whole class of bipartite quantum states, separable and nonseparable, that admit an LHV description under any bipartite quantum correlation scenario with two settings per site and never violate the ‘perfect correlation’ version of the original Bell inequality—though do not necessarily

\(^{17}\) Here, \( \{e_n, n = 1, 2\} \) is an orthonormal basis in \( C^2 \).
exhibit perfect correlations and may even have a negative correlation function if the same quantum observable is measured at both sites. Separable and nonseparable bipartite quantum states specified by us earlier in [5–8] are included into the new state class introduced in this paper only as a particular subclass.

Our comparative analysis of classical and quantum measurement situations indicates that an arbitrary classical state \( \pi \) satisfies the inequality

\[
\left| \langle \lambda_1 \lambda_2 \rangle^{(A_1,B_1)}_{\pi} - \langle \lambda_1 \lambda_2 \rangle^{(A_1,B_1)}_{\pi} \right| \leq 1 - \langle \lambda_1 \lambda_2 \rangle^{(B_1,B_2)}_{\pi} \tag{62}
\]

under any of Alice’s and Bob’s measurements, ideal or non-ideal, of arbitrary classical observables \( A_1, A_2 = B_1, B_2 \) with values in \([-1, 1]\) whereas an arbitrary separable state \( \rho \) does not need to satisfy the inequality

\[
\left| \langle \lambda_1 \lambda_2 \rangle^{(A_1,B_1)}_{\rho} - \langle \lambda_1 \lambda_2 \rangle^{(A_1,B_1)}_{\rho} \right| \leq 1 - \langle \lambda_1 \lambda_2 \rangle^{(B_1,B_2)}_{\rho} \tag{63}
\]

under any of Alice’s and Bob’s measurements, ideal or non-ideal, of arbitrary quantum observables \( A_1, A_2 = B_1, B_2 \) with eigenvalues in \([-1, 1]\).

This, in particular, means\(^{19}\) that, under a \(2 \times 2\) setting correlation experiment, performed at both sites by identical measurement devices with settings \( a_1, a_2 = b_1, b_2 \) and upon a pair of non-interacting, identically prepared, indistinguishable physical particles, quantum or classical, the ‘perfect correlation’ version of the original Bell inequality does not need to hold in an arbitrary separable quantum case but is always fulfilled in every classical case, that is, for any initial state of classical particles and any type of Alice’s and Bob’s classical measurements, ideal (necessarily exhibiting perfect correlations if the same classical observables are measured at both sites) or non-ideal (not necessarily exhibiting perfect correlations).

Thus, under classical and quantum correlation experiments with the same physical context, a classical state and a separable quantum state may exhibit statistically different correlations and the original Bell inequality reveals a gap between classicality and quantum separability. This observation agrees with our arguments in [5, 15] that an arbitrary separable quantum state does not need to satisfy every probabilistic constraint inherent to bipartite measurements on a classical system and also with the statement of Ollivier and Zurek: ‘absence of entanglement does not imply classicality’ [16], built up on the notion of a quantum discord.

The results of this paper disprove in rigorous mathematical terms the faulty claims of Simon [3] and Zukowski [4] that, in any bipartite case, classical or quantum, the perfect correlation version of the original Bell inequality holds only under the assumption on ‘perfect correlations if the same observable is measured at both sites’ ([3], abstract), and that, for the validity of the original Bell inequality in an LHV frame, the assumption on perfect correlations or anticorrelations is ‘minimal’ ([4], p 544) and without this assumption ‘the original Bell inequality cannot be derived’ ([4], p 544).

We note that it is specifically due to the latter misconception that the original Bell inequality has been disregarded in physical applications. Our results, however, indicate that, for a variety of quantum states, not necessarily exhibiting any correlation between Alice’s and Bob’s outcomes whenever the same quantum observable is measured at both sites, the ‘perfect correlation’ version of the original Bell inequality holds for any three bounded quantum observables \( A_1, A_2 = B_1, B_2 \) measured ideally or non-ideally at the sites of Alice and Bob and that, in contrast to the Clauser–Horne–Shimony–Holt (CHSH) inequality [17] does not distinguish between classicality and quantum separability, the original Bell inequality does distinguish between these two physical concepts.

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18 Any separable quantum state admits an LHV description.
19 See the discussions at the end of sections 3 and 4.
Appendix

Proof of theorem 2. Let a state $\rho$ have a source operator $R_\bullet \otimes 0$. Then, due to equations (41), (45), the joint distributions $P^{(a_i, b_i)}_\rho$, $P^{(a_i, b_i)}_\rho$, $\forall i = 1, 2$, admit representations

\begin{align}
    P^{(a_i, b_i)}_\rho(dx_1 \times dx_2) &= \text{tr}\left[R_\bullet \left[M_1^{(a_i)}(dx_1) \otimes M_2^{(b_i)}(dx_2) \otimes I_{H_1}\right]\right], \\
    P^{(a_i, b_i)}_\rho(dx_1 \times dx_2) &= \text{tr}\left[R_\bullet \left[M_1^{(a_i)}(dx_1) \otimes I_{H_1} \otimes M_2^{(b_i)}(dx_2)\right]\right],
\end{align}

(A.1)

where $M_1^{(a_i)}(A) = I_{H_1}$, $M_2^{(b_i)}(A) = I_{H_2}$, $i = 1, 2$. From these representations and property (47) it follows that, for any index $i = 1, 2$, distributions $P^{(a_i, b_i)}_\rho$, $P^{(a_i, b_i)}_\rho$ are marginals of the normalized joint probability measure

\begin{align}
    \tau^{(1)}_{\rho_\bullet}(dx_1 \times dx_2 \times dx'_2) &= \text{tr}\left[R_\bullet \left[M_1^{(a_i)}(dx_1) \otimes M_2^{(b_i)}(dx_2) \otimes M_2^{(a_i)}(dx'_2)\right]\right].
\end{align}

(A.2)

and measures $\tau^{(1)}_{\rho_\bullet}$, $\tau^{(2)}_{\rho_\bullet}$ are compatible in the sense

\begin{align}
    \tau^{(1)}_{\rho_\bullet}(A \times dx_2 \times dx'_2) &= \text{tr}\left[R_\bullet \left[I_{H_1} \otimes M_2^{(b_i)}(dx_2) \otimes M_2^{(a_i)}(dx'_2)\right]\right] \\
    &= \tau^{(2)}_{\rho_\bullet}(A \times dx_2 \times dx'_2).
\end{align}

(A.3)

Due to theorem 2 in [9], the existence of compatible probability measures $\tau^{(i)}_{\rho_\bullet}$, $i = 1, 2$, each returning distributions $P^{(a_i, b_i)}_\rho$, $P^{(a_i, b_i)}_\rho$ as marginals, implies the existence for this correlation experiment of the LHV model (18) where distribution (21) takes the form

\begin{align}
    \mu(dx_1' \times dx_2 \times dx'_2) &= \tau^{(2)}_{\rho_\bullet}(dx_1' \times dx_2 \times dx'_2).
\end{align}

(A.4)

Substituting this expression into the left-hand side of equation (20) and taking into account equations (A.2), (48), (49), we derive the relations

\begin{align}
    \int \lambda_2'(\lambda_2 \mp \lambda_1') \mu(dx_1' \times dx_2 \times dx'_2) \\
    &= \int \lambda_2'(\lambda_2 \mp \lambda_1') \text{tr}\left[R_\bullet \left[M_1^{(a_i)}(dx_1') \otimes M_2^{(b_i)}(dx_2) \otimes M_2^{(a_i)}(dx'_2)\right]\right] \\
    &= \text{tr}[\sigma_{\rho_\bullet}(B_1 \otimes B_2)] = \text{tr}[\rho(A \otimes B_2)] \geq 0,
\end{align}

(A.5)

meaning the validity of condition (20). This proves the statement. \hfill \Box

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