WILLMORE LEGENDRIAN SURFACES IN $S^5$ ARE MINIMAL LEGENDRIAN SURFACES

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ABSTRACT. In this paper we continue to consider Willmore Legendrian surfaces and csL Willmore surfaces in $S^5$, notions introduced by Luo in [11]. We will prove that every complete Willmore Legendrian surface in $S^5$ is minimal and construct nontrivial examples of csL Willmore surfaces in $S^5$.

1. Introduction

Let $\Sigma$ be a Riemann surface, $(M^n, g) = S^n$ or $\mathbb{R}^n (n \geq 3)$ the unit sphere or the Euclidean space with standard metrics and $f$ an immersion from $\Sigma$ to $M$. Let $B$ be the second fundamental form of $f$ with respect to the induced metric, $H$ the mean curvature vector field of $f$ defined by

$$H = \text{trace } B,$$

$k_M$ the Gauss curvature of $df(T\Sigma)$ with respect to the ambient metric $g$ and $d\mu_f$ the area element on $f(\Sigma)$. The Willmore functional of the immersion $f$ is then defined by

$$W(f) = \int_\Sigma \left( \frac{1}{4} |H|^2 + k_M \right) d\mu_f,$$

For a smooth and compactly supported variation $f : \Sigma \times I \mapsto M$ with $\phi = \partial_t f$ we have the following first variational formula (cf. [24, 25])

$$\frac{d}{dt} W(f) = \int_\Sigma \left( \overline{W}(f), \phi \right) d\mu_f,$$

with $\overline{W}(f) = \sum_{\alpha=3}^n \overline{W}(f)^\alpha e_\alpha$, where $\{e_\alpha : 3 \leq \alpha \leq n\}$ is a local orthonormal frame of the normal bundle of $f(\Sigma)$ in $M$ and

$$\overline{W}(f)^\alpha = \frac{1}{2} \left( \Delta H^\alpha + \sum_{i,j} h_i^\alpha h_j^\beta H^\beta - 2 |H|^2 H^\alpha \right), \quad 3 \leq \alpha \leq n,$$

where $h_i^\alpha$ is the component of $B$ and $H^\alpha$ is the trace of $\left( h_i^\alpha \right)$.

A smooth immersion $f : \Sigma \mapsto M$ is called a Willmore immersion, if it is a critical point of the Willmore functional $W$. In other words, $f$ is a Willmore immersion if and only if it
satisfies
\begin{equation}
\Delta H^\alpha + \sum_{i,j} h^\alpha_{ij} h^\beta_{ij} H^\beta - 2 |H|^2 H^\alpha = 0, \quad 3 \leq \alpha \leq n. \tag{1.1}
\end{equation}

When \((M, g) = \mathbb{R}^3\), Willmore [27] proved that the Willmore energy of closed surfaces are larger than or equal to \(4\pi\) and equality holds only for round spheres. When \(\Sigma\) is a torus, Willmore conjectured that the minimum is \(2\pi^2\) and it is attained only by the Clifford torus, up to a conformal transformation of \(\mathbb{R}^3\) [6, 26], which was verified by Marques and Neves in [15]. When \((M, g) = \mathbb{R}^n\), Simon [22], combined with the work of Bauer and Kuwert [1], proved the existence of an embedded surface which minimizes the Willmore functional among closed surfaces of prescribed genus. Motivated by these mentioned papers, Minicozzi [16] proved the existence of an embedded torus which minimizes the Willmore functional in a smaller class of Lagrangian tori in \(\mathbb{R}^4\). In the same paper Minicozzi conjectured that the Clifford torus minimizes the Willmore functional in its Hamiltonian isotropic class, which he verified has a close relationship with Oh’s conjecture [19, 20]. We should also mention that before Minicozzi, Castro and Urbano proved that the Whitney sphere in \(\mathbb{R}^4\) is the only minimizer for the Willmore functional among closed Lagrangian sphere. This result was further generalized by Castro and Urbano in [4] where they proved that the Whitney sphere is the only closed Willmore Lagrangian sphere (a Lagrangian sphere which is also a Willmore surface) in \(\mathbb{R}^4\). Examples of Willmore Lagrangian tori (Lagrangian tori which also are Willmore surfaces) in \(\mathbb{R}^4\) were constructed by Pinkall [21] and Castro and Urbano [5]. Motivated by these works, Luo and Wang [13] considered the variation of the Willmore functional among Lagrangian surfaces in \(\mathbb{R}^4\) or variation of a Lagrangian surface of the Willmore functional among its Hamiltonian isotropic class in \(\mathbb{R}^4\), whose critical points are called LW or HW surfaces respectively. We should also mention that Willmore type functional of Lagrangian surfaces in \(\mathbb{CP}^2\) were studied by Montiel and Urbano [18] and Ma, Mironov and Zuo [14].

Inspired by the study of the Willmore functional for Lagrangian surfaces in \(\mathbb{R}^4\), Luo [11] naturally considered the Willmore functional of Legendrian surfaces in \(\mathbb{S}^5\).

**Definition 1.1.** A Willmore and Legendrian surface in \(\mathbb{S}^5\) is called a Willmore Legendrian surface.

**Definition 1.2.** A Legendrian surface in \(\mathbb{S}^5\) is called a contact stationary Legendrian Willmore surface (in short, a csL Willmore surface) if it is a critical point of the Willmore functional under contact deformations.

Luo [11] proved that Willmore Legendrian surfaces in \(\mathbb{S}^5\) are csL surfaces (see Definition 2.6). In this paper, we continue to study Willmore Legendrian surfaces and csL Willmore surfaces in \(\mathbb{S}^5\). Surprisingly we will prove that every complete Willmore Legendrian surface in \(\mathbb{S}^5\) must be a minimal surface (Theorem 3.2). We will also construct nontrivial examples of csL Willmore surfaces from csL surfaces in \(\mathbb{S}^5\) for the first time, by exploring relationships between them (Proposition 4.1).

The method here we used to construct nontrivial csL Willmore surfaces in \(\mathbb{S}^5\) in Section 4 should also be useful in constructing nontrivial HW surfaces in \(\mathbb{R}^4\) introduced by Luo and Wang in [13]. We will consider this problem in a forthcoming paper.
2. Basic material and formulas

In this section we record some basic material of contact geometry. We refer the reader to consult [7] and [2] for more materials.

2.1. Contact Manifolds.

Definition 2.1. A contact manifold $M$ is an odd dimensional manifold with a one form $\alpha$ such that $\alpha \wedge (d\alpha)^n \neq 0$, where $\dim M = 2n + 1$.

Assume now that $(M, \alpha)$ is a given contact manifold of dimension $2n + 1$. Then $\alpha$ defines a $2n$–dimensional vector bundle over $M$, where the fibre at each point $p \in M$ is given by

$$\xi_p = \ker \alpha_p.$$ 

Since $\alpha \wedge (d\alpha)^n$ defines a volume form on $M$, we see that $\omega := d\alpha$ is a closed nondegenerate 2-form on $\xi \oplus \xi$ and hence it defines a symplectic product on $\xi$, say $(\xi, \omega|_{\xi \oplus \xi})$ becomes a symplectic vector bundle. A consequence of this fact is that there exists an almost complex bundle structure $\tilde{J}$: $\xi \mapsto \xi$ compatible with $d\alpha$, i.e. a bundle endomorphism satisfying:

1. $\tilde{J}^2 = -\text{id}_\xi$,
2. $d\alpha(\tilde{J}X, \tilde{J}Y) = d\alpha(X, Y)$ for all $X, Y \in \xi$,
3. $d\alpha(X, \tilde{J}X) > 0$ for $X \in \xi \setminus 0$.

Since $M$ is an odd dimensional manifold, $\omega$ must be degenerate on $T M$, and so we obtain a line bundle $\eta$ over $M$ with fibres

$$\eta_p := \{ V \in TM | \omega(V, W) = 0, \forall W \in \xi \}.$$ 

Definition 2.2. The Reeb vector field $R$ is the section of $\eta$ such that $\alpha(R) = 1$.

Thus $\alpha$ defines a splitting of $TM$ into a line bundle $\eta$ with the canonical section $R$ and a symplectic vector bundle $(\xi, \omega|_{\xi \oplus \xi})$. We denote the projection along $\eta$ by $\pi$, i.e.

$$\pi : TM \mapsto \xi, \quad V \mapsto \pi(V) := V - \alpha(V)R.$$ 

Using this projection we extend the almost complex structure $\tilde{J}$ to a section $J \in \Gamma(T^* M \otimes TM)$ by setting

$$J(V) = \tilde{J}(\pi(V)),$$

for $V \in TM$.

We have special interest in a special class of submanifolds in contact manifolds.

Definition 2.3. Let $(M, \alpha)$ be a contact manifold, a submanifold $\Sigma$ of $(M, \alpha)$ is called an isotropic submanifold if $T_x\Sigma \subseteq \xi$ for all $x \in \Sigma$.

For algebraic reasons the dimension of an isotropic submanifold of a $2n + 1$ dimensional contact manifold can not bigger than $n$.

Definition 2.4. An isotropic submanifold $\Sigma \subseteq (M, \alpha)$ of maximal possible dimension $n$ is called a Legendrian submanifold.
2.2. Sasakian manifolds. Let \((M, \alpha)\) be a contact manifold. A Riemannian metric \(g_\alpha\) defined on \(M\) is said to be associated, if it satisfies the following three conditions:

1. \(g_\alpha(R, R) = 1,\)
2. \(g_\alpha(V, R) = 0, \forall V \in \xi,\)
3. \(\omega(V, JW) = g_\alpha(V, W), \forall V, W \in \xi.\)

We should mention here that on any contact manifold there exists an associated metric on it, because we can construct one in the following way. We introduce a bilinear form \(b\) by

\[b(V, W) := \omega(V, JW),\]

then the tensor

\[g := b + \alpha \otimes \alpha\]
defines an associated metric on \(M\).

Sasakian manifolds are the odd dimensional analogue of Kähler manifolds.

**Definition 2.5.** A contact manifold \(M\) with an associated metric \(g_\alpha\) is called Sasakian, if the cone \(CM\) equipped with the following extended metric \(\tilde{g}\)

\[(CM, \tilde{g}) = (\mathbb{R}^+ \times M, dr^2 + r^2 g_\alpha)\]
is Kähler with respect to the following canonical almost complex structure \(J\) on \(TCM = \mathbb{R} \oplus \langle R \rangle \oplus \xi:\)

\[J(r\partial r) = -R, \quad J(R) = r\partial r.\]

Furthermore if \(g_\alpha\) is Einstein, \(M\) is called a Sasakian Einstein manifold.

We record more several lemmas which are well known in Sasakian geometry. These lemmas will be used in the subsequent sections.

**Lemma 2.1.** Let \((M, \alpha, g_\alpha, J)\) be a Sasakian manifold. Then

\[\bar{\nabla}_X R = -JX,\]

and

\[(\bar{\nabla}_X J)(Y) = g(X, Y)R - \alpha(Y)X,\]

for \(X, Y \in TM\), where \(\bar{\nabla}\) is the Levi-Civita connection on \((M, g_\alpha)\).

**Lemma 2.2.** Let \(\Sigma\) be a Legendrian submanifold in a Sasakian Einstein manifold \((M, \alpha, g_\alpha, J)\), then the mean curvature form \(\omega(H, \cdot)|_\Sigma\) defines a closed one form on \(\Sigma\).

For a proof of this lemma we refer to \([10, \text{Proposition A.2}],\) and \([23, \text{lemma 2.8}].\) In fact they proved this result under the weaker assumption that \((M, \alpha, g_\alpha, J)\) is a weakly Sasakian Einstein manifold, where weakly Einstein means that \(g_\alpha\) is Einstein only when restricted to the contact hyperplane.

**Lemma 2.3.** Let \(\Sigma\) be a Legendrian submanifold in a Sasakian manifold \((M, \alpha, g_\alpha, J)\) and \(A\) be the second fundamental form of \(\Sigma\) in \(M\). Then we have

\[g_\alpha(A(X, Y), R) = 0.\]
Proof. For any $X, Y \in T\Sigma$, 
\[
\langle A(X, Y), R \rangle = \langle \bar{\nabla}_X Y, R \rangle = -\langle Y, \bar{\nabla}_X R \rangle = \langle Y, JX \rangle = 0,
\]
where in the third equality we used (2.1).

In particular this lemma implies that the mean curvature $H$ of $\Sigma$ is orthogonal to the Reeb field $R$. This fact is important in our following argument.

Lemma 2.4. For any $Y, Z \in \ker \bar{\alpha}$, we have 
\[
\bar{g}_\sigma(\bar{\nabla}_X (JY), Z) = \bar{g}_\sigma(J\bar{\nabla}_X Y, Z).
\]

A canonical example of Sasakian Einstein manifolds is the standard odd dimensional sphere $S^{2n+1}$.

Example 2.1 (The standard sphere $S^{2n+1}$). Let $\mathbb{C}^n = \mathbb{R}^{2n+2}$ be the Euclidean space with coordinates $(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1})$ and $S^{2n+1}$ be the standard unit sphere in $\mathbb{R}^{2n+2}$. Define 
\[
a_0 = \sum_{j=1}^{n+1} (x_j dy_j - y_j dx_j),
\]
then 
\[
\alpha := a_0|_{S^{2n+1}}
\]
defines a contact one form on $S^{2n+1}$. Assume that $g_0$ is the standard metric on $\mathbb{R}^{2n+2}$ and $J_0$ is the standard complex structure of $\mathbb{C}^n$. We define $g_\sigma = g_0|_{S^{2n+1}}$, then $(S^{2n+1}, \alpha, g_\sigma)$ is a Sasakian Einstein manifold. The contact hyperplane is characterized by 
\[
\ker \alpha_x = \{Y \in T_x S^{2n+1} | \langle Y, J_0 x \rangle = 0 \}.
\]

2.3. Legendrian submanifolds in the unit sphere. Assume $\phi : \Sigma^n \mapsto S^{2n+1} \subseteq \mathbb{C}^{n+1}$ is a Legendrian immersion. Let $B$ be the second fundamental form, $A'$ be the shape operator with respect to the norm vector $v \in T^\perp \Sigma$ and $H$ be the mean curvature vector. The shape operator $A'$ is a symmetric operator on the tangent bundle and satisfies the following Weingarten equations 
\[
\langle B(X, Y), v \rangle = \langle A'(X), Y \rangle, \quad \forall X, Y \in T\Sigma, v \in T^\perp \Sigma.
\]
The Gauss equations, Codazzi equations and Ricci equations are given by 
(2.3) 
\[
R(X, Y, Z, W) = \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle + \langle B(X, Z), B(Y, W) \rangle - \langle B(X, W), B(Y, Z) \rangle,
\]
(2.4) 
\[
(\nabla^\perp_X B)(Y, Z) = (\nabla^\perp_Y B)(X, Z),
\]
\[
R^+(X, Y, \mu, v) = \langle A'^\perp(X), A'^\perp(Y) \rangle - \langle A'^\perp(Y), A'^\perp(X) \rangle,
\]
where $X, Y, Z, W \in T\Sigma, \mu, v \in T^\perp \Sigma$.

Let $\{e_1, e_2\}$ be a local orthonormal frame of $\Sigma$. Then $\{Je_1, Je_2, J\phi\}$ is a local orthonormal frame of the normal bundle $T^\perp \Sigma$, where $J$ is the complex structure of $\mathbb{C}^{n+1}$. Set 
\[
\sigma_{ijk} := \langle B(e_i, e_j), Je_k \rangle, \quad \mu_j := \langle H, Je_j \rangle = \sum_{i=1}^{n} \sigma_{ij},
\]
then by Lemma 2.2, Lemma 2.4 and the Codazzi equation (2.4) we have
\[
\sigma_{ijk} = \sigma_{jik} = \sigma_{ikj}, \quad \sigma_{ijk,l} = \sigma_{ijl,k}.
\]
(2.5)
\[
d\mu = 0, \quad \delta\mu = \text{div}(JH).
\]

Recall that

**Definition 2.6.** \(\Sigma\) is a csL submanifold if it is a critical point of the volume functional among Legendrian submanifolds.

CsL submanifolds satisfy the following Euler-Lagrangian equation ([3, 8]):
\[
\text{div}(JH) = 0.
\]

It is obvious that \(\Sigma\) is csL when \(\Sigma\) is minimal. The following observation is very important for the study of csL submanifolds.

**Lemma 2.5.** \(\Sigma\) is csL iff \(\mu\) is a harmonic 1-form iff \(JH\) is a harmonic vector field.

By using the Bochner formula for harmonic vector fields (cf. [9]), we get

**Lemma 2.6.** If \(\Sigma\) is csL, then
\[
\frac{1}{2} \Delta |H|^2 = |\nabla(JH)|^2 + \text{Ric}(JH, JH).
\]

From (2.6) it is easy to see that we have

**Lemma 2.7.** If \(\Sigma \subset S^5\) is csL and non-minimal, then the zero set of \(H\) is isolate and
\[
\Delta \log |H| = \kappa
\]
provided \(H \neq 0\), where \(\kappa\) is the Gauss curvature of \(\Sigma\).

### 3. Willmore Legendrian surfaces in \(S^5\)

In this section we prove that every complete Willmore Legendrian surface in \(S^5\) must be a minimal surface. Firstly, we rewrite the Willmore operator acting on Legendrian surfaces, i.e., we prove the following

**Proposition 3.1.** Assume that \(\Sigma\) is a Legendrian surface in \(S^5\), then its Willmore operator can be written as
\[
\hat{W}(\Sigma) = \frac{1}{2} \left\{ -J\nabla \text{div}(JH) + B(JH, JH) - \frac{1}{2} |H|^2 H - 2 \text{div}(JH)R \right\}.
\]

In particular, the Euler-Lagrangian equation of Willmore Legendrian surfaces in \(S^5\) is
\[
-\nabla \text{div}(JH) + B(JH, JH) - \frac{1}{2} |H|^2 H - 2 \text{div}(JH)R = 0.
\]

**Proof.** Let \(\{\nu_1, \nu_2, R\}\) be a local orthonormal frame of the normal bundle of \(\Sigma\), then the Willmore equation (1.1) can be rewritten as
\[
\Delta' H + \sum_{\alpha} \langle A^\nu, A^H \rangle \nu_\alpha - \frac{1}{2} |H|^2 H = 0.
\]

Note that by (2.2) we have
\[
\nabla_X(JY) = J\nabla_X Y + g(X, Y)R
\]
for $X, Y \in \Gamma (T\Sigma)$. Choose a local orthonormal frame field around $p$ with $\nabla e_i\big|_p = 0$, then

$$J\nabla_{e_i}(JH) = \nabla_{e_i}(J(JH)) - g(e_i, JH)\mathbf{R}$$

$$= -\nabla_{e_i}H - g(e_i, JH)\mathbf{R}$$

and

$$J\nabla_{e_i}(\nabla_{e_i}(JH)) = \nabla_{e_i}(J\nabla_{e_i}(JH)) - g(e_i, \nabla_{e_i}JH)\mathbf{R}$$

$$= \nabla_{e_i}(-\nabla_{e_i}H - g(e_i, JH)\mathbf{R}) - g(e_i, \nabla_{e_i}JH)\mathbf{R}$$

$$= -\nabla_{e_i}\nabla_{e_i}H - 2g(e_i, \nabla_{e_i}(JH))\mathbf{R} - g(e_i, JH)\left(\nabla_{e_i}\mathbf{R}\right)$$

$$= -\nabla_{e_i}\nabla_{e_i}H - 2g(e_i, \nabla_{e_i}(JH))\mathbf{R} - g(e_i, JH)\mathbf{R}$$

where in the last equality we also used (2.1). Therefore we obtain

$$\Delta' H = -J\Delta(JH) - H - 2\text{ div}(JH)\mathbf{R},$$

which implies that $\Sigma$ satisfies the following equation

$$-J\Delta(JH) + \sum_\alpha \langle A^\alpha, A^\alpha \rangle \nu_\alpha - \frac{1}{2} (2 + |H|^2) H - 2\text{ div}(JH)\mathbf{R} = 0.$$

In addition, by Lemma 2.2, the dual one form of $JH$ is harmonic. By the Ricci identity we have

$$\Delta(JH) = \nabla \text{ div}(JH) + \kappa JH.$$

The Proposition is then a consequence of the following Claim together with above two identities.

Claim.

$$2\kappa = 2 + |H|^2 - |B|^2,$$

$$\sum_\alpha \langle A^\alpha, A^\alpha \rangle \nu_\alpha - \frac{1}{2} |B|^2 H = B(JH, JH) - \frac{1}{2} |H|^2 H.$$

Proof: The first equation is obvious by the Gauss equation (2.3). The second equation can be proved by the Gauss equation (2.3) and the tri-symmetry of the tensor $\sigma$ (see (2.5)). To be precise, for every tangent vector field $Z \in T\Sigma$ we have

$$\langle B(JH, JH), JZ \rangle - \sum_\alpha \langle A^\alpha, A^\alpha \rangle \nu_\alpha, JZ \rangle$$

$$= -\langle B(Z, JH), H \rangle - \sum_{i,j} \langle B(e_i, e_j), JZ \rangle \langle B(e_i, e_j), H \rangle$$

$$= \sum_{i,j} \langle B(Z, e_j), Je_i \rangle \langle B(JH, e_j), e_i \rangle - \langle B(Z, JH), H \rangle$$

$$= \sum_j \langle B(Z, e_j), B(JH, e_j) \rangle - \langle B(Z, JH), H \rangle$$

$$= Ric(Z, JH) - \langle Z, JH \rangle$$

$$= (\kappa - 1) \langle Z, JH \rangle$$
\[ \frac{1}{2} (|H|^2 - |B|^2) \langle Z, JH \rangle \]
\[ = \frac{1}{2} (|B|^2 - |H|^2) \langle H, JZ \rangle. \]

This completes the proof of the second equation. \( \square \)

Now we are in position to prove the following

**Theorem 3.2.** Every complete Willmore Legendrian surface in \( S^5 \) is a minimal surface.

**Proof.** We prove by a contradiction argument. Assume that \( \Sigma \) is a complete Willmore Legendrian surface in \( S^5 \) which is not a minimal surface. If \( H \neq 0 \), then let \( \{ e_1 = \frac{JH}{|H|}, e_2 \} \) be a local orthonormal frame field of \( \Sigma \). From (3.1) we have
\[
B(e_1, e_1) = -\frac{1}{2} |H| J e_1,
\]
which also implies that
\[
B(e_2, e_2) = -\frac{1}{2} |H| J e_1, \quad h_{11}^2 = 0.
\]
Then by the Gauss equation (2.3) we have
\[
\kappa = 1 + \langle B(e_1, e_1), B(e_2, e_2) \rangle - |B(e_1, e_2)|^2
\]
\[
= 1 + \frac{1}{4} |H|^2 - |h_{12}^2|^2 - |h_{12}^2|^2
\]
\[
= 1 + \frac{1}{4} |H|^2 - |h_{12}^2|^2
\]
\[
= 1.
\]
Since \( \Sigma \) is a Willmore Legendrian surface, from (3.1) we see that \( \text{div}(JH) = 0 \). By Lemma 2.7 the minimal points of \( \Sigma \) are discrete and so the Gauss curvature of \( \Sigma \) equals 1 everywhere on \( \Sigma \), therefore \( \Sigma \) is compact by Bonnet-Myers theorem. Apply Lemma 2.6 to obtain that on \( \Sigma \)
\[
\frac{1}{2} \Delta |H|^2 = |\nabla (JH)|^2 + |H|^2.
\]
Then the maximum principle implies that \( H \equiv 0 \) which is a contradiction. Therefore \( \Sigma \) is a minimal surface. \( \square \)

4. **Examples of csL Willmore surfaces in \( S^5 \)**

From the definition we see that complete Willmore Legendrian surfaces, which are minimal surface by Theorem 3.2 in the last section, are trivial examples of csL Willmore surfaces in \( S^5 \). Thus it is very natural and important to construct nonminimal csL Willmore surfaces in \( S^5 \). This will be done in this section by analyzing a very close relationship between csL Willmore surfaces and csL surfaces in \( S^5 \).

Assume that \( \Sigma \) is a csL Willmore surface in \( S^5 \), then since the variation vector field on \( \Sigma \) under Legendrian deformations can be written as \( J \nabla u + \frac{1}{2} u R \) for smooth function \( u \) on \( \Sigma \) (cf.
Therefore we can find the following examples of csL Willmore surfaces from csL surfaces in $\mathbb{S}^5$.

**Remark 4.1.** Note that the coefficient of the Euler-Lagrangian equation (4.1) for csL Willmore surfaces in $\mathbb{S}^5$ is slightly different with (1.7) in [11]. That is because here we use the notation $H = \text{trace } B$, whereas in [11] we defined $H = \frac{1}{2} \text{trace } B$.

Then by (3.1), $\Sigma$ satisfies the following equation.

$$\text{div} \left( \nabla \text{div}(J H) + J B(J H, J H) - \frac{1}{2} |H|^2 J H - 4 J H \right) = 0.$$ 

In addition, by the four-symmetric of $(\tau_{ijkl})$ (see (2.5)), a direct computation shows

$$\text{div}(J B(J H, J H)) = 2 \text{ trace } \langle B(\cdot, \nabla(J H)), H \rangle + \frac{1}{2} \nabla_{J H} |H|^2.$$ 

Therefore $\Sigma$ satisfies the following equation

$$\Delta \text{div}(J H) + 2 \text{ trace } \langle B(\cdot, \nabla(J H)), H \rangle - \frac{1}{2} |H|^2 \text{div}(J H) - 4 \text{div}(J H) = 0.$$ 

**Proposition 4.1.** Assume that $\Sigma$ is a csL surface in $\mathbb{S}^5$ and trace $\langle B(\cdot, \nabla(J H)), H \rangle = 0$, then $\Sigma$ is a csL Willmore surface.

With the aid of Proposition 4.1, we can find the following examples of csL Willmore surfaces from csL surfaces in $\mathbb{S}^5$. Firstly, according to Proposition 4.1, all closed Legendrian surfaces with parallel tangent vector field $J H$, which are exactly minimal surfaces or the Calabi tori (cf. [12, Proposition 3.2]), are csL Willmore surfaces. For reader’s convenience, we give some detailed computations as follows.

**Example 4.1** (Calabi tori). For every four nonzero real numbers $r_1, r_2, r_3, r_4$ with $r_1^2 + r_2^2 = r_3^2 + r_4^2 = 1$, the Calabi torus $\Sigma$ is a csL surface in $\mathbb{S}^5$ defined as follows.

$$F : [0,1] \times [0,1] \rightarrow \mathbb{S}^5, \quad (t, s) \mapsto \left( r_1 r_3 \exp \left( \sqrt{-1} \left( \frac{r_2}{r_1} t + \frac{r_4}{r_3} s \right) \right), r_1 r_4 \exp \left( \sqrt{-1} \left( \frac{r_2}{r_1} t - \frac{r_3}{r_4} s \right) \right), r_2 \exp \left( - \sqrt{-1} \frac{r_4}{r_2} t \right) \right).$$

Denote

$$\phi_1 = \exp \left( \sqrt{-1} \left( \frac{r_2}{r_1} t + \frac{r_4}{r_3} s \right) \right), \quad \phi_2 = \exp \left( \sqrt{-1} \left( \frac{r_2}{r_1} t - \frac{r_3}{r_4} s \right) \right), \quad \phi_3 = \exp \left( - \sqrt{-1} \frac{r_4}{r_2} t \right),$$

and

$$\phi_4 = \exp \left( \sqrt{-1} \left( \frac{r_2}{r_1} t + \frac{r_3}{r_4} s \right) \right), \quad \phi_5 = \exp \left( \sqrt{-1} \left( - \frac{r_3}{r_4} t + \frac{r_2}{r_1} s \right) \right), \quad \phi_6 = \exp \left( - \sqrt{-1} \frac{r_4}{r_2} t \right).$$

Then

$$\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 = 1.$$

Therefore $\Sigma$ is a csL Willmore surface.

We can find the following examples of csL Willmore surfaces from csL surfaces in $\mathbb{S}^5$.

**Example 4.1 (Calabi tori).** For every four nonzero real numbers $r_1, r_2, r_3, r_4$ with $r_1^2 + r_2^2 = r_3^2 + r_4^2 = 1$, the Calabi torus $\Sigma$ is a csL surface in $\mathbb{S}^5$ defined as follows.

$$F : [0,1] \times [0,1] \rightarrow \mathbb{S}^5, \quad (t, s) \mapsto \left( r_1 r_3 \exp \left( \sqrt{-1} \left( \frac{r_2}{r_1} t + \frac{r_4}{r_3} s \right) \right), r_1 r_4 \exp \left( \sqrt{-1} \left( \frac{r_2}{r_1} t - \frac{r_3}{r_4} s \right) \right), r_2 \exp \left( - \sqrt{-1} \frac{r_4}{r_2} t \right) \right).$$

Denote

$$\phi_1 = \exp \left( \sqrt{-1} \left( \frac{r_2}{r_1} t + \frac{r_4}{r_3} s \right) \right), \quad \phi_2 = \exp \left( \sqrt{-1} \left( \frac{r_2}{r_1} t - \frac{r_3}{r_4} s \right) \right), \quad \phi_3 = \exp \left( - \sqrt{-1} \frac{r_4}{r_2} t \right),$$

and

$$\phi_4 = \exp \left( \sqrt{-1} \left( \frac{r_2}{r_1} t + \frac{r_3}{r_4} s \right) \right), \quad \phi_5 = \exp \left( \sqrt{-1} \left( - \frac{r_3}{r_4} t + \frac{r_2}{r_1} s \right) \right), \quad \phi_6 = \exp \left( - \sqrt{-1} \frac{r_4}{r_2} t \right).$$

Then

$$\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 = 1.$$
then \( F(t, s) = (r_1 r_3 \phi_1, r_1 r_4 \phi_2, r_2 \phi_3) \). Since
\[
\frac{\partial F}{\partial t} = \left( \sqrt{-1} r_2 r_3 \phi_1, \sqrt{-1} r_2 r_4 \phi_2, -\sqrt{-1} r_1 \phi_3 \right),
\]
\[
\frac{\partial F}{\partial s} = \left( \sqrt{-1} r_1 r_4 \phi_1, -\sqrt{-1} r_1 r_3 \phi_2, 0 \right),
\]
the induced metric in \( \Sigma \) is given by
\[
g = dt^2 + r_1^2 ds^2.
\]
Let \( E_1 = \frac{\partial F}{\partial t}, E_2 = \frac{1}{r_1} \frac{\partial F}{\partial s} \), then \( \{E_1, E_2, \nu_1 = \sqrt{-1} E_1, \nu_2 = \sqrt{-1} E_2, R = -\sqrt{-1} F \} \) is a local orthonormal frame of \( S^3 \) such that \( \{E_1, E_2\} \) is a local orthonormal tangent frame and \( R \) is the Reeb field. A direct calculation yields
\[
\frac{\partial \nu_1}{\partial t} = \left( -\sqrt{-1} \frac{r_2^2 r_3}{r_1} \phi_1, -\sqrt{-1} \frac{r_2^2 r_4}{r_1} \phi_2, -\sqrt{-1} \frac{r_1^2}{r_2} \phi_3 \right),
\]
\[
\frac{\partial \nu_1}{\partial s} = \left( -\sqrt{-1} \frac{r_2 r_3^2}{r_4} \phi_1, -\sqrt{-1} \frac{r_2 r_4^2}{r_3} \phi_2, 0 \right),
\]
\[
\frac{\partial \nu_2}{\partial t} = \left( -\sqrt{-1} \frac{r_2^2 r_4}{r_1} \phi_1, -\sqrt{-1} \frac{r_2^2 r_3}{r_1} \phi_2, 0 \right),
\]
\[
\frac{\partial \nu_2}{\partial s} = \left( -\sqrt{-1} \frac{r_1^2 r_4}{r_3} \phi_1, -\sqrt{-1} \frac{r_1^2 r_3}{r_4} \phi_2, 0 \right),
\]
\[
\frac{\partial R}{\partial t} = (r_2 r_3 \phi_1, r_2 r_4 \phi_2, -r_1 \phi_3),
\]
\[
\frac{\partial R}{\partial s} = (r_1 r_4 \phi_1, -r_1 r_3 \phi_2, 0).
\]

Hence,
\[
A^\nu_1 = - \Re \langle dF, d\nu_1 \rangle = \left( \frac{r_2}{r_1} - \frac{r_1}{r_2} \right) dt^2 + r_1 r_2 ds^2,
\]
\[
A^\nu_2 = - \Re \langle dF, d\nu_2 \rangle = 2r_2 dt ds + r_1 \left( \frac{r_4}{r_3} - \frac{r_3}{r_4} \right) ds^2,
\]
\[
A^R = 0.
\]

Thus
\[
H = \left( \frac{2r_2}{r_1} - \frac{r_1}{r_2} \right) \nu_1 + \frac{1}{r_1} \left( \frac{r_4}{r_3} - \frac{r_3}{r_4} \right) \nu_2.
\]
Moreover \( E_1 \) and \( E_2 \) are two parallel tangent vector field. It is obvious that \( \Sigma \) is a csL Willmore surface.

Secondly, we give some examples that \( JH \) is not parallel. Mironov [17] constructed some new csL surfaces in \( S^3 \). We can verify that Mironov’s examples are in fact csL Willmore surfaces.
Example 4.2 (Mironov’s examples [17]). Let $F : \Sigma^2 \hookrightarrow S^5$ be an immersion. Then $F$ is a Legendrian immersion if

$$\langle F_x, F \rangle = \langle F_y, F \rangle = 0.$$ 

Here $\{x, y\}$ is a local coordinates of $\Sigma$ and $\langle , \rangle$ stands for the hermitian inner product in $\mathbb{C}^3$. Set

$$G = \begin{pmatrix} F \\ F_x \\ F_y \end{pmatrix},$$

then

$$GG^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \langle F_x, F_x \rangle & \langle F_x, F_y \rangle \\ 0 & \langle F_y, F_x \rangle & \langle F_y, F_y \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}.$$ 

where $g$ is a real positive matrix which is the induce metric of $\Sigma$. There is a hermitian matrix $\Theta$ such that

$$G = \left( \begin{array}{cc} 1 & 0 \\ 0 & g^{1/2} \end{array} \right) e^{-\sqrt{-1} \Theta}.$$

We compute

$$GG^T_x = \begin{pmatrix} 0 & -\langle F_x, F_x \rangle & -\langle F_x, F_y \rangle \\ \langle F_x, F_x \rangle & \langle F_x, F_{xx} \rangle & \langle F_x, F_{xy} \rangle \\ \langle F_y, F_x \rangle & \langle F_y, F_{xx} \rangle & \langle F_y, F_{xy} \rangle \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{g} \end{pmatrix} e^{-\sqrt{-1} \Theta} \left( e^{-\sqrt{-1} \Theta} \right)_x \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{g} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{g} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Hence

$$\mathcal{R} \left( \sqrt{-1} GG^T_x \right) = \mathcal{R} \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & \langle F_x, F_{xx} \rangle & \langle F_x, F_{xy} \rangle \\ 0 & \langle F_y, F_{xx} \rangle & \langle F_y, F_{xy} \rangle \end{pmatrix} \right)$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & A^{-1} F_x \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{g} \end{pmatrix} \mathcal{R} \left( \sqrt{-1} e^{\sqrt{-1} \Theta} \left( e^{-\sqrt{-1} \Theta} \right) \right) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{g} \end{pmatrix},$$

which implies

$$\begin{pmatrix} 0 & 0 \\ 0 & g^{-1/2} A^{-1} F_x g^{1/2} \end{pmatrix} = \mathcal{R} \left( \sqrt{-1} e^{\sqrt{-1} \Theta} \left( e^{-\sqrt{-1} \Theta} \right) \right).$$

Similarly,

$$\begin{pmatrix} 0 & 0 \\ 0 & g^{-1/2} A^{-1} F_y g^{1/2} \end{pmatrix} = \mathcal{R} \left( \sqrt{-1} e^{\sqrt{-1} \Theta} \left( e^{-\sqrt{-1} \Theta} \right) \right).$$

The Lagrangian angle is then given by $\theta = tr \mathcal{R} \Theta$. The above discussion implies that

$$J \nabla \theta = H.$$
Let $a, b, c$ are three positive constants and consider the following immersion

$$F : \mathbb{S}^1 \times \mathbb{S}^1 \mapsto \mathbb{S}^3, \\
(x, y) \mapsto (\phi(x)e^{-\sqrt{\tan}y}, \psi(x)e^{-\sqrt{\tanh}y}, \zeta(x)e^{-\sqrt{\tan}y}),$$

where

$$\phi(x) = \sqrt{\frac{c}{a + c}} \sin x,$$

$$\psi(x) = \sqrt{\frac{c}{b + c}} \cos x,$$

$$\zeta(x) = \frac{a \sin^2 x}{a + c} + \frac{b \cos^2 x}{b + c} = \sqrt{\frac{ab + u(x)}{(a + c)(b + c)}},$$

where

$$u(x) = \frac{c(a + b + (b - a) \cos(2x))}{2}.$$

One can check that $F$ is a Legendrian immersion. Denote $\Sigma := F(\mathbb{S}^1 \times \mathbb{S}^1)$. Notice that

$$F_x = \left(\sqrt{\frac{c}{a + c}} \cos x e^{-\sqrt{\tan}y}, -\sqrt{\frac{c}{b + c}} \sin x e^{-\sqrt{\tanh}y}, \frac{-c(b - a) \sin(2x)}{2 \sqrt{(a + c)(b + c)(ab + u(x))}} e^{-\sqrt{\tan}y}\right),$$

$$F_y = \left(\sqrt{-1}a\phi(x)e^{-\sqrt{\tan}y}, \sqrt{-1}b\psi(x)e^{-\sqrt{\tanh}y}, -\sqrt{-1}c\zeta(x)e^{-\sqrt{\tan}y}\right).$$

The induced metric $g$ is given by

$$g = \left[\frac{c \cos^2 x}{a + c} + \frac{c \sin^2 x}{b + c} + \frac{c^2(b - a)^2 \sin^2(2x)}{4(a + c)(b + c)(ab + u(x))}\right] dx^2$$

$$+ \left[\frac{a^2 \sin^2 x}{a + c} + \frac{b^2 \cos^2 x}{b + c} + \frac{c^2(a \sin^2 x)}{a + c} + \frac{b \cos^2 x}{b + c}\right] dy^2$$

$$= \frac{(ab + u(x))}{dx^2 + u(x) dy^2}$$

$$=: e^{2p(x)} dx^2 + e^{2q(x)} dy^2.$$

A strait forward calculation yields that

$$A^{-1}F_x = \Re \begin{pmatrix} 0 & \sqrt{-1}\langle F_x, F_x \rangle \\
-\sqrt{-1}\langle F_y, F_x \rangle & 0 \end{pmatrix} = \begin{pmatrix} 0 & c(1 - e^{2p(x)}) \\
c(1 - e^{2p(x)}) & 0 \end{pmatrix},$$

$$A^{-1}F_y = \Re \begin{pmatrix} \sqrt{-1}\langle F_x, F_y \rangle & 0 \\
0 & \sqrt{-1}\langle F_y, F_y \rangle \end{pmatrix} = \begin{pmatrix} c(1 - e^{2p(x)}) & 0 \\
(a + b - c)e^{2q(x)} & 0 \end{pmatrix}.$$

We get

$$\Re \begin{pmatrix} \sqrt{-1}e^{\sqrt{-1}\theta}(e^{-\sqrt{-1}\theta}) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & \frac{abc}{u \sqrt{ab + u}} \\
0 & \frac{abc}{u \sqrt{ab + u}} & 0 \end{pmatrix},$$
\[
\Re \left( \sqrt{-1} e^{\sqrt{-1}\Theta} \left( e^{-\sqrt{-1}\Theta} \right)_y \right) = \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{abc}{u} & 0 \\
0 & 0 & (a + b - c) - \frac{abc}{u}
\end{pmatrix}.
\]

Thus
\[
H^{\sqrt{-1}F_i} = 0, \quad H^{\sqrt{-1}F_j} = a + b - c.
\]

We get
\[
H = \frac{a + b - c}{u(x)} \sqrt{-1} \frac{\partial}{\partial y},
\]
and
\[
\nabla_{\partial_y} \left( \sqrt{-1} H \right) = \frac{(a + b - c)u_x}{2u^2} \frac{\partial}{\partial y}, \quad \nabla_{\partial_x} \left( \sqrt{-1} H \right) = \frac{(ab + u)(a + b - c)u_y}{2u^2} \frac{\partial}{\partial x}.
\]

In particular
\[
\text{div} \left( \sqrt{-1} H \right) = 0.
\]

Hence \(\Sigma\) is csL. Moreover
\[
\sum_{i=1}^{2} \langle B(e_i, \nabla e_i(JH)), H \rangle = 0.
\]

Therefore, \(\Sigma\) is a csL Willmore surface in \(S^5\).

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Complete Willmore Legendrian surfaces in $S^5$ are minimal Legendrian surfaces✩

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Abstract

In this paper we continue to consider Willmore Legendrian surfaces and csL Willmore surfaces in $S^5$, notions introduced by Luo in [9]. We will prove that every complete Willmore Legendrian surface in $S^5$ is minimal and find nontrivial examples of csL Willmore surfaces in $S^5$.

Keywords: Willmore Legendrian surface, csL surface, csL Willmore surface

2010 MSC: 53C24, 53C42, 53C44

1. Introduction

Let $Σ$ be a Riemann surface, $(M^n, g) = S^n$ or $\mathbb{R}^n (n ≥ 3)$ the unit sphere or the Euclidean space with standard metrics and $f$ an immersion from $Σ$ to $M$. Let $B$ be the second fundamental form of $f$ with respect to the induced metric, $H$ the mean curvature vector field of $f$ defined by

$$H = \text{tr} \, B,$$

$κ_M$ the Gauss curvature of $df(TΣ)$ with respect to the ambient metric $g$ and $dμ_f$ the area element on $f(Σ)$. The Willmore functional of the immersion $f$ is then defined by

$$W(f) = \int_{Σ} \left( \frac{1}{4} |H|^2 + κ_M \right) dμ_f.$$

For a smooth and compactly supported variation $f : Σ×I \mapsto M$ with $φ = \partial_t f$ we have the following first variational formula (cf. [22, 23])

$$\frac{d}{dt} W(f) = \int_{Σ} \left( \langle \vec{W}(f), φ \rangle \right) dμ_f,$$

with $\vec{W}(f) = \sum_{α=3}^n \vec{W}(f)^α e_α$, where $\{e_α : 3 ≤ α ≤ n\}$ is a local orthonormal frame of the normal bundle of $f(Σ)$ in $M$ and

$$\vec{W}(f)^α = \frac{1}{2} \left( ΔH^α + \sum_{i,j,k} h_{ij}^α h_{kj}^α H^β - 2 |H|^2 H^α \right), \quad 3 ≤ α ≤ n,$$

where $h_{ij}^α$ is the component of $B$ and $H^α$ is the trace of $(h_{ij}^α)$.

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A smooth immersion $f : \Sigma \mapsto M$ is called a Willmore immersion, if it is a critical point of the Willmore functional $W$. In other words, $f$ is a Willmore immersion if and only if it satisfies
\[
\Delta H^\alpha + \sum_{i,j} h^i_j h^j_i H^\alpha - 2 |H|^2 H^\alpha = 0, \quad 3 \leq \alpha \leq n.
\] (1.1)

When $(M, g) = \mathbb{R}^3$, Willmore [25] proved that the Willmore energy of closed surfaces are larger than or equal to $4\pi$ and equality holds only for round spheres. When $\Sigma$ is a torus, Willmore conjectured that the minimum is $2\pi^2$ and it is attained only by the Clifford torus, up to a conformal transformation of $\mathbb{R}^3$ [24, 6], which was verified by Marques and Neves in [13]. When $(M, g) = \mathbb{R}^4$, Simon [20], combined with the work of Bauer and Kuwert [1], proved the existence of an embedded surface which minimizes the Willmore functional among closed surfaces of prescribed genus. Motivated by these mentioned papers, Minicozzi [14] proved the existence of an embedded torus which minimizes the Willmore functional in a smaller class of Lagrangian tori in $\mathbb{R}^4$. In the same paper Minicozzi conjectured that the Clifford torus minimizes the Willmore functional in its Hamiltonian isotropic class, which he verified has a close relationship with Oh’s conjecture [17, 18]. We should also mention that before Minicozzi, Castro and Urbano proved that the Whitney sphere in $\mathbb{R}^4$ is the only minimizer for the Willmore functional among closed Lagrangian sphere. This result was further generalized by Castro and Urbano in [4] where they proved that the Whitney sphere is the only closed Willmore Lagrangian sphere (a Lagrangian sphere which is also a Willmore surface) in $\mathbb{R}^4$. Examples of Willmore Lagrangian tori (Lagrangian tori which also are Willmore surfaces) in $\mathbb{R}^4$ were constructed by Pinkall [19] and Castro and Urbano [5]. Motivated by these works, Luo and Wang [11] considered the variation of the Willmore functional among Lagrangian surfaces in $\mathbb{R}^4$ or variation of a Lagrangian surface of the Willmore functional among its Hamiltonian isotropic class in $\mathbb{R}^4$, whose critical points are called LW or HW surfaces respectively. We should also mention that Willmore type functional of Lagrangian surfaces in $\mathbb{C}P^2$ were studied by Montiel and Urbano [16] and Ma, Mironov and Zuo [12].

Inspired by the study of the Willmore functional for Lagrangian surfaces in $\mathbb{R}^4$, Luo [9] naturally considered the Willmore functional of Legendrian surfaces in $S^5$.

**Definition 1.1.** A Willmore and Legendrian surface in $S^5$ is called a Willmore Legendrian surface.

**Definition 1.2.** A Legendrian surface in $S^5$ is called a contact stationary Legendrian Willmore surface (in short, a csL Willmore surface) if it is a critical point of the Willmore functional under contact deformations.

Luo [9] proved that Willmore Legendrian surfaces in $S^5$ are csL surfaces (see **Definition 2.1**). In this paper, we continue to study Willmore Legendrian surfaces and csL Willmore surfaces in $S^5$. Surprisingly we will prove that every complete Willmore Legendrian surface in $S^5$ must be a minimal surface (**Theorem 2.5**). We also find nontrivial examples of csL Willmore surfaces from csL surfaces in $S^5$ for the first time, by exploring relationships between them (**Proposition 3.1**).

The method here we used to find nontrivial csL Willmore surfaces in $S^5$ in Section 4 should also be useful in discovering nontrivial HW surfaces in $\mathbb{R}^4$ introduced by Luo and Wang in [11]. We will consider this problem in the future.

2. Willmore Legendrian surfaces in $S^5$

In this section we will prove that every complete Willmore Legendrian surfaces in $S^5$ is minimal. Firstly we briefly record several facts about Legendrian surfaces in $S^5$. We refer the reader to consult [2] for more materials about the contact geometry.

Let $S^5$, the 5-dimensional unit sphere, be the standard Sasakian Einstein manifold with contact one form $\alpha$, almost complex structure $J$, Reed field $R$ and canonical metric $g$. Let $\Sigma$ be a closed surface of $S^5 \subset C^5$. We say that $\Sigma$ is Legendrian if

$$JT \Sigma \subset T^\alpha \Sigma, \quad JF \in \Gamma (T^\alpha \Sigma)$$
where $F : \Sigma \rightarrow S^5$ is the position vector and $T\Sigma, T^\perp \Sigma$ are tangent and normal bundles of $\Sigma$ respectively. We say that $\Sigma$ is a minimal Legendrian surface of $S^5$ if $\Sigma$ is a minimal and Legendrian surface of $S^5$. Define

$$\sigma(X, Y, Z) := \langle B(X, Y), JZ \rangle, \quad \forall X, Y, Z \in T\Sigma.$$  

The Weingarten equation implies that

$$\sigma(X, Y, Z) = \sigma(Y, X, Z).$$

Moreover, by definition, one can check that $\sigma$ is a three order symmetric tensor, i.e.,

$$\sigma(X, Y, Z) = \sigma(Y, X, Z) = \sigma(X, Z, Y).$$  \hspace{1cm} (2.1)

The Gauss equation, Codazzi equation and Ricci equation becomes

$$R(X, Y, Z, W) = \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle + \sigma(X, Z, e_i) \sigma(Y, W, e_i) - \sigma(X, W, e_i) \sigma(Y, Z, e_i),$$  

$$\nabla X \sigma(Y, Z, W) = \nabla Y \sigma(X, Z, W),$$  

where $\{e_i\}$ is an orthonormal basis of $T\Sigma$. The Codazzi equation implies

$$\nabla X \sigma(Y, Z, W) = \nabla Y \sigma(X, Z, W) = \nabla X \sigma(Z, Y, W) = \nabla Y \sigma(Y, W, Z),$$  \hspace{1cm} (2.3)

i.e., $\nabla \sigma$ is a fourth order symmetric tensor.

Recall that

**Definition 2.1.** $\Sigma$ is a csL surface in $S^5$ if it is a critical point of the volume functional among Legendrian surfaces.

CsL surfaces in $S^5$ satisfy the following Euler-Lagrange equation ([3, 7]):

$$\text{div}(JH) = 0.$$  

It is obvious that $\Sigma$ is csL in $S^5$ when $\Sigma$ is minimal. The following observation is very important for the study of csL surfaces.

**Lemma 2.1.** $\Sigma$ is csL in $S^5$ iff $JH$ is a harmonic vector field.

By using the Bochner formula for harmonic vector fields (cf. [8]), we get

**Lemma 2.2.** If $\Sigma$ is csL in $S^5$, then

$$\frac{1}{2} \Delta |H|^2 = |\nabla (JH)|^2 + \text{Ric}(JH, JH).$$

From Lemma 2.2 it is easy to see that we have

**Lemma 2.3.** If $\Sigma \subset S^5$ is csL and non-minimal, then the zero set of $H$ is isolate and

$$\Delta \log |H| = \kappa$$

provided $H \neq 0$, where $\kappa$ is the Gauss curvature of $\Sigma$.

We then prove that every complete Willmore Legendrian surface in $S^5$ must be a minimal surface. Firstly, we rewrite the Willmore operator acting on Legendrian surfaces, i.e., we prove the following
Proposition 2.4. Assume that \( \Sigma \) is a Legendrian surface in \( S^5 \), then its Willmore operator can be written as

\[
\tilde{W}(\Sigma) = \frac{1}{2} \left\{ -J \nabla \text{div}(JH) + B(JH, JH) - \frac{1}{2} |H|^2 H - 2 \text{div}(JH)R \right\}.
\]

In particular, the Euler-Lagrange equation of Willmore Legendrian surfaces in \( S^5 \) is

\[
-J \nabla \text{div}(JH) + B(JH, JH) - \frac{1}{2} |H|^2 H - 2 \text{div}(JH)R = 0. \tag{2.4}
\]

Proof. Let \( \{v_1, v_2, R\} \) be a local orthonormal frames of the normal bundle of \( \Sigma \), then the Willmore equation (1.1) can be rewritten as

\[
\Delta^\nu H + \sum_a \langle A^a, A^H \rangle v_a - \frac{1}{2} |H|^2 H = 0.
\]

Note that by (2.8) in [9] we have

\[
\nabla_X^e (JY) = (\tilde{\nabla}_X^e (JY))^\nu = ((\tilde{\nabla}_X^e JY + J\tilde{\nabla}_X^e Y)^\nu = J \nabla_X Y + g(X, Y)R
\]

for \( X, Y \in \Gamma(T\Sigma) \), where \( \tilde{\nabla} \) denotes the covariant derivative of \( S^5 \). Choose a local orthonormal frame field around \( p \) with \( \nabla_{e_i} e_j \mid_p = 0 \), then

\[
J \nabla_{e_i} (JH) = \nabla_{e_i}^e (J(JH)) - g(e_i, JH)R
= -\nabla_{e_i}^e H - g(e_i, JH)R
\]

and

\[
J \nabla_{e_i} (\nabla_{e_i} (JH)) = \nabla_{e_i}^e (J \nabla_{e_i} (JH)) - g(e_i, \nabla_{e_i} JH)R
= \nabla_{e_i}^e (-\nabla_{e_i}^e H - g(e_i, JH)R - g(e_i, \nabla_{e_i} JH)R
= -\nabla_{e_i}^e H - 2g(e_i, \nabla_{e_i} (JH)R - g(e_i, JH)R)
\]

where in the last equality we used (2.7) in [9]. Therefore we obtain

\[
\Delta^\nu H = -J \Delta(JH) - H - 2 \text{div}(JH)R,
\]

which implies that \( \Sigma \) satisfies the following equation

\[
-J \Delta(JH) + \sum \langle A^a, A^H \rangle v_a - \frac{1}{2} \left( 2 + |H|^2 \right) H - 2 \text{div}(JH)R = 0.
\]

In addition, by [9, Lemma 2.9], the dual one form of \( JH \) is closed, thus by the Ricci identity we have

\[
\Delta(JH) = \nabla \text{div}(JH) + \kappa JH.
\]

The Proposition is then a consequence of the following Claim together with above two identities.

Claim.

\[
2\kappa = 2 + |H|^2 - |B|^2;
\]

\[
\sum \langle A^a, A^H \rangle v_a - \frac{1}{2} |B|^2 H = B(JH, JH) - \frac{1}{2} |H|^2 H.
\]
Proof. The first equation is obvious by the Gauss equation (2.2). The second equation can be proved by the Gauss equation (2.2) and the tri-symmetry of the tensor $\sigma$ (see (2.1)). To be precise, for every tangent vector field $Z \in T\Sigma$ we have

$$\langle B(JH, JH), JZ \rangle - \sum_{i,j} \langle A_{ij} JH, JZ \rangle = -\langle B(Z, HJ), H \rangle - \sum_i \langle B(Z, e_i), (B(JH, e_i)) - \langle B(Z, HJ), H \rangle$$

This completes the proof of the second equation.

Now we are in position to prove the following

**Theorem 2.5.** Every complete Willmore Legendrian surface in $S^5$ is a minimal surface.

**Proof.** We prove by a contradiction argument. Assume that $\Sigma$ is a complete Willmore Legendrian surface in $S^5$ which is not a minimal surface. If $H \neq 0$, then let $\{e_1 = \frac{JH}{|H|}, e_2\}$ be a local orthonormal frame field of $T\Sigma$. From (2.4) we have

$$B(e_1, e_1) = -\frac{1}{2} |H| Je_1,$$

which also implies that

$$B(e_2, e_2) = -\frac{1}{2} |H| Je_1, \quad h_{11}^2 = 0.$$

Then by the Gauss equation (2.2) we have

$$\kappa = 1 + \langle B(e_1, e_1), B(e_2, e_2) \rangle - |B(e_1, e_2)|^2$$

$$= 1 + \frac{1}{4} |H|^2 - |h_{12}|^2 - |h_{22}|^2$$

$$= 1 + \frac{1}{4} |H|^2 - |h_{22}|^2$$

$$= 1.$$

Since $\Sigma$ is a Willmore Legendrian surface, from (2.4) we see that $\text{div}(JH) = 0$. By Lemma 2.3 the minimal points of $\Sigma$ are discrete and so the Gauss curvature of $\Sigma$ equals one everywhere on $\Sigma$, therefore $\Sigma$ is compact by Bonnet-Myers theorem. Apply Lemma 2.2 to obtain that on $\Sigma$

$$\frac{1}{2} \Delta |H|^2 = |\nabla(JH)|^2 + |H|^2.$$

Then the maximum principle implies that $H \equiv 0$, which is a contradiction. Therefore $\Sigma$ is a minimal Legendrian surface in $S^5$. 

□
3. Examples of csL Willmore surfaces in $\mathbb{S}^5$

From the definition we see that complete Willmore Legendrian surfaces, which are minimal surfaces by Theorem 2.5 in the last section, are trivial examples of csL Willmore surfaces in $\mathbb{S}^5$. Thus it is very natural and important to find nonminimal csL Willmore surfaces in $\mathbb{S}^5$. This will be done in this section by analyzing a very close relationship between csL Willmore surfaces and csL surfaces in $\mathbb{S}^5$.

Assume that $\Sigma$ is a csL Willmore surface in $\mathbb{S}^5$, then since the variation vector field on $\Sigma$ under Legendrian deformations can be written as $J\nabla u + \frac{1}{2}u R$ for smooth function $u$ on $\Sigma$ (cf. [21, Lemma 3.1]), we have

$$0 = \int_{\Sigma} \left( \nabla (\nabla u) + JH u \right) d\mu_{\Sigma}$$

$$= \int_{\Sigma} \left( \nabla (\nabla u) + \frac{1}{2}u R \right) d\mu_{\Sigma}$$

$$= \int_{\Sigma} \nabla \left( \nabla u + JH u \right) d\mu_{\Sigma}$$

where in the last equality we used $(\nabla (\nabla u), R) = -2 \text{div}(JH)$, by Proposition 2.4. Therefore $\Sigma$ satisfies the following Euler-Lagrange equation:

$$\text{div} \left( J\nabla (\nabla u) + JH u \right) = 0.$$  \hspace{1cm} (3.1)

Remark 3.1. Note that the coefficient of the Euler-Lagrange equation (3.1) for csL Willmore surfaces in $\mathbb{S}^5$ is slightly different with [9, equation (1.7)]. That is because here we use the notation $H = \text{tr} \ B$, whereas in [9] we defined $H = \frac{1}{2} \text{tr} \ B$.

Then by (2.4), $\Sigma$ satisfies the following equation.

$$\text{div} \left( \nabla \text{div}(JH) + JB(JH, JH) - \frac{1}{2}|H|^2 JH - 4JH \right) = 0.$$  \hspace{1cm} (3.2)

In addition, by the four-symmetric of $(\sigma_{ijkl})$ (see (2.3)), a direct computation shows

$$\text{div}(JB(JH, JH)) = 2 \text{tr} \left( B(\cdot, \nabla(JH)), H \right) + \frac{1}{2} \nabla_{JH} |H|^2.$$  \hspace{1cm} (3.3)

Therefore $\Sigma$ satisfies the following equation

$$\Delta \text{div}(JH) + 2 \text{tr} \left( B(\cdot, \nabla(JH)), H \right) - \frac{1}{2}|H|^2 \text{div}(JH) - 4 \text{div}(JH) = 0.$$  \hspace{1cm} (3.4)

Therefore we have

**Proposition 3.1.** Assume that $\Sigma$ is a csL surface in $\mathbb{S}^5$ and $\text{tr} \left( B(\cdot, \nabla(JH)), H \right) = 0$, then $\Sigma$ is a csL Willmore surface.

With the aid of Proposition 3.1, we can find the following examples of csL Willmore surfaces from csL surfaces in $\mathbb{S}^5$. Firstly, according to Proposition 3.1, all closed Legendrian surfaces with parallel tangent vector field $JH$, which are exactly minimal surfaces or the Calabi tori (cf. [10, Proposition 3.2]), are csL Willmore surfaces. For reader’s convenience, we give some detailed computations as follows.

**Example 3.1** (Calabi tori). For every four nonzero real numbers $r_1, r_2, r_3, r_4$ with $r_1^2 + r_2^2 = r_3^2 + r_4^2 = 1$, the Calabi torus $\Sigma$ is a csL surface in $\mathbb{S}^5$ defined as follows.

$$F: \mathbb{S}^1 \times \mathbb{S}^1 \mapsto \mathbb{S}^5,$$

$$(t, s) \mapsto \left( r_1 r_3 \exp \left( \sqrt{-1} \left( \frac{r_2}{r_1} + \frac{r_4}{r_3} \right) \right), r_1 r_4 \exp \left( \sqrt{-1} \left( \frac{r_2}{r_1} - \frac{r_3}{r_4} \right) \right), r_2 \exp \left( -\sqrt{-1} \frac{r_1}{r_2} \right) \right).$$
Denote
\[ \phi_1 = \exp \left( \sqrt{-1} \left( \frac{r_2}{r_1} + \frac{r_3}{r_4} \right) \right), \quad \phi_2 = \exp \left( \sqrt{-1} \left( \frac{r_2}{r_1} - \frac{r_3}{r_4} \right) \right), \quad \phi_3 = \exp \left( -\sqrt{-1} \frac{r_4}{r_2} \right), \]
then \( F(t, s) = (r_1 r_3 \phi_1, r_1 r_4 \phi_2, r_2 \phi_3) \). Since
\[ \frac{\partial F}{\partial t} = \left( \sqrt{-1} r_2 r_3 \phi_1, \sqrt{-1} r_2 r_4 \phi_2, -\sqrt{-1} r_1 \phi_3 \right), \]
\[ \frac{\partial F}{\partial s} = \left( \sqrt{-1} r_1 r_4 \phi_1, -\sqrt{-1} r_1 r_3 \phi_2, 0 \right), \]
the induced metric in \( \Sigma \) is given by
\[ g = dt^2 + r_1^2 ds^2. \]
Let \( E_1 = \frac{\partial F}{\partial t}, E_2 = \frac{1}{r_1} \frac{\partial F}{\partial r_1} \), then \( \{ E_1, E_2, v_1 = \sqrt{-1} E_1, v_2 = \sqrt{-1} E_2, R = -\sqrt{-1} F \} \) is a local orthonormal frame of \( S^5 \) such that \( \{ E_1, E_2 \} \) is a local orthonormal tangent frame and \( R \) is the Reeb field. A direct calculation yields
\[ \frac{\partial v_1}{\partial t} = \left( -\sqrt{-1} \frac{r_2 r_3}{r_1} \phi_1, -\sqrt{-1} \frac{r_2 r_4}{r_1} \phi_2, -\sqrt{-1} \frac{r_2}{r_1} \phi_3 \right), \]
\[ \frac{\partial v_1}{\partial s} = \left( -\sqrt{-1} \frac{r_2 r_3}{r_4} \phi_1, \sqrt{-1} \frac{r_2 r_4}{r_4} \phi_2, 0 \right), \]
\[ \frac{\partial v_2}{\partial t} = \left( -\sqrt{-1} \frac{r_2 r_4}{r_1} \phi_1, \sqrt{-1} \frac{r_2 r_3}{r_1} \phi_2, 0 \right), \]
\[ \frac{\partial v_2}{\partial s} = \left( -\sqrt{-1} \frac{r_2}{r_4} \phi_1, -\sqrt{-1} \frac{r_2}{r_4} \phi_2, 0 \right), \]
\[ \frac{\partial R}{\partial t} = (r_2 r_3 \phi_1, r_2 r_4 \phi_2, -r_1 \phi_3), \]
\[ \frac{\partial R}{\partial s} = (r_1 r_4 \phi_1, -r_1 r_3 \phi_2, 0). \]
Hence,
\[ A^r = -\Re \langle dF, dv_1 \rangle = \left( \frac{r_2}{r_1} - \frac{r_1}{r_2} \right) dt^2 + r_1 r_2 ds^2, \]
\[ A^{v_1} = -\Re \langle dF, dv_2 \rangle = 2 r_2 dt ds + r_1 \left( \frac{r_4}{r_3} - \frac{r_3}{r_4} \right) ds^2, \]
\[ A^R = 0. \]
Thus
\[ H = \left( \frac{2 r_2}{r_1} - \frac{r_1}{r_2} \right) v_1 + \frac{1}{r_1} \left( \frac{r_4}{r_3} - \frac{r_3}{r_4} \right) v_2. \]
Moreover \( E_1 \) and \( E_2 \) are two parallel tangent vector field. It is obvious that \( \Sigma \) is a csL Willmore surface.

Secondly, we give some examples that \( JH \) is not parallel. Mironov [15] constructed the following new csL surfaces in \( S^5 \). We will verify that Mironov’s examples are in fact csL Willmore surfaces.

**Example 3.2 (Mironov’s examples [15]).** Let \( F : \Sigma^2 \mapsto S^5 \) be an immersion. Then \( F \) is a Legendrian immersion iff
\[ \langle F_*, F \rangle = \langle F_y, F \rangle = 0. \]
Here \( \{x, y\} \) is a local coordinates of \( \Sigma \) and \( \langle, \rangle \) stands for the hermitian inner product in \( \mathbb{C}^3 \). Set

\[
G = \begin{pmatrix} F \\ F_x \\ F_y \end{pmatrix},
\]

then

\[
GG^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \langle F_x, F_x \rangle & \langle F_x, F_y \rangle \\ 0 & \langle F_y, F_x \rangle & \langle F_y, F_y \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix},
\]

where \( g \) is a real positive matrix which is the induce metric of \( \Sigma \). There is a hermitian matrix \( \Theta \) such that

\[
G = \begin{pmatrix} 1 & 0 \\ 0 & g^{1/2} \end{pmatrix} e^{-\sqrt{-1} \Theta}.
\]

We compute

\[
GG^T = \begin{pmatrix} 0 & -\langle F_x, F_x \rangle & -\langle F_x, F_y \rangle \\ \langle F_x, F_x \rangle & \langle F_x, F_x \rangle & \langle F_x, F_y \rangle \\ \langle F_y, F_x \rangle & \langle F_y, F_x \rangle & \langle F_y, F_y \rangle \end{pmatrix} \implies \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{pmatrix} x \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{pmatrix} e^{\sqrt{-1} \Theta}
\]

Hence

\[
\mathbb{R} \left( \sqrt{-1} GG^T \right) = \begin{pmatrix} 0 & 0 \\ 0 & A \sqrt{-1} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{pmatrix} \mathbb{R} \left( \sqrt{-1} e^{-\sqrt{-1} \Theta} \right) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{pmatrix},
\]

which implies

\[
\begin{pmatrix} 0 & 0 \\ 0 & g^{-1/2} A \sqrt{-1} g^{1/2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A \sqrt{-1} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{pmatrix} \mathbb{R} \left( \sqrt{-1} e^{-\sqrt{-1} \Theta} \right) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{pmatrix}.
\]

Similarly,

\[
\begin{pmatrix} 0 & 0 \\ 0 & g^{-1/2} A \sqrt{-1} g^{1/2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A \sqrt{-1} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{pmatrix} \mathbb{R} \left( \sqrt{-1} e^{-\sqrt{-1} \Theta} \right) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{pmatrix}.
\]

The Lagrangian angle is then given by \( \theta = tr \mathbb{R} \Theta \). The above discussion implies that

\[
J \nabla \theta = H.
\]

Let \( a, b, c \) are three positive constants and consider the following immersion

\[
F : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^5,
\]

\[
(x, y) \mapsto \left( \phi(x)e^{\sqrt{-1} \gamma}, \psi(x)e^{\sqrt{-1} \delta}, \xi(x)e^{\sqrt{-1} \gamma} \right),
\]

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where
\[ \phi(x) = \sqrt{\frac{c}{a+c}} \sin x, \]
\[ \psi(x) = \sqrt{\frac{c}{b+c}} \cos x, \]
\[ \zeta(x) = \sqrt{\frac{a \sin^2 x}{a+c} + \frac{b \cos^2 x}{b+c}}. \]

where
\[ u(x) = \frac{c(a + b + (b - a) \cos(2x))}{2}. \]

One can check that \( F \) is a Legendrian immersion. Denote \( \Sigma := F(\mathbb{S}^1 \times \mathbb{S}^1) \). Notice that
\[ F_x = \left( \sqrt{\frac{c}{a+c}} \cos x e^{\sqrt{-1} \theta_0} - \sqrt{\frac{c}{b+c}} \sin x e^{\sqrt{-1} \theta_0}, \frac{-c(b-a) \sin(2x)}{2 \sqrt{(a+c)(b+c)(ab+u(x))}} e^{-\sqrt{-1} \theta_0} \right), \]
\[ F_y = \left( -1a \phi(x)e^{\sqrt{-1} \theta_0}, -1b \phi(x)e^{\sqrt{-1} \theta_0}, -1c \zeta(x)e^{-\sqrt{-1} \theta_0} \right). \]

The induced metric \( g \) is given by
\[ g = \left[ \begin{array}{cc} \frac{c \cos^2 x}{a+c} + \frac{c \sin^2 x}{b+c} + \frac{c^2(b-a)^2 \sin^2(2x)}{4(a+c)(b+c)(ab+u(x))} & \frac{2}{a+c} \frac{a \sin^2 x}{a+c} + \frac{b \cos^2 x}{b+c} \\ \frac{2}{a+c} \frac{a \sin^2 x}{a+c} + \frac{b \cos^2 x}{b+c} & \frac{2}{a+c} \frac{a \sin^2 x}{a+c} + \frac{b \cos^2 x}{b+c} \end{array} \right] dx^2 + \frac{u(x)}{ab+u(x)} dy^2 \]
\[ := e^{2\varphi(x)} dx^2 + e^{2\varphi(x)} dy^2. \]

A strait forward calculation yields that
\[ A^{\sqrt{-1} F_x} = \Re \left( \begin{array}{cc} 0 & \sqrt{-1}(F_{x}, F_{x}) \\ 0 & \sqrt{-1}(F_{y}, F_{y}) \end{array} \right) = \left( \begin{array}{cc} 0 & c \left(1 - e^{2\varphi(x)}\right) \\ c \left(1 - e^{2\varphi(x)}\right) & 0 \end{array} \right), \]
\[ A^{\sqrt{-1} F_y} = \Re \left( \begin{array}{cc} -\sqrt{-1}(F_{x}, F_{y}) & 0 \\ 0 & \sqrt{-1}(F_{y}, F_{y}) \end{array} \right) = \left( \begin{array}{cc} c \left(1 - e^{2\varphi(x)}\right) & 0 \\ 0 & (a + b - c)e^{2\varphi(x)} - abc \end{array} \right). \]

We get
\[ \Re \left( \sqrt{-1} e^{\sqrt{-1} \theta_0} \left( e^{-\sqrt{-1} \theta_0} \right) \right) = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & \frac{abc}{u \sqrt{ab+u}} & 0 \\ \frac{abc}{u \sqrt{ab+u}} & 0 & 0 \end{array} \right), \]
\[ \Re \left( \sqrt{-1} e^{\sqrt{-1} \theta_0} \left( e^{-\sqrt{-1} \theta_0} \right) \right) = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & \frac{abc}{u} & 0 \\ \frac{abc}{u} & 0 & 0 \end{array} \right), \]
\[ \Re \left( \sqrt{-1} e^{\sqrt{-1} \theta_0} \left( e^{-\sqrt{-1} \theta_0} \right) \right) = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & \frac{abc}{u} & 0 \\ \frac{abc}{u} & 0 & 0 \end{array} \right). \]
Thus
\[ H \nabla^2 F_c = 0, \quad H \nabla^2 F_c = a + b - c. \]

We get
\[ H = \frac{a + b - c}{u(x)} \sqrt{-1} \frac{\partial}{\partial y}, \]
and
\[ \nabla_{\partial_i} \left( \sqrt{-1} H \right) = \frac{(a + b - c)u}{2u^2} \frac{\partial}{\partial y}, \quad \nabla_{\partial_i} \left( \sqrt{-1} H \right) = \frac{(ab + u)(a + b - c)u}{2u^2} \frac{\partial}{\partial x}. \]

In particular
\[ \text{div} \left( \sqrt{-1} H \right) = 0. \]

Hence \( \Sigma \) is csL. Moreover
\[ \sum_{i=1}^{2} \langle B(e_i, \nabla_{e_i} (JH)), H \rangle = 0. \]

Therefore, \( \Sigma \) is a csL Willmore surface in \( S^5 \).

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