The two envelopes paradox in non-Bayesian and Bayesian statistics

Shiro Ishikawa

Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1, Hiyoshi, Kouhoku-ku Yokohama, 223-8522, Japan. E-mail: ishikawa@math.keio.ac.jp

Abstract
The purpose of this paper is to clarify the (non-Bayesian and Bayesian) two-envelope problems in terms of quantum language (or, measurement theory), which was recently proposed as a linguistic turn of quantum mechanics (with the Copenhagen interpretation). The two envelopes paradox is only a kind of high school student’s probability puzzle, and it may be exaggerated to say that this is an unsolved problem. However, since we are convinced that quantum language is just statistics of the future, we believe that there is no clear answer without the description by quantum language. In this sense, the readers are to find that quantum language provides the final answer (i.e., the easiest and deepest understanding) to the two envelope-problems in both non-Bayesian and Bayesian statistics. Also, we add the discussion about St. Petersburg two-envelope paradox.

1 Introduction

1.1 Two-envelope paradox

In what follows, we firstly introduce the two-envelope problem (cf. [16,18]), which is well known as a kind of high school students’ mathematical puzzle.

Problem 1 [The two envelope problem]. The host presents you with a choice between two envelopes (i.e., Envelope A and Envelope B). You know one envelope contains twice as much money as the other, but you do not know which contains more. You choose randomly (by a fair coin toss) one envelope, for example, call it Envelope A. Suppose that you find $\alpha$ dollars inside your envelope $A$.

Now the host says "You are offered the options of keeping your A or switching to my B". What should you do?

[(P1):Why is it paradoxical?]. You reason that, with probability 1/2, the other envelope $B$ has either $\alpha/2$ or $2\alpha$ dollars. Thus the expected value (denoted $E_{\text{other}}(\alpha)$ at this moment) of the other envelope is

$$E_{\text{other}}(\alpha) = (1/2)(\alpha/2) + (1/2)(2\alpha) = 1.25\alpha$$

(1)

This is greater than the $\alpha$ in your current envelope $A$. Therefore, you should switch to $B$. But this seems clearly wrong, as your information about $A$ and $B$ is symmetrical. This is the famous two-envelope paradox (i.e., "The Other Person’s Envelope is Always Greener").

Further consider the following problem, which is quite easy.

Problem 1’ [The trivial two envelope problem]. The host presents you with a choice between two envelopes (i.e., Envelope A and Envelope B). You know Envelope A [resp. Envelope B] includes 10 dollars [resp. 20 dollars]. Define

$$\mathcal{X} = \begin{cases} 
20, & \text{(if } x = 10) \\
10, & \text{(if } x = 20) 
\end{cases}$$
You choose randomly (by a fair coin toss) one envelope, and you get $x_1$ dollars (i.e., if the envelope is A [resp. B], $x_1$ is equal to 10 dollars [resp. 20 dollars]). And the host gets $x_2$ dollars. Next, by a similar way, you choose randomly (by a fair coin toss) one envelope, and you get $x_2$ dollars. Repeating the trial, you get

$$x_1, x_2, x_3, \cdots$$

Then, you can assure, by the law of large numbers (cf. [5, 10, 15]), that

$$\lim_{N \to \infty} \frac{x_1 + x_2 + \cdots + x_N}{N} = \lim_{N \to \infty} \frac{\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_N}{N} = \frac{10 + 20}{2}$$

Is it true? Of course, it is true and not paradoxical.

We consider that it is well known that the above two problems are essentially the same, in spite that the two are superficially different. Thus, the purpose of this paper is not to show the equality of the two problems, but to show that

- the equivalence of two problems (i.e., Problem 1 (the two-envelope paradox) and Problem 1' (the trivial two-envelope paradox)) is automatically clarified, if Problems 1 is described in terms of quantum language.

This will be done in Section 3 (non-Bayesian two envelope paradox). Also, we add Section 4 (Bayesian two envelope paradox) and Section 5 (non-Bayesian St. Petersburg two envelope paradox). In the following section 2, according to refs. [3]-[14], we review quantum language.

## 2 Measurement theory (= Quantum language)

### 2.1 The motivation of quantum language

In [17], N.D. Mermin introduced Feynman’s two words about quantum mechanics as follows (cf. The Character of Physical Law (Cambridge: MIT Press, 1965)).

(A1) There was a time when the newspapers said that only twelve men understood the theory of relativity. I do not believe there ever was such a time. There might have been a time when only one man did, because he was the only guy who caught on, before he wrote his paper. But after people read the paper a lot of people understood the theory of relativity in some way or other, certainly more than twelve. On the other hand, I think I can safely say that nobody understands quantum mechanics.

(A2) We have always had a great deal of difficulty understanding the world view that quantum mechanics represents. ····· I cannot say the real problem, therefore I suspect there’s no real problem, but I’m not sure there’s no real problem.

For this significant Feynman’s words, we assert that

(A3) If we start from the declaration "there’s no real problem", it is a matter of course that nobody understands quantum mechanics, but we can necessarily discover "quantum language", which is located such as in Figure 1 (world views) below.

This is all my opinion concerning quantum language.
2.2 The classifications of measurement theory

In this section, we introduce measurement theory (or in short, MT). This theory is a kind of language, and thus, it is also called quantum language (or in short, QL).

Measurement theory (cf. refs. [3]-[14]) is, by an analogy of quantum mechanics (or, as a linguistic turn of quantum mechanics), constructed as the scientific theory formulated in a certain C*-algebra \( \mathcal{A} \) (i.e., a norm closed subalgebra in the operator algebra \( \mathcal{B}(H) \) composed of all bounded linear operators on a Hilbert space \( H \), cf. [19, 20]).

When \( \mathcal{A} = \mathcal{B}_c(H) \), the C*-algebra composed of all compact operators on a Hilbert space \( H \), the MT is called quantum measurement theory (or, quantum system theory), which can be regarded as the linguistic aspect of quantum mechanics. Also, when \( \mathcal{A} \) is commutative (that is, when \( \mathcal{A} \) is characterized by \( \mathcal{C}_0(\Omega) \), the C*-algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space \( \Omega \) (cf. [20])), the MT is called classical measurement theory. Thus, we have the following classification:

\[
\text{(B1) MT} \begin{cases} 
\text{quantum MT} & \text{(when non-commutative } \mathcal{A} = \mathcal{B}_c(H) ) \\
\text{classical MT} & \text{(when commutative } \mathcal{A} = \mathcal{C}_0(\Omega) ) 
\end{cases}
\]

In this paper, we devote ourselves to CMT(= classical MT), which is classified as follows.

\[
\text{(B2) CMT} \begin{cases} 
(\text{B2}_1): \text{PCMT (= pure classical measurement theory) in } \S 2.3 \\
(\text{B2}_2): \text{SCMT (= statistical classical measurement theory) in } \S 2.4 
\end{cases}
\]
2.3 The preparation of CMT

For the general theory of measurement theory, see refs. [3]-[14]. In order to read this paper, it suffices to know the following.

Let $\Omega$ be a locally compact space. Define the continuous functions space $C_0(\Omega)$ such that

$$C_0(\Omega) = \{ f \mid f \text{ is a complex valued continuous function on } \Omega \text{ such that } \lim_{\omega \to \infty} f(\omega) = 0 \}$$

which is a Banach space (or precisely, commutative C*-algebra) with the norm $\|f\| = \max_{\omega \in \Omega} |f(\omega)|$.

Let $M(\Omega)$ be the dual Banach space of $C_0(\Omega)$, i.e., $M(\Omega) = C_0(\Omega)^*$. Riesz theorem (cf. ref. [21]) says that $M(\Omega)$ is the space of all finite complex-valued measures on $\Omega$. Thus, we denote that

$$M^m(\Omega) (\text{i.e., the space of all probability measures on } \Omega)$$

is called a mixed state class. An element $\rho \in M^m(\Omega)$ is called a mixed state.

For each $\omega \in \Omega$, define the point measure $\delta_\omega \in \mathcal{M}^m(\Omega)$ such that

$$M^p(\Omega) (\text{i.e., the space of all point measures on } \Omega)$$

is a pure state class. An element $\rho \in M^p(\Omega)$ is called a pure state (or in short, state). Under the identification: $M^p(\Omega) \ni \delta_\omega \leftrightarrow \omega \in \Omega$, the $\omega(\in \Omega)$ is also called a state (or precisely, pure state).

Let $\nu$ be a fixed ($\sigma$-finite) measure on $\Omega$ such that

$$\nu(K) < \infty, 0 < \nu(D) \quad (\forall \text{ compact set } K, \forall \text{ open set } D(\in \mathcal{B}_\Omega : \text{the Borel field in } \Omega))$$

Define the Banach space $L^p(\Omega, \nu) (1 \leq p \leq \infty)$ such that

$$f \in L^p(\Omega) \iff f \text{ is a complex-valued measurable function on } \Omega \text{ such that } \|f\|_{L^p(\Omega)} < \infty$$

where $\|f\|_{L^p(\Omega)} = \left( \int_{\Omega} |f(\omega)|^p \nu(d\omega) \right)^{1/p} (1 \leq p < \infty), = \inf \{ a \geq 0 : \nu(\{ \omega : |f(\omega)| > a \}) = 0 \}$ (p = $\infty$).

Motivated by a nice idea in ref. [2], an observable $O \equiv (X, \mathcal{F}, F)$ in the $L^\infty(\Omega, \nu)$ is defined as follows:

(C1) [σ-field] $X$ is a set, $\mathcal{F}(\subseteq 2^X$, the power set of $X)$ is a σ-field of $X$, that is, $\{ \Xi_1, \Xi_2, \Xi_3, \ldots \in \mathcal{F} \Rightarrow \cup_{k=1}^\infty \Xi_k \in \mathcal{F} \}$, “$X \in \mathcal{F}$” and “$\Xi \in \mathcal{F} \Rightarrow X \setminus \Xi \in \mathcal{F}$”.

(C2) [Countably additivity] $F$ is a mapping from $\mathcal{F}$ to $L^\infty(\Omega, \nu)$ satisfying: (a): for every $\Xi \in \mathcal{F}$, $F(\Xi)$ is a non-negative element in $L^\infty(\Omega, \nu)$ such that $0 \leq F(\Xi) \leq I$, (b): $F(\emptyset) = 0$ and $F(X) = I$, where 0 and I is the 0-element and the identity in $L^\infty(\Omega, \nu)$ respectively. (c): for any countable decomposition $\{ \Xi_1, \Xi_2, \ldots \}$ of $\Xi \in \mathcal{F}$ (i.e., $\Xi_k, \Xi \in \mathcal{F}$ such that $\bigcup_{k=1}^\infty \Xi_k = \Xi, \Xi_i \cap \Xi_j = \emptyset (i \neq j)$), it holds that

$$\lim_{K \to \infty} L^1(\Omega) \langle \rho, F(\bigcup_{k=1}^K \Xi_k) \rangle_{L^\infty(\Omega, \nu)} = L^1(\Omega) \langle \rho, F(\Xi) \rangle_{L^\infty(\Omega, \nu)} \left( \equiv \int_{\Omega} \rho(\omega) \cdot [F(\Xi)](\omega) \nu(d\omega) \right) \quad (2)$$

(\forall \rho \in L^1(\Omega, \nu))

i.e., $\lim_{K \to \infty} F(\bigcup_{k=1}^K \Xi_k) = F(\Xi)$ in the sense of weak* convergence in $L^\infty(\Omega, \nu)$. 


Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence in \( L^1(\Omega, \nu) \). And let \( \rho_0 \in \mathcal{M}^m(\Omega) \). Here, "\( w^* - \lim_{n \to \infty} f_n = \rho_0 \)" means that

\[
\lim_{n \to \infty} \int_{\Omega} f_n(\omega) \cdot \phi(\omega) \nu(d\omega) = \int_{\Omega} \phi(\omega) \rho_0(\nu) \quad (\forall \phi \in C_0(\Omega))
\]

And, we say that "\( F(\in L^\infty(\Omega, \nu)) \) is essentially continuous at \( \rho_0(\in \mathcal{M}^m(\Omega)) \)", if there uniquely exists a complex number \( \gamma \) such that

"\( w^* - \lim_{n \to \infty} f_n = \rho_0 \)" \( \implies \lim_{n \to \infty} \int_{\Omega} F(\omega) \cdot f_n(\omega) \nu(d\omega) = \gamma \)

And we denote that \( \rho_0(F) \) \( (= c_0(\Omega)^* \langle \rho_0, F \rangle_{L^\infty(\Omega, \nu)}) = \gamma \).

**Remark 1.** Without loss of generality, we can assume that \( \Omega \) is compact, and \( \nu(\Omega) = 1 \).

### 2.4 Pure Classical Measurement Theory

With any classical system \( S \), a fundamental structure \( [C_0(\Omega) \subseteq L^\infty(\Omega, \nu)] \) can be associated in which the pure measurement theory (B2) of that system can be formulated. A pure state of the system \( S \) is represented by an element \( \delta_\omega(\in \mathcal{M}^P(\Omega)="\text{pure state class}"(\text{cf. ref. [6]}) \) and an observable is represented by an observable \( \phi = (X, \mathcal{F}, F) \) in \( L^\infty(\Omega, \nu) \). Also, the measurement of the observable \( O \) for the system \( S \) with the pure state \( \delta_\omega \) is denoted by \( \mathcal{M}_{L^\infty(\Omega, \nu)}(O, S_{[\delta_\omega]}) \) \( (\text{or more precisely,} \mathcal{M}_{L^\infty(\Omega, \nu)}(O = (X, \mathcal{F}, F), S_{[\delta_\omega]}) \) \). An observer can obtain a measured value \( x \in X \) by the measurement \( \mathcal{M}_{L^\infty(\Omega, \nu)}(O, S_{[\delta_\omega]}) \).

The AxiomPCMT 1 presented below is a kind of mathematical generalization of Born’s probabilistic interpretation of quantum mechanics.

**AxiomPCMT 1** [Pure Measurement]. The probability that a measured value \( x \in X \) obtained by the measurement \( \mathcal{M}_{L^\infty(\Omega, \nu)}(O = (X, \mathcal{F}, F), S_{[\delta_\omega]}) \) belongs to a set \( \Xi \in (F) \) is given by \( \delta_{\omega_0}(F(\Xi)) \)

\[
(= c_0(\Omega)^* \langle \omega_0, F(\Xi) \rangle_{L^\infty(\Omega, \nu)}), \text{ if } F(\Xi) \text{ is essentially continuous at } \omega_0.
\]

Let \( [C_0(\Omega, \nu) \subseteq L^\infty(\Omega, \nu)] \) be a fundamental structure. We shall introduce the following notation: It is usual to consider that we do not know the pure state \( \delta_\omega(\in \mathcal{M}^P(\Omega)) \) when we take a measurement \( \mathcal{M}_{L^\infty(\Omega, \nu)}(O, S_{[\delta_\omega]}) \). That is because we usually take a measurement \( \mathcal{M}_{L^\infty(\Omega, \nu)}(O, S_{[\delta_\omega]}) \) in order to know the state \( \delta_\omega \). Thus,

(D1) when we want to emphasize that we do not know the state \( \delta_\omega \), \( \mathcal{M}_{L^\infty(\Omega, \nu)}(O, S_{[\delta_\omega]}) \) is denoted by \( \mathcal{M}_{L^\infty(\Omega, \nu)}(O, S_{[\delta_\omega]}) \)

\( \text{D2) also, when we know the distribution } \rho_0 \in \mathcal{M}^m(\Omega) \) \( \text{of the unknown state } \delta_\omega \) \( \text{, the } \mathcal{M}_{L^\infty(\Omega, \nu)}(O, S_{[\delta_\omega]}) \) \( \text{is denoted by } \mathcal{M}_{L^\infty(\Omega, \nu)}(O, S_{[\delta_\omega]}(\rho_0)) \). The \( \rho_0 \) is called a mixed state.

We have the following fundamental theorem in measurement theory:

**Theorem 1** [Fisher’s maximum likelihood method (cf. [4]- [11])]. Assume that a measured value \( x \in X \) is obtained by a measurement \( \mathcal{M}_{L^\infty(\Omega, \nu)}(O := (X, \mathcal{F}, F), S_{[\omega]}) \). Put

\[
f(x, \omega) = \inf_{\omega_1 \in \Omega} \left[ \lim_{\Xi \to \{x\}, \Xi \ni x} \frac{\mathcal{F}(\Xi)(\omega_1)}{|F(\Xi)(\omega_1)|} \right] \quad (\forall \omega \in \Omega)
\]
Then, there is a reason to infer that the unknown state \([\ast] \) is equal to \(\delta_{\omega_0} \in \Omega \) such that
\[
f(x, \omega_0) = 1
\]
Also, if \(f(x, \omega_1) = 0\), then there is no possibility that \([\ast] = \delta_{\omega_1}\).

**Definition 1** [Parallel measurement]. Consider two measurements: \(M_{L^\infty(\Omega_1, \mu_1)}(O_1 := (X_1, F_1, F_1), S_{[\delta_{\omega_1}]}), \) and \(M_{L^\infty(\Omega_2, \mu_2)}(O_2 := (X_2, F_2, F_2), S_{[\delta_{\omega_2}]}), \) Let \((\Omega_1 \times \Omega_2, \nu_1 \otimes \nu_2)\) be the product measure space of \((\Omega_1, \nu_1)\) and \((\Omega_2, \nu_2)\). And consider the parallel measurement \(M_{L^\infty(\Omega_1 \times \Omega_2, \nu_1 \otimes \nu_2)}(O_1 \otimes O_2 := (X_1 \times X_2, F_1 \otimes F_2, F_1 \otimes F_2), S_{[\delta_{(\omega_1, \omega_2)}]}),\) which is denoted by \(\bigotimes_{n=1}^2 M_{L^\infty(\Omega_n, \nu_n)}(O_n := (X_1, F_n, F_n), S_{[\delta_{\omega_n}]}),\) Here, \(F_1 \otimes F_2\) is the product field of \(F_1\) and \(F_2\). And, \(F_1 \otimes F_2\) is defined by
\[
[(F_1 \otimes F_2)(\Xi_1 \times \Xi_2)](\omega_1, \omega_2) = [F_1(\Xi_1)](\omega_1) \cdot [F_2(\Xi_2)](\omega_2)
\]
\((\forall (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2, \forall \Xi_1 \in F_1, \forall \Xi_2 \in F_2)\). Here, the linguistic interpretation of quantum mechanics (cf. \([6, 7]\)) asserts the identification: "(D3)+(D4)" \(\Rightarrow"(D5):"

\begin{align*}
(D3) & \text{ a measured value } x_1(\in X_1) \text{ is obtained by a measurement } M_{L^\infty(\Omega_1, \mu_1)}(O_1 := (X_1, F_1, F_1), S_{[\delta_{\omega_1}]}) \\
(D4) & \text{ a measured value } x_2(\in X_2) \text{ is obtained by a measurement } M_{L^\infty(\Omega_2, \mu_2)}(O_2 := (X_2, F_2, F_2), S_{[\delta_{\omega_2}]}) \\
(D5) & \text{ a measured value } (x_1, x_2)(\in X_1 \times X_2) \text{ is obtained by a measurement } M_{L^\infty(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)}(O_1 \otimes O_2 := (X_1 \times X_2, F_1 \otimes F_2, F_1 \otimes F_2), S_{[\delta_{(\omega_1, \omega_2)}]})
\end{align*}

This definition is generalized as follows. For each \(n \in \mathbb{N} = \{1, 2, \cdots\}\), consider a measurement: \(M_{L^\infty(\Omega_n, \nu_n)}(O_n := (X_n, F_n, F_n), S_{[\delta_{\omega_n}]})\). Let \((X_n \in \Omega_n, \otimes_{n \in \mathbb{N}} \nu_n)\) be the infinite product probability measure space (cf. Remark 1 and \([15]\)). Let \((X_n \in \Omega_n, \otimes_{n \in \mathbb{N}} F_n)\) be the infinite product measurable space. Then, we have the parallel observable \(\bigotimes_{n \in \mathbb{N}} O_n = (X_n \in \Omega_n, \otimes_{n \in \mathbb{N}} F_n, \otimes_{n \in \mathbb{N}} F_n)\) in \(L^\infty(\Omega_n, \otimes_{n \in \mathbb{N}} \nu_n)\) such that
\[
[(\otimes_{n \in \mathbb{N}} F_n)(\otimes_{n \in \mathbb{N}} \Xi_n)](\omega_1, \omega_2, \cdots) = \times_{n \in \mathbb{N}} [F_n(\Xi_n)](\omega_n) \quad (\forall \Xi_n \in F_n, \forall (\omega_1, \omega_2, \cdots) \in \times_{n \in \mathbb{N}} \Omega_n)
\]
where a set \(\{n \in \mathbb{N} \mid \Xi_n \neq X_n\}\) is finite.

Thus, we have the infinite parallel measurement: \(\bigotimes_{n=1}^\infty M_{L^\infty(\Omega_n, \nu_n)}(O_n := (X_n, F_n, F_n), S_{[\delta_{\omega_n}]})\), i.e.,
\[
M_{L^\infty(\times_{n \in \mathbb{N}} \Omega_n, \otimes_{n \in \mathbb{N}} \nu_n)}(\bigotimes_{n \in \mathbb{N}} O_n) = (X_n \in \times_{n \in \mathbb{N}} \Omega_n, \otimes_{n \in \mathbb{N}} F_n, \otimes_{n \in \mathbb{N}} F_n, S_{[\delta_{(\omega_1, \omega_2, \cdots)}]}) \tag{4}
\]

### 2.5 Statistical Classical Measurement Theory

The Axiom\textsuperscript{SCMT 1} presented below is also a kind of mathematical generalization of Born's probabilistic interpretation of quantum mechanics.

**Axiom\textsuperscript{SCMT 1}** [Statistical measurement]. Recall the \((D2)\). The probability that a measured value \(x (\in X)\) obtained by the measurement \(M_{L^\infty(\Omega, \nu)}(O := (X, F, F), S_{[\ast]}(\rho_0))\) belongs to a set \(\Xi(\in F)\) is given by \(\rho_0(F(\Xi)) = \langle \rho_0, F(\Xi) \rangle_{L^\infty(\Omega, \nu)} \) if \(F(\Xi)\) is essentially continuous at \(\rho_0\).

**Theorem 2** [Bayes' method (cf. \([3]-[11]\)]). Assume that a measured value \(x(\in X)\) is obtained by a measurement \(M_{L^\infty(\Omega, \nu)}(O := (X, F, F), S_{[\ast]}(\rho_0))\). Thus, we can assert that:
(E) When we know a measured value $x(\in X)$ obtained by a statistical measurement $M_{L^{\infty}(\Omega, \nu)}(O \equiv (X, \mathcal{F}, F), S_{\nu}(\rho_0))$, there is a reason to infer that the post-state (i.e., the mixed state after the measurement) is equal to $\rho_{\text{post}}^{x}(\in \mathcal{M}^{m}(\Omega, \nu))$, where

$$\rho_{\text{post}}^{x} = \lim_{\Xi \to \{x\}} \frac{\int_{\Omega} [F(\Xi)](\omega) \rho_0(\omega)(d\omega)}{\int_{\Omega} [F(\Xi)](\omega)(d\omega)}$$

(5)

Remark 2 [Bayesian statistics]. When Bayes’ theorem is used in SCMT, SCMT is called Bayesian statistics. In Bayesian statistics, the mixed state $\rho_0$ may be called a ”pretest state” (cf. refs. [4]–[11]).

Remark 3 [Overview; quantum language (cf. [3]-[14])]. Although, in order to read this paper, it suffices to understand Axiom 1 (measurement: Axiom $^{\text{PCMT}}$ 1 and Axiom $^{\text{SCMT}}$ 1), we want to mention the overview of quantum language as follows. Quantum language (=QL) is a kind of metaphysics (i.e., language) that has the following structure:

$$\text{QL} = \text{measurement} + \text{causality} + \text{Linguistic interpretation}$$

And quantum language says that

(F1) Follow examples of the wordings in Axioms 1 and 2, and describe every phenomenon!

Applying a trial-and-error method repeatedly, you may make progress without the manual (i.e., the linguistic interpretation). In fact, the author has mastered the linguistic interpretation now at last by the trial and error for about twenty years. In this sense, the manual (i.e., the linguistic interpretation) is not absolutely indispensable for quantum language. That is, we consider that the term ”interpretation” should not exist in physics but in metaphysics. However, it is earlier for progress to know the manual. For example, the following two:

(F2) consider the dualism (i.e., observer and object)

(F3) only one measurement is permitted

are leading indicators. Although we believe that the linguistic interpretation should be determined uniquely and naturally, this is not guaranteed. However, if it is not determined uniquely, it suffices to add something to axioms. This is the fate of metaphysics. Also, note that

(F4) the competitor of quantum language (i.e., the linguistic interpretation of quantum mechanics) is statistics and is not physics (i.e., the several interpretations of quantum mechanics).

Of course, we believe that quantum language is forever.
3 Non-Bayesian approach to the two envelopes problem

3.1 The simple answer in which it is hard to notice the mistake (1)

Consider the classical fundamental structure such that
\[ C_0(\Omega) \subseteq L^\infty(\Omega, \nu) \]

Put \( X = \mathbb{R}_+ = \{ x \mid x \text{ is a non-negative real number} \} \). Let \( V_1 : \Omega \to \mathbb{R}_+ \) and \( V_2 : \Omega \to \mathbb{R}_+ \) be continuous maps. You may think that \( V_2(\omega) = 2V_1(\omega) \) \( (\forall \omega \in \Omega) \).

For each \( k = 1, 2 \), define the observable \( O_k = (X(= \mathbb{R}_+), F(= B_{\mathbb{R}_+} : \text{the Borel field}), F_k) \) in \( L^\infty(\Omega, \nu) \) such that
\[
[F_k(\Xi)](\omega) = \begin{cases} 
1 & (\text{if } V_k(\omega) \in \Xi) \\
0 & (\text{if } V_k(\omega) \notin \Xi)
\end{cases}
\]
(\( \forall \omega \in \Omega, \forall \Xi \in F = B_{\mathbb{R}_+} \) i.e., the Bore field in \( X(= \mathbb{R}_+) \) )

Here we identify \( V_k \) with \( O_k \). Further, define the observable \( O = (X, F) \) in \( L^\infty(\Omega, \nu) \) such that
\[
F(\Xi) = \frac{1}{2} \left( F_1(\Xi) + F_2(\Xi) \right) \quad (\forall \Xi \in F)
\]
that is,
\[
[F(\Xi)](\omega) = \begin{cases} 
1 & (\text{if } V_1(\omega) \in \Xi, V_2(\omega) \in \Xi) \\
1/2 & (\text{if } V_1(\omega) \in \Xi, V_2(\omega) \notin \Xi) \\
1/2 & (\text{if } V_1(\omega) \notin \Xi, V_2(\omega) \in \Xi) \\
0 & (\text{if } V_1(\omega) \notin \Xi, V_2(\omega) \notin \Xi)
\end{cases}
\]
(\( \forall \omega \in \Omega, \forall \Xi \in F = B_{\mathbb{R}_+} \) i.e., the Bore field in \( X(= \mathbb{R}_+) \) )

In what follows, we shall present the three answers to Problem 1 such that

| Simple answer \( \text{ (in §3.1.1) } \) | (more strict) | Usual answer \( \text{ (in §3.1.2) } \) | (more strict) | Strict answer \( \text{ (in §3.1.3) } \) |
---|---|---|---|---|

which are essentially equivalent, and thus, these are true.

3.1.1 The simplest answer to Problem 1

Fix any \( \omega_0(\in \Omega) \), which is assumed to be unknown. Consider the measurement \( M_{L^\infty(\Omega, \nu)}(O = (X, F, F), S_{[\delta_{\omega_0}]} \) ). Then, Axiom \( \text{PCMT } 1 \) says that

\[
(G1) \quad \text{the probability that a measured value } \left\{ \frac{V_1(\omega_0)}{V_2(\omega_0)} \right\} \text{ obtained by the measurement } M_{L^\infty(\Omega, \nu)}(O = (X, F, F), S_{[\delta_{\omega_0}]}) \text{ is given by } \left\{ \frac{1}{2}, \frac{1}{2} \right\}.
\]
Then, by the switching to \( \left\{ \begin{array}{l} V_2(\omega_0) \\ V_1(\omega_0) \end{array} \right\} \), you gain \( \left\{ \begin{array}{l} V_2(\omega_0) - V_1(\omega_0) \\ V_1(\omega_0) - V_2(\omega_0) \end{array} \right\} \) dollars. This means that the expectation of the switching gain is equal to
\[
\frac{(V_2(\omega_0) - V_1(\omega_0))/2 + (V_1(\omega_0) - V_2(\omega_0))/2}{2} = 0,
\]
which is independent of \( \omega_0 \).

This implies that the swapping is even, i.e., no advantage and no disadvantage.

Since \( \omega_0(\in \Omega) \) is assumed to be unknown, the measurement \( M_{L^\infty(\Omega,\nu)}(O = (X,\mathcal{F},F), S_{[\delta_{\omega_0}]} \) is also denoted by \( M_{L^\infty(\Omega,\nu)}(O = (X,\mathcal{F},F), S_{[\delta_{\omega_0}]} \) and \( \mathcal{F} \). Thus, when you obtain a measured value \( \alpha(\in X) \) by \( M_{L^\infty(\Omega,\nu)}(O = (X,\mathcal{F},F), S_{[\delta_{\omega_0}]} \) (and you do not have a way for getting to know whether the money included in the other envelope is more or less than \( \alpha \) dollars), you should conclude that the swapping is even. That is, we can not believe in the proverb: "The Other Person's Envelope is Always Greener".

### 3.1.2 The usual answer to Problem 1

Define the quasi-product observable \( O \times O = (X \times X,\mathcal{F} \otimes \mathcal{F}, F^q \times F) \) in \( L^\infty(\Omega,\nu) \) such that
\[
[(F \times F)(\Xi \times \Gamma)](\omega) = \left\{ \begin{array}{ll} 1 & \text{if } V_1(\omega) \in \Xi, V_2(\omega) \in \Xi, V_1(\omega) \in \Gamma, V_2(\omega) \in \Gamma \\ 1/2 & \text{if } V_1(\omega) \in \Xi, V_2(\omega) \notin \Xi, V_1(\omega) \notin \Gamma, V_2(\omega) \in \Gamma \\ 1/2 & \text{if } V_1(\omega) \notin \Xi, V_2(\omega) \in \Xi, V_1(\omega) \in \Gamma, V_2(\omega) \notin \Gamma \\ 0 & \text{if } V_1(\omega) \notin \Xi, V_2(\omega) \notin \Xi, V_1(\omega) \notin \Gamma, V_2(\omega) \notin \Gamma \end{array} \right.
\]

(\( \forall \omega \in \Omega, \forall \Xi, \forall \Gamma \in \mathcal{F} = B_{\mathbb{R}_+} \) i.e., the Bore field in \( X(= \mathbb{R}_+) \))

Fix any \( \omega_0(\in \Omega) \), which is assumed to be unknown. Consider the measurement \( M_{L^\infty(\Omega,\nu)}(O \times O = (X \times X,\mathcal{F} \otimes \mathcal{F}, F^q \times F), S_{[\delta_{\omega_0}]} \). Then, Axiom\( \text{PCMT} 1 \) says that

\begin{align*}
\text{(G2) the probability that a measured value } & \left\{ \begin{array}{l} (V_1(\omega_0), V_2(\omega_0)) \\ (V_2(\omega_0), V_1(\omega_0)) \end{array} \right\} \text{ obtained by the measurement } \\
M_{L^\infty(\Omega,\nu)}(O \times O = (X \times X,\mathcal{F} \otimes \mathcal{F}, F^q \times F), S_{[\delta_{\omega_0}]} \text{ is given by } & \left\{ \frac{1}{2} \right\}.
\end{align*}

Here,

\begin{align*}
\text{(G3) "a measured value } & \left\{ \begin{array}{l} (V_1(\omega_0), V_2(\omega_0)) \\ (V_2(\omega_0), V_1(\omega_0)) \end{array} \right\} \text{ is obtained" means } \\
\text{"you and the host respectively get } & \left\{ \begin{array}{l} V_1(\omega_0) \\ V_2(\omega_0) \end{array} \right\} \text{ dollars and } \left\{ \begin{array}{l} V_2(\omega_0) \\ V_1(\omega_0) \end{array} \right\} \text{ dollars."
}\end{align*}

Therefore,

\begin{align*}
\text{(G4) your expectation } & \left[ \frac{V_1(\omega_0) + V_2(\omega_0)}{2} \right] \text{ and the host's expectation } \left[ \frac{V_2(\omega_0) + V_1(\omega_0)}{2} \right] \text{ are equal.}
\end{align*}

This implies that the swapping is even in the sense of (G4), i.e., no advantage and no disadvantage. That is, we can not believe in the proverb: "The Other Person's Envelope is Always Greener".

### 3.1.3 The strict answer ((F3): only one measurement is permitted)

Fix any \( \omega_0(\in \Omega) \), which is assumed to be unknown. Consider the measurement \( M_{L^\infty(\Omega,\nu)}(O \times O = (X \times X,\mathcal{F} \otimes \mathcal{F}, F^q \times F), S_{[\delta_{\omega_0}]} \). And further, consider the infinite parallel measurement
\[ \bigotimes_{n \in \mathbb{N}} M_{L^\infty(\Omega, \nu)}(O \times O) = (X \times X, F \otimes F, F \times F), S_{[\delta_{\omega_0}]} \], which is, by (4), characterized as follows.

Then, Axiom PCMT 1 says that

\[ (G5) \text{ the probability } P(\hat{\Xi}) \text{ that a measured value obtained by the infinite parallel measurement } \bigotimes_{n \in \mathbb{N}} M_{L^\infty(\Omega, \nu)}(O \times O) = (X \times X, F \otimes F, F \times F), S_{[\delta_{\omega_0}]} \text{ belongs to } \hat{\Xi}(\omega_0, \omega_0, \cdots) \]

Here, put

\[ \hat{\Xi} = \left\{ (x_n, y_n)_{n \in \mathbb{N}} \in (X \times X)^\mathbb{N} \mid \lim_{N} \frac{1}{N} \sum_{n=1}^{N} x_n = \lim_{N} \frac{1}{N} \sum_{n=1}^{N} y_n = \frac{V_1(\omega_0) + V_2(\omega_0)}{2} \right\} \]

Then we see, by (G2) and the law of large numbers (cf. [5, 10, 15]), that

\[ P(\hat{\Xi}) = \left[ \left( \bigotimes_{n \in \mathbb{N}} (F \times F) \right) (\hat{\Xi}) \right] (\omega_0, \omega_0, \cdots) = 1 \]

\[ \lim_{N} \frac{1}{N} \sum_{n=1}^{N} x_n = \lim_{N} \frac{1}{N} \sum_{n=1}^{N} y_n = \frac{V_1(\omega_0) + V_2(\omega_0)}{2} \quad (\forall (x_n, y_n)_{n \in \mathbb{N}} \in \hat{\Xi}) \]

This implies that the swapping is even, in the above sense, i.e., no advantage and no disadvantage. That is, we can not believe in the proverb: "The Other Person’s Envelope is Always Greener".

Remark 4. If it holds that \( V_1(\omega_0) = V_2(\omega_0) \), it is clear that the swapping is even. Therefore, without loss of generality, we can assume that \( V_1(\omega_0) \neq V_2(\omega_0) \). Also, it should be noted that the above argument is applicable to the simplest case that \( \Omega = \{\omega_0\} \), i.e., the one-point space.

3.2 The answer in which it is easy to notice the mistake (1)

The answer in Section 3.1 is best, but it does not explain why we make a mistake (1). Thus we add the following.
3.2.1 The simplest answer to Problem 1

Put \( \Omega = \{ (\omega, 2\omega) \mid \omega \in \mathbb{R}_+ \} \). Here note that the \( \Omega \) can be identified with \( \mathbb{R}_+ \), i.e.,

\[
\Omega \ni (\omega, 2\omega) \iff \omega \in \mathbb{R}_+ \tag{8}
\]

and assume that it has the Lebesgue measure \( \nu(d\omega) \), which is simply denoted by \( d\omega \) from here.

Define the observable \( O = (X(= \mathbb{R}_+), \mathcal{F}(= B_{\mathbb{R}_+} : \text{the Borel field}), F) \) in \( L^\infty(\Omega, d\omega) \) such that

\[
[F(\Xi)](\omega, 2\omega) \equiv [F(\Xi)](\omega) = \begin{cases} 
1 & \text{(if } \omega \in \Xi, \ 2\omega \in \Xi) \\
1/2 & \text{(if } \omega \in \Xi, \ 2\omega \notin \Xi) \\
1/2 & \text{(if } \omega \notin \Xi, \ 2\omega \in \Xi) \\
0 & \text{(if } \omega \notin \Xi, \ 2\omega \notin \Xi) 
\end{cases} (\forall (\omega, 2\omega) \in \Omega, \forall \Xi \in \mathcal{F}) \tag{9}
\]

Thus, for any unknown state \((\omega_0, 2\omega_0)(\in \Omega)\), we have the measurement \( M_{L^\infty(\Omega, d\omega)}(O = (X, \mathcal{F}, F), S_{[\delta(\omega_0, 2\omega_0)]}) \). And, we see, by Axiom PCMT 1, that

\[(H1) \text{ the probability that a measured value } x(\in X(= \mathbb{R}_+)) \text{ obtained by } M_{L^\infty(\Omega, d\omega)}(O = (X, \mathcal{F}, F), S_{[\delta(\omega_0, 2\omega_0)]}) \text{ is equal to } \{ \frac{\omega_0}{2\omega_0} \} \text{ is given by } \{ \frac{1}{2} \}. \]

Here, assume that \( \{ x = \omega_0 \\
x = 2\omega_0 \} \). Then, by the switching to \( \{ \frac{2\omega_0}{\omega_0} \} \), you gain \( \{ \frac{2\omega_0 - \omega_0}{\omega_0 - 2\omega_0} \} \).

This implies that the expectation of the switching gain is equal to

\[
(2\omega_0 - \omega_0)/2 + (\omega_0 - 2\omega_0)/2 = 0
\]

which implies that the swapping is even, i.e., no advantage and no disadvantage.

Since \((\omega_0, 2\omega_0)(\in \Omega)\) is assumed to be unknown, the measurement \( M_{L^\infty(\Omega, d\omega)}(O = (X, \mathcal{F}, F), S_{[\delta(\omega_0, 2\omega_0)]}) \) is also denoted by \( M_{L^\infty(\Omega, d\omega)}(O = (X, \mathcal{F}, F), S_{[\alpha]}) \). Thus, when you obtain a measured value \( \alpha(\in X) \) by \( M_{L^\infty(\Omega, d\omega)}(O = (X, \mathcal{F}, F), S_{[\alpha]}) \), you should conclude that the swapping is even.

3.2.2 Why do we make a mistake (1)?
Now we can explain why we make a mistake (1). Let $M_{L^\infty(\Omega,d\omega)}(O=(X,\mathcal{F},F),S_{[s]})$ be the measurement considered in Section 3.2.1. Assume that a measured value $\alpha$ is obtained by $M_{L^\infty(\Omega,d\omega)}(O=(X,\mathcal{F},F),S_{[s]})$. Here, note that, the likelihood function (3) is calculated as follows:

$$f(\alpha,\omega) \equiv \inf_{\omega_1 \in \Omega} \lim_{z \to \{\alpha\},\exists \omega_1,|F(\Xi)(\omega)|\neq 0} \frac{|F(\Xi)(\omega)|}{|F(\Xi)|}\omega_1 = \begin{cases} 1 & (\omega = (\alpha/2,\alpha) \text{ or } (\alpha,2\alpha)) \\ 0 & \text{(elsewhere)} \end{cases}$$

Therefore, we can infer, by Theorem 1 (Fisher’s maximum likelihood method), that

(H2) the unknown state $[s]$ is equal to $(\alpha/2,\alpha)$ or $(\alpha,2\alpha)$,

(1) if $[s] = (\alpha/2,\alpha)$ [resp. $[s] = (\alpha,2\alpha)$], the switching gain is $(\alpha/2 - \alpha)$ [resp. $(2\alpha - \alpha)$].

However, it is not guaranteed that

(H3) \["the probability that $[s] = (\alpha/2,\alpha)"=1/2,\]

\["the probability that $[s] = (\alpha,2\alpha)"=1/2,\]

\["the probability that $[s]$ is elsewhere"=0\]

That is, the phrase "with probability 1/2" in [(P1): Why is it paradoxical?] is wrong, and therefore, the expectation of the switching gain "$E_{\text{other}}(\alpha) - \alpha = (1/2)(\alpha/2) + (1/2)(2\alpha) - \alpha = \alpha/4 > 0$" is wrong. That is, it is impossible to calculate the expected value $E_{\text{other}}(\alpha)$. In other words, the expected value $E_{\text{other}}(\alpha)$ in the formula (1) is meaningless.

4 Bayesian approach to the two envelopes paradox

In the framework of the pure measurement $M_{L^\infty(\Omega,d\omega)}(O=(X,\mathcal{F},F),S_{[s]})$ (defined in Section 3.2), we can not derive the statement (H3). Thus, next, consider another situation of Problem 1 (Bayesian approach to the two envelopes paradox), i.e., the statistical measurement $M_{L^\infty(\Omega,d\omega)}(O=(X,\mathcal{F},F),S_{[s]}(\rho_0))$. Recalling the identification (8): $\Omega \ni (\omega,2\omega) \leftrightarrow \omega \in \mathbb{R}_+$, assume that

$$\rho_0(D) = \int_D h(\omega)d\omega \quad (\forall D \in \mathcal{B}_\Omega = \mathcal{B}_{\mathbb{R}_+})$$

where the probability density function $h : \Omega(\approx \mathbb{R}_+) \to \Omega(= \mathbb{R}_+)$ is assumed to be continuous positive function. That is, the mixed state $\rho_0(\in \mathcal{M}^\infty(\Omega(= \mathbb{R}_+)))$ has the probability density function $h$.

AxiomSCMT 1 says that

(I1) The probability $P(\Xi)$ ($\Xi \in \mathcal{B}_X = \mathcal{B}_{\mathbb{R}_+}$) that a measured value obtained by the statistical measurement $M_{L^\infty(\Omega,d\omega)}(O=(X,\mathcal{F},F),S_{[s]}(\rho_0))$ belongs to $\Xi(\in \mathcal{B}_X = \mathcal{B}_{\mathbb{R}_+})$ is given by

$$P(\Xi) = \int_\Omega |F(\Xi)(\omega)|\rho_0(d\omega) = \int_\Omega |F(\Xi)(\omega)|h(\omega)d\omega$$

$$= \int_{\Xi} \frac{h(x/2)}{4} + \frac{h(x)}{2} dx \quad (\forall \Xi \in \mathcal{B}_{\mathbb{R}_+})$$

(10)

Therefore, the expectation is given by

$$\int_{\mathbb{R}_+} xP(dx) = \frac{1}{2} \int_0^\infty x \cdot \left(\frac{h(x/2)}{2} + h(x)\right)dx = \frac{3}{2} \int_{\mathbb{R}_+} xh(x)dx$$

(11)

Further, Theorem 2 (Bayes’ theorem) says that
When a measured value $\alpha$ is obtained by the statistical measurement $M_{L^\infty}(\Omega,d\omega)(\Omega = (X,F,F))$, then the post-state $\rho^{\alpha}_{\text{post}}(\in \mathcal{M}^m(\Omega))$ is given by

\[ (5) = \rho^{\alpha}_{\text{post}} = \frac{h(\alpha)}{\frac{h(\alpha)}{2} + h(\alpha)} \delta(\frac{\alpha}{2},\alpha) + \frac{h(\alpha)}{\frac{h(\alpha)}{2} + h(\alpha)} \delta(\alpha,2\alpha) \]  

Hence,

\[ (12) \]

If $[\ast] = \left\{ \delta(\frac{\alpha}{2},\alpha), \delta(\alpha,2\alpha) \right\}$, then we have to change $\left\{ \alpha \rightarrow \frac{\alpha}{2}, \alpha \rightarrow 2\alpha \right\}$, and thus we get the switching gain $\left\{ \frac{\alpha}{2} - \alpha = -\frac{\alpha}{2} \right\}$. Therefore, the expectation of the switching gain is calculated as follows:

\[
\int_{\mathbb{R}_+ \times \mathbb{R}_+} \left( -\frac{\alpha}{2} \right) \frac{h(\alpha)}{\frac{h(\alpha)}{2} + h(\alpha)} + \alpha \frac{h(\alpha)}{\frac{h(\alpha)}{2} + h(\alpha)} P(d\alpha) = 0 \]

Therefore, if $\int \omega h(\omega) d\omega < \infty$, we see, by (11), that the swapping is even, i.e., no advantage and no disadvantage, in the sense of (13).

### 5 The St. Petersburg two-envelope paradox in quantum language

#### 5.1 The St. Petersburg two-envelope paradox

In what follows, we introduce the St. Petersburg two-envelope problem (cf. [1]), which is well known as a kind of high school students’ mathematical puzzle.

**Problem 2** [The St. Petersburg two envelope problem]. You are presented with two envelopes, A and B. You are told that each of them contains an amount determined by the following procedure, performed separately for each envelope: a coin was flipped until it came up heads, and if it came up heads on the nth trial, $2^n$ is put into the envelope. This procedure is performed separately for each envelope. You are given envelope A, and you find $2^m$ dollars in envelope A. Now you are offered the options of keeping A or switching to B. What should you do?

[(P2);Why is it paradoxical?]. You reason that, before opening the envelopes A and B, the expected values $E(x)$ and $E(y)$ in A and B is infinite respectively. For any $2^m$, if you knew that A contained $x = 2^m$ dollars, then the expected value $E(y)$ in B would still be infinite. Therefore, you should switch to B. But this seems clearly wrong, as your information about A and B is symmetrical. This is the famous St. Petersburg two-envelope paradox (i.e., ”The Other Person’s Envelope is Always Greener”).

#### 5.2 (P2): The St. Petersburg two-envelope paradox in Statistical CMT (without Bayes’ method)

Here, let us explain the St. Petersburg two-envelope paradox in Statistical CMT (without Bayes’ method).
Define the state space $\Omega$ such that $\Omega = \{\omega \mid \omega = 1, 2, \cdots\}$ with the counting measure $\nu$, that is, the set of all natural numbers. And define the observable $O = (X, \mathcal{F}, F)$ such that

\[ X = \{k \mid k = 1, 2, \cdots\}, \quad \mathcal{F} = 2^X \]

\[
[F(\Xi)](\omega) = \begin{cases} 
1 & \text{if } \omega \in \Xi \\
0 & \text{elsewhere} 
\end{cases} \quad (\forall \Xi \in \mathcal{F}, \forall \omega = 1, 2, \cdots)
\]

Define the mixed state $\rho_0$ (i.e., the probability measure on $\Omega$) such that

\[
\rho_0(\{\omega\}) = \begin{cases} 
1/2^m & \text{if } \omega = 2^m, m = 1, 2, \ldots \\
0 & \text{elsewhere} 
\end{cases}
\]

Consider the statistical measurement $M_{L,\infty}(\Omega, \nu)(O = (X, \mathcal{F}, F), S_{[y]}(\rho_0))$. Axiom$^{SCMT}$ 1 says that \( \text{(J1)} \) the probability that a measured value $x(\in X)$ obtained by $M_{L,\infty}(\Omega, \nu)(O = (X, \mathcal{F}, F), S_{[y]}(\rho_0))$ is equal to $2^k$ is given by $2^{-k}$.

Therefore, the expectation $E(x)$ of a measured value is equal to

\[
E(x) = \sum_{k=1}^{\infty} 2^k \cdot 2^{-k} = \infty
\]

Now consider the parallel measurement $M_{L,\infty}(\Omega, \nu)(O \otimes O = (X \times X, \mathcal{F} \boxtimes \mathcal{F}, F \otimes F), S_{[y]}(\rho_0 \otimes \rho_0))$, where $\rho_0 \otimes \rho_0$ is the product measure on $\Omega \times \Omega$. By the similar way of Definition 1, we consider that this parallel measurement is the same as taking a measurement $M_{L,\infty}(\Omega, \nu)(O = (X, \mathcal{F}, F), S_{[y]}(\rho_0))$ twice. Let $(x, y)(\in X \times X)$ be the measured value obtained by the parallel measurement $M_{L,\infty}(\Omega, \nu)(O \otimes O = (X \times X, \mathcal{F} \boxtimes \mathcal{F}, F \otimes F), S_{[y]}(\delta_{(0,0)}))$. We of course see that the expectation $E(x, y) = (\infty, \infty)$. Problem 2 says that you got a measured value $2^m$ (i.e., 2$^m$ dollars in Envelope A). Namely, $x = 2^m$. However, since $E(y) = \infty$, in the next measurement, you are expected to get a measured value $y$ such that $E(y) = \infty > 2^m$. That is, you are expected to find $y$ dollars (i.e., $E(y) = \infty > 2^m$) in Envelope B. Thus, you should switch to Envelope B.

**Remark 5.** (i): Note that, in the above argument, Axiom$^{SCMT}$ 1 is used, and not Bayes’ theorem (Theorem 2) (cf. the formula (5)).

(ii): Recall the statement ”as your information about A and B is symmetrical” in [(P2): Why is it paradoxical?]. This statement is not true, since you find $2^m$ dollars in Envelope A, but you do not open Envelope B yet. Therefore, Problem 2 is not paradoxical. That is, we can believe in the proverb: “The Other Person’s Envelope is Always Greener”. However, if you do not open both envelopes, Envelopes A and B are even.

(iii): The probability $P(y > 2^m)$ such that ”$y > 2^m$” is easily calculated as follows.

\[
P(y > 2^m) = \frac{1}{2^m}
\]

Concerning the St. Petersburg two-envelope paradox, this ”probability criterion” may be rather reasonable.

### 5.3 (P2): The St. Petersburg two-envelope paradox in Pure CMT ($\approx$ non-Bayesian statistics)

Let us explain the St. Petersburg two-envelope paradox in Pure CMT, which is essentially the same as the argument in the previous section 5.2.
Define the state space $\Omega$ such that $\Omega = \{\omega_0\}$, that is, the set composed of one element, where $\nu(\{\omega_0\}) = 1$. And define the observable $O = (X, F, F')$ such that

$$X = \{2^k \mid k = 1, 2, \cdots\}, \quad F = 2^X$$

$$[F(\{2^k\})](\omega_0) = 2^{-k} \quad (k = 1, 2, \cdots)$$

Consider the measurement $M_{L^\infty(\Omega, \nu)}(O = (X, F, F'), S[\delta_{\omega_0}])$. Then, Axiom PCMT 1 says that

(J2) the probability that a measured value $x \in X$ obtained by $M_{L^\infty(\Omega, \nu)}(O = (X, F, F'), S[\delta_{\omega_0}])$ is equal to $2^k$ is given by $2^{-k}$.

Therefore, the expectation $E(x)$ of a measured value $x$ is equal to

$$E(x) = \sum_{k=1}^{\infty} 2^k \cdot 2^{-k} = \infty$$

Now consider the parallel measurement $M_{L^\infty(\Omega \times \Omega, \nu \otimes \nu)}(O \otimes O = (X \times X, F \boxdot F, F \otimes F), S[\delta_{(\omega_0, \omega_0)}])$. Recalling Definition 1, we consider that this parallel measurement is the same as taking a measurement $M_{L^\infty(\Omega, \nu)}(O = (X, F, F'), S[\delta_{\omega_0}])$ twice. Let $(x, y) \in X \times X$ be the measured value obtained by the parallel measurement $M_{L^\infty(\Omega \times \Omega, \nu \otimes \nu)}(O \otimes O = (X \times X, F \boxdot F, F \otimes F), S[\delta_{(\omega_0, \omega_0)}])$. We of course see that the expectation $E(x, y) = (\infty, \infty)$. Problem 2 says that you got a measured value $2^m$ (i.e., $2^m$ dollars in Envelope A). Namely, $x = 2^m$. However, since $E(y) = \infty$, in the next measurement, you are expected to get a measured value $y$ such that $E(y) = \infty > 2^m$. That is, you are expected to find $y$ dollars (i.e., $E(y) = \infty > 2^m$) in Envelope B. Thus, you should switch to Envelope B.

**Remark 6.** The above answer may be rather fit for the following problem.

(K) In the envelope A and the envelope B, there are infinite pins $\{P_k\}_{k=1}^{\infty}$ with the length $2^{-k}$. The pin $P_k$ may be identified with the interval $(2^{-k}, 2^{1-k}] (\subseteq (0, 1])$. Assume that the pin $P_k$ is $2^k$ dollars. And further assume that the probability that a pin $P_k$ will be picked from Envelope A (and Envelope B) is given by $2^{-k}$. You are given envelope A, and, from the envelope A, you pick up a pin $P_m$, which is $2^m$ dollars. Now you are offered the options of keeping A or switching to B. What should you do?

Although this and Problem 2 are similar but somewhat different, we consider that two answers (in Sections 5.2 and 5.3) are valid.

**6 Conclusions**

In order to show the great descriptive power of quantum language (i.e., “quantum language is future statistics”), we want to assert that

(L1) if a probabilistic problem is described in terms of quantum language, the problem will be automatically solved.

As one of examples (L1), in this paper we showed that the two envelope problem is automatically solved in Section 3 (non-Bayesian two envelopes paradox) and Section 4 (Bayesian two envelopes paradox).

The readers may ask the following question:

(L2) Why is it hard to make a mistake in quantum language method?
We consider that this is due to the fact:

(L3) Quantum language has visible key-words: ”measurement”, ”observable”, ”state”, ”measured value”. And these concepts are motivated by quantum mechanics.

On the other hand, statistics has invisible key-words: ”probability space”, ”random variable”, ”parameter”.

This is our answer to the question (L2). Also, it should be noted that the sum (6) of observables has not appeared once throughout our research [3]- [14], that is, it is rare in the usual situations. In this sense, the two envelopes problem may be tricky and paradoxical. After all, we conclude that quantum language provides the final answer (i.e., the easiest and deepest understanding ) to the two envelope-problem.

Also, we add:

(L4) In Section 5, we see that the St. Petersburg two-envelope paradox has two formulations (i.e., Classical SMT and Classical PMT), and also, the St. Petersburg two-envelope paradox is independent of Bayes’ method, and thus it is not related to Bayesian statistics (cf. Remark 2).

For completeness, our main assertion (G6) is again rewritten as follows.

(L5) quantum language says that, if Problem 1 is a scientific statement, Problem 1 should be essentially the same as Problem 1’. If the reader wants to assert that these are different, he has to propose another language (except quantum) by which Problem 1 and Problem 1’ are described as the different problems. That is because we believe Wittgenstein’s words (i.e., the spirit of the philosophy of language):”The limits of my language mean the limits of my world.”

We hope that our proposal will be discussed and examined from various view-points.

References

[1] D.J. Chalmers, “The St. Petersburg Two-Envelope Paradox,” Analysis, Vol.62, 155-157, 2002.

[2] E. B. Davies, “Quantum Theory of Open Systems,” Academic Press, 1976.

[3] S. Ishikawa, A Quantum Mechanical Approach to Fuzzy Theory, Fuzzy Sets and Systems, Vol. 90, No. 3, 277-306, 1997, doi: 10.1016/S0165-0114(96)00114-5

[4] S. Ishikawa, Statistics in measurements, Fuzzy sets and systems, Vol. 116, No. 2, 141-154, 2000 doi:10.1016/S0165-0114(98)00280-2

[5] S. Ishikawa, Mathematical Foundations of Measurement Theory, Keio University Press Inc. 335pages, 2006, (http://www.keio-up.co.jp/kup/mfmt/)

[6] S. Ishikawa, A New Interpretation of Quantum Mechanics, Journal of quantum information science, Vol. 1, No. 2, 35-42, 2011, doi: 10.4236/jqis.2011.12005 (http://www.scirp.org/journal/PaperInformation.aspx?paperID=7610)

[7] S. Ishikawa, Quantum Mechanics and the Philosophy of Language: Reconsideration of traditional philosophies, Journal of quantum information science, Vol. 2, No. 1, 2-9, 2012 doi: 10.4236/jqis.2012.21002 (http://www.scirp.org/journal/PaperInformation.aspx?paperID=18194)

[8] S. Ishikawa, A Measurement Theoretical Foundation of Statistics, Applied Mathematics, Vol. 3, No. 3, 283-292, 2012, doi: 10.4236/am.2012.33044 (http://www.scirp.org/journal/PaperInformation.aspx?paperID=18109&
[9] S. Ishikawa, *The linguistic interpretation of quantum mechanics*, arXiv:1204.3892v1 [physics.hist-ph], (2012) (http://arxiv.org/abs/1204.3892)

[10] S. Ishikawa, *Measurement Theory in the Philosophy of Science*, arXiv:1209.3483 [physics.hist-ph] 2012. (http://arxiv.org/abs/1209.3483)

[11] S. Ishikawa, *What is Statistics?: The Answer by Quantum Language*, arXiv:1207.0407 [physics.data-an] 2012. (http://arxiv.org/abs/1207.0407)

[12] S. Ishikawa, “Monty Hall Problem and the Principle of Equal Probability in Measurement Theory,” *Applied Mathematics*, Vol. 3 No. 7, 2012, pp. 788-794, doi: 10.4236/am.2012.37117. (http://www.scirp.org/journal/PaperInformation.aspx?PaperID=19884)

[13] S. Ishikawa, *The Final Solutions of Monty Hall Problem and Three Prisoners Problem*, arXiv:1408.0963 [stat.OT] 2014. (http://arxiv.org/abs/1408.0963)

[14] S. Ishikawa, K. Kikuchi, *Kalman filter in quantum language*, arXiv:1404.2664 [math.ST] 2014. (http://arxiv.org/abs/1404.2664)

[15] A. Kolmogorov, “Foundations of the Theory of Probability (Translation),” Chelsea Pub Co. Second Edition, New York, 1960.

[16] G. Martin, “Aha! Gotcha: Paradoxes to Puzzle and Delight” Freeman and Company, 1982

[17] N.D. Mermin, “Boojums all the way through, Communicating Science in a Prosaic Age” Cambridge university press, 1994.

[18] B. Nalebuff, “The other person’s envelope is always greener,” Journal of economic Perspectives Vol.3(1), 171-181, 1989

[19] J. von Neumann, “Mathematical Foundations of Quantum Mechanics,” Springer Verlag, Berlin, 1932.

[20] S. Sakai, $C^*$-algebras and $W^*$-algebras, Ergebnisse der Mathematik und ihrer Grenzgebiete (Band 60), Springer-Verlag, (1971)

[21] K. Yosida, “Functional Analysis,” Springer-Verlag, 6th edition, 1980.