OPERATORS WHOSE CONJUGATION ORBITS SATISFY POLYNOMIAL GROWTH CONDITIONS

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Abstract. Let $A$ be a bounded linear operator on a complex Banach space $X$. For a given $\alpha \geq 0$, we consider the class $D^\alpha_A(\mathbb{R})$ of all bounded linear operators $T$ on $X$ for which there exists a constant $C_T > 0$, such that
$$\left\| e^{\alpha A} T e^{-\alpha A} \right\| \leq C_T (1 + |t|)^\alpha, \quad \forall t \in \mathbb{R}.$$ We present complete description of the class $D^\alpha_A(\mathbb{R})$ in the case when the spectrum of $A$ consists of one point. These results are linked to the decomposability of $A$. Some estimates for the norm of the commutator $AT - TA$ are obtained in the case $0 \leq \alpha < 1$.

1. Introduction

Let $H$ be an infinite dimensional separable Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on $H$. A family $\{E\}$ of subspaces of $H$ is called a nest if it is totally ordered by inclusion. Given a nest $\{E\}$, Ringrose [10] introduced the concept of the associated nest algebra $\text{Alg}\{E\}$ defined by
$$\text{Alg}\{E\} = \{T \in B(H) : TE \subseteq E, \forall E \in \{E\}\}.$$ In [10], Leobl and Muhly show that every nest algebra is the algebra of all analytic operators with respect to the one-parameter representation $T \rightarrow e^{-itA}T e^{itA}$ of inner $*$-automorphisms of $B(H)$, where $A$ is a self-adjoint operator on $H$.

For an invertible operator $A$ on $H$, Deddens [4] introduced the set
$$D_A := \left\{ T \in B(H) : \sup_{n \geq 0} \left\| A^nTA^{-n} \right\| < \infty \right\}.$$ Notice that $D_A$ is a normed (not necessarily closed) algebra with identity and contains the commutant $\{A\}'$ of $A$. In the same paper, Deddens shows that if $A \geq 0$, then $D_A$ coincides with the nest algebra associated with the nest of spectral subspaces of $A$. This gives a new and convenient characterization of nest algebras. In [4], Deddens conjectured that the equality $D_A = \{A\}'$ holds if the spectrum of $A$ is reduced to $\{1\}$. In [17], Roth gave a negative answer to Deddens conjecture. He shows existence of a quasinilpotent operator $A$ for which $D_{I+A} \neq \{I+A\}'$.

Let $X$ be a complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on $X$. In [13], Williams proved that if the spectrum of an invertible operator $A \in B(X)$ is reduced to $\{1\}$ and $\sup_{n \in \mathbb{Z}} \left\| A^nTA^{-n} \right\| < \infty$, then $AT = TA$.

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For a fixed $A \in B(\mathcal{X})$ and $\alpha \geq 0$, we define the class $\mathcal{D}_A^\alpha(\mathbb{R})$ of all operators $T \in B(\mathcal{X})$ for which the growth of the function
\[ t \mapsto \|e^{tA}Te^{-tA}\| \]
is at most polynomial in $t \in \mathbb{R}$, explicitly, there exists a constant $C_T > 0$ such that
\[ \|e^{tA}Te^{-tA}\| \leq C_T (1 + |t|)^\alpha, \quad \forall t \in \mathbb{R}. \]

Notice that $\mathcal{D}_A^\alpha(\mathbb{R})$ is a linear (not necessarily closed) subspace of $B(\mathcal{X})$ and contains the commutant of $A$. In the case $\alpha = 0$, instead of $\mathcal{D}_A^\alpha(\mathbb{R})$ we will use the notation $\mathcal{D}_A(\mathbb{R})$. Notice also that $\mathcal{D}_A(\mathbb{R})$ is a normed (not necessarily closed) algebra with identity.

In Section 2, we give complete characterization of the class $\mathcal{D}_A^\alpha(\mathbb{R})$ in the case when the spectrum of $A$ consists of one point. Section 3 contains results related to the decomposability of $A$. In the case $0 \leq \alpha < 1$, some estimates for the norm of the commutator $AT - TA$ are obtained in Section 4, where $T \in \mathcal{D}_A^\alpha(\mathbb{R})$.

2. The class $\mathcal{D}_A^\alpha(\mathbb{R})$

In this section, we give complete characterization of the class $\mathcal{D}_A^\alpha(\mathbb{R})$ in the case when the spectrum of $A$ consists of one point. As usual, $\sigma(T)$ will denote the spectrum of the operator $T \in B(\mathcal{X})$. Throughout, $[\alpha]$ denotes the integer part of $\alpha \in \mathbb{R}$.

Let $A \in B(\mathcal{X})$ and $\Delta_A$ be the inner derivation on $B(\mathcal{X})$:
\[ \Delta_A : T \mapsto AT - TA, \quad T \in B(\mathcal{X}). \]

Then, we can write
\[ \Delta_A^\alpha(T) = \sum_{k=0}^n (-1)^k \binom{n}{k} A^{n-k} T A^k \quad (n \in \mathbb{N}). \]

We have the following:

**Theorem 2.1.** If the spectrum of the operator $A \in B(\mathcal{X})$ consists of one point, then
\[ \mathcal{D}_A^\alpha(\mathbb{R}) = \ker \Delta_A^{[\alpha]+1}. \]
In particular, if $0 \leq \alpha < 1$, then $\mathcal{D}_A^\alpha(\mathbb{R}) = \{A\}'$.

Before to prove this theorem, we first prove the following:

**Theorem 2.2.** If the spectrum of the operator $A \in B(\mathcal{X})$ lies on the real line, then
\[ \mathcal{D}_A^\alpha(\mathbb{R}) = \ker \Delta_A^{[\alpha]+1}. \]
In particular, if $0 \leq \alpha < 1$, then $\mathcal{D}_A^\alpha(\mathbb{R}) = \{A\}'$.

For related results see also, [1] [13]. For the proof of Theorem 2.2, we need some preliminary results.

For an arbitrary $T \in B(\mathcal{X})$ and $x \in \mathcal{X}$, we define $\rho_T(x)$ to be the set of all $\lambda \in \mathbb{C}$ for which there exists a neighborhood $U_\lambda$ of $\lambda$ with $u(z)$ analytic on $U_\lambda$ having values in $\mathcal{X}$ such that $(zI - T)u(z) = x$ for all $z \in U_\lambda$. This set is open and contains the resolvent set $\rho(T)$ of $T$. By definition, the local spectrum of $T$ at $x \in \mathcal{X}$, denoted by $\sigma_T(x)$ is the complement of $\rho_T(x)$, so it is a compact subset of $\sigma(T)$. This object is most tractable if the operator $T$ has the single-valued extension property (SVEP), i.e. for every open set $U$ in $\mathbb{C}$, the only analytic function $u : U \rightarrow \mathcal{X}$ for
which the equation \((zI - T)u(z) = 0\) holds is the constant function \(u \equiv 0\). If \(T\) has SVEP, then \(\sigma_T(x) \neq \emptyset\), whenever \(x \in X \setminus \{0\}\) \([9\) Proposition 1.2.16\]. It can be seen that an operator \(T \in B(X)\) having spectrum without interior points has the SVEP. Ample information about local spectra can be found in \([2, 3, 5, 9]\).

The local spectral radius of \(T \in B(X)\) at \(x \in X\) is defined by

\[ r_T(x) = \sup \{|\lambda| : \lambda \in \sigma_T(x)\}. \]

If \(T\) has SVEP, then

\[ r_T(x) = \lim_{n \to \infty} \|T^n x\|^\frac{1}{n} \quad \text{\cite{9} Proposition 3.3.13}. \]

Recall that a weight function (shortly a weight) \(\omega\) is a continuous function on \(\mathbb{R}\) such that \(\omega(t) \geq 1\) and \(\omega(t+s) \leq \omega(t) \omega(s)\) for all \(t, s \in \mathbb{R}\). For a weight function \(\omega\), by \(L^1_\omega(\mathbb{R})\) we will denote the Banach space of the functions \(f \in L^1(\mathbb{R})\) with the norm

\[ \|f\|_{1, \omega} = \int_{\mathbb{R}} |f(t)| \omega(t) \, dt < \infty. \]

The space \(L^1_\omega(\mathbb{R})\) with convolution product and the norm \(\|\cdot\|_{1, \omega}\) is a commutative Banach algebra and is called Beurling algebra. The dual space of \(L^1_\omega(\mathbb{R})\), denoted by \(L^\infty_\omega(\mathbb{R})\), is the space of all measurable functions \(g\) on \(\mathbb{R}\) such that

\[ \|g\|_{\infty, \omega} := \text{ess sup}_{t \in \mathbb{R}} \frac{|g(t)|}{\omega(t)} < \infty. \]

The duality being implemented by the formula

\[ \langle g, f \rangle = \int_{\mathbb{R}} g(-t) f(t) \, dt, \forall f \in L^1_\omega(\mathbb{R}), \forall g \in L^\infty_\omega(\mathbb{R}). \]

We say that the weight \(\omega\) is regular if

\[ \int_{\mathbb{R}} \frac{\log \omega(t)}{1 + t^2} \, dt < \infty. \]

For example, \(\omega(t) = (1 + |t|)^\alpha (\alpha \geq 0)\) is a regular weight and is called polynomial weight. If \(\omega\) is a regular weight, then

\[ \lim_{t \to +\infty} \frac{\log \omega(t)}{t} = \lim_{t \to +\infty} \frac{\log \omega(-t)}{t} = 0. \tag{2.1} \]

Consequently, the maximal ideal space of the algebra \(L^1_\omega(\mathbb{R})\) can be identified with \(\mathbb{R}\) (for instance, see \([6, 11, 14]\)). The Gelfand transform of \(f \in L^1_\omega(\mathbb{R})\) is just the Fourier transform of \(f\). Moreover, the algebra \(L^1_\omega(\mathbb{R})\) is regular in the Shilov sense \([6\) Ch.VI\]. Notice also that \(L^1_\omega(\mathbb{R})\) is Tauberian, that is, the set \(\{f \in L^1_\omega(\mathbb{R}) : \text{supp} \hat{f} \text{ is compact}\}\) is dense in \(L^1_\omega(\mathbb{R})\) \([11\) Ch.5\]. Below, we will assume that \(\omega\) is a regular weight.

Denote by \(M_\omega(\mathbb{R})\) the Banach algebra (with respect to convolution product) of all complex regular Borel measures on \(\mathbb{R}\) such that

\[ \|\mu\|_{1, \omega} := \int_{\mathbb{R}} \omega(t) \, d|\mu|(t) < \infty. \]

The algebra \(L^1_\omega(\mathbb{R})\) is naturally identifiable with a closed ideal of \(M_\omega(\mathbb{R})\). By \(\hat{f}\) and \(\hat{\mu}\), we will denote the Fourier and the Fourier-Stieltjes transform of \(f \in L^1_\omega(\mathbb{R})\) and \(\mu \in M_\omega(\mathbb{R})\), respectively.
As usual, to any closed subset $S$ of $\mathbb{R}$, the following two closed ideals of $L^1_\omega(\mathbb{R})$ associated:

$$I_\omega (S) := \left\{ f \in L^1_\omega(\mathbb{R}) : \hat{f}(S) = \{0\} \right\}$$

and

$$J_\omega (S) := \left\{ f \in L^1_\omega(\mathbb{R}) : \text{supp} \hat{f} \text{ is compact and supp} \hat{f} \cap S = \emptyset \right\}.$$  

The ideals $J_\omega (S)$ and $I_\omega (S)$ are respectively, the smallest and the largest closed ideals in $L^1_\omega(\mathbb{R})$ with hull $S$. When these two ideals coincide, the set $S$ is said to be a set of synthesis for $L^1_\omega(\mathbb{R})$ (for instance, see [3, Sect. 8.3]).

Notice that $I_\omega (\{\infty\}) = L^1_\omega(\mathbb{R})$ and

$$J_\omega (\{\infty\}) = \left\{ f \in L^1_\omega(\mathbb{R}) : \text{supp} \hat{f} \text{ is compact} \right\}.$$  

Since the algebra $L^1_\omega(\mathbb{R})$ is Tauberian, we have $I_\omega (\{\infty\}) = J_\omega (\{\infty\})$. Hence, $\{\infty\}$ is a set of synthesis for $L^1_\omega(\mathbb{R})$. Notice also that if $\omega(t) = (1 + |t|)^\alpha$ ($0 \leq \alpha < 1$), then each point of $\mathbb{R}$ is a set of synthesis for $L^1_\omega(\mathbb{R})$ [15, Ch.6].

Let $M$ be a non-void subset of $L^\infty_\omega(\mathbb{R})$. A point $\lambda \in \mathbb{R}$ is said to be Beurling spectrum of $M$ if the character $e^{-i\lambda t}$ belongs to the weak*-closed translation invariant subspace of $L^\infty_\omega(\mathbb{R})$ generated by $M$. By $\sp_B \{M\}$, we will denote the set of all Beurling spectrum of $M$. It is easy to verify that

$$\sp_B \{M\} = \text{hull} \left(\mathcal{I}(M)\right),$$

where

$$\mathcal{I}(M) = \left\{ f \in L^1_\omega(\mathbb{R}) : f * g = 0, \; \forall g \in M \right\}$$

is a closed ideal of $L^1_\omega(\mathbb{R})$. Notice also that

$$\sp_B \{M\} = \bigcup_{g \in M} \sp_B \{g\}.$$  

For $g \in L^\infty_\omega(\mathbb{R})$, we put $g^\vee(t) := g(-t)$. Clearly,

$$\sp_B \{g^\vee\} = \{-\lambda : \lambda \in \sp_B \{g\}\}.$$  

Recall that the Carleman transform of $g \in L^\infty_\omega(\mathbb{R})$ is defined as the analytic function $G(z)$ on $\mathbb{C} \setminus i\mathbb{R}$, given by

$$G(z) = \begin{cases} \int_0^\infty e^{-zt}g(t)\; dt, & \text{Re} \; z > 0; \\ -\int_{-\infty}^0 e^{-zt}g(t)\; dt, & \text{Re} \; z < 0. \end{cases}$$

It is known [7] that $\lambda \in \sp_B \{g\}$ if and only if the Carleman transform $G(z)$ of $g$ has no analytic extension to a neighborhood of $i\lambda$.

Let $\omega$ be a weight function, $T \in B(X)$ and let

$$E_T^\omega := \left\{ x \in X : \exists C > 0, \; \|e^{tT}x\| \leq C\omega(t), \; \forall t \in \mathbb{R} \right\}.$$  

Then, $E_T^\omega$ is a linear (non-closed, in general) subspace of $X$. If $x \in E_T^\omega$, then for an arbitrary $\mu \in M_\omega(\mathbb{R})$, we can define $x_\mu \in X$ by

$$x_\mu = \int_{\mathbb{R}} e^{tT}xd\mu(t).$$

Clearly, $\mu \mapsto x_\mu$ is a bounded linear map from $M_\omega(\mathbb{R})$ into $X$;

$$\|x_\mu\| \leq C \|\mu\|_{1,\omega}, \; \forall \mu \in M_\omega(\mathbb{R}).$$
Further, from the identity
\[ e^{tT}x_\mu = \int e^{(t+s)T} x d\mu(s) , \]
we can write
\[
\|e^{tT}x_\mu\| \leq \int \|e^{(t+s)T}x\|d|\mu|(s) \\
\leq C \int \omega(t+s)d|\mu|(s) \\
\leq C \int \omega(t)\omega(s)d|\mu|(s) \\
= C \|\mu\|_{1,\omega}(t), \ \forall t \in \mathbb{R}.
\]
This shows that \( x_\mu \in E_\omega^T \) for every \( \mu \in M_\omega(\mathbb{R}) \). It is easy to check that
\[
(x_\mu)_\nu = x_{\mu*\nu}, \ \forall \mu, \nu \in M_\omega(\mathbb{R}).
\]
It follows that if \( x \in E_\omega^T \), then
\[
I_x := \left\{ f \in L_\omega^1(\mathbb{R}) : x_f = 0 \right\}
\]
is a closed ideal of \( L_\omega^1(\mathbb{R}) \), where
\[
x_f = \int f(t)e^{tT}x dt.
\]
For a given \( x \in E_\omega^T \), consider the function
\[
(2.2) \quad u(z) := \begin{cases} 
\int_0^\infty e^{-zt}e^{tT}x dt, & \text{Re } z > 0; \\
0 & \text{Re } z = 0; \\
-\int_{-\infty}^0 e^{-zt}e^{tT}x dt, & \text{Re } z < 0.
\end{cases}
\]
It follows from (2.1) that \( u(z) \) is a function analytic on \( \mathbb{C} \setminus \mathbb{i}\mathbb{R} \). Let \( a := \text{Re } z > 0 \).

Then, for an arbitrary \( s > 0 \), we can write
\[
(zI-T)^s \int_0^se^{-zt}e^{tT}x dt = -\int_0^s \frac{d}{dt} e^{(T-zI)t}x dt + e^{s(T-zI)}x.
\]
Since
\[
\|e^{s(T-zI)}x\| = e^{-as}\|e^{sT}x\| \leq Ce^{-as}\omega(s)
\]
and
\[
\lim_{s \to +\infty} e^{-as}\omega(s) = 0,
\]
we have
\[
(zI-T)u(z) = x, \ \forall z \in \mathbb{C} \text{ with } \text{Re } z > 0.
\]
Similarly,
\[
(zI-T)u(z) = x, \ \forall z \in \mathbb{C} \text{ with } \text{Re } z < 0.
\]
Hence
\[
(2.3) \quad (zI-T)u(z) = x, \ \forall z \in \mathbb{C} \setminus i\mathbb{R}.
\]
This clearly implies that \( \sigma_T(x) \subset i\mathbb{R} \).

Thus we have the following:
Proposition 2.3. Let \( \omega \) be a regular weight. Assume that \( T \in B(X) \) and \( x \in X \) satisfy the condition \( \| e^{tT}x \| \leq C \omega(t) \) for all \( t \in \mathbb{R} \) and for some \( C > 0 \). Then, \( \sigma_T(x) \subset i\mathbb{R} \).

Now, assume that \( T \) has SVEP. We claim that \( \sigma_T(x) \) consists of all \( \lambda \in i\mathbb{R} \) for which the function \( u(z) \) has no analytic extension to a neighborhood of \( \lambda \). To see this, let \( v(z) \) be the analytic extension of \( u(z) \) to a neighborhood \( U_\lambda \) of \( \lambda \in i\mathbb{R} \). It follows from the identity (2.3) that the function
\[
 w(z) := (zI - T)v(z) - x
\]
vanishes on \( U_\lambda^+ := \{ z \in U_\lambda : \text{Re} z > 0 \} \) and on \( U_\lambda^- := \{ z \in U_\lambda : \text{Re} z < 0 \} \). By uniqueness theorem, \( w(z) = 0 \) for all \( z \in U_\lambda \). So we have
\[
(zI - T)v(z) = x, \quad \forall z \in U_\lambda.
\]
This shows that \( \lambda \in \rho_T(x) \). If \( \lambda \in \rho_T(x) \cap i\mathbb{R} \), then there exists a neighborhood \( U_\lambda \) of \( \lambda \) with \( v(z) \) analytic on \( U_\lambda \) having values in \( X \) such that
\[
(zI - T)v(z) = x, \quad \forall z \in U_\lambda.
\]
By (2.3),
\[
(zI - T)(u(z) - v(z)) = 0, \quad \forall z \in U_\lambda^+, \quad \forall z \in U_\lambda^-.
\]
Since \( T \) has SVEP, we have
\[
u(z) = \nu(z), \quad \forall z \in U_\lambda^+, \quad \forall z \in U_\lambda^-.
\]
This shows that \( u(z) \) can be analytically extended to a neighborhood of \( \lambda \).

Let \( x \in E_T^\sigma \). For a given \( \varphi \in X^* \), define a function \( \varphi_x \) on \( \mathbb{R} \) by
\[
\varphi_x(t) = \langle \varphi, e^{tT}x \rangle.
\]
Then, \( \varphi_x \) is continuous and
\[
|\varphi_x(t)| \leq C \|\varphi\| \omega(t), \quad \forall t \in \mathbb{R}.
\]
Consequently, \( \varphi_x \in L^\infty_\omega(\mathbb{R}) \). Taking into account the identity (2.2), we have
\[
\langle \varphi, u(z) \rangle = \begin{cases} 
\int_0^\infty e^{-zt} \varphi_x(t) \, dt, & \text{Re } z > 0; \\
- \int_{-\infty}^0 e^{-zt} \varphi_x(t) \, dt, & \text{Re } z < 0.
\end{cases}
\]
This shows that the function \( z \to \langle \varphi, u(z) \rangle \) is the Carleman transform of \( \varphi_x \). It follows that
\[
isp_B \{ \varphi_x \} \subseteq \sigma_T(x), \quad \forall \varphi \in X^*,
\]
and so
\[
\bigcup_{\varphi \in X^*} \sp_B \{ \varphi_x \} \subseteq -i\sigma_T(x).
\]
To show the reverse inclusion, assume that \( \lambda_0 \in \mathbb{R} \) and
\[
\lambda_0 \notin \bigcup_{\varphi \in X^*} \sp_B \{ \varphi_x \}.
\]
Then, there exist a neighborhood \( U \) of \( \bigcup_{\varphi \in X^*} \sp_B \{ \varphi_x \} \) and \( \varepsilon > 0 \) such that
\[
(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \cap U = \emptyset.
\]
Since the algebra \( L^1_\omega(\mathbb{R}) \) is regular, there exists a function \( f \in L^1_\omega(\mathbb{R}) \) such that \( \tilde{f} = 1 \) on \( [\lambda_0 - \varepsilon/2, \lambda_0 + \varepsilon/2] \) and \( \tilde{f} = 0 \) on \( \overline{U} \). Notice that
\( \hat{f} \) vanishes in a neighborhood of \( \text{sp}_B \{ \varphi_x \} \) and \( \text{supp} \hat{f} \subseteq [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \). Consequently, \( f \) belongs to the smallest ideal of \( L^1_\omega (\mathbb{R}) \) whose hull is \( \text{sp}_B \{ \varphi_x \} \). Since

\[
\text{sp}_B \{ \varphi_x \} = \text{hull} \{ f \in L^1_\omega (\mathbb{R}) : f \ast \varphi_x = 0 \},
\]

we have \((\lambda_0 - \varepsilon/2, \lambda_0 + \varepsilon/2) \subseteq \mathbb{R} \setminus \text{sp}_B \{ \varphi_x \} \) for every \( \varphi \in X^* \). It follows that the function \( z \to \langle \varphi, u(z) \rangle \) can be analytically extended to \( i(\lambda_0 - \varepsilon/2, \lambda_0 + \varepsilon/2) \) for every \( \varphi \in X^* \). Hence, \( u(z) \) can be analytically extended to \( i(\lambda_0 - \varepsilon/2, \lambda_0 + \varepsilon/2) \) and therefore, \( i\lambda_0 \notin \sigma_T(x) \) or \( \lambda_0 \notin -i\sigma_T(x) \). Thus we have

\[
\bigcup_{\varphi \in X^*} \text{sp}_B \{ \varphi_x \} = -i\sigma_T(x).
\]

Further, it is easy to check that

\[
I_x = \bigcap_{\varphi \in X^*} \mathcal{I}_{(\varphi_x^\gamma)},
\]

where

\[
\mathcal{I}_{(\varphi_x^\gamma)} := \{ f \in L^1_\omega (\mathbb{R}) : f \ast \varphi_x^\gamma = 0 \}.
\]

Taking into account that

\[
\text{sp}_B \{ \varphi_x^\gamma \} = \{-\lambda : \lambda \in \text{sp}_B \{ \varphi_x \} \},
\]

we can write

\[
\text{hull} \left( I_x \right) = \bigcup_{\varphi \in X^*} \text{hull} \left( \mathcal{I}_{(\varphi_x^\gamma)} \right) = \bigcup_{\varphi \in X^*} \text{sp}_B \{ \varphi_x^\gamma \} = i\sigma_T(x).
\]

Thus we have the following:

**Proposition 2.4.** Let \( \omega \) be a regular weight. Assume that \( T \in B(X) \) has SVEP and \( x \in X \) satisfies the condition \( \| e^{iT}x \| \leq C_\omega(t) \) for all \( t \in \mathbb{R} \) and for some \( C > 0 \). Then,

\[
i\sigma_T(x) = \text{hull} \left( I_x \right).
\]

For \( f \in L^1_\omega(\mathbb{R}) \) and \( s \in \mathbb{R} \), let \( f_s(t) := f(t - s) \). Let \( e_n := 2n \chi\left[ -1/n, 1/n \right] \) \((n \in \mathbb{N}) \), where \( \chi\left[ -1/n, 1/n \right] \) is the characteristic function of the interval \([-1/n, 1/n]\). If \( K := \sup_{t \in [-1, 1]} \omega(t) \), then \( \| e_n \|_{\omega} \leq K \) for all \( n \in \mathbb{N} \). On the other hand, by continuity of the mapping \( s \mapsto f_s \), we have

\[
\lim_{n \to \infty} \| f \ast e_n - f \|_{1,\omega} = 0.
\]

Consequently, \( \{ e_n \} \) is a bounded approximate identity (b.a.i.) for \( L^1_\omega(\mathbb{R}) \). If \( x \in E^\omega_T \), then from the identity

\[
x_{e_n} - x = \int_{-1/n}^{1/n} e_n(t) (e^{iT}x - x) \, dt,
\]

it follows that \( x_{e_n} \to x \). Similarly, \( x_{f_{e_n}} \to x_f \) for all \( f \in L^1_\omega(\mathbb{R}) \).

**Proposition 2.5.** Let \( \omega \) be a regular weight and \( x \in X \). Assume that \( T \in B(X) \) has SVEP and \( \| e^{iT}x \| \leq C_\omega(t) \) for all \( t \in \mathbb{R} \) and for some \( C > 0 \). For an arbitrary \( f \in L^1_\omega(\mathbb{R}) \), the following assertions hold:

a) If \( x_f = 0 \), then \( \hat{f} \) vanishes on \( i\sigma_T(x) \).

b) If \( \hat{f} \) vanishes in a neighborhood of \( i\sigma_T(x) \), then \( x_f = 0 \).

c) If \( \hat{f} = 1 \) in a neighborhood of \( i\sigma_T(x) \), then \( x_f = x \).
Proof. a) By Proposition 2.4, \(i\sigma_T(x) = \text{hull}(I_x)\) and therefore \(I_x \subseteq L_\omega(i\sigma_T(x))\). This clearly implies a).

b) Let \(g \in L^1_\omega(\mathbb{R})\) be such that \(\text{supp}\hat{g}\) is compact. Then, \(f \ast g \in J_\omega(i\sigma_T(x))\) and therefore, \(f \ast g \in I_x\). So we have \(x_{f \ast g} = 0\). Since the algebra \(L^1_\omega(\mathbb{R})\) is Tauberian, \(x_{f \ast g} = 0\) for all \(g \in L^1_\omega(\mathbb{R})\). It follows that \(x_{f \ast e_n} = 0\) for all \(n\), where \(\{e_n\}\) be a b.a.i. for \(L^1_\omega(\mathbb{R})\). As \(n \to \infty\), we have \(x_f = 0\).

c) Since the Fourier transform of \(f \ast e_n - e_n\) vanishes in a neighborhood of \(i\sigma_T(x)\), by b), \(x_{f \ast e_n} = x_{e_n}\). As \(n \to \infty\), we have \(x_f = x\). \(\square\)

By \(S(\mathbb{R})\), we denote the set of all rapidly decreasing functions on \(\mathbb{R}\), i.e. the set of all infinitely differentiable functions \(\phi\) on \(\mathbb{R}\) such that

\[
\lim_{|t| \to \infty} |t^n\phi^{(k)}(t)| = 0, \forall n, k = 0, 1, 2, \ldots
\]

(in this definition, \(n\) can be replaced by any \(\alpha \geq 0\)). It can be seen that if \(\omega\) is a polynomial weight, then \(S(\mathbb{R}) \subseteq L^1_\omega(\mathbb{R})\).

**Lemma 2.6.** Assume that \(T \in B(X)\) and \(x \in X\) satisfy the condition \(\|e^{tX}x\| \leq C (1 + |t|)^\alpha (\alpha \geq 0)\) for all \(t \in \mathbb{R}\) and for some \(C > 0\). Then, for an arbitrary \(\phi \in S(\mathbb{R})\), we have

\[
x_{\phi^{(k)}} = (-1)^k T^k x_{\phi}, \forall k \in \mathbb{N}.
\]

**Proof.** For an arbitrary \(a, b \in \mathbb{R} (a < b)\), we can write

\[
\int_a^b \phi'(t) e^{tX}x dt = \phi(b) e^{bX}x - \phi(a) e^{aX}x - T \int_a^b \phi(t) e^{tX}x dt.
\]

On the other hand,

\[
\|\phi(b) e^{bX}x - \phi(a) e^{aX}x\| \leq C |\phi(b)| (1 + |b|)^\alpha + C |\phi(a)| (1 + |a|)^\alpha \\
\leq 2^\alpha C (|\phi(b)| + |\phi(b)| |b|^\alpha + |\phi(a)| + |\phi(a)| |a|^\alpha).
\]

Since \(\phi \in S(\mathbb{R})\), it follows that

\[
\lim_{b \to +\infty} \lim_{a \to -\infty} \|\phi(b) e^{bX}x - \phi(a) e^{aX}x\| = 0.
\]

Hence \(x_{\phi'} = -Tx_{\phi}\). By induction we obtain our result. \(\square\)

Next, we have the following:

**Proposition 2.7.** Let \(\omega(t) = (1 + |t|)^\alpha (\alpha \geq 0)\). Assume that \(T \in B(X)\) has SVEP and \(x \in X\) satisfies the condition \(\|e^{tX}x\| \leq C \omega(t)\) for all \(t \in \mathbb{R}\) and for some \(C > 0\). If \(\sigma_T(x) = \{0\}\), then for an arbitrary \(f \in L^1_\omega(\mathbb{R})\), we have

\[
x_f = \hat{f}(0) x + \frac{\hat{f}(0)}{1!} (iT) x + \ldots + \frac{\hat{f}(k)(0)}{k!} (iT)^k x,
\]

where \(k = [\alpha]\). In particular, we have \(T^{k+1} x = 0\).

**Proof.** We know (for instance, see [8, Ch.VI, §41] and [13, Theorem 3.2]) that if \(f \in L^1_\omega(\mathbb{R})\), then the first \(k\) derivatives of the Fourier transform of \(f\) exist and

\[
J_\omega(\{0\}) = \left\{ f \in L^1_\omega(\mathbb{R}) : \hat{f}(0) = \hat{f}'(0) = \ldots = \hat{f}^{(k)}(0) = 0 \right\},
\]

where \(k = [\alpha]\). Recall that \(J_\omega(\{0\})\) is the smallest closed ideal of \(L^1_\omega(\mathbb{R})\) whose hull is \(\{0\}\). On the other hand, by Proposition 2.4, \(\text{hull}(I_x) = \{0\}\). Hence we have \(J_\omega(\{0\}) \subseteq I_x\).
Let $\phi \in S(\mathbb{R})$ be such that $\hat{\phi}(\lambda) = 1$ in a neighborhood of 0. For a given $f \in L^1_\omega(\mathbb{R})$, consider the function
\[
g := f - \hat{f}(0) \phi - \frac{\hat{f}'(0)}{1!} (i\lambda)^1 \phi - \ldots - \frac{\hat{f}^{(k)}(0)}{k!} (i\lambda)^k \phi.
\]
As
\[
\hat{\phi}^{(k)}(\lambda) = (i\lambda)^k \hat{\phi}(\lambda),
\]
we have
\[
\hat{g}(\lambda) = \hat{f}(\lambda) - \left[ \hat{f}(0) + \frac{\hat{f}'(0)}{1!} \lambda + \ldots + \frac{\hat{f}^{(k)}(0)}{k!} \lambda^k \right] \hat{\phi}(\lambda).
\]
It can be seen that the first $k$ derivatives of $\hat{g}$ at 0 are zero and therefore $g \in J_\omega(\{0\})$. Consequently, $g \in I_x$ and so $x_g = 0$. On the other hand, by Lemma 2.6 and Proposition 2.5,
\[
x_{\phi^{(k)}} = (-1)^k T^k x
\]
which implies
\[
x_g = x_f - \hat{f}(0) x - \frac{\hat{f}'(0)}{1!} (iT)x - \ldots - \frac{\hat{f}^{(k)}(0)}{k!} (iT)^k x.
\]
Hence,
\[
x_f = \hat{f}(0) x + \frac{\hat{f}'(0)}{1!} (iT)x + \ldots + \frac{\hat{f}^{(k)}(0)}{k!} (iT)^k x.
\]
If $f := \phi^{(k+1)}$, then as
\[
\hat{f}(0) = \hat{f}'(0) = \ldots = \hat{f}^{(k)}(0) = 0,
\]
we get
\[
0 = x_f = (-1)^{k+1} T^{k+1} x.
\]

Now, we are in a position to prove Theorem 2.2.

Proof of Theorem 2.2. If $T \in D_\alpha(A)(\mathbb{R})$, then as
\[
e^{tA} T e^{-tA} = e^{t\Delta_A}(T),
\]
we have
\[
\|e^{t\Delta_A}(T)\| \leq C_T (1 + |t|)^\alpha, \quad \forall t \in \mathbb{R}.
\]
By Proposition 2.3, $\sigma_{\Delta_A}(T) \subset i\mathbb{R}$. Further, since
\[
\sigma(\Delta_A) = \{\lambda - \mu : \lambda, \mu \in \sigma(A)\}
\]
[9 Theorem 3.5.1] and $\sigma(A) \subset \mathbb{R}$, we have $\sigma(\Delta_A) \subset \mathbb{R}$. Therefore, $\Delta_A$ has SVEP. On the other hand, as
\[
\sigma_{\Delta_A}(T) \subset \sigma(\Delta_A) \subset \mathbb{R},
\]
we obtain that
\[
\sigma_{\Delta_A}(T) \subset \mathbb{R} \cap i\mathbb{R} = \{0\}.
\]
Since $\Delta_A$ has SVEP, $\sigma_{\Delta_A}(T) \neq \emptyset$, so that $\sigma_{\Delta_A}(T) = \{0\}$. Applying now Proposition 2.7 to the operator $\Delta_A$ on the space $B(X)$, we get
\[
\Delta_A^{[\alpha]+1}(T) = 0.
\]
For the reverse inclusion, assume that \( T \in B (X) \) satisfies the equation \( \Delta_A^n (T) = 0 \) for some \( n \in \mathbb{N} \). Then, we can write
\[
\| e^{tA} T e^{-tA} \| = \| e^{tA} (T) \| = \| I + t \Delta_A (T) + \frac{t^2}{2} \frac{\Delta_A (T)}{1} + \cdots + \frac{t^n}{n!} \Delta_A^n (T) \| = O (1 + |t|)^{n-1} .
\]
This shows that \( T \in \mathcal{D}_A^{-1} ( \mathbb{R} ) \). The proof is complete. \( \square \)

Next, we will prove Theorem 2.1.

**Proof of Theorem 2.1.** Assume that \( \sigma (A) = \{ \lambda \} \). If \( T \in \mathcal{D}_A ( \mathbb{R} ) \), then \( T \in \mathcal{D}_B ( \mathbb{R} ) \), where \( B = A - \lambda I \). Since \( \sigma (B) = \{ 0 \} \) and \( \Delta_B^n (T) = \Delta_A^n (T) (\forall n \in \mathbb{N} ) \), by Theorem 2.2 we obtain our result. \( \square \)

Note that if \( \alpha \geq 1 \), then \( \mathcal{D}_A ( \mathbb{R} ) \neq \{ A \}' \), in general. To see this, let \( A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) and \( T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) be two \( 2 \times 2 \) matrices on \( 2 \)-dimensional Hilbert space. As \( A^2 = 0 \), we have \( \sigma (A) = \{ 0 \} \) and
\[
eq e^{tA} T e^{-tA} = (I + tA) T (I - tA) = \begin{pmatrix} 1 & 0 \\ t & 0 \end{pmatrix} , \forall t \in \mathbb{R} .
\]
Since
\[
\| e^{tA} T e^{-tA} \| = \left( 1 + |t|^2 \right)^{\frac{1}{2} ,}
\]
we have \( T \in \mathcal{D}_A ( \mathbb{R} ) \), but \( AT \neq TA \).

For a given \( \alpha \geq 0 \), we define the class \( \mathcal{D}^\alpha_A ( \mathbb{Z} ) \) of all operators \( T \in B (X) \) for which there exists a constant \( C_T > 0 \) such that
\[
\| e^{nA} T e^{-nA} \| \leq C_T (1 + |n|)^\alpha , \forall n \in \mathbb{Z} .
\]
Clearly, \( \mathcal{D}_A^0 ( \mathbb{R} ) \subseteq \mathcal{D}_A^\alpha ( \mathbb{Z} ) \). We claim that \( \mathcal{D}_A^\alpha ( \mathbb{R} ) = \mathcal{D}_A^\alpha ( \mathbb{Z} ) \). Indeed, if \( T \in \mathcal{D}_A^\alpha ( \mathbb{Z} ) \) and \( t \in \mathbb{R} \), then \( t = n + r \), where \( n \in \mathbb{Z} \), \( |r| < 1 \) and \( |n| \leq |t| \). Consequently, we can write
\[
\| e^{tA} x \| = \| e^{tA} e_n A x \| \leq e^{\| T \|} \| e^{nA} x \| \leq C_T e^{\| T \|} (1 + |n|)^\alpha \leq C_T e^{\| T \|} (1 + |t|)^\alpha .
\]
Therefore, in Theorems 2.1 and 2.2, the class \( \mathcal{D}_A^\alpha ( \mathbb{R} ) \) can be replaced by \( \mathcal{D}_A^\alpha ( \mathbb{Z} ) \).

As a consequence of Proposition 2.7, we will need the following:

**Corollary 2.8.** Assume that \( T \in B (X) \) has SVEP and \( x \in X \) satisfies the condition
\[
\| e^{nT} x \| \leq C (1 + |n|)^\alpha , \forall n \in \mathbb{Z} , \alpha \geq 0 ,
\]
for all \( n \in \mathbb{Z} \) and for some \( C > 0 \). If \( \sigma_T (x) = \{ 0 \} \), then \( T^{[\alpha]+1} x = 0 \).

By \( K (X) \) we will denote the space of compact operators on a Banach space \( X \). Next, we have the following:
Proposition 2.9. Assume that the spectrum of the operator \( A \in B(X) \) consists of one point. If \( K(X) \subset \mathcal{D}_A^3(\mathbb{R}) \), then
\[
\mathcal{D}_A^{2[\alpha]}(\mathbb{R}) = B(X).
\]

Proof. We have
\[
\|e^{n\Delta A}(T)\| = \|e^{nA}Te^{-nA}\| \leq C_\|T\| (1 + n)^\alpha, \quad \forall T \in K(X), \; \forall n \in \mathbb{N}.
\]
Applying uniform boundedness principle to the sequence of operators
\[
\left\{ \frac{1}{(1+n)}e^{n\Delta A} \right\}_{n \in \mathbb{N}},
\]
we obtain existence of a constant \( C > 0 \) such that
\[
\|e^{nA}Te^{-nA}\| \leq C (1 + n)^\alpha \|T\|, \quad \forall T \in K(X), \; \forall n \in \mathbb{N}.
\]
Consequently, we have
\[
\|e^{nA}Te^{-nA}\| \leq C (1 + |n|)^\alpha \|T\|, \quad \forall T \in K(X), \; \forall n \in \mathbb{Z}.
\]
For a given \( x \in X \) and \( \varphi \in X^* \), let \( x \otimes \varphi \) be the one dimensional operator on \( X \);
\[
x \otimes \varphi : y \mapsto \varphi(y) x, \quad y \in X.
\]
By taking \( T = x \otimes \varphi \) in the preceding inequality, we can write
\[
\|e^{nA}x\| \|e^{-nA}\varphi\| \leq C (1 + |n|)^\alpha \|x\| \|\varphi\|, \quad \forall x \in X, \; \forall \varphi \in X^*,
\]
which implies
\[
\|e^{nA}\| \|e^{-nA}\| \leq C (1 + |n|)^\alpha, \quad \forall n \in \mathbb{Z}.
\]
Now, assume that \( \sigma(A) = \{\lambda\} \). Then as \( \sigma(e^{n(A-I)}) = \{1\} \), we have
\[
\|e^{n(A-I)}\| \geq 1, \quad \forall n \in \mathbb{Z},
\]
so that
\[
\|e^{n(A-I)}\| \leq \|e^{n(A-I)}\| \|e^{-n(A-I)}\| = \|e^{nA}\| \|e^{-nA}\| \leq C (1 + |n|)^\alpha, \quad \forall n \in \mathbb{Z}.
\]
Thus, we obtain that
\[
\|e^{n(A-I)}\| \leq C (1 + |n|)^\alpha, \quad \forall n \in \mathbb{Z}.
\]
By Corollary 2.8, \( (A - \lambda I)^{k+1} = 0 \), where \( k = [\alpha] \). If \( N := A - \lambda I \), then \( A = \lambda I + N \), where \( N \) is a nilpotent of degree \( \leq k + 1 \). Further, for an arbitrary \( T \in B(X) \) and \( t \in \mathbb{R} \), from the identity
\[
e^{tA}Te^{-tA} = e^{t(\lambda I + N)}Te^{-t(\lambda I + N)} = \left( I + \frac{tN}{1!} + \ldots + \frac{t^kN^k}{k!} \right)\left( I - \frac{tN}{1!} + \ldots + (-1)^k \frac{t^kN^k}{k!} \right),
\]
we can write
\[
e^{tA}Te^{-tA} = T + tf_1(N,T) + \ldots + t^{2k}f_{2k}(N,T),
\]
where the functions \( f_1, \ldots, f_{2k} \) do not depend from \( t \). It follows that
\[
\|e^{tA}Te^{-tA}\| = O \left( (1 + |t|)^{2k} \right)
\]
and so $T \in \mathcal{D}^{2k}_{A}(\mathbb{R})$. Thus we have $\mathcal{D}^{2k}_{A}(\mathbb{R}) = B(X)$. □

As a consequence of Proposition 2.9, we have the following.

**Corollary 2.10.** Assume that $\Delta_A$ is a quasinilpotent for some $A \in B(X)$. If $K(X) \subset \mathcal{D}^{\alpha}_{A}(\mathbb{R})$, then $\Delta_A$ is a nilpotent of degree $\leq 2[\alpha] + 1$.

**Proof.** It follows from the identity

$$\sigma(\Delta_A) = \{\lambda - \mu : \lambda, \mu \in \sigma(A)\} = \{0\}$$

that $\sigma(A)$ consists of one point. By Proposition 2.9, $\mathcal{D}^{2[\alpha]}_{A}(\mathbb{R}) = B(X)$. On the other hand, by Theorem 2.1,

$$\ker \Delta^{2[\alpha]+1}_{A} = \mathcal{D}^{2[\alpha]}_{A}(\mathbb{R}) = B(X).$$

□

We conclude this section with the following result:

**Proposition 2.11.** The following assertions hold:

a) For an arbitrary $T \in \mathcal{D}_A(\mathbb{R})$,

$$\text{dist}(T, \{A\}^t) = \sup_{t \in \mathbb{R}} \|T - e^{tA}Te^{-tA}\|.$$  

b) If $\mathcal{D}_A(\mathbb{R})$ is closed, then there exists a constant $C > 0$ such that

$$\sup_{t \in \mathbb{R}} \|T - e^{tA}Te^{-tA}\| \leq C \text{dist}(T, \{A\}^t), \forall T \in \mathcal{D}_A(\mathbb{R}).$$

**Proof.** a) Let $T \in \mathcal{D}_A(\mathbb{R})$ and $\delta := \sup_{t \in \mathbb{R}} \|T - e^{tA}Te^{-tA}\|$. Define a mapping $\pi : \mathcal{D}_A(\mathbb{R}) \to B(X)$ by

$$\langle \pi(T)x, \varphi \rangle = \Phi_t(e^{tA}Te^{-tA}x, \varphi) \quad (x \in X, \varphi \in X^*),$$

where $\Phi$ is a fixed invariant mean on $\mathbb{Z}$. We claim that $\pi(T) \in \{A\}^t$. Indeed, for an arbitrary $s \in \mathbb{R}$, from the identities

$$\langle e^{sA} \pi(T) e^{-sA}x, \varphi \rangle = \langle \pi(T) e^{-sA}x, e^{sA} \varphi \rangle = \Phi_t(e^{tA}Te^{-tA}e^{-sA}x, e^{sA} \varphi) = \Phi_t(e^{(t+s)A}Te^{-(t+s)A}x, \varphi) = \Phi_t(e^{tA}Te^{-tA}x, \varphi) = \langle \pi(T)x, \varphi \rangle,$$

we have $e^{sA} \pi(T) = \pi(T) e^{sA}$. This clearly implies $A \pi(T) = \pi(T) A$. Notice that $\langle \pi(T)x, \varphi \rangle$ belongs to the closure of convex combination of the set

$$\{\langle e^{tA}Te^{-tA}x, \varphi \rangle : t \in \mathbb{R}\}.$$

Now, from the inequality

$$|\langle Tx, \varphi \rangle - \langle e^{tA}Te^{-tA}x, \varphi \rangle| \leq \delta \|x\| \|\varphi\|,$$

we have

$$|\langle Tx, \varphi \rangle - \langle \pi(T)x, \varphi \rangle| \leq \delta \|x\| \|\varphi\|, \forall x \in X, \forall \varphi \in X^*.$$ 

Hence $\|T - \pi(T)\| \leq \delta$. Since $\pi(T) \in \{A\}^t$, the result follows.

b) Applying uniform boundedness principle to the family of the operators

$$f_t : \mathcal{D}_A(\mathbb{R}) \to B(X); \ f_t(T) = e^{tA}Te^{-tA} \quad (t \in \mathbb{R}),$$
Proposition 3.1. Assume that $A \in B(X)$ is decomposable and $T \in B(X)$ satisfies the condition
\[
\|e^{nA} T e^{-nA}\| \leq C_T (1 + |n|)^\alpha, \quad (\alpha \geq 0),
\]
for all $n \in \mathbb{Z}$ and for some $C_T > 0$. Then the following conditions are equivalent:
(a) $T X_A (F) \subseteq X_A (F)$ for every closed set $F \subset \mathbb{C}$.
(b) $\Delta_{[\alpha]+1}^{\alpha} (T) = 0$.
In particular, if $0 \leq \alpha < 1$, then $AT = TA$ if and only if $T X_A (F) \subseteq X_A (F)$, for every closed set $F \subset \mathbb{C}$.

Proof. (a)⇒(b) We have
\[
\|e^{nA} (T)\| = \|e^{nA} T e^{-nA}\| \leq C_T (1 + |n|)^\alpha, \quad \forall n \in \mathbb{Z}.
\]
Since $A$ is decomposable, $\Delta_A$ has SVEP [9] Proposition 3.4.6] and therefore,
\[
r_{\Delta_A} (T) = \lim_{n \to \infty} \|\Delta_A^n (T)\|^{\frac{1}{n}}.
\]
On the other hand, $T X_A (F) \subseteq X_A (F)$ for every closed set $F \subset \mathbb{C}$ if and only if
\[
\lim_{n \to \infty} \|\Delta_A^n (T)\|^{\frac{1}{n}} = 0
\]
[9] Corollary 3.4.5. Now, since $\sigma_{\Delta_A} (T) = \{0\}$, by Corollary 2.8, $\Delta_{[\alpha]+1}^{\alpha} (T) = 0$.
In fact, (b)⇒(a) follows from [9] Proposition 3.4.2. Here, we present more simple proof. Now, it suffices to show that $\sigma_A (T x) \subseteq \sigma_A (x)$ for every $x \in X$. If $x \in X$
and $\lambda \in \rho_A (x)$, then there is a neighborhood $U_\lambda$ of $\lambda$ with $u (z)$ analytic on $U_\lambda$ having values in $X$, such that

$$(zI - A) u (z) = x, \ \forall z \in U_\lambda.$$ 

Using this identity, it is easy to check that the function

$$v (z) := Tu (z) - \Delta_A (T) u' (z) + \ldots + (-1)^k \Delta^k (T) u^{(k)} (z) \ (k = [\alpha]),$$

satisfies the equation

$$(zI - A) v (z) = Tx, \ \forall z \in U_\lambda.$$ 

This shows that $\lambda \in \rho_A (Tx)$.

Next, we have the following:

**Proposition 3.2.** Assume that the operators $A, T \in B (X)$ satisfy the following conditions:

1. $A$ is decomposable and $\sigma (A) \subset \{ z \in \mathbb{C} : \text{Re} \ z > 0 \}$;
2. $\| A^n T A^{-n} \| \leq C_T (1 + |n|)^\alpha$ ($0 \leq \alpha < 1$) for all $n \in \mathbb{Z}$ and for some $C_T > 0$.

Then, $AT = TA$ if and only if $TX_A (F) \subseteq X_A (F)$ for every closed set $F \subseteq \mathbb{C}$.

**Proof.** Assume that $TX_A (F) \subseteq X_A (F)$ for every closed set $F \subseteq \mathbb{C}$. We can write $A = e^B$, where $B = \log A$. Then, $B$ is decomposable [3, Theorem 3.3.6] and

$$\| e^{nB} Te^{-nB} \| \leq C_T (1 + |n|^\alpha), \ \forall n \in \mathbb{Z}.$$ 

Moreover, for every closed set $F \subseteq \mathbb{C}$,

$$TX_B (F) = TX_A (f^{-1} (F)) \subseteq X_A (f^{-1} (F)) = X_B (F),$$

where $f (z) = \log z$. By Proposition 3.1, $BT = TB$ which implies $e^B T = T e^B$. Hence $AT = TA$.

Assume that $AT = TA$. It suffices to show that $\sigma_A (Tx) \subseteq \sigma_A (x)$ for every $x \in X$. If $x \in X$ and $\lambda \in \rho_A (x)$, then there is a neighborhood $U_\lambda$ of $\lambda$ with $u (z)$ analytic on $U_\lambda$ having values in $X$, such that

$$(zI - A) u (z) = x, \ \forall z \in U_\lambda.$$ 

It follows that

$$(zI - A) Tu (z) = Tx, \ \forall z \in U_\lambda.$$ 

This shows that $\lambda \in \rho_A (Tx)$.

4. The Norm of the Commutator $AT - TA$

In this section, we give some estimates for the norm of the commutator $AT - TA$, where $T \in D_A^\alpha (\mathbb{R}) \ (0 \leq \alpha < 1)$.

**Lemma 4.1.** Let $\mu \in M_\omega (\mathbb{R})$, where $\omega (t) = (1 + |t|)^\alpha$ ($\alpha \geq 0$). Assume that $T \in B (X)$ and $x \in X$ satisfy the following conditions:

1. $\| e^{iT} x \| \leq C \omega (t)$ for all $t \in \mathbb{R}$ and for some $C > 0$;
2. $T$ has SVEP.

If $\mu (\lambda) = \lambda$ in a neighborhood of $i \sigma_T (x)$, then

$$x_\mu = iTx.$$
Proof. Let \( g \in S(\mathbb{R}) \) be such that \( \hat{g}(\lambda) = 1 \) in a neighborhood of \( i\sigma_T(x) \). By Proposition 2.5, \( xg = x \). On the other hand, by Lemma 2.6,

\[
xg' = -Tx.
\]

Since

\[
\hat{g'}(\lambda) = i\lambda \hat{g}(\lambda),
\]

the Fourier transform of the function \(-ig' - \mu * g\) vanishes in a neighborhood of \( i\sigma_T(x) \). By Proposition 2.5,

\[
-ixg' = x\mu * g = (xg)\mu = x\mu.
\]

Hence \( x\mu = iTx \). \( \square \)

Note that in the preceding lemma, the weight function \( \omega (t) = (1 + |t|^\alpha) \) \((\alpha \geq 0)\) can be replaced by the weight \( \omega (t) = 1 + |t|^{\alpha} \).

**Theorem 4.2.** Assume that \( T \in B(X) \) has SVEP and \( x \in X \) satisfies the condition

\[
\|e^{\lambda T} x\| \leq C (1 + |t|^\alpha) \quad (0 \leq \alpha < 1),
\]

for all \( t \in \mathbb{R} \) and for some \( C > 0 \). Then we have

\[
\|Tx\| \leq C \left[ r_T(x) + C (\alpha) r_T(x)^{1-\alpha} \right],
\]

where

\[
C(\alpha) = \left( \frac{2}{\pi} \right)^{2-\alpha} \sum_{k \in \mathbb{Z}} \frac{1}{|2k+1|^{2-\alpha}}.
\]

**Proof.** We basically follow the proof of Lemma 3.4 in [12]. Let an arbitrary \( a > r_T(x) \) be fixed. Consider the function \( f \), defined by \( f(\lambda) = \lambda \) for \( -a \leq \lambda \leq a \) and \( f(\lambda) = 2a - \lambda \) for \( a \leq \lambda \leq 3a \). We extend this function periodically to the real line by putting \( f(\lambda + 4a) = f(\lambda) \) \((\lambda \in \mathbb{R})\). A few lines of computation show that the Fourier coefficients of \( f \) are given by the equalities:

\[
c_{2k}(f) = 0, \quad c_{2k+1}(f) = \frac{1}{i \pi^2} \left( -1 \right)^k \frac{1}{(2k+1)^2} \quad (k \in \mathbb{Z}).
\]

Let \( \mu \) be a discrete measure on \( \mathbb{R} \) concentrated at the points

\[
\lambda_k := -\frac{1}{a} \left( 2k + 1 \right) \frac{\pi}{2} \quad (k \in \mathbb{Z}),
\]

with the corresponding weights

\[
c_k := \frac{1}{i \pi^2} \left( -1 \right)^k \frac{1}{(2k+1)^2} \quad (k \in \mathbb{Z}).
\]

Since

\[
\sum_{k \in \mathbb{Z}} |c_k| < \infty,
\]

it follows from the uniqueness theorem that

\[
\hat{\mu}(\lambda) = f(\lambda) = \frac{1}{i \pi^2} \sum_{k \in \mathbb{Z}} (-1)^k \frac{1}{(2k+1)^2} \exp \left[ \frac{1}{a} (2k+1) \frac{\pi}{2} \lambda \right].
\]
Now, if \( \omega(t) := 1 + |t|^\alpha \) \((0 \leq \alpha < 1)\), then as
\[
\sum_{k \in \mathbb{Z}} \frac{1}{(2k + 1)^2} = \frac{\pi^2}{4},
\]
we can write
\[
\|\mu\|_\omega = \int \frac{(1 + |t|)^\alpha}{|\tau_\mu (t)|} \, dt = \sum_{k \in \mathbb{Z}} |c_k| (1 + |\lambda_k|^\alpha)
\]
\[
= \frac{4a}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{1 + \frac{1}{|k|} |2k + 1|^\alpha}{(2k + 1)^2}
\]
\[
= a + C(\alpha) a^{1-\alpha},
\]
where
\[
C(\alpha) = \left( \frac{2}{\pi} \right)^{2-\alpha} \frac{1}{2^{2-\alpha}}.
\]
Since \( \hat{\mu}(\lambda) = \lambda \) in a neighborhood of \( i\sigma_T(x) \), by Lemma 4.1, \( x_\mu = iTx \). Therefore, we get
\[
\|Tx\| = \|x_\mu\| \leq C \|\mu\|_\omega \leq C \left[ a + C(\alpha) a^{1-\alpha} \right].
\]
Since \( a > r_T(x) \) is arbitrary, we obtain our result.
\[ \square \]

As an application of Theorem 4.2, we have the following quantitative version of Theorem 2.1 in the case \( 0 \leq \alpha < 1 \).

**Corollary 4.3.** Let \( A \in B(X) \) and assume that \( \Delta_A \) has SVEP. If \( T \in B(X) \) satisfies the condition
\[
\|e^{tA}T e^{-tA}\| \leq C_T (1 + |t|^{\alpha}) \quad (0 \leq \alpha < 1),
\]
for all \( t \in \mathbb{R} \) and for some \( C_T > 0 \), then
\[
\|AT - TA\| \leq C_T \left[ r_{\Delta_A}(T) + C(\alpha) r_{\Delta_A}(T)^{1-\alpha} \right],
\]
where \( C(\alpha) \) is defined by (4.1).

**Proof.** Noting that
\[
\|e^{t\Delta_A}(T)\| = \|e^{tA}T e^{-tA}\| \leq C_T (1 + |t|^{\alpha}) \quad (\forall t \in \mathbb{R}),
\]
by Theorem 4.2,
\[
\|AT - TA\| \leq C_T \left[ r_{\Delta_A}(T) + C(\alpha) r_{\Delta_A}(T) \right].
\]
\[ \square \]

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