Abstract

We analyse the chain fountain effect – the chain siphoning when falling from a container onto the floor. We argue that the main reason for this effect are the inertial forces that appear in the chain and not the momentum received by the beads of the chain from the bottom of the container, as it was considered before. The inertia of the chain leads to an effect similar to pulling the chain over a pulley placed up in the air, above the container. This effect have been overlooked until now because of the method of calculating the chain velocity. In the method used before, the momentum conservation was apparently imposed, which led to the apparent dissipation of half of the energy of the chain. Because of this large “dissipation”, this approach cannot explain the formation of the fountain effect unless part of the “wasted” energy is recovered in the form of “kicks” from the bottom of the container. Here we show that in a correct approach, if there is no explicit dissipation, then both the momentum and the energy are conserved and therefore the velocity of the chain is high enough to produce the fountain effect without relying on eventually unimportant effects, such as “kicks” from the container. We propose an experiment, which may validate our model by producing the highest fountain chain, while eliminating the kicks the previous model relied upon.

We also analyse the equations of motion and observe that the stationary solution is unstable. For this reason, the stationary solution is not expected to be formed in an experiment, but the trajectory of the chain should rather fluctuate around it. Whether the fluctuating trajectory averages to the stationary solution or not, remains to be studied.

1 Introduction

The fountain chain [1], also called the Mould effect [2], has attracted quite some interest in the past few years (Mould’s youtube videos on the siphoning beads has attracted millions of views). When a chain falls from a container (for example, a beaker) over its rim, onto the floor, sometimes forms a high arch in the air, which is called the fountain chain. This effect has been investigated both, theoretically and experimentally (see, for example, Refs. [3] [4] [5] [6] [7] [8] [9] [10] and references therein), and although it is hard to isolate and observe the main
physical phenomenon which produces it, it is quite generally accepted that the
reaction from the bottom of the container, which “kicks” up the beads of the
chain as they start flying, is the culprit \[3, 4\]. As a consequence of this belief,
one would expect that the more flexible the chain is, the less likely it is to
siphon from the beaker. This is apparently supported by experiments showing
ropes or chains of loosely connected beads crawling over the rim of the container
and falling on the other side, onto the floor, pulled by their own weights \[3, 7\].
But such experiments are misleading (as also observed, for example, in the
numerical simulations of Ref. \[9\]) and the small velocity of the chain (or rope),
which leads to the failure of the formation of the fountain chain are due to
dissipation phenomena which are not taken into account.

In the approach of Refs. \[3, 4\], an apparent application of the momentum
conservation law led to the wrong conclusion that half of the mechanical work
done by the tension in the chain is wasted. Because of this waste, the velocity
acquired by the chain in the process of falling over the rim of the beaker is not
enough to produce the fountain effect. Therefore, in order to explain the effect,
at least some part of this wasted energy should be recovered and the solution
proposed by the authors of Refs. \[3, 4\] (and adopted since then) was that the
chain is kicked off from the bottom of the container with enough energy, so
that it is able to jump in the air high enough, siphoning from the beaker. We
argue here that the kicks received by the beads from the bottom of the container
are not the main reason (assuming they have any relevance) for the fountain
chain formation. If energy and momentum conservation are properly taken into
account (while the dissipation effects are correctly evaluated), the chain gains
enough speed to siphon from the container. Then, due to its own inertia, the
chain cannot simply change from moving upwards to moving downwards at the
rim of the beaker, but forms up in the air something that is better described
as a “spontaneous pulley”, which pulls the chain high above the rim. Such a
phenomenon can be observed in many activities, like when a rope or a chain
is pulled with high velocity over a pulley or an obstacle. If the tension is not
strong enough, the chain (or rope) simply detaches from the pulley and turns
at a certain distance from it, as if a second, imaginary pulley is formed. Our
model is supported not only by the simplicity of the argument, but also by
some unpublished observations \[11\], as we shall show in the Appendix. Further
experiments are suggested, to test our solution and disprove the previous method
of calculation.

The paper is organized at follows. In Section \[2\] we describe the chain and
write the main equations governing its dynamics. These equations are generally
known, but we include them here to make the paper self-contained and to put
them in a convenient form. We write the differential equations that describe
the stationary regime and the fluctuations around it in Section \[2.1\]. Using these
equations, we find the shape of the chain in the stationary case and show that it
is unstable. This implies that the chain deviates in general from the stationary
solution, which, in principle, cannot to be observed directly. Eventually, the
stationary solution may represent the average over many experimental realiza-
tions, but we did not study how the average converges – this should be a separate
analysis. In Section \[2\] we discuss the idea of the “spontaneous pulley” and in
Section \[4\] we draw the conclusions. In the Appendix \[A\] we compare different
models and give arguments for our approach. We also propose and experiment
which should produce the highest fountain chain (for the same tension in the
chain) while totally eliminating the kick-off process. Such an experiment should confirm our calculations and disprove the older models.

2 The description of the chain

A non-extensible chain, like the one presented in Fig. 1 of constant linear density $\lambda$, is described by the position vector $r(l, t) \equiv [x(l, t), y(l, t), z(l, t)]$, which depends on two parameters: the position along the chain $l$ and time $t$. The local velocity (with respect to the laboratory frame) and acceleration are $v(l, t) \equiv [v_x(l, t), v_y(l, t), v_z(l, t)]$ and $a(l, t) \equiv [a_x(l, t), a_y(l, t), a_z(l, t)]$, respectively.

2.1 The stationary case and fluctuations

In the stationary case, the chain is moving along a path which does not depend on $t$, $r(l) \equiv [x(l), y(l), z(l)]$. The chain is not extensible, so its velocity $v(l) \equiv v[x(l), y(l), z(l)]$ has the same modulus and is tangent to the path in all its points. The differential of $r$ is

$$dr = \left( \frac{dx}{dl} + \frac{dy}{dl} + \frac{dz}{dl} \right) dl,$$ 

where

$$dr = dl \quad \text{and} \quad \left( \frac{dx}{dl} \right)^2 + \left( \frac{dy}{dl} \right)^2 + \left( \frac{dz}{dl} \right)^2 = 1. \quad (1)$$

Since $\frac{dx}{dl} + \frac{dy}{dl} + \frac{dz}{dl}$ has unit modulus, we shall denote it by $\hat{l} \equiv \hat{l}(l)$. From (1) we get

$$v(l) \equiv \frac{dr(l)}{dt} = \hat{l}(l)v \quad \text{and} \quad a(l) \equiv \frac{dv}{dl} = \left( \frac{d^2x}{dl^2} + \frac{d^2y}{dl^2} + \frac{d^2z}{dl^2} \right) v^2. \quad (2)$$

From the invariance of $\delta l$ we obtain $va = 0$, that is, the acceleration is always perpendicular on the local velocity and, therefore, on the path. We introduce the notations $c(l) \hat{l}_\perp(l) \equiv \hat{x} \frac{d^2x}{dl^2} + \hat{y} \frac{d^2y}{dl^2} + \hat{z} \frac{d^2z}{dl^2}$, where $|\hat{l}_\perp(l)| = 1$, $c(l) \geq 0$, and $\hat{l}_\perp(l)\hat{l}(l) = 0$ for any $l$. Using Fig. 2 one can prove that $c(l)\hat{l}_\perp(l) = d\hat{l}/dl = \hat{l}_\perp(l)/R$, where $R$ is the radius of the curvature of the chain trajectory at $l$.

The forces that act on the chain are the tension $T(l) \equiv T(l)\hat{l}(l)$ (we assume that the chain is not stiff, so $T$ acts along the chain) and the external force density $f_{ext}(l)$. The resultant force that acts on the chain element $\delta l$ is (see Fig. 3)

$$\delta F = \delta l \left( f_{ext} + \frac{dT}{dl} \hat{l} + T c \hat{l}_\perp \right), \quad (3)$$

so, using Eqs. (2) and (3), we can write Newton’s law $\delta F = \delta m a \equiv \delta l \lambda a$, from where we obtain

$$\lambda v^2 \hat{l}_\perp = f_{ext} + \frac{dT}{dl} \hat{l} + T c \hat{l}_\perp. \quad (4)$$

To analyze the situation, let us introduce a local (and right handed) system of coordinates $(\hat{l}, \hat{l}_\perp, \hat{l}_t)$, where $\hat{l}_t = \hat{l}_\perp \times \hat{l} \times \hat{l} = 0$. Then, we may write
Figure 1: The chain in stationary conditions. A chain element $\delta l$, located at $r[x(l), y(l), z(l)]$, has the velocity $v$ and acceleration $a$, such that $va = 0$. The forces that act on this chain element are the tension $T$ and the external force $\delta F_{ext} \equiv f_{ext}\delta l$, where $f_{ext}$ is the external force density.

Figure 2: A detail from the chain trajectory. Taking the limit $\delta l \to 0$ and using the fact that $|\hat{l}_1| = |\hat{l}_2| = 1$, one can easily prove that $d\hat{l}/dl \equiv c(l) = 1/R$, where $R$ is the curvature radius.

For $f_{ext} \equiv f_{\perp} + f_{\parallel} + f_{t} \equiv l_{\perp}f_{\perp} + l_{t}f_{t} + l_{\parallel}f_{\parallel}$, to obtain the conditions for equilibrium,

$$
\begin{align*}
    f_{\parallel}(l) &= -\frac{dT(l)}{dl}, \\
    f_{\perp}(l) &= [\lambda v^2 - T(l)]c(l), \quad \text{and} \\
    f_{t}(l) &= 0.
\end{align*}
\tag{5a}
$$

From Eq. (5a) we see that the variation of $T$ along the chain is only caused by the component of the external force parallel to the chain and compensates it.

Equation (5b) is more interesting. First, we observe that if $f_{\perp}(l) = 0$, then either $\lambda v^2 - T(l) = 0$, or $c(l) = 0$. If $c(l) \neq 0$ and $T(l) < \lambda v^2$ (see Fig. 3), the tension is not strong enough to keep the chain on a curved path, so the centrifugal force that acts on the moving chain will tend to displace it even further. Any disturbance will evolve in the direction of the convexity and the chain is unstable. Vice-versa, if $c(l) \neq 0$ and $T(l) > \lambda v^2$, the tension is too strong, the centrifugal force cannot compensate the tension and the chain straightens (the concavities and convexities are removed). In either case, the chain is unstable at $c(l) \neq 0$, whereas if $\lambda v^2 - T(l) = 0$, the chain is in local (unstable) equilibrium.
The forces that act on a small chain element \( \delta l \) and the acceleration produced. \( T_1 \) and \( T_2 \) are the tensions that act on the element’s ends; \( \delta F_{ext} \) is the external force caused by the force density \( f_{ext} \).

Similarly, we analyse the case \( c = 0 \). In this situation, we observe that the system is in unstable equilibrium if \( T < \lambda v^2 \), because, as we explained above, any disturbance tends to be amplified by the centrifugal force, whereas if \( T > \lambda v^2 \) the system is in stable equilibrium, because the tension is strong enough to reduce any disturbance back to zero.

All the discussion of Eq. (5b) for \( f_{\perp} = 0 \) (the discussion about stability) applies also to Eq. (5c), since \( f_t(l) = 0 \) for any \( l \).

If \( f_{\perp} \neq 0 \) and Eq. (5b) is satisfied, then the equilibrium of the chain in motion is realized with the participation of the external force. If we deform the chain from the equilibrium shape moving it in a direction opposite to \( \hat{l}_{\perp} \) – that is, increasing locally \( c(l) \) – we have two situations. (1) If \( T < \lambda v^2 \), we obtain \( [\lambda v^2 - T(l)]c(l) - f_{\perp}(l) > 0 \), which means that the centrifugal force becomes stronger than the resultant force of \( T \) and \( f_{ext} \) (the constraining forces) and the chain is further displaced from the equilibrium shape (see Fig. 3). (2) If \( T > \lambda v^2 \), we obtain \( [\lambda v^2 - T(l)]c(l) - f_{\perp}(l) < 0 \), which means that the centrifugal force becomes weaker than the constraining forces and the chain is pushed back to equilibrium.

Similarly, if we deform the chain in the direction \( \hat{l}_{\perp} \) – that is, decreasing locally \( c(l) \) – we have again the situations (1) and (2). Therefore, we can conclude in general that, if \( T < \lambda v^2 \) the equilibrium chain shape is unstable, whereas if \( T > \lambda v^2 \) the equilibrium is stable. In the next section we shall see that the stationary shape of the chain is unstable. It is also important to note that the chain cannot be in stationary conditions in the region around the pick-up point, where the chain from the beaker is set in motion.

### 2.2 Solutions for the stationary shape of the chain

The stationary shape of the chain was found before (see, for example, [4]) and may be obtained from Eqs. (5), with proper boundary conditions. The situation is schematically illustrated in Fig. 4, where we represent a chain falling from a beaker onto the floor. The beaker is at height \( h \) above the floor and the force that acts on it is the gravity: \( F_{ext} = \lambda g \), where \( g = -g\hat{z} \) is the gravitational acceleration, acting along the \( z \) direction, downwards. Due to the symmetry of the problem, we can choose the coordinate axes \((x, y, z)\) such that \( y \equiv 0 \) – that is, we work in the \((x, z)\) plane. The parametric curve \([x(l), z(l)]\), that describes
the chain in this plane will be changed into the function \(z(x)\), which is single valued and defined in the interval \([0, x_{\text{max}}]\). From Eqs (11) and (29), together with (see Fig. 4) \(\frac{dz}{dx} \equiv \frac{z'}{(dz/dl)/(dx/dl)} = \tan(\alpha)\), we obtain

\[
\frac{dx}{dl} = \frac{1}{\sqrt{1 + (z')^2}}, \quad \frac{dz}{dl} = \frac{|z'|}{\sqrt{1 + (z')^2}} \quad \frac{|d^2z|}{dl^2} = \frac{|z''|}{[1 + (z')^2]^{3/2}}, \quad \text{and} \quad c = \frac{|z''|}{[1 + (z')^2]^{3/2}} \equiv \frac{1}{R}.
\]

From Eq. (5a) we obtain

\[
f_{\parallel} = g\lambda \sin(\alpha) = -T'(dx/dl) = -T'\cos(\alpha), \quad \text{where} \quad T' \equiv dT/dx.
\]

Taking into account that \(z' = \tan(\alpha)\), we further obtain

\[
T' = -g\lambda \tan(\alpha) = -g\lambda z' \quad \text{or} \quad T(z(x)) = g\lambda z + T_0, \quad (7)
\]

where \(T_0\) is the tension in the chain at \(z = 0\). Assuming that the tension becomes zero when the chain reaches the table \((z = 0)\), in the rest of the calculations we shall assume \(T_0 = 0\) \([3, 4, 9]\). If \(T_0 \neq 0\), one can adjust the position \(z = 0\) below or above the table, such that the condition \(T(z = 0) \equiv T_0 = 0\) is still satisfied.

From Eq. (5b) we obtain

\[
g\lambda = \left[\lambda v^2 - T(x)\right] \frac{|z''|}{1 + (z')^2}.
\]

We immediately observe that \(\lambda v^2 - T(x) > 0\) along the whole trajectory, so the chain is in an unstable state.

Taking into account that along the chain trajectory \(z'' < 0\) and replacing \(T[z(x)]\) from Eqs. (7) and (8) we get

\[
g = -\left[v^2 - g\lambda z'' \frac{z''}{1 + (z')^2}\right]. \quad (9)
\]

We see that the stationary trajectory does not depend on \(\lambda\). Furthermore, if we replace the coordinates \((x, z)\) by the dimensionless coordinates \((x_1, z_1)\), where

\[
x_1 \equiv \frac{gx}{v^2} \quad \text{and} \quad z_1 \equiv \frac{gz}{v^2}, \quad (10)
\]

Eq. (10) simplifies to

\[
z''_1(z_1 - 1) - (z'_1)^2 - 1 = 0, \quad (11)
\]

with the solution

\[
z_1(x_1) = C_1 \cosh \left(\frac{x_1 + C_2}{C_1}\right) + 1, \quad (12)
\]

where \(C_1\) and \(C_2\) are two constants that have to be determined.

We determine the value of \(g/v^2\) from the energy conservation. As we mentioned in the Introduction, in earlier works (see, for example \([3, 4, 9, 10]\)) the energy conservation was assumed to be intrinsically “violated” in the process of setting the chain in motion, by a generic process of dissipation, such that no more than half of the mechanical work done by the tension force may recovered in the form of kinetic energy of the chain. We consider this to be incorrect and we shall show in Appendix A how a chain may be put in motion wasting neither energy nor momentum. In this section we use the results of Appendix A.
The approximately vertical pulling force that acts on the chain at rest on the bottom of the beaker (which is at height \( h \)) is \( T_b = g \lambda h \) and the mechanical work done along a distance of vertical component \( \delta l \) is \( L = T_b \delta l = g \lambda h \delta l \). \( L \) should be recovered in the form of kinetic energy of a chain segment of the same length which is put into motion, \( \delta E_k = \delta l \nu^2/2 \). This gives the relations (see Appendix A for details)

\[
T_b = \frac{\lambda \nu^2}{2} \quad \text{and} \quad g/\nu^2 = 1/(2h), \quad (13)
\]

but we should keep in mind that the process of setting the chain in motion cannot be a stationary process, as we already mentioned in Section 2.1.

To determine the constants \( C_1 \) and \( C_2 \), let’s choose the coordinates as in Fig. 4 with \( z(0) = h \). First, from the Eqs. (12) and (13) we obtain

\[
-\frac{1}{2} = C_1 \cosh \left( \frac{C_2}{C_1} \right) ; \quad (14)
\]

Eq. (14) implies \( C_1 < 0 \). Similarly, the point of maximum height \( H \) corresponds to \( x_1 = -C_2 \) (so \( C_2 < 0 \) also) and

\[
\frac{H}{2h} - 1 = C_1. \quad (15)
\]

We notice that \( h < H < 2h \), which implies \(-0.5 < C_1 < 0 \). The point \( x_{\text{max}} \equiv 2hx_{1, \text{max}} \) where the chain reaches the floor is determined from Eq. (12),

\[
\frac{x_{\text{max}}}{2h} = x_{1, \text{max}} = -C_2 - C_1 \operatorname{arccosh} \left( \frac{-1}{C_1} \right) \quad (16)
\]

From Eqs. (14)–(16) we see that the shape of the chain in the stationary state is determined only by the constant \( C_1 \), which is further determined by the ratio \( H/h \), by Eqs. (14) and (15). To understand what physical conditions determine this ratio, let us calculate the derivative \( dz/dx \) in \( x = 0 \), using Eqs. (14) and (15):

\[
\left. \frac{dz}{dx} \right|_{x=0} = \left. \frac{dz_1}{dx_1} \right|_{x_1=0} = \sinh \left( \frac{C_2}{C_1} \right) = \sqrt{\frac{1}{4[H/(2h) - 1]^2} - 1}. \quad (17)
\]

Equation (17) may be inverted and \( H/h \) is obtained as a function of the direction followed by the chain when it is set in motion, which represents the initial condition (this is only an approximation, since, as we mentioned above, the dynamics of the chain at the bottom of the beaker cannot be described as a stationary process).

Using Eq. (17), we obtain the tension along the chain,

\[
T(x) = g \lambda z(x) = 2hg\lambda \left( C_1 \cosh \left( \frac{x/(2h) + C_2}{C_1} \right) + 1 \right), \quad (18)
\]

with its maximum value at the maximum height \( H \),

\[
T_{\text{max}} = T(2hC_2) = g\lambda H. \quad (19)
\]

Since \( \nu^2 = 2gh \) (Eq. 13) and \( H < 2h \) (Eq. 15), from Eq. (19) we observe again that \( T(x) < \lambda \nu^2 \) for any \( x \), so the stationary trajectory is unstable.
3 The spontaneous pulley

It is generally argued that the chain raises above the rim of the beaker due to the kicks received by the bids from the bottom of the beaker (see, for example, [3, 4, 5, 7, 8, 9, 10]). We argue that indeed, if proper conditions are met, the chain is pulled upwards from the container by its own inertia – that is, by the centrifugal force that is strongest in the region where the chain bends downwards, as if it would be passed over a pulley. From Eq. (5b) we see that as $v$ increases, $T$ has to increase also in order to keep the system in equilibrium, if we keep the same curvature and since the gravitational force remains the same.

Let’s prove that the centrifugal force is the main cause of the chain fountain phenomenon by reductio ad absurdum, assuming that the chain is soft enough, so that the reaction from the beaker is too weak to produce siphoning and the trajectory of the chain is similar to curve (2) of Fig. 4. Then, the highest point of the chain is $H'$ (at the rim of the beaker) and, if the dissipation is negligible, in the stationary condition, $T(H') = T'_{\text{max}} = \lambda gH'$, $(v')^2 = 2gh$, and from the equilibrium condition (5b) we get

$$R(H') = 2h - H',$$

where $R(H')$ is the curvature radius at the rim of the beaker – notice that the curvature is minimum at the point where the chain is horizontal, because in this case $f_{\perp} = g\lambda \cos(\alpha = 0) = g\lambda$ is maximum. But from Eq. (20) we notice that $2h - H' \equiv h - h_b$ is the difference between the height of the bottom of the beaker and the height of the beaker $h_b$, which may be orders of magnitude larger than any of the dimensions of the beaker. This implies that this equilibrium condition cannot be satisfied in the absence of dissipation and therefore the dissipation (friction) is the main reason for the absence of the fountain chain, for reducing the velocity of the chain, and for eventually producing extra, external forces at the rim of the beaker. If for the same experimental conditions, one chain is siphoning and another one is not, this means that the dissipation is larger in the second case, not (necessarily) that the kicks are stronger in the first case. Keep in mind that in some experiments [3] the chains may consist of beads so loosely tied together that the assumption that the chain is homogeneous may not hold and this brings extra complications to the problem – eventually one should treat this as a chain of discreet masses.

If the siphoning mechanism appears in an experiment, this is due to the fact that the condition (5b) cannot be satisfied at the rim of the beaker (as argued after Eq. (20) and the resultant of the centrifugal force, tension, and gravitational force pull the chain upwards, as if a pulley is present in the upper part of the chain. This phenomenon is the main cause of the fountain chain effect, not the “kicks” from the bottom of the container, which are introduced to theoretically compensate for the large and unphysical dissipation assumed in other models.

4 Conclusions

We analyzed the fountain chain and observed that the stationary trajectory is unstable. The stability of the trajectory may be increased by the lateral stiffness of the chain. The fluctuating trajectory may not average (in time or over many
realization of similar experiments) to the stationary solution – this remains to be studied.

The most important aspect of our study is that we clarified that the reason for the formation of the fountain chain are not the (eventually small) “kicks” received by the beads of the chain from the bottom of the container (a beaker, for example), but the centrifugal forces that appear in the chain when the trajectory is bent. These centrifugal forces have an effect equivalent to the existence of a pulley, up in the air, above the beaker. With the aid of this pulley, the chain is lifted from the beaker and left to fall onto the floor. Here we called this phenomenon the spontaneous pulley effect, since it may be observed in a wider class of phenomena, when a chain or a rope moving with high velocity is forced to change direction.

We also theoretically proposed a special type of setup, which may exhibit the strongest fountain effect and eliminates the kicks from the container (see Appendix A and Fig. 6). If this prediction would be confirmed, it would validate our theoretical model.

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A Conservation of energy vs “momentum transfer”

In Section 2.2 we applied the conservation of energy to obtain the velocity of the chain, as a function of the beaker height $h$ (Eq. 13). Other authors (see, for example, Refs. [3, 4, 9]) calculate $v$ apparently by the momentum transfer from the moving part of the chain to the beads that are set in motion. The difference between the two methods of calculation (that we shall call as our method and the second method) is a factor of 2 in the energy transfer, which prohibits the formation of the fountain chain in the second method, unless part of the “wasted” energy is restored by a (supposedly exaggerated) interaction of the chain with the bottom of the beaker. So, here we shall argue in favor of our method, by showing examples of ideal chains which are set in motion and where not only both, energy and momentum are conserved, but also the velocity is high enough to ensure the formation of the fountain without having to rely on additional (and eventually irrelevant) phenomena, like the kick-off effect.

First, let us discuss the second method. If $T_b$ is the tension in the chain at the beaker bottom, near the beads that are to be set in motion in the next time interval $\delta t$, then, following [9], the momentum transfer to the beads is $\Delta P = \lambda v' \delta l = \lambda (v')^2 \delta t$ (where we used the notation $v'$ to differentiate the velocity in this formalism from the one obtained by our method). If this momentum variation is only due to the tension $T_b$, which acts during the same interval of time $\delta t$, one obtains

$$T_b = \lambda (v')^2$$

(21a)
at the bottom of the container, whereas above the bottom one should have
\( \lambda (v')^2 < T_b \). But this inequality cannot satisfy the equilibrium conditions [5] and it implies that the chain cannot rise above the bottom of the beaker [3, 4, 9]. Furthermore, by comparing Eqs. (13) and (21a) one can see that the kinetic energy acquired by the chain in this formalism is half the kinetic energy obtained by us. In order to cure the chain’s incapacity, one should add to \( T_b \) a net upwards force, \( \Delta F_z = F_z - G \), where \( G \) is gravity and \( F_z \) “the force from the container bottom or other beads” [9]. Acting for the same interval of time \( \delta t \), one then writes \( \Delta P = (T_b + \Delta F_z) \delta t \) and obtains

\[
T_b + \Delta F_z = g \lambda h + \Delta F_z = \lambda (v')^2, \tag{21b}
\]

This argument further implies that \( \Delta F_z = \lambda g (H'' - h) \), where \( H'' \) is the maximum height predicted by this method [3, 4, 9].

To clarify the situation, let’s analyse the movement of the chain depicted in Fig. 5. The vertical part of the chain always moves at constant speed \( v_1 \), whereas the oblique part is at rest. The part at rest represents the part of the chain that lies on the bottom of the beaker, whereas the part that moves represents the part that it picked-up. The part of the chain which is at rest makes an angle \( \alpha \) with the part that is moving. Geometrical considerations lead to (see Fig. 5)

\[
\frac{v}{v_1} = \frac{\delta l (1 + \cos \alpha)}{\delta l \sin \alpha} = \frac{1 + \cos \alpha}{\sin \alpha} \quad \text{and} \quad v_1^2 = \frac{2 \nu_1^2}{1 + \cos \alpha}. \tag{22}
\]

On the other hand, if the traction force is \( T \), whereas the tension in the chain at rest is \( T_1 \), then momentum conservation along the \( x \) axis gives

\[
T + T_1 \cos \alpha = \frac{\lambda v_1^2}{1 + \cos \alpha}, \tag{23a}
\]

whereas the momentum conservation along the \( y \) axis gives

\[
T_1 \sin^2 \alpha = \lambda v_{1y}^2, \quad \text{so} \quad T_1 = \frac{\lambda v_1^2}{(1 + \cos \alpha)^2}. \tag{23b}
\]

From Eqs. (22) and (23) we get

\[
T = \frac{\lambda v_1^2}{(1 + \cos \alpha)^2} = \frac{\lambda v_{1y}^2}{2(1 + \cos \alpha)} = \frac{\epsilon_k}{1 + \cos \alpha}, \tag{24}
\]

where \( \epsilon_k \) is the linear density of kinetic energy of the moving chain and which, for \( \alpha = \pi/2 \), is twice the one predicted by the second method (Eq. 21a).

To analyse the system from the energetic perspective, let us observe that the work done by \( T \) is \( L = T \delta l (1 + \cos \alpha) \), whereas the kinetic energy gained by the segment \( \delta l \) is \( E_k = \delta \lambda v_{1y}^2/2 \). Equating \( L \) and \( E_k \) and using Eqs. (24), we re-obtain Eq. (24), which certifies the consistency of our formalism and the conservation of both, energy and momentum. If \( \alpha = \pi/2 \), as it happens in the case of (almost) vertical pick-up, Eq. (24) becomes Eq. (13). We can also see that the condition for the formation of the fountain chain, namely \( \lambda v_{1y}^2 > T \), is satisfied for any \( \alpha < \arccos(-0.5) = 2\pi/3 \).

In conclusion, we exemplified that, if we do not take into account other dissipation mechanisms, the chain pick-up process conserves both, energy and
momentum. The velocity of the chain satisfies in rather general configurations the condition for the formation of the fountain chain, without the necessity of additional kicks from the container, contrary to what it was claimed before (see, for example, [3, 4, 5, 7, 8, 9, 10]). If we assume that the formation of the fountain would be mainly due to the reaction from the bottom of the beaker, this should be experimentally observable, as an extra force on the beaker. Unpublished experimental works suggest that such a force is absent [11], but more careful investigations are needed for a clear conclusion (that is why we do not discuss here these experiments). Furthermore, Eq. (24) shows that the configuration most favorable for the formation of the fountain chain is the one with the angle \( \alpha = 0 \), which is a chain hanging from a wall and siphoning vertically over it, as schematically shown in Fig. 6. For the same tension in the chain, this configuration produces maximum velocity and therefore highest fountain, plus it eliminates the possibility of kicks from the bottom of the chain. The observation of such a fountain would represent an experimental confirmation of our model.

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Figure 4: The schematic representation of a chain falling from the beaker onto the floor in the cases when the chain fountain is formed (1) and when it is not formed (2). The bottom of the beaker is at height \( h \) above the floor and the chain is moving with velocity \( v \) (1) or \( v' \) (2). The gravity force that acts on the chain segment \( \delta l \) is \( \delta \mathbf{G} = -\lambda g \delta l \mathbf{z} \) and \( \mathbf{l} = \cos(\alpha) \).

Figure 5: An ideal chain, of mass density \( \lambda \), is being pulled by one end by the force \( \mathbf{T} \). The other end is held fixed, by a force \( \mathbf{T}_1 \) (it is attached to a wall, for example). The moving part of the chain always has the constant speed \( v_1 \) and the chain’s shapes at time \( t \) and \( t + \delta t \) are represented by solid and dashed brown lines, respectively.

Figure 6: The configuration that may lead to maximum fountain effect: a vertically hanging chain is pulled (also vertically) over an obstacle.