Deterministic limit of mean field games associated with nonlinear Markov processes

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Abstract

The paper is concerned with the deterministic limit of mean field games with the nonlocal coupling. It is assumed that the dynamics of mean field games are given by nonlinear Markov processes. This type of games includes stochastic mean field games as well as mean field games with finite state space. We consider the limiting deterministic mean field game within the framework of minimax approach. The concept of minimax solutions is close to the probabilistic formulation. In this case the Hamilton–Jacobi equation is considered in the minimax/viscosity sense, whereas the flow of probabilities is determined by the probability on the set of solutions of the differential inclusion associated with the Hamilton-Jacobi equation such that those solutions are viable in the graph of the minimax solution. The main result of the paper is the convergence (up to subsequence) of the solutions of the mean field games to the minimax solution of deterministic mean field game in the case when the underlying dynamics converge to the deterministic evolution.

Keywords: mean field games, deterministic limit, minimax solutions.
MSC codes: 91A23, 91A13, 49L20, 3E20

1 Introduction

The mean field game theory initiated independently by Lasry and Lions (see [30]–[33]) and Huang, Malhamé, and Caines (see [18]–[21]) provides a way for describing a control system with a large number of independent players by studying the limit system when the number of players tends to infinity. It is assumed that the players are identical and the dynamics and award of each player depend on her state, her control and the distribution of all players.

There are several approaches to analysis of mean field games. A first way consists in studying the coupled system made of a Hamilton-Jacobi equation for the value function and a Kolmogorov equation for the law of probabilities. This approach was used for the stochastic mean field games (see [3], [16], [32], [33] and reference therein),

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for deterministic mean field games (see [32], [33]) and for the mean field games with a finite number of states (see [15]). The most general case when the dynamics of each player is controlled by a nonlinear Markov process is considered in the framework of this approach in [24]–[26].

The second approach is based on probabilistic formulation of mean field games. In this case the Hamilton-Jacobi equation is replaced with the optimization problem whereas the probability law is given by the distribution of states of the players choosing their optimal strategies. This approach was developed for stochastic mean field games in [8], [12], [28] and for deterministic mean field games in [17]. Note that the probabilistic approach is convenient for the convergence problem. It is used in [13], [28] to analyse the limit behaviour of symmetric equilibria for $N$-player games.

The concept of minimax solutions proposed in [2] for the deterministic case is close to the probabilistic approach. In this case the Hamilton–Jacobi equation is considered in the minimax/viscosity sense, whereas the flow of probabilities is determined by the probability on the set of solutions of the differential inclusion associated with the Hamilton-Jacobi equation such that those solutions are viable in the graph of the minimax solution. The notion of minimax solution to mean field game goes back to the minimax solution introduced for Hamilton-Jacobi PDE by Subbotin [35]. The definitions of minimax and viscosity solutions for Hamilton–Jacobi PDEs are equivalent. The minimax solutions deal primarily with the case when the Hamiltonian enjoys sublinear growth.

The third approach developed in [3], [4], [7], [11], [17], [25] relies on consideration of the mean field game as a dynamical system in the space of probabilities. In the framework of this approach the study of mean field game is performed by examining the master equation that is a Hamilton–Jacobi equation involving probability as a state variable. This approach permits to prove that a limit of feedback Nash equilibria when the number of player tends to infinity is a solution of a mean field game [7].

The paper is concerned with the deterministic limit of the mean field games. The deterministic limit of the second order mean field games was studied by Lasry and Lions in [32], and Lions in [33] (see also [1]). We consider the case when the dynamics are controlled by nonlinear Markov processes as exposed in the monograph [23]. These systems include processes described by stochastic differential equations or continuous time Markov chains.

The main result of the paper is the convergence (up to subsequence) of the solutions of mean field games with nonlocal coupling to the minimax solution of the first order mean field game in the case when the corresponding dynamics governed by nonlinear Markov processes converge to the deterministic evolution. As a corollary we obtain the equivalence of the minimax solutions and the solutions in the probabilistic sense for first-order mean field games.

The results of the paper are based on the relative compactness of solutions of mean field games. This idea was used in several papers to construct the solution of the second-order mean field game with unbounded coefficients (see [9], [29]). The difference between those paper and this one is as follows. In [9], [29] the sequence of mean field games is constructed by the given mean field game with unbounded coefficients. Thus, the underlying probability spaces of all games coincide. Moreover,
the mentioned papers are concerned only with the case of second-order mean field games. In the present paper we consider the mean field games with dynamics given by nonlinear Markov processes. Thus, the probability spaces for original mean field games include the sets of càdlàg functions; whereas the solution of the limiting mean field game is determined by the probability on the set of continuous functions. To overcome these difficulties we introduce auxiliary stochastic processes with the dynamics describing by dynamics of the deterministic (limiting) mean field game and control policy borrowed from solutions of the original mean field games. The main idea of the proof of the main result is to evaluate the distance between original and auxiliary processes and to show the relative compactness of the auxiliary stochastic processes.

The outline of the paper is as follows. Section 2 provides main notations. In this section we introduce the definition of solution to mean field game system in the probabilistic sense for the case when the dynamics of each player is governed by nonlinear Markov process. Additionally, we recall the definition of minimax solution to first-order mean field game system. In Section 3 the main result of the paper is formulated. In Section 4 we introduce a auxiliary stochastic processes and obtain the bounds for moments of the original and auxiliary stochastic processes. The distance between original and auxiliary stochastic processes is estimated in Section 5. Finally, Section 6 provides the proof of the main result. It is based on the relative compactness of the auxiliary stochastic processes and analysis of the limiting probability on the set of trajectories.

2 Preliminaries

2.1 Main notations

If $A$ is a Banach space, then denote by $P(A)$ the set of probabilities on $A$. Further let $P^2(A)$ denote the set of probabilities $m$ on $A$ satisfying

$$
\zeta^2(m) \triangleq \int_A \|a\|^2 m(da) < \infty.
$$

Denote

$$
\zeta(m) \triangleq \sqrt{\zeta^2(m)}.
$$

We consider the 2-Wasserstein distance between $m_1, m_2 \in P^2(A)$ i.e.

$$
W_2(m_1, m_2) \triangleq \left( \inf \left\{ \int_{A \times A} \|a_1 - a_2\|^2 \pi(d(a_1, a_2)) : \pi \in \Pi(m_1, m_2) \right\} \right)^{1/2}.
$$

Here

$$
\Pi(m_1, m_2) \triangleq \{ \pi \in P^2(A \times A) : \pi(\Gamma \times A) = m_1(\Gamma), \ \pi(A \times \Gamma) = m_2(\Gamma) \}.
$$

The space $P^2(A)$ with the metric $W_2$ is a complete metric space [1, 36].
A measurable function \( \mu : [0, T] \rightarrow \mathcal{P}^2(\mathbb{R}^d) \) is called a flow of probabilities. Below we denote the set of flows of probabilities by \( \mathcal{M} \).

If \((\Omega^1, \mathcal{F}^1), (\Omega^2, \mathcal{F}^2)\) are measurable spaces, \( h : \Omega^1 \rightarrow \Omega^2 \) is measurable, \( m \) is a measure on \( \Omega^1 \), then \( h_# m \) denotes the push-forward measure on \( \Omega^2 \) i.e. for \( \Gamma \in \mathcal{F}^2 \),

\[
(h_# m)(\Gamma) = m(h^{-1}(\Gamma)).
\]

We consider the space \( \mathbb{R}^{d+1} \) as the set of pairs \( w = (x, z) \), where \( x \in \mathbb{R}^d, z \in \mathbb{R} \).

To simplify the designation put

\[
\mathcal{C} \triangleq C([0, T], \mathbb{R}^{d+1}).
\]

Denote by \( e_t \) the following projection from \( \mathcal{C} \) onto \( \mathbb{R}^d \)

\[
e_t(x(\cdot), z(\cdot)) \triangleq x(t).
\]

Note that if \( \chi_1, \chi_2 \in \mathcal{P}^2(\mathcal{C}) \), then

\[
W^2_2(e_t_# \chi_1, e_t_# \chi_2) \leq W^2_2(\chi_1, \chi_2).
\]

Indeed, if \( \pi \in \Pi(\chi_1, \chi_2) \), then

\[
\int_{\mathcal{C} \times \mathcal{C}} \|e_t(w_1(\cdot)) - e_t(w_2(\cdot))\|^2 \pi(dw_1(\cdot), w_2(\cdot))
\]

\[
\leq \int_{\mathcal{C} \times \mathcal{C}} \left( \sup_{\tau \in [0, T]} \|w_1(\tau) - w_2(\tau)\| \right)^2 \pi(dw_1(\cdot), w_2(\cdot)).
\]

Moreover,

\[
W^2_2(e_t_# \chi_1, e_t_# \chi_2) \leq \inf_{\pi \in \Pi(\chi_1, \chi_2)} \int_{\mathcal{C} \times \mathcal{C}} \|e_t(w_1(\cdot)) - e_t(w_2(\cdot))\|^2 \pi(dw_1(\cdot), w_2(\cdot))
\]

Thus, (I) holds true.

### 2.2 Stochastic mean field games

For a parameter \( n \in \mathbb{N} \), probability \( m \in \mathcal{P}^2(\mathbb{R}^d) \), and a constant control \( u \in U \) let \( L^n_t[m, u] \) be a generator of Lévy-Khintchine type, i.e.

\[
L^n_t[m, u] \varphi(x) = \frac{1}{2}\langle G^n(t, x, m, u) \nabla, \nabla \rangle \varphi(x) + \langle f^n(t, x, m, u), \nabla \rangle \varphi(x)
\]

\[
+ \int_{\mathbb{R}^d} [\varphi(x + y) - \varphi(x) - \langle y, \nabla \varphi(x) \rangle 1_{B_1}(y)] \nu^n(t, x, m, u, dy).
\]

Here \( U \) is a control space, \( G^n(t, x, m, u) \) is a nonnegative symmetric matrix, \( B_1 \) denotes the unit ball centered at the origin, \( \nu^n(t, x, m, u, \cdot) \) is a measure on \( \mathbb{R}^d \) such that \( \nu^n(t, x, m, u, \{0\}) = 0 \) and

\[
\int_{\mathbb{R}^d} \min\{1, y^2\} \nu^n(t, x, m, u, dy) < \infty.
\]
Let $\mathcal{D}^n \subset C^2(\mathbb{R}^d)$ be a Banach subspace such that $C^2_b(\mathbb{R}^d) \subset \mathcal{D}^n$ and, for any $\xi \in \mathbb{R}^d$, the functions $x \mapsto \langle \xi, x \rangle$, $x \mapsto \|x - \xi\|^2$ belong to $\mathcal{D}^n$.

Note that the dynamics of second order mean field games can be expressed in the form (2) with $\nu^n = 0$ [24]. Analogously, pure jump processes are specified by the generator of form (2) with $f^n = 0$, $G^n = 0$ [24].

Extending definition introduced in [14, p. 135] to the mean field game case we say that a 6-tuple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P, u, Y)$ is admissible for a generator $L^n$ and a flow of probabilities $\zeta$ control process if

1. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P)$ is a filtered probability space;
2. $u$ is a $\{\mathcal{F}_t\}_{t \in [0,T]}$-progressively measurable process with values in $U$; $Y$ is a $\{\mathcal{F}_t\}_{t \in [0,T]}$-adapted càdlàg process with values in $\mathbb{R}^d$;
3. for any $\varphi \in \mathcal{D}^n$, the process
\[
\varphi(Y(t)) - \int_0^t L^n_\tau[\zeta(\tau), u(\tau)]\varphi(Y(\tau))d\tau
\]
is a $\{\mathcal{F}_t\}_{t \in [0,T]}$-martingale.

Below we assume the following condition.

(A0) The generator $L^n$ is such that, for any flow of probabilities $\zeta$, any measurable deterministic function $v : [0, T] \to U$ and any $(s, \xi) \in [0, T] \times \mathbb{R}^d$, there exist a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P)$ and a process $Y$ such that $Y(s) = \xi$ $P$-a.s. and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P, v, Y)$ is admissible for $L^n$ and $\zeta$.

Remark 1. The conditions on $G^n$, $f^n$ and $\nu^n$ which guarantee assumption (A0) can be derived from [22, Theorem 5.4.2, Theorem 5.5.1]. It suffices to assume that

- the function
\[
(t, x, u, \xi) \mapsto \frac{1}{2} (G^n(t, x, m, u)\xi, \xi) - i(f^n(t, x, m, u), \xi)
+ \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, y \rangle} + i1_{B_1}(y)\langle \xi, y \rangle\right)\nu^n(t, x, m, u, dy)
\]
is continuous (here $i$ stays for imaginary unit);

- for any $C > 0$,
\[
\sup_{t \in [0,T], x \in \mathbb{R}^d, m \in P^2(\mathbb{R}^d), \|m\| < C, u \in U} \left( \frac{\|G^n(t, x, m, u)\|}{1 + \|x\|^2} + \frac{\|f^n(t, x, m, v(t))\|}{1 + \|x\|} + \frac{\int_{B_1} y^2 \nu^n(t, x, m, u)}{1 + \|x\|^2} + \int_{\mathbb{R}^d \setminus B_1} \nu^n(t, x, m, u, dy) \right) < \infty,
\]

- for any $C > 0$,
\[
\sup_{t \in [0,T], x \in \mathbb{R}^d, m \in P^2(\mathbb{R}^d), \|m\| < C, u \in U} \int_{\mathbb{R}^d \setminus B_1} \ln(\|y\|)\nu^n(t, x, m, u, dy) < \infty.
\]
Given the generator $L^n$, we consider the following mean field game system \[24\]:

$$
\frac{\partial V(t, x)}{\partial t} + \max_{u \in U}(L^n_t[\zeta[t], u]V(t, x) + g(t, x, \zeta[t], u)) = 0, \quad (3)
$$

$$
\frac{d}{dt} \langle \phi, \zeta[t] \rangle = \langle L^n_t[\zeta[t], u^*(t, \cdot, \zeta[t], V(t, \cdot))]\phi, \zeta[t] \rangle \quad \forall \phi \in D^n, \quad (4)
$$

$$
V(T, x) = \sigma(x, \zeta[T]), \quad \zeta[0] = m^n_0. \quad (5)
$$

Here $u^*(t, x, m, \varphi) : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ and a flow of probabilities $\zeta^n$ solve system \(3\)–\(5\) in the probabilistic sense if

- $\zeta^n[0] = m^n_0$;
- there exists a control process $(\Omega^n, \mathcal{F}^n, \{\mathcal{F}^n_t\}_{t \in [0, T]}, P^n, u^n, Y^n)$ that is admissible for $L^n$ and $\zeta^n$ and satisfies

$$
V^n(s, \xi) = \mathbb{E}^n_{s, \xi} \left[ \sigma(Y^n(T), \zeta^n[T]) + \int_s^T g(\tau, Y^n(\tau), \zeta^n[\tau], u^n(\tau)) d\tau \right], \quad (6)
$$

$$
\zeta^n[t] = \text{Law}(Y^n(t)), \quad (7)
$$

- for any tuple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P, u, Y)$ that is admissible for $L^n$ and $\zeta^n$, the following inequality holds true:

$$
V^n(s, \xi) \geq \mathbb{E}_{s, \xi} \left[ \sigma(Y(T), \zeta^n[T]) + \int_s^T g(\tau, Y(\tau), \zeta^n[\tau], u(\tau)) d\tau \right]. \quad (8)
$$

Here $\mathbb{E}^n_{s, \xi}$ (respectively, $\mathbb{E}_{s, \xi}$) denotes the conditional expectation for the probability $P^n$ (respectively, $P$) under condition $Y^n(s) = \xi$ (respectively, $Y(s) = \xi$).

The link between solution in probabilistic approach and classical solutions to mean field games is given in the following proposition.

**Proposition 1.** Assume that $(V^n, \zeta^n)$ is a classical solution to system \(3\)–\(5\) and there exists a solution to the martingale problem specified by the operator $L^n_t[\zeta^n[t], u^*(t, \cdot, \zeta^n[t], V^n(t, \cdot))] : D^n \rightarrow C(\mathbb{R}^d)$. Then the pair $(V^n, \zeta^n)$ solves \(3\)–\(5\) in the probabilistic sense.
Proof. The proof is close to the proof of the verification theorem [14, Theorem 8.1]. There exist a probability space \((\Omega^n, \mathcal{F}^n, \{\mathcal{F}_t^n\}_{t\in[0,T]}, P^n)\) and a stochastic process \(Y^n(\cdot)\) such that, for any \(\varphi \in D^n\),

\[
\varphi(Y^n(t)) - \int_0^t L^n[\zeta^n[\tau], u^*(\tau, \cdot, \zeta^n[\tau], V^n(\tau, \cdot))\varphi(Y^n(\tau))]d\tau
\]

is a \(\{\mathcal{F}_t^n\}_{t\in[0,T]}\)-martingale and \(\text{Law}(Y^n(0)) = \zeta^n[0]\). Further, we have that

\[
\text{Law}(Y^n(t)) = \zeta^n[t]. \quad (9)
\]

Put, \(u^n(t, \omega) \triangleq u^*(t, Y(t, \omega), \zeta^n[t], V^n(t, Y(t, \omega)))\). We have that the 6-tuple \((\Omega^n, \mathcal{F}^n, \{\mathcal{F}_t^n\}_{t\in[0,T]}, P^n, u^n, Y^n)\) is admissible for the generator \(L^n\) and the flow of probabilities \(\zeta^n\).

Since \(V^n\) is a solution to equation (3) we have that \(V^n\) is a value function for the optimization problem

\[
\text{maximize } \mathbb{E}_{\sigma \in \mathcal{G}} \left[ \sigma(Y(T), \zeta^n[T]) + \int_0^T g(t, Y(t), \zeta^n[t], u(t))dt \right]
\]

over the set of 6-tuples \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P, u, Y)\) that are admissible for the generator \(L^n\) and the flow of probabilities \(\zeta^n\).

From this and (9) we conclude the pair \((V^n, \zeta^n)\) solves (3)–(5) in the probabilistic sense.

2.3 Deterministic mean field game

Consider the deterministic mean field game system

\[
\frac{\partial V}{\partial t} + H(t, x, \mu[t], \nabla V) = 0, \quad V(T, x) = \sigma(x, \mu[T]), \quad (10)
\]

\[
\frac{d}{dt}\mu[t] = \left< \frac{\partial H(t, x, \mu[t], \nabla V)}{\partial p}, \nabla \right> \mu[t], \quad \mu[0] = m_0. \quad (11)
\]

Here \(p\) stands for the 4-th coordinate in

\[
H(t, x, m, p) = \max\{(p, f(t, x, m, u) + g(t, x, m, u) : u \in U)\}.
\]

System (10), (11) can be rewritten in the form (3)–(5) for the generator

\[
L^*_\mu[u]\varphi(x) = \langle f(t, x, u), \nabla \varphi(x) \rangle.
\]

Note that, for a given flow of probabilities \(\mu\), equation (10) provides the value function of the following optimization problem:

\[
\text{maximize } \sigma(x(T), \mu(T)) + \int_0^T g(t, x(t), u(t), \mu(t))dt \quad (12)
\]
subject to \( \frac{d}{dt} x(t) = f(t, x(t), \mu(t), u(t)), \quad x(s) = \xi, \quad x(t) \in \mathbb{R}^d, \quad u(t) \in U \) \hspace{1cm} (13)

The solution for optimization problem (12), (13) does not exist in the general case [37]. To guarantee the existence of the optimal trajectory we consider the relaxation of this problem based on differential inclusion.

Given a flow of probabilities \( \mu \), denote the set of solutions of the differential inclusion
\[
(\dot{x}(t), \dot{z}(t)) \in \co\{(f(t, x, \mu[t], u), g(t, x, \mu[t], u)) : u \in U\}, \quad x(s) = \xi, \quad z(s) = 0
\]
by \( \text{Sol}(\mu, s, \xi) \). Here \( \co \) denotes the convex hull. Additionally, put
\[
\text{Sol}_0(\mu) \triangleq \bigcup_{\xi \in \mathbb{R}^d} \text{Sol}(\mu, 0, \xi).
\]

Under assumptions made in this paper (see conditions (A1)–(A5) below) the problem
\[
\text{maximize } \sigma(x(T), \mu(T)) + z(T) \hspace{1cm} (14)
\]
\[
\text{subject to } (x(\cdot), z(\cdot)) \in \text{Sol}(\mu, s, \xi) \hspace{1cm} (15)
\]
is a relaxation of original problem (12), (13). Moreover, any element of \( \text{Sol}(\mu, s, \xi) \) can be approximated by trajectories generated by usual controls. Denote by \( U \) the set of measurable functions \( v : [0, T] \to U \). If \( s \in [0, T], \xi \in \mathbb{R}^d, v \in U \), then let \( x[\cdot, \mu, s, \xi, v] \) be a solution of the initial value problem
\[
\dot{x}(t) = f(t, x(t), \mu[t], v(t)), \quad x(s) = \xi.
\]
Moreover, put
\[
z[t, \mu, s, \xi, v] = \int_s^t g(\tau, x[\cdot, \mu, s, \xi, v], \mu[\tau], v(\tau))d\tau. \hspace{1cm} (16)
\]

By [37, Theorem VI.3.1] we have that
\[
\text{Sol}(\mu, s, \xi) = \text{cl}\{(x[\cdot, \mu, s, \xi, v], z[\cdot, \mu, s, \xi, v]) : v \text{ is measurable}\}. \hspace{1cm} (17)
\]
Here closure is taken in the space \( C = C([0, T], \mathbb{R}^{d+1}) \). Moreover, Gronwall’s inequality implies that
\[
\|x[t, \mu, s, \xi, v]\| \leq \left(\|\xi\| + MT + MT \sup_{t} \varsigma(\mu[t])\right)e^{MT}. \hspace{1cm} (18)
\]

Note that the relaxation based on differential inclusions is equivalent to the approaches based on measure-valued controls or on control measures. This means that \( (x(\cdot), z(\cdot)) \in \text{Sol}(\mu, s, \xi) \) if and only if there exists a measure-valued control \( \gamma : [0, T] \to \mathcal{P}(U) \) such that
\[
\dot{x}(t) = \int_U f(t, x(t), \mu(t), u)\gamma(t, du), \quad x(s) = \xi,
\]
\[ \dot{z}(t) = \int_U g(t, x(t), \mu(t), u) \gamma(t, du), \quad z(s) = 0. \]

Here \( \mathcal{P}(U) \) denotes the set of all probabilities on \( U \). The equivalence between approaches based on measure-valued controls and on control measures can be find, for example, in [2].

We use minimax solutions first proposed in [2]. Note that the first order mean field games can be considered within the framework of probabilistic approach. The advantage of the minimax solutions to mean field games is that they do not depend on the choice of the probability space and the representation of the Hamiltonian \( H \). The link between minimax solutions and solution in the probabilistic sense is given in Corollary 2 below.

The definition of a minimax solution to a deterministic mean field game involves the definition of a minimax solution to a Hamilton–Jacobi PDE. There exist several (equivalent) definitions of a minimax solution [35] to a Hamilton–Jacobi PDE. They are based either on viability theory or on the nonsmooth analysis. However, for our purposes the definition involving optimization problem is more useful.

For \( t \in [0, T], x \in \mathbb{R}^d, m \in \mathcal{P}^2(\mathbb{R}^d) \) and \( y \in \mathbb{R}^d \) put

\[
H^*(t, x, m, y) = \sup_{p \in \mathbb{R}^n} \{ \langle y, p \rangle - H(t, x, m, p) \},
\]

\[
\mathcal{H}(t, x, m) = \{ \xi \in \mathbb{R}^n : H^*(t, x, m, \xi) < \infty \}.
\]

The function \( V \) is a minimax solution to equation (10) for the given flow of probabilities \( \mu \) if \( V(s, \xi) \) is a value of the optimization problem

\[
\begin{align*}
\text{maximize} & \quad \sigma(x(T), \mu[T]) + z(T) - z(s) \\
\text{subject to} & \quad \dot{x}(t) \in \mathcal{H}(t, x(t), \mu[t]), \quad \dot{z}(t) \leq -H^*(t, x(t), \mu[t], \dot{x}(t)), \quad x(s) = \xi. \tag{20}
\end{align*}
\]

This is equivalent to the following property: \( V(s, \xi) \) is a value of the optimization problem (14), (15).

Denote

\[
S[V, \mu] \triangleq \{ (x(\cdot), z(\cdot)) : \dot{x}(t) \in \mathcal{H}(t, x(t), \mu[t]), \quad \dot{z}(t) = H^*(t, x(t), \mu[t], \dot{x}(t)), \quad z(t) = V(t, x(t)) \}.
\]

**Definition 2.** We say that the pair \((V^*, \mu^*) \in C([0, T] \times \mathbb{R}^d) \times \mathcal{M}\) is a minimax solution to system (10), (11) if

1. \( V^* \) is a minimax solution of equation (10);
2. \( \mu^*[0] = m_0 \)
3. there exists a probability \( \chi \) on \( \mathcal{C} \) such that \( \mu^*[t] = e_{t\#} \chi \) and \( \text{supp}(\chi) \) \( \subset S[V^*, \mu^*] \).

Note that the definition of the minimax solution to mean field game does not depend on the representation of the Hamiltonian.

The embedding of the set of optimal trajectories of problem (14), (15) into the set \( S[V, \mu] \) implies the following statement.
Proposition 2. Assume that the pair \((V^*, \mu^*) \in C([0,T] \times \mathbb{R}^d) \times \mathcal{M}\) satisfies the following conditions:

1. for any \((s, \xi) \in [0,T] \times \mathbb{R}^d\), \(V^*(s, \xi)\) is a value of the optimization problem \((14), (15)\);
2. \(\mu^*[0] = m_0\); 
3. there exists a probability \(\chi \in \mathcal{P}^2(\mathcal{C})\) such that \(\mu^*[t] = e_t \# \chi\) and 
   \[\text{supp} (\chi) \subset \{(x(\cdot), z(\cdot)) \in \text{Sol}_0(\mu) : V^*(s, x(s)) = \sigma(x(T), \mu^*[T]) + z(T) - z(s), \ s \in [0,T]\}. \] (21)

Then \((V^*, \mu^*)\) is a minimax solution to system \((10), (11)\).

3 Main result

To simplify the notations put
\[
\Sigma^n(t, x, m, u) \triangleq \sum_{i=1}^{d} G^n_{ni}(t, x, m, u) + \int_{\mathbb{R}^d} \|y\|^2 \nu^n(t, x, m, u, dy),
\]
\[
b^n(t, x, m, u) \triangleq f^n(t, x, m, u) + \int_{\mathbb{R}^d \setminus B_1} y \nu^n(t, x, m, u, dy).
\]

Below, if \(\phi\) is a function of \(k\) vectors and takes values in \(\mathbb{R}\), then we assume that \(L^n_t\) (respectively, \(L^*_t\)) acts only on the first variable. Using this convention we get that if \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P, X, u)\) is admissible for \(L^n\) and \(\zeta, s, t \in [0,T]\), \(s < t\), then
\[
\mathbb{E} \phi(X(t), \eta_1, \ldots, \eta_k) = \mathbb{E} \phi(X(s), \eta_1, \ldots, \eta_k) + \mathbb{E} \int_s^t L^n_\tau \phi(X(\tau), \eta_1, \ldots, \eta_k) d\tau. \] (22)

Here \(\mathbb{E}\) denote the expectation according to \(P\). Analogue formula is fulfilled for the generator \(L^*\) i.e. if \(\mu\) is a flow of probabilities, \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P, X, u)\) is such that
\[
\frac{d}{dt}X(t, \omega) = f(t, X(t, \omega), \mu[t], u(t, \omega),
\]
then
\[
\mathbb{E} \phi(X(t), \eta_1, \ldots, \eta_k) = \mathbb{E} \phi(X(s), \eta_1, \ldots, \eta_k) + \mathbb{E} \int_s^t L^*_\tau \phi(X(\tau), \eta_1, \ldots, \eta_k) d\tau. \] (23)

Note that if \(q(x_1, x_2) = \|x_1 - x_2\|^2\), then
\[
L^n_t[m, u]q(x_1, x_2) = \Sigma^n(t, x_1, m, u) + 2\{b^n(t, x_1, m, u), x_1 - x_2\}. \] (24)
Moreover, if $\lambda(x_1, x_2, x_3) = \langle x_1 - x_2, x_3 \rangle$

$$L^u[m, u]\lambda(x_1, x_2, x_3) = \langle b^n(t, x_1, m, u), x_3 \rangle, \quad (25)$$

$$L^u[m, u]\lambda(x_1, x_2, x_3) = \langle f(t, x_1, m, u), x_3 \rangle. \quad (26)$$

We put the following assumptions on the set $U$, functions $f^n, f, g$ and $\sigma$.

(A1) $U$ is compact;

(A2) There exists a function $\alpha(\cdot)$ such that $\alpha(\delta) \to 0$ as $\delta \to 0$ and, for any $t', t'' \in [0, T], x \in \mathbb{R}^d, m \in \mathcal{P}^2(\mathbb{R}^d), u \in U$,

$$\|f(t', x, m, u) - f(t'', x, m, u)\| \leq \alpha(t' - t'');$$

(A3) there exists a constant $K$ such that, for any $t \in [0, T], x', x'' \in \mathbb{R}^d, u \in U, m', m'' \in \mathcal{P}^2(\mathbb{R}^d)$,

$$\|f(t, x', m', u) - f(t, x'', m'', u)\| \leq K(\|x' - x''\| + W_2(m', m'')),$$

(A4) there exists a constant $M$ such that, for any $t \in [0, T], x \in \mathbb{R}^d, m \in \mathcal{P}^2(\mathbb{R}^d), u \in U$,

$$\|f(t, x, m, u)\| \leq M(1 + \|x\| + \varsigma(m)),$$

$$\|b^n(t, x, m, u)\| \leq M(1 + \|x\| + \varsigma(m)),$$

$$|\Sigma^n(t, x, m, u)| \leq M(1 + \|x\|^2 + \varsigma^2(m)),$$

$$|g(t, x, m, u)| \leq M(1 + \|x\| + \kappa(\varsigma(m))),$$

where $\kappa : [0, +\infty) \to [0, +\infty)$ is a strictly increasing function;

(A5) there exists a constant $R$ such that

$$|\sigma(x', m') - \sigma(x'', m'')| \leq R(\|x' - x''\| + W_2(m', m''))$$

$$\cdot(1 + \|x'\| + \|x''\| + \kappa(\varsigma(m')) + \kappa(\varsigma(m''))),$$

$$|g(t, x', m', u) - g(t, x'', m'', u)| \leq R(\|x' - x''\| + W_2(m', m''))$$

$$\cdot(1 + \|x'\| + \|x''\| + \kappa(\varsigma(m')) + \kappa(\varsigma(m''))).$$

We assume that the generators $L^n$ converge to the generator of the deterministic mean field game $L^*$ in the following sense:

$$\sup_{t \in [0, T], x \in \mathbb{R}^d, m \in \mathcal{P}^2(\mathbb{R}^d)} \frac{\Sigma^n(t, x, m, u)}{1 + \|x\|^2 + \varsigma^2(m)} \to 0 \text{ as } n \to \infty; \quad (27)$$

$$\sup_{t \in [0, T], x \in \mathbb{R}^d, m \in \mathcal{P}^2(\mathbb{R}^d)} \frac{\|b^n(t, x, m, u) - f(t, x, m, u)\|}{(1 + \|x\| + \varsigma(m))} \to 0 \text{ as } n \to \infty. \quad (28)$$

Additionally, we assume that the initial distributions $m^n_0$ converge to the probability $m_0$ in the 2-Wasserstein metric i.e.

$$W_2(m^n_0, m_0) \to 0 \text{ as } n \to \infty. \quad (29)$$

The main result of the paper is the following.

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Theorem 1. Assume that conditions (27)–(29) hold true and, for each natural number \(n\), the pair \((V_n, \zeta_n)\) solves system (3)–(5) in the probabilistic sense.

Then there exist a pair \((V^*, \mu^*)\) such that \(\{m_l\}_{l=1}^{\infty}\) such that

1. \(\sup_{t \in [0,T]} W_2(\zeta_n[t], \mu^*[t]) \to 0\) as \(l \to \infty\);

2. \(\lim_{l \to \infty} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |V_{n_l}(t, x) - V^*(t, x)| + \|x\|^2 = 0\)

Corollary 1. If, for each natural number \(n\), the pair \((V_n, \zeta_n)\) is a classical solution to system (3)–(5), then the conclusion of Theorem 1 holds true.

Corollary 2. Any solution in the probabilistic sense is a minimax solution.

If \((V^*, \mu^*)\) satisfies conditions of Proposition 2 and the set \(\{(f(t, x, m, u), g(t, x, m, u)) : u \in U\}\) is convex for all \(t \in [0,T]\), \(x \in \mathbb{R}^d\), \(m \in \mathcal{P}_2(\mathbb{R}^d)\), then \((V^*, \mu^*)\) is a solution to system (10), (11) in the probabilistic sense.

4 Uniform bounds for flows of probabilities

We choose and fix the control process \((\Omega^n, \mathcal{F}^n, \{\mathcal{F}^n_t\}_{t \in [0,T]}, P^n, u^n, Y^n)\) that is admissible for the generator \(L^n\) and the flow of probabilities \(\zeta^n\). Further, assume that the tuple \((\Omega^n, \mathcal{F}^n, \{\mathcal{F}^n_t\}_{t \in [0,T]}, P^n, u^n, Y^n)\) satisfies conditions (6)–(8). Below \(\mathbb{E}^n\) denotes expectation corresponding to the probability \(P^n\).

There exists (see [34]) a \(\{\mathcal{F}_t\}_{t \in [0,T]}\)-adapted càdlàg stochastic process \(X^n\) with values in \(\mathbb{R}^d\) and continuous sample paths and a flow of \(\mu^n\) such that

\[
\frac{d}{dt} X^n(t, \omega) = f(t, X^n(t, \omega), \mu^n[t], u^n(t, \omega)), \quad X^n(0, \omega) = X^n_0(\omega),
\]

\[
\mu^n[t] = \text{Law}(X^n(t, \cdot)).
\]

Here \(X^n_0(\omega) \triangleq Y^n(0, \omega)\). Note that \(\text{Law}(X^n_0) = m^n_0\). Further, put

\[
\mathcal{X}^n(t, \omega) \triangleq \int_0^t g(\tau, X^n(\tau, \omega), \mu^n[\tau], u^n(\tau, \omega))d\tau.
\]

Let \(\text{traj}^n : \Omega^n \to \mathcal{C}\) be defined by the following rule:

\[
\text{traj}^n(\omega) \triangleq (X^n(\cdot, \omega), \mathcal{X}^n(\cdot, \omega)).
\]

Put

\[
\chi^n \triangleq \text{traj}^n \# \Omega^n.
\]

We have that

\[
\mu^n[t] = e_t \# \chi^n.
\]
Note that,
\[ \varsigma^2(m_0^n) \leq 2W^2(m_0^n, m_0) + 2\varsigma^2(m_0). \]
Therefore there exists a constant \( M_0 \) such that
\[ \varsigma^2(m_0^n) \leq M_0. \]

**Lemma 1.** There exists a constant \( C_1 \) such that, for any \( t \in [0, T] \), \( \varsigma^2(\zeta^n[t]) \leq C_1. \)

**Proof.** Formulas (22), (24) and condition (A4) yield that
\[ \mathbb{E}\|Y^n(t)\|^2 = \mathbb{E}\|Y^n(0)\|^2 + \mathbb{E} \int_0^t (\Sigma^n(\tau, Y^n(\tau), \zeta^n[\tau], u^n(\tau)) d\tau 
+ 2\mathbb{E} \int_0^t \langle b^n(\tau, Y^n(\tau), \zeta^n[\tau], u^n(\tau), Y^n(\tau)) d\tau 
\leq \varsigma^2(m_0^n) + M \int_0^t \mathbb{E}(1 + \|Y^n(\tau)\|^2 + \varsigma^2(\zeta^n[\tau])) d\tau 
+ 2M \int_0^t \mathbb{E}((1 + \|Y^n(\tau)\| + \varsigma(\zeta^n[\tau])) \|Y^n(\tau)\|) d\tau 
\leq \varsigma^2(m_0^n) + M \int_0^t \mathbb{E}(2 + 5\|Y^n(\tau)\|^2 + 2\varsigma^2(\varsigma^n[\tau])) d\tau. \]

Since \( \mathbb{E}\|Y^n(t)\|^2 = \varsigma^2(\zeta^n[t]) \) we get the inequality
\[ \varsigma^2(\zeta^n[t]) \leq \varsigma^2(m_0^n) + 2Mt + 7M \int_0^t \varsigma^2(\varsigma^n[\tau]) d\tau. \]

Using Gronwall inequality we get the conclusion of the Lemma with \( C_1 = (M_0 + 2MT)e^{7MT}. \)

**Lemma 2.** There exists a constant \( C_2 \) such that, for any \( t \in [0, T] \), \( \varsigma^2(\mu^n[t]) \leq C_2. \)

The proof of Lemma 2 is analogous to the proof of Lemma 1.

**Lemma 3.** There exists a constant \( C_3 \) such that if \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P, u, Y) \) is admissible for the generator \( L^n \) and the flow of probabilities \( \zeta^n \), then, for any \( s, t \in [0, T] \), \( t \geq s, \)
\[ \mathbb{E}\|Y(t)\|^2 \leq C_3(1 + \mathbb{E}\|Y(s)\|^2). \]

**Proof.** By (22), (24) and condition (A4) we have that
\[ \mathbb{E}\|Y(t)\|^2 = \mathbb{E}\|Y(s)\|^2 + \mathbb{E} \int_s^t (\Sigma^n(\tau, Y(\tau), \zeta^n[\tau], u(\tau)) d\tau 
+ 2\mathbb{E} \int_s^t \langle b^n(\tau, Y(\tau), \zeta^n[\tau], u(\tau), Y(\tau)) d\tau 
\leq \mathbb{E}\|Y(s)\|^2 + M \int_s^t \mathbb{E}(1 + \|Y(\tau)\|^2 + \varsigma^2(\zeta^n[\tau])) d\tau 
+ 2M \int_s^t \mathbb{E}((\|Y(\tau)\| + \|Y(\tau)\| + \varsigma(\zeta^n[t])) \|Y(\tau)\|) d\tau 
\leq \mathbb{E}\|Y(s)\|^2 + M \int_s^t \mathbb{E}(2 + 5\|Y(\tau)\|^2 + 2\varsigma^2(\varsigma^n[\tau])) d\tau. \]

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Using Lemma 1 and Gronwall’s inequality we obtain the conclusion of the Lemma with $C_3 \triangleq 2MT(1 + C_1)e^{5MT}$.

**Lemma 4.** There exists a constant $C_4$ such that if $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P)$ is a probability space, control $u$ is a $\{\mathcal{F}_t\}_{t \in [0,T]}$-progressively measurable control, and $X$ is a stochastic process satisfying

$$\frac{d}{dt}X(t, \omega) = f(t, X(t, \omega), \mu^n[t], u(t, \omega)), \quad (35)$$

then, for any $s, t \in [0, T]$, $t \geq s$,

$$\mathbb{E}\|X(t)\|^2 \leq C_4 (1 + \mathbb{E}\|X(s)\|^2).$$

The proof of this Lemma is analogous to the proof of Lemma 3.

### 5 Distance between flows of probabilities

Without loss of generality we may assume that there exists a sequence $\{\varepsilon^n\}_{n=1}^\infty$ such that

- $\varepsilon^n \leq 1$, $\varepsilon^n \to 0$ as $n \to \infty$;
- $\sup_{t \in [0,T], x \in \mathbb{R}^d, m \in \mathcal{P}^2(\mathbb{R}^d)} \frac{\Sigma^n(t, x, m, u)}{1 + \|x\|^2 + \varsigma^2(m)} \leq \varepsilon^n;
- \sup_{t \in [0,T], x \in \mathbb{R}^d, m \in \mathcal{P}^2(\mathbb{R}^d)} \frac{\|b^n(t, x, m, u) - f(t, x, m, u)\|}{1 + \|x\| + \varsigma(m)} \leq \sqrt{\varepsilon^n};
- W_2^2(\delta^n_0, \delta_0) \leq \varepsilon^n.$

**Lemma 5.** There exists a constant $C_5$ such that, for any $t \in [0, T]$,

$$W_2^2(\zeta^n[t], \mu^n[t]) \leq C_5\varepsilon^n.$$

**Proof.** To simplify the designations let us introduce the following stochastic processes

- $\hat{\Sigma}^n(t) = \Sigma^n(t, Y^n(t), \zeta^n[t], u^n(t))$,
- $\hat{b}^n(t) \triangleq b^n(t, Y^n(t), \zeta^n[t], u^n(t))$,
- $\hat{f}(t) \triangleq f(t, X^n(t), \mu^n[t], u^n(t)).$

For $t, s \in [0, T]$, $t \geq s$ we have that

$$\|Y^n(t) - X^n(t)\|^2 = \|(Y^n(t) - Y^n(s)) - (X^n(t) - X^n(s)) + (Y^n(s) - X^n(s))\|^2$$

$$= \|Y^n(s) - X^n(s)\|^2 + \|Y^n(t) - Y^n(s)\|^2 + \|X^n(t) - X^n(s)\|^2$$

$$+ 2\langle Y^n(s) - X^n(s), Y^n(t) - Y^n(s) \rangle - 2\langle Y^n(s) - X^n(s), X^n(t) - X^n(s) \rangle$$

$$\leq \|Y^n(s) - X^n(s)\|^2 + 2\|Y^n(t) - Y^n(s)\|^2 + 2\|X^n(t) - X^n(s)\|^2$$

$$+ 2\|Y^n(s) - X^n(s), Y^n(t) - Y^n(s) \rangle - 2\langle Y^n(s) - X^n(s), X^n(t) - X^n(s) \rangle.$$
Therefore, we conclude that for $c$

$$\begin{align*}
\mathbb{E}^n \|Y^n(t) - X^n(t)\|^2 \\
\leq \mathbb{E}^n \|Y^n(s) - X^n(s)\|^2 + 2\mathbb{E}^n \|Y^n(t) - Y^n(s)\|^2 \\
+ 2\mathbb{E}^n \|X^n(t) - X^n(s)\|^2 + 2\mathbb{E}^n \langle Y^n(s), Y^n(t) - Y^n(s) \rangle \\
- 2\mathbb{E}^n \langle Y^n(s), X^n(t) - X^n(s) \rangle.
\end{align*}$$

(36)

Hence,

$$\begin{align*}
\mathbb{E}^n \|Y^n(t) - Y^n(s)\|^2 \\
&\leq \mathbb{E}^n \|Y^n(s) - X^n(s)\|^2 + 2\mathbb{E}^n \|Y^n(t) - Y^n(s)\|^2 \\
&+ 2\mathbb{E}^n \|X^n(t) - X^n(s)\|^2 + 2\mathbb{E}^n \langle Y^n(s), Y^n(t) - Y^n(s) \rangle \\
&- 2\mathbb{E}^n \langle Y^n(s), X^n(t) - X^n(s) \rangle
\end{align*}$$

Using Lemma 1 we get the inequality

$$\begin{align*}
\mathbb{E}^n \|Y^n(t)\|^2 &= \zeta^2(\zeta^n[t]), \\
\mathbb{E}^n \|X^n(t)\|^2 &= \zeta^2(\mu^n[t]).
\end{align*}$$

(37)

Further, (22), (24) and condition (A4) imply that

$$\begin{align*}
\mathbb{E}^n \|Y^n(t) - Y^n(s)\|^2 \\
&= \mathbb{E}^n \int_s^t \tilde{\Sigma}^n(\tau) d\tau + 2\mathbb{E}^n \int_s^t \langle \tilde{b}^n(\tau), Y^n(\tau) - Y^n(s) \rangle d\tau \\
&\leq \varepsilon^n \left( (t-s) + 2 \int_s^t \zeta^2(\zeta^n(\tau)) d\tau \right) + 2\mathbb{E}^n \int_s^t \langle \tilde{b}^n(\tau), Y^n(\tau) - Y^n(s) \rangle d\tau.
\end{align*}$$

Using Lemma 1 we get the inequality

$$\begin{align*}
\mathbb{E}^n \|Y^n(t) - Y^n(s)\|^2 &\leq \varepsilon^n c'_1 (t-s) + 2\mathbb{E}^n \int_s^t \langle \tilde{b}^n(\tau), Y^n(\tau) - Y^n(s) \rangle d\tau,
\end{align*}$$

(38)

where $c'_1 = 1 + 2C_1$. It follows from condition (A4) and Lemma 1 that

$$\begin{align*}
\mathbb{E}^n \|\tilde{b}^n(\tau)\|^2 &\leq M^2(2 + 3\zeta^2(\zeta^n[\tau])) + 3\mathbb{E}^n \|Y^n(\tau)\|^2 \\
&\leq M^2(2 + 6\zeta^2(\zeta^n[\tau]))\leq c'_2. 
\end{align*}$$

(39)

Here $c'_2 \triangleq M^2(2+6C_1)$. Therefore

$$\begin{align*}
\mathbb{E}^n \|Y^n(t) - Y^n(s)\|^2 &\leq (\varepsilon^n c'_1 + c'_2) (t-s) + \int_s^t \mathbb{E}^n \|Y^n(\tau, \cdot) - Y^n(s, \cdot)\|^2 d\tau.
\end{align*}$$

Using Gronwall inequality we obtain the following estimate

$$\begin{align*}
\mathbb{E}^n \|Y^n(t) - Y^n(s)\|^2 &\leq c'_3 (t-s).
\end{align*}$$

(39)

Here constant $c'_3$ is equal to $(c'_1 + c'_2)e^T$. From this, (38) and (39) we get that

$$\begin{align*}
\mathbb{E}^n \|Y^n(t) - Y^n(s)\|^2 &\leq \varepsilon^n c'_1 (t-s) + 2 \int_s^t \sqrt{\mathbb{E}^n \|\tilde{b}^n(\tau)\|^2} \sqrt{\mathbb{E}^n \|Y^n(\tau) - Y^n(s)\|^2} d\tau \\
&\leq \varepsilon^n c'_1 (t-s) + 2 \int_s^t \sqrt{c'_2 \sqrt{c'_3 \sqrt{\tau - s}}} d\tau.
\end{align*}$$

Therefore, we conclude that for $c'_4 \triangleq 2\sqrt{c'_2 c'_3}/3$

$$\begin{align*}
\mathbb{E}^n \|Y^n(t) - Y^n(s)\|^2 &\leq \varepsilon^n c'_1 (t-s) + c'_4 (t-s)^{3/2}.
\end{align*}$$

(40)
Now let us estimate $E^n\|X^n(t) - X^n(s)\|^2$.

By Jensen's inequality and Lemma 2 we have that

$$E^n\|X^n(t)\| \leq \sqrt{E^n(\|X^n(t)\|^2)} = \varsigma(\mu^n[\tau]) \leq \sqrt{C_2}.$$ 

Thus, we obtain that

$$E^n\|X^n(t) - X^n(s)\|^2 = E^n\left\| \int_s^t f(\tau, X^n(\tau), \mu^n[\tau], u^n(\tau)) d\tau \right\|^2$$

$$\leq M^2 \int_s^t E^n(1 + \|X^n(\tau)\| + \varsigma(\mu^n[\tau]))^2 d\tau \leq c_5'(t - s)^2.$$ 

Here $c_5' \equiv M^2(3 + 5C_2)$.

Now let us estimate the remaining terms of the right-hand side of (36).

Since the control process $(\Omega^n, F^n, \{F^n_t\}_{t \in [0, T]}, P^n, u^n, Y^n)$ is admissible for $L^n$ and $\zeta^n$ by (22), (23), (25), (26) we have that

$$E^n\langle Y^n(s) - X^n(s), Y^n(t) - Y^n(s) \rangle - E^n\langle Y^n(s) - X^n(s), X^n(t) - X^n(s) \rangle$$

$$= E^n \int_s^t \langle Y^n(s) - X^n(s), \hat{b}^n(\tau) - \hat{f}(\tau) \rangle d\tau.$$ 

Further,

$$\|\hat{b}^n(\tau) - \hat{f}(\tau)\| = \|b^n(\tau, Y^n(\tau), \zeta^n[\tau], u^n(\tau)) - f(\tau, X^n(\tau), \mu^n[\tau], u^n(\tau))\|$$

$$\leq \|b^n(\tau, Y^n(\tau), \zeta^n[\tau], u^n(\tau)) - f(\tau, Y^n(\tau), \zeta^n[\tau], u^n(\tau))\|$$

$$+ \|f(\tau, Y^n(\tau), \zeta^n[\tau], u^n(\tau)) - f(\tau, Y^n(\tau), \zeta^n[\tau], u^n(\tau))\|$$

$$+ \|f(s, Y^n(s), \zeta^n[\tau], u^n(\tau)) - f(s, X^n(s), \mu^n[\tau], u^n(\tau))\|$$

$$+ \|f(s, X^n(s), \mu^n[\tau], u^n(\tau)) - f(\tau, X^n(\tau), \mu^n[\tau], u^n(\tau))\|$$

$$\leq \sqrt{\epsilon^n}(1 + \|Y^n(\tau)\| + \varsigma(\zeta^n[\tau])) + 2\alpha(\tau - s)$$

$$+ K\|Y^n(s) - X^n(s)\| + KW_2(\zeta^n[\tau], \mu^n[\tau])$$

$$+ K\|Y^n(\tau) - Y^n(s)\| + K\|X^n(\tau) - X^n(s)\|.$$ 

Hence, using Lemma 1 and estimates (40), (41) we get

$$E^n\langle Y^n(s) - X^n(s), Y^n(t) - Y^n(s) \rangle - E^n\langle Y^n(s) - X^n(s), X^n(t) - X^n(s) \rangle$$

$$\leq \frac{5K + 5}{2}(t - s)\|Y^n(s) - X^n(s)\|^2 + \frac{K}{2} \int_s^t W_2^2(\zeta^n[\tau], \mu^n[\tau]) d\tau$$

$$+ \varepsilon^n c_6'(t - s) + \alpha_1(t - s)(t - s).$$ 

where $c_6' = 1/2 + C_1$ and $\alpha_1(\delta) = (\alpha(\delta))^2 + K(1' \delta + c_4'\delta^3/2 + c_5'\delta^2) / 2$. We have that $\alpha_1(\delta) \to 0$ as $\delta \to 0$.

Combining (36) and (40)-(42) we get the inequality

$$E^n\|Y^n(t) - X^n(t)\|^2$$

$$\leq (1 + c_7'(t - s))E^n\|Y^n(s) - X^n(s)\|^2 + K \int_s^t W_2^2(\zeta^n[\tau], \mu^n[\tau]) d\tau$$

$$+ \alpha_2(t - s)(t - s) + \varepsilon^n c_8'(t - s)$$ 

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for 
\[ c'_7 = 5K + 5, \] (44)
\[ c'_8 = c'_1 + c'_6 \] and the function \( \alpha_2(\delta) \triangleq \alpha_1(\delta) + c'_9 \delta^{1/2} + c'_3 \delta \). Note that \( \alpha_2(\delta) \to 0 \) as \( \delta \to 0 \).

Now let \( t_* \in [0, T] \). For a natural number \( N \) set \( t'_N \triangleq it_*/N \). Applying inequality \( \Box \) sequentially for \( s = t'_N, t = t'_{N+1} \) we conclude that
\[
\mathbb{E}\| Y^n(t_*) - X^n(t_*) \|^2 \leq e^{c'_9 T} \mathbb{E}\| Y^n(0) - X^n(0) \|^2 + c'_1 \int_0^{t_*} W^2_2(\zeta^n[\tau], \mu^n[\tau]) d\tau + c'_3 t_* + c'_4 \mathbb{E}\| Y^n(0) - X^n(0) \|^2 = 0,
\]
we have that
\[
\mathbb{E}\| Y^n(t_*) - X^n(t_*) \|^2 \leq c'_9 \int_0^{t_*} W^2_2(\zeta^n[\tau], \mu^n[\tau]) d\tau + c'_4 t_* + c'_3 (T/N)t_*.
\]
Passing to the limit as \( N \to \infty \) we get the inequality
\[
\mathbb{E}\| Y^n(t_*) - X^n(t_*) \|^2 \leq c'_9 \int_0^{t_*} W^2_2(\zeta^n[\tau], \mu^n[\tau]) d\tau + c'_4 t_*.
\]
Since
\[
W^2_2(\zeta^n[\tau], \mu^n[\tau]) \leq \mathbb{E}\| Y^n(t_*) - X^n(t_*) \|^2,
\]
we have that
\[
W^2_2(\zeta^n[\tau], \mu^n[\tau]) \leq c'_9 \int_0^{t_*} W^2_2(\zeta^n[\tau], \mu^n[\tau]) d\tau + c'_4 t_*.
\]
Using Gronwall inequality we obtain the conclusion of the Lemma with \( C_5 = c'_4 Te^{c'_9 T} \).

The following Lemma is concerned with the distance between admissible control process and the control process governed by the differential equation depending on probability.

**Lemma 6.** There exists a constant \( C_6 \) such that, for any \( t \in [0, T] \), \( \xi \in \mathbb{R}^d \), if

- the control process \( (\Omega, \mathcal{F}, \mathcal{F}_{t \in [0, T]}, P, u, Y) \) is admissible for \( L^n \) and \( \zeta^n \),
- \( Y(s) = \xi \) \( P \)-a.s.,
- the process \( X \) satisfies the condition
  \[
  \frac{d}{dt} X(t, \omega) = f(t, X(t, \omega), \mu^n[\cdot], u(t, \omega)), \quad X(s, \omega) = \xi,
  \]
then
\[
\mathbb{E}\| Y(t) - X(t) \|^2 \leq C_6 \| \xi \|^2 (1 + \| \xi \|^2).
\]
Here \( \mathbb{E} \) denotes the expectation corresponding to probability \( P \).
Proof. By Lemmas 3 and 4 we get that there exist constants $c_1'$ and $c_2'$ such that, for any $t, s \in [0, T], t \geq s,$

$$\mathbb{E}\|Y(t) - Y(s)\|^2 \leq c_1'[\varepsilon^n(1 + \|\xi\|^2) + (t - s)^{1/2}](t - s),$$

$$\mathbb{E}\|X(t) - X(s)\|^2 \leq c_2'(1 + \|\xi\|^2)(t - s)^2.$$  

These inequalities can be deduced in the same way as inequalities (40), (41).

For $t, s \in [0, T], t \geq s$ we have that

$$\mathbb{E}\|Y(t) - X(t)\|^2$$

$$\leq \mathbb{E}\|Y(s) - X(s)\|^2 + 2\mathbb{E}\|Y(t) - Y(s)\|^2 + 2\mathbb{E}\|X(t) - X(s)\|^2$$

$$+ 2\mathbb{E}\langle Y(s) - X(s), Y(t) - Y(s)\rangle - 2\mathbb{E}\langle Y(s) - X(s), X(t) - x(s)\rangle.$$  

Estimating the right-hand side as in the proof of Lemma 5 using (22), (23), (25), (26) and Lemmas 1–4 we get that

$$\mathbb{E}\|Y(t) - X(t)\|^2$$

$$\leq (1 + c_4'(t - s))\mathbb{E}\|Y(t) - X(t)\|^2 + K \int_s^t W^2(\zeta^n[\tau], \mu^n[\tau])d\tau$$

$$+ \varepsilon^n c_3'(1 + \|\xi\|^2)(t - s) + \alpha_4(t - s, \xi).$$  

Here $c_4'$ is given by (44), $c_3'$ is a constant, $\alpha_4(\cdot, \cdot)$ is a function such that $\alpha_4(\delta, \xi) \to 0$ as $\delta \to 0$ for any $\xi \in \mathbb{R}^d$. Hence, by Lemma 5 we obtain the following estimate

$$\mathbb{E}\|Y(t) - X(t)\|^2 \leq (1 + c_4'(t - s))\mathbb{E}\|Y(s) - X(s)\|^2$$

$$+ \alpha_4(t - s)(t - s) + \varepsilon^n(c_3'(1 + \|\xi\|^2) + KC_5)(t - s).$$  

(45)

Now let $t_* \in [0, T], t_N^* \triangleq i t_*/N$. Using the inequality (45) sequentially we get the inequality

$$\mathbb{E}\|Y(t_N^*) - X(t_N^*)\|^2 \leq e^{c_5(t_N^* - t_0^*)}\mathbb{E}\|Y(t_0^*) - X(t_0^*)\|^2$$

$$+ e^{c_5(t_N^* - t_0^*)}\alpha_4(T/N, \xi)(t_N^* - t_0^*) + \varepsilon^n(c_3'(1 + \|\xi\|^2) + K C_5)e^{c_5(t_N^* - t_0^*)}(t_N^* - t_0^*).$$  

Hence,

$$\mathbb{E}\|Y(t_*) - X(t_*)\|^2 \leq e^{c_5^* T\alpha_4(T/N, \xi)T} + \varepsilon^n(c_3'(1 + \|\xi\|^2) + K C_5)e^{c_5^* T} T.$$  

Letting $N \to \infty$ we get the conclusion of the Lemma. 

\[\square\]

6 Limit flow of probabilities

This section is devoted to the proofs of Theorem 1 and Corollaries 1, 2.
Proof of Theorem 1. First, let us show that the family of probabilities \( \{\chi^n\}_{n=1}^{\infty} \) is tight. Indeed, by (33), (34)

\[
\chi^n = \text{traj}^n \#P^n, \quad \mu^n[t] = e_t \#\chi^n = X(t, \cdot) \#P^n.
\]

This means that, for any continuous function \( \varphi : C \to \mathbb{R} \),

\[
\int_C \varphi(x(\cdot), z(\cdot))\chi^n(d(x(\cdot), z(\cdot))) = \mathbb{E}^n\varphi(X^n(\cdot), \mathcal{X}(\cdot)).
\]

We have that

\[
\mathbb{E}^n(\|X^n(0)\|^2 + \|\mathcal{X}^n(0)\|^2) = \mathbb{E}^n\|X^n(0)\|^2 \leq \varsigma^2(m_0^n).
\]

Further,

\[
\|X^n(t') - X^n(t)\| \leq M \left( 1 + \sup_t \|X^n(t)\| + \sup_t \varsigma(\mu^n(t)) \right) |t' - t|.
\]

In addition,

\[
|\mathcal{X}^n(t') - \mathcal{X}^n(t)| \leq M \left( 1 + \sup_t \|X^n(t)\| + \sup_t \kappa(\mu^n(t)) \right) (t' - t).
\]

Since \( \mathbb{E}\|X^n(t)\| \leq \sqrt{\mathbb{E}\|X^n(t)\|^2} = \varsigma(\mu^n[t]) \) by Lemma 2 there exists a constant \( c_1^* \) such that

\[
\mathbb{E}^n(\|X^n(t') - X^n(t)\| + \|\mathcal{X}^n(t') - \mathcal{X}^n(t)\|) \leq c_1^*|t' - t|.
\]

Therefore by [3, Theorem 7.3] the sequence \( \{\chi^n\}_{n=1}^{\infty} \) is tight.

Now let us prove that \( \{\chi^n\} \) have uniformly integrable second moments. Condition (A(\(H\)) and Lemma 2 imply that there exists a constant \( c_1^* \) such that, for any \( \omega \in \Omega \),

\[
\|X^n(\cdot, \omega)\|^2 \leq c_1^*(1 + \|X^n_0(\omega)\|^2).
\]

Further, we have that

\[
\|\mathcal{X}^n(\cdot, \omega)\|^2 \leq c_1^*(1 + \|X^n_0(\omega)\|^2).
\]

Thus,

\[
\int_\|w(\cdot)\|^2 \geq k \|w(\cdot)\|^2\chi^n(d(w(\cdot)))
\]

\[
= \int_{\Omega^n: \|X_n(\cdot, \omega)\|^2 + \|\mathcal{X}^n(\cdot, \omega)\|^2 \geq k} \left[ \|X^n(\cdot, \omega)\|^2 + \|\mathcal{X}^n(\cdot, \omega)\|^2 \right] P^n(d\omega)
\]

\[
\leq \int_{\Omega^n: \|X^n_0(\omega)\|^2 \geq k/c_1^* + c_2^*} (c_1^* + c_2^*)(1 + \|X^n_0(\omega)\|^2) P^n(d\omega)
\]

\[
= \int_{\|\xi\| \geq k/(c_1^* + c_2^*)} (c_1^* + c_2^*)(1 + \|\xi\|^2)m_0^n(d\xi). \quad (46)
\]
Since \( \{m^n_0\} \) converges to \( m_0 \) in 2-Wasserstein metric, we have that the sequence \( \{m^n_0\} \) has uniformly integrable second moments. This and (46) implies that \( \{\chi^n\} \) have uniformly integrable second moments.

Using this and tightness of \( \{\chi^n\} \) we have that that there exist a sequence \( n_l \) and a probability \( \chi^* \in P^2(C) \) such that

\[
W_2(\chi^n, \chi^*) \to 0 \text{ as } l \to \infty.
\] (47)

Denote \( \mu^*[t] \triangleq e_{t\#}\chi^* \). Further, let \( V^*(s, \xi) \) be a value function of problem (14), (15) for \( \mu = \mu^* \). Below we show that

(i) \( \mu^n, V^n \) converge to \( \mu^*, V^* \), respectively (see statements 1, 2 of the Theorem);

(ii) \( (V^*, \mu^*) \) is a solution to mean field game (10), (11).

Inequality (1) yields that

\[
\sup_{t \in [0, T]} W_2(\mu^*[t], \mu^n[t]) \to 0 \text{ as } l \to \infty.
\]

Thus, statement 1 of the Theorem is proved.

To prove the second statement of the Theorem we introduce the auxiliary function \( \hat{V}^n(s, \xi) \) that is equal to the value function of problem (14), (15) for \( \mu = \mu^n \). Now let us estimate \( |V^*(s, \xi) - \hat{V}^n(s, \xi)| \).

By (17) we have that

\[
\hat{V}^n = \sup \{ \sigma(x[T, \mu^n, s, \xi, v], \mu^n[T]) - z[T, \mu^n, s, \xi, v] : v \in \mathcal{U} \},
\] (49)

\[
V^* = \sup \{ \sigma(x[T, \mu^*, s, \xi, v], \mu^*[T]) - z[T, \mu^*, s, \xi, v] : v \in \mathcal{U} \}.
\] (50)

Using Gronwalls’ inequality, condition (A3), and inequality (11) we obtain, for every \( v \in \mathcal{U} \), the following estimate:

\[
\|x[t, \mu^n, s, \xi, v] - x[t, \mu^*, s, \xi, v]\| \leq KTW_2(\chi^n, \chi^*) e^{KT}.
\] (51)

This, condition (A5), formula (16), inequality (18) and Lemma 2 imply that there exist a constant \( c_3^* \) such that

\[
|\sigma(x[T, \mu^n, s, \xi, v], \mu^n[T]) - z[T, \mu^n, s, \xi, v]) - (\sigma(x[T, \mu^*, s, \xi, v], \mu^*[T]) - z[T, \mu^*, s, \xi, v])| \\
\leq c_3^*(1 + \|\xi\|)W_2(\chi^n, \chi^*).
\]

This inequality and representations (49), (50) yield the following estimate:

\[
|\hat{V}^n(s, \xi) - V^*(s, \xi)| \leq c_3^*(1 + \|\xi\|)W_2(\chi^n, \chi^*).
\] (52)
Further, let us estimate $|\hat{V}^n(s, \xi) - V^n(s, \xi)|$. To this end we consider a control process $(\Omega, F, \{F_t\}_{t \in [0,T]}, P, u, Y)$ admissible for $L^n$ and $\xi^n$ such that $Y(s) = \xi$ P-a.s. Let, for each $\omega \in \Omega$, $X(\cdot, \omega)$ satisfy (53) and initial condition $X(s, \omega) = \xi$.

Condition (A5) implies that

$$|\sigma(X(T), \mu^n(T)) - \sigma(Y(T), \xi^n(T))|$$

$$\leq R(|X(T) - Y(T)| + W_2(\mu^n(T), \xi^n(T)))$$

$$\cdot (1 + \|X(T)\| + \|Y(T)\| + \varkappa(\sigma^m[T]) + \varkappa(\xi^n[T])).$$

These inequalities and Lemmas 3–6 yield that, for some constant $c_4$, the following inequality holds true:

$$\mathbb{E}_{s, \xi} \left| \sigma(X(T), \mu^n(T)) + \int_s^T g(t, X(t), \mu^n(t), u(t)) dt 
- \sigma(Y(T), \xi^n(T)) - \int_s^T g(t, Y(t), \xi^n(t), u(t)) dt \right| \leq c_4 \sqrt{\varepsilon^n} (1 + \|\xi\|^2).$$

Further, by (50) we have that

$$\hat{V}^n(s, \xi) \geq \mathbb{E}^{s, \xi} \sigma(X^n(T), \mu^n(T)) + \int_s^T g(t, X^n(t), \mu^n(t), u(t)) dt.$$

Using this inequality, definition of $V^n$, and estimate (53) for control space $(\Omega^n, F^n, \{F^n_t\}_{t \in [0,T]}, P^n, u^n, Y^n)$ and $X = X^n$ we get the following:

$$\hat{V}^n(s, \xi) \geq V^n(s, \xi) - c_4 \sqrt{\varepsilon^n} (1 + \|\xi\|^2).$$

Now we turn to the opposite estimate. Let $v \in U$ be a deterministic control. By condition (A0) there exists a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P)$ and a stochastic process $Y$ such that $Y(s) = \xi$ P-a.s., and a control space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P, v, Y)$ is admissible for $L^n$, $\xi^n$. In this case if $X(\cdot, \omega)$ satisfies (35) and $X(s, \omega) = \xi$, then $X(t, \omega) = x[t, \mu^n, s, \xi, v]$,

$$\int_s^T g(t, X(t), \mu^n(t), v(t)) dt = z[t, \mu^n, s, \xi, v].$$

Using this, (49), and (53) we get:

$$\hat{V}^n(s, \xi) \leq V^n(s, \xi) + c_4 \sqrt{\varepsilon^n} (1 + \|\xi\|^2).$$

Combining this and (54) we conclude that

$$|\hat{V}^n(s, \xi) - V^n(s, \xi)| \leq c_4 \sqrt{\varepsilon^n} (1 + \|\xi\|^2).$$
This estimate, (52), and convergence of \( W_2(\chi^n, \chi^*) \) to zero imply the second statement of the Theorem.

Now we shall prove that \((V^*, \mu^*)\) is a minimax solution to (10), (11). First we show that the support of \( \chi^* \) lies in the set \( \text{Sol}_0(\mu^*) \). Recall that \( \text{Sol}_0(\mu^*) \) is the set of functions \((x(\cdot), z(\cdot)) : [0, T] \to \mathbb{R}^d \) satisfying

\[
(\dot{x}(t), \dot{z}(t)) \in \co\{ (f(t, x(t), \mu^*[t], u), g(t, x, \mu^*[t], u)) : u \in U \}. 
\]

(56)

Let \((x^*(\cdot), z^*(\cdot)) \in \text{supp}(\chi^*)\). Since the sequence \(\{\chi^n\}_{n=1}^\infty\) converges to \(\chi^*\) by Proposition 5.1.8 there exists a sequence of motions \((x^n(\cdot), z^n(\cdot)) \in \text{supp}(\chi^n)\) such that \((x^n(\cdot), z^n(\cdot)) \to (x^*(\cdot), z^*(\cdot))\) as \(t \to \infty\). We have that \((x^n(\cdot), z^n(\cdot))\) satisfies the differential inclusion

\[
(\dot{x}(t), \dot{z}(t)) \in \co\{ (f^n(t, x(t), \mu^n[t], u), g^n(t, x(t), \mu^n[t], u)) : u \in U \}. 
\]

From this and (18) it follows that the pair \((x^*(\cdot), z^*(\cdot))\) satisfies inclusion (56). Consequently,

\[
\text{supp}(\chi^*) \subset \text{Sol}_0(\mu^*). 
\]

By construction we have that \(V^*\) is a minimax solution to (10) and \(\mu^* = e_t\#\chi^*\).

It remains to prove (21). To this end let us introduce the functions \(\mathcal{R}^n(s)\) and \(\mathcal{R}^*(s)\) by the following rules:

\[
\mathcal{R}^n(s) = \int_C [\sigma(x(T), \mu^n[T]) + z(T) - z(s) - \hat{V}^n(s, x(s))] \chi^n(d(x(\cdot), z(\cdot))), 
\]

\[
\mathcal{R}^*(s) = \int_C [\sigma(x(T), \mu^*[T]) + z(T) - z(s) - V^*(s, x(s))] \chi^*(d(x(\cdot), z(\cdot))).
\]

We have that, for \((x(\cdot), z(\cdot)) \in \text{supp}(\chi^*) \subset \text{Sol}_0(\mu_*)\),

\[
\sigma(x(T), \mu^*[T]) + z(T) - z(s) - V^*(s, x(s)) \leq 0. 
\]

(57)

Consequently,

\[
\mathcal{R}^*(s) \leq 0. 
\]

(58)

The construction of probabilities \(\chi^n\) yields that

\[
\mathcal{R}^n(s) = \mathbb{E}^n \left[ \sigma(X^n(T), \mu^n[T]) + \int_s^T g(t, X^n(t), \mu^n[t], u^n(t)) dt - \hat{V}^n(s, X^n(s)) \right]. 
\]

By (55) we have that

\[
\mathcal{R}^n(s) \geq \mathbb{E}^n \left[ \sigma(X^n(T), \mu^n[T]) + \int_s^T g(t, X^n(t), \mu^n[t], u^n(t)) dt - V^n(s, X^n(s)) \right] - c_5^\varepsilon \sqrt{n}.
\]

Here (see (55) and Lemma 2) \(c_5^\varepsilon \overset{\Delta}{=} c_3^\varepsilon (1 + C_2)\). Using Lemma 2 definition of the function \(V^n\), and (53) for \(\Omega = \Omega^n\), \(\mathcal{F} = \mathcal{F}^n\), \(\mathcal{F}_t = \mathcal{F}_t^n\), \(P = P^n\), \(u = u^n\), \(Y = Y^n\), and \(X = X^n\) we conclude that

\[
\mathcal{R}^n(s) \geq -2c_5^\varepsilon \sqrt{n}. 
\]

(59)
From \((\ref{47}), (\ref{52})\) and Lemma \(2\) it follows that 
\[
\mathcal{R}^n_l \to \mathcal{R}.
\]
Hence, using \((\ref{59})\) we get that 
\[
\mathcal{R}^*(s) \geq 0.
\]
This and \((\ref{58})\) imply the equality 
\[
\mathcal{R}^*(s) = 0.
\]
Therefore, by inequality \((\ref{57})\) we get that 
\[
\sigma(x(T), \mu^*[T]) + z(T) - z(s) = V^*(s, x(s))
\]
for \(\chi^*\)-a.e. \((x(\cdot), z(\cdot))\). This implies that inclusion \((\ref{21})\) is fulfilled. Thus, the pair 
\((V^*, \mu^*)\) is a minimax solution to \((\ref{10}), (\ref{11})\).

**Proof of Corollary 1.** The conclusion of the Corollary directly follows from Theorem \(1\) and Proposition \(1\).

**Proof of Corollary 2.** Note that if the pair \((V^*, \mu^*)\) solves \((\ref{10}), (\ref{11})\) in the probabilistic sense, then by Theorem \(1\) it is a solution to \((\ref{10}), (\ref{11})\) in the minimax sense.

Now assume that \((V^*, \mu^*)\) satisfies conditions of Proposition \(2\) and the set 
\[
\{(f(t, x, m, u), g(t, x, m, u)) : u \in U\}
\]

is convex. We have that if \((x(\cdot), z(\cdot)) \in \text{Sol}_0(\mu^*)\), then there exists a deterministic control \(u_{x(\cdot),z(\cdot)}\) such that

\[
\dot{x}(t) = f(t, x(t), \mu^*[t], u_{x(\cdot),z(\cdot)}(t)), \quad \dot{z}(t) = g(t, x(t), \mu^*[t], u_{x(\cdot),z(\cdot)}(t)).
\]

Put \(\Omega \triangleq C, \mathcal{F}_t \triangleq \mathcal{B}(\mathbb{R}^{d+1})\) (here \(\mathcal{B}\) states for the Borel \(\sigma\)-algebra), \(\mathcal{F} \triangleq \mathcal{F}_T,\)

\(P \triangleq \chi\). For \(\omega = (x(\cdot), z(\cdot))\), set \(u(t, \omega) \triangleq u_{x(\cdot),z(\cdot)}, Y(t, \omega) \triangleq x(t)\). Thus, \((V^*, \mu^*)\) solves \((\ref{10}), (\ref{11})\) in the probabilistic sense.

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