Frank-Wolfe Methods in Probability Space

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Abstract

We introduce a new class of Frank-Wolfe algorithms for minimizing differentiable functionals over probability measures. This framework can be shown to encompass a diverse range of tasks in areas such as artificial intelligence, reinforcement learning, and optimization. Concrete computational complexities for these algorithms are established and demonstrate that these methods enjoy convergence in regimes that go beyond convexity and require minimal regularity of the underlying functional. Novel techniques used to obtain these results also lead to the development of new complexity bounds and duality theorems for a family of distributionally robust optimization problems. The performance of our method is demonstrated on several nonparametric estimation problems.

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1 Introduction.

Problems in artificial intelligence, statistics, and optimization often find a common root as an infinite dimensional optimization problem in the form

\[
\inf \left\{ J(\mu) : \mu \in \mathcal{P}(\mathbb{R}^d) \right\},
\]

for the space \(\mathcal{P}(\mathbb{R}^d)\) of Borel probability measures over \(\mathbb{R}^d\). In recent years, quantitative statistical and algorithmic treatments of these formulations have produced insights into modern computational methods—resulting in novel approaches to difficult, open problems. Recent works in robust optimization \([6, 46, 57, 59]\), probabilistic fairness \([62, 56]\), reinforcement learning \([68, 69]\), and generalized adversarial networks \([44, 19, 20]\) highlight these gains and are linked by the following theme: problems in the form of (1) provide access to rich infinite dimensional structure that sidesteps brittle artifacts of finite dimensional formulations. In this paper, we develop a Frank-Wolfe algorithm for (1) that operates from this infinite dimensional perspective and provides concrete convergence and complexity guarantees for a sub-family of (1) which are well-behaved with respect to the Wasserstein distance of order 2.

Development of our Frank-Wolfe method is inspired by efforts in distributionally robust optimization \([46, 6, 27, 57]\) which have considered variants of (1) in the form

\[
\sup \left\{ \int f \, d\mu : D_c(\mu, \mu_0) \leq \delta \right\},
\]

where \(D_c(\mu, \mu_0)\) is the optimal transport cost between \(\mu\) and \(\mu_0\) (a reference measure) under some cost function \(c\). The form of (2), itself, immediately suggests the basis of an infinite dimensional Frank-Wolfe procedure since it provides a “linear” objective subject to a local, “trust-region” constraint—centered at \(\mu_0\). More generally, one can even consider variants of (1) in the form

\[
\inf \left\{ \int f \, d\mu + \psi(D_c(\mu, \mu_0)) : \mu \in \mathcal{P}(\mathbb{R}^d) \right\},
\]

where \(\psi : \mathbb{R} \to \mathbb{R}\) is a convex penalty function. The benefit of this formulation is suggested by its finite dimensional analogue

\[
\left\{ \inf_{y \in \mathbb{R}^d} s^T y + \tilde{\psi}(y) : y \in \mathbb{R}^d \right\}
\]

where common instantiations of \(\tilde{\psi}\) (including powers of norms, Bregman divergences, and indicator functions of convex sets) allow one to express an array of first-order methods and account for a variety of non-trivial geometries. By appropriately configuring the cost \(c\) and penalty \(\psi\) in (3), similar benefits can be realized in the context of (1).

These considerations, motivated by the extent to which (1) proliferates data-related fields, give rise to the following investigation for this work. First, to what extent can a Frank-Wolfe method for (1) be formulated within the framework of (2)—such that quantitative bounds on complexity and convergence can be obtained. Second, how can problems in the form of (2) or (3) be efficiently solved—subject to assumptions that are compatible with an infinite-dimensional, first-order framework?

1.1 Previous work.

The relevance of (2) in distributionally robust optimization (DRO) and mathematical finance results in notably more literature for the latter of these issues than for the former. Indeed, \([46, 37, 39, 57, 59]\),
all highlight computational schemes for solving (2) that are similar in objective to this work. What makes such efforts notable and solution of (2) non-trivial is: without particular assumptions, (2) can disguise an NP-hard problem—despite being convex in the usual Banach sense on \( \mathcal{P}(\mathbb{R}^d) \). In fact, even in the case where the cost is the squared Euclidean norm \( c(x, y) = \|x - y\|^2 \) (the case of primary concern for this work), computational trouble can lie dormant—an artifact of inherently difficult problems in unconstrained optimization [15]. These issues are discussed in further detail in Section 6.2, but this should not be surprising given specters of computational hardness dating back to early formulations of DRO [24].

Such computational pitfalls are not realized in practice, however, and two relevant approaches have emerged for removing these concerns from quantitative analyses. First, is to consider particular instances of (2) where the objective and constraints are sufficiently structured to preclude computational intractability and permit solution via methods adapted to the provided structure. Early work with this line [30, 24, 65], has recently been supplemented by approaches [14, 29, 7, 49, 39, 72, 38] which focus directly on DRO formulations from particular contexts in machine learning and operations research. Unfortunately, the techniques offered by these efforts require assumptions which are too restrictive for this work. These assumptions typically relate to a specific form for the objective function or constraints in (2) (e.g. linear/convex functions/piecewise-convex objectives or constraints with support or density requirements, see [34, 71, 46, 37, 70, 5, 63] for additional examples). In this instance, such limitations preclude their applicability since, in general, a “gradient object” for a functional \( J \) (see Section 1) need not satisfy these conditions. A second, more relevant, approach to perform quantitative analyses of DRO problems (2) is to restrict the level of robustness for which the problem is solved. In the context of (2), this reduces to preventing \( \delta \) from being too large. Such an approach is substantially more befitting of our purposes since a Frank-Wolfe procedure need only solve a sequence of local problems.

This technique has been used by works such as [6, 57] and the approach presented in this work (for establishing computationally tractability of (3)) is similar to ideas appearing in [57]. In that work, smoothness of the objective in (2) is used to (qualitatively) argue that a sufficiently small \( \delta \) will regularize the dual of (2) sufficiently strongly to produce a computationally-tractable optimization problem. In contrast, however, we provide quantification of the level of robustness required to achieve such a goal and do so in the scope of a more general problem class (3).

Formulation of a Frank-Wolfe method for (1) with quantitative bounds on complexity and convergence has, to the best of the authors' knowledge, failed to appear in previous literature. Perhaps the most closely related effort is [41] where similar, infinite dimensional conditions to those appearing in this work (Section 3.5) are used to study a particle-based methods for computing Nash equilibria of zero-sum games. It should be noted that, as a special case, our Frank-Wolfe method can produce a particle-based optimization procedure and this hints at possible connections with other particle techniques [42, 26, 12, 11]. Such connections are beyond the scope of this work, however, and left for future consideration.

2 Main result.

This work considers the problem

\[
\min_{\nu \in \mathcal{P}_2(\mathbb{R}^d)} J(\nu)
\]
for functionals $J : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ over (11) that possess a “gradient object” (Definition 2) with respect to $\mathcal{W}$—the Wasserstein distance of order 2

$$\mathcal{W}^2(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d} \|x - y\|^2 d\pi(x, y)$$

(6)

Our main result (Theorem 2) provides a Frank-Wolfe algorithm for (5) which operates on $\mathcal{P}_2(\mathbb{R}^d)$ and obtains quantitative iteration and sample complexities. This yields an intuitive, non-parametric algorithm for (5) with the guarantee:

**Theorem 1** (Informal; see Theorem 2). For a differential functional $J$ whose “gradient” $F_\nu$ provides an approximation that is slightly more than first order accurate

$$\min_{\mathcal{W}(\mu, \nu) \leq \delta} J(\mu) = \min_{\mathcal{W}(\mu, \nu) \leq \delta} (F_\nu, \mu - \nu) + O \left( \delta^{1+\alpha} \right), \quad 0 < \alpha \leq 1$$

(7)

and obeys the domination condition

$$\tau \left( J(\mu) - \inf_{\nu \in \mathcal{P}_2(\mathbb{R}^d)} J(\nu) \right)^\theta \leq \|F_\nu\|, \quad \tau, \theta \in \mathbb{R}^*_+$$

(8)

there is a Frank-Wolfe procedure which obtains an $\epsilon$-optimal solution of (5) in $O \left( \epsilon^{1-\alpha^*/\theta} \right)$ iterations where $\alpha^* = (1 + \alpha)/\alpha$ is the dual exponent.

When $J$ is convex in a Wasserstein sense (Definition 1) and has at least one minimizer, (8) holds with $\theta = 1$. Hence, for smooth $J$ ($\alpha = 1$ in (7)), Theorem 1 recovers a intuitive $O(k^{-1})$ convergence rate (accelerated rates are difficult in this context due to the difficulty of averaging in Wasserstein spaces, see Remark 6). A highlight of Theorem 1 is that the assumptions needed for quantitative convergence are relatively weak. Indeed, the condition (8) (properly known as a Łojasiewicz inequality; Section 3.5) is generally broader than convexity. The condition (7) is less stringent than smoothness, particularly as utilized in other literature [20, 41, 3, 16].

Supplementary to Theorem 1, we also construct a scalable implementation of our Frank-Wolfe method and demonstrate it’s performance on several non-parametric estimation problems (Section 5). We also detail algorithms with novel complexity guarantees for (2) and (3) (Theorem 4) and provide a new strong duality result for (3) (Theorem 3). These results are of independent interest due to the relevance of (2) and (3) for distributionally robust optimization, mathematical finance, and stochastic processes [5, 4, 6]. All technical proofs of these results are given in the appendix.

3 Preliminaries on Wasserstein geometry.

3.1 Notation and terminology

Denote the set of real numbers by $\mathbb{R}$, the set of extended real numbers by $\mathbb{R}$, and their respective subsets of non-negative numbers by $\mathbb{R}_+$ and $\mathbb{R}_+$. For a general function $f$, $\text{Dom}(f)$ and $\text{Ran}(f)$ denote the domain and range (respectively) while, for a convex function $t : X \to \mathbb{R}$ over some vector space $X$, the notation is overloaded so that $\text{Dom}(t)$ denotes the effective domain of $t$. That is,

$$\text{Dom}(t) = \{ x \in X : t(x) < \infty \}$$

We further say that the convex function $t$ is proper if $-\infty < t(x)$ for all $x \in X$ and $t(y) < \infty$ for some $y \in X$. For a concave function $z : X \to \mathbb{R}$, these terms are likewise defined by considering the
convex function $-z$. A convex function $t : X \to \bar{R}$ is called \textit{closed} if it is lower-semicontinuous with respect to the topology on $X$. Likewise, a concave function $z : X \to \bar{R}$ will be called closed if it is upper-semicontinuous.

Unless otherwise specified, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$ and a function $\phi : \mathbb{R}^d \to \bar{R}$ is called \textit{semiconvex} (or \textit{weakly convex}) if
\[
x \longrightarrow \phi(x) + \frac{\lambda}{2} \|x - x_0\|^2
\] (9)
is convex for some for some $\lambda \geq 0$. The choice of $x_0$ in (9) is largely irrelevant: if (9) is convex for one such $x_0$, it is convex for all $x_0 \in \mathbb{R}^d$. It is clear that a semiconvex function possesses a minimal $\lambda \geq 0$ such that (9) is a convex function. This value will be denoted by $\rho_*$ and a semiconvex function with such a value will be termed a $\rho_*$-semiconvex function. Clearly, any convex function is 0-semiconvex.

A continuously differentiable function $\phi : \mathbb{R}^d \to \bar{R}$ will be called $\alpha$-\textit{Hölder smooth} with parameter $T$ if it has Hölder continuous gradients with parameter $T$ and exponent $\alpha$. That is:
\[
\|\nabla \phi(y) - \nabla \phi(x)\| \leq T \|y - x\|^\alpha
\] (10)
When (10) holds for $\alpha = 1$, $\phi$ will simply be called $T$-smooth. Further, the notation $\mathcal{P}(\mathbb{R}^d)$ denotes the set of Borel probability measures on $\mathbb{R}^d$ while
\[
\mathcal{P}_2(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \|x\|^2 \, d\mu(x) < \infty \right\}
\] (11)
The expression $C_\infty^c(\mathbb{R}^d)$ denotes the space of all compactly supported, smooth functions on $\mathbb{R}^d$.

3.2 Functionals on probability measures

Before providing a rigorous specification of a “gradient” with respect to Wasserstein distance consider the following possible instances of $J$, for illustrative purposes.

\textbf{Example 1 (Divergences).} A common functional on $\mathcal{P}(\mathbb{R}^d)$ is KL-divergence with respect to a fixed, reference measure on $\nu$:
\[
J(\mu) := D_{KL}(\mu || \nu) = \int_{\mathbb{R}^d} \log \left( \frac{d\mu}{d\nu} \right) d\mu
\] (12)
More generally, for any convex, lower-semicontinuous function $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f(1) = 0$, one can consider a “$f$-divergence” of the form
\[
J(\mu) := D_f(\mu || \nu) = \int_{\mathbb{R}^d} f \left( \frac{d\mu}{d\nu} \right) d\nu
\] (13)
Canonical dual formulations show that such functionals (13) are lower-semicontinuous with respect to the weak topology on $\mathcal{P}(\mathbb{R}^d)$ [55]. This helps make these functionals amenable to our analyses—since lower-semicontinuity is at least necessary for an iterative optimization procedures (such as a Frank-Wolfe algorithm) to converge to an optimizer. As the Wasserstein topology on $\mathcal{P}_2(\mathbb{R}^d)$ is finer than the weak topology, this means that weak lower-semicontinuity is at least sufficient for our purpose.
In many cases, divergences can also be supplemented with a potential \( v : \mathbb{R}^d \to \mathbb{R} \) and interaction function \( w : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \)

\[
J(\mu) := \int_{\mathbb{R}^d} v(x) \mu(dx) + \int_{\mathbb{R}^d} w(x,y) \mu(dx) \nu(dy) + D_f(\mu||\nu) \tag{14}
\]
to yield “energy functionals” on \( \mathcal{P}(\mathbb{R}^d) \) [55].

**Example 2** (Integral Probability Metrics). For a set of real valued functions \( F \) on \( \mathbb{R}^d \) one can define the discrepancy

\[
J(\mu) := \text{IPM} (\mu, \nu) = \sup_{f \in F} \int_{\mathbb{R}^d} f \mu(dx) - \int_{\mathbb{R}^d} f \nu(dx) \tag{15}
\]
for \( \mu, \nu \in \mathcal{P}(\mathbb{R}^d) \), where \( \nu \) is a fixed, reference measure. Such discrepancies are termed Integral Probability Metrics (IPMs), although they may not strictly satisfy the requirements of a metric—say, by failing to distinguish all pairs of measures. Instead, for a pair of measures \( \mu, \nu \in \mathcal{P}(\mathbb{R}^d) \), IPMs can be interpreted as measuring the extent to which \( \mu \) and \( \nu \) differ on functions in \( F \) or, rather, measuring the extent to which \( \mu \) and \( \nu \) can be distinguished by \( F \).

**Example 3** (Markov Decision Process). Consider a set of states \( S = \mathbb{R}^m \) and a set of actions \( A = \mathbb{R}^n \). At a denumerable set of times \( t = 1, 2, 3, \ldots \) an agent which occupies state \( s_{t-1} \) chooses an action \( a_t \) and randomly transitions to a new state \( s_t \), while receiving a reward \( r_t \in \mathbb{R} \). For transitions which are Markovian, this process can be described by a set of Markov transition kernels \( p_t(s_t, r_t|s_{t-1}, a_t) \) which give the probability of obtaining state \( s_t \) and reward \( r_t \) for an agent which was most recently in state \( s_{t-1} \) and chose action \( a_t \).

The goal of the agent to choose a distribution \( \mu^* \in \mathcal{P}(\mathbb{R}^{m+n}) \), termed a policy, so as to maximize his or her expected reward:

\[
\mu^* := \arg \min_{\mu \in \mathcal{P}(\mathbb{R}^{m+n})} J(\mu) = \arg \min_{\mu \in \mathcal{P}(\mathbb{R}^{m+n})} \mathbb{E}_{\mu, p_t|_{t=1}} \left[ \sum_{i=1}^{\infty} r_i \right] \tag{16}
\]
Here, the expectation is taken with the transition kernels \( p_t \) and a an agent that chooses actions which are distributed according to the conditional distribution of \( \mu \). Note that, in most works, the policy is specified in terms of a (potentially infinite) set of conditional distributions over actions: \( \mu(a|s) \). Hence, the expected reward is, instead, a functional over the product space \( \otimes_{s \in \mathbb{R}^m} \mu(a|s) \). However, by choosing an arbitrary distribution \( \alpha \in \mathcal{P}(\mathbb{R}^m) \) and considering the joint distribution \( \mu(s, a) = \alpha(s) \mu(a|s) \), this formulation can be seen to be equivalent to (16) see [19] for further details.

### 3.3 Properties of Wasserstein space

Under Wasserstein distance, \( \mathcal{P}_2(\mathbb{R}^d) \) is a Polish space [64] and, via it’s kinematic characterizations (Proposition 1), provides a natural structure for studying stochastic optimization. For a Frank-Wolfe method to meet a stated goal of minimizing local, linear approximations, one requires an appropriate definition of a “gradient.” This requires providing rigorous meaning to the expression

\[
\lim_{\alpha \to 0} \frac{J(\mu_\alpha) - J(\mu)}{\alpha} \tag{17}
\]
where \( \mu_\alpha \) denotes a (purely formal) perturbation from \( \mu \) of Wasserstein distance \( \alpha \). To this end, consider the following properties of Wasserstein space that are essential for this work— a basic proof is given in Appendix J.
**Proposition 1** (Properties of Wasserstein space).

- **Under the Wasserstein metric** $W$, $\mathcal{P}_2(\mathbb{R}^d)$ is a geodesic space. That is, for every $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, there exists a constant-speed geodesic curve $\mu_t : [0, 1] \to \mathcal{P}_2(\mathbb{R}^d)$ where $\mu_0 = \mu$, $\mu_1 = \nu$ and

$$W(\mu_t, \mu_s) = |t - s| W(\mu_0, \mu_1) \quad (18)$$

Moreover, there is a bijection between constant-speed geodesics and optimal transport plans. Every geodesic corresponds to a unique, optimal transport plan $\gamma \in \Pi(\mu, \nu)$

$$W(\mu, \nu)^2 = \int \|x - y\|^2 \, d\gamma(x, y) \quad (19)$$

such that

$$\mu_t = ((1 - t)x + ty)_{\#} \gamma \quad (20)$$

Conversely, every optimal transport plan gives rise to a unique geodesic via (20).

- For a constant-speed geodesic $\mu_t : [0, 1] \to \mathcal{P}_2(\mathbb{R}^d)$, there exists a ($\mu_t$-almost surely) unique Borel vector field $v_t : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d$ which satisfies

$$W(\mu_0, \mu_1)^2 = \int_0^1 \int_{\mathbb{R}^d} \|v_t(x)\|^2 \, d\mu_t(x) \, dt = \min_{v_t \in V_\mu} \int_0^1 \int_{\mathbb{R}^d} \|v_t(x)\|^2 \, d\mu_t(x) \quad (21)$$

for

$$V_\mu := \left\{ v_t : \frac{d\mu_t}{dt} + \nabla \cdot (v_t \mu_t) = 0 \right\} \quad (22)$$

defined as the set of all Borel vector fields which solve the continuity equation for $\mu_t$. The continuity equation is understood in duality with $C^\infty_c(\mathbb{R}^d)$.

- For any constant-speed geodesic $\mu_t$, the corresponding optimal transport plan $\gamma \in \Pi(\mu_0, \mu_1)$ and the corresponding vector field $v_t$ (given by (21)) satisfy the relation

$$v_t((1 - t)x + ty) = y - x, \quad \gamma\text{-almost surely} \quad (23)$$

for Lebesgue-almost every $t$.

- The space $\mathcal{P}_2(\mathbb{R}^d)$ is positively curved under $W$ and at each point $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the tangent space

$$\text{Tan}(\mu) := \left\{ \nabla \psi : \psi \in C^\infty_c(\mathbb{R}^d) \right\}^{L^2(\mu)} \quad (24)$$

is the closure in $L^2(\mu)$ of the gradients of smooth functions with compact support. Via the Riesz isomorphism, $\text{CoTan}(\mu) = \text{Tan}(\mu)$ where $\text{CoTan}(\mu)$ denotes the cotangent space. The tangent and cotangent bundles will be denoted $\text{Tan}_{\mathcal{P}_2(\mathbb{R}^d)}$ and $\text{CoTan}_{\mathcal{P}_2(\mathbb{R}^d)}$, respectively.

### 3.4 Differentiability in Wasserstein space

Proposition 1 clarifies that $\mathcal{P}_2(\mathbb{R}^d)$ has a non-Euclidean geometry with respect to $W$. Unfortunately, this complicates the notion of a “gradient” in the sense of (17). Since the tangent space (24) varies from point to point, the notion of linear approximation varies from point to point. Hence, one must define gradients in terms of a selections in the cotangent bundle. Despite these complications, however, the theory of Proposition 1 now yields a direct expression of the “gradients” that our Frank-Wolfe algorithm will utilize.
Definition 1 (Geodesic convexity). A set $S \subseteq \mathcal{P}_2(\mathbb{R}^d)$ is said to be convex or geodesically convex if for any $\mu, \nu \in S$ one has $\mu_t \in S$ for any geodesic curve $\mu_t$ between $\mu$ and $\nu$. Similarly, a functional $J : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is said to be convex or geodesically convex if, for any $\mu, \nu \in S$ in a convex set $S$, 

$$J(\mu_t) \leq tJ(\nu) + (1-t)J(\mu)$$ \hspace{1cm} (25) 

for all geodesics $\mu_t$ between $\mu$ and $\nu$.

Definition 2 (Wasserstein differentiability). Let $S$ be a geodesically convex set. A functional $J$ is Wasserstein differentiable on $S$ if there is a map $F : \mathcal{P}_2(\mathbb{R}^d) \to \text{CoTan}\mathcal{P}_2(\mathbb{R}^d)$ such that for all $\mu, \nu \in S$ and any constant-speed geodesic $\mu_t : [0,1] \to \mathcal{P}_2(\mathbb{R}^d)$ between $\mu$ and $\nu$, one has

$$\lim_{\alpha \to 0} \frac{J(\mu_\alpha) - J(\mu)}{\alpha} = \int_{\mathbb{R}^d \times \mathbb{R}^d} F(\mu; x)^T (y - x) \, d\gamma(x,y)$$ \hspace{1cm} (26) 

where $\gamma$ is the unique optimal transport plan (20) corresponding to $\mu_t$. Note that $F(\mu; x) = (F(\mu)) (x)$ provides a more aesthetic way of representing the evaluation at $x \in \mathbb{R}^d$ of the output of $F$ at $\mu$. The map $F$ will be called the Wasserstein derivative of $J$.

Remark 1. The description of differentiability provided by Definition 2 falls within the general framework of metric derivatives and Wasserstein gradient flows, originally codified in [1]. This framework is now a well-established component of the theory of Wasserstein spaces, while the relation (26), itself, presents only a narrow structuring of ideas from this framework. Definition 2, however, is often how works in statistical and algorithmic fields interact with this broader area [61, 17, 40, 41]. Moreover, this literature demonstrates the most motivating feature of (26): a large number of functionals of interest for machine learning and statistical inference exhibit Wasserstein gradients in the sense of (26). The curious reader is referred to [1, 55, 10] for precise statements of conditions under which (26) is guaranteed. However, let it suffice to say that $F$ typically arises from the Gateaux differential for $J$ [55, 61]. Recall, the Gateaux differential for a functional $J$ exists when there is an appropriate, dual space $D^* \supseteq C_b(\mathbb{R}^d)$ on a closed subspace $D \subseteq \mathcal{P}(\mathbb{R}^d)$ such that

$$\langle dJ(\mu), \nu - \mu \rangle = \lim_{\alpha \to 0} \frac{J(\mu + \alpha(\nu - \mu)) - J(\mu)}{\alpha}$$ \hspace{1cm} (27) 

for some $dJ(\mu) \in D^*$ and all $\mu$ in some set $S$ such that $S - S \subseteq D$. In instances where the Gateaux differential $dJ(\mu)$ exists, the Wasserstein derivative $F$ will often also exist and be given by $\nabla dJ(\mu) \in \text{Tan}(\mu)$. Here, we use the gradient operator formally, and omit a rigorous exposition on this operation in the context of $\text{Tan}(\mu)$.

Remark 2. The notion of geodesic convexity given in Definition 1 is standard for Wasserstein spaces and dates back to at least [45]. It has appeared ubiquitously in subsequent works [1, 22]. What is surprising, however, is that functionals which are non-convex with respect to canonical vector space structure on $\mathcal{P}(\mathbb{R}^d)$ are convex in the sense of Definition 1.

It should be noted that computation of the Wasserstein derivative might be difficult. Indeed, for a $J$ in a variational form such as (15), computation of the Wasserstein derivative is equivalent to finding a witness function that achieves the supremum [55]. In the case of a pathological $F$ (in (15)), such a task might be intractable. To resolve this issue, this work utilizes the existence of an oracle for the computation of a Wasserstein gradient. This oracle permits a unified description of our Frank-Wolfe algorithm and abstracts away variation in functional-specific computational cost.
Definition 3 (Wasserstein Derivative Oracle). Let $J : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be a Wasserstein differentiable functional on a set $S$ with Wasserstein derivative $F : \mathcal{P}_2(\mathbb{R}^d) \to \text{Tan}_{\mathcal{P}_2(\mathbb{R}^d)}$. A $L$-smooth Wasserstein derivative oracle over $S$ is an oracle which, given sample access to a distribution $\mu \in S$ and an error parameter $\epsilon$, returns an $L$-smooth function $\hat{\phi}_\mu \in C^1(\mathbb{R}^d)$ satisfying

$$\left\| \nabla \phi_\mu - F(\mu) \right\|_{L^2(\mu)} \leq \epsilon$$

(28)

Remark 3. The qualification that the Wasserstein derivative oracle return an $L$-smooth function is necessary to exclude the, aforementioned, possibility of a pathological Wasserstein derivative— which would be intractable for use in a computational procedure. In some ways, this is representative of the fact that the cotangent space $\text{CoTan}(\mu)$ at a point is too large; the $L^2(\mu)$ closure of gradients of smooth, compactly supported functions still contains vector fields that are stubbornly complex. Such a condition is common in other variational methods [3, 20, 69, 21] and is relatively superficial— when coupled with the degree of approximation afforded by $\epsilon$. Indeed, via smoothing techniques [54, 12, 40], functionals can often be assumed to have Wasserstein derivatives which are $C^1(\mathbb{R}^d)$ or are well-approximable by $C^1(\mathbb{R}^d)$ functions.

3.5 Smoothness and Łojasiewicz inequalities

In finite dimensions, iterative, gradient-based methods typically require the specification of two conditions in order to achieve convergence.

- The accuracy of local, linear approximations that are provided by the gradient.
- The extent to which local descent makes global progress on the objective.

Here, we state these conditions in the context of functionals over Wasserstein space.

Definition 4 ($\alpha$-Holder smoothness). Let $S$ be a geodesically convex set and let $J : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be a functional which is continuously Wasserstein differentiable on the set $S$. $J$ is said to be locally $\alpha$-Holder smooth on $S$ with parameters $T$ and $\Delta$ if for all $\mu \in S$ and all $\nu \in S$ such that $W(\mu, \nu) \leq \Delta$, there exists an optimal transport plan $\gamma \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$J(\nu) \leq J(\mu) + \int_{\mathbb{R}^d \times \mathbb{R}^d} F(\mu;x)^T (y-x) \, d\gamma(x,y) + \frac{T}{1+\nu} \mathcal{W}^{1+\alpha}(\nu,\mu)$$

(29)

Definition 5 (Łojasiewicz inequality). A Wasserstein differentiable functional $J$ on a set $S \subseteq \mathcal{P}_2(\mathbb{R}^d)$ is said to satisfy a Łojasiewicz inequality with parameter $\tau$ and exponent $\theta$ if for all $\mu \in S$ and $J_* := \inf_{\mu \in S} J(\mu)$

$$\tau (J(\mu) - J_*)^\theta \leq \| F(\mu) \|_{L^2(\mu)}$$

(30)

where $F$ is the Wasserstein derivative (26) of $J$.

Remark 4. More restrictive versions of both (29) and (30) commonly appear in previous literature [3, 36, 41, 20, 16]. In most cases, the $\alpha$-Holder smoothness condition (29) is stated for $\alpha = 1$ and required to hold globally ($\Delta = \infty$). This smoothness criterion is considerably weaker since it requires that the Wasserstein gradient only provide a local approximation that is slightly more than first-order accurate. Further, statement of the Łojasiewicz inequality (30) is broader than canonical treatments due to the presence of the auxiliary power $\theta$. Most often, the specific instances of either $\theta = 1/2$ or $\theta = 1$ are considered, since they are implied by various forms [1] of geodesic convexity (25)— for instance, see Lemma 18.
4 The Frank-Wolfe algorithm.

Algorithm 1 provides our Frank-Wolfe procedure along with its associated convergence guarantees and sample complexities (Theorem 2). To obtain these guarantees, we require the following assumptions on the objective \( J \)-phrased in the language of the previous theory.

**Assumption 1** (Smoothness assumption). The functional \( J : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) is Wasserstein differentiable (Definition 2) and locally \( \alpha \)-Holder smooth (4) on a set \( S \subseteq \mathcal{P}_2(\mathbb{R}^d) \) with parameters \( T \) and \( \Delta_1 > 0 \) (Definition 4). Further, a \( L \)-smooth Wasserstein derivative oracle (Definition 3) for \( J \) exists.

**Assumption 2** (Local richness). The set \( S \) is rich enough to contain the solution to (43) for \( \mu \in S \), \( L \)-smooth \( f \), and \( \delta \leq \Delta_2 \).

**Assumption 3** (Łojasiewicz assumption). The functional \( J \) satisfies a Łojasiewicz inequality (30) on \( S \subseteq \mathcal{P}_2(\mathbb{R}^d) \) with parameters \( \tau > 0 \) and \( \theta \).

**Theorem 2.** Under Assumptions 1, 2, 3, and an appropriate choice of input parameters, Algorithm 1 computes a distribution \( \mu^* \) satisfying

\[
r(\mu^*) := J(\mu^*) - \inf_{\mu \in S} J(\mu) \leq \epsilon
\]

in at most

\[
k = \tilde{O} \left( r(\mu_0)^{p^+} \epsilon^{-p^-} \right)
\]

iterations, where \( \mu_0 \) is the initial iterate and \( p^+, p^- \) denote the positive and negative parts of \( p = 1 - \alpha^* \theta \) for the dual exponent \( \alpha^* = (1 + \alpha)/\alpha \). Further, each iteration of Algorithm 1 can be performed using at most \( \tilde{O} \left( \epsilon^{-2\alpha^* \theta} \right) \) independent samples from the initial distribution \( \mu_0 \). Note that the notation \( \tilde{O}(\cdot) \) obscures logarithmic factors in it’s arguments.

**Remark 5.** Since the computation of the Frank Wolfe step (31) is performed using Algorithm 3, the result of Algorithm 1 is a bi-level procedure with inner and outer iteration loops. Further, since Algorithm 3 requires only sample access to it’s inputs and can return an oracle providing sample access to it’s output, all operations in Algorithm 1 can be implemented with only sample access to
the underlying distributions $\mu_i$. Via a simple induction argument, it also follows that all operations in Algorithm 1 can be implemented using only sample access to the initial distribution $\mu_0$; this analysis provides the stated sample complexity of Theorem 2. Further, since the chief consumer of these samples (Algorithm 3), uses them to compute $O(\log \epsilon^{-1})$ sample averages, it is clear that nearly all of the $O(\epsilon^{-2\alpha^*})$ samples in Theorem 2 can be drawn in parallel. That is, (31) can be computed with low parallel depth.

Practically, it is often more efficient to directly maintain approximations to the $\mu_i$ via a nonparametric estimator— as opposed to a exact sampling oracle. When this is done, it results in an additional, additive error in the residual (32) at each step of Algorithm 1. However, so long as this error is on the order of the additive error produced by the Wasserstein derivative oracle $\Theta$, the iteration complexity (33) remains unaffected. Moreover since analysis of the error induced by a particular approximation of the $\mu_i$ is highly problem dependent, we do not consider it in the context of these results.

Remark 6. The dependence on the dual exponent $\alpha^*$ in (33) can be rather punishing for small $\alpha$. It is natural to ask if this exponent could be improved within the scope of Assumptions 1, 2, and 3— perhaps under the auspice of the class of first order methods presented in Section 6. Moreover, in finite dimensions, it is well known that first-order methods for convex and $\alpha$-Hölder smooth functions (also known as weakly smooth functions) can obtain $\epsilon$-optimal solutions in $O(\epsilon^{-2/(1+3\alpha)})$ iterations [50]. Hence, it could even be considered whether, given geodesic-convexity assumptions on $J$, a better iteration complexity for Algorithm 1 would be obtainable.

We conjecture that such improvements are unlikely, however. Particularly those that would draw on analogy from finite dimensional techniques; the motivation for this is as follows. A common approach to establishing improved iteration complexities for convex, $\alpha$-Hölder smooth functions in finite dimensions is to consider their gradient oracles as inexact oracles for convex, 1-Hölder smooth functions [25]. Using either averaging arguments or accelerated methods, more rapid progress on an underlying objective can then be made with these inexact oracles. Our Frank-Wolfe method already utilizes an inexact step (31), thus it is conceivable that such an approach could be applied to Algorithm 1.

Unfortunately, this finite dimensional analogy fails due to the fact that averaging is difficult in Wasserstein space. Indeed to prevent error accumulation from outpacing objective progress, averaging iterates is crucial— either directly or in the form of an accelerated method. Since Wasserstein space is positively curved (Proposition 1) computing analogous convex combinations of the $\mu_i$ in Algorithm 1 is itself a variational problem and could be as expensive to compute.

5 Computational experiments.

In this section, we demonstrate the application our Frank-Wolfe algorithm to several non-parametric estimation problems in statistics and machine learning.

5.1 Gaussian deconvolution

A classical task in nonparametric statistics [13, 9] is to infer a latent, data-generating distribution $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ from a set of observations that are corrupted by independent, additive Gaussian noise. For observations $Y_1, \ldots, Y_n$ such that

$$Y_i = X_i + Z_i \quad \text{where} \quad X_i \sim \nu, \ Z_i \sim N(0, \sigma^2)$$

(34)

one seeks to compute a non-parametric estimate of $\nu$— the variance of the noise $\sigma^2$ is considered known. Since $Z_i$ is independent of $X_i$, this task amounts to “deconvolving” $\nu$ from the distribution
of $Z_i$. A natural candidate for $\nu$ is the maximum-likelihood estimator (MLE)
\[
\hat{\nu} := \arg \max_{\nu \in \mathcal{P}_2(\mathbb{R}^d)} \sum_{i=1}^{n} \log (\phi_{\sigma} \ast d\mu(Y_i)) \quad \text{where} \quad \phi_{\sigma} = \int_{\mathbb{R}^d} \phi_{\sigma}(Y_i - x) \, d\mu(x)
\]
where $\phi_{\sigma}$ is the density of $Z_i$. In [52], it was shown that $\hat{\nu}$ has an equivalent characterization as
\[
\hat{\nu} = \arg \min_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} \mathcal{W}_2^2(\mu, \hat{P}_Y)
\]
where
\[
\mathcal{W}_2^2(\mu_1, \mu_2) := \inf_{\pi \in \Pi(\mu_1, \mu_2)} \frac{1}{2} \int \|x - y\|^2 \, d\pi(x, y) + \sigma^2D(\pi || \mu_1 \otimes \mu_2)
\]
is the entropic optimal transportation distance [23] and $\hat{P}_Y$ is the empirical distribution of the $Y_i$. The problem (36) readily lies within the framework of (5) for $J(\mu) := \mathcal{W}_2^2(\mu, \hat{P}_Y)$. Moreover, it is known [43] that the Wasserstein derivative (26) of $\mathcal{W}_2^2(\mu, \hat{P}_Y)$ with respect $\mu$ is given by
\[
\phi_{\mu}(x) = \sigma^2 \log \left( \frac{1}{n} \sum_{i=1}^{n} \exp \left( \left( v_i^* - \|x - y_i\|^2 / 2 \right) / \sigma^2 \right) \right)
\]
where $v^* \in \mathbb{R}^d$ is dual variable (corresponding to $\hat{P}_Y$) which is optimal for $\mathcal{W}_2^2(\mu, \hat{P}_Y)$. This provides a Wasserstein derivative oracle for (36) as the vector $v^*$ can be readily approximated using stochastic gradient methods [28].

A simple, two dimensional instance of this problem is shown in Figure 1 on a dataset $Y_i$ of 50 samples with four distinct modes—illustrated by the kernel density estimator of the $Y_i$, shown in red. The behavior of Algorithm 1 is depicted over the course of several iterations, where the foreground contours provide the density of the iterate, $\mu_i$, that is maintained by the algorithm. In this setting, $\mu_i$ is approximated as a mixture of $N$-gauassians of fixed bandwidth (for $N = 200$); as opposed to maintaining a full sampling oracle for each $\mu_i$. This approximation induces an additional, additive error in the residual of each iterate. So long as this error is of the same order as the error in the Wasserstein gradient, however, the analysis of Theorem 2 is unaffected. Moreover, empirically, this is consistent with the performance of the Frank-Wolfe algorithm. Instead, performance appears to be dominated by the accuracy of the Wasserstein derivative computation; which consumes the majority of the computational time for this problem. Figure 2 provides a quantitative demonstration of the convergence of Algorithm 1 for a similar, multi-modal data in 64 dimensions.

### 5.2 Maximum mean discrepancy

For a reproducing kernel Hilbert space (RKHS) $H$ on a space on a space $X$, the maximum mean discrepancy (MMD) [32] is the integral probability metric (IPM) between distributions $\mu, \nu \in \mathcal{P}(X)$ generated by the unit ball of $H$. That is,
\[
\text{MMD}(\mu, \nu) := \sup_{\|f\|_H \leq 1} \int_X f(x) \, d\mu - \int_X f(x) \, d\nu
\]
where MMD$(\mu, \nu)$ quantifies the degree to which $\mu$ and $\nu$ can be distinguished by functions in $H$. Indeed, for an $H$ which is universal and an $X$ which is compact, MMD provides a metric on $\mathcal{P}(X)$ [32]. The rise of generalized adversarial networks (GANs) [31] and efforts connecting neural networks and kernel regression [18], have generated interest in MMD, particularly with respect to it’s role in constructing high-dimensional, distributional embeddings [21, 48]. This development is
Figure 1: Deconvolution of a multi-modal dataset via the Frank-Wolfe algorithm. Background contours (in red) provide an illustration of the underlying data distribution, while the foreground contours provide the density of the iterate maintained by Algorithm 1.

Figure 2: Estimated entropic Wasserstein distance computed at each iteration of the Frank-Wolfe algorithm (1) for a 64-dimensional deconvolution problem with multi-modal data. Displayed is the average distance over 10 independent runs with random initializations.
predicated on the observation that any neural network \((x, \theta) \to \psi(x, \theta)\), which produces an output \(\psi(x, \theta) \in \mathbb{R}^d\) from input data \(x \in X \subseteq \mathbb{R}^d\) and parameters \(\theta \in \Theta \subseteq \mathbb{R}^m\), yields a kernel on the parameter set \(\Theta\):

\[
 k(\theta_1, \theta_2) := E_x [\psi(x, \theta_1)^T \psi(x, \theta_2)] \tag{40}
\]

where the expectation over \(x\) is taken with respect to a data generating distribution. Via MMD, \(k\) induces a natural discrepancy measure between distributions over network parameters \(\theta\) and, therefore, learning of a generative image model can be expressed as minimizing (39) with respect to latent, generative distribution for \(\nu\). We refer to [48, 3] for further descriptions of these applications.

With respect to the variational framework of this paper (5) minimization of (39) against a latent, target distribution \(\nu\) provides a natural fit for (5). Indeed, for

\[
 J(\mu) := MMD^2(\mu, \nu) \tag{41}
\]

the Wasserstein derivative (Definition 2) of \(J\) is the unique witness function \(f^*_\mu\) achieving (39) [3]. Moreover, \(f^*\) has a natural expression as the difference between the mean embeddings of \(\mu\) and \(\nu\)

\[
 f^*_\mu(x) = E_{z \sim \mu} [k(z, x)] - E_{z \sim \nu} [k(z, x)] \tag{42}
\]

and can be computed via sampling methods, even when \(\mu\) or \(\nu\) are continuous or are large, discrete distributions [32]. Perhaps the most advantageous consequence of (42), however, is that the Wasserstein gradient directly inherits regularity present in \(k\). Indeed, should \(\nabla_x k(x, y)\) be \(L\)-Lipschitz in \(x\) (uniformly for all \(y\)), \(J\) (41) is naturally \(L\)-smooth [3]. This has led to the development of several variational or particle-based methods for minimizing (41) [3, 48, 21].

Figure 3 contrasts the performance of our Frank-Wolfe algorithm with two of these methods on the student-teacher network problem showcased in [3]. Our method is shown on the left, the center plot shows the “MMD gradient flow” algorithm from [3], and the right plot provides the “Sobolev Descent” algorithm of [48]. Performance is evaluated in terms of MMD error on a validation dataset and is shown as a function of the total gradient evaluations performed by each method. This provides a better proxy for relative performance and convergence since an iteration of Algorithm 1 performs multiple solves that are, each, similar in terms of gradient complexity to a single iteration of MMD gradient flow or Sobolev descent. Further, the total number of gradient evaluations should not be viewed as a proxy for wall-time as, for each gradient evaluation, the number of operations performed by each method can vary widely. Indeed, for each gradient evaluation in Sobolev descent an entire linear system solve is performed. Also, note that, as both MMD gradient flow and Sobolev descent are particle-based, Algorithm 1 was, for the purposes of comparison, instantiated with a particle distribution of equal size.

### 6 Duality and computational procedures.

The focus of this section is to provide a complete analysis of the Frank-Wolfe method in Section 4 by furnishing a concrete, computational procedure (and complexity guarantee) for the subroutine in Algorithm 1: compute a \(\nu^* \in \mathcal{P}_2(\mathbb{R}^d)\) such that \(W(\nu^*, \mu) \leq \delta\) and

\[
 \int f \, d\nu^* - \inf_{W(\nu, \mu) \leq \delta} \int f \, d\nu \leq \epsilon \tag{43}
\]

This problem and its computational solution are, themselves, of independent interest since they frequently arise in distributionally robust optimization (DRO) [6, 70, 46, 57]—typically, phrased as
Frank-Wolfe

MMD Flow (with noise injection)

Kernel Schölkopf Descent

Figure 3: Maximum mean discrepancy (with respect to a validation dataset) between a latent, “teacher” neural network and a distribution over “student” networks as computed by each algorithm. The discrepancy, also referred to as validation error, is shown as a function of the total number of gradient evaluations performed by each algorithm.

a maximization problem. One can take an even broader view, however, that (43) is a particular instance of

$$\inf_{\pi \in \Pi(\mu)} \int f \, d\pi + \psi \left( \int c \, d\pi \right)$$

(44)

where $\Pi(\mu)$ is the set of couplings whose first marginal is given by $\mu$, $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ is now an arbitrary, non-negative, Borel-measurable cost function (having replaced the Wasserstein cost $\|\cdot\|^2$), and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is now a proper, closed, and convex function (having replaced the trust-region constraint $W(\mu, \nu) \leq \delta$).

Beyond the expanded relevance that (44) has for stochastic processes and gradient flows [5, 1, 4], the purpose of this consideration is two-fold. First, (44) provides a template for a wide class of infinite-dimensional, first-order optimization methods. In finite dimensional optimization, first-order procedures are often expressed as solving a sequence of problems in the form

$$\inf_{y \in \mathbb{R}^d} s^T y + \psi(y)$$

(45)

for a convex, lower-semicontinuous, function $\psi : \mathbb{R} \to \mathbb{R}$. Under this same token, usage of (43) in Algorithm 1 could be replaced with another instance of (44) for, say, a problem-specific cost function $c$. This would yield an alternate variational procedure that could be better suited for a particular problem at hand. Second, the tool enabling a computational procedure for (43), duality, exists with the same level of utility for (44) and yields the same structures that facilitate computation: supergradients.

In the hope that these considerations elucidate how further variational procedures could be derived from our techniques, we resolve a computational procedure for (43) in the following manner. In Section 6.1, we show that (44) exhibits a dual formulation that makes it approachable for computation. We do this under more general assumptions than are available in previous works [6, 27, 5] to highlight the breadth of possible extensions to our Frank Wolfe procedure. In Section 6.2, we then specialize our techniques to (43) and provide a sampling-based algorithm for (43) with complexity bounds.
6.1 Duality

The full generalization of (43) to be considered is

\[ \mathcal{P}_\mu (f) := \inf_{\pi \in \Pi(\mu)} L_f(\pi) = \inf_{\pi \in \Pi(\mu)} \int_{S_1} f d\pi + \psi \left( \int_{S_0 \times S_1} c d\pi \right) \]  

(46)

where \( S_0, S_1 \) are Polish spaces, \( \Pi(\mu) \in \mathcal{P}(S_0 \times S_1) \) is the set of joint couplings with first marginal given by \( \mu \), \( f : S_1 \to \mathbb{R} \) and \( c : S_0 \times S_1 \to \mathbb{R}_+ \) are Borel-measurable, and \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) is proper, closed, and convex. The objective (46) follows the convention that \( \infty - \infty = \infty \) and has a value of \( \infty \) if \( \int f d\pi \) is not defined. The dual of (46) is

\[ \mathcal{D}_\mu (f) := \sup_{\lambda \in \mathbb{R}} \int_{S_0} f^{\lambda c} d\mu - \psi^*(\lambda), \quad f^{\lambda c}(x) := \inf_{y \in S_1} f(y) + \lambda c(x, y) \]  

(47)

where \( f^{\lambda c} : S_0 \to \mathbb{R} \) is canonically called the “c-transform” of \( f \) [64]. Note, (46) induces the convention \( f(y) + \lambda c(x, y) = \infty \) if \( c(x, y) = \infty \).

Remark 7. Rigorously, the dual of (46) is better defined as

\[ \mathcal{D}_\mu (f) := \sup_{\lambda \in \mathbb{R}} \left( \sup_{\phi \in \Lambda_\mu (f+\lambda c)} \int_{S_1} \phi d\mu - \psi^*(\lambda) \right) \]  

(48)

where \( \psi^* \) is the convex conjugate of \( \psi \) and, for any \( g : S_0 \times S_1 \to \mathbb{R} \),

\[ \Lambda_\mu (g) := \{ \phi \in L^1(\mu) \mid \phi(x) \leq g(x, y) \quad \forall y \in S_1 \} \]

This definition side steps the technicality that \( f^{\lambda c} \) is not necessarily Borel-measurable and keeps the dual variables within the space of integrable functions. However, under conditions for strong duality (Theorem 3), the formulations (47) and (48) are equivalent and the lack of Borel-measurability in \( f^{\lambda c} \) is a formality since \( f^{\lambda c} \) is universally measurable– therefore it is measurable with respect to the completion of \( \mu \). These details are discussed in greater length in Appendix B.

Define the functionals \( \tau_c : \Pi(\mu) \to \mathbb{R}_+ \) and \( \tau_f : \Pi(\mu) \to \mathbb{R} \)

\[ \tau_c(\pi) := \int c d\pi \quad \text{and} \quad \tau_f(\pi) := \int f d\pi \]  

(49)

where \( \tau_f(\pi) \) is set to be \( \infty \) if the integral is undefined; \( \tau_c \) is always well-defined by the non-negativity of \( c \). Since both functionals are linear on \( \Pi(\mu) \) there is flexibility in defining their effective domains (Section 3.1). For the sake of Theorem 3, the effective domains of \( \tau_c \) and \( \tau_f \) are defined by regarding them to be convex.

Theorem 3 (Strong Duality). Let

\[ D := \text{Dom} (\tau_f) \cap \text{Dom} (\tau_c) \quad \text{and} \quad t_c(D) := \{ t_c(\pi) : \pi \in D \} \]  

(50)

If

\[ \text{Dom} (\psi) \cap \text{rel-int} (\tau_c(D)) \neq \emptyset \quad \text{and} \quad 0 \in \text{Dom} (\psi) \]  

(51)

where \text{rel-int}(\cdot) denotes the relative interior of a set, then

\[ \inf_{\pi \in \Pi(\mu)} \int f d\pi + \psi \left( \int c d\pi \right) = \sup_{\lambda \in \mathbb{R}} \int_{S_0} f^{\lambda c} d\mu - \psi^*(\lambda) \]  

(52)
Remark 8. The key consequence of Theorem 3 that facilitates the development of computational methods for (43) is: the primary decision variable of an equivalent dual problem (47) is a single, scalar number. Granted, (47) also depends on the c-transform $f^{\lambda c}$. However, $f^{\lambda c}$ is given by an optimization problem (on the ambient spaces $S_0 \times S_1$) which is regularized by $\lambda$ and $c$. This is a setting which is now significantly more amenable to computation using iterative procedures.

Remark 9. Strong duality of the form (52) has been previously noted in [5], under more stringent conditions and assumptions. Most notably, [5, Section 2] requires the cost function to be lower-semicontinuous, satisfy growth conditions, and approach certain values on subsets of $S_0 \times S_1$. Additional restrictions are also placed on $\psi$. Related work [6, Theorem 1] (a special case of Theorem 3 in this work) makes similar assumptions: the cost function must attain a specific value on a subset of $S_0 \times S_1$, and $f$ and $c$ must be upper and lower-semicontinuous, respectively. Theorem 3 eliminates all of these assumptions and replaces them with a natural, Fenchel-type condition (51). Stated simply, (51) requires the objective is finite on a set with suitable “interior.” This is essentially what one would anticipate from analogs in finite dimensional optimization. Moreover, Fenchel-type are often more precise because the primarily tend to fail when the primal is already infinite/infeasible or when it is a pathological limit of infinite/infeasible problems.

Example 4.

- When $\psi(x) = \infty 1_{(\delta, \infty]}(x)$ for $\delta > 0$ ($\psi$ is zero on $[0, \delta]$ and $\infty$ outside), then $\psi^*(\lambda) = (\delta \lambda)_+$ and Theorem 3 gives
  \[
  \inf \left\{ \int f \, d\pi : \pi \in \Pi(\mu), \int c \, d\pi \leq \delta \right\} = \sup_{\lambda \in \mathbb{R}} \int f^{\lambda c} \, d\mu(x) - (\delta \lambda)_+ \tag{53}
  \]
  provided that there exists a $\pi \in \Pi(\mu)$ such that $\int c \, d\pi < \delta$ and $\int f \, d\pi < \infty$. Note, when $c$ is lower-semicontinuous, the infimum in (53) can be taken over optimal couplings between $\mu$ and any Borel measure $\nu \in \mathcal{P}(S_1)$—resulting in the optimal-transport-based, robust optimization problem (2).

- When $\psi(x) = x^{1+\alpha}/(1 + \alpha)$ for $\alpha \geq 0$,
  \[
  \inf_{\pi \in \Pi(\mu)} \int f \, d\pi + \frac{1}{1 + \alpha} \left( \int c \, d\pi \right)^{1+\alpha} = \sup_{\lambda \in \mathbb{R}} \int f^{\lambda c} \, d\mu(x) - \frac{\alpha}{1 + \alpha} (\lambda)^{(1+\alpha)/\alpha} \tag{54}
  \]
  provided there exists a $\pi \in \Pi(\mu)$ such that $\int c \, d\pi < \infty$ and $\int f \, d\pi < \infty$. Duality holds for other, commonly-used, smooth penalties (such as $x \mapsto e^x$) under the same condition.

6.2 Computational procedures.

The dual (47) can be re-expressed as

\[
\mathcal{D}_\mu(f) = \sup_{\lambda \in \mathbb{R}} g(\lambda) - \psi^*(\lambda), \quad g(\lambda) := E_{x \sim \mu} \left[ f^{\lambda c}(x) \right] \tag{55}
\]

where, henceforward, sufficient conditions for strong duality (51) are assumed. The function $g$ is concave, non-decreasing, and upper-semicontinuous (Lemma 11). Therefore, (55) makes sense as a one-dimensional, stochastic, convex optimization problem. A standard approach to solve (55) is to notice that supergradients/subgradients of $g$ and $\psi^*$ exist at every point in rel-int $(\text{Dom}(g))$ and rel-int $(\text{Dom}(\psi^*))$ [53]. If one can compute estimates of these supergradients, then a supergradient-ascent procedure in $\lambda$ will provide a suitable algorithm for computing (55). See Appendix D for a more detailed description of the supergradients of $g$. 

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6.2.1 Computing supergradients

Computation of supergradients for $g$ is where meaningful computational difficulty arises. This difficulty is the result of the inner minimization problem defining the $c$-transform $f^{\lambda c}$. Estimating $f^{\lambda c}$, at even a single point $x \in S_0$, suggests the need to solve

$$
\inf_{y \in S_1} f(y) + \lambda c(x, y) \quad (56)
$$

which, without additional regularity in $f$ and $c$, could be NP-hard– even for relatively simple $f$ and $c$. Indeed, consider the case $S_0 = S_1 = \mathbb{R}^d$, $c(x, y) = 1_{\Delta_d}(y)$ is the indicator function of the simplex $\Delta_d$, and $f(y) = y^T (I + A) y$ for the adjacency matrix $A$ of any graph $G$. Then, for any $x \in \mathbb{R}^d$ and $\lambda > 0$, (56) is the maximum independent set problem for the graph $G$ [47].

In the interest of developing a computational procedure for (43), we consider computation of supergradients for (55) when $S_0, S_1 \subseteq \mathbb{R}^d$ and $c(x, y) = \|x - y\|^2 / 2$. In this case, the dual (55) becomes

$$
\sup_{\lambda \in \mathbb{R}} \int \inf_{y \in \mathbb{R}^d} f(y) + \frac{\lambda}{2} \|y - x\|^2 \, d\mu - \psi^*(\lambda) \quad (57)
$$

and (56) provides the Moreau-Yosida envelope for the function $f$ [67]. If $f$ is semiconvex (9), then (56) is computationally tractable for large enough $\lambda$.

**Definition 6** (Supergradient oracle with high probability). A function $\theta_g : \mathbb{R} \to \mathbb{R}$ is called a $(\epsilon, \delta)$-supergradient oracle with high probability for $g$ (on the interval $[l, u]$) if, when queried with a $\lambda \in [l, u]$, it returns an independent random sample $\theta_g(\lambda)$ satisfying

$$
\mathbb{P}\left( \left[ \min_{z \in \partial g(\lambda)} |\theta_g(\lambda) - z| \geq \frac{\epsilon}{\max(\lambda - l, 1)} \right] \right) \leq \delta \quad (58)
$$

**Algorithm 2** Supergradient oracle (8)

**Input:** Distribution $\mu$, point $\lambda$, semi-convexity parameter $\rho_*$, smoothness parameter $L$, error tolerance $\epsilon$

Sample $x \sim \mu$

\[
y_0 \leftarrow x, \quad \kappa \leftarrow \sqrt{(\lambda + L)/(\lambda - \rho_*)}, \quad k \leftarrow \max\left(\left\lfloor 4\kappa \log(12\kappa \|\nabla f(x)\|/\epsilon) \right\rfloor, 0\right)\]
\]

for $1 \leq i \leq k$ do

\[
z_i = y_{i-1} - \frac{1}{\kappa} (\nabla f(y_{i-1}) + \lambda(y_{i-1} - x))
\]

\[
y_i = z_i + \frac{\kappa - 1}{\kappa + 1} (z_i - z_{i-1})
\]

return $\theta = \frac{1}{k} \|y_k - x\|^2$

**Proposition 2.** For a $\rho_*$-semiconvex function $f : \mathbb{R}^d \to \mathbb{R}$, which is also $L \geq \rho_*$ smooth (10), the mean of

$$
K \geq \frac{64\mathbb{E}_\mu \left\| \nabla f(x) \right\|^4}{(\lambda - \rho_*)^2 \min((\lambda - \rho_*)^2, 1) \delta^2} \quad (59)
$$

independent calls to Algorithm 2 with inputs $\lambda > \rho_*$ and $\epsilon = \tilde{\epsilon}/(2 \max(\lambda - \rho_*, 1))$, provides a $(\tilde{\epsilon}, \delta)$-supergradient oracle with high probability (6) for $g$ in (55) on the interval $(\rho_*, \infty)$.  

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6.2.2 A primal-dual algorithm

The supergradient oracle of Proposition 2 provides a mechanism to perform ascent steps in $\lambda$ to solve (55) (for $c = \| x - y \|^2 / 2$). Previous work [49, 29], regarding related, distributionally robust optimization problems, has focused on mirror ascent and bisection search to perform these ascent steps. For completeness, these algorithms (along with their complexities) are provided in the context of (55) in Appendices G and H.

The caveat to these procedures is that they only provide well-founded ascent methods for (55) when $\lambda$ is sufficiently large. Previous works [49, 8] have noted this in the context of (2); that it results in (2) only being computable for small to moderate values of $\delta$. The following condition will be used to actually quantify these values.

Definition 7. For a proper, closed, and convex function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$, let $\partial_- \psi$ denote it’s left derivative. The function $\psi$ is said to provide $R$-regularization at $x \in \mathbb{R}$ if

$$\partial_- \psi^*(x) \leq R^{-1}$$

Remark 10. In the context of problem (53) with $\psi = \infty 1_{(\delta, \infty]}$, one has $\psi^*(x) = (\delta x)_-$ and therefore $\partial_- \psi^*(x) = \delta 1_{(0, \infty]}(x)$. Hence, $R$-regularization at $x > 0$ induces the requirement $\delta \leq R^{-1}$ and asserts that the level of robustness in (2), $\delta$, is moderate. A broader understanding of (60), results from considering: since $x \in \partial \psi(y)$ for $y = \partial_- \psi^*(x)$ (where $\partial \psi(y)$ denotes the subgradient set at $y$), enforcement of (60) for large values of $x$ and $R$ constrains $\psi$ to attain large subgradients on small neighborhoods of 0. Thus, (60) quantifies the degree of regularization provided by $\psi$ in (57) and ensures that the level of regularization meets a given threshold.

With this mechanism, an algorithm with concrete computational guarantees for solving (57) can be furnished. This algorithm performs bisection ascent, using the supergradient oracle provided by Algorithm 2. For the sake of our Frank-Wolfe procedure, it is of importance that the algorithm implicitly maintains a primal-feasible iterate for

$$\inf_{\pi \in \Pi(\mu)} \int f \, d\pi + \psi \left( \int \frac{1}{2} \| y - x \|^2 \, d\pi \right)$$

and that the algorithm makes progress on the primal-dual gap between (61) and (57). For this reason, we title the algorithm a “primal-dual” algorithm.

Algorithm 3 Primal-dual algorithm

Input: Supergradient oracle $\theta_y$, error tolerance $\epsilon$, termination width $B$

$\eta \leftarrow \infty$, $b \leftarrow l$

while $u - l > \epsilon / B$ do

$\lambda \leftarrow (l + u) / 2$

$\eta \leftarrow \theta_y(\lambda)$, $\eta \leftarrow (\eta - \psi^*(\lambda))$

if $\eta < -\epsilon / \max (\lambda - b, 1)$ then $u \leftarrow \lambda$

else $l \leftarrow \lambda$

return $u$

Remark 11. The primal iterate that this algorithm maintains can be clarified by remarking that Assumption 4 enforces (60) with appropriate constants to guarantee that $\lambda^* > \rho_*$ for a $\rho_*$-semiconvex $f$ in (61) and optimal $\lambda^*$ (57). Since the function $y \mapsto f(y) + \lambda / 2 \| x - y \|^2$ is strictly convex for $\lambda > \rho_*$, the distribution $\pi_{\lambda, \mu} \in \Pi(\mu)$ given by

$$(X, m(X)) \sim \pi_{\lambda, \mu}, \quad X \sim \mu, \quad m_{\lambda}(x) = \arg \min_{y \in \mathbb{R}^d} f(y) + \frac{\lambda}{2} \| y - x \|^2$$

where $\lambda \in (0, \infty)$ and $\mu$ is the distribution that the algorithm is performing ascent on.
is the unique distribution in \( \Pi(\mu) \) such that
\[
\int f(y) + \frac{\lambda^*}{2} \|y - x\|^2 \, d\pi_{\lambda, \mu} = \int \min_{y \in \mathbb{R}^d} \left( f(y) + \frac{\lambda^*}{2} \|y - x\|^2 \right) \, d\mu(x)
\] (63)

Hence, \( \pi_{\lambda, \mu} \) is the implicit distribution that is maintained by Algorithm 3. The criterion that is used for bisection of an interval in Algorithm 3 is designed to make progress on the primal-dual gap between the current dual iterate \( \lambda_i \) and \( \pi_{\lambda_i, \mu} \):
\[
G(\lambda_i) := \int f \, d\pi_{\lambda_i, \mu} + \psi \left( \int \|y - x\|^2 \, d\pi_{\lambda_i, \mu} \right) - (g(\lambda_i) - \psi^*(\lambda_i))
\] (64)

This stands contrary to the sequence of iterates that are maintained by, say, Algorithm 5 where, \( \pi_{\lambda_i, \mu} \) need not even be primal feasible for a dual feasible \( \lambda_i \).

**Assumption 4.** The function \( f : \mathbb{R}^d \to \mathbb{R} \) is \( L \)-smooth (10), \( \rho_* \)-semiconvex, and \( \psi : \mathbb{R} \to \mathbb{R} \) provides \( C / \left( E_{\mu} \left[ \|\nabla f(x)\|^2 \right] \right) \)-regularization (60) at \( \rho_* + 1 \), for some \( C \geq 8L^2 \) Further, \( \psi \) is minimized at 0 and \( \psi^* \) is \( M \)-smooth (\( (\psi^*)' \) exists and is \( M \)-Lipschitz) on the interval \([l, u]\) where
\[
l := \rho_* + 1 \quad \text{and} \quad u := \rho_* + 1 + \sqrt{2C}
\] (65)

**Theorem 4.** Under Assumption 4 and a correct configuration of it’s inputs, Algorithm 3 returns a \( \lambda^* \) such that the primal-dual gap (64) satisfies \( G(\lambda^*) \leq \epsilon \) with probability \( 1 - \delta \). Moreover, the algorithm draws at most
\[
\tilde{O} \left( \frac{\rho^2 E_{\mu} \left[ \|\nabla f(x)\|^4 \right]}{\delta \epsilon^2} \right)
\] (66)
independent samples from \( \mu \) and performs \( \tilde{O} \left( \frac{\rho^2 L^{1/2} E_{\mu} \left[ \|\nabla f(x)\|^4 \right]}{(\delta \epsilon^2)} \right) \) expected gradient evaluations of \( f \) where \( \tilde{O} \) suppresses logarithmic factors in \( \rho_* \), \( L \), \( C \), \( M \), \( E_{\mu} \left[ \|\nabla f(x)\|^2 \right] \) and \( \epsilon \).

**Corollary 5.** If \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \), \( f \) is \( L \)-smooth (10), and \( \delta \leq \|\nabla f\|_{L^2(\mu)} / (2L) \), there exists a stochastic algorithm which (for any probability \( \gamma < 1 \)) computes a \( \lambda^* \) such that \( \mathbb{W}(\nu_{\lambda^*}, \mu) \leq \delta \) and
\[
\int f \, d\nu_{\lambda^*} - \inf_{\mathbb{W}(\nu, \mu) \leq \delta} \int f \, d\nu \leq \epsilon
\] (67)

where \( \nu_{\lambda^*} \) is second marginal of \( \pi_{\lambda^*, \mu} \) in (62). This algorithm requires at most \( \tilde{O}(L^2 \|\nabla f\|_{L^2(\mu)}^4 / ((1 - \gamma) \epsilon^2)) \) independent samples from \( \mu \) and executes \( \tilde{O}(L^{5/2} \|\nabla f\|_{L^4(\mu)}^4 / ((1 - \gamma) \epsilon^2)) \) gradient evaluations of \( f \) in expectation.

**Remark 12.** The conclusion of Theorem 4 is: regularization from \( \psi \) enables the computational solution of (61) when it occurs at the level specified by Assumption 4. When this conclusion is specialized to the instance (43), it results in Corollary 5 and a bound on the magnitude of \( \delta \). Such a result is quite befitting of our purposes, however, since the Frank-Wolfe procedure (Algorithm 1) need only solve local problems, not global ones. Further, since the instruction of these results is that an appropriate \( \delta \) should necessarily depend on \( \mu \) and \( f \) (43), Algorithm 1 adapts it’s choice of \( \delta \) per iteration.

It should also be noted that restriction of \( \delta \) to provide computational tractability for (2) has been used both qualitatively [6] and quantitatively [57] in previous works. Indeed, the techniques
presented in this work most closely resemble ideas from [57], where smoothness (10) was used similarly. In contrast, however, Assumption 4 and Theorem 4 provide actual quantification of the level of robustness required to achieve tractability (through (60)) and they do so for a more general set of problems (3). Moreover, Theorem 4 provides guarantees with respect to the primal-dual gap of these problems— a more elusive criterion than considered in previous work.

A Proof of weak duality

Proposition 3 (Weak Duality). Weak duality always holds for the pair (46) and (48). That is,

\[ D_\mu (f) \leq P_\mu (f) \]  

(68)

Proof. It is sufficient to show that, for any primal variable \( \pi \in \Pi(\mu) \) and any dual variables \( \lambda \in \mathbb{R} \) and \( \phi \in \Lambda_\mu(f + \lambda c) \)

\[
\int f \, d\pi + \psi \left( \int c \, d\pi \right) \geq \int \phi \, d\mu - \psi^*(\lambda)
\]

This nearly follows by definition:

\[
\int f \, d\pi + \psi \left( \int c \, d\pi \right) = \int f \, d\pi + \sup_{\eta \in \mathbb{R}} \eta \int c \, d\pi - \psi^*(\eta)
\]

\[
= \sup_{\eta \in \mathbb{R}} \int (f + \eta c) \, d\pi - \psi^*(\eta)
\]

\[
\geq \int \phi \, d\mu - \psi^*(\lambda)
\]  

(69)

where the first line is justified by the fact that \( \psi \) is convex and closed with \( \text{Dom}(\psi) \subseteq \mathbb{R} \). Therefore, \( \psi(x) = \psi^{**}(x) \) for all \( x \in \mathbb{R}_+ \cup \{\infty\} \).

\[\square\]

B Properties of the dual (48)

This section establishes properties of the dual problem (48) that are necessary to prove Theorem 3. Define

\[ K(g, \mu) := \sup_{\phi \in \Lambda_\mu(g)} \int_{S_1} \phi \, d\mu \quad \text{where} \quad D_\mu (f) = \sup_{\lambda \in \mathbb{R}} K (f + \lambda c, \mu) - \psi^*(\lambda) \]  

(70)

where we begin with the ansatz

\[ K(g, \mu) = \int h_g \, d\mu, \quad h_g (x) := \inf_{y \in S_1} g (x, y) \]  

(71)

A small technicality that occurs when writing the relation (71): the function \( h_g \) need not be Borel measurable even when \( g \) is Borel measurable. This arises from the fact that the sets

\[ h_g^{-1} ((-\infty, a)) = \{x : g(x, y) < a\} \]

are projections of Borel sets and therefore not necessarily Borel. The sets \( h_g^{-1} ((-\infty, a)) \) are analytic, however, which makes them universally measurable and therefore measurable with respect to the
completion of \( \mu \) or any other Borel measure [58]. For our purposes, this means that the lack of Borel measurability is superfluous. One can always define the right-hand side of (71) to be the integral of \( h_\phi \) under the completion of \( \mu \) – assuming the integral is well-defined.

The following lemmas establish (71) and the conditions under which it’s right-hand side is well defined.

**Lemma 6.** For any universally measurable set \( U \) and Borel measure \( \mu \), there exist Borel sets \( B, N \) and a set \( T \) such that
\[
U = B \cup T, \quad T \subseteq N \quad \text{and} \quad \mu(N) = 0
\]

**Proof.** Since \( U \) is contained in the completion of the Borel \( \sigma \)-algebra under \( \mu \), we have

\[
\mu(U) = \inf_{S_n \subseteq \bigcup_{n \in \mathbb{N}} S_n} \sum_{n \in \mathbb{N}} \mu(S_n) \quad \text{(72)}
\]

This implies that there exists a Borel set \( S \) such that \( U \subseteq S \) and \( \mu(U) = \mu(S) \). Defining the universally measurable set \( D := S \setminus U \) and noticing that \( \mu(D) = 0 \), one can again apply (72) to obtain a Borel measurable \( N \) such that \( D \subseteq N \) and \( \mu(N) = 0 \). Setting \( B = S \setminus N \) it is easy to that \( B \subseteq U \) and that this set is Borel. Moreover, for \( T = U \setminus B \) we have \( T \subseteq N \).

**Lemma 7.** Let \( g : S_0 \times S_1 \to \overline{\mathbb{R}} \) be any Borel measurable function and let \( g_+ \) be it’s non-negative part. If there exists a \( \pi \in \Pi(\mu) \) such that \( \int g_+ d\pi < \infty \), the integral \( \int h_\phi d\mu \) is well defined and

\[
K(g, \mu) = \sup_{\phi \in \Lambda_\mu(g)} \int \phi d\mu = \int h_\phi d\mu \quad \text{(73)}
\]

**Proof.** Note the following trivial inequality
\[
g(x, y) \geq h_\phi(x) \geq \phi(x) \quad (x, y) \in S_0 \times S_1, \ \phi \in \Lambda_\mu(g) \quad \text{(74)}
\]

and consider the functions
\[
p_k := \max(g, -k) \quad z_k(x) := \mathbb{E}_\pi[p_k \mid x]
\]

Clearly, \( z_k \) exists \( \mu \) almost everywhere and is integrable for all \( k \in \mathbb{N} \) since \( \int g_+ d\pi < \infty \) for some \( \pi \in \Pi(\mu) \). Notice that (74) implies
\[
z_k(x) \geq h_\phi(x) \quad \mu \text{ a.s}, \quad \forall k \in \mathbb{N}
\]

Thus, \( \int h_\phi d\mu \) is well defined and, by Fatou’s lemma

\[
\int g \, d\pi \geq \limsup_{k \to \infty} \int z_k \, d\mu \geq \int h_\phi \, d\mu \quad \text{(75)}
\]

Now, observe that \( \Lambda_\mu(g) = \emptyset \) implies that \( \int g \, d\pi = -\infty \). Hence, without loss of generality, we can assume that \( \Lambda_\mu(g) \neq \emptyset \) and consider a sequence \( \phi_n \in \Lambda_\mu(g) \) such that

\[
\int \phi_n \, d\mu \geq \sup_{\phi \in \Lambda_\mu(g)} \int \phi \, d\mu - \frac{1}{n}
\]
From (74) and the fact that $\int g_+ \, d\pi < \infty$, it follows that for $\phi^* = \sup_{n \in \mathbb{N}} \phi_n$ one has $\phi^* \in \Lambda_\mu(g)$. Thus, the supremum in (70) is achieved for some $\phi^* \in \Lambda_\mu(g)$. Since, (74) implies

$$\int h \, d\mu \geq \sup_{\phi \in \Lambda_\mu(g)} \int \phi \, d\mu$$

if it can shown that $\int h_g \, d\mu = \int \phi^* \, d\mu$, the desired conclusion (73) will hold.

Let $\epsilon > 0$ and consider the universally measurable set

$$A_\epsilon := \{x \in S_0 : \phi^*(x) < h_g(x) - \epsilon\}$$

By Lemma 6, there exist Borel measurable $B_\epsilon$ and $N_\epsilon$ such that $B_\epsilon \subseteq A_\epsilon$, $A_\epsilon \subseteq B_\epsilon \cup N_\epsilon$ and $\mu(N_\epsilon) = 0$. Additionally, observe that $\mu(B_\epsilon) = 0$ by the optimality of $\phi^*$. Thus,

$$\mu(Z_0) = 0 \quad \text{where} \quad Z_0 := \bigcup_{k \in \mathbb{N}} (B_{1/k} \cup N_{1/k})$$

and $\phi^* = h_g$ on the complement of the Borel measurable set $Z_0$. This gives

$$\int \phi^* \, d\mu = \int h_g \, d\mu$$

One can now derive an approximation property for $K(g, \mu)$ using Lemma 7.

**Lemma 8.** Let $g : S_0 \times S_1 \rightarrow [\bar{\mathbb{R}}$ be Borel measurable such that there exists a $\pi \in \Pi(\mu)$ for which $\int g_+ \, d\pi < \infty$. Then, there exists a sequence of $\pi_n \in \Pi(\mu)$ such that

$$K(g, \mu) = \int h_g \, d\mu = \lim_{n \rightarrow \infty} \int g \, d\pi_n \quad (76)$$

**Proof.** We give a proof following the design of Lemma 8 in [6]. First, observe that, since $g$ dominates $h_g$, it is sufficient to show that there exists a sequence of distributions $\pi_n \in \Pi(\mu)$ such that

$$\lim_{n \rightarrow \infty} \int g \, d\pi_n \leq \int h_g \, d\mu$$

Again, one can consider the notation overloaded so that $\int g \, d\pi_n$ denotes both the integral of $g$ with respect to $\pi_n$ and the integral with respect to the completion of $\pi_n$.

Let $n \in \mathbb{N}$ and for any $i \leq 2n^2$, define the sets

$$G_k^{(n)} := \{(x, y) \in S_0 \times S_1 : \frac{k - 1}{n} - n \leq g(x, y) \leq \frac{k}{n} - n\}$$

Also, define

$$G_0^{(n)} := \{(x, y) \in S_0 \times S_1 : g(x, y) \leq -n\} \quad \text{and} \quad G_{2n^2+1}^{(n)} := \{(x, y) \in S_0 \times S_1 : g(x, y) \geq n\}$$

Denoting the projection operation onto $S_0$ by $\text{Proj}_{S_0}(\cdot)$, set

$$Z_i^{(n)} := G_i^{(n)} \setminus \bigcup_{j<i} G_j^{(n)} \quad \text{and} \quad A_i^{(n)} := \text{Proj}_{S_0}(Z_i^{(n)})$$

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and notice that the $Z_i^{(n)}$ are Borel and, therefore, the $A_i^{(n)}$ are universally measurable. Also, notice that the $Z_i$ form a partition of $S_0 \times S_1$ and the $A_i$ form a partition of $S_0$.

From the von-Neumann selection theorem \[58\], it follows that, for each $i \leq 2n^2 + 2$, there exists a universally measurable selection $\xi_i : A_i^{(n)} \to S_1$ such that $(\xi_i(x), x) \in Z_{i}^{(n)}$ for all $x \in A_i^{(n)}$. Since the $A_i^{(n)}$ form a partition of $S_0$, define $\gamma_n : S_0 \to S_1$ to be the unique, universally measurable extension of the $2n^2 + 2$ selections $\xi_i$ to all of $S_0$.

Now, notice that for $D_n := \bigcup_{i=0}^{2n^2} A_i^{(n)}$ we have $D_j \subseteq D_k$ for $j \leq k$ and

$$h_g(x) \leq g(x, \gamma_n(x)) \leq \max(h_g(x), -n) + \frac{1}{n} \quad \forall x \in D_n$$

Moreover, let $(X, Y) \sim \pi$ (where, without loss of generality, we assume that $\pi$ is complete) and consider the law $\pi_n$ of the random variable given by

$$(X_n, Y_n) = \begin{cases} 
(X, \gamma_n(X)) & \text{if } X \in D_n \\
(X, Y) & \text{otherwise}
\end{cases}$$

Observe that $\pi_n$ induces a unique Borel measure in $\Pi(\mu)$, which we also denote by $\pi_n$.

By (77) and the construction of $(Y_n, X_n)$, we have, for all $n \in \mathbb{N}$,

$$g(X_n(\omega), Y_n(\omega)) \leq \max(g(X(\omega), Y(\omega)), -n) + \frac{1}{n}$$

Moreover, taking (77) in the limit as $n \to \infty$, we get

$$\limsup_{n \to \infty} g(X_n(\omega), Y_n(\omega)) \leq h_g(X(\omega))$$

Technically, it should be noted that taking (77) in the limit as $n \to \infty$ does not cover $X(\omega) \in D_\infty$, where $D_\infty := \left(\bigcup_{n \in \mathbb{N}} D_n\right)^C$. However, this is a trivial technicality since

$$x \in D_\infty \quad \Rightarrow \quad g(x, y) = \infty \quad \forall y \in S_1$$

and therefore (79) still holds for $X(\omega) \in D_\infty$. Since (78) implies that the $g(X_n, Y_n)$ have a common, integrable upper bound, Fatou’s Lemma applies and one obtains

$$\limsup_{n \to \infty} \int g \, d\pi_n \leq \int h_g \, d\mu$$

As a trivial consequence of Lemma 8, one has the aesthetic result:

**Corollary 9.** If $g : S_0 \times S_1 \to \mathbb{R}$ is Borel measurable such that there exists a $\pi \in \Pi(\mu)$ where $\int g \, d\pi < \infty$, then

$$\sup_{\phi \in \Lambda_\mu(g)} \int \phi \, d\mu = \inf_{\pi \in \Pi(\mu)} \int g \, d\pi$$

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Proof. The equality follows from Lemma 8 and the fact that
\[ \int h_y \, d\mu \leq \int g \, d\pi \]
for all \( \pi \in \Pi(\mu) \) such that \( \int g \, d\pi < \infty \). \( \square \)

This culminates in the chief regularity result needed to establish strong duality:

**Proposition 4.** If there exists a \( \pi \in \Pi(\mu) \) such that \( \int f \, d\pi < \infty \) and \( \int c \, d\pi < \infty \) then
\[ D_{\mu}(f) = \sup_{\lambda \in \mathbb{R}} \inf_{\pi \in \Pi(\mu)} \int_{S_0 \times S_1} (f(y) + \lambda c(x, y)) \, d\pi(x, y) - \psi^*(\lambda) \quad (80) \]

Proof. Since, for each \( \lambda \in \mathbb{R} \),
\[ \int f(y) + \lambda c(x, y) \, d\pi(x, y) < \infty \]
applying Corollary 9 to the function \( g(x, y) = f(y) + \lambda c(x, y) \) gives the result. \( \square \)

## C  Proof of Theorem 3

Proof. Define the function \( \tilde{f} : \mathbb{R} \to \mathbb{R} \)
\[ \tilde{f}(x) = \inf_{\pi \in M(x)} \tau_f(\pi) \quad \text{where} \quad M(x) := \{ \pi \in \Pi(\mu) : \tau_c(\pi) = x \} \]
and notice that \( \tilde{f} \) is convex. Indeed, for any \( \pi_x, \pi_y \in \Pi(\mu) \) such that \( \tau_c(\pi_x) = x \) and \( \tau_c(\pi_y) = y \), one can construct
\[ \pi_{\alpha x + (1-\alpha)y} := \alpha \pi_x + (1-\alpha)\pi_y, \quad \alpha \in [0, 1] \]
such that \( \pi_{\alpha x + (1-\alpha)y} \in \Pi(\mu) \) and \( \tau_c(\pi_{\alpha x + (1-\alpha)y}) = \alpha x + (1-\alpha)y \). Hence,
\[ \tilde{f}(\alpha x + (1-\alpha)y) \leq \alpha \int f \, d\pi_x + (1-\alpha) \int f \, d\pi_y = \int f \, d\pi_{\alpha \pi_x + (1-\alpha)\pi_y} \quad (81) \]
and convexity follows by taking an infimum of the right-hand side of (81). Note that the effective domain of \( \tilde{f} \) is \( \text{Dom}(\tilde{f}) = \tau_c(D) \) where \( D \) is as defined in (50).

As a first step, we will show that if \( f(y) = -\infty \) for some \( y \in \mathbb{R}_+ \) then (52) holds. Let \( \pi_n \in \Pi(\mu) \) be a sequence such that \( \tau_c(\pi_n) = y \) for all \( n \) and \( \lim_{n \to \infty} \tau_f(\pi_n) = -\infty \). By the hypothesis of Theorem 3, there also exists a \( \pi^* \in \Pi(\mu) \) such that
\[ x = \tau_c(\pi^*) \in S \quad \text{and} \quad \tau_f(\pi^*) < \infty \quad (82) \]
Moreover, if \( y \neq x \), then \( \text{rel-int}(\tau_c(D)) \) is an open interval; and combined with the fact that \( 0 \in \text{Dom}(\psi) \) and \( S \neq \emptyset \), \( S \) must contain an open interval. Hence, no matter if \( y = x \) or \( y \neq x \), there exists an \( \alpha^* \neq 0 \) such that
\[ \tau_c(\pi_{\alpha^* x + (1-\alpha^*)\pi_n}) = \alpha^* x + (1-\alpha^*)y \in S \quad \text{and} \quad \lim_{n \to \infty} \tau_f(\pi_{\alpha^* x + (1-\alpha^*)\pi_n}) = -\infty \]
This gives \( \tilde{f}(\alpha^* x + (1-\alpha^*)y) = -\infty \) and \( \psi(\alpha^* x + (1-\alpha^*)y) < \infty \); or, in other words, \( \mathcal{P}(f, \mu, c) = -\infty \). In this case, weak duality (68) implies strong duality. Thus, without loss of generality, one can assume that \( -\infty < f(y) \) for all \( y \in \mathbb{R}_+ \). Additionally, since \( \tau_f(\pi^*) < \infty \), \( \tilde{f} \) is finite at the point
x. Therefore, \( \tilde{f} \) is a proper, convex function; this also implies that the function \(-\tilde{f}\) is a proper, concave function.

Now, to finish the proof, observe that, since \( \pi^* \) satisfies the conditions of Proposition 4, the dual problem (80) can be rewritten as

\[
\mathcal{D}_\mu(f) = \sup_{\lambda \in \mathbb{R}} \inf_{\pi \in \Pi(\mu)} \int_{S_0 \times S_1} (f(y) + \lambda c(x,y)) \, d\pi(x,y) - \psi^*(\lambda)
\]

\[
= \sup_{\lambda \in \mathbb{R}} \inf_{x \in \text{Ran}(\tau_c)} \tilde{f}(x) + \lambda x - \psi^*(\lambda)
\]

\[
= \sup_{\lambda \in \mathbb{R}} \tilde{f}_\ast^*(\lambda) - \psi^*(\lambda)
\]

where \( \tilde{f}_\ast^* : \mathbb{R} \to \bar{\mathbb{R}} \) denotes the concave conjugate of the concave function \(-\tilde{f}\). Since the primal problem trivially has the expression

\[
\mathcal{P}_\mu(f) = \inf_{x \in \mathbb{R}} \psi(x) - (-\tilde{f}(x))
\]

showing (52) is equivalent to

\[
\inf_{x \in \mathbb{R}} \psi(x) - (-\tilde{f}(x)) = \sup_{\lambda \in \mathbb{R}} \tilde{f}_\ast^*(\lambda) - \psi^*(\lambda)
\]

By Fenchel-Rockafeller duality [53, Theorem 31.1], (83) holds if \( \psi \) is proper convex, \(-\tilde{f}\) is proper concave, and

\[
\text{rel-int (Dom}(\psi)) \cap \text{rel-int (Dom}(-\tilde{f})) \neq \emptyset
\]

(84)

However, since \( 0 \in \text{Dom}(\psi) \) but \( 0 \notin \text{rel-int}(\tau_c(D)) \subseteq \mathbb{R}_+ \), one has

\[
\text{Dom}(\psi) \cap \text{rel-int}(\tau_c(D)) \neq \emptyset \quad \Rightarrow \quad \text{rel-int}(\text{Dom}(\psi)) \cap \text{rel-int}(\tau_c(D)) \neq \emptyset
\]

As \( \tau_c(D) = \text{Dom}(-\tilde{f}) \), it follows that (84) holds, giving strong duality via (83).

\[\square\]

D Characterization of gradients for \( g \) (55) and primal-dual gap bounds

This section establishes key properties of (55) that are used to provide the results of sub-section 6.2.2. These results are of independent interest, however, since (through the similar arguments) they could be used in establishing analogous computational bounds for other cost functions in (55).

Recalling (47), define the set of \( \delta \)-optimizers of the \( c \)-transform as

\[
Y_\delta(\lambda, x) := \{ y \in S_1 : f(y) + \lambda c(x, y) \leq f^\lambda(x) + \delta \}
\]

and let \( z_x(\lambda) := f^\lambda(x) \) denote the \( c \)-transform of \( f \) as a function of \( \lambda \). Clearly, this function is concave, upper-semicontinuous, and non-decreasing.

**Lemma 10.** If \( z_x \) is proper, then the right derivative of \( z_x \) satisfies

\[
\partial_+ z_x(\lambda) = \lim_{\delta \to 0} \inf_{y \in Y_\delta(\lambda, x)} c(x, y)
\]

**Proof.** Let

\[
T := \{(x, y) \in S_0 \times S_1 : f(y) \text{ and } c(x, y) \text{ are finite}\}
\]

Since \( z_x \) is proper, one can write

\[
z_x(\lambda) = \inf_{(x, y) \in T} w_{x,y}(\lambda) \quad \text{where} \quad w_{x,y}(\lambda) := f(y) + \lambda c(x, y)
\]

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Moreover, the \( w_{x,y} \) are closed/upper-semicontinuous so one has \( z_x = z_x^{**} = \inf_{(x,y) \in T} w_{x,y} \), where \((\cdot)^{**}\) denotes the biconjugate in the concave sense.

From this, Theorem 4 in [35] permits the characterization:

\[
\partial z_x(\lambda) = \bigcap_{\delta > 0} \text{cl conv} \left( \bigcup_{(x,y) \in T_\delta(\lambda)} \partial_\delta w_{x,y}(\lambda) - N_{\text{Dom}(z_x)}(\lambda) \right) \tag{86}
\]

where \text{cl conv} \((\cdot)\) denotes the convex closure, \( N_{\text{Dom}(z_x)} \) is the normal cone of \( \text{Dom}(z_x) \), \( \partial_\delta w_{x,y}(\lambda) \) denotes the \( \delta \)-superdifferential of \( w_{x,y}(\lambda) \), and

\[ T_\delta(\lambda) = \{ (x,y) \in T : w_{x,y}(\lambda) \leq z_x(\lambda) + \delta \} \]

is the set of \((x,y) \in T \) which are \( \delta \) optimal. Recall that \( \partial_\delta w_{x,y}(\lambda) \) is the set of \( t \in \mathbb{R} \) such that

\[ w_{x,y}(\xi) \leq w_{x,y}(\lambda) + t(\xi - \lambda) + \delta, \quad \forall \xi \in \mathbb{R} \]

Since \( w_{x,y} \) is affine for all \((x,y) \in T \), this set is identical to the usual superdifferential of \( w_{x,y} \) and one has

\[ \partial_\delta w_{x,y}(\lambda) = c(x,y), \quad \forall \delta \geq 0 \tag{87} \]

Additionally, observe that either \( \text{Dom}(z_x) = \mathbb{R} \) or \( \text{Dom}(z_x) = [a, \infty) \) for some \( a \in \mathbb{R} \), since \( z_x \) is proper, closed, concave, and non-decreasing. Hence,

\[ N_{\text{Dom}(z_x)}(\lambda) = \begin{cases} \mathbb{R}_- & \text{if } \lambda = a \\ 0 & \text{otherwise} \end{cases} \tag{88} \]

Using (87) and (88) to simplify (86), we obtain

\[ \partial z_x(\lambda) = \bigcap_{\delta > 0} \text{cl conv} \left( A_\lambda \right) \tag{89} \]

where

\[ A(a) := \bigcup_{y \in \mathcal{Y}_\lambda(a,x)} \{ c(x,y), \infty \} \quad \text{and otherwise} \quad A(\lambda) := \bigcup_{y \in \mathcal{Y}_\lambda(\lambda,x)} \{ c(x,y) \} \tag{90} \]

Since \( \partial_+ z_x(\lambda) = \min \partial z_x(\lambda) \), (85) directly follows from (89) and (90).

\[ \Box \]

**Lemma 11.** If sufficient conditions for strong duality (51) hold, \( g(\lambda) \) (55) is upper-semicontinuous. Further, for any \( \lambda \in \mathbb{R} \) at which the right derivative of \( g \) exists (denoted \( \partial_+ g(\lambda) \)):

\[ \partial_+ g(\lambda) = E_{x \sim \mu} \left[ \lim_{\delta \to 0} \inf_{y \in \mathcal{Y}_\lambda(\lambda,x)} c(x,y) \right] \tag{91} \]

**Proof.** Since \( g \) is non-decreasing, it is sufficient to show that \( g \) is continuous from the right in order to prove that it is upper-semicontinuous. To this end, observe that duality conditions (51) imply

\[ g(\lambda) = E_{x \sim \mu} \left[ z_x(\lambda) \right] < \infty \tag{92} \]

for all \( \lambda \in \mathbb{R} \). Hence, for any \( \lambda_n \downarrow a \), the monotone convergence theorem applies to the sequence \( z_x(\lambda_n) \downarrow z_x(a) \). This gives \( \lim_{\lambda \to a^+} g(\lambda) = g(a) \).

To show (91), define

\[ D_{-\infty} := \{ x : \forall \lambda \in \mathbb{R}, \ z_x(\lambda) = -\infty \} \quad \text{and} \quad D_{\infty} := \{ x : \exists \lambda \in \mathbb{R}, \ z_x(\lambda) = \infty \} \]

27
The first claim is that $D_{-\infty}$ and $D_\infty$ are universally measurable. Indeed, since $z_x(\lambda)$ is point-wise non-decreasing in $\lambda$:
\[
D_{-\infty} = \bigcap_{q \in Q} \{ x : z_x(q) = -\infty \} \quad \text{and} \quad D_\infty = \bigcup_{q \in Q} \{ x : z_x(q) = \infty \}
\]
Thus, the universal measurability of $z_x(\lambda)$ (see Appendix B) gives measurability of $D_{-\infty}$ and $D_\infty$.

Now, without loss of generality, assume $\text{Dom}(g) \neq \emptyset$. Since $g(\lambda) > -\infty$ for some $\lambda \in \mathbb{R}$, $\mu(D_{-\infty}) = 0$. Further, by using (92), one has $\mu(\{ x : z_x(q) = \infty \}) = 0$ for all $q \in Q$, implying $\mu(D_\infty) = 0$. Thus,
\[
\mu(D_{-\infty} \cup D_\infty) = 0
\]
Since $z_x(\cdot)$ is a proper, concave function for $x \in D_{C_{-\infty}} \cap D_{C_{\infty}}$, this implies that $z_x$ is a proper, concave function in $\lambda$ for $\mu$-almost every $x$.

To complete the proof, notice that Lemma 10 can now be used to conclude that $\partial_+ z_x(\lambda)$ is given by (85) for $\mu$-almost every $x$. Moreover, from [60], one has
\[
\partial_+ g(\lambda) = E_\mu[\partial_+ z_x(\lambda)], \quad \lambda \in \text{int}(\text{Dom}(g))
\] (93)
since $g$ is finite on $\text{int}(\text{Dom}(g))$. This gives the conclusion
\[
\partial_+ g(\lambda) = E_\mu \left[ \lim_{\delta \to 0} \inf_{y \in \lambda(\lambda,y)} c(x,y) \right], \quad \lambda \in \text{int}(\text{Dom}(g))
\]
Finally, to show the desired conclusion (91) on the boundary $\partial(\text{Dom}(g))$, let $a \in \partial(\text{Dom}(g))$ and assume $\partial_+ g(a)$ exists. Since $g$ is upper-semicontinuous and concave,
\[
\partial_+ g(a) = \lim_{\lambda \downarrow a} \partial_+ g(\lambda) = \lim_{\lambda \downarrow a} E_\mu[\partial_+ z_x(\lambda)]
\] (94)
where the second equality follows from (93) and $\text{int}(\text{Dom}(g)) = (a, \infty)$. Noticing that $z_x(\lambda)$ is concave and non-decreasing in $\lambda$, it follows that $\partial_+ z_x(\lambda)$ is non-negative and non-increasing in $\lambda$. Hence, (94) and monotone convergence now give the desired result (91). \hfill \square

**Lemma 12.** Let $\lambda \in \text{Dom}(g)$ and assume that there exists a $\pi \in \Pi(\mu)$ which satisfies
\[
\int_{S_0 \times S_1} f(y) + \lambda c(x,y) \, d\pi = \int_{S_0} h_{f+\lambda c}(x) \, d\mu(x)
\]
Then, for any $\lambda^* \in \partial \psi \left( \int c \, d\pi \right)$ and any $t \in \partial \psi^*(\lambda)$, one has
\[
\int f \, d\pi + \psi \left( \int c \, d\pi \right) - (\lambda - \lambda^*) \left( t - \int c \, d\pi \right) \leq g(\lambda) - \psi^*(\lambda)
\] (95)
Additionally, if $\psi$ is $M$-Holder smooth with exponent $\nu$ then
\[
\int f \, d\pi + \psi \left( \int c \, d\pi \right) - M \left| t - \int c \, d\pi \right|^{1+\nu} \leq g(\lambda) - \psi^*(\lambda)
\] (96)
**Proof.** Define $z := \int c \, d\pi$ and let $\lambda \in \text{Dom}(g)$ and $\lambda^* \in \partial \psi(z)$. Since $\lambda^*$ is a subgradient at $z$, one has the identity
\[
\int \lambda^* c(x,y) \, d\pi = \psi^*(\lambda^*) + \psi(z)
\]
From this, it follows that

\[
g(\lambda) - \psi^*(\lambda) = \int f(y) + \lambda c(x, y) \, d\pi - \psi^*(\lambda) \\
= \int f \, d\pi + \psi(z) + \psi^*(\lambda^*) + (\lambda - \lambda^*) z - \psi^*(\lambda) \\
= \int f \, d\pi + \psi(z) - (\psi^*(\lambda) - \psi^*(\lambda^*) - z(\lambda - \lambda^*)) \\
\geq \int f \, d\pi + \psi(z) - (\lambda - \lambda^*)(t - z)
\]

which gives (95). If \( \psi \) is also \( M \)-Holder smooth with exponent \( \nu \), then \( \lambda = \psi'(t) \) and \( \lambda^* = \psi'(z) \). Hence, one obtains (96) from (95) and the inequality

\[
|\psi'(t) - \psi'(z)| \leq M|t - z|^\nu
\]

\[\square\]

E Optimality conditions for (57)

This section provides bounds (101) on the magnitude of a near-optimal decision variable for (57). These bounds are used to establish the computational complexities of Theorem 4.

**Lemma 13.** If \( f \) is \( L \)-smooth (10) and \( L < \lambda \) then, for any \( \epsilon > 0 \) there exists \( \delta > 0 \), such that all \( \delta \)-optimizers

\[
f(y_\delta) + \frac{\lambda}{2} \|y_\delta - x\|^2 \leq \inf_{y \in \mathbb{R}^d} f(y) + \frac{\lambda}{2} \|y - x\|^2 + \delta
\]

satisfy \( \|y_\delta - x\| \geq \frac{\|\nabla f(x)\|}{2\lambda} - \epsilon \).

**Proof.** From \( L \)-smoothness and the fact \( \lambda > L \), the function

\[
v(y) := f(y) + \frac{\lambda}{2} \|y - x\|^2
\]

is \((\lambda - L)\)-strongly convex. Thus, it has a unique minimizer \( y^* \) and for any \( \epsilon \) there exists a \( \delta > 0 \) such that

\[
\|y^* - y_\delta\| \leq \epsilon
\]

for any \( \delta \)-optimizer \( y_\delta \). Hence, it is sufficient to show that

\[
\frac{\|\nabla f(x)\|}{2\lambda} \leq \|y^* - x\|
\]

to prove the desired result. To do this, notice that

\[
\nabla f(y^*) + \lambda(y^* - x) = 0
\]

(97)

by first-order optimality conditions for \( y^* \). Combining (97) with the \( L \)-smoothness of \( f \), one obtains

\[
\|\nabla f(x) - \nabla f(y^*)\|^2 \leq L^2 \|x - y^*\|^2 \\
\Rightarrow \|\nabla f(x)\|^2 + (\lambda^2 - L^2) \|x - y^*\|^2 \leq 2\lambda \nabla f(x)^T(x - y^*) \leq 2\lambda \|\nabla f(x)\| \|x - y^*\|
\]

(98)

Using the fact that \( \lambda > L \), the desired result then follows directly from (98). \[\square\]
Proposition 5. Let $S_0, S_1 = \mathbb{R}^d$ and $c(x,y) = \|x - y\|^2 / 2$. If $f$ is differentiable and $\rho_\ast$-semiconvex then, for any $\epsilon > 0$, there exists a $\lambda_\epsilon \leq \rho_\ast + E_{\mu} \left[ \|\nabla f(x)\|^2 \right] / (2\epsilon)$ such that

$$
(\sup_{\lambda \in \mathbb{R}} g(\lambda) - \psi^*(\lambda)) - (g(\lambda_\epsilon) - \psi^*(\lambda_\epsilon)) \leq \epsilon
$$

(99)

Further, if $f$ is $L$-smooth and

$$
\partial_-\psi^*(\rho_\ast) = \frac{E_{\mu} \left[ \|\nabla f(x)\|^2 \right]}{C}
$$

(100)

for $C \geq 8L^2$, then $\lambda_\epsilon$ can be chosen in the interval $[l, u] \subseteq \mathbb{R}$ for

$$
l = \rho_\ast \quad \text{and} \quad u = \min \left( \beta, \rho_\ast + \sqrt{2C} \right)
$$

(101)

where $\beta = \rho_\ast + E_{\mu} \left[ \|\nabla f(x)\|^2 \right] / (2\epsilon)$

Proof. For any $\lambda \geq \rho_\ast$, $\rho_\ast$-semiconvexity of $f$ provides the lower bound

$$
g(\lambda) = E_{\mu} \left[ \inf_{y \in \mathbb{R}^d} f(y) + \frac{\lambda}{2} \|y - x\|^2 \right] \geq E_{\mu} [f(x)] - \frac{1}{2(\lambda - \rho_\ast)} E_{\mu} \left[ \|\nabla f(x)\|^2 \right]
$$

Since

$$
g(\lambda) = E_{\mu} \left[ \inf_{y \in \mathbb{R}^d} f(y) + \frac{\lambda}{2} \|y - x\|^2 \right] \geq E_{\mu} [f(x)] \quad \forall \lambda \in \mathbb{R}
$$

one obtains the identity

$$
g(\hat{\lambda}) - \psi^*(\hat{\lambda}) \geq (g(\lambda) - \psi^*(\lambda)) - \frac{1}{2(\lambda - \rho_\ast)} E_{\mu} \left[ \|\nabla f(x)\|^2 \right] + \left( \psi^*(\lambda) - \psi^*(\hat{\lambda}) \right)
$$

(102)

for any $\hat{\lambda} \geq \rho_\ast$ and $\lambda \in \text{Dom}(\psi^*)$. Via (102), Proposition 5 can be easily established; indeed let us first show (99).

Define $\lambda_n \in \mathbb{R}$ be an optimizing sequence for (55)

$$
\lim_{n \to \infty} g(\lambda_n) - \psi^*(\lambda_n) = \sup_{\lambda \in \mathbb{R}} g(\lambda) - \psi^*(\lambda)
$$

and set $\beta := \rho_\ast + E_{\mu} \left[ \|\nabla f(x)\|^2 \right] / (2\epsilon)$. Since $\psi^*$ is lower-semicontinuous and $g$ is upper-semicontinuous ($L$-smoothness of $f$ (10) guarantees that Lemma 11 applies), it is sufficient to show that there exists a $\lambda_\epsilon \leq \beta$ satisfying (99) if $\beta < \lim_{n \to \infty} \lambda_n$.

Since $\beta < \liminf_{n \to \infty} \lambda_n$, one can assume without loss of generality that $\beta < \lambda_n$ for all $n \in \mathbb{N}$. As $\psi^*$ is non-decreasing (the domain of $\psi$ is $\mathbb{R}_+$), this gives

$$
\psi^*(\beta) \leq \psi^*(\lambda_n) \quad \forall n \in \mathbb{N}
$$

(103)

Substituting $\hat{\lambda} = \beta$ and $\lambda = \lambda_n$ in (102), (103) and algebraic simplification provide

$$
g(\beta) - \psi^*(\beta) \geq g(\lambda_n) - \psi^*(\lambda_n) - \epsilon
$$

(104)

Taking the limit in (104) and setting $\lambda_\epsilon = \beta$ gives the desired result (99).
To show the second half of Proposition 5, observe that the previous result implies one can assume 
\( \liminf_{n \to \infty} \lambda_n \leq \beta \) for an optimizing sequence \( \lambda_n \). Otherwise, \( \beta \) is \( \epsilon \)-optimal and the second half of Proposition 5 is trivially true. The immediate consequence of this assumption is that an optimizer \( \lambda^* \) of (55) exists. Indeed, \( L \)-smoothness of \( f \) provides \( g(\lambda) = -\infty \) for any \( \lambda < -L \) and, combined with \( \liminf_{n \to \infty} \lambda_n \leq \beta \), the optimizing sequence \( \lambda_n \) can be assumed to be bounded. Via Bolzano-Weierstrass, the sequence is therefore convergent to some \( \lambda^* \leq \beta \) and upper-semicontinuity of \( g \) along with lower-semicontinuity of \( \psi^* \) then imply that \( \lambda^* \) is an optimizer of (55).

The main consequence of the existence of \( \lambda^* \) is that, in combination with (102), one has the upper bound
\[
\psi^*(\lambda^*) - \psi^*(\lambda) - \frac{1}{2(\lambda - \rho_\star)_+} E_\mu \left[ \|\nabla f(x)\|^2 \right] \geq g(\lambda^*) - \psi^*(\lambda^*) \leq g(\lambda) - \psi^*(\lambda)
\]
for any \( \lambda \in \text{Dom}(\psi^*) \), where \((\cdot)_+\) denotes the non-negative part. If \( \lambda \leq \lambda^* \), then the convexity of \( \psi^* \) gives
\[
(\lambda - \rho_\star)_+ \psi^*(\lambda^*) - \psi^*(\lambda) \leq \frac{1}{2} E_\mu \left[ \|\nabla f(x)\|^2 \right]
\]
Taking \( \lambda = (\lambda^* + \rho_\star)/2 \) in (105) will lead to the desired conclusion of Proposition 5 so long as \( \rho_\star \leq \lambda^* \). To show that (100) implies \( \rho_\star \leq \lambda^* \), observe that, in the notation of Lemma 11,
\[
\frac{\|\nabla f(x)\|^2}{2(2\lambda)^2} \leq \liminf_{\delta \to 0} \inf_{y \in \mathcal{Y}(\lambda,x)} c(x,y), \quad \lambda > L
\]
by Lemma 13. In combination with the result of Lemma 11, this yields
\[
\frac{1}{8\lambda^2} E_\mu \left[ \|\nabla f(x)\|^2 \right] \leq \partial_+ g(\lambda), \quad \lambda \geq L
\]
(107)
Indeed, since \( g \) is upper-semicontinuous by Lemma 11, \( \lim_{\lambda \downarrow L} \partial_+ g(\lambda) = \partial_+ g(L) \) and it is sufficient that (106) hold for \( \lambda > L \) to obtain (107) for \( \lambda \geq L \). Under (100), (107) produces the relation
\[
\partial_- \psi^*(\rho_\star) \leq \frac{1}{8L^2} E_\mu \left[ \|\nabla f(x)\|^2 \right] \leq \partial_+ g(L) \leq \partial_+ g(\rho_\star)
\]
(108)
since \( \rho_\star \leq L \). As \( g \) is concave and \( \psi \) is convex, (108) immediately gives \( g(\rho_\star) - \psi^*(\rho_\star) \geq g(\lambda) - \psi^*(\lambda) \) for all \( \lambda < \rho_\star \). Hence, \( \lambda^* \) can be chosen so that \( \rho_\star \leq \lambda^* \).

Finally, using the fact that \( \rho_\star \leq \lambda^* \) and substituting \( \lambda = (\lambda^* + \rho_\star)/2 \) into (105), one obtains
\[
\lambda^* \leq \rho_\star + \left( \frac{2 E_\mu \left[ \|\nabla f(x)\|^2 \right]}{\partial_+ \psi^*((\lambda^* + \rho_\star)/2)} \right)^{1/2} \leq \rho_\star + \sqrt{2C}
\]
(109)
where the last inequality is a result of the fact that \( \psi^* \) is convex. After combining (109) with the bounds \( \rho_\star \leq \lambda^* \) and \( \lambda^* \leq \beta \), the final conclusion of Proposition 5 follows.
F Proof of Proposition 2

This section establishes the guarantees of Algorithm 2 and the desired result of Proposition 2. Since Algorithm 2 provides a more general oracle than described in Definition 6, we first give a definition of this oracle. Showing that Algorithm 2 fulfills this broader definition is necessary to analyze the mirror ascent procedure of Appendix G.

Definition 8 (Supergradient oracle in expectation). A function \( \theta_g : \mathbb{R} \rightarrow \mathbb{R} \) is called a \((\epsilon, V)\)-supergradient oracle in expectation for \( g \) (on the interval \([l, u]\)) if, when queried with a \( \lambda \in [l, u] \), it returns an independent random sample \( \theta_g(\lambda) \) satisfying

\[
\min_{z \in \partial g(\lambda)} |E[\theta_g(\lambda)] - z| \leq \epsilon \quad \text{and} \quad E[\theta_g(\lambda)^2] \leq V(\lambda) \tag{110}
\]

for \( \epsilon \geq 0 \) and some function \( V : \mathbb{R} \rightarrow \mathbb{R}_+ \).

Proposition 6. If \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is \( L \)-smooth and \( \rho_s\)-semiconvex (9), then Algorithm 2 implements a \((\epsilon, V)\)-supergradient oracle in expectation (Definition 8) for \( g \) in (55) on the interval \((\rho_s, \infty)\) where

\[
V(\lambda) := \frac{256}{(\lambda - \rho_s)^4} E_{\mu}\left[\|\nabla f(x)\|^4\right] \tag{111}
\]

To prove Proposition 6, the following lemma is required.

Lemma 14. If \( S_0, S_1 = \mathbb{R}^d, c(x, y) = \|x - y\|^2 / 2 \) and \( f \) is differentiable and \( \rho_s\)-semiconvex (9), then the function \( g \) (55) is differentiable on \((\rho_s, \infty)\) and

\[
g'(\lambda) = E_{\mu}\left[\frac{1}{2} \|y_{\lambda, x}^* - x\|^2\right], \quad y_{\lambda, x}^* := \arg\min_{y \in \mathbb{R}^d} f(y) + \frac{\lambda}{2} \|y - x\|^2 \tag{112}
\]

where the unique minimizer \( y_{\lambda, x}^* \) satisfies

\[
\frac{1}{2} \|y_{\lambda, x}^* - x\|^2 \leq \frac{2}{(\lambda - \rho_s)^2} \|\nabla f(x)\|^2 \tag{113}
\]

Additionally, for any \( \rho_s < \lambda_1 \leq \lambda_2 \) one has

\[
\left(1 - 2\sqrt{\frac{\lambda_2 - \lambda_1}{\lambda_2 - \rho_s}}\right) g'(\lambda_1) \leq g'(\lambda_2) \tag{114}
\]

This implies that for any \( t^* > \rho_s, g' \) is \( 1/2 \)-Holder continuous on \([t^*, \infty)\) with a constant depending only on \( t^* \) and \( \rho_s \).

Proof. Define the functions

\[
a_\lambda(y; x) := f(y) + \frac{\lambda}{2} \|y - x\|^2 \quad \text{and} \quad z_\lambda := \inf_{y \in \mathbb{R}} a_\lambda(y; x)
\]

Since \( f \) is \( \rho_s\)-semiconvex (9), \( a_\lambda(y; x) \) is \( \lambda - \rho_s \) strongly convex in \( y \) for \( \lambda > \rho_s \). Therefore, the minimizer \( y_{\lambda, x} \) is unique. Further, semiconvexity and differentiability of \( f \) provide the lower bound

\[
a_\lambda(y; x) \geq f(x) + l_\lambda(y; x) \quad \text{where} \quad l_\lambda(y; x) := \nabla f(x)^T (y - x) + \frac{\lambda - \rho_s}{2} \|y - x\|^2
\]

Noticing \( l_\lambda(y; x) > 0 \) for any \( y \in \mathbb{R}^d \) such that \( \|y - x\| > (2 \|\nabla f(x)\|) / (\lambda - \rho_s) \), one obtains (113).
For open subsets $O \subset (\rho_*, \infty)$ whose closure does not contain $\rho_*$, (113) implies that the radius of the ball containing $y^*_{x,\lambda}$ is uniformly bounded for all $\lambda \in O$. Danskin’s theorem [33] can, therefore, be applied to the function $z_x(\lambda) := f^{\lambda}(x) \ (47)$ to conclude that $z_x(\lambda)$ is differentiable on $(\rho_*, \infty)$ with derivative

$$z'_x(\lambda) = \frac{1}{2} \|y^*_{x,\lambda} - x\|^2$$

Observing that $g(\lambda) = \mathcal{E}_{x \sim \mu}[z_x(\lambda)]$, the conclusion (112) then follows from (113) and dominated convergence.

Finally, let $\rho_* < \lambda_1 \leq \lambda_2$. Since $z_x(\lambda)$ is concave in $\lambda$

$$|z'_x(\lambda_1) - z'_x(\lambda_2)| = z'_x(\lambda_1) - z'_x(\lambda_2)$$

and it is enough to show a one-sided bound on the quantity $z'_x(\lambda_1) - z'_x(\lambda_2)$. To this end, observe

$$z'_x(\lambda_1) - z'_x(\lambda_2) \leq \|y^*_{x,\lambda_1} - x\| \|y^*_{x,\lambda_2} - y^*_{x,\lambda_1}\| \tag{115}$$

Hence, (114) can be provided by producing a bound on $\|y^*_{x,\lambda_2} - y^*_{x,\lambda_1}\|$. Strong convexity of $a_\lambda(y; x)$ in $y$ yields the identity

$$a_{\lambda_2}(y^*_{x,\lambda_2}; x) + \frac{\lambda_2 - \rho_*}{2} \|y^*_{x,\lambda_2} - y^*_{x,\lambda_1}\|^2 \leq a_{\lambda_2}(y^*_{x,\lambda_1}; x) = a_{\lambda_1}(y^*_{x,\lambda_1}; x) + \frac{\lambda_2 - \lambda_1}{2} \|y^*_{x,\lambda_1} - x\|^2$$

which, when combined with the fact that $a_{\lambda_1}(y^*_{x,\lambda_1}; x) \leq a_{\lambda_2}(y^*_{x,\lambda_2}; x)$ ($z_x(\lambda)$ is non-decreasing in $\lambda$), gives

$$\|y^*_{x,\lambda_2} - y^*_{x,\lambda_1}\| \leq \sqrt{\frac{\lambda_2 - \lambda_1}{\lambda_2 - \rho_*}} \|y^*_{x,\lambda_1} - x\| \tag{116}$$

Applying (116) to (115) and rearranging produces

$$\left(1 - 2 \sqrt{\frac{\lambda_2 - \lambda_1}{\lambda_2 - \rho_*}}\right) z'_x(\lambda_1) \leq z'_x(\lambda_2) \tag{117}$$

Taking the expectation with respect to $x$ on both sides of (117) yields (114). \hfill \square

Proof of Proposition 6. Consider the sample $x$ which is computed by Algorithm 2. In light of Lemma 14, it is clear that

$$\theta^* := \frac{1}{2} \|y^*_{x,\lambda} - x\|^2$$

is an unbiased estimate of $g'(\lambda)$. Hence, to prove the conclusion of Proposition 6, it need only be shown that the output of Algorithm 2, $\theta$, satisfies

$$|\theta - \theta^*| \leq \epsilon \quad \text{and} \quad \theta \leq \left(\frac{4 \|\nabla f(x)\|}{\lambda - \rho_*}\right)^2 \tag{118}$$

when $\lambda \in (\rho_*, \infty)$.

To this end, notice that Algorithm 2 performs Nesterov’s accelerated gradient descent [51] on the $\lambda - \rho_*$-strongly convex and $\lambda + L$-smooth function $a_\lambda(y; x)$. Strong convexity yields the identity

$$\frac{\lambda - \rho_*}{2} \|y^*_{x,\lambda} - y\|^2 \leq a_\lambda(y; x) - a_\lambda(y^*_{x,\lambda}; x) \tag{119}$$
while the convergence guarantees of accelerated gradient descent [51, Theorem 2.2.3] give
\[
a_{\lambda}(y_k; x) - a_{\lambda}(y_{\lambda,x}^*; x) \leq (1 - \kappa)^k (\lambda + L) \|y_{\lambda,x}^* - x\|^2 \tag{120}
\]
for \( \kappa = \sqrt{(\lambda + L)/(\lambda - \rho_x)} \). Combining these relations and setting \( C = 2\|\nabla f(x)\|/(\lambda - \rho_x) \)
\[
\|y_{\lambda,x}^* - y_k\|^2 \leq 2 \frac{\left( a_{\lambda}(y_k; x) - a_{\lambda}(y_{\lambda,x}^*; x) \right)}{\lambda - \rho_x} \leq 2 (1 - \kappa)^k \kappa^2 \|y_{\lambda,x}^* - x\|^2 \leq 2 \left( \frac{\epsilon}{6C} \right)^2 \tag{121}
\]
since \( k \geq 4\kappa \log (6\kappa C/\epsilon) \) and \( \|y_{\lambda,x}^* - x\| \leq C \) via (113). Completing the analysis,
\[
|\theta - \theta^*| = \frac{1}{2} \left| \|y_k - x\|^2 - \|y_{\lambda,x}^* - x\|^2 \right| \leq \frac{1}{2} \|y_k - y_{\lambda,x}^*\| \left( \|y_k - x\| + \|y_{\lambda,x}^* - x\| \right) \tag{122}
\]
\[
\leq \frac{3}{2} \|y_k - y_{\lambda,x}^*\| \|y_{\lambda,x}^* - x\| \leq \epsilon \tag{123}
\]
where triangle inequality provides both (122) and
\[
\|y_k - x\| \leq 2 \|y_{\lambda,x}^* - x\| \leq 2C \tag{124}
\]
Moreover, (123) is the desired left-hand inequality of (118) while (124) contains the desired right-hand inequality– this completes the proof.

With the guarantee on Algorithm 2 established by Proposition 6, Proposition 2 becomes an immediate corollary.

Proof of Proposition 2. This is a straightforward consequence of Chebyshev’s inequality. Indeed, the proof of Proposition 6 shows that the output \( \theta \) of Algorithm 2 satisfies
\[
|\mathbb{E} [\theta] - g'(\lambda)| \leq \frac{\tilde{\epsilon}}{2 \max(\lambda - \rho_x, 1)} \quad \text{and} \quad \theta \leq \frac{16}{(\lambda - \rho_x)^2} \|\nabla f(x)\|^2 \tag{125}
\]
when \( \epsilon = \tilde{\epsilon}/(2 \max(\lambda - \rho_x, 1)) \). Letting \( \tilde{\theta} \) be the average of \( K \) independent calls to Algorithm 2, Chebyshev’s inequality gives
\[
\mathbb{P} \left( |\theta - \mathbb{E} [\theta]| \geq \frac{\tilde{\epsilon}}{2 \max(\lambda - \rho_x, 1)} \right) \leq \frac{64 \mathbb{E}_\mu \left[ \|\nabla f(x)\|^4 \right]}{(\lambda - \rho_x)^2 \min \left( (\lambda - \rho_x)^2, 1 \right) \tilde{\epsilon}^2 K} \leq \delta \tag{126}
\]

G Stochastic mirror ascent for (55).

For completeness with respect to previous approaches [49, 29], this section provides an analysis of mirror ascent in the context of (55) and (57). The main result of this analysis is: usage of stochastic mirror ascent, under slightly weaker assumptions than those used to obtain Theorem 4, provides an output whose expected objective value (over the randomness of the algorithm) is nearly optimal. Further, the computational complexity of this procedure has better dependence on the smoothness of the objective function– compare (66) to (135). The sacrifice is that only an estimate
of the optimal value of the dual (55) is produced. The output of the algorithm does not provide a primal-feasible distribution with guarantees on the primal-dual gap (64).

**Algorithm 4** Stochastic Mirror Ascent for (55)

**Input:** Supergradient oracle $\theta_g$, initial iterate $\lambda_1$, step-size $\alpha$, number of iterations $k$

**for** $1 \leq i \leq k$ **do**

Sample $\eta_i \leftarrow \theta_g(\lambda_i)$

$\xi_i \leftarrow \text{Proj}_{\partial\psi^*(\lambda_i)}(\eta_i)$

For $\lambda_{i+1} \leftarrow \text{Proj}_{[l,u]}\left(\lambda_i + \frac{\alpha(u-l)}{\sqrt{2k}} (\eta_i - \xi_i)\right)$

**return** $\lambda^* = \frac{1}{k}\sum_{i=1}^{k} \lambda_i$

**Proposition 7** (Convergence of Algorithm 4). For the problem (55), let $\theta_g$ be a $(\epsilon, V)$-supergradient oracle in expectation (Definition 8) for $g$ on $[l, u]$. If $\sup_{\lambda \in [l, u]} |\partial_+ \psi^*(\lambda)| \leq D$ (where $\partial_+$ denotes the right-derivative), $\sup_{\lambda \in [l, u]} V(\lambda) \leq C^2$, and $\alpha = 1/\sqrt{(C^2 + D^2)}$ then

$$
\sup_{\lambda \in [l, u]} g(\lambda) - \psi^*(\lambda) - E[g(\lambda^*) - \psi^*(\lambda^*)] \leq (u - l) \left(\frac{2(C^2 + D^2)}{k} + \epsilon\right)
$$

where $\lambda^*$ is the output of Algorithm 4 and the expectation is taken with respect to the randomness of the oracle $\theta_g$.

**Proof.** Let $\lambda_i$ be the $i$th iterate computed by Algorithm 4 and let $\eta_i$ and $\xi_i$ be the corresponding, computed supergradient and subgradients for $g$ and $\psi^*$. By construction, $\lambda_{i+1}$ solves

$$
\lambda_{i+1} = \arg\max_{\lambda \in [l, u]} \alpha_k \gamma_i (\lambda - \lambda_i) - \frac{1}{2} (\lambda - \lambda_i)^2
$$

where $\alpha_k = \alpha(u-l)/\sqrt{2k}$ and $\gamma_i = \eta_i - \xi_i$. From first-order optimality condition

$$(\alpha_k \gamma_i - \lambda_{i+1} - \lambda_i) (\lambda - \lambda_{i+1}) \leq 0 \quad \forall \lambda \in [l, u]$$

one obtains

$$
\alpha_k \gamma_i (\lambda - \lambda_{i+1}) \leq \frac{1}{2} (\lambda - \lambda_i)^2 - \frac{1}{2} (\lambda - \lambda_{i+1})^2 - \frac{1}{2} (\lambda_{i+1} - \lambda_i)^2
$$

for any fixed $\lambda \in [l, u]$. Adding $\alpha_k \gamma_i (\lambda_{i+1} - \lambda_i)$ to both sides of (128) and applying Young’s inequality on the right provides the relation

$$
\alpha_k \gamma_i (\lambda - \lambda_i) \leq \frac{(\alpha_k \gamma_i)^2}{2} + \frac{1}{2} (\lambda - \lambda_i)^2 - \frac{1}{2} (\lambda - \lambda_{i+1})^2
$$

The equation (129) can then be summed over $i \leq k$ to give

$$
\sum_{i=1}^{k} \gamma_i (\lambda - \lambda_i) \leq \alpha_k \sum_{i=1}^{k} \frac{\gamma_i^2}{2} + \frac{(\lambda - \lambda_1)^2}{2\alpha_k}
$$

Essentially, what has been obtained is an upper bound on the quantities $\gamma_i (\lambda - \lambda_i)$. These quantities, themselves, roughly upper bound the difference between the objective value (55) at $\lambda$ and the value
at $\lambda_i$. Taking expectations (with respect to the randomness of the oracle $\theta_g$) on both sides of (130),

$$
\mathbb{E} \left[ \sum_{i=1}^{k} (\eta_i - \xi_i)(\lambda - \lambda_i) \right] \leq \alpha_k \sum_{i=1}^{k} \mathbb{E} \left[ \gamma_i^2 \right] / 2 + \frac{(\lambda - \lambda_1)^2}{2\alpha_k}
$$

$$
\leq k\alpha_k \left( C^2 + D^2 \right) + \frac{(\lambda - \lambda_1)^2}{2\alpha_k}
$$

(131)

where (110), $\sup_{\lambda \in [l,u]} V(\lambda) \leq C^2$ and $\sup_{\lambda \in [l,u]} \partial_+ \psi^*(\lambda) \leq D$ were used.

Notice that (110) implies there exists a $z^* \in \partial g(\lambda_i)$ such that $|\mathbb{E} [\eta_i | \eta_j, j < i] - z^*| \leq \epsilon$. Since $\xi_i = \text{Proj}_{\partial \psi^*(\lambda_i)}(\eta_i)$, one has $\mathbb{E} \left[ \xi_i | \eta_j, j < i \right] \in \partial \psi^*(\lambda_i)$ and this gives

$$
\mathbb{E} \left[ (\eta_i - \xi_i)(\lambda - \lambda_i) | \eta_j, j < i \right] = (\mathbb{E} \left[ \eta_i | \eta_j, j < i \right] - \mathbb{E} \left[ \xi_i | \eta_j, j < i \right]) (\lambda - \lambda_i)
$$

$$
\geq (z^* - \mathbb{E} \left[ \xi_i | \eta_j, j < i \right]) (\lambda - \lambda_i) - \epsilon (u - l)
$$

$$
\geq g(\lambda) - \psi^*(\lambda) - (g(\lambda_i) - \psi^*(\lambda_i)) - \epsilon (u - l)
$$

(133)

where (132) is a result of the fact that $\lambda_i$ depends only on $\eta_j$ for $j < i$ and (133) follows from the concavity of the objective $\lambda \mapsto g(\lambda) - \psi^*(\lambda)$.

Applying the relation (133) to (131),

$$
\mathbb{E} \left[ \sum_{i=1}^{k} g(\lambda) - \psi^*(\lambda) - (g(\lambda_i) - \psi^*(\lambda_i)) \right] \leq k\alpha_k \left( C^2 + D^2 \right) + \frac{(\lambda - \lambda_1)^2}{2\alpha_k} + \epsilon (u - l)
$$

Dividing both sides by $1/k$ and substituting $\alpha_k = (u - l) / \sqrt{2k(C^2 + D^2)}$, one obtains

$$
g(\lambda) - \psi^*(\lambda) - \mathbb{E} \left[ \frac{1}{k} \sum_{i=1}^{k} (g(\lambda_i) - \psi^*(\lambda_i)) \right] \leq (u - l) \left( \sqrt{2\frac{(C^2 + D^2)}{k}} + \epsilon \right)
$$

(134)

Since $\lambda \in [l,u]$ is arbitrary and the output of Algorithm 4 is given by $\lambda^* = \left( \sum_{i=1}^{k} \lambda_i \right) / k$, the desired result (127) follows from (134) and concavity of $\lambda \mapsto g(\lambda) - \psi^*(\lambda)$.

\begin{proposition}
Let $f$ be $L$-smooth (10), $\rho_*$-semiconvex, and assume that $\psi$ provides $C/\mathbb{E}_\mu \left[ \|\nabla f(x)\| \right]^2$-regularization (60) at $\rho_* + 1$ for $C \geq 8L^2$. If $\sup_{\lambda \in [l,u]} |\partial_+ \psi^*(\lambda)| \leq D$ for $l$ and $u$ given by (65), then there exists an stochastic algorithm which returns a $\lambda^*$ such that, recall (55),

$$
\sup_{\lambda \in \mathbb{R}} g(\lambda) - \psi^*(\lambda) - \mathbb{E} [g(\lambda^*) - \psi^*(\lambda^*)] \leq \epsilon
$$

where the expectation is taken with respect to the randomness of the algorithm. This algorithm draws at most

$$
O \left( C \max \left( \mathbb{E}_\mu \left[ \|\nabla f(x)\| \right]^4, D^2 \right) \right)
$$

independent samples from $\mu$ and performs $\widetilde{O} \left( L^{1/2} C \max \left( \mathbb{E}_\mu \left[ \|\nabla f(x)\| \right]^4, D^2 \right) / \epsilon^2 \right)$ expected gradient evaluations of $f$ – where $\widetilde{O}$ suppresses logarithmic factors in $L, C, \mathbb{E}_\mu \left[ \|\nabla f(x)\| \right]$ and $\epsilon$.

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Proof. By Proposition 5, it is enough for an algorithm to return a value \( \lambda^* \) which satisfies

\[
\sup_{\lambda \in [l, u]} g(\lambda) - \psi^*(\lambda) - \mathbb{E}[g(\lambda^*) - \psi^*(\lambda^*)] \leq \frac{\epsilon}{2}
\]  

(136)

for \( l \) and \( u \) given by (65). Without loss of generality, it will be assumed that \( u - l > 0 \).

Apply Algorithm 4 to the interval \([l, u]\) with the supergradient oracle given by Algorithm 2; where the error tolerance in Algorithm 2 is set to \( \epsilon/(4(u-l)) \). For all \( \lambda \in [l, u] \), the variance bound (111) gives

\[
V(\lambda) \leq 256 \mathbb{E}_\mu \left[ \|\nabla f(x)\|^4 \right]
\]

since \( \min_{\lambda \in [l, u]} (\lambda - \rho_*) \geq 1 \). Thus, by Proposition 7, running Algorithm 4 for

\[
k = \left\lceil \frac{32(u-l)^2}{\epsilon^2} \left( 256 \mathbb{E}_\mu \left[ \|\nabla f(x)\|^4 \right] + D^2 \right) \right\rceil
\]

iterations will produce a \( \lambda^* \) satisfying (136). Further, as each iteration of Algorithm 4 executes a single call to the supergradient oracle provided by Algorithm 2, it is clear that

\[
O \left( \frac{C}{\epsilon^2} \max \left( \mathbb{E}_\mu \left[ \|\nabla f(x)\|^4 \right], D^2 \right) \right)
\]

samples are drawn from \( \mu \).

To compute a bound on the expected number of gradient evaluations of \( f \) that are performed, observe that each of the \( k \) calls to Algorithm 2 (with error tolerance \( \epsilon/(4(u-l)) \)) executes at most

\[
t = \max \left( \left\lceil 4\kappa \log \left( \frac{48\kappa \|\nabla f(x)\|(u-l)}{\epsilon} \right) \right\rceil, 0 \right)
\]

(137)

gradient evaluations of \( f; x \) and \( \kappa \) are the random sample and condition number, respectively, which are used in Algorithm 2. Both \( x \) and \( \kappa \) are random variables, but \( \kappa = ((\lambda + L)/(\lambda - \rho_*))^{1/2} \leq (1 + 2L)^{1/2} \) and (due to Jensen)

\[
\mathbb{E} \left[ \max (\log(z), 0) \right] \leq \log \mathbb{E} \left[ \max (z, 1) \right]
\]

for any non-negative random variable \( z \). Hence, the expected number of gradient evaluations performed by Algorithm 2 obeys the bound

\[
\mathbb{E}_\mu \left[ t \right] \leq 4(1 + 2L)^{1/2} \log \mathbb{E}_\mu \left[ \left( \max \left( \frac{48(1 + 2L)^{1/2} \|\nabla f(x)\|(u-l)}{\epsilon}, 1 \right) \right) \right]
\]

(138)

Summing over the \( k \) calls to Algorithm 2 and using the identity \( u \leq l + \sqrt{2C} \leq \rho_* + 1 + (2C)^{1/2} \), one obtains

\[
\mathbb{E}_\mu \left[ t \right] \leq \tilde{O} \left( \frac{(1 + 2L)^{1/2}C}{\epsilon^2} \max \left( \mathbb{E}_\mu \left[ \|\nabla f(x)\|^4 \right], D^2 \right) \right)
\]

where \( \tilde{O} \) suppresses logarithmic factors in \( L, C, \mathbb{E}_\mu \left[ \|\nabla f(x)\|^4 \right] \) and \( \epsilon \). \( \square \)
This section provides an analysis of a stochastic bisection procedure for (55) under slightly weaker assumptions than those used to obtain Theorem 4. This demonstrates that, if only estimation of the value of (55) is required, then slightly modified version of Algorithm 3 provides a computational complexities with better dependence on the smoothness of the objective–compare (66) with (146).

**Algorithm 5 Stochastic bisection**

**Input:** Supergradient oracle \( \theta_g \), error tolerance \( \epsilon \), termination width \( B \)

\[
\eta \leftarrow \infty, \ b \leftarrow l
\]

**while** \(|\eta| > \epsilon / \max(\lambda - b, 1)\) and \(u - l > \epsilon / B\) do

\[
\lambda \leftarrow (l + u) / 2
\]

\[
\eta \leftarrow \theta_g(\lambda), \ \eta \leftarrow \left( \eta - \text{Proj}_{\partial\psi^*(\lambda)}(\eta) \right)
\]

**if** \( \eta > 0 \) **then** \( l \leftarrow \lambda \)

**else** \( u \leftarrow \lambda \)

**return** \( \lambda \)

**Proposition 9** (Convergence of Algorithm 5). For the problem (55), let \( \theta_g \) be a \((\epsilon, \delta)\)-supergradient oracle with high probability for \( g \) on \([l, u]\). If \( \sup_{\lambda \in [l, u]} |\partial_- g(\lambda)| + |\partial_+ \psi^*(\lambda)| \leq B \) (where \( \partial_- \) and \( \partial_+ \) denote the left derivative and right derivatives respectively) then the output \( \lambda^* \) of Algorithm 5 satisfies

\[
\sup_{\lambda \in [l, u]} g(\lambda) - \psi^*(\lambda) - g(\lambda^*) - \psi^*(\lambda^*) \leq 2\epsilon
\]

with probability at least \( 1 - \delta (\log_2 (B(u - l)/\epsilon) + 1) \).

**Proof.** Let \( \lambda_i, u_i, l_i \) and \( \eta_i \) denote the \( i \)th values of \( \lambda, u, l \) and \( \eta \) which are computed by Algorithm 5–the indexes \( l_0, u_0 \) denote the initial values of these variables. Let \( k \) denote the total number of iterations performed by the loop of Algorithm 5. Since \( u_i - l_i = (u_{i-1} - l_{i-1}) / 2 \), it is clear that \( k \leq \log_2 (B(u_0 - l_0)/\epsilon) + 1 \). Thus, using (58) and the fact that \( \lambda_i \) depends only on \( \theta_g(\lambda_j) \) for \( j < i \), one obtains the union bound

\[
\mathbb{P} \left( \bigcup_{i \leq k} \left[ \min_{z \in \partial\psi^*(\lambda_i)} |\theta_g(\lambda_i) - z| \geq \frac{\epsilon}{\max(\lambda_i - l_0, 1)} \right] \right) \leq \delta (\log_2 (B(u_0 - l_0)/\epsilon) + 1)
\]

Hence, it need only be shown that (139) holds when

\[
\min_{z \in \partial\psi^*(\lambda_i)} |\theta_g(\lambda_i) - z| \leq \frac{\epsilon}{\max(\lambda_i - l_0, 1)} \quad \forall i \leq k
\]

For brevity, set \( \epsilon_{\lambda_i} = \epsilon / \max(\lambda_i - l_0, 1) \) and let \( z_i^* = \text{Proj}_{\partial\psi^*(\lambda_i)}(\theta_g(\lambda_i)) \). Recall \( \eta_i = \theta_g(\lambda_i) - \text{Proj}_{\partial\psi^*(\lambda_i)}(\theta_g(\lambda_i)) \) and define \( \eta_i^* := z_i^* - \text{Proj}_{\partial\psi^*(\lambda_i)}(z_i^*) \) to be the true supergradient of (55) which \( \eta_i \) approximates. From (141)

\[
\eta_i \eta_i^* \leq 0 \Rightarrow \max(|\eta_i|, |\eta_i^*|) \leq \epsilon_{\lambda_i}
\]

Hence, at all iterations prior to the last iteration (iteration \( k \)) of the loop in Algorithm 5, \( \eta_i \) and \( \eta_i^* \) have the same sign. Since \( \lambda \mapsto (g(\lambda) - \psi^*(\lambda)) \) is concave, this gives

\[
\sup_{\lambda \in [l_i, u_i]} g(\lambda) - \psi^*(\lambda) = \sup_{\lambda \in [l_{i-1}, u_{i-1}]} g(\lambda) - \psi^*(\lambda)
\]
for all \(1 < i < k\). Additionally, if \(\eta_k \eta_k^* > 0\) then (143) also holds for \(i = k\).

Now, at iteration \(k\), either \(|\eta_k^*| \leq 2\epsilon_{\lambda_k}\) or \(|\eta_k^*| > 2\epsilon_{\lambda_k}\). If \(|\eta_k^*| \leq 2\epsilon_{\lambda_k}\), then

\[
\sup_{\lambda \in [l_u, u]} g(\lambda) - \psi^*(\lambda) - (g(\lambda_k) - \psi^*(\lambda_k)) \leq \sup_{\lambda \in [l_u, u]} \eta_k^*(\lambda - \lambda_k) \leq \epsilon_{\lambda_k}(u_{k-1} - l_{k-1}) \leq 2\epsilon
\]  

(144)

where the first inequality of (144) is a result of (143) and concavity and the second inequality follows from the observation \(u_{k-1} - l_{k-1} \leq 2(\lambda_k - l_0)\). Observe that (144) immediately gives the desired result (139).

To show (139) when \(|\eta_k^*| > 2\epsilon_{\lambda_k}\), notice that (141) implies \(|\eta_k^*| - |\eta_k| \leq \epsilon_{\lambda_k}\). Hence, \(|\eta_k^*| > 2\epsilon_{\lambda_k}\) entails that the second termination condition \((u_k - l_k \leq \epsilon/B)\) of Algorithm 5 was reached and \(\eta_k \eta_k^* \geq 0\). Then, by (143),

\[
\sup_{\lambda \in [l_u, u]} g(\lambda) - \psi^*(\lambda) - (g(\lambda_k) - \psi^*(\lambda_k)) \leq \sup_{\lambda \in [l_u, u]} \eta_k^*(\lambda - \lambda_k) \leq \epsilon
\]  

(145)

where the second inequality is a consequence of \(|\eta_k^*| \leq \sup_{\lambda \in [l_u, u]} |\partial_- g(\lambda)| + |\partial_+ \psi^*(\lambda)| \leq B\). The desired result (139) then follows.

\[\square\]

**Proposition 10.** Let \(f\) be \(L\)-smooth (10), \(\rho_\ast\)-semiconvex, and assume that \(\psi\) provides \(C/\mathbb{E}_\mu \left[ \left\| \nabla f(x) \right\|^2 \right]\) - regularization (60) at \(\rho_\ast + 1\) for \(C \geq 8L^2\). If \(\sup_{\lambda \in I} |\partial_+ \psi^*(\lambda)| \leq D\) for \(l\) and \(u\) given by (65) then, for any \(\delta > 0\), there exists an stochastic algorithm which returns a \(\lambda^\ast\) such that, recall (55),

\[
\sup_{\lambda \in \mathbb{R}} g(\lambda) - \psi^*(\lambda) - (g(\lambda^\ast) - \psi^*(\lambda^\ast)) \leq \epsilon
\]

with probability \(1 - \delta\). Moreover, this algorithm draws at most

\[
\tilde{O} \left( \frac{\mathbb{E}_\mu \left[ \left\| \nabla f(x) \right\|^4 \right]}{\delta \epsilon^2} \right)
\]  

(146)

independent samples from \(\mu\) and performs \(\tilde{O} \left( L^{1/2} \mathbb{E}_\mu \left[ \left\| \nabla f(x) \right\|^4 \right] / (\delta \epsilon^2) \right)\) expected gradient evaluations of \(f\) – where \(\tilde{O}\) suppresses logarithmic factors in \(L, C, D, \mathbb{E}_\mu \left[ \left\| \nabla f(x) \right\|^2 \right]\) and \(\epsilon\).

**Proof.** Similarly to the proof of Proposition 8, it is enough for an algorithm to return a value \(\lambda^\ast\) which (with probability \(1 - \delta\)) satisfies

\[
\sup_{\lambda \in [l, u]} g(\lambda) - \psi^*(\lambda) - (g(\lambda^\ast) - \psi^*(\lambda^\ast)) \leq \frac{\epsilon}{2}
\]  

(147)

where \(l\) and \(u\) are given by (65). Again, without loss of generality, it will be assumed that \(u - l > 0\).

Apply Algorithm 5 to the interval \([l, u]\) with the supergradient oracle given by Proposition 2; where the error tolerance in Proposition 2 is set to \(\epsilon/4\) and the error probability in is set to \(\delta/(\log_2 (4B(u - l)/\epsilon) + 1)\) for

\[
B := 2\mathbb{E}_\mu \left[ \left\| \nabla f(x) \right\|^2 \right] + D
\]

By Lemma 14 and the fact that \(\psi^*\) is non-decreasing, one has

\[
\sup_{\lambda \in [l, u]} |\partial_- g(\lambda)| + |\partial_- \psi^*(\lambda)| \leq B
\]
Consequently, Proposition 9 guarantees that the value $\lambda^*$ returned by Algorithm 5 satisfies (147) with probability $1 - \delta$.

To compute a bound on the number of samples from $\mu$ which are drawn under this procedure note that, by definition of Algorithm 5, at most $\lceil \log_2 (4B(u - l)/\epsilon) \rceil$ calls are made to the supergradient oracle given by Proposition 2. Via (59), it follows that at most

$$\frac{(32 \log_2 (4B(u - l)/\epsilon) + 1)^2 \mathbb{E}_\mu \|\nabla f(x)\|^4}{\delta \epsilon^2}$$

(148)

invocations of Algorithm 2 are performed with an error parameter which is at least $\epsilon/(8 \max(u - \rho_s, 1))$. Since each invocation of Algorithm 2 requires a single sample of $\mu$, the above procedure therefore draws

$$\widetilde{O} \left( \frac{\mathbb{E}_\mu \|\nabla f(x)\|^4}{\delta \epsilon^2} \right)$$

independent samples from $\mu$ where $\widetilde{O}$ suppresses logarithmic factors in $C, D, \mathbb{E}_\mu \|\nabla f(x)\|^2$ and $\epsilon$.

Finally, since the expected number of gradient evaluations of $f$ performed by Algorithm 2 obeys the bound (138) for an error parameter of $\epsilon$, the expected number of gradient evaluations of $f$ used by Algorithm 2 with an error parameter of at least $\epsilon/(8 \max(u - \rho_s, 1))$ is $\widetilde{O} \left( L^{1/2} \right)$. In combination with (148), it follows that

$$\widetilde{O} \left( \frac{L^{1/2} \mathbb{E}_\mu \|\nabla f(x)\|^4}{\delta \epsilon^2} \right)$$

gradient evaluations of $f$ are performed in expectation. \(\square\)

I Proof of Theorem 4 and Corollary 5

Lemma 15. Under the assumptions of Theorem 4, let $M$ be the smoothness parameter of $\psi^*$ and let $\theta_g$ be a $(\epsilon, \delta)$-supergradient oracle with high probability for $g$ on the interval $[l, u]$. If $1 \leq l - \rho_s$, $g'(u) - (\psi^*)'(u) \leq 0 \leq g'(l) - (\psi^*)'(l)$, and Algorithm 3 is run with $B = \max \{M, 4(g'(l))^2\}$, the output $\lambda^*$ of satisfies (recalling (62))

$$\int f \, d\pi_{\lambda^*, \mu} + \psi \left( \int \frac{1}{2} \|y - x\|^2 \, d\pi_{\lambda^*, \mu} \right) - (g(\lambda^*) - \psi^*(\lambda^*)) \leq (4 + l)\epsilon$$

(149)

with probability at least $1 - \delta (\log_2 (B(u - l)/\epsilon) + 1)$.

Proof of Lemma 15. Notice that $g'$ and $\psi'$ can be written since $g'$ exists on the interval $[l, u]$ by Lemma 14 and $\psi'$ exists under the assumptions of Theorem 4. The proof proceeds in similar style to the proof of Proposition 9 in Appendix H. Indeed, as before, let $\lambda_i, u_i, l_i$ and $\eta_i$ denote the $i$th values of $\lambda, u, l$ and $\eta$ that are computed – where the index $0$ denotes the initial value of the variable. The natural number $k$ denotes the total number of iterations performed by the loop of Algorithm 3 and clearly $k \leq \log_2 (B(u_0 - l_0)/\epsilon) + 1$. Thus, the same union bound argument (140) implies it is sufficient to show (149) when

$$|\theta_g(\lambda_i) - g'(\lambda_i)| \leq \frac{\epsilon}{\max (\lambda_i - l_0, 1)} \quad \forall i \leq k$$

(150)
to guarantee that (149) occurs with probability at least \(1 - \delta (\log_2 (B(u_0 - l_0)/\epsilon) + 1).\)

For brevity, denote \(\epsilon_{\lambda_i} = \epsilon / \max(\lambda_i - l_0, 1)\) and let \(\eta^*_i := g'(\lambda_i) - (\psi^*)'(\lambda_i)\) be the true gradient of (55) which \(\eta_i\) approximates. As before, (150) provides

\[
\eta_i \eta^*_i \leq 0 \implies \max(|\eta_i|, |\eta^*_i|) \leq \epsilon_{\lambda_i} \tag{151}
\]

which will produce the desired guarantee (149) on \(u_k = \lambda^*\).

Indeed, from Algorithm 3, it is clear that either \(u_k = \lambda_i\) for some \(i > 0\) such that \(\eta_i < -\epsilon_{\lambda_i}\) or \(u_k = u_0\). Similarly, \(l_k = \lambda_j\) for some \(j > 0\) such that \(\eta_j > -\epsilon_{\lambda_j}\) or \(l_k = l_0\). One can assume, without loss of generality, that \(l_k = \lambda_j\) for some \(j > 0\) and, in combination with (151) and \(g'(u_0) - \psi'(u_0) \leq 0 \leq g'(l_0) - \psi'(l_0)\), this gives

\[
-\epsilon_{\lambda_j} \leq g'(l_k) - \psi'(l_k) \quad \text{and} \quad g'(u_k) - \psi'(u_k) \leq 0 \tag{152}
\]

Moreover, since \(0 \in \text{Dom}(\psi)\), the inequality \(g'(u_k) - (\psi^*)'(u_k) \leq 0\) implies \(\int 1/2 \|y - x\|^2 d\pi_{u_k,\mu} = g'(u_k) \in \text{Dom}(\psi)\). Hence, denoting the primal-dual gap,

\[
T := \int f d\pi_{u_k,\mu} + \psi \left( \int 1/2 \|y - x\|^2 d\pi_{u_k,\mu} \right) - (g(u_k) - \psi^*(u_k))
\]

Lemma 12 yields

\[
T \leq (u_k - \lambda^*) \left( (\psi^*)'(u_k) - g'(u_k) \right)
\]

for any \(\lambda^* \in \partial \psi \left( \int 1/2 \|y - x\|^2 d\pi_{u_k,\mu} \right)\). If \(\psi\) is minimized at 0 then \(\lambda^* \geq 0\) and

\[
T \leq u_k \left( (\psi^*)'(u_k) - g'(u_k) \right) \tag{153}
\]

Using (114) and that \((\psi^*)'\) is \(M\)-Lipschitz on \([l_0, u_0]\), one obtains

\[
T \leq u_k \left( (\psi^*)'(u_k) - g'(u_k) \right)
\]

\[
\leq u_k \left( (\psi^*)'(l_k) + M(u_k - l_k) - \left( 1 - \frac{2}{\sqrt{u_k - \rho_*}} (u_k - l_k)^{1/2} \right) g'(l_k) \right)
\]

\[
\leq u_k \epsilon_{\lambda_j} + u_k \left( M(u_k - l_k) + \frac{2}{\sqrt{u_k - \rho_*}} (u_k - l_k)^{1/2} g'(l_k) \right) \tag{154}
\]

where the last inequality also used (152). To bound the first term on the left side of (154), observe that \(l_k \neq l_0\) implies there exists a minimal \(t > 0\) such that \(l_t \neq l_0\). Clearly,

\[
u_k \epsilon_{\lambda_j} = \frac{u_k \epsilon}{\max(\lambda_j - l_0, 1)} \leq \frac{u_k \epsilon}{\max(\lambda_t - l_0, 1)} \leq \epsilon \frac{u_t - l_0}{\max(\lambda_t - l_0, 1)} + l_0 \epsilon \leq (2 + l_0) \epsilon
\]

Combining this with the termination condition

\[
u_k - l_k \leq \frac{\epsilon}{B} \leq \frac{\epsilon}{\max(M, 4g(l_0)^2)} \leq \frac{\epsilon}{\max(M, 4g(l_0)^2/(u_k - \rho_*))}
\]

to bound the second term of (154) and one obtains the desired result

\[
T \leq (4 + l_0) \epsilon
\]
Proof of Theorem 4. Let $l = \rho_s + 1$ and $u = \rho_s + 1 + \sqrt{2C}$ and apply Algorithm 3 to the interval $[l, u]$ with the supergradient oracle given by Proposition 2. Set the error tolerance used by Algorithm 3 to $\epsilon/(4 + \rho_s + 1)$ and the termination width to

$$B := \max \left( M, 16 \left( \mathbb{E}_\mu \left[ \|\nabla f(x)\|^2 \right] \right)^2 \right)$$

Likewise, the error tolerance used in Proposition 2 should be $\epsilon/(4 + \rho_s + 1)$ and the error probability should be $\delta/(\log_2 (B(u - l) (4 + \rho_s + 1)/\epsilon) + 1)$.

Under this setting of parameters, Lemma 15 establishes that the output of $\lambda^*$ of Algorithm 3 satisfies

$$\int f d\pi_{\lambda^*, \mu} + \psi \left( \int \frac{1}{2} \|y - x\|^2 d\pi_{\lambda^*, \mu} \right) - (g(\lambda^*) - \psi^*(\lambda^*)) \leq \epsilon$$

with probability $\delta$ so long as

$$g'(u) - (\psi^*)'(u) \leq 0 \leq g'(l) - (\psi^*)'(l)$$

To see that (156) is fulfilled for the chosen $l$ and $u$, notice that, by Assumption 4, (100) holds for $C \geq 8(\rho_s + 1)^2$. Hence, (108) gives

$$g'(l) - (\psi^*)'(l) \geq 0$$

Similarly, (100) combined with (113) provides

$$g'(u) - (\psi^*)'(u) \leq \frac{2}{(u - \rho_s)^2} \mathbb{E}_\mu \left[ \|\nabla f(x)\|^2 \right] - (\psi^*)'(u) \leq \frac{1}{C} \mathbb{E}_\mu \left[ \|\nabla f(x)\|^2 \right] - (\psi^*)'(l) \leq 0$$

Hence, (156) holds and the output $\lambda^*$ of Algorithm 3 obeys (155) with probability $\delta$.

It remains to compute a bound on the number of samples from $\mu$ which are required by this procedure. Clearly, by the definition of Algorithm 3, at most $\lceil \log_2 (B(u - l) (4 + \rho_s + 1)/\epsilon) \rceil$ calls are made to the supergradient oracle given by Proposition 2. Via (59), this yields that at most

$$\frac{8(4 + \rho_s + 1)(\log_2 (B(u - l) (4 + \rho_s + 1)/\epsilon) + 1)^2 \mathbb{E}_\mu \left[ \|\nabla f(x)\|^4 \right]}{\delta \epsilon^2}$$

invocations of Algorithm 2 are performed with an error parameter which is at least $\epsilon/(2(4 + \rho_s + 1)(\max(u - \rho_s, 1)))$. Since each invocation of Algorithm 2 requires a single sample of $\mu$, it follows from (157) that

$$\tilde{O} \left( \frac{\rho_s^2 \mathbb{E}_\mu \left[ \|\nabla f(x)\|^4 \right]}{\delta \epsilon^2} \right)$$

samples are used by Algorithm 3—where $\tilde{O}$ suppresses logarithmic factors in $\rho_s, C, M, \mathbb{E}_\mu \left[ \|\nabla f(x)\|^2 \right]$, and $\epsilon$.

Finally, as the expected number of gradient evaluations performed by a call to Algorithm 2 obeys (138) for an error parameter of $\epsilon$, the expected number of gradient evaluations executed by each call to Algorithm 2 is at most $\tilde{O} \left( L^{1/2} \right)$. In combination with (157), one obtains that at most

$$\tilde{O} \left( \frac{\rho_s^2 L^{1/2} \mathbb{E}_\mu \left[ \|\nabla f(x)\|^4 \right]}{\delta \epsilon^2} \right)$$

expected gradient evaluations are performed. \qed
Proof of Corollary 5. Apply Theorem 4 to (61) with \( \psi \) given by
\[
\psi(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq \frac{\delta^2}{2} \\
\infty & \text{otherwise}
\end{cases} \tag{158}
\]
The trust-region problem (43) is obtained and, as \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \), sufficient conditions for strong duality (51) hold. By Theorem 4, it suffices to show that \( \psi \) provides \( C / \left( E_\mu \left[ \| \nabla f(x) \|^2 \right] \right) \)-regularization (60) at \( L + 1 \) for \( C \geq 8L^2 \). Indeed, the other suppositions of Theorem 4 are clearly true since: any \( L \)-smooth function \( f \) is also \( L \)-semiconvex, \( \psi \) is minimized at 0, and
\[
\psi^*(\lambda) = \left( \delta^2 / 2 \right) (\lambda)_+ \tag{159}
\]
is trivially smooth on any interval not containing 0. From (159) and the guarantee \( \delta \leq \| \nabla f \|_{L^2(\mu)} / 2L \), however, this level of regularization is clear since
\[
\partial_- \psi^*(L + 1) \leq \frac{E_\mu \left[ \| \nabla f(x) \|^2 \right]}{8L^2}.
\]

\( \Box \)

J  Proof of Proposition 1

The first bullet is essentially a restatement of Theorem 7.2.2 in [1]. To verify the second bullet, we first establish the existence of such a \( v_t \). Let \( \mu_t \) be the constant speed geodesic and define the set of functions
\[
A_\mu := \left\{ z \in L^2([0,1]) : W(\mu_t, \mu_s) \leq \int_s^t z(r) \, dr \quad \forall \ 0 \leq s \leq t \leq 1 \right\}
\]
It is clear that the function \( m(r) := W(\mu_0, \mu_1) \) is in \( A \) and satisfies
\[
m = \arg \min_{z \in A} \int_0^1 z^p(r) \, dr \tag{160}
\]
for any \( p \geq 1 \). Hence, the metric derivative \( |\mu'| \) of \( \mu_t \) fulfills
\[
|\mu'|(t) = d(\mu_0, \mu_1) \quad \text{Lebesgue almost everywhere for } t \in [0,1]
\]
By Theorem 8.3.1 in [1], there exists Borel vector field \( v_t : [0,1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) satisfying the continuity equation (22) such that
\[
\| v_t \|_{L^2(\mu_t)} = |\mu'|(t) = W(\mu_0, \mu_1) \quad \text{Lebesgue almost everywhere for } t \in [0,1] \tag{161}
\]
Combined with (160), this implies that \( v_t \) is a solution of (21). Uniqueness of \( v_t \) follows directly from the third bullet.

For a constant-speed geodesic \( \mu_t \) from \( \mu \) to \( \nu \). Theorem 2.4 in [2] gives that, for any \( \sigma \in \mathcal{P}_2(\mathbb{R}^d) \),
\[
\frac{d}{dt} \left[ \frac{1}{2} \mathcal{W}^2(\mu_t, \sigma) \right] = \int \langle v_t(x), x - y \rangle \, d\gamma(x, y) \quad \forall \ \gamma \in \Pi_o(\mu_t, \sigma) \tag{162}
\]
where \( \Pi_o(\mu_t, \sigma) \subseteq \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \) is the set of optimal transport plans between \( \mu_t \) and \( \sigma \). Setting \( \sigma = \nu \), the fact that \( \mu_t \) is a geodesic implies that there is a unique optimal coupling \( \gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \) between \( \mu \) and \( \nu \) such that
\[
((1 - t)x + ty, y) \# \gamma \in \Pi_o(\mu_t, \sigma)
\]
Hence, (162) gives
\[-(1 - t)W^2(\mu, \nu) = \int \langle v_t(x), x - y \rangle \, d\gamma(x, y) = -(1 - t) \int \langle v_t((1 - t)x + ty), y - x \rangle \, d\gamma(x, y)\]

\[\Rightarrow \quad W^2(\mu, \nu) = \int \langle v_t((1 - t)x + ty), y - x \rangle \, d\gamma(x, y) \tag{164}\]

For $t$ satisfying (161), the fact that $\|v_t\|_{L^2(\mu_t)} = W(\mu, \nu)$ and $\|y - x\|_{L^2(\gamma)} = W(\mu, \nu)$ means that (164) gives equality for Cauchy-Schwarz. Thus, $v_t((1 - t)x + ty) = y - x$, $\gamma$-almost surely and (23) follows for Lebesgue almost every $t \in [0, 1]$.

The final bullet is a direct restatement of the results of Section 8.4 in [1].

\[\square\]

**K Proof of Theorem 2**

**Lemma 16.** Let $\gamma \in \Pi(\mu, \nu)$ be an optimal transport plan between $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\nu \in \mathcal{P}_2(\mathbb{R}^d)$. If $\phi_\mu \in C^1(\mathbb{R}^d)$ is $L$-smooth

\[\|\nabla \phi_\mu(x) - \nabla \phi_\mu(y)\| \leq L \|x - y\| \tag{165}\]

then

\[\left| \int_{\mathbb{R}^d} \langle \nabla \phi_\mu(x), y - x \rangle \, d\gamma(x, y) - \left( \int_{\mathbb{R}^d} \phi_\mu \, d\nu - \int_{\mathbb{R}^d} \phi_\mu \, d\mu \right) \right| \leq \frac{L}{2} W^2(\nu, \mu) \tag{166}\]

**Proof.** First, it will be shown that

\[\left| \int_{\mathbb{R}^d} \langle \nabla \phi_\mu(x), y - x \rangle \, d\gamma(x, y) - \int_0^1 \langle \nabla \phi_\mu(v_t), v_t \rangle \, dt \right| \leq \frac{L}{2} W^2(\nu, \mu) \tag{167}\]

for $\mu_t$ and $v_t$ which correspond (21) to the unique-constant speed geodesic given by $\gamma \in \Pi(\mu, \nu)$ (20). Notice that, since $\nabla \phi_\mu$ has at most linear growth, therefore both terms in the left-hand side of (167) are finite. Moreover, by (23), one has

\[\int_0^1 \langle \nabla \phi_\mu, v_t \rangle_{\mu_t} \, dt = \int_0^1 \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \nabla \phi_\mu((1 - t)x + ty), y - x \rangle \, d\gamma(x, y) \, dt \tag{168}\]

Thus, Cauchy-Schwarz and (165) give

\[\left| \int_0^1 \langle \phi_\mu, v_t \rangle_{\mu_t} \, dt - \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \nabla \phi_\mu(x), y - x \rangle \, d\gamma(x, y) \right| =
\]

\[\left| \int_0^1 \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \nabla \phi_\mu((1 - t)x + ty) - \nabla \phi_\mu(x), y - x \rangle \, d\gamma(x, y) \, dt \right| \leq
\]

\[\int_0^1 \int_{\mathbb{R}^d} tL \|x - y\|^2 \, d\gamma(x, y) \, dt = \frac{L}{2} W^2(\nu, \mu)
\]

To obtain (166), it only remains to show that

\[\int_0^1 \langle \nabla \phi_\mu, v_t \rangle_{\mu_t} \, dt = \int \phi_\mu \, d\nu - \int \phi_\mu \, d\mu \tag{169}\]

Moreover, since $v_t$ satisfies (22), Lemma 8.1.2 in [1] gives

\[\int_0^1 \langle \nabla \psi, v_t \rangle_{\mu_t} \, dt = \int \psi \, d\nu - \int \psi \, d\mu \tag{170}\]
for every $\psi \in C^1_c(\mathbb{R}^d)$—where $C^1_c(\mathbb{R}^d)$ denotes the space of continuously differentiable functions on $\mathbb{R}^d$ with compact support. Hence, (169) will be obtained from (170) by the following approximation argument.

Define the functions:

$$\beta_-(x) := \left(\sqrt{\|x\|^2 + 1} - \sqrt{2}\right)^{-1}$$

and

$$\beta_+(x) := \left(\sqrt{3} - \sqrt{\|x\|^2 + 1}\right)^{-1}$$

and

$$\eta(x) := \begin{cases} 
1 & \text{if } \|x\| \leq 1 \\
\frac{e^{\beta_-(x)}}{e^{\beta_-(x)} + e^{\beta_+(x)}} & \text{if } 1 < \|x\| < 2 \\
0 & \text{if } \|x\| \geq 2
\end{cases}$$

It is easy to verify that $\eta \in C^\infty_c(\mathbb{R}^d)$ and $\|\nabla \eta(x)\| \leq B$ for all $x \in \mathbb{R}^d$ and some constant $B$. Moreover, $\eta$ provides a sequence of mollified approximations of $\phi_\mu$

$$\psi_k(x) := \phi_\mu(x) \eta_k(x) \quad \text{for} \quad \eta_k(x) := \eta\left(\frac{x}{k}\right)$$

where $\psi_k \in C^1_c(\mathbb{R}^d)$. Clearly, (170) holds for all such $\psi_k$. Thus, if

$$\lim_{k \to \infty} \int \psi_k \, d\nu - \int \psi_k \, d\mu = \int \phi_\mu \, d\nu - \int \phi_\mu \, d\mu$$

then (169) will follow directly from (170).

The relations (171) and (172) are straight-forward consequences of dominated convergence. Indeed, as $\eta_k \to 1$ and $\nabla \eta_k \to 0$ (pointwise), clearly

$$\psi_k \to \phi_\mu \quad \text{and} \quad \nabla \psi_k \to \nabla \phi_\mu$$

Quadratic growth of $\phi_\mu$ yields $\phi_\mu \in L^2(\mu) \cap L^2(\nu)$ and combined with

$$|\psi_k(x)| \leq |\phi_\mu(x)| \quad \forall x \in \mathbb{R}^d$$

(171) clearly holds via dominated convergence. One also has

$$\|\nabla \psi_k(x)\| \leq \|\nabla \phi_\mu(x)\| + \frac{B|\phi_\mu(x)|}{k} \{1_{\{\|x\| < 2k\}}$$

(174)

Using the quadratic growth of $\phi_\mu$, linear growth of $\|\nabla \phi_\mu\|$, and the bound $\|x\| \{1_{\{\|x\| < 2k\}}/k \leq 2$, (174) yields

$$\|\nabla \psi_k(x)\| \leq \|\nabla \phi_\mu(x)\| + C \|x\| \{1_{\{\|x\| < 2k\}} + D \leq E \|x\| + F$$

(175)

for some constants $C, D, E \in \mathbb{R}_+$. Recalling (168), (175) provides

$$\int \mathbb{R}^d \times \mathbb{R}^d \left|\langle \nabla \psi_k((1-t)x + ty), y - x \rangle\right| \, d\gamma(x, y) \leq \int \mathbb{R}^d \times \mathbb{R}^d \|\nabla \psi_k((1-t)x + ty)\| \|y - x\| \, d\gamma(x, y) \leq \int \mathbb{R}^d \times \mathbb{R}^d \left(E \|(1-t)x + ty\| + F\right) \|y - x\| \, d\gamma(x, y) \leq H$$

(176)

for some $H \in \mathbb{R}_+$; where the last inequality is a result of Cauchy-Schwarz. The combination of pointwise convergence (173) and (176) then immediately yield (172) by dominated convergence and (168).
Lemma 17. Let \( r_i \in \mathbb{R}_+ \) be a sequence of non-negative numbers satisfying
\[
    r_{i+1} \leq r_i - \kappa r_i^p
\]
for some constants \( \kappa > 0 \) and \( p \geq 0 \). Then,
\[
    r_n \leq \begin{cases} 
        e^{-\kappa n/r_0^{1-p}}r_0 & \text{if } p \leq 1 \\
        (\kappa n + r_0^{1-p})^{-1/(p-1)} & \text{if } p > 1 
    \end{cases}
\]

Proof. If \( p \leq 1 \), then (177) combined with the fact that \( r_i \) is a non-increasing sequence implies
\[
    r_i \leq \left(1 - \frac{\kappa}{r_0^{1-p}}\right)r_{i-1}
\]
Iterating this inequality from 1 to \( n \) yields the first part of (178). Next, let \( p > 1 \) and notice that, by taking the reciprocals of both sides of (177) and rearranging, one obtains
\[
    \frac{\kappa r_i^{p-2}}{1 - \kappa r_i^{p-1}} \leq r_i^{-1} - r_{i-1}^{-1}
\]
Summing this inequality over \( i \) (from 1 to \( n \)),
\[
    \kappa n r_k^{p-2} \leq \sum_{i=1}^{n} \frac{\kappa r_i^{p-2}}{1 - \kappa r_i^{p-1}} \leq r_k^{-1} - r_0^{-1}
\]
where the first inequality is a result of \( r_i \) being non-increasing. Algebraic manipulation then provides
\[
    r_n \leq (\kappa n + r_0^{1-p})^{-1/(p-1)}
\]
\( \square \)

Proof of Theorem 2. Recall the parameters specified in Assumptions 1 and 3 and let \( \epsilon \) be the desired tolerance with which (32) should hold. Let Algorithm 1 be run with the following parameters:
\[
    \beta_1 = \min(\Delta_1, \Delta_2), \quad \beta_2 = \alpha(4L)^{-1}, \quad \beta_3 = (1 - \alpha/2)^{1/\alpha}T^{-1/\alpha}
\]
and
\[
    r = \tau \epsilon^{\theta}/2, \quad \hat{r} = (2\alpha^*)^{-1} r, \quad \bar{r} = \alpha r/2, \quad \bar{\epsilon} = (4\alpha^*)^{-1} r, \quad k = \lceil M \rceil
\]
where \( \alpha^* = (1 + \alpha)/\alpha \) is the dual exponent of \( 1 + \alpha \) and \( M \) is defined in (195). It will be shown that the last iterate, \( \mu_t \), computed by Algorithm 1 satisfies (32).

First, we bound the decrease in \( J \) at each step of Algorithm 1. Let \( \delta_i \) be the \( i \)th value of \( \delta \) that is computed by Algorithm 1 and let \( s_i \) denote the \( i \)th value of \( s \). One has the relation
\[
    \delta_i = \min\left(\beta_1, \beta_2 s_i, \beta_3 s_i^{\alpha^* - 1}\right)
\]
and, since \( \delta_i \leq \Delta_2 \) for all \( i \), \( \mu_0 \in S \) implies \( \mu_i \in S \) for all \( i \). Via the smoothness of \( J \) on \( S \) and \( \delta_i \leq \Delta_1 \), it follows that
\[
    J(\mu_i) \leq J(\mu_{i-1}) + \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle F(\mu_{i-1}; x), y - x \rangle \, d\gamma(x, y) + \frac{T}{1 + \alpha} \delta_i^{1+\alpha}
\]
for any optimal transport plan \( \gamma \in \Pi(\mu_i, \mu_{i-1}) \) between \( \mu_i \) and \( \mu_{i-1} \). Recognizing (28),
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \left\langle F(\mu_{i-1}; x) - \nabla \hat{\phi}_{\mu_{i-1}}(x), y - x \right\rangle \, d\gamma(x, y) \leq \left\| F(\mu_{i-1}; x) - \nabla \hat{\phi}_{\mu_{i-1}} \right\|_{L^2(\mu_{i-1})} W(\mu_i, \mu_{i-1}) \leq \delta_1 \epsilon
\]
and therefore
\[
J(\mu_i) \leq J(\mu_{i-1}) + \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla \hat{\phi}_{\mu_{i-1}}(x)^T (y - x) \, d\gamma(x, y) + \frac{T}{1 + \alpha} \delta_1^{1+\alpha} + \delta_1 \epsilon
\]
Via Lemma 16,
\[
J(\mu_i) \leq J(\mu_{i-1}) + \int \hat{\phi}_{\mu_{i-1}} \, d\mu_i - \int \hat{\phi}_{\mu_{i-1}} \, d\mu_{i-1} + \frac{T}{1 + \alpha} \delta_1^{1+\alpha} + \frac{L^2}{2} \delta_1^2 + \delta_1 \epsilon \tag{182}
\]
Now, since \( \hat{\phi}_{\mu_{i-1}} \) is \( L \)-smooth, it is a Kantorovich potential \([1, \text{Section 6.1}] \) for \( \mu_{i-1} \), under the cost function \( L \| x - y \|^2 / 2 \). Thus, there exists a geodesic \( \nu_t \) (Proposition 1) such that: \( \nu_0 = \mu_{i-1} \) and the transport plan \( \gamma_t \in \Pi(\mu_{i-1}, \nu_t) \) between \( \mu_{i-1} \) and \( \nu_t \) satisfies \([1, \text{Section 8.3}] \)
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \left\langle \nabla \phi_{\mu_{i-1}}(x), y - x \right\rangle \, d\gamma_t(x, y) = -\frac{t}{L} \left\| \nabla \hat{\phi}_{\mu_{i-1}} \right\|^2_{L^2(\mu_{i-1})} \quad \text{and} \quad W(\nu_t, \mu_{i-1}) = \frac{t}{L} \left\| \nabla \hat{\phi}_{\mu_{i-1}} \right\|_{L^2(\mu_{i-1})}
\]
for \( 0 \leq t \leq 1 \). For the sake of notation, define \( g_{i-1} := \left\| \nabla \hat{\phi}_{\mu_{i-1}} \right\|_{L^2(\mu_{i-1})} \) and set \( t = L \delta_i / g_{i-1} \).

Clearly, \( t \leq 1 \) since \( \delta_i \leq \beta_2 s_i \leq \beta_2 g_{i-1} \).

By construction, \( \mu_i \) also satisfies
\[
\int \hat{\phi}_{\mu_{i-1}} \, d\mu_i - \int \hat{\phi}_{\mu_{i-1}} \, d\mu_{i-1} \leq \int \hat{\phi}_{\mu_{i-1}} \, d\nu_t - \int \hat{\phi}_{\mu_{i-1}} \, d\mu_{i-1} + \zeta_i
\]
for \( \zeta_i = \delta_i \epsilon \). Hence, with another application of Lemma 16, one obtains
\[
\int \hat{\phi}_{\mu_{i-1}} \, d\mu_i - \int \hat{\phi}_{\mu_{i-1}} \, d\mu_{i-1} \leq \int_0^t \left\langle \nabla \hat{\phi}_{\mu_{i-1}}, \nu_s \right\rangle_{\nu_s} \, ds + \zeta_i
\]
\[
\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \left\langle \nabla \phi_{\mu_{i-1}}(x), y - x \right\rangle \, d\gamma(x, y) + \frac{L}{2} W(\nu_t, \mu_{i-1})^2 + \zeta_i
\]
\[
= -\frac{t}{L} \left( 1 - \frac{t}{2} \right) g_{i-1}^2 + \zeta_i \tag{183}
\]
Combining (183) with (182) and recalling \( \delta_i = t g_{i-1} / L \) gives
\[
J(\mu_i) \leq J(\mu_{i-1}) - \frac{t}{L} \left( C - t - \frac{D}{1 + \alpha} \right) g_{i-1} + \zeta_i \tag{184}
\]
for the values
\[
C := 1 - \epsilon \quad \text{and} \quad D := \frac{T}{L^\alpha g_{i-1}^{1-\alpha}}
\]
Rewriting (184) using the residual term
\[
r(\nu) := J(\nu) - \inf_{\mu \in S} J(\mu) \tag{185}
\]
one obtains

\[ r(\mu_i) \leq r(\mu_{i-1}) - \frac{t}{L} \left( C - t - \frac{D}{1 + \alpha} t^\alpha \right) g_{i-1} + \zeta_i \]  

(186)

This relation will now be used to show that Algorithm 1 makes sufficient progress on \( J \), prior to the termination of its loop.

Let \( l \) be the index of the last iterate \( \mu_i \) which is computed by Algorithm 1. First, observe that if \( s_{l+1} \leq r \), then early termination of the loop in Algorithm 1 has occurred. Using (30) and the definitions (180), it follows that

\[ \tau (r(\mu_l))^\theta \leq \| F(\mu_l) \|_{L^2(\mu_l)} \leq g_l + \hat{\epsilon} \]

\[ \leq r + \hat{\epsilon} + \dot{\epsilon} \leq \tau \epsilon^\theta \]  

(187)

and, hence, sufficient progress on \( J \) has been made– \( \mu_l \) satisfies (32). Thus, we need only analyze the case where early termination in Algorithm 1 does not occur and \( l = k \) (180).

If \( l = k \), then \( s_i > r \) for all \( i \leq k \) and, by extension, \( g_{i-1} > r \) for all \( i \leq k \) since \( s_i \) is a lower bound for \( g_{i-1} \). In this case, the definitions of \( \dot{\epsilon} \) and \( r \) (180) imply \( C \geq 1 - \alpha / (2(1 + \alpha)) \) and the choices for \( \beta_2 \) and \( \beta_3 \) (179) provide

\[ t \leq \min \left( \frac{\alpha}{2(1 + \alpha)}, \frac{(1 - \alpha/2)^{1/\alpha}}{D^{1/\alpha}} \right) \]

This gives

\[ C - t - \frac{D}{1 + \alpha} t^\alpha \geq (2\alpha^*)^{-1} \]

from which substitution into (186) yields

\[ r(\mu_i) \leq r(\mu_{i-1}) - \frac{t}{2L\alpha} g_{i-1} + \zeta_i \]

\[ \leq r(\mu_{i-1}) - \frac{\delta_i}{2\alpha^*} g_{i-1} + \zeta_i \]

\[ \leq r(\mu_{i-1}) - \frac{\delta_i}{4\alpha^*} g_{i-1} \]  

(188)

where the last inequality is a result of the definition of \( \bar{\epsilon} \) (180), \( \zeta_i \), and \( g_{i-1} > r \). As \( \delta_i \) is the minimum of three different terms (181), (188) will be used to analyze the amount of progress, that is made on the objective \( J \), corresponding to each of these three terms. Note, the following identities that will be used in the analysis of each term:

\[ \left( 1 - \frac{\alpha}{2} \right) g_{i-1} \leq g_{i-1} - \frac{\alpha r}{2} \leq g_{i-1} - \bar{\epsilon} \leq s_i \]  

(189)

and

\[ -g_{i-1}^p \leq - \left( \| F(\mu_{i-1}) \|_{L^2(\mu_{i-1})} - \dot{\epsilon} \right)^p \]

\[ \leq - \left( 1 - \frac{\alpha}{2 + \alpha} \right)^p \| F(\mu_{i-1}) \|_{L^2(\mu_{i-1})}^p \leq - \frac{1}{2e} \| F(\mu_{i-1}) \|_{L^2(\mu_{i-1})}^p \]  

(190)

for all \( 1 \leq p \leq \alpha^* \). The relation (189) simply observes that \( s_i \) is a multiplicative approximation to \( g_{i-1} \) in Algorithm 1, while (190) is a consequence of \( r - \dot{\epsilon} \leq \| F(\mu_{i-1}) \|_{L^2(\mu_{i-1})} \).
First, consider the case where \( \delta_i = \beta_1 \). Substitution into (188), coupled with (190), provides

\[
 r(\mu_i) \leq r(\mu_{i-1}) - \frac{\beta_1}{8\epsilon_0^\alpha} \| F(\mu_{i-1}) \|_{L^2(\mu_{i-1})}
\]  
(191)

Applying (30) to (191) and defining \( r_i := r(\mu_i) \) (for the sake of notation) yields

\[
 r_i \leq r_{i-1} - \kappa^{(1)} r_{i-1}^{\theta} \quad \text{for} \quad \kappa^{(1)} := \omega \beta_1
\]  
(192)

for the constant \( \omega = (8\epsilon_0^\alpha)^{-1} \tau \). In the cases (181) corresponding to \( \beta_2 \) and \( \beta_3 \), similar applications of the previous identities (along with (189)) give

\[
 r_i \leq r_{i-1} - \kappa^{(2)} r_{i-1}^{\theta} \quad \text{for} \quad \kappa^{(2)} := \omega \tau (1 - \alpha/2) \beta_2
\]  
(193)

\[
 r_i \leq r_{i-1} - \kappa^{(3)} r_{i-1}^{\theta} \quad \text{for} \quad \kappa^{(3)} := \omega (1 - \alpha/2)^{1/\alpha} \beta_3
\]  
(194)

Now, for the sake of notation, define the function

\[
 z(u, v) := u^{-1} \epsilon^{-(1-v)} \left( r_0 \log \frac{1}{1-v} \right)^{1-v} +
\]

where \((\cdot)_+\) and \((\cdot)_-\) denote the positive and negative parts. Using Lemma 17, it follows that, if (192) occurs for more than \( \omega^{-1} z(\beta_1, \theta) \) iterations of Algorithm 1, then \( r_k \leq \epsilon \), where \( k \) is the index of the last loop iteration in Algorithm 1. Similar deductions for (193) and (194) lead to the conclusion that, if

\[
 k \geq \omega^{-1} \left( z(\beta_1, \theta) + z(\tau (1 - \alpha/2) \beta_2, 2 \theta) + z((\tau (1 - \alpha/2))^{1/\alpha} \beta_3, \alpha^* \theta) \right) \quad := M
\]  
(195)

then either (192), (193), or (194) has occurred sufficiently many times during the execution of Algorithm 1 to guarantee \( r_k \leq \epsilon \). As \( k \) has been chosen exactly so that \( k = \lceil M \rceil \) (180), one obtains that \( \mu_k \) satisfies (32). The desired complexity bound (33) on \( M \) now follows by plugging in for \( \beta_1, \beta_2, \) and \( \beta_3 \) in (195) and then, taking asymptotic estimates as \( \epsilon \to 0 \); the term \( z((\tau (1 - \alpha/2))^{1/\alpha} \beta_3, \alpha^* \theta) \) clearly dominates.

To obtain the stated sample complexities of Theorem 2 notice that each iteration requires sampling from \( \mu_{i-1} \) to estimate both \( s_i \) and \( \mu_i \). Computing \( s_i \) is a simple mean estimation task and can be performed (with the necessary accuracy \( \epsilon \)) using \( \tilde{O}(\epsilon^{-2\alpha}) \) samples. From Corollary 5, recall that a \( \lambda_i \) such that \( \pi_{\lambda_i, \mu_{i-1}} \) (62) yields \( \mu_i \) can be computed using \( O(\epsilon_i^{-2}) \) samples from \( \mu_{i-1} \) where \( \epsilon_i = \delta_i \epsilon \). Utilizing the definition of \( \epsilon \) and the previously computed lower bounds on \( \delta \), it follows that sample access to \( \mu_i \) can be obtained with \( O(\epsilon_i^{-2}) = O(\epsilon^{-2\alpha}) \) samples from \( \mu_{i-1} \). Clearly, this dominates the number of samples required to estimate \( s_i \) since \( \alpha^* \geq 2 \). Thus, an iteration of Algorithm 1 requires \( O(\epsilon^{-2\alpha}) \) samples from \( \mu_{i-1} \).

To reduce this to a sample complexity in terms of \( \mu_0 \), notice that (provided a computed \( \lambda_{i-1} \) and \( \phi_{\mu_{i-1}, \lambda_{i-1}} \) a draw from \( \mu_{i-1} \) can be obtained using accelerated gradient descent and a draw from \( \mu_{i-2} \); in only \( O(\log \epsilon^{-1}) \) gradient evaluations of \( \phi_{\mu_{i-1}} \) (see the proof of Proposition 6 for this analysis). Chaining this procedure, it follows that, if \( \lambda_j \) has been computed for all \( j \leq i - 1 \), a sample from \( \mu_{i-1} \) can be produced using a sample from \( \mu_0 \) and \( O(i \log \epsilon^{-1}) \) total gradient evaluations. Hence, each iteration of Algorithm 1 can be performed using \( O(\epsilon^{-2\alpha}) \) samples from \( \mu_0 \).

\[ \square \]

### L Geodesic convexity and Łojasiewicz inequalities

**Lemma 18.** If \( J : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) is geodesically convex (Definition 1) and Wasserstein differentiable (Definition 2) then the Wasserstein gradient field \( F : \mathcal{P}_2(\mathbb{R}^d) \to \text{CoTan}\mathcal{P}_2(\mathbb{R}^d) \) satisfies

\[
 J(\mu) + \int_{\mathbb{R}^d \times \mathbb{R}^d} F(\mu; x)^T (y - x) d\gamma(x, y) \leq J(\nu)
\]  
(196)
where $\gamma \in \Pi(\mu, \nu)$ is any optimal transport plan between $\mu$ and $\nu$. Consequently, if $J$ has a bounded (with respect to $W$) level set with diameter $R$

$$\text{Diam} (Q_p) \leq R, \quad Q_p := \left\{ \mu \in P_2(\mathbb{R}^d) : J(\mu) \leq p \right\} \quad (197)$$

then $J$ satisfies a Łojasiewicz inequality (30) on $Q_p$ with parameters $\tau = R^{-1}$ and $\theta = 1$.

**Proof.** Let $\gamma \in \Pi(\mu, \nu)$ and let $\mu_t$ be the constant-speed geodesic (20) corresponding to $\gamma$. Rearranging the definition of geodesic convexity (25), one obtains

$$\frac{J(\mu_t) - J(\mu)}{t} \leq J(\nu) - J(\mu)$$

Taking the limit as $t \to 0$ and applying (26) provides (196). To obtain a Łojasiewicz inequality on a bounded level set $Q$, simply recognize that (196) and Cauchy-Schwarz imply

$$J(\mu) - J(\nu) \leq \|F(\mu)\|_{L^2(\mu)} W(\mu, \nu)$$

for any $\mu, \nu \in Q$. $\square$

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