Uniqueness and nonuniqueness of fronts for degenerate diffusion-convection-reaction equations

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Abstract

We consider a scalar parabolic equation in one spatial dimension. The equation is constituted by a convective term, a reaction term with one or two equilibria, and a positive diffusivity which can however vanish. We prove the existence and several properties of traveling-wave solutions to such an equation. In particular, we provide a sharp estimate for the minimal speed of the profiles; for wavefronts, we improve previous results about their regularity; we discover a new family of semi-wavefronts.

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1 Introduction

We study the existence and qualitative properties of traveling-wave solutions to the scalar diffusion-convection-reaction equation

$$\rho_t + f(\rho)x = \left(D(\rho)\rho_x\right)_x + g(\rho), \quad t \geq 0, \ x \in \mathbb{R}. \quad (1.1)$$

Here \(\rho = \rho(t, x)\) is the unknown variable and takes values in the interval \([0, 1]\). The flux function \(f\) satisfies the condition

(f) \(f \in C^1[0, 1], \ f(0) = 0.\)

Notice that the requirement \(f(0) = 0\) is not a real assumption, since \(f\) is defined up to an additive constant; we denote \(h(\rho) = \dot{f}(\rho)\), where with a dot we intend the derivative with respect to the state variable \(\rho\) (or \(\varphi\) later on). About the diffusivity \(D\) and the source term \(g\) we consider two different scenarios, where the assumptions are made on the pair \(D, g\); more precisely, we assume either

(D1) \(D \in C^1[0, 1], \ D > 0 \text{ in } (0, 1) \text{ and } D(1) = 0,\)

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(g0) \( g \in C^0[0,1], g > 0 \) in \((0,1], g(0) = 0, \)

or else

(D0) \( D \in C^1[0,1], D > 0 \) in \((0,1) \) and \( D(0) = 0, \)

(g01) \( g \in C^0[0,1], g > 0 \) in \((0,1), g(0) = g(1) = 0. \)

In the above notation, the numbers suggest where it is mandatory that the corresponding function vanishes. Notice that (D1) leaves open the possibility for \(D\) to vanish or not at 0, and (D0) for \(D\) at 1. We refer to Figure 1 for a graphical illustration of these assumptions. Notice that the product \(Dg\) always vanishes at both 0 and 1 under both set of assumptions.

![Figure 1: Typical plots of the functions \(f, D\) and \(g\). In the plots of \(D\) and \(g\), solid or dashed lines depict pairs of functions \(D\) and \(g\) that are considered together in the following. The possibility that \(D\) vanishes at the other extremum is left open.](image)

We also require the following condition on the product of \(D\) and \(g\):

\[
\limsup_{\varphi \to 0^+} \frac{D(\varphi)g(\varphi)}{\varphi} < +\infty.
\]

Condition (1.2) is equivalent to \(D(\varphi)g(\varphi) \leq L\varphi\), for some \(L > 0\) and \(\varphi\) in a right neighborhood of 0, and it is satisfied by minimal regularity assumptions on \(D\) and/or \(g\).

In (1.1), the notation \(\rho = \rho(t,x)\) suggests a density; this is indeed the case. In the last twenty years, the modeling of vehicular traffic flows or pedestrian dynamics has attracted the interest of several mathematicians, providing new and challenging problems [11, 12, 26]. This paper was partly motivated by such a research stream and carries on the analysis of a scalar parabolic model begun in [6, 7, 8]. Indeed, if \(f(\rho) = \rho v(\rho)\), where the velocity \(v\) is an assigned function, then equation (1.1) can be understood as a simplified model for a crowd walking with velocity \(v\) along a straight path with side entries for other pedestrians, which are modeled by \(g\); here \(\rho\) is understood as the crowd normalized density. Assumption (g01), for instance, means that pedestrians do not enter if the road is empty \((g(0) = 0,\) modeling an aggregative behavior) or if it is fully occupied \((g(1) = 0,\) because of lack of space). If the diffusivity is small, then the diffusion term accounts for some “chaotic” behavior, which is common in crowds movements. In this framework it is usual to assume that \(D\) degenerates at the extrema of the interval where it is defined [2, 3, 5, 24]. For more details on this modeling we refer to [7] and references there, in particular for the dependence of \(D\) on \(\rho\).

On the other hand, the assumption (g0) is better motivated by population dynamics. In
this case \( g \) is a growth term which, for instance, increases with the population density \( \rho \). We refer to [23] for analogous modelings in biology. Anyhow, apart from the above possible applications, equation (1.1) is a quite general diffusion-convection-reaction equation that deserves to be full understood.

A \textit{traveling-wave solution} is, roughly speaking, a solution to (1.1) of the form \( \rho(t,x) = \varphi(x - ct) \), for some profile \( \varphi = \varphi(\xi) \) and constant wave speed \( c \), see [13] for general information. In this case the profile must satisfy, in some sense, the equation

\[
(D(\varphi)\varphi')' + (c - h(\varphi))\varphi' + g(\varphi) = 0,
\]

(1.3)

where \( ' \) denotes the derivative with respect to \( \xi \). A unique (up to shifts) solution to (1.3) is usually determined by imposing conditions on \( \varphi \) at \( \pm \infty \), which must coincide with the equilibria of (1.1), i.e., with the zeros of \( g \). We consider in this paper non-constant, monotone profiles, and focus on the case they are decreasing. This leads to require either

\[
\varphi(-\infty) = 1, \quad \varphi(+\infty) = 0,
\]

(1.4)

or simply

\[
\varphi(+\infty) = 0,
\]

(1.5)

according to we make assumption (g01) or (g0). The former profiles are called \textit{wavefronts}, the latter are \textit{semi-wavefronts}, see Figure 2. Precise definitions are provided in Definition 2.1 Notice that in both cases the equilibria may be reached for a finite value of the variable \( \xi \) as a consequence of the degeneracy of \( D \) at those points. These solutions represent single-shape smooth transitions between the two constant densities 0 and 1. In the case of wavefronts, their interest lies in the fact that they are viscous approximations of shock waves to the inviscid version of equation (1.1), i.e., when \( D = 0 \). Semi-wavefronts lack of this motivation but are nevertheless meaningful for applications [7]; moreover, wavefronts connecting “nonstandard” end states can be constructed by pasting semi-wavefronts [8], see also the end of this Introduction. At last, we point out that assumption (1.2) is usual in this framework, when looking for decreasing profiles, see e.g. [1], and implies that the dynamical system underlying (1.3) has a node at the origin.

Figure 2: Left: a wavefront joining 1 with 0; right: a semi-wavefront to 0.

If \( D(\rho) \geq 0 \), the existence of solutions to the initial-value problem for (1.1) is more or less classical [28]; however, the \textit{fine structure} of traveling waves reveals a variety of different patterns. We refer to [19] [20], respectively, for the cases where \( D \) is non degenerate, i.e., \( D > 0 \), and for the degenerate case, where \( D \) can vanish at either 0 or 1. The main results
of those papers is that there is a critical threshold $c^*$, depending on both $f$ and the product $Dg$, such that traveling waves satisfying \((1.4)\) exist if and only if $c \geq c^*$. The smoothness of the profiles depend on $f$, $D$ and $c$ but not on $g$. In both papers the source term satisfies \((g01)\); see \([6, 7]\) for the case when $g$ has only one zero.

The case when $D$ changes sign, which is not studied in this paper, also has strong motivations: we quote \([17, 25]\) for biological models, \([8]\) for applications to collective movements, and \([9, 14, 15]\) for other models. Several results about traveling waves have been obtained in \([8, 10, 16, 17, 18]\).

In this paper we study semi-wavefronts and wavefronts for equation \((1.1)\); in particular we complete the analysis that began in \([6, 7]\). We prove that in both cases there is a threshold $c^*$, as above, such that profiles only exist for $c \geq c^*$; we also study their regularity and strict monotonicity, namely whether they are classical (i.e., $C^1$) or sharp (and then reach an equilibrium at a finite $\xi$ in a no more than continuous way). Several explicit examples are scattered throughout the paper to show that our assumptions are necessary in most cases.

This research has some important novelties. First, we give a refined estimate for $c^*$, which allows to better understand the meaning of this threshold. Second, we improve a result obtained in \([20]\) about the appearance of wavefronts with a sharp profile. Third, in the case of semi-wavefronts, we make the surprising discovery of a whole family of profiles which was previously unknown, to the best of our knowledge. This is a consequence of a detailed study of a singular first-order problem, as we briefly explain now.

The main tool to investigate \((1.3)\) is the analysis of singular first-order problems as

\[
\begin{aligned}
\dot{z}(\varphi) &= h(\varphi) - c - \frac{D(\varphi)u(\varphi)}{z(\varphi)}, \quad \varphi \in (0, 1), \\
z(\varphi) &< 0, \quad \varphi \in (0, 1), \\
z(0) &= 0.
\end{aligned}
\]  

Problem \((1.6)\) is deduced by problem \((1.3)-(1.5)\) by the singular change of variables $z(\varphi) := D(\varphi)\varphi'$, where the right-hand side is understood to be computed at $\varphi^{-1}(\varphi)$, see e.g. \([7, 19]\). Notice that $\varphi^{-1}$ exists by the assumption of monotony of $\varphi$. The use of \((1.6)\) to tackle problem \((1.3)-(1.5)\) is not new, but in this paper the analysis is pushed to a detail that was never considered in previous papers and reveals unexpected solutions, which lead to the loss of uniqueness of semi-wavefronts to \((1.3)\).

On the other hand, the analysis of problem \((1.6)\) is fully exploited in the forthcoming paper \([4]\), which deals with the case in which $D$ changes sign once. In that paper we show that there still exist wavefronts joining 1 with 0, which travel across the region where $D$ is negative; they are constructed by pasting two semi-wavefronts obtained in the current paper. Similar results in the case $g = 0$ are proved in \([8]\).

Here follows an account of the content of the paper. In Section 2 we provide some basic definitions, state our main results and make several comments. The analysis of problem \((1.6)\) and of other related singular problems occupies Sections 3 to 8. There, we study in great detail the existence, uniqueness and qualitative properties of the solutions to \((1.6)\). Then, in Sections 9 and 10 we exploit such results to construct semi-wavefronts and wavefronts, respectively; there, we prove our main results, show some consequences and provide further comments.
2 Main results

We begin this section with some definitions on traveling waves and the related profiles, under assumptions somewhat weaker than those stated in the Introduction. We denote with $I \subseteq \mathbb{R}$ an open interval.

**Definition 2.1.** Assume $f, D, g \in C[0, 1]$. Consider a function $\varphi \in C(I)$ with values in $[0, 1]$, which is differentiable a.e. and such that $D(\varphi)\varphi' \in L^1_{\text{loc}}(I)$; let $c$ be a real constant. The function $\rho(x, t) := \varphi(x - ct)$, for $(x, t)$ with $x - ct \in I$, is a traveling-wave solution of equation (1.1) with wave speed $c$ and wave profile $\varphi$ if, for every $\psi \in C_0^\infty(I)$,

$$
\int_I \left( D(\varphi(\xi))\varphi'(\xi) - f(\varphi(\xi)) + c\varphi(\xi) \right) \psi'(\xi) - g(\varphi(\xi)) \psi(\xi) d\xi = 0. \quad (2.1)
$$

The previous definition can be made more precise as follows. Below, monotonic means that $\xi_1 < \xi_2$ implies $\varphi(\xi_1) \leq \varphi(\xi_2)$; in the fourth item we assume $g(0) = g(1) = 0$, while in the two last ones we only require that $g$ vanishes at the point which is specified by the semi-wavefront. A traveling-wave solution is

- **global** if $I = \mathbb{R}$ and **strict** if $I \neq \mathbb{R}$ and $\varphi$ is not extendible to $\mathbb{R};$
- **classical** if $\varphi$ is differentiable, $D(\varphi)\varphi'$ is absolutely continuous and (1.3) holds a.e.;
- **sharp at $\ell$** if there exists $\xi_\ell \in I$ such that $\varphi(\xi_\ell) = \ell$, with $\varphi$ classical in $I \setminus \{\xi_\ell\}$ and not differentiable at $\xi_\ell;$
- **a wavefront** if it is global, with a monotonic, non-constant profile $\varphi$ satisfying either (1.4) or the converse condition.
- **a semi-wavefront to 1** (or to 0) if $I = (a, \infty)$ for $a \in \mathbb{R}$, the profile $\varphi$ is monotonic, non-constant and $\varphi(\xi) \to 1$ (respectively, $\varphi(\xi) \to 0$) as $\xi \to \infty.$
- **a semi-wavefront from 1** (or from 0) if $I = (-\infty, b)$ for $b \in \mathbb{R}$, the profile $\varphi$ is monotonic, non-constant and $\varphi(\xi) \to 1$ (respectively, $\varphi(\xi) \to 0$) as $\xi \to -\infty.$

In the last two items we say that $\varphi$ connects $\varphi(a^+)$ (1 or 0) with 1 or 0 (resp., with $\varphi(b^-)$).

The smoothness of a profile is related to the degeneracy of $D$, see (8.10). More precisely, assume (f), and either (D1), (g0) or (D0), (g01); let $\rho$ be any traveling-wave solution of (1.1) with profile $\varphi$ defined in $I$ and speed $c$. Then $\varphi$ is classical in every interval $I_{\pm} \subseteq I$ where $D(\varphi(\xi)) \gtrless 0$ for $\xi \in I_{\pm}$; moreover, $\varphi \in C^2(I_{\pm})$. Notice that profiles always are determined up to a space shift.

Our first main result concerns semi-wavefronts.

**Theorem 2.1.** Assume (f), (D1), (g0) and (1.2). Then, there exist $c^* \in \mathbb{R}$, which satisfies

$$
\max \left\{ \sup_{\varphi \in (0, 1]} \frac{f(\varphi)}{\varphi}, h(0) + 2 \liminf_{\varphi \to 0^+} \frac{D(\varphi)g(\varphi)}{\varphi} \right\} \leq c^* \leq 2 \sup_{\varphi \in (0, 1]} \frac{D(\varphi)g(\varphi)}{\varphi} + \sup_{\varphi \in (0, 1]} \frac{f(\varphi)}{\varphi},
$$

such that (1.1) has strict semi-wavefronts to 0, connecting 1 to 0, if and only if $c \geq c^*$.

Moreover, if $\varphi$ is the profile of one of such semi-wavefronts, then it holds that

$$
\varphi'(\xi) < 0 \quad \text{for any} \quad 0 < \varphi(\xi) < 1. \quad (2.3)
$$
For a fixed $c > c^*$, the profiles of Theorem 2.1 are not unique. This lack of uniqueness is not due only to the action of space shifts but, more intimately, to the non-uniqueness of solutions to problem (1.6) that is proved in Proposition 5.1 below. Roughly speaking, these profiles depend on a parameter $b$ ranging in the interval $[\beta(c), 0]$, for a suitable threshold $\beta(c) \leq 0$. As a conclusion, the family of profiles can be precisely written as
\begin{equation}
\varphi_b = \varphi_b(\xi), \quad \text{for } b \in [\beta(c), 0].
\end{equation}

Moreover, $\beta(c) < 0$ if $c > c^*$ and $\beta(c) \to -\infty$ as $c \to +\infty$. The threshold $\beta(c)$ essentially corresponds to the minimum value that the quantity $D(\varphi_b)\varphi'_b$ may achieve when $\varphi_b$ reaches 1, for $b \in [\beta(c), 0]$. As a conclusion, the family of profiles can be precisely written as
\begin{equation}
\varphi_b = \varphi_b(\xi), \quad \text{for } b \in [\beta(c), 0].
\end{equation}

The estimates (2.2) deserve some comments. First, the left estimate improves analogous bounds (see [22] for a rather comprehensive list) by including the term $\sup_{\varphi(0,1)} f(\varphi)/\varphi \geq h(0)$ on the left-hand side. This improvement looks more significative if we also assume $(Dg)(0) = 0$, as we do in the following Theorem 2.2; the latter condition holds, for instance, if $D(0) = 0$ or $\dot{g}(0) = 0$. In this case (2.2) reduces to
\begin{equation}
\sup_{\varphi(0,1)} \frac{f(\varphi)}{\varphi} \leq c^* \leq 2 \sqrt{\sup_{\varphi(0,1)} \frac{D(\varphi)g(\varphi)}{\varphi} + \sup_{\varphi(0,1)} \frac{f(\varphi)}{\varphi}},
\end{equation}
which can be written with obvious notation as
\begin{equation}
c_a \leq c^* \leq c_{dr} + c_a,
\end{equation}
where the indexes label velocities related to the convection (we use the letter $a$ to suggest advection, in order to avoid the awkward notation $c_c$ for convection) or diffusion-reaction components. In spite of several different bounds provided for $c^*$ in the literature [22], in (2.5) the same term, accounting for the dependence on $f$, occurs in both the lower and upper bound. This symmetry, which shows the shift of the critical threshold as a consequence of the convective term $f$, occurs in none of the previous estimates.

The interpretation of $c_{dr}$ is well known since H: by a shooting argument, only profiles with a sufficiently high speed can reach 1 (a saddle for the dynamical system) starting from 0 (which consequently turns out to be a node instead of a center).

We now comment on $c_a$. In the diffusion-convection case (i.e., when $g = 0$), there exist profiles connecting $\ell \in (0, 1)$ to 0 if and only
\begin{equation}
s_\ell(\varphi) := \frac{f(\ell)}{\ell} \varphi > f(\varphi), \quad \text{for } \varphi \in (0, \ell),
\end{equation}
i.e., if the line joining the points $(0,0)$ and $(\ell, f(\ell))$ lies strictly above the graph of $f$ for $\varphi \in (0, \ell)$, see [13, Theorem 9.1]. In this case, then, the quantity $c_a$ represents the maximal speed that can be reached by the profiles connecting $\ell$ to 0, for $\ell$ ranging in $(0,1]$. Notice
that condition (2.6) is also necessary and sufficient in the purely hyperbolic case (i.e., when also \( D = 0 \)) in order that the equation \( u_t + f(u)_x = 0 \) admits a shock wave of speed \( f(\ell)/\ell \) with \( \ell \) as left state and 0 as right state. This is not surprising since the viscous profiles approximate the shock wave and converge to it in the vanishing viscosity limit. Indeed, condition (2.6) does not depend on \( D \).

The presence of the positive reaction term \( g \) satisfying (g01) (notice that when (g0) holds, and then \( g(1) > 0 \), we only have semi-wavefronts, but nevertheless the same bounds still hold) does not allow profile speeds to be less than \( c_a \): assuming that \( z \) satisfies Equation (1.6), by the positivity of both \( D \) and \( g \) we deduce

\[
\begin{align*}
\sup_{\varphi \in (0,1]} \left( \frac{f(\varphi)}{\varphi} - \frac{z(\varphi)}{\varphi} \right) & \geq c_a. 
\end{align*}
\]

Then, \( c_a \) now becomes a bound for the minimal speed of the profiles. Notice that the bound (2.7) is strict (i.e., there is a gap between \( c_a \) and \( c^* \)) if \( \dot{D}g(0) > 0 \); this occurs for instance if \( D(0) > 0 \) and \( \dot{g}(0) > 0 \) and follows by integrating (1.6) from 0 to \( \varphi \) and (2.2), see Remark 5.2. If \( f = 0 \), the corresponding strict bound \( c^* > 0 \) occurs for any positive and continuous \( D \) and \( g \): if \( c^* = 0 \) then \( z \) should be an increasing function by (3.11), a contradiction.

The following corollary investigates the qualitative properties of the profiles when they reach the equilibrium 0; the classification is complete, apart from some possibilities corresponding to \( c^* = h(0) \); in these minor sub-cases, further assumptions are needed, see e.g. Remark 10.1. Notice that below the existence of the \( \lim_{\xi \to a^+} D(\varphi(\xi)) \varphi'(\xi) \) is a consequence of the definition (9.20) and Lemma 3.1.

**Corollary 2.1.** Under the assumptions of Theorem 2.1, let \( c \geq c^* \) and \( \varphi \) be a strict semi-wavefront to 0 of (1.1), connecting 1 to 0, defined in its maximal-existence interval \((a, +\infty)\). Then, for \( c > c^* \), there exists \( \hat{\beta}(c) \in [\beta(c), 0] \) such that the following results hold.

(i) \( D(0) > 0 \) implies that \( \varphi \) is classical and strictly decreasing.

(ii) \( D(0) = 0, \ c > c^* \) and

\[
\lim_{\xi \to a^+} D(\varphi(\xi)) \varphi'(\xi) > \hat{\beta}(c),
\]

imply that \( \varphi \) is classical; moreover, \( \varphi \) reaches 0 at some \( \xi_0 > a \), and then it is not strictly decreasing, if

\[
c > h(0) + \sup_{\varphi \to 0^+} \frac{g(\varphi)}{\varphi}. \tag{2.9}
\]

(iii) \( D(0) = 0, \ c^* > h(0) \) and

either \( c = c^* \) or \( \lim_{\xi \to a^+} D(\varphi(\xi)) \varphi'(\xi) \leq \hat{\beta}(c) \) \( \tag{2.10} \)

imply that \( \varphi \) is sharp at 0 (reached at some \( \xi_0 > a \)) with

\[
\lim_{\xi \to \xi_0} \varphi'(\xi) = \begin{cases} 
\frac{h(0) - c}{D(0)} & \text{if } \dot{D}(0) > 0, \\
-\infty & \text{if } \dot{D}(0) = 0.
\end{cases} \tag{2.11}
\]
Figure 3: Examples of profiles occurring in Corollary 2.1. From the left to the right, they depict, respectively, what stated in Part (i), (ii) and (iii).

We shall prove in Proposition 8.2 that \( \hat{\beta}(c) = \beta(c) \) under suitable assumptions, but in general it is still open whether the two thresholds differ. Notice that \( \beta \) is related to the existence of the semi-wavefronts while \( \hat{\beta} \) deals with their smoothness (see Figure 3).

We now present our result on wavefronts; we assume that \( D \) and \( g \) satisfy (D0) and (g01). The goal is to extend results contained in [20, Theorems 2.1 and 6.1] regarding the existence and, more importantly, the regularity of wavefronts of Equation (1.1). In particular, the next theorem has the merit to derive the classification of wavefronts under (D0), merely, without additional assumptions (which were instead required in [20, Theorems 2.1 and 6.1]). Notice that in the following result we require that \( D \) vanishes at 0; this assumption leads to improve not only the left-hand bound (2.2) on \( c^\ast \) by (2.5), but also the right-hand bound, by means of a recent integral estimate provided in [22].

**Theorem 2.2.** Assume (f), (D0) and (g01) and (1.2). Then there exists \( c^\ast \), satisfying

\[
\sup_{\varphi \in [0,1]} \frac{f(\varphi)}{\varphi} \leq c^\ast \leq \sup_{\varphi \in [0,1]} \frac{f(\varphi)}{\varphi} + 2 \sqrt{\sup_{\varphi \in (0,1]} \frac{1}{\varphi} \int_{0}^{\varphi} \frac{D(\varphi)g(\varphi)}{\sigma} d\sigma},
\]

such that Equation (1.1) admits a (unique up to space shifts) wavefront, whose wave profile \( \varphi \) satisfies (1.4), if and only if \( c \geq c^\ast \).

Moreover, we have \( \varphi'(\xi) < 0 \), for any \( 0 < \varphi(\xi) < 1 \), and

(i) if \( c > c^\ast \), then \( \varphi \) is classical at 0;

(ii) if \( c = c^\ast \) and \( c^\ast > h(0) \), then \( \varphi \) is sharp at 0. Furthermore, \( \varphi \) reaches 0 at some \( \xi_0 \in \mathbb{R} \) and it holds that

\[
\lim_{\xi \to \xi_0^-} \varphi'(\xi) = \begin{cases} 
\frac{h(0) - c^\ast}{D(0)} & \text{if } D(0) > 0, \\
-\infty & \text{if } D(0) = 0.
\end{cases}
\]

As in analogous cases [7], Theorem 2.2 provides no information about the smoothness of the profiles when \( c = c^\ast = h(0) \). We show in Remark 10.1 that in such a case profiles may be either sharp or classical. Further assumptions on the joint behavior of \( f, D \) and \( g \) are needed to establish whether which of the possibility occurs.
3 Singular first-order problems

In this section we begin the analysis of the auxiliary problem (1.6), which shall be concluded
with Section 8. First, we consider, for $c \in \mathbb{R}$, the problem

$$
\begin{aligned}
\dot{z}(\varphi) &= h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, \quad \varphi \in (0, 1), \\
z(\varphi) &= 0, \quad \varphi \in (0, 1),
\end{aligned}
$$

where we assume

$$
q \in C^0[0, 1] \quad \text{and} \quad q > 0 \quad \text{in} \quad (0, 1).
$$

We point out that the differential equation (3.1) generalizes (1.6) since the assumptions
on $q$ are a bit less strict than the ones on $Dg$, under (D1)-(g0) or (D1)-(g01).

In the following lemma we prove that a solution of (3.1) can be extended continuously
up to the boundary.

**Lemma 3.1.** Assume (3.2). If $z \in C^1(0, 1)$ is a solution of (3.1), then it can be extended
continuously to the interval $[0, 1]$.

**Proof.** Since $q/z < 0$ in $(0, 1)$, then for any $0 < \varphi < \varphi_1 < 1$ the function

$$
\varphi \to \int_{\varphi}^{\varphi_1} \frac{q(\sigma)}{z(\sigma)} \, d\sigma
$$

is strictly increasing. Hence, we can pass to the limit as $\varphi \to 0^+$ in the expression

$$
z(\varphi) = z(\varphi_1) - \int_0^{\varphi_1} \left( h(\sigma) - c \right) \, d\sigma + \int_0^{\varphi_1} \frac{q(\sigma)}{z(\sigma)} \, d\sigma,
$$

which is obtained by integrating (3.1) in $(\varphi, \varphi_1)$. Then $z(0^+)$ exists and necessarily lies in
$[\varphi_1^{-}, 0]$ because of (3.1). If $z(0^+) = -\infty$, then by passing to the limit for $\varphi \to 0^+$ in (3.3)
we find a contradiction, since the last integral converges as $\varphi \to 0^+$. Hence, $z(0^+) \in (-\infty, 0]$.

For $z(1^-)$ the proof is even simpler: by integrating (3.1) in $(\varphi_2, \varphi_1)$, for $0 < \varphi_2 < \varphi_1 < 1$, we obtain (3.3) with $\varphi_2$ replacing $\varphi_1$. As before, we deduce that $z(1^-)$ exists. Also, since the last integral in (3.3) is now positive, we get $z(\varphi) > z(\varphi_2) + \int_{\varphi_2}^{\varphi} \left( h(\sigma) - c \right) \, d\sigma$, for any

$$
\varphi \in (\varphi_2, 1).$$

This directly rules out the alternative $z(1^-) = -\infty$ and concludes the proof.

We summarize here below [7, Lemmas 4.1 and 4.3] in a version ad hoc for our purposes,
by also exploiting Lemma 3.1. These technical tools were obtain in [7] under a bit more
specific assumptions on $q$. Nonetheless it is easy to verify that they also apply in the current
case, in virtue of (3.2).

A function $\eta \in C^1(\sigma_1, \sigma_2)$, for some $0 \leq \sigma_1 < \sigma_2 \leq 1$, is called an upper-solution of (3.1) in $(\sigma_1, \sigma_2)$ if

$$
i\eta(\varphi) \geq h(\varphi) - c - \frac{q(\varphi)}{\eta(\varphi)} \quad \text{for any} \quad \sigma_1 < \varphi < \sigma_2.
$$

The upper-solution $\eta$ is said strict if the inequality in (3.4) is strict. A function $\omega \in C^1(\sigma_1, \sigma_2)$
is a (strict) lower-solution of (3.1) in $(\sigma_1, \sigma_2)$ if the (strict) inequality in (3.4) is reversed.

**Lemma 3.2.** Assume (3.2) and consider equation (3.1): the following results hold.
1. Set \( \mu < 0 \). Then,

(a) let \( \sigma \in (0,1] \); the problem

\[
\begin{aligned}
\dot{z}(\varphi) &= h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, \quad \varphi < \sigma, \\
z(\sigma) &= \mu,
\end{aligned}
\]

admits a unique solution \( z \in C^0[0,\sigma] \cap C^1(0,\sigma) \);

(b) let \( \sigma \in [0,1) \); the problem

\[
\begin{aligned}
\dot{z}(\varphi) &= h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, \quad \varphi > \sigma, \\
z(\sigma) &= \mu,
\end{aligned}
\]

admits a unique solution \( z \in C^0[\sigma,\delta] \cap C^1(\sigma,\delta) \), for some maximal \( \sigma < \delta \leq 1 \). Moreover, either \( \delta = 1 \) or \( z(\delta) = 0 \).

2. Set \( 0 \leq \sigma_1 < \sigma_2 \leq 1 \); let \( z \) be a solution of (3.1) in \( (\sigma_1,\sigma_2) \). It holds that:

(a) if \( \eta \) is a strict upper-solution of (3.1) in \( (\sigma_1,\sigma_2) \), then

(i) if \( \eta(\sigma_2) \leq z(\sigma_2) < 0 \), then \( \eta < z \) in \( (\sigma_1,\sigma_2) \); moreover, if \( \eta \) is defined in \( [0,1] \), then \( z \) must be defined in \( [\sigma_1,1] \) and \( \eta > z \) in \( (\sigma_1,1) \);

(ii) if \( 0 > \eta(\sigma_1) \geq z(\sigma_1) \) then \( \eta > z \) in \( (\sigma_1,\sigma_2) \); moreover, if \( \eta \) is defined in \( [0,1] \), then \( z \) must be defined in \( [0,\sigma_2] \) and \( \eta > z \) in \( (0,\sigma_2) \);

(b) if \( \omega \) is a strict lower-solution of (3.1) in \( (\sigma_1,\sigma_2) \), then

(i) if \( 0 > \omega(\sigma_2) \geq z(\sigma_2) \), then \( \omega > z \) in \( (\sigma_1,\sigma_2) \); moreover, if \( \omega \) is defined in \( [0,1] \), then \( z \) must be defined in \( [0,\sigma_2] \) and \( \omega > z \) in \( (0,\sigma_2) \);

(ii) if \( \omega(\sigma_1) \leq z(\sigma_1) < 0 \) then \( \omega < z \) in \( (\sigma_1,\sigma_2) \).

\[\text{Figure 4: An illustration of Lemma 3.2 (2). Left: supersolutions } \eta; \text{ right: subsolutions } \omega.\]

In the context of equations as (3.1), proper limit arguments are often needed. For the reader’s convenience, we gather in Lemma 3.3 the ones we shall exploit.

**Lemma 3.3.** Assume (3.2). Let \( \{c_n\}_n \) be a sequence of real numbers and \( c \in \mathbb{R} \) such that \( c_n \to c \) as \( n \to \infty \). Let \( z_n \in C^0[0,1] \cap C^1(0,1) \) satisfy (3.1) corresponding to \( c_n \). If \( \{z_n\}_n \) is increasing and there exists \( v \in C^0[0,1] \) such that

\[
z_n(\varphi) \leq v(\varphi) < 0 \quad \text{for any } n \in \mathbb{N} \quad \text{and} \quad \varphi \in (0,1),
\]

(3.6)
then $z_n$ converges (uniformly on $[0,1]$) to a solution $\bar{z} \in C^0[0,1] \cap C^1(0,1)$ of (3.1).

The same conclusion holds if $\{z_n\}_n$ is decreasing and there exists $w \in C^0[0,1]$ such that

$$z_n(\varphi) \geq w(\varphi) \quad \text{for any } n \in \mathbb{N} \text{ and } \varphi \in (0,1).$$

**Proof.** Take first $\{z_n\}_n$ increasing. From (3.6), we can define $\bar{z} = \bar{z}(\varphi)$ as

$$\lim_{n \to \infty} z_n(\varphi) =: \bar{z}(\varphi), \quad \varphi \in (0,1).$$

It is obvious that $z_1 \leq \bar{z} \leq v < 0$ in $(0,1)$. By integrating (3.1), we have

$$z_n(\varphi) - z_n(\varphi_0) = \int_{\varphi_0}^\varphi \left( h(\sigma) - c_n + \frac{q(\sigma)}{-z_n(\sigma)} \right) \, d\sigma \quad \text{for any } \varphi_0, \varphi \in (0,1).$$

Since, for every $\sigma \in (0,1)$, the sequence

$$\left\{ \frac{q(\sigma)}{-z_n(\sigma)} \right\}_n$$

is increasing, then the Monotone Convergence Theorem implies that

$$\bar{z}(\varphi) - \bar{z}(\varphi_0) = \int_{\varphi_0}^\varphi \left( h(\sigma) - c - \frac{q(\sigma)}{\bar{z}(\sigma)} \right) \, d\sigma \quad \text{for any } \varphi_0, \varphi \in (0,1),$$

where all the involved quantities are finite. This tells us that $\bar{z}$ is absolutely continuous in every compact interval $[a,b] \subset (0,1)$. By differentiating, we then obtain that $\bar{z} \in C^1(0,1)$ satisfies (3.1). From Lemma 3.1, we also have that $\bar{z} \in C^0[0,1]$. To conclude that $z_n$ converges to $\bar{z}$ uniformly on $[0,1]$, it only remains to prove that

$$\bar{z}(0^+) = \lim_{n \to \infty} z_n(0) \quad \text{and} \quad \bar{z}(1^-) = \lim_{n \to \infty} z_n(1). \quad (3.7)$$

Indeed, if (3.7) holds, then $\{z_n\}_n$ turns out to be a monotone sequence of continuous functions converging pointwise to $\bar{z} \in C^0[0,1]$ on a compact set. Then, by Dini’s monotone convergence theorem (see [27, Theorem 7.13]), $z_n$ must converge uniformly to $\bar{z}$ on $[0,1]$. We prove only (3.7) since the proof of the other identity runs similarly. If $z_n(0) \to 0$, as $n \to \infty$, then $\bar{z}(0^+) = 0$, because $z_n \leq \bar{z} < 0$ in $(0,1)$. Hence (3.7), is verified. If instead $z_n(0) \to \mu < 0$, we argue as follows. Consider $\delta \in \mathbb{R}$ such that $c_n > \delta$, for any $n \in \mathbb{N}$, and let $\eta = \eta(\varphi)$ satisfy

$$\begin{cases}
\dot{\eta}(\varphi) = h(\varphi) - \delta - \frac{q(\varphi)}{\eta(\varphi)}, \quad \varphi > 0, \\
\eta(0) = \mu.
\end{cases} \quad (3.8)$$

An application of Lemma 3.2 (1.b) informs us that such an $\eta$ exists in its maximal-existence interval $[0,\sigma)$, for some $\sigma \in (0,1]$. Moreover, we have

$$\dot{\eta}(\varphi) > h(\varphi) - c_n - \frac{q(\varphi)}{\eta(\varphi)}, \quad \varphi \in (0,\sigma).$$

Hence, in $(0,\sigma)$, $\eta$ is a strict upper-solution of (3.1) with $c = c_n$ and $z_n(0) \leq \eta(0) < 0$. Thus, Lemma 3.2 (2.a.ii) implies that $z_n \leq \eta$ in $(0,\sigma)$. By passing to the pointwise limit, for
\( n \to \infty \), it is clear that \( \bar{z} \leq \eta \) in \((0, \sigma)\). Since \( \bar{z}, \eta \) are continuous up to \( \varphi = 0 \), then \( \bar{z}(0^+) \leq \mu \).

On the other hand we have \( \bar{z}(0^+) \geq \mu \) because \( z_n \leq \bar{z} \) in \((0, 1)\) and \( z_n, \bar{z} \in C^0[0, 1] \). Then \( \bar{z}(0^+) = \mu \) and this concludes the proof of (3.7).

Consider \( \{z_n\}_n \) decreasing. By adapting the arguments used in the first part of this proof, we can show that \( z_n \) converges pointwise in \((0, 1)\) to \( \bar{z} \in C^0[0, 1] \cap C^1(0, 1) \) satisfying \( (3.1) \). As before we need \( (3.7) \) to conclude. To this end, we again observe that similarly to the case of \( \{z_n\}_n \) increasing, we have \( (3.7) \) if both \( z_n(0) \to \mu < 0 \) and \( z_n(1) \to \nu < 0 \). Instead, the proofs of either \( (3.7) \) when \( z_n(0) \to 0 \) and \( (3.7) \) when \( z_n(0) \to \) are now more subtle. We provide them both. First, since \( z_n < 0 \) in \((0, 1)\), observe that requiring that \( z_n(0) \to 0 \) (or \( z_n(1) \to 0 \)) corresponds to have \( z_n(0) = 0 \) (or \( z_n(1) = 0 \), for every \( n \in \mathbb{N} \).

Take \( z_n(0) = 0 \), for every \( n \in \mathbb{N} \). Let \( n \in \mathbb{N} \) and for \( \varphi \in (0, 1) \), let \( \sigma_\varphi \in (0, \varphi) \) be defined by

\[
\hat{z}_n(\sigma_\varphi) = \frac{z_n(\varphi)}{\varphi}.
\]

Take \( \delta_1 \in \mathbb{R} \) such that \( \delta_1 > c_n \), for each \( n \in \mathbb{N} \). By using \( (3.1)_1 \) and the fact that \( q/z_n < 0 \) in \((0, 1)\), we deduce, for any \( \varphi \in (0, 1) \),

\[
\frac{z_n(\varphi)}{\varphi} = \hat{z}_n(\sigma_\varphi) > h(\sigma_\varphi) - c_n > \inf_{\varphi \in (0, 1)} h(\varphi) - \delta_1 =: C < 0. \tag{3.9}
\]

The sign of \( C \) is justified by the fact that \( c_n \geq h(0) \), for every \( n \in \mathbb{N} \); otherwise, it would not be possible to have \( z_n \) satisfying \( (3.1) \) and \( z_n(0) = 0 \). Inequality (3.9) implies that \( z_n(\varphi) > C\varphi \) for \( \varphi \in (0, 1) \). Hence, letting \( n \to \infty \), this leads to \( \bar{z}(\varphi) \geq C\varphi \), for \( \varphi \in (0, 1) \).

Passing to the limit as \( \varphi \to 0^+ \) gives \( \bar{z}(0^+) \geq 0 \), which in turn implies that \( \bar{z}(0^+) = 0 \). Thus, \( (3.7)_1 \) is verified.

Lastly, let \( z_n(1) = 0 \), for any \( n \in \mathbb{N} \). Fix \( \varepsilon > 0 \) and consider \( \eta_2 = \eta_2(\varphi) \) such that

\[
\begin{cases}
\eta_2(\varphi) = h(\varphi) - \delta - \frac{q(\varphi)}{\eta_2(\varphi)}, & \varphi > 0, \\
\eta_2(1) = -\varepsilon < 0,
\end{cases} \tag{3.10}
\]

where \( \delta \in \mathbb{R} \) is such that \( \delta < c_n \), for any \( n \in \mathbb{N} \). Such an \( \eta_2 \) exists and is defined and continuous in \([0, 1]\), because of Lemma 3.2 (1.a) and Lemma 3.1. Take an arbitrary \( n \in \mathbb{N} \). From \( 0 = z_n(1) > \eta_2(1) \), it follows that \( \eta_2 < z_n \) in \([\sigma_n, 1]\), for some \( \sigma_n > 0 \), with \( z_n(\sigma_n) < 0 \).

Thus, since

\[
\eta_2(\varphi) > h(\varphi) - c_n - \frac{q(\varphi)}{\eta_2(\varphi)}, \quad \varphi \in (0, 1),
\]

then \( \eta_2 \) is a strict upper-solution of \( (3.1)_1 \) with \( c = c_n \) in \((0, \sigma_n)\) and \( \eta_2(\sigma_n) < z_n(\sigma_n) < 0 \). An application of Lemma 3.2 (2.a.i) implies that \( \eta_2 < z_n \) in \((0, \sigma_n)\). Thus, \( z_n > \eta_2 \) in \((0, 1)\), for any \( n \in \mathbb{N} \). By passing to the pointwise limit, as \( n \to \infty \), we then have \( \bar{z}(\varphi) \geq \eta_2(\varphi) \), for \( \varphi \in (0, 1) \). By the continuity of both \( \bar{z} \) and \( \eta_2 \) at \( \varphi = 1 \), we obtain \( 0 \geq \bar{z}(1^-) \geq -\varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, we deduce that necessarily \( \bar{z}(1^-) = 0 \). \( \square \)

Because of Lemmas 3.1 and 3.3, in the following we always look for (and understand) solutions \( z \) to problem \( (3.1) \), and analogous ones, in the class \( C[0, 1] \cap C^1(0, 1) \), without any further mention.
Motivated by Lemma 3.1 in the next sections we focus the following problem, where the boundary condition is given on the left extremum of the interval of definition:

$$
\begin{cases}
\dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, \quad \varphi \in (0, 1), \\
z(\varphi) < 0, \quad \varphi \in (0, 1), \\
z(0) = 0.
\end{cases}
$$

(3.11)

This problem is exploited in the case of semi-wavefronts. Notice that the value of $z$ at $\varphi = 1$ is not prescribed; obviously, from (3.11), we have $z(1) \leq 0$. Later on, we shall also briefly deal with an analogous problem, see (6.1), where however the boundary condition is given on the right extremum of the interval of definition.

The extremal case, i.e., $z(1) = 0$, has a peculiar role in what follows. It is then worth displaying explicitly that occurrence:

$$
\begin{cases}
\dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, \quad \varphi \in (0, 1), \\
z(\varphi) < 0, \quad \varphi \in (0, 1), \\
z(0) = z(1) = 0.
\end{cases}
$$

(3.12)

Such a system is needed in the study of wavefronts.

### 4 The singular problem with two boundary conditions

Problems (3.11) and (3.12) have solutions only when $c$ is larger than a critical threshold $c^*$. In this section we first give a new estimate to $c^*$ under mild conditions on $q$; then, we obtain a result of existence and uniqueness of solutions to (3.12) if $c \geq c^*$.

Recalling (D1), (g0) and (1.2) and (D0)-(g01), throughout the next sections (until Section 8) we need to strengthen the very weak assumptions (3.2) of the previous section; for commodity we gather them all here below. We assume

(q) $q \in C^0[0, 1], \ q > 0 \text{ in } (0, 1), \ q(0) = q(1) = 0, \text{ and } \limsup_{\varphi \to 0^+} \frac{q(\varphi)}{\varphi} < +\infty.$

We improve, in the same spirit of [22, Theorem 3.1], a well-known result [1, 13, 19]. More precisely, in the case that $q$ is differentiable at 0, in [22, Theorem 3.1] it is proved that Problem (3.11) has a solution if

$$
c > \sup_{\varphi \in (0,1]} \frac{f(\varphi)}{\varphi} + 2 \sqrt{\sup_{\varphi \in (0,1]} \frac{1}{\varphi} \int_0^{\varphi} \frac{q(\sigma)}{\sigma} d\sigma}.
$$

(4.1)

The last assumption in (q) is a bit weaker than the differentiability of $q$ at 0 and, as a consequence, our result below is less stronger than the one in [22]. It is an open problem whether the existence of solutions to Problem (3.12) under (4.1) can be achieved by only assuming $\limsup_{\varphi \to 0^+} q(\varphi)/\varphi < +\infty$.

**Lemma 4.1.** Assume (q) and suppose that

$$
c > \sup_{\varphi \in (0,1]} \frac{f(\varphi)}{\varphi} + 2 \sqrt{\sup_{\varphi \in (0,1]} \frac{q(\varphi)}{\varphi}}.
$$

(4.2)

Then Problem (3.12) admits a solution.
Proof. We follow mainly the line of the proof of [22, Theorem 3.1]. A plain inspection of (4.2) implies that there exists $K > 0$ and $\varepsilon > 0$ such that

$$K^2 + \left( \sup_{\varphi \in (0,1)} \frac{f(\varphi)}{\varphi} - c \right) K + \sup_{\varphi \in (0,1)} \frac{q(\varphi)}{\varphi} < -\varepsilon < 0 \text{ for } \varphi \in (0,1].$$

For every $\tau > 0$, we get, for any $\varphi > \tau$,

$$\frac{1}{\varphi - \tau} \int_{\tau}^{\varphi} \frac{q(s)}{s} \, ds = \frac{q(s_{\varphi, \tau})}{s_{\varphi, \tau}} \leq \sup_{\varphi \in (0,1]} \frac{q(\varphi)}{\varphi},$$

where $s_{\varphi, \tau} \in (\tau, \varphi)$ is detected by the Mean Value Theorem. As a consequence, for any $\tau > 0$,

$$K^2 + \left( \sup_{\varphi \in (0,1]} \frac{f(\varphi)}{\varphi} + \varepsilon - c \right) K + \frac{1}{\varphi - \tau} \int_{\tau}^{\varphi} \frac{q(s)}{s} \, ds < 0 \text{ for every } \varphi \in (\tau, 1].$$

A fine continuity argument in [22] implies that there exists $\overline{\tau}$ such that for any $\tau < \overline{\tau}$ we have

$$\frac{f(\varphi) - f(\tau)}{\varphi - \tau} \leq \frac{f(\varphi)}{\varphi} + \varepsilon \leq \sup_{\varphi \in (0,1]} \frac{f(\varphi)}{\varphi} + \varepsilon, \; \varphi \in (\tau, 1],$$

and thus, for such values of $\tau$, it must hold

$$K^2 + \left( \frac{f(\varphi) - f(\tau)}{\varphi - \tau} - c \right) K + \frac{1}{\varphi - \tau} \int_{\tau}^{\varphi} \frac{q(s)}{s} \, ds < 0 \text{ for every } \varphi \in (\tau, 1].$$

This, after straightforward computations, implies that the function $\eta_\tau = \eta_\tau(\varphi)$, defined for $\varphi \in [\tau, 1]$ by

$$\eta_\tau(\varphi) := -K\tau + \int_{\tau}^{\varphi} \left\{ h(\sigma) - c \frac{q(\sigma)}{-K\sigma} \right\} \, d\sigma,$$

is an upper-solution of (3.11), such that $\eta_\tau(\varphi) < -K\varphi$, for $\varphi \in (\tau, 1]$, and $\eta_\tau(\tau) = -K\tau < 0$. Arguments based essentially on Lemma 3.2 (2.a.ii) imply that it results defined in $[\tau, 1]$ a function $z_\tau$ which solves (3.5) with $\mu = -K\tau$; we extend continuously $z_\tau$ to $[0, \tau]$ by setting $z_\tau(\varphi) = -K\varphi$, for $\varphi \in [0, \tau]$. This gives a family $\{z_\tau\}_{\tau > 0}$ of decreasing functions as $\tau \to 0^+$ (in the sense that $z_{\tau_1} \leq z_{\tau_2}$ in $[0, 1]$ for $0 < \tau_1 < \tau_2$). After direct manipulations of (3.11), based essentially on the sign of $q/\varphi$ and on the fact that $\eta_\tau(\varphi) < -K\varphi$, for $\varphi \in (\tau, 1]$, we deduce that

$$f(\varphi) - c\varphi \leq z_\tau(\varphi) \leq -K\varphi, \; \varphi \in [0, 1].$$

Hence, applying Lemma 3.3 in each interval $(a, b) \subset [0, 1]$ we finally deduce that $\bar{z}$, the limit of $z_\tau$ for $\tau \to 0^+$, solves (3.11), $\bar{z} < 0$ in $(0, 1)$ and $\bar{z}(0) = 0$. Hence, $\bar{z}$ is a solution of (3.11). Finally, as observed in [22, Lemma 2.1] implies the conclusion.

We now give a result about existence and uniqueness of solutions to (3.12). We point out that Proposition 3.1, apart from the estimate due to Lemma 3.1, was already given in [21, Proposition 1]. For the reader’s convenience, we give its complete proof.
Proposition 4.1. Assume \((q)\). Then, there exists \(c^*\) satisfying

\[
h(0) + 2 \sqrt{\liminf_{\phi \to 0^+} \frac{q(\phi)}{\phi}} \leq c^* \leq 2 \sqrt{\sup_{\phi \in (0,1]} q(\phi)} + \sup_{\phi \in (0,1]} f(\phi),
\]

(4.3)
such that there exists a unique \(z\) satisfying (3.12) if and only if \(c \geq c^*\).

Proof. Consider the following problem

\[
\begin{align*}
\phi'' - (c - h(\phi)) \phi' + q(\phi) &= 0, \\
\phi(-\infty) &= 0, \quad \phi(+\infty) = 1.
\end{align*}
\]

(4.4)

Since \((q)\) holds, we observe that (4.4) was studied in [19]. In fact, (4.4) corresponds to [19, Equation (1.2)] with \(D = 1\) and \(g = q\). With this in mind, by applying [19, Theorem 4.1], we conclude that (4.4) admits a unique (up to space shifts) non-decreasing solution if and only if \(c \geq c^*\), with \(c^*\) satisfying

\[
h(0) + 2 \sqrt{\liminf_{\phi \to 0^+} \frac{q(\phi)}{\phi}} \leq c^* \leq 2 \sqrt{\sup_{\phi \in (0,1]} q(\phi)} + \max_{\phi \in [0,1]} h(\phi).
\]

As a consequence, by applying [19, Lemma 2.2], we have that also

\[
\begin{align*}
\dot{w}(\phi) &= c - h(\phi) - \frac{q(\phi)}{w(\phi)}, \quad \phi \in (0,1), \\
w(\phi) &> 0, \quad \phi \in (0,1), \\
w(0) &= w(1) = 0,
\end{align*}
\]

(4.5)
is uniquely solvable if and only if \(c \geq c^*\). A direct check shows that \(w\) solves (4.5) if and only if \(z := -w\) solves (3.12); we conclude that also such a \(z\) exists (uniquely) if and only if \(c \geq c^*\).

Finally, with Lemma 4.1 in mind and because of \(\sup_{\phi \in [0,1]} f(\phi)/\phi \leq \max_{\phi \in [0,1]} h(\phi)\), then (4.3) follows.

5 The singular problem with left boundary condition

In this section we face problem (3.11), where the value at 1 of the solution is not constrained. We always put ourselves under the assumptions of Proposition 4.1 and then assume \((q)\). Moreover, we always refer to the speed threshold \(c^*\) introduced in that proposition and denote by \(z^*\) the corresponding unique solution to (3.12). We refer to Figure 5 for an illustration of Propositions 4.1 and 5.1.

Proposition 5.1. Assume \((q)\). For every \(c > c^*\), there exists \(\beta = \beta(c) < 0\) satisfying

\[
\beta \geq f(1) - c,
\]

(5.1)
such that problem (3.11) with the additional condition \(z(1) = b < 0\) admits a unique solution \(z\) if and only if \(b \geq \beta\).
The existence of a solution $\hat{\psi}$ implies that $(1, \hat{\psi})$ is a maximal-existence interval of $\hat{\psi}$. Since $\hat{\psi}$, we get $\hat{\psi}(0^+) = 0$. Since $\hat{\psi}$ in $(0, \delta)$, we obtain that $\hat{\psi}(\delta^-) \leq z^*(\delta^-)$. Thus $\delta = 1$, otherwise $\hat{\psi}(\delta^-) < 0$, in contradiction with the fact that $(0, \delta)$ is the maximal-existence interval of $\hat{\psi}$.

From Lemma 6.2, $\hat{\psi}(1) \in \mathbb{R}$. It remains to prove that $\hat{\psi}(0^+) = 0$. From what we observed above, it follows that $z^* > \hat{\psi}(0^+)$. Hence, for any $\varphi \in (0, 1)$, we have

$$z^*(\varphi) - \hat{\psi}(\varphi) = c - c^* + \frac{q(\varphi)}{z^*(\varphi)\hat{\psi}(\varphi)} (z^* - \hat{\psi}(\varphi)) (\varphi) > \frac{q(\varphi)}{z^*(\varphi)\hat{\psi}(\varphi)} (z^* - \hat{\psi}(\varphi)) (\varphi) > 0.$$

This implies that $(z^* - \hat{\psi})$ is strictly increasing in $(0, 1)$ and hence

$$-\hat{\psi}(1) = z^*(1) - \hat{\psi}(1) > z^*(0) - \hat{\psi}(0) = 0.$$
Figure 6: The functions $\hat{z}_{\varphi_0}$ and $z^*$ of Step (i).

which means $\hat{z}_{\varphi_0}(1) < 0$. Thus, $\hat{z}_{\varphi_0}(1) \in A_c$.

Step (ii): if $b \in A_c$ then $[b, 0) \subset A_c$. Suppose that there exists $b \in A_c$ and let $z_b$ be the solution of (3.11) and $z_b(1) = b$. Take $b < b_1 < 0$. For Lemma 3.2 (1.a) there exists $z_{b_1}$ defined in $(0, 1)$ satisfying (3.11) and $z_{b_1}(1) = b_1 < 0$.

We claim that $z_b < z_{b_1}$ in $(0, 1)$. If not, then $z_b(\varphi_0) = z_{b_1}(\varphi_0) =: y_0 < 0$, for some $\varphi_0 \in (0, 1)$. Without loss of generality we can assume $z_b < z_{b_1}$ in $(\varphi_0, 1]$. We denote by $f_c(\varphi, y) = h(\varphi) - c - q(\varphi)/y$ the right-hand side of the differential equation in (3.11); the function $f_c$ is continuous in $[0, 1] \times (-\infty, 0)$ and locally Lipschitz-continuous in $y$. Hence, $z_b$ and $z_{b_1}$ are two different solutions of

$$\begin{cases} y' = f_c(\varphi, y), \varphi \in (\varphi_0, 1), \\ y(\varphi_0) = y_0, \end{cases}$$

which contradicts the uniqueness of the Cauchy problem. Thus, $z_b < z_{b_1} < 0$ in $(0, 1)$. Since $z_b$ satisfies (3.11) then $z_{b_1}(0^+) = 0$ and hence $b_1 \in A_c$.

Step (iii): $\inf A_c \in \mathbb{R}$. Suppose that $z$ satisfies Equation (3.11). As already observed, this implies $\dot{z}(\varphi) > h(\varphi) - c, \varphi \in (0, 1)$. Thus, for any $\varphi \in (0, 1)$,

$$z(\varphi) = z(\varphi) - z(0) \geq \int_0^\varphi h(\sigma) - c \, d\sigma = f(\varphi) - c\varphi. \quad (5.4)$$

This implies that $z(1) \geq f(1) - c$. Define $\beta = \beta(c)$ by

$$\beta := \inf A_c.$$

Thus, $\beta \geq f(1) - c > -\infty$, which also proves (5.1).

Step (iv): $\beta \in A_c$. Let $\{b_n\}_n \subset A_c$ be a strictly decreasing sequence such that $b_n \to \beta^+$. Since $b_n \in A_c$, each $b_n$ is associated with a solution $z_n$ of (3.11) and $z_n(1) = b_n$. From the uniqueness of the solution of Cauchy problem for (3.11), the sequence $z_n$ is decreasing.

For any given $\delta < \beta$, let $y$ be defined by

$$\begin{cases} \dot{y}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{y(\varphi)}, \varphi < 1 \\ y(1) = \delta < \beta. \end{cases}$$

Such a $y$ exists and is defined in $[0, 1]$ from Lemma 3.2 (1.a). Also, $b_n > \delta$, for any $n \in \mathbb{N}$. Thus, for any $n \in \mathbb{N}$, $z_n \geq y$ in $[0, 1]$. Lemma 3.3 implies that there exists $\varepsilon$ satisfying (3.1)
such that $z_n \to \bar{z}$ uniformly in $[0, 1]$ (see Figure 7). In particular, we deduce that $\bar{z}(0) = 0$ and $\bar{z}(1) = \beta$. Hence, we conclude that $\beta \in A_c$.

Putting together Steps (i) – (iv), we conclude that $A_c = [\beta, 0)$.

A monotonicity of solutions of (3.11) follows. We omit its proof since it is quite standard, once that Lemma 3.2 (2) is given. (See [7, Lemma 5.1].)

**Corollary 5.1** (Monotonicity of solutions). Assume (q). Let $c_2 > c_1 \geq c^*$ and assume that $z_1$ and $z_2$ satisfy (3.11) with $c = c_1$ and $c = c_2$, respectively. Then, if $z_1(1) \leq z_2(1)$ it occurs that $z_1 < z_2$ in $(0, 1)$.

A monotony property of $\beta(c)$ now follows.

**Corollary 5.2.** Under (q) we have:

(i) $\beta(c_2) < \beta(c_1)$ for every $c_2 > c_1 > c^*$;

(ii) $\beta(c) \to -\infty$ as $c \to +\infty$.

**Proof.** To prove (i), let $z_1$ be a solution of (3.11) corresponding to $c = c_1$ and such that $z_1(1) = b_1 \in A_{c_1}$. As a consequence of Lemma 3.2 (1.a), the problem

$$\begin{cases}
\dot{z}(\varphi) = h(\varphi) - c_2 - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0, 1), \\
z(1) = b_1 < 0,
\end{cases}$$

admits a (unique) solution $z_2$ defined in $[0, 1]$. Moreover, from the monotonicity of solutions given by Corollary 5.1, we have $z_1 < z_2 < 0$ in $(0, 1)$. Since $z_1(0) = 0$, then we have $z_2(0) = 0$. Thus, $A_{c_1} \subseteq A_{c_2}$ and hence $\beta(c_1) \geq \beta(c_2)$. To prove $\beta(c_1) > \beta(c_2)$ we argue as follows.

For any $\varphi_0 \in (0, 1)$ we can repeat the same arguments as in Step (i) of Proposition 5.1 by replacing $c$ with $c_2$ and $z^*$ with $z_1$ in (5.2). Thus, the problem

$$\begin{cases}
\dot{z}(\varphi) = h(\varphi) - c_2 - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0, 1), \\
z(\varphi_0) = z_1(\varphi_0) < 0,
\end{cases}$$

admits a unique solution $\hat{z}_{c_2}$ defined in $[0, 1]$, because necessarily any solution of the last problem must be bounded from above by $z_2$, see Figure 8. Moreover, by applying Lemma 3.2 (2.b.ii), $\hat{z}_{c_2} < z_1$ in $(\varphi_0, 1)$, which implies that $\hat{z}_{c_2}(1) < z_1(1)$, since

$$\dot{z}_{c_2}(\varphi) - \dot{z}_1(\varphi) = c_1 - c_2 + \frac{q(\varphi)}{z_1(\varphi)z_{c_2}(\varphi)} (\hat{z}_{c_2}(\varphi) - z_1(\varphi)) < 0 \text{ for any } \varphi \in (\varphi_0, 1).$$

Figure 7: The functions $z_n$, $y$ and $\bar{z}$ of Step (iv).
Since \( \beta(c_2) \leq \hat{z}_{c_2}(1) < z_1(1) = b_1 \) then we proved (ii) since \( b_1 \) is arbitrary in \( A_{c_1} \).

Finally, we prove (ii). For \( c > c^* \), let \( z_c \) be the solution of (3.11) such that \( z_c(1) = \beta(c) \).

For any fixed \( c_1 > c^* \), we have \( z_c < z_{c_1} \) in \((0,1)\), if \( c > c_1 \). Thus, for any \( c > c_1 \),

\[
\dot{z}_c(\varphi) = h(\varphi) - c + \frac{q(\varphi)}{-z_c(\varphi)} < h(\varphi) - c + \frac{q(\varphi)}{-z_{c_1}(\varphi)}, \quad \varphi \in (0,1).
\]

In particular, since \( z_{c_1} < 0 \) in \((0,1]\), then, for any \( 0 < \delta < 1 \), there exists \( M > 0 \) such that

\[
\frac{q(\varphi)}{-z_{c_1}(\varphi)} \leq M \quad \text{for any} \quad \varphi \in (\delta,1].
\]

Thus, for any \( \varphi \in (\delta,1),

\[
z_c(\varphi) \leq z_c(\delta) + f(\varphi) - f(\delta) + (M - c)(\varphi - \delta) < f(\varphi) - f(\delta) + (M - c)(1 - \delta),
\]

which in turn implies \( \beta(c) = z_c(1) \leq f(1) - f(\delta) + (M - c)(1 - \delta) \). Hence, (ii) is proved.

At last, we collect some important consequences of (5.4) and Lemma 4.1 (or [22, Theorem 3.1]), concerning a sharper estimate to \( c^* \). To the best of our knowledge these estimates are new, and we provide a comment to their meaning.

**Corollary 5.3.** Assume (q). It holds that

\[
c^* \geq \max \left\{ \sup_{\varphi \in (0,1]} \frac{f(\varphi)}{\varphi}, h(0) + 2 \sqrt{\liminf_{\varphi \to 0^+} \frac{q(\varphi)}{\varphi}} \right\}. \tag{5.5}
\]

**Proof.** Formula (5.5) in Step (iii) implies in particular that \( f(\varphi) < c\varphi \), for \( \varphi \in (0,1) \). Thus, \( f(\varphi) \leq c^*\varphi \), for \( \varphi \in (0,1) \). This implies that we obtain the following estimate from below for \( c^* \):

\[
c^* \geq \sup_{\varphi \in (0,1]} \frac{f(\varphi)}{\varphi}.
\]

This formula, together with (4.3) implies (5.5).

**Remark 5.1.** It is worth noting that Lemma 4.1 and Corollary 5.3 imply that, under (q), the threshold \( c^* \) verifies (2.2). Moreover, make the assumption \( \dot{q}(0) = 0 \), which is valid if for instance \( q = Dg \) under (D1), with \( D(0) = 0 \), (g0) or under (D0) and (g01). In this case,
the estimates in (2.12) hold true. Indeed, the assumptions on $q$ are covered by [22, Theorem 3.1] and hence it follows that

$$c^* \leq \sup_{\varphi \in (0,1]} \frac{f(\varphi)}{\varphi} + 2 \sqrt{\sup_{\varphi \in (0,1]} \frac{1}{\varphi} \int_0^\varphi \frac{q(\sigma)}{\sigma} d\sigma}.$$

The bound from above in (2.12) is then proved. The bound from below in (2.12) is instead due directly to (5.5), because of $\dot{q}(0) = 0$.

**Remark 5.2.** We can now make precise the statement following formula (2.7) about the gap between $c_a$ and $c^*$. Indeed, if the value $c_a$ is obtained at some $\varphi \in (0,1]$, then the sup in the right-hand side of (2.7) is strictly larger than $c_a$ because $z < 0$ in $(0,1)$. Then $c^* > c_a$. Otherwise, if $\sup_{\varphi \in (0,1]} f(\varphi)(\varphi) = h(0)$, then $c_a = h(0)$ and by (5.5) we still deduce $c^* > c_a$.

6 Further existence and non-existence results

Propositions 4.1 and 5.1 completely treat the existence of solutions of (3.12) and (3.11), respectively, in the cases $c \geq c^*$ and $c > c^*$. In this section, we investigate the remaining cases and show that such propositions are somehow optimal.

Preliminarily, we give a lemma. It deals with the following problem, where $c \in \mathbb{R}$ but, differently from (3.11), the boundary condition is imposed on the right extremum of the interval of definition:

$$\begin{align*}
\dot{\zeta}(\varphi) &= h(\varphi) - c - \frac{q(\varphi)}{\zeta(\varphi)}, \quad \varphi \in (0,1), \\
\zeta(\varphi) &= 0, \quad \varphi \in (0,1), \\
\zeta(1) &= 0.
\end{align*}$$

The differential equation in (3.11) and (6.1) is the same, and inherits the properties of the dynamical system underlying equation (1.3), which has a center or a node at $(0,0)$ and a saddle at $(1,0)$; no wonder then that the corresponding results, cf. Proposition 5.1 and Lemma 6.1, are different. This is well known, see e.g. Lemma 3.2 (1).

Moreover, while in problem (3.11) the threshold $c^*$ discriminated the existence of solutions, for problem (6.1) solutions will be proved to exist for every $c \in \mathbb{R}$; instead, the threshold $c^*$ enters into the problem to discriminate whether solutions reach 0 or not (see Figure 9). A related behavior was pointed out in [7, Theorem 2.6]. On the contrary, the monotonicity properties stated in Corollary 5.1 and in Lemma 6.1 are the same.

![Figure 9](image)

**Lemma 6.1.** Assume (q). For any $c \in \mathbb{R}$, Problem (6.1) admits a unique solution $\zeta_c$. If $c \geq c^*$ then $\zeta_c(0) = 0$ and if $c < c^*$ then $\zeta_c(0) < 0$. Moreover, we have:
(i) if \( c_2 > c_1 \) then \( \zeta_{c_2} > \zeta_{c_1} \) in \((0,1)\);

(ii) it holds that

\[
z^*(\varphi) = \lim_{c \to c^*} \zeta_c(\varphi) \text{ for any } \varphi \in [0,1].
\]

(6.2)

Proof. The part regarding the existence and the uniqueness was proved in [7, Theorem 2.6], while the monotonicity as stated in (i) was given in [7, Lemma 5.1]. It remains to prove (ii). We show that, for any \( \varphi \in [0,1] \),

\[
\lim_{\delta \to 0^+} \zeta_{c^*-\delta}(\varphi) = \lim_{\delta \to 0^+} \zeta_{c^*+\delta}(\varphi) = z^*(\varphi).
\]

For any \( \varphi \in [0,1] \), by (i) we have

\[
\zeta_{c^*-\delta_2}(\varphi) < \zeta_{c^*+\delta_2}(\varphi) < z^*(\varphi) < \zeta_{c^*+\delta_1}(\varphi) < \zeta_{c^*+\delta_1}(\varphi) \text{ for any } 0 < \delta_1 < \delta_2.
\]

(6.3)

These inequalities and Lemma 3.3 imply that there exist two functions \( \overline{w}, \underline{w} \in C^0[0,1] \cap C^1(0,1) \) such that

\[
\overline{w}(\varphi) = \lim_{\delta \to 0^+} \zeta_{c^*+\delta}(\varphi) \text{ and } \underline{w}(\varphi) = \lim_{\delta \to 0^+} \zeta_{c^*-\delta}(\varphi), \quad \varphi \in [0,1],
\]

and that both \( \underline{w} \) and \( \overline{w} \) satisfy (3.1) with \( c = c^* \). Since \( \underline{w}(1) = \overline{w}(1) = 0 \), both of them then solve (5.1). By the uniqueness of solutions of \( (5.1) \) it follows that \( \underline{w} = \overline{w} = z^* \).

Remark 6.1. Note that, because of the uniqueness stated in Lemma 6.1, it follows that, for any \( c \geq c^* \), the solution \( z \) given by Proposition 4.1 corresponds to \( \zeta_c \) of Lemma 6.1. Moreover, for \( c < c^* \) fixed, there exists a bound from below for \( \zeta_c(0) < 0 \). We have

\[
\zeta_c(0) \geq -1 - A_c,
\]

where

\[
A_c := \max \left\{ \max_{\varphi \in [0,1]} h(\varphi) - c, 0 \right\} + \max_{\varphi \in [0,1]} q(\varphi) > 0.
\]

Indeed, the function \( \eta(\varphi) := A_c (\varphi - 1) - 1 \), for \( \varphi \in [0,1] \), is a strict upper-solution of \( (6.1) \). Therefore, if \( \zeta_c(\varphi_0) \leq \eta(\varphi_0) \), for some \( \varphi_0 \in (0,1) \), then \( \zeta_c < \eta \) in \((\varphi_0,1)\) by Lemma 3.2 (2.a.ii), which is in contradiction with \( \zeta_c(1) = 0 > \eta(1) \). Thus, \( \zeta_c(0) \geq \eta(0) = -A_c - 1 \).

Notice that, for \( c \geq \max h \), \( A_c = \max q \) does not depend on \( c \), while \( A_c \to \infty \), as \( c \to -\infty \).

The following result shows that \( \beta(c^*) = 0 \) under two light additional assumptions. The first one strengthens the last condition in (q) and is satisfied if \( \hat{q}(\varphi) = O(\varphi^\alpha) \) for \( \varphi \to 0^+ \), for some \( \alpha > 0 \).

Proposition 6.1. Assume (q) and also

\[
\int_0^\infty \frac{q(\sigma)}{\sigma^2} d\sigma < +\infty \quad \text{and} \quad c^* > h(0).
\]

(6.4)

Then Problem \( (3.11) \) with \( c = c^* \) admits a unique solution \( z \), which satisfies \( z(1) = 0 \).
Figure 10: The functions $z^*$, $\zeta_c$, $y^*$ and $z_c^*$, for $c < c^*$.

*Proof.* Suppose, by contradiction, that there exists $y^*$ which solves (3.11) and $y^*(1) < 0$; observe that

$$z^* > y^* \quad \text{in} \quad (0, 1].$$  

(6.5)

We show that $y^*$ is an upper bound for the family of functions \{\(z_c^*\)\}_{c < c^*} defined as follows, see Figure 10. For any $c < c^*$, let $\zeta_c$ be the solution of (6.1), given in Lemma 6.1. Consider the initial-value problem

$$
\begin{aligned}
\dot{z}(\varphi) &= h(\varphi) - c - q(\varphi), \quad \varphi \in (0, 1), \\
z(0) &= \zeta_c(0) < 0.
\end{aligned}
$$

(6.6)

By Lemma 3.2 (1.b), problem (6.6) admits a unique solution $z_c^*$ in $[0, \delta]$ for some $\delta \leq 1$. Moreover, since $z_c^*(0) < 0$ and $z_c^*$ satisfies (6.6), then $z_c^* < z^*$ in $[0, \delta)$. Thus, if $\delta < 1$ then we have $-\infty < z_c^*(\delta) \leq z^*(\delta) < 0$ by Lemma 5.1 and hence $\delta = 1$. Since the same argument holds with $y^*$ in place of $z^*$, then

$$y^* > z_c^* \quad \text{in} \quad [0, 1).$$

(6.7)

By both (6.2) and (6.7) we now find a contradiction, which implies that such a $y^*$ cannot exist. For this, for any $c < c^*$, define $\eta_c$ by

$$\eta_c(\varphi) = \zeta_c(\varphi) - z_c^*(\varphi), \quad \varphi \in [0, 1].$$

Since $z_c^*$ is a strict lower-solution of (3.11), then Lemma 3.2 (2.b.ii) implies $\eta_c > 0$ in $(0, 1)$. We claim that, for any fixed $\varphi_0 \in (0, 1)$, $\eta_c(\varphi_0)$ is uniformly bounded from below for $c$ close to $c^*$. Indeed, for any $0 < \delta < (z^* - y^*)(\varphi_0)$, we clearly have, by (6.7) and (6.5),

$$\eta_c(\varphi_0) > \zeta_c(\varphi_0) - y^*(\varphi_0) = (\zeta_c - z^*) (\varphi_0) + (z^* - y^*) (\varphi_0) > (\zeta_c - z^*) (\varphi_0) + \delta.$$ 

Thus, in virtue of (6.2), for any $c$ sufficiently close to $c^*$, we have

$$\eta_c(\varphi_0) \geq \frac{\delta}{2} > 0,$$  

(6.8)

which proves our claim. On the other hand, define $k = k(\varphi) > 0$ by

$$k(\varphi) := \frac{q(\varphi)}{(z^* y^*)(\varphi)}, \quad \varphi \in (0, 1).$$
From assumption \((6.4)_1\), we deduce \(\dot{z}^*(0) = h(0) - c^*\) (see \[7\] Proposition 5.2). Also, by \((6.5)\) we deduce that \(y^* z^* > z^*^2\) in \((0, 1]\). Thus,

\[
k(\varphi) < \frac{q(\varphi)}{\varphi^2} \left( \frac{\varphi}{z^*(\varphi)} \right)^2 = \frac{q(\varphi)}{\varphi^2} \left\{ \frac{1}{(c^* - h(0))^2} + o(1) \right\} \quad \text{for} \quad \varphi \to 0^+.
\]

This leads to

\[
\int_0^{\varphi_0} k(\sigma) d\sigma =: M < +\infty
\]

by means of \((6.4)\). Since \(\zeta_c\) and \(z_c^*\) satisfy \((3.11)_1\) with \(c < c^*\) and \(c = c^*\), respectively, and since \(\zeta_c z_c^* > z^* y^*\) by the monotonicity stated in Lemma 6.1 and \((6.7)\), then

\[
\eta_c(\varphi) = c^* - c - \frac{q(\varphi)}{\zeta_c(\varphi) z_c^*(\varphi)} (z_c^*(\varphi) - \zeta_c(\varphi)) < c^* - c + k(\varphi) \eta_c(\varphi),
\]

for \(\varphi \in (0, 1)\). After some straightforward manipulations, this gives

\[
\frac{d}{d\varphi} \left( \eta_c(\varphi) e^{-\int_0^{\sigma} k(\sigma) d\sigma} \right) \leq (c^* - c) e^{-\int_0^{\sigma} k(\sigma) d\sigma}, \quad \varphi \in (0, 1).
\]

By integrating in \((0, \varphi_0)\) (where \(\varphi_0\) is the point for which \((6.8)\) holds) we obtain

\[
0 < \eta_c(\varphi_0) \leq \left( c^* - c \right) e^{\int_0^{\varphi_0} k(\sigma) d\sigma} \left[ e^{-\int_0^{\varphi_0} \int_0^{\sigma} k(s) ds d\sigma} \right] \leq (c^* - c) e^M \varphi_0, \quad \text{for any } \varphi_0 < \varphi < 0,
\]

where we used that \(e^{-\int_0^{\sigma} k(s) ds} \leq 1\), for any \(0 < \sigma < \varphi_0\), because of \(k > 0\). Since \(M\) does not depend on \(c\), from \((6.9)\), we conclude that \(\eta_c(\varphi_0) \to 0\), for \(c \to c^*\). This clearly contradicts \((6.8)\).

We notice that if \(q = Dg\), with \(D \in C^1[0, 1]\), then \((6.4)\) follows if we have both \(D(0) = 0\) and there exists \(L \geq 0\) such that \(g(\varphi) \leq L \varphi^\alpha\) for any \(\varphi\) in a right neighborhood of 0 and some \(\alpha > 0\). We point out that \(\dot{q}(0) = 0\) does not imply \((6.4)\), necessarily. About \((6.4)_2\), more subtle observations are needed. The next remark takes care of them.

**Remark 6.2.** We now comment on \((6.4)_2\).

First, from \((1.3)\), we have \(c^* \geq \sup_{\varphi \in (0, 1]} \frac{f(\varphi)}{\varphi} \geq h(0)\). We show that the case \(c^* = h(0)\) can indeed occur and then \((6.4)_2\) is a real assumption. Set, for \(\varphi \in (0, 1)\),

\[
q(\varphi) = \varphi^3 (1 - \varphi), \quad h(\varphi) = 3\varphi (\varphi - 1),
\]

and

\[
z(\varphi) = \varphi^2 (\varphi - 1).
\]

Direct computations show that \(z\) satisfies \((3.11)\) with \(c = 0 = h(0)\). Hence, \(c^* = h(0)\), because of \(c^* \geq h(0)\).

Second, in the spirit of \([20]\) Theorems 1.2 and 1.3, which concerns a similar case, we claim that \((6.4)_2\) occurs if there exists \(\delta > 0\) such that

\[
h(\varphi) \geq h(0) \quad \text{for all } \varphi \in [0, \delta].
\]
Indeed, if \( z \) is a solution of (3.11) with \( c = c^* \), then from (3.11) we have \( \dot{z}(\varphi) > h(\varphi) - c^* \geq h(0) - c^* \), for \( \varphi \in (0, \delta) \). This implies \( h(0) - c^* \leq \inf_{\varphi \in (0, \delta)} \dot{z}(\varphi) < 0 \), because of (3.11) and (3.11)3, which proves our claim.

Lastly, we show by a counter-example that the conclusion of Proposition 6.1 fails when (6.4)1 holds but (6.4)2 does not. Consider, for \( \varphi \in [0, 1] \),

\[
q(\varphi) = \varphi^4 (1 - \varphi) \quad \text{and} \quad y^*(\varphi) = -\varphi^2
\]

Clearly, \( y^* < 0 \) in \((0, 1)\) and \( y^*(0) = 0 \). Furthermore, we have

\[
\dot{y}^*(\varphi) + \frac{q(\varphi)}{y^*(\varphi)} = -2\varphi - \varphi^2 (1 - \varphi), \quad \varphi \in (0, 1).
\]

This implies that \( y^* \) satisfies (3.11) with

\[
h(\varphi) = -2\varphi - \varphi^2 (1 - \varphi) \quad \text{and} \quad c = 0.
\]

As a consequence, by exploiting the minimality of \( c^* \) together with \( h(0) = 0 \) and \( c^* \geq h(0) \), we deduce \( c^* = h(0) = 0 \). Thus, we proved that there exists \( q \) satisfying (6.4)1 such that (3.11) with \( c = c^* = h(0) \) admits a solution \( y^* \) such that \( y^* \neq z^* \). (Recall that \( z^* \) is the solution of (3.12) corresponding to \( c = c^* \).)

**Proposition 6.2.** Assume \((q)\). For no \( c < c^* \) problem (3.11) admits solutions.

**Proof.** Take \( c < c^* \) and assume by contradiction that problem (3.11) has a solution \( z \). Let \( \zeta = \zeta_c \) be the solution of (6.1) given by Lemma 6.1. Note that necessarily such a \( \zeta \) must satisfy \( \zeta(0) < 0 \), by Proposition 4.1. Then it holds that \( \zeta(\varphi_0) = z(\varphi_0) =: y_0 < 0 \), for some \( \varphi_0 \in (0, 1) \); see Figure 11. This contradicts the uniqueness of the Cauchy problem associated to (6.1)1. Thus, the proof is concluded.

![Figure 11: The functions z and ζ.](image)

\[
\theta(1) \quad \text{exists and is finite.} \quad (7.1)
\]

7 The behavior of \( z \) near 1

In this section and in the next one we investigate the behavior of the solutions \( z \) to (3.11) at 1 and 0. We now deal with the former case. In the following proposition, under a bit more regularity on the term \( q \) at the right extremum, we prove that \( \dot{z}(1) \) exists and explicitly compute its value. We suppose that

\[
\dot{q}(1) \quad \text{exists and is finite.} \quad (7.1)
\]
Proposition 7.1. Assume \((q)\) and \((7.1)\); consider \(c \geq c^*\) and let \(z\) be a solution of \((3.11)\). Then, \(\dot{z}(1)\) exists and it holds that

(i) if \(z(1) \in [\beta, 0)\), then
\[
\dot{z}(1) = h(1) - c;
\]

(ii) if \(z(1) = 0\), then
\[
\dot{z}(1) = \begin{cases} 
\frac{1}{2} \left[ h(1) - c + \sqrt{(h(1) - c)^2 - 4\dot{q}(1)\gamma} \right] & \text{if } \dot{q}(1) < 0, \\
\max \{0, h(1) - c\} & \text{if } \dot{q}(1) = 0.
\end{cases}
\]

(7.2)

Proof. Case (i). Since \(z(1) < 0\), taking the limit for \(\varphi \to 1^-\) in \((3.11)_1\) gives
\[
\lim_{\varphi \to 1^-} \dot{z}(\varphi) = h(1) - c.
\]

By the Mean Value Theorem and the uniqueness of the limit it follows \(\dot{z}(1) = h(1) - c\).

Case (ii). Notice that if \(\dot{z}(1)\) exists then \(\dot{z}(1) \geq 0\); we claim that \(\dot{z}(1)\) exists and is finite. Suppose instead that \(\dot{z}(1)\) does not exist and then
\[
\ell := \liminf_{\varphi \to 1^-} \frac{z(\varphi)}{\varphi - 1} < \limsup_{\varphi \to 1^-} \frac{z(\varphi)}{\varphi - 1} =: L,
\]
for \(0 \leq \ell < L \leq \infty\). Take \(\gamma \in (\ell, L)\). It is plain to verify that there exist two sequences \(\{\sigma^1_n\}_n, \{\sigma^2_n\}_n \subset (0,1)\) satisfying \(\lim_{n \to \infty} \sigma^i_n = 1\), for \(i = 1, 2\), and such that, for any \(n \in \mathbb{N}\),
\[
\frac{z(\sigma^i_n)}{\sigma^i_n - 1} = \gamma, \text{ for } i = 1, 2,
\]
and
\[
\frac{d}{d\varphi} \left\{ \frac{z(\varphi)}{\varphi - 1} \right\}_{\varphi = \sigma^1_n} \geq 0, \quad \frac{d}{d\varphi} \left\{ \frac{z(\varphi)}{\varphi - 1} \right\}_{\varphi = \sigma^2_n} \leq 0.
\]
(7.3)
(7.4)

Focus on \(\sigma^1_n\). We have
\[
\frac{d}{d\varphi} \left\{ \frac{z(\varphi)}{\varphi - 1} \right\}_{\varphi = \sigma^1_n} = \frac{1}{\sigma^1_n - 1} \left( \dot{z}(\varphi) - \frac{z(\varphi)}{\varphi - 1} \right),
\]
which, for \(\varphi = \sigma^1_n\), by \((6.3)\) and \((6.4)_1\) gives
\[
0 \leq \frac{1}{\sigma^1_n - 1} \left( \dot{z}(\sigma^1_n) - \frac{z(\sigma^1_n)}{\sigma^1_n - 1} \right) = \frac{1}{\sigma^1_n - 1} \left( \dot{z}(\sigma^1_n) - \gamma \right).
\]

Since \(z\) satisfies the differential equation in \((3.11)\) and \(\sigma^1_n - 1 < 0\), we deduce
\[
h(\sigma^1_n) - c - \frac{q(\sigma^1_n)}{\gamma (\sigma^1_n - 1)} \leq \gamma.
\]

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Hence, by passing to the limit as \( n \to \infty \), we deduce \( (h(1) - c) \gamma - \dot{q}(1) \leq \gamma^2 \). Analogously, taking \( \sigma_n^2 \) instead of \( \sigma_n^1 \) gives the reverse inequality for \( \gamma \). Hence, we have

\[
\gamma^2 - (h(1) - c) \gamma + \dot{q}(1) = 0.
\]

This is absurd, since \( \gamma \) is arbitrary in \( (\ell, L) \), and then \( \ell = L \); therefore

\[
\lim_{\varphi \to 1^-} \frac{z(\varphi)}{\varphi - 1} =: \mu \in [0, \infty].
\]

Now, for any \( \varphi \in (0, 1) \), by the Mean Value Theorem there exists \( \sigma_{\varphi} \in (\varphi, 1) \) satisfying

\[
\dot{z}(\sigma_{\varphi}) = \frac{z(\varphi)}{\varphi - 1}. \tag{7.6}
\]

By the definition of \( \dot{z}(1) \) it then follows that

\[
\lim_{\varphi \to 1^-} \dot{z}(\sigma_{\varphi}) = \mu \quad \text{and} \quad \lim_{\varphi \to 1^-} \frac{z(\sigma_{\varphi})}{\sigma_{\varphi} - 1} = \mu. \tag{7.7}
\]

If we evaluate (3.11) at each \( \sigma_{\varphi} \), some plain manipulations give

\[
(\dot{z}(\sigma_{\varphi}) - h(\sigma_{\varphi}) + c) \frac{z(\sigma_{\varphi})}{\sigma_{\varphi} - 1} + \frac{\dot{q}(\sigma_{\varphi})}{\sigma_{\varphi} - 1} = 0 \quad \text{for any} \quad \varphi \in (0, 1). \tag{7.8}
\]

We infer that \( \mu \neq +\infty \). If not, by letting \( \varphi \to 1^- \) we obtain that the left-hand side of (7.8) diverges to \(+\infty\) (recall (7.1)), while its right-hand side equals 0. Thus, \( \mu \in \mathbb{R} \) and \( \dot{z}(1) = \mu \).

Also, by means of (7.7), letting \( \varphi \to 1^- \) in (7.8) shows that \( \mu \) must satisfy (7.5). This implies that \( \mu \in \{r_-, r_+\} \), where \( r_- \leq r_+ \) denote the two real roots of (7.5), given by

\[
r_\pm := \frac{h(1) - c \pm \sqrt{(h(1) - c)^2 - 4\dot{q}(1)}}{2}.
\]

A direct check shows that the right-hand side of (7.2) corresponds exactly to \( r_+ \). Thus, if we prove that \( \mu = r_+ \) then we conclude the proof.

If \( \dot{D}(1) < 0 \), the fact that \( r_- < 0 \) implies necessarily that \( \mu = r_+ \), because of \( \mu \geq 0 \). Let \( \dot{D}(1) = 0 \). From (3.11)_1, the sign conditions on \( q \) and \( z \) given (respectively) in (3.2)_2 and (3.11)_2 imply that

\[
\dot{z}(\sigma_{\varphi}) > h(\sigma_{\varphi}) - c, \quad \varphi \in (0, 1). \tag{7.9}
\]

By (7.7), passing to the limit as \( \varphi \to 1^- \) gives \( \mu \geq h(1) - c \), because of the continuity of \( h \) at 1. Moreover, since \( \mu \geq 0 \) it holds that

\[
\mu \geq \max\{0, h(1) - c\} = r_+.
\]

This concludes the proof, since it necessarily follows that \( \mu = r_+ \) also in this case. \( \square \)

**Remark 7.1.** We shall prove in Remark 9.1 that \( z \in C^1(0, 1] \) under the assumptions of Proposition 7.1. More importantly, we now show that (7.1) is indeed necessary for the existence of \( \dot{z}(1) \). For \( \varphi \in [0, 1] \), define

\[
q(\varphi) = \varphi^3 (1 - \varphi) \left[ (\sin (\log (1 - \varphi)) + 2)^2 + 2 \cos (\log (1 - \varphi)) + \frac{1}{2} \sin (2 \log (1 - \varphi)) \right].
\]
The function \( q \) satisfies \((q)\), while \( \dot{q}(1) \) does not exist. Direct computations show that the function \( z = z(\varphi) \) defined by

\[
z(\varphi) = - \left( 2 + \sin \left( \log (1 - \varphi) \right) \right) (1 - \varphi) \varphi^2
\]
satisfies \((3.11)\) with \( c = 0 \) and

\[
h(\varphi) = \varphi (\varphi - 1) \left[ \cos \left( \log (1 - \varphi) \right) + 3 \sin \left( \log (1 - \varphi) \right) + 6 \right].
\]

It is easy to verify that \( \dot{z}(1) \) does not exist.

8 The behavior of \( z \) near 0

In this section we prove that \( \dot{z}(0) \) exists and compute its value. We first give a lemma. For \( \varphi_0 \in (0,1) \) consider the problem, see Figure 12,

\[
\begin{cases}
\dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, \; \varphi \in (0,1), \\
z(\varphi_0) = z^*(\varphi_0).
\end{cases}
\]

(8.1)

**Lemma 8.1.** Assume \((q)\). Fix \( c > c^* \). For every \( \varphi_0 \in (0,1) \) there is a unique solution \( \hat{z}_{\varphi_0} \in C[0,1] \cap C^1(0,1) \) to problem \((8.1)\). We have \( \hat{z}_{\varphi_0}(0) = 0 \), and also

\[
\hat{z}_{\varphi_0}(\varphi) \leq \hat{z}_{\varphi_0} \leq z^*(\varphi_0) \quad \text{in} \quad (0,1),
\]

where \( z^*_\beta \) is the solution to \((3.11)\) with \( z^*_\beta(1) = \beta \). If \( 0 < \varphi_1 < \varphi_0 \) then \( \hat{z}_{\varphi_1} < \hat{z}_{\varphi_0} \) in \((0,1]\).

**Proof.** The existence and uniqueness of solutions is proved by Step (i) in the proof of Proposition 5.1. Inequality \((8.2)\) follows from the arguments contained in Step (i) of the proof of Proposition 5.1 while \((8.2)\) is obvious.

At last, if \( 0 < \varphi_1 < \varphi_0 \) then \( \hat{z}_{\varphi_1} < \hat{z}_{\varphi_0} \), because \( \hat{z}_{\varphi_1} < z^* \) in \((\varphi_1,1)\) and \( \varphi_0 \in (\varphi_1,1] \). The monotony follows by the uniqueness of solutions to the Cauchy problem associated to \((3.11)\). The regularity of \( \hat{z}_{\varphi_0} \) follows from both \((8.1)\) and Lemma 3.1. Directly from \((8.2)\), we deduce \( \hat{z}_{\varphi_0}(0) = 0 \).

For every \( c > c^* \), by the monotonicity of \( \{\hat{z}_{\varphi_0}\}_{\varphi_0} \) and \((8.2)\), Lemma 3.3 implies that there exists \( \hat{\varphi} \in C^0[0,1] \cap C^1(0,1) \) which solves \((3.11)\) such that

\[
\hat{\varphi}(\varphi) = \lim_{\varphi_0 \to 0^+} \hat{z}_{\varphi_0}(\varphi), \; \varphi \in [0,1].
\]

Moreover, from \((8.2)\), such a \( \hat{\varphi} \) satisfies \( z_\beta \leq \hat{\varphi} \leq z^* \) in \((0,1)\). Thus, \( \hat{\varphi} \) satisfies Problem \((3.11)\). Define \( \hat{\beta} \in [\beta,0) \) by

\[
\hat{\beta} := \hat{\varphi}(1).
\]

(8.4)

In the following proposition we assume that \( \dot{q}(0) \) exists and denote

\[
\dot{q}(0) = \lim_{\varphi \to 0^+} \frac{q(\varphi)}{\varphi} = M,
\]

(8.5)
for some $M \geq 0$. We shall prove in Remark 8.1 that this condition is necessary for the existence of $\dot{z}(0)$. From (4.3) we deduce $(h(0) - c)^2 - 4M \geq 0$ for any $c \geq c^*$; we can then denote

$$s_\pm(c) := \frac{h(0) - c \pm \sqrt{(h(0) - c)^2 - 4M}}{2}, \quad \text{for } c \geq c^*.$$  

The next proposition both generalizes and extends [7, Proposition 5.2] to the case of a bit more generic $q$, and, more deeply, to the case of $z(1) < 0$. It is worth noting that this latter case reveals the behavior detected by (8.7), and shown in Figure 13, which was not contained in [7] at all.

**Proposition 8.1.** Assume (q) and (8.5) for some $M$. If $c \geq c^*$ and $z$ is a solution of (3.11), then, $\dot{z}(0)$ exists. Moreover, it holds that

$$\dot{z}(0) = \begin{cases} s_+(c) & \text{if } c > c^* \text{ and } z(1) > \hat{\beta}, \\ s_-(c^*) & \text{if } c = c^*, \\ \end{cases} \quad \text{(8.6)}$$

and, if $c^* > h(0)$,

$$\dot{z}(0) = s_-(c) \quad \text{if } c > c^* \text{ and } z(1) \in [\beta, \hat{\beta}]. \quad \text{(8.7)}$$

**Proof.** Let $c \geq c^*$. First, we show that $\dot{z}(0)$ exists. From (3.11) it follows that

$$\limsup_{\varphi \to 0^+} \frac{z(\varphi)}{\varphi} = L \leq 0. \quad \text{(8.8)}$$

For any $\varphi \in (0, 1)$ let $\sigma_\varphi \in (0, \varphi)$ be defined by

$$\dot{z}(\sigma_\varphi) = \frac{z(\varphi)}{\varphi}. \quad \text{(8.9)}$$

Formula (7.9) still holds, and implies

$$\liminf_{\varphi \to 0^+} \frac{z(\varphi)}{\varphi} \leq \liminf_{\varphi \to 0^+} \frac{z(\sigma_\varphi)}{\varphi} =: \ell \geq h(0) - c. \quad \text{(8.10)}$$

Figure 12: The functions $\hat{\varphi}$, $\hat{z}$ and $z_\beta$. 

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Figure 13: An illustration of Proposition 8.1 for fixed $c > c^*$. Solutions are labelled according to their right-hand limit; $s_{\pm}$ denote the slope of the tangent of $z$ at $0$. The dashed curve is the plot of $z^*$.

Suppose by contradiction that $\ell < L$ and take $\gamma \in (\ell, L)$. We argue as in the proof of Case (ii) in Proposition 7.1. Let $\{\sigma_n^1\}_{n=0}^\infty \subset (0, 1)$, for $i = 1, 2$ be such that $\sigma_n^i \to 0$, for $n \to \infty$ and $z(\sigma_n^1) = \gamma$ and $\dot{z}(\sigma_n^1) \leq \gamma \leq \dot{z}(\sigma_n^2)$ for any $n \in \mathbb{N}$.

This implies

$$h(\sigma_n^1) - c - \frac{q(\sigma_n^1)}{\gamma_{\sigma_n^1}} \leq \gamma \leq h(\sigma_n^2) - c - \frac{q(\sigma_n^2)}{\gamma_{\sigma_n^2}} \quad \text{for any } n \in \mathbb{N}.$$

Thus, by (8.5), passing to the limit for $n \to \infty$ implies that $\gamma < 0$ must satisfy

$$\gamma^2 - (h(0) - c) \gamma + M = 0. \tag{8.11}$$

This clearly contradicts the fact that $\gamma$ is arbitrary in $(\ell, L)$. Thus $\ell = L$, which means that $\hat{z}(0)$ exists. Moreover, since $h(0) - c \leq \ell = L \leq 0$ by (8.8) and (8.10), it follows that $h(0) - c \leq \hat{z}(0) \leq 0$.

Let $\sigma_{\varphi}$ be given by (8.9). We have $\dot{z}(\sigma_{\varphi}) \to \hat{z}(0)$, as $\varphi \to 0^+$. By letting $\varphi \to 0^+$ in

$$(\dot{z}(\sigma_{\varphi}) - h(\sigma_{\varphi}) + c) \frac{z(\sigma_{\varphi})}{\sigma_{\varphi}} + \frac{q(\sigma_{\varphi})}{\sigma_{\varphi}} = 0, \quad \varphi \in (0, 1),$$

from (8.6) we obtain that $\dot{z}(0)$ must satisfy (8.11). This means that $\dot{z}(0) \in \{s_-(c), s_+(c)\}$ for every $c \geq c^*$.

Straightforward computations give

$$s_-(c) < s_-(c^*) \leq s_+(c^*) \leq s_+(c) \leq 0 \quad \text{for any } c > c^* \tag{8.12}$$

and $h(0) - c \leq s_-(c)$, for any $c \geq c^*$. We denote $s_\pm := s_\pm(c^*)$.

Take $c > c^*$. Let $\tilde{z}_{\varphi_0}$ and $\hat{z}$ be defined as in the beginning of Subsection 8, see Figure 12. If $z(1) > \hat{\beta}$ then necessarily $z(1) > \tilde{z}_{\varphi_1}(1)$, for some $\varphi_1 \in (0, 1)$, because of (8.3). Thus,
$z > \hat{z}_{\varphi_1}$ in $(0, 1]$. We already observed in (5.3) that $\hat{z}_{\varphi_1} > z^* in (0, \varphi_1)$. Thus, $z > z^*$ in $(0, \varphi_1)$ and hence $\dot{z}(0) \geq \hat{z}(0)$. Since $s_-(c) < s_-^* \leq 0$ by (8.12), we deduce $\dot{z}(0) = s_-(c)$. Thus, we proved (8.6)

Now, we prove (8.6)\#2. If $z = z^*$, then (8.6)\#2 was obtained in [7, Proposition 5.2] under some specific assumptions on $q$. Since the relevant ones were (3.2) and (8.5), we deduce that (8.6)\#2 occurs also in the current case. If $z = y^*$ is a solution of (3.11), different from $z^*$ (note that such a $y^*$ can exist since (6.4) does not necessarily follow), then $y^* < z^*$ in $(0, 1]$ by Proposition 4.1. Since $\dot{y}^*(0) \in \{s^*_-, s^*_+\}$ and $\dot{z}^*(0) = s^*_{-}$ then we have $\dot{y}^*(0) = s^*_+$. Hence, (8.6)\#2 holds.

It remains to prove (8.7) under the additional condition $h(0) - c^* < 0$. Since $\beta \leq z(1) \leq \hat{\beta}$ then $z \leq \hat{z}$ and hence $z < z^*$, which implies $\dot{z}(0) \leq \hat{z}(0)$. Since, under the additional condition $h(0) - c^* < 0$, we have $s^*_+ < s^*_-$ and since we proved that $\dot{z}^*(0) = s^*_-$, we conclude that necessarily $\dot{z}(0) = s_-(c)$, which is (8.7). This concludes the proof.

**Remark 8.1.** The following example shows that (8.5) is necessary for the existence of $\dot{z}(0)$.

For $\varphi \in [0, 1]$ define

$$q(\varphi) = \varphi(1 - \varphi)^4 \left(2 + \sin(\log \varphi) \right)\left(3 - \cos(\log \varphi) - \sin(\log \varphi)\right).$$

The function $q$ satisfies (q), while $\dot{q}(0)$ does not exist, since

$$\lim_{\varphi \to 0^+} \frac{q(\varphi)}{\varphi} < \lim_{\varphi \to 0^+} \frac{q(\varphi)}{\varphi}.$$  

Direct computations show that the function $z = z(\varphi)$ defined by

$$z(\varphi) = -\left(2 + \sin(\log \varphi) \right) \left(1 - \varphi\right)^2 \varphi$$

solves (3.1) with $c = 0$ and

$$h(\varphi) = 2 \left(2 + \sin(\log \varphi) \right) \left(1 - \varphi\right) \varphi - 5 \left(1 - \varphi\right)^2.$$  

Clearly, $\dot{z}(0)$ does not exists.

We now show that, under (6.4), the threshold $\hat{\beta}(c)$ defined in (8.4) and occurring in Proposition 8.1 coincides with the threshold $\beta(c)$ introduced in Proposition 5.1.

**Proposition 8.2.** Assume (q); take $c > c^*$ and assume (6.4). Then $\beta(c) = \hat{\beta}(c)$.

**Proof.** Consider $\varepsilon > 0$ and let $z_\varepsilon$ be the solution of

$$\begin{cases} 
\dot{z}_\varepsilon(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z_\varepsilon(\varphi)}, \varphi > 0, \\
\dot{z}_\varepsilon(0) = -\varepsilon < 0.
\end{cases}$$

Lemma 3.2 (1.b) implies that $z_\varepsilon$ exists and it is defined in its maximal-existence interval $[0, \delta]$, for some $0 < \delta \leq 1$. By the uniqueness of solutions of the Cauchy problem associated to (3.1)\#1, we have necessarily $z_\varepsilon < z_\beta$ in $[0, \delta]$, where $z_\beta$ was defined in the statement of Lemma 8.1. Since $z_\beta(\delta) < 0$ then $\delta = 1$.  

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We claim that \( z_\varepsilon \) converges for \( \varepsilon \to 0^+ \) to both \( \hat{z} \) and \( z_\beta \), where \( \hat{z} \) is defined in (8.3), see Figure 12. From the uniqueness of the limit, it follows then that \( \hat{z} \) and \( z_\beta \) must coincide and hence that \( \beta = \hat{\beta} \).

To prove the claim, consider
\[
\eta_\varepsilon(\varphi) := \hat{z}(\varphi) - z_\varepsilon(\varphi), \quad \varphi \in [0, 1].
\]
Since \( \hat{z} \geq z_\beta > z_\varepsilon \) in \([0, 1]\), then \( \eta_\varepsilon > 0 \) in \([0, 1]\). Moreover, \( \eta_\varepsilon(0) = \varepsilon \). We have
\[
\frac{\eta_\varepsilon'(\varphi)}{\eta_\varepsilon(\varphi)} = \frac{q(\varphi)}{z_\varepsilon(\varphi)\hat{z}(\varphi)}, \quad \varphi \in (0, 1)
\]
and hence, for any \( 0 < \tau < \varphi \),
\[
\log(\eta_\varepsilon(\varphi)) - \log(\eta_\varepsilon(\tau)) = \int_{\tau}^{\varphi} \frac{q(s)}{z_\varepsilon(s)\hat{z}(s)} ds \leq \int_{\tau}^{1} \frac{q(s)}{z_\varepsilon(s)\hat{z}(s)} ds.
\]
(8.13)
Notice, from (6.4) it follows that we can apply (8.7) with \( M = 0 \) and obtain \( z_\beta(s)\hat{z}(s) = (h(0) - c)^2 s^2 + o(s^2) \), as \( s \to 0^+ \). Hence, from (6.4),
\[
\sup_{\tau > 0} \int_{\tau}^{1} \frac{q(s)}{z_\beta(s)\hat{z}(s)} ds =: C < +\infty.
\]
From (8.13), direct manipulations and taking the limit as \( \tau \to 0^+ \) give then
\[
\eta_\varepsilon(\varphi) \leq \varepsilon e^C, \quad \varphi \in [0, 1),
\]
which means that
\[
\lim_{\varepsilon \to 0^+} z_\varepsilon(\varphi) = \hat{z}(\varphi), \quad \varphi \in [0, 1).\]
(8.14)
On the other hand, we apply Lemma 3.3 to deduce that \( z_\varepsilon \) converges (uniformly on \([0, 1]\)) to a solution \( \bar{z} \) of (3.1) in \((0, 1)\) such that \( \bar{z} < 0 \) in \((0, 1)\) and \( \bar{z}(0) = 0 \). Since \( z_\varepsilon < z_\beta \) and \( z_\beta \) lies below every solution of (3.1), by the very definition of \( z_\beta \), we conclude that \( \bar{z} \) must coincide with \( z_\beta \), that is
\[
\lim_{\varepsilon \to 0^+} z_\varepsilon(\varphi) = z_\beta(\varphi), \quad \varphi \in [0, 1].
\]
(8.15)
From (8.14) and (8.15), we clearly have \( z_\beta = \hat{z} \).

9 Strongly non-unique strict semi-wavefronts

In this section, we apply some of the results showed above, regarding Problems (3.11) and (3.12), to study semi-wavefronts of Equation (1.1) when \( D \) and \( g \) satisfy (D1), (g0) and (1.2); in particular, we prove Theorem 2.1. Indeed, all the results obtained in Sections 4–8 apply when we set
\[
q := Dg,
\]
(9.1)
since clearly such \( q \) fulfills (q) We highlight that throughout all the present section, when we refer to \( c^* \) we always intend the threshold given by Proposition 4.1 for \( q \) given by (9.1), for which it holds (2.2), as observed in Remark 5.1.

The next technical lemma provides a property of solutions of (3.12) which shall turn out crucial in the proof of Theorem 2.1.

**Lemma 9.1.** Assume (D1), (g0) and (1.2). Consider \( c \geq c^* \) and let \( z \) be the solution of (3.12) when (9.1) occurs. Then, it holds that

\[
\lim_{\varphi \to 1^-} \frac{D(\varphi)}{z(\varphi)} = \begin{cases} 
\frac{h(1) - c - \sqrt{(h(1) - c)^2 - 4D(1)g(1)}}{2g(1)} & \text{if } \dot{D}(1) < 0, \\
\min\left\{0, \frac{h(1) - c}{g(1)}\right\} & \text{if } \dot{D}(1) = 0.
\end{cases}
\]  

(9.2)

**Proof.** First, observe that Proposition 7.1 applies to the current case.

If either \( \dot{D}(1) < 0 \) or \( \dot{D}(1) = 0 \) and \( c < h(1) \), then \( \dot{z}(1) > 0 \), as (7.2) informs us, because \( \dot{q}(1) = \dot{D}(1)g(1) \). As a consequence, we have

\[
\lim_{\varphi \to 1^-} \frac{D(\varphi)}{z(\varphi)} = \lim_{\varphi \to 1^-} \frac{D(\varphi)}{\varphi - 1} = \dot{D}(1) = 0.
\]

which, together with (7.2), implies both (9.2), after trivial manipulations, and the first half of (9.2), directly.

If \( \dot{D}(1) = 0 \) and \( c \geq h(1) \), we need a refined argument based on strict upper- and lower-solutions of (3.11). We split the proof in two subcases.

(i) Assume first \( \dot{D}(1) = 0 \) and \( c > h(1) \). Fix \( \varepsilon > 0 \) and define \( \omega = \omega(\varphi) \) by

\[
\omega(\varphi) := \frac{g(1)}{c - h(1) + \varepsilon g(1)} D(\varphi), \quad \text{for } \varphi \in (0, 1).
\]  

(9.3)

First, we observe that \( \omega < 0 \) in \((0, 1)\). Moreover, we get

\[
\hat{\omega}(\varphi) = -\frac{g(1)}{c - h(1) + \varepsilon g(1)} \dot{D}(\varphi),
\]

which in turn implies \( \hat{\omega}(1) = 0 \), since \( \dot{D}(1) = 0 \). Now, if we compute the right-hand side of (3.11) applied to \( \omega \), we obtain

\[
h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{\omega(\varphi)} = h(\varphi) - c - \frac{g(\varphi)}{g(1)} \left[ c - h(1) + \varepsilon g(1) \right], \quad \text{for } \varphi \in (0, 1),
\]

which tends to \( \varepsilon g(1) > 0 \) as \( \varphi \to 1^- \). Hence, there exists \( \sigma \in (0, 1) \) such that

\[
\hat{\omega}(\varphi) < h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{\omega(\varphi)}, \quad \varphi \in [\sigma, 1),
\]  

(9.4)

that is, \( \omega \) is a (strict) lower-solution of (3.11) in \([\sigma, 1)\).
Since \( \dot{z}(1) = 0 \), we can take a sequence \( \{ \varphi_n \} \subset (\sigma, 1) \), with \( \varphi_n \to 1 \) as \( n \to \infty \), such that \( \dot{z}(\varphi_n) \to 0 \) as follows. Let \( \{ \sigma_n \} \subset (\sigma, 1) \) be such that \( \sigma_n \to 1 \). For any \( n \in \mathbb{N} \), the Mean Value Theorem implies that there exists \( \varphi_n \in (\sigma_n, 1) \) for which it holds

\[
\dot{z}(\varphi_n) = \frac{z(\sigma_n)}{\sigma_n - 1}.
\]

Since the sequence in the right-hand side of this last identity tends to \( \dot{z}(1) = 0 \), as \( n \to \infty \), we obtained the desired \( \{ \varphi_n \} \). With this in mind, from (3.11), we obtain

\[
\lim_{n \to \infty} \frac{D(\varphi_n)g(\varphi_n)}{z(\varphi_n)} = h(1) - c,
\]

and then

\[
\lim_{n \to \infty} \frac{\omega(\varphi_n)}{z(\varphi_n)} = \frac{c - h(1)}{c - h(1) + \varepsilon g(1)} = 1 - \frac{\varepsilon g(1)}{c - h(1) + \varepsilon g(1)} < 1.
\]

Hence, there exists \( \pi \) such that \( \omega(\varphi_n) > z(\varphi_n) \) for \( n \geq \pi \). Without loss of generality we assume that \( \pi = 1 \). We claim that

\[
\omega(\varphi) > z(\varphi), \quad \text{for} \quad \varphi \in (\varphi_1, 1).
\]

We reason by contradiction, see Figure [13]. Suppose that there exists \( \tilde{\varphi} \in (\varphi_1, 1) \) such that \( \omega(\tilde{\varphi}) \leq z(\tilde{\varphi}) \). There exists \( n \in \mathbb{N} \) for which \( \tilde{\varphi} \in (\varphi_n, \varphi_{n+1}) \). Since

\[
\omega(\varphi_n) > z(\varphi_n) \quad \text{and} \quad \omega(\varphi_{n+1}) > z(\varphi_{n+1})
\]

the existence of such a \( \tilde{\varphi} \) implies that the function \( (\omega - z) \) in \( (\varphi_n, \varphi_{n+1}) \) admits a non-positive minimum at \( \tilde{\varphi}_2 \in (\varphi_n, \varphi_{n+1}) \), that is

\[
\dot{\omega}(\tilde{\varphi}_2) = \dot{z}(\tilde{\varphi}_2) \quad \text{and} \quad \omega(\tilde{\varphi}_2) \leq z(\tilde{\varphi}_2).
\]

Thus, from (3.11) and (9.4) we have that

\[
h(\tilde{\varphi}_2) - c - \frac{(Dg)(\tilde{\varphi}_2)}{z(\tilde{\varphi}_2)} = \dot{z}(\tilde{\varphi}_2) = \dot{\omega}(\tilde{\varphi}_2) < h(\tilde{\varphi}_2) - c - \frac{(Dg)(\tilde{\varphi}_2)}{\omega(\tilde{\varphi}_2)},
\]

which in turn implies

\[
\frac{1}{z(\tilde{\varphi}_2)} > \frac{1}{\omega(\tilde{\varphi}_2)}
\]

because of \( (Dg)(\tilde{\varphi}_2) > 0 \). Hence, \( z(\tilde{\varphi}_2) < \omega(\tilde{\varphi}_2) \) which contradicts the existence of \( \tilde{\varphi}_2 \). Then (9.6) is proved. At last, we have

\[
\frac{D(\varphi)}{z(\varphi)} > \frac{D(\varphi)}{\omega(\varphi)} = -\frac{c - h(1)}{g(1)} - \varepsilon, \quad \varphi \in (\varphi_1, 1).
\]

(9.7)

Analogously, we define \( \eta = \eta(\varphi) \), for \( \varphi \in (0, 1) \), by

\[
\eta(\varphi) := -\frac{g(1)}{c - h(1) - \varepsilon g(1)}D(\varphi),
\]

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where $\varepsilon > 0$ is small enough to satisfy $c > h(1) + \varepsilon g(1)$. We have $\eta < 0$ in $(0, 1)$ and $\dot{\eta}(1) = 0$. Also, $\eta$ satisfies

$$h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{\eta(\varphi)} = h(\varphi) - c + \frac{g(\varphi)[c - h(1) - \varepsilon g(1)]}{g(1)}, \quad \text{for } \varphi \in (0, 1).$$

The right-hand side tends to $-\varepsilon g(1) < 0$ as $\varphi \to 1^-$. Hence, there exists $\sigma_2 \in (0, 1)$ such that

$$\dot{\eta}(\varphi) > h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{\eta(\varphi)}, \quad \varphi \in [\sigma_2, 1),$$

that is, $\eta$ is a (strict) upper-solution of \((3.11)\) in $[\sigma_2, 1)$.

Let $\{\varphi_n\} \subset (\sigma_2, 1)$ be such that $\dot{z}(\varphi_n) \to 0$ as $n \to \infty$, similarly to above. We have

$$\lim_{n \to \infty} \eta(\varphi_n) \equiv \frac{c - h(1)}{c - h(1) - \varepsilon g(1)} = 1 + \frac{\varepsilon g(1)}{c - h(1) - \varepsilon g(1)} > 1.$$

Again, without loss of generality we can suppose $\varphi_1 > \sigma_2$ and $\eta(\varphi_n) < z(\varphi_n)$ for $n \geq 1$. After obvious adjustments, the argument which provides \((9.6)\) leads now to $\eta(\varphi) < z(\varphi)$ for $\varphi \in (\varphi_1, 1)$. Thus, we obtain

$$\frac{D(\varphi)}{z(\varphi)} < \frac{D(\varphi)}{\eta(\varphi)} = \frac{c - h(1)}{g(1)} + \varepsilon, \quad \varphi \in (\varphi_1, 1). \quad (9.8)$$

Finally, putting together \((9.7)\) and \((9.8)\), since $\varepsilon > 0$ is arbitrary, we deduce

$$\lim_{\varphi \to 1^-} \frac{D(\varphi)}{z(\varphi)} = \frac{h(1) - c}{g(1)}. \quad (9.9)$$

Thus, we proved \((9.2)\) with $c > h(1)$.

(ii) Now, we consider the case $\dot{D}(1) = 0$ and $c = h(1)$. Fix $\varepsilon > 0$. Set

$$\omega(\varphi) := -\frac{D(\varphi)}{\varepsilon}, \quad \varphi \in (0, 1), \quad (9.10)$$

which coincides with \((9.3)\) in the current case. By proceeding exactly as in the case (ii), we obtain \((9.3)\) for $\omega$ defined as in \((9.10)\), namely $0 > \omega(\varphi) > z(\varphi)$, for $\varphi \in (\varphi_1, 1)$, for some $\varphi_1 \in (0, 1)$. This implies, as in \((9.7)\),

$$0 > \frac{D(\varphi)}{z(\varphi)} > \frac{D(\varphi)}{\omega(\varphi)} = -\varepsilon, \quad \varphi \in (\varphi_1, 1).$$
A consequence of the last formula is that
\[ \frac{D(\varphi)}{\varphi} \to 0^-, \quad \text{as} \quad \varphi \to 1^- \]
which is (9.2) in the case \( c = h(1) \).

**Remark 9.1.** Let \( c \geq c^* \) and \( z \) be any solution of (3.11). We infer that \( z \in C^{1}(0,1) \). In fact, if \( z(1) = b < 0 \), in the proof of case (i) of Proposition 7.1, we already checked that this is true, since \( \lim_{\varphi \to 1^-} \dot{z}(\varphi) = \dot{z}(1) \). If \( z(1) = 0 \), from (9.2) it follows that the right-hand side of (3.11) still has a finite limit, as \( \varphi \to 1^- \). As observed, this means that \( z \in C^{1}(0,1) \).

We now prove Theorem 2.1.

**Proof of Theorem 2.1.** To begin with, we prove that there exists a semi-wavefront to 0 of (1.1) if \( c \geq c^* \). To this end, for \( q = Dg \), consider one of the solutions \( z = z(\varphi) \) of (3.11) (which is (1.6)), provided by Propositions 4.1 and 5.1. Consider the Cauchy problem

\[
\begin{cases}
\varphi' = \frac{z(\varphi)}{D(\varphi)}, \\
\varphi(0) = \frac{1}{2}.
\end{cases}
\]  

The right-hand side of (9.11) is of class \( C^1 \) in a neighborhood of \( \frac{1}{2} \), and then there exists a unique solution \( \varphi \) in its maximal-existence interval \( (a,\xi_0) \), for \( -\infty \leq a < \xi_0 \leq \infty \). Since \( z(\varphi)/D(\varphi) < 0 \) for \( \varphi \in (0,1) \), we deduce that \( \varphi \) is decreasing and then (see Figure 15)

\[
\lim_{\xi \to a^+} \varphi(\xi) = 1, \quad \lim_{\xi \to \xi_0^-} \varphi(\xi) = 0.
\]

![Figure 15: Plot of the profile \( \varphi \) in the case \( \xi_0 \in \mathbb{R} \).](image)

A direct consequence of (9.11) is that \( \varphi \) satisfies (1.3) in \( (a,\xi_0) \). We show that, if \( \xi_0 \in \mathbb{R} \), we can extend \( \varphi \) and obtain a solution of (1.3), in the sense of Definition 2.1, defined in all the half-line \( (a, +\infty) \).

Assume \( \xi_0 \in \mathbb{R} \) and set \( \varphi(\xi) = 0 \), for any \( \xi \geq \xi_0 \). The new function (which without any ambiguity we still call \( \varphi \)) is clearly of class \( C^0(a, +\infty) \cap C^2 ((a, +\infty) \setminus \{\xi_0\}) \) and is a classical solution of (1.3) in \( (a, +\infty) \setminus \{\xi_0\} \). Moreover, observe that, as a consequence of both the fact that \( z \) satisfies (3.11), (9.2), and (9.11), we have

\[
\lim_{\xi \to \xi_0^-} D \left( \varphi(\xi) \right) \varphi'(\xi) = 0.
\]  

(9.12)
This implies that $D(\varphi)\varphi' \in L^1_{\text{loc}}(a, +\infty)$.

To show that $\varphi$ is a solution of (1.3) according to Definition 2.1 it remains to prove (2.1). For this purpose, consider $\psi \in C_0^\infty(a, +\infty)$, and let $a < \xi_1 < \xi_2 < \infty$ such that $\psi(\xi) = 0$, for any $\xi \geq \xi_2$ or $\xi \leq \xi_1$. Our goal is then to prove the following:

$$\int_{\xi_1}^{\xi_2} (D(\varphi)\varphi' - f(\varphi) + c\varphi) \psi' - g(\varphi)\psi \, d\xi = 0. \quad (9.13)$$

Identity (9.13) is obvious if $\xi_2 < \xi_0$, since $\varphi$ solves (1.3) in $(a, \xi_0)$. Assume $\xi_2 \geq \xi_0$. In the interval $(\xi_0, \xi_2)$ we have $\varphi = 0$, and since $g(0) = f(0) = 0$ we deduce

$$\int_{\xi_0}^{\xi_2} (D(\varphi)\varphi' - f(\varphi) + c\varphi) \psi' - g(\varphi)\psi \, d\xi = 0. \quad (9.14)$$

In the interval $(\xi_1, \xi_0)$ we have, by (9.12),

$$\int_{\xi_1}^{\xi_0} (D(\varphi)\varphi' - f(\varphi) + c\varphi) \psi' - g(\varphi)\psi \, d\xi = \left(\int (D(\varphi)\varphi') \psi(\xi)\right)(\xi_0) = 0. \quad (9.15)$$

Thus, identities (9.14) and (9.15) imply (9.13).

At last, we claim that $a \in \mathbb{R}$, i.e., that $\varphi$ is strict. In order to prove the claim, it is sufficient to prove

$$\lim_{\xi \to a^+} \varphi' (\xi) < 0. \quad (9.16)$$

We stress that the case $\lim_{\xi \to a^+} \varphi'(\xi) \to -\infty$, for short $\varphi'(a^+) = -\infty$, is included in (9.16). To prove (9.16), we notice that, from (9.11),

$$\lim_{\xi \to a^+} \varphi' (\xi) = \lim_{\varphi \to 1^-} \frac{z(\varphi)}{D(\varphi)}.$$  

Thus, (9.16) easily follows from either a direct check, in the case $z(1) < 0$, or the application of Lemma 9.1 in the case $z(1) = 0$.

This concludes the first part of the proof.

Conversely, we prove that if there exists a semi-wavefront $\varphi$ in $0$ defined in $(a, +\infty)$, then $c \geq c^*$. Let $\bar{b}$ be defined by

$$\bar{b} := \sup \{\xi > a : \varphi(\xi) > 0\} \in (a, +\infty]. \quad (9.17)$$

We observe that $0 < \varphi < 1$ in $(a, \bar{b})$. Thus, $\varphi$ is a classical solution of (1.3) in $(a, \bar{b})$. We claim that

$$\lim_{\xi \to \bar{b}^-} D(\varphi(\xi)) \varphi'(\xi) = 0. \quad (9.18)$$

Suppose $\bar{b} \in \mathbb{R}$. Take $\xi_1 > a$ and $\xi_2 > \bar{b}$. By choosing, in Definition 2.1 $\psi \in C_0^\infty(a, +\infty)$ with support in $(\xi_1, \xi_2)$ such that $\psi(\bar{b}) \neq 0$, (2.1) reads as

$$0 = \int_{\xi_1}^{\xi_2} (D(\varphi)\varphi' + c\varphi - f(\varphi)) \psi' - g(\varphi) \psi \, d\xi = \int_{\xi_1}^{\bar{b}} (D(\varphi)\varphi' + c\varphi - f(\varphi)) \psi' - g(\varphi) \psi \, d\xi = (D(\varphi)\varphi')(\bar{b}^-)\psi(\bar{b}).$$
Hence, we obtained (9.18) in this case. If \( \bar{b} = +\infty \), by integrating (1.3) in \([\eta, \xi] \subset (a, +\infty)\), we have

\[
D (\varphi (\xi)) \varphi' (\xi) = D (\varphi (\eta)) \varphi' (\eta) - c (\varphi (\xi) - \varphi (\eta)) + (f (\varphi (\xi)) - f (\varphi (\eta))) - \int_{\eta}^{\xi} g (\varphi (\sigma)) \, d\sigma.
\]  

(9.19)

Since the function

\[
\xi \mapsto \int_{\eta}^{\xi} g (\varphi (\sigma)) \, d\sigma
\]

is increasing (because \( \varphi \) is decreasing and \( g > 0 \) in \((0, 1)\)), then \( \lim_{\xi \to -\infty} D (\varphi (\xi)) \varphi' (\xi) = \ell \) for some \( \ell \in [-\infty, 0] \). If \( \ell < 0 \), then, \( \varphi' (\xi) \) tends either to some negative value or to \(-\infty\) as \( \xi \to -\infty \). In both cases, this contradicts the boundedness of \( \varphi \), and so (9.18) is proved.

We show now (2.3). Suppose by contradiction the (2.3) does not occur, there exists \( \xi_0 \in (a, \bar{b}) \), with \( 0 < \varphi (\xi_0) < 1 \), such that \( \varphi' (\xi_0) = 0 \). Then (1.3) implies \( \varphi'' (\xi_0) = -g (\varphi (\xi_0)) / D (\varphi (\xi_0)) < 0 \) and hence \( \xi_0 \) is a local maximum point of \( \varphi \). It is plain to see that, in turn, this implies that there exists \( a < \xi_1 < \xi_0 \) which is a local minimum point of \( \varphi \). From what we said about \( \xi_0 \), we necessarily have \( \varphi (\xi_1) = \varphi' (\xi_1) = 0 \).

Take \( \xi \in (\xi_1, \bar{b}) \). Integrating (1.3) in \([\xi_1, \xi] \) gives (9.19) with \( \xi_1 \) replacing \( \eta \). By passing to the limit for \( \xi \to \bar{b}^- \), from (9.18), we obtain the contradiction \( 0 < 0 \). This proves (2.3).

From (2.3), we can define the function \( z = z (\varphi) \), for \( \varphi \in (0, 1) \), by

\[
z (\varphi) := D (\varphi) \varphi' (\xi (\varphi)),
\]

(9.20)

where \( \xi = \xi (\varphi) \) is the inverse function of \( \varphi \). Again by (2.3), it follows also that \( z < 0 \) in \((0, 1)\). From (9.18), we clearly have \( z (0^+) = 0 \); furthermore, a direct computation shows that \( z \) solves equation (1.6). Thus, \( z \) solves problem (1.6), which is (4.11) with \( q = D g \). At last, Proposition 6.2 implies \( c \geq c^* \).

Remark 9.2. In the proof of Theorem 2.1 we have showed en passant that there exists a bijection between solutions \( z \) of (1.6) and strict semi-wavefronts to 0 (modulo space shifts) \( \varphi \) of (1.1), connecting 1 to 0. In particular, this bijection is given by (9.11) and (9.20).

Remark 9.3. The proof of Theorem 2.1 provides a formula for \( \varphi' (a^+) \). If \( z (1) < 0 \), then \( \varphi' (a^+) = -\infty \). If \( z (1) = 0 \), Lemma 5.1 leads to

\[
\lim_{\xi \to a^+} \varphi' (\xi) = \begin{cases} 
\frac{2g (1)}{h (1) - c - \sqrt{(h (1) - c)^2 - 4D (1) g (1)}} & \text{if } \dot{D} (1) < 0, \\
\frac{g (1)}{h (1) - c} & \text{if } \dot{D} (1) = 0 \text{ and } c > h (1), \\
-\infty & \text{if } \dot{D} (1) = 0 \text{ and } c \leq h (1).
\end{cases}
\]  

(9.21)

We can now prove Corollary 2.1.

Proof of Corollary 2.1. Define \( \xi_0 := \sup \{ \xi > a : \varphi (\xi) > 0 \} \in (a, +\infty] \). We assume without loss of generality that \( a < 0 < \xi_0 \) and \( \varphi (0) = 1/2 \), see Figure 15. Let \( z \) be the function defined in (9.20). Notice, \( 1 = D (\varphi) \varphi' / z (\varphi) \) if \( \varphi \in (0, 1) \). Thus, for any \( \xi > 0 \), it follows that

\[
\xi = \int_{0}^{\xi} \frac{D (\varphi (s))}{z (\varphi (s))} \varphi' (s) \, ds = \int_{1/2}^{\varphi (\xi)} \frac{D (\sigma)}{z (\sigma)} \, d\sigma = \int_{\varphi (\xi)}^{1/2} \frac{D (\sigma)}{z (\sigma)} \, d\sigma.
\]

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Therefore, \( \xi_0 \in \mathbb{R} \) if and only if it holds that
\[
\int_0^{1/2} \frac{D(\sigma)}{-z(\sigma)} d\sigma := \lim_{\varphi \to 0^+} \int_0^{1/2} \frac{D(\sigma)}{-z(\sigma)} d\sigma < +\infty.
\] (9.22)
For \( c > c^* \), let \( \hat{\beta}(c) \) be given by \( (8.3) \). We prove \( (i) \). In the proof of Theorem 2.1, we already showed that \( z \) is a solution of \( (1.6) \) (or \( (3.11) \) with \( q = Dg \)). Thus, by arguing as in \( (8.10) \), we deduce that
\[
\lim \inf_{\sigma \to 0^+} \frac{z(\sigma)}{\sigma} \geq h(0) - c,
\]
which in turn implies that \( z(\sigma) \geq (h(0) - c) \sigma + o(\sigma) \), as \( \sigma \to 0^+ \). This, together with \( D(0) > 0 \), implies that \( (9.22) \) cannot be verified. Then, \( \xi_0 = +\infty \) and consequently \( \varphi \) results strictly decreasing. This, and the fact that \( \varphi \) is of class \( C^2 \) when \( \varphi \in (0, 1) \), implies that \( \varphi \) is of class \( C^2(a, +\infty) \) and hence is classical. Part \( (i) \) is hence showed.

Assume \( D(0) = 0 \). In this case, Formula \( (8.5) \) holds with \( M = 0 \), and Proposition 8.1 informs us that \( \dot{\varphi}(0) \) exists. We show \( \dot{\varphi} \). Since \( (2.8) \) holds then \( (8.6) \) reads as \( \dot{\varphi}(0) = 0 \).

We treat separately the cases \( \dot{D}(0) > 0 \) or \( \dot{D}(0) = 0 \). Suppose that \( \dot{D}(0) > 0 \). Therefore,
\[
\lim_{\varphi \to 0^+} \varphi' (\xi_0) = \lim_{\varphi \to 0^+} \frac{\dot{\varphi}(0)}{D(0)} = 0
\]
and hence \( \varphi \) (not necessarily strictly monotone) is classical. Suppose then \( D(0) = \dot{D}(0) = 0 \).

Fix \( \varepsilon > 0 \) and define \( \eta(\varphi) := -\varepsilon D(\varphi), \varphi \in (0, 1) \). We have
\[
\dot{\eta}(\varphi) - h(\varphi) + c + \frac{D(\varphi)g(\varphi)}{\eta(\varphi)} \to -h(0) + c > 0, \quad \text{as} \quad \varphi \to 0^+.
\]
Therefore \( \eta \) is a strict upper-solution of \( (1.6) \), in \( (0, \delta] \), for some \( \delta > 0 \). Also, since \( \dot{\varphi}(0) = 0 \), there exists a sequence \( \{\varphi_n\}_n \), with \( \delta \geq \varphi_n \to 0^+ \), such that \( \dot{\varphi}(\varphi_n) \to 0 \). From \( (1.6) \), this implies that
\[
\lim_{n \to \infty} \frac{\varepsilon D(\varphi_n)}{-z(\varphi_n)} = \varepsilon \lim_{n \to \infty} \frac{\dot{\varphi}(\varphi_n) + c - h(\varphi_n)}{g(\varphi_n)} = \infty.
\]
Hence, \(-\eta(\delta_1) = \varepsilon D(\delta_1) > -z(\delta_1)\), for some \( 0 < \delta_1 \leq \delta \) small enough. An application of Lemma 3.2 (2.a.i) then gives
\[
z(\varphi) > -\varepsilon D(\varphi), \varphi \in (0, \delta_1].
\] (9.24)
This clearly implies that
\[
0 > \frac{z(\varphi)}{D(\varphi)} > -\varepsilon, \varphi \in (0, \delta_1].
\]
Since \( \varepsilon > 0 \) is arbitrary, then we have \( \varphi'(\xi) \to 0 \) for \( \xi \to \xi_0^- \) and hence \( \varphi \) is classical, that is we showed the first part of \( (ii) \). Define \( \eta(\varphi) := -\varphi D(\varphi) \). We have, for any \( \varphi \in (0, 1) \),
\[
\dot{\eta}(\varphi) - h(\varphi) + c + \frac{D(\varphi)g(\varphi)}{\eta(\varphi)} = -\dot{D}(\varphi)\varphi - D(\varphi) - h(\varphi) + c - \frac{g(\varphi)}{\varphi}.
\]
Thus, by means of \( (2.4) \), we get
\[
\lim \inf_{\varphi \to 0^+} \left[ \dot{\eta}(\varphi) - h(\varphi) + c + \frac{D(\varphi)g(\varphi)}{\eta(\varphi)} \right] = c - h(0) - \lim \sup_{\varphi \to 0^+} \frac{g(\varphi)}{\varphi} > 0.
\]
Therefore, \( \eta \) is a strict upper-solution of (1.6) in \((0, \delta]\), for some \( \delta > 0 \). Furthermore, taking the same sequence \( \phi_n \to 0^+ \) as above such that \( \dot{\phi}(\phi_n) \to 0 \), as \( n \to \infty \), then we have
\[
\liminf_{n \to \infty} \frac{D(\phi_n)\phi_n}{-z(\phi_n)} = \liminf_{n \to \infty} \frac{\dot{\phi}(\phi_n) + c - h(\phi_n)}{g(\phi_n)/\phi_n} = \frac{c - h(0)}{\limsup_{n \to \infty} g(\phi_n)/\phi_n} > 1,
\]
since (2.19) holds. Thus, as in (9.24), we deduce that \( D(\phi) > -z(\phi) \) in \((0, \delta]\), after choosing \( 0 < \delta \leq 1/2 \) small enough. Hence,
\[
\int_{1/2}^0 \frac{D(\sigma)}{-z(\sigma)} d\sigma > \int_{0}^\delta \frac{d\sigma}{\sigma} = +\infty,
\]
which concludes the proof of (ii), by means of (9.22).

We show (iii). With (8.6) and (8.7) in mind, from \( c^* > h(0) \) and (2.10) we obtain \( \dot{z}(0) = h(0) - c < 0 \). Therefore,
\[
\frac{D(\sigma)}{-z(\sigma)} = \frac{\dot{D}(0) + o(1)}{c - h(0) + o(1)} \quad \text{as} \quad \sigma \to 0^+,
\]
and consequently (9.22) is verified. Thus, \( \xi_0 \in \mathbb{R} \). Furthermore, from (9.20),
\[
\lim_{\xi \to \xi_0} \frac{\phi'(\xi)}{\phi(\xi)} = \lim_{\phi \to 0^+} \frac{z(\phi)/\phi}{D(\phi)/\phi} = \frac{h(0) - c}{D(0)} \in [-\infty, 0),
\]
which implies that \( \phi \) is sharp at 0 and that (2.11) holds. □

10 New regularity classification of wavefronts

In this section we assume that \( D \) and \( g \) satisfy (D0) and (g01) and prove Theorem 2.2. Analogously to Section 9, but now thanks to assumptions (D0) - (g01), we apply results of Sections 4 – 8 to the case \( q = Dg \).

Proof of Theorem 2.2. We first show that wavefronts are allowed if and only if \( c \geq c^* \) for some \( c^* \) which satisfies (2.12). The proof is quite standard and is mostly contained in the proof of Theorem 2.1. Then, we prove (i) and (ii), by making use of some of the arguments detailed in the proof of Corollary 2.1.

Set \( q = Dg \). Clearly, \( q \) satisfies (q), with in particular \( \dot{q}(0) = 0 \). We apply Proposition 4.1. Problem (3.12) admits a unique solution \( z \) if and only if \( c \geq c^* \) where for \( c^* \) the estimates in (1.3) hold. As already observed in Remark 5.1 since (D0) and (g01) hold true, in this case \( c^* \) satisfies (2.12).

To the solution \( z \) there is associated the solution \( \varphi = \varphi(\xi) \) of the problem
\[
\begin{cases}
\varphi' = \frac{z(\varphi)}{D(\varphi)}, \\
\varphi(0) = \frac{1}{2}.
\end{cases}
\]
(10.1)

Such a \( \varphi \) exists and satisfies (10.1) in some maximal interval \((\xi_1, \xi_0)\), so that
\[
\lim_{\xi \to \xi_1^+} \varphi(\xi) = 1 \quad \text{and} \quad \lim_{\xi \to \xi_0^-} \varphi(\xi) = 0.
\]
Also, \( \phi \) satisfies (1.3) in \((\xi_1, \xi_0)\). As already discussed in the proof of Theorem 2.1, if \( \xi_0 \in \mathbb{R} \), then \( \phi \) can be extended continuously to a solution of (1.3) according to Definition 2.1 in \((\xi_0, +\infty)\), by setting \( \phi(\xi) = 0 \), for \( \xi \geq \xi_0 \). Since \( g(1) = 0 \), it also holds that if \( \xi_1 \in \mathbb{R} \) then we can extend \( \phi \) to a solution of (1.3) (in the sense of Definition 2.1) in \((\xi_0, +\infty)\), by setting \( \phi(\xi) = 1 \) for \( \xi \leq \xi_1 \). Thus, we can always consider \( \phi \) satisfying weakly (1.3) in \( \mathbb{R} \); moreover \( \phi \) solves (10.1) in \((\xi_1, \xi_0)\) with

\[
\xi_1 = \inf \{ \xi \in \mathbb{R} : \phi(\xi) < 1 \} \in [-\infty, 0), \quad \xi_0 = \sup \{ \xi \in \mathbb{R} : \phi(\xi) > 0 \} \in (0, +\infty],
\]

and it is constant in \( \mathbb{R} \setminus (\xi_1, \xi_0) \). Thus, we showed that if \( c \geq c^* \) then there exists a wavefront \( \phi \) whose profile satisfies (1.3).

By reasoning as in the proof of Theorem 2.1, also the converse implication holds. Indeed, if \( \phi \) is a profile of a wavefront satisfying (1.4), then the function \( z \) defined by

\[
z(\varphi) := D(\varphi) \frac{\varphi'(-1(\varphi))}{\varphi}, \quad 0 < \varphi < 1,
\]

is a solution of (3.12). Thus, \( c \geq c^* \).

We now prove the latter part of the statement. We prove (i). Assume \( c > c^* \). From (8.6) in Proposition 8.1, we have \( \dot{z}(0) = 0 \). Hence, if \( \dot{D}(0) \neq 0 \) then it holds

\[
\lim_{\xi \to \xi_0^-} \phi'(\xi) = \lim_{\varphi \to 0^+} \frac{z(\varphi)}{D(\varphi)} = 0. \quad (10.2)
\]

Indeed, we have

\[
\lim_{\xi \to \xi_0^-} \phi'(\xi) = \lim_{\varphi \to 0^+} \frac{\dot{z}(0)}{\dot{D}(0)} = 0.
\]

If \( \dot{D}(0) = 0 \), then we argue as in the proof of Corollary 2.1, see (9.24), to show that, for any \( \varepsilon > 0 \) there exists \( \delta \in (0, 1) \) such that

\[
z(\varphi) > -\varepsilon D(\varphi), \quad \varphi \in (0, \delta].
\]

Hence,

\[
\lim_{\xi \to \xi_0^-} \phi'(\xi) = \lim_{\varphi \to 0^+} \frac{z(\varphi)}{D(\varphi)} \geq -\varepsilon.
\]

Since \( \phi' < 0 \) in \((\xi_1, \xi_0)\) and \( \varepsilon \) is arbitrarily small, it follows again (10.2).

We prove now (ii). With (8.6) in mind, from \( c = c^* > h(0) \) we obtain \( \dot{z}(0) = h(0) - c^* < 0 \). Therefore,

\[
\frac{D(\sigma)}{-z(\sigma)} = \frac{\dot{D}(0) + o(1)}{c - h(0) + o(1)} \quad \text{as} \quad \sigma \to 0^+,
\]

and consequently (9.22) is verified. Thus, \( \xi_0 \in \mathbb{R} \). Furthermore, from (9.20),

\[
\lim_{\xi \to \xi_0^-} \phi'(\xi) = \lim_{\varphi \to 0^+} \frac{z(\varphi)/\varphi}{D(\varphi)/\varphi} = \frac{h(0) - c^*}{D(0)} \in [-\infty, 0),
\]

and thus the conclusions hold. □
Remark 10.1 (Case \(c = c^* = \text{h}(0)\)). Part (i) and (ii) of Theorem 2.2 do not cover the case \(c = c^* = \text{h}(0)\). The following discussion shows that, to classify the behavior in that case, further assumptions are needed. More precisely, either a classical and a sharp wavefront can indeed occur under (D0) and (g01). Take \(q\) and \(h\) as in (6.10) in Remark 6.2. There, we proved that in this case it holds \(c^* = \text{h}(0) = 0\). Consider

\[
\begin{aligned}
D_1(\varphi) &= \varphi^2, \\
g_1(\varphi) &= \varphi(1 - \varphi), \\
D_2(\varphi) &= \varphi, \\
g_2(\varphi) &= \varphi^2(1 - \varphi).
\end{aligned}
\]

Clearly, \(D_1\) and \(g_1\) satisfy (D0) and (g01) and so \(D_2\) and \(g_2\). Also, since \(D_1g_1 = q = D_2g_2\), then \(c^*_1 = c^*_2 = \text{h}(0) = 0\), where \(c^*_1\) and \(c^*_2\) are the thresholds given by Proposition 4.1 associated with \(D_1g_1\) and \(D_2g_2\), respectively.

Define, for \(\xi \in \mathbb{R}\),

\[
\varphi_1(\xi) := \begin{cases} 
1 - \frac{e^{\xi}}{2}, & \xi < \log(2), \\
0, & \text{otherwise},
\end{cases}
\]

and \(\varphi_2(\xi) := \frac{1}{1 + e^{\xi}}\).

Direct computations show that \(\varphi_1\) and \(\varphi_2\) are two wave profiles defining two wavefronts, both of them associated with \(c = \text{h}(0)\). Plainly, \(\varphi_1\) is sharp at \(\xi = \log(2)\) while \(\varphi_2\) is classical.

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