On some comparison of the multistep hybrid methods and their application solving of the Volterra integro-differential equations

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Abstract. As is known, there are some classes of numerical methods for solving of the initial-value problem for the Volterra integro-differential equations. Here, by comparison of the known methods have constructed the methods with the new properties which have applied to solve the initial-value problem for the ODE and for the Volterra integra-differential equations. By the construction of some relation between of these equations have established the direct connection among them which have called as the $p$-equivalents between the initial-value problem for ODE and for the Volterra integro-differential equations. Constructed here the stable methods with the high order of accuracy show some advantages of them. Some of them are applied to solving of the initial-value problem for the Volterra integro-differential equations. And also for the illustration of the received results here constructed have applied one of these methods to solve the model problem.

1. Introduction

As is known the necessity of investigation of Volterra integro-differential equations arises in solving many problems of the different industrials of the natural science. One of the popular methods for solving this problem is the quadrature methods. In the last time some authors to prefer to apply the multistep methods with constant coefficients to solve the above mentioned problem. Therefore here have considered the comparison of the known methods and as the resulting of which recommended some of the multistep hybrid methods to solve the initial-value problem for the Volterra integro-differential equations.

Let us consider the following problem:

$$y'(x) = f(x, y) + \int_{x_0}^{x} K(x, s, y(s)) ds, \quad y(x_0) = y_0, \quad x \in [x_0, X].$$ (1)

This problem usually is called as the initial-value problem for the Volterra integro-differential equations of the first order.

Here suppose that the continuous on the totality of the arguments the functions $f(x, y)$ and $K(x, s, y)$ are defined on some close sets. For the determined the values of the solution of the problem (1) let us denote by $y_i$ the approximately and by $y(x_i)$ the exact values of the solution of the problem (1) at the mesh point $x_{i+1} = x_i + h(i = 0, 1, ..., N - 1)$.

As is known to solve the problem of (1) in usually have used the following ways:

$$y'(x) = f(x, y) + v(x), \quad y(x_0) = y_0, \quad x \in [x_0, X],$$ (2)


If we suppose that the function \( v(x) \) is known, then from the correlation of (3) is the Volterra integral equation to respect of the functions \( y(x) \). Thus receive that the solution of the problem (1) can be found as the solution of the system (2)-(3). To solve the equation of (3) one can use some modification of the quadrature method. But for solving the initial-value problem of (2) can be used the following method:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} \quad (\alpha_k \neq 0, n = 0, 1, ..., N - k),
\]

which is called the multistep method with constant coefficients. Note that the method of (4) fundamentally have investigated by some authors (see for examples [1]-[7]). As is known using the quadrature methods in solving of the Volterra integral equation the volume of the computational works increases. Therefore here have constructed the methods which are freed from the mentioned disadvantage.

It follows to remark that solving of the problem (1) we have replaced calculation of the values of the function \( v(x) \) and to solving of the problem (2). Thus we receive that to solve the problem (1) one can use the system consisting of two methods. For the simplicity of the presentation of the constructing here method let us replace the equation (3) by the following:

\[
v(x) = \int_{x_0}^{x} \varphi(s, y(s))ds.
\]

It is not difficult to define that the solving of the equation of (5) is equivalent to the following problem:

\[
v'(x) = \varphi(x, y(x)), \quad v(x_0) = 0.
\]

As follows from the above description, solving of the problem (1) have replaced by solving of the system of initial-value problem for the ODE of the first order which consists of the two ODEs. Thus in the case of (5) here have shown that the determination of the solution of the initial-value problem can be replaced by the determination of the solving of the initial-value problem for the ODE. And then by using one of the known methods from the class of methods which have proposed to solve the initial-value problem for the ODE. One can solve the considering problem for example by using the following method:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} \quad (m \geq 0; n = 0, 1, ..., N - k),
\]

Here \( s \) and \( k \) -integer values and hold one of them: \( k \geq s \) or \( k \leq s \). This method usually called as the finite-difference method or multistep method with constant coefficients fundamentally investigated by the many authors (see for example [8]-[10]). It is evident that the method of (7) can be applied to solve the initial-value problem both for the Volterra integro-differential equations and for the ODE. But let us note that to solve the problem (1) it is needed to use the system which consist from the finite-difference methods. For the comparison of this situation let us consider solving of the following problem:

\[
y'' = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad x \in [x_0, X].
\]

In usually this problem is called as the initial-value problem for the ODE of the second order received by the higher derivative. As is known this problem can reduce to the following:

\[
y'(x) = z(x), \quad y(x_0) = y_0, \quad z'(x) = f(x, y, z), \quad z(x_0) = z'_0.
\]

But this way we receive that to solving of the problem (1) and to solving of the problem (8) is equivalent. If we compare the system of initial-value problem consists from the problems of (1) and
(6) with the problem consists from the problems (9) and (10), then receive that to solving of these problems one can be applied one of the same methods. For example the method of (7). As is known to applied of the method (7) to solving of the problem (8) is not effective because if the method (7) is stable and has the order of accuracy \( p \), then the following holds (see for example [1], [5]): 
\[ p \leq 2[k/2] + 2. \]
And there exist the stable methods with the degree \( p = 2[k/2] + 2 \) for the all values of the order of \( k \). But as is known for solving problem (8) one can use the following
\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i y''_{n+i}. \tag{11}
\]
It is known that if the method (11) is stable and has the degree of \( p \), then \( p \leq 2k+2 \) holds. There exist the stable methods with the degree \( p = 2k + 2 \) for the all values of \( k \) (see for example [1], [5]).

The conceptions of stability and the degree are defined as follows:
If the roots of the following polynomial
\[
\rho(\lambda) = \alpha_k \lambda^k + \alpha_{k-1} \lambda^{k-1} + \ldots + \alpha_1 \lambda + \alpha_0
\]
lie in the unit circle on the boundary of which there are no multiple roots. The method (11) has the degree of \( p \) if for the sufficiently smooth function \( y(x) \) the following holds:
\[
\sum_{i=0}^{k} (\alpha_i y(x + ih) - h\beta_i y'(x + ih) - h^2 \gamma_i y''(x + ih)) = O(h^{p+1}), h \rightarrow 0. \tag{12}
\]

It follows from here that to application of the method (11) to solving problem (1) is the preferable than the using of the method (7).

By using this recommendation let us consider the following problem in the case when
\[
K(x, s, y) = \varphi(s, y):
\]
\[
y''(x) = f''(x, y) + f'(x, y)y'(x) + \varphi(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = f(x_0, y_0). \tag{13}
\]
Thus receive that by application of the method (11) to solving of the problem (13) one can be find the approximately values for the solution of the problem (1) with the order \( p = 2k + 2 \), if there use the methods for calculation of the values \( y'_m(m = 0, 1, \ldots) \) with the corresponding exactness. For this we cannot use the method (7) because if the method (7) is stable then \( p \leq k + 2 \). Therefore the construction of the methods with the higher degrees let us use other way.

Remark, that for the finding of the solution of the problem (13) as the problem (8) can be reduced that to finding of the solution of the following system of ODEs of the first order:
\[
y' = z(x), \quad y(x_0) = y_0; \quad z'' = f'_x(x, y) + f'_y(x, y)z(x) + \varphi(x, y), \quad z(x_0) = f(x_0, y_0). \tag{14}
\]

2. Constructing of the stable methods with more exactness
By the above described way have established the direct relation between of the initial-value problem for the Volterra integro-differential equation and for the ODE in the case when \( K(x, s, y) = \varphi(s, y) \) holds. And taking into account that one of the basic problems for the modern computational mathematics is the construction of the stable methods with the high degree and having the wide region of stability, therefore here have proposed the class of the methods with the high degrees. For this aim have proposed to use the multistep second derivative method (11) with the constant coefficients which were fundamentally investigated by the many authors (see for example [11]-[13]). It is evident that if for the finding of the solution of the problem (1) to use the system (14) and solving that by using of the method of (7) then receive that one can be found the values of the solutions of the problem (14) with the degree \( p \leq k + 2 \). Therefore to receive more exact results one can be applied the method of (11) to solving of the problem (13) which is the initial-value problem for the ODE of second order. As was noted above in the problem (13) has participated the function \( y'(x) \) and for the calculation of the values \( y'_i(i \geq 0) \) with more exactness, we need to construct the new methods for calculation those
values \( y'_i \) with the high order of accuracy. For this aim let us use the following multistep method with constant coefficients:

\[
\sum_{i=0}^{k} \alpha_i y'_{n+i} = h \sum_{i=0}^{k} \beta_i y_{n+i} + h \sum_{i=0}^{k} \gamma_i y'_{n+i + v_i}.
\]

(15)

This is the general multistep hybrid method with constant coefficients. For the sake of noted objectivity that one can propose to calculation of the values \( y'_m (m > 0) \), to using of the problems (6) and (2). But in this case and also arises the necessity for construction of the methods with the degree \( p \leq 2k \) for calculation of the values \( y'_m (m \geq 0) \), if to calculation of the values \( y'_m (m \geq 0) \) to use the method of (11). For solving this problem here proposed to use the methods of type (15), which have investigated by many authors (see for example [16]-[20]).

Thus receive that for the finding of the values of the solution of the problem (1) we must construct the new algorithm. For this aim let us consider comparison of the solution of problem (1) and the solution of the following model problem:

\[
y' = f(x, y) + \int_{x_0}^{x} \varphi(s, y(s))ds, y(x_0) = y_0.
\]

(16)

As is known by using Lagrange interpolation polynomial the function \( K(x, s, y) \) can be presented in the next form:

\[
K(x, s, y) = \sum_{j=0}^{k} l_j (x)K(x_j, s, y) + R_k (x), x \in [x_0, x_m].
\]

(17)

In this case, the Volterra integro-differential equation can be rewritten as follows:

\[
\int_{x_0}^{x} K(x, s, y(s))ds = \sum_{j=0}^{k} l_j (x) \int_{x_0}^{x} K(x_j, s, y(s))ds + \int_{x_0}^{x} R_k (s)ds, x \in [x_0, x_m].
\]

(18)

By taking this into account in the problem of (1), receive:

\[
y' = f(x, y) + \sum_{j=0}^{k} l_j (x) \int_{x_0}^{x} K(x_j, s, y(s))ds + \int_{x_0}^{x} R_k (s)ds, x \in [x_0, x_m],
\]

(19)

here \( l_j (x) \)-the Lagrange interpolation basis functions. If use the following denotation:

\[
v_j (x) = \int_{x_0}^{x} K(x_j, s, y(s))ds (j = 0, 1, \ldots, k),
\]

then after discarding of the remainder term in the equality (19), receive:

\[
y' = f(x, y) + \sum_{j=0}^{k} l_j (x) v_j (x), \quad y(x_0) = y_0,
\]

(20)

\[
v'_j (x) = K(x_j, x, y), \quad v_j (x_0) = 0, \quad (j = 0, 1, \ldots, k).
\]

(21)

Thus we have reduced the finding of the solution of the initial-value problems for the Volterra integro-differential equations to the finding of the solution of the initial-value problem for the ODEs. Let us note that from here follows that the problem (1) and the problem (20)-(21) are equivalents. They can be called the equivalents of the order of \( p \). For this let us consider the following definition.

**Definition 1.** The problem called as equivalent with the order of \( p \) if the solution of one of them is different from the solution of the other problem with the remainder term having the order of \( p \).

By using this definition receive that the problems (1) and (16) are equivalent with the order of \( p \).

Remark that the values of \( p \) can become greater by replacement in the equality of (17) the Lagrange interpolation polynomials by the Hermit or Gauss interpolation polynomials.
As follows from the above-mentioned the investigation of the problem (1) can be replaced by the investigation of the problem (16). The problem (16) receive from the problem (20) in the case when \( l_j(x) \equiv \text{const}(j = 0,1,\ldots,k) \).

Let us note that by solving of the problem (16) we can find the approximate solution of the problem (1) don’t take into account the properties of the kernel \( K(x,s,y) \) with the related of variable of \( x \).

Therefore let us construct the methods for solving the problem (1) by using the following two parts:

I. Construct the methods for solving the problem (16). By using the equivalence of the problem (1) and (16) receive that the exactness of the proposed methods for solving of the problem (1) in basically dependence from the exactness of the methods, which have constructed for solving of the problem (16).

II. Adapting the constructed methods to solving for the problem (20). On this part, by choosing some free coefficients one can extend of the stable region for the proposed methods.

And now let us apply the multistep method of (4) to solving of the problem (16). In this case receive:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h \sum_{i=0}^{k} \beta_i v_{n+i},
\]

\[
\sum_{i=0}^{k} \alpha_i v_{n+i} = h \sum_{i=0}^{k} \beta_i \varphi_{n+i}.
\]

And now consider the adapted method of (22). For this aim, the method (22) has applied to solve the problem (20). It is not difficult to understand that the method (22) to be changed and can be written as the following:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i,j} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h \sum_{i=0}^{k} \sum_{j=0}^{k} \beta^{(j)}_i \varphi_{n+i,j}.
\]

It is evident that in the initial-value problems (20) and (21) doesn't violate the generality the initial value \( x_0 \) to be changed by the \( x_n \). In this case, the solution of the problems not changed. Therefore the method (23) can be written as:

\[
\sum_{i=0}^{k} \alpha_i v_{j,n+i} = h \sum_{i=0}^{k} \beta_i K(x_{n+j},x_{n+i},y_{n+i})(j = 0,1,\ldots,k).
\]

Thus we prove that by using one of the same methods may be solved the problems (20) and (21). But in the application of the above described way to solve the problem (1) arise some difficulties in using the interpolation polynomials. Therefore let us consider the following method:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i,j} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h \sum_{j=0}^{k} \beta^{(j)}_i v_{n+i,j},
\]

\[
\sum_{i=0}^{k} \alpha_i v_{j,n+i} = h \sum_{i=0}^{k} \sum_{j=0}^{k} \beta^{(j)}_i K(x_{n+j},x_{n+i},y_{n+i}).
\]

It follows to note that to solve the problem (1) by using the multistep methods have dedicated many known authors (see for example [21-[26]), but methods are proposed here differ from the known to them that these methods can be applied to solving of the initial-value problem for ODE.

The method (27) is the modification of the method (23). From the formula (27) it follows that the discreet function \( v_{n+i} \) is a certain sum of the functions \( v_{j,n+i}(j = 0,1,\ldots,k) \). This receives from the interpolation polynomials.

As was noted above the aim of our investigation is the construction of the stable methods with the higher degree by using the constant volume of the computational works at each step. Note that method constructed by the help of the formulas (26) and (27) are to guarantee the constant volume of the computational works at each step. Therefore let us consider the construction of stable methods with the high degrees having the type of (26) and (27). And also to remark that our aim is using one of the same methods to solving of the problem (2) and of the equation (3). Therefore let us consider to
determine the values of the coefficients $\beta_{ij}^{(i)}(i, j = 0, 1, ..., k)$. For this it is enough to put $l(y) = \text{const}$ $(j = 0, 1, ..., k)$ in the equality of (24).

In this case by comparison of equalities of (23) and (27) receive that:

$$\sum_{j=i}^{k} \beta_{ij}^{(i)} = \beta_{ij}(i = 0, 1, ..., k),$$  \hspace{1cm} (28)

here have used the condition $s \leq x$ (see (3)) which is related of the properties of the Volterra integral equation.

Thus we have proved that by using one of the same methods can be solved the initial-value problem for the Volterra integro-differential equations and ODE. Taking into account the existence of the wide class of numerical method for solving the initial-value problem of the ODE, the coefficients $\alpha_i, \beta_i(i = 0, 1, ..., k)$ can be considered are known. And the system (28) always has the solution more than one.

It follows from here that the stable methods constructed by using the above described way will have the degree $p \leq 2[k / 2] + 2$. Therefore for construction of the stable methods with the degree $p > k + 2$, here propose to use the general hybrid method, which can be written as:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h \sum_{i=0}^{k} \gamma_i y'_{n+i+v_i}, \quad |v_i| < 1, i = 0, 1, ..., k).$$  \hspace{1cm} (29)

This method fundamentally investigated by some authors (see for example [24],[27]-[30]) and have constructed concrete methods with the degree $p \leq 9$. As was noted above, here we have adapted some of them to solving equations (3) by using the solution of the system (28). Let us note that for construction of methods of type (29) one can be used the Lagrange and Gauss interpolation polynomials. In the results of which receive the following formula:

$$\int_{x_n}^{x_{n+2}} f(s)ds = h \sum_{i=0}^{k} \beta_i f_{n+i} + h \sum_{i=0}^{k} \gamma_i f_{n+i+v_i}, \hspace{1cm} (30)$$

here the points $x_{i+v_i} = x_n + (i + v_i)h$ are called the hybrid points and there are a direct relation between of them and Gauss node points.

Thus prove that the method (29) can be applied to solving of the initial-value problem for the Volterra integro-differential equation and to solving of the initial-value problem for the ODEs. For the illustration of the advantages proposed the method to consider of the application of the following method:

$$y_{n+1} = y_n + h(y'_{n+1} + y'_{n+1+1})(l_0 = 1 / 2 - \sqrt{3} / 6; l_1 = 1 / 2 + \sqrt{3} / 6)$$  \hspace{1cm} (31)

to solving of the problem:

$$y' = \left(4 \exp(-y) - x^3 \right) / 3 + 4 \frac{x^2}{3} \exp(y(s))ds, 1 \leq x \leq 2, y(1) = 0,$$  \hspace{1cm} (32)

the exact solution for which can be written as: $y(x) = \ln x$.

By taking into account that the method has the degree $p = 4$ for the comparison of results received by that method with the results received by using of the known methods have considered the methods to application of method (31) and the method of Simpson to solving of the problem (32). As follows from the results receiving in the solving of the problem (32) the method (31) gives the best results (see table 1).

**Table 1. Results for methods with the same degrees (p=4).**
And now let us apply the method of (31) to solving of the following problem:

\[ y'(x) = \lambda y(x) - m(1 - \exp(-\lambda x)) + \lambda m \int_0^x \exp(-\lambda s)y(s)ds, \quad y(0) = 1, 0 \leq s \leq x \leq 1 \quad (33) \]

with the exact solution \( y(x) = \exp(\lambda x) \).

This problem was solved by using the method (31). It is clear that for the value \( m = 0 \) corresponds initial-value problem for the ODEs. Therefore the problem (33) have solved for the case \( m = 0 \) and \( m = 1 \).

**Table 2. Results for initial value-problem for integro-differential and ordinary-differential equations**

| Variable x | Integro-differential equation | Ordinary-differential equation |
|------------|-------------------------------|-------------------------------|
|            | \( h = 1/32, \lambda = 1, m = 1 \) | \( h = 1/32, \lambda = 1, m = 0 \) |
| 1.031      | 4.44E-10                      | 4.679E-10                     |
| 1.250      | 6.12E-9                       | 4.72E-9                       |
| 1.500      | 3.49E-8                       | 1.21E-8                       |
| 1.750      | 8.75E-8                       | 2.34E-8                       |
| 2.000      | 1.664E-7                      | 4.00E-8                       |

As is shown from this table the results received here for both cases \( m = 0 \) and \( m = 1 \) are almost the same.

**3. Conclusion**

The aim of our investigation is contained in the construction of the multistep hybrid methods with the best properties. Therefore we decided to compare some of the known methods and to define priority direction in the theory of numerical methods. It was determined that one of the priority directions of numerical methods with the best properties is the use of hybrid methods in their construction.

By showing the advantages of the hybrid methods here constructed and applied the general hybrid methods to solve the initial-value problem for the Volterra integro-differential equations. For the proving of the advantages of these methods have considered using of some simple hybrid methods for solving of the model problem. The model problem has constructed so that from it can be received the initial-value problem for the ODE and for the Volterra integro-differential equation. As it follows from here that by the same method one can be solved the above mentioned problems use of sections to divide the text of the paper is optional and left as a decision for the author. Where the author wishes to divide the paper into sections the formatting shown in table 2 should be used.

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Acknowledgments

The authors express their thanks to the academicians Telman Aliyev and Ali Abbasov for their suggestion to investigate the computational aspects of our problem. This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan Grant № EIF-KETPL-2-20151(25)-56/07/1.