GENERALIZED NEHARI MANIFOLD AND SEMILINEAR SCHRÖDINGER EQUATION WITH WEAK MONOTONICITY CONDITION ON THE NONLINEAR TERM

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Abstract. We study the Schrödinger equations $-\Delta u + V(x)u = f(x,u)$ in $\mathbb{R}^N$ and $-\Delta u - \lambda u = f(x,u)$ in a bounded domain $\Omega \subset \mathbb{R}^N$. We assume that $f$ is superlinear but of subcritical growth and $u \mapsto f(x,u)/|u|$ is nondecreasing. In $\mathbb{R}^N$ we also assume that $V$ and $f$ are periodic in $x_1, \ldots, x_N$. We show that these equations have a ground state and that there exist infinitely many solutions if $f$ is odd in $u$. Our results generalize those in [11] where $u \mapsto f(x,u)/|u|$ was assumed to be strictly increasing. This seemingly small change forces us to go beyond methods of smooth analysis.

1. Introduction

We consider the semilinear Schrödinger equations

\[(1.1)\quad -\Delta u + V(x)u = f(x,u), \quad u \in H^1(\mathbb{R}^N)\]

and

\[(1.2)\quad -\Delta u - \lambda u = f(x,u), \quad u \in H^1_0(\Omega),\]

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $H^1(\mathbb{R}^N)$, $H^1_0(\Omega)$ are the usual Sobolev spaces. In both problems we make the following assumptions on $f$:

(F1) $f$ is continuous and $|f(x,u)| \leq C(1 + |u|^{p-1})$ for some $C > 0$ and $p \in (2, 2^*)$, where $2^* := 2N/(N-2)$ if $N \geq 3$ and $2^* := +\infty$ if $N = 1$ or 2,

(F2) $f(x,u) = o(u)$ uniformly in $x$ as $u \to 0$,

(F3) $F(x,u)/u^2 \to \infty$ uniformly in $x$ as $|u| \to \infty$, where $F(x,u) := \int_0^u f(x,s) \, ds$,

(F4) $u \mapsto f(x,u)/|u|$ is non-decreasing on $(-\infty, 0)$ and on $(0, \infty)$.

The assumptions (F1)–(F3) appear in [11] while a condition corresponding to (F4) is a little stronger there:

(F4) $u \mapsto f(x,u)/|u|$ is strictly increasing on $(-\infty, 0)$ and on $(0, \infty)$.  

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As we shall see, this slightly weaker hypothesis will force us to go beyond methods of smooth analysis, and introducing a non-smooth approach in this context is in fact our main purpose. In what follows we shall frequently refer to different results and arguments in \[11\] \[12\]. When such reference is made, it should be understood that no stronger conditions than \((F_1)\)–\((F_4)\) were needed there.

The main results of this paper are the following two theorems:

**Theorem 1.1.** Suppose \(f\) satisfies \((F_1)\)–\((F_4)\), \(V\) and \(f\) are 1-periodic in \(x_1, \ldots, x_N\) and \(0 \notin \sigma(-\Delta + V)\), where \(\sigma(\cdot)\) denotes the spectrum in \(L^2(\mathbb{R}^N)\). Then equation \((1.1)\) has a ground state solution. If moreover \(f\) is odd in \(u\), then equation \((1.1)\) has infinitely many pairs of geometrically distinct solutions.

**Theorem 1.2.** (i) Suppose \(f\) satisfies \((F_1)\)–\((F_4)\) and \(\lambda \neq \lambda_k\) for any \(k\), where \(\lambda_k\) is the \(k\)-th eigenvalue of \(-\Delta\) in \(H^1_0(\Omega)\). Then equation \((1.2)\) has a ground state solution. If moreover \(f\) is odd in \(u\), then equation \((1.1)\) has infinitely many pairs of geometrically distinct solutions \(\pm u_k\) such that the \(L^\infty(\Omega)\)-norm of \(u_k\) tends to infinity with \(k\).

(ii) If \(\lambda = \lambda_k\) for some \(k\), then the above results remain valid under the additional assumption that \(f(x, u) \neq 0\) unless \(u = 0\).

Similar results, but under the stronger condition \((F_4)\), have been proved in \[11\].

As usual, a *ground state* is a solution which minimizes the functional corresponding to the problem over the set of all nontrivial \((u \neq 0)\) solutions. Later in this section we shall define what we mean by geometrically distinct solutions.

Existence of a ground state solution under the assumptions of Theorem \[11\] has been shown by S. Liu in \[7\]; since this result is an easy consequence of our approach, we include it here anyway. See also \[16\] where a number of results on ground states for problems similar to \((1.1)\) and \((1.2)\) has been proved and \[13\] where \((F_4)\) has been further weakened. Existence of ground states for *systems* of equations has been discussed in \[8\]. Concerning existence of infinitely many solutions we know of a result by Tang \[14\] where a condition different from \((F_4)\) has been introduced for \((1.2)\), and by Zhong and Zou \[16\] where \((1.1)\) and \((1.2)\) have been considered under the same hypotheses as in Theorems \[11\] and \[12\]. However, they needed an additional assumption which is not easy to verify unless \(u \mapsto f(x, u)/|u|\) is “most times” strictly increasing.

Consider equation \((1.1)\) under the assumptions of Theorem \[11\]. Let \(E := H^1(\mathbb{R}^N)\). The functional corresponding to \((1.1)\) is

\[
\Phi(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx - \int_{\mathbb{R}^N} F(x, u) \, dx.
\]

It is well known (see e.g. \[15\]) that \(\Phi \in C^1(E, \mathbb{R})\) and critical points of \(\Phi\) are solutions for \((1.1)\). Let \(E = E^+ \oplus E^-\) be the decomposition corresponding to the positive and the negative part of the spectrum of \(-\Delta + V\). Since \(0 \notin \sigma(-\Delta + V)\), there exists an equivalent
inner product $\langle \cdot, \cdot \rangle$ in $E$ such that
\begin{equation}
\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}^N} F(x, u) \, dx,
\end{equation}
where $u^\pm \in E^\pm$.

For equation (1.2) under the assumptions of Theorem 1.2 we put $E = H^1_0(\Omega)$ and we have the spectral decomposition $E = E^+ \oplus E^0 \oplus E^-$, where $E^0$ is the nullspace of $-\Delta - \lambda$ in $E$ and $0 \leq \dim(E^0 \oplus E^-) < \infty$. Also here we can choose an equivalent inner product such that the corresponding functional $\Phi$ is of the form (1.3), with $\mathbb{R}^N$ replaced by $\Omega$.

The following set introduced by Pankov [9] is called the generalized Nehari manifold or the Nehari-Pankov manifold:
\begin{equation}
\mathcal{M} := \{ u \in E \setminus (E^0 \oplus E^-) : \Phi'(u)u = 0 \text{ and } \Phi'(u)v = 0 \text{ for all } v \in E^0 \oplus E^- \}.
\end{equation}
(E^0$ is necessarily trivial in Theorem 1.1). $(F_4)$ implies $f(x, u)u \geq 0$, and the assumptions of Theorem 1.2 imply that if dim $E^0 > 0$, then $f(x, u)u > 0$ for $u \neq 0$. Hence $\mathcal{M}$ contains all nontrivial critical points of $\Phi$. Note that if $E^0 \oplus E^- = \{0\}$, then $\mathcal{M}$ is the usual Nehari manifold [12]. Since this case is considerably easier to handle, we assume in what follows that $\sigma(-\Delta + V) \cap (-\infty, 0) \neq \emptyset$ in Theorem 1.1 and $\lambda \geq \lambda_1$ in Theorem 1.2. As in [11], for $u \notin E^0 \oplus E^-$ we define
\begin{equation}
E(u) := E^0 \oplus E^- \oplus \mathbb{R}u = E^0 \oplus E^- \oplus \mathbb{R}u^+
\end{equation}
and
\begin{equation}
\hat{E}(u) := E^0 \oplus E^- \oplus \mathbb{R}^+ u = E^0 \oplus E^- \oplus \mathbb{R}^+ u^+,
\end{equation}
where $\mathbb{R}^+ = [0, \infty)$. It has been shown there that if $(F_4)$ is replaced by $(F'_4)$, then $\hat{E}(u)$ intersects $\mathcal{M}$ at a unique point which is the unique global maximum of $\Phi|_{\hat{E}(u)}$. It has been shown in [16] by an explicit example that if $(F_4)$ but not $(F'_4)$ holds, then (in the framework of Theorem 1.2) $\hat{E}(u)$ and $\mathcal{M}$ may intersect on a finite line segment. In the next section we shall show that $\hat{E}(u) \cap \mathcal{M} \neq \emptyset$ and if $w \in \hat{E}(u) \cap \mathcal{M}$, then there exist $\sigma_w > 0$, $\tau_w \geq \sigma_w$ such that $\hat{E}(u) \cap \mathcal{M} = [\sigma_w, \tau_w]w$. In other words, $\hat{E}(u) \cap \mathcal{M}$ is either a point or a finite line segment. We also show that a point $\hat{w} \in [\sigma_w, \tau_w]w$ is critical for $\Phi$ if and only if the whole segment $[\sigma_w, \tau_w]w$ consists of critical points.

In Theorem 1.1 the functional $\Phi$ is invariant with respect to the action of $\mathbb{Z}^N$ given by the translations $k \mapsto u(\cdot - k)$, $k \in \mathbb{Z}^N$. Hence if $u$ is a solution of (1.1), then so is $u(\cdot - k)$. This and the preceding paragraph justify the following definition: Two solutions $u_1$ and $u_2$ are called geometrically distinct if $u_2 \neq u_1(\cdot - k)$ for any $k \in \mathbb{Z}^N$ and $u_2 \notin [\sigma_{u_1}, \tau_{u_1}]u_1$. In Theorem 1.2 there is no $\mathbb{Z}^N$-invariance but we still want to identify solutions in $\hat{E}(u) \cap \mathcal{M}$. So $u_1, u_2$ are geometrically distinct if $u_2 \notin [\sigma_{u_1}, \tau_{u_1}]u_1$.

2. Preliminaries

In this section we assume that the hypotheses of Theorem 1.1 or 1.2 are satisfied. In particular, $(F_1)$–(F_4) hold. To simplify notation, $\Omega$ will stand for $\mathbb{R}^N$ or for a bounded domain in $\mathbb{R}^N$. 
Lemma 2.1. If \( f(x, u) \neq 0 \), then \( F(x, u) < \frac{1}{2} f(x, u) u \).

Proof. Suppose \( u > 0 \). Since \( f(x, t)/t \to 0 \) as \( t \to 0 \) and \( f(x, u)/u > 0 \),
\[
F(x, u) = \int_0^u \frac{f(x, t)}{t} t \, dt < \frac{f(x, u)}{u} \int_0^u t \, dt = \frac{1}{2} f(x, u) u.
\]
For \( u < 0 \) the proof is similar. \( \Box \)

The following result will be crucial for studying the structure of the set \( \widehat{E}(u) \cap \mathcal{M} \).

Proposition 2.2. Let \( x \in \Omega \) be fixed and let \( s, v \in \mathbb{R} \) be such that \( s \geq 0 \) and \( f(x, u) \neq 0 \). Then:

(i) \[
g(s, v) := f(x, u) \left[ \frac{1}{2} (s^2 - 1) u + sv \right] + F(x, u) - F(x, su + v) \leq 0
\]
for all \( x \).

(ii) There exist \( s_u \in (0, 1] \), \( t_u \geq 1 \) such that \( g(s, v) = 0 \) if and only if \( s \in [s_u, t_u] \) and \( v = 0 \) \((s_u = t_u \text{ not excluded})\). Moreover, for such \( s \) we have \( f(x, su) = sf(x, u) \).

Part (i) of this proposition has been shown in [7] and it extends a similar result in [11] where \((F_4')\) has been assumed (however, our \( s \) corresponds to \( s + 1 \) in [7, 11]). Here we provide a different argument which will be needed in order to show part (ii).

Proof. Obviously, \( g(1, 0) = 0 \). We shall show that \( g(s, v) \to -\infty \) as \( s + |v| \to \infty \). Put \( z = z(s) := su + v \). Using Lemma 2.1 we obtain
\[
g(s, v) = f(x, u) \left[ \frac{1}{2} (s^2 - 1) u + sv \right] + F(x, u) - F(x, z)
\]
\[
< f(x, u) \left[ \frac{1}{2} (s^2 - 1) u + s(z - su) \right] + \frac{1}{2} f(x, u) u - F(x, z)
\]
\[
= -\frac{1}{2} s^2 f(x, u) u + sf(x, u) z - Az^2 + (Az^2 - F(x, z)).
\]
Since the quadratic form (in \( s \) and \( z \)) above is negative definite if \( A > 0 \) is a constant large enough and since \( Az^2 - F(x, z) \) is bounded above according to \((F_3)\), \( g(s, v) \to -\infty \) as \( s + |v| \to \infty \) as claimed.

It follows that \( g \) has a maximum \( \geq 0 \) on the set \( \{(s, v) : s \geq 0\} \). As
\[
g(0, v) = -\frac{1}{2} f(x, u) u + F(x, u) - F(x, v) < -F(x, v) \leq 0
\]
(by Lemma 2.1), the maximum is attained at some \( (s, v) \) with \( s > 0 \). Then
\[
g'_v(s, v) = sf(x, u) - f(x, su + v) = 0
\]
and
\[
g'_s(s, v) = (su + v)f(x, u) - uf(x, su + v) = 0.
\]
Using (2.2) in (2.3) we obtain \( vf(x, u) = 0 \). Hence \( v = 0 \) and
\[
g'_s(s, 0) = su^2 \left( \frac{f(x, u)}{u} - \frac{f(x, su)}{su} \right) = 0.
\]
By $(F_4)$, there must exist $s_u, t_u$ such that $s_u \in (0, 1]$, $t_u \geq 1$ and $g'(s, 0) = 0$ if and only if $s \in [s_u, t_u]$. For such $s$ we have $g(s, 0) = g(1, 0) = 0$ and $f(x, su) = sf(x, u)$. □

**Corollary 2.3.** Suppose $u \in \mathcal{M}$ and let $s \geq 0$, $v \in E^0 \oplus E^-$. Then

$$\int_{\Omega} \left( f(x, u) \left[ \frac{1}{2} (s^2 - 1) u + sv \right] + F(x, u) - F(x, su + v) \right) dx \leq 0$$

and there exist $0 < s_u \leq 1 \leq t_u$ such that equality holds if and only if $s \in [s_u, t_u]$, $v = 0$. Moreover, for such $s$ and almost all $x \in \Omega$, $f(x, su) = sf(x, u)$.

**Proof.** If $u \in \mathcal{M}$, then $f(x, u(x)) \neq 0$ for $x$ on a set of positive measure. According to Proposition 2.2, inequality (2.1) holds for such $x$ and there exist $s_u(x) \in (0, 1]$, $t_u(x) \geq 1$ such that the left-hand side of (2.1) is zero if and only if $s \in [s_u(x), t_u(x)]$ and $v(x) = 0$. Moreover, for such $s$, $f(x, su(x)) = sf(x, u(x))$. Now one takes $s_u := \text{ess sup} \{s_u(x) : f(x, u(x)) \neq 0\}$ and $t_u := \text{ess inf} \{t_u(x) : f(x, u(x)) \neq 0\}$.

Note that if $f(x, u(x)) = 0$, then $F(x, u(x)) = \int_0^{u(x)} f(x, t) dt = 0$ because $f(x, t) = 0$ for $t$ between 0 and $u(x)$ according to $(F_4)$. Hence the integrand above is $\leq 0$ also in this case. □

**Proposition 2.4.** (i) If $u \in E \setminus (E^0 \oplus E^-)$, then $\hat{E}(u) \cap \mathcal{M} \neq \emptyset$.

(ii) If $w \in \hat{E}(u) \cap \mathcal{M}$, then there exist $0 < s_w \leq 1 \leq t_w$ such that $\hat{E}(u) \cap \mathcal{M} = [s_w, t_w]w$. Moreover, $\Phi(sw) = \Phi(w)$, $\Phi'(sw) = s\Phi'(w)$ for all $s \in [s_w, t_w]$ and $\Phi(z) < \Phi(w)$ for all other $z \in \hat{E}(u)$.

(iii) $\mathcal{M}$ is bounded away from $E^0 \oplus E^-$, closed and $c := \inf_{w \in \mathcal{M}} \Phi(w) > 0$. Moreover, $\Phi|_{\mathcal{M}}$ is coercive, i.e., $\Phi(u) \to \infty$ as $u \in \mathcal{M}$ and $\|u\| \to \infty$.

Note that an immediate consequence is that if $w$ is a critical point of $\Phi$, then the whole line segment $[s_w, t_w]w$ consists of critical points.

**Proof.** (i) The conclusion can be found in [11] Lemma 2.6 and Theorem 3.1], see also [12] Proposition 39]. The proof is by showing that $\Phi(z) \leq 0$ for $z \in \hat{E}(u)$ and $\|z\|$ large enough, and then weak upper semicontinuity of $\Phi|_{\hat{E}(u)}$ implies that there exists a positive maximum.

(ii) For each $z \in \hat{E}(u)$ we have $z = sw + v$, where $s \geq 0$ and $v = v^0 + v^- \in E^0 \oplus E^-$. It has been shown in the course of the proof of [11] Proposition 2.3 and [12] Proposition 39] that

$$\Phi(z) - \Phi(w) = \Phi(sw + v) - \Phi(w) = \frac{1}{2} \|v^\prime\|^2$$

$$+ \int_{\Omega} \left( f(x, w) \left[ \frac{1}{2} (s^2 - 1) w + sv \right] + F(x, w) - F(x, sw + v) \right) dx$$

(again, keep in mind that our $s$ corresponds to $s + 1$ in [11] [12]). Hence according to Corollary 2.3 $\Phi(z) \leq \Phi(w)$ for all $z \in \hat{E}(u)$ and $\Phi(z) = \Phi(w)$ if and only if $z \in [s_w, t_w]w$. That $\Phi(sw) = \Phi(w)$ for $s \in [s_w, t_w]$ is clear and since $\Phi(sw) = \max_{\hat{E}(u)} \Phi(z)$, it is also clear
that $\hat{E}(u) \cap \mathcal{M} = [s_w, t_w]w$ and $\Phi(z) < \Phi(w)$ for other $z$. The equality $\Phi'(sw) = s\Phi'(w)$ follows immediately from the fact that $f(x, sw) = sf(x, w)$.

(iii) That $c > 0$ has been shown in [11] Lemma 2.4 and is an immediate consequence of the fact that $\Phi(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2)$ as $u \to 0$, $u \in E^+$. Since $\Phi|_{E^0 \oplus E^-} \leq 0$, $\mathcal{M}$ is bounded away from $E^0 \oplus E^-$ and hence closed. Finally, according to Proposition 2.7 and the proof of Theorem 3.1 in [11], $\Phi|_{\mathcal{M}}$ is coercive. \hfill $\square$

**Remark 2.5.** If $f$ satisfies $(F_1)$–$(F_4)$ and is of the form $f(x, u) = a(x)h(u)$, where $h(u) \neq 0$ for $u \neq 0$, then $s_w = t_w = 1$ in Proposition 2.4, i.e. $\hat{E}(u)$ intersects $\mathcal{M}$ at a unique point. Assuming the contrary, suppose $t_w > 1$ and $w > 0$ on a set of positive measure (other cases are treated similarly). So meas$\{x : w(x) > d\}$ is positive for some $d > 0$. We claim that $h(t)/t$ is constant for $0 < t < d$. Otherwise there exist $s \in (1, t_w)$, $t_0$ and $\varepsilon > 0$ such that $\varepsilon < t_0 < d - \varepsilon$ and

$$\frac{h(t)}{t} < \frac{h(st)}{st} \quad \text{for all } t \in (t_0 - \varepsilon, t_0 + \varepsilon).$$

Since the sets $\{x : w(x) > t_0 + \varepsilon\}$ and $\{x : w(x) < t_0 - \varepsilon\}$ have positive measure, so does the set $\{x : w(x) \in (t_0 - \varepsilon, t_0 + \varepsilon)\}$, see [11]. But this contradicts the last statement of Corollary 2.3. Hence $h(t)/t$ is constant for $0 < t < d$ and $h(t)/t \to 0$ as $t \to 0$. So $h(t) = 0$ on $(0, d)$ which is impossible.

According to Proposition 2.4, for each $u \in E^+ \setminus \{0\}$ there exist $w$ and $0 < \sigma_w \leq \tau_w$ such that

$$m(u) := [\sigma_w, \tau_w]w = \hat{E}(u) \cap \mathcal{M} \subset E.$$ 

This is a multivalued map from $E^+ \setminus \{0\}$ to $E$. However, the map $\hat{\Psi} : E^+ \setminus \{0\} \to \mathbb{R}$ given by

$$\hat{\Psi}(u) := \Phi(m(u)) = \max_{z \in \hat{E}(u)} \Phi(z)$$

is single-valued because $\Phi$ is constant on $\hat{E}(u) \cap \mathcal{M}$. In fact more is true:

**Proposition 2.6.** The map $\hat{\Psi}$ is locally Lipschitz continuous.

**Proof.** If $u_0 \in E^+ \setminus \{0\}$, then there exist a neighbourhood $U \subset E^+ \setminus \{0\}$ of $u_0$ and $R > 0$ such that $\Phi(w) \leq 0$ for all $u \in U$ and $w \in \hat{E}(u)$, $\|w\| \geq R$. For otherwise we can find sequences $(u_n)$, $(w_n)$ such that $u_n \to u_0$, $w_n \in \hat{E}(u_n)$, $\Phi(w_n) > 0$ and $\|w_n\| \to \infty$. But $u_0, u_1, u_2, \ldots$ is a compact set, hence according to [11] Lemma 2.5, $\Phi(w) \leq 0$ for some $R$ and all $w \in \hat{E}(u_j)$, $j = 0, 1, 2, \ldots$, $\|w\| \geq R$, which is a contradiction.

Let $U, R$ be as above and $s_1u_1 + v_1 \in m(u_1)$, $s_2u_2 + v_2 \in m(u_2)$, where $u_1, u_2 \in U$ and $v_1, v_2 \in E^0 \oplus E^-$. Then $\|m(u_1)\|, \|m(u_2)\| \leq R$. By the maximality property of $m(u)$ and the mean value theorem,

$$\hat{\Psi}(u_1) - \hat{\Psi}(u_2) = \Phi(s_1u_1 + v_1) - \Phi(s_2u_2 + v_2) \leq \Phi(s_1u_1 + v_1) - \Phi(s_1u_2 + v_1) \leq s_1 \sup_{t \in [0,1]} \|\Phi'(s_1(tu_1 + (1-t)u_2) + v_1)\| \|u_1 - u_2\| \leq C\|u_1 - u_2\|,$$
where the constant $C$ depends on $R$ but not on the particular choice of points in $m(u_1)$, $m(u_2)$. Similarly, $\tilde{\Psi}(u_2) - \tilde{\Psi}(u_1) \leq C\|u_1 - u_2\|$ and the conclusion follows. □

**Remark 2.7.** It has been shown in [11] that if $(F'_1)$ holds instead of $(F_3)$, then $\tilde{\Psi} \in C^1(E^+ \setminus \{0\}, \mathbb{R})$. An easy inspection of the arguments in [11] or [12] shows that if for each $u \in E^+ \setminus \{0\}$ there exists a unique positive maximum of $\Phi|_{E(u)}$, then $\tilde{\Psi}$ is still of class $C^1$. Hence in particular, if $f$ is as in Remark 2.5 then the conclusions of Theorems 1.1 and 1.2 hold with the same proofs as in [11].

However, under our assumptions we can in general only assert that $\tilde{\Psi}$ is locally Lipschitz continuous (because $u \mapsto m(u)$ may not be single-valued). Therefore, instead of the derivative of $\tilde{\Psi}$ we shall use Clarke’s subdifferential [4]. The study of minimax methods for differential equations whose associated functional is merely locally Lipschitz continuous has been initiated by Chang in [3]. We recall some notions and facts taken from [3, 4]. They may also be found conveniently collected in Section 7.1 of [2]. The *generalized directional derivative* of $\tilde{\Psi}$ at $u$ in the direction $v$ is defined by

$$\tilde{\Psi}^0(u; v) := \limsup_{h \to 0} \frac{\tilde{\Psi}(u + h + tv) - \tilde{\Psi}(u + h)}{t}.$$ 

The function $v \mapsto \tilde{\Psi}^0(u; v)$ is convex and its subdifferential $\partial \tilde{\Psi}(u)$ is called the *generalized gradient* (or Clarke’s subdifferential) of $\tilde{\Psi}$ at $u$, that is,

$$(2.4) \quad \partial \tilde{\Psi}(u) := \{w \in E^+ : \tilde{\Psi}^0(u; v) \geq \langle w, v \rangle \text{ for all } v \in E^+\}.$$ 

In [2] $E$ is a Banach space and the generalized gradient is in the dual space $E^*$. Since here we work in a Hilbert space, we may assume via duality that $\partial \tilde{\Psi}(u)$ is a subset of $E$ (or more precisely, of $E^+$). A point $u$ is called a *critical point* of $\tilde{\Psi}$ if $0 \in \partial \tilde{\Psi}(u)$, i.e. $\tilde{\Psi}^0(u; v) \geq 0$ for all $v \in E^+$, and a sequence $(u_n)$ is called a *Palais-Smale sequence* for $\tilde{\Psi}$ (PS-sequence for short) if $\tilde{\Psi}(u_n)$ is bounded and there exist $w_n \in \partial \tilde{\Psi}(u_n)$ such that $w_n \to 0$. The functional $\tilde{\Psi}$ satisfies the *PS-condition* if each PS-sequence has a convergent subsequence. Below we collect some notation which we shall need:

$$S^+ := \{u \in E^+ : \|u\| = 1\}, \quad T_u S^+ := \{v \in E^+ : \langle u, v \rangle = 0\}, \quad \Psi := \tilde{\Psi}|_{S^+},$$

$$\Psi^d := \{u \in S^+ : \Psi(u) \leq d\}, \quad \Psi_c := \{u \in S^+ : \Psi(u) \geq c\}, \quad \Psi_c := \Psi_c \cap \Psi^d,$$

$$K := \{u \in S^+ : 0 \in \partial \tilde{\Psi}(u)\}, \quad K_c := \Psi_c \cap K, \quad \partial \Psi(u) := \partial \tilde{\Psi}(u), \text{ where } u \in S^+.$$ 

Note that the symbol $\partial \Psi(u)$ stands for $\partial \tilde{\Psi}(u)$ when $u$ is restricted to $S^+$. This is in consistency with the notation $\Psi = \tilde{\Psi}|_{S^+}$. As we shall see in the proof of the next proposition, $\tilde{\Psi}^0(u; su) = 0$ for all $s \in \mathbb{R}$. Hence $\partial \Psi(u) \subset T_u S^+$.

**Proposition 2.8.** (i) $u \in S^+$ is a critical point of $\tilde{\Psi}$ if and only if $m(u)$ consists of critical points of $\Phi$. The corresponding critical values coincide.

(ii) $(u_n) \subset S^+$ is a PS-sequence for $\tilde{\Psi}$ if and only if there exist $w_n \in m(u_n)$ such that $(w_n)$ is a PS-sequence for $\Phi$. 
Proof. (i) Let $u \in S^+$. We shall show that $\tilde{\Psi}^\circ(u; v) \geq 0$ for all $v \in E^+$ if and only if $m(u)$ consists of critical points. Note first that there exists an orthogonal decomposition $E = E(u) \cap T_u S^+$, and by the maximizing property of $m(u)$, $\Phi'(w)v = 0$ for all $w \in m(u)$ and $v \in E(u)$. Let $s \in \mathbb{R}$ be fixed. Since $\tilde{\Psi}(u) = \tilde{\Psi}(su)$ for all $s > 0$ and $\tilde{\Psi}$ is locally Lipschitz continuous,

$$\|\tilde{\Psi}(u + h + t(su) - \tilde{\Psi}(u + h)) - \tilde{\Psi}((1 + ts)u + h) - \tilde{\Psi}((1 + ts)(u + h))\| \leq C t|s||h|$$

for $\|h\|$ and $t > 0$ small. Hence $\tilde{\Psi}^\circ(u; su) = 0$ for all $s \in \mathbb{R}$. So we only need to consider $v \in T_u S^+$.

Let $s_n u + z_n$, where $s_n > 0$ and $z_n \in E^0 \oplus E^-$, denote an (arbitrarily chosen) element of $m(u)$. Then, using the maximizing property of $m(u)$ and the mean value theorem,

$$\tilde{\Psi}(u + h + tv) - \tilde{\Psi}(u + h) = \Phi(s_n u + hv + t(z_n + tv)) - \Phi(s_n u + hv + z_n)$$

$$\leq \Phi(s_n u + hv + z_n + tv) - \Phi(s_n u + hv + z_n) + tv$$

for some $\theta \in (0, 1)$. Dividing by $t$ and letting $h \to 0$ and $t \downarrow 0$ via subsequences we obtain

$$\tilde{\Psi}^\circ(u; v) \leq s^* \Phi'(s^* u + z^*) v,$$

where $s_n := s_n u + h_n + t_n v \to s^* > 0$ and $z_n := z_n u + h_n + t_n v \to z^*$. This follows because $\mathcal{M}$ is bounded away from 0 and $\Phi|_\mathcal{M}$ coercive, hence $s_n$ and $z_n$ must be bounded. We claim that $s^* u + z^* \in \mathcal{M}$. Indeed, taking subsequences once more, writing $z_n = z_n^0 + z_n^\infty \in E^0 \oplus E^-$ and using Fatou’s lemma,

$$\tilde{\Psi}(u) = \lim_{n \to \infty} \tilde{\Psi}(u + h_n + t_n v) = \lim_{n \to \infty} \Phi(s_n(u + h_n + t_n v) + z_n)$$

$$= \lim_{n \to \infty} \left(\frac{1}{2}\|s_n(u + h_n + t_n v)\|^2 - \frac{1}{2}\|z_n\|^2 - \int_{\Omega} F(x, s_n(u + h_n + t_n v) + z_n) \, dx \right)$$

$$\leq \frac{1}{2}\|s^* u\|^2 - \frac{1}{2}\|z^*\|^2 - \int_{\Omega} F(x, s^* u + z^*) \, dx \leq \tilde{\Psi}(u).$$

This implies that $\|z_n\| \to \|z^*\|$ (recall dim $E^0 < \infty$), hence $z_n \to z^*$ and $s_n(u + h_n + t_n v) + z_n \to s^* u + z^*$. As $\mathcal{M}$ is closed, the claim follows. Since $\tilde{E}(u) \cap \mathcal{M}$ may be a line segment, it is not sure that $s^*$ and $z^*$ are the same for different $v$. However, if $s_1^*, s_2^*$ and $z_1^*, z_2^*$ correspond to $v_1$ and $v_2$, then by Proposition 2.4 $s_1^* u + z_1^* = \tau(s_2^* u + z_2^*)$ and $\Phi'(s_1^* u + z_1^*)v_2 = \tau\Phi'(s_2^* u + z_2^*)v_2$ for some $\tau > 0$. Taking this into account, we see from (2.5) that if $y \in \partial \tilde{\Psi}(u)$, then

$$\langle y, v \rangle \leq \tilde{\Psi}^\circ(u; v) \leq \tau(v) \Phi'(s^* u + z^*) v,$$

where $\tau$ is bounded and bounded away from 0 (by constants independent of $v$). It follows immediately that $u$ is a critical point of $\Psi$ if and only if $m(u)$ consists of critical points of $\Phi$.

(ii) The proof is very similar here. We take $y_n \in \partial \tilde{\Psi}(u_n)$ and $w_n \in m(u_n)$. Since $\Phi|_\mathcal{M}$ is coercive, boundedness of $\Phi(m(u_n))$ implies that $(m(u_n))$ is bounded. As in (2.6), we
see that
\begin{equation}
\langle y_n, v \rangle \leq \hat{\Psi}^n(u_n; v) \leq \tau_n(v) \Phi'(w_n) v,
\end{equation}
where \( \tau_n \) is bounded and bounded away from 0 because so is \( m(u_n) \). So the conclusion follows. \( \Box \)

Note that if \( (w_n) \subset (m(u_n)) \) is a PS-sequence for \( \Phi \), then so is any sequence \( (w'_n) \subset (m(u_n)) \).

Finally for this section we construct a pseudo-gradient vector field \( H : S^+ \setminus K \to TS^+ \) for \( \Psi \). For \( u \in S^+ \), let
\[
\partial^{-}\Psi(u) := \left\{ p \in \partial \Psi(u) : \|p\| = \min_{a \in \partial \Psi(u)} \|a\| \right\} \text{ and } \mu(u) := \inf_{a \in S^+} \{ \|\partial^{-}\Psi(a)\| + \|u - a\| \}.
\]
Since \( \partial \Psi(u) \) is closed and convex, \( p \) as above exists and is unique, cf. [2, Proposition 7.1.1(vi)]. Hence
\[
K = \{ u \in S^+ : \partial^{-}\Psi(u) = 0 \}.
\]
The map \( u \mapsto \|\partial^{-}\Psi(u)\| \) is lower semicontinuous [2 Proposition 7.1.1(vi)] but not continuous in general. The reason for introducing the function \( \mu \) is that it regularizes \( \|\partial^{-}\Psi(u)\| \).

The idea comes from [5] where a similar function has been defined.

**Lemma 2.9.** The function \( \mu \) is continuous and \( u \in K \) if and only if \( \mu(u) = 0 \).

**Proof.** Let \( u, v, a \in S^+ \). Then
\[
\mu(u) \leq \|\partial^{-}\Psi(a)\| + \|u - a\| \leq \|\partial^{-}\Psi(a)\| + \|v - a\| + \|u - v\|,
\]
and taking the infimum over \( a \) on the right-hand side we obtain \( \mu(u) \leq \mu(v) + \|u - v\| \).

Reversing the roles of \( u \) and \( v \) we see that \( |\mu(u) - \mu(v)| \leq \|u - v\| \). Hence \( \mu \) is (Lipschitz) continuous.

Since \( 0 \leq \mu(u) \leq \|\partial^{-}\Psi(u)\| \), it is clear that \( \mu(u) = 0 \) if \( u \in K \). Suppose \( \mu(u) = 0 \). Then there exist \( a_n \) such that \( \partial^{-}\Psi(a_n) \to 0 \) and \( a_n \to u \), so \( u \in K \) by the lower semicontinuity of \( u \mapsto \|\partial^{-}\Psi(u)\| \). \( \Box \)

**Proposition 2.10.** There exists a locally Lipschitz continuous function \( H : S^+ \setminus K \to TS^+ \) such that \( \|H(u)\| \leq 1 \) and \( \inf\{ \langle p, H(u) \rangle : p \in \partial \Psi(u) \} > \frac{1}{2} \mu(u) \) for all \( u \in S^+ \setminus K \). If \( \Phi \) is even, then \( H \) may be chosen to be odd.

**Proof.** Let \( u \in S^+ \setminus K \) and put \( v_u := \partial^{-}\Psi(u)/\|\partial^{-}\Psi(u)\| \). Consider the map
\[
\chi : w \mapsto \inf_{p \in \partial \Psi(u)} \langle p, v_u - \langle v_u, w \rangle w \rangle - \frac{1}{2} \mu(w), \quad w \in S^+ \setminus K
\]
(note that \( v_u - \langle v_u, w \rangle w \in T_w(S^+) \)). Since \( \partial \Psi(u) \) is convex, \( \inf_{p \in \partial \Psi(u)} \langle p, v_u \rangle \geq \|\partial^{-}\Psi(u)\| \geq \mu(u) \) and therefore \( \chi(u) \geq \frac{1}{2} \mu(u) > 0 \). Moreover, since
\[
\inf_{p \in \partial \Psi(w)} \langle p, v_u - \langle v_u, w \rangle w \rangle = - \sup_{p \in \partial \Psi(w)} \langle p, \langle v_u, w \rangle w - v_u \rangle = - \hat{\Psi}(w; \langle v_u, w \rangle w - v_u)
\]
(see Proposition 7.1.1(vii) and property (c) on p. 168 in [2]) and $\hat{\Psi}^0$ is upper semicontinuous in both arguments [2 Proposition 7.1.1(vii)], $\chi$ is lower semicontinuous. Hence there exists a neighbourhood $U_w$ of $w$ such that $\chi(w) > 0$ for all $w \in U$.

The remaining part of the proof is standard. Take a locally finite open refinement $(U_n)_{n \in \mathbb{N}}$ of the open cover $(U_w)_{w \in S^+ \setminus K}$ and a subordinated locally Lipschitz continuous partition of unity $\{\lambda_i\}_{i \in I}$. Define

$$H(u) := \sum_{i \in I} \lambda_i(u)v_{u_i}, \quad u \in S^+ \setminus K.$$ 

It is easy to see that $H$ satisfies the required conclusions.

If $\Phi$ is even, then so is $\Psi$ and we may replace $H(u)$ with $\frac{1}{2}(H(u) - H(-u))$. □

3. Proofs of Theorems 1.1 and 1.2

Since the arguments are very similar to those appearing in [11, 12], we shall describe them rather briefly and concentrate on pointing out the main differences.

We start with Theorem 1.1. First we want to show that there exists a minimizer for $\Psi$ on $S^+$. It follows from the results of Section 2 that

$$c := \inf_{w \in \mathcal{M}} \Phi(w) = \inf_{u \in S^+} \Psi(u) > 0.$$ 

According to Ekeland’s variational principle [6], there exists a sequence $(u_n) \subset S^+$ such that $\Psi(u_n) \to c$ and

$$\Psi(w) \geq \Psi(u_n) - \frac{1}{n} \|w - u_n\| \quad \text{for all} \ w \in S^+.$$ 

(3.1)

For a given $v \in T_{u_n}S^+$, let $z_n(t) := (u_n + tv)/\|u_n + tv\|$. Since $\|u_n + tv\| - 1 = O(t^2)$ as $t \to 0$ and $\hat{\Psi}(u_n + tv) = \Psi(z_n(t))$, it follows from (3.1) that

$$\hat{\Phi}^0(u_n; v) \geq \limsup_{t \downarrow 0} \frac{\hat{\Psi}(u_n + tv) - \hat{\Psi}(u_n)}{t} = \limsup_{t \downarrow 0} \frac{\Psi(z_n(t)) - \Psi(u_n)}{t} \geq -\frac{1}{n} \|v\|.$$ 

Since $m(u_n)$ is bounded by coercivity of $\Phi|_\mathcal{M}$, the second inequality in (2.7) implies that

$$-\frac{1}{n} \|v\| \leq \hat{\Phi}^0(u_n; v) \leq \tau_n(v)\Phi'(w_n)v,$$

where $w_n \in m(u_n) \subset \mathcal{M}$ and $\tau_n$ is bounded and bounded away from 0. So recalling $\Phi'(w_n)v = 0$ for all $v \in E(w_n)$, it follows that $(w_n)$ is a bounded PS-sequence for $\Phi$.

Now we may proceed exactly as in the proof of Theorem 1.1 in [11], pp. 3811–3812 (or in the proof of Theorem 40 in [12]). More precisely, one shows invoking Lions’ lemma [15, Lemma 1.21] in a rather standard way that there exists a sequence $(y_n) \subset \mathbb{R}^N$ such that

$$\int_{|x - y_n| < \varepsilon} w_n^2 \, dx \geq \varepsilon \quad \text{for} \ n \text{ large enough and some} \ \varepsilon > 0,$$

and since $\Phi$ and $\mathcal{M}$ are invariant by translations $u(\cdot) \mapsto u(\cdot - k), k \in \mathbb{Z}^N$, we may assume $(y_n)$ is bounded. So passing to a subsequence, $w_n \to w \neq 0$. This $w$ is a solution and an additional argument shows it is a ground state, see [11] or [12] for more details.
Suppose now $f$ is odd in $u$ and note that $\Psi$ is even and invariant by translations by elements of $\mathbb{Z}^N$. To prove that there exist infinitely many geometrically distinct solutions we assume the contrary. Since to each $[\sigma_w, \tau_w]w \subset \mathcal{M}$ there corresponds a unique point $u \in S^+, K$ consists of finitely many orbits $\mathcal{O}(u) := \{u(\cdot - k) : u \in K, k \in \mathbb{Z}^N\}$. We choose a subset $\mathcal{F} \subset K$ such that $\mathcal{F} = -\mathcal{F}$ and each orbit has a unique representative in $\mathcal{F}$. Now an easy inspection shows that Lemmas 2.11 and 2.13 in [11] hold, i.e. the mapping $t \to T$ for $\Psi$ defined on the set $K, v \neq F \subset$ a subset such that $\lim\inf_{t \to \infty} d\eta(t, w) \to 0$ as $n \to \infty$ or $\limsup_{n \to \infty} d\eta(t, w) = \rho(d) > 0$, where $\rho$ depends on $d$ but not on the particular choice of PS-sequences in $\Psi^d$.

The argument is exactly the same as in [11], taking into account that by Proposition 2.14 in [11] to $(v_n^1) \subset \Psi^d$ there correspond PS-sequences $(u_n^1)$ with $u_n^1 \in m(v_n^1), j = 1, 2$. Once $u_n^1$ have been chosen, one follows the lines of [11].

Let $H$ be the vector field constructed in Proposition 2.10 and consider the flow given by

$$\frac{d}{dt} \eta(t, w) = -H(\eta(t, w)), \quad \eta(0, w) = w,$$

defined on the set

$$\mathcal{G} := \{(t, w) : w \in S^+ \setminus K, \ T^-(w) < t < T^+(w)\},$$

where $(T^-(w), T^+(w))$ is the maximal existence time for the trajectory passing through $w$ at $t = 0$.

**Proposition 3.2** (cf. Lemma 2.15 in [11]). For each $w \in S^+ \setminus K$ the limit $\lim_{t \to T^+(w)} \eta(t, w)$ exists and is a critical point of $\Psi$.

**Proof.** We adapt the argument in [11].

If $T^+(w) < \infty$, then for $0 \leq s < t < T^+(w)$ we have

$$\|\eta(t, w) - \eta(s, w)\| \leq \int_s^t \|H(\eta(\tau, w))\| d\tau \leq t - s,$$

hence $\lim_{t \to T^+(w)} \eta(t, w)$ exists and must be a critical point (or the flow can be continued for $t > T^+(w)$).

Let $T^+(w) = \infty$. It suffices to show that for each $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that $\|\eta(t_\varepsilon, w) - \eta(t, u)\| < \varepsilon$ for all $t \geq t_\varepsilon$. Assuming the contrary, we find $\varepsilon \in (0, \rho(d)/2)$ and $t_n \to \infty$ such that $\|\eta(t_n, w) - \eta(t_{n+1}, w)\| = \varepsilon$ for all $n$. Choose the smallest $t_n^1 \in (t_n, t_{n+1})$ such that $\|\eta(t_n, w) - \eta(t_n^1, w)\| = \varepsilon/3$. Recall from Lemma 2.9 that $\mu$ is continuous and set

$$\mu(\eta(t_n^1, w)) := limsup_{n \to \infty} \mu(\eta(t_n, w)).$$
\[ \kappa_n := \min_{s \in [t_n, \tau_n]} \mu(\eta(s, w)) \]. Then, using Proposition 2.10 and Proposition 7.1.1(viii),

\[
\frac{\varepsilon}{3} = \| \eta(t_1^{n}, w) - \eta(t_n, w) \| \leq \int_{t_n}^{t_1^n} \| H(\eta(s, w)) \| \, ds \leq t_1^n - t_n
\]

\[
\leq \frac{2}{\kappa_n} \int_{t_n}^{t_1^n} \inf_{p \in \partial \Psi(\eta(s, w))} \langle p, H(\eta(s, w)) \rangle \, ds = -\frac{2}{\kappa_n} \int_{t_n}^{t_1^n} \sup_{p \in \partial \Psi(\eta(s, u))} \langle p, -H(\eta(s, w)) \rangle \, ds
\]

\[
\leq -\frac{2}{\kappa_n} \int_{t_n}^{t_1^n} \frac{d}{ds} \Psi(\eta(s, w)) \, ds = \frac{2}{\kappa_n} (\Psi(\eta(t_n, w)) - \Psi(\eta(t_1^n, w))).
\]

Since \( \Psi \) is bounded below, \( \Psi(\eta(t_n, w)) - \Psi(\eta(t_1^n, w)) \leq 0 \), hence \( \kappa_n \to 0 \) and we may find \( s_1^n \in [t_n, t_1^n] \) such that if \( z_1^n := \eta(s_1^n, w) \), then \( \mu(z_1^n) \to 0 \). By the definition of \( \mu \) there exist \( w_1^n \) such that \( w_1^n - z_1^n \to 0 \) and \( \partial^{-} \Psi(w_1^n) \to 0 \). So \( \limsup_{n \to \infty} \| w_1^n - \eta(t_n, w) \| = \varepsilon/3 \).

Similarly, there exists a largest \( t_2^n \in (t_n, t_{n+1}] \) with \( \| \eta(t_{n+1}, w) - \eta(t_2^n, w) \| = \varepsilon/3 \) and we find a \( w_2^n \) with \( \partial^{-} \Psi(w_2^n) \to 0 \) and \( \limsup_{n \to \infty} \| w_2^n - \eta(t_{n+1}, w) \| = \varepsilon/3 \). It follows that \( \varepsilon/3 \leq \limsup_{n \to \infty} \| w_1^n - w_2^n \| \leq 2\varepsilon < \rho(d) \), a contradiction to Proposition 3.1.

**Proposition 3.3** (cf. Lemma 2.16 in [11]). Let \( d \geq c \). Then for each \( \delta > 0 \) there exists \( \varepsilon > 0 \) such that \( \Psi_{d+\varepsilon} \cap K = K_d \) and \( \lim_{t \to T^+(w)} \Psi(\eta(t, w)) < d-\varepsilon \) for all \( w \in \Psi_{d+\varepsilon} \setminus U_\delta(K_d) \), where \( U_\delta(K_d) \) is the open \( \delta \)-neighbourhood of \( K_d \).

The proof requires changes which, in view of the arguments of Proposition 3.2, are rather obvious (in particular, \( \nabla \Psi \) in the definition of \( \tau \) in [11] should be replaced by \( \mu \)).

With all these prerequisites, existence of infinitely many solutions is obtained by repeating the arguments on pp. 3817–3818 in [11]. Let

\[ c_k := \inf \{ d \in \mathbb{R} : \gamma(\Psi^d) \geq k \}, \] where \( \gamma \) denotes Krasnosel’skii’s genus [10]. Using the flow \( \eta \) and Proposition 3.3 one shows \( K_{c_k} \neq 0 \) and \( c_k < c_{k+1} \) for all \( k \). This contradicts our assumption that there are finitely many geometrically distinct solutions.

Now we turn our attention to Theorem 1.2. Here there is no \( \mathbb{Z}^N \)-symmetry but instead there is a compact embedding \( H_0^1(\Omega) \hookrightarrow L^q(\Omega) \) for \( q \in [1, 2^*) \). Using this, one sees as in the proof of Theorem 3.1 in [11] or Theorem 37 in [12] that the ground state exists. The minimizing PS-sequence is extracted by using Ekeland’s variational principle in the same way as at the beginning of this section. To obtain infinitely many solutions for odd \( f \) one first shows as in [11] Theorem 3.2] (or in [12] Section 4.2) that \( \Psi \) satisfies the PS-condition. Now a standard minimax argument as in [11] Theorem 3.2] can be employed. Note that with the aid of the vector field \( H \) and suitable cutoff functions one can construct a deformation in the usual way as e.g. in [10] (see also [3]). We leave the details to the reader.
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