Inverse Scattering Problem
for Sturm–Liouville Operators

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Abstract. On the space $L^2(\mathbb{R})$ the Sturm-Liouville operator $L$ with certain behavior of the potential at infinity is considered. It is proved that $L$ is uniquely determined by its scattering data. The recovery of $L$ is reduced to the solving of a certain linear integral equation.

Consider the differential operation (expression) $l(y) = -y'' + qy$ on $\mathbb{R} = (-\infty, +\infty)$, where the coefficient (the potential) $q$ is a real-valued measurable function of $x \in \mathbb{R}$, such that

$$
\int_{-\infty}^{0} |q(x) - a^-| dx + \int_{0}^{\infty} |q(x) - a^+| dx < \infty
$$

with some constants $a^\pm \in \mathbb{R}$.

We say that $l$ is applicable to a function $y : \mathbb{R} \rightarrow \mathbb{C}$, if $y$ has absolutely continuous first derivative in any closed interval $[\alpha, \beta] \subset \mathbb{R}$. We set

$$
[y(x), z(x)] = i(y(x)\overline{z'(x)} - y'(x)\overline{z(x)})
$$

for any functions $y$ and $z$ in $\mathbb{R}$, to which $l$ is applicable. Besides, denote

$$
\mu_0 = -\infty, \quad \mu_1 = \min\{a^+, a^-\}, \quad \mu_2 = \max\{a^+, a^-\}, \quad \mu_3 = \infty,
$$

$$
\lambda_j^\pm(\mu) = (-1)^{j-1}\sqrt{\mu - a^\pm} \quad (\mu \in \mathbb{C}, \quad j = 1, 2)
$$

(where we take the principal value of the root). For $\mu \in \mathbb{R}$, by $r^\pm(\mu)$ we denote the half of the number of real roots of the equation $\lambda^2 + a^\pm = \mu$. Obviously, the functions $r^\pm(\cdot)$ are constant in each interval.
For each $k = 0, 1, 2$, by $r_k^±$ we denote the value of the function $r^±$ in the interval $(\mu_k, \mu_{k+1})$.

**Theorem 1.** For any $\mu \in \mathbb{C} \setminus \{a^+\}$ (or $\mu \in \mathbb{C} \setminus \{a^-\}$) the equation $l(y) = \mu y$ has linearly independent solutions $y_j^+(x, \mu), y_j^−(x, \mu)$ (correspondingly, $y_j^+(x, \mu), y_j^−(x, \mu)$) such that they and their derivatives with respect to $x$ have the following asymptotics as $x \to \infty$ (correspondingly, as $x \to -\infty$):

\[
y_j^±(x, \mu) = e^{i\bar{\lambda}_j^±(\mu)}[1 + o(1)],
\]
\[
y_j'±(x, \mu) = i\bar{\lambda}_j^±(\mu)e^{i\bar{\lambda}_j^±(\mu)}[1 + o(1)].
\]

Moreover:

1. for any $\mu \in \mathbb{R} \setminus \{a^+, a^-\}$ and $j, k = 1, 2$

\[
[y_j^±(x, \mu), y_k^±(x, \mu)] = \begin{cases} 2\bar{\lambda}_j^±(\mu) & \text{if } \lambda_j^±(\mu) = \bar{\lambda}_k^±(\mu) \\ 0 & \text{if } \lambda_j^±(\mu) \neq \bar{\lambda}_k^±(\mu). \end{cases}
\]

2. for each $k = 0, 1, 2$, the functions $y_j^±(x, \mu), y_j'm_{j-k}(x, \mu)$ ($1 \leq j \leq 1 + r_k^+$) and $y_j^−(x, \mu), y_j'm_{j-k}(x, \mu)$ ($2 - r_k^− \leq j \leq 2$) are continuous in the variables $x \in \mathbb{R}$, $\mu \in (\mu_k, \mu_{k+1})$.

3. the functions $y_1^+(x, \mu), y_1^′+(x, \mu), y_2^+(x, \mu), y_2^′+(x, \mu)$ are differentiable in $\mu \in (\mu_0, \mu_1)$, and the derivatives $\frac{\partial y_1^±(x, \mu)}{\partial \mu}, \frac{\partial y_2^±(x, \mu)}{\partial \mu}, \frac{\partial y_2^±(x, \mu)}{\partial \mu}$ are continuous functions of two variables $x \in \mathbb{R}$, $\mu \in (\mu_0, \mu_1)$.

4. if the potential $q$ satisfies the condition

\[
\int_{-\infty}^0 (1 - x)|q(x) - a^-| \, dx + \int_0^\infty (1 + x)|q(x) - a^+| \, dx < \infty,
\]

then for each $k = 1, 2$ the functions $y_j^±(x, \mu), y_j^′+(x, \mu)$ ($1 \leq j \leq 1 + r_k^+$) and $y_j^−(x, \mu), y_j'm_{j-k}(x, \mu)$ ($2 - r_k^− \leq j \leq 2$) are differentiable in $\mu \in (\mu_k, \mu_{k+1})$, and the derivatives $\frac{\partial y_j^±(x, \mu)}{\partial \mu}, \frac{\partial y_j^′+(x, \mu)}{\partial \mu}, \frac{\partial y_j^−(x, \mu)}{\partial \mu}, \frac{\partial y_j'm_{j-k}(x, \mu)}{\partial \mu}$ ($1 \leq j \leq 1 + r_k^+$) as well as $\frac{\partial y_j^+(x, \mu)}{\partial \mu}, \frac{\partial y_j^−(x, \mu)}{\partial \mu}$ ($2 - r_k^− \leq j \leq 2$) are continuous in the variables $x \in \mathbb{R}$, $\mu \in (\mu_k, \mu_{k+1})$.

**Proof:** We outline it for $y_j^+(x, \mu)$ ($j = 1, 2$). For $\text{Im} \lambda_j^+(\mu) \geq 0$ the solutions $y_j^+(x, \mu)$ are obtained from the integral equations

\[
y_j^+(x, \mu) = e^{i\lambda_j^+(\mu)x} + \frac{1}{\lambda_j^+(\mu)} \int_x^\infty \sin[\lambda_j^+(\mu)(t-x)] \left[q(t) - a^+\right] y_j^+(t, \mu) \, dt,
\]
and, if Im$\lambda_j^+(\mu) < 0$, from the integral equations

\[
y_j^+(x, \mu) = e^{i\lambda_j^+(\mu)x} - \frac{1}{2i\lambda_j^+(\mu)} \int_a^x e^{-i\lambda_j^+(\mu)(x-t)} [q(t) - a^+] y_j^+(t, \mu) \, dt - \frac{1}{2i\lambda_j^+(\mu)} \int_x^\infty e^{-i\lambda_j^+(\mu)(t-x)} [q(t) - a^+] y_j^+(t, \mu) \, dt,
\]

where the number $a$ is chosen to be large enough. These equations are solved using the method of successive approximations. The relations (2), (3) and the assertion 1 are obtained from these integral equations, and the remaining assertions are proved using the first equation.

Note that for differential operators of order $m \geq 2$ the existence of solutions $y_j^\pm(x, \mu) \; (j = 1, 2)$ satisfying the asymptotic relations (2), (3) and the assertion 1 of Theorem 1, is established in [7].

The operator $L$ acting in the space $L^2(\mathbb{R})$ is defined as follows (see [6], p. 60). The domain $D$, where the operator $L$ is defined, is the set of all functions $y \in L^2(\mathbb{R})$ for which the expression $l(y)$ is meaningful and $l(y) \in L^2(\mathbb{R})$, and for any $y \in D$ we define $Ly = l(y)$. Under the condition (1), the operator $L$ is self-adjoint (see [7]) and has a bounded point spectrum. Moreover, under the condition (1), all eigenvalues of $L$ (if they exist) are simple, lie in the interval $(-\infty, \mu_1]$ and as a limit point can have only $\mu_1$, see [1]. In the same work, it is shown that under the condition (4) the set of eigenvalues of the operator $L$ is finite and lies in the interval $(-\infty, \mu_1)$.

The inverse scattering problem for $L$ was considered by L. D. Faddeev (see [3] and [5], pp. 264 – 283) for $a^+ = a^- = 0$, and the case $a^+ = a^- \neq 0$ is easily the same. We shall assume that $a^+ \neq a^-$ and note that our approach is applicable to the case $a^+ = a^-$, too.

For $k = 1, 2$ and $\mu \in (\mu_k, \mu_{k+1})$ the equation $l(\varphi) = \mu \varphi$ has $k$ linearly independent bounded solutions $\varphi_j(x, \mu) \; (x \in \mathbb{R}, 1 \leq j \leq k)$, and the following asymptotic relations hold [8]:

\[
\varphi_j(x, \mu) = \frac{1}{\sqrt{2\pi}} \sum_{r=1}^{1+r_k^+} \sqrt{|\lambda_{j,r}^+(\mu)|} A_{j,r}^+(\mu) e^{i\lambda_{j,r}^+(\mu)(1 + o(1))} \quad (x \to \infty),
\]

\[
\varphi_j(x, \mu) = \frac{1}{\sqrt{2\pi}} \sum_{r=2-r_k^-}^{2} \sqrt{|\lambda_{j,r}^-(\mu)|} A_{j,r}^-(\mu) e^{i\lambda_{j,r}^-(\mu)(1 + o(1))} \quad (x \to -\infty).
\]

Assuming the notations

\[
B_{j,r}(\mu) = \begin{cases} 
A_{j,r}^+(\mu) & \text{for } 1 \leq r \leq r_k^+ \\
A_{j,r-r_k^+,r_k^-}^-(\mu) & \text{for } r_k^+ < r \leq k, 
\end{cases}
\]

\[
C_{j,r}(\mu) = \begin{cases} 
A_{j,r}^-(\mu) & \text{for } 1 \leq r \leq r_k^- \\
A_{j,r+r_k^-,r_k^-}^+(\mu) & \text{for } r_k^- < r \leq k, 
\end{cases}
\]
the matrices

\[ B(\mu) = B_j(\mu)_{j,v=1}^k, \quad C(\mu) = C_j(\mu)_{j,v=1}^k \]  

(6)

are nondegenerate and satisfy the relation

\[ B(\mu)B^*(\mu) = C(\mu)C^*(\mu). \]  

(7)

An arbitrary nondegenerate matrix can be taken as one of these matrices, and this will uniquely

determine the other matrix as well as the solutions \( \varphi_j(x,\mu) \). From (7), it follows that if one of the

matrices (6) is unitary, then so is the other one, too.

**Lemma 1.** For the matrices (6) the following statements are true:

1. If one of the matrices (6) is measurable (continuous) on the intervals \((\mu_1, \mu_2)\) and \((\mu_2, \mu_3)\), then

   the other is measurable (continuous) on the same intervals.

2. If the potential \( q \) satisfies the condition (4) and one of the matrices (6) is differentiable (continuously differentiable) on the intervals \((\mu_1, \mu_2)\) and \((\mu_2, \mu_3)\), then the other one is also differentiable (continuously differentiable) on the same intervals.

**Proof:** is based on Theorem 1 and explicit formulas representing the entries of one of the matrices (6)

by means of the entries of the other.

**Remark 1.** For \( r_k^+ = 1 \), the coefficients \( A^+_j(\mu) \) in (5) are entries of one of the matrices (6). This is

not true for \( r_k^+ = 0 \). However, one can prove that in the latter case measurability, continuity and,

under (4), differentiability and continuous differentiability of one of the matrices (6) extends to these

coefficients, too. The same is true for the coefficients \( A^-_j(\mu) \).

Henceforth we assume that \( B(\mu) \) and \( C(\mu) \) are unitary matrices and their elements are measurable

functions (in particular, one of them could be the identity matrix). Normalizing in this way, we call

the system of solutions \( \varphi_j(x,\mu) \) \( \mu \in (\mu_k, \mu_{k+1}) \), \( 1 \leq j \leq k \) the normalized system of generalized

eigenfunctions of the operator \( L \), corresponding to the value \( \mu \).

Consider the matrices

\[ A^+_j(\mu) = (A^+_j(\mu))_{1 \leq j \leq k, 1 \leq v \leq r_k^+}, \quad A^-_j(\mu) = (A^-_j(\mu))_{1 \leq j \leq k, 2-r_k^+ \leq v \leq 2}. \]

For any \( x \in \mathbb{R} \) and any \( \mu \in (\mu_k, \mu_{k+1}) \) \( (k = 1, 2) \), \( \varphi(x,\mu) \) will denote the vector-column, consisting of the solutions \( \varphi_j(x,\mu) \) \( (1 \leq j \leq k) \).

**Lemma 2.** Let \( \varphi_j(x,\mu) \) \( \mu \in (\mu_k, \mu_{k+1}) \), \( 1 \leq j \leq k \) be a normalized system of generalized eigenfunctions of the operator \( L \). Then for any measurable unitary matrix \( U(\mu) = (U_{ij}(\mu))_{1 \leq j \leq k} \), the functions

\( \tilde{\varphi}_j(x,\mu) \) \( \mu \in (\mu_k, \mu_{k+1}) \), \( 1 \leq j \leq k \) determined by the relation

\[ \tilde{\varphi}(x,\mu) = U(\mu)\varphi(x,\mu) \quad (\tilde{\varphi}(x,\mu) = (\tilde{\varphi}_j(x,\mu))_{1 \leq j \leq k}), \]

(8)
form a normalized system of generalized eigenfunctions of $L$.

Conversely, for any normalized systems $\varphi_j(x, \mu)$ and $\tilde{\varphi}_j(x, \mu)$ ($\mu \in (\mu_k, \mu_{k+1})$, $1 \leq j \leq k$) of generalized eigenfunctions of $L$, there exists a unique measurable, unitary matrix $U(\mu) = (U_{ij}(\mu))_{k,j=1}^k$ satisfying (8). Moreover, the following equalities hold:

$$\tilde{A}^\pm(\mu) = U(\mu)A^\pm(\mu), \quad \tilde{B}(\mu) = U(\mu)B(\mu), \quad \tilde{C}(\mu) = U(\mu)C(\mu).$$

**Proof:** is simple and directly follows from the corresponding definitions.

If the operator $L$ has eigenvalues, we consider an orthonormal system of eigenfunctions $\{\psi_j, j = 1, 2, \ldots\}$ of $L$, complete in the closure of the span of all eigenfunctions of $L$. If the point spectrum $T$ of the operator $L$ is finite (countable), then the system of its eigenfunctions $\psi_j$ is finite (countable). Since the eigenvalues of $L$ are simple, to each eigenvalue of $L$ corresponds exactly one eigenfunction from $\{\psi_j : j = 1, 2, \ldots\}$.

According to the results of [8], any function $f \in L^2(\mathbb{R})$ has Fourier expansion

$$f(x) = \sum_j \psi_j(x) \int_{-\infty}^{\infty} f(t) \overline{\psi_j(t)} \, dt + \sum_{k=1}^2 \sum_{j=1}^{\mu_{k+1}} \int_{\mu_k}^{\mu_{k+1}} F_j(\mu) \varphi_j(x, \mu) \, d\mu \quad (x \in \mathbb{R}),$$

and for any $f, g \in L^2(\mathbb{R})$ the generalized Parseval equality is true:

$$\int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx = \sum_j \int_{-\infty}^{\infty} f(t) \overline{\psi_j(t)} \, dt \int_{-\infty}^{\infty} g(t) \overline{\psi_j(t)} \, dt + \sum_{k=1}^2 \sum_{j=1}^{\mu_{k+1}} \int_{\mu_k}^{\mu_{k+1}} F_j(\mu) \overline{G_j(\mu)} \, d\mu,$$

where

$$F_j(\mu) = \int_{-\infty}^{\infty} f(t) \overline{\varphi_j(t, \mu)} \, dt, \quad G_j(\mu) = \int_{-\infty}^{\infty} g(t) \overline{\varphi_j(t, \mu)} \, dt,$$

and the last integral in (9) converges in the norm of $L^2(\mathbb{R})$, and for $\mu \in (\mu_k, \mu_{k+1})$ the integrals in (11) converge in the norm of the space $L^2(\mu_k, \mu_{k+1})$ (if $L$ has no eigenvalues, then the first sums in (9) and (10) vanish).

Now we introduce the scattering data for the operator $L$. To this end, for $\mu \in (\mu_k, \mu_{k+1})$ and $1 \leq j, v \leq 1 + r_k^+$ ($k = 1, 2$) we set

$$S_{jv}^+(\mu) = \sum_{i=1}^{k} \sqrt{\lambda_i^+(\mu)} \sqrt{\lambda_i^+(\mu)|A_{i+v}^+(\mu)A_{i+j}^+(\mu)},$$

and consider the square matrix $S^+(\mu) = (S_{jv}^+(\mu))_{j,v=1}^{1+r_k^+(\mu)}$ of order $1 + r_k^+(\mu)$. By Lemma 2, this matrix is independent of the choice of normalized system of generalized eigenfunctions. Hence Lemma 1 and Remark 1 imply that under condition (1) the matrix–function $S^+$ is continuous and under condition (4) it is continuously differentiable in the intervals $(\mu_1, \mu_2)$ and $(\mu_2, \mu_3)$. 

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Let \( T \) be the point spectrum of \( L \), \( \mu \in T \) and \( \psi(\cdot, \mu) \) be a corresponding normalized eigenfunction. Then the following asymptotic relation holds:

\[
\psi(x, \mu) = c^+(\mu)e^{ix\lambda^+(\mu)}[1 + o(1)] \quad \text{as} \quad x \to \infty,
\]

where \( c^+(\mu) \) is a nonzero complex number. One can easily see that the numbers

\[
N^+(\mu) = |c^+(\mu)|^2 \quad (\mu \in T)
\]

do not depend on the choice of the normalized eigenfunctions \( \psi \).

Now consider the data

\[
\{T, N^+(\mu) \ (\mu \in T), S^+(\mu) \ (\mu \in (\mu_1, \mu_2) \cup (\mu_2, \mu_3))\}
\]

called the right scattering data of the operator \( L \).

The inverse scattering problem for the operator \( L \) consists in recovery of \( L \) from the knowledge of the right scattering data (15).

It is known (see [5], pp. 162–166) that if for each \( a \in \mathbb{R} \) the function \( q \) satisfies the condition

\[
\int_a^\infty (1 + |x|)|q(x) - a^+| \, dx < \infty
\]

with some number \( a^+ \in \mathbb{R} \), then for any \( \lambda \in \mathbb{C}, \ \text{Im} \lambda \geq 0 \) the equation \( l(y) = (\lambda^2 + a^+)y \) has a solution \( y^+(x, \lambda) \) for which the representations

\[
y^+(x, \lambda) = e^{i\lambda x} + \int_x^\infty e^{i\lambda t}K^+(x, t) \, dt \quad (\infty < x < \infty),
\]

\[
e^{i\lambda x} = y^+(x, \lambda) + \int_x^\infty y^+(t, \lambda)H^+(x, t) \, dt \quad (\infty < x < \infty)
\]

hold. Here the kernels \( K^+(x, t) \) and \( H^+(x, t) \ (\infty < x \leq t < \infty) \) do not depend on \( \lambda \), are real-valued continuous functions of two variables \( x, t \) and satisfy the relations

\[
H^+(x, t) + K^+(x, t) + \int_x^t H^+(x, \xi)K^+(\xi, t) \, d\xi = 0,
\]

\[
K^+(x, t) + H^+(x, t) + \int_x^t K^+(x, \xi)H^+(\xi, t) \, d\xi = 0.
\]
The estimates
\[
|K^+(x,t)| \leq \frac{1}{2} h^+ \left( \frac{x+t}{2} \right) \exp \left[ h_1^+ (x) - h_1^+ \left( \frac{x+t}{2} \right) \right],
\]
\[
|H^+(x,t)| \leq \frac{1}{2} h^+ \left( \frac{x+t}{2} \right) \exp \left\{ h_1^+ (x) - h_1^+ \left( \frac{x+t}{2} \right) + \exp \left[ h_1^+ (x) - h_1^+ \left( \frac{x+t}{2} \right) \right] - 1 \right\}
\]
hold, where \(-\infty < x \leq t < \infty\) and
\[
h^+(x) = \int_x^\infty |q(t) - a^+| \, dt, \quad h_1^+(x) = \int_x^\infty h^+(t) \, dt.
\]

For \(a \in \mathbb{R}\) and \(1 \leq p \leq \infty\) the operators \(K_a^+, H_a^+\), defined by
\[
(K_a^+ f)(x) = \int_x^\infty K^+(x,t) f(t) \, dt \quad (f \in L^p(a,\infty), \, x > a),
\]
\[
(H_a^+ f)(x) = \int_x^\infty H^+(x,t) f(t) \, dt \quad (f \in L^p(a,\infty), \, x > a),
\]
are bounded in \(L^p(a,\infty)\), \(I + K_a^+\) is invertible and
\[
(I + K_a^+)^{-1} = I + H_a^+,
\]

\[
K^+(x,x) = \frac{1}{2} \int_x^\infty (q(t) - a^+) \, dt \quad (x \in \mathbb{R}).
\]

From now on, the condition (4) is assumed to be satisfied.

In analogy with [4], [2], we consider the function
\[
\tilde{F}^+(x,t) = \sum_{\mu \in T} \frac{N^+(\mu)}{|\lambda_{1+}^+(\mu)|^2} \left( e^{i\lambda_1^+(\mu)} - 1 \right) \left( e^{2i\lambda_1^+(\mu)} - 1 \right) +
\]
\[
+ \frac{1}{2\pi} \int_{\mu_1}^{1+r^+(\mu)} \sum_{\nu,j=1} \frac{S_{\nu j}^+(\mu)}{\lambda_{\nu j}^+(\mu) \lambda_{\nu j}^+(\mu)} \left( e^{i\lambda_{\nu j}^+(\mu)} - 1 \right) \left( e^{2i\lambda_{\nu j}^+(\mu)} - 1 \right) \, d\mu - \omega(x,t) \quad (x, t \in \mathbb{R}),
\]
where
\[
\omega(x,t) = \begin{cases} 
\min\{|x|, |t|\} & \text{for } xt \geq 0, \\
0 & \text{for } xt < 0.
\end{cases}
\]

According to what was said above, the point spectrum \(T\) of the operator \(L\) is a finite set. Therefore the first sum in (23) contains a finite number of summands. Since \(T \subset (-\infty, \mu_1)\), for \(\mu \in T\) the numbers \(\lambda_{1+}^+(\mu)\) lie in the upper part of the imaginary axis. Hence the mentioned sum is a real number. We will see below that the integral in (23) is convergent (in the usual sense).
Theorem 2. The derivative

\[ F^+(x,t) = \frac{\partial^2 \tilde{F}^+(x,t)}{\partial x \partial t} \quad (x,t \in \mathbb{R}) \]

exists and is continuous, real-valued and symmetric:

\[ F^+(x,t) = F^+(t,x) \quad (x,t \in \mathbb{R}). \] (24)

Moreover, the following equalities hold:

\[ F^+(x,t) = H^+(x,t) + \int_t^\infty H^+(x,\xi)H^+(t,\xi) \, d\xi \quad (-\infty < x \leq t < \infty), \] (25)

\[ F^+(x,t) = H^+(t,x) + \int_x^\infty H^+(x,\xi)H^+(t,\xi) \, d\xi \quad (-\infty < t < x < \infty). \] (26)

Proof: For \( \mu \in (\mu_k, \mu_{k+1}) \), \( 1 \leq l \leq k \) \((k = 1, 2)\) we denote

\[ F_N^+(x,t,\mu) = N^+(\mu)e^{ix\lambda_i^+(\mu)}e^{it\lambda_i^+(\mu)} \quad (\mu \in T), \] (27)

\[ v_l(x,\mu) = \frac{1}{\sqrt{2\pi}} \sum_{\nu=1}^{1+r^+(\mu)} \sqrt{|\lambda_{\nu}^+|^2(\mu)|A_{j\nu}^{+\dagger}(\mu)e^{ix\lambda_i^+(\mu)}}. \] (28)

The following equalities hold

\[ \frac{N^+(\mu)}{|\lambda_i^+(\mu)|^2} \left( e^{ix\lambda_i^+(\mu)} - 1 \right) \left( e^{it\lambda_i^+(\mu)} - 1 \right) = \int_0^x \int_0^t F_N^+(\xi,\eta,\mu) \, d\eta \, d\xi \quad (\mu \in T), \]

\[ \frac{1}{2\pi} \sum_{\nu,j=1}^{1+r^+(\mu)} \frac{S_{j\nu}^{+\dagger}(\mu)}{\lambda_{\nu}^+(\mu)\lambda_j^+(\mu)} \left( e^{ix\lambda_i^+(\mu)} - 1 \right) \left( e^{it\lambda_j^+(\mu)} - 1 \right) = \]

\[ = \sum_{l=1}^k \int_0^x v_l(\xi,\mu) \, d\xi \int_0^t v_l(\eta,\mu) \, d\eta \quad (\mu \in (\mu_k, \mu_{k+1}), \ k = 1, 2) \]

(to obtain the first equality, it is necessary to take into account that \(|\lambda_i^+(\mu)|^2 = -[\lambda_i^+(\mu)]^2\) since \(i\lambda_i^+(\mu)\) are real numbers). Consequently, formula (23) takes the form

\[ \tilde{F}^+(x,t) = \sum_{\mu \in T} \int_0^x \int_0^t F_N^+(\xi,\eta,\mu) \, d\eta \, d\xi \]

\[ + \sum_{k=1}^{2} \sum_{l=1}^{k} \int_0^{\mu_{k+1}} \left[ \int_0^x v_l(\xi,\mu) \, d\xi \int_0^t v_l(\eta,\mu) \, d\eta \right] \, d\mu - \omega(x,t). \] (29)
(13) implies the equality

$$\psi(x, \mu) = c^+(\mu)y^+(x, \lambda_1^+(\mu)).$$  \hfill (30)

Since the function $y^+(x, \lambda_1^+(\mu))$ is real-valued, we have

$$\overline{\psi(x, \mu)} = c^+(\mu)y^+(x, \lambda_1^+(\mu)).$$  \hfill (31)

Further, in view of (27) and (16), we have

$$F_N^+(x, t, \mu) = N^+(\mu) \left[ y^+(x, \lambda_1^+(\mu)) + \int_x^\infty y^+(\xi, \lambda_1^+(\mu)) H^+(x, \xi) \, d\xi \right] \times$$

$$\times \left[ y^+(t, \lambda_1^+(\mu)) + \int_t^\infty y^+(\eta, \lambda_1^+(\mu)) H^+(t, \eta) \, d\eta \right].$$

From this equality and (14), (30), (31) we obtain

$$F_N^+(x, t, \mu) = \left[ \psi(x, \mu) + \int_x^\infty \psi(\xi, \mu) H^+(x, \xi) \, d\xi \right] \left[ \overline{\psi(t, \mu)} + \int_t^\infty \overline{\psi(\eta, \mu)} H^+(t, \eta) \, d\eta \right].$$  \hfill (32)

(5) implies that

$$\phi_l(x, \mu) = \frac{1}{\sqrt{2\pi}} \sum_{\nu=1}^{1+x^+(\mu)} \sqrt{|\lambda_\nu'\mu(\mu)|} A_{1\nu}^+(\mu) y^+(x, \lambda_\nu^+(\mu)),$$

hence, using (28) and (16), we get

$$v_l(x, \mu) = \phi_l(x, \mu) + \int_x^\infty \phi_l(\xi, \mu) H^+(x, \xi) \, d\xi \quad (1 \leq l \leq k, \quad \mu \in (\mu_k, \mu_{k+1})).$$  \hfill (33)

Consider the function $f_x(\cdot)$ ($x \in \mathbb{R}$) defined by the formula

$$f_x(\xi) = \begin{cases} 0 & \text{if } \xi < 0 \\ 1 + \int_0^\xi H^+(\eta, \xi) \, d\eta & \text{if } 0 \leq \xi < x \\ \int_0^x H^+(\eta, \xi) \, d\eta & \text{if } \xi \geq x \end{cases}$$

for $x \geq 0$ and by the formula
for $x < 0$. The integration of (32) and (33) yields

\[
\int_0^x \int_0^t F_N^+(\xi, \eta, \mu) \, d\eta \, d\xi = \int_{-\infty}^\infty \psi(\xi, \mu) f_x(\xi) \int_{-\infty}^{\infty} \overline{\psi(\xi, \mu)} f_t(\xi) \, d\xi,
\]

\[
\int_0^x \nu_l(\xi, \mu) \, d\xi = \int_{-\infty}^\infty \varphi_l(\xi, \mu) f_x(\xi) \, d\xi.
\]

In view of these equalities, (29) takes the form

\[
\tilde{F}^+(x, t) = \sum_{\mu \in T} \int_{-\infty}^\infty \psi(\xi, \mu) f_x(\xi) \int_{-\infty}^\infty \overline{\psi(\xi, \mu)} f_t(\xi) \, d\xi + \sum_{k=1}^{\mu_k+1} \left[ \int_{-\infty}^\infty \varphi_l(\xi, \mu) f_x(\xi) \int_{-\infty}^\infty \overline{\varphi_l(\xi, \mu)} f_t(\xi) \, d\xi \right] \, d\mu - \omega(x, t).
\]

By the estimate (20), $f_x(\cdot) \in L^2(\mathbb{R})$ for any $x \in \mathbb{R}$. Therefore, by (34) and the generalized Parseval equality (10), we get

\[
\tilde{F}^+(x, t) = \int_{-\infty}^\infty f_x(\xi) f_t(\xi) \, d\xi - \omega(x, t).
\]

This equality implies the existence of the continuous mixed derivative $F^+(x, t) = \frac{\partial^2 \tilde{F}^+(x, t)}{\partial x \partial t}$ on $\mathbb{R}^2$ and the representations (25), (26). These representations show that $F^+$ is real-valued and symmetric. The proof is complete.

**Theorem 3.** The function $F^+(x, t)$ and the kernel $K^+(x, t)$ satisfy

\[
F^+(x, t) + K^+(x, t) + \int_{x}^{\infty} K^+(x, \xi) F^+(\xi, t) \, d\xi = 0 \quad (-\infty < x \leq t < \infty), \tag{35}
\]

\[
F^+(x, t) + \int_{x}^{\infty} K^+(x, \xi) F^+(\xi, t) \, d\xi = H^+(t, x) \quad (-\infty < t \leq x < \infty). \tag{36}
\]
Proof: For a fixed \( a \in \mathbb{R} \) in \( L^2(a, \infty) \) consider the operators \( K^+_a, H^+_a \) and the operator \( F^+_a \) defined by

\[
(F^+_a f)(x) = \int_a^\infty F^+(x, t) f(t) \, dt \quad (f \in L^2(a, \infty), \, x > a).
\]

(25), (26) and (20) show that there exists a decreasing summable function \( \sigma \) on \([a, \infty)\) such that

\[
|F^+(x, t)| \leq \sigma \left( \frac{x + t}{2} \right) \quad (a \leq x, \, t < \infty).
\]

(37)

Hence the operator \( F^+_a \) is bounded. Moreover, by (25) and (26),

\[
I + F^+_a = (I + H^+_a)(I + H^+_a^*),
\]

(38)

and by (38) and (21),

\[
K^+_a + F^+_a + K^+_a F^+_a = H^+_a^*.
\]

For the corresponding kernels we get

\[
F^+(x, t) + K^+(x, t) + \int_x^\infty K^+(x, \xi) F^+(\xi, t) \, d\xi = 0 \quad (a < x \leq t < \infty),
\]

(39)

\[
F^+(x, t) + \int_x^\infty K^+(x, \xi) F^+(\xi, t) \, d\xi = H^+(t, x) \quad (a < t \leq x < \infty).
\]

(40)

Since \( a \) was arbitrary, the equalities (35) and (36) follow from (39) and (40). The proof is complete.

**Lemma 3.** For any \( p \in [1, \infty] \) and \( x \in \mathbb{R} \) the function \( K^+(x, \cdot) \) is the unique solution of the equation (35) in the space \( L^p(x, \infty) \).

**Proof:** Fix \( p \in [1, \infty] \) and \( x \in \mathbb{R} \). Consider the following integral operators on \( L^p(0, \infty) \):

\[
(K_x f)(\xi) = \int_{\xi}^\infty K^+(x + \xi, x + \eta) f(\eta) \, d\eta \quad (\xi > 0),
\]

\[
(K^*_x f)(\xi) = \int_0^\xi K^+(x + \eta, x + \xi) f(\eta) \, d\eta \quad (\xi > 0),
\]

\[
(H_x f)(\xi) = \int_{\xi}^{\infty} H^+(x + \xi, x + \eta) f(\eta) \, d\eta \quad (\xi > 0),
\]

\[
((K_x f)(\xi)) = \int_{\xi}^\infty K^+(x + \xi, x + \eta) f(\eta) \, d\eta \quad (\xi > 0),
\]

\[
((K^*_x f)(\xi)) = \int_0^\xi K^+(x + \eta, x + \xi) f(\eta) \, d\eta \quad (\xi > 0),
\]

\[
((H_x f)(\xi)) = \int_{\xi}^{\infty} H^+(x + \xi, x + \eta) f(\eta) \, d\eta \quad (\xi > 0),
\]
\[(H_x f)(\xi) = \int_{0}^{\xi} H^+(x + \eta, x + \xi) f(\eta) \, d\eta \quad (\xi > 0),\]

\[(G_x f)(\xi) = \int_{0}^{\infty} F^+(x + \xi, x + \eta) f(\eta) \, d\eta \quad (\xi > 0).\]

By the estimates (19), (20) and (37) for \(K^+(x, t), H^+(x, t)\) and \(F^+(x, t)\), the operators \(K_x, K_x^*, H_x, H_x^*\) and \(G_x\) are bounded. It is obvious that \(K_x^*\) and \(H_x^*\) are the conjugate operators of \(K_x\) and \(H_x\) in the case \(p = 2\).

The equalities (17) and (18) imply

\[ (I + H_x)(I + K_x) = I, \quad (I + K_x)(I + H_x) = I, \]

i.e., the operator \(I + H_x\) is invertible and

\[ (I + H_x)^{-1} = I + K_x. \quad \text{(41)} \]

Similarly, \(I + H_x^*\) is an invertible operator and

\[ (I + H_x^*)^{-1} = I + K_x^*. \quad \text{(42)} \]

It follows from (25) and (26) that

\[ I + G_x = (I + H_x)(I + H_x^*). \]

Using this and (41), (42), we conclude that \(I + G_x\) is an invertible operator.

To show that equation (35) can have no more that one solution in \(L^p(x, \infty)\), suppose that

\[ g_1(t) + F^+(x,t) + \int_{x}^{\infty} g_1(\xi) F^+(\xi, t) \, d\xi = 0 \quad (x < t < \infty), \]

\[ g_2(t) + F^+(x,t) + \int_{x}^{\infty} g_2(\xi) F^+(\xi, t) \, d\xi = 0 \quad (x < t < \infty), \]

for some functions \(g_j \in L^p(x, \infty)\) \((j = 1, 2)\). Subtract the second equation from the first one to obtain

\[ g_1(t) - g_2(t) + \int_{x}^{\infty} [g_1(\xi) - g_2(\xi)] F^+(\xi, t) \, d\xi = 0 \quad (x < t < \infty). \quad \text{(43)} \]
Put \( g(u) = g_1(u + x) - g_2(u + x) \) \((u > 0)\). Using (24), we bring (43) to the form
\[
g(t - x) + \int_x^\infty F^+(t, \xi) g(\xi - x) d\xi = 0 \quad (x < t < \infty).
\]
Substituting \( u = t - x \) and \( v = \xi - x \), we get
\[
g(u) + \int_0^\infty F^+(u + x, v + x) g(v) dv = 0 \quad (0 < u < \infty)
\]
which is the same as
\[
(I + G_x) g = 0.
\]
By the invertability of the operator \( I + G_x \), this means that \( g(u) = 0 \) \((0 < u < \infty)\), i.e.,
\[
g_1(u + x) - g_2(u + x) = 0 \quad (0 < u < \infty).
\]
Taking \( u = t - x \) \((x < t < \infty)\), we complete the proof:
\[
g_1(t) = g_2(t) \quad (x < t < \infty).
\]
The above lemma shows that the kernel \( K^+(x,t) \) can be uniquely recovered by the right scattering data. This implies that the potential \( q \) can also be uniquely recovered by the right scattering data. Indeed, by the unitarity of the matrices (6) and (12), for \( \mu > \mu_1 \)
\[
S_{11}^+(\mu) = \lambda_{t_1}^+(\mu) = \frac{1}{2\sqrt{\mu - a^+}}.
\]
Therefore the constant \( a^+ \) is determined by the right scattering data:
\[
a^+ = \lim_{\mu \to \infty} \left\{ \mu - \frac{1}{4|S_{11}^+(\mu)|^2} \right\}.
\]
Moreover, (22) shows that
\[
q(x) = a^+ - 2 \frac{d}{dx} K^+(x,x),
\]
and hence the potential \( q \) is uniquely recovered by the right scattering data. Practically, the recovery of \( q \) reduces to solution of the linear integral equation (35).

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