CONTINUITY OF THE MIXING OPERATOR

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Abstract. Mixed distributions are considered as a results of application of a linear operator, which maps mixing measures to mixed measures. The main result is a proof of continuity of this mixing operator. Corollaries for parametric families of distributions (usually considered in literature) are also discussed.

1. Introduction

Mixed distributions appear for the first time near the end of 19th century, in the works of [Newcomb (1886)] and [Pearson (1894)]. The first problem solved in this context was finding a mixture of two normal distributions. In the middle of the 20th century, interest in mixed distributions was renewed in works of [Robbins (1948)] and [Teicher (1960, 1961, 1963)], where basic definitions were made and fundamental properties proved.

These and subsequent works were concentrated on mixtures of parametric families of distributions. In the parametric approach, a mixing distribution is defined on the parameter space. Alternatively, the mixing measure may be considered as defined on the space of measures being mixed. This allows us to consider operation of mixing as an operator $\text{Mix}: \mathcal{M}(\Omega) \rightarrow \mathcal{M}(\Omega)$, where $\mathcal{M}(\Omega)$ is the space of measures being mixed (and where mixtures take values), and $\mathcal{M}(\mathcal{M}(\Omega))$ is the space of mixing measures. This operator is obviously linear; the proof of its continuity is the main topic of the present article.

To proceed in this way, one needs to specify a class of basic spaces $\Omega$, a class of $\sigma$-algebras, and a class of measures. We propose to consider Radon measures on Borel $\sigma$-algebras on polish spaces. This choice, on the one hand, covers most of practically important cases, and, on the other hand, provides a natural way to obtain all required structures: the topology on basic space defines a narrow topology on the first-level space of measures, which, in turn, provides a Borel $\sigma$-algebra for the second-level measures and defines a topology on the second-level space.

We introduce required constructions and list related facts in section 2. A proof of continuity of operator $\text{Mix}$ is given in section 3. Section 4 discusses corollaries for parametric families.

2. Spaces of measures

For a topological space $\Omega$, $C(\Omega)$ denotes the space of continuous real-valued functions on $\Omega$, and $C^*(\Omega)$ denotes its dual, a space of continuous real-valued linear functionals on $C(\Omega)$.

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If \( \Omega \) is a Hausdorff topological space, the space of Radon charges (signed measures) \( \mathcal{M}(\Omega) \) may be imbedded into \( C^*(\Omega) \). We are interested in the narrow topology on \( \mathcal{M}(\Omega) \), i.e. the weakest topology in which functionals \( \phi_f : \mu \mapsto \int f(\omega) \mu(d\omega) \) are continuous for all \( f \in C(\Omega) \). (This topology is usually called “weak” in probability theory; we use term “narrow” to distinguish it from the weak topology on \( C^*(\Omega) \).)

When \( \Omega \) is a compact metric space, the following strong facts take place:

1. \( C(\Omega) \) is separable metric space in norm topology (Dunford and Schwartz, 1958, IV.13.16).
2. \( \mathcal{M}(\Omega) \) is isomorphic to \( C^*(\Omega) \) (Riesz representation theorem; Dunford and Schwartz, 1958, IV.6.3), and the narrow topology on \( \mathcal{M}(\Omega) \) coincides with the weak* topology on \( C^*(\Omega) \).
3. The weak* topology on \( C^*(\Omega) \) is metrizable on every bounded in norm subset of \( C^*(\Omega) \) (Kolmogorov and Fomin, 1972, IV.3.4.4).
4. Every bounded and closed in norm topology subset of \( C^*(\Omega) \) is compact in weak* topology (Kolmogorov and Fomin, 1972, IV.3.4.5).
5. There exists a countable subset of \( C^*(\Omega) \) possessing the following property: its intersection with every ball \( B_r \) is everywhere dense in \( B_r \) in weak* topology (follows from the compactness of the balls \( B_r \)).

However, the requirement for \( \Omega \) to be compact is too restrictive, as many interesting spaces (from applicational point of view) are excluded from the consideration. One possibility to relax this restriction is to allow \( \Omega \) to be a polish (i.e., complete separable metric) space.

As a separable metric space, \( \Omega \) has a metric compactification \( b\Omega \) (Kuratowski, 1966, 2.22.II). Thus, \( \mathcal{M}(b\Omega) \) possesses all the above properties. The natural embedding of \( \mathcal{M}(\Omega) \) into \( \mathcal{M}(b\Omega) \) (a charge on \( \Omega \) corresponds to a charge on \( b\Omega \) carried by \( \Omega \)) has an important property: the topology on \( \mathcal{M}(\Omega) \) generated by \( C(\Omega) \) coincides with the narrow topology inherited from \( \mathcal{M}(b\Omega) \) (Dellacherie and Meyer, 1978, theorem III.58). Every bounded closed subset of \( \mathcal{M}(\Omega) \) is also complete (Dellacherie and Meyer, 1978, theorem III.60).

Thus, every bounded in norm closed subset \( \mathcal{M} \subseteq \mathcal{M}(\Omega) \) is a polish space in the narrow topology (but not necessarily compact metric space). In particular, all probabilistic measures compose a polish space. This allows us to consider a space of Radon charges on \( \mathcal{M} \), i.e., \( \mathcal{M}(\mathcal{M}) \); it will possess the same topological properties as \( \mathcal{M}(\Omega) \). For our purposes, \( \mathcal{M}(\Omega) \) serves as a first-level space of measures, which contains measures being mixed, and \( \mathcal{M}(\mathcal{M}) \) serves as a second-level space, which contains mixing measures.

Below, \( I_A \) denotes the indicator function of the set \( A \), \( \partial A \) denotes a topological boundary of the set \( A \), and \( \mathcal{M}^+(\Omega) \) denotes a space of non-negative measures on \( \Omega \).

The mixing operator \( \text{Mix} : \mathcal{M}(\mathcal{M}) \to \mathcal{M}(\Omega) \) is defined as:

\[
\text{Mix}(\nu)(A) = \int_{\mathcal{M}} \mu(A) \nu(d\mu) \quad \text{for every Borel set } A \subseteq \Omega
\]

(1)

To ensure the correctness of this definition, one needs to prove the existence of the integral on the right-hand side. The latest follows from the fact that the mapping
CONTINUITY OF THE MIXING OPERATOR

3. Continuity of operator Mix

For the sake of simplicity, we restrict ourselves to the case $M \subseteq \mathfrak{M}(\Omega)$, and prove continuity of $\text{Mix}$ only on positive cone $\mathfrak{M}(M)$. Such a restriction, on the one hand, eliminates the necessity of technical details, while, on the other hand, still covers the most important case of probabilistic mixtures of probabilistic measures.

Our proof of continuity of operator $\text{Mix}$ is based on the following lemma (Dellacherie and Meyer, 1978, III.57):

Lemma 3.1. Let $f : \Omega \rightarrow \mathbb{R}$ be a bounded Borel function. If a charge $\lambda$ is carried by the set of points of continuity of $f$, the mapping $\mu \mapsto \int f(\omega) \mu(d\omega)$ is continuous at the point $\lambda$.

We also need:

Corollary 3.2. For every Borel $A \subseteq \Omega$, if a measure $\mu'$ is a point of discontinuity of mapping $\phi_A : \mu \mapsto \mu(A)$, then $\mu'(\partial A) \neq 0$.

Proof. One has $\phi_A(\mu) = \int I_A(\omega) \mu(d\omega)$, and $\partial A$ is the set of points of discontinuity of $I_A$. If, in contrary, $\mu'(\partial A) = 0$, then by lemma 3.1 $\mu'$ is a point of continuity of $\phi_A$, which contradicts the assumption. □

Theorem 3.3. For every bounded in norm closed set $M \subseteq \mathfrak{M}(\Omega)$, operator $\text{Mix} : \mathfrak{M}(M) \rightarrow \mathfrak{M}(\Omega)$ is continuous.

Proof. As both $\mathfrak{M}(\Omega)$ and $\mathfrak{M}(M)$ are separable metric spaces, it is sufficient to show that for every narrowly converging sequence $\{\nu_k\}_k$ in $\mathfrak{M}(M)$, $\nu_k \rightarrow \nu_0$, its image converges to the image of its limit, $\text{Mix}(\nu_k) \rightarrow \text{Mix}(\nu_0)$.

For this, it is necessary and sufficient to show that for every Borel $A \subseteq \Omega$ satisfying

$$\text{Mix}(\nu_0)(\partial A) = 0$$

one has

$$\text{Mix}(\nu_k)(A) \rightarrow \text{Mix}(\nu_0)(A)$$

(2)

The convergence (2) means that the mapping $\nu \mapsto \int_M \mu(A) \nu(d\mu)$ is continuous at the point $\nu_0$. By lemma 3.1 one needs to show that $\nu_0$ is carried by the set of points of continuity of mapping $\phi_A : \mu \mapsto \mu(A)$.

Let $M_A$ be the set of points of discontinuity of mapping $\phi_A$. Suppose, in contrary, that $\nu_0(M_A) > 0$. By corollary 3.2 $\mu \in M_A \Rightarrow \mu(\partial A) > 0$; thus, $\int_M \mu(\partial A) \nu_0(d\mu) > 0$. But $\int_M \mu(\partial A) \nu_0(d\mu) = \text{Mix}(\nu_0)(\partial A)$, and we come to contradiction with condition (2). Thus, the convergence (2) takes place, and the theorem is proved. □

4. Parametric families

In parametric approach, one considers a measurable space of parameters $(\Theta, F)$ and a mapping $\psi : \Theta \rightarrow \mathfrak{M}(\Omega)$; the required property is that for every Borel $A \subseteq \Omega$:

$$\psi_A : \Theta \rightarrow \mathbb{R} : \theta \mapsto \psi(\theta)(A)$$

is measurable

(4)
The approach considered above can be reduced to the parametric one by taking Θ = \mathcal{M}^+(\mathcal{M}) and ψ = id_{\mathcal{M}^+(\mathcal{M})}. The condition (1) follows from the fact that the mapping \( \mu \mapsto \mu(A) = \int I_A(\omega) \mu(d\omega) \) is Borel for every Borel \( A \) (see end of the section [2]).

In turn, parametric approach can be reduced to the one considered above by taking \( \mathcal{M} = \operatorname{Rng}(\psi) \). The following fact is a straightforward corollary of requirement (4):

**Proposition 4.1.** The mapping \( \psi \) is measurable w.r.t. Borel σ-algebra of \( \mathcal{M} \).

In many practical cases, mapping \( \psi \) is even continuous (for example, parameterization of normal distributions by means of mean and variance defines a continuous mapping from real half-plane to the space of probabilistics measures on real line).

Proposition 4.1 allows us to defined the mapping \( \hat{\psi} : \mathcal{M}(\Theta) \to \mathcal{M}(\mathcal{M}) \) by letting \( \hat{\psi}(\nu)(A) = \nu(\psi^{-1}(A)) \). When \( \Theta \) is a polish space, a narrow topology can be defined on \( \mathcal{M}(\Theta) \). If the mapping \( \psi \) is continuous, the mapping \( \hat{\psi} \) is continuous, too (Shiryaev, 2004, theorem III.8.2). Thus, theorem 3.3 implies:

**Theorem 4.2.** The operator \( \operatorname{Mix}_\Theta = \operatorname{Mix} \circ \hat{\psi} : \mathcal{M}^+(\Theta) \to \mathcal{M}^+(\Omega) \) is continuous, whenever the mapping \( \psi \) is continuous.

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