Can one hear the spanning trees of a quantum graph?

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Abstract
Kirchhoff showed that the number of spanning trees of a graph is the spectral determinant of the combinatorial Laplacian divided by the number of vertices; we reframe this result in the quantum graph setting. We prove that the spectral determinant of the Laplace operator on a finite connected metric graph with standard (Neumann–Kirchhoff) vertex conditions determines the number of spanning trees when the lengths of the edges of the metric graph are sufficiently close together. To obtain this result, we analyze an equilateral quantum graph whose spectrum is closely related to spectra of discrete graph operators and then use the continuity of the spectral determinant under perturbations of the edge lengths.

Keywords Quantum graphs \& Spectral determinant \& Zeta functions \& Spanning trees

Mathematics Subject Classification 81Q10 \& 81Q35 \& 05C05 \& 34B45

1 Introduction
The question “Can one hear the shape of a quantum graph?” was answered in the affirmative by Gutkin and Smilansky [12] for a quantum graph where the set of edge lengths are incommensurate. Their work was inspired by the famous question of Kac [16] which also sowed the seeds of results on isospectral billiards [5, 11], graphs [2] and Riemannian manifolds [25], along with many other related works. The connection

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between these questions is the extent to which properties of the spectrum can be used to recover information on the system’s geometry.

Isospectral domains were also studied in the context of discrete graphs [4]. In graph theory, a historic spectral geometric connection is provided by a theorem of Kirchhoff [18]. Kirchhoff’s matrix tree theorem expresses the number of spanning trees of a connected graph in terms of the spectral determinant of the combinatorial Laplacian of the graph which is the matrix $L = D - A$ where $D$ is a diagonal matrix of the vertex degrees and $A$ is the adjacency matrix, see Sect. 2.

**Theorem 1** (Kirchhoff’s Matrix Tree Theorem) For a connected graph $G$ with $V$ vertices,

$$\# \text{ spanning trees } = \frac{1}{V} \det'(L) = \det(L[i])$$

for any $i = 1, 2, \ldots, V$ where $L[i]$ is the matrix $L$ with row $i$ and column $i$ removed.

In this article, we reframe Kirchhoff’s theorem in the setting of quantum graphs. Quantum graphs were introduced to model electrons in organic molecules [24] and are widely employed in mathematical physics as a model of quantum mechanics in systems with complex geometry, see [3, 19] for a review. In a quantum graph, the edges of the graph consist of intervals connected at the vertices with a self-adjoint differential operator on the set of intervals. Here we consider the most widely studied case of the Laplace operator with Neumann–Kirchhoff (or standard) vertex conditions, denoted $L$. For Neumann–Kirchhoff conditions, functions on the graph are continuous and outgoing derivatives at the vertices sum to zero. We obtain the following relationship between the spectral determinant of $L$ and the number of spanning trees of the graph.

**Theorem 2** Let $\Gamma$ be a connected metric graph with edge lengths in the interval $[\ell, \ell + \delta)$ and $L$ the Laplacian with Neumann–Kirchhoff vertex conditions. If

$$\delta < \frac{\ell}{VV'2E + V\sqrt{2EV}}$$

then the number of spanning trees is the closest integer to

$$T_\Gamma = \frac{\prod_{v \in V} d_v}{E 2^E \ell^{\beta+1}} \det'(L),$$

where $V$ is the set of vertices, $d_v$ is the degree of vertex $v$, $E$ is the number of edges and $\beta$ is the first Betti number of $\Gamma$.

If we have an equilateral quantum graph $\widetilde{\Gamma}$ where all the edge lengths are $\ell$, then $T_{\widetilde{\Gamma}}$ is precisely the number of spanning trees. We use $\widetilde{L}$ to denote the Laplace operator of an equilateral quantum graph.
Theorem 3  Let \( \tilde{\Gamma} \) be a connected equilateral metric graph with edge length \( \ell \) and \( \tilde{\mathcal{L}} \) the Laplacian with Neumann–Kirchhoff vertex conditions. Then,

\[
\# \text{spanning trees} = \prod_{v \in V} d_v \frac{E}{2E^2 \ell^{\beta+1}} \det' (\tilde{\mathcal{L}}),
\]

(4)

where \( V \) is the set of vertices, \( d_v \) is the degree of vertex \( v \), \( E \) is the number of edges and \( \beta \) is the first Betti number of \( \tilde{\Gamma} \).

In the formula for \( T_{\Gamma} (3) \), the number of edges \( E \) can be determined from the spectrum of \( \mathcal{L} \) via the Weyl law, where the mean spacing of the eigenvalues is given by \( \pi^{-1} \sum_{e \in \mathcal{E}} \ell_e \), see [3], as the edge lengths are tightly constrained to \( [\ell, \ell + \delta) \). For a connected graph \( \Gamma \), the first Betti number is determined by the number of edges and vertices, \( \beta = E - V + 1 \). Hence, it would be interesting to know if there are also spectral interpretations of the number of vertices and the product of the degrees in (3).

The article is organized as follows. In Sect. 2, we introduce discrete graph and metric graph notation and operators. We relate the spectral determinants of equilateral quantum graphs and discrete graph operators in Sect. 3. In Sect. 4, we use Kirchhoff’s matrix tree theorem and the relationship between spectral determinants to prove Theorem 3. In Sect. 5, we compare spectral determinants of equilateral and non-equilateral quantum graphs, and in Sect. 6, we prove Theorem 2. Finally we summarize the results in Sect. 7.

2 Background

A discrete graph \( G \) is comprised of a set of vertices \( \mathcal{V} \) and a set of edges \( \mathcal{E} \) such that each edge connects a pair of vertices; see, for example, Fig. 1. Two vertices \( u, v \in \mathcal{V} \) are adjacent if \( (u, v) \in \mathcal{E} \) and we write \( u \sim v \). We will denote the number of vertices by \( V = |\mathcal{V}| \) and the number of edges by \( E = |\mathcal{E}| \). The degree of a vertex \( v \), denoted \( d_v \), is the number of vertices that are adjacent to \( v \). We let \( \mathcal{E}_v \) denote the set of edges containing the vertex \( v \) so \( d_v = |\mathcal{E}_v| \). If \( d_v = d \) for all \( v \in \mathcal{V} \), the graph is regular. In this paper, we only consider simple graphs where there are no loops or multiple edges and the number of edges and vertices are finite. We also assume that \( G \) is connected, so there is a path between every pair of vertices. The distance between a pair of vertices \( u, v \in \mathcal{V} \) on a connected discrete graph is the number of edges in a shortest path joining \( u \) and \( v \). The diameter \( D \) of the graph is the maximum distance between a pair of vertices. The first Betti number of \( G \) is \( \beta = E - V + 1 \), the number of independent cycles on \( G \). A tree is a connected graph with no cycles, \( \beta = 0 \). A spanning tree of \( G \) is a subgraph of \( G \) that is a tree and contains all the vertices of \( G \).
A function on a discrete graph \( G \) takes values at the vertices and therefore can be viewed as a vector \( f \in \mathbb{C}^V \). The combinatorial Laplace operator, \( L \), is defined by

\[
(Lf)(v) = d_v f(v) - \sum_{u \sim v} f(u). \tag{5}
\]

This means that we can define the self-adjoint \( V \times V \) matrix \( L \) where

\[
L_{uv} = \begin{cases} 
  d_u & \text{if } u = v \\
  -1 & \text{if } u \sim v \\
  0 & \text{otherwise}. 
\end{cases} \tag{6}
\]

We note that \( L \) has zero as an eigenvalue (with multiplicity one for a connected graph), and we will index the eigenvalues of \( L \) in increasing order,

\[
0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_V. \tag{7}
\]

The harmonic Laplacian is

\[
\Delta = D^{-1}L, \tag{8}
\]

where \( D \) is a diagonal matrix of the degrees, \( D_{uv} = d_v \delta_{uv} \). We denote the eigenvalues of \( \Delta \) with \( \lambda_j \) for \( j = 1, \ldots, V \).

A metric graph \( \Gamma \) is a discrete graph where every edge \( e \in E \) is assigned a length \( \ell_e > 0 \). We associate the edge \( e \) with the interval \([0, \ell_e]\). For \( e = (u, v) \), we set \( x_e = 0 \) at \( u \) and \( x_e = \ell_e \) at \( v \); the choice of orientation of the coordinate is arbitrary and will not affect the results. The total length of a metric graph is the sum of the lengths of each edge, \( \ell_{\text{tot}} = \sum_{e \in E} \ell_e \). An equilateral metric graph \( \tilde{\Gamma} \) is a metric graph where all the edge lengths are equal. We will use the term generic metric graph to distinguish a non-equilateral graph. A function \( f \) on \( \Gamma \) is a collection of functions \( \{f_e\}_{e \in E} \) where \( f_e \) is a function on the interval \([0, \ell_e]\).

A quantum graph is a metric graph \( \Gamma \) along with a self-adjoint differential operator. In this paper, we consider the Laplace operator, denoted by \( \mathcal{L} \), which acts as \( -\frac{d^2}{dx^2} \) on functions defined on \([0, \ell_e]\). We will use the notation \( \tilde{\mathcal{L}} \) if \( \tilde{\Gamma} \) is an equilateral metric graph. At the vertices, functions will satisfy the Neumann–Kirchhoff (or standard) vertex conditions,

\[
\begin{align*}
  f(x) & \text{ is continuous on } \Gamma \text{ and } \\
  \sum_{e \in \mathcal{E}_v} \frac{df_e}{dx_e}(v) & = 0 \text{ at each vertex } v. \tag{9}
\end{align*}
\]

By convention, \( \frac{df_e}{dx_e}(v) \) is taken to be the outgoing derivative at the end of the interval \([0, \ell_e]\) corresponding to \( v \). The second Sobolev space on \( \Gamma \) is the direct sum of the
second Sobolev spaces on the set of intervals,
\[ H^2(\Gamma) = \bigoplus_{e \in \mathcal{E}} H^2([0, \ell_e]). \]  
(10)

The domain of \( \mathcal{L} \) is the set of functions \( f \in H^2(\Gamma) \) that satisfy (9). With these vertex conditions, \( \mathcal{L} \) has an infinite number of nonnegative eigenvalues [3] which we will denote as
\[ 0 \leq k_1^2 \leq k_2^2 \leq \cdots \]  
(11)
with \( k_j \in \mathbb{R} \).

Some of these eigenvalues may correspond to eigenfunctions where \( f_e(0) = f_e(\ell_e) = 0 \) for all \( e \in \mathcal{E} \). In this case, the eigenfunction is also an eigenfunction of the Laplacian with Dirichlet vertex conditions. The spectrum of the graph with Dirichlet vertex conditions is just the union of the spectra of the Laplacian on \( E \) disconnected intervals \([0, \ell_e]\) with Dirichlet boundary conditions. We call this the Dirichlet spectrum of \( \Gamma \).

### 3 Spectral determinants of equilateral graphs

In this section, we relate the spectral determinants of \( \Delta \) and \( \tilde{\mathcal{L}} \). This is used in Sect. 4 to prove Theorem 3. The spectral determinant of the harmonic Laplacian \( \Delta \) is the product of its nonzero eigenvalues,
\[ \det'(\Delta) = \prod_{j=1}^{V'} \lambda_j. \]  
(12)

The prime shows that eigenvalues of zero are omitted from the product. Correspondingly, the spectral determinant of the Laplace operator \( \tilde{\mathcal{L}} \) on an equilateral metric graph is, formally,
\[ \det'(\tilde{\mathcal{L}}) = \prod_{j=1}^{\infty'} k_j^2. \]  
(13)

The spectral determinant of quantum graphs has been studied in a number of situations [8–10, 13, 15]. Here we use a zeta function regularization of the product. The spectral zeta function, \( Z(s) \), is a generalization of the Riemann zeta function where the nonzero eigenvalues take the place of the integers. The spectral zeta function corresponding to \( \tilde{\mathcal{L}} \) is
\[ Z(s) = \sum_{j=1}^{\infty} k_j^{-2s}, \]  
(14)
which converges for \( \text{Re}(s) > 1 \). Making an analytic continuation to the left of \( \text{Re}(s) = 1 \), the regularized spectral determinant is defined as

\[
det'(\tilde{\mathcal{L}}) = \exp(-Z'(0)).
\] (15)

There is a correspondence between the eigenvalues of \( \Delta \) and \( \tilde{\mathcal{L}} \) [26] (see also [3, 19, 23]).

**Proposition 1** Suppose \( G \) is a discrete graph and \( \tilde{\Gamma} \) is the corresponding equilateral metric graph with edge length \( \ell \). If \( k^2 \) is not in the Dirichlet spectrum of \( \tilde{\mathcal{L}} \), then

\[
k^2 \in \sigma(\tilde{\mathcal{L}}) \iff 1 - \cos(k\ell) \in \sigma(\Delta)
\] (16)

where \( \sigma(\cdot) \) denotes the spectrum of the operator.

By proposition 1, every eigenvalue of \( \Delta \) can be written as \( 1 - \cos(t\ell) \) for some \( t \in [0, \pi/\ell] \). Let \( T = \{t_j\}_{j=1}^V \) be the set \( 0 = t_1 < t_2 \leq \ldots \leq t_V \leq \pi/\ell \) such that \( 1 - \cos(t_j\ell) \in \sigma(\Delta) \) (including multiplicity). We know \( 0 = t_1 < t_2 \) as zero is an eigenvalue of \( \Delta \) with multiplicity one when \( G \) is connected. This allows us to write the spectral zeta function of \( \tilde{\mathcal{L}} \) in terms of the finite set \( T \). The following lemma is a combination of equations (20) and (24) from [14].

**Lemma 1** Suppose \( G \) is a discrete graph and \( \tilde{\Gamma} \) is the corresponding equilateral metric graph with edge length \( \ell \). For \( \text{Re}(s) \neq \frac{1}{2} \), the spectral zeta function of \( \tilde{\mathcal{L}} \) is

\[
Z(s) = (4^s(\beta - 1) + 2) \left( \frac{\ell}{2\pi} \right)^{2s} \zeta_R(2s) + \left( \frac{\ell}{2\pi} \right)^{2s} \sum_{j=2}^V \left( \zeta_H \left( 2s, \frac{t_j\ell}{2\pi} \right) + \zeta_H \left( 2s, 1 - \frac{t_j\ell}{2\pi} \right) \right),
\] (17)

where \( \zeta_R(z) \) is the Riemann zeta function, \( \zeta_H(z, a) \) is the Hurwitz zeta function and \( t_j \in T \).

Using Lemma 1, we can obtain a relationship between the spectral determinants of the harmonic Laplacian of a discrete graph and the Laplacian of the corresponding equilateral quantum graph.

**Proposition 2** Suppose \( G \) is a connected discrete graph and \( \tilde{\Gamma} \) is the corresponding equilateral metric graph with edge length \( \ell \). Then

\[
det'(\tilde{\mathcal{L}}) = 2^{E-1} \ell^{\beta + 1} \det'(\Delta).
\] (18)

**Proof** Since \( \zeta_H(0, a) = \frac{1}{2} - a \) [22], it follows that,

\[
\zeta_H(0, a) + \zeta_H(0, 1 - a) = 0.
\] (19)
Using Lemma 1,

\[ Z'(0) = -\ln \left( 2^{\beta-1} \right) - \ln \left( \ell^{\beta+1} \right) + 2 \sum_{j=2}^{V} \left[ \ln \left( \frac{t_j \ell}{2\pi} \right) - \ln 2 \pi \right] \]

(20)

since \( \zeta_R(0) = -1/2, \zeta'_R(0) = -\ln(2\pi)/2, \) and \( \zeta'_H(0, a) = \ln(\Gamma(a)) - \frac{1}{2} \ln(2\pi) \) [22]. Additionally \( \Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z) \) [22] and \( 4 \sin^2 \left( \frac{t_j \ell}{2} \right) = 2(1 - \cos(t_j \ell)) = 2\lambda_j, \) so

\[ -Z'(0) = \ln \left( 2^{\beta-1} \right) + \ln(\ell^{\beta+1}) + \sum_{j=2}^{V} \ln \left( 4 \sin^2 \left( \frac{t_j \ell}{2} \right) \right) \]

(21)

\[ = \ln \left( 2^V + \ell^{\beta+1} \prod_{j=2}^{V} \lambda_j \right) . \]

(22)

Taking the exponential of this expression yields the result.

\[ \square \]

3.1 Example: complete bipartite graphs

Proposition 2 simplifies for complete bipartite graphs. The complete bipartite graph, \( K_{m,p}, \) consists of \( m + p \) vertices and \( mp \) edges. The vertices can be divided into two disjoint sets, \( M \) of size \( m \) and \( P \) of size \( p, \) such that every vertex in \( M \) is connected to every vertex in \( P \) and no vertex is connected to another vertex from the same set; see, for example, Fig. 1. A star graph is a complete bipartite graph with \( m = 1 \) and \( p = V - 1 = E. \)

Corollary 1 For an equilateral complete bipartite graph \( K_{m,p} \) with edge length \( \ell, \)

\[ \det'(\tilde{\Delta}) = 2^{mp} \ell^{mp-m-p+2}. \]

(23)

In particular, for a star graph,

\[ \det'(\tilde{\Delta}) = 2^E \ell . \]

(24)

Proof The eigenvalues of a discrete complete bipartite graph \( K_{m,p} \) are 0 with multiplicity one, 1 with multiplicity \( m + p - 2 \) and 2 with multiplicity one [6], and therefore,

\[ \det'(\Delta) = 2. \]

(25)
The complete bipartite graph $K_{m,p}$ has $\beta = E - V + 1 = mp - (m + p) + 1$. Substituting into Proposition 2 produces the result. The case of a star graph is obtained by setting $m = 1$ and $p = V - 1 = E$.

Corollary 1 agrees with [14] where the spectral determinant was obtained from the quantum spectral zeta function directly. The star graph formula (24) agrees with [13] where the spectral zeta functions of quantum star graphs were calculated using a contour integral approach.

4 Spanning trees of equilateral graphs

Kirchhoff’s matrix tree theorem [18] gives the number of spanning trees of a discrete graph in terms of the spectral determinant of the combinatorial Laplacian. In this section, we prove Theorem 3 which reframes this result in terms of the spectral determinant of an equilateral quantum graph.

4.1 Regular equilateral graphs

We start by proving Theorem 3 for a $d$-regular graph where the proof is straightforward.

Proposition 3 Suppose $\tilde{\Gamma}$ is a connected $d$-regular equilateral metric graph with edge length $\ell$. Then

$$\# \text{spanning trees} = \frac{d^{V-1}}{2E-1\ell\beta+1V} \det'(\tilde{\Delta}).$$

(26)

Proof Since $\tilde{\Gamma}$ is a $d$-regular graph,

$$\Delta = D^{-1}L = \frac{1}{d}L.$$  

(27)

If $\mu$ is an eigenvalue of $L$, then $\mu/d$ is an eigenvalue of $\Delta$ so

$$\det'(\Delta) = \prod_{i=2}^{V} \frac{\mu_i}{d} = d^{1-V} \det'(L).$$  

(28)

By Kirchhoff’s matrix tree theorem,

$$\# \text{spanning trees} = \frac{d^{V-1}}{V} \det'(\Delta).$$  

(29)

Combining Eq. (29) with proposition 2 produces the result. □
4.2 General equilateral graphs

Theorem 3 follows from Proposition 2 and the next lemma, which is an adaptation of a theorem by Chung and Yau [7] and generalizes Eq. (29).

**Lemma 2** For a connected discrete graph $G$,

$$\# \text{ spanning trees} = \prod_{v \in V} d_v \frac{\det'(\Delta)}{2E}.$$

**Proof** Since $\Delta$ has one eigenvalue of zero, its characteristic polynomial is

$$p(x) = \det(\Delta - x I) = -x \prod_{i=2}^{V} (\lambda_i - x).$$

Therefore, the coefficient of the linear term in $p(x)$ is

$$-\prod_{i=2}^{V} \lambda_i = -\det'(\Delta).$$

On the other hand, since $\Delta = D^{-1}L$,

$$\Delta - x I = D^{-1}(L - x D),$$

and the characteristic polynomial can also be written as

$$p(x) = \left(\prod_{v \in V} d_v\right)^{-1} \det(L - x D).$$

Determinants are linear along the rows of a matrix so

$$\det(A + B) = \sum_{S \subseteq [n]} \det B_S,$$

where $A$ and $B$ are square matrices of size $n$, $[n] = \{1, 2, \ldots, n\}$, and $B_S$ is the matrix $B$ whose rows that are indexed by $S$ are replaced with the corresponding rows from $A$. Hence,

$$\det(L - x D) = \det(L) + \sum_{S \subseteq [V]} \det(x D_S) + \sum_{S \subseteq [V]} \det(x D_S) + \cdots + \det(x D)$$

$$= \det(L) - x \sum_{i=1}^{V} d_i \det(L[i]) + c_2 x^2 + \cdots + c_n x^n,$$
where \( L[i] \) is the matrix \( L \) with both row \( i \) and column \( i \) removed. By theorem 1, \( \det(L[i]) \) is the number of spanning trees of \( G \). Therefore, we can see that

\[
\det'(\Delta) = \frac{\sum_{v \in V} d_v}{\prod_{v \in V} d_v} \times \# \text{ spanning trees},
\]

which proves the lemma as \( \sum_{v \in V} d_v = 2E \). \( \square \)

We have now established Theorem 3. So, for example, a star graph has \( \beta = 0 \) and one vertex has degree \( E \) while the others have degree 1. Then, using the spectral determinant of an equilateral star graph (Corollary 1), we see

\[
\# \text{ spanning trees} = \frac{\prod_{v \in V} d_v}{E^2E^{\ell+1}} 2E^{\ell} = 1
\]

as required.

5 Spectral determinants of generic quantum graphs

In this section, we compare the spectral determinants of equilateral and generic quantum graphs where the corresponding discrete graphs are the same. In particular, given an equilateral graph \( \tilde{\Gamma} \) with edge length \( \ell \), we will consider a generic graph whose edge lengths lie in the interval \([\ell, \ell + \delta)\). Friedlander computed the spectral determinant of a generic quantum graph [10].

**Theorem 4** Suppose \( \Gamma \) is a connected metric graph. Then

\[
\det'(L) = \frac{2E^2}{\det'(R)},
\]

where \( R \) is the \( V \times V \) matrix defined by

\[
R_{uv} = \begin{cases} \sum_{w \sim v} \ell_{(w,v)}^{-1} & \text{if } u = v \\ -\ell_{(u,v)}^{-1} & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases}
\]

First, we note that the proof of Lemma 2 also showed

\[
\det'(L) = \frac{V}{2E} \prod_{v \in V} d_v \det'(\Delta).
\]

In the case of an equilateral graph with edge length \( \ell \), we will define \( \tilde{R} = \ell^{-1}L \), which agrees with (41) when \( \ell_e = \ell \) for every edge \( e \). In fact, setting the edge lengths equal
in Theorem 4,

\[ \det'(L)|_{\ell_e=\ell} = \frac{2^E E \ell^{E-V+2}}{V \prod_{v \in V} d_v} \det'(L) \]

(43)

\[ = 2^{E-1} \ell^{E+1} \det'(\Delta) \]

(44)

\[ = \det'(\tilde{L}) \]

(45)

where we used Proposition 2 for the last step. This demonstrates that evaluating (40) with equal edge lengths produces the spectral determinant of the equilateral graph as expected.

5.1 Bound on the spectral norm of $\tilde{R} - R$

Let $\ell_e = \ell + \delta_e$ where $\delta_e \in [0, \delta)$. Then $|\ell^{-1} - \ell_e^{-1}| < \delta \ell^{-2}$. To bound the spectral norm $||\tilde{R} - R||_2$, we use the fact that $||A||_2 \leq \sqrt{\sum_{ij} |A_{ij}|^2}$ for a matrix $A$. From (41),

\[ \tilde{R}_{uv} - R_{uv} = \begin{cases} \sum_{r \sim u} \frac{\delta_{(r,u)}}{(\ell + \delta_{(r,u)})\ell} & \text{if } u = v \\ -\frac{\delta_{(u,v)}}{(\ell + \delta_{(u,v)})\ell} & \text{if } u \sim v \\ 0 & \text{otherwise.} \end{cases} \]

(46)

Hence, by Bergström’s inequality,

\[ \sum_v |\tilde{R}_{uv} - R_{uv}|^2 \leq \left( \sum_{r \sim u} \frac{\delta_{(r,u)}}{(\ell + \delta_{(r,u)})\ell} \right)^2 + \sum_{u \sim v} \left( \frac{\delta_{(u,v)}}{(\ell + \delta_{(u,v)})\ell} \right)^2 \]

(47)

\[ < (d_u + 1) \sum_{u \sim v} \delta^2 \ell^{-4}. \]

(48)

From this, we see,

\[ ||\tilde{R} - R||_2 < \frac{\delta \sqrt{2E(d_{\max} + 1)}}{\ell^2} < \frac{\delta \sqrt{2EV}}{\ell^2} \]

(49)

where $d_{\max}$ is the maximum degree of any vertex.

Consequently, if the eigenvalues of $R$ are $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_V$ and the eigenvalues of $\tilde{R}$ are $0 = \tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots < \tilde{\lambda}_V$ then

\[ |\tilde{\lambda}_j - \lambda_j| < ||\tilde{R} - R||_2 < \frac{\delta \sqrt{2EV}}{\ell^2}. \]

(50)

5.2 A bound on the change in a spectral determinant

Given a set of real numbers $0 < a < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$,
\[
\prod_{j=1}^{n}(\alpha_j + a) - \prod_{j=1}^{n}\alpha_j = a^n + a^{n-1} \sum_{i=1}^{n} \alpha_i + a^{n-2} \sum_{i_1 \neq i_2} \alpha_{i_1} \alpha_{i_2} + \cdots + a \sum_{i_1, \ldots, i_{n-1}: i_j \neq i_k} \alpha_{i_1} \cdots \alpha_{i_{n-1}} \tag{51}
\]

\[
< a \left( \prod_{i=2}^{n} \alpha_i \right) \left( \sum_{j=0}^{n-1} \binom{n}{j} \right) < a^2 \left( \prod_{i=2}^{n} \alpha_i \right). \tag{52}
\]

\[
< a^2 \left( \prod_{i=2}^{n} \alpha_i \right). \tag{53}
\]

Now we compare the spectral determinants of \( R \) and \( \tilde{R} \). From (50), we know that the difference between the corresponding eigenvalues of \( R \) and \( \tilde{R} \) is at most

\[
a = \frac{\delta \sqrt{2EV}}{\ell^2}. \tag{54}
\]

In our situation, we have \( 0 = \lambda_1 < a < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_V \) (assuming that \( \delta \) is small enough so that \( a < \lambda_2 \)). Then from (53),

\[
|\text{det}'(R) - \text{det}'(\tilde{R})| < \delta 2^{V-1} \sqrt{2EV} \frac{\ell^2}{\ell^2 \lambda_2} \text{det}'(R). \tag{55}
\]

6 Spanning trees of generic quantum graphs

For a generic quantum graph \( \Gamma \) with edge lengths in \([\ell, \ell + \delta]\), let

\[
T_{\Gamma} = \frac{\prod_{v \in V} d_v}{E^2 \ell^{2\beta + 1}} \text{det}'(L). \tag{56}
\]

Notice that, by Theorem 3, \( T_{\tilde{\Gamma}} \) is the number of spanning trees of an equilateral graph. For \( \delta \) sufficiently small, the value of \( T_{\Gamma} \) will be close enough to \( T_{\tilde{\Gamma}} \) to determine the number of spanning trees of a generic quantum graph. Using Theorem 4,

\[
T_{\Gamma} = \frac{\ell_{\text{tot}}}{E V \ell^{2\beta + 1}} \text{det}'(R), \tag{57}
\]

and similarly,

\[
T_{\tilde{\Gamma}} = \frac{\ell^{E-\beta}}{V} \text{det}'(\tilde{R}). \tag{58}
\]

We will determine a bound on \( \delta \) such that \( |T_{\Gamma} - T_{\tilde{\Gamma}}| < 1/2 \), and hence, the number of spanning trees is the closest integer to \( T_{\Gamma} \). To this end, we employ two bounds on
the spectrum of $L$. The first is a lower bound on the second smallest eigenvalue in terms of the graph diameter $D$, the maximum distance between a pair of vertices, due to McKay \cite{21}.

**Theorem 5** The second smallest eigenvalue of $L$ is bounded below, $\mu_2 \geq \frac{4}{DV}$.

The second is an upper bound on the eigenvalues of $L$, see \cite{1, 17, 20}.

**Theorem 6** The eigenvalues of $L$ are bounded above, $\lambda_j \leq V$ for $j = 1, \ldots, V$.

**Proof of Theorem 2** From Eqs. \eqref{57} and \eqref{58},

$$|T_\Gamma - T_{\tilde{\Gamma}}| \leq \frac{\ell_{\text{tot}}}{E} \prod_{e \in E} \ell_e \{ |\text{det}'(R) - \text{det}'(\tilde{R})| + \frac{|\text{det}'(\tilde{R})|}{V} \left| \frac{\ell_{\text{tot}}}{E} \prod_{e \in E} \ell_e - \frac{V}{E} \right| \right|.$$

\hspace*{1cm} (59)

Using \eqref{55} and assuming that $\delta < \ell$,

$$|T_\Gamma - T_{\tilde{\Gamma}}| < \left( \frac{2\ell}{V} \right)^{E+1} \frac{2^{V-1}\sqrt{2EV} \delta |\text{det}'(\tilde{R})|}{\ell^2 \tilde{\lambda}_2} + \frac{|\text{det}'(\tilde{R})|}{V^{\beta+1}} \left| (\ell + \delta)^{E+1} - \ell^{E+1} \right|.$$

\hspace*{1cm} (60)

Using \eqref{53}, we can see that

$$|(\ell + \delta)^{E+1} - \ell^{E+1}| < \delta 2^{E+1} \ell E.$$

\hspace*{1cm} (61)

Consequently,

$$|T_\Gamma - T_{\tilde{\Gamma}}| < \delta |\text{det}'(L)| \frac{2^{E+1}}{\ell V} \left[ \frac{2^{V-1}\sqrt{2EV}}{\ell \tilde{\lambda}_2} + 1 \right]$$

\hspace*{1cm} (62)

since $\text{det}'(\tilde{R}) = \ell^{-(V-1)} \text{det}'(L)$.

We can conclude from Theorem 5 that $\tilde{\lambda}_2 > \frac{4}{V^2 \ell}$, and we know by applying Theorem 6 that $\text{det}'(L) \leq V^{V-1}$. Using these inequalities, we see that

$$|T_\Gamma - T_{\tilde{\Gamma}}| < \frac{\delta}{\ell} V^{V-2} 2^{E+1} \left[ V^{2V-3} \sqrt{2EV} + 1 \right] < \frac{\delta}{\ell} V^V 2^{E+V-1} \sqrt{2EV},$$

\hspace*{1cm} (63)

which establishes Theorem 2. \hfill \Box

### 6.1 Star graph example

For comparison, we write an equivalent bound in the case of the star graph where there are explicit formulae for the spectral determinants of the equilateral and generic
quantum graphs. From [13],

$$\det'(L) = \frac{2^E}{E} \sum_{e \in E} \ell_e$$  \hspace{1cm} (64)$$

which agrees with the spectral determinant of the equilateral graph, Corollary 1. For $\ell_e \in [\ell, \ell + \delta)$,

$$|\det'(\tilde{L}) - \det'(L)| < 2^E \delta.$$  \hspace{1cm} (65)$$

Hence,

$$|T_\Gamma - T_{\tilde{\Gamma}}| < \frac{\delta}{\ell}.$$  \hspace{1cm} (66)$$

Therefore, the closest integer to $T_\Gamma$ is the number of spanning trees if $\delta \leq \ell/2$.

If we have a star graph, then $E = V - 1$ and the condition on $\delta$ from Theorem 2 is

$$\delta < \frac{\ell}{\sqrt{V(2V - 1)}}.$$  \hspace{1cm} (67)$$

Clearly, the demand on the edge lengths in Theorem 2 is suboptimal so, in fact, one may expect that the nearest integer to $T_\Gamma$ gives the number of spanning trees even when the edge lengths are less tightly constrained.

7 Discussion

In this paper, we proved an analog of Kirchhoff’s matrix tree theorem for quantum graphs. In particular, we determined the number of spanning trees of an equilateral quantum graph from its spectral determinant. To do this, we related the spectral determinant of an equilateral quantum graph to the spectral determinant of the harmonic Laplacian of the corresponding discrete graph. We extended this to non-equilateral quantum graphs where the edge lengths are sufficiently constrained.

The bound on the permitted variance in the edge lengths in Theorem 2 is suboptimal but requires minimal information on the structure of the graph. However, the constraint may be loosened in some situations, even if there is no additional information about the graph’s structure. For example, all eigenvalues of $L$ satisfy [1],

$$\lambda_j \leq \max_{(u, v) \in E} (d_u + d_v).$$  \hspace{1cm} (68)$$

As we assume in Theorem 2 that we know the product of the vertex degrees, if we also know that $\max_{(u, v) \in E} (d_u + d_v) < V$, then we can use (68) in place of Theorem 6 which weakens the constraint on the spread of the edge lengths.
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Declarations

Conflict of interest  The authors declare that they have no conflict of interest.

References

1. Anderson, W.N., Morley, T.D.: Eigenvalues of the Laplacian of a graph. Linear Multilinear Algebra (1985). https://doi.org/10.1080/03081088508817681
2. Band, R., Parzanchevski, O., Ben-Shach, G.: The isospectral fruits of representation theory: quantum graphs and drums. J. Phys. A (2009). https://doi.org/10.1088/1751-8113/42/17/175202
3. Berkolaiko, G., Kuchment, P.: Introduction to Quantum Graphs, Volume 186 of Mathematical Surveys and Monographs. American Mathematical Society, Providence (2013)
4. Brooks, R.: Non-Sunada graphs. Ann. Inst. Fourier (1999). https://doi.org/10.5802/afif.1688
5. Chapman, J.S.: Drums that sound the same. Am. Math. Month. (1995). https://doi.org/10.2307/2975346
6. Chung, F.: Spectral Graph Theory, Volume 92 of CBMS Regional Conference Series in Mathematics. American Mathematical Society, Providence (1997)
7. Chung, F., Yau, S.T.: Coverings, heat kernels and spanning trees. Electron. J. Combin. 6, R12 (1999)
8. Desbois, J.: Spectral determinant of Schrödinger operators on graphs. J. Phys. A (2000). https://doi.org/10.1088/0305-4470/33/7/103
9. Desbois, J.: Spectral determinant on graphs with generalized boundary conditions. Eur. Phys. J. B (2001). https://doi.org/10.1007/s100510170013
10. Friedlander, L.: Determinant of the Schrödinger operator on a metric graph. In: Berkolaiko, G., Carlson, R., Fulling, S., Kuchment, P. (eds.) Quantum graphs and their applications, pp. 151–160. American Mathematical Society, Providence (2006). https://doi.org/10.1090/conm/415/07866
11. Gordon, C., Webb, D.L., Wolpert, S.: One cannot hear the shape of a drum. Bull. Am. Math. Soc. (1992). https://doi.org/10.1090/S0273-0979-1992-00289-6
12. Gutkin, B., Smilansky, U.: Can one hear the shape of a graph? J. Phys. A (2001). https://doi.org/10.1088/0305-4470/34/31/301
13. Harrison, J.M., Kirsten, K.: Zeta functions of quantum graphs. J. Phys. A (2011). https://doi.org/10.1088/0305-4470/44/23/235301
14. Harrison, J.M., Weyand, T.: Relating zeta functions of discrete and quantum graphs. Lett. Math. Phys. (2018). https://doi.org/10.1007/s11005-017-1017-0
15. Harrison, J.M., Kirsten, K., Texier, C.: Spectral determinants and zeta functions of Schrödinger operators on metric graphs. J. Phys. A (2012). https://doi.org/10.1088/1751-8113/45/12/125206
16. Kac, M.: Can one hear the shape of a drum? Am. Math. Mon. (1966). https://doi.org/10.2307/2313748
17. Kelmans, A.K.: On properties of the characteristic polynomial of a graph. Kibern. službu kommunizmu 4, 27–41 (1967)
18. Kirchhoff, G.: Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Strme geführt wird. Ann. Phys. Chem. 72, 497–508 (1847)
19. Kuchment, P.: Quantum graphs I. Some basic structures. Waves Random Media (2003). https://doi.org/10.1088/0959-7174/14/1/014
20. Merris, R.: Laplacian matrices of graphs: a survey. Linear Algebra Appl. (1994). https://doi.org/10.1016/0024-3795(94)90486-3
21. Mohar, B.: Eigenvalues, diameter, and mean distance in graphs. Graphs Combin. 7, 53–64 (1991)
22. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W. (eds): NIST handbook of mathematical functions, U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC. Cambridge University Press, Cambridge, U.K. (2010)
23. Pankrashkin, K.: Spectra of Schrödinger operators on equilateral quantum graphs. Lett. Math. Phys. (2006). https://doi.org/10.1007/s11005-006-0088-0
24. Pauling, P.: The diamagnetic anisotropy of aromatic molecules. J. Chem. Phys. (1936). https://doi.org/10.1063/1.1749766
25. Sunada, T.: Riemannian coverings and isospectral manifolds. Ann. Math. (1985). https://doi.org/10.2307/1971195
26. von Below, J.: A characteristic equation associated to an eigenvalue problem on $c^2$-networks. Linear Algebra Appl. (1985). https://doi.org/10.1016/0024-3795(85)90258-7

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