Goldstone bosons in the massless Thirring model.
Witten’s criterion

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Abstract

We discuss the Ward identity and the low–energy theorem for the divergence of
the axial–vector current in the massless Thirring model with fermion fields quantized
in the chirally broken phase (Eur. Phys. J. C 20, 723 (2001)). The Ward identity
and the low–energy theorem are analysed in connection with Witten’s criterion for
Goldstone bosons (Nucl. Phys. B 145, 110 (1978)). We show that the free massless
(pseudo)scalar field, bosonizing the massless Thirring model in the chirally broken
phase, satisfies Witten’s criterion to interpret quanta of this field as Goldstone
bosons. As has been shown in hep–th/0210104 and hep–th/0212226, Goldstone’s
criterion, the non–invariance of the wave function of the ground state, is also fulfilled.

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1 Introduction

The massless Thirring model \( \{1\} \) is an exactly solvable quantum field theoletic model of fermions with a non–trivial four-fermion interaction embedded in 1+1–dimensional space–time. A solution of a quantum field theoretic model requires the development of a procedure for the calculation of any correlation function. In the chiral symmetric phase this has been carried out in \( \{2\} \) (see also \( \{3\} \)).

The existence of the chirally broken phase of the massless Thirring model has been recently pointed out in \( \{4\} \). It has been shown that the chirally broken phase of the massless Thirring model is identical to the superconducting phase of the BCS theory of superconductivity. In the chirally broken phase the massless Thirring model bosonizes to the quantum field theory of a free massless (pseudo)scalar field \( \vartheta(x) \). According to \( \{4\}–\{11\} \), the quanta of the free massless (pseudo)scalar field \( \vartheta(x) \) are Goldstone bosons (see also \( \{12\} \)).

In the paper \( \{13\} \) Witten has investigated the \( 1/N \) expansion within the massless Thirring model with \( SU(N) \) symmetry. He has confirmed the existence of spontaneously broken chiral symmetry and massless (pseudo)scalar bosons. However, according to Witten these bosons are massless but non–Goldstone. Indeed, Witten has written \( \{13\} \): “A Goldstone boson is a massless boson whose singular contributions to the Ward identities enable the identities to be satisfied even though some symmetry breaking Green functions are non–zero. This is not possible in two dimensions. But massless bosons are possible. The massless boson in this theory is not a Goldstone boson; it satisfies no pertinent low–energy theorems; in this theory the chirality violating Green functions are zero and there is no room for Goldstone–boson contributions in the Ward identities.”

In \( \{14\} \) Wightman has pointed out that due to infrared divergences of the two–point Wightman functions, leading to the violation of Wightman’s positive definiteness condition, one cannot construct a mathematically correct quantum field theory of a free massless (pseudo)scalar field in 1+1–dimensional space–time with Wightman’s observables defined on the test functions \( h(x) \) from the Schwartz class \( \mathcal{S}(\mathbb{R}^2) \). In other words, Wightman has asserted that such a theory does not merely exist. In \( \{15\} \) Coleman has reformulated Wightman’s assertion in the form of the theorem suppressing the existence of Goldstone bosons and spontaneous breaking of continuous symmetry in quantum field theories with Wightman’s observables defined on the test functions \( h(x) \) from the Schwartz class \( \mathcal{S}(\mathbb{R}^2) \). This is the well–known theorem of Coleman \( \{15\} \).

As has been shown in \( \{3\}, \{9\} \), due to infrared divergences the vacuum–to–vacuum transition amplitude (or the generating functional of Green functions), induced by an external source in Schwinger’s approach to the quantum field theory of the free massless (pseudo)scalar field, vanishes. This confirms Wightman’s statement \( \{14\} \).

In \( \{3\} \) we have shown how one can construct a quantum field theory of the free massless (pseudo)scalar field in 1+1–dimensional space–time, which bosonizes the Thirring model with fermion fields quantized in the chirally broken phase, with a non–vanishing vacuum–to–vacuum transition amplitude and the two–point Wightman functions without infrared divergences. As has been shown in \( \{9\} \), this is equivalent to Wightman’s quantum field theory with Wightman’s observables, defined on the test functions from the Schwartz class \( \mathcal{S}_0(\mathbb{R}^2) = \{h(x) \in \mathcal{S}(\mathbb{R}^2); \tilde{h}(0) = 0\} \). Since Wightman’s observables are defined on the test functions from \( \mathcal{S}_0(\mathbb{R}^2) \), there is no contradiction to Coleman’s.
Theorem [10, 11].

Such a quantum field theory of the free massless (pseudo)scalar field \( \vartheta(x) \) can be applied to the description of massless (pseudo)scalar bosons in Witten’s analysis of the massless Thirring model.

Now the problem left is: “Whether one can call the quanta of this free massless (pseudo)scalar field \( \vartheta(x) \) Goldstone bosons or not?” As has been shown in [10, 11], Goldstone’s criterion, the non–invariance of the wave function of the ground state [16], is fulfilled. Therefore, in order to gain an allowance to interpret the quanta of the free massless (pseudo)scalar field \( \vartheta(x) \) as Goldstone bosons, one has to show that they satisfy Witten’s criterion. In order words, one has to demonstrate that the contributions of the quanta of the \( \vartheta \) field play an important role for the fulfillment of the low–energy theorems and the Ward identities quoted by Witten.

The paper is organized as follows. In Section 2 we derive the Ward identity and the low–energy theorem for the divergence of the axial–vector current. We show that in the chiral symmetric phase, when the axial–vector current is conserved, the Ward identity and the low–energy theorem are trivially fulfilled as \( 0 = 0 \). In turn, in the chirally broken phase the Ward identity and the low–energy theorem are fulfilled only due to a non–trivial contribution of the free massless (pseudo)scalar field \( \vartheta(x) \). According to Witten’s criterion this testifies that the quanta of the free massless (pseudo)scalar field \( \vartheta(x) \) are Goldstone bosons. Since the ground state of a free massless (pseudo)scalar field \( \vartheta(x) \) obtained in [10] is not invariant under chiral symmetry transformations, the quanta of the free massless (pseudo)scalar field \( \vartheta(x) \) satisfy also Goldstone’s criterion for Goldstone bosons [16]. In Section 3 we show that the Ward identity and the low–energy theorem derived in Section 2 are consistent with the gap–equation defining the dynamical mass of the Thirring fermion fields quantized in the chirally broken phase [4]. In the Conclusion we discuss the obtained results.

2 Ward’s identity and low–energy theorem in the massless Thirring model

To clarify the problem whether the quanta of the free massless (pseudo)scalar field \( \vartheta(x) \), the quantum field theory of which bosonizes the massless Thirring model in the chirally broken phase [4, 5, 10], are Goldstone bosons we suggest to treat the following vacuum expectation value

\[
\langle 0 | T(j^5_5(x)\psi(y)\bar{\psi}(z)) | 0 \rangle_{\text{conn.}},
\]

where \( j^5_5(x) = \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x) \) is the axial–vector current, \( \psi(x) \) and \( \bar{\psi}(x) \) are massless Thirring fermion fields\(^1\). For the derivation of the Ward identity we need to write down explicitly the time–ordered product. It reads

\[
T(j^5_5(x)\psi_a(y)\bar{\psi}_b(z)) = \theta(x^0 - y^0)\theta(y^0 - z^0) j^5_5(x)\psi_a(y)\bar{\psi}_b(z)
- \theta(z^0 - x^0)\theta(x^0 - y^0) \bar{\psi}_b(z)j^5_5(x)\psi_a(y)
\]

\(^1\)The Dirac \( \gamma \)–matrices are defined by [4]: \( \gamma^\mu = (\gamma^0 = \sigma_1, \gamma^1 = -i\sigma_2) \) and \( \gamma^5 = \gamma^0\gamma^1 = \sigma_3 \), where \( \sigma_i (i = 1, 2, 3) \) are Pauli matrices.
The divergence of this time–ordered product is equal to
\[ \partial_\mu T(j_5^\mu(x)\psi_a(y)\bar{\psi}_b(z)) = T(\partial_\mu j_5^\mu(x)\psi_a(y)\bar{\psi}_b(z)) + \delta(y^0 - x^0) \ T([j_5^0(x), \psi_a(y)]\bar{\psi}_b(z)) \]
\[ + \delta(x^0 - y^0) \ T(\psi_a(y)[\bar{j}_5^0(x), \bar{\psi}_b(z)]). \tag{2.3} \]
This relation is valid for Thirring fermion fields quantized in both chiral symmetric and chirally broken phases.

The equal–time commutation relations can be calculated using the canonical anti–commutation relations
\[ \delta(x^0 - y^0) \{ \psi_a(x), \psi_b^\dagger(y) \} = \delta_{ab} \delta^{(2)}(x - y). \tag{2.4} \]
This gives
\[ \delta(x^0 - y^0)[j_5^0(x), \psi_a(y)] = -\delta^{(2)}(x - y) \ \gamma_5^{\psi_5} \psi_c(y) \]
\[ \delta(x^0 - z^0)[\bar{j}_5^0(x), \bar{\psi}_b(z)] = -\delta^{(2)}(x - z) \ \bar{\psi}_c(z) \gamma_5^{\psi_c}. \tag{2.5} \]
where we have used the relation \[ [AB,C] = A\{B,C\} - \{A,C\}B. \]
Substituting (2.5) in (2.3) we get
\[ \partial_\mu T(j_5^\mu(x)\psi_a(y)\bar{\psi}_b(z)) = T(\partial_\mu j_5^\mu(x)\psi_a(y)\bar{\psi}_b(z)) \]
\[ -\delta^{(2)}(x - y) \ T(\gamma_5^{\psi_5}\psi_c(y)\bar{\psi}_b(z)) - \delta^{(2)}(x - z) \ T(\psi_a(y)\bar{\psi}_c(z)\gamma_5^{\psi_c}). \tag{2.6} \]
The corresponding Ward identity reads
\[ \partial_\mu \langle 0|T(j_5^\mu(x)\psi_a(y)\bar{\psi}_b(z))|0\rangle_{\text{conn.}} = \langle 0|T(\partial_\mu j_5^\mu(x)\psi_a(y)\bar{\psi}_b(z))|0\rangle_{\text{conn.}} \]
\[ -\delta^{(2)}(x - y) \langle 0|T(\gamma_5^{\psi_5}\psi_c(y)\bar{\psi}_b(z))|0\rangle - \delta^{(2)}(x - z) \langle 0|T(\psi_a(y)\bar{\psi}_c(z)\gamma_5^{\psi_c})|0\rangle. \tag{2.7} \]
Integrating both sides over \( x \) we obtain
\[ \int d^2x \partial_\mu \langle 0|T(j_5^\mu(x)\psi_a(y)\bar{\psi}_b(z))|0\rangle_{\text{conn.}} = \int d^2x \langle 0|T(\partial_\mu j_5^\mu(x)\psi_a(y)\bar{\psi}_b(z))|0\rangle_{\text{conn.}} \]
\[ -\langle 0|T(\gamma_5^{\psi_5}\psi_c(y)\bar{\psi}_b(z))|0\rangle - \langle 0|T(\psi_a(y)\bar{\psi}_c(z)\gamma_5^{\psi_c})|0\rangle. \tag{2.8} \]
Assuming that the surface term does not give a contribution, the Ward identity (2.8) can be reduced to the form
\[ \int d^2x \langle 0|T(\partial_\mu j_5^\mu(x)\psi_a(y)\bar{\psi}_b(z))|0\rangle_{\text{conn.}} = \langle 0|T(\gamma_5^{\psi_5}\psi_c(y)\bar{\psi}_b(z))|0\rangle \]
\[ + \langle 0|T(\psi_a(y)\bar{\psi}_c(z)\gamma_5^{\psi_c})|0\rangle. \tag{2.9} \]
For the subsequent analysis it is convenient to set \( z = 0 \). This yields
\[ \int d^2x \langle 0|T(\partial_\mu j_5^\mu(x)\psi_a(y)\bar{\psi}_b(0))|0\rangle_{\text{conn.}} = \langle 0|T(\gamma_5^{\psi_5}\psi_c(y)\bar{\psi}_b(0))|0\rangle \]
\[ + \langle 0|T(\psi_a(y)\bar{\psi}_c(0)\gamma_5^{\psi_c})|0\rangle. \tag{2.10} \]
The vacuum expectation values in the r.h.s. of (2.10) are related to the two–point fermion Green function, which we define using the Källen–Lehmann representation

\[ \langle 0 | T(\psi_a(y)\bar{\psi}_b(0)) | 0 \rangle = \int \frac{d^2k}{(2\pi)^2} e^{-ik \cdot y} \int_0^\infty \frac{dm^2}{m^2 - k^2 - i0} \frac{\hat{k}_{ab} \rho_1(m^2) + \delta_{ab} \rho_2(m^2)}{m^2 - k^2 - i0}, \quad (2.11) \]

where \( \rho_1(m^2) \) and \( \rho_2(m^2) \) are Källen–Lehmann spectral functions. It is obvious that in the chiral symmetric phase \( \rho_2(m^2) = 0 \).

In terms of the Källen–Lehmann representation the r.h.s. of (2.10) can be written as

\[ \langle 0 | T(\gamma^5 \psi_c(y)\bar{\psi}_c(0)) | 0 \rangle + \langle 0 | T(\psi_a(y)\bar{\psi}_c(0)\gamma^5_{ab}) | 0 \rangle = 2\gamma^5_{ab} \int \frac{d^2k}{(2\pi)^2} e^{-ik \cdot y} \int_0^\infty \frac{dm^2}{m^2 - k^2 - i0} \{(\gamma^5 \hat{k} + \hat{k} \gamma^5)_{ab} \rho_1(m^2) + 2 \gamma^5_{ab} \rho_2(m^2)\} = 2\gamma^5_{ab} \int \frac{d^2k}{(2\pi)^2} e^{-ik \cdot y} \int_0^\infty \frac{dm^2}{m^2 - k^2 - i0} \rho_2(m^2). \quad (2.12) \]

Substituting (2.12) in (2.10), multiplying by \( e^{ip \cdot y} \) and integrating over \( y \) we get

\[ i \int d^2x d^2y e^{ip \cdot y} \langle 0 | T(\partial_{\mu} j^\mu_5(x)\psi_a(y)\bar{\psi}_b(0)) | 0 \rangle_c = 2\gamma^5_{ab} \int_0^\infty \frac{dm^2}{m^2 - p^2 - i0} \rho_2(m^2). \quad (2.13) \]

Now we suggest to multiply both sides of (2.13) by \( \gamma^5_{ba} \), to sum over \( a \) and \( b \) and to take the low–energy limit \( p \to 0 \). This gives the following low–energy theorem

\[ \frac{i}{4} \int d^2x d^2y \langle 0 | T(\partial_{\mu} j^\mu_5(x)\psi_a(y)\bar{\psi}_b(0)\gamma^5_{ba}) | 0 \rangle_{\text{conn.}} = \int_0^\infty \frac{dm^2}{m^2} \rho_2(m^2). \quad (2.14) \]

In the chiral symmetric phase when the axial–vector current is conserved, \( \partial_{\mu} j^\mu_5(x) = 0 \), and \( \rho_2(m^2) = 0 \) the low–energy theorem (2.14) reduces to a trivial equality \( 0 = 0 \).

Now let us consider the fulfillment of the low–energy theorem in the chirally broken phase. According to [4], in the chirally broken phase the Thirring fermion fields obey the equations of motion

\[ \gamma^\mu \partial_{\mu} \psi(x) = -iM e^{i\gamma^5 \beta \theta(x)} \psi(x), \]
\[ \partial_{\mu} \bar{\psi}(x) \gamma^\mu = +iM \bar{\psi}(x) e^{i\gamma^5 \beta \theta(x)}, \quad (2.15) \]

where \( M \) is a dynamical mass of the massless Thirring fermion fields quantized relative to the non–perturbative vacuum [4]. The coupling constant \( \beta \) is related to the coupling constant \( g \) of the massless Thirring model as [4]

\[ \frac{8\pi}{\beta^2} = 1 - e^{-2\pi/g}. \quad (2.16) \]

The divergence of the axial–vector current is equal to

\[ \partial_{\mu} j^\mu_5(x) = 2M \bar{\psi}(x)i\gamma^5 e^{i\gamma^5 \beta \theta(x)} \psi(x). \quad (2.17) \]
The vacuum expectation value in the l.h.s. of (2.14) we represent in terms of the generating functional of Green functions of the massless Thirring fermion fields [4]

\[-M \frac{1}{2} \int \int d^2x d^2y \left( \gamma^5 \exp \left\{ \gamma^5 \beta \frac{\delta}{\delta J(x)} \right\} \right) \delta \eta_c(x) \delta \eta_d(x) \delta \eta_a(y) \delta \eta_b(0) \]

\[\times Z[\eta, \bar{\eta}, J] \bigg|_{\eta = \bar{\eta} = J = 0} = \int_0^{\infty} \frac{dm^2}{m^2} \rho_2(m^2), \quad (2.18)\]

where \(Z[\eta, \bar{\eta}, J] \) is the generating functional of Green functions defined by [4, 5]

\[Z[\eta, \bar{\eta}, J] = \int \mathcal{D} \vartheta \exp \left\{ i \int d^2z \left[ \frac{1}{2} \partial_{\alpha} \vartheta(z) \partial^\alpha \vartheta(z) + \vartheta(z) J(z) \right] \right\} + i \int \int d^2z_1 d^2z_2 \bar{\eta}(z_1) S_F(z_1, z_2; \vartheta) \eta(z_2). \quad (2.19)\]

Here \(S_F(z_1, z_2; \vartheta)\) is the fermion Green function obeying the equation

\[\left( i \gamma^\alpha \frac{\partial}{\partial z_1^\alpha} - M e^{i \gamma^5 \beta \vartheta(z_1)} \right) S_F(z_1, z_2; \vartheta) = -\delta^{(2)}(z_1 - z_2), \quad (2.20)\]

and \(J(x)\) is an external source of the \(\vartheta\)–field satisfying the constraint [5]

\[\int d^2x J(x) = \tilde{J}(0) = 0. \quad (2.21)\]

Here \(\tilde{J}(0)\) is the Fourier transform of the external source \(J(x)\) at momentum zero, \(k = 0\). Due to the constraint (2.21) the collective zero–mode, describing the motion of the “center of mass” of the free massless (pseudo)scalar field \(\vartheta(x)\), does not contribute to correlation functions [5, 10]. This is important, since, as has been shown in [11], the collective zero–mode is responsible for the infrared divergences of the two–point Wightman functions in the quantum field theory of the free massless (pseudo)scalar field \(\vartheta(x)\).

Differentiating in (2.18) with respect to the external sources of fermion fields we obtain

\[M \frac{1}{2} \int \int d^2x d^2y \left\{ \text{tr} \left\{ S_F(y, x; \vartheta) \gamma^5 e^{i \gamma^5 \beta \vartheta(y)} S_F(x, 0; \vartheta) \gamma^5 \right\} \right\}_{\text{conn.}} = \int_0^{\infty} \frac{dm^2}{m^2} \rho_2(m^2). \quad (2.22)\]

Since the dynamical mass \(M\) is proportional to the ultra–violet cut–off \(\Lambda\) [4], we can neglect the contribution of the gradient term and use the Green function equal to

\[S_F(y, x; \vartheta) = \frac{1}{M} e^{-i \gamma^5 \beta \vartheta(y)} \delta^{(2)}(y - x). \quad (2.23)\]

Substituting (2.23) in (2.22) and calculating the trace over Dirac matrices we arrive at the low–energy theorem

\[\frac{1}{M} \langle \cos \beta \vartheta(0) \rangle = \int_0^{\infty} \frac{dm^2}{m^2} \rho_2(m^2). \quad (2.24)\]

Recall that the vacuum expectation value \(\langle \cos \beta \vartheta(0) \rangle\) has the meaning of a spontaneous magnetization [5]. It is related to the fermion condensate \(\langle \bar{\psi} \psi \rangle\) by [4, 5],

\[\langle \bar{\psi} \psi \rangle = -\frac{M}{g} \langle \cos \beta \vartheta(0) \rangle, \quad (2.25)\]
Thus, the low–energy theorem (2.24) defines the fermion condensate as
\[ \langle \bar{\psi} \psi \rangle = -\frac{M^2}{g} \int_0^\infty \frac{dm^2}{m^2} \rho_2(m^2). \] (2.26)

As has been shown in [5], \( \langle \bar{\psi} \psi \rangle = -\frac{M}{g} \), therefore we get
\[ \frac{1}{M} = \int_0^\infty \frac{dm^2}{m^2} \rho_2(m^2). \] (2.27)

This testifies that the quanta of the free massless (pseudo)scalar field \( \vartheta(x) \), describing the bosonized form of the massless Thirring model in the chirally broken phase, behave like Goldstone bosons and saturate the low–energy theorem defining the spontaneous magnetization and the fermion condensate.

In order to argue that the quanta of the massless (pseudo)scalar field \( \vartheta(x) \) satisfy Witten’s criterion for Goldstone bosons [13] we have to show that the low–energy theorem (2.27) agrees with the gap–equation defining the dynamical mass of the Thirring fermion fields in the chirally broken phase [4].

3 Gap–equation and low–energy theorem

As has been shown in [4], the dynamical mass of the Thirring fermion fields quantized in the chirally broken phase obeys the gap–equation
\[ M = M \frac{g}{2\pi} \frac{\Lambda^2}{M^2}, \] (3.1)
where \( \Lambda \) is an ultra–violet cut–off.

A consistency of the low–energy theorem (2.27) with the gap–equation should confirm the interpretation of the quanta of the free massless (pseudo)scalar field \( \vartheta(x) \) as Goldstone bosons.

In order to proof the agreement of the low–energy theorem (2.27) and the gap–equation (3.1) we suggest to turn to the analysis of the spontaneously broken chiral symmetry in the massless Thirring model in terms of the normal ordering [4]. The Lagrangian of the massless Thirring model reads [4]
\[ \mathcal{L}(x) = : \bar{\psi}(x)i\gamma^\nu \partial_\nu \psi(x) :_\mu - \frac{1}{2} g : \bar{\psi}(x)\gamma_\nu \psi(x) \bar{\psi}(x) \gamma^\nu \psi(x) :_\mu, \] (3.2)
where : \ldots : denotes the normal ordering at the infrared scale \( \mu \to 0 \). The Lagrangian (3.2) can be also rewritten as [4]
\[ \mathcal{L}(x) = : \bar{\psi}(x)i\gamma^\nu \partial_\nu \psi(x) :_\mu - \frac{1}{2} g : \bar{\psi}(x)\gamma_\nu \psi(x) :_\mu : \bar{\psi}(x) \gamma^\nu \psi(x) :_\mu. \] (3.3)

Changing the scale of the normal ordering from \( \mu \to 0 \) to \( M \) we get
\[ \mathcal{L}(x) = : \bar{\psi}(x)(i\gamma^\nu \partial_\nu - M)\psi(x) :_M + (M - g\gamma_\nu (-i)S_F(0)\gamma^\nu) : \bar{\psi}(x)\psi(x) :_M \]
\[ - \frac{1}{2} g : \bar{\psi}(x)\gamma_\nu \psi(x) \bar{\psi}(x) \gamma^\nu \psi(x) :_M, \] (3.4)
where the normal ordering should be carried out at the scale $M$; $S_F(0)$ is the total two-point Green function, which we define in the Källen–Lehmann representation (2.11). This yields

$$
\gamma_\nu(-i)S_F(0)\gamma^\nu = \int_0^\infty dm^2 \rho_2(m^2) \int \frac{d^2k}{2\pi^2 i} \frac{1}{m^2 - k^2 - i0} = \frac{1}{2\pi} \int_0^\infty dm^2 \rho_2(m^2) \ln \left( 1 + \frac{\Lambda^2}{m^2} \right).
$$

(3.5)

Substituting (3.5) in (3.4) we arrive at

$$
\mathcal{L}(x) = : \bar{\psi}(x)(i\gamma^\nu \partial_\nu - M)\psi(x) :_M 
+ \left[ M - \frac{g}{2\pi} \int_0^\infty dm^2 \rho_2(m^2) \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right] : \bar{\psi}(x)\psi(x) :_M

- \frac{1}{2} g : \bar{\psi}(x)\gamma_\nu \psi(x) \bar{\psi}(x)\gamma^\nu \psi(x) :_M.
$$

(3.6)

The scale $M$ acquires the meaning of a dynamical mass if it satisfies the gap–equation

$$
M - \frac{g}{2\pi} \int_0^\infty dm^2 \rho_2(m^2) \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) = 0.
$$

(3.7)

Due to the gap–equation (3.7) the Lagrangian (3.6) takes the form

$$
\mathcal{L}(x) = : \bar{\psi}(x)(i\gamma^\nu \partial_\nu - M)\psi(x) :_M - \frac{1}{2} g : \bar{\psi}(x)\gamma_\nu \psi(x) \bar{\psi}(x)\gamma^\nu \psi(x) :_M,
$$

(3.8)

where $M$ can be identified with the mass of the fermion fields.

The gap–equation (3.7) and the low–energy theorem (2.27) are consistent if the Källen–Lehmann spectral function $\rho_2(m^2)$ is equal to

$$
\rho_2(m^2) = M \delta(m^2 - M^2).
$$

(3.9)

In the chiral symmetric phase when $M = 0$ the spectral function $\rho_2(m^2)$ vanishes.

This completes the proof of the agreement of the low–energy theorem (2.27) (or more general (2.14)) and the gap–equation (3.1) for the dynamical mass of the Thirring fermion fields quantized in the chirally broken phase.

Thus, we can argue that the quanta of the free massless (pseudo)scalar field $\vartheta(x)$, bosonizing the massless Thirring model in the chirally broken phase [4, 5] and described by the quantum field theory without infrared divergences [5]–[11], satisfy Witten’s criterion for Goldstone bosons.

4 Conclusion

The quantum field theory of the free massless (pseudo)scalar field $\vartheta(x)$, described by the Lagrangian $\mathcal{L}(x) = \frac{1}{2} \partial_\mu \vartheta(x) \partial^\mu \vartheta(x)$, bosonizes the massless Thirring model with fermion fields quantized in the chirally broken phase. The main aim of this paper is to show that the quanta of this field satisfy Witten’s criterion for Goldstone bosons [13].
According to Witten [13], the quanta of the massless (pseudo)scalar field $\vartheta(x)$ should saturate the low–energy theorems. Following Witten’s requirement we have derived a low–energy theorem for the divergence of the axial–vector current. We have shown that in the chiral symmetric phase of the massless Thirring model, when the divergence of the axial–vector current vanishes, this low–energy theorem reduces to a trivial identity $0 = 0$. In turn, in the chirally broken phase with a non–vanishing divergence of the axial–vector current, the fulfillment of the low–energy theorem is fully caused by the contribution of the free massless (pseudo)scalar field $\vartheta(x)$ in terms of the spontaneous magnetization $\langle \cos \beta \vartheta(0) \rangle$. In the quantum field theory of the free massless (pseudo)scalar field $\vartheta(x)$ without infrared divergences this magnetization does not vanish, $\langle \cos \beta \vartheta(0) \rangle = 1$ [5–10]. This defines a non–vanishing fermion condensate in the massless Thirring model with fermion fields quantized in the chirally broken phase, $\langle \bar{\psi} \psi \rangle = -\langle \cos \beta \vartheta(0) \rangle M/g = -M/g [4, 5]$.

The consistency of the obtained results with the existence of the chirally broken phase of the massless Thirring model has been confirmed by the agreement of the low–energy theorem under consideration with the gap–equation, defining a dynamical mass of the Thirring fermion fields quantized in the chirally broken phase. Hence, one can conclude that the quanta of the free massless (pseudo)scalar field $\vartheta(x)$ satisfy Witten’s criterion for Goldstone bosons.

As has been shown in [10, 11], the wave function of the ground state of the free massless (pseudo)scalar field $\vartheta(x)$ is described by the bosonized BCS–type wave function of the ground state of the massless Thirring model in the chirally broken phase. This wave function is not invariant under chiral symmetry transformations [10, 11]. Such a non–invariance agrees with Goldstone’s criterion [16] allowing to interpret the quanta of the $\vartheta$–field as Goldstone bosons.

Thus, the quanta of the free massless (pseudo)scalar field $\vartheta(x)$, bosonizing the massless Thirring model with fermion fields quantized in the chirally broken phase [4–11], satisfy both Witten’s and Goldstone’s criteria for Goldstone bosons.

We would like to emphasize that this is not a counterexample to Coleman’s theorem [15]. Indeed, the quantum field theory of the free massless (pseudo)scalar field $\vartheta(x)$ in 1+1–dimensional space–time is equivalent to Wightman’s quantum field theory with Wightman’s observables defined on the test functions $h(x)$ from the Schwartz class $S_0(\mathbb{R}^2) = \{ h(x) \in S(\mathbb{R}^2); \tilde{h}(0) = 0 \} [9–11]$, whereas Coleman’s theorem [15] has been formulated for the quantum field theories in 1+1–dimensional space–time with Wightman’s observables defined on the test functions from the Schwartz class $S(\mathbb{R}^2)$.

References

[1] W. Thirring, Ann. Phys. (N.Y.) 3, 91 (1958).

[2] K. Johnson, Nuovo Cim. 20, 773 (1961); F. L. Scarf and J. Wess, Nuovo Cim. 26, 150 (1962); C. R. Hagen, Nuovo Cim. 51, 169 (1967); B. Klaiber, in LECTURES IN THEORETICAL PHYSICS, Lectures delivered at the Summer Institute for Theoretical Physics, University of Colorado, Boulder, 1967, edited by A. Barut and W. Brittin, Gordon and Breach, New York, 1968, Vol. X, part A, pp.141–176.
[3] M. Faber and A. N. Ivanov, *On the solution of the massless Thirring model with fermion fields quantized in the chiral symmetric phase*, hep–th/0112183.

[4] M. Faber and A. N. Ivanov, Eur. Phys. J. C **20**, 723 (2001).

[5] M. Faber and A. N. Ivanov, Eur. Phys. J. C **24**, 653 (2002).

[6] M. Faber and A. N. Ivanov, *Comments no Coleman’s paper “There are no Goldstone Bosons in Two Dimensions”*, hep–th/0204237.

[7] M. Faber and A. N. Ivanov, *Is the energy of the ground state of the sine–Gordon model unbounded from below for $\beta^2 > 8\pi$?*, hep–th/0205249, 2002.

[8] M. Faber and A. N. Ivanov, *Massless Thirring fermion fields in the boson field representation*, hep–th/0206034, 2002.

[9] M. Faber and A. N. Ivanov, *Quantum field theory of a free massless (pseudo)scalar field in 1+1–dimensional space–time as a test for the massless Thirring model*, hep–th/0206244, 2002.

[10] M. Faber and A. N. Ivanov, *Bosonic vacuum wave functions from the BCS–type wave function of the ground state of the massless Thirring model*, hep–th/0210104 (to appear in Physics Letters B).

[11] M. Faber and A. N. Ivanov, *On the ground state of the massless (pseudo)scalar field in two dimensions*, hep–th/0212226, 2002.

[12] C. Itzykson and J.–B. Zuber, in *QUANTUM FIELD THEORY*, McGraw–Hill Book Company, New York, 1980, p.521.

[13] E. Witten, Nucl. Phys. B **145**, 110 (1978).

[14] A. S. Wightman, *Introduction to Some Aspects of the Relativistic Dynamics of Quantized Fields*, in *HIGH ENERGY ELECTROMAGNETIC INTERACTIONS AND FIELD THEORY*, Cargèse Lectures in Theoretical Physics, edited by M. Levy, 1964, Gordon and Breach, 1967, pp.171–291; R. F. Streater and A. S. Wightman, in *PCT, SPIN AND STATISTICS, AND ALL THAT*, Princeton University Press, Princeton and Oxford, Third Edition, 1980.

[15] S. Coleman, Comm. Math. Phys. **31**, 259 (1973).

[16] J. Goldstone, Nuovo Cimento **19**, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962); (see [12] pp.519–526).

[17] (see [12] p.214).