SOME TERNARY CUBIC TWO-WEIGHT CODES

MINJIA SHI
School of Mathematical Sciences, Anhui University, Anhui, 230601, P. R. China, National Mobile Communications Research Laboratory, Southeast University, 210096, Nanjing, P. R. China

DAITAO HUANG
School of Mathematical Sciences, Anhui University, Anhui, 230601, P. R. China

PATRICK SOLÉ
CNRS/LAGA, University Paris 8, 93 526 Saint-Denis, France

Abstract. We study trace codes with defining set $L$, a subgroup of the multiplicative group of an extension of degree $m$ of the alphabet ring $\mathbb{F}_3 + u\mathbb{F}_3 + u^2\mathbb{F}_3$, with $u^3 = 1$. These codes are abelian, and their ternary images are quasi-cyclic of co-index three (a.k.a. cubic codes). Their Lee weight distributions are computed by using Gauss sums. These codes have three nonzero weights when $m$ is singly-even and $|L| = \frac{3^{3m} - 3^{2m}}{2}$. When $m$ is odd, and $|L| = \frac{3^{3m} - 3^{2m}}{2}$, or $|L| = 3^{3m} - 3^{2m}$ and $m$ is a positive integer, we obtain two new infinite families of two-weight codes which are optimal. Applications of the image codes to secret sharing schemes are also given.

Keywords: three-weight codes, two-weight codes, Gauss sums, trace codes

1. Introduction

Linear codes with few weights are of special interest in secret sharing schemes, as the associated access structures can be completely determined analytically [12, 21]. Due to their connections with strongly regular graphs, association schemes [6] and difference sets [3, 5], two-weight codes and three-weight codes have been studied in many articles, see for instance [7, 8, 9, 13, 22]. However, most constructions, so far, have used cyclic codes [9, 22].

A recent study introduced trace codes over rings. In a series of papers [17, 18, 20], the authors have extended the notion of trace codes from fields to rings. The

1991 Mathematics Subject Classification. Primary: 94 B25; Secondary: 05 E30.

The first author is supported by NNSF of China (61672036), Technology Foundation for Selected Overseas Chinese Scholar, Ministry of Personnel of China (05015133) and the Open Research Fund of National Mobile Communications Research Laboratory, Southeast University (2015D11) and Key projects of support program for outstanding young talents in Colleges and Universities (gxyqZD2016008).

** Corresponding author.
alphabet ring we consider here is \( R = \mathbb{F}_3 + u\mathbb{F}_3 + u^2\mathbb{F}_3 \) with \( u^3 = 1 \). This ring is instrumental in constructing quasi-cyclic codes of co-index 3 \([2, 14]\), called cubic codes in \([2]\). While the construction we use here is not exactly the cubic construction of \([14]\), the ternary codes we construct here are still cubic. In \([19]\), the authors constructed optimal ternary codes from one-Lee-weight codes and two-Lee-weight codes over the ring \( R \).

In the present paper, we construct a trace code with a defining set

\[
L = \{d_1, d_2, \ldots, d_n\} \subseteq \mathbb{R}^*,
\]

by the formula \( C_L = \{(\text{Tr}(ax))_{a \in L} \mid a \in \mathbb{R}\} \), where \( \mathbb{R} \) denotes an extension of \( R \) of degree \( m \), and \( \text{Tr}() \) denotes a linear function from \( \mathbb{R} \) down to \( R \). This code has two or three weights, depending on the choice of the parameters \( L \) and \( m \). By varying \( L \) and \( m \), various codes can be constructed. Compared with the linear codes in \([19]\), the two-weight codes we construct here are different, and the method is also different. The localizing set of our abelian code is not a cyclic group, but it is an abelian group. It is related to quadratic residues in an extension of degree \( m \) of \( \mathbb{F}_3 \), which makes quadratic Gauss sums appear naturally in the weight distribution analysis. When \( m \) is odd, \( L = L' \), or \( L = \mathbb{R}^* \) and \( m \) is an integer, we obtain an infinite family of two-weight codes which satisfy the optimality. We show that, both in the three-weight and two-weight cases, the ternary image has a very nice support structure of the ternary image and describes an application to secret sharing schemes. Section 9 concludes this paper.

The paper is organized as follows. Section 2 collects the basic notions and notations needed. Section 3 shows that the trace codes are abelian. Section 4 recalls and reproves some results on Gaussian periods. Section 5 computes the weight distributions of our codes, building on the character sum evaluation of the preceding section. Section 6 determines the optimality of the ternary linear codes. The minimum distance of the dual codes is discussed in Section 7. Section 8 determines the support structure of the ternary image and describes an application to secret sharing schemes. Section 9 concludes this paper.

### 2. Basic notions and notation

#### 2.1. Rings

We consider the ring \( R = \mathbb{F}_3 + u\mathbb{F}_3 + u^2\mathbb{F}_3 \) with \( u^3 = 1 \). Note that, by Fermat little theorem, \( u^3 - 1 = (u - 1)^3 \). This implies that \( R \) is a local ring with the following lattice of ideals:

\[
0 \subseteq (1 + u + u^2) = \{0, 1 + u + u^2, 2 + 2u + 2u^2\} \subseteq \langle u - 1 \rangle = \{(u - 1)a : a \in \mathbb{R}\} \subseteq R.
\]

Hence, \( \langle u - 1 \rangle \) is the unique maximum ideal of \( R \). Given a positive integer \( m \), we can construct the ring extension of \( R \) of degree \( m \) given by \( \mathcal{R} = \mathbb{F}_3^m + u\mathbb{F}_3^m + u^2\mathbb{F}_3^m \). There is a Frobenius operator \( F \) which maps \( a + ub + u^2c \) onto \( a^3 + ub^3 + u^2c^3 \), for all \( a, b, c \in \mathbb{F}_3^m \). The Trace function, denoted by \( \text{Tr} \), is defined as

\[
\text{Tr} = \sum_{j=0}^{m-1} F^j.
\]

It is easy to check that

\[
\text{Tr}(a + ub + u^2c) = \text{tr}(a) + utr(b) + u^2tr(c)
\]

for \( a, b, c \in \mathbb{F}_3^m \). Here \( tr() \) denotes the standard trace of \( \mathbb{F}_3^m \).
The ring $\mathcal{R}$ is local with maximal ideal $M = \langle u - 1 \rangle$ and $\mathcal{R}/M \cong \mathbb{F}_{3^m}$. The group of units $\mathcal{R}^* = \mathbb{F}_{3^m}^\times \times \mathbb{F}_{3^m}^\times \times \mathbb{F}_{3^m}^\times$, as a multiplicative group, is isomorphic to the product of a cyclic group of order $3^m - 1$ by two elementary abelian groups of order $3^m$. Denoting by $\mathcal{Q}$, and $\mathcal{N}$, respectively, the squares and the nonsquares of $\mathbb{F}_{3^m}$. For simplicity, $L' = \mathcal{Q} \times \mathbb{F}_{3^m} \times \mathbb{F}_{3^m}$. Thus $L'$ is a subgroup of $\mathcal{R}^*$, of index 2.

2.2. Gray map. The Gray map $\phi$ from $R$ to $\mathbb{F}_3^n$ is defined by
\[ \phi(a' + ub + u^2c') = (a', b', c'), \]
for $a', b', c' \in \mathbb{F}_3$. It is a one to one map from $R$ to $\mathbb{F}_3^n$. The Lee weight of a vector $a + ub + u^2c$ is defined as the Hamming weight of its Gray image. That is to say,
\[ w_L(a + ub + u^2c) = w_H(a) + w_H(b) + w_H(c), \]
for $a, b, c \in \mathbb{F}_3^n$. The Lee distance of $x, y \in \mathbb{F}_3^n$ is defined as $w_L(x - y)$. So the Gray map is, by construction, a linear isometry from $(\mathbb{R}^n, d_L)$ to $(\mathbb{F}_3^n, d_H)$, where $d_L, d_H$ means Lee distance and Hamming distance, respectively. For simplicity, we let throughout $N = 3n$. For future use, we note that scalars of weight one in $R$ comprise $au^2$, $a \in \mathbb{F}_3^*$, $j = 0, 1, 2$.

2.3. Codes. A linear code $C$ over $R$ of length $n$ is an $R$-submodule of $\mathbb{R}^n$. If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are two elements of $\mathbb{R}^n$, their Euclidean inner product is defined by $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$, where the operation is performed in $R$. The dual code of $C$ is denoted as $C^\perp = \{ y \in \mathbb{R}^n | \langle x, y \rangle = 0, \forall x \in C \}$. By definition, $C^\perp$ is also a linear code over $R$. Given a finite abelian group $G$, a code over $R$ is said to be abelian if it is an ideal of the group ring $R[G]$. Namely, the coordinates of $C$ are indexed by elements of $G$ and $G$ acts regularly on this set. In the special case when $G$ is cyclic, the code is a cyclic code in the usual sense [15]. A code of length $\ell m$, is said to be quasi-cyclic of index $\ell$ and co-index $m$, if it is linear and invariant under a shift of $\ell$ places. In particular, if the co-index is equal to three the code is said to be cubic [2].

3. Symmetry

For $a \in \mathcal{R}$, the vector $ev(a)$ is defined by the following evaluation map
\[ ev(a) = (Tr(ax))_{x \in L}, \]
where $L = L'$ or $L = \mathcal{R}^*$.

Define the code $C(m)$ by the formula $C(m) = \{ ev(a) | a \in \mathcal{R} \}$. Thus $C(m)$ is a code of length $|L|$ over $R$. Note that $|L'| = \frac{3^m - 2^m}{2}$ and $|\mathcal{R}^*| = 3^m - 3^m$.

Lemma 3.1. If for all $x \in L$, we have $Tr(ax) = 0$, then $a = 0$.

Proof. When $L = L'$, write $x = x_1 + x_2(u - 1) + x_3(u - 1)^2$ and $a = a_1 + a_2(u - 1) + a_3(u - 1)^2$ with $x_1 \in \mathcal{Q}, x_2, x_3, a_1, a_2, a_3$ in $\mathbb{F}_{3^m}$. By a simple calculation we get
\[ ax = a_1 x_1 - a_1 x_2 + a_1 x_3 - a_2 x_2 + a_2 x_1 + a_2 x_3 + a_2 x_2 + a_3 x_1 \]
\[ + (u - 1)(a_1 x_2 + a_1 x_3 + a_2 x_1 + a_2 x_2 + a_3 x_1) \]
\[ + (u - 1)^2(a_1 x_3 + a_2 x_2 + a_3 x_1) \]
\[ =: A_1 + (u - 1) A_2 + (u - 1)^2 A_2 \]
and $Tr(ax) = 0$ is equivalent to $tr(A_i) = 0$, where $i = 1, 2, 3$. The trace function $tr()$ [15] is nondegenerate, so we have $a_j = 0$ where $j = 1, 2, 3$. Consequently, $a = 0$. The case of $L = \mathcal{R}^*$ is similar to that of $L = L'$. Thus the proof is proved. □
Proposition 3.2. The subgroup $L$ of the group of units $\mathcal{R}^*$ acts regularly on the coordinates of $C(m)$.

Proof. For any $v,v' \in L$, the change of variables $x \mapsto (v'/v)x$ permutes the coordinates of $C(m)$, and maps $v$ to $v'$. This defines a transitive action of $L$ on itself. Such a permutation is unique, given $v, v'$. This shows that the action is regular. $\square$

The code $C(m)$ is thus an abelian code with respect to the group $L$. In other words, it is an ideal of the group ring $\mathcal{R}[L]$. As observed in the previous section that neither $\mathcal{R}^*$, nor $L'$ are a cyclic group, hence $C(m)$ may be not cyclic.

It is immediate to see that, by the definition of Gray map, the ternary code $\phi(C(m))$ is quasi-cyclic of co-index 3.

4. Character sums

This section is similar to that in [20]. For completeness, we collect it here. Let $\chi$ denote an arbitrary multiplicative character of $F_q$. Assume $q$ is odd. Denote by $\eta$ the quadratic multiplicative character defined by $\eta(x) = 1$, if $x$ is a square and $\eta(x) = -1$, if not. Let $\psi$ denote the canonical additive character of $F_q$. The classical Gauss sum can be defined by

$$G(\chi) = \sum_{x \in \mathbb{F}_q^*} \psi(x)\chi(x).$$

Now, we give the following character sums

$$\overline{Q} = \sum_{x \in Q} \psi(x), \quad \overline{N} = \sum_{x \in N} \psi(x).$$

By orthogonality of characters [15, Lemma 9, p. 143], it is not difficult to check that $\overline{Q} + \overline{N} = -1$. Since the characteristic function of $Q$ is $\frac{1+\eta}{2}$, we obtain then

$$\overline{Q} = \frac{G(\eta) - 1}{2}, \quad \overline{N} = \frac{-G(\eta) - 1}{2}.$$

It is well known [11], that if $q = 3^m$, the quadratic Gauss sum evaluates as

$$G(\eta) = (-1)^{m-1}i^m \sqrt{q}.$$  

In particular, if $m$ is singly-even, these formulas can be simplified to $G(\eta) = \epsilon(3)\sqrt{q}$, with $\epsilon(3) = (-1)^{\frac{m+1}{2}} = 1$, which implies

$$\overline{Q} = \frac{\sqrt{q} - 1}{2}, \quad \overline{N} = -\frac{\sqrt{q} + 1}{2}.$$  

In fact $\overline{Q}$ and $\overline{N}$ are examples of Gaussian periods, and these relations could have been deduced from [11, Lemma 11].

5. Weight distributions of trace codes

Before we calculate the weight distribution of the trace code, let us first introduce a correlation lemma. Let $\omega = \exp(\frac{2\pi i}{3})$. If $y = (y_1, y_2, \ldots, y_N) \in \mathbb{F}_3^N$, let

$$\Theta(y) = \sum_{j=1}^{N} \omega^{y_j}.$$  

For convenience, we let $\theta(a) = \Theta(\phi(ev(a)))$. By linearity of the Gray map, and of the evaluation map, we see that $\theta(sa) = \Theta(\phi(ev(sa)))$ for any $s \in \mathbb{F}_3^*$.  

4 MINJIA SHI, DAITAO HUANG, AND PATRICK SÓLÉ
Lemma 5.1 ([20], Lemma 1). For all \( y = (y_1, y_2, \cdots, y_N) \in \mathbb{F}_3^N \), we have
\[
\sum_{s=1}^{2} \Theta(sy) = 2N - 3w_H(y).
\]

5.1. The case when \( L = L' \). From Lemma 5.1, for \( ev(a) \in C(m) \), by the definition of the Gray map, we have
\[
w_L(ev(a)) = \frac{2N - \sum_{s=1}^{2} \Theta(\phi(ev(a)))}{3} = \frac{2N - \sum_{s=1}^{2} \theta(sa)}{3}.
\]
According to the value of \( m \), we can obtain ternary codes with different weights. Now we present the weights of codewords of \( C(m) \) by using Equation (1). The following theorem tells us that when \( L = L' \) and \( m \) is singly-even, \( \phi(C(m)) \) is a three-weight ternary linear code.

5.1.1. \( m \) is singly-even.

Theorem 5.2. Assume \( m \) is singly-even. For \( a \in \mathcal{R} \), the Lee weight of codewords of \( C(m) \) is given below:
\begin{enumerate}
  \item If \( a = 0 \), then \( w_L(ev(a)) = 0 \);
  \item If \( a = (u-1)^2a_3 \), then if
    \begin{enumerate}
      \item \( a_3 \in \mathbb{Q} \), then \( w_L(ev(a)) = 3^{3m} - 3^{5m/2} \);
      \item \( a_3 \in \mathcal{N} \), then \( w_L(ev(a)) = 3^{3m} + 3^{5m/2} \);
    \end{enumerate}
  \item If \( a \in \mathcal{R} \setminus ((u-1)^2) \), then \( w_L(ev(a)) = 3^{3m} - 3^{2m} \).
\end{enumerate}

Proof. Let \( a = a_1 + a_2(u-1) + a_3(u-1)^2 \) with \( a_1, a_2, a_3 \in \mathbb{F}_3^m \), \( x = x_1 + x_2(u-1) + x_3(u-1)^2 \) with \( x_1 \in \mathbb{Q} \), \( x_2, x_3 \in \mathbb{F}_3^m \), by a direct calculation we get
\[
ax = a_1x_1 - a_1x_2 + a_1x_3 - a_2x_1 + a_2x_2 + a_3x_1 + (a_1x_2 + a_1x_3 + a_2x_2 + a_2x_3 + a_3x_1)u + (a_1x_3 + a_2x_2 + a_3x_1)u^2 =: b_1 + b_2u + b_3u^2.
\]
Thus we have
\[
\phi(ev(\alpha)) = (tr(b_1), tr(b_2), tr(b_3))x_1, x_2, x_3, \\
\theta(\alpha) = \sum_{x_1, x_2, x_3} \omega^{tr(b_1)} + \sum_{x_1, x_2, x_3} \omega^{tr(b_2)} + \sum_{x_1, x_2, x_3} \omega^{tr(b_3)}.
\]
Since \( m \) is singly-even, \( s \in \mathbb{F}_3^* \) is a square in \( \mathbb{F}_3^m \), we have \( \theta(sa) = \theta(a) \) for any \( s \in \mathbb{F}_3^* \).
\begin{enumerate}
  \item If \( a = 0 \), then \( Tr(ax) = 0 \). So \( w_L(ev(a)) = 0 \).
  \item When \( a = (u-1)^2a_3 \) with \( a_3 \in \mathbb{Q} \), then \( \theta(a) = 3^{2m+1}Q \). Thus \( w_L(ev(a)) = 3^{3m} - 3^{5m/2} \) by Equation (1).
    When \( a = (u-1)^2a_3 \) with \( a_3 \in \mathcal{N} \), \( \theta(a) = 3^{2m+1}Q \). Then we deduce from the Equation (1) that \( w_L(ev(a)) = 3^{3m} + 3^{5m/2} \).
  \item When \( a \in \mathcal{R} \setminus ((u-1)^2) \), \( \theta(a) = 0 \). Then we have \( w_L(ev(a)) = 3^{3m} - 3^{2m} \) by Equation (1).
\end{enumerate}

Remark 5.1 By the point of Lemma 3.1 and Theorem 5.2, a family of ternary linear code of length \( N = (3^{3m+1} - 3^{2m+1})/2 \), dimension \( 3m \), with three nonzero weights \( w_1 < w_2 < w_3 \) of values has been constructed. In detail, we list the weight distribution of \( \phi(C_m) \) in Table I. Notice that the parameters are different from those
in [7, 9, 20]. Thus, the obtained code in Theorem 5.2 are new.

Table I. weight distribution of $\phi(C_m)$

| Weight | Frequency |
|--------|-----------|
| $0$    | $1$       |
| $w_1 = 3^{3m} - 3^{\frac{5m}{2}}$ | $f_1 = \frac{3^{m-1}}{2}$ |
| $w_2 = 3^{3m} - 3^{2m}$ | $f_2 = 3^{5m} - 3^m$ |
| $w_3 = 3^{3m} + 3^{3m/2}$ | $f_3 = 3^{m-1}$ |

Example 5.1 Let $m = 2$. We obtain a ternary code of parameters [972, 6, 486]. The nonzero weights are 486, 648 and 972, of frequencies 4, 720 and 4, respectively.

5.1.2. $m$ is odd. Note that $G(\eta)$ is imaginary, which implies that $\Re(\overline{\eta}) = \Re(\eta) = -\frac{1}{2}$. The following lemma is a special case of Lemma 2 in [20].

Lemma 5.3 ([20], Lemma 2). \(\sum_{s=1}^{2} \theta(sa) = 2\Re(\theta(a))\).

By a similar approach in the proof of Theorem 5.2 and combining Lemma 5.3, it is not difficult to obtain the following theorem.

Theorem 5.4. Assume $m$ is odd. For $a \in \mathcal{R}$, the Lee weight of codewords of $C(m)$ is given below:

(a) If $a = 0$, then $w_L(ev(a)) = 0$;
(b) If $a = (u - 1)^2a_3$, with $a_3 \in \mathbb{F}_{3}^{m}$, then $w_L(ev(a)) = 3^{3m}$;
(c) If $a \in \mathcal{R} \setminus \langle (u - 1)^2 \rangle$, then $w_L(ev(a)) = 3^{3m} - 3^{2m}$.

Proof. The proofs of the cases (a) and (c) are similar to that of Theorem 5.2. The case (b) follows from Lemma 5.3 applied to the correlation lemma. Thus $\Re(\theta(a)) = -3^{2m+1}/2$, and $3w_L(ev(a)) = 2N - 2\Re(\theta(a))$, which implies $w_L(ev(a)) = 3^{3m}$. The result follows. \(\square\)

Example 5.2 Let $m = 1$. We obtain a ternary code of parameters [27, 3, 18]. The nonzero weights are 18 and 27, of frequencies 24 and 2, respectively.

5.2. The case when $L = \mathcal{R}^*$. In this subsection, we will construct an infinite family of two-weight codes on the condition that $L = \mathcal{R}^*$.

Theorem 5.5. For $a \in \mathcal{R}$, the Lee weight of codewords of $C(m)$ is given below:

(a) If $a = 0$, then $w_L(ev(a)) = 0$;
(b) If $a = (u - 1)^2a_3$, with $a_3 \in \mathbb{F}_{3}^{m}$, then $w_L(ev(a)) = 2 \cdot 3^{2m}$;
(c) If $a \in \mathcal{R} \setminus \langle (u - 1)^2 \rangle$, then $w_L(ev(a)) = 2(3^{3m} - 3^{2m})$.

Proof. By a similar approach in the proof of Theorem 5.2, the result follows, so we omit the details here. \(\square\)

Example 5.3 Let $m = 1$. We obtain a ternary code of parameters [54, 3, 36]. The nonzero weights are 36 and 54, of frequencies 24 and 2, respectively.

Example 5.4 Let $m = 2$. We obtain a ternary code of parameters [1944, 6, 1296]. The nonzero weights are 1296 and 1458, of frequencies 720 and 8, respectively.

Remark 5.2 Comparing with [4, 19, 20], we obtain two new families of ternary two-weight codes based on the obtained trace codes over $R$ in Theorem 5.4 and Theorem 5.5. The weight distribution is listed in Table II.
Table II. weight distribution of $\phi(C_m)$ from Theorems 5.4 and 5.5

| Weight  | Frequency |
|---------|-----------|
| $0$     | 1         |
| $w'_1 = 3^{3m} - 3^{2m}$ | $f'_1 = 3^{3m} - 3^m$ |
| $w'_2 = 3^{3m}$ | $f'_2 = 3^m - 1$ |
| $0$     | 1         |
| $w''_1 = 2(3^{3m} - 3^{2m})$ | $f''_1 = 3^{3m} - 3^m$ |
| $w''_2 = 2 \cdot 3^{3m}$ | $f''_2 = 3^m - 1$ |

6. Optimality of the image codes

In the previous section, we have constructed two new infinite families of ternary two-weight codes and a new family of three-weight ternary codes. Now we study their optimality.

**Theorem 6.1.** The image codes $\phi(C_m)$ of length $3|L|$ are optimal based on the following cases

(i) $m$ is odd and $L = L'$ in Theorem 5.4;

(ii) $m$ is a positive integer and $L = R^*$ in Theorem 5.5.

**Proof.** Recall the 3-ary version of the Griesmer bound. If $[N, K, d]$ are the parameters of a linear ternary code, then

$$K - 1 \sum_{j=0}^{K-1} \left\lceil \frac{d}{3^j} \right\rceil \leq N.$$ 

In the case of (i), $N = (3^{3m+1} - 3^{2m+1})/2$, $K = 3m$, $d = 3^{3m} - 3^{2m}$. The ceiling function takes two values depending on the position of $j$.

- $0 \leq j \leq 2m \Rightarrow \left\lceil \frac{d+1}{3^j} \right\rceil = 3^{3m-j} - 3^{2m-j} + 1$.
- $2m < j \leq 3m - 1 \Rightarrow \left\lceil \frac{d+1}{3^j} \right\rceil = 3^{3m-j}$.

$$\sum_{j=0}^{K-1} \left\lceil \frac{d+1}{3^j} \right\rceil = 2m (3^{3m-j} - 3^{2m-j} + 1) + \sum_{j=2m+1}^{3m-1} 3^{3m-j}$$

$$= (3^{3m+1} - 3^{2m+1})/2 + 2m - 1 > N,$$ which collapses to $m \geq 1$. Thus, this completed the proof of the case (i).

For case (ii), we have $N = (3^{3m+1} - 3^{2m+1})$, $K = 3m$, $d = 2(3^{3m} - 3^{2m})$. By a simple calculation, we can easily obtain that

$$\sum_{j=0}^{K-1} \left\lceil \frac{d+1}{3^j} \right\rceil = 2m (2(3^{3m-j} - 3^{2m-j}) + 1) + \sum_{j=2m+1}^{3m-1} 2 \cdot 3^{3m-j}$$

$$= 3^{3m+1} - 3^{2m+1} + 2m - 1 > N,$$ which collapses to $m \geq 1$. $\square$

7. The dual Lee distance of the trace code

Similar to Lemma 3 and Theorem 4 in [20], we can calculate the dual distance of the obtained trace code. We still prove them for completeness.

**Lemma 7.1.** If for all $a \in R$, we have that $Tr(ax) = 0$, then $x = 0$.

**Proof.** The proof is similar to that of Lemma 3.1, we omit it here. $\square$
Theorem 7.2. For all \( m \geq 1 \), the dual Lee distance \( d' \) of \( C(m) \) is 2.

Proof. First, we check that \( d' \geq 2 \). The approach is the same as Theorem 7.2 in [20]. Now we will prove it by showing that \( C(m)^+ \) does not contain a codeword of Lee weight one. If it does, let us assume first that it has value \( \alpha u \neq 0 \) at some \( x \in L \), where \( j = 0, 1, 2 \). This implies that \( \forall a \in R, \alpha u' Tr(ax) = 0 \), or, \( Tr(\alpha u' x) = 0 \), and by Lemma 7.1, we have \( x = 0 \), which contradicts the assumption. So \( d' \geq 2 \).

Next, we show that \( d' < 3 \). If not, we can apply the sphere-packing bound to \( \phi(C(m)^+) \). Naturally, we obtain
\[
3^{3m} \geq 1 + |L|(3 - 1),
\]
where \( |L| = |L'| \) or \( |L| = |R^*| \). That implies \( (3 - 2 \cdot 3^m)3^{2m} \geq 1 \) or \( 3^{2m+1}(2 - 5 \cdot 3^{m-1}) \geq 1 \), respectively. It is a contradiction when \( m \geq 1 \). In summary, \( d = 2 \). □

8. Application to secret sharing schemes

8.1. Determining minimal vectors. It is interesting to determine minimal vectors of a given \( p \)-ary linear code. Minimal vectors in linear codes arise in numerous applications, particularly, in studying linear secret sharing schemes (SSS). We say that a minimal vector of a linear code \( C \) is a nonzero codeword that does not cover any other nonzero codeword. The basic property minimal vector of a given \( p \)-ary linear code was described by the following lemma [1].

Lemma 8.1. (Ashikhmin-Barg) Denote by \( w_0 \) and \( w_\infty \) the minimum and maximum nonzero weights of a \( p \)-ary linear code \( C \), respectively. If
\[
\frac{w_0}{w_\infty} > \frac{p - 1}{p},
\]
then every nonzero codeword of \( C \) is minimal.

Next, under Lemma 8.1, we investigate the minimal vectors of the three-weight and two-weight codes which constructed in Section 5.

Proposition 8.2. All the nonzero codewords of the image codes \( \phi(C(m)) \) are minimal based on the following three cases

(i) \( m(\geq 6) \) is singly-even and \( L = L' \) in Theorem 5.2.

(ii) \( m(\geq 1) \) is odd and \( L = L' \) in Theorem 5.4.

(iii) \( m \) is a positive integer and \( L = R^* \) in Theorem 5.5.

Proof. In the case of (1), by the preceding lemma with \( w_0 = w_1 \), and \( w_\infty = w_3 \). Rewriting the inequality of the lemma as \( 3w_1 > (3 - 1)w_3 \), we end up with the condition
\[
3 \cdot 3^{3m} - 3 \cdot 3^{5m/2} > 2 \cdot 3^{3m} + 2 \cdot 3^{5m/2}.
\]
Since \( m \) is singly-even. The condition follows from the fact that \( 3^{m/2} \geq 5 \).

The proof of other cases are similar to those of the case (1), we omit them here. □
8.2. Secret sharing schemes. The purpose of determining minimal vectors is to determine the sets of all minimal access sets of a secret sharing scheme (SSS). An SSS were first introduced by Blakly and Shamir at the end of 70s twentieth antury. Massey’s scheme is one of the famous SSS. Massey’s scheme is a construction of such a scheme where a code $C$ of length $N$ over $\mathbb{F}_p$. On the other hand, it is worth mentioning when all nonzero codewords are minimal, it was shown in [10] that there is the following alternative, depending on $d'$:

- If $d' \geq 3$, then the SSS is “democratic”: every user belongs to the same number of coalitions.
- If $d' = 2$, then there are users who belong to every coalition: the “dictators”.

Depending on the application, one or the other situation might be more suitable.

By Proposition 8.2 and Theorem 7.2, we see that three Secret Sharing Schemes built on the image codes $\phi(C(m))$ in Theorems 5.2, 5.4 and 5.5 are dictatorial.

9. Conclusion

This paper is devoted to the study of trace codes with defining set $L$ included in an extension of degree $m$ of the alphabet, the local ring $\mathbb{F}_3 + u\mathbb{F}_3 + u^2\mathbb{F}_3$ with $u^3 = 1$. These codes are abelian, and their ternary images are quasi-cyclic of co-index three (a.k.a. cubic codes). Their Lee weight distributions are computed by using Gauss sums (see Table I and Table II). These codes have three nonzero weights when $m$ is singly-even and $|L| = \frac{3^m - 3^2m}{2}$. When $m$ is odd, and $|L| = \frac{3^m - 3^2m}{2}$, or $|L| = 3^m - 3^2m$ and $m$ is a positive integer, we obtain two new infinite family of two-weight codes. Both are shown to be optimal, by application of the Griesmer bound. Applications of the image codes to secret sharing schemes are also given.

It would be interesting to replace our Gaussian periods $Q, N$ by other character sums that are amenable to exact evaluation, in the vein of the sums which appear in the study of irreducible cyclic codes [11, 16]. This would lead to other enumerative results of codes with few weights. The parameters of the two-weight codes we constructed are different from those in the classic paper [4], and also from more recent constructions like [12, 17, 18]. Writing a new survey on two-weight codes seems like a challenging but very useful project.

10. Acknowledgement

This research is supported by National Natural Science Foundation of China (61672036), the Open Research Fund of National Mobile Communications Research Laboratory, Southeast University (2015D11), Technology Foundation for Selected Overseas Chinese Scholar, Ministry of Personnel of China (05015133) and Key Projects of Support Program for outstanding young talents in Colleges and Universities (gxyqZD2016008).

References

[1] Ashikhmin, A., Barg, A.: Minimal vectors in linear codes, IEEE Transactions on Information Theory, 44, 2010–2017 (1998)
[2] Bonnecaze, A., Bracco, A.D., Dougherty, S.T., Nochefranca, L.R., Solé, P.: Cubic self-dual binary codes, IEEE Transactions on Information Theory, 49, 2253–2259 (2003)
[3] Calderbank, A.R., Goethals, J.M.: Three weight codes and association schemes, Philips J. Res. 39, 143–152 (1984)
[4] Calderbank, R., Kantor, W.M.: The geometry of two-weight codes, Bull. London Math. Soc., 18, 97–122 (1986)
[5] Courteau, B., Wolfmann, J.: On triple sum sets and three weight codes, Discrete Mathematics 50, 179–191 (1984)
[6] Delsarte, P.: Weights of linear codes and strongly regular normed spaces, Discrete Mathematics 3, 47–64 (1972)
[7] Ding, C., Gao, Y., Zhou, Z.: Five families of three-weight ternary cyclic codes and their duals, IEEE Transactions on Information Theory, 59, 7940–7946 (2013)
[8] Ding, C., Kløve, T., Sica, F.: Two classes of ternary codes and their weight distributions, Discrete Applied Mathematics, 111, 37–53 (2001)
[9] Ding, C., Li, C., Li, N., Zhou, Z.: Three weight cyclic codes and their weight distribution, Discrete Mathematics, 339, 415–427 (2016)
[10] Ding, C., Yuan, J.: Covering and secret sharing with linear codes, Springer LNCS 2731, 11–25 (2003)
[11] Ding, C., Yang, J.: Hamming weights in irreducible cyclic codes, Discrete Mathematics, 313, 434–446 (2015)
[12] Ding, K., Ding, C.: A class of two-weight and three-weight codes and their applications in secret sharing, IEEE Transactions on Information Theory, 61, 5835–5842 (2015)
[13] Li, N., Li, C., Helleseth, T., Tang, X.: Optimal ternary cyclic codes with minimum distance four and five, Finite Fields and Their Applications, 30, 100–120 (2013)
[14] Ling, S., Solé, P.: On the algebraic structure of quasi-cyclic codes I, Finite fields, IEEE Transactions on Information Theory, 47, 2751–2760 (2001)
[15] MacWilliams, F. J., Sloane, N.J.A.: The theory of error-correcting codes, North-Holland (1977)
[16] McEliece, R.J., Rumsey, H.Jr.: Euler products, cyclotomy, and coding, J. Number Theory, 4, 302–311 (1972)
[17] Shi, M., Liu, Y., Solé, P.: Optimal two weight codes from trace codes over a non-chain ring, Discrete Applied Mathematics, (Accepted).
[18] Shi, M., Liu, Y., Solé, P.: Optimal two weight codes from trace codes over $\mathbb{F}_2 + u\mathbb{F}_2$, IEEE Communications Letters, http://ieeexplore.ieee.org/document/7582413/
[19] Shi, M., Wang, D., Solé, P.: Linear codes over $\mathbb{F}_3 + u\mathbb{F}_3 + u^2\mathbb{F}_3$, MacWilliams identities, optimal ternary codes from one-Lee weight codes and two-Lee weight codes, Journal of Applied Mathematics and Computing, 51, 527–544 (2016)
[20] Shi, M., Wu, R., Liu, Y., Solé, P.: Two and three weight codes over $\mathbb{F}_p + u\mathbb{F}_p$, Cryptography and Communications-Discrete Structures, Boolean Functions and Sequences, DOI 10.1007/s12095-016-0206-5
[21] Yuan, J., Ding, C.: Secret sharing schemes from three classes of linear codes, IEEE Transactions on Information Theory, 52, 206–212 (2006)
[22] Zhou, Z., Ding C.: Seven classes of three-weight cyclic codes, IEEE Transactions on Communications, 25, 4120–4126 (2013)

E-mail address: smjwu163.com
E-mail address: dtHuang@163.com
E-mail address: sole@enst.fr