We analyze nonnegative solutions of the nonlinear elliptic problem $\Delta u = \frac{\lambda f(x)}{u^2} + P$, where $\lambda > 0$ and $P \geq 0$, on a bounded domain $\Omega$ of $\mathbb{R}^N$ ($N \geq 1$) with a Dirichlet boundary condition. This equation models an electrostatic–elastic membrane system with an external pressure $P \geq 0$, where $\lambda > 0$ denotes the applied voltage. First, we completely address the existence and nonexistence of positive solutions. The classification of all possible singularities at $|x| = 0$ for nonnegative solutions $u(x)$ satisfying $u(0) = 0$ is then analyzed for the special case where $\Omega = B_1(0) \subset \mathbb{R}^2$ and $f(x) = |x|^\alpha$ with $\alpha \geq 0$. In particular, we show that for some $\alpha$, $u(x)$ admits only the “isotropic” singularity at $|x| = 0$, and otherwise $u(x)$ may admit the “anisotropic” singularity at $|x| = 0$. When $u(x)$ admits the “isotropic” singularity at $|x| = 0$, the refined singularity of $u(x)$ at $|x| = 0$ is further investigated, depending on whether $P > 0$, by applying Fourier analysis.

Keywords: electrostatic MEMS; classification; singular solution; anisotropic singularity; Lojasiewicz–Simon method; convergence rate

Mathematics Subject Classification (2010): 35J75, 35A01, 35C20, 74K15, 74F15

1 Introduction

In this study, we consider nonnegative solutions of the following singular elliptic equation:

$$\begin{cases} \Delta u = \frac{\lambda f(x)}{u^2} + P & \text{in } \Omega, \\ 0 \leq u \leq 1 & \text{in } \Omega, \\ u = 1 & \text{on } \partial \Omega, \end{cases}$$

where $\lambda > 0$, $P \geq 0$, and $0 \leq f \in L^\infty(\Omega)$ with $\Omega \subset \mathbb{R}^N$ ($N \geq 1$). This equation models (cf. [2],[14]) an electrostatic–elastic membrane system with an external pressure denoted by $P \geq 0$. This device consists of an elastic membrane suspended over a rigid ground plate, where the normalized distance between the membrane and the ground plate is described by $u$ in equation $(S)_{\lambda,P}$. When a voltage, represented here by $\lambda$, is applied, the membrane deflects toward the ground plate and a snap-through may occur when it exceeds a certain critical value $\lambda^*$ (pull-in voltage). This creates a so-called “pull-in instability,” which greatly affects the design of many devices. Therefore, we note from [2] that the study of such systems is...
important, not only in the field of electro-hydrodynamics, but also in the study of electrostatic actuators and their importance to the design of MEMS devices in which they are used. The permittivity profile $f$ of $(S)_{\lambda,P}$ is assumed to vanish somewhere and satisfies

$$0 \leq f \in C^r(\Omega) \text{ for some } \epsilon \in (0,1] \text{ and } f \not\equiv 0,$$

where $\Omega \subset \mathbb{R}^N$ and $N = 1$ or 2 for an electrostatic–elastic membrane system. When $P = 0$, the elliptic problem $(S)_{\lambda,0}$ has been widely investigated over the past few years, see [3,7,18,21] and the references therein. We remark that Beckham and Pelesko [2] recently studied positive solutions of the elliptic problem $(S)_{\lambda,P}$ in certain special domains $\Omega$, where the interesting mathematical structures, including the existence and nonexistence, bifurcation behavior, and stability, of positive solutions were successfully analyzed and computed. Moreover, for the parabolic problem related to $(S)_{\lambda,0}$, the dependence on $f$ of quenching behavior, including the case $f(x) = |x|^\alpha$, was studied in [10].

Stimulated by [2,4,5,7,18,21], the main purposes of this study are to address the complete description, in terms of $(\lambda,P)$, of the existence and nonexistence of positive solutions $u$ for $(S)_{\lambda,P}$, and the investigate the possible singular behavior at $|x| = 0$ of nonnegative solutions $u$ satisfying $u(0) = 0$. Toward the first purpose, we denote for convenience $0 < \Phi \in H^1_0(\Omega)$ to be the unique positive solution of

$$-\Delta \Phi = 1 \text{ in } \Omega; \quad \Phi = 0 \text{ on } \partial \Omega,$$

and $P^* := \frac{1}{|\Phi|_\infty} > 0$. Note that $P^*$ depends only on $\Omega$. By making full use of $\Phi$, the following theorem is concerned with the existence and nonexistence of positive solutions $u$ for $(S)_{\lambda,P}$.

**Theorem 1.1.** Suppose that $f$ satisfies (1.1). Then we have

1. If $P \geq P^*$, then there is no positive solution for $(S)_{\lambda,P}$ as soon as $\lambda > 0$.
2. If $0 \leq P < P^*$, then there exists a constant $\lambda_P^* = \lambda_P^*(f,\Omega)$ satisfying

$$\frac{4}{27(P^*)^2 \sup_{\Omega} f(P^* - P)^3} \leq \lambda_P^* \leq \frac{|\Omega| - P \int_\Omega \Phi dx}{\int_\Omega \Phi f dx}, \quad \text{where } P \int_\Omega \Phi dx < \frac{P}{P^*} |\Omega|,$$

such that

(a) if $0 \leq \lambda < \lambda_P^*$, there exists at least one positive solution for $(S)_{\lambda,P}$.
(b) if $\lambda > \lambda_P^*$, there is no positive solution for $(S)_{\lambda,P}$.
3. The critical constant $\lambda_P^* = \lambda_P^*(f,\Omega)$ is nonincreasing in $P$ for $0 \leq P < P^*$.

Note that once the solution $u > 0$, then $u$ is a classical solution to $(S)_{\lambda,P}$. We remark that the existence and nonexistence results of Theorem 1.1 are proved for classical solutions and by a standard process. The arguments of [5,7] and the references therein can be actually used to prove that for any fixed $0 \leq P < P^*$, there exists a unique minimal (positive) solution $w_{\lambda,P}$ of $(S)_{\lambda,P}$ for any $0 < \lambda < \lambda_P^*$, which is called the extremal solution of $(S)_{\lambda,P}$. Moreover, $w_P^*(x) = \lim_{\lambda \to \lambda_P^*} w_{\lambda,P}(x)$ exists and solves $(S)_{\lambda_P^*,P}$ uniquely, which is called the extremal solution of $(S)_{\lambda,P}$. Of course, the unique extremal solution $w_P^*$ of $(S)_{\lambda,P}$ may be either regular or singular (in the sense $\|1 - w_P^*\|_\infty = 1$), which depends on the dimension $N$ and the profile $f$ as well. More precisely, following [5,7] and the references therein, the following properties of extremal solutions can be further established: if $0 < N \leq 7$, then the extremal solution $w_P^*$ of $(S)_{\lambda,P}$ exists and is regular, whereas for $N > 7$, such an extremal solution $w_P^*$ of $(S)_{\lambda,P}$ exists, but $w_P^*$ may be singular, depending on the profile $f$. Following Theorem 1.1 the existence and nonexistence of positive solutions for $(S)_{\lambda,P}$ are illustrated by Figure 1.1 below. One can also check that if $\Omega = B_1(0)$ is a ball in $\mathbb{R}^2$, then $\Phi(x) = \frac{1}{2}(1 - |x|^2)$ and $P^* = 4$, and hence Theorem 1.1 seems consistent with the numerical observations of [2].

The second main purpose of this study is to discuss the singular behavior of solutions $u$ for $(S)_{\lambda,P}$. This is motivated by the fact that when $f(x) = |x|^4$ and $\Omega = B_1(0) \subset \mathbb{R}^2$, then $u(x) = |x|^2$ is a singular solution of $(S)_{\lambda,P}$ for any $(\lambda,P)$ satisfying $\lambda + P = 4$. In the following, we consider the equations for
Figure 1: Existence and nonexistence of positive solutions for $$(S)_{\lambda,p}$$ at different values $$(P,\lambda)$$. 

$$f(x) = |x|^\alpha$$ ($$\alpha \geq 0$$) and $$\Omega = B_1(0) \subset \mathbb{R}^2$$. More precisely, next we are concerned with the local behavior near the origin for solutions to the singular elliptic equation

$$\begin{cases}
\Delta u = \frac{\lambda |x|^{\alpha}}{u^2} + P & \text{in } B_1(0) \subset \mathbb{R}^2, \\
u \geq 0 & \text{in } B_1(0),
\end{cases} \tag{1.4}$$

where $$\lambda > 0$$, $$\alpha \geq 0$$, and $$P \geq 0$$.

In fact, we rewrite equation (1.4) as a semilinear evolution elliptic problem

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = \frac{\lambda r^\alpha}{u^2} + P, \tag{1.5}$$

where $$u(x) = u(r, \theta)$$, by using the polar coordinate $$(r, \theta) \in [0, 1] \times S^1$$ in $$B_1(0)$$. Furthermore, we define $$v(t, \theta)$$ by

$$v(t, \theta) := r^{-\frac{2+\alpha}{3}} u(r, \theta), \text{ where } t = -\ln r \text{ and } r = |x|, \tag{1.6}$$

such that $$v(t, \theta)$$ satisfies the following evolution elliptic problem

$$-v_t + \frac{2(2+\alpha)}{3} v_t = v_{\theta \theta} + \left(\frac{2+\alpha}{3}\right)^2 v - \frac{\lambda}{v^2} - Pe^{-\frac{4-\alpha}{3} t}, \quad (t, \theta) \in (0, +\infty) \times S^1. \tag{1.7}$$

We assume that

$$\alpha \in \mathcal{A} := \begin{cases} [0, +\infty), & \text{for } P = 0, \\ [0, 4), & \text{for } P > 0, \end{cases} \tag{1.8}$$

throughout the remainder of this paper. In association with the stationary problem of (1.7), we also denote $$w(\theta)$$ to be a solution of

$$w'' + \left(\frac{2+\alpha}{3}\right)^2 w - \frac{\lambda}{w^2} = 0 \text{ on } S^1, \tag{1.9}$$

and define the solution set by

$$\mathcal{S} = \{w > 0 : w \text{ is a solution of (1.9)}\}. \tag{1.10}$$

The following analysis of the structure of $$\mathcal{S}$$ in terms of $$\alpha$$ plays a fundamental role in classifying the singularities of solutions for (1.4).

**Theorem 1.2.** Consider the set $$\mathcal{S}$$ defined in (1.10), where $$\lambda > 0$$ is arbitrary. The following results hold:
1. If \[ \alpha \in A_s := \begin{cases} \mathcal{A}_0 := [0,1] \cup \bigcup_{k=3}^7 \left( (k - 1)\sqrt{3} - 2, \frac{3k - 4}{2} \right), & \text{for } P = 0, \\ \mathcal{A}_0 \cap [0,4) = [0,1] \cup [2\sqrt{3} - 2, \frac{2\sqrt{3} - 4}{2}] \cup [3\sqrt{3} - 2, 4), & \text{for } P > 0, \end{cases} \] then we have \( \mathcal{S} = \left\{ \left( \frac{9\alpha}{r + \alpha} \right) \right\} \).

2. If \( \alpha \in A \setminus A_s \) with \( A \) defined in (1.8), then \( \mathcal{S} \) contains precisely \( 1 + N_0(\alpha) \) connected components:
\[ \mathcal{S}_0 = \left\{ \left( \frac{9\alpha}{r + \alpha} \right) \right\}, \mathcal{S}_1, \cdots, \mathcal{S}_i, \cdots, \mathcal{S}_{N_0(\alpha)}. \] Here,
\[ \mathcal{S}_1 = \{ w_{j_i}(\cdot + a) : a \in S^1 \}, \quad i = 1, 2, \cdots, N_0(\alpha), \] where \( j_i = \left\lfloor \frac{\sqrt{3}(2 + \alpha)}{3} + i \right\rfloor, w_{j_i}(\theta) \) satisfying \( \min_{\theta \in \mathbb{R}} w_{j_i}(\theta) = w_{j_i}(0) \) is a \( \frac{2\pi}{j_i} \)-periodic positive solution of (1.9), and \( 1 \leq N_0(\alpha) < +\infty \) denotes the number of integers in \( \left( \frac{\sqrt{3}(2 + \alpha)}{3}, \frac{2(2 + \alpha)}{3} \right) \).

We can get more precise information about case 2 of Theorem 1.2. In fact, we have

**Remark 1.1.** Under the assumption of Theorem 1.2 if \( \alpha \in (1,2\sqrt{3} - 2) \), then \( \left[ \frac{\sqrt{3}(2 + \alpha)}{3} \right] = 1, N_0(\alpha) = 1, j_1 = 2, \mathcal{S}_1 = \{ w_2(\cdot + a) : a \in S^1 \} \), and \( \mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 = \left\{ \left( \frac{9\alpha}{r + \alpha} \right) \right\} \cup \{ w_2(\cdot + a) : a \in S^1 \} \); if \( \alpha \in \left( \frac{\sqrt{3}}{3}, 3\sqrt{3} - 2 \right) \), then \( \left[ \frac{\sqrt{3}(2 + \alpha)}{3} \right] = 2, N_0(\alpha) = 1, j_1 = 3, \mathcal{S}_1 = \{ w_3(\cdot + a) : a \in S^1 \} \), and \( \mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 = \left\{ \left( \frac{9\alpha}{r + \alpha} \right) \right\} \cup \{ w_3(\cdot + a) : a \in S^1 \} \), etc.

We now return to classify nonnegative solutions of the original equation (1.4), for which we have the following theorem.

**Theorem 1.3.** Suppose \( u(x) = u(r, \theta) \) is a solution of (1.4) that is continuous on \( B_1(0) \subset \mathbb{R}^2 \). Assume there exist constants \( \beta \in (0,1] \) and \( C_\beta > 0 \) such that \( v \) defined in (1.5) satisfies the assumption
\[ \sup_{0 < r \leq 1} \frac{1}{r^{\beta + \beta}} \int_{B_r(x)} \frac{1}{v} dy \leq C_\beta, \quad \forall x = (t, \theta) \in [t^0, +\infty) \times S^1 \] (1.13)
for some \( t^0 \in \mathbb{R} \). Then, either
1. \( u(0) > 0 \) and \( u \in C^2(B_{r^0}(0)) \) for some \( 0 < r^0 < 1 \),
2. or \( u(0) = 0 \) and there exists \( w \in \mathcal{S} \) such that
\[ \| r^{-2 + \alpha} u(r, \theta) - w(\theta) \|_{C^2(S^1)} \leq C(1 - \ln r)^{-\frac{\theta}{r(1 - 2\theta)}}, \quad as \ r \to 0^+, \] (1.14)
for some \( \theta \in (0, \frac{1}{2}) \) depending on \( w \).

By combining Theorem 1.3 with Theorem 1.4 (2), we immediately obtain the following:

**Corollary 1.4.** Suppose \( u(x) = u(r, \theta) \) is a singular solution of (1.4) with \( u(0) = 0 \) and (1.13) holds.

1. If \( \alpha \in A_s \), then \[ \lim_{r \to 0^+} r^{2 + \alpha} u(r, \theta) = \left( \frac{9\lambda}{(2 + \alpha)^2} \right)^{\frac{1}{2}} \text{ in } C^2(S^1). \] (1.15)

2. If \( \alpha \in A \setminus A_s \), then either (1.13) holds, or there exists \( w_{j_i}(\theta + a), 1 \leq i \leq N_0(\alpha) \) such that
\[ \lim_{r \to 0^+} r^{2 + \alpha} u(r, \theta) = w_{j_i}(\theta + a) \] (1.16)
in \( C^2(S^1) \), where \( A, N_0(\alpha), j_i, w_{j_i} \) are as in Theorem 1.3.
According to Remark 1.1, we emphasize that we can obtain more precise information about case 2 of the above corollary: that is, if $\alpha \in (1, 2\sqrt{3} - 2)$, then either (1.15) holds or there exists $w_2(\theta + a)$ such that $\lim_{r \to 0} r^{-\frac{3}{2}+\alpha} u(r, \theta) = w_2(\theta + a)$; if $\alpha \in (\frac{3}{2}, 3\sqrt{3} - 2)$, then either (1.15) holds or there exists $w_3(\theta + a)$ such that $\lim_{r \to 0} r^{-\frac{3}{2}+\alpha} u(r, \theta) = w_3(\theta + a)$, etc.

Here, we make several remarks concerning the above results. First, if $\alpha \in A_s$, then Corollary 1.4 (1) shows that $u$ satisfying $u(0) = 0$ admits only the “isotropic” singularity at $|x| = 0$ in the sense of (1.15). However, Corollary 1.4 (2) implies that $u(x)$ satisfying $u(0) = 0$ may admit the “anisotropic” singularity at $|x| = 0$ if $\alpha \in A\setminus A_s$. Second, one can easily check that $u_c = |x|^{-\frac{\alpha}{2}}$, which admits the “isotropic” singularity at $|x| = 0$, is always a singular solution to (1.14) with the boundary condition $u|_{\partial B_1(0)} = 1$ for $(\lambda, P) = (\frac{(2+\alpha)^2}{2}, 0)$. Third, we remark that the method of [4, Lemma 1.6] does not work for proving the convergence result in Theorem 1.3. To overcome this difficulty for the case $P = 0$, where (1.17) is an autonomous evolution equation, the convergence result of Theorem 1.3 can be followed from (1.7) by employing the Łojasiewicz–Simon method. For the case $P > 0$, where (1.17) is an asymptotically autonomous evolution equation, the method used in (1.17) does not work directly. We shall combine the techniques of [3, 13] with the methods of (1.7) to overcome this difficulty.

Next, we follow Corollary 1.4 to analyze the refined “isotropic” singularity of $u$ further for the case that (1.15) holds with

$$\alpha \in \mathcal{A} := \mathcal{A}\setminus \{(k - 1)\sqrt{3} - 2 : k = 3, 4, \ldots\},$$

where $\mathcal{A}$ is defined in (1.8). We set a new transformation

$$V(t, \theta) = r^{-\frac{2+\alpha}{3}} u(r, \theta) - \left(\frac{9\lambda}{(2 + \alpha)^2}\right)^{\frac{1}{3}} \text{ in } (t_0, +\infty) \times S^1, \quad t = -\ln r \text{ and } r = |x|.$$  

By carefully analyzing the asymptotic behavior of $V(t, \theta)$ as $t \to +\infty$, the following refined singular behavior is proved in Section 4:

**Theorem 1.5.** Assume $u(x) = u(r, \theta)$ is a singular solution of (1.4) with $u(0) = 0$ such that (1.15) holds. Then we have the following refined singular behavior:

1. If $\alpha \in \mathcal{A}\setminus \{(2\sqrt{3} - 2, \frac{3}{2}\sqrt{10} - 2) \cup (3\sqrt{3} - 2, 4)\}$, then for both cases $P = 0$ and $P > 0$, once $\alpha \in ((k - 1)\sqrt{3} - 2, k\sqrt{3} - 2) \cap [0, +\infty)$, there exist $A_k \in \mathbb{R}$ and $\theta_k \in S^1$ such that

$$\lim_{r \to 0} r^{-\frac{3}{2}+\alpha} u(r, \theta) - \left(\frac{9\lambda}{(2 + \alpha)^2}\right)^{\frac{1}{3}} r^{\frac{3}{2}+\alpha} = A_k \sin(k\theta + \theta_k) \text{ in } C^2(S^1),$$

where $k = 2, 3, 4, \ldots$ for $P = 0$, and $k = 2, 3$ for $P > 0$.

2. If $\alpha \in (2(\sqrt{3} - 1), \frac{3}{2}\sqrt{10} - 2) \cup (3\sqrt{3} - 2, 4)$ and $P = 0$, then (1.19) still holds.

3. If $\alpha \in (2(\sqrt{3} - 1), \frac{3}{2}\sqrt{10} - 2) \cup (3\sqrt{3} - 2, 4)$ and $P > 0$, then

$$\lim_{r \to 0} r^{-2} \left[u(r, \theta) - \left(\frac{9\lambda}{(2 + \alpha)^2}\right)^{\frac{1}{3}} r^{\frac{3}{2}+\alpha}\right] = \frac{9P}{36 + 2(2 + \alpha)^2} \text{ in } C^2(S^1).$$

4. If $\alpha = \frac{3}{2}\sqrt{10} - 2$ and $P > 0$, then there exist $A_3 \in \mathbb{R}$ and $\theta_3 \in S^1$ such that

$$\lim_{r \to 0} r^{-3} \left[u(r, \theta) - \left(\frac{9\lambda}{(2 + \alpha)^2}\right)^{\frac{1}{3}} r^{\frac{3}{2}+\alpha}\right] = A_3 \left(\sin(3\theta + \theta_3) + \frac{P}{9}\right) \text{ in } C^2(S^1).$$

Under the assumptions of Theorem 1.4 if $\alpha \in (2(\sqrt{3} - 1), \frac{3}{2}\sqrt{10} - 2) \cup (3\sqrt{3} - 2, 4)$, then Theorem 1.5 reveals the following refined “isotropic” singularity on singular solutions $u$ of (1.3): for the case $P > 0$, $u$ admits the “strongly isotropic” singularity at $|x| = 0$, in the sense that for some $\gamma > \frac{3}{2}+\alpha$, the limit

$$\lim_{r \to 0} \left[u(r, \theta) - \left(\frac{9\lambda}{(2 + \alpha)^2}\right)^{\frac{1}{3}} r^{\frac{3}{2}+\alpha}\right]^{\frac{\gamma}{r}}$$




Theorem 1.5, which is concerned with the refined singular behavior near the origin of nonnegative solutions 1.2 and 1.3 on the classification of singular solutions for (1.4). In Section 4, we complete the proof of the equivalent problem in this section for convenience:

\[ P > \text{pressure} \]

one can obtain that existence and nonexistence of positive solutions for (\ref{1.3}). In Appendix B, which plays an important role in the proof of Theorem 1.3, type inequality is established in Appendix B.

2. Existence and Nonexistence of Positive Solutions

In this section, we focus on the proof of Theorem 1.1. We denote \( \Phi_P = P\Phi \), where \( \Phi \) is the unique positive solution in \( H^1_0(\Omega) \) of \( \{2.1\} \); \( \Phi_P \) satisfies

\[ 0 < \Phi_P = P\Phi \leq \frac{P}{P^*} \quad \text{in } \Omega, \quad \text{and } \|\Phi_P\|_\infty = \frac{P}{P^*}, \]  

where \( P^* := \frac{1}{\|\Phi\|_\infty} > 0 \).

To consider positive solutions of \((S)_{\lambda,P}\), by setting \( \tilde{u} = 1 - u \) in \((S)_{\lambda,P}\), we work on the following equivalent problem in this section for convenience:

\[
\begin{aligned}
-\Delta \tilde{u} &= \frac{\lambda f}{(1 - u)^2} + P \quad \text{in } \Omega; \\
0 &\leq \tilde{u} < 1 \quad \text{in } \Omega; \\
\tilde{u} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

\((\tilde{S})_{\lambda,P}\)

Lemma 2.1. For any \( 0 \leq P < P^* \), there exists a finite positive constant \( \lambda_P = \lambda^*_P(f, \Omega) \) satisfying \( \{3.3\} \) such that

1. If \( 0 \leq \lambda < \lambda^*_P \), there exists at least one solution for \((\tilde{S})_{\lambda,P}\).

2. If \( \lambda > \lambda^*_P \), there is no solution for \((\tilde{S})_{\lambda,P}\).

Proof. For any fixed \( 0 \leq P < P^* \), define

\[ 
\lambda_P^* := \lambda^*_P(f, \Omega) = \sup \{ \lambda > 0 \mid (\tilde{S})_{\lambda,P} \text{ possesses at least one solution} \}. 
\]  

We first prove that \( \lambda_P^* > 0 \) holds for any \( 0 \leq P < P^* \). It is clear that \( u \equiv 0 \) is a subsolution of \((\tilde{S})_{\lambda,P}\) for any \( \lambda > 0 \). To construct a supersolution of \((\tilde{S})_{\lambda,P}\) with \( 0 \leq P < P^* \), set \( \bar{u} = s\Phi_{P^*} \) with \( \frac{P}{P^*} < s < 1 \), such that \( \bar{u} \leq s \) in \( \Omega \) and \( \bar{u} = 0 \) on \( \partial \Omega \). We then have

\[
-\Delta \bar{u} = P + \frac{P^*}{\sup_{\Omega} f} \left( s - \frac{P}{P^*} \right) (1 - s)^2 \sup_{\Omega} f \left( 1 - s \right)^2 \geq P + \frac{P^*}{\sup_{\Omega} f} \left( s - \frac{P}{P^*} \right) (1 - s)^2 \frac{f(x)}{(1 - \bar{u})^2} \quad \text{in } \Omega. 
\]

Setting

\[ g(s) = \frac{P^*}{\sup_{\Omega} f} \left( s - \frac{P}{P^*} \right) (1 - s)^2, \]  

where \( \frac{P}{P^*} < s < 1 \),

one can obtain that

\[ g(s) \leq g \left( \frac{P^* + 2P}{3P^*} \right) = \frac{4}{27(P^*)^2 \sup_{\Omega} f} (P^* - P)^3 := \lambda_P. \]

This implies from \( \{2.3\} \) by taking \( s = \frac{P^* + 2P}{3P^*} \), that

\[ -\Delta \tilde{u} \geq \frac{\lambda_P f}{(1 - \bar{u})^2} + P \quad \text{in } \Omega, \]
which shows that for any fixed $0 \leq P < P^*$, $\tilde{u} = \frac{P^* + P}{\lambda^*} \Phi > 0$ is a supersolution of $(\bar{S})_{\lambda,P}$ for $0 \leq \lambda \leq \lambda_P$. By the method of sub-supersolutions, we conclude that for any fixed $0 \leq P < P^*$, there is a solution $\tilde{u}_{\lambda,P}$ of $(\bar{S})_{\lambda,P}$ for every $\lambda \in (0, \lambda_P)$, which implies that $\lambda_P > \lambda > 0$ holds for any $0 < P < P^*$.

We next prove the finiteness of $\lambda_P^*$ for any fixed $0 < P < P^*$. Suppose that $(\bar{S})_{\lambda,P}$ has a solution $\tilde{u}$. Multiplying $(\bar{S})_{\lambda,P}$ by $\Phi$ and integrating over $\Omega$, we obtain

$$|\Omega| \geq \int_{\Omega} \tilde{u} dx = -\int_{\Omega} \Phi \Delta \tilde{u} dx = \int_{\Omega} \frac{\lambda f}{(1 - \tilde{u})^2} dx + P \int_{\Omega} \Phi dx \geq \lambda \int_{\Omega} \Phi f dx + \int_{\Omega} \Phi dx,$$

which then implies that

$$\lambda_P^* \leq \frac{|\Omega| - P \int_{\Omega} \Phi dx}{\int_{\Omega} \Phi f dx} < +\infty, \text{ since } \frac{P}{\Phi} \int_{\Omega} \Phi dx < |\Omega| < |\Omega|.$$

For any fixed $0 < P < P^*$, because $\lambda_P^*$ is positive and finite, we choose any $\lambda \in (0, \lambda_P^*)$ and use the definition of $\lambda_P^*$ to find $\tilde{\lambda} \in (\lambda, \lambda_P^*)$ such that $(\bar{S})_{\tilde{\lambda},P}$ has a solution $\tilde{u}_{\tilde{\lambda},P}$ satisfying

$$-\Delta \tilde{u}_{\tilde{\lambda},P} = \frac{\tilde{\lambda} f}{(1 - \tilde{u}_{\tilde{\lambda},P})^2} + P \text{ in } \Omega; \ 0 \leq \tilde{u}_{\tilde{\lambda},P} < 1 \text{ in } \Omega; \ \tilde{u}_{\tilde{\lambda},P} = 0 \text{ on } \partial \Omega,$$

which implies that $-\Delta \tilde{u}_{\tilde{\lambda},P} \geq \frac{\lambda f}{(1 - \tilde{u}_{\tilde{\lambda},P})^2} + P$ in $\Omega$. This shows that $\tilde{u}_{\tilde{\lambda}}$ is a supersolution of $(\bar{S})_{\lambda,P}$. Because $\tilde{u} \equiv 0$ is a subsolution of $(\bar{S})_{\lambda,P}$, by the method of sub-supersolutions, we deduce that there is a solution $\tilde{u}_{\lambda}$ of $(\bar{S})_{\lambda,P}$ for every $\lambda \in (0, \lambda_P^*)$. Note from the definition of $\lambda_P^*$ that there is no solution of $(\bar{S})_{\lambda,P}$ for any $\lambda > \lambda_P^*$. This completes the proof of Lemma 2.1. \hfill \Box

**Lemma 2.2.** If $P \geq P^*$, then there is no solution for $(\bar{S})_{\lambda,P}$ as soon as $\lambda > 0$.

**Proof.** On the contrary, suppose there exists $P \geq P^*$ such that $(\bar{S})_{\lambda,P}$ has a solution $0 \leq \tilde{u}_{\lambda,P} < 1$ for some $\lambda > 0$. This implies that

$$-\Delta (\tilde{u}_{\lambda,P} - P) = \frac{\lambda f}{(1 - \tilde{u}_{\lambda,P})^2} \geq 0 \text{ in } \Omega; \ \tilde{u}_{\lambda,P} - P = 0 \text{ on } \partial \Omega. \quad (2.4)$$

Applying the strong maximum principle to $(2.4)$, we then obtain that $\Phi_P < \tilde{u}_{\lambda,P} \leq 1$ in $\Omega$, which is however a contradiction to the fact that $\|\Phi_P\|_{\infty} = \frac{P}{\Phi} \geq 1$ by (2.4). This completes the proof of Lemma 2.2. \hfill \Box

**Proof of Theorem 1.1.** We complete the proof of Theorem 1.1 by Lemmas 2.1 and 2.2 and part (3) of Theorem 1.1(3) can be easily established. \hfill \Box

### 3 Classification of Singularities

This section is concerned with the proof of Theorem 1.2 and Theorem 1.3 which handle the classification of singular solutions for (1.1). We reduce (1.1) into the semilinear evolution elliptic problem (1.7), such that it suffices to analyze the long-time profile of $v(t,\theta)$ for (1.7) as $t \to +\infty$. By using the phase-plane method, we first give the proof of Theorem 1.2 in Subsection 3.1. The proof of Theorem 1.3 is then completed in Subsection 3.2 by employing the theory of infinite dimensional dynamical systems as well as the Łojasiewicz–Simon method.

We start with the following crucial local estimates of singular solutions.

**Lemma 3.1.** Suppose that $u$ is a nonnegative singular solution of (1.4) satisfying $u(0) = 0$ and (1.13), where $\alpha$ satisfies (1.8). Then there exist constants $0 < C_1 < C_2 < +\infty$ such that

$$C_1 |x|^{\frac{4+\alpha}{4}} \leq u(x) \leq C_2 |x|^{\frac{4+\alpha}{4}} \quad \text{as } |x| \to 0. \quad (3.1)$$
Because Lemma 3.1 can be established in a similar way to that in [10], [11], we sketch the proof in Appendix A for simplicity. Recall that the function $v(t, \theta)$ defined in (3.2) satisfies the evolution elliptic equation given by

$$
- v_{tt} + \frac{2(2 + \alpha)}{3} v_t = v_{\theta\theta} + \left(\frac{2 + \alpha}{3}\right)^2 v - \frac{\lambda}{v^2} - Pe^{-\frac{\lambda}{v^2}} t, \quad (t, \theta) \in (t_0, +\infty) \times S^1.
$$

(3.2)

It follows from Lemma 3.1 that $0 < C_1 \leq v \leq C_2 < +\infty$ uniformly in $(t_0, +\infty) \times S^1$ for some $t_0 > 0$. Therefore, we need only investigate the long-time behavior of the bounded solution $v$ of (3.2) as $t \to +\infty$. The following results give some analytic properties of the evolution equation (3.2).

**Lemma 3.2.** Under the assumptions of Theorem 1.3, the following results hold:

1. There exists $\delta \in (0, 1)$ such that $v(t, \cdot), v_t(t, \cdot), v_{\theta}(t, \cdot), v_{tt}(t, \cdot), v_{t\theta}(t, \cdot), v_{\theta\theta}(t, \cdot), v_{tt\theta}(t, \cdot),$ and $v_{tt\theta\theta}(t, \cdot)$ all remain bounded in $C^0(S^1)$ for any $t \in [t_0, +\infty)$, where $C^0(S^1)$ denotes the usual Hölder continuous space on $S^1$.

2. Both $v_t(t, \cdot)$ and $v_{tt}(t, \cdot)$ tend to 0 in $C^0(S^1)$ as $t \to +\infty$.

3. The “orbit” $\mathcal{L} := \{v(t, \cdot): t \geq t_0\}$ of $v$ is relatively compact in $C^2(S^1)$.

**Proof.** We prove this lemma in a similar way to [4]. Because all coefficients of (3.2) are bounded and Lemma 3.1 gives that $0 < C_1 \leq v \leq C_2 < +\infty$ holds in $(t_0, +\infty) \times S^1$, Lemma 3.2(1) is an immediate consequence of $L^p$ and the Schauder estimates for (3.2). Moreover, Lemma 3.2(3) follows by directly applying Lemma 3.2(1) and the Ascoli–Arzelà Theorem as well.

As for Lemma 3.2(2), multiplying (3.2) by $v_t$, and integrating it by parts with respect to $\theta$ and $t$, we obtain that

$$
\int_{t_0}^{+\infty} \int_{S^1} v_t^2 d\theta dt < +\infty,
$$

(3.3)

owing to the boundedness of $v, v_{\theta}$, and $v_t$. Set

$$
k(t) = \int_{S^1} v_t^2 d\theta.
$$

We thus deduce from the boundedness of $v_t$ and $v_{tt}$ that $k'(t)$ is bounded uniformly on $[t_0, +\infty)$. Therefore, we derive from (3.3) that

$$
k(t) \geq 0, \quad \int_{t_0}^{+\infty} k(t) dt < +\infty,
$$

which implies that $k(t) \to 0$ as $t \to +\infty$. It then follows from the boundedness of $v_{t\theta}$ that

$$
v_t \to 0 \quad \text{as} \quad t \to +\infty \quad \text{(uniformly in} \ \theta \in S^1). \quad (3.4)
$$

Therefore, the convergence $v_{tt} \to 0$ as $t \to +\infty$ holds by applying (3.4), and the boundedness of $v_{tt}, v_{tt\theta},$ and $v_{tt\theta\theta}$, where the details of the proof are omitted for simplicity. This completes the proof of Lemma 3.2(2), and we have finished. \qed

### 3.1 Structure of set $\mathcal{G}$

In this subsection, we analyze the set $\mathcal{G}$ defined in (1.10), where the constant $\frac{2 + \alpha}{3}$ is replaced by a generic constant $A$. Our main results are given by the following proposition, from which Theorem 1.2 can be established immediately.

**Proposition 3.3.** Consider the set

$$
\mathcal{G} := \{w(\theta) \in C^2(S^1) : w'' + A^2 w - \frac{\lambda}{w^2} = 0, \quad w > 0\},
$$

(3.5)

where $A > 0$, $\lambda > 0$ are given constants. Then we have the following results:
1. If
\[ A \in A_c := (0, \frac{1}{2}] \cup \bigcup_{k=1}^{6} \left[ \frac{k}{\sqrt{3}}, \frac{k+1}{2} \right], \] (3.6)
then \( \mathcal{S} = \left\{ \left( \frac{1}{A^2} \right)^{i/3} \right\} \).

2. If \( A \notin A_c \), then \( \mathcal{S} \) contains precisely \( 1 + N_0(A) \) connected components \( \mathcal{S}_0 = \left\{ \left( \frac{1}{A^2} \right)^{i/3} \right\}, \mathcal{S}_1, \ldots, \mathcal{S}_{N_0(A)} \), where \( \mathcal{S}_i \) is defined by
\[ \mathcal{S}_i = \{ w_j(\cdot + a); 0 \leq a < 2\pi \}, \; i = 1, 2, \ldots, N_0(A). \] (3.7)
Here, \( 1 \leq N_0(A) < +\infty \) denotes the number of integers in \( (\sqrt{3}A, 2A) \), \( j_i = [\sqrt{3}A] + i \), and \( w_j(\theta) \) satisfying \( \min \theta w_j(\theta) = w_j(0) \) is the \( \frac{2\pi}{j_i} \)-periodic positive solution of
\[ w'' + A^2w - \frac{\lambda}{w^2} = 0 \; \text{in} \; \mathbb{R}. \] (3.8)

To prove Proposition 3.3, we use the standard phase-plane method (cf. [15, 11, 22, 23]). Note that a first integral of (3.8) is given by
\[ (w')^2 + A^2w^2 + \frac{2\lambda}{w} = E \] (3.9)
for some constant \( E \). Define
\[ g(w) = A^2w^2 + \frac{2\lambda}{w}, \] where \( w > 0 \), such that
\[ g(w) \geq g(w_0) := E_0 = g\left( \left( \frac{1}{A^2} \right)^{1/3} \right) = 3\lambda A^{\frac{2}{3}} > 0, \]
and
\[ g'(w) < 0, \; 0 < w < w_0; \; g'(w) > 0, \; w_0 < w < +\infty; \]
\[ g(w) \to +\infty \; \text{as either} \; w \to 0 \; \text{or} \; w \to +\infty. \]

As a consequence, (3.8) has nontrivial positive solutions if and only if \( E > g(w_0) \). Moreover, it is easy to see that any nontrivial solution of problem (3.8) has the following two properties: (i) it is periodic; (ii) if \( w(\theta) \) is a solution of (3.8), then \( w(\theta + a) \) is also a solution of (3.8) for any \( a \in \mathbb{R} \).

Suppose now that \( w(\theta) \) is a nontrivial positive solution of (3.8). Denote \( w_1 \) (resp. \( w_2 \)) the minimum (resp. maximum) value of \( w(\theta) \). Then \( w_1 \) and \( w_2 \) are two roots of
\[ g(w) = E \; \text{for some} \; E > g(w_0), \]
i.e.,
\[ A^2w_1^2 + \frac{2\lambda}{w_1} = E, \; A^2w_2^2 + \frac{2\lambda}{w_2} = E. \] (3.10)

Therefore, by setting \( \tau = \frac{w_2}{w_1} \), we conclude from the above that
\[ w_1^3 = \frac{2\lambda}{A^2\tau(1+\tau)}. \] (3.11)

We can assume without loss of generality that \( \theta = 0 \) is a minimum point of \( w(\theta) \) and \( \theta = L > 0 \) is a maximum point of \( w(\theta) \), such that \( w'(\theta) > 0 \) for any \( \theta \in (0, L) \). Thus, there holds \( w'(0) = w'(L) = 0 \), where \( L > 0 \) is the half-minimum period of \( w \). Note also from (3.9) that
\[ d\theta = \frac{dw}{\sqrt{E - A^2w^2 - \frac{\lambda}{w}}}, \]
which implies that
\[ L(E) = \int_{w_1}^{w_2} \frac{dw}{\sqrt{E - A^2 w^2 - \lambda^2 w}} \quad (3.12) \]

By setting \( y = \frac{w}{w_1} \), \( L(E) \) can be rewritten as
\[ L(\tau) = \int_1^\tau \frac{dy}{\sqrt{\frac{E}{w_1^2} - A^2 y^2 - \lambda^2 y}} = \frac{1}{A} \int_1^\tau \frac{dy}{\sqrt{1 + \tau(1 + \tau) - y^2 - \frac{1}{2} \tau(1 + \tau)}} \quad (3.13) \]
where (3.10) and (3.11) are used. We next address some analytic properties of \( L(\tau) \).

**Lemma 3.4.** \( L(\tau) \) is continuous on \((1, +\infty)\) and satisfies
\[ \lim_{\tau \to 1} L(\tau) = \frac{\pi}{\sqrt{3A}}, \quad \lim_{\tau \to +\infty} L(\tau) = \frac{\pi}{2A}. \quad (3.14) \]
Moreover, \( L(\tau) \) is strictly decreasing in \( \tau \).

**Proof.** Denote \( Q(w) = -A^2 w + \frac{\lambda^2}{w_1^2} \), and let \( w_0 = \left( \frac{\lambda}{A} \right)^2 \) be the unique root of \( Q(w) \). By [19, Lemma 3.2] we then have
\[ L(E) \xrightarrow{E \to E_0} Q(w_0) = \frac{\pi}{\sqrt{3A}}. \quad (3.15) \]
Thus, the first equation of (3.14) follows directly from (3.15), because \( \tau \to 1 \) is equivalent to \( E \to E_0 = g(w_0) \). By setting \( \xi = \frac{w-1}{w_1} \), we rewrite (3.13) as
\[ L(\tau) = \frac{1}{A} \int_0^1 \frac{\tau - 1}{\sqrt{1 + \tau(1 + \tau) - (\xi(\tau-1) + 1)^2 - \frac{1}{(\xi(\tau-1)+1)^2}}} d\xi. \]
We then obtain that
\[ \lim_{\tau \to +\infty} L(\tau) = \frac{1}{A} \int_0^1 \frac{d\xi}{\sqrt{1 - \xi^2}} = \frac{\pi}{2A}, \]
i.e., the second equation of (3.14) holds. Finally, the monotonicity of \( L \) in \( \tau \) follows directly from the case of \( \alpha = \frac{1}{2} \) of [11 Corollary 5.6]. \( \square \)

**Proof of Proposition 3.3.** By Lemma 3.4 the range of the half period \( L \) is \( I_A := \left( \frac{\pi}{2A}, \frac{\pi}{\sqrt{3A}} \right) = \left( \frac{\pi}{A}, \frac{\pi}{\sqrt{3A}} \right) \). Thus (3.8) has no nontrivial solution if and only if the interval \( I_A \) does not contain \( \frac{\pi}{j} \) for any integer \( j \geq 1 \), which implies that either \( 0 < A \leq 1/2 \) or
\[ \frac{\pi}{j+1} \leq \frac{\pi}{2A} \quad \text{and} \quad \frac{\pi}{\sqrt{3A}} \leq \frac{\pi}{j} \quad \text{for some} \quad j \geq 1, \]
i.e., either \( 0 < A \leq 1/2 \) or
\[ \frac{j}{\sqrt{3A}} \leq A \leq \frac{j+1}{2} \quad \text{for some} \quad j \geq 1. \quad (3.16) \]
Note that \( \frac{j}{\sqrt{3A}} \leq \frac{j+1}{2} \) holds only for \( j \leq 6 \). Thus, (3.8) has no nontrivial solution if and only if
\[ A \in A_c := \left( 0, \frac{1}{2} \right) \cup \bigcup_{k=1}^6 \left( \frac{k}{\sqrt{3A}}, \frac{k+1}{2} \right). \]
Therefore, if \( A \in A_c \), we then have \( S = \left\{ \left( \frac{j}{\sqrt{3}} \right)^{\pm} \right\} \).

Suppose now that \( A \not\in A_c \). Then it is clear that \( S_0 = \left\{ \left( \frac{j}{\sqrt{3}} \right)^{\pm} \right\} \subset S \), and \( S \) also contains nontrivial solutions because the interval \( I_A \) contains \( \frac{\pi}{j} \) for some integer \( j \geq 1 \). In other words, there exists \( j \geq 1 \) such that \( \frac{\pi}{2A} < \frac{j}{\sqrt{3A}} \leq \frac{\pi}{\sqrt{3A}} \), i.e., \( \sqrt{3A} < j < 2A \). More precisely, if we denote by \( 1 \leq N_0(A) < +\infty \)
the number of integers in $({\sqrt 3}A,2A)$, then $I_A$ contains \{ \[ \frac{\pi}{j} \] \] \_{j = 1} = \[ \sqrt 3A \] + i, i = 1, 2, \ldots, N_0(A) \}. This implies that \[ \{ \{ \frac{\pi}{j} \}_j \] \ has $N_0(A)$ periodic solutions $w_{j_1}(\theta), w_{j_2}(\theta), \ldots, w_{j_{N_0(A)}}(\theta)$, where each $w_{j_i}(\theta)$ has the $\frac{\pi}{j_i}$-period and $\min_{\theta} w_{j_i}(\theta) = w_{j_i}(0)$. Therefore, if $A \notin A_c$, then $\mathfrak{S}$ contains precisely $1 + N_0(A)$ connected components $\mathfrak{S}_0 = \{ \{ \frac{\pi}{j} \}_j \}, \mathfrak{S}_1, \ldots, \mathfrak{S}_{N_0(A)}$, where $\mathfrak{S}_1$ is defined by \[ \mathfrak{S}_1 \]. This completes the proof of the proposition. \hfill \Box

We finally remark that Theorem 1.2 follows immediately from Proposition 3.3 because if
\[ \alpha \in \left[ 0, 1 \right] \cup \bigcup_{k=3}^7 \left[ (k-1)\sqrt 3 - 2, \frac{3k-4}{2} \right] \],
then $0 < A := \frac{2+2\alpha}{3} \in A_c$, where the set $A_c$ is as in Proposition 3.3.

### 3.2 Proof of Theorem 1.3

This subsection is devoted to the proof of Theorem 1.3 for which we still suppose that $u$ is a singular solution of \[ (1.4) \] satisfying $u(0) = 0$ and \[ (1.13) \]. Let $v$ be a solution of \[ (3.2) \] such that $0 < C_1 \leq v \leq C_2 < \infty$ holds. We define the “$\omega$-limit set” $\omega(v)$ of $v$ by
\[ \omega(v) = \{ w \mid w \in C^2(S^1), \exists t_n \to +\infty, \lim_{n \to +\infty} \|v(t_n, \cdot) - w(\cdot)\|_{C^2(S^1)} = 0 \}. \]

A standard argument of dynamical systems then gives that $\omega(v)$ is nonempty, compact, and connected in $C^2(S^1)$. Note from Lemma 3.2 that $\omega(v) \subset \mathfrak{S}$, where $\mathfrak{S}$ is given by \[ \mathfrak{S} \] with $A = \frac{2+2\alpha}{3}$ and $\alpha$ satisfying \[ \mathfrak{S} \].

Following the above analysis, inspired by \[ \mathfrak{S} \], \[ \mathfrak{S} \], \[ \mathfrak{S} \], we further obtain the following convergence result.

**Proposition 3.5.** Under the assumption of Theorem 1.3, let $0 < C_1 \leq v \leq C_2 < \infty$ be a solution of the evolution equation
\[ -v_{tt} + 2A v_t = v_{\theta \theta} + A^2 v - \frac{\lambda}{v^2} - Pe^{-\beta t}, \quad (t, \theta) \in (t_0, +\infty) \times S^1, \]
where $A, \beta, \lambda > 0, P \geq 0$ are given constants. Then there exists a positive solution $w$ of
\[ w_{\theta \theta} + A^2 w - \frac{\lambda}{w^2} = 0 \quad \text{in} \ S^1, \]
such that
\[ \|v(t, \cdot) - w(\cdot)\|_{C^2(S^1)} \leq C(1 + t) - \frac{\theta}{t^{1 - \alpha}} \quad \text{as} \ t \to \infty, \]
where $\theta \in (0, \frac{1}{2})$ is a constant depending on $w$.

In order to prove Proposition 3.5 we need to borrow the following technical lemma, which was established in \[ \mathfrak{S} \].

**Lemma 3.6.** Let $0 \leq Z \in L^2((t_0, +\infty))$ be a measurable function on $(t_0, +\infty)$ and $\zeta \in (0, \frac{1}{2})$. If there exist two constants $C > 0$ and $T_0 \geq t_0$ such that
\[ \int_{t_0}^{+\infty} Z^2(s)ds \leq C \mathfrak{Z}^{1 - \zeta}(t) \quad \text{for a.e.} \ t \geq T_0, \]
then $Z \in L^1(T_0, +\infty)$.

**Proof of Proposition 3.5.** Because $\omega(v)$ defined by \[ \omega(v) \] is a nonempty, compact, and connected subset of $\mathfrak{S}$, we take $w \in \omega(v)$ and a sequence $\{t_n\}$ such that
\[ v(t_n, \cdot) \to w \quad \text{as} \ t_n \to +\infty \]
in $C^2(S^1)$. For convenience, we denote
\[ j(v) = A^2v - \frac{\lambda}{v^2}, \quad z(t) = Pe^{-\beta t}. \] (3.23)

In the following, we shall prove that $\omega(v)$ contains a single element $w$, and it satisfies the estimate \(3.21\) for large $t$. The proof is divided into the following four steps:

**Step 1:** For any $\varepsilon > 0$, define for all $t \geq t_0$,
\[ H(v) = \frac{1}{2} \int_{S^1} |v_t|^2 d\theta + (1 + 2A\varepsilon)E(v) + \varepsilon(v_{\theta\theta} + j(v), v_t), \] (3.24)
where
\[ E(v) = \int_{S^1} \left( \frac{1}{2}v_{\theta}^2 - J(v) \right) d\theta, \quad J(v) = \frac{A^2}{2}v^2 + \frac{\lambda}{v}. \]
We claim that for $\varepsilon > 0$ small enough,
\[ H(v) \equiv H_\infty \quad \text{and} \quad E(v) \equiv E_\infty \quad \text{on} \ \omega(v), \] (3.25)
where $H_\infty$ and $E_\infty$ are two constants depending on $\varepsilon$.

To prove the above claim, we first note that (3.19) can be rewritten as
\[-v_{tt} + 2Av_t = v_{\theta\theta} + j(v) - z(t), \quad (t, \theta) \in (t_0, +\infty) \times S^1, \] (3.26)
by (3.23). Multiplying (3.26) by $v_1$ and integrating on $S^1$, we obtain that
\[ \frac{d}{dt} \left( \frac{1}{2}||v_t||_{L^2(S^1)}^2 + E(v) \right) = -2A||v_t||_{L^2(S^1)}^2 - \int_{S^1} z(t)v_t d\theta, \] (3.27)
which implies that
\[ \frac{dH}{dt} = -2A||v_t||_{L^2(S^1)}^2 - \int_{S^1} z(t)v_t d\theta + 2A\varepsilon \frac{dE}{dt} + \varepsilon(v_{\theta\theta} + j(v), v_t). \] (3.28)

By (3.26), we also have
\[ (v_{\theta\theta} + j(v), v_t)_t = -\|v_{\theta t}\|_{L^2(S^1)}^2 + \int_{S^1} j'(v)v_t^2 d\theta - \|v_{\theta t} + j(v)\|_{L^2(S^1)}^2 + (v_{\theta t} + j(v), z(t)) \]
\[ + 4A^2\|v_t\|_{L^2(S^1)}^2 + 2A(z(t), v_t) - (v_{tt}, 2Av_t). \] (3.29)

Together with (3.28), this yields
\[ \frac{dH}{dt} = \int_{S^1} (-2A + \varepsilon j'(v))v_t^2 d\theta - \varepsilon\|v_{\theta t} + j(v)\|_{L^2(S^1)}^2 - \varepsilon\|v_{\theta t}\|_{L^2(S^1)}^2 \]
\[ - (z(t), v_t) + \varepsilon(v_{\theta t} + j(v), z(t)) \]
\[ \leq \int_{S^1} (-2A + \varepsilon j'(v) + \varepsilon)v_t^2 d\theta - \frac{\varepsilon}{2}\|v_{\theta t} + j(v)\|_{L^2(S^1)}^2 - \varepsilon\|v_{\theta t}\|_{L^2(S^1)}^2 + C\|z(t)\|_{L^2(S^1)}^2. \] (3.30)

Therefore, there exists a constant $K > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the following holds
\[ \frac{dH}{dt} - K\|z(t)\|_{L^2(S^1)}^2 \leq 0. \] (3.31)

That is, $\frac{d}{dt}H(t) \leq 0$, where $\tilde{H}(t) = H(v(t)) + K\int_{t}^{+\infty} \|z(s)\|_{L^2(S^1)}^2 ds$. Because $\tilde{H}$ is bounded from below by Lemmas 3.1 and 3.2, we infer that for any $\varepsilon \in (0, \varepsilon_0)$, $\tilde{H} \rightarrow H_\infty$ as $t \rightarrow +\infty$ for some constant $H_\infty$. Because
\[ \lim_{t \rightarrow +\infty} \int_{t}^{+\infty} \|z(s)\|_{L^2(S^1)}^2 ds = \lim_{t \rightarrow +\infty} 2\pi P^2 \int_{t}^{+\infty} e^{-2\beta s} ds = 0, \]
we obtain that
\[ \lim_{t \to +\infty} H(v(t)) = H_{\infty}, \] (3.32)
and hence
\[ \lim_{t \to +\infty} E(v(t)) = \frac{1}{1 + 2A\varepsilon} \lim_{t \to +\infty} H(v(t)) = \frac{1}{1 + 2A\varepsilon} H_{\infty} := E_{\infty}, \]
where Lemma 3.2 is used. By the definition of $\omega(v)$, it is easy to see that $E(v) \equiv E_{\infty}$, which implies that $H(v) \equiv H_{\infty}$ on $\omega(v)$, and the claim is therefore proved.

**Step 2:** We claim that there exist $\theta \in (0, \frac{1}{2})$ and $T_1 > 0$ such that
\[ |H(v) - H_{\infty}|^{1-\theta} \leq C\left(\|v_t\|_{L^2(S^1)} + \|v_{\theta \theta} + j(v)\|_{L^2(S^1)}\right), \quad \text{for all } t > T_1. \] (3.33)
To prove (3.33), we first note from Lemma 3.1 and Step 1 that for each $v_{\infty} \in \omega(v)$, there exist constants $\sigma_{v_{\infty}} > 0$ and $\theta_{v_{\infty}} \in (0, \frac{1}{2})$ depending on $v_{\infty}$, such that
\[ \|v_{\theta \theta} + j(v)\|_{L^2(S^1)} \geq |E(v) - E(v_{\infty})|^{1-\theta_{v_{\infty}}} = |E(v) - E_{\infty}|^{1-\theta_{v_{\infty}}}, \quad v \in B_{\sigma_{v_{\infty}}}(v_{\infty}), \] (3.34)
where we denote the set
\[ B_{\sigma_{v_{\infty}}}(v_{\infty}) := \{ v \in C^2(S^1) : \|v - v_{\infty}\|_{C^2(S^1)} < \sigma_{v_{\infty}} \}. \]
Because the union of balls \{ $B_{\sigma_{v_{\infty}}}(v_{\infty}) : v_{\infty} \in \omega(v)$ \} forms an open cover of $\omega(v)$, by the compactness of $\omega(v)$ in $C^2(S^1)$, there exist $v_{\infty}^{i} \in \omega(v)$ ($i = 1, 2, \cdots, m$) such that $\bigcup_{i=1}^{m} B_{\sigma_{i}}(v_{\infty}^{i})$ (for $i = 1, 2, \cdots, m$) is a subcover of $\omega(v)$, where the constants $\sigma_{i} := \sigma_{v_{\infty}^{i}}$ and $\theta_{i} = \theta_{v_{\infty}^{i}}$ corresponding to $v_{\infty}^{i}$ are as in (3.34).
From the definition of $\omega(v)$, there exists a sufficiently large $T_0 > t_0$ such that
\[ v(t) \in \bigcup_{i=1}^{m} B_{\sigma_{i}}(v_{\infty}^{i}), \quad t \geq T_0. \]
Because $v_{\infty}^{i} \in \omega(v) \subset \mathcal{G}$, by taking
\[ \theta = \min\{\theta_{i}, i = 1, 2, \cdots, m\} \in (0, \frac{1}{2}), \] (3.35)
we deduce from (3.24) and (3.34) that
\[ \|v_{\theta \theta} + j(v)\|_{L^2(S^1)} \geq |E(v(t)) - E_{\infty}|^{1-\theta}, \quad t \geq T_0. \] (3.36)
Using the Hölder inequality, we obtain from (3.24) that for any $w \in \omega(v)$,
\[ |H(v) - H(w)|^{1-\theta} \leq C_1 \left(\|v_t\|_{L^2(S^1)}^{2(1-\theta)} + |E(v) - E(w)|^{1-\theta} + \|v_{\theta \theta} + j(v)\|_{L^2(S^1)}^{1-\theta} \|v_t\|_{L^2(S^1)}^{1-\theta}\right) \] (3.37)
holds for some constant $C_1 > 0$. Because Young’s inequality yields that
\[ \|v_{\theta \theta} + j(v)\|_{L^2(S^1)}^{1-\theta} \|v_t\|_{L^2(S^1)}^{1-\theta} \leq \|v_{\theta \theta} + j(v)\|_{L^2(S^1)} + C\|v_t\|_{L^2(S^1)}, \]
we obtain from (3.37) that
\[ |H(v) - H(w)|^{1-\theta} \leq C \left(\|v_t\|_{L^2(S^1)}^{2(1-\theta)} + |E(v) - E(w)|^{1-\theta} + \|v_{\theta \theta} + j(v)\|_{L^2(S^1)} + \|v_t\|_{L^2(S^1)}^{1-\theta}\right). \] (3.38)
Recall from Lemma 3.2 that $\|v_t\|_{L^2(S^1)} \to 0$ as $t \to \infty$. Because $\frac{1-\theta}{\theta} > 1$ and $2(1-\theta) > 1$, we conclude from (3.25), (3.30), and (3.33) that there exist $T_1 > T_0$ and $C > 0$ such that (3.33) holds for all $t > T_1$, and Step 2 is therefore proved.

**Step 3:** We claim that
\[ \|v(t) - w\|_{C^2(S^1)} \to 0 \quad \text{as } t \to \infty, \] (3.39)
which implies that $\omega(v)$ contains a single element $w$, where $w$ is as in (3.22).

Denote

\[ Y(t) = \|v_t\|_{L^2(S^1)} + \|v_{\theta\theta} + j(v)\|_{L^2(S^1)}. \] (3.40)

Note first from (3.30) that

\[ \frac{dH}{dt} + C_1 Y^2(t) \leq C_2 \|z(t)\|^2_{L^2(S^1)}. \] (3.41)

Integrating (3.41) over $(t, \infty)$, where $t > T_1$, we obtain from (3.23) and (3.32) that

\[ H(\infty) - H(t) + C_1 \int_t^{\infty} Y^2(s) ds \leq C_2 \int_t^{\infty} \|z(s)\|^2_{L^2(S^1)} ds = C_3 e^{-2\beta t}, \] (3.42)

that is,

\[ C_1 \int_t^{\infty} Y^2(s) ds \leq H(t) - H(\infty) + C_3 e^{-2\beta t}. \] (3.43)

Because it follows from (3.33) that

\[ H(t) - H(\infty) \leq C Y^1(t), \quad t > T_1, \] (3.43)

we then have

\[ C_1 \int_t^{\infty} Y^2(s) ds \leq C Y^1(t) + C_3 e^{-2\beta t}, \quad t > T_1. \] (3.44)

By noting $0 < \theta < \frac{1}{2}$, there exists $T_2 > T_1$ such that

\[ \int_t^{\infty} e^{-4\beta(1-\theta)s} ds = C e^{-4\beta(1-\theta)t} \leq C e^{-2\beta t}, \quad t > T_2. \] (3.45)

Define

\[ Z(t) = Y(t) + e^{-2\beta(1-\theta)t}. \]

We then deduce from (3.44) and (3.45) that

\[ \int_t^{\infty} Z^2(s) ds \leq C \left( \int_t^{\infty} Y^2(s) ds + \int_t^{\infty} e^{-4\beta(1-\theta)s} ds \right) \leq C Z^1(t), \quad t > T_2. \] (3.46)

Applying Lemma 3.6 we thus conclude from (3.46) that

\[ \int_{T_2}^{\infty} Z(t) dt < +\infty, \]

which further implies that

\[ \int_{T_2}^{\infty} \|v_t\|_{L^2(S^1)} dt < +\infty. \] (3.47)

Because

\[ \|v(t) - v(s)\|_{L^2(S^1)} \leq \int_s^t \|v_t(\tau)\|_{L^2(S^1)} d\tau, \]

we obtain from (3.47) that

\[ v(t) \to w \quad \text{in} \quad L^2(S^1) \quad \text{as} \quad t \to \infty, \] (3.48)

where $w$ is the same as that of (3.22). By the relative compactness of the orbit $\{v(t, \cdot) : t \geq t_0\}$, we obtain the desired conclusion (3.39).

**Step 4:** We proceed to prove that the convergence rate of (3.21) holds true. Essentially, combining (3.33) with (3.41) yields that

\[ \frac{d}{dt}[H(v) - H(\infty)] + C_1 [H(v) - H(\infty)]^{2(1-\theta)} \leq C_2 e^{-2\beta t}, \quad t > T_1, \] (3.49)
where $0 < \theta < \frac{1}{2}$ is as in Step 2. By (3.49), direct calculations give that
\[
H(v) - H_\infty \leq C(1 + t)^{-\frac{1}{1-2\theta}} \quad \text{for sufficiently large} \ t > 0.
\] (3.50)

We then infer from (3.30) and (3.50) that for sufficiently large $t > 0$,
\[
\int_t^{2t} \mathcal{Y}(s)ds \leq \tilde{H}(t) - \frac{\tilde{H}(2t)}{2} \leq \tilde{H}(t) - H_\infty \leq H(t) + Ce^{-2\beta t} - H_\infty \leq C(1 + t)^{-\frac{1}{1-2\theta}},
\]
which yields that
\[
\int_t^{2t} \mathcal{Y}(s)ds \leq t^\frac{\theta}{2} \left( \int_t^{2t} \mathcal{Y}^2(s)ds \right)^{\frac{1}{2}} \leq C(1 + t)^{-\frac{6}{1-2\theta}}.
\]

We thus have
\[
\int_t^{+\infty} \mathcal{Y}(s)ds \leq \sum_{j=0}^{+\infty} \int_{2^j t}^{2^{j+1} t} \mathcal{Y}(s)ds \leq C \sum_{j=0}^{+\infty} (2^j t)^{-\frac{6}{1-2\theta}} \leq C(1 + t)^{-\frac{6}{1-2\theta}},
\]
which implies that
\[
\|v(t) - w\|_{L^2(S^1)} \leq \int_t^{+\infty} \|v_\circ\|_{L^2(S^1)}ds \leq \int_t^{+\infty} \mathcal{Y}(s)ds \leq C(1 + t)^{-\frac{6}{1-2\theta}}.
\]

Using the Sobolev imbedding theorem and Gagliardo–Nirenberg inequality, by Lemma 3.2 we thus conclude that for sufficiently large $t > 0$,
\[
\|v(t) - w\|_{C^2(S^1)} \leq C \|v(t) - w\|_{H^3(S^1)} \leq C_1 \|D^3(v(t) - w)\|_{L^2(S^1)} + C_2 \|v(t) - w\|_{L^2(S^1)} \leq C(1 + t)^{-\frac{6}{1-2\theta}},
\]
and the estimate (3.21) is then proved. The proof of this proposition is therefore completed. \(\square\)

By applying Proposition 3.3, we are ready to complete the proof of Theorem 1.3.

**Proof of Theorem 1.3** For all $P > 0$ and $\alpha \geq 0$, we first note that if $u(0) > 0$, then the continuity of $u$ implies that $0 < a \leq u(x) \leq b$ holds in $B_r(0)$ for some $0 < r < 1$. Furthermore, one can employ the elliptic $L^p$ theory and Schauder theory to conclude that $u \in C^{1,\alpha}(B_{\rho}(0)) \cap C^\infty(B_{r}(0))$, which further implies that Theorem 1.3(1) holds true.

In the following, we consider the alternative case where $u(0) = 0$. Under the assumption (1.13), Lemma 3.1 is then applicable. In (3.19), denote $A := \frac{2+\alpha}{2+\alpha} > 0$ for the case where $P = 0$, and further choose $\beta := \frac{4-\alpha}{4-\alpha}$ for the case where $P > 0, 0 \leq \alpha < 4$. Then the convergence of Theorem 1.3(2) follows directly from Proposition 3.5 in view of (1.6). This completes the proof of Theorem 1.3 \(\square\)

### 4 Refined Singular Behavior

In this section, we prove Theorem 1.5 on the refined singular behavior of solutions $u$ satisfying $\lim_{r \to 0^+} r^{-\frac{2+\alpha}{2+\alpha}} u(r, \theta) = \left( \frac{9\lambda}{(2+\alpha)^2} \right)^{\frac{1}{4}}$. Throughout this entire section, we define
\[
m := \left( \frac{9\lambda}{(2+\alpha)^2} \right)^{\frac{1}{4}} > 0, \quad \mu := \frac{2+\alpha}{3} > 0,
\]
and always suppose that $\alpha$ satisfies (1.17). We use the transformation
\[
V(t, \theta) = r^{-\frac{2+\alpha}{2+\alpha}} u(r, \theta) - m.
\] (4.2)
Therefore, \( \lim_{t \to +\infty} V(t, \cdot) = 0 \) and \( V(t, \theta) \) is a uniformly bounded solution of the following evolution elliptic equation

\[
- V_{tt} + 2\mu V_t = V_{\theta\theta} + \mu^2 V + \frac{\lambda V(V + 2m)}{m^2(V + m)^2} - P e^{-(2-\mu)t}, \quad (t, \theta) \in (t_0, +\infty) \times S^1, \tag{4.3}
\]

where \( \mu > 0 \) and \( m > 0 \) are as in (4.1). In the following, we investigate the asymptotic behavior of the Fourier coefficients of \( V(t, \theta) \) satisfying (4.3). We start with the following exponential decay of \( V(t, \cdot) \) as \( t \to +\infty \).

**Lemma 4.1.** Under the assumptions of Theorem 1.3 suppose that \( \alpha \) satisfies (1.17). Then there exists some constant \( \varepsilon > 0 \) such that

\[
\sup_{t \geq t_0} e^{\varepsilon t} \| V(t, \cdot) \|_{C^0(S^1)} < +\infty. \tag{4.4}
\]

**Proof.** Inspired by [4], on the contrary, suppose that (4.4) is false. Set \( \rho(t) = \| V(t, \cdot) \|_{C^0(S^1)} \); then \( \rho(t) \in C^0([t_0, +\infty)) \), and

\[
\lim_{t \to +\infty} \rho(t) = 0, \quad \lim_{t \to +\infty} \sup_{t \geq t_0} e^{\varepsilon t} \rho(t) = +\infty \tag{4.5}
\]

for any constant \( \varepsilon > 0 \). By applying [4, Lemma A.1], there exists a function \( \eta(t) \in C^\infty([t_0, +\infty)) \) such that

\[
\eta(t) > 0, \quad \eta'(t) < 0, \quad \lim_{t \to +\infty} \eta(t) = 0, \quad \lim_{t \to +\infty} e^{\varepsilon t} \eta(t) = +\infty, \quad \text{for any } \varepsilon > 0, \tag{4.6a}
\]

\[
0 < \lim_{t \to +\infty} \sup_{t \geq t_0} \frac{\rho(t)}{\eta(t)} < +\infty, \tag{4.6b}
\]

\[
(\eta'/\eta)'(t), \quad (\eta''/\eta)'(t) \in L^1((t_0, +\infty)), \quad \lim_{t \to +\infty} \frac{\eta'(t)}{\eta(t)} = 0, \quad \lim_{t \to +\infty} \frac{\eta''(t)}{\eta(t)} = 0. \tag{4.6c}
\]

Define \( w(t, \theta) = \frac{V(t, \theta)}{\eta(t)} \), such that \( w \) is bounded uniformly in \( [t_0, +\infty) \times S^1 \). Then by (4.3), we have \( w \) satisfies

\[
-w_{tt} + \left( \frac{\mu - \eta'}{\eta} \right) w_t = w_{\theta\theta} - P e^{-(2-\mu)t} + \left[ \frac{\mu^2 + \eta''}{\eta} + \frac{\lambda(2m + w\eta)}{m^2(m + w\eta)^2} \right] w, \tag{4.7}
\]

where \( (t, \theta) \in (t_0, +\infty) \times S^1 \). Note from (4.6a)-(4.6c) that all coefficients of equation (4.7) are bounded uniformly in \( (t_0, +\infty) \times S^1 \), and \( \lim_{t \to +\infty} P e^{-(2-\mu)t} = 0 \) in view of assumption (1.17). By applying \( L^p \) and the Schauder estimates of (4.7), similar to Lemma 6.2(1) one can deduce that there exists \( \delta \in (0, 1) \) such that \( w(t, \cdot), w_t(t, \cdot), w_{\theta}(t, \cdot), w_{tt}(t, \cdot), w_{\theta\theta}(t, \cdot), w_{tt\theta}(t, \cdot), w_{\theta\theta\theta}(t, \cdot), w_{tt\theta\theta}(t, \cdot) \), and \( w_{\theta\theta\theta\theta}(t, \cdot) \) all remain bounded in \( C^\delta(S^1) \) for all \( t \in [t_0, +\infty) \). Applying (4.6d), as in Lemma 6.2(2), one can further prove that \( w_t(t, \cdot) \) and \( w_{tt}(t, \cdot) \) tend to 0 in \( C^0(S^1) \)-topology as \( t \to +\infty \). So if we define the “\( \omega \)-limit set” \( \Gamma(\mathcal{L}') \) of the “orbit” \( \mathcal{L}' := \{ w(t, \cdot) : t \geq t_0 \} \) for (4.7) as

\[
\Gamma(\mathcal{L}') := \bigcap_{t \geq t_0} \left\{ w(\cdot, \cdot) : t \geq t_0 \right\},
\]

where the closure is with respect to the topology of \( C^2(S^1) \), then a standard argument of dynamical systems shows that \( \Gamma(\mathcal{L}') \) is a nonempty, compact, and connected set in \( C^2(S^1) \). Moreover, \( \Gamma(\mathcal{L}') \subset \mathcal{S}' \), where \( \mathcal{S}' \) is the set of stationary solutions of (4.7), i.e.,

\[
\mathcal{S}' := \left\{ w(\theta) \in C^2(S^1) : w'' + 3\mu^2 w = 0 \right\}.
\]

Because \( \sqrt{3\mu} > 0 \) is not an integer for \( \alpha \) satisfying (1.17), we obtain \( \mathcal{S}' = \{ 0 \} \), which contradicts (4.6d). This completes the proof of this lemma.

The following Fourier analysis gives better estimates of the power \( \varepsilon \) in (4.4), depending on the specific range of \( \alpha \) and \( P \).
Lemma 4.2. Under the assumptions of Theorem 4.1, suppose $V(t, \theta)$ satisfies (4.3) and $\mu$ is given by (4.4). Then there exists a constant $M > 0$ such that

1. If $\alpha \in A \setminus \{(2\sqrt{3} - 2, 3\sqrt{3} - 2) \cup (3\sqrt{3} - 2, 4)\}$, then for both cases $P = 0$ and $P > 0$, there holds
   \[
   \|V(t, \cdot)\|_{C^0(S^1)} \leq Me^{-(\sqrt{k^2 - 2\mu^2} - \mu)t}, \quad \alpha \in \{(k - 1)\sqrt{3} - 2, k\sqrt{3} - 2\} \cap [0, +\infty),
   \]
   where $k = 2, 3, 4, \cdots$ for $P = 0$, and $k = 2, 3$ for $P > 0$.

2. If $\alpha \in \{(2(\sqrt{3} - 1), 2\sqrt{3} - 2) \cup (3\sqrt{3} - 2, 4)\}$ and $P = 0$, then (4.8) still holds.

3. If $\alpha \in \{(2(\sqrt{3} - 1), 2\sqrt{3} - 2) \cup (3\sqrt{3} - 2, 4)\}$ and $P > 0$, then
   \[
   \|V(t, \cdot)\|_{C^0(S^1)} \leq MPe^{-(2\mu)t}.
   \]

Proof. Using Fourier analysis, we denote the Fourier series of $V$ and $\frac{\lambda(3mV^2 + 2V^3)}{m^3(V + m)^2}$ as follows:

\[
V(t, \theta) = (2\pi)^{-\frac{1}{2}} \sum_{k \in \mathbb{Z}} A_k(t)e^{ik\theta},
\]

where $m > 0$ is defined as in (4.1). It then follows from Lemma 3.4 that $w(t, \theta) = e^{\epsilon t}V(t, \theta)$ is bounded uniformly in $[0, +\infty) \times S^1$, where $\epsilon > 0$ is the same as that of Lemma 3.4. We thus obtain from (4.8) that $w$ satisfies the following evolution elliptic equation

\[
-w_{tt} + 2(\mu + \epsilon)w_t + w_{\theta\theta} - Pe^{-(2\mu - \epsilon)t} + \left[m^2 + \epsilon^2 + 2\mu \epsilon + \frac{\lambda(2m + \epsilon)^2}{m^2(m + \epsilon)^2}\right]w
\]

in $(t_0, \infty) \times S^1$. Similar to Lemma 3.2, one can derive from a priori estimates that $w$ and its derivatives, up to the third order, remain bounded on $[t_0, +\infty) \times S^1$, i.e.,

\[
\|e^{\epsilon t}V(t, \theta)\|_{C^3([t_0, +\infty) \times S^1)} < +\infty.
\]

We thus obtain that there exists $C > 0$ such that

\[
\sum_{k \in \mathbb{Z}} (k^2 + 1)|A_k(t)|^2 \leq Ce^{-2\epsilon t}, \quad \sum_{k \in \mathbb{Z}} (k^2 + 1)|A_k(t)|^2 \leq Ce^{-4\epsilon t}.
\]

In view of (4.1), equation (4.3) can be rewritten as

\[
-V_{tt} + 2\mu V_t + V_{\theta\theta} + 3\mu^2 V - \frac{\lambda(3mV^2 + 2V^3)}{m^3(V + m)^2} - Pe^{-(2\mu)t}.
\]

This implies that $a_k(t)$ is a bounded solution of

\[
a_k''(t) - 2\mu a_k'(t) + (3\mu^2 - k^2)a_k(t) = g_k(t) := \begin{cases} A_0(t) + \sqrt{2\pi}Pe^{-(2\mu)t}, & k = 0, \\ A_k(t), & k \neq 0, \end{cases}
\]

where $g_k(t)$ satisfies

\[
|g_k(t)| \leq \begin{cases} M_0e^{-2\epsilon t} + \sqrt{2\pi}Pe^{-(2\mu)t}, & k = 0, \\ \frac{M_0}{\sqrt{k^2 + 1}}e^{-2\epsilon t}, & k \neq 0, \end{cases}
\]

for some constant $M_0 > 0$. 

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Denote $d_{k,\mu} = k^2 - 2\mu^2$. By applying (4.13), the integration of (4.15) yields that

$$a_k(t) = \frac{1}{\sqrt{-d_{k,\mu}}} \left( e^{\mu t} \cos \sqrt{-d_{k,\mu}} t \int_{t}^{\infty} e^{-\mu s} g_k(s) \sin \sqrt{-d_{k,\mu}} s \, ds ight. $$

$$\left. - e^{\mu t} \sin \sqrt{-d_{k,\mu}} \int_{t}^{\infty} e^{-\mu s} g_k(s) \cos \sqrt{-d_{k,\mu}} s \, ds \right), \text{ for } |k| < \sqrt{2}\mu,$$

$$a_k(t) = e^{\mu t} \int_{t}^{\infty} (s-t) e^{-\mu s} g_k(s) \, ds, \text{ for } |k| = \sqrt{2}\mu,$$

$$a_k(t) = \frac{1}{2\sqrt{d_{k,\mu}}} e^{\mu(t-t_0)} \int_{t_0}^{\infty} g_k(s) e^{-(\mu+\sqrt{d_{k,\mu}}) s} \, ds$$

$$- \frac{1}{2\sqrt{d_{k,\mu}}} g_k(t_0) e^{-(\mu+\sqrt{d_{k,\mu}}) (t-t_0)} \int_{t_0}^{\infty} e^{-(\mu+\sqrt{d_{k,\mu}}) s} \, ds,$$

$$|k| > \sqrt{3}\mu.$$

It then follows from the above that there exists $M_1 > 0$, depending only on $\alpha$ and $P$, such that

$$|a_0(t)| \leq M_1 \left( Pe^{-2\mu t} + e^{-2\varepsilon t} \right),$$

$$|a_k(t)| \leq M_1 e^{-2\varepsilon t} \text{ for } 0 < |k| \leq \sqrt{3}\mu, \quad (4.17)$$

$$|a_k(t)| \leq |a_k(t_0)| e^{-(\sqrt{d_{k,\mu}} - \mu)(t-t_0)} + \frac{M_1}{|k|^2 + 1} \left( e^{-(\sqrt{d_{k,\mu}} - \mu)(t-t_0)} + e^{-2\varepsilon t} \right), \text{ for } |k| > \sqrt{3}\mu.$$  

In the following, we present proofs only for two special cases, as other cases can be proved in a similar way.

**Case 1:** $\alpha \in [0, 2\sqrt{3} - 2]$. In this case, we have

$$1 < \sqrt{3}\mu < 2, \quad \sqrt{2^2 - 2\mu^2} - \mu < 2 - \mu. \quad (4.18)$$

It then follows from (4.13) and (4.17) that there exist positive constants $C_4, C_5$, and $C_6$ such that

$$\|V(t, \cdot)\|_{L^2(S_1)}^2 = \sum_{k \in \mathbb{Z}} (k^2 + 1)|a_k(t)|^2$$

$$= |a_0(t)|^2 + \sum_{0 < |k| < 2} (k^2 + 1)|a_k(t)|^2 + \sum_{|k| \geq 2} (k^2 + 1)|a_k(t)|^2$$

$$\leq C_4 P^2 e^{-2\mu t} + C_5 e^{-4\varepsilon t} + \sum_{|k| \geq 2} (k^2 + 1)|a_k(t_0)|^2 e^{-2(\sqrt{2^2 - 2\mu^2} - \mu)(t-t_0)} \quad (4.19)$$

$$+ \sum_{|k| \geq 2} (k^2 + 1) \frac{C}{k^2 (k^2 + 1)} \left( e^{-2(\sqrt{2^2 - 2\mu^2} - \mu)(t-t_0)} + e^{-4\varepsilon t} \right)$$

$$\leq C_4 P^2 e^{-2\mu t} + C_5 e^{-4\varepsilon t} + C_6 e^{-2(\sqrt{2^2 - 2\mu^2} - \mu)t}.$$ 

By (4.18), this further implies that both for $P = 0$ and $P > 0$,

$$\|V(t, \cdot)\|_{C^0(S_1)} \leq N_1 e^{-2\varepsilon t} + N_2 e^{-(\sqrt{2^2 - 2\mu^2} - \mu)t}, \quad (4.20)$$

where $N_1$ and $N_2$ are positive constants. If $\sqrt{2^2 - 2\mu^2} - \mu \leq 2\varepsilon$, then (4.20) implies (4.18), and the proof of Case 1 is thus complete. Otherwise, we repeat the above procedure with $\varepsilon$ replaced by $2\varepsilon$ in (4.12). By taking finite similar steps, we reach a finite integer $n$ such that $2^n \varepsilon \geq \sqrt{2^2 - 2\mu^2} - \mu$, and the estimate (4.20) holds for $2\varepsilon$ replaced by $2^n \varepsilon$. Therefore, we conclude that there exists $C > 0$ such that

$$\|V(t, \cdot)\|_{C^0(S_1)} \leq C e^{-(\sqrt{2^2 - 2\mu^2} - \mu)t},$$
and hence the estimate (4.8) is proved in this case.

**Case 2:** \( \alpha \in (2\sqrt{3} - 2, 3\sqrt{3} - 2) \). In this case, we have \( 2 < \sqrt{3}\mu < 3 \) and

\[
2 - \mu \begin{cases} < \sqrt{3^2 - 2\mu^2 - \mu}, & \text{for } \alpha \in (2\sqrt{3} - 2, \frac{4}{3}\sqrt{10} - 2), \\ = \sqrt{3^2 - 2\mu^2 - \mu}, & \text{for } \alpha = \frac{4}{3}\sqrt{10} - 2, \\ > \sqrt{3^2 - 2\mu^2 - \mu}, & \text{for } \alpha \in (\frac{4}{3}\sqrt{10} - 2, 3\sqrt{3} - 2). \\
\end{cases} \tag{4.21}
\]

We then deduce from (4.17) that there exist constants \( C_i > 0 \) \( (i = 9, 10, 11, 12) \) such that

\[
\| V(t,\cdot) \|_{H^1(S^1)}^2 = \sum_{k \in \mathbb{Z}} |k(t) + 1| |a_k(t)|^2 = |a_0(t)|^2 + \sum_{0 < |k| < 3} |k(t) + 1| |a_k(t)|^2 = \sum_{|k| \geq 3} |k(t) + 1| |a_k(t)|^2 \leq C_9 P^2 e^{-2(1-\mu)t} + C_{10} e^{-4\mu t} + C_{11} e^{-2(1-\mu)t}.
\]

By (4.21), this further implies that

\[
\| V(t,\cdot) \|_{C^0(S^1)} \leq \begin{cases} N_3 e^{-2\mu t} + N_4 e^{-(\sqrt{3^2 - 2\mu^2 - \mu})t}, & \text{for } \alpha \in (2\sqrt{3} - 2, \frac{4}{3}\sqrt{10} - 2] \text{ and } P = 0, \\
N_5 e^{-2\mu t} + N_6 P e^{-(1-\mu)t}, & \text{for } \alpha \in (\frac{4}{3}\sqrt{10} - 2, 3\sqrt{3} - 2] \text{ and } P \geq 0, \\
N_7 e^{-2\mu t} + N_8 P e^{-(1-\mu)t}, & \text{for } \alpha \in (\frac{4}{3}\sqrt{10} - 2, 3\sqrt{3} - 2) \text{ and } P > 0,
\end{cases}
\]

where \( N_i \), \( i = 3, 4, 5, 6 \) are positive constants. Furthermore, similar to Case 1, one can obtain the estimate (4.8) for \( P = 0 \) and (4.9) for \( P > 0 \). This completes the proof of Lemma 4.2. \( \square \)

**Proposition 4.3.** Under the assumptions of Theorem 4.2, suppose \( V(t,\theta) \) satisfies (4.2) and \( \mu \) is given by (4.7). Then we have the following results:

1. If \( \alpha \in (\sqrt{3} - k - 1) \sqrt{3} - k\cdot \cdot \cdot \) for \( P = 0, \) and \( k = 2, 3, 4 \) for \( P > 0 \). \( \square \)

2. If \( \alpha \in (2\sqrt{3} - 2, \frac{4}{3}\sqrt{10} - 2] \cup (3\sqrt{3} - 2, 4) \) and \( P = 0 \), then (4.21) still holds.

3. If \( \alpha \in (2\sqrt{3} - 1, \frac{4}{3}\sqrt{10} - 2) \cup (3\sqrt{3} - 2, 4) \) and \( P > 0, \) then

\[
\lim_{t \to +\infty} e^{(2-\mu)t} V(t,\theta) = \frac{9P}{36 + 2(2 + \alpha)^2} \quad \text{in } C^2(S^1).
\]

4. If \( \alpha = \frac{4}{3}\sqrt{10} - 2 \) and \( P > 0 \), then there exist \( A_3 \in \mathbb{R} \) and \( \theta_3 \in S^1 \) such that

\[
\lim_{t \to +\infty} e^{(\sqrt{3^2 - 2\mu^2 - \mu})t} V(t,\theta) = A_3 \left( \sin(3\theta + \theta_3) + \frac{P}{9} \right) \quad \text{in } C^2(S^1).
\]

**Proof.** Define

\[
w(t,\theta) = e^{\gamma t} V(t,\theta),
\]

where \( \gamma > 0 \) is to be chosen later, such that \( w \) satisfies

\[
-w_{tt} + 2(\gamma + \mu)w_t = w_{\theta\theta} - P e^{(\gamma - (2-\mu))t} + \left[ \mu^2 + \gamma(\gamma + 2\mu) + \frac{\lambda(2m + we^{-\gamma t})}{m^2(m + we^{-\gamma t})^2} \right] w
\]

in \( (t_0, +\infty) \times S^1 \). Note that

\[
\frac{\lambda(2m + we^{-\gamma t})}{m^2(m + we^{-\gamma t})^2} - 2\mu^2 = O(e^{-\gamma t}) \quad \text{as } t \to +\infty.
\]
In the following, we only provide proofs for two special cases, as other cases can be proved in a similar way.

**Case 1:** $\alpha \in [0, 2\sqrt{2} - 2)$. In this case, we take $\gamma = \sqrt{2^2 - 2\mu^2} - \mu < 2 - \mu$. Then $\mu^2 + \gamma(\gamma+2\mu) = 4 - 2\mu^2$ and $w$ is bounded uniformly in $[t_0, +\infty) \times S^1$ in view of Lemma 4.2. Moreover, it follows from (4.28) that

$$-w_t + 2\sqrt{2^2 - 2\mu^2}w_t = w_{\theta\theta} + 4w - e^{-\gamma t}f(t, \theta) - Pe^{-[2^2 - 2\mu^2]t} \quad \text{in} \quad (0, +\infty) \times S^1. \quad (4.30)$$

Similar to Lemma 3.2, if we define the “$w$-limit set” $\Gamma(\mathcal{L}'')$ of the “orbit” $\mathcal{L}'' := \{w(t, \cdot) : t \geq t_0\}$ for (4.28) with $\gamma = \sqrt{2^2 - 2\mu^2} - \mu$ as

$$\Gamma(\mathcal{L}'') := \bigcap_{t \geq t_0} \{w(t_0) : t \geq t\},$$

where the closure is with respect to the topology of $C^2(S^1)$, then we obtain that $\Gamma(\mathcal{L}'')$ is a nonempty, compact, and connected set in $C^2(S^1)$. Moreover, $\Gamma(\mathcal{L}'') \subset \mathcal{S}'$, where $\mathcal{S}'$ is a nonempty, compact, and connected subset of

$$\{\psi(\theta) \in C^2(S^1) : \frac{d^2\psi}{d\theta^2} + 4\psi = 0\} = \{A_0 \sin(2\theta + \theta_0) : A_0 \in \mathbb{R}, \ \theta_0 \in S^1\}. \quad (4.31)$$

We next further analyze the limit behavior of $w(t, \cdot)$ as $t \to \infty$. Consider the bounded Fourier coefficients $a_k(t)$ of $w(t, \theta)$, which are defined by

$$a_k(t) := (2\pi)^{-1/2} \int_{S^1} w(t, \theta)e^{-ik\theta}d\theta, \quad k \in \mathbb{Z}.$$

It then follows from (4.30) that

$$a_k'(t) - 2\sqrt{2^2 - 2\mu^2}a_k(t) + (4 - k^2)a_k(t) = F_k(t) := \begin{cases} e^{-\gamma t}f_0(t) + \sqrt{2\pi}Pe^{-[2^2 - 2\mu^2]t}, & k = 0, \\ e^{-\gamma t}f_k(t), & k \neq 0, \end{cases} \quad (4.32)$$

where $f_k(t) = (2\pi)^{-1/2} \int_{S^1} f(t, \theta)e^{-ik\theta}d\theta$ is bounded uniformly in $[t_0, +\infty)$. Also denote $d_{k, \mu}^2 = k^2 - 2\mu^2$ as in Lemma 4.2. By the uniform boundedness of $a_k(t)$, the integration of (4.32) yields that

$$a_k(t) = \frac{1}{2\sqrt{d_{k, \mu}}} \int_{t}^{+\infty} (s-t)e^{-\sqrt{d_{k, \mu}}s}F_k(s)ds, \quad |k| = \sqrt{2}\mu,$$

$$a_k(t) = e^{\sqrt{d_{k, \mu}}t} \int_{t}^{+\infty} (s-t)e^{-\sqrt{d_{k, \mu}}s}F_k(s)ds, \quad |k| = \sqrt{2}\mu,$$

$$a_k(t) = \frac{1}{2\sqrt{d_{k, \mu}}} \int_{t}^{+\infty} \left[ e^{2\sqrt{d_{k, \mu}}s} - e^{-2\sqrt{d_{k, \mu}}s} \right] e^{-(\sqrt{d_{k, \mu}}+\sqrt{d_{k, \mu}})s}F_k(s)ds, \quad \sqrt{2}\mu < |k| < 2,$$

$$a_{\pm 2}(t) = B_{\pm 2} + \frac{1}{2\sqrt{d_{2, \mu}}} \int_{t}^{+\infty} \left( 1 - e^{2\sqrt{d_{2, \mu}}(t-s)} \right) F_{\pm 2}(s)ds, \quad |k| = 2,$$

$$a_k(t) = a_k(t_0)e^{(\sqrt{d_{k, \mu}}-\sqrt{d_{k, \mu}})(t-t_0)} + e^{(\sqrt{d_{k, \mu}}-\sqrt{d_{k, \mu}})(t-t_0)} \frac{e^{(\sqrt{d_{k, \mu}}+\sqrt{d_{k, \mu}})t_0}}{2\sqrt{d_{k, \mu}}} \int_{t_0}^{+\infty} e^{-(\sqrt{d_{k, \mu}}+\sqrt{d_{k, \mu}})s}F_k(s)ds$$

$$-e^{(\sqrt{d_{k, \mu}}-\sqrt{d_{k, \mu}})t} \int_{t_0}^{t} e^{-(\sqrt{d_{k, \mu}}-\sqrt{d_{k, \mu}})s}F_k(s)ds.$$
where \( B_{\pm 2} \) are two complex constants satisfying
\[
B_{\pm 2} = a_{\pm 2}(t_0) - \frac{1}{2\sqrt{d_{2,\mu}}} \int_{t_0}^{+\infty} e^{-(\sqrt{d_{2,\mu}} + \sqrt{d_{3,\mu}})s} F_k(s) ds, \quad \text{for } |k| > 2,
\]
We calculate from the above that there exists a constant \( C_{13} > 0 \) such that
\[
|a_0(t)| \leq C_{13} e^{-\beta t}, \quad \text{where } \beta = \min \{ \sqrt{d_{2,\mu}} - \mu, 2 - \sqrt{d_{2,\mu}} \},
\]
\[
|a_k(t)| \leq C_{13} e^{-\gamma t}, \quad |k| < 2,
\]
\[
|a_{\pm 2}(t) - B_{\pm 2}| \leq C_{13} e^{-\gamma t},
\]
\[
|a_k(t)| \leq C_{13} \left[ e^{-(\sqrt{d_{2,\mu}} - \sqrt{d_{3,\mu}})t} + e^{-\gamma t} \right], \quad |k| > 2.
\]
Therefore, \( a_k(t) \to 0 \) exponentially as \( t \to +\infty \) if \( |k| \neq 2 \), and \( a_{\pm 2}(t) \to a_{\pm} \in \mathbb{R} \) exponentially as \( t \to +\infty \). We then conclude from (4.31) and the above that there exist \( A_2 \in \mathbb{R} \) and \( \theta_2 \in S^1 \) such that
\[
\lim_{t \to +\infty} e^{(\sqrt{2^2 - 2^2 - \mu^2})^t} V(t, \theta) = A_2 \sin(2\theta + \theta_2) \quad \text{as } t \to +\infty,
\]
which is a special case of (4.24).

**Case 2:** \( \alpha \in (2\sqrt{3} - 2, \frac{3}{2}\sqrt{10} - 2) \) and \( P > 0 \). In this case, we take \( \gamma = 2 - \mu \), such that \( w \) is bounded uniformly in \([t_0, +\infty) \times S^1\) in view of Lemma 4.28. We then derive from (4.28) that
\[
-w_{tt} + 4w_t = w_{\theta\theta} + P + (4 + 2\mu^2)w - e^{-(2 - \mu)t} f(t, \theta) \quad \text{in } (0, +\infty) \times S^1,
\]
where \( f(t, \theta) \) is bounded uniformly in \([0, +\infty) \times S^1\). Moreover, we know that in this case, with \( \gamma = 2 - \mu \), \( \Gamma(L'') \subset \mathcal{G}'' \), where \( \mathcal{G}'' \) is a nonempty, compact, and connected subset of
\[
\left\{ \psi(\theta) \in C^2(S^1) : \frac{d^2 \psi}{d\theta^2} + (4 + 2\mu^2)\psi - P = 0 \right\}
\]
\[
= \left\{ A_0 \sin(\sqrt{4 + 2\mu^2}\theta + \theta_0) + \frac{P}{4 + 2\mu^2} : A_0 \in \mathbb{R}, \ \theta_0 \in S^1 \right\}
\]
\[
= \left\{ \frac{P}{1 + 2\mu^2}, \quad \text{for } \alpha \in (2\sqrt{3} - 2, \frac{3}{2}\sqrt{10} - 2), \right\}
\]
\[
= \left\{ A_0 \sin(3\theta + \theta_0) + \frac{P}{7} : A_0 \in \mathbb{R}, \ \theta_0 \in S^1 \right\}, \quad \text{for } \alpha = \frac{3}{2}\sqrt{10} - 2,
\]
because \( \sqrt{4 + 2\mu^2} \) is not an integer for \( \alpha \in (2\sqrt{3} - 2, \frac{3}{2}\sqrt{10} - 2) \), and \( \sqrt{4 + 2\mu^2} = 3 \) for \( \alpha = \frac{3}{2}\sqrt{10} - 2 \). Therefore, similar to Case 1, one can further derive from (4.30) that \( V(t, \theta) \) satisfies (4.24) for \( \alpha \in (2\sqrt{3} - 2, \frac{3}{2}\sqrt{10} - 2) \), and satisfies (4.26) for \( \alpha = \frac{3}{2}\sqrt{10} - 2 \). This completes the proof of Proposition 4.3.

We finally remark that Theorem 1.5 follows immediately from Proposition 4.3 and (4.2).

## A Proof of Lemma 3.1

In this appendix, inspired by [10, 11] we complete the proof of Lemma 3.1. We first establish the following estimates.

**Lemma A.1.** Suppose \( \phi \) is a nonnegative smooth function satisfying
\[
-\Delta \phi + \frac{2(2 + \alpha)^2}{9} \phi_x \leq \lambda \phi^4 + P \phi^2 \quad \text{in } B_R(x_0) \subset \mathbb{R}^2,
\]
where \( x_0 \in \mathbb{R}^2 \), \( 0 < R \leq 1 \), \( \alpha \geq 0 \), \( \lambda > 0 \), and \( P \geq 0 \) are given constants. Then there exists a constant \( \eta_0 > 0 \) depending only on \( \alpha \), \( \lambda \), and \( P \), such that the estimate

\[
\frac{1}{r^{1/3}} \int_{B_r(x)} \phi dx \leq \eta_0, \quad \forall B_r(y) \subset B_R(x_0),
\]

implies that

\[
\phi(x) \leq \frac{2}{R} \quad \text{in} \quad B_{\frac{R}{2}}(x_0).
\]

Proof. Inspired by [10, Lemma 2.1] and [11, Lemma 2.2], we denote \( K = \max_{|x-x_0| \leq R} (R-|x-x_0|)\phi(x) > 0 \), where \( 0 < R \leq 1 \). Choose \( \xi \in B_R(x_0) \) such that \( (R-|\xi-x_0|)\phi(\xi) = K \), and set \( \sigma = R - |\xi-x_0| \). Then we have \( \phi(x) \leq \frac{2K}{\sigma} \) for \( x \in B_{\frac{R}{2}}(\xi) \). Denote \( \mu = \frac{K}{\sigma} = \phi(\xi) \) and consider \( \psi(x) = \frac{1}{\mu} \phi(\xi + \frac{1}{\mu}x) \), such that \( \psi(x) \) satisfies

\[
\begin{aligned}
-\Delta \psi + \frac{1}{\mu^3} \frac{2(2+\alpha)^2}{9} \psi_x &\leq \lambda \psi^4 + \frac{P}{\mu^2} \psi^2 \quad \text{in} \quad B_{\frac{R}{2},\mu,\frac{1}{\mu}}(0), \\
0 \leq \psi &\leq 2 \quad \text{in} \quad B_{\frac{R}{2},\mu,\frac{1}{\mu}}(0), \\
\psi(0) &= 1.
\end{aligned}
\]

One can note that if \( K \leq 1 \) holds, then (A.2) follows immediately.

It now suffices to prove that \( K \leq 1 \) holds. First, if \( \mu \leq 1 \), then it is clear that \( K = (R-|\xi-x_0|)\mu \leq R \leq 1 \). It only remains therefore to prove that \( K \leq 1 \) for the case where \( \mu > 1 \). On the contrary, suppose \( \mu > 1 \) and \( K > 1 \). We then have \( \sigma \mu^{1/2} = K \mu^{1/2} > 1 \) and it thus follows from (A.3) that

\[
\begin{aligned}
-\Delta \psi + \frac{1}{\mu^3} \frac{2(2+\alpha)^2}{9} \psi_x &\leq (8\lambda + 2P)\psi \quad \text{in} \quad B_{\frac{R}{2},\mu,\frac{1}{\mu}}(0), \\
0 \leq \psi &\leq 2 \quad \text{in} \quad B_{\frac{R}{2},\mu,\frac{1}{\mu}}(0), \\
\psi(0) &= 1, \quad \frac{1}{\mu^{3/2}} \in (0,1) \quad \text{is bounded}.
\end{aligned}
\]

Moreover, we obtain from (A.1) that

\[
\int_{B_{\frac{R}{2}}(0)} \psi(x)dx = \mu^2 \int_{B_{\frac{R}{2},\mu,\frac{1}{\mu}}(\xi)} \phi(y)dy \leq 2^{-\frac{3}{2}} \eta_0,
\]

because \( \frac{1}{\mu^{3/2}} < \frac{1}{\mu^2} = \frac{\sigma}{2K} < \frac{\sigma}{2} \) for \( \mu > 1 \) and \( K > 1 \). By the elliptic estimate [8, p. 244], we then deduce from (A.3) that

\[
\psi(0) \leq C \int_{B_{\frac{R}{2},\mu,\frac{1}{\mu}}(0)} \psi(y)dy \leq 2^{-\frac{3}{2}} C \eta_0,
\]

where \( C \) is a constant depending only on \( \alpha \), \( \lambda \), and \( P \). By choosing \( \eta_0 > 0 \) small enough that \( 2^{-\frac{3}{2}} C \eta_0 < 1 \), we conclude from (A.3) that \( \psi(0) < 1 \), which is a contradiction in view of (A.3). This shows that \( K \leq 1 \) also holds for the case where \( \mu > 1 \), and we have finished.

Proof of Lemma 3.1. Under the assumption (1.13), we first prove that there exist \( C > 0 \) and \( t_1 \geq 0 \) such that

\[
v(t,\theta) \geq C, \quad \forall (t,\theta) \in (t_1, +\infty) \times S^1,
\]

where \( C \) depends only on \( \alpha \), \( \lambda \), and \( \beta \), and \( C_\beta \) given in (1.13). By taking \( w = \frac{1}{v} \), it follows from (1.17) that

\[
-(w_{tt} + w_{\theta\theta}) + \frac{2(2+\alpha)^2}{9} w_t - \frac{\lambda}{9} w^4 + P e^{-\frac{\lambda}{9} \theta} w^2 - \frac{2+\alpha}{9} w^2 - \frac{w_t^2 + w_{\theta\theta}^2}{w} \leq \lambda w^4 + P w^2.
\]

Let \( \eta_0 > 0 \) be the same as that of Lemma 1.1. Under the assumption (1.13), we then have

\[
\frac{1}{r^{1/3}} \int_{B_{r}(x)} w(y)dy \leq C r^{3/2} \eta_0 \leq \eta_0, \quad \forall r < \bar{r}, \quad x = (t,\theta) \in (t_0, +\infty) \times S^1,
\]
where the constant \( \bar{r} = \min\{\left(\frac{2\eta}{\nu}\right)^{\frac{1}{\beta}}, 1\} > 0 \) depends only on \( \beta, \alpha, \lambda, \) and \( P \). Applying Lemma A.3, we then derive that there exists \( t_1 > 0 \) such that

\[
 w(t, \theta) \leq \frac{2}{r}, \quad \forall (t, \theta) \in (t_1, +\infty) \times \mathbb{S}^1,
\]

from which the estimate (A.6) follows.

To complete the proof of Lemma 3.1 we only need to prove that \( u(x) \leq C|x|^{\frac{2+\alpha}{3}} \) near the origin. By applying (A.6), it is standard to derive (e.g., [10, Lemma 2.3]) that there exists a constant \( C > 0 \), depending only on \( \alpha, \lambda, \beta, \) and \( C_\beta \), such that the following spherical Harnack inequality holds

\[
 \sup_{|x|=r} u(x) \leq C \inf_{|x|=r} u(x), \quad \forall r \in (0, \frac{1}{2}). \tag{A.7}
\]

Define

\[
 \bar{u}(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) d\theta,
\]

where \( u(r, \theta) = u(x) \), such that by (1.5), \( \bar{u} \) satisfies

\[
 (r\bar{u}_r)_r = \frac{r}{2\pi} \int_0^{2\pi} \left( \lambda \frac{\alpha - 1}{v^2} + P \right) d\theta = \frac{r}{2\pi} \int_0^{2\pi} \left( \lambda \frac{\alpha - 1}{v^2} + \frac{P}{v^2} \right) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left( \lambda \frac{\alpha - 1}{v^2} + Pr \right) d\theta.
\]

This implies that \( r\bar{u}_r \) is increasing monotonically near the origin, and thus the limit \( \lim_{r \to 0^+} r\bar{u}_r \) exists. Moreover, it cannot be negative because otherwise \( \bar{u} \) must be negative near the origin, which is impossible. We now prove that \( \lim_{r \to 0^+} r\bar{u}_r = 0 \). If it is false, then \( r\bar{u}_r \geq C > 0 \), i.e., \( \bar{u}_r \geq \frac{C}{r} \), near the origin. This implies that

\[
 +\infty > \bar{u}(1) = \bar{u}(1) - \bar{u}(0) = \int_1^1 \bar{u}_r dr > C \int_0^{1} \frac{1}{r} dr = +\infty,
\]

which is a contradiction. Therefore, we have \( \lim_{r \to 0^+} r\bar{u}_r = 0 \).

By estimate (A.6), we have

\[
 (r\bar{u}_r)_r = \frac{1}{2\pi} \int_0^{2\pi} \left( \lambda \frac{\alpha - 1}{v^2} + Pr \right) d\theta \leq C\lambda \frac{\alpha - 1}{r} + Pr.
\]

Given any \( \varepsilon > 0 \), integrating the above inequality from \( \varepsilon \) to \( r \), we obtain that

\[
 r\bar{u}_r(r) - \varepsilon \bar{u}_r(\varepsilon) \leq \frac{3CA}{\alpha + 2} \left( r^{\frac{\alpha + 2}{2}} - \varepsilon^{\frac{\alpha + 2}{2}} \right) + \frac{P}{2} r^2 - \frac{P}{2} \varepsilon^2.
\]

Because \( \lim_{r \to 0^+} r\bar{u}_r = 0 \), the above estimate gives that

\[
 r\bar{u}_r(r) \leq \frac{3CA}{\alpha + 2} r^{\frac{\alpha + 2}{2}} + \frac{P}{2} r^2,
\]

i.e.,

\[
 \bar{u}_r(r) \leq \frac{3CA}{\alpha + 2} r^{\frac{\alpha - 1}{2}} + \frac{P}{2} r.
\]

We thus obtain that

\[
 \bar{u}(r) = \bar{u}(r) - \bar{u}(0) = \int_0^r \bar{u}(r), dr \leq \frac{9CA}{(\alpha + 2)^2} r^{\frac{\alpha + 2}{2}} + \frac{P}{4} r^2. \tag{A.8}
\]

By (A.7) and (A.8) we conclude that

\[
 u(x) \leq \sup_{|y|=|x|} u(y) \leq C \inf_{|y|=|x|} u(y) \leq C\bar{u}(|x|) \leq \frac{9CA}{(\alpha + 2)^2} |x|^{\frac{\alpha + 2}{2}} + \frac{P}{4} |x|^2 = \left( \frac{9CA}{(\alpha + 2)^2} + \frac{P}{4} |x|^{\frac{4+\alpha}{3}} \right) |x|^{\frac{2+\alpha}{3}}.
\]

Because either \( \alpha \geq 0 \) for \( P = 0 \) or \( 0 \leq \alpha < 4 \) for \( P > 0 \), the above estimate gives that \( u(x) \leq C|x|^{\frac{2+\alpha}{3}} \) holds near the origin, and the proof is therefore complete. \[\square\]
B  Lojasiewicz–Simon-type inequality

Recall that the set $\mathcal{S}$, which is defined by (3.10), denotes the set of all positive solutions for (1.9). Define the following functional

$$E(v) = \int_{S^1} \left( \frac{1}{2} v^2 - \frac{A^2}{2} v^2 - \frac{\lambda}{v^2} \right) d\theta,$$

(B.1)

where $A > 0$ and $\lambda > 0$ are as in Section 3. In this appendix, we derive the following Lojasiewicz–Simon-type inequality in terms of $E(\cdot)$.

**Lemma B.1.** For any $w \in \mathcal{S}$, which is defined by (3.10), there exist positive constants $\sigma > 0$ and $\bar{\theta} \in (0, \frac{1}{2})$, depending only on $w$, such that for all $v \in H^2(S^1)$ and $\|v - w\|_{H^2(S^1)} < \sigma$,

$$\| - v_{\theta\theta} - A^2 v + \frac{\lambda}{v^2} \|_{L^2(S^1)} \geq |E(v) - E(w)|^{1-\delta},$$

(B.2)

where $E(v)$ is defined by (B.1).

**Proof.** Our proof is inspired by [24]. We first consider the linearized problem of (3.8) near the equilibrium $w \in \mathcal{S}$:

$$L\varphi = -\varphi_{\theta\theta} - A^2 \varphi - \frac{2\lambda}{w^3} \varphi, \quad \varphi \in H^2(S^1).$$

It is easy to see that the operator $L$ defined on $H^2(S^1) \subset L^2(S^1)$ is a self-adjoint operator. Define the bilinear form $B[\cdot, \cdot]$ by

$$B[h, k] = (Lh, k)_{L^2(S^1)} = \int_{S^1} h_{\theta\theta} k_{\theta\theta} d\theta - A^2 \int_{S^1} h k_{\theta\theta} d\theta - 2\lambda \int_{S^1} \frac{1}{w^3} h k_{\theta\theta} d\theta, \quad h, k \in H^1(S^1).$$

Because it follows from (3.1) that $0 < C_1 < w(\bar{\theta}) < C_2$ on $S^1$, we have

$$|B[h, k]| \leq \|h_{\theta\theta}\|_{L^2(S^1)} \|k_{\theta\theta}\|_{L^2(S^1)} + C \|h\|_{L^2(S^1)} \|k\|_{L^2(S^1)} \leq C \|h\|_{H^2(S^1)} \|k\|_{H^2(S^1)},$$

and

$$\|h_{\theta\theta}\|_{L^2(S^1)}^2 \leq B[h, h] + C \|h\|_{L^2(S^1)}^2,$$

(B.3)

which then implies that

$$\|h\|_{H^2(S^1)}^2 \leq B[h, h] + (C + 1) \|h\|_{L^2(S^1)}^2.$$  

Thus, there exists a real constant $\gamma > 0$ such that the operator $\gamma I + L$ is coercive on $H^1(S^1)$. Using the Lax–Milgram theorem, a Fredholm alternative result then holds for the problem

$$L\varphi = h, \quad \varphi \in H^2(S^1) \subset L^2(S^1).$$

More precisely, we have either $\ker(L) = \emptyset$ or $\dim(\ker L) = m > 0$ for some $m \in \mathbb{N}$, in which case the equation $L\varphi = h$ has a solution if and only if $h \in (\ker L)^\perp$.

We now focus on the case $\dim(\ker L) = m > 0$ to finish the proof of the lemma. Let $(\varphi_1, \varphi_2, \cdots, \varphi_m)$ be the normalized orthogonal basis of $\ker(L)$ in $L^2(S^1)$, and denote by $\Pi$ the projection from $L^2(S^1)$ onto $\ker(L)$. Define the operator $\mathcal{L}$ from $H^2(S^1)$ onto $L^2(S^1)$ as follows:

$$\mathcal{L}\varphi = \Pi\varphi + L\varphi, \quad \varphi \in H^2(S^1).$$

Then

$$\mathcal{L} : H^2(S^1) \rightarrow L^2(S^1)$$

is a one-to-one and onto operator. Define $\psi = v - w$ and

$$\mathcal{M}\psi = -v_{\theta\theta} - A^2 v + \frac{\lambda}{v^2} : H^2(S^1) \rightarrow L^2(S^1).$$

(B.4)
It is easy to see that
\[ D.\mathcal{M}(0) = L, \]
where \( D.\mathcal{M} \) denotes the Frechet derivative of \( \mathcal{M} \). Denote
\[ \mathcal{N} \psi = \mathcal{M} \psi + \Pi \psi, \ \psi \in H^2(S^1), \]
such that
\[ D.\mathcal{N}(0) = \mathcal{L}. \]
Because \( \mathcal{L} \) is a one-to-one and onto operator, by the local inversion theorem in nonlinear analysis, there exist a small neighborhood \( W_1(0) \) of the origin in \( H^2(S^1) \), a small neighborhood \( W_2(0) \) of the origin in \( L^2(S^1) \), and an inverse mapping
\[ \mathcal{T} : W_2(0) \to W_1(0), \]
such that
\[ \mathcal{N}(\mathcal{T}(g)) = g, \ \forall g \in W_2(0), \]
and
\[ \mathcal{T}(\mathcal{N}(\psi)) = \psi, \ \forall \psi \in W_1(0). \]
Because \( 0 < C_1 < w \) on \( S^1 \), \( \frac{1}{w} = \frac{1}{\|w\|_{L^2}} \) is analytic in \( \psi \in W_1(0) \). Thus, the operator \( \mathcal{N} \) and its inverse mapping \( \mathcal{T} \) are all analytic. Furthermore, there exists a positive constant \( C > 0 \) such that
\[ \|\mathcal{T}(g_1) - \mathcal{T}(g_2)\|_{H^2(S^1)} \leq C\|g_1 - g_2\|_{L^2(S^1)}, \quad \forall g_1, g_2 \in W_2(0), \] (B.5)
and
\[ \|\mathcal{N}(\psi_1) - \mathcal{N}(\psi_2)\|_{L^2(S^1)} \leq C\|\psi_1 - \psi_2\|_{H^2(S^1)}, \quad \forall \psi_1, \psi_2 \in W_1(0). \] (B.6)
Denote
\[ \xi = (\xi_1, \xi_2, \cdots, \xi_m), \quad \Pi \psi = \sum_{j=1}^{m} \xi_j \varphi_j, \]
such that \( \sum_{j=1}^{m} \xi_j \varphi_j \in W_2(0) \) when \( |\xi| \) is sufficiently small. We now define \( \Gamma : \mathbb{R}^m \to \mathbb{R} \) as follows:
\[ \Gamma(\xi) = E(\mathcal{T}(\sum_{j=1}^{m} \xi_j \varphi_j) + w). \] (B.7)
It is clear that \( \Gamma(\xi) \) is analytic in a small neighborhood of the origin in \( \mathbb{R}^m \).

Straightforward calculations show that
\[ \frac{\partial \Gamma}{\partial \xi_j} = DE\left(\sum_{j=1}^{m} \xi_j \varphi_j\right) + w \cdot D.\mathcal{T}\left(\sum_{j=1}^{m} \xi_j \varphi_j\right) \varphi_j. \] (B.8)
We infer from (B.1) that
\[ DE(u) \cdot v = \int_{S^1} (u_\theta v_\theta - A^2 u v + \frac{\lambda}{u^2} v) d\theta, \ u, v \in H^2(S^1), \]
which implies that
\[ DE(u) \cdot v = \int_{S^1} (-u_\theta \bar{v}_\theta - A^2 u v + \frac{\lambda}{u^2} v) d\theta, \ u, v \in H^2(S^1). \] (B.9)
Because \( w \) is an equilibrium, it follows from (B.8) and (B.9) that \( \xi = 0 \) is a critical point of \( \Gamma(\xi) \). By \( \|\varphi_j\|_{L^2(S^1)} = 1 \), we infer from (B.4), (B.8), and (B.9) that
\[ \left| \frac{\partial \Gamma}{\partial \xi_j} \right| \leq \|\mathcal{M}\left(\sum_{j=1}^{m} \xi_j \varphi_j\right)\|_{L^2(S^1)} \|D.\mathcal{T}\left(\sum_{j=1}^{m} \xi_j \varphi_j\right)\|_{L^2(S^1)} \|\varphi_j\|_{L^2} \leq C \|\mathcal{M}\left(\sum_{j=1}^{m} \xi_j \varphi_j\right)\|_{L^2(S^1)} \|. \]
Because $\Pi \psi = \sum_{j=1}^{m} \xi_j \varphi_j$, we derive from the above that
\[
|\nabla \Gamma(\xi)| \leq C \{ \| \mathcal{M}(\mathcal{T}(\Pi \psi)) \|_{L^2(S^1)} \leq C \{ \| \mathcal{M}(\psi) \|_{L^2(S^1)} + \| \mathcal{M}(\mathcal{T}(\Pi \psi)) - \mathcal{M}(\psi) \|_{L^2(S^1)} \}.
\] (B.10)
Recall from $\mathcal{N}(\psi) = \mathcal{M}(\psi) + \Pi \psi$ and (B.6) that
\[
\| \mathcal{M}(\psi_1) - \mathcal{M}(\psi_2) \|_{L^2(S^1)} \leq \| \Pi \psi_1 - \Pi \psi_2 \|_{L^2(S^1)} + \| \mathcal{N}(\psi_1) - \mathcal{N}(\psi_2) \|_{L^2(S^1)} \leq C \| \psi_1 - \psi_2 \|_{H^2(S^1)}.
\] (B.11)
Note also from (B.5) that
\[
\| \mathcal{T}(\Pi \psi) - \psi \|_{H^2(S^1)} = \| \mathcal{T}(\Pi \psi) - \mathcal{T}(\mathcal{N}(\psi)) \|_{H^2(S^1)} \leq C \| \mathcal{N}(\psi) \|_{L^2(S^1)} = C \| \mathcal{M}(\psi) \|_{L^2(S^1)}.
\] (B.12)
We then deduce from (B.10), (B.11), and (B.12) that
\[
|\nabla \Gamma(\xi)| \leq C \| \mathcal{M}(\psi) \|_{L^2(S^1)}.
\] (B.13)
If $v \in W_1(0)$, then $v + t(\mathcal{T}(\Pi \psi) - v) \in W_1(0)$ for $t \in [0, 1]$. We thus infer from (B.7), (B.11), and (B.12) that for $v = w + \psi$,
\[
|E(v) - \Gamma(\xi)| = |E(v) - E(\mathcal{T}(\Pi \psi) + w)| = | \int_0^1 \frac{d}{dt} E(v + (1 - t)(\mathcal{T}(\Pi \psi) - \psi)) dt |
\leq \max_{0 \leq t \leq 1} \| \mathcal{M}(\psi + (1 - t)(\mathcal{T}(\Pi \psi) - \psi)) \|_{L^2(S^1)} \| \mathcal{T}(\Pi \psi) - \psi \|_{L^2(S^1)}
\leq \left( \| \mathcal{M}(\psi) \|_{L^2(S^1)} + C \| \mathcal{T}(\Pi \psi) - \psi \|_{H^2(S^1)} \right) \| \mathcal{T}(\Pi \psi) - \psi \|_{H^2(S^1)} \leq C \| \mathcal{M}(\psi) \|_{L^2(S^1)}^2.
\] (B.14)
By the Lojasiewicz inequality (cf. [13]), there exist a small constant $\sigma > 0$ and $\Theta \in (0, \frac{1}{2})$ such that
\[
|\nabla \Gamma(\xi)| \geq |\Gamma(\xi) - \Gamma(0)|^{1-\Theta} \quad \text{for} \quad |\xi| \leq \sigma.
\] (B.15)
Note that we can choose $\sigma > 0$ small enough that $\Pi \psi = \sum_{j=1}^{m} \xi_j \varphi_j \in W_2(0)$ holds for $|\xi| \leq \sigma$. By the definition of $\Gamma(\xi)$, we infer from (B.15) that if $\sigma > 0$ is small enough, then
\[
|\nabla \Gamma(\xi)| \geq |\Gamma(\xi) - E(w)|^{1-\Theta} \quad \text{for} \quad |\xi| \leq \sigma.
\] (B.16)
Applying the elementary inequality $|a + b|^{1-\Theta} \geq \frac{1}{2} |a|^{1-\Theta} - \frac{1}{2} |b|^{1-\Theta}$, we deduce from (B.13), (B.14), and (B.16) that for $\psi \in W_1(0)$,
\[
C \| \mathcal{M}(\psi) \|_{L^2(S^1)} \geq |\nabla \Gamma(\xi)| \geq |\Gamma(\xi) - E(w)|^{1-\Theta} \geq \frac{1}{2} |E(v) - E(w)|^{1-\Theta} - \frac{1}{2} C^{1-\Theta \Theta} \| \mathcal{M}(\psi) \|_{L^2(S^1)}^{2(1-\Theta)}.
\] (B.17)
Because $0 < \Theta < \frac{1}{2}$, we have $2(1 - \Theta) > 1$. It then follows from (B.17) that
\[
\| \mathcal{M}(\psi) \|_{L^2(S^1)} \geq C |E(v) - E(w)|^{1-\Theta}, \quad \psi \in W_1(0).
\] (B.18)
Choose both $\varepsilon > 0$ and $\sigma > 0$ sufficiently small, such that
\[
C |E(v) - E(w)|^{-\varepsilon} \geq 1, \quad \text{if} \quad \| \psi \|_{H^2(S^1)} \leq \sigma.
\] (B.19)
Combining (B.18) with (B.19), there exist sufficiently small constants $\varepsilon > 0$ and $\sigma > 0$ such that
\[
\| \mathcal{M}(\psi) \|_{L^2(S^1)} \geq |E(v) - E(w)|^{1-\Theta}
\]
holds for $\| \psi \|_{H^2(S^1)} \leq \sigma$, where $0 < \theta' := \theta - \varepsilon < \frac{1}{2}$. This therefore completes the proof for the case where $\dim(\ker L) = m > 0$.

As for the case where $\dim(\ker L) = 0$, similar to (B.14), one can also obtain that there exists $\sigma > 0$ depending on $w$ such that

$$|E(v) - E(w)| \leq C\| \mathcal{M}(\psi) \|^2_{L^2(S^1)}, \quad \| v - w \|_{H^2(S^1)} = \| \psi \|_{H^2(S^1)} < \sigma,$$

which can be rewritten as

$$\| \mathcal{M}(\psi) \|_{L^2(S^1)} \geq C|E(v) - E(w)|^{1-\frac{1}{2}}.$$  

Then (B.2) follows by an analysis similar to that above. This completes the proof of this lemma. □

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