Critical behaviour of (2+1)-dimensional QED: $1/N$-corrections

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Abstract. We present recently obtained results for dynamical chiral symmetry breaking studied within (2 + 1)-dimensional QED with $N$ four-component fermions. The leading and next-to-leading orders of the $1/N$ expansion are computed exactly in an arbitrary non-local gauge.

1 Introduction

In these Proceedings we present the results of our recent papers [1, 2], where the critical behavior of Quantum Electrodynamics in 2 + 1 dimensions (QED$_3$) has been studied. QED$_3$ is described by the Lagrangian:

$$L = \bar{\Psi}(i\hat{\nabla} - e\hat{A})\Psi - \frac{1}{4}F_{\mu\nu}^2,$$

where $\Psi$ is taken to be a four component complex spinor. In the presence of $N$ fermion flavours, the model has a $U(2N)$ symmetry. A (parity-invariant) fermion mass term, $m\bar{\Psi}\Psi$, breaks this symmetry to $U(N) \times U(N)$. In the massless case, loop expansions are plagued by infrared divergences. The latter soften upon analyzing the model in a $1/N$ expansion [3]. Since the theory is super-renormalizable, the mass scale is then given by the dimensionless coupling constant: $a = Ne^2/8$, which is kept fixed as $N \to \infty$. Early studies of this model [4, 5] suggested that the physics is rapidly damped at momentum scales $p \gg a$ and that a fermion mass term breaking the flavour symmetry is dynamically generated at scales which are orders of magnitude smaller than the intrinsic scale $a$. Since then, dynamical chiral symmetry breaking (D$\chi$SB) in QED$_3$ and the dependence of the dynamical fermion mass on $N$ have been the subject of extensive studies, see, e.g., [1, 2, 4–13].

One of the central issue is related to the value of the critical fermion number, $N_c$, which is such that D$\chi$SB takes place only for $N < N_c$. An accurate determination of $N_c$ is of crucial importance to understand the phase structure of QED$_3$ with far reaching implications from particle physics to planar condensed matter physics systems having relativistic-like low-energy excitations [14]. It turns out that the values that can be found in the literature vary from $N_c \to \infty$ [4, 6] corresponding to D$\chi$SB for all values of $N$, all the way to $N_c \to 0$ in the case where no sign of D$\chi$SB is found [7].
Of importance to us in the following, is the approach of Appelquist et al. [5] who found that $N_c = 32/\pi^2 \approx 3.24$ by solving the Schwinger-Dyson (SD) gap equation using a leading order (LO) $1/N$-expansion. Lattice simulations in agreement with a finite non-zero value of $N_c$ can be found in [8]. Soon after the analysis of [5], Nash approximately included next-to-leading order (NLO) corrections and performed a partial resummation of the wave-function renormalization constant at the level of the gap equation; he found [9]: $N_c \approx 3.28$.

Recently, in [1], the NLO corrections could be computed exactly in the Landau gauge upon refining the analysis of [10]. This led to $N_c \approx 3.29$, a value which is surprisingly close to the one of Nash in [9]. More recently, in [2], the results of [1] were generalized to an arbitrary non-local gauge [15]. Moreover, it was shown in [2] that a resummation of the wave-function renormalization yields a strong suppression of the gauge dependence of the critical fermion flavour number, $N_c(\xi)$ where $\xi$ is the gauge fixing parameter, which is such that $D_\chi$SB takes place for $N < N_c(\xi)$. Neglecting the gauge-dependent terms yields $N_c = 2.8469$, that coincides with results in [11]. In the general case, it is found that: $N_c(1) = 3.0084$ in the Feynman gauge, $N_c(0) = 3.0844$ in the Landau gauge and $N_c(2/3) = 3.0377$ in the $\xi = 2/3$ gauge where the leading order fermion wave function is finite. These results suggest that $D_\chi$SB should take place for integer values $N \leq 3$. Using a very different method, Herbut obtained [12] a close value: $N_c \approx 2.89$.

It is the purpose of this work to review some of the basic steps of papers [1, 2] which represent an essential improvement with respect to Nash’s approximate NLO results derived some 30 years ago.

## 2 Schwinger-Dyson equations

With the conventions of [1], the inverse fermion propagator is defined as: $S^{-1}(p) = [1 + A(p)][(i\hat{p} + \Sigma(p))]$ where $A(p)$ is the fermion wave function and $\Sigma(p)$ is the dynamically generated parity-conserving mass which is taken to be the same for all the fermions. The SD equation for the fermion propagator may be decomposed into scalar and vector components as follows:

$$\Sigma(p) = \frac{2a}{N} \int \frac{d^3k}{(2\pi)^3} \frac{\gamma^\mu D_{\mu\nu}(p-k)\Sigma(k)\Gamma^\nu(p,k)}{[1 + A(k)][k^2 + \Sigma^2(k)]},$$

$$A(p)p^2 = -\frac{2a}{N} \int d^3k \frac{D_{\mu\nu}(p-k)p^{\mu\nu}(k)}{[1 + A(k)][k^2 + \Sigma^2(k)]},$$

where $\Sigma(p) = \Sigma(p)[1 + A(p)]$, $D_{\mu\nu}(p)$ is the photon propagator in the non-local $\xi$-gauge:

$$D_{\mu\nu}(p) = \frac{p_{\mu\nu}(p)}{p^2 [1 + \Pi(p)]}, \quad p_{\mu\nu}(p) = g_{\mu\nu} - (1 - \xi) \frac{P_{\mu}P_{\nu}}{p^2},$$

$\Pi(p)$ is the polarization operator and $\Gamma^\nu(p,k)$ is the vertex function. In the following, (2) will be studied for an arbitrary value of the gauge-fixing parameter $\xi$. All calculations will be performed with the help of the standard rules of perturbation theory for massless Feynman diagrams as in [16], see also the recent short review [17]. For the most complicated diagrams, the Gegenbauer polynomial technique will be used following [18].
3 Gap equation at leading order

The LO approximations in the $1/N$ expansion are given by: $A(p) = 0$, $\Pi(p) = a/|p|$ and $\Gamma^\nu(p,k) = \gamma^\nu$, where the fermion mass has been neglected in the calculation of $\Pi(p)$. A single diagram contributes to the gap equation (2) at LO, see figure 1, and the latter reads:

$$\Sigma(p) = \frac{8(2 + \xi) a}{N} \int \frac{d^3 k}{(2\pi)^3} \frac{\Sigma(k)}{(k^2 + \Sigma^2(k))[|p - k| + a|p|]}.$$  \hspace{0.5cm} (4)

Following [5], we consider the limit of large $a$ and linearize (4) which yields:

$$\Sigma(p) = \frac{8(2 + \xi)}{N} \int \frac{d^3 k}{(2\pi)^3} \frac{\Sigma(k)}{k^2|p - k|}.$$  \hspace{0.5cm} (5)

The mass function may then be parameterized as [5]: $\Sigma(k) = B(k^2)^{-\alpha}$, where $B$ is arbitrary and the index $\alpha$ has to be self-consistently determined. Using this Ansatz, (5) leads to the LO gap equation: $(\beta^{-1} = \alpha(1/2 - \alpha)$ and $L = \pi^2 N$)

$$1 = \frac{(2 + \xi)\beta}{L} \quad \text{and} \quad \alpha_\xi = \frac{1}{4} \left( 1 \pm \sqrt{1 - \frac{16(2 + \xi)}{L}} \right),$$  \hspace{0.5cm} (6)

which reproduces the solution given by Appelquist et al. [5]. The gauge-dependent critical number of fermions: $N_c = N_c(\xi) = 16(2 + \xi)/\pi^2$, is such that $\Sigma(p) = 0$ for $N > N_c$ and $\Sigma(0) \approx \exp[-2\pi/(N_c/N - 1)^2]$, for $N < N_c$. Thus, $D_N$SB occurs when $\alpha$ becomes complex, that is for $N < N_c$.

The gauge-dependent fermion wave function may be computed in a similar way. At LO, (2) simplifies as:

$$A(p)p^2 = \frac{2a}{N} \text{Tr} \int \frac{d^4 k}{(2\pi)^4} \frac{P_{\mu\nu}(p-k)\gamma^\mu k^\nu}{k^2|p-k|},$$  \hspace{0.5cm} (7)

where the integral has been dimensionally regularized with $D = 3 - 2\varepsilon$. Taking the trace and computing the integral on the r.h.s. yields:

$$A(p) = \overline{\mu} \frac{2\pi^2}{p^{2\varepsilon}} C_1(\xi) + O(\varepsilon), \quad C_1(\xi) = \frac{1}{3\pi^2} \left( \frac{2}{3} - 3\xi \right) \left[ \frac{1}{\varepsilon} - 2 \ln 2 \right] + \frac{1}{3} - 6\varepsilon,$$  \hspace{0.5cm} (8)

where the $\overline{MS}$ parameter $\overline{\mu}$ has the standard form $\overline{\mu}^2 = 4\pi e^{-\gamma_E} \mu^2$ with the Euler constant $\gamma_E$. We note that in the $\xi = 2/3$-gauge, the value of $A(p)$ is finite and $C_1(\xi = 2/3) = +4/(9\pi^2 N)$. From (8), the LO wave-function renormalization constant may be extracted: $\lambda_A = \mu(d/d\mu)A(p) = 4(2 - 3\xi)/(3\pi^2 N)$ a result which coincides with the one of [19].

4 Next-to-leading order

We now consider the NLO contributions and parametrize them as:

$$\Sigma^{(\text{NLO})}(p) = \left( \frac{8}{N} \right)^2 B \frac{(p^2)^{-\alpha}}{(4\pi)^3} \left( \Sigma_A + \Sigma_1 + 2\Sigma_2 + \Sigma_3 \right),$$  \hspace{0.5cm} (9)

where each contribution to the linearized gap equation is represented graphically in figure 2. The gap equation has the following general form:

$$1 = \frac{(2 + \xi)\beta}{L} + \frac{\Sigma_1(\xi) + \Sigma_2(\xi) + 2\Sigma_3(\xi)}{L^2}, \quad \Sigma_i = \pi \Sigma_i, \quad (i = 1, 2, 3, A)$$  \hspace{0.5cm} (10)
Performing the calculation of the diagrams shown in figure 2 (see [1, 2]), the gap equation (10) may be written in an explicit form as:

$$1 = \frac{(2 + \xi)\beta}{L} + \frac{1}{L^2} \left[ 8S(\alpha, \xi) - 2(2 + \xi)\hat{\Pi}\beta + \left( -\frac{5}{3} + \frac{26}{3}\xi - 3\xi^2 \right) \beta^2 - 8\beta^3 \left( \frac{2}{3}(1 - \xi) - \xi^2 \right) \right],$$

where $\hat{\Pi} = 92/9 - \pi^2$ arises from the two-loop polarization operator in dimension $D = 3$ [20, 21] and $S(\alpha, \xi)$ contains the contributions of complicated diagrams. Considering (11) directly at the critical point $\alpha = 1/4$, $\beta = 16$, we have

$$L_c^2 - 16(2 + \xi)L_c - 8 \left[ S(\xi) - 4(2 + \xi)\hat{\Pi} - 16 \left( 4 - 50\xi/3 + 5\xi^2 \right) \right] = 0,$$

where $S(\xi) = S(\alpha = 1/4, \xi)$ and

$$8S(\xi) = (8 - \xi)R_1 + (\xi^2 - 1)R_2 - (7 + 16\xi - 3\xi^2) \frac{P_2}{16}, \quad R_1 = 163.7428, \quad R_2 = 209.175, \quad P_2 = 1260.720$$

Solving (12), we have two standard solutions:

$$L_{c,\pm} = 8(2 + \xi) \pm \sqrt{d_1(\xi)}, \quad d_1(\xi) = 8 \left[ S(\xi) - 8 \left( 4 - \frac{112}{3}\xi + 9\xi^2 + \frac{2 + \xi}{2} \hat{\Pi} \right) \right].$$

Combining these values with the one of $\hat{\Pi}$, yields: $N_c(\xi = 0) = 3.29, N_c(\xi = 2/3) = 3.09$, where “−” solutions are unphysical and there is no solution in the Feynman gauge. The range of $\xi$-values for which there is a solution corresponds to $\xi_- \leq \xi \leq \xi_+$, where $\xi_+ = 0.88$ and $\xi_- = -2.36$.

5 Resummation

Following [9], we would like to resum the LO term together with part of the NLO corrections containing terms $\sim \beta^2$. In order to do so, we will now rewrite the gap equation (11) in a form which is suitable for resummation. This amounts to extract the terms $\sim \beta$ and $\sim \beta^2$ from the complicated part of the fermion self-energy, $S(\alpha, \xi)$, yielding:

$$S(\alpha, \xi) = \frac{1}{4} (1 - \xi)\beta(3\beta - 8) - \frac{1}{2} \xi(4 + \xi)\beta + \tilde{S}(\alpha, \xi).$$

At the critical point $\alpha = 1/4 (\beta = 16)$, $\tilde{S}(\xi) = \tilde{S}(\alpha = 1/4, \xi)$ has the following form:

$$8\tilde{S}(\xi) = 8(1 - \xi)\tilde{R}_1 + (\xi^2 - 1)\tilde{R}_2 - (7 + 16\xi - 3\xi^2) \frac{\tilde{P}_2}{16}, \quad \tilde{R}_1 = 3.7428, \quad \tilde{R}_2 = 1.175, \quad \tilde{P}_2 = -19.28.$$
With the help of the results (16), the gap equation (11) may be written as:

\[
1 = \frac{(2 + \xi)\beta}{L} + \frac{1}{L^2} \left[ 8\xi(\alpha, \xi) - 2(2 + \xi)\hat{\Pi}\beta + \left(\frac{2}{3} - \xi\right)(2 + \xi)\hat{\beta}^2 + 4\beta \left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right) \right].
\]  

(17)

At this point (11) and (17) are strictly equivalent to each other and yield the same values for \(N_c(\xi)\). Equation (17) is the convenient starting point to perform a resummation of the wave function renormalization constant. To do it (see details in [2]) (17) can now be expressed as:

\[
1 = \frac{8\beta}{3L} + \frac{1}{L^2} \left[ 8\xi(\alpha, \xi) - \frac{16}{3}\beta \left(\frac{40}{9} + \hat{\Pi} \right) \right].
\]  

(18)

which displays a strong suppression of the gauge dependence even at NLO as \(\xi\)-dependent terms do exist but they enter the gap equation only through the rest, \(\hat{S}\), which is very small numerically.

We now consider (18) at the critical point, \(\alpha = 1/4 (\beta = 16)\), which yields:

\[
L_c^2 - \frac{128}{3}L_c - \left[ 8\xi(\xi) - \frac{256}{3}\left(\frac{40}{9} + \hat{\Pi} \right) \right] = 0.
\]  

(19)

Solving (19), we have two standard solutions:

\[
L_{c,\pm} = \frac{64}{3} \pm \sqrt{d_2(\xi)}, \quad d_2(\xi) = \left(\frac{64}{3}\right)^2 + \left[ 8\xi(\xi) - \frac{256}{3}\left(\frac{40}{9} + \hat{\Pi} \right) \right].
\]  

(20)

In order to provide a numerical estimate for \(N_c\), we have used the values of \(R_1, R_2, \) and \(P_2\) of (16). Combining these values together with the value of \(\hat{\Pi}\), yields, for \(N_c(\xi)\) ("--" solutions being unphysical):

\[
N_c(0) = 3.08, \quad N_c(2/3) = 3.04, \quad N_c(1) = 3.01.
\]  

(21)

Actually, solutions exist for a broad range of values of \(\xi\): \(\tilde{\xi}_- \leq \xi \leq \tilde{\xi}_+\), where \(\tilde{\xi}_+ = 4.042\) and \(\tilde{\xi}_- = -8.412\); this is consistent with the weak \(\xi\)-dependence of the gap equation. Moreover, following [22], we think that the "right(est)" gauge choice is one close to \(\xi = 2/3\) where the LO fermion wave function is finite. Indeed, upon resumming the theory, the value of \(N_c(\xi)\) increases (decreases) for small (large) values of \(\xi\). For \(\xi = 2/3\), the value of \(N_c\) is very stable, decreasing only by 1-2% during resummation. Finally, if we neglect the rest, i.e., \(\hat{S}(\xi) = 0\) in (19), the gap equation becomes \(\xi\)-independent and we have: \(\overline{L}_c = 28.0981\) and therefore: \(\overline{N}_c = 2.85\), a value that coincides with the one in [11].

### 6 Conclusion

We have presented the studies [1, 2] of DfSB in QED\(_3\) by including \(1/N^2\) corrections to the SD equation exactly and taking into account the full \(\xi\)-dependence of the gap equation. Following Nash, the wave function renormalization constant has been resummed at the level of the gap equation leading to a very weak gauge-variance of the critical fermion number \(N_c\). The value obtained for the latter, (21), suggests that DfSB takes place for integer values \(N \leq 3\) in QED\(_3\).

Notice that the large-\(N\) limit of the photon propagator in QED\(_3\) has precisely the same momentum dependence as the one in the so-called reduced QED, see [22]. One difference is that the gauge fixing parameter in reduced QED is twice less than the one in QED\(_3\). Such a difference can be taken into account with the help of our present results for QED\(_3\) together with the multi-loop results obtained in [20, 23]. The case of reduced QED, and its relation with dynamical gap generation in graphene which is the subject of active ongoing research, see, e.g., the reviews [24], was considered in our paper [25].
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