On Lusztig’s asymptotic Hecke algebra for SL₂

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Abstract

Let $H$ be the Iwahori-Hecke algebra and let $J$ be Lusztig’s asymptotic Hecke algebra, both specialized to type $A_1$. For $SL_2$, when the parameter $q$ is specialized to a prime power, Braverman and Kazhdan showed recently that a completion of $H$ has codimension two as a subalgebra of a completion of $J$, and described a basis for the quotient in spectral terms. In this note we write these functions explicitly in terms of the basis $\{t_w\}$ of $H$, and further invert the canonical isomorphism between the completions of $H$ and $J$, obtaining explicit formulas for the each basis element $t_w$ in terms of the basis $T_w$ of $H$. We conjecture some properties of this expansion for more general groups. We conclude by using our formulas to prove that $J$ acts on the Schwartz space of the basic affine space of $SL_2$, and produce some formulas for this action.

1 Introduction

1.1 Notation and conventions

We write $H$ for the Iwahori-Hecke algebra over the ring $\mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ corresponding to an affine Weyl group $W$ with length function $\ell$ and set $S$ of simple reflections. We recall that $H$ has a basis $\{T_w\}_{w \in W}$, where multiplication is defined by relations $T_wT_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$ and quadratic relation $(T_s + 1)(T_s - q) = 0$ for $s \in S$. Additionally, we have the Kazhdan-Lusztig basis

$$C_w = \sum_{y \leq w} (-1)^{\ell(w) - \ell(y)} q^{\frac{\ell(w) - \ell(y)}{2}} P_{y,w}(q^{-1}) q^{-\frac{\ell(w)}{2}} T_y$$

and the basis $\{C'_w\}_{w \in W}$, which we recall is related to the $\{C_w\}_{w \in W}$ basis by $C'_w = (-1)^{\ell(w)} j(C_w)$. Here $j$ is the algebra involution on $\mathcal{H}$ defined in [KL79] by $j(\sum a_w T_w) = \sum a_w (-1)^{\ell(w)} q^{-\ell(w)} T_w$. Several definitions will be given in terms of the structure constants of $\mathcal{H}$ in the basis $\{C'_w\}$, and we write $h_{x,y,z}$ to mean these elements of $\mathcal{A}$ such that $C_x C_y = \sum h_{x,y,z} C_z$.

Throughout, $\pi$ is a uniformizer of a fixed non-archimedean local field $F$ with ring of integers $\mathcal{O}$, and $q$ is the cardinality of the residue field $\mathcal{O}/\pi \mathcal{O}$. We shall write $G = SL_2$ as algebraic groups. When there is no room for confusion, we write $G$ for $G(F)$ as well. We fix the Borel subgroup $B$ of upper triangular matrices, and write $I$ for the corresponding Iwahori subgroup.

1.2 The asymptotic Hecke algebra

In [Lus87], Lusztig defined the asymptotic Hecke algebra $J$, which is a $\mathbb{Z}$-algebra equipped with an injection $\phi: H \hookrightarrow J \otimes_{\mathbb{Z}} \mathcal{A}$. Multiplication (see remark 3.3 in $J$, and the definition of the map $\phi$ is given combinatorially in terms of the structure constants for $\mathcal{H}$ written in the $\{C_w\}$ basis. As an abelian group, $J$ is free with basis $\{t_w\}_{w \in W}$.

It was also shown in [Lus87] that $\phi$ is an isomorphism after a certain completion, whose details we now recall. Let $\mathcal{A}$ be the ring of formal Laurent series in $q^{\frac{1}{2}}$, and let $\mathcal{A}^+$ be the ring of formal power series in $q^{\frac{1}{2}}$. 

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We obtain a completion $H$ of $H$ whose elements are (possibly infinite) $\mathbb{A}$-linear combinations $\sum b_x C_x$ such that $b_x \to 0$ in the Krull topology on $\mathbb{A}^+$, i.e. such that for any $N > 0$, $b_x \in (q^{\frac{1}{2}})^N \mathbb{A}^+$ for $\ell(x)$ sufficiently large. When working with the basis $\{C_{w'}\}_{w' \in W}$, we complete with respect to the negative powers of $q$. The involution $j$ naturally extends to a homeomorphism between these different completions. In the same way, we obtain a completion $J$ of $J$. The definition of $\phi$ (see definition 1) carries over verbatim, yielding an isomorphism $\phi : H \xrightarrow{\sim} J$.

The purpose of this paper is to study the map $\phi$ in more detail (in the case of $SL_2$). In what follows it will be convenient to twist $\phi$ by the involution $j$. Then our first main result is as follows: we give a formula for $(\phi \circ j)^{-1} (t_w)$ for all $w$ by an explicit calculation in a self-contained way. The resulting formulas are given in Theorem 3 and Corollary 1. As a byproduct we obtain the following result:

**Theorem 1.**

1. For any $w$ the element $(\phi \circ j)^{-1} (t_w) \in H$ has the form

$$\sum a_{w,x} C_x''$$

where $a_{w,x}$ is a polynomial in $q^{-\frac{1}{2}}$. Moreover, $(-1)^{\ell(x)} a_{w,x}$ has nonpositive integer coefficients.

2. For any $w$ the element $(\phi \circ j)^{-1} (t_w) \in H$ has the form

$$\sum b_{w,x} T_x$$

where $(q + 1) b_{w,x}$ is a polynomial in $q^{-\frac{1}{2}}$.

Let us remark that if we work with a finite Coxeter group instead of an affine one, then while the second assertion of Theorem 1 remains true (in general $q + 1$ must be replaced by the Poincaré polynomial of the corresponding flag variety), the first assertion is wrong in that case (it is in fact clear that for finite Coxeter groups if some of the coefficients $b_{w,x}$ are genuine rational functions (i.e. not polynomials) then the same will also be true for some of the $a_{w,x}$).

We conjecture that similar statements hold for any $G$. Namely, we conjecture that for any $G$ and any $w \in \hat{W}$ we have

$$(\phi \circ j)^{-1} (t_w) = \sum a_{w,x} C_x'$$

where $a_{w,x}$ is a polynomial in $q^{-\frac{1}{2}}$ such that $(-1)^{\ell(x)} a_{w,x}$ has nonpositive coefficients. If the conjecture is true, one can hope that these coefficients carry representation-theoretic information. Similarly, we conjecture that

$$(\phi \circ j)^{-1} (t_w) = \sum b_{w,x} T_x$$

where $(\sum w \in W q^{\ell(w)}) b_{w,x}$ is a polynomial in $q^{-1/2}$ (note that the sum in the 1st factor is over the finite Weyl group).

The above Conjecture (if true) is very interesting from a geometric point of view. Namely, it would be extremely interesting to categorify $J$ with its basis $t_w$. By this we mean the following. Let $K = \mathbb{C}((z))$, $O = \mathbb{C}[[z]]$. Consider the ind group-scheme $G(K)$. Let $I$ denote its Iwahori subgroup (which consists of all elements of $G(O)$ which at $z = 0$ lie in a fixed Borel subgroup $B$ of $G$). Let $Fl = G(K)/I$ denote the affine flag variety. Then the Iwahori-Hecke algebra $H$ is the Grothendieck ring of the bounded derived category of mixed $I$-equivariant constructible sheaves on $Fl$. Under this isomorphism the elements $C_x'$ correspond to the classes of irreducible perverse sheaves. The above conjecture suggests that the elements $t_w$ correspond to some canonical ind-objects in the above derived category. Moreover, these objects should have the property that every particular simple perverse sheaf appears there with finite multiplicity. It would be extremely interesting to find a construction of these objects.

1.3 The functions $f$ and $g$

In [BK18], Braverman and Kazhdan gave a spectral definition of two functions $f$ and $g$ in $J$ which span $J/H$ when $g$ is specialized to a prime power.
They are
\[ f = T_1 + T_{s_0} + \sum_{n=1}^{\infty} q^{-2n} (T_{(s_0 s_0)^n} + T_{s_0(s_1 s_0)^n} - q (T_{(s_0 s_1)^n} + T_{s_1(s_0 s_1)^n})) \]
and
\[ g = \sum_{w \in \mathcal{W}} (-1)^{\ell(w)} q^{-\ell(w)} T_w \]

We find their images under $\phi$ and show they lie in $J$ by explicit calculation in theorem 2

Remark 1. The function $f$ is defined in $[\text{BKIS}]$ directly as a function on $\text{SL}_2(\mathcal{O})/\text{SL}_2(F)/I$. Our definition is equivalent, as can be seen by writing
\[ \text{SL}_2(\mathcal{O}) \cdot \text{diag}(t^n, t^{-n}) \cdot I = I \cdot \text{diag}(\pi^n, \pi^{-n}) \cdot I \prod_I \left( \begin{smallmatrix} 1 & -1 \\ n & 1 \end{smallmatrix} \right) \text{diag}(\pi^{-n}, \pi^n) \cdot I. \]

1.4 Further Results

In $[\text{K}]$ we show in an elementary way that $J$ acts on $C^\infty_c(G/N)^I$, reproping a result of $[\text{BKIS}]$, and that $J$ lies in the Harish-Chandra Schwartz space of $G$. These results are recorded as Propositions 3 and 4, and Theorem 4.

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2 Formulas for the map $\phi$

2.1 The map $\phi$

From now on, $\tilde{W} = W \ltimes X_s(A) = W \ltimes \mathbb{Z} (\alpha^\vee)$ is the affine Weyl group for $G = \text{SL}_2$, with fixed presentation $\tilde{W} = \langle s_0, s_1 \mid s_0^2 = s_1^2 = 1 \rangle$. We write $S = \{s_0, s_1\}$, with $s_1$ the affine reflection, so that $W = \langle s_0 \rangle$ is the finite Weyl group. The identification between this presentation and the semidirect product realization of $\tilde{W}$ sends $s_0$ to the simple reflection $s_0$ corresponding to the positive root $\alpha$ of $\text{SL}_2$, and $s_1$ corresponds to $s_0 \pi$, where $\pi = \pi^\vee$. Our convention is that $\alpha^\vee$ is dominant, so that dominant coweights correspond to positive integers, with $\pi^n = \pi^{n \alpha^\vee} = (s_0 s_1)^n$ being dominant and $\pi^{-n} = (s_1 s_0)^n$ being antidominant. The distinguished involutions in $\tilde{W}$ are $\mathcal{D} = \{1, s_0, s_1\}$. We have
\[ C'_w = q^{-\ell(w)} \sum_{y \leq w} T_y, \]
where $\leq$ is the strong Bruhat order i.e. $y \leq w$ iff after writing $w$ in terms of the fixed presentation and deleting some letters, we obtain a word for $y$.

Example 1. We have $C'_1 = 1 = T_1$, and
\[ C'_{s_0 s_1 s_0} = q^{-2} (T_{s_0 s_1 s_0} + T_{s_1 s_0 s_0} + T_{s_0 s_1} + T_{s_0} + T_{s_1} + 1). \]

We now recall and specialize some definitions from [Lus87]. Recall that given two elements $w$ and $y$, we write $y \prec w$ if $y < w$, $\ell(w) - \ell(y)$ is odd, and $P_{y,w}(q) = \mu(y,w) q^{|\ell(w) - \ell(y)|}/2$ for $\mu(y,w)$ a nonzero integer. Two elements $y$ and $w$ are said to be joined if either $y < w$ or $w < y$. We write $w - x$ for joined elements and $\mu(y,w)$ for $\mu(y,w)$ or $\mu(w,y)$, whichever is defined. In our case, all the nonzero Kazhdan-Lusztig polynomials are constant with constant term 1, so that $w - y$ if and only if $|\ell(w) - \ell(y)| = 1$. Write $\mathcal{R}(w) = \{s \in S \mid ws < w\}$. If $w = rs_1$ is nontrivial, $\mathcal{R}(w) = \{s_1\}$ is a singleton.
Definition 1 (Lusztig’s $a$ function.). For $w \in \tilde{W}$, define $a(w)$ to be the smallest integer such that $(-q)^{a(w)} \in A^+$. 

Definition 2 ([Lus87]). The morphism of algebras $\phi: H \to J$ is defined by 

$$
\phi: \sum_{x \in \tilde{W}} b_x C_x \mapsto \sum_{x,z \in \tilde{W}} b_x h_{x,d,z} t_z.
$$

Over the course of the next three lemmas, we shall see this definition simplifies considerably in our case.

Lemma 1. Let $w \in \tilde{W}$ and $s = s_i$. Then 

$$
C_w C_s = \begin{cases} 
- \left( q^{\frac{1}{2}} + q^{-\frac{1}{2}} \right) C_w & \text{if } s \in R(w) \\
\sum_{\ell(w) - \ell(y) \text{ is odd}} C_y & \text{if } s \notin R(w) 
\end{cases}.
$$

Proof. This is just a special case of 4.3.2 in [Lus85], where there second case is given in general as 

$$
C_w C_s = \sum_{y \prec w \prec y} \tilde{\mu}(y,w) C_y.
$$

We have seen above that $y - w$ reduces to the condition that $\ell(w) - \ell(y)$ is odd. The integers $\tilde{\mu}(y,w)$ are all 1, as they are the leading coefficients of some Kazhdan-Lusztig polynomials. \qed

Lemma 2. Let $w \in \tilde{W}$. If $w = 1$, then $a(w) = 0$. Otherwise $a(w) = 1$.

Proof. It follows from inequality 1.3 (a) of [Lus87] that $a(1) = 0$, and by the remark on the top of p.538 of [Lus87], we have $0 \leq a(w) \leq 1$ for all $w \in \tilde{W}$; $s_0 s_1$ has infinite order. If $w \neq 1$ there is $s_i \in R(w)$, and by lemma 1 $h_{w,s,w} = - \left( q^{\frac{1}{2}} + q^{-\frac{1}{2}} \right)$, so $a(w) \neq 0$. \qed

Assembling Lemmas 1 and 2 we can describe $\phi$ explicitly.

Lemma 3. Let $i \neq j$ and $i, j \in \{0, 1\}$. Then 

$$
\phi(C_{s_i}) = - \left( q^{\frac{1}{2}} + q^{-\frac{1}{2}} \right) t_{s_i} + t_{s_is_j}.
$$

More generally, if $\ell(w) \geq 2$ and $w = rs_i$, then 

$$
\phi(C_w) = - \left( q^{\frac{1}{2}} + q^{-\frac{1}{2}} \right) t_{rs_i} + t_{r} + t_{rs_is_j}.
$$

Proof. We need only note that the condition $ys_j < y$ from Lemma 1 implies $y$ ends in $s_j$. \qed

Recall that $\phi$ preserves units, i.e. we have $\phi(C_1) = 1 + t_{s_1} + t_{s_0}$.

2.2 Preparation of the functions $f$ and $g$

It is easy to rewrite elements given in the $T_w$ basis to elements given in the $C'_w$ basis; the change of basis is “upper-triangular with power series entries.” Precisely, we have the following

Proposition 1. We have 

$$
T_w = \sum_{y \leq w} q^{\ell(y)} (-1)^{\ell(w) - \ell(y)} C'_y.
$$
Proof. Clearly the proposition is true for \( \ell(w) = 0 \), and for \( \ell(w) = 1 \). Now write \( w = s_ir_sj \), so that

\[
C'_w = q^{-\ell(w)} \left( T_w + T_{rsj} + T_{sir} + \cdots \right) = q^{-\ell(w)} \left( T_w + T_{rsj} + q^{\ell(w)} C'_{sir} \right)
\]

whence

\[
q^{\ell(w)} C'_w - q^{-\ell(w)} C'_{sir} = T_w + T_{rsj}.
\]

The claim follows by induction on \( \ell(w) \).

We can now rewrite the functions \( f \) and \( g \) in the \( C'_w \) basis, in preparation for writing them in \( C_w \) basis and applying \( \phi \) to them. In the case of \( g \), we have

\[
g = \sum_{w \in \tilde{W}} (-1)^{\ell(w)} q^{-\ell(w)} T_w = \sum_{w \in \tilde{W}} (-1)^{\ell(w)} q^{-\ell(w)} \left( \sum_{y \leq w} q^{\ell(w)} (-1)^{\ell(w) - \ell(y)} C_y \right),
\]

and we see that the coefficient \( b_w \) of \( C'_w \) is a power series in \( q^{-\frac{1}{2}} \) of order \( q^{\ell(w)} \). Indeed, \( C'_w \) will appear once in the expansion of \( T_w \), and then twice for each length greater than \( \ell(w) \), and thus

\[
b_w = (-1)^{\ell(w)} q^{-\ell(w)} q^{-\frac{1}{2}} + 2 \left( \sum_{n=\ell(w)+1}^{\infty} (-1)^{n-\ell(w)} q^{\ell(w)} q^{-n} \right).
\]

For \( z \in \tilde{W} \) such that \( \ell(z) = n \geq \ell(w) \), \( (-1)^n q^{-n} \) is the coefficient of \( T_z \) in rewriting \( g \), and \( (-1)^{n-\ell(w)} q^{\ell(w)} \) is the coefficient of \( C_w \) in the expansion of \( T_z \) according to proposition \( \square \)

Therefore

\[
b_w = (-1)^{\ell(w)} q^{-\ell(w)} \left( 1 + 2q^{-1} \frac{q^{-1}}{1-q^{-1}} \right),
\]

and so

\[
g = \left( 1 + 2q^{-1} \frac{q^{-1}}{1-q^{-1}} \right) \sum_{w \in \tilde{W}} (-1)^{\ell(w)} q^{-\ell(w)} C'_w.
\]  

(1)

We note that \( 1 + 2q^{-1} \frac{q^{-1}}{1-q^{-1}} = 1 + 2q^{-1} + 2q^{-2} + \cdots = \sum_{w \in \tilde{W}} q^{-\ell(w)} \) is a unit in \( \mathbb{Z}[q^{-\frac{1}{2}}] \).

Rewriting the function \( f \) is simpler, in the sense that no infinite series coefficients appear. In order to simplify the eventual calculation, we will work with a related function

\[
\tilde{f} = f - T_1 - T_{s_0} = \sum_{m=1}^{\infty} q^{-2m} \left( T_{s_0(s_1s_0)^m} + T_{(s_1s_0)^m} - q \left( T_{s_0(s_1)}^m + T_{s_1(s_0s_1)}^m \right) \right).
\]  

(2)

The first thing is again to calculate the coefficients \( b_w \) such that \( \tilde{f} = \sum_{w \in \tilde{W}} b_w C'_w \). For coefficients \( b_{s_0s_1} \), we see that instances of \( C_w \) are contributed by the \( A \)- and \( B \)-type terms starting from \( m = n \), and that, for length reasons, almost all the contributions cancel, leaving just \( -qq^{-n} \). The type \( A \) terms contribute starting from \( m = n \), and the type \( B \) terms, from \( m = n + 1 \). For the same reason, only the first instance of \( C'_{(s_1s_0)^m} \) coming from \( T_{s_0(s_1s_0)}^m \) fails to cancel, so that \( b_{s_0s_1} = q^n(1-q) \).

No terms \( C'_{(s_1s_0)^m} \) appear. Indeed, \( A \)- and \( B \)-type terms both begin contributing at \( m = n \), but have contributions with opposite signs. The same goes for \( C \)- and \( D \)-type terms, which both start contributing from \( m = n + 1 \). For exactly the same reasons (except the \( A \) and \( B \)-type terms start to contribute at \( m = n + 1 \) as well).

For \( b_{s_0(s_1s_0)^m} \), the \( A \)-type terms contribute from \( m = n \) onwards, and the \( B \)-type terms, from \( m = n + 1 \). All contributions except the first cancel, leaving \( q^{-n+1} \). The type \( C \) and \( D \) terms contribute from \( m = n + 1 \) and \( m = n + 2 \), respectively, with opposite signs as usual. Their contribution simplifies to \( qq^{-n+\frac{1}{2}} \), making \( b_{s_0(s_1s_0)^m} = q^{-n}(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \).
Therefore
\[ \hat{f} = \sum_{n=1}^{\infty} q^{-n}(1 - q)C_{(s_0,s_1)}^n + q^{-n} \left(q^{\frac{r}{2}} + q^{-\frac{r}{2}}\right) C_{s_0(s_1,s_0)}^n. \] (3)

**Definition 3.** If \( w \) and \( y \) are elements in \( \tilde{W} \), we say that \( w \) starts with \( y \) if we have reduced expressions \( y = s_{i_1} \cdots s_{i_n} \) and \( w = s_{i_1} \cdots s_{i_n} s_{i_{n+1}} \cdots s_{i_{n+m}} \) for some \( m \geq 0 \).

**Lemma 4.** We have
\[ \phi: \sum_{w \in \tilde{W}} q^{\ell(w)} C_w \mapsto -t_{s_0}, \]
and likewise
\[ \phi: \sum_{w \in \tilde{W}} q^{\ell(w)} C_w \mapsto -t_{s_1}. \]

**Proof.** Under \( \phi \), \( \sum_{w \in \tilde{W}} q^{\ell(w)} C_w \) is sent to
\[ q^{\frac{r}{2}} \left(- \left(q^{\frac{r}{2}} + q^{-\frac{r}{2}}\right) t_{s_0} + t_{s_0,s_1}\right) \] (4)
\[ + q \left(- \left(q^{\frac{r}{2}} + q^{-\frac{r}{2}}\right) t_{s_0,s_1} + t_{s_0} + t_{s_0,s_1,s_0}\right) \] (5)
\[ + q^{\frac{r}{2}} \left(- \left(q^{\frac{r}{2}} + q^{-\frac{r}{2}}\right) t_{s_0,s_1,s_0} + t_{s_0,s_1} + t_{s_0,s_1,s_0,s_1}\right) \] (6)
\[ + \cdots. \]

By Lemma 3 again, cancellation of terms appearing in \( \phi(C_w) \) with \( \ell(w) = n \) can occur only against terms appearing in \( \phi(C_{m}) \) with \( |n - m| = 1 \), and we see that after cancellations between the terms on lines 4 through 6, the sum stands as
\[ -t_{s_0} - q^2 t_{s_0,s_1} + t_{s_0,s_1,s_0,s_1} + \text{terms from longer words}. \]

Further, if \( r \) starts with \( s_0 \) and \( w = rs_0 \), the term \( q^{\frac{r}{2}} t_r \) from \( \phi(C_r) \) cancels with the term \( q^{\frac{r}{2}} t_r \) coming from \( \phi(C_w) \), and the term \( q^{\frac{\ell(w)-1}{2}} t_w \) from \( \phi(C_{w}) \) cancels with the term \( -q^{\frac{\ell(w)}{2}} q^{-\frac{r}{2}} t_w \) in \( \phi(C_{w}). \)

Likewise the terms \( -q^{\frac{\ell(w)-1}{2}} q^{\frac{r}{2}} t_w \) cancels with a term from \( \phi(C_{w,s_1}) \) and \( q^{\frac{\ell(w)}{2}} t_{ws_1} \) cancels with the term \( -q^{\frac{\ell(w)+1}{2}} q^{-\frac{r}{2}} t_{ws_1} \) from \( \phi(C_{ws_1}). \) The case for \( w \) ending in \( s_1 \) is identical, and cancellations happen between terms from two words ending both in \( s_0 \). The calculation for \( t_{s_1} \) is identical. \[ \square \]

### 2.3 Images under \( \phi \)

**Theorem 2.** We have

1. \( \phi(j(g)) = \left(1 + 2 \frac{q}{1-q}\right) t_1; \)
2. \( \phi(j(\hat{f})) = \left(q^{\frac{r}{2}} + q^{-\frac{r}{2}}\right) t_{s_0,s_1} - (q + 1)t_{s_0}. \)

**Proof.** Applying \( j \) to equation 1, we get \( j(g) = \left(1 + 2 \frac{q}{1-q}\right) \sum_{w \in \tilde{W}} q^{\ell(w)} C_w \). We conclude by adding the results of lemma 3 together and recalling that \( \phi \) preserves units.

Applying \( j \) to expression 6, we obtain
\[ j(\hat{f}) = (1 - q^{-1}) \sum_{n=1}^{\infty} q^n C_{(s_0,s_1)}^n + q^n \frac{r}{2} C_{s_0(s_1,s_0)}^n, \]
to which we apply Lemma 5 which appears below. \[ \square \]
3 The functions $t_w$

3.1 Inverting the map $\phi$

The formula for $\phi^{-1}$ is implicit in the proof of Lemma 4 Indeed, the lemma upgrades to

**Lemma 5.** Let $y = s_1 \cdots s_n$, and let $i = i_n$. Then

$$\phi: \sum_{w \in W} q^{\ell(w)} C_w \mapsto -q^{\frac{\ell(y)}{2}} t_y + q^{\frac{\ell(y)}{2}} t_{y s_i}.$$  

**Proof.** Direct calculation as in Lemma 4. Let $s_j$ be the generator that is not $s_i$. Then the first terms are

$$q^{\frac{\ell(y)}{2}} \left( -\left(q^\frac{1}{2} + q^{-\frac{1}{2}}\right)t_y + t_y s_i + t_{y s_j} \right) + q^{\frac{\ell(y)+1}{2}} \left( -\left(q^\frac{1}{2} + q^{-\frac{1}{2}}\right)t_{y s_j} + t_y + t_{y s_j s_i} \right) + \cdots,$$

and the cancellations in the proof of Lemma 4 pick up from this point, leaving only $-q^{\frac{\ell(y)-1}{2}} t_y + q^{\frac{\ell(y)}{2}} t_{y s_i}$. □

We can therefore calculate $\phi^{-1}(t_y)$ up to an error term of length $\ell(y) - \ell(y_i) < \ell(y)$. Given that we can calculate $\phi(t_{s_i})$, we can cancel the error terms inductively, yielding a formula for $\phi^{-1}$.

**Theorem 3.** Let $y = s_1 s_2 \cdots s_n$ so that $\ell(y) = n > 0$, and for $k \leq n$, write $y_k = s_1 s_2 \cdots s_k$. Then

$$-q^{\frac{n-k}{2}} \phi^{-1}(t_y) = \sum_{k=1}^{n} q^{n-k} \sum_{w \in W} q^{\ell(w)} C_w$$

**Proof.** Apply Lemma 5 to the last $\ell(y) - 1$ summands and Lemma 4 to the first. □

**Example 2.** We calculate $\phi^{-1}(t_{s_0 s_1 s_2})$, where $n = 4$. Under $\phi$,

$$q^2 C_{s_0 s_1 s_2} + q^\frac{3}{2} C_{s_0 s_1 s_2 s_0} + q^3 C_{s_0 s_1 s_2 s_0 s_1} + \cdots$$

$$+ q \left( q^\frac{3}{2} C_{s_0 s_1 s_2} + q^3 C_{s_0 s_1 s_2 s_0} + q^\frac{3}{2} C_{s_0 s_1 s_2 s_0 s_1} + \cdots \right)$$

$$+ q^2 \left( q^{C_{s_0 s_1}} + q^\frac{3}{2} C_{s_0 s_1 s_2} + q^3 C_{s_0 s_1 s_2 s_0} + q^\frac{3}{2} C_{s_0 s_1 s_2 s_0 s_1} + \cdots \right)$$

$$+ q^3 \sum_{w \in W} q^{\ell(w)} C_w$$

is sent to

$$q^2 t_{s_0 s_1 s_2} - q^\frac{3}{2} t_{s_0 s_1 s_2 s_0} + q^3 t_{s_0 s_1 s_2} - q^2 t_{s_0 s_1 s_2} + q^3 t_{s_0} - q^\frac{3}{2} t_{s_0 s_1} - q^3 t_{s_0} = -q^\frac{3}{2} t_{s_0 s_1 s_2}.$$

**Corollary 1.** If $y$ is as above, we have

$$-q^{\frac{n-1}{2}} (\phi \circ j)^{-1}(t_y) = \sum_{k=1}^{n} q^{k-n} \left( \sum_{w \in W} \frac{(-1)^{\ell(w)} q^{-\ell(w)+1}}{1+q} T_w + \sum_{w \in W} \frac{(-1)^{\ell(w)+1} q^{-\ell(w)}}{1+q} T_w \right)$$

$$+ \frac{(-1)^k q^{-k+1}}{1+q} \sum_{w \in W} \frac{T_w}{\ell(w) < k}.$$
The constant factor $q(1 + q)^{-1}$ in each summand appears as $\sum_{n=0}^{\infty} (-1)^n q^{-n}$. We may conjecture that some features of the above expansion are general.

**Conjecture 1.** For any reductive group $G$, after identification via $\phi \circ j$, the coefficient of any $T_y$ in the expansion of any $t_w$ is a rational function $b_{w,y}$ such that $(\sum_{w \in W} q^{\ell(w)})b_{w,x}$ is a polynomial in $q^{-\frac{1}{2}}$. If we expand $t_w$ in terms of the basis \{C'_x\}, the coefficient $a_{w,x}$ of $C'_x$ is a polynomial in $q^{-\frac{1}{2}}$ such that $(-1)^{\ell(x)}a_{w,x}$ has nonpositive integer coefficients.

### 3.2 The Harish-Chandra Schwartz space

From now on, we write $t_w$ for $(\phi \circ j)^{-1}(t_w)$.

Recall that we can interpret $\mathcal{H}$ as the convolution algebra $C_c(I\backslash G/I)$. Using Corollary 1, we can see in an elementary way that the functions $t_y$ lie in the Harish-Chandra Schwartz space $\mathcal{C}(G)$, whose definition we now recall.

Write $G = KAK$ where $K = \text{SL}_2(\mathcal{O})$ is a maximal compact subgroup and $A$ is the maximal torus of diagonal matrices. We can write any $g \in G$ as $g = k_1 \pi^\lambda(k_2)$, where $k_1, k_2 \in K$ and $\lambda(g)$ is a dominant coweight depending on $g$ i.e. in our case identifiable with a nonnegative integer. Define $\Delta(g) = q^{\langle \lambda, \rho \rangle}$, where $\rho$ is the half-sum of positive roots. The *Harish-Chandra Schwartz space* is then the space of functions $f: G \to \mathbb{C}$ such that $f$ is bi-invariant with respect to some open compact subgroup, and such that for all polynomial functions $p: G \to F$ and $m > 0$, we have

$$\Delta(g)|f(g)| \leq \frac{C}{(\log(1 + |p(g)|))^m}$$

for some constant $C$ depending on $m$ and $p$.

**Proposition 2.** The functions defined in Corollary 1 all lie in $\mathcal{C}(G)$.

**Proof.** Clearly the $t_y$ are all bi-invariant with respect to the Iwahori subgroup, which is open and closed in the compact subgroup $K$, as it is the preimage of the discrete group $B(\mathbb{F}_q)$, hence is open compact. Fix $y$ and let $f = t_y$.

Let $g \in K\pi^\lambda K = I\pi^\lambda I \sqcup I\pi^\lambda I \sqcup I\pi^\lambda I$ for $\lambda = \lambda(g) = n > 0$. Thus $g$ lies in an Iwahori double coset corresponding to an element of $\tilde{W}$ of length $2n \pm 1$. Here $\pi^\lambda$ is $(s_0 s_1)^n$. In our case, $\Delta(g) = q^{\lambda(g)}$, and so by Corollary 1 up to a multiplicative scalar depending on $f$ we have $\Delta(g)|f(g)| \leq q^{-\frac{n}{2} \pm 1}$ if $\lambda$ is identified with $n$. We must therefore bound the right-hand side of (7) uniformly in $n$. If $\lambda(g) = 0$, then $\Delta(g)|f(g)| \leq q$ up to the same scalar. Let $p$ and $m$ be given. Then

$$p(g) = p(k_1 a k_2) = \sum_{i=-N_1}^{N_2} (\pi^\lambda)^i p_i(k_1, k_2)$$

where the $p_i$ are polynomials in the eight entries of $k_1$ and $k_2$, and $N_1, N_2 \in \mathbb{N}$. Therefore

$$|p(g)| \leq \max_i |(\pi^\lambda)^i p_i(k_1, k_2)| = \max_i |\pi|^{|i|} C_p \leq q^{nM_p} C_p$$

for $C_p > 0$ and $M_p \in \mathbb{N}$ depending on $p$. Then

$$\log(1 + |p(g)|) \leq \log(q^{nM_p} + q^{nM_p} C_p)$$

$$\leq \log(q^{nM_p}(1 + C_p))$$

$$= nM_p \log(q(1 + C_p)^{1/nM_p})$$

$$\leq nM_p \log(q(1 + C_p))$$

$$= nM_p D_p$$
with $D_p > 0$. Therefore $D^m(\log(1 + |p(g)|)^m \geq n^{-m}$. By elementary calculus, there is $F_m > 0$ such that $n^m \leq F_m q^n$ for all $n \in \mathbb{N}$. It follows that

$$\frac{1}{q^{n+1}} \leq \frac{1}{q^{n-1}} \leq \frac{qF_m D^m_p}{(\log(1 + |p(g)|)^m}$$

as required.

3.3 Action on functions on the plane

3.3.1 The plane

Let $G = N(F)$ be subgroup of upper triangular matrices with 1s on the diagonal, and recall that $G/N = F^2 \setminus \{0\}$. Recalling the Iwasawa decomposition $G = KAN$, we see that $K$-orbits in $F^2 \setminus \{0\}$ are labelled by $Z = X_*(A)$, and are of the form

$$K\pi^n(1) = (\pi^n e \pi^n g).$$

if elements of $K$ are written $k = \left( \begin{smallmatrix} e & f \\ g & h \end{smallmatrix} \right)$. Note that we cannot have both $e$ and $g$ divisible by $\pi$, and therefore $K$-orbits are precisely of the form $\pi^n\mathcal{O} \setminus \pi^{n+1}\mathcal{O}$. Indeed, $e$ and $g$ are not both in $\pi\mathcal{O}$, so one is a unit. If $e$ is a unit, then $k = \left( \begin{smallmatrix} e & 0 \\ g & e^{-1} \end{smallmatrix} \right)$ is a suitable matrix. If $g$ is a unit, we can use $k = \left( \begin{smallmatrix} e & -g^{-1} \\ 0 & 1 \end{smallmatrix} \right)$.

Each $K$-orbit decomposes into two $I$-orbits. The two cases that partition the points $k\pi^n(1,0)^T$ are $k \in I$ and $k \notin I$. If $k \in I$, then the $I$-orbit consists of points of the form

$$\left( \begin{smallmatrix} \pi^n e \\ \pi^{n+1} g \end{smallmatrix} \right) \in \left( \begin{smallmatrix} \pi^n \mathcal{O}^X \\ \pi^{n+1} \mathcal{O} \end{smallmatrix} \right) \subset \pi^n\mathcal{O} \setminus \pi^{n+1}\mathcal{O}.$$

We denote the characteristic functions of such orbits $\psi_n$. The remaining orbit consists of points of the form

$$\left( \begin{smallmatrix} \pi^n e \\ \pi^{n+1} g \end{smallmatrix} \right) \in \left( \begin{smallmatrix} \pi^n \mathcal{O} \\ \pi^n \mathcal{O}^X \end{smallmatrix} \right) \subset \pi^n\mathcal{O} \setminus \pi^{n+1}\mathcal{O}.$$

We denote the characteristic functions of such orbits $\varphi_n$. The characteristic functions of the closures of these orbits are

$$\varphi_n := \sum_{k=n}^{\infty} \varphi_k + \psi_k$$

and

$$\bar{\psi}_n := \sum_{k=n}^{\infty} \psi_k + \varphi_{k+1}.$$

The Iwahori subgroup acts on functions on $G/N$ by translation as $(g \cdot f)(x) = f(g^{-1}x)$, and the functions $\varphi_n$ and $\psi_n$ give a basis for $C^\infty_c(F^2)$. Note that we have e.g. $\varphi_0 = \varphi_0 - \psi_0$. The functions $\varphi_n$ give a basis for $C^\infty_c(F^2)$. K.

Recall also that $I/G/N \simeq \tilde{W}$, hence $I$-invariant functions on the plane are the same as functions on the set of alcoves; in our case, intervals in $\mathbb{R}$ with integer endpoints. A basis for $C^\infty_c(F^2)$ is given under this identification by half lines with integer boundary points, corresponding to $I$-orbit closures. We now fix some relevant notation and identifications for alcoves. We identify the alcove corresponding to $\varphi_0$ with the interval $[-1,0]$ and the alcove corresponding $\psi_0$ with the interval $[0,1]$, so that e.g. $\varphi_2$ corresponds to $[3,4]$. 

9
3.3.2 Convolutions

We can now describe how the affine Hecke algebra acts on functions on the plane. The content of the below lemmas are well known; for a general combinatorial description of them with different normalizations, see [Lus97]. It will be useful to observe that the convolution action commutes with the right action of $2\mathbb{Z}$ on the set of alcoves.

We view the convolution action as follows: given $T_w$ and the characteristic function $\chi_X$ of an $I$-orbit $X$, we have a multiplication map

$$G \times X \to G/N,$$

which descends to the quotient of the left-hand side by the equivalence relation $(g, x) \sim (gi, i^{-1}x)$ for $i \in I$, yielding a map

$$G \times X \to G/N.$$

The image of this map is finitely-many $I$-orbits, and the coefficient of the characteristic function of each orbit is the number of points in the fibre over any point in that orbit.

It will be useful to note that $T_{s_0}$ and $T_{s_1}$ are related by the following automorphism $\Phi$ of $G$. Let $\Theta$ be the automorphism given by conjugate-transpose, $\Psi$ be conjugation by diag$(1, \pi) \in \text{GL}_2(F)$, and then $\Phi = \Psi \circ \Theta$. Observe that $\Phi$ preserves $I$, and therefore induces an automorphism of $H$, which exchanges $T_{s_0}$ and $T_{s_1}$. In particular, $T_{s_1}$ can be realized as the characteristic function of $K' \setminus I$, where $K'$ is the maximal compact subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bigg| a, d \in \mathcal{O}, \ c \in \pi\mathcal{O}, \ b \in \pi^{-1}\mathcal{O} \right\}.$$

The complement of $I$ is then the subset of such matrices with $b \in \pi^{-1}\mathcal{O}^\times$.

**Lemma 6.** We have

1. $T_{s_0} \ast \psi_n = \varphi_n$;
2. $T_{s_0} \ast \varphi_n = (q-1)\varphi_n + q\psi_n$;
3. $T_{s_1} \ast \varphi_n = \psi_{n-1}$;
4. $T_{s_1} \ast \psi_n = (q-1)\psi_n + q\varphi_{n+1}$.

**Proof.** It suffices to prove the formulas in the case $n = 0$. To prove the first formula, let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \setminus I := Y$ i.e. with $c \in \mathcal{O}^\times$ and $x$ be an element in the orbit $X$ corresponding to $\psi_0$. Then $x = (x, y)$ with $x \in \mathcal{O}^\times$ and $y \in \pi\mathcal{O}$, and

$$gx = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

so that $cx + dy \in \mathcal{O}^\times$, and $ax + by$ is obviously integral. Thus $T_{s_0} \ast \psi_0$ is proportional to $\varphi_0$. To prove the formula it remains to show that all fibres have size one. Without loss of generality the situation is $g_1(1, 0) = g_2(1, 0)$ i.e. the first columns of $g_1$ and $g_2$ agree. It follows that $g_2^{-1}g_1 \in \mathcal{N}(\mathcal{O})$, which stabilizes $(1, 0)$ in $\mathcal{O}^+ \cap I$. Therefore all fibres have size one.

To prove the second formula, let $g$ be as above and let $x = (x, y) \in \mathcal{O}^2$ with $y \in \mathcal{O}^\times$. Then $gx$ is an integral vector, and does not lie in $\pi\mathcal{O}^2$ as $x$ is nonzero modulo $\pi$, and $g$ is invertible modulo $\pi$. Therefore $T_{s_0} \ast \varphi_0$ is a linear combination of $\varphi_0$ and $\psi_0$. Consider the map

$$\xi: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{a}{c} \mod \pi$$

into $\mathbb{F}_q$, which descends to the quotient $I \setminus Y$. Therefore the fibre over any integral point $(x, y)$ in either orbit injects into $\mathbb{F}_q$. In the case where $y \in \mathcal{O}^\times$, then taking the fibre over $x = (0, -1)$ we see that $a \in \mathcal{O}^\times$, so that $\xi$ is into $\mathbb{F}_q^\times$ in this case. If $a \in \mathbb{F}_q^\times$, then

$$\begin{pmatrix} a & 0 \\ 1 & a^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \begin{pmatrix} \mathcal{O} \\ \mathcal{O}^\times \end{pmatrix}$$

10
is a product of a matrix in $K \setminus I$ with a vector in the orbit corresponding to $\varphi_0$. This shows that the coefficient of $\varphi_0$ is $q - 1$. For any $a \in \mathbb{F}_q$, we have

$$
\begin{pmatrix}
a & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
-1
\end{pmatrix} =
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\in \left(\mathcal{O}^\times, \pi \mathcal{O}\right).
$$

Therefore the coefficient of $\psi_0$ is $q$.

The case for the third formula is similar: if the matrices with entries $a_i, b_i, c_i, d_i$ are in $I$, then

$$
\begin{pmatrix}
a_1 & b_1 \\
c_1 & d_1
\end{pmatrix}
\begin{pmatrix}
-\pi \\
\pi^{-1}
\end{pmatrix}
\begin{pmatrix}
a_2 & b_2 \\
c_2 & d_2
\end{pmatrix}
\begin{pmatrix}
0 \\
1
\end{pmatrix} =
\begin{pmatrix}
\pi^{-1}a_1d_1 - \pi b_1b_2 \\
\pi^{-1}c_1d_2 - \pi b_2d_1
\end{pmatrix}
\in \left(\mathcal{O}^\times, \pi \mathcal{O}\right) \quad \text{(8)}
$$

has top entry in $\pi^{-1}\mathcal{O}^\times$ and bottom entry in $\mathcal{O}$. Indeed, $\pi \nmid a_1$ and $\pi \nmid d_2$, and $\pi \mid c_1$, so the bottom row of $\text{(8)}$ is integral. Therefore $T_{s_1} \ast \varphi_0$ is proportional to $\psi_{-1}$. To show the fibres all have size one, we can again calculate that any two matrices of the above form whose right columns agree are in the same $N^-(\mathcal{O}) \cap I = \text{Stab}_I((0, 1))$ coset.

For the fourth formula, the fact that we have

$$
\begin{pmatrix}
a_1 & b_1 \\
c_1 & d_1
\end{pmatrix}
\begin{pmatrix}
-\pi \\
\pi^{-1}
\end{pmatrix}
\begin{pmatrix}
a_2 & b_2 \\
c_2 & d_2
\end{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix} =
\begin{pmatrix}
a_1c_2\pi^{-1} - a_2b_1\pi \\
c_1c_2\pi^{-1} - a_2d_1\pi
\end{pmatrix}
\in \left(\mathcal{O}, \pi \mathcal{O}\right) \quad \text{(9)}
$$

is clear, as the top row of $\text{(9)}$ is integral, and therefore not in $\pi^{-r}\mathcal{O}^\times$ for $r > 0$, and $a_sd_1\pi$ has valuation exactly one, so taking $c_1 = 0$ excludes $\pi^2\mathcal{O}^\times$. We want to see that these products lie in

$$
\left(\mathcal{O}^\times, \pi \mathcal{O}\right) \bigg/ \left(\mathcal{O}, \pi \mathcal{O}\right),
$$

i.e. that these products cannot land in the orbits corresponding to $\varphi_0$ or $\psi_1$. But the bottom row of $\text{(9)}$ vanishes modulo $\pi$, and taking $c_1 = 0$, does not lie in $\pi^2\mathcal{O}$. Therefore $T_{s_1} \ast \varphi_0$ is a linear combination of $\psi_0$ and $\varphi_1$. To count points in the fibre, we will use that $T_{s_1} = \chi_{K' \setminus I}$. Define $\xi' : K' \setminus I \to \mathbb{F}_q$ by

$$
\xi' : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{d}{\pi b} \mod \pi,
$$

and note this function is right $I$-invariant. For any $d \in \mathbb{F}_q$, we have that

$$
\begin{pmatrix}
0 & t^{-1} \\
-t & d
\end{pmatrix}
\begin{pmatrix}
-1 \\
0
\end{pmatrix} =
\begin{pmatrix}
0 \\
\pi
\end{pmatrix}
\in \left(\pi \mathcal{O}, \pi \mathcal{O}^\times\right)
$$

is the product of a matrix in $K' \setminus I$ and a vector in $X$. Therefore the coefficient of $\varphi_1$ is $q$. Taking the fibre over $(1, 0)$, we see that $d \in \mathcal{O}^\times$, so that $\xi'$ is into $\mathbb{F}_q^\times$ in this case. If $d \in \mathbb{F}_q^\times$, then

$$
\begin{pmatrix}
d^{-1} & t^{-1} \\
0 & d
\end{pmatrix}
\begin{pmatrix}
d \\
0
\end{pmatrix} =
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\in \left(\mathcal{O}^\times, \pi \mathcal{O}\right)
$$

shows that the coefficient of $\psi_0$ is $q - 1$. \hfill \Box

Assembling these formulas recovers the following fact.

**Corollary 2.** The Iwahori-Hecke algebra $H$ acts on $C_c^\infty(F^2)$. We have

1. $T_{s_0} \ast \bar{\varphi}_n = q \bar{\varphi}_n$;
2. $T_{s_1} \ast \psi_n = q \psi_n$;
3. $T_{s_0} \ast \bar{\psi}_n = \bar{\varphi}_n - \bar{\psi}_n + q \bar{\varphi}_{n+1}$;
4. $T_{s_1} \star \tilde{\varphi}_n = \tilde{\psi}_{n-1} + \tilde{\varphi}_n + q\tilde{\psi}_{n+1}$.

**Lemma 7.** We have

1. $T_{s_1(s_1 s_0)}^n \star \psi_m = \psi_{m-n}$;
2. $T_{s_0(s_1 s_0)}^n \star \psi_m = \varphi_{m-n}$;
3. $T_{s_0(s_1)}^n \star \varphi_m = \varphi_{m-n}$;
4. $T_{s_1(s_0 s_1)}^n \star \varphi_m = \psi_{m-n-1}$;
5. $T_{s_1(s_1 s_0)}^n \star \varphi_m = q^{2n} \varphi_{m+n} + (q - 1) \sum_{k=1}^{2n} q^{2n-k} \psi_{m+n-k}$;
6. $T_{s_0(s_1 s_0)}^n \star \varphi_m = q^{2n+1} \varphi_{m+n} + (q - 1) \sum_{k=0}^{2n} q^{2n-k} \varphi_{m+n-k}$;
7. $T_{s_0(s_1)}^n \star \psi_m = q^{2n} \psi_{m+n} + (q - 1) \sum_{k=1}^{2n} q^{2n-k} \varphi_{m+n+1-k}$;
8. $T_{s_1(s_0 s_1)}^n \star \psi_m = q^{2n+1} \varphi_{m+n+1} + (q - 1) \sum_{k=0}^{2n} q^{2n-k} \psi_{m+n-k}$.

**Proof.** Formulas 1–4 follow directly from Lemma 6 and the remaining formulas follow from 1–4 and another application of the lemma.

**Remark 2.** Observe that the formulas in Lemma 7 recover those of Lemma 5 upon specifying $n$, provided that sums with decreasing indices are interpreted as empty.

We can now describe the action of $J$ on functions on the plane. To begin with, we present an elementary proof of the result from the discussion following equation 4.1 in [BK18], namely that $t_1$ acts trivially.

**Proposition 3.** We have $t_1 \star \psi_m = t_1 \star \varphi_m = 0$ for all $m$.

**Proof.** It suffices to check that $g$ (as identified with a scalar multiple of $t_1$) acts trivially, and for this it suffices to check that $g \star \varphi_0 = g \star \psi_0 = 0$. Now, $g$ sends $\psi_0$ to

$$
\psi_0 - q^{-1}(q - 1)(\varphi_0 + q\varphi_1 + (q - 1)\psi_0) + q^{-2}(q^2\psi_1 + (q - 1)(q\varphi_1 + \varphi_0) + \psi_1)
- q^{-3}(\varphi_{-1} + q^3\varphi_2 + (q - 1)(q^2\psi_1 + q\psi_0 + \psi_1))
+ q^{-4}(\psi_{-2} + q^4\psi_2 + (q - 1)(q^3\varphi_2 + q^2\varphi_1 + q\varphi_0 + \varphi_1))
- q^{-5}(\varphi_{-3} + q^5\varphi_3 + (q - 1)(q^4\psi_2 + q^3\psi_1 + q^2\varphi_0 + q\psi_1 + \psi_1))
+ \cdots
$$

and after cancellations between these terms we are left with

$$
-q^4(q^3\varphi_2 + q^2\varphi_1 + q\varphi_0 + \varphi_{-1}) - q^{-5}(\varphi_{-2} + q^5\varphi_3 - (q^4\psi_2 + q^3\psi_1 + q^2\varphi_0 + q\psi_{-1} + \psi_{-2})) + \cdots
$$
Further, all cancellation of terms corresponding to elements of length \( l \) occurs between terms corresponding to lengths \( l \pm 2 \), and proceeds as follows. We have

\[
-q^{-2n+1} \left( \varphi_{n+1} + q^{2n-1} \varphi_n + (q - 1) \sum_{k=0}^{2n-2} q^{2n-2-k} \psi_{n-1-k} \right) \tag{10}
\]

\[
+ q^{-2n} \left( \psi_n + q^{2n} \psi_n + (q + 1) \sum_{k=1}^{2n} q^{2n-k} \varphi_{n+1-k} \right) \tag{11}
\]

\[
-q^{-2n-1} \left( \varphi_n + q^{2n+1} \varphi_{n+1} + (q - 1) \sum_{k=0}^{2n} q^{2n-k} \psi_{n-k} \right) \tag{12}
\]

\[
+ q^{-2n-2} \left( \psi_{n-1} + q^{2n+2} \psi_{n+1} + (q - 1) \sum_{k=1}^{2n+2} q^{2n+2-k} \varphi_{n+2-k} \right) \tag{13}
\]

\[
-q^{-2n-3} \left( \varphi_{n-1} + q^{2n+3} \varphi_{n+2} + (q - 1) \sum_{k=0}^{2n+2} q^{2n+2-k} \psi_{n+1-k} \right) \tag{14}
\]

where line (10) corresponds to \( T_{s_0(s_0s_0)^n-1} \ast \psi_0 + T_{s_1(s_0s_1)^n-1} \ast \psi_0 \), line (11) corresponds to \( T_{(s_1s_0)^n} \ast \psi_0 + T_{(s_0s_1)^n} \ast \psi_0 \) and so on up to line (14) corresponding to \( T_{(s_0s_0s_0)^{n+1}} \ast \psi_0 + T_{s_1(s_0s_1)^{n+1}} \ast \psi_0 \).

We will explain the cancellation of the terms in line (12), the cancellation of terms in even power lines follows the same pattern. The lead term in line (12) cancels with the final term in \( q \) times the sum in line (13), and the second cancels with the first term in \( q \) times the sum. The first and last terms in \( q \) times the sum in line (12) cancel with the leading terms of line (11), and the middle terms cancel with \(-1\) times the sum in line (10). The terms in \(-1\) times the sum in line (12) cancel with the middle terms of \( q \) times the sum in line (14).

The cancellations in \( g \ast \varphi_0 \) follow the same pattern. \( \square \)

**Lemma 8.** We have (note that none of the sums below contains a \( T_1 \) term)

1. \[
\sum_{\substack{w \in W \\text{w starts with } s_0}} (-1)^{\ell(w)} q^{-\ell(w)} T_w \ast \varphi_m = -\varphi_m;
\]

2. \[
\sum_{\substack{w \in W \\text{w starts with } s_1}} (-1)^{\ell(w)} q^{-\ell(w)} T_w \ast \varphi_m = -\varphi_m;
\]

3. \[
\sum_{\substack{w \in W \\text{w starts with } s_0}} (-1)^{\ell(w)} q^{-\ell(w)} T_w \ast \psi_m = \varphi_{m+1};
\]

4. \[
\sum_{\substack{w \in W \\text{w starts with } s_1}} (-1)^{\ell(w)} q^{-\ell(w)} T_w \ast \psi_m = -\varphi_n.
\]

**Proof.** By periodicity it suffices to prove the lemma for \( m = 0 \). We evaluate each convolution term-by-term, and then explain the cancellations that occur between adjacent terms. After accounting for the contributions of the first few terms, this gives the results of the lemma.
In the case of formula [1] we have adjacent terms of the form
\[
-q^{-2n+1} \left( \frac{D}{q^{2n+1} \varphi_{n-1}} + (q - \frac{A}{1}) \sum_{k=0}^{2n-2} q^{2n-2-k} \varphi_{n-1-k} + \frac{C}{T_{(s_0^{(1)})^{n-1}}} \right) + q^{-2n} \left( \frac{B}{T_{(s_0^{(1)})^{n}}} \varphi_{n-1} \right) + q^{-2n-1} \left( q^{2n+1} \varphi_n + \left( \frac{B}{q} - 1 \right) \sum_{k=0}^{2n} q^{2n-k} \varphi_{n-k} \right).
\]

Adding the contributions \( A + B + C + D \) gives \(-(\varphi_n + \psi_n)\). The other terms cancel out similarly by induction. Starring this procedure from \( n = 1 \) captures the contributions of all terms starting from \( T_{s_0} \), although we must add the contribution of the first \( D \)- and \( B \)-type terms. Thus formula [1] is proved.

In the case of formula [2] we have adjacent terms of the form
\[
q^{-2n+2} \left( \frac{H}{q^{2n-2} \varphi_{n-1}} + (q - \frac{E}{1}) \sum_{k=1}^{2n-2} q^{2n-2-k} \varphi_{n-1-k} \right) + q^{-2n+1} \left( \frac{L}{T_{(s_0^{(1)})^{n-1}}} \psi_n \right) + q^{-2n} \left( q^{2n} \varphi_n + \left( \frac{F}{q} - 1 \right) \sum_{k=1}^{2n} q^{2n-k} \psi_{n-k} \right).
\]

Adding terms \( E + F + L + H \) gives \( \varphi_{n-1} + \psi_{n-1} \). We can start this cancellation from \( n = 2 \), adding the contributions of the first type \( L \) and \( E \) terms. This proves formula [2].

The remaining formulas follow the same pattern.

**Proposition 4.** For all \( m \):

1. We have \( t_{s_0} \ast \varphi_m = \varphi_m \), and \( t_{s_0} \ast \psi_m = 0 \). Thus \( t_{s_0} \) acts by a projector
   \[
   C_c^\infty (F^2) \rightarrow C_c^\infty (F^2)^K.
   \]

2. We have: \( t_{s_1} \ast \psi_m = \psi_m \), and \( t_{s_1} \ast \varphi_m = 0 \). Therefore \( t_{s_1} \) acts as \( \text{id} - t_{s_0} \).

**Proof.** It is enough to prove the proposition for \( m = 0 \). We first calculate \( t_{s_0} \ast (\varphi_0 + \psi_0) \), then using periodicity we will obtain formulas for \( t_{s_0} \ast (\varphi_n + \psi_n) \). The last step will be to take
   \[
   t_{s_0} \ast (\varphi_0) = \sum_{n=0}^{\infty} t_{s_0} \ast (\varphi_n + \psi_n).
   \]

Indeed, it follows from Corollary [1] and Lemma [5] that
\[
-q^{-1} (1 + q) (t_{s_0} \ast (\varphi_0 + \psi_0)) = -(1 + q^{-1}) (\varphi_0 + \psi_0)
\]
so that \( t_{s_0} \ast \varphi_0 = \varphi_0 \). The first statement follows. Again by periodicity used to calculate \( t_{s_0} \ast (\varphi_1) \), we get that \( t_{s_0} \ast \psi_1 = 0 \). Therefore \( t_{s_0} \) kills all basis functions that are not \( K \)-invariant.

The calculation for \( t_{s_1} \) is similar.
Remark 3. Thus far we have not used the ring structure on $J$, but it is in fact easy to see that $t_{s_0}$ is idempotent: the multiplication in $J$ is defined by $t_x \cdot t_y = \sum_{z \in \tilde{W}} \gamma_{x,y,z} t_z^{-1}$. The integers $\gamma_{x,y,z}$ are defined by the property that

$$q^{\frac{\ell(z)}{2}} h_{x,y,z} - \gamma_{x,y,z} \in q^\frac{1}{2} \mathbb{Z}[q^\frac{1}{2}].$$

We want to calculate $\gamma_{s_0,s_0,s_0}$. We have $h_{s_0,s_0,s_0} = -\left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right)$ if $z^{-1} = s_0 = z$ and is zero otherwise, by Lemma 3. Therefore $\gamma_{s_0,s_0,s_0}$ is the only nonzero structure constant, and is equal to minus the constant term of $-q^{\frac{1}{2}} \left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right) = -q - 1$.

That is, we have $t_{s_0}^2 = t_{s_0}$. For the same reason, the same holds for $t_{s_1}$, and for general groups, for any simple reflection.

Theorem 4. The algebra $J$ acts on $C^\infty_c(F^2)^I$.

Proof. The last sentence of Proposition 3 says that the identity in $J$ acts as the identity endomorphism; recall we have shown $t_1$ acts trivially in Proposition 3. By Corollary 2 the action of $H$ is well-defined. It follows by induction on $\ell(w)$ using Lemma 3 and Proposition 4 that each $t_w$ has a well-defined action.

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