I. INTRODUCTION

The study of multiband superconductors started from old works by Moskalenko, Suhl et al., Peretti and Kondo. They were regarded as a generalization of the Bardeen-Cooper-Schrieffer (BCS) theory to a multiband superconductor. They are, however, not only a simple generalization but also contain many interesting and fruitful properties. There appear many interesting properties in multi-component (multi-band) superconductors which are not found in a single-band superconductor. They are, for example, time-reversal symmetry breaking (TRSBD), the existence of massless modes, the existence of vortices with fractionally quantized-flux, unconventional isotope effect, and a new type of superconductors called the type-1.5 superconductivity. The existence of fractional quantum-flux vortices has been examined theoretically and is now an interesting subject. It is expected to be hard to observe fractional-flux vortices in real superconductors and there have been several experimental attempts to demonstrate them. There are theoretical and experimental proposals concerning the stabilization of fractionally quantized flux in a two-band superconductor. Recently, the experimental observation of fractional vortices in a thin superconducting bi-layer was reported as a magnetic flux distribution image taken by a scanning superconducting quantum interference device (SQUID) microscope. It is important to study physical phenomena concerning fractional-flux vortices.

In a two-band superconductor, a solution of the half-flux vortex exists. The dynamics of phase difference \( \varphi \) between two gaps is described by the sine-Gordon model:

\[
\nabla^2 \varphi - \alpha \sin \varphi = 0, \tag{1}
\]

where \( \alpha \) is a constant proportional to the Josephson coupling constant. We adopt that the phase difference \( \varphi \) has spatial dependence only in one direction \( x \). We have a kink solution for the boundary condition that \( \varphi \to 0 \) as \( x \to -\infty \) and \( \varphi \to 2\pi \) as \( x \to \infty \). We assume that we have two equivalent bands with the same intraband coupling constants. A half-quantum flux vortex exists at the edge of the kink. This is shown in Fig. 1, where the phase variables \( \theta_1 \) and \( \theta_2 \) change across the kink from 0 to \( \pi \) and 0 to \(-\pi \), respectively. A net change of \( \theta_1 \) is \( 2\pi \) by a counterclockwise encirclement of the vortex, and that of \( \theta_2 \) vanishes. Here the magnetic field is perpendicular to the plane. The phase \( \varphi \) changes from 0 to \( 2\pi \) as going across the kink. The other end of the kink is at the boundary of a superconductor. Two half-flux vortices are connected by the kink solution of the sine-Gordon equation. In a superconductor in three dimensions, the kink forms a surface (two-dimensional plane) between two fractional vortices.

Let us discuss a stability of fractional-flux vortices. We need energy to form the kink between fractional vortices, where the energy is proportional to the square root of the Josephson coupling. Since the energy of magnetic field is proportional to the square of the flux, the total energy of a pair of half-flux vortices is just half of that of a vortex with the unit flux. However, in general, the energy of the kink is larger than that, and thus the pair will be glued to be a unit-flux vortex. A pair of half-flux and anti-half-flux vortices will vanish completely because of the magnetic energy \( \propto \phi_0^2 \).

Our discussion is based on the effective action derived from the BCS theory for a multi-component superconductor. We focus on excitation modes concerning kinks and fractional vortices since the Nambu-Goldstone and Leggett modes have been studied by many authors. The derived action contains the second derivative with respect to time. In this paper we do not introduce the dissipation term which is the first derivative term with respect to time.

In this paper, we first consider a stability of fractional-flux vortices in multi-component superconductors. We argue that fractional-flux vortices indeed can be stabilized in a three-band superconductor with time-reversal symmetry breaking. Second, we examine excitation modes of kinks in two- and three-gap superconductors, and show that there are zero-energy mode and quantized
excitation modes. Generally, there are zero-energy modes when there are fractional vortices and kinks. The paper is organized as follows. In Section II, we discuss a stability of fractional-flux vortices on a kink in a superconductor with time-reversal symmetry breaking. In Section III, we investigate the quantization of a kink solution of the sine-Gordon model for two- and three-gap superconductors. We give a summary in the last Section.

Let us turn to examine the time reversal symmetry broken superconducting state. In a three-band superconductor, the time reversal symmetry (TRS) is broken when \( \Gamma_{12}\Gamma_{23}\Gamma_{31} > 0 \). In this case, a fractional vortex has two kinks because we have two phase-difference modes [13]. When three bands are equivalent, a fractional vortex with the flux \( \phi_0/3 \) exists with two kinks. It is possible to have a pair of fractional vortices which is the bound state of fractional vortices that corresponds to the meson under the duality transformation between charge and magnetic flux [13]. The attractive interaction works between two vortices because the energy of kink is proportional to the distance between fractional vortices.

A fractional-vortex pair on the kink can be stable in a superconductor with time reversal symmetry breaking. This is shown in Fig.3. Let us consider a three-band superconductor with equivalent bands and thus the Josephson potential \( V \) has \( |\Gamma_{12}| = |\Gamma_{23}| = |\Gamma_{31}| \) with \( \Gamma_{12}\Gamma_{23}\Gamma_{31} > 0 \). The ground state has a \( 2\pi/3 \) structure in this case. We adopt that there is a kink that connects two states with \((\theta_1, \theta_2, \theta_3) = (2\pi/3, 0, -2\pi/3)\) and \((\theta_1, \theta_2, \theta_3) = (-2\pi/3, 0, 2\pi/3)\). Now we put a vortex with unit flux \( \phi_0 \) on the kink. This vortex will be separated into two fractional vortices as shown in Fig.3(c) because the flux energy proportional to \((\phi_0/3)^2 + (2\phi_0/3)^2\) is less that of the unit flux proportional to \(\phi_0^2\). The energy of the kink remains the same after the separation of vortices. The phases of gaps are shown in Fig.3(c). Thus a pair of vortices can be stabilized in a TRSB superconductor. This is because the kink structure is not trivial in the TRSB case compared to that in the TRS case.

II. KINKS (DEFECTS) AND STATES OF FRACTIONAL-FLUX VORTICES

In a two-component (two-gap) superconductor, there may exist a kink that runs from one end of the boundary to the other end as shown in Fig.2(a). We call this type of kink the transverse kink in a superconductor. This is also the case for superconductors with bands more than two. We write the Josephson potential in the form

\[
V = \sum_{i>j} \Gamma_{ij} \cos(\theta_i - \theta_j),
\]

where \( \theta_j \) is the phase of the gap function in the \( j \)-th gap for \( j = 1, 2, \cdots, N \) (\( N \) is the number of gaps) and \( \Gamma_{ij} \) are constants. We assume that \( \Gamma_{ij} = \Gamma_{ji} \). When all the \( \Gamma_{ij} \) are negative, the ground state is at \( \theta_i - \theta_j = 0 \) mod\( 2\pi \). We have a kink that connects two states, for example, \((\theta_1, \theta_2, \cdots) = (0, 0, \cdots, 0)\) and \((\pi, -\pi, \pi, -\pi, \cdots)\). A fractional-flux vortex can exist at the edge of the kink as in the two-band case.

We assume that the kink surface is a plane and the magnetic field is applied being parallel to the plane. When the vortex with unit flux \( \phi_0 \) exists just at the plane (Fig.2(b)), the flux \( \phi_0 \) is separated into two half-flux vortices (Fig.2(c)). The kink will disappear to reduce the energy of the kink.

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(a) A kink with in a three-band superconductor with equivalent bands. We assume that the time reversal symmetry is broken. The phases of gap functions below the kink is \( \Phi_1 \equiv (2\pi/3,0,-2\pi/3) \) and above the kink is \( \Phi_2 \equiv (-2\pi/3,0,2\pi/3) \). Two states with \( \Phi_1 \) and \( \Phi_2 \) are degenerate with the same energy. (b) A vortex with the unit flux approaching the kink in (a). (c) The unit-flux vortex in (b) is separated into two fractional vortices on the kink. We defined \( \Phi_3 \equiv (0,-2\pi/3,2\pi/3) \), \( \Phi_4 \equiv (0,2\pi/3,-2\pi/3) \) and \( \phi_{1/3} = (2\pi/3,2\pi/3,2\pi/3) \). Note that \( \Phi_2 = \Phi_4 + 2\phi_{1/3} \text{ mod } 2\pi \). The phases of gaps are shown in the figure.

III. QUANTIZATION OF KINKS

A. Multi-component effective model

We consider the Hamiltonian given by

\[
H = \sum_{n} \int d\mathbf{r} \psi_{n\sigma}^\dagger(r) K_n(r) \psi_{n\sigma}(r) - \sum_{n} g_{nm} \int d\mathbf{r} \psi_{n1\uparrow}^\dagger(r) \psi_{m1\uparrow}^\dagger(r) \psi_{m\uparrow}(r) \psi_{n\uparrow}(r),
\]

where \( n \) and \( m \) stands for band indices and \( K_n(r) \) indicates \( K_n(r) = \mathbf{p}^2/(2m_n) - \mu \) where \( \mu \) is the chemical potential and \( m_n \) is the mass of electrons in the \( n \)-th band.

For a two-band superconductor, the Lagrangian density concerning phase variables is given by [61]

\[
\mathcal{L} = \frac{\rho_1}{4} (\partial_i \theta_1)^2 + \frac{\rho_2}{4} (\partial_i \theta_2)^2 - \frac{n_1}{8m_1} (\nabla \theta_1)^2 - \frac{n_2}{8m_2} (\nabla \theta_2)^2 - 2\gamma_{12} \Delta_1 \Delta_2 \cos(\theta_1 - \theta_2),
\]

where \( \theta_i \) is the phase of the \( i \)-th gap \( \Delta_i \), \( \rho_i \) is the density of states in the \( i \)-th band, \( n_i \) is the electron density and \( m_i \) indicates the mass of electrons in the \( i \)-th band. \( \gamma_{12} \) is the Josephson coupling constant given by \( \gamma_{12} = (g^{-1})_{12} \) for the BCS coupling constants \( g = (g_{ij}) \). We adopt that \( \gamma_{12} < 0 \) and put \( K = n_i/(2m_i) \).

We define the phase difference \( \varphi \) and the total phase \( \phi \) as \( \varphi = \theta_1 - \theta_2 \) and \( \phi = \theta_1 + a\theta_2 \) where \( a \) is a real constant. We put \( v_0 \) as

\[
v_0^2 = \frac{K_1 + K_2}{\rho_1 + \rho_2}.
\]

The Lagrangian is written as

\[
\mathcal{L} = \frac{K_1 + K_2}{4(a + 1)^2} \left( \frac{1}{v_0^2} (\partial_i \phi)^2 - (\nabla \phi)^2 \right)
\]

\[
+ \frac{1}{4(a + 1)^2} ((a^2 \rho_1 + \rho_2) \partial_i \partial_i \varphi - (a K_1 - K_2) \nabla \phi \nabla \cdot \varphi)
\]

\[
+ \frac{1}{4(a + 1)^2} ((a^2 \rho_1 + \rho_2) (\partial_i \varphi)^2 - (a^2 K_1 + K_2) (\nabla \varphi)^2)
\]

\[- 2\gamma_{12} \Delta_1 \Delta_2 \cos(\theta_1 - \theta_2) \cos(\varphi).
\]

Here we set \( a K_1 = K_2 \) and integrate out the field \( \phi \) to obtain the effective action \( \mathcal{L}_\varphi \) for \( \varphi \). We have

\[
\mathcal{L}_\varphi = \frac{\rho_1 \rho_2}{4(\rho_1 + \rho_2)} (\partial_i \varphi)^2 - \frac{K_1 K_2}{4(K_1 + K_2)} (\nabla \varphi)^2
\]

\[- 2\gamma_{12} \Delta_1 \Delta_2 \cos(\varphi)
\]

\[
= \frac{K_1 K_2}{4(K_1 + K_2)} \left( \frac{1}{v_0^2} (\partial_i \varphi)^2 - (\nabla \varphi)^2 \right) - 2\gamma_{12} \Delta_1 \Delta_2 \cos(\varphi),
\]

\[
= A \left( \frac{1}{2v_0^2} (\partial_i \varphi)^2 - \frac{1}{2} (\nabla \varphi)^2 + \alpha \cos(\varphi) \right),
\]

where \( v \) is the velocity defined by

\[
v_0^2 = \frac{\rho_1 + \rho_2}{\rho_1 \rho_2} \frac{K_1 K_2}{K_1 + K_2} = \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \left( \frac{1}{K_1} + \frac{1}{K_2} \right)^{-1},
\]

and we put

\[
A = \frac{K_1 K_2}{2(K_1 + K_2)},
\]

\[
\alpha = \frac{K_1 + K_2}{K_1 K_2} \gamma_{12} \Delta_1 \Delta_2.
\]

We assume that \( \gamma_{12} < 0 \). The coefficient \( \alpha \) has the dimension of the inverse of square of the distance.

In the three-gap case, the Lagrangian for the phase part is given as

\[
\mathcal{L} = \frac{1}{4} \rho_1 (\partial_i \theta_1)^2 + \frac{1}{4} \rho_2 (\partial_i \theta_2)^2 + \frac{1}{4} \rho_3 (\partial_i \theta_3)^2
\]

\[- \frac{n_1}{8m_1} (\nabla \theta_1)^2 - \frac{n_2}{8m_2} (\nabla \theta_2)^2 - \frac{n_3}{8m_3} (\nabla \theta_3)^2
\]

\[- 2\gamma_{12} \Delta_1 \Delta_2 \cos(\theta_1 - \theta_2) - 2\gamma_{23} \Delta_2 \Delta_3 \cos(\theta_2 - \theta_3)
\]

\[- 2\gamma_{31} \Delta_3 \Delta_1 \cos(\theta_3 - \theta_1),
\]

where we assume that \( \gamma_{ij} = \gamma_{ji} \). We define the total phase and phase differences as follows.

\[
\phi = \theta_1 + a \theta_2 + b \theta_3,
\]

\[
\varphi_1 = \theta_1 - \theta_2,
\]

\[
\varphi_2 = \theta_2 - \theta_3.
\]
We choose a and b as $a = K_2/K_1$ and $b = K_3/K_1$. We obtain the effective Lagrangian after integrating out the total phase $\phi$ as

$$L_\varphi = \frac{\rho_1(\rho_2 + \rho_3)}{4(\rho_1 + \rho_2 + \rho_3)}(\partial_t \varphi_1)^2 + \frac{\rho_3(\rho_1 + \rho_2)}{4(\rho_1 + \rho_2 + \rho_3)}(\partial_t \varphi_2)^2 + \frac{\rho_1 \rho_3}{2(\rho_1 + \rho_2 + \rho_3)}\partial_t \varphi_1 \partial_t \varphi_2$$

$$- \frac{K_1(K_2 + K_3)}{4(K_1 + K_2 + K_3)}(\nabla \varphi_1)^2 - \frac{K_3(K_1 + K_2)}{4(K_1 + K_2 + K_3)}(\nabla \varphi_2)^2$$

$$- \frac{K_1 K_3}{2(K_1 + K_2 + K_3)} \nabla \varphi_1 \cdot \nabla \varphi_2$$

$$- 2\gamma_{12} \Delta_1 \Delta_2 \cos(\varphi_1) - 2\gamma_{23} \Delta_2 \Delta_3 \cos(\varphi_2)$$

$$- 2\gamma_{31} \Delta_3 \Delta_1 \cos(\varphi_1 + \varphi_2).$$

B. Moduli approximation and mass of a kink and a monopole

Let us consider the static kink in a two-gap superconductor and examine the energy functional given by

$$E[\varphi] = A \int d^4x \left[ \frac{1}{2} (\nabla \varphi)^2 - \alpha \cos(\varphi) \right].$$

We consider a two-dimensional superconductor where there is a one-dimensional kink with variable $y$ being independent of the other space variable $x$. The kink satisfies the equation

$$\frac{\partial^2 \varphi}{\partial y^2} = \alpha \sin \varphi.$$

A one-kink solution is

$$\varphi = \pi + 2 \sin^{-1}(\tanh(\sqrt{\alpha}(y - y_0))),$$

with the boundary condition $\varphi \to 0$ as $y \to -\infty$ and $\varphi \to 2\pi$ as $y \to \infty$. $y_0$ indicates the position where the kink exists and take any real value. A set $\{y_0\}_{y_0 \in \mathbb{R}}$ represents a moduli space of one-kink solution. We here regard $y_0$ as a variable that depends on time $t$: $y_0 = y_0(t)$. We obtain

$$\left( \frac{\partial \varphi}{\partial t} \right)^2 = 4\alpha \cosh^2(\sqrt{\alpha}(y - y_0)) \left( \frac{dy_0}{dt} \right)^2.$$  \hspace{1cm} (19)

Then the action for $y_0(t)$ is given by

$$S_{\text{kink}}[y_0] = \int dt dx A \frac{1}{2v^2} \left( \frac{\partial \varphi}{\partial t} \right)^2 = \int dt \frac{4A}{v^2} \sqrt{\alpha} \left( \frac{dy_0}{dt} \right)^2.$$  \hspace{1cm} (20)

This indicates that the kink motion exhibits a zero-energy mode (massless mode) with the mass

$$M_\alpha = \frac{8A}{v^2} \sqrt{\alpha} L_x = \frac{4K_1 K_2}{v^2(K_1 + K_2)} \sqrt{\alpha} L_x,$$

where $L_x$ is the length of the kink in the $x$ direction. The kink would move with a constant velocity $u$: $y_0(t) = y_0(0) + ut$. When two bands are equivalent in a simple case with $K_1 = K_2 = K = n/2m$ and $\rho_1 = \rho_2 = \rho_F$, $M_\alpha$ is given by

$$M_\alpha = 2\rho_F \sqrt{\alpha} L_x,$$  \hspace{1cm} (22)

with

$$\alpha = \frac{2d}{\pi^2} \frac{\gamma_{12}}{\rho_F} \frac{1}{\xi_0^2},$$  \hspace{1cm} (23)

where $d$ is the spatial dimension and $\xi_0$ is the coherence length $\xi_0 = h v_F / (\pi \Delta)$. $1/\sqrt{\alpha}$ indicates the spread of kink that is proportional to $\xi_0$. In two dimensions, $M_\alpha$ is proportional to the electron mass $m$,

$$M_\alpha = \frac{m}{\pi} \sqrt{\alpha} L_x.$$  \hspace{1cm} (24)

This massless mode is the sliding motion of kink.$^63$

A fractional-flux vortex is regarded as a monopole in a superconductor where the fractional vortex exists at the end of the kink, for instance, at $(x, y) = (x_0, 0)$ and the other end is at the boundary of superconductor. The phase of the second gap is represented as

$$\theta_2 = \frac{1}{2} \text{Im} \log(x - x_0 + iy),$$  \hspace{1cm} (25)

and that of the first gap is $\theta_1 = -\theta_2 + \theta$ where $\theta$ is the rotation angle about the point $(x_0, 0)$. The gauge field corresponding to this phase configuration has a singularity and represents a monopole in the phase space.$^23$ The phase difference is given by $\varphi = 2\theta_1 - \theta \equiv 0 \mod 2\pi$. The fractional vortex with flux $\phi_0/2$ exists at $(x, y) = (x_0, 0)$. Let $x_0$ be dependent on time $t$: $x_0 = x_0(t)$. Then we have

$$\left( \frac{\partial \theta_1}{\partial t} \right)^2 = \frac{1}{4} y^2 \frac{4((x - x_0)^2 + y^2)^2}{(dx_0/dt)^2},$$  \hspace{1cm} (26)

The action for $\varphi$ is given as

$$S_{\text{mpole}}[x_0] = \int dx dy A \frac{1}{2v^2} \left( \frac{\partial \varphi}{\partial t} \right)^2 = \frac{1}{2} M_{\text{mp}} \left( \frac{dx_0}{dt} \right)^2,$$  \hspace{1cm} (27)

where the mass $M_{\text{mp}}$ is given by

$$M_{\text{mp}} = \frac{\pi}{2} \log \left( \frac{L_x}{a} \right) \cdot \frac{\rho_1 \rho_2}{\rho_1 + \rho_2},$$  \hspace{1cm} (28)

where $a$ is a cutoff. In two spatial dimensions, using $\rho_i = m_i/(2\pi)$, $M_{\text{mp}}$ is proportional to the effective mass of electron:

$$M_{\text{mp}} = \frac{m^*}{4} \log \left( \frac{L_x}{a} \right),$$  \hspace{1cm} (29)

with $m^* = m_1 m_2 / (m_1 + m_2)$. A fractional-flux vortex, namely, the monopole exhibits a massless mode with the mass proportional to $\text{log} L_x$. 
C. Quantization of kink

Let us examine the excitation modes of kinks in two-gap and three-gap superconductors. The kink solution has a continuous parameter called a moduli which forms a space of solution. Since the kink solution is a classical solution of the equation of motion, there is a zero-energy mode in the tangent space of the classical solution. We write the solution of kink as \( \varphi = \varphi_0 + \varphi^{(1)} \) where \( \varphi_0 \) is the kink state given in eq. (31) and \( \varphi^{(1)} \) represents a fluctuation to the classical kink solution. \( \varphi^{(1)} \) describes excitation modes of kink in a two-gap superconductor. We obtain the energy functional \( E[\varphi] \) up to the second order of \( \varphi^{(1)} \). \( \varphi^{(1)}(x) \) satisfies the eigenvalue equation given as (we use \( x \) as a variable)

\[
\left( -\frac{\partial^2}{\partial x^2} + \alpha \cos \varphi_0 \right) \varphi^{(1)} = \lambda \varphi^{(1)}, \quad (30)
\]

Since the eigenvalue is zero or positive, we put \( \lambda = \omega^2 \) where \( \omega \) (times \( A \)) indicates the excitation energy. The eigenfunctions and eigenvalues are given by

\[
\varphi^{(1)}_0 = \frac{1}{\cosh(\sqrt{\omega} x)}, \quad \lambda_0 = 0, \quad (31)
\]

\[
\varphi^{(1)}_1 = \tanh(\sqrt{\omega} x), \quad \lambda_1 = \alpha, \quad (32)
\]

\[
\varphi^{(1)}_q = e^{iqx\sqrt{\omega}}(q + i \tanh(\sqrt{\omega} x)), \quad \lambda_q = (q^2 + 1)\alpha, \quad (33)
\]

where we set \( x_0 = 0 \) for simplicity and \( q \) in the third equation is quantized according to the boundary condition. We adopt that the system is the box of length \( L \) and we impose the periodic boundary condition at \( x = -L/2 \) and \( L/2 \):

\[
q_n L = 2\eta(q_n) = 2n\pi, \quad (34)
\]

where \( \eta(q) \) is the phase difference when \( x \) is large given by \( \tan \eta(q) = 1/q \). The parameter \( q \) is quantized as \( q = q_n \) for \( n = 0, 1, \ldots \), \( \varphi^{(1)}_0 \) represents the zero energy mode, and \( \varphi^{(1)}_1 \) and \( \varphi^{(1)}_q \) are quantized excitation modes.

Let us consider excitation modes of the kink in the three-gap case. We examine the case where two bands 1 and 3 are equivalent and we set \( \gamma_{12} = \gamma_{23} = 2, \Delta_1 = \Delta_3 \) and \( K_1 = K_3 \). In this case we have \( \varphi_1 = \varphi_2 \equiv \varphi \). The energy functional is written as

\[
E[\varphi] = \frac{1}{g} \int d^d x \left[ \frac{1}{2} (\nabla \varphi)^2 - \alpha (\cos \varphi + \frac{u}{2} \cos(2\varphi)) \right], \quad (35)
\]

where \( u = \gamma_{13} \Delta_1/(\gamma_{12} \Delta_2) \) and \( g = A^{-1} \) given as

\[
\frac{1}{g} = \frac{K_1(K_1 + 2K_2)}{2K_1 + K_2}. \quad (36)
\]

The parameter \( \alpha \) is

\[
\alpha = -4 \frac{2K_1 + K_2}{K_1(K_1 + 2K_2)} \gamma_{12} \Delta_1 \Delta_2. \quad (37)
\]

We assume that \( \gamma_{12} < 0 \) and \( u > -1/2 \). We consider a one-dimensional kink with variable \( x \) as before and the solution is given by

\[
\varphi_{kink} = \varphi_0 = \cos^{-1} \left( \frac{2 \sinh^2(s(x - x_0))}{\cosh^2(s(x - x_0)) + 2u} - 1 \right), \quad (38)
\]

where we put \( s = \sqrt{\alpha(1 + 2u)} \). When we regard the parameter \( x_0 \) as a time-dependent parameter, \( x_0(t) \) represents a gapless mode with the mass

\[
M_n = 2p_F \sqrt{\alpha(1 + 2u)} \left( 1 + \frac{1}{2u(1 + 2u)} \cosh^{-1}(\sqrt{1 + 2u}) \right) L, \quad (39)
\]

where \( p_F^* \) is the effective density of states and \( L \) is the length of the kink.

The kink solution is written as \( \varphi = \varphi_0 + \varphi^{(1)} \) as before where \( \varphi^{(1)} \) represents a correction to the kink state. We examine the following eigenvalue equation:

\[
\left( -\frac{\partial^2}{\partial x^2} + \alpha (\cos \varphi + 2u \cos(2\varphi)) \right) \varphi^{(1)}(x) = \lambda \varphi^{(1)}(x), \quad (40)
\]

where \( \nu = \lambda/s^2 \). The zero-energy mode is easily found as

\[
\varphi^{(1)}_0(x) = \frac{\cosh(s(x - x_0))}{\cosh^2(s(x - x_0)) + 2u}, \quad (42)
\]

with the eigenvalue \( \nu_0 = \lambda_0 = 0 \). It is not easy to obtain an exact expression for excited states with positive eigenvalues. An approximate solution of the first excited state for small \( u \) and large \( x \) is obtained as

\[
\varphi^{(1)}_1(z) \approx \frac{\sinh(z)}{\cosh(z)} - 4u \frac{\sinh(z)}{\cosh^2(z)} \quad (43)
\]

The eigenvalue is \( \nu_1 \approx 1 \), namely, \( \lambda_1 \approx s^2 \). The third and higher excited states have continuous spectra when the system size \( L \) is very large. Under the same condition, \( \varphi^{(1)}_q \) will be approximated as

\[
\varphi^{(1)}_q(z) \approx e^{iqz} (q + i\varphi^{(1)}_1(z)), \quad (44)
\]

with the eigenvalue \( \nu_q \approx q^2 + 1 \).

We solve the eigen-equation in eq. (41) numerically by reducing it to a tridiagonal matrix on a lattice. For \( u = 0 \) we have \( \nu = 0, 1 \) and \( q_n^2 + 1 \), \( n = 0, 1, \cdots \), and for \( u > 0 \), \( \nu = 0, \nu_1 \) and \( q_n(u)^2 + 1 \) \( n = 0, 1, \cdots \). We found \( \nu_1 \leq 1 \) for \( u \geq 0 \) and \( q_n(u) \) is almost independent of \( u \). We here mention that \( \lambda_1/\alpha = \nu_1 s^2/\alpha = \nu_1 (1 + 2u) \geq 1 \) although \( \nu_1 \leq 1 \). We show the zero energy mode as a function of
$x$ in Fig.4 where $\xi_{\text{kink}} = 1/s$ is the characteristic length of kink and is the unit of length. This mode is localized near $x = x_0$ and its spreading width is about $\xi_{\text{kink}}$. The excitation mode is a spread state. The first and second excited states are shown in Fig.5 and Fig.6, respectively. We show eigenvalues $\nu_n = \lambda_n/s^2$ as a function of $u$ in Fig.7. The first excited state shows strong dependence on $u$, and other states, however, are mostly independent of $u$. The size dependence of eigenvalues is shown as a function of $1/L$ in Fig.8. The eigenvalues of higher excited modes constitute a continuous spectrum when $L$ is large.

![FIG. 4. Zero-energy mode as a function of $x$ with $x_0 = 0$ for $u = 0$ and 0.5.](image)

![FIG. 5. First excited mode as a function of $x$ with $x_0 = 0$ for $u = 0$ and 0.5.](image)

![FIG. 6. Second excited mode as a function of $x$ with $x_0 = 0$ for $u = 0$ and 0.5. PBC and Open indicate the periodic and open boundary condition, respectively. The second and higher excited states depend on the boundary condition.](image)

![FIG. 7. Eigenvalues $\nu_n = \lambda_n/s^2$ as a function of $u$ ($n = 0, 1, \cdots, 9$ from the bottom) where $\nu_0 = 0$.](image)

**D. Energy of Kink**

Here let us discuss the energy of kink. The energy of the solution $\varphi_0$ is given by

$$E[\varphi_0] = \frac{1}{g} \int dx \left[ \frac{1}{2} \left( \frac{d\varphi_0}{dx} \right)^2 - \alpha \left( \cos \varphi_0 + \frac{u}{2} \cos(2\varphi_0) - 1 - \frac{u}{2} \right) \right],$$

(45)
The Lagrangian for mononic oscillators. The field functions in Fig. 7, FIG. 8. Eigenvalues \( \nu \) where a constant \(- (1 + u/2)\) is introduced to subtract the divergence. This is evaluated as

\[
E[\varphi_0] = \frac{1}{g} 4s \left( 1 + \frac{1}{\sqrt{2u(1+2u)}} \cosh^{-1}(\sqrt{1+2u}) \right) .
\]  

(46)

The excitation modes are approximated as a set of harmonic oscillators. The field \( \varphi \) is written as

\[
\varphi = \varphi_0(x - x_0(t)) + \sum_n c_n(t) \varphi_n^{(1)}(x - x_0(t)),
\]

(47)

where we adopt that \( x_0 \) and the coefficients \( \{c_n\} \) are time dependent. \( x_0(t) \) and \( c_0(t) \) give the zero-energy mode. The Lagrangian for \( c_n \) with \( n \neq 0 \) is given by

\[
L[c_n] = \frac{1}{2g\omega_n^2} \sum_n (\dot{c}_n^2 - v^2 \omega_n^2 c_n^2) ,
\]

(48)

where we set \( \lambda_n = \omega_n^2 \). The excitation modes are described by harmonic oscillators with frequencies \( v\omega_n \). The corrections to the energy from quantized modes are written as

\[
E_{kink} = E[\varphi_0] + \left( N_1 + \frac{1}{2} \right) v\omega_1 + \sum_{q_n} \left( N_{q_n} + \frac{1}{2} \right) v\omega_{q_n} ,
\]

(49)

where \( N_1 \) and \( N_{q_n} \) are integers, and \( v^2 = (\rho_1 + \rho_2)/(\rho_1 \rho_2) \). \( K_1 K_2/(K_1 + K_2) \) in the two-gap case \( (u = 0) \) and

\[
v^2 = \frac{2\rho_1 + \rho_2}{\rho_1 (\rho_1 + 2\rho_2)} \frac{K_1 (K_1 + 2K_2)}{2K_1 + K_2} ,
\]

(50)

in the three-gap case. We put \( \omega_1 = \sqrt{\lambda_1} = s\sqrt{\nu_1} \) and \( \omega_{q_n} = \sqrt{\lambda_{q_n}} = s\sqrt{\nu_{q_n}} \). According to numerical calculations in Fig. 7, \( \nu_{q_n} \) is mostly independent of \( u \). Thus, we use the formula for \( u = 0 \): \( \nu_{q_n} \approx q_n^2 + 1 \) for \( n = 0, 1, 2, \ldots \). \( q_n \) is determined from the boundary condition given by

\[
q_n L + 2\eta(q_n) = 2n\pi \equiv k_n L. \]

(51)

Then, \( E_{kink} \) is approximated as

\[
E_{kink} \simeq E[\varphi_0] + \left( N_1 + \frac{1}{2} \right) vs\sqrt{\nu_1} + \sum_{q_n} \left( N_{q_n} + \frac{1}{2} \right) vs\sqrt{\nu_{q_n}^2 + 1} .
\]

(52)

The ground state energy of kink is given as

\[
E_{kink,0} \simeq E[\varphi_0] + \frac{1}{2} vs\sqrt{\nu_1} + \sum_{q_n} vs\sqrt{\nu_{q_n}^2 + 1} .
\]

(53)

Since the summation with respect to \( q_n \) diverges, we subtract \( \sqrt{k_n^2 + s^2} \) from \( s\sqrt{\nu_{q_n}^2 + 1} \). We obtain for large \( L \)

\[
\frac{1}{2} vs \sum_{k_n} \left( \sqrt{(q_n s)^2 + s^2} - \sqrt{k_n^2 + s^2} \right) = -\frac{L}{4\pi} \frac{1}{v} \int_{-\infty}^{\infty} dk \frac{k}{\sqrt{k^2 + s^2}} \eta(k) 
\]

\[
= -\frac{vs}{\pi} \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{k^2 + s^2}}
\]

(54)

where we used partial integration and \( \eta(k) \sim s/k \) for large \( k \). The last integral diverges, which is removed by the renormalization of \( \alpha \propto s^2 \). In fact, the same divergence appears from the interaction term \(-\alpha/g\cos \varphi \). The one-loop contribution is given by

\[
(\alpha/g)(\langle \varphi^2 \rangle/2) \cos \varphi.
\]

This is estimated by using

\[
\langle \varphi^2 \rangle = \int \frac{d\omega dk}{(2\pi)^2} \frac{g}{\omega^2 + k^2 + s^2} = \frac{gv}{4\pi} \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{k^2 + s^2}}
\]

(55)

cos \varphi is approximated as

\[
(\cos \varphi_0 - 1) = -2\int_{-\infty}^{\infty} dx \frac{1}{\cosh^2(sx)} = -\frac{4}{s},
\]

(56)

by subtracting 1 form \cos \varphi_0 \) to regularize the integral. This gives the same correction as in eq. \( \text{[53]} \) and is cancelled by introducing the counter term. For the interaction \(-\alpha/g)[\cos \varphi + (u/2)\cos(2\varphi)] \), the contribution to the ground-state energy is given by evaluating

\[
\langle \cos(\varphi_0) + 2u \cos(2\varphi) \rangle - 1 - 2u
\]

\[
= (1 + 2u) \int_{-\infty}^{\infty} dx \left[ -\frac{2}{\cosh^2(sx)} + \frac{16u(1+2u)}{\cosh^2(sx) + (2u)^2} \right]
\]

\[
= \frac{2}{s} (1 + 2u) \left[ -4 + \frac{2}{\sqrt{2u(1+2u)}} \cosh^{-1} \sqrt{1+2u} \right]
\]

\[
= \frac{4}{s} (1 + 2u) (1 + O(u)).
\]

(57)

Then the ground-state energy of kink is

\[
E_{kink,0} \simeq E[\varphi_0] + \left( \frac{1}{2} \sqrt{\nu_1} - \frac{1}{\pi} \right) vs.
\]

(58)
IV. SUMMARY

The observation of fractional-flux vortices in a layered superconductor indicates that there are indeed kinks in the phase space of a multi-component superconductor. Thus, it is important to investigate physical properties of kinks in superconductors.

We investigated quantized excitation modes of the kink in a two-gap and a three-gap superconductor. There are zero-energy mode (massless mode) and quantized excitation modes with energy gaps proportional to $s \propto \sqrt{\alpha}$. Second and higher excitation modes have almost continuous spectra when the system size $L$ is large. Since $\alpha$ is proportional to the Josephson coupling, properties of excitation modes are determined by the Josephson coupling between different gap states. Although the energy of kink has a divergence, it is regularized by the renormalization of coupling constants in the Lagrangian. The zero-energy mode has a peak at the position of the kink, which indicates a fluctuation of the amplitude of domain wall at the position of the kink. We expect that this mode can be observed by some experimental equipments such as scanning tunneling spectroscopy.

There are two massless modes so far; one is the massless mode of the kink and the other is that of sliding motion of the kink. The former is a small oscillation mode of the kink, which may be called the ripple mode. A fractional vortex itself also exhibits a massless mode as a motion of a monopole.

We also examined a stability of fractional-flux vortices in a multiband superconductor. In a two-band superconductor, a fractional-flux vortex is unstable because of the energy cost of kink. Two fractional vortices connected by a kink are glued to be a vortex with the unit flux. In a real superconductor, however, fractional-flux vortices may be stabilized due to some vortex pinning effect and magnetic interaction between vortices and external magnetic field outside the superconductor. An interesting possibility of stable fractional vortices was found in a three-gap superconductor with time-reversal symmetry breaking. It was shown that fractional vortices can exist on the transverse kink that connects two states with the same energy. The unit flux is decomposed into two vortices with flux $\phi_0/3$ and $2\phi_0/3$ along the transverse kink. This is a vortex pinning by kinks. The vortex pinning by kinks indicates a possibility of high critical current of a superconducting wire.

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