A SELF-DUAL HOPF ALGEBRA ON DOUBLE PARTIALLY ORDERED SETS

CLAUDIA MALVENUTO AND CHRISTOPHE REUTENAUER

ABSTRACT. Let $D$ be the set of isomorphism types of finite double partially ordered sets, that is sets endowed with two partial orders. On $\mathbb{Z}D$ we define a product and a coproduct, together with an internal product, that is, degree-preserving. With these operations $\mathbb{Z}D$ is a Hopf algebra, self-dual with respect to a scalar product which counts the number of pictures (in the sense of Zelevinsky) between two double posets. The product and coproduct correspond respectively to disjoint union of posets and to a natural decomposition of a poset into order ideals. Restricting to special double posets (meaning that the second order is total), we obtain a notion equivalent to Stanley’s labelled posets, and obtain a sub-Hopf-algebra already considered by Blessenohl and Schocker. The mapping which maps each double poset onto the sum of the linear extensions of its first order, identified via its second (total) order with permutations, is a Hopf algebra homomorphism, which is isometric and preserves the internal product, onto the Hopf algebra of permutations, previously considered by the two authors. Finally, the scalar product between any special double poset and double posets naturally associated to integer partitions is described by an extension of the Littlewood-Richardson rule.

1. Introduction

We define a (combinatorial) Hopf algebra based on ”double posets”, with a scalar product based on ”pictures” between double posets, in analogy to pictures of tableaux as defined by Zelevinsky in [Z]. Pictures had been introduced previously by James and Peel in [JP] p.351-352. Zelevinsky’s definition extends straightforwardly to double posets. The results we prove show that pictures are fundamentally linked to scalar products, a point of view already present in Zelevinsky’s work, who proved that the scalar product of two skew Schur functions is equal to the number of pictures between their shapes. See also [FG] and [L] for the study of pictures between skew shapes.

We call double poset a set which is endowed with two partial orders $<_1$ and $<_2$. We consider isomorphism classes of double posets: on the $\mathbb{Z}$-module with basis the set of (isomorphism classes of) double posets, we define combinatorially a product and a coproduct, which will make it a graded Hopf algebra, self-dual with respect to the scalar product $\langle x, y \rangle$ defined as the number of pictures from $x$ to $y$; in other words,

$$\langle xy, z \rangle = \langle x \otimes y, \delta(z) \rangle.$$
Recall that self-dual Hopf algebras play a great role in representation theory, see [Gei, Z1].

When the second order $<_2$ of a double poset is total, one obtains the notion which we call special double poset; this bialgebra has already been considered by Blessenohl and Laue, see [BISc] p.41-42. There is a natural homomorphism into the bialgebra of permutations of [MR]. This mapping is implicit in Stanley’s work (see also [G]). The fact that one has a sub-bialgebra and a homomorphism is already due to [BlSc] (see also [M1] and [HP]). We give further properties of this homomorphism: it is an isometry, and preserves the internal product.

To each integer partition is naturally associated a special double poset; this construction is described in [G]. We describe the scalar product of such a double poset and any special double poset by a rule which extends the Littlewood-Richardson rule.

Note that all the bialgebras in this article are $\mathbb{Z}$-algebras, are graded and connected (that is, the 0-component is $\mathbb{Z}$), hence these bialgebras are Hopf algebras.

2. The bialgebra on double posets

### 2.1. The self-dual bialgebra on double posets

A double poset (a notion which is implicit in [MR]) is a triple $(E,<_1,<_2)$, where $E$ is a finite set, and $<_1$ and $<_2$ are two partial orders on $E$. When no confusion arises, we denote $(E,<_1,<_2)$ simply by $E$. We call $<_1$ the first order of $E$ and $<_2$ the second order of $E$.

As expected, we say that two double posets $(E,<_1,<_2)$ and $(F,<_1,<_2)$ are isomorphic if there exists a bijection $\phi : E \to F$ which is an isomorphism from the partial order $(E,<_1)$ to $(F,<_1)$ and from $(E,<_2)$ to $(F,<_2)$, i.e.

$$\forall x, y \in E : x <_i y \text{ in } E \iff \phi(x) <_i \phi(y) \text{ in } F, \text{ for } i = 1, 2.$$  

Rather than on double posets, we want to work on isomorphism classes of double posets: to avoid too much notation, we simply say double poset, meaning its isomorphism class.

Let $D$ denote the set of double posets. We define several combinatorial operations on this set, which will serve to define the bialgebra structure on $\mathbb{Z}D$, the set of $\mathbb{Z}$-linear combinations of double posets.

If $E$ and $F$ are two double posets, their composition, denoted $EF$, is the double poset $(E \cup F,<_1,<_2)$, where the union is disjoint and where

- the first order $<_1$ of $EF$ is the extension to $E \cup F$ of the first orders $<_1$ of $E$ and $F$, and no element of $E$ is comparable to any element of $F$;
- the second order $<_2$ of $EF$ is the extension to $E \cup F$ of the second orders $<_2$ of $E$ and $F$, together with the relations $e <_2 f$ for any $e \in E, f \in F$.

The product on $\mathbb{Z}D$ is obtained by extending linearly the composition on $D$.

Recall that an inferior ideal of a poset $(E,<)$ is a subset $I \subseteq E$ such that if $y \in I$ and $x < y$, then $x \in I$. A superior ideal of $E$ is a subset $S \subseteq E$ such that if $x \in J$ and $x < y$, then $y \in S$. Clearly, the complement of an inferior ideal is a superior ideal and conversely. A decomposition of a poset $(E,<)$ is a couple $(I,S)$ where $I$ is an inferior ideal and $S$ its complement.

We call decomposition of a double poset $(E,<_1,<_2)$ a pair

$$(I,<_1,<_2), (S,<_1,<_2),$$
where \((I, S)\) is a decomposition of the poset \((E, <_1)\), and where the first and second orders \(<_1, <_2\) for \(I\) and \(S\) are obtained by restricting the orders \(<_1, <_2\) of the double poset \(E\).

Now let \(\delta : \mathbb{Z}D \to \mathbb{Z}D \otimes \mathbb{Z}D\) be the linear map defined on \(D\) by
\[
\delta((E, <_1, <_2)) = \sum (I, <_1, <_2) \otimes (S, <_1, <_2),
\]
where the sum is extended to all decompositions of \((E, <_1, <_2)\).

A picture between double posets \((E, <_1, <_2)\) and \((F, <_1, <_2)\) is a bijection \(\phi : E \to F\) such that:
- \(e <_1 e' \Rightarrow \phi(e) <_2 \phi(e')\) and
- \(f <_1 f' \Rightarrow \phi^{-1}(f) <_2 \phi^{-1}(f')\).

In other words, a picture is a bijection \(\phi\) of \(E\) to \(F\) which is increasing from the first order of \(E\) to the second order of \(F\) and such that its inverse \(\phi^{-1}\) is increasing from the first order of \(F\) to the second order of \(E\). We define a pairing \(\langle , \rangle : \mathbb{Z}D \times \mathbb{Z}D \to \mathbb{Z}\) for any double posets \(E, F\) by:
\[
\langle E, F \rangle = |\{\alpha : E \to F, \alpha\text{ is a picture}\}|,
\]
and extend it bilinearly to obtain a scalar product on \(\mathbb{Z}D\), which we call \textit{Zelevinsky scalar product}.

\textbf{Theorem 2.1.} \(\mathbb{Z}D\) is a graded self-dual Hopf algebra.

Note that a similar bialgebra structure has been defined on posets (not double posets) by Schmitt in \[Sch\] p. 27-28; see also \[ABSo\] Example 2.3. Formally it means that the mapping of \(\mathbb{Z}D\) into the bialgebra of Schmitt, which sends a double poset onto the poset with only the first order \(<_1\), is a bialgebra homomorphism.

Self-duality means that for any double posets \(E, F, G\),
\[
\langle EF, G \rangle = \langle E \otimes G, \delta G \rangle.
\]

\textbf{Proof.} We omit the easy verification of the associativity of the product and of the coassociativity of the coproduct. Similarly for the homogeneity of both, where the degree of a double poset is the number of its elements. In order to show that \(\mathbb{Z}D\) is a bialgebra, we show that the coproduct \(\delta\) is a homomorphism for the product. This is rather a tautology, since isomorphic double posets are identified. It amounts to show that there is a bijection between the set of decompositions of the double poset \(EF\) and the set of pairs \((E, F_1, E_sF_s)\), where \((E_s, F_s)\) is a decomposition of \(E\) and \((F_1, F_s)\) is a decomposition of \(F\). The bijection is the natural one: take a decomposition \((I, S)\) of \(EF\); then \(I\) is an inferior ideal of \((E \cup F, <_1)\); hence, \(I \cap E\) is an inferior ideal of \((E, <_1)\), \(I \cap F\) is an inferior ideal of \((E, <_1)\) and \(I = (I \cap E)(I \cap F)\). Similarly, \(S \cap E\) is a superior ideal of \((E, <_1)\), \(S \cap F\) is a superior ideal of \((F, <_1)\) and \(S = (S \cap E)(S \cap F)\). Hence the mapping is the identity mapping (modulo double posets isomorphisms):
\[
(I, S) \mapsto ((I \cap E)(I \cap F), (S \cap E)(S \cap F)).
\]
To see that it is a bijection, note that if \((E_i, E_s)\) is a decomposition of \(E\) and \((F_1, F_s)\) is a decomposition of \(F\), then \(E_i \cup F_1\) is an inferior ideal \(I\), and \(E_s \cup F_s\) the complementary superior ideal \(S\) of \((E \cup F, <_1)\); moreover, \(I = E_iF_1\) and \(S = E_sF_s\).

We prove now self-duality. In view of the identity stated before the proof, this amounts to give, for any double posets \(E, F, G\), a bijection between pictures from
EF to G and 4-tuples \((\phi, \psi, I, S)\), where \(I\) is an inferior ideal of G, \(S\) the complementary superior ideal, \(\phi\) a picture of \(E\) onto \(I\) and \(\psi\) a picture of \(F\) onto \(S\). So, let \(\alpha\) be a picture from \(EF\) onto \(G\). Define \(I = \alpha(E), S = \alpha(F)\) and the bijections \(\phi : E \to I, \psi : F \to S\) obtained by restriction of \(\alpha\) to \(E\) and \(F\). We verify first that \(I\) is an inferior ideal of \(G\); take \(g, g'\) in \(G\) with \(g <_1 g'\) and \(g' \in I\), hence \(\alpha^{-1}(g') \in E\). Then, \(\alpha\) being a picture, we have \(\alpha^{-1}(g) <_2 \alpha^{-1}(g')\). Now, if we had \(g \notin I\), then \(g \in S\), hence \(\alpha^{-1}(g) \in F\), hence \(\alpha^{-1}(g') <_2 \alpha^{-1}(g)\), by definition of the second order of \(EF\): contradiction. Similarly, \(S\) is a superior ideal of \(G\). Now, the restriction of a picture is a picture, so \(\phi\) and \(\psi\) are pictures.

Conversely, given a 4-tuple as above, we glue together the two bijections \(\phi\) and \(\psi\) and obtain a bijection \(\alpha : E \to G\). Since in \(EF\), elements of \(E\) and elements of \(F\) are \(_1\)-incomparable, the fact that \(\alpha\) is increasing from \((EF, <_1)\) onto \((G, <_2)\) follows from the similar property for \(\phi\) and \(\psi\). Now let \(g, g' \in G\) with \(g <_1 g'\); if they are both in \(I\) or both in \(S\), then \(\alpha^{-1}(g) <_2 \alpha^{-1}(g')\), by the similar property for \(\phi\) and \(\psi\); otherwise, we have \(g \in I\) and \(g' \in S\), since \(I, S\) are ideals. Then \(\alpha^{-1}(g) \in E\) and \(\alpha^{-1}(g') \in F\), and consequently, \(\alpha^{-1}(g) <_2 \alpha^{-1}(g')\), by the definition of the second order of \(EF\). Thus \(\alpha\) is a picture.

The counity map \(\epsilon : \mathbb{ZD} \to \mathbb{Z}\) maps the empty double poset on 1, and all other doubles posets onto 0. It is a morphism for the product. By a well–known fact, a graded connected bialgebra is a Hopf algebra.

\[\square\]

2.2. A homomorphism into quasi-symmetric functions. Let \(\pi = (E, <_1, <_2)\) be a double poset. Similarly to \(\text{S}\) and \(\text{G}\), we call \(\pi\)-partition, a function \(x\) from \(E\) into a totally ordered set \(X\), such that:

- \(e <_1 e'\) implies \(x(e) \leq x(e')\);
- \(e <_1 e'\) and \(e >_2 e'\) implies \(x(e) < x(e')\).

Note that if \(x\) is injective, the first condition suffices.

Now suppose that \(X\) is an infinite totally ordered set of commuting variables. Then the generating quasi-symmetric function of \(\pi\) is the sum, over all \(\pi\-partitions\), of the monomials \(\prod_{e \in E} x(e)\). We denote it \(\Gamma(\pi)\). By extending linearly \(\Gamma\) to \(\mathbb{ZD}\), we obtain a linear mapping into the algebra of quasi-symmetric function. For quasi-symmetric functions, see \(\text{S1}\) 7.19. They form a bialgebra denoted \(\text{QSym}\), see e.g. \(\text{MR}\).

**Theorem 2.2.** \(\Gamma : \mathbb{ZD} \to \text{QSym}\) is a homomorphism of bialgebras.

This result is implicit in \(\text{M}\), Proposition 4.6 and Théorème 4.16. We therefore omit the proof, recalling only the definition of the coproduct of the bialgebra of quasi-symmetric functions (\(\text{G}\) p.300). Take a second infinite totally ordered set of commuting variables \(Y\). Now order \(X \cup Y\) by \(x < y\) for any \(x \in X\) and \(y \in Y\). Then a quasi-symmetric function \(F\) on \(X\) defines a unique quasi-symmetric function on \(X \cup Y\), which may be rewritten as a finite linear combination \(\sum_i F_i(X)G_i(Y)\), for some quasi-symmetric functions \(F_i, G_i\). Then the coproduct \(\delta(F)\) is defined as \(\sum_i F_i \otimes G_i\).

Note also that a bialgebra constructed on posets has been considered in \(\text{E}\) and \(\text{BeSo}\), together with a bialgebra homomorphism into quasi-symmetric functions. There seems to be no evident link between their construction and ours, although the coproduct is essentially the same, while the product rests on product of posets.
2.3. An internal product. Let \((E, <_1, <_2)\) and \((F, <_1, <_2)\) be two double posets. Let \(\phi : (E, <_1) \to (F, <_2)\) be a bijection and denote its graph by

\[
E \times F = \{(e, f) : \phi(e) = f\}.
\]

This set becomes a double poset by: \((e, f) <_1 (e', f')\) if and only if \(f <_1 \phi(e')\); \((e, f) <_2 (e', f')\) if and only if \(e <_2 \phi(e')\). In other words, denoting by \(p_1, p_2\) the first and second projections, \(p_1\) is an order isomorphism \((E \times F, <_2) \to (E, <_2)\) and \(p_2\) is an order isomorphism \((E \times F, <_1) \to (F, <_1)\). Note that the inverse isomorphisms are \(p_1^{-1} = (id, \phi)\) and \(p_2^{-1} = (\phi^{-1}, id)\).

Define the internal product of \((E, <_1, <_2)\) and \((F, <_1, <_2)\) as the sum of the double posets \(E \times F\) for all increasing bijections \(\phi : (E, <_1) \to (F, <_2)\). It is denoted \(E \circ F\).

Note that this product has been chosen, among several symmetrical ones, so that the following holds: let \(\sigma\) be a permutation in \(S_n\) and denote by \(P_\sigma\) the double poset with underlying set \(\{1, \ldots, n\}\), with \(<_2\) as the natural order of this set, and \(<_1\) the order defined by \(\sigma(1) <_1 \sigma(2) <_1 \ldots <_1 \sigma(n)\). Then given two permutations \(\tau\) and \(\sigma\), one has

\[
P_\sigma \circ P_\tau = P_{\sigma \circ \tau},
\]

as the reader may easily verify.

The internal product is compatible with the Zelevinsky scalar product, as follows.

**Proposition 2.1.** Let \(E, F, G\) be double posets. Then \(\langle E \circ F, G \rangle = \langle E, F \circ G \rangle\).

The proposition immediately follows from the following lemma.

**Lemma 2.1.** Let \((E, <_1, <_2), (F, <_1, <_2), (G, <_1, <_2)\) be three double posets. There is a natural bijection between

(i) the set of pairs \((\phi, \alpha)\), where \(\phi\) is an increasing bijection \((E, <_1) \to (F, <_2)\) and \(\alpha\) is a picture from \(E \times F\) into \(G\);

(ii) the set of pairs \((\psi, \beta)\), where \(\psi\) is an increasing bijection \((F, <_1) \to (G, <_2)\) and \(\beta\) a picture from \(E \times F\) into \(F \times G\).

**Proof.** We show that the bijection is defined by \(\psi = \alpha \circ (\phi^{-1}, id)\) and \(\beta = (id, \psi) \circ \phi\), the inverse bijection being defined by \(\phi = p_1 \circ \beta\), with \(p_1\) the projection \(F \times G \to F\) and \(\alpha = \psi \circ p_2\) with \(p_2\) the projection \(E \times F \to F\).

1. Let \((\phi, \alpha)\) be as in (i) and define \(\psi = \alpha \circ (\phi^{-1}, id)\) and \(\beta = (id, \psi) \circ \phi\). Now notice that \((\phi^{-1}, id)\) is a mapping \(F \to E \times F\) and by definition of the first order of \(E \times F\), it is increasing for the first orders on \(F\) and \(E \times F\). Since \(\alpha\) is increasing \((E \times F, <_1) \to (G, <_2)\), we see that \(\psi\) is increasing \((F, <_1) \to (G, <_2)\). Now \(\beta\) maps bijectively \(E\) into \(F \times G\), as desired, and we verify that it is a picture. Note that \((id, \psi) : F \to F \times G\) is increasing for the second orders, by the definition of the latter on \(F \times G\). Hence \(\beta\) is increasing \((E, <_1) \to (F \times G, <_2)\).

Moreover, let \((f, g), (f', g') \in F \times G\) with \((f, g) <_1 (f', g')\); then \(\psi(f) = g\), \(\psi(f') = g'\) and \(g <_1 g'\). Thus \(\psi(f) <_1 \psi(f')\). We have for some \(e, e' \in E\), \(\beta(e) = (f, g)\) and \(\beta(e') = (f', g')\). Note that the definition of \(\psi\) implies that \(\alpha^{-1} \circ \psi = (\phi^{-1}, id)\), thus \(\alpha^{-1}(\psi(f)) = (\phi^{-1}(f), f)\). Since \(\alpha\) is a picture, \(\alpha^{-1}\) is increasing \((G, <_1) \to (E \times F, <_2)\), so that \(\alpha^{-1}(\psi(f)) <_2 \alpha^{-1}(\psi(f'))\) and therefore by definition of the second order on \(E \times F\), we have \(\phi^{-1}(f) <_2 \phi^{-1}(f')\); since \(\beta(e) = (\phi(e), \psi(\phi(e)))\), we obtain \(f = \phi(e)\) and \(e = \phi^{-1}(f)\). Finally \(e <_2 e'\) showing that \(\beta\) is a picture.
Define now $\phi' = p_1 \circ \beta$ and $\alpha' = \psi \circ p_2$. We must show that $\phi' = \phi$ and $\alpha' = \alpha$. We have $\phi' = p_1 \circ (id, \psi) \circ \phi$, which is equal to $\phi$ since $p_1 \circ (id, \psi)$ is the identity of $F$. Moreover, $\alpha' = \alpha \circ (\phi^{-1}, id) \circ p_2$, and we are done, since $(\phi^{-1}, id) \circ p_2$ is the identity of $E \times_\phi F$.

2. Let $(\psi, \beta)$ be as in (ii), and define $\phi = p_1 \circ \beta$ and $\alpha = \psi \circ p_2$. Now, $\beta$ is an increasing bijection $(E, \prec_1) \to (F \times_\psi G, \prec_2)$ and $p_1$ is an increasing bijection $(F \times_\psi G, \prec_2) \to (F, \prec_2)$ by definition of the second order on $F \times_\psi G$. Thus $\phi$ is an increasing bijection $(E, \prec_1) \to (F, \prec_2)$. Moreover, $p_2$ is an increasing bijection $(E \times_\psi F, \prec_1) \to (F, \prec_1)$ by definition of the first order of $E \times_\psi F$, and $\psi$ is an increasing bijection $(F, \prec_1) \to (G, \prec_2)$. Thus $\alpha$ is an increasing bijection:

$$\alpha : (E \times_\psi F, \prec_1) \to (G, \prec_2).$$

We show that $\alpha^{-1}$ is also increasing $(G, \prec_1) \to (E \times_\psi F, \prec_2)$. Indeed, let $g, g' \in G$ with $g \prec_1 g'$. Then $g = \alpha(e, f), g' = \alpha(e', f')$ with $\phi(e) = f, \phi(e') = f'$ and we must show that $(e, f) \prec_2 (e', f')$, that is, $e \prec_2 e'$. Since $\beta$ is a picture, $\beta^{-1}$ is increasing $(F \times_\psi G, \prec_1) \to (E, \prec_2)$. We have $g = \alpha(e, f) = \psi(f)$ and similarly $g' = \psi(f')$. Hence $(f, g), (f', g') \in F \times_\psi G$ and $(f, g) \prec_1 (f', g')$ since $g \prec_1 g'$. Thus $\beta^{-1}(f, g) \prec_2 \beta^{-1}(f', g')$. Now $f = \phi(e) = p_1(\beta(e)) \Rightarrow \beta(e) = (f, \psi(f)) = (f, g) \Rightarrow e = \beta^{-1}(f, g)$. Similarly, $e' = \beta^{-1}(f', g')$. It follows that $e \prec_2 e'$.

Define now $\psi' = \alpha \circ (\phi^{-1}, id)$ and $\beta' = (id, \psi) \circ \phi$. We need to show that $\psi' = \psi$ and $\beta' = \beta$. We have $\psi' = \psi \circ p_2 \circ (\phi^{-1}, id)$ which is clearly equal to $\psi$. Moreover, $\beta' = (id, \psi) \circ p_1 \circ \beta$ and we are done since $(id, \psi) \circ p_1$ is the identity of $F \times_\psi G$.

3. The sub-bialgebra of special double posets

3.1. Special double posets. We call a double poset special if its second order is total. Since we identify isomorphic double posets, a special double poset is nothing else than a labelled poset in the sense of [S]. Given a labelled poset, the labelling (which is a bijection of the poset into $\{1, \ldots, n\}$) defines a second order, which is total, on the poset. We denote by ZDS the submodule of ZD spanned by the special double posets.

A linear extension of a special double poset $\pi = (E, \prec_1, \prec_2)$ is a total order on $E$ which extends the first order $\prec_1$ of $E$. We may identify a total order on $E$ with the word obtained by listing the elements increasingly for this order: let $e_1 e_2 \ldots e_n$ be this word, with $|E| = n$. Let moreover $\omega$ be the labelling of $\pi$, that is the unique order isomorphism from $(E, \prec_2)$ onto $\{1, \ldots, n\}$. Then we identify the linear extension with the permutation $\sigma = \omega(e_1) \ldots \omega(e_n)$ in $S_n$; in other words $\sigma(i) = \omega(e_i)$ and the mapping $\sigma^{-1} \circ \omega$ is an increasing bijection $(E, \prec_1) \to \{1, \ldots, n\}$, since $\sigma^{-1} \circ \omega(e_i) = i$. In this way, a linear extension of $\pi$ is a permutation $\sigma \in S_n$ such that $\sigma^{-1} \circ \omega$ is an increasing bijection $(E, \prec_1) \to \{1, \ldots, n\}$.

In [MR] a bialgebra structure on $ZS = \oplus_{n \in \mathbb{N}} ZS_n$ has been constructed. We recall it briefly. Recall that, for any word $w$ of length $n$ on a totally ordered alphabet, the standard permutation of $w$, denoted by $st(w)$, is the permutation which is obtained by giving the numbers $1, \ldots, n$ to the positions of the letters in $w$, starting with the smallest letter from left to right, then the second smallest, and so on. For example, if $w = 432134233$, then $st(w) = 84219151011367$. The product $\ast$ for two permutations $\sigma \in S_n$ and $\tau \in S_p$ is defined as the sum of permutations in $S_{n+p}$ which are in the shifted shuffle product of the words $\sigma$ and $\tau$, that is, the shuffle product of $\sigma$ and $\bar{\tau}$, where the latter word is obtained from $\tau$ by replacing
in it each digit $j$ by $j + n$. The coproduct $\delta$ on $\mathbb{Z}S$ is defined on a permutation $\sigma \in S_n$ by: $\delta(\sigma)$ is the sum, over all factorizations (as concatenation) $\sigma = uv$ of the word $\sigma$, of $st(u) \otimes st(v)$, where $st$ denotes standardization of a word. See [MR] for these definitions.

On $\mathbb{Z}S$ put the Jöllenbeck scalar product, defined by

$$(\sigma, \tau) = \begin{cases} 1 & \text{if } \sigma = \tau^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

It turns $\mathbb{Z}S$ into an self-dual bialgebra, see [BlSc] 5.14 (where an isomorphic algebra is considered, obtained by replacing each permutation by its inverse).

Define the linear mapping $L$ from $\mathbb{Z}DS$ into $\mathbb{Z}S$ by sending each special double poset on the sum of its linear extensions. The following result is already in [BlSc], see 4.18, 5.5 and 5.10. Note that what we call here special double poset is called shape by Blessenohl and Schocker. Also, what we call composition is called semi-direct product by them. For sake of completeness, we give here an alternative proof of the coalgebra property; we construct a bijection between the appropriate sets of decompositions. The proof in [BlSc] (different from ours) uses an argument, due to Gessel [G], which is inductive on the number of edges in the Hasse diagramm of the poset.

**Theorem 3.1.** $\mathbb{Z}DS$ is a sub-bialgebra of $\mathbb{Z}D$ and $L : \mathbb{Z}DS \to \mathbb{Z}S$ is homomorphism of bialgebras.

**Proof.** It is straightforward to see that the composition of two special double posets is special, and that a decomposition of a special double poset is a pair of special double posets (the class of special double posets is a hereditary family in the sense of Schmitt [Sch]: it is closed under taking disjoint unions and ideals). Thus, $\mathbb{Z}DS$ is a sub-bialgebra of $\mathbb{Z}D$. Moreover, the set of linear extensions of the composition of $\pi$ and $\pi'$ is classically the shifted shuffle product of the set of linear extensions of $\pi$ by that of $\pi'$. Hence $L$ is a homomorphism of algebras.

The fact that it is also a homomorphism of coalgebras is proved as follows. Let $\pi = (E, <_1, <_2)$ be a special double poset. Then

$$\delta \circ L(\pi) = \sum_{\sigma, u, v} st(u) \otimes st(v),$$

where the summation is over all triples $(\sigma, u, v)$, with $\sigma$ a linear extension of $(E, <_1)$ and where $\sigma$ is the concatenation $uv$; moreover,

$$(L \otimes L) \circ \delta(\pi) = \sum_{I, S, \alpha, \beta} \alpha \otimes \beta,$$

where the summation is over all quadruples $(I, S, \alpha, \beta)$ with $I$ an inferior ideal of $(E, <_1)$ and $S$ its complementary superior ideal, and $\alpha, \beta$ are respectively linear extensions of $I, S$ for the induced order $<_1$. We show that there is a bijection between the set of such triples and quadruples. To simplify, take $\pi = (E, <_1, <_2)$ with $E = \{1, ..., n\}$ and $<_2 = <$ the natural total order on $E$. Then the labelling $\omega$ of $\pi$ is the identity mapping. We show that

$$(\sigma, u, v) \mapsto (I, S, st(u), st(v)),$$

with $I$ the set of naturals appearing in $u$ and $S$ the set of naturals appearing in $v$, is the desired bijection. Note first that, since $\sigma$ is a linear extension of $(E, <_1)$ and $\sigma = uv$, then $I$ and $S$ as defined are a lower and a superior ideal of $(E, <_1)$;
moreover, $st(u)$ and $st(v)$ are linear extensions of $(I, <_1)$ and $(S, <_1)$. This mapping is injective, since any permutation $\sigma = uv$ is determined by $st(u)$, $st(v)$ and the sets of digits in $u$ and $v$. We show that it is also surjective: let $(I, S, \alpha, \beta)$ a quadruple as above, and define uniquely $\sigma = uv$ with $st(u) = \alpha$, $st(v) = \sigma$ and $I, S$ the set of digits in $u, v$. All we have to show is that $\sigma$ is a linear extension of $(E, <_1)$. That is: if $e = \sigma(j)$ and $e' = \sigma(k)$, with $e <_1 e'$, then $j < k$. Since $\sigma$ is the concatenation of $u$ and $v$, and since $st(u), st(v)$ are linear extensions of $I, S$ for the order $<_1$, this is clear if $j, k$ are both digits in $\{1, ..., i\}$ or $\{i+1, ..., i+s\}$, with $i, s$ the cardinality of $I, S$; also, if $j$ is in the first set and $k$ in the second. Suppose by contradiction that $j$ is in the second set and $k$ in the first; then $e \in S$ and $e' \in I$, contradicting the ideal property.

The homomorphism $L$ has two other properties.

**Theorem 3.2.** The homomorphism $L : ZDS \rightarrow ZS$ preserves the Zelevinsky scalar product and the internal product.

Recall that the *internal product* of $ZS$ is simply the product which extends the product on permutations. The next lemma extends known results on classical pictures between skew shapes, cf. [GR], [BSc] Remark 13.6.

**Lemma 3.1.** Let $\pi, \pi'$ be special double posets. There is a natural bijection between pictures from $\pi$ into $\pi'$ and linear extensions of $\pi$ whose inverse is a linear extension of $\pi'$.

**Proof.** Let $\phi$ be a picture from $\pi$ to $\pi'$, and denote by $\omega, \omega'$ the respective labellings. Then, the first condition on a picture means that $\omega' \circ \phi$ is an increasing bijection of $(P, <_1)$ into $\{1, ..., n\}$; hence $\omega' \circ \phi^{-1} \circ \omega^{-1}$ is a linear extension of $(P, <_1)$. Similarly, the second condition means that $\omega' \circ \phi \circ \omega^{-1}$ is a linear extension of $(P', <_1)$. Thus the lemma follows.

**Proof of theorem.** The fact that $L$ preserves the scalar product is immediate from the lemma.

It remains to show that $L$ is a homomorphism for the internal product. Let $\pi, \pi'$ be two special double posets with underlying sets $E, F$. We may assume that $E = F = \{1, ..., n\}$, such that their second order $<_2$ is the natural order on $\{1, ..., n\}$. Then, since the labellings of $E$ and $F$ are the identity mappings, a linear extension of $\pi$ (resp. $\pi'$) is a permutation $\alpha$ (resp. $\beta$) in $S_n$ such that $\alpha^{-1}$ (resp. $\beta^{-1}$) is increasing from $(E, <_1)$ (resp. $(F, <_1)$) into $\{1, ..., n\}$.

Now, let $\phi$ be increasing $(E, <_1) \rightarrow (F, <_2) = \{1, ..., n\}$. We construct the double poset $\Pi = E \times_\phi F$ as in Section 2.2. Since we identify isomorphic double posets, we may take $F$ as underlying set, with the first order $<_1$ of $F$ as first order of $\Pi$, and with second order defined by the labelling $\phi^{-1} : F \rightarrow E = \{1, ..., n\}$. Then a linear extension $\sigma$ of $\Pi$ is a permutation $\sigma$ such that $\sigma^{-1} \circ \phi^{-1}$ is increasing $(F, <_1) \rightarrow \{1, ..., n\}$. Define $\alpha = \phi^{-1}$ and $\beta = \phi \circ \sigma$. Then $\alpha^{-1} = \phi$ (resp. $\beta^{-1} = \sigma^{-1} \circ \phi^{-1}$) is increasing $(E, <_1) \rightarrow \{1, ..., n\}$ (resp. $(F, <_1) \rightarrow \{1, ..., n\}$), and therefore $\alpha$ and $\beta$ are linear extensions of $\pi$ and $\pi'$ with $\alpha \circ \beta = \sigma$.

Conversely let $\alpha$ and $\beta$ be linear extensions of $\pi$ and $\pi'$. Put $\sigma = \alpha \circ \beta$ and $\phi = \alpha^{-1}$. Then $\phi$ is increasing $(E, <_1) \rightarrow (F, <_2) = \{1, ..., n\}$ and $\sigma^{-1} \circ \phi^{-1} = \beta^{-1}$ is increasing $(F, <_1) \rightarrow \{1, ..., n\}$. Hence $\sigma$ is a linear extension of $\Pi$.

All this implies that $L(\pi)L(\pi')$, which is the sum of all $\alpha \circ \beta$’s, is equal to $L(\pi \circ \pi')$, which is equal to the sum of all $\sigma$’s, for all possible $\phi$’s.
It is easy to prove also that the submodule of $\mathbb{ZDS}$ spanned by the naturally labelled special double posets, that is, those whose second order is a linear extension of the first order, is a sub-bialgebra of $\mathbb{ZDS}$ (the class of naturally labelled posets is closed under taking disjoint unions and ideals). It may be possible that one could compute the antipode of this subalgebra, and that of $\mathbb{ZDS}$, by extending the techniques of Aguiar and Sottile [ASo], who computed the antipode of $\mathbb{ZS}$ using the weak order on the symmetric group, and a recursive method in posets, as [G] proof of Th.1 and [BSc] Lemma 4.11.

Recall that for a permutation $\sigma \in S_n$, its descent composition $C(\sigma)$ is the composition of $n$ equal to $(c_1,\ldots,c_k)$, if $\sigma$ viewed as a word has $k$ consecutive ascending runs of length $c_1,\ldots,c_k$ and $k$ is minimum. For example $C(51247836) = (1,5,2)$, the ascending runs being 5, 1247, 83, 6. Recall from [G] p.291 the definition of the fundamental quasi-symmetric function $F_C$, for any composition $C$ (see also [S1] 7.19 where it is denoted $L_a$). Then it follows from [MR] Th.3.3 that the linear function $F : \mathbb{ZS} \to \mathbb{QSym}$ defined by $\sigma \mapsto F_C(\sigma)$ is a homomorphism of bialgebras. Recall that the bialgebra homomorphism $\Gamma : \mathbb{ZD} \to \mathbb{QSym}$ has been defined in Section 3. Then the following result is merely a reformulation of a result of Stanley (see [G] Th.1 and Eq.(1) page 291, or [S1] Cor.7.19.5).

**Corollary 3.3.** The mapping $F \circ L$ is equal to $\Gamma$ restricted to $\mathbb{ZDS}$.

3.2. **Littlewood-Richardson rule.** A lattice permutation (or Yamanouchi word) is a word on the symbols in $\mathbb{P} = \{1, 2, 3, \ldots\}$ such that, for any $i$, in each left factor, the number of $i$'s is not less than the number of $i+1$'s. For example, 11122132 is such a lattice permutation. Given a word $w$ on the symbols 1, 2, 3, ..., with $k$ the greatest symbol appearing in it, we call complement of $w$, the word obtained from $w$ by exchanging 1 and $k$ in $w$, then 2 and $k-1$, and so on. For the word of the above example, its complement is therefore 33322312. The weight of a word $w$ is the partition $\nu = 1^{n_1}2^{n_2}\ldots$, where $n_i$ is the number of $i$'s in $w$. For the word above, it is the partition $1^42^33^1$.

Given a special double poset $\pi = (E, <_1, <_2)$ of cardinality $n$ with labelling $\omega$, and a word $a_1a_2\ldots a_n$ of length $n$ over a totally ordered alphabet $A$, we say that $w$ fits into $\pi$ (we take the terminology from [GR]) if the function $E \to A$ defined by $e \mapsto a_\omega(e)$ is a $\pi$-partition. In other words, given a function $f : E \to A$, call reading word of $f$ the word $f(\omega(1))\ldots f(\omega(n))$. Then $f$ is a $\pi$-partition if and only if its reading word fits into $\pi$. Note the case where the word is a permutation $\tau$: we have that $\tau$ fits into $\pi$ if and only if the word $\tau(1)\ldots \tau(n)$ fits into $\pi$. This means that the mapping $e \mapsto \tau(\omega(e))$ is a $\pi$-partition, that is, since it is a bijection, is increasing $(E, <_1) \to \{1, \ldots , n\}$.

Given a partition $\nu$ of $n$, we define (as in [G]) a special double poset $\pi_\nu = (E_\nu, <_1', <_2')$ where $E_\nu$ is the Ferrers diagram of $\nu$, where $<_1$ is the order induced on $E_\nu$ by the natural partial order of $\mathbb{N} \times \mathbb{N}$, and where $<_2$ is given on the elements of $E_\nu$ by $(x, y) <_2 (x', y')$ if and only if either $y > y'$, or $y = y'$ and $x < x'$. Recall that there is a well-known bijection between standard Young tableaux of shape $\nu$ and lattice permutations of weight $\nu$, see [S1] Prop.7.10.3 (d). In this bijection, the shape of the tableau is equal to the weight of the lattice permutation.

If $\pi$ is a special double poset, we denote by $\hat{\pi}$ the special double poset obtained by replacing the two orders of $\pi$ by their opposite. Clearly, a permutation $\sigma$ fits into $\pi$ (assumed to be special) if and only if $w_0 \circ \sigma \circ w_0$ fits into $\hat{\pi}$ ($w_0$ is defined in Lemma 3.3 below).
Theorem 3.4. Let $\pi$ be a special double poset and $\nu$ be some partition. Then, the scalar product $(\pi, \pi_{\nu})$, that is, the number of pictures from $\pi$ to $\pi_{\nu}$, is equal to:

(i) the number of lattice permutations of weight $\nu$ whose complements fit into $\pi$;
(ii) the number of lattice permutations of weight $\nu$ whose mirror images fit into $\tilde{\pi}$.

Note that part (ii) of this is the classical formulation of the Littlewood-Richardson rule (see [Ma] (9.2) or [S1] Th.A1.3.3), once one realizes that a skew Schur function indexed by a skew shape is equal to the skew Schur function obtained by rotating by 180 degrees that shape (cf. [BlSc] Chapter 11 p.109-110).

We first need the following:

Lemma 3.2. Let $\pi$ be a special double poset. A permutation $\sigma$ is a linear extension of $\pi$ (with respect to $<_1$) if and only if its inverse fits into $\pi$.

Proof. We may assume that $\pi = (E, <_1, <_2)$ with $E = \{1, \ldots, n\}$ and $<_2$ the natural order of $E$. Then $\omega$ is the identity and therefore a permutation $\sigma$ is a linear extension of $\pi$ if and only if $\sigma^{-1}$ is increasing $(E, <_1) \to \{1, \ldots, n\}$. On the other hand, $\tau$ fits into $\pi$ if and only if it is increasing $(E, <_1) \to \{1, \ldots, n\}$, as noted previously. □

Proposition 3.1. Let $\pi$ be a special double poset. A word $w = a_1 \ldots a_n$ fits into $\pi$ if and only if its standard permutation does.

This result is equivalent to a result of Stanley, see [G] Th.1 or [S1] Th.7.19.14.

A standard Young tableau of shape $\nu$ is the same thing as a $\pi_{\nu}$-partition which is a bijection from the Ferrers diagram of $\nu$ onto $\{1, \ldots, n\}$. Thus we can speak of the reading word of a tableau, which is a classical notion. We denote it $\text{read}(T)$. We consider also the mirror reading word of $T$, which is the mirror image of $\text{read}(T)$. The following result must be well-known, but we give a proof for the convenience of the reader. Part (i) is proved in [BlSc] page 109.

Proposition 3.2. Let $T$ be a standard Young tableau and $w$ the associated lattice permutation.

(i) Let $u$ be the complement of $w$. Then the reading word of $T$ is equal to the inverse of the standard permutation of $u$.

(ii) Let $v$ be the mirror image of $w$. Then the mirror reading word of $T$ is equal to the complement of the inverse of the standard permutation of $v$.

Consider for instance the Young tableau

\[
\begin{array}{ccc}
5 \\
3 & 9 \\
2 & 6 & 10 & 11 \\
1 & 4 & 7 & 8
\end{array}
\]

Then its lattice permutation is $w = 1 \ 2 \ 3 \ 1 \ 4 \ 2 \ 1 \ 1 \ 3 \ 2 \ 2$. The complement of the latter is $u = 4 \ 3 \ 2 \ 1 \ 3 \ 4 \ 4 \ 2 \ 3 \ 3$. Standardizing this latter word, we obtain the permutation $8 \ 4 \ 2 \ 9 \ 1 \ 5 \ 10 \ 11 \ 3 \ 6 \ 7$, whose inverse is $5 \ 3 \ 9 \ 2 \ 6 \ 10 \ 11 \ 1 \ 4 \ 6 \ 7$, which is indeed the reading word of the given tableau, obtained by concatenating its rows, beginning with the last row. This illustrates (i). For (ii), the mirror image of $w$ is $v = 2 \ 2 \ 3 \ 1 \ 1 \ 2 \ 4 \ 1 \ 3 \ 2 \ 1$. The standard permutation of $v$ is $5 \ 6 \ 9 \ 1 \ 2 \ 7 \ 11 \ 3 \ 10 \ 8 \ 4$. The inverse of this permutation is $4 \ 5 \ 8 \ 11 \ 1 \ 2 \ 6 \ 10 \ 3 \ 9 \ 7$. Finally, the complement
of this permutation is $874111062935$, which is indeed the mirror reading word of $T$.

We need a lemma. For this, we use a variant of the reading word of a tableau. Call row word of a standard Young tableau $T$ the permutation, in word form, obtained by reading in increasing order the first row of $T$, then the second, and so on. Denote it by $\text{row}(T)$. For example, the row word of the previous example is

$$\text{row}(T) = 1478261011395.$$ 

Lemma 3.3. Let $w$ be a word on the alphabet $\mathbb{P}$ of weight equal to the partition $\nu = (\nu_1 > ... > \nu_k > 0)$ of $n$. Let $w_0 = n - 1 ... 2 1$ be the longest element in the group $S_n$ and $\gamma$ the longest element in the Young subgroup $S_{\nu_1} \times ... \times S_{\nu_k}$.

(i) Let $w$ be the complement of $w$. Then $\text{st}(u) = w_0 \circ \gamma \circ \text{st}(w)$.

(ii) Let $v$ be the mirror image of $w$. Then $\text{st}(v) = \gamma \circ \text{st}(w) \circ w_0$.

(iii) Suppose that $w$ is a lattice permutation and let $T$ be the tableau of shape $\nu$ corresponding to $w$. Then $\text{st}(w)$ is the inverse of $\text{row}(T)$.

Proof. (i) Note that $\gamma$ as word is equal to

$$\nu_1 ... 1 (\nu_1 + \nu_2) ... (\nu_1 + 1) ... n (\nu_1 + ... + \nu_{k-1} + 1).$$

Hence

$$w_0 \circ \gamma = (\nu_2 + ... + \nu_k + 1) ... n (\nu_2 + ... + \nu_k + 1) ... (\nu_2 + ... + \nu_k) ... 1 ... \nu_k.$$ 

The proof of (i) then follows by inspection.

(ii) Likewise, one proves that $\text{st}(v) \circ w_0 = \gamma \circ \text{st}(w)$ by inspection.

(iii) Let $I_1, ..., I_k$ denote the successive intervals of $\{1, ..., n\}$ of cardinality $\nu_1, ..., \nu_k$. Let $L_1, ..., L_k$ be the set of elements in the successive rows of $T$. If $I, L$ are two subsets of equal cardinality of $\mathbb{P}$, we denote by $I \supset L$ the unique increasing bijection from $I$ into $L$. We denote also $f_1 \cup ... \cup f_k$ the function which restricts to $f_i$ on its domains, assuming the domains are disjoint. Then $\text{row}(T)$ is the permutation $\cup_{j=1,...,k} I_j \supset L_j$, and its inverse is therefore $\cup_{j=1,...,k} L_j \supset I_j$. Moreover, the word $w$ is defined by the following condition: for each position $p \in \{1, ..., n\} = \cup_{j=1,...,k} L_j$, the $p$-th letter of $w$ is $j$ if and only if $p \in L_j$. Recall that $\text{st}(w)$, viewed as word, is obtained by giving the numbers $1, ..., n$ to the positions of the letters in $w$, starting with the 1’s from left to right, then 2’s, and so on. Therefore $\text{st}(w) = \cup_{j=1,...,k} L_j \supset I_j$. \hfill $\square$

Proof of proposition. (i) We have to prove that $\text{read}(T) = \text{st}(u)^{-1}$. We know by Lemma 5.2.(i) and (iii) that $\text{st}(u) = w_0 \circ \gamma \circ \text{st}(w)$ and $\text{st}(w)^{-1} = \text{row}(T)$. Clearly $\text{read}(T) = \text{row}(T) \circ \delta$, where $\delta$ is the permutation

$$(\nu_1 + ... + \nu_{k-1} + 1) ... n (\nu_1 + ... + \nu_{k-2} + 1) ... (\nu_1 + ... + \nu_{k-1}) ... 1 ... \nu_1.$$ 

Now $\delta = \gamma \circ w_0$. Therefore, $\text{read}(T) = \text{row}(T) \circ \gamma \circ w_0 = \text{st}(w)^{-1} \circ \gamma \circ w_0 = \text{st}(u)^{-1}$, since $w_0$ and $\gamma$ are involutions.

(ii) We have to show that $\text{read}(T) \circ w_0 = w_0 \circ \text{st}(v)^{-1}$. We know by Lemma 5.2.(ii) that $\text{st}(v) = \gamma \circ \text{st}(w) \circ w_0$, hence $\text{st}(w) = \gamma \circ \text{st}(v) \circ w_0$. Using what we have done in (i), we have therefore

$$\text{read}(T) = \text{st}(w)^{-1} \circ \gamma \circ w_0 = w_0 \circ \text{st}(v)^{-1} \circ \gamma \circ \gamma \circ w_0 = w_0 \circ \text{st}(v)^{-1} \circ w_0.$$ \hfill $\square$

Proof of theorem. (i) By Lemma 3.1 and Lemma 3.2, the indicated scalar product is equal to the number of permutations $\sigma$ which fit into $\pi$ and whose inverse fits
into $\pi_{\nu}$. Let $H$ denote the set of complements of lattice permutation of weight $\nu$. By Proposition 3.2.(i), the mapping $H \rightarrow \text{RWSYT}_{\nu}, \, w \mapsto st(w)^{-1}$ is a bijection, where we denote by $\text{RWSYT}_{\nu}$ the set of reading words of standard Young tableaux of shape $\nu$. By Proposition 3.1, part (i) of the theorem follows.

(ii) Let $K$ denote the set of mirror images of lattice permutations of weight $\nu$. By Prop. 3.2.(ii), the mapping $K \rightarrow \text{RWSYT}_{\nu}, \, v \mapsto \sigma = w_0 \circ st(v)^{-1} \circ w_0$ is a bijection. Moreover, $\sigma^{-1} = w_0 \circ st(v) \circ w_0$ fits into $\tilde{\pi}$ if and only if $st(v)$ fits into $\pi$, that is, by Prop. 3.1, if and only if $v$ fits into $\pi$. □

References

[ABSo] M. Aguiar, N. Bergeron and F. Sottile: Combinatorial Hopf algebras and generalized Dehn-Sommerville relations, Compositio Mathematica 142, 1-30 (2006).
[ASo] M. Aguiar and F. Sottile: Structure of the Malvenuto-Reutenauer Hopf algebra of permutations, Advances in Mathematics 191, 225-275 (2005).
[BeSo] N. Bergeron and F. Sottile: Hopf algebras and edge-labelled posets, Journal of Algebra 216, 641–651 (1999).
[BiSc] D. Blesseohol and M. Schocker: Noncommutative character theory of the symmetric group, Imperial College Press, London (2005).
[E] R. Ehrenborg: On posets and Hopf algebras, Advances in Mathematics 138, 211–262 (1996).
[FG] S. Fomin and C. Greene: A Littlewood-Richardson miscellany, European Journal of Combinatorics 14 n. 3, 191–212 (1993).
[GR] A. Garciá and J. Remmel: Shuffles of permutations and the Kronecker product, Graphs and Combinatorics 1, 217-263 (1985).
[Gei] L. Geissinger: Hopf algebras of symmetric functions and class functions, in Combinatoire et Représentation du Groupe Symétrique, Springer Lecture notes in Mathematics 579, 168–181 (1977).
[G] I. Gessel: Multipartite P-partitions and inner product of skew Schur functions, in Combinatorics and Algebra, Contemporary Mathematics 34, 289–301 (1984).
[H] S. K. Hsiao and T. Kyle Petersen: Colored posets and colored quasi-symmetric functions, Annals of Combinatorics (to appear).
[JP] G. D. James and M. H. Peel: Specht series for skew representations of symmetric groups, Journal of Algebra 56, 343–364 (1979).
[L] M. A. A. van Leeuwen: Tableau algorithms defined naturally for pictures, Discrete Mathematics 157, 321-362 (1996).
[Lo] M. Lothaire: Algebraic combinatorics on words, Cambridge University Press (2002).
[Ma] I. Macdonald: Symmetric functions and Hall polynomials, Oxford University Press (1995).
[MR] C. Malvenuto and C. Reutenauer: Duality between quasi-symmetric functions and the Solomon descent algebra, Journal of Algebra 177, 967–982 (1995).
[MR1] C. Malvenuto and C. Reutenauer: Plethysm and conjugation of quasi-symmetric functions, Discrete Mathematics 193, 225–233 (1998).
[M] C. Malvenuto: Produits et coproduits des fonctions quasi symétriques et de l’algèbre des descentes, Publications du Laboratoire de Combinatoire et d’Informatique Mathématique 16 (1994).
[M1] C. Malvenuto: A Hopf algebra of labelled partially ordered sets, AMS Meeting #976, Special Session on Combinatorial Hopf Algebras, http://www.ams.org/notices/200206/mtl-montreal-prog.pdf Montreal (2002).
[PR] S. Poirier and C. Reutenauer: Algèbre de Hopf des tableaux, Annales des Sciences Mathématiques du Québec 19 n. 1, 79–90 (1995).
[Sc] W. Schmitt: Incidence Hopf algebras, Journal of Pure and Applied Algebra 96, 299–330 (1994).
[S] R. Stanley: Ordered structures and partitions, Memoirs of the American Mathematical Society 119 (1972).
[St] R. Stanley: Enumerative Combinatorics Vol 2, Cambridge University Press (1999).
[Z] A. V. Zelevinsky: A generalization of the Littlewood-Richardson rule and the Robinson-Schensted-Knuth correspondence, Journal of Algebra 69, 82-94 (1981).
[Z1] A.V. Zelevinsky: Representations of finite classical groups. A Hopf algebra approach, Lecture Notes in Mathematics, 869 (1981).

Claudia Malvenuto, Dipartimento di Informatica, Università La Sapienza, Via Salaria 113, 00198 Roma Italia
E-mail address: claudia@di.uniroma1.it

Christophe Reutenauer, Département de Mathématiques UQAM, Case Postale 8888 Succ. Centre-ville, Montréal (Québec) H3C 3P8 Canada
E-mail address: Reutenauer.Christophe@uqam.ca