Branching rules for $S_{2N} \to W(B_N)$

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Abstract

This note presents a procedure to determine the reduction of the irreducible and the induced characters of the symmetric group $S_{2N}$ in terms of the irreducible and induced characters of the hyperoctahedral group $W(B_N) = Z_2^N \sim S_N$.

Mathematical Subject Classification

20Bxx, 20Cxx, 20Exx

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1 Introduction

To each classical Lie group corresponds a finite group generated by the reflections of its root system, called the Weyl group. There has been a number of situations in which the Weyl groups have played an important role. This importance grew out of the various possibilities of application to physical problems i.e., particle physics, discrete $\sigma$ models, lattice gauge theories, chiral models (Ref. [1, 2]).

The Symmetric group $S_N$ is the Weyl group of the Unitary Group. For $B_N = SO(2N + 1)$ and $C_N = Sp(2N)$, the Weyl Groups $W(B_N)$ and $W(C_N)$ are isomorphic. $W(B_N)$ is $Z_2^N \sim S_N$, the wreath product of the abelian group $Z_2^N$ generated by the $N$ sign changes $(+i, -i), 1 \leq i \leq N$, and the symmetric group $S_N$. The order of $W(B_N)$ is $2^N N!$ (Ref. [3, 4]). Let $K_N$ be defined as the convex hull of points $\pm e_i, 1 \leq i \leq N$, where $e_1, \ldots e_N$ are the unit coordinate vectors in $R^N$. It is the N-dimensional generalization of the octahedron $K_3$. The group of symmetries of $K_N$, called the hyperoctahedral group is $W(B_N)$. The structure and representation of this group have been studied (Ref. [5, 6, 7]). Moreover the hyperoctahedral groups appear in numerous applications such as weakly bound water clusters, non-rigid molecules, disordered proteins and the enumeration of isomers (Ref. [8, 9]). The hyperoctahedral group $Z_2^N \sim S_N$ is a subgroup
of the symmetric group $S_{2N}$. The purpose of this note is to propose a procedure to solve the reduction $S_{2N} \rightarrow (Z_2^N \sim S_N)$. Although there are already computer codes available to generate the character tables of $S_N$ for any $N$, and their wreath products (Ref. [10]), to the best of my knowledge this branching case has not been treated as yet.

In order to make this article reasonably self-contained some pertinent results already published will be exposed anew. In Section 2 and Section 3, respectively, algorithms for the irreducible and induced characters of $S_{2N}$ and $W(B_N)$ are treated. Section 4 deals with the reduction $S_{2N} \rightarrow W(B_N)$.

## 2 The induced and the irreducible characters

Consider a partition $(\lambda) = (\lambda_1, \ldots, \lambda_p)$ of $2N$, where $\lambda_1 + \lambda_2 + \ldots + \lambda_p = 2N$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p = 0$; $p(2N)$ is the number of partitions of $2N$.

Corresponding to each partition of $2N$ we can construct $S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_p}$. Such subgroups are called the canonical subgroups of $S_N$. Let $C$ be a class of $S_{2N}$ characterized by its cycle structure $(1^\alpha, 2^\beta, 3^\gamma, \ldots)$. This symbol denotes that the permutations in $C$ contain $\alpha$ 1-cycles, $\beta$ 2-cycles, $\gamma$ 3-cycles, etc., where $\alpha + 2\beta + 3\gamma + \ldots = 2N$. Besides for each $S_{\lambda_i}$ we have

$$\alpha_i + 2\beta_i + 3\gamma_i + \ldots = \lambda_i \quad (A1)$$

The character induced in $S_{2N}$ by the identity representation of a canonical subgroup is

$$\phi^{(\lambda)}_{(1^\alpha, 2^\beta, \ldots)} = \sum \frac{\alpha!}{\alpha_1!\alpha_2!\ldots} \frac{\beta!}{\beta_1!\beta_2!\ldots} \frac{\gamma!}{\gamma_1!\gamma_2!\ldots} \ldots$$

Where

$$\sum \alpha_i = \alpha, \quad \sum \beta_i = \beta, \quad \sum \gamma_i = \gamma, \ldots \quad (A2)$$

The sum is over all the integer solutions of the system of Eqs. (A1) and (A2). These characters may be arranged as the entries of a $p(2N) \times p(2N)$ matrix $\phi$ whose rows and columns are labeled, respectively, by partitions of $2N$ arranged in lexicographical order and by the classes (Ref. [11]).

The table of irreducible characters of $S_{2N}$ may be derived from $\phi$ (Ref. [11]). Each row $\phi_i$ must be considered as a vector; it suffices to orthonormalize them via the Gram-Schmidt method to get the rows $x_i$ of the irreducible characters table $X$, i.e.,

$$x_i = \phi_i - \sum_{k=1}^{i-1} (\phi_i K x_k) x_k \quad (1)$$
(for \( i = 1 \), \( x_i = \phi_1 \)), where \( x_i \) and \( \phi_i \) are the \( i \)-th rows of \( X \) and \( \phi \) respectively, and \( K \) is a diagonal matrix whose elements are

\[
[K_{ij}] = \delta_{jk} \frac{C}{(2N)!}
\]

\( C \) is the order of the class \((1^\alpha, 2^\beta, 3^\gamma, \ldots)\) of \( S_{2N} \), \( C = \frac{(2N)!}{1^{\alpha_1} \cdot 2^{\beta_1} \cdot \alpha ! ...} \).

Expression (1) may be written as

\[
\phi_i = x_i + \sum_{k=1}^{i-1} (\phi_i K x_k) x_k \tag{2}
\]

Considering the coefficients of the \( x_k \) we get a lower triangular matrix \( \Delta \) such that \( \det \Delta = 1 \). In general we have for \( S_{2N} \)

\[
\phi = \Delta X \tag{3}
\]

As an example for \( S_4 \) we have:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
4 & 2 & 0 & 1 \\
6 & 2 & 2 & 0 \\
12 & 2 & 0 & 0 \\
24 & 0 & 0 & 0 \\
\end{array}
= \begin{array}{cccc}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 2 & 1 & 1 \\
1 & 3 & 2 & 3 \\
\end{array}
= \begin{array}{cccc}
1 & 1 & 1 & 1 \\
3 & 1 & -1 & 0 \\
2 & 0 & 2 & -1 \\
3 & -1 & -1 & 0 \\
1 & -1 & 1 & 1 \\
\end{array}
\]

### 3 The Induced and the Irreducible characters of \( W(B_N) \)

The set of all \( g = (\sigma; f) \), where \( \sigma \in S_{2N} \) and \( f \) is a mapping of \([1, 2N]\) into \( \mathbb{Z}_2 \), together with the composition defined by

\[
(\sigma'; f') (\sigma; f) = (\sigma' \sigma; f' (f \sigma^{-1}))
\]

form the group \( W(B_N) = Z_2^N \sim S_N \).

The cycles of the permutation are called “cycles of \( g \)”. A cycle \((a_1, \ldots, a_\beta)\) of \( g \) is positive or negative if \( f(a_1) \ldots f(a_\beta) = +1 \) or \(-1\). Let \( \beta = (\beta_1, \ldots, \beta_k) \) be the \( \beta \) system of cycles of \( \sigma \), and suppose the cycles are arranged in such a way that a negative cycle necessarily precedes a positive cycle of equal length. Then \((\beta, b)\) is called the \( \beta \) system of cycles of \( g \), where \( b := (b_1, \ldots, b_k) \) with \( b_i := 1 \) or \( 0 \) if the \( i \)-th cycle is positive or negative (remark: if \( \beta_i = \beta_{i+1} \), then \( b_i \leq b_{i+1} \)). Moreover if \( \alpha_+^i \) and \( \alpha_-^i \) denote then number of positive and negative cycles, respectively, of length \( i \) of \( g \), then
\[ \alpha = (\alpha_1^+, \alpha_1^-, \alpha_2^+, \alpha_2^-, \ldots, \alpha_\ell^+, \alpha_\ell^-) \]

is called the \( \alpha \) system of cycles of \( g \) (remark: if \( \alpha_i := \alpha_i^+ + \alpha_i^- \) then \( \sum \alpha_i^+ = N \)).

The elements of \( W(B_N) \) are conjugates \( i f f \) they have the same \( \alpha \) system of cycles and \( i f f \) they have the same \( \beta \) system of cycles. The class of elements with \( \alpha \) system \( \alpha = (\alpha_1^+, \ldots, \alpha_\ell^-) \) is denoted \( C(\alpha) \).

Let \( \lambda = (\lambda_1, \ldots, \lambda_k) \) be a partition of \( N \) \((\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k) \) and \( b = (b_1, \ldots, b_k) \) be such that \( b_i = 1 \) or \( 0 \) (remark: if \( \lambda_i = \lambda_{i+1} \), then \( b_i \leq b_{i+1} \)). The subgroup \( \left( Z_2^{(\lambda_1-b_1)} \sim S_{\lambda_1} \right) \times \left( Z_2^{(\lambda_2-b_2)} \sim S_{\lambda_2} \right) \ldots \), denoted by \( S(\lambda, b) \), is a canonical subgroup of \( W(B_N) \). Then, for the class \( C(\alpha) \) and the canonical subgroup \( S(\lambda, b) \) the algorithm giving the character \( I_{S(\lambda, b)}^{(C(\alpha))} \) of the representation of \( W(B_N) \) induced by the identity representation of \( S(\lambda, b) \) is:

\[
I_{S(\lambda, b)}^{(C(\alpha))} = \left( \sum_{i=1}^{\Sigma b_i} \prod_{i=1}^{\ell} \frac{\Pi^+_{\ell} (\alpha_i^+)! (\alpha_i^-)!}{\Pi^+_{j=1} (\alpha_{ij}^+)! (\alpha_{ij}^-)!} \right)
\]

Then sum concerns the matrices \( \left( \alpha_{ij}^{+/-} \right) \) of dim \( \ell \times k \times 2 \) where

\[
\forall i_0, \quad \sum_{j=1}^{k} \alpha_{i_0j}^+ = \alpha_{i_0}^+
\]

and

\[
\sum_{j=1}^{k} \alpha_{i_0j}^- = \alpha_{i_0}^-
\]

\[
\forall j_0, \quad \sum_{j=1}^{\ell} \left( \alpha_{ij_0}^+ + \alpha_{ij_0}^- \right) = \lambda j_0
\]

Besides \( \forall j_0 \), if \( b_{j_0} = 1 \), then \( \sum_i \alpha_{ij_0}^- \) is an even number. The order of the class \( C(\alpha) \) is

\[
|C(\alpha)| = N! \prod_{i=1}^{\ell} \left( \frac{2^{\alpha_i(i-1)}}{\alpha_i^+! (\alpha_i^-)!} \right)
\]

By means of such an algorithm, the induced character table \( I \{ W(B_N) \} \) is obtained. Each row of the table is given by the corresponding \( I_{S(\lambda, b)}(C(\alpha)) \).

For \( N = 2 \), the table of induced characters is:

4
The table of irreducible characters $Y \{W(B_N)\}$ can be obtained from $I \{W(B_N)\}$. As before each row of $I \{W(B_N)\}$ must be considered as a vector and via the Gram-Schmidt procedure the rows of $Y \{W(B_N)\}$ are obtained. In general

$$Y_i = I_i - \sum_{k=1}^{i-1} (I_i D Y_k) Y_k \quad \text{for } i = 1, \quad Y_1 = I_1$$

where $Y_i$ and $I_i$ are the $i$-th row of $Y \{W(B_N)\}$ respectively and $D$ is a triangular matrix whose elements are $(D_{\alpha\beta}) = \delta_{\alpha\beta} \frac{|C(\alpha)|}{2 N!}$, $|C(\alpha)|$ is the order of the class $C(\alpha)$ of $W(B_N)$. Here

$$I \{W(B_N)\} = DY \{W(B_N)\}. \quad (4)$$

For instance, for $W(B_2)$, the Weyl group of $SO(5)$, we have

$$\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 0 & 2 & 2 \\
2 & 2 & 2 & 0 \\
4 & 2 & 0 & 0 \\
8 & 0 & 0 & 0 \\
\end{array} \quad = \quad \begin{array}{c}
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 2 \\
1 & 1 & 1 & 2 \\
\end{array} \quad \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 \\
\end{array} \\
\end{array}$$

i.e., $I \{W(B_2)\} = DY \{W(B_2)\}$.

4 The Reduction $S_{2N} \rightarrow W(B_N)$

In this section we expose a procedure to express the content of the irreducible and the induced characters of $S_{2N}$ in terms of the irreducible and the induced characters of its subgroup $W(B_N)$. Such an algorithmic process is valid in general i.e., for any $N$. However it must be pointed out that every branching case must be treated with due regard to its own structural traits (see Appendix (I)).
As a matter of fact we shall envisage the reduction from two different points of view (hereafter Method (A) and Method (B)).

4.1 Method (A)

We already know that for $S_{2N}$ we have $\phi = \Delta X$ (Section 2) and for $W(B_N) I = DY$ (Section 3). Besides, in order to carry out the reduction use must be made of the modified characters tables $X'$ and $\phi'$ (see Appendix (I)). The characters of $S_{2N}$ can be expressed in terms of the characters of $W(B_N)$ by means of reduction matrices. We denote the reduction matrices for the irreducible and induced characters as

$$R_{Y_{W(B_N)}}^{X_{S_{2N}}}$$ and $$R_{I_{W(B_N)}}^{\phi_{S_{2N}}}$$

(in short, $R_1$ and $R_2$ respectively) Then:

$$X' = R_1 Y$$  \hspace{1cm} (5)

$$\phi' = R_2 I$$  \hspace{1cm} (6)

To obtain the entries of the reduction matrices a system of $P(N)$ linear equations with $K(W(B_N))$ unknowns must be solved via $K(W(B_N))$ independent linear equations.

$K(W(B_N))$ is the number of classes of $W(B_N)$. A simple expression for $K(W(B_N))$ appears in ref [9].

Let us note that (6) can be written as

$$\phi' = R_2 I = R_2 DY$$  \hspace{1cm} (7)

and

$$\phi' = \Delta' X' = \Delta' R_1 Y$$

then

$$R_2 DY = \Delta' R_1$$

hence

$$R_2 D = \Delta' R_1$$  \hspace{1cm} (8)
This equation establishes a direct relation between the two branching matrices. To illustrate equation (8), we shall consider the simplest reduction case $S_4 \rightarrow W(B_2)$:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 3 \\
3 & 1 & 1 & 1 \\
\end{array}
\begin{array}{cccc}
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 2 & 1 \\
\end{array}
= \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 3 & 2 & 3 & 1 \\
\end{array}
\begin{array}{cccc}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{array}
\]

4.2 Method (B)

This approach relies on two branching rules which have been solved. The first one is the classic Weyl’s rule for $S_N \rightarrow S_{N-1}$: "The irreducible representations of $S_N$ with the symmetry pattern $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ reduces on restricting $S_N$ to the subgroup $S_{N-1}$ associated with the patterns $(\lambda_1 - 1, \lambda_2, \lambda_3, \ldots); (\lambda_1, \lambda_2 - 1, \lambda_3, \ldots); (\lambda_1, \lambda_2, \lambda_3 - 1, \ldots)$ and so on. Those patterns in which the rows are not arranged in decreasing length are to be omitted" (Ref. [13]). Such a reduction may be written as a matrix whose rows and columns are indexed by the partitions of $N$ and $N-1$ ordered in lexicographic order. For example the matrix corresponding to $S_4 \rightarrow S_3$ is:

\[
\begin{array}{cccc}
3 & 21 & 111 \\
4 & 1 & & \\
31 & 1 & 1 & \\
22 & 1 & & \\
211 & 1 & 1 & \\
1111 & 1 & & \\
\end{array}
\]

The second one is the reduction rule for the hyperoctahedral group (Ref. [12]). We have then:

(a) $S_N \rightarrow S_{N-1} \rightarrow \ldots \rightarrow S_2 \rightarrow S_1$

(b) $W(B_N) \rightarrow W(B_{N-1}) \rightarrow \ldots \rightarrow W(B_1)$

Since $S_2$ and $W(B_1)$ are isomorphic, from (a) and (b) we deduce

$$\{S_{2N} \rightarrow W(B_N)\} \{W(B_N) \rightarrow W(B_1)\} = S_{2N} \rightarrow W(B_N)$$
For $N = 2$

$$\{S_4 \to W(B_2)\} \{W(B_2) \to W(B_1)\} = S_4 \to W(B_1)$$

(i) \(W(B_2) \to W(B_1)\)

(ii) \(S_4 \to S_3 \to S_2\)

(iii) Finally

Let us remark that Method (B) can be employed to verify the branching result obtained by following Method (A).

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Appendix (I)

(i) Let \( g \) be an element of \( S_{2N} \) and \( C(g) \) the conjugacy class of \( g \) in \( S_{2N} \). The character \( F_{W(B_N)}^{S_{2N}} \) may be defined as follows:

\[
F_{W(B_N)}^{S_{2N}} = \frac{|S_{2N}|}{|W(B_N)|} \frac{|C(g) \cap W(B_N)|}{|C(g)|}
\]

where \(|S_{2N}|\) and \(|W(B_N)|\) are the orders of \( S_{2N} \) and \( W(B_N) \) and \(|C(g) \cap W(B_N)|\) and \(|C(g)|\) are, respectively, the orders of the class \( g \) in \( W(B_N) \) and the order of the class \( g \) of \( S_{2N} \). Hence

\[
F_{W(B_N)}^{S_{2N}} = \frac{(2N)!}{2^N N!} \frac{|C(g) \cap W(B_N)|}{|C(g)|}.
\]

(ii) For an even number \( 2N \) the number of partitions whose subpartitions are even numbers is \( P(N) \). For instance for \( N = 4 \),

\[
P(8) = (8) + (6,2) + (4,4) + (4,2,2) + (2,2,2,2) = 5 = P(4).
\]

(iii) Let the irreducible characters of \( S_{2N} \) corresponding to such partitions compose \( F_{W(B_N)}^{S_{2N}} \). For \( N = 4 \), \( F_4 = x(4) + x(2,2) \). By means of the irreducible character table of \( S_4 \) it is possible to write:

| order | \( x(4) \) | \( x(2,2) \) | \( F_{W(B_2)}^{S_4} \) |
|-------|------------|------------|----------------|
| 1     | 1^4        | 1          | 2              |
| 6     | 1^22       | 1          | 0              |
| 3     | 2^2        | 1          | 2              |
| 8     | 1^3        | -1         | 0              |
| 6     | 1^4        | 1          | 0              |

From the formulas stated in (i), \(|C(g) \cap W(B_2)|\) can be evaluated:

- order of \( C_1 \) in \( W(B_2) \) = 1
- order of \( C_2 \) in \( W(B_2) \) = 2
- order of \( C_3 \) in \( W(B_2) \) = 3
- order of \( C_4 \) in \( W(B_2) \) = 0
- order of \( C_5 \) in \( W(B_2) \) = 2

The order of \( W(B_2) \) is \( 2^22! = 8 \). The order of \( C_3 \) does not divide the order of \( W(B_2) \). So the class \( C_3 \) of \( W(B_2) \) must be decomposed in the character table of \( S_4 \) and the class \( C_4 \) must be omitted. The resulting irreducible character table of \( S_4 \) (denoted \( X' \)) is:
\[ X' = \begin{array}{cccccc}
1^4 & 1^2 2 & 2^2 & 2^2 & 4 \\
1 & 1 & 1 & 1 & 1 \\
3 & 1 & -1 & -1 & -1 \\
2 & 0 & 2 & 2 & 0 \\
3 & -1 & -1 & -1 & 1 \\
1 & -1 & 1 & 1 & -1 \\
\end{array} \]

Remarks (1): For the identity class \( 1^{2N} \) the character \( F_{W(B_N)}^{S_{2N}} \) is:

\[ \begin{array}{c}
N = 2 & F_{W(B_2)}^{S_4} = 3 = 3 \cdot 1 \\
N = 3 & F_{W(B_3)}^{S_6} = 15 = 5 \cdot 3 \cdot 1 \\
N = 4 & F_{W(B_4)}^{S_8} = 105 = 7 \cdot 5 \cdot 3 \cdot 1 \\
N = 5 & F_{W(B_5)}^{S_{10}} = 945 = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 \\
\end{array} \]

Accordingly:

\[ F_{W(B_N)}^{S_{2N}} = (2N - 1)(2N - 3) \ldots 1 \]

(2): In general if \( |C(g) \cap W(B_N)| \) is not a divisor of \( |W(B_N)| \) the corresponding class in \( X'(S_{2N}) \) must be divided.

For \( N = 3 \) this occurs for the classes \((1^2 2^2)\) and \((2^3)\); for \( N = 4 \), the classes \((1^4 2^2)\), \((1^2 2^3)\), \((2^2 4)\) and \((4^2)\) are decomposed. It must be emphasized that for each \( N \) the procedure must be carried out. Perhaps this is the main difficulty of the present algorithm for the reduction \( S_{2N} \rightarrow W(B_N) \).

(3): The induced character table of \( S_{2N} \), \( \phi \), is treated in an analogous manner. A modified character table, \( \phi' \), results. So for \( S_4 \):

\[ \phi' = \begin{array}{cccccc}
1^4 & 1^2 2 & 2^2 & 2^2 & 1^4 \\
1 & 1 & 1 & 1 & 1 \\
4 & 2 & 0 & 0 & 0 \\
6 & 2 & 2 & 2 & 0 \\
12 & 2 & 0 & 0 & 0 \\
24 & 0 & 0 & 0 & 0 \\
\end{array} \]

\( \phi' \) and \( X' \) are related by the equation:

\[ \phi' = \Delta' X' \]
\[
\begin{array}{cccc}
 1 & 1 & 1 & 1 \\
 4 & 2 & 0 & 0 \\
 6 & 2 & 2 & 2 \\
12 & 2 & 0 & 0 \\
24 & 0 & 0 & 0
\end{array}
\quad = \quad
\begin{array}{cccc}
 1 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 \\
 1 & 1 & 1 & 0 \\
 1 & 2 & 1 & 1 \\
 1 & 3 & 2 & 3
\end{array}
\quad = \quad
\begin{array}{cccc}
 1 & 1 & 1 & 1 \\
 3 & 1 & -1 & -1 \\
 2 & 0 & 2 & 2 \\
 3 & -1 & -1 & 1 \\
 1 & -1 & 1 & 1
\end{array}
\]

Note that $\Delta = \Delta'$.
Appendix (II)
The Reduction $S_6 \rightarrow W(B_3)$ (Method (B))

(1) $W(B_3) \rightarrow W(B_2) \rightarrow W(B_1)$

\[
\begin{array}{c|c|c}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\begin{array}{c|c|c}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\begin{array}{c|c|c}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

(2) $S_6 \rightarrow S_2$

\[
\begin{array}{c|c|c}
1 & 1 & 1 \\
3 & 1 & 1 \\
3 & 3 & 1 \\
3 & 3 & 1 \\
1 & 2 & 1 \\
2 & 6 & 2 \\
1 & 3 & 3 \\
2 & 1 & 3 \\
3 & 3 & 1 \\
1 & 2 & 1 \\
\end{array}
\begin{array}{c|c|c}
1 & 1 & 1 \\
4 & 1 & 1 \\
6 & 3 & 1 \\
6 & 4 & 1 \\
3 & 2 & 1 \\
8 & 8 & 1 \\
4 & 6 & 1 \\
2 & 3 & 1 \\
3 & 6 & 1 \\
1 & 4 & 1 \\
\end{array}
\begin{array}{c|c|c}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

$\begin{array}{c|c|c}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}$
(3) \( \{S_6 \to W(B_3)\} \{W(B_3) \to W(B_1)\} = \{S_6 \to S_2\} \)
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