SELFIMPROVEMENT OF THE INEQUALITY BETWEEN ARITHMETIC AND GEOMETRIC MEANS

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It is well known that the AM-GM inequality has selfimproving properties. Let \( x_i \geq 0 \) for \( i = 1, \ldots, n \). The classical, equal weights case, states that

\[
\prod_{i=1}^{n} x_i^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i.
\]

Let \( \alpha_i > 0 \) satisfy \( \sum_{i=1}^{n} \alpha_i = 1 \). Inequality (1) selfimproves to the rational weights case simply via repetition of terms, and to the case of real weights \( \alpha_i \) just by taking limits. So the general AM-GM inequality

\[
\prod_{i=1}^{n} x_i^{\alpha_i} \leq \sum_{i=1}^{n} \alpha_i x_i
\]

follows. There is a second way in which the AM-GM inequality selfimproves. Let \( s > 0 \) and use the change of variables \( x_i = y_i^s \). Substituting in (2) and taking \( s \)-th roots we get

\[
\prod_{i=1}^{n} y_i^{\alpha_i} \leq \left( \sum_{i=1}^{n} \alpha_i y_i^s \right)^{1/s}.
\]

Now for \( 0 < s < 1 \), Jensen’s inequality tells us that \( \left( \sum_{i=1}^{n} \alpha_i y_i^s \right)^{1/s} \leq \sum_{i=1}^{n} \alpha_i y_i^s \) since \( t^s \) is concave, and furthermore the inequality is strict unless \( y_1 = \cdots = y_n \) (this follows from the equality case in Jensen’s inequality). So (2) automatically proves a family of better inequalities; it “pulls itself by its bootstraps”. The particular case \( s = 1/2 \) immediately leads to a natural and useful refinement of (2).

**Theorem 0.1.** For \( i = 1, \ldots, n \), let \( x_i \geq 0 \), and let \( \alpha_i > 0 \) satisfy \( \sum_{i=1}^{n} \alpha_i = 1 \). Then

\[
\prod_{i=1}^{n} x_i^{\alpha_i} \leq \sum_{i=1}^{n} \alpha_i x_i - \sum_{i=1}^{n} \alpha_i \left( x_i^{1/2} - \sum_{k=1}^{n} \alpha_k x_k^{1/2} \right)^2.
\]

Note that the right most term of (4) is the variance \( \text{Var}(x^{1/2}) \) of the vector \( x^{1/2} = (x_1^{1/2}, \ldots, x_n^{1/2}) \) with respect to the probability \( \sum_{i=1}^{n} \alpha_i \delta_{x_i} \). So a large variance (of \( x^{1/2} \)) pushes the arithmetic and geometric means apart.

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Remark 0.5. Let \( n \geq m \) be how to extend this sharper version to \( n \) case weight. This would lead to the same type of application as above. In fact, for the special case \( \sum_{k=1}^{n} \alpha_k x_k^{1/2} \) is much larger than the other. But perhaps it is possible to give an upper bound for \( \sum_{k=1}^{n} \alpha_k x_k^{1/2} \) of the variance of \( x \), comparable to \( \text{Var}(x^{1/2}) \). One would need to obtain the same homogeneity on both sides of (4).

The motivation to search for variants of (6) comes the fact that it is not well suited to the particular application considered here (refining Hölder’s inequality). One would need to assume that \( |f_i| \leq M \) almost everywhere. We give bounds using the variance of \( x^{1/2} \) instead of the variance of \( x \) in order to ensure the integrability of the functions involved, and also to obtain the same homogeneity on both sides of (4).

Corollary 0.2. For \( i = 1, \ldots, n \), let \( 1 < p_i < \infty \) be such that \( p_1^{-1} + \cdots + p_n^{-1} = 1 \), and let \( 0 \leq f_i \in L^{p_i} \) satisfy \( ||f_i||_{p_i} > 0 \). Then

\[
\left\| \prod_{i=1}^{n} f_i \right\|_{1} \leq \prod_{i=1}^{n} \| f_i \|_{p_i} \left( 1 - \sum_{i=1}^{n} \frac{1}{p_i} \| f_i \|_{p_i}^{p_i/2} - \sum_{k=1}^{n} \frac{1}{p_k} \| f_k \|_{p_k}^{p_k/2} \right)^{1/2}.
\]

Proof. Let \( \alpha_i = p_i^{-1} \) and \( x_i = f_i^{p_i}(u)/\| f_i \|_{p_i}^{p_i} \) in (4). To obtain (5), integrate and multiply both sides by \( \prod_{i=1}^{n} \| f_i \|_{p_i} \).

Remark 0.3. Inequality (4) was suggested by the following result of D. I. Cartwright and M. J. Field (cf. [CaFi]; cf. also [Alz] and [Me] for additional refinements along these lines). Let \( 0 < m = \min\{x_1, \ldots, x_n\} \) and let \( M = \max\{x_1, \ldots, x_n\} \). Then

\[
\frac{1}{2M} \sum_{i=1}^{n} \alpha_i \left( x_i - \sum_{k=1}^{n} \alpha_k x_k \right)^2 \leq \sum_{i=1}^{n} \alpha_i x_i - \prod_{i=1}^{n} x_i^{\alpha_i} \leq \frac{1}{2m} \sum_{i=1}^{n} \alpha_i \left( x_i - \sum_{k=1}^{n} \alpha_k x_k \right)^2.
\]

The motivation to search for variants of (6) comes the fact that it is not well suited to the particular application considered here (refining Hölder’s inequality). One would need to assume that \( |f_i| \leq M \) almost everywhere. We give bounds using the variance of \( x^{1/2} \) instead of the variance of \( x \) in order to ensure the integrability of the functions involved, and also to obtain the same homogeneity on both sides of (4).

Remark 0.4. The difference between the arithmetic and geometric means is in general not comparable to \( \text{Var}(x^{1/2}) \). To see this, it is enough to consider the equal weights case, with \( n \gg 1 \), \( x_1 = 0 \), and \( x_2 = \cdots = x_n = 1 \). Or the case where \( n = 2 \), and one of the weights is much larger than the other. But perhaps it is possible to give an upper bound for \( \sum_{i=1}^{n} \alpha_i x_i - \prod_{i=1}^{n} x_i^{\alpha_i} \) using \( \text{Var}(x^{1/2}) \) times some polynomial function of 1 over the smallest weight. This would lead to the same type of application as above. In fact, for the special case \( n = 2 \) a two sided, sharper version of (4) appears in Lemma 2.1 of [Al]. It is not clear to me how to extend this sharper version to \( n > 2 \).

Remark 0.5. When \( n = 2 \), inequality (5) reduces to

\[
\|fg\| \leq \| f \|_p \| g \|_q \left( 1 - \frac{1}{pq} \| f \|_p^{p/2} - \frac{g^{q/2}}{\| g \|_q^{q/2}} \right)^2,
\]
where $p$ and $q$ are conjugate exponents, $0 \leq f \in L^p$, $0 \leq g \in L^q$, $\|f\|_p > 0$, and $\|g\|_q > 0$. In addition to providing a lower bound, with $1/\min\{p, q\}$ instead of $1/(pq)$, Lemma 2.1 of [Al] yields a slightly better upper bound: $1/(pq) = 1/(p + q)$ can be replaced by $1/\max\{p, q\}$. But we note that (7) suffices, via the standard argument, to give a refinement of the triangle inequality for $L^p$ spaces, $1 < p < \infty$, which in turn leads to a fairly straightforward proof of uniform convexity in the real valued case (arguing as in [Al]). So the selfimproving properties of the AM-GM inequality have repercussions beyond what one might expect.

Remark 0.6. Note that $f_i^{p_i/2}/\|f_i\|_{p_i}^{p_i/2}$ is just a unit vector in $L^2$. The strategy underlying inequality (5) is to normalize all functions and map them into $L^2$, which becomes the common measuring ground where dispersion around the mean is determined. When $n = 2$, the correction term reduces to a function of the angular distance between $f_i^{p_i/2}$ and $g_i^{q_i/2}$.

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