PRUNING LÉVY TREES VIA AN ADMISSIBLE FAMILY OF BRANCHING MECHANISMS

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Abstract. By studying an admissible family of branching mechanisms introduced in Li (2014), we obtain a pruning procedure on Lévy trees. Then we could construct a decreasing Lévy-CRT-valued process \( \{T_t\} \) by pruning Lévy trees and an analogous process \( \{T^*_t\} \) by pruning a critical Lévy tree conditioned to be infinite. Under a regular condition on the admissible family of branching mechanisms, we show that the law of \( \{T_t\} \) at the ascension time can be represented by \( \{T^*_t\} \). The results generalize those studied in Abraham and Delmas (2012).

1. Introduction

By pruning Lévy trees, Abraham and Delmas in [1] constructed and studied decreasing continuum-tree-valued Markov processes which correspond to a family of branching mechanisms obtained by shifting a given branching mechanism. More precisely, let \( \psi \) be a branching mechanism defined by

\[
\psi(\lambda) = b\lambda + c\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda z} - 1 + \lambda z\right) m(dz), \quad \lambda \geq 0,
\]

where \( b \in \mathbb{R}, c \geq 0 \) and \( m \) is a \( \sigma \)-finite measure on \((0, +\infty)\) such that \( \int_0^\infty (z \wedge z^2) m(dz) < +\infty \).

Define \( \psi^\theta(\lambda) = \psi(\theta + \lambda) - \psi(\theta) \). Denote by \( \Theta^\psi \) the set of \( \theta \) such that \( \int_{(1,\infty)} e^{-\theta z} m(dz) < \infty \). The family of branching mechanisms \( \{\psi^\theta : \theta \in \Theta^\psi\} \) was considered in [1].

Li in [20] introduced the admissible family of branching mechanisms. Roughly, the model is described as follows: Given a time interval \( \mathcal{I} \subset \mathbb{R} \), let \( (\theta, \lambda) \mapsto \zeta_\theta(\lambda) \) be a continuous function on \( \mathcal{I} \times [0, \infty) \) with the representation

\[
\zeta_\theta(\lambda) = \beta_\theta(\lambda) + \int_{(0,\infty)} (1 - e^{-z\lambda}) n_\theta(dz), \quad \theta \in \mathcal{I}, \lambda \geq 0,
\]

where \( \beta_\theta \geq 0 \) and \( (1 \wedge z)n_\theta(dz) \) is a finite kernel from \( \mathcal{I} \) to \((0, \infty)\). Then \( \{\psi_\theta : \theta \in \mathcal{I}\} \) is called an admissible family if

\[
\psi_q(\lambda) = \psi_t(\lambda) + \int_t^q \zeta_\theta(\lambda) d\theta, \quad q \geq t \in \mathcal{I}, \lambda \geq 0.
\]

Then it is easy to see that \( \{\psi^\theta : \theta \in \Theta^\psi\} \) considered in [1] is an admissible family with

\[
\zeta_\theta(\lambda) = 2c\lambda + \int_{(0,\infty)} (1 - e^{-z\lambda}) e^{-z\theta} z m(dz).
\]

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By using the techniques of stochastic equations and measure-valued processes, Li [20] studied a class of increasing path-valued Markov processes which correspond to the admissible family. Those path-valued processes could be regarded as counterparts of the tree-valued processes constructed in [1] (However, to the best of our knowledge, no link is actually pointed out between tree-valued processes and path-valued branching processes). It is natural to ask whether there exists a continuum-tree-valued process corresponding to a given admissible family by pruning Lévy trees.

The second motivation of our work is the study of so-called ascension process and its representation. The pioneer works were given by Aldous and Pitman in [9] where, according to the marks on edges, they constructed a tree-valued Markov process \( \{G(u)\} \) by pruning Galton-Watson trees and an analogous process \( \{G^*(u)\} \) by pruning a critical or subcritical Galton-Watson tree conditioned to be infinite. It was shown in [9] that the process \( \{G(u)\} \) run until its ascension time has a representation in terms of \( \{G^*(u)\} \) in the special case of Poisson offspring distributions. By using the pruning procedure defined in [6] and exploration processes introduced in [17], Abraham and Delmas in [1] extended the above results to Lévy trees, where a decreasing Lévy-tree-valued process \( \{T_\theta : \theta \in \Theta^\psi\} \) was constructed such that \( T_\theta \) is a \( \psi^\theta \)-Lévy tree. They also showed that \( \{T_\theta\} \) run until its ascension time can be represented in terms of another tree-valued process obtained by applying the same pruning procedure to a Lévy tree conditioned on non-extinction. Similar results can be found in [2] for Galton-Watson trees where the trees are pruned according to the marks on nodes. Similar cases for sub-trees of Lévy trees were also studied in [3].

Motivated by the above mentioned works, in this paper, we shall show that given an admissible family of branching mechanisms \( \{\psi_t : t \in \mathcal{T}\} \), one can define a pruning procedure on Lévy trees which will lead to a deceasing tree-valued Markov process \( \{T_t : t \in \mathcal{T}\} \) such that \( T_t \) is a \( \psi_t \)-Lévy tree; see Theorem 4.2 in Section 4. Furthermore, under a regular condition on the admissible family, we also prove that the law of the tree-valued process at its ascension time can be represented in terms of another tree-valued process obtained by pruning a critical Lévy tree conditioned to be infinite; see Theorem 6.1 and Corollary 6.4 in this paper. Since \( \{\psi^\theta : \theta \in \Theta^\psi\} \) considered in [1] is a special case of admissible family, our results generalize those in [1]. However, we use the framework of real trees but not exploration processes.

Let us mention that the study of theory of continuum random trees was initiated by Aldous in [7] and [8]. Lévy trees, also known as Lévy continuum random trees or Lévy CRTs, were first studied by Le Gall and Le Jan in [17] and [18], where it was shown that Lévy trees code the genealogy of continuous state branching processes (CSBPs). Later, in [10], it was shown that Galton-Watson trees which code the genealogy of Galton-Watson processes, suitably rescaled, converge to Lévy trees, as rescaled Galton-Watson processes converge to CSBPs. Then based on [20] and the present work, one may expect to introduce the notation “admissible family” to study the Galton-Watson processes and Galton-Watson trees. And a general pruning procedure on Galton-Watson trees may be developed. Possibly such a pruning procedure is a combination of Aldous and Pitman’s pruning procedure in [9] and Abraham et.al.’s pruning procedure in [2]. This gives the third motivation of the present work. We will explore those questions in future.

The remaining of this paper is organized as follows. In Section 2, we introduce and study the admissible family of branching mechanisms. In Section 3, we recall some notation and collect some known results on real trees and Lévy trees. In Section 4 based on the study of admissible family, the pruning procedure will be given and the marginal distributions of the pruning process are studied. The evolution of the tree-valued process will be explored in Section 5. In the last section, Section 6, we construct a tree-valued process by pruning a critical Lévy tree conditioned to be infinite and get the representation of the tree at the ascension time.
PRUNING AND ADMISSIBLE FAMILY

2. Admissible family of branching mechanisms

Throughout the paper, for $-\infty \leq a \leq b \leq +\infty$, we make the convention

$$\int_a^b = \int_{(a,b)}.$$

The admissible family of branching mechanisms was first introduced by Li in [20]. Suppose that $\mathcal{T} \subset \mathbb{R}$ is an interval and $\Psi = \{\psi_q : q \in \mathcal{T}\}$ is a family of branching mechanisms, where $\psi_q$ is given by

$$\psi_q(\lambda) = b_q \lambda + c \lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z) m_q(dz), \quad \lambda \geq 0,$$

with the parameters $(b, m) = (b_q, m_q)$ depending on $q \in \mathcal{T}$ such that $b_q \in \mathbb{R}$ and $\int z \wedge z^2 m_q(dz) < \infty$.

**Definition 2.1.** [Li(2014)] We call $\{\psi_q : q \in \mathcal{T}\}$ an admissible family if for each $\lambda > 0$ the function $q \mapsto \psi_q(\lambda)$ is increasing and continuously differentiable with

$$\zeta_q(\lambda) := \frac{\partial}{\partial q} \psi_q(\lambda) = \beta_q \lambda + \int_0^\infty (1 - e^{-\lambda z}) n_q(dz), \quad q \in \mathcal{T}, \quad \lambda > 0,$$

where $\beta_q \geq 0$ and $(1 \wedge z) n_q(dz)$ is a finite kernel from $\mathcal{T}$ to $(0, \infty)$ satisfying

$$\int_t^q \beta_q d\theta + \int_t^q d\theta \int_0^\infty z n_q(dz) < \infty, \quad q \geq t \in \mathcal{T}.$$

**Remark 2.2.** In fact, in Li [20], it is assumed that $q \mapsto \psi_q(\lambda)$ is decreasing and $\zeta_q(\lambda) = -\frac{\partial}{\partial q} \psi_q(\lambda)$. In that case, we will get an increasing tree-valued process.

**Remark 2.3.** For the purpose in this work, we also weaken the assumptions on $\beta_q$ and $n_q(dz)$. In [20], it is assumed that

$$\sup_{t \leq \theta \leq q} \left( \beta_q + \int_0^\infty z n_q(dz) \right) < \infty, \quad q \geq t \in \mathcal{T},$$

which is essential there. If we assume (4), some interesting cases of pruning Lévy trees may be excluded. See Example 2.5 below for some cases.

**Remark 2.4.** It is also possible to assume that

$$\psi_q(\lambda) = b_q \lambda + c \lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z \mathbf{1}_{z \leq 1}) m_q(dz),$$

with the parameters $(b, m) = (b_q, m_q)$ depending on $q \in \mathcal{T}$ such that $b_q \in \mathbb{R}$ and $\int 1 \wedge z^2 m_q(dz) < \infty$. Then (3) would be replaced by

$$\int_t^q \beta_q d\theta + \int_t^q d\theta \int_{(0,1]} z n_q(dz) < \infty, \quad q \geq t \in \mathcal{T}.$$

We assume further that $\psi_q$ is conservative; i.e.; $\int_{(0,\epsilon)} \frac{d\lambda}{|\psi_q(\lambda)|} = +\infty$ for all $\epsilon > 0$. We conjecture that all results in this work could be deduced in this framework.

In the following we shall give some examples of admissible family of branching mechanisms.
Example 2.5. Let $\psi$ be defined in (II). Abraham and Delmas in [II] considered the case of $\psi_q(\lambda) = \psi(q + \lambda) - \psi(q)$, $q \in \Theta^\psi$, where $\Theta^\psi$ is set of $\theta \in \mathbb{R}$ such that $\int_1^\infty ze^{-\theta z}m(dz) < \infty$. Then \{\psi_q : q \in \Theta^\psi\} is an admissible family with

$$b_q = b + 2cq + \int_0^\infty (1 - e^{-zq})zm(dz), \quad m_q(dz) = e^{-zq}m(dz).$$

and

$$\beta_q = 2c, \quad n_q(dz) = e^{-zq}zm(dz), \quad m_z(t, dq) = z1_{\{q \geq t\}}dq.$$  

Note that $\Theta^\psi = [\theta_\infty, +\infty)$ or $(\theta_\infty, +\infty)$ for some $\theta_\infty \in [\infty, 0]$. However, in the case of $\Theta^\psi = [\theta_\infty, +\infty)$, $n_{\theta_\infty}(dz)$ may fail to satisfy (II). A sufficient condition for that (II) holds is $\int_1^\infty z^2e^{-\theta_\infty z}m(dz) < \infty$. We also remark here that for the study of the ascension process, we always exclude the case of $\Theta^\psi = [\theta_\infty, +\infty)$; see Remark [5.4] in Section [5] below.

Example 2.6. Let $\psi$ be defined in (II). Let $f \geq 0$ be a bounded decreasing function on $\mathbb{R}$ with bounded derivative and $\sup_{x \geq 0} |xf'(x)| < +\infty$. Let $g$ be a differentiable increasing function on $\mathbb{R}$. For $q \in \mathbb{R}$, let $\psi_q$ be a branching mechanism with parameters ($b_q, m_q$) defined by

$$b_q = b + g(q) + \int_0^\infty (f(0) - f(zq))zm(dz), \quad m_q(dz) = f(qz)m(dz).$$

Then one can check that \{\psi_q : q \in \mathbb{R}\} is an admissible family of branching mechanisms with

$$\frac{\partial}{\partial q}\psi_q(\lambda) = g'(q)\lambda - \int_0^\infty (1 - e^{-z\lambda})zf'(qz)m(dz), \quad q \in \mathbb{R}, \quad \lambda \leq 0,$$

and

$$\beta_q = g'(q), \quad n_q(dz) = -zf'(qz)m(dz).$$

Typically, if $m = 0$, then $\psi_q(\lambda) = (b + g(q))\lambda + c\lambda^2$. If $f \equiv 1$, then $\psi_q(\lambda) = \psi(\lambda) + g(\lambda)$.

Example 2.7. Let $\psi$ be defined in (II) with $m(dz) = f(z)dz$ for some nonnegative measurable function $f$ on $\mathbb{R}$, where $dz$ is the Lebesgue measure. Let $g$ be a differentiable increasing function on $\mathbb{R}$. Let $h$ be a differentiable decreasing function on $\mathbb{R}$ such that $h(z) > 0$ for all $z \in \mathbb{R}$. For $q \in \mathbb{R}$, define

$$b_q = g(q) + \int_{h(q)}^\infty zm(dz), \quad m_q = 1_{\{z \leq h(q)\}}m(dz).$$

Then \{\psi_q : q \in \mathbb{R}\} is an admissible family of branching mechanisms with parameters ($b_q, m_q$) such that

$$\frac{\partial}{\partial q}\psi_q(\lambda) = g'(q)\lambda + (e^{-\lambda h(q)} - 1)h'(q)f(h(q)), \quad q \in \mathbb{R}, \quad \lambda \geq 0,$$

and

$$\beta_q = g'(q), \quad n_q(dz) = -h'(q)f(h(q))\delta_h(q)(dz).$$

Example 2.8. In the above example, if $h(\cdot) = -(\cdot \wedge a)$ for some constant $a \leq 0$, then \{\psi_q : q \in (-\infty, a)\} is an admissible family of branching mechanisms.

Example 2.9. Let $\mathcal{S}_- \subset (-\infty, 0]$ be an interval and let \{\psi_q : q \in \mathcal{S}_-\} be an admissible family of branching mechanisms with the parameters ($b_q, m_q$). Assume that $0 \in \mathcal{S}_-$ and $\psi_0$ is critical. Let $\eta_q$ denote the largest root of $\psi_q(s) = 0$. For $q \in -\mathcal{S}_- := \{-t : t \in \mathcal{S}_-\}$, define $\psi_q(\cdot) = \psi_{-q}(\eta_{-q} + \cdot)$. Then we have \{\psi_q : q \in \mathcal{S}_- \cup (-\mathcal{S}_-)\} is an admissible family of branching mechanisms such that for $q \in -\mathcal{S}_-$

$$b_q = b_{-q} + 2c\eta_{-q} + \int_0^\infty (1 - e^{-z\eta_{-q}})zm_{-q}(dz), \quad m_q = e^{-z\eta_{-q}}m_{-q}(dz).$$
Next, we shall show how to get pruning parameters from a given admissible family of branching mechanisms. Without loss of generality, we always assume that $\psi_t \neq \psi_q$ for $t \neq q \in \mathcal{T}$. It follows from the Definition 2.1 that for $q \geq t \in \mathcal{T}$,

$$b_q = b_t + \int_t^q \beta_\theta d\theta + \int_t^q d\theta \int_0^\infty z n_\theta(dz)$$

and

$$m_t(dz) = m_q(dz) + \int_{t \leq \theta < q} n_\theta(dz)d\theta.$$  

(5)

and

(6)

Remark 2.10. By (5) one can see $q \mapsto b_q$ is a continuous decreasing function on $\mathcal{T}$. Typically, $b_t = b_q$ implies $\psi_t = \psi_q$ and vice versa.

For $t \in \mathcal{T}$, define $\mathcal{T}_t = \mathcal{T} \cap [t, +\infty)$. Then we shall see from (6) that for any $q \in \mathcal{T}_t$,

$$m_q(dz) \ll m_t(dz), \quad \text{on } (0, \infty).$$

Denote by $m_z(t, q)$ the corresponding Radon-Nikodym derivative; i.e.,

$$m_q(dz) = m_z(t, q)m_t(dz).$$

Then we see

$$\int_{t \leq \theta < q} n_\theta(dz)d\theta = (1 - m_z(t, q)) m_t(dz), \quad q \in \mathcal{T}_t,$$

which implies $m_t(dz)$-a.e.

(7)

Furthermore, we have for $t \leq \theta \leq q$, $m_t(dz)$-a.e.

(8)

$$m_z(t, q) \leq 1 \quad \text{and} \quad q \mapsto m_z(t, q)(z) \text{ is decreasing}.$$

Then we make the following assumptions:

(H1) For every $z \in (0, \infty)$ and $t \in \mathcal{T}$,

$$m_z(t, q) \leq 1 \quad \text{and} \quad q \mapsto m_z(t, q)(z) \text{ is decreasing}.$$

(H2) For every $z \in (0, \infty)$ and $t \leq \theta \leq q \in \mathcal{T}$,

$$m_z(t, q) = m_z(t, \theta)m_z(\theta, q).$$

(H3) For every $q \in \mathcal{T}$,

$$\int_{\psi_q(\lambda)}^\infty d\lambda < +\infty.$$  

By (H1), we could define a measure $m_z(t, dq)$ on $\mathcal{T}_t$ by

$$m_z(t, [t, q]) = -\ln(m_z(t, q)),$$

where induces the pruning measure on branching nodes of infinite degree of a $\psi_t$-Lévy tree. By (H2) we could have a tree-valued Markov processes on $\mathcal{T}$. (H3) is used to ensure that all trees are locally compact.

From now on, we assume that (H1-3) are in force.
3. Real trees and Lévy trees

3.1. Notations. Let \((E,d)\) be a metric Polish space. We denote by \(M_f(E)\) (resp. \(M^\text{loc}_f(E)\)) the space of all finite (resp. locally finite) Borel measures on \(E\). For \(x \in E\), let \(\delta_x\) denote the Dirac measure at point \(x\). For \(\mu \in M^\text{loc}_f(E)\) and \(f\) a non-negative measurable function, we set \(\langle \mu, f \rangle = \int f(x) \mu(dx) = \mu(f)\).

3.2. Real trees. We refer to [13] or [16] for a general presentation of random real trees. A metric space \((T,d)\) is a real tree if the following properties are satisfied:

1. For every \(s, t \in T\), there is a unique isometric map \(f_{s,t}\) from \([0, d(s,t)]\) to \(T\) such that \(f_{s,t}(0) = s\) and \(f_{s,t}(d(s,t)) = t\).
2. For every \(s, t \in T\), if \(q\) is a continuous injective map from \([0,1]\) to \(T\) such that \(q(0) = s\) and \(q(1) = t\), then \(q([0,1]) = f_{s,t}([0, d(s,t)])\).

If \(s, t \in T\), we will note \([s, t]\) the range of the isometric map \(f_{s,t}\) described above and \([s, t]\) for \([s, t]\). We say that \((T, d, \emptyset)\) is a rooted real tree with root \(\emptyset\) if \((T, d)\) is a real tree and \(\emptyset \in T\) is a distinguished vertex. For every \(x \in T\), \([\emptyset, x]\) is interpreted as the ancestral line of vertex \(x\) in the tree.

Let \((T, d, \emptyset)\) be a rooted real tree. The degree \(n(x)\) of \(x \in T\) is the number of connected components of \(T \setminus \{x\}\) and the number of children of \(x \neq \emptyset\) is \(\kappa_x = n(x) - 1\) and of the root is \(\kappa_\emptyset = n(\emptyset)\). We shall consider the set of leaves \(L\text{f}(T) = \{x \in T, \kappa_x = 0\}\), the set of branching points \(\text{Br}(T) = \{x \in T, \kappa_x \geq 2\}\) and the set of infinite branching points \(\text{Br}_\infty(T) = \{x \in T, \kappa_x = \infty\}\). The skeleton of \(T\) is the set of points in the tree that aren’t leaves: \(\text{Sk}(T) = T \setminus \text{Lf}(T)\). The trace of the Borel \(\sigma\)-field of \(T\) restricted to \(\text{Sk}(T)\) is generated by the sets \([s, s']\); \(s, s' \in \text{Sk}(T)\).

One defines uniquely a \(\sigma\)-finite Borel measure \(\ell^T\) on \(T\), called the length measure of \(T\), such that

\[
\ell^T(\text{Lf}(T)) = 0 \quad \text{and} \quad \ell^T([s, s']) = d(s, s').
\]

3.3. Measured rooted real trees. According to [5], one can define a Gromov-Hausdorff-Prohorov metric on the space of rooted measured metric space as follows; see also [12] and [14] for some related works.

Let \((X, d)\) be a Polish metric space. For \(A, B \in \mathcal{B}(X)\), set

\[
d_H(A, B) = \inf\{\varepsilon > 0, A \subset B^\varepsilon \text{ and } B \subset A^\varepsilon\},
\]

the Hausdorff distance between \(A\) and \(B\), where \(A^\varepsilon = \{x \in X, \inf_{y \in A} d(x, y) < \varepsilon\}\). If \(\mu, \nu \in M_f(X)\), we define

\[
d_P(\mu, \nu) = \inf\{\varepsilon > 0, \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \text{ for all closed set } A\},
\]

the Prohorov distance between \(\mu\) and \(\nu\).

A rooted measured metric space \(X = (X, d, \emptyset, \mu)\) is a metric space \((X, d)\) with a distinguished element \(\emptyset \in X\) and a locally finite Borel measure \(\mu \in M^\text{loc}_f(X)\). Let \(X = (X, d, \emptyset, \mu)\) and \(X' = (X', d', \emptyset', \mu')\) be two compact rooted measured metric spaces, and define:

\[
d_{\text{GHP}}(X, X') = \inf_{\Phi, \Phi', Z} \left( d^2_H(\Phi(X), \Phi'(X')) + d^2(\Phi(\emptyset), \Phi'(\emptyset')) + d^2_P(\Phi_* \mu, \Phi'_* \mu') \right),
\]

where the infimum is taken over all isometric embeddings \(\Phi : X \hookrightarrow Z\) and \(\Phi' : X' \hookrightarrow Z\) into some common Polish metric space \((Z, d^2)\) and \(\Phi_* \mu\) is the measure \(\mu\) transported by \(\Phi\).
If $\mathcal{X} = (X, d, \emptyset, \mu)$ is a rooted measured metric space, then for $r \geq 0$ we will consider its restriction to the ball of radius $r$ centered at $\emptyset$, $\mathcal{X}^{(r)} = (X^{(r)}, d^{(r)}, \emptyset, \mu^{(r)})$, where

$$X^{(r)} = \{x \in X; d(\emptyset, x) \leq r\},$$

with $d^{(r)}$ and $\mu^{(r)}$ defined in an obvious way.

By a measured rooted real tree $(\mathcal{T}, d, \emptyset, \mathbf{m})$, we mean $(\mathcal{T}, d, \emptyset)$ is a locally compact rooted real tree and $\mathbf{m} \in \mathcal{M}_f^{loc}(\mathcal{T})$ is a locally finite measure on $\mathcal{T}$. When there is no confusion, we will simply write $\mathcal{T}$ for $(\mathcal{T}, d, \emptyset, \mathbf{m})$. We define for two measured rooted real trees $\mathcal{T}_1, \mathcal{T}_2$:

$$d_{\text{GHP}}(\mathcal{T}_1, \mathcal{T}_2) = \int_0^\infty e^{-r} \left(1 \wedge d_{\text{GHP}}(\mathcal{T}_1^{(r)}, \mathcal{T}_2^{(r)})\right) \, dr.$$  

$\mathcal{T}_1$ and $\mathcal{T}_2$ are said GHP-isometric if $d_{\text{GHP}}(\mathcal{T}_1, \mathcal{T}_2) = 0$. Denote by $\mathcal{T}$ the set of (GHP-isometry classes of) measured rooted real trees $(\mathcal{T}, d, \emptyset, \mathbf{m})$. According to Corollary 2.8 in [5], $(\mathcal{T}, d_{\text{GHP}})$ is a Polish metric space.

3.4. Grafting procedure. Let $\mathcal{T}$ be a measured rooted real tree and let $((\mathcal{T}_i, x_i), i \in I)$ be a finite or countable family of elements of $\mathcal{T} \times \mathcal{T}$. We define the real tree obtained by grafting the trees $\mathcal{T}_i$ on $\mathcal{T}$ at point $x_i$. We set $\mathcal{T} = \mathcal{T} \cup (\bigsqcup_{i \in I} \mathcal{T}_i \setminus \{\emptyset \mathcal{T}_i\})$ where the symbol $\cup$ means that we choose for the sets $(\mathcal{T}_i)_{i \in I}$ representatives of GHP-isometry classes in $\mathcal{T}$ which are disjoint subsets of some common set and that we perform the disjoint union of all these sets. We set $\emptyset \mathcal{T} = \emptyset \mathcal{T}$. The set $\hat{\mathcal{T}}$ is endowed with the following metric $d^{\hat{\mathcal{T}}}$: if $s, t \in \hat{\mathcal{T}}$,

$$d^{\hat{\mathcal{T}}}(s, t) = \begin{cases} d^\mathcal{T}(s, t) & \text{if } s, t \in \mathcal{T}, \\ d^\mathcal{T}(s, x_i) + d^\mathcal{T}(\emptyset \mathcal{T}_i, t) & \text{if } s \in \mathcal{T}, t \in \mathcal{T}_i \setminus \{\emptyset \mathcal{T}_i\}, \\ d^\mathcal{T}(s, t) & \text{if } s, t \in \mathcal{T}_i \setminus \{\emptyset \mathcal{T}_i\}, \\ d^\mathcal{T}(x_i, x_j) + d^\mathcal{T}(\emptyset \mathcal{T}_i, s) + d^\mathcal{T}(\emptyset \mathcal{T}_j, t) & \text{if } i \neq j \text{ and } s \in \mathcal{T}_j \setminus \{\emptyset \mathcal{T}_j\}, \ t \in \mathcal{T}_i \setminus \{\emptyset \mathcal{T}_i\}. \end{cases}$$

We define the mass measure on $\hat{\mathcal{T}}$ by:

$$\mathbf{m}^{\hat{\mathcal{T}}} = \mathbf{m}^\mathcal{T} + \sum_{i \in I} \left(\mathbf{1}_{\mathcal{T}_i \setminus \{\emptyset \mathcal{T}_i\}} \mathbf{m}^{\mathcal{T}_i} + \mathbf{m}^{\mathcal{T}_i}(\{\emptyset \mathcal{T}_i\}) \delta_{x_i}\right).$$

Then $(\hat{\mathcal{T}}, d^{\hat{\mathcal{T}}}, \emptyset \mathcal{T})$ is still a rooted complete real tree. (Notice that it is not always true that $\hat{\mathcal{T}}$ remains locally compact or that $\mathbf{m}^{\hat{\mathcal{T}}}$ is a locally finite measure on $\hat{\mathcal{T}}$). We will use the following notation for the grafted tree:

$$\mathcal{T} \otimes_{i \in I} (\mathcal{T}_i, x_i) = (\hat{\mathcal{T}}, d^{\hat{\mathcal{T}}}, \emptyset \mathcal{T}, \mathbf{m}^{\hat{\mathcal{T}}}),$$

with convention that $\mathcal{T} \otimes_{i \in I} (\mathcal{T}_i, x_i) = \mathcal{T}$ for $I = \emptyset$. If $\varphi$ is an isometry from $\mathcal{T}$ onto $\mathcal{T}'$, then $\mathcal{T} \otimes_{i \in I} (\mathcal{T}_i, x_i)$ and $\mathcal{T}' \otimes_{i \in I} (\mathcal{T}_i, \varphi(x_i))$ are also isometric. Therefore, the grafting procedure is well defined on $\mathcal{T}$.

3.5. Sub-trees above a given level. For $\mathcal{T} \in \mathcal{T}$, define $H_{\text{max}}(\mathcal{T}) = \sup_{x \in \mathcal{T}} d^\mathcal{T}(\emptyset \mathcal{T}, x)$ the height of $\mathcal{T}$ and for $a \geq 0$:

$$\mathcal{T}^{(a)} = \{x \in \mathcal{T}; d(\emptyset, x) \leq a\} \quad \text{and} \quad \mathcal{T}(a) = \{x \in \mathcal{T}; d(\emptyset, x) = a\}$$

the restriction of the tree $\mathcal{T}$ under level $a$ and the set of vertices of $\mathcal{T}$ at level $a$ respectively. We denote by $(\mathcal{T}^{i, \varnothing}, i \in I)$ the connected components of $\mathcal{T} \setminus \mathcal{T}^{(a)}$. Let $\emptyset_i$ be the most recent common ancestor of all the vertices of $\mathcal{T}^{i, \varnothing}$. We consider the real tree $\mathcal{T}^i = \mathcal{T}^{i, \varnothing} \cup \{\emptyset_i\}$ rooted at point $\emptyset_i$. 

with mass measure $m^T$ defined as the restriction of $m^T$ to $\mathcal{T}^i,\infty$ and $m^T(\emptyset_i) = 0$. Notice that $\mathcal{T} = (\mathcal{T}^i)_{i \in I}$ $(\mathcal{T}_i, \emptyset_i)$. We will consider the point measure on $\mathcal{T} \times \mathbb{T}$:

$$N^T_a = \sum_{i \in I} \delta_{(\emptyset_i, \mathcal{T}_i)}.$$  

3.6. **Excursion measure of a Lévy tree.** Recall (1). We say $\psi$ is subcritical, critical or super-critical if $b > 0$, $b = 0$ or $b < 0$, respectively. Typically, we say $\psi$ is (sub)critical, if $b \geq 0$. We assume the Grey condition holds:

$$\int^{+\infty} \frac{d\lambda}{\psi^{\infty}((\lambda))} < +\infty.$$  

**Remark 3.1.** The Grey condition is used to ensure that the corresponding Lévy tree is locally compact and also implies $c > 0$ or $\int_{(0,1)} \ell_m (d\ell) = +\infty$ which is equivalent to the fact that the Lévy process with index $\psi$ is of infinite variation.

Let $v^\psi$ be the unique non-negative solution of the equation:

$$\int v^\psi(a) \frac{d\lambda}{\psi^{\infty}(\lambda)} = a.$$  

Results from (11) in the (sub)critical cases, where height functions are introduced to code the compact real trees, can be extended to the super-critical cases; see [4]. We state those results in the following form. There exists a $\sigma$-finite measure $N^\psi[d\mathcal{T}]$ on $\mathbb{T}$, or excursion measure of a Lévy tree. A $\psi$-Lévy tree is a “random” tree with law $N^\psi$ and the following properties.

(i) **Height.** For all $a > 0$, $N^\psi[H_{\max}(\mathcal{T}) > a] = v^\psi(a)$.

(ii) **Mass measure.** The mass measure $m^T$ is supported by $\operatorname{Lf}(\mathcal{T})$, $N^\psi[d\mathcal{T}]$-a.e.

(iii) **Local time.** There exists a $\mathcal{T}$-measure-valued process $(\ell^a, a \geq 0)$ which is càdlàg for the weak topology on the set of finite measures on $\mathcal{T}$ such that $N^\psi[d\mathcal{T}]$-a.e.:

$$\ell^0 = 0, \inf \{a > 0; \ell^a = 0\} = \sup \{a \geq 0; \ell^a \neq 0\} = H_{\max}(\mathcal{T}) \text{ and for every fixed } a \geq 0, N^\psi[d\mathcal{T}] \text{-a.e.:}$$  

- The measure $\ell^a$ is supported on $\mathcal{T}(a)$.
- We have for every bounded continuous function $\phi$ on $\mathcal{T}$:

$$\langle \ell^a, \phi \rangle = \lim_{\varepsilon \downarrow 0} \frac{1}{v^\psi(\varepsilon)} \int \phi(x) 1_{\{H_{\max}(\mathcal{T}) \geq \varepsilon\}} N^T_a(dx, d\mathcal{T}').$$  

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{v^\psi(\varepsilon)} \int \phi(x) 1_{\{H_{\max}(\mathcal{T}) \geq \varepsilon\}} N^T_{a-\varepsilon}(dx, d\mathcal{T}'), \text{ if } a > 0.$$  

Under $N^\psi$, the real valued process $(\langle \ell^a, 1 \rangle, a \geq 0)$ is distributed as a CSBP with branching mechanism $\psi$ under its canonical measure.

(iv) **Branching property.** For every $a > 0$, the conditional distribution of the point measure $N^T_a(dx, d\mathcal{T}')$ under $N^\psi[d\mathcal{T}]|_{H_{\max}(\mathcal{T}) > a}$, given $\mathcal{T}(a)$, is that of a Poisson point measure on $\mathcal{T}(a) \times \mathbb{T}$ with intensity $\ell^a(dx)N^\psi[d\mathcal{T}]$.

(v) **Branching points.**

- $N^\psi[d\mathcal{T}]$-a.e., the branching points of $\mathcal{T}$ have 2 children or an infinity number of children.
- The set of binary branching points (i.e. with 2 children) is empty $N^\psi$-a.e if $c = 0$ and is a countable dense subset of $\mathcal{T}$ if $c > 0$.  

• The set $\text{Br}_\infty(T)$ of infinite branching points is nonempty with $\mathbb{N}^{\psi}$-positive measure if and only if $m \neq 0$. If $\langle m, 1 \rangle = +\infty$, the set $\text{Br}_\infty(T)$ is $\mathbb{N}^{\psi}$-a.e. a countable dense subset of $T$.

(vi) **Mass of the nodes.** The set $\{d(\emptyset, x), x \in \text{Br}_\infty(T)\}$ coincides $\mathbb{N}^{\psi}$-a.e. with the set of discontinuity times of the mapping $a \mapsto \ell^a$. Moreover, $\mathbb{N}^{\psi}$-a.e., for every such discontinuity time $b$, there is a unique $x_b \in \text{Br}_\infty(T)$ with $d(\emptyset, x_b) = b$ and $\Delta_b > 0$, such that:

$$\ell^b = \ell^b + \Delta_b \delta_{x_b},$$

where $\Delta_b$ is called the mass of the node $x_b$. Furthermore $\Delta_b$ can be obtained by the approximation:

$$\Delta_b = \lim_{\varepsilon \to 0} \frac{1}{v^\psi(\varepsilon)} n(x_b, \varepsilon),$$

where $n(x_b, \varepsilon) = \int 1_{\{x_b\}}(x) 1_{\{\max(T') > \varepsilon\}} N^\psi_b(dx, dT')$ is the number of sub-trees with MRCA $x_b$ and height larger than $\varepsilon$.

In order to stress the dependence in $T$, we may write $\ell^a,T$ for $\ell^a$. We set $\sigma_T$ or simply $\sigma$ when there is no confusion, for the total mass of the mass measure on $T$:

$$\sigma = m^T(T).$$

Notice that (15) readily implies that $m^T(\{x\}) = 0$ for all $x \in T$.

### 3.7. Related measures on Lévy trees

We define a probability measure on $T$ as follow. Let $r > 0$ and $\sum_{k \in K} \delta_{T^k}$ be a Poisson random measure on $T$ with intensity $r N^{\psi}$. Consider $\emptyset$ as the trivial measured rooted real tree reduced to the root with null mass measure. Define $T = \emptyset \otimes_{k \in K} (T^k, \emptyset)$. Using Property (i) as well as (20) below, one easily get that $T$ is a measured locally compact rooted real tree, and thus belongs to $T$. We denote by $\mathbb{P}_r^\psi$ its distribution. Its corresponding local time and mass measure are respectively defined by $\ell^a = \sum_{k \in K} \ell^a, T^k$ for $a \geq 0$, and $m^T = \sum_{k \in K} m^T^k$. Furthermore, its total mass is defined by $\sigma = \sum_{k \in K} \sigma^T_k$. By construction, we have $\mathbb{P}_r^\psi(dT)$-a.s. $\emptyset \in \text{Br}_\infty(T)$, $\Delta_0 = r$ (see definition (16) with $b = 0$) and $\ell^0 = r \delta_0$. Under $\mathbb{P}_r^\psi$ or under $\mathbb{N}^\psi$, we define the process $Z = \{Z_a, a \geq 0\}$ by:

$$Z_a = (\ell^a, 1).$$

Notice that (under $\mathbb{N}$ or $\mathbb{P}_r^\psi$):

$$\sigma = \int_0^{+\infty} Z_a da = m^T(T).$$

In particular, as $\sigma$ is distributed as the total mass of a CSBP under its canonical measure, we have that for $\lambda \geq 0$

$$\mathbb{N}^\psi \left[1 - e^{-\lambda \sigma}\right] = \psi^{-1}(\lambda), \quad \mathbb{N}^\psi[1 - e^{-\lambda Z_a}] = u^\psi(a, \lambda),$$

where $(u^\psi(a, \lambda), a \geq 0, \lambda > 0)$ is the unique non-negative solution to

$$\int_0^\lambda \frac{dr}{u^\psi(a, \lambda) \psi(r)} = a; \quad u^\psi(0, \lambda) = \lambda.$$

By results in Chapter 3 in [19], we further have

$$u^\psi(a, u^\psi(a', \lambda)) = u^\psi(a + a', \lambda), \quad \lim_{\lambda \to 0} u^\psi(a, \lambda) = v^\psi(a).$$
Finally, we recall the Girsanov transformation from [1]. Let \( \theta \in \Theta^\psi \) and \( a > 0 \). We set:
\[
M_{a}^{\psi,\theta} = \exp \left\{ \theta Z_{0} - \theta Z_{a} - \psi(\theta) \int_{0}^{a} Z_{s} ds \right\}.
\]

Recall that \( Z_{0} = \langle \ell_{0}, 0 \rangle = 0 \) under \( \mathbb{N}^\psi \). For any non-negative measurable functional \( F \) defined on \( T \), we have for \( \theta \in \Theta^\psi \) and \( a \geq 0 \):
\[
E_{\psi}^{\theta} \left[ F(T^{(a)}) \right] = \mathbb{E}_{\psi} \left[ F(T^{(a)}) M_{a}^{\psi,\theta} \right] \quad \text{and} \quad \mathbb{N}^\psi F(T^{(a)}) = \mathbb{N}^\psi \left[ F(T^{(a)}) M_{a}^{\psi,\theta} \right].
\]

Typically,
\[
\mathbb{N}^\psi \left[ F(T) \right] = \mathbb{N}^\psi \left[ F(T) e^{-\psi(\theta) \sigma} 1_{\{\sigma < +\infty\}} \right].
\]

We have that under \( \mathbb{P}^\psi_{t}(d \mathcal{T}) \), the random measure \( \mathcal{N}^\psi_{0} \left( dx, d \mathcal{T}^\prime \right) \), defined by (12) with \( a = 0 \), is a Poisson point measure on \( \{ \emptyset \} \times T \) with intensity \( r \delta_{0}(dx) \mathbb{N}^\psi \left[ d \mathcal{T}^\prime \right] \). Then, using the first equality in (23) with \( F = 1 \), we get that for \( \theta > 0 \) such that \( \psi(\theta) \geq 0 \),
\[
\mathbb{N}^\psi \left[ 1 - \exp \left\{ \theta Z_{a} + \psi(\theta) \int_{0}^{a} Z_{s} ds \right\} \right] = -\theta.
\]

4. A GENERAL PRUNING PROCEDURE

In this section we define a pruning procedure on a Lévy tree according to the admissible family of branching mechanisms in Definition 2.1.

Recall (2) and (10). Also recall that \( \mathcal{T}_{t} = \mathcal{S} \cap [t, \infty) \) for \( t \in \mathcal{S} \). For \( \mathcal{T} \in T \), we consider two Poisson random measures \( M_{t}^{\text{skel}}(d \theta, dy) \) and \( M_{t}^{\text{nod}}(d \theta, dy) \) on \( \mathcal{T}_{t} \times T \) whose intensity is
\[
\beta_{d \theta \ell^T(dy)} \quad \text{and} \quad \sum_{x \in \text{Br}_{\infty}(T) \setminus \{\emptyset\}} m_{\Delta}(t, d \theta) \delta_{x}(dy),
\]
respectively. Then \( M_{t}^{\text{skel}}(d \theta, dy) \) gives the marks on the skeleton and \( M_{t}^{\text{nod}}(d \theta, dy) \) gives the marks on the nodes of infinite degree.

We define a new Poisson random measure on \( \mathcal{T}_{t} \times T \) by
\[
M_{t}^{\text{s}}(d \theta, dy) = M_{t}^{\text{skel}}(d \theta, dy) + M_{t}^{\text{nod}}(d \theta, dy).
\]

Using this measure of marks, we define the pruned tree at time \( q \) as:
\[
T_{q}^{\text{s}} := \{ x \in T \mid M_{t}^{\text{s}}([t, q] \times [\emptyset, x]) = 0 \}, \quad q \in \mathcal{T}_{t},
\]
with the induced metric, root \( \emptyset \) and mass measure which is the restriction of the mass measure \( m^{T} \).

Remark 4.1. Note that \( \mathbb{N}^{\text{skel}} \cdot n(\emptyset) = 1 \) and \( \mathbb{P}^{\psi}_{t} \cdot n(\emptyset) = \infty \) with \( \Delta_{\emptyset} = r \). The above definition of \( T_{q}^{\text{s}} \) means that we do not put marks on the root even if \( \emptyset \) is a node of infinite degree with mass \( r \). Or \( \mathbb{P}^{\psi}_{t} \left( T_{q}^{\text{s}} = \emptyset \right) > 0 \). However, \( \mathbb{P}^{\psi}_{t} \left( \mathcal{T} = \emptyset \right) = 0 \).

For fixed \( q \in \mathcal{T}_{t} \), \( M_{t}^{\text{s}}([t, q], dy) = M_{t}^{\text{skel}}([t, q], dy) + M_{t}^{\text{nod}}([t, q], dy) \) is also a point measure on tree \( T \):

(i) \( M_{t}^{\text{skel}}([t, q], dy) \) is a Poisson point measure on the skeleton of \( T \) with intensity \( \int_{t}^{q} \beta_{d \theta \ell^T(dy)} \); 
(ii) The atoms of \( M_{t}^{\text{nod}}([t, q], dy) \) give the marked nodes: each node of infinite degree is marked (or pruned) independently from the others with probability
\[
\mathbb{P} \left( M_{t}^{\text{nod}}([t, q], \{ y \}) > 0 \right) = 1 - \exp \left\{ -m_{\Delta_{y}}([t, q]) \right\} = 1 - m_{\Delta_{y}}(t, q),
\]
where \( \Delta_{y} \) is the mass associated with the node.
Thus for fixed $q \in \mathcal{T}$, there exists a measurable functional $\mathcal{M}_{\alpha_t,q,p_t,q}$ on $\mathcal{T}$ such that:

$$T_q^t = \mathcal{M}_{\alpha_t,q,p_t,q}(T),$$

where $\alpha_{t,q} = \int_t^q \beta d\theta, p_{t,q} = 1 - m_z(t,q)$. Our main result in this section is the following theorem.

**Theorem 4.2.** Assume that $\{\psi_t : t \in \mathcal{T}\}$ is an admissible family satisfying (H1-3). Then we have

(a) The tree-valued process $(T_q^t, q \in \mathcal{T})$ is a Markov process under $\mathbb{N}^{\psi_t}$;

(b) For fixed $q \in \mathcal{T}$, the distribution of $T_q^t$ under $\mathbb{N}^{\psi_t}$ is $\mathbb{N}^{\psi_t}$;

(c) Given $(T, m^T) \in \mathcal{T}$, let $\mathcal{M}(dx,dT) = \sum_{i \in \mathbb{T}} \delta(x_i,T_i)$ be a Poisson random measure on $\mathcal{T} \times \mathcal{T}$ with intensity

$$m^T(dx) \left( \int_t^q \beta d\theta \mathbb{N}^{\psi_t}[dT] + \int_t^q d\theta \int_0^\infty \gamma_0(dz) \mathbb{P}^{\psi_t}(dT) \right).$$

Then for $q \in \mathcal{T}$, $(T, \mathcal{M}_{\alpha_t,q,p_t,q}(T))$ under $\mathbb{N}^{\psi_t}$ has the same distribution as $(\tilde{T}, T)$ under $\mathbb{N}^{\psi_t}$, where

$$\tilde{T} = T \otimes_{i \in I} (T_i, x_i).$$

**Remark 4.3.** (c) in Theorem 4.2 is the so-called special Markov property. One may follow the proof in Appendix A in [15] to extend (c) to have pruning times in (sub)critical cases and then follow the arguments in Step 4 below to extend the result to super-critical cases.

**Proof.** The proof will be divided into five steps:

**Step 1:** We shall prove (a). We first study the behavior of $M_t^{nod}(d\theta, dy)$. Given a branching node $y \in Br_{\infty}(\mathcal{T})$, for $t \leq \theta \leq q \in \mathcal{T}$, we have

$$\mathbb{P}\left( M_t^{nod}(\theta, q), \{y\} > 0 \big| M_t^{nod}(\theta, q), \{y\} = 0 \right) = \frac{\mathbb{P}(M_t^{nod}(\theta, q), \{y\} > 0, M_t^{nod}(\theta, q), \{y\} = 0)}{\mathbb{P}(M_t^{nod}(\theta, q), \{y\} = 0)} = \frac{\mathbb{P}(M_t^{nod}(\theta, q), \{y\} > 0) - \mathbb{P}(M_t^{nod}(\theta, q), \{y\} > 0)}{\mathbb{P}(M_t^{nod}(\theta, q), \{y\} = 0)} = \frac{m_{\Delta_p}(\theta, \{y\})}{m_{\Delta_p}(\theta, \{y\})} = 1 - m_{\Delta_p}(\theta, q),$$

where we used (27) for the third equality and assumption (H2) for the fourth equality. Similarly, one can prove that for $x \in \mathcal{T}$ and $t \leq \theta \leq q \in \mathcal{T}$,

$$\mathbb{P}\left( M_t^{skc}(\theta, q), [0, x] \right) > 0 \big| M_t^{skc}(\theta, q), [0, x] = 0 \right) = \mathbb{P}\left( M_t^{skc}(\theta, q), [0, x] > 0 \right).$$

Then (a) follows readily.

**Step 2:** For the second and the third assertions, if $\psi_t$ is (sub)critical, then an application of Theorem 1.1 in [11] gives the desired results. So we only need to study the super-critical case. Without loss of generality, we may assume $t = 0$. From now on we shall assume that $\psi_0$ is super-critical. For this proof only, we set

$$|x| = d_T(0, x), \quad x \in \mathcal{T}.$$ We also write $T_q$ for $T_q^0$. 
In this step, we prove the desired results for a special sub-critical case. Let $\eta_0$ be the maximum root of $\psi_0(s) = 0$. Define
\begin{equation}
\psi_q^{\eta_0}(\lambda) = \psi_q(\lambda + \eta_0) - \psi_q(\eta_0), \quad \lambda \geq 0, \quad q \in \mathcal{I}_0.
\end{equation}

One can check that if $\{\psi_q : q \in \mathcal{I}_0\}$ is an admissible family satisfying (H1-3), then $\{\psi_q^{\eta_0} : q \in \mathcal{I}_0\}$ is also an admissible family with parameter $(b_q^{\eta_0}, m_q^{\eta_0}, q \in \mathcal{I}_0)$ satisfying (H1-3) such that
\begin{equation}
b_q^{\eta_0} = b_q + 2c\eta_0 + \int_0^\infty (1 - e^{-\eta_0 z})m_q(dz), \quad m_q^{\eta_0}(dz) = e^{-\eta_0 z}m_q(dz).
\end{equation}

Typically, by (32) and (33),
\begin{align*}
b_q^{\eta_0} &= b_q^{\eta_0} + \int_0^q \beta_0 d\theta + \int_0^q d\theta \int_0^\infty z\eta_0(d\eta) - \int_0^q d\theta \int_0^\infty (1 - e^{-\eta_0 z})\eta_0(d\eta) \\
&= b_q^{\eta_0} + \int_0^q \beta_0 d\theta + \int_0^q d\theta \int_0^\infty e^{-\eta_0 z}\eta_0(d\eta) \\
&\geq b_q^{\eta_0}.
\end{align*}

Since $b_0^{\eta_0} = \psi_0'(\eta_0) > 0$, $\psi_q^{\eta_0}$ is subcritical for all $q \in \mathcal{I}_0$. Moreover, by (32) and (33), we have
\begin{equation}
\frac{\partial}{\partial q} \psi_q^{\eta_0}(\lambda) = \zeta_q(\lambda + \eta_0) - \zeta_q(\eta_0) = \beta_q\lambda + \int_0^\infty (1 - e^{-\lambda z})e^{-\eta_0 z}n_q(dz)
\end{equation}
and
\begin{equation}
m_q^{\eta_0}(dz) = \frac{m_q(dz)}{m_0(dz)} = m_z(0, q), \quad q \in \mathcal{I}_0.
\end{equation}

Thus $\{\psi_q : q \in \mathcal{I}_0\}$ and $\{\psi_q^{\eta_0} : q \in \mathcal{I}_0\}$ induce the same pruning parameters $\beta_q$ and $m_z(0, q)$. Typically according to assertions (b) and (c) for (sub)critical case, we have
\begin{enumerate}
\item[(b')] $\mathcal{T}_a = \mathcal{M}_{\alpha_0,q_0, q_0} (\mathcal{T})$ is a $\psi_q^{\eta_0}$-Lévy tree under $\mathbb{N}_{\psi_q^{\eta_0}}$;
\item[(c')] Given $(\mathcal{T}, \mathbf{m}^\mathcal{T}) \in \mathcal{T}$, let $\mathcal{M}^{\eta_0}(dx, d\mathcal{T}) = \sum_{i \in I_0} \delta_{(x_i, T_i)}$ be a Poisson point measure on $\mathcal{T} \times \mathcal{T}$ with intensity
\begin{equation}
\mathbf{m}^\mathcal{T} (dx) \left( \int_0^q \beta_0 d\theta \mathbb{N}_{\psi_q^{\eta_0}}[d\mathcal{T}] + \int_0^q d\theta \int_0^\infty e^{-\eta_0 z}n_q(dz) \mathbb{P}_{\psi_q^{\eta_0}}(d\mathcal{T}) \right).
\end{equation}
\end{enumerate}

Then for $q \in \mathcal{I}_0$, $(\mathcal{T}, \mathcal{M}_{\alpha_0,q_0, q_0}(\mathcal{T}))$ under $\mathbb{N}_{\psi_q^{\eta_0}}$ has the same distribution as $(\hat{\mathcal{T}}_{\eta_0}, \mathcal{T})$ under $\mathbb{N}_{\psi_q}$, where
\begin{equation}
\hat{\mathcal{T}} = \mathcal{T} \otimes i \in I_0 (\mathcal{T}_i, x_i).
\end{equation}

Step 3: We shall prove (b) for $\psi_0$ is super-critical. Recall $\mathcal{T}(a) = \{x \in \mathcal{T} : d^\mathcal{T}(\emptyset, x) \leq a\}$. By Girsanov transformation (23), for any nonnegative function $F$ on $\mathcal{T}$, we have
\begin{align}
\mathbb{N}_{\psi_q}[F(\mathcal{T}_q^{(a)})] &= \mathbb{N}_{\psi_q}^0[F(\mathcal{M}_{\alpha_0,q_0, q_0}(\mathcal{T}^{(a)}))] \\
&= \mathbb{N}_{\psi_q}^0[\exp_{\eta_0} Z_a F(\mathcal{M}_{\alpha_0,q_0, q_0}(\mathcal{T}^{(a)}))] \\
&= \mathbb{N}_{\psi_q}^0[\exp_{\eta_0} \hat{Z}_a F(\mathcal{T}^{(a)})],
\end{align}
where the last equality follows from Special Markov property (c') and
\begin{equation}
\hat{Z}_a = \langle t^a, \hat{\mathcal{T}}, 1 \rangle = Z_a + \sum_{i \in I_0 \setminus \{a\}} 1_{\{x_i \leq a\}} Z_{a-x_i}^T.
\end{equation}
with $Z_t^a = (t^a, T_t, 1)$. Thus
\begin{equation}
N_{\psi^0}^\theta [e^{\rho_0 \tilde{Z}_u} F(T^{(a)})] = N_{\psi^0}^\theta [e^{\rho_0 Z_u} F(T^{(a)}) H(a, \eta_0)],
\end{equation}
where, by property of Poisson random measure,
\begin{equation}
H(a, \eta_0) = N_{\psi^0}^\theta \left[ e^{\rho_0 \sum_{i \in I_{00}} 1_{\{\xi_i \leq a\}} Z_t^i} \bigg| \mathcal{T} \right] = \exp \left\{ - \int_{T^{(a)}} m(T) (dx) \left( \int_0^q \beta_0 d\theta N_{\psi^0}^\theta \left[ 1 - e^{\rho_0 Z_{a-|x|}} \right] + \int_0^q d\theta \int_0^\infty e^{-\rho_0 z} n_\theta (dz) \left( 1 - e^{-z N_{\psi^0}^\theta [1 - e^{\rho_0 Z_{a-|x|}}]} \right) \right) \right\}.
\end{equation}
Noting that $\psi_0(\eta_0) = 0$, together with (25), yields
\begin{equation}
N_{\psi^0}^\theta \left[ 1 - e^{\rho_0 Z_{a-|x|}} \right] = -\eta_0.
\end{equation}
Then (6) and (7) imply
\begin{equation}
H(a, \eta_0) = \exp \left\{ - \int_{T^{(a)}} m(T) (dx) \left( - \int_0^q \beta_0 d\theta \eta_0 + \int_0^q d\theta \int_0^\infty e^{-\rho_0 z} n_\theta (dz) (1 - e^{z \eta_0}) \right) \right\}.
\end{equation}
Since $m(T^{(a)}) = \int_0^a Z_s ds$ and
\begin{equation}
\psi_q(\eta_0) = \psi_q(\eta_0) - \psi_0(\eta_0) = \int_0^q \beta_0 d\theta \eta_0 - \int_0^q d\theta \int_0^\infty e^{-\rho_0 z} n_\theta (dz) (1 - e^{z \eta_0}),
\end{equation}
we have
\begin{equation}
H(a, \eta_0) = \exp \left\{ \psi_q(\eta_0) m(T^{(a)}) \right\} = \exp \left\{ \psi_q(\eta_0) \int_0^a Z_s ds \right\}.
\end{equation}
By (37), (38) and Girsanov transformation (23), one can see that
\begin{equation}
N_{\psi^0}^\theta [F(T_q^{(a)})] = N_{\psi^0}^\theta \left[ e^{\rho_0 Z_u + \psi_q(\eta_0) \int_0^a Z_s ds} F(T^{(a)}) \right] = N_{\psi^0}^\theta [F(T^{(a)})],
\end{equation}
which implies that under $N_{\psi^0}^\theta$, $T_q$ is a $\psi_q$-Lévy tree. We have completed the proof of assertion (b).

**Step 4:** We shall prove (c) for $\psi_0$ is super-critical. Recall (30) and (36). Note that
\begin{equation}
\tilde{T}^{(a)} = T^{(a)} \otimes_{i \in I_{0, \xi_i \leq a}} (T_t^{(a-|x_i|)}, x_i).
\end{equation}
We only need to show that for all $a \geq 0$, $(T^{(a)}, \mathcal{M}_{\alpha_{0, q}, p_{0, q}}(T^{(a)}))$ under $N_{\psi^0}^\theta$ has the same distribution as $(\tilde{T}^{(a)}, T^{(a)})$ under $N_{\psi^0}^\theta$. By (23), we have for any nonnegative functional $F$ on $\mathbb{T}^2$:
\begin{equation}
N_{\psi^0}^\theta \left[ F(T^{(a)}, \mathcal{M}_{\alpha_{0, q}, p_{0, q}}(T^{(a)})) \right] = N_{\psi^0}^\theta \left[ e^{\rho_0 Z_a} F(T^{(a)}, \mathcal{M}_{\alpha_{0, q}, p_{0, q}}(T^{(a)})) \right]
\end{equation}
By (c’ in Step 2), we have $(T^{(a)}, \mathcal{M}_{\alpha_{0, q}, p_{0, q}}(T^{(a)}))$ under $N_{\psi^0}^\theta$ has the same distribution as $(\tilde{T}^{(a)}, T^{(a)})$ under $N_{\psi^0}^\theta$, where
\begin{equation}
\tilde{T}^{(a)} = T^{(a)} \otimes_{i \in I_{00}, \xi_i \leq a} (T_t^{(a-|x_i|)}, x_i).
\end{equation}
Thus (41) implies
\begin{equation}
N_{\psi^0}^\theta \left[ F(T^{(a)}, \mathcal{M}_{\alpha_{0, q}, p_{0, q}}(T^{(a)})) \right] = N_{\psi^0}^\theta \left[ e^{\rho_0 Z_a} F(\tilde{T}^{(a)}, T^{(a)}) \right].
\end{equation}
We CLAIM that for all \( a > 0 \) and any nonnegative measurable functional \( \Phi \) on \( T^{(a)} \times T \),
\begin{equation}
N^{\psi_0}_\nu \left[ e^{\eta_0 \hat{T}_a} F(T^{(a)}) \exp \{- \langle M^{\psi_0}_\nu, \Phi \rangle \} \right] = N^{\psi_\nu} \left[ F(T^{(a)}) \exp \{- \langle M_{a}, \Phi \rangle \} \right],
\end{equation}
where
\[ M^{\psi_0}_\nu(dx, dT) = \sum_{i \in I_{\nu_0}} 1_{|x_i| \leq a} \delta \left( x_i, T^{(a)}(x_i) \right)(dx, dT) \]
and
\[ M_{a}(dx, dT) = \sum_{i \in I} 1_{|x_i| \leq a} \delta \left( x_i, T^{(a)}(x_i) \right)(dx, dT). \]

Then with (43) in hand, we have
\begin{equation}
N^{\psi_0}_\nu \left[ e^{\eta_0 \hat{T}_a} F(\hat{T}^{(a)}, T^{(a)}) \right] = N^{\psi_0}_\nu \left[ e^{\eta_0 \hat{Z}_a} F(\hat{T}^{(a)}) \otimes_{i \in I_{\nu_0}, |x_i| \leq a} (T^{(a)}(x_i), T^{(a)}) \right] = N^{\psi_\nu} \left[ F(\hat{T}^{(a)}, T^{(a)}) \right].
\end{equation}
which, together with (12), gives that
\[ N^{\psi_0}_\nu \left[ F(\hat{T}^{(a)}, M_{\alpha_0 q, p_0 q}(T^{(a)})) \right] = N^{\psi_\nu} \left[ F(\hat{T}^{(a)}, T^{(a)}) \right]. \]
Since \( a \) is arbitrary, assertion (c) follows readily.

**Step 5:** The remainder of this proof is devoted to (43). Define
\[ g(a, x) = N^{\psi_0}_\nu \left[ 1 - e^{-\Phi(a, x)} \right]. \]
Then we have
\[ P^{\psi_\nu}_\xi \left[ 1 - e^{-\Phi(a, x)} \right] = 1 - e^{-2g(a, x)}. \]
First, by property of Poisson random measure,
\begin{equation}
N^{\psi_\nu} \left[ F(T^{(a)}) \exp \{- \langle M_{a}, \Phi \rangle \} \right] = N^{\psi_\nu} \left[ F(T^{(a)}) \exp \left\{ - \int_{T^{(a)}} m^{T^{(a)}} dx G(a, x) \right\} \right],
\end{equation}
where
\begin{equation}
G(a, x) = \int_0^q \beta_\theta d\theta g(a, x) + \int_0^q d\theta \int_0^\infty n_\theta(dz) \left( 1 - e^{-2g(a, x)} \right).
\end{equation}
By (23),
\begin{align}
g(a, \theta, x) &= N^{\psi_0}_\nu \left[ 1 - e^{-\Phi(a, x, T^{(a)})} \right] \\
&= N^{\psi_0}_\nu \left[ e^{\eta_0 Z_{a - |x|}} \left( 1 - e^{-\Phi(a, x, T^{(a)})} \right) \right] \\
&= N^{\psi_0}_\nu \left[ e^{\eta_0 Z_{a - |x|}} - 1 + 1 - e^{-\Phi(a, x, T^{(a)}) + \eta_0 Z_{a - |x|}} \right] \\
&= \eta_0 + N^{\psi_0}_\nu \left[ 1 - e^{-\Phi(a, x, T^{(a)}) + \eta_0 Z_{a - |x|}} \right] = \eta_0 + g_{\eta_0}(a, x),
\end{align}
where the last equality follows from (39). With (40) in hand, substituting (46) into (45) yields
\begin{align}
G(a, x) &= \int_0^q \beta_\theta d\theta \eta_0 - \int_0^q d\theta \int_0^\infty e^{-\eta_0 n_\theta}(dz) \left( 1 - e^{2\eta_0} \right) \\
&\quad + \int_0^q \beta_\theta d\theta g_{\eta_0}(a, x) + \int_0^q d\theta \int_0^\infty e^{-\eta_0 n_\theta}(dz) \left( 1 - e^{-2g_{\eta_0}(a, x)} \right)
\end{align}
we will use the version of \( \{T_t\} \) of \( \Psi \space{T_t} \) for \( t \in \mathbb{T} \) such that \( \Psi_t \) is a non-increasing process (for the inclusion of trees) and is càdlàg. Theorem 4.2 implies that \( \{T_t\} \) is a non-increasing \( \mathbb{P}^q_{\Psi} \)-Markov process under \( \mathbb{P}^q_{\Psi} \). In the sequel, we also have

\[
\Psi_t = m^{T_t}(T_t), \quad t \in \mathbb{T}.
\]

which, together with (48), implies (43). We have completed the proof. \( \square \)

A direct consequence of Theorem 4.2 is that

**Corollary 4.4.** Assume that \( \{\psi_t : t \in \mathbb{T}\} \) is an admissible family satisfying (H1-3). Then for \( r > 0 \) we have the tree-valued process \( \{T_t^q, q \in \mathbb{T}_t\} \) is a Markov process under \( \mathbb{P}^q_{\Psi} \) and for fixed \( q \in \mathbb{T}_t \), the distribution of \( T_t^q \) under \( \mathbb{P}^q_{\Psi} \) is \( \mathbb{P}^q_{\Psi} \).

5. A TREE-VALUED PROCESS

Because of the pruning procedure, we have \( T_t^q \subset T_{t_0}^q \) for \( p \leq q \in \mathbb{T}_t \). The process \( \{T_t^q, q \in \mathbb{T}_t\} \) is a non-increasing process (for the inclusion of trees) and is càdlàg. Theorem 4.2 implies that for \( t_1 \leq t_2 \in \mathbb{T}, \{T_t^{q_1} : q \in \mathbb{T}_{t_2}\} \) under \( \mathbb{N}_{\Psi_1}^{q_2} \) has the same distribution as \( \{T_t^{q_1} : q \in \mathbb{T}_{t_2}\} \) under \( \mathbb{N}_{\Psi_1}^{q_1} \). By Kolmogorov’s theorem, there exists a tree-valued Markov process \( \{T_t : t \in \mathbb{T}\} \) such that \( \{T_t^q : q \in \mathbb{T}_t\} \) has the same finite dimensional distribution as \( \{T_t^q : q \in \mathbb{T}_t\} \) under \( \mathbb{N}_{\Psi_1}^{q_1} \). Denote by \( \mathbb{N}^{\Psi} \) the law of \( \{T_t : t \in \mathbb{T}\} \). We have for any nonnegative measurable functional \( F \),

\[
\mathbb{N}^{\Psi}[F(T)] = \mathbb{N}^{\Psi}_{\Psi}[F(T)].
\]

Set

\[
\sigma_t = m^{T_t}(T_t), \quad t \in \mathbb{T}.
\]

Then one can check that \( \{\sigma_t : t \in \mathbb{T}\} \) is a non-increasing \( [0, \infty]\)-Markov process. In the sequel, we will use the version of \( \{T_t : t \in \mathbb{T}\} \) such that \( \sigma_t : t \in \mathbb{T} \) is càdlàg. For \( t \in \mathbb{T} \), set \( \Psi_t = \{\psi_t : q \in \mathbb{T}_t\} \) and \( \Psi_t^{\psi} = \{\psi_t : q \in \mathbb{T}_t\} \).

**Proposition 5.1.** For \( t \in \mathbb{T} \) and any non-negative measurable functional \( F \),

\[
\mathbb{N}^{\Psi_t}[F(T_t^q)\mathbb{1}_{\{\sigma_t < \infty\}}] = \mathbb{N}^{\Psi_t^\psi}[F(T_t^q) : q \in \mathbb{T}_t].
\]
Proof. According to arguments in Step 2 in the proof of Theorem 4.2, \( \{\psi_q : q \in \mathcal{T}_t\} \) and \( \{\psi_q^n : q \in \mathcal{T}_t\} \) induce the same pruning parameters. Then the desired result is a direct consequence of the fact \( N^{\psi_q^N}[F(T)1_{\{\sigma < \infty\}}] = N^{\psi_q^N}[F(T)] \); see \((24)\). \( \square \)

We then study the behavior of \( \{\sigma_t : t \in \mathcal{T}\} \).

**Lemma 5.2.** For \( t \leq q \in \mathcal{S} \) and \( \lambda \geq 0 \),

\[
N^{\Psi}[e^{-\lambda \sigma_t}|T_q] = \exp\{-\psi_q(\psi_q^{-1}(\lambda))\sigma_q\}
\]

and \( N^{\Psi}[\sigma_t < +\infty|T_q] = \exp\{-\psi_q(\psi_q^{-1}(0))\sigma_q\} \). Moreover, if \( \psi_t \) is subcritical, then

\[
N^{\Psi}[\sigma_t|T_q] = \psi_q'(0)\sigma_q/\psi_q'(0).
\]

**Proof.** Recall \((2)\) and \((20)\). By \((c)\) in Theorem 4.2

\[
N^{\Psi}[e^{-\lambda \sigma_t}|T_q] = N^{\Psi}[e^{-\lambda \sigma_q - \lambda \sum_{i=1} \sigma_i}|T_q] = e^{-\lambda \sigma_q} e^{-\lambda f_{T_q} m_{T_q}(dx) G(\lambda)},
\]

where \( \sigma_i = m_{T_i}(T_i) \) and

\[
G(\lambda) = \int_t^q \beta_0 d\theta \nu_{\psi_t} [1 - e^{-\lambda \sigma}] + \int_t^q n_\theta (dz) \nu_{\psi_t} (1 - e^{-\lambda \sigma}) = \int_t^q \beta_0 d\theta \psi_t^{-1}(\lambda) + \int_t^q n_\theta (dz) (1 - e^{-e \psi_t^{-1}(\lambda)}) = \int_t^q \zeta_\theta (\psi_t^{-1}(\lambda)).
\]

Thus

\[
N^{\Psi}[e^{-\lambda \sigma_t}|T_q] = \exp\{-\psi_q(\psi_q^{-1}(\lambda))\sigma_q\}.
\]

Then,

\[
N^{\Psi}[\sigma_t < +\infty|T_q] = \lim_{\lambda \to 0} N^{\Psi}[e^{-\lambda \sigma_t}|T_q] = \exp\{-\psi_q(\psi_q^{-1}(0))\sigma_q\}.
\]

If \( \psi_t \) is subcritical, then \( N^{\Psi}\)-a.e. \( \sigma_t < \infty \). We obtain

\[
N^{\Psi}[\sigma_t|T_q] = \frac{d}{d\lambda} N^{\Psi}[e^{-\lambda \sigma_t}|T_q]|_{\lambda=0} = \psi_q'(0)\sigma_q/\psi_q'(0).
\]

Recall that \( \eta_q \) the largest root of \( \psi_q(\lambda) = 0 \). Thus

\[
\eta_q = \lim_{\lambda \to 0^+} \psi_q^{-1}(\lambda) = \psi_q^{-1}(0).
\]

Define the **ascension time**

\[
A = \inf\{t \in \mathcal{S} : \sigma_t = +\infty\}
\]

with the convention that \( \inf\{\emptyset\} = \inf\mathcal{S} = t_\infty \in [-\infty, 0] \). In the sequel of this paper, we always assume that

\[
0 \in \mathcal{S}, \quad t_\infty < 0.
\]

Recall \((2)\). Let us consider the following condition:

\[
\lim_{t \to t_\infty^+} \int_t^0 \zeta_\theta (\lambda) d\theta = \psi_0 (\lambda) - \lim_{t \to t_\infty^+} \psi_t (\lambda) < +\infty, \quad \text{for some } \lambda > 0.
\]

(51)
Proposition 5.3. \( \lim_{q \to t_\infty} \psi_q^{-1}(0) < \infty \) if and only if (51) holds.

Proof. “if” part: (51) implies

\[
\begin{align*}
\lim_{t \to t_\infty} \int_t^0 \beta_0 d\theta + \lim_{t \to t_\infty} \int_0^\infty (1 + z) n_\theta (dz) d\theta < +\infty.
\end{align*}
\]

Since \( 1/z^2 \leq 1/z \), by (60) and (52), we have \( \sup_{q \in \mathcal{T}} \int_q^\infty (1 + z^2) m_q (dz) < \infty \). Then monotonicity of \( q \mapsto (1 + z^2) m_q (dz) \) yields that there exists a \( \sigma \)-finite measure \( m_{t_\infty} (dz) \) on \( (0, \infty) \) such that

\[
\int (1 + z^2) m_{t_\infty} (dz) < +\infty
\]

and as \( q \to t_\infty \),

\[
(1 + z^2) m_q (dz) \to (1 + z^2) m_{t_\infty} (dz) \quad \text{in } M_f ((0, \infty)).
\]

Then we can define

\[
\psi_{t_\infty} (\lambda) = b_{t_\infty} \lambda + c \lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z \mathbf{1}_{\{z \leq 1\}}) m_{t_\infty} (dz), \quad \lambda \geq 0,
\]

where

\[
b_{t_\infty} = b_0 + \int_1^\infty m_0 (dz) - \lim_{q \to t_\infty} \left( \int_q^0 \beta_0 d\theta + \int_0^q \int_0^1 z m_\theta (dz) d\theta \right).
\]

\( \psi_{t_\infty} (\lambda) \) is a convex function. Since \( \psi_q \) satisfies (H3) which implies \( c > 0 \) or \( \int_0^1 z m_q (dz) = \infty \), we have \( \lim_{\lambda \to \infty} \psi_{t_\infty} (\lambda) = \infty \) and \( \psi_{t_\infty}^{-1} (0) < \infty \) which is the largest root of \( \psi_{t_\infty} (\lambda) = 0 \). Meanwhile, note that \( e^{-\lambda z} - 1 + \lambda z \mathbf{1}_{\{z \leq 1\}} \leq 1 + z^2 \). By (5) and (6),

\[
\psi_q (\lambda) = \left( b_0 + \int_1^\infty z m_0 (dz) - \int_q^0 \beta_0 d\theta - \int_0^q \int_0^1 z m_\theta (dz) d\theta \right) \lambda
\]

\[
+ c \lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z \mathbf{1}_{\{z \leq 1\}}) m_q (dz)
\]

\[
\to \psi_{t_\infty} (\lambda), \quad \text{as } q \to t_\infty + .
\]

Then the fact \( \lim_{q \to t_\infty} \psi_q (\lambda) \to \psi_{t_\infty} (\lambda) \) yields

\[
\psi_{t_\infty}^{-1} (0) = \lim_{q \to t_\infty} \psi_q^{-1} (0) < \infty.
\]

“only if” part: If \( \int_{t_\infty} \zeta_\theta (\lambda) d\theta = +\infty \) for some \( \lambda > 0 \) ( hence for all \( \lambda > 0 \) ), by (5) and (6),

\[
\psi_q (\lambda) = \psi_0 (\lambda) - \int_q^0 \zeta_\theta (\lambda) d\theta \to -\infty, \quad \text{as } q \to t_\infty.
\]

Then we have \( \lim_{q \to t_\infty} \psi_q^{-1} (0) = +\infty \).

Define \( \psi_{t_\infty}^{-1} (0) = \lim_{q \to t_\infty} \psi_q^{-1} (0) \) and

\[
\mathcal{T}_\infty = \begin{cases} 
\mathcal{T} \cup \{t_\infty\}, & \psi_{t_\infty}^{-1} (0) < +\infty \\
\mathcal{T}, & \psi_{t_\infty}^{-1} (0) = +\infty.
\end{cases}
\]

Remark 5.4. From the proof of Proposition [5.3] we see that it is possible to extend the definition of a given admissible family to \( \mathcal{T}_\infty \). For some results in the sequel of this paper, we need to avoid this case by assuming \( t_\infty \notin \mathcal{T}_\infty \) (hence \( t_\infty \notin \mathcal{T} \)).

Next, we study the distribution of \( A \) and \( \mathcal{T}_A \).
Lemma 5.5. For $q \in \mathcal{T} \cup \{t_\infty\}$,

$$\mathbf{N}^\psi[A > q] = \psi_q^{-1}(0).$$

and

$$\mathbf{N}^\psi[A = t_\infty] = \begin{cases} 0, & t_\infty \notin \mathcal{T}_\infty \\ \infty, & t_\infty \in \mathcal{T}_\infty. \end{cases}$$

(53)

Proof. Recall (20). By Lemma 5.2, for $q > t_\infty$

$$\mathbf{N}^\psi[A > q] = \mathbf{N}^\psi[\sigma_q = +\infty] = \lim_{\lambda \to 0} \mathbf{N}^\psi[(1 - e^{-\lambda\sigma}] = \lim_{\lambda \to 0} \psi_q^{-1}(\lambda) = \psi_q^{-1}(0).$$

And letting $q \to t_\infty$ gives the case of $q = t_\infty$. By applying Lemma 5.2 again, we obtain

$$\mathbf{N}^\psi[A = t_\infty] = \mathbf{N}^\psi[\forall q > t_\infty, \sigma_q < +\infty | T_0] = \lim_{q \to t_\infty} \mathbf{N}^\psi[\sigma_q < +\infty | T_0] = \lim_{q \to t_\infty} e^{-\sigma_0 \psi_t(0)} \psi_q^{-1}(0).$$

Then noting that $\forall \lambda > 0, \mathbf{N}^\psi[e^{-\lambda\sigma}] = +\infty$ gives the desired result. \qed

Remark 5.6. (53) implies that if $t_\infty \in \mathcal{T}_\infty$, then $\mathbf{N}^\psi[T_q] is compact for all $t > t_\infty] = +\infty$. If $t_\infty \notin \mathcal{T}_\infty$, then $\mathbf{N}^\psi$-a.e. there exists $t \in \mathcal{T}$ such that $T_q$ is not compact ($\sigma_q = \infty$) for $t > q \in \mathcal{T}$.

Theorem 5.7. Assume that $\psi_0$ is critical. For $q \in (t_\infty, 0)$ and any nonnegative measurable functional $F$ on $\mathcal{T}$,

$$\mathbf{N}^\psi[F(T) | A = q] = \psi_q'(\eta_q) \mathbf{N}^\psi[F(T) \sigma 1_{\{\sigma < \infty\}}]$$

and for $\lambda \geq 0$

$$\mathbf{N}^\psi[e^{-\lambda\sigma} | A = q] = \frac{\psi_q'(\eta_q)}{\psi_q'(\psi_t^{-1}(\lambda))}.$$ (55)

In particular, we have

$$\mathbf{N}^\psi[\sigma < \infty | A = q] = 1.$$ (56)

Proof. By Lemma 5.2 we have for every $t_\infty < t \leq q < 0$,

$$\mathbf{N}^\psi[F(T_q) 1_{\{\sigma \geq t\}}] = \mathbf{N}^\psi[F(T_q) 1_{\{\sigma = +\infty\}}] = \mathbf{N}^\psi[F(T_q) \mathbf{N}^\psi[\sigma = +\infty | T_q]] = \mathbf{N}^\psi[F(T_q) \left(1 - e^{-\sigma_t \psi_t(0))}\right)] = \mathbf{N}^\psi[F(T_q) \left(1 - e^{-\sigma_t \psi_t(\eta_t))}\right)].$$
Since $\eta_t$ is the largest root of $\psi_t(s) = 0$, we have the mapping $t \mapsto \eta_t$ is differentiable with
\begin{equation}
\frac{d\eta_t}{dt} = -\frac{\zeta_t(\eta_t)}{\psi'_t(\eta_t)}.
\end{equation}
Then we get
\begin{align*}
\frac{d}{dt} N^\Psi \left[F(T_q)1_{\{A > t\}}\right] &= N^\Psi \left[F(T_q)\sigma_q e^{-\sigma_q \psi_q(\eta_t)}\right] \frac{d\psi_q(\eta_t)}{dt} \\
&= -N^\Psi \left[F(T_q)\sigma_q e^{-\sigma_q \psi_q(\eta_t)}\right] \frac{\psi'_q(\eta_t)\zeta_t(\eta_t)}{\psi'_t(\eta_t)}.
\end{align*}
The right-continuity of $\sigma$ gives
\begin{equation}
N^\Psi \left[F(T_A), A \in dq\right] = -\left.\frac{d}{dt} \left(N^\Psi \left[F(T_q)1_{\{A > t\}}\right]\right)\right|_{t=q} = \zeta_t(\eta_t)N^\Psi \left[F(T_q)\sigma_q 1_{\{\sigma_q < +\infty}\}\right].
\end{equation}
Thus
\begin{equation}
N^\Psi \left[F(T_A)\right] = q = \frac{N^\Psi \left[F(T_q)\sigma_q 1_{\{\sigma_q < +\infty}\}\right]}{N^\Psi \left[\sigma_q 1_{\{\sigma_q < +\infty}\}\right]} = \frac{N^{\psi_q} \left[F(T)\sigma 1_{\{\sigma < +\infty\}}\right]}{N^{\psi_q} \left[\sigma 1_{\{\sigma < +\infty\}}\right]}.
\end{equation}
Meanwhile, note that
\begin{equation}
N^{\psi_q} \left[\sigma e^{-r\sigma}\right] = \frac{d}{dr} N^{\psi_q} \left[1 - e^{-r\sigma}\right] = \frac{d}{dr} \psi_q^{-1}(r) = \frac{1}{\psi'_q(\psi_q^{-1}(r))}.
\end{equation}
Then we have
\begin{equation}
N^{\psi_q} \left[\sigma 1_{\{\sigma < +\infty\}}\right] = \lim_{r \to 0} N^{\psi_q} \left[\sigma e^{-r\sigma}\right] = \frac{1}{\psi'_q(\eta_q)},
\end{equation}
which, together with (58), implies (54). Using (58) again, we also get
\begin{equation}
N^\Psi \left[e^{-\lambda \sigma_A} | A = q\right] = \frac{N^{\psi_q} \left[e^{-\lambda \sigma q}\right]}{N^{\psi_q} \left[\sigma 1_{\{\sigma < +\infty\}}\right]} = \frac{\psi'_q(\eta_q)}{\psi'_q(\psi_q^{-1}(\lambda))},
\end{equation}
which is just (59). Then (56) follows readily by letting $\lambda$ in (59) go to 0.

**Proposition 5.8.** Assume $t_\infty \in \mathcal{F}$ and $\psi_0$ is critical. Then for any nonnegative measurable functional $F$ on $\mathbb{T}$,
\begin{equation}
N^\Psi \left[F(T_A)1_{\{A = t_\infty\}}\right] = N^{\psi_{t_\infty}}_\infty \left[F(T)\right],
\end{equation}
where $\psi_{t_\infty}^\infty(\cdot) = \psi_{t_\infty}(\eta_{t_\infty} + \cdot)$. Typically, for $\lambda \geq 0$
\begin{equation}
N^\Psi \left[(1 - e^{-\lambda \sigma_A})1_{\{A = t_\infty\}}\right] = \psi_{t_\infty}^{-1}(\lambda) - \eta_{t_\infty}.
\end{equation}
**Proof.**
\begin{align*}
N^\Psi \left[F(T_A)1_{\{A = t_\infty\}}\right] &= N^\Psi \left[F(T_{t_\infty})1_{\{\sigma_{t_\infty} < +\infty\}}\right] \\
&= N^{\psi_{t_\infty}}_\infty \left[F(T)1_{\{\sigma < +\infty\}}\right] \\
&= N^{\psi_{t_\infty}}_\infty \left[F(T)\right],
\end{align*}
where the last equality follows from (24). Thus
\begin{equation}
N^\Psi \left[(1 - e^{-\lambda \sigma_A})1_{\{A = t_\infty\}}\right] = N^{\psi_{t_\infty}}_\infty \left[1 - e^{-\lambda \sigma}\right]
\end{equation}
Proposition 5.10. \( h_q = \phi \) with convention sup

\[ T \]

Note that applying (24) to (54), we have for any nonnegative measurable functional \( \psi \). The second equality follows from the fact (61)

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as \( q \) corresponds to the tree \( N \). We only need to prove the first one. Recall (18). We write \( \xi \) and the desired result follows from the fact that \( \psi_t^q : \eta_q + \cdot - \psi_t(\eta_q) \). Set \( T_0 = \{ \theta : 0 : q \in T_q \} \) and \( \Psi_q = \{ \psi_t^q : \theta \in T_q \} \). Then we have the following corollary.

Corollary 5.9. Assume that \( \psi_0 \) is critical. For \( q \in (t_{\infty}, 0) \), for any nonnegative measurable functional \( F \)

\[ N^\Psi[F(T_{A+t}) : t \in T_0^q] |A = q] = \psi_q(\eta_q)N^\Psi[F(t \in T_0^q)A] \]

Proof. Applying (24) to (54), we have for any nonnegative measurable functional \( F \) on \( T \),

\[ N^\Psi[F(T_A) : t \in T_0^q] |A = q] = \psi_q(\eta_q)N^\Psi[F(T_A)A] \]

Note that \( T_0^q = T_q - q \). Then the desired result follows from the fact that \( \{ \psi_t : t \in T_q \} \) and \( \{ \psi_t^q : t \in T_0^q \} \) induce the same pruning parameters.

An application of Corollary [5.9] is to study the distribution of exit times. Define

\[ A_h = \text{sup} \{ t \in T : H_{\text{max}}(T_t) > h \}, \quad h > 0, \]

with convention \( \text{sup} \emptyset = t_{\infty} \). Then \( A_h \) is the first time that the height of the trees is larger than \( h \).

Proposition 5.10. Assume that \( \psi_0 \) is critical. For \( t_{\infty} < q < q_0 < 0 \), we have

\[ N^\Psi[A_h > q_0 | A = q] = 1 - \psi_q(\eta_q)\psi_q^\Psi(\psi_q^\Psi(h)) \int_{\psi_q^\Psi(h)}^{\infty} \frac{dr}{\psi_q^\Psi(r)^2}. \]

\[ N^\Psi[A_h = A | A = q] = \psi_q(\eta_q)\psi_q^\Psi(\psi_q(\eta_q)) \int_{\psi_q(\eta_q)}^{\infty} \frac{dr}{\psi_q^\Psi(r)^2}. \]

Proof. The second equality follows from the fact \( N^\Psi[A_h > q_0 | A = q] = 1 \) and the first equality as \( q_0 \to q \). We only need to prove the first one. Recall (18). We write \( Z_a(T) \) to emphasize that \( Z_a \) corresponds to the tree \( T \). Note that

\[ N^\Psi[A_h > q_0 | A = q] = N^\Psi[Z_h(T_{q_0}) > 0 | A = q] = N^\Psi[Z_h(T_{A+q_0-q}) > 0 | A = q] \]

which, by Corollary [5.9] is equal to \( \psi_q^\Psi(\eta_q)N^\Psi[1_{\{Z_h(T_{q_0-q}) > 0\}}] \). Since for every \( t \in T_q \), \( \psi_t^q \) is subcritical, by (49), we have

\[ N^\Psi[A_h > q_0 | A = q] = \psi_q(\eta_q)N^\Psi[1_{\{Z_h(T_{q_0-q}) > 0\}}]N^\Psi[1_{\{Z_h > 0\}}] \]

\[ = \psi_q(\eta_q)N^\Psi[1_{\{Z_h(T_{q_0-q}) > 0\}}]N^\Psi[1_{\{Z_h > 0\}}] \]

\[ = \psi_q(\eta_q)N^\Psi[1_{\{Z_h(T_{q_0-q}) > 0\}} + \psi_q(\eta_q)N^\Psi[1_{\{Z_h = 0\}}] \int_0^h Z_a \frac{da}{Z_a} \]

\[ = 1 - \psi_q(\eta_q) \int_0^h \frac{da}{Z_a} \lim_{\lambda \to 0} N^\Psi[1_{\{Z_h = 0\}}] \int_0^h Z_a e^{-\lambda Z_a} \].
Recall (14), (21) and (22). Then by (20) and branching property (iv), conditioning on $\mathcal{Z}_a$, we see

$$\lim_{\lambda \to 0} \mathbb{N}^q_{\lambda} \left[ Z_a e^{-\lambda Z_h} \right] = \lim_{\lambda \to 0} \mathbb{N}^q_{\lambda} \left[ Z_a e^{-Z_a \psi^q_{\lambda} (h-a)} \right] = \frac{\partial}{\partial \lambda} \psi^q_{\lambda} (a, \psi^q_{\lambda} (h-a)).$$

Typically,

$$\frac{\partial}{\partial \lambda} \psi^q_{\lambda} (a, \psi^q_{\lambda} (h-a)) = \frac{\psi^q_{\lambda} (\psi^q_{\lambda} (a, \psi^q_{\lambda} (h-a)))}{\psi^q_{\lambda} (\psi^q_{\lambda} (h-a))} = \frac{\psi^q_{\lambda} (\psi^q_{\lambda} (h))}{\psi^q_{\lambda} (\psi^q_{\lambda} (h-a))} \frac{\partial}{\partial \lambda} \psi^q_{\lambda} (h-a).$$

Thus

$$\mathbb{N}^q_{\lambda} [A_h > q_0 | A = q] = 1 - \psi^q_{\lambda} (\eta_q) \int_0^h \frac{\partial}{\partial \lambda} \psi^q_{\lambda} (a, \psi^q_{\lambda} (h-a)) da$$

$$= 1 - \psi^q_{\lambda} (\eta_q) \psi^q_{\lambda} (\psi^q_{\lambda} (h)) \int_0^\infty \frac{dr}{\psi^q_{\lambda} (r)^2}.$$

We have completed the proof.

Remark 5.11. It is easy to see that $\mathbb{N}^q_{\lambda} [A_h \geq q] = \psi^q_{\lambda} (h)$.

Remark 5.12. With Theorem 4.2 and Remark 4.3 in hand, similarly to the work in [4], one may give an explicit construction of an increasing tree-valued process which has the same distribution as $\{T_q : q \in \Sigma_q\}$ under $\mathbb{N}^q_{\lambda}$. Then by this construction and similar arguments as in [4] (Theorem 4.6 there), one could know the joint distribution of $(T_{A_h}, T_{A_h})$ (and hence $(T_{A_h}, T_{A_h})$). We left these to interested readers.

6. TREE AT THE ASCENSION TIME

In this section, we study the representation of the tree at the ascension time. We shall always assume that

$$0 \in \Sigma, \quad t_\infty < 0, \quad \sup \Sigma > 0.$$

We first consider an infinite CRT and its pruning. An infinite CRT was constructed in [1] which is the Lévy CRT conditioned to have infinite height. Before recalling its construction, we stress that under $\mathbb{P}^\psi_0$, the root $\emptyset$ belongs to $\text{Br}_\infty$ and has mass $\Delta_\emptyset = \alpha$. We identify the half real line $[0, +\infty)$ with a real tree denoted by $[0, \infty]$ with the null mass measure. We denote by $d x$ the length measure on $[0, \infty]$. Let $\sum_{i \in I_1} \delta_{(x_i^*, T^* i)}$ and $\sum_{i \in I_2} \delta_{(x_i^*, T^* i)}$ be two independent Poisson random measures on $[0, \infty] \times \mathbb{T}$ with intensities

$$d x 2c\mathbb{N}^\psi_0 [d \mathbb{T}] \quad \text{and} \quad d x \int_0^\infty \lambda_0 (d l) \mathbb{P}^\psi_0 (d \mathbb{T}),$$

respectively. The infinite CRT from [1] is defined as:

$$\mathbb{T}^* = \emptyset \cup \sum_{i \in I_1} \delta_{(x_i^*, T^* i)}.$$

We denote by $\mathbb{P}^\psi_0 (d \mathbb{T}^*)$ the distribution of $\mathbb{T}^*$. Similarly to the setting in Section 4, we consider on $\mathbb{T}^*$ the mark processes $M_{skr}^* (d q, d y)$ and $\tilde{M}_{skr}^* (d q, d y)$ which are Poisson random measures on $\Sigma_0 \times \mathbb{T}^*$ with intensities:

$$\mathbb{1}_{\{ q \in \Sigma_0 \}} \beta_q dq \delta^*_T (d y) \quad \text{and} \quad \mathbb{1}_{\{ q \in \Sigma_0 \}} \sum_{i \in I_1} \sum_{x \in \text{Br}_\infty (T^* i)} m_{\Delta_x} (0, dq) \delta^*_x (d y),$$

where

$$m_{\Delta_x} (0, dq) \delta^*_x (d y) = m_{\Delta_x} (0, dq) \delta^*_x (d y).$$
respectively, with the identification of \( x^{\ast,i} \) as the root of \( T^{\ast,i} \). In particular nodes in \([0, \infty]\) with infinite degree will be charged by \( M_{\text{node}}^{T^{\ast}} \). Then set

\[
M^{T^{\ast}}(dq, dy) = M_{\text{ske}}^{T^{\ast}}(dq, dy) + M_{\text{node}}^{T^{\ast}}(dq, dy).
\]

For every \( q \in \mathfrak{T}_0 \), the pruned tree at time \( q \) is defined as:

\[
T_q^* = \{ x \in T^{\ast} \mid M^{T^{\ast}}([0, q] \times [\emptyset, x]) = 0 \},
\]

with the induced metric, root \( \emptyset \) and mass measure which is the restriction to \( T_q^* \) of the mass measure \( m^{T^{\ast}} \). Our main result in this section is the following theorem whose proof will be given at the end of this section.

**Theorem 6.1.** Suppose \( \psi_0 \) is critical. Given \( q \in (t_{\infty}, 0) \), if there exists \( \bar{q} \in \mathfrak{T}_0 \) such that \( \psi_q(\cdot) = \psi_\bar{q}(\eta_q + \cdot) \), then conditioned on \( \{ A = q \} \), \( T_A \) is distributed as \( T_{\bar{q}}^* \).

Proof. Delayed. \( \square \)

Before proving Theorem 6.1, we give some applications of it. Recall \( \mathfrak{T}_0^q = \{ \theta \geq 0 : \theta + q \in \mathfrak{T}_q \} \). By a similar reasoning as Corollary 5.9 we have the following corollary.

**Corollary 6.2.** Suppose \( \psi_0 \) is critical. Given \( q \in (t_{\infty}, 0) \), if there exists \( \bar{q} \in \mathfrak{T}_0 \) such that \( \psi_{q+t}(\cdot) = \psi_{\bar{q}+t}(\cdot) \) for all \( t \in \mathfrak{T}_0^q \), then conditioned on \( \{ A = q \} \), \( \{ T_{\bar{q}+t} : t \in \mathfrak{T}_0^q \} \) is distributed as \( \{ T_{\bar{q}+t}^* : t \in \mathfrak{T}_0^q \} \).

**Lemma 6.3.** Assume that \( \psi_0 \) is critical and for every \( t \in (t_{\infty}, 0) \), there exists \( \bar{t} \in \mathfrak{T} \) such that \( \psi_{\bar{t}}(\cdot) = \psi_t(\eta_t + \cdot) \). We further suppose \( t_{\infty} \notin \mathfrak{T}_\infty \). Then \( t \rightarrow \bar{t} \) is differentiable and

\[
\frac{d\bar{t}}{dt} = \frac{\zeta_t(\eta_t)\psi_t'(\eta_t) - \psi_t'(\eta_t)\zeta_t(\eta_t)}{\zeta_t'(0)\psi_t'(\eta_t)} =: -\gamma_t, \quad t \in (t_{\infty}, 0).
\]

Proof. It is obvious that \( t \rightarrow \bar{t} \) is differentiable. By \( \psi_{\bar{t}}(\cdot) = \psi_t(\eta_t + \cdot) \) and (57), we have for all \( \lambda > 0 \),

\[
\frac{d\bar{t}}{dt} = \frac{\zeta_t(\eta_t + \lambda)\psi_t'(\eta_t) - \psi_t'(\eta_t + \lambda)\zeta_t(\eta_t)}{\zeta_t(\lambda)\psi_t'(\eta_t)}.
\]

Letting \( \lambda \rightarrow 0 \) in above equality gives the desired result. \( \square \)

Define \( \bar{t}_{\infty} = \sup\{ \bar{t} : t \in \mathfrak{T}, b_t < 0 \} \). For \( t \in (0, \bar{t}_{\infty}) \), let \( \bar{t} \) be the unique negative number such that

\[
\bar{t} = \bar{t}.
\]

Let \( U \) be a positive “random” variable with nonnegative “density” with respect to the Lebesgue measure given by

\[
1_{\{t \in (0, \bar{t}_{\infty})\}} \frac{\zeta_t(\eta_t)\zeta_t'(0)}{\psi_t'(\eta_t)\gamma_t}.
\]

Assume that \( U \) is independent of \( T^* \).

**Corollary 6.4.** Suppose that all assumptions in Lemma 6.3 hold. Then \( T_A \) is distributed under \( N^\Psi \) as \( T_{\bar{U}}^* \).

**Remark 6.5.** If \( U \) has the same distribution as \( A \), then we have \( T_A \) is distributed as \( T_{\bar{U}}^* \).
Proof. Recall (6.7). By Lemma 6.5 if we have the law of $A$ under $N^\Psi$ has a density with respect to the Lebesgue measure on $\mathbb{R}$ give by

$$1 \{t \in (t_{\infty}, 0)\} \frac{\zeta_t(\eta_t)}{\psi'_t(\eta_t)}.$$ 

Thus for any nonnegative measurable function $F$ on $\mathbb{T}$, by Theorem 6.1

$$N^\Psi[F(T_A)] = \int_{(t_{\infty}, 0)} E^{s, \psi_0}[F(T_t^\ast)] \frac{\zeta_t(\eta_t)}{\psi'_t(\eta_t)} dt = \int_{(t_{\infty}, 0)} E^{s, \psi_0}[F(T_t^\ast)] \frac{\zeta_t(\eta_t) dt}{\psi'_t(\eta_t) \gamma_t} = \int_{0, t_{\infty}} E^{s, \psi_0}[F(T_t^\ast)] \frac{\zeta_t(\eta_t) \zeta'_t(0) dt}{\psi'_t(\eta_t) \gamma_t} = E^{s, \psi_0}[F(T_{t_{\infty}}^\ast)],$$

where the fourth equality follows from Lemma 6.3. We have completed the proof. \hfill \Box

By Corollary 6.2 and Corollary 6.4 we have the following result which is a generalization of Corollary 8.2 in [1].

**Corollary 6.6.** Suppose that $\psi_0$ is critical and $[0, \infty) \subset \Sigma$. If for every $q \in (t_{\infty}, 0)$, there exists $\bar{q} \in \Sigma_0$ such that $\psi_{q+t} = \psi_{\bar{q}+t}$ for all $t \in \Sigma_0$. We further assume that $t_{\infty} \notin \Sigma_\infty$. Then $\{T_{q+t} : t \geq 0\}$ is distributed under $N^\Psi$ as $\{T^\ast_{q+t} : t \geq 0\}$.

In the following we give some examples.

**Example 6.7.** We first consider the case studied by Abraham and Delmas in [1]. Let $\psi$ be defined in (1). Recall $\psi^q(\lambda) = \psi(q + \lambda) - \psi(q), q \in \Theta^\psi$, where $\Theta^\psi$ is set of $q \in \mathbb{R}$ such that $\int_{1, \infty} re^{-qz} m(dr) < \infty$. In [1], it was assumed that $0 > \theta_{\infty} := \inf \Theta^\psi \notin \Theta^\psi$ and $\psi$ is critical. $\{\psi^q : q \in \Theta^\psi\}$ satisfies all assumptions in Corollary 6.6. Then for $t \in (\theta_{\infty}, 0), \eta_t = t - t$ and

$$\zeta_t(\lambda) = 2c\lambda + \int_0^\infty (1 - e^{-z\lambda}) e^{-zt} zm(dz) = \psi(t + \lambda) - \psi(t).$$

Using $\eta_t = t - t$, we shall see

$$1 \{t \in (0, \delta_{\infty})\} \frac{\zeta_t(\eta_t) \zeta'_t(0)}{\psi'_t(\eta_t) \gamma_t} = 1 \{t \in (0, \delta_{\infty})\} \left( 1 - \frac{\psi'(t)}{\psi'(t)} \right).$$

Then we go back to Corollary 8.2 in [1].

**Example 6.8.** Let $b > 0$ and $c > 0$ be two constants. Define $\psi_q(\lambda) = q b + c \lambda^2, q \in \mathbb{R}, \lambda \geq 0$. Then $\{\psi_q : q \in \mathbb{R}\}$ satisfies all assumptions in Corollary 6.6. Typically, if $b = 2c$, we have $\psi_q(\cdot) = \psi_0(q + \cdot) - \psi_0(q)$.

**Example 6.9.** Let $\Sigma^- \subset (-\infty, 0]$ be an interval and let $\{\psi_q : q \in \Sigma^-\}$ be an admissible family of branching mechanisms with the parameters $(b_q, m_q)$. Assume that $0 \in \Sigma^-, \psi_0$ is critical and $\inf \Sigma^- \notin \Sigma^\infty$. Let $\eta_q$ denote the largest root of $\psi_q(s) = 0$. For $q \in -\Sigma^- := \{t : t \in \Sigma^-\}$, define $\psi_q(\cdot) = \psi_q(\eta_q + \cdot)$. Then we have $\{\psi_q : q \in \Sigma^- \cup (-\Sigma^-)\}$ is an admissible family of branching mechanisms such that for $q \in -\Sigma^-

$$b_q = b_{-q} + 2c\eta_{-q} + \int_0^\infty (1 - e^{-z\eta_q}) zm_{-q}(dz), \quad m_q = e^{-z\eta_q} m_{-q}(dz).$$
Typically, \( \{ \psi_q : q \in \mathbb{Z}^+ \cup \{-\mathbb{Z}^+\} \} \) satisfies all assumptions in Corollary \[6.1\]

The end of this section is devoted to the proof of Theorem \[6.1\].

**Proposition 6.10.** Suppose \( \psi_0 \) is critical. Then for any nonnegative measurable functional \( F \) on \( \mathbb{T} \) and for every \( q \in \mathbb{R}_0 \),

\[
\psi'_q(0) N^\Psi [\sigma_q F(T_q)] = E^*_{\psi_q} [F(T_q^*)] .
\]

**Proof.** First, by Bismut decomposition (Theorem 4.5 in \[11\] or Theorem 2.17 in \[3\]), we have that there exists some measurable functional \( \bar{F} \) on \([0, \infty) \times \mathbb{T} \) such that

\[
\psi'_q(0) N^\Psi [\sigma_q F(T_q)] = \psi'_q(0) N^\Psi_q [\sigma F(T)]
\]

where under \( E \), \( \sum_{i \in I} \delta_{(z_i, \hat{T}_i)}(dz, dT) \) is a Poisson random measure on \([0, \infty) \times \mathbb{T} \) with intensity

\[
dz \left( 2cN^{\Psi_0}[dT] + \int_0^\infty l\eta_q(m_d)(dr) P^{\psi_0}_q(dT) \right).
\]

For \( i \in I^*_1 \cup I^*_2 \), define

\[
T_q^* = \{ x \in T^* : M^{T^*}_q([0, q] \times [0, x]) = 0 \}, \quad q \in \mathbb{R}_0.
\]

With abuse of notation, we have

\[
T_q^* = \{ [0, x], \xi \} \otimes_{i \in I^*_1 \cup I^*_2}, \xi < x \left( x^* \downarrow, T_q^* \downarrow \right),
\]

where

\[
\xi = \sup \{ x \in [0, +\infty] : M^{T^*}_q([0, q] \times [0, x]) = 0 \}
\]

\[
= \sup \{ x \in [0, +\infty] : M^{T^*}_{\text{ske}}([0, q] \times [0, x]) = 0 \} \land \inf \{ x^* \downarrow : M^{T^*}_{\text{node}}([0, q] \times \{ x^* \}) > 0 \}
\]

\[
=: \xi^*_1 \land \xi^*_2.
\]

By \[63\] and \[65\], to prove \[62\], we only need to show that \( \xi \) is exponentially distributed with parameter \( \psi'_q(0) \). Obviously, \( \xi^*_1 \) is exponentially distributed with parameter \( \int_0^q \beta_\theta d\theta \). According to Corollary \[1.4\] and property of Poisson random measure, we have

\[
\sum_{i \in I^*_2} \{ M^{T^*}_{\text{node}}([0, q] \times \{ x^* \}) > 0 \} \delta_{(x^* \downarrow, T_q^* \downarrow)}(dx, dT)
\]

is a Poisson random measure with intensity \( dx \int_0^q d\theta \int_0^\infty z\eta(dz) P^{\psi_0}_q(dT) \). Then one can deduce that \( \xi^*_2 \) is exponentially distributed with parameter \( \int_0^q d\theta \int_0^\infty z\eta(dz) \). Hence \( \xi \) is exponentially distributed with parameter

\[
\int_0^q \beta_\theta d\theta + \int_0^q d\theta \int_0^\infty z\eta(dz),
\]

which, by \[5\], is just \( b_q = \psi'_q(0) \). We have completed the proof. \( \square \)

**Proof of Theorem 6.1.** For any nonnegative measurable function \( F \) on \( \mathbb{T} \), by \[58\], we have for \( q < 0 \),

\[
N^\Psi [F(T_A)|A = q] = \psi'_q(\eta_q) N^\Psi_q [F(T)\sigma 1_{\sigma < \infty}]
\]

\[
= \psi'_q(\eta_q) N^\Psi_q [F(T)\sigma], \quad q < 0.
\]

(66)
where the last equality follows from (24). Since \( \psi_q(\cdot) = \psi_q(\eta_q + \cdot) = \psi_q(\eta_q) = \psi'_q(0) \), Proposition 6.10 yields

\[
N^\Psi[F(T_A)|A = q] = \psi'_q(0)N^\Psi[\sigma_q F(T_q)] = E^{\ast,\psi_0}[F(T^\ast_q)].
\]

We have completed the proof. \( \square \)

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