Additivity of the Classical Capacity of Entanglement-Breaking Quantum Channels

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Abstract

We show that for the tensor product of an entanglement-breaking quantum channel with an arbitrary quantum channel, both the minimum entropy of an output of the channel and the Holevo-Schumacher-Westmoreland capacity are additive. In addition, for the tensor product of two arbitrary quantum channels, we give a bound involving entanglement of formation for the amount of subadditivity (for minimum entropy output) or superadditivity (for classical capacity) that can occur.

One of the more important open questions of quantum information theory is the determination of the capacity of a quantum channel for carrying classical information. This question has been only partially resolved. If entanglement between multiple inputs to the channel is not allowed, a formula for the classical capacity of a quantum channel has indeed been discovered \[ \chi^*(\Psi) = \max_{p_i, \rho_i} H\left( \sum_i \Psi(p_i \rho_i) \right) - \sum_i p_i H\left( \Psi(\rho_i) \right) \] (1)

where \( H \) is the von Neumann entropy \( H(\rho) = -\operatorname{Tr} \rho \log \rho \), and where the maximization is over probability distributions \( p_i \) on density matrices \( \rho_i \) over the input space of the channel. This maximum can be attained because we need at most \( d^2 \) density matrices \( \rho_i \) to achieve any attainable value of

\[ \chi(\{p_i, \rho_i\}) = H\left( \sum_i \Psi(p_i \rho_i) \right) - \sum_i p_i H\left( \Psi(\rho_i) \right) \] (2)

and are thus maximizing over a compact space. The general capacity of a quantum channel \( \Psi \), without feedback or prior entanglement between sender and receiver, but possibly using entangled inputs, is

\[ C(\Psi) = \lim_{n \to \infty} \frac{1}{n} \chi^*(\Psi^\otimes n), \] (3)

i.e., the limit for large \( n \) of the capacity when we permit the input to be entangled over blocks of \( n \) channel uses. This limit can be shown to exist because \( \chi^* \) satisfies the superadditivity condition

\[ \chi^*(\Psi \otimes \Phi) \geq \chi^*(\Psi) + \chi^*(\Phi). \] (4)

It is conjectured that equality holds, i.e., that \( \chi^* \) is additive, in which case \( \chi^* \) would give the classical capacity of a quantum channel without feedback. Substantial work has been done
on this conjecture [3, 4], and it has been proven for several special cases. In particular, it has been proven when one of the channels is the identity channel [4, 5], when one of the channels is what A. S. Holevo calls a c-q or q-c channel (these terms will be defined later) [6, 7], and when one of the channels is a unital qubit channel [8].

We will prove additivity for the special case where one of the two channels is entanglement breaking. Entanglement breaking channels are channels which destroy entanglement with other quantum systems. That is, when the input state is entangled between the input space $H_{in}$ and another quantum system $H_{ref}$, the output of the channel is no longer entangled with the system $H_{ref}$. Both c-q and q-c channels are special cases of entanglement breaking channels. A c-q channel is a channel which can be expressed by the composition of a complete von Neumann measurement on the input space followed by an arbitrary completely positive trace-preserving (CPT) map. A q-c channel can be expressed as the composition of a CPT map followed by a complete von Neumann measurement on the output space. Stated more intuitively, for c-q maps, the input can be treated as being classical, and for q-c maps, the output can be taken to be classical. In either case, the von Neumann measurement eliminates any entanglement between the input space and another system, so c-q and q-c maps are both special cases of entanglement breaking channels. In a conversation with the author, Michal Horodecki [9] gave a simple proof that any entanglement breaking channel can be expressed as a q-c-q channel; that is, the composition of a CPT operator followed by a complete von Neumann measurement followed by another CPT operator. (See also [10] for details of this proof.) As a consequence, the action of an entanglement breaking channel $\Phi$ on a state $\rho$ can always be written in the following form introduced by Holevo [6]:

$$\Phi(\rho) = \sum_i \text{Tr}(X_i \rho) \theta_i$$

where $\{X_i\}$ form a general POVM and $\{\theta_i\}$ are arbitrary states. For a c-q map, $X_i = |i\rangle \langle i|$ where $|i\rangle$ form an orthonormal basis, and for a q-c map $\theta_i = |i\rangle \langle i|$. The additivity problem for capacity is closely related to another additivity problem; that of the minimum entropy output of a channel [1]. For the case of entanglement breaking channels, we first found the additivity proof for the minimum entropy output, and then discovered a straightforward way to extend this additivity proof to cover the classical capacity. In this paper, we first give the proof for additivity of minimum entropy output, as this proof contains the important ideas for the capacity proof, but has significantly fewer technicalities.

**Theorem 1** For an arbitrary quantum channel $\Psi$, and an entanglement breaking channel $\Phi$

$$\min_{\rho_{AB}} H((\Psi \otimes \Phi)(\rho_{AB})) = \min_{\rho_A} H(\Psi(\rho_A)) + \min_{\rho_B} H(\Phi(\rho_B)).$$  \hspace{1cm} (6)

**Proof:** The left-hand side is clearly at most the right-hand side, as can be seen by choosing $\rho_{AB} = \rho_A \otimes \rho_B$. We would like to show that it is at least the right-hand side. We use the strong subadditivity property of von Neumann entropy [3]. Consider the minimum obtainable value of $H((\Psi \otimes \Phi)(\rho_{AB}))$. Because $\Phi$ is entanglement breaking,

$$(I \otimes \Phi)(\rho_{AB}) = \sum_j q_j |a_j\rangle \langle a_j| \otimes |b_j\rangle \langle b_j|$$

for some $q_j, |a_j\rangle \in H_A$ and $|b_j\rangle \in H_B$. Now, we apply to the state

$$\sigma_{ABC} = \sum_j q_j \Psi(|a_j\rangle \langle a_j|) \otimes |b_j\rangle \langle b_j| \otimes |j\rangle \langle j|$$

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the property of strong subadditivity in the form
\[ H(\sigma_{AB}) \geq H(\sigma_{ABC}) - H(\sigma_{BC}) + H(\sigma_B). \] (9)

We have
\[ \sigma_{AB} = \sum_j q_j \Psi(\langle a_j | a_j \rangle) \otimes | b_j \rangle \langle b_j | \] (10)
\[ = (\Psi \otimes \Phi)(\rho_{AB}), \]
the quantity for the entropy of which we would like a lower bound. Now, note that
\[ H(\sigma_{ABC}) - H(\sigma_{BC}) = H(\sigma_{AC}) - H(\sigma_C) \] (11)
\[ = \sum_j q_j H(\Psi(\langle a_j | a_j \rangle)). \] (12)

Now, note that
\[ \sigma_B = \sum_j q_j | b_j \rangle \langle b_j | \] (13)
\[ = \text{Tr}_A(I \otimes \Phi)(\rho_{AB}) \]
\[ = \Phi(\text{Tr}_A\rho_{AB}) \]

Putting the above equalities together, we see that
\[ H((\Psi \otimes \Phi)(\rho_{AB})) \geq \sum_j q_j H(\Psi(\langle a_j | a_j \rangle)) + H(\Phi(\text{Tr}_A\rho_{AB})). \] (14)

Since \( \sum_j q_j = 1 \), the right-hand side is clearly at least the sum of the minimum output entropies of \( \Psi \) and of \( \Phi \). We have thus shown that the minimum output entropy is additive for the tensor product of two channels if one of the channels is an entanglement breaking channel. \( \square \)

We now prove the corresponding additivity result for the Holevo-Schumacher-Westmoreland capacity \( \chi^* \); recall
\[ \chi^*(\Psi) = \max_{p_i, \rho_i} H\left(\Psi\left(\sum_i p_i \rho_i\right)\right) - \sum_i p_i H\left(\Psi(\rho_i)\right) \] (15)
over probability distributions \( p_i \) and density matrices \( \rho_i \).
Theorem 2  For an arbitrary quantum channel \( \Psi \), and an entanglement breaking channel \( \Phi \)

\[
\chi^*(\Psi \otimes \Phi) = \chi^*(\Psi) + \chi^*(\Phi)
\]  

(16)

Proof: The capacity \( \chi^* \) is composed of two terms. We will be treating these two terms separately. For the second term, additivity is shown in essentially the same way as in the proof of additivity for minimum entropy, and for the first term, additivity follows from the subadditivity of von Neumann entropy.

Again, we assume that we have an arbitrary quantum channel \( \Psi \), and an entanglement breaking channel \( \Phi \). We use strong subadditivity. Consider the optimal signal states for \( \Psi \otimes \Phi \), i.e., the \( p_i \) and \( \rho_i \) such that

\[
\chi^*(\Psi \otimes \Phi) = H((\Psi \otimes \Phi)(\rho)) - \sum p_i H((\Psi \otimes \Phi)(\rho_i))
\]  

(17)

where \( \rho = \sum p_i \rho_i \). Let us consider the state \( (I \otimes \Phi)(\rho_i) \). Because \( \Phi \) is an entanglement breaking map, this state is separable, and so

\[
(I \otimes \Phi)(\rho_i) = \sum_j q_{ij} |a_{ij} \rangle \langle a_{ij}| \otimes |b_{ij} \rangle \langle b_{ij}|
\]  

(18)

for some \( q_{ij}, |a_{ij} \rangle, |b_{ij} \rangle \). Now, we apply strong subadditivity to the state

\[
\sigma_{ABC} = \sum_j q_{ij} |a_{ij} \rangle \langle a_{ij}| \otimes |b_{ij} \rangle \langle b_{ij}| \otimes |j \rangle \langle j|
\]  

(19)

To simplify notation, we let the dependence of \( \sigma \) on \( i \) be implicit. Again, we apply strong subadditivity in the form

\[
H(\sigma_{AB}) \geq H(\sigma_{ABC}) - H(\sigma_{BC}) + H(\sigma_B)
\]  

(20)

As before,

\[
H(\sigma_{AB}) = H((\Psi \otimes \Phi)(\rho_i)).
\]  

(21)

We also have that

\[
H(\sigma_B) = H(\Phi(\text{Tr}_A p_i))
\]  

(22)

and

\[
H(\sigma_{ABC}) - H(\sigma_{BC}) = \sum_j q_{ij} H(\Psi(|a_{ij} \rangle \langle a_{ij}|)).
\]  

(23)

We let \( |a_{ij} \rangle \langle a_{ij}| = \tau_{ij} \). Then \( \text{Tr}_B p_i = \sum_j q_{ij} \tau_{ij} \). Combining the terms, we observe

\[
H((\Psi \otimes \Phi)(\rho_i)) \geq \sum_j q_{ij} H(\Psi(\tau_{ij})) + H(\Phi(\text{Tr}_A p_i))
\]  

(24)

Now, let us sum over all the states \( \rho_i \). We obtain

\[
\sum_i p_i H((\Psi \otimes \Phi)(\rho_i)) \geq \sum_{i,j} p_i q_{ij} H(\Psi(\tau_{ij})) + \sum_i p_i H(\Phi(\text{Tr}_A p_i)).
\]  

(25)

Using subadditivity of von Neumann entropy and the above inequality (25), we get that

\[
\chi^*(\Psi \otimes \Phi) = H((\Psi \otimes \Phi)(\rho)) - \sum_i p_i H((\Psi \otimes \Phi)(\rho_i))
\]  

\[
\leq H(\Psi(\text{Tr}_B \rho)) + H(\Phi(\text{Tr}_A p_i)) - \sum_{i,j} p_i q_{ij} H(\Psi(\tau_{ij})) - \sum_i p_i H(\Phi(\text{Tr}_A p_i)).
\]  

(26)
However, since
\[ \sum_{i,j} p_i q_j \tau_{ij} = \sum_i p_i \text{Tr}_B \rho_i = \text{Tr}_B \rho \quad \text{and} \quad \sum_i p_i \text{Tr}_A \rho_i = \text{Tr}_A \rho, \] (27)
we see that
\[ \chi^*(\Psi \otimes \Phi) \leq \chi^*(\Psi) + \chi^*(\Phi). \] (28)

As the opposite inequality is easy, we have additivity of \( \chi^* \) for entanglement breaking channels. \( \square \)

We finally give a bound on the amount of superadditivity for general channels. For this, we need to define the entanglement of formation of a bipartite state. This is another quantity that is conjectured to be additive, but for which additivity has not been proved. Entanglement of formation for a bipartite state \( \rho_{AB} \) is defined
\[ E_F(\rho_{AB}) = \min_{p_i, \rho_i} \sum_i p_i H(\text{Tr}_A \rho_i) \] (29)
where the minimization is over probability distributions \( p_i \) on rank-one density matrices \( \rho_i \) such that \( \sum_i p_i \rho_i = \rho_{AB} \).

**Theorem 3** Suppose we have two quantum channels, i.e., completely positive trace preserving maps, \( \Psi \) and \( \Phi \). Then
\[ \min_{\rho_{AB}} H\left( (\Psi \otimes \Phi)(\rho_{AB}) \right) \geq \min_{\rho_A} H\left( \Psi(\rho_A) \right) + \min_{\rho_B} H\left( \Phi(\rho_B) \right) \] (30)
and
\[ \chi^*(\Psi \otimes \Phi) \leq \chi^*(\Psi) + \chi^*(\Phi) + \max_{\rho_{AB}} E_F\left( (I \otimes \Phi)(\rho_{AB}) \right). \] (31)

Note that the formulation of the theorem is asymmetric with respect to \( \Psi \) and \( \Phi \). Thus, to bound the amount of sub- or superadditivity, one can use either the entanglement of formation of \( (I \otimes \Phi)(\rho_{AB}) \) or of \( (\Psi \otimes I)(\rho_{AB}) \), whichever is smaller.

**Proof:** We first give the proof of the first part of Theorem 3. Let
\[ (I \otimes \Phi)(\rho_{AB}) = \sum_i q_i \nu_i \] (32)
be the decomposition of \( (I \otimes \Phi)(\rho_{AB}) \) into pure states \( \nu_i \) that minimizes entanglement of formation, i.e., so that \( \sum_i q_i H(\text{Tr}_A \nu_i) \) is minimum. Now, we consider
\[ \sigma_{ABC} = \sum_j q_j (\Psi \otimes I)(\nu_j) \otimes |j\rangle \langle j| \] (33)
and apply strong subadditivity to this state. We obtain
\[ H(\sigma_{AB}) \geq \left( H(\sigma_{ABC}) - H(\sigma_C) \right) + \left( H(\sigma_B) - H(\sigma_{BC}) - H(\sigma_C) \right). \] (34)
As in (10), we have
\[ H(\sigma_{AB}) = H\left( (\Psi \otimes \Phi)(\rho_{AB}) \right). \] (35)
Similar to (13), we get
\[ H(\sigma_B) = H\left( \sum_j q_j \text{Tr}_A \nu_j \right) = H\left( \Phi(\text{Tr}_A \rho_{AB}) \right) \] (36)
Furthermore, the choice of $\nu_j$ and the definition of $E_F$ gives
\[
H(\sigma_{BC}) - H(\sigma_C) = E_F \left( (I \otimes \Phi)(\rho_{AB}) \right)
\]
Finally application of the entropy chain rule (12) gives
\[
H(\sigma_{ABC}) - H(\sigma_C) = \sum_j q_j H \left( (\Psi \otimes I)(\nu_j) \right)
\]
The expression (36) is bounded below by $\min_\rho H(\Phi(\rho))$. The second expression (37) is bounded above by $\max_\rho E_F ((I \otimes \Phi)(\rho))$. The third expression (38) is bounded below by $\min_\rho H(\Psi(\rho))$. Combining these three expressions give the first part of Theorem 3.

To prove the second part of the theorem, (38) must be replaced by
\[
H(\sigma_{ABC}) - H(\sigma_C) \geq \sum_{j,k} q_j r_{jk} H \left( (\Psi \otimes I)(\nu_{jk}) \right)
\]
for states $|v_{jk}\rangle$ and probabilities $q_j r_{jk}$ such that
\[
\sum_{j,k} q_j r_{jk} |v_{jk}\rangle = \text{Tr}_B \nu_j.
\]
We then consider the signal states $\rho_i$ and the associated probabilities $p_i$ which give the value of $\chi^*(\Psi \otimes \Phi)$ in Equation (1), and let $\sum_i p_i \rho_i = \rho$. We now use expressions (36), (37), (39) with $\rho_i$ in the place of $\rho_{AB}$. Combining these three expressions yields
\[
H \left( (\Psi \otimes \Phi)(\rho_i) \right) \geq H \left( \Phi(\text{Tr}_A \rho_i) \right) + \sum_{j,k} q_j r_{jk} H \left( (\Psi(|v_{jk}\rangle|v_{jk}\rangle)) \right)
\]

The second part of Theorem 3 then follows in a way entirely analogous to the proof of Theorem 2. We use the equalities
\[
\text{Tr}_B \rho = \sum_i p_i \text{Tr}_A \rho_i
\]
and
\[
\text{Tr}_A \rho = \sum_{i,j,k} p_i q_j r_{jk} |v_{jk}\rangle |v_{jk}\rangle |v_{jk}\rangle.
\]
and expand $\chi^*(\Psi \otimes \Phi)$ similarly to Eq. (26) to obtain Eq. (31).

We still must prove the inequality (39). The left hand side of (39) is
\[
\sum_j q_j H \left( (\Psi \otimes I)(\nu_j) \right)
\]
Now, $\nu_j$ is a purification of $\text{Tr}_B \nu_j$, and $H \left( (\Psi \otimes I)(\nu_j) \right) = H \left( (\Psi \otimes I)(\nu_j) \right)$ for any other quantum state $\tau$ which is a purification of $\sigma_j = \text{Tr}_B \nu_j$. Let $\sigma_j = \sum_k q_j |v_{jk}\rangle |v_{jk}\rangle |v_{jk}\rangle$ be the eigenvector decomposition of $\sigma_j$. A different purification is
\[
\tau_j = \left( \sum_k q_j |v_{jk}\rangle \otimes |k\rangle \otimes |k\rangle \right) \left( \sum_k q_j |v_{jk}\rangle \otimes |k\rangle \otimes \langle k| \right)
\]
It suffices to show that
\[
H\left(\left(\Psi \otimes I\right)(\tau_j)\right) \geq H\left(\text{Tr}_3\left(\Psi \otimes I\right)(\tau_j)\right) - H\left(\text{Tr}_{12}\left(\Psi \otimes I\right)(\tau_j)\right)
\] (46)
as the first term in the above equation is \(H\left(\left(\Psi \otimes I\right)(\nu_j)\right)\), the second is \(H\left(\left\{q_{jk}\right\}\right) + \sum_k q_{jk} H\left(\Psi(\left|v_{jk}\right\rangle \langle v_{jk}\right|)\right)\), and the third is \(H\left(\left\{q_{jk}\right\}\right)\). However, the above equation follows from the inequality \(H(\rho_{34}) \geq H(\rho_3) - H(\rho_4)\), which is a consequence (after another purification) of the subadditivity property of entropy \([13]\). □

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