Dispersionless DKP hierarchy and the elliptic Löwner equation

V Akhmedova\(^1\) and A Zabrodin\(^{1,2,3}\)

\(^1\)National Research University Higher School of Economics, International Laboratory of Representation Theory and Mathematical Physics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russia
\(^2\)Institute of Biochemical Physics, 4 Kosygina street, Moscow 119334, Russia
\(^3\)ITEP, 25 B. Cheremushkinskaya, Moscow 117218, Russia

E-mail: valeria-58@yandex.ru and zabrodin@itep.ru

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Abstract

We show that the dispersionless DKP hierarchy (the dispersionless limit of the Pfaff lattice) admits a suggestive reformulation through elliptic functions. We also consider one-variable reductions of the dispersionless DKP hierarchy and show that they are described by an elliptic version of the Löwner equation. With a particular choice of the driving function, the latter appears to be closely related to the Painlevé VI equation with a special choice of parameters.

Keywords: integrable hierarchies, Löwner equation, elliptic functions

1. Introduction

The DKP hierarchy is one of the integrable hierarchies with \(\infty\) symmetries introduced by Jimbo and Miwa in 1983 [1]. It was subsequently rediscovered and came to be also known as the coupled KP hierarchy [2] and the Pfaff lattice [3, 4], see also [5–7]. The latter name is motivated by the fact that some solutions to the hierarchy are expressed through Pfaffians. The solutions and the algebraic structure were studied in [8–10], the relation to matrix integrals was elaborated in [3–5, 11, 12]. Bearing certain similarities with the KP and Toda chain hierarchies, the DKP one is essentially different and less well understood.

The dispersionless version of the DKP hierarchy (the dDKP hierarchy) was suggested in [13, 14]. It is an infinite system of differential equations

\[
e^{D(z)D(\zeta)F} \left( 1 - \frac{1}{\zeta} e^{2\partial_z (2\partial_{\zeta} + D(z) + D(\zeta))F} \right) = 1 - \frac{\partial_z D(z)F - \partial_{\zeta} D(\zeta)F}{\zeta - \zeta},
\]

(1)
\[ e^{-D(z)D(\zeta)F} \frac{z^2 e^{-2\partial_0 D(z)F} - \zeta^2 e^{-2\partial_0 D(\zeta)F}}{z - \zeta} = z + \zeta - \partial_0 \left( 2\partial_0 + D(z) + D(\zeta) \right) F \]  

(2)

for the function \( F = F(t) \) of the infinite number of (real) 'times' \( t = \{ t_0, t_1, t_2, \ldots \} \), where

\[ D(z) = \sum_{k=1}^{\infty} \frac{z^{-k}}{k} \partial_t. \]  

(3)

The differential equations are obtained by expanding equations (1), (2) in powers of \( z, \zeta \). For example, the first two equations of the hierarchy are

\[
\begin{align*}
6F_{11}^2 + 3F_{22} - 4F_{13} &= 12e^{4F_{00}}, \\
2F_{03} + 4F_{01}^3 + 6F_{01}F_{11} - 6F_{01}F_{02} &= 3F_{12}.
\end{align*}
\]

Here and below we use the short-hand notation \( F_{mn} \equiv \partial_m \partial_n F \). Note that the first equation with 0 in the right-hand side, i.e., \( 6F_{11}^2 + 3F_{22} - 4F_{13} = 0 \), is the dispersionless KP (dKP) (Khokhlov–Zabolotskaya) equation written in the Hirota form.

In this paper we study the dDKP hierarchy. The aim of the paper is two-fold. First, we show that somewhat unsightly looking equations (1), (2), when rewritten in an elliptic parametrization in terms of Jacobi’s theta-functions \( \theta_\tau(u|\tau) \), assume a nice and suggestive form which looks like a natural elliptic extension of the dKP hierarchy

\[ (\zeta^{-1} - \xi^{-1}) e^{\partial_0 + D(z) + D(\zeta)} F = \frac{\theta'(u(z) - u(\zeta)|\tau)}{\theta_4(u(z) - u(\zeta)|\tau)}. \]  

(4)

Here the function \( u(z) \) is defined by

\[ e^{\partial_0 + D(z)} F = z \frac{\theta_4'(u(z)|\tau)}{\theta_4(u(z)|\tau)}. \]  

(5)

The modular parameter \( \tau \) is a dynamical variable: \( \tau = \tau(t) \). This feature suggests some similarities with the genus 1 Whitham equations [15] and the integrable structures behind boundary value problems in plane doubly-connected domains [16].

Second, we investigate one-variable reductions of the dDKP hierarchy assuming that all dynamical variables depend on the times \( t \) through a single variable which in a generic case can be identified with the modular parameter \( \tau \). We show that such reductions are classified by solutions of a differential equation which is an elliptic analogue of the famous Löwner equation (see, e.g., [17 chapter 6]). In complex analysis, this ‘elliptic Löwner equation’ is also known as the Goluzin–Komatu equation [18, 19], see also [20–23]

\[ 4\pi \partial_\tau u(z, \tau) = -E^{(1)} \left( u(z, \tau) + \xi(\tau) \right) \left[ \frac{\xi'}{\xi} \right] + E^{(1)} \left( \xi(\tau) \right), \]  

(6)

where \( E^{(1)}(u, \tau) := \partial_\tau \log \theta_4(u|\tau) \) and \( \xi(\tau) \) is an arbitrary (continuous) function of \( \tau \) (the ‘driving function’). This equation is the basic element of the theory of parametric conformal maps from doubly connected slit domains to annuli. During the last decade, the interest to this topic was renewed in connection with the Schramm–Löwner evolution (SLE); for the SLE in an annulus see [24, 25]. A similar relation between the chordal Löwner equation and one-variable reductions of the dKP hierarchy was known since the seminal papers by Gibbons and Tsarev [26, 27]. Further developments are discussed in [28–32].

Finally, we point out an unexpected connection with the Painlevé VI equation. Namely, we show that the second \( \tau \)-derivative of the elliptic Löwner equation (6), with a particular
choice of the driving function, gives the Painlevé VI equation with special values of the parameters written in the elliptic (‘Calogero-like’) form.

2. The dispersionless DKP hierarchy

**Algebraic formulation.** In what follows we will use the differential operator

\[ V(z) = \partial_z + D(z) \]  

which in the dDKP case is more convenient than \( D(z) \). Introducing the functions

\[ p(z) = z - \partial_z V(z) F, \quad w(z) = z^2 e^{-2\partial_z V(z) F}, \]

we can rewrite equations (1), (2) in a more compact form

\[ e^{D(z)D(\xi) F} \left( 1 - \frac{1}{w(z)w(\xi)} \right) = \frac{p(z) - p(\xi)}{z - \xi}, \]

\[ e^{-D(z)D(\xi) F + 2\partial_z F} \frac{w(z) - w(\xi)}{z - \xi} = p(z) + p(\xi). \]

Multiplying the two equations, we get the relation

\[ p^2(z) - e^{2F(w(z) + w^{-1}(z))} = p^2(\xi) - e^{2F(w(\xi) + w^{-1}(\xi))} \]

from which it follows that \( p^2(z) - e^{2F(w(z) + w^{-1}(z))} \) does not depend on \( z \). Tending \( z \) to infinity, we find that this expression is equal to \( F_{02} - 2F_{11} - F_{01}^2 \). Therefore, we conclude that \( p(z), w(z) \) satisfy the algebraic equation [14]

\[ p^2(z) = R^2 (w(z) + w^{-1}(z)) + V, \]  

where

\[ R = e^{F_01}, \quad V = F_{02} - 2F_{11} - F_{01}^2. \]

This equation defines an elliptic curve, with \( w, p \) being algebraic functions on this curve. The functions \( w \) and \( p \) have respectively a double pole and a simple pole at infinity.

**Elliptic formulation.** A natural further step is to uniformize the curve through elliptic functions. To this end, we use the standard Jacobi theta-functions \( \theta_a(u) = \theta_a(u|\tau) \) \((a = 1, 2, 3, 4)\). Their definition and basic properties are listed in the appendix.

The elliptic parametrization of (11) is as follows:

\[ w(z) = \frac{\theta^2_4(u(z))}{\theta^2_2(u(z))}, \quad p(z) = \gamma \theta^2_4(0) \theta_2(u(z)) \theta_4(u(z)) \theta_1(u(z)) \theta_3(u(z)), \]

where \( u(z) = u(z, \tau) \) is some function of \( z \) and \( \gamma \) is a \( z \)-independent factor, and

\[ R = \gamma \theta_2(0) \theta_4(0), \quad V = -\gamma^2 (\theta^2_2(0) + \theta^2_4(0)). \]

At this stage \( \gamma \) is an arbitrary parameter but we will see that it can not be put equal to a fixed number like 1 because it is a dynamical variable, as well as the modular parameter \( \tau; \gamma = \gamma(\tau), \tau = \tau(\tau) \). In this parametrization, the equation of the curve is equivalent to the identity
\[ \theta_2^2(0) \frac{\partial_2^2(u)}{\partial_2^2(u)} \theta_2^2(u) = \theta_2^2(0) \frac{\partial_2^2(u)}{\partial_2^2(u)} \theta_2^2(u) \left( \frac{\partial_2^2(u)}{\partial_2^2(u)} + \frac{\partial_2^2(u)}{\partial_2^2(u)} \right) - (\theta_2^2(0) + \theta_2^2(0)) \]

which can be proved either by using some standard identities for theta-functions or by comparing analytical properties of the both sides. It is convenient to normalize \( u(z) \) by the condition \( u(\infty) = 0 \), then the expansion around \( \infty \) is

\[ u(z, t) = \frac{c_1(t)}{z} + \frac{c_2(t)}{z^2} + \cdots. \quad (15) \]

It is not difficult to check the identity

\[ \frac{w(z_1) - w(z_2)}{p(z_1) + p(z_2)} = \frac{1}{\gamma} \frac{\theta_2(u_1)\theta_2(u_2)}{\theta_1(u_1)\theta_1(u_2)} - \frac{\theta_1(u_1 - u_2)}{\theta_4(u_1 - u_2)}, \]

where \( u_i \equiv u(z_i) \). This identity allows one to represent equations (9), (10) as a single equation

\[ \left( z_1^{-1} - z_2^{-1} \right) e^{V(z_1)V(z_2)} = \frac{\theta_1(u(z_1) - u(z_2))}{\theta_4(u(z_1) - u(z_2))}. \quad (16) \]

Note that the limit \( z_2 \to \infty \) in (16) gives the definition of the function \( u(z) \)

\[ e^{\theta_2V(z)} = \frac{\theta_2(u(z))}{\theta_4(u(z))} \quad (17) \]

(equivalent to the first formula in (13)). In addition, we see from (14) that

\[ \frac{V}{R^2} = e^{-\theta_2w}2F_1 + F_0 - F_0 \quad = \frac{\theta_2^2(0)\theta_2^2(0)}{\theta_4^2(0)\theta_4^2(0)} + \frac{\theta_4^2(0)}{\theta_2^2(0)\theta_4(0)}. \quad (18) \]

The \( z \to \infty \) limit of equation (17) yields: \( e^{\theta_2w} = R = \pi c_1(0)\theta_2(0) \), hence

\[ c_1(t) = \frac{\gamma(t)}{\pi}. \quad (19) \]

Yet another useful form of equation (16) can be obtained by passing to logarithms and applying \( \partial_\nu \) to both sides. It is convenient to introduce the function

\[ S(u|\nu) := \log \frac{\theta_4(u|\nu)}{\theta_2(u|\nu)}, \quad (20) \]

which has the following quasiperiodicity properties:

\[ S(u + 1|\nu) = S(u|\nu) + i\pi, \quad S(u + \tau|\nu) = S(u|\nu). \quad (21) \]

In terms of this function, the equation reads

\[ V(z_1)S(u(z_1)|\tau) = \partial_\nu S(u(z_1) - u(z_2)|\tau). \quad (22) \]

In particular, this equation means that the left-hand side is symmetric with respect to the permutation \( z_1 \leftrightarrow z_2 \): \( V(z_1)S(u(z_1)|\tau) = V(z_2)S(u(z_2)|\tau) \). This symmetry is a manifestation of integrability. In the limit \( z_2 \to \infty \) equation (22) gives

\[ V(z) \log R = \partial_\nu S(u(z)|\tau). \quad (23) \]
In order to connect this with the algebraic formulation, we note that
\[ S(u(z)|r) = -\frac{1}{2} \log w(z), \quad c_1S'(u(z)|r) = p(z), \quad (24) \]
where \( S'(u|\tau) \equiv \partial_\tau S(u|\tau) \). The first formula directly follows from the definitions. To derive the second one, we use equation \((A.13)\) from the appendix.

3. One-variable reductions

One may look for solutions of the hierarchy such that \( u(z, t) \) and \( \tau(t) \) depend on the times through a single variable \( \lambda = \lambda(t) \): \( u(z, t) = u(z, \lambda(t)), \tau(t) = \tau(\lambda(t)) \). Such solutions are called one-variable reductions of the hierarchy. The function of two variables, \( u(z, \lambda) \), can not be arbitrary. Our next goal is to characterize the class of functions \( u(z, \lambda), \tau(\lambda) \) that are consistent with the structure of the hierarchy and can be used for one-variable reductions.

The consistency condition for one-variable reductions. Applying the chain rule of differentiation to \( S(u(z, t)|\tau(t)) = S(u(z, \lambda(t))|\tau(\lambda(t))) \), we get from equation \((22)\)
\[
V(z_1)S(u(z_2)) = \left[ V(z_1)\lambda \right] \left( \partial_\lambda u(z_2)S'(u(z_2)) + \partial_\lambda \tau S(u(z_2)) \right).
\]
Hereafter, we write simply \( u(z) := u(z, \lambda), S(u) := S(u|\tau) \) and denote \( S'(u) = \partial_\tau S(u|\tau), \quad \tilde{S}(u) = \partial_\tau S(u|\tau) \). Next, we have, using \((23)\)
\[
V(z_1)\lambda = \frac{d\lambda}{d\log R} V(z_1) \log R = \frac{d\lambda}{d\log R} \partial_\lambda S(u(z_1))
\]
\[
= \frac{d\lambda}{d\log R} \left( \partial_\lambda u(z_1)S'(u(z_1)) + \partial_\lambda \tau \tilde{S}(u(z_1)) \right).
\]
In a similar way, we get
\[
\partial_\lambda S(u(z_1) - u(z_2)) = \partial_\lambda \left[ \left( \partial_\lambda u(z_1) - \partial_\lambda u(z_2) \right)S'(u(z_1) - u(z_2)) + \partial_\lambda \tau \tilde{S}(u(z_1) - u(z_2)) \right].
\]
The formulas simplify a bit if we choose \( \lambda = \tau \). Assuming that \( \partial_\lambda \tau \) is not identically zero, we arrive at the following relation:
\[
\left[ \partial_\lambda u(z_1)S'(u(z_1)) + \tilde{S}(u(z_1)) \right] \left[ \partial_\lambda u(z_2)S'(u(z_2)) + \tilde{S}(u(z_2)) \right] = \frac{d\log R}{d\tau} \left[ \left( \partial_\lambda u(z_1) - \partial_\lambda u(z_2) \right)S'(u(z_1) - u(z_2)) + \tilde{S}(u(z_1) - u(z_2)) \right]. \quad (25)
\]
Note that this relation can be written in the compact form
\[
\frac{dS(u(z_1))}{d\tau} \frac{dS(u(z_2))}{d\tau} = \frac{d\log R}{d\tau} \frac{dS(u(z_1) - u(z_2))}{d\tau}, \quad (26)
\]
where \( d/d\tau \) is the total \( \tau \)-derivative.

To proceed, we need to know \( S(u) \). It is given by the formula
\[
2\pi i\tilde{S}(u) = S'(u)E^{(2)}(u) + \frac{\pi^2}{2} \theta_4^4(0), \quad (27)
\]
which is proved in the appendix. Here and below we use the notation\textsuperscript{4}

\[ E^{(a)}(u) = E^{(a)}(u|\tau) = \partial_u \log \theta_a(u|\tau). \]

The properties of these Eisenstein-like functions that we need for calculations are listed in the appendix (see (A.8), (A.9)). Using (27), we rewrite (25) in the form

\[
S'(u_1) \left[ 4\pi i \partial_u u_1 + 2E^{(2)}(u_1) + \frac{\pi^2 \theta_4^4(0)}{S'(u_1)} \right] S'(u_2) \left[ 4\pi i \partial_u u_2 + 2E^{(2)}(u_2) + \frac{\pi^2 \theta_4^4(0)}{S'(u_2)} \right] = 4\pi i \frac{\text{dlog} R}{\text{dr}} S'(u_1-u_2) \left[ 4\pi i \left( \partial_u u_1 - \partial_u u_2 \right) + 2E^{(2)}(u_1-u_2) + \frac{\pi^2 \theta_4^4(0)}{S'(u_1-u_2)} \right],
\]

where \( u_j \equiv u(z_j) \) for brevity. Now, one can see that the substitutions

\[
\begin{align*}
4\pi i \partial_u \equiv & -E^{(1)}(u + \xi) - E^{(4)}(u + \xi) + E^{(1)}(\xi) + E^{(4)}(\xi), \\
4\pi i \partial_\tau \equiv & (S'(\xi))^2,
\end{align*}
\]

where \( \xi \) is an arbitrary parameter, convert equation (28) into identity. Some details are given in the appendix. This means that the function \( u(z, \tau) \) is compatible with the infinite hierarchy if it satisfies the differential equation

\[
4\pi i \partial_\tau u(z) = -E^{(1)}(u(z) + \xi(\tau)|\tau) - E^{(4)}(u(z) + \xi(\tau)|\tau) + E^{(1)}(\xi(\tau)|\tau) + E^{(4)}(\xi(\tau)|\tau),
\]

where \( \xi(\tau) \) can be arbitrary function of \( \tau \). The identity \( E^{(1)}(u|\tau) + E^{(4)}(u|\tau) = E^{(1)}(u|\tau^2) \) allows one to write this equation in a more compact form

\[
4\pi i \partial_\tau u(z) = -E^{(1)} \left( u(z) + \xi(\tau) \frac{z}{2} \right) + E^{(1)} \left( \xi(\tau) \frac{\tau}{2} \right).
\]

This is the elliptic analogue of the Löwner equation known also as the Goluzin–Komatu equation \textsuperscript{[18, 19]}. One can also see that the equation

\[
4\pi i \partial_\tau \log R = (S'(\xi(\tau)))^2
\]

emerges as the limiting case of (30) when \( z \to \infty \). The function \( \xi(\tau) \) is the ‘driving function’ that encodes the shape of the slit in the Löwner theory. In our setting, it specifies the reduction.

Two remarks are in order.

(a) Using the identity proved in the appendix, it is possible to show that for one-variable reductions the total \( \tau \)-derivative of \( S(u(z)) \) is given by

\[
4\pi i \frac{\text{d} S(u(z))}{\text{d} \tau} = S'(\xi(\tau))S'(u(z) + \xi(\tau)).
\]

(b) The \( u(z) \)-independent second term in the right-hand side of equation (31) can be eliminated by another choice of normalization. Indeed, let us consider the function \( \tilde{u}(z) = u(z) + c_0(\tau) \), i.e.

\textsuperscript{4} Note that the standard notation for the Eisenstein function \( E^{(1)} \) is \( E_1 \).
\[ u(z) = c_0(\tau) + \frac{c_1(\tau)}{z} + \frac{c_2(\tau)}{z^2} + \cdots \quad \text{with} \quad c_0(\tau) = -\frac{1}{4\pi i} \int_0^\infty E^{(1)}(\xi(\tau)) \frac{\tau'}{2} d\tau' \]

and set \( \xi = \xi - c_0 \). Then the elliptic Löwner equation (31) acquires the form

\[ 4\pi i \partial \hat{u}(z) = -E^{(1)}(\hat{u}(z) + \xi(\tau)) \left( \frac{\tau}{2} \right). \]  

(34)

**The system of reduced equations and their solution.** In order to complete the description of one-variable reductions, we should derive the equation satisfied by \( \tau(t) \) and find its solution. Following the way we have used to derive relation (26), we write

\[ V(z)\tau = \frac{\partial u(z)S'(u(z)) + \dot{S}(u(z))}{\text{dlog} R/\text{d}\tau} \partial_0 \tau = \frac{\text{d}S(u(z))/\text{d}\tau}{\text{dlog} R/\text{d}\tau} \partial_0 \tau. \]

Substituting (32) and (33), we get \( V(z)\tau = \frac{S'(u(z) + \xi(r))}{S'((\xi(r))}\partial_0 \tau \). This is a generating equation for a hierarchy of equations of the hydrodynamic type. To write them explicitly, we use the expansion

\[ S'(u(z) + u) = S'(u) + \sum_{k \geq 1} \frac{z^{-k}}{k} B_k(u) \]

(35)

which defines the functions \( B_k(u) = B_k(u(r)) \). By analogy with the dKP case, one may think of them as elliptic analogues of the Faber polynomials. To make the analogy precise, one should develop the elliptic version of the Lax representation for the Pfaff lattice\(^5\). In terms of the functions \( B_k(u) \), the equations of the reduced hierarchy are as follows:

\[ \frac{\partial \tau}{\partial t_k} = \phi_k(\xi(\tau)|r) \frac{\partial \tau}{\partial_0}, \quad \phi_k(\xi(\tau)|r) = \frac{B_k(\xi(\tau)|r)}{S'(\xi(\tau)|r)}, \quad k \geq 1. \]

(36)

The common solution to these equations can be written in the hodograph form:

\[ \sum_{k=1}^\infty t_k \phi_k(\xi(\tau)|r) = \Phi(\tau), \]

(37)

where \( \Phi(\tau) \) is an arbitrary function of \( \tau \). In the simplest case, when \( \Phi(\tau) = 0 \), we conclude from (37) that \( \sum_{k=1}^\infty t_k \frac{\partial \tau}{\partial_0} = 0 \), i.e., \( \tau(t) \) is a homogeneous function of the times of degree 0.

**A connection with Painlevé VI.** Here we work with the elliptic Löwner equation in the normalization (34) skipping tilde from the notation and changing \( \tau \to 2\tau \)

\[ 2\pi i \partial \hat{u}(z) = -E^{(1)}(u(z) + \xi(\tau)). \]

(38)

As an example, consider the simplest possible case when \( \xi \) does not depend on \( \tau : \xi = \text{const} \) (in the dKP case, such a choice of the driving function means the reduction to the dispersionless KdV hierarchy or hierarchies equivalent to it). Assume that \( u(z) = u(z, \tau) \) satisfies this equation. An easy calculation with the use of (A.11) shows that the function \( f(\xi, \tau) := E^{(1)}(u(z, \tau) + \xi(\tau)) \) obeys the heat equation\(^6\)

\[ 4\pi i \partial f(\xi, \tau) = \partial_2^2 f(\xi, \tau). \]

(39)

\(^5\) We thank the referee who draw our attention to this yet unsolved problem.

\(^6\) We thank A Levin who conjectured this fact.
Applying $\partial_\tau$ to the both sides of equation (38), we get, using the heat equation (39)

$$(2\pi i)^2 \partial_\tau^2 u = \frac{1}{2} \wp'(u + \xi),$$

(40)

where $\wp(u) = -\partial_\tau E^{(1)}(u) + \text{const}$ is the Weierstrass $\wp$-function with periods 1 and $\tau$. If $\xi = 0$ or $\xi = \frac{1}{2}$, this is the Painlevé VI equation written in the elliptic form with a special choice of the parameters [33].

4. Concluding remarks

We have demonstrated that the Pfaff lattice (an infinite integrable hierarchy with the $D_{\infty}$ symmetry) in the dispersionless limit can be naturally reformulated as an ‘elliptic deformation’ of the usual dKP hierarchy. This seems to be a rather surprising and somewhat mysterious fact. What could be the hidden link between Pfaffians and elliptic functions?

Once the elliptic reformulation has been done, the description of one-variable reductions of the dDKP hierarchy obtained in this paper looks rather natural if one keeps in mind the corresponding Gibbons–Tsarev result for the dKP case. To wit, the one-variable reductions, i.e., reductions with only one independent function, are obtained from solutions to the elliptic analogue of the Löwner equation (the Goluzin–Komatu equation), well known in the theory of conformal maps of doubly-connected slit domains. We hope to clarify the geometric meaning of the reductions, and of the hierarchy in general, in subsequent publications.

It should be noted that we have found only sufficient conditions for the consistent one-variable reductions. In order to find the necessary conditions and to give a complete description, one should find all solutions to the functional relation (26), which is the consistency condition for the reductions.

A more complicated problem is to describe multi-variable reductions. Here one can anticipate that an elliptic analogue of the system of the Gibbons–Tsarev equations should come into play as consistency conditions.

An unexpected observation, which seems to be especially interesting, is the close connection with the Painlevé VI equation. For a particular (simplest possible?) choice of the driving function, the elliptic Löwner equation appears to be the integrated Painlevé VI with special values of the parameters. Note that in the dKP case, the simplest possible driving function (equal to zero) corresponds to the most familiar and explicit reduction, the one to the dispersionless KdV hierarchy. This is one of the very few cases when the chordal Löwner equation can be explicitly solved. It would be very interesting to find such solvable cases for the elliptic version of the Löwner equation.

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Appendix

Theta-functions. The Jacobi’s theta-functions $\theta_a(u) = \theta_a(u|\tau)$, $a = 1, 2, 3, 4$, are defined by the formulas

$$
\begin{align*}
\theta_1(u) &= -\sum_{k \in \mathbb{Z}} \exp \left( \pi i r \left( k + \frac{1}{2} \right)^2 + 2\pi i \left( u + \frac{1}{2} \right) \left( k + \frac{1}{2} \right) \right), \\
\theta_2(u) &= \sum_{k \in \mathbb{Z}} \exp \left( \pi i r \left( k + \frac{1}{2} \right)^2 + 2\pi i u \left( k + \frac{1}{2} \right) \right), \\
\theta_3(u) &= \sum_{k \in \mathbb{Z}} \exp \left( \pi i r^2 + 2\pi i u k \right), \\
\theta_4(u) &= \sum_{k \in \mathbb{Z}} \exp \left( \pi i r^2 + 2\pi i \left( u + \frac{1}{2} \right) k \right).
\end{align*}
$$

(A.1)

where $\tau$ is a complex parameter (the modular parameter) such that $\text{Im} \, \tau > 0$. The function $\theta_1(u)$ is odd, the other three functions are even. The infinite product representation for the $\theta_1(u)$ reads

$$
\theta_1(u) = i \exp \left( \frac{ir}{4} - i\pi u \right) \prod_{k=1}^{\infty} \left( 1 - \exp \left( 2\pi i (k-1) r + u \right) \right) \left( 1 - \exp \left( 2\pi i (k-1) r - u \right) \right).
$$

(A.2)

We also mention the identity

$$
\theta_1'(0) = \pi \theta_2(0) \theta_3(0) \theta_4(0).
$$

(A.3)

In order to unify some formulas given below, it is convenient to understand the index $a$ modulo 4, i.e., to identify $\theta_a(z) \equiv \theta_{a+4}(z)$. Set $\omega_0 = 0$, $\omega_1 = \frac{1}{2}$, $\omega_2 = \frac{1+r}{2}$, $\omega_3 = \frac{r}{2}$ then the function $\theta_a(u)$ has simple zeros at the points of the lattice $\omega_{a-1} + \mathbb{Z} + \mathbb{Z} \tau$.

The theta-functions have the following quasi-periodic properties under shifts by 1 and $\tau$

$$
\begin{align*}
\theta_a(u + 1) &= e^{\pi i (1+2a) \omega_{a-1}} \theta_a(u), \\
\theta_a(u + \tau) &= e^{\pi i (a+2) \omega_{a-1}} e^{-\pi i 2\pi u} \theta_a(u).
\end{align*}
$$

(A.4)

Shifts by the half-periods relate the different theta-functions to each other

$$
\begin{align*}
\theta_1(u + \omega_1) &= \theta_2(u), & \theta_3(u + \omega_1) &= \theta_4(u), \\
\theta_1(u + \omega_2) &= e^{-\pi i \omega_{a-1}} \theta_3(u), & \theta_2(u + \omega_2) &= -ie^{-\pi i \omega_{a-1}} \theta_4(u) \\
\theta_1(u + \omega_3) &= ie^{-\pi i \omega_{a-1}} \theta_4(u), & \theta_2(u + \omega_3) &= e^{-\pi i \omega_{a-1}} \theta_3(u).
\end{align*}
$$

(A.5)

(A.6)

(A.7)

In the main text we use the special notation for the Eisenstein-like functions

$$
E^{(a)}(u) = E^{(a)}(u|\tau) = \partial_u \log \theta_a(u|\tau).
$$

Using (A.4), (A.5), it is easy to prove the following properties of the functions $E^{(a)}(u)$:

$$
\begin{align*}
E^{(a)}(u + 1) &= E^{(a)}(u), & E^{(a)}(u + \tau) &= E^{(a)}(u) - 2\pi i.
\end{align*}
$$

(A.8)
and

\[ E^{(1)}(u + \frac{\tau}{2}) = E^{(4)}(u) - \pi i, \]
\[ E^{(4)}(u + \frac{\tau}{2}) = E^{(1)}(u) - \pi i. \]  

(A.9)

For calculations we also need \( E^{(2)}(0) = 0, \ E^{(2)}(\frac{\tau}{2}) = -\pi i. \)

All formulas for derivatives of elliptic functions with respect to the modular parameter follow from the ‘heat equation’ satisfied by the theta-functions

\[ 4\pi i \partial_\tau \theta_j(u) = \partial^2_{uu} \theta_j(u). \]  

(A.10)

In particular, the \( \tau \)-derivative of the Eisenstein function is given by

\[ 4\pi i \partial_\tau E^{(1)}(u|\tau) = 2E^{(1)}(u|\tau)E^{(1)}(u|\tau) + E^{(1)}(u|\tau) \]  

(A.11)

(see, e.g., [35]).

**Proof of equation (27).** Here we prove formula (27) for the \( \tau \)-derivative of the function

\[ S(u|\tau) = \log \frac{\theta_4(u|\tau)}{\theta_4(u|\tau)}, \]  

(A.12)

A similar formula has been derived in [34, 35] in the context of the Painlevé–Calogero correspondence.

We start with the following factorized representation of \( S'(u) \)

\[ S'(u) = E^{(1)}(u) - E^{(4)}(u) = \pi \theta_4^2(0) \frac{\theta_2(u)\theta_3(u)}{\theta_1(u)\theta_4(u)}. \]  

(A.13)

which can be easily proved, with the help of equation (A.3), by comparing analytical properties of the both sides. We will also need the particular case of this identity obtained by shifting \( u \rightarrow u + \frac{1}{2}, \) taking the \( u \)-derivative and tending \( u \rightarrow 0: \)

\[ \frac{\theta_2^2(0)}{\theta_3(0)} = \frac{\theta_3^2(0)}{\theta_2(0)} = \pi^2 \theta_4^4(0). \]  

(A.14)

From (21) we see that \( S(u + 1) = S(u), \ S(u + \tau) = S(u) - S'(u), \) so both sides of equation (A.12) are periodic under the shift \( u \rightarrow u + 1 \) and gain the additive contribution \(-2\pi i S'(u) \) under the shift \( u \rightarrow u + \tau \) (see (A.8)). Therefore, the function

\[ g(u) := 4\pi i S(u) - 2S'(u)E^{(2)}(u) - \pi^2 \theta_4^4(0) \]  

is doubly-periodic with primitive periods 1, \( \tau. \) Using the heat equation (A.10), we have

\[ g(u) = \frac{\theta_1^2(u)}{\theta_1(u)} - \frac{\theta_4^2(u)}{\theta_4(u)} - 2\pi \theta_4^2(0) \frac{\theta_2(u)\theta_3(u)}{\theta_1(u)\theta_4(u)} = \pi^2 \theta_4^4(0). \]

In order to prove that \( g(u) \equiv 0, \) it is enough to show that it is regular at \( u = 0, \ u = \frac{\tau}{2} \) (zeros of the denominators) and \( g(u_0) = 0 \) at some point \( u_0 \) (it is convenient to choose \( u_0 = \frac{1+\tau}{2} \)). The regularity at \( u = 0 \) is obvious since \( \theta_1^2(u) \) and \( \theta_4^2(u) \) have simple zeros at \( u = 0 \) which cancel zeros in the denominators. The regularity at \( u = \frac{\tau}{2} \) is less obvious but holds due to identity (A.3). Finally, \( g\left(\frac{1+\tau}{2}\right) \) can be found to be zero with the help of (A.14). Clearly, \( g(u) \equiv 0 \) is equivalent to (A.12).
Proof of the key identity. Here we prove the key identity which allows one to derive the elliptic Löwner equation from (28). Set
\[ \varphi(x_1, x_2) := -E^{(1)}(x_1) - E^{(4)}(x_1) + E^{(1)}(x_2) + E^{(4)}(x_2) + 2E^{(2)}(x_1 - x_2). \]
The identity is
\[ S'(x_1 - x_2) \varphi(x_1, x_2) + \pi^2 \theta_4^4(0) = S'(x_1)S'(x_2). \] (A.15)
To prove it, we note that \( \varphi(x_1, x_2) \) admits the following factorized representation:
\[ \varphi(x_1, x_2) = \pi \theta_2(0) \theta_3(0) \theta_2^2(0) \theta_1(x_1 - x_2) \theta_4(x_1 + x_2) - \theta_1(x_1) \theta_4(x_1) \theta_4(x_2) \theta_2(x_1 - x_2). \] (A.16)
The proof is standard in the theory of elliptic functions. We should check that: (a) the both sides are doubly periodic as functions of \( x_1 \) with periods 1 and \( \tau \), b) the both sides have the same zeros and poles. Therefore, they differ by an \( x_1 \)-independent factor (actually equal to 1) which can be found by tending \( x_1 \to 0 \). Next, substitute the explicit form of \( S'(x) \) (A.13) to the left-hand side of (A.15). We get
\[ \text{lhs} = \pi^2 \theta_4^4(0) \left( \frac{\theta_2(0) \theta_3(0) \theta_2^2(0) \theta_1(x_1 - x_2) \theta_4(x_1 + x_2)}{\theta_1(x_1) \theta_4(x_1) \theta_4(x_2) \theta_2(x_1 - x_2)} + 1 \right). \]
The same argument as above shows that this function is equal to \( S'(x_1)S'(x_2) \).

Derivation of the elliptic Löwner equation from (28). We should show that the substitution (29),
\[ \pi \xi \tau = -E^{(1)}(u + \xi) - E^{(4)}(u + \xi) + E^{(1)}(\xi) + E^{(4)}(\xi), \]
\[ 4\pi i \delta_z \log R = (S'(\xi))^2, \] (A.17)
converts (28) into identity. Indeed, after this substitution equation (28) acquires the form
\[ S'(u_1) \left[ \varphi(u_1 + \xi, \xi) + \frac{\pi^2 \theta_4^4(0)}{S'(u_1)} \right] S'(u_2) \left[ \varphi(u_2 + \xi, \xi) + \frac{\pi^2 \theta_4^4(0)}{S'(u_2)} \right] \]
\[ = (S'(\xi))^2 S'(u_1 - u_2) \left[ \varphi(u_1 + \xi, u_2 + \xi) + \frac{\pi^2 \theta_4^4(0)}{S'(u_1 - u_2)} \right]. \]
It remains to employ the identity (A.15) for \( (x_1, x_2) = (u_1 + \xi, \xi), (x_1, x_2) = (u_2 + \xi, \xi) \) and \( (x_1, x_2) = (u_1 + \xi, u_2 + \xi) \).

Finally, we should check that equation (32), \( 4\pi i \delta_z \log R = (S'(\xi(\tau)))^2 \), is the limiting case of (30) as \( z \to \infty \). Substituting the series (15) into (30) and comparing the leading terms, we get
\[ 4\pi i \delta_z \log c_1 = -E^{(1)}(\xi(\tau)) - E^{(4)}(\xi(\tau)), \]
where \( E^{(\nu)}(u) = \delta_z E^{(\nu)}(u) \). Recall that \( \log R = \log(\tau c_1) + \log(\theta_2(0)\theta_3(0)) \) (see (14), (19)), so
\[ 4\pi i \delta_z \log R = -E^{(1)}(\xi) - E^{(4)}(\xi) + 4\pi i \delta_z \log(\theta_2(0)\theta_3(0)). \]
The last term can be transformed using the heat equation (A.10) for theta-functions. Taking into account that \( \theta_2(0) = \theta_3(0) = 0 \), we have...
where the well known identities

\[
2 \theta_1(u|x) \theta_4(u|x) = \theta_2(0|x) \theta_4(u|x),
\]

\[
2 \theta_2(u|x) \theta_3(u|x) = \theta_2(0|x) \theta_2(u|x)
\]

are used. We see that the equality that we are going to prove, i.e.

\[
4 \pi i \partial_t \log R = \theta_1^2 \left( \frac{x}{2} \right) - \theta_1^2 \left( \frac{x}{2} \right)
\]

is equivalent to the identity

\[
- \partial_t^2 \log \theta_1 \left( \frac{x}{2} \right) + \partial_t^2 \log \theta_1 \left( \frac{x}{2} \right) = \pi^2 \theta_2^4 (0|x) \theta_2^2 \left( \frac{x}{2} \right).
\]

The latter is proved by the standard argument. The both sides are elliptic functions with periods 1 and \( \frac{\tau}{2} \) and a pole of order 2 at \( x = 0 \) of the form \( x^{-2} + O(1) \) (to see this, one should use the identities (A.3) and \( \theta_2^2 (0|x) = \theta_3(0|x) \theta_4(0|x) \)). Therefore, their difference is a constant. Evaluating both sides at \( x = \frac{1}{2} \), we find that the constant is 0.

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