ON FLAT TRIGONOMETRIC SUMS AND ERGODIC FLOW WITH
SIMPLE LEBESGUE SPECTRUM

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ABSTRACT. A complex polynomial \( P(z) = c_0 + c_1 z + \ldots + c_n z^n \) is called unimodular if \( |c_j| = 1 \), \( j = 0, \ldots, n \). Littlewood asked the question (1966) on how close a unimodular polynomial come to satisfying \( |P(z)| \approx \sqrt{n + 1} \) if \( n \geq 1 \)? In this paper we show that for a given \( 0 < a < b \) and \( \varepsilon > 0 \) there exist trigonometric sums \( P(t) = n^{-1/2} \sum_{j=0}^{n-1} \exp(2\pi i t \omega(j)) \) with a real frequency function \( \omega(j) \) which are \( \varepsilon \)-flat on segment \([a, b]\) according to the norm in \( L^1([a, b]) \) (as well as in \( L^2([a, b]) \)). We apply this method to construct a dynamical system having simple spectrum and Lebesgue spectral type in the class of rank-one flows.

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1. Introduction

In this paper we study the spectral properties of a class of ergodic dynamical systems satisfying certain approximation properties and called system of finite rank. In brief we say that a measure preserving transformation \( T \) on a Lebesgue probability space has rank one if there exists a sequence of Rokhlin towers \( T^B \sqcup \ldots \sqcup T^{h_n} B \) approximating any measurable set. Rank one dynamical systems are known to have simple spectrum. We show that there exist rank one flows of measure preserving transformations having Lebesgue spectral type which gives positive answer to Banach question on simple Lebesgue spectrum for ergodic \( \mathbb{R} \)-actions. The method we use is actually based on purely analytic problem on existence of flat polynomials and trigonometric sums which goes back to classical Littlewood question about how close a unimodular polynomial

\[ P(z) = c_0 + c_1 z + \ldots + c_n z^n, \quad |c_j| = 1, \]

come to satisfying \( |P(z)| \approx \sqrt{n + 1} \) if \( n \geq 1 \)? In particular case of restriction \( c_j = \pm 1 \) the question still remains open. Bourgain [13] discovered that the spectral type of a rank one transformation is given by generalized Riesz product, and as an ingredient it contains polynomials with coefficients 0 and 1 which are similar and connected to polynomials with Littlewood type coefficient constraints. The purpose of our investigation is to construct \( \varepsilon \)-flat trigonometric sums in order to find a class of rank one flows having simple Lebesgue spectrum.

1.1. Spectral invariants of dynamical systems. Given \( T: X \to X \) a measure preserving invertible transformation on a standard Lebesgue space \((X, \mathcal{X}, \mu)\) we consider a unitary operator \( \hat{T} \) on \( L^2(X, \mathcal{X}, \mu) \) called Coopman operator acting as shift along the trajectory, \( \hat{T}f(x) = f(Tx) \). Since constants are invariant under \( \hat{T} \) the operator \( \hat{T} \) is usually restricted to the space \( L^2_0(X) \) of functions with zero mean, \( \int f d\mu = 0 \). For each \( f \in L^2_0(X) \) consider the spectral measure \( \sigma_f \) on the unit circle \( S^1 = \{ z \in \mathbb{C}: |z| = 1 \} \) defined by

\[ \int_{S^1} z^k d\sigma_f = \langle T^k f, f \rangle. \]

According to spectral theorem \( \hat{T} \) is uniquely determined up to spectral equivalence by two invariants, the spectral type \( \sigma \) on the unit circle \( S^1 \) and the multiplicity function \( \tilde{m}(z) \). For any \( f \) measure \( \sigma_f \)
is absolutely continuous with respect to the spectral type measure \( \sigma \), we denote it as \( \sigma_f \ll \sigma \), and \( \sigma \) is the minimal measure having this property up to the following measure equivalence: \( \lambda_1 \sim \lambda_2 \) iff \( \lambda_1 \ll \lambda_2 \) and \( \lambda_2 \ll \lambda_1 \) (see [3]).

We say that \( T \) has simple spectrum if \( T \) possesses a cyclic vector, an element \( h \in L^0_0(X) \) such that \( Z(h) = L^0_0(X) \), where \( Z(h) = \text{Span} \{ T^k h \colon k \in \mathbb{Z} \} \). The equivalent way to define simple spectrum property is to require \( \tilde{m}(z) = 1 \) for the multiplicity function. Well known examples of simple spectrum maps are ergodic dynamical system with purely discrete spectrum studied by von Neuman (see [5]). In this case the spectral measure is supported on a discrete subgroup \( \Lambda \) of \( S^1 \).

A variety of constructions of dynamical systems with finite spectral multiplicity and different dynamic effects comes from approximation theory inspired by Rokhlin ideas on tower approximation and developed by Katok, Oseledec and Stepin (see [7, 39, 48]). It was discovered that spectral multiplicity of \( r \)-interval exchange maps is bounded by \( r - 1 \), and further Robinson [42] proved that there exist interval exchange maps realizing maximal multiplicity of \( r - 1 \).

The question on existence of an ergodic transformation having homogeneous spectral multiplicity \( m > 1 \) known as Rokhlin problem was open by 1999 when Ryzhikov and Ageev constructed first examples of ergodic maps with homogeneous spectral multiplicity two (see [1]).

### 1.2. Rank one transformations and flows.

**Definition 1.1.** Let \( T \) be a measure preserving invertible transformation on a Lebesgue space \((X, \mathcal{X}, \mu)\). Given a set \( B \in \mathcal{X} \) and integer \( h \) assume that \( B, TB, T^2 B, \ldots, T^{h-1} B \) are disjoint. The set \( T_{h,B} = \bigcup_{j=0}^{h-1} T^k B \) is called Rokhlin tower of height \( h \), and \( B \) is called the base of the tower. We can draw a tower as a sequence of levels \( T^k B \) such that \( T \) lifts \( k \)-th level \( T^k B \) to the next level \( T^{k+1} B \) (except the top level). We refer to the following partition as level partition of the tower

\[
\xi_{h,B} = \{ B, TB, \ldots, T^{h-1} B, X \setminus T_{h,B} \}.
\]

The well-known Rokhlin–Halmos’ lemma (see [5]) states that given aperiodic transformation \( T \) for any height \( h \) and arbitrary \( \varepsilon > 0 \) there exists a Rokhlin tower \( T_{h,B} \) satisfying \( \mu(T_{h,B}) > 1 - \varepsilon \).

**Definition 1.2.** A measure preserving invertible map \( T \) is called rank one transformation if \( \mu(T_{h_n,B_n}) \to 1 \) and there exists a sequence of Rokhlin towers \( T_{h_n,B_n} \) approximating \( \sigma \)-algebra in the following sense. For any measurable set \( A \) we can find \( \xi_{h_n,B_n} \)-measurable sets \( A_n \) such that \( \mu(A_n \triangle A) \to 0 \) as \( n \to \infty \).

Rank one transformations was introduced by Chacon [21], Ornstein [38] and Fridman [26]. Some classical ergodic maps, for example discrete spectrum maps appear to be rank one (see survey [25]). A series of dynamical system with finite rank are found in the class of adic and substitution systems [12]. In paper [38] Ornstein constructed a randomized family of rank one maps known to be almost surely mixing which means that \( \mu(T^j A \cap B) \to \mu(A) \mu(B) \) as \( j \to \infty \) for all measurable sets \( A \) and \( B \). For the class of Ornstein transformations Bourgain proved that almost surely they have purely singular spectrum [18].

Today all known examples of rank one maps and flows have spectral type which is combination \( \sigma = \sigma_d + \sigma_s \) of discrete \( \sigma_d \) and singular \( \sigma_s \) measures (see [22, 30, 31]).

The question on existence of a map (or a flow) having simple spectrum (in orthocomplement to constants) and Lebesgue spectral type is known as Banach problem on simple Lebesgue spectrum (see [6, 8]). Mathew and Nadkarni [36] answering a question by Helson and Parry [27] has constructed a transformation having a Lebesgue component in spectrum of multiplicity 2 with a discrete component. Guenais [35] has found a criterion for existence of Lebesgue component in spectrum of generalized Morse dynamical systems.

In this paper we focus on spectral type of rank one flows (see [43, 41]).
Definition 1.3. We say that a set \( T_{h,B} = \bigcup_{t \in [0,h]} T^tB \) is a Rokhlin tower for a flow \( T^t \) on a standard Lebesgue space \((X, \mathcal{X}, \mu)\) if the sets \( T^tB \) are disjoint and for any Borel set \( J \) the union \( \bigcup_{t \in J} T^tB \) is measurable. Let us define measurable partitions \( \xi_{h,B} \) of the phase space \( X \) into levels \( T^tB \) of the tower and the set complementary to the tower.

Definition 1.4. A flow \( T^t \) is called rank one flow if there exist a sequence of Rokhlin towers \( T_{h,B_n} \) such that \( \mu(T_{h,B_n}) \to 1 \) and for any measurable \( A \) we can find \( \xi_{h,B_n} \)-measurable sets \( A_n \) such that \( \mu(A_n \triangle A) \to 0 \) as \( n \to \infty \).

We will assume that additionally partition \( \xi_{h_n+1,B_n+1} \) refines partition \( \xi_{h_n,B_n} \). In this case we can define the rank one flow by the following construction.

Definition 1.5. Cutting-and-stacking construction: Consider a sequence \( q_n \in \mathbb{N} \) and spacer parameters \( s_{n,k} \geq 0 \). At \( n \)-th step we cut the Rokhlin tower \( T_{h_n,B_n} \) into \( q_n \) equal subtowers called columns, add spacer \( s_{n,k} \) to \( k \)-th columns and stack it together ordered from the first to the \( q_n \)-th. Repeating this procedure infinitely many times we construct a Lebesgue probability space if the following condition holds:

\[
\prod_{n=1}^{\infty} \frac{h_{n+1}}{q_n h_n} < \infty,
\]

where \( h_n \) is the height of the \( n \)-th tower. A point \( a \in X \) moves up with unit velocity along any tower under action of the flow \( T^t \) and it makes jump reaching the roof of the tower.

Equivalently cutting-and-stacking construction can be represented as follows.

Definition 1.6. Consider segments \( X_n = [0, h_n] \) and define a space \( X \) to be the inverse limit of \( X_n \) up to projections \( \phi_n : X_{n+1} \to X_n \),

\[
\phi_n(a_{n,k} + t) = t, \quad 0 \leq t < h_n,
\]

where \( a_{n,k} = \sum_{j < k} (h_n + s_{n,j}) \), and \( \phi_n(0) = 0 \) otherwise. Namely,

\[
X_1 \xleftarrow{\phi_1} X_2 \xleftarrow{\phi_2} \ldots \xleftarrow{\phi_{n-1}} X_n \xleftarrow{\phi_n} \ldots.
\]

If (3) holds we can normalize Lebesgue measure on \( X_n \) so that \( \phi_n \) become measure preserving and \( X \) becomes Lebesgue probability space. A point \( x \in X \) is now a sequence

\[
x = (x_1, x_2, \ldots) \quad \text{such that} \quad \phi_n(x_{n+1}) = x_n.
\]

To define \( T^t \) for a point \( x \) we can write \( (T^t x)_n = x_n + t \) if the point \( x_n + t \) do not go out the right boundary of the segment \([0, h_n] \), elsewise we have to watch the point at some upper level \( n' > n \). It can be easily seen that almost surely \( x_n \) is \( t \)-far from the right boundary of \([0, h_n] \) starting from some index \( n_0 \), and the definition of the flow is correct.

1.3. Unimodular polynomials and trigonometric sums. Complex polynomial

\[
P(z) = c_0 + c_1 z + \ldots + c_n z^n, \quad c_j \in \mathbb{C}, \quad z \in \mathbb{C}, \quad |z| = 1,
\]

is called unimodular if all the coefficients \( |c_j| = 1, \ j = 0, 1, \ldots, n \). Let \( K_n \) be the class containing unimodular polynomials of degree \( n \), and let \( L_n \) be the class of polynomials with coefficients \( \pm 1 \),

\[
L_n = \left\{ Q(z) = \sum_{j=0}^{n} c_j z^j, \ c_j \in \{-1, +1\} \right\}.
\]
Consider a function $|P(z)|$ on a unit circle $S^1 = \{ |z| = 1 \}$. J. Littlewood \cite{Littlewood1946}, cf. \cite{Kahane1962, Korner1964, Körner1966} has formulated a question on how close a unimodular polynomial come to satisfying $|P(z)| \approx \sqrt{n+1}$ when $P$ range over the class $\mathcal{K}_n$ or the class $\mathcal{L}_n$? We say that a polynomial $P(z)$ is $\varepsilon_n$-flat if
\begin{equation}
\left\| \frac{1}{\sqrt{n+1}} |P(z)| - 1 \right\| \leq \varepsilon_n
\end{equation}
according to some norm. T. Körner \cite{Körner1964, Körner1966} has discovered a unimodular polynomial $P(z)$ with the property $A \leq (n+1)^{-1/2} |P(z)| \leq B$ for some absolute constants $0 < A < 1 < B$. J. Kahane \cite{Kahane1962} has constructed a polynomial satisfying the following estimate at any point $z \in S^1$
\begin{equation}
| (n+1)^{-1/2} |P(z)| - 1 | \leq \varepsilon_n, \quad \varepsilon_n = O(n^{1/17} \log n).
\end{equation}
It is also an open question if there exist a flat polynomial in the class
\begin{equation}
\mathcal{M}_n = \left\{ P(z) = q^{-1/2} (z^{a_0} + z^{a_1} + \cdots + z^{a_{n-1}}), \ a_k \in \mathbb{Z}, \ a_k < a_{k+1} \right\}
\end{equation}
where $n \geq 2$.

1.4. Flat trigonometric sums. We will study trigonometric sums
\begin{equation}
\mathcal{P}(t) = \frac{1}{\sqrt{q}} \sum_{y=0}^{q-1} \exp(2\pi i t \omega(y)), \quad \omega(y) \in \mathbb{R}, \quad t \in \mathbb{R},
\end{equation}
where $\omega(y)$ is referred to as frequency function. The main target of our investigation is the following special class of functions $\omega(y)$.

**Definition 1.7.** We call exponential staircase frequency function the function $\omega: \mathbb{Z} \to \mathbb{R}$ given by the following expression:
\begin{equation}
\omega(y) = E_0 + Ae^{\epsilon y}, \quad E_0, A, \epsilon \in \mathbb{R}.
\end{equation}
Observe that $\omega(y)$ is just a solution of the differential equation
\begin{equation}
\omega'' = \epsilon \omega'
\end{equation}
if we consider it for a while as a function on $\mathbb{R}$.

**Theorem 1.1.** For given $0 < a < b$, $\varepsilon > 0$ and $\delta > 0$ there exists $m_0$ such that for any $m \geq m_0$ there exists an infinite sequence $q_j$ generating trigonometric sums with exponential staircase frequency function
\begin{equation}
\omega(y, m) = m \frac{q_j}{\varepsilon^2} e^{\varepsilon y/q_j},
\end{equation}
which are $\delta$-flat in $L^1([a, b])$.

1.5. Exponential staircase flows. Main result. In this paper we study spectral type of a class of rank one flows given the following definition.

**Definition 1.8.** We say that a rank one flow is given by exponential staircase construction if the roof function over the $n$-th tower is given by the graph of the discrete derivative $\omega_n(y+1) - \omega_n(y)$ of an exponential function $\omega_n(y)$, in other words if there exist $\varepsilon_n$ and $m_n$ such that
\begin{equation}
h_n + s_{n,y} = \omega_n(y+1) - \omega_n(y), \quad \omega_n(y) = m_n \frac{q_n}{\varepsilon_n^2} (e^{\varepsilon_n y/q_n} - 1), \quad h_n = \frac{m_n}{\varepsilon_n}.
\end{equation}
Remark that $-1$ in the parenthesis is used to provide standard position of the segment $[0, h_n]$ in $\mathbb{R}$, namely, $\omega_n(0) = 0$, and if $\varepsilon_n \to 0$ as $n \to \infty$ then
\begin{equation}
s_{n,y} \sim \omega_n'(y) - h_n = m_n \frac{1}{\varepsilon_n} e^{\varepsilon_n y/q_n} - h_n = m_n \frac{1}{\varepsilon_n} (e^{\varepsilon_n y/q_n} - 1).
\end{equation}
It would be also interesting to observe that the spacer function $s_{n,y}$ ranges over $[0, m_n(1 + O(\varepsilon_n))]$.

Remark 1. Terminology we use is explained by the note that spacer function $s_{n,k}$ approximates the spacer function $\tilde{s}_{n,y} = \alpha y$ which leads to staircase construction of a rank one flow, generalizing the concept of rank one staircase transformation introduced by Adams and Smorodinsky (see [15]) providing the first explicit example of mixing rank one transformation (see also [45]).

The main purpose of the paper is to study the spectral properties of exponential staircase constructions. It appears that in many cases the spectral type of such flows can be calculated in a certain sense, and we can find large variety of flows with Lebesgue spectral type having cardinality of continuum.

**Theorem 1.2.** Let us consider the exponential staircase construction of rank one flow $T^t$ given by the sequence of functions

\begin{equation}
\omega_n(y) = m_n \frac{q_n}{\varepsilon_n^2} \varepsilon_n^{y/q_n}, \quad h_n = \frac{m_n}{\varepsilon_n}.
\end{equation}

Suppose that for an $\alpha$ with $0 < \alpha < 1/4$

\begin{equation}
h_n^{1+\alpha} \leq q_n, \quad m_n \leq h_n^{1/2-\alpha}, \quad \sum_{n=1}^{\infty} \frac{1}{m_n^{1/4-\alpha}} < \infty,
\end{equation}

and $q_n$ has the following property:

\begin{equation}
\forall k \in \mathbb{Z} \cap \left[ \frac{m_n^{1-\alpha}}{\varepsilon_n}, \frac{m_n^{1+\alpha}}{\varepsilon_n} e^{\varepsilon_n} \right], \quad \left\| (q_n + 1) \frac{1}{\varepsilon_n} k \ln k \right\|_{\mathbb{Z}} \leq \varepsilon_n,
\end{equation}

where

\begin{equation}
\|x\|_{\mathbb{Z}} = \min\{|\ell - x| : \ell \in \mathbb{Z}\}.
\end{equation}

Then the flow $T^t$ has Lebesgue spectral type.

**Sketch of the proof.** Consider a function $f$ which is measurable up to partition $\xi_{h_n^0, B_n^0}$ into levels of the $n_0$-th tower in the cutting-and-stacking construction of rank one flow. We may consider $f$ as a function on $\mathbb{R}$ defining $f(t)$ to be the constant value $f$ keeps on the level $T^t B$. Using observation due
to Bourgain on approximation of the spectral measure \( \sigma_f \) we can say that it is given by a generalized Riesz product converging in weak sense,

\[
\sigma_f = |\hat{f}|^2 \cdot \prod_{n \geq n_0} |\mathcal{P}_n(t)|^2,
\]

where

\[
\mathcal{P}_n(t) = \frac{1}{\sqrt{q_n}} \sum_{y=0}^{q_n-1} \exp 2\pi i t \omega_n(y),
\]

and \( \omega_n(y) \) is connected with the sequence of spacers as follows: \( h_n + s_{n,y} = \omega_n(y+1) - \omega_n(y) \).

In brief it easily follows from the observation that the lift \( f_n(x) \) of the function \( f \) to the space \( X_n = [0, h_n] \) and the lift \( f_{n+1} \) to larger space \( X_{n+1} \) are connected in the following simple way

\[
f_{n+1}(x) = \sum_{y=0}^{q_n-1} f_n(x - \omega_n(y)),
\]

thus,

\[
\hat{f}_{n+1}(t) = \hat{f}_n(t) \cdot \frac{1}{\sqrt{q_n}} \sum_{y=0}^{q_n-1} e^{2\pi i t \omega_n(y)}.
\]

The idea is to find a sequence of trigonometric sums \( \mathcal{P}(t) \) which are \( \delta_n \)-flat on expanding intervals \( (a_n, b_n) \), \( a_n \to 0 \), \( b_n \to \infty \), with \( \delta_n \) to 0 sufficiently fast, so that we can control convergence of the Riesz product to a regular density, and by ergodicity we see that \( \sigma_f \) is absolutely continuous with respect to Lebesgue measure, and choosing appropriate \( f \) we can assume that the density of \( \sigma_f \) is not vanishing for any given interval in \( \mathbb{R} \).

2. **Stationary phase method**

2.1. **Preliminary lemmas. Van der Corput’s method.** Consider a smooth real-valued frequency function \( f(x) \) on the real line \( \mathbb{R} \) and look at the integral

\[
\int_{x_0}^{x_1} e^{itf(x)} \, dx
\]

An extremal point of the function \( f \) defined by equation \( f' = 0 \) is called a stationary phase of \( f \).

For example, a frequency function \( f(x) = (a + cx^2) \) has one stationary phase point \( x_0 = -a/(2c) \).

Let us mention two well-known lemmas on oscillatory integrals with quadratic frequency.

**Lemma 2.1. Oscillatory integral with quadratic frequency function and zero stationary phase can be estimated as follows**

\[
\int_{-1}^{1} e^{ikx^2} \, dx = \sqrt{\frac{\pi}{k}} \exp \left( \frac{\pi i}{4} \right) + O \left( \frac{1}{k} \right), \quad k \to \infty.
\]

**Lemma 2.2. For real \( a, c \neq 0 \) and \( b > 0 \)**

\[
\int_{0}^{b} e^{it(a+cx^2)} \, dx = A_0 \frac{e^{iat}}{2(|c|t)^{1/2}} - \frac{i}{2bct} e^{i(a+cb^2)t} + O \left( \frac{1}{b^3(ct)^2} \right)
\]

as \( t \to \infty \), where \( A_0 \) is defined as

\[
A_0 = \int_{0}^{\infty} u^{-1/2} e^{i\pi \text{sgn}(c)} \, du = e^{\pm \pi i \text{sgn}(c)} \sqrt{\pi}.
\]
Now let us consider a sum over an interval in the integer line
\[ S = \sum_{a \leq y \leq b} e^{2\pi i f(y)}, \quad y \in \mathbb{Z}, \]
where \( f \in C^2([a, b]) \). The following procedure is called van der Corput’s method. The sum \( S \) can be approximated by a special sum over stationary phase set \([2, 10, 34]\), namely, let \( f' \) be non-decreasing on \([a, b], \alpha = f'(a), \beta = f'(b)\), then (van der Corput, see \(34\), Lemma 2.3).
\[ S = \sum_{\alpha < k < \beta} |f''(y_k)|^{-1/2} e^{2\pi i (f(y_k) - ky_k + 1/8)} + \mathcal{E}, \]
where \( y_k \) are solutions of the equation \( f'(y_k) = k \), and \( \mathcal{E} \) is error term.

**Lemma 2.3.** Suppose \( \lambda_2 \leq f''(y) \leq \text{const} \cdot \lambda_2, \ |f^{(3)}(y)| \leq \lambda_3, \lambda_2, \lambda_3 > 0 \), and \( \mathcal{E} \) is defined in (31) then (van der Corput, see \([34]\) and \([10]\) Theorem 4.9)
\[ \mathcal{E} = O(\lambda_2^{-1/2}) + O(\ln(2 + (b - a)\lambda_2))) + O((b - a)\lambda_2^{1/5})\lambda_3^{1/5}). \]

In order to get more accurate estimate for oscillatory sums, notice that if the values \( \alpha, \beta \) of \( f' \) at the points \( a, b \) are located not too much close to integers, then the above estimate is more precise in first term. This estimate was established by Min in case of algebraic function \( f \). We will use the following more general result due to Liu (\([34]\), Theorem 1).

**Lemma 2.4.** Let \( f(y) \in C^5([a, b]) \) be a real function, \( f''(y) > 0 \), and \( R, U, C_k, k = 1, \ldots, 6 \), be positive constants such that
\[ C_1R^{-1} \leq |f''(y)| \leq C_2R^{-1}, \quad |\beta_k(y)| \leq C_kU^{2-k}, \quad U \geq 1, \quad 3 \leq k \leq 5, \]
\[ \beta_k(y) = f^{(k)}(y)/f''(y). \]

Suppose \( |3\beta_3(y) - 5\beta_3^2(y)| \geq C_6U^{-2} \) for any \( y \in [a, b] \); then
\[ \sum_{a \leq y \leq b} e^{2\pi i f(y)} = \sum_{\alpha < k < \beta} |f''(y_k)|^{-1/2} e^{2\pi i (f(y_k) - ky_k + 1/8)} + \mathcal{E}, \]
moreover,
\[ \mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + O(\ln(2 + (b - a)R^{-1})) + O((b - a + R)U^{-1}) + O(\min\{\sqrt{R}, \max\left(\frac{1}{|\langle \alpha \rangle|}, \frac{1}{|\langle \beta \rangle|}\right)\}), \]
where
\[ \mathcal{E}_1 = b_\alpha |f''(y_\alpha)|^{-1/2} e^{2\pi i (f(x_\alpha) - \alpha x_\alpha + 1/8)}, \]
\[ \mathcal{E}_2 = b_\beta |f''(y_\beta)|^{-1/2} e^{2\pi i (f(x_\beta) - \beta x_\beta + 1/8)}, \]
\[ b_\alpha = 1/2 \text{ if } \alpha \in \mathbb{Z}, \text{ otherwise } b_\alpha = 0, \text{ and } b_\beta \text{ is defined in the similar way; } \langle \alpha \rangle = \beta - \alpha \text{ if } \alpha \in \mathbb{Z} \text{ and } \]
\[ \langle \alpha \rangle = \min_{n \in \mathbb{Z}} |n - \alpha|, \quad \langle \beta \rangle = \min_{n \in \mathbb{Z}} |n - \beta| \]
if \( \alpha \not\in \mathbb{Z} \) (respectively, \( \beta \not\in \mathbb{Z} \)).

3. Free quantum particle with Hamiltonian \( H(p) = E_0 + Ae^{\epsilon p} \)

3.1. **Outline of the method.** Consider a free quantum particle moving along the real line according to classical Hamiltonian \( H(p) = \frac{p^2}{2} \) (where \( p \) is impulse variable). It generates an action \( R_t: \mathbb{R} \rightarrow \mathbb{R} \) of multiplicative group \( \mathbb{R}_+ = \{ t > 0 \} \) as follows. Solving equation
\[ t \frac{\partial}{\partial p} H(p) = k, \]
we get \( p = k/t \), and define \( R_t: p \mapsto t^{-1}p \). Actually \( R_t \) controls the dynamics of the stationary phase set for the sum

\[
(41) \quad \mathcal{P}(t) = \sum_{p \in [0,p^*] \cap \delta \mathbb{Z}} e^{i t H(p)}, \quad \delta > 0.
\]

Obviously acting on \( p^* \)-periodic functions the map \( R_2 \) is dual to the map \( \mathbb{T}_{p^*} \to \mathbb{T}_{p^*}: p \mapsto 2p \pmod{p^*} \) which is a kind of hyperbolic transformation, where \( \mathbb{T}_{p^*} = \mathbb{R}/p^* \mathbb{Z} \). This is the point which complicates the behavior of the trigonometric sum \( \mathcal{P}(t) \).

The idea is to change the dynamics of the stationary phase set just by a small alteration of Hamiltonian adding terms for higher derivatives. This leads to exponential Hamiltonian

\[
(42) \quad \tilde{H}(p) = E_0 + \frac{1}{\varepsilon^2} e^{\varepsilon p}
\]

generating action \( \tilde{R}_t(p) = p - \ln t \). Observe that (cf. \[46\])

\[
(43) \quad \tilde{H}(p) = E_0 + \frac{1}{\varepsilon^2} + \frac{p^2}{2} + \frac{\varepsilon p^3}{6} + \ldots
\]

Using Van der Corput’s method (see \[2,3\]), we get

\[
(44) \quad S(t) \approx \int \tilde{R}_t \phi \cdot \mathcal{L}_t,
\]

where \( \tilde{R}_t \phi(p) = \phi(\tilde{R}_t(p)) \), \( \phi(p) \) is a test function supported on \([a,b]\) and \( \mathcal{L}_t \) is generalized Legendre transform given by a discrete distribution on \( \mathbb{R} \). Again using Van der Corput’s method, we see that the sum \( S(t) \) is flat.

3.2. Main construction. Consider trigonometric sum

\[
(45) \quad S(t, \varepsilon, q) = \frac{1}{\sqrt{q}} \sum_{y=0}^{q-1} e^{2\pi i t \omega(y)}, \quad \omega(y) = \omega_0 + \frac{q}{\varepsilon^2} e^{\varepsilon y/q},
\]

where \( q \in \mathbb{N}, \ 0 < \varepsilon < 1, \ \varepsilon^{-1} \in \mathbb{N} \). Adding a constant to \( \omega(y) \) has no effect on \(|\mathcal{P}(t)|\), and during this section we assume that \( \omega_0 = 0 \). Our usual assumption on correlation of \( t \) and \( q \) will be \( 1 \ll t \ll q^{1/3} \).

Let us list several simple properties of the function \( \omega(y) \):

(a) \( \omega^{(r)}(y) \) increases for any \( r \);
(b) \( \omega'(y) \geq \varepsilon^{-1} \);
(c) \( \omega''(y) = \frac{1}{q}(1 + O(\varepsilon)) \), where large “O” is considered relatively to \( \varepsilon \to 0 \), where \( 0 \leq y < q \).

Let \( \mathcal{K}(t) \) be the set of indexes \( k \) such that \( y_k(t) \) appears in \((0,q)\),

\[
(46) \quad \mathcal{K}(t) = \{ k \in \mathbb{Z}: y_k(t) \in (0,q) \}.
\]

The set \( \mathcal{K}(t) \) is an integer segment,

\[
(47) \quad \mathcal{K}(t) = [k_0(t), k_1(t)],
\]

\[
(48) \quad K_0(t) = \frac{t}{\varepsilon}, \quad K_1(t) = \frac{t}{\varepsilon} e^\varepsilon,
\]

\[
(49) \quad k_0(t) = \min\{k > K_0(t)\}, \quad k_1(t) = \max\{k < K_1(t)\}.
\]

Lemma 3.1. Let \( S(t, \varepsilon, q) \) be the oscillatory sum defined in \[3.2\] and assume that \( \varepsilon^{-1} \in \mathbb{N} \), then \( S(t, \varepsilon, q) \) can be approximated by a sum over the set of stationary phases \( \{y_k\} \)

\[
(50) \quad S(t, \varepsilon, q) = \sum_{K_0(t) < k < K_1(t)} \frac{e^{2\pi i (-ky_k)}}{\sqrt{t}} \gamma(y_k) + \frac{\varepsilon(t)}{\sqrt{q}}, \quad \gamma(y) = e^{\frac{1}{2}\varepsilon y/q},
\]
where \( \mathcal{E}(t) \) is estimated as follows

\[
\left| \mathcal{E}(t) \right| = O \left( \frac{\max\{t^{-1}, \ln t\}}{\sqrt{q}} \right) + O \left( \frac{1}{\sqrt{q}} \min\{\sqrt{\frac{q}{t}}, \nu_\varepsilon(t)\} \right),
\]

\[
\nu_\varepsilon(t) = \frac{1}{\|\frac{t}{\varepsilon}\|_Z} + \frac{1}{\|\varepsilon \varepsilon\|_Z},
\]

and \( \|x\|_Z = \min_{n \in \mathbb{Z}} |n - x| \). For any \( t \) the sum \( S(t, \varepsilon, q) \) satisfies the evident condition \( |S(t, \varepsilon, q)| \leq \sqrt{q} \).

Remark 2. Investigating the sum \( S(t, \varepsilon, q) \) small and large O’s are considered with respect to \( \varepsilon \to 0 \), \( t \to \infty \) and \( q \to \infty \).

Remark 3. We will apply this lemma to find upper bounds for the error term with respect to different integral norms. So we can skip some countable set of points \( t \) in our considerations.

Proof. Let \( f(y) = tw(y) \) and note that \( \omega'(y) = \varepsilon q^{-1} \omega(y) \) and \( f'(y) = \varepsilon q^{-1} f(y) \). Applying the main Lemma \[2.3\] with \( a = 0 \), \( b = q \), we have

\[
S(t, \varepsilon, q) = \sum_{K_0(t) < k < K_1(t)} \frac{e^{2\pi i (f(y_k) - ky_k)}}{\sqrt{q |f'(y_k)|}} + q^{-1/2} \mathcal{E}(t) = \sum_{K_0(t) < k < K_1(t)} \frac{e^{2\pi i (f(y_k) - ky_k)}}{\sqrt{t}} \gamma(y_k) + q^{-1/2} \mathcal{E}(t) = \sum_{K_0(t) < k < K_1(t)} \frac{e^{2\pi i (-ky_k)}}{\sqrt{t}} \gamma(y_k) + q^{-1/2} \mathcal{E}(t),
\]

since \( f(y_k) = \varepsilon^{-1} q f'(y_k) = \varepsilon^{-1} q k \in \mathbb{Z} \),

\[
q |f'(y_k)| = q \frac{t}{q} e^{\varepsilon y/q} = t \gamma^{-2}(y_k), \quad \gamma(y) = e^{-\frac{1}{2} \varepsilon y/q}, \quad e^{\varepsilon y/q} = 1 + O(\varepsilon),
\]

and \( \mathcal{E}(t) \) has the following form (notation from Lemma \[2.3\])

\[
\mathcal{E}(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t) + O(\ln(2 + (b - a)R^{-1})) + O((b - a + R)U^{-1}) + O(\min\{\sqrt{R}, \max\left(\frac{1}{(\alpha)^j}, \frac{1}{(\beta)^j}\right)\}),
\]

where \( a = 0 \), \( b = q \) and values \( R \) and \( U \) are calculated below:

\[
b - a = q, \quad K_1(t) - K_0(t) = t + O(t \varepsilon) \sim t,
\]

\[
\frac{t}{q} \leq f''(y) \leq \frac{t}{q} (1 + O(\varepsilon)), \quad R = \frac{q}{t},
\]

\[
f^{(2 + j)}(y) = \frac{\varepsilon^j}{q^3} f''(y), \quad \beta_{2+j} = \frac{\varepsilon^j}{q^j}, \quad j = 1, 2, 3,
\]

\[
U = \frac{q}{\varepsilon} > 1,
\]

\[
|3\beta_4 - 5\beta_3^2| = \left| 3 \frac{\varepsilon^2}{q^2} - 5 \left( \frac{\varepsilon}{q} \right)^2 \right| = 2 \frac{\varepsilon^2}{q^2} = 2U^{-2}.
\]

So, applying Lemma \[2.3\] we obtain

\[
\mathcal{E}(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t) + O(\ln(2 + t)) + O \left( (q + t^{-1}q) \frac{\varepsilon}{q} \right) + \mathcal{E}_3(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t) + O(\max\{t^{-1}, \ln(t)\}) + \mathcal{E}_3(t),
\]
where
\[ E_3(t) = O(\min\{\frac{\sqrt{q}}{t}, \max\left(\frac{1}{\langle \alpha \rangle}, \frac{1}{\langle \beta \rangle}\right)\}). \]

The terms \( E_1(t) \) and \( E_2(t) \) are supported on a countable set. To compute \( E_3(t) \) look at the set
\[ \Lambda(t) = \{y_k(t)\}, \quad y_k(t) = \frac{q}{\varepsilon} \ln \frac{\varepsilon k}{t}. \]

The set \( \Lambda(t) \) is moving rigidly along the line when \( t \) changes. The term \( E_3(t) = O(\sqrt{q/t}) \) is large whenever some point in \( \Lambda(t) \) is close to border of \([0, q]\). At the same time for a typical \( t \) we have \( E_3(t) = O(1) \). Further, we have the following explicit representation for \( \langle \alpha \rangle \) and \( \langle \beta \rangle \):

\[ \langle \alpha \rangle = O(\|f'(0)\|_Z) = O\left(\left\|\frac{t}{\varepsilon} \right\|_Z\right), \quad \|x\|_Z = \min_{n \in \mathbb{Z}} |n - x|. \]

Similarly,
\[ \langle \beta \rangle = O(\|t^\varepsilon\|_Z). \]

Thus, omitting the terms \( E_1(t) \) and \( E_2(t) \) with countable supports we have
\[ \frac{|E(t)|}{\sqrt{q}} = O\left(\frac{\max\{t^{-1}, \ln t\}}{\sqrt{q}}\right) + O\left(\frac{1}{\sqrt{q}} \min\{\sqrt{\frac{q}{t}}, \nu_\varepsilon(t)\}\right). \]

Lemma 3.2. Given \( \varepsilon \) and \( q \) consider the sum \( S(t, \varepsilon, q) \) for \( t \in [t_1, t_2] \), and suppose that all conditions of the previous lemma are satisfied as well as the following requirements
\[ t_1 \geq 1, \quad t_2 - t_1 \geq \varepsilon, \quad t_2 \leq q, \quad \text{and} \quad \varepsilon^{-1} \leq q. \]

Then
\[ \|q^{-1/2} E|_{[t_1, t_2]}\|_\infty = O\left(\frac{\max\{1, \ln t_2\}}{\sqrt{q}}\right) + O\left(\frac{1}{\sqrt{q}} \ln \frac{1}{\sqrt{t_1}}\right), \]
\[ \|q^{-1/2} E|_{[t_1, t_2]}\|_1 \lesssim (t_2 - t_1) \frac{\ln q}{\sqrt{q}}. \]

Proof. Since \( t \geq \varepsilon \)
\[ \int_{t_1}^{t_2} |E(t)| \, dt \lesssim \frac{t_2 - t_1}{\varepsilon} \cdot \int_0^{\varepsilon} \min\left\{\sqrt{\frac{q}{t}}, \frac{\varepsilon}{t}\right\} \, d\tau \lesssim (t_2 - t_1) \ln q, \]
hence,
\[ \int_{t_1}^{t_2} \frac{|E(t)|}{\sqrt{q}} \, dt \lesssim \frac{(t_2 - t_1) \ln t_2 + (t_2 - t_1) \ln q}{\sqrt{q}} \lesssim (t_2 - t_1) \frac{\ln q}{\sqrt{q}}. \]
Our purpose is to apply this series of lemmas around van der Corput’s method to prove that $|S(t,\varepsilon,q)| \approx 1$ for most points $t$. Therefore let us formulate the following improved lemma.

**Lemma 3.3.** In the scope of Lemma 3.2 the expression for $S(t,\varepsilon,q)$ can be simplified as follows:

$$S(t,\varepsilon,q) = \frac{1}{\sqrt{t}} \sum_{K_0(t)<k<K_1(t)} e^{2\pi i (-ky_k)} + O(\varepsilon\sqrt{t}) + \frac{\mathcal{E}(t)}{\sqrt{t}},$$

where $\frac{\mathcal{E}(t)}{\sqrt{t}}$ is estimated as shown in Lemmas 3.1 and 3.2.

**Proof.** The lemma follows from the statement of lemma 3.2 and simple observation that the sum over $k$ contains no more than $K_1(t) - K_0(t) \sim t$ terms and the fact that $\gamma(y)$ is $\varepsilon$-close to 1 for $y \in [0,q]$, namely, $\gamma(y) = e^{-\frac{1}{2} \varepsilon y/q} = 1 + O(\varepsilon)$. □

At this point notice that in the expression for $S(t,\varepsilon,q)$ we see a reduced (with smaller number of terms) oscillatory sum of the same form as $S(t,\varepsilon,q)$, and we can iterate the reduction procedure applying van der Corput’s method to that sum. Denoting the new frequency function as $\eta_{\varepsilon,q}(k)$,

$$\eta_{\varepsilon,q}(k) = -ky_k,$$

let us see how it looks like for the sequence of stationary phases $y_k$ generated by an exponential Hamiltonian $\omega(y)$:

$$\eta_{\varepsilon,q}(k) = -ky_k = -\frac{qk}{\varepsilon} \ln \frac{\varepsilon k}{t} = q(\ln t - \ln \varepsilon) \ln k = x_{\varepsilon,q}(t)k - q \cdot \Omega_{\varepsilon}(k),$$

where

$$x_{\varepsilon,q}(t) = \frac{q(\ln t - \ln \varepsilon)}{\varepsilon}$$

and

$$\Omega_{\varepsilon}(k) = \frac{1}{\varepsilon} k \ln k.$$

It appears that starting from the property of constant distance between adjacent stationary phases we can find a variety of functions $\omega(y)$ (having continuum cardinality) generating flat reduced sums. At the same time we will see that surprisingly there are infinite subsequence of flat sums in the sequence of frequency functions $\eta_{\varepsilon,q}(k)$ without any modification in the construction. To see that we observe that modulo one $Y(q) = q \cdot \Omega_{\varepsilon}(k)$ is a point in the torus $\prod_k \mathbb{T}$ ruled by the dynamics of rigid shift by the vector $\Omega_{\varepsilon}(k)$.

**Lemma 3.4.** Let $k$ range over a finite set $\mathcal{K}$. For any $\delta > 0$ there exist infinitely many $q = q_j \rightarrow \infty$ such that for each $k \leq m\varepsilon^{-1}$

$$q_j \cdot \Omega_{\varepsilon}(k) = -\Omega_{\varepsilon}(k) + O(\delta) \pmod{1}.$$

**Proof.** The statement follows from Poincaré recurrence theorem for torus shift on $\prod_{k \in \mathcal{K}} \mathbb{T}$ by the vector $\Omega_{\varepsilon}(k)$.

□

**Corollary 1.** It is important that if $t$ ranges over a bounded segment, i.e. $t \in [t_1,t_2]$, then $k$ ranges over a finite set $[K_0(t_1),K_1(t_2)] \cap \mathbb{Z}$. Thus, the statement of the previous lemma concerning the dynamics of $Y(q)$ is true for a sequence $q_j$ simultaneously for all integer $k$ that can appear in $[K_0(t),K_1(t)]$ for all $t \in [t_1,t_2]$. 
Lemma 3.5. The oscillatory sum defined by $\Omega_\varepsilon(k)$ approximates the reduced sum given by the function $\eta_{\varepsilon,q}(k)$. If
\begin{equation}
q \cdot \Omega_\varepsilon(k) = -\Omega_\varepsilon(k) + O(\delta) \pmod{1}
\end{equation}
then
\begin{equation}
\left| \frac{1}{\sqrt{t}} \sum_{K_0(t) < k < K_1(t)} e^{2\pi i \eta_{\varepsilon,q}(k)} - \frac{1}{\sqrt{t}} \sum_{K_0(t) < k < K_1(t)} e^{2\pi i (x_{\varepsilon,q}(k) - \Omega_\varepsilon(k))} \right| = O(\sqrt{t}).
\end{equation}

Proof. We know that $\eta_{\varepsilon,q}(k) = x_{\varepsilon,q}(k) - q \Omega_\varepsilon(k)$ and the lemma follows from the estimate of the total number of terms in the sum, $K_1(t) - K_0(t) \sim t$. □

Now let us study the oscillatory sum defined by $\Omega_\varepsilon(k)$ with additional linear term $xk$
\begin{equation}
\Phi(t,x,\varepsilon) = \frac{1}{\sqrt{t}} \sum_{K_0(t) < k < K_1(t)} \exp(2\pi i (xk + \Omega_\varepsilon(k))).
\end{equation}

Remark 4. The fundamental observation concerning $\Omega_\varepsilon(k)$ consists of the following. When $k$ passes the interval $(K_0(t), K_1(t))$,
\begin{equation}
k \in (K_0(t), K_1(t)) = \left( \frac{t}{\varepsilon}, \frac{t}{\varepsilon} e^\varepsilon \right)
\end{equation}
the derivative $\Omega_\varepsilon(k) = \frac{1}{\varepsilon} (1 + \ln k)$ increases exactly by one:
\begin{equation}
\Omega'_\varepsilon(K_1(t)) - \Omega'_\varepsilon(K_0(t)) = \frac{1}{\varepsilon} \ln \frac{t}{\varepsilon} - \frac{1}{\varepsilon} \ln \left( \frac{t}{\varepsilon} e^\varepsilon \right) = 1,
\end{equation}
hence, we get (almost surely) exactly one stationary phase $k^* \in (K_0(t), K_1(t))$. Also let us point that the terms of the sum do not depend on $t$, just $K_0$ and $K_1$ do.

Lemma 3.6. Let the following conditions be satisfied:
\begin{equation}
t \geq 1, \quad 0 < \varepsilon < 1, \quad \varepsilon^{-1} \in \mathbb{N},
\end{equation}
and let $(k^*, \ell)$ be a unique solution of equation $x + \Omega'_\varepsilon(k^*) = \ell, \ell \in \mathbb{Z}, k^* \in (K_0(t), K_1(t))$ (if exists). Then
\begin{equation}
\Phi(t,x,\varepsilon) = e^{2\pi i (\Omega_\varepsilon(k^*) + xk^* - \ell k^*)} + t^{-1/2} \mathcal{E}_\Phi(t,x),
\end{equation}
\begin{equation}
t^{-1/2} \mathcal{E}_\Phi(t,x) = O(\varepsilon) + \frac{1}{\sqrt{t}} O\left( \min\{\sqrt{t}, \frac{1}{\sqrt{\|\ln \varepsilon + x\|_Z}}\} \right).
\end{equation}

Proof. Let us apply again Lemma 2.4 to $xk + \Omega_\varepsilon(k)$. Calculating
\begin{equation}
(xk + \Omega_\varepsilon(k))' = x + \frac{1}{\varepsilon} (1 + \ln k), \quad R^{-1} = \frac{1}{\varepsilon} \frac{1}{\varepsilon} - \frac{1}{\varepsilon}, \quad R = t.
\end{equation}
notice that $k \sim t/\varepsilon$ and we can take
\begin{equation}
R^{-1} = \frac{1}{\varepsilon} \frac{1}{\varepsilon} - \frac{1}{\varepsilon}, \quad \text{and} \quad R = t.
\end{equation}
Further, let us define
\begin{equation}
\beta_3(k) = -\frac{1}{k}, \quad \beta_4(k) = \frac{2}{k^2}, \quad \beta_5(k) = -\frac{6}{k^3},
\end{equation}
\begin{equation}
U = \frac{t}{\varepsilon} \sim k, \quad |3\beta_4 - 5\beta_5^2| = \frac{1}{k^2} = U^{-2},
\end{equation}
Recall that $\Omega_\varepsilon'(K_1(t)) - \Omega_\varepsilon(K_0(t)) = 1$. So, we the following representation for the sum $\Phi(t, x, \varepsilon)$:

$$
\Phi(t, x, \varepsilon) = \frac{1}{\sqrt{t|\Omega_\varepsilon(k^*)|}} e^{2\pi i (\Omega_\varepsilon(k^*) + x\varepsilon - t\ell k^*)} + q^{-1/2} \mathcal{E}_0(t, x),
$$

where $(k^*, \ell)$ are found as a unique solution (if exists) of the equation $x + \Omega_\varepsilon'(k^*) = \ell$ with $\ell \in \mathbb{Z}$, $k^* \in (K_0(t), K_1(t))$. Clearly the coefficient near exponent in the right-hand side is close to 1 in absolute value,

$$
\frac{1}{\sqrt{t|\Omega_\varepsilon'(k^*)|}} = \left( t \frac{1}{\varepsilon k^*} \right)^{-1/2} = \left( t \frac{1}{\varepsilon \varepsilon^{-1} t (1 + O(\varepsilon))} \right)^{-1/2} = 1 + O(\varepsilon),
$$

and using Lemma 2.4 we can compute the following estimate for the error term $\mathcal{E}_0$

$$
\mathcal{E}_0(t, x) = \mathcal{E}_1 + \mathcal{E}_2 + O(1) + \mathcal{E}_3, \quad \mathcal{E}_3 = O(\min\{\sqrt{t}, \tilde{\nu}\}),
$$

with

$$
\tilde{\nu} = \frac{1}{\|K_0(t) + x\|_Z} = \frac{1}{\frac{1}{\varepsilon} \ln \frac{t}{\varepsilon} + x\|_Z}.
$$

Finally, we have the following expression for the common error term

$$
t^{-1/2} \mathcal{E}_\Phi(t, x) = O(\varepsilon) + \frac{1}{\sqrt{t}} O \left( \min\{\sqrt{t}, \frac{1}{\|\frac{1}{\varepsilon} \ln \frac{t}{\varepsilon} + x\|_Z} \} \right).
$$

Now to join the reduction procedure from $S(t, \varepsilon, q)$ to $\Phi(t, x, \varepsilon)$ and approximation given by Lemma 3.6 we have to calculate boundary effect term $\frac{1}{\varepsilon} \ln \frac{t}{\varepsilon} + x_{\varepsilon, q}(t)$ as well as the argument of the exponent, namely, $\Omega_\varepsilon(k^*) + x_{\varepsilon, q}(t)k^* - \ell k^*$ (which is of independent interest but does not influence to flatness). So,

$$
v_{\varepsilon, q}(t) = \frac{1}{\varepsilon} \ln \frac{t}{\varepsilon} + x_{\varepsilon, q}(t) = (q - 1) \frac{1}{\varepsilon} \ln \frac{t}{\varepsilon}.
$$

Further, remark that

$$
\Omega_\varepsilon(k^*) + x_{\varepsilon, q}(t)k^* - \ell k^* = k^*(\ell - \frac{1}{\varepsilon})
$$

do not depend “directly” on $x_{\varepsilon, q}$ but via $k^*(t)$ and $\ell(t)$. The rate of grows of this doubly reduced frequency function can be estimated as follows

$$
\Omega_\varepsilon(k^*) + x_{\varepsilon, q}(t)k^* - \ell k^* = O(k^*(t)\ell(t)) \sim \frac{1}{\varepsilon} k^*(t) \ln k^*(t) \sim \frac{t}{\varepsilon^2} \ln \frac{t}{\varepsilon}.
$$

**Lemma 3.7.** In the scope of Lemmas 3.1 and 3.6 for an arbitrary $\delta > 0$ there exist a sequence $q = q_j \to \infty$ such that for $t \geq 1$

$$
S(t, \varepsilon, q) = \exp \left( 2\pi i k^*(t)(\ell(t) - \varepsilon^{-1}) \right) + \tilde{\mathcal{E}}(t)
$$

with an error term (defined up to countable set of points $t$)

$$
\tilde{\mathcal{E}}(t) = O(\delta \sqrt{t}) + O(\varepsilon \sqrt{t}) + O \left( \frac{\max\{1, \ln t\}}{\sqrt{q}} \right) + O \left( \frac{1}{\sqrt{q}} \min\{\sqrt{t}, \nu_{\varepsilon}(t)\} \right) +
$$

$$
\frac{1}{\sqrt{t}} O \left( \min\{\sqrt{t}, \frac{1}{\|q - 1\|_Z} \} \right).
$$

**Proof.** The proof is just a compilation of Lemmas 3.1 and 3.6
Lemma 3.8. Preserving conditions of Lemma 3.7 suppose that \( t \in [t_1, t_2] \) and \( t_2 - t_1 \geq (q - 1)^{-1} \varepsilon t_2 \). Then
\[
\left\| S(t, \varepsilon, q) \big|_{[t_1, t_2]} - 1 \right\|_1 = O \left( (t_2 - t_1) \frac{\sqrt{2} \max \{ \delta, \varepsilon \} + O \left( t_2 - t_1 \frac{\ln q}{\sqrt{q}} + O \left( t_2 - t_1 \frac{\ln t_2}{\sqrt{t_1}} \right) \right)}{\varepsilon} \right).
\]
Proof. We have to estimate the influence of the last term in the error given by the previous lemma,
\[
\int_{t_1}^{t_2} \frac{1}{\sqrt{t}} \min \left\{ \sqrt{t}, \frac{1}{\|q - 1\|_{L_2} \ln \frac{q}{\varepsilon}} \right\} \, dt \lesssim (t_2 - t_1) \frac{\ln t_2}{t_1}.
\]
\( \square \)

Lemma 3.9. If the parameters \( t_1, t_2, \varepsilon, q \) satisfy the following requirements
\[
t_1 \geq 1, \quad t_2 - t_1 \geq \varepsilon, \quad t_2 \leq \frac{q}{\ln q}, \quad \delta \leq \varepsilon,
\]
then for a sequence \( q_j \to \infty \)
\[
\left\| S(t, \varepsilon, q_j) \big|_{[t_1, t_2]} \right\|_1^2 - 1 \lesssim \left\| S(t, \varepsilon, q_j) \big|_{[t_1, t_2]} \right\|_1 - 1 = O \left( (t_2 - t_1) \frac{\ln t_2}{\sqrt{t_1}} \right).
\]
Proof. To established the improved estimate of the error term we just have to observe that all the components of the estimate are dominated by the term \( O \left( (t_2 - t_1) t_1^{-1/2} \ln t_2 \right) \). The first inequality of the lemma follows from the fact that \( \tilde{E}(t) = O(1) \).
\( \square \)

Recall that the initial frequency function \( \omega(y) \) satisfies differential equation \( \omega'' = a \omega' \) with solution \( \omega(y) = E_0 + e^{a(y - y_0)} \) for arbitrary \( E_0, y_0 \). By this point we studied \( \omega \) not taking into account correlation of additional multiplier \( m = e^{-ay_0} \) near \( t \) and the length \( q \) of the segment \([0, q] \).

Lemma 3.10. Let us consider the sum
\[
S_m(t, \varepsilon, q) = \frac{1}{\sqrt{q}} \sum_{y=0}^{q-1} e^{2\pi i \tau \omega(y)}
\]
with
\[
\omega(y, m) = m \frac{q}{\varepsilon^2} e^{y/q}.
\]
Suppose that \( t_1 = m \tau_1, \ t_2 = m \tau_2 \) as well as \( \varepsilon \) and \( q \) meet the requirements of the previous lemma. Then for a sequence \( q_j \to \infty \)
\[
\left\| S(t, \varepsilon, q_j) \big|_{[\tau_1, \tau_2]} \right\|^2_1 - 1 \lesssim O \left( (\tau_2 - \tau_1) \frac{\ln(m \tau_2)}{\sqrt{m \tau_1}} \right).
\]

Applying this lemma in the situation when \( \tau_1 \) and \( \tau_2 \) are fixed and \( m \) is sufficiently large we come to the following theorem.

Theorem 3.11. For given \( 0 < a < b, \varepsilon > 0 \) and \( \delta > 0 \) there exists \( m_0 \) such that for any \( m \geq m_0 \) there exists an infinite sequence \( q_j \) generating trigonometric sums with exponential staircase frequency function
\[
\omega(y, m) = m \frac{q_j}{\varepsilon^2} e^{y/q},
\]
which are \( \delta \)-flat in \( L^1([a, b]) \).
4. CONVERGENCE OF RIESZ PRODUCTS FOR EXPONENTIAL STAIRCASE FLOW

4.1. Preliminary lemmas. Let $R(t)$ be the correlation function of a measurable flow $T^t$. Observe that if the flow acts on a Lebesgue space then it obeys the following continuity property. If $f, g \in L^2(X)$ then

$$ (T^t f, g) \rightarrow (f, g) \quad \text{as} \quad t \rightarrow 0. $$

**Lemma 4.1.** Suppose that $R(t)$ is characteristic function of a spectral measure $\sigma_f$, i.e.

$$ R(t) = \hat{\sigma}_f(t) = \int \overline{e^{2\pi i tx}} \, d\sigma_f(x). $$

Let $\nu_n$ be a sequence of finite positive measures and let $R_n = \hat{\nu}_n$. Then $\nu_n$ converges weakly to $\sigma_f$ iff $R_n(t) \rightarrow R(t)$ for any $t$.

**Proof.** The statement follows from continuity of $R(t)$ at point $t = 0$. \qed

Throughout this section let $\nu_n$ be a sequence of finite positive measures. Evidently if we have to prove weak convergence of $\nu_n$ it is sufficient to watch $\int \psi \, d\nu_n$ for a dense set of functions $\psi$ in $C_0(\mathbb{R})$, for example a dense set of smooth functions with bounded support. Let us denote $C_0(a,b)$ the set of continuous functions $\psi(t)$ on $(a,b)$ such that $\psi(t) \rightarrow 0$ if $t \rightarrow a$ or $t \rightarrow b$.

**Lemma 4.2.** Let $\Phi(t)$ be a function such that it is locally $L^1$ and for any function $\psi \in C_0(a,b)$ with $0 < a < b$ (or $a < b < 0$)

$$ \int \psi \, d\nu_n \rightarrow \int \psi(t) \Phi(t) \, dt. $$

Suppose that $\nu_n \rightarrow \eta$ weakly. Then $\Psi \in L^1(\mathbb{R})$ and $\|\Psi\|_1 \leq \|\eta\|_1$.

**Proof.** Let us denote

$$ \langle \eta, \psi \rangle = \int \psi \, d\eta. $$

By definition of weak convergence $\langle \nu_n, \psi \rangle \rightarrow \langle \eta, \psi \rangle$, and, so

$$ \lim_{n \rightarrow \infty} \left| \langle \nu_n, \psi \rangle \right| \leq \left| \langle \eta, \psi \rangle \right| \leq \left\| \eta \right\|_1 \left\| \psi \right\|_\infty. $$

At the same time we know that $\langle \nu_n, \psi \rangle \rightarrow \langle \Phi, \psi \rangle$, hence,

$$ \left| \langle \Phi, \psi \rangle \right| \leq \left\| \eta \right\|_1 \left\| \psi \right\|_\infty, $$

and the statement follows from the equality $\|\Phi\|_1 = \|\lambda_\Phi\|$ where $\lambda_\Phi$ is the measure with density $\Phi$. \qed

**Lemma 4.3.** Let $\eta$ be a finite positive measure on the line. If $\int \psi \, d\eta = \int \psi(t) \Phi(t) \, dt$ for any $\psi \in C_0(a,b)$ then $\eta = \eta(\{0\}) \delta_0 + \lambda_\Phi$ where $\delta_0$ is the unit atomic measure in $0$ and $\lambda_\Phi$ is the measure with density $\Phi$.

**Proof.** It is possible to check coincidence on positive continuous functions $\psi$ with bounded support since its linear combinations are dense in $C_0(\mathbb{R})$. For any $\varepsilon > 0$ there exists $\alpha$ such that

$$ \int_{-\alpha}^{\alpha} \Phi(t) \, dt < \varepsilon $$

and

$$ \eta|_{[-\alpha, \alpha]} = \eta(\{0\}) \delta_0 + \eta(\alpha) \quad \text{with} \quad \|\eta(\alpha)\| < \varepsilon. $$
Let us split \( \psi \) in two parts, \( \psi = \psi_0 + \psi_1 \) with the following properties:

\[
\text{(116)} \quad \text{supp } \psi_0 \subseteq [-\alpha, \alpha], \quad \text{supp } \psi_1 \subseteq [-L, -\alpha/2] \cup [\alpha/2, L]
\]

and \( 0 \leq \psi_0(t), \psi_1(t) \leq \psi(t) \) for any \( t \). We have

\[
\text{(117)} \quad \langle \eta, \psi \rangle = \langle \eta, \psi_0 \rangle + \langle \eta, \psi_1 \rangle = \eta(\{0\}) \psi(0) + \langle \eta(\alpha), \psi_0 \rangle + \langle \Phi, \psi_1 - \psi \rangle + \langle \Phi, \psi \rangle.
\]

Using inequality \( |\psi(t)| \leq \| \psi \|_\infty \), we can estimate small terms in the above formula:

\[
\text{(118)} \quad |\langle \eta(\alpha), \psi_0 \rangle| \leq \varepsilon \cdot \| \psi \|_\infty, \quad |\langle \Phi, \psi_1 - \psi \rangle| \leq |\langle \Phi, \psi \rangle| < \varepsilon \cdot \| \psi \|_\infty.
\]

Since \( \varepsilon \) is arbitrary we get \( \langle \eta, \psi \rangle = \eta(\{0\}) \psi(0) + \langle \Phi, \psi \rangle. \)

Lemma 4.4. Consider positive functions \( f(t) \) and \( Q_n(t) \) on the real line \( \mathbb{R} \) satisfying the following conditions:

(i) \( f \in L^1(\mathbb{R}) \);

(ii) \( Q_n \) are continuous;

(iii) \( \int_{a_n}^{b_n} |Q_n(t) - 1|\,dt < \varepsilon_n^2 \) and \( \int_{-a_n}^{-b_n} |Q_n(t) - 1|\,dt < \varepsilon_n^2 \), where \( 0 < a_n < b_n \);

(iv) intervals \((a_n, b_n)\) monotonously increase and exhaust \((0, +\infty)\), exactly, \(a_n+1 < a_n, b_n < b_{n+1}, \lim_n a_n = 0 \) and \( \lim_n b_n = +\infty \);

(v) \( \sum_{n=1}^{\infty} \varepsilon_n < \infty \).

The infinite product of \( f \) and \( Q_n \) converges in \( L^1 \) on any interval \((a^*, b^*)\) or \((-b^*, -a^*)\), where \( 0 < a^* < b^* \).

Proof. Without lost of generality, passing from \( f(t) \) to \( \tilde{f}(t) = fQ_1 \cdots Q_{n_1}(t) \) we can assume that \((a^*, b^*) \subseteq (a_1, b_1)\). Let us consider sets

\[
\text{(119)} \quad E_n = \{ t \in (a^*, b^*): |Q_n(t) - 1| > \varepsilon_n \}.
\]

By Chebyshev inequality \( \lambda(E_n) < \varepsilon_n^2/\varepsilon_n = \varepsilon_n \). Using Borel–Cantelli lemma we can deduce from convergence \( \sum_n \lambda(E_n) = \infty \) that almost surely a point \( t \in (a^*, b^*) \) belongs to finitely many sets \( E_n \), hence, the product \( f(t) \prod_n Q_n \) converges by Lebesgue dominated convergence theorem. Indeed, if we restrict functions to the set

\[
\text{(120)} \quad A_N = (a^*, b^*) \setminus \bigcup_{n>N} E_n, \quad \lambda(A_n) \to b^* - a^*,
\]

then by pointwise convergence

\[
\text{(121)} \quad \prod_{n=N+1}^{m} Q_n(t) \to \prod_{n>N} Q_n(t), \quad |Q_n(t) - 1| < \varepsilon_n \quad \text{if} \quad n > N,
\]

we get convergence in \( L^1 \)

\[
\text{(122)} \quad \Pi_m = f \prod_{n=1}^{m} Q_n \to f \prod_{n} Q_n
\]

for the restrictions to the interval \((a^*, b^*)\).

Lemma 4.5. Preserving conditions of the previous lemma let us assume that the partial products \( \Pi_m(t) \) are \( L^1 \)-densities of measures on \( \mathbb{R} \) which converge weakly to a finite positive measure \( \eta \), i.e. for any \( \psi \in C_0(\mathbb{R}) \)

\[
\text{(123)} \quad \int \psi(t) \Pi_m(t)\,dt \to \langle \eta, \psi \rangle.
\]
Then there is a function $\Phi \in L^1(\mathbb{R})$ such that $\eta = \eta(\{0\})\delta_0 + \lambda_\Phi$ and for Lebesgue almost every $t \in \mathbb{R}$ we have $\Phi(t) = f(t) \prod_n Q_n(t)$.

Proof. Consider an interval $(a^*, b^*)$ with $0 < a^* < b^*$. By $L^1$-convergence $\Pi_n|_{(a^*, b^*)}$ converges weakly to the measure with density $f(t) \prod_n Q_n(t)$ restricted to $(a^*, b^*)$. This implies that the weak limit $\eta$ of $\Pi_n$ acting on functions $\psi \in C_0(a^*, b^*)$ coincide with $\Phi(t) = f(t) \prod_n Q_n(t)$. Using above lemmas we found that $\Phi \in L^1(\mathbb{R})$ and $\eta = \eta(\{0\})\delta_0 + \lambda_\Phi$. \hfill $\square$

4.2. Spectral type of exponential staircase flows. Consider exponential staircase construction of rank one flow which is characterized by the following parameters: numbers of subcolumns $q_n$ and staircase grade values $m_n$. We define $\varepsilon_n$ by the equality

$$(124) \quad h_n = \frac{m_n}{\varepsilon_n},$$

and require that $\varepsilon^{-1} \in \mathbb{Z}$.

**Lemma 4.6.** The spectral measure $\sigma_f$ of a function $f$ measurable up to $n_0$-th level partition is given by the Riesz product

$$(125) \quad \sigma_f = |\hat{f}|^2 \prod_{n \geq n_0} |P_n(t)|^2,$$

where

$$(126) \quad P_n(t) = \frac{1}{\sqrt{q_n}} \sum_{y=0}^{q_n-1} e^{2\pi i \omega_n(y)},$$

with exponential staircase frequency function

$$(127) \quad \omega_n(y) = m_n \frac{q_n}{\varepsilon_n} (e^{\varepsilon_n y/q_n} - 1).$$

Remark that in $|P_n(t)|^2$ we can omit $-1$ in the parenthesis, and recall that $h_{n+1} \geq q_n h_n$, hence, $h_n$ are uniquely determined by $q_n$. Let us define $Q_n(t) = |P_n(t)|^2$. We know that if we restrict $Q_n(t)$ to $(a_n, b_n)$ then

$$(128) \quad \|Q_n(t)_{(a_n, b_n)} - 1\|_1 = O\left( \frac{b_n - a_n}{\sqrt{a_n m_n} \ln(b_n m_n)} \right).$$

Thus to apply machinery described in the beginning of this section we need estimate

$$(129) \quad \sum_{n=1}^{\infty} b_n \frac{(\ln m_n)^{1/2}}{m_n^{1/4}} < \infty,$$

where for simplicity we assume that $a_n = b_n^{-1}$.

**Theorem 4.7.** Let us consider the exponential staircase construction of rank one flow $T^t$ given by the sequence of functions

$$(130) \quad \omega_n(y) = m_n \frac{q_n}{\varepsilon_n} e^{\varepsilon_n y/q_n}, \quad h_n = \frac{m_n}{\varepsilon_n}.$$ 

Suppose that for an $\alpha$ with $0 < \alpha < 1/4$

$$(131) \quad h_n^{1+\alpha} \leq q_n, \quad m_n \leq h_n^{1/2-\alpha}, \quad \sum_{n=1}^{\infty} \frac{1}{m_n^{1/4-\alpha}} < \infty.$$
and $q_n$ has the following property:

$$\forall k \in \mathbb{Z} \cap [m_n^{1-\alpha} \varepsilon_n, m_n^{1+\alpha} \varepsilon_n] \quad \| (q_n + 1)^{1/k} \ln k \|_\mathbb{Z} \leq \varepsilon_n,$$

where

$$\| x \|_\mathbb{Z} = \min \{ |\ell - x| : \ell \in \mathbb{Z} \}.$$

Then the flow $T^t$ has Lebesgue spectral type.

**Proof.** The theorem follows directly from lemmas 3.10 and 4.5 as well as ergodicity of the flow. Let us check necessary conditions $m_n, q_n$ and $h_n$ should satisfy. First, notice that it follows from inequality $q_n \geq h_n$ that $h_n \geq \text{const} \cdot 2^{c^2 n}$. Since $h_n = \frac{m_n}{\varepsilon_n}$, condition $m_n b_n \leq \varepsilon_n^{-1}$ follows from comparison of $m_n$ and $h_n$,

$$m_n \leq h_n^{1/2-\varepsilon}, \quad \frac{1}{\varepsilon_n} = \frac{h_n}{m_n} \geq h_n^{1/2+\varepsilon},$$

and $\varepsilon^{-1} \leq h_n \leq \frac{1}{2 n^{1/(1+\varepsilon)}}$. Thus, the series $\sum_{n=1}^{\infty} b_n \frac{\ln m_n}{m_n^{1/2}}$ converges and all other requirements of $m_n, \varepsilon_n$ and $q_n$ are satisfied, so we can apply the above flatness lemma 3.10 concerning exponential staircase trigonometric sums with convergence lemmas 4.4 and 4.5. To complete the proof we have to mention that it follows from ergodicity of our rank one flow that the spectral measure has no atom at zero.

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**References**

[1] D.V. Anosov, *On spectral multiplicity in ergodic theory*, in Modern problems in Mathematics, Steklov Institute, Issue 3 (2003).

[2] J. Dieudonné, *Infinitesimal Calculus*, Hermann, 1971.

[3] T. Erdelyi, *Polynomials with Littlewood-type coefficient constraints*, Approximation Theory X: Abstract and Classical Analysis, Charles K. Chui, Larry L. Schumaker, and Joachim Stockler (Eds.), Vanderbilt University Press, Nashville, TN, 2002, 153-196, ISBN 0-8265-1415-4.

[4] G.R. Goodson, *A Survey of Recent Results in the Spectral Theory of Ergodic Dynamical Systems*, Journal of Dynamical and Control Systems, Vol. 5, Issue 2 (1999), 173–226.

[5] P.R. Halmos, *Lectures on ergodic theory*, Publications of the Mathematical Society of Japan, 3, Math. Soc. Japan, Tokyo, 1956.

[6] I.P. Kornfel’d, Ya. G. Sinaïand S.V. Fomin. *Ergodic theory*, Nauka, Moscow 1980; English transl., Springer–Verlag, Berlin–Heidelberg–New York 1982.

[7] A.B. Katok, A.M. Stepin. “Approximations in ergodic theory”. Uspekhi Mat. Nauk 22:5 (1967), 81–106; English transl. in *Russian Math. Surveys* 22:5 (1967).

[8] M.G. Nadkarni. *Spectral Theory of Dynamical Systems*, Hindustan Book Agency, New Delhi, (1998); Birkhäuser Advanced Texts : Basler Lehrb?her. [Birkhäuser Advanced Texts: Basel Textbooks] Birkhäuser Verlag, Basel, 1998.

[9] V.A. Rokhlin, *Selected topics in metric theory of dynamical systems*, UMN, 42:2(230) (1949), 57-128.

[10] E.C. Titchmarsh, *The Theory of the Riemann zeta-Function*, second edition Revised by D.R. Heath-Brown, Oxford, 1986.

[11] J.-P. Thouvenot, *Some properties and applications of joinings in ergodic theory*, Ergodic Theory and its Connections with Harmonic Analysis, (K.E. Petersen, ed.), Cambridge Univ. Press, Cambridge, 1995, pp. 207–235.
