Abstract

We discuss questions arising from the recent work of Schellekens\[17\], and also from an earlier paper by Schellekens and Yankielowicz\[19\]. We summarise Schellekens’ results, and proceed to discuss the uniqueness of the $c = 24$ self-dual conformal field theory with no weight one states, i.e. the Monster module $V^\natural[8]$. After introducing the concept of complementary representations, we examine $\mathbb{Z}_2$-orbifold constructions in general, and then proceed to apply our considerations firstly to the specific case of the FKS constructions $\mathcal{H}(\Lambda)$ and then to the reflection twisted theories $\tilde{\mathcal{H}}(\Lambda)[5]$. Our techniques provide evidence for the existence of several new theories beyond those proven to exist in [3] and conjectured to exist in [19].

1 Introduction

Recently much progress has been made towards the classification of CFT’s. One approach to this problem is to study the algebra of the fusion rules of representations of some chiral algebra (an extension of the Virasoro algebra)\[22\]. However, we are interested here in CFT’s which are overlooked by this technique, i.e. theories whose fusion rules are trivial, though they themselves are not necessarily without an interesting structure. (Indeed, one such theory is the natural module $V^\natural$ for the Monster, first constructed by Frenkel, Lepowsky and Meurman\[3, 4, 8\].) Such theories clearly must be classified separately from the mainstream if fusion rule techniques are to be used.

We consider chiral bosonic meromorphic CFT’s defined on the Riemann sphere (see [11, 13] for the relevant definitions, and also [10] for a mathematician’s view). We define a CFT $\mathcal{H}$ to be self-dual if the partition function

$$\chi_\mathcal{H}(\tau) = \text{Tr}_\mathcal{H} q^{L_0 - c/24},$$

\[1\]
where \( q = e^{2\pi i \tau} \), is covariant under modular transformations of \( \tau \), i.e. invariant under
\[ S : \tau \mapsto -1/\tau \]
and invariant up to a phase under
\[ T : \tau \mapsto \tau + 1. \]
(So that the full partition function when the antichiral sector is included is then modular invariant, as required for the theory to be physically well-defined on the torus described by the parameter \( \tau \).) This restricts us to \( c \in 8\mathbb{Z} \). The theories for \( c = 8 \) and \( c = 16 \) are easily classified[11]. They are simply the FKS constructions \( \mathcal{H}(\Lambda) \) from the even self-dual lattices in the corresponding dimensions, i.e. the root lattice of \( E_8 \) in 8 dimensions and the lattices \( E_8^2 \) and \( D_{16}^+ \) (an extension of the root lattice of \( D_{16} \) by adding in one of the spinor weights) in 16 dimensions.

Only in 24 dimensions does the problem of classification first become non-trivial. Indeed, it may be argued that, since the classification of even the even self-dual lattices in more than 24 dimensions is intractable at present due to the rapid increase in their number, then \( c = 24 \) is really the only case worthy of consideration. There are 24 inequivalent even self-dual lattices in 24 dimensions[3]. The constructions \( \mathcal{H}(\Lambda) \) and \( \tilde{\mathcal{H}}(\Lambda) \) of \[5\] would thus be naively expected to produce 48 CFT’s. However, it is shown in [3] and [4] that the constructions produce equivalent theories if and only if there is a corresponding doubly-even self-dual binary code. There are 9 such codes in 24 dimensions[4], and so we obtain 39 distinct self-dual \( c = 24 \) CFT’s. These are, so far, the only such theories which have been constructed explicitly, though in [19] two further theories were postulated to exist. We shall discuss these further in the following section.

2 Schellekens’ results

In [17] Schellekens proves results for conformal field theories analogous to those obtained by Venkov in his reformulation of Niemeier’s classification of even self-dual lattices[21, 3]. In particular, it is shown that the Kac-Moody algebra generated by the modes of the weight one states[11] is restricted to contain components whose central charges sum to 24 and moreover have a common value for the ratio of the Coxeter number to the level, given in terms of the number \( N \) of weight one states by

\[
\frac{g}{k} = \frac{N}{24} - 1. \tag{2}
\]

The possible combinations of algebras thus allowed in a \( c = 24 \) self-dual conformal field theory with \( N \) weight one states are shown below. The rank is also indicated for convenient reference. A * indicates that the theory is one of the 39 obtained by the FKS construction or a reflection twist of such a theory. The two theories marked by a † are those claimed (but not proven) to exist in [19]. Their evidence consisted of the construction of a modular invariant combination of characters of the corresponding Kac-Moody algebras, though no explicit constructions were given. We shall return to this question in a later section. We mark by ⊕ new theories proposed/constructed in this work.

| \( N \) | algebra | rank | \( N \) | algebra | rank |
|---|---|---|---|---|---|
| 0 | \( \emptyset \) | 0 | 24 | \( U(1)^{24} \) | 24 |
| 25 | \( A_{3,96}C_{2,72} \) | 5 | 25 | \( A_{1,48}A_{2,72}G_{2,96} \) | 5 |
| 25 | \( A_{1,48}^3A_{2,72}^2 \) | 7 | 25 | \( A_{1,48}A_{2,72} \) | 5 |
| 26 | \( A_{2,36}^2C_{2,36} \) | 6 | 26 | \( A_{1,24}A_{2,36}A_{3,48} \) | 6 |
| 26 | $A_{1,24}^2C_{2,36}^2$ | 6 | 26 | $A_{1,24}^4G_{2,48}$ | 6 |
| 26 | $A_{1,24}^6A_{2,36}$ | 8 | 27 | $A_{1,16}C_{2,24}G_{2,32}$ | 5 |
| 27 | $A_{1,16}A_{4,40}$ | 5 | 27 | $A_{1,16}A_{2,24}^3$ | 7 |
| 27 | $A_{1,16}^2C_{3,32}$ | 5 | 27 | $A_{1,16}^2B_{3,30}$ | 5 |
| 27 | $A_{1,16}^3A_{2,24}C_{2,24}$ | 7 | 27 | $A_{1,16}^4A_{3,32}$ | 7 |
| 28 | $G_{2,48}$ | 2 | 28 | $G_{2,24}$ | 4 |
| 28 | $A_{1,12}A_{2,18}C_{2,18}$ | 6 | 28 | $A_{1,12}^2A_{2,18}G_{2,24}$ | 6 |
| 28 | $A_{1,12}^4A_{2,18}^2$ | 8 | 28 | $A_{1,12}^6C_{2,18}$ | 8 |
| 30 | $C_{2,12}$ | 6 $\oplus$ | 30 | $A_{3,16}$ | 6 |
| 30 | $A_{2,12}^2G_{2,16}$ | 6 | 30 | $A_{1,16}^2C_{2,12}G_{2,16}$ | 6 |
| 30 | $A_{1,16}A_{4,20}$ | 6 | 30 | $A_{1,16}^2A_{2,12}^3$ | 8 |
| 30 | $A_{1,16}^3C_{3,16}$ | 6 | 30 | $A_{1,16}^3B_{3,20}$ | 6 |
| 30 | $A_{1,16}^4A_{3,12}C_{2,12}$ | 8 | 30 | $A_{1,16}^5A_{3,16}$ | 8 |
| 30 | $A_{1,16}^{10}A_{2,12}$ | 10 | 32 | $A_{2,9}$ | 8 |
| 32 | $A_{2,9}A_{4,15}$ | 6 | 32 | $A_{1,16}A_{2,9}C_{3,12}$ | 6 |
| 32 | $A_{1,16}A_{2,9}B_{3,15}$ | 6 | 32 | $A_{1,16}^2A_{2,9}^2C_{2,9}$ | 8 |
| 32 | $A_{1,16}^3A_{2,9}^3C_{2,12}$ | 8 | 32 | $A_{1,16}^4C_{2,9}^2$ | 8 |
| 32 | $A_{1,16}^6G_{2,12}$ | 8 | 36 | $A_{1,16}^8A_{2,9}$ | 10 |
| 36 | $C_{4,10}$ | 4 | 36 | $B_{4,14}$ | 4 |
| 36 | $A_{3,8}A_{3,8}$ | 6 | 36 | $A_{3,8}A_{3,10}$ | 6 |
| 36 | $A_{2,6}G_{2,8}^2$ | 6 | 36 | $A_{2,6}D_{4,12}$ | 6 |
| 36 | $A_{2,6}^2C_{2,6}^2$ | 8 | 36 | $A_{1,4}A_{2,6}A_{3,8}C_{2,6}$ | 8 |
| 36 | $A_{1,4}^2C_{2,6}^3$ | 8 | 36 | $A_{1,4}^2A_{3,8}^2$ | 8 |
| 36 | $A_{1,4}^2A_{2,6}G_{2,8}$ | 8 | 36 | $A_{1,4}^4C_{2,6}G_{2,8}$ | 8 |
| 36 | $A_{1,4}^4A_{4,10}$ | 8 | 36 | $A_{1,4}^4A_{2,6}$ | 10 |
| 36 | $A_{1,4}^5C_{3,8}$ | 8 | 36 | $A_{1,4}^5B_{3,10}$ | 8 |
| 36 | $A_{1,4}^6A_{2,6}C_{2,6}$ | 10 | 36 | $A_{1,4}^7A_{3,8}$ | 10 |
| 36 | $A_{1,4}^{12}C_{3,8}$ | 12 * | 40 | $A_{1,3}^4G_{2,6}^2$ | 8 |
| 40 | $A_{1,3}^4D_{4,19}$ | 8 | 42 | $A_{2,4}^4C_{2,4}$ | 10 $\oplus$ |
| 48 | $C_{2,3}^2G_{2,4}^2$ | 8 $\oplus$ | 48 | $C_{2,3}^2D_{4,6}$ | 8 |
| 48 | $A_{6,7}$ | 6 | 48 | $A_{4,5}C_{2,3}G_{2,4}$ | 8 |
| 48 | $A_{4,5}^2$ | 8 | 48 | $A_{2,3}C_{2,3}^4$ | 10 |
| 48 | $A_{2,3}A_{3,4}C_{2,3}^2$ | 10 | 48 | $A_{2,3}A_{2,3}^3C_{2,3}G_{2,4}$ | 10 |
| 48 | $A_{2,3}A_{3,5}^2$ | 10 | 48 | $A_{2,3}^6$ | 12 |
| 48 | $A_{1,2}D_{5,8}$ | 6 | 48 | $A_{1,2}C_{2,3}C_{3,4}G_{2,4}$ | 8 |
| 48 | $A_{1,2}B_{3,5}^2C_{2,3}G_{2,4}$ | 8 | 48 | $A_{1,2}A_{5,6}C_{2,3}$ | 8 |
| 48 | $A_{1,2}A_{4,5}C_{3,4}$ | 8 | 48 | $A_{1,2}A_{4,5}B_{3,5}$ | 8 |
| 48 | $A_{1,2}A_{3,4}C_{2,3}^3$ | 10 | 48 | $A_{1,2}A_{3,4}^3$ | 10 |
| 48 | $A_{1,2}A_{2,3}^2A_{3,4}G_{2,4}$ | 10 | 48 | $A_{1,2}A_{2,3}^3C_{3,4}$ | 10 |
| 48 | $A_{1,2}A_{2,3}^3B_{3,5}$ | 10 | 48 | $A_{1,2}^2G_{2,4}^3$ | 8 |
| 48 | $A_{1,2}^2D_{4,6}G_{2,4}$ | 8 | 48 | $A_{1,2}^2C_{3,4}^2$ | 8 |
| 48 | $A_{1,2}^2B_{3,5}C_{3,4}$ | 8 | 48 | $A_{1,2}^2B_{3,5}^2$ | 8 |
| 48  | $A_{1,2}A_2^2 A_{2,3}C_{2,3}G_{2,4}$ | 10 | 48  | $A_{1,2}A_2^2 A_{2,3}A_{4,5}C_{2,3}$ | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,1}A_{2,3}C_{2,3}$          | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}C_{2,3}G_{2,4}$ | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_{3,4}A_{2,5}$          | 10 | 48  | $A_{1,2}A_2^2 A_{2,3}C_{2,3}G_{2,4}$ | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{2,3}B_{3,5}C_{2,3}$ | 10 | 48  | $A_{1,2}A_2^2 A_{3,4}A_{2,3}C_{2,3}$ | 12 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}B_{4,5}$                  | 8  | 48  | $A_{1,2}A_2^2 A_{3,4}A_{2,3}G_{2,4}$ | 12 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}C_{3,4}$    | 10 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{2,3}G_{2,4}$    | 10 | 48  | $A_{1,2}A_2^2 A_{3,4}D_{4,6}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}G_{2,4}$    | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}C_{2,3}$ | 12 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{2,3}C_{2,3}$    | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}G_{2,4}$ | 12 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{2,3}G_{2,4}$    | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}C_{2,3}$ | 12 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{2,3}C_{2,3}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}G_{2,4}$ | 12 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$    | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}G_{2,4}$    | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{2,3}C_{2,3}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}C_{2,3}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}G_{2,4}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}C_{2,3}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}G_{2,4}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}C_{2,3}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}G_{2,4}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}C_{2,3}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}G_{2,4}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}C_{2,3}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}G_{2,4}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}C_{2,3}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}G_{2,4}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}C_{2,3}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}G_{2,4}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}C_{2,3}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}G_{2,4}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}C_{2,3}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}G_{2,4}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
|-----|-----------------------------------|--|-----|-----------------------------------|--|
| 48  | $A_{1,2}A_2^2 A_{3,4}A_{3,4}C_{2,3}$ | 12 | 48  | $A_{1,2}A_2^2 A_{3,4}B_{3,5}$      | 10 |
### 3 Uniqueness of the $N = 0$ theory

It is to be expected, by analogy with the situation for codes and also lattices, that the $N = 0$ theory would be unique. (Note that such analogies are not perfect, since in Venkov's work on the even self-dual lattices, the previous section mirrored, there is one and only one lattice corresponding to each possible combination of algebras, whereas Schellekens has observed that there are some algebras in the above list for which it is impossible to obtain even a modular invariant combination of characters, let alone a fully consistent conformal field theory. Nevertheless, we expect uniqueness when the theories do exist, and almost certainly uniqueness is to be expected in the case $N = 0$ [in the case of lattices, the uniqueness proofs for $N = 0$ and $N > 0$ are distinct]). Let us consider this problem.

Suppose that $\mathcal{H}$ is a self-dual $c = 24$ conformal field theory with no weight one states. Suppose an involution $g$ exists, and consider the orbifold constructed using it (which we also suppose to exist). We may evaluate the partition function for the orbifold in terms of the Thompson series for the involution, i.e.

\[
\chi_{\mathcal{H}_g}(\tau) = \frac{1}{2} \left( (1 + g^2) \right) \quad (3)
\]

\[
= \frac{1}{2} (J(\tau) + T_g(\tau) + T_g(S(\tau)) + T_g(ST(\tau))) . \quad (4)
\]

Clearly, $T_g(\tau) = g^2$ is invariant under $\Gamma_0(2)$, i.e. under $T$ and $ST^2S$. If it has the correct behaviour at $q = 1$ (i.e. if the ground state of the twisted sector has energy $\geq 1$ [20]) then

| $N$ | $120$ | $A_{3,1}^8$ | $132$ | $A_{8,2}F_4,2$ | $12$ |
|-----|-------|-------------|-------|----------------|-----|
| $144$ | $C_{4,1}^4$ | $16$ | $144$ | $B_{3,1}C_{4,1}D_{6,2}$ | $16$ |
| $144$ | $A_{4,1}B_{3,1}^4$ | $20$ | $144$ | $A_{4,1}A_{9,2}B_{3,1}$ | $16$ |
| $144$ | $A_{4,1}^3C_{4,1}$ | $20$ | $144$ | $A_{4,1}^6$ | $24$ |
| $156$ | $B_{6,2}$ | $12$ | $168$ | $D_{4,1}^6$ | $24$ |
| $168$ | $A_{5,1}E_{7,3}$ | $12$ | $168$ | $A_{5,1}C_{5,1}D_{6,2}$ | $16$ |
| $168$ | $A_{5,1}^4D_{4,1}$ | $24$ | $192$ | $B_{4,1}C_{6,1}^2$ | $16$ |
| $192$ | $B_{4,1}^2D_{8,2}$ | $16$ | $192$ | $A_{6,1}B_{4,1}^4$ | $22$ |
| $192$ | $A_{6,1}^4$ | $24$ | $216$ | $A_{7,1}D_{9,2}$ | $16$ |
| $216$ | $A_{7,1}^2D_{5,1}^2$ | $24$ | $240$ | $C_{8,1}F_{4,1}^2$ | $16$ |
| $240$ | $B_{5,1}E_{7,2}F_{4,1}$ | $16$ | $240$ | $A_{8,1}^3$ | $24$ |
| $264$ | $D_{6,1}^4$ | $24$ | $264$ | $A_{9,1}^2D_{6,1}$ | $24$ |
| $288$ | $B_{6,1}C_{10,1}$ | $16$ | $300$ | $B_{12,2}$ | $12$ |
| $312$ | $E_{6,1}^4$ | $24$ | $312$ | $A_{11,1}D_{7,1}E_{6,1}$ | $24$ |
| $336$ | $A_{12,1}^2$ | $24$ | $360$ | $D_{8,1}^3$ | $24$ |
| $384$ | $B_{8,1}E_{8,2}$ | $16$ | $408$ | $A_{15,1}D_{9,1}$ | $24$ |
| $456$ | $D_{10,1}E_{7,1}^2$ | $24$ | $456$ | $A_{17,1}E_{7,1}$ | $24$ |
| $552$ | $B_{12,1}^2$ | $24$ | $624$ | $A_{24,1}$ | $24$ |
| $744$ | $E_{8,1}^3$ | $24$ | $744$ | $D_{16,1}E_{8,1}$ | $24$ |
| $1128$ | $D_{24,1}$ | | | | }
it is a $\Gamma_0(2)$ hauptmodul. Thus, it is known explicitly:

\[ T_g(\tau) = \left( \frac{\eta(\tau)}{\eta(2\tau)} \right)^{24} + 24. \quad (5) \]

We find that

\[ \chi_{H_g}(\tau) = J(\tau) + 24. \quad (6) \]

Thus, in the above notation $N = 24$, and we see from Schellekens’ list that the only possibility is to have algebra $U(1)^{24}$. Now, we have the theorem from [4] that when the rank is equal to the central charge, the theory is equivalent to an FKS lattice theory $\mathcal{H}(\Lambda)$ for some even lattice $\Lambda$, and further that the CFT is self-dual if and only if the lattice is self-dual. So $\mathcal{H}_g \cong \mathcal{H}(\Lambda_{24})$, the Leech lattice being the unique even self-dual lattice in 24 dimensions having no vectors of length squared two (thus giving 24 weight 1 states in the CFT).

To proceed now in a way analogous to Venkov’s proof of the uniqueness of the Leech lattice (i.e. following the spirit of Schellekens’ approach in the previous section), we would assume the existence of an automorphism $h$ of our new theory $\mathcal{H}_g$ such that $(\mathcal{H}_g)_h$ exists and is isomorphic to the original theory $\mathcal{H}$, i.e. we assume the existence of an inverse to the orbifold construction. The projection onto $h$ invariant states removes all the weight one states, by definition. Hence we must have that $h$ acts as $-1$ on the states $a_1^j|0\rangle$ (using the notation of [3]). We deduce from this that $h$ is simply the automorphism of $\mathcal{H}(\Lambda_{24})$ induced by the reflection twist on $\Lambda_{24}$. But we know that $\mathcal{H}(\Lambda_{24})_{-1} \cong V^\natural$, the natural Monster module $[9, 4]$. Hence, $\mathcal{H} \cong V^\natural$ as required.

Note that this is not a proof of the uniqueness of the $N = 0$ theory. What we have demonstrated is that the uniqueness problem, modulo some general results on orbifold theory not specific to the Monster, is equivalent to the hauptmodul property, or, as Tuite has shown, to the nature of the energy and degeneracy of the twisted sector ground state. This exercise has merely demonstrated how Schellekens’ work allows us to tackle problems by giving us data on the possible $c = 24$ theories. What we have done is essentially analogous to Venkov’s proof of the uniqueness of the Monster in the same way that Schellekens’ work itself is simply an analogue of (part of) Venkov’s reformulation of Niemeier’s classification of lattices. The power of Schellekens’ result is that, we can say what the theory $\mathcal{H}_g$ must be without an explicit construction (though such a construction would of course be needed to demonstrate the actual consistency of the orbifold theory). The problem, as always, is that we lack any general formulation of orbifold construction, but can only do it in “simple” cases such as in [3].

If, on the other hand, we do not want to follow Venkov’s method too closely, then we can provide a more convincing “proof” of the uniqueness of the $N = 0$ conformal field theory. We have that $\mathcal{H}_g = \mathcal{H}_g^0 \oplus U$, with $\mathcal{H}_g^0$ the invariant sector of $\mathcal{H}$ under the action of $g$ and $U$ a representation of $\mathcal{H}_g^0$. Now, the states $a_1^j|0\rangle$ lie in $U$, and so the states $a_{-m}^j (a_{-n}^i |0\rangle)$ lie in $\mathcal{H}_g^0$. Thus, we see that $\mathcal{H}_g^0 \cong \mathcal{H}(\Lambda_{24})_+$, that part of $\mathcal{H}(\Lambda_{24})$ invariant under the reflection twist introduced in [3] (which we can easily check is consistent, as we know its partition function is $\frac{1}{2} \left( 1 + g \right) = \frac{1}{2} (J(\tau) + T_g(\tau))$). We have proved [4] that there are only two representations of $U$ of $\mathcal{H}(\Lambda)_+$ for $\Lambda$ self-dual such that $\mathcal{H}(\Lambda)_+ \oplus U$ is a consistent
(self-dual) conformal field theory. These representations are distinguished by the number of weight one states. So we must have that $\mathcal{H} \cong V^2$, as required. Note that this proof circumvents the need to invoke any argument about inverting an orbifold construction, though still relies upon the assumption of the existence of a suitable involution $g$ of the original theory.

The same technique may be used instead beginning with a higher order automorphism, though of course we would then need to know about higher order twisted constructions of the Monster conformal field theory. This is work which is still in progress[13].

4 Complementary representations

Consider a bosonic meromorphic hermitian conformal field theory $S$ which may be extended to form a self-dual theory by adding in a representation $U$ (real, hermitian and satisfying the additional locality requirement as detailed in [13, 4]). Thus $\mathcal{H} = S \oplus U$ is self-dual.

4.1 Definition

Now, let $\theta$ be the automorphism of $\mathcal{H}$ defined to be 1 on $S$ and $-1$ on $U$. The invariant subtheory is simply $S$, and we shall assume that we may construct a corresponding self-dual orbifold theory $\mathcal{H}_\theta = S \oplus U'$, with $U'$ a representation of $S$.

We have

$$\chi_{\mathcal{H}_\theta}(\tau) = \chi_S(\tau) + \chi_{U'}(\tau), \tag{7}$$

where

$$\chi_{U'}(\tau) = \frac{1}{2} \left( \chi_{\mathcal{H}}(\tau) + \theta \chi_{\mathcal{H}}(\tau) \right) \tag{8}$$

and

$$\chi_S(\tau) = \frac{1}{2} \left( \chi_{\mathcal{H}}(\tau) + \theta \chi_{\mathcal{H}}(\tau) \right) \tag{9}$$

and all the boxes are to be understood with reference to $\mathcal{H}$.

Thus

$$\chi_S(S(\tau)) = \frac{1}{2} \left( \chi_{\mathcal{H}}(\tau) + \theta \chi_{\mathcal{H}}(\tau) \right) \tag{10}$$

and

$$\chi_S(ST(\tau)) = \frac{1}{2} \left( \chi_{\mathcal{H}}(\tau) + \theta \chi_{\mathcal{H}}(\tau) \right). \tag{11}$$

Hence

$$\chi_{\mathcal{H}}(\tau) + \chi_{\mathcal{H}_\theta}(\tau) = \chi_S(\tau) + \chi_S(S(\tau)) + \chi_S(ST(\tau)). \tag{12}$$

In other words, the sum of the partition functions of $\mathcal{H}$ and $\mathcal{H}_\theta$ is determined solely in terms of the partition function of the invariant sub-theory $S$. For $c = 24$, the partition functions are restricted to be of the form $J(\tau) + N$, where $J$ is the elliptic modular function.
with zero constant term and $N$ is the number of weight one states. So the above result
simply states that the sum of the number of weight one states in $\mathcal{H}$ and $\mathcal{H}_\theta$ is determined
solely by the partition function of $S$.

**Definition.** We shall say that $U'$ is the complementary representation of $S$ to $U$. Further,
if $U$ and $U'$ are equivalent as representations of $S$, we say that $U$ is self-complementary
with respect to $S$.

### 4.2 Example

Consider $\mathcal{S} = \mathcal{H}(\Lambda)_+$, $\Lambda$ an even self-dual lattice. We have shown in [4]
that there exist two representations $U_1$ and $U_2$ extending $S$ to a self-dual $c = 24$ conformal field theory,
i.e. $\mathcal{H}(\Lambda) \cong S \oplus U_1$ and $\tilde{\mathcal{H}}(\Lambda) \cong S \oplus U_2$. Consider taking $U = U_1$ in the above nota-
tion, and construct $U'$. We can argue that $U'_1 \cong U_2$. Suppose not. Then $U_1$ must be
self-complementary, and so the number of weight one states in $U_1$ would then be fixed by
the partition function of $S$. Now, assuming that the notion of complementarity is sym-
metric, i.e. $U'' \cong U$ (see below for more discussion), $U_2$ must also be self-complementary
(otherwise $U'_2 \cong U_1$, and hence $U'_1 \cong U''_2 \cong U_2$). So (12) then implies that $U_2$ must have
the same number of weight one states as $U_1$. We know from the explicit constructions of
these representations that this is a contradiction, as required.

Note the advantage of this point of view is that it puts $\mathcal{H}(\Lambda)$ and $\tilde{\mathcal{H}}(\Lambda)$ on an equal
footing, unlike the conventional approach.

### 4.3 Symmetry of the definition

We discuss here the property alluded to in the above example. Suppose $U'$ is the comple-
mentary representation of the conformal field theory $\mathcal{S}$ to a representation $U$. We can then
construct a representation $U''$ of $\mathcal{S}$ complementary to $U'$. We see from (12) that it must
have the same partition function as $U$ (which is more than saying merely that the number
of weight one states is the same, at least if $c > 24$). This observation alone is enough in the
case of our example above, since we know that $U_1$ and $U_2$ have distinct partition functions
and so symmetry is assured. The question in general of whether $U \cong U''$ remains an open
one however. Even for $c = 24$, we have examples of distinct theories with the same number
of weight one states, and so the equality of the partition functions alone is not sufficient,
although it is a conjecture which we still expect to be true.

### 4.4 Application to self-dual $c = 24$ conformal field theories

We may ask exactly what we must know of the partition function of the theory $\mathcal{S}$ in order
to calculate the sum in (12). Restricting our considerations to central charge 24, we need
only worry about the number of states of conformal weight one.

Clearly, $\chi_S(\tau)$ is a $\Gamma_0(2)$ invariant. So we can write [20]

$$
\chi_S(\tau) = N_S + \frac{1}{2} \left\{ J(\tau) + \alpha \left( \frac{\theta_3(\tau)^8 \theta_4(\tau)^8 + 2^{-4} \theta_2(\tau)^16}{\eta(\tau)^8 \eta(2\tau)^8} + 24 \right) + \beta \left( \left( \frac{\eta(\tau)}{\eta(2\tau)} \right)^{24} + 24 \right) \right\},
$$

(13)
using the $\Gamma_0(2)$ and $\Gamma_0(2)+$ hauptmoduls, where $N_S$ is the number of states of weight 1 in $S$ and $\alpha + \beta = 1$ (since $\chi_S(\tau) \sim q^{-1}$ as $q \to 0$). (Note that if $\alpha = 0$ or 1 then the Moonshine Conjecture\cite{1} holds in this case.) Considering the transformation $S : \tau \mapsto -1/\tau$ shows that $\alpha$ is to be interpreted as the number of weight $\frac{1}{2}$ states in the twisted sector, and is given by

$$\alpha = \frac{N_{S,2} - 98580}{2048}, \quad (14)$$

where $N_{S,2}$ is the number of states of weight 2 in $S$. We also find, using (12), that

$$N_H + N_{H'} = 3N_S + 24(1 - \alpha), \quad (15)$$

where $N_H$ is the number of weight one states in $H = S \oplus U$, and similarly for $N_{H'}$.

In the following sections, we will apply our arguments to cases in which the theory $S$ is constructed as the invariant sub-theory of a self-dual theory under an involution, and we are thus simply considering the orbifold of the first theory with respect to this involution. The representations of the sub-theory giving us the orbifold and the original theory are complementary, allowing us to use (14) and (15) to calculate the number of weight one states in the orbifold theory. Together with a knowledge of the Kac-Moody algebra corresponding to the weight one states of $S$, the number of weight one states allows us to look up in the table given by the results of Schellekens in section 2 and identify the most likely candidate for the theory. An absence of a suitable candidate will imply that the orbifold is not consistent.

[The motivation for the above approach is based upon an extension of the analogies between constructions of lattices from binary codes and constructions of CFT’s from lattices which were summarised in \cite{9}. In \cite{10} constructions for all of the Niemeier lattices from ternary codes were given, suggesting the existence of corresponding constructions of CFT’s from (Eisenstein) lattices by some form of $\mathbb{Z}_2$-orbifold approach. The exact nature of the orbifold theory would be difficult to write down. Instead it is considerably easier to postulate some sub-theory with $c = 24$ of the $c = 48$ theory $\mathcal{H}(\Lambda)$ ($\Lambda$ the 48-dimensional even self-dual lattice corresponding to a 24-dimensional Eisenstein lattice) which is to form the invariant space upon which the orbifold is constructed. The above argument would then give us the sum of the number of weight one states in two (potentially distinct) orbifolds that may be then formed by the addition of complementary representations, and reference to the Kac-Moody algebra of the invariant theory and Schellekens’ results would provide information hopefully sufficient to identify the theories. This program is still in progress.

In any case, note that (12) is analogous to the situation we have in constructing lattices from ternary codes in \cite{10}. In that case we had a lattice $\Lambda_C^0$ and formed a pair $\Lambda_C^\pm$ by adding appropriate vectors. The final number of length squared 2 vectors in the even self-dual lattices $\Lambda_C^0 \cup \Lambda_C^\pm$ turned out to be $3n_3 + n_6 + n_{24}^\pm$ respectively, the $3n_3 + n_6$ coming from $\Lambda_C^0$ ($n_m$ is the number of codewords of weight $m$—see \cite{10} for a full description of the notation). In general, $n_{24}^\pm$ are independent of $n_3$ and $n_6$, but $n_{24}$ is not\cite{11}, i.e. $|\Lambda_C^+(2)| + |\Lambda_C^-(2)|$ is fixed by $\Lambda_C^0$. Similarly, we note that (15) is again analogous to the lattice situation, for which we have $|\Lambda_C^+(2)| + |\Lambda_C^-(2)| = 3|\Lambda_C^0(2)| + 48(1 - \frac{1}{2}n_3).$]
4.5 Application to the theories $\mathcal{H}(\Lambda)$

Let us apply the above considerations to some simple examples. Consider projections by arbitrary involutions of the Niemeier lattice theories $\mathcal{H}(\Lambda)$. Schellekens and Yankielowicz in [19] have already observed that this is a useful thing to consider, since one of their two proposed new theories they claimed could be regarded as a $\mathbb{Z}_2$-orbifold of the theory $\mathcal{H}(E_8^3)$ induced by the involution on the lattice which interchanges two of the $E_8$ factors and shifts the third by a $D_8$ weight vector (although we feel that they should really consider a reflection on the third factor instead, since the shift is not a lattice automorphism!).

Note that it seems to be known, at least to the mathematicians [16], how to construct the twisted vertex operators corresponding to an arbitrary lattice automorphism. For an involution, the techniques applied in [5] may then be used to construct the corresponding intertwining vertex operators which enable one to unite the invariant theory and its representation into a consistent CFT. The question of the consistency of such a theory still has to be resolved by direct calculation though, analogous to that carried out in [5]. In any case, we can certainly calculate the ground state energy and degeneracy of the twisted sector, assuming consistency. Thus, use of the above techniques may seem unnecessary, though they do provide a quicker route to the answer and are really the only tool which can be used in the more complex situation considered in the next subsection where no explicit construction of the orbifold is yet known.

We need to consider the automorphism groups of the Niemeier lattices [8]. The lattices are specified by giving the root system and specifying a set of glue vectors, i.e. a set of vectors in the dual of the root system which, taken together with the root system, span the Niemeier lattice. The automorphism groups are composed of three pieces. One permutes the different components in the root system. One permutes the glue within a given component but leaves the components fixed, while the third leaves both the glue and the components fixed. Note that we shall frequently refer to the Niemeier lattices simply by specifying the semi-simple Lie algebras corresponding to their root system. This should not be confused with the root lattice of the algebra.

Note that not all involutions give rise to a consistent orbifold theory. All cases of this which we will come across can be eliminated simply by observing that the energy of the twisted sector ground state is not in $\frac{1}{2}\mathbb{Z}$. We will see an example of this in constructing our chosen example.

Let us consider an example. The glue code for the Niemeier lattice $E_6$ is the tetracode $C_4$. This has an involution of the form $(\leftrightarrow 1 - 1)$, using the obvious notation. If we try to take this as our involution, it gives a twist invariant algebra $E_6^2C_4$, and the argument of section 4.4 tells us that the number of weight one states in the orbifold theory is $288 - 24\alpha$ (we can work out $\alpha$ if required). The only possible theory from Schellekens’ list of sufficient rank would need $\alpha = 5$ and have algebra $A_5C_5E_6$, which is inconsistent. The reason for failure is that the ground state energy of the twisted sector is $6/16$ from the $-1$ on one $E_6$ plus $6/16$ from the interchange of the pair of $E_6$’s, which is not half-integral.

Let us try replacing the 1 in the specification of the automorphism with an involution of $E_6$ (which leaves the glue fixed). We try a one with invariant algebra $A_5A_1[16]$, which we can check leaves the glue unaffected. Thus, the twist invariant algebra is $E_6A_5A_1C_4$, and we find the number of weight one states to be $168 - 24\alpha$. This is consistent with the
new theory $A_5C_5E_{6,2}$, if we demonstrate that $\alpha = 0$. Note that, from [13], we see that the extra automorphism in this case contributes $1/4$ to the ground state energy in the twisted sector, making it integral (and also incidentally implying that $\alpha = 0$, since there are no weight $1/2$ states). Also, note that the weight one state argument implies that 16 new weight one states must come from the twisted sector, as an explicit construction along the lines discussed in [16] would verify.

One may proceed systematically through the Niemeier lattices and the involutions of each. We present $E_8^3$ as the simplest example. Note that we merely consider the lattice involutions, whereas the automorphism group of the conformal field theory is extended by the presence of the cocycles. The implication of the presence of additional involutions due to these must also be taken into account. We will discuss this briefly below. We have three possibilities for contributions to the lattice involution of $E_8^3$. We have transposition of a pair of the $E_8$’s and also two involutions of the $E_8$ root lattice itself. We refer to these as $\theta_1$ and $\theta_2$. $\theta_1$ is simply the reflection, and gives a contribution of $1/2$ to the twisted sector ground state energy with degeneracy $16$ (see for example [3]). The corresponding invariant algebra is $D_8$. $\theta_2$ has invariant subalgebra $E_7A_1$, and contributes $1/4$ to the energy with degeneracy $2$ [16].

The results are tabulated in table 1. We label the orbifold by the algebra, though of course we only know the corresponding theory to (exist and) be unique in the rank $24$ case [14]. We see from applying our above arguments to the calculation of $\alpha$ that we may interpret the transposition of two components of the lattice as contributing $1/2$ to the energy of the ground state in the twisted sector with unit degeneracy (independent of whether we attach another automorphism). Note that theory 3 would give $N = 432 - 24\alpha$, with a subalgebra $E_8E_7A_1$. We can eliminate this without even bothering to calculate $\alpha$ as we see that there is no suitable theory in Schellekens’ list. Note also that we can eliminate it immediately since it would require a ground state energy of $3/4$ in the twisted sector. This appears to be characteristic of the theories, i.e. that the unusual or even inconsistent looking algebras can be eliminated simply by consideration of the ground state energy. Note that in some places we use known results for the reflection twist [3], i.e. for the involutions 8 and 11, whereas for 1, 10, 13 and 16 we do not have sufficient knowledge from these simple calculations to identify the conformal field theory completely. Thus, we obtain the four distinct theories $\mathcal{H}(D_{10}E_7^2)$, $\mathcal{H}(E_8^3)$, $\mathcal{H}(E_8D_{16})$ and $\mathcal{H}(D_8^3)$ and at least one theory with algebra $E_{8,2}B_{8,1}$ by orbifolding $\mathcal{H}(E_8^3)$ with respect to the lattice induced involutions.

We may briefly consider the extension of the automorphism group of the lattice due to the presence of the cocycles. We have automorphisms $ua^2iu^{-1} = R_{ij}a^\mu_n, u|\lambda\rangle = (-1)^{\lambda\mu}|R\lambda\rangle, R \in \text{Aut } (\Lambda), \mu \in \Lambda/2\Lambda$. Take the simple case $R = 1$. We use the construction for the root lattice $E_8$ given in the appendix of Myhill’s thesis. Then there are two inequivalent choices for $\mu$, i.e. $e_1 + e_2$ and $\frac{1}{2} \sum_{i=1}^{8} e_i$. These are both found to give 136 invariant states, i.e. they behave in the same way as $\theta_2$ on each component.

Let us consider what we may say about the twisted sector from our point of view, even though we know its structure explicitly from [16]. We find that, in the cases where $\alpha = 0$ (which appear to be most common), that the partition function is of the form $2^{12}\eta(\tau)/\eta(\tau/2) + M$, where $M$ is the number of weight one states, i.e. it appears that we have $M$ states at weight 1 with no descendants, and then the usual $2^{12}$ dimensional
Table 1: Involutions of $E_8^3$ and the corresponding orbifolds of $\mathcal{H}(E_8^3)$.

| Label | Involution | Orbifold | $\alpha$ |
|-------|------------|----------|----------|
| 1     | $\leftrightarrow \cdot$ | $E_8^3$ or $E_8 D_{16}$ | 1        |
| 2     | $\leftrightarrow \theta_1$ | $E_{8,2} B_{8,1}$ | 0        |
| 3     | $\leftrightarrow \theta_1$ | incorrect vacuum energy | -        |
| 4     | $\theta_1 \theta_2 \theta_2$ | $D_{10,1} E_{7,1}^2$ | 0        |
| 5     | $\theta_1 \theta_1 \theta_2$ | incorrect vacuum energy | -        |
| 6     | $\theta_1 \theta_1 \theta_1$ | $D_8^3$ | 0        |
| 7     | $\theta_2 \theta_2 \theta_2$ | incorrect vacuum energy | -        |
| 8     | $\cdot \theta_1 \theta_1$ | $E_8 D_{16}$ | 0        |
| 9     | $\cdot \theta_1 \theta_2$ | incorrect vacuum energy | -        |
| 10    | $\cdot \theta_2 \theta_2$ | $E_8^3$ or $E_8 D_{16}$ | 4        |
| 11    | $\cdot \theta_1$ | $E_8^3$ | 16       |
| 12    | $\cdot \theta_2$ | incorrect vacuum energy | -        |
| 13    | $(\theta_1 \leftrightarrow \theta_1) \cdot$ | $E_8^3$ or $E_8 D_{16}$ | 1        |
| 14    | $(\theta_1 \leftrightarrow \theta_1) \theta_1$ | $E_{8,2} B_{8,1}$ | 0        |
| 15    | $(\theta_1 \leftrightarrow \theta_1) \theta_2$ | incorrect vacuum energy | -        |
| 16    | $(\theta_2 \leftrightarrow \theta_2) \cdot$ | $E_8^3$ or $E_8 D_{16}$ | 1        |
| 17    | $(\theta_2 \leftrightarrow \theta_2) \theta_1$ | $E_{8,2} B_{8,1}$ | 0        |
| 18    | $(\theta_2 \leftrightarrow \theta_2) \theta_2$ | incorrect vacuum energy | -        |

ground state at level $3/2$ with the remainder of the states being created by the action of 24 half-integrally graded bosonic creation operators on these. This situation of course seems ridiculous. Let us concentrate on the theory corresponding to involution 2 of table 1, i.e. the “new” Schellekens theory [19], and try to resolve this.

We have the partition function in the twisted sector as

$$1 \square_\theta (\tau) = 2^{12} \left( \frac{\eta(\tau)}{\eta(\tau/2)} \right)^{24} + 16. \quad (16)$$

However, we know that the $\theta_1$ part of the involution would be expected to give a contribution of $2^4 \left( \frac{\eta(\tau)}{\eta(\tau/2)} \right)^8 q^{1/3}$ to this. Considering the first few terms in the expansion of the ratio of the two, we arrive at the conjecture

$$1 \square_\theta (\tau) = 2^4 \left( \frac{\eta(\tau)}{\eta(\tau/2)} \right)^8 \cdot \frac{\theta_{E_8}(\tau/2)}{\eta(\tau/2)^8}, \quad (17)$$

which is trivial to check.

We postulate that the twisted sector is composed of a degeneracy 16 spinor ground state with creation operators $c_{-r}, r \in \mathbb{Z} + \frac{1}{2}$, tensored with states built up by creation oscillators $d_t, t \in \mathbb{Z} \cup (\mathbb{Z} + \frac{1}{2})$, from momentum states $|\lambda\rangle, \lambda \in E_8$. Thus, we have two pieces to the Virasoro operator on the second factor, a piece constructed from the integer
graded oscillators and a piece disjoint from the rest. Both give \( c = 8 \) and so sum to give \( c = 16 \), as required.

This is precisely the structure we would obtain by the approach of [16], which gives us a form for the vertex operators. Note that these are incorrect due to a normal ordering problem, though they may be corrected by the argument used in [13], which also specifies the procedure for obtain the intertwining operators and hence the full orbifold theory.

It may seem that the application of the above techniques was unnecessary, since we knew the explicit structure in any case. We have merely used it as a simple example to demonstrate the power of our approach, and in the next section will consider situations where no similar argument can be applied. Note that we may turn the above argument around, if desired, to show that the orbifold theory which we may postulate explicitly in fact leads to a self-dual partition function, as a consequence of the identity between ([16] and [17]).

### 4.6 Application to the theories \( \tilde{\mathcal{H}}(\Lambda) \)

As we remarked above, the application to theories \( \mathcal{H}(\Lambda) \) is in some sense trivial, while the attempt at an analogue of the ternary code \( \mathbb{Z} \)-lattice constructions, for which we have already stated that we developed this approach, is work which is still in progress. Let us instead consider projecting out by a lattice induced involution the theories \( \tilde{\mathcal{H}}(\Lambda) \) for \( \Lambda \) self-dual. There are two cases to consider.

The first is that in which \( \tilde{\mathcal{H}}(\Lambda) \cong \mathcal{H}(\Lambda') \) for some even self-dual lattice \( \Lambda' \) (of which there are 9 cases for \( c = 24 \), i.e. one for each doubly-even self-dual binary linear code in 24 dimensions [8]). In this case, it may be that the lattice induced automorphism is not lattice induced from the point of view of the theory \( \mathcal{H}(\tilde{\Lambda}) \), and so the above arguments would yield non-trivial information about a new orbifold theory. However, it seems likely that all the automorphisms of the theories \( \mathcal{H}(\Lambda) \) are given by the (extended) lattice automorphism group, and so this case is probably not too promising, though the full automorphism group of \( \mathcal{H}(\Lambda) \) does still remain to be determined definitively.

The second possibility is that \( \tilde{\mathcal{H}}(\Lambda) \not\cong \mathcal{H}(\Lambda') \) for any \( \Lambda' \) (15 instances in \( c = 24 \)). This means that we are certainly in a non-trivial situation, i.e. the explicit twisted vertex operator construction of [16] is not valid.

A group of automorphisms of the theory (though not necessarily the full automorphism group) \( \tilde{\mathcal{H}}(\Lambda) \) is given by the exact sequence

\[
1 \to \Gamma(\Lambda) \to C(\Lambda) \to \text{Aut}(\Lambda)/\mathbb{Z}_2 \to 1,
\]

where \( \Gamma(\Lambda) = \{\pm \gamma_\lambda : \lambda \in \Lambda\} \), i.e.

\[
u_{R,S} a_n^i u_{R,S}^{-1} = R_{ij} a_n^j; \quad u_{R,S} c_r^i u_{R,S}^{-1} = R_{ij} c_r^j \quad (19)
\]

\[
u_{R,S} \langle \lambda \rangle = v_{R,S}(\lambda) \langle R\lambda \rangle; \quad u_{R,S} \chi = S\chi, \quad (20)
\]

where \( R \in \text{Aut}(\Lambda) \) and \( S\gamma_\lambda S^{-1} = v_{R,S}(\lambda) \gamma_{R\lambda} \). (See [3] for a full explanation of the notation. The \( a_n \)'s are the usual integer graded oscillators in the untwisted sector of \( \mathcal{H}(\Lambda) \) acting on the momentum ground states \( |\lambda\rangle \), while the \( c_r \)'s are half-integer graded oscillators in the twisted sector acting on the spinor ground states \( \chi \), which form a representation.
space for the gamma matrix algebra \( \gamma_\lambda \gamma_\mu = (-1)^{\lambda \cdot \mu} \gamma_\mu \gamma_\lambda \). Note that we only need to know the matrix \( S \) for the evaluation of \( \alpha \), since the twisted states first appear at level 2 [We have seen in the above that we can usually guess the value of \( \alpha \) in any case from Schellekens’ list, but here we should really calculate it as we are unable to work out the vacuum energy (and degeneracy) like we were able to do in the previous case (where we knew the twisted vertex operators explicitly), and so it would provide a check, though not necessarily an independent one.] However, we do need \( v_{R,S}(\lambda) \) for evaluating the number of invariant weight one states, except in some special cases which we shall consider below.

Let us consider an example which falls into the second possibility mentioned above. We have an automorphism given by 6 pairs on transpositions on the root system components of the Niemeier lattice \( A_{2,12} \). Studying the glue code for this lattice[3], we find that this automorphism is of order 4. So, in particular, it would be unsuitable for the construction of a \( Z_2 \)-orbifold from the theory \( H(A_{2,12}) \). However, in the case of \( \tilde{H}(A_{2,12}) \), the automorphism \( \theta \) becomes of order 2, since it squares to the lattice reflection, which has trivial action on the twisted theory. The number of invariant weight one states is given by

\[
\frac{1}{2} \text{Tr}_{H_1} (1 + \theta),
\]

(21)

where \( H_1 \) is the space of states at level one. Noting that \( v(\lambda) = v(-\lambda) \) (since we have chosen the “gauge” such that \( \gamma_\lambda = \gamma_{-\lambda}[3] \)), we find that this becomes

\[
\frac{1}{4} \sum_{\lambda \in \Lambda(2)} (1 + v(\lambda)(\langle \lambda|\theta\lambda \rangle + \langle -\lambda|\theta\lambda \rangle)).
\]

(22)

In this case, \( \langle \pm\lambda|\theta\lambda \rangle = 0 \), and so we obtain \( \frac{1}{4} |\Lambda(2)| \) invariant weight one states, leading to \( \frac{1}{4} |\Lambda(2)| + 24(1 - \alpha) \) weight one states in the orbifold theory. Since \( \frac{1}{4} |\Lambda(2)| = 18 \), the only possible value for \( \alpha \) (assuming the orbifold theory is consistent!) is 0. Then the only possibility for the algebra is the new theory \( A_{2,12}^4 C_{2,4} \). We know the invariant weight one states are of the form \(|\lambda\rangle + |\gamma\lambda\rangle \). We can easily work out the corresponding algebra. In the original twisted theory, the \( A_{2,12} \) breaks down to \( A_{12}^4 \), and it is trivial to observe that the invariant algebra in the new orbifold theory must be \( A_{12}^4 \). The partition function argument tells us that 24 states arise from the twisted sector to enhance this algebra in the final orbifold theory. The algebra \( A_{2,12}^4 C_{2,4} \) is thus at least consistent, in that it contains the invariant algebra.

Consider the theory \( \tilde{H}(A_{4,6}) \). This is also an example of the second case referred to above, since it has algebra \( C_{2,6}^4 \). We have that set of pairwise transpositions on the 6 components of the root lattice exists in the automorphism group of the lattice. (This exists since the glue code for \( A_{4,6} \) is such that the group \( G_2(A_{4,6}) \) is isomorphic to \( PGL_2(5) \) acting on \( \{\infty, 0, 1, 2, 3, 4\} \) \[3\]. It is also an involution acting on the twisted theory, as in the case of the above example.) This automorphism will give us 54-24\( \alpha \) weight one states in the new orbifold theory (assuming it is consistent). There are no theories in Schellekens’ list with \( N = 6 \) or 54, and so we must have \( \alpha = 1 \), \( N = 30 \), i.e. we have no algebra enhancement. Thus the algebra must be just the invariant subalgebra, which is clearly \( C_{2,12}^3 \), an algebra which does appear on the list of section \[3\]. This provides strong evidence of the existence of another new theory.
Let us now make an attempt at a more systematic consideration. One possibility would be to consider all automorphisms given by transpositions of the components of the root system of the Niemeier lattice on the 15 non-trivial twisted lattice theories. For these, there is no need to know the matrix $S$, except to confirm the value of $\alpha$, which typically will be uniquely determined from Schellekens’ list in any case. In general, in order to avoid consideration of the $v(\lambda)$ in the evaluation of the number of invariant weight one states, we require $\theta\lambda$ to be distinct from both $\lambda$ and $-\lambda$ for the vectors of length squared two. In particular, distinction from $\lambda$ is equivalent to saying that, if the automorphism does not interchange lattice components, it acts as a no-fixed-point automorphism (NFPA) on the root lattice of one component. (Conversely, an NFPA on $\Lambda(2)$, the vectors of length squared 2 in $\Lambda$, is also one on $\langle \Lambda(2) \rangle$.) Distinction from $-\lambda$ means that we take all NFPA’s modulo the NFPA $-1$. The NFPA’s of the root lattices have been classified in [16]. The number of weight one states in the new orbifold theory is $\frac{1}{4} |\Lambda(2)| + 24(1-\alpha) = 6h + 24(1-\alpha)$. This must be greater than the number of invariant weight one states, which is $\frac{1}{4} |\Lambda(2)|$, and so we must have $\alpha = 0$ or 1 if the theory is to be consistent. In other words, consistency of the orbifold theory requires that Moonshine holds!

Now, [16] tells us that the number of non-reflection NFP involutions of the ADE algebras is zero. Let us therefore consider the 15 lattices giving rise to non-trivial reflection twisted theories and try to orbifold with respect to an NFPA of order 2 induced by transpositions and NFPA’s of order 4 on the root systems of the components. We summarise the results in table 2.

From [16], we have that all automorphisms of the ADE root systems of the same order are conjugate. Thus, all NFPA’s of order 4 square to give the reflection involution, and so act as involutions on the twisted theory, as required. The only root systems of type ADE admitting fourth order NFPA’s are $E_8$, $A_4$ and $D_{2n}$ for $2 \leq n \leq 11$ except for $n = 9$ [16]. This, coupled with the simple observation that many of the 15 theories cannot have transposition automorphisms, allows us to immediately exclude a large number of no-fixed-point involutions, indicated by a $\times$ in the “Involution” column of the table. The $\circ$ by the Leech lattice just indicates that, since it cannot give any interesting theories, we are not concerned with this case. (The theories it produces must have either 24 or 0 weight one states. The arguments of section 3 and [13] then identify the theories uniquely in this case, as we have indicated in the appropriate columns of the table.)

As noted above, the orbifold theory will have a number of weight one states given by $\frac{1}{4} |\Lambda(2)| + 24 - 24\alpha$, with $\alpha = 0$ or $\alpha = 1$. We list both possible numbers in the table. Those for which no corresponding algebra exists on Schellekens’ list have a $\times$ by them. In those cases where there is a unique algebra, we have noted that (in fact we know the unique theory in the case of 24 weight one states [4] and have noted that). The $\times$ by the algebra $A_2^4 C_2$ in the $A_6^4$ entry indicates that, though this is the only algebra in Schellekens’ list with the appropriate value of $N$, it cannot be the correct one since the involution must be a pure transposition if it exists at all (i.e. if the automorphism of the root system extends to an automorphism of the glue code), and so we must have an algebra of the form $X^2$ as the invariant algebra is not enhanced in the case $\alpha = 1$. The ? by the algebra $C_2^3$ in the $A_4^6$ entry indicates that, though there is no unique algebra for this number of weight one states, this is the only one of the form $X^3$, which we know the non-enhanced invariant algebra must be.
Referring to the glue codes for the theories which we have not yet excluded from having involutions, we get transposition involutions as indicated. Two of the cases marked by a ? in the “Involution” column could have involutions given by combining transpositions with fourth order root system automorphisms. It remains to check both the glue code and the action of the fourth order map on the corresponding glue vectors. The remaining ? indicates that we have yet to investigate the corresponding glue code for transposition automorphisms, though we see from the other columns for this entry in the table that the orbifold cannot be consistent.

The $E_6^4$ invariant algebra under $(1 - 1)(1 - 1)$ is $C_4^2$. There is no such theory at $N = 72$. We may ask if it can be enhanced to one of the $N = 96$ theories. We first note that we need at least 2 rank $\geq 4$ algebras or one of rank $\geq 8$. This narrows down the list. Then a simple consideration of dimensions of the appropriate algebras shows that $C_4$ cannot be embedded, except possibly as $C_4^2$ in $A_8$ to give the theory $A_2^2A_8$. This embedding though is clearly impossible, and so the orbifold must be inconsistent.

The invariant algebra for $A_7^2D_5^2$ under $(1 - 1)(1 - 1)$ is $C_2^2D_4$. Such an algebra exists at $N = 48$. We can eliminate the possibility of enhancement to a theory at $N = 72$ by trivially considering the possibilities with the restriction of one component of rank at least 4 and total rank 12. The postulated orbifold theory is indicated in the table.

| Lattice     | Involution           | Orbifold I | Orbifold II |
|-------------|----------------------|------------|-------------|
| $A_{24}$    | $\circ$              | 24 $\mathcal{H}(A_{24})$ | 0 $V^2$    |
| $A_3^8$     | $(1 \leftrightarrow -1)\ldots(1 \leftrightarrow -1)$ | 48         | $24 \mathcal{H}(A_{24})$ |
| $A_7^2D_5^2$| $(1 \leftrightarrow -1)(-1 \leftrightarrow 1)$ | 72 $\times$| 48 $C_2^4D_4?$ |
| $A_{24}$    | $\times$             | 174 $\times$ | 150 $\times$ |
| $A_{17}E_7$ | $\times$             | 132 $A_8F_4$ | 108 $B_4^3$ |
| $A_{15}D_9$ | $\times$             | 120         | 96          |
| $A_{12}^2$  | $(1 \leftrightarrow -1)$ | 102 $\times$ | 78 $\times$ |
| $A_{11}D_7E_6$ | $\times$         | 96          | 72          |
| $A_9^2D_6$  | ?                    | 84          | 60          |
| $A_8^3$     | $\times$             | 78 $\times$ | 54 $\times$ |
| $A_6^4$     | ?                    | 66 $\times$ | 42 $A_2^4C_2$ $\times$ |
| $A_5^4D_4$  | ?                    | 60          | 36          |
| $A_4^6$     | $(1 \leftrightarrow -1)(1 \leftrightarrow -1)(1 \leftrightarrow -1)$ | 54 $\times$ | 30 $C_2^3?$ |
| $A_2^{12}$  | $(1 \leftrightarrow -1)\ldots(1 \leftrightarrow -1)$ | 42 $A_2^4C_2$ | 18 $\times$ |
| $E_6^4$     | $(1 \leftrightarrow -1)(-1 \leftrightarrow 1)$ | 96          | 72          |

Table 2: No-fixed-point involution orbifolds of the 15 non-trivial reflection twisted lattice theories.
5 Conclusions

Starting from results for CFT’s by Schellekens analogous to some of those proved for the Niemeier lattices by Venkov, we have shown firstly how to reformulate the uniqueness problem for the “Monster” CFT. The notion of complementary representations coupled with the restrictions on Kac-Moody algebras in the theories derived by Schellekens then enabled us, for the first time, to investigate with some degree of confidence orbifold theories for which no explicit construction is known.

It remains to present a more systematic survey of the orbifolds of the theories $\tilde{\mathcal{H}}(\Lambda)$ for arbitrary involutions. This will require explicit knowledge of the action of the involution on the twisted sector ground states, as discussed in [12]. Also, we need to understand the conditions under which the orbifolds are consistent. For the theories $\mathcal{H}(\Lambda)$, it seemed to be sufficient simply to calculate the twisted sector ground state energy, though for the theories $\tilde{\mathcal{H}}(\Lambda)$ we need to find some condition which will eliminate the $A_{12}^2$ theory in table 2 say while preserving the $A_{4}^6$ theory. We could investigate this by evaluating the first few terms in the partition function of the invariant sector, since our assumption of $\Gamma_0(2)$ invariance assumes an orbifold-like behaviour. In order to do this, we must again know about the action of the automorphism on the twisted sector ground state.

Finally, the obvious hope for the future is to complete the analogue of Venkov’s results for CFT’s. Venkov showed that, for each of the possible semi-simple algebras which his conditions selected, that there was one and only one corresponding even self-dual lattice. His approach was based on coding theory via the idea of glue codes. (Any relation to the code structures investigated in [3, 15, 13] is as yet not understood.) However, the analogue cannot be exact, for Schellekens has demonstrated that for some of the algebras listed in section 2 there is no modular invariant combination of Kac-Moody algebras, and hence there can be no consistent orbifold theory. Nevertheless, as an extension to the result of section 3, it might be argued that, where a theory does exist, it is unique. There are, as yet, no counterexamples to such a claim.

6 Note

After completion of this paper, the paper [18] was received, in which the number of possible combinations of Kac-Moody algebras listed in section 2 was reduced to 71 by consideration of higher order trace identities. The results would appear to eliminate some of the new theories claimed in this paper, although further checks remain to be done. Nevertheless, the techniques discussed in this paper when applied to the remaining involutions of the twisted lattice theories should be rendered more powerful as a result of this more restrictive set of possibilities. This is work which is currently in progress.

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