ON THE HIGHER ORDER CORRECTIONS TO
HEAVY QUARK EFFECTIVE THEORY

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Abstract

We show that the different ways of deriving the Heavy Quark Effective Theory (HQET) lead to equivalent theories. The equivalence can be established through a careful redefinition of the field variables. We demonstrate the equivalence to order $\frac{1}{m^2}$ in the presence of a constant electric field.
I. Introduction:

In recent years, there has been a lot of work in the context of Heavy Quark Effective Theories (HQET) [1-6]. When a quark is heavy, it can be effectively described by a two component spinor and there are essentially two different ways of obtaining an effective theory for such a quark. The most logical way to take the heavy quark mass limit, in some sense, is through the Foldy-Wouthuysen transformations [7,8] where one essentially diagonalizes the quark Lagrangian such that the upper and lower two component spinors decouple after which one restricts to the positive energy two component upper spinors. (In the functional language, one would simply integrate out the lower component spinors after diagonalization and absorb the result into the normalization factor.) The other approach to an effective theory is the traditional way [4,9] where one decomposes the four component spinor into a “large” and a “small” two component spinor. One then eliminates the “small” component through its equation of motion to obtain an effective theory for the “large” component spinors. (In the functional language, one would integrate out the “small” components and absorb the result into the normalization factor of the path integral.) While the two methods give the same identical theory in the lowest order in \( \frac{1}{m} \), their structures appear to be different at higher orders. This disagreement has justifiably raised concern in the recent literature [10] mainly because higher order corrections to various physical processes are currently being calculated [11] using the theory resulting from the traditional method of eliminating the “small” components.

The discrepancy between the resulting theories in the two approaches is nothing new. It was already noted in connection with an electron interacting with an external electric field [12] as well as in the context of the Tamm-Dancoff method in nuclear physics [13], that eliminating the “small” component naively leads to a nonhermitian Hamiltonian. (In the case of the electron interacting with an external electric field, the lowest order manifestation of the nonhermiticity is in an imaginary electric dipole moment.) To the best of our knowledge, the resolution of this puzzle was first proposed in ref. 14 where
it was noted that a renormalization (redefinition) of the large components is essential for the hermiticity of the Hamiltonian. In the case of the Dirac electron interacting with an electric field, the equivalence of the two approaches was thereby demonstrated to order $\frac{1}{m^2}$ [14].

It is in general believed that an appropriate field redefinition will lead to an equivalence of the two approaches. In this note, we wish to demonstrate up to order $\frac{1}{m^3}$ that both the methods indeed give the same theory with appropriate field redefinitions. Both the methods have their advantages and disadvantages and we comment on this in the conclusion. The equivalence of the two approaches is quite important and so is the understanding of the field redefinition since otherwise the higher order corrections calculated with the naive effective theory may not represent the true physical effects. In sec. II, we review the known results up to order $\frac{1}{m^2}$ and try to bring out the necessity for a field redefinition. In sec. III, we show that, in the case of the free theory, the two approaches lead to the same effective theory upon field redefinition. We then demonstrate to order $\frac{1}{m^3}$ that in the presence of a constant electric field, the two approaches also give the same effective theory through a careful redefinition of fields. In sec. IV, we show the equivalence of the two approaches in the functional approach and present our conclusions in sec. V.

II. The Problem of Imaginary Dipole Moment:

In this section, we consider a Dirac electron interacting with a constant, external electric field. While the discussion can be carried out equally well in the first or second quantized language we will follow a first quantized approach for simplicity and clarity. Our discussion in this section will follow closely the work in ref. 14. The Dirac equation in the present case has the form (we use Bjorken-Drell metric, $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$, $\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$)

$$
(i\gamma^0 (\partial_0 + ieA_0) + i\vec{\gamma} \cdot \vec{\nabla} - m) \psi = 0 \quad (2.1)
$$

with

$$
\vec{E} = \vec{\nabla} A_0 = \text{constant} \quad (2.2)
$$
If we let
\[ \psi \rightarrow e^{-imt}\psi \]  
then Eq. (2.1) takes the form
\[ \left( i\gamma^0 (\partial_0 + ieA_0) + i\gamma^i \cdot \vec{\nabla} - m (1 - \gamma^0) \right) \psi = 0 \] (2.4)

Introducing the “large” and the “small” components as (In the current terminology of the subject our entire discussion will be with the choice \( v^\mu = (1, 0, 0, 0) \) for simplicity.)

\[ \psi = \begin{pmatrix} \psi_L \\ \psi_S \end{pmatrix} \] (2.5)
we note that the Dirac equation (2.4) separates into two equations

\[ \psi_S = (i\partial_0 - eA_0 - 2m)^{-1} (-i\vec{\sigma} \cdot \vec{\nabla}) \psi_L = A\psi_L \] (2.6)
\[ (i\partial_0 - eA_0) \psi_L = -i\vec{\sigma} \cdot \vec{\nabla}\psi_S \] (2.7)

Upon substituting Eq. (2.6) into Eq. (2.7) and expanding up to order \( \frac{1}{m^2} \), the equation for the “large” component takes the form

\[ (i\partial_0 - eA_0) \psi_L = \left( -\frac{1}{2m} \vec{\nabla}^2 - \frac{ie}{4m^2} \vec{\sigma} \cdot (\vec{E} \times \vec{\nabla}) + \frac{e}{4m^2} \vec{E} \cdot \vec{\nabla} + O\left( \frac{1}{m^3} \right) \right) \psi_L \] (2.8)

It is the last term on the right hand side in Eq. (2.8) which represents an imaginary electric dipole moment and arises naturally as a consequence of eliminating the “small” component spinors. While this may be puzzling, it’s origin is not hard to understand. As explained beautifully in ref. 14, one can view the process of eliminating the “small” component also equivalently as finding a transformation which will take

\[ \psi = \begin{pmatrix} \psi_L \\ \psi_S \end{pmatrix} = \begin{pmatrix} \psi_L \\ A\psi_L \end{pmatrix} \xrightarrow{S} \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \] (2.9)

Such a transformation is generated by the matrix
\[ S = \begin{pmatrix} I & 0 \\ -A & I \end{pmatrix} \] (2.9’)

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It is clear that this transformation is not unitary and as a result the Hamiltonian does not remain hermitian under such a transformation. Another manifestation of the transformation not being unitary is to note that under such a transformation, the norm of the state is not invariant. In fact, let us note that

\[
\int d^3x \, \psi^\dagger \psi = \int d^3x \left( \psi_L^\dagger \psi_L + \psi_S^\dagger \psi_S \right)
\]

\[
= \int d^3x \, \psi_L^\dagger (1 + A^\dagger A) \psi_L \neq \int d^3x \, \psi_L^\dagger \psi_L
\]  

(2.10)

Therefore, if

\[
\int d^3x \, \psi^\dagger \psi = 1
\]  

(2.11)

then,

\[
\int d^3x \, \psi_L^\dagger \psi_L \neq 1
\]  

(2.12)

It is clear, however, that the norm can be maintained (state will be normalized) if we redefine

\[
\tilde{\psi}_L = (1 + A^\dagger A)^{1/2} \psi_L
\]  

(2.13)

As is described in ref. 14, this redefinition also restores hermiticity of the Hamiltonian. (The norm will, of course, be time independent.) In fact, let us note from Eqs. (2.6) and (2.13) that if we define up to order \( \frac{1}{m^2} \)

\[
\tilde{\psi}_L = \left( 1 - \frac{1}{4m^2} \vec{\nabla}^2 \right)^{1/2} \psi_L \simeq \left( 1 - \frac{1}{8m^2} \vec{\nabla}^2 \right) \psi_L
\]  

(2.14)

or,

\[
\psi_L \simeq \left( 1 + \frac{1}{8m^2} \vec{\nabla}^2 \right) \tilde{\psi}_L
\]

then Eq. (2.8) would take the form

\[
(i\partial_0 - eA_0) \tilde{\psi}_L = \left(-\frac{1}{2m} \vec{\nabla}^2 - \frac{ie}{4m^2} \vec{\sigma} \cdot (\vec{E} \times \vec{\nabla}) + O \left( \frac{1}{m^3} \right) \right) \tilde{\psi}_L
\]  

(2.15)

The imaginary electric dipole term has now been absorbed into the field redefinition and as we will see in the next section this is exactly the same equation which we would get through the Foldy-Wouthuysen transformation up to order \( \frac{1}{m^2} \).
III. Equivalence up to Order $\frac{1}{m^2}$:

As a warm up, let us consider a free Dirac particle satisfying

$$\left( i\gamma^0 \partial_0 + i\vec{\gamma} \cdot \vec{\nabla} - m \right) \psi = 0 \quad (3.1)$$

The idea behind the Foldy-Wouthuysen transformations [7] is to block diagonalize the Dirac operator such that the upper and the lower two component spinors are decoupled. It is well known [7] that under the unitary transformation

$$\psi \rightarrow \psi' = U\psi = e^{\frac{im}{2} \vec{\gamma} \cdot \vec{\nabla}} \alpha \left( \frac{|\vec{\nabla}|}{m} \right) \psi \quad (3.2)$$

where

$$\alpha \left( \frac{|\vec{\nabla}|}{m} \right) = \frac{m}{|\vec{\nabla}|} \tanh^{-1} \left( \frac{|\vec{\nabla}|}{m} \right) \quad (3.3)$$

the Dirac operator transforms into the diagonal form

$$\left( i\gamma^0 \partial_0 + i\vec{\gamma} \cdot \vec{\nabla} - m \right) \rightarrow U \left( i\gamma^0 \partial_0 + i\vec{\gamma} \cdot \vec{\nabla} - m \right) U^\dagger$$

$$= i\gamma^0 \partial_0 - \left( m^2 - \vec{\nabla}^2 \right)^{1/2} \quad (3.4)$$

Thus the transformed equation has the form

$$\left( i\gamma^0 \partial_0 - \left( m^2 - \vec{\nabla}^2 \right)^{1/2} \right) \psi' = 0 \quad (3.5)$$

If we further let

$$\psi' \rightarrow e^{-imt} \psi' \quad (3.6)$$

then Eq. (3.5) will take the form

$$\left( i\gamma^0 \partial_0 + m\gamma^0 - \left( m^2 - \vec{\nabla}^2 \right)^{1/2} \right) \psi' = 0 \quad (3.7)$$

Writing

$$\psi' = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (3.8)$$
and restricting to the upper two component spinor, we obtain from Eq. (3.7)

\[
\left( i\partial_0 + m - \left( m^2 - \vec{\nabla}^2 \right)^{1/2} \right) \psi_1 = 0 \quad (3.9)
\]

A power series expansion of Eq. (3.9) in \( \frac{1}{m} \) would then give the effective dynamical equation in the Foldy-Wouthuysen approach.

In contrast, let us note from Eqs. (2.6) and (2.7) that in the absence of interactions

\[
\psi_S = A\psi_L = (i\partial_0 + 2m)^{-1}(-i\vec{\sigma} \cdot \vec{\nabla})\psi_L \quad (3.10)
\]

\[
i\partial_0 \psi_L = -i\vec{\sigma} \cdot \vec{\nabla} \psi_S = -\vec{\nabla} (i\partial_0 + 2m)^{-1} \vec{\sigma} \cdot \vec{\nabla} \psi_L = -(i\partial_0 + 2m)^{-1} \vec{\nabla}^2 \psi_L \quad (3.11)
\]

The equation for the “large” component, Eq. (3.11), of course, does not at all resemble Eq. (3.9) obtained through the other approach. Let us also note that if we define

\[
\tilde{\psi}_L = \left(1 + A^\dagger A\right)^{1/2} \psi_L = \left(1 - (i\partial_0 + 2m)^{-1} \vec{\nabla}^2 \right)^{1/2} \psi_L \quad (3.12)
\]

as we should to maintain normalization, the form of the equation does not change

\[
i\partial_0 \tilde{\psi}_L = -(i\partial_0 + 2m)^{-1} \vec{\nabla}^2 \tilde{\psi}_L \quad (3.13)
\]

This is mainly because the redefinition factor in Eq. (3.12) is field independent, commutes with \( \partial_0 \) as well as \( \vec{\nabla} \) and that Eq. (3.11) is linear in \( \psi_L \). However, we note that Eq. (3.13) can also be written as

\[
(i\partial_0 + (i\partial_0 + 2m)^{-1} \vec{\nabla}^2) \tilde{\psi}_L = 0 \quad (3.14)
\]

Therefore, if we further redefine

\[
\tilde{\psi}_L \rightarrow e^{-i(m - (m^2 - \vec{\nabla}^2)^{1/2} - (i\partial_0 + 2m)^{-1} \vec{\nabla}^2)t)} \tilde{\psi}_L \quad (3.15)
\]

then Eq. (3.14) will take the form

\[
\left( i\partial_0 + m - \left( m^2 - \vec{\nabla}^2 \right)^{1/2} \right) \tilde{\psi}_L = 0 \quad (3.16)
\]
which is, of course, what we have in the Foldy-Wouthuysen approach (see Eq. (3.9)).

The point of this exercise is to note that even after the spinor is normalized properly, there still remains an arbitrariness up to unitary transformations [15] and this is crucial in establishing the equivalence of the two approaches. Alternately, we note that Eq. (3.14) can also be written as

\[(i\partial_0 + 2m)^{-1}\left(i\partial_0 + m + \left(m^2 - \nabla^2\right)^{1/2}\right)\left(i\partial_0 + m - \left(m^2 - \nabla^2\right)^{1/2}\right)\tilde{\psi}_L = 0 \quad (3.16')\]

Consequently, if we redefine

\[\tilde{\psi}_L \rightarrow (i\partial_0 + 2m)^{-1/2}\left(i\partial_0 + m + \left(m^2 - \nabla^2\right)^{1/2}\right)^{1/2}\tilde{\psi}_L \quad (3.16'')\]

then the dynamical equation would take the form

\[\left(i\partial_0 + m - \left(m^2 - \nabla^2\right)^{1/2}\right)\tilde{\psi}_L = 0\]

which is the equation obtained through the Foldy-Wouthuysen approach. It would seem, therefore, that at the level of the equation of motion, there is an arbitrariness in the choice of field redefinition. Furthermore, the redefinition in Eq. (3.16'') would seem to imply a change in the norm of the state. However, in the next section, where we will show the equivalence of the two approaches for the free theory in the functional approach, it will become clear that such a field redefinition is essential for the matching of the functional determinants.

Let us next consider a Dirac particle interacting with a constant, external electric field. This is chosen to avoid the unnecessary complexities associated with a generalized non Abelian gauge potential which do not provide any new insight into the problem. The equation of motion is

\[\left(i\gamma^0 (\partial_0 + ieA_0) + i\vec{\gamma} \cdot \vec{\nabla} - m\right)\psi = 0 \quad (3.17)\]

with

\[\vec{E} = \vec{\nabla}A_0 = \text{constant} \quad (3.18)\]
In the presence of interactions, the Foldy-Wouthuysen transformations have to be implemented order by order. We refer the readers to the literature [16] for details and merely quote here the result that the equation satisfied by the upper two component spinor in the present case, up to order \( \frac{1}{m^5} \), after a phase redefinition of the form in Eq. (3.6) is given by

\[
(i\partial_0 - eA_0)\psi_1 = \left( -\frac{1}{2m} \vec{\nabla}^2 - \frac{ie}{4m^2} \vec{\sigma} \cdot (\vec{E} \times \vec{\nabla}) - \frac{1}{8m^3} (\vec{\nabla}^2)^2 + \frac{e^2}{8m^3} \vec{E}^2 - \frac{3ie}{16m^4} \vec{\sigma} \cdot (\vec{E} \times \vec{\nabla}) \vec{\nabla}^2 - \frac{1}{16m^5} (\vec{\nabla}^2)^3 + \frac{5e^2}{48m^5} \vec{E}^2 \vec{\nabla}^2 - \frac{e^2}{24m^5} (\vec{E} \cdot \vec{\nabla})^2 + O\left(\frac{1}{m^6}\right) \right)\psi_1
\]

(3.19)

Note here that up to order \( \frac{1}{m^2} \), it is the same equation as in Eq. (2.15).

To analyze the effective equation in the other approach, let us recall from Eqs. (2.6) and (2.7) that in the present case

\[
\psi_S = A\psi_L = -(i\partial_0 - eA_0 + 2m)^{-1} \vec{\sigma} \cdot \vec{\nabla}\psi_L
\]

\[
(i\partial_0 - eA_0)\psi_L = -i\vec{\sigma} \cdot \vec{\nabla}\psi_S = -\vec{\sigma} \cdot \vec{\nabla} (i\partial_0 - eA_0 + 2m)^{-1} \vec{\sigma} \cdot \vec{\nabla}\psi_L
\]

The normalization factor, in this case,

\[
(1 + A^\dagger A)^{1/2} = \left( 1 - \vec{\sigma} \cdot \vec{\nabla} (i\partial_0 - eA_0 + 2m)^{-2} \vec{\sigma} \cdot \vec{\nabla} \right)^{1/2}
\]

(3.20)

is field dependent and involves the Dirac operator. Expanding to order \( \frac{1}{m^6} \), we note that the redefined field will be given by

\[
\tilde{\psi}_L = (1 + A^\dagger A)^{1/2} \psi_L
\]

or,

\[
\psi_L = (1 + A^\dagger A)^{-1/2} \tilde{\psi}_L
\]

\[
= \left[ 1 + \frac{1}{8m^2} \vec{\nabla}^2 - \frac{1}{8m^3} \vec{\sigma} \cdot \vec{\nabla} (i\partial_0 - eA_0) \vec{\sigma} \cdot \vec{\nabla} + \frac{3}{128m^4} (\vec{\nabla}^2)^2 + \frac{3}{32m^4} \vec{\sigma} \cdot \vec{\nabla} (i\partial_0 - eA_0)^2 \vec{\sigma} \cdot \vec{\nabla} - \frac{1}{16m^5} \vec{\sigma} \cdot \vec{\nabla} (i\partial_0 - eA_0)^3 \vec{\sigma} \cdot \vec{\nabla} - \frac{3}{128m^5} \vec{\nabla}^2 \vec{\sigma} \cdot \vec{\nabla} (i\partial_0 - eA_0) \vec{\sigma} \cdot \vec{\nabla} - \frac{3}{128m^5} \vec{\sigma} \cdot \vec{\nabla} (i\partial_0 - eA_0) \vec{\sigma} \cdot \vec{\nabla} \vec{\nabla}^2 + O\left(\frac{1}{m^6}\right) \right] \psi_L
\]

(3.21)
Substituting this back into the equation for the “large” component we obtain after some algebra

\[
(i\partial_0 - eA_0)\tilde{\psi}_L = \left[ -\frac{1}{2m} \tilde{\nabla}^2 - \frac{ie}{4m^2} \tilde{\sigma} \cdot (\tilde{E} \times \tilde{\nabla}) - \frac{1}{16m^2} (\tilde{\nabla}^2)^2 + \frac{e^2}{8m^3} \tilde{E}^2 \\
+ \frac{e}{32m^4} (2\tilde{E} \cdot \tilde{\nabla} - 3i\tilde{\sigma} \cdot (\tilde{E} \times \tilde{\nabla})) \tilde{\nabla}^2 - \frac{3}{256m^5} (\tilde{\nabla}^2)^3 \\
+ \frac{e^2}{16m^5} \tilde{E}^2 \tilde{\nabla}^2 + \frac{e^2}{16m^5} (\tilde{E} \cdot \tilde{\nabla})^2 + \frac{ie^2}{32m^5} \tilde{E} \cdot \tilde{\nabla} \tilde{\sigma} \cdot (\tilde{E} \times \tilde{\nabla}) \\
+ \frac{1}{8m^2} \tilde{\nabla}^2 (i\partial_0 - eA_0) + \frac{e}{8m^3} (\tilde{E} \cdot \tilde{\nabla} + i\tilde{\sigma} \cdot (\tilde{E} \times \tilde{\nabla})) (i\partial_0 - eA_0) \\
+ \frac{9}{128m^4} (\tilde{\nabla}^2)^2 (i\partial_0 - eA_0) - \frac{3e^2}{16m^4} \tilde{E}^2 (i\partial_0 - eA_0) \\
- \frac{3e}{16m^4} \tilde{E} \cdot \tilde{\nabla} (i\partial_0 - eA_0)^2 - \frac{1}{32m^4} \tilde{\nabla}^2 (i\partial_0 - eA_0)^3 \\
+ \ldots \right] \tilde{\psi}_L
\]

(3.22)

We note here that since the normalization factor involves the Dirac operator, Eq. (3.22) has to be solved iteratively. This is crucial because this implies that one must keep terms consistently up to any given order. We note that this is a new feature not present at order $\frac{1}{m^2}$. Iterating Eq. (3.22) up to order $\frac{1}{m^5}$ gives

\[
(i\partial_0 - eA_0)\tilde{\psi}_L = \left[ -\frac{1}{2m} \tilde{\nabla}^2 - \frac{ie}{4m^2} \tilde{\sigma} \cdot (\tilde{E} \times \tilde{\nabla}) - \frac{1}{8m^3} (\tilde{\nabla}^2)^2 + \frac{e^2}{8m^3} \tilde{E}^2 \\
- \frac{3ie}{16m^4} \tilde{\sigma} \cdot (\tilde{E} \times \tilde{\nabla}) \tilde{\nabla}^2 - \frac{1}{16m^5} (\tilde{\nabla}^2)^3 + \frac{15e^2}{64m^5} \tilde{E}^2 \tilde{\nabla}^2 \right] \tilde{\psi}_L
\]

(3.23)

Comparing with Eq. (3.19), we see that the two equations almost agree – in fact, only the coefficients of the last two terms are different. However, as discussed earlier, we also recognize that $\tilde{\psi}_L$ is unique only up to a unitary transformation. Taking advantage of this, we let

\[
\tilde{\psi}_L \rightarrow e^{\frac{25\alpha}{192m^5}} \tilde{\psi}_L
\]

(3.24)
With this redefinition, we note that Eq. (3.23) takes the form

\[
(i\partial_0 - eA_0)\tilde{\psi}_L = \left[ -\frac{1}{2m} \vec{\nabla}^2 - \frac{ie}{4m^2} \vec{\sigma} \cdot (\vec{E} \times \vec{\nabla}) - \frac{1}{8m^3} (\vec{\nabla}^2)^2 + \frac{e^2}{8m^3} \vec{E}^2 \\
- \frac{3ie}{16m^4} \vec{\sigma} \cdot (\vec{E} \times \vec{\nabla})\vec{\nabla}^2 - \frac{1}{16m^5} (\vec{\nabla}^2)^3 + \frac{5e^2}{48m^5} \vec{E}^2 \vec{\nabla}^2 \\
- \frac{e^2}{24m^5} (\vec{E} \cdot \vec{\nabla})^2 + O\left(\frac{1}{m^6}\right) \right] \tilde{\psi}_L
\] (3.25)

This is exactly the same as Eq. (3.19) which is what we would obtain in the Foldy-Wouthuysen approach. Thus, the two approaches give the same effective equation up to order \(\frac{1}{m^5}\) with appropriate field redefinition.

IV. Functional Equivalence:

In this section, we will show the equivalence of the two approaches in the functional method. Let us start with a free Dirac theory described by

\[
\mathcal{L} = \bar{\psi} (i\partial_0 - m) \psi
\] (4.1)

If we make the phase transformation

\[
\psi \rightarrow e^{-imt} \psi
\] (4.1')

then the Lagrangian will take the form

\[
\mathcal{L} = \bar{\psi} (i\partial_0 - m(1 - \gamma^0)) \psi
\] (4.2)

The generating functional in this case is given by

\[
Z = N \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \, e^{i\int d^4x \, \mathcal{L}}
\] (4.2)

In the Foldy-Wouthuysen approach if we make a unitary redefinition of the field variables

\[
\psi' = e^{\frac{i\gamma^\mu}{2m}} \gamma^\nu \alpha(\sqrt{m}) \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\] (4.3)
with
\[
\alpha \left( \frac{|\nabla|}{m} \right) = \frac{m}{|\nabla|} \tanh^{-1} \left( \frac{|\nabla|}{m} \right)
\] (4.4)
then the Lagrangian diagonalizes and takes the form
\[
\mathcal{L} = \psi_1^\dagger (i \partial_0 + m - (m^2 - \nabla^2)^{1/2}) \psi_1 + \psi_2^\dagger (i \partial_0 + m + (m^2 - \nabla^2)^{1/2}) \psi_2
\] (4.5)
+ surface terms

The functional measure does not change under the unitary redefinition of the fields in Eq. (4.3). Consequently, the generating functional has the form
\[
Z = N \int \mathcal{D}\psi_1^\dagger \mathcal{D}\psi_1 \mathcal{D}\psi_2^\dagger \mathcal{D}\psi_2 e^{i \int d^4x \left( \psi_1^\dagger (i \partial_0 + m - (m^2 - \nabla^2)^{1/2}) \psi_1 + \psi_2^\dagger (i \partial_0 + m + (m^2 - \nabla^2)^{1/2}) \psi_2 \right)}
\] (4.6)

We note here that the propagator in the Foldy-Wouthuysen approach would have the form
\[
S_{FW} = \frac{1}{E + m - (\vec{k}^2 + m^2)^{1/2}}
\] (4.7)
Clearly, this has a pole at
\[
E = (\vec{k}^2 + m^2)^{1/2} - m
\] (4.8)
with unit residue.

On the other hand, if we write
\[
\psi = \begin{pmatrix} \psi_L \\ \psi_S \end{pmatrix}
\] (4.9)
then the Lagrangian of Eq. (4.1') will have the form
\[
\mathcal{L} = \psi_L^\dagger i \partial_0 \psi_L + \psi_L^\dagger i \vec{\sigma} \cdot \nabla \psi_S + \psi_S^\dagger i \vec{\sigma} \cdot \nabla \psi_L + \psi_S^\dagger (i \partial_0 + 2m) \psi_S
\] (4.10)
Since the Lagrangian is at the most quadratic in the fermions, we can integrate out the “small” component and obtain
\[
Z = N \int \mathcal{D}\psi_L^\dagger \mathcal{D}\psi_L \mathcal{D}\psi_S^\dagger \mathcal{D}\psi_S e^{i \int d^4x \mathcal{L}}
\] (4.11)
\[
= N [\det (i \partial_0 + 2m)] \int \mathcal{D}\psi_L^\dagger \mathcal{D}\psi_L e^{i \int d^4x \psi_L^\dagger (i \partial_0 + (i \partial_0 + 2m)^{-1} \vec{\nabla}^2) \psi_L}
\] (4.11)
\[
= N [\det (i \partial_0 + 2m)] Z_{\text{CONV}}
\]
From the structure of the Lagrangian in Eq. (4.11) we note that the conventional (naive) propagator for the “large” component will have the form

\[ S_{\text{CONV}} = \frac{1}{E - \frac{k^2}{E^2 + 2m}} \]  

(4.12)

We note that the propagator has poles at

\[ E = \pm \left( \sqrt{k^2 + m^2} \right)^{1/2} - m \]  

(4.13)

The location of the positive energy pole, in which we are interested, coincides with that of the other approach. But we note that the residue at the positive energy pole is \( \frac{m + \sqrt{k^2 + m^2}}{2(k^2 + m^2)^{1/2}} \). This suggests that the field \( \psi_L \) needs to be redefined. Note that if we redefine (compare this with Eq. (3.16′′))

\[ \psi_L \rightarrow \tilde{\psi}_L = (i\partial_0 + 2m)^{-1/2} \left( i\partial_0 + m + \left( m^2 - \nabla^2 \right)^{1/2} \right)^{1/2} \psi_L \]  

(4.14)

then the generating functional will have the form

\[ Z = N[\det(i\partial_0 + 2m)] \int D\psi^\dagger_L D\psi_L e^{i \int d^4x \psi^\dagger_L (i\partial_0 + (i\partial_0 + 2m)^{-1} \nabla^2)\psi_L} \]

\[ = N[\det(i\partial_0 + 2m)][\det(i\partial_0 + 2m)]^{-1} \times [\det(i\partial_0 + m + (m^2 - \nabla^2)^{1/2})] \int D\tilde{\psi}_L^\dagger D\tilde{\psi}_L e^{i \int d^4x \tilde{\psi}_L^\dagger (i\partial_0 + m - (m^2 - \nabla^2)^{1/2})\tilde{\psi}_L} \]

\[ = N[\det(i\partial_0 + m + (m^2 - \nabla^2)^{1/2})] \tilde{Z}_{\text{CONV}} \]  

(4.15)

Comparing Eqs. (4.6) and (4.15), we conclude that

\[ Z_{FW} = \tilde{Z}_{\text{CONV}} \]  

(4.16)

Namely, the two approaches lead to equivalent theories but only after appropriate field redefinition. For an interacting theory, Foldy-Wouthuysen transformations do not have a closed form. Therefore, the equivalence in the functional approach has to be shown order by order. An example, let us consider the Dirac particle interacting with a constant
electric field up to order $\frac{1}{m^2}$. The Lagrangian with the phase redefinition of Eq. (4.1') is given by

$$\mathcal{L} = \overline{\psi} \left( i\gamma^0 (\partial_0 + ieA_0) + i\vec{\gamma} \cdot \vec{\nabla} - m \left( 1 - \gamma^0 \right) \right) \psi$$

(4.17)

In the Foldy-Wouthuysen approach, if we diagonalize the Lagrangian up to order $\frac{1}{m^2}$, it will have the form (FW transformations are unitary and hence the functional measure does not change.)

$$\mathcal{L} = \overline{\psi'} \left( i\gamma^0 (\partial_0 + ieA_0) - m(1 - \gamma^0) \right.
\left. + \frac{1}{2m} \vec{\nabla}^2 - \frac{ie}{4m^2} \gamma_5 \gamma^0 \cdot (\vec{E} \times \vec{\nabla}) \right) \psi'$$

(4.18)

Writing $\psi'$ as in Eq. (4.3), the Lagrangian density has the form

$$\mathcal{L} = \psi_1^\dagger \left( i\partial_0 - eA_0 + \frac{1}{2m} \vec{\nabla}^2 + \frac{ie}{4m^2} \vec{\sigma} \cdot (\vec{E} \times \vec{\nabla}) + O\left( \frac{1}{m^3} \right) \right) \psi_1
+ \psi_2^\dagger \left( i\partial_0 - eA_0 + 2m - \frac{1}{2m} \vec{\nabla}^2 + \frac{ie}{4m^2} \vec{\sigma} \cdot (\vec{E} \times \vec{\nabla}) + O\left( \frac{1}{m^3} \right) \right) \psi_2$$

(4.19)

The generating functional, therefore, has the form

$$Z = N \int \mathcal{D}\psi_1^\dagger \mathcal{D}\psi_1 \mathcal{D}\psi_2^\dagger \mathcal{D}\psi_2 e^i \int d^4 x \mathcal{L}$$

$$= N \left[ \det \left( i\partial_0 - eA_0 + 2m - \frac{1}{2m} \vec{\nabla}^2 + \frac{ie}{4m^2} \vec{\sigma} \cdot (\vec{E} \times \vec{\nabla}) + O\left( \frac{1}{m^3} \right) \right) \right] \times \int \mathcal{D}\psi_1^\dagger \mathcal{D}\psi_1 e^i \int d^4 x \psi_1^\dagger \left( i\partial_0 - eA_0 + \frac{1}{2m} \vec{\nabla}^2 + \frac{ie}{4m^2} \vec{\sigma} \cdot (\vec{E} \times \vec{\nabla}) + O\left( \frac{1}{m^3} \right) \right) \psi_1$$

$$= N \left[ \det \left( 2m(1 + \frac{1}{2m} (i\partial_0 - eA_0) - \frac{1}{4m^2} \vec{\nabla}^2 + O\left( \frac{1}{m^3} \right)) \right) \right] Z_{FW}$$

(4.20)

On the other hand, the Lagrangian in terms of “large” and “small” components has the form

$$\mathcal{L} = \psi_L^\dagger (i\partial_0 - eA_0) \psi_L + \psi_S^\dagger i\vec{\sigma} \cdot \vec{\nabla} \psi_S + \psi_S^\dagger i\vec{\sigma} \cdot \vec{\nabla} \psi_L + \psi_S^\dagger (i\partial_0 - eA_0 + 2m) \psi_S$$

(4.21)
Integrating out the “small” components we have

\[ Z = N \left[ \det(i\partial_0 - eA_0 + 2m) \right] \int \mathcal{D}\psi_L \mathcal{D}\psi_L e^{i \int d^4x \psi_L^\dagger \left(i\partial_0 - eA_0 + \vec{\sigma} \cdot \vec{\nabla}(i\partial_0 - eA_0 + 2m)^{-1} \vec{\sigma} \cdot \vec{\nabla} \right) \psi_L} \]

\[ = N \left[ \det \left( 2m(1 + \frac{1}{2m}(i\partial_0 - eA_0)) \right) \right] \int \mathcal{D}\psi_L \mathcal{D}\psi_L \]

\[ \times e^{i \int d^4x \psi_L^\dagger \left(i\partial_0 - eA_0 + \frac{1}{4m^2} \vec{\nabla}^2 - \frac{ie}{4m^2} \vec{\nabla} \cdot \vec{\sigma} \cdot \vec{\nabla} + \frac{1}{4m^2} \vec{\nabla}^2 \right) \psi_L} \]

\[ = N \left[ \det \left( 2m(1 + \frac{1}{2m}(i\partial_0 - eA_0)) \right) \right] Z_{\text{CONV}} \]  

(4.22)

If we now redefine (as in Eq. (2.14))

\[ \psi_L \rightarrow \tilde{\psi}_L = \left(1 - \frac{1}{8m^2} \vec{\nabla}^2 \right) \psi_L \]  

(4.23)

then the Lagrangian in Eq. (4.22) would take the form

\[ \mathcal{L}_{\text{CONV}} = \tilde{\psi}_L^\dagger \left(i\partial_0 - eA_0 + \frac{1}{2m} \vec{\nabla}^2 + \frac{ie}{4m^2} \vec{\sigma} \cdot \left( \vec{E} \times \vec{\nabla} \right) + O \left( \frac{1}{m^3} \right) \right) \tilde{\psi}_L \]  

(4.24)

Furthermore, with the Jacobian arising from the redefinition in Eq. (4.23), the generating functional would become

\[ Z = N \left[ \det \left( 2m(1 + \frac{1}{2m}(i\partial_0 - eA_0)) \right) \right] \left[ \det \left( 1 - \frac{1}{8m^2} \vec{\nabla}^2 \right) \right]^2 \]

\[ \times \int \mathcal{D}\tilde{\psi}_L \mathcal{D}\tilde{\psi}_L e^{i \int d^4x \tilde{\psi}_L^\dagger \left(i\partial_0 - eA_0 + \frac{2m}{4m^2} \vec{\nabla}^2 + \frac{ie}{4m^2} \vec{\sigma} \cdot \left( \vec{E} \times \vec{\nabla} \right) + O \left( \frac{1}{m^3} \right) \right) \tilde{\psi}_L} \]

\[ = N \left[ \det \left( 2m(1 + \frac{1}{2m}(i\partial_0 - eA_0) - \frac{1}{4m^2} \vec{\nabla}^2 + O \left( \frac{1}{m^3} \right)) \right) \right] \tilde{Z}_{\text{CONV}} \]  

(4.25)

Comparing Eqs. (4.20) and (4.25), we conclude that

\[ Z_{\text{FW}} = \tilde{Z}_{\text{CONV}} \]  

(4.26)

This establishes equivalence of the two approaches up to order \( \frac{1}{m^2} \) in the presence of a constant electric field. Once again, the significance of the field redefinition cannot be over emphasized. It is tedious, but as the discussion of the earlier section shows, the equivalence
can be carried out to any given order with appropriate field redefinitions. The higher order equivalence, however, can only be shown through a careful iterative procedure.

V. Conclusion:

In this note we have shown up to order $\frac{1}{m^2}$ that the two ways of obtaining an interacting heavy quark effective theory yield the same result. From a practical point of view, it is the Foldy-Wouthuysen method that is simpler for the derivation of the effective theory. However, since it involves field dependent unitary transformations, unless the fields vanish sufficiently rapidly, it is conceptually unclear whether the S-matrix elements will remain the same [17]. The traditional method of eliminating the “small” components, on the other hand, is quite tricky. Here the “large” component fields must be renormalized (redefined) and this must be carried out carefully in an iterative manner. This is essential since otherwise the calculations may not represent the true physical effects. This procedure of field redefinition is quite tedious but has the virtue that the existence of the S-matrix elements is never in question in this approach since the asymptotic form of the wave function remains unchanged in general. Thus, in some sense, one can view the two approaches to be complementary and equivalent.

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