SCHUBERT POLYNOMIALS FOR THE AFFINE GRASSMANNIAN OF THE SYMPLECTIC GROUP

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Abstract. We study the Schubert calculus of the affine Grassmannian \( \text{Gr} \) of the symplectic group. The integral homology and cohomology rings of \( \text{Gr} \) are identified with dual Hopf algebras of symmetric functions, defined in terms of Schur’s \( P \) and \( Q \) functions. An explicit combinatorial description is obtained for the Schubert basis of the cohomology of \( \text{Gr} \), and this is extended to a definition of the affine type \( C \) Stanley symmetric functions. A homology Pieri rule is also given for the product of a special Schubert class with an arbitrary one.

1. Introduction

Let \( G \) be a simply-connected simple complex algebraic group and let \( \text{Gr}_G \) denote the affine Grassmannian of \( G \). Following Peterson [22] and Lam [15] we study the homology and cohomology Schubert calculus of \( \text{Gr}_G \).

The structure of \( H^*_s(\text{Gr}_G) \) and \( H^*(\text{Gr}_G) \) is particularly rich because of the interaction of two phenomena. On the one hand, \( \text{Gr}_G \) inherits free \( \mathbb{Z} \)-module Schubert bases \( \{ \xi_x \in H^*_s(\text{Gr}_G) \} \) and \( \{ \xi^*_x \in H^*(\text{Gr}_G) \} \) from its presentation \( \text{Gr}_G = \mathcal{G}/\mathcal{P} \) where \( \mathcal{G} \) is the affine Kac-Moody group associated to \( G \) and \( \mathcal{P} \subset \mathcal{G} \) is a maximal parabolic subgroup. On the other hand, it is a classical result of Quillen [24] (see also [7] and [23]) that \( \text{Gr}_G \) is homotopy equivalent to the based loops \( \Omega K \) into the maximal compact subgroup \( K \subset G \). The group structure on \( \Omega K \) endows \( H^*_s(\text{Gr}_G) \) and \( H^*(\text{Gr}_G) \) with the structure of dual Hopf algebras.

The dual Hopf algebras \( H^*_s(\text{Gr}_G) \) and \( H^*(\text{Gr}_G) \) were first studied intensively by Bott [2]. Bott gave an algorithm to compute these Hopf algebras in terms of the Cartan data of \( G \), essentially by transgressing elements of \( H^*(K) \) to obtain the primitive elements in \( H^*(\text{Gr}_G) \). With \( \mathbb{Q} \)-coefficients, \( H^*(K, \mathbb{Q}) \) is an exterior algebra with odd-dimensional generators so \( H^*(\text{Gr}_G, \mathbb{Q}) \) is a polynomial algebra on even-dimensional generators. The situation is even more favorable when \( G = \text{Sp}_{2n}(\mathbb{C}) \) since \( \text{Sp}_{2n}(\mathbb{C}) \) is torsion-free and \( H^*_s(\text{Gr}_{\text{Sp}_{2n}(\mathbb{C})}) \) is a polynomial algebra over \( \mathbb{Z} \). Bott comments that his description does not give polynomial generators for \( H^*_s(\text{Gr}_{\text{Sp}_{2n}(\mathbb{C})}) \). We resolve this by producing \( n \) special Schubert classes which are polynomial generators over \( \mathbb{Z} \).

Our main result identifies \( H^*_s(\text{Gr}_{\text{Sp}_{2n}(\mathbb{C})}) \) and \( H^*(\text{Gr}_{\text{Sp}_{2n}(\mathbb{C})}) \) with certain dual Hopf algebras \( \Gamma^{(n)} \) and \( \Gamma^{(n)} \) of symmetric functions, defined in terms of Schur’s \( P \)- and \( Q \)-functions [20]. We explicitly describe symmetric functions \( Q^{(n)}_w \in \Gamma^{(n)} \) which represents the cohomology Schubert basis. These symmetric functions are constructed using the combinatorics of a remarkable subset \( Z \subset \tilde{C}_n \) of the affine
Weyl group $\tilde{C}_n$ of $SP_{2n}(\mathbb{C})$. In fact the cohomology representatives $Q_w^{(n)}$ extend to a larger family of symmetric functions: the type $C$ affine Stanley symmetric functions.

1.1. **Peterson’s work on affine Schubert calculus.** Our results rely heavily on the work of Peterson [22] who defined a Hopf embedding $j : H_T(Gr_G) \rightarrow A$ of the $T$-equivariant cohomology of $Gr_G$ as a commutative subalgebra of the nil-Hecke ring $A = kC$ of Kostant and Kumar [12]. Here $T \subset K$ is a maximal torus. Peterson characterizes the image $j(x)$ of the Schubert basis of $H_T(Gr_G)$ in terms of certain identities inside $A$. In the non-equivariant case, Lam [15] showed that Peterson’s embedding specializes to a Hopf isomorphism $j_0 : H_*(Gr_G) \rightarrow B$ with an algebra which he called the affine Fomin-Stanley subalgebra. We give an explicit combinatorial formula for generators of $B$ in the case $G = SP_{2n}(\mathbb{C})$.

1.2. **Earlier work for $G = SL_n(\mathbb{C})$.** For $G = SL_n(\mathbb{C})$, Lam [15] identified the Schubert basis of $H_*(Gr_{SL_n(\mathbb{C})})$ with symmetric functions, called $k$-Schur functions, of Lapointe, Lascoux and Morse [18]; these arose in the study of Macdonald polynomials. The Schubert basis of $H^*(Gr_{SL_n(\mathbb{C})})$ are the dual $k$-Schur functions [19] which are generalized by the affine Stanley symmetric functions [14]. In [16] Pieri rules were given for the multiplication of Bott’s generators on the Schubert bases of Bott’s realization of $H_*(Gr_{SL_n(\mathbb{C})})$ and $H^*(Gr_{SL_n(\mathbb{C})})$. Furthermore, a combinatorial interpretation of the pairing between $H_*(Gr_{SL_n(\mathbb{C})})$ and $H^*(Gr_{SL_n(\mathbb{C})})$ is given.

1.3. **Two Hopf algebras of symmetric functions.** Let $\Lambda$ denote the ring of symmetric functions over $\mathbb{Z}$. Let $P_i$ and $Q_i$ denote the Schur $P$- and $Q$-functions with a single part [20] III.8. Define the Hopf subalgebras of $\Lambda$ given by $\Gamma_* = \mathbb{Z}[P_1, P_2, \ldots]$ and $\Gamma^* = \mathbb{Z}[Q_1, Q_2, \ldots]$. There is a natural pairing (see (2.14)) $[\cdot, \cdot]: \Gamma_* \times \Gamma^* \rightarrow \mathbb{Z}$ making $\Gamma_*$ and $\Gamma^*$ into dual Hopf algebras. For $n \geq 1$ the subspace $\Gamma_{(n)} = \mathbb{Z}[P_1, P_2, \ldots, P_{2n}] \subset \Gamma_*$ is a Hopf subalgebra and we let $\Gamma^* \rightarrow \Gamma^{(n)}$ be the dual quotient Hopf algebra.

1.4. **Special classes.** The affine Weyl group of $SP_{2n}(\mathbb{C})$, denoted $\tilde{C}_n$, has simple generators $s_0, s_1, \ldots, s_n$ with the relations (3.2). Let $\tilde{C}_n^0$ denote the minimal length coset representatives of $\tilde{C}_n/C_n$, also called the Grassmannian elements of $\tilde{C}_n$. Define $\rho_i \in \tilde{C}_n^0$ by

\begin{align}
\rho_i &= \begin{cases} 
  s_{i-1}s_{i-2}\cdots s_1s_0 & \text{for } 1 \leq i \leq n \\
  s_{2n+1-i}s_{2n+2-i}\cdots s_{n}s_{n+1-i}s_0 & \text{for } n+1 \leq i \leq 2n.
\end{cases}
\end{align}

The homology Schubert classes $\xi_{\rho_i} \in H_*(Gr_{SP_{2n}(\mathbb{C})})$ for $1 \leq i \leq 2n$, are called special classes.

1.5. **Zee-s.** Let $Z$ be the Bruhat order ideal in $\tilde{C}_n$ generated by the conjugates of the element $\rho_{2n}$, that is, the set of $w \in \tilde{C}_n$ which have a reduced word that is a subword of a rotation of the unique reduced word of $\rho_{2n}$. An element of $Z$ is called a $Z$. Let $Z_r = \{ w \in Z \mid \ell(w) = r \}$ denote the set of $Z$-s of length $\ell(w)$ equal to $r$.

**Example 1.1.** Let $n = 2$. Then $Z$ consists of the elements of $\tilde{C}_2$ which have a reduced word that is a subword of one of the words 1210, 2101, 1012, 0121.
Given a word $u$ with letters in $I_{af}$, its support $\text{Supp}(u) \subset I_{af}$ is by definition the set of letters appearing in $u$. For $w \in \tilde{C}_n$ define $\text{Supp}(w) = \text{Supp}(u)$ for any reduced word $u$; this is independent of the choice of $u$. A component of a subset of $I_{af}$ is by definition a maximal nonempty subinterval. Let $c(w)$ denote the number of components of $\text{Supp}(w)$.

1.6. **Affine type $C$ Stanley symmetric functions.** For $w \in \tilde{C}_n$ we define the generating function

$$Q^{(n)}_w[Y] = \sum_{(v^1, v^2, \ldots)} \prod_i 2^{c(v^i)} y_i^{\ell(v^i)},$$

where the sum runs over the factorizations $v^1 v^2 \cdots = w$ of $w$ such that $v^i \in \mathcal{Z}$ and $\ell(v^1) + \ell(v^2) + \cdots = \ell(w)$.

**Theorem 1.2.** The series $Q^{(n)}_w$ is symmetric and defines an element of $\Gamma^{(n)}$. The subset $\{Q^{(n)}_v | v \in \tilde{C}_n\}$ forms a basis of $\Gamma^{(n)}$ such that all product and coproduct structure constants are positive and every $Q^{(n)}_w$ for $w \in \tilde{C}_n$ is positive in this basis.

The symmetric functions $Q^{(n)}_w$ are of type $C$ analogues of the affine Stanley symmetric functions studied in [14]. Some examples for the type $C$ affine Stanley symmetric functions are given in Appendix B. They have the following geometric interpretation.

Let $LSp(n)$ and $\Omega Sp(n)$ denote respectively the space of all loops and based loops, into the maximal compact subgroup $Sp(n) \subset Sp_{2n}(\mathbb{C})$ and let $T \subset Sp(n)$ be the maximal torus. Let $p : \Omega Sp(n) \rightarrow LSp(n)/T$ denote the composition $\Omega Sp(n) \hookrightarrow LSp(n) \rightarrow LSp(n)/T$ of the inclusion and natural projection. The type $C$ affine Stanley symmetric functions $Q^{(n)}_w$ can be identified via Theorem 1.3 (see below) with the pullbacks $p^*(\xi^w)$ of the Schubert classes $\xi^w \in H^*(LSp(n)/T)$. This follows from [5.7] and [15] Remark 8.6. See also [15] Remark 4.6. For $w \in \tilde{C}_n$, $p^*(\xi^w)$ is itself a Schubert class in $H^*(\Omega Sp(n)) \cong H^*(Gr_{Sp_{2n}(\mathbb{C})})$ as detailed below.

1.7. **(Co)homology Schubert polynomials.** The Hopf algebras $H^*(Gr_{Sp_{2n}(\mathbb{C})})$ and $H_*(Gr_{Sp_{2n}(\mathbb{C})})$ are dual via the cap product. The Schubert bases $\{\xi_x \in H_*(Gr_{G})\}$ and $\{\xi^x \in H^*(Gr_{G})\}$ are dual under the cap product and are both indexed by the Grassmannian elements $x \in \tilde{C}_n^0$.

**Theorem 1.3.** There are dual Hopf algebra isomorphisms

$$\Phi : \Gamma^{(n)} \rightarrow H_*(Gr_{Sp_{2n}(\mathbb{C})})$$
$$\Psi : H^*(Gr_{Sp_{2n}(\mathbb{C})}) \rightarrow \Gamma^{(n)}$$

such that

$$\Phi(P_i) = \xi_{\rho_i} \quad \text{for } 1 \leq i \leq 2n,$$
$$\Psi(\xi^w) = Q^{(n)}_w \quad \text{for } w \in \tilde{C}_n^0.$$

Since $\Gamma^{(n)} = \mathbb{Z}[P_1, P_2, \ldots, P_{2n-1}]$, we obtain in particular that $H_*(Gr_{Sp_{2n}(\mathbb{C})})$ is a polynomial algebra on $\xi_{\rho_1}, \xi_{\rho_3}, \ldots, \xi_{\rho_{2n-1}}$. It also follows that the basis $\{P^{(n)}_w | w \in \tilde{C}_n^0\}$ of $\Gamma^{(n)}$ dual to $\{Q^{(n)}_w | w \in \tilde{C}_n^0\} \subset \Gamma^{(n)}$ maps to the homology Schubert classes $\xi_w \in H^*(Gr_{Sp_{2n}(\mathbb{C})})$. The symmetric functions $P^{(n)}_w$ are Schubert polynomials for
$H_*(\text{Gr}_{\text{Sp}_{2n}}(\mathbb{C}))$ and are the type $C$ analogue of $k$-Schur functions \cite{15, 18}. Examples are given in Appendix C.

1.8. **Pieri rule for $H_*(\text{Gr}_{\text{Sp}_{2n}}(\mathbb{C}))$.** We also give a positive formula for the multiplication of an arbitrary homology class by a special class.

**Theorem 1.4.** Let $w \in \tilde{\mathcal{C}}_n^0$. Then in $H_*(\text{Gr}_{\text{Sp}_{2n}}(\mathbb{C}))$ we have

$$\xi_{\rho}, \xi_{w} = \sum_{v \in \mathbb{Z}_i} 2^{c(v)-1} \xi_{vw}$$

where the sum is taken over all $v \in \mathbb{Z}_i$ such that $\ell(vw) = \ell(v) + \ell(w)$ and $vw \in \tilde{\mathcal{C}}_n^0$.

1.9. **Future work and other directions.**

1.9.1. **Pieri rule for $H^*(\text{Gr}_{\text{Sp}_{2n}}(\mathbb{C}))$ and explicit description of homology Schubert basis.** We hope to describe the symmetric functions $\{P_{w}^{(n)} \mid w \in \tilde{\mathcal{C}}_n^0 \} \subset \Gamma_{(n)}$ explicitly in the future, perhaps in a manner similar to the *strong tableaux* in \cite{16}. As is explained in \cite{16}, the description of $P_{w}^{(n)}$ is essentially equivalent to the description of a Pieri rule for $H^*(\text{Gr}_{\text{Sp}_{2n}}(\mathbb{C}))$.

1.9.2. **The special orthogonal groups.** A generalization of our work to the special orthogonal groups $G = SO_n(\mathbb{C})$, together with the $G = SL_n(\mathbb{C})$ case in \cite{15}, would complete the analysis of the classical groups. The symmetric function description of $H_*(\text{Gr}_{SO_n}(\mathbb{C}))$ is likely to be more involved as it is not a polynomial algebra over $\mathbb{Z}$.

1.9.3. **Comparison with finite case.** We hope to explore the relationship between our symmetric functions and the “type $B$” Stanley symmetric functions and classical type Schubert polynomials of Fomin and Kirillov \cite{6}, and of Billey and Haiman \cite{1}. In particular, specializing $A_n = 0$ in Theorem 5.1 we obtain an expression nearly the same as the formula \cite{6, 4.1].

1.9.4. **Embedding of groups and branching of Schubert classes.** We intend to study the behavior of the affine Schubert classes studied here and in \cite{15} induced by the inclusions of compact groups:

$$SU(n) \subset SU(n + 1), \quad Sp(n) \subset Sp(n + 1), \quad Sp(n) \subset SU(2n), \quad SU(n) \subset Sp(n).$$

In particular, the symmetric functions $P_{w}^{(n)}$ and $Q_{w}^{(n)}$ have positivity properties with respect to expansions involving Schur $P$-functions, Schur $Q$-functions, and ordinary Schur functions.

1.9.5. **Work of Ginzburg and Bezrukavnikov, Finkelberg and Mirkovic.** The rings $H_*(\text{Gr}_{G})$ and $H^*(\text{Gr}_{G})$ were also studied by Ginzburg \cite{8} and by Bezrukavnikov, Finkelberg and Mirkovic \cite{3} from the point of view of geometric representation theory. The connection with our point of view is unclear since the Schubert basis is as yet unavailable in their descriptions, although part of the Schubert basis is described by Ginzburg.
1.10. **Organization.** In section 2 we give notation for symmetric functions and describe the dual Hopf algebras $\Gamma^{(n)}$ and $\Gamma^{(n)}$. In section 3 we fix notation concerning affine root systems and Weyl groups. In section 4 we explain the connection between the Peterson and the Fomin-Stanley subalgebras, and the homology of the affine Grassmannian. The material in sections 3–4 are valid for the affine Grassmannian of any simply-connected simple complex algebraic group $G$.

Apart from Proposition 7.1, the remainder of the paper specializes to the case $G = SP_{2n}(C)$. In section 5 we prove our main results (Theorems 1.2, 1.3 and 1.4) modulo two nilHecke algebra calculations – Theorems 5.1 and 5.5. Section 6 is devoted to the study of the Bruhat order of $C_n$ restricted to $Z$, and to the proof of Theorem 6.1. Section 7 presents a general formula (Proposition 7.1) for the coproduct in a nilHecke algebra and uses it to prove Theorem 5.5. Some data, in particular for the type $C$ affine Stanley symmetric functions $Q^{(n)}_w$ and $k$-Schur functions $F^{(n)}_w$, is given in Appendices B and C.

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2. **Symmetric functions**

In this section we study a subring $\Gamma^{(n)}$ and subquotient $\Gamma^{(n)}$ of the ring of symmetric functions. Let $\Lambda$ be the Hopf algebra of symmetric functions over $\mathbb{Z}$. It has a number of bases indexed by partitions $\lambda$:

$$
\begin{align*}
\Lambda &\to \mathbb{Z} \\
\langle h_\lambda, m_\mu \rangle &\to \delta_{\lambda\mu}.
\end{align*}
$$

(2.1)

It has reproducing kernel [20, I.4.1,4.2]

$$
\Omega := \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}
$$

(2.2)
where $z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!$ and $m_i(\lambda)$ is the number of times the part $i$ occurs in $\lambda$. Here and elsewhere, unless otherwise specified the sum runs over the set of all partitions.

The Schur $P$ and $Q$ functions are defined by \cite[III.8]{20}

\begin{equation}
\begin{aligned}
P_\lambda[X] &= P_\lambda[X; -1] \\
Q_\lambda[X] &= Q_\lambda[X; -1] = 2^\ell(\lambda) P_\lambda[X]
\end{aligned}
\end{equation}

where $\ell(\lambda)$ is the number of nonzero parts of $\lambda$. We have \cite[III.8.7]{20}

\begin{equation}
P_\lambda[X] = Q_\lambda[X] = 0 \quad \text{if} \quad \lambda \not\in \mathcal{SP}
\end{equation}

where $\mathcal{SP}$ is the set of strict partitions $\lambda$, those with $\lambda_1 > \lambda_2 > \cdots$; see (2.10) and (2.13).

2.1. Homology ring. Define the Hopf subalgebra $\Gamma_* \subset \Lambda$ by

\begin{equation}
\Gamma_* = \mathbb{Z}[P_1, P_3, P_5, \ldots].
\end{equation}

The $P_i$ for $i$ odd, are algebraically independent, so that

\begin{equation}
\Gamma_* = \bigoplus_{\lambda \in \mathcal{OP}} \mathbb{Z} P_{\lambda_1} P_{\lambda_2} \cdots
\end{equation}

where $\mathcal{OP}$ is the set of partitions with odd parts. The Hopf structure on $\Gamma_*$ is given by

\begin{equation}
\Delta(P_r) = 1 \otimes P_r + P_r \otimes 1 + 2 \sum_{0 \leq s < r} P_s \otimes P_{r-s},
\end{equation}

where the $P_i$ for $i$ even, satisfy only the relations \cite[III.8.2']{20}

\begin{equation}
P_{2i} = 2 \left( P_1 P_{2i-1} - P_2 P_{2i-2} + \cdots + (-1)^{i-2} P_{i-1} P_{i+1} \right) + (-1)^{i-1} P_i^2.
\end{equation}

Iterating \cite[III.8.15]{20} yields the relation

\begin{equation}
P_{\lambda_1} P_{\lambda_2} \cdots = \sum_{\mu \in \mathcal{SP} \atop \mu \supseteq \lambda} L_{\mu \lambda} P_\mu,
\end{equation}

where $L_{\mu \lambda} \in \mathbb{Z}_{\geq 0}$ and $L_{\mu \mu} = 1$. Here $\supseteq$ denotes the dominance partial order on partitions \cite[I.1]{20}. It follows that

\begin{equation}
\Gamma_* = \bigoplus_{\lambda \in \mathcal{SP}} \mathbb{Z} P_\lambda.
\end{equation}

Define the Hopf subalgebra $\Gamma_{(n)} \subset \Gamma_*$ by

\begin{equation}
\Gamma_{(n)} = \mathbb{Z}[P_1, P_2, \ldots, P_{2n}] = \mathbb{Z}[P_1, P_3, \ldots, P_{2n-1}] = \bigoplus_{\lambda \in \mathcal{OP} \atop \lambda_1 \leq 2n-1} \mathbb{Z} P_{\lambda_1} P_{\lambda_2} \cdots.
\end{equation}
2.2. Cohomology ring. Define

\[ \Gamma^* = \mathbb{Z}[Q_1, Q_2, \ldots] \subset \Lambda. \]

By (2.9) we have

\[ \Gamma^* = \bigoplus_{\lambda \in SP} \mathbb{Z}Q_\lambda. \]

Define the pairing \([\cdot, \cdot] : \Gamma^* \times \Gamma^* \to \mathbb{Z}\) by \([P_\lambda, Q_\mu] = \delta_{\lambda\mu}\) for \(\lambda, \mu \in SP\).

The pairing \([\cdot, \cdot]\) has reproducing kernel

\[ \Omega_{-1} := \prod_{i,j \geq 1} \frac{1 + x_iy_j}{1 - x_iy_j} \]

\[ = \sum_{\lambda \in SP} P_\lambda[X]Q_\lambda[Y] \]

\[ = \sum_{\lambda} P_{\lambda_1}[X]P_{\lambda_2}[X] \cdots P_{\lambda_{2n}}[Y] \]

\[ = \sum_{\lambda \in OP} z_\lambda^{-1} 2^{\ell(\lambda)} p_\lambda[X]p_\lambda[Y], \]

where \(M_\lambda = 2^{\ell(\lambda)} m_\lambda\). These equalities hold by definition, \([20, \text{III.8.13}]\), setting \(t = -1\) in \([20, \text{III.4.2}], \) and \([20, \text{III.8.12}]\).

Let \(J_n \subset \Gamma^*\) be the ideal given by the annihilator of \(\Gamma (n) \subset \Gamma^*\) with respect to \([\cdot, \cdot]\).

Define \(\Gamma^{(n)} = \Gamma^*/J_n\)

which is a Hopf quotient algebra of \(\Gamma^*\). The pairing \([\cdot, \cdot]\) descends to a perfect pairing \(\Gamma^{(n)} \otimes \Gamma^{(n)} \to \mathbb{Z}\) which by (2.15) has reproducing kernel

\[ \Omega_{-1}^{(n)} = \sum_{\lambda_1 \leq 2n} P_{\lambda_1}[X]P_{\lambda_2}[X] \cdots P_{\lambda_{2n}}[Y]. \]

2.3. Comparing \(\Lambda\) with \(\Gamma_\ast\) and \(\Gamma^*\). Since \(\Lambda = \bigoplus_{\lambda} \mathbb{Z} h_\lambda\) \([20, \text{I.2.8}]\) one may define a surjective ring homomorphism \(\theta : \Lambda \to \Gamma^*\) defined by \(\theta(h_i) = Q_i\) for \(i \in \mathbb{Z}_{>0}\).

Over \(\mathbb{Q}\) it may be defined by \(\theta(p_{2i}) = 0\) and \(\theta(p_{2i-1}) = 2p_{2i-1}\) for \(i \in \mathbb{Z}_{>0}\) \([20, \text{Ex. III.8.10}]\).

Let \(\iota : \Gamma_\ast \to \Lambda\) be the inclusion map.

Lemma 2.1.

\[ \langle \iota(f), g \rangle = [f, \theta(g)] \quad \text{for } f \in \Gamma_\ast, g \in \Lambda. \]

Proof. By linearity one may reduce to the case \(f = p_\lambda\) and \(g = p_\mu\) for \(\lambda, \mu \in OP\).

By (2.2) and (2.15) we have

\[ [p_\lambda, \theta(p_\mu)] = 2^{\ell(\mu)}[p_\lambda, p_\mu] \]

\[ = 2^{\ell(\mu) - \ell(\lambda)} z_\lambda \delta_{\lambda\mu} \]

\[ = z_\lambda \delta_{\lambda\mu} \]

\[ = \langle \iota(p_\lambda), p_\mu \rangle. \]

\[ \square \]
Lemma 2.1 can be restated as
\[ \theta^Y \Omega = \Omega^{-1} \]
where \( \theta^Y \) means the operator \( \theta \) applied to the \( Y \) variables.

**Lemma 2.2.** For \( \nu \in SP \)
\[ Q_\nu = \sum_\lambda L_{\nu\lambda} M_\lambda. \]

In particular \( \Gamma^* \subset \bigoplus_\lambda \mathbb{Z} M_\lambda \).

**Proof.** Since both \( \Omega \) and \( \Omega^{-1} \) are invariant under exchanging the \( X \) and \( Y \) variables, by (2.19) and (2.9) we have
\[ \theta^Y \Omega = \Omega^{-1} = \theta^X \Omega \]
\[ = \theta^X \sum_\lambda h_\lambda[X] m_\lambda[Y] \]
\[ = \sum_\lambda Q_{\lambda_1}[X] Q_{\lambda_2}[X] \cdots m_\lambda[Y] \]
\[ = \sum_\lambda P_{\lambda_1}[X] P_{\lambda_2}[X] \cdots M_\lambda[Y] \]
\[ = \sum_\lambda \sum_{\nu \in SP} L_{\nu\lambda} P_\nu[X] M_\lambda[Y] \]
\[ = \sum_{\nu \in SP} P_\nu[X] \sum_\lambda L_{\nu\lambda} M_\lambda[Y]. \]

By (2.15) and (2.10), taking the coefficient of \( P_\nu[X] \), the Lemma follows. \( \square \)

2.4. A monomial-like basis for \( \Gamma^{(n)} \). For a partition \( \lambda \), define \( T_\lambda = \theta(m_\lambda) \). We shall give a “monomial” basis of \( \Gamma^{(n)} \) using the \( T_\lambda \). Let \( \chi(\text{true}) = 1 \) and \( \chi(\text{false}) = 0 \).

**Lemma 2.3.** For every partition \( \lambda \),
\[ T_\lambda \in \chi(\lambda \in \mathcal{OP}) M_\lambda + \sum_{\mu \triangleright \lambda} \mathbb{Z} M_\mu. \]

**Proof.** Define \( y_\lambda = \prod_{i \geq 1} m_i(\lambda)! \) where \( m_i(\lambda) \) is the multiplicity of the part \( i \) in \( \lambda \). By expanding \( p_\lambda \) it is easy to see that \( p_\lambda \in y_\lambda m_\lambda + \sum_{\mu \triangleright \lambda} \mathbb{Z} y_\mu m_\mu \). It follows that
\[ m_\lambda \in y_\lambda^{-1} p_\lambda + \sum_{\mu \triangleright \lambda} Q p_\mu \]
and
\[ T_\lambda \in y_\lambda^{-1} \theta(p_\lambda) + \sum_{\mu \triangleright \lambda} Q \theta(p_\mu) \]
\[ = \chi(\lambda \in \mathcal{OP}) y_\lambda^{-1} \theta(p_\lambda) + \sum_{\mu \triangleright \lambda \in \mathcal{OP}} Q p_\mu \]
\[ = \chi(\lambda \in \mathcal{OP}) M_\lambda + \sum_{\mu \triangleright \lambda} Q m_\mu. \]

By Lemma 2.2 \( T_\lambda = \theta(m_\lambda) \in \Gamma^* \) is a \( \mathbb{Z} \)-linear combination of the \( M_\mu \). Since by definition the \( M_\mu \) are integer multiples of the \( m_\mu \), (2.20) expresses \( T_\lambda \) as a \( \mathbb{Q} \)-linear combination of the \( M_\mu \). Since the \( M_\mu \) are independent, the coefficients must then be integers. \( \square \)
Lemma 2.4. For \( \lambda, \mu \in OP \),
\[
[P_\lambda, P_\mu] = 0 \quad \text{unless} \quad \lambda \succeq \mu, \tag{2.21}
\]
\[
[P_\lambda, P_\mu] = 1. \tag{2.22}
\]

Proof. Let \( A = (a_{\lambda \nu}) \) and \( B = (b_{\lambda \nu}) \) be the change of basis matrices
\[
h_\lambda = \sum_{\nu \leq \lambda} a_{\lambda \nu} p_\nu \quad \text{and} \quad p_\nu = \sum_{\rho \leq \nu} b_{\nu \rho} h_\rho.
\]
They are unitriangular and mutually inverse. We have
\[
P_\lambda, P_\lambda, \ldots = 2^{-\ell(\lambda)} \theta(h_\lambda)
\]
\[
= \sum_{\nu \leq \lambda} \sum_{\rho \leq \nu} 2^{\ell(\nu) - \ell(\lambda)} a_{\lambda \nu} b_{\nu \rho} h_\rho.
\]
For any partition \( \mu \), by Lemmata 2.1 and 2.3 we have
\[
[P_\lambda, P_\lambda, \ldots, T_\mu] = \langle P_\lambda, P_\lambda, \ldots, m_\mu \rangle
\]
\[
= \sum_{\nu \leq \lambda} \sum_{\rho \leq \nu} 2^{\ell(\nu) - \ell(\lambda)} a_{\lambda \nu} b_{\nu \rho} \delta_{\mu \rho}
\]
by (2.21). But this sum is zero unless \( \lambda \succeq \mu \), proving (2.21). When \( \mu = \lambda \) we have
\[
[P_\lambda, P_\lambda, \ldots, T_\lambda] = a_{\lambda \lambda} h_\lambda = 1, \text{ since } AB = I. \]
\[
\Box
\]

Proposition 2.5. We have
\[
\Gamma^* = \bigoplus_{\lambda \in OP} ZT_\lambda, \quad \Gamma^{(n)} = \bigoplus_{\lambda \in OP, \lambda_1 \leq 2n} ZT_\lambda, \quad J_n = \bigoplus_{\lambda \in OP, \lambda_1 \geq 2n+1} ZT_\lambda.
\]

Proof. This follows from Lemma 2.4, which says that \( \{T_\mu \mid \mu \in OP\} \) is unitriangularly related (over \( Z \)) to the \( Z \)-basis of \( \Gamma^* \) that is \([\cdot, \cdot]\)-dual to the \( Z \)-basis of \( \Gamma^* \) given by \( \{P_\lambda, P_\lambda, \ldots \mid \lambda \in OP\} \).
\[
\Box
\]

2.5. Another realization of \( \Gamma^{(n)} \). Let \( I_k \subset \Lambda \) be the ideal generated by \( m_\lambda \) for \( \lambda_1 \geq k \). There is a natural ring isomorphism
\[
\Gamma^*/(\Gamma^* \cap I_{2n+1}) \cong (\Gamma^* + I_{2n+1})/I_{2n+1}.
\]
By Proposition 2.5, we have \( \Gamma^* \cap I_{2n+1} = J_n \). Therefore
\[
\Gamma^{(n)} \cong (\Gamma^* + I_{2n+1})/I_{2n+1}. \tag{2.23}
\]
It follows from (2.23) and Lemma 2.1 that
\[
[P_\lambda, P_\lambda, \ldots, f] \text{ is the coefficient of } M_\lambda \text{ in } f \tag{2.24}
\]
for \( f \in \Gamma^{(n)} \) and \( \lambda \) satisfying \( \lambda_1 \leq 2n \).
3. Affine root systems

3.1. Weyl group. A Cartan datum is a pair \((I, A)\) where \(I\) is a finite set (the set of Dynkin nodes) and \(A = \{a_{ij} \mid i, j \in I\}\) is a generalized Cartan matrix, which by definition satisfies \(a_{ii} = 2\) for \(i \in I\), \(a_{ij} \leq 0\) for \(i \neq j\), and \(a_{ij} < 0\) if and only if \(a_{ji} < 0\). The Cartan datum \((I, A)\) is of finite type if \(A\) is nonsingular and of affine type if \(A\) has corank one.

Given a Cartan datum \((I, A)\), for \(i, j \in I\) with \(i \neq j\), define the integers \(m_{ij} = 2, 3, 4, 6, \infty\) according as \(a_{ij}a_{ji} = 0, 1, 2, 3,\) or \(4\). The Weyl group \(W = W(I, A)\) is the Coxeter group with generators \(s_i\) for \(i \in I\) such that \(s_i^2 = 1\) for all \(i \in I\) and braid relations

\[
\prod_{m_{ij}} s_i s_j \cdots = \prod_{m_{ij}} s_j s_i \cdots \quad \text{for} \ i \neq j.
\]

The length function \(\ell : W \to \mathbb{Z}\) is given by \(\ell(w) = l\) if a shortest expression \(w = s_{i_1}s_{i_2} \cdots s_{i_l}\) of \(w\) as a product of the \(s_{i_i}\) is of length \(l\). We call such an expression \(w = s_{i_1}s_{i_2} \cdots s_{i_l}\) a reduced expression. The word \(i_1i_2 \cdots i_l\) consisting of the indices of a reduced expression is called a reduced word for \(w\). We denote by \(R(w)\) the set of reduced words for \(w\). We write \(w = u^t\) if \(u, u^t \in R(w)\) for some \(w \in W\).

An element \(s \in W\) is a reflection if \(s = ws_iw^{-1}\) for some \(i \in I\) and \(w \in W\).

The Bruhat order on \(W_{af}\) is defined by \(v \leq w\) if some (equivalently every) reduced word of \(w\) has a subword that is a reduced word for \(v\). Alternatively \(v < w\) if \(v^{-1}w\) is a reflection and \(\ell(w) = \ell(v) + 1\).

We now fix notation for an affine Cartan datum. Let \((I, A)\) be the finite Cartan datum associated with the Lie algebra \(g\) of a simple simply-connected complex algebraic group \(G\). Let \(I = \{1, 2, \ldots, n\}\) where \(n\) is the rank of \(g\). Let \((I_{af}, A_{af})\) be the affine Cartan datum for the un twisted affine algebraic group \(g_{af} = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}K \oplus \mathbb{C}d\) \([11] \text{§7.2}\). We write \(I_{af} = \{0\} \sqcup I\) where \(0 \in I_{af}\) is the distinguished Kac 0 node \([11] \text{§4.8}\). Let \(W = W(I, A)\) be the finite Weyl group and \(W_{af} = W(I_{af}, A_{af})\) the affine Weyl group. We denote by \(W_{af}^0 \subset W_{af}\) the set of Grassmannian elements, which by definition are the minimal length coset representatives of \(W_{af}/W\).

Example 3.1. If \(g = sp_{2n}(\mathbb{C})\), the Cartan matrix for \(g_{af}\) is given by \(a_{ii} = 2\) for \(i \in I_{af}\), \(a_{i,i+1} = a_{i+1,i} = -1\) for \(1 \leq i \leq n - 2\), \(a_{01} = -1\), \(a_{10} = -2\), \(a_{n-1,n} = -2\), \(a_{n,n-1} = -1\), and \(a_{ij} = 0\) if \(|i - j| \geq 2\). The affine Weyl group \(W_{af} = \tilde{C}_n\) has generators \(\{s_0, s_1, \ldots, s_n\}\) and relations

\[
s_i^2 = 1 \\
s_i s_j = s_j s_i \quad \text{if} \ |i - j| > 1 \\
s_i s_{i+1}s_i = s_{i+1}s_is_{i+1} \quad \text{if} \ 1 \leq i \leq n - 2 \\
s_0s_1s_0s_1 = s_1s_0s_1s_0 \\
s_{n-1}s_n s_{n-1}s_n = s_n s_{n-1}s_n s_{n-1}.
\]

For \(W_{af} = \tilde{C}_n\), we use the notation \(W_{af}^0 = \tilde{C}_n^0\) and \(W = C_n\).

3.2. Affine root, coroot, and weight lattices. Let \(P_{af} = \mathbb{Z} \delta \oplus \bigoplus_{i \in I_{af}} \mathbb{Z} \Lambda_i\) be the affine weight lattice, where \(\delta\) is the null root and the \(\Lambda_i\) are the fundamental weights. Let \(P_{af}^* = \text{Hom}_\mathbb{Z}(P_{af}, \mathbb{Z})\) be the dual weight lattice and \(\langle \cdot, \cdot \rangle : P_{af}^* \times P_{af} \to \mathbb{Z}\)
the natural perfect pairing. Let \( \{d\} \cup \{ \alpha_i^\vee | i \in I_{af} \} \) be the basis of \( P_{af}^* \) dual to the above basis of \( P_{af} \); in particular,

\[
\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij} \quad \text{for } i, j \in I_{af}
\]

\[
\langle \alpha_i^\vee, \delta \rangle = 0 \quad \text{for } i \in I_{af}
\]

where \( \delta_{ij} \) is the Kronecker delta. The \( \alpha_i^\vee \) are called simple coroots. For \( j \in I_{af} \) define the simple root \( \alpha_j \in P_{af} \) by

\[
\alpha_j = \sum_{i \in I_{af}} a_{ij} \Lambda_i + \delta_{j0} \delta.
\]

Note that

\[
\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij} \quad \text{for all } i, j \in I_{af}.
\]

Due to a linear dependence among the columns of the Cartan matrix, we have

\[
\delta = \alpha_0 + \theta
\]

where \( \theta \) is the highest root of \( g \). Let \( Q_{af} = \bigoplus_{i \in I_{af}} \mathbb{Z} \alpha_i \subset P_{af} \) and \( Q_{af}^\vee = \bigoplus_{i \in I_{af}} \mathbb{Z} \alpha_i^\vee \) be the affine root and coroot lattices. The nullroot satisfies

\[
\langle \mu, \delta \rangle = 0 \quad \text{for all } \mu \in Q_{af}^\vee.
\]

Similarly a dependence among the rows of the Cartan matrix, yields the canonical central element \( K \in Q_{af}^\vee \) defined by

\[
K = \alpha_0^\vee + \theta^\vee
\]

where \( \theta^\vee \) is the coroot associated to \( \theta \) (defined in the next subsection). \( K \) satisfies

\[
\langle K, \lambda \rangle = 0 \quad \text{for all } \lambda \in Q_{af}.
\]

**Example 3.2.** For \( g = sp_{2n}(\mathbb{C}) \), \( g_{af} \) has nullroot \( \delta = \alpha_0 + 2(\alpha_1 + \cdots + \alpha_{n-1}) + \alpha_n \) and canonical central element \( K = \alpha_0^\vee + \cdots + \alpha_n^\vee \).

The affine Weyl group \( W_{af} \) acts on \( P_{af} \) and \( P_{af}^\vee \) by

\[
s_i \lambda = \lambda - \alpha_i \langle \alpha_i^\vee, \lambda \rangle \quad \text{for } \lambda \in P_{af}
\]

\[
s_i \mu = \mu - \alpha_i^\vee \langle \mu, \alpha_i \rangle \quad \text{for } \mu \in P_{af}^\vee.
\]

One may show that

\[
\langle w\mu, w\lambda \rangle = \langle \mu, \lambda \rangle \quad \text{for } w \in W_{af}, \lambda \in P_{af}, \mu \in P_{af}^\vee.
\]

By (3.3) and (3.8) we have

\[
w\delta = \delta, \quad wK = K \quad \text{for all } w \in W_{af}.
\]

### 3.3. Finite root, coroot, and weight lattices.

The finite coroot lattice is defined by \( Q^\vee = \bigoplus_{i \in I} \mathbb{Z} \omega_i^\vee \subset Q_{af}^\vee \). The finite root and weight lattices \( Q \) and \( P \) are quotients of their affine counterparts \( Q_{af} \) and \( P_{af} \), but by abuse we will define them as sublattices. The finite root lattice is defined by \( Q = \bigoplus_{i \in I} \mathbb{Z} \omega_i \subset Q_{af} \subset P_{af} \). The finite weight lattice is defined by \( P = \bigoplus_{i \in I} \mathbb{Z} \omega_i \subset P_{af} \) where

\[
\omega_i = \Lambda_i - \langle K, \Lambda_i \rangle \Lambda_0
\]

for \( i \in I \); these are the fundamental weights of \( g \). We have

\[
\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij} \quad \text{for } i, j \in I.
\]
3.4. **Roots.** The root system of \( \mathfrak{g} \) may be defined by
\[
R = W \cdot \{ \alpha_i \mid i \in I \}.
\]
Given \( \alpha \in R \) with \( \alpha = u \alpha_i \) for some \( u \in W \) and \( i \in I \), its associated coroot is defined by \( \alpha^\vee = u \alpha_i^\vee \in Q^\vee \). Its associated reflection is \( s_\alpha = u s_i u^{-1} \). Both are independent of the choice of \( u \) and \( i \). There is a decomposition \( R = R^+ \cup -R^+ \) where \( R^+ = R \cap \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \) is the set of positive roots.

The set of affine roots \( R_{af} \subset Q_{af} \) is given by the set of nonzero elements in the set \( R + \mathbb{Z} \delta \). We have \( R_{af} = R^+_{af} \cup -R^+_{af} \) where \( R^+_{af} \) is the set of positive affine roots, which have the form \( \alpha + m \delta \) where either \( m > 0 \) or both \( m = 0 \) and \( \alpha \in R^+ \). Equivalently, \( R^+_{af} = R_{af} \cap \bigoplus_{i \in I_{af}} \mathbb{Z}_{\geq 0} \alpha_i \).

The set of real affine roots is defined by
\[
R^+_{af} = W_{af} \cdot \{ \alpha_i \mid i \in I_{af} \}.
\]
For \( \alpha = u \alpha_i \in R^+_{af} \) for \( u \in W_{af} \) and \( i \in I_{af} \) define the associated coroot by \( \alpha^\vee = u \alpha_i^\vee \) and associated reflection \( s_\alpha \in W_{af} \) by \( s_\alpha = u s_i u^{-1} \); as before one may show these definitions are independent of \( u \) and \( i \).

Let \( v < w \) in \( W_{af} \). Then \( s = v^{-1} w \) is a reflection \( s = u s_i u^{-1} \) for some \( i \in I_{af} \) and \( u \in W_{af} \). Let \( u \) be shortest so that \( \alpha = u \alpha_i \) is a positive real root. For later use we denote this root \( \alpha \) by \( \alpha_{vw} \) and its associated coroot by \( \alpha^\vee_{vw} \).

**Example 3.3.** Let \( W_{af} = \tilde{C}_3 \), \( w = s_1 s_2 s_3 s_2 s_1 s_0 \), and \( v = s_1 s_3 s_2 s_1 s_0 \); this defines a cover \( v < w = vs_\alpha \). Then \( s_\alpha = (s_0 s_1 s_2 s_3)(s_3 s_2 s_1 s_0) \) and
\[
\begin{align*}
\alpha^\vee_{vw} &= s_0 s_1 s_2 s_3(\alpha_2^\vee) \\
&= 2\alpha_0^\vee + \alpha_1^\vee + \alpha_2^\vee + 2\alpha_3^\vee.
\end{align*}
\]

3.5. **Level 0 action.** There is a surjective group homomorphism \( W_{af} \to W \) given by \( s_i \mapsto s_i \) for \( i \in I \) and \( s_0 \mapsto s_0 \) where \( \theta \in R^+ \) is the highest root. Since \( W \) acts on \( P \), \( W_{af} \) acts on \( P \) via the above homomorphism; this is called the level zero action. It is not faithful since \( s_0 \) and \( s_\theta \) are different elements of \( W_{af} \).

4. **NilHecke algebra and affine Grassmannian**

4.1. **(Co)homology of affine Grassmannian.** For this section we fix \( G \) a simple and simply-connected complex algebraic group with Weyl group \( W \), and Cartan datum \((I, A)\) as in section 3.1. Let \( K \) denote a maximal compact subgroup of \( G \) and \( T \) denote a maximal torus in \( K \).

Let \( \mathbb{F} = \mathbb{C}((t)) \) and \( \mathbb{O} = \mathbb{C}[[t]] \). The affine Grassmannian \( Gr_G \) is the ind-scheme \( G(\mathbb{F})/G(\mathbb{O}) \) (see [13]). It is a homogeneous space for the affine Kac-Moody group \( G \) associated to \( W_{af} \). It is a classical result due to Quillen that the space \( Gr_G \) is homotopy-equivalent to the space \( \Omega K \) of based loops in \( K \); see for example [7, 23].

The group \( G \) possesses a **Bruhat decomposition** \( G = \bigcup_{w \in W_{af}} BwB \) where \( B \) denotes the Iwahori subgroup. The Bruhat decomposition induces a decomposition of \( Gr_G \) into **Schubert cells** \( \Omega_w = BwG(\mathbb{O}) \subset G(\mathbb{F})/G(\mathbb{O}) \). Thus the equivariant homology \( H_\ast(Gr_G) \) and cohomology \( H^\ast(Gr_G) \) of \( Gr_G \) are free \( S = H^\ast(\text{pt}) \)-modules with Schubert bases \( \xi^w_\ast \in H_\ast(Gr_G) \) and \( \xi^w_\ast \in H^\ast(Gr_G) \). Similarly, the homology \( H_\ast(Gr_G) \) and cohomology \( H^\ast(Gr_G) \) of \( Gr_G \) are free \( \mathbb{Z} \)-modules with Schubert bases \( \xi_\ast \in H_\ast(Gr_G) \) and \( \xi^\ast \in H^\ast(Gr_G) \). The index \( x \) varies over the Grassmannian elements \( W_{af}^0 \). We refer the reader to [12, 13] for the general construction and properties of Schubert bases in the Kac-Moody setting.
The pointwise multiplication of loops on $K$ induces the structure of dual Hopf algebras over $\mathbb{Z}$ to $H_\epsilon(Gr_G)$ and $H^*(Gr_G)$, and the structure of dual Hopf algebras over $S$ to $H_T(Gr_G)$ and $H^T(Gr_G)$. This is a special feature of the affine Grassmannian unavailable in the more general Kac-Moody setting.

4.2. NilCoxeter algebra. The nilCoxeter algebra $\mathcal{A}_0$ is the associative $\mathbb{Z}$-algebra with generators $A_i$ for $i \in I$ and relations $A_i^2 = 0$ for $i \in I$ and braid relations

\[
A_i A_j A_i = A_i A_j \quad \text{for } \text{all } i \neq j.
\]

Since these are the same braid relations $[3,1]$ satisfied by $s_i \in W$, for $w \in W$ one may define $A_w = A_{i_1} A_{i_2} \cdots A_{i_k}$ for any $i_1, i_2, \ldots, i_k \in R(w)$.

The algebra $\mathcal{A}_0$ is a free $\mathbb{Z}$-module with basis $\{A_w \mid w \in W\}$. In this basis, the multiplication is given by

\[
A_v A_u = \begin{cases} 
A_{vu} & \text{if } \ell(v) + \ell(u) = \ell(vu) \\
0 & \text{otherwise}. 
\end{cases}
\]

Example 4.1. For the affine Cartan datum of Example 3.1, $\mathcal{A}_0$ has generators $A_i$ for $i \in I_{\text{aff}}$ and relations

- $A_i^2 = 0$
- $A_i A_j = A_j A_i$ \quad if $|i - j| > 1$
- $A_i A_{i+1} A_i = A_{i+1} A_i A_{i+1}$ \quad if $1 \leq i \leq n - 2$
- $A_0 A_1 A_0 A_1 = A_1 A_0 A_1 A_0$
- $A_{n-1} A_n A_{n-1} A_n = A_n A_{n-1} A_n A_{n-1}$

4.3. Kostant and Kumar’s NilHecke algebra. Let $P$ be the weight lattice of $\mathfrak{g}$ and $S = \text{Sym}(P)$ the symmetric algebra. The Peterson affine nilHecke algebra $\mathcal{H}$ is by definition the associative $\mathbb{Z}$-algebra generated by $S$ and the nilCoxeter algebra $\mathcal{A}_0$ for the affine Cartan datum $(I_{\text{af}}, I_{\text{af}})$ with

\[
A_i \lambda = (s_i \cdot \lambda) A_i + \langle \alpha_i^\vee, \lambda \rangle 1 \quad \text{for } i \in I_{\text{af}} \text{ and } \lambda \in P.
\]

Consequently $\mathcal{H}$ is a free left $S$-module with basis $\{A_w \mid w \in W_{\text{af}}\}$.

Iterating (4.2) produces the following relation.

Lemma 4.2. For $x \in W_{\text{af}}$ and $\lambda \in P$,

\[
A_x \lambda = (x \cdot \lambda) A_x + \sum_{y \leq x} \langle \alpha_y^\vee, \lambda \rangle A_y
\]

where $\alpha_y^\vee$ is defined in section 3.4.

Proposition 4.3. [22] Let $M$ and $N$ be left $\mathcal{A}$-modules. Define $M \otimes_S N = (M \otimes Z N)/(sm \otimes n - m \otimes sn \mid s \in S; m \in M; n \in N)$. Then $M \otimes_S N$ is a left $\mathcal{A}$-module via

\[
A_i \cdot (m \otimes n) = (A_i \cdot m) \otimes n + m \otimes (A_i \cdot n) - \alpha_i (A_i \cdot m) \otimes (A_i \cdot n)
\]

\[
s \cdot (m \otimes n) = sm \otimes n.
\]

\footnote{The nilHecke algebra of Kostant and Kumar [22] for the affine Cartan datum, uses a larger weight lattice than Peterson’s nilHecke algebra. See [15] for a comparison of the two.}
To verify the braid relations, it is convenient to introduce the elements $s_1 \otimes m \otimes n$. We verify (4.2).

We verify (4.2).

$$A_i \cdot (\lambda \cdot (m \otimes n)) = (A_i \cdot \lambda m) \otimes n + \lambda m \otimes (A_i \cdot n) - \alpha_i (A_i \cdot \lambda m) \otimes (A_i \cdot n) = (s_1 \lambda) (A_i m \otimes n) + (\alpha_i \lambda) m \circ n + \lambda m \otimes A_i n - \alpha_i(s_1 \lambda) A_i m \otimes A_i n - \alpha_i(\alpha_i \lambda) m \otimes A_i n = (s_1 \lambda)(A_i m \otimes n) + (\lambda - \alpha_i(\alpha_i \lambda)) m \otimes A_i n - (s_1 \lambda) \alpha_i A_i m \otimes A_i n + (\alpha_i \lambda) m \otimes n = (s_1 \lambda) \cdot A_i \cdot (m \otimes n) + (\alpha_i \lambda) m \otimes n.$$  

We verify $A_i^2 = 0$.

$$A_i \cdot (A_i \cdot (m \otimes n)) = 2A_i m \otimes A_i n - (A_i \alpha_i A_i m) \otimes n = 2A_i m \otimes A_i n + \alpha_i A_i m \otimes A_i n - (\alpha_i \lambda, \alpha_i) A_i m \otimes A_i n = 0.$$  

To verify the braid relations, it is convenient to introduce the elements $r_i = 1 - \alpha_i A_i \in \mathbb{A}$. It is not difficult to see that given (4.2) the relation $(A_i A_i)^{m_{ij}} = (A_j A_j)^{m_{ij}}$ is equivalent to $(r_i r_j)^{m_{ij}} = (r_j r_i)^{m_{ij}}$. An easy calculation shows that the $r_i$ act on $M \otimes S N$ by $r_i \cdot (m \otimes n) = r_i m \otimes r_i n$. It is clear that this action satisfies the braid relations for the $r_i$.

Thus there is a left $S$-module homomorphism $\Delta : \mathbb{A} \to \mathbb{A} \otimes S \mathbb{A}$ defined by

$$\Delta(a) = a \cdot (1 \otimes 1) \quad \text{for } a \in \mathbb{A}.$$  

By (4.2) we have

$$\Delta(A_i) = A_i \otimes 1 + 1 \otimes A_i - A_i \otimes \alpha_i A_i,$$

$$\Delta(s) = s \otimes 1.$$  

The map $\Delta$ is injective so there is a linear map $\Delta(\mathbb{A}) \otimes (\mathbb{A} \otimes S \mathbb{A}) \to (\mathbb{A} \otimes S \mathbb{A})$ defined by

$$\Delta(a) \otimes (x \otimes y) \to a \cdot (x \otimes y)$$  

using the left $\mathbb{A}$-module structure on $\mathbb{A} \otimes S \mathbb{A}$ afforded by Proposition 4.3. We deduce that this map yields a ring structure on $\Delta(\mathbb{A})$ and an action of $\Delta(\mathbb{A})$ on $\mathbb{A} \otimes S \mathbb{A}$.

It follows by induction using Proposition 4.3 that this action is computed explicitly as follows. Let $a \in \mathbb{A}$ and $\Delta(a) = \sum_{w, v} A_w \otimes a_{wv} A_v$. Then

$$\Delta(a) \cdot (x \otimes y) = \sum_{w, v} A_w x \otimes a_{wv} A_v y.$$  

In particular, if $b \in \mathbb{A}$ and $\Delta(b) = \sum_{w, v} A_w \otimes b_{wv} A_v$ for $b_{wv} \in S$, then

$$\Delta(ab) = \Delta(a)\Delta(b) = \sum_{w, v, w', v'} A_w A_{w'} \otimes a_{wv} A_v b_{w'v'} A_{w'}.$$  

The ring structure on $\Delta(\mathbb{A})$ does not extend to all of $\mathbb{A} \otimes S \mathbb{A}$ by the formula $(a \otimes b)(c \otimes d) = ac \otimes bd$, because if it did, then since $s \otimes 1 = 1 \otimes s$ we would have $fs \otimes g = (f \otimes g)(s \otimes 1) = (f \otimes g)(1 \otimes s) = f \otimes gs$, which is false in general (say, for $g = 1$ and $fs \neq sf$). Equation (4.10) says that when this “obvious” generally
ill-defined multiplication formula is applied to expressions coming from the action of $\Delta(\lambda)$ on $A \otimes S_A$, the result is well-defined.

4.4. The Peterson subalgebra and equivariant cohomology of affine Grassmannian. The Peterson subalgebra of $A$ is the centralizer $Z_A(S)$ of $S$. It is a Hopf algebra over $S$ since the factorwise product on $Z_A(S) \otimes S_A$ gives it an $S$-algebra structure under which the restriction of $\Delta$ to $Z_A(S)$, is an $S$-algebra homomorphism.

Theorem 4.4. [22] [15, Theorem 4.4] There is an $S$-Hopf algebra isomorphism $j : H_T(Gr_G) \rightarrow Z_A(S)$ which is characterized by the property that for all $x \in W_{af}^0$, $j(\xi_T^x)$ is the unique element of $Z_A(S) \cap (A_x + \sum_{y \in W_{af}} S_A y)$.

For $x \in W_{af}^0$ and $y \in W_{af}$ let $j^y_x \in S$ be defined by

$$j(\xi_T^x) = \sum_{y \in W_{af}} j^y_x y.$$  \hspace{1cm} (4.11)

Proposition 4.5. [22] [17, Theorem 6.3]

1. For $x \in W_{af}^0$ and $y \in W_{af}$, the polynomial $j^y_x$ is either zero or homogeneous of degree $\ell(y) - \ell(x)$; in particular it is zero if $\ell(y) < \ell(x)$.

2. For $x, z \in W_{af}^0$ we have

$$\xi_T^x \xi_T^z = \sum_y j^y_x \xi_T^y$$  \hspace{1cm} (4.12)

where $y$ runs over the $y \in W_{af}$ such that $yz \in W_{af}^0$ and $\ell(yz) = \ell(y) + \ell(z)$.

We wish to compute $j^y_x$ in the “nonequivariant case” $\ell(x) = \ell(y)$, when $j^y_x \in Z_{\geq 0}$. For this purpose we consider the maps that forget the $T$-equivariance.

4.5. Affine Fomin-Stanley subalgebra. Let $\phi_0 : S \rightarrow \mathbb{Z}$ be the map that sends a polynomial to its evaluation at 0. By abuse of notation define $\phi_0 : A \rightarrow A_0$ by $\phi_0(\sum_w s_w A_w) = \sum_w \phi_0(s_w) A_w$ for $s_w \in S$. Peterson’s $j$-map induces an injective ring homomorphism $j_0 : H_*(Gr_G) \rightarrow A_0$ such that the diagram commutes:

$$\begin{array}{ccc}
H_T(Gr_G) & \xrightarrow{j} & A \\
\downarrow & & \downarrow \phi_0 \\
H_*(Gr_G) & \xrightarrow{j_0} & A_0
\end{array}$$  \hspace{1cm} (4.13)

where $\epsilon : H_T(Gr_G) \rightarrow H_*(Gr_G)$ is obtained by $\xi_T^x \mapsto \xi_x$ and the evaluation $\phi_0$.

By (4.11) and (4.13) we have

$$j_0(\xi_w) = \sum_{\substack{u \in W_{af} \\ell(u) = \ell(w)}} j^u_w A_u$$  \hspace{1cm} (4.14)

for $w \in W_{af}^0$.

The affine Fomin-Stanley subalgebra is defined in [15] by

$$\mathbb{B} = \{ a \in A \mid \phi_0(s)a = \phi_0(as) \text{ for every } s \in S \}.$$  \hspace{1cm} (4.15)
Define \( \phi_0^{(2)} : A \otimes_S A \to A_0 \otimes_Z A_0 \) by
\[
\phi_0^{(2)} \left( \sum_{w,v \in W_{af}} a_{w,v} A_w \otimes A_v \right) = \sum_{w,v \in W_{af}} \phi_0(a_{w,v}) A_w \otimes A_v
\]
for \( a_{w,v} \in S \). Then \( \mathbb{B} \) is a Hopf algebra with coproduct given by the restriction of \( \phi_0^{(2)} \circ \Delta \) to \( \mathbb{B} \).

**Theorem 4.6** ([15] Prop. 5.4, Thm. 5.5). The map \( j_0 \) is a Hopf algebra isomorphism \( H_\ast(\text{Gr}_G) \cong \mathbb{B} \). Moreover, for every \( w \in W_{af}^0 \), \( j_0(\xi_w) \) is the unique element of \( \mathbb{B} \cap (A_w + \sum_{u \in W_{af}^0} Z A_u) \).

\[ \mathbb{B} \text{ has a basis } \{ P_w \mid w \in W_{af}^0 \} \text{ defined by} \]
\[ (4.16) \quad P_w = j_0(\xi_w) \quad \text{for } w \in W_{af}^0. \]

For \( G = SL_n(\mathbb{C}) \) these are the noncommutative \( k \)-Schur functions of [15]. The following Lemma is an aid for computing the elements \( P_w \).

**Lemma 4.7.** Let \( a = \sum_{w \in W_{af}} c_w A_w \in A_0 \) with \( c_w \in Z \). Then \( a \in \mathbb{B} \) if and only if \( \sum_{w > v} c_w \alpha^\vee_{w,v} \in ZK \) for all \( v \in W_{af}^0 \).

**Proof.** The following are equivalent:

1. Equation (4.10) holds for \( a \).
2. \( \phi_0(a \lambda) = 0 \) for all \( \lambda \in P \).
3. \( \sum_w c_w \sum_{v < w} \langle \alpha_{w,v}, \lambda \rangle A_v = 0 \) for all \( \lambda \in P \).
4. \( \sum_{w > v} c_w \langle \alpha_{w,v}^\vee, \lambda \rangle = 0 \) for all \( v \in W_{af}^0 \) and all \( \lambda \in P \).
5. \( \sum_{w > v} c_w \alpha_{w,v} \in ZK \) for all \( v \in W_{af}^0 \).

(1) and (2) are easily seen to be equivalent. The equivalence of (2) and (3) follows from equation (4.2). (3) and (4) are equivalent because the \( A_v \) form a basis of \( A_0 \).

(4) and (5) are equivalent because \( ZK = \{ \mu \in Q_{af}^\vee \mid \langle \mu, P \rangle = 0 \} \).

5. **Schubert polynomials for \( H_\ast(\text{Gr}_{Sp_{2n}}(\mathbb{C})) \) and \( H^\ast(\text{Gr}_{Sp_{2n}}(\mathbb{C})) \)**

In this section we outline the proofs of Theorems [12, 13, 14] and [15] relegating two technical calculations to sections [6, 7].

### 5.1. Special generators of Fomin-Stanley subalgebra.

Recall the special elements \( \rho_i \) defined in [11]. For \( 1 \leq i \leq 2n \) define
\[ (5.1) \quad P_i = P_{\rho_i} \]
where \( P_i \) is defined in (4.6).

We now state the explicit expansion of the elements \( P_i \in \mathbb{B} \) that correspond to homology generators. Recall the set \( Z \) defined in section [13]. Note that \( \rho_r \) is the unique Grassmannian element in \( Z_r \) for \( 1 \leq r \leq 2n \).

**Theorem 5.1.** For \( 1 \leq r \leq 2n \),
\[ (5.2) \quad P_r = \sum_{w \in Z_r} \rho(w)^{-1} A_w. \]

This result is proved in section [6]. Some examples for \( P_r \) are given in Appendix A.

**Remark 5.2.** It follows from Theorem [13] that the elements \( P_r \in \mathbb{B} \) generate the affine Fomin-Stanley subalgebra \( \mathbb{B} \).
5.2. Relations among special generators. Let $P^0_C$ be the set of partitions $\lambda$ with $\lambda_1 \leq 2n$, which have at most one part of size $i$ for all $i \leq n$. We first note the following result which is essentially [4, Lemma 24]. The bijection of Lemma 5.3 was first brought to our attention by Morse [21] who discovered it independently.

Lemma 5.3. Let $w \in \tilde{C}_n^0$. Then $w$ has a unique length-additive factorization

$$w = \rho_{\lambda_1} \cdots \rho_{\lambda_s} \rho_{\lambda_1}$$

into Grassmannian $Z$-s such that every left factor $\rho_{\lambda_1} \cdots \rho_{\lambda_s}$ is Grassmannian. Furthermore the map $w \mapsto \lambda(w)$ is a bijection $\tilde{C}_n^0 \to P^0_C$, such that $\ell(w) = |\lambda(w)|$.

Proof. The result follows nearly immediately from [4, Lemma 24]. In [4] one associates to $w \in \tilde{C}_n$ the window

$$[-w_n, \ldots, -w_1, 0, w_1, \ldots, w_n]$$

of an affine permutation. This corresponds to the embedding of $\tilde{C}_n$ into the affine symmetric group $S_{2n+2}$. In [4] the parabolic subgroup is generated by $\{s_0, \ldots, s_{n-1}\}$ rather than by $\{s_1, \ldots, s_n\}$ so we must apply the notational involution $s_i \mapsto s_{n-i}$ to be compatible with [4].

In any case, for $w \in \tilde{C}_n^0$ it is shown in [4, Lemma 24] that the window of $w$ can be successively sorted to become the identity. Each sorting operation corresponds to right multiplication by a factor $\rho_{\lambda_i}$. The requirement that every left factor $\rho_{\lambda_1} \cdots \rho_{\lambda_s}$ is Grassmannian corresponds to asking for the window of $w$ to be completely sorted at each step. The rest of the statement now follows from [4].

Proposition 5.4. The elements $P_i \in B$ satisfy

$$P_{2m} = 2 (P_1 P_{2m-1} - P_2 P_{2m-2} + \cdots + (-1)^{m-2} P_{m-1} P_{m+1}) + (-1)^{m-1} P_m$$

for $1 \leq m \leq n$.

Proof. We use the explicit computation of the $P_i$ given in Theorem 5.1. By evaluating the statement of Proposition 5.3 at 0, we observe that

$$P_i P_j = \sum_{w=\rho_{j_i}} 2^{c(v)-1} P_w$$

where the summation is over all $w = v \rho_{j_i}$ such that (a) $v \in Z_i$, (b) $\ell(w) = i + j$, and (c) $w \in \tilde{C}_n^0$. Now any reduced expression for $v \in Z_i$ can have at most one occurrence of $s_0$, so by Lemma 5.3 we deduce that the set of $w$ such that $P_w$ can occur in a product of the form $P_i P_j$ has the form $P_a P_b$ where $a + b = i + j$.

Now fix $1 \leq m \leq n$ and let us compute

$$S = 2 (P_1 P_{2m-1} - P_2 P_{2m-2} + \cdots + (-1)^{m-2} P_{m-1} P_{m+1}) + (-1)^{m-1} P_m.$$

First via a direct calculation we note that $\rho_m \rho_m \notin \tilde{C}_n^0$ for $1 \leq m \leq n$. We claim that for $1 \leq j \leq m$ and $w = \rho_{i} \rho_{2m-i}$ satisfying $w \in C_0^0$ and $\ell(w) = 2m$ we have

$$[P_w] P_j P_{2m-j} = \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i = j \\ 2 & \text{if } 0 < i < j \\ 1 & \text{if } i = 0. \end{cases}$$
where \([P_w]b\) denotes the coefficient of \(P_w\) in \(b \in \mathbb{B}\). The case \(i > j\) follows from Lemma 5.3. The case \(i = j\) is immediate since \(c(\rho_i) = c(\rho_{2m-i}) = 1\). For the case \(i < j\) we must consider \(v = \rho_i \rho_{2m-i} \rho_{2m-j}^{-1}\). We observe that

\[
\text{Supp}(v) = [0, i - 1] \cup \text{Supp}(\rho_{2m-i} \rho_{2m-j}^{-1})
\]

and \(i\) is smaller than all elements of \(\text{Supp}(\rho_{2m-i} \rho_{2m-j}^{-1})\) (we use the inequality \(2m - j > i\)). Thus \(v\), being a product two “non-touching” \(Z\)-s, is itself a \(Z\) and we have \(c(v) = 2\). The final case \(i = 0\) is trivial.

Now it follows that the \(S = P_{2m}\), as required. \(\Box\)

5.3. Coproduct formula for special generators. The following result is proved in section 7.

**Theorem 5.5.** For \(1 \leq r \leq 2n\)

\[
\phi_0^{(2)}(\Delta(P_r)) = 1 \otimes P_r + P_r \otimes 1 + 2 \sum_{0 < s < r} P_s \otimes P_{r-s}.
\]

**Remark 5.6.** An alternative formulation of Theorem 5.5 is that the coefficient of \(\xi^{\rho_r} \in \xi^{\rho_r} \xi^{\rho_r^{-1}} \in H^*_{\text{GrSp}_{2n}(\mathbb{C})}\) is equal to 2, for \(1 \leq s \leq r - 1\).

5.4. Affine type \(C\) Cauchy kernel. Define \(\Phi : \Gamma(n) \to H_{\text{GrSp}_{2n}(\mathbb{C})}\) by \(P_i \mapsto \xi^{\rho_i}\) for \(1 \leq i \leq 2n\) as in Theorem 1.3. By Proposition 5.4 and Theorem 4.6 this map is well-defined. Define \(\Omega_{-1}^B \in \mathbb{B} \otimes \Gamma(n)\) by taking the image of \(\Omega_{-1}^n\) under the composition \(\Phi_B = j_0 \circ \Phi : \Gamma(n) \to \mathbb{B}\):

\[
\Omega_{-1}^B = \sum_{\lambda_1 \leq 2n} P_{\lambda_1} \cdots \otimes M_{\lambda}[Y]
\]

\[
= \sum_{\alpha \leq 2n} P_{\alpha_1} \cdots \otimes 2^{\ell(\alpha)} y^\alpha
\]

(5.4)

where \(\alpha\) runs over compositions whose parts have size at most \(2n\). The second equality holds since \(\mathbb{B}\) is a commutative ring. For \(w \in \mathcal{C}_n\), the type \(C\) affine Stanley function \(Q_w^{(n)}\) is defined by

\[
\Omega_{-1}^B = \sum_{w \in \mathcal{C}_n} A_w \otimes Q_w^{(n)}[Y].
\]

(5.5)

A straightforward computation shows that this definition agrees with (1.2). Note that (1.2) defines an element of the ring \(\Gamma(n)\) via (2.23). By Theorem 4.6 we have

\[
\Omega_{-1}^B = \sum_{w \in \mathcal{C}_n} P_w \otimes Q_w^{(n)}[Y]
\]

(5.6)

where \(P_w\) is defined by (4.16).

5.5. Proof of Theorem 1.4. Theorem 1.4 follows immediately from applying the non-equivariant part of Proposition 4.5 to Theorem 5.1.
5.6. Proof of Theorem 1.3. It follows from Proposition 5.4, Theorem 5.5 and Theorem 4.6 that $\Phi : \Gamma(n) \to H_*(\text{Gr}_{SP_{2n}(C)})$ is a bialgebra morphism. Since both $\Gamma(n)$ and $H_*(\text{Gr}_{SP_{2n}(C)})$ are graded commutative and cocommutative Hopf algebras, $\Phi$ must in addition be a Hopf morphism. Recall that we define $\Psi : H^*(\text{Gr}_{SP_{2n}(C)}) \to \Gamma(n)$ by the linear map $\xi^w \mapsto Q^{(n)}_w$ for $w \in \tilde{C}^0_n$.

We first show that $\Psi : H^*(\text{Gr}_{SP_{2n}(C)}) \to \Gamma(n)$ and $\Phi : \Gamma(n) \to H_*(\text{Gr}_{SP_{2n}(C)})$ are dual with respect to the pairing $\langle \cdot , \cdot \rangle : H_*(\text{Gr}_{SP_{2n}(C)}) \times H^*(\text{Gr}_{SP_{2n}(C)}) \to \mathbb{Z}$ induced by the cap product and the pairing $[\cdot , \cdot ] : \Gamma(n) \times \Gamma(n) \to \mathbb{Z}$ of section 2.4. It suffices to show that for each $w \in \tilde{C}^0_n$ we have $\langle \Phi(f), \xi^w \rangle = [f, \Psi(\xi^w)]$ as $f$ varies over the spanning set $\{P_{\lambda_1} \cdots P_{\lambda_r} \mid \lambda_1 \leq 2n\}$ of $\Gamma(n)$. Identifying $\mathbb{B}$ with $H_*(\text{Gr}_{SP_{2n}(C)})$ via the map $j_0$ of Theorem 4.6 we calculate

$$[P_{\lambda_1} \cdots P_{\lambda_r}, \Psi(\xi^w)] = [P_{\lambda_1} \cdots P_{\lambda_r}, Q^{(n)}_w]$$

$$= [P_{\lambda_1} \cdots P_{\lambda_r}, \langle \Omega^{B_1}_w, \xi^w \rangle]$$

$$= \langle [P_{\lambda_1} \cdots P_{\lambda_r}, \Omega^{B_1}_w], \xi^w \rangle$$

$$= \langle P_{\lambda_1} \cdots P_{\lambda_r}, \xi^w \rangle$$

$$= \langle \Phi_B(P_{\lambda_1} \cdots P_{\lambda_r}), \xi^w \rangle.$$ 

The second equality holds by (5.6). The fourth holds by (5.4) and (2.24). The other equalities hold by definition.

Since $\Phi$ is a Hopf-morphism, we deduce that $\Psi$ is also a Hopf-morphism. It only remains to prove that $\Psi$ is a bijection. For surjectivity, since the $Q_r$ generate $\Gamma(n)$ as an algebra, it suffices to show that $Q^{(n)}_w = Q_r$ in $\Gamma(n)$, where $c_r \in \tilde{C}^0_n$ is the length $r$ element of the form

$$c_r = \cdots s_1 s_0 s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_2 s_1 s_0.$$ 

It is easy to see that $c_r$ has a unique reduced word. So a length-additive factorization of $c_r$ into a product $c_r = \prod_i v^i$ with each $v^i \in \mathbb{Z}$, is equivalent to a composition $(a_1, a_2, \ldots, a_s)$ of $r$ into parts of size less than $2n$, where each $v^i$ is either the identity or has one component. After multiplying by $2^t$ where $t = \# \{i \mid a_i > 0\}$, we see that $Q^{(n)}_c$ is the generating function of shifted tableaux $T$ whose shape is a single row of length $r$ where no letter can be used more than $2n$ times. The tableau $T$ is obtained from the composition $a$ by setting $a_i$ letters equal to $i$. The factor $2^t$ comes from the two possible choices of marking for the leftmost occurrence of each letter. This matches $Q^{(n)}_c$ to the combinatorial definition of $Q_r$ using tableaux [20 III.8.16].

For injectivity, it suffices to show that $\{Q^{(n)}_w \mid w \in \tilde{C}^0_n\}$ is linearly independent. We shall establish the triangularity property

$$Q^{(n)}_w = \sum_{\mu \leq \lambda(w)} a_{\mu, w} M_\mu$$

where $w \mapsto \lambda(w)$ is the bijection between $\tilde{C}^0_n$ and $\mathcal{P}_n$ of Lemma 6.3 and $\leq$ is the lexicographic order on partitions. Furthermore $a_{\mu, w}$ is unitriangular.

We first observe that if $w \in \tilde{C}^0_n$ and $w = v^s \cdots v^1$ is a factorization into $Z$’s then $v^1$ must be Grassmannian, so it is one of the $\rho_r$’s for $r \in [1, 2n]$. But if $w = \rho_{\lambda_1} \cdots \rho_{\lambda_2} \rho_{\lambda_1}$ where $\rho_{\lambda_1} \cdots \rho_{\lambda_2}$ is Grassmannian then $w \rho_r^{-1}$ cannot be length subtractive for $2n \geq r > \lambda_1$. This is because every reduced expression for
\( \rho_1 \cdots \rho_2 \) ends in \( s_0 \). Repeating this, we see that the matrix of coefficients \( a_{\mu,w} \) is triangular with respect to the lexicographic order. We are using the fact that if \( a_{\mu,w} \neq 0 \) then the factorization \( w = v^* \cdots v^1 \) can be chosen so that \( \ell(v^i) = \mu_i \).

Finally, the factorization \( w = \rho_1 \cdots \rho_2 \rho_1 \) of Lemma 5.3 shows that \( a_{\lambda(w),w} = 1 \) since \( c(\rho_1) = 1 \).

5.7. Proof of Theorem 1.2. The fact that \( Q^{(n)}_w \) is symmetric and defines an element of \( \Gamma^{(n)} \) follows from the definition via the affine type \( C \) Cauchy kernel.

The statement that \( \{ Q^{(n)}_w \mid w \in \tilde{C}^0_n \} \) forms a basis follows from Theorem 1.3 and the fact that \( \{ \xi^w \mid w \in \tilde{C}^0_n \} \) is a basis for \( H^*(\text{Gr}_{Sp_{2n}(\mathbb{C})}) \). The positivity of the product structure constants is a general theorem due to Graham [9] and Kumar [13].

The coproduct structure constants of \( \{ Q^{(n)}_w \mid w \in \tilde{C}^0_n \} \) are the same as those of \( \{ \xi^w \mid w \in \tilde{C}^0_n \} \). By the duality of \( H_*(\text{Gr}_{Sp_{2n}(\mathbb{C})}) \) and \( H^*(\text{Gr}_{Sp_{2n}(\mathbb{C})}) \) and their Schubert bases, the above constants are the same as the product structure constants for the homology classes \( \{ \xi^w \mid w \in \tilde{C}^0_n \} \). Using the nonequivariant case \( \ell(y) = \ell(x) \) of (4.12), these constants are given by the coefficients \( j^w_y \) of (4.11). But these are known to be nonnegative from the work of Peterson [22] and Lam and Shimozono [17]; they are equal to certain three-point genus zero Gromov-Witten invariants of the (finite) flag variety.

For the final positivity statement we claim that

\[
(5.7) \quad \text{the coefficient of } Q^{(n)}_v \text{ where } v \in \tilde{C}^0_n \text{ in } Q^{(n)}_w \text{ is equal to } j^w_v
\]

that is, the coefficient of \( A_w \) in \( \mathbb{P}_v \). But this follows from expanding (5.6) using the definition of \( \mathbb{P}_w \).

6. THE COMBINATORICS OF ZEE-S

It is obvious that \( Z_{cr} \) contains a unique Grassmannian element, namely, \( \rho_r \), and that \( c(\rho_r) = 1 \). To prove Theorem 5.1 by Theorem 4.6 it remains to show that the right hand side of (5.2) is an element of \( B \). By Lemma 4.7 and Example 3.2 it suffices to prove the following result, whose proof occupies the rest of this section.

Proposition 6.1. For any \( v \in Z \) with \( \ell(v) < 2n \), let \( C_v = \{ w \in Z \mid w \triangleright v \} \). Then

\[
(6.1) \quad \sum_{w \in C_v} 2^{c(w)-1} \alpha^\vee_{vw} = 2^{c(v)} K.
\]

Example 6.2. Let \( n = 3 \) and \( v = s_0 s_2 s_3 s_2 \in Z \). Every \( w \in C_v \) is obtained by putting a 1 into some reduced word for \( v \). For each \( w \in C_v \), a reduced word and the coroot \( \alpha^\vee_{vw} \) is given below. They may be computed as in Example 3.3.

| red. word | \( \alpha^\vee_{vw} \) |
|-----------|----------------------|
| 10232     | \( 2\alpha_0^\vee + \alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee \) |
| 01232     | \( \alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee \) |
| 23210     | \( 2\alpha_0^\vee + \alpha_1^\vee \) |
| 23201     | \( \alpha_1^\vee \) |

The sum of these coroots is \( 4K \), which agrees with the fact that \( \text{Supp}(v) \) has two components, \( \{ 0 \} \) and \( \{ 2, 3 \} \).
Let $w \in Z$. Since $s_is_j = s_js_i$ for $i$ and $j$ in different components of $\text{Supp}(w)$, there exists a factorization $w = w_{I_1} \cdots w_{I_r}$ where $I_1, I_2, \ldots, I_r$ are the components of $\text{Supp}(w)$ and $\text{Supp}(w_{I_p}) = I_p$. Let us index the components so their elements are ordered consistently with the total order on $I_{sd}$. Then the above factorization is unique. For a component $C = I_{p}$ of $\text{Supp}(w)$ define $w_C = w_{I_p}$, which is called the $C$-component of $w$.

**Example 6.3.** Let $n = 9$ and $u = 4689852102$: we have $u \in R(w)$ for some $w \in Z$. We have $I_1 = \{0, 1, 2\}$, $I_2 = \{4, 5, 6\}$, and $I_3 = \{8, 9\}$, and $w_{I_1} = s_2s_1s_0s_2$, $w_{I_2} = s_4s_6s_5$, and $w_{I_3} = s_8s_9$.  

6.1. Bruhat covers in $Z$. To prove Proposition 6.1, we study in detail the Bruhat order of $\hat{C}_n$ when restricted to the subset $Z$. The results in these subsections may be of independent combinatorial interest.

We construct the set of covers $\mathcal{C}_v$ in $Z$, of a fixed element $v \in Z$. For $k', k \in I_{af}$ with $k' < k$ let

$$N_{k,k'} = k(k+1) \cdots (n-1)n(n-1) \cdots (k' - 1)k'$$

$$\text{N}\text{E} k,k' = k'(k'-1) \cdots (n-1)n(n-1) \cdots (k+1)k.$$

For $w \in Z$, we define

$$R^N(w) = \{u \in R(w) \mid u \subset N_{k,k-1} \text{ for some } 1 \leq k \leq n\}$$

$$\text{R}\text{O}^N(w) = \{u \in R(w) \mid u \subset \text{N}_{k-1,k} \text{ for some } 1 \leq k \leq n\}$$

where $u \subset u'$ denotes a specific embedding of a word $u$ as a subword of a word $u'$. Then by definition $w \in Z$ if and only if $R^N(w) \cup \text{R}\text{O}^N(w) \neq \emptyset$.

Therefore $w \in \mathcal{C}_v$ if and only if either (1) there is a word $u \in R^N(v)$ with an embedding of the form $u \subset N_{k,k-1}$ and a letter $j \subset N_{k,k-1}$ that is missing from $u$, such that the word $\hat{u}$ obtained by inserting $j$ into $u$, is a reduced word of $w$, or (2) there is a $u \in \text{R}\text{O}^N(v)$ with an embedding of the form $u \subset \text{N}_{k-1,k}$ and a letter $j \in \text{N}_{k-1,k}$ missing from $u$, such that inserting $j$ into $u$ yields $\hat{u} \in R(w)$.

**Lemma 6.4.** Let $v \in Z$ and $u \in R^N(v)$ with $u \subset N_{k,k-1}$ (resp. $u \in \text{R}\text{O}^N(v)$ with $u \subset \text{N}_{k-1,k}$). Let $j \subset N_{k,k-1}$ (resp. $j \subset \text{N}_{k-1,k}$) be a letter that is not in $u$. Then adding this copy of $j$ to $u$, produces a word in $R^Z(w)$ for some $w \in \mathcal{C}_v$, if and only if (1) $j \notin \text{Supp}(u)$ or (2) $j+1 \in \text{Supp}(u)$ for $j \geq k$ or $j-1 \in \text{Supp}(u)$ for $j \leq k-1$.

**Proof.** This follows directly from the Coxeter relations for $\hat{C}_n$. \hfill \Box

We define the reduced words

$$V_{k,k'} = k(k-1) \cdots 101 \cdots (k'-1)k'$$

$$A_{k,k'} = k(k+1) \cdots (n-1)n(n-1) \cdots (k'+1)k'$$

$$I_{k,k'}^1 = k'(k'+1) \cdots (k-1)k$$

$$I_{k,k'}^1 = k(k-1) \cdots (k'+1)k'$$

A word is an N if it is a subword of $N_{k,k-1}$ for some $1 \leq k \leq n$ and a reverse N (abbreviated by the symbol $\text{N}\text{E}$) if it is a subword of $\text{N}_{k-1,k}$ for some $1 \leq k \leq n$. The name N is suggested by the definition: the values in such a word go up, then down, and then up, like the letter N. A word $v$ is a Z if it is an N or a $\text{N}\text{E}$. For
$w \in \hat{C}_n$ let $\mathcal{R}^Z(w)$ be the set of reduced words for $w$ that are Zs. Then by definition, $w \in \hat{C}_n$ is a Z if and only if $\mathcal{R}^Z(w) \neq \emptyset$. Let $\mathcal{R}^N(w)$ (resp. $\mathcal{R}^\overline{N}(w)$) be the subset of reduced words of $w$ that are Ns or (resp. $\overline{N}$s).

A saturated $N$ (resp. $\overline{N}$) is a word of the form $N_{k,k'}$ (resp. $\overline{N}_{k',k}$). An N or $\overline{N}$ is proper if it contains both the letters 0 and $n$. We emphasize the important fact that if $u$ is a proper N, then first$(u) >$ last$(u)$, where first$(u)$ and last$(u)$ are the first and last letters of $u$ respectively. Similarly if $u$ is a proper $\overline{N}$ then first$(u) <$ last$(u)$.

Let $u = i_1i_2\cdots i_M$ be a word with letters in $I$. We say that $u$ has a peak at $p$ if $1 < p < M$ and $i_{p-1} < i_p > i_{p+1}$ or if $p = 1$ and $i_1 > i_2$ or if $p = M$ and $p_{M-1} < i_p$ or if $M = 1$. We say that $u$ has a valley at $p$ if $1 < p < M$ and $i_{p-1} > i_p < i_{p+1}$ or if $p = 1$ and $i_1 < i_2$ or if $p = M$ and $p_{M-1} > i_p$ or if $M = 1$.

We say that a word is a $V$ (resp. $\Lambda$) if it is either empty or has exactly one valley (resp. peak). Note that only the empty word has no valleys (resp. peaks). Note that $V$s and $\Lambda$s are both Ns and $\overline{N}$s. Write $\mathcal{R}^V(w)$ and $\mathcal{R}^\Lambda(w)$ for the sets of reduced words of $w$ that are respectively $V$s and $\Lambda$s. A saturated $V$ (resp. $\Lambda$) is one of the form $V^{k,k'}$ (resp. $\Lambda_{k,k'}$).

**Example 6.5.** Let $n = 4$. Then $234101$ is a proper $N$, $20143$ is a proper $\overline{N}$, $312$ is a $V$, and $24321$ is a $\Lambda$.

### 6.2. Equivalences for reduced words and rotation

The following Lemma is essentially a special case of Edelman-Greene insertion [5]. It says that a $\Lambda$ with no $(n-1)n(n-1)$ is equivalent to a $V$. Similarly a $V$ with no $101$ is equivalent to a $\Lambda$.

**Lemma 6.6.** Suppose $i_1i_2\cdots i_nj_1j_2\cdots j_q \in \mathcal{R}(w)$ for some $w \in Z$ such that $i_1 < i_2 < \cdots < i_p < j_1 > j_2 > \cdots > j_q$ and $(n-1)n(n-1)$ is not a subword. Then there is a $k_1k_2\cdots k_q1_l_2\cdots l_p \in \mathcal{R}(w)$ such that $i_1$ occurs in $k_1k_2\cdots k_q$, $k_1 > k_2 > \cdots > k_q < l_1 < l_2 < \cdots < l_p$ and $k_s \leq j_s$ for $1 \leq s \leq q$ and $l_s < i_s$ for $1 \leq s \leq p$.

**Proof.** The result is trivial if $p = 0$ or $q = 0$. Suppose $p = 1$. If both $i_1 + 1$ and $i_1$ occur in $j_1j_2\cdots j_q$ then $i_1 > 0$ and $i_1j_1\cdots j_q \equiv j_1\cdots j_q(i_1 + 1)$ using the braid relations. We take $k_s = j_s$ for $1 \leq s \leq q$ and $l_1 = i_1 + 1$, which satisfies $l_s \geq i_s$. Otherwise let $r$ be maximal such that $i_1 < j_r$. It cannot be the case that $j_r+1 = i_1$ for then $i_1j_1\cdots j_q$ is not reduced. We have $i_1j_1\cdots j_q \equiv (j_1\cdots j_{r-1}i_1j_r+1\cdots j_q)j_r$ and the latter word has the desired form. Note that in the case $p = 1$, $i_1$ occurs in $k_1\cdots k_q$. Finally suppose $p > 1$. By induction $i_2\cdots i_nj_1j_2\cdots j_q \equiv k'_1\cdots k'_q1_l_2\cdots l_p$ with $k'_1 > \cdots > k'_q < l_2 < \cdots < l_p$ with $j_s \leq k'_s$ for $1 \leq s \leq q$ and $l_s > i_s$ for $2 \leq s \leq p$. Since $i_1 < i_2$ and $i_2$ occurs in $k'_1\cdots k'_q$, we may apply the $p = 1$ case and obtain $i_1j'_1\cdots j'_q \equiv k_1\cdots k_q1_l_2\cdots l_p$ with $k_1 > \cdots > k_q < l_1$ and $k_s \leq k'_s$ for $1 \leq s \leq q$. Since $i_2$ was in $k'_1\cdots k'_q$ and $i_1 < i_2$, it follows by considering the $p = 1$ case that $l_1 \leq i_2 < l_2$. It follows that $k_1\cdots k_q1_l_2\cdots l_p$ is the desired reduced word. □

**Lemma 6.7.** Suppose $u$ and $u'$ are two $V$s (resp. $\Lambda$s) such that all letters of $u$ are greater than those in $u'$. Then $uu'$ and $u'uu$ are both equivalent to a $V$ (resp. $\Lambda$).
Proof. Let \( u \) and \( u' \) be Vs with \( u = u_1 m u_2 \) where \( m \) is the valley of \( u \). Then \( u_1 m u' u_2 \) and \( u_1 u' m u_2 \) are Vs that are equivalent to \( uu' \) and \( u'u \) respectively. The proof for \( \Lambda \) is similar. \( \square \)

Given a word \( u \), let \( u^+ \) (resp. \( u^- \)) be the word obtained by adding (resp. subtracting) one from each letter in \( u \).

**Lemma 6.8.** Let \( w \in \mathbb{Z} \) and \( J = \text{Supp}(w) \).

1. Suppose \( n \notin J \). Then \( \mathcal{R}^V(w) \neq \emptyset \). Moreover if \( J \) is an interval then \( \mathcal{R}^V(w) \) is a singleton.
2. Suppose \( 0 \notin J \). Then \( \mathcal{R}^\Lambda(w) \neq \emptyset \). Moreover if \( J \) is an interval then \( \mathcal{R}^\Lambda(w) \) is a singleton.
3. Suppose \( J \) is an interval \([m, M]\) with \( 0 < m \leq M < n \). Let \( u_1 M u_2 \in \mathcal{R}^\Lambda(w) \) and \( u'_2 m u'_1 \in \mathcal{R}^V(w) \). Then \( u'_2 = u_2^\pm \) and \( u'_1 = u_1^\pm \).

**Proof.** We shall prove (1) as (2) is similar. Let \( u \in \mathcal{R}^\Lambda(w) \). Suppose that \( n \notin J \) and that \( u \) is an \( N \); the case of a \( \overline{N} \) is similar. Say \( u \) is embedded in \( N_{k,k-1} \). Then \( u = u_1 u_2 \) where \( u_1 \) is a \( \Lambda \) such that \( \text{Supp}(u_1) \subset [k,n-1] \) and \( u_2 \) is a \( V \) with \( \text{Supp}(u_2) \subset [0,k-1] \). Then \( \mathcal{R}^V(w) \neq \emptyset \) by Lemmata 6.6 and 6.7.

Let \( u \in \mathcal{R}^V(w) \) with \( J \) an interval. We prove its uniqueness by induction on \( \ell(w) \). For \( \ell(w) \leq 3 \) this is evident from (3.2). Let \( M = \max(J) < n \). Suppose first that \( u \) contains a single \( M \). Then \( u \) has the form \( u = Mu \) or \( u = \hat{u} M \). We assume the former as the latter has an analogous proof. We have \( \hat{u} \in \mathcal{R}^V(s_Mw) \) and \( s_M w \in \mathbb{Z} \). By induction \( \hat{u} \) is unique. Now let \( u' \in \mathcal{R}^V(w) \). Since \( \mathcal{R}(w) \) is connected by the braid relations (3.2), every reduced word for \( w \) (and in particular \( u' \)) has a single \( M \) which precedes every \( M-1 \). Since \( u' \) is a \( V \) it must start with \( M \). Therefore \( u' = M \hat{u} = u \) by the uniqueness of \( \hat{u} \).

Otherwise \( u \) must have the form \( u = M \hat{u} M \). Let \( u' \in \mathcal{R}^V(w) \). Clearly \( u' \) must contain an \( M \) which must be at the beginning or end. We suppose \( u' \) has the form \( u' = Mu'' \) as the case \( u' = u'' M \) is similar. By induction \( \mathcal{R}^V(s_M w) \) is a singleton. Therefore \( u'' = \hat{u} M \) and \( u' = u \) as desired.

(3) is proved by induction on the length of \( u_1 M u_2 \). If either \( u_1 \) or \( u_2 \) is empty then the result certainly holds. Write \( u_1 = u_3 x \) and \( u_2 = y u_4 \) where \( x \) and \( y \) are letters. Since \( \text{Supp}(u_1 M u_2) = [m, M] \), \( x = M-1 \) or \( y = M-1 \). Suppose \( x = y = M-1 \). By induction we have \( u_1 M u_2 = u_3 (M-1) M u_4 = u_3 M (M-1) M u_4 = M u_3 (M-1) u_4 M = M u_3 m u_4^\pm M = u_2^\pm m u_1^\pm \). Suppose next that \( x = M-1 > y \). Then again by induction we have \( u_1 M u_2 = u_3 (M-1) M u_2 = u_3 (M-1) u_2 M = u_2^\pm m u_1^\pm M = u_2^\pm m u_1^\pm \). The case \( y = M-1 > x \) is similar. \( \square \)

**Example 6.9.** For \( n > 7 \) the \( N \) is equivalent to a \( V \): \( 676545 \equiv 76545 \equiv 765457 \).

Suppose \( u \subset N_{k,k-1} \) is a subword and \( \ell \subset N_{k,k-1} \) is a subletter (resp. \( u \subset \overline{N}_{k-1,k} \) is a subword and \( \ell \subset \overline{N}_{k-1,k} \) is a subletter) with \( \ell \) missing from \( u \). We give an explicit way to obtain another embedded word \( u' \in \mathcal{R}^V(v) \) such that \( \ell \) is at the beginning or end of the ambient \( N \) or \( \overline{N} \). We call this process rotation. The only cases not treated in Lemma 6.10 are \( \ell = 0 \) or \( \ell = n \), in which case we may use Lemma 6.8 to obtain an equivalent reduced word that is a \( \Lambda \) or \( V \) respectively, and these can be embedded into an \( N \) or \( \overline{N} \) with the missing letter at the beginning or end.
Lemma 6.10. Suppose $w \in \mathcal{Z}$ and $u \in \mathcal{R}^N(w)$ (resp. $u \in \mathcal{R}^\mathcal{N}(w)$) with $u \subset N_{k,k-1}$ (resp. $u \subset \overline{N}_{k,k-1}$).

1. If there is an $\ell$ such that $k \leq \ell < n$ and $\ell$ does not appear in the part of $u$ that is embedded in $I_{0}^{\ell} \subset N_{k,k-1}$ (resp. $I_{0}^{\ell} \subset \overline{N}_{k,k-1}$), then there is a $u' \in \mathcal{R}^N(w)$ (resp. $u' \in \mathcal{R}^\mathcal{N}(w)$) such that $u' \subset N_{\ell+1,\ell}$ (resp. $u' \subset \overline{N}_{\ell+1,\ell}$).

2. If there is an $\ell$ such that $0 < \ell \leq k-1$ and $\ell$ does not appear in the part of $u$ that is embedded in $I_{0}^{k-1} \subset N_{k,k-1}$ (resp. $I_{0}^{k-1} \subset \overline{N}_{k,k-1}$), then there is a $u' \in \mathcal{R}^N(w)$ (resp. $u' \in \mathcal{R}^\mathcal{N}(w)$) such that $u' \subset N_{\ell,\ell-1}$ (resp. $u' \subset \overline{N}_{\ell,\ell-1}$).

3. If there is an $\ell$ such that $0 < \ell < n$ and $\ell$ does not appear in the part of $u$ that is embedded in $I_{0}^{\ell} \subset N_{k,k-1}$ (resp. $I_{0}^{\ell} \subset \overline{N}_{k,k-1}$) then there is a $u' \in \mathcal{R}^N(w)$ (resp. $u' \in \mathcal{R}^\mathcal{N}(w)$) such that $u' \subset \overline{N}_{\ell,n-1}$ (resp. $u' \subset N_{\ell,n-1}$). Moreover $k-1$ or $k$ is missing in the increasing (resp. decreasing) part of $u'$, according as $\ell < k$ or $\ell \geq k$.

Proof. We prove (1) for $u \in \mathcal{R}^N(w)$; the other cases of (1) and (2) are similar. Let $u = u_1u_2u_3u_4$ where $u_1 \subset I_{k}^{\ell+1}$, $u_2 \subset I_{k+1,\ell+1}$, $u_3 \subset I_{k}^{\ell}$, and $u_4 \subset \mathcal{V}^{k-1,\ell-1}$. We have $u \equiv u_2u_1u_3u_4$. $u_1u_3$ is reduced since it is a factor of a reduced word. Since $u_3u_3$ is a $\Lambda$ with no $n$, by Lemma 5.8 it is equivalent to a $V$: $u_3u_3 \equiv u'_3u'_1$ where $u'_3u'_1$ is a $V$ with valley last($u'_3$) such that $u'_3 \subset I_{k}^{\ell}$ and $u'_1 \equiv u'_1 \subset I_{k+1}^{\ell}$. Then $u \equiv u_2u_1u_3u_4 \equiv u'_2u'_1u'_4 \equiv u_2u'_3u'_1u'_4 \subset N_{\ell+1,\ell}$.

We prove (3) for $u \in \mathcal{R}^N(w)$ and $\ell < k$; the cases that $\ell \geq k$ and $u \in \mathcal{R}^\mathcal{N}(w)$, are similar. Let $u = u_1u_2u_3u_4$ where $u_1 \subset I_{k}^{k-1}$, $u_2 \subset I_{k+1,\ell-1}$, $u_3 \subset I_{k}^{\ell-1}$, and $u_4 \subset I_{k}^{\ell-1}$. We have $u \equiv u_1u_3u_2u_4$. $u_2u_4$ is a reduced word supported on $[\ell,k-1]$ that is a $V$. By Lemma 5.8 there is an equivalent $\Lambda$: $u_2u_4 \equiv u'_2u'_4$ where $u'_2u'_4$ is a $\Lambda$ with peak first($u'_2$) such that $u'_2 \subset I_{k}^{k-2}$ and $u'_4 \subset I_{k}^{k-1}$. Then $u \equiv u_1u_3u_2u_4 \equiv u_1u_3u'_2u'_4 \equiv u_3u'_2u_1u'_4 \subset \overline{N}_{\ell-1,\ell}$.

Example 6.11. Let $n = 6$. We start with a reduced word for an element of $w \in \mathcal{Z}$ and apply rotations, choosing $\ell$ to be the first break from the left, indicated by the symbol $\bullet$, in the given reduced word.
Rotating the last word yields the first one. There are two other words in $\mathcal{R}^Z(w)$, which are obtained from the first and third words above, by commutations.

6.3. Normal words. The set $\mathcal{Z}$ has a partition into three subsets: the elements $w$ with $\mathcal{R}^N(w) = \emptyset$, those with $\mathcal{R}^N(w) = \emptyset$, and those with both $\mathcal{R}^N(w) \neq \emptyset$ and $\mathcal{R}^N(w) \neq \emptyset$. We give a criterion for membership in these subsets.

Lemma 6.12. Let $w \in \mathcal{Z}$.

(1) $\mathcal{R}^N(w) = \emptyset$ (resp. $\mathcal{R}^N(w) = \emptyset$) if and only if some word in $\mathcal{R}^N(w)$ (resp. $\mathcal{R}^N(w)$) contains $I^n_0$ (resp. $I^n_0$) as a factor, if and only if every word in $\mathcal{R}^N(w)$ (resp. $\mathcal{R}^N(w)$) does.

(2) There is a $u \in \mathcal{R}^N(w)$ that does not contain $I^n_0$ as a factor, if and only if there is a $u' \in \mathcal{R}^N(w)$ that does not contain $I^n_0$ as a factor.

Proof. (2) follows from Lemma 6.10(3).

For (1) we observe that the property of having $I^n_k$ as a subword, is invariant under the braid relations, which connect $\mathcal{R}(w)$.

Suppose $\mathcal{R}^N(w)$ contains a word with factor $I^n_0$. In particular it contains $I^n_0$ as a subword. Therefore the same is true for all of $\mathcal{R}(w)$. Now every $N$ that contains $I^n_0$ as a subword must contain it as a factor. This proves the second equivalence in (1). Moreover no $\mathcal{N}$ contains $I^n_0$ as a subword, so $\mathcal{R}^N(w) = \emptyset$. Conversely, suppose $\mathcal{R}^N(w) = \emptyset$. Then $\emptyset \neq \mathcal{R}^Z(w) = \mathcal{R}^N(w)$. Let $u \in \mathcal{R}^N(w)$. Then $u$ must contain $I^n_0$ as a factor, for otherwise (2) yields a contradiction.

Let $w \in \mathcal{Z}$ satisfy $\text{Supp}(w) = I_n^d$. A normal word for $w \in \mathcal{Z}$ is an element $u \in \mathcal{R}^Z(w)$ such that:

(1) If $\mathcal{R}^N(w) \neq \emptyset$ then $u$ has the form $u = I^n_k \cdots$.

(2) If $\mathcal{R}^N(w) = \emptyset$, then $u \in \mathcal{R}^N(w)$ has the form $u = I^n_k \cdots$.

Lemma 6.13. Let $v \in \mathcal{Z}$ be such that $\text{Supp}(v) = I_n^d$. Then $v$ has a unique normal word, denoted $v_{\text{nor}}$.

Proof. Existence holds by Lemma 6.10. Suppose that $\mathcal{R}^N(v) \neq \emptyset$. The case $\mathcal{R}^N(v) = \emptyset$ is analogous. Let $u = I^n_k u_1$ and $u' = I^n_{k'} u_1'$ be normal words for $v$.

Suppose first that $k' < k$. We cannot have $k = n$, because the form of $u$ implies that $s_n v < v$ while that of $u'$ implies $s_n v > v$. So $k < n$. We have $v' = s_k v < v$. By the Exchange Property there is a letter in $u'$ whose removal gives a reduced word $u''$ for $v'$. Since $k I^n_{k'}$ is a reduced word, the removed letter does not occur in $I^n_{k'}$. In particular $k \in \text{Supp}(v')$. But $\text{Supp}(I^n_{k+1}) \supset (I_n^d \setminus \{k\})$, so $\text{Supp}(v') = \text{Supp}(v) = I_n^d$. By induction on length, $u'' = I^n_{k+1} u_1$, which is a contradiction. Similarly $k < k'$ leads to a contradiction. Therefore $k = k'$. But
then $u_1$ and $u'_1$ are reduced words and $Vv$ for the same element of $Z$, so by Lemma
\[ u_1 = u'_1 \] and therefore $u = u'$.
\[ \square \]

### 6.4 Special words

In this section we assume that $v \in Z$ is such that $\text{Supp}(v) = I_{af}$. By Lemma 6.13 $v$ has a unique normal word $v_{nor}$. We say that an embedded subword $u \subset N_{k,k-1}$ (resp. $u \subset \overline{N}_{k,k-1}$) is normally embedded if $u = v_{nor}$ for some $v \in Z$ with $\text{Supp}(v) = I_{af}$ and $u = I_{k}^{n} \cdots N_{k,k-1}$ (resp. $u = I_{k}^{n} \cdots \overline{N}_{k,k-1}$).

Suppose $u \subset u'$ and $u \neq u'$. Define firstgap($u \subset u'$) (resp. lastgap($u \subset u'$)) to be the first (resp. last) letter $j \subset u'$ that is not in $u$.

We say that $u \in R^Z(v)$ is special if it has a special embedding, that is, an embedding of the form $u \subset u'$ where $u' = N_{a,a-1}$ or $u' = \overline{N}_{a-1,a}$ for some $1 \leq a \leq n$ such that, if $j = \text{firstgap}(u \subset u')$, then adding $j$ to $u$ produces the normally embedded word $w_{nor} \subset u'$ for some $w \in C_v$. More specifically, one of the following holds:

\begin{enumerate}
\item $u \in R^N(v)$ and $u \subset N_{a,a-1}$ for some $1 \leq a \leq n$ such that $u$ contains all but exactly one of the letters in $I_{a}^{n} \subset N_{a,a-1}$, or
\item $u \in R^{N}(v)$ with $u \subset \overline{N}_{a-1,a}$ for some $1 \leq a \leq n$ and $u$ contains all but exactly one of the letters in $I_{a}^{n-1} \subset \overline{N}_{a-1,a}$.
\end{enumerate}

**Lemma 6.14.** Let $v \in Z$ with $\text{Supp}(v) = I_{af}$ and $\ell(v) < 2n$. Then $v$ has a unique specially embedded word, denoted $v_{sp} \subset v''$, which is obtained by rotating the normal embedding $v_{nor} \subset u'$ at $p = \text{lastgap}(v_{nor} \subset u')$. This given, we define the special cover $v^* \subset C_v$ of $v$, to be the unique cover $w \subset C_v$ such that the normally embedded word $w_{nor}$ is obtained from the specially embedded word $v_{sp} \subset u''$ by inserting firstgap($v_{sp} \subset u''$). Moreover, if $\ell = \text{firstgap}(v_{nor} \subset u')$ and $\ell(v) < 2n - 1$ then $\ell = \text{firstgap}(v_{nor} \subset u'')$, except when $u' = N_{k,k-1}$ and lastgap($v_{nor} \subset u') < \ell \leq k - 1$, in which case firstgap($v_{nor} \subset u''$) = $\ell - 1$.

**Proof.** Suppose $v_{nor} \subset N_{k,k-1}$. Let $\ell = \text{firstgap}(v_{nor} \subset N_{k,k-1})$.

Suppose $p \subset I_{1}^{k-1} \subset N_{k,k-1}$ is missing from $v_{nor}$. Let $v_{nor} = u_{1}u_{2}u_{3}u_{4} \subset N_{k,k-1}$ where $u_{1} \subset \Lambda_{k,k}$, $u_{2} \subset I_{p+1}^{k-1}$, $u_{3} \subset V_{p-1}^{p-1}$, and $u_{4} \subset I_{p+1}^{k-1}$. Then using Lemma 6.13 we have $v_{nor} \equiv u_{1}u_{2}u_{3}u_{4} \equiv u_{1}u_{2}(k-1)u_{3}u_{4} \equiv u_{1}(k-1)u_{2}u_{3}u_{4} =: u \subset N_{p,p-1}$. Now $u_{2} \subset I_{p+1}^{k-2}$. In this case, $u$ is special if and only if $u_{2} = I_{p+1}^{k-1}$, that is, $p = \text{lastgap}(v_{nor} \subset N_{k,k-1})$. Suppose so then firstgap($v_{nor} \subset N_{p,p-1}$) is $\ell$ unless $\ell \subset I_{p+1}^{k-1} \subset N_{k,k-1}$, in which case the answer is $\ell - 1$.

Suppose $p \subset I_{1}^{k-1} \subset N_{k,k-1}$ is missing from $v_{nor}$. Let $v_{nor} = u_{1}u_{2}u_{3}u_{4} \subset N_{k,k-1}$ where $u_{1} \subset \Lambda_{k,k}$, $u_{2} \subset I_{p+1}^{k-1}$, $u_{3} \subset V_{p-1}^{p-1}$, and $u_{4} \subset I_{p+1}^{k-1}$. Then $v_{nor} \equiv u_{1}u_{2}u_{3}u_{4} \equiv u_{1}(k-1)u_{2}u_{3}u_{4} =: u \subset N_{p,p-1}$. In this case $u$ is special if and only if $u_{3} = V_{p-1}^{p-1}$ and $u_{4} = I_{p+1}^{k-2}$, that is, $p = \text{lastgap}(v_{nor} \subset N_{k,k-1})$.

Suppose so. Then firstgap($v_{nor} \subset N_{p,p-1}$) is $\ell$ unless $\ell \subset I_{1}^{k-1} \subset N_{k,k-1}$ and $p = \text{lastgap}(v_{nor} \subset N_{k,k-1})$ in which case the answer is $\ell - 1$.

Suppose $p \subset I_{1}^{k-1} \subset N_{k,k-1}$ is missing from $v_{nor}$. Let $v_{nor} = I_{p+1}^{p}u_{1}u_{2}u_{3} \subset N_{k,k-1}$ where $u_{1} \subset \Lambda_{p+1,p+1}$, $u_{2} \subset I_{p+1}^{p}$, and $u_{3} \subset V_{k-1,k-1}$. We have $v_{nor} \equiv u_{1}u_{2}u_{3}u_{4} \equiv u_{1}u_{2}^{l}u_{3}u_{4} \equiv u_{1}^{l}u_{2}^{k-1}u_{3}u_{4} \equiv u_{1}^{l+1}u_{2}^{k-1}u_{3}u_{4} =: u \subset N_{p,p+1}$. In this case $u$ is special if and only if $u_{3} = V_{p-1}^{p-1}$ and $u_{4} = I_{p+1}^{k-2}$, that is, $p = \text{lastgap}(v_{nor} \subset N_{k,k-1})$.

Suppose so. Then $\ell \subset I_{p+1}^{p} \subset N_{k,k-1}$, and $\ell = \text{firstgap}(v_{nor} \subset N_{p,p+1})$. 


Suppose \( v_{\text{nor}} \subset \overline{N}_{k-1,k} \). Let \( \ell = \text{firstgap}(v_{\text{nor}} \subset \overline{N}_{k-1,k}) \). Let \( p \subset \overline{N}_{k-1,k} \) be missing for \( v_{\text{nor}} \). Then from the definitions we have \( p \subset I_{1,k}^{n-1} \subset \overline{N}_{k-1,k} \).

Write \( v_{\text{nor}} = I_{1,k}^{n-1} I_{0,k}^{n-1} u_1 u_2 \) where \( u_1 \subset I_{p+1}^{n-1} \) and \( u_2 \subset I_{k-1,k}^{p+1} \). Therefore \( v_{\text{nor}} = u_2 \overline{I}_{1,k}^{n-1} I_{0,k}^{n-1} u_1 =: u \subset N_{p,p+1} \). \( u \) is special if and only if \( u_2 = I_{p+1}^{n-1} \), that is, \( p = \text{lastgap}(v_{\text{nor}} \subset \overline{N}_{k-1,k}) \). Suppose so. Then \( \ell = \text{firstgap}(u_{\text{nor}}^* \subset \overline{N}_{p,p+1}) \).

Thus rotation at lastgap(\( v_{\text{nor}} \subset u' \)) creates a particular specially embedded word which we shall denote by \( v_{sp} \subset u'' \). It remains to show that \( v_{sp} \) is unique. Suppose \( u \in R(v) \) is such that \( u \subset u' \) is a special embedding. Rotating \( u \subset u' \) at firstgap(\( u \subset u' \)), we obtain the normal embedding of \( v_{\text{nor}} \), which is unique. The explicit computation of this rotation shows that it is the inverse of the rotation at the last gap of the normal embedding of \( v_{\text{nor}} \) (which was given above explicitly in all cases). It follows that there is a unique specially embedded word for \( v \). \( \square \)

For later use we summarize the construction of Lemma 6.14 in the following table, where \( p \) is the last gap. We have indicated the form of \( v_{sp} \), and used the symbol * to indicate where a letter (either \( k \) or \( k-1 \)) can be added to obtain \( v_{\text{nor}}^* \).

| \( p \subset \) | \( v_{sp} \) | \( u_1 \subset \) | \( u_2 \subset \) | \( u_3 \subset \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( I_{1,k}^{n-1} \subset \overline{N}_{k-1,k} \) | \( I_{1,k}^{n-1} \ast u_2(k-1)u_3 \) | \( \Lambda_{k,k} \) | \( I_{p+1}^{k-1} \ast u_2 \) | \( V_{p-1,p-1} \) |
| \( I_{1,k}^{n-1} \subset \overline{N}_{k-1,k} \) | \( I_{1,k}^{n-1} I_{0,k}^{n-1} \ast u_2(k-1)u_2^* \) | \( \Lambda_{k,k} \) | \( I_{p+1}^{k-1} \) | \( I_{p+1}^{k-1} \) |
| \( I_{1,k}^{n-1} \subset \overline{N}_{k-1,k} \) | \( I_{1,k}^{n-1} I_{0,k}^{n-1} \ast u_2(k-1)u_2^* \) | \( \Lambda_{p+1,p+1} \) | \( I_{p+1}^{k-1} \) | \( I_{p+1}^{k-1} \) |
| \( I_{1,k}^{n-1} \subset \overline{N}_{k-1,k} \) | \( I_{1,k}^{n-1} I_{0,k}^{n-1} \ast u_2(k-1)u_2^* \) | \( \Lambda_{p+1,p+1} \) | \( I_{p+1}^{k-1} \) | \( I_{p+1}^{k-1} \) |

Example 6.15. Take \( n = 7 \) and \( v_{\text{nor}} = 56754310124 \subset N_{5,4} \). In this case \( p = 3, v_{sp} = 35675431012 \), and \( v_{\text{nor}}^* = 345675431012 \).

6.5. Kinds of covers. Let \( v \in Z \) be fixed. The set \( I_{af} \) is divided into four kinds of letters. Let \( j \) be \( v \)-internal if \( j \in \text{Supp}(v) \). If \( j \notin \text{Supp}(v) \), let \( j \) be \( v \)-isolated, \( v \)-adjoining, and \( v \)-merging if the number of components of \( \text{Supp}(v) \) adjacent to \( j \) is 0, 1, or 2, that is, \( |\{j-1, j+1\} \cap \text{Supp}(v)| \) is 0, 1, or 2.

Let \( w \in C \) with a reduced word \( \tilde{w} \in R^Z(w) \) and a letter \( j \subset \tilde{w} \) whose omission leaves a reduced word \( w \in R^Z(v) \). Then we call the cover \( w \) internal, isolated, adjoining, or merging, according as \( j \) is (with respect to \( v \)). Such \( w \) have \( c(w) \) equal to \( c(v), c(v) + 1, c(v), \) and \( c(v) - 1 \) respectively.

In the case of an internal cover the omitted letter \( j \) may vary if the reduced word \( \tilde{w} \) is changed; however the component \( C \) of \( j \in \text{Supp}(v) \) depends only on \( w \). Moreover \( u_C = v_C \) and \( w_C = v_C' \) for components \( C' \) of \( \text{Supp}(v) \) with \( C' \neq C \).

If \( j \notin \text{Supp}(v) \) then the omitted letter \( j \) is uniquely determined by \( w \).

Lemma 6.16. Let \( v \in Z \) with \( \ell(v) < 2n \).

1. For each \( v \)-isolated letter \( j \in I_{af} \) there is a unique cover \( w \in C_v \) that omits \( j \), namely, \( s_j v \).
2. For each \( v \)-adjoining letter \( j \in I_{af} \) there are exactly two covers \( w \in C_v \) that omit \( j \), namely, \( s_j v \) and \( v s_j \).
3. For each \( v \)-merging letter \( j \in I_{af} \) there are exactly four covers \( w \in C_v \) that omit \( j \). Let \( u \in R(v) \) and \( u_+ \) and \( u_- \) the subwords of \( u \) given by the restriction to the letters greater and less than \( j \) respectively and let \( v_+ \) and \( v_- \) be the corresponding elements of \( Z \). Then the four covers of \( v \) that omit \( j \) are \( s_j v + v_-, v_+ s_j v_- \).
Proof. We prove (3) as the other cases are easier. We observe that $v_+$ and $v_-$ are defined independent of the reduced word $u$. The four given elements of $\mathcal{C}_n$ are all covers of $v$ that omit $j$, and are distinct since $j-1 \in \text{Supp}(v_-)$ and $j+1 \in \text{Supp}(v_+)$. We now realize each of them by reduced words that are $Z$s. By Lemma 6.8 let $u_+ \in \mathcal{R}^A(v_+)$ and $u_- \in \mathcal{R}^V(v_-)$. Then $ju_+u_-, u_+ju_-$, and $u_-u_j$ are all $N$s, and $u_-ju_+$ is a $\overline{N}$, and they are reduced words for the above elements of $\mathcal{C}_n$. It remains to show that if $w \in \mathcal{C}_v$ omits $j$ then $w$ is one of the four given covers. Let $\tilde{u} \in \mathcal{R}^Z(w)$ and $j \in \tilde{u}$ such that the omission of $j$ from $\tilde{u}$ leaves $u \in \mathcal{R}^Z(v)$. Suppose $\tilde{u} \subset N_{k,k-1}$. Suppose $j \geq k$. Let $u = u_1u_2u_3$ where $u_2 \subseteq \Lambda_{j+1,j+1}$, so that $\tilde{u} = u_1ju_2u_3$ or $u_2ju_1u_3$. Here $\text{Supp}(u_1) \subset [0,j-1]$ and $\text{Supp}(u_3) \subset [0,j-1]$ while $\text{Supp}(u_2) \subset [j+1,n]$. If $\tilde{u} = u_1ju_2u_3$ then $\tilde{u} = u_1ju_3u_2$. But not both $u_1$ and $u_3$ can contain $j-1$, for if they did then $u_1$ ends with $j-1$ and $u_3$ starts with $j-1$ and $u \equiv u_1u_3u_2$ is not reduced. If $u_1$ does not contain $j-1$ then $\tilde{u} = ju_1u_3u_2$ and $w = s_jv_-v_+$. If $u_2$ does not contain $j-1$ then $\tilde{u} = u_1u_3u_2$ and $w = v_-s_jv_+$. The cases that $\tilde{u} = u_1u_2u_3$, $j < k$ and $\tilde{u} \subset \overline{N}_{k-1,k}$ are similar. \qed

We now classify the internal covers of $v$. For this purpose we may assume $\text{Supp}(v)$ has a single component. For $k, \ell \leq M$ let $\Lambda_{k,\ell}^M = I_{\ell}^M I_{k}^{M-1}$ and for $m \leq k, \ell$ let $V_{m,\ell} = I_{m}^{k} I_{m+1}^{\ell}$. Lemma 6.17. Suppose $v \in \mathcal{Z}$ is such that $\text{Supp}(v)$ consists of a single component $[m, M]$.

1. If $M < n$ (resp. $m > 0$) then the internal covers of $v$ are precisely those obtained by inserting missing letters into $u \subset V_{m,\ell}^M$ (resp. $u \subseteq \Lambda_{m,m}$) where $u$ is the unique element of $\mathcal{R}^V(v)$ (resp. $\mathcal{R}^A(v)$).

2. If $m = 0$ and $M = n$, consider the normal embedding $v_{n_{\text{or}}} \subset u'$. Then the internal covers of $v$ are precisely those obtained by inserting missing letters into $v_{n_{\text{or}}} \subset u'$ (normal covers), plus the special cover, which is obtained from the special embedding of $v_{sp}$ by inserting the first missing letter.

Proof. Since internal covers do not change the support and the support is assumed to be an interval, by Lemma 6.3 adding any missing letter of $[m, M]$ creates a cover. Any internal cover $w \in \mathcal{C}_v$ has the same support as $v$. If $M < n$ then $w$ has a reduced word that is a $V$, and removing one of its letters yields a reduced word for $v$ that is a $V$. By uniqueness this word must be $u$. This proves (1) for $M < n$, and $m > 0$ is similar. For (2) suppose $\text{Supp}(v) = I_{\text{af}}$. Let $w \in \mathcal{C}_v$. Consider the normal embedding of $w_{\text{nor}}$, which is unique since $\text{Supp}(w) = I_{\text{af}}$. There is a unique letter in $w_{\text{nor}}$ whose removal yields an embedded reduced word $u$ for $v$. It is easy to check that $u$ is either $v_{\text{nor}}$ normally embedded or $v_{sp}$ specially embedded. \[ \Box \]

6.6. Associated coroots. Let $v \leq v'$ with $v, v' \in \mathcal{C}_n$ and let $u \in \mathcal{R}(v)$ and $u' \in \mathcal{R}(v')$ be such that $u \subseteq u'$. For $j \subseteq u'$, define $\alpha^V(u \subseteq u', j)$ to be $\alpha^V_{uv}$ if adding the given occurrence of $j$ to $u$ creates a reduced word for a cover $w \triangleright v$, and 0 otherwise. In particular the value is 0 if $j \notin u$. Define $\alpha^V(u \subseteq u') = \sum_{j \subseteq u'} \alpha^V(u \subseteq u', j)$. The following Lemma holds by the definitions.

Lemma 6.18. Let $v_1 \leq v'_1, v_2 \leq v'_2, u_1, u'_1, u_2, u'_2$ reduced words for $v_1, v'_1, v_2, v'_2$ such that $u_1 \subseteq u'_1$ and $u_2 \subseteq u'_2$. Then

$$\alpha^V(u_1u_2 \subseteq u'_1u'_2) = v_2^{-1} \alpha^V(u_1 \subseteq u'_1) + \alpha^V(u_2 \subseteq u'_2).$$
It is straightforward to compute sums of associated coroots for subwords of increasing or decreasing reduced words.

**Lemma 6.19.** Let $0 \leq m \leq M \leq n$ and $u \subset I^M_m$ or $u \subset I^M_m$. Then

\[
\begin{align*}
\alpha & (u \subset I^M_m) = \alpha_k + \alpha_{k+1} + \cdots + \alpha_M & \text{if } M < n \\
\alpha & (u \subset I^M_m) = \alpha_k + \alpha_{k+1} + \cdots + \alpha_{n-1} + 2\alpha_n & \text{if } m > 0 \\
\alpha & (u \subset I^M_m) = \alpha_k + \alpha_{k-1} + \cdots + \alpha_m & \text{if } m = 0 \\
\alpha & (u \subset I^M_m) = \alpha_k + \alpha_{k-1} + \cdots + \alpha_1 + 2\alpha_0 & \text{if } m > 0
\end{align*}
\]

where $k = \text{firstgap}(u \subset u')$ for $u' = I^M_m$ or $u' = I^M_m$. If $k$ does not exist (that is, $u = u'$) then the sum is 0.

Next we compute sums of associated coroots for subwords of $V_s$ and $\Lambda_s$ whose support are intervals. We assume there is a letter missing in the initial monotonic part of the embedded word; otherwise the result is given by Lemma 6.19.

**Lemma 6.20.** Let $v \in Z$ have $\text{Supp}(v) = [m, M] \subseteq I_M$. Suppose $u \in \mathcal{R}(v)$ with $M < n$ (resp. $u \in \mathcal{R}(v)$ with $M > n$) of the form $u = u_1u_2$ with $u_1 \subset I^M_m$ (resp. $u_1 \subset I^M_m$) and $u_2 \subset I^{M+1}_m$ (resp. $u_2 \subset I^{M-1}_m$) so that $u \subset V^M_m,M$. Suppose that $u_1 \neq u'$ for $u' = I^M_m$ (resp. $u' = I^M_m$) so that $k = \text{firstgap}(u_1 \subset u')$ is well-defined. Let $k' = \text{firstgap}(u_2 \subset u'')$ where $u'' = I^{M+1}_m$ (resp. $u'' = I^{M-1}_m$); if $u_2 = u''$ then set $k' = M + 1$ (resp. $k' = m - 1$). Then

\[
\begin{align*}
\alpha & (u \subset V^M_m,M) = (\alpha_m + \cdots + \alpha_{k-1}) + (\alpha_k + \cdots + \alpha_M) & \text{if } m > 0 \\
\alpha & (u \subset V^M_m,M) = 2(\alpha_0 + \cdots + \alpha_{k-1}) + (\alpha_k + \cdots + \alpha_M) & \text{if } m = 0 \\
\alpha & (u \subset \Lambda^m_m) = (\alpha_m + \cdots + \alpha_{k'}) + (\alpha_{k+1} + \cdots + \alpha_M) & \text{if } M < n \\
\alpha & (u \subset \Lambda^m_m) = (\alpha_m + \cdots + \alpha_{k'}) + 2(\alpha_{k+1} + \cdots + \alpha_n) & \text{if } M = n
\end{align*}
\]

**Proof.** Since $\text{Supp}(v)$ is an interval and we are adding letters in that same interval, adding any missing letter creates a reduced word by Lemma 6.19. Let $v_2 \in Z$ be such that $u_2 \in \mathcal{R}(v_2)$.

Let $M < n$ and $u \subset V^M_m,M$. Suppose $m > 0$. We have

\[
\alpha (u_1 \subset I^M_m) = \alpha_m + \cdots + \alpha_{k-1} + \alpha_k.
\]

By the assumption on support, since $k \notin \text{Supp}(u_1)$ we have $k \in \text{Supp}(u_2)$ and $k \neq k'$. Therefore

\[
v_2^{-1}\alpha (u_1 \subset I^M_m) = \alpha_m + \cdots + \alpha_{k-1}.
\]

By Lemma 6.18 the desired expression is obtained.

Suppose $m = 0$. Since $0 \in \text{Supp}(w)$ we have

\[
\alpha (u_1 \subset I^M_0) = \alpha_k + \alpha_{k-1} + \cdots + \alpha_1 + 2\alpha_0
\]

\[
v_2^{-1}\alpha (u_1 \subset I^M_0) = \begin{cases} 2(\alpha_0 + \cdots + \alpha_{k-1}) + (\alpha_k + \cdots + \alpha_{k'-1}) & \text{if } k > k' \\
2(\alpha_0 + \cdots + \alpha_{k-1}) + (\alpha_k + \cdots + \alpha_{k'-1}) & \text{if } k < k'.
\end{cases}
\]

By Lemma 6.18 we obtain the desired formula.

The other computations are similar.
Lemma 6.21. Suppose \( v \in Z \) is such that \( \text{Supp}(v) = I_{af} \) with normal embedding \( v_{nor} \subset N_{k,k-1} \), \( v_{nor} \) does not contain \( I_{1,0}^k \subset N_{k,k-1} \), and \( \ell = \text{firstgap}(v_{nor} \subset N_{k,k-1}) \). Then

\[
\alpha^v (u \subset N_{k,k-1}) = \begin{cases} 
2(\alpha_0^v + \cdots + \alpha_{k-1}^v) + (\alpha_k^v + \cdots + \alpha_0^v) & \text{if } \ell \geq k \\
2(\alpha_0^v + \cdots + \alpha_{k-1}^v) + (\alpha_k^v + \cdots + \alpha_{k-1}^v) & \text{if } \ell < k.
\end{cases}
\]

Proof. Follows from Lemmata 6.18, 6.19 and 6.20 \( \Box \)

Lemma 6.22. Suppose \( v \in Z \) is such that \( \ell(v) < 2n \) and \( \text{Supp}(v) = [m, M] \) is an interval. Then the sum of \( \alpha_{uv}^v \) as \( w \) runs over the internal covers in \( C_v \), is given by

\[
2(\alpha_0^v + \cdots + \alpha_{m+1}^v + \cdots + \alpha_M^v) - \chi(M < n)(-\alpha_M^v + v^{-1} \alpha_{M-1}^v) - \chi(m > 0)(-\alpha_{m-1}^v + v^{-1} \alpha_{m-2}^v).
\]

Proof. We begin with the most involved case, when \( \text{Supp}(v) = I_{af} \). In this case we must show that \( \sum_{w \in C_v} \alpha_{uv}^w = 2(\alpha_0^v + \cdots + \alpha_n^v) = 2K \). By Lemma 6.17 \( \sum_{w \in C_v} \alpha_{uv}^w = \alpha^v (v_{nor} \subset u^v) + \alpha_{uv}^v \) where \( v_{nor} \subset u^v \) is the normal embedding. For the computation of the special coroot \( \alpha_{uv}^v \), we shall refer back to the proof of Lemma 6.14 without further mention, for the explicit computations of the special embedding \( v_{sp} \subset u^v \) given by rotating the normal embedding \( v_{nor} \subset u^v \) at \( p = \text{lastgap}(v_{nor} \subset u^v) \). The reader may find the table after Lemma 6.14 helpful.

Suppose that \( v_{nor} \subset N_{k,k-1} \) is the normal embedding for some \( 1 \leq k \leq n \). Let \( \ell = \text{firstgap}(v_{nor} \subset N_{k,k-1}) \) and \( p = \text{lastgap}(v_{nor} \subset N_{k,k-1}) \).

Suppose \( \ell \in I_{1,1}^{k-1} \subset N_{k,k-1} \). We have \( p \in I_{1,1}^{k-1} \subset N_{k,k-1} \), \( v_{nor} \subset N_{p,p-1} \) is normally embedded, and \( \ell = \text{firstgap}(v_{nor} \subset N_{p,p-1}) \). We compute \( \alpha_{uv}^v = s_{\ell-1} \cdots s_{1} s_0 s_1 \cdots s_{n-1} s_n \cdots s_k (\alpha_{k-1}^v) = 2(\alpha_0^v + \cdots + \alpha_n^v) - (\alpha_0^v + \cdots + \alpha_k^v) \). Combining this with \( \alpha^v (v_{nor} \subset N_{k,k-1}) = \alpha_0^v + \cdots + \alpha_k^v \), we obtain the total \( 2K \).

Suppose \( \ell \in I_{1,1}^{k-1} \subset N_{k,k-1} \). Since \( \text{Supp}(v) = I_{af} \), \( \ell \in I_{1,1}^{k-1} \subset N_{k,k-1} \) appears in \( v_{nor} \). Suppose \( p \in I_{1,1}^{k-1} \subset N_{k,k-1} \). Suppose first that \( p > \ell \). Then \( v_{nor} \subset N_{p,p-1} \) is normally embedded with firstgap \( v_{nor} \subset N_{p,p-1} = \ell \). We have \( \alpha_{uv}^v = s_{\ell-1} \cdots s_{\ell-1} \cdots s_{n-1} s_n \cdots s_k (\alpha_{k-1}^v) = 2(\alpha_0^v + \cdots + \alpha_n^v) - (\alpha_0^v + \cdots + \alpha_k^v) \). By Lemma 6.21 for \( \ell < k \) we have \( \alpha^v (v_{nor} \subset N_{k,k-1}) = 2(\alpha_0^v + \cdots + \alpha_{\ell-1}^v) + (\alpha_\ell^v + \cdots + \alpha_k^v) \), and the total is \( 2K \). Suppose next that \( p < \ell \). Again \( v_{nor} \subset N_{p,p-1} \) is normally embedded and firstgap \( v_{nor} \subset N_{p,p-1} = \ell - 1 \). The coroot computation is similar to the previous case. By definition \( p \) occurs after \( \ell \) in \( N_{k,k-1} \) so the remaining subcase is \( p \in I_{1,1}^{k-1} \subset N_{k,k-1} \). Then \( v_{nor} \subset \overline{N}_{p-1,p} \) is normally embedded and firstgap \( v_{nor} \subset \overline{N}_{p-1,p} = \ell - 1 \subset I_{1,1}^{p-1} \). The coroot computation is similar.

Suppose \( \ell \in I_{1,1}^{k-1} \subset N_{k,k-1} \). Since \( \text{Supp}(v) = I_{af} \), \( k - 1 \) must occur in \( v_{sp} \) after \( \ell \). In all cases firstgap \( v_{nor} \subset u^v \subset u^v \) = \( \ell \). If \( p \subset I_{1,1}^{k-1} \subset N_{k,k-1} \), then \( v_{nor} \subset N_{p,p-1} \) is normally embedded with \( \alpha_{uv}^v = s_{k-1} s_\ell \cdots s_{n-1} s_n \cdots s_k (\alpha_{k-1}^v) = \alpha_{k-1}^v(2(\alpha_\ell^v + \cdots + \alpha_n^v) + (\alpha_{k-1}^v + \cdots + \alpha\ell_k^v)) = 2(\alpha_{\ell+1}^v + \cdots + \alpha_n^v) + (\alpha_k^v + \cdots + \alpha_{k-1}^v) \). Combined with \( \alpha^v (v_{nor} \subset N_{k,k-1}) = 2(\alpha_0^v + \cdots + \alpha_k^v) + (\alpha_k^v + \cdots + \alpha_{k-1}^v) \) from Lemma 6.21 we obtain a total of \( 2K \). If \( p \subset I_{1,1}^{k-1} \subset N_{k,k-1} \), then \( v_{nor} \subset \overline{N}_{p-1,p} \) is the normal embedding with coroot computation proceeding as in the previous case. If \( p \subset I_{1,1}^{k-1} \subset N_{k,k-1} \), then \( v_{nor} \subset \overline{N}_{p-1,p} \) and the coroot computation proceeds in the same way.
The case \( \nu_{\text{nor}} \subset N_{k-1,k} \) is very similar to the case \( \nu_{\text{nor}} \subset N_{k,k-1} \) with \( \ell \subset I_{\ell+1}^{k-1} \subset N_{k,k-1} \) and \( p \subset I_{\ell+1}^{k-1} \).

This finishes the case \( \text{Supp}(v) = I_{af} \).

Next we consider the case \( m = 0 \) and \( M < n \). Let \( u \in \mathcal{R}(v) \) with \( u \subset V^{M,M} \). In this case the sum of \( \alpha^\vee \) for \( w \in \mathcal{C}_v \) an internal cover of \( v \), is equal to \( \alpha^\vee(u \subset V^{M,M}) \).

Let \( u = u_10u_2 \) where \( u_1 \subset I_1^{M} \) and \( u_2 \subset I_1^{M} \). Let \( a \subset I_1^{M} \) (resp. \( b \subset I_1^{M} \)) be the first missing letter from \( u_1 \) (resp. \( u_2 \)), which exists if \( u_1 \neq I_1^{M} \) (resp. \( u_2 \neq I_1^{M} \)). By Lemma 6.20 we have

\[
\alpha^\vee(u \subset V^{M,M}) = \begin{cases} 
2(\alpha^\vee_0 + \cdots + \alpha^\vee_{n-1}) + (\alpha^\vee_M + \cdots + \alpha^\vee_1) & \text{if } u_1 \neq I_1^{M} \\
\alpha^\vee_M + \cdots + \alpha^\vee_1 & \text{if } u_1 = I_1^{M} \text{ and } u_2 \neq I_1^{M} \\
0 & \text{if } u_1 = I_1^{M} \text{ and } u_2 = I_1^{M}.
\end{cases}
\]

Consider \( \beta = -\alpha^\vee_{M+1} + v^{-1}\alpha^\vee_M \). Suppose first that \( u_1 \neq I_1^{M} \). Since \( a \) is missing from \( u_1 \) and \( \text{Supp}(v) = [0,M] \) is an interval, \( a \in u_2 \). Therefore \( \beta = -\alpha^\vee_{M+1} + s_a s_{a+1} \cdots s_M \alpha^\vee_{M+1} = \alpha^\vee_0 + \cdots + \alpha^\vee_M \), which yields the desired total. Suppose \( u_1 = I_1^{M} \) and \( u_2 \neq I_1^{M} \). Then \( \beta = -\alpha^\vee_{M+1} + s_{b-1} \cdots s_0 s_1 \cdots s_M \alpha^\vee_{M+1} = 2(\alpha^\vee_0 + \cdots + \alpha^\vee_M) \) as desired. If \( u_1 = I_1^{M} \) and \( u_2 = I_1^{M} \) then \( \beta = -\alpha^\vee_{M+1} + s_{M} \cdots s_1 s_0 s_1 \cdots s_M \alpha^\vee_{M+1} = 2(\alpha^\vee_0 + \cdots + \alpha^\vee_M) \) as desired.

The case that \( m > 0 \) and \( M = n \) is entirely similar to the previous case. The remaining case is \( 0 < m < M < n \). Using \( u \in \mathcal{R}(v) \) and \( v_1mu_2 = u \subset V^{M,M}_m \), the proof is similar to the case for \( m = 0 \) and \( M = n \) except that one must also compute \( -\alpha^\vee_{m-1} + v^{-1}\alpha^\vee_{m-1} \), which equals \( \alpha^\vee_0 + \cdots + \alpha^\vee_M \) if \( u_2 \neq I_{m+1} \) and equals \( \alpha^\vee_0 + \cdots + \alpha^\vee_M \) if \( u_2 = I_{m+1} \).

\( \square \)

6.7. Proof of Proposition 6.1 We fix \( v \in \mathcal{Z} \) with \( \ell(v) < 2n \) and \( i \in I_{af} \). Let

\[
\mathcal{C}'_v = \{ w \in \mathcal{C}_v \mid \alpha^\vee_i \text{ occurs in } \alpha^\vee_w \}.
\]

**Case 1.** \( i \notin \text{Supp}(v) \). Let \( w \in \mathcal{C}'_v \). Since \( \alpha^\vee_i \) occurs in \( \alpha^\vee_w \) and \( i \notin \text{Supp}(v) \) it follows that \( i \in \text{Supp}(w) \). It is easy to check that \( \alpha^\vee_i \) occurs in \( \alpha^\vee_w \) with coefficient 1. The desired multiplicity is obtained by Lemma 6.16.

**Case 2.** \( i \in \text{Supp}(v) \). Let \( C = [m,M] \) be the component of \( i \) in \( \text{Supp}(v) \). The covers in \( \mathcal{C}'_v \) add letters that are either in \( C \) or adjacent to \( C \).

**Case 2a.** \( C = I_{af} \). In this case there are only internal covers. Therefore \( \sum_{w \in \mathcal{C}_v} \alpha^\vee_{w} = 2(\alpha^\vee_0 + \cdots + \alpha^\vee_n) \) by Lemma 6.22. Since \( c(w) = c(v) \) for all such \( w \), Proposition 6.1 is verified in this case.

**Case 2b.** \( C = [0,M] \) with \( M < n \). (The case \( C = [m,n] \) with \( m > 0 \) is similar.) Write \( v = v_C v' \) where \( v' \) is the product of the components of \( v \) other than \( v_C \). Then the internal covers in \( C'_v \) consist of the \( w \in \mathcal{C}_v \) such that \( w_C \gg v_C \) and \( w'_C = v'_C \) for components \( C' \) of \( \text{Supp}(v) \) with \( C' \neq C \). The sum of \( \alpha^\vee_{w} \) for internal covers of \( v \) in \( C'_v \), is given by Lemma 6.22. For such \( w \) we have \( c(w) = c(v) \). Suppose \( M + 2 \notin \text{Supp}(v) \), so that \( M + 1 \) is \( v \)-adjoining. Then all the noninternal covers \( w \in \mathcal{C}_v \) adjoin the letter \( M + 1 \) to \( C \); such \( w \) satisfy \( c(w) = c(v) \) also. By Lemma 6.10 there are exactly two adjoining covers in \( \mathcal{C}_v \), namely, \( s_{M+1}v = v's_{M+1}v_C \) and \( v_{s_{M+1}} \). The latter has associated coroot \( \alpha^\vee_{M+1} \) and therefore does not contribute \( \alpha^\vee_i \) for \( i \in C \). For \( w = s_{M+1}v \) we have \( \alpha^\vee_{w} = v_C \alpha^\vee_{M+1} \). Combining this with the sum of coroots for internal covers associated to the component \( C \) of \( \text{Supp}(v) \), by Lemma 6.22 the coefficient of \( \alpha^\vee_i \) is 2 as desired. Suppose \( M + 2 \in \text{Supp}(v) \). Then
$M + 1$ is $v$-merging. By Lemma 6.16 there are four covers $w \in C_v$ that add $M + 1$; each has $c(w) = c(v) - 1$. Their associated coroots are

\[
\begin{align*}
\alpha^\vee_{v, v; M+1} &= \alpha^\vee_{M+1} \\
\alpha^\vee_{v, v'; M+1} &= v_{C-1}^{-1} \alpha_{M+1} \\
\alpha^\vee_{v, s_{M+1}} &= -\alpha_{M+1} + v_{C-1}^{-1} \alpha_{M+1} + (v')^{-1} \alpha_{M+1} \\
\alpha^\vee_{v, v; M+1} &= (v')^{-1} \alpha_{M+1}.
\end{align*}
\]

The sum of these coroots, forgetting the $\alpha^\vee_j$ for $j \not\in C$, is $2(-\alpha^\vee_{M+1} + v_{C-1}^{-1} \alpha_{M+1})$. Together with the coroots corresponding to internal covers given by Lemma 6.22 which receive a relative factor of 2 since $c(w) = c(v)$ for internal covers and $c(w) = c(v) - 1$ for merging covers, gives the desired result.

**Case 2c.** $0 < m < M < n$. The computations for this case are similar to those above.

This completes the proof of Proposition 6.1

### 7. Hopf Property of $\Phi$

In this section we prove Theorem 5.5.

#### 7.1. A coproduct formula for nilHecke algebras.

In Proposition 7.1 below, we give a complicated but explicit formula for $\phi^{(2)}_0(\Delta(A_w))$ for $w \in W_{af}$. This formula is valid for the nilHecke algebra for any Cartan datum.

Let $v \in R(w)$ and consider the tuples $v = [v^{(1)}, v^{(2)}, \ldots, v^{(k)}]$ consisting of subwords $v^{(i)} \subset v$ (the embedding of the $v^{(i)}$ are fixed). Let $x_i$ (resp. $y_i$) be the first (resp. last) letters of $v^{(i)}$, considered as subletters of $v$ via the embedding $x_i \subset v^{(i)} \subset v$. Define $S_v$ to be the set of (possibly empty) tuples $v = [v^{(1)}, \ldots, v^{(k)}]$ such that:

1. $v^{(i)}$ is a subword of length at least two of $v \setminus \{x_1, \ldots, x_{i-1}\}$, which is the word $v$ with the letters $x_1, \ldots, x_{i-1}$ removed;
2. $y_i = y_i y_2 \cdots y_k$ is a subword of $v$; and
3. the letters $x_i$ are distinct from the letters $y_j$ as subwords in $v$.

For a word $u = u_1 u_2 \cdots u_t$ and a tuple $v = [v^{(1)}, \ldots, v^{(k)}]$ let

\[
b_u = \prod_{i=1}^{t-1} b_{u_i u_{i+1}} \quad \text{and} \quad b_v = \prod_{i=1}^{k} b_{v^{(i)}},
\]

where $b_{ij} = -\langle \alpha^\vee_i, \alpha_j \rangle = -a_{ij}$ is the negative of the entry of the Cartan matrix.

For a given $v = [v^{(1)}, \ldots, v^{(k)}] \in S_v$ let $x = \{x_1, \ldots, x_k\}$ and $y = \{y_1, \ldots, y_k\}$. Then set $v \setminus (x \cup y)$ to be the word $v$ with the letters in $x$ and $y$ removed. For a subword $u \subset v \setminus (x \cup y)$ (again with a fixed embedding), define $u.y$ to be the word $u$ with the letters in $y$ added in the correct order of $v$.

**Proposition 7.1.** For $w \in W_{af}$ and $v \in R(w)$,

\[
\phi^{(2)}_0(\Delta(A_w)) = \sum_{v = [v^{(1)}, \ldots, v^{(k)}] \in S_v} b_v \sum_{u \in v \setminus (x \cup y)} A_{u,y} \otimes A_{u^+, y}
\]

where $u^+$ is the complement word of $u$ in $v \setminus (x \cup y)$. 

Example 7.2. Take \( v = i j k l \) and \( v = [v^{(1)}] \) with \( v^{(1)} = v \), so that \( v \setminus (x \cup y) = jk \).

Take the subword \( u = j \) of \( v \setminus (x \cup y) \). Then \( u \cdot y = jl \) and \( u \cdot y = kl \), so that the term \( A_{u \cdot y} \otimes A_{u \cdot y} = A_{jl} \otimes A_{kl} \) appears with coefficient \( b_{jl} b_{kl} \) for this particular \( v \) and \( u \) in the sum. Of course such a term can appear in other summands. For example taking \( v = [ikl] \), we also have \( v \setminus (x \cup y) = jk \). Taking again \( u = j \), we get the term \( A_{jl} \otimes A_{kl} \) with a coefficient of \( b_{jk} b_{kl} \).

Proof of Proposition 7.1. The proof proceeds by induction on \( \ell(w) \). For \( \ell(w) = 1 \), let \( v = i \in \mathcal{R}(w) \). We have \( \mathcal{S}_0 = \{[]\} \), so that \( \phi_0'(\Delta(A_w) = \sum_{u \in v} A_u \otimes A_u \). From (4.3) and (3.9) we have, for an element \( A \)

\[
\phi_0'(\Delta A_w) = (\Delta A_w) \Delta A_i
\]

where to obtain the first term in the last equation we have merged the terms obtained from \( A_i \otimes 1 + 1 \otimes A_i \) which correspond to \( i \in u \) and \( i \notin u \) respectively.

From (4.3) and (3.9) we have, for an element \( w \) with reduced word \( z = z_1 z_2 \cdots z_k \)

\[
\phi_0[A_w(-\alpha_i)] = \sum_{u \cdot y \cdot z \in u \cdot y \cdot z_i} (\beta')'(\alpha_i) A_{u \cdot y \cdot z_i}
\]

where in the last equation \( p = p_1 \cdots p_\ell \) is a subword of \( zi \) satisfying: (a) \( \ell \geq 2 \), (b) \( p_\ell = i \), and (c) \( p_1 = z_j \). Applying this equation to the last summand of (4.2) with \( z = (u') \perp y \) we see that it suffices to find a bijection \( \Phi \) from the set of triples

\[
(\beta', \alpha_i, A_{z_i})
\]

such that \( (\alpha') \in \mathcal{S}_1 \), (b) \( u' \subset v' \setminus (x' \cup y') \), and (c) \( p = p_1 \cdots p_\ell \) is a subword of \( (A_{z_i})' \) satisfying \( p_\ell = i \) and \( \ell \geq 2 \), to the set of pairs

\[
(v = [v^{(1)}, \ldots, v^{(r)}], u)
\]
such that (a) \( v \in \mathcal{S}_v \), (b) \( u \subset v \setminus (x \cup y) \), and (c) \( y_r = i \). Furthermore under \( \Phi \) we must have (a) \( b_r = b_v \), (b) \( u.y = u'.y'i \) and (c) \( u^+y \) equal to the word \( (u')^+y'i \) with the letter \( p_1 \) removed.

Given \( (v' = [v'(^1), \ldots, v'(^k)], u', p) \) we consider two cases. If \( p_1 \not\in y' \) we define \( v = [v(^1), \ldots, v(^k), p] \) and \( u = u' \). It is clear then that \( v \in \mathcal{S}_v \) and we have \( x = x' \cup \{p_1\} \) and \( y = y' \cup \{i\} \). Since \( p_1 \in (u')^+ \) we see that \( u \subset v \setminus (x \cup y) \) and that \( u^+ \) is \( (u')^+ \) with \( p_1 \) removed.

Now suppose \( p_1 = y_j \) for some \( 1 \leq j \leq k \). We define the fusion of two words \( z \) and \( iz' \), where \( z, z' \) are words and \( i \) is a letter to be \( z \ast iz' = ziz' \). Here, all words and letters are considered subwords of \( v \). Let \( \hat{v} = v'(^j) \ast p \) be the fusion of \( v'(^j) \) and \( p \), which is defined since the last letter of \( v'(^j) \) and the first letter of \( p \) are the same. Define \( \check{v} = [v'(^1), \ldots, v'(^j), v(^k), \hat{v}] \) where the hat denotes omission. Now \( \check{v} \) satisfies all the conditions of \( \mathcal{S}_v \) except possibly condition (1). We produce \( v \) from \( \check{v} \) by the following shuffling procedure. Suppose \( \hat{v} = \hat{v}_1 \cdots \hat{v}_s \) and let \( t \in (1,s) \) be the maximal index (if it exists) such that \( \hat{v}_t \in x' \), say \( \hat{v}_t = x'_m \), where \( m \in (j, k) \).

We define \( v(k) = \hat{v}_t \hat{v}_{t+1} \cdots \hat{v}_s \) and replace \( v(m) \) with \( \hat{v} = (\hat{v}_1 \cdots \hat{v}_{t-1}) \). Now repeat the procedure with the new \( \hat{v} \), searching for some \( m' \in (j, m) \) such that \( \hat{v} = x'_{m'} \), for \( t' \in (1, t-1) \). When no more shuffling occurs, we label the subwords \( v(^1), \ldots, v(^k) \) in order. Note that the \( y'_t \) for \( r \neq j \) are always kept in order. By construction \( v \in \mathcal{S}_v \) and we have \( x = x' \) and \( y = (y' \setminus \{y_j\}) \cup \{i\} \). We define \( (u = u' \cup \{y_j\}) \subset v \setminus (x \cup y) \) and check that \( u.y = u'.y'i \) and \( u^+ = (u')^+ \). We now make the crucial observation: shuffling is invertible if the letter \( y_j \in v(k) \) is given – we will call this “performing inverse shuffling at \( y_j \)”. This completes the definition of \( \Phi \).

We now define \( \Phi^{-1} \). Given \( (v = [v(^1), \ldots, v(^r)], u) \) we consider again two cases. If \( v(r) \cap u = \emptyset \) we proceed by defining \( (v' = [v'(^1), \ldots, v'(^{r-1})], u' = u, p = v(r)) \). Otherwise, suppose \( v(r) = v_1(r) \cdots v_s(r) \) and let \( t \in (1, s) \) be the maximal index such that \( v_t(r) \in u \). We define \( p = v_1(r) \cdots v_{t-1}(r) \cup v_t(r) \), \( u' = u \setminus \{v_t(r)\} \) and to produce \( v' \) we perform inverse shuffling at \( v_t(r) \). It is straightforward to show that this process well-defines a map that is inverse to \( \Phi \).

\[ \square \]

7.2. Proof of Theorem 5.5 Recall that by Theorem 5.1

\[ P_r = A_{\rho_r} + \text{non-Grassmannian terms} . \]

By Theorems 4.6 and 5.1 in order to prove Theorem 5.5 it suffices to show that

\[ \phi_0^{(2)}(\Delta(A_{\rho_r})) = 1 \otimes A_{\rho_r} + A_{\rho_r} \otimes 1 + 2 \sum_{1 \leq s < r} A_{\rho_s} \otimes A_{\rho_{r-s}} + \text{non-Grassmannian terms} . \]

We have used the fact that if \( w \) is not Grassmannian then any term \( A_x \otimes A_y \) occurring in \( \phi_0^{(2)}(\Delta(A_w)) \) has either \( x \) or \( y \) non-Grassmannian.

We apply Proposition 7.1 to the case \( w = \rho_r \) and \( v \) the unique reduced word of \( \rho_r \), which by (1.1) is given by

\[ v = \begin{cases} (r - 1) \cdots 1 \otimes 0 & \text{ for } 1 \leq r \leq n, \\ \frac{2n + 1 - r}{2} \otimes n - 1 & \text{ for } n < r \leq 2n, \end{cases} \]

where, by convention, if a letter occurs twice in \( \rho_r \) the left occurrence is distinguished by a bar.
The terms $1 \otimes A_{\rho_r} + A_{\rho_r} \otimes 1$ come from $v = [] \in S_v$ and $u = \emptyset$ and $u = v$ in (7.1). All other $u \subset v$ yield non-Grassmannian terms.

Now we calculate the coefficient of the term $A_{\rho_s} \otimes A_{\rho_{r-s}}$ for $s \geq 1$. Since the operation $\phi_0^{(2)} \circ \Delta$ is cocommutative, it suffices to consider the case $s \leq r - s$.

Define $R$ to be the set of letters occurring in $\rho_r \rho_{r-s}^{-1}$ (together with the bars, if any). If $r - s - 1 \in R$ (in particular $r - s - 1 < n$) define $\overline{R}$ to be $R$ with $r - s - 1$ replaced by $r - s - 1$; otherwise set $\overline{R} = R$. If $s - 1 \in R$ define $R_-$ to be $R$ with $s - 1$ removed. If $r - s - 1 \in R_-$ define $\overline{R}_-$ to be $R_-$ with $r - s - 1$ replaced by $r - s - 1$; otherwise set $\overline{R}_- = R_-$. 

**Lemma 7.3.** Suppose $s \leq r - s$. The terms in Proposition 7.1 which give $A_{\rho_s} \otimes A_{\rho_{r-s}}$ are exactly the following tuples $v \in S_v$ and $u \subset v \setminus (x \cup y)$:

Case 1: $r \leq n$ or $r > n$ and $s \leq 2n + 1 - r$:

$v = s - 1 s - 2 \cdots 1 0$, $x = x_1 x_2 \cdots x_s$ is a permutation of the letters in $R$ or $\overline{R}$, and $u = \emptyset$.

Case 2: $r > n$, $s < r - s$ and $s > 2n + 1 - r$:

In addition to the possibilities in Case 1 we may also have $v = s - 2 \cdots 1 0$, $x = x_1 x_2 \cdots x_{s-1}$ is a permutation of the letters in $R_-$ or $\overline{R}_-$, and $u = s - 1$.

Case 3: $r > n$, $s = r - s$ and $s > 2n + 1 - r$:

We have either $y = s - 1 s - 2 \cdots 1 0$, $u = \emptyset$, and $x$ is a permutation of $R_-$; or $y = s - 1 s - 2 \cdots 1 0$, $u = \emptyset$, and $x$ is a permutation of $\overline{R}_-$; or $y = s - 2 \cdots 1 0$, $u = s - 1$, $u^+ = s - 1$, and $x$ is a permutation of $R_- = \overline{R}_-$.

**Proof.** $u, y = \rho_s$ is embedded as a subword of $\rho_r$ in two ways: $s - 1 s - 2 \cdots 1 0$ or $s - 1 s - 2 \cdots 1 0$. Similarly if $r - s > n$ then $u^+, y = \rho_{r-s} \subset \rho_r$ and if $r - s \leq n$ then $u^+, y$ is either $r - s - 1 r - s - 2 \cdots 1 0$ or $r - s - 1 r - s - 2 \cdots 1 0$.

Suppose $0 \leq p \leq s - 2$ is such that $p \not\in y$. Then $p \in u$ and $p \not\in u^+$, so that $p \not\in u^+ y$, a contradiction. Therefore $y \supset s - 2 \cdots 1 0$. This gives four cases.

1. $u = \emptyset$ and $y = s - 1 s - 2 \cdots 1 0$.

Here $x$ can be a permutation of $R$ or also of $\overline{R}$ provided $r - s - 1 \in R$.

2. $u = \emptyset$ and $y = s - 1 s - 2 \cdots 1 0$.

Suppose $r - s > n$. Since $u^+, y$ contains $s - 1$ we have $2n + 1 - (r - s) < s$, giving the contradiction $2n + 1 < r$. Therefore $r - s \leq n$. Again since $u^+, y$ contains $s - 1$ we must have $r - s = s$ and $u^+ = \emptyset$. Then $x$ is a permutation of $\overline{R}$.

3. $u = s - 1$ and $y = s - 2 \cdots 1 0$.

$s - 1 \not\in u^+, y$. This can only occur if $r - s = s$, $u^+, y = s - 1 s - 2 \cdots 1 0$, and $u^+ = s - 1$. Then $x$ is a permutation of $R_-$.

4. $u = s - 1$ and $y = s - 2 \cdots 1 0$.

If $r - s > n$ then $x$ is a permutation of $R_-$. Suppose $s < r - s \leq n$. If $u^+, y = r - s - 1 r - s - 2 \cdots 1 0$ then $x$ is a permutation of $\overline{R}_-$ and if $u^-, y = r - s - 1 r - s - 2 \cdots 1 0$ then $x$ is a permutation of $\overline{R}_-$. If $s = r - s$ then $u^+ = s - 1$ and $x$ is a permutation of $R_-$.

In particular the last three cases only occur if $s - 1 \in \rho_r$, that is, $2n + 1 - r < s$. This given, the Lemma follows. □
By Lemma 7.3 the possibilities for the tuples \( v \in S_u \) which contribute to the coefficient of \( A_{\rho} \otimes A_{\rho_{-x}} \) in \( \phi_0^{(2)}(\Delta(A_{\rho_r})) \) are determined by whether \( u = \emptyset \), \( u = s-1 \) and \( u = \frac{s-1}{2} \). We denote the corresponding subsets of \( S_{\rho_r} \) by \( S_u \). Note that \( S_{s-1} = S_{\frac{s-1}{2}} \). Define \[ T'_u = \sum_{v \in S_u} b_v. \]

**Proposition 7.4.** Depending on the case of Lemma 7.3, we have \( T'_0 = 2 \) or \( T'_0 + T'_{s-1} = 2 \) or \( T'_0 + T'_{s-1} + T'_{\frac{s-1}{2}} = 2 \).

Proposition 7.4 shows that the coefficient of \( A_{\rho_r} \otimes A_{\rho_{-x}} \) in \( \phi_0^{(2)}(\Delta(A_{\rho_r})) \) is equal to 2, thereby proving Theorem 5.5.

The proof of Proposition 7.4 is given in Section 7.9, after first preparing some technical preliminary results in Sections 7.8.

### 7.3. Notation.

For the evaluation of \( T_u' \) we require slightly more general functions.

Let \( \hat{x}\hat{y} \) be an embedded subword of \( \rho_r \) with \( \hat{x} \) and \( \hat{y} \) subletters, and \( I \) a subword. Let \( S^{\hat{y}}_I \) be the set of all subwords \( p \) of \( \rho_r \) with first letter \( \hat{x} \) and last letter \( \hat{y} \) such that \( p \cap I = \emptyset \). Then define \[ T_{\hat{x}\hat{y}} := \sum_{p \in S^{\hat{y}}_I} b_p. \]

Given sequences \( x = x_1x_2\cdots x_k \) and \( y = y_1y_2\cdots y_k \) of subletters of \( \rho_r \) we define \[ T'(x, y) = \prod_{i=1}^k T_{\hat{x}_i\hat{y}_i}^{x_i, y_i} \cap(x_i, y_i) \]
where \( (x_i, y_i) \) denotes the subword of \( \rho_r \) occurring between the letters \( x_i \) and \( y_i \). Finally, let \( T(x, y) \) be obtained from \( T'(x, y) \) by ignoring the extra power of 2 (if any) which arises when \( y_k = 0 \).

Given an interval \([t, q]\) of unbarred letters and a set \( X \) of barred letters, let \( f(t, q, X) \) denote the sum of \( T(x, y) \) as \( x \) varies over all permutations of \( X \) and \( y = q(q-1)\cdots t \). We always assume \( ||t, q|| = |X| \), and write \( k = |X| \). Given \( (t, q, X) \), we partition \( X \) into subsets \( A, B, C \) where

1. \( A \subset X \) consists of letters greater than \( q \),
2. \( B \subset X \) is a subset of \([t, q]\),
3. \( C \subset X \) consists of letters less than \( t \).

Thus \( X \) is the disjoint union of \( A, B, \) and \( C \). In the following we will write \( X - a \) to mean the set \( X - \{a\} \) with the element \( a \in X \) removed. For a set \( S \) of (barred) integers let \( S^- \) denote \( S \) with its maximum element removed.

Now let us suppose that we are given a set \( X \) of barred and unbarred letters, such that \( X \) is the disjoint union of sets \( A, B, C, U, \) and \( U' \), satisfying:

1. \( U' \) is a set of unbarred letters including \( n \),
2. \( A \) consists of some barred letters in \( X \) greater than \( q \),
3. \( B \) consists of the barred letters in \( X \) in the interval \([t, q]\),
4. \( C \) consists of the barred letters in \( X \) less than \( t \),
5. \( U \) consists of some barred letters in \( X \) greater than \( q \),
6. every letter in \( U \) or in \( U' \) is greater than every letter in \( A, B, \) and \( C \),
7. the minimum element of \( U \) is smaller than or equal to the minimum element of \( U' \).
Denote by \( g(t, q, X) \) the sum of \( T(x, y) \) as \( x \) varies over all permutations of \( X \) such that all letters in \( U \) occur to the left of all letters in \( U' \) and \( y = q(q-1) \cdots t \). Let us call \( X \) balanced if \( \min(U') = \min(U) \). Note that if \( X \) is unbalanced it will stay unbalanced if \( \min(U') \) is removed from it.

Let us now suppose that \( y_1 = \bar{q} \) (instead of \( q \)), but the rest of \( y \) is as before. Denote the answers by \( f'(t, q, X) \) and \( g'(t, q, X) \) and use all the same conventions as before.

Results about these various functions are proven in Sections 7.4-7.7. We are ultimately interested in the case that \( X \) is one of \( R, \bar{R}, R_- \), or \( \bar{R}_- \) and \( y \) is one of the words described in Lemma 7.3. In Section 7.8 we explain how to construct the various subsets \( A, B, C, U, \) and \( U' \), before proving Proposition 7.4 in Section 7.9.

7.4. Results for \( T_i^{x\bar{y}} \). In this section we calculate \( T_i^{x\bar{y}} \), which gives the contribution from a single pair of letters \( x \) and \( \bar{y} \).

**Lemma 7.5.** Let \( \bar{x} i \bar{y} \) be an embedded subword of \( \rho_r \), with \( x \) and \( \bar{y} \) subletters, \( I \) a subword, and \( \bar{y} < n \). Then
\[
T_i^{x\bar{y}} = \sum_{p \in S_i^{x\bar{y}}} b_p = -\chi(\bar{x} = \bar{y}) + 2\chi(y = 0) \begin{cases} 1 & \text{if } I = \emptyset \text{ or } \min(I) = \max(\bar{x}, \bar{y}) \\ -1 & \text{if } I = \{n\} \text{ or } \min(I) = \{i, \bar{i}\} \\ 0 & \text{else} \end{cases}
\]
where \( \min(I) \) consists of the 0, 1, or 2 smallest letters of \( I \) and in the comparison \( \min(I) = \max(\bar{x}, \bar{y}) \) we ignore bars.

**Proof.** Let \( p = p_1 p_2 \cdots p_t \in S_i^{x\bar{y}} \) be such that \( b_p \neq 0 \). Then \( p_i - p_{i+1} \in \{\pm 1, 0\} \) for all \( 1 \leq i < t \). Since \( 0 \) never occurs in \( \rho_r \), \( \bar{y} = 0 \) and \( \bar{x} = \bar{y} \) cannot both hold. If \( \bar{y} = 0 \) then since \( b_{10} = 2 \), \( b_p \) contains a factor of two, giving rise to the overall factor \( 2\chi(y = 0) \).

Since \( p \) is a subword of the unique reduced word of \( \rho_r \) and the latter word is a \( \Lambda \) (see subsection 6.1), \( p \) crosses at most once from barred to unbarred letters. Suppose \( ij \) are consecutive letters in \( p \). Then \( j \in \{i-1, i, i+1\} \) and the corresponding values of \( b_{ij} \) are \( 1, -2, 1 \) provided that \( j \neq n \).

The contributions of the paths that only differ by \( \cdots i-1, \cdots i, i, \cdots i-1, \cdots \) and \( \cdots i+1, i, i-1, \cdots \) all cancel out. For \( \bar{y} > \bar{x} \), the contributions of the paths which differ in \( \cdots \bar{y} - 1 \bar{y}, \cdots \bar{y} - 1 \bar{y} \bar{y}, \cdots, \cdots \bar{y} - 1 \bar{y} \bar{y} + 1 \bar{y} \bar{y} \bar{y} \bar{y} \) all cancel out. For \( \bar{x} = \bar{y} \) the paths \( \bar{y} \bar{y} \bar{y} \bar{y} \bar{y} \bar{y} \bar{y} \bar{y} \) leave a net contribution of \( -1 \).

The case \( ij = n - 1n \) is special since \( b_{n-1n} = 2 \). Given the above discussion, it is tedious but not difficult to check all cases claimed in the lemma explicitly. \( \square \)

In the following sections, we will use Lemma 7.5 repeatedly without mention.

7.5. Results for \( f(t, q, X) \): Barred letters only. In this section we derive results for \( f(t, q, X) \), which is defined for a pair of unbarred letters \( t, q \) and a set \( X \) of barred letters. It follows from its definition that \( f(t, q, X) \) only depends on \( B \) and the sizes \( |A| \) and \( |C| \). By definition \( f(q, q, \emptyset) = 0 \).

Let \( \epsilon(n) \) denote the function which is 1 when \( n \) is even and 0 when \( n \) is odd.

**Lemma 7.6.** If \( |B| = 0 \) then \( f(t, q, X) = 1 \).

**Proof.** By Lemma 7.5 the only nonzero contributions to \( f(t, q, X) \) occur when \( I = \{x_1, \ldots, x_{i-1}\} \cap (x_i, y_i) = \emptyset \) for all \( i \). For this to hold for all \( i \in [1, q-t+1] \), the
letters in \( x = (x_1, x_2, \ldots, x_{q-t+1}) \) must occur in the same order as in \( \rho_r \). This term gives contribution 1 to \( f(t, q, X) \).

**Lemma 7.7.** Suppose \( X \neq \emptyset \). If \( t \notin B \) then \( f(t, q, X) = f(t+1, q, X^−) \).

**Lemma 7.8.** Suppose \( |C| > 0 \). Then \( f(t, q, X) = f(t+1, q, X^−) \).

**Proof.** By Lemma 7.11, this holds if \( t \notin B \). Suppose \( t \in B \). We have two cases \( t = \max(X) \) or \( t \neq \max(X) \). In the first case, a sequence \( x \) can only contribute to \( f(t, q, X) \) if \( x_k = \max(C) \). But \( f(t+1, q, X^− - \max(C)) = f(t+1, q, X^−) = 1 \) by Lemma 7.6. In the second case \( x_k \) may be \( \max(X) \), \( t \) or \( \max(C) \), all of which are distinct. Hence we have \( f(t, q, X) = f(t+1, q, X^−) + f(t+1, q, X^− - \max(C)) - f(t+1, q, X^− - t) \) by Lemma 7.7. Since \( f(t+1, q, X^− - t) = f(t+1, q, X^− - \max(C)) \), we conclude that \( f(t, q, X) = f(t+1, q, X^−) \).

**Lemma 7.9.** Suppose \( |C| > 0 \). Then \( f(t, q, X) = 1 \).

**Proof.** This follows from Lemmas 7.6 and 7.8 by induction.

**Lemma 7.10.** Suppose \( |C| = 0 \). Then \( f(t, q, X) = \epsilon(|B|) \).

**Proof.** We proceed by induction on the size of \( X \). Suppose \( t \notin B \). Then \( |A| \) must be non-empty. The inductive step holds by Lemma 7.7. Suppose \( t \in B \). If \( t = \max(X) \) then \( |X| = 1 \) and the result is trivial. Thus we may assume that \( t \neq \max(X) \). Then \( f(t, q, X) = f(t+1, q, X^−) - f(t+1, q, X - t) \) by Lemma 7.7. \( f(t+1, q, X^−) = 1 \). Also by the induction hypothesis \( f(t+1, q, X - t) = \epsilon(|B| - 1) \). The result follows.

### 7.6 Results for \( g(t, q, X) \): Unbarred letters as well.

In this section we prove results for the function \( g(t, q, X) \), where \( X \) is a set of barred and unbarred letters.

**Lemma 7.11.** Suppose \( |A| = |B| = |C| = 0 \). Then \( g(t, q, X) = 1 \).

**Proof.** Since by assumption all letters in \( U \) occur to the left of \( U' \), again by Lemma 7.7, the only contribution to \( g(t, q, X) \) occurs when all letters in \( x \) occur in the same order as in \( \rho_r \).

**Lemma 7.12.** Suppose \( X \) is unbalanced and \( |C| > 0 \). Then \( g(t, q, X) = 1 \).

**Proof.** We proceed by induction on \( |X| \). If \( t \notin B \) then for a non-zero contribution we must have \( x_k = \min(U') \). If \( |U'| = 1 \) the result follows from Lemma 7.9. Otherwise it is the inductive hypothesis. If \( t \in B \) then \( x_k \) may be equal to \( \max(C) \), \( t \), or \( \min(U') \), which are distinct. Hence we have \( g(t, q, X) = g(t+1, q, X - \min(U')) + g(t+1, q, X - \max(C)) - g(t+1, q, X - t) \) by Lemma 7.7. Since \( g(t+1, q, X - t) = g(t+1, q, X - \max(C)) \), we conclude that \( g(t, q, X) = g(t+1, q, X - \min(U')) \). The argument now proceeds as for \( t \notin B \).

**Lemma 7.13.** Suppose \( X \) is unbalanced and \( |C| = 0 \). Then \( g(t, q, X) = \epsilon(|B|) \).

**Proof.** We proceed by induction on \( |X| \). If \( t \notin B \) the argument is as for Lemma 7.12. If \( t \in B \), we may have \( x_k \in \{t, \min(U')\} \). Thus \( g(t, q, X) = g(t+1, q, X - \min(U')) - g(t+1, q, X - t) \). By induction and Lemma 7.12 this is equal to \( 1 - \epsilon(|B| - 1) \), as required.

**Lemma 7.14.** Suppose \( X \) is balanced and \( |C| > 0 \). Then \( g(t, q, X) = \epsilon(|A| + |B| + |C|) \).
Proof. We proceed by induction on $|X|$. Let $a = \max(A \cup B \cup C)$. If $t \notin B$ then $g(t, q, X) = g(t + 1, q, X - \min(U')) - g(t + 1, q, X - a)$. By Lemma 7.12, $g(t + 1, q, X - \min(U')) = 1$. If $a \in C$ and $|C| = 1$ then the result follows from Lemma 7.11 otherwise it follows by induction. If $t \in B$ then $g(t, q, X) = g(t + 1, q, X - \min(U')) - g(t + 1, q, X - a) - g(t + 1, q, X - t) + g(t + 1, q, X - \max(C))$. Note that this formula holds even if $t = a$. Clearly, the last two terms cancel, so the result follows by induction and Lemma 7.12.

Let $\theta(n)$ be the function with values $1, -1, 2, -2, 3, -3, \ldots$ on the nonnegative integers.

**Lemma 7.15.** Suppose $X$ is balanced, $|C| = 0 = |A|$, and $t \in B$. Then $g(t, q, X) = \theta(|B|)$.

**Proof.** We proceed by induction on $|X|$. We have

$$g(t, q, X) = g(t + 1, q, X - \min(U')) - g(t + 1, q, X - t) - g(t + 1, q, X - b)$$

where $b = \max(B)$. Again this holds even if $t = b$. By Lemma 7.12, $g(t + 1, q, X - \min(U')) = 1$. The result follows from induction if $t = b$ since $g(t + 1, q, X - t) = \theta(0) = 1$. Assume $t \neq b$. By Lemma 7.14, $g(t + 1, q, X - b) = \epsilon(|B| - 1)$. By the inductive hypothesis, $g(t + 1, q, X - t) = \theta(|B| - 1)$. But $\theta(|B|) = 1 - \theta(|B| - 1) - \epsilon(|B| - 1)$, proving the lemma.

**Lemma 7.16.** Suppose $X$ is balanced and $|C| = 0$. Furthermore suppose that either we have $|B| = 0$ or we have $B = [t', t'']$ for some $t' \geq t$ and $|A| \leq t' - t$. Then $g(t, q, X) = (-1)^{|A|}\theta(|B|) + \epsilon(|B|)\epsilon(|A| - 1)$.

**Proof.** First suppose $|A| = 0$. If $|B| = 0$ then $g(t, q, X) = g(t + 1, q, X - \min(U'))$ and we are done by Lemma 7.13. Otherwise let $t' = \min(B)$. If $t = t'$ we are done by Lemma 7.13. So let $t' > t$. We proceed by induction on $t' - t \geq 1$. Thus $t \notin B$ and $g(t, q, X) = g(t + 1, q, X - \min(U')) - g(t + 1, q, X - b)$ where $b = \max(B)$. By inductive hypothesis and Lemma 7.13, $g(t, q, X) = \epsilon(|B|) - \theta(|B| - 1) = \theta(|B|)$, as required.

Now suppose that $|A| > 0$. Then $t \notin B$ and by Lemma 7.13 and induction we have

$$g(t, q, X) = g(t + 1, q, X - \min(U')) - g(t + 1, q, X - \max(A))$$

$$= \epsilon(|B|) - \left((-1)^{|A|-1}\theta(|B|) + \epsilon(|B|)\epsilon(|A| - 2)\right)$$

$$= (-1)^{|A|}\theta(|B|) + \epsilon(|B|)\epsilon(|A| - 1)$$

as required.

For the next lemma we assume that $\min(U) > \min(U')$. Suppose that $\min(U) \in U'$ (though one is barred and the other unbarred). Define $U'_- = \{u \in U' \mid u < \min(U)\}$.

**Lemma 7.17.** If $|C| = 0$, $B = [t', t'']$ and $|U'_-| + |A| \leq t' - t$, then Lemma 7.16 holds as stated.

**Proof.** The statements follow from the fact that a non-zero contribution only occurs with $\sigma_k = \min(U'_-)$. When $|U'_-|$ becomes zero, $X$ is balanced and we are in the situation of Lemma 7.16.
7.7. Results for \( f'(t, q, X) \) and \( g'(t, q, X) \): One barred letter in \( y \). Let us now suppose that \( y_1 = \bar{q} \) (instead of \( q \)), but the rest of \( y \) is as before. Denote the answers by \( f'(t, q, X) \) and \( g'(t, q, X) \) and use all the same conventions as before. We state the relevant results. The proofs are identical to before. Let \( \epsilon'(n) = 1 - \epsilon(n) \) and \( \theta'(n) \) be defined on nonnegative integers by the sequence \( 0, 1, -1, 2, -2, \ldots \).

**Lemma 7.18.** Suppose \( |C| > 0 \). Then \( f'(t, q, X) = 1 \).

**Lemma 7.19.** Suppose \( |C| = 0 \). Then \( f'(t, q, X) = \epsilon'(|B|) \).

**Lemma 7.20.** Suppose \( X \) is unbalanced and \( |C| > 0 \). Then \( g'(t, q, X) = 1 \).

**Lemma 7.21.** Suppose \( X \) is unbalanced and \( |C| = 0 \). Then \( g'(t, q, X) = \epsilon'(|B|) \).

**Lemma 7.22.** Suppose \( X \) is balanced and \( |C| > 0 \). Then \( g'(t, q, X) = \epsilon'(|A| + |B| + |C|) \).

**Lemma 7.23.** Suppose \( X \) is balanced and \( |C| = 0 \). Then \( g'(t, q, X) = \theta'(|B|) \).

7.8. Defining \( U, U', A, B, C \). Let us set \( X \) to be one of \( R, \bar{R}, R_-, \bar{R}_- \) and pick \( y \) to be one of the words described in Lemma 7.3. Thus \( t = 0 \) and \( q \in \{s - 1, s - 2\} \). We describe how to construct \( U, U', A, B, \) and \( C \). First we must have \( C = \emptyset \) and there is no choice for \( B \) and \( U' \). We let \( A \) be the set of barred letters in \( X \) which are greater than \( q \) and less than all unbarred letters in \( X \). We let \( U \) be any remaining barred letters in \( X \).

For example, in Case 1 of 7.3 when \( X = R, r > n, \) and \( 2n + 1 - r \leq r - s, \) we would have \( B = \emptyset \) and

\[
U' = \{r - s, r - s + 1, \ldots, n\}
\]

\[
U = \{r - s, r - s + 1, \ldots, n - 1\}
\]

\[
A = \{r - s - 1, r - s - 2, \ldots, 2n + 1 - r\}.
\]

We claim that the sum \( g(t, q, X) \) of subsection 7.3 is equal to the same sum, but without the assumption that letters in \( U \) occur to the left of letters in \( U' \).

This is proved as follows. In the case that \( X \in \{R, R, R_-, R_-\} \), the letters in \( U \) are all present in \( U' \). Let us take a permutation \( x \) of \( X \) such that \( T(x, y) \neq 0 \). Then it is clear that unbarred letters of \( X \) occur in \( x \) in the same order as in \( \rho_t \). Suppose \( x_i = n \) and there is a barred letter \( j \in U \) occurring to the right of \( x_i \), say \( x_p = j \) where \( p > i \). Then one checks that in \( x \), (a) barred letters greater than \( j \) occur before \( x_i \), (b) letters in \( \{n - 1, \ldots, j + 1\} \) occur between \( x_i \) and \( x_p \), and (c) no unbarred letter less than \( j \) occurs between \( x_i \) and \( x_p \). We now define a sign-reversing involution: let \( \tilde{x} \) be obtained from \( x \) by swapping the locations of \( j \) with \( \bar{j} \). By Lemma 7.23, \( T(x, y) = -T(\tilde{x}, y) \). In other words, to calculate the sum of \( T(x, y) \) as \( x \) varies over all permutations of \( X \) we need only consider permutations \( x \) such that letters in \( U \) occur to the left of letters in \( U' \).

7.9. Proof of Proposition 7.4.

7.9.1. Notation. Since \( t = 0 \) and \( q \in \{s - 1, s - 2\} \) are known for the cases of Lemma 7.3 from now on we will denote the total contributions by \( f(X), g(X), f'(X), g'(X) \), where \( X \in \{R, R, R_-, R_-\} \) is one of the sets in Lemma 7.3. We prove Proposition 7.4 by splitting into the cases of Lemma 7.3. Note that the terms \( T_k \) of Proposition 7.4 always differ from the corresponding numbers \( f(X) \) and \( g(X) \) by a factor of 2 (arising from \( y_k = 0 \)).
7.9.2. Case 1. If $R = \overline{R}$ then $T_0' = 2f(R)$. The result follows from Lemma 7.17.

Suppose $R \neq \overline{R}$. By Lemma 7.17 we have $g(R) = (-1)^a + \epsilon(a - 1)$, where $a = |A|$ for $X = R$. Similarly, we have $g(\overline{R}) = (-1)^{a-1} + \epsilon(a)$. Thus $g(R) + g(\overline{R}) = 1$, so $T_0' = 2$.

7.9.3. Case 2. If $R$ consists only of barred letters, then we have $f(R) + f(R_-) = 1$ by Lemma 7.17.

Otherwise we need to calculate $g(R) + g(R_-) + g(\overline{R}) + g(\overline{R_-})$. Let $a$ and $b$ be the sizes of $|A|$ and $|B|$ when $X = R$. Then by Lemma 7.17

$$g(R) = (-1)^a \theta(b) + \epsilon(b)\epsilon(a - 1)$$
$$g(\overline{R}) = (-1)^{a-1} \theta(b) + \epsilon(b)\epsilon(a)$$
$$g(R_-) = (-1)^a \theta(b - 1) + \epsilon(b - 1)\epsilon(a - 1)$$
$$g(\overline{R_-}) = (-1)^{a-1} \theta(b - 1) + \epsilon(b - 1)\epsilon(a)$$

and the sum is 1.

In both cases $T_0' + T_{s-1}' = 2$ as required.

7.9.4. Case 3. Let $b = 3s - 2n - 1$ be the size of $B$ in $R$. Then $g(R) = \theta(b)$ and $g(R_-) = \theta(b - 1)$ by Lemma 7.17. For the case $y = s - 1s - 2 \cdots 0$ we have a contribution of $g'(\overline{R}) = \theta'(b - 1)$ by Lemma 7.23. We calculate $T_0' + T_{s-1}' + T_{s-1}' = 2(\theta(b) + \theta'(b - 1)) + 2\theta(b - 1) + 2\theta(b - 1) = 2$.

Appendix A. $\mathbb{F}_i$

In the data that follows, elements of $\tilde{C}_n$ are indicated by reduced words.

Some $\mathbb{F}_i$ are expressed in the $A_w$ basis of $\mathbb{A}_0$.

A.1. $n = 2$.

$\mathbb{F}_1 = A_0 + A_1 + A_2$
$\mathbb{F}_2 = A_{01} + A_{10} + A_{12} + 2A_{20} + A_{21}$
$\mathbb{F}_3 = A_{012} + A_{101} + A_{120} + A_{121} + A_{201} + A_{210}$
$\mathbb{F}_4 = A_{0121} + A_{1012} + A_{1210} + A_{2101}$

A.2. $n = 3$.

$\mathbb{F}_1 = A_0 + A_1 + A_2 + A_3$
$\mathbb{F}_2 = A_{01} + A_{10} + A_{12} + 2A_{20} + A_{21} + A_{23} + 2A_{30} + 2A_{31} + A_{32}$
$\mathbb{F}_3 = A_{012} + A_{101} + A_{120} + A_{121} + A_{123} + A_{201} + A_{210} + 2A_{230} + A_{231} + A_{232} + 2A_{301} + 2A_{310} + A_{312} + 2A_{320} + A_{321}$
$\mathbb{F}_4 = A_{0121} + A_{0123} + A_{1012} + A_{1210} + A_{1230} + A_{1231} + A_{1232} + A_{2101} + A_{2310} + 2A_{2320} + A_{2321} + A_{3012} + 2A_{3101} + A_{3120} + A_{3121} + A_{3201} + A_{3210}$
$\mathbb{F}_5 = A_{01231} + A_{01232} + A_{10123} + A_{12310} + A_{12320} + A_{12321} + A_{21012}$
$\mathbb{F}_6 = A_{01231} + A_{01232} + A_{10123} + A_{12310} + A_{12320} + A_{12321} + A_{21012} + A_{23101} + A_{23201} + A_{23210} + A_{30121} + A_{31012} + A_{31210} + A_{32101} + A_{32102}$
A.3. $n = 4.$

\[ \mathbb{P}_1 = A_0 + A_1 + A_2 + A_3 + A_4 \]
\[ \mathbb{P}_2 = A_{01} + A_{10} + A_{12} + 2A_{20} + A_{21} + A_{23} + 2A_{30} + 2A_{31} + A_{32} + A_{34} + 2A_{40} + 2A_{41} + 2A_{42} + A_{43} \]
\[ \mathbb{P}_3 = A_{012} + A_{101} + A_{120} + A_{121} + A_{123} + A_{1201} + A_{210} + 2A_{230} + A_{231} + A_{232} + A_{234} + 2A_{301} + 2A_{310} + A_{312} + 2A_{320} + A_{321} + 2A_{340} + 2A_{341} + A_{342} + A_{343} + 2A_{401} + 2A_{412} + 4A_{420} + 2A_{421} + A_{423} + 2A_{430} + 2A_{431} + A_{432} \]
\[ \mathbb{P}_4 = A_{0123} + A_{0102} + A_{0121} + A_{1210} + A_{1230} + A_{1231} + A_{1232} + A_{1234} + A_{2101} + A_{2301} + A_{2310} + 2A_{2320} + A_{2321} + 2A_{2340} + A_{2341} + A_{2342} + A_{2343} + A_{3012} + 2A_{3101} + A_{3120} + A_{3121} + A_{3201} + A_{3210} + 2A_{3401} + 2A_{3410} + A_{3412} + 2A_{3420} + A_{3421} + 2A_{3430} + 2A_{3431} + A_{3432} + 2A_{4012} + 2A_{4101} + 2A_{4120} + 2A_{4121} + A_{4123} + 2A_{4201} + 2A_{4210} + 2A_{4230} + A_{4231} + A_{4232} + 2A_{4301} + 2A_{4310} + A_{4312} + 2A_{4320} + A_{4321} \]

### APPENDIX B. $Q_{w}^{(n)}$

In the following tables, for $w \in \hat{C}_n$ and $\lambda$ a partition, the $(w, \lambda)$-th entry is the coefficient of $M_\lambda = 2^{(\langle \lambda \rangle \cdot m_\lambda)}$ in $Q_{w}^{(n)}$. We work in the quotient in $\mathcal{O}(2, 23)$ and hence we expand in $M_\lambda$ for $\lambda_1 \leq 2n$.

#### B.1. $n = 2$.

|   | 1   | 2   | 11  | 3   | 21  | 11  | 4   | 31  | 22  | 21  | 1   |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0 | 1   | 1   | 10  | 1   | 1   | 1   | 10  | 1   | 2   | 2   | 2   |
| 0 | 2   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
| 4 | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
| 0 | 2   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
B.2. $n = 3$.

|   | 1 | 2 | 11 | 3 | 21 | 111 | 4 | 31 | 22 | 211 | 1^5 |
|---|---|---|----|---|----|-----|---|----|----|-----|-----|
| 0 | 1 | 10 | 1 | 1 | 010 | 1 | 1 | 0210 | 1 | 2 | 2 | 2 |
|   | 210 | 1 | 1 | 1 | 3210 | 1 | 1 | 1 | 1 | 1 |

|   | 5 | 41 | 32 | 31 | 221 | 21^3 | 1^5 |
|---|---|----|----|----|-----|------|-----|
| 10210 | 1 | 1 | 2 | 2 | 2 |
| 03210 | 1 | 2 | 2 | 3 | 3 | 3 |
| 23210 | 1 | 1 | 1 | 1 | 1 | 1 |

|   | 6 | 51 | 42 | 41 | 33 | 321 | 31^3 | 222 | 2211 | 21^4 | 1^6 |
|---|---|----|----|----|----|-----|------|-----|------|------|-----|
| 010210 | | | | | | 1 | 1 | 2 | 2 | 2 | 2 |
| 103210 | 1 | 1 | 2 | 3 | 3 | 5 | 5 | 5 | 5 | 5 | 5 |
| 023210 | 1 | 2 | 2 | 2 | 3 | 3 | 4 | 4 | 4 | 4 | 4 |
| 123210 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

|   | 61 | 52 | 51 | 43 | 421 | 41^3 | 331 | 322 | 3211 | 31^4 | 2^4 | 221^3 | 21^5 | 1^6 |
|---|----|----|----|----|-----|------|-----|-----|------|------|----|------|------|-----|
| 0103210 | | | | | | 1 | 1 | 2 | 4 | 4 | 4 | 7 | 7 | 7 | 7 |
| 2103210 | 1 | 1 | 1 | 2 | 3 | 3 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 1023210 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 4 | 4 | 4 |
| 0123210 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

**Appendix C. $P_w^{(n)}$**

In the following tables, for $w \in \hat{C}_n^0$ and $\lambda$ a strict partition, the $(w, \lambda)$-th entry is the coefficient of the Schur $P$-function $P_\lambda$ in $P_w^{(n)}$. Again $w$ is given as a reduced word.

C.1. $n = 2$.

|   | 1 | 2 | 3 | 21 | 4 | 31 | 5 | 41 | 32 |
|---|---|---|---|----|---|----|---|----|-----|
| 0 | 1 | 10 | 1 | 010 | 1 | 0210 | 1 | 10210 | 1 |
|   | 210 | 1 | 1210 | 1 | 01210 | 1 |

|   | 6 | 51 | 42 | 321 | 7 | 61 | 52 | 43 | 421 |
|---|---|----|----|-----|---|----|----|----|-----|
| 010210 | 1 | 1 | 1 | 0210210 | 1 | 1 | 1 |
| 210210 | 1 | 1 | 0101210 | 1 | 1 | 1 |
| 101210 | 1 | 2 | 2101210 | 1 | 2 | 2 | 1 |
n = 3.

\[\emptyset \ 1 \ \ \ 5 \ 41 \ 32\]
\[0 \ 1 \ 10 \ 1 \ 210 \ 1 \ 3210 \ 1\]
\[010210 \ 1 \ 103210 \ 1 \ 023210 \ 1 \ 123210 \ 1\]

\[\begin{array}{c|c|c|c|c}
2 & 3 & 21 & 4 & 31 \\
010 & 1 & 210 & 1 & 3210 & 1
\end{array}\]

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