HOMOLOGY STABILITY FOR SYMPLECTIC GROUPS

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ABSTRACT. In this paper the homology stability for symplectic groups over a ring with finite stable rank is established. First we develop a 'nerve theorem' on the homotopy type of a poset in terms of a cover by subposets, where the cover is itself indexed by a poset. We use the nerve theorem to show that a poset of sequences of isotropic vectors is highly connected, as conjectured by Charney in the eighties.

1. Introduction

Interest in homological stability problems in algebraic $K$-theory started with Quillen, who used it in [14] to study the higher $K$-groups of a ring of integers. As a result of stability he proved that these groups are finitely generated (see also [8]). After that there has been considerable interest in homological stability for general linear groups. The most general results in this direction are due to the second author [20] and Suslin [19].

Parallel to this, similar questions for other classical groups such as orthogonal and symplectic groups have been studied. For work in this direction, see [23], [1], [2], [6], [11], [12]. The most general result is due to Charney [6]. She proved the homology stability for orthogonal and symplectic groups over a Dedekind domain. Panin in [12] proved a similar result but with a different method and with better range of stability. There he gave a complete formulation of the homology stability conjecture for these groups. Charney in [6] also showed that knowing the higher connectivity of the poset of isotropic unimodular sequences $\mathcal{IU}(R^{2n})$, with respect to the given bilinear form on $R^{2n}$, one can get the homology stability of the corresponding family of classical groups.

Our goal in this paper is to prove that homology stabilizes of the symplectic groups over rings with finite stable rank. To do so we prove that the poset of isotropic unimodular sequences is indeed highly connected. Recall that Panin in [11] had already sketched how one can do this for a finite dimensional affine algebra over an infinite field. However, while the assumption about the infinite field provides a significant simplification, it excludes cases of primary interest, namely rings that are finitely generated over the integers.

Our approach is as follows. We first extend a theorem of Quillen [15, Thm 9.1] which was his main tool to prove that certain posets are highly connected. We use it to develop a quantitative analogue for posets of the
nerve theorem, which expresses the homotopy type of a space in terms of the
the nerve of a suitable cover. In our situation both the elements of the cover
and the nerve are replaced with posets. We work with posets of ordered
sequences 'satisfying the chain condition', as this is a good replacement
for simplicial complexes in the presence of group actions. (This roughly
corresponds with the passage to a barycentric subdivision of a simplicial
complex.) The new nerve theorem allows us to exploit the higher connec-
tivity of the poset of unimodular sequences due to the second author. The
higher connectivity of the poset of isotropic unimodular sequences follows
inductively. We conclude with the homology stability theor em.

2. Preliminaries

Recall that a topological space $X$ is $(-1)$-connected if it is non-empty,
0-connected if it is non-empty and path connected, 1-connected if it is non-
empty and simply connected. In general for $n \geq 1$, $X$ is called $n$-connected
if $X$ is nonempty, $X$ is 0-connected and $\pi_i(X,x) = 0$ for every base point
$x \in X$ and $1 \leq i \leq n$. For $n \geq -1$ a space $X$ is called $n$-acyclic if it is
nonempty and $\tilde{H}_i(X,\mathbb{Z}) = 0$ for $0 \leq i \leq n$. For $n < -1$ the conditions of
$n$-connectedness and $n$-acyclicness are vacuous.

**Theorem 2.1** (Hurewicz). For $n \geq 0$, a topological space $X$ is $n$-connected
if and only if the reduced homology groups $\tilde{H}_i(X,\mathbb{Z})$ are trivial for $0 \leq i \leq n$
and $X$ is 1-connected if $n \geq 1$.

**Proof.** See [25], Chap. IV, Corollaries 7.7 and 7.8. \hfill $\square$

Let $X$ be a partially ordered set or briefly a poset. Consider the simplicial
complex associated to $X$, that is the simplicial complex where vertices or
0-simplices are the elements of $X$ and the $k$-simplices are the $(k+1)$-tuples
$(x_0,\ldots, x_k)$ of elements of $X$ with $x_0 < \cdots < x_k$. We denote it again by $X$.
We denote the geometric realization of $X$ by $|X|$ and we consider it with
the weak topology. It is well known that $|X|$ is a CW-complex [10]. By a
morphism or map of posets $f : X \to Y$ we mean an order-preserving map
i. e. if $x \leq x'$ then $f(x) \leq f(x')$. Such a map induces a continuous map
$|f| : |X| \to |Y|$. 

**Remark 2.2.** If $K$ is a simplicial complex and $X$ the ordered set of simplices
of $K$, then the space $|X|$ is the barycentric subdivision of $K$. Thus every
simplicial complex, with weak topology, is homeomorphic to the geometric
realization of some, and in fact many, posets. Furthermore since it is
well known that any CW-complex is homotopy equivalent to a simplicial
complex, it follows that any interesting homotopy type is realized as the
geometric realization of a poset.

**Proposition 2.3.** Let $X$ and $Y$ be posets.

(i) (Segal [16]) If $f, g : X \to Y$ are maps of posets such that $f(x) \leq g(x)$
for all $x \in X$, then $|f|$ and $|g|$ are homotopic.
(ii) If the poset $X$ has a minimal or maximal element then $|X|$ is contractible.

(iii) If $X^{op}$ denotes the opposite poset of $X$, i.e. with opposite ordering, then $|X^{op}| \simeq |X|$.

Proof. (i) Consider the poset $I = \{0, 1 : 0 < 1\}$ and define the poset map $h : I \times X \to Y$ as $h(0, x) = f(x)$, $h(1, x) = g(x)$. Since $|I| \simeq [0, 1]$, we have $|h| : [0, 1] \times |X| \to |Y|$ with $|h|(0, x) = |f|(x)$ and $|h|(1, x) = |g|(x)$. This shows that $|f|$ and $|g|$ are homotopic.

(ii) Suppose $X$ has a maximal element $z$. Consider the map $f : X \to Y$ with $f(x) = z$ for every $x \in X$. Clearly for every $x \in X$, $\text{id}_X(x) \leq f(x)$. This shows that $\text{id}_X$ and the constant map $f$ are homotopic. So $X$ is contractible. If $X$ has a minimal element the proof is similar.

(iii). This is natural and easy. ☐

The construction $X \mapsto |X|$ allows us to assign topological concepts to posets. For example we define the homology groups of a poset $X$ to be those of $|X|$, we call $X$ $n$-connected or contractible if $|X|$ is $n$-connected or contractible etc. Note that $X$ is connected if and only if $X$ is connected as a poset. By the dimension of a poset $X$, we mean the dimension of the space $|X|$, or equivalently the supremum of the integers $n$ such that there is a chain $x_0 < \cdots < x_n$ in $X$. By convention the empty set has dimension $-1$.

Let $X$ be a poset and $x \in X$. Define $\text{Link}_X^+(x) := \{ u \in X : u > x \}$ and $\text{Link}_X^-(x) := \{ u \in X : u < x \}$. Given a map $f : X \to Y$ of posets and an element $y \in Y$, define subposets $f/y$ and $y \backslash f$ of $X$ as follows

$$f/y := \{ x \in X : f(x) \leq y \} \quad y \backslash f := \{ x \in X : f(x) \geq y \}.$$ 

In fact $f/y = f^{-1}(Y_{\leq y})$ and $y \backslash f = f^{-1}(Y_{\geq y})$ where $Y_{\leq y} = \{ z \in Y : z \leq y \}$ and $Y_{\geq y} = \{ z \in Y : z \geq y \}$. Note that by 2.3 (ii), $Y_{\leq y}$ and $Y_{\geq y}$ are contractible. If $\text{id}_Y : Y \to Y$ is the identity map, then $\text{id}_Y/y = Y_{\leq y}$ and $\text{id}_Y \backslash y = Y_{\geq y}$.

Let $F : X \to \text{Ab}$ be a functor from a poset $X$, regarded as a category in usual way, to the category of abelian groups. We define the homology groups $H_i(X, F)$ of $X$ with coefficient $F$ to be the homology of the complex $C_\bullet(X, F)$ given by

$$C_n(X, F) = \bigoplus_{x_0 < \cdots < x_n} F(x_0)$$

where the direct sum is taken over all $n$-simplices in $X$, with differential $\partial_n = \sum_{i=0}^n (-1)^i d^n_i$ where $d^n_i : C_n(X, F) \to C_{n-1}(X, F)$ and $d^n_i$ takes the $(x_0 < \cdots < x_n)$-component of $C_n(X, F)$ to the $(x_0 < \cdots < \hat{x}_i < \cdots < x_n)$-component of $C_{n-1}(X, F)$ via $d^n_i = \text{id}_F(x_0)$ if $i > 0$ and $d^n_0 : F(x_0) \to F(x_1)$.

In particular, for the empty set we have $H_i(\emptyset, F) = 0$ for $i \geq 0$.

Let $F$ be the constant functor $\mathbb{Z}$. Then the homology groups with this coefficient coincide with the integral homology of $|X|$, that is $H_k(X, \mathbb{Z}) = \ldots$
$H_k(|X|, \mathbb{Z})$ for all $k \in \mathbb{Z}$, [7, App. II]. Let $\tilde{H}_i(X, \mathbb{Z})$ denote the reduced integral homology of the poset $X$, that is $\tilde{H}_i(X, \mathbb{Z}) = \ker \{ H_i(X, \mathbb{Z}) \to H_i(pt, \mathbb{Z}) \}$

if $X \neq \emptyset$ and $\tilde{H}_i(\emptyset, \mathbb{Z}) = \{ \mathbb{Z} \}$ if $i = -1$
$0$ if $i \neq -1$. So $\tilde{H}_i(X, \mathbb{Z}) = H_i(X, \mathbb{Z})$ for $i \geq 1$
and for $i = 0$ we have the exact sequence

$0 \to \tilde{H}_0(X, \mathbb{Z}) \to H_0(X, \mathbb{Z}) \to \mathbb{Z} \to \tilde{H}_{-1}(X, \mathbb{Z}) \to 0$

where $\mathbb{Z}$ is identified with the group $H_0(pt, \mathbb{Z})$. Notice that $H_0(X, \mathbb{Z})$ is identified with the free abelian group generated by the connected components of $X$.

A local system of abelian groups on a space (resp. poset) $X$ is a functor $F$ from the groupoid of $X$ (resp. viewed $X$ as a category), to the category of abelian groups which is morphism-inverting, i.e. such that the map $F(x) \to F(x')$ associated to a path from $x$ to $x'$ (resp. $x \leq x'$) is an isomorphism. Clearly, a local system $F$ on a path connected space (resp. 0-connected poset) is determined, up to canonical isomorphism, by the following data: if $x \in X$ is a base point, it suffices to be given the group $F(x)$ and an action of $\pi_1(X, x)$ on $F(x)$.

The homology groups $H_k(X, F)$ of a space with a local system $F$ are a generalization of the ordinary homology groups [25, Chap. VI]. In fact if $X$ is a 0-connected space and if $F$ is a constant local system on $X$, then $H_k(X, F) \simeq H_k(X, F(x_0))$ for every $x_0 \in X$ [25, Chap. VI, 2.1].

Let $X$ be a poset and $F$ a local system on $|X|$. Then the restriction of $F$ to $X$ is a local system on $X$. Considering $F$ as a functor from $X$ to the category of abelian groups, we can define $H_k(X, F)$ as in the above. Conversely if $F$ is a local system on the poset $X$, then there is a unique local system, up to isomorphism, on $|X|$ such that the restriction to $X$ is $F$ [25, Chap. VI, Thm 1.12], [13 I, Prop. 1]. We denote both local systems by $F$.

**Theorem 2.4.** Let $X$ be a poset and $F$ a local system on $X$. Then the homology groups $H_k(|X|, F)$ are isomorphic with the homology groups $H_k(X, F)$.

*Proof.* See [25, Chap. VI, Thm. 4.8] or [13 I, p. 91].

**Theorem 2.5.** Let $X$ be a path connected space with a base point $x$ and let $F$ be a local system on $X$. Then the inclusion $\{x\} \hookrightarrow X$ induces an isomorphism $F(x)/G \xrightarrow{\sim} H_0(X, F)$ where $G$ is the subgroup of $F(x)$ generated by all the elements of the form $a - \beta a$ with $a \in F(x)$, $\beta \in \pi_1(X, x)$.

*Proof.* See [25, Chap. VI, Thm. 2.8* and Thm. 3.2].

We need the following interesting and well known lemma about the covering spaces of the space $|X|$, where $X$ is a poset (or more generally a simplicial set). For a definition of a covering space, useful for our purpose, and some more information, see [17 Chap. 2].
Lemma 2.6. The category of the covering spaces of the space $|X|$ of a poset $X$ is equivalent to the category $\mathcal{L}_{\text{Set}}(X)$, the category of functors $F : X \to \text{Set}$, where $\text{Set}$ is the category of sets, such that $F(x) \to F(x')$ is a bijection for every relation $x \leq x'$.

Proof. See [13, I, p. 90]. For the same proof with more details see [18, lem. 6.1]. □

3. Homology and homotopy of posets

Theorem 3.1. Let $f : X \to Y$ be a map of posets. Then there is a first quadrant spectral sequence

$$E^2_{p,q} = H_p(Y, y \mapsto H_q(f/y, Z)) \Rightarrow H_{p+q}(X, Z).$$

The spectral sequence is functorial, in the sense that if there is a commutative diagram of posets

$$\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow g_X & & \downarrow g_Y \\
X & \xrightarrow{f} & Y
\end{array}$$

then there is a natural map from the spectral sequence arising from $f'$ to the spectral sequence arising from $f$. Moreover the map $g_X_* : H_*(X', Z) \to H_*(X, Z)$ is compatible with this natural map.

Proof. Let $C_{*,*}(f)$ be the double complex such that $C_{p,q}(f)$ is the free abelian group generated by the set $\{(x_0 < \cdots < x_q, f(x_q) < y_0 < \cdots < y_p) : x_i \in X, y_i \in Y\}$. The first spectral sequences of this double complex has as $E^1_{p,q}(I) = \bigoplus_{y_0 < \cdots < y_p} H_q(f/y, Z)$. By the general theory of double complexes (see for example [24, Chap. 5]), we know that $E^2_{p,q}(I)$ is the homology of the chain complex $C_*(Y, G_q) = E^1_{*,q}(I)$ where $G_q : Y \to \text{Ab}$, $G_q(y) = H_q(f/y, Z)$ and hence $E^2_{p,q}(I) = H_p(Y, y \mapsto H_q(f/y, Z))$. The second spectral sequence has as $E^1_{p,q}(II) = \bigoplus_{f(x_q) < y_0 < \cdots < y_p} H_p(id_Y \backslash f(x_q), Z)$. But by [23] (ii), $id_Y \backslash f(x_q) = f(x_q) \backslash Y$ is contractible, so $E^1_{*,0}(II) = C_*(X^{op}, Z)$ and $E^1_{*,q}(II) = 0$ for $q > 0$. Hence $H_*(C_{*,*}(f)) \simeq H_*(X^{op}, Z) \simeq H_*(X, Z)$. This completes the proof of existence and convergence of the spectral sequence. The functorial behavior of the spectral sequence follows from the functorial behavior of the spectral sequence of a filtration. □

Remark 3.2. The above spectral sequence is a special case of a more general Theorem [7, App. II]. The above proof is taken from [9, Chap. I] where the functorial behavior of the spectral sequence is more visible. For more details see [9].

Definition 3.3. Let $X$ be a poset. A map $ht_X : X \to \mathbb{Z}_{\geq 0}$ is called height function if it is a strictly increasing map.
For example the height function \( \text{ht}_X(x) = 1 + \dim(\text{Link}^+_X(x)) \) is the usual one considered in [15], [9] and [6].

**Lemma 3.4.** Let \( X \) be a poset such that \( \text{Link}^+_X(x) \) is \((n-\text{ht}_X(x)-2)\)-acyclic, for every \( x \in X \), where \( \text{ht}_X \) is a height function on \( X \). Let \( \mathcal{F} : X \to \mathbb{A}^{\mathbb{H}} \) be a functor such that \( \mathcal{F}(x) = 0 \) for all \( x \in X \) with \( \text{ht}_X(x) \geq m \), where \( m \geq 1 \). Then \( H_k(X, \mathcal{F}) = 0 \) for \( k \leq n - m \).

**Proof.** First consider the case of a functor \( \mathcal{F} \) such that \( \mathcal{F}(x) = 0 \) if \( \text{ht}_X(x) \neq m - 1 \). Then \( C_k(X, \mathcal{F}) = \bigoplus_{\text{ht}_X(x)=m-1} \mathcal{F}(x_0) \). Clearly \( 0 = d^0_k = \mathcal{F}(x_0 < x_1) = \mathcal{F}(x_0) \rightarrow \mathcal{F}(x_1) \). Thus \( \partial_k = \sum_{i=1}^k (-1)^i d^i_k \). Define \( C_{-1}(\text{Link}^+_X(x_0), \mathcal{F}(x_0)) = \mathcal{F}(x_0) \) and complete the singular complex of \( \text{Link}^+_X(x_0) \) with coefficient in \( \mathcal{F}(x_0) \) to

\[
\cdots \to C_0(\text{Link}^+_X(x_0), \mathcal{F}(x_0)) \overset{\varepsilon}{\to} C_{-1}(\text{Link}^+_X(x_0), \mathcal{F}(x_0)) \to 0
\]

where \( \varepsilon((g_i)) = \sum_i g_i \). Then

\[
C_k(X, \mathcal{F}) = \bigoplus_{\text{ht}_X(x)=m-1} \left( \bigoplus_{x_0 < \cdots < x_k} \mathcal{F}(x_0) \right) = \bigoplus_{\text{ht}_X(x)=m-1} C_{k-1}(\text{Link}^+_X(x_0), \mathcal{F}(x_0)).
\]

The complex \( C_{k-1}(\text{Link}^+_X(x_0), \mathcal{F}(x_0)) \) is the standard complex for computing the reduced homology of \( \text{Link}^+_X(x_0) \) with constant coefficient \( \mathcal{F}(x_0) \). So

\[
H_k(X, \mathcal{F}) = \bigoplus_{\text{ht}_X(x)=m-1} \tilde{H}_{k-1}(\text{Link}^+_X(x), \mathcal{F}(x)).
\]

If \( \text{ht}_X(x_0) = m - 1 \) then \( \text{Link}^+_X(x_0) \) is \((n-(m-1)-2)\)-acyclic, and by the universal coefficient theorem [17] Chap. 5, Thm. 8, \( \tilde{H}_{k-1}(\text{Link}^+_X(x_0), \mathcal{F}(x_0)) = 0 \) for \(-1 \leq k - 1 \leq n - (m - 1) - 2 \). This shows that \( H_k(X, \mathcal{F}) = 0 \) for \( 0 \leq k \leq n - m \). To prove the lemma in general, we argue by induction on \( m \). If \( m = 1 \) then for \( \text{ht}_X(x) \geq 1 \), \( \mathcal{F}(x) = 0 \). So the lemma follows from the special case above. Suppose \( m \geq 2 \). Define \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) to be the functors

\[
\mathcal{F}_0(x) = \begin{cases} \mathcal{F}(x) & \text{if } \text{ht}_X(x) < m - 1 \\ 0 & \text{if } \text{ht}_X(x) \geq m - 1 \end{cases}, \quad \mathcal{F}_1(x) = \begin{cases} \mathcal{F}(x) & \text{if } \text{ht}_X(x) = m - 1 \\ 0 & \text{if } \text{ht}_X(x) \neq m - 1 \end{cases}
\]

respectively. Then there is a short exact sequence \( 0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_0 \to 0 \). By the above discussion, \( H_k(X, \mathcal{F}_1) = 0 \) for \( 0 \leq k \leq n - m \) and by induction for \( m - 1 \), we have \( H_k(X, \mathcal{F}_0) = 0 \) for \( k \leq n - (m - 1) \). The long exact sequence for the above short exact sequence of functors it is easy to see that \( H_k(X, \mathcal{F}) = 0 \) for \( 0 \leq k \leq n - m \). \( \square \)

**Theorem 3.5.** Let \( f : X \to Y \) be a map of posets and \( \text{ht}_Y \) a height function on \( Y \). Assume for every \( y \in Y \), that \( \text{Link}^+_Y(y) \) is \((n - \text{ht}_Y(y) - 2)\)-acyclic.
and $f/y$ is $(\mathrm{ht}_Y(y) - 1)$-acyclic. Then $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$ is an isomorphism for $0 \leq k \leq n - 1$.

Proof. By theorem [3.1] we have the first quadrant spectral sequence

$$E^2_{p,q} = H_p(Y, y \mapsto H_q(f/y, \mathbb{Z})) \Rightarrow H_{p+q}(X, \mathbb{Z}).$$

Since $H_q(f/y, \mathbb{Z}) = 0$ for $0 < q \leq \mathrm{ht}_Y(y) - 1$, the functor $G_q : Y \to \text{Ab}$, $G_q(y) = H_q(f/y, \mathbb{Z})$ is trivial for $\mathrm{ht}_Y(y) \geq q + 1$, $q > 0$. By lemma [3.4], $H_p(Y, G_q) = 0$ for $p \leq n - (q + 1)$. Hence $E^2_{p,q} = 0$ for $p + q \leq n - 1$, $q > 0$. If $q = 0$, by writing the long exact sequence for the short exact sequence $0 \to \tilde{H}_0(f/y, \mathbb{Z}) \to H_0(f/y, \mathbb{Z}) \to \mathbb{Z} \to 0$, valid because $f/y$ is nonempty, we have

$$\cdots \to H_0(Y, \mathbb{Z}) \to H_{-1}(Y, y \mapsto \tilde{H}_0(f/y, \mathbb{Z})) \to E^2_{0,-1,0} \to \cdots$$

$$\to H_{-2}(Y, y \mapsto \tilde{H}_0(f/y, \mathbb{Z})) \to E^2_{0,-2,0} \to \cdots$$

$$\to H_1(Y, \mathbb{Z}) \to H_0(Y, y \mapsto \tilde{H}_0(f/y, \mathbb{Z})) \to E^2_{0,0,0} \to H_0(Y, \mathbb{Z}) \to 0.$$

If $\mathrm{ht}_Y(y) \geq 1$, then $\tilde{H}_0(f/y, \mathbb{Z}) = 0$. By lemma [3.4], $H_k(Y, y \mapsto \tilde{H}_0(f/y, \mathbb{Z})) = 0$ for $0 \leq k \leq n - 1$. Thus

$$E^2_{p,q} = \begin{cases} H_p(Y, \mathbb{Z}) & \text{if } q = 0, 0 \leq p \leq n - 1 \\ 0 & \text{if } p + q \leq n - 1, q > 0. \end{cases}$$

This shows that $E_{p,q}^2 \simeq E_{p,q}^{2} \simeq \cdots \simeq E_{p,q}^{\infty}$ for $0 \leq p + q \leq n - 1$.

For every $k$, we have a filtration $0 = F_{-1}H_k \subseteq F_0H_k \subseteq \cdots \subseteq F_kH_k = H_k(X, \mathbb{Z})$ of $H_k(X, \mathbb{Z})$, such that $E_{p,q}^{\infty} \simeq F_pH_{p+q}/F_{p-1}H_{p+q}$ [24, Chap. 5, 5.2.6]. Let $0 \leq k \leq n - 1$. For $0 \leq i < k$, we have $0 = E_{i,k-i}^{\infty} \simeq F_iH_k/F_{i-1}H_k$, so $F_iH_k = F_{i-1}H_k$. Hence $0 = F_{-1}H_k = F_0H_k = \cdots = F_{k-1}H_k$. On the other hand $E_{k,0}^{\infty} \simeq F_kH_k/F_{k-1}H_k$. Therefore $H_k(X, \mathbb{Z}) \simeq H_k(Y, \mathbb{Z})$. This shows that $H_k(X, \mathbb{Z}) \simeq H_k(Y, \mathbb{Z})$ for $0 \leq k \leq n - 1$. Now consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow f \downarrow & & \downarrow \text{id}_Y \\ Y & \xrightarrow{\text{id}_Y} & Y. \end{array}$$

By functoriality of the spectral sequence, and the above calculation we get the diagram

$$\begin{array}{ccc} H_k(Y, y \mapsto H_0(f/y, \mathbb{Z})) & \xrightarrow{\sim} & H_k(X, \mathbb{Z}) \\ \downarrow \text{id}_Y \downarrow & & \downarrow f_* \\ H_k(Y, y \mapsto H_0(id_Y/y, \mathbb{Z})) & \xrightarrow{\sim} & H_k(Y, \mathbb{Z}). \end{array}$$

Since $\text{id}_Y/y = Y_{\leq 0}$ is contractible, we have $H_k(Y, y \mapsto H_0(\text{id}_Y/y, \mathbb{Z})) = H_k(Y, \mathbb{Z})$. The map $\text{id}_Y$ is an isomorphism for $0 \leq k \leq n - 1$, from the above long exact sequence. This shows that $f_*$ is an isomorphism for $0 \leq k \leq n - 1$. 

$\square$
Lemma 3.6. Let X be a 0-connected poset. Then X is 1-connected if and only if for every local system $\mathcal{F}$ on X and every $x \in X$, the map $\mathcal{F}(x) \to H_0(X, \mathcal{F})$, induced from the inclusion $\{x\} \hookrightarrow X$, is an isomorphism (or equivalently, every local system on X is isomorphic with a constant local system).

Proof. If X is 1-connected then by theorem [2.5] and the connectedness of X, one has $\mathcal{F}(x) \xrightarrow{\sim} H_0(X, \mathcal{F})$ for every $x \in X$. Now let every local system on X be isomorphic with a constant local system. Let $\mathcal{F} : X \to \text{Set}$ be in $L_S(X)$. Define the functor $\mathcal{G} : X \to \text{Ab}$ where $\mathcal{G}(x)$ is the free abelian group generated by $\mathcal{F}(x)$. Clearly $\mathcal{G}$ is a local system and so it is constant system. This shows that $\mathcal{F}$ is isomorphic to a constant functor. So by lemma [2.6] any connected covering space of $[X]$ is isomorphic to $[X]$. This shows that the universal covering of $[X]$, is $[X]$. Note that the universal covering of a connected simplicial simplex exists and is simply connected [17, Chap. 2, Cor. 14 and 15]. Therefore X is 1-connected. □

Theorem 3.7. Let $f : X \to Y$ be a map of posets and $\text{ht}_Y$ a height function on Y. Assume for every $y \in Y$, that $\text{Link}_Y^+(y)$ is $(n - \text{ht}_Y(y) - 2)$-connected and $f/y$ is $(\text{ht}_Y(y) - 1)$-connected. Then X is $(n - 1)$-connected if and only if $Y$ is $(n - 1)$-connected.

Proof. By [2.1] and [3.5] it is enough to prove that X is 1-connected if and only if $Y$ is 1-connected, when $n \geq 2$. Let $\mathcal{F} : X \to \text{Ab}$ be a local system. Define the functor $\mathcal{G} : Y \to \text{Ab}$ with

$$\mathcal{G}(y) = \begin{cases} H_0(f/y, \mathcal{F}) & \text{if } \text{ht}_Y(y) \neq 0 \\ H_0(\text{Link}_Y^+(y), y' \mapsto H_0(f/y', \mathcal{F})) & \text{if } \text{ht}_Y(y) = 0. \end{cases}$$

We prove that $\mathcal{G}$ is a local system. If $\text{ht}_Y(y) \geq 2$ then $f/y$ is 1-connected and by [3.5] $\mathcal{F}_{f/y}$ is a constant system, so by [3.6] $H_0(f/y, \mathcal{F}) \simeq \mathcal{F}(x)$ for every $x \in f/y$. If $\text{ht}_Y(y) = 1$, then $f/y$ is 0-connected and $\text{Link}_Y^+(y)$ is nonempty. Choose $y' \in Y$ such that $y < y'$. Now $f/y'$ is 1-connected and so $\mathcal{F}_{f/y'}$ is a constant system on $f/y'$. But $f/y \subset f/y'$, so $\mathcal{F}_{f/y}$ is a constant system. Since $f/y$ is 0-connected, by [2.5] and the fact that we mentioned before theorem [2.4] $H_0(f/y, \mathcal{F}) \simeq \mathcal{F}(x)$ for every $x \in f/y$. Now let $\text{ht}_Y(y) = 0$. Then $\text{Link}_Y^+(y)$ is 0-connected, $f/y$ is nonempty and for every $y' \in \text{Link}_Y^+(y)$, $H_0(f/y', \mathcal{F}) \simeq H_0((f/y)^\circ, \mathcal{F})$ where $(f/y)^\circ$ is a component of $f/y$, which we fix. This shows that the local system $\mathcal{F}' : \text{Link}_Y^+(y) \to \text{Ab}$ with $y' \mapsto H_0(f/y', \mathcal{F})$ is isomorphic to a constant system, so $H_0(\text{Link}_Y^+(y), y' \mapsto H_0(f/y', \mathcal{F})) = H_0(\text{Link}_Y^+(y), \mathcal{F}') \simeq \mathcal{F}'(y') \simeq \mathcal{F}(x)$ for every $x \in f/y'$. This shows that $\mathcal{G}$ is a local system.

If Y is 1-connected, by [3.6] $\mathcal{G}$ is a constant system. But it is easy to see that $\mathcal{F} \simeq \mathcal{G} \circ f$. Therefore $\mathcal{F}$ is a constant system. Since X is connected by our homology calculation, by [3.6] we conclude that X is 1-connected. Now let X be 1-connected. If $\mathcal{E}$ is a local system on Y, then $f^* \mathcal{E} := \mathcal{E} \circ f$ is a local system on X. So it is a constant local system. As above we can
construct a local system $G'$ on $Y$ from $F' := \mathcal{E} \circ f$. This gives a natural transformation from $G'$ to $\mathcal{E}$ which is an isomorphism. Since $\mathcal{E} \circ f$ is constant, by \ref{2.10} and \ref{3.8} and an argument as above one sees that $G'$ is constant. Therefore $\mathcal{E}$ is isomorphic to a constant local system and \ref{3.6} shows that $Y$ is 1-connected. \hfill \blackqed

\textbf{Remark 3.8.} In the proof of the above theorem \textbf{3.7} we showed in fact that: Let $f : X \to Y$ be a map of posets and $\text{ht}_Y$ a height function on $Y$. Assume for every $y \in Y$, that $\text{Link}_Y^+(y)$ is $(n - \text{ht}_Y(y) - 2)$-connected and $f/y$ is $(\text{ht}_Y(y) - 1)$-connected. Then $f^* : \mathcal{L}_S(Y) \to \mathcal{L}_S(X)$, with $\mathcal{E} \mapsto \mathcal{E} \circ f$ is an equivalence of categories.

\textbf{Remark 3.9.} Theorem \textbf{3.7} is a generalization of a theorem of Quillen [15, Thm. 9.1]. We proved that the converse of that theorem is also valid. Our proof is similar in outline to the proof by Quillen. Furthermore, lemma \textbf{3.4} is a generalized version of lemma 1.3 from [6]. With more restrictions, Maazen, in [9, Chap. II] gave an easier proof of Quillen’s theorem.

4. HOMOLOGY AND HOMOTOPY OF POSETS OF SEQUENCES

Let $V$ be a nonempty set. We denote by $\mathcal{O}(V)$ the poset of finite ordered sequences of distinct elements of $V$, the length of each sequence being at least one. The partial ordering on $\mathcal{O}(V)$ is defined by refinement: $(v_1, \ldots, v_m) \leq (w_1, \ldots, w_n)$ if and only if there is a strictly increasing map $\phi : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that $v_i = w_{\phi(i)}$, in other words, if $(v_1, \ldots, v_m)$ is an order preserving subsequence of $(w_1, \ldots, w_n)$. If $v = (v_1, \ldots, v_m)$ we denote by $|v|$ the length of $v$, that is $|v| = m$. If $v = (v_1, \ldots, v_m)$ and $w = (w_1, \ldots, w_n)$, we write $(v_1, \ldots, v_m, w_1, \ldots, w_n)$ as $vw$. Define $F_v$ to be the set of $w \in F$ such that $vw \in F$. Note that $(F_v)_w = F_{vw}$. A subset $F$ of $\mathcal{O}(V)$ is said to satisfy the \textit{chain condition} if $v \in F$ whenever $w \in F$ and $v \in \mathcal{O}(V)$ and $v \leq w$. The subposets of $\mathcal{O}(V)$ which satisfy the chain condition are extensively studied in [9], [20] and [5]. In this section we will study them some more.

Let $F \subseteq \mathcal{O}(V)$. For a nonempty set $S$ we define the poset $F \langle S \rangle$ as

$$F \langle S \rangle := \{((v_1, s_1), \ldots, (v_r, s_r)) \in \mathcal{O}(V \times S) : (v_1, \ldots, v_r) \in F\}.$$ 

Assume $s_0 \in S$ and consider the injective poset map $l_{s_0} : F \to F \langle S \rangle$ with $(v_1, \ldots, v_r) \mapsto ((v_1, s_0), \ldots, (v_r, s_0))$. We have clearly a projection $p : F \langle S \rangle \to F$ with $((v_1, s_1), \ldots, (v_r, s_r)) \mapsto (v_1, \ldots, v_r)$ such that $p \circ l_{s_0} = \text{id}_F$.

\textbf{Lemma 4.1.} Suppose $F \subseteq \mathcal{O}(V)$ satisfies the chain condition and $S$ is a nonempty set. Assume for every $v \in F$, that $F_v$ is $(n - |v|)$-connected.

(i) If $s_0 \in S$ then $(l_{s_0})_* : H_k(F, \mathbb{Z}) \to H_k(F \langle S \rangle, \mathbb{Z})$ is an isomorphism for $0 \leq k \leq n$.

(ii) If $F$ is $\min\{1, n - 1\}$-connected, then $(l_{s_0})_* : \pi_k(F, v) \to \pi_k(F \langle S \rangle, l_{s_0}(v))$ is an isomorphism for $0 \leq k \leq n$.

\textbf{Proof.} This follows by [5, Prop. 1.6] from the fact that $p \circ l_{s_0} = \text{id}_F$. \hfill \blackqed
Lemma 4.2. Let $F \subseteq \mathcal{O}(V)$ satisfies the chain condition. Then $|\text{Link}_F(v)| \simeq S^{[v]-2}$ for every $v \in F$.

Proof. Let $v = (v_1, \ldots, v_n)$. By definition $\text{Link}_F(v) = \{w \in F : w < v\} = \{ (v_1, \ldots, v_i) : k < n, i_1 < \cdots < i_k \}$. Hence $|\text{Link}_F(v)|$ is isomorphic to the barycentric subdivision of the boundary of the standard simplex $\Delta_{n-1}$. It is well known that $\partial \Delta_{n-1} \simeq S^{n-2}$, hence $|\text{Link}_F(v)| \simeq S^{[v]-2}$. \qed

Theorem 4.3 (Nerve Theorem for Posets). Let $V$ and $T$ be two nonempty sets, $F \subseteq \mathcal{O}(V)$ and $X \subseteq \mathcal{O}(T)$. Assume $X = \bigcup_{v \in F} X_v$ such that if $v \leq w$ in $F$, then $X_w \subseteq X_v$. Let $F$, $X$ and $X_v$, for every $v \in F$, satisfy the chain condition. Also assume

(i) for every $v \in F$, $X_v$ is $(l - |v| - 1)$-acyclic (resp. $(l - |v| + 1)$-connected),

(ii) for every $x \in X$, $A_x := \{ v \in F : x \in X_v \}$ is $(l - |x| - 1)$-acyclic (resp. $(l - |x| + 1)$-connected).

Then $H_k(F, Z) \simeq H_k(X, Z)$ for $0 \leq k \leq l$ (resp. $F$ is $l$-connected if and only if $X$ is $l$-connected).

Proof. Let $F_{\leq l+2} = \{ v \in F : |v| \leq l + 2 \}$ and let $i : F_{\leq l+2} \to F$ be the inclusion. Clearly $|F_{\leq l+2}|$ is the $(l + 1)$-skeleton of $|F|$, if we consider $|F|$ as a cell complex whose $k$-cells are the $|F_v|$ with $|v| = k + 1$. It is well known that $i_* : H_k(F_{\leq l+2}, Z) \to H_k(F, Z)$ and $i_* : \pi_k(F_{\leq l+2}, v) \to \pi_k(F, v)$ are isomorphisms for $0 \leq k \leq l$ (see \cite{25}, Chap. II, corollary 2.14, and \cite{25}, Chap. II, Corollary 3.10 and Chap. IV lemma 7.12.) So it is enough to prove the theorem for $F_{\leq l+2}$ and $X_{\leq l+2}$. Thus assume $F = F_{\leq l+2}$ and $X = X_{\leq l+2}$. We define $Z \subseteq X \times F$ as $Z = \{ (x, v) : x \in X_v \}$. Consider the projections

$$f : Z \to F, (x, v) \mapsto v, \quad g : Z \to X, (x, v) \mapsto x.$$

First we prove that $f^{-1}(v) \sim v \setminus f$ and $g^{-1}(x) \sim x \setminus g$, where $\sim$ means homotopy equivalence. By definition $v \setminus f = \{ (x, w) : w \geq v, x \in X_v \}$. Define $\phi : v \setminus f \to f^{-1}(v), (x, w) \mapsto (x, v)$. Consider the inclusion $j : f^{-1}(v) \to v \setminus f$. Clearly $\phi \circ j(x, v) = \phi(x, v) = (x, v)$ and $j \circ \phi(x, v) = j(x, v) = (x, v) \leq (x, w)$. So by \cite{23}ii), $v \setminus f$ and $f^{-1}(v)$ are homotopy equivalent. Similarly $x \setminus g \sim g^{-1}(x)$.

Now we prove that the maps $f^{op} : Z^{op} \to Y^{op}$ and $g^{op} : Z^{op} \to X^{op}$ satisfy the conditions of \cite{3.3}. First $f^{op} : Z^{op} \to Y^{op}$; define the height function $\text{ht}_{F^{op}}$ on $F^{op}$ as $\text{ht}_{F^{op}}(v) = |v| + 2 - |v|$. It is easy to see that $f^{op}/v \simeq v \setminus f \simeq f^{-1}(v) \simeq X_v$. Hence $f^{op}/v$ is $(l - |v| - 1)$-acyclic (resp. $(l - |v| + 1)$-connected). But $l - |v| + 1 = (l + 2 - |v|) - 1 = \text{ht}_{F^{op}}(v) - 1$, so $f^{op}/v$ is $(\text{ht}_{F^{op}}(v) - 1)$-acyclic (resp. $(\text{ht}_{F^{op}}(v) - 1)$-connected). Let $n := l + 1$. Clearly $\text{Link}_{F^{op}}(v) = \text{Link}_F(v)$. By lemma \cite{4.2} $|\text{Link}_F(v)|$ is $(|v| - 3)$-connected. But $(|v| - 3) = l + 1 - (l + 2 - |v|) - 2 = n - \text{ht}_{F^{op}}(v) - 2$. Thus $\text{Link}_{F^{op}}(v)$ is $(n - \text{ht}_{F^{op}}(v) - 2)$-acyclic (resp. $(n - \text{ht}_{F^{op}}(v) - 2)$-connected). Therefore by theorem \cite{3.3} $f_* : H_i(Z, Z) \to H_i(F, Z)$ is an isomorphism for $0 \leq i \leq l$ (resp. by \cite{3.7} $F$ is $l$-connected if and onl if $Z$ is $l$-connected). Now
consider $g^{op} : Z^{op} \to X^{op}$. We saw in the above that $g^{op}/x \simeq x \setminus g \sim g^{-1}(x)$ and $g^{-1}(x) = \{(x, v) : x \in X_v\} \simeq \{v \in F : x \in X_v\}$. It is similar to the case of $f^{op}$ to see that $g^{op}$ satisfies the conditions of theorem 3.3, hence $g_* : H_*(Z, \mathbb{Z}) \to H_*(X, \mathbb{Z})$ is an isomorphism for $0 \le i \le l$ (resp. by 3.7, $X$ is $l$-connected).

This completes the proof.

Let $K$ be a simplicial complex and $\{K_i\}_{i \in I}$ a family of subcomplexes such that $K = \bigcup_{i \in I} K_i$. The nerve of this family of subcomplexes of $K$ is the simplicial complex $\mathcal{N}(K)$ on the vertex set $I$ so that a finite subset $\sigma \subseteq I$ is in $\mathcal{N}(K)$ if and only if $\bigcap_{i \in \sigma} K_i \neq \emptyset$. The nerve $\mathcal{N}(K)$ of $K$, with the inclusion relation, is a poset. As we already said we can consider a simplicial complex as a poset of its simplices. We leave it to the interested reader to derive the next result from the above theorem.

**Corollary 4.4 (Nerve Theorem).** Let $K$ be a simplicial complex and $\{K_i\}_{i \in I}$ a family of subcomplexes such that $K = \bigcup_{i \in I} K_i$. Suppose every nonempty finite intersection $\bigcap_{j=1}^t K_{i_j}$ is $(l-t+1)$-acyclic (resp. $(l-t+1)$-connected). Then $H_k(K, \mathbb{Z}) \simeq H_k(\mathcal{N}(K), \mathbb{Z})$ for $0 \le k \le l$ (resp. $K$ is $l$-connected if and only if $\mathcal{N}(K)$ is $l$-connected).

**Remark 4.5.** In [8], a special case of the theorem 4.3 is proved. The nerve theorem for a simplicial complex [4.4] in the stated generality, is proved for the first time in [4], see also [3] p. 1850. For more information about different types of nerve theorem and more references about them see [9] p. 1850.

**Lemma 4.6.** Let $F \subseteq O(V)$ satisfy the chain condition and let $\mathcal{G} : F^{op} \to \text{Ab}$ be a functor. Then the natural map $\psi : \bigoplus_{v \in F, \ |v|=1} \mathcal{G}(v) \to H_0(F^{op}, \mathcal{G})$ is surjective.

**Proof.** By definition $C_0(F^{op}, \mathcal{G}) = \bigoplus_{v \in F^{op}} \mathcal{G}(v)$, $C_1(F^{op}, \mathcal{G}) = \bigoplus_{v < v' \in F^{op}} \mathcal{G}(v)$ and we have the chain complex

$$
\cdots \to C_1(F^{op}, \mathcal{G}) \xrightarrow{\partial_1} C_0(F^{op}, \mathcal{G}) \to 0,
$$

where $\partial_1 = d_0^1 - d_1^1$. Again by definition $H_0(F^{op}, \mathcal{G}) = C_0(F^{op}, \mathcal{G})/\partial_1$. Now let $w \in F$ and $|w| \ge 2$. Then there is a $v \in F$, $v \le w$, with $|v| = 1$. So $w < v$ in $F^{op}$, and we have the component $\partial_1|_{\mathcal{G}(w)} : \mathcal{G}(w) \to \mathcal{G}(w) \oplus \mathcal{G}(v)$, $x \mapsto d_0^1(x) - d_1^1(x) = d_0^1(x) - x$. This shows that $\mathcal{G}(w) \subseteq \text{im} \partial_1 + \text{im} \psi$. Therefore $H_0(F^{op}, \mathcal{G})$ is generated by the groups $\mathcal{G}(v)$ with $|v| = 1$.

**Theorem 4.7.** Let $V$ and $T$ be two nonempty sets, $F \subseteq O(V)$ and $X \subseteq O(T)$. Assume $X = \bigcup_{v \in F} X_v$ such that if $v \le w$ in $F$, then $X_w \subseteq X_v$ and let $F, X$ and $X_v$, for every $v \in F$, satisfy the chain condition. Also assume

(i) for every $v \in F, X_v$ is $\min\{l - 1, l - |v| + 1\}$-connected,

(ii) for every $x \in X, A_x := \{v \in F : x \in X_v\}$ is $(l - |x| + 1)$-connected,
(iii) $F$ is $l$-connected.
Then $X$ is $(l - 1)$-connected and the natural map
$$\bigoplus_{v \in F, |v| = 1} (i_v)_* : \bigoplus_{v \in F, |v| = 1} H_1(X_v, \mathbb{Z}) \to H_1(X, \mathbb{Z})$$
is surjective, where $i_v : X_v \to X$ is the inclusion. Moreover, if for every $v$ with $|v| = 1$, there is an $l$-connected $Y_v$ with $X_v \subseteq Y_v \subseteq X$, then $X$ is also $l$-connected.

Proof. If $l = -1$, then everything is easy. If $l = 0$, then for $v$ of length one, $X_v$ is nonempty, so $X$ is nonempty. This shows that $X$ is $(-1)$-connected. Also, every connected component of $X$ intersects at least one $X_w$ and therefore also contains a connected component of an $X_v$ with $|v| = 1$. This gives the surjectivity of the homomorphism
$$\bigoplus_{v \in F, |v| = 1} (i_v)_* : \bigoplus_{v \in F, |v| = 1} H_0(X_v, \mathbb{Z}) \to H_0(X, \mathbb{Z}).$$

Now assume that, for every $v$ of length one, $X_v \subseteq Y_v$ where $Y_v$ is connected. We prove, in a combinatorial way, that $X$ is connected. Let $x, y \in X$, $x \in X_{(v_1)}$ and $y \in X_{(v_2)}$ where $(v_1), (v_2) \in F$. Since $F$ is connected, there is a sequence $(w_1), \ldots, (w_r) \in F$ such that they give a path, in $F$, from $(v_1)$ to $(v_2)$, that is
$$(v_1) \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (v_2)$$
$$(v_1, w_1) \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (w_r, (v_2).$$

Since $Y_{(v_1)}$ is connected, $x \in X_{(v_1)} \subseteq Y_{(v_1)}$ and $X_{(v_1, w_1)} \neq \emptyset$, there is an element $x_1 \in X_{(v_1, w_1)}$ such that there is a path, in $Y_{(v_1)}$, from $x$ to $x_1$. Now $x_1 \in Y_{(w_1)}$. Similarly we can find $x_2 \in X_{(w_1, w_2)}$ such that there is a path, in $Y_{(w_1)}$, from $x_1$ to $x_2$. Now $x_2 \in Y_{(w_2)}$. Repeating this process finitely many times, we find a path from $x$ to $y$. So $X$ is connected.

Hence we assume that $l \geq 1$. As we said in the proof of theorem 4.3, we can assume that $F = F_{\leq l+2}$ and $X = X_{\leq l+2}$ and we define $Z$, $f$ and $g$ as we defined them there. Define the height function $h_{F^{op}}$ on $F^{op}$ as $h_{F^{op}}(v) = l+2-|v|$. As we proved in the proof of theorem 4.3 $f^{op}/v \simeq v \setminus f \sim f^{-1}(v) \simeq X_v$. Thus $f^{op}/v$ is $(h_{F^{op}}(v) - 1)$-connected if $|v| > 1$ and it is $(h_{F^{op}}(v) - 2)$-connected if $|v| = 1$ and also $|\text{Link}_{F^{op}}^+(v)|$ is $(l+1 - h_{F^{op}}(v) - 2)$-connected. By theorem 3.1 we have the first quadrant spectral sequence
$$E^2_{p,q} = H_p(F^{op}, v \mapsto H_q(f^{op}/v, \mathbb{Z})) \Rightarrow H_{p+q}(Z^{op}, \mathbb{Z}).$$
For $0 < q \leq h_{F^{op}}(v) - 2$, $H_q(f^{op}/v, \mathbb{Z}) = 0$. Define $G_q : F^{op} \to Ab$, $G_q(v) = H_q(f^{op}/v, \mathbb{Z})$. Then $G_q(v) = 0$ for $h_{F^{op}}(v) \geq q + 2$, $q > 0$. By lemma 3.4 $H_{p}(F^{op}, G_q) = 0$ for $p \leq l + 1 - (q + 2)$. Therefore $E^2_{p,q} = 0$ for $p + q \leq l - 1$, $q > 0$. If $q = 0$, arguing similarly to the proof of theorem 3.5, we get $E^2_{p,0} = 0$ if $0 < p \leq l - 1$ and $E^2_{0,0} = \mathbb{Z}$. Also by the fact that $F^{op}$ is $l$-connected we get the surjective homomorphism $H_l(F^{op}, v \mapsto$
\[ \hat{H}_0(f^{op}/v, \mathbb{Z}) \to E^2_{l,0}. \] Since \( l \geq 1 \), \( \hat{H}_0(f^{op}/v, \mathbb{Z}) = 0 \) for all \( v \in F^{op} \) with \( \text{ht}_{F^{op}}(v) \geq 1 \) and so \( H_1(F^{op}, v \mapsto \hat{H}_0(f^{op}/v, \mathbb{Z})) = 0 \) by lemma 3.4. Therefore \( E^2_{l,0} = 0 \). Let \( \mathcal{G}'_q : F^{op} \to \mathbb{Ab}, \mathcal{G}'_q(v) = \begin{cases} 0 & \text{if } \text{ht}_{F^{op}}(v) < l + 1 \\ H_q(f^{op}/v, \mathbb{Z}) & \text{if } \text{ht}_{F^{op}}(v) = l + 1 \end{cases} \) and \( \mathcal{G}''_q : F^{op} \to \mathbb{Ab}, \mathcal{G}''_q(v) = \begin{cases} H_q(f^{op}/v, \mathbb{Z}) & \text{if } \text{ht}_{F^{op}}(v) < l + 1 \\ 0 & \text{if } \text{ht}_{F^{op}}(v) = l + 1 \end{cases} \). Then we have the short exact sequence \( 0 \to \mathcal{G}'_q \to \mathcal{G}_q \to \mathcal{G}''_q \to 0 \) and the associated long exact sequence

\[ \cdots \to H_{l+q}(F^{op}, \mathcal{G}'_q) \to H_{l-q}(F^{op}, \mathcal{G}_q) \to H_{l-q}(F^{op}, \mathcal{G}''_q) \to \cdots. \]

If \( q > 0 \), then \( \mathcal{G}''_q(v) = 0 \) for \( 0 < q \leq \text{ht}_{F^{op}}(v) - 1 \) and so by lemma 3.4, \( H_p(F^{op}, \mathcal{G}''_q) = 0 \) for \( p + q \leq l \), \( q > 0 \). Also if \( |v| = 1 \) then \( H_0(f^{op}/v, \mathbb{Z}) = 0 \) for \( 0 < q \leq \text{ht}_{F^{op}}(v) - 2 = l - 1 \). This shows \( \mathcal{G}'_q = 0 \) for \( 0 < q \leq l - 1 \). From the long exact sequence and the above calculation we get, \( E^2_{p,q} = \begin{cases} \mathbb{Z} & \text{if } p = q = 0 \\ 0 & \text{if } 0 < p + q \leq l, q \neq l \end{cases} \)

\[
\begin{array}{c|cccc}
 & l+1 & l & \cdots & E^2_{p,q} \\
\hline
l & * & 0 & \cdots & * \\
0 & 0 & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & * \\
0 & 1 & l & l+1 & 0
\end{array}
\]

Thus for \( 0 \leq p + q \leq l, q \neq l \), \( E^2_{p,q} \simeq E^3_{p,q} \simeq \cdots \sim E^\infty_{p,q} \) and we have a filtration of \( H_i(Z^{op}, \mathbb{Z}) \), \( 0 = F_{-1}H_i \subseteq F_0H_i \subseteq \cdots \subseteq F_lH_i = H_i(Z^{op}, \mathbb{Z}) \) such that \( E^\infty_{p,q} \simeq F_pH_{p+q}/F_{p-1}H_{p+q} \). If \( i \neq 0 \) then \( 0 = E^\infty_{i,0} \simeq F_iH_i/F_{i-1}H_i \), so \( F_iH_k = F_{i-1}H_k \). Therefore \( 0 = F_{-1}H_i \subseteq F_0H_i = \cdots = F_lH_i = H_i(Z^{op}, \mathbb{Z}) \) and \( E^\infty_{0,1} \simeq F_0H_{i+1}/F_{i-1}H_i \simeq H_i(Z^{op}, \mathbb{Z}) \). By definition \( E^r_{0,1} = \ker(d^r_{0,1})/\text{im}(d^r_{r-1,1}) \). Thus there exist \( r \in \mathbb{Z} \) such that \( E^2_{0,1} \to E^3_{0,1} \to \cdots \to E^r_{0,1} \simeq E^{r+1}_{0,1} \sim \cdots \simeq E^\infty_{0,1} \). Hence we get a surjective map \( H_0(f^{op}, v \mapsto H_i(f^{op}/v, \mathbb{Z})) \to H_i(Z^{op}, \mathbb{Z}) \). By lemma 4.6 we have a surjective map \( \bigoplus_{v \in F, |v| = 1} H_1(f^{op}/v, \mathbb{Z}) \to H_1(Z^{op}, \mathbb{Z}) \).

Now consider the map \( g^{op} : Z^{op} \to X^{op} \) and define the height function \( \text{ht}_{X^{op}}(x) = l + 2 - |x| \) on \( X^{op} \). Arguing similarly to the proof of theorem 4.3 one sees that \( g_* : H_k(Z, \mathbb{Z}) \to H_k(X, \mathbb{Z}) \) is an isomorphism for \( 0 \leq k \leq l \). Therefore we get a surjective map \( \bigoplus_{v \in F, |v| = 1} H_1(X_v, \mathbb{Z}) \to H_1(X, \mathbb{Z}) \). We call it \( \psi \). We prove that this map is the same map that we claimed. For \( v \)
of length one consider the commutative diagram of posets
\[
\begin{array}{ccc}
\{v\}^{op} & \longrightarrow & F^{op} \\
\downarrow & & \downarrow \\
f^{-1}\{v\}^{op} & \longrightarrow & Z^{op} \\
\downarrow & & \downarrow \\
X_v^{op} & \longrightarrow & X^{op}
\end{array}
\]
By functoriality of the spectral sequence for the above diagram and the lemma 4.6 we get the commutative diagram
\[
\begin{array}{cccc}
H_l(f^{op}_* / v, Z) & \longrightarrow & \bigoplus_{v \in F, |v| = 1} H_l(f^{op}_* / v, Z) \\
\downarrow & & \downarrow \\
H_0(\{v\}^{op}, v \mapsto H_l(f^{op}_* / v, Z)) & \longrightarrow & H_0(F^{op}, v \mapsto H_l(f^{op}_* / v, Z)) \\
\downarrow & & \downarrow \\
H_l(f^{-1}(v)^{op}, Z) & \longrightarrow & H_l(Z^{op}, Z) \\
\downarrow & & \downarrow \\
H_l(X_v^{op}, Z) & \longrightarrow & H_l(X^{op}, Z)
\end{array}
\]
where \(j_v : f^{op}_* / v \rightarrow f^{op}_* / v\) is the inclusion which is a homotopy equivalence as we already mentioned. It is not difficult to see that the composition of homomorphisms in the left column of the above diagram induces the identity map from \(H_l(X_v, Z)\), the composition of homomorphisms in the right column of above diagram induces the surjective map \(\psi\) and the last row induces the homomorphism \((i_v)_*\). This show that \((i_v)_* = \psi|_{H_l(X_v, Z)}\). This completes the proof of surjectiveness.

Now let for \(v\) of length one \(X_v \subseteq Y_v\) where \(Y_v\) is \(l\)-connected. Then we have the commutative diagram
\[
\begin{array}{ccc}
H_l(X_v, Z) & (i_v)_* & H_l(X, Z) \\
\downarrow & & \downarrow \\
H_l(Y_v, Z)
\end{array}
\]
By the assumption \(H_l(Y_v, Z)\) is trivial and this shows that \((i_v)_*\) is the zero map. Hence by the surjectivity, \(H_l(X, Z)\) is trivial. If \(l \geq 2\), the nerve theorem 4.3 says that \(X\) is simply connected and by the Hurewicz theorem 2.1 \(X\) is \(l\)-connected. So the only case that is left is when \(l = 1\). By theorem 3.7 \(X\) is 1-connected if and only if \(Z\) is 1-connected. So it is enough to prove that \(Z^{op}\) is 1-connected. Note that as we said, we can assume that \(F = F_{\leq 3}\) and \(X = X_{\leq 3}\). Suppose \(F\) is a local system on \(Z^{op}\). Define the functor \(G : F^{op} \rightarrow Ab\), as
\[
G(y) = \begin{cases} 
H_0(f^{op}_* / v, F) & \text{if } |v| = 1, 2 \\
H_0(Link_{F^{op}}^+(v), v' \mapsto H_0(f^{op}_* / v', F)) & \text{if } |v| = 3
\end{cases}
\]
We prove that \(G\) is a local system on \(F^{op}\). Put \(Z_w := g^{-1}(Y_w)\) for \(|w| = 1\). If \(|v| = 1, 2\), then \(f^{op}_* / v\) is 0-connected and \(f^{op}_* / v \subseteq Z_w^{op}\), where \(w \leq v, |w| = 1\). By remark 3.8 we can assume that \(F = \mathcal{E} \circ g^{op}\) where \(\mathcal{E}\) is a local system
on $X^{op}$. Then $\mathcal{F}|_{Z^{op}} = \mathcal{E}|_{Y^{op}} \circ g^{op}|_{Z^{op}}$. Since $Y^{op}_w$ is 1-connected, $\mathcal{E}|_{Y^{op}}$ is a constant local system. This shows that $\mathcal{F}|_{Z^{op}}$ is a constant local system. So $\mathcal{F}|_{f^{op}/v}$ is a constant local system and since $f^{op}/v$ is 0-connected we have $H_0(f^{op}/v, \mathbb{Z}) \simeq \mathcal{F}(x)$, for every $x \in f^{op}/v$. If $|v| = 3$, with an argument similar to the proof of the theorem 3.6 and the above discussion one can get $\mathcal{G}(v) \simeq \mathcal{F}(x)$ for every $x \in f^{op}/v$. This shows that $\mathcal{G}$ is a local system on $F^{op}$. Hence it is a constant local system, because $F^{op}$ is 1-connected. It is easy to see that $\mathcal{F} \simeq \mathcal{G} \circ f$. Therefore $\mathcal{F}$ is a constant system. Since $X$ is connected by our homology calculation, by 3.6 we conclude that $X$ is 1-connected. This completes the proof. □

5. Posets of isotropic and hyperbolic unimodular sequences

Let $R$ be an associative ring with unit. A vector $(r_1, \ldots, r_n) \in R^n$ is called unimodular if there exist $s_1, \ldots, s_n \in R$ such that $\sum s_i r_i = 1$, or equivalently if the submodule generated by this vector is a free summand of the right $R$-module $R^n$. We denote the standard basis of $R^n$ by $e_1, \ldots, e_n$. If $n \leq m$, we assume that $R^n$ is the submodule of $R^m$ generated by $e_1, \ldots, e_n \in R^m$.

We say that a ring $R$ satisfies the stable range condition $(S_m)$, if $m \geq 1$ is an integer so that for every unimodular vector $(r_1, \ldots, r_m, r_{m+1}) \in R^{m+1}$, there exist $t_1, \ldots, t_m$ in $R$ such that $(r_1 + t_1 r_{m+1}, \ldots, r_m + t_m r_{m+1}) \in R^m$ is unimodular. We say that $R$ has stable rank $m$, we denote it with $sr(R) = m$, if $m$ is the least number such that $(S_m)$ holds. If such a number does not exist we say that $sr(R) = \infty$.

An $n \times k$-matrix $B$ with $n < k$ is called unimodular if $B$ has a right inverse. If $B$ is an $n \times k$-matrix and $C \in GL(k, R)$, then $B$ is unimodular if and only if $CB$ is unimodular. A matrix of the form $\begin{pmatrix} 1 & 0 \\ u & B \end{pmatrix}$, where $u$ is a column vector with coordinates in $R$, is unimodular if and only if the matrix $B$ is unimodular.

We say that the ring $R$ satisfies the stable range condition $(S_k^R)$ if for every $n \times (n + k)$-matrix $B$, there exists a vector $r = (r_1, \ldots, r_{n+k-1})$ such that $B\begin{pmatrix} 1 & r \\ 0 & I_{n+k-1} \end{pmatrix} = \begin{pmatrix} u & B' \end{pmatrix}$, where the $n \times (n + k - 1)$-matrix $B'$ is unimodular and $u$ is the first column of the matrix $B$.

**Theorem 5.1** (Vaserstein). For every $k \geq 1$ and $n \geq 1$, a ring $R$ satisfies $(S_k)$ if and only if it satisfies $(S_k^R)$.

**Proof.** The definition of $(S_k^R)$ and this theorem are just the transposed version of theorem [22], Thm. 3′ of Vaserstein. □

Let $\mathcal{U}(R^n)$ denote the subposet of $\mathcal{O}(R^n)$ consisting of unimodular sequences. Recall that a sequence of vectors $v_1, \ldots, v_k$ in $R^n$ is called unimodular when $v_1, \ldots, v_k$ is basis of a free direct summand of $R^n$. Note that if $(v_1, \ldots, v_k) \in \mathcal{O}(R^n)$ and if $n \leq m$, it is the same to say that $(v_1, \ldots, v_k)$
is unimodular as a sequence of vectors in \( R^n \) or as a sequence of vectors in \( R^m \). We call an element \((v_1, \ldots, v_k)\) of \( U(R^n) \) a \( k \)-frame.

**Theorem 5.2** (Van der Kallen). Let \( R \) be a ring with \( sr(R) < \infty \) and \( n \leq m \). Then

(i) \( O(R^n) \cap U(R^m) \) is \((n - sr(R) - 1)\)-connected.

(ii) \( O(R^n) \cap U(R^m)_v \) is \((n - sr(R) - |v| - 1)\)-connected for all \( v \in U(R^m) \).

**Proof.** See [20, Thm. 2.6]. \( \square \)

From now on let \( R \) be a commutative ring. Let \( e_{i,j}(r) \) be the \( 2n \times 2n \)-matrix with \( r \in R \) in the \((i,j)\) place and zero elsewhere. Consider \( Q = \Sigma_{i=1}^n (e_{2i-1,2i}(1) - e_{2i,2i-1}(1)) \in GL(2n,R) \). By definition the symplectic group is the group

\[
Sp(2n,R) := \{ A \in GL(2n,R) : \, ^tAQ A = Q \}.
\]

Let \( \sigma \) be the permutation of the set of natural numbers given by \( \sigma(2i) = 2i - 1 \) and \( \sigma(2i - 1) = 2i \). For \( 1 \leq i, j \leq 2n, \ i \neq j \), and every \( r \in R \) define

\[
E_{i,j}(r) = \begin{cases} 
I_{2n} + e_{i,j}(r) & \text{if } i = \sigma(j) \\
I_{2n} + e_{i,j}(r) - (-1)^{i+j} e_{\sigma(i),\sigma(j)}(r) & \text{if } i \neq \sigma(j) \text{ and } i < j
\end{cases}
\]

where \( I_{2n} \) is the identity element of \( GL(2n,R) \). It is easy to see that \( E_{i,j}(r) \in Sp(2n,R) \), for every \( r \in R \). Let \( ESp(2n,R) \) be the group generated by \( E_{i,j}(r) \), \( r \in R \). We call it elementary symplectic group. Define the bilinear map \( h : R^{2n} \times R^{2n} \to R \) by \( h(x,y) = ^tAQ y = \Sigma_{i=1}^n (x_{2i-1}y_{2i} - y_{2i-1}x_{2i}) \) where \( x = (x_1, \ldots, x_{2n}) \) and \( y = (y_1, \ldots, y_{2n}) \). Clearly \( h(x,x) = 0 \) for every \( x \in R^{2n} \). We say that a subset \( S \) of \( R^{2n} \) is isotropic if for every \( x,y \in S \), \( h(x,y) = 0 \). So every element of \( R^{2n} \) is isotropic. If \( h(x,y) = 0 \), then we say that \( x \) is perpendicular to \( y \). We denote by \( (S) \) the submodule of \( R^{2n} \) generated by \( S \), and by \( (S)^\perp \) the submodule consisting of all the elements of \( R^{2n} \) which are perpendicular to all the elements of \( S \).

Let \( U(R^{2n}) \) be the set of sequences \((x_1, \ldots, x_k)\), \( x_i \in R^{2n} \), such that \( x_1, \ldots, x_k \) form a basis for an isotropic direct summand of \( R^{2n} \). Let \( HU(R^{2n}) \) be the set of sequences \((x_1, y_1), \ldots, (x_k, y_k)\) such that \( (x_1, \ldots, x_k), (y_1, \ldots, y_k) \in U(R^{2n}) \), \( h(x_i,y_j) = \delta_{i,j} \), where \( \delta_{i,j} \) is the Kronecker delta. We call \( U(R^{2n}) \) and \( HU(R^{2n}) \) the poset of isotropic unimodular sequences and the poset of hyperbolic unimodular sequences, respectively. For \( 1 \leq k \leq n \), let \( U(R^{2n},k) \) and \( HU(R^{2n},k) \) be the set of all elements of length \( k \) of \( U(R^{2n}) \) and \( HU(R^{2n}) \) respectively. We call the elements of \( U(R^{2n},k) \) and \( HU(R^{2n},k) \) the isotropic \( k \)-frames and the hyperbolic \( k \)-frames, respectively. Define the poset \( MU(R^{2n}) \) as the set of \((x_1, y_1), \ldots, (x_k, y_k)\) \in \( O(R^{2n} \times R^{2n}) \) such that, (i) \((x_1, \ldots, x_k) \in U(R^{2n})\), (ii) for each \( i \), either \( y_i = 0 \) or \( (x_i, y_i) = \delta_{ii} \), (iii) \((y_1, \ldots, y_k) \) is isotropic. We identify \( U(R^{2n}) \) with \( MU(R^{2n}) \cap O(R^{2n} \times \{0\}) \) and \( HU(R^{2n}) \) with \( MU(R^{2n}) \cap O(R^{2n} \times (R^{2n} \setminus \{0\})) \).
Lemma 5.3. Let $R$ be a ring with $sr(R) < \infty$. If $n \geq sr(R) + k$ then $ESp(2n,R)$ acts transitively on $\mathcal{U}(R^{2n}, k)$ and $H\mathcal{U}(R^{2n}, k)$.

Proof. The proof is by induction on $k$. If $k = 1$, the fact that $ESp(2n,R)$ acts transitively on $\mathcal{U}(R^{2n}, 1)$, is due to Vaserstein [21 Thm. 2.]. The transitivity of action of $H\mathcal{U}(R^{2n}, 1)$ is also well known and easy. In fact if $(x, y)$ is a hyperbolic pair then there is an $E \in ESp(2n,R)$ such that $Ex = e_1$. Thus $E y = (r_1, 1, r_3, \ldots, r_{2n})$. Now if $E' = E_{1,2}(-r_1) \prod_{i=3}^{2n} E_{i,2}(-r_i)$, then $E'Ex = e_1$ and $E'E y = e_2$. This shows that $ESp(2n,R)$ acts transitively on $H\mathcal{U}(R^{2n}, 1)$. The rest is an easy induction and the fact that for every isotropic $k$-frame $(x_1, \ldots, x_k)$ there is an isotropic $k$-frame $(y_1, \ldots, y_k)$ such that $((x_1, y_1), \ldots, (x_k, y_k))$ is a hyperbolic $k$-frame. □

Lemma 5.4. Let $R$ be a ring with $sr(R) < \infty$, and let $n \geq sr(R) + k$. Let $((x_1, y_1), \ldots, (x_k, y_k)) \in H\mathcal{U}(R^{2n})$, $(x_1, \ldots, x_k) \in \mathcal{U}(R^{2n})$ and $V = \langle x_1, \ldots, x_k \rangle$. Then

(i) $\mathcal{U}(R^{2n})_{\langle x_1, \ldots, x_k \rangle} \simeq \mathcal{U}(R^{2(n-k)}) V$,

(ii) $H\mathcal{U}(R^{2n}) \cap H\mathcal{U}(R^{2n})_{\langle x_1, \ldots, x_k \rangle} \simeq H\mathcal{U}(R^{2n})_{\langle x_1, \ldots, x_k \rangle} V$,

(iii) $H\mathcal{U}(R^{2n})_{\langle x_1, y_1, \ldots, x_k, y_k \rangle} \simeq H\mathcal{U}(R^{2(n-k)})$.

Proof. See [6], the proof of lemma 3.4 and the proof of Thm. 3.2. □

Lemma 5.5. Let $R$ be a ring with $sr(R) < \infty$. Assume $n \geq sr(R) + k$ and $(v_1, \ldots, v_k) \in U(R^{2n})$. Then there is a hyperbolic basis $(x_1, y_1, \ldots, x_n, y_n)$ of $R^{2n}$ such that $v_1, \ldots, v_k \in \langle x_1, y_1, \ldots, x_{k-1}, y_{k-1}, x_k \rangle$.

Proof. By lemma 5.3 we can assume that $v_1 = e_1$. The proof is by induction on $k$. If $k = 1$, everything is trivial. Consider the $k \times 2n$-matrix $A$, whose $i$-th row is the vector $v_i$. Let $B$ be the $(k - 1) \times (2n - 1)$-matrix obtained from $A$ by deleting the first column and the first row. Since $A$ is unimodular, $B$ is unimodular too. By theorem 5.1 there exist a vector $s = (s_3, \ldots, s_{2n})$ such that $B \left( \begin{array}{cc} 1 & s \\ 0 & I_{2n-2} \end{array} \right) = \left( \begin{array}{cc} u & C \end{array} \right)$, where the $(k - 1) \times (2n - 2)$-matrix $C$ is unimodular and $u$ is the first column of the matrix $B$. Now let $E = \prod_{i=3}^{2n} E_{2,i}(s_i) \in ESp(2n,R)$. An easy computation shows that

\[ AE = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \ast & \ast & \cdots & \ast \\ \vdots & \vdots & \ddots & \vdots \\ \ast & \ast & \cdots & C \end{pmatrix}. \]

Denote the rows of $C$ by $w_1, \ldots, w_{k-1} \in R^{2n-2}$. Since $n - 1 \geq sr(R) + (k - 1)$, by induction hypothesis there exist a hyperbolic basis $(x_2, y_2, \ldots, x_n, y_n)$ of $R^{2n-2}$ such that $w_1 \in \langle x_2, y_2, \ldots, x_{k-1}, y_{k-1}, \rangle$. If we consider $R^{2n-2}$ as a subspace of $R^{2n}$ generated by $e_3, \ldots, e_{2n}$ and if $x_1 = e_1$ and $y_1 = e_2$, then $v_1 \in \langle x_1, y_1, \ldots, x_{k-1}, y_{k-1}, x_k \rangle$. This completes the proof. □
Lemma 5.6. Let $R$ be a ring with $sr(R) < \infty$. Let $(v_1, \ldots, v_k) \in \mathcal{U}(R^{2n})$. If $n \geq sr(R) + k$ then

(i) $\mathcal{U}(R^{2n})_{(v_1, \ldots, v_k)} \cap \mathcal{O}(\langle v_1, \ldots, v_k \rangle^\perp)$ is $(2n - sr(R) - 2k - 1)$-connected.

(ii) $\mathcal{U}(R^{2n})_{(v_1, \ldots, v_k, w_1, \ldots, w_r)} \cap \mathcal{O}(\langle v_1, \ldots, v_k \rangle^\perp)$ is $(2n - sr(R) - 2k - r - 1)$-connected, for every $(w_1, \ldots, w_r) \in \mathcal{U}(R^{2n})_{(v_1, \ldots, v_k)}$.

Proof. It is well known and easier than lemma 5.3 that $GL(2n, R)$ acts transitively on $k$-frames. Therefore let us choose a basis $b_1, \ldots, b_{2n}$ of $R^{2n}$ so that $(b_{2n-k+1}, \ldots, b_{2n}) = (v_1, \ldots, v_k)$. Let $c_1, \ldots, c_{2n}$ be the dual basis: $h(b_i, c_j) = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta. Then $\langle v_1, \ldots, v_k \rangle^\perp = c_1 R + \cdots + c_{2n-k} R$, so we may identify $\mathcal{O}(\langle v_1, \ldots, v_k \rangle^\perp)$ with $\mathcal{O}(R^{2n-k})$ after a change of basis. Now by theorem 5.2, the poset $\mathcal{U}(R^{2n})_{(v_1, \ldots, v_k)} \cap \mathcal{O}(R^{2n-k})$ is $(2n - sr(R) - 2k - 1)$-connected. The proof of (ii) is similar to the proof of (i). 

For a real number $l$, by $\lfloor l \rfloor$ we mean the largest integer $n$ with $n \leq l$.

Theorem 5.7. The poset $\mathcal{IU}(R^{2n})$ is $\lfloor \frac{n - sr(R) - 2}{2} \rfloor$-connected and $\mathcal{IU}(R^{2n})_x$ is $\lfloor \frac{n - sr(R) - \lfloor x \rfloor - 2}{2} \rfloor$-connected for every $x \in \mathcal{IU}(R^{2n})$.

Proof. If $n \leq sr(R)$, the result is clear, so let $n > sr(R)$. Let $X_v = \mathcal{IU}(R^{2n}) \cap \mathcal{U}(R^{2n})_v \cap \mathcal{O}(\langle v \rangle^\perp)$, for every $v \in \mathcal{U}(R^{2n})$, and put $X := \bigcup_{v \in F} X_v$ where $F = \mathcal{U}(R^{2n})$. It follows from lemma 5.3 that $\mathcal{IU}(R^{2n})_{\leq n - sr(R)} \subseteq X$. So to treat $\mathcal{IU}(R^{2n})$, it is enough to prove that $X$ is $\lfloor \frac{n - sr(R) - 2}{2} \rfloor$-connected. First we prove that $X_v$ is $\lfloor \frac{n - sr(R) - \lfloor v \rfloor - 2}{2} \rfloor$-connected for every $v \in F$. The proof is by descending induction on $|v|$. If $|v| > n - sr(R)$, then $\lfloor \frac{n - sr(R) - |v| - 2}{2} \rfloor < -1$. In this case there is nothing to prove. If $n - sr(R) - 1 \leq |v| \leq n - sr(R)$, then $\lfloor \frac{n - sr(R) - |v| - 2}{2} \rfloor = -1$, so we must prove that $X_v$ is nonempty. This follows from lemma 5.5. Now assume $|v| \leq n - sr(R) - 2$ and assume by induction that $X_w$ is $\lfloor \frac{n - sr(R) - |w| - 2}{2} \rfloor$-connected for every $w$, with $|w| > |v|$. Let $l = \lfloor \frac{n - sr(R) - |v| - 2}{2} \rfloor$, and observe that $n - |v| - sr(R) \geq l + 2$. Put $T_w = \mathcal{IU}(R^{2n}) \cap \mathcal{U}(R^{2n})_w \cap \mathcal{O}(\langle vw \rangle^\perp)$ where $w \in G_v = \mathcal{U}(R^{2n})_v \cap \mathcal{O}(\langle v \rangle^\perp)$ and put $T := \bigcup_{w \in G_v} T_w$. It follows by lemma 5.5 that $(X_v)_{\leq n - |v| - sr(R)} \subseteq T$. So it is enough to prove that $T$ is $l$-connected. The poset $G_v$ is $l$-connected by lemma 5.6. By induction, $T_w$ is $\lfloor \frac{n - sr(R) - |v| - |w| - 2}{2} \rfloor$-connected. But $\min\{l - 1, l - |w| + 1\} \leq \lfloor \frac{n - sr(R) - |v| - |w| - 2}{2} \rfloor$, so $T_w$ is $\min\{l - 1, l - |w| + 1\}$-connected. For every $y \in T$, $A_y = \{w \in G_v : y \in T_w\}$ is isomorphic to $\mathcal{U}(R^{2n})_{vy} \cap \mathcal{O}(\langle vy \rangle^\perp)$ so by lemma 5.6, it is $(l - |y| + 1)$-connected. Let $w \in G_v$ with $|w| = 1$. For every $z \in T_w$, we have $wz \in X_v$, so $T_w$ is contained in a cone, call it $C_w$, inside $X_v$. Put $C(T_w) = T_w \cup (C_w)_{\leq n - |v| - sr(R)}$. Thus $C(T_w) \subseteq T$. The poset $C(T_w)$ is $l$-connected because $C(T_w)_{\leq n - |v| - sr(R)} = (C_w)_{\leq n - |v| - sr(R)}$. By theorems 5.2 and 4.7, $T$ is $l$-connected. In other words, we have now shown that $X_v$ is $\lfloor \frac{n - sr(R) - |v| - 2}{2} \rfloor$-connected. By knowing
this one can prove, in a similar way, that $X$ is $\left\lfloor \frac{n - sr(R) - 2}{2} \right\rfloor$-connected. (Just pretend that $|x| = 0$.)

Now consider the poset $\mathcal{IU}(R^{2n})_x$ for an $x = (x_1, \ldots, x_k) \in \mathcal{IU}(R^{2n})$. The proof is by induction on $n$. If $n = 1$, everything is easy. Similarly, we may assume $n - sr(R) - |x| \geq 0$. Let $l = \left\lfloor \frac{n - sr(R) - |x| - 2}{2} \right\rfloor$. By lemma 5.4, $\mathcal{IU}(R^{2n})_x \cong \mathcal{IU}(R^{2(n-|x|)})(V)$, where $V = (x_1, \ldots, x_k)$. In the above we proved that $\mathcal{IU}(R^{2(n-|x|)})$ is $l$-connected and by induction, the poset $\mathcal{IU}(R^{2(n-|x|)})_y$ is $\left\lfloor \frac{n - |y| - sr(R) - |y| - 2}{2} \right\rfloor$-connected for every $y \in \mathcal{IU}(R^{2(n-|x|)})$. But $l - |y| \leq \left\lfloor \frac{n - |x| - sr(R) - |y| - 2}{2} \right\rfloor$. So $\mathcal{IU}(R^{2(n-|x|)})(V)$ is $l$-connected by lemma 4.1. Therefore $\mathcal{IU}(R^{2n})_x$ is $l$-connected. □

**Theorem 5.8.** The poset $\mathcal{HU}(R^{2n})$ is $\left\lfloor \frac{n - sr(R) - 3}{2} \right\rfloor$-connected and $\mathcal{HU}(R^{2n})_x$ is $\left\lfloor \frac{n - sr(R) - |x| - 3}{2} \right\rfloor$-connected for every $x \in \mathcal{HU}(R^{2n})$.

**Proof.** The proof is by induction on $n$. If $n = 1$, then everything is trivial. Let $F = \mathcal{HU}(R^{2n})$ and $X_v = \mathcal{HU}(R^{2n}) \cap \mathcal{IU}(R^{2n})_v$, for every $v \in F$. Put $X := \bigcup_{v \in F} X_v$. It follows from lemma 5.4 that $\mathcal{HU}(R^{2n})_{\leq n-sr(R)} \subseteq X$.

Thus to treat $\mathcal{HU}(R^{2n})$, it is enough to prove that $X$ is $\left\lfloor \frac{n - sr(R) - 3}{2} \right\rfloor$-connected, and we may assume $n \geq sr(R) + 1$. Take $l = \left\lfloor \frac{n - sr(R) - 3}{2} \right\rfloor$ and $V = (v_1, \ldots, v_k)$, where $v = (v_1, \ldots, v_k)$. By lemma 5.4 there is an isomorphism $X_v \cong \mathcal{HU}(R^{2(n-|v|)})(V \times V)$, if $n \geq sr(R) + |v|$. By induction $\mathcal{HU}(R^{2(n-|v|)})(V \times V)$ is $\left\lfloor \frac{n - |v| - sr(R) - |v| - 3}{2} \right\rfloor$-connected and again by induction $\mathcal{HU}(R^{2(n-|v|)})(V \times V)$ is $\left\lfloor \frac{n - |v| - sr(R) - |v| - 3}{2} \right\rfloor$-connected for every $y \in \mathcal{HU}(R^{2(n-|v|)})(V \times V)$. So by lemma 4.1, $X_v$ is $\left\lfloor \frac{n - |v| - sr(R) - 3}{2} \right\rfloor$-connected. Thus the poset $X_v$ is $\min\{l - 1, l - |v| + 1\}$-connected. Let $x = ((x_1, y_1), \ldots, (x_k, y_k))$. It is easy to see that $A_x = \{v \in F : x \in X_v \} \cong \mathcal{IU}(R^{2n})_{(x_1, \ldots, x_k)}$. By the above theorem $A_x$ is $\left\lfloor \frac{sr(R) - k - 2}{2} \right\rfloor$-connected. But $l - |x| + 1 \leq \left\lfloor \frac{n - sr(R) - k - 2}{2} \right\rfloor$, so $A_x$ is $\left\lfloor \frac{n - sr(R) - k - 2}{2} \right\rfloor$-connected. Let $v = (v_1) \in F$, $|v| = 1$, and let $D_v := \mathcal{HU}(R^{2n})(v_1, w_1) \cong \mathcal{HU}(R^{2(n-1)})$ where $w_1 \in R^{2n}$ is a hyperbolic dual of $v_1 \in R^{2n}$. Then $D_v \subseteq X_v$ and $D_v$ is contained in a cone, call it $C_v$, inside $\mathcal{HU}(R^{2n})$. Take $C(D_v) := D_v \cup (C_v)_{\leq n-sr(R)}$. By induction $D_v$ is $\left\lfloor \frac{n - 1 - sr(R) - 3}{2} \right\rfloor$-connected and so $(l - 1)$-connected. Let $Y_v = X_v \cup C(D_v)$. By the Mayer-Vietoris theorem and the fact that $C(D_v)$ is $l$-connected, we get the exact sequence

$$\tilde{H}_l(D_v, \mathbb{Z}) \xrightarrow{(i_v)_*} \tilde{H}_l(X_v, \mathbb{Z}) \to \tilde{H}_l(Y_v, \mathbb{Z}) \to 0.$$
lemma 4.1(ii), \( \pi_1(Y_v, x) \) is trivial. Thus by the Hurewicz theorem 2.1, \( Y_v \) is \( l \)-connected. By having all this we can apply theorem 4.7 and so \( X \) is \( l \)-connected. The fact that \( H\mathcal{U}(R^{2n})_x \) is \( \lfloor \frac{n - \text{sr}(R) - |x| - 3}{2} \rfloor \)-connected follows from the above and lemma 5.4. □

Remark 5.9. Charney in [6, 2.10], conjectured that \( \mathcal{I}\mathcal{U}(R^{2n}) \), with respect to certain bilinear forms, is highly connected. Our theorem 5.7 proves this conjecture for the above bilinear form. Also, by assuming the high connectivity of the \( \mathcal{I}\mathcal{U}(R^{2n}) \), she proved that \( H\mathcal{U}(R^{2n}) \) is highly connected. Our proof is different and relies on our theory, but we use ideas from her paper, such as the lemma 5.4 and her lemma 4.1, which is a modified version of work of Maazen.

Remark 5.10. One should notice that in the above, we used heavily the fact that every unimodular vector (in fact any vector) in \( R^{2n} \) is isotropic. So our arguments don’t work for orthogonal groups. One needs to do more. Consider the bilinear form \( h': R^{2n} \times R^{2n} \to R \) defined by \( h'(x, y) = \sum_{i=1}^{n} (x_{2i-1} y_{2i} + y_{2i-1} x_{2i}) \). Let \( T = \{ x \in R^{2n} : h'(x, x) = 0 \} \) and \( \mathcal{U}'(R^{2n}) = \mathcal{U}(R^{2n}) \cap O(T) \). If \( \mathcal{U}'(R^{2n}) \) is highly connected then we have similar results for the poset \( \mathcal{I}\mathcal{U}(R^{2n}) \) arising from the form \( h' \). So we formulate the following question:

Question. Is \( \mathcal{U}'(R^{2n}) \) is highly connected for example \( (n - \text{sr}(R) - 1) \)-connected?

6. Homology stability

From theorem 5.8 one can get the homology stability of symplectic groups as Charney proved in [6, Sec. 4]. Here we only formulate the theorem and for the proof we refer to Charney’s paper.

Theorem 6.1. Let \( R \) be a commutative ring with \( \text{sr}(R) < \infty \). Then for every abelian group \( L \) the homomorphism \( \psi_n : H_i(\text{Sp}(2n, R), L) \to H_i(\text{Sp}(2n + 2, R), L) \) is surjective for \( n \geq 2i + \text{sr}(R) + 2 \) and bijective for \( n \geq 2i + \text{sr}(R) + 3 \), where \( \psi_n : \text{Sp}(2n, R) \to \text{Sp}(2n + 2, R), A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_2 \end{pmatrix} \).

Proof. See [6, Sec. 4]. □

Remark 6.2. To prove homology stability of this type one only needs high acyclicity of the corresponding poset. But usually this type of posets are also highly connected and it looks that it is a tradition to give a proof of high connectivity of the posets. In this paper we follow the tradition. In particular we wished to confirm the conjecture of Charney [6, 2.10] for the particular bilinear form that we considered.

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