COUNTING CURVES ON SURFACES: 
A GUIDE TO NEW TECHNIQUES AND RESULTS

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1. INTRODUCTION

1.1. Abstract and summary. A series of recent results solving classical enumerative problems for curves on rational surfaces is described. Impulse to the subject came from recent ideas from quantum field theory leading to the definition of quantum cohomology. As a by-product, formulas enumerating rational curves on certain varieties were derived from the properties of certain generating functions representing the free energy of certain topological field theories. A mathematically acceptable construction of quantum cohomology came soon afterwards ([RT] and [KM]-[K]). Different proofs of some of these formulas were provided later (in [CH1] and [CH2]) using different methods that could be generalized to cases (such as Hirzebruch surfaces) for which the quantum cohomology theory did not give enumerative results. For higher genera, the connection between enumerative geometry and quantum cohomology or quantum field theory is still largely conjectural. On the other hand, a recursive formula enumerating plane curves of any genus has been recently proved using purely algebro-geometric techniques ([CH3]). Moreover a generating function exists together with a differential equation implying such a recursion ([G]).

The enumerative problem is precisely stated in the introduction.

The second chapter contains a short description of the relation with Quantum Cohomology and Kontsevich’s formula for plane rational curves. A discussion of enumerative problems for Hirzebruch surfaces concludes this part, which is entirely dedicated to rational curves.

The general case of plane curves of any genus is described in the third chapter, focussing on the results of [CH3]. The main recursive formula of that paper is explained, together with an outline of the proof and a description of the generating function found in [G]. At the end, there is a discussion of generalizations of the procedure of [CH3] to other varieties.

Chapters 2 and 3 are completely independent from each other.

In the fourth and last chapter various enumerative techniques are applied to the concrete example of counting rational plane cubics through 8 general points.

1.2. Terminology. We work with complex, projective algebraic varieties. Let $S$ be a smooth, minimal rational surface (that is, $S$ is either the projective plane $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n$) and let $D$ be a curve on $S$. The linear system $|D|$ of all curves linearly equivalent to $D$ is a projective space whose dimension is given by Riemann-Roch theorem (if $S$ is the plane then $|D|$ is the $\frac{d(d+3)}{2}$-dimensional projective space of all curves
of degree \( d = \deg D \). Moreover the genus \( p_a(D) \) of a smooth curve in \( |D| \) is constant, computable by the adjunction formula (equal to \( \left( \frac{d-1}{2} \right) \) for a smooth plane curve of degree \( d \)).

We study the geometry of certain closed subvarieties of \( |D| \), the so-called “Severi varieties”. The following notation is used throughout the paper. Severi varieties are denoted by the symbol “\( V \)”, suitably decorated; for example, by \( V_g(D) \) we denote the Severi variety defined to be the closure of the locus of all irreducible curves in \( |D| \) having fixed geometric genus \( g \) (see below). All of these Severi varieties have pure dimension (that is, all of their irreducible components have the same dimension); we use a decorated “\( r \)” for the dimension of \( V \) (equally decorated); for example, \( r_g(D) := \dim V_g(D) \). Analogously, we use a decorated “\( N \)” for the degree of \( V \) as a subvariety of \( |D| \), for example, \( N_g(D) := \deg V_g(D) \).

1.3. The problem. For a given integer \( g \) (with \( 0 \leq g \leq p_a(D) \) to avoid trivial cases), consider \( V_g(D) \subset |D| \) defined above. Also, we denote by \( V_g(d) \) the variety of irreducible plane curves of genus \( g \) and degree \( d \). These varieties were first introduced by Severi for \( \mathbb{P}^2 \) (in [S]) and have been object of much study. For example, it is known that their general point represents a curve with \( \delta = p_a(D) - g \) nodes (and no other singularities, a fortiori), and that their dimension satisfies \( r_g(D) = -(K_S \cdot D) + g - 1 = \dim |D| - \delta \), where \( K_S \) is the canonical class of \( S \).

This is to say that for any \( r_g(D) \) general points on the surface \( S \) there is a finite number of curves of \( |D| \) having geometric genus \( g \) and passing through such points. Such a number is the degree of \( V_g(D) \) as a subvariety of \( |D| \) and it is precisely what we want to compute. More generally, we will consider a larger class of Severi varieties and we will describe various ways to arrive at formulas for their degrees.

The simplest example is that of the plane, where we are asking:

**Question.** How many plane curves of degree \( d \) and genus \( g \) pass through \( 3d + g - 1 \) general points?

Until recently there was an explicit formula answering this classical problem, only in some special cases, namely for plane curves having few nodes (up to 6 nodes, as far as the author knows; due to [KP] and [Va]; see [DI] for a list of formulas). For example, it is not hard to deal with curves with only one node, and to show that for each \( d \) there are \( 3(d-1)^2 \) of them through the appropriate number of points (see 4.1).

An interesting recursive procedure to compute \( N_g(d) \) in general was suggested in [R1], (but notice that the formula there is not correct, see also [R2] and [Ch]). Ran constructs a family where the plane degenerates to a reducible surface (called a “fan”). Correspondingly he gets a family of Severi varieties of which he studies the flat limit. This procedure can be viewed as a recursion because such a degeneration of \( \mathbb{P}^2 \) to a reducible surface induces a family of irreducible curves specializing to a reducible one.

In fact, the common part to most ways of approaching these problems is the use of a technique where the irreducible curves that one is trying to enumerate degenerate to reducible curves, so that one obtains an inductive formula.
2. RATIONAL CURVES

2.1. Relation with Quantum Cohomology. Interest and enthusiasm for these problems was revived by recent ideas from quantum field theory which led to various enumerative predictions for rational curves on varieties.

It was proposed by Gromov to study a series of new invariants of a given variety \( V \). This was also done by Witten (cf. [W]) on the base of physical intuition, using intersection theory on \( \overline{M}_{g,n} \), the moduli space of Deligne-Mumford stable curves of genus \( g \), with \( n \) marked points, a space well known by string theorists.

These invariants, which we call “Gromov-Witten invariants” (they are also called “topological \( \sigma \)-models” or “mixed invariants” depending on context) depend on the geometry of curves lying on \( V \). For certain varieties (such as projective spaces) a subclass of them corresponds to enumerative invariants; for example, the degrees \( N_0(d) \) for plane rational curves are Gromov-Witten invariants.

We restrict this brief description to the case of curves of genus zero, to avoid parts of the theory that are still at a conjectural state, and because this is where there is a clear link with algebraic geometry: the above mentioned enumerative predictions all have to do with rational curves. (See [G] for some very recent developments for genus one.)

It was conjectured (and it is now proved) that the Gromov-Witten invariants satisfy a series of properties; the most important of them is the so-called “splitting principle” or “composition law”, which gives a way of computing these invariants recursively. On their existence one can base the construction of a family of quantum ring structures on the cohomology ring of \( V \), deforming the standard cup product; the associativity of this quantum product is a consequence of the splitting principle.

2.2. A formula for plane rational curves. In 1993 Kontsevich derived a beautiful formula for rational curves in the plane, assuming the associativity of the quantum product for \( \mathbb{P}^2 \) (not yet proved at the time).

Kontsevich’s formula. For \( d \geq 2 \)

\[
N_0(d) = \sum_{d_1+d_2=d} N_0(d_1)N_0(d_2)d_1d_2 \left[ \binom{3d-4}{3d_1-2} d_1d_2 - \binom{3d-4}{3d_1-3} d_2^2 \right].
\]

This, together with the basic fact \( N_0(1) = 1 \), (i.e. there is a unique line through two distinct points) allows one to compute degrees of all Severi varieties of rational curves in the plane.

Here is, briefly, how such a formula was discovered. Using the Gromov-Witten invariants, one defines a generating function (the “potential”) on the cohomology ring of \( V \), which completely encodes the quantum ring structure. This is part of the basic set-up of Quantum Cohomology, I will not say much about it and refer to [KM] for the details. For the special case of \( \mathbb{P}^2 \) such a generating function is, for \( \Delta \in H^*(\mathbb{P}^2) \):

\[
\Phi(\Delta) = \Phi^{cl} + \Phi^q = \frac{1}{2}(x_0^2x_2 + x_0x_1^2) + \sum_{d=1}^{\infty} N_0(d) \frac{x_0^{3d-1}}{(3d-1)!} e^{dx_1}
\]
where the variables $x_0, x_1, x_2$ are the coefficients of $\Delta = x_0 \Delta_0 + x_1 \Delta_1 + x_2 \Delta_2$, with $\Delta_0$ the identity, and $\Delta_1$ and $\Delta_2$ the duals of the class of the line and the point respectively.

For each $\Delta$ one gets a quantum ring structure on $H^*(\mathbb{P}^2)$ which is defined using the rank-3 tensor of all derivatives $(\partial_j \partial_i \partial_k \Phi)_{|\Delta}$. The summand $\Phi^{cl} = \frac{1}{2}(x_0^2 x_2 + x_0 x_1^2)$ gives the “generating function” for the classical cup product, which of course does not depend on $\Delta$.

It is worth mentioning that a “full” potential should be defined as to include a term for every genus (so that $\Phi^q$ above corresponds to the genus zero part), but it is still an open question how to do that in a geometrically meaningful way.

The crucial observation is that the quantum product is associative if and only if the potential $\Phi$ satisfies the WDVV differential equation, that is to say, if and only if the following identity holds:

$$\Phi_{222}^q = (\Phi_{112}^q)^2 - \Phi_{111}^q \Phi_{122}^q.$$  

Finally, as the reader can check by a straightforward computation, the above identity implies Kontsevich’s formula.

Complete proofs of Kontsevich’s formula were given independently by Ruan and Tian (in [RT]) and by Kontsevich and Manin (in [KM] and [K]), using rather sophisticated techniques. In both cases the goal was to give a mathematically rigorous definition of the Gromov-Witten invariants, so that they satisfy the required properties (especially the composition law!). In [RT] this is done using symplectic topology and the Gromov theory of pseudo-holomorphic curves. In [KM] the authors follow an algebro-geometric approach and use the existence of a good compactification of the moduli space of maps from $\mathbb{P}^1$ to $V$ (constructed in the later paper [K]). We notice that these techniques work for a larger class of varieties; in the specific case of surfaces they give enumerative results for the plane, for $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, for $\mathbb{F}_1$ and for blow-ups of $\mathbb{P}^2$ at general points.

2.3. Hirzebruch surfaces In the First Reconstruction Theorem (in [KM]) there is the sketch for a heuristic argument to obtain the formula above. More recently we showed (in [CH1]) that such a heuristic argument could be made into a completely rigorous proof involving only classical tools, with the advantage that this old fashioned approach (which we call the “cross ratio” method) leads to formulas for rational curves on any rational surface $S$, that could not be found otherwise.

The methods of [KM] and [RT] do not answer our enumerative questions for the Hirzebruch surfaces $\mathbb{F}_n$. In fact $\mathbb{F}_n$ is not “convex” which implies that there does not exist a well-behaved compact moduli space of maps as it exists for $\mathbb{P}^2$ (convexity here is defined follows: a variety $V$ is convex if for every stable map $f: \mathbb{P}^1 \to V$ we have $H^1(f^* T_V) = 0$ where $T_V$ is the tangent bundle of $V$). On the other hand, the techniques of [RT] only depend on the symplectic type of the surface; the degrees $N_0(D)$ coincide with certain Gromov-Witten numbers as long as $S$ is $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$. But while the Gromov-Witten numbers are symplectic-invariants, the Severi degrees $N_0(D)$ are not, as we can see in the following example. There are only two symplectic types of Hirzebruch surfaces $\mathbb{F}_n$, depending on the parity of $n$. So that we can compare $\mathbb{F}_0$ with $\mathbb{F}_2$, which must have the same Gromov-Witten invariants. On both surfaces, consider $D$ equal to the anticanonical class; then one can show (using any of the techniques described in 4.1 or 4.2)
that $N_0(-K_{\mathbb{P}_1}) = 12$ while $N_0(-K_{\mathbb{P}_2}) = 10$.

We describe the cross ratio method in an example later (cf. 4.2). Here we just give a summary of the results that it gives. First, for $\mathbb{F}_n$, with $n \leq 2$, and blow-ups of the plane at general points one obtains an inductive formula (completely analogous to Kontsevich’s formula) expressing $N_0(D)$ as a function of simply $N_0(D')$, with $D' < D$ (that is $D - D'$ effective and non-zero). This corresponds to the fact that for such surfaces one can construct degenerations of rational curves in $|D|$ whose degenerate fibers will be reducible rational curves all of whose components are general points of Severi varieties $V_0(D')$.

For $n \geq 3$ a new phenomenon complicates things: this is the occurrence in codimension one of degenerate loci that are no longer of type $V_0(D')$; these will instead be loci of curves satisfying certain tangency conditions. More precisely, for $\mathbb{F}_n$ let us define the tangential Severi varieties $V^i_0(D) \subset V_0(D)$ as the closure of the set of curves in $V_0(D)$ that have a point of contact of order $i$ with the exceptional curve $E$ of $\mathbb{F}_n$ (that is, the unique curve $E$ having self intersection $-n$). Then the cross ratio method gives a formula for $N_0(D)$ in terms of the degree $N^i_0(D')$ of $V^i_0(D')$, for suitable $i$ (see [CH1]). Therefore to have a complete picture one should also compute the degrees of these tangential Severi varieties.

There is another ad-hoc technique to deal with rational curves, the so-called “rational fibration method” (illustrated in example 4.2). This is described in [CH2] where it is applied to obtain a complete set of formulas for $\mathbb{F}_3$. The picture appears to be essentially the same as for the cross ratio approach: we get inductive formulas expressing $N_0(D)$ in terms of degrees of tangential Severi varieties. Such a method again focuses on the study of one-parameter families of rational curves, and it is based on the very basic fact that two line bundles on $\mathbb{P}^1$ are isomorphic if they have the same degree. See 4.2 for an example illustrating this method.

3. HIGHER GENERA

3.1. The degeneration method of [CH3]. We now consider curves of any genus, for which none of the methods described in the previous chapter seem to work so far. We shall give an answer to the Question in 1.2; from now on, we take $S = \mathbb{P}^2$. Fix a line $L \subset \mathbb{P}^2$ once and for all, the inductive technique now uses degenerations whose special fiber is forced to contain an increasing number of points of $L$, until it becomes reducible, having to contain $L$ itself as a component. What remains of the special fiber is a curve of degree $d - 1$ (which may very well be reducible) so that one can then use induction.

This procedure is different from the ones used for rational curves for a crucial reason. For curves of genus 0 one uses generic degenerations (such as: the family of all rational plane curves of degree $d$ through $3d - 1$ general points). Here we use a special type of degeneration, by using the device of placing some of the points on a fixed line.

Therefore we start by considering a larger class of Severi varieties, including certain tangential loci. This is inspired by the case of Hirzebruch surfaces described above, the role of the exceptional curve $E$ is played by the line $L$. For rational curves on $\mathbb{F}_n$ we were
forced to consider curves satisfying tangency conditions with respect to \( E \), getting more complicated sets of recursions. For curves of any genus in the plane we actually choose to take tangential loci from the beginning, and this way we get a rather simple formula.

3.2. A formula for plane curves of any genus. Let us define generalized Severi varieties. Let \( \alpha = (\alpha_1, \ldots, \alpha_h) \) and \( \beta = (\beta_1, \ldots, \beta_k) \) be strings of nonnegative integers. Fix \( \sum \alpha_j \) general points on \( L \) denoted by \( \{p_j^{(i)}\}_{1 \leq j \leq \alpha_i} \). Assume that \( \sum i\alpha_i + \sum i\beta_i = d \).

The generalized Severi variety \( V^{d, \delta}[\alpha, \beta] \) is defined as the closure of the locus of reduced plane curves of degree \( d \) with \( \delta \) nodes (hence the geometric genus is \( g = \binom{d-1}{2} - \delta \)) which

(i) do not contain \( L \),
(ii) have contact of order \( i \) with \( L \) at \( p_j^{(i)} \) for \( 1 \leq j \leq \alpha_i \) (briefly: have \( \alpha_i \) assigned points of contact of order \( i \) with \( L \))
(iii) have \( \beta_i \) points of contact of order \( i \) with \( L \) at some “unassigned” points.

Notice that we do not assume the curves to be irreducible, this is why we change notation and label by the number of nodes \( \delta \) instead of the geometric genus.

Some examples: \( V^{3,1}[(0), (3)] \) is the variety of rational cubics, denoted by \( V_0(3) \) with the notation of the previous sections (we omit the 0 entries in \( \alpha \) and \( \beta \) whenever this does not create confusion); and \( V^{d, \delta}[(0), (d)] = V_g(d) \); also \( V^{4,3}[(0), (4)] \) the 11-dimensional variety of quartics with 3 nodes; this has two irreducible components \( V_1 \) and \( V_2 \), where \( V_1 = V_0(4) \) and \( V_2 \) parametrizes all reducible quartics made of the union of a line and a cubic.

One can show that

(1) each irreducible component of \( V^{d, \delta}[\alpha, \beta] \) has the expected dimension

\[
\dim V^{d, \delta}[\alpha, \beta] = \frac{d(d + 3)}{2} - \delta - \sum i\alpha_i - \sum (i - 1)\beta_i = 2d + g - 1 + |\beta|;
\]

(2) the general point of \( V^{d, \delta}[\alpha, \beta] \) parametrizes a curve having only nodes as singularities and smooth along \( L \).

Set now \( N^{d, \delta}[\alpha, \beta] := \deg V^{d, \delta}[\alpha, \beta] \). Let us introduce the notation \( \alpha! \) := \( \alpha_1!\alpha_2! \ldots \),

\[
\left( \frac{\alpha}{\alpha'} \right) := \left( \frac{\alpha_1}{\alpha_1'} \right) \left( \frac{\alpha_2}{\alpha_2'} \right) \ldots \text{ and if } S = \{s_1, s_2, \ldots \} \text{ is any ordered set (for example, } S = \mathbb{N} \text{ the set of positive integers) } S^{\alpha} := s_1^{\alpha_1} s_2^{\alpha_2} s_3^{\alpha_3} \ldots ; \text{ let also } \epsilon^{(j)} \text{ be defined as the string of integers having 1 at the } j\text{th place and 0 elsewhere. We can then state the main enumerative result (See Theorem 1.1 in [CH3]):}

**Theorem.**

\[
N^{d, \delta}[\alpha, \beta] = \sum_{j; \beta_j > 0} j N^{d, \delta}[\alpha + \epsilon^{(j)}, \beta - \epsilon^{(j)}]
+ \sum_{\beta' - \beta} N^{d-1, \delta'}[\alpha', \beta']
\]

where the second sum is taken over all \( \alpha', \beta' \) and \( \delta' \geq 0 \) satisfying \( \alpha' \leq \alpha, \beta' \geq \beta, \delta' \leq \delta \) and \( \delta - \delta' + |\beta' - \beta| = d - 1 \).

The proof of this theorem uses techniques of deformation theory together with semistable reduction. The degeneration technique is as follows. Let \( V = V^{d, \delta}[\alpha, \beta] \); if \( p \in \mathbb{P}^2 \) is a
point we denote by \( H_p \) the hyperplane of \(|D|\) parametrising curves through \( p \). Let now \( p_1, \ldots, p_t \) be points on \( L \). We consider the scheme theoretic intersection

\[
V_t := V \cap (\cap_{i=1}^t H_{p_i})
\]

which has the same degree as \( V \). The formula above can be read as a statement describing the hyperplane section \( V \cap H_p \) as a scheme. Notice in fact that \( V_{d,\delta}[\alpha + \epsilon(j), \beta - \epsilon(j)] \) has codimension 1 in \( V \), while the last condition \( \delta - \delta' + |\beta' - \beta| = d - 1 \) is precisely saying that the codimension of \( V_{d-1,\delta'}[\alpha', \beta'] \) in \( V \) must be 1. The coefficients of the formula have different meanings. We will illustrate the procedure on an example (cf. 4.3). All details can be found in [CH3].

We conclude with a nice and hopefully inspiring new way of writing the above formula; this was found by Getzler and appears in [G]. Let \( z \) be a variable and let \( u = (u_1, u_2, \ldots) \) and \( v = (v_1, v_2, \ldots) \) be sets of variables. Then we define a generating function using the degrees \( N_{d,\delta}^{\alpha,\beta} \)

\[
G = \sum_{\alpha, \beta} \frac{u^\alpha}{\alpha!} \frac{v^\beta}{\beta!} N_{d,\delta}^{\alpha,\beta} \frac{z^{r_{d,\delta}^{\alpha,\beta}}}{r_{d,\delta}^{\alpha,\beta}}!
\]

then an easy computation shows that the recursion in the above Theorem is equivalent to the following identity

\[
\frac{\partial G}{\partial z} = \sum_{k=0}^\infty k v_k \frac{\partial G}{\partial u_k} + \text{Res}_{t=0} \exp \left( \sum_{k=0}^\infty \frac{u_k}{t^k} + \sum_{k=0}^\infty k t^k \frac{\partial}{\partial u_k} \right) G
\]

where clearly the first summand \( \sum_{k=0}^\infty k v_k \frac{\partial G}{\partial u_k} \) corresponds to the first summand in the above formula and the remaining part corresponds to the second summand. We just notice that taking the residue for \( t = 0 \) (that is, the coefficient of \( t^{-1} \)) is the analog of the codimension-one condition \( \delta - \delta' + |\beta' - \beta| = d - 1 \).

3.3. Generalizations. The recursive procedure that we just described can easily be applied to Hirzebruch surfaces, by replacing the line \( L \) with the exceptional curve \( E \). Then one obtains a completely similar formula for curves of any genus (see [V]). A more subtle problem is generalizing the method to higher dimensional projective spaces. More precisely, one could ask for the number of (smooth) curves of given degree and genus in \( \mathbb{P}^n \) that pass through the appropriate number of points, or more generally which satisfy a given set of intersection conditions with respect to linear subspaces of varying dimension. For example, it is not hard to show that, in \( \mathbb{P}^3 \), there are finitely many rational curves of degree \( d \) satisfying \( 4d \) linear conditions (ie, passing through \( 4d \) points, or meeting \( 4d \) lines, and so on).

The natural way of generalizing our technique is to place the “linear conditions” on a fixed hyperplane one at the time so as to get a recursion. This has been shown to work by Vakil, for curves of genus 0 in \( \mathbb{P}^n \). The higher genus case is still under investigation.
Consider the following well known

**Proposition.** There are 12 rational plane cubics passing through 8 general points.

In symbols: \( N_0(3) = N^{3,1}(0, 3) = 12 \). We give here a few different ways to prove such a result, to illustrate how the techniques we talked about work. We first prove it by classical arguments (two of them) in 4.1, then we use the cross ratio method and the rational fibration method in 4.2 (these are ad-hoc techniques for rational curves, inspired by the First Reconstruction Theorem of [KM]). Finally, we use the recursive procedure of [CH3] which led to the Theorem stated in the previous chapter.

We let \( V = V_0(3) \subset \mathbb{P}^9 \) and \( N = N_0(3) \); we have \( \dim V = 8 \).

**Remark.** The following fact about the geometry of \( V \) will be needed (cf. [CH3] and loc. cit.). \( V \) is irreducible and smooth at its general point (corresponding to an irreducible cubic with one node). \( V \) contains a codimension 1 irreducible subvariety \( W \) whose general points parametrize reducible cubics given by the union of a conic and a line. This is the unique degenerate locus of \( V \) having codimension 1 and parametrizing reducible curves. \( V \) is singular along \( W \), looking like two smooth sheets crossing transversally. The two sheets correspond to deformations of the reducible curves that are locally trivial at either one of the nodes.

### 4.1. Classical proofs.

These proofs are well known. We actually prove the more general fact that the degree of the Severi variety of curves of degree \( d \) with one node is \( 3(d - 1)^2 \).

So, let \( V = V^{d,1}[0, d] \) and \( N \) be its degree. \( V \) has codimension 1 in the space \( \mathbb{P}^r \) of all curves of degree \( d \), so that \( N = \ell \cap V \) where \( \ell \) is a general line in \( \mathbb{P}^r \).

We can identify \( \ell \) with a general pencil of curves of degree \( d \); that is, a family given by a polynomial equation \( F(t_0, t_1; x_0, x_1, x_2) = 0 \) where \( F \) is homogeneous of degree 1 in \( t_0, t_1 \) and of degree \( d \) in \( x_0, x_1, x_2 \). We obtain this way a family

\[ \mathcal{Y} \longrightarrow \mathbb{P}^1 \cong \ell \]

where \( \mathcal{Y} \subset \mathbb{P}^1_{t_0, t_1} \times \mathbb{P}^2_{x_0, x_1, x_2} \). Then \( N \) corresponds exactly to the total number of nodes of the fibers of such a family.

There are now two ways of computing such a number, an algebraic way and a topological way. Algebraically, the nodes of the fibers are given by the (non-zero) solutions of the system

\[ F_{x_0} = F_{x_1} = F_{x_2} = 0, \]

that is, by the number of points in which the three surfaces given by \( F_{x_i} = 0 \) in \( \mathbb{P}^1_{t_0, t_1} \times \mathbb{P}^2_{x_0, x_1, x_2} \) intersect. Now, these surfaces are of class \( h_1 + (d - 1)h_2 \), where \( h_i \) is the pull-back of the generator of \( \text{Pic}(\mathbb{P}^i) \), \( i = 1, 2 \). Hence the basic relations \( h_1^3 - h_2^2h_2 = h_2^3 = 0 \) and \( h_1h_2^2 = 1 \) imply that \( (h_1 + (d - 1)h_2)^3 = 3(d - 1)^2 \), which is what we wanted to prove. We obtain the proposition as a special case.
The same result can be obtained by computing the Euler characteristic of \( \mathcal{Y} \) in two different ways. First, \( \mathcal{Y} \) is the blow up of \( \mathbb{P}^2 \) at the \( d^2 \) base points of the pencil, hence

\[
\chi(\mathcal{Y}) = 3 + d^2.
\]

On the other hand, the family \( \mathcal{Y} \to \mathbb{P}^1 \) has general fiber \( F \) a Riemann surface of topological characteristic \( \chi(F) = 2 - 2g = 2 - 2(d - 2) \). If \( N \) is the total number of nodes, we have (cf. [GH] Chapter 4)

\[
\chi(\mathcal{Y}) = \chi(F)\chi(\mathbb{P}^1) + N
\]

which implies \( N = 3(d - 1)^2 \).

4.2. The cross ratio and the rational fibration methods. Here we prove the proposition by using the fact that the curves in question are rational (see also [DI]). Fix 7 general points in the plane and let \( \Gamma \subseteq V \) be the irreducible curve parametrizing all nodal cubics through such points. Let \( \mathcal{X} \to \Gamma \) be the corresponding family, so that \( \mathcal{X} \subseteq \Gamma \times \mathbb{P}^2 \).

Observe that the family has exactly \( \binom{7}{2} \) reducible fibers, corresponding to all reducible cubics of type \( C_1 \cup C_2 \), where \( C_1 \) is a line through two of the base points and \( C_2 \) is the conic through the remaining five points. If \( t \in \Gamma \) is a point such that the fiber \( X_t \) is one of these reducible curves, then \( t \) will be a node of \( \Gamma \) by the introductory remark. We then let \( B \) be the normalization of \( \Gamma \) and we let \( \mathcal{Y} \) be the normalization of the base change family \( \mathcal{X} \times_{\Gamma} B \). It turns out that \( \mathcal{Y} \) is a smooth surface and that all of the fibers of \( \mathcal{Y} \to B \) are at most nodal. We do not really need this; if it were not the case that all fibers were nodal (there may be a priori some cuspidal curves) we could make a base change and perform semistable reduction; if \( \mathcal{Y} \) were not smooth, we could take its minimal desingularization. None of these two operations would affect the rest of the procedure.

Finally \( \mathcal{Y} \to B \) is a family of generically smooth rational curves, having \( 2\binom{7}{2} \) reducible nodal fibers made of two (smooth, rational) components, and no other singular fiber. We let \( \pi : \mathcal{Y} \to \mathbb{P}^2 \) be the natural map.

This part of the set-up is common to both the cross ratio method and the rational fibration method. Now we concentrate on the first. From \( \mathcal{Y} \to B \) we want to obtain a family of rational curves having 4 sections. We do that as follows: the first two sections will be two of the 7 base points, call them \( p_1 \) and \( p_2 \); the other two sections will be given by intersecting the curves of the family with 2 fixed lines \( L_3 \) and \( L_4 \) (which are chosen to be general with respect to the base points of the family). This will clearly be possible after a base change of order 9; in fact, \( L_3 \) intersects each curve of the family in 3 points, hence \( \pi^*L_3 \) is a curve in \( \mathcal{Y} \) which is a 3 to 1 cover of \( B \). Therefore if we perform the base change of degree 3 given by \( \pi^*L_3 \to B \) we can define a (single-valued) section of \( \mathcal{Y} \times_B \pi^*L_3 \) representing the intersection of \( L_3 \) with the curves of the family. Then we repeat the same process to obtain a (single-valued) section out of the intersection with \( L_4 \). In total, we made a degree 9 base change \( A \to B \) and we get a new family \( \mathcal{W} \to A \) where \( \mathcal{W} := \mathcal{Y} \times_B A \). We call again \( \pi \) the natural map from \( \mathcal{W} \) to \( \mathbb{P}^2 \). By construction, this is a family of generically smooth rational curves, all of whose fibers are at most nodal, and such that there are 4 sections \( p_i : A \to \mathcal{W} \), for \( i = 1, 2, 3, 4 \). These sections are such that the first two correspond to \( p_1 \) and \( p_2 \) respectively, while \( \pi(a) \in L_i \) for \( i = 3, 4 \) and
for all $a \in A$. The reader familiar with the theory of moduli of curves will immediately see that this family has a canonical morphism $\phi : A \to \overline{M}_{0,4}$ where $\overline{M}_{0,4}$ denotes the Deligne-Mumford moduli space of stable curves of genus 0 with 4 marked points. There is a completely equivalent way of describing this map. Define $\varphi : A \to \mathbb{P}^1$ via the cross ratio:

$$\varphi(a) = \frac{(p_1(a) - p_3(a))(p_2(a) - p_4(a))}{(p_1(a) - p_2(a))(p_3(a) - p_4(a))}.$$ 

Notice that $\overline{M}_{0,4} \cong \mathbb{P}^1$ so that if we want $\phi$ to coincide with $\varphi$ we just have to identify the 3 boundary points of $\overline{M}_{0,4}$ with the degenerate values 0, 1, and $\infty$ of the cross ratio as follows. The value 0 is identified with the point corresponding to the isomorphism class of stable, nodal, rational curves having $p_1$ and $p_3$ on one component and $p_2$, $p_4$ on the other component (we call these curves of type $(13,24)$). The value $\infty$ with the stable nodal rational curve having $p_1$ and $p_2$ on one component and $p_3, p_4$ on the other component.

Now we prove the Proposition using the basic fact that $\deg \varphi^*(0) = \deg \varphi^*(\infty)$.

To compute the degree of the zero divisor of $\varphi$, we know that the value 0 is achieved on curves of type $(13,24)$ (notice that, by the genericity assumption, the sections $p_1$ and $p_3$ are disjoint, and so are the sections $p_2$ and $p_4$). To count them, we observe that there are 10 reducible cubics through the seven base points, such that $p_1$ and $p_2$ lie on different components: 5 have $p_1$ on the line (because we need one more point, out of the remaining 5 base points, to pin down the line) and 5 have $p_1$ on the conic (as before the line is determined by $p_2$ and any one of the remaining 5 base points). The corresponding 10 points of $\Gamma$ will be nodes, by the remark at the beginning of the chapter, correspondingly we get 20 points on $B$. Then we have to take into account the base change of order 9, and, more important, the fact that $L_4$ meets the conic in two points. Finally we get

$$\deg \varphi^*(0) = 9 \cdot 2 \cdot 20.$$ 

The poles of $\varphi$ are assumed on points of $A$ corresponding to reducible fibers of type $(12,34)$ and also on points of $A$ where the section $p_3$ crosses the section $p_4$ (the sections $p_1$ and $p_2$ do not intersect). Notice that $p_3$ and $p_4$ intersect in all points $a \in A$ such that the fiber $W_a$ is mapped by $\pi$ to a nodal cubics passing through the 7 base points and through the point of intersection of $L_3$ and $L_4$. There are $N$ such plane curves, and hence 9$N$ corresponding points in $A$, where $\varphi$ has a pole. To count the fibers of type $(12,34)$ we proceed as before: there is a unique plane reducible cubic of our family having $p_1$ and $p_2$ on a line, then we have to account for the node of $\Gamma$, for the base change of order 9, and for the fact that $L_3$ and $L_4$ meet the conic in two points. We get a contribution of $2 \cdot 2 \cdot 2 \cdot 9$ to $\deg \varphi^*(\infty)$. Finally, we find $(\binom{5}{3})$ curves having $p_1$ and $p_2$ on a conic, and a total contribution of $(\binom{5}{3}) \cdot 2 \cdot 9$ to $\deg \varphi^*(\infty)$. We get

$$\deg \varphi^*(\infty) = 9N + 2 \cdot 2 \cdot 2 \cdot 9 + \binom{5}{3} \cdot 2 \cdot 9 = \deg \varphi^*(0) = 9 \cdot 2 \cdot 20$$

which yields $N = 12$.

The rational fibration method is somewhat simpler, because it does not involve any further construction, once we arrive at the above family $\mathcal{Y} \to B$. We let $Y$ be the class of
the fiber, so that \( Y^2 = 0 \). Then we let \( A \) be the class of a section corresponding to one of the seven base points, call it \( q \). Then \( (A \cdot Y) = 1 \). Let \( B' \subset B \) be the set of points such that the corresponding fiber is reducible (so that \( B' \) contains exactly \( 2 \cdot \binom{7}{2} \) points), for \( b \in B' \) let \( Z_b \) and \( W_b \) be the two components. Choose the names \( Z \) and \( W \) so that \( (A \cdot Z_b) = 1 \) and \( (A \cdot W_b) = 0 \) (in other words, \( A \) always intersects the \( Z_b \) component, for every \( b \in B' \)).

Let \( L \) be the hyperplane class in \( \mathbb{P}^2 \). Then we prove the proposition using\\

\[
(\pi^*L \cdot \pi^*L) = (\deg \pi)(L \cdot L) = N.
\]

Let us write \( \pi^*L = c_Y Y + c_A A + \sum_{b \in B'} c_b W_b \) and now notice that

\[
\begin{align*}
(i) \quad (\pi^*L \cdot Y) = 3 & \implies c_A = 3 \\
(ii) \quad (\pi^*L \cdot A) = 0 & \implies c_Y = -3A^2 \\
(iii) \quad (\pi^*L \cdot W_b) = (L \cdot \pi_* W_b) = \deg \pi_* W_b & \implies c_b = -\deg \pi_* W_b.
\end{align*}
\]

Using all of these relations we obtain

\[
(\pi^*L \cdot \pi^*L) = -9A^2 - \sum_{b \in B'} (\deg \pi_* W_b)^2.
\]

To compute \( A^2 \), pick another of the base points, call it \( q' \), and let \( A' \) be the corresponding section of the family. Since \( A^2 = (A')^2 \) and \( (A \cdot A') = 0 \) we get \( 2A^2 = (A - A')^2 \). Now \( A - A' \) is supported exactly on those reducible fibers such that \( q \) and \( q' \) lie on different components (in other words, \( q' \in Z_b \)). The number of them is \( 2 \cdot 10 \), in fact there will be 5 curves of the family such that \( q \) lies on a line and \( q' \) on a conic, and 5 such that \( q \) lies on a conic and \( q' \) on a line; the factor of 2 comes from the fact that \( \Gamma \) has a node at such curves. So we conclude that \( A^2 = 10 \).

The last thing to compute is \( \sum (\deg \pi_* W_b)^2 \). This amounts to count how many reducible fibers have \( q \) on the line, and how many have \( q \) on the conic. Clearly \( q \) is on a line for \( 2 \cdot 6 \) fibers and on a conic for \( 2 \cdot \binom{6}{2} \) fibers. Hence \( \sum (\deg \pi_* W_b)^2 = 2 \cdot 6 \cdot 4 + 2 \cdot \binom{6}{2} = 78 \) and we conclude \( N = 9 \cdot 10 - 78 = 12 \).

4.3. The proof using the Theorem of [CH3]. As we said, for the inductive technique we fix a line \( L \) in \( \mathbb{P}^2 \) and we pick points \( p_1, p_2, p_3, p_4, \ldots \) on \( L \); then we study successive scheme-theoretic intersections

\[
V_t = V \cap (\cap_{i=1}^t H_{p_i})
\]

for which, of course, \( \deg V = \deg V_t \).
By dimension count we have $V \cap H_{p_i} = V^{3,1}[(1), (2)]$ and $V_2 = V \cap H_{p_1} \cap H_{p_2} = V^{3,1}[(2), (1)]$ and these intersections are transverse, hence there are no extra factors in the degree computation. The next step gives two components of dimension 5, namely

$$V_3 = V^{3,1}[(3), (0)] \cup W$$

where the general point of $W$ is a reducible cubic $X = L \cup C$ where $C$ is a conic. Hence $\deg W = 1$, but there will be a coefficient of 2 in the degree computation. This is because $V_2$ is singular along $W$, by the remark at the beginning of this chapter. This coefficient 2 corresponds to $\binom{3}{2}$ in the formula.

Now we have to compute the degree of $V' := V^{3,1}[(3), (0)]$. By intersecting with a fourth hyperplane we now get five different irreducible components of dimension 4, three of which are of the same type. Let us write

$$V' \cap H_{p_4} = U_1 \cup U_2 \cup U_3 \cup W' \cup W''.$$

The general point of $U_i$ is a curve $X = L \cup C$ where $C$ is a conic through $p_i$. To understand this degeneration in more detail, consider a generic one-parameter family of curves of $V'$ degenerating to $X$. This is a family whose general fiber has one node while the special fiber has two, one of which is $p_i$.

We claim that the limit on $X$ of the node of the general fiber must be $p_i$. To see this, we normalize the total space of the family so as to obtain a family of generically smooth curves. Let $Y$ be the curve lying over $X$, so that $Y$ is the partial normalization at the limit node. Now the total space of this normalized family has an obvious map to $\mathbb{P}^2$, call it $\pi$. The preimage $\pi^{-1}L$ of the fixed line $L$ cannot have isolated points, because $L$ is irreducible of codimension 1. Hence $\pi^{-1}L = L' \cup S_1 \cup S_2 \cup S_3$ where $L'$ is a subcurve of $Y$ and $S_i$ is a curve such that $\pi S_i = p_i$. Let $q \in Y$ be the point lying over the node of $X$ such that $q \notin L'$. Clearly $q \in \pi^{-1}L$, therefore there must be a curve of $\pi^{-1}L$ passing through $q$. Such a curve can only be one of the $S_i$, and this implies our claim.

We conclude that, in the degree computation, we shall find no further coefficient, as the intersection is transverse and $V'$ is smooth along $U_i$. We have of course $\deg U_i = N^{2,0}[(1), (1)] = 1$.

The general curve parametrized by $W'$ is the union of $L$ and a conic $C$ tangent to $L$ at some unassigned point. The degree of the variety of conics tangent to $L$ is 2, and $H_{p_4}$ is tangent to $V'$ along $W'$, hence we get a factor of 2 which correspond to $\mathbb{N}^{3'-\beta}$ in the Theorem.

Finally $W''$ parametrizes curves of type $X = L \cup L_1 \cup L_2$ where $L_i$ is a line; the same argument used before shows that the limit node of such a curve can only be $L_1 \cap L_2$, so that $V'$ is smooth along $W''$ and the intersection is transverse. The degree of the variety of nodal conics is 3.

We can summarize the above description of the (scheme intersection) $V^{3,1}[(3)(0)] \cap H_{p_4}$ by looking at the following three possibilities for the limit of the node of the general fiber. If such a limit is an assigned point, say $p_i$, then the limit curve lies in $U_i$; if the limit node is an unassigned point of $L$, then the limit curve is in $W'$; finally, if the limit node is not on $L$, we get a curve in $W''$. 

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We conclude that \( N^{3,1}[(3)(0)] = 3 + 2 \cdot 2 + 3 = 10 \) and hence \( N_0(3) = 12 \). The reader might find the following explicit formula useful

\[
N_0(3) = 2 \cdot N^{2,0}[(0), (2)] + 3 \cdot N^{2,0}[(1), (1)] + 2 \cdot N^{2,0}[(0), (1), (0)] + N^{2,1}[(0), (0)]
\]

where the coefficient 3 in front of the second summand corresponds to \( \binom{\alpha}{\alpha'} \).

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