Breaking Rayleigh’s curse for two unbalanced point sources by BLESS technique

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According to the Rayleigh criterion, it is impossible to resolve two statistically independent point sources separated by a distance below the width of the point spread function (PSF). Almost twenty years ago it was shown that the distance between two point sources can be statistically estimated with an accuracy better than the PSF width. However, the estimation error increases with decreasing distance. This effect was informally named Rayleigh’s curse. Next, it was demonstrated that PSF shaping allows breaking the curse provided that the sources are balanced in intensity. In this work, we propose a new imaging technique based on the Target Beam modulation and the Examination of Shot Statistics (BLESS). Using the Fisher information approach, we show that the technique can break Rayleigh’s curse even for unbalanced point sources.

The standard diffraction theory claims that the far-field linear optical imaging resolution is restricted by the Rayleigh limit [1]: two point sources cannot be resolved if the distance between them is smaller than the point spread function (PSF) width, which is proportional to the radiation wavelength. There are many techniques allowing to overcome the Rayleigh limit [2], but most of them are based on nonlinear light-matter interaction [3], complex systems of excitation and suppression of luminescence [4], and consequently have a very limited field of application.

A common strategy allowing super-resolved imaging is based on the usage of a prior information about the object. Describing the whole object with a few parameters reduces the imaging problem to the problem of statistical estimation. In particular, van den Bos group considered the problem of two point sources localization [5, 6]. Using the model of Gaussian PSF, they concluded that if the distance $d$ between sources is larger than PSF width $\sigma$, its estimation error $\Delta d$ is proportional to $\frac{\sigma}{\sqrt{K}}$, where $K$ is the number of registered particles. If $d < \sigma$ then $\Delta d \propto \frac{\sigma}{\sqrt{Kd/\sigma}}$. Hence, the distance between two close point sources cannot be accurately estimated with a limited amount of data. This problem was named Rayleigh’s curse [7].

Later Tsang et al showed the possibility to overcome Rayleigh’s curse [7, 9]. They considered the problem of resolving two equal point sources in terms of quantum Fisher information (QFI). They proved that QFI for two point sources is independent of the distance $d$ value, which means that the parameter can in principle be precisely estimated beyond the Rayleigh limit. Moreover, the QFI limit can almost be saturated by practical measurement protocols: SPAtial-Mode DEmultiplexing (SPADE) [7] and SuperLocalization by Image inVERsion interferometry (SLIVER) [8]. Both protocols make use of the PSF shaping. In particular, the use of odd PSF instead of even Gaussian PSF breaks Rayleigh’s curse. This has been demonstrated in the set of proof-of-principle experiments [10–13].

But it was later shown that the Rayleigh’s curse can be overcome for the two-parameter object model only [14–20]. Two unbalanced point sources [15, 17] or more than two equal sources [14] cannot be precisely localized beyond the Rayleigh limit: the position estimation error increases polynomially with decreasing distance between sources. Generally, any object can be parameterized by its intensity moments. It was shown that the estimation errors of the first and the second intensity moments are independent of the object size, but the Rayleigh’s curse is still valid for higher $k$-order moments $M_k$, resulting in $\Delta M_k \propto d^{2-k/2}$, so they cannot be well estimated beyond the Rayleigh limit [18–20].

To perform precise imaging of complex objects one needs to extract additional information from the measurements. Previously, most of the imaging statistical estimation problems was considered in the weak source approximation, where the number of detected photons was restricted by unity, and the mean photon number was measured [7, 20]. However, higher order intensity (or photon number) moments give benefits for solving imaging problems. One of the first demonstrations of this was done by Brown and Twiss in their stellar interferometer experiment [21]. It was later shown that PSF for $N$-order intensity moment measurement was $\sqrt{N}$ times narrower than the first-order one. [22]. This was experimentally applied to single-photon emitters imaging [23, 24]. Also, photon statistics analysis was used in
stochastic optical reconstruction microscopy (STORM) \cite{20} and photoactivated localization microscopy (PALM) \cite{21} techniques, where the positions of independently blinking point sources were estimated from a set of frames obtained at different time intervals. Using higher-order correlations for statistical reconstruction was recently reported \cite{28,29}, but the break of the Rayleigh’s curse was not demonstrated.

Below we propose an approach that combines statistical estimation of image parameters, PSF shaping and the examination of photon statistics distribution. We will show that this technique allows breaking the Rayleigh’s curse for the problem of two unbalanced point sources.

Consider a 1D imaging problem for two incoherent point sources of light \(S_a\) and \(S_b\) located at \(x_a\) and \(x_b\) respectively (Fig. 1). Each source has its own photon number distribution \(P_{a,d}(n)\) and \(P_{b,d}(n)\) defined by corresponding mean photon numbers \(\mu_a\) and \(\mu_b\). One can describe this object using the following set of 4 parameters: distance \(d = x_a - x_b\), total mean photon number \(\mu = \mu_a + \mu_b\), centroid \(c = (\mu_a x_a + \mu_b x_b) / \mu\), and sources relative intensity \(\gamma = (\mu_a - \mu_b) / \mu \in [-1,1]\).

The light is coupled into a single-mode fiber (SMF) projecting the beam onto the HG0 spatial mode with center at \(x_d\). The photon number resolving detector (PNRD) reads the output signal. For simplicity, we consider quantum efficiency of the detector to be 100%. The probability \(\pi_a\) of detecting a single-photon emitted by the point source \(S_a\) is then \(|\Psi_0(x_a - x_d)|^2\), where \(\Psi_0(x) \propto \exp(-x^2/(2\sigma^2))\) and \(\sigma\) is the PSF width. One can transform the target beam with a spatial light modulator (SLM) and convert mode HG0 to HG1 resulting in \(\pi_a = |\Psi_1(x_a - x_d)|^2\), where \(\Psi_1(x) \propto x \exp(-x^2/(2\sigma^2))\).

Then the total probability of detecting \(k\) photons from the source \(S_a\) is

\[
P_{a,d}(k) = \sum_{n=k}^{\infty} \binom{n}{k} P_{a,n}(n) \pi_a^k (1 - \pi_a)^{n-k}. \tag{1}\]

A similar probability distribution \(P_{b,d}\) could be found for the source \(S_b\).

Since the sources are incoherent, the convolution of \(P_{a,d}\) and \(P_{b,d}\) gives the total photon number distribution \(P_d(k|x_d,\theta)\). It depends on the target beam position and image parameters \(\theta = \{d,\gamma,\mu, x_c\}\). The mean photon number for this distribution is \(\mu_d = \mu_a \pi_a + \mu_b \pi_b\). The variance \(\sigma_d^2\) depends on the particular form of \(P_a\) and \(P_b\). We study two types of image sources: spontaneously emitting single-photon sources and thermal sources. Table I shows the detector photon number distribution and its statistical properties: the variance \(\sigma_d^2\) and the second order correlation function \(g_d^{(2)}\).

The statistical reconstruction of image parameters is done by measuring photocounts for some set of detector position values and fitting \(P_D\) over \(\theta\) with respect to the observed data. Thus, we are using both spatial beam modulation and examination of shot statistics to statistically reconstruct the image parameters.

Meanwhile, the standard approach exploits only the integrated number \(N\) of registered photons. If one performs a \(K\)-shot measurement for detector position \(x_d\) and gets \(K\) photon numbers, then according to the central limit theorem the distribution of \(N\) tends to normal:

\[
\frac{N - K \mu_d}{\sigma_d \sqrt{K}} \xrightarrow{K \to \infty} N(0,1). \tag{2}\]

Note that \(K \mu_d \geq 100\) was enough to use the limit \(K \to \infty\) in simulation. Experimentally, the use of integrated statistics corresponds to a single measurement with a long exposition time, while the shot statistics analysis assumes many measurements with a short exposition time.

Below we will see that the examination of shot statistics gives an improvement over the integrated one. Consider an efficient statistical estimate \(\hat{\theta}\). For each imaging experiment, \(\hat{\theta}\) is a random vector having multivariate normal distribution \(f(\hat{\theta})\) located at point \(\theta_T\) of the

\[
\begin{array}{|c|c|c|}
\hline
 & \text{Single-photon sources} & \text{Thermal sources} \\
\hline
P_{a,b}(n) & \delta_{0,n}(1 - \mu_a, b) + \delta_{n,\mu_a, b} & Q_{\mu_a, b}(n) \\
\hline
P_{d}(k) & \lambda_b Q_{\lambda_b}(k) - \lambda_b Q_{\lambda_b}(k) & \lambda_a(1 + \lambda_a) + \lambda_b(1 + \lambda_b) \\
\hline
\sigma_d^2 & 2 \frac{(1 + \lambda_a)(1 + \lambda_b)}{(\lambda_a + \lambda_b)^2} & 2 \frac{(1 + \lambda_a)(1 + \lambda_b)}{(\lambda_a + \lambda_b)^2} \\
\hline
\end{array}
\]
true parameters values. According to Cramér–Rao lower bound, the covariance matrix Σ of \( \hat{f}(\theta) \) is the inverse of the Fisher information matrix

\[
I_{\alpha\beta} = \sum_k \frac{1}{P_d(k)} \left| \frac{\partial P_d(k)}{\partial \theta_\alpha} \frac{\partial P_d(k)}{\partial \theta_\beta} \right|_{\theta = \theta_t}.
\] (3)

Since the Fisher information is additive over independent trials, we define the complete information matrix as \( I = K \sum_{x_d} I_{x_d} \). Here we take the sum over various target beam positions \( x_d \) with corresponding Fisher information matrices \( I_{x_d} \). For the integrated statistics we calculate the Fisher information for the normal distribution (see Eq. (2)).

Non-efficient estimators give \( \Sigma > I^{-1} \) \((\Sigma - I^{-1} \) is a positive-definite matrix\). Below we consider maximum likelihood estimator (MLE). It is asymptotically efficient, which means its convergence to \( f(\theta) \) with \( K \to \infty \). In practice, \( f(\theta) \) turns out to be a good approximation for finite \( K \). For this, it is usually sufficient to satisfy two conditions:

- all the eigenvalues of \( I \) are \( \gg 1 \);
- distribution \( f(\theta) \) almost certainly lies within the parameters domain.

Then the estimation error for each parameter is a square root of corresponding diagonal elements of the covariance matrix diagonal elements: \( \Delta_\alpha = \sqrt{\Sigma_{\alpha\alpha}} \). We have verified it by conducting a number of Monte Carlo simulation experiments. Fig. 2 shows the results of 100 numerical experiments and the corresponding marginal distributions of \( f(\theta) \) calculated using Cramér–Rao bound. The image source consisted of two single-photon emitters with parameters \( d = 0.1, \gamma = 0.5, \mu = 0.1, x_c = 0 \) (here and below all the spatial quantities are given in the units of PSF width \( \sigma \)). The estimation has been done using MLE. As the likelihood function is non-convex we have used the SciPy package in Python to do the global optimization [32].

Similar plots for other light sources and model parameters have also shown a close correspondence between MLE estimates and theoretical distributions. Note, however, that the numerical optimization problem becomes far more complex when \( d \ll 1 \) because of the large amount of local maximums. Moreover, the complexity tremendously increases with the amount of light sources and consequently the number of parameters to estimate. In this regard, it is important to discover efficient methods for solving this optimization problem.

Next, we have analyzed lower bounds for the estimation accuracy of image parameters depending on the distance \( d \) between two unbalanced point sources. Fig. 3 shows the estimation errors for parameters \( d \) and \( \gamma \). Note that we normalize them by \( \sqrt{K} \) to make them sample size independent. As expected, considering the integrated statistics results in \( \Delta_d \propto 1/d \) law being in correspondence with Rayleigh’s curse. Turning to shot statistics analysis significantly relaxes the dependence of \( \Delta_d \) on \( d \). If the target beam is converted to HG1 mode, \( \Delta_d \) becomes independent of \( d \) for close light sources. Thus, the method could provide finite accuracy for infinitely small \( d \), which breaks Rayleigh’s curse.

The estimation error \( \Delta_\gamma \) of sources relative intensity \( \gamma \) is almost insensitive to beam modulation: considering the shot statistics instead of the integrated one is enough to make \( \Delta_\gamma \) independent of the distance for small \( d \). A similar behavior has been observed for \( \Delta_{x_c} \), while \( \Delta_\mu \sqrt{K} \) has been below 1 in all the cases. Finally, note that these results for thermal and single-photon sources are much the same, even the latter one gives slightly better accuracy.

The considered approach can be used in various imaging applications from astronomy [33, 34] to fluorescence microscopy [35, 36]. Moreover, it can be utilized in other problems like axial resolution of point sources [37, 40], temporal [41] and spectral [42] resolution, etc. There are many ways for the further research: extending this approach to more complex objects [14, 43], taking into account partial coherence of point sources [41, 44, 45], and experimental imperfections like detector noise [17, 49], making use of adaptive measurement strategies [50], etc. Particularly we should note, that photon statistics analysis may give additional information about point sources coherence and help to separate the signal radiation from the background.

In conclusion, we have proposed a novel imaging technique. This technique is based on the multi-shot photon number measurements in the modulated target beam scanning the sample. Image parameters are then estimated by fitting the photon number distribution model.
FIG. 3. Normalized estimation errors of the distance $d$ between the point sources and their relative intensity $\gamma$ versus $d$. Simulation parameters: $\gamma = 0.1, \mu = 0.1, x_c = 10^{-4}, x_d = -2, -1.9, \ldots, 1.9, 2.$

to the collected data. The approach has been theoretically studied on the example of a couple of unbalanced point sources (thermal or single-photon). The Fisher information analysis has shown that even for infinitely close sources the estimation error of the distance between them is limited. Thus, we have demonstrated that the shape of photon number distribution should be used in the statistical estimation of image parameters since it provides a huge amount of additional information and in particular allows one to break Rayleigh’s curse.

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