Simple model of self-organized biological evolution as a completely integrable dissipative system

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Abstract

The Bak-Sneppen model of self-organized biological evolution of an infinite ecosystem of randomly interacting species is represented in terms of an infinite set of variables which can be considered as an analog to the set of integrals of motion of completely integrable system. Each of this variables remains to be constant but its influence on the evolution process is restricted in time and after definite moment its value is excluded from description of the system dynamics.

1 Introduction

Investigations of complex dynamical systems play an important role in the modern theoretical and mathematical physics. In the recent years, the essential achievements have been arrived in studies of two dimensional field theoretical models. A large class of the exactly solvable and completely integrable models has been found [1]. It enabled one to understand the laws of the soliton dynamics creating many non-ordinary phenomena in several physical systems.

The non-trivial completely integrable models like sine-Gordon or nonlinear Schrödinger equations are characterized by the infinite set of the conservation laws [1]. If the dynamics of such a system is considered in the framework of Hamilton’s formalism and the integrals of motions are chosen as momenta, then the evolution in such a description appears to be similar to one of the mechanical system of non-interacting material points: the momenta are constant and the coordinates are linear functions of time.

Many theoretical physicists hope that there are universal types of dynamics of natural processes, which can be classified. A possible theoretical approach for constructing such a classification could be the following. For the model under consideration one needs to find its ”principal” representation, i.e. to formulate the dynamics of the system in terms of ”principal” variables for which the dynamics is as simple as possible. If
the principal representations of two models are the same, it is naturally to consider these models as representatives of the same type of dynamics. From this point of view the system of free particles is the representative of the class of all the completely integrable systems.

If the construction of such a typology could be possible it would be very useful for understanding the most essential features in dynamics of complex natural systems. However, to find what we call the principal representation is not an easy problem for complex dynamical system. Probably, the set of ones for which this problem can be solved is restricted but it seems to be important to reveal the non-trivial systems of such a kind.

In the present paper the proposed by Bak and Sneppen model (BSM) of biological evolution [2] is considered. It is a dynamical system describing the ecosystem evolution as mutation and natural selection of interacting species. The most essential property of dynamics in this model is the self-organized criticality (SOC). The main characteristic of the SOC dynamics is that the system evolves to a critical state without fine tuning of its parameters. The SOC type in the BSM has the main specific features of real biological evolution considered in the framework of the Gould-Eldridge “punctuated equilibrium” conception [3]. One can hope that this model represent an important universal type of critical dynamics of avalanche-like processes, realizing in many systems [4].

The simplest version of the BSM with random interaction structure [5] appeared to be exact solvable both in the thermodynamical limit and for finite system [6], [7], [8]. For infinite system it is possible to construct its principal representation. It is done in this paper. The dynamics in terms of principal variables looks as follows: each variable conserves its initial value the definite (different for several variables) length of time and after that becomes to be equal to zero. Thus the principal variables in this model are not exact integral of motion but in the process of the system evolution they transform (to zero) only once. For an alternative set of principal variables, one of them is lost at each time step and the others are renumbered.

2 Formulation of the model

In the framework of the BSM the biological evolution of ecosystem is described as follows [1]. The state of the ecosystem of \( N \) species is characterized by a set \( \{x_1, ..., x_N\} \) of \( N \) number, \( 0 \leq x_i \leq 1 \). The number \( x_i \) represents the barrier of the \( i \)-th species toward further evolution. Initially, each \( x_i \) is set to a randomly chosen value. At each time step the barrier \( x_i \) with minimal value and \( K - 1 \) other barriers are replaced by \( K \) new random numbers. In the random neighbor model (RNM) [5] which will be considered in this paper the \( K - 1 \) replaced non-minimal
barriers are chosen at random. In the local or nearest neighbor model (LM) these are the barriers of the nearest neighbors to the species with minimal barrier. For each species in the LM the nearest neighbors are assumed to be defined.

For the LM the most of results are obtained by numerical experiments or in the framework of mean field approximation. The RNM is more convenient for analytical studies. The master equations obtained in [6] for RNM are very useful for this aim. These equations appeared to be exact solvable. The stationary solution was found in [6]. The time dependent solutions were obtained in [7] (infinite system), [8] (finite system).

In this paper we construct the principal representation of the RNM master equation for infinite system. The obtained in [6] master equations for the RNM are of the form:

\[ P_n(t+1) = \sum_{l=0}^{K} C^l_n \theta(K-l)\lambda^l(1-\lambda)^{K-l} P_{n-l+1}(t) + \theta(K-n)C^K_n \lambda^n(1-\lambda)^{K-n} P_0(t) \] (1)

Here, \( P_n(t) \) is the probability that \( n \) is the number of barriers having values less than a fixed parameter \( \lambda \) at the time \( t \); \( 0 \leq n \leq N, \ 0 \leq \lambda \leq 1, \ t \geq 0. \) For compact writing we used the threshold function \( \theta(n) \) defined as follows

\[ \theta(n) = 1 \text{ for } n \geq 0 \text{ and } \theta(n) = 0 \text{ for } n < 0. \]

The initial probability distribution \( P_n(0) \) is supposed to be given. In virtue of the definition of \( P_n(t) \),

\[ P_n(t) \geq 0, \quad \sum_{n=0}^{\infty} P_n(t) = 1. \] (2)

Making summation in (1) over \( n \) it is easy to prove that

\[ \sum_{n=0}^{N} P_n(t+1) = \sum_{n=0}^{N} P_n(t). \] (3)

In virtue of (3) if \( P_n(0) \) are chosen in such a way that equalities (2) are fulfilled for \( t = 0 \), then for the solution of (1) it is the case for \( t > 0 \) too. For analysis of (1) it is convenient to introduce the generating function \( q(z, u) \):

\[ q(z, u) = \sum_{t,n=0}^{\infty} P_n(t)z^n u^t. \] (4)

In virtue of (4) the function \( q(z, u) \) is analytical in \( z \) and \( u \) for \( |z| < 1, \ |u| < 1. \) The master equations (1) can be rewritten for the generating function \( q(z, u) \) as follows:

\[ (z-u(1-\lambda+\lambda z)^K)q(z, u) = (z-1)u(1-\lambda+\lambda z)^K q(0,u) + zq(z, 0). \] (5)
In (5) the function

\[ q(z, 0) = \sum_{n=0}^{N} P_n(0) z^n \]

is assumed to be given. If the generating function \( q(z, u) \) (4) is known, \( P_n(t) \) can be obtained as

\[ P_n(t) = \frac{\partial^{n+t}}{\partial x^n \partial u^t} q(x, u) \bigg|_{x=u=0} = \oint \oint \frac{q(z, u) dz du}{z^{n+1} u^{t+1}} \]

where \( \epsilon_1 < 1 \) and \( \epsilon_2 < 1 \).

3 Principal representation

Let us denote \( \alpha(u) \) the analytical in \( u = 0 \) solution of the algebraic equation

\[ \alpha(u) - u(1 + \lambda(\alpha(u) - 1))^K = 0. \]  

(6)

It has the form

\[ \alpha(u) = u(1 - \lambda)^K + Ku^2 \lambda(1 - \lambda)^{K-1} + \ldots. \]

This series converges and \( |\alpha(u)| < 1 \) if \( |u| < u_0 \) and parameter \( u_0 \) is chosen small enough. For \( K = 2 \) \( \alpha(u) \) can be written as follows

\[ \alpha = \alpha(u) = \frac{1 - 2\lambda(1 - \lambda)u - \sqrt{(1 - 4\lambda(1 - \lambda)u)}}{2\lambda^2 u}. \]

Let us define the function \( \beta(z) \):

\[ \beta(z) = \frac{z}{(1 - \lambda + \lambda z)^K}. \]  

(7)

It is analytical and \( |\beta(z)| < 1 \) if \( |z| \) is small enough. In virtue of definitions (3), (7)

\[ \alpha(\beta(z)) = z, \; \beta(\alpha(u)) = u. \]  

(8)

Now, we define the function \( d(y, u) \)

\[ d(y, u) = \frac{q(\alpha(y), u)}{1 - \alpha(y)} \]  

(9)

which is analytical in \( y \) and \( u \) in the neighborhood of \( y = 0 \) and \( u = 0 \):

\[ d(y, u) = \sum_{n,t=0}^{\infty} C_n(t) y^n u^t. \]  

(10)
Substituting in (9) \( y = \beta(z) \) and taking unto account (8) we obtain

\[ q(z, u) = (1 - z)d(\beta(z), u). \]  

(11)

The functions \( C_n(t) \) defined by (10) we consider as new dynamical variables of the RNM. The equalities (9), (11) are the compact formulas of variable transformations from \( P_n(t) \) to \( C_n(t) \) and backward. They are not dependent in an evident way on the time what is seen if (9), (11) is written in the more detailed form:

\[
\sum_{n=0}^{\infty} P_n(t)z^n = (1 - z) \sum_{n=0}^{\infty} C_n(t)\beta(z)^n, \quad \sum_{n=0}^{\infty} C_n(t)y^n = \sum_{n=0}^{\infty} P_n(t)\frac{\alpha(y)^n}{1 - \alpha(y)}.
\]

By substitution (11) in (5) the following equation is obtained

\[ (\beta(z) - u)d(\beta(z), u) = \beta(z)d(\beta(z), 0) - ud(0, u). \]  

(12)

Setting in (12) \( z = \alpha(y) \) we have

\[ (y - u)d(y, u) = yd(y, 0) - ud(0, u). \]  

(13)

It follows from (13) that

\[
\frac{\partial^{n+t+2}}{\partial y^{n+1}\partial u^{t+1}} (y - u)d(y, u) \bigg|_{y=u=0} = 0 \quad \text{for} \quad n \geq 0, \ t \geq 0.
\]

what looks in terms of the functions \( C_n(t) \) as

\[ C_n(t + 1) = C_{n+1}(t) \quad \text{for} \quad n \geq 0, \ t \geq 0. \]  

(14)

The initial conditions

\[ C_n(0) = c_n \]  

(15)

are defined by \( q(z, 0) \):

\[ d(y, 0) = \sum_{n=0}^{\infty} c_n y^n = \frac{q(\alpha(y), 0)}{1 - \alpha(y)}. \]  

(16)

The equations (14) has the simple solution:

\[ C_n(t) = c_{n+t}. \]  

(17)

The variables \( C_n(t) \) can be considered as the principal ones for the RNM. Principal representation of its dynamics is defined by equations (14), with initial conditions (15), (16). Setting in (13) \( u = y \) we see that \( d(y, 0) = d(0, y) \) and

\[ d(y, u) = \frac{yd(y, 0) - ud(0, u)}{y - u}. \]  

(18)
Setting \( y = \beta(z) \) in (18) and taking into account (9), (11) we obtain the solution of the master equation (5).

\[
q(z, u) = \frac{z(1 - \alpha(u))q(z, 0) + u(z - 1)(1 + \lambda(z - 1))^K q(\alpha(u), u)}{(1 - \alpha(u))[z - u(1 + \lambda(z - 1))^K]}
\]

If time dependent transformations of variables are allowed, an alternative principal representation can be constructed. Let us define the variables \( S_n(t) \) as follows

\[ S_n(t) = C_{n-t(t)} \text{ for } t \leq n, \quad \text{and } S_n(t) = 0 \text{ for } t > n; \quad n \geq 0, \ t \geq 0. \]

The backward variable transformation can be written in the form

\[ C_n(t) = Z_{n+t(t)}, \quad n \geq 0, \ t \geq 0. \]

It follows from (17) that

\[ S_n(t) = c_n \theta(n - t), \]

i.e. the variable \( S_n(t) \) conserves its initial value \( S_n(0) = c_n \) until \( t \leq n \) and \( S_n(t) = 0 \) from the moment \( t = n + 1 \).

4 Conclusion

We have obtained the following results. In terms of variables chosen in a special way the SOC dynamics in the BSM can be described by very simple equations (13), (14). These variables are expressed straightforwardly in terms of infinite set of the constants defined explicitly by the initial conditions. On each time step one of these constants is forgotten, i.e its value does not influence the further stages of the system evolution. The consequent loose of the information about initial state is all what happens in the self-organization process described by RNM. The system of such a kind could be called completely integrable dissipative system. It would be interesting to understand how robust this dynamics is. Is it the inherent property of the SOC processes or an artifact of the considered model?

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