NONPARAMETRIC GRAPHON ESTIMATION

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We propose a nonparametric framework for the analysis of networks, based on a natural limit object termed a graphon. We prove consistency of graphon estimation under general conditions, giving rates which include the important practical setting of sparse networks. Our results cover dense and sparse stochastic blockmodels with a growing number of classes, under model misspecification. We use profile likelihood methods, and connect our results to approximation theory, nonparametric function estimation, and the theory of graph limits.

1. Introduction. Networks are fast becoming part of the modern statistical landscape (Durrett, 2007; Diaconis and Janson, 2008; Bickel and Chen, 2009; Choi, Wolfe and Airoldi, 2012; Fienberg, 2012; Zhao, Levina and Zhu, 2012; Arias-Castro and Grimmett, 2013; Ball, Britton and Sirl, 2013; Choi and Wolfe, 2013). Yet we lack a full understanding of their large-sample properties in all but the simplest settings, hindering the development of models and inference tools that admit theoretical performance guarantees.

In this article we introduce a nonparametric framework for the analysis of networks, which relates to kernel-based random graph models (Janson, 2010; Sussman, Tang and Priebe, 2013), stochastic blockmodels (Airoldi et al., 2008; Rohe, Chatterjee and Yu, 2011), and degree-based models (Chatterjee, Diaconis and Sly, 2011; Bickel, Chen and Levina, 2011). We use this framework to establish consistency of likelihood-based network inference under general conditions, and to show convergence rates across a range of network regimes, from dense to sparse. Our framework thus addresses one of the biggest factors limiting the use of statistical network models in practice: a lack of flexible and transparent analysis tools that admit coherent statistical interpretations (Fienberg, 2012).

Our methodology derives from a large-sample theory tailored to network data, in which well-defined limiting objects play a role akin to the infinite-dimensional functions that underpin classical nonparametric statistics (Bickel and Chen, 2009). An exchangeable stochastic network can be

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viewed as a partial observation of this limiting object under Bernoulli sampling (Diaconis and Janson, 2008). Hence our theory is closely related to that of generalized linear models (Green and Silverman, 1994) and of contingency tables (Fienberg and Rinaldo, 2012), as well as to nonparametric function approximation. High-dimensional statistical theory in this setting is nascent, and so the linkages we develop below provide for a foundational understanding of nonparametric statistical network analysis.

2. Model elicitation. A network can be represented by an \( n \times n \) data matrix \( A \), whose \( ij \)th entry describes the relation between node \( i \) and node \( j \) of the network. In the most fundamental setting of graph theory, \( A \) is a symmetric, binary-valued contingency table: it is sparse yet structured, with \( A_{ij} \in \{0, 1\} \) denoting the absence or presence of an edge between nodes \( i \) and \( j \), and with fixed, structural zeros along the main diagonal.

We call \( A \) an adjacency matrix, and model it as a realization of \( \binom{n}{2} \) independent Bernoulli trials. Independently for \( 1 \leq i < j \leq n \), we have

\[
A_{ij} | p_{ij} \sim \text{Bernoulli}(p_{ij}), \quad A_{ji} = A_{ij}, \quad A_{ii} = 0.
\]

Each Bernoulli trial \( A_{ij} \) has success probability \( p_{ij} \), which in turn we model using a bivariate function termed a graphon that derives from the theory of graph limits (Lovász, 2012).

A graphon is a nonnegative symmetric function, measurable and bounded, that represents a discrete network as an infinite-dimensional analytic object. It is a basic characterization, allowing us to go from the discrete set of probabilities \( \{p_{ij}\}_{i<j} \) to a limit object \( f(x, y) \) defined on \((0, 1)^2\), independently of the network size. Various summaries of the network can be calculated as functionals of the graphon; for example, a network’s degree distribution is characterized by its graphon marginal \( \int_0^1 f(\cdot, y) \, dy \).

To model both dense and sparse networks, we allow the success probabilities \( p_{ij} \) appearing in (2.1) to depend on \( n \). We link these to a scaled graphon \( \rho_n f(x, y) \) through a random sample \( \{\xi_i\}_{i=1}^n \) of uniform variates, via a scale parameter \( \rho_n > 0 \) that specifies the expected probability of a network edge:

\[
p_{ij} = \rho_n f(\xi_i, \xi_j) ; \quad \{\xi_1, \ldots, \xi_n\} \overset{iid}{\sim} \text{Uniform}(0, 1), \quad \int \int f(x, y) \, dx \, dy = 1.
\]

Observe that \( E A_{ij} = E \xi p_{ij} = \rho_n \) for all \( 1 \leq i < j \leq n \), and so \( \rho_n \) specifies the sparsity of the generated network. We assume the sequence \( \{\rho_n\}_{n=2,3,\ldots} \) to be fixed and monotone non-increasing.

This is a canonical model based on exchangeable random networks (Bickel and Chen, 2009; Bickel, Chen and Levina, 2011), and is also strongly related
to other statistical modeling paradigms. It relates the infinite-dimensional graphon \( f(x, y) \) to the set of probabilities \( \{p_{ij}\}_{i<j} \) sampled via \( \xi \). This modeling strategy is similar to time series analysis, where a sampled autocovariance is related to an infinite-dimensional spectral representation. As with an independent increments process, we may think of each \( \xi_i \) in (2.2) as a latent variable. Furthermore, \( \xi_i \) is associated with the \( i \)th network node, acting as a latent random index into the graphon. This reflects the fact that the observed ordering of the network nodes conveys no information.

Similarly, the ordering of a given graphon \( f(x, y) \) along the \( x \) and \( y \) axes has no inherent meaning; that is, \( f(x, y) \) has a built-in invariance to “rearrangements” of the \( x \) and \( y \) axes. This is similar to statistical shape analysis, where we seek to describe objects in a manner that is invariant to their orientation in Euclidean space. Thus \( f(x, y) \) represents an equivalence class of all symmetric functions that can be obtained from one another through measure-preserving transformations of \([0,1]\).

This notion was formalized by Aldous (1981) and Hoover (1979) in the context of exchangeable infinite arrays. Their eponymous theorem asserts that any such array admits a representation in terms of some \( f(x, y, \alpha) \). This representation is unique up to measure-preserving transformation (Diaconis and Janson, 2008), and the value of \( \alpha \) is not identifiable from a single network observation (Bickel and Chen, 2009). The Aldous–Hoover representation thus relates (2.2) to an exchangeable infinite array \( \{A_{ij}\}_{i,j=1}^{\infty} \) of binary random variables, such that for all \( n = 1, 2, \ldots, \) all permutations \( \Pi \) of \( \{1, \ldots, n\} \) and all \( a \in \{0,1\}^{n \times n} \), we have that \( \Pr(A_{ij} = a_{ij}, 1 \leq i < j \leq n) = \Pr(A_{ij} = a_{\Pi(i)\Pi(j)}, 1 \leq i, j \leq n) \).

By putting an observed \( n \times n \) adjacency matrix \( A \) in correspondence with a finite set of rows and columns of \( \{A_{ij}\}_{i,j=1}^{\infty} \), we arrive at a model for exchangeable networks, or for sub-networks thereof. Exchangeability implies that once we condition on the latent variable \( \xi_i \) associated to network node \( i \), then all linkages \( A_i \) to node \( i \) are conditionally independent and identically distributed. This follows from de Finetti’s representation of a sum of exchangeable indicator variables (Diaconis, 1977).

3. Main result. Our main result is that whenever a graphon \( f \) is Hölder continuous, and maximum likelihood fitting is used to derive a non-parametric estimator of \( f \) from \( A \), then this estimator will be consistent as long as \( \rho_n = \omega(n^{-1} \log^3 n) \), and its rate of convergence can be established.

To construct our estimator, we will calculate group averages after forming \( k \) groups from \( n \) nodes. Any such grouping can be represented as an integer partition of \( n \) via a vector \( h \in \{2, \ldots, n\}^k \), such that \( \sum_{a=1}^{k} h_a = n \). Thus
may view $n^{-1}h$ as the probability mass function of a random variable with range \( \{1, \ldots, k\} \), indexed via a cumulative distribution function \( H \) and its generalized inverse \( H^{-1} \):

\[
\begin{align*}
(3.1a) \quad & H(u) = \frac{1}{n} \sum_{a=1}^{u} h_a, \quad u \in [0, k], \quad H(u) \in \left\{ 0, \frac{h_1}{n}, \frac{h_1+h_2}{n}, \ldots, 1 \right\}, \\
(3.1b) \quad & H^{-1}(x) = \inf_{u \in [0, k]} \{ H(u) \geq x \}; \quad x \in (0, 1], \quad H^{-1}(x) \in \{1, \ldots, k\}.
\end{align*}
\]

The central difficulty in constructing a nonparametric graphon estimator is that we do not know the ordering of our observed adjacency matrix \( A \), relative to the ordered sample \( \{\xi(i)\}_{i=1}^{n} \) indexing the graphon \( f \). We thus define an estimator \( \hat{f} \) as a composition of two operations: first we re-index the rows and columns of \( A \) according to some permutation \( \Pi \) of \( \{1, \ldots, n\} \), and then we group them in accordance with \( H \):

\[
\hat{f}(x, y; h) = \hat{\rho}_{n}^{-1} \tilde{A}_{H^{-1}(x)H^{-1}(y)}, \quad \hat{\rho}_{n} = \left( \frac{n}{2} \right)^{-1} \sum_{i<j} A_{ij}, \quad (x, y) \in (0, 1)^2;
\]

\[
\tilde{A}_{ab} = \frac{1}{h_a \{h_b - \mathbb{1}(a = b)\}} \sum_{j=nH(b)-1+1}^{nH(b)} \sum_{i=nH(a)-1+1}^{nH(a)} A_{\Pi(i)\Pi(j)}, \quad 1 \leq a, b \leq k.
\]

We then define the mean-squared error of \( \hat{f} \) relative to \( f \) as

\[
\inf_{\sigma \in \mathcal{M}} \int_{(0,1)^2} \left| f(\sigma(x), \sigma(y)) - \hat{f}(x, y; h) \right|^2 dx \, dy,
\]

where \( \mathcal{M} \) is the set of all measure-preserving bijections of the form \( \sigma : [0, 1] \to [0, 1] \). This error criterion is based on the so-called cut distance in the theory of graph limits (Lovász, 2012), and allows for all possible rearrangements of the axes of \( f \) (Choi and Wolfe, 2013).

Any estimator \( \hat{f} \) can be viewed as a Riemann sum approximation of \( f \), and thus we must understand when such sums converge. Lebesgue’s criterion asserts that a bounded graphon on \( (0, 1)^2 \) is Riemann integrable if and only if it is almost everywhere continuous. A sufficient condition is that \( f \) is \( \alpha \)-Hölder continuous for some \( 0 < \alpha \leq 1 \), where we write

\[
(3.2) \quad f \in \text{Hölder}^{\alpha}(M) \iff \sup_{(x, y) \neq (x', y') \in (0,1)^2} \frac{|f(x, y) - f(x', y')|}{|(x, y) - (x', y')|^\alpha} \leq M < \infty.
\]

This assumption ensures that \( f \) is uniformly continuous, so that its approximation error can be controlled through Riemann sums.

Under this model specification, we obtain our main result, which we prove in Appendix A.
Theorem 3.1 (Consistency of smooth graphon estimation). Assume a sequence of graphon estimators $\hat{f}(x, y; h)$ is fitted under the model of (2.2), with $k = \omega(1)$ and $\bar{h} = n/k$ the average group size, where

1. The graphon $f$ is symmetric, bounded away from zero and $\alpha$-Hölder continuous, $0 < \alpha \leq 1$;
2. The scaling sequence $\rho_n$ satisfies $\rho_n = \omega(n^{-1} \log^3 n)$, and $\max_n \rho_n f$ is bounded away from unity;
3. Every admissible partition $H$ has group sizes bounded uniformly above and below by $h_\vee = o(n)$, $h_\wedge = \omega(\log^{1/2} n)$, and may be composed with any permutation $\Pi$ of $\{1, \ldots, n\}$ to yield $\hat{f}(x, y; h)$.

Suppose furthermore that the minimum effective sample size of every possible fitted grouping, $(h_\wedge^2)\rho_n$, and the average effective sample size across all groupings, $\bar{h}^2\rho_n$, both grow sufficiently rapidly in $n$:

$$h_\wedge^2\rho_n = \omega(\log n), \quad \bar{h}^2\rho_n = \omega(\max \{\bar{h}^2/n, 1\} \log^3 n).$$

Then if $\hat{f}(x, y; h)$ is fitted by blockmodel maximum profile likelihood estimation as described in Section 4 below, the mean-squared error of $\hat{f}$ satisfies

$$O_P \left( \frac{\log \bar{h}}{\bar{h}^2\rho_n} + \sqrt{\frac{\log^2 (1/\rho_n) \log (n/\bar{h})}{n\rho_n}} + \left( \frac{h_\vee}{n} \right)^{2\alpha} + \frac{\log (h_\vee/\rho_n)}{n^{\alpha/2}} \right).$$

The terms appearing in this expression each stem from a different portion of the nonparametric inference problem of graphon estimation, and will be derived and discussed in Section 5–7 below.

4. Nonparametric graphon approximation via blockmodels. To understand Theorem 3.1, we must first describe how a particular class of statistical network model—the stochastic blockmodel—lends itself naturally to nonparametric approximation. Later, in Section 5, we will establish blockmodel consistency under model misspecification, in settings ranging from dense (Chatterjee, 2012; Choi and Wolfe, 2013) to very sparse networks.

4.1. Stochastic blockmodels and nonparametric graphon approximation. A $k$-community blockmodel $(k, z, \theta)$ is a statistical network model that consists of two main components:

1. A community assignment function $z: \{1, \ldots, n\} \to \{1, \ldots, k\}$. This mapping assigns each of $n$ network nodes to exactly one of $k$ groupings or “communities,” each of size $h_a, 1 \leq a \leq k$. 
2. A block mean estimator $\theta: \{1, \ldots, k\}^n \times [0, 1]^{n \times n} \to [0, 1]^{k \times k}$. This assigns an interaction rate $\theta_{ab}$ to every pair $(a, b)$ of communities, based on the observations $\{A_{ij} : i \in z^{-1}(a), j \in z^{-1}(b)\}$.

Any community assignment function $z$ thus has two components: a vector $h(z) = (h_1, \ldots, h_k)$ of community sizes equivalent to some $H$ as defined in (3.1a), and a permutation $\Pi_z$ of $\{1, \ldots, n\}$ that re-orders the set of network nodes prior to applying the quantile function $H^{-1}(\cdot/n)$ as defined in (3.1b). Thus the community to which $z$ assigns node $i$ is determined by the composition $H^{-1} \circ \Pi_z$:

$$z_i = H^{-1}\{\Pi_z(i)/n\}, \hspace{0.5cm} 1 \leq i \leq n.$$  

(4.1)

Each $z$ thus represents a re-ordering of the network nodes, followed by a partitioning of the unit interval. Each $\theta_{ab}$ in turn describes the expected rate of interaction between the nodes in communities $a$ and $b$.

If $k$ grows with $n$, then the nonparametric properties of blockmodels come to the fore (Rohe, Chatterjee and Yu, 2011; Choi, Wolfe and Airoldi, 2012; Fishkind et al., 2013; Zhao, Levina and Zhu, 2012). In the theory of graph limits (Lovász, 2012), such a model is known as the “blowup” of a weighted graph to the domain $(0, 1)^2$, or as a “stepfunction approximation” of a given graphon $f(x, y)$.

There are strong theoretical reasons why an arbitrary graphon should be well approximated by blocks (Lovász, 2012). These reasons stem from a fundamental result in combinatorics known as Szemerédi’s regularity lemma, which cuts across graph theory, analysis and number theory. In our context, this lemma suggests that any sufficiently large graph behaves approximately like a $(k, z, \theta)$-blockmodel for some $k$. However, this value of $k$ may potentially be very large, and so regularizing strategies are needed to infer a blockmodel approximation with good risk properties while requiring relatively few degrees of freedom.

4.2. Fitting blockmodels to inhomogeneous random graphs. Once $f(x, y)$ has been specified and a uniform random sample $\{\xi_i\}_{i=1}^n$ realized, our network reduces to a set of $\binom{n}{2}$ Bernoulli($p_{ij}$) trials that are conditionally independent given $\{\xi_i\}_{i=1}^n$. We refer to this as an inhomogeneous random graph model (Bollobás, Janson and Riordan, 2007) for the observed data matrix $A \in \{0, 1\}^{n \times n}$. From (2.2), the conditional log-probability of observing a given adjacency matrix $A$ is

$$\log \Pr(A | \{p_{ij}\}_{i<j}) = \sum_{i<j} \{A_{ij} \log (p_{ij}) + (1-A_{ij}) \log (1-p_{ij})\}. $$
Adopting the notation of Choi, Wolfe and Airoldi (2012), we write the log-likelihood function of a blockmodel \((k, z, \theta)\) with respect to an observed data matrix \(A\) as

\[
L(A; z, \theta) = \sum_{i<j} \left\{ A_{ij} \log \theta_{z_iz_j} + (1 - A_{ij}) \log \left(1 - \theta_{z_iz_j}\right) \right\}, \quad 1 \leq i, j \leq n
\]

\[
= \sum_{a \leq b} \sum_{i \in z^{-1}(a), \ j \in z^{-1}(b)} \left\{ A_{ij} \log \theta_{ab} + (1 - A_{ij}) \log (1 - \theta_{ab}) \right\}, \quad 1 \leq a, b \leq k
\]

\[
= \sum_{a \leq b} \log \theta_{ab} \sum_{i \in z^{-1}(a), \ j \in z^{-1}(b)} A_{ij} + \sum_{a \leq b} \log (1 - \theta_{ab}) \sum_{i \in z^{-1}(a), \ j \in z^{-1}(b)} (1 - A_{ij})
\]

\[
(4.2) = \sum_{a \leq b} h_{ab}^2 \left\{ \bar{A}_{ab} \log \theta_{ab} + (1 - \bar{A}_{ab}) \log (1 - \theta_{ab}) \right\},
\]

where \(\bar{A}_{ab}\) is the arithmetic average of the values of \(A\) in the \((a,b)\)th block:

\[
(4.3) \bar{A}_{ab} = \frac{1}{h_{ab}^2} \sum_{i \in z^{-1}(a), \ j \in z^{-1}(b)} A_{ij}, \quad h_{ab}^2 = \begin{cases} \frac{1}{2} & \text{if } a = b, \\ h_a h_b & \text{if } a \neq b. \end{cases}
\]

and \(h_a\) is the size of the \(a\)th community. Note that this aligns with our earlier definition of \(\bar{f}\), and that the quantities \(h_{ab}^2, \bar{A}_{ab}, \theta_{ab}\) all depend on the community assignment function \(z\). The structural zeros along the main diagonal of \(A\) imply that \(h_{ab}\) differs for diagonal blocks \((a = b)\) relative to off-diagonal blocks. We see from (4.2) that for any fixed assignment \(z \in \{1, \ldots, k\}^n\), the log-likelihood \(L(A; z, \theta)\) of \(A\) will be maximized in \(\theta \in [0, 1]^{k \times k}\) by taking \(\theta_{ab} = \bar{A}_{ab}\). This is because each sample proportion \(\bar{A}_{ab}\) is an extended maximum likelihood estimator for its expectation; “extended”, because we include the boundary \([0, 1]^{k \times k}\) of the parameter space, allowing for the possibility that \(\theta_{ab} = \bar{A}_{ab} \in (0, 1)\). Thus the extended maximum likelihood estimator coincides with the method of moments estimator for \(\theta_{ab}\).

Note that (4.2) is a continuous function in \(\theta\), and so (by the extreme value theorem) \(L(A; z, \theta)\) attains its supremum over the compact set \([0, 1]^{k \times k}\). Thus we “profile out” \(\theta\) from the log-likelihood \(L(A; z, \theta)\):

\[
L(A; z) = \max_{\theta \in [0, 1]^{k \times k}} L(A; z, \theta)
\]

\[
= \sum_{a \leq b} h_{ab}^2 \left\{ \bar{A}_{ab} \log \bar{A}_{ab} + (1 - \bar{A}_{ab}) \log (1 - \bar{A}_{ab}) \right\}
\]

\[
(4.4) = \sum_{i<j} \left\{ A_{ij} \log \bar{A}_{z_i z_j} + (1 - A_{ij}) \log (1 - \bar{A}_{z_i z_j}) \right\}.
\]
Any maximizer of (4.4) over a fixed, non-empty subset $Z_k \subseteq \{1, \ldots, k\}^n$ is a maximum profile likelihood estimator (MPLE) of $z$ with respect to $Z_k$. We may equivalently re-cast the problem of likelihood maximization as one of Bernoulli Kullback–Leibler divergence minimization, with

$$D(p \mid p') = p \log \left( \frac{p}{p'} \right) + (1 - p) \log \left( \frac{1 - p}{1 - p'} \right)$$

denoting the Kullback–Leibler divergence of a Bernoulli($p'$) distribution from a Bernoulli($p$) one.

Equipped with this definition, observe that any MPLE $\hat{z}(A, Z_k)$ satisfies

$$\hat{z}(A, Z_k) = \arg\max_{z \in Z_k} \sum_{i<j} \left\{ A_{ij} \log \bar{A}_{z_i z_j} + (1 - A_{ij}) \log (1 - \bar{A}_{z_i z_j}) \right\}$$

(4.5)

$$= \arg\max_{z \in Z_k} \max_{\theta \in [0,1]^{k \times k}} L(A; z, \theta)$$

$$= \arg\min_{z \in Z_k} \min_{\theta \in [0,1]^{k \times k}} \sum_{i<j} D(A_{ij} \mid \theta_{z_i z_j})$$

$$= \arg\min_{z \in Z_k} \sum_{i<j} D(A_{ij} \mid \bar{A}_{z_i z_j}).$$

Maximizing the profile log-likelihood of (4.4) to obtain an MPLE $\hat{z}(A, Z_k)$ is thus equivalent to minimizing the sum of divergences $\sum_{i<j} D(A_{ij} \mid \bar{A}_{z_i z_j})$. This sum serves as a proxy for its “oracle” counterpart based on the matrix $p \in [0,1]^{n \times n}$ of Bernoulli parameters of the underlying generative model. This corresponds to an idealized “best blockmodel approximation” of $p$.

With this in mind, we define an “oracle MPLE” $\tilde{z}(p, Z_k)$ in direct analogy to (4.5). Let $\bar{p}(z)_{ab}$ denote the arithmetic average of the $h^2_{ab}$ elements of $p$ in the $(a,b)$th block induced by $z$:

$$\bar{p}(z)_{ab} = \frac{1}{h^2_{ab}} \sum_{i \in z^{-1}(a), j \in z^{-1}(b)} p_{ij},$$

(4.6)

where we recall that $h^2_{ab}$ also depends on the choice of community assignment function $z$. We then have

$$\tilde{z}(p, Z_k) = \arg\max_{z \in Z_k} \sum_{i<j} \left\{ p_{ij} \log \bar{p}_{z_i z_j} + (1 - p_{ij}) \log (1 - \bar{p}_{z_i z_j}) \right\}$$

(4.7)

$$= \arg\min_{z \in Z_k} \sum_{i<j} D(p_{ij} \mid \bar{p}_{z_i z_j}).$$

Observe that neither $\hat{z}(A, Z_k)$ nor $\tilde{z}(p, Z_k)$ is unique, since permuting the community labels $\{1, \ldots, k\}$ does not affect the likelihood of community
assignment in (4.5) or (4.7). Even aside from the issue of label switching, we are not guaranteed uniqueness; see Chatterjee, Diaconis and Sly (2011) and Rinaldo, Petrović and Fienberg (2013) for discussion of this issue in the specific context of network modeling, as well as Fienberg and Rinaldo (2012) in the general setting of log-linear models for sparse contingency tables.

5. Sparse blockmodel consistency under model misspecification.

We now establish that an observed matrix $A \in \{0, 1\}^{n \times n}$ of binary adjacencies yields “oracle” information on its generative $p \in \{0, 1\}^{n \times n}$ at a rate that depends both on the sparsity of the network and on the speed at which the admissible network community sizes grow with $n$. We show that for suitable sequences of sets $Z_k(n) \subseteq \{1, \ldots, k\}^n$ of admissible blockmodels, the maximum profile likelihood assignment method $\hat{z}(A, Z_k)$ implies that the likelihood risk of a fitted blockmodel, as measured by summing the divergences $D(p_{ij} \| \hat{A}_{ij})$, approaches the risk $\sum_{i,j} D(p_{ij} \| \bar{p}_{ij})$ of the best possible blockmodel approximation as $n$ grows large.

Theorem 5.1 (proving Appendix B) makes this statement precise and provides a set of sufficient conditions, driven primarily by the effective sample size of each fitted block.

**Theorem 5.1 (Controlling excess blockmodel risk).** For each $n = 2, 3, \ldots$, let $A \in \{0, 1\}^{n \times n}$ be the adjacency matrix of a simple random graph with independent Bernoulli($p_{ij}$) edges, and consider a corresponding sequence of $k$-community blockmodel estimators, with $k = k(n)$ a function of $n$. Assume:

1. The expected edge density $(n)^{-1} \sum_{i,j} p_{ij}(n)$ of $A$ does not approach 0 or 1 too rapidly in $n$: there exists a monotone non-increasing, strictly positive sequence $\bar{p}(n)$, such that for all $n$ sufficiently large, $\bar{p}(n) \leq (n)^{-1} \sum_{i<j} p_{ij}(n) \leq 1 - \sqrt{\bar{p}(n)}$.

2. Likewise, no block density $\{\bar{p}_{z_{i,j}}(n)\}_{i<j,z \in Z_k(n)}$ approaches 0 or 1 too rapidly in $n$: there exists a monotone non-increasing, strictly positive sequence $\bar{p}_h(n)$, such that $\bar{p}_h(n) \leq \bar{p}(n)$ and $\bar{p}_h(n) \leq \bar{p}_{z_{i,j}}(n) \leq 1 - \sqrt{\bar{p}_h(n)}$ for all $z \in Z_k(n)$, $1 \leq i < j \leq n$ and $n$ sufficiently large.

3. The sizes $\{h_i(n)\}_{1 \leq i \leq n, z \in Z_k(n)}$ of all possible communities grow sufficiently rapidly in $n$: there exists a monotone strictly increasing sequence $h_l(n)$ taking values in $\{2, \ldots, \lceil n/k(n) \rceil \}$ such that for all $n$ sufficiently large, $h_l(n) \leq \min_{z \in Z_k(n)} \{\min_{1 \leq i \leq n} h_i(n)\}$.

Assume that the sequences $Z_k, \bar{p}, \bar{p}_h, h_l$ are fixed in advance and independent of all other quantities. Let $h = n/k \in [1, n]$, and suppose that the minimum effective sample size of every possible fitted block, $(h)^2 \bar{p}_h$, and the
average effective sample size across all blocks, \( \bar{h}^2 \tilde{\rho} \), both grow sufficiently rapidly in \( n \):

\[
h_\Lambda^2 \rho_\Lambda = \omega(\log n), \quad \bar{h}^2 \tilde{\rho} = \omega(\max\{\bar{h}^2/n, 1\} \log^3 n).
\]

Then for all sequences of subsets \( Z_k \subseteq \{1, \ldots, k\}^n \) that respect condition 3, we have as \( n \to \infty \) that for any choice of \( z \in Z_k \), deterministic or random,

\[
\sum_{i<j: \bar{A}_{zi}z_j \notin \{0,1\}} D(\bar{p}_{ij} \mid \bar{A}_{zi}z_j) / \sum_{i<j} \bar{p}_{ij} = \mathcal{O}_P\left( \max\left\{ \frac{\log (n/\bar{h})}{\bar{h} \tilde{\rho}}, \sqrt{\frac{\log^2 (1/\rho_\Lambda) \log (n/\bar{h})}{n \bar{\rho}}} \right\} \right).
\]

For \( \hat{z}(A, Z_k) = \arg\max_{z \in Z_k} \sum_{i<j} \{ A_{ij} \log \bar{A}_{zi}z_j + (1 - A_{ij}) \log (1 - \bar{A}_{zi}z_j) \} \),

\[
\sum_{i<j: \bar{A}_{zi}z_j \notin \{0,1\}} D(\bar{p}_{ij} \mid \bar{A}_{zi}z_j) / \sum_{i<j} \bar{p}_{ij} = \mathcal{O}_P\left( \max\left\{ \frac{\log \bar{h}}{h^2 \tilde{\rho}}, \sqrt{\frac{\log^2 (1/\rho_\Lambda) \log (n/\bar{h})}{n \bar{\rho}}} \right\} \right).
\]

These results also hold marginally with respect to the model of (2.2).

Theorem 5.1 is significant because it gives conditions under which the excess risk of a fitted blockmodel converges to zero, implying that blockmodel parameters can be estimated consistently even when the true generative model giving rise to \( A \) is unknown. It predicts different rates of convergence for different network sparsity regimes. Depending on the growth of \( k \) with \( n \), either the first or the second of two rate terms in (5.2) will dominate.

We may summarize these regimes as follows:

1. **Dense networks**: If \( \rho_\Lambda \) and \( \tilde{\rho} \) remain constant in \( n \), and \( k \) grows with \( n \) as \( k = \mathcal{O}(n^{3/4}) \), then Theorem 5.1 predicts a convergence rate of at least \( \sqrt{\log(n)/n} \). If instead \( k \) grows like \( n^{\delta} \) for \( 3/4 < \delta < 1 \), then this rate will decrease to \( \log n/n^{2(1-\delta)} \).

2. **Sparse networks**: If \( \rho_\Lambda \) and \( \tilde{\rho} \) decrease like \( n^{-2\gamma} \) for \( 0 < \gamma < 1/2 \), and \( k = \mathcal{O}(n^{3/4 - \gamma/2}) \), then Theorem 5.1 predicts the rate \( \log(n)^{3/2}/n^{1/2-\gamma} \). If \( k \) grows like \( n^\delta \) for \( 3/4 - \gamma/2 < \delta < 1 - \gamma \), then this rate will decrease to \( \log n/n^{2(1-\delta-\gamma)} \).
3. **Ultra-sparse networks:** If $\rho_{\land}$ and $\bar{\rho}$ decrease like $\log(n)^{3+\beta}/n$ for $\beta > 0$, then Theorem 5.1 predicts rate $\log(n)^{-\beta/2}$ whenever $k = \mathcal{O}(n^{1/2})$, matching the regime of Choi, Wolfe and Airoldi (2012).

In each of these cases, the given conditions on $\rho_{\land}$ can be relaxed accordingly.

Theorem 5.1 is the first such result known for sparse or ultra-sparse networks—those for which $\bar{\rho} = o(1)$, so that the average number of connections per node can grow sublinearly, here as slowly as logarithmically in $n$. This complements the recent result of Choi and Wolfe (2013) for fixed-$k$ fitting of dense bipartite graphs—those for which $\rho_{\land}$ and $\bar{\rho}$ remain constant, so that the average number of connections per node grows linearly in $n$. Theorem 5.1 extends this regime, allowing for the growth of $k$ with $n$, while also yielding an improved convergence rate of $\sqrt{\log(k)/n}$ for dense graphs.

To understand why Theorem 5.1 holds in this setting, we begin by conditioning on a choice of community assignment function $z$. Blocks of network edges then comprise independent sets of independent Bernoulli trials. Conditionally upon $z$, sample proportions $\bar{A}_{zi\bar{z}_j} \mid z$ of these blocks are thus independent Poisson–Binomial variates. Without additional restrictions, however, a fitted block could be any size—even as small as a single Bernoulli trial. Thus it is necessary to constrain the set $Z_k \subseteq \{1, \ldots, k\}$ of admissible blockmodels, and also to constrain the allowable global and local sparsity of the network, so that the effective sample size of every possible $\bar{A}_{zi\bar{z}_j} \mid z$ grows in $n$. This ensures that all block-wise sample proportions $\bar{A}_{zi\bar{z}_j} \mid z$ behave like Normal variates in the large-sample limit, when appropriately standardized.

There are then two main technical challenges:

1. **Double randomness:** While every $\bar{A}_{zi\bar{z}_j} \mid z$ is amenable to analysis, choosing $\hat{z}$ by profile likelihood maximization introduces “double randomness,” coupling all blocks and precluding a direct analysis of $\bar{A}_{\hat{z}_i\hat{z}_j}$. Instead, we take the approach of Choi, Wolfe and Airoldi (2012), and show that results for $\bar{A}_{zi\bar{z}_j} \mid z$ hold uniformly for any choice of $z$ — and therefore that they also hold for $\bar{A}_{\hat{z}_i\hat{z}_j}$.

2. **Likelihood zeros:** The assumption that all $p_{ij} \in (0, 1)$ ensures that each $D(p_{ij} \mid \bar{p}_{zi\bar{z}_j})$ is finite. However, $D(p_{ij} \mid \bar{A}_{\hat{z}_i\hat{z}_j})$ will fail to be finite if $\bar{A}_{\hat{z}_i\hat{z}_j} \in \{0, 1\}$, in which case the $(\hat{z}_i, \hat{z}_j)$th block has saturated. Such blocks add 0 to the likelihood; their parameters are not estimable (Fienberg and Rinaldo, 2012). The theorem conditions allow us to control the probability of these likelihood zeros, by requiring the effective sample size of each block to grow sufficiently rapidly in $n$.

This latter point is particularly important, since only values in the interior of the parameter space $[0, 1]^{k \times k}$ are estimable (Fienberg and Rinaldo, 2012).
As in the case of additional structural zeros (Fienberg and Rinaldo, 2012, Corollary 8), the Fisher information matrix will be rank-deficient, and the degrees of freedom must be adjusted accordingly in order to obtain correct inferential conclusions. This explains why the random denominator term is necessary in the left-hand side of (5.2).

We may connect this understanding to the three sparsity regimes described above: the case of dense networks, corresponding to the setting of exchangeable random graphs; that of sparse networks, where the density of network edges \(\binom{n}{2}^{-1}\sum_{i<j} p_{ij}\) decays as some power of \(n\); and that of ultra-sparse networks, where the edge density decays at a rate approaching \(\log(n)/n\). This is the so-called connectivity threshold, above which an inhomogeneous random graph will be fully connected with probability approaching 1 as \(n \to \infty\) (Alon, 1995). If the edge density were instead to decay at a rate of \(1/n\)—the extremely sparse setting of Bollobás and Riordan (2009)—then the resulting networks would fail in general to be connected, and Poisson rather than Normal limiting behavior would hold for each block (Olhede and Wolfe, 2013).

### 6. From blockmodels to smooth graphon estimation

We now present our final result leading to consistent graphon estimation. To go beyond conditional estimation of inhomogeneous random graphs via blockmodels, we will assume additional structure via graphon smoothness. This smoothness will in turn allow us to control estimation risk, by sending the main term in Theorem 5.1 to zero.

A blockmodel first orders the rows and columns of \(A\), and then groups its entries according to a vector of community sizes \(h \in \{2, \ldots, n\}^k\). This specifies a partition \(H\) in accordance with (3.1a), which in turn induces a piecewise-constant approximation of the graphon \(f(x, y)\) along blocks. To see this, define the domain \(\omega_{ab} \subseteq [0,1)^2\) of the \((a, b)\)th block as

\[
\omega_{ab} = [H(a - 1), H(a)] \times [H(b - 1), H(b)), \quad 1 \leq a, b \leq k,
\]

and define the blockmodel approximation \(\hat{f}(x, y; h)\) of \(f(x, y)\) via the local averages \(\hat{f}_{ab}, 1 \leq a, b \leq k:\)

\[
(6.1) \quad \hat{f}(x, y; h) = \hat{f}_{H^{-1}(x)H^{-1}(y)}, \quad \hat{f}_{ab} = \frac{1}{|\omega_{ab}|} \int_{\omega_{ab}} f(x, y) \, dx \, dy.
\]

If \(f(x, y)\) is smooth as well as bounded, then results from approximation theory allow the error \(\|f - \hat{f}\|\) to be controlled in any \(L_p\) norm, as a function of the maximum over all block diameters \((h_a^2 + h_b^2)^{1/2}/n\) for \(1 \leq a, b \leq k\) (DeVore, 1998, see also Lemma C.6).
Recall from (4.1) that any blockmodel community assignment vector \( z \) is a composition \( H^{-1} \circ \Pi_z \) for some partition \( H \) of \([0, 1]\) and permutation \( \Pi_z \) of \( \{1, \ldots, n\} \), so that \( z_i = H^{-1}\{\Pi_z(i)\}/n \), \( 1 \leq i \leq n \). From (4.6), we may express \( \bar{p}(z) \) for any \( 1 \leq a, b \leq k \) as

\[
\bar{p}(z)_{ab} = \frac{1}{h_{ab}^2} \sum_{i<j} p_{ij} \mathbb{I}[H^{-1}\{\Pi_z(j)/n\} = b] \mathbb{I}[H^{-1}\{\Pi_z(i)/n\} = a]
\]

(6.2)

Thus \( \bar{p}(z)_{ab} \) is an average over \( h_{ab}^2 \) graphon evaluations \( f(\xi_{\Pi_z^{-1}(i)}, \xi_{\Pi_z^{-1}(j)}) \), since the model of (2.2) asserts that \( p_{ij}(n) \propto f(\xi_i, \xi_j) \). These evaluations occur at random points determined by \( \{\xi_1, \ldots, \xi_n\} \) according to the inverse of the permutation \( \Pi_z \), while \( H \) determines the size of each block.

From this simple observation, we will show that it is possible to relate \( \bar{p}(z)_{ab} \) to \( f(x, y) \) by choosing an “oracle” permutation \( \Pi_z(i) \) whose inverse yields the ordered sample \( \{\xi_{(1)}, \ldots, \xi_{(n)}\} \). To see this, first note that whenever the Hölder condition of (3.2) is satisfied, we have by Lemma C.7 that

\[
f(\xi_{(i)}, \xi_{(j)}) = f\left(\frac{i}{n+1}, \frac{j}{n+1}\right) + \mathcal{O}_P\left(n^{-\alpha/2}\right),
\]

because each \( \xi_{(i)} \) converges in probability to its expectation \( i/(n+1) \) at a rate no worse than \( n^{-1/2} \), and (3.2) relates this to \( |f(\xi_{(i)}, \xi_{(j)}) - f\left(\frac{i}{n+1}, \frac{j}{n+1}\right)| \).

Now take \( \Pi_z(i) = (i)^{-1} \), where \((i)^{-1}\) denotes the rank of \( \xi_i \) from smallest to largest, and observe that \( f(\xi_{\Pi_z^{-1}(i)}, \xi_{\Pi_z^{-1}(j)}) \) evaluates to \( f(\xi_{(i)}, \xi_{(j)}) \).

The key point is that when \( f \) is \( \alpha \)-Hölder continuous, then convergence of the ordered sample \( \{\xi_{(i)}\}_{i=1}^n \) governs convergence of the random averages comprising \( \bar{p}(z)_{ab} \) in (6.2). Indeed, if \( h_\nu \) uniformly upper-bounds the largest possible community size, then by Lemma C.5, we have that

\[
\Pi_z = (\cdot)^{-1} \Rightarrow \rho_n^{-1} \bar{p}_{z(\cdot)\xi_{(j)}} - \bar{f}(\xi_{(i)}, \xi_{(j)}; h) = \mathcal{O}_P\left(n^{-\alpha/2} + (n/h_\nu)^{-\alpha}\right),
\]

where we recall from (6.1) that \( \bar{f}(x, y; h) \) is the local block average of \( f \).

As a consequence, we can control the oracle estimation risk featured in Theorem 5.1 as follows.

**Theorem 6.1 (Controlling absolute risk).** Assume in the scaled exchangeable graph model of (2.2) that:

1. The graphon \( f \) is a positive, symmetric function on \([0, 1]^2\), and is \( \alpha \)-Hölder continuous, \( 0 < \alpha \leq 1 \);
Furthermore, $f$ is bounded away from zero and $\max_n \rho_n f$ is bounded away from unity;

3. Each set $Z_k(n) \subseteq \{1, \ldots, k\}^n$ of admissible blockmodel assignments has the following property: If $H$ is generated by some $z \in Z_k$, then $H^{-1} \circ \Pi \in Z_k$ for every permutation $\Pi$ of $\{1, \ldots, n\}$.

Then for $h_\vee(n)$ the largest community size in each $Z_k(n)$, the oracle likelihood risk in Theorem 5.1 satisfies

\[
\min_{z \in Z_k} \frac{\sum_{i<j} D(p_{ij} \mid \bar{p}_{z(i)z(j)})}{\sum_{i<j} p_{ij}} = O_P\left(n^{-\alpha} + (n/h_\vee)^{-2\alpha}\right).
\]

We prove this theorem in Appendix C by using the oracle choice of permutation $\cdot^{-1}$ to upper-bound the risk via a block approximation $\bar{f}(x, y; h)$ of $f(x, y)$, based on some $z^*$ which achieves the minimum in (6.3). Conditions 1 and 2 are then sufficient to guarantee the claimed rate of approximation. Condition 3 ensures that $H^{-1} \circ \cdot^{-1} \in Z_k$, since we do not know $z^*$ or the requisite ordering $\cdot^{-1}$ in advance.

7. Rates of convergence. We see directly that the rate of convergence in Theorem 6.1 depends on the Hölder continuity of $f$ in two ways: through the convergence of the ordered sample $\{\xi(i)\}_{i=1}^n$ (variance), and through the rate at which $h_\vee/n$ goes to zero in $n$ (bias). This rate is also self-scaling relative to the sparsity of the network, as it does not depend on $\rho_n$.

In contrast, Theorem 5.1 depends strongly both on the network sparsity factor $\rho_n$, as well as the minimum and average admissible block sizes, $h_\wedge$ and $\bar{h}$. The conditions of Theorem 5.1 ensure that excess blockmodel risk can be controlled under model misspecification, enabling groupings of nodes with good risk properties to be estimated, despite the variability of the data.

Together, the results of Theorems 5.1 and 6.1 enable us to establish mean-square graphon consistency at the rates indicated in Theorem 3.1, namely

\[
\mathcal{O}_{P}\left(\frac{\log \bar{h}}{\bar{h}^2 \rho_n} + \sqrt{\frac{\log^2(1/\rho_n) \log(n/\bar{h})}{np_n}} + \left(\frac{h_\vee}{n}\right)^{2\alpha} + \frac{\log(h_\vee/\rho_n)}{n^{\alpha/2}}\right).
\]

The first two terms come directly from Theorem 5.1, while the third is from Theorem 6.1. The final term comes from relating the discrete quantities featured in these theorems to the graphon itself, and is driven in part by the fact that we do not know the ordering of the data relative to the Uniform$(0, 1)$ variates $\{\xi(i)\}_{i=1}^n$ by which the graphon is sampled. The $\mathcal{O}(n^{-1/2})$ variance of the ordered sample $\{\xi(i)\}_{i=1}^n$ subsequently appears, and is modulated by the regularity of the graphon through its Hölder continuity exponent $\alpha$. 

8. Conclusion. In this article we have established a number of new results within a nonparametric framework for network inference, based on graphons as natural limiting objects. Understanding graphons as analytic objects, as well as the behavior of dense and sparse networks based on them, is fundamental to advancing our nonparametric understanding of networks.

To this end, we have established consistency of graphon estimation under general conditions, giving rates which include the important practical setting of sparse networks. By treating dense and sparse stochastic blockmodels with a growing number of classes, under model misspecification, our results improve substantially upon what is currently known in the literature.

Our results link strongly to approximation theory, nonparametric function estimation, and the theory of graph limits, and thus provide for a foundational understanding of nonparametric statistical network analysis.

APPENDIX A: PROOF OF THEOREM 3.1 AND ITS LEMMAS

A.1. Proof of Theorem 3.1.

Proof. We note from Lemma A.1 that for \((x, y) \in (0, 1)^2\)

\[
\hat{f}(x, y; h) = \hat{\rho}_n^{-1} \bar{A}_{H^{-1}(x)H^{-1}(y)} = \{1 + O_P(n^{-1/2})\} \rho_n^{-1} \bar{A}_{H^{-1}(x)H^{-1}(y)}.
\]

Recalling the definition of \(\bar{A}_{ab}\), we see that uniformly for all choices of \(H\) and \(\Pi\), and for all \(1 \leq a, b \leq k\), we have \(0 \leq E \bar{A}_{ab} \leq \rho_n \sup_{(x,y) \in (0,1)^2} f(x, y)\) and \(0 \leq E \bar{A}^2_{ab} \leq \rho_n^2 \sup_{(x,y) \in (0,1)^2} f^2(x, y)\).

Since \(f\) is by hypothesis Hölder continuous on a bounded domain, it is bounded, and thus \(\bar{A}_{ab} = O_P(\rho_n)\) and \(\bar{A}^2_{ab} = O_P(\rho_n^2)\) by Markov’s inequality. We will thus expand the squared error term in the integrand of the graphon mean-squared error pointwise, using the fact that the error term should be evaluated at the infimum over measure preserving bijections. Therefore this error be upper-bounded by its evaluation at some \(\sigma^* \in \mathcal{M}\), which we will choose in accordance with the proof of Lemma A.3 below:

\[
\inf_{\sigma \in \mathcal{M}} \int_{(0,1)^2} \int \left| f(\sigma(x), \sigma(y)) - \{1 + O_P(n^{-1/2})\} \rho_n^{-1} \bar{A}_{H^{-1}(x)H^{-1}(y)} \right|^2 \, dx \, dy
\leq \int_{(0,1)^2} \int \left| f(\sigma^*(x), \sigma^*(y)) - \rho_n^{-1} \bar{A}_{H^{-1}(x)H^{-1}(y)} \right|^2 \, dx \, dy + O_P(n^{-1/2})
\leq \int_{f \notin (0,1)} \int \left| f(\sigma^*(x), \sigma^*(y)) - \rho_n^{-1} \bar{A}_{H^{-1}(x)H^{-1}(y)} \right|^2 \, dx \, dy + O_P(n^{-1/2})
\leq 2 (\sup f) \int_{f \notin (0,1)} \rho_n^{-1} \int \rho_n^{-1} D \left\{ \rho_n f(\sigma^*(x), \sigma^*(y)) \left| \rho_n f(x, y; h) \right\} dx \, dy + O_P(n^{-1/2}),
\]

where \(D\) denotes the total variation distance.
where the last two lines follow from Lemmas A.2 and C.9, respectively. By Lemma A.3, we have

\[
2 \left( \sup f \right) \iint \rho_n^{-1} D \left\{ \rho_n f (\sigma^*(x), \sigma^*(x)) \left| \rho_n \hat{f} (x, y; h) \right. \right\} dx dy = 2 \left( \sup f \right)
\]

\[
\cdot \iint f (x, y) \; dx \; dy \sum_{i < j, \hat{A}_{ij} \notin \{0, 1\}} \frac{D(p_{ij} || \hat{A}_{ij})}{\sum_{i < j, \hat{A}_{ij} \notin \{0, 1\}} p_{ij}} \left\{ 1 + O_P \left( n^{-\alpha/2} \right) \right\}
\]

\[
+ O_P \left( \frac{\log (h \sqrt{\rho_n})}{n^{\alpha/2}} + \frac{\log h}{\rho_n n} \right),
\]

uniformly in z. The conditions of Theorem 3.1 are sufficient for Theorems 5.1 and 6.1 to hold, and so if \( f \) is fitted by maximum profile likelihood, then we may substitute terms from Theorems 5.1 and 6.1 to obtain

\[
2 \left( \sup f \right) \iint \rho_n^{-1} D \left\{ \rho_n f (\sigma^*(x), \sigma^*(x)) \left| \rho_n \hat{f} (x, y; h) \right. \right\} dx dy = 2 \left( \sup f \right)
\]

\[
\cdot \iint f (x, y) \; dx \; dy \left[ O_P \left( n^{-\alpha} + \left( \frac{n}{h^2} \right)^{-2\alpha} \right) + O_P \left( \max \left\{ \frac{\log h}{h^2 \rho_n}, \sqrt{\frac{\log^2 h \rho_n}{n \rho_n}} \right\} \right) \right]
\]

\[
+ O_P \left( \frac{\log (h \sqrt{\rho_n})}{n^{\alpha/2}} + \frac{\log h}{\rho_n n} \right). \tag*{\Box}
\]

### A.2. Auxiliary lemmas needed for Theorem 3.1.

**Lemma A.1.** Assume the setting of Theorem 3.1. Then \( \hat{\rho}_n = \rho_n \), \( \text{var} \; \hat{\rho}_n = O \left( \rho_n^2 / n \right) \).

**Proof.** Since \( i < j \) and \( k < l \), we have that \( E A_{ij} | \xi = \rho_n f (\xi_i, \xi_j) \) and \( \text{cov} (A_{ij}, A_{kl} | \xi) = \rho_n f (\xi_i, \xi_j) \{ 1 - \rho_n f (\xi_i, \xi_j) \} \mathbb{1} (i = k) \mathbb{1} (j = l) \). We first use the law of total expectation to deduce

\[
E \rho_n = \left( \frac{n}{2} \right)^{-1} \sum_{i < j} E \xi \{ \rho_n f (\xi_i, \xi_j) \} = \rho_n \iint_{(0, 1)^2} f(x, y) \; dx \; dy = \rho_n.
\]

The necessary marginal variances and covariances can then be established hierarchically:

\[
\text{var}(A_{ij}) = E \xi \{ \text{var} (A_{ij} | \xi) \} + \text{var} \{ E(A_{ij} | \xi) \}
\]

\[
= \{ E \rho_n f (\xi_i, \xi_j) \} \{ 1 - E \rho_n f (\xi_i, \xi_j) \} = \rho_n \left( 1 - \rho_n \right),
\]

\[
\text{cov}(A_{ij}, A_{kl}) = E \xi \{ \text{cov} (A_{ij}, A_{kl} | \xi) \} + \text{cov} \{ E(A_{ij} | \xi), E(A_{kl} | \xi) \}, (i, j) \neq (k, l).
\]
Since \( E f (\xi_i, \xi_j) f (\xi_k, \xi_l) = \int_{(0,1)^2} f^2 (x, y) d x d y \) if \( i = k \) and \( j = l \), and 
\[ \left\{ \int_{(0,1)^2} f (x, y) d x d y \right\}^2 \] if \( i \neq k \) and \( j \neq l \), we obtain when either \( i \neq k \) or \( j \neq l \) that

\[
\text{cov}_\xi (A_{ij}, A_{kl}) = \text{cov}_\xi \{ E (A_{ij} \mid \xi), E (A_{kl} \mid \xi) \} \\
= E_\xi \{ \rho_n f (\xi_i, \xi_j) \rho_n f (\xi_k, \xi_l) \} - E_\xi \{ \rho_n f (\xi_i, \xi_j) \} E_\xi \{ \rho_n f (\xi_k, \xi_l) \}
\leq \rho_n^2 \max \{ \text{var} f (\xi_i, \xi_j), \text{var} f (\xi_k, \xi_l) \}
\leq \rho_n^2 \left( \int_{(0,1)^2} \{ f (x, y) \}^2 d x d y - \left\{ \int_{(0,1)^2} f (x, y) d x d y \right\}^2 \right).
\]

Because \( \text{cov}_\xi \{ A_{ij}, A_{kl} \} = 0 \) when all \( i, j, k, \) and \( l \) are distinct, and since \( i \neq j \) and \( k \neq l \), we obtain

\[
\text{var} \rho_n = \begin{pmatrix} n \end{pmatrix}^{-2} \sum_{i < j} \text{var} A_{ij} + \begin{pmatrix} n \end{pmatrix}^{-2} \sum_{i \neq k \land j \neq l} \text{cov} (A_{ij}, A_{kl})
\leq \begin{pmatrix} n \end{pmatrix}^{-2} \rho_n (1 - \rho_n) + \begin{pmatrix} n \end{pmatrix}^{-2} \sum_{i \neq k \land j \neq l} \text{cov} (A_{ij}, A_{kl}) [\mathbb{I} (i = k) + \mathbb{I} (i = l)
+ \mathbb{I} (j = k) + \mathbb{I} (j = l)]
\leq \begin{pmatrix} n \end{pmatrix}^{-2} \rho_n (1 - \rho_n) + 4n \begin{pmatrix} n \end{pmatrix}^{-2} \rho_n^2 \left( \int_{(0,1)^2} \{ f (x, y) \}^2 d x d y - 1 \right).
\]

The order term of \( \mathcal{O} (\rho_n^2 / n) \) follows, as \( \rho_n^2 / n \geq \rho_n / n^2 \Leftrightarrow \rho_n \geq 1 / n \), since \( \rho_n = \omega (n^{-1} \log^3 n) \).

**Lemma A.2.** Assume the setting of Theorem 3.1. Then

\[
\sup_{\sigma \in \mathcal{M}} \int_{\mathcal{J} \in \{0,1\}} | f (\sigma (x), \sigma (x)) - \hat{f} (x, y; h) |^2 d x d y = \mathcal{O}_p \left( e^{-\left( \begin{pmatrix} n \end{pmatrix}^2 \right) \rho_n + 2 \log (1 / \rho_n) \right).\]

**Proof.** We apply Lemma B.2 to control \( \sum_{i < j} \mathbb{I} (\tilde{A}_{ab} \in \{0,1\}) \) marginally, after observing that

\[
\sup_{\sigma \in \mathcal{M}} \int_{\mathcal{J} \in \{0,1\}} | f (\sigma (x), \sigma (x)) - \hat{f} (x, y; h) |^2 d x d y \leq \int_{\mathcal{J} \in \{0,1\}} 2 \rho_n^{-2} d x d y
= 2 (\rho_n)^{-2} \sum_{a,b: A_{ab} \in \{0,1\}} h_a h_b \leq 2 (\rho_n)^{-2} \sum_{a \leq b: \tilde{A}_{ab} \in \{0,1\}} 4 h_{ab}^2
= 8 (\rho_n)^{-2} \sum_{i < j} \mathbb{I} (\tilde{A}_{ab} \in \{0,1\}).
\]

\( \square \)
Lemma A.3. Assume the setting of Theorem 3.1. Then for any $z \in \mathbb{Z}_k,$

$$\inf_{\sigma \in \mathcal{M}} \iint \mathbb{E}_\mathbb{P} \rho_n^{-1} D \left\{ \rho_n f (\sigma(x), \sigma(y)) \bigg| \rho_n \hat{f} (x, y; h) \right\} \, dx \, dy$$

(A.1)  

$$= \sum_{i < j} \frac{A_{ij}}{A_{ii} \Delta z} \mathcal{D} \left( \hat{\varphi}_{ij} \right) \left\{ 1 + \mathcal{O}_P \left( \frac{1}{n^{\alpha/2}} \right) \right\} + \mathcal{O}_P \left( \frac{\log \left( \frac{h_{ij}}{\rho_n} \right)}{n \rho_n} \right) + \mathcal{O}_P \left( \frac{\log h_{ij}}{\rho_n n} \right).$$

Proof. We first treat the numerator of (A.1), whose infimum is over $\mathcal{M},$ the set of all measure-preserving bijective maps of the form $\sigma : [0, 1] \to [0, 1].$ We may write

$$0 \leq \inf_{\sigma \in \mathcal{M}} \iint \mathbb{E}_\mathbb{P} \rho_n^{-1} D \left\{ \rho_n f (\sigma(x), \sigma(y)) \bigg| \rho_n \hat{f} (x, y; h) \right\} \, dx \, dy$$

(A.2)  

$$= \inf_{\sigma \in \mathcal{M}} \sum_{a, b} \frac{A(z)_{ab}}{A(z)_{ii}} \iint \mathbb{E}_\mathbb{P} \rho_n^{-1} D \left\{ \rho_n f (\sigma(x), \sigma(y)) \bigg| \hat{A}(z)_{ab} \right\} \, dx \, dy,$$

since $\hat{f}$ is constant on blocks. Observe that for each individual summand in (A.2), we may write

$$\iint \mathbb{E}_\mathbb{P} \rho_n^{-1} D \left\{ \rho_n f (\sigma(x), \sigma(y)) \bigg| \hat{A}(z)_{ab} \right\} \, dx \, dy$$

(A.3)  

$$= \int_{H(b)} \int_{H(a)} \rho_n^{-1} D \left\{ \rho_n f (\cdot) \bigg| \hat{A}(z)_{ab} \right\} \, dx \, dy$$

$$= \sum_{j = nH(b) + 1}^{nH(b)} \sum_{i = nH(a) + 1}^{nH(a)} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \rho_n^{-1} D \left\{ \rho_n f (\sigma(x), \sigma(y)) \bigg| \hat{A}(z)_{ab} \right\} \, dx \, dy.$$

We now restrict our choice of $\sigma \in \mathcal{M}$ to satisfy the following property:

$$\int_{i/n}^{(i+1)/n} \int_{j/n}^{(j+1)/n} f (\sigma(x), \sigma(y)) \, dx \, dy = \int_{j/(n+1)}^{(j+1)/(n+1)} \int_{i/(n+1)}^{(i+1)/(n+1)} f (x, y) \, dx \, dy, \quad 1 \leq i, j \leq n,$$

(A.4)  

for some permutation $\Pi$ of $\{1, \ldots, n\}.$ Such a choice of measure-preserving bijection can always be made, as it simply partitions the unit interval into $n+1$ subintervals of the form $\left[ (i-1)/n, i/n \right), \quad 1 \leq i \leq n,$ and permutes their order in accordance with $\Pi.$ We make this choice in order to preserve the Hölder continuity of $f$ on each domain $(x, y) \in \left[ (i-1)/n, i/n \right) \times \left[ (j-1)/n, j/n \right),$ as will be shown below.
Thus we may write, combining (A.2)–(A.6),

\[
\inf_{\sigma \in \mathcal{M}} \int_{f \notin \{0,1\}} \rho_n^{-1} D \left\{ \rho_n f(\sigma(x), \sigma(y)) \bigg| \rho_n \hat{f}(x, y; h) \right\} \, dx \, dy \\
\leq \min_{\Pi \in S_n} \sum_{a,b} \sum_{j=nH(b-1)+1}^{nH(b)} \sum_{i=nH(a-1)+1}^{nH(a)} \rho_n^{-1} D \left\{ \rho_n f(x, y) \bigg| \tilde{A}(z)_{ab} \right\} \, dx \, dy,
\]

with \( S_n \) the set of permutations of \( \{1, \ldots, n\} \). From Lemma A.4 we then obtain

\[
n^2 \int_{\Pi(i)}^{\Pi(i)} \int_{\Pi(j)}^{\Pi(j)} \rho_n^{-1} D \left\{ \rho_n f(x, y) \bigg| \tilde{A}(z)_{ab} \right\} \, dx \, dy \\
= \rho_n^{-1} D \left[ \rho_n f \left( \xi(\Pi(i)), \xi(\Pi(j)) \right) \bigg| \tilde{A}(z)_{ab} \right] + O_P \left( \left\{ \log \left( 1/\rho_n \right) + \log \left( h^2 \right) \right\} n^{-\alpha/2} \right),
\]

where \( \xi(\Pi(i)) \) is the \( \Pi(i) \)-th element of the ordered sample \( \{\xi(i)\}_{i=1}^n \). Starting from (A.5), we then have

\[
\inf_{\sigma \in \mathcal{M}} \int_{f \notin \{0,1\}} \rho_n^{-1} D \left\{ \rho_n f(\sigma(x), \sigma(y)) \bigg| \rho_n \hat{f}(x, y; h) \right\} \, dx \, dy \\
\leq \min_{\Pi \in S_n} \frac{1}{n^2} \sum_{a,b} \sum_{j=nH(b-1)+1}^{nH(b)} \sum_{i=nH(a-1)+1}^{nH(a)} \left[ 1 + \mathbb{I} \left( a = b \right) + \mathbb{I} \left( i = j \right) \right] \\
\cdot \rho_n^{-1} D \left[ \rho_n f \left( \xi(\Pi(i)), \xi(\Pi(j)) \right) \bigg| \tilde{A}(z)_{ab} \right] + O_P \left( \left\{ \log \left( 1/\rho_n \right) + \log \left( h^2 \right) \right\} n^{-\alpha/2} \right) \\
\leq \frac{1}{n^2} \sum_{i<j} \rho_n^{-1} D \left\{ \rho_{ij} \bigg| \tilde{A}(z)_{ij} \right\} + \sum_{1 \leq i \leq n: \tilde{A}(z)_{ij} \notin \{0,1\}} \rho_n^{-1} D \left\{ \rho_n f \left( \xi(i), \xi(j) \right) \bigg| \tilde{A}(z)_{ij} \right\} \\
+ O_P \left( \left\{ \log \left( 1/\rho_n \right) + \log \left( h^2 \right) \right\} n^{-\alpha/2} \right),
\]

where we have chosen \( \Pi = (\cdot)^{-1} \circ \Pi_i^{-1} \), so that \( \Pi(i) = (\Pi_i^{-1}(i))^{-1} \), with \( (i)^{-1} \) the rank of \( \xi(i) \), from smallest to largest. This choice allows us to match each \( \xi(\Pi(i)) \) to the corresponding group assignment \( z_i \) of the \( i \)-th network node. To see this, recall from (4.1) that \( z_i = H^{-1} \{ \Pi_i(i) / n \} \), \( 1 \leq i \leq n \), and...
from (4.3) and (C.6) respectively that

$$
\bar{A}(z)_{ab} = \frac{1}{b_{ab}} \sum_{j=nH(b)-1}^{nH(b)} \sum_{i=nH(a)-1}^{nH(a)} A_{\Pi^{-1}(i)\Pi^{-1}(j)},
$$

$$
\bar{p}(z)_{ab} = \frac{1}{b_{ab}} \sum_{j=nH(b)-1}^{nH(b)} \sum_{i=nH(a)-1}^{nH(a)} \rho_n f \left( \xi_{\Pi^{-1}(i)}, \xi_{\Pi^{-1}(j)} \right).
$$

Note that $\bar{p}(z)_{ab} = E \{ A(z)_{ab} | \xi, z \}$. Thus we relate each $p_{ij} = \rho_n f (\xi_i, \xi_j)$ to the average $A(z)_{zi,zj}$ of the block to which it is assigned by $z$.

Continuing from (A.6), we appeal to Lemma A.5 to bound the diagonal term, thereby obtaining

$$
\inf_{\sigma \in \mathcal{M}} \int_{f \notin \{0,1\}} \rho_n^{-1} D \left\{ \rho_n f (\sigma(x), \sigma(y)) \parallel \rho_n f (x, y; h) \right\} dx dy
$$

$$
\leq \frac{1}{(n)} \sum_{2 < j < i, A(z)_{zi,zj} \notin \{0,1\}} \rho_n^{-1} D \left\{ p_{ij} \parallel A(z)_{zi,zj} \right\} + \mathcal{O}_P \left( \left\{ \log (1/\rho_n) + \log \left( \frac{h}{\rho} \right) \right\} n^{-\alpha/2} + \log \left( \frac{h}{\rho} \right) (\rho_n)^{-1} \right).
$$

Lemma A.6 yields the denominator of (A.1), and the result follows by taking the ratio of these terms.

**Lemma A.4.** Assume the setting of Theorem 3.1. Then for $1 \leq i, j \leq n$, $(a, b) : A(z)_{ab} \notin \{0, 1\}$

$$
\bar{A}(z)_{ab} = \frac{1}{b_{ab}} \sum_{j=nH(b)-1}^{nH(b)} \sum_{i=nH(a)-1}^{nH(a)} A_{\Pi^{-1}(i)\Pi^{-1}(j)},
$$

$$
= \rho_n^{-1} D \left\{ \rho_n f (\xi_{\Pi^{-1}(i)}, \xi_{\Pi^{-1}(j)}) \parallel A(z)_{ab} \right\} + \mathcal{O}_P \left( \left\{ \log (1/\rho_n) + \log \left( \frac{h}{\rho} \right) \right\} n^{-\alpha/2} \right).
$$

**Proof.** The result follows from a Taylor series of the integrand of (A.7), which we will show to converge everywhere on the domain of integration, as long as $A(z)_{ab} \notin \{0, 1\}$. We begin by noting that whenever $f \in H^\alpha(M)$, we have from Lemma C.7 that for all $(x, y) \in \left( \frac{i-1}{n}, \frac{i}{n} \right) \times \left( \frac{j-1}{n}, \frac{j}{n} \right)$,

$$
E \left| f(x, y) - f(\xi_{(i)}, \xi_{(j)}) \right| \leq E \left| f(x, y) - f\left( \frac{i}{n+1}, \frac{j}{n+1} \right) \right|
$$

$$
+ E \left| f\left( \frac{i}{n+1}, \frac{j}{n+1} \right) - f(\xi_{(i)}, \xi_{(j)}) \right| \leq M \left\{ 2^{-1/2} (n+1)^{-\alpha} + M \{2(n+2)^{-\alpha/2} \right.
$$
From Markov's inequality, $f(\xi_{ij}, \xi_{ij}) = f(x, y) + \mathcal{O}_P(n^{-\alpha/2})$ for every fixed $(x, y)$ in the domain of interest. Thus the following Taylor series holds whenever $f \in \text{Hölder}^\alpha(M)$ and $\bar{A}(z)_{ab} \notin \{0, 1\}$:

\begin{align}
(A.8) & \quad \rho_n^{-1} \mathbb{D} \left\{ \rho_n f(\xi_{ij}, \xi_{ij}) \bigg| \bar{A}(z)_{ab} \right\} = \rho_n^{-1} \mathbb{D} \left\{ \rho_n f(x, y) \bigg| \bar{A}(z)_{ab} \right\} \\
& \quad + \log \left\{ \frac{\rho_n f(x, y)}{1 - \rho_n f(x, y)} \cdot \frac{1 - \bar{A}(z)_{ab}}{A(z)_{ab}} \right\} \{ f(\xi_{ij}, \xi_{ij}) - f(x, y) \} + \mathcal{O}_P(n^{-\alpha/2}).
\end{align}

To bound the second term in (A.8), let $l = \inf_{x \in (0, 1)} f(x, x)$ and $u = \sup_{x \in (0, 1)} f(x, x)$. Since $\bar{A}(z)_{aa} \notin \{0, 1\}$, we may bound the magnitudes of $\log \bar{A}(z)_{aa}$, $\log \{1 - \bar{A}(z)_{aa}\}$ via $\log \left(\frac{h_v}{2}\right) \leq \log \left(\frac{h_v}{2}\right)$. Then

\begin{align}
(A.9) & \quad \mathbb{E} \left| \log \left\{ \frac{\rho_n f(x, y)}{1 - \rho_n f(x, y)} \cdot \frac{1 - \bar{A}(z)_{ab}}{A(z)_{ab}} \right\} \right| \leq \log \left\{ (\rho_n l)^{-1} \right\} \\
& \quad + \log \left\{ (1 - \rho_n u)^{-1} \right\} + 2 \log \left(\frac{h_v}{2}\right).
\end{align}

The first two terms in (A.9) are bounded by hypothesis, and then we apply Markov’s inequality to (A.8).

**Lemma A.5.** Assume the setting of Theorem 3.1. Then

\begin{align}
(A.10) & \quad n^{-2} \sum_{1 \leq i \leq n: A(z)_{zi, zi} \notin \{0, 1\}} \rho_n^{-1} \mathbb{D} \left\{ \rho_n f(\xi_i, \xi_i) \bigg| A(z)_{zi, zi} \right\} \\
& \quad = \mathcal{O}_P \left( \left\{ \log (1/\rho_n) + \rho_n^{-1} \log \left(\frac{h_v}{2}\right) \right\} n^{-1} \right).
\end{align}

**Proof.** Let $l = \inf_{x \in (0, 1)} f(x, x)$ and $u = \sup_{x \in (0, 1)} f(x, x)$. Since $\bar{A}(z)_{aa} \notin \{0, 1\}$, we may bound the magnitudes of $\log \bar{A}(z)_{aa}$ and $\log \{1 - \bar{A}(z)_{aa}\}$ via $\log \left(\frac{h_v}{2}\right) \leq \log \left(\frac{h_v}{2}\right)$. We bound the expectation of each summand in (A.10) for $1 \leq i \leq n$

\begin{align*}
\mathbb{E} \left[f(\xi_i, \xi_i) \log \left\{ \frac{\rho_n f(\xi_i, \xi_i)}{A(z)_{zi, zi}} \right\} \right] & \leq u \left\{ \log (\rho_n l)^{-1} + \log \left(\frac{h_v}{2}\right) \right\} + \rho_n^{-1} \left[ \log \left\{ (1 - \rho_n u)^{-1} \right\} + \log \left(\frac{h_v}{2}\right) \right] \\
& \leq u \left\{ \log (\rho_n l)^{-1} + \log \left(\frac{h_v}{2}\right) \right\} + \rho_n^{-1} \left[ \log \left\{ (1 - \rho_n u)^{-1} \right\} + \log \left(\frac{h_v}{2}\right) \right] \\
& = \mathcal{O}_P \left( \log (1/\rho_n) + \rho_n^{-1} \log \left(\frac{h_v}{2}\right) \right).
\end{align*}

The result then follows from linearity of expectation and Markov’s inequality, as per Lemma A.4. \qed
LEMMA A.6. Assume the setting of Theorem 3.1. Then

\[
\int \int_{f \notin \{0,1\}} f(x, y) \, dx \, dy = \frac{1}{\rho_n(n^2)} \sum_{i<j: \bar{A}(z)_{zi}z_j \notin \{0,1\}} p_{ij} + O_P(n^{-\alpha/2}).
\]

PROOF. We start by discretizing the integral. We therefore write that

\[
\int \int_{f \notin \{0,1\}} f(x, y) \, dx \, dy = \sum_{a,b: \bar{A}(z)_{ab} \notin \{0,1\}} \sum_{j=\min(nH(b)+1,i=nH(a)+1)}^{nH(a)} \sum_{i=\min(nH(b)+1,j=nH(a)+1)}^{nH(b)} \rho_n n^2 + \sum_{a,b: \bar{A}(z)_{ab} \notin \{0,1\}} \sum_{j=\min(nH(b)+1,i=nH(a)+1)}^{nH(a)} \sum_{i=\min(nH(b)+1,j=nH(a)+1)}^{nH(b)} \rho_n n^2
\]

\[
\cdot \int \int_{f \notin \{0,1\}} f(x, y) \, dx \, dy = \sum_{a,b: \bar{A}(z)_{ab} \notin \{0,1\}} \sum_{j=\min(nH(b)+1,i=nH(a)+1)}^{nH(a)} \sum_{i=\min(nH(b)+1,j=nH(a)+1)}^{nH(b)} \rho_n n^2 \cdot \int \int_{f \notin \{0,1\}} f(x, y) \, dx \, dy,
\]

where the latter term may be bounded using the technique of Lemma A.4, yielding

\[(A.11) \quad \left| \int \int_{f \notin \{0,1\}} f(x, y) \, dx \, dy - \sum_{i,j: \bar{A}(z)_{zi}z_j \notin \{0,1\}} p_{ij} \rho_n n^2 \right| = O_P(n^{-\alpha/2}).\]

Note \(\sum_{i,j: \bar{A}(z)_{zi}z_j \notin \{0,1\}} p_{ij} = 2 \sum_{i<j: \bar{A}(z)_{zi}z_j \notin \{0,1\}} p_{ij} + \sum_{1 \leq i \leq n: \bar{A}(z)_{zi}z_j \notin \{0,1\}} p_{ii},\)
so that

\[
E \rho_n^{-1} n^{-2} \sum_{1 \leq i \leq n: \bar{A}(z)_{zi}z_j \notin \{0,1\}} p_{ii} \leq n^{-2} \sum_{i=1}^n E f(x_i, x_i) = O(n^{-1}).
\]

Applying Markov’s theorem and combining the result with (A.11) then yields the stated result. \(\square\)

APPENDIX B: PROOF OF THEOREM 5.1 AND LEMMAS

B.1. Proof of Theorem 5.1. The proof is divided into four steps, with each the subject of a technical lemma proved in Section B.2.

Lemma B.1 yields the key first step, which is to relate \(D(p_{ij} \mid \bar{A}_{zi}z_j)\) to \(D(p_{ij} \mid \bar{p}_{zi}z_j)\) for any \(z \in Z_k\), assuming that \(\bar{A}_{zi}z_j \notin \{0,1\}\). This ensures that both terms are finite, and hence comparable. To obtain sufficient variance reduction in this setting, every \(\bar{A}_{zi}z_j\) must concentrate to its mean \(\bar{p}_{zi}z_j\), in that the ratio of mean to standard deviation must shrink. The minimum effective block sample size \((h/\lambda)^{\rho_h}\) must grow quickly enough that this takes place, even for the sparsest of all possible fitted blocks.
**Lemma B.1.** Assume conditions 1–3 of Theorem 5.1, and that \((h^\lambda)_{\rho_\lambda} = \omega(\log(\frac{h^\lambda}{2}))\). Then

\[
0 \leq \sum_{i<j: \bar{A}_{izj} \notin \{0,1\}} \left\{ D(p_{ij} \mid \bar{A}_{izj}) - D(p_{ij} \mid \bar{p}_{izj}) \right\} = \mathcal{O}_P \left( \frac{2 \log |Z_k| + \frac{(k+1)}{2} \sum_{i<j} p_{ij}}{(\frac{a}{2})^2} \right), \forall z \in Z_k.
\]

Our next step relies on controlling \(\Pr(\bar{A}_{izj} \in \{0,1\})\) uniformly in \(z\), via Lemma B.2.

**Lemma B.2.** Assume conditions 1–3 of Theorem 5.1. Then

\[
\sum_{i<j} \mathbb{I}(\bar{A}_{izj} \in \{0,1\}) = \mathcal{O}_P \left( e^{-\left(\frac{h^\lambda}{2}\right)_{\rho_\lambda} + \log(1/\bar{\rho})} \sum_{i<j} p_{ij} \right), \forall z \in Z_k.
\]

This result shows that the set of all \(\bar{A}_{izj} \in \{0,1\}\) has vanishing relative cardinality relative to \(\sum_{i<j} p_{ij}\), no matter which \(z \in Z_k\) is chosen. It is a direct consequence of condition 3 of Theorem 5.1, which ensures that the minimum fitted block size is uniformly lower-bounded by \(h^\lambda = \omega(1)\).

Lemma B.2 has two immediate consequences. First, we may apply it to conclude that

\[(B.1) \quad \frac{\sum_{i<j: \bar{A}_{izj} \notin \{0,1\}} p_{ij}}{\sum_{i<j} p_{ij}} = 1 + \mathcal{O}_P \left( e^{-\left(\frac{h^\lambda}{2}\right)_{\rho_\lambda} + \log(1/\bar{\rho})} \right), \forall z \in Z_k.
\]

Second, it enables us to substitute for the term \(\sum_{i<j: \bar{A}_{izj} \notin \{0,1\}} D(p_{ij} \mid \bar{p}_{izj})\) in Lemma B.1 as follows.

**Lemma B.3.** Assume conditions 1–3 of Theorem 5.1. Then uniformly for all \(z \in Z_k\),

\[
0 \leq \sum_{i<j} D(p_{ij} \mid \bar{p}_{izj}) - \sum_{i<j: \bar{A}_{izj} \notin \{0,1\}} D(p_{ij} \mid \bar{p}_{izj}) = \mathcal{O}_P \left( e^{-\left(\frac{h^\lambda}{2}\right)_{\rho_\lambda} + \log(1/\bar{\rho})} \sum_{i<j} p_{ij} \right).
\]

Thus whenever all of the above quantities are \(o_P(1)\), we may combine Lemmas B.1 and B.3 with (B.1) to obtain our first claimed result: for any choice of \(z \in Z_k\), deterministic or random, we have that

\[(B.2) \quad \frac{\sum_{i<j: \bar{A}_{izj} \notin \{0,1\}} D(p_{ij} \mid \bar{A}_{izj}) - \sum_{i<j: \bar{A}_{izj} \notin \{0,1\}} D(p_{ij} \mid \bar{p}_{izj})}{\sum_{i<j: \bar{A}_{izj} \notin \{0,1\}} p_{ij}} = \mathcal{O}_P \left( \frac{2 \log |Z_k| + \frac{(k+1)}{2}}{(\frac{a}{2})^2} + e^{-\left(\frac{h^\lambda}{2}\right)_{\rho_\lambda} + \log(1/\bar{\rho})} \right)
\]
whenever conditions 1–3 of Theorem 5.1 hold, \((h_\lambda)\rho_\lambda = \omega(\log(h_\lambda))\) and the argument of the right-hand side of (B.2) is \(o_p(1)\). Under these conditions, the numerator term of (B.2), when scaled by \(\sum_{i<j} p_{ij}\), converges in probability to 0 and hence in law, whereas (B.1) converges in probability to a non-zero constant. Thus by Slutsky’s theorem, their ratio converges in law, and hence also in probability as per (B.2). Separating terms on the left-hand side of (B.2), and then multiplying the latter numerator term by \(\sum_{i<j} p_{ij}/\sum_{i<j} p_{ij}\), we obtain the first result of result of Theorem 5.1, as stated in (5.1).

We now establish sufficient conditions for (B.2). We see immediately that \((h_\lambda)\rho_\lambda = \omega(\log(1/\rho))\) must hold. Since Lemma B.1 requires that \((h_\lambda)/2\rho_\lambda = \omega(\log(h_\lambda))\), we obtain the combined requirement

\[
(B.3) \quad h_\lambda^2 \rho_\lambda = \omega(\log \{h_\lambda^2, \log(1/\rho)\}) \iff h_\lambda^2 \rho_\lambda = \omega(\log n).
\]

To see that this condition will be satisfied if the effective sample size of every possible fitted block is \(\omega(\log n)\), first note that \(h_\lambda \leq n\), and so \(h_\lambda^2 = O(\log n)\). Now observe that because \(\rho_\lambda \leq \rho\), it follows that \(h_\lambda^2 \rho_\lambda = \omega(\log h_\lambda^2)\) implies \(h_\lambda^2 \rho = \omega(\log h_\lambda^2)\), or equivalently, \(\log(1/\rho) = o(\log(h_\lambda^2/\log h_\lambda^2))\). Since \(h_\lambda \leq n\), this in turn implies \(\log(1/\rho) = o(\log n)\). Thus \(h_\lambda^2 \rho_\lambda = \omega(\log n)\) implies (B.3) as claimed.

To achieve convergence in probability, (B.2) also requires \(n^2 \rho = \omega(\log |Z_k| + \binom{k}{2})\). To simplify this requirement and obtain a sufficient condition, observe that \(\log |Z_k| \leq n \log k\), since \(Z_k \subseteq \{1, \ldots, k\}^n\). Now write \(\binom{k+1}{2} = k^2 \{1/2 + O(1)\}\), and let \(\tilde{h} = n/k\). From these simplifications we obtain \(\rho = \omega(\log(n/\tilde{h})/n + \tilde{h}^{-2})\), which is implied by \(\tilde{h}^2 \rho = \omega(\max \{\tilde{h}^2 / n, 1\} \log n)\).

Finally, observe that since the results above hold uniformly over all \(z \in Z_k\), they also hold for \(z = \bar{z}(A, Z_k)\), the maximum profile likelihood estimator of \(z\). The following lemma relates this choice to its oracle counterpart \(\bar{z}(p, Z_k)\)—the best choice of \(z \in Z_k\)—enabling us to strengthen (B.2).

**Lemma B.4.** Assume conditions 1 and 2 of Theorem 5.1. Then it follows from the arguments of Theorems 2 and 3 of Choi, Wolfe and Airoldi (2012) that for any \(\bar{z}(A, Z_k)\) and \(\bar{z}(p, Z_k)\) as per (4.5) and (4.7),

\[
0 \leq \sum_{i<j} \{D(p_{ij} || \bar{p}_{i|z_j}) - D(p_{ij} || \bar{p}_{i|z_j})\} = O_P\left(\frac{\log |Z_k| + \binom{k+1}{2} \log \binom{n}{k}^{(k+1)/2+1}}{(\rho)^2}\right)
\]

\[
+ O_P\left(\frac{\log(1/\rho) \log |Z_k|}{\tilde{h}}\right) \left(1 + \frac{18 \binom{n}{k} \rho}{\log |Z_k|} \sum_{i<j} p_{ij}\right).
\]
Since \( \tilde{z}(p, Z_k) \) results in the minimum value of \( \sum_{i<j} D(p_{ij} \mid \bar{z}_{i;j}) \), this difference is nonnegative. Its convergence in probability to 0 when suitably normalized is due to the maximizing properties of \( \tilde{z}(A, Z_k) \) and \( \tilde{z}(p, Z_k) \). Thus we conclude that \( \hat{z} \) together with a sufficient condition being that \( \bar{z} \) normalized is due to the maximizing properties of \( \hat{z} \).

To complete the proof, set \( z = \tilde{z}(A, Z_k) \) in (B.2) and combine it with Lemma B.4. Comparing terms, we see that the latter’s will dominate the rate of convergence, and so we upper-bound them using \( \bar{h} = n/k = \omega(1) \), subadditivity of the square root and the fact that \( (n)/(k+1) \leq \bar{h}^2 \). We thus obtain

\[
\sum_{i<j} \bar{z}_{i;j} \notin \{0, 1\} D(p_{ij} \mid \bar{z}_{i;j}) - \min_z \in Z_k \sum_{i<j} D(p_{ij} \mid \bar{z}_{i;j})
\]

(B.4)

where the final line follows because \( \log(n/\bar{h}) = o(n\bar{\rho}) \) is needed for (B.4) to be \( o_P(1) \), whereas \( \rho \leq \rho < 1/2 \) implies that \( \log(1/\rho) > (2)^2 = \omega(\log(n/\bar{h})/(n\bar{\rho})) \). Thus we have derived the claimed rate of convergence, with a sufficient condition being that \( \bar{h}^2 \bar{\rho} = \omega( 1 + o(1) ) \), since together \( \bar{h}^2 \bar{\rho} = \omega(\log n) \) and \( \bar{\rho} = \omega(\log(n)^3/n) \) imply that (B.4) is \( o_P(1) \).

To complete the proof of Theorem 5.1, we now re-interpret the above results under the scaled exchangeable random graph model of (2.2). Lemmas B.1–B.4 then hold for every realized value of \( \xi \), and thus the implicit conditioning on \( \xi \) inherent to these results can be removed. Specifically, in Lemmas B.1 and B.4, we may marginalize (B.7) and (B.12) respectively via the law of total probability, noting that their right-hand sides do not depend on \( \xi \). For Lemmas B.2 and B.3, we simply note that the bound of (B.8) holds for all \( \xi \).
B.2. Proofs and auxiliary lemmas needed for Theorem 5.1.

**Lemma B.1.** We write

\[
\sum_{i<j: \bar{A}_{izj} \notin \{0,1\}} \left\{ D(p_{ij} \mid \bar{A}_{izj}) - D(p_{ij} \mid \bar{p}_{izj}) \right\}
\]

\[
= \sum_{i<j: \bar{A}_{izj} \notin \{0,1\}} \left\{ p_{ij} \log \left( \frac{\bar{p}_{izj}}{A_{izj}} \right) + (1 - p_{ij}) \log \left( \frac{1 - \bar{p}_{izj}}{1 - A_{izj}} \right) \right\}
\]

\[
= \sum_{a \leq b: A_{ab} \notin \{0,1\}} \sum_{i \in z^{-1}(a), j \notin z^{-1}(b)} \left\{ p_{ij} \log \left( \frac{\bar{p}_{ab}}{A_{ab}} \right) + (1 - p_{ij}) \log \left( \frac{1 - \bar{p}_{ab}}{1 - A_{ab}} \right) \right\}
\]

\[
= \sum_{a \leq b: A_{ab} \notin \{0,1\}} \log \left( \frac{\bar{p}_{ab}}{A_{ab}} \right) \sum_{i \in z^{-1}(a), j \notin z^{-1}(b)} p_{ij} + \log \left( \frac{1 - \bar{p}_{ab}}{1 - A_{ab}} \right) \sum_{i \in z^{-1}(a), j \notin z^{-1}(b)} (1 - p_{ij})
\]

(B.5)

\[
= \sum_{a \leq b: A_{ab} \notin \{0,1\}} h_{ab}^2 D\left( \bar{p}_{ab} \mid \bar{A}_{ab} \right).
\]

Since (B.5) is a sum of Kullback–Leibler divergences, it is nonnegative. To show its convergence when suitably normalized, we appeal to Lemma B.5 below, which implies the following under conditions 1–3 of Theorem 5.1 and the hypothesis \((h_0^2/2) \rho_\Lambda = \omega(\log(h_0^2/2))\):

For every \(\epsilon > 0\), eventually in \(n\) and with \(1^+/2\) approaching arbitrarily
closely to 1/2,
\[
\Pr\left(\max_{z \in \mathcal{Z}_k} \sum_{a \leq b: A_{ab} \notin \{0,1\}} h_{ab}^2 D(\bar{p}_{ab} \mid \bar{A}_{ab}) \geq \epsilon \sum_{i<j} P_{ij} \right) \\
\leq \exp\left(\log |\mathcal{Z}_k| - \frac{\left\{\epsilon \sum_{i<j} P_{ij} - \frac{k+1}{2}\right\}^2}{2\epsilon \sum_{i<j} P_{ij} + \frac{k+1}{2}}\right)
\]
(B.6)
\[
\leq \exp\left(\log |\mathcal{Z}_k| - \frac{\epsilon \sum_{i<j} P_{ij} \max\{\epsilon \sum_{i<j} P_{ij} - 1,0\}}{2\epsilon \sum_{i<j} P_{ij} + \frac{k+1}{2}}\right)
\]
\[
\leq \exp\left(\log |\mathcal{Z}_k| - \frac{\max\{\epsilon \sum_{i<j} P_{ij} - 1,0\}}{2\epsilon + \frac{k+1}{2} / (\epsilon \sum_{i<j} P_{ij})}\right)
\]
(B.7)
where (B.6) follows as \(\epsilon \sum_{i<j} P_{ij} \geq 0\) and \((1/2)(k+1) \geq 0\) eventually in \(n\), and (B.7) follows from condition 1 of Theorem 5.1, by which \(\sum_{i<j} P_{ij}(n) \geq \frac{n}{2}\bar{\rho}(n)\) eventually in \(n\).

**Lemma B.2.** We will bound \(\Pr(\bar{A}_{z_i z_j} \in \{0,1\})\) uniformly in \(z\). Observe that for any \(1 \leq a \leq b \leq k\), conditionally on any \(z \in \mathcal{Z}_k\), we have by the arithmetic-geometric mean inequality that
\[
\Pr(\bar{A}_{ab} \in \{0,1\} \mid Z = z) = \Pr(\bar{A}_{ab} = 0 \mid Z = z) + \Pr(\bar{A}_{ab} = 1 \mid Z = z) \\
= \prod_{i \in z^{-1}(a), \ j \in z^{-1}(b)} (1 - p_{ij}) + \prod_{i \in z^{-1}(a), \ j \in z^{-1}(b)} p_{ij} \\
\leq (1 - \bar{p}(z)_{ab}) h_{ab}^2 + (\bar{p}(z)_{ab}) h_{ab}^2.
\]

(B.8)

Conditions 2 and 3 of Theorem 5.1 stipulate that for every pair \((a,b)\) and every \(z \in \mathcal{Z}_k\), eventually in \(n\), \(\rho_\Lambda(n) \leq \bar{p}_{ab}(n) \leq 1 - \sqrt{\rho_\Lambda(n)}\) and \(h_\Lambda(n) \leq h_{ab}(n)\). Hence (B.8) implies that, eventually in \(n\), for \(1 \leq a \leq b \leq k\)
\[
\Pr(\bar{A}_{ab} \in \{0,1\} \mid Z = z) \leq (1 - \rho_\Lambda)^{h_{ab}^2} + (1 - \sqrt{\rho_\Lambda})^{h_{ab}^2};
\]
\[
\Rightarrow \max_{a \leq b} \Pr(\bar{A}_{ab} \in \{0,1\} \mid Z = z) \leq 2(1 - \rho_\Lambda)^{h_{\Lambda}^2}.
\]
(B.9)

Since the conditional probability \(\Pr(\bar{A}_{z_i z_j} \in \{0,1\} \mid Z = z)\) is upper-bounded by (B.9) uniformly for every value of \(z \in \mathcal{Z}_k\), this same bound also
holds after marginalizing out $Z$. Thus, eventually in $n$,  
\begin{equation}
\text{Pr} \left( \bar{A}_{zizj} \in \{0, 1\} \right) \leq 2(1 - \rho_\Lambda)^{\left(\frac{h_\Lambda}{2}\right)}.
\end{equation}

Applying Markov’s inequality, we see that for any $\epsilon > 0$, eventually in $n$,  
\begin{align*}
\text{Pr} \left( \sum_{i<j} I(\bar{A}_{zizj} \in \{0, 1\}) \geq \epsilon \sum_{i<j} p_{ij} \right) &\leq \frac{\sum_{i<j} \text{Pr} \left( \bar{A}_{zizj} \in \{0, 1\} \right)}{\epsilon \sum_{i<j} p_{ij}} \\
&\leq \frac{\binom{n}{2} 2(1 - \rho_\Lambda)^{\left(\frac{h_\Lambda}{2}\right)}}{\epsilon \sum_{i<j} p_{ij}} \\
&\leq \frac{2(1 - \rho_\Lambda)^{\left(\frac{h_\Lambda}{2}\right)}}{\epsilon \bar{\rho}} \\
&\leq \frac{2 \exp \left\{ -\left(\frac{h_\Lambda}{2}\right) \rho_\Lambda \right\}}{\epsilon \bar{\rho}} \\
&\leq \frac{\exp \left\{ -\left(\frac{h_\Lambda}{2}\right) \rho_\Lambda \left(\log(1/\bar{\rho}) \right) \right\}}{(\epsilon/2)},
\end{align*}

where the second inequality follows directly from (B.10), the third inequality follows from condition 1 of Theorem 5.1, by which $\sum_{i<j} p_{ij}(n) \geq \binom{n}{2} \bar{\rho}(n)$ eventually in $n$, and the final inequality follows from the fact that $\log\left\{ (1 - \rho_\Lambda)^{\left(\frac{h_\Lambda}{2}\right)} \right\} = \left(\frac{h_\Lambda}{2}\right) \log(1 - \rho_\Lambda) \leq -\left(\frac{h_\Lambda}{2}\right) \rho_\Lambda$. \hfill \Box

**Lemma B.3.** First, we express the term of interest as a sum of nonnegative random variables:
\begin{align*}
\sum_{i<j} D \left( p_{ij} \mid \bar{p}_{zizj} \right) - \sum_{i<j: A_{zizj} \notin \{0,1\}} D \left( p_{ij} \mid \bar{p}_{zizj} \right) &= \sum_{i<j} D \left( p_{ij} \mid \bar{p}_{zizj} \right) I(\bar{A}_{zizj} \in \{0, 1\}).
\end{align*}
To show the claimed convergence in probability, we write
\[
0 \leq \sum_{i<j} D \left( p_{ij} \mid \bar{p}_{z_{i}z_{j}} \right) \mathbb{I}(A_{z_{i}z_{j}} \in \{0, 1\}) \\
= -\sum_{i<j} \left\{ p_{ij} \log (\bar{p}_{z_{i}z_{j}}) + (1 - p_{ij}) \log (1 - \bar{p}_{z_{i}z_{j}}) \right\} \mathbb{I}(A_{z_{i}z_{j}} \in \{0, 1\}) \\
+ \sum_{i<j} \left\{ p_{ij} \log (p_{ij}) + (1 - p_{ij}) \log (1 - p_{ij}) \right\} \mathbb{I}(\bar{A}_{z_{i}z_{j}} \in \{0, 1\}) \\
\leq -\sum_{i<j} \left\{ p_{ij} \log (\bar{p}_{z_{i}z_{j}}) + (1 - p_{ij}) \log (1 - \bar{p}_{z_{i}z_{j}}) \right\} \mathbb{I}(\bar{A}_{z_{i}z_{j}} \in \{0, 1\}) \\
= -\sum_{a \leq b} \sum_{i \in z^{-1}(a), j \in z^{-1}(b)} \left\{ p_{ij} \log (\bar{p}(z)_{ab}) + (1 - p_{ij}) \log (1 - \bar{p}(z)_{ab}) \right\} \mathbb{I}(\bar{A}_{ab} \in \{0, 1\}) \\
\leq \sum_{a \leq b} h_{ab}^2 \mathbb{I}(\bar{A}_{ab} \in \{0, 1\}) \\
= (\log 2) \sum_{i<j} \mathbb{I}(\bar{A}_{z_{i}z_{j}} \in \{0, 1\}).
\]

The result then follows from Lemma B.2, which establishes that for every \( z \in \mathcal{Z}_k \), we have \( \sum_{i<j} \mathbb{I}(\bar{A}_{z_{i}z_{j}} \in \{0, 1\}) = \mathcal{O}_P\left(e^{-h_{2}^z}\rho + \log(1/\rho) \sum_{i<j} p_{ij}\right) \) under conditions 1–3 of Theorem 5.1.

**Lemma B.4.** In the notation of Choi, Wolfe and Airoldi (2012), define for any fixed \( z \in \mathcal{Z}_k \)
\[
\bar{L}(z) = \sum_{i<j} \left\{ p_{ij} \log \bar{p}_{z_{i}z_{j}} + (1 - p_{ij}) \log (1 - \bar{p}_{z_{i}z_{j}}) \right\} ; \\
\Rightarrow \bar{z}(p, \mathcal{Z}_k) = \text{argmax}_{z \in \mathcal{Z}_k} \bar{L}(z) = \arg \min_{z \in \mathcal{Z}_k} \sum_{i<j} D \left( p_{ij} \mid \bar{p}_{z_{i}z_{j}} \right).
\]
where the implication follows directly from the definition of the “oracle” MPLE in \( \bar{z}(p, \mathcal{Z}_k) \) in (4.7). Thus
\[
0 \leq \sum_{i<j} \left\{ D \left( p_{ij} \mid \bar{p}_{z_{i}z_{j}} \right) - D \left( p_{ij} \mid \bar{p}_{z_{i}z_{j}} \right) \right\} = \bar{L}(\bar{z}) - \bar{L}(\check{z}), \quad \bar{z}, \check{z} \in \mathcal{Z}_k.
\]
By construction, since \( \check{z}(p, \mathcal{Z}_k) \) maximizes \( \bar{L}(z) \) over \( \mathcal{Z}_k \), this difference is nonnegative. Similarly, from (4.5) we see that \( \check{z}(A, \mathcal{Z}_k) \) maximizes \( L(A; z) \)
over $\mathcal{Z}_k$, and so $L(A; \hat{z}) - L(A; \bar{z}) \geq 0$. Hence,

$$0 \leq \bar{L}(\bar{z}) - \bar{L}(\hat{z}) \leq \bar{L}(\hat{z}) - \bar{L}(\hat{z}) + \{L(A; \hat{z}) - L(A; \bar{z})\}, \quad \bar{z}, \hat{z} \in \mathcal{Z}_k$$

$$= \bar{L}(\hat{z}) - L(A; \hat{z}) + L(A; \bar{z}) - \bar{L}(\bar{z})$$

(B.11)

\[
\leq |\bar{L}(\hat{z}) - L(A; \hat{z})| + |L(A; \bar{z}) - \bar{L}(\bar{z})|,
\]

and so the result will follow from (B.11) if we can show that $|\bar{L}(\hat{z}) - L(A; \hat{z})|$ and $|L(A; \bar{z}) - \bar{L}(\bar{z})|$ both converge in probability to zero when suitably renormalized. We accomplish this in the manner of Choi, Wolfe and Airoldi (2012, Theorem 2), who establish that $\max_{z \in \mathcal{Z}_k} |\bar{L}(z) - L(A; z)| / \sum_{i < j} p_{ij}$ converges as required. Since this result holds for the maximum over all $z \in \mathcal{Z}_k$, then it must also hold for both $\hat{z}$ and $\bar{z}$, and we can therefore apply this same result twice.

In particular, Theorem 2 of Choi, Wolfe and Airoldi (2012) shows that for any fixed $n$, whenever $\max_{ij} |\logit \bar{p}_{zi|z_j}|$ is finite for all $z \in \mathcal{Z}_k$, it holds that for all nonempty $\mathcal{Z}_k \subseteq \{1, \ldots, k\}^n$ and any $\epsilon > 0$,

(B.12) \[
\Pr \left( \max_{z \in \mathcal{Z}_k} |L(A; z) - \bar{L}(z)| \geq 2 \epsilon \sum_{i < j} p_{ij} \right)
\]

\[
\leq |\mathcal{Z}_k| \exp \left( (k+1) \log \left\{ \binom{n}{2} / \binom{k+1}{2} \right\} + 1 \right) - \epsilon \sum_{i < j} p_{ij}
\]

$$+ \sum_{z \in \mathcal{Z}_k} 2 \exp \left\{ - \epsilon \sum_{i < j} p_{ij} \right\} \left( 1 + \left[ \frac{\epsilon \sum_{i < j} p_{ij}}{\max_{i < j} |\logit \bar{p}_{zi|z_j}|} \right]^2 \right) \max_{i < j} |\logit \bar{p}_{zi|z_j}| \right\}.
\]

From condition 2 of Theorem 5.1, we have that each $p_{ij}(n) \in (0, 1)$ eventually in $n$. This implies that $\max_{ij} |\logit \bar{p}_{zi|z_j}(n)|$ will eventually be finite for all $z \in \mathcal{Z}_k$, and thus (B.12) holds eventually in $n$.

To simplify the right-hand side of (B.12), we upper-bound $|\logit \bar{p}_{zi|z_j}|$ via $\max_{i < j} |\logit \bar{p}_{zi|z_j}|$, which allows a factor of $\sum_{i < j} p_{ij}$ to be canceled:

$$\Pr \left( \max_{z \in \mathcal{Z}_k} |L(A; z) - \bar{L}(z)| \geq 2 \epsilon \sum_{i < j} p_{ij} \right) \leq |\mathcal{Z}_k|$$

\[
\exp \left( \frac{(k+1)}{2} \log \left\{ \binom{n}{2} / \binom{k+1}{2} \right\} + 1 \right) - \epsilon \sum_{i < j} p_{ij}
\]

$$+ \sum_{z \in \mathcal{Z}_k} 2 \exp \left\{ - \frac{(\epsilon^2)}{2} \sum_{i < j} p_{ij} \right\} \max_{i < j} |\logit \bar{p}_{zi|z_j}| \right\}.$$
$p_{ij}$ away from 0 and 1. We may now sum over $z \in \mathcal{Z}_k$ to obtain

$$
\Pr \left( \max_{z \in \mathcal{Z}_k} |L(A; z) - \bar{L}(z)| \geq 2\epsilon \sum_{i<j} p_{ij} \right) \leq |\mathcal{Z}_k| \exp \left[ \frac{1}{2} \log \left( \frac{\binom{n}{2}}{\binom{k+1}{2}} + 1 \right) - \epsilon \sum_{i<j} p_{ij} \right]
$$

$$
+ 2 |\mathcal{Z}_k| \exp \left\{ - \frac{(\epsilon^2/2) \sum_{i<j} p_{ij}}{\max_{z \in \mathcal{Z}_k} \left\{ \max_{i<j} |\log \hat{p}_{zi}(n)| \right\} + (\epsilon/3) \max_{z \in \mathcal{Z}_k} \left\{ \max_{i<j} |\log \hat{p}_{zi}(n)| \right\} \right\}.
$$

Condition 2 stipulates that every $\hat{p}_{zi}(n)$ satisfies $\rho(n) \leq \hat{p}_{zi}(n) \leq 1 - \sqrt{\rho(n)}$ eventually in $n$, so

$$
\max_{z \in \mathcal{Z}_k} \left\{ \max_{i<j} |\log \hat{p}_{zi}(n)| \right\} = \max_{z \in \mathcal{Z}_k} \left\{ \max_{i<j} \left| \log \left( \frac{\hat{p}_{zi}(n)}{1 - \hat{p}_{zi}(n)} \right) \right| \right\} = \max_{z \in \mathcal{Z}_k} \left[ \max_{i<j} \left\{ \max \left( \log \left( \frac{\hat{p}_{zi}(n)}{1 - \hat{p}_{zi}(n)} \right), \log \left( \frac{1 - \hat{p}_{zi}(n)}{\hat{p}_{zi}(n)} \right) \right) \right\} \right] \leq \log \left\{ \frac{1}{\rho(n)} \right\},
$$

which is finite, as condition 1 specifies that $0 < \rho(n) < 1/2$ for all $n$.

Finally, condition 1 of Theorem 5.1 ensures that $\binom{n}{2} \hat{\rho}(n) \leq \sum_{i<j} p_{ij}$ eventually in $n$. Thus, recalling (B.11), we obtain the claimed result, since we have shown that for all $n$ sufficiently large,

$$
\Pr \left( \max_{z \in \mathcal{Z}_k} |L(A; z) - \bar{L}(z)| \geq 2\epsilon \sum_{i<j} p_{ij} \right)
$$

$$
\leq \exp \left( \log |\mathcal{Z}_k| + \binom{n+1}{2} \log \left( \frac{\binom{n}{2}}{\binom{k+1}{2}} + 1 \right) - \epsilon \binom{n}{2} \hat{\rho} \right)
$$

$$
+ 2 \exp \left\{ \log |\mathcal{Z}_k| - \frac{\binom{n}{2} \hat{\rho}}{\log (1/\rho(n))^2} \left( \frac{\epsilon^2/2}{1 + (\epsilon/3)/\log (1/\rho(n))} \right) \right\}
$$

$$
\leq 4 \exp \left\{ \log |\mathcal{Z}_k| + \max \left( \binom{k+1}{2} \log \left( \frac{\binom{n}{2}}{\binom{k+1}{2}} + 1 \right) - \epsilon \binom{n}{2} \hat{\rho}, \frac{-\binom{n}{2} \hat{\rho}}{\log (1/\rho(n))^2} \left( \frac{\epsilon^2/2}{1 + (\epsilon/3)/\log (1/\rho(n))} \right) \right) \right\}.
$$
Lemma B.5. Assume conditions 1–3 of Theorem 5.1 and the hypothesis $(h_{\lambda}^{2})_{\lambda} = o\left(\log \left(h_{\lambda}^{2}\right)\right)$, which together ensure that for every $z \in \mathbb{Z}_k$, 

$$\frac{\log \left(h_{ab}^{2}/\min(\hat{p}_{ab}, 1 - \hat{p}_{ab})/\sqrt{p_{ab}}\right)}{\min(\hat{p}_{ab}, 1 - \hat{p}_{ab})/\sqrt{p_{ab}}} = o(1), \quad 1 \leq a \leq b \leq k.$$  

(B.13) Then for every $\epsilon > 0$, we have eventually in $n$ that

$$\Pr \left( \max_{z \in \mathbb{Z}_k} \sum_{a \leq b : \hat{A}_{ab} \notin \{0, 1\}} h_{ab}^{2} \left( \hat{p}_{ab} \mid \hat{A}_{ab} \right) \geq \epsilon \right) \leq \exp \left( \log |\mathbb{Z}_k| - \frac{\epsilon - 1^{+}(k+1)/2}{2\epsilon + 1^{+}(k+1)/2} \right),$$

with $1^{+}/2$ approaching arbitrarily closely to $1/2$ from above, at the rate given by (B.13).

Proof. Observe that for any fixed $z \in \mathbb{Z}_k$, we may re-express

$$\sum_{a \leq b : \hat{A}_{ab} \notin \{0, 1\}} h_{ab}^{2} \left( \hat{p}_{ab} \mid \hat{A}_{ab} \right)$$

as a sum of the terms whose moments will be bounded by Lemma B.6:

$$\sum_{a \leq b : \hat{A}_{ab} \notin \{0, 1\}} h_{ab}^{2} \left( \hat{p}_{ab} \mid \hat{A}_{ab} \right) = \sum_{a \leq b} g \left( h_{ab}^{2} \hat{A}_{ab} \right), \quad z \in \mathbb{Z}_k \text{ fixed.}$$

Here, setting $X_{n} = h_{ab}^{2} \hat{A}_{ab}$ in (B.17) of Lemma B.6, we define $g \left( h_{ab}^{2} \hat{A}_{ab} \right)$ as

$$g \left( h_{ab}^{2} \hat{A}_{ab} \right) = \begin{cases} h_{ab}^{2} \left\{ \hat{p}_{ab} \log \left( \frac{\hat{p}_{ab}}{\hat{A}_{ab}} \right) + (1 - \hat{p}_{ab}) \log \left( \frac{1 - \hat{p}_{ab}}{1 - \hat{A}_{ab}} \right) \right\} & \text{if } h_{ab}^{2} \hat{A}_{ab} \in \{1, \ldots, h_{ab}^{2} - 1\}, \\ 0 & \text{if } h_{ab}^{2} \hat{A}_{ab} \in \{0, h_{ab}^{2}\}. \end{cases}$$

By hypothesis, the conditions of Lemma B.6 apply for all $1 \leq a \leq b \leq k$ and every $z \in \mathbb{Z}_k$, and so each $g \left( h_{ab}^{2} \hat{A}_{ab} \right)$ behaves like a chi-square variate on 1 degree of freedom in terms of its $m$th moment where $m = 1, 2, \ldots$

(B.14) 

$$\mathbb{E} \left\{ g \left( h_{ab}^{2} \hat{A}_{ab} \right)^{m} \right\} \leq \frac{\Gamma \left( m + \frac{1}{2} \right)}{\sqrt{\pi}} \left\{ 1 + O \left( \frac{\sqrt{\log \left(h_{ab}^{2}/\min(\hat{p}_{ab}, 1 - \hat{p}_{ab})/\sqrt{p_{ab}}\right)}}{\min(\hat{p}_{ab}, 1 - \hat{p}_{ab})/\sqrt{p_{ab}}} \right) \right\}. $$

Controlling the moments of $g \left( h_{ab}^{2} \hat{A}_{ab} \right)$ enables us to apply a Bernstein concentration inequality due to Birgé and Massart (1998, Lemma 8). To do so requires the existence of constants $v^{2}$ and $c$ such that

(B.15) 

$$\left( k+1 \right)^{-1} \sum_{a \leq b} \mathbb{E} \left\{ g \left( h_{ab}^{2} \hat{A}_{ab} \right)^{m} \right\} \leq \frac{m!}{2} v^{2} c^{m-2}, \quad m = 2, 3, \ldots$$
By hypothesis,
\[
\frac{\Gamma \left( m + \frac{1}{2} \right)}{\sqrt{\pi}} \left\{ 1 + O \left( \frac{\sqrt{\log (h^2_{ab}) / h^2_{ab}}}{\min (\bar{p}_{ab}, 1 - \bar{p}_{ab}) / \sqrt{\bar{p}_{ab}}} \right) \right\} = \frac{\Gamma \left( m + \frac{1}{2} \right)}{\sqrt{\pi}} \{1 + o(1)\}
\]
\[
< \frac{3}{4} + \delta,
\]
eventually in \( n \), for every \( \delta > 0 \). Thus we fix \( v^2 \) arbitrarily close to \( 3/4 \), and write \( v^2 = 3^+/4 \). To ensure that (B.15) is satisfied for each \( m \), we then let \( c = 1 \).

We can see from (B.14) that these choices of \( v^2, c \) yield
\[
\left( \begin{array}{l}
\frac{1}{2}
\end{array} \right)_{-1} \sum_{a \leq b} E \{g \left( h^2_{ab} A_{ab} \right)^m \} \leq \frac{\Gamma \left( m + \frac{1}{2} \right)}{\sqrt{\pi}} \{1 + o(1)\}, \quad m = 2, 3, \ldots
\]
\[
< \frac{\Gamma (m + 1)}{\sqrt{\pi}}, \quad \text{eventually in} \ n,
\]
\[
\leq \frac{m!}{2} v^2 c^{m-2}, \quad m = 2, 3, \ldots,
\]
and thus (B.15) holds eventually in \( n \). Lemma 8 of Birgé and Massart (1998) then shows that for
\[
Y = \sum_{a \leq b} g \left( h^2_{ab} A_{ab} \right), \quad \text{with} \ z \in \mathcal{Z}_k \ \text{fixed},
\]
the following concentration inequality holds for any \( \epsilon > 0 \):
\[
\Pr \left( Y - EY \geq \left( \begin{array}{l}
\frac{1}{2}
\end{array} \right) \epsilon \right) \leq \exp \left( -\frac{\left( \begin{array}{l}
\frac{1}{2}
\end{array} \right) \epsilon^2 / 2}{v^2 + \epsilon \epsilon} \right)
\]
(B.16)
\[
\Rightarrow \Pr \left( Y \geq \epsilon \right) \leq \exp \left( -\frac{(\epsilon - EY)^2 / 2}{\left( \begin{array}{l}
\frac{1}{2}
\end{array} \right) v^2 + c(\epsilon - EY)} \right)
\]

Observe that since \( EY \geq 0 \), (B.16) still holds if we replace \( EY \) with an upper bound \( u \), because for any \( u \geq EY \geq 0 \), the event \( Y - u \geq \epsilon \) implies the event \( Y - EY \geq \epsilon \), and so \( \Pr (Y - u \geq \epsilon) \leq \Pr (Y - EY \geq \epsilon) \). Thus, we may substitute the eventual upper bound \( u = (1^+ / 2) \left( \begin{array}{l}
\frac{1}{2}
\end{array} \right) \geq EY \) from (B.14) into (B.16), where \( (1^+ / 2) \) is arbitrarily close to \( 1/2 \). Substituting \( (1^+ / 2) \left( \begin{array}{l}
\frac{1}{2}
\end{array} \right) \) in place of \( EY \) in (B.16), along with the constants \( v^2 = 3^+/4 \) and \( c = 1 \), we see that for any \( \epsilon > 0 \), eventually in \( n \),
\[
\Pr \left( Y \geq \epsilon \right) \leq \exp \left( -\frac{\left\{ \epsilon - \frac{1^+}{2} \left( \begin{array}{l}
\frac{1}{2}
\end{array} \right) \right\}^2 / 2}{\left( \begin{array}{l}
\frac{1}{2}
\end{array} \right) \frac{3^+}{4} + \left\{ \epsilon - \frac{1^+}{2} \left( \begin{array}{l}
\frac{1}{2}
\end{array} \right) \right\}} \right).
\]
Simplifying this expression and applying a union bound over all $z \in \mathbb{Z}_k$ then yields the stated result.

**Lemma B.6.** Let $X_n$ denote a sequence of Poisson–Binomial variates, each with mean $\mu_n$, and define

$$g(X_n) = \begin{cases} \mu_n \log \left( \frac{\mu_n}{X_n} \right) + (n - \mu_n) \log \left( \frac{n - \mu_n}{n - X_n} \right) & \text{if } X_n \in \{1, 2, \ldots, n - 1\}, \\ 0 & \text{if } X_n \in \{0, n\}. \end{cases}$$

If $\min(\mu_n, n - \mu_n) = \omega\left( \sqrt{\mu_n \log \{\max(\mu_n, n - \mu_n)\}} \right)$, then the moments of $g(X_n)$ satisfy for $m = 1, 2, \ldots$

$$\mathbb{E}(g(X_n)^m) \leq \frac{\Gamma \left( \frac{m + \frac{1}{2}}{\sqrt{\pi}} \right)}{\frac{1}{\sqrt{\pi}}} \left\{ 1 + O \left( \frac{\sqrt{\mu_n \log \{\max(\mu_n, n - \mu_n)\}}}{\min(\mu_n, n - \mu_n)} \right) \right\}. $$

**Proof.** To simplify notation, we suppress the dependence of $X$ and $\mu$ on $n$ throughout; note, however, that $m \in \{1, 2, \ldots\}$ is fixed and so does not depend on $n$. Using the fact that $g(0) = g(n) = 0$, we write

$$\mathbb{E}(g(X)^m) = \sum_{k=0}^{n} g(k)^m \mathbb{P}(X = k), \quad m = 1, 2, \ldots$$

$$= \sum_{k=1}^{n-1} g(k)^m \mathbb{P}(X = k)$$

$$= \left( \sum_{k=1}^{k_1} + \sum_{k=k_1+1}^{k_2-1} + \sum_{k=k_2}^{n-1} \right) g(k)^m \mathbb{P}(X = k),$$

with $k_1, k_2$ chosen to balance the contribution of the central sum in (B.18) with that of the tail sums in (B.18):

$$(B.19a) \quad k_1 = \max \left\{ 1, \left\lfloor \mu - \sqrt{2\mu(m + \delta) \log \mu} \right\rfloor \right\},$$

$$(B.19b) \quad k_2 = \min \left\{ \left\lceil \mu + \sqrt{2\mu(m + \delta) \log(n - \mu)} \right\rceil, n - 1 \right\}.$$
for any fixed $\delta > 0$. Since $g(k) \geq 0$ for every value of $k$, (B.18) implies that

$$E \{g(X)^m\} \leq \left\{ \max_{1 \leq k \leq k_1} g(k)^m \right\} \sum_{k=1}^{k_1} \Pr(X = k) + \sum_{k=k_1+1}^{k_2-1} g(k)^m \Pr(X = k)$$

$$+ \left\{ \max_{k_2 < k < n} g(k)^m \right\} \sum_{k=k_2}^{n-1} \Pr(X = k)$$

$$\leq \left\{ \max_{1 \leq k \leq k_1} g(k)^m \right\} \Pr(X \leq k_1) + \sum_{k=k_1+1}^{k_2-1} g(k)^m \Pr(X = k)$$

$$+ \left\{ \max_{k_2 < k < n} g(k)^m \right\} \Pr(X \geq k_2). \tag{B.20}$$

We now bound the two tail terms in (B.20). From the definitions of $k_1$ and $k_2$ in (B.19), our hypothesis $\min(\mu, n - \mu) = \omega(\sqrt{\mu \log\{\max(\mu, n - \mu)\}})$ implies that eventually in $n$,

$$k_1 = \mu - \epsilon_1, \quad \epsilon_1 \geq \sqrt{2\mu(m + \delta) \log(\mu)}, \tag{B.21a}$$

$$k_2 = \mu + \epsilon_2, \quad \epsilon_2 \geq \sqrt{2\mu(m + \delta) \log(n - \mu)}. \tag{B.21b}$$

Now recall the standard Chernoff bounds for Poisson–Binomial variates, which hold for any $\epsilon > 0$:

$$\Pr(X \leq \mu - \epsilon) \leq \exp\left( -\frac{\epsilon^2}{2\mu} \right),$$

$$\Pr(X \geq \mu + \epsilon) \leq \exp\left\{ -\frac{\epsilon^2}{2\mu} \left( 1 + \frac{\epsilon}{3\mu} \right)^{-1} \right\}. \tag{B.22a}$$

Applying these bounds to $X \leq \mu - \epsilon_1$ and $X \geq \mu + \epsilon_2$, respectively, we conclude that eventually in $n$,

$$\Pr(X \leq k_1) \leq \mu^{-(m+\delta)}, \tag{B.22a}$$

$$\Pr(X \geq k_2) \leq \exp\left\{ -(m + \delta) \log(n - \mu) \left( 1 + \frac{\sqrt{2(m+\delta)}}{3} \sqrt{\frac{\log(n-\mu)}{\mu}} \right)^{-1} \right\}$$

$$= (n - \mu)^{-(m+\delta)} \left\{ 1 + \mathcal{O}\left( \sqrt{\frac{\log(n-\mu)}{\mu}} \right) \right\}, \tag{B.22b}$$
with the hypothesis \( \min(\mu, n - \mu) = \omega\left(\sqrt{\mu \log\{\max(\mu, n - \mu)\}}\right) \) implying that \( \mu = \omega\left(\log(n - \mu)\right) \).

This hypothesis also implies that \( 1 < \mu < n - 1 \) eventually in \( n \). Since \( g(k) \) is strictly decreasing on \( 1 \leq k < \mu \) and strictly increasing on \( \mu < k \leq n - 1 \), we have for \( m = 1, 2, \ldots \) that \( \max_{1 \leq k \leq k_1} g(k)^m = g(1)^m \leq (\mu \log \mu)^m \) and \( \max_{k_2 < k < n} g(k)^m = g(n - 1)^m \leq \{(n - \mu) \log(n - \mu)\}^m \) eventually in \( n \).

Combining these two upper bounds with (B.20) and (B.22), we conclude that, eventually in \( n \),

\[
\mathbb{E}\{g(X)^m\} \leq \log(\mu)^m \mu^{-\delta} + \sum_{k = k_1 + 1}^{k_2 - 1} g(k)^m \Pr(X = k)
+ \log(n - \mu)^m (n - \mu)^{-\delta} \left\{ 1 + O\left(\sqrt{\log(n - \mu)^m \min(\mu, n - \mu)}\right) \right\}.
\]

As a final step, we bound \( \sum_{k = k_1 + 1}^{k_2 - 1} g(k)^m \Pr(X = k) \) in (B.23). Recognizing \( g(k) \) from (B.17) as a scaled form of a Bernoulli Kullback–Leibler divergence, we have by the Taylor expansion of Lemma C.9 that

\[
g(k) \leq n(k - \mu)^2 \left\{ 1 + \frac{2}{\min(\mu, n - \mu)} (1 - \frac{|k - \mu|}{\min(\mu, n - \mu)})^{-3} \right\}, \quad |k - \mu| < \min(\mu, n - \mu).
\]

Now, (B.21) implies that for all \( n \) sufficiently large, \( |k - \mu| \leq \sqrt{2\mu(m + \delta) \log\{\max(\mu, n - \mu)\}} + 1 \) whenever \( k \in \{k_1, \ldots, k_2\} \), and so

\[
\frac{|k - \mu|}{\min(\mu, n - \mu)} \leq \sqrt{2(m + \delta) \frac{\sqrt{\mu \log\{\max(\mu, n - \mu)\}}}{\min(\mu, n - \mu)}} \cdot \left( 1 + \frac{1}{\sqrt{2\mu(m + \delta) \log\{\max(\mu, n - \mu)\}}} \right)
= O\left(\frac{\sqrt{\mu \log\{\max(\mu, n - \mu)\}}}{\min(\mu, n - \mu)}\right), \quad k_1 \leq k \leq k_2,
\]

since the hypothesis \( \min(\mu, n - \mu) = \omega\left(\sqrt{\mu \log\{\max(\mu, n - \mu)\}}\right) \) implies that \( \mu = \omega\left(\log(n)\right) \). From (B.25), we see that this hypothesis also implies that the Lagrange remainder term in (B.24) is \( o(1) \).
Therefore, we may use the Taylor expansion of (B.24) to obtain the upper bound

\[
\sum_{k=k_1+1}^{k_2-1} g(k)^m \Pr(X = k) \leq \sum_{k=k_1+1}^{k_2-1} \left\{ \frac{n(k - \mu)^2}{2\mu(n - \mu)} \right\}^m \left\{ 1 + \mathcal{O}\left( \frac{|k - \mu|}{\min(\mu, n - \mu)} \right) \right\}^m \Pr(X = k) = \left\{ \frac{n}{2\mu(n - \mu)} \right\}^m \left\{ 1 + \mathcal{O}\left( \frac{\sqrt{\mu \log \max(\mu, n - \mu)}}{\min(\mu, n - \mu)} \right) \right\}
\]

\[
\cdot \sum_{k=k_1+1}^{k_2-1} (k - \mu)^{2m} \Pr(X = k).
\]

Noting that each term appearing in the sum of (B.26) is nonnegative, we see that

\[
\sum_{k=k_1+1}^{k_2-1} (k - \mu)^{2m} \Pr(X = k) \leq \left( \sum_{k=0}^{k_1} + \sum_{k=k_1+1}^{k_2-1} + \sum_{k=k_2}^{n} \right) (k - \mu)^{2m} \Pr(X = k) = E \{(X - \mu)^{2m}\}, \quad m = 1, 2, \ldots,
\]

with each $E \{(X - \mu)^{2m}\}$ an even-order central moment of the Poisson–Binomial random variable $X$.

Shaked and Shanthikumar (1994, Theorem 3.A.37) show that $Y \sim \text{Binomial}(n, \mu/n)$ is larger than $X$ in the convex order, meaning that $E \phi(X) \leq E \phi(Y)$ holds for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$ for which the expectations exist. Since the even-order central moments $E(Y - \mu)^{2m}$ exist and are convex for all $m = 1, 2, \ldots$, it follows that

\[
E \{(X - \mu)^{2m}\} \leq E \{(Y - \mu)^{2m}\}, \quad m = 1, 2, \ldots,
\]

where $X$ is the Poisson–Binomial variate under study and the random variable $Y \sim \text{Binomial}(n, \mu/n)$ has a matched mean.

As observed by Romanovsky (1923), the central moments of the Binomial distribution admit a recurrence relation that allows each of their leading-order terms to be expressed in closed form:

\[
E \{(Y - \mu)^{2m}\} = (2m - 1)!! \text{var} Y^m \left\{ 1 + \mathcal{O}\left( \frac{1}{\text{var} Y} \right) \right\},
\]

with
Thus we have from (B.26) that
\[
\sum_{k=k_1+1}^{k_2-1} g(k)^m \Pr (X = k) \leq \left\{ \frac{n}{2\mu(n-\mu)} \right\}^m \left\{ 1 + \mathcal{O} \left( \frac{\sqrt{\mu \log \{ \max (\mu, n-\mu) \}}}{\min (\mu, n-\mu)} \right) \right\} \cdot \left( 2m-1 \right)! \left\{ \frac{\mu(n-\mu)}{n} \right\}^m \left\{ 1 + \mathcal{O} \left( \frac{1}{\min (\mu, n-\mu)} \right) \right\}
\]

or
\[
=B.27
\]
where the combination of the \( \mathcal{O}(\cdot) \) terms follows because \( \mu = \omega(\log n) \) is implied by the hypothesis that \( \min (\mu, n-\mu) = \omega(\sqrt{\mu \log \{ \max (\mu, n-\mu) \}) \).

Finally, combining (B.23) with (B.27), and noting that \( (2m-1)!/2^m = \Gamma(m+1/2)/\sqrt{\pi} \), we obtain for any choice of \( \delta > 0 \) and every fixed \( m = 1, 2, \ldots \) that
\[
E \{ g(X)^m \} \leq \log (\mu)^m \mu^{-\delta} + \frac{\Gamma(m+1/2)}{\sqrt{\pi}} \left\{ 1 + \mathcal{O} \left( \frac{\sqrt{\mu \log \{ \max (\mu, n-\mu) \}}}{\min (\mu, n-\mu)} \right) \right\}^m \left\{ 1 + \mathcal{O} \left( \frac{\log (n-\mu)}{\mu} \right) \right\}
\]
eventually in \( n \). To complete the proof, observe that \( \delta > 0 \) can be chosen for each \( m \) such that the terms \( \log (\mu)^m \mu^{-\delta} \) and \( \log (n-\mu)^m (n-\mu)^{-\delta} \) tend to 0 arbitrarily quickly in \( n \), thus yielding the theorem. \( \square \)

APPENDIX C: PROOF OF THEOREM 6.1 AND LEMMAS

C.1. Proof of Theorem 6.1.

PROOF. Recall that our aim is to establish (6.3), which asserts that \( \min_{z \in Z_k} \sum_{i<j} D (p_{ij} \mid \bar{p}_{i|z_j}) = \mathcal{O}_P \left\{ n^{-a} + (n/h)^{-2a} \right\} \cdot \sum_{i<j} p_{ij} \). We will do so by upper-bounding this risk in terms of a random community assignment vector \( \tilde{z}^* \) that depends on the ordered sample \( \{ \xi(i) \}_{i=1}^n \) of Uniform(0, 1) variates that index the graphon \( f \). Convergence of this ordered sample to
the lattice \((n + 1)^{-1}(1, \ldots, n)\), coupled with the uniform continuity of \(f\), as enforced by a Hölder assumption, will yield the result.

We proceed as follows. Let \(z^*\) be any minimizer of \(\sum_{i<j} D \left( p_{ij} \,||\, \bar{p}_{zi z_j} \right)\) over the set \(Z_k\) of admissible blockmodel assignment vectors, and define \(\tilde{z}^*_i = H_{k, z^*}^{-1} \{(i)^{-1}/n\}\), with \((i)^{-1}\) the rank of \(\xi_i\) from smallest to largest. Thus \(\tilde{z}^*_i = H_{k, z^*}^{-1} \circ (\cdot)^{-1}\), and therefore by construction, condition 3 of the theorem ensures that \(\tilde{z}^* \in Z_k\) for any \(z^* \in Z_k\). Hence we have the following upper bound:

\[
\min_{z \in Z_k} \sum_{i<j} D \left( p_{ij} \,||\, \bar{p}_{zi z_j} \right) \leq \sum_{i<j} D \left( p_{ij} \,||\, \bar{p}_{\tilde{z}^*_i \tilde{z}^*_j} \right) = \sum_{i<j} D \left( p_{(i)(j)} \,||\, \bar{p}_{\tilde{z}^*_i \tilde{z}^*_j} \right),
\]

with equality stemming from the fact that the sum over all \(i < j\) is invariant to permutation, and hence we may re-order it in accordance with the ordered sample \(\{\xi_i\}_{i=1}^n\).

Conditions 1 and 2 of the theorem then imply that Lemma C.1 holds, thereby completing the proof.

**C.2. Auxiliary lemmas needed for Theorem 6.1.**

**Lemma C.1.** If \(r_n \to 0\) in Lemma C.4, then

\[
\frac{\sum_{i<j} D \left( p_{(i)(j)} \,||\, \bar{p}_{\tilde{z}^*(i) \tilde{z}^*(j)} \right)}{\sum_{i<j} \rho_n f(\xi_i, \xi_j)} = O_P\left( r_n^2 \right).
\]

**Proof.** This follows from via Slutsky’s theorem, after combining the results of Lemmas C.2 and C.3:

\[
\binom{n}{2}^{-1} \sum_{i<j} f(\xi_i, \xi_j) = \int \int_{(0,1)^2} f(x, y) \,dx \,dy + O_P(n^{-1/2}),
\]

\[
\binom{n}{2}^{-1} \sum_{i<j} D \left( p_{(i)(j)} \,||\, \bar{p}_{\tilde{z}^*(i) \tilde{z}^*(j)} \right) = O_P\left( r_n^2 \right).
\]

Since the denominator term converges in probability to a constant, it also converges in law. Thus by Slutsky’s theorem, the ratio converges in law to a constant, and hence it also converges in probability.

**Lemma C.2.** Let \(f\) be a symmetric measurable function on \((0,1)^2\) with bounded magnitude, and let \(\{\xi_i\}_{i=1}^n\) be a random sample of \(\text{Uniform}(0,1)\) variates. Then

\[
\binom{n}{2}^{-1} \sum_{i<j} f(\xi_i, \xi_j) = \int \int_{(0,1)^2} f(x, y) \,dx \,dy + O_P(n^{-1/2}).
\]
Proof. The result follows from Chebyshev’s inequality. We obtain the necessary moments as
\[ \mathbb{E}(\mathrm{Var}(\xi))^2 = \int f(x) \, dx \, dy, \]
(C.1)
\[ \mathrm{Var}(\mathrm{Var}(\xi)) = \sum_{i < j} \mathbb{E}(\xi_i \xi_j) = \sum_{i < j} \mathbb{E}(\xi_i \xi_j), \]
where \( \xi \) are independent, any individual covariance term appearing in the sum of (C.1) can be nonzero only if \((i = k) \cup (i = l) \cup (j = k) \cup (j = l)\). Thus we conclude that
\[ \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} = \mathcal{O}(n^{-1}), \]
and so Chebyshev’s inequality yields the result. \( \square \)

**Lemma C.3.** Whenever \( r_n \to 0 \) in (C.3) from Lemma C.4, we have that
\[ \{\rho_n\}. \]
Proof. The result follows by combining Lemmas C.4 and C.8. From Lemma C.4, we have directly that
\[ \rho_n^{-1} \mathbb{E}(\rho(\xi)) = \mathcal{O}(r_n^2) \]
under the hypothesis that \( r_n \to 0 \), and thus
\[ \{\rho_n\}. \]
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\[ \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} = \mathcal{O}(n^{-1}), \]
and so Chebyshev’s inequality yields the result. \( \square \)
Since (C.21) of Lemma C.8 upper-bounds each $\Delta_{ab}$ by the ratio of terms $\rho_n M \left( \sqrt{2 \max_a h_a / n} \right)^2 / \min \left( \rho_n \bar{f}_{ab}, 1 - \rho_n \bar{f}_{ab} \right)$, we see that $\Delta_{ab} = O(r_n)$, and so the hypothesis $r_n \to 0$ is sufficient to imply that $\max_a \Delta_{ab} = o(1)$.

We also see that the main term in (C.2) is $O(r_n^2)$, since the quantity $\min_{1 \leq a, b \leq k} \left\{ \min \left( \bar{f}_{ab}, \rho_n^{-1} - \bar{f}_{ab} \right) \right\} \leq \sup_{(x,y) \in (0,1)^2} f(x,y)$, and thus after applying Markov’s inequality via (C.2), we obtain the result.

**Lemma C.4.** Let $f$ be a symmetric Hölder-$\alpha(M)$ function on $(0,1)^2$, with $\bar{f}(x,y) = H_{-1}(x) H_{-1}(y)$ its stepfunction approximation, and let $\{\xi(i)\}_{i=1}^n$ be an ordered sample of independent Uniform$(0,1)$ random variables. Assume $\rho_n > 0$ and $0 < \rho_n f(x,y) < 1$ everywhere on $(0,1)^2$. Then for any $\bar{z}$ such that $\Pi_{\bar{z}} = (\cdot)^{-1}$, with $i^{-1}$ denoting the rank of $\xi_i$ from smallest to largest, we have

$$
\rho_n^{-1} D \left( p(i|j) \left\| \tilde{p}_{\bar{z}(i) \bar{z}(i)} \right\| \right) = \rho_n^{-1} D \left( p(i|j) \left\| \rho_n \tilde{f} (\xi(i), \xi(j)) \right\| \right) + O(r_n^2)
$$

whenever

$$
r_n = \rho_n M 2^{\alpha/2} \left\{ \frac{2^{1-a}}{n^a} + \frac{2^{(\max_{1 \leq a \leq k} h_a)^a + 1 + 2^a (\max_{1 \leq a \leq k} h_a)^a \max_{1 \leq a \leq k} h_a^{a+1}}{n^a} \right\} \to 0.
$$

**Proof.** We apply Taylor’s theorem, after first establishing via Markov’s inequality that

$$
\delta_n = \frac{\tilde{p}_{\bar{z}(i) \bar{z}(i)} - \rho_n \tilde{f} (\xi(i), \xi(j))}{\min \{ \rho_n \tilde{f} (\xi(i), \xi(j)), 1 - \rho_n \tilde{f} (\xi(i), \xi(j)) \}} = O_P(r_n).
$$

To show (C.4), we lower-bound the denominator of $\delta_n$, and then apply Lemma C.5 to upper-bound $E|\delta_n|:

$$
E|\delta_n| \leq E \frac{\rho_n \left| \rho_n^{-1} \tilde{p}_{\bar{z}(i) \bar{z}(i)} - \tilde{f} (\xi(i), \xi(j)) \right|}{\min_{1 \leq a, b \leq k} \{ \min (\rho_n \tilde{f}_{ab}, 1 - \rho_n \tilde{f}_{ab}) \}} \leq r_n.
$$

We now apply Taylor’s theorem to expand $D \left( p(i|j) \left\| \tilde{p}_{\bar{z}(i) \bar{z}(i)} \right\| \right)$ as a function of $\delta_n$ about the point $\rho_n \tilde{f} (\xi(i), \xi(j))$, writing $\tilde{p}_{\bar{z}(i) \bar{z}(i)}$ for $\rho_n \tilde{f} (\xi(i), \xi(j))$, etc.
we have that if \( r_n \to 0 \), then

\[
\begin{aligned}
&\left| D \left( \tilde{p}(i)(j) \right) - D \left( \tilde{p}(i)(j) \right) \right| = \left| \tilde{p}(i)(j) - \tilde{p}(i)(j) \right| \\
&\quad + \frac{1}{2} \left\{ p(i)(j) \left( 1 - 2\tilde{p}(i)(j) \right) + \tilde{p}(i)(j) \right\} \left\{ \frac{\tilde{p}(i)(j) - \tilde{p}(i)(j)}{\tilde{p}(i)(j) \left( 1 - \tilde{p}(i)(j) \right)} \right\}^2 + o_P \left( r_n^2 \right)
\end{aligned}
\]

(C.5)

\[
< 2\rho_n M \left( \sqrt{2} \max_{1 \leq a \leq k} h_a / n \right) \alpha |\delta_n| + 3\rho_n \sup_{(x,y) \in (0,1)^2} f (x, y) \delta_n^2 + o_P \left( \rho_n r_n^2 \right),
\]

where the terms in (C.5) follow because, by Lemma C.6, \( \left| \tilde{p}(i)(j) - \tilde{p}(i)(j) \right| \leq \rho_n M \left( \sqrt{2} \max_{1 \leq a \leq k} h_a / n \right) \alpha, \) since \( f \in \text{Hölder}^\alpha (M) \); also, since \( 0 < \tilde{p}(i)(j) < 1 \), we have that \( \left| 1 - 2\tilde{p}(i)(j) \right| / \max \left( \tilde{p}(i)(j), 1 - \tilde{p}(i)(j) \right) < 1 \); and likewise we have \( \max \left( \tilde{p}(i)(j), 1 - \tilde{p}(i)(j) \right) \geq 1/2 \). Since \( f \in \text{Hölder}^\alpha (M) \) is bounded by hypothesis, the right-hand side of (C.5) is \( O_P \left( \rho_n r_n^2 \right) \). The lemma follows from multiplying both sides of (C.5) by \( \rho_n^{-1} \).

**Lemma C.5.** Let \( f \) be a symmetric Hölder\(^\alpha \)(\( M \)) function on \((0,1)^2\), and let \( \{\xi(i)\}_{i=1}^n \) be an ordered sample of independent Uniform\((0,1)\) variates. Let \( \rho_n > 0 \) and define for \( z_i = H^{-1} \{ \Pi_z(i) / n \} \):

\[
\bar{p}(z)_{ab} = \frac{1}{\rho_n^2} \sum_{j=nH(b-1)+1}^{nH(b)} \sum_{i=nH(a-1)+1}^{nH(a)} \rho_n \bar{f} \left( \xi_{\Pi_z^{-1}(i)}, \xi_{\Pi_z^{-1}(j)} \right).
\]

Then for any \( \tilde{z} \) such that \( \Pi_z = (\cdot)^{-1} \), with \((\cdot)^{-1}\) denoting the rank of \( \xi_i \) from smallest to largest, we have

\[
\begin{aligned}
&\left| \rho_n^{-1} \bar{p}(z)(\tilde{z}, \xi(i), \xi(j)) \right| \\
&\leq M 2^{\alpha/2} \left\{ 2^{1-\alpha} \frac{nH(a)}{n^{\alpha/2}} + 2 (\max_{1 \leq a \leq k} h_a) \alpha + 1 + 2^{\alpha} \frac{\tilde{z}(i) - \tilde{z}(j)}{n^{\alpha}} \right\}.
\end{aligned}
\]

**Proof.** Define the \( k \times k \) matrix \( \bar{f} \) such that \( \rho_n^{-1} \bar{p}(z)_{ab} = \bar{f}(z)_{ab} + O_P \left( n^{-\alpha/2} \right) \) when \( f \) is \( \alpha \)-Hölder:

\[
\bar{f}(z)_{ab} = \frac{1}{\rho_n^2} \sum_{j=nH(b-1)+1}^{nH(b)} \sum_{i=nH(a-1)+1}^{nH(a)} \bar{f} \left( \frac{\Pi_z^{-1}(i) - 1}{n+1}, \frac{\Pi_z^{-1}(j) - 1}{n+1} \right).
\]
Note that $\tilde{f}(\tilde{z})$ is deterministic, since the set of admissible $\tilde{z}$ has been chosen such that $\Pi^{-1} \{ (i)^{-1} \} = i$ for all $1 \leq i \leq n$. We will then obtain our claimed result by bounding the expectation of

\[(C.9) \quad \left| \rho_n^{-1} \bar{p}_{\tilde{z}(i)\tilde{z}(j)} - \tilde{f}(\xi(i),\xi(j)) \right| \]

\[\leq \left| \rho_n^{-1} \bar{p}_{\tilde{z}(i)\tilde{z}(j)} - \tilde{f}(\xi(i)\tilde{z}(j)) \right| + \left| \tilde{f}(\xi(i)\tilde{z}(j)) - \tilde{f}(i_n,j_n) \right| + \left| \tilde{f}(i_n,j_n) - \tilde{f}(\xi(i),\xi(j)) \right|. \]

We begin with the final term in (C.9), for which Lemma C.7 immediately yields

\[(C.10) \quad E \left| \tilde{f}(i_n,j_n) - \tilde{f}(\xi(i),\xi(j)) \right| \leq M \{ 2(n+2) \}^{-\alpha/2} + 2M \left( \sqrt{2} \max_{1 \leq a \leq k} h_a/n \right)^{\alpha}. \]

Next we consider the first term in (C.9). To bound its expectation, note that both $\rho_n^{-1} \bar{p}(\tilde{z})_{ab}$ and $\tilde{f}(\tilde{z})_{ab}$ are averages over the same subset of indices $(i,j)$. From (C.6) and (C.8), we then have that

\[(C.11) \quad E \left| \rho_n^{-1} \bar{p}(\tilde{z})_{ab} - \tilde{f}(\tilde{z})_{ab} \right| \]

\[\leq \frac{1}{\rho^2_{ab}} \sum_{j=nH(b-1)+1}^{nH(b)} \sum_{i=nH(a-1)+1}^{nH(a)} E \left| f(\xi(i),\xi(j)) - f\left(\frac{i}{n+1},\frac{j}{n+1}\right) \right| \]

\[(C.12) \quad \leq 1 \cdot M \{ 2(n+2) \}^{-\alpha/2}, \]

with the final inequality following again from Lemma C.7. Since (C.12) holds uniformly over all $\tilde{z}$ and every $1 \leq a,b \leq k$, we have bounded $E \left| \rho_n^{-1} \bar{p}_{\tilde{z}(i)\tilde{z}(j)} - \tilde{f}(\xi(i)\tilde{z}(j)) \right|$. It remains only to bound $E \left| \tilde{f}(\tilde{z}(i)\tilde{z}(j)) - \tilde{f}(i_n,j_n) \right|$. We will do so using the following deterministic upper bound, which we prove below, and which holds uniformly over all $\tilde{z}$ and every $1 \leq a,b \leq k$:

\[(C.13) \quad \left| \tilde{f}(\tilde{z})_{ab} - \tilde{f}(\tilde{z})_{ab} \right| \leq M \{ \sqrt{2}/(n+1) \}^\alpha + M \left( \sqrt{2} h_a/n \right)^\alpha (h_a - 1)^{-1} \mathbb{I}(a = b) \]

\[(C.14) \quad \leq M 2^{\alpha/2} n^{-\alpha} \{ 1 + 2^\alpha \mathbb{I}(a = b) \}. \]

Here the second inequality following because, by definition, any $H(\cdot)$ has $\min_{1 \leq a \leq k} h_a \geq 2$. 

Lemma C.10 yields \((i_n, j_n) \in \omega \tilde{z}(i) \tilde{z}(j)\) for any \(\tilde{z}\); thus \(\tilde{f}(i_n) \tilde{z}(j) = \tilde{f}(i_n, j_n)\), and so if (C.13) holds, then it applies to \(|\tilde{f}(i_n) \tilde{z}(j) - \tilde{f}(i_n, j_n)|\). Finally, summing (C.10), (C.12) and (C.14) to obtain (C.7) completes the proof.

To establish (C.13), let \(i_n = i/(n+1)\), and multiply \(\tilde{f}(\tilde{z})_{ab}\) from (C.8) by \(1 = n^2/n^2\) to obtain

\[
\tilde{f}(\tilde{z})_{ab} = \frac{n^2}{h_{ab}} \sum_{j=H(b-1)+1}^{nH(b)} \sum_{i=H(a(1)+1)}^{nH(a)} \left( \frac{1}{n^2} \right) f(i_n, j_n), \quad 1 \leq a < b \leq k,
\]

(C.15)

\[
= \frac{n^2}{h_{ab}} \sum_{j=H(b(1)+1)}^{nH(b)} \sum_{i=H(a(1)+1)}^{nH(a)} \left( \int_{\frac{j}{n}}^{\frac{j+1}{n}} \int_{\frac{i}{n}}^{\frac{i+1}{n}} dx \, dy \right) f(i_n, j_n)
\]

(C.16)

\[
\int_{\frac{j}{n}}^{\frac{j+1}{n}} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \left[ f(x, y) + \{ f(i_n, j_n) - f(x, y) \} \right] \, dx \, dy.
\]

From (C.15) we will obtain the left-hand side of (C.13), plus a remainder term when \(a = b\), by writing

(C.17)

\[
\tilde{f}(\tilde{z})_{ab} = \frac{n^2}{h_{ab}} \sum_{j=H(b(1)+1)}^{nH(b)} \sum_{i=H(a(1)+1)}^{nH(a)} \int_{\frac{j}{n}}^{\frac{j+1}{n}} \int_{\frac{i}{n}}^{\frac{i+1}{n}} f(x, y) \, dx \, dy
\]

\[
= \tilde{f}(\tilde{z})_{ab} - \begin{cases} 
\frac{n^2}{h_{ab}} \int_{H(b(1)+1)}^{H(b)} \int_{H(a(1)+1)}^{H(a)} f(x, y) \, dx \, dy & a \neq b, \\
\frac{n^2}{h_{ab}} \sum_{j=H(b(1)+1)}^{nH(b)} \left( \int_{\frac{j}{n}}^{\frac{j+1}{n}} f(x) \, dx - \int_{\frac{j}{n}}^{\frac{j+1}{n}} f(x, y) \, dx \right) \, dy & a = b.
\end{cases}
\]

We recognize the first case in (C.17) as \(\tilde{f}(\tilde{z})_{ab,a \neq b}\). Since \(f\) is symmetric, the
second case can be written

\[ \tilde{f}(\bar{z})_{bb} + \sum_{j=nH(b-1)+1}^{nH(b)} \left[ \sum_{i=1}^{n \frac{j}{2}} \left\{ \frac{n^2}{(b^2)} - \frac{2n^2}{h_b^2} \right\} \int_{H(b-1)}^{y} - \frac{n^2}{h_b^2} \int_{j-1}^{y} \right] f(x, y) \, dx \, dy \]

\[ = \tilde{f}(\bar{z})_{bb} + \frac{1}{h_b - 1} \sum_{j=nH(b-1)+1}^{nH(b)} \left[ n^2 \int_{j-1}^{y} \left\{ \frac{2n^2}{h_b^2} \int_{H(b-1)}^{y} \right. \right. \right. \]

\[ = \tilde{f}(\bar{z})_{bb} + \frac{1}{h_b - 1} \left[ \frac{1}{h_b} \sum_{j=nH(b-1)+1}^{nH(b)} 2n^2 \int_{j-1}^{y} \left\{ \tilde{f}(\bar{z})_{bb} - f(x, y) \right\} \, dx \, dy \right] \]

Since \( \tilde{f}(x, y; h) = \tilde{f}(\bar{z})_{bb} \) on the domain of interest \( \omega_{bb} = [H(b-1), H(b)]^2 \), we conclude

\[ \frac{1}{h_b} \sum_{j=nH(b-1)+1}^{nH(b)} 2n^2 \int_{j-1}^{y} \left\{ \tilde{f}(\bar{z})_{bb} - f(x, y) \right\} \, dx \, dy \leq 1 \cdot 1 \cdot \| \tilde{f} - f \|_{L_\infty(\omega_{bb})} \]

\[ \leq M \left( \sqrt{2} h_b / n \right)^\alpha, \]

with the latter inequality from (C.19) of Lemma C.6, since \( f \in \text{Hölder}^\alpha(M) \).

This yields the upper bound term in (C.13) specific to \( a = b \). To derive the main term in (C.13), we return to (C.15), noting from Lemma C.7:

\[ \frac{n^2}{h_{ab}^2} \sum_{j=nH(b-1)+1}^{nH(b)} \left[ \sum_{i=nH(a-1)+1}^{nH(a)} \int_{j-1}^{y} \int_{i-1}^{y} \{ f(i_n, j_n) - f(x, y) \} \, dx \, dy \right] \]

\[ \leq \frac{1}{h_{ab}^2} \sum_{j=nH(b-1)+1}^{nH(b)} \left[ \sum_{i=nH(a-1)+1}^{nH(a)} \left[ \int_{j-1}^{y} \int_{i-1}^{y} |f(i_n, j_n) - f(x, y)| \, dx \, dy \right] \right] \]

\[ \leq 1 \cdot 1 \cdot M \left\{ \sqrt{2}/(n + 1) \right\}^\alpha. \]

\[ \square \]

**Lemma C.6.** Let \( f \) be a Hölder^\alpha(M) function on \((0, 1)^2\), with \( \tilde{f}(x, y; h) = \tilde{f}_{H^{-1}(x)H^{-1}(y)} \) its stepfunction approximation. Then for all \( 0 < p \leq \infty \),

\[ \| f - \tilde{f} \|_{L_p((0, 1)^2)} \leq M \left( \sqrt{2} \max_{1 \leq a \leq k} \frac{h_a}{n} \right)^\alpha. \]
Thus for any \( 0 < p < 1 \) since (C.18) \( E \parallel \) and so we immediately conclude where 
\[
\begin{align*}
|f_{ab} - f(x, y)| &= \left| \frac{1}{\omega_{ab}} \int_{\omega_{ab}} f(x', y') \, dx' \, dy' - f(x, y) \right|, \quad (x, y) \in (0, 1)^2 \\
\Rightarrow |\tilde{f}(x, y) - f(x, y)| &\leq \frac{1}{\omega_{ab}} \int_{\omega_{ab}} |f(x', y') - f(x, y)| \, dx' \, dy', \quad (x, y) \in \omega_{ab} \\
\Rightarrow \|\tilde{f} - f|_{\omega_{ab}}\|_{L^\infty(\omega_{ab})} &\leq \frac{1}{\omega_{ab}} \int_{\omega_{ab}} |f(x', y') - f(x, y)| \, dx' \, dy', \quad (x, y) \in \omega_{ab}
\end{align*}
\]

since \(|f(x, y) - f(x', y')| \leq M \|x, y) - (x', y')\|^\alpha = M \{(x - x')^2 + (y - y')^2\}^{\alpha/2} \) holds on \((0, 1)^2\).

To simplify (C.18), note that the diameter \( \sup_{\omega_{ab}}(x, y) \in (x, y) - (x', y') \) of the rectangular domain \( \omega_{ab} \) evaluates to \( \sqrt{h_a^2 + h_b^2/n} \), where \( h_a = H(a) - H(a - 1) \). Thus (C.18) implies
\[
\|\tilde{f} - f|_{\omega_{ab}}\|_{L^\infty(\omega_{ab})} \leq M \left( \sqrt{h_a^2 + h_b^2/n} \right)^\alpha, \quad 1 \leq a, b \leq k,
\]
and so we immediately conclude \( \|\tilde{f} - f\|_{L^\infty((0, 1)^2)} \leq M \left( \sqrt{2} \max a h_a/n \right)^\alpha \).

Thus for any \( 0 < p < \infty \),
\[
\begin{align*}
\|\tilde{f} - f\|_{L^p((0, 1)^2)}^p &= \int_{(0, 1)^2} |\tilde{f}(x, y) - (x, y)|^p \, dx \, dy \\
&\leq \int_{(0, 1)^2} \left\{ \|\tilde{f} - f\|_{L^\infty((0, 1)^2)} \right\}^p \, dx \, dy.
\end{align*}
\]

\(\square\)

**Lemma C.7.** Let \( f \) be a Hölder-\( \alpha \) \( (M) \) function on \((0, 1)^2\), and let \( \{\xi(i)\}_{i=1}^n \) be an ordered sample of independent Uniform\((0, 1)\) random variables. Then, recalling that \( E \xi(i) = i/(n + 1) \), we have for \( 1 \leq i, j \leq n \):

\[
\begin{align*}
E \left| f(\xi(i), \xi(j)) - f\left(\frac{i}{n + 1}, \frac{j}{n + 1}\right) \right|^\beta &\leq M^\beta \{2(n + 2)\}^{-\alpha \beta / 2}, \quad 0 < \beta \leq 2; \\
E \left| \tilde{f}(\xi(i), \xi(j)) - \tilde{f}\left(\frac{i}{n + 1}, \frac{j}{n + 1}\right) \right| &\leq M \{2(n + 2)\}^{-\alpha / 2} + 2M \left( \sqrt{2} \max_{1 \leq a \leq k} h_a/n \right)^\alpha,
\end{align*}
\]

where \( \tilde{f}(x, y; h) = \tilde{f}_{H^{-1}(x), H^{-1}(y)} \) is the stepfunction approximation of \( f \). Furthermore, we have for \( 1 \leq i, j \leq n \) that
\[
\left| f\left(\frac{i}{n + 1}, \frac{j}{n + 1}\right) - f(x, y) \right| \leq M \left\{ \sqrt{2}/(n + 1) \right\}^\alpha, \quad (x, y) \in \left(\frac{i - 1}{n}, \frac{i}{n}\right) \times \left(\frac{j - 1}{n}, \frac{j}{n}\right).
\]
PROOF. Let $i_n = E \xi(i) = i/(n + 1)$. Since $f \in \text{Hölder}^\alpha(M)$, it holds everywhere on $(0, 1)^2$ that

$$
|f(\xi(i), \xi(j)) - f(i_n, j_n)|^\beta \leq \{M |(\xi(i), \xi(j)) - (i_n, j_n)|^\alpha \}^\beta, \quad 1 \leq i, j \leq n,
$$

where $|\cdot|$ is the Euclidean metric on $\mathbb{R}^2$. By Jensen’s inequality, we have for any $0 < \alpha \beta \leq 2$ that for $1 \leq i, j \leq n$,

$$
E \{(\xi(i) - i_n)^2 + (\xi(j) - j_n)^2 \}^{\alpha \beta / 2} \leq (\text{var} \xi(i) + \text{var} \xi(j))^{\alpha \beta / 2} \leq \{2(n + 2)^{-\alpha \beta / 2},
$$

with the latter inequality via $\text{var} \xi(i) = i_n(1 - i_n)/(n + 2) \leq (1/4)/(n + 2)$. This proves the first result. For the second, we use Lemma C.6 and a chaining argument, since $f$ is piecewise-constant on blocks:

$$
|\bar{f}(\xi(i), \xi(j)) - \bar{f}(i_n, j_n)| \leq |(\bar{f} - f)(\xi(i), \xi(j))| + |f(\xi(i), \xi(j)) - f(i_n, j_n)|
$$

$$
+ |(f - \bar{f})(i_n, j_n)|
$$

$$
\leq |f(\xi(i), \xi(j)) - f(i_n, j_n)| + 2M (\sqrt{2} \max_{1 \leq a \leq k} h_a/n)^\alpha.
$$

Finally, $f \in \text{Hölder}^\alpha(M)$ implies for $(x, y) \in \left(\frac{i-1}{n}, \frac{i}{n}\right) \times \left(\frac{j-1}{n}, \frac{j}{n}\right)$ the uniform upper bound for $1 \leq i, j \leq n$:

$$
\sup_{(x, y) \in \left(\frac{i-1}{n}, \frac{i}{n}\right) \times \left(\frac{j-1}{n}, \frac{j}{n}\right)} \{(i_n - x)^2 + (j_n - y)^2 \}^{\alpha / 2}
$$

$$
\leq M \left[\max_{1 \leq i \leq n} \left\{2 \max\left(\frac{(i_n)^2}{n^2}, \frac{(1 - i_n)^2}{n^2}\right)\right\}\right]^{\alpha / 2}.
$$

\[ \square \]

**Lemma C.8.** Let $f$ be a symmetric Hölder $^\alpha(M)$ function on $(0, 1)^2$, with stepfunction approximation $\tilde{f}(x, y; h) = \tilde{f}_{H^{-1}(x), H^{-1}(y)}$, and let $\{\xi(i)\}_{i=1}^n$ be an ordered sample of independent Uniform(0, 1) random variables. Then whenever $\rho_n > 0$ and $0 < \rho_n f(x, y) < 1$ everywhere on $(0, 1)^2$,

$$(C.20) \quad \{\rho_n (n/2) \}^{-1} E \sum_{i < j} D \{\rho_n f(\xi_i, \xi_j) \| \rho_n \tilde{f}(\xi_i, \xi_j)\}$$

$$
\leq \frac{\rho_n M^2 \left(\sqrt{2} \max_{1 \leq a \leq k} h_a/n\right)^{2\alpha}}{\min_{1 \leq a, b \leq k} \left\{\min\left(\rho_n f_{ab}, 1 - \rho_n \tilde{f}_{ab}\right)\right\}} \cdot \max_{1 \leq a, b \leq k} \left[1 + \frac{1 + 2\Delta_{ab}}{3(1 - \Delta_{ab})^2}\right],
$$

where for $f|_{\omega_{ab}}$ the restriction of $f$ to $\omega_{ab} = [H(a - 1), H(a)] \times [H(b - 1), H(b)]$, we define

$$(C.21) \quad \Delta_{ab} = \frac{\rho_n \|f|_{\omega_{ab}} - \tilde{f}_{ab}\|_{L^\infty(\omega_{ab})}}{\min\left(\rho_n \tilde{f}_{ab}, 1 - \rho_n \tilde{f}_{ab}\right)} \leq \frac{\rho_n M \left(\sqrt{2} \max_{1 \leq a \leq k} h_a/n\right)^{\alpha}}{\min\left(\rho_n f_{ab}, 1 - \rho_n \tilde{f}_{ab}\right)}, \quad 1 \leq a, b \leq k.
Proof. Since \(\{\xi_i\}_{i=1}^n\) is a random sample of Uniform(0, 1) variates, and \(f\) is symmetric, we have

\[
(C.22) \quad \left\{ \rho_n (\frac{n}{2}) \right\}^{-1} E \sum_{i<j} D \left\{ \rho_n f (\xi_i, \xi_j) \mid \rho_n \tilde{f} (\xi_i, \xi_j) \right\} = \int \int_{(0,1)^2} \rho_n^{-1} D \left\{ \rho_n f (x,y) \mid \rho_n \tilde{f} (x,y) \right\} \, dx \, dy.
\]

Let \(p = \rho_n \tilde{f}\) and \(\delta = \rho_n (f - \tilde{f})\) pointwise on \((0,1)^2\), in order to apply Lemma C.9 to the integrand of \((C.22)\), and define the following ratio: \(\Delta_{ab} = \rho_n \| f \omega_{ab} - \tilde{f}_{ab} \|_{L^\infty (\omega_{ab})} / \min (\rho_n \tilde{f}_{ab}, 1 - \rho_n \tilde{f}_{ab})\). We may then write

\[
\int \int_{(0,1)^2} \rho_n^{-1} D \left\{ \rho_n f (x,y) \mid \rho_n \tilde{f} (x,y) \right\} \, dx \, dy
= \sum_{a=1}^k \sum_{b=1}^k \int_{\omega_{ab}} \rho_n^{-1} D \left\{ \rho_n f (x,y) \mid \rho_n \tilde{f}_{ab} \right\} \, dx \, dy
\leq \sum_{a=1}^k \sum_{b=1}^k \int_{\omega_{ab}} \rho_n^{-1} \left\{ \frac{\rho_n f (x,y) - \rho_n \tilde{f}_{ab}}{2 \rho_n \tilde{f}_{ab} (1 - \rho_n \tilde{f}_{ab})} \right\}^2 \left[ 1 + \Delta_{ab} \left\{ 1 + \frac{2}{3} (1 - \Delta_{ab})^2 \right\} \right] \, dx \, dy
\leq \max_{1 \leq a, b \leq k} \left[ \frac{1 + \Delta_{ab} \left\{ 1 + \frac{2}{3} (1 - \Delta_{ab})^2 \right\}}{2 \rho_n \tilde{f}_{ab} (1 - \rho_n \tilde{f}_{ab})} \right] \rho_n \| f - \tilde{f} \|_{L^2 ((0,1)^2)}^2.
\]

Our final step is to control the norms \(\| f \omega_{ab} - \tilde{f}_{ab} \|_{L^\infty (\omega_{ab})}\) and \(\| f - \tilde{f} \|_{L^2 ((0,1)^2)}^2\) in this bound. To do so, we apply Lemma C.6, which asserts that whenever \(f \in \text{Hölder}^\alpha (M)\), we have for all \(1 \leq a, b \leq k\) that

\[
(C.23) \quad \| f \omega_{ab} - \tilde{f}_{ab} \|_{L^\infty (\omega_{ab})} \leq \| f - \tilde{f} \|_{L^2 ((0,1)^2)} \leq M \left( \sqrt{2} \max_{1 \leq a \leq k} h_a / n \right)^\alpha.
\]

The result follows from \((C.23)\), since by hypothesis \(\max (\rho_n \tilde{f}_{ab}, 1 - \rho_n \tilde{f}_{ab}) \geq 1/2\) for every \((a,b)\), and so

\[
\frac{\rho_n \| f - \tilde{f} \|_{L^2 ((0,1)^2)}^2}{2 \rho_n \tilde{f}_{ab} (1 - \rho_n \tilde{f}_{ab})} \leq \frac{\rho_n \| f - \tilde{f} \|_{L^2 ((0,1)^2)}^2}{\min (\rho_n \tilde{f}_{ab}, 1 - \rho_n \tilde{f}_{ab})} \leq \frac{\rho_n M^2 \left( \sqrt{2} \max_{1 \leq a \leq k} h_a / n \right)^{2\alpha}}{\min (\rho_n \tilde{f}_{ab}, 1 - \rho_n \tilde{f}_{ab})}.
\]

Lemma C.9. Consider the Bernoulli Kullback–Leibler divergence quantities \(D (p \mid \mid p + \delta)\) and \(D (p + \delta \mid \mid p)\), where \(0 < p < 1\) and \(-p \leq \delta \leq 1 - p\).
If $|\delta| < \min(p, 1 - p)$, then the following bounds hold:

$$\left| D(p \parallel p + \delta) - \frac{\delta^2}{2p(1-p)} \right| \leq \frac{3}{2 \min(p, 1-p)} \left( 1 - \frac{|\delta|}{\min(p, 1-p)} \right)^{-3},$$

$$\left| D(p \parallel p - \delta) - \frac{\delta^2}{2p(1-p)} \right| \leq \frac{|\delta|}{\min(p, 1-p)} \left( 1 + \frac{2|\delta|}{\min(p, 1-p)} \right) \left( 1 - \frac{|\delta|}{\min(p, 1-p)} \right)^{-3}. \tag{C.24}$$

Now consider $\rho_n, f, g > 0$ such that $0 < \rho_n f, \rho_n g < 1$. Then $|f - g|^2 \leq 2f\rho_n^{-1} D(\rho_n f \parallel \rho_n g)$.

**Proof.** The first result follows by manipulating a Taylor series expansion of $D(p \parallel p + \delta)$ using the Lagrange form of the remainder. For some $\delta', \delta''$ satisfying $0 < |\delta'| < |\delta|$ and $0 < |\delta''| < |\delta|$, we have

$$D(p \parallel p + \delta) = \frac{\delta^2}{2p(1-p)} \left[ 1 + \frac{\delta}{3 \min(p, 1-p)} \left\{ p^2 (1 - \frac{\delta''}{1-p})^{-3} - (1-p)^2 (1 + \frac{\delta'}{p})^{-3} \right\} \right].$$

The first result then follows by controlling the scaled difference of the remainder terms appearing in (C.24), both of which are non-negative. We upper-bound this difference by the maximum of these two quantities, writing

$$\max\left\{ p^2 (1 - \frac{\delta''}{1-p})^{-3}, (1-p)^2 (1 + \frac{\delta'}{p})^{-3} \right\} \leq \left\{ \max(p, 1-p) \right\}^2 \left( 1 - |\delta| / \min(p, 1-p) \right)^{-3}.$$

The second result follows similarly, by manipulating a Taylor series expansion of $D(p + \delta \parallel p)$.

The final result follows from rewriting $D(\rho_n f \parallel \rho_n g)$ as $D(\rho_n (g + d) \parallel \rho_n g)$, with $d = f - g$. We first bound the second derivative of $D(\rho_n (g + d) \parallel \rho_n g)$ in $d$ below by $\rho_n / f$, and then integrate twice, using that $D(\rho_n (g + d) \parallel \rho_n g) = 0$ if $d = 0$.

**Lemma C.10.** Let $i_n = i/(n + 1)$ and $j_n = j/(n + 1)$. Then $(i_n, j_n) \in \omega_{a_i b_j}$, where $a_i$ and $b_j$ are defined by

$$a_i = H^{-1}(i/n), \quad b_j = H^{-1}(j/n), \quad 1 \leq a, b \leq k, \quad 1 \leq i, j \leq n.$$

**Proof.** From the definition of $a_i$ we may directly compute

$$H\{a_i\} = H\{H^{-1}(i/n)\} = n^{-1} \sum_{a=1}^{\min\{H^{-1}(i/n), k\}} h_a \begin{cases} = i/n & \text{if } \sum_{a=1}^{\min\{H^{-1}(i/n), k\}} h_a = i, \\ \geq (i+1)/n & \text{if } \sum_{a=1}^{\min\{H^{-1}(i/n), k\}} h_a \neq i. \end{cases}$$
We also have that

\[ H(a_i - 1) = H \{ H^{-1} \left( \frac{i}{n} \right) - 1 \} \]

\[
= n^{-1} \sum_{a=1}^{\min \{ H^{-1}(i/n) - 1, k \}} h_a \begin{cases} 
(i - 1)/n & \text{if } \sum_{a=1}^{i-1} h_a = i - 1, \\
(i - 2)/n & \text{if } \sum_{a=1}^{i-1} h_a \neq i - 1.
\end{cases}
\]

We have by definition that \( \omega_{a_i b_j} = \left[ H \{ H^{-1} \left( \frac{i}{n} \right) - 1 \}, H \left\{ H^{-1} \left( \frac{j}{n} \right) \right\} \right] \times \left[ H \{ H^{-1} \left( \frac{j}{n} \right) - 1 \}, H \left\{ H^{-1} \left( \frac{i}{n} \right) \right\} \right] \). Since \( H(\cdot) \) and its inverse \( H^{-1}(\cdot) \) are non-decreasing functions, it follows that \( H \{ H^{-1} \left( \frac{i}{n} \right) \} \geq i/n \geq i/(n+1) = i_n \). Thus the claimed upper bound is respected. Furthermore, for the lower limit, \( H \{ H^{-1} \left( \frac{i}{n} \right) - 1 \} \leq (i - 1)/n \leq i_n \), as \( (i - 1)/n \leq i/(n + 1) = i_n \iff i \leq n + 1 \). Thus the claimed lower bound is also respected, and so by symmetry, we conclude that \((i_n, j_n) \in \omega_{a_i b_j}\) .

\medskip

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