Envelope of Mid-Planes of Surfaces in $\mathbb{R}^3$

Ady Cambraia Junior · Marcos Craizer

Abstract Given two points of a smooth convex planar curve, the mid-line is the line passing through the mid-point and the intersection of tangent lines. The Envelope of Mid- Lines is a well-known affine invariant set. It consists of the union of three subsets: The Affine Envelope Symmetry Set, which corresponds to non-parallel tangent lines, the Mid-Parallel Tangents Locus, corresponding to parallel tangent lines at non-coincident points and the Affine Evolute, corresponding to coincident points.

In this paper we generalize this concept to convex surfaces as an envelope of mid-planes EMP, where a mid-plane is the plane containing the mid-point and the intersection line of the tangent planes at two given points. The EMP can similarly be divided into three affine invariant subsets. In case of non-parallel tangent planes, we show that the geometry of the EMP is contained in a distinguished plane. In case of parallel tangents at non-coincident points, the EMP coincides with a known set, the Mid-Parallel Tangents Surface. The case of coincident points is particularly interesting, since it connects to very old topics of affine differential geometry, like Transon planes, cones of B.Su and Moutard’s quadrics. In the first and third cases we describe conditions under which the EMP is a regular surface.

Keywords affine envelope symmetry set · mid-parallel tangent locus · affine evolute · Transon planes · Moutard’s quadrics · cone of B.Su

1 Introduction

Consider a smooth convex planar curve $\gamma$ and let $p_1, p_2$ be points of $\gamma$. The mid-line $L(p_1, p_2)$ is the line that passes through the mid-point $M$ of $p_1$ and $p_2$ and the intersection of the tangent lines at $p_1$ and $p_2$. If these tangent lines are parallel, the mid-line is the line through $M$ parallel to both tangents. When $p_1 = p_2$, the mid-line is just the affine normal at the point. The envelope of these mid-lines is an important affine invariant set associated with the curve. It has been studied by many authors ([1], [2], [3], [4], [10]).

In this paper we generalize this concept to a surface $S$ in $\mathbb{R}^3$ by considering the envelope of its mid-planes. For $p_1, p_2 \in S$, the mid-plane $F(p_1, p_2)$ is the plane that passes through the mid-point $M$ of $p_1$ and $p_2$ and the intersection line of the tangent planes at $p_1$ and $p_2$. If these tangent planes are parallel, the mid-plane is the plane through $M$ parallel to both tangent planes. When $p_1 = p_2$, we have to consider Transon planes, a classical concept in affine differential geometry.

The second author wants to CNPq for financial support during the preparation of this manuscript

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The Envelope of Mid-Lines of planar curves can be divided into 3 parts: The Affine Envelope Symmetry Set (AESS), corresponding to pairs \((p_1, p_2)\) with non-parallel tangent planes, the Mid-Points Parallel Tangent Locus (MPTL), corresponding to pairs \((p_1, p_2), p_1 \neq p_2\), with parallel tangent planes, and the Affine Evolute, corresponding to coincident points. The Envelope of Mid-Planes (EMP) is also naturally divided into three subsets: The EMP1, corresponding to pairs \((p_1, p_2)\) with non-parallel tangent planes, the EMP2 corresponding to pairs \((p_1, p_2), p_1 \neq p_2,\) with parallel tangent planes, and the EMP3 corresponding to the limit case of coincident points.

For curves, the AESS is very well studied and coincides with the locus of center of conics having contact of order \(\geq 3\) with the curve at \(p_1\) and \(p_2\). Moreover, conditions for the AESS to be a regular at a given point are known \([2, 4]\). In this article we prove corresponding results for the set EMP1. The interesting fact here is that all relevant points of this construction belong to one distinguished plane, which is generated by tangent vectors orthogonal in the Blaschke metric to the intersection line of the tangent planes. We prove that a point of the EMP1 is the center of a conic contained in this plane having contact of order \(\geq 3\) with the surface. Moreover, under certain conditions, the EMP1 is a regular surface. Another interesting property of the AESS is that if it is contained in a straight line, then the curve is invariant by an affine reflection with axis containing the AESS. We give an example that shows that this property does not remain true for the EMP1.

The Mid-Parallel Tangent Surface (MPTS) of a surface is the locus of mid-points of pairs \((p_1, p_2)\) with parallel tangents \([10]\). It turns out that the MPTS coincides with the set EMP2. As in the curve case, the EMP1 and MPTS meet at center of conics with contact of order \(\geq 3\) and parallel tangents, where both sets are expected to be singular. Based on \([10]\), we give conditions under which, at such a point, the MPTS is equivalent to a cuspidal edge.

The Affine Evolute of a planar curve is the limit of the Envelope of Mid-Lines when \(p_1 = p_0\). We study in this paper the corresponding limit set EMP3 for surfaces in \(\mathbb{R}^3\). We verify that if we fix a tangent vector \(T\) and make \(p_1 \rightarrow p_0\) in this direction, the mid-plane of \((p_1, p_0)\) converges to the Transon plane of \(T\) at \(p_0\). We define the medial curve \(\gamma\) of \(T\) as the intersection of \(S\) with the plane containing \(T\) and the line of the cone of B.Su associated to \(T\). Then the affine normal to \(\gamma\) belongs to the cone of B.Su and the affine center of curvature of \(\gamma\) coincides with the center of the Moutard’s quadric. The notions of Transon Plane, cone of B.Su and Moutard’s quadric are very old, but there are some modern references \([5, 8]\). We verify that the tangents \(T\) that generates some point for the EMP3 are the solutions of a polynomial equation of degree 6, thus they are at most 6. We also give conditions for the regularity of each branch of the EMP3.

The paper is organized as follows: In section 2 we review some basic facts of affine differential geometry of surfaces in \(\mathbb{R}^3\). In section 3 we study the EMP1, giving necessary and sufficient conditions for a pair \((p_1, p_2)\) to contribute to this set. We give also conditions for the EMP1 to be a regular surface. In section 4 we describe some properties of the MPTS. In section 5, we describe the limit equations for the EMP and relate them with the classical concepts of Transon plane, cone of B.Su and Moutard’s quadric. Then we give conditions for the EMP3 to be a regular surface.

This work is part of the doctoral thesis of the first author under the supervision of the second author.

2 Affine differential geometry of surfaces in \(\mathbb{R}^3\)

In this section we review the basic concepts of affine differential geometry of surfaces in \(\mathbb{R}^3\) (for details, see \([7]\)). Denote by \(D\) the canonical connection and by \(\Omega\) the standard volume form in \(\mathbb{R}^3\). Let \(S \subset \mathbb{R}^3\) be a surface and denote by \(\mathfrak{X}(S)\) the tangent bundle of \(S\). Given a transversal vector field \(\xi\), write the Gauss equation

\[
D_\xi Y = \nabla_\xi Y + h(X, Y)\xi,
\]

\(X, Y \in \mathfrak{X}(S)\), where \(h\) is a symmetric bilinear form and \(\nabla\) is a torsion free connection in \(S\). We shall assume that \(h\) is non-degenerate, which is independent of the choice of \(\xi\). For \(X \in \mathfrak{X}(S)\), we have Weingarten formula

\[
D_\xi \xi = -AX + \tau(X)\xi,
\]

where \(A\) is a \((1, 1)\)-tensor and \(\tau\) a 1-form.
The volume form $\Omega$ induces a volume form in $S$ by the relation

$$\theta(X_1, X_2) = \Omega(X_1, X_2, \xi).$$

The metric $h$ also defines a volume form in $S$: Given $X_i \in \mathfrak{X}(S)$ para $1 \leq i \leq 2$, denote by $H := (h_{i,j})$ the $2 \times 2$ matrix whose entries are $h_{i,j} := h(X_i, X_j)$ and define

$$\theta_h(X_1, X_2) := |\det H|^\frac{1}{2}.$$

Next theorem is fundamental in affine differential geometry ([7], th.3.1, ch.II):

**Theorem 1** There exists, up to signal, a unique transversal vector field $\xi$ such that $\nabla \theta = 0$ and $\theta = \theta_h$.

Let $\mathbb{R}^3$ denote the dual vector space of $\mathbb{R}^3$. For $x \in S$, let $\nu_x$ be the linear functional in $\mathbb{R}^3$ such that

$$\nu_x(\xi) = 1 \quad \text{and} \quad \nu_x(X) = 0 \quad \forall \ X \in T_xS.$$

(1)

The differentiable map $\nu : S \rightarrow \mathbb{R}^3 - \{0\}$ is called the conormal map. It satisfies the following property ([7], prop.5.1, ch.II):

**Proposition 1** Let $S \subset \mathbb{R}^3$ be a non-degenerate surface and $\nu$ the conormal map. Then

$$D_Y \nu(\xi) = 0 \quad \text{and} \quad D_Y \nu(X) = -h(Y, X), \quad \forall \ X, Y \in \mathfrak{X}(S)$$

**Corollary 1** If $X \in \mathfrak{X}(\mathbb{R}^3)$ is any vector field in $\mathbb{R}^3$, then

$$D_Y \nu(X) = -h(Y, X^T), \quad Y \in \mathfrak{X}(S),$$

where $X = X^T + \lambda \xi$, $\lambda \in \mathbb{R}$ and $X^T$ is the tangent component of $X$.

**Proof.** We have

$$D_Y \nu(X) = D_Y \nu(X^T + \lambda \xi) = D_Y \nu(X^T) + \lambda D_Y \nu(\xi) = -h(Y, X^T),$$

thus proving the corollary.  

\[\Box\]

3 Envelope of Mid-Planes - Non-Parallel Tangent Planes

Let $S$ be a non-degenerate convex surface. Take points $p_1, p_2 \in S$ and let $S_1 \subset S$ and $S_2 \subset S$ be open subsets around $p_1$ and $p_2$, respectively. Denote $h_1$ and $h_2$ the Blaschke metrics of $S_1$ and $S_2$, respectively. In this section we shall assume that the tangent planes at $p_1$ and $p_2$ are non-parallel.

3.1 Basic definitions

Denote by $M(p_1, p_2) = \frac{p_1 + p_2}{2}$ the mid-point of $p_1$ and $p_2$ and by $C(p_1, p_2) = p_1 - p_2$ the chord connecting these points. The mid-plane $F(p_1, p_2)$ is the plane that contains $M(p_1, p_2)$ and the line $r(p_1, p_2)$ of intersection of the tangent planes at $p_1$ and $p_2$.

**Lemma 1** The kernel of

$$\nu = \nu_2(C) \nu_1 + \nu_1(C) \nu_2$$

is exactly the mid-plane.

**Proof.** Take $R \in r$. Since $M - R = \frac{C}{2} + p_1 - R$ and $M - R = -\frac{C}{2} + p_2 - R$, we obtain from equation (2) that

$$\nu(M - R) = \nu_2(C) \nu_1 \left(\frac{C}{2} + p_1 - R\right) + \nu_1(C) \nu_2 \left(-\frac{C}{2} + p_2 - R\right) = \frac{\nu_2(C) \nu_1(C)}{2} - \frac{\nu_1(C) \nu_2(C)}{2} = 0,$$

thus proving the lemma.  

\[\Box\]
From lemma 1 we conclude that the equation of the mid-plane is given by

$$F(p_1, p_2, X) = 0,$$

where $$F : S_1 \times S_2 \times \mathbb{R}^3 \rightarrow \mathbb{R}$$ is given by

$$F(p_1, p_2, X) = (\nu_2(C)\nu_1 + \nu_1(C)\nu_2)(X - M).$$

(3)

Denote

$$\overline{EPM} = \{(p_1, p_2, X) \in S_1 \times S_2 \times \mathbb{R}^3 | F = F_{p_1} = F_{p_2} = 0\}$$

and

$$\overline{EPM}_1 = \{(p_1, p_2, X) \in \overline{EPM} | T_{p_1}S_1 \not\parallel T_{p_2}S_2\}.$$

Denoting by $$\pi$$ the projection in the third coordinate, the envelope EMP of the family of mid-planes is the set $$\pi(\overline{EPM})$$. In this section we shall study the properties of the set $$EPM_1 = \pi(\overline{EPM}_1)$$.

Consider a smooth function

$$Z : S_1 \times S_2 \rightarrow \mathbb{R}^3$$

such that $$Z(p_1, p_2)$$ is parallel to $$r(p_1, p_2)$$. Consider also, for $$i = 1, 2$$, smooth functions

$$Y_i : S_1 \times S_2 \rightarrow \mathbb{R}^3$$

such that $$Y_i(p_1, p_2)$$ is tangent to $$S_i$$ and $$h_i(Y_i, Z) = 0$$. We want to find $$X$$ satisfying $$F = F_{p_1} = F_{p_2} = 0$$, for some $$p_1 \in S_1$$, $$p_2 \in S_2$$. Since $$Y_i$$ and $$Z$$ are $$h_i$$-orthogonal, $$\{Y_i, Z\}$$ is a basis of $$T_{p_i}S_i$$, $$i = 1, 2$$. Thus we have to find $$X$$ in the following system:

$$\begin{cases}
F(p_1, p_2, X) = 0 \\
F_{p_1}(p_1, p_2, X)(Y_1) = 0 \\
F_{p_2}(p_1, p_2, X)(Y_2) = 0 \\
F_{p_1}(p_1, p_2, X)(Z) = 0 \\
F_{p_2}(p_1, p_2, X)(Z) = 0
\end{cases}$$

(4)

The notation $$F_{p_i}(p_1, p_2, X)(W)$$ corresponds to the partial derivative of $$F$$ with respect $$p_i$$ in the direction $$W \in T_{p_i}S_i$$, thus keeping $$p_j$$, $$j \neq i$$ and $$X$$ fixed.

3.2 Main result

We begin with the following simple lemma:

**Lemma 2** We have that

$$D_{Y_1}\nu_1 = (\nu_1)_{Y_1} = a\nu_1 + b\nu_2$$

and

$$D_{Y_2}\nu_2 = (\nu_2)_{Y_2} = \bar{a}\nu_1 + \bar{b}\nu_2,$$

where $$a, b, \bar{a}, \bar{b}$$ are given by

$$a = -\frac{h_1(Y_1, X_2)}{\nu_1(X_2)}, \quad b = -\frac{h_1(Y_1, X_1)}{\nu_2(X_1)}, \quad \bar{a} = -\frac{h_2(Y_2, X_2)}{\nu_1(X_2)}, \quad \bar{b} = -\frac{h_2(Y_2, X_1)}{\nu_2(X_1)},$$

for any $$X_1 \in T_{p_1}S_1$$, $$X_2 \in T_{p_2}S_2$$.

**Proof.** Take a basis $$\{\nu_1, \nu_2, \zeta\}$$ of the dual space $$\mathbb{R}^3$$. Thus we can write the linear functional $$D_{Y_1}\nu_1$$ as a linear combination of the basis vector, i.e., $$D_{Y_1}\nu_1 = a\nu_1 + b\nu_2 + c\zeta$$. Since $$D_{Y_1}\nu_1(Z) = -h_1(Y_1, Z) = 0$$ we obtain $$c = 0$$ and so $$D_{Y_1}\nu_1 = a\nu_1 + b\nu_2$$. In an analogous way we show that $$D_{Y_2}\nu_2 = (\nu_2)_{Y_2} = \bar{a}\nu_1 + \bar{b}\nu_2$$. Applying $$D_{Y_1}\nu_1$$ to any tangent vector field $$X_1$$ on $$S_1$$ we get $$D_{Y_1}\nu_1(X_1) = b\nu_2(X_1)$$, thus proving the formula for $$b$$. The other formulas are proved similarly.

For the sake of clarity, we divide the main result in two parts, beginning with the following:
Proposition 2  The first three equations of the system \( (4) \) admit a solution if and only if

\[
\nu_1(C) = -\lambda \nu_2(C), \quad \text{ where } \quad \lambda = \left( \frac{\nu_2(Y_2) \ h_1(Y_1, Y_2)}{\nu_2(Y_1) \ h_2(Y_2, Y_2)} \right)^{\frac{1}{2}}. \tag{5}
\]

Proof. Note that \( C_{p_1} \cdot Y_1 = Y_1 \) and that \( M_{p_1} \cdot Y_1 = \frac{Y_1}{2} \). Since \( \nu_1(Y_1) = \nu_2(Y_2) = 0 \), it follows that the derivative \( F_{p_1}(Y_1) \) is given by

\[
F_{p_1}(Y_1) = (d \nu_1(p_1) \cdot Y_1) (C) \nu_2(X - M) + \nu_2(Y_1) \nu_1(X - M) + \nu_2(C) (d \nu_1(p_1) \cdot Y_1) (X - M),
\]

where \( d \nu_1(p_1) \cdot Y_1 \) denotes the derivative of the co-normal map \( \nu_1 \) at \( p_1 \) in the direction \( Y_1 \). By lemma \( (2) \) we obtain

\[
F_{p_1}(Y_1) = (2b \nu_2(C) \nu_2 + \nu_2(Y_1) \nu_1)(X - M) - \frac{1}{2} \nu_1(C) \nu_2(Y_1) + a F.
\]

Similarly

\[
F_{p_2}(Y_2) = (2 \bar{a} \nu_1(C) \nu_1 - \nu_1(Y_2) \nu_2)(X - M) - \frac{1}{2} \nu_2(C) \nu_1(Y_2) + \bar{b} F.
\]

Since \( F = 0 \), it follows that

\[
F_{p_1}(Y_1) = (2b \nu_2(C) \nu_2 + \nu_2(Y_1) \nu_1)(X - M) - \frac{1}{2} \nu_1(C) \nu_2(Y_1),
\]

\[
F_{p_2}(Y_2) = (2 \bar{a} \nu_1(C) \nu_1 - \nu_1(Y_2) \nu_2)(X - M) - \frac{1}{2} \nu_2(C) \nu_1(Y_2).
\]

Taking \( \nu_1(X - M) \) from the equation \( F = 0 \) and substituting in \( F_{p_1}(Y_1) = 0 \) and \( F_{p_2}(Y_2) = 0 \) we get

\[
\left( - \frac{\nu_2(Y_1) \nu_1(C) \nu_2}{\nu_2(C)} + 2b \nu_2(C) \nu_2 \right)(X - M) = \frac{1}{2} \nu_1(C) \nu_2(Y_1), \tag{6}
\]

\[
\left( - \frac{2 \bar{a} \nu_1^2(C) \nu_2}{\nu_2(C)} - \nu_1(Y_2) \nu_2 \right)(X - M) = \frac{1}{2} \nu_2(C) \nu_1(Y_2). \tag{7}
\]

Taking \( \nu_2(X - M) \) from equation \( (6) \) and substituting in equation \( (7) \), after some simplifications we obtain

\[
2b \nu_2^3(C) \nu_2(Y_2) = -2 \bar{a} \nu_1^3(C) \nu_2(Y_1). \tag{8}
\]

It follows that \( \nu_1(C) = -\lambda \nu_2(C) \), where \( \lambda = \left( \frac{\nu_2(Y_2) \ h_1(Y_1, Y_2)}{\nu_2(Y_1) \ h_2(Y_2, Y_2)} \right)^{\frac{1}{2}} \), which proves the proposition. \( \square \)

Next lemma is a preparation for the main result:

Lemma 3  We have that

\[
C = A \left( Y_1 - \frac{\lambda \nu_2(Y_1)}{\nu_1(Y_2)} Y_2 \right) + \alpha Z
\]

and

\[
X - M = B \left( Y_1 + \frac{\lambda \nu_2(Y_1)}{\nu_1(Y_2)} Y_2 \right) + \beta Z,
\]

where \( A, \alpha, \beta \in \mathbb{R} \) and \( B = -\frac{\lambda A}{2(\lambda + 2 A b)} \).

Proof. Writing \( C = A Y_1 + \zeta Y_2 + \alpha Z \), we obtain \( \nu_1(C) = \zeta \nu_1(Y_2) \) and \( \nu_2(C) = A \nu_2(Y_1) \). Since \( \nu_1(C) = -\lambda \nu_2(C) \), it follows that \( \zeta = -\frac{\lambda \nu_2(Y_1)}{\nu_1(Y_2)} A \), thus proving the first equation. Since \( \nu_1(C) = -\lambda \nu_2(C) \), it follows from \( F = 0 \) that \( \nu_1(X - M) = \lambda \nu_2(X - M) \). The same argument now proves the second equation. From equation \( (6) \) we have

\[
\nu_2(X - M) = \frac{-\lambda \nu_2(C) \nu_2(Y_1)}{2(\lambda \nu_2(Y_1) + 2 b \nu_2(C))}.
\]

We conclude that \( B = -\frac{\lambda \nu_2(C)}{2(\lambda \nu_2(Y_1) + 2 b \nu_2(C))} = -\frac{\lambda A}{2(\lambda + 2 A b)} \), thus proving the lemma. \( \square \)
Next theorem is our main result and says that $\alpha$ and $\beta$ in lemma 3 are zero when the system 4 admits a solution.

**Theorem 2** The system 4 has a solution if and only if

$$\nu_1(C) = -\lambda \nu_2(C) \quad \text{and} \quad C, Y_1 \text{ e } Y_2 \text{ are co-planar.}$$

Moreover, the solution of the system is given by

$$X - M = B \left( Y_1 + \frac{\lambda \nu_2(Y_1)}{\nu_1(Y_2)} Y_2 \right),$$

where $B = -\frac{\lambda \nu_2(C)}{2(\lambda \nu_2(Y_1) + 2b \nu_2(C))}$.

**Proof.** To show that $\alpha = \beta = 0$, we shall consider the last two equations of the system 4. The derivative $F_{p_1}(Z)$ is given by

$$F_{p_1}(Z) = \left( (dv_1(p_1) \cdot Z) (C) + \nu_1(C_p) \cdot Z \right) \nu_2(X - M) + \nu_1(C) \nu_2(-M_{p_1} \cdot Z) + \nu_2(C_p) \cdot Z \nu_1(X - M) +$$

$$+ \nu_2(C) ((dv_1(p_1) \cdot Z) (X - M) + \nu_1(-M_{p_1} \cdot Z)).$$

From lemma 3 we have $(dv_1(p_1) \cdot Z) (C) = -h_1(Z, C^T)$ and $(dv_1(p_1) \cdot Z) (X - M) = -h_1(Z, (X - M)^T)$. Thus

$$F_{p_1}(Z) = -h_1(Z, C^T) \nu_2(X - M) - \nu_2(C) h_1(Z, (X - M)^T).$$

From lemma 3 $C^T e (X - M)^T$ are given by

$$C^T = A \left( Y_1 - \frac{\lambda \nu_2(Y_1)}{\nu_1(Y_2)} Y_2 \right) + \alpha Z$$

and

$$(X - M)^T = B \left( Y_1 + \frac{\lambda \nu_2(Y_1)}{\nu_1(Y_2)} Y_2 \right) + \beta Z.$$

Thus

$$F_{p_1}(Z) = \left( -\frac{\lambda \nu_2(Y_1)}{\nu_1(Y_2)} Ah_1(Z, Y_2^T) + \alpha h_1(Z, Z) \right) B \nu_2(Y_1) + A \nu_2(Y_1) \left( \frac{\lambda \nu_2(Y_1)}{\nu_1(Y_2)} Bh_1(Z, Y_2^T) + \beta h_1(Z, Z) \right)$$

and so

$$F_{p_1}(Z) = h_1(Z, Z) \nu_2(Y_1)(\alpha B + \beta A).$$

Similarly we obtain

$$F_{p_2} Z = \lambda h_2(Z, Z) \nu_2(Y_1)(-\alpha B + \beta A).$$

Since the metrics $h_1, h_2$ are positive, $A, B, \nu_2(Y_1)$ and $\lambda$ are non-zero, since the tangent planes are non-parallel, it follows from equations $F_{p_1} Z = F_{p_2} Z = 0$ that $\alpha = \beta = 0$. \qed

Theorem 2 says that the chord $C$, $Y_1$ and $Y_2$ are co-planar. This implies that all the geometry of the envelope of mid-planes occurs in the plane generated by $Y_1$ and $Y_2$ (see figure 1).

**Remark 1** In section 3.2 we have used the co-normals defined by equations 4. In fact, we can use any multiple of the co-normal vector field to define the mid-planes. We have only to take care and use the correct multiple of the Blaschke metric. More precisely, let $\tilde{\nu}_i = f_i \nu_i$, $i = 1, 2$, be an arbitrary co-normal vector field, where $f_i \neq 0$. Define

$$\tilde{h}_i(X, Y) = - (D_X \tilde{\nu}_i)(Y),$$

for $X, Y \in T_{p_i} S_i$. Then

$$\tilde{h}_i(X, Y) = - (D_X \tilde{\nu}_i)(Y) = - (X(f_i) \nu_i + f_i D_X \nu_i)(Y) = f_i h_i(X, Y).$$
3.3 Centers of 3 + 3 conics

Given a non-degenerate convex surface $S$, consider conics that makes contact of order $\geq 3$ with $S$ at two different points $p_1, p_2$ in directions $h$-orthogonals to the intersection of $T_{p_1}S$ and $T_{p_2}S$, which are assumed to be non-parallel. We shall prove in this section that the set of centers of these $3 + 3$ conics coincides with the set EMP1.

Along the paper, we shall denote by $O(n)$ terms of degree $\geq n$ in $(x, y)$. We begin with the following simple lemma:

**Lemma 4** Let $S$ be the graph of a function $f$ given by

$$f = f_0 + f_{1,0}x + f_{0,1}y + f_{2,0}x^2 + f_{1,1}xy + f_{0,2}y^2 + O(3),$$

Then, at $(x, y) = (0, 0)$, the tangent vectors $(1, 0, f_{1,0})$ and $(0, 1, f_{0,1})$ are orthogonal in the Blaschke metric if and only if $f_{1,1} = 0$.

**Proof.** The Blaschke metric at $(0, 0)$ is conformal with the quadratic form given by the hessian matrix of $f$ at $(0, 0)$.

**Lemma 5** Assume that the pair $(p_1, p_2)$ generates a point of EMP1. Then by an affine change of coordinates, we may assume that $p_1 = (0, 0, 1)$, $p_2 = (0, 0, -1)$ and $S_1$ and $S_2$ are graphs of

$$f_1(u_1, v_1) = 1 + \frac{p}{c}u_1 - \frac{(p^2 + c)}{2c^2}u_1^2 + f_{0,2}v_1^2 + \sum_{i=0}^{3} f_{3-i,i}u_1^{3-i}v_1^i + O(4),$$

$$f_2(u_2, v_2) = -1 - \frac{p}{c}u_2 + \frac{(p^2 + c)}{2c^2}u_2^2 + g_{0,2}v_2^2 + \sum_{i=0}^{3} g_{3-i,i}u_2^{3-i}v_2^i + O(4),$$

where $(p, 0, 0)$ is the corresponding point in EMP1. As a consequence, $(p, 0, 0)$ is the center of a $3 + 3$ conic.

**Proof.** Consider $p_1 \in S_1$ and $p_2 \in S_2$ with non-parallel tangent planes. By an adequate affine change of variables, we may assume that $p_1 = (0, 0, 1)$, $p_2 = (0, 0, -1)$ and the mid-plane is $z = 0$. We may also assume that $Y_1$ and $Y_2$ are in the $xz$-plane and that the line $r$ of intersection of the tangent planes $T_{p_1}S_1$ and $T_{p_2}S_2$ is the $y$-axis. By lemma 4 these conditions implies that the coefficients of $u_1v_1$ and $u_2v_2$ are zero. Then $S_1$ and $S_2$ are graphs of

$$f_1(u_1, v_1) = 1 + \frac{p}{c}u_1 - \frac{(p^2 + c)}{2c^2}u_1^2 + f_{0,2}v_1^2 + \sum_{i=0}^{3} f_{3-i,i}u_1^{3-i}v_1^i + O(4),$$

$$f_2(u_2, v_2) = -1 - \frac{p}{c}u_2 + \frac{(p^2 + c)}{2c^2}u_2^2 + g_{0,2}v_2^2 + \sum_{i=0}^{3} g_{3-i,i}u_2^{3-i}v_2^i + O(4).$$
\[ f_2(u_2, v_2) = -1 - \frac{p}{c} u_2 + \delta \frac{(p^2 + c)}{2c^2} u_2^2 + g_{0,2} v_2^2 + \sum_{i=0}^{3} g_{3-i, i} u_2^{3-i} v_2^i + O(4), \]  \hspace{1cm} (13)

At the origin, the system \( F = F_{u_1} = F_{v_1} = F_{u_2} = F_{v_2} = 0 \) is given by

\[
\begin{align*}
-2z &= 0 \\
x + pz - p &= 0 \\
2f_{0,2} y &= 0 \\
(p^2 - p^2\delta - \delta)x + pz + p &= 0 \\
2g_{0,2} y &= 0
\end{align*}
\]  \hspace{1cm} (14)

Since, by hypothesis, this system admit a solution, \((p, 0, 0)\) satisfies the above system. Thus we conclude that \( \delta = 1 \). It is not difficult to verify now that there exists a conic centered at \((p, 0, 0)\) and making contact of order \( \geq 3 \) with \( S_1 \) at \( p_1 \) and \( S_2 \) at \( p_2 \), thus proving the lemma.

**Lemma 6** Assume that \((p_1, p_2)\) are tangent points of a 3+3 conic. Then, by an affine change of coordinates, we may assume that \( p_1 = (0, 0, 1) \), \( p_2 = (0, 0, -1) \) and \( S_1 \) and \( S_2 \) are graphs of functions \( f_1 \) and \( f_2 \) given by equations (10) and (11), where \((p, 0, 0)\) is the center of the conic and \( c \in \mathbb{R} \). As a consequence, \((p, 0, 0)\) belongs to EMP1.

**Proof.** Consider \( p_1 \in S_1 \) and \( p_2 \in S_2 \) with non-parallel tangent planes. By an adequate affine change of variables, we may assume that \( p_1 = (0, 0, 1) \), \( p_2 = (0, 0, -1) \) and the mid-plane is \( z = 0 \). We may also assume that the conic is contained in the \( xz \)-plane and that the line \( r \) of intersection of the tangent planes \( T_{p_1}S_1 \) and \( T_{p_2}S_2 \) is the \( y \)-axis. By lemma 4, these conditions imply that the coefficients of \( u_1 v_1 \) and \( u_2 v_2 \) are zero.

By the same argument of lemma 6, \( S_1 \) and \( S_2 \) are graphs of functions given by equations (12) and (13). Now the hypothesis that the conic makes a contact of order \( \geq 3 \) with \( S_2 \) at \( p_2 \) implies that \( \delta = 1 \). Thus \((p, 0, 0)\) satisfies the system (14), which implies that \((p, 0, 0)\) belongs to EMP1.

From the above two lemmas we can conclude the main result of this section.

**Proposition 3** The set of centers of 3+3 conics coincides with the set EMP1.

### 3.4 Regularity of the EPM1

In this section, we shall study the regularity of the set EPM1. Let \( F(u_1, v_1, u_2, v_2, X) = 0 \) be the equation of the mid-plane at \((p_1, p_2)\) and consider the map

\[
H : \mathbb{R}^4 \times \mathbb{R}^3 \rightarrow \mathbb{R}^5 \quad (u_1, v_1, u_2, v_2, X) \rightarrow H(u_1, v_1, u_2, v_2, X) = (F, F_{u_1}, F_{v_1}, F_{u_2}, F_{v_2})
\]

Then the set EPM1 is the projection in \( \mathbb{R}^3 \) of the set \( H = 0 \). If \( 0 \in \mathbb{R}^5 \) is a regular value of \( H \), then \( H^{-1}(0) \) is a 2-dimensional submanifold of \( \mathbb{R}^7 \). We want to find conditions under which \( \pi_2(H^{-1}(0)) \) becomes smooth, where \( \pi_2(u_1, v_1, u_2, v_2, X) = X \).

The jacobian matrix of \( H \) is

\[
JH = 
\begin{pmatrix}
F_{u_1} & F_{v_1} & F_{u_2} & F_{v_2} & F_x & F_y & F_z \\
F_{u_1u_1} & F_{u_1v_1} & F_{u_1u_2} & F_{u_1v_2} & F_{u_1x} & F_{u_1y} & F_{u_1z} \\
F_{v_1u_1} & F_{v_1v_1} & F_{v_1u_2} & F_{v_1v_2} & F_{v_1x} & F_{v_1y} & F_{v_1z} \\
F_{u_2u_1} & F_{u_2v_1} & F_{u_2u_2} & F_{u_2v_2} & F_{u_2x} & F_{u_2y} & F_{u_2z} \\
F_{v_2u_1} & F_{v_2v_1} & F_{v_2u_2} & F_{v_2v_2} & F_{v_2x} & F_{v_2y} & F_{v_2z}
\end{pmatrix}
\]

Denote by \( JH_1 \) the matrix of second derivatives of \( F \) with respect to the parameters \( u_1, v_1, u_2, v_2 \), which is the \( 4 \times 4 \) matrix consisting of the elements \( JH(i, j) \), \( 2 \leq i \leq 5 \), \( 1 \leq j \leq 4 \). Denote \( \det(JH_1(u_1, v_1, u_2, v_2, X)) \) by \( \Delta(u_1, v_1, u_2, v_2) \).
Assuming that $S_1$ and $S_2$ are graphs of functions $f_1$ and $f_2$ given by equations (10) and (11), straightforward calculations show that $(F_x, F_y, F_z) = (0, 0, 2)$ and the Jacobian matrix $JH$, in point $(0, 0, 0, 0, 0, 0, 0)$, is given by

\[
\begin{pmatrix}
3p^2 + 3p^4 - 6pf_{3,0} & -2pf_{2,1} & 0 & 0 \\
-2pf_{2,1} & -2f_0p^2 - 2f_{2,1}p & 0 & (f_0,2 + 2go,2)(p^2 + 1) \\
0 & 0 & 0 & 2pg_{2,1} \\
0 & (f_0,2 + 2go,2)(p^2 + 1) & 2pg_{2,1} & 2go,2p^2 - 2pg_{1,2}
\end{pmatrix}.
\]

**Theorem 3** If $\Delta \neq 0$, then the EMP is smooth at the point $X$.

**Proof.** We may assume that $S_1$ and $S_2$ are graphs of functions $f_1$ and $f_2$ given by equations (10) and (11). Since $F_{u_1} = F_{v_1} = F_{u_2} = F_{v_2} = 0$, the determinant of the $5 \times 5$ matrix obtained from $JH$ at $(0, 0, 0, 0, 0, 0)$ by excluding the columns 5 and 6 is $2\Delta$. Thus the hypothesis implies that $JH$ has rank 5 and so $H^{-1}(0)$ is a regular surface in $\mathbb{R}^7$. Moreover, the hypothesis $\Delta \neq 0$ implies that the differential of $\pi$ restricted to $H^{-1}(0)$ is an isomorphism. We conclude that the EMP is smooth at this point.

3.5 An example

In [11], it is proved that if the AESS of a pair of planar curves is contained in a line, then there exists an affine reflection taking one curve into the other. This fact is not true for the EMP of a pair of surfaces as the following example shows us.

Consider $\gamma_1(t) = (t, 0, f(t))$ a smooth convex curve and let $\gamma_2(t) = (t - \lambda f(t), 0, -f(t))$, $\lambda \in \mathbb{R}$, be obtained from $\gamma_1$ by an affine reflection. Let $S_1$ and $S_2$ be rotational surfaces obtained by rotating $\gamma_1$ and $\gamma_2$ around the $z$-axis. $S_1$ and $S_2$ can be parameterized by

\[
\phi_1(t, \theta) = (t \cos(\theta), t \sin(\theta), f(t))
\]

and

\[
\phi_2(t, \theta) = ((t - \lambda f(t)) \cos(\theta), (t - \lambda f(t)) \sin(\theta), -f(t)).
\]

The intersection of the tangent planes at $\phi_1(t, \theta)$ and $\phi_2(t, \theta)$ has direction $Z = (\sin(\theta), -\cos(\theta), 0)$.

Observe that the vectors $Y_1 = (\phi_1)_t$ and $Y_2 = (\phi_2)_t$ are orthogonal to $Z$ in the Blaschke metric. This implies that the EMP of this pair of surfaces is contained in the plane $z = 0$. But it is clear that $S_2$ is not an affine reflection of $S_1$.

4 Parallel Tangent Planes

In this section, we shall consider the set EMP2 of points of the EMP obtained from pairs $(p_1, p_2)$, $p_1 \neq p_2$, with parallel tangent planes. In this case, the mid-plane of $(p_1, p_2)$ is the plane through $M$ parallel to the tangent planes at $p_1$ and $p_2$. The Mid-Parallel Tangents Surface (MPTS) is the set of mid-points $M$ of all such pairs $(p_1, p_2) \in S_1 \times S_2$. The following proposition is proved in [10].

**Proposition 4** The tangent plane at any point of the MPTS is parallel to the tangent planes to $T_{p_1}S_1$ and $T_{p_2}S_2$.

It follows from this proposition that the MPTS is the envelope of the mid-planes with parallel tangents. In other words, the set EMP2 coincides with the MPTS.

One relevant property is that the MPTL and the AESS are both singular at the center of a conic with contact of order $\geq 3$ at $p_1$ and $p_2$, and, under certain conditions, they are ordinary cusps (14). Similarly, we expect singularities at points of the EMP2 that are limit of points of the EMP1. In what follows we shall give some conditions under which, at such points, the MPTS is equivalent to a cuspidal edge. Assume that $S_1$ and $S_2$ are graphs of function $f$ and $g$ of the form

\[
f(u_1, v_1) = 1 + \sum_{i=0}^{i=2} f_{2-i,i}u_1^{2-i}v_1^i + \sum_{i=0}^{i=3} f_{3-i,i}u_1^{3-i}v_1^i + O(4)
\]
and
\[ g(u_2, v_2) = -1 + 2 \sum_{i=0}^{2} g_{2-i, i} u_2^{2-i} v_2^i + \sum_{i=0}^{3} g_{3-i, i} u_2^{3-i} v_2^i + O(4). \] (16)

The following two propositions are proved in \cite{10}.

**Proposition 5** The MPTS is smooth at \((0, 0, 0)\) if \( \det(\text{Hess}(f) + \text{Hess}(g)) \neq 0 \).

By a rotation, we may assume that \( f_{1,1} = 0 \) and we shall do it for the next proposition.

**Proposition 6** Assume that \( f_{1,1} = 0 \), \( f_{0,2} + g_{0,2} \neq 0 \), \( \zeta = 0 \) and \( \eta \neq 0 \), where
\[ \zeta = g_{1,1}^2 - 4(f_{0,2} + g_{0,2})(f_{0,2} - g_{0,2}) \]
and
\[ \eta = f_{0,3} - g_{0,3} - \left( \frac{f_{1,2} - g_{1,2}}{2(f_{0,2} + g_{0,2})} \right) g_{1,1} + \left( \frac{f_{2,1} - g_{2,1}}{4(f_{0,2} + g_{0,2})^2} \right) g_{1,1}^2 - \left( \frac{f_{3,0} - g_{3,0}}{8(f_{0,2} + g_{0,2})^3} \right) g_{1,1}^3. \]

Then the MPTS is locally equivalent to a cuspidal edge at the point \((0,0)\).

Next corollary is the main result of the section:

**Corollary 2** Assume that \( S_1 \) and \( S_2 \) are given by equations \((15)\) and \((16)\) and that \((0,0,0)\) is a limit point of the set EMP1. If \( f_{0,3} - g_{0,3} \neq 0 \) and \( f_{0,2} + g_{0,2} \neq 0 \), then the MPTS is locally equivalent to a cuspidal edge at \((0,0,0)\).

**Proof.** By a rotation, assume that \( f_{1,1} = 0 \). Since \((0,0,0)\) is a limit point of the EMP1, there are directions \( Y_1 \in T_{p_0} S_1, Y_2 \in T_{p_0} S_2 \), orthogonal to a direction \( Z \) in the corresponding Blaschke metric, such that \( \{Y_1, Y_2, C\} \) are coplanar. We may assume that \( Z \) is parallel to \((0,1,0)\). Since \( f_{1,1} = 0 \), it follows from lemma \([4]\) that \( Y_1 \) is parallel to \((1,0,0)\). Now the co-planarity of \( Y_1, Y_2 \) and \( C \) implies that \( Y_2 \) is also parallel to \((1,0,0)\). Using again lemma \([4]\) we conclude that \( g_{1,1} = 0 \).

Since \((0,0,0)\) is the center of a \(3 + 3\) conic, it follows that \( f_{2,0} + g_{2,0} = 0 \). By proposition \([6]\) the MPTS is locally equivalent to a cuspidal edge at this point. \(\square\)

5 Envelope of Mid-Planes at coincident points

In this section we shall describe the limit of the EMP when we make \( p_1 \) tend to \( p_2 \). In the case of planar curves, this set is the usual affine evolute. For surfaces, we obtain an interesting new set that, under certain conditions, is locally a regular surface with at most 6 branches.

5.1 Transon planes, cone of B.Su, Moutard’s quadric and the medial curve

We begin with a two-hundred years old result of A.Transon \([9]\). Consider a regular surface \( S \subset \mathbb{R}^3 \), \( p_0 \in S \) and \( T \in T_{p_0} S \). Then the affine normal lines of the planar curves obtained as the intersection of \( S \) with planes containing \( T \) form a plane, which is called the Transon plane of the tangent \( T \) at \( p_0 \).

Assume that \( p_0 = (0,0,0) \) and that the tangent plane at \( p_0 \) is \( z = 0 \). Then, \( S \) is the graph of a function \( f \) that can be written as
\[ f(x, y) = \frac{1}{2}(x^2 + y^2) + \sum_{i=0}^{3} f_{3-i, i} x^{3-i} y^i + \sum_{i=0}^{4} f_{4-i, i} x^{4-i} y^i + O(5). \] (17)

In this case the Transon plane at \( p_0 \) of a direction \( T = (\xi, \eta) \) is given explicitly by
\[ A(\xi, \eta)x + B(\xi, \eta)y + C(\xi, \eta)z = 0, \] (18)

where
\[ A = \frac{\xi}{2}(\xi^2 + \eta^2), \quad B = \frac{\eta}{2}(\xi^2 + \eta^2), \quad C = \sum_{i=0}^{3} f_{3-i, i} \xi^{3-i} \eta^i. \]
Envelope of Mid-Planes of Surfaces in $\mathbb{R}^3$

The Transon plane of the tangent $T$ is formed by the affine normal lines of the planar sections that contain $T$. (see [5]). Observe that, as expected, the Transon plane depends only on the direction of $T$.

Consider now the family of all Transon planes obtained as the direction of $T$ varies. The envelope of this family is called cone of B.Su. Thus two infinitesimally close Transon planes meet at a line of the cone of B.Su. We shall denote by $N$ a vector generating this line. When $S$ is the graph of a function $f$ given by equation (17), the cone of B.Su at $p_0 = (0,0,0)$ can be parameterized by $\mu (x(\xi,\eta), y(\xi,\eta), z(\xi,\eta))$, where

$$
\begin{align*}
x(\xi,\eta) &= -2(-2f_{0,3}\eta^3\xi + 2f_{2,1}\eta^3\xi + 3f_{3,0}\eta^2\xi^2 - f_{1,2}\eta^2\xi^2 + f_{3,0}\xi^4 + f_{1,2}\eta^4), \\
y(\xi,\eta) &= -2(f_{0,3}\eta^3 + 3\eta^2 f_{0,3}\eta^2 - f_{2,1}\eta^2\xi^2 - 2\eta f_{3,0}\xi^3 + 2\xi^2 f_{1,2}\eta + \xi^4 f_{2,1}), \\
z(\xi,\eta) &= 2\xi^2\eta^2 + \xi^4 + \eta^4,
\end{align*}
$$

(see [5]). For $(\xi,\eta) = (1,0)$ we obtain

$$(x, y, z)(1,0) = (-2f_{3,0}, -2f_{2,1}, 1).$$

(20)

We shall consider also the osculating conics of all planar sections obtained from planes containing $T$. It is an old result of T.Moutard ([6]) that the union of these osculating conics form a quadric, which is called the Moutard’s quadric of the tangent $T$. In case $S$ is the graph of a function $f$ given by (17), the Moutard’s quadric of $T = (1,0)$ at the origin is given by

$$
0 = \frac{z}{8} - (x y z) \begin{pmatrix} 1 & 0 & f_{30} \\ 0 & 1 & f_{21} \\ f_{30} & f_{21} & f_{40} - f_{30} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}
$$

(21)

(see [5]). The center of this quadric is thus

$$
X = \frac{1}{2(5f_{3,0}^2 - 2f_{4,0} + f_{2,1}^2)} \begin{pmatrix} f_{3,0}, f_{2,1}, -\frac{1}{2} \end{pmatrix},
$$

(22)

We shall call medial curve of the tangent $T$ the intersection of $S$ with the plane generated by $T$ and $N$.

**Proposition 7** The center of the Moutard’s quadric of a tangent $T$ coincides with the center of affine curvature of the medial curve of $T$. 

The affine center of curvature at the origin is given by (see formula (20)). The plane generated by $T = (1, 0, 0)$ has equation $y + 2f_{2,1}z = 0$. Thus the medial curve is defined by this equation and $z = f(x, y)$. The projection of the medial curve in the $xz$-plane is then

$$
z = \frac{1}{2}x^2 - 2f_{2,1}^2z^2 - f_{3,0}x^3 + 2f_{2,1}^2x^2z - 4f_{1,2}f_{2,1}^2xz^2 + 8f_{0,3}f_{2,1}^3z^3 - 4f_{4,0}x^4 + 2f_{3,1}f_{2,1}^3x^2z - 4f_{2,2}f_{2,1}^2x^2z^2 + 8f_{1,3}f_{2,1}^3xz^3 - 16f_{0,4}f_{2,1}^3z^4 = 0.
$$

In a neighborhood of the origin we can write

$$
z = \frac{1}{2}x^2 + f_{3,0}x^3 + \left(f_{4,0} - \frac{f_{2,1}^2}{2}\right)x^4 + \cdots
$$

Since the affine curvature of $\gamma$ is the same as the affine curvature of the projection of $\gamma$ in the $xz$-plane we obtain

$$
\mu_\gamma(0) = -4(5f_{3,0}^2 - 2f_{4,0} + f_{2,1}^2).
$$

The affine center of curvature at the origin is given by

$$
(0, 0, 0) + \frac{1}{\mu_\gamma(0)}N(1, 0) = \frac{1}{2(5f_{3,0}^2 - 2f_{4,0} + f_{2,1}^2)}\left(f_{3,0}f_{2,1}^2 - \frac{1}{2}\right),
$$

which coincides with formula (22) that gives the center of Moutard’s quadric of $T = (1, 0)$ at the origin.

5.2 The limit behavior of the EMP

Consider points $p_1, p_2 \in S$. The mid-plane for this pair of points is given by $F(u_1, v_1, u_2, v_2, X) = 0$, where

$$
F(u_1, v_1, u_2, v_2, X) = N_1(C)N_2(X - M) + N_2(C)N_1(X - M),
$$

$X = (x, y, z) \in \mathbb{R}^3$, $C$ is the chord connecting $p_1$ and $p_2$, $M$ is the mid-point of $p_1$ and $p_2$ and $N_i(x, y, z)$, $i = 1, 2$, are co-normals of $S$ at $p_1$ and $p_2$.

Assume that $S$ is the graph of a function $f$ defined by equation (17) and consider the co-normal $N_i$ given by the cross product of the tangent vectors $(1, 0, f_u)(p_i)$ and $(1, 0, f_u)(p_i)$. Then long but straightforward calculations show that the equation of the mid-plane of the pair $(p_1, p_2)$ is given by

$$
\frac{1}{4}\left\{(u_1 - u_2)\left[(u_1 - u_2)^2 + (v_1 - v_2)^2\right] + O(4)\right\}x + \frac{1}{4}\left\{(v_1 - v_2)\left[(u_1 - u_2)^2 + (v_1 - v_2)^2\right] + O(4)\right\}y + \frac{1}{4}\left\{[(u_1 - u_2)^2 + (v_1 - v_2)^2] + O(4)\right\}z = 0.
$$
Proof. Fixing \( p_0 = (0,0) \), the equation of the mid-plane of \((p_0, p_1)\) is

\[
\frac{1}{4} \left( u_1^2 + O(4) \right) x + \frac{1}{4} \left( \frac{v_1}{u_1} + \frac{v_1^3}{u_1^3} + O(4) \right) y + \frac{1}{2} \left( f_{3,0} u_1^3 + f_{2,1} u_1^2 v_1 + f_{1,2} u_1 v_1^2 + f_{0,3} v_1^3 + O(4) \right) z = O(4).
\]

Assume \( \xi \neq 0 \). Since \((u_1, v_1) \to (0,0)\) along the direction \((\xi, \eta)\), we may assume also that \( u_1 \neq 0 \). Thus the above equation becomes

\[
\frac{1}{4} \left( 1 + \frac{v_1^3}{u_1^3} + O(4) \right) x + \frac{1}{4} \left( \frac{v_1}{u_1} + \frac{v_1^3}{u_1^3} + O(4) \right) y + \frac{1}{2} \left( f_{3,0} + f_{2,1} \frac{v_1}{u_1} + f_{1,2} \frac{v_1^2}{u_1^2} + f_{0,3} \frac{v_1^3}{u_1^3} + O(4) \right) z = O(4).
\]

Taking the limit \((u_1, v_1) \to (0,0)\) we have \( \frac{O(4)}{u_1^3} \to 0 \) since \( \frac{v_1}{u_1} = \frac{\eta}{\xi} \). Thus we obtain

\[
\frac{1}{4} \left( 1 + \lambda^2 \right) x + \frac{1}{4} \lambda \left( 1 + \lambda^2 \right) y + \frac{1}{2} \left( f_{3,0} + f_{2,1} \lambda + f_{1,2} \lambda^2 + f_{0,3} \lambda^2 \right) z = 0,
\]

which is the Transon plane of the tangent \((\xi, \eta)\) at the origin. \(\square\)

We have to consider the system

\[
\begin{align*}
F(u_1, v_1, u_2, v_2, X) &= 0 \\
F_{u_1}(u_1, v_1, u_2, v_2, X) &= 0 \\
F_{v_1}(u_1, v_1, u_2, v_2, X) &= 0 \\
F_{u_2}(u_1, v_1, u_2, v_2, X) &= 0 \\
F_{v_2}(u_1, v_1, u_2, v_2, X) &= 0
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
F(u_1, v_1, u_2, v_2, X) &= 0 \\
F_{v_1}(u_1, v_1, u_2, v_2, X) - F_{v_2}(u_1, v_1, u_2, v_2, X) &= 0 \\
F_{u_1}(u_1, v_1, u_2, v_2, X) - F_{u_2}(u_1, v_1, u_2, v_2, X) &= 0 \\
F_{u_2}(u_1, v_1, u_2, v_2, X) + F_{v_2}(u_1, v_1, u_2, v_2, X) &= 0 \\
F_{v_1}(u_1, v_1, u_2, v_2, X) + F_{v_2}(u_1, v_1, u_2, v_2, X) &= 0
\end{align*}
\]

(23)

Define the \(5 \times 4\) matrix \( L(\xi, \eta) \) by

\[
\begin{pmatrix}
\frac{\xi(\xi^2 + \eta^2)}{2} & \frac{\eta(\xi^2 + \eta^2)}{2} & f_{3,0}\xi^2 + f_{2,1}\xi^2\eta + f_{1,2}\xi\eta^2 + f_{0,3}\eta^2 & 0 \\
\xi & \eta & \xi^2 + 3\eta^2 & f_{2,1}\xi^2 + 2f_{1,2}\xi\eta + 3f_{0,3}\eta^2 \\
\frac{3\xi^2 + \eta^2}{2} & \frac{\xi^2 + 3\eta^2}{2} & \xi & 3f_{3,0}\xi^2 + 2f_{2,1}\xi\eta + f_{1,2}\eta^2 \\
L_{41}(\xi, \eta) & L_{42}(\xi, \eta) & L_{43}(\xi, \eta) & L_{44}(\xi, \eta) \\
L_{51}(\xi, \eta) & L_{52}(\xi, \eta) & L_{53}(\xi, \eta) & L_{54}(\xi, \eta)
\end{pmatrix}
\]
where

\[
L_{41} = \frac{f_{3,0}}{2}(5\xi^2 + 3\eta^2)\xi + \frac{f_{2,1}}{2}(2\xi^2 + \eta^2)\eta - \frac{f_{0,3}}{2}\eta^3,
\]
\[
L_{42} = 3\frac{f_{3,0}}{2}\eta\xi^2 + \frac{f_{2,1}}{2}(\xi^2 + 3\eta^2)\xi + \frac{f_{1,2}}{2}(\xi^2 + 2\eta^2)\eta,
\]
\[
L_{43} = 2f_{4,0}\xi^2 + 3\frac{f_{3,1}}{2}\eta\xi^2 + f_{2,2}\eta^2\xi + \frac{f_{1,3}}{2}\eta^3,
\]
\[
L_{44} = \frac{1}{4}(\xi^2 + \eta^2)\xi,
\]
\[
L_{51} = \frac{f_{3,0}}{2}(5\xi^2 + 3\eta^2)\xi + \frac{f_{2,1}}{2}(2\xi^2 + \eta^2)\eta - \frac{f_{0,3}}{2}\eta^3,
\]
\[
L_{52} = 3\frac{f_{3,0}}{2}\eta\xi^2 + \frac{f_{2,1}}{2}(\xi^2 + 3\eta^2)\xi + \frac{f_{1,2}}{2}(\xi^2 + 2\eta^2)\eta,
\]
\[
L_{53} = 2f_{4,0}\xi^2 + 3\frac{f_{3,1}}{2}\eta\xi^2 + f_{2,2}\eta^2\xi + \frac{f_{1,3}}{2}\eta^3,
\]
\[
L_{54} = \frac{1}{4}(\xi^2 + \eta^2)\xi.
\]

We shall denote by \(L_1, L_2, L_3, L_4\) and \(L_5\) the lines of \((\xi, \eta)\).

**Proposition 8** Fix \(u_2 = v_2 = 0\) and make \((u_1, v_1) \to (0, 0)\) along the direction \((\xi, \eta)\). Then the limit of the matrix of (23) is \(L(\xi, \eta)\).

**Proof.** The proof is similar to the proof of lemma 7 for each line of the matrix. Our calculations were done with the software Maple. \(\square\)

Proposition 8 says that for each \(p \in S\) and \(T \in T_pS\), the system (23) converges to a limit system \(L(p, T)\) when we fix \(p_2 = p\) and make \(p_1 \to p\) along the tangent \(T\). From now on we shall be interested in the properties of the system \(L(p, T)\). Assuming that \(S\) is the graph of \(f\) given by (17), \(p = (0, 0)\) and \(T = (\xi, \eta)\), the matrix of \(L(p, T)\) is exactly \(L(\xi, \eta)\).

**Remark 2** Observe that \(L_2 = L_1\), \(L_3 = L_1\xi\). Since \(L_1\) is homogeneous of degree 3, Euler’s lemma implies that \(3L_1 = \eta L_1\eta + \xi L_1\xi\). Hence the lines \(L_1, L_2\) and \(L_3\) are linearly dependent. If \((\xi, \eta) = (1, 0)\), then \(3L_1 = L_1\xi = L_3\) and thus it is natural to discard \(L_1\) or \(L_3\).

**Lemma 8** \(L_1, L_2\) and \(L_3\) define the direction \(N\) of the cone of B.Su associated with the tangent \(T = (\xi, \eta)\) at the origin.

**Proof.** Assuming \((\xi, \eta) = (1, 0)\), the system defined by \(L_1\) and \(L_2\) is given by

\[
\begin{align*}
\frac{1}{4}x + \frac{1}{2}f_{3,0}z &= 0 \\
\frac{1}{2}y + f_{2,1}z &= 0.
\end{align*}
\]

Observe that the director vector \(N = (-2f_{0,3}, -2f_{2,1}, 1)\) of the solution line is exactly the direction of the cone of B. Su at the origin in the direction \((1, 0)\) (see formula (20)). \(\square\)

**Lemma 9** \(L_1, L_2, L_3\) and \(L_4\) define the center of the Moutard’s quadric of \(T = (\xi, \eta)\).

**Proof.** We may assume that the tangent \((\xi, \eta)\) is \((1, 0)\). Taking \(\xi = 1, \eta = 0\) in the system \(L_1, L_2, L_4\) we get

\[
\begin{align*}
\frac{1}{4}x + \frac{1}{2}f_{3,0}z &= 0 \\
\frac{1}{2}y + f_{2,1}z &= 0, \\
\frac{5}{2}f_{3,0}x + \frac{f_{2,1}}{2}y + 2f_{4,0}z &= -\frac{1}{4}.
\end{align*}
\]
The solution of this system is

\[(x, y, z) = \frac{1}{2(5f_{2,0}^2 + f_{2,1}^2 - 2f_{4,0})} \left( f_{3,0}, f_{2,1}, -\frac{1}{2} \right),\]

which is exactly the center of the Moutard’s quadric of \( T = (1, 0, 0) \) at the origin (see formula 22).

Denote by \( D : S \times S^1 \rightarrow \mathbb{R} \) the determinant of \((L_1, L_2, L_4, L_5)\).

**Lemma 10** The system \( L(\xi, \eta) \) admit solutions for at most 6 values of the direction \((\xi, \eta)\).

**Proof.** The system \( L_1 = L_2 = L_4 = L_5 = 0 \) admit a non-trivial solution if and only if \( D = 0 \). Calculations with the software Maple show that

\[D((0, 0), (\xi, \eta)) = (\xi^2 + \eta^2)^2 \cdot q(\xi, \eta),\]

where

\[
q(\xi, \eta) = (f_{1,3} - 2f_{2,1}f_{1,2} - 2f_{1,2}f_{0,3})\eta^6 + (2f_{2,1}f_{1,2} - f_{3,1} + 2f_{3,0}f_{2,1})\xi^6 + \\
+(-4f_{2,1}^2 - 6f_{1,2}f_{3,0} + 2f_{0,3}f_{2,1} - 4f_{0,4} + 2f_{2,2} + 6f_{0,5} + 2f_{1,2}^2)\eta^5 + \\
+(-9f_{3,0}f_{2,1} + 5f_{2,1}f_{1,2} - f_{3,1} + \frac{3}{2}f_{3,1} + 6f_{0,3}f_{3,0} - 4f_{1,2}f_{0,3})2\xi^2\eta^4 + \\
+(2f_{3,1} + 18f_{1,2}f_{0,3} - 12f_{3,0}f_{3,0} - 3f_{3,1} + 8f_{0,3}f_{2,1} - 10f_{2,1}f_{1,2})\eta^3\xi^4 + \\
+(-9f_{2,1}^2 - 8f_{1,2}f_{3,1} - 3f_{2,1}^2 + 2f_{4,0} + 9f_{1,2}f_{3,0} + 8f_{1,2}f_{3,0} - 3f_{2,1}^2)2\xi^3\eta^3 - \\
-4f_{0,4}f_{3,0}^3 + (-2f_{2,2} - 6f_{2,0}^2 + 4f_{4,0} - 2f_{2,1} - 2f_{1,2}f_{3,0})\xi^5\eta^5 + \\
+(6f_{0,3}f_{2,1} + 4f_{1,2}^2)\xi^5\eta
\]

is a homogeneous polynomial of degree 6 in \((\xi, \eta)\). Thus \( D = 0 \) admit at most 6 solutions.

**Corollary 3** For each \( p \in S \), the system \( L(p, T) \) admits a solution for at most 6 values of the direction \( T \).

For each \( p \in S \) and \( T_i(p), 1 \leq i \leq 6 \), given by the above corollary, define \( X_i(p) \) as the solution of the system \( L(p, T_i(p)) \) and write \( E(p) = \bigcup_{i=1}^{6} X_i(p) \). We shall call the set \( E(p), p \in S \), the Medial Curves Evolute of \( S \). It is clear from this definition that the Medial Curves Evolute has at most six branches.

5.3 Regularity of the Medial Curves Evolute

In this section we study the regularity of the branches of the Medial Curves Evolute \( E(p) \). We begin by showing that, under certain conditions, the vector fields \( T_i(p), 1 \leq i \leq 6 \), are smooth.

**Proposition 9** Assume that \( D(p, T) = 0 \). If \( T \) is not critical for the map \( T \rightarrow (p, T) \) then locally \( T \) is a differentiable function of \( p \).

**Proof.** Use the implicit function theorem.

When \( S \) is the graph of \( f \) given by (17), \( p = (0, 0) \) and \( T = (1, 0) \), we can explicitly obtain a condition that guarantees that \( T \) is not critical for the map \( T \rightarrow D(p, \xi, \eta) \). First observe that since \( q(\xi, \eta) \) is homogeneous of degree 6, we have \( 6q = \xi q_\xi + \eta q_\eta \). But \( \eta = 0 \) and \( q = 0 \), which implies \( q(1, 0) = 0 \). Thus we must guarantee the condition \( D_\eta(1, 0) \neq 0 \). Straightforward calculations show that

\[q(1, 0) = 2f_{3,0}f_{2,1} - f_{3,1} + 2f_{3,0}f_{2,1},\]

and

\[D_\eta(1, 0) = 4f_{4,0} - 2f_{3,0}f_{1,2} - 6f_{3,0}f_{2,1} - 2f_{2,1}f_{0,3} - 2f_{2,1}^2 + 4f_{1,2}^2 - 2f_{2,2}.\]

Thus, if \( q(1, 0) = 0 \) but \( D_\eta(1, 0) \neq 0 \), we can write \( T = T(p) \) for a differentiable function \( T \) defined in a neighborhood of \( p = (0, 0) \).

From now on, we shall assume that, for a given branch of the Medial Curves Evolute, \( T = T(p) \) is a smooth function of \( p \). Next step is to give conditions under which the map \( X = X(p) \) is smooth. Define \( H : S \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) by \( H(p, X) = (L_2, L_3, L_4)(p, X) \), where \( L_k(p, X) \) denotes line \( k \) of the system \( L(p, T(p)) \).
**Proposition 10** Assume that $X$ is not critical for the map $X \to H(p, X)$. Then locally we can write $X$ as a differentiable function of $p$.

**Proof.** Use the implicit function theorem. □

When $S$ is the graph of $f$ given by (19), $p = (0, 0)$ and $T(p) = (1, 0)$, we can explicitly obtain a condition that guarantees that $X$ is not critical for the map $X \to H(p, X)$. Observe that $H((0, 0), X_0) = (0, 0, 0)$, where $X_0$ is a solution of $L2, L3, L4$ at the origin. The Jacobian matrix of the map $X \to H((0,0), X)$ at $X_0$ is given by

$$J = \begin{pmatrix} 0 & 1 & 0.5 & f_{2,1} \\ 3 & 2 & 0 & 3f_{3,0} \\ 15f_{3,0} & f_{2,1} & 2 & 2f_{4,0} \end{pmatrix}$$

and its determinant is $\frac{1}{4}(3f_{2,1}^2 + 15f_{3,0}^2 - 6f_{4,0})$. Thus if we assume that $3f_{2,1}^2 + 15f_{3,0}^2 - 6f_{4,0} \neq 0$, we guarantee that $X$ is a differentiable function of $p$.

From now on, we shall assume that for a given branch of the Medial Curves Evolute, $T$ and $X$ are differentiable functions of $p$. Our last step is to describe conditions under which this branch of the Medial Curves Evolute becomes a regular surface.

**Lemma 11** For any $W \in T_pS$, $X_W$ belongs to the Transon plane of $T$. For $W = T$, $X_W$ belongs to the cone of $B.Su$.

**Proof.** Assume that $S$ is given by (17), $p = (0, 0)$ and $T(p) = (1, 0)$. Differentiating $L1((\xi, \eta)(p))(X(p)) = 0$ in the direction $W$ we obtain

$$L1_\eta(X)\eta_W + L1(X_W) = 0.$$ But $L1_\eta(X) = L2(X) = 0$ and so $L1(X_W) = 0$, which implies that $X_W$ belongs to the Transon plane of $T$, for any $W$. For $W = T$, since $X$ is a point of the affine evolute of the medial curve and $T$ is tangent to this curve, $X_T$ is in the direction of the its affine normal, which belongs to the cone of $B.Su$. □

**Lemma 12** If $p$ is not critical for the map $p \to D(p,T)$, with $T$ fixed, and $L2_T(X) \neq 0$, then, for some $W \in T_pS$, $X_W$ does not belong to the cone of $B.Su$.

**Proof.** Assume that $S$ is given by (17), $p = (0, 0)$ and $T(p) = (1, 0)$. Differentiating $L2(X) = 0$ in the direction $W$ we obtain

$$L2_\eta(X)\eta_W + L2(X_W) = 0.$$  

Since by hypothesis $L2_\eta(X) \neq 0$ we must show that $\eta_W \neq 0$ for some $W \in T_pS$. Since $p$ is not critical for the map $p \to D(p,T)$, it follows that $D_W(p, T) \neq 0$, for some $W$, where $D_W(p, T)$ denotes the partial derivative of $D$ with respect to $p$ in the direction $W$. Taking the total derivative of $D(p, T(p)) = 0$ in the direction $W$, we get

$$D_W + D_TT_W = 0.$$ This implies that $T_W \neq 0$, or equivalently, $\eta_W \neq 0$. □

**Theorem 4** Assume that $p$ is not critical for the map $p \to D(p,T)$, with $T$ fixed, and $L2_T(X) \neq 0$. If $\mu'(0) \neq 0$, where $\mu'$ denotes the affine curvature of the medial curve $\gamma$, the branch of the Medial Curves Evolute is a regular surface and its tangent plane is the Transon plane.

**Proof.** The hypothesis $\mu'(0) \neq 0$ implies that $X_T \neq 0$. By lemmas 11 and 12 $X_T$ and $X_W$ are linearly independent vector in the Transon plane. We conclude that the branch of the Medial Curves Evolute is a regular surface in a neighborhood of $p$ and its tangent plane is the Transon plane. □
Remark 3 Assume that $S$ is given by (17), $p = (0,0)$ and $T(p) = (1,0)$. At the origin, 
\[ L_2 p(X) = x + 2f_{1,2}z = \frac{f_{3,0} - f_{1,2}}{4(5f^2_{3,0} + f^2_{2,1} - 2f_{4,0})}, \]
where $X = (x, y, z)$ is the center of Moutard’s quadric of $T$ whose coordinates are given by (22). Thus the condition $L_2 T(X) \neq 0$ can be written as $f_{3,0} - f_{1,2} \neq 0$. Also, since $\mu'(0) = 8f^2_{3,0} - 3(2f_{4,0} - f^2_{2,1})$, the condition $\mu' \neq 0$ can be written as $8f^2_{3,0} - 3(2f_{4,0} - f^2_{2,1}) \neq 0$.

The condition $p$ not critical for $p \to D(p, T)$ can be described as follows: Since $D(p, T) = 0$, we have necessarily $f_{3,1} = f_{2,1}(f_{1,2} + f_{3,0})$. The condition $p$ not critical for $p \to D(p, T)$ is equivalent to $D_v((0,0),(1,0)) \neq 0$ which can be written as
\[-2f_{3,2} - f^2_{2,1}f_{1,2} - 3f_{1,2}f_{3,0} - 3f_{1,2}f^2_{3,0} + 3f_{1,3}f_{2,1} + 2f_{1,2}f_{2,2} + 4f_{1,2}f_{4,0} + 2f_{2,2}f_{3,0} - 3f_{0,3}f_{2,1}f_{1,2} \neq 0.\]

![Fig. 4](image)

Fig. 4 The surface, its Transon plane and the branch of the EPM3 of example. Observe that the Transon plane is tangent plane to the EMP3.

Example 1 Consider the surface $S$ given by the graph of
\[ f(x, y) = x^2 + 4xy + 2y^2 + x^3. \]

The Transon plane of $(\xi, \eta)$ is given by
\[(2\xi + \eta)(\xi^2 + 4\xi\eta + 2\eta^2)x + 2(\xi + \eta)(\xi^2 + 4\xi\eta + 2\eta^2)y + \frac{\xi^3}{2} = 0.\]

and the determinant of the matrix $L(\xi, \eta)$ is
\[ D(0, \xi, \eta) = -12\xi^4(\xi + \eta)(5\xi^2 + 12\xi\eta + 6\eta^2)(\xi^2 + 4\xi\eta + 2\eta^2)^2. \]

Thus $D \left(0, \frac{-6 + \sqrt{6}}{5}, 1\right) = 0$. Moreover, $(D_u, D_v)(0,0) = (-0.1512, 0)$ and thus $(0,0)$ is not critical for $(u,v) \to D(u,v)$. It is straightforward to check that the conditions of theorem are satisfied and so this branch of the Medial Curves Evolute is a regular surface at $(0,0)$. In figure we see a sketch of this branch and its tangent plane.
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