A method to solve the heat transfer problem for laminar flow of Phan-Tien-Tanner fluid in a flat channel

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Abstract. The paper presents a solution to the heat transfer problem for a flow of a four-mode viscoelastic linear Phan-Tien Tanner fluid in flat channels at a constant wall temperature. A special feature of this solution is the parametric representation of dependence of an axial velocity and temperature on coordinates. To obtain a solution to the energy transfer equation, we used the method of separating variables and decomposing the sought-for function into series by parameter. For a specific liquid, the profiles of dimensionless velocity and distribution of dimensionless temperature, Nusselt numbers, and other heat transfer characteristics for various Weissenberg numbers are presented.

1. Introduction
For the description of nonlinear and viscoelastic properties of materials, lately, priority has been given to rheological models based on the structure of molecules of polymer materials. The equations corresponding to these models include time derivatives of stress tensors and are called relaxation equations of state. These include the Phan-Tien Tanner model [1], which has proven itself in practice. The use of single-mode equations of state [2] also reduces the accuracy of an adequate description of rheological behavior of materials in a wide range of changes in regime parameters. The papers [3-4] propose analytical formulas for calculating hydrodynamic characteristics obtained with additional assumptions limiting the type of the rheological equation of state. The use of numerical calculation methods [5] in the presence of nonlinear terms in the equations requires additional study of issues related to approximation, stability and accuracy of the applied computational schemes, and is even more difficult than obtaining the solutions themselves. Therefore, obtaining the main characteristics of heat transfer based on semi-analytical methods is going to reduce the calculation time and improve the accuracy of the results.

In this paper, we obtain an analytical solution to the heat transfer problem for a flow of a viscoelastic liquid in a plane slit using a complete linear four-mode rheological Phan-Tien–Tanner equation of state with a Newtonian stress component. Also, the necessary numerical coefficients of this solution are calculated using numerical methods. This solution, in contrast to others, for example [6], was obtained in a parametric form similar to what was done in the paper [7] for the exponential Phan-Tien–Tanner fluid when solving the hydrodynamic part of the problem. The main feature of this approach is the transition in the energy transfer equation to a new independent variable, which is a parameter, instead of using the traditional spatial variable.
2. Problem statement

Let us consider the non-isothermal flow of a viscoelastic linear Phan-Tien-Tanner (PTT) fluid in a plane slit using the full linear four-mode rheological equation of state of PTT with the Newtonian stress component as:

\[
\sigma = \sum_{i=1}^{n} \sigma_{vi} + \sigma_N, \quad \sigma_N = 2\mu_N D, \\
\lambda_i \left( \sigma_{vi} + \xi_i (D \cdot \sigma_{vi} + \sigma_{vi} \cdot D) \right) + g_i \sigma_{vi} = 2\mu_N D, \quad g_i = 1 + \frac{\varepsilon_i}{\mu_N} \text{tr}(\sigma_{vi}),
\]

(1)

where \( \sigma \) is an upper convective derivative of the tensor \( \sigma \); \( D \) is the tensor of velocities of deformations; \( \sigma_N, \sigma_{vi} \) are viscous and elastic components of the stress tensor deviator \( \sigma \); \( \mu_N, \mu_{vi} \) are viscosities; \( \lambda_i \) is relaxation time; \( \xi_i, \varepsilon_i \) are rheological parameters; \( V \) is a velocity vector; \( i = 1 \ldots 4 \).

We assume the following: the only axial component of the velocity \( V_z \) depends only on the transverse variable \( y \), where \( x, y, z \) are the independent variables of the Cartesian coordinate system with the \( z \) axis directed along the channel axis; the thermophysical properties of the liquid (density, thermal conductivity and specific heat capacity) and the parameters of the rheological model of the PTT change slightly in a given temperature range; energy transfer in the axial direction due to thermal conductivity is negligible in comparison with the same transfer in the transverse direction and in comparison with convective heat transfer; dissipative heat generation is not taken into account.

These assumptions allow us to obtain a separate solution of the hydrodynamic part of the problem, which is then used to solve the energy transfer equation.

2.1. Solution of hydrodynamic part of main problem

With these assumptions, the system of equations for the transfer of the amount of motion and continuity describing the laminar flow of a liquid in a flat channel is going to take the following form:

\[
0 = \partial P \partial z + \frac{d}{dy} \left( \sigma_N \right), \quad 0 = \partial P \partial y + \frac{d}{dy} \left( \sigma_N \right), \quad 0 = \partial P \partial x, \quad \frac{dV_z}{dz} = 0
\]

(2)

we will assume that the liquid adheres to the walls of the channel. Let’s also assume that the value of the axial component of the pressure gradient (\( C_0 \)) is set. Then it follows from (2) that

\[
\frac{\partial P}{\partial z} = C_0 = \text{const}, \quad P = \sigma_{y} (y) + C_0 z, \quad \sigma_{yz} = C_0 y.
\]

(3)

Taking into account the assumptions made above, the rheological equation of state (1) in the Cartesian coordinate system will be written as

\[
\sigma_{y} = \sum_{i=1}^{n} \sigma_{vi} + \sigma_N, \quad \sigma_N = 2\mu_N D, \\
\lambda_i \left( \sigma_{vi} + \xi_i (D \cdot \sigma_{vi} + \sigma_{vi} \cdot D) \right) + g_i \sigma_{vi} = 2\mu_N D, \quad g_i = 1 + \frac{\varepsilon_i}{\mu_N} \text{tr}(\sigma_{vi}),
\]

(4)
Taking into account (3) and (1), analysis of the structure of equation (4) has shown that it is convenient to enter a parameter $\alpha$ which all the required functions depend on.

$$\alpha = \alpha_i = \frac{\lambda_i \xi_i (2-\xi_i) \sigma_{ii}}{2(1-\xi_i) \mu_{ii}},$$  \hspace{1cm} (5)

then

$$\frac{dV_i}{dy} = \frac{1 + k_i \alpha}{\lambda_i \sqrt{\xi_i (2-\xi_i)}} \sqrt{\frac{\alpha}{(1-\alpha)}}, \quad \sigma_{yz} = \mu_N \frac{dV_i}{dy} + \sum_{j=1}^{n} \sigma_{ijyz},$$  \hspace{1cm} (6)

$$\sigma_{yz} = -\mu_N \frac{1 + k_i \alpha}{\lambda_i \sqrt{\xi_i (2-\xi_i)}} \sqrt{\frac{\alpha}{(1-\alpha)}} - \sum_{j=1}^{m} \frac{\mu_{ij}}{\lambda_j \sqrt{\xi_j (2-\xi_j)}} \sqrt{\frac{\alpha_j}{(1-\alpha_j)}}, \quad k_i = 2\xi_i (1-\xi_i)$$  \hspace{1cm} (7)

in this relation, the values $\alpha_j$, $j = 2, 3, 4$ are functions of parameter $\alpha$. When we enter the dimensionless variable $\eta = y/H$ and dimensionless velocity $v = V \sqrt{2Hb}$, where $V = Q/(2Hb)$ is the average flow rate and $Q$ is the volume flow rate of a liquid through the cross section of the channel, the parametric dependence of the dimensionless shear rate $dV_i/d\eta$ on the independent variable is written as

$$\eta = \frac{1}{G} \left( \beta_j \sqrt{(1-\alpha_j)} + \psi_j \sqrt{(1-\alpha_j)} \right),$$  \hspace{1cm} (8)

$$\left(1 + k_i \alpha \right)^2 \frac{\alpha}{(1-\alpha)} = \frac{1}{\psi_j^2} \left(1 + k_i \alpha_i \right)^2 \frac{\alpha_i}{(1-\alpha_i)}, \quad j = 2, ..., n.$$

where $\beta_N = \frac{\mu_N}{\mu_N + \sum_{i=1}^{n} \mu_{ii}}, \beta_i = \frac{\mu_{ii}}{\mu_N + \sum_{i=1}^{n} \mu_{ii}}, i = 2, ..., n, \quad \psi_j = \frac{\lambda_j \sqrt{\xi_j (2-\xi_j)}}{\lambda_i \sqrt{\xi_i (2-\xi_i)}}, j = 2, ..., m,$

$$G = \frac{\lambda_i \sigma_{ii}}{H \sqrt{\xi_i (2-\xi_i)}}, \quad \kappa = \sqrt{\xi_i (2-\xi_i)} W e^*, \quad W e^* = \frac{\lambda_i V_0}{H}$$ - Weissenberg number, $0 \leq \eta = \frac{y}{H} \leq 1$.

Then the obtained system of equations (8) was solved by Fourier method. To do this, the dependencies of coordinates and the axial component of speed $(dV_i/d\eta = -f(\alpha_i)/\kappa)$ on the parameter were presented as series for this parameter.

2.2. Solution of energy equation
The energy transfer equation for the considered case in dimensionless form takes the following form:

$$v_i(\eta) \frac{\partial \theta}{\partial z} \frac{d^2 \theta}{d\eta^2},$$  \hspace{1cm} (9)
where \( Pe_H = \rho \gamma C_{pl} H \nu / \lambda_f \) is Peclet number with characteristic dimension which equals half of the channel width, \( \eta = y / H \) and \( \zeta = z (Pe_H H) \) are dimensionless variables, \( \theta = (T - T_e) / (T_0 - T_e) \) is dimensionless temperature.

Boundary conditions are the following:

\[
\theta(\eta, 0) = 1, \ 0 \leq \eta \leq 1, \ \frac{d \theta}{d \eta}(0, \zeta) = 0, \ \theta(1, \zeta) = 0, \ 0 \leq \zeta \leq \infty
\]

(10)

The solution to this problem can be found in the form of a series:

\[
\theta = \sum_{p=1}^{\infty} A_p f_p(\eta) e^{-\lambda_p \zeta}
\]

(11)

After substituting this series in equation (9) and equating the coefficients for the same eigenfunctions, we obtain the following Sturm-Liouville problem in parametric form:

\[
X(t) \frac{d^2 f_p}{dt^2} + Y(t) \frac{df_p}{dt} + \lambda_p Z(t) f_p = 0, \quad \frac{1}{d\eta/dt} \frac{df_p}{dt}(0) = 0, \ f_p(t_w) = 0.
\]

(12)

where \( X(t) = \frac{d\eta(t)}{dt}, \ Y(t) = -\frac{d^2\eta(t)}{dt^2}, \ Z(t) = v_i(t) \left( \frac{d\eta(t)}{dt} \right)^3, \ t = \sqrt{\alpha} \) is a new parameter.

Dependence \( \eta(t) \) and \( v_i(t) \) can also be written in the form of a series \( \eta = \frac{1}{G} \sum_{i=0}^{\infty} \eta_i t^{2i}, \ v_i = \frac{1}{G} \sum_{i=0}^{\infty} v_i t^{2i} \), where the coefficients and are determined according to (9) and (10). Thus, the solution of the original Sturm-Liouville problem for \( f_p(t) \) is sought in the form of a series \( f_p(t) = \sum_{i=0}^{\infty} \phi_{ip} t^{2i} \). The coefficients \( A_p \) in the expansion (11) are found from the condition at the entrance to the pipe (10) and from the condition of orthogonality of eigenfunctions \( f_p(\eta) \) on a segment of \( 0 \leq \eta \leq 1 \) with weight of \( v_i(\eta) \)

\[
\int_0^1 v_i(\eta) f_p(\eta) f_q(\eta) d\eta = 0 \quad \text{for} \quad p \neq q
\]

(13)

Local and limit Nusselt numbers for a flat channel are defined as:

\[
Nu = \frac{2}{\theta_a} \left. \frac{\partial \theta}{\partial \zeta} \right|_{z \to \infty}, \ \ Nu_x = \lim_{z \to \infty} \frac{2}{\theta_a} \left. \frac{\partial \theta}{\partial \zeta} \right|_{z \to \infty} = 2 \lambda_i
\]

(14)

where \( \theta_a = \sum_{p=1}^{\infty} E_p e^{-\lambda_p \zeta}, \ E_p = A_p \int_0^1 f_p v_i(\eta) d\eta d\tau = \frac{A_p}{\kappa G^2} \sum_{i=0}^{\infty} \left( \sum_{j=0}^{\infty} \phi_{ip} v_{ij} \right) (2i+1)^{2i+1} t_w^{2i+1} \).

3. Results

To check the adequacy of the obtained results, the developed solution method was tested on the flow of a simplified 4 mode viscoelastic PTT fluid in a round pipe with a linear stress coefficient \( \varepsilon_i = 0.02 \ (i = 1...4) \) and a Newtonian solvent contribution equals \( \eta_N = 0 \). The comparison results are shown in figure 1, which indicates the adequacy of the developed solution method.
Figure 1. Normalized axial velocity in pipe: solid line – [4], dashed line – present study.

Four mode Phan-Thien-Tanner model for the PAA500 solution (500 ppm by weight of polyacrilamide in a 85% glycerin-15% water mixture) was used to investigate the non-isothermal viscoelastic flow in the flat channel with a linear stress function having \( \varepsilon = 0.02 \) and \( \xi = 0.04 \) [8]. The linear viscoelastic spectra are the following: \( (\lambda_1 = 30.0 \ [s], \mu_1 = 2.50 \ [Pa \cdot s]), \ (\lambda_2 = 3.0 \ [s], \mu_2 = 0.9 \ [Pa \cdot s]), \ (\lambda_3 = 0.3 \ [s], \mu_3 = 0.3 \ [Pa \cdot s]), \ (\lambda_4 = 0.03 \ [s], \mu_4 = 0.1 \ [Pa \cdot s]), \mu_N = 0.27 \ [Pa \cdot s] \).

Figure 2 presents tangent (\( \sigma_{yz} \)) and normal (\( \sigma_{yy}, \sigma_{zz} \)) stresses at the flow of the considered liquid in a flat channel for various Weissenberg numbers. It can be seen that \( \sigma_{zz} \) makes the greatest contribution to the stress tensor. Also, the tangent stresses of the viscous component in the considered range of Weissenberg numbers are significantly less than the elastic components of the stress.

In contrast to the flow in a round tube, the axial velocity profile in a plane slit is more filled (Fig. 3a). At the same time, the tendency remains that the velocity distribution for a viscoelastic liquid approaches the Newtonian one when the Weissenberg number decreases.
Figure 3. (a) Profiles of the axial velocity in a cross section; (b) Distribution of the ratio $Nu/Nu_0$ along the length of the pipe for different values of the Weissenberg number: 1 - Newtonian fluid ($Nu_0$), 2 - $We = 0.71$, 3 - $We = 1.24$; 4 - $We = 2.14$; 5 - $We = 10.26$.

Conclusions
A method is proposed to solve heat transfer problem for a flow of a viscoelastic PTT fluid in a plane slit at a constant temperature. The solution is found in parametric form, where the axial velocity and temperature are expressed through a parameter. Tests of the developed method have demonstrated a good agreement of obtained results with data from literature. The results of calculations have shown that the largest contribution to the stress tensor is made by the component $\sigma_{zz}$. An increase in Weissenberg number leads to an increase in the intensity of heat transfer in relation to the flow of a Newtonian fluid.

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