The many faces of optimism
Extended version

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Abstract

The exploration-exploitation dilemma has been an intriguing and unsolved problem within the framework of reinforcement learning. “Optimism in the face of uncertainty” and model building play central roles in advanced exploration methods. Here, we integrate several concepts and obtain a fast and simple algorithm. We show that the proposed algorithm finds a near-optimal policy in polynomial time, and give experimental evidence that it is robust and efficient compared to its ascendants.

1 Introduction

Reinforcement learning (RL) is the art of maximizing long-term rewards in a stochastic, unknown environment. In the construction of RL algorithms, the choice of exploration strategy is of central significance.

We shall examine the problem of exploration in the Markov decision process (MDP) framework. While simple methods like $\epsilon$-greedy and Boltzmann exploration are commonly used, it is known that their behavior can be extremely poor (?). Recently, a number of efficient exploration algorithms have been published, and for some of them, formal proofs of efficiency also exist. We review these methods in Section 2. By combining ideas from several sources, we construct a new algorithm for efficient exploration. The new algorithm, optimistic initial model (OIM), is described in Section 3. In Section 4 we show that many of the advanced algorithms, including ours, can be treated in a unified way. We use this fact to sketch a proof that OIM finds a near-optimal policy in polynomial time with high probability. Section 5 provides experimental comparison between OIM and a number of other methods on some benchmark problems. Our results are summarized in Section 6. In the rest of this section, we review the necessary preliminaries, Markov decision processes and the exploration task.

1.1 Markov decision processes (MDPs)

Markov decision processes are the standard framework for RL, and the basis of numerous extensions (like continuous MDPs, partially observable MDPs or
factored MDPs). An MDP is characterized by a quintuple \((X, A, R, P, \gamma)\), where \(X\) is a finite set of states; \(A\) is a finite set of possible actions; \(R : X \times A \times X \rightarrow \mathbb{R}\) is the reward distribution, \(R(x, a, y)\) denotes the mean value of \(R(x, a, y)\), \(P : X \times A \times X \rightarrow [0, 1]\) is the transition function; and finally, \(\gamma \in [0, 1)\) is the discount rate on future rewards. We shall assume that all rewards are nonnegative and bounded from above by \(R_0\max\).

A (stationary) policy of the agent is a mapping \(\pi : X \times A \rightarrow [0, 1]\). For any \(x_0 \in X\), the policy of the agent and the parameters of the MDP determine a stochastic process experienced by the agent through the instantiation \(x_0, a_0, r_0, x_1, a_1, r_1, \ldots, x_t, a_t, r_t, \ldots\)

The goal is to find a policy that maximizes the expected value of the discounted total reward. Let us define the state-action value function (value function for short) of \(\pi\) as \(Q^\pi(x, a) := E\left(\sum_{t=0}^{\infty} \gamma^t r_t \mid x=x_0, a=a_0\right)\) and the optimal value function as \(Q^*(x, a) := \max_\pi Q^\pi(x, a)\) for each \((x, a)\) \(\in X \times A\). Let the greedy action at \(x\) w.r.t. value function \(Q\) be \(a^Q_x := \arg \max_a Q(x, a)\). The greedy policy of \(Q\) deterministically takes the greedy action in each state. It is well-known that the greedy policy of \(Q^*\) is an optimal policy and \(Q^*\) satisfies the Bellman equations:

\[
Q^*(x, a) = \sum_y P(x, a, y) \left( R(x, a, y) + \gamma Q^*(y, a^Q_y) \right).
\]

### 1.2 The exploration problem

In the classical reinforcement learning setting, it is assumed that the environment can be modelled as an MDP, but its parameters (that is, \(P\) and \(R\)) are unknown to the agent, and she has to collect information by interacting with the environment. If too little time is spent with the exploration of the environment, the agent will get stuck with a suboptimal policy, without knowing that there exists a better one. On the other hand, the agent should not spend too much time visiting areas with low rewards and/or accurately known parameters.

What is the optimal balance between exploring and exploiting the acquired knowledge and how could the agent concentrate her exploration efforts? These questions are central for RL. It is known that the optimal exploration policy in an MDP is non-Markovian, and can be computed only for very simple tasks like \(k\)-armed bandit problems.

### 2 Related literature

Here we give a short review about some of the most important exploration methods and their properties.
2.1 \(\epsilon\)-greedy and Boltzmann exploration

The most popular exploration method is \(\epsilon\)-greedy action selection. The method works without a model, only an approximation of the action value function \(Q(x,a)\) is needed. The agent in state \(x\) selects the greedy action \(a^Q_x\) or an explorative move with a random action with probabilities \(1-\epsilon\) and \(\epsilon\), respectively. Sooner or later, all paths with nonzero probability will have been visited many times, so, a suitable learning algorithm can learn to choose the optimal path. It is known, for example, that Q-learning with nonzero exploration converges to the optimal value function with probability 1 (?), and so does SARSA (?), if the exploration rate diminishes according to an appropriate schedule.

Boltzmann-exploration selects actions as follows: the probability of choosing action \(a\) is

\[
\frac{\exp(Q(s,a)/T)}{\sum_{a'\in A} \exp(Q(s,a')/T)},
\]

where ‘temperature’ \(T(>0)\) regulates the amount of explorative actions. Convergence results of the \(\epsilon\)-greedy method carry through to this case.

Unfortunately, for the \(\epsilon\)-greedy and the Boltzmann method, exploration time may scale exponentially in the number of states (?).

2.2 Optimistic initial values (OIV)

One may boost exploration with a simple trick: the initial value of each state action pair can be set to some overwhelmingly high number. If a state \(x\) is visited often, then its estimated value will become more exact, and therefore, lower. Thus, the agent will try to reach the more rarely visited areas, where the estimated state values are still high. This method, called ‘exploring starts’ or ‘optimistic initial values’, is a popular exploration heuristic (?), sometimes combined with others, e.g., the \(\epsilon\)-greedy exploration method. Recently, ? (?) gave theoretical justification for the method: they proved that if the optimistic initial values are sufficiently high, Q-learning converges to a near-optimal solution. One apparent disadvantage of OIV is that if initial estimations are too high, then it takes a long to fix them.

2.3 Bayesian methods

We may assume that the MDP (with the unknown values of \(P\) and \(R\)) is drawn from a parameterized distribution \(M_0\). From the collected experience and the prior distribution \(M_0\), we can calculate successive posterior distributions \(M_t, t = 1, 2, \ldots\) by Bayes’ rule. Furthermore, we can calculate (at least in principle) the policy that minimizes the uncertainty of the parameters (?). ? (?) approximates the distribution of state values directly. Exact computation of the optimal exploration policy is infeasible and Bayesian methods are computationally demanding even with simplifying assumptions about the distributions, e.g., the independencies of certain parameters.
2.4 Confidence interval estimation

Confidence interval estimation algorithms are between Bayesian exploration and OIV. It assumes that each state value is drawn from an independent Gaussian distribution and it computes the confidence interval of the state values. The agent chooses the action with the highest upper confidence bound. Initially, all confidence intervals are very wide, and shrink gradually towards the true state values. Therefore, the behavior of the technique is similar to OIV. The IEQL+ method of ? (7) directly estimates confidence intervals of Q-values, while ? (7) calculate confidence intervals for P and R, and obtain Q-value bounds indirectly. ? (7) improve the method and prove a polynomial-time convergence bound. Both algorithms are called model-based interval estimation. To avoid confusion, we will refer to them as MBIE(WS) and MBIE(SL).

? (7) give a confidence interval-based algorithm, for which the online regret is only logarithmic in the number of steps taken.

2.5 Exploration Bonus Methods

The agent can be directed towards less-known parts of the state space by increasing the value of ‘interesting’ states artificially with bonuses. States can be interesting given their frequency, recency, error, etc. (?; ?).

The balance of exploration and exploitation is usually set by a scaling factor \( \kappa \), so that the total immediate reward of the agent at time \( t \) is \( r_t + \kappa \cdot b_t(x_t, a_t, x_{t+1}) \), where \( b_t \) is one of the above listed bonuses. The bonuses are calculated by the agent and act as intrinsic motivating forces. Exploration bonuses for a state can vary swiftly and model-based algorithms (like prioritized sweeping or Dyna) are used for spreading the changes effectively. Alas, the weight of exploration \( \kappa \) needs to be annealed according to a suitable schedule.

Alternatively, the agent may learn two value functions separately: a regular one, \( Q^r_t \) which is based on the rewards \( r_t \) received from the environment, and an exploration value function \( Q^e_t \) which is based on the exploration bonuses. The agent’s policy will be greedy with respect to their combination \( Q^r_t + \kappa Q^e_t \). Then the exploration mechanism may remain the same, but several advantages appear. First of all, the changes in \( \kappa \) take effect immediately. As an example, we can immediately switch off exploration by setting \( \kappa \) to 0. Furthermore, \( Q^r_t \) may converge even if \( Q^e_t \) does not.

Confidence interval estimation can be phrased as an exploration bonus method: see IEQL+ (7) or MBIE-EB (7). ? (7) have shown that \( \epsilon \)-greedy and Boltzmann explorations can be formulated as exploration bonus methods although rewards are not propagated through the Bellman equations.

2.6 \( E^3 \) and R-max

The Explicit explore or exploit (\( E^3 \)) algorithm of ? (7) and its successor, R-max (7) were the first algorithms that have polynomial time bounds for finding near-
optimal policies. R-max collects statistics about transitions and rewards. When visits to a state enable high precision estimations of real transition probabilities and rewards then state is declared known. R-max also maintains an approximate model of the environment. Initially, the model assumes that all actions in all states lead to a (hypothetical) maximum-reward absorbing state. The model is updated each time when a state becomes known. The optimal policy of the model is either the near-optimal policy in the real environment or enters a not-yet-known state and collects new information.

3 Construction of the algorithm

Our agent starts with a simple, but overly optimistic model. By collecting new experiences, she updates her model, which becomes more realistic. The value function is computed over the approximate model with (asynchronous) dynamic programming. The agent always chooses her action greedily w.r.t. her value function. Exploration is induced by the optimism of the model: unknown areas are believed to yield large rewards. Algorithmic components are detailed below.

Separate exploration values. Similarly to the approach of ?? (?), we shall separate the ‘true’ state values from exploration values. Formally, the value function has the form

\[ Q(x, a) = Q^r(x, a) + Q^e(x, a) \]

for all \((x, a) \in X \times A\), where \(Q^r\) and \(Q^e\) will summarize external and exploration rewards, respectively.

‘Garden of Eden’ state. Similarly to R-max, we introduce a new hypothetical ‘garden of Eden’ state \(x_E\), and assume an extended state space \(X' = X \cup \{x_E\}\). Once there, then, according to the inherited model, the agent remains in \(x_E\) indefinitely and receives \(R_{\text{max}}\) reward for every step, which may exceed \(R_{\text{max}}^0 =: \max_{x,a,y} R(x, a, y)\), the maximal reward of the original environment.

Model approximation. The agent builds an approximate model of the environment. For each \(x, y \in X\) and \(a \in A\), let \(N_t(x, a, y)\) and \(C_t(x, a, y)\) denote the number of times when \(a\) was selected in \(x\) up to step \(t\), the number of times when transition \(x \xrightarrow{a} y\) was experienced, and the sum of external rewards for \(x \xrightarrow{a} y\) transitions, respectively. With these notations, the approximate model parameters are

\[ \hat{P}_t(x, a, y) = \frac{N_t(x, a, y)}{N_t(x, a)} \quad \text{and} \quad \hat{R}_t(x, a, y) = \frac{C_t(x, a, y)}{N_t(x, a, y)}. \]

Suitable initializations of \(N_t(x, a, y)\) and \(C_t(x, a, y)\) will ensure that the ratios are well-defined everywhere. The exploration rewards are defined as

\[ R^e(x, a, y) := \begin{cases} R_{\text{max}}, & \text{if } y = x_E; \\ 0, & \text{if } y \neq x_E, \end{cases} \]
for each \(x, y \in X \cup \{x_E\}, a \in A\), and are not modified during the course of learning.

**Optimistic initial model.** The initial model assumes that \(x_E\) has been reached once for each state-action pairs: for each \(x \in X \cup \{x_E\}, y \in X\) and \(a \in A\),

\[
N_0(x, a) = 1, \\
N_0(x, a, y) = 0, \\
C_0(x, a, y) = 0. \\
N_0(x, a, x_E) = 1, \\
C_0(x, a, x_E) = 0.
\]

Then, the optimal initial value function equals

\[
Q_0(x, a) = Q^r_0(x, a) + Q^e_0(x, a) = 0 + \frac{1}{1-\gamma}R_{\max} := V_{\max}
\]

for each \((x, a) \in X' \times A\), analogously to OIV.

**Dynamic programming.** Both value functions can be updated using the approximate model. For each \(x \in X\), let \(a_x\) be the greedy action according to the combined value function, i.e.,

\[
a_x := \arg \max_{a \in A} (Q^r(x, a) + Q^e(x, a)).
\]

The dynamic programming equations for the value function components are

\[
Q_{t+1}^r(x, a) := \sum_{y \in X} \hat{P}_t(x, a, y) \left( \hat{R}_t(x, a, y) + \gamma Q_t^r(y, a_y) \right)
\]

\[
Q_{t+1}^e(x, a) := \gamma \sum_{y \in X} \hat{P}_t(x, a, y) Q_t^e(y, a_y)
\]

\[
+ \hat{P}_t(x, a, x_E)V_{\max}.
\]

Episodic tasks can be handled as usual way; we introduce an absorbing final state with 0 external reward.

**Asynchronous update.** The algorithm can be online, if instead of full update sweeps over the state space updates are limited to state set \(L_t\) in the ‘neighborhood’ of the agent’s current state. Neighborhood is restricted by computation time constraints; any asynchronous dynamic programming algorithm suffices. It is implicitly assumed that the current state is always updated, i.e., \(x_t \in L_t\). In this paper, we used the improved prioritized sweeping algorithm of \(?\) (?).

**Putting it all together.** The method is summarized as Algorithm 1.

4 Analysis

In the first part of this section, we analyze the similarities and differences between various exploration methods, with an emphasis on OIM. Based on this analysis, we sketch the proof that OIM finds a near-optimal policy in polynomial time.
Algorithm 1 The Optimistic initial model algorithm

\textbf{Input:} $x_0 \in X$ initial state, $\epsilon > 0$ required precision, optimism parameter $R_{\max}$

\textbf{Model initialization:} $t := 0; \forall x, y \in X, \forall a \in A:$

\[ N(x, a, y) := 0, \quad N(x, a, x_E) := 1, \quad C(x, a, y) := 0, \quad Q^r(x, a) := 0, \quad Q^e(x, a) := \frac{R_{\max}}{1 - \gamma}; \]

\textbf{repeat}

\[ a_t := \text{greedy action w.r.t. } Q^r + Q^e; \text{ apply } a_t \text{ and observe } r_t \text{ and } x_{t+1} \]

\[ C(x_t, a_t, x_{t+1}) := C(x_t, a_t, x_{t+1}) + r_t; \quad N(x_t, a_t, x_{t+1}) := N(x_t, a_t, x_{t+1}) + 1; \]

\[ L_t := \text{list of states to be updated} \]

\textbf{for each} $x \in L_t$ \textbf{do}

\[ Q_{t+1}^r(x, a) := \sum_{y \in X} \hat{P}(x, a, y) \left( \hat{R}(x, a, y) + \gamma Q_t^r(y, a_y) \right) \]

\[ Q_{t+1}^e(x, a) := \hat{P}(x, a, x_E) R_{\max} / (1 - \gamma) + \gamma \sum_{y \in X} \hat{P}(x, a, y) Q_t^e(y, a_y). \]

\textbf{end for}

\[ t := t + 1 \]

\textbf{until} Bellman-error $> \epsilon$

4.1 Relationship to other methods

‘Optimism in the face of uncertainty’ is a common point in exploration methods: the agent believes that she can obtain extra rewards by reaching the unexplored parts of the state space.

Note that as far as the combined value function $Q$ is concerned, OIM is an asynchronous dynamic programming method augmented with model approximation.

**Optimistic initial values.** Apparently, OIM is the model-based extension of the OIV heuristic. Note however, that optimistic initialization of $Q$-values is not effective with a model: the more updates are made, the less effect the initialization has and it fully diminishes if value iteration is run until convergence. Therefore, naive combination of OIV and model construction is contradictory: the number of DP-updates should be kept low in order to save the initial boost, but it should be as high as possible in order to propagate the real rewards quickly.

OIM resolves this paradox by moving the optimism into the model. The optimal value function of the initial model is $Q_0 \equiv V_{\max}$, corresponding to OIV. However, DP updates can not, but only model updates may lower the exploration boost.

Note that we can set the initial model value as high as we like, but we do not have to wait until the initial boost diminishes, because $Q^r$ and $Q^e$ are separated.

**R-max.** The ‘Garden of Eden’ state $x_E$ of OIM is identical to the fictitious max-reward absorbing state of R-MAX (and $E^3$). In both cases, the agent’s model tells that all unexplored $(x, a)$ pairs lead to $x_E$. R-MAX, however, updates the model only when the transition probabilities and rewards are known with
high precision, which is only after many visits to \((x, a)\). In contrast, OIM updates the model after each single visit, employing each bit of experience as soon as it is obtained. As a result, the approximate model can be used long before it becomes accurate.

**Exploration bonus methods.** The extra reward offered by the Garden of Eden state can be understood as an exploration bonus: for each visit of the pair \((x, a)\), the agent gets the bonus \(b_t(x, a) = \frac{1}{N_t(x, a)}(V_{\text{max}} - Q_t(x, a))\). It is insightful to contrast this formula with those of the other methods like the frequency-based bonus \(b_t = -\alpha \cdot N_t(x, a)\) or the error-based bonus \(b_t = \alpha \cdot |Q_{t+1}(x, a) - Q_t(x, a)|\).

**Model-based interval exploration.** The exploration bonus form of the MBIE method of \(? (? )\) sets \(b_t = \frac{\alpha}{N_t(x, a)}\). MBIE-EB is not an ad-hoc method: the form of the bonus comes from confidence interval estimations. The comparison to MBIE-EB will be especially valuable, as it converges in polynomial-time and the proof can be transported to OIM with slight modifications.

### 4.2 Polynomial-time convergence

**Theorem 4.1** For any \(\epsilon > 0, \delta > 0, \epsilon_1 := \epsilon/6, \epsilon_2 := \frac{(1-\gamma)^2}{|X|(1-\gamma+R_{\text{max}})}, \epsilon_3 := \frac{1}{1-\gamma} \ln \frac{R_0}{\epsilon_3 (1-\gamma)}, m := \frac{2\max\{1, R_0^2\}^2}{\epsilon_2^2} \ln \frac{8}{\delta}, OIM\) converges almost surely to a near-optimal policy in polynomial time if started with \(R_{\text{max}} = 2\frac{(R_{\text{max}}^2)^2 \ln(2|X||A|m/\delta)}{\epsilon_3 (1-\gamma)}\), that is, with probability \(1 - \delta\), the number of timesteps where \(Q_{t+1}^{\text{OIM}}(x_t, a_t) > Q^*(x_t, a_t) - \epsilon\) does not hold, is at most \(2m|X||A|HR_{\text{max}}^2 \ln \frac{4}{\delta} \).

The proof can be found in the Appendix.

### 5 Experiments

To assess the practical utility of OIM, we compared its performance to other exploration methods. Experiments were run on several small benchmark tasks challenging exploration algorithms.

For fair comparisons, benchmark problems were taken from the literature without changes, nor did we change the experimental settings or the presentation of experimental data. It also means that the presentation format varies for different benchmarks.

#### 5.1 RiverSwim and SixArms

The first two benchmark problems, *RiverSwim* and *SixArms*, were taken from \(? (? )\).

The *RiverSwim* MDP has 6 states, representing the position of the agent in a river. The agent has two possible actions: she can swim either upstream or downstream. Swimming down is always successful, but swimming up succeeds...
Table 1: Results on the RiverSwim task.

| Method   | Cumulative reward          |
|----------|---------------------------|
| $E^3$    | $3.020 \times 10^6 \pm 0.027 \times 10^6$ |
| R-MAX    | $3.014 \times 10^6 \pm 0.039 \times 10^6$ |
| MBIE(SL) | $3.168 \times 10^6 \pm 0.023 \times 10^6$ |
| MBIE-EB  | $3.093 \times 10^6 \pm 0.023 \times 10^6$ |
| OIM      | $3.201 \times 10^6 \pm 0.016 \times 10^6$ |

Table 2: Results on the SixArms task.

| Method   | Cumulative reward          |
|----------|---------------------------|
| $E^3$    | $1.623 \times 10^6 \pm 0.244 \times 10^6$ |
| R-MAX    | $2.819 \times 10^6 \pm 0.256 \times 10^6$ |
| MBIE(SL) | $9.205 \times 10^6 \pm 0.559 \times 10^6$ |
| MBIE-EB  | $9.486 \times 10^6 \pm 0.587 \times 10^6$ |
| OIM      | $10.007 \times 10^6 \pm 0.654 \times 10^6$ |

only with a 30% chance and there is a 10% chance of slipping down. The lowermost position yields +5 reward per step, while the uppermost position yields +10000.

The SixArms MDP consists of a central state and six ‘payoff states’. In the central state, the agent can play 6 one-armed bandits. If she pulls arm $k$ and wins, she is transferred to payoff state $k$. Here, she can get a reward in each step, if she chooses the appropriate action. The winning probabilities range from 1 to 0.01, while the rewards range from 50 to 6000 (for the exact values, see ?).

Data for $E^3$, R-MAX, MBIE and MBIE-EB are taken from ? (?). Parameters of all four algorithms were chosen optimally. Following a coarse search in parameter space, the $R_{max}$ parameter for OIM was set to 2000 for RiverSwim and to 10000 for SixArms. State spaces are small and value iteration instead of prioritized sweeping was completed in each step.

On both problems, each algorithm ran for 5000 time steps and the undiscounted total reward was recorded. The averages and 95% confidence intervals are calculated over 1000 test runs (Tables 1 and 2).

5.2 50 × 50 maze with subgoals

Another benchmark problem, MazeWithSubgoals, was suggested by ? (?). The agent has to navigate in a 50 × 50 maze from the start position at (2, 2) to the goal (with +1000 reward) at the opposite corner (49, 49). There are suboptimal goals (with +500 reward) at the other two corners. The maze has blocked places
Table 3: Results on the MazeWithSubgoals task. The number of steps required to learn $p$-optimal policies ($p=0.95, 0.99, 0.998$) on the $50 \times 50$ maze task with suboptimal goals. In parentheses: how many runs out of 20 have found the goal. ‘$k$’ stands for 1000.

| Method          | 95%  | 99%  | 99.8% |
|-----------------|------|------|-------|
| $\epsilon$-GReedy, $\epsilon = 0.2$ | $-$ (0) | $-$ (0) | $-$ (0) |
| $\epsilon$-GReedy, $\epsilon = 0.4$ | 43k (4) | 52k (4) | 68k (4) |
| Recency-bonus   | 27k (19) | 55k (18) | 69k (9) |
| Freq.-bonus     | 24k (20) | 50k (18) | 66k (10) |
| MBIE(WS)        | 25k (20) | 42k (19) | 66k (18) |
| OIM             | 19k (20) | 29k (20) | 31k (20) |

and punishing states ($-10$ reward), set randomly in 20-20% of the squares. The agent can move in four directions, but with a 10% chance, its action is replaced by a random one. If the agent tries to move to a blocked state, it gets a reward of $-2$. Reaching any of the goals resets the agent to the start state. In all other cases, the agent gets a $-1$ reward for each step.

Each algorithm was run on 20 different mazes for 100,000 steps. After every 1000 steps, we tested the learned value functions by averaging 20 test runs, in each one following the greedy policy for 10,000 steps, and averaging cumulated (undiscounted) rewards. We measured the number of test runs needed for the algorithms to learn to collect 95%, 99% and 99.8% of the maximum possible rewards in 100,000 steps, and the number of steps this takes on average, if the algorithms can meet the challenge.

The algorithms that we compared were the recency based and frequency based exploration bonus methods, two versions of $\epsilon$-greedy exploration, MBIE(WS) and OIM. All exploration rules applied the improved prioritized sweeping of $\tilde{Q}$ (7). OIM’s $R_{\text{max}}$ was set to 1000. The results are summarized in Table 3.

### 5.3 Chain, Loop and FlagMaze

The next three benchmark MDPs, the Chain, Loop and FlagMaze tasks were investigated, e.g., by ? (3), ? (3) and ? (3). In the Chain task, 5 states are lined up along a chain. The agent gets $+2$ reward for being in state 1 and $+10$ for being in state 5. One of the actions advances one state ahead, the other one resets the agent to state 1. The Loop task has 9 states in two loops (arranged in a 8-shape). Completing the first loop (using any combination of the two actions) yields $+1$ reward, while the second loop yields $+2$, but one of the actions resets the agent to the start. The FlagMaze task consists of a 6 $\times$ 7 maze with several walls, a start state, a goal state and 3 flags. Whenever the agent reaches the goal, her reward is the number of flags collected.

The following algorithms were compared: Q-learning with variance-based
Table 4: Average accumulated rewards on the *Chain* task. Optimal policy gathers 3677.

| Method          | Phase 1 | Phase 2 | Phase 8 |
|-----------------|---------|---------|---------|
| QL+var.-bonus   | –       | 2570\textsuperscript{1} | –       |
| QL+err.-bonus   | –       | 2530\textsuperscript{1} | –       |
| QL \(\epsilon\)-greedy | 1519 | 1611 | 1602 |
| QL Boltzmann    | 1606 | 1623 | –       |
| IEQL+           | 2344 | 2557 | –       |
| Bayesian QL     | 1697 | 2417 | –       |
| Bayesian DP\textsuperscript{2} | 3158 | 3611 | 3643 |
| OIM             | 3510 | 3628 | 3643 |

Table 5: Average accumulated rewards on the *Loop* task. Optimal policy gathers 400.

| Method          | Phase 1 | Phase 2 | Phase 8 |
|-----------------|---------|---------|---------|
| QL+var.-bonus   | –       | 179\textsuperscript{1} | –       |
| QL+err.-bonus   | –       | 179\textsuperscript{1} | –       |
| QL \(\epsilon\)-greedy | 337 | 392 | 399 |
| QL Boltzmann    | 186 | 200 | –       |
| IEQL+           | 264 | 293 | –       |
| Bayesian QL     | 326 | 340 | –       |
| Bayesian DP\textsuperscript{2} | 377 | 397 | 399 |
| OIM             | 393 | 400 | 400 |

and TD error-based exploration bonus (model-free variants), \(\epsilon\)-greedy exploration, Boltzmann exploration, IEQL+, Bayesian Q-learning, Bayesian DP and OIM. Data were taken from ? (?), ? (?), and ? (?). According to the sources, parameters for all algorithms were set optimally. OIM’s \(R_{\text{max}}\) parameter was set to 0.5, 10 and 0.005 for the three tasks, respectively.

Each algorithm ran for 8 learning phases. The total cumulated reward over each learning phase was measured. One phase lasted for 1000 steps for the first two tasks and 20,000 steps for the *FlagMaze* task. We carried out 256 parallel runs for the first 2 tasks and 20 for the third one.

\textsuperscript{1}Results for Phase 5.

\textsuperscript{2}Augmented with limited amount of pre-wired knowledge (the list of successor states).
Table 6: Average accumulated rewards on the FlagMaze task. Optimal policy gathers approximately 1890.

| Method          | Phase 1 | Phase 2 | Phase 8 |
|-----------------|---------|---------|---------|
| QL $\epsilon$-greedy | 655     | 1135    | 1147    |
| QL Boltzmann    | 195     | 1024    | -       |
| IEQL+           | 269     | 253     | -       |
| Bayesian QL     | 818     | 1100    | -       |
| Bayesian DP$^2$ | 750     | 1763    | 1864    |
| OIM             | 1133    | 1169    | 1171    |

6 Summary of the results

We proposed a new algorithm for exploration and reinforcement learning in Markov decision processes. The algorithm integrates concepts from other advanced exploration methods. The key component of our algorithm is an optimistic initial model. The optimal policy according to the agent’s model will either explore new information that helps to make the model more accurate, or follows a near-optimal path. The extent of optimism regulates the amount of exploration. We have shown that with a suitably optimistic initialization, our algorithm finds a near-optimal policy in polynomial time. Experiments were conducted on a number of benchmark MDPs. According to the experimental results our novel method is robust and compares favorably to other methods.

Acknowledgments

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A Proof of Polynomial-time convergence

For the proof, we shall follow the technique of \cite{KS} and \cite{SL}, and will use the shorthands \cite{KS} and \cite{SL} for referring to them. We will proceed by a series of lemmas.

Throughout the proof, note the difference between $R_{\text{max}}$ and $R_{\text{max}}^0$. Value estimates of our model start from $R_{\text{max}}$. However, all actual rewards observed by the agent are bounded by $R_{\text{max}}^0$, which is smaller than $R_{\text{max}}$.

Lemma A.1 (Azuma’s Lemma) If the random variables $X_1, X_2, \ldots$ form a
martingale difference sequence, meaning that $E[X_k|X_1, X_2, \ldots, X_{k-1}] = 0$ for all $k$, and $|X_k| \leq b$ for each $k$, then

$$
\Pr \left[ \sum_{i=1}^{k} X_i \geq a \right] \leq \exp \left( -\frac{a^2}{2b^2 k} \right)
$$

and

$$
\Pr \left[ \sum_{i=1}^{k} X_i \geq a \right] \leq 2 \exp \left( -\frac{a^2}{2b^2 k} \right)
$$

The following lemma is similar to Lemma 5 of [KS] (with the modification that $R(x, a, y)$ values are learnt instead of $R(x, a)$-values, and tells that if a state-action pair is visited many times, then its parameter estimates become accurate.

**Lemma A.2** Consider an MDP $M = (X, A, P, R, \gamma)$, and let $(x, a)$ be a state-action pair that has been visited at least $m$ times. Let $\hat{P}(x, a, y)$ and $\hat{R}(x, a, y)$ denote the obtained empirical estimates, let $\epsilon > 0$ and $\delta > 0$ be arbitrary positive values. If

$$
m \geq \frac{2 \max \{1, R_0^{\max} \}^2}{\epsilon^2} \ln \frac{2}{\delta},
$$

then for all $y \in X$,

$$
\left| P(x, a, y)R(x, a, y) - \hat{P}(x, a, y)\hat{R}(x, a, y) \right| \leq \epsilon \quad \text{and} \quad \left| P(x, a, y) - \hat{P}(x, a, y) \right| \leq \epsilon
$$

holds with probability at least $1 - \delta$.

**Proof.** Suppose that $(x, a)$ is visited $k$ times at steps $t_1, \ldots, t_k$. Define the random variables

$$
Z_i(y) = \begin{cases} 
1, & \text{if } x_{t_i+1} = y; \\
0, & \text{otherwise}.
\end{cases}
$$

Clearly, $E[Z_i(y)] = P(x, a, y)$ and $Z_i(y) - P(x, a, y)$ is a martingale, so we can apply Azuma’s lemma with $a = k\epsilon$ to get

$$
\Pr \left[ \frac{1}{k} \sum_{i=1}^{k} Z_i - P(x, a, y) \geq \epsilon \right] \leq 2 \exp \left( -\frac{\epsilon^2 k}{2} \right) \leq 2 \exp \left( -\frac{\epsilon^2 m}{2} \right).
$$

The right-hand side is less than $\delta$ for

$$
m \geq \frac{2}{\epsilon^2} \ln \frac{2}{\delta}.
$$

Similarly, define the random variables

$$
W_i(y) = \begin{cases} 
r_{t_i+1}, & \text{if } x_{t_i+1} = y; \\
0, & \text{otherwise}.
\end{cases}
$$

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In this case, \( E[W_i(y)] = P(x, a, y)R(x, a, y) \), \( W_i(y) - P(x, a, y)R(x, a, y) \) is a martingale and is bounded by \( R_0^{\max} \) (note that we are considering only states \( x, y \in X \), that is, the garden-of-Eden state \( x_E \) is excluded. Therefore, \( R_0^{\max} \) is indeed an upper bound on \( R(x, a, y) \), so we can apply Azuma’s lemma with \( a = k\epsilon \) to get

\[
\Pr \left[ \frac{1}{k} \sum_{i=1}^{k} W_i - P(x, a, y)R(x, a, y) \geq \epsilon \right] \leq 2 \exp \left( -\frac{\epsilon^2 k}{2(R_0^{\max})^2} \right) \leq 2 \exp \left( -\frac{\epsilon^2 m}{2(R_0^{\max})^2} \right).
\]

The right-hand side is less than \( \delta \) for

\[
m \geq \frac{2(R_0^{\max})^2}{\epsilon^2} \ln \frac{2}{\delta}.
\]

Unifying the two requirements for \( m \) completes the proof of the lemma.

The following is a minor modification of [KS] lemma 4, and [SL] Lemma 1. The result tells that if the parameters of two MDPs are very close to each other, then the value functions in the two MDPs will also be similar.

**Lemma A.3** Let \( \epsilon > 0 \), and consider two MDPs \( M = (X, A, P, R, \gamma) \) and \( \bar{M} = (X, A, \bar{P}, \bar{R}, \gamma) \) that differ only in their transition and reward functions, furthermore, their difference is bounded:

\[
|P(x, a, y)R(x, a, y) - \bar{P}(x, a, y)\bar{R}(x, a, y)| \leq \epsilon' \quad \text{and} \quad |P(x, a, y) - \bar{P}(x, a, y)| \leq \epsilon'
\]

for all \((x, a, y) \in X \times A \times X\) and

\[
\epsilon' := \frac{(1 - \gamma)^2}{|X| (1 - \gamma + R_0^{\max})} \cdot \epsilon.
\]

Then for any policy \( \pi \) and any \((x, a) \in X \times A\),

\[
|Q^\pi(x, a) - \bar{Q}^\pi(x, a)| \leq \epsilon.
\]

**Proof.** Let \( \Delta := \max_{(x, a) \in X \times A} |Q^\pi(x, a) - \bar{Q}^\pi(x, a)| \), and note that for any \( x \in X \),

\[
|V^\pi(x) - \bar{V}^\pi(x)| = \left| \sum_a \pi(x, a)(Q^\pi(x, a) - \bar{Q}^\pi(x, a)) \right| \leq \sum_a \pi(x, a)\Delta = \Delta
\]

For a fixed \((x, a)\) pair,

\[
\Delta = |Q^\pi(x, a) - \bar{Q}^\pi(x, a)|
\]

\[
= \left| \sum_{y \in X} P(x, a, y) \left( R(x, a, y) + \gamma V^\pi(y) \right) - \sum_{y \in X} \bar{P}(x, a, y) \left( \bar{R}(x, a, y) + \gamma \bar{V}^\pi(y) \right) \right|
\]

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\[
\leq \sum_{y \in X} |P(x, a, y)R(x, a, y) - \bar{P}(x, a, y)\bar{R}(x, a, y)| \\
+ \sum_{y \in X} \left[\left|P(x, a, y) - \bar{P}(x, a, y)\right| (\gamma V^\pi(y)) + \left|\bar{P}(x, a, y)(\gamma[V^\pi(y) - \bar{V}^\pi(y)]\right|\right] \\
\leq |X| \epsilon' + \sum_{y \in X} \epsilon' |\gamma V^\pi(y)| + \sum_{y \in X} \bar{P}(x, a, y)(\gamma \Delta) \\
\leq |X| \epsilon' + |X| \epsilon \frac{R_0}{1 - \gamma} + \gamma \Delta.
\]

Therefore,
\[
\Delta \leq \frac{|X| \epsilon' (1 - \gamma + R_0^\delta)}{(1 - \gamma)^2} = \epsilon
\]

Let us introduce a modified version of OIM that behaves exactly like the old one, except that in each \((x, a)\) pairs, it performs at most \(m\) updates. If a pair is visited more than \(m\) times, the modified algorithm leaves the counters unchanged.

The following result is a modification of [SL]'s Lemma 7.

**Lemma A.4** Suppose that the modified OIM (stopping after \(m\) updates) is executed on an MDP \(M = (X, A, P, R, \gamma)\) with

\[
m := \frac{2 \max\{1, R_0^\delta\}^2}{\epsilon^2} \ln \frac{2}{\delta},
\]

\[
\beta := \frac{R_0^\delta}{1 - \gamma} \sqrt{2 \ln(2 |X| |A| m / \delta)}.
\]

Then, with probability at least \(1 - \delta\),
\[
Q^*(x, a) - \sum_{y \in X} \bar{P}_t(x, a, y) \left[\bar{R}_t(x, a, y) + \gamma V^*(y)\right] \leq \beta / \sqrt{k}
\]

for all \(t = 1, 2, \ldots\)

**Proof.** Fix a state-action pair \((x, a)\) and suppose that it has been visited \(k \leq m\) times until time step \(t\), at steps \(t_1, \ldots, t_k\). Define the random variables \(X_1, \ldots, X_k\) by

\[
X_i := r_{t_i} + \gamma V^*(x_{t_i + 1}).
\]

Note that \(E[X_i] = Q^*(x, a)\) and \(0 \leq X_i \leq R_0^\delta/(1 - \gamma)\) for all \(i = 1, \ldots, k\), and the sequence \(Q^*(x, a) - X_i\) is a martingale difference sequence. Applying Azuma’s lemma yields

\[
\Pr \left[ E[X_1] - \frac{1}{k} \sum_{i=1}^{k} X_i \geq a/k \right] \leq \exp \left( -\frac{a^2(1 - \gamma)^2}{2(R_0^\delta)^2 k} \right)
\]

(1)
for any \( a \). Let the right-hand side be equal to \( \frac{\delta}{2|X||A|m} \), corresponding to
\[
a = \beta \sqrt{k}
\]
with
\[
\beta := R_{\text{max}}^0 / (1 - \gamma) \sqrt{2 \ln(2|X||A|m/\delta)}.
\]
Note that by the construction of the OIM algorithm,
\[
\sum_{y \in X'} \hat{P}_t(x, a, y) \left[ \hat{R}_t(x, a, y) + \gamma V^*(y) \right]
= \sum_{y \in X : N_t(x, a, y) > 0} \frac{N_t(x, a, y)}{N_t(x, a)} \left[ \frac{C_t(x, a, y)}{N_t(x, a, y)} + \gamma V^*(y) \right] + \frac{1}{N_t(x, a)} \left[ R_{\text{max}} + \gamma V^*(x_{\text{GOE}}) \right]
= \frac{N_t(x, a) - 1}{N_t(x, a)} \sum_{y \in X : N_t(x, a, y) > 0} \frac{N_t(x, a, y)}{N_t(x, a, y) - 1} \left[ \frac{C_t(x, a, y)}{N_t(x, a, y)} + \gamma V^*(y) \right] + \frac{1}{N_t(x, a)} \frac{R_{\text{max}}}{1 - \gamma}
= \frac{k}{k + 1} \sum_{y \in X : N_t(x, a, y) > 0} \frac{N_t(x, a, y)}{k} \left[ \frac{C_t(x, a, y)}{N_t(x, a, y)} + \gamma V^*(y) \right] + \frac{1}{k + 1} \frac{R_{\text{max}}}{1 - \gamma},
\]
where we exploited the fact that \( k = N_t(x, a) - 1 \). Therefore,
\[
\frac{1}{k} \sum_{i=1}^{k} X_i = \frac{k + 1}{k} \sum_{y \in X'} \hat{P}_t(x, a, y) \left[ \hat{R}_t(x, a, y) + \gamma V^*(y) \right] - \frac{1}{k} \frac{R_{\text{max}}}{1 - \gamma}.
\]
Substituting this to (1), we get that
\[
Q^*(x, a) - \frac{k + 1}{k} \sum_{y \in X'} \hat{P}_t(x, a, y) \left[ \hat{R}_t(x, a, y) + \gamma V^*(y) \right] + \frac{1}{k} \frac{R_{\text{max}}}{1 - \gamma} < \frac{\beta}{\sqrt{k}}
\]
with high probability, but we will use only the slightly looser inequality
\[
Q^*(x, a) - \sum_{y \in X} \hat{P}_t(x, a, y) \left[ \hat{R}_t(x, a, y) + \gamma V^*(y) \right] \leq \beta / \sqrt{k}. \tag{2}
\]
For each \((x, a)\), the modified OIM algorithm changes the parameters at most \( m \) times, which is at most \( m|X||A| \) changes in total. Each different approximation fails with probability less than \( \frac{\delta}{2|X||A|m} \), so, by the union bound, the total probability that (2) fails (at any time, for any state-action pair) is still less than \( \delta/2 \).

The following result shows that the modified OIM algorithm preserves the optimism of the value function with high probability.
Lemma A.5 Let $\epsilon_1 > 0$ and suppose that the modified OIM is executed on an MDP $M = (X, A, P, R, \gamma)$ with

$$R_{\text{max}} \geq \frac{\beta^2}{\epsilon_1}$$

where

$$m := \frac{2 \max\{1, R_{\text{max}}^0\}^2}{\epsilon^2} \ln \frac{2}{\delta},$$

$$\beta := \frac{R_{\text{max}}^0}{1 - \gamma} \sqrt{2 \ln(2 |X||A| m/\delta)}.$$  

Then, with probability at least $1 - \delta/2$, $Q_{t}^{\text{OIM}}(x,a) > Q^*(x,a) - \epsilon_1$ for all $t = 1, 2, \ldots$

According to the previous lemma,

$$\sum_y \tilde{P}_t(x,a,y)(\tilde{R}_t(x,a,y) + \gamma V^*(y)) - Q^*(x,a) \geq -\beta/\sqrt{N_t(x,a)}$$  

(3)

with probability $1 - \delta/2$.

We will show that

$$\frac{R_{\text{max}}}{N_t(x,a)(1 - \gamma)} + (1 - \gamma)\epsilon_1 \geq \frac{\beta}{\sqrt{N_t(x,a)}}.$$  

(4)

For $N_t(x,a) \leq \frac{R_{\text{max}}}{(1 - \gamma)^2 \epsilon_1}$, the first term dominates the l.h.s. and we can omit the second term (and prove the stricter inequality). In the following, we proceed by a series of equivalent transformations:

$$\frac{R_{\text{max}}}{N_t(x,a)(1 - \gamma)} \geq \frac{\beta}{\sqrt{N_t(x,a)}},$$

$$\frac{R_{\text{max}}}{\beta(1 - \gamma)} \geq \sqrt{N_t(x,a)},$$

$$\frac{R_{\text{max}}^2}{\beta^2(1 - \gamma)^2} \geq N_t(x,a),$$

which is implied by the stricter inequality

$$\frac{R_{\text{max}}^2}{\beta^2(1 - \gamma)^2} \geq \frac{R_{\text{max}}}{(1 - \gamma)^2 \epsilon_1},$$

$$R_{\text{max}} \geq \frac{\beta^2}{\epsilon_1},$$

which holds by the assumption of the lemma. If the relation is reversed, then the first term can be omitted, leading to

$$(1 - \gamma)\epsilon_1 \geq \frac{\beta}{\sqrt{N_t(x,a)}}.$$
\[
\frac{\beta}{(1 - \gamma)\epsilon_1} \leq \sqrt{N_t(x, a)},
\]
\[
\frac{\beta^2}{(1 - \gamma)^2\epsilon_1^2} \leq N_t(x, a),
\]

which is implied by the stricter inequality
\[
\frac{\beta^2}{(1 - \gamma)^2\epsilon_1^2} \leq \frac{R_{\text{max}}}{(1 - \gamma)^2\epsilon_1},
\]
\[
R_{\text{max}} \geq \frac{\beta^2}{\epsilon_1},
\]
similarly to the previous case.

At step \( t \), a number of DP updates are carried out. We proceed by induction on the number of DP-updates. Initially,
\[ Q^{(0)}(x, a) \geq Q^*(x, a) - \epsilon_1, \]
then
\[ Q^{(i+1)}(x, a) = \sum_y \hat{P}_t(x, a, y)(\hat{R}_t(x, a, y) + \gamma V^{(i)}(y)) + \frac{V_{\text{max}}}{N_t(x, a)} \]
\[ \geq \sum_y \hat{P}_t(x, a, y)(\hat{R}_t(x, a, y) + \gamma (V^*(y) - \epsilon_1)) + \frac{V_{\text{max}}}{N_t(x, a)} \]
\[ \geq Q^*(x, a) - \beta/\sqrt{N_t(x, a)} - \gamma \epsilon_1 + \frac{V_{\text{max}}}{N_t(x, a)} \]
\[ \geq Q^*(x, a) - \gamma \epsilon_1 - (1 - \gamma)\epsilon_1 = Q^*(x, a) - \epsilon_1, \]
where we applied (3), (4) and the induction assumption. \( \blacksquare \)

Define the \( H \)-step truncated value function of policy \( \pi \) as
\[ Q^\pi(x, a, H) := \mathbb{E}\left[ \sum_{t=0}^{H} \gamma^t r_t \mid x=x_0, a=a_0 \right]. \]

**Lemma A.6 ([KS] Lemma 2)** Let \( \epsilon > 0 \) and consider an MDP \( M = (X, A, P, R, \gamma) \). If
\[ H \geq \frac{1}{1 - \gamma} \log \frac{P_0}{\epsilon(1 - \gamma)}, \]
then
\[ Q^\pi(x, a, H) \leq Q^\pi(x, a) \leq Q^\pi(x, a, H) + \epsilon \]
for any \((x, a) \in X \times A\).

**Proof.** Let \( \Xi(x, a) \) denote the set of infinite trajectories starting in \((x, a)\), and for any trajectory \( \xi \in \Xi(x, a) \), let \( \xi_H \) denote its \( H \)-step truncation. Furthermore, denote the discounted total reward along a trajectory \( \xi \) by \( v(\xi) \). Clearly,
\[ Q^\pi(x, a) = \mathbb{E}_\xi[v(\xi)] = \sum_{\xi \in \Xi(x, a)} \Pr(\xi)v(\xi) \]
and
\[ Q^\pi(x, a, H) = \mathbb{E}_\xi[v(\xi_H)] = \sum_{\xi \in \Xi(x, a)} \Pr(\xi)v(\xi_H). \]
Fix a trajectory $p$, along which the agent receives rewards $r_1, r_2, \ldots$, for which

$$v(\xi_H) = \sum_{t=0}^{H-1} \gamma^t r_{t+1} \quad \text{and}$$

$$v(\xi) = \sum_{t=0}^{\infty} \gamma^t r_{t+1} = v(\xi_H) + \sum_{t=H}^{\infty} \gamma^t r_{t+1}.$$ 

It is trivial that $v(\xi) \geq v(\xi_H)$, as the additional terms are all nonnegative by assumption. On the other hand,

$$\sum_{t=H}^{\infty} \gamma^t r_{t+1} \leq \frac{\gamma^H}{1 - \gamma} R_{\max}^0,$$

which is smaller than $\epsilon$ if $H \geq \log \frac{\epsilon (1 - \gamma)}{R_{\max}^0}$ (which follows from the assumption of the lemma and the inequality $-\log \gamma > 1 - \gamma$), that is,

$$v(\xi) \leq v(\xi_H) + \epsilon.$$

As the relations hold for each trajectory in $\Xi(x, a)$, they hold for the expected value, too.

The following lemma tells that OIM and its modified version learn almost the same values with high probability.

**Lemma A.7** For any $\epsilon > 0$, $\delta > 0$,

$$\epsilon' := \frac{(1 - \gamma)^2}{|X| (1 - \gamma + R_{\max}^0)} \cdot \epsilon,$$

$$m \geq \frac{2 \max \{1, R_{\max}^0 \}^2}{\epsilon'^2 \ln \frac{2}{\delta}},$$

for any MDP $M$ and any $(x, a) \in X \times A$,

$$\left| Q_{M}^{\text{mOIM}}(x, a) - Q_{M}^{\text{OIM}}(x, a) \right| \leq 2\epsilon'$$

with probability at least $1 - 2\delta$.

**Proof.** The model estimates of the two algorithm-variants are identical on not-yet-known states where the visit count is less than $m$. On known pairs, we can apply Lemma A.2 to both model-estimates to see that they are $\epsilon'$-close to the true model parameters with probability at least $1 - \delta$. Consequently, they are $2\epsilon'$-close to each other with at least $1 - 2\delta$ probability. Applying Lemma A.3 proves the statement of the lemma.

**Lemma A.8 (Lemma 3 of [SL])** Let $M = (X, A, P, R, \gamma)$ be an MDP, $K$ a set of state-action pairs, $\tilde{M}$ an MDP equal to $M$ on $K$ (identical transition and reward functions), $\pi$ a policy, and $H$ some positive integer. Let $A_M$ be the event
that a state-action pair not in $K$ is encountered in a trial generated by starting from $(x, a)$ and following $\pi$ for $H$ steps in $M$. Then,

$$Q^\pi_M(x, a) \geq Q_{\bar{M}}^\pi(x, a) - \frac{R_{\max}^0}{1 - \gamma} \Pr(A_M).$$

(5)

**Proof.** Let $\Xi$ be the set of $H$-step long trajectories, and let $\Xi^K \subset \Xi$ be the set of trajectories for which all occurring $(x_t, a_t)$ pairs are in $K$. For any $\xi \in \Xi$, let $\Pr_M(\xi)$ denote the probability of that trajectory happening in MDP $M$.

Let $v(\xi)$ be the discounted total reward received by the agent along the $H$-step trajectory $\xi \in \Xi$. Now, we have the following:

$$Q^\pi_{\bar{M}}(x, a) = \sum_{\xi \in \Xi} \Pr_{\bar{M}}(\xi)v(\xi)$$

$$= \sum_{\xi \in \Xi^K} \Pr_{\bar{M}}(\xi)v(\xi) + \sum_{\xi \in \Xi \setminus \Xi^K} \Pr_{\bar{M}}(\xi)\frac{R_{\max}^0}{1 - \gamma}$$

$$\leq \sum_{\xi \in \Xi^K} \Pr_{\bar{M}}(\xi)v(\xi) + \sum_{\xi \in \Xi \setminus \Xi^K} \Pr_{\bar{M}}(\xi)\frac{R_{\max}^0}{1 - \gamma}$$

$$\leq \sum_{\xi \in \Xi^K} \Pr_{\bar{M}}(\xi)v(\xi) + \Pr(A_{\bar{M}})\frac{R_{\max}^0}{1 - \gamma}$$

$$= \sum_{\xi \in \Xi^K} \Pr_{\bar{M}}(\xi)v(\xi) + \Pr(A_M)\frac{R_{\max}^0}{1 - \gamma}$$

$$\leq Q^\pi_M(x, a) + \Pr(A_M)\frac{R_{\max}^0}{1 - \gamma}.$$

\[\blacksquare\]

**Theorem A.9** For any $\epsilon > 0$, $\delta > 0$, let

$$\epsilon_1 := \epsilon/6$$

$$\epsilon_2 := \frac{(1 - \gamma)^2}{|X| (1 - \gamma + R_{\max}^0)}, \epsilon_1,$$

$$H := \frac{1}{1 - \gamma} \ln \frac{R_{\max}^0}{\epsilon_1 (1 - \gamma)}$$

$$m := \frac{2 \max\{1, R_{\max}^0\}^2}{\epsilon_2^2} \ln \frac{8}{\delta}.$$

OIM converges almost surely to a near-optimal policy in polynomial time if started with

$$R_{\max} = \frac{2(R_{\max}^0)^2 \ln(2 |X| |A| m/\delta)}{\epsilon_1 (1 - \gamma)^3},$$

that is, with probability $1 - \delta$, the number of timesteps where $Q^{OIM}_t(x_t, a_t) > Q^*(x_t, a_t) - \epsilon$ does not hold, is at most

$$\frac{2m |X| |A| H R_{\max}^0}{\epsilon_1 (1 - \gamma)} \ln \frac{4}{\delta}.$$
Remark A.10 When expressed in terms of MDP parameters, time requirement is

\[
\frac{864 |X|^3 |A| R_0^0 \max \{1, R_0^0\}^2 (1 - \gamma + R_0^0)^2}{\epsilon^3 (1 - \gamma)^4} \ln \frac{6 R_0^0 \max}{\epsilon (1 - \gamma)} \ln \frac{4}{\delta} \ln \frac{8}{\delta}
\]

and the required initialization value is

\[
R_\max = \frac{12 (R_0^0)^2}{\epsilon (1 - \gamma)^3} \ln \left( \frac{12 |X|^3 |A| \max \{1, R_0^0\}^2 (1 - \gamma + R_0^0)^2}{\delta \epsilon (1 - \gamma)^2} \ln \frac{8}{\delta} \right)
\]

Proof.

Let \( M \) denote the true (and unknown) MDP, let \( \hat{M} \) be the approximate model of OIM. An \((x, a)\) pair is considered known if it has been visited at least \( m \) times. According to Lemma A.2, for a known pair \((x, a)\), the model estimates \( \hat{P}(x, a, \cdot) \) and \( \hat{R}(x, a, \cdot) \) are \( \epsilon_2 \)-close to the true values with probability at least \( 1 - \delta/4 \).

Define the MDP \( \bar{M} \) so that it is identical to \( M \) for known pairs, and equals \( \hat{M} \) for unknown pairs. The parameters of \( \hat{M} \) and \( \bar{M} \) are identical on unknown pairs and \( \epsilon_2 \)-close for known pairs (with probability \( 1 - \delta/4 \)), so, by Lemma A.3,

\[
|Q^\pi_{\hat{M}}(x, a) - Q^\pi_{\bar{M}}(x, a)| < \epsilon_1
\]

for any policy \( \pi \) and any \((x, a) \in X \times A\).

Let

\[
H := \frac{1}{1 - \gamma} \ln \frac{R_\max}{\epsilon_1 (1 - \gamma)}
\]

By Lemma A.6,

\[
|Q^\pi_{\hat{M}}(x, a, H) - Q^\pi_{\bar{M}}(x, a)| < \epsilon_1
\]

holds for the \( H \)-step truncated value function for any \((x, a), \pi\).

Consider a state-action pair \((x_1, a_1)\) and a \( H \)-step long trajectory generated by \( \pi \). Let \( K \) be the set of known \((x, a)\) pairs and let \( A_M \) be the event that an unknown pair is encountered along the trajectory. Then, by Lemma A.8,

\[
Q^\pi_M(x_1, a_1) \geq Q^\pi_{\hat{M}}(x_1, a_1) - \frac{R_0^0}{1 - \gamma} \Pr(A_M).
\]

By applying Lemma A.7 to \( \epsilon_1, \delta/4 \), we get that the above setting of \( R_\max \) ensures that the original and the modified version of OIM behaves similarly:

\[
|Q_M^{\text{OIM}}(x, a) - Q_M^{\text{OIM}}(x, a)| \leq 2 \epsilon_1
\]
with probability at least $1 - \delta/2$. Furthermore, by Lemma A.5 (with $\epsilon \leftarrow \epsilon_1$ and $\delta \leftarrow \delta/4$), the modified algorithm preserves the optimism of the value function with probability at least $1 - \delta/4$: 

$$Q^m_{OIM}(x, a) > Q^*(x, a) - \epsilon_1$$

To conclude the proof, we separate two cases (following the line of thoughts of Theorem 1 in [SL]). In the first case, an exploration step will occur with high probability: Suppose that $\Pr(A_M) > \epsilon_1(1 - \gamma)/R^0_{\max}$, that is, an unknown pair is visited in $H$ steps with high probability. This can happen at most $m |X| |A|$ times, so by Azuma’s bound, with probability $1 - \delta/4$, all $(x, a)$ will become known after $\frac{2m|X||A|H R^0_{\max}}{\epsilon_1(1 - \gamma)} \ln \frac{4}{\delta}$ exploration steps.

On the other hand, if $\Pr(A_M) \leq \epsilon_1(1 - \gamma)/R^0_{\max}$, then the policy is near-optimal with probability $1 - \delta$:

$$Q^o_{OIM}(x_1, a_1) \geq Q^o_{OIM}(x_1, a_1, H)$$
$$\geq Q^o_{OIM}(x_1, a_1, H) - \frac{R^0_{\max}}{1 - \gamma} \Pr(A_M)$$
$$\geq Q^o_{OIM}(x_1, a_1) - \epsilon_1 \geq Q^o_{OIM}(x_1, a_1) - 2\epsilon_1$$
$$\geq Q^o_{OIM}(x_1, a_1) - 3\epsilon_1$$
$$\geq Q^o_{OIM}(x_1, a_1) - 5\epsilon_1$$
$$\geq Q^o(x_1, a_1) - 6\epsilon_1$$
$$= Q^o(x_1, a_1) - \epsilon,$$

where we applied (in this order) the property that truncation decreases the value function; Eq. (8); our assumption; Eq. (7); Eq. (6); Eq. (9); Lemma A.5 and the definition of $\epsilon_1$.

B A dimension-respecting version of the proof

For the proof, we shall follow the technique of [KS] and [SL], and will use the shorthands [KS] and [SL] for referring to them. We will proceed by a series of lemmas.

**Lemma B.1** Consider an MDP $M = (X, A, P, R, \gamma)$, and let $(x, a)$ be a state-action pair that has been visited at least $m$ times. Let $\hat{P}(x, y)$ and $\hat{R}(x, a, y)$ denote the obtained empirical estimates, let $\epsilon > 0$ and $\delta > 0$ be arbitrary positive values. If

$$m \geq \frac{2}{\epsilon^2} \ln \frac{2}{\delta},$$

then

$$Q^m_{OIM}(x_1, a_1) \geq Q^*(x_1, a_1) - \epsilon,$$
then for all \( y \in X \),

\[
\left| P(x, a, y) R(x, a, y) - \hat{P}(x, a, y) \hat{R}(x, a, y) \right| \leq \epsilon R^0_{\text{max}} \quad \text{and} \quad \left| P(x, a, y) - \hat{P}(x, a, y) \right| \leq \epsilon
\]

holds with probability at least \( 1 - \delta \).

**Proof.** The second statement is already proven, so let us consider the first one. Suppose that \((x, a)\) is visited \( k \) times at steps \( t_1, \ldots, t_k \). Define the random variables

\[
W_i(y) = \begin{cases} r_{t_i+1}, & \text{if } x_{t_i+1} = y; \\ 0, & \text{otherwise.} \end{cases}
\]

In this case, \( E[W_i(y)] = P(x, a, y) R(x, a, y) \), \( W_i(y) - P(x, a, y) R(x, a, y) \) is a martingale and is bounded by \( R^0_{\text{max}} \) (note that we are considering only states \( x, y \in X \), that is, the garden-of-Eden state \( x_E \) is excluded. Therefore, \( R^0_{\text{max}} \) is indeed an upper bound on \( R(x, a, y) \)), so we can apply Azuma’s lemma with \( a = k \epsilon R^0_{\text{max}} \) to get

\[
\Pr \left[ \left| \frac{1}{k} \sum_{i=1}^{k} W_i - P(x, a, y) R(x, a, y) \right| \geq \epsilon R^0_{\text{max}} \right] \leq 2 \exp \left( -\frac{(\epsilon R^0_{\text{max}})^2 k}{2 R^0_{\text{max}}^2} \right) \leq 2 \exp \left( -\frac{\epsilon^2 m}{2} \right).
\]

The right-hand side is less than \( \delta \) for

\[
m \geq \frac{2}{\epsilon^2} \ln \frac{2}{\delta}.
\]

The following is a minor modification of [KS] lemma 4, and [SL] Lemma 1. The result tells that if the parameters of two MDPs are very close to each other, then the value functions in the two MDPs will also be similar.

**Lemma B.2** Let \( \epsilon > 0 \), and consider two MDPs \( M = (X, A, P, R, \gamma) \) and \( \bar{M} = (X, A, \bar{P}, \bar{R}, \gamma) \) that differ only in their transition and reward functions, furthermore, their difference is bounded:

\[
\left| P(x, a, y) R(x, a, y) - \bar{P}(x, a, y) \bar{R}(x, a, y) \right| \leq \epsilon' R^0_{\text{max}} \quad \text{and} \quad \left| P(x, a, y) - \bar{P}(x, a, y) \right| \leq \epsilon'
\]

for all \((x, a, y) \in X \times A \times X\) and

\[
\epsilon' := \frac{(1 - \gamma)^2}{|X|} \epsilon.
\]

Then for any policy \( \pi \) and any \((x, a) \in X \times A\),

\[
\left| Q^\pi(x, a) - \bar{Q}^\pi(x, a) \right| \leq \epsilon R^0_{\text{max}}.
\]
Proof. Let $\Delta := \max_{(x,a) \in X \times A} |Q^\pi(x,a) - \bar{Q}^\pi(x,a)|$, and note that for any $x \in X$,

$$
|V^\pi(x) - \bar{V}^\pi(x)| = \left| \sum_a \pi(x,a)(Q^\pi(x,a) - \bar{Q}^\pi(x,a)) \right| \leq \sum_a \pi(x,a)\Delta = \Delta
$$

For a fixed $(x,a)$ pair,

$$
\Delta = |Q^\pi(x,a) - \bar{Q}^\pi(x,a)|
= \left| \sum_{y \in X} P(x,a,y)\left( R(x,a,y) + \gamma V^\pi(y) \right) - \sum_{y \in X} \bar{P}(x,a,y)\left( \bar{R}(x,a,y) + \gamma \bar{V}^\pi(y) \right) \right|
\leq \sum_{y \in X} |P(x,a,y)R(x,a,y) - \bar{P}(x,a,y)\bar{R}(x,a,y)|
+ \left| \sum_{y \in X} P(x,a,y) - \bar{P}(x,a,y) \right| \left( \gamma V^\pi(y) \right)
+ \left| \sum_{y \in X} \bar{P}(x,a,y) \right| \left( \gamma \left[ V^\pi(y) - \bar{V}^\pi(y) \right] \right)
\leq |X| \epsilon P^0_{\max} + \sum_{y \in X} \epsilon \left| \gamma V^\pi(y) \right| + \sum_{y \in X} \bar{P}(x,a,y) \left( \gamma \Delta \right)
\leq |X| \epsilon P^0_{\max} + |X| \epsilon \frac{\beta R^0_{\max}}{1 - \gamma} + \gamma \Delta.
$$

Therefore,

$$
\Delta \leq \frac{|X| \epsilon P^0_{\max}}{(1 - \gamma)^2} = \epsilon P^0_{\max}
$$

Let us introduce a modified version of OIM that behaves exactly like the old one, except that in each $(x,a)$ pairs, it performs at most $m$ updates. If a pair is visited more than $m$ times, the modified algorithm leaves the counters unchanged.

The following result is a modification of [SL]'s Lemma 7.

**Lemma B.3** Suppose that the modified OIM (stopping after $m$ updates) is executed on an MDP $M = (X, A, P, R, \gamma)$ with

$$
m := \frac{2}{\epsilon^2} \ln \frac{2}{\delta},
\beta := \frac{R^0_{\max}}{1 - \gamma} \sqrt{2 \ln(2 |X| |A| m/\delta)}.
$$

Then, with probability at least $1 - \delta$,

$$
Q^*(x,a) - \sum_{y \in X} P_t(x,a,y) \left[ \bar{R}_t(x,a,y) + \gamma V^*(y) \right] \leq \beta / \sqrt{t}
$$

for all $t = 1, 2, \ldots$
Proof. The proof is identical to the proof of Lemma A.4.

The following result shows that the modified OIM algorithm preserves the optimism of the value function with high probability.

**Lemma B.4** Let $\epsilon_1 > 0$ and suppose that the modified OIM is executed on an MDP $M = (X, A, P, R, \gamma)$ with

$$R_{\text{max}} \geq \frac{\beta^2}{\epsilon_1 R_{0\text{max}}}$$

where

$$m := \frac{2}{\epsilon^2} \ln \frac{2}{\delta},$$

$$\beta := \frac{R_{0\text{max}}}{R_{\text{max}}} \sqrt{2 \ln(2 |X| |A| m/\delta)}.$$

Then, with probability at least $1 - \delta/2$, $Q_{t\text{OIM}}^m(x, a) > Q^*(x, a) - \epsilon_1 R_{0\text{max}}$ for all $t = 1, 2, \ldots$

According to the previous lemma,

$$\sum_y \hat{P}_t(x, a, y)(\hat{R}_t(x, a, y) + \gamma V^*(y)) - Q^*(x, a) \geq -\beta/\sqrt{N_t(x, a)} \quad (10)$$

with probability $1 - \delta/2$.

We will show that

$$\frac{R_{\text{max}}}{N_t(x, a)(1 - \gamma)} + (1 - \gamma)\epsilon_1 R_{0\text{max}} \geq \frac{\beta}{\sqrt{N_t(x, a)}}. \quad (11)$$

For $N_t(x, a) \leq \frac{R_{\text{max}}}{R_{0\text{max}}(1 - \gamma)\epsilon_1}$, the first term dominates the l.h.s. and we can omit the second term (and prove the stricter inequality). In the following, we proceed by a series of equivalent transformations:

$$\frac{R_{\text{max}}}{N_t(x, a)(1 - \gamma)} \geq \frac{\beta}{\sqrt{N_t(x, a)}},$$

$$\frac{R_{\text{max}}}{\beta(1 - \gamma)} \geq \sqrt{N_t(x, a)},$$

$$\frac{R_{\text{max}}^2}{\beta^2(1 - \gamma)^2} \geq N_t(x, a),$$

which is implied by the stricter inequality

$$\frac{R_{\text{max}}^2}{\beta^2(1 - \gamma)^2} \geq \frac{R_{\text{max}}}{R_{0\text{max}}(1 - \gamma)^2\epsilon_1},$$

$$R_{\text{max}} \geq \frac{\beta^2}{\epsilon_1 R_{0\text{max}}},$$
which holds by the assumption of the lemma. If the relation is reversed, then the first term can be omitted, leading to

\[(1 - \gamma)\epsilon_1 R_{0_{\text{max}}}^0 \geq \frac{\beta}{\sqrt{N_t(x, a)}},\]

\[\frac{\beta}{(1 - \gamma)\epsilon_1 R_{0_{\text{max}}}^0} \leq \sqrt{N_t(x, a)},\]

\[\frac{\beta^2}{(1 - \gamma)^2 \epsilon_1^2 (R_{0_{\text{max}}}^0)^2} \leq N_t(x, a),\]

which is implied by the stricter inequality

\[
\frac{\beta^2}{(1 - \gamma)^2 \epsilon_1^2 (R_{0_{\text{max}}}^0)^2} \leq \frac{R_{\text{max}}}{R_{\text{max}}^0 (1 - \gamma)^2 \epsilon_1},
\]

\[R_{\text{max}} \geq \frac{\beta^2}{\epsilon_1 R_{\text{max}}^0},\]

similarly to the previous case.

At step \(t\), a number of DP updates are carried out. We proceed by induction on the number of DP-updates. Initially, \(Q^{(0)}(x, a) \geq Q^*(x, a) - \epsilon_1 R_{0_{\text{max}}}^0\), then

\[
Q^{(i+1)}(x, a) = \sum_y \hat{P}_t(x, a, y)(\hat{R}_t(x, a, y) + \gamma V^{(i)}(y)) + \frac{V_{\text{max}}}{N_t(x, a)}
\]

\[
\geq \sum_y \hat{P}_t(x, a, y)(\hat{R}_t(x, a, y) + \gamma(V^*(y) - \epsilon_1 R_{0_{\text{max}}}^0)) + \frac{V_{\text{max}}}{N_t(x, a)}
\]

\[
\geq Q^*(x, a) - \beta/\sqrt{N_t(x, a)} - \gamma \epsilon_1 R_{0_{\text{max}}}^0 + \frac{V_{\text{max}}}{N_t(x, a)}
\]

\[
\geq Q^*(x, a) - \gamma \epsilon_1 R_{0_{\text{max}}}^0 - (1 - \gamma)\epsilon_1 R_{0_{\text{max}}}^0 = Q^*(x, a) - \epsilon_1 R_{0_{\text{max}}}^0,
\]

where we applied (3), (4) and the induction assumption. \(\blacksquare\)

Define the \(H\)-step truncated value function of policy \(\pi\) as \(Q^\pi(x, a, H) := E\left(\sum_{t=0}^{H} \gamma^t r_t \mid x=x_0, a=a_0\right)\).

**Lemma B.5** Let \(\epsilon > 0\) and consider an MDP \(M = (X, A, P, R, \gamma)\). If

\[H \geq \frac{1}{1 - \gamma} \log \frac{1}{\epsilon(1 - \gamma)},\]

then

\[Q^\pi(x, a, H) \leq Q^\pi(x, a) \leq Q^\pi(x, a, H) + \epsilon R_{\text{max}}^0\]

for any \((x, a) \in X \times A\).

**Proof.** Let \(\Xi(x, a)\) denote the set of infinite trajectories starting in \((x, a)\), and for any trajectory \(\xi \in \Xi(x, a)\), let \(\xi_H\) denote its \(H\)-step truncation. Furthermore,
denote the discounted total reward along a trajectory $\xi$ by $v(\xi)$. Clearly,

$$Q^\pi(x, a) = E_{\xi}[v(\xi)] = \sum_{\xi \in \Xi(x, a)} \Pr(\xi)v(\xi) \quad \text{and}$$

$$Q^\pi(x, a, H) = E_{\xi}[v(\xi_H)] = \sum_{\xi \in \Xi(x, a)} \Pr(\xi)v(\xi_H).$$

Fix a trajectory $p$, along which the agent receives rewards $r_1, r_2, \ldots$, for which

$$v(\xi_H) = \sum_{t=0}^{H-1} \gamma^t r_{t+1} \quad \text{and}$$

$$v(\xi) = \sum_{t=0}^{\infty} \gamma^t r_{t+1} = v(\xi_H) + \sum_{t=H}^{\infty} \gamma^t r_{t+1}.$$

It is trivial that $v(\xi) \geq v(\xi_H)$, as the additional terms are all nonnegative by assumption. On the other hand,

$$\sum_{t=H}^{\infty} \gamma^t r_{t+1} \leq \sum_{t=H}^{\infty} \gamma^t R_0^0 = \frac{\gamma H}{1 - \gamma} R_0^0,$$

which is smaller than $\epsilon R_0^0$ if $H \geq \log (1 - \gamma)/ \log \gamma$ (which follows from the assumption of the lemma and the inequality $-\log \gamma > 1 - \gamma$), that is,

$$v(\xi) \leq v(\xi_H) + \epsilon R_0^0.$$

As the relations hold for each trajectory in $\Xi(x, a)$, they hold for the expected value, too.

The following lemma tells that OIM and its modified version learn almost the same values with high probability.

**Lemma B.6** For any $\epsilon > 0$, $\delta > 0$,

$$\epsilon' := \frac{(1 - \gamma)^2}{|X|} \epsilon,$$

$$m \geq \frac{2}{\epsilon^2} \ln \frac{2}{\delta},$$

for any MDP $M$ and any $(x, a) \in X \times A$,

$$\left| Q_{m^{OIM}}^M(x, a) - Q_{OIM}^M(x, a) \right| \leq 2\epsilon R_0^0$$

with probability at least $1 - 2\delta$.

**Proof.** The model estimates of the two algorithm-variants are identical on not-yet-known states where the visit count is less than $m$. On known pairs, we can apply Lemma B.1 to both model-estimates to see that they are $\epsilon'$-close to the true model parameters with probability at least $1 - \delta$. Consequently, they are $2\epsilon'$-close to each other with at least $1 - 2\delta$ probability. Applying Lemma B.2 proves the statement of the lemma.

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Lemma B.7 Let $M = (X, A, P, R, \gamma)$ be an MDP, $K$ a set of state-action pairs, $M$ an MDP equal to $M$ on $K$ (identical transition and reward functions), $\pi$ a policy, and $H$ some positive integer. Let $A_M$ be the event that a state-action pair not in $K$ is encountered in a trial generated by starting from $(x, a)$ and following $\pi$ for $H$ steps in $M$. Then,

$$Q^*_M(x, a) \geq Q^*_\bar{M}(x, a) - \frac{R^0_{\max}}{1 - \gamma} \Pr(A_M).$$

(12)

Proof. The lemma is identical to Lemma A.8.

Theorem B.8 For any $\epsilon > 0$, $\delta > 0$, let

$$\epsilon_1 := \frac{\epsilon}{6},$$

$$\epsilon_2 := \frac{(1 - \gamma)^2}{|X|} \cdot \epsilon_1,$$

$$H := \frac{1}{1 - \gamma} \ln \frac{1}{\epsilon_1 (1 - \gamma)},$$

$$m := \frac{2}{\epsilon_2^2} \ln \frac{8}{\delta}.$$

OIM converges almost surely to a near-optimal policy in polynomial time if started with

$$R^0_{\max} = \frac{2R^0_{\max} \ln(2 |X| |A| m/\delta)}{\epsilon_1 (1 - \gamma)^2},$$

that is, with probability $1 - \delta$, the number of timesteps where $Q^{*OIM}(x_t, a_t) > Q^*(x_t, a_t) - \epsilon R^0_{\max}$ does not hold, is at most

$$\frac{2m |X| |A| H}{\epsilon_1 (1 - \gamma)} \ln \frac{4}{\delta}.$$

Remark B.9 When expressed in terms of MDP parameters, time requirement is

$$\frac{864 |X|^3 |A|}{\epsilon^3 (1 - \gamma)^4} \ln \frac{6}{\epsilon (1 - \gamma)} \ln \frac{4}{\delta} \ln \frac{8}{\delta}$$

$$= O \left( \frac{|X|^3 |A|}{\epsilon^3 (1 - \gamma)^4} \ln \frac{1}{\epsilon (1 - \gamma)} \ln \frac{1}{\delta} \right)$$

and the required initialization value is

$$R^0_{\max} = \frac{12R^0_{\max}}{\epsilon (1 - \gamma)^2} \ln \left( \frac{144 |X|^3 |A|}{\delta \epsilon (1 - \gamma)^4} \ln \frac{8}{\delta} \right)$$

$$= O \left( \frac{R^0_{\max}}{\epsilon (1 - \gamma)^2} \ln \left( \frac{|X|^3 |A|}{\delta \epsilon (1 - \gamma)^3} \ln \frac{1}{\delta} \right) \right).$$
Proof.

Let \( M \) denote the true (and unknown) MDP, let \( \hat{M} \) be the approximate model of OIM.

An \((x, a)\) pair is considered known if it has been visited at least \( m \) times. According to Lemma A.2, for a known pair \((x, a)\), the model estimates \( \hat{P}(x, a, \cdot) \) and \( \hat{P}(x, a, \cdot)\hat{R}(x, a, \cdot) \) are \( \epsilon_2 \)-close and \( \epsilon_2 R_\max^0 \) to the true values with probability at least \( 1 - \delta/4 \).

Define the MDP \( \bar{M} \) so that it is identical to \( M \) for known pairs, and equals \( \hat{M} \) for unknown pairs. The parameters of \( \hat{M} \) and \( \bar{M} \) are identical on unknown pairs and \( \epsilon_2 \)-close for known pairs (with probability \( 1 - \delta/4 \)), so, by Lemma A.3,

\[
|Q_{\pi}^\pi(x, a) - Q_{\bar{M}}^\pi(x, a)| < \epsilon_1 R_\max^0
\]

for any policy \( \pi \) and any \((x, a) \in X \times A\).

Let

\[
H := \frac{1}{1 - \gamma} \ln \frac{1}{\epsilon_1(1 - \gamma)}.
\]

By Lemma B.5,

\[
|Q_{\pi}^\pi(x, a, H) - Q_{\bar{M}}^\pi(x, a)| < \epsilon_1 R_\max^0
\]

holds for the \( H \)-step truncated value function for any \((x, a), \pi\).

Consider a state-action pair \((x_1, a_1)\) and a \( H \)-step long trajectory generated by \( \pi \). Let \( K \) be the set of known \((x, a)\) pairs and let \( A_M \) be the event that an unknown pair is encountered along the trajectory. Then, by Lemma B.7,

\[
Q_{\pi}^\pi(x_1, a_1) \geq Q_{\bar{M}}^\pi(x_1, a_1) - \frac{R_\max^0}{1 - \gamma} \Pr(A_M).
\]

By applying Lemma B.6 to \( \epsilon_1, \delta/4 \), we get that the above setting of \( R_\max^0 \) ensures that the original and the modified version of OIM behaves similarly:

\[
\left|Q_{\pi}^{\pi_{OIM}}(x, a) - Q_{\bar{M}}^{\pi_{OIM}}(x, a)\right| \leq 2\epsilon_1 R_\max^0
\]

with probability at least \( 1 - \delta/2 \). Furthermore, by Lemma B.4 (with \( \epsilon \leftarrow \epsilon_1 \) and \( \delta \leftarrow \delta/4 \)), the modified algorithm preserves the optimism of the value function with probability at least \( 1 - \delta/4 \):

\[
Q_{\pi}^{\pi_{OIM}}(x, a) > Q^*(x, a) - \epsilon_1 R_\max^0
\]

To conclude the proof, we separate two cases (following the line of thoughts of Theorem 1 in [SL]). In the first case, an exploration step will occur with high probability: Suppose that \( \Pr(A_M) > \epsilon_1 R_\max^0(1 - \gamma) \), that is, an unknown pair is visited in \( H \) steps with high probability. This can happen at most \( m |X| |A| \) times, so by Azuma’s bound, with probability \( 1 - \delta/4 \), all \((x, a)\) will become known after \( \frac{2m|X||A|H}{\epsilon_1(1-\gamma)} \ln \frac{4}{\delta} \) exploration steps.
On the other hand, if $\Pr(A_M) \leq \epsilon_1(1 - \gamma)R^0_{\text{max}}$, then the policy is near-optimal with probability $1 - \delta$:

$$Q_{\pi}^{\text{OIM}}(x_1, a_1) \geq Q_{\pi}^{\text{OIM}}(x_1, a_1, H)$$

$$\geq Q_{\pi}^{\text{OIM}}(x_1, a_1, H) - \frac{R^0_{\text{max}}}{1 - \gamma} \Pr(A_M)$$

$$\geq Q_{\pi}^{\text{OIM}}(x_1, a_1, H) - \epsilon_1R^0_{\text{max}} \geq Q_{\pi}^{\text{OIM}}(x_1, a_1) - 2\epsilon_1R^0_{\text{max}}$$

$$\geq Q_{\pi}^{\text{OIM}}(x_1, a_1) - 3\epsilon_1R^0_{\text{max}}$$

$$\geq Q_{\pi}^{\text{OIM}}(x_1, a_1) - 5\epsilon_1R^0_{\text{max}}$$

$$\geq Q^*(x_1, a_1) - 6\epsilon_1R^0_{\text{max}}$$

$$= Q^*(x_1, a_1) - \epsilon R^0_{\text{max}},$$

where we applied (in this order) the property that truncation decreases the value function; Eq. (15); our assumption; Eq. (14); Eq. (13); Lemma B.4 and the definition of $\epsilon_1$. ■

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