ZARISKI-LOCAL FRAMED $\mathbb{A}^1$-HOMOTOPY THEORY

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Abstract. For any (not necessarily perfect) field $k$ we obtain equivalences of $\infty$-categories

$$H^{fr, sp}(k) \simeq H^{fr, sp}_d(k) \text{ and } DM(k) \simeq DM_{zar}(k).$$

We also construct an equivalence of $\infty$-categories

$$H^{fr, sp}(S) \simeq H^{fr, sp}_d(S)$$

of group-like framed motivic spaces over a separated noetherian scheme $S$ of finite Krull dimension with respect to the Nisnevich topology at one side and the Zariski fibre topology $zf$ generated by the Zariski one and the trivial fibre topology (introduced by Druzhinin, Kolderup and Østvær) on the other side. Over a field, the Zariski fibre topology equals the Zariski topology and the result follows from the previous one. To prove it in the case of a general base scheme, we prove a localisation theorem for $H^{fr, sp}(-)$ employing the ideas from the proof of the affine localisation theorem for the trivial fibre topology by the first author, Kolderup and Østvær.

Introduction

It is known that the Nisnevich topology on the category of smooth schemes is very natural in the context of motivic homotopy theory. In particular, the constructions of Voevodsky motives [10, 34, 38, 41], unstable and stable Morel-Voevodsky’s motivic homotopy categories [20, 32, 35, 36], and framed motivic categories [22, 27, 29] are all based on Nisnevich sheaves of spaces. In this note, we show that the Zariski topology over a field and a modification of it over finite-dimensional separated noetherian schemes called the Zariski fibre topology lead to the same $\infty$-categories of Voevodsky motives $DM(S)$ and of the framed motivic spectra $SH^{fr}(S)$ [22]. Consequently, this allows us to improve the reconstruction theorem from [31] that states $SH(S) \simeq SH^{fr}(S)$, and write $SH(S) \simeq SH^{fr}_{zar}(S)$.

Regarding the categories of Voevodsky motives over a field, we prove that for any field $k$ the obvious fully faithful embedding is an equivalence

$$DM(k) \simeq DM_{zar}(k). \quad (0.1)$$

As shown in [39, Theorem 5.7] the strict homotopy invariance [39, Theorem 5.6] and the injectivity [39, Cor. 4.18, 4.19] theorems imply the isomorphism

$$H^n_{nis}(X, F) \simeq H^n_{zar}(X, F)$$

for an $\mathbb{A}^1$-invariant presheaf of abelian groups with transfers. While the latter isomorphism implies the categorical equivalence [7,4], and moreover, conversely, the categorical equivalence [0.4] together with the strict homotopy invariance theorem imply the isomorphism, the known proof of [31, Theorem 5.6] uses perfectness of $k$. Our proof of (0.1) does not use the assumption; in fact our proof is self-contained apart from the only external ingredient – the étale excision theorem, which is provided by [32] for all fields $k$.

0.1. We expect that the direct generalization of the equivalence (0.1) to positive-dimensional base schemes does not hold, because $DM_{zar}(S)$ most likely does not satisfy the Localisation Theorem, while $DM(S)$ does (see [10]). The Zariski fibre topology $zf$ on $Sm_S$ is defined to fix this problem of the Zariski topology. It coincides with the Zariski topology over fields and satisfies the Localisation Theorem over general bases. Concretely, this topology is generated by the Zariski topology and the

2020 Mathematics Subject Classification. 14F42.

Key words and phrases. stable motivic homotopy category, framed correspondences, Zariski topology, localisation theorem.
trivial fibre topology introduced in [17, Definition 3.1] by the first author, Kolderup and Østvær. The latter is the topology generated by the Nisnevich covers coming from the Nisnevich squares of the form

\[
\begin{array}{ccc}
X' \times_S (S - Z) & \longrightarrow & X' \\
\downarrow & & \downarrow \\
X \times_S (S - Z) & \longrightarrow & X
\end{array}
\]

where \(Z\) is a closed subscheme of the base.

Remark. Zariski fibre topology is the strongest subtopology of the Nisnevich topology on \(\text{Sch}_S\) that equals the Zariski topology over the residue fields. It is expected that the Zariski fibre topology is the weakest topology that contains Zariski topology and satisfies the Localisation Theorem for categories \(\text{DM}(S)\) or \(\text{SH}(S)\).

0.2. Recall that Voevodsky motives \(\text{DM}_1(S)\), motivic spectra \(\text{SH}_1(S)\) and framed motivic spectra \(\text{SH}^F_1(S)\) can all be defined via a general construction of taking the \(\infty\)-category of \(F^1\)-spectra in \(\text{H}^\bullet_\Sigma(S)\) – the \(\infty\)-category of \(\tau\)-sheaves on a given \(\infty\)-category of correspondences \(\text{Corr}_S\) (see Definition 3.40). In the case of \(\text{DM}(S)\) one takes \(\text{Corr}_S\) to be the category of finite correspondences [41] and in the case of motivic spectra one takes \(\text{Corr}_S = \text{Sm}_S\). In the case of framed motivic spectra one takes the \(\infty\)-category of framed correspondences [1, 7, 22].

We employ an axiomatic treatment of the localisation theorem, defining it as the property of a topology in the context of a given family of categories of correspondences (see Definition 3.45). This allows to apply our result both in the setting of Voevodsky motives and motivic spectra.

Example 0.2. The localisation property for the Nisnevich topology in the context of the category of framed correspondences is proved in [31], furthermore, it follows that it holds when \(\text{Corr}_S\) is taken to be \(\text{Sm}_S\) from the works of Ayoub [4, 36]. In the context of finite correspondences the property follows from Cisinski-Déglise's work [10]. In [17] the localisation property for the trivial fibre topology was proved, both in the context of framed correspondences and when \(\text{Corr}_S = \text{Sm}_S\).

In the present paper, based on the ideas of [17], we show that the localisation theorem also holds for the Zariski fibre topology. More generally, we prove the following:

Theorem A (Theorem 3.42 and 3.43). Let \(i: Z \to S\) be a closed immersion of affine schemes. Let \(\tau\) be a topology on \(\text{Sch}_S\) such that \(\tau \supset \mathfrak{f}\) and satisfies the additional technical assumption: \(w\tau = \tau\mathfrak{f}\) on \(\text{SmAff}_{S,Z}\) (see Definition 3.7).

Denote by \(H^\bullet_{\Sigma,r}(\text{SmAff}_S)\) one of \(H^\bullet_{\Sigma,r}(\text{SmAff}_S)\), \(H^\bullet_{\Sigma,r}(\text{SmAff}_S)\). Denote by \(H^\bullet_{\Sigma,r}(\text{SmAff}_S)\) one of \(H^\bullet_{\Sigma,r}(\text{SmAff}_S)\), \(H^\bullet_{\Sigma,r}(\text{SmAff}_S)\). Then we have adjunctions \(i^! \dashv i_*\), \(j^! \dashv j_*\), \(i^* \dashv i_*\), \(j^\# \dashv j^\ast\)

\[
\begin{array}{ccc}
H^\bullet_{\Sigma,r}(\text{SmAff}_Z) & \xrightarrow{i^\ast} & H^\bullet_{\Sigma,r}(\text{SmAff}_S) \\
\downarrow & & \downarrow \\
H^\bullet_{\Sigma,r}(\text{SmAff}_S) & \xleftarrow{i_*} & H^\bullet_{\Sigma,r}(\text{SmAff}_{S-Z})
\end{array}
\]

\[
\begin{array}{ccc}
H^\bullet_{\Sigma,r}(\text{SmAff}_Z) & \xrightarrow{j^\ast} & H^\bullet_{\Sigma,r}(\text{SmAff}_S) \\
\downarrow & & \downarrow \\
H^\bullet_{\Sigma,r}(\text{SmAff}_S) & \xleftarrow{j_*} & H^\bullet_{\Sigma,r}(\text{SmAff}_{S-Z})
\end{array}
\]

inducing a pullback square and, respectively, a pushout square

\[
\begin{array}{ccc}
i_*i^\ast F & \longrightarrow & F \\
\downarrow & & \downarrow \\
j_*j^\ast F & \longrightarrow & i_*i^\ast G
\end{array}
\]

\[
\begin{array}{ccc}
j^\#j^\ast G & \longrightarrow & G \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & i_*i^\ast G
\end{array}
\]
for any $F \in H_{\Sigma, \tau}^{fr}(\text{SmAff}_S)$ and for any $G \in H_{\Sigma, \tau}^{fr}(\text{SmAff}_S)$. In particular, the claim holds for $\tau$ being the Zariski fibre topology $\mathcal{Z} = \text{zar} \cup \text{tf}$, as well as for the Nisnevich topology and the trivial fibre topology $\text{tf}$.

As discussed above, using the étale excision theorem from [18, 19, 24], we prove the equivalence

$$H_{\text{nis}}^{fr, \text{gp}}(k) \simeq H_{\text{zar}}^{fr, \text{gp}}(k).$$

Combining this with the above localisation result, we obtain the following generalization to the relative setting.

**Theorem B.** For any noetherian separated scheme $S$ of finite Krull dimension, there is a canonical equivalence of categories

$$H_{\text{nis}}^{fr, \text{gp}}(S) \simeq H_{\text{zar}}^{fr, \text{gp}}(S).$$

**Remark 0.3.** It is likely true that the above equivalence holds as well in the context of other families of $\infty$-categories of correspondences, although we do not discuss it here. Some interesting examples to consider include $K$-correspondences, GW-correspondences [15, 28, 42], Milnor-Witt correspondences [5, 8, 11, 12] and finite $A$-correspondences in the sense of [16, 21].

**Structure of the text.** We start in Section 1.1 with a formalism of families of $\infty$-categories of correspondences $\text{Corr}_S$ and respective motivic $\infty$-categories

$$H_{\Sigma, \tau}^{fr}(S), \quad \mathbf{SH}_{\Sigma, \tau}^{S, fr}(S), \quad \mathbf{SH}_{\Sigma, \tau}^{fr}(S)$$

over base schemes $S \in \text{Sch}$. In the rest of Section 1 we discuss various properties of families of $\infty$-categories of correspondences and their consequences. In Section 2 we discuss general fibre topologies, that include the Zariski fibre topology $\mathcal{Z}$, Nisnevich topology and the trivial fibre topologies as examples. In Section 3 we prove the localisation theorem for $H_{\Sigma, \tau}^{fr}(S)$ for fibre topologies. In Section 4.1 we discuss sheaves that satisfy the property of being excisive with respect to varying topologies. In Lemma 4.10 we consider a pair of topologies $Z \subset N$, such that $Z$ have enough set of points, and $N$ is completely decomposable. We prove that a $Z$-sheaf is an $N$-sheaf whenever it is excisive with respect to $N$-squares on $Z$-points. In Section 4.2 we discuss the example given by the étale excision for $\mathcal{A}$-invariant framed presheaves. Finally, in Section 4.3 we prove Theorem B over fields and in Section 4.4 we deduce the result over base schemes $S \in \text{Sch}^{\text{aff}}$.

**Notation.** Throughout the paper we make use of the following conventions and notation.

1. $\text{Sch}$ is the category of schemes, and $\text{Aff}$ is the category of affine schemes.
2. $\text{Sch}^{\text{fin}}$, and $\text{Aff}^{\text{fin}}$ are the categories of noetherian separated schemes of finite Krull dimension, and affine noetherian schemes of finite Krull dimension.
3. $\text{Sch}_S$ and $\text{Aff}_S$ are the categories of schemes over a given scheme $S$.
4. $\text{Sm}_S$ and $\text{SmAff}_S$ are the full subcategories of $\text{Sch}_S$ and $\text{Aff}_S$ spanned by smooth $S$-schemes; $\text{Sm}_S^{\text{fr}}$ is the subcategory of $\text{SmAff}_S$ spanned by schemes whose tangent bundle is stably trivial.
5. For a scheme $S \in \text{Sch}$, we identify a point $z \in S$ with the spectrum of its residue field. We write $S_{(-)} = \text{Spec} \mathcal{O}_{S,z}$, and $S^b_{(-)} = \text{Spec} \mathcal{O}^b_{S,z}$.
6. For any $X \in \text{Sch}_S$, $z \in S$, and $S' \to S$, we write $X_z = X \times_S z$, and $X_{S'} = X \times_S S'$.
7. $\text{EssSm}_S$ is the category of essentially smooth schemes over $S$, namely, filtered limits in $\text{Sch}_S$ of diagrams in $\text{Sm}_S$ with étale affine transition maps. For a functor $F$ on $\text{Sm}_S$, denote by the same symbol the continuous functor on $\text{EssSm}_S$ given by $F(\lim_{\alpha} X_{\alpha}) \cong \lim_{\alpha} F(X_{\alpha})$.
8. For any presheaf on the small Zariski site over a scheme $X$, and a closed immersion $i : Z \hookrightarrow X$, $F(X_{Z})$ equals the global sections of the presheaf $i^{-1}F$ on $Z$.
9. Staring from Definition 1.4, we write $S_{(-)}$ for $S_S$, when $S \subset \text{Sch}$ is clear from the context.
10. We use the standard language of $\infty$-categories following [32]. We denote by $\text{Cat}^{\text{add}}_{\infty}$ the $\infty$-categories of small preadditive $\infty$-categories and $\text{Cat}^{\text{add}}_{\infty}$ is its subcategory consisting of additive $\infty$-categories. We denote by $(-)^{\text{gp}}$ the left adjoint to the obvious embedding functor $\text{Cat}^{\text{add}}_{\infty} \to \text{Cat}^{\text{add}}_{\infty}$. 


(11) $\text{PSh}(\mathcal{S})$ and $\text{PSh}^*(\mathcal{S})$ are the $\infty$-categories of presheaves of spaces or pointed spaces on $\mathcal{S}$. $\text{PSh}_C(\mathcal{S})$ and $\text{PSh}^*_{C}(\mathcal{S})$ are the subcategories of additive presheaves.

(12) Given an $\infty$-category denoted by $\text{Corr}(\mathcal{S})$, we write

$$\text{PSh}^{tr}(\mathcal{S}) = \text{PSh}(\text{Corr}(\mathcal{S})),$$

$$\text{PSh}^{tr}(\mathcal{S}) = \text{PSh}^*(\text{Corr}(\mathcal{S}))$$

when there is no confusion.

(13) We denote by $h^{tr}_h(X) \in \text{PSh}^{tr}(\mathcal{S}_h)$ the presheaf represented by $X \in \text{Corr}(\mathcal{S}_h)$. We write $h^{tr}(X)$ for $h^{tr}_h(X)$, when the base scheme is clear from the context.

(14) $h^{tr}(X)$ for $h^{tr}_h(X)$, when the base scheme is clear from the context.

(15) In Section 3.4, we use notation $PSh^{tr}(\mathcal{S}_h) = \text{PSh}(\text{Corr}^h(\mathcal{S}_h))$ in sense of Definition 3.25.

(16) $H^{tr}_{1\text{intr}}(-)$ or $H^{tr}_{1\text{intr}}(-)$ denotes the subcategory of $PSh^{tr}(-)$ or $PSh^{tr}(-)$ spanned by $\Delta^1$-invariant objects.

(17) Throughout the text we consider subcanonical topologies $\tau$ on categories $\mathcal{S}_h$: $\text{Shv}^{tr}_h(\mathcal{S}_h) = \text{Shv}_h(\text{Corr}(\mathcal{S}_h))$, that is the subcategory in $\text{PSh}^{tr}(\mathcal{S}_h)$ spanned by the objects that go to $\tau$-sheaves in $\text{PSh}(\mathcal{S}_h)$, and similarly for $\text{Shv}^{tr}_h(\mathcal{S}_h)$.

(18) $H^{tr}_h(-) = H^{tr}_h(\mathcal{S}_h)$, and similarly for $\text{PSh}^{tr}(\mathcal{S}_h)$.

(19) We write $\text{SH}^{\text{r}, \text{tr}}(\mathcal{S}_h) = \text{SH}^{\text{r}}(\text{Corr}(\mathcal{S}_h))$, $\text{SH}^{\text{r}}(\mathcal{S}_h) = \text{SH}^*(\text{Corr}(\mathcal{S}_h))$.

(20) We denote the corresponding localisation functors $L_h: \text{PSh}^{tr}(\mathcal{S}_h) \to \text{H}^{\text{tr}}(\mathcal{S}_h)$, $L_{\tau}: \text{PSh}^{tr}(\mathcal{S}_h) \to \text{Shv}^\tau(\mathcal{S}_h)$, and $L_{h, \tau}: \text{PSh}^{tr}(\mathcal{S}_h) \to \text{H}^{\tau}(\mathcal{S}_h)$, and similarly for pointed categories.

(21) We write $\text{PSh}(\mathcal{S})$ for $\text{PSh}(\text{Spec}(\mathcal{S}))$, where $\mathcal{S} \in \text{Sch}$, and write $\text{PSh}(\mathcal{R})$ for $\text{PSh}(\text{Spec}(\mathcal{R}))$ for a ring $\mathcal{R}$. We use similar notation for $\text{Shv}^\tau(\mathcal{S})$, $\text{H}^{\tau}(\mathcal{S})$, $\text{H}^{\tau}(\mathcal{S})$, $\text{SH}^{\tau}_{\text{tr}}(\mathcal{S})$, $\text{SH}^{\tau}_{\text{tr}}(\mathcal{S})$.

(22) We denote by $\tau \cup \nu$ the topology generated by topologies $\tau$ and $\nu$.

(23) We write $\text{PSh}^{\text{r}, \text{tr}}(\mathcal{S}) = \text{PSh}(\text{Corr}^{\text{r}, \text{tr}}(\mathcal{S}))$, when the $\infty$-category $\text{Corr}(\mathcal{S}_h)$ is as above is preadditive.

(24) $\text{Corr}^\tau(\mathcal{S}) = \text{Corr}^{\text{r}, \text{tr}}(\mathcal{S})$ is the $\infty$-category of framed correspondences over $\mathcal{S}$ (constructed in [22]). We write $\tau$ for $\tau$ in the above notation when $\text{Corr}(\mathcal{S}) = \text{Corr}^\tau(\mathcal{S})$.

1. Acknowledgement. The article combines two parts of research with separate support: (1) one part is dedicated to the localisation theorem for motives with preadditive transfers and the equivalence $\text{SH}^{\text{r}}(\mathcal{S}) \simeq \text{SH}^{\text{r}}(\mathcal{S})$, see Theorem 3.12 and Theorem 4.11. (2) the second part is dedicated to the localisation theorem for unpointed motivic homotopy categories with or without transfers, see Section 6.3.1, Theorem 6.4.3 and Corollary 6.4.4.

The research except Section 6.3.1 and Theorem 6.4.3 is supported by the Russian Science Foundation grant 20-41-04401 only. The first author is supported for Section 6.3.1 and Theorem 6.4.3 by a Young Russian Mathematics award, the second author is supported by the SFB 1085 “Higher Invariants” funded by the Deutsche Forschungsgesellschaft (DFG).

1. Families of categories and topologies

1.1. Categories of correspondences.

Definition 1.1. Let $\mathcal{S} \in \text{Sch}$, $\mathcal{S}_h$ be a monoidal subcategory of $\text{Sch}(\mathcal{S})$ with respect to the monoidal structure $\times_\mathcal{S}$. An $\infty$-category of correspondences $\text{Corr}_{\mathcal{S}}$ on $\mathcal{S}$ is an $\infty$-category $\text{Corr}_{\mathcal{S}} \in \text{Cat}_{\infty}$ with the pointed object $\emptyset \in \text{Corr}_{\mathcal{S}}$, and a given object $\text{pt}_{\mathcal{S}} \in \text{Corr}_{\mathcal{S}}$, and an action of the monoidal category $\mathcal{S}_h$,

$$\mathcal{S}_h \times \text{Corr}_{\mathcal{S}} \to \text{Corr}_{\mathcal{S}}; \quad (X, T) \mapsto (X \times_\mathcal{S} T),$$

such that the induced functor

$$\text{corr}_{\mathcal{S}}: \mathcal{S}_h \to \text{Corr}_{\mathcal{S}} \quad X \mapsto X \times \text{pt}_{\mathcal{S}}$$

(1.2) is essentially surjective. The correspondences $\text{Corr}_{\mathcal{S}}$ are called preadditive if the presheaves $h^\tau_{\mathcal{S}}(X) = \text{Corr}_{\mathcal{S}}(\mathcal{S}_h, X)$ on $\text{Sm}_{\mathcal{S}}$ are additive for all $X \in \text{Corr}_{\mathcal{S}}$.

Example 1.3. (i) The category $\text{Sch}^\mathcal{S}$ for $\mathcal{S} \in \text{Sch}$ is a category of correspondences over $\mathcal{S}$ in $\text{Sch}$ itself. (2) The $\infty$-category of framed correspondences $\text{Corr}_{\mathcal{S}}^{\text{r}, \text{tr}}$ from [22].
Denote by $- \times G_m : \text{PSh}(\text{Corr}_S) \to \text{PSh}(\text{Corr}_S)$ the direct image endofunctor induced by the endofunctor $- \times G_m$ on Corr$S$. Then the endofunctor $- \times G_m$ on Corr$S$ induces the endofunctors $\Omega_{G_m}$ and $\Sigma_{G_m}$ on PSh(Corr$S$)

$$\Omega_{G_m} F(X) \simeq \text{fib}(F(X \times G_m) \to F(X \times \{1\}), \quad \Sigma_{G_m} F = \text{cofib}(F \times \{1\} \to F \times G_m),$$

where $F \times \{1\} \simeq F$, and the right-side morphisms in (1.4) are induced by the embedding morphism $\{1\} \to G_m$. Functors (1.4) agree with the $G_m$-loop and $G_m$-suspension on PSh(Sm$S$) in the sense of natural equivalences

$$\Sigma_{G_m} r \simeq r^* \Sigma_{G_m}, \quad \Omega_{G_m} r \simeq r_* \Omega_{G_m}. \quad (1.5)$$

**Definition 1.6.** Let Corr$S$ be an $\infty$-category of correspondences on $S_S$ over $S \in \text{Sch}$ such that $\Sigma^1_S \in S_S$, and let $\tau$ be a topology on $S_S$. A presheaf $F \in \text{PSh}(\text{Corr}_S)$ is called $\tau$-sheaf, for a topology $\tau$ on $S_S$ (resp. $\Sigma^1$-invariant), if it goes to the object of such type along the forgetful functor $\text{PSh}(\text{Corr}_S) \to \text{PSh}(S_S)$. Define $H^0_{\Sigma^1,\tau}(S_S) = H_{\Sigma^1,\tau}(\text{Corr}_S)$ as the subcategory spanned by $\Sigma^1$-invariant $\tau$-sheaves in $\text{PSh}_{\Sigma^1}(\text{Corr}_S)$, and $H^*_{\Sigma^1,\tau}(S_S) = H^*_{\Sigma^1,\tau}(\text{Corr}_S)$ for pointed ones,

$$\text{SH}^1_{\Sigma^1,\tau}(\text{Corr}_S) = H_{\Sigma^1,\tau}(\text{Corr}_S)[(\Sigma^1)^{-1}], \quad \text{SH}^\infty_{\Sigma^1,\tau}(\text{Corr}_S) = \text{SH}^\infty_{\Sigma^1,\tau}(\text{Corr}_S)[(\Sigma^1_m)^{-1}].$$

A family of $\infty$-categories over a subcategory $S$ in the category $\text{Sch}$, is a functor $S \to \text{Cat}_\infty$. Denote by Sm$S$ the contravariant functor

$$\text{Sm}_S : S \to \text{Cat}; S \mapsto \text{Sm}_S.$$

**Definition 1.7.** Let $S$ be a subcategory of $\text{Sch}$, and $S_S : S \to \text{Cat}$ be a functor equipped with a natural embedding $S_S \hookrightarrow \text{Sch}_S$. A family of $\infty$-categories of correspondences Corr$S$ on $S_S$ is a contravariant functor

$$\text{Corr}_S : S \to \text{Cat}_\infty; S \mapsto \text{Corr}_S; f \mapsto f^\star \quad (1.8)$$

equipped with a natural structure of an $\infty$-category of correspondences over $S$ for each $S \in S$, and a natural isomorphism

$$f^\star(-) \times_{S'} (X \times_S S') \simeq f^\star(- \times_S X),$$

for each morphism $f : S' \to S$, and $X \in S_S$. We write $S_{(-)}$ for $S_S$, when $S$ is clear from the context, and write Corr$_{S_{(-)}}$ or Corr$_{(-)}$ for Corr($S_S$).

Given a family of topologies $\tau$ on $S_S$ as above that is compatible with the base change functors, Definition 1.6 gives the respective families of $\infty$-categories.

**Definition 1.9.** An $\infty$-category $A$ is called preadditive if it admits a zero object and the map

$$X \coprod Y \to X \times Y$$

is an equivalence for any objects $X, Y \in A$. In this case, it makes sense to talk about the direct sum of objects, which we denote by $X \oplus Y$.

Preadditive categories are canonically enriched over $\mathbb{E}_\infty$-monoids (see [30 Proposition 2.3]). An $\infty$-category $A$ is called additive if all mapping spaces are group-like with respect to the $\mathbb{E}_\infty$-monoid structure.

**Example 1.10.**

1. The 2-category of spans of morphisms of $G$-sets Span$_G$ is preadditive, [8 Def. 5.7].

2. Various $\infty$-categories of correspondences in motivic homotopy theory are preadditive. In particular, Corr$^p_S$ is preadditive, see [22], as well as the discrete category Corr$S$ from [41].

**Definition 1.11.** If the functor (1.8) lands in Cat$^\text{padd}$ or Cat$^\text{add}_\infty$ the family of $\infty$-categories of correspondences will be called additive or preadditive. For a preadditive family of $\infty$-categories of correspondences Corr$S$, we denote by Corr$^p_S$ the corresponding additive one.
1.2. Continuous families. Let $\mathbf{S} \subset \text{Sch}$, $\mathbf{S}_S \subset \text{Sch}_S$ be subcategories closed with respect to filtered limits and coproducts and such that for any $S \in \mathbf{S}$, $z \in S$, $S(z) \in \mathbf{S}$, and there is a filtered system of Zariski neighbourhoods $U_\alpha \in S$ of $z$ in $S$ such that $\lim_{\alpha} U_\alpha = S(z)$, and any scheme in $\mathbf{S}_S(z)$ equals $X_{S(z)} = X \times U_\alpha S(z)$ of a scheme $X \in \mathbf{S}_S$ for some $U_\alpha$.

**Definition 1.12.** A family of correspondences $\text{Corr}(\mathbf{S}_S)$ satisfies the property (Embed), if for any open immersion $U \to S$ in $\mathbf{S}$, there is a fully faithful functor $j_\# : \text{Corr} U \to \text{Corr} S$ that is left adjoint to $j^* : \text{Corr} S \to \text{Corr} U$.

**Definition 1.13.** Given a point $z \in S$, for $S \in \mathbf{S}$, and $X, Y \in \mathbf{S}_S$, consider the morphism in $\text{Spc}$,

$$\text{Corr}_{S(z)}(X \times_S S(z), Y \times_S S(z)) \leftarrow \lim_{\alpha} \text{Corr}_{U_\alpha}(X \times S U_\alpha, Y \times S U_\alpha), \quad (1.14)$$

induced by inverse image functors along the morphisms $S(z) \to U_\alpha$, where $U_\alpha$ runs over the filtered set of Zariski neighbourhoods of $z$. We say that $\text{Corr}_S$ is continuous, if it satisfies (Embed) and for any $S \in \mathbf{S}$, and a point $z \in S$, for any $X, Y \in \mathbf{S}_S$, the morphism (1.14) is an isomorphism in the $\infty$-category $\text{Spc}$.

**Example 1.15.** (0) The family of the categories $\text{Sch}(\_\_)$ over $\text{Sch}$ satisfies (Embed). (1) The family of the $\infty$-category of correspondences $\text{Corr}_S(\_\_)$ satisfies (Embed).

**Definition 1.16.** Let $\tau$ be a family of topologies on the categories $\mathbf{S}_S$. We say that $\tau$ satisfies the property (Embed), if for any open immersion $U \to S$ in $\mathbf{S}$, the canonical embedding functor

$$\tau_U : \pi \to \tau_S$$

preserves and detects $\nu$-coverings. In other words, a morphism $\tilde{X} \to X$ in $\tau_U$ is a $\tau$-covering, if its image $\tilde{X} \to X$ in $\tau_S$ is a $\tau$-covering.

**Lemma 1.17.** Let $\tau$ be a family of topologies on $\mathbf{S}_S$ that has the property (Embed). Then for any open immersion of schemes $j : U \to S$ in $\mathbf{S}$, the functor

$$j^* : \text{PSh}(\mathbf{S}_S) \to \text{PSh}(\mathbf{S}_U)$$

preserves $\tau$-sheaves.

**Proof.** Let $F \in \text{Sh}_\tau(\mathbf{S}_S)$. Let $v : \tilde{X} \to X$ be a $\tau$-covering in $\tau_U$. Then $v$ is a $\tau$-covering in $\tau_S$. Denote by $h(X)$ the representable presheaves, and by $h_{\tilde{X}}(X)$ the covering sieves in both categories $\mathbf{S}_S$ and $\tau_U$. Then $j^*(h(X)) = h(X)$, and $j^* h_{\tilde{X}}(X) = h_{\tilde{X}}(X)$. The sequence of isomorphisms

$$\text{Map}_{\text{PSh}(\mathbf{S}_U)}(h_{\tilde{X}}(X), j^* F) \simeq \text{Map}_{\text{PSh}(\mathbf{S}_S)}(h_{\tilde{X}}(X), F) \\simeq \text{Map}_{\text{PSh}(\mathbf{S}_S)}(h(X), F) \simeq \text{Map}_{\text{PSh}(\mathbf{S}_S)}(h(X), j^* F)$$

implies that $j^* F$ is a $\tau$-sheaf. \qed

**Definition 1.18.** Let $\tau$ be a family of topologies on $\mathbf{S}_S$. We say that $\tau$ is continuous if $\tau$ satisfies (Embed), and for any $S \in \mathbf{S}$, $X \in \mathbf{S}_S$, point $z \in S$, and $\nu$-covering $\tilde{X}_{S(z)} \to X_{S(z)}$, where $X_{S(z)} = X \times S(z)$, there is a Zariski neighbourhood $U$ of $z$ and a $\nu$-covering $\tilde{X}_U \to X \times S U$ such that the induced morphism $\tilde{X}_U \times S U S(z) \to X_{S(z)}$ is a refinement of $\tilde{X}_{S(z)} \to X_{S(z)}$.

**Example 1.19.** The Zariski, Nisnevich, and the trivial fibre topologies are continuous in sense of Definition 1.18.

**Lemma 1.20.** Let $\tau$ be a continuous family of topologies on $\mathbf{S}_S$. Then the functor $j^* : \text{PSh}(\mathbf{S}_S) \to \text{PSh}(\mathbf{S}_{S(z)})$ preserves $\tau$-sheaves for each point $z$ of a scheme $S \in \mathbf{S}$.

**Proof.** Any scheme in $\mathbf{S}_{S(z)}$ equals $X_{S(z)} = X \times S(z)$ of a scheme $X \in \mathbf{S}_S$. Given a $\tau$-covering $\tilde{X}_{S(z)} \to X_{S(z)}$, by Definition 1.18, there is a $\tau$-covering $\tilde{X}_V \to X \times S V$, for an Zariski neighbourhood $V$ of $z$ such that $\tilde{X}_{S(z)} = \tilde{X}_V \times V S(z) \to X_{S(z)}$ is a refinement of $\tilde{X}_{S(z)} \to X_{S(z)}$. If $F \in \text{PSh}(\mathbf{S}_S)$ is a $\nu$-sheaf, then for any Zariski neighbourhood $U$ of $z$ in $V$,

$$F(X \times \tau_U) \simeq F(C(\tilde{X}_V \times \tau U)).$$
Hence for a filtered system of $U_\alpha$ such that $\varprojlim \alpha U_\alpha = S(\cdot)$, we have

$$j^*F(X_{S(\cdot)}) \simeq \varprojlim \alpha F(XV \times U_\alpha) \simeq \varprojlim \alpha F(\tilde{C}(XV \times U)) \simeq j^*F(\tilde{C}(X_{S(\cdot)})).$$

Thus $L_\nu j^*F \simeq F$, and consequently $j^*F$ is a $\tau$-sheaf.

\[ \square \]

**Lemma 1.21.** Let $S \in S$, and $S(0)$ denote the union of generic points of $S$, and $U_\alpha$ be a filtered system of Zariski neighbourhoods, such that $\varprojlim \alpha U_\alpha = S(0)$. Denote $j_\alpha : U_\alpha \to U$, $j : S(0) \to S$. The following natural equivalence of the endofunctors on $\mathrm{PSh}(\text{Shv}_S)$ holds

$$\varprojlim \alpha (j_\alpha)_*(j_\alpha)^*(F)(X) \simeq \varprojlim \alpha F(X \times S U_\alpha) \simeq F(X \times S S(0)) \simeq j_* j^*(F)(X).$$

So equivalence (1.22) for $\mathrm{PSh}(\text{Shv}_S)$ follows. By Lemma 1.20 the functors $j^*$ and $j_*$ on $\mathrm{PSh}(\cdot)$ induces the ones on $\mathrm{Shv}_S(\cdot)$ by the restriction along the embedding $\mathrm{Shv}_S(\cdot) \to \mathrm{PSh}(\cdot)$. So equivalence (1.22) for $\text{Shv}_S(\cdot)$ follows from the one for $\mathrm{PSh}(\cdot)$. Equivalence (1.22) for $\text{Shv}_S(\cdot)$ holds as well because of the canonical equivalence

$$r_* j_* \simeq j^* r_* ,$$

where $r : S_S \to \text{Corr}(S_S)$, $r_* : \text{Shv}_S^r (S_S) \to \text{Shv}(S_S)$, for any $\text{Corr}(\cdot)$ that is continuous in sense of Definition 1.13.

\[ \square \]

1.3. Lifting properties with respect to the affine henselian pairs. We discuss the lifting property for discrete presheaves on $\text{EssSmAff}_S$, $S \in \text{Sch}$, with respect to affine henselian pairs, and apply this to framed correspondences.

**Definition 1.23.** We say that a discrete presheaf of sets $c$ on $\text{Aff}_S$ has the lifting property with respect to affine henselian pairs, whenever for any $X \in \text{Aff}_S$ and a closed subscheme $Y \subset X$, the morphism

$$c(X_Y^h) \to c(Y)$$

is surjective.

**Lemma 1.25.** For any $X \in \text{Aff}_S$, a closed subscheme $Y \subset X$, and a closed subscheme $W \subset X$, the closed immersion $Y \hookrightarrow W$ $W^h_{Y \cap W} \leftarrow Y$ is a henselian pair, and the closed immersion $X^h_Y \hookrightarrow Y \hookrightarrow W$ is a henselian pair.

**Proof.** The claims follow from the universal property of the henselisation.

\[ \square \]

**Lemma 1.26.** Suppose a discrete presheaf of sets $c$ on $\text{Aff}_S$ has the lifting property with respect to affine henselian pairs. Then for any $X \in \text{Aff}_S$, a closed subscheme $Y \subset X$, and a closed subscheme $W \subset X$, the morphism

$$c(X^h_Y) \rightarrow c(Y \hookrightarrow W \rightarrow W^h_{Y \cap W})$$

is surjective.

**Proof.** Applying the surjectivity of (1.22) to $X$ being $X^h_Y$, and $Y$ being $Y \hookrightarrow W \rightarrow W^h_{Y \cap W}$ by Lemma 1.25, we get the claim.

\[ \square \]

For any $U \in \text{Sch}$, we consider the functor

$$\text{sSet} \to \text{Aff}_U; K \mapsto K_U$$

that is the left extension of the functor $\Delta : \text{Aff}_U$ given by the cosimplicial object $\Delta^*_U$. 
Corollary 1.27. Suppose a discrete presheaf of sets $c$ on $\text{Aff}_S$ has the lifting property with respect to affine henselian pairs in sense of Definition 1.23 and satisfies closed gluing on $\text{Aff}_S$. Let $U \in \text{SmAff}_S$, and $Y \subset U$ be a closed subscheme. Then the morphism of simplicial sets
$$c((\Delta^n)_{U}^h) \to c((\Delta^n)_{Y}^h)$$
is a trivial fibration, where $(\Delta^n)_{U}^h$ denotes the cosimplicial object with terms $(\Delta^n)_{U}^h(\Delta^n)_{Y}^h$.

Proof. Let $K \to N$ be an injection of simplicial sets. Then there are the induced closed immersions $K_U \to N_U$, $N_Y \to N_U$, $K_Y \to K_U$, see Section 0.2. Then since $c$ satisfies closed gluing, the morphism of sets
$$c((\Delta^n)_{U}^h)^N \to c((\Delta^n)_{Y}^h)^K \times (c((\Delta^n)_{Y}^h)^N$$
equals
$$c((N_U)^h) \to c((K_U)^h) \times c(N_Y) \cong c((K_U)^h \amalg_K (N_Y)),$$
where $(N_U)^h = (N_U)^h_{N_Y}$, $(K_U)^h_{K_Y} = (K_U)^h_{K_Y}$. The last morphism is surjective by Lemma 1.26 applied to $X = N_U$, $Y = N_Y$, $W = K_U$.

Example 1.28. The following presheaves satisfy the lifting property with respect to affine henselian pairs:

0) The representable presheaves of the category $\text{SmAff}_S$, $S \in \text{Aff}$.

1) The presheaves $\text{Fr}_{-}((-,-)X$ for $X \in \text{SmAff}_S$ of framed correspondences from [25, 40], see Lemma 1.21.

2) The presheaves of normally framed correspondences $h^{nfr}(X)$ for $X \in \text{SmAff}_S$ defined in [22] and [23] because of the representability by the smooth affine scheme for smooth affine $X$.

3) The presheaves $h^{fr}(X) = \text{Fr}^{fr, id}((-,-)X$ for $X \in \text{SmAff}_S$ defined in [14, Definition 7] by the respective representability result.

Definition 1.29. A family of correspondences $\text{Corr}(\text{EssSmAff}_S)$ satisfies the property (AHP)(1), if for any $S \in \text{Aff}$, and $X \in \text{EssSmAff}_S$ there is a discrete presheaf of sets $c$ on $\text{EssSmAff}_S$ such that $c$ satisfies the lifting property with respect to affine henselian pairs in sense of Definition 1.23, and there is an isomorphism in $\text{PSh}(\text{EssSmAff}_S)$
$$L_{\mathbb{A}^1;}h^{fr}(X) \cong L_{\mathbb{A}^1;}c$$
in $\text{PSh}(\text{EssSmAff}_S)$.

Example 1.30. (0) The family of the categories $\text{EssSmAff}_{\mathbb{A}^1}$ satisfies (AHP)(1) by Example 1.28(0).

(1) The family of the $\infty$-category of correspondences $\text{Corr}^E(\text{EssSmAff}_{\mathbb{A}^1})$ satisfies (AHP)(1), because by Corollary 2.3.25 it follows the equivalence $L_{\mathbb{A}^1;}h^{fr}(E) \cong L_{\mathbb{A}^1;}h^{fr}(E)$ on the category $\text{EssSmAff}_S$, while $h^{fr}(E)$ satisfies the lifting property with respect to affine henselian pairs by Example 1.28(2). Similar argument holds with the use any one of Example 1.28(1), and Example 1.28(3) as well.

Definition 1.31. A family of $\infty$-categories of correspondences $\text{Corr}(\text{EssSmAff}_{\mathbb{A}^1})$ satisfies the property (AHP)(2), if for any $S \in \text{Aff}$, and a closed subscheme $Z$, such that $S_X \simeq S$,
$$\text{Corr}_{S}(S, X^h_Z) \simeq \text{Corr}_{S}(S, X)$$
for any $X \in \text{EssSmAff}_S$.

Example 1.32. (0) The family of the categories $\text{EssSmAff}_{\mathbb{A}^1}$ satisfies (AHP)(2).

(1) The family of the $\infty$-category of correspondences $\text{Corr}^E(\text{EssSmAff}_{\mathbb{A}^1})$ satisfies (AHP)(2). Indeed, the spaces $\text{Corr}^E_{S}(U, X)$ are the spaces of the data
$$U \xleftarrow{f} Z \xrightarrow{\tau} X, \quad \tau: \mathcal{L}_f \simeq 0,$$
(1.33)
where $f$ is finite flat lci, and $\mathcal{L}_f$ is the cotangent complex of $f$ in the K-theory $\infty$-groupoid $K(Z)$. So the claim follows by point (0) applied to morphisms $v$ in $\text{EssSmAff}_S$ from [1833].
Definition 1.34. A family of ∞-categories of correspondences $\text{Corr}(\text{EssSmAff}_{\text{AR}})$ satisfies the property $(\text{AHP})$, if it satisfies both $(\text{AHP})$,(1), and $(\text{AHP})$,(2).

1.4. Localisation property. In this section,

- $S$ is a category of schemes that contains all open and all closed subschemes of any $S \in S$.
- $\text{Corr}(S_S)$ is a continuous family of $\infty$-categories of radditive correspondences in sense of Definition 1.13.
- $\tau$ is a continuous family of topologies on $S_S$ in sense of Definition 1.18.

Definition 1.35. We say that $\tau$ satisfies the localisation property with respect to $\text{Corr}(S_S)$, if for any $S \in S$, a closed immersion $i : Z \to S$, and the open embedding $j : S - Z \to S$, for any $F \in H^*_S(S_S)$, the canonical sequence

$$i_*i^*F \to F \to j_*j^*F,$$

provided by the adjunctions $i_* \dashv i^*, j^* \dashv j_*$,

$$H^*_S(Z) \rightleftarrows H^*_S(S) \rightleftarrows H^*_S(S - Z),$$

is a fibre sequence, see Section 5 for the definitions of $i_*, i^*, j_*, j^*$.

Lemma 1.36. Suppose $\text{Corr}(S_S)$ is preadditive, and suppose that $\tau$ satisfies the localisation property with respect to $\text{Corr}(S_S)$. Let $\nu$ be a continuous family of topologies that contains $\tau$. Then for any $S \in S$ of finite Krull dimension, the functor

$$SH^S_{\Sigma, \tau}(S_S) \to \prod_{\alpha \in S} SH^S_{\Sigma, \tau}(S_{z}); F \mapsto (i^*_z j^* F)_{z \in S},$$

where $j : S_z \to S$, $i_z : z \to S_z$, detects $\nu$-sheaves, and similarly for $H^*_S(\Sigma)$, and $SH^S_{\Sigma, \tau}(-)$.

Proof. The claim is tautological for fields. Assume the claim for all base schemes of dimension less than $\dim S$. Let $F \in SH^S_{\Sigma, \tau}(S_S)$ whose image in $SH^S_{\Sigma, \tau}(S_z)$ is a $\nu$-sheaf for each $z \in S$. Consider the cofiltered system of closed subschemes $Z_\alpha$ of positive codimension in $S$. Then

$$(i_\alpha)_*(i_\alpha)^*F \simeq \text{fib}(F \to (j_\alpha)_*(j_\alpha)^*F),$$

where $j_\alpha : S - Z_\alpha \to S$, and $i_\alpha : Z_\alpha \to S$ are the canonical embeddings. Then

$$\lim_\alpha (i_\alpha)_*(i_\alpha)^*F \simeq \text{fib}(F \to \lim_\alpha (j_\alpha)_*(j_\alpha)^*F), \quad (1.37)$$

The projective limit of the schemes $S - Z_\alpha$ equals the union of generic points of $S$. Denote it by $S^{(0)}$ and by $j : S^{(0)} \to S$ the canonical embedding, then by Lemma 1.21

$$\lim_\alpha (j_\alpha)_*(j_\alpha)^*F \simeq j_*j^*F.$$

By the assumption it follows that $(i_\alpha)_*(i_\alpha)^*F$, and $j_*j^*F$ are $\nu$-sheaves. By (1.37) it follows that

$$F \simeq \text{fib}(j_*j^*F \to \lim_\alpha i_*i^*F[1]).$$

Hence $F$ is a $\nu$-sheaf. Thus the claim for $SH^S_{\Sigma, \tau}(-)$ is proved. The claim on $H^*_{\Sigma, \tau}(\Sigma)$ follows, because of the embedding $H^*_{\Sigma, \tau}(\Sigma) \hookrightarrow SH^S_{\Sigma, \tau}(-)$. The claim on $SH^S_{\Sigma, \tau}(-)$ follows, because the functor

$$\prod_{\ell \in Z} \Omega^\ell : SH^S_{\Sigma, \tau}(S_S) \to \prod_{\ell \in Z} SH^S_{\Sigma, \tau}(S_S)$$

detects $\nu$-sheaves.

---

2 affine henselian pairs
2. Zariski fibre topology

Throughout this section

- $S_{(-)}$ is any one from the list $\text{Sch}_{(-)}$, $\text{Aff}_{(-)}$, $\text{Sm}_{(-)}$, $\text{SmAff}_{(-)}$,
- $S = \text{Sch}^\text{ins}$, $\text{Aff}^\text{fin}$.

Let $\nu$ be a family of subtopologies of the étale topology on the family of categories $S_S$.

**Definition 2.1.** Given a family of subtopologies $\nu$ of the étale topology on $S_S$, define the family of subtopologies $\nu f$ of the étale topology on $S_S$ as the strongest one such that for each reduced zero-dimensional scheme $z \in S$ the topology $\nu f$ on $S_z$ is contained in $\nu$.

**Remark 2.2.** A $\nu f$-covering $\tilde{X} \to X$ in $S_S$ for $S \in S$ is a Nisnevich covering that base change along the morphism $z \to S$ for each point $z \in S$ is a $\nu$-covering.

**Example 2.3.** Let $zf$ be the topology $\tilde{X} \to X$ such that the base change $z \to S$ is a $\nu f$-refinement of $\nu$-covering over $S$ for each point $z \in S$ is a $\nu$-covering.

Let us recall the definition of so called $tf$-topology, the trivial fibre topology introduced in [17, Definition 3.1].

**Definition 2.4.** (1) We say that a morphism $v: \tilde{X} \to X$ in $S_S$ is a $tf$-covering, if it is étale affine, and for each $z \in S$, there is a morphism $X \times_S z \to \tilde{X}$ that composite with $w$ equals the morphism $X \times_S z \to X$.

(2) We define a $tf$-square to be a pullback square of the form

$$
\begin{array}{ccc}
U'' & \to & X' \\
\downarrow & & \downarrow f \\
U & \to & X \\
\end{array}
$$

where $f$ is étale affine, and there is a closed subscheme $Z$ in $S$ such that $U = X - Y$, and $Y \times_X X' \simeq Y$, where $Y = Z \times_S X$, and $j$ is the open immersion.

**Remark 2.5.** It is shown in [17, §3] $tf$-squares from point (2) define a regular bounded cd-structure on $\text{Aff}_S$ or $\text{Sch}_S$. For $S \in \text{Sch}^\text{ins}$, the induced completely decomposable topology coincides with the topology defined by $tf$-coverings from point (1) similarly to Nisnevich topology [36, Prop. 3.4.1].

According to [17, Remark 3.7] the $tf$-topology on $\text{Aff}_S$ over affine $S$ is the strongest subtopology of the Nisnevich topology that is trivial over the residue fields. We prove the following general statement.

**Lemma 2.6.** Suppose that $\nu$ is continuous, and furthermore, for any $S \in S$, $X \in S_S$, $z \in S$, and a $\nu$-covering $u: Y \to X_z$ over $z$, there is a $\nu$-covering $v: \tilde{X} \to X_{S_z}^b$ over $S_z^b$ that base change $v_z: \tilde{X}_z \to X_z$ along the morphism $z \to S_z^b$ is a $\nu$-refinement of $u$. Then $\nu f = \nu \cup tf$.

**Proof.** By Definition 2.4, we have $\nu \cup tf \subset \nu f$. We prove the converse implication by the induction on $\dim S$. The claim is trivial for $S = \emptyset$, and suppose that claim for all because schemes of dimension less then $\dim S$. Given a $\nu f$-covering $v: \tilde{X} \to X$ over $S$, we are going to show that there is a $\nu f$-refinement of $v$ that is a $\nu \cup tf$-covering. Since and Nisnevich covering of $S$ is a $tf$-covering over $S$, and since $\nu$ is continuous, without loss of generality we can assume that $S = S_z^b$ for $z \in S$.

Consider the $\nu$-covering $v_z: \tilde{X}_z \to X_z$. By assumption, there is a $\nu$-covering $\tilde{X}' \to \tilde{X}$ such that the morphism $\tilde{X}'_z \to X_z$ is a $\nu$-refinement of $v_z$. Then the morphism

$$
\tilde{X}' \times_X \tilde{X} \to \tilde{X}'
$$

is a $\nu f$-covering over $S$, and consequently it is étale. The base change of $\tilde{X}' \to S$ equals $\tilde{X}'_z \times_X \tilde{X}_z \to \tilde{X}_z$ and has a section provided by the morphism $\tilde{X}'_z \to \tilde{X}_z$. Then there is an open
subscheme $V \subset \tilde{X}' \times_X \tilde{X}$ such that the morphism $V \to \tilde{X}'$ is an étale neighbourhood of $\tilde{X}'$. So the morphism

$$w: W = V \cup (\tilde{X}' \times (S - z)) \to \tilde{X}'$$

is a tf-covering. The morphism $\tilde{X} \times_S V \to V$ is a tf-covering, because it is étale and admits a section given by $V \to \tilde{X}' \times_X \tilde{X} \to \tilde{X}$. The morphism $\tilde{X} \times_S (\tilde{X}' \times (S - z)) \to (\tilde{X}' \times (S - z))$ is a $\nu$ tf-covering by the inductive assumption since $\dim(S - z) < \dim S$. Thus the morphisms in the sequence

$$\tilde{X} \times_S W \to W \to \tilde{X}' \to X$$

are $\nu$ tf-coverings, while the morphisms in the sequence

$$\tilde{X} \times_S W \to \tilde{X} \to X$$

are $\nu$ coverings. So the claim follows because the composites of (2.8) and Equation (2.9) are equal. □

**Example 2.10.** $zf = zar \cup tf$.

### 3. Localisation Theorem for $\text{SH}_{zf}^\mu(\text{SmAff}_S)$.

In the section, generalizing the localisation theorems from [4, 17, 31, 36], we prove a localisation theorem for $\infty$-categories $\text{H}^\bullet_S(\text{SmAff}_S)$ and $\text{H}^\mu_S(\text{SmAff}_S)$ for some class of topologies $\tau$ on $\text{Sm}_S$ that includes tf, and $zf$, and $nis$.

The proof is based on the principles of [17] that covers the case $\tau = tf$. In fact, in this case [17] proves two results [17, Theorem 11.2, Theorem 11.3], that in combination are much stronger and more involved than the discussed here Localisation Theorem. Though we think that a similar enhancement of the Localisation Theorem holds for all topologies $\tau$ considered here, we do not study it here, since classical form of Localisation Theorem is enough for the purposes of the text.

This restriction helps us keep the text short and self-contained at the same time. The presented here general argument is formally independent at least from the most complicated results of [17] including the proof of the Localisation Theorem, and from more classical proofs of Localisation Theorems. Throughout this section

- $S$ is Aff, or Aff$^\infty$.
- Corr$(-) = \text{Corr}(\text{Sch}_S)$ is a family of $\infty$-categories of radditive correspondences that satisfies (AHP) in sense of Definition [1.31] and satisfies the closed gluing on affine schemes.
- $S \in \text{S}$, and $Z$ is a closed subscheme of $S$.

#### 3.1. Following [17], for a given scheme $X \in \text{SmAff}_S$, denote $X_Z = X \times Z$, denote $X_{S - Z} = X \times (S - Z)$, and denote by $X^h_Z = X^{h\text{hs}}_Z$, the henselisation of $X$ along $X_Z$.

As in [17], we consider the subcategory $\text{Aff}_{S,Z}$ in $\text{Aff}_S$ spanned by the schemes of the form $X^h_Z$ for all $X \in \text{Aff}_S$. For any subcategory $\mathcal{S}_S$ in $\text{Aff}_S$, define $\mathcal{S}_{S,Z}$ in $\text{Aff}_{S,Z}$ as the subcategory in $\text{Aff}_{S,Z}$ spanned by the objects $X^h_Z$, where $X \in \mathcal{S}_S$.

For any $X, Y \in \mathcal{S}_{S,Z}$, we write $X \times_{S,Z} Y$ for $(X \times_S Y)^h_Z$.

**Definition 3.1.** Let $\mathcal{S}_S$ be a family of subcategories of $\text{Aff}_S$.

For a topology $\tau$ on $\mathcal{S}_S$, define the topology $w\tau$ on $\mathcal{S}_{S,Z}$ as the weakest topology such that the functor $\mathcal{S}_S \to \mathcal{S}_{S,Z}: X \mapsto X^h_Z$ is continuous.

For a topology $\tau$ on $\mathcal{S}_Z$, define the topology $s\tau$ on $\mathcal{S}_{S,Z}$ as the strongest topology such that the functor $\mathcal{S}_{S,Z} \to \mathcal{S}_Z: X \mapsto X_Z$ is continuous.

**Remark 3.2.** Note that the topology $s\tau$ above is stronger then $w\tau$, since $\text{Sch}_S$ the base change functor $\mathcal{S}_S \to \mathcal{S}_Z$ is continuous with respect to $\tau$.

**Lemma 3.3.** Let $\mathcal{S}_S$ be $\text{SmAff}_S$ or $\text{Sm}_{S,Z}^c$, see Section [0.2]. Let $\tau$ be a topology on $\mathcal{S}_Z$. A morphism $\tilde{X} \to X$ is a $s\tau$-covering if and only if $X_Z \to X_Z$ is a $\tau$-covering.

**Proof.** The property of coverings in $\mathcal{S}_{S,Z}$ described in the lemma defines a pretopology $s\tau'$ on $\mathcal{S}_Z$, that defines a topology on $\text{Aff}_Z$, and moreover, the latter topology restricts to $\text{Sm}_{Z}^c$. Any topology on $\mathcal{S}_{S,Z}$ such that the functor $\mathcal{S}_{S,Z} \to \mathcal{S}_Z$ is continuous is contained in $s\tau'$. Hence $s\tau' = s\tau$. □
Lemma 3.4. Let $S = \mathbb{A}^n_f$, $\tau = \nu f$ be provided by Definition 3.4 for a subtopology $\nu$ of the étale topology on $\mathbb{A}^n_f$. Then any $\tau$-covering in $\mathbb{A}^{n \times Z}_S$ is a $\nu f$-covering.

Proof. We repeat the argument of [14, Proposition 4.2] that covers $\tau = \nu f$ being Nisnevich topology. Every $\tau$-covering $X^h_{Z} \to X^h_{Z}$ in $\mathbb{A}^{n \times Z}_S$ is obtained from a morphism $f: X \to X$ in $\mathbb{A}^{n}_N$ such that $f_{Z}: X^h_{Z} \to X^h_{Z}$ is a $\tau$-covering in $\mathbb{A}^{n}_Z$. Since $X$, $X \in \mathbb{A}^{n}_N$ and $f_{Z}$ is étale, the morphism $f$ is étale over $X$. Moreover, since $f_{Z}$ is a $\tau$-covering, it follows that $\tilde{X} \amalg X_{S \sim Z} \to X$ is a $\tau$-covering, where $X_{S \sim Z} = X \times_S (S \sim Z)$, since $\tau$ is the strongest subtopology of the étale topology that coincides with $\nu f$ on $\mathbb{A}^{n}_Z$ for each $z \in S$. Summarising, if $X^h_{Z} \to X^h_{Z}$ is an $\tau f$-covering, then $\tilde{X}^h_{Z} \amalg (X_{S \sim Z})^h_{Z} \to X^h_{Z}$ is a $\tau f$-covering in $\mathbb{A}^{n \times Z}_{S,Z}$. This implies $\tilde{X}^h_{Z} \to X^h_{Z}$ is a $\tau f$-covering in $\mathbb{A}^{n \times Z}_{S,Z}$ since $(\tilde{X}_{S \sim Z})^h_{Z} = \emptyset$.

3.2. For any category of correspondences $\text{Corr}_S$ on $\text{EssSmAff}_S$, we define $\text{Corr}_{S,Z}$ as the subcategory of $\text{Corr}_S$ spanned by the objects of $\text{EssSmAff}_{S,Z}$. 

Definition 3.5. A presheaf $F \in \text{PSH}^{tr}(\text{SmAff}_{S,Z})$ is called $\mathbb{A}^1$-invariant if $F((\mathbb{A}^{1 \times Z}_X)^h_{Z}) \simeq F(X^h_{Z})$ for all $X \in \text{SmAff}_{S,Z}$.

As mentioned in [14, §4] and [14, §2.3.4] the equality $X^h_{Z} \times_{S,Z} Y^h_{Z} = (X \times Y)^h_{Z}$ defines the monoidal structure on $\mathbb{A}^{n \times Z}_S$, and the subcategory of $\mathbb{A}^1$-invariant presheaves in $\text{PSH}^{tr}(\text{SmAff}_{S,Z})$ is reflective, where the localisation functor $L_{\mathbb{A}^1}$ is given by $F \mapsto F((\mathbb{A}^{1 \times Z}_X)^h_{Z})$. As in [14, §2.3.4] we use the notation $\Delta_{S,Z}^h = (\Delta_S^h)^h_{Z}$, so $F((\Delta_{S,Z}^h)^h_{Z}) = F(\Delta_{S,Z}^h)$. 

Remark 3.6. We note that $F(\Delta_{S,Z}^h)$ is not equivalent to the restriction of $F(\Delta^h_{S,Z})$ along the embedding functor $\text{EssSmAff}_{S,Z} \to \text{SmAff}_{S,Z}$, because $(\Delta^h_{S,Z})^h_{Z} \neq \Delta^h_{S,Z}$. Precisely, $F_{\Delta_{S,Z}^h}(U^h_{Z}) \neq F_{\Delta^{h}_{S,Z}}(U^h_{Z})$.

Lemma 3.7. There is the isomorphism $\text{Corr}_{S}(U^h_{Z}, X^h_{Z}) \simeq \text{Corr}_{S}(U^h_{Z}, X)$ for any $U, X \in \text{EssSmAff}_S$.

Proof. For any $U^h_{Z}$-scheme $E$ there is an isomorphism $E^h_{U \times S,Z} \simeq E^h_{Z}$, where $E$ is considered as $S$-scheme at the right side. So write $E^h_{Z}$ for $E^h_{U \times S,Z}$. Further, there is an isomorphism of $U^h_{Z}$-schemes $(X^h_{Z} \times S U^h_{Z})_{Z} \simeq (X \times S U^h_{Z})_{Z}$. Then due to (AHP)(2)

$$
\text{Corr}_{S}(U^h_{Z}, X^h_{Z}) \simeq \text{Corr}_{U^h_{Z}}(U^h_{Z}, (X \times S U^h_{Z})) \\
\simeq \text{Corr}_{U^h_{Z}}(U^h_{Z}, (X \times S U^h_{Z})),
$$

Lemma 3.8. For any $X \in \text{SmAff}_{S,Z}$, there is a vector bundle $\tilde{X}$ over $X$ such that $\tilde{X} \in \text{SmAff}^{\text{frei}}_{S,Z}$.

Proof. The claim follows by the argument of [14, Lemma 5.4]. Indeed, there is $\tilde{X}$ such that $\tilde{X} \times X T_X \simeq 1^n_X$, for some $N \in \mathbb{Z}_{\geq 0}$, where $T_X$ and $1^n_X$ are tangent and trivial bundles.

Corollary 3.9. The adjunction $r: H^{tr}_{S}(\text{SmAff}_{S,Z}) \rightleftharpoons H^{tr}_{S}(\text{SmAff}^{\text{frei}}_{S,Z}): l$, where $l \dashv r$, is an equivalence.

Proof. Since the functor $\text{Corr}(\text{SmAff}^{\text{frei}}_{S,Z}) \to \text{Corr}(\text{SmAff}_{S,Z})$ is fully faithful, the adjunction $l \dashv r$, is a coreflection. By Lemma 3.8 for any $X$ in $\text{SmAff}_{S,Z}$, there is an $\mathbb{A}^1$-equivalence $\tilde{X} \to X$ such that $\tilde{X} \in \text{SmAff}^{\text{frei}}_{S,Z}$. Hence $r$ is conservative, and the claim follows.

Remark 3.10. The argument of [14, Proposition 5.8] proves even $\text{PSH}^{tr}(\text{SmAff}_{S,Z}) \simeq \text{PSH}^{tr}(\text{SmAff}^{\text{frei}}_{S,Z})$.

3.3. We proceed with the localisation property for the homotopy categories of sheaves on $\text{SmAff}_{S,Z}$, $\text{SmAff}_S$, $\text{SmAff}_{S,Z}$. Throughout this section $\tau$ is a topology on $\text{EssSmAff}_S$, and speaking about the topology $\tau$ on $\text{SmAff}_{S,Z}$ we mean the topology $\mathbb{A}^1$.
3.3.1. We prove localisation theorem for functors $i^*$, $i_*$, $j^*$, $j_#$ on unpointed $\infty$-categories.

**Lemma 3.11.** There are well defined adjunctions $i^* \dashv i_*$, $j_# \dashv j^*$,

$$PSh^h_S(SmAff_{S,Z}) \rightleftarrows PSh^h_S(SmAff_S) \rightleftarrows PSh^h_S(SmAff_{S-Z}) \tag{3.12}$$

$$i^*(h^{|}(X)) = h^{|}(X_{\frac{1}{2}}) \quad i_*(F)(X) = F(X_{\frac{1}{2}}) \quad j^*(F)(X) = F(X) \quad j_#(h^{|}(X)) = h^{|}(X).$$

**Proof.** The adjunction $i^* \dashv i_*$ is given by the inverse and the direct image functors along the functor $\text{Corr}(SmAff_S) \to \text{Corr}(SmAff_{S-Z}) \colon X \mapsto X_{\frac{1}{2}}$.

The adjunction $j_# \dashv j^*$ is given by the inverse and the direct image functors along the functor $\text{Corr}(SmAff_{S-Z}) \to \text{Corr}(SmAff_S) \colon X \mapsto X$.

Since $i^*$ in (3.12) preserves $\tau$-local equivalences, and $i_*$ in (3.12) preserves $\tau$-sheaves, they induce the adjunction

$$i^* : Shv^h_{\Sigma}(SmAff_S) \rightleftarrows Shv^h_{\Sigma}(SmAff_{S-Z}) : i_*.$$

**Lemma 3.13.** Suppose $\tau \supset tf$. Let $X \in SmAff_S$. Then there is a $\tau$-local equivalence $i_*^*i^*(h^{|}(X)) \simeq \text{cofib}(h^{|}(X - X_Z) \to h^{|}(X))$.

**Proof.** For any $U \in SmAff_{S-Z}$, there is the isomorphism $U_{\frac{1}{2}} \cong \emptyset$, and consequently,

$$i_*^*i^*(h^{|}(X))(U) \simeq \ast.$$  

Hence the unit $h^{|}(X) \to i_*^*i^*(h^{|}(X))$ of the adjunction $i^* \dashv i_*$ induces the morphism

$$\text{cofib}(h^{|}(X - X_Z) \to h^{|}(X)) \to i_*^*i^*(h^{|}(X)), \tag{3.14}$$

and moreover, (3.14) is an isomorphism on $SmAff_{S-Z}$. Let $U \in SmAff_S$ be such that $U \times_S Z \not\cong \emptyset$, then $h^{|}(X - X_Z)(U_{\frac{1}{2}}) \cong \emptyset$, and consequently,

$$\text{cofib}(h^{|}(X - X_Z) \to h^{|}(X))(U_{\frac{1}{2}}) \cong i_*^*i^*(h^{|}(X))(U_{\frac{1}{2}}) \cong i_*^*i^*(h^{|}(X))(U_{\frac{1}{2}}).$$

So (3.14) is an isomorphism on non-empty objects of $SmAff_{S-Z}$.

Thus since $\tau \supset tf$, it follows that (3.14) is a $\tau$-local equivalence.

**Lemma 3.15.** Suppose $\tau \supset tf$.

(0) There are the natural equivalences

$$i^*(h^{|}(X)) \simeq h^{|}(X_{\frac{1}{2}}) \quad j^*(h^{|}(X)) \simeq h^{|}(X - X_Z) \quad j_#(h^{|}(X)) \simeq h^{|}(X).$$

(1) The functors $i_*, i^*, j^*, j_#$ in (3.12) preserve $\tau$-local equivalences.

(2) The functors $i^*$, $j^*$, $j_#$ in (3.12) preserve $\mathcal{A}^1$-equivalences. The functor induced by $i_*$ on $Shv^h_{\Sigma}(\tau)(-)$. preserve $\mathcal{A}^1$-equivalences.

Consequently, there are adjunctions

$$H^h_{\Sigma}(SmAff_{S,Z}) \rightleftarrows H^h_{\Sigma}(SmAff_S) \rightleftarrows H^h_{\Sigma}(SmAff_{S-Z}) \tag{3.16}$$

induced by (3.12) via the localisation with respect to $\tau$-local equivalences, and $\mathcal{A}^1$-equivalences.

**Proof.** Point (0) is provided by Lemmas 3.11 and 3.13.

Points (1) and (2) for $i^*$, $j^*$, $j_#$ follows by Point (0).

We proceed with Point (1) for $i_*$. The functor $i_*$ preserves tf-equivalences because the functor $X \mapsto X_{\frac{1}{2}}$ preserves tf-points. Hence $i_*$ induces functor $H^h_{\Sigma}(SmAff_{S,Z}) \to H^h_{\Sigma}(SmAff_S)$ via the localisation with respect to tf-local equivalences. The induced functor preserves $\tau$-local equivalences by Lemma 3.13. Since $\tau \supset tf$, it follows that $i_*$ preserves $\tau$-local equivalences.

Point (2) for $i_*$ follows from Lemma 3.13.

□
Proposition 3.17. For any $F \in \mathcal{H}^{tr}_{\Sigma, \tau}(\text{SmAff}_S)$, the square

$$
\begin{array}{c}
\tilde{i}_* \tilde{j}^* F \\
\downarrow \quad \downarrow
\end{array}
\quad
\begin{array}{c}
\tilde{i}_* F \\
\downarrow \quad \downarrow
\end{array}
\quad
\begin{array}{c}
\tilde{j}_# \tilde{j}^* F
\end{array}

$$

is a pushout.

Proof. The claim follows by Lemmas 3.13 and 3.15. \qed

3.3.2. We prove localisation theorem for functors $\tilde{i}^!, \tilde{i}_*, \tilde{j}^*, \tilde{j}_*$ on pointed $\infty$-categories.

Recall that according to Section 3.2 $\mathcal{PSh}^{tr}_{\Sigma}(\cdot) = \mathcal{PSh}_{\Sigma}^\tau(\text{Corr}(\cdot))$ is the $\infty$-category of pointed radditive presheaves on the respective $\infty$-category of correspondences. Then Lemma 3.18 holds for the $\infty$-category of pointed radditive $\tau$-sheaves $\mathcal{Sh}^{tr}_{\Sigma, \tau}(\text{SmAff}_S)$ as well.

Lemma 3.18. (0) There are well defined functors

$$
\mathcal{PSh}^{tr}(\text{SmAff}_{S,Z}) \cong \mathcal{PSh}^{tr}(\text{SmAff}_S) \cong \mathcal{PSh}^{tr}(\text{SmAff}_{S-Z})
$$

(3.19)

that preserve the subcategories $\mathcal{PSh}^{tr}_\tau(\cdot)$.

(1) All the functors $\tilde{i}^!, \tilde{i}_*, \tilde{j}^*, \tilde{j}_*$ in (3.19) preserve $\tau$-sheaves; so there are functors

$$
\mathcal{Sh}^{tr}_{\Sigma, \tau}(\text{SmAff}_{S,Z}) \cong \mathcal{Sh}^{tr}_{\Sigma, \tau}(\text{SmAff}_S) \cong \mathcal{Sh}^{tr}_{\Sigma, \tau}(\text{SmAff}_{S-Z}),
$$

(3.20)

induced by (3.19) by the restriction to the subcategories of sheaves.

(2) The functors $\tilde{i}_*, \tilde{j}^*, \tilde{j}_*$ in (3.19) preserve $\mathcal{A}^{1}$-invariant presheaves. If $\tau \supset \mathcal{A}$, then the functor $\tilde{i}^!$ in (3.20) preserves $\mathcal{A}^{1}$-invariant sheaves.

Proof. Functionalities of the scheme operations in the definition imply that the formulas in point (0) define functors $\tilde{i}^!, \tilde{i}_*, \tilde{j}^*, \tilde{j}_*$. Further, the claims (1) and (2) for $\tilde{j}^*$ and $\tilde{j}_*$ are well known.

We are going to prove (1) for $\tilde{i}^!$. Let $F \in \mathcal{Sh}^{tr}_{\Sigma, \tau}(\text{SmAff}_S)$. By definition the topology $\mathcal{A}$ is generated by coverings in the category $\text{SmAff}_{S-Z}$ of the form $X^h_Z \to X_Z$ for a $\tau$-covering $\tilde{X} \to X$ in $\text{SmAff}_S$. Since for any $\tau$-covering $\tilde{X} \to X, \tilde{X} \to X$ and $\tilde{X}_{S-Z} \to X_{S-Z}$ are $\tau$-coverings, it follows that $\tilde{i}^!(F)$ is $\mathcal{A}$-sheaf. Thus $\tilde{i}^!$ preserves $\tau$-sheaves.

We proceed with (2) for $\tilde{i}^!$. If $F$ is $\mathcal{A}^{1}$-invariant, and $\mathcal{A} \subset \tau$, then for any $X \in \text{SmAff}_S$

$$
\begin{array}{c}
\text{fib}(F((\mathcal{A} \times X)^h_Z) \to F((\mathcal{A} \times X)^h_Z \times_S (S-Z)) \cong \\
\text{fib}(F(\mathcal{A} \times X) \to F((\mathcal{A} \times X) \times_S (S-Z)) \cong \\
\text{fib}(F(X) \to F(X \times_S (S-Z)) \cong \\
\text{fib}(F(X^h_Z) \to F(X^h_Z \times_S (S-Z)).
\end{array}
$$

Thus $\tilde{i}^!F$ is $\mathcal{A}^{1}$-invariant.

To prove that claims (1) and (2) for $\tilde{i}_*$ consider $F \in \mathcal{Sh}^{tr}_{\Sigma, \tau}(\text{SmAff}_{S,Z})$ is a $\tau$-sheaf, because for any $\tau$-covering $\tilde{X} \to X$ in $\text{SmAff}_S$ the morphism $\tilde{X}^h_Z \to X^h_Z$ is a $\tau$-covering in $\text{SmAff}_{S-Z}$. If $F$ is $\mathcal{A}^{1}$-invariant, then for any $X \in \text{SmAff}_S$, $\tilde{i}_*F((\mathcal{A} \times X) = F((\mathcal{A} \times X)^h_Z) \cong F(X^h_Z) = \tilde{i}_*F(X)$. \qed

Lemma 3.21. If $\tau \supset \mathcal{A}$, the functors in (3.20) from the adjunctions $\tilde{i}_* \dashv \tilde{i}^!, \tilde{j}^* \dashv \tilde{j}_*$, and consequently, there are the adjunctions

$$
\mathcal{H}^{tr}_{\Sigma, \tau}(\text{SmAff}_{S,Z}) \cong \mathcal{H}^{tr}_{\Sigma, \tau}(\text{SmAff}_S) \cong \mathcal{H}^{tr}_{\Sigma, \tau}(\text{SmAff}_{S-Z})
$$

induced by (3.19) and (3.20) by the restriction.
Proof. Let $G \in \text{Shv}_{\Sigma, \tau}^{tr}(\text{SmAff}_{S, Z})$ be the sheaf associated with the presheaf represented by $X^h_Z \in \text{SmAff}_{S, Z}$, where $X \in \text{SmAff}_S$. Denote $iG = \text{colim}(h^tr(X - X_Z) \to h^tr(X)) \in \text{Shv}_{\Sigma, \tau}^{tr}(\text{SmAff}_S)$. By Lemma 3.13, $i_*(G) \cong iG$. Since $\tau \succ \text{tf}$, it follows that $i^! F(X^h_Z) \cong \text{fib}(F(X) \to F(X - X_Z)) \cong \text{Map}(iG, F)$ for any $F \in \text{Shv}_{\Sigma, \tau}(\text{SmAff}_S)$, and there are the isomorphisms
\[
\text{Map}(i_! G, F) \cong i^! F(X^h_Z) \cong \text{Map}(G, i^! F).
\]
Since representable objects generate the category $\text{Shv}_{\Sigma, \tau}(\text{SmAff}_S)$, we get the adjunction.  

\[\Box\]

Proposition 3.22. Consider the pair of adjunctions $i_* \dashv i^!$, $j^* \dashv j_*$, 
\[H^r_{\Sigma, \tau}(\text{SmAff}_{S, Z}) \rightleftarrows H^r_{\Sigma, \tau}(\text{SmAff}_S) \rightleftarrows H^r_{\Sigma, \tau}(\text{SmAff}_{S - Z}),\]
provided by Lemmas 3.18 and 3.21. For any $F \in H^r_{\Sigma, \tau}(\text{SmAff}_S)$, the square
\[
\begin{array}{ccc}
\tilde{i}^! \tilde{i}^* F & \longrightarrow & F \\
\downarrow & & \downarrow \\
* & \longrightarrow & j_* j^* F
\end{array}
\]
is a pullback.

Proof. For any $F \in \text{Shv}_{\Sigma, \tau}^{tr}(\text{SmAff}_S)$,
\[
\tilde{i}^! \tilde{i}^* F \cong \text{fib}(F(X^h_Z) \to F(X^h_Z - X_Z)) \cong \text{fib}(F(X) \to F(X - X_Z)) \cong \text{fib}(F(X) \to j_* j^* F(X)).
\]
Hence the corresponding square is pullback in $\text{Shv}_{\Sigma, \tau}^{tr}(\text{SmAff}_S)$. The claim for $H^r_{\Sigma, \tau}(\text{SmAff}_S)$ follows by Lemma 3.18. 

3.4. Throughout this subsection

- $\mathcal{S}_S$ is $\text{Sm}^c_S$ or $\text{SmAff}_S$, and
- $\mathcal{S}$ denotes a category from the list: $\mathcal{S}_S, \mathcal{S}_Z, \mathcal{S}_{S, Z}$.

Lemma 3.24. Suppose a discrete presheaf of sets $c$ on $\text{Aff}_S$ satisfies lifting property with respect to affine henselian pairs in sense of Definition 1.26 and satisfies closed gluing on $\text{Aff}_S$. Denote by $c^h_Z$ and $c_Z$ the restrictions of $c$ to the categories $\text{SmAff}_{B, Z}$ and $\text{SmAff}_Z$.

Let $U \in \text{SmAff}_S$, and $Y \subset U$ be a closed subscheme. Then the morphism of simplicial sets
\[L^h c^h_Z(U^h_Z) \to L^h c_Z(U)\]
is a trivial fibration.

Proof. By the construction of $L^h$ on $\text{PSh}_{\Sigma}(\text{SmAff}_{S, Z})$ and $\text{PSh}_{\Sigma}^{tr}(\text{SmAff}_Z)$, see Section 3.1, there are weak equivalences of simplicial sets $L^h c^h_Z(U^h_Z) \cong c((\Delta^*_Z \times_S U)^h_Z), L^h c_Z(U) \cong c(\Delta^*_Z \times_Z U)$. Since $(\Delta^*_Z \times_S U)^h_Z \cong (\Delta^*_Z)^h_Y, \Delta^*_Z \times_S U \cong \Delta^*_Y$, where $Y = U \times_S Z$, the claim follows by Corollary 1.24.  

Corollary 3.25. For any $E \in \text{SmAff}_S, U \in \text{SmAff}_S$, and closed subscheme $Z \subset S$, the morphism $L^h h^tr(E)(U^h_Z) \to L^h h^tr(E)(U_Z)$ is a weak equivalence.

Proof. The claim follows by Definition 1.29 and Lemma 3.23.  

Definition 3.26. Define the $\mathcal{S}$-category $\text{Corr}^A(\mathcal{S})$ as the full subcategory in $H_\Sigma(A)(\mathcal{S})$ spanned by the objects of the form $L^h h^tr(X), X \in \mathcal{S}$. Denote by
\[l: \text{Corr}(\mathcal{S}) \to \text{Corr}^A(\mathcal{S})\]
the functor $X \mapsto L^h h^tr(X)$. We write $X \in \text{Corr}^A(\mathcal{S}_\tau)$, where $X$ denotes $L^h h^tr(X)$, so we may write $l(X) \cong X$. Denote by $h^tr(X) = \text{Corr}^A(-, X)$ the representable presheaves.

Remark 3.27. The representable presheaves $h^tr(X)$ being restricted to $\mathcal{S}_\tau$ are equivalent to the presheaves $h^tr(X)^{\Delta^*_Z} \cong \text{Corr}(- \times \Delta^*_Z, X)$. 


Lemma 3.28. The adjunction of ∞-categories

\[ l^* : \text{PSh}^\Sigma(S_{S,Z}) \rightleftarrows \text{PSh}^\Sigma(S_{S,Z}) : l_\ast \]  

restricts to the equivalence

\[ \text{H}^\Sigma(S_{S,Z}) \rightleftarrows \text{PSh}^\Sigma(S_{S,Z}). \]  

Proof. Since the \( A^1 \)-localisation functors on \( \text{PSh}^\Sigma(S_{S,Z}) \) and \( \text{PSh}^\Sigma(S_{S,Z}) \) are both given by \( L_{A^1} \cong (-)^{\Delta_{S,Z}} \), and the functor \( l \) commutes with the functors \(- \times \Delta_{S,Z}\), i.e.

\[ l_\ast L_{A^1} \cong L_{A^1} l_\ast : \text{PSh}^\Sigma(S_{S,Z}) \rightarrow \text{PSh}^\Sigma(S_{S,Z}). \]  

Further, by the definition \( l_\ast h^\Sigma(E) \cong L_{A^1} h^\Sigma(E) \), and \( l^* h^\Sigma(E) \cong h^\Sigma(E) \). Hence

\[ l_\ast l^* h^\Sigma(E) \cong L_{A^1} h^\Sigma(E) \subset \text{PSh}^\Sigma(S_{S,Z}), \]  

and

\[ L_{A^1}, l_\ast h^\Sigma(E) \cong L_{A^1} h^\Sigma(E) \cong l_\ast h^\Sigma(E). \]  

So by Lemma 3.28, the unit of the adjunction (3.29) is equivalent to the endofunctor \( L_{A^1} \) on \( \text{PSh}^\Sigma(S_{S,Z}) \). Since the functor \( \text{PSh}^\Sigma(S_{S,Z}) \rightarrow \text{PSh}^\Sigma(S_{S,Z}) \) is conservative, \( l_\ast \) in (3.32) is conservative, and hence Lemma 3.28 implies that the counit is equivalent to the identity endofunctor on \( \text{PSh}^\Sigma(S_{S,Z}) \). Thus (3.29) is a reflection, and induces the equivalence of \( \text{H}^\Sigma(S_{S,Z}) \) and \( \text{PSh}^\Sigma(S_{S,Z}). \) □

Lemma 3.33. The functor \( \text{Corr}^A(S_{Z,S}) \rightarrow \text{Corr}^A(S_{Z,C}) \) is an equivalence.

Proof. The claim follows because morphisms \( \text{Corr}^A(S_{Z,S}) \rightarrow \text{Corr}^A(S_{Z,S}) \) are weak equivalences by Corollary 3.24 and for any scheme \( X \in \text{Sm}_{Z,C} \) there is a scheme \( X \in \text{Sm}_{S,C} \) such that \( X \times S \cong X \). □

The mapping \( X^b \rightarrow X \) defines the functor \( \overline{t}^\ast_{\text{Corr}} : \text{Corr}(S_{S,Z}) \rightarrow \text{Corr}(S_{Z}) \) that induces the functors

\[ \overline{t} : \text{PSh}^\Sigma(S_{S,Z}) \rightleftarrows \text{PSh}^\Sigma(S_{Z}) : \overline{t}_\ast \]  

given by \( \overline{t}(h^\Sigma(X^b)) = h^\Sigma(X) \), \( \overline{t}_\ast(F)(X^b) = F(X) \). Since \( \overline{t}^\ast_{\text{Corr}} \) takes \( A^1 \times X \) to \( A^1 \times X \), the functor \( \overline{t}_\ast \) preserves \( A^1 \)-invariant objects, and \( \overline{t}^\ast \) preserves \( A^1 \)-equivalences, and (3.34) induces the adjunction

\[ \overline{t}^\ast : \text{H}^\Sigma(S_{S,Z}) \rightleftarrows \text{H}^\Sigma(S_{Z}) : \overline{t}_\ast. \]  

Proposition 3.36. The adjunction (3.35) is an equivalence.

Proof. Consider the commutative square

\[ \begin{array}{c}
\text{Corr}^A(S_{Z,S}) \rightarrow \text{Corr}^A(S_{Z,C}) \\
\text{Corr}(\text{Sm}_{S,C}) \rightarrow \text{Corr}(\text{Sm}_{Z,C})
\end{array} \]  

where the horizontal arrows take \( X^b \) to \( X \), and similar one for \( \text{Sm}^c_{Z,C} \). Then (3.35) for \( S_* = \text{Sm}^c_{Z,C} \) is an equivalence since by Lemma 3.28 and Lemma 3.33 there are the equivalences

\[ \begin{array}{c}
\text{PSh}^\Sigma(S_{Z,S}) \rightleftarrows \text{PSh}^\Sigma(S_{Z,C}) \\
\text{H}^\Sigma(S_{S,Z}) \rightleftarrows \text{H}^\Sigma(S_{Z,C})
\end{array} \]  

Then the claim for \( \text{Sm}_{A^1} \text{Aff} \) follows because of Corollary 3.9 □
Lemma 3.38. (1) For any $X \in \text{Sm}_{\mathbb{Z}}^\text{cfi}$ there is $\tilde{X} \in \text{Sm}_{S,Z}^\text{cfi}$ such that $X = \tilde{X} \times_{S} Z$. (2) For any given morphism $X \to Y \in \text{Sm}_{\mathbb{Z}}^\text{cfi}$, there is a morphism $\tilde{X} \to \tilde{Y}$ that goes to the given one along the functor $- \times_{S} Z$.

Proof. Point (1) follows because of [17, Lemma A.11]. Point (2) follows then by [17, Lemma E.10].

Lemma 3.39. Let $\tau$ be a topology on $\text{Sm}_{\mathbb{Z}}^\text{cfi}$, see Section 3.4.

(1) Let $\tau$ be a topology on $\text{Sm}_{\mathbb{Z}}^\text{cfi}$. A morphism $\tilde{X} \to X$ is a $\tau$-covering if and only if there is a morphism $\tilde{X} \to X$ such that $\tilde{X} \times_{S} Z \to \tilde{X} \times_{S} Z$ is a $\tau$-covering.

(2) A presheaf $F \in \text{PSh}_{\tau}^\text{tr}(\text{Sm}_{\mathbb{Z}}^\text{cfi})$ is a $\tau$-sheaf if and only if $\tilde{\tau} F \in \text{PSh}_{\tau}^\text{tr}(\text{Sm}_{S,Z}^\text{cfi})$ is $\tau$-sheaf.

Proof. Point (1) follows from Lemma [3.38] because of Lemma 3.39. We proceed with point (2). Since the functor $\text{Sm}_{S,Z}^\text{cfi} \to \text{Sm}_{\mathbb{Z}}^\text{cfi}$ is continuous, the functor $\text{PSh}_{\tau}^\text{tr}(\text{Sm}_{\mathbb{Z}}^\text{cfi}) \to \text{PSh}_{\tau}^\text{tr}(\text{Sm}_{S,Z}^\text{cfi})$ preserves the sheaves, and it detects the sheaves by point (1).

Proposition 3.40. The adjunction (3.35) induces the equivalence

$$\tilde{\tau} : \text{H}_{S,Z}^\text{tr}(\text{Sm}_{Z}) \cong \text{H}_{S,Z}^\text{tr}(\text{Sm}_{S,Z}) : \tau_* \quad (3.41)$$

for any topology $\tau$ on $S_Z$.

Proof. Let $S_S$ be $\text{Sm}_{\mathbb{Z}}^\text{cfi}$. The equivalence proved in Proposition 3.30 restricts to the equivalence on the subcategories of sheaves because of Lemma 3.39 (2).

By Lemmas 3.33 and 3.39 a morphism in $\text{Sm}_{S,Z}^\text{cfi}$ is a $\tau$-covering if and only if it is an $\tau$-covering in $\text{Sm}_{S,Z}^\text{cfi}$. So the functor $\text{PSh}_{\tau}^\text{tr}(\text{Sm}_{S,Z}^\text{cfi}) \to \text{PSh}_{\tau}^\text{tr}(\text{Sm}_{\mathbb{Z}}^\text{cfi})$ preserves $\tau$-sheaves, and the claim for $S_S = \text{Sm}_{S,Z}$ follows.

3.5. We conclude the main results of the section. Let $\tau$ be a family of topologies on $\text{EssSmAff}_{S}$. Consider the functor

$$i_* : \text{H}_{S,Z}^\text{tr}(\text{Sm}_{S,Z}) \to \text{H}_{S,Z}^\text{tr}(\text{Sm}_{S,Z}); i_* F(X) = F(X_Z).$$

Then $i_* \cong \tilde{i}_* \tilde{\tau}_*$, and the functor $i^! := \tilde{i}^! \tilde{\tau}^*$ is the right adjoint, where $\tilde{\tau}^*$ is from (3.41). The letter adjunctions induce the ones for the pointed $\infty$-categories $\text{H}_{S,Z}^\text{tr}(\cdot)$.

Theorem 3.42. Suppose that $\tau \supset \tau_{\text{eff}}$, and $\tau_{\text{eff}} = \tau_{\text{eff}}$ on $\text{Sm}_{S,Z}$, see Definition 3.1. Consider the pair of adjunctions

$$\text{H}_{S,Z}^\text{tr}(\text{Sm}_{S,Z}) \cong \text{H}_{S,Z}^\text{tr}(\text{Sm}_{S,Z}) \cong \text{H}_{S,Z}^\text{tr}(\text{Sm}_{S,Z})$$

given by $i_* \mapsto i^!, j^* \mapsto j_*$. Then for any $F \in \text{H}_{S,Z}^\text{tr}(\text{Sm}_{S,Z})$, there is a pullback square

$$\begin{array}{ccc}
& i_* i^! F & \to & F \\
\downarrow & & \downarrow & \\
& j_* j^* F & \to & \\
\end{array}$$

Proof. The claim follows by Propositions 3.22 and 3.40.

Theorem 3.43. Suppose that $\tau \supset \tau_{\text{eff}}$, and $\tau_{\text{eff}} = \tau_{\text{eff}}$ on $\text{Sm}_{S,Z}$, see Definition 3.1. Consider the pair of adjunctions

$$\text{H}_{S,Z}^\text{tr}(\text{Sm}_{S,Z}) \cong \text{H}_{S,Z}^\text{tr}(\text{Sm}_{S,Z}) \cong \text{H}_{S,Z}^\text{tr}(\text{Sm}_{S,Z})$$

given by $i_* \mapsto i^*, j^* \mapsto j_*$. Then for any $F \in \text{H}_{S,Z}^\text{tr}(\text{Sm}_{S,Z})$, there is a pushout square

$$\begin{array}{ccc}
\tilde{i}_* \tilde{i}^* F & \to & F \\
\downarrow & & \downarrow \\
& j_\# j^* F & \to & \\
\end{array}$$
Proof. The claim follows by Propositions \ref{prop:3.17} and \ref{prop:3.40}. \hfill \Box

Corollary 3.44. The claims of Theorems \ref{thm:3.42} and \ref{thm:3.43} hold for $S = \text{Aff}^\text{fn}$, and the topology $\tau = \nu \mathcal{F}$ for any family of subtopologies $\nu$ of the étale topology, see Definition 2.1.

Proof. The property $s\tau = w\tau$ holds by Lemma 3.4. \hfill \Box

Example 3.45. The required properties on $\text{Corr}_\mathcal{W}(\cdot)$ in Theorem \ref{thm:3.42} are satisfied for the category $\text{SmAff}_\mathcal{W}(\cdot)$, and the $\infty$-category $\text{Corr}^\mathcal{W}(\cdot)$.

Example 3.46. The example of the topologies $\nu$ in Corollary \ref{cor:3.44} are the trivial topology, Zariski topology, and the Nisnevich, or étale topologies. Then $\tau$ is the trivial fibre, Zariski fibre, and the Nisnevich, or étale topology respectively.

4. Étale excision and motivic localisation.

4.1. General lemma on the excision property.

Definition 4.1. For an $\infty$-category $\mathcal{S}$ and an object $X \in \mathcal{S}$ denote by $\mathcal{S}_X$ the comma category over $X$. Define the functor

$$e_X : \text{PSh}(\mathcal{S}_X) \to \text{PSh}(\mathcal{S})$$

that takes a presheaf $F$ on $\mathcal{S}_X$ to a presheaf $G$ on $\mathcal{S}$

$$G(W) = \prod_{W \to X \in \mathcal{S}_X} F(W \to X).$$

Lemma 4.2. For an $\infty$-category $\mathcal{S}$ and $X \in \mathcal{S}$ the functor $e_X : \text{PSh}(\mathcal{S}_X) \to \text{PSh}(\mathcal{S})$ is conservative.

Proof. Let $F \to F'$ be a morphism in $\text{PSh}(\mathcal{S}_X)$. Let $w : W \to X \in \mathcal{S}_X$. If $e_X(F) \simeq e_X(F')$ then the canonical morphism $\coprod_{s : W \to X \in \mathcal{S}_X} F(s) \to \coprod_{s : W \to X \in \mathcal{S}_X} F'(s)$ is an equivalence. Since the coproduct functor in $\text{Spc}$ is conservative the equivalence $F(w) \simeq F'(w)$ follows. \hfill \Box

Definition 4.3. Let $Z$ be a Grothendieck topology on a category $\mathcal{S}$. We say that an object $F$ of $\text{PSh}(\mathcal{S})$ is a $Z$-sheaf if for any $Z$-covering sieve $c \in \text{PSh}(\mathcal{S})$ of $X \in \mathcal{S}$ the map $F(X) \to \lim_{X^c} F(X')$ is an equivalence, where $c$ in the subscript the fibrant category over $\mathcal{S}$ corresponding to $c$.

For a Grothendieck topology $Z$ on a category $\mathcal{S}$ denote by the same symbol the induced topology on $\mathcal{S}_X$.

Lemma 4.4. The functor $e_X$ from Def. 4.1 preserves $Z$-sheaves.

Proof. The claim follows form the definitions. \hfill \Box

For a category $\mathcal{S}$ denote by pro-$\mathcal{S}$ the category of pro-objects. For any $F \in \text{PSh}(\mathcal{S})$, we denote by the same symbol the corresponding continuous presheaf on pro-$\mathcal{S}$.

Lemma 4.5. Let $\mathcal{T}$ be a family of pro-objects in $\mathcal{S}$ that define enough set of points for a Grothendieck topology $Z$ on $\mathcal{S}$ such that the topos of sheaves $\text{Shv}_Z(\mathcal{T})$ is hypercomplete. Then for $X \in \mathcal{S}$ the family $\mathcal{T}_X$ of pro-objects in $\mathcal{S}_X$ given by morphisms of the form $T \to X$, $T \in \mathcal{T}$, gives enough set of points for the topology $Z$ on $\mathcal{S}_X$, and the topos of sheaves $\text{Shv}_Z(\mathcal{S}_X)$ is hypercomplete.

Proof. Since a pro-object $T \in \mathcal{T}$ defines a point of $Z$ on $\mathcal{S}$, for any $Z$-covering $W \to W$, $W \in \mathcal{S}$, the morphism $e_W : W \times_W T \to T$ has a right inverse $l_W$. Since $e_W \circ l_W = \text{id}_W$, for any morphism $\mathcal{W} \to X$ the morphism $l_W$ defines the morphism in $\mathcal{S}_X$. Then it follows that any pro-object $T \to X \in \mathcal{T}_X$ is a point of the topology $Z$ on $\mathcal{S}_X$.

We are going to show that the set of points $\mathcal{T}_X$ is enough for the topology $Z$ on $\mathcal{S}_X$, and the topos is hypercomplete. Let $F \to F'$ be any morphism of $Z$-sheaves on $\mathcal{S}_X$ such that

$$F(t) \simeq F'(t)$$

(4.6)
for any $t: T \to X \in \mathcal{T}_X$. By Lemma 1.4, $\epsilon_X(F) \to \epsilon_X(F')$ is a morphism of $Z$-sheaves on $\mathcal{S}$. By the above $t: T \to X \in \mathcal{T}_X$ defines a $Z$-point in $\mathcal{S}_X$. Hence for any $T \in \mathcal{T}$, and $F \to F'$ as above, there are isomorphisms

$$
\epsilon_X(F)(T) \simeq \prod_{t: T \to X} F(t) \simeq \prod_{t: T \to X} F'(t) \simeq \epsilon_X(F')(T)
$$

where the middle one follows by (4.6) because for any $T \in \mathcal{T}$ and $t: T \to X$, we have $t \in \mathcal{T}_X$. Since $\mathcal{T}$ is an enough set of $Z$-points for $Z$ on $\mathcal{S}$, it follows that $\epsilon_X(F) \simeq \epsilon_X(F')$, and consequently $F \simeq F'$ by Lemma 4.2.

**Definition 4.7.** Let $R$ be a square

$$
\begin{array}{ccc}
W' & \xrightarrow{t'} & X' \\
\downarrow & & \downarrow \\
W & \xrightarrow{t} & X
\end{array}
$$

in $\mathcal{S}$. We say that a presheaf $F \in \text{PSh}(\mathcal{S})$ is $R$-excisive if

$$
F(X) \simeq F(W) \times_{F(W \times_X V)} F(X').
$$

(4.9)

Given a morphism $V \to X$ in $\mathcal{S}$, we say that $F$ is $R$-excisive over $V$ if $F$ is excisive with respect to the square $R \times_X V$.

Given a cd-structure $N$ on a category $\mathcal{S}$, denote by the same symbol $N$ the completely decomposable topology $N$ on $\mathcal{S}$ defined by the cd-structure, and call the squares of $N$ by $N$-squares. Any $N$-sheaf is $N$-excisive and under the set of assumptions of the criterion [3, Theorem 3.2.5] a presheaf $F \in \text{PSh}(\mathcal{S})$ is an $N$-sheaf if and only if it is $N$-excisive for each $N$-square.

**Lemma 4.10** (Main lemma). Let $N$ be a Grothendieck topology on a category $\mathcal{S}$ and $Z$ be a subtopology of $N$. Assume that $N$ is complete decomposable, and $Z$ admits enough set of points given by a family of pro-objects in $\mathcal{S}$ and the associated topos is hypercomplete.

Let $F$ be a $Z$-sheaf on $\mathcal{S}$ such that for each $Z$-point $U$ the presheaf $F$ is $N$-excisive over $U$. Then $F$ is $N$-excisive, and it is an $N$-sheaf under the assumptions of [3, Theorem 3.2.5].

**Proof.** Let $R$ be an $N$-square (4.8) in $\mathcal{S}$ over the scheme $X$. Consider the presheaf $F'$ on the category of pro-objects in $\mathcal{S}_X$ defined by the assignment

$$
\{ V \to X \} \mapsto F(W \times_X V) \times_{F(W \times_X V)} F(X' \times_X V).
$$

Since finite limits commute with filtered colimits, $F'$ is continuous, and since finite limits commute with limits, $F'$ is a $Z$-sheaf on $\mathcal{S}_X$.

It follows by the assumption on $F$ and Lemma 1.3 that $F$ is $R$-excisive over any $T \in \mathcal{T}_X$. So we have equivalences $F(T) \simeq F'(T)$ for each $T \in \mathcal{T}_X$. Hence by Lemma 1.5, the canonical morphism $F \to F'$ is $Z$-local equivalence. Thus $F \simeq F'$ since both presheaves are $Z$-sheaves. Since $F'$ is $R$-excisive by the definition, $F$ is $R$-excisive.

Thus $F$ is $R$-excisive for each $N$-square $R$ in $\mathcal{S}$.

**4.2. Étale excision theorem for framed presheaves.** Let $\text{Corr}_k$ be an additive $\infty$-category of correspondences over a field $k$ in the sense of Definition 1.4 and suppose that the canonical functor $\text{Sm}_k \to \text{Corr}_k$ passes through the graded classical category of framed correspondences $\text{Fr}_+^0(k)$ defined in [25], and the functor $\text{Fr}_+^0(k) \to \text{Corr}_k$ takes the morphisms $\sigma_X$ for all $X \in \text{Sm}_k$ to the elements in $\text{Map}_{\text{Corr}}(X, X)$ equivalent to the identity morphisms.

**Proposition 4.11** (Étale excision). Given a smooth scheme $X$ over a field $k$, a Nisnevich square $R$ of the form (1.3), and a point $x \in X$, we have that any $k^1$-invariant presheaf $F \in \text{PSh}^0_k(X)$ is $R$-excisive over the local scheme $X_x$. i.e. the map

$$
F(X_x) \to F(W \times_X X_x) \times_{F(W \times_X X_x)} F(X_x)
$$

is an equivalence.
Proof. It suffices to show that the map
\[ \pi_i \mathrm{Fib}(F(X_x)) \to \pi_i \mathrm{Fib}(F(W' \times X, X'_p) \to \pi_i \mathrm{Fib}(F(W' \times X, X'_p)) \]

is an isomorphism for any \( i \in \mathbb{Z} \). There is a long exact sequence
\[ \cdots \to \pi_i \mathrm{Fib}(F(W \times X, \to \pi_i \mathrm{Fib}(F(W' \times X, X')) \to \pi_i (F(X_x)) \to \cdots \quad (4.12) \]

Note that we are implicitly using here that taking fibres commutes with taking filtered colimits. The long exact sequence (4.12) together with the similar one for \( X' \) instead of \( X, X'_p \) instead of \( X, \) and \( W' \) instead of \( W, \) imply the short exact sequences
\[ \text{Coker}(\pi_{i+1}(i^*)) \to \pi_i \mathrm{Fib}(F(X_x)) \to \pi_i (F(X_x)) \to \text{Ker}(\pi_{i+1}(i^*)) \]
\[ \text{Coker}(\pi_{i+1}(i^*)) \to \pi_i \mathrm{Fib}(F(X'_p) \to \pi_i (F(X'_p)) \to \text{Ker}(\pi_{i+1}(i^*)) \]

The kernels in the above short exact sequences are trivial by [26, Theorem 2.15(3)] and [18, Corollary 3.6(2)], and the maps
\[ \text{Coker}(\pi_{i+1}(i^*)) \to \text{Coker}(\pi_{i+1}(i^*)) \]

are isomorphisms by [26, Theorem 2.15(5)] in combination with the main result of [19] and [18, Corollary 3.6(4)]. \( \square \)

4.3. Zariski and Nisnevich motivic localisations. Let \( k \) be a field.

Definition 4.13. An \( \infty \)-category of correspondences \( \text{Corr}_k \) over \( k \), see Definition [1.1], satisfies the property \( \text{\acute{E}t} \text{Ex} \) if any \( \Delta^1 \)-invariant presheaf \( F \in \text{PSh}_k^\Delta(S) \) is \( R \)-excisive over the essentially smooth scheme \( X_S \), for any Nisnevich square \( R \) of the form [1.8] over \( k \). See Definition [4.7]

Example 4.14. Proposition [1.13] equivalently claims that the \( \infty \)-category \( \text{Corr}^{\mathrm{gp}, \text{fr}}(k) \) equipped with the composite functor \( \text{Sm}_k \to \text{Corr}^\Delta(k) \to \text{Corr}^{\mathrm{gp}, \text{fr}}(k) \) satisfies \( \text{\acute{E}t} \text{Ex} \).

Lemma 4.15. Given an \( \infty \)-category of correspondences \( \text{Corr}_k \) over \( k \), see Definition [4.7] that satisfies \( \text{\acute{E}t} \text{Ex} \), then an \( \Delta^1 \)-invariant presheaf \( F \in \text{PSh}_k^\Delta(S) \) is a Nisnevich sheaf if and only if \( F \) is a Zariski sheaf. In particular, any group-like \( \Delta^1 \)-invariant Zariski sheaf \( F \in \text{PSh}_k^\Delta(S) \) is a Nisnevich sheaf.

Proof. The first claim follows by Definition [4.13] from Lemma [4.10] the second by Example [4.14] \( \square \)

Theorem 4.16. Let \( k \) be a field, and \( \text{Corr}_k \) be an \( \infty \)-category of correspondences over \( k \) that satisfies \( \text{\acute{E}t} \text{Ex} \). There is an equivalence of \( \infty \)-categories \( \text{H}^{\text{fr}}(k) \simeq \text{H}_\text{zar}^{\text{fr}}(k) \). In particular, \( \text{H}_\text{nis}^{\text{fr}}(k) \simeq \text{H}^{\text{fr}, \text{gp}}(k), \text{SH}^{\text{fr}}_\text{nis}(k) \simeq \text{SH}^{\text{fr}}_\text{zar}(k). \)

Proof. The first claim follows from Lemma [4.15] since the \( \infty \)-categories \( \text{H}^{\text{fr}}(k) \) and \( \text{H}_\text{nis}^{\text{fr}}(k) \) are the full subcategories of in \( \text{PSh}_k^\Delta(S) \) whose objects are group-like \( \Delta^1 \)-invariant Nisnevich and Zariski sheaves, respectively. The case of framed presheaves follows because of Example [4.13] and the equivalence \( \text{PSh}^{\text{fr}, \text{gp}}(k) \simeq \text{PSh}(\text{Corr}^{\text{fr}, \text{gp}}(k)) \) provided by [30, Lemma 1.6]. Note that the case of \( \text{H}^{\text{fr}}(k) \) implies the equivalences for \( \text{SH}^{\text{fr}, \text{gp}}(k), \text{SH}^{\text{fr}}(k). \) \( \square \)

4.4. Result over base schemes.

Lemma 4.17. Let \( \nu \) be the Zariski or the Nisnevich topology on \( \text{Sm}_S \) over a noetherian separated scheme \( S \in \text{Sch} \) of finite Krull dimension. Then

1. the canonical restriction
\[ \text{Shv}_\nu^{\text{fr}}(S) \to \text{Shv}_\nu^{\text{fr}}(\text{SmAff}_S) \]
is an equivalence;
2. for a Zariski covering \( \tilde{S} \to S \) the inverse image functor
\[ \text{Shv}_\nu^{\text{fr}}(S) \to \text{Shv}_\nu^{\text{fr}}(\tilde{S}) \]
is conservative.
**Proof.** (1) Any scheme in $\text{Sm}_{S}$ has a Zariski covering $\nu: \tilde{S} \rightarrow S$ in $\text{SmAff}_{S}$, and consequently, there is a $\nu$-local equivalence $h(S) \simeq \nu \colim_{[n] \in A} h(\tilde{S}^{\times n})$. Applying $\gamma^{\ast}: \text{PSh}(S) \rightarrow \text{PSh}^{\text{tr}}(S)$, we get $h^{\text{tr}}(S) \simeq \nu \colim_{[n] \in A} h^{\text{tr}}(\tilde{S}^{\times n})$. Hence the fully faithful Kan extension functor $\text{Shv}_{\nu}^{\text{tr}}(\text{SmAff}_{S}) \rightarrow \text{Shv}_{\nu}^{\text{tr}}(S)$ is essentially surjective. So the claim follows.

(2) A morphism $F \rightarrow G \in \text{Shv}_{\nu}^{\text{tr}}(S)$ is an equivalence if the induced map $\text{Map}(h^{\text{tr}}(X), F) \rightarrow \text{Map}(h^{\text{tr}}(X), G)$ is an equivalence for any $X \in \text{Sm}_{S}$. Using the equivalence shown above, this induced map can be rewritten as

$$\lim_{[n] \in A} F(X \times S \tilde{S}^{\times n}) \rightarrow \lim_{[n] \in A} G(X \times S \tilde{S}^{\times n}).$$

Each map in the simplicial diagram is an equivalence by assumption. □

**Theorem 4.18.** Let $\text{Corr}(-,-)$ be a family of preadditive $\infty$-categories of correspondences on $\text{Sch}_{\text{Sch}^{\text{h}}}$, see Definition 1.4.7, that is continuous, see Definition 1.13, satisfies (AHP), see Definition 1.23, satisfies (\text{\acute{E}tEx}) for each field $k$, see Definition 1.7, and satisfies the closed gluing on affine schemes, satisfies (H), see Definition 1.28, and satisfies (\text{\acute{E}tEx}) for each field $k$, see Definition 1.7.

Then the canonical functor

$$\text{H}^{\text{tr}}_{\text{sp}}(S) \rightarrow \text{H}^{\text{tr}}_{\text{nis}}(S),$$

is an equivalence for any noetherian separated scheme $S$ of finite Krull dimension.

**Proof.** By Theorem 4.10 the claim holds over fields. Let $S$ be affine scheme of finite Krull dimension. Let $F \in \text{H}^{\text{tr}}_{\text{sp}}(\text{SmAff}_{S})$. The claim is that $F \in \text{H}^{\text{tr}}_{\text{nis}}(\text{SmAff}_{S})$. By Theorem 4.10 $F$ goes to the Nisnevich sheaf under the functor

$$\text{H}^{\text{tr}}_{\text{sp}}(\text{SmAff}_{S}) \rightarrow \prod_{\tau \in \mathcal{F}} \text{H}^{\text{tr}}_{\tau}(\text{SmAff}_{\tau}),$$

By Lemma 4.39 applied to the topologies $\text{zf}$ and $\text{nis}$ in view of Example 1.19 the claim follows. So the claim holds for the $\infty$-categories $\text{H}^{\text{tr}}_{\text{sp}}(\text{SmAff}_{S})$, where $\tau = \text{zf}$, $\text{nis}$.

By the first point of Lemma 4.17 the claim for the $\infty$-categories $\text{H}^{\text{tr}}_{\text{sp}}(\text{SmAff}_{S})$ follows. The claim for an arbitrary scheme $S$ of finite Krull dimension follows by the second point of Lemma 4.17 applied to a Zariski covering $\coprod_{\beta} S_{\beta} \rightarrow S$ with affine schemes $S_{\beta}$.

**Remark 4.19.** It is expected that Theorem 4.18 holds for all qcqs schemes for the appropriate definition of the Zariski topology.

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