An alternative explanation of the conflict between $1/R$ gravity and solar system tests

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Abstract
Recently the $1/R$ gravity has been proposed in order to explain the accelerated expansion of the universe. However, it was argued that the $1/R$ gravity conflicts with solar system tests. While this statement is true if one views the $1/R$ gravity as an effective theory, we find that this difficulty might be avoided if one treats the $1/R$ term as a correction to the scalar curvature term in the high curvature limit $R \gg \mu^2$.

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The growing evidence indicates that the universe is undergoing a period of accelerated expansion [1-3], which presents one of the greatest problems in theoretical physics today: what drives the accelerated expansion of the universe? The accelerated expansion is usually explained through violations of strong energy condition by introducing an extra component in the Einstein equations in the form of dark energy with an equation of state $\omega < -1/3$. The simplest possibility for dark energy is a cosmological constant. Unfortunately, this explanation is plagued with theoretical problems such as why the observed value is 120 orders in magnitude less than the theoretical estimates [4]. More recently there have been a number of different attempts to modify gravity to yield accelerated cosmologies at late times [5-10]. One of famous modifications is the so-called 1/R gravity suggested by Carroll, Duvvuri, Trodden, and Turner (CDTT) [5,11], with the following action

$$S = \frac{\kappa^2}{2} \int d^4x \sqrt{-g}(R - \frac{\mu^4}{R}) + \int d^4x \sqrt{-g}L_M$$

(1)

where $\mu$ is a parameter with dimensions of mass and $L_M$ is the Lagrangian density for matter. When the scalar curvature $R \gg \mu^2$, one expects that effect of the corrected term $\mu^4/R$ can be neglected. In this case, the theory reduces to the usual general relativity. When the corrected term $\mu^4/R$ and the Hilbert-Einstein term $R$ can be comparable to each other, this theory significantly deviates from general relativity. In particular, for the low curvature with $R \sim \mu^2$, it was found that the term $\mu^4/R$ can lead to an accelerated expansion in late-time cosmologies [5]. If one chooses $\mu$ to be in the order of current Hubble scale, this gravity theory can describe current epoch of the accelerated universe very well [10].

However, it was argued that such a gravity theory is equivalent to the Brans-Dicke theory with a vanishing Brans-Dicke parameter $\omega$ and a potential. Base on this equivalence it was subsequently proved by a number of authors that this theory is in conflict with solar system tests [12-20]. Although some others suggested the Palatini form by treating the metric and connection as independent dynamical variables in the variational principle, which seems to give a hope of constructing a viable model to describe currently accelerated expansion of the universe, the post-Newtonian approximation shows that it is still incompatible with solar system observations [15,17,21].

For this theory to explain the cosmic acceleration, it requires $\mu \sim H_0$, where $H_0 \sim 1.5 \times 10^{-33}$eV is the current Hubble scale. In the equivalent Brans-Dicke theory with $\omega=0$, the effective mass of scalar field is in the order of $\mu$. Since the Brans-Dicke theory has been extensively studied in the literature and its post-Newtonian limit is well known, the smallness of $\mu$ is obviously inconsistent.
with experiments [12].

From Eq. (1) it is expected that as $\mu \to 0$, the general relativity can be recovered. From the equivalent Brans-Dick theory, however, the general relativity cannot be recovered due to $\omega = 0$. This seems not a consistent result. Since we wish that in the low curvature limit (or in the cosmic scale), the term $1/R$ makes sense; on the other hand, in the high curvature limit (or in the solar system scale), the general relativity can be recovered, that is the term $1/R$ can be neglected in this case. In this sense, one may view that the term $1/R$ in (1) appears just as a corrected term to the Hilbert-Einstein term $R$ in the high curvature limit. If we go in this way, the post-Newtonian approximation of the theory (1) should be reconsidered since so far all investigations about the post-Newtonian approximation on the $1/R$ gravity are based on the expansion about a de Sitter space with constant curvature, $R_0 = 3^{1/2}\mu^2$. This is true if one views the $1/R$ gravity as an effective theory and it holds for the whole range of space-time curvature. However, we want to emphasize here that in the high curvature limit, the term $1/R$ appears just as a corrected term and the theory described by the action (1) is not an effective theory. In this sense, $\mu^4$ can be regarded as an expansion parameter in the high curvature limit. Therefore the Newtonian approximation of general relativity still holds here and the term $\mu^4/R$ just gives a tiny modification to the post-Newtonian approximation of general relativity.

In this short note we will give the modification, due to the term $\mu^4/R$, to the post-Newtonian approximation of general relativity. Since the effect of the term $\mu^4/R$ is very tiny in the solar system, it will not give rise to any conflict with solar system test of gravity. This expanding technique reduced the number of the degrees of freedom and replaced the $1/R$ gravity theory (which contains an extra scalar degree of freedom [22]) with a different theory without extra degrees of freedom. Our Lagrangian agrees to that of the $1/R$ theory only in the first two terms, and it contains additional terms with higher orders in $\mu^4/R$. However, we will not include such higher order terms in following discussion since such terms make no sense in the high curvature limit, for the purpose of demonstration, considering the term $1/R$ is enough.

We start by varying Eq. (1) with respect to the metric, which yields the following equations of motion

$$(1 + \frac{\mu^4}{R^2})R_{\mu\nu} - \frac{1}{2}(1 - \frac{\mu^4}{R^2})Rg_{\mu\nu} - \mu^4(\nabla_\mu \nabla_\nu \frac{1}{R^2} + g_{\mu\nu} \nabla_\alpha \nabla_\alpha \frac{1}{R^2}) = \kappa^2 T_{\mu\nu}$$

(2)

Here the metric satisfies a system of fourth-order partial differential equations. Eq. (2) has two constant curvature vacuum solutions. For the interest from cosmological point of view, one usually considers the positive constant-curvature solution with scalar curvature $R_0 = 3^{1/2}\mu^2$, so that the
The universe can accelerated expand at late times. This is the de Sitter solution mentioned above. The fourth-order equations are very complicated and it turns out convenient to consider the trace equation of Eq. (2)

$$-R + 3\mu^4 \left( \frac{1}{R} + \nabla^\alpha \nabla_\alpha \frac{1}{R^2} \right) = \kappa^2 T,$$

(3)

where the curvature is dynamic and is equivalent to a scalar field $\phi = 1 + \mu^4/R^2$. The vacuum de Sitter solution corresponds to the case with $\phi_0 = 4/3$.

In order to compare this theory with the solar system experiments, one can calculate the approximately static solutions. The usual post-Newtonian approximation is performed around the de Sitter vacuum, where the curvature is expressed as $R = R_0 + \delta R$. The corresponding scalar field is expressed in the form $\phi = \phi_0 + \delta \phi$. $\delta R$ (or $\delta \phi$) represents the local deviation from the background curvature $R_0$ (or $\phi_0$) and vanishes far from the local system. Linearizing Eq. (3) requires $R_0 \gg \delta R$. However, this does not hold for the high curvature limit $R \gg \mu^2$. Note that the solar system just belongs to the high curvature case. That is, the breaking down of $R_0 \gg \delta R$ in the solar system invalidates the post-Newtonian approximation around the de Sitter vacuum, which is also noticed in Refs.[20,21].

Since we regard the term $\mu^4/R$ as a corrected term to the Hilbert-Einstein action $R$, we therefore expand the curvature around the parameter $\mu$, not around the de Sitter background. If $\mu = 0$, we have from Eqs. (2) and (3) that $R = -T \approx \rho$ in the lowest post-Newtonian approximation, where we have taken $\kappa = 1$ and $\rho$ the energy density of the local system. In the local solar system, we have $\rho \gg \mu^2$ and the solution of Eq. (3) can be expanded in orders of $\mu^4/\rho^2$ as

$$R \approx \rho + \frac{\mu^4}{\rho^2} R_1 + \frac{\mu^8}{\rho^4} R_2 + \cdots. \tag{4}$$

Substituting Eq. (4) into Eq. (3) and expanding in orders of $\mu^4/\rho^2$, we obtain the first two terms of corrections

$$R_1 = 3\rho + 3\rho^2 \nabla^\alpha \nabla_\alpha \frac{1}{\rho^2}, \quad R_2 = -3R_1 - 6\rho^4 \nabla^\alpha \nabla_\alpha \frac{R_1}{\rho^5}. \tag{5}$$

By using the expansion Eq. (4), we can rewrite the equations of motion for metric in orders of $\mu^4/\rho^2$ as

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T - \frac{\mu^4}{\rho^2} (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \rho) + \mu^4 (\nabla_\mu \nabla_\nu \frac{1}{\rho^2} + \frac{1}{2} g_{\mu\nu} \nabla^\alpha \nabla_\alpha \frac{1}{\rho^5}). \tag{6}$$

Next we calculate the post-Newtonian approximation around the Minkowskian space with $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$. To the order $\mu^4/\rho^2$ and the lowest post-Newtonian approximation, the metric satisfies
the following equations

\[ \nabla^2 h_{00} = -(1 - 3\frac{\mu^4}{\rho^2})\rho + \mu^4\nabla^2 \frac{1}{\rho^2}, \]
\[ \nabla^2 h_{ij} = -\delta_{ij}(1 + \frac{\mu^4}{\rho^2})\rho - \delta_{ij}\mu^4\nabla^2 \frac{1}{\rho^2} - 2\mu^4 \partial_i \partial_j \frac{1}{\rho^2}, \]  

(7)

where the harmonic coordinate condition has been used. For simplicity, we consider the field of a static spherically symmetric mass source, where the mass density \( \rho \) is a function of \( r \) only. Then the metric can be easily integrated to give

\[ h_{00} \approx 2U + \frac{\mu^4}{\rho^2} \]
\[ h_{ij} \approx \delta_{ij}(2U - \frac{\mu^4}{\rho^2}) - 2\mu^4[(\delta_{ij} - \frac{3x_i x_j}{r^2}\frac{1}{\rho^2})\frac{1}{\rho^2}\int_0^r \frac{r^2}{\rho^2} dr + \frac{x_i x_j}{\rho^2}] \]  

(8)

where we have kept the first order of \( U \) and \( \mu^4/\rho^2 \) with \( U \) the Newtonian potential. It might be worth stressing here that the expansion (4) does not hold for point sources and it should be kept in mind that the term \( \mu^4/\rho^2 \) in (8) is always less than \( U \). For the solar system, taking the mean density as \( \rho \sim 10^{-10}\text{g/cm}^3 \), we have \( \mu^4/\rho^2 \sim 10^{-36} \), which is far smaller than the Newtonian potential \( U \). Therefore, the 1/R gravity is compatible with solar system experiments in our setup.

In summary, while it was claimed that the 1/R gravity as an effective theory is not consistent with the solar system observation of gravity, we have shown that if we view the term 1/R as a corrected term to the Hilbert-Einstein scalar curvature term \( R \) in the high curvature limit, namely in the case of \( R \gg \mu^2 \), the term \( \mu^4/R \) will not make any conflict with the solar system test of gravity. We have given the modification of the post-Newtonian approximation of the general relativity due to the term \( \mu^4/R \). In our setup, the 1/R gravity is compatible with the solar system test. However we would like to stress here that these modified models of gravity contain higher order terms of 1/R may or may not be viable for the expansion of the Universe, since these higher order terms will be as large as the \( \mu^4/R \) term when the Universe begins to accelerate. Finally, it is an interesting issue to see whether in our setup the 1/R gravity could be ruled out by observation of gravitational force in the laboratory [22]. At first look from (6), one might worry that the 1/R gravity will give very strongly gravitational force at low densities since 1/\( \rho \) appears in (6). In fact, we can see from (4) that our approximation holds only for the case \( \mu^4/\rho^2 \ll 1 \). When \( \rho^2 \sim \mu^4 \), we need to consider the complete effective gravity, which contains other forms of corrections. The main purpose of this note is just to point out that the usual treatment of post-Newtonian approximation of the 1/R gravity is not applicable to the solar system tests of gravity.
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