Research article

Quantum Montgomery identity and quantum estimates of Ostrowski type inequalities

Mehmet Kunt\textsuperscript{1}, Artion Kashuri\textsuperscript{2}, Tingsong Du\textsuperscript{3,∗} and Abdul Wakil Baidar\textsuperscript{1,4}

\textsuperscript{1} Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, 61080, Trabzon, Turkey
\textsuperscript{2} Department of Mathematics, Faculty of Technical Science, University Ismail Qemali, Vlora, Albania
\textsuperscript{3} Department of Mathematics, College of Science, China Three Gorges University, 443002, Yichang, P. R. China
\textsuperscript{4} Kabul University, Department of Mathematics, Kabul, Afghanistan

* Correspondence: Email: tingsongdu@ctgu.edu.cn.

Abstract: In this paper, the new version of the celebrated Montgomery identity is determined via quantum integral operators. By using it, certain quantum integral inequalities of Ostrowski type are established. Moreover, the relevant connection of the obtained results of this work with the derived results in previously published works is discussed.

Keywords: convex functions; quantum differentiable; quantum integrable; Ostrowski type inequality

Mathematics Subject Classification: 26D15, 26A51, 05A30

1. Introduction

The following inequality is named the Ostrowski type inequality [30].

\textbf{Theorem 1.} [10] Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on $(a, b)$ and $f' \in L[a, b]$ (i.e. $f'$ be an integrable function on $[a, b]$). If $|f'(x)| < M$ on $[a, b]$, then the following inequality holds:

$$
\left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(t) \, dt \right| \leq \frac{M}{b-a} \left[ \frac{(x-a)^2 + (b-x)^2}{2} \right] \quad (1.1)
$$

for all $x \in [a, b]$. 

To prove the Ostrowski type inequality in (1.1), the following identity is used, (see [26]):

\[ f(x) = \frac{1}{b-a} \int_a^b f(t) \, dt + \int_a^b \frac{t-a}{b-a} f'(t) \, dt + \int_a^b \frac{b-t}{b-a} f'(t) \, dt, \]

(1.2)

where \( f(x) \) is a continuous function on \([a, b]\) with a continuous first derivative in \((a, b)\). The identity (1.2) is known as Montgomery identity.

By changing variable, the Montgomery identity (1.2) can be expressed as:

\[ f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt = (b-a) \int_0^1 K(t) f'(tb + (1-t)a) \, dt, \]

(1.3)

where

\[ K(t) = \begin{cases} t, & t \in \left[0, \frac{x-a}{b-a}\right], \\ t-1, & t \in \left(\frac{x-a}{b-a}, 1\right]. \end{cases} \]

A number of different identities of the Montgomery and many inequalities of Ostrowski type were obtained by using these identities. Through the framework of Montgomery’s identity, Cerone and Dragomir [9] developed a systematic study which produced some novel inequalities. By introducing some parameters, Budak and Sarıkaya [8] as well as Özdemir et al. [31] established the generalized Montgomery-type identities for differentiable mappings and certain generalized Ostrowski-type inequalities, respectively. Aljinović in [1], presented another simpler generalization of the Montgomery identity for fractional integrals by utilizing the weighted Montgomery identity. Furthermore, the generalized Montgomery identity involving the Ostrowski type inequalities in question with applications to local fractional integrals can be found in [32]. For more related results considering the different Montgomery identities, [2, 4, 7, 11–13, 15, 16, 21–25, 33, 34, 36] and the references therein can be seen.

In the related literature of Montgomery type identity, it was not considered via quantum integral operators. The aim of this work is to set up a quantum Montgomery identity with respect to quantum integral operators. With the help of this new version of Montgomery identity, some new quantum integral inequalities such as Ostrowski type, midpoint type, etc are established. The absolute values of the derivatives of considered mappings are quantum differentiable convex mappings.

Throughout this paper, let \(0 < q < 1\) be a constant. It is known that quantum calculus constructs in a quantum geometric set. That is, if \(q x \in A\) for all \(x \in A\), then the set \(A\) is called quantum geometric.

Suppose that \(f(t)\) is an arbitrary function defined on the interval \([0, b]\). Clearly, for \(b > 0\), the interval \([0, b]\) is a quantum geometric set. The quantum derivative of \(f(t)\) is defined with the following expression:

\[ D_q f(t) := \frac{f(t) - f(qt)}{(1-q)t}, \quad t \neq 0, \]

\[ D_q f(0) := \lim_{t \to 0} D_q f(t). \]

(1.4)

Note that

\[ \lim_{q \to 1^-} D_q f(t) = \frac{df(t)}{dt}, \]

(1.5)

if \(f(t)\) is differentiable.

The quantum integral of \(f(t)\) is defined as:

\[ \int_0^b f(t) \, dq t = (1-q) b \sum_{n=0}^{\infty} q^n f(q^n b) \]

(1.6)
and
\[ \int_c^b f(t) \, dq_t = \int_0^b f(t) \, dq_t - \int_0^c f(t) \, dq_t, \]
where \(0 < c < b\) (see [3, 14]).

Note that if the series in right-hand side of (1.6) is convergence, then \(\int_0^b f(t) \, dq_t\) is exist, i.e., \(f(t)\) is quantum integrable on \([0, b]\). Also, provided that if \(\int_0^b f(t) \, dt\) converges, then one has
\[ \lim_{q \to 1} \int_0^b f(t) \, dq_t = \int_0^b f(t) \, dt. \] (see [3, page 6]).

These definitions are not sufficient in establishing integral inequalities for a function defined on an arbitrary closed interval \([a, b] \subset \mathbb{R}\). Due to this fact, Tariboon and Ntouyas in [37, 38] improved these definitions as follows:

**Definition 1.** [37, 38]. Let \(f : [a, b] \to \mathbb{R}\) be a continuous function. The \(q\)-derivative of \(f\) at \(t \in [a, b]\) is characterized by the expression:
\[ aD_q f(t) = \frac{f(t) - f(qt + (1-q)a)}{(1-q)(t-a)}, \quad t \neq a, \]
\[ aD_q f(a) = \lim_{t \to a} aD_q f(t). \]
The function \(f\) is said to be \(q\)-differentiable on \([a, b]\), if \(aD_q f(t)\) exists for all \(t \in [a, b]\).

Clearly, if \(a = 0\) in (1.9), then \(0D_q f(t) = D_q f(t)\), where \(D_q f(t)\) is familiar quantum derivatives given in (1.4).

**Definition 2.** [37, 38]. Let \(f : [a, b] \to \mathbb{R}\) be a continuous function. Then the quantum definite integral on \([a, b]\) is defined as
\[ \int_a^b f(t) \, dq_t = (1-q)(b-a) \sum_{n=0}^{\infty} q^n f(q^n b + (1-q^n)a) \] (1.10)
and
\[ \int_c^b f(t) \, dq_t = \int_a^b f(t) \, dq_t - \int_a^c f(t) \, dq_t, \] (1.11)
where \(a < c < b\).

Clearly, if \(a = 0\) in (1.10), then
\[ \int_0^b f(t) \, dq_t = \int_0^b f(t) \, dt, \]
where \(\int_0^b f(t) \, dq_t\) is familiar definite quantum integrals on \([0, b]\) given in (1.6).

Definition 1 and Definition 2 have actually developed previous definitions and have been widely used for quantum integral inequalities. There is a lot of remarkable papers about quantum integral inequalities based on these definitions, including Kunt et al. [19] in the study of the quantum Hermite–Hadamard inequalities for mappings of two variables considering convexity and quasi-convexity on the
Lemma 1. (Similar case with AIMS Mathematics Volume 5, Issue 6, 5439–5457. (2\textsuperscript{t} for all \textit{t}.)

Since \textit{f} is called quantum differentiable on \textit{t}, it is not necessary that the function \textit{f} must be continuous on \textit{[a, b]}.

2. Main results

Firstly, we discuss the assumptions of the continuity of the function \( f(t) \) in Definition 1 and Definition 2. Also, under these conditions, we want to discuss that similar cases with (1.5) and (1.8) can exist.

By considering the Definition 1, it is not necessary that the function \( f(t) \) must be continuous on \textit{[a, b]}. Indeed, for all \( t \in [a, b] \), \( qt + (1-q)a \in [a, b] \) and \( f(t) - f(qt + (1-q)a) \in \mathbb{R} \). It means that \( \frac{f(t)-f(qt+(1-q)a)}{(1-q)(t-a)} \in \mathbb{R} \) exists for all \( t \in (a, b) \), so the Definition 1 should be as follows:

**Definition 3.** (Quantum derivative on \textit{[a, b]}) Let \( f : [a, b] \to \mathbb{R} \) be an arbitrary function. Then \( f \) is called quantum differentiable on \textit{[a, b]} with the following expression:

\[
_{a}D_{q}f(t) = \frac{f(t) - f(qt + (1-q)a)}{(1-q)(t-a)} \in \mathbb{R}, \ t \neq a
\]

and \( f \) is called quantum differentiable on \( t = a \), if the following limit exists:

\[
_{a}D_{q}f(a) = \lim_{t \to a} _{a}D_{q}f(t).
\]

**Lemma 1.** (Similar case with (1.5)) Let \( f : [a, b] \to \mathbb{R} \) be a differentiable function. Then we have

\[
\lim_{q \to 1^{-}}_{a}D_{q}f(t) = \frac{df(t)}{dt}.
\]

**Proof.** Since \( f \) is differentiable on \textit{[a, b]}, clearly we have

\[
\lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = \frac{df(t)}{dt}
\]

for all \( t \in (a, b) \). Since \( 0 < q < 1 \), for all \( a < t \leq b \), we have \((1-q)(a-t) < 0 \). Changing variable in (2.2) as \((1-q)(a-t) = h \), then \( q \to 1^{-} \) we have \( h \to 0^{-} \) and \( qt + (1-q)a = t + h \). Using (2.3), we have

\[
\lim_{q \to 1^{-}}_{a}D_{q}f(t) = \lim_{q \to 1^{-}} \frac{f(t) - f(qt + (1-q)a)}{(1-q)(t-a)}
\]
\[
\begin{align*}
&= \lim_{q \to 1^-} \frac{f(qt + (1 - q)a) - f(t)}{(1 - q)(a - t)} \\
&= \lim_{h \to 0^+} \frac{f(t + h) - f(t)}{h} \\
&= \lim_{h \to 0^+} \frac{f(t + h) - f(t)}{h} \\
&= \frac{df(t)}{dt}
\end{align*}
\]
for all \( t \in (a, b) \). On the other hand, for \( t = a \) we have
\[
\begin{align*}
\lim_{q \to 1^-} aD_q f(a) &= \lim_{q \to 1^-} \lim_{t \to a} aD_q f(t) \\
&= \lim_{t \to a} \lim_{q \to 1^-} aD_q f(t) \\
&= \lim_{t \to a} \frac{df(t)}{dt} \\
&= \lim_{h \to 0^+} \frac{f(t + h) - f(t)}{h} \\
&= \lim_{h \to 0^+} \frac{f(t + h) - f(t)}{h} \\
&= \lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h} \\
&= \frac{df(a)}{dt},
\end{align*}
\]
which completes the proof. \( \square \)

In Definition 2, the condition of the continuity of the function \( f(t) \) on \([a, b]\) is not required. For this purpose, it is enough to construct an example in which a function is discontinuous on \([a, b]\), but quantum integrable on it.

**Example 1.** Let \( 0 < q < 1 \) be a constant, and the set \( A \) is defined as
\[
A := \{q^n 2 + (1 - q^n) (-1) : n = 0, 1, 2, \ldots \} \subset [-1, 2].
\]
Then the function \( f : [-1, 2] \to \mathbb{R} \) defined as
\[
f(t) := \begin{cases} 
1, & t \in A, \\
0, & t \in [-1, 2] \setminus A.
\end{cases}
\]
Clearly, it is not continuous on \([-1, 2]\). On the other hand
\[
\int_{-1}^{2} f(t) \, dq = (1 - q) (2 - (-1)) \sum_{n=0}^{\infty} q^n f(q^n 2 + (1 - q^n) (-1))
\]
\[
= 3 (1 - q) \sum_{n=0}^{\infty} q^n = 3 (1 - q) \frac{1}{1 - q} = 3,
\]
i.e., the function \( f(t) \) is quantum integrable on \([-1, 2]\).
Hence the Definition 2 should be described in the following way.

**Definition 4.** *(Quantum definite integral on \([a, b]\))* Let \(f : [a, b] \to \mathbb{R}\) be an arbitrary function. Then the quantum integral of \(f\) on \([a, b]\) is defined as

\[
\int_a^b f(t) \, \text{d}q_t = (1 - q)(b - a) \sum_{n=0}^{\infty} q^n f(q^n b + (1 - q^n) a).
\] (2.4)

If the series in right-hand side of (2.4) is convergent, then \(\int_a^b f(t) \, \text{d}q_t\) exist, i.e., \(f(t)\) is quantum integrable on \([a, b]\).

**Lemma 2.** *(Similar case with (1.8))* Let \(f : [a, b] \to \mathbb{R}\), be an arbitrary function. It provided that if \(\int_a^b f(t) \, dt\) converges, then we have

\[
\lim_{q \to 1} \int_a^b f(t) \, \text{d}q_t = \int_a^b f(t) \, dt.
\] (2.5)

**Proof.** If \(\int_a^b f(t) \, dt\) converges, then \(\int_0^1 f(tb + (1 - t)a) \, dt\) also converges. Using (1.8), we have that

\[
\lim_{q \to 1} \int_a^b f(t) \, \text{d}q_t = \lim_{q \to 1} \left[ (1 - q)(b - a) \sum_{n=0}^{\infty} q^n f(q^n b + (1 - q^n) a) \right]
\]

\[
= (b - a) \lim_{q \to 1} \int_0^1 f(tb + (1 - t)a) \, \text{d}q_t
\]

\[
= (b - a) \int_0^1 f(tb + (1 - t)a) \, dt
\]

\[
= \int_a^b f(t) \, dt.
\]

Next we present an important quantum Montgomery identity, which is similar with the identity in (1.3).

**Lemma 3.** *(Quantum Montgomery identity)* Let \(f : [a, b] \to \mathbb{R}\), be an arbitrary function with \(aD_q f\) is quantum integrable on \([a, b]\), then the following quantum identity holds:

\[
f(x) - \frac{1}{b-a} \int_a^b f(t) \, \text{d}q_t = (b - a) \int_0^1 K_q(t) \, aD_q f(tb + (1 - t)a) \, \text{d}q_t,
\] (2.6)

where

\[
K_q(t) = \begin{cases} 
qt, & t \in [0, \frac{a}{b-a}], \\
qt - 1, & t \in (\frac{a}{b-a}, 1].
\end{cases}
\]

**Proof.** By the Definition 3, \(f(t)\) is quantum differentiable on \((a, b)\) and \(aD_q f\) is exist. Since \(aD_q f\) is quantum integrable on \([a, b]\), by the Definition 4, the quantum integral for the right-side of (2.6) is exist. The integral of the right-side of (2.6), with the help of (2.1) and (2.4), is equal to

\[
(b - a) \int_0^1 K_q(t) \, aD_q f(tb + (1 - t)a) \, \text{d}q_t
\]
\[
\begin{align*}
&= (b - a) \left[ \int_0^{\frac{x_a}{x+b+a}} qr_a f (tb + (1 - t) a) \quad 0d_q t + \int_{\frac{x_a}{x+b+a}}^1 (qt - 1) \quad 0D_q f (tb + (1 - t) a) \quad 0d_q t \right] \\
&= (b - a) \left[ \int_0^{\frac{x_a}{x+b+a}} qr_a f (tb + (1 - t) a) \quad 0d_q t + \int_{\frac{x_a}{x+b+a}}^1 (qt - 1) \quad 0D_q f (tb + (1 - t) a) \quad 0d_q t \\
&\quad - \int_{\frac{x_a}{x+b+a}}^{\frac{x_a}{x+b+a}} (qt - 1) \quad 0D_q f (tb + (1 - t) a) \quad 0d_q t \right] \\
&= (b - a) \left[ \int_0^1 (qt - 1) \quad 0D_q f (tb + (1 - t) a) \quad 0d_q t + \int_{\frac{x_a}{x+b+a}}^1 (qt - 1) \quad 0D_q f (tb + (1 - t) a) \quad 0d_q t \\
&\quad - \int_{\frac{x_a}{x+b+a}}^{\frac{x_a}{x+b+a}} (qt - 1) \quad 0D_q f (tb + (1 - t) a) \quad 0d_q t \right] \\
&= (b - a) \left[ \int_0^1 qr_a f (tb + (1 - t) a) - f (qttb + (1 - qt) a) \quad 0d_q t \\
&\quad - \int_0^1 f (tb + (1 - t) a) - f (qttb + (1 - qt) a) \quad 0d_q t \\
&\quad + \int_{\frac{x_a}{x+b+a}}^1 \frac{f(t(t-1)a) - f(qtb + (1 - qt) a)}{(1 - q) t (b - a)} \quad 0d_q t \right] \\
&= \frac{1}{1 - q} \left[ q \left[ \int_0^1 f (tb + (1 - t) a) \quad 0d_q t - \int_0^1 f (qttb + (1 - qt) a) \quad 0d_q t \right] \\
&\quad - \left[ \int_0^1 \frac{f(t(t-1)a)}{t} \quad 0d_q t - \int_0^1 \frac{f(qtb + (1 - qt) a)}{t} \quad 0d_q t \right] \\
&\quad + \left[ \int_{\frac{x_a}{x+b+a}}^1 \frac{f(t(t-1)a)}{t} \quad 0d_q t - \int_{\frac{x_a}{x+b+a}}^1 \frac{f(qtb + (1 - qt) a)}{t} \quad 0d_q t \right] \right]
\end{align*}
\]
which completes the proof. □

Remark 1. If one takes limit $q \to 1^-$ on the Quantum Montgomery identity in (2.6), one has the Montgomery identity in (1.3).

The following calculations of quantum definite integrals are used in next result:

\[
\int_{0}^{\frac{x-a}{b-a}} q^{t} \, dt = q(1-q) \frac{x-a}{b-a} \sum_{n=0}^{\infty} q^n \left( \frac{x-a}{b-a} q^n \right)
= q(1-q) \left( \frac{x-a}{b-a} \right)^2 \frac{1}{1-q^2}
= \frac{q}{1+q} \left( \frac{x-a}{b-a} \right)^2,
\]

\[
\int_{0}^{\frac{x-a}{b-a}} q^{t^2} \, dt = q(1-q) \frac{x-a}{b-a} \sum_{n=0}^{\infty} q^n \left( \frac{x-a}{b-a} q^n \right)^2
= q(1-q) \left( \frac{x-a}{b-a} \right)^3 \frac{1}{1-q^2}
= \frac{q}{1+q+q^2} \left( \frac{x-a}{b-a} \right)^3.
\]
\[
\int_{\frac{x}{b}}^{1} (1 - qt) \quad \text{d}q \cdot t = \int_{0}^{1} (1 - qt) \quad \text{d}q \cdot t - \int_{0}^{\frac{x}{b}} (1 - qt) \quad \text{d}q \cdot t
\]

\[
= (1 - q) \sum_{n=0}^{\infty} q^n (1 - q^n) - (1 - q) \frac{x - a}{b - a} \sum_{n=0}^{\infty} q^n \left( 1 - q^n \frac{x - a}{b - a} \right)
\]

\[
= (1 - q) \left( \frac{1}{1 - q} - \frac{q}{1 - q^2} \right) - (1 - q) \frac{x - a}{b - a} \left( \frac{1}{1 - q} - \frac{q}{1 - q^2} \frac{x - a}{b - a} \right)
\]

\[
= \frac{1}{1 + q} - \frac{x - a}{b - a} \left( 1 - \frac{q}{1 + q} \frac{x - a}{b - a} \right)
\]

\[
= \frac{1}{1 + q} - \left( 1 - \frac{b - x}{b - a} \right) \left( \frac{1}{1 + q} + \frac{q}{1 + q} \frac{b - x}{b - a} \right)
\]

\[
= \left[ \frac{1}{1 + q} - \frac{1}{1 + q} - \frac{q}{1 + q} \frac{b - x}{b - a} + \frac{q}{1 + q} \frac{b - x}{b - a} + \frac{q}{1 + q} \frac{(b - x)^2}{b - a} \right]
\]

\[
= \frac{q}{1 + q} \frac{(b - x)^2}{b - a},
\]

and

\[
\int_{\frac{x}{b}}^{1} (t - qt^2) \quad \text{d}q \cdot t = \int_{0}^{1} (t - qt^2) \quad \text{d}q \cdot t - \int_{0}^{\frac{x}{b}} (t - qt^2) \quad \text{d}q \cdot t
\]

\[
= (1 - q) \sum_{n=0}^{\infty} q^n (q^n - q^{2n})
\]

\[
- (1 - q) \frac{x - a}{b - a} \sum_{n=0}^{\infty} q^n \left( q^n \frac{x - a}{b - a} - q^{2n} \left( \frac{x - a}{b - a} \right)^2 \right)
\]

\[
= (1 - q) \left( \frac{1}{1 - q^2} - \frac{q}{1 - q^3} \right)
\]

\[
- (1 - q) \frac{x - a}{b - a} \left( \frac{1}{1 - q^3} \frac{x - a}{b - a} - \frac{q}{1 - q^3} \left( \frac{x - a}{b - a} \right)^2 \right)
\]

\[
= \left( \frac{1}{1 + q} - \frac{q}{1 + q + q^2} \right)
\]

\[
- \frac{x - a}{b - a} \left( \frac{1}{1 + q} \frac{x - a}{b - a} - \frac{q}{1 + q + q^2} \left( \frac{x - a}{b - a} \right)^2 \right)
\]

\[
= \frac{1}{(1 + q)(1 + q + q^2)} - \frac{1}{1 + q} \left( \frac{x - a}{b - a} \right)^2 + \frac{q}{1 + q + q^2} \left( \frac{x - a}{b - a} \right)^3.
\]

Let us introduce some new quantum integral inequalities by the help of quantum power mean inequality and Lemma 3.
Theorem 3. Let $f : [a, b] \rightarrow \mathbb{R}$ be an arbitrary function with $\|aD_q f\|'$ is quantum integrable on $[a, b]$. If $\|aD_q f\|' r$, $r \geq 1$ is a convex function, then the following quantum integral inequality holds:

$$
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, _a d_q t \right| \leq (b-a) \left[ K_1^{1-\frac{1}{r}} (a, b, x, q) \left|_{aD_q f (a)} K_2 (a, b, x, q) + \left|_{aD_q f (b)} K_3 (a, b, x, q) \right|^{ \frac{1}{r} } \right] + K_4^{1-\frac{1}{r}} (a, b, x, q) \left[ \left|_{aD_q f (a)} K_5 (a, b, x, q) + \left|_{aD_q f (b)} K_6 (a, b, x, q) \right|^{ \frac{1}{r} } \right] 
$$

(2.11)

for all $x \in [a, b]$, where

$$
K_1 (a, b, x, q) = \int_0^\frac{x-a}{b-a} qt \, d_q t = \frac{q}{1+q} \left( \frac{x-a}{b-a} \right)^2,
$$

$$
K_2 (a, b, x, q) = \int_0^\frac{x-a}{b-a} q t^2 \, d_q t = \frac{q}{1+q+q^2} \left( \frac{x-a}{b-a} \right)^3,
$$

$$
K_3 (a, b, x, q) = \int_0^\frac{x-a}{b-a} qt - q t^2 \, d_q t = K_1 (a, b, x, q) - K_2 (a, b, x, q),
$$

$$
K_4 (a, b, x, q) = \int_\frac{x-a}{b-a}^1 (1 - qt) \, d_q t = \frac{q}{1+q} \left( \frac{b-x}{b-a} \right)^2,
$$

$$
K_5 (a, b, x, q) = \int_\frac{x-a}{b-a}^1 (t - q t^2) \, d_q t = \frac{1}{(1+q)(1+q+q^2)} - \frac{1}{1+q} \left( \frac{x-a}{b-a} \right)^2 + \frac{q}{1+q+q^2} \left( \frac{x-a}{b-a} \right)^3,
$$

and

$$
K_6 (a, b, x, q) = \int_\frac{x-a}{b-a}^1 \left( 1 - qt - t + q t^2 \right) \, d_q t = K_4 (a, b, x, q) - K_5 (a, b, x, q).
$$

Proof. Using convexity of $\|aD_q f\|'$, we have that

$$
\|aD_q f (tb + (1-t) a)\|^r \leq t \|aD_q f (a)\|^r + (1-t) \|aD_q f (b)\|^r. \tag{2.12}
$$

By using Lemma 3, quantum power mean inequality and (2.12), we have that
Using (2.13), we obtain the desired result in (2.11). This ends the proof. □

**Corollary 1.** In Theorem 3, the following inequalities are held by the following assumptions:

1. If one takes \( r = 1 \), one has

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, \alpha d\gamma t \right| \leq (b-a) \left[ \left| \alpha D_q f (a) \right| K_2 (a, b, x, q) + \left| \alpha D_q f (b) \right| K_3 (a, b, x, q) \right. \\
+ \left. \left| \alpha D_q f (a) \right| K_5 (a, b, x, q) + \left| \alpha D_q f (b) \right| K_6 (a, b, x, q) \right].
\]

2. If one takes \( r = 1 \) and \( |\alpha D_q f (x)| < M \) for all \( x \in [a, b] \), then one has (a quantum Ostrowski type inequality, see [27, Theorem 3.1])

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, \alpha d\gamma t \right| \leq M (b-a) \left[ K_2 (a, b, x, q) + K_3 (a, b, x, q) \right. \\
+ \left. K_5 (a, b, x, q) + K_6 (a, b, x, q) \right] \\
\leq M (b-a) \left[ K_1 (a, b, x, q) + K_4 (a, b, x, q) \right] \\
\leq M (b-a) \left[ \frac{q}{1+q} \left( \frac{x-a}{b-a} \right)^2 + \frac{q}{1+q} \left( \frac{b-x}{b-a} \right)^2 \right]
\]
3. If one takes \( r = 1 \), \( |\mathcal{D}_q f(x)| < M \) for all \( x \in [a, b] \) and \( q \to 1^- \), then one has (Ostrowski inequality (1.1)).

4. If one takes \( r = 1 \) and \( x = \frac{qa + b}{1 + q} \), then one has (a new quantum midpoint type inequality)

\[
\left| f \left( \frac{qa + b}{1 + q} \right) - \frac{1}{b-a} \int_a^b f(t) \ |a_d q(t)\right| \leq (b-a) \left[ |\mathcal{D}_q f(a)| K_2 \left( a, b, \frac{qa + b}{1 + q}, q \right) + |\mathcal{D}_q f(b)| K_3 \left( a, b, \frac{qa + b}{1 + q}, q \right) \right]
\]

\[
\leq (b-a) \left[ \frac{q}{(1+q)^3 (1+q+q^2)} + |\mathcal{D}_q f(b)| \frac{q^2 + q^3}{(1+q)^3 (1+q+q^2)} \right]
\]

5. If one takes \( r = 1 \), \( x = \frac{qa + b}{1 + q} \) and \( q \to 1^- \), then one has (a midpoint type inequality, see [17, Theorem 2.2])

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq (b-a) \left[ |f'(a)| + |f'(b)| \right] \frac{1}{8}
\]

6. If one takes \( r = 1 \) and \( x = \frac{a+b}{2} \), then one has (a new quantum midpoint type inequality)

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) \ |a_d q(t)\right| \leq (b-a) \left[ \frac{q}{8 (1+q+q^2)} + |\mathcal{D}_q f(b)| \frac{q + q^2 + 2q^3}{8 (1+q)(1+q+q^2)} \right]
\]

\[
\leq (b-a) \left[ \frac{6}{8 (1+q)(1+q+q^2)} + |\mathcal{D}_q f(b)| \frac{3q + 3q^2 + 2q^3 - 6}{8 (1+q)(1+q+q^2)} \right]
\]

7. If one takes \( |\mathcal{D}_q f(x)| < M \) for all \( x \in [a, b] \), then one has (a quantum Ostrowski type inequality, see [27, Theorem 3.1])

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \ |a_d q(t)\right| \leq \frac{4q + 4q^2 + 4q^3 - 6}{8 (1+q)(1+q+q^2)}
\]
\[\begin{align*}
\leq (b - a) M \left[ K_1^{1 - \frac{1}{r}} (a, b, x, q) [ K_2 (a, b, x, q) + K_3 (a, b, x, q)] \right] ^{\frac{1}{2}} \\
+ K_4^{1 - \frac{1}{r}} (a, b, x, q) [ K_5 (a, b, x, q) + K_6 (a, b, x, q)] ^{\frac{1}{2}}
\end{align*}\]
\[\begin{align*}
\leq (b - a) M \left[ K_1^{1 - \frac{1}{r}} (a, b, x, q) K_1^{\frac{1}{2}} (a, b, x, q) + K_4^{1 - \frac{1}{r}} (a, b, x, q) K_4^{\frac{1}{2}} (a, b, x, q) \right]
\end{align*}\]
\[\begin{align*}
\leq (b - a) M [ K_1 (a, b, x, q) + K_4 (a, b, x, q) ]
\end{align*}\]
\[\begin{align*}
\leq M (b - a) \left[ \frac{q}{1 + q} \left( \frac{x - a}{b - a} \right)^2 + \frac{q}{1 + q} \left( \frac{b - x}{b - a} \right)^2 \right]
\end{align*}\]
\[\begin{align*}
\leq \frac{qM}{b - a} \left[ \frac{(x - a)^2 + (b - x)^2}{1 + q} \right].
\end{align*}\]

8. If one takes \( x = \frac{qa + b}{1 + q} \), then one has (a new quantum midpoint type inequality)
\[\begin{align*}
\left| f \left( \frac{qa + b}{1 + q} \right) - \frac{1}{b - a} \int_a^b f (t) \, dq_t \right| 
\leq (b - a) \left[ K_1^{1 - \frac{1}{r}} \left( a, b, \frac{qa + b}{1 + q}, q \right) \left| a D_q f (a) \right|^r K_2 \left( a, b, \frac{qa + b}{1 + q}, q \right) + \left| a D_q f (b) \right|^r K_3 \left( a, b, \frac{qa + b}{1 + q}, q \right) \right] ^{\frac{1}{2}} \\
+ K_4^{1 - \frac{1}{r}} \left( a, b, \frac{qa + b}{1 + q}, q \right) \left[ K_5 \left( a, b, \frac{qa + b}{1 + q}, q \right) + \left| a D_q f (b) \right|^r K_6 \left( a, b, \frac{qa + b}{1 + q}, q \right) \right] ^{\frac{1}{2}}
\end{align*}\]
\[\begin{align*}
\leq (b - a) \left[ \frac{q}{1 + q} \left( \frac{1}{3} \right) \left| a D_q f (a) \right|^r \left( \frac{2q}{1 + q} \right) \left( \frac{q}{3} \right) \left( 1 + q + q^2 \right) + \left| a D_q f (b) \right|^r \left( \frac{-2q + q^2 + q^4 + q^5}{(1 + q)^3 (1 + q + q^2)} \right) \right] ^{\frac{1}{2}}
\end{align*}\]

9. If one takes \( x = \frac{qa + b}{1 + q} \) and \( q \to 1^- \), then one has (a midpoint type inequality, see [6, Corollary 17])
\[\begin{align*}
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f (t) \, dt \right| \leq (b - a) \left( \frac{1}{2^{1 - \frac{1}{r}}} \left[ \left( \left| f' (a) \right|^r \frac{1}{12} + \left| f' (b) \right|^r \frac{1}{12} \right) ^{\frac{1}{r}} \right] \right)^{\frac{1}{2}}
\end{align*}\]

10. If one takes \( x = \frac{a + b}{2} \), then one has (a new quantum midpoint type inequality)
\[\begin{align*}
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f (t) \, dq_t \right| 
\leq (b - a) \left[ K_1^{1 - \frac{1}{r}} \left( a, b, \frac{a + b}{2}, q \right) \left| a D_q f (a) \right|^r K_2 \left( a, b, \frac{a + b}{2}, q \right) + \left| a D_q f (b) \right|^r K_3 \left( a, b, \frac{a + b}{2}, q \right) \right] ^{\frac{1}{2}} \\
+ K_4^{1 - \frac{1}{r}} \left( a, b, \frac{a + b}{2}, q \right) \left[ K_5 \left( a, b, \frac{a + b}{2}, q \right) + \left| a D_q f (b) \right|^r K_6 \left( a, b, \frac{a + b}{2}, q \right) \right] ^{\frac{1}{2}}
\end{align*}\]
\[\begin{align*}
\leq (b - a) \left( \frac{q}{4 (1 + q)} \right) ^{\frac{1}{2}} \left[ \left| a D_q f (a) \right|^r \frac{q}{8 (1 + q + q^2)} + \left| a D_q f (b) \right|^r \frac{q + q^2 + 2q^3}{8 (1 + q) (1 + q + q^2)} \right] ^{\frac{1}{2}}
\end{align*}\]
Finally, we give the following calculated quantum definite integrals used as the next Theorem 4.

11. If one takes \( x = \frac{a + qb}{1 + q} \), then one has (a new quantum midpoint type inequality)

\[
\left| f \left( \frac{a + qb}{1 + q} \right) - \frac{1}{b - a} \int_a^b f(t) \, d_q t \right| \\
\leq (b - a) \left[ K_1^{1, \frac{1}{2}} \left( a, b, \frac{a + qb}{1 + q}, q \right) \right] \\
+ K_3^{1, \frac{1}{2}} \left( a, b, \frac{a + qb}{1 + q}, q \right) \left[ a D_q f (a) \right] \left( a, b, \frac{a + qb}{1 + q}, q \right) + K_3 \left( a, b, \frac{a + qb}{1 + q}, q \right) \left[ a D_q f (b) \right] \left( a, b, \frac{a + qb}{1 + q}, q \right)
\]

Finally, we give the following calculated quantum definite integrals used as the next Theorem 4.

\[
\int_0^{x-a} \frac{t \, d_q t}{b-a} = (1-q) \frac{x-a}{b-a} \sum_{n=0}^{\infty} q^n \left( \frac{x-a}{b-a} \right)
\]

\[
= (1-q) \left( \frac{x-a}{b-a} \right)^2 \frac{1}{1-q^2}
\]

\[= \frac{1}{1+q} \left( \frac{x-a}{b-a} \right)^2, \tag{2.14} \]

\[
\int_0^{x-a} (1-t) \, d_q t = (1-q) \frac{x-a}{b-a} \sum_{n=0}^{\infty} q^n \left( 1-q^n \frac{x-a}{b-a} \right)
\]

\[
= (1-q) \frac{x-a}{b-a} \left( \frac{1}{1-q} - \frac{x-a}{b-a} \frac{1}{1-q^2} \right)
\]

\[= \frac{x-a}{b-a} \left( 1 - \frac{1}{1+q} \frac{x-a}{b-a} \right)
\]

\[= \frac{x-a}{b-a} - \frac{1}{1+q} \left( \frac{x-a}{b-a} \right)^2, \tag{2.15} \]

\[
\int_0^{\frac{1}{1+q}} t \, d_q t = \int_0^1 t \, d_q t - \int_0^{\frac{1}{1+q}} t \, d_q t
\]

\[= \frac{1}{1+q} - \frac{1}{1+q} \left( \frac{x-a}{b-a} \right)^2
\]

\[= \frac{1}{1+q} \left( 1 - \left( \frac{x-a}{b-a} \right)^2 \right), \tag{2.16} \]
\begin{align}
\int_{\frac{r}{b-a}}^{1} (1-t) \, d_q t &= \int_{0}^{1} (1-t) \, d_q t - \int_{0}^{\frac{r}{b-a}} (1-t) \, d_q t \\
&= \frac{q}{1+q} - \frac{x-a}{b-a} + \frac{1}{1+q} \left( \frac{x-a}{b-a} \right)^2 \tag{2.17}
\end{align}

**Theorem 4.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be an arbitrary function with \( _aD_q f \) is quantum integrable on \([a, b]\). If \( \| _aD_q f \| _r' \), \( r > 1 \) and \( \frac{1}{r} + \frac{1}{p} = 1 \) is a convex function, then the following quantum integral inequality holds:

\[
\left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(t) \, d_q t \right| \\
\leq (b-a) \left( \int_{0}^{\frac{r}{b-a}} q t \, d_q t \right)^{\frac{1}{p}} \left( \int_{0}^{\frac{r}{b-a}} \left| _aD_q f(a) \right|^{r'} \left[ \frac{\frac{1}{1+q} \left( \frac{x-a}{b-a} \right)^2}{x-a} - \frac{1}{1+q} \left( \frac{x-a}{b-a} \right)^2 \right] \right)^{\frac{1}{r'}} + \left( \int_{\frac{r}{b-a}}^{1} (1-qt) \, d_q t \right)^{\frac{1}{p}} \left( \int_{\frac{r}{b-a}}^{1} \left| _aD_q f(b) \right|^{r'} \left[ \frac{\frac{1}{1+q} \left( 1- \frac{x-a}{b-a} \right)^2}{x-a} + \frac{1}{1+q} \left( \frac{x-a}{b-a} \right)^2 \right] \right)^{\frac{1}{r'}} \tag{2.18}
\]

for all \( x \in [a, b] \).

**Proof.** By using Lemma 3, quantum Hölder inequality and (2.8), we have that

\[
\left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(t) \, d_q t \right| \\
\leq (b-a) \int_{0}^{1} \left| K_q(t) \right| \left| _aD_q f(tb + (1-t)a) \right| \, d_q t \\
\leq (b-a) \left[ \int_{0}^{\frac{r}{b-a}} q t \left| _aD_q f(tb + (1-t)a) \right| \, d_q t + \int_{\frac{r}{b-a}}^{1} (1-qt) \left| _aD_q f(tb + (1-t)a) \right| \, d_q t \right] \\
\leq (b-a) \left[ \left( \int_{0}^{\frac{r}{b-a}} (qt)^p \, d_q t \right)^{\frac{1}{p}} \left( \int_{0}^{\frac{r}{b-a}} \left| _aD_q f(tb + (1-t)a) \right|^{r'} \, d_q t \right)^{\frac{1}{r'}} \right]^{\frac{1}{p}} + \left( \int_{\frac{r}{b-a}}^{1} (1-qt)^p \, d_q t \right)^{\frac{1}{p}} \left( \int_{\frac{r}{b-a}}^{1} \left| _aD_q f(tb + (1-t)a) \right|^{r'} \, d_q t \right)^{\frac{1}{r'}} \]

AIMS Mathematics
\[
\leq (b-a) \left[ \left( \int_0^{\frac{x-a}{b-a}} (qt)^p \, dq \right)^{\frac{1}{p}} \left( \int_0^{\frac{x-a}{b-a}} \left| \frac{1}{a} D_q f (a) \right| ^{p} + (1-t) \left| \frac{1}{a} D_q f (b) \right| ^{p} \, dq \right)^{\frac{1}{p}} \right]^{\frac{1}{p}} + \left( \int_0^{\frac{x-a}{b-a}} (1-qt)^p \, dq \right)^{\frac{1}{p}} \left( \int_0^{\frac{x-a}{b-a}} \left| \frac{1}{a} D_q f (a) \right| ^{p} + (1-t) \left| \frac{1}{a} D_q f (b) \right| ^{p} \, dq \right)^{\frac{1}{p}} \\
\leq (b-a) \left[ \left( \int_0^1 qt \, dq \right)^{\frac{1}{p}} \left( \int_0^1 \left| \frac{1}{a} D_q f (a) \right| ^{p} + \left| \frac{1}{a} D_q f (b) \right| ^{p} \, dq \right)^{\frac{1}{p}} \right]^{\frac{1}{p}} + \left( \int_0^1 (1-qt)^p \, dq \right)^{\frac{1}{p}} \left( \int_0^1 \left| \frac{1}{a} D_q f (a) \right| ^{p} + \left| \frac{1}{a} D_q f (b) \right| ^{p} \, dq \right)^{\frac{1}{p}} \\
+ \left( \int_0^1 (1-qt)^p \, dq \right)^{\frac{1}{p}} \left( \int_0^1 \left| \frac{1}{a} D_q f (a) \right| ^{p} + \left| \frac{1}{a} D_q f (b) \right| ^{p} \, dq \right)^{\frac{1}{p}} \right]
\]

(2.19)

Using (2.14)–(2.17) in (2.19), we obtain the desired result in (2.18). This ends the proof. \( \square \)

**Remark 2.** In Theorem 4, many different inequalities could be derived similarly to Corollary 1.

### 3. Conclusions

In the terms of quantum Montgomery identity, some quantum integral inequalities of Ostrowski type are established. The establishment of the inequalities is based on the mappings whose first derivatives absolute values are quantum differentiable convex. Furthermore, the important relevant connection obtained in this work with those which were introduced in previously published papers is investigated. By considering the special value for \( x \in [a,b] \), some fixed value for \( r \), and as well as \( q \to 1^- \), many sub-results can be derived from the main results of this work. It is worthwhile to mention that certain quantum inequalities presented in this work are generalized forms of the very recent results given by Alp et al. (2018) and Noor et al. (2016). With the contribution of this work, the interested researchers will be motivated to explore this fascinating field of the quantum integral inequality based on the techniques and ideas developed in this article.

### Acknowledgments

The first author would like to thank Ondokuz Mayis University for being a visiting professor and providing excellent research facilities.

### Conflict of interest

The authors declare that they have no competing interests.
References

1. A. A. Aljinović, Montgomery identity and Ostrowski type inequalities for Riemann–Liouville fractional integral, J. Math., 2014 (2014), Article ID 503195, 1–6.
2. G. A. Anastassiou, Ostrowski type inequalities, Proc. Amer. Math. Soc., 123 (1995), 3775–3781.
3. M. H. Annaby, Z. S. Mansour, q-Fractional Calculus and Equations, Berlin: Springer, 2012.
4. M. Alomari, M. Darus, S. S. Dragomir, et al. Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, Appl. Math. Lett., 23 (2010), 1071–1076.
5. N. Alp, M. Z. Sarıkaya, A new definition and properties of quantum integral which calls q-integral, Konuralp J. Math., 5 (2017), 146–159.
6. N. Alp, M. Z. Sarıkaya, M. Kunt, et al. q-Hermite–Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions, J. King Saud Univ. Sci., 30 (2018), 193–203.
7. Y. Bascı, D. Baleanu, Ostrowski type inequalities involving ψ-hilfer fractional integrals, Mathematics, 2019 (2019), Article ID 770, 1–10.
8. H. Budak, M. Z. Sarıkaya, On generalized Ostrowski-type inequalities for functions whose first derivatives absolute values are convex, Turkish J. Math., 40 (2016), 1193–1210.
9. P. Cerone, S. S. Dragomir, On some inequalities arising from Montgomery’s identity, J. Comput. Anal. Appl., 5 (2003), 341–367.
10. S. S. Dragomir, T. M. Rassias, Ostrowski type inequalities and applications in numerical integration, Netherlands: Springer, 2002.
11. G. Farid, Some new Ostrowski type inequalities via fractional integrals, Int. J. Anal. Appl., 14 (2017), 64–68.
12. M. GÜRBÜZ, Y. TAŞDAN, E. SET, Ostrowski type inequalities via the Katugampola fractional integrals, AIMS Math., 5 (2019), 42–53.
13. İ. İŞCAN, Ostrowski type inequalities for p-convex functions, New Trends Math. Sci., 4 (2016), 140–150.
14. V. Kac, P. Cheung: Quantum calculus, New York: Springer, 2002.
15. H. Kavurmacı, M. E. Özdemir, M. Avci, New Ostrowski type inequalities for m-convex functions and applications, Hacet. J. Math. Stat., 40 (2011), 135–145.
16. M. E. Kiriş, M. Z. Sarıkaya, On Ostrowski type inequalities and Čebyšev type inequalities with applications, Filomat, 29 (2015), 1695–1713.
17. U. S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comput., 147 (2004), 137–146.
18. M. Kunt, İ. İşcan, N. Alp, et al. (p, q)-Hermite–Hadamard inequalities and (p, q)-estimates for midpoint type inequalities via convex and quasi-convex functions, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, 112 (2018), 969–992.
19. M. Kunt, M. A. Latif, İ. İşcan, et al. *Quantum Hermite-Hadamard type inequality and some estimates of quantum midpoint type inequalities for double integrals*, Sigma J. Eng. Nat. Sci., 37 (2019), 207–223.

20. W. J. Liu, H. F. Zhuang, *Some quantum estimates of Hermite-Hadamard inequalities for convex functions*, J. Appl. Anal. Comput., 7 (2017), 501–522.

21. Z. Liu, *Some Ostrowski type inequalities*, Math. Comput. Model., 48 (2008), 949–960.

22. W. Liu, X. Gao, Y. Wen, *Approximating the finite Hilbert transform via some companions of Ostrowski’s inequalities*, Bull. Malays. Math. Sci. Soc., 39 (2016), 1499–1513.

23. W. Liu, A. Tuna, *Diamond-α weighted Ostrowski type and Grüss type inequalities on time scales*, Appl. Math. Comput., 270 (2015), 251–260.

24. W. Liu, A. Tuna, Y. Jiang, *On weighted Ostrowski type, trapezoid type, Grüss type and Ostrowski–Grüss like inequalities on time scales*, Appl. Anal., 93 (2014), 551–571.

25. M. Matloka, *Ostrowski type inequalities for functions whose derivatives are h-convex via fractional integrals*, J. Sci. Res. & Rep., 3 (2014), 1633–1641.

26. D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Inequalities for functions and their integrals and derivatives*, Netherlands: Springer, 1991.

27. M. A. Noor, M. U. Awan, K. I. Noor, *Quantum Ostrowski inequalities for q-differentiable convex functions*, J. Math. Inequal., 10 (2016), 1013–1018.

28. M. A. Noor, K. I. Noor, M. U. Awan, *Some quantum estimates for Hermite-Hadamard inequalities*, Appl. Math. Comput., 251 (2015), 675–679.

29. M. A. Noor, K. I. Noor, M. U. Awan, *Some quantum integral inequalities via preinvex functions*, Appl. Math. Comput., 269 (2015), 242–251.

30. A. Ostrowski, *Über die absolutabweichung einer differenzierbaren funktion von ihrem integralmittelwert*, Comment. Math. Helv., 10 (1938), 226–227.

31. M. E. Özdemir, H. Kavurmacı, M. Avci, *Ostrowski type inequalities for convex functions*, Tamkang J. Math., 45 (2014), 335–340.

32. M. Z. Sarıkaya, H. Budak, *Generalized Ostrowski type inequalities for local fractional integrals*, Proc. Amer. Math. Soc., 145 (2017), 1527–1538.

33. E. Set, A. O. Akdemir, A. Gözpınar, et al. *Ostrowski type inequalities via new fractional conformable integrals*, AIMS Math., 4 (2019), 1684–1697.

34. E. Set, M. E. Özdemir, M. Z. Sarıkaya, *New inequalities of Ostrowski’s type for s-convex functions in the second sense with applications*, Facta Univ. Ser. Math. Inform., 27 (2012), 67–82.

35. W. Sudsutad, S. K. Ntouyas, J. Tariboon, *Quantum integral inequalities for convex functions*, J. Math. Inequal., 9 (2015), 781–793.

36. S. F. Tahir, M. Mushtaq, M. Muddassar, *A note on integral inequalities on time scales associated with Ostrowski’s type*, J. Funct. Spaces, 2019 (2019), Article ID 4748373, 1–6.

37. J. Tariboon, S. K. Ntouyas, *Quantum calculus on finite intervals and applications to impulsive difference equations*, Adv. Difference Equ., 2013 (2013), Article ID 282, 1–19.
38. J. Tariboon, S. K. Ntouyas, *Quantum integral inequalities on finite intervals*, J. Inequal. Appl., 2014 (2014), Article ID 121, 1–13.

39. M. Tunç, E. Göv, S. Balgeçti, *Simpson type quantum integral inequalities for convex functions*, Miskolc Math. Notes, 19 (2018), 649–664.

40. Y. Zhang, T. S. Du, H. Wang, et al. *Different types of quantum integral inequalities via (α, m)-convexity*, J. Inequal. Appl., 2018 (2018), Article ID 264, 1–24.

41. H. F. Zhuang, W. J. Liu, J. Park, *Some quantum estimates of Hermite-Hadamard inequalities for quasi-convex functions*, Mathematics, 7 (2019), Article ID 152, 1–18.