The Classical and Quantum Theory of Relativistic p-Branes without Constraints*

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Abstract

It is shown that a relativistic (i.e. a Poincaré invariant) theory of extended objects (called p-branes) is not necessarily invariant under reparametrizations of corresponding p-dimensional worldsheets (including worldlines for $p = 0$). Consequently, no constraints among the dynamical variables are necessary and quantization is straightforward. Additional degrees of freedom so obtained are given a physical interpretation as being related to membrane’s elastic deformations (“wiggleness”). In particular, such a more general, unconstrained theory implies as solutions also those p-brane states that are solutions of the conventional theory of the Dirac-Nambu-Gotto type.

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1. Introduction

Quantization of relativistic extended object such as $p$-dimensional membranes (often called $p$-branes) has not yet been satisfactory solved in general. There has been much progress in dealing with strings ($p = 1$) and point particles ($p = 0$)\footnote{General theory of relativistic point particles and strings is now a standard knowledge, therefore it is difficult to cite selected particular works among so many important original contributions. For a review and list of references see e.g. M.Kaku \cite{1}} but the treatment of quantized higher dimensional objects, with $p > 2$, in spite of important particular results \cite{2} is not yet completed in general. Such intricacies arise because the $p + 1$ dimensional worldsheet swept by a $p$-brane is invariant with respect to reparametrizations; a consequence is the existence of constraints.

An alternative approach, which has been elaborated in the case of point particle, is to dispense with constraint and formulate the classical and quantum relativistic theory by assuming that all coordinates and momenta are independent \cite{3}-\cite{6}. In the unconstrained theory mass is not fixed but occurs as a constant of motion and a free particle still follows a straight line with uniform speed \cite{4}-\cite{6}. Even in the presence of an electromagnetic field it turns out that a solution of the constrained (conventional) theory is also a solution of the unconstrained theory \cite{4}.

In the quantized unconstrained theory the parameter $\tau$ of evolution is explicitly present. Therefore the theory is also called the parametrized relativistic quantum mechanics. This elegant theory (manifestly covariant under Poincaré group at every step) has been initialized by Fock and followed by many workers \cite{3}. It is more general than the conventional (constrained) quantum theory, since mass is not definite. But in particular, the theory admits also the existence of definite mass eigenstates.

In the present paper I propose to extend the unconstrained theory of a point particle to extended objects. For this purpose I first reformulate the constrained classical theory of a $p$-brane by using the generalized Howe-Tucker action \cite{7} in which I isolate $d = p + 1$ independent Lagrange multipliers by splitting the metric tensor $\gamma_{ab}$ in the ADM-like manner \cite{8}. So we obtain an action and a Hamiltonian which look like those of a point particle.
except for the integration over a space-like hypersurface on the worldsheet. This reformulation of the classical constrained $p$-brane theory is interesting in itself and possibly important for quantization even without recourse to the unconstrained theory which is given in Sec.4. In Sec.5 I discuss $p$-branes with variable tension (wiggly membranes) and in Sec.6 I compare them with the unconstrained membranes.

2. The unconstrained point particle theory

The idea that space-time coordinates $x^\mu$ of a relativistic point particle should be considered as independent has been pursued by many authors $[3]-[6]$. Formally this has been achieved $[4]-[6]$ by replacing the first order action (called also phase space action) where $\lambda$ is a Lagrange multiplier

$$I[x, p, \lambda] = \int d\tau \left( p_\mu \dot{x}^\mu - \frac{\lambda}{2} (p^2 - m^2) \right)$$

with another action, similar in form but different in content,

$$I[x, p] = \int d\tau \left( p_\mu \dot{x}^\mu - \frac{\Lambda}{2} (p^2 - m^2) \right)$$

in which $\Lambda$ is not a quantity to be varied, but it is a fixed function of the evolution parameter $\tau$. The latter action is not invariant with respect to reparametrizations of $\tau$, therefore there is no constraint, and all $x^\mu$ and $p_\mu$ are independent dynamical variables. And yet (2) and all equations derived from it are invariant under Poincaré transformations.

Analogous procedure can be used in the second order (Howe-Tucker) action

$$I[x, \lambda] = \frac{1}{2} \int d\tau \left( \frac{\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}}{\Lambda} + \Lambda m^2 \right)$$

We can replace it with another action which is solely a functional of $x^\mu$:

$$I[x] = \frac{1}{2} \int d\tau \left( \frac{\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}}{\Lambda} + \Lambda m^2 \right)$$

where $\Lambda$ is a fixed function of $\tau$ or a constant and $g_{\mu\nu}$ the metric tensor of spacetime.
The equation of motion derived from (4) is
\[ \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{\Lambda} \right) - \frac{1}{2\Lambda} g_{\alpha\beta,\mu} \dot{x}^\alpha \dot{x}^\beta = 0 \] (5)

This can be recast into a more familiar geodetic-like equation
\[ \frac{1}{\Lambda} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{\Lambda} \right) + \Gamma^\mu_{\alpha\beta} \frac{\dot{x}^\alpha \dot{x}^\beta}{\Lambda^2} = 0 \] (6)

where \( \Gamma^\mu_{\alpha\beta} \) is the affinity composed of the spacetime metric \( g_{\mu\nu} \). In eq.(5) and (6) all the variables \( x^\mu \) are independent.

Let us now consider the quadratic form
\[ M^2(\tau) = g_{\alpha\beta} \frac{\dot{x}^\alpha \dot{x}^\beta}{\Lambda^2} \] (7)

and calculate its derivative with respect to \( \tau \). We find
\[ M \dot{M} = \frac{1}{2} \frac{d}{d\tau} \left( g_{\alpha\beta} \frac{\dot{x}^\alpha \dot{x}^\beta}{\Lambda^2} \right) = g_{\alpha\beta} \frac{\ddot{x}^\alpha}{\Lambda} \frac{d}{d\tau} \left( \frac{\dot{x}^\beta}{\Lambda} \right) + \frac{1}{2} g_{\alpha\beta,\mu} \frac{\dot{x}^\mu \dot{x}^\alpha \dot{x}^\beta}{\Lambda^2} = \dot{x}_\mu \left( \frac{1}{\Lambda} \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{\Lambda} \right) + \Gamma^\mu_{\alpha\beta} \frac{\dot{x}^\alpha \dot{x}^\beta}{\Lambda^2} \right) = 0 \] (8)

In equating the above expression (8) to zero we have used the equation of motion (6).

From eq.(8) we conclude that
\[ M^2 = g_{\mu\nu} \frac{\dot{x}^\mu \dot{x}^\nu}{\Lambda^2} = g_{\mu\nu} p^\mu p^\nu = \text{constant} \] (9)

where \( p_\mu = \partial L / \partial \dot{x}^\mu = \dot{x}^\mu / \Lambda \) is the canonical momentum. \( M^2 \) is thus a constant of motion even in the presence of the background gravitational field. We may call \( M \) mass, but mass is here not a fixed constant (entering the Lagrangian, like in the conventional theory); it is an arbitrary constant of motion, and there is no constraint among the momenta \( p_\mu \).

By expressing \( \Lambda \) in terms of \( M \) and \( \dot{x}^2 \) (see eq. (7)) we find that eq.(8) becomes indistinguishable from the usual geodetic equation of the constraint theory:
\[ \frac{1}{\sqrt{x^2}} \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{\sqrt{x^2}} \right) + \Gamma^\mu_{\alpha\beta} \frac{\dot{x}^\alpha \dot{x}^\beta}{x^2} = 0 \] (10)

\[ \text{Eq.(8) follows directly from eq.(3) if we insert } \dot{x}_\mu = g_{\mu\nu} \dot{x}^\nu \text{ into (3) and use } \frac{d}{d\tau} \left( \frac{g_{\mu\nu} \dot{x}^\nu}{\Lambda} \right) = g_{\mu\nu} \frac{d}{d\tau} \left( \frac{\dot{x}^\nu}{\Lambda} \right) + g_{\mu\nu,\alpha} \frac{\dot{x}^\nu \dot{x}^\alpha}{\Lambda}. \] The equation so obtained must then be multiplied by \( g^{\mu\rho} \) (and summed over \( \mu \)) and the definition of the affinity \( \Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\rho} (g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}) \) has to be taken into account.
The last equation is reparametrization invariant, because we used the equation (9) for a fixed value of the constant of motion $M$ and because for a fixed $M$ eq. (11) acts as a constraint. But the original equation (5) is not reparametrization invariant. For a constant $\Lambda$ eq. (5) has the same form as the geodetic equation expressed in terms of proper time. The trajectory of spatial coordinates $x^r, r = 1, 2, ..., D$ is the same in both the constrained and unconstrained theory. But in the unconstrained theory the zero component, $\mu = 0$, of equation (5) has also a dynamical meaning, it is not a redundant equation. In a previous paper [4] I proposed a physical meaning of coordinate time $x^0$ evolving in terms of the evolution or historical time $\tau$. According to that interpretation this expresses the fact that an observer doesn’t perceive a worldline all at once, but instead he perceives it point by point (i.e. event by event) along the increasing $x^0 \equiv t$. It is indeed true that in the way we perceive the world there is something more than in the way the conventional relativity describes it. I can only perceive the events close to the intersection point of a time-like hypersurface (time slice or simultaneity hypersurface) with the worldline of my body. That is, I perceive my "now", but I cannot perceive past or future events. And in order to be able to denote this momentary time slice being perceived right now we need an additional parameter besides the coordinate time $x^0$. The additional parameter is just $\tau$, the evolution or historical time. An observer then infers that the time slice intersects also other worldlines besides his own one and that it is moving forward in space time, the intersection points (events) progressing along worldlines. Only the progression of events and not the whole worldline is perceived. The relation $x^0 = x^0(\tau)$ traces such a progression of events on a worldline as it is perceived by an observer.

The above interpretation obtains an even more transparent meaning in the quantized theory. First of all, the parametrized relativistic first and second quantized theory is very elegant [3]-[6]. Hamiltonian is not zero and it generates the true evolution which is governed by the Schrödinger equation:

$$i \frac{\partial \psi}{\partial \tau} = \mathcal{H} \psi, \quad \mathcal{H} = \frac{\Lambda}{2} \left( (-i)^2 \partial_\mu \partial^\mu - m^2 \right)$$

(11)
A general solution of eq.(11) is given by

\[ \psi(\tau, x) = \int d^D p \, c(p) e^{ip_{\mu}x^\mu - \frac{i\Lambda}{2}(p^2 - m^2)\tau} \tag{12} \]

and is normalized in spacetime:

\[ \int \psi^* (\tau, x) \psi (\tau, x) \, d^4 x = 1 \tag{13} \]

It may represent a wave packet which is localized in spacetime and which moves in spacetime. The probability of finding a particle (or better an event) at a given value of \( \tau \) is different from zero in a certain region \( \Omega \) of spacetime and negligibly small (or zero) outside \( \Omega \). At later value of \( \tau \) the wave packet is shifted into another spacetime region. Thus a wave packet center sweeps a worldline in spacetime. But at every particular value of \( \tau \) a particle (event) is most likely to be found within a particular region of spacetime, and thus at a particular value of coordinate time \( t = x^0 \).

Localization of wave function in spacetime has been usually considered as problematic, just because it represents an instantaneous event, and therefore it could not have been associated with a physical particle for which probability must be conserved and unitarity of evolution operator assured. This is indeed the case within the conventional constrained quantum theory without a physical evolution parameter \( \tau \), since in such a theory a wave function which is localized in spacetime is frozen for ever within a spacetime region \( \Omega \). On the contrary, in the parametrized quantum theory wave function is not frozen, and a wave packet moves in spacetime. If a packet moves then also the probability of observing a particle does move; at a value of the evolution parameter \( \tau_1 \) a particle is likely to be observed at the value of the coordinate time \( t_1 \), and at a later value \( \tau_2 \) the particle is likely to be observed at another value \( t_2 \). Since the wave function is normalized in spacetime (eq.(13)), one immediately finds that the \( \tau \)-evolution operator \( U \) which brings \( \psi(\tau) \) into \( \psi(\tau') = U\psi(\tau) \) is unitary. A more concised and detailed explanation of the interpretation of the parametrized quantum theory is given in ref [4]. Both first and second quantized parametrized theories are straightforward and elegant. They are more general than the conventional constrained theories, nevertheless, they contain states with definite masses.
and all other results of a conventional free field theory. Extension of such a second quantized unconstrained theory to include interactions has not yet been fully elaborated, but significant success has been achieved at the first quantized level \([10]\).

3. The separation of true Lagrange multipliers in the conventional (constrained) p-brane action

As a preparation for the next section in which we describe the unconstrained p-brane theory we are now going to elaborate an ADM-like splitting of the metric on the worldsheet swept by a p-brane. The content of this section is intended to be self-consistent and need not be applied to an unconstrained theory. It might be interesting to and bring new insight to those researchers who will keep working on constrained p-brane theories.

We start from the Howe-Tucker action generalized to a membrane of arbitrary dimension \(p\) (p-brane):

\[
I[X^\mu, \gamma^{ab}] = \frac{\kappa_0}{2} \int \sqrt{|\gamma|} (\gamma^{ab} \partial_a X^\mu \partial_b X_\mu + 2 - d) \tag{14}
\]

Besides the variables \(X^\mu(\xi), \mu = 0, 1, 2, \ldots, D-1\) which denote position of a \(d\)-dimensional \((d = p + 1)\) worldsheet \(V_d\) in the embedding spacetime \(V_D\), the above action contains also the auxiliary variables \(\gamma^{ab}\) (with a role of Lagrange multipliers) which have to be varied independently from \(X^\mu\). The worldsheet parameters are \(\xi^a, a = 0, 1, 2, \ldots, d - 1\).

By varying (14) with respect to \(\gamma^{ab}\), we arrive at the equation for the induced metric on a worldsheet:

\[
\gamma_{ab} = \partial_a X^\mu \partial_b X_\mu, \quad \partial_a \equiv \frac{\partial}{\partial \xi^a} \tag{15}
\]

Inserting (15) into (14) we obtain the Dirac-Nambu-Gotto action for a minimal surface:

\[
I[X^\mu] = \kappa_0 \int d^d \xi \sqrt{|f|}, \quad f \equiv \det f_{ab}, \quad f_{ab} \equiv \partial_a X^\mu \partial_b X_\mu \tag{16}
\]

The actions (14) and (16) are equivalent, but for the purpose of quantization, the form (14) is more convenient.
In eq. (14) \( \gamma^{ab} \) are the Lagrange multipliers, but they are not all independent. The number of worldsheet constraints is \( d \) and the same is the number of independent Lagrange multipliers. In order to separate out of \( \gamma^{ab} \) the independent multipliers we proceed as follows. Let \( \Sigma \) be a space like hypersurface on the worldsheet, and \( n^a \) the normal vector field to \( \Sigma \). Then the worldsheet metric tensor can be written as

\[
\gamma^{ab} = \frac{n^a n^b}{n^2} + \bar{\gamma}^{ab}, \quad \gamma_{ab} = \frac{n_a n_b}{n^2} + \bar{\gamma}_{ab}
\]

(17)

where \( \bar{\gamma}^{ab} \) is projection tensor, satisfying

\[
\bar{\gamma}^{ab} n_b = 0, \quad \bar{\gamma}_{ab} n^b = 0
\]

(18)

It projects any vector into the hypersurface to which \( n^a \) is the normal. For instance, using (17) we can introduce the tangent derivatives

\[
\bar{\partial}_a X^\mu = \bar{\gamma}^b_a \partial_b X^\mu = \gamma^b_a \partial_b X^\mu - \frac{n_a n_b}{n^2} \partial_b X^\mu
\]

(19)

An arbitrary derivative \( \partial_a X^\mu \) is thus decomposed into a normal and tangential part (relative to \( \Sigma \)):

\[
\partial_a X^\mu = n_a \partial X^\mu + \bar{\partial}_a X^\mu
\]

(20)

where

\[
\partial \equiv \frac{n^a \partial_a X^\mu}{n^2}, \quad n^a \bar{\partial}_a X^\mu = 0
\]

(21)

Details about using and keeping the \( d \)-dimensional covariant notation as far as possible are given in ref. [9]. Here I shall present a shorter and more transparent procedure, but without the covariant notation in \( d \)-dimensions.

Let us take such a coordinate system in which covariant components of normal vectors are

\[
n_a = (1, 0, 0, ..., 0)
\]

(22)

From eqs. (17) and (22) we have

\[
n^2 = \gamma_{ab} n^a n^b = \gamma^{ab} n_a n_b = n^0 = \gamma^{00}
\]

(23)
\[ \tilde{\gamma}^{00} = 0, \quad \tilde{\gamma}^{0i} = 0 \]  \hspace{1cm} (24)

\[ \gamma_{00} = \frac{1}{n^0} + \tilde{\gamma}_{ij} n^i n^j / (n^0)^2 \]  \hspace{1cm} (25)

\[ \gamma_{0i} = -\frac{\tilde{\gamma}_{ij} n^j}{n^0} \]  \hspace{1cm} (26)

\[ \gamma_{ij} = \tilde{\gamma}_{ij} \]  \hspace{1cm} (27)

\[ \gamma^{00} = n^0 \]  \hspace{1cm} (28)

\[ \gamma^{0i} = n^i \]  \hspace{1cm} (29)

\[ \gamma^{ij} = \tilde{\gamma}^{ij} + \frac{n^i n^j}{n^0}, \quad i, j = 1, 2, ..., p \]  \hspace{1cm} (30)

The decomposition (20) then becomes

\[ \partial_0 X^\mu = \partial X^\mu + \bar{\partial}_0 X^\mu \]  \hspace{1cm} (31)

\[ \partial_i X^\mu = \bar{\partial}_i X^\mu \]  \hspace{1cm} (32)

where

\[ \partial X^\mu = \dot{X}^\mu + \frac{n^i \partial_i X^\mu}{n^0}, \quad \dot{X}^\mu \equiv \partial_0 X^\mu \equiv \frac{\partial X^\mu}{\partial \xi^0} \]  \hspace{1cm} (33)

\[ \bar{\partial}_0 X^\mu = -\frac{n^i \partial_i X^\mu}{n^0} \]  \hspace{1cm} (34)

As \( d \) independent Lagrange multipliers can be taken \( n^a = (n^0, n^i) \). We can now rewrite our action in terms of \( n^0 \) and \( n^i \). We insert (28-30) into (14) and take into account that

\[ |\gamma| = \frac{\tilde{\gamma}}{n^0} \]  \hspace{1cm} (35)

where \( \gamma = \det \gamma_{ab} \) is the determinant of the worldsheet metric and \( \tilde{\gamma} = \det \tilde{\gamma}_{ij} \) the determinant of the metric \( \tilde{\gamma}_{ij} = \gamma_{ij}, \ i, j = 1, 2, ..., p \) on the hypersurface \( \Sigma \).

So our action (14) after using (28-30) becomes

\[ I[X^\mu, n^a, \tilde{\gamma}^{ij}] = \frac{\kappa_0}{2} \int d^d \xi \frac{\sqrt{\gamma}}{\sqrt{n^0}} \left( n^0 \dot{X}^\mu \dot{X}_\mu + 2n^i \dot{X}^\mu \partial_i X^\mu + (\tilde{\gamma}^{ij} + \frac{n^i n^j}{n^0}) \partial_i X^\mu \partial_j X^\mu + 2 - d \right) \]  \hspace{1cm} (36)
Variation of the latter action with respect to $\tilde{\gamma}^{ij}$ gives the expression for the induced metric on the surface $\Sigma$:

$$\tilde{\gamma}^{ij} = \partial_i X^\mu \partial_j X_\mu, \quad \tilde{\gamma}^{ij} \tilde{\gamma}_{ij} = d - 1 \quad (37)$$

We can eliminate $\tilde{\gamma}^{ij}$ from the action (36) by using the relation (37):

$$I[X^\mu, n^a] = \frac{\kappa_0}{2} \int d^d\xi \frac{\sqrt{\tilde{\gamma}}}{\sqrt{n^0}} \left( \frac{1}{n^0} (n^0 \dot{X}^\mu + n^i \partial_i X^\mu)(n^0 \dot{X}_\mu + n^i \partial_i X_\mu) + 1 \right) \quad (38)$$

The latter action is a functional of the worldsheet variables $X^\mu$ and $d$ independent Lagrange multipliers $n^a = (n^0, n^i)$. Varying (38) with respect to $n^0$ and $n^i$ we obtain the worldsheet constraints:

$$\delta n^0 : \ (\dot{X}^\mu + \frac{n^i \partial_i X^\mu}{n^0}) \dot{X}_\mu = \frac{1}{n^0} \quad (39)$$

$$\delta n^i : \ (\dot{X}^\mu + \frac{n^i \partial_i X^\mu}{n^0}) \partial_j X_\mu = 0 \quad (40)$$

Using (33) the constraints can be written as

$$\partial X^\mu \partial X_\mu = \frac{1}{n^0} \quad (41)$$

$$\partial X^\mu \partial_i X_\mu = 0 \quad (42)$$

The action (38) contains the expression for the normal derivative $\partial X^\mu$ and can be written in the form

$$I = \frac{\kappa_0}{2} \int d\tau dp \sigma \sqrt{|f|} \left( \frac{\partial X^\mu \partial X_\mu}{\lambda} + \lambda \right), \quad \lambda \equiv \frac{1}{\sqrt{n^0}} \quad (43)$$

where we have written $d^d\xi = d\tau dp \sigma$, since $\xi^a = (\tau, \sigma^i)$.

So we arrived at an action which looks like the well known Howe-Tucker action for a point particle, except for the integration over space-like hypersurface $\Sigma$, parametrized by coordinates $\sigma^i, i = 1, 2, ..., p$.

The equations of motion for variables $X^\mu$ derived from (38) are exactly the equations of a minimal surface give by (16).

4. Relativistic membranes (p-branes) without constraints
In the previous section we arrived at an action (38) or (43) which is equivalent to the well known Dirac-Nambu-Gotto action for a minimal surface. Let us now consider a new action which has the same form as (38), but now instead of the Lagrange multipliers \( n^0, n_i \) it contains fixed functions \( N^0(\tau, \sigma), N^i(\tau, \sigma) \):

\[
I[X^\mu, n^a] = \frac{\kappa_0}{2} \int d\tau d^p\sigma \sqrt{\frac{\gamma}{N^0}} \left( \frac{1}{N^0}(\dot{N}^0 + N^i \partial_i X^\mu)(\dot{N}^0 \dot{X}_\mu + N^i \partial_i X_\mu) + 1 \right) \tag{44}
\]

To different choices of \( N^0(\tau, \sigma), N^i(\tau, \sigma) \) there correspond physically different actions, describing physically different systems. A particularly simple action we have if we take \( N^i = 0 \):

\[
I[X^\mu] = \frac{\kappa_0}{2} \int d\tau d^p\sigma \sqrt{|f|} \left( \frac{\dot{X}_\mu \dot{X}_\mu}{\Lambda} + \Lambda \right), \quad \Lambda \equiv \frac{1}{\sqrt{N^0}} \tag{45}
\]

This action describes a continuous collection of *unconstrained point particles*, each being described by the action (4). Individual particles are labeled by the indices \( \sigma^i \) and they all together form a fluid localized on a continuous membrane (p-brane). Choice of labels \( \sigma^i \) is arbitrary. Indeed, the action (45) is invariant with respect to arbitrary reparametrizations of membrane coordinates \( \sigma^i \). The freedom of choice of a parametrization on a given, say initial surface \( V_\Sigma \), is trivial and it does not impose any local gauge group (and constraints) among the dynamical variables \( X^\mu \) which depend also on the evolution parameter \( \tau \).

The action (44) or (45) is not equivalent to the action of the Dirac-Nambu-Gotto \( p \)-dimensional membrane. In (44) and (45) all components \( X^\mu(\tau, \sigma) \) are *independent dynamical variables*. They describe motion of fluid particles in spacetime.

Initial data may be specified on any \( p \)-dimensional space-like surface \( V_\Sigma \) embedded in \( D \)-dimensional spacetime. They are given by

\[
X^\mu(0, \sigma^i), \quad \dot{X}^\mu(0, \sigma^i) \tag{46}
\]

Once \( X^\mu(0, \sigma) \) on a chosen initial \( V_\Sigma \) are determined, also a parametrization of \( V_\Sigma \) (i.e. choice of coordinates \( \sigma^i \)) is determined. The dynamical equations of motion (which can

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3 A suitable redefinition of \( \Lambda \) and \( \tau \) brings (4) into the form \( I = \frac{\kappa_0}{2} \int d\tau \left( \frac{\dot{X}^\mu \dot{X}_\mu}{\Lambda} + \Lambda \right) \).

4 Analogous situation occurs in the description of non-relativistic (Newtonian) motion of a usual 1-dimensional string or 2-dimensional membrane in 3-dimensional space, with the ordinary time \( t \) as evolution parameter. The fact that one can arbitrarily parametrize string or membrane does not imply any constraints in such a non-relativistic motion.
be straightforwardly derived from (44) or (45)) then determine $X^\mu(\tau,\sigma)$ at arbitrary $\tau$. 

Had we chosen different initial velocities $\dot{X}'^\mu(0,\sigma)$ then we would have obtained different $X'^\mu(\tau,\sigma)$. In particular we can choose $\dot{X}'^\mu(0,\sigma)$ so that $X'^\mu(\tau,\sigma)$ describes from the mathematical point of view the same manifold $V_d$ as it is represented by $X^\mu(\tau,\sigma)$. But physically, $X^\mu(\tau,\sigma)$ and $X'^\mu(\tau,\sigma)$ represent motions of different objects: the first membrane is elastically deformed in a certain way, and the second membrane is elastically deformed in some other way. This illustrates that our system is a ”wiggly” membrane (see Sec.5).

The canonically conjugate variables belonging to the action (45) are

$$X^\mu(\sigma),\quad p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{\kappa_0 \sqrt{|f|}}{\Lambda} \dot{X}_\mu$$

(47)

The Hamiltonian is

$$H = \frac{\Lambda}{2} \int d^p \sigma \sqrt{f} \left( \frac{p^\mu p_\mu}{|f|} - \kappa_0^2 \right)$$

(48)

There are $D$ independent functions $X^\mu(\sigma)$ and $D$ independent functions $p_\mu(\sigma)$, and no constraints. Therefore the Poisson brackets and the Hamiltonian formalism can be written down straightforwardly (according to the lines initiated e.g. in [9]).

The theory can be straightforwardly quantized by considering $X^\mu(\sigma)$, $p_\mu(\sigma)$ as operators, satisfying the commutation relations

$$[X^\mu(\sigma), p_\nu(\sigma')] = \delta^\mu_\nu \delta(\sigma - \sigma')$$

(49)

In the representation in which operators $X^\mu(\sigma)$ are diagonal the momentum operator is given by the functional derivative

$$p_\mu = -i \left. \frac{\delta}{\delta X^\mu(\sigma)} \right|_{{X^\mu(\sigma)}}$$

(50)

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5 Again we have the analogy with a usual non-relativistic elastic string or membrane. It can be elastically deformed in such a way that the manifold $V_p$ ($p = 1$ or 2) remains the same, but nevertheless a deformed object, described by $x'(\sigma)$, is physically different from the ”original” object described by $x(\sigma)$. Both $x(\sigma)$ and $x'(\sigma)$ describe the same manifold $V_p$, but $x'(\sigma)$ now represents positions of an elastically deformed string or membrane.
A quantum state is represented by a wave-functional $\psi[\tau, X^\mu(\sigma)]$ which depends on the evolution parameter $\tau$ and the coordinates $X^\mu(\sigma)$ of our unconstrained membrane. It satisfies the functional Schrödinger equation

$$i \frac{\partial \psi}{\partial \tau} = H \psi$$

(51)

where the Hamiltonian operator $H$ is given by eq.(48) in which $p_\mu$ are now operators (50).

The parameter $\tau$ is invariant with respect to Lorentz transformations and general transformations of spacetime coordinates, and $H$ is also invariant. Therefore (51) is a relativistically invariant equation, yet it implies a state evolution (and no constraints).

A general solution to eq.(51) is given by

$$\psi[\tau, X(\sigma)] = \int \mathcal{D} p(\sigma) e^{i p(\sigma) \cdot X(\sigma)} e^{i H_\tau e^{i \int p_\mu(\sigma) X^\mu(\sigma) d\sigma}}$$

(52)

where $H$ is given by (48) and $p_\mu(\sigma)$ are now eigenvalues of the corresponding operators.

A generic wave functional, such as given in eq.(52) represents a wave packet which is a superposition of states with definite momentum $p_\mu(\sigma)$. It is localized in spacetime around a p-brane and the region of localization proceeds, with increasing $\tau$, forward along a time-like direction and thus sweeps a $(p+1)$-dimensional worldsheet.

A wave packet is normalized according to

$$\int \mathcal{D} X(\sigma) \psi^*[\tau, X^\mu(\sigma)] \psi[\tau, X^\mu(\sigma)] = 1$$

(53)

which is a straightforward extension of the corresponding point particle relation (13). Since (53) is satisfied at any $\tau$, the evolution operator $U$ which brings $\psi(\tau) \rightarrow \psi(\tau') = U \psi(\tau)$ is unitary.

Expectation value of an operator $A$ is given by

$$< A > = \int \mathcal{D} X(\sigma) \psi^*[\tau, X^\mu(\sigma)] A \psi[\tau, X^\mu(\sigma)]$$

(54)

The amplitude for transition from a state with definite $X^\mu_1(\sigma)$ at $\tau_1$ to a state with definite $X^\mu_2(\sigma)$ at $\tau_2$ is given by the Feynman functional integral

$$< X_2(\sigma), \tau_2 | X_1(\sigma), \tau_1 > = \int \mathcal{D} X^\mu(\tau, \sigma) e^{i H [X^\mu]}$$

(55)
The functions $X^\mu(\tau, \sigma)$ in the expression (53) represent various kinematically possible motions of an elastically deformed membrane. Since all $X^\mu(\tau, \sigma)$ are physically distinguishable there is no gauge group of transformations connecting equivalent functions $X^\mu(\tau, \sigma)$. Consequently, the functional integration in (53) is straightforward, and there is no need to introduce ghosts.

5. Relativistic membranes with variable tension - wiggly membranes; a conventional, reparametrization invariant, description

Let us now consider a generalization of the usual Dirac-Nambu-Gotto $p$-dimensional membranes such that tension in general is no more a constant. Tension is admitted to vary and this is determined by the equations of motion. A theory of wiggly strings was considered by Hong et al. [11] and they derived equations of motion - without using an action - by writing the spacetime stress-energy tensor and then requiring its vanishing divergence, $T^\mu_{\nu, \nu} = 0$. Here I extend the theory to an arbitrary $p$-brane.

A reparametrization invariant action for a wiggly membrane (which to my knowledge has not yet been explicitly written down) is:

$$I_W = \frac{1}{2} \int d^d \xi \sqrt{|\gamma|} \left[ t_{ab} \partial_a X^\mu \partial_b X_\mu - \epsilon + \kappa (3 - d) \right]$$

(56)

where

$$t_{ab} = (\epsilon - \kappa) u^a u^b + \kappa \gamma^{ab}$$

(57)

is the stress-energy tensor on our $d$-dimensional worldsheet, $\kappa(\xi)$ the tension and $u^a(\xi)$ the fluid velocity satisfying $u^a u_a = 1$. The variables $X^\mu$ describe position of the worldsheet in embedding spacetime.

Action (56) is invariant with respect to arbitrary reparametrizations of worldsheet coordinates $\xi^a$. The theory of wiggly membranes that we are now describing is just a straightforward extension of the usual membrane theory in which tension $\kappa$ is constant and equal to the energy density $\epsilon$. In the latter case the expression (56) reduces to (14).
If we vary (56) with respect to $\gamma^{ab}$ then we obtain the worldsheet constraints which imply the expression for the induced metric

$$\gamma_{ab} = \partial_a X^\mu \partial_b X_\mu$$  \hspace{1cm} (58)

The equations of motion for $X^\mu$ derived from (56) are

$$\frac{1}{\sqrt{|\gamma|}} \partial_a \left( \sqrt{|\gamma|} t^{ab} \partial_b X^\mu \right) \equiv D_a (t^{ab} \partial_b X^\mu) = 0$$  \hspace{1cm} (59)

where $D_a$ is covariant derivative with respect to the worldsheet metric $\gamma_{ab}$.

Eq. (59) can be rewritten as

$$D_a t^{ab} \partial_b X^\mu + t^{ab} D_a D_b X^\mu = 0$$  \hspace{1cm} (60)

Contracting the latter equation by $\partial_c X^\mu$ and using the identity (which comes from the expression for the induced metric)

$$D_a D_b X^\mu \partial_c X^\mu \equiv 0$$  \hspace{1cm} (61)

we obtain

$$D_a t^{ab} = 0$$  \hspace{1cm} (62)

Eq. (60) thus becomes simply

$$t^{ab} D_a D_b X^\mu = 0$$  \hspace{1cm} (63)

We see that the equations of motion (59) imply the law of motion (62) for the fluid velocity $u^a$ and energy density $\epsilon$ besides the law of motion (63) for the embedding variables $X^\mu$.

In order to provide a complete description of the membrane's dynamics eq. (59) must be supplemented by an equation of state

$$\kappa = \kappa(\epsilon)$$  \hspace{1cm} (64)

Let us now count the number of independent equations. Because of the identities (61) there are only $D - d$ independent equations (63) besides $d$ independent equations (62). Altogether there are are $D$ independent equation (62), (63) or equivalently (59).
This is the same as the number of independent variables: $d - 1$ independent $u^a$, $D - d$ independent $X^\mu$, and $\kappa$ (or equivalently $\epsilon$, since the relation (64) holds).

An equation of state (64) can be arbitrarily chosen. Various equations of state hold for various kinds of wiggly membranes. In particular, we may choose the equation of state

$$\epsilon = \kappa$$

(65)

Then the stress-energy tensor obtains the simple form

$$t^{ab} = \kappa \gamma^{ab}$$

(66)

and the equation of motion (62) reads

$$D_a(\kappa \gamma^{ab}) = \gamma^{ab} \partial_a \kappa + \kappa D_a \gamma^{ab} = \partial^b \kappa = 0$$

(67)

which implies that tension $\kappa$ must be a constant. \footnote{In the last step of eq.(67) we used the property that the covariant derivative of the metric tensor is zero.}

Then eq.(59) or (63) becomes the equation of motion for a Dirac-Nambu-Gotto membrane (i.e. an equation of a minimal surface):

$$\frac{1}{\sqrt{|\gamma|}} \partial_a(\sqrt{|\gamma|} \gamma^{ab} \partial_b X^\mu) = \gamma^{ab} D_a D_b X^\mu = 0$$

(68)

The theory of wiggly membranes (p-branes) is just a straightforward interesting extension of the well known theory of membranes or p-branes (with constraints) and it contains the latter as a particular case.

6. Comparison of an unconstrained membrane with a wiggly membrane

In Secs. 4 and 5 we find that an unconstrained membrane has the same number of independent variables as a wiggly membrane. \footnote{By "wiggly membrane" from now on I mean one described in Sec.5, and by "unconstrained membrane" one described in Sec. 4.} Also both kinds of membranes suffer deformations during their motion. Unconstrained membrane $\nu_p$ can be elastically...
deformed which necessarily causes the energy density on it to vary, like on a wiggly membrane. Therefore we expect a close relationship between both kinds of objects.

We are now going to compare the action (44) with (56). For this purpose we apply to the action (56) the decomposition of the worldsheet metric as done in eqs. (17)-(35). After a straightforward calculation we find the following form for the action of a wiggly membrane:

\[ I_W = \frac{1}{2} \int d^2 \xi \sqrt{f} \left[ \left( (\epsilon - \kappa)u^a u^b + \kappa \frac{n^a n^b}{n^2} \right) \partial_a X^\mu \partial_b X_\mu + 2\kappa - \epsilon \right] \] (69)

Writing \( n^a = (n^0, n^i) \), \( i = 1, 2, ..., p \) and using (23), eq.(69) assumes a longer form:

\[ I_W = \frac{1}{2} \int d^d \xi \sqrt{f} \left[ \left( (\epsilon - \kappa)u^0 u^0 + \kappa n^0 \right) \partial_0 X^\mu \partial_0 X_\mu + 2 \left( (\epsilon - \kappa)u^0 u^i + \kappa n^i \right) \partial_0 X^\mu \partial_i X_\mu + \left( (\epsilon - \kappa)u^i u^j + \kappa n^i n^j n^0 \right) \partial_i X^\mu \partial_j X_\mu + 2\kappa - \epsilon \right] \] (70)

where \( \partial_0 X^\mu \equiv \partial X^\mu / \partial \xi^0 \), \( \partial_i X^\mu \equiv \partial X^\mu / \partial \xi^i \) and \( \xi^a = (\xi^0, \xi^i) \). Here \( n^a \) are Lagrange multipliers, and varying (69) or (70) with respect to \( n^a \) gives the worldsheet constraints.

Let us now take a particular choice of coordinates \( \xi^a \), such that the fluid velocity becomes

\[ u^a = (u^0, 0, 0, ..., 0) \] (71)

This means that the coordinate lines \( \xi^i = \text{constant} \) coincide with the worldlines of the fluid particles. Then, from the spacetime point of view, \( \partial_0 X^\mu \) are the tangent vectors to the fluid worldlines. In other words, \( \partial_0 X^\mu \) is spacetime velocity of a fluid particle. At this point let us recall that in the case of an unconstrained membrane (treated in Sec.4) the quantity \( \dot{X}^\mu \equiv \partial X^\mu / \partial \tau \) is also velocity of a fluid particle, and the set \( \dot{X}^\mu (\tau, \sigma) \) for all \( \tau \), \( \sigma \) is a velocity field belonging to the bundle of fluid worldlines forming the membrane.

Coordinates \( \xi^a \) chosen so that (71) is satisfied are thus identical with the parameters \( (\tau, \sigma^i) \) used in the description of an unconstrained membrane, and \( \partial_0 X^\mu \) in eq.(70) is the same thing as \( \dot{X}^\mu \) in eq.(14). Putting \( u^i = 0 \) in eq.(70) we may now identify the action (14) of an unconstrained membrane with the action (70) of a wiggly membrane (remembering that both objects have the same number of independent dynamical variables). By
comparing coefficients at $\dot{X}^{\mu} \dot{X}_{\mu}$, $\dot{X}^{\mu} \partial_{\mu} X$ and $\partial_i X^\mu \partial_j X_{\mu}$ we find the following relations

$$\frac{1}{\sqrt{n^0}} \left( (\epsilon - \kappa)(u^0)^2 + \kappa n^0 \right) = \kappa_0 \sqrt{N^0} \quad (72)$$

$$\frac{\kappa n^i}{\sqrt{n^0}} = \frac{\kappa_0 N^i}{\sqrt{N^0}} \quad (73)$$

$$\frac{\kappa n^i n^j}{(n^0)^{3/2}} = \frac{\kappa_0 N^i N^j}{(N^0)^{3/2}} \quad (74)$$

$$\frac{1}{\sqrt{n^0}} (2\kappa - \epsilon) = \frac{\kappa_0}{\sqrt{N^0}} \quad (75)$$

In addition we have also the following relation

$$(u^0)^2 = \left( \frac{1}{n^0} + \gamma_{ij} \frac{n^i n^j}{(n^0)^2} \right)^{-1} \quad (76)$$

which comes from $\gamma_{ab} u^a u^b = 1$ using (25) and (71). The quantities $n^0$, $n^i$ in (72)-(76) are no more arbitrary (as they were in the action (69) or (70)), but are fixed by the chosen parametrization $\xi^a = (\tau, \sigma)$ in which eq.(71) holds. If we eliminate $n^0$, $n^i$ and $u^0$ from eqs. (72)-(75) we obtain the following relation between $\epsilon$ and $\kappa$:

$$(\epsilon - \kappa)^2 = \kappa^2 - \kappa_0^2 - \gamma_{ij} \frac{N^i N^j}{N^0} \left( \frac{\kappa_0}{\kappa} \right)^2 (\kappa_0^2 + (\epsilon - 2\kappa) \kappa) \quad (77)$$

This is the equation of state that a wiggly membrane must satisfy in order to be equivalent to an unconstrained membrane with given $N^0$ and $N^j$.

The relation between $\epsilon$ and $\kappa$ in eq.(77) is not unique, since $\kappa$ occurs in the 4th order. Therefore we must decide which of the 4 branches we shall take into account. For this purpose we consider the property of a wiggly membrane given in eq.(65)-(67) stating that the equation of state $\epsilon = \kappa$ implies constant $\kappa$. Let therefore insert $\epsilon = \kappa$ into eq.(77). We obtain

$$(\kappa^2 - \kappa_0^2) \left( 1 - \gamma_{ij} \frac{N^i N^j}{N^0} \left( \frac{\kappa_0}{\kappa} \right)^2 \right) = 0 \quad (78)$$

Among 4 solutions to eq.(78) there are two solutions in which tension $\kappa$ is a constant. This solutions are

$$\kappa = \pm \kappa_0 \quad (79)$$
In order to have always positive tension we therefore choose positive sign in eq. (77). The requirement that for \( \epsilon = \kappa \) it is \( \kappa = \kappa_0 \) then selects the right equation of state among four relations between \( \epsilon \) and \( \kappa \) contained in eq. (77).

Since the relation (77) indeed implies that, when \( \epsilon = \kappa \), the tension \( \kappa \) is constant it is therefore consistent with the requirement of the theory of wiggly membranes. On the other hand, eq. (77) is a consequence of the relations (72)-(75) which come from the identification of the two actions, namely the one of an unconstrained membrane and the one of a wiggly membrane. We have thus proved that any unconstrained membrane is equivalent to a wiggly membrane for which the equation of state (77) holds.

The above relations become very simple if we consider a special subclass of unconstrained membranes given by \( N^i = 0 \) (see action (15)). Then from (73) \( n^i = 0 \) and from (74) \((u^0)^2 = n^0\) so that eqs. (72) and (75) read

\[
\begin{align*}
\epsilon \sqrt{n^0} &= \kappa_0 \sqrt{N^0} \\
\frac{1}{\sqrt{n^0}} (2\kappa - \epsilon) &= \frac{\kappa_0}{\sqrt{N^0}}
\end{align*}
\]

The equation of state is simply

\[
(\epsilon - \kappa)^2 = \kappa^2 - \kappa_0^2
\]

giving \( \kappa = \kappa_0 \) when \( \epsilon = \kappa \).

7. Conclusion

I have investigated relativistic extended objects, called membranes or p-branes, of arbitrary dimension \( p \), including point particles when \( p = 0 \). When such an object moves in spacetime it sweeps a \((p + 1)\)-dimensional manifold called worldsheet. In conventional approaches the properties of such a worldsheet are considered. The variables describing position of a worldsheet in spacetime are not independent but satisfy \( p + 1 \) constraints. In the present work I take into account the fact that our observer does not perceive the whole
worldsheet but only a space-like slice on it (this is just what we call membrane or p-brane) and the fact that such a slice is not at rest in spacetime but it *moves* into a time-like direction. The speed of such a motion is taken as a dynamical variable and all variables describing position of a membrane in spacetime are independent, so that there are no constraint relations among them. The classical and quantum theory so constructed is much more straightforward - both conceptually and technically - and easier to handle than the constrained formalisms of conventional p-brane theories which for $p \geq 2$ become nearly intractable because of technical obstacles. The usual Dirac-Nambu-Gotto membranes (p-branes) belong to a subclass of solutions to such a theory of unconstrained $p$-dimensional membranes.

Further investigations then reveal that the unconstrained membranes are equivalent to the so called wiggly membranes which have variable tension $\kappa$, different from energy density $\epsilon$, provided that the latter quantities satisfy a special equation of state $\epsilon = \epsilon(\kappa)$.

My final conclusion is therefore that instead of the theory of the Dirac-Nambu-Gotto membranes we should rather consider the theory of unconstrained or wiggly membranes as an appropriate candidate for a "final" physical theory.
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