ENERGETICS IN CONDENSATE STAR AND WORMHOLES

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Abstract

It is known that the total gravitational energy in localized sources having static spherical symmetry and satisfying energy conditions is negative (attractive gravity). A natural query is how the gravitational energy behaves under circumstances where energy conditions are violated. To answer this, the known expression for the gravitational energy is suitably adapted to account for situations like the ones occurring in wormhole spacetime. It is then exemplified that in many cases the modified expression yields desirable answers. The implications are discussed.

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1. Introduction

Classical wormholes, just as black holes, represent self consistent solutions of Einstein’s theory of general relativity. Topologically they are like handles connecting two distant regions of spacetime. Wormhole solutions were conceived as particle models by Einstein himself (Einstein-Rosen bridge \cite{1}) in 1935. (A 1916 predecessor of wormholes is Flamm \cite{2} paraboloid). The seminal theoretical framework laid in 1988 by Morris, Thorne and Yurtsever (MTY) \cite{3} has since led to serious investigations into the topic of wormhole physics. In addition to the traditional method of solving Einstein’s equations, there exist what is well known as Synge’s method (It is a reverse method employed by MTY in constructing wormholes) in which one first fixes the spacetime geometry and then computes, via field equations, the stress components needed to support such a geometry. The resulting stress components automatically satisfy local conservation laws in virtue of Bianchi identities. Either method has led to several wormhole solutions in well known theories such as in Brans-Dicke theory \cite{4},
scalar field theory with potential [5], low energy string theory [6], braneworld model [7], phantom model [8], Chaplygin gas model [9], Thin-Shell model [10] and in cosmology [11]. Configurations resulting from these theories could be potential candidates to occur in a natural way and are (perhaps) of some astrophysical interest [12,13]. A largely unnoticed but important work was carried out in 1948 by Fisher [14] who discovered formal solutions to minimally coupled scalar field Einstein equations with a positive sign kinetic term. Thereafter, in 1973, Ellis [15] and independently, Bronnikov [16] found wormhole solutions of the Einstein minimally coupled theory with a negative sign kinetic term. All wormhole solutions require exotic material for their construction. However, to our knowledge the gravitational energy content in the interior of exotic matter distribution has not yet been studied. An initiative along this direction can be taken by employing the formulation of gravitational energy provided by the works of Lynden-Bell, Katz and Bičák [17-19]. A Maxwellian analogy together with a conformal factor interpretation of gravitational energy density, which is new, is also given in [17].

The energy formulation in references [17-19] is intended for isolating and calculating the total attractive gravitational energy $E_G$ of stationary gravity fields. In our view, their formulation did not require any compelling restriction on the energy conditions of the source matter. We shall therefore allow the source matter to violate one or more energy conditions, particularly the Weak Energy Condition (WEC) $\rho > 0$ and/or the Null Energy Condition (NEC) $\rho + p_r \geq 0$ where $\rho$ is the matter energy density and $p_r$ is the radial pressure. (Transverse pressures $p_\perp$ are not considered as they refer strictly to ordinary matter.) The violation of Null Energy Condition (NEC) is a minimal requirement to have defocussing of light trajectories (repulsive gravity) passing across the wormhole throat [20]. The necessity of NEC violation in wormholes is provided by the Topological Censorship Theorem [21] and by dynamical circumstances [22].

In this paper, we first adapt $E_G$ to wormhole geometry and distinguish it by $\tilde{E}_G$. Then we investigate the behavior of $\tilde{E}_G$ in certain static spherically symmetric model solutions that violate the energy conditions as stated above. The examples we consider, one star and three wormholes, are somewhat critical in nature. Explicit calculations in the Ellis III wormhole show that the definition of $\tilde{E}_G$ is robust enough. In the case of de Sitter star, exact calculation shows it has attractive gravity which then admits a plausible physical interpretation. In the case of phantom wormhole, we find that $\tilde{E}_G$ is repulsive around the throat, which is necessary for defocussing effect. Such a behavior of $\tilde{E}_G$ may serve as a constraint with regard to the practical feasibility of localized wormholes.

The paper is organized as follows: In Sec.2, we adapt the equation for total gravitational energy so as to account for the wormhole geometry. In Sec.3, we work out a specific solution to show that the conformal factor interpretation holds also in the case of wormhole spacetime. The same example is used to show the robustness of $\tilde{E}_G$ in Sec.4. In Sec.5, we apply the energetics to the Mazur-Mottola gravastar [23] and interpret the result. In Sec.6, we calculate contribution to $\tilde{E}_G$ coming from the thin shell. In Sections 7 and 8, we
investigate wormholes which are localized by spacetime cut-off at some radii. Sec.9 summarizes the results. Units are chosen so that \(8\pi G = c = 1\), unless specifically restored.

2. Gravitational energy

We shall consider spherically symmetric static spacetime with the metric expressed in “standard” coordinates \((x^0, x^1, x^2, x^3) \equiv (t, r, \theta, \psi)\) as

\[
ds^2 = -e^{2\Phi(r)}dt^2 + g_{rr}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\psi^2). \tag{1}
\]

The stress components resulting from the metric via Einstein’s equations are denoted by

\[
T^0_0 = \rho, \quad T^1_1 = p_r, \quad T^2_2 = p_\theta, \quad T^3_3 = p_\psi. \tag{2}
\]

in which \(\rho\) is the matter energy density and \(p_r\) is the radial pressure and \(p_\theta\), \(p_\psi\) are transverse pressures of the fluid in its rest frame. Because of spherical symmetry, \(p_\theta = p_\psi\). The total gravitational energy \(E_G\) appropriate for ordinary matter is given in [17,18] as

\[
E_G = Mc^2 - E_M = \frac{1}{2} \int_0^r \left[ 1 - \left(\frac{g_{rr}}{r}\right)^\frac{1}{2} \right] T^0_0 r^2 dr \tag{3}
\]

where the total mass-energy within the standard coordinate radius \(r\) is provided by Einstein’s equations as

\[
Mc^2 = \frac{1}{2} \int_0^r T^0_0 r^2 dr \tag{4}
\]

and the sum of other forms of energy like rest energy, kinetic energy, internal energy etc is defined by

\[
E_M = \frac{1}{2} \int_0^r T^0_0 \left(\frac{g_{rr}}{r}\right)^\frac{1}{2} r^2 dr. \tag{5}
\]

The factor \(\frac{1}{2}\) comes from \(\frac{4\pi}{8\pi}\). The \(E_M\) is similar to the geometric definition given by Wald [24]. Since \((g_{rr})^\frac{1}{2} > 1\) by definition (proper radial length larger than the Euclidean length), one immediately deduces the criteria that \(E_G < 0\) (attractive) if \(T^0_0 > 0\) [25] and that \(E_G > 0\) (repulsive) if \(T^0_0 < 0\).

For wormhole spacetime, we consider the spherically symmetric spacetime metric in the generic MTY form

\[
ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\psi^2) \tag{6}
\]

where \(\Phi(r)\) and \(b(r)\) are redshift and shape functions respectively. Throughout the wormhole \(0 < \frac{b(r)}{r} < 1\) and that \(\frac{b(r)}{r} \to 0\) as \(r \to \infty\). The expressions for
the stress components are $[3$

\[ \rho = \frac{b'}{r^2} \]

\[ p_r = 2 \left( 1 - \frac{b}{r} \right) \frac{\Phi'}{r^2} - \frac{b}{r^3} \]

\[ p_\theta = p_\psi = \left( 1 - \frac{b}{r} \right) \left[ \Phi'' + \Phi' \frac{2}{r} - \frac{b'}{2r} \Phi' - \frac{b'r - b}{2r^2(r - b)} \right] \] (9)

where primes denote differentiation with respect to $r$.

By construction, the wormhole geometry has a hole instead of a center and so we shall change the lower limit of integration in Eq.(3) to the minimum allowed radius or throat $r_0$ defined by $b(r_0) = r_0$. The radius $r$ has the significance that it is the embedding space radial coordinate; it decreases from $+\infty$ to $r = r_0$ in the lower side and again increases to $+\infty$ in the upper side. This requires us to change the integrals (4) and (5) to

\[ M c^2 = \frac{1}{2} \int_{r_0}^{r} T_0^0 r^2 dr + \frac{r_0}{2} \] (10)

\[ E_M = \frac{1}{2} \int_{r_0}^{r} T_0^0 (g_{rr})^{\frac{1}{2}} r^2 dr \] (11)

where $g_{rr} = \left( 1 - \frac{b(r)}{r} \right)^{-1}$, the entire spacetime geometry being assumed to be free of singularities. The constant $\frac{r_0}{2}$ in Eq.(10) comes from the integration of Einstein’s equation $\frac{\partial M}{\partial r} = \frac{1}{2} T_0^0 r^2$ and we shall choose it so as to offset the inner boundary term $\frac{b(r_0)}{2}$ coming from the integration. When $T_0^0 = 0$, we should fix $r_0 = 0$ in order to recover $M = 0$. In geometries with a regular center, one has $r_0 = 0$, the above then reproduces Eqs.(4) and (5) respectively. The difference between the above integrals, viz.,

\[ \bar{E}_G = M c^2 - E_M = \frac{1}{2} \int_{r_0}^{r} \left[ 1 - (g_{rr})^{\frac{1}{2}} \right] T_0^0 r^2 dr + \frac{r_0}{2} \] (12)

is what we call the total gravitational energy of wormholes within the region of integration. Clearly, it is a straightforward adaptation of Eq.(3) to wormholes. However, one immediately notices that due to the presence of the nonzero last term, the sign of $T_0^0$ does not necessarily determine the sign of $\bar{E}_G$, as would be the case otherwise. Eq.(12) is the main proposition of our paper and will be implemented in the sequel.

In Ref.[17] it is shown that the gravitational energy density in a stationary attractive gravity field can be written in remarkable analogy with electrical energy density of Maxwell electrodynamics: The total gravitational energy $E_G$ can be written as a volume integral of a perfect square of the gravitational field strength $F_G$, that is, $E_G = -\int_0^\infty F_G^2 dV$ where $dV$ is the element of spatial 3-volume. In case of wormhole spacetime, the expression for $E_G$ can be re-written.
as (see Appendix):

\[ \tilde{E}_G = - \int_{r_0}^{\infty} \frac{1}{r^2} \left[ 1 - (g_{rr})^{-\frac{1}{2}} \right]^2 dV + r_0 = - \int_{r_0}^{\infty} \widetilde{F}_G^2 dV + r_0. \]  

Furthermore, Lynden-Bell et al [17] have shown that one can then introduce a function \( \Psi \) defined by

\[ \widetilde{F}_G \equiv \frac{1}{r} \left[ 1 - (g_{rr})^{-\frac{1}{2}} \right] = \pm (g_{rr})^{-\frac{1}{2}} \frac{\partial \Psi}{\partial r}. \]  

and that there is then a change of coordinates that will make the spatial slices conformally flat with conformal factor \( e^{2\Psi} \). The \( \pm \) sign in Eq.(14) corresponds respectively to repulsive and attractive nature of gravitational potential \( \Psi \) and the choice is generally open unless the information is provided by independent physical observations.

A typical wormhole solution may be derived from source \( (T_{\mu \nu}) \) that has before it an overall wrong sign (negative) so that all energy conditions are violated. A well known example is the Ellis-Bronnikov wormhole. Then the equation for \( \Psi \) [Eq.(9) of [17], (A6) below] should be rephrased as

\[ \nabla^2 \Psi = -\frac{1}{2} T_{00} + \frac{1}{2} (\nabla \Psi)^2 \]  

which shows that a positive gravitational energy density \( \frac{1}{2} (\nabla \Psi)^2 \) is acting alongside negative exotic matter density \( -\frac{1}{2} T_{00} \) as a source of \( \Psi \).

### 3. Conformal factor interpretation

A good example to demonstrate the conformal factor interpretation is the Ellis III wormhole which is a solution of Einstein minimally coupled equation with an overall negative source term. It has the metric [15]

\[ ds^2 = -f(l)dt^2 + \frac{1}{f(l)} \left[ dl^2 + (l^2 + m^2) \left( d\theta^2 + \sin^2 \theta d\psi^2 \right) \right], \]  

\[ f(l) = \exp\left[ -2\beta \left( \frac{\pi}{2} - \arctan \left( \frac{l}{m} \right) \right) \right] \]  

\[ \varphi(l) = \left[ \sqrt{2} \sqrt{1 + \beta^2} \left( \frac{\pi}{2} - \arctan \left( \frac{l}{m} \right) \right) \right] \]  

where \( m \) and \( \beta \) are two constant arbitrary parameters. The throat appears at \( l_0 = m\beta \). The spacetime (16) is singularity free and Taylor expansion of \( f(l) \) gives asymptotic mass-energy \( M^+ = m\beta \) on one side and \( M^- = -m\beta e^{\beta \pi} \) on the other. These masses follow from the definition (10) as well. For a recent study of geodesics in the \( \beta = 0 \) case, see [26] and for its stability, see [27]. A remarkable feature of this solution is that the parameters can be adjusted to make the wormhole both macroscopic and microscopic satisfying quantum energy conditions [28,29].
The metric (16) can be rewritten in the standard MTY form by redefining the radial variable as

\[ r^2 = (l^2 + m^2) \exp\{2\beta\left[\frac{\pi}{2} - \arctan\left(\frac{l}{m}\right)\right]\} \] (19)

(Note that \( l \to \pm\infty \) implies \( r \to +\infty \) and conversely). Then the redshift function \( \Phi(r) \) is given by

\[ \Phi(r) = \beta \left[ \arctan \left( \frac{l(r)}{m} \right) - \frac{\pi}{2} \right] \] (20)

and the shape function \( b(r) \) is given by

\[ b(r) = r \left[ 1 - \frac{[l(r) - m\beta]^2}{r^2} \exp\{2\beta\left[\frac{\pi}{2} - \arctan\left(\frac{l(r)}{m}\right)\right]\} \right] \] (21)

such that \( \frac{b(r)}{r} \to 0 \) as \( r \to \infty \). Throat occurs at the minimum of \( r \) where \( b(r_0) = r_0 \). Putting \( l_0 = m\beta \) in Eq.(19), we find

\[ r_0 = m(1 + \beta^2)^{\frac{1}{2}} \exp\{\beta\left[\frac{\pi}{2} - \arctan\beta\right]\}. \] (22)

Now consider the following transformation \( l \to R \) where \( R \) is the isotropic radial coordinate

\[ l = \frac{R^2 - m^2}{2R} \] (23)

with its inverse

\[ R = l + \sqrt{l^2 + m^2}. \] (24)

It can be verified that the original metric (16) goes into its isotropic form as follows

\[ ds^2 = -f[l(R)]dt^2 + \frac{1}{f} \left[ \frac{R^2 + m^2}{2R^2} \right]^2 \left[ dR^2 + R^2 \left( d\theta^2 + \sin^2 \theta d\psi^2 \right) \right] \] (25)

\[ = -f[l(R)]dt^2 + e^{2\Psi(R)} \left[ dl^2 \right] \] (26)

giving the conformal factor

\[ \Psi(R) = \frac{1}{2} \ln \left\{ \frac{1}{f} \left[ l(R) \right] \left[ \frac{R^2 + m^2}{2R^2} \right]^2 \right\}. \] (27)

Putting this \( \Psi \) in Eq.(14) and integrating as in Eq.(13) plus a bit of algebra taking care of coordinate changes \( r \to l \to R \), we get the same expression for \( \tilde{E}_G \) as defined in Eq.(12) with \( T_0^0 = \rho \) from Eq.(28) below. By itself, the result is no surprise as the calculation in Ref.[17] is quite generic. What is of interest here is that the Maxwell analogy is valid even in the case of exotic matter associated with geometry peculiar to wormholes. This exercise lends reliability to \( \tilde{E}_C \) as defined in Eq.(12). We now exemplify that the definition is quite robust as well.
4. Robustness of $\tilde{E}_G$

To show it, we allow deviations from the conditions behind the original definition of $E_G$ in Eq. (3), the wormhole reincarnation of which is $\tilde{E}_G$. The same wormhole metric (16) is a good candidate for this purpose. Our aim is to see if $\tilde{E}_G$ still produces known results. The deviation lies in the following features. In (16), the matter-energy content is not localized in a finite region though the stress quantities do fall off to zero with radial distance. As required of wormholes, the solution has two asymptotically flat regions ($+ve$: $l \in [m\beta, +\infty)$ and ($-ve$: $l \in [m\beta, -\infty)$) connected by the throat at $l_0 = m\beta$. The stress components, given below, identically vanish in the asymptotic limit $l \to \pm\infty$.

Thus, using Eqs.(20) and (21) in Eqs.(7)-(9), we obtain

\[
\rho = -\frac{m^2(1 + \beta^2)}{(l^2 + m^2)^2}e^{-\beta\pi - 2\arctan(\beta)} \tag{28}
\]

\[
p_r = \rho \tag{29}
\]

\[
p_\theta = p_\psi = -\rho \tag{30}
\]

which shows that both WEC and NEC violated everywhere since $\rho < 0$ and $\rho + p_r = 2\rho < 0$ respectively. Using Eqs.(28), (29), (21), (22) and definitions (10) and (11), we get on the +ve side, noting that $\rho_r = \int_{r_0}^{b(+\infty)} dr = m\beta$, $b(r_0) = r_0 = m\beta:

\[
M^+c^2 = \frac{1}{2} \int_{r_0}^{\infty} T_{00} g^{00} r^2 dr + \frac{r_0}{2} = m\beta \tag{31}
\]

\[
E^+_M = \frac{1}{2} \int_{r_0}^{\infty} T_0^0 (g_{rr}) \frac{1}{2} r^2 dr dl \tag{32}
\]

\[
= \frac{m}{2} \left( \frac{1 + \beta^2}{\beta} \right) \left( 1 - \sqrt{e^{2\beta(\pi - 2\arctan(\beta))}} \right). \tag{33}
\]

We always find that $\tilde{E}_G^+ = M^+c^2 - E^+_M > 0$ or repulsive gravity for $m > 0$, $\beta > 0$. Proceeding in similar manner for the other side, we get

\[
M^-c^2 = -m\beta e^{\beta\pi} \tag{34}
\]

\[
E^-_M = m \left( \frac{1 + \beta^2}{\beta} \right) \left( e^{\beta\pi} - \sqrt{e^{\beta\pi - 2\arctan(\beta)}} \right) \tag{35}
\]

showing that $\tilde{E}_G^- = M^-c^2 - E^-_M < 0$ or attractive gravity for $m > 0$, $\beta > 0$. We recall that the Ellis solution (16) describes a Janus-faced wormhole that sucks in test particles in one mouth and pumps out at the other. The $\tilde{E}_G^\pm$ just calculated nicely describe this scenario despite the deviations mentioned above.

We can have some additional insight about the wormhole with zero Keplerian mass, $M = m\beta = 0 \Rightarrow \beta = 0$, for which the metric can be written in standard coordinates as

\[
ds^2 = -dt^2 + \frac{dr^2}{1 - \frac{m^2}{r^2}} + r^2 (d\theta^2 + \sin^2 \theta d\psi^2) \tag{36}
\]
where $r^2 = l^2 + m^2$. The shape function is $b(r) = \frac{m^2}{r}$ and the throat appears at $r_0 = m$ or equivalently at $l_0 = 0$. This is a well discussed single parameter symmetric wormhole made entirely of the massless scalar field $\varphi$ [cf. Eqs.(17),(18)]. We obtain from above, in the limit $\beta \to 0$,

$$M^+ c^2 = 0, E^+_M = -\frac{m\pi}{4} \Rightarrow E^+_G = \frac{m\pi}{4}$$

$$M^- c^2 = 0, E^-_M = \frac{m\pi}{4} \Rightarrow E^-_G = -\frac{m\pi}{4}.$$  

We see that $E^+_G > 0$ and $E^-_G < 0$, and vanishing mass-energy $M^\pm c^2$ on both sides, but nonvanishing $E^\pm_G$ contributed by the scalar field $\varphi$. The nonvanishing of $E^\pm_G$ explains why the wormhole is able to capture test particles despite the fact that it has zero Keplerian mass [15].

5. Energetics in the Mazur-Mottola star

Consider the static spherically symmetric vacuum condensate star (also called gravastar) devised by Mazur and Mottola [23]. The star has an isotropic de-Sitter vacuum in the interior, the matter marginally satisfying the NEC and strictly violating the Strong Energy Condition (SEC) $\rho + 3p \geq 0$. The star has an interior boundary at $r = r_1$ containing de Sitter vacuum ($p = -\rho$) and an exterior boundary at $r = r_2$ beyond which the spacetime is described by the Schwarzschild exterior ($p = 0, \rho = 0$) of mass $M$. The intermediate region is covered by a thin shell of stiff matter ($p = +\rho$).

The self-consistent interior de Sitter metric for a constant density vacuum $\rho = \rho_{\text{vac}} = \frac{3H_0^2}{8\pi G} = \text{const.} > 0$ is given by

$$d\tau^2 = -\left(1 - \frac{r^2}{\hat{R}^2}\right) dt^2 + \left(1 - \frac{r^2}{\hat{R}^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\psi^2)$$

where $\hat{R}^2 = \frac{3}{8\pi \epsilon \rho_{\text{vac}}} = \frac{1}{H_0^2}$. The transverse pressures in the thin shell serve to act more like a Roman arch supporting the star than making any substantial contribution to mass-energy. The shell contribution has actually been shown [23] to be negligible, $M_{\text{shell}} \sim \epsilon M$ where $0 < \epsilon \ll 1$. Israel-Darmois junction conditions then imply a negative surface tension at the inner interface of the shell which balances the outward force exerted by the repulsive vacuum within. Likewise, the positive surface tension at the outer interface balances the inward force from without. Using the thin shell approach, Visser and Wiltshire [31] studied dynamic stability of similar type of configurations. The mass-energy contained within the boundary radius $r = r_b$ is given by

$$M = \frac{4\pi}{3} r_b^3 \rho_{\text{vac}} > 0.$$  

Physics begins to become interesting in the region where horizon $r_{\text{hor}}$ ought to have formed. This is the region where the inner and outer boundaries tend to meet, viz.,

$$r_1 \sim r_2 \sim 2M \sim \frac{1}{H_0} = \hat{R} \sim r_{\text{hor}}.$$  

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at which the $g_{rr}$ from either side tend to approach arbitrarily close to infinity. Since $r_0 = 0$ (the star has a regular center), our $\tilde{E}_G$ coincides with $E_G$. Thus, putting $g_{rr} = \left(1 - \frac{r^2}{R^2}\right)^{-1}$ and $T_0^0 = \rho_{\text{vac}}$ in Eq.(12), we get the exact expression:

$$E_G = \frac{1}{2} \int_0^{r_b} \left[1 - (g_{rr})^2\right]T_0^0 r^2 dr$$

$$= \rho_{\text{vac}} \left[\frac{r_b^4}{6} - \frac{\tilde{R}^2}{4} \left\{\tilde{R}^2 \arcsin \left(\frac{r_b}{\tilde{R}}\right) - r_b \sqrt{\tilde{R}^2 - r_b^2}\right\}\right]. \quad (42)$$

Taking the boundary close to the horizon, viz., $r_b \to \tilde{R} \sim r_{\text{hor}}$, we find that (in units $8\pi G = 1$):

$$E_G = \left(\frac{4 - 3\pi}{24}\right) \rho_{\text{vac}} \tilde{R}^3$$

$$= \left(\frac{4 - 3\pi}{24}\right) \times 3 \tilde{R} = -0.678 \left(\frac{1}{H_0}\right) = -1.356 M. \quad (43)$$

The result $E_G < 0$ implies that the total gravitational energy inside the de Sitter star is attractive whereas independent physical information is that the de Sitter space has repulsive gravity (because $\rho + 3p < 0$). So one might conclude that the sign of $E_G$ is conveying a wrong result. This need not be so. We have to recall that the de Sitter expansion means that the entire 3-space is expanding. On the other hand, by construction the de Sitter gravastar has a finite boundary close to the horizon of an exterior Schwarzschild metric, the inner boundary exerting inward force balancing the outward force from within.

The whole scenario can be given a metric equivalent description replacing the inner region by the interior Schwarzschild solution for constant density $\rho_{\text{vac}}$. That this replacement is indeed possible can be seen by looking at the interior Schwarzschild $g_{rr}$ (Remember: for $E_G$ we need to consider only $g_{rr}$) which is given by

$$g_{rr} = \left(1 - \frac{2Mr^2}{r_b^3}\right)^{-1} \quad (44)$$

which matches the exterior at $r = r_b$. Putting $M$ from Eq.(40) and using $\tilde{R}^2 = \frac{3}{8\pi G \rho_{\text{vac}}}$, we get exactly the $g_{rr}$ of metric (39). The interior Schwarzschild metric always has $E_G < 0$. By the same token, the gravastar too can have attractive gravity in the interior via the interpretation of metric equivalence. This explanation seems feasible since the gravastar is after all a stable Schwarzschild-like star (as viewed from outside) with an arbitrarily thin layer of quasi-normal matter at a place where horizon would have formed. In the next sections, we shall consider truncated wormholes which are constructed in a manner very similar to that of gravastar.

6. Thin shell contribution
Several asymptotically flat wormholes are known in the literature with matter threading the wormhole all the way to infinity with radial fall-offs in the stress quantities. Such wormholes might be existing in nature as an end result of some past astrophysical phenomena or might be artificially constructed by truncation. For completeness, we shall calculate the thin shell contribution to $\tilde{E}_G$ although the contribution can be made arbitrarily small.

The idea of a truncated wormhole is the following. One wants to artificially create a wormhole by localizing the exotic matter within a finite radius around the throat $r = r_0$ of a given solution. This can be achieved by taking a cut-off at any finite radius away from $r = r_0$, say, at $r = a > r_0$ and matching the surface at $r = a$ to an exterior Schwarzschild vacuum. The matching brings into play junction conditions as follows: The induced metric on the spacelike junction interface $\Sigma$ is given by

$$ds^2_{\Sigma} = -d\tau^2 + a^2(d\theta^2 + \sin^2\theta d\psi^2)$$

where $\tau$ is the proper time on the surface. On this surface the matter energy density $\sigma$ and transverse pressures are calculated from the jump in the extrinsic curvature $[K_{ij}]^+ = K_{ij}^+ - K_{ij}^-$ as $r \to a \pm$. The result is

$$\sigma = -\frac{1}{4\pi a} \left[ \sqrt{1 - \frac{2M}{a}} - \sqrt{1 - \frac{b(a)}{a}} \right]$$

$$P_\theta = P_\psi = \frac{1}{8\pi a} \left[ \frac{1 - \frac{M}{a}}{\sqrt{1 - \frac{2M}{a}}} - \zeta \sqrt{1 - \frac{b(a)}{a}} \right]$$

where $\zeta = 1 + a \frac{d\Phi}{dr}|_{r=a}$, $\sigma$ is the surface energy density and $P_\theta, P_\psi$ are transverse pressures on the surface. When $\sigma = 0$, $P_\theta = P_\psi = 0$, the surface $r = a$ is called the boundary. However, a more interesting possibility is to consider an arbitrarily thin shell of quasi-normal matter (that is, matter satisfying both WEC and NEC) at $r = a$. Then the total mass-energy is given by

$$M = b(a) \left[ \sqrt{1 - \frac{b(a)}{a}} - \frac{M_{\text{shell}}}{2a} \right]$$

where $M_{\text{shell}} = 4\pi a^2 \sigma$ is the shell mass contribution. If $b(a) = 2M$, then $\sigma = 0$. To have an idea of how $\sigma \neq 0$ contributes to the gravitational energy $\tilde{E}_G$, we should fix the shape function $b(a)$ to a value slightly away from $2M$. For instance, we can fix $b(a) = 2M - \epsilon M$ where $0 < \epsilon \ll 1$ is a dimensionless parameter related to the infinitesimally thin thickness of the shell. In this case we get, to leading order in $\epsilon$,

$$M_{\text{shell}} \approx \frac{\epsilon M}{2}.$$  

Up to a factor $(\frac{1}{4})$, this is exactly the same result as that obtained in Ref.[23]. To get an idea of the measure of $E_M$ in the shell, we can regard the density to
be approximately the constant $\sigma$ throughout the shell while the spacetime can be approximately described by a Schwarzschild metric for mass $M$. Then

$$E_{M}^{shell} = \frac{1}{2} \sigma \int_{a}^{a+\epsilon} \left( 1 - \frac{2M}{r} \right)^{-\frac{1}{2}} r^2 dr$$

$$\simeq \frac{\epsilon M}{2} \frac{\epsilon}{4\pi a^2} \left[ 3a^2 + aM \right] = O(\epsilon^2)$$

The total gravitational energy of the truncated wormhole therefore becomes

$$\tilde{E}_G = M - E_M = \frac{1}{2} \int_{r_0}^{a} \left[ 1 - (g_{rr})^{\frac{1}{2}} \right] T_{0}^{0} r^2 dr + \frac{r_0}{2}$$

$$+ M_{shell} \left[ \sqrt{1 - \frac{b(a)}{a}} - \frac{M_{shell}}{2a} \right] - E_{M}^{shell}.$$ 

The term $M_{shell}^2$ as well as $E_{M}^{shell}$ may be neglected as being of order $\epsilon^2$. To calculate $M_{shell}$ for a given shape function $b(r)$, we express Eq.(49) in terms of $b(a)$ as follows

$$M_{shell} = \left( \frac{\epsilon}{2} \left[ \frac{b(a)}{2 - \epsilon} \right] \right) \simeq \frac{\epsilon b(a)}{4}$$

(to first order in $\epsilon$). So the contribution to mass-energy coming from the thin shell reduces to $\frac{\epsilon b(a)}{4}$ which is always positive. This may be added to the right hand side of Eq.(12). So, in all, we can write

$$\tilde{E}_G = M - E_M = \frac{1}{2} \int_{r_0}^{a} \left[ 1 - (g_{rr})^{\frac{1}{2}} \right] T_{0}^{0} r^2 dr + \frac{r_0}{2} + \frac{\epsilon b(a)}{4}.$$ 

The contribution to energy from thin shell of quasi-normal matter is essentially of academic interest rather than anything substantial because of the limit $\epsilon \to 0$ and is generally ignored. (See for instance the second reference in [32].) We too shall ignore it in what follows.

The physical situation in any wormhole is that the cross-sectional area of a bundle of light rays entering one mouth must decrease and then increase while emerging at the other mouth. This can be produced only by the gravitational repulsion of matter [20] at or in the vicinity of the throat. Let us now analyze a couple of known truncated wormhole solutions to see if this criterion is satisfied by the definition of $\tilde{E}_G$.

### 7. Lobo phantom wormhole

The metric is given by

$$ds^2 = - \left[ 1 - \left( \frac{r_0}{r} \right)^{1-\alpha} \right]^{\frac{1+\alpha}{1-\alpha}} dt^2 + \frac{dr^2}{1 - \left( \frac{r_0}{r} \right)^{1-\alpha}} + r^2 \left( d\theta^2 + \sin^2 \theta d\psi^2 \right)$$

(55)
where $\alpha$ and $\omega$ are constant parameters and $0 < \alpha < 1$. The phantom equation of state further demands that $p_r/\rho = \omega < -1$. The shape function and the redshift function respectively are

$$b(r) = r^{\alpha} r_0^{1-\alpha}$$

$$\Phi(r) = \left(1 + \frac{\alpha \omega}{1 - \alpha}\right) \ln \left[1 - \left(\frac{r_0}{r}\right)^{1-\alpha}\right].$$

The density and radial pressure for this wormhole are

$$\rho = \frac{\alpha r_0}{r^3} \left(\frac{r_0}{r}\right)^{-\alpha}$$

$$p_r = \frac{\alpha \omega r_0}{r^3} \left(\frac{r_0}{r}\right)^{-\alpha}.$$  

To have the spacetime free of singularities, we must impose a constraint $1 + \alpha \omega = 0$. We thus obtain the NEC violating condition

$$\rho + p_r = \frac{\alpha r_0}{r^3} \left(\frac{r_0}{r}\right)^{-\alpha} \left(\frac{\alpha - 1}{\alpha}\right) < 0$$  

satisfied for all $r$.

As discussed by Lobo [8], this wormhole can be truncated at some finite radius at $r = a$ away from the throat $r = r_0$ to match to an exterior Schwarzschild spacetime. Neglecting the thin shell contribution $O(\epsilon)$, we can explicitly do the integration in Eq.(12) to get $\bar{E}_G$. Taking for example, $\alpha = \frac{1}{3}$, so that $\omega = -3$, we obtain

$$\rho = \frac{1}{3} \left(\frac{1}{r}\right)^{-\frac{1}{3}} \frac{1}{r^3} > 0.$$  

Choosing mass units in which $r_0 = 1$ and using the metric (55), we get

$$\bar{E}_G = \frac{1}{2} \int_1^a \left[1 - \left(1 - r^{-\frac{2}{3}}\right)^{-\frac{3}{2}}\right] r^2 dr + \frac{1}{2}$$

$$= \frac{a^{\frac{1}{3}}}{2} \left[a^{-\frac{1}{3}} - \sqrt{1 - a^{-\frac{2}{3}}}\right].$$

For any value of $a > 1$, it is evident that $\bar{E}_G > 0$ (Fig.1). That is, there is the expected repulsion around the throat, and elsewhere within the cut off boundary. One may take any other value in the range $0 < \alpha < 1$ and $\omega < -1$ consistent with $1 + \alpha \omega = 0$ to see that the same repulsion continues to occur.

8. Lemos - Lobo - Oliveira wormhole (LLO)

The metric inside $r_0 \leq r \leq a$ is given by [33]

$$ds^2 = - \left[1 - \left(\frac{r_0}{a}\right)^{\frac{2}{3}}\right] dt^2 + \frac{dr^2}{1 - \left(\frac{r_0}{a}\right)^{\frac{2}{3}}} + r^2 \left(d\theta^2 + \sin^2 \theta d\psi^2\right).$$
which gives

\[ b(r) = \sqrt{rr_0} \quad (65) \]

\[ \Phi = \frac{1}{2} \ln \left[ 1 - \left( \frac{r_0}{a} \right)^{1/2} \right] = \text{const.} \quad (66) \]

The exterior vacuum is described by the Schwarzschild metric in \( a \leq r < \infty \) as follows

\[ ds^2 = -\left[ 1 - \left( \frac{r_0 a}{r} \right)^{1/2} \right] dt^2 + \frac{dr^2}{1 - \left( \frac{r_0 a}{r} \right)^{1/2}} + r^2 \left( d\theta^2 + \sin^2 \theta d\psi^2 \right). \quad (67) \]

The energy density and radial pressure are

\[ \rho = \frac{1}{r^2} \frac{db}{dr} = \frac{\sqrt{r_0}}{2r^{5/2}} > 0, \quad (68) \]

\[ p_r = -\frac{\sqrt{r_0}}{r^{5/2}} < 0 \quad (69) \]

\[ \frac{p_r}{\rho} = -2. \quad (70) \]

We have a phantom equation of state \( (\omega = -2) \) here although the metric properties are quite different from the earlier example. NEC is violated everywhere, including at the throat, since \( \rho + p_r = -\frac{\sqrt{r_0}}{2r^{5/2}} \). The throat appears at \( r = r_0 \), and the spacetime is perfectly regular there. To get an estimate, we again choose mass units in which \( r_0 = 1 \) with the cut-off at \( r = a \). Using the metric (64), we can calculate \( \tilde{E}_G \) as follows:

\[ \tilde{E}_G = \frac{1}{2} \int_1^a \left[ 1 - \left( 1 - r^{-\frac{1}{2}} \right)^{-\frac{1}{2}} \right] \rho r^2 dr + \frac{1}{2} \]

\[ = \left( \frac{1}{2a^{\frac{3}{2}} \sqrt{1 - a^{-\frac{1}{2}}}^2} \right) \left( a^{\frac{1}{4}} + a^{\frac{3}{4}} \left( \sqrt{1 - a^{-\frac{1}{2}}} - 1 \right) \right) \]

\[ - \left( \sqrt{a^{\frac{7}{2}} - 1} \right) \ln \left( a^{\frac{1}{4}} + \sqrt{a^{\frac{3}{2}} - 1} \right). \quad (71) \]

From Fig.1, it is evident that \( \tilde{E}_G > 0 \) for \( 1 < a < 2.15 \) while \( \tilde{E}_G \leq 0 \) for \( a \geq 2.15 \). One also sees exactly where \( \tilde{E}_G \) changes sign. The main thing however is that there is the desired repulsion (defocussing) in the vicinity of the throat which lie within the range \( 1 < a < 2.15 \).

9. Summary

The original derivation of the formula for \( E_G \) for a static spherically symmetric asymptotically flat spacetime, as given in [17,18], is adapted to exotic matter sources that automatically satisfy local conservation laws. There is a
statement in [19] to the effect that $E_G < 0$ for localized sources satisfying energy conditions. The statement is certainly true for ordinary fluids. However, the converse question, namely, whether $E_G > 0$ in case of energy condition violating matter such as occurring in gravastar or wormholes, remained essentially open. The present article is a primary initiative to answer the question.

To handle wormhole configurations, which require repulsion, we proposed the expression $\tilde{E}_G$ consistent with wormhole geometry without center. Subsequent implementation of it not only supported the Maxwellian analogy in a wider regime but also correctly produced the gravitational energy picture in the Ellis, Lobo and LLO phantom wormholes. The definition of $\tilde{E}_G$ was also shown to be robust in the sense that it did reproduce the expected behavior under slightly deviating circumstances. The explicit analysis of truncated wormhole lends force to the notion that a condition weaker than WEC violation, namely, NEC violation is sufficient to cause defocussing of light rays. The Mazur-Mottola gravastar does not have wormhole topology but the exact result for $\tilde{E}_G (\equiv E_G)$ admits a plausible physical interpretation.

What are the possible implications of these results? We recall that $\rho > 0$ wormholes are not ruled out [3] but there needs to be defocussing of light rays, hence repulsion, at or in the vicinity of the throat. Looking at Eq.(12) we realize that the integral can, in principle, result in values having either signs depending on the wormhole model chosen. If it so happens that the integral is large and negative overcoming the additive factor $\frac{r_0^2}{2}$, then we end up with $\tilde{E}_G < 0$ or lack of repulsion everywhere. We might rule out such wormhole configurations as physically unrealistic or unrealizable, though they might be technically valid solutions.

(Note added: It has been brought to our notice that wormholes in ghost scalar field theories are unstable under both linear and nonlinear perturbations [34-36], which refutes the result of Ref. [27]).

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Appendix

The basic idea of Lynden-Bell et al [17] is to draw an energy analogy between electrodynamics and general relativity: The total electrical energy $E_{em}$ of a spherical charge distribution $Q(r)$ can be derived in various ways but the true electrical energy density $F_{em}^2 (\equiv \frac{Q^2}{r^2})$ can be found only from the expression which due to Maxwell, and given by

$$E_{em} = \int F_{em}^2 dV = \int \left( \frac{Q}{r^2} \right)^2 dV$$

where $dV$ is the elementary volume of flat 3-space. The integral over the perfect square evidently gives the electrical field strength $F_{em} = \frac{Q}{r}$.

The question is whether a similar notion of gravitational energy density can be developed within the framework of general relativity. Misner, Thorne and Wheeler [25] deny the existence of localized gravitational field energy density in general. Nevertheless they give an expression for it but only in the exceptional case of spherical symmetry. Lynden-Bell et al developed the gravitational field energy density in a form which is remarkably analogous to the above Maxwell expression. They further extended the notion to axisymmetric spacetimes.

Adapting their derivation to spherically symmetric wormholes, we define $x = \frac{2M(r)}{r} = \frac{\kappa(r)}{r}$, and noting $x \to 0$ as $r \to \infty$, and $x = 1$ at the throat $r = r_0$, and further using $dV = (1-x)^{-\frac{3}{2}} \times 4\pi r^2 dr$, we obtain

$$\tilde{E}_G = -\int_{r_0}^{\infty} F_G^2 dV + r_0$$

which is Eq.(13) in the text.