Tunneling time in attosecond experiments and time-energy uncertainty relation

Ossama Kullie

Theoretical Physics, Institute for Physics,
Department of Mathematics and Natural Science, University of Kassel, Germany

In this work we present a theoretical model supported with a physical reasoning leading to a relation which performs an excellent estimation for the tunneling time in attosecond and strong field experiments, where we address the important case of the He-atom [1, 2]. Our tunneling time estimation is found by utilizing the time-energy uncertainty relation and represents a quantum clock. The tunneling time is also featured as the time of passage through the barrier similarly to the Einstein’s photon box Gedanken experiment. Our work tackles an important study case for the theory of time in quantum mechanics, and is very promising for the search for a (general) time operator in quantum mechanics. The work can be seen as a new fundamental step in dealing with the tunneling time in strong field and ultra-fast science, and is appealing for more elaborate treatments using quantum wave packet dynamics and especially for complex atoms and molecules.

Keywords: Tunneling time in strong field and ultra-fast science, time measurement in attosecond experiments, quantum clock, time-energy uncertainty relation, time and time-operator in quantum mechanics, photon box Gedanken experiment.

*Electronic mail: kullie@uni-kassel.de
I. INTRODUCTION

A comprehensive theory of time measurement in quantum mechanics is missing to date [3] (chap. 3). Often is said that time plays a role essentially different from the role of the position in quantum mechanics. In contrast Hilgevoord [4] argued that there is nothing in the formalism of the quantum mechanics that forces us to treat time and position differently. Observables as position, velocity, etc. both in classical mechanics as well as in quantum mechanics, are relative observables, and one never measures the absolute position of a particle, but the distance in between the particle and some other object [4, 5]. Indeed there is many attempts to consider time as a dynamical intrinsic, or an observable time called event time. Hilgevoord concluded [4] that when looking to a time operator a distinction must be made between universal time coordinate t, a c-number like a space coordinate, and the dynamical time variable of a physical system suited in space-time, i.e. clocks. Busch [6, 7] argued that the comundrum of the time-energy uncertainty relation (TEUR) in quantum mechanics is related in first place to the fact that the time is identified as a parameter in Schrödinger equation (SEQ). He classified three types of time in quantum mechanics: external time (parametric or laboratory time), intrinsic or dynamical time and observable time. External time are carried out with clocks that are not dynamically connected with the object studied in the experiment, and usually called parametric time. The intrinsic or dynamical time is measured in term of the physical system undergoing dynamically a change, where every dynamical variable marks the passage of time, we will see this is important for our time invention where the energy serves as the dynamical variable in question, which enables a quantitative measure for the length of the time interval of the tunneling or the tunneling time (T-time) in strong field experiments. The third type according to Busch is the observable time or event time, for example the time of arrival of the decay products at a detector.

In the history of the quantum mechanics, the earliest attempt, which causes one of the most impressive debates, is the Einstein’s photon box Gedanken experiment [8] or the Bohr-Einstein weighing photon box Gedanken experiment [5] (and the references therein). A photon is allowed to escape from a box through a hole, which is closed and opened temporarily by a shutter, the period of time is determined by a clock, which is part of the box system, that means the time is intrinsic and dynamically connected with the system under consideration. The total mass of the box before and after a photon passes is measured. Bohr showed that the process of weighting introduces a quantum uncertainty (in the gravitational field) leading to an uncertainty in time $\tau$, which is the time needed to pass out of the box that usually called the time of passage [6, 7], in accordance with the TEUR, eq (1) below. Aharonov and Rezinik [5] offer a similar interpretation, that the weighing leads, due to the back reaction of the system underlying a perturbation (energy measurement), to an uncertainty in the time of the internal clock relative to the external time [5]. Hence for quantum systems it is important to observe the time from within the system or using an internal clock. Busch [6] presented an argument which makes no assumptions concerning the method of measurement, and simply based only on a version of quantum clock uncertainty relation as follows, if the energy of the escaping photons is determined with an accuracy $\delta E$ from the difference of energy before and after the opening period of the shutter, then these energies must be defined within an uncertainty $\delta E$, i.e the box energy uncertainty $\Delta E$ must satisfy $\Delta E \leq \delta E$. Then the clock uncertainty allows to conclude that the box system needs at least a time $t_0 = \frac{\hbar}{\Delta T}$ in order to evolve from the initial state ‘shutter closed’ to the orthogonal state ‘shutter open’. Accordingly, the time interval within which a photon can pass the shutter is indeterminate by an amount $\Delta T = t_0$. This leads to Bohr’s TEUR $\Delta E \approx \hbar$ [3] (chap. 3).

In this work we use similar ideas, where the T-times under the barrier (denoted $\tau_{T,d}$) is suggested to be similar to the time of passage through the barrier (and escaping at the exit of the barrier), and the (quantum) particle (an electron) undergoes this process spends a time that is the time needed from the moment of entering the barrier to the moment of escaping from under the barrier in the tunneling direction. In addition we suggest a time interval needed to reach the entrance of the barrier (denoted $\tau_{T,i}$), after it is shaken off by the laser field at its initial position $x_i$. $\tau_{T,d}$ is similar to the traversal time used in context of the tunneling approaches [9, 10] or the Feynman path integral (FPI) approach [11–13] (and [3] chap. 7). But in contrast we do not make any assumption about paths inside the barrier, where as well-known, the FPI approach is based on all paths starting at the entrance of the barrier at $t = 0$ and end at the exit of the barrier at time $t = \tau$, which defines a time duration $\tau$. A second type of T-time that we invent is what we call the symmetrical T-time or the total T-time (denoted $\tau_{T,sym}$). We will see that can be easily calculated from the symmetry property of the T-time but then later we found $\tau_{T,sym} = \tau_{T,i} + \tau_{T,d}$, that is the time is accounted from the moment of starting the interaction process, where the electron getting a shake-off, responses...
and jumping up to the tunneling “entrance” point taking the (opposite) orientation of the field, moving then under the barrier (the tunneling) to the “exit” point and escapes the barrier to the continuum. The key issue of the present work is [in the words of Busch [3]] a case study, the T-time in attosecond experiment and ultra-fast science derived by utilizing the time-energy uncertainty relation (TEUR):

$$\Delta T \cdot \Delta E \geq \frac{\hbar}{2}$$  \hspace{1cm} (1)

In sec II we present our theoretical model, in sec III we offer a convincing physical reasoning for our theoretical model, in IV we discuss our result with a comparison to the experiment and finally we give a conclusion to our work.

### II. THEORY AND MODEL

#### A. Preview

In the following we suggest a way to approximate the T-time in attosecond experiment based on simple mathematical and quantum mechanical rules. Our start is a model of Augst et al. [14, 15], where the appearance intensity of a laser pulse for the ionization of the noble gases is predicted. The appearance intensity is defined [14] as the intensity at which a small number of ions is produced. In this model (in atomic units) the effective potential of the atom-laser system is given by

$$V_{\text{eff}}(x) = V(x) - xF = -\frac{Z_{\text{eff}}}{x} - xF,$$  \hspace{1cm} (2)

where $F = F_m$ is the field strength at maximum of the laser pulse (in this work in all our formulas $F$ stands for $F_m$), and $Z_{\text{eff}}$ is the effective nuclear charge that can be found by treating the (active) electron orbital as hydrogen-like, similarly to the well-known single-active-electron (SAE) model [16, 17]. The choice of $Z_{\text{eff}}$ is easy recognized for many electron system and well-known in atomic, molecular and plasma physics [18–20]. We take one dimensional model along the x-axis as justified by Klaiber and Yakaboylu et al [22, 23]. Augst et al [14] calculated the position of the barrier maximum $x_m$ by setting $\partial V_{\text{eff}}(x)/\partial x = 0 \Rightarrow x_a = x_m(F_a) = (\sqrt{Z_{\text{eff}}/F_a}$) and by equating $V_{\text{eff}}(x_m)$ to the ionization potential $V_{\text{eff}}(x_a) = -I_p$ (compare fig 1, the lower green curve) they found an expression for the atomic field strength $F_a$,

$$-\frac{Z_{\text{eff}}}{x_a} - x_aF = -I_p \Rightarrow F_a = \frac{F_a^2}{4Z_{\text{eff}}}$$  \hspace{1cm} (3)

and the appearance intensity $I_a = F_a^2$. Now we take this idea and relate our argumentation to this model for $F \leq F_a$, i.e. for the tunnel ionization in the regime of the well-known Keldysh [24] parameter $\gamma_K < 1$. It is easy to see (see fig 1) that the tunnel exit obeyes $x_{+,}(F) \geq x_m(F)$ (note the equality is valid for $F = F_a$, and for the subscript + see below) and the energy of the tunneling electron is not sufficient to reach the the top of the barrier (as for $F_a$) suggesting an energy uncertainty, which is determined by the energy the electron needs to overcome the barrier and appears in the continuum at the exit point. It appears with zero velocity at the exit point $x_{+,}$ according to the strong-field approximation (SFA) due to Keldysh-Faisal-Reiss theory [24–26]. Indeed the barrier height at a position $x$ is, compare fig 1:

$$\bar{h}_B(x) = | h_B(x) | = | E - V_{\text{eff}}(x) | = | -I_p + \frac{Z_{\text{eff}}}{x} + xF |$$  \hspace{1cm} (4)

that is equal to the difference between the ionization potential and effective potential $V_{\text{eff}}(x)$ of the system (atom+laser) at the position $x$, where $E = -I_p$ is the binding energy of the electron before interacting with the laser. Note we can also get $x_m$ and the maximum $h_B(x_m)$ from the derivative of eq (4), $\partial h_B(x)/\partial x = 0$. An immediately arising question is, what about its maximum $h_B(x_m)$ and the energy uncertainty when the electron passes the barrier and is shifted to the continuum or a “quasi” energy level?. First in regard to the derivation of $x_a = x_m(F_a)$ and the eq (3) of Augst et al, it turns out (compare fig 1) that the maximum of the barrier height $h_B(x_m)$ for arbitrary filed strength lies at $x_m(F) = \sqrt{Z_{\text{eff}}/F}$. This follows immediately from the fact that $x_m$ is determined by the maximum of the effective potential for arbitrary filed strength and that is the intersection point of the two potentials, $V(x) = -\frac{Z_{\text{eff}}}{x}$ and $-xF$ (see fig 1), then

$$-\frac{Z_{\text{eff}}}{x} = -xF \Rightarrow x_m(F) = \sqrt{\frac{Z_{\text{eff}}}{F}}$$  \hspace{1cm} (5)
Otherwise (to the both sides) one of the two potentials \((-\frac{Z_{eff}}{2F})\) or \(-xF\) slopes down stronger than the other slopes up (which can be easily gathered from fig 1 and eq (5)), leading to \(V_{eff}(x) < V_{eff}(x_m)\) for \(x \neq x_m\). Indeed eq (3) can be generalized as the following, for a filed strength \(F \leq F_a\) we get

\[
F \leq \frac{I_p^2}{4Z_{eff}F} \Rightarrow \delta_x^2 = I_p^2 - 4Z_{eff}F \geq 0
\]

The equality \(\delta_x = 0\) is valid for \(F = F_a\). We will see that \(\delta_x = \delta_x(F) = \sqrt{I_p^2 - 4Z_{eff}F}\) is a key quantity, it controls the tunneling process, and determines the time “delay” under the barrier \(\tau_{T,d}\) and the total or the symmetrical T-time \(\tau_{T,sym}\), subsec. II C. From fig 1 we see that the barrier height at \(x_m\):

\[
h_B(x_m) = | -I_p - V_{eff}(x_m) | = | -I_p + \sqrt{4Z_{eff}F} |
\]

This is the maximum of the barrier height, and by setting \(h_B(x_m) = 0\) one obtains \(F_a = \frac{I_p^2}{4Z_{eff}}\), which is equivalent to the setting \(V_{eff}(x_m) = -I_p\) as done by Augst et el [14], and can be easily verified from eqs (4), (7). Here we indicate one of the failure of the Keldysh time primary resulting form its inadequate definition. If we recall the definition of the Keldysh time, the time it takes a classical electron (with an average velocity) to cross a static barrier of a length \(l\) [27]. For \(h_B = 0 \Rightarrow l = 0\) we get the T-time \(\tau_0 = 0\) (meaning the ejection of an electron happen instantaneously at \(F = F_a\) because the barrier width vanishes for \(F \rightarrow F_a\) and \(|x_{e,-} - x_{e,+}| = 0\). But we know, at appearance intensity \(I_a = F_a^2\) the ionization time is equal to \(\frac{1}{I_p}\) (in au) [28] (chap. 8) and is not zero. Which seems naturally because the energy gap that has been overcome is \(I_p\), and as we will see later this follows immediately from our model, \(\tau_{T,d}(F_a) = \frac{1}{2I_p}\) and to a total time \(\tau_{T,sym}(F_a) = \frac{1}{I_p}\), see below. As a consequence Keldysh time represents a laboratory clock (parametric time), whereas in our following model and time-relation(s) the time is dynamically connected to the system (to observe the time form within the system and consider the quantum nature of the particle), thus it represents a quantum clock.

**B. Sketch of the model**

The (tunneling-) ionization happens according to SFA with zero kinetic energy at the exit point \(x_e\). Our idea is that the uncertainty in the energy can be quantitatively discerned from the atomic potential energy at the exit point \(\Delta E \sim |V(x_e)| = | -\frac{Z_{eff}}{2x_e} |\) for arbitrary field strength \(F \leq F_a\). Then when the electron moves under the barrier in the \(x\)-direction [22, 23], its kinetic energy getting smaller, at the same time it moves upwards on the potential energy scale losing potential energy \(( -\frac{Z_{eff}}{2} \) getting smaller in absolute value). The change happens simultaneously in the potential and the kinetic energy while staying at the level \(-I_p\)-line when tunneling, compare fig 1, until its kinetic energy becomes zero at the exit point, although its (atomic) potential energy \(\frac{Z_{eff}}{2x_e} \neq 0\). This can be also gathered from the analysis of the short-range Yukawa and

**Figure 1.** Graphic display of the potential curves, the barrier width and the two inner and outer points \(x_{e,\pm} = (I_p \pm \delta_e)/2F\), the “classical exit” point \(x_e = I_p/F\) and the \(x_m(F) = \sqrt{\frac{2I_p}{Z_{eff}F}}\) the position at maximum of the barrier height, (note \(x_a = x_m(F = F_a)\) see text.
long-range Coulomb potentials given by Torlina et al. [29]. Their conclusion supported with ab initio numerical tests, is that for long range potentials, ionization is not yet completed at the "moment" the electron exits the tunneling barrier in contrast to the usual assumption that ionization is completed once the electron emerges from the barrier. Indeed it is not difficult to see that the electric field of the laser pulse shift the electrons along the x-axis direction [22, 23], compare fig 1, reaching the exit point with zero velocity, i.e. the electron is forced by the electric field to take and move along a preferred direction, and to reduce its kinetic energy to zero at the exit point with zero velocity, i.e. the electron is forced by the electric field to take and move along a preferred direction, and to reduce its kinetic energy to zero at the exit point. (the field interacts only kinematically with the electron since no photon absorbing), where it is still underling the attraction of the atomic potential \( V(x_e) = -\frac{Ze\phi}{x_e} \), that defines the uncertainty in the energy and acts as a shutter open/closed like in the photon box Gedanken experiment with an uncertainty proportional to \( \Delta E \sim |V(x_e)| \). As we will see in IV the result is very convincing.

First it is straightforward to show from eq (4) that for the classical exit point \( x_{e,c} \), setting \( \Delta E \sim |V(x_{e,c})| \) leads to incorrect T-time (i.e. it fails to predict T-time measured by the experiment). This is because the atomic potential is neglected in calculating \( x_{e,c} \) [30]:

\[
v_{e,c}^2 - v_0^2 = 0 - 2I_p = -2F(x_e - x_0) \Rightarrow x_{e,c}F = I_p
\]

where \( x_e \approx (x_e - x_0) \), \( x_0 \approx 0 \) is the initial point of the electron and assuming the electron moves along the x-axis direction [22, 23]. It is easy to see, from fig 1, that the "classical exit" point is the intersection of the electric field line \( -xF \) with the ionization potential of the electron \( -I_p \)-line, hence \( x_{e,c} = \frac{I_p}{F} \), which is far from the "correct" exit point. Therefor to use the "classical exit" point, \( x_{e,c} \), to determine the T-time will never give a correct answer. It is important to use a correct exit point. As seen in fig 1 the relation to other intersection points is simple. The crossing points \( x_{e,\pm} \) of \( V_{eff}(x) \) with the \(-I_p\)-line are given by \( h_B(x) = 0 \), which leads to

\[
x_{e,\pm} = \frac{I_p \pm \delta_z}{2F} \Rightarrow x_{e,\pm} = x_{e,c} - x_{e,-}
\]

where \( \delta_z = \delta_z(F) \) is given in eq (6), we emphasize the dependence of \( \delta_z \) on \( Z_{eff} \) as done by [20]. Note the origin of the axes is at 0. We will see later that using the "classical exit" point \( x_{e,c} \) in the uncertainty relation gives the first term in the expansion of the T-time obtained using the "correct" points \( x_{e,\pm} \), eq (8).

A first arising question in our model is, what happens in the limit of the appearance intensity, i.e. for \( F \rightarrow F_a = I_p^2/(4Z_{eff}) \), where the electron is shifted from the ground state \( E_0 = -I_p \) to \( E_f \approx 0 \) appearing at \( x_{e,-} = x_a = x_{e,+} \) with zero velocity. Its energy uncertainty (since no photon absorption) is then (according to our model) \( \Delta E(F_a) \sim |-Z_{eff}/x_a| = \frac{I_p}{4} \). One sees that for atomic field strength \( (F_a) \) the electron is heavily disturbed but appearing not far from the nucleus at \( x_a = x_m(F_a) = \sqrt{\frac{2Z_{eff}}{I_p}} \) with an ionization time that follows immediately from TEUR in eq (1):

\[
\tau_a = \frac{1}{2 \Delta E(F_a)} = \frac{1}{I_p}
\]

as it should be for the ionization process at the \( F_a \) [28] (chap. 8). However we will see later that \( F_a \) is a special case because \( x_{e,-} = x_a = x_{e,+} \) (a double solution of \( h_B(x) = 0 \)) and the limit \( \tau_{T,imp}(F \rightarrow F_a) = \frac{1}{I_p} \) is a sum of two equally terms \( \frac{1}{2I_p} \) [see sec. II C, and III 1]. In conclusion, our model is meaningful and the atomic potential energy of the electron at the tunnel exit \( x_e \) (instead of the gravitational potential in the Einstein-Bohr Gedanken experiment) amounts to calculate the uncertainty of the energy in the tunneling process, and hence the T-time by the virtue of eq (1) which leads to an excellent result as we will see sec. IV.

### C. Tunneling time

Our goal now to find an expression to calculate the T-time, and what we need is the correct exit point, where many approximations exist. The most used one in the literature can be obtained from eq (4), where the barrier height at the entrance and the exit vanishes \( h_B(x_{e,\pm}) = 0 \), they are the crossing points of \( V_{eff} \) curves with the \(-I_p\)-line, see fig 1,

\[
h_B(x) = 0 \Leftrightarrow -\frac{Z_{eff}}{x} - xF = -I_p
\]
Solving eq (10) gives immediately eq (8). As seen in fig 1, \( x_{e,-} \) is the inner crossing point, the “entrance” point, and \( x_{e,+} \) is the outer crossing point, the “exit” point. Physically is argued that the electron escapes the barrier at \( x_{e,+} \), when it moves in the direction \( x_{e,-} \to x_{e,+} \) and vice versa for the opposite direction, and we will see this presents a useful symmetry property of the tunneling process when deriving the T-time.

Now we can calculate the uncertainty in the energy \( \Delta E(x_{e,+}) \) by using the exit point \( x_{e,+} \) (in the direction \( x_{e,-} \to x_{e,+} \)) and from this the T-time. From eqs (8), (1) and according to our model we get:

\[
\Delta E(x_{e,+}) = \left| -\frac{Z_{eff}}{x_{e,+}} \right| = \frac{2F}{(I_p + \delta_z)} = \frac{(I_p - \delta_z)}{2}
\]

\[
\tau_{T,\text{unsy}} = \frac{1}{2} \frac{1}{\Delta E} = \frac{1}{I_p - \delta_z}
\]

which we call the unsymmetrical T-time \( \tau_{T,\text{unsy}} \). We will see later that a factor \((1/2)\) is missing that can be recovered by a symmetry consideration. We first show that the first order of eq (12) is equal to the T-time resulting from using \( x_{e,c} \), the “classical exit” point, and then we look to the symmetry of the tunneling process. Expanding eq (12) in term of \( \xi = (\frac{2F}{I_p}) \), we get in the first order \( \tau_{T,c} \), the T-time at the “classical exit” point \( x_{e,c} \) as mentioned above, then

\[
O^1(\tau_{T,\text{unsy}}) = \frac{I_p}{2F} = \frac{1}{2} \frac{1}{\Delta E_c} = \tau_{T,c}
\]

where \( \Delta E_c = |\frac{Z_{eff}}{x_{e,c}}| \). What about the inner point \( x_{e,-} \)? we could assume, due to the \( \delta_z \)-symmetry between \( x_{e,+}, x_{e,-} \), that the electron enter the barrier backwards from \( x_{e,+} \) “entrance” to \( x_{e,-} \) “exit” with an uncertainty (according to our model)

\[
\Delta E(x_{e,-}) = |\frac{-Z_{eff}}{x_{e,-}}|, \text{which leads to} (\text{compare eq (11), (12)})
\]

\[
\frac{1}{2\Delta E(x_{e,-})} = \frac{(I_p - \delta_z)}{4I_p Z_{eff} F} = \frac{1}{I_p - \delta_z}
\]

This symmetry is deduced in a similar way the Aharonov-Bohm time operator [31] is defined for a free particle \( \hat{T} = \frac{1}{2}(\hat{e}^{-1} + \hat{p}^{-1} \hat{e}) \) or in more elaborate (the so-called bilinear form) and detailed treatment given by Olkhovsky and Recami [32, 33]. Such operators (given by Aharonov-Bohm or Olkhovsky) have the property of maximally symmetric in the case of the continuous energy spectra, and the property of quasi-self-adjoint [34] operators in the case of the discrete energy spectra [32] (and the references therein), and are the nearest best thing to self-adjoint operators and satisfy the conjugate relation with the Hamiltonian and therefore implies an ordinary TEUR [32] and [3] (chap. 1). We use this property, i.e. we assume that the maximally symmetric (or almost self-adjoint [33]) property holds for the T-time, for more details see [32], which leads using eqs (12), (14) to a simple relation for what we call the symmetrical T-time given by:

\[
\tau_{T,sym} = \tau_{T,+} + \tau_{T,-} = \tau_{T,i} + \tau_{T,d}
\]

\[
= \frac{1}{2} \left( \frac{1}{\Delta E^+} + \frac{1}{\Delta E^-} \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{(I_p + \delta_z)} + \frac{1}{(I_p - \delta_z)} \right) = \frac{I_p}{4Z_{eff} F}
\]

where we defined \( \tau_{T,\pm} = 1/(2\Delta E^\mp) = (2(Ip \pm \delta_z))^{-1}, \text{or} (1/2)\Delta E^\pm = \Delta E(x_{e,\pm}) \). We call the first term of eq (15) \( \tau_{T,i} = \tau_{T,+} \) and the second term \( \tau_{T,d} = \tau_{T,-} \) for a reason that will be clear in the next section, sec. III. Relation (15) has again (clearly because \( \delta_z = 0 \)) the correct limit for atomic field strength (compare eq (9) and the discussion after it):

\[
\tau_{T,sym}(F \to F_a) = \frac{1}{2I_p} + \frac{1}{2I_p} = \frac{1}{4Z_{eff} F} \frac{I_p}{4Z_{eff} F} = \frac{1}{4Z_{eff} F}
\]

Note that the limit \( F \to F_a \) gives \( x_{e,+} = x_{e,-} = x_m(F_a) = x_a \) which means that the two points coincide at the top of the barrier (a double solution of eq 10). The question is whether this means a symmetry break of the tunneling process? because at this limit there is only one direction, namely towards the continuum, so that the “tunneling” becomes a “real” ionization (or an ejection) process at the appearance intensity \( I_a = F_a^2 \), and \( \delta_z \) becomes imaginary for super-atomic field strength \( F > F_a \) (whereas \( F < F_a \) is called the subatomic field strength), see below sec IV. A reasonable question in this case, whether the
time to reach the “entrance” of the barrier and under the barrier, intrinsic or dynamical to be measured by a quantum clock, becomes after the tunneling a classical, external (or parametric) time due to the break of some symmetry property? so that (only) under such a symmetry break a quantum clock (the internal time) coincides with a laboratory clock (the external time).

In sec. IV we will see that eq (15) (especially $\tau_{T,d}$) gives an excellent agreement with the experimental result of $[1]$. It worthwhile to mention that the $T$-time measured by the experiment is the time that the electron spends when it moves under the barrier from $x_{e,-}$ to $x_{e,+}$, which corresponds to the second term of eq (15) $\tau_{T,d} = \frac{1}{2} \frac{1}{(I_p - \delta_z)}$, see further sec. IV result and discussion.

III. PHYSICAL REASONING

1. Tunneling time and a model of a shutter

The theoretical mathematical model developed in sec. II can be supported and derived by physical arguments, the only difference is that we try to figure a physical insight that helps us to put physics in mathematical relations. From fig 1 and eq (8) we get ($d_B(F)$ is the barrier width):

$$d_B(F) = x_{e,+} - x_{e,-} = \left( \frac{I_p + \delta_z}{2F} \right) - \left( \frac{I_p - \delta_z}{2F} \right) = \frac{\delta_z}{F}$$

(16)

from this it follows $\delta_z = (x_{e,+} - x_{e,-}) F$. Because for atomic field strength $F = F_a \Rightarrow \delta_z = 0$, $d_B = 0$, we can interpret $\delta_z$ as the field’s (kinetic) energy exerting in the tunneling process between the “entrance” (or the “inner”) point and the “outer” exit point. The uncertainty in the energy as the electron moves on the $-I_p$ level to $x_{e,+}$, i.e. when tunneling the barrier and escapes at the exit to the continuum is then $\Delta E^+ = \text{abs}(-I_p + \delta_z)$. That means the barrier itself causes a delaying time relative to the atomic field strength $F_a$ ($\delta_z = 0$). The $T$-time is then obtained from eq (1),

$$\tau_{T,d} \equiv \frac{1}{2\Delta E^+} = \frac{1}{2(I_p - \delta_z)} \quad \text{(for } F \leq F_a)$$

(17)

which we call $\tau_{T,d}$ (compare eq 15) meaning that is the time duration (time interval) to cross the distance under the barrier (in the direction $x_{e,-} \rightarrow x_{e,+}$) and escapes at the exit point $x_{e,+}$ to the continuum. The term $\tau_{T,d}(F \rightarrow F_a) = \frac{1}{2I_p}$ at the limit $F = F_a$ account for the turning off the wave packet (or shack-off step, see discussion below) at the “entrance-exit” point $x_a$ to escape to the continuum, which indicates that the shack-off step or turning off at $x_{e,+}$ for $F < F_a$ happens with a time delay as given in eq (17). Expanding $\delta_z = I_p \sqrt{1 - (4Z_{eff}F/I_p^2)}$ and taking the first we get eq (15), $\tau \approx \frac{I_p}{2Z_{eff}F} = \tau_{T,sym}$. It means using the symmetrization gives a linearized time duration (linearized T-time). As we already mentioned the limit of eq (17), $1/(2I_p)$, is only one part of the total tunneling time $1/I_p$.

Now we argue that, this picture fits well in the Gedanken experiment of Einstein and the (intrinsic) time $\tau_{T,d}$ (the second term in eq (15) or (18)) is rather the time of passage with the shutter open/closed time interval (generated in the internal time [5]) is related by the virtue of eq (1) to the uncertainty $(1/2)\Delta E^+ = \Delta E = V(x_{e,+})$ (note we recovered in eq (15), (17) the factor $1/2$ missing in eq (12)), and we think that the attosecond experiment with the help of our model represents a realization of the photon-box Gedanken experiment (with the electron as a particle instead of the photon) with an uncertainty being determined from the (Coulomb) atomic potential due to the disturbing by the field $F$, instead of (the disturbing by) the weighting process and as a result an uncertainly in the gravitational potential [5], as done by the famous proof of Bohr to the uncertainty (or indeterminacy) of time in the photon-box Gedanken experiment [5–7].

2. Totale time. ($F = F_a$)

At the moment one can obtain the total time, i.e. including the time to reach the entrance of the barrier $x_{e,-}$, by adding the term $1/(2I_p)$

$$\tau_{T,t} \approx \frac{1}{2} \left( \frac{1}{I_p} + \frac{1}{\text{abs}(-I_p + \delta_z)} \right)$$

(18)
where the index $t$ only to distinguish between different notations. Nevertheless the term $1/(2I_p)$ we added is exact only when

$$F = F_a$$

or $x_{e,-}(F_a) = x_a = x_{e,+}(F_a)$, hence we call it $\tau(x_i, x_a)$

$$\tau(x_i, x_a) = \frac{1}{2I_p}$$  \hspace{1cm} (19)

where $x_i$ is the initial point. It can be viewed as the response time of the electron to the field, that is, the electron received a kick by the field, and is polarized along the field direction, and (jumping up [30]) is moving from $x_i$ to the “entrance-exit” point $x_a$ to the continuum, $x_i \rightarrow x_{e,-}(= x_a = x_{e,+}) \rightarrow \infty$. In this case ($\hbar B(F = F_a) = 0$, $\delta_0 = 0$) the most probable “tunneling” mechanism is that the laser field distorts the electron, shakes it up (moving from $x_i$ to $x_a$), and shakes it off (moving to the continuum) at a (total) time given in eqs (15) or (18) \text{.}

$$\tau = \frac{1}{2} \left( \frac{1}{I_p} + \frac{1}{I_p} \right) = \frac{1}{I_p}$$  \hspace{1cm} (19)

In this model, for $F = F_a$ the (illustrative) two steps are not strictly separated, whereas, see further below, for $F < F_a$ they are well separated due to the barrier $d_B(F) > 0 = d_B(F_a)$.

3. Total time for subatomic field $F < F_a$

However, $x_a = x_{e,-} = x_{e,+}$ is the maximum “entrance” point (most right see fig 1), the electron is less disturbed for $F < F_a$ and moved to a point $x_{e,-} < x_a$ that is closer to the initial point $x_i$ [35], this shortens the time to reach the “entrance” point, and we expect that the response of the electron to a small field strength $F$ is weaker than to a stronger field $F \rightarrow F_a$. The time reduction in $\tau(x_i, x_{e,-})$ for $F$ to $\tau(x_i, x_a) = 1/(2I_p)$ for $F_a$, eq (19), is a factor depending on $\delta_x$ (see discussion after eq (16)), because the kinetic energy experiences a change proportional to $(x_{e,-} - x_i) F < (x_a - x_i) F$ A (weaker) field $F < F_a$ is not sufficient to compensate for the kinetic energy at the (shake-up) step, instead the electron is at the “entrance” $x_{e,-} < x_a$ with a velocity that is sufficient to inter under the barrier and reaches the exit point $x_{e,+}$ with zero velocity. Here is again another failure of the Keldysh time (see also discussion after eq (7)) that the electron did not cross the barrier with its initial velocity $\sqrt{2I_p}$ suggested by many authors [30].

Now we give the following relation for the time needed to reach the “entrance” point $x_{e,-}$, and show an explanation further below.

$$\tau_{T,i} \equiv \tau(x_i, x_{e,-}) = \frac{1}{2(I_p + \delta_x)} \equiv \frac{1}{2\Delta E}$$

The factor $\delta_x$ (comparing to eq (19) for $F_a$) in the denominator results from two parts. Indeed we follow [5] in that the uncertainty is a result of the different reactions or responses of the electron to different filed strengths. The one part comes from the difference of moving along the $x$-axis. i.e. the difference in shifting the electron to $x_a$ with $F_a$ or to $x_{e,-} < x_a$ for $F < F_a$, which leads to $\Delta_1 = x_a F_a - x_{e,-} F = \frac{I_p}{2} - \frac{(I_p - \delta_x)}{2} = \frac{\delta_x}{2}$. The other part can be deduced from the change on the vertical (potential energy) scale. When the electron receives a kick, changing its potential (on a vertical scale), its atomic potential has a different changes between $V(x_i)$ and $V(x)$ for $x = x_a$ or $x_{e,-}$. But this is a result of different responses (on energy scale), while the electron is forced to follow an orientation along the (opposite) field direction at $x_a$, or at $x_{e,-}$. Therefore this part can be approximated from the difference $\Delta V(x)$ in the atomic potential at the different “entrance” points, which gives $\Delta_2 = V(x_a) - V(x_{e,-}) = -\sqrt{Z_{eff} F_a} + \frac{2Z_{eff} F}{(I_p - \delta_x)} = -\frac{I_p}{2} + \frac{(I_p+\delta_x)}{2} = \frac{\delta_x}{2}$. We are led to a difference equal to $\Delta_1 + \Delta_2 = \delta_x$ between reaching the “entrance” point $x_{e,-}$ (for $F_a$) leading to an energy uncertainty $\Delta E = abs(-I_p - \delta_x) = (I_a + \delta_x)$, for $F \leq F_a$ and with a time $\tau_{T,i}(F) = I_p \frac{1}{2\Delta E}$ for arbitrary subatomic field strength $F$, hence one obtains eq (20) instead of eq (19). We have explained so far eq (17) and eq (20), in doing so we explained the physical meaning of the symmetry consideration done above (see eq (15) and the discussion before) and we obtain from eqs (17), (20) the result obtained in eq (15):

$$\tau_{T,x_{sym}} = \tau_{T,i} + \tau_{T,d} \equiv \tau_{T,+} + \tau_{T,-}$$

$$= \frac{1}{2} \left( \frac{1}{I_p + \delta_x} + \frac{1}{(I_p - \delta_x)} \right) = \frac{I_p}{4Z_{eff} F}$$  \hspace{1cm} (21)

the first term $\tau_{T,+} = 1/(2E^-)$ corresponds to the first step, where the electron is shacked up and moved to the “entrance” $x_{e,-}$ (or $x_a$ for $F_a$) that takes the times $\tau_{T,i} = \tau_{T,+}$. The second term $\tau_{T,-} = 1/(2E^+)$ corresponds to the actual T-time or moving
under the barrier and shaken off to the continuum, with a time delaying \( \tau_{T,d} = \tau_{T,i} \) due to the barrier relative to the atomic field strength \( F_a \), where \( \delta_x = 0 \) and \( \tau_{d,T} = \frac{I}{27p} \).

For \( F_a \), as already mentioned, the second or shake-off step follows immediately the first or shack-up step and the two steps are not strictly separated. For \( F < F_a \) the two steps model of the tunneling process are well separated. They happen with opposite directions at the time scale, the first step is less time consuming since \( F \) causes a smaller disturbance relative to \( F_a \), and the electron moves not far from its initial position for small \( F \), \( x_{e,-} < x_a \), whereas the second step happens with a time delaying, \( x_{e,+} > x_a \), relative to the ionization at atomic field strength. So far our theoretical model is assisted with an explanation through a physical reasoning. In the following we show and discuss our result for He-atom with a comparison to the experiment [1, 2].

### IV. RESULT AND DISCUSSION

In fig 2 we show the results of eq (12) the unsymmetrical \( \tau_{T,unsym} \), and eq (15) the symmetrical T-time \( \tau_{T,sym} \). The results for \( \tau_{T,d} \) eq (17), and again the the symmetrical (or total) T-time \( \tau_{T,sym} \) of eq (21) are shown in fig 3. Note eq (21) is identical with eq (15), whereas eq (17), is the second term of eq (21) or (15), which is the actual T-time, i.e. the time needed to pass the under barrier region \( (x_{e,-} \rightarrow x_{e,+}) \) and escape at the exit point to the continuum, and that is usually the T-time measured in the experiment. The results are for the He-atom in a comparison with the experimental result of [1]. The experimental data and the error bars in the figure were kindly sent by A. Landsman [36]. We plotted the relations (12), (15), (17) and (21) at the values of the field strength at maximum of the elliptically polarized laser pulse \( (\lambda = 735, \text{elliptical parameter } \varepsilon = 0.87, F = F_0/\sqrt{1 + \varepsilon^2}) \) used by the experiment exactly as given in [36]. In fig 2, the upper two curves are the unsymmetrical T-time \( \tau_{T,unsym} \), eq (12), for two different models of the effective nuclear charge \( Z_{eff} \), that the tunneling electron experiences during the tunneling process. Accordingly the lower two curves are the symmetrical T-time, \( \tau_{T,sym} \), eq (15) for the two different models of \( Z_{eff} \). We mention that eq (18) (not plotted) gives a closer result to eqs (15), (21). The two different effective charge models are from Kullie [37], with \( Z_{eff,K} = 1.375 \) and Clementi [38] with \( Z_{eff,C} = 1.6875 \). In fig 2 we see that our \( \tau_{T,unsym} \) is not close to the experimental data of [1]. Whereas \( \tau_{T,sym} \) is close for both models of the \( Z_{eff} \). But we notice, see discussion further below, that the first term in eqs (15), (21) is much smaller than the second \( \tau_{T,i} < \tau_{T,d} \) for small \( F \) (relative to \( F_a \)).

Concerning \( Z_{eff} \) we see for small field strength \( F \lesssim 0.05 \) that \( \tau_{T,sym} \) with \( Z_{eff,K} \) is closer to the experimental data (and especially for \( \tau_{T,d} \), see discussion below fig 3). The reason is that the \( Z_{eff,K} \) model is a H-atom-like model, based on the assumption that the first electron of the He-atom occupies the 1s-orbital (with probability density \( |\Psi(r)|^2 \), which screens the electronic charge and the second electron is treated as an active electron or a “valence” electron [37] similar to the SAE approximation. This is a good approximation when the tunneling electron moves far from the left atomic core (He\(^+\)), or the barrier width is large \( x_{e,+} > 15a_u \), hence the better agreement, and possibly this is important for smaller field strength in the region where \( \gamma_K \approx 1 \). In the range of larger field strength multielectron effects are expected and the model \( Z_{eff,C} \) based on Hartree-Fock calculation is more reliable, where the electron moves not far from the left atomic core (small barrier width) and hence the better agreement in this region. It is likely that a model depending on the \( x \)-coordinate \( Z_{eff}(x_{e,+}) \) will achieve a better agreement that smoothly fits the two regions.

Now we look to fig 3, where \( \tau_{T,sym} \) eq (21) and \( \tau_{T,d} \) eq (17) are shown. Eq (21) is the same as eq (15) (shown in fig 2). For the \( \tau_{T,d} \) we see an excellent agreement with the experiment. As already mentioned \( \tau_{T,d} \) corresponds to the T-time measured in the experiment, that is the time (interval) needed to move under the barrier from the entrance to the exit point and escape to the continuum with a shack-off, or between the instant of orientation at \( x_{e,-} \) an the instant of ionization at \( x_{e,+} \), which is the time spent in the classically forbidden region [1]. For small \( F \lesssim 0.055a_u \), \( Z_{eff,K} \) gives better agreement with the experiment, whereas for larger field strength \( Z_{eff,C} \) is more reliable, where multielectron effects are expected due to the decreasing width of the barrier and the tunneling electron is closer to the first one when it traverses the barrier.

In fig 3 we see that the difference between the total or symmetrical T-time \( \tau_{T,sym} \) and the (actual) T-time \( \tau_{T,d} \) is small, because the second term \( \tau_{T,d} \) in eq (21), incorporates the delaying time caused by the barrier and is the main time of the tunneling process for large barrier. Whereas the first part term \( \tau_{T,i} \), is due to the shake-up of the electron by the field moving it to the "entrance" \( x_{e,-} \), which is small for small \( F \). For large field strength the two parts become closer because the barrier
Figure 2. T-time $\tau_{T, \text{unsy}}$ eq (12), and $\tau_{T, \text{sym}}$ eq (15), for two $Z_{\text{eff}}$ models [37] and [38]. Time is in attosecond units vs laser field strength in atomic units, corresponds to the tunneling ionization of the He-Atom in strong field, in good agreement with experimental result [1, 2, 36]. Experimental values are kindly sent by A. Landsman [36].

Figure 3. T-time $\tau_{T, d}$ eq (17) and $\tau_{T, \text{sym}}$ eq (21), (note eq (21) is identical to eq (15), see fig 2), for two different $Z_{\text{eff}}$ models as in fig 2. Time vs laser field strength as in fig 2, corresponds to the tunneling ionization of the He-Atom in strong field, in excellent agreement with experimental result [1, 2, 36]. Experimental values as in fig 2.

width getting smaller $\delta_z/F = (x_{e,+} - x_{e,-}) \to 0$ and for appearance intensity ($\delta_z = 0$) they become equal. Concerning $Z_{\text{eff}}$ in fig 3, we readily see for $\tau_{T, d}$ the same behavior as in fig 2.

In fig 4 we plotted our result $\tau_{T, d}$ eq(17) together with the FPI result of [36] (data were kindly sent by A. Landsman and C. Hofmann [36, 39]). From the figure we see a good agreement between the two results, and the difference is smaller than the experimental error bars. Indeed we expect that the FPI would agree better for large field strength $F > 0.055$ with lower curve ($Z_{\text{eff}, C}$, green). For small field strength, FPI is more or less close to both curves (green, blue), but our upper curve ($Z_{\text{eff}, B}$, blue) tends to be in a better agreement with the experimental values for smaller field strength. An important point is that our model and result(s) predict a (real) T-time (time of passage) of a single particle, it is not distributive of an ensemble (although indeterminately in regard to the uncertainty relation) and we make no assumption about the path of the particle inside the barrier, whereas the FPI treatment is probabilistic/distributive that makes a use of all possible (classical) paths inside the barrier that have a traversal tunneling time $\tau$. Furthermore, Landsman et al [40] uses the time $\tau_0$, which is determined by the measurement to coarse-grained the probability distribution of the T-times to achieve the aimed results, although Sokolovski [13], [3] (chap. 7) claims (in regard to his FPI description) that no real time is associated with the tunneling. We think that the two views are rather complementary as it is usual in quantum mechanics: wave/particle or individual (single particle)/statistical (distributive) etc. The Larmer Clock should in principle agrees with our result, the result of Landsman [36] shows that the agreement is good only for $F \approx 0.06 - 0.1$, hence Larmer Clock values are inferior for $F < 0.06$. The same holds for $F > 0.1$,
In quantum mechanical the tunneling-ionization time has a real part limit time with the system arbitrarily small, the real part of the time can be made arbitrary small, and an imaginary part is absent.

For "real" time tunnel-ionization process. Indeed in this case the atomic potential is heavily disturbed and the imaginary part of the time is considered, i.e. when assuming the maximally symmetrical (or quasi-self-adjoint) property discussed in details by Olkhovsky [32]. It is now the question to which extend the above relation of the time is considered, i.e. when assuming the maximally symmetrical (or quasi-self-adjoint) property discussed in details by Olkhovsky [32]. It is now the question to which extend the above relation of the time is considered, i.e. when assuming the maximally symmetrical (or quasi-self-adjoint) property discussed in details by Olkhovsky [32].

γ >> physically a different character. It certainly also loses its validity in the multiphoton regime, i.e. for large Keldysh parameter, preserves its validity for $F > F_a$ (or for small $F << F_a$, where $\gamma \ll 1$, where or what is/are the limit(s) of its validity? Probably a break of some symmetry for

At the limit $F = F_a$ of the sub-atomic field strength the tunneling process is out and an ionization process called the "above the barrier decay" is beginning. For supper-atomic field strength $F > F_a$, $\delta_z$ becomes imaginary (and so the crossing points, compare eq (8), but still a real $x_m = \sqrt{\epsilon F}$), which indicates that the real part of $\frac{1}{\text{eff}}$ of $\tau_{T,d}$ or $\tau_{T,i}$, is the limit for "real" time tunnel-ionization process. Indeed in this case the atomic potential is heavily disturbed and the imaginary part of the time $\tau_{T,d}$ is then due to the release of or escaping the electron (at $x_m(F)$) from a lower energy level than $-Ip$ (and possibly escaping with a high velocity), where the ionization happens mainly by a shock-off step [28] (chap. 9). Here we see the clear difference between the quantum mechanical and the classical clocks [5, 31]. Classically we can make the interaction time with the system arbitrarily small, the real part of the time can be made arbitrary small, and an imaginary part is absent.

In quantum mechanical the tunneling-ionization time has a real part limit $\tau_{T,d} = 1/(2Ip)$, an imaginary part arise when the field strength is larger than the atomic field strength $F_a$, in both terms $\tau_{T,i}$ and $\tau_{T,d}$.

However in our treatment, although, $\tau_{T,i}, \tau_{T,d}$ have an imaginary part, we get a real total or symmetrical T-time $\tau_{T,\text{sym}} = \frac{1}{\text{eff}}$ for ionization processes with an arbitrary field strength. It becomes very small for very large field strength, and probably it loses its validity in this regime suspecting a break of some symmetry and non-linear effects arise, and the interaction becomes physically a different character. It certainly also loses its validity in the multiphoton regime, i.e. for large Keldysh parameter $\gamma >> 1$, where $F << F_a$. It is apparent from $\tau_{T,\text{sym}}$, eqs (15), (21) that the T-time has no imaginary part when the symmetry of the time is considered, i.e. when assuming the maximally symmetrical (or quasi-self-adjoint) property discussed in details by Olkhovsky [32]. It is now the question to which extend the above relation $\frac{1}{\text{eff} F}$ preserves its validity for $F > F_a$ (or for small $F << F_a$, where $\gamma \ll 1$, where or what is/are the limit(s) of its validity? Probably a break of some symmetry for $F \geq F_a$ (or $F << F_a$) can give a hint to answer this question. Finally we mention that for $F > F_a$ or intensities $I > I_a$ Stark-shift, relativistic and non-linear effects become large, the perturbation theory breaks down (which is valid for small parameter $F/F_a$ [28] and several regions appear at intensities larger than the appearance intensity $I_a$, such as the critical $I_c$ and the saturation $I_s$ intensities, $I_s > I_c > I_a$, where multiple ionization occurs [28] (chap. 7, 9).

Conclusion We presented in this work an analysis for the tunneling process and the tunneling time in attosecond experiment and found an accurate and simple relation to calculate the tunneling time for the important case of He-atom, where a reliable experimental data are available. Our result (especially the T-time $\tau_{T,d}$) was shown to be in excellent agreement with the experiment [1, 2] and with the Feynman path integral treatment of [36] although for small field strength our result of
Figure 5. T-time $\tau_{T,d} \text{eq (17)}$ in as vs barrier width $d_B(F)$ in au. for two different $Z_{eff}$ models as in fig 2. The lines at the bottom of the figure show the time spent by a particle (a photon) traversing the same barrier with the speed of light.

$Z_{eff,K}$ tends to agree better with experimental values. Note in figs 2-5 we use for the evaluation of our result the same values of the field strengths used by the experiment, i.e the filed strength at maximum, see [36]. The T-time in our treatment is dynamical or intrinsic-time type and represents a quantum clock, i.e. to observe the time form within the system and consider the quantum nature of the (bound) particle, in contrast to the classical Keldysh time which is external or parametric, where we indicated (one or two of) its failures to treat the T-time in our (study) case.

Further we suggest a model of a shutter to the tunneling process in attosecond experiment, and we think the experiment together with our tunneling model (subsec. II B, II C) offer a realization of the Bohr-Einstein’s photon box Gedanken experiment, with the electron as a particle instead of the photon and with the uncertainty being determined from the (Coulomb) atomic potential instead of the gravitational potential. Our treatment suggests that a symmetry (maximally symmetrical or quasi-self-adjoint) [32] assumption to calculate the T-time is important and gives a hint to the search for a time operator in the tunneling process and maybe for a general time operator in quantum mechanics. Our result uses two models of the effective charge $Z_{eff}$ of the left core $He^+$ that the tunneling electron experiences. The $Z_{eff,K} = 1.375$ of Kullie [37], that based on a similar model to the single-active-electron, is better for small field strength $F \lesssim 0.055$, whereas $Z_{eff,C} = 1.6875$ of Clementi et al [38] is more reliable for larger field strength, because it is based on the Hartree-Fock calculation, and that is justified, when the multielectron effects are not negligible.

[1] P. Eckle, A. N. Pfeiffer, C. Cirelli, A. Staudte, R. Dörner, H. G. Muller, M. Büttiker, and U. Keller, Science 322, 1525 (2008)
[2] P. Eckle, M. Smolarski, F. Schlup, J. Biegert, A. Staudte, M. Schöffler, H. Muller, R. Dörner, and U. Keller, Nat. phys. 4, 565 (2008)
[3] in Time in Quantum Mechanics, Lecture Notes in Physics 734, Vol. I, edited by G. Muga, R. S. Mayato, and I. Egusquiza (Springer-Verlag Berlin, 2008)
[4] J. Hilgevoord, Am. J. of Phys. 70, 301 (2002)
[5] Y. Aharonov and B. Reznik, Phys. Rev. Lett. 84, 1368 (2000)
[6] P. Busch, Found. Phys. 20, 1 (1990)
[7] P. Busch, Found. Phys. 20, 33 (1990)
[8] L. F. Cooke, Am. J. of Phys. 48, 142 (1980)
[9] E. H. Huge and Stovneng, Rev. Mod. Phys. 61, 917 (1989)
[10] R. Landauer and T. Martin, Rev. Mod. Phys. 66, 217 (1994)
[11] N. Yamada, Phys. Rev. Lett. 93, 170401 (2004)
[12] D. Sokolovski, S. Brouard, and J. N. L. Connor, Phys. Rev. A 50, 1240 (1994)
[13] D. Sokolovski, S. Brouard, and J. N. L. Connor, Phys. Rev. A 42, 6512 (1990)
[14] S. Augst, D. Strickland, D. D. Meyerhofer, S. L. Chin, and J. H. Eberly, Phys. Rev. Lett. 63, 2212 (1989)
[15] S. Augst, D. D. Meyerhofer, D. Strickland, and S. L. Chin, J. Opt. Soc. Am. B 8, 858 (1991)
[16] H. Muller, Phys. Rev. A 60, 1341 (1999)
[17] X. Tong and C. Lin, J. Phys. B 38, 593 (2005)
[18] D. Schlüter, Z. Phys. D 6, 249 (1987)
[19] R. Lange and D. Schlüter, J. Quant. Spectros. Radiat. Transfer 48, 153 (1992)
[20] I. Dreissigacker and M. Lein, Chem. Phys. 414, 69 (2013)
[21] N. B. Delone and V. P. Krainov 41, 469 (1998)
[22] M. Klaiber, E. Yakaboylu, H. Bauke, K. Z. Hatsagortsyan, and C. H. Keitel, Phys. Rev. Lett. 110, 153004 (2013)
[23] E. Yakaboylu, M. Klaiber, H. Bauke, K. Z. Hatsagortsyan, and C. H. Keitel, Phys. Rev. A 88, 063421 (2013)
[24] L. V. Keldysh, Zh. eksp. teor. Fiz. 47, 1945 (1964), [English translation: 1965, Soviet Phys. JETP, 20, 1307]
[25] F. H. M. Faisal, J. Phys. B 6 (1973)
[26] H. R. Reiss, Phys. Rev. A 22, 1782 (1980)
[27] G. Orlando, C. R. McDonald, N. H. Protik, and T. Brabec, Phys. Rev. A 89, 014102 (2014)
[28] N. B. Delone and V. P. Krainov. Multiphoton Processes in Atoms, 2-edition (Springer-Verlag 2000)
[29] Torlina, F. Morales, J. Kaushal, I. I. H. Geert Muller, A. Kheifets, A. Zielinski, A. Scrinzi, M. Ivanov, and O. Smirnova, arXiv: 402.5620v2 (2015)
[30] M. Y. Ivanov, M. Spanner, and O. Smirnova, J. Mod. Opt. 52, 165 (2005)
[31] Y. Aharonov and D. Bohm, Phys. Rev. 122, 1649 (1961)
[32] V. S. Olkhovsky, Adv. in Math. Phys. 2009, 1 (2009), article ID 859710
[33] V. S. Olkhovsky and E. Recami, Lettere Al Nuovo Cimento 4, 1165 (1970)
[34] It can be chosen as an almost self-adjoint operator with practically almost any degree of the accuracy [32]
[35] Note that for a small field strength \( F \ll F_a \) the electron absorbs one or more photons and moves vertically on the energy scale emerging at exit point very close to the initial point \( x_i \) on the coordinate scale
[36] A. Landsman, M. Weger, J. Maurer, R. Boge, A. Ludwig, S. Heuser, C. Cirelli, L. Gallmann, and U. Keller, Optica 1, 343 (2014)
[37] O. Kullie, diploma thesis (1997), presented at the Mathematics-Natural Science department, Christian-Albrecht-university of Kiel, Germany. And O. Kullie and D. Schlüter, work in preparation
[38] E. Clementi and D. L. Raimondi, J. Chem. Phys. 38, 2686 (1963)
[39] C. Hofmann, , A. S. Landsman, C. Cirelli, A. N. Pfeiffer, and U. Keller, J. Phys. B 44, 125601 (2013)
[40] A. Landsman and U. Keller, J. Phys. B 47, 1 (2014)