Fine grading of $sl(p^2, \mathbb{C})$ generated by tensor product of generalized Pauli matrices and its symmetries

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Abstract

Study of the normalizer of the MAD-group corresponding to a fine grading offers the most important tool for describing symmetries in the system of non-linear equations connected with contraction of a Lie algebra. One fine grading that is always present in any Lie algebra $sl(n, \mathbb{C})$ is the Pauli grading. The MAD-group corresponding to it is generated by generalized Pauli matrices. For such MAD-group, we already know its normalizer; its quotient group is isomorphic to the Lie group $SL(2, \mathbb{Z}_n) \times \mathbb{Z}_2$.

In this paper, we deal with a more complicated situation, namely that the fine grading of $sl(p^2, \mathbb{C})$ is given by a tensor product of the Pauli matrices of the same order $p$, $p$ being a prime. We describe the normalizer of the corresponding MAD-group and we show that its quotient group is isomorphic to $Sp(4, \mathbb{F}_p) \times \mathbb{Z}_2$, where $\mathbb{F}_p$ is the finite field with $p$ elements.

1 Introduction

A grading $\Gamma$ of a Lie algebra $L$ is a decomposition $\Gamma : L = \oplus_{i \in J} L_i$ into non-trivial subspaces $L_i$ such that, for each pair of indices $i, j \in J$, there exists an index $k \in J$ fulfilling the property $[L_i, L_j] \subset L_k$. Among all the gradings of a Lie algebra, the most important ones are fine gradings, since any grading is created from some fine grading.

It was shown (in [10]) that there is a one-to-one correspondence between fine gradings of a simple Lie algebra over $\mathbb{C}$ and maximal Abelian groups of diagonalizable automorphisms (so called MAD-groups) in $Aut L$. Each fine grading of a simple Lie algebra over $\mathbb{C}$ is obtained as a decomposition of $L$ into eigensubspaces of automorphisms from a MAD-group. In [9], all MAD-groups of $Aut sl(n, \mathbb{C})$ were described.

Let us recall that the Lie algebra $sl(n, \mathbb{C})$ has inner and outer automorphisms. An inner automorphism is given by a non-singular matrix $A$ of order $n$ by the prescription

$$Ad_{A}X := A^{-1}XA, \text{ for } X \in sl(n, \mathbb{C}).$$

An outer automorphism is connected with a non-singular matrix $A$ of order $n$ as well, and it is given by the prescription

$$Out_{A}X := -(A^{-1}XA)^\top, \text{ for } X \in sl(n, \mathbb{C}),$$

where $\top$ denotes the transpose.
where \(A^\top\) denotes the transposed matrix.

An important role in the description of MAD-groups without outer automorphism is played by generalized Pauli matrices. They were introduced in \([\text{11}]\):

\[\text{Definition 1.1.}\] For a given \(n \in \mathbb{N}\), put \(\omega = \omega_n = e^{\frac{2\pi i}{n}}\). A group of matrices

\[\mathcal{P}_n := \{\omega^j P_n^k Q_n^l \mid j, k, l \in \{0, 1, \ldots, n-1\}\}, \quad \text{where}
\]

\[
P_n = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & \omega & 0 & \ldots & 0 \\
0 & 0 & \omega^2 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & \omega^{n-1}
\end{pmatrix} \in \mathbb{C}^{n \times n}
\]

and \(Q_n = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix} \in \mathbb{C}^{n \times n},
\]

is called the \textbf{Pauli group} of order \(n\); \(P_n\) and \(Q_n\) are the generalized Pauli matrices of order \(n\).

Let us note that the matrices \(P_n\) and \(Q_n\) do not commute, since \(Q_n P_n = \omega P_n Q_n\). Nevertheless, the inner automorphisms corresponding to these matrices do commute: \(\text{Ad}_{Q_n} \text{Ad}_{P_n} = \text{Ad}_{P_n} \text{Ad}_{Q_n}\).

In order to describe MAD-groups of the algebra \(\mathfrak{sl}(n, \mathbb{C})\), we need further notation:

The group of non-singular diagonal matrices of order \(n\) will be denoted by \(\mathcal{D}_n\), i.e.

\[\mathcal{D}_n = \{\text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) \mid \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C} \setminus \{0\}\}.\]

If \(G_1\) and \(G_2\) are groups of matrices, then \(G_1 \otimes G_2\) denotes the group of all tensor product \(\mathbb{C}A \otimes B\), where \(A \in G_1\) and \(B \in G_2\).

The MAD-groups of \(\text{Aut} \mathfrak{sl}(n, \mathbb{C})\) can be divided into two classes, depending whether or not they contain an outer automorphism. It is proved in \([\text{9}]\) that any MAD-group in the automorphism group \(\text{Aut} \mathfrak{sl}(n, \mathbb{C})\) containing only inner automorphisms is isomorphic to a group of the following form

\[\mathcal{G} = \{\text{Ad}_A \mid A \in \mathcal{P}_{n_1} \otimes \mathcal{P}_{n_2} \otimes \ldots \otimes \mathcal{P}_{n_{r-1}} \otimes \mathcal{D}_{n_r}\},\]

where \(n_1 n_2 \ldots n_r = n\) and \(n_{r-1}\) divides \(n_i\) for any \(i = 2, 3, \ldots, r - 1\).

A grading \(\Gamma : L = \oplus_{i \in J} L_i\) of a Lie algebra \(L\) is a starting point for searching for graded contractions of the Lie algebra. This method for finding contractions of Lie algebras was used by several authors \([\text{11} \text{ 12} \text{ 13} \text{ 14}]\). In this type of contraction, we define new Lie brackets by prescription

\[\left[x, y\right]_{\text{new}} := \varepsilon_{jk}[x, y]\text{, \ where } x \in L_j, y \in L_k.\]

The complex or real parameters \(\varepsilon_{jk}\), for \(j, k \in J\), must be determined in such way that the vector space \(L\) with the binary operation \([\ldots]\)\text{new} forms again a Lie algebra. Antisymmetry of Lie brackets demands that \(\varepsilon_{jk} = \varepsilon_{kj}\). Compliance with the Jacobi identity, however, already implies that the coefficients \(\varepsilon_{jk}\) fulfill a complicated system of quadratic equations, which is in general difficult to solve. For description of symmetries of this system, it is important to know the symmetries of the original grading \(\Gamma\). By a symmetry of a grading of the Lie algebra \(L\) we mean such an automorphism \(g \in \text{Aut} L\) that

\[
\text{for each } j \in J \text{ there exists } k \in J \text{ fulfilling } gL_j = L_k. \tag{1}
\]

\(\text{1If } A \in \mathbb{C}^{n \times n} \text{ and } B \in \mathbb{C}^{m \times m}, \text{ then the tensor product } A \otimes B \in \mathbb{C}^{nm \times nm} \text{ is defined by } (A \otimes B)_{ij} = A_{i1} B_{j1},\]

where \(i_1, i_2 \in \{0, 1, \ldots, n - 1\}, j_1, j_2 \in \{0, 1, \ldots, m - 1\}, I, J \in \{0, 1, \ldots, mn - 1\} \text{ and } I = i_1 m + j_1, J = i_2 m + j_2.\)
Let us suppose that a fine grading $\Gamma : \mathfrak{sl}(n, \mathbb{C}) = \bigoplus_{j \in J} L_j$ corresponds to a MAD-group $G \subset \text{Aut} \, \mathfrak{sl}(n, \mathbb{C})$. It means that

$$hL_k = L_k \quad \text{for all } h \in G \text{ and } k \in J.$$  \hspace{1cm} (2)

Combining (2) and (1), we obtain

$$ghg^{-1}L_j = L_j \quad \text{for any } j \in J.$$  

The maximality of $G$ implies that $ghg^{-1} \in G$ for any $h \in G$. This means, in other words, that the symmetries of the grading $\Gamma$ corresponding to the MAD-group $G$ form a group

$$\mathcal{N}(G) = \{ g \in \text{Aut} \, \mathfrak{sl}(n, \mathbb{C}) \mid gGg^{-1} \subseteq G \}.$$  

This group is usually called the normalizer of the subgroup $G$ in $\text{Aut} \, \mathfrak{sl}(n, \mathbb{C})$.

The article [4] studied the normalizer of one MAD-group of $\mathfrak{sl}(n, \mathbb{C})$, namely

$$G = \{ \text{Ad}_A \mid A \in \mathcal{P}_n \}.$$  

It was shown that $\mathcal{N}(G)/G$ is isomorphic to the matrix group $\{ A \in \mathbb{Z}_n^{2 \times 2} \mid \det A = \pm 1 \}$, $\mathbb{Z}_n$ being a cyclic group of order $n$. This result was used for obtaining all the graded contractions (see [5]) of Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ that arise from the Pauli grading

$$\mathfrak{sl}(3, \mathbb{C}) = \bigoplus_{(j, k) \in J} \{ P^j Q^k \}_{\text{lin}}, \quad J = \mathbb{Z}_3 \times \mathbb{Z}_3 \setminus \{(0, 0)\}.$$  

In this article, we are going to study the symmetries of the grading corresponding to the MAD-group

$$G = \{ \text{Ad}_A \mid A \in \mathcal{P}_n \otimes \mathcal{P}_n \} \subset \text{Aut} \, \mathfrak{sl}(n^2, \mathbb{C}), \; n \text{ prime}. \hspace{1cm} (3)$$

In the sequel we will use notation $P$ and $Q$ instead of $P_n$ and $Q_n$, and by the letter $G$ we will denote only the group given by (3).

If $n$ and $m$ are coprime integers, then the tensor product $\mathcal{P}_n \otimes \mathcal{P}_m$ of the Pauli groups $\mathcal{P}_n$ and $\mathcal{P}_m$ is isomorphic to the Pauli group $\mathcal{P}_{nm}$, therefore it is a natural step in investigation of symmetries of gradings to devote attention to the MAD-group given by $\mathcal{P}_n \otimes \mathcal{P}_n$.

2 The normalizer of the MAD-group corresponding to the tensor product $\mathcal{P}_n \otimes \mathcal{P}_n$

If $(g_l)_{l \in I}$ is a set of generators of a group $H \subset \text{Aut} \, L$, then $\varphi \in \text{Aut} \, L$ belongs to the normalizer $\mathcal{N}(H)$ if and only if $\varphi_{g_l} \varphi^{-1} \in H$ for all the generators $g_l$. Since $\mathcal{P}_n \otimes \mathcal{P}_n = \{ P^i Q^j \otimes P^k Q^l \}$, our MAD-group

$$G = \{ \text{Ad}_{P^i Q^j \otimes P^k Q^l} \mid i, j, k, l \in \mathbb{Z}_n \}$$

has four generators (we use the letter $I$ for unit matrix of order $n$):
$Ad_{A_1}$, where $A_1 = P \otimes I$.
$Ad_{A_2}$, where $A_2 = Q \otimes I$.
$Ad_{A_3}$, where $A_3 = I \otimes P$.
$Ad_{A_4}$, where $A_4 = I \otimes Q$.

Any element of the MAD-group $\mathcal{G}$ is characterized by a quadruple of indices in $Z_n$. We know that an automorphism $\varphi \in Aut sl(n^2, \mathbb{C})$ belongs to $\mathcal{N}(\mathcal{G})$ if and only if $\varphi Ad_{A_i} \varphi^{-1} \in \mathcal{G}$ for $i = 1, 2, 3, 4$. Thus each $\varphi \in \mathcal{N}(\mathcal{G})$ is characterized by a set of 16 coefficients $(a_{ij})_{i,j=1}^4$ such that

$$\varphi Ad_{A_j} \varphi^{-1} = Ad_{p^{a_{11}}; Q^{a_{21}} \otimes P^{a_{31}} Q^{a_{41}}} = Ad_{A_1^{a_{11}} A_2^{a_{21}} A_3^{a_{31}} A_4^{a_{41}}},$$

for $j = 1, 2, 3, 4$.

We order these 16 parameters into a matrix $C(\varphi) \in \mathbb{Z}_n^{4 \times 4}$ as follows:

$$\varphi \mapsto C(\varphi) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}. \quad (4)$$

**Remark 2.1.** Obviously, the assignment $\varphi \mapsto C(\varphi)$ implies that $C(\varphi) = I_4$ if and only if $\varphi$ commutes with each generator of the MAD-group $\mathcal{G}$, and thus with the whole MAD-group. This means that $\varphi$ necessarily belongs to $\mathcal{G}$ (due to the maximality of $\mathcal{G}$). Shortly, we have

$$C(\varphi) = I_4 \iff \varphi \in \mathcal{G}. \quad (5)$$

The advantage of such ordering of the 16 coefficients corresponding to $\varphi \in \mathcal{N}(\mathcal{G})$ is obvious from the following statement:

**Proposition 2.2.** Let $\varphi, \psi \in \mathcal{N}(\mathcal{G})$. Then $C(\varphi \psi) = C(\varphi)C(\psi)$.

**Proof.** We denote the coefficient matrices by $C(\varphi) = (a_{ij})_{i,j=1}^4$, and $C(\psi) = (b_{ij})_{i,j=1}^4$. Let us apply the automorphism $\varphi \psi$ on an element $Ad_{A_p} \in \mathcal{G}$ as follows:

$$(\varphi \psi) Ad_{A_p} (\varphi \psi)^{-1} = \varphi (\psi Ad_{A_p} \psi^{-1}) \varphi^{-1} = \varphi (Ad_{A_1^{a_{1p}} A_2^{a_{2p}} A_3^{a_{3p}} A_4^{a_{4p}}}) \varphi^{-1} = (\varphi Ad_{A_1} \varphi^{-1})^{a_{1p}} (\varphi Ad_{A_2} \varphi^{-1})^{a_{2p}} (\varphi Ad_{A_3} \varphi^{-1})^{a_{3p}} (\varphi Ad_{A_4} \varphi^{-1})^{a_{4p}} = (Ad_{A_1^{a_{11}} A_2^{a_{21}} A_3^{a_{31}} A_4^{a_{41}}})^{a_{1p}} (Ad_{A_1^{a_{12}} A_2^{a_{22}} A_3^{a_{32}} A_4^{a_{42}}})^{a_{2p}} (Ad_{A_1^{a_{13}} A_2^{a_{23}} A_3^{a_{33}} A_4^{a_{43}}})^{a_{3p}} (Ad_{A_1^{a_{14}} A_2^{a_{24}} A_3^{a_{34}} A_4^{a_{44}}})^{a_{4p}}.$$

Since $A_j A_k = \text{const} A_k A_j$, we have $Ad_{A_j} A_i = Ad_{A_j A_i}$ for any $j, k = 1, 2, 3, 4$. Therefore

$$(\varphi \psi) Ad_{A_p} (\varphi \psi)^{-1} = Ad_{A_1^{c_{1p}} A_2^{c_{2p}} A_3^{c_{3p}} A_4^{c_{4p}}},$$

where

$$c_{1p} = a_{11} b_{1p} + a_{12} b_{2p} + a_{13} b_{3p} + a_{14} b_{4p},$$
$$c_{2p} = a_{21} b_{1p} + a_{22} b_{2p} + a_{23} b_{3p} + a_{24} b_{4p},$$
$$c_{3p} = a_{31} b_{1p} + a_{32} b_{2p} + a_{33} b_{3p} + a_{34} b_{4p},$$
$$c_{4p} = a_{41} b_{1p} + a_{42} b_{2p} + a_{43} b_{3p} + a_{44} b_{4p}.$$

This means, in brief notation, that $C(\varphi \psi) = C(\varphi)C(\psi)$. \[\square\]
We prove below that the matrix \( C(\varphi) \) assigned to the element \( \varphi \) of the normalizer \( \mathcal{N}(\mathcal{G}) \) characterizes a coset belonging to the quotient group \( \mathcal{N}(\mathcal{G})/\mathcal{G} \):

**Proposition 2.3.** Let \( \varphi, \psi \) belong to the normalizer \( \mathcal{N}(\mathcal{G}) \) of the MAD-group \( \mathcal{G} \). Then \( C(\varphi) = C(\psi) \) if and only if there exists \( h \in \mathcal{G} \) such that \( \varphi = h\psi \).

**Proof.** Let \( \varphi, \psi \in \mathcal{N}(\mathcal{G}) \) such that \( C(\varphi) = C(\psi) \). Since \( \psi^{-1} \in \mathcal{N}(\mathcal{G}) \) as well, we obtain from Proposition 2.2:

\[
C(\varphi\psi^{-1}) = C(\varphi)C(\psi^{-1}) = C(\psi)C(\psi^{-1}) = C(\psi\psi^{-1}) = C(Id) = I_{4}.
\]

By Remark 2.1 \( \varphi\psi^{-1} \) commutes with all elements of \( \mathcal{G} \), which is only possible when \( \varphi\psi^{-1} \in \mathcal{G} \).

The opposite implication follows directly from Remark 2.1. \( \square \)

**Lemma 2.4.** The outer automorphism \( \text{Out}_1 \) belongs to the normalizer \( \mathcal{N}(\mathcal{G}) \), and

\[
C(\text{Out}_1) = \text{diag}(-1, 1, -1, 1).
\]

**Proof.** Let us denote \( \varphi_0 = \text{Out}_1 \). As \( \varphi_0 \) is given by the prescription \( \varphi_0 X = -X^\top \), clearly \( \varphi_0^{-1} = \varphi_0 \).

We can derive for any inner automorphism \( \text{Ad}_A \) that

\[
(\varphi_0\text{Ad}_A\varphi_0^{-1})(X) = (\varphi_0\text{Ad}_A)(-X^\top) = \varphi_0(-A^{-1}X^\top A)
\]

\[
= (A^{-1}X^\top A)^\top = A^\top X A^{-\top} = (A^{-\top})^{-1}X A^{-\top} = \text{Ad}_{A^{-\top}}(X),
\]

where we used abbreviated notation \( A^{-\top} \) instead of \( (A^{-1})^\top \). This notation is used in the sequel as well.

Thus we have shown that the action of \( \text{Out}_1 \) on any inner automorphism \( \text{Ad}_A \) is

\[
\text{Out}_1\text{Ad}_A\text{Out}_1^{-1} = \text{Ad}_{A^{-\top}}.
\]

Now, for each generator \( \text{Ad}_{A_j}, j = 1, 2, 3, 4 \), of the MAD-group \( \mathcal{G} \), we prove that \( \varphi_0\text{Ad}_{A_j}\varphi_0^{-1} \) belongs to \( \mathcal{G} \):

Let us recall the following trivial properties of matrices \( P \) and \( Q \) and the properties of tensor product:

\[
\begin{align*}
i) &\quad (A \otimes B)^{-1} = A^{-1} \otimes B^{-1}; \\
ii) &\quad (A \otimes B)^\top = A^\top \otimes B^\top; \\
iii) &\quad P^\top = P, \text{ as } P \text{ is diagonal}; \\
iv) &\quad Q^{-1} = Q^\top, \text{ as } Q \text{ is a permutation matrix}.
\end{align*}
\]

Using these relations, we obtain:

1. \( A_1^{-\top} = (P \otimes I)^{-\top} = P^{-\top} \otimes I = P^{-1} \otimes I = (P \otimes I)^{-1} = A_1^{-1} \);
2. \( A_2^{-\top} = (Q \otimes I)^{-\top} = Q^{-\top} \otimes I = Q \otimes I = A_2 \);
3. \( A_3^{-\top} = (I \otimes P)^{-\top} = I \otimes P^{-\top} = I \otimes P^{-1} = (I \otimes P)^{-1} = A_3^{-1} \);
4. \( A_4^{-\top} = (I \otimes Q)^{-\top} = I \otimes Q^{-\top} = I \otimes Q = A_4 \).

Statements 1.-4. together with equation (6) already prove the lemma. \( \square \)

**Remark 2.5.** Product of two outer automorphisms is an inner automorphism. Thus, when describing the set of all automorphisms in \( \mathcal{N}(\mathcal{G}) \), we can focus on the subgroup \( \mathcal{N}_{in}(\mathcal{G}) \) containing all the inner automorphisms in \( \mathcal{N}(\mathcal{G}) \). The whole \( \mathcal{N}(\mathcal{G}) \) can then be described as

\[
\mathcal{N}(\mathcal{G}) = \mathcal{N}_{in}(\mathcal{G}) \cup \text{Out}_1\mathcal{N}_{in}(\mathcal{G}).
\]
The following theorem shows the connection between normalizers of these MAD-groups and the symplectic groups over finite field which were introduced in [2].

**Theorem 2.6.** Let \( Ad_A \) be an inner automorphism contained in the normalizer \( N(G) \) of the MAD-group \( G \), and let \( C(Ad_A) \) be the coefficient matrix corresponding to \( Ad_A \). Then

\[
C(Ad_A) \in Sp(4, \mathbb{Z}_n) := \{ X \in \mathbb{Z}_n^{4 \times 4} \mid X^TJX = J \},
\]

where \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

**Proof.** Let us denote \( C(Ad_A) = (a_{ij})_{i,j=1} \). The definition of the matrix \( C(Ad_A) \) implies that

\[
Ad_A Ad_A (Ad_A)^{-1} = Ad_{A^{-1} A_A} = Ad_A^{a_{i1} A_A^{a_{i2} A_A^{a_{i3} A_A^{a_{i4} A_A^{-1}}}}}
\]

for \( p = 1, 2, 3, 4 \).

As \( Ad_K = Ad_H \) if and only if \( K = \alpha H \) for some \( \alpha \in \mathbb{C} - \{0\} \), we obtain from (8) existence of four non-zero constants \( \alpha_p, p = 1, 2, 3, 4 \), such that

\[
A_p = \alpha_p A_A^{a_{p1} A_A^{a_{p2} A_A^{a_{p3} A_A^{a_{p4} A_A^{-1}}}}}.
\]

We derive easily from the basic relation \( PQ = \omega PQ \) that

\[
A_1 A_2 = \omega^{-1} A_2 A_1 \quad (10)
\]

\[
A_3 A_4 = \omega^{-1} A_4 A_3 \quad (11)
\]

The remaining pairs \( A_i, A_j \) commute:

\[
A_1 A_3 = A_3 A_1 \quad (12)
\]

\[
A_1 A_4 = A_4 A_1 \quad (13)
\]

\[
A_2 A_3 = A_3 A_2 \quad (14)
\]

\[
A_2 A_4 = A_4 A_2 \quad (15)
\]

By inputting \( A_1, A_2 \) expressed in the form (9) into the relation (10), we obtain

\[
\alpha_1 A_1^{a_{11} A_1^{a_{12} A_1^{a_{13} A_1^{a_{14} A_1^{-1}}}}}
\]

\[
= \omega^{-1} \alpha_2 A_2^{a_{21} A_2^{a_{22} A_2^{a_{23} A_2^{a_{24} A_2^{-1}}}}}
\]

\[
= \omega^{-1} \alpha_1 A_1^{a_{11} A_1^{a_{12} A_1^{a_{13} A_1^{a_{14} A_1^{-1}}}}}
\]

and, after simplification and using relations (10)-(15),

\[
\omega^{a_{11} a_{12} + a_{13} a_{14} + a_{11} a_{21} + a_{22} a_{23} + a_{31} a_{32} a_{41} + a_{42}}
\]

This implies that \( \omega^{a_{11} a_{12} + a_{13} a_{14}} = \omega^{-1} + a_{21} a_{22} + a_{31} a_{32} \), and therefore

\[
1 = a_{11} a_{22} - a_{21} a_{12} + a_{31} a_{42} - a_{41} a_{32} \quad (mod \ n).
\]

Analogously, the equations (11)-(15) result in

\[
a_{13} a_{24} = a_{14} a_{23} + a_{33} a_{44} - a_{34} a_{43} \quad (mod \ n)
\]

\[
a_{11} a_{23} - a_{13} a_{21} + a_{31} a_{43} - a_{33} a_{41} \quad (mod \ n)
\]

\[
a_{11} a_{24} - a_{14} a_{23} + a_{31} a_{44} - a_{34} a_{41} \quad (mod \ n)
\]

\[
a_{12} a_{23} - a_{13} a_{22} + a_{32} a_{43} - a_{33} a_{42} \quad (mod \ n)
\]

\[
a_{12} a_{24} - a_{14} a_{22} + a_{32} a_{44} - a_{34} a_{42} \quad (mod \ n).
\]
It can be easily verified (by a direct calculation) that the matrix \((a_{ij})_{i,j=1}^4\) belongs to the group \(Sp(4,\mathbb{Z}_n)\) if and only if the matrix elements \(a_{ij}\) fulfill equations \((16) - (21)\).

**Remark 2.7.** We were notified \([9]\) that the set \(Sp(4,\mathbb{Z}_n)\) defined analogously to \((7)\) is a group even in the case when \(\mathbb{Z}_n\) is not a field. All our previous considerations hold therefore for any positive integer \(n\). But our deductions in the sequel already need \(n\) to be a prime number.

We are going to prove that the mapping given by \((3)\) is in fact the mapping on the whole group \(Sp(4,\mathbb{F}_n)\). To show it we need to find for any element of \(Sp(4,\mathbb{F}_n)\) its preimage, or equivalently for any generator of \(Sp(4,\mathbb{F}_n)\) its preimage.

To simplify the proof we need to find the smallest possible set of generators of \(Sp(4,\mathbb{F}_n)\). In \([8]\] a set of generators of the group \(Sp(2m,K)\) over a finite field \(K\) is described. In case \(m = 2\), the set of generators contains \(n^4 - n^2 + n + 1\) elements, where \(n\) is the cardinality of \(K\). As we show in Appendix B, it is possible to reduce the number of generators of the group \(Sp(4,\mathbb{F}_n)\) to four matrices. In formal notation

\[
Sp(4,\mathbb{F}_n) = \langle D_1, D_2, D_3, D_4 \rangle,
\]

where

\[
D_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad D_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

We show that these four matrices are images of inner automorphisms which belong to the normalizer of the group \(G\).

**Proposition 2.8.** Let \(n\) be a prime. Then for the four matrices \(D_j\) as introduced in \((22)\), there exist inner automorphisms \(\varphi_j = Ad_{B_j} \in N_{in}(G)\) such that \(D_j = C(\varphi_j) = C(Ad_{B_j}).\)

The proof is postponed to the Appendix A since it is rather technical and we do not want to interrupt coherency in the content of the article.

The immediate consequence of the previous proposition is the following main result of the article.

**Theorem 2.9.** Let \(n\) be a prime. The mapping \(\varphi \mapsto C(\varphi)\) defined in \((3)\) is an isomorphism between groups

\[
N_{in}(G)/G \simeq Sp(4,\mathbb{F}_n) = \{ X \in \mathbb{Z}_{n}^{4x4} \mid X^TJX = J \},
\]

and

\[
N(G)/G \simeq \{ X \in \mathbb{Z}_{n}^{4x4} \mid X^TJX = \pm J \},
\]

where \(J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = I_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

**Proof.**

- The mapping \(\varphi \mapsto C(\varphi)\) from \(N_{in}(G)/G\) to \(Sp(4,\mathbb{F}_n)\) is a homomorphism, as \(C(\varphi \psi) = C(\varphi)C(\psi)\), which was proved in Proposition \(2.2\).

- The mapping \(\varphi \mapsto C(\varphi)\) from \(N_{in}(G)/G\) to \(Sp(4,\mathbb{F}_n)\) is injective, as shown in Proposition \(2.3\).

---

\(^{2}\)We recall that for elements \(k_1, k_2, \ldots, k_s\) of a group \(G\), the notation \(\langle k_1, k_2, \ldots, k_s \rangle\) means the smallest subgroup of the group \(G\) containing \(k_1, k_2, \ldots, k_s\).
The group $Sp(4, \mathbb{F}_n)$ is generated by four matrices $D_1, D_2, D_3, D_4$ (see Theorem 3.1 in Appendix).

All the matrices $D_j$, $j = 1, 2, 3, 4$, have their inverse images $\varphi_j \in N_{in}(G)/G$, such that $C(\varphi_j) = D_j$ (see Proposition 2.8). This implies that the mapping $\varphi \mapsto C(\varphi)$ is also surjective.

In total, we see that the mapping $\varphi \mapsto C(\varphi)$ is an isomorphism from $N_{in}(G)/G$ onto $Sp(4, \mathbb{F}_n)$. To show isomorphism between $N(G) / G$ and $Sp(4, \mathbb{F}_n) \otimes \mathbb{Z}_2$, it is enough to use Remark 2.5 and the fact that the matrix $M := C(Out_I) = \text{diag}(-1, 1, -1, 1)$ corresponding to the outer automorphisms $Out_I \in N(G)$ satisfies the equality $M^TJM = -J$.

3 Conclusions

Let us summarize the content of the article:

1. The normalizer $N(G)$ of the MAD-group $G = \{Ad_{P_i \otimes Q_j \otimes P_k \otimes Q_l} \mid i, j, k, l \in \mathbb{Z}_n\} \subset \text{Aut}sl(n^2, \mathbb{C})$ consists of two subsets:

   - $N_{in}(G)$: the group of all inner automorphisms in $N(G)$,
   - $Out_I N_{in}(G)$: the set of all outer automorphisms in $N(G)$.

2. We provide an explicit expression of the four generators of $N_{in}(G)/G$, namely the inner automorphisms $Ad_{B_j}$, $j = 1, 2, 3, 4$ (see Proposition 2.8).

3. Altogether, we can write the set of generators

   $\mathcal{G} = \langle Ad_{P_i \otimes Q_j \otimes P_k \otimes Q_l} \rangle$,
   $N_{in}(G)/G = \langle Ad_{B_1}, Ad_{B_2}, Ad_{B_3}, Ad_{B_4} \rangle$,
   $N(G) = N_{in}(G) \cup Out_I N_{in}(G)$.

4. Thus, one can generate each element of the normalizer from the set $\{Ad_{B_1}, Ad_{B_2}, Ad_{B_3}, Ad_{B_4}, Out_I, Ad_{P_i \otimes P_j} \}$. In formal notation,

   $N(G) = \langle Ad_{B_1}, Ad_{B_2}, Ad_{B_3}, Ad_{B_4}, Out_I, Ad_{P_i \otimes P_j} \rangle$.

   where the matrices $B_j$ were defined in the proof of Proposition 2.8.

The description of the normalizer as done in this article was only possible for $n$ prime. For $n$ non-prime, the problem is still open.

It was shown previously that for the Pauli grading, the normalizer of the respective MAD-group is isomorphic to $SL(2, \mathbb{Z}_n)$, which is isomorphic to $Sp(2, \mathbb{Z}_n)$, for any positive integer $n > 1$. This suggests that in the case of a MAD-group formed by inner automorphisms generated by $P_n \otimes P_n \otimes \ldots \otimes P_n$, the normalizer may be isomorphic to $Sp(2k, \mathbb{Z}_n)$.

Let us mention that the normalizer has not yet been described for any MAD-group containing outer automorphisms.
Appendix A

This section contains a proof of Proposition 2.8. The matrices $D_j$’s considered in the proof are defined by (22).

Proof. In order to prove that an automorphism $\varphi_j = Ad_{B_j}$ is an inverse image of $D_j$, we must express the action of $\varphi_j$ on the basis elements $P \otimes I, Q \otimes I, I \otimes P, I \otimes Q$ of $G$ again in terms of $P \otimes I$, $Q \otimes I, I \otimes P, I \otimes Q$. The coefficients $a_{kl}$ describing the action of $\varphi_j$ (as introduced in (3)) then form the matrix $D_j$. In the following we set the four matrices $B_j$, and verify that each satisfies the equation $D_j = C(Ad_{B_j})$.

Throughout the proof, we use the coefficient $\omega$, which is, as defined previously, the $n$-th root of unity: $\omega = \omega_n = e^{2\pi i/n}$. We also shorten the notation of $I_n$ to $I$.

And, finally, the elements of matrices $P, Q$ (whose indices are also counted modulo $n$) can be written in terms of the Kronecker symbol as

$$P_{ij} = \delta_{ij}\omega^j, \quad Q_{ij} = \delta_{(j-1)}, \quad Q'_{ij} = \delta_{(j+1)}.$$ (23)

1) We define $\varphi_1 = Ad_{B_1}$, where

$$B_1 = \bar{B}_1 \otimes I, \quad \bar{B}_1 = \text{diag}(b_0, b_1, \ldots, b_{(n-1)}), \quad b_j = \varepsilon^j\omega^{\frac{n(n-1)}{2}}, \quad \varepsilon = \omega^{\frac{n-1}{2}}.$$

As $\bar{B}_1$, $\bar{B}_1^\dagger$, and $P$ are diagonal, they all mutually commute, and thus $B_1^\dagger (P \otimes I)B_1 = (\bar{B}_1 \otimes I)^\dagger (P \otimes I)(\bar{B}_1 \otimes I) = (\bar{B}_1^\dagger \otimes I)(P\bar{B}_1 \otimes I) = (\bar{B}_1^\dagger \bar{B}_1 P) \otimes I = P \otimes I$. In other words,

$$\varphi_1 Ad_{PQ} \varphi_1^{-1} = Ad_{B_1} Ad_{PQ} Ad_{B_1}^{-1} = Ad_{B_1^\dagger (P \otimes I) B_1} = Ad_{P \otimes I},$$

which means, according to the definition of $C(\varphi_1)$, that the first column of the matrix $C(\varphi_1)$ is

$$\bullet (a_{11}, a_{21}, a_{31}, a_{41})^T = (1, 0, 0, 0)^T.$$

Now we apply $\varphi_1$ on the second generator of the group $G$, which is the inner automorphism defined by the matrix $Q \otimes I$. In fact, we need to express $B_1^\dagger (Q \otimes I)B_1$ in terms of the basis matrices $P \otimes I, Q \otimes I, I \otimes P, I \otimes Q$. Using the notation of elements of $P$ and $Q$ introduced in (23), we obtain

$$(\bar{B}_1^\dagger Q \bar{B}_1)_{ij} = \sum_{k=0}^{n-1} \delta_{ik} \varepsilon^{-k} \omega^{-\frac{k(k-1)}{2}} \delta_{j(i-1)} \delta_{ij} \varepsilon^j \omega^\frac{(n-1)j}{2}$$

$$= \sum_{k=0}^{n-1} \delta_{ik} \varepsilon^{-k} \omega^{-\frac{k(k-1)}{2}} \delta_{j(i-1)} \delta_{ij} \varepsilon^j \omega^\frac{(n-1)j}{2}$$

$$= \sum_{k=0}^{n-1} \delta_{ik} \varepsilon^{-k} \omega^{-\frac{k(k-1)}{2}} \delta_{j(i-1)} = \varepsilon \delta_{(j-1)} \omega^j.$$

$$(PQ)_{ij} = \sum_{k=0}^{n-1} P_{ik} Q_{kj} = \sum_{k=0}^{n-1} \delta_{ik} \omega^k \delta_{j(k-1)} = \delta_{(j-1)} \omega^j.$$

We see that the matrix $B_1^\dagger (Q \otimes I) B_1$ is just an $\varepsilon$ multiple of $PQ$, and it follows that

$$\varphi_1 Ad_{PQ} \varphi_1^{-1} = Ad_{B_1} Ad_{PQ} Ad_{B_1}^{-1} = Ad_{B_1^\dagger (Q \otimes I) B_1} = Ad_{(\bar{B}_1^\dagger Q \bar{B}_1) \otimes I} = Ad_{P \otimes I}.$$
The second column of the matrix $C(\varphi_1)$ is thus equal to

- $(a_{12}, a_{22}, a_{32}, a_{42})^\top = (1, 1, 0, 0)^\top$.

By simple matrix multiplication, we see that

$$B_1^{-1}(I \otimes P)B_1 = (\tilde{B}_1^{-1} \otimes I)(I \otimes P)(\tilde{B}_1 \otimes I) = (\tilde{B}_1^{-1} \tilde{B}_1) \otimes P = I \otimes P;$$

which means $\varphi_1 Ad_{I \otimes P} \varphi_1^{-1} = Ad_{B_1} Ad_{I \otimes P} Ad_{B_1^{-1}} = Ad_{B_1^{-1}(I \otimes P)B_1} = Ad_{I \otimes P}$, and therefore

- $(a_{13}, a_{23}, a_{33}, a_{43})^\top = (0, 0, 1, 0)^\top$.

Analogously, putting $Q$ on the place of $P$, we have

$$B_1^{-1}(I \otimes Q)B_1 = (\tilde{B}_1^{-1} \otimes I)(I \otimes Q)(\tilde{B}_1 \otimes I) = (\tilde{B}_1^{-1} \tilde{B}_1) \otimes Q = I \otimes Q;$$

which means $\varphi_1 Ad_{I \otimes Q} \varphi_1^{-1} = Ad_{B_1} Ad_{I \otimes Q} Ad_{B_1^{-1}} = Ad_{B_1^{-1}(I \otimes Q)B_1} = Ad_{I \otimes Q}$, and therefore

- $(a_{14}, a_{24}, a_{34}, a_{44})^\top = (0, 0, 0, 1)^\top$.

Thus, we have shown that

$$C(\varphi_1) = D_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2) We define $\varphi_2 = Ad_{B_2}$, where

$$B_2 = \tilde{B}_2 \otimes I, \quad (\tilde{B}_2)_{ij} = \omega^{ij}, \quad i, j = 0, 1, \ldots, n - 1.$$

Note that $\tilde{B}_2$ is the famous Sylvester matrix.

In preparation for describing the action of $Ad_{B_2}$, we apply the matrix $\tilde{B}_2$ on $P$, $Q$, and $Q^\top$ from both right and left:

$$(P\tilde{B}_2)_{ij} = \sum_{k=0}^{n-1} P_{ik}(\tilde{B}_2)_{kj} = \sum_{k=0}^{n-1} \delta_{ik}\omega^k\omega^k = \omega^i\omega^j = \omega^{(i+1)}$$

$$(\tilde{B}_2Q^\top)_{ij} = \sum_{k=0}^{n-1} (\tilde{B}_2)_{ik}(Q^\top)_{kj} = \sum_{k=0}^{n-1} \omega^{ik}\delta_{k(j+1)} = \omega^{(j+1)}$$

$$(Q\tilde{B}_2)_{ij} = \sum_{k=0}^{n-1} Q_{ik}(\tilde{B}_2)_{kj} = \sum_{k=0}^{n-1} \delta_{i(k-1)}\omega^k = \omega^{(i+1)}$$

$$(\tilde{B}_2P)_{ij} = \sum_{k=0}^{n-1} (\tilde{B}_2)_{ik}P_{kj} = \sum_{k=0}^{n-1} \omega^{ik}\delta_{kj}\omega^j = \omega^i\omega^j = \omega^{(i+1)}.$$

We easily conclude that

$$P\tilde{B}_2 = \tilde{B}_2Q^\top \quad \Rightarrow \quad \tilde{B}_2^{-1}P\tilde{B}_2 = Q^\top = Q^{-1},$$

$$Q\tilde{B}_2 = \tilde{B}_2P \quad \Rightarrow \quad \tilde{B}_2^{-1}Q\tilde{B}_2 = P.$$

Using these relations, the way to find the coefficients of $C(\varphi_2) = C(Ad_{B_2})$ is quite straightforward:
\[ B_2^{-1}(P \otimes I)B_2 = (\tilde{B}_2 \otimes I)^{-1}(P \otimes I)(\tilde{B}_2 \otimes I) = (\tilde{B}_2^{-1} \otimes I)(P \tilde{B}_2 \otimes I) = (\tilde{B}_2^{-1}P \tilde{B}_2) \otimes I = Q^{-1} \otimes I. \]

Therefore, \( \varphi_2 Ad_{P \otimes I} \varphi_2^{-1} = Ad_{B_2^{-1}(P \otimes I)}B_2 = Ad_{(Q \otimes I)^{-1}}, \) and we have found the coefficients

\[ (a_{11}, a_{21}, a_{31}, a_{41})^T = (0, -1, 0, 0)^T. \]

\[ B_2^{-1}(Q \otimes I)B_2 = (\tilde{B}_2 \otimes I)^{-1}(Q \otimes I)(\tilde{B}_2 \otimes I) = (\tilde{B}_2^{-1} \otimes I)(Q \tilde{B}_2 \otimes I) = (\tilde{B}_2^{-1}Q \tilde{B}_2) \otimes I = P \otimes I. \]

Therefore, \( \varphi_2 Ad_{Q \otimes I} \varphi_2^{-1} = Ad_{B_2^{-1}(Q \otimes I)}B_2 = Ad_{(P \otimes I)} \), and the respective coefficients of \( D_2 \) are

\[ (a_{12}, a_{22}, a_{32}, a_{42})^T = (0, 0, 0, 0)^T. \]

\[ B_2^{-1}(I \otimes P)B_2 = (\tilde{B}_2 \otimes I)^{-1}(I \otimes P)(\tilde{B}_2 \otimes I) = (\tilde{B}_2^{-1} \otimes I)(\tilde{B}_2 \otimes P) = (\tilde{B}_2^{-1} \tilde{B}_2) \otimes P = I \otimes P. \]

In this case, \( \varphi_2 Ad_{I \otimes P} \varphi_2^{-1} = Ad_{B_2^{-1}(I \otimes P)}B_2 = Ad_{(I \otimes P)}, \) and the respective coefficients of \( D_2 \) are

\[ (a_{13}, a_{23}, a_{33}, a_{43})^T = (0, 0, 0, 0)^T. \]

\[ B_2^{-1}(I \otimes Q)B_2 = (\tilde{B}_2 \otimes I)^{-1}(I \otimes Q)(\tilde{B}_2 \otimes I) = (\tilde{B}_2^{-1} \otimes I)(\tilde{B}_2 \otimes Q) = (\tilde{B}_2^{-1} \tilde{B}_2) \otimes Q = I \otimes Q. \]

So, \( \varphi_2 Ad_{I \otimes Q} \varphi_2^{-1} = Ad_{B_2^{-1}(I \otimes Q)}B_2 = Ad_{(I \otimes Q)}, \) and obviously,

\[ (a_{14}, a_{24}, a_{34}, a_{44})^T = (0, 0, 0, 1)^T. \]

Thus, the matrix \( C(\varphi_2) \) is equal to

\[ C(\varphi_2) = D_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

3) We define \( \varphi_3 = Ad_{B_3}, \) where

\[ (B_3)_{pq} = \delta_{p_1 q_2} \delta_{p_2 q_1}, \quad p = p_1 n + p_2, q = q_1 n + q_2, \quad p_1, p_2, q_1, q_2 \in \{0, 1, \ldots, n - 1\}. \]

One can easily verify by a direct calculation that \( B_3^{-1} = B_3. \)

In order to satisfy the relation \( C(\varphi_3) = C(Ad_{B_3}) = D_3, \) it is sufficient to show that the matrix \( B_3 \) fulfills the following equations:

\[ B_3^{-1}(P \otimes I)B_3 = I \otimes P \quad B_3^{-1}(I \otimes P)B_3 = P \otimes I \]
\[ B_3^{-1}(Q \otimes I)B_3 = I \otimes Q \quad B_3^{-1}(I \otimes Q)B_3 = Q \otimes I. \]

In fact, we have found \( B_3 \) such that even a more general relation is satisfied:

\[ B_3^{-1}(K \otimes L)B_3 = L \otimes K \text{ for any matrices } K, L \in \mathbb{C}^{n \times n}. \quad (24) \]

The matrix elements of \( B_3 \) are \( \delta_{p_1 q_2} \delta_{p_2 q_1} \) as introduced above. In order to prove the equation (24), it is sufficient to express the \((pq)-th\) element of the tensor product \( K \otimes L \) as \((K \otimes L)_{pq} = K_{p_1 q_1} L_{p_2 q_2}, \)
and proceed by

\[ [B^{-1}_3(K \otimes L)B_3]_{pq} = [B^{-1}_3(K \otimes L)B_3]_{(p_1n+p_2)(q_1n+q_2)} \]

\[ = \sum_{r_1,r_2=0}^{n-1} (B^{-1}_3)_{(p_1n+p_2)(r_1n+r_2)}([K \otimes L]B_3)_{(r_1n+r_2)(q_1n+q_2)} \]

\[ = \sum_{r_1,r_2=0}^{n-1} \sum_{s_1,s_2=0}^{n-1} (B^{-1}_3)_{(p_1n+p_2)(r_1n+r_2)}(K \otimes L)_{(r_1n+r_2)(s_1n+s_2)}(B_3)_{(s_1n+s_2)(q_1n+q_2)} \]

\[ = \sum_{r_1,r_2=0}^{n-1} \sum_{s_1,s_2=0}^{n-1} \delta_{p_1r_2} \delta_{p_2r_1} K_{r_1s_1} L_{r_2s_2} \delta_{s_2q_1} \delta_{s_1q_2} \]

\[ = K_{pq} L_{pq} = L_{p,q_1} K_{p,q_2} = (L \otimes K)_{pq}. \]

4) We define \( \varphi_4 = Ad_{B_4} \), where

\[ (B_4)_{pq} = \delta_{(p_1-p_2)q_1} \delta_{p_2q_2}, \quad p = p_1n + p_2, q = q_1n + q_2, \quad p_1, p_2, q_1, q_2 \in \{0, 1, \ldots, n-1\}. \]

Remember that, with the matrix coefficients \( p_1, p_2, q_1, q_2 \), we count modulo \( n \).

We first express the matrices relevant for the proof by means of their \( pq \)-th elements:

\[
(P \otimes I)_{pq} = \omega^{p_1} \delta_{p_1q_1} \delta_{p_2q_2}
\]

\[
(P \otimes P)_{pq} = \omega^{p_1+p_2} \delta_{p_1q_1} \delta_{p_2q_2}
\]

\[
(I \otimes P)_{pq} = \omega^{p_2} \delta_{p_1q_1} \delta_{p_2q_2}
\]

\[
(Q \otimes I)_{pq} = \delta_{(p_1+1)q_1} \delta_{p_2q_2}
\]

\[
(Q^{-1} \otimes Q)_{pq} = \delta_{p_1q_1+1} \delta_{(p_2+1)q_2}
\]

\[
(I \otimes Q)_{pq} = \delta_{p_1q_1} \delta_{(p_2+1)q_2}
\]

Then, we proceed by showing that the elements of matrix \( D_4 \) indeed reflect the action of the automorphism \( \varphi_4 = Ad_{B_4} \):

\[
[(P \otimes I)B_4]_{pq} = \sum_{r_1,r_2=0}^{n-1} (P \otimes I)_{(p_1n+p_2)(r_1n+r_2)}(B_4)_{(r_1n+r_2)(q_1n+q_2)}
\]

\[ = \sum_{r_1,r_2=0}^{n-1} \omega^{p_1} \delta_{p_1r_1} \delta_{p_2r_2} \delta_{(r_1-r_2)q_1} \delta_{r_2q_2} \]

\[ = \omega^{p_1} \delta_{(p_1-p_2)q_1} \delta_{p_2q_2}, \]
\[
[B_4(P \otimes P)]_{pq} = \sum_{r_1, r_2=0}^{n-1} (B_4)_{(p_1, n+p_2)(r_1, n+r_2)} (P \otimes P)_{(r_1, n+r_2)(q_1, n+q_2)}
\]

\[
= \sum_{r_1, r_2=0}^{n-1} \delta_{(p_1-p_2)r_1} \delta_{p_2r_2} \omega^{r_1+r_2} \delta_{r_1q_1} \delta_{r_2q_2}
\]

\[
= \delta_{(p_1-p_2)q_1} \delta_{p_2q_2} \omega^{q_1+q_2} = \omega^{p_1} \delta_{(p_1-p_2)q_1} \delta_{p_2q_2}.
\]

Thus we have shown that \((P \otimes I)B_4 = B_4(P \otimes P)\), and consequently, \(B_4^{-1}(P \otimes I)B_4 = P \otimes P\), which gives the coefficients

\[
\begin{align*}
(a_{11}, a_{21}, a_{31}, a_{41})^\top & = (1, 0, 1, 0)^\top. \\
\end{align*}
\]

\[
[(Q \otimes I)B_4]_{pq} = \sum_{r_1, r_2=0}^{n-1} (Q \otimes I)_{(p_1, n+p_2)(r_1, n+r_2)} (B_4)_{(r_1, n+r_2)(q_1, n+q_2)}
\]

\[
= \sum_{r_1, r_2=0}^{n-1} \delta_{(p_1+1)r_1} \delta_{p_2r_2} \delta_{(r_1-r_2)q_1} \delta_{r_2q_2}
\]

\[
= \delta_{p_2q_2} \delta_{(p_1+1)(q_1+q_2)}.
\]

\[
[B_4(Q \otimes I)]_{pq} = \sum_{r_1, r_2=0}^{n-1} (B_4)_{(p_1, n+p_2)(r_1, n+r_2)} (Q \otimes I)_{(r_1, n+r_2)(q_1, n+q_2)}
\]

\[
= \sum_{r_1, r_2=0}^{n-1} \delta_{(p_1-p_2)r_1} \delta_{p_2r_2} \delta_{(r_1+1)q_1} \delta_{r_2q_2}
\]

\[
= \delta_{p_2q_2} \delta_{(p_1-p_2)(q_1+1)} = \delta_{p_2q_2} \delta_{(p_1+1)(q_1+q_2)}.
\]

Combining the two equations, we obtain \((Q \otimes I)B_4 = B_4(Q \otimes I)\), and \(B_4^{-1}(Q \otimes I)B_4\). In terms of the matrix coefficients of \(D_4\), it writes as

\[
\begin{align*}
(a_{12}, a_{22}, a_{32}, a_{42})^\top & = (0, 1, 0, 0)^\top. \\
\end{align*}
\]

\[
[(I \otimes P)B_4]_{pq} = \sum_{r_1, r_2=0}^{n-1} (I \otimes P)_{(p_1, n+p_2)(r_1, n+r_2)} (B_4)_{(r_1, n+r_2)(q_1, n+q_2)}
\]

\[
= \sum_{r_1, r_2=0}^{n-1} \omega^{p_2} \delta_{p_1r_1} \delta_{p_2r_2} \delta_{(r_1-r_2)q_1} \delta_{r_2q_2}
\]

\[
= \omega^{p_2} \delta_{(p_1-p_2)q_1} \delta_{p_2q_2}.
\]

\[
[B_4(I \otimes P)]_{pq} = \sum_{r_1, r_2=0}^{n-1} (B_4)_{(p_1, n+p_2)(r_1, n+r_2)} (I \otimes P)_{(r_1, n+r_2)(q_1, n+q_2)}
\]

\[
= \sum_{r_1, r_2=0}^{n-1} \delta_{(p_1-p_2)r_1} \delta_{p_2r_2} \omega^{r_2} \delta_{r_1q_1} \delta_{r_2q_2}
\]

\[
= \omega^{p_2} \delta_{(p_1-p_2)q_1} \delta_{p_2q_2}.
\]

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From \((I \otimes P)B_4 = B_4(I \otimes P)\) we obtain \(B_4^{-1}(I \otimes P)B_4 = I \otimes P\), i.e. the matrix \(I \otimes P\) remains intact by the action of \(Ad_{B_4}\), which means that

\[
\begin{align*}
(I \otimes Q)B_4 &= \sum_{r_1, r_2 = 0}^{n-1} (I \otimes Q)_{(p_1n+p_2),(r_1n+r_2)} (B_4)_{(r_1n+r_2),(q_1n+q_2)} \\
&= \sum_{r_1, r_2 = 0}^{n-1} \delta_{p_1r_1} \delta_{(p_2+1)r_2} \delta_{(r_1-r_2)q_1} \delta_{r_2q_2} \\
&= \delta_{(p_2+1)q_2} \delta_{p_1(q_1+q_2)}.
\end{align*}
\]

\[
\begin{align*}
[B_4(Q^{-1} \otimes Q)]_{pq} &= \sum_{r_1, r_2 = 0}^{n-1} (B_4)_{(p_1n+p_2),(r_1n+r_2)} (Q^{-1} \otimes Q)_{(r_1n+r_2),(q_1n+q_2)} \\
&= \sum_{r_1, r_2 = 0}^{n-1} \delta_{(p_1-p_2)r_1} \delta_{p_2r_2} \delta_{r_1(q_1+1)} \delta_{r_2q_2} \\
&= \delta_{(p_2+1)q_2} \delta_{(p_1-q_1)q_2} = \delta_{(p_2+1)q_2} \delta_{p_1(q_1+q_2)}.
\end{align*}
\]

Here, we see the action of \(Ad_{B_4}\) on the last of the four matrices generating \(G\): \((I \otimes Q)B_4 = B_4(Q^{-1} \otimes Q)\) is equivalent to \(B_4^{-1}(I \otimes Q)B_4 = Q^{-1} \otimes Q\). As a result,

\[
\begin{align*}
(a_{14}, a_{24}, a_{34}, a_{44})^T &= (0, -1, 0, 1)^T.
\end{align*}
\]

Thus, we have shown that

\[
C(\varphi_4) = D_4 = \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{array} \right).
\]

\(\blacksquare\)

**Appendix B**

**Theorem 3.1.** Let \(n\) be a prime. Then the four matrices

\[
D_1 = \left( \begin{array}{cccc}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array} \right), \quad D_2 = \left( \begin{array}{cccc}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{array} \right), \quad D_3 = \left( \begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{array} \right), \quad \text{and} \quad D_4 = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array} \right)
\]

generate the group \(Sp(4, \mathbb{F}_n)\).

For a better overview, we first sketch the main framework of the proof, and only afterwards we prove the individual steps in detail.

**Remark 3.2.** The fact that \(n\) is a prime ensures that \(\mathbb{Z}_n\) is a field \(\mathbb{F}_n\), and consequently that the set \(Sp(4, \mathbb{F}_n) = \{X \in \mathbb{Z}_n^{4 \times 4} \mid X^TX = J\}\), where \(J = \left( \begin{array}{cc}
0 & 1 \\
-1 & 0 \\
\end{array} \right) \oplus \left( \begin{array}{cc}
0 & 1 \\
-1 & 0 \\
\end{array} \right)\), is a group.

**Remark 3.3.** It is a well known fact that, for any integer \(n\), the two matrices \(\left( \frac{1}{n} \frac{1}{n} \right)\) and \(\left( \frac{-1}{n} \frac{1}{n} \right)\) generate the group \(SL(2, \mathbb{Z}_n) = \{A \in \mathbb{Z}_n^{2 \times 2} \mid \det A = 1\}\) (see [3]).

**Remark 3.4.** It is easy to verify that \(D_1, D_2, D_3,\) and \(D_4\) belong to \(Sp(4, \mathbb{F}_n)\).
For a group $K$ and any group elements $k_1, k_2, \ldots, k_r \in K$, we denote by $\langle k_1, \ldots, k_r \rangle$ the smallest subgroup of $K$ containing $k_1, \ldots, k_r$.

Under the framework of this notation and Remark 3.3, we have

$$\langle D_1, D_2 \rangle = \left\{ \begin{pmatrix} A & 0 \\ 0 & I_2 \end{pmatrix} \bigm| A \in SL(2, \mathbb{Z}_n) \right\},$$

$$\langle D_1, D_2, D_3 \rangle = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \bigm| A, B \in SL(2, \mathbb{Z}_n) \right\}.$$

Our aim is to prove that $\langle D_1, D_2, D_3, D_4 \rangle = Sp(4, \mathbb{Z}_n)$. For this purpose, it is enough to verify statements of the next two steps:

**Step 1.** Let us note $H := \langle D_1, D_2, D_3 \rangle$ and $M \in Sp(4, \mathbb{F}_p)$, $M \notin H$. There exist $k \in \mathbb{Z}_n$ and matrices $G_1, G_2 \in H$ such that $G_1 MG_2 = S(k)$, where

$$S(k) := \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & k & 0 & 1-k \\ k-1 & 0 & k & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

**Step 2.** The matrix $S(k)$ belongs to $\langle D_1, D_2, D_3, D_4 \rangle$ for any $k \in \mathbb{Z}_n$.

**Lemma 3.5.** Let $n$ be a prime and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}_n^{2 \times 2}$, $A \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then there exist

(i) matrices $B, C \in SL(2, \mathbb{Z}_n)$ such that $BAC = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, where $k = det A$;

(ii) matrices $D, E \in SL(2, \mathbb{Z}_n)$ such that $DAE = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, where $k = det A$.

Moreover, if $det A = k \neq 0$, then we have $C = E = I_2$.

**Proof.** a) Firstly, we consider the case $det A = k \neq 0$.

(i) We assume, without loss of generality, that $a \neq 0$ (otherwise, we would consider matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -b & a \\ d & c \end{pmatrix}$, as $b \neq 0$ when $a = 0$). The desired matrices $B$ and $C$ are as follows:

\[
B = B_3 B_2 B_1 = \begin{pmatrix} 1 & -ba^{-1}k^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -ac & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},
\]

\[
C = I_2,
\]

\[
BAC = B_3 B_2 \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = B_3 \begin{pmatrix} 1 & 0 \\ -ac & 1 \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ ac & ad \end{pmatrix} = \begin{pmatrix} 1 & -ba^{-1}k^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}.
\]

(ii) Alternatively, we multiply the result by one more matrix, in order to obtain the desired matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$:

\[
D = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} B,
\]

\[
E = C = I_2,
\]

\[
DAE = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} BAC = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}.
\]
b) Secondly, we have the situation \( \det A = k = 0 \), but still \( A \neq 0 \).

(i) If \( a \neq 0 \), we put
\[
B = B_2B_1, \quad C = \begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix},
\]
\[
BAC = B_2 \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} C = \begin{pmatrix} 1 & 0 \\ -ac & 1 \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ ac & ad \end{pmatrix} C
\]
\[
= \begin{pmatrix} 1 & a^{-1}b \\ 0 & k \end{pmatrix} \begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

If \( a = 0 \), then at least one of the remaining three matrix elements is non-zero. Thus, we can analogously work with one of the matrices
\[
A \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A, \quad A \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (25)
\]
namely with the one whose element in the first row and first column is non-zero.

(ii) Again, without loss of generality, we assume that \( d \neq 0 \) (otherwise we would transform \( A \) into one of the three matrices given in (25); one of which would have a non-zero element in its second row and second column). We put
\[
D = D_2D_1 = \begin{pmatrix} 1 & -bd \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix},
\]
\[
E = \begin{pmatrix} 1 & 0 \\ -cd^{-1} & 1 \end{pmatrix},
\]
\[
DAE = D_2 \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} E = \begin{pmatrix} 1 & -bd \\ 0 & 1 \end{pmatrix} \begin{pmatrix} da & db \\ d^{-1}c & 1 \end{pmatrix} E
\]
\[
= \begin{pmatrix} k & 0 \\ d^{-1}c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -cd^{-1} & 1 \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

\[\square\]

**Proof of Step 1.** Let us express a matrix \( M \in Sp(4, \mathbb{F}_n) \) in blocks:
\[
M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad \text{where } M_{ij} \in \mathbb{Z}_n^{2 \times 2}.
\]
The equality (17) means that \( \det M_{12} + \det M_{22} = 1 \). We denote \( \det M_{22} = k \), and consequently \( \det M_{12} = 1 - k \).

(i) Let us assume that \( k \neq 0 \). We take matrices \( B, C \in SL(2, \mathbb{Z}_n) \) as described in Lemma 3.5, so as to obtain \( BM_{12}C = \begin{pmatrix} 1 & 0 \\ 0 & 1-k \end{pmatrix} \). Afterwards, we apply Lemma 3.5 on matrix \( M_{22}C \), whose determinant is \( \det(M_{22}C) = (\det M_{22})(\det C) = k \cdot 1 = k \neq 0 \). Thus, we find \( D \in SL(2, \mathbb{Z}_n) \).
such that \( DM_{22}C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). At this moment, we are able to transform \( M \) by means of matrices \( F_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), \( F_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \) into the following:

\[
F_1MF_2 = \begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & C \end{pmatrix} = \begin{pmatrix} BM_{11} & BM_{12} \\ DM_{21} & DM_{22} \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & C \end{pmatrix}
\]

\[
= \begin{pmatrix} BM_{11} & BM_{12} \\ DM_{21} & DM_{22} \end{pmatrix} = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} \\ \tilde{m}_{21} & \tilde{m}_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 - k \end{pmatrix} = \tilde{M},
\]

where we denote matrices \( BM_{11} = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} \\ \tilde{m}_{21} & \tilde{m}_{22} \end{pmatrix} \) and \( DM_{21} = \begin{pmatrix} \tilde{m}_{31} & \tilde{m}_{32} \\ \tilde{m}_{41} & \tilde{m}_{42} \end{pmatrix} \) by \( \tilde{M}_{11}, \tilde{M}_{21} \), respectively.

As the matrix \( \tilde{M} \) is a multiple of \( F_1, F_2 \), and \( M \), which all belong to \( Sp(4,F_n) \), then also \( \tilde{M} \) is an element of \( Sp(4,F_n) \), and so its elements fulfill equations \([18]–[21]\):

\[
0 = -\tilde{m}_{21} - k\tilde{m}_{41} \pmod{n}
\]

\[
0 = (1 - k)\tilde{m}_{11} + \tilde{m}_{31} \pmod{n}
\]

\[
0 = -\tilde{m}_{22} - k\tilde{m}_{42} \pmod{n}
\]

\[
0 = (1 - k)\tilde{m}_{12} + \tilde{m}_{32} \pmod{n}.
\]

These relations ensure that we can transform \( \tilde{M} \) into \( S(k) \) by means of matrix \( N = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} \\ -\tilde{m}_{41} & -\tilde{m}_{42} \end{pmatrix} \).

\[
\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} N = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} \\ -k\tilde{m}_{41} & -k\tilde{m}_{42} \end{pmatrix} = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} \\ \tilde{m}_{21} & \tilde{m}_{22} \end{pmatrix} = \tilde{M}_{11},
\]

\[
\begin{pmatrix} k - 1 & 0 \\ 0 & -1 \end{pmatrix} N = \begin{pmatrix} (k - 1)\tilde{m}_{11} & (k - 1)\tilde{m}_{12} \\ \tilde{m}_{41} & \tilde{m}_{42} \end{pmatrix} = \begin{pmatrix} \tilde{m}_{31} & \tilde{m}_{32} \\ \tilde{m}_{41} & \tilde{m}_{42} \end{pmatrix} = \tilde{M}_{21}.
\]

Therefore, \( \tilde{M}_{11}N^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}, \tilde{M}_{21}N^{-1} = \begin{pmatrix} k - 1 & 0 \\ 0 & -1 \end{pmatrix} \), and consequently

\[
\tilde{M} \begin{pmatrix} N^{-1} & 0 \\ 0 & I_2 \end{pmatrix} = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} \\ \tilde{m}_{21} & \tilde{m}_{22} \\ \tilde{m}_{31} & \tilde{m}_{32} \\ \tilde{m}_{41} & \tilde{m}_{42} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 - k \\ 0 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N^{-1} & 0 \\ 0 & I_2 \end{pmatrix} = S(k).
\]

As an element of \( Sp(4,F_n) \), the matrix \( \tilde{M} \) also fulfills equation \([16]\), which implies that \( det\tilde{M}_{11} + det\tilde{M}_{21} = 1 = k detN + (1 - k) detN = detN \), and thus \( N \in SL(2,Z_n) \). Hence, we have found the desired matrices \( G_1 = F_1 = B \oplus D, G_2 = F_2(N^{-1} \oplus I_2) = N^{-1} \oplus C \), transforming \( M \) into \( G_1MG_2 = S(k) \).

(ii) The case when \( k = 0 \), we treat as follows. Firstly, we find \( D, E \in SL(2,Z_n) \) such that \( DM_{22}E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) and \( B \in SL(2,Z_n) \) such that \( BM_{21}E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), according to Lemma \([5,6]\).

Analogously to the case \( k \neq 0 \), we set \( G_1 = F_1 = B \oplus D, G_2 = F_2(N^{-1} \oplus I_2) = N^{-1} \oplus E \), and we reach the result \( G_1MG_2 = S(k) \).
Proof of Step 2. We need to express each matrix $S(k)$ as an element of $\langle D_1, D_2, D_3, D_4 \rangle$. We begin by showing (by induction) that

$$D_4^j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -j \\ j & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = D_4 D_4^{j-1}. $$

We make use of the fact that $J = (0 \ 1 \ -1 \ 0) \oplus (0 \ 1 \ -1 \ 0) \in \mathcal{H}$, $J^T \in \mathcal{H}$, and $D_4^T D_4 D_4 \in Sp(4, \mathbb{F}_n)$ (this can be verified by a simple matrix multiplication); and we generate $S(k)$ from $D_4$ and elements from $\mathcal{H}$:

$$J^T (D_4^{1-k})^T JD_4^j = S(k). $$

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References

[1] M. A. Abdelmalek, X. Leng, J. Patera, P. Winternitz, Grading refinements in the contraction of Lie algebras and their invariants, J. Phys. A: Math. Gen., 29 (1996), 7519-7543.

[2] E. Artin, Geometric algebra, Interscience Publishers, New York, 1957.

[3] M. Havlíček, J. Patera, E. Pelantová, On Lie gradings II, Lin. Alg. Appl. 277 (1998), 97–125.

[4] M. Havlíček, J. Patera, E. Pelantová, J. Tolar, Automorphisms of the fine grading of $\mathfrak{sl}(n, \mathbb{C})$ associated with the generalized Pauli matrices, J. Math. Phys. 43 (2002), 1083–1094.

[5] J. Hrivnák, P. Novotný, J. Patera, J. Tolar, Graded contraction of the Pauli graded $\mathfrak{sl}(3, \mathbb{C})$, math-ph/0509033, 49 pp.

[6] M. de Montigny, J. Patera, Discrete and continuous graded contractions of Lie algebras and superalgebras, J. Phys. A: Math. Gen. 24 (1991), 525-549.

[7] R. V. Moody, J. Patera, Discrete and continuous graded contractions of representations of Lie algebras, J. Phys. A: Math. Gen. 24 (1991), 2227-2258.
[8] M. Neuhauser, *An Explicit Construction of the Metaplectic Representation over a Finite field*, J. Lie Theory, **12** (2002), 15-30.

[9] P. Novotný, oral communication.

[10] J. Patera, H. Zassenhaus, *On Lie gradings I*, Lin. Alg. Appl. **112** (1989), 87–159.

[11] J. Patera, H. Zassenhaus, *The Pauli matrices in n dimensions and finest gradings of simple Lie algebras of type $A_{n-1}$*, J. Math. Phys. **29** (1988), 665–673.