Number and phase: complementarity and joint measurement uncertainties

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Abstract

We show that number and canonical phase (of a single mode optical field) are complementary observables. We also bound the measurement uncertainty region for their approximate joint measurements.

Keywords: number observable, canonical phase observable, complementary observables, uncertainty, joint measurement

1. Introduction

Analogously to position and momentum of a quantum object, number and phase of a single mode optical field are often considered as an example of a pair of observables which is complementary and for which the uncertainty relations put severe limitations both for preparations and measurements. However, since there is no phase shift covariant spectral measure solution to the quantum phase problem it has remained a challenge to formulate the exact content of these intuitive ideas for this pair of observables.

The notion of complementarity, which goes back to the 1927 Como lecture of Niels Bohr [1] and which was strongly advocated also by Wolfgang Pauli [2], is often discussed only rather vaguely and mostly in connection with Werner Heisenberg’s uncertainty relations [3]. However, the notion of mutual exclusiveness which is associated with the idea of complementarity has rather straightforward independent formulations in quantum mechanics, and, like uncertainty, it has both probabilistic and measurement theoretical aspects. Along with Bohr [4], we say that two observables are complementary if all the instruments (measurements) which allow their unambiguous definitions are mutually exclusive. The notion of mutual exclusiveness of measurements is easily expressed with respect to the order structure of the set of quantum effects, sharp or unsharp. Following Pauli [2], one may also say that two observables are probabilistically complementary if certain predictions concerning the measurement outcomes of these observables are mutually exclusive. In addition, the notion of value complementarity of two
observables is often used to refer to the case where sharply defined value (exact knowledge) of one observable implies uniform distribution (complete ignorance) on the values of the other observable. All these notions have natural expressions in terms of the measurement outcome probabilities of quantum mechanics and their straightforward formulations have been proposed and extensively studied in the literature, see, for instance [5, 6] and references therein.

Concerning number and phase, it is well known that they are probabilistically complementary as well as value complementary, see, for instance [7, propositions 16.2 and 16.3], but it has remained an open question if among the phase shift covariant phase observables there is any which would be complementary with the number [8]. This question is now settled in section 3 where it is shown that the canonical phase and number form a complementary pair.

Complementary observables are necessarily incompatible, that is, they do not have any (exact) joint measurements. This leads one to study their approximate joint measurements, a topic which has gained a substantial clarification in recent years. Rather than digging in the extensive history of this topic, we refer to the relevant chapters of the monograph [7]. In section 4 we follow the ideas and methods initiated in [10, 11] and further developed, for instance, in [12–14], to bound the measurement uncertainty domain of the complementary pair of number and canonical phase.

Throughout the paper we use freely the standard notions and terminology of Hilbert space quantum mechanics. Yet, we start with a short account of the main terminology and the basic results concerning the canonical phase observable.

### 2. Basic notions

Let $\mathcal{H}$ be a Hilbert space, $\{|n⟩ | n ∈ \mathbb{N}\}$ an orthonormal basis of $\mathcal{H}$, and $N = \sum_{n=0}^{∞} n|n⟩⟨n|$ the corresponding number operator. Let $\mathcal{L}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$ denote, respectively, the sets of bounded and trace class operators on $\mathcal{H}$. We also let $S(\mathcal{H}) ⊂ \mathcal{T}(\mathcal{H})$ denote the set of positive, trace one operators (states). We denote by $N : 2^\mathbb{N} → \mathcal{L}(\mathcal{H})$ the spectral measure of $N$ and call it the number observable. With any observable, like $N$, we let $N_\rho$ denote the probability measure $Y → N_\rho(Y) = \text{tr}[\rho N(Y)]$ defined by the observable and a state $\rho ∈ S(\mathcal{H})$. Occasionally, we refer to such a probability measure as the measurement outcome distribution of the observable in the given state.

Let $\mathcal{B}([0, 2\pi])$ be the Borel sigma algebra of $[0, 2\pi]$. By a phase observable we mean any normalized positive operator measure (semispectral measure) $E : \mathcal{B}([0, 2\pi]) → \mathcal{L}(\mathcal{H})$ which is covariant under the phase shifts generated by the number observable, that is, satisfies the condition $e^{iθN}E(X)e^{-iθN} = E(X+θ)$ for all $θ ∈ [0, 2\pi]$ and $X ∈ \mathcal{B}([0, 2\pi])$, where $+$ denotes addition modulo $2\pi$. The structure of such observables is completely known, see, for instance, [7, 15, 16]. Among them there is the one referred to the canonical phase observable, which we denote by $Φ : \mathcal{B}([0, 2\pi]) → \mathcal{L}(\mathcal{H})$ and which has the effects

$$Φ(X) = \sum_{m,n=0}^{∞} \int_{X} e^{i(m−n)θ} \frac{dθ}{2\pi} |m⟩⟨n| \tag{2.1}$$

There are several properties which distinguish $Φ$ as the canonical phase among all the phase observables. Without entering the whole list of such properties\(^2\), we mention here only the fact that, up to unitary equivalence, the canonical phase is the only phase observable which generates number shifts: $V^{(k)}N(Y + k)(V^{(k)})^* = N(Y)$, where $V^{(k)} = \int_{0}^{2\pi} e^{ikθ} dΦ(θ)$.

\(^2\) A reader interested in those properties of $Φ$ may check the list of 19 items of [16, section 4.8] together with some further properties obtained in [17, 18].
are the cyclic moment operators of \( \Phi \). We recall also that the spectrum of the effect \( \Phi(X) \), \( 0 \neq \Phi(X) \neq I \), is the whole interval \([0, 1]\) with no eigenvalues. In particular, for any \( \theta \in [0, 2\pi] \) and for any \( \epsilon > 0 \), the (operator) norm of the effect \( \Phi((\theta - \epsilon, \theta + \epsilon) \cap [0, 2\pi]) \) equals one. Thus, for each point \( \theta \in [0, 2\pi] \) there is a sequence of unit vectors \( (\psi_i)_{i\in\mathbb{N}} \) such that the probability measures \( X \mapsto \langle \psi_i|\Phi(X)|\psi_i \rangle \) tend, with increasing \( i \), to the point measure \( \delta_\theta \) at \( \theta \). In such a case, the number probabilities \( |\langle \psi_i|n \rangle|^2 \) tend to zero for all \( n \). Observing, in addition, that in the number states \( |n \rangle \) the phase distribution is uniform, \( \langle n|\Phi(X)|n \rangle = \int_X \frac{d\theta}{2\pi} = \ell(X) \), the probabilistic and the value complementarity of the pair \((N, \Phi)\) become obvious (as discussed in a greater detail, for instance, in section 16.4 of [7].)

As well-known, number \( N \) and phase \( \Phi \) are incompatible observables, that is, they cannot be measured jointly. Indeed, since \( N \) is a spectral measure, their joint measurement \( M \) would necessarily be of the product form, that is, \( M(n, X) = |n\rangle\langle n|\Phi(X) = \Phi(X)|n\rangle\langle n| \) for any \( n \in \mathbb{N}, X \in \mathcal{B}([0, 2\pi]) \) (see, for instance, [7, proposition 4.8]). But this would imply that \( \Phi(X) = \ell(X)I \), which contradicts (2.1).

Though \( \Phi \) and \( N \) have no joint observable, there are observables \( M : \mathcal{B}([0, 2\pi] \times \mathbb{N}) \to \mathcal{L}(\mathcal{H}) \) having either \( \Phi \) or \( N \) as a margin, that is, either \( M_1 = \Phi \), with \( M_1(X) = M(X \times \mathbb{N}) \), or \( M_2 = N \), with \( M_2(Y) = M([0, 2\pi] \times Y) \). In either case the joint observable is a smearing of the exact margin. Indeed, if \( N = M_2 \), then \( M(X \times Y) = M_1(X)N(Y) \) (see above) and each \( M_1(X) \) is a function of the number operator \( N \) so that \( M(X \times Y) = \sum_{n\in\mathbb{N}} p(X,n)|n\rangle\langle n| \), with a Markov kernel \( B([0, 2\pi]) \times \mathbb{N} \ni (X,n) \mapsto p(X,n) \in [0,1] \). On the other hand, if \( M_1 = \Phi \), then again there is a kernel \( p : [0, 2\pi] \times 2^\mathbb{N} \to [0,1] \) such that \( M \) is obtained as

\[
M(X \times Y) = \int_X p(\theta, Y) \, d\Phi(\theta),
\]

so that, in particular, for each \( Y \in 2^\mathbb{N} \), \( M_2(Y) = \int_0^{2\pi} p(\theta, Y) \, d\Phi(\theta) \). The structural similarity of the two cases is due to the fact that both \( \Phi \) and \( N \) are rank-1 observables, for details, see [18, 19].

The above incompatibility results contain also the following well-known facts. In any of the sequential measurements (in either order), if the first measurement is exact, that is, measures either \( N \) or \( \Phi \), then any information on the other observable coded in the initial state of the measured system is completely lost in the following precise sense: if, say, \( N \) is measured first, with an instrument \( \mathcal{I} \), in a state \( \rho \), then the subsequent phase probabilities are \( \text{tr} [\mathcal{I}(|n\rangle\langle n|)\Phi(X)] = \text{tr} [\rho \mathcal{I}(|n\rangle\langle n|)\Phi(X)] \), where the ‘distorted’ phase effects \( \mathcal{I}(|n\rangle\langle n|)\Phi(X) \) are smearings of the number observable for some kernel \( n \mapsto p(X,n) \). Similarly, if one first performs an exact phase measurement, with an instrument \( \mathcal{J} \), say, then the subsequent number probabilities are \( \text{tr} [\mathcal{J}([0, 2\pi])|\rho\rangle\langle \rho|] = \text{tr} [\rho \mathcal{J}([0, 2\pi])|\phi\rangle\langle \phi|] \), where the ‘distorted’ number effects \( \mathcal{J}([0, 2\pi])|\phi\rangle\langle \phi| \) are smearings of the phase observable \( \Phi \) with a kernel \( \theta \mapsto p(\theta, Y) \).

We now turn to study the complementarity of the number and the canonical phase.

### 3. Complementarity of the pair \((N, \Phi)\)

As already pointed out, the pair \((N, \Phi)\) is known to be both probabilistically complementary and value complementary, but it has remained an open question if they are also complementary. This question will now be settled with theorem 1 which shows that for each finite subset

\cite{footnote}{footnote: We recall that this integral simply means that for each state \( \rho \), \( \text{tr}[\rho M(X \times Y)] = \int_X p(\theta, Y) \, d\Phi(\theta) \), the integral of the (measurable) function \( \theta \mapsto p(\theta, Y) \) with respect to the probability measure \( \Phi_\rho \).}
$Y \subset \mathbb{N}$ and $X \in \mathcal{B}(\{0, 2\pi\})$, for which $\Phi(X) \neq I$, the greatest lower bound of the effects $\Phi(X)$ and $\mathcal{N}(Y)$ exists in the partially ordered set of effects $\mathcal{E}(\mathcal{H}) = \{E \in \mathcal{L}(\mathcal{H}) \mid 0 \leq E \leq I\}$ and equals the null effect, that is

$$\Phi(X) \land \mathcal{N}(Y) = 0. \quad (3.1)$$

It is this relation which we take to express the complementarity of the pair $(\mathcal{N}, \Phi)$ in the sense that all the measurements which serve to define these observables are mutually exclusive. In fact, if (3.1) were not true, then for some such $X$ and $Y$ there would be an effect $E$ below both $\mathcal{N}(Y)$ and $\Phi(X)$, so that, in any state $\rho$, the probability $\text{tr}[\rho E]$ would also be a common lower bound for the corresponding number and the phase probabilities. Thus, with measuring the effect $E$ in any state one would also get information from the effects $\mathcal{N}(Y)$ and $\Phi(X)$ in that state. Relation (3.1) excludes such measurements.

The restriction to bounded subsets in the definition of complementarity of observables in (3.1) is justified by the fact that each observable (with a separable value space, like $[0, 2\pi]$ or $\mathbb{N}$) is uniquely defined by the effects associated with such sets. We note further that complementarity of any two observables can also be defined in terms of any pair of instruments uniquely defining such observables. We say that instruments $\mathcal{I}_1$ and $\mathcal{I}_2$ are mutually exclusive if for any two bounded sets $X$ and $Y$ for which neither of the operations $\mathcal{I}_1(X)$ nor $\mathcal{I}_2(Y)$ is maximal (with respect to the order structure of operations), their greatest lower bound exists and equals the zero operation. Then, any two observables are complementary (in the sense of (3.1)) exactly when any pair of their defining instruments are mutually exclusive. For further discussion of this formulation of complementarity we refer to [6].

The order structure of the set of effects is known to be quite complicated when compared with the order structure of the set of projections. However, a characterization of pairs of effects $E, F \in \mathcal{E}(\mathcal{H})$ for which $E \land F$ exists has been obtained [20], and, in particular, it is known that if one of them is a projection then their greatest lower bound always exists [20, corollary 3.1]. Therefore, $\Phi(X) \land \mathcal{N}(Y)$ exists for any $X \in \mathcal{B}(\mathbb{T})$ and $Y \subset \mathbb{N}$, and it remains to be shown that all these meets are zero whenever $Y$ is a finite set and $X$ such that $\ell(X) < 1$ (i.e. $\Phi(X) \neq I$). Clearly, such a result depends on the explicit properties of the number and the canonical phase.

From now on we identify the phase interval $[0, 2\pi)$ (addition modulo $2\pi$) with the unit circle $\mathbb{T}$ in the usual way through the map $\theta \mapsto e^{i\theta}$, denoting still by $d\ell(\theta) = \frac{d\theta}{2\pi}$ the normalized measure on $\mathbb{T}$. Let $Q$ be the canonical spectral measure of the Hilbert space $\mathcal{H} = L^2(\mathbb{T})$ and let $\{e_k \mid k \in \mathbb{Z}\}$ be its Fourier basis, that is, $e_k(\theta) = e^{-ik\theta}$. Let $P_N$ be the projection $\sum_{n=-N}^{N} |e_n\rangle \langle e_n|$. The Naimark projection of $Q$ onto $P_N(\mathcal{H})$, that is, the map $X \mapsto P_NQ(X)\big|_{P_N(\mathcal{H})}$ is exactly of the form (2.1). In fact, $Q$ is the minimal Naimark dilation of $\Phi$ [7, theorem 8.1].

We identify $\mathcal{H}$ with the subspace $P_N(\mathcal{H})$ of $\mathcal{H}$ via the isometry $V : |n\rangle \mapsto e_n$, so that $P_N = WV^*$ and

$$\Phi(X) = V^*Q(X)V = V^*P_NQ(X)V$$

for all $X \in \mathcal{B}(\mathbb{T})$.

**Remark 1.** Let $P$ be the spectral measure with the (atomic) projections $|e_k\rangle \langle e_k|$, $k \in \mathbb{Z}$. In [21, example 4.2] it was shown that the pair $(Q, P)$ of $L^2(\mathbb{T})$ is complementary, that is, $Q(X) \land P(Y) = 0$ for all $X \in \mathcal{B}(\mathbb{T})$, for which $Q(X) \neq I_{\mathcal{H}}$, and for all finite $Y \subset \mathbb{Z}$. The corresponding result for the position-momentum pair $(Q, P)$ of $L^2(\mathbb{R})$ is well known, see, e.g. [7, proposition 8.2]. Though $\Phi(X) = V^*Q(X)V$ and $N(Y) = V^*P(Y)V$ (= $P(Y)$, for $Y \subset \mathbb{N}$), the noncommutativity of $P_N$ and $Q(X)$ prevents one to conclude the disjointness of the effects $\Phi(X)$ and $\mathcal{N}(Y)$ directly from the disjointness of the projections $Q(X)$ and $P(Y)$. 

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Lemma 1. Let \( \alpha \geq 0 \) and \( X \in \mathcal{B}(\mathbb{T}) \) such that \( \Phi(X) \neq I \). Then \( \alpha |0\rangle \langle 0 | \leq \Phi(X) \) implies \( \alpha = 0 \).

Proof. Suppose that \( \alpha |0\rangle \langle 0 | \leq \Phi(X) = V^*Q(X)V \), that is, \( \alpha |e_0\rangle \langle e_0 | \leq V\Phi(X)V^* = P_NQ(X)P_N = [Q(X)P_N]^{*}\tilde{Q}(X)P_N \) and note that \( \Phi(X) \neq I \) if and only if \( Q(X) \neq I \H \) if and only if \( \ell(X) < 1 \). Let \( \mathcal{K} = Q(X)P_N(\tilde{\mathcal{H}}) \). Define an operator \( D \in \mathcal{L}(\mathcal{H}) \) by \( D(Q(X)\tilde{\psi}) = \sqrt{\alpha}e_0\tilde{\psi}e_0, \tilde{\psi} \in P_N(\tilde{\mathcal{H}}) \), and \( D\varphi = 0, \varphi \in \mathcal{K}^\perp \). Indeed, \( D \) is clearly linear and well defined since, if \( Q(X)\psi = Q(X)\psi' \), \( \psi, \psi' \in P_N(\tilde{\mathcal{H}}) \), i.e. \( Q(X)\psi_+ = 0, \psi_+ = \psi - \psi' \), then

\[
0 \leq \|D(Q(X)\psi_+)|^2 = \langle \psi_+ |\alpha e_0\rangle \langle e_0 | \psi_+ \rangle \leq \langle \psi_- |P_NQ(X)P_N\psi_- \rangle = \langle \psi_- |Q(X)\psi_- \rangle = 0
\]

so that \( D(Q(X)\psi) = D(Q(X)\psi') \). Similarly, \( \|D(Q(X)\psi')\| \leq \|Q(X)\psi'\|, \psi' \in P_N(\tilde{\mathcal{H}}) \), showing that \( D \) is bounded and thus extends to the whole \( \tilde{\mathcal{H}} \). Since the range of \( D \) is \( \mathbb{C}e_0 \), one has \( D = |e_0\rangle \langle \eta | \) for some \( \eta \in \tilde{\mathcal{H}} \). In addition, since \( DQ(X)P_N = \sqrt{\alpha}|e_0\rangle \langle e_0 | \)

\[
\alpha |e_0\rangle \langle e_0 | = [Q(X)P_N]^{*}\tilde{Q}(X)P_N = |\eta'\rangle \langle \eta' |
\]

where \( \eta' = P_NQ(X)\eta \) and also \( \eta' = ze_0 \). Now \( \langle e_m|Q(X)\eta \rangle = \langle e_m|P_NQ(X)\eta \rangle = \langle e_m|\eta' \rangle = 0 \) for all \( m > 0 \) so that \( Q(X)\eta = \sum_{n=0}^{\infty}c_ne_{-n} \) for some square summable sequence of complex numbers \( c_n \), i.e. \( Q(X)\eta \) is a Hardy function which vanishes on a set \( T \setminus X \) of measure \( 1 - \ell(X) > 0 \). As well known, a Hardy function which vanishes on a set of positive measure is identically zero (see, e.g. [22, theorem 1]). Therefore, \( Q(X)\eta = 0, \eta' = 0 \), and \( \alpha |e_0\rangle \langle e_0 | = 0 \), yielding \( \alpha = 0 \).

Lemma 2. Let \( E \in \mathcal{L}(\mathcal{H}) \) be a positive operator such that \( |n|E|n \rangle = 0 \) for all \( n > r \) where \( r \in \mathbb{N} \), and let \( X \in \mathcal{B}(\mathbb{T}) \) be such that \( \Phi(X) \neq I \). Then \( E \leq \Phi(X) \) implies \( E = 0 \).

Proof. The proof is by induction on \( r \). First we note that, by positivity, if \( \langle n|E|n \rangle = 0 \) for some \( n \), then \( \langle m|E|m \rangle = \langle n|E|m \rangle = 0 \) for all \( m \in \mathbb{N} \). The condition \( E \leq \Phi(X) \) implies \( \langle r|E|r \rangle = 0 \), \( \langle 0|E|0 \rangle = W \Phi(X)W^* = \Phi(X) \)

\[
W = \sum_{k=0}^{\infty} |k\rangle \langle k + r |
\]

From lemma 1 one gets \( \langle r|E|r \rangle = 0 \) and by induction \( \langle n|E|n \rangle = 0 \) for all \( n \in \mathbb{N} \), i.e. \( E = 0 \).

Theorem 1. For any finite subset \( Y \) of \( \mathbb{N} \) and \( X \in \mathcal{B}(\mathbb{T}) \) such that \( \Phi(X) \neq I \), \( \Phi(X) \wedge N(Y) = 0 \).

Proof. Clearly, the claim holds if \( Y = \emptyset \) (i.e. \( N(Y) = 0 \)) so that we assume that \( Y \) is finite and non-empty. Assume that there exists an effect \( E \) such that \( E \leq \Phi(X) \) and \( E \leq N(Y) \). Thus, \( r = \max Y \in \mathbb{N}, N(Y) \leq R = \sum_{n=0}^{\infty} |n| \langle n|E|n \rangle = 0 \) for all \( n > r \). Since also \( E \leq \Phi(X) \), lemma 2 now implies that \( E = 0 \), that is, 0 is the only lower bound of \( \Phi(X) \) and \( N(Y) \).

We note that (3.1) is equivalent with the seemingly weaker requirement that this condition holds for all singletons \( Y = \{n\} \). Finally, we give bounds for the joint predictability of number and phase, noting that from (3.1) it only follows that for any \( \rho \), one has \( \Phi_{\rho}(X) + N_{\rho}(Y) < 2 \).

Proposition 1. For any \( X \in \mathcal{B}(\mathbb{T}) \), with \( \ell(X) < 1 \), and for any finite \( Y \subset \mathbb{N} \),

\[
\sup_{\rho \in \mathcal{S}(\mathcal{H})} \left( \Phi_{\rho}(X) + N_{\rho}(Y) \right) \leq 1 + \sqrt{\alpha} + \sqrt{\alpha} < 2,
\]
where \( a_+ \) is the largest eigenvalue the (finite rank) operator \( N(Y)\Phi(X)N(Y) \).

**Proof.** Considering \( \Phi \) and \( N \) as the Naimark projections of \( Q \) and \( P \) on the subspace \( P_{\pi}(\mathcal{H}) \) of \( L^2(\mathbb{T}) \), we have

\[
\sup_{\rho \in \mathcal{S}(\mathcal{H})} (\Phi_\rho(X) + N_\rho(Y)) \leq \sup_{\rho \in \mathcal{S}(\mathcal{H})} (Q_\rho(X) + P_\rho(Y)).
\]

Using the results of \[9\] the numerical range \( \{ (\langle f|P(Y)f \rangle, \langle f|Q(X)f \rangle) \mid f \in \mathcal{H}, \|f\| = 1 \} \) of the pair of projections \( P(Y), Q(X) \) can completely be determined. Since \( P(Y) \wedge Q(X) = 0 \), the point \((1, 1)\) is now excluded from this range. It suffice to recall here that the numerical range is a convex subset of \([0, 1] \times [0, 1] \) \[9, proposition 1\] and that for any unit vector \( f \in L^2(\mathbb{T}) \), the sum \( \langle f|P(Y)f \rangle + \langle f|Q(X)f \rangle \) is bounded by the number \( 1 + \sqrt{a_+} \), where \( a_+ \) is the maximal eigenvalue of the positive finite rank operator \( P(Y)Q(X)P(Y) \) \[9, proposition 5\]. Note that the spectra of the operators \( N(Y)\Phi(X)N(Y) \) and \( P(Y)Q(X)P(Y) \) are identical. Since \( \text{tr} [\rho \Phi(X)] < 1 \) for any state \( \rho \in \mathcal{S}(\mathcal{H}) \) (see, for instance, \[7, proposition 16.2\]), the eigenvalue \( a_+ \) is strictly less than one.

\[\square\]

4. Errors in approximate joint measurements of \( N \) and \( \Phi \)

We study next the necessary errors appearing in an approximate joint measurement of number and canonical phase. We follow the idea, carefully expounded already in \[10, pp 197–8\], that in quantum mechanics ‘measurement error’ is to be found by comparing a ‘real’ measurement outcome statistics with the desired, ‘ideal’ one. More explicitly, if, for instance, \( M_1 \) represents an approximate measurement of, say, the canonical phase \( \Phi \), then the measurement error should be found out by comparing the measurement outcome distributions \( M_{1,\rho} \) and \( \Phi_\rho \) in an arbitrary state \( \rho \).

**Remark 2.** There are also other proposals to define ‘measurement error’ in quantum mechanics than the one followed in this paper. In particular, the approach developed, for instance, by Ozawa \[24\], see also Hall \[25\], has received extensive publicity. To express the relevant notion, assume that \( M_2 \) represents an approximate measurement of the number observable \( N \). The measurement error in a state \( \rho \) is then defined as follows

\[
\epsilon(N, M_2, \rho)^2 = \text{tr} [\rho(M_2[2] - M_2[1]^2)] + \text{tr} [\rho(N - M_2[1]^2)].
\]  

(4.1)

Here \( M_2[k] = \int x^k \text{d}M_2(dx) \) is the \( k \)th moment operator of \( M_2, k = 1, 2 \) (see footnote 2). The number \( \epsilon(N, M_2, \rho) \) consists of two non-negative terms, the expectation of the noise operator \( M_2[2] - M_2[1]^2 \) of the approximating observable \( M_2 \) and the expectation of the square of the difference operator \( N - M_2[1] \) in the state \( \rho \). If the approximating measurement is unbiased, that is \( N = M_2[1] \), then the error consists only of the expectation of the noise operator. As evident, the number \( \epsilon(N, M_2, \rho) \) is not determined by the distributions \( N_\rho \) and \( M_{2,\rho} \). Neither is the expectation of the noise operator determined by the distribution \( M_{2,\rho} \). For a further discussion of this approach together with a comparison with the present approach we refer to \[13\]. Finally, since there is no phase shift covariant selfadjoint operator representing a phase observable the notion of error as defined in (4.1) is not directly applicable here.

\[\footnote{An excellent general introduction of this topic is given in section 2.1 of \[14\].}\]

\[\footnote{Actually, the original definition is given in terms of an explicit measurement scheme which however reduces to the above given form \[26\].}\]
The comparison between the 'real' and 'ideal' distributions can naturally be based on various methods ranging from the metric distances between the involved probability measures to the relative entropy describing the loss of information when, say, $M_{1,\rho}$ is used instead of $\Phi$; for information-theoretic approaches, see, for instance [27, 28] and the references given therein. Here we follow the approach initiated in [11] and further developed in [12, 13] where the error is quantified using the Wasserstein distance between probability measures.

For simplicity, we only use the Wasserstein-2 distances and fix the metrics to be the arc distance on $\mathbb{T}$, $d(\theta, \theta') = \min_{n \in \mathbb{Z}} |\theta - \theta' - 2\pi n|$, and the standard distance on $\mathbb{N}$, $d(m, n) = |m - n|$.

Let $M_1 : \mathcal{B}(\mathbb{T}) \to \mathcal{L}(\mathcal{H})$ and $M_2 : \mathcal{B}(\mathbb{N}) \to \mathcal{L}(\mathcal{H})$ be any two observables (semispectral measures) which approximate measurements of $\Phi$ and $\mathcal{N}$, respectively. The error in approximating $\Phi$ by $M_1$ is now defined as

$$d(M_1, \Phi) = \sup_{\rho} D((M_{1,\rho}, \Phi_\rho),$$

where $D((M_{1,\rho}), \Phi_\rho)$ is the Wasserstein-2 distance between the probability measures $(M_{1,\rho})$ and $\Phi_\rho$, that is,

$$D((M_{1,\rho}), \Phi_\rho) = \inf_{\gamma} \int_{\mathbb{T} \times \mathbb{T}} d(\theta, \theta')^2 \, d\gamma(\theta, \theta'),$$

where the infimum is taken over all couplings (joint probabilities) $\gamma : \mathcal{B}(\mathbb{T} \times \mathbb{T}) \to [0, 1]$ of $(M_{1,\rho})$ and $\Phi_\rho$. We recall that the existence of a minimizing coupling is known [23, theorem 4.1]. In (4.2) the supremum ensures that the error is the worst of the deviations $D((M_{1,\rho}), \Phi_\rho)$, meaning that in any state the error in using a measurement of $M_1$ as an approximate to $\Phi$ is at most $d(M_1, \Phi)$. Similarly, one defines the error $d(M_2, \mathcal{N})$.

**Remark 3.** Canonical phase $\Phi$ is not a spectral measure. Still, as pointed out above, it resembles a spectral measure in many respects. In particular, the notion of calibration error

$$d^\epsilon(M_1, \Phi) = \lim_{\epsilon \to 0} \sup \{D((M_{1,\rho}), \delta_\epsilon) \mid D(\Phi_\rho, \delta_\epsilon) \leq \epsilon\}$$

makes sense, along with all spectral measure observables, also to canonical phase and one has $d^\epsilon(M_1, \Phi) \leq d(M_1, \Phi)$. Moreover, if $M_1$ is a smearing of $\Phi$ in the sense of a convolution, that is, $M_1 = \mu * \Phi$ for a probability measure $\mu$, then $d^\epsilon(M_1, \Phi)^2 = d(M_1, \Phi)^2 = \int_{\mathbb{T}} d(\theta, 0)^2 \, d\mu = \int_{\mathbb{T}} \min_{n \in \mathbb{Z}} |\theta - 2\pi n|^2 \, d\mu = \int_{\mathbb{T}} \theta^2 \, d\mu = \mu[2]$. Similarly, if $M_2 = \nu * \mathcal{N}$ for some probability measure $\nu$, then $d^\epsilon(M_2, \mathcal{N})^2 = d(M_2, \mathcal{N})^2 = \sum_{k=0}^{\infty} d(k, 0)^2 \nu\{k\} = \sum_{k=0}^{\infty} k^2 \nu\{k\} = \nu[2][12]$, lemmas 7 and 11.

For an approximate joint measurement of $\Phi$ and $\mathcal{N}$, the approximators $M_1$ and $M_2$ must be compatible, that is, they have to be margins of a joint observable $M : \mathcal{B}(\mathbb{T} \times \mathbb{N}) \to \mathcal{L}(\mathcal{H})$. The basic problem is thus to characterize the joint measurement error set

$$\mathcal{M}U(\mathbb{T} \times \mathbb{N}) = \{ (d(M_1, \Phi), d(M_2, \mathcal{N})) \mid M : \mathcal{B}(\mathbb{T} \times \mathbb{N}) \to \mathcal{L}(\mathcal{H}) \},$$

(4.3)

where $M_j$ are the cartesian margins of $M$. Here we use the notation $\mathcal{M}U(\mathbb{T} \times \mathbb{N})$ to indicate explicitly the value space of the approximate joint observables.

The incompatibility of $\Phi$ and $\mathcal{N}$ implies that the point $(0, 0)$ is not in the set $\mathcal{M}U(\mathbb{T} \times \mathbb{N})$. On the other hand, if one of the errors is zero, then $M$ is a smearing of the exact margin $M_1$ or $M_2$. From the below proposition 2 we then conclude that if $d(M_1, \Phi) = 0$, that is, $M_1 = \Phi$, then $d(M_2, \mathcal{N})$ cannot be finite. On the other hand, if $M_2 = \mathcal{N}$, then $\pi/\sqrt{3} \leq d(M_1, \Phi) \leq \pi$ where

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6 See [7, theorem 11.1] for several alternative definitions.
the lower bound is attained with the kernel \( p_k = \ell, k \in \mathbb{N} \), and the upper bound with \( p_k = \delta_{\beta}, k \in \mathbb{N} \), where \( \beta \in [0, 2\pi) \).

The semigroup structure of the outcome space of the number measurements has thwarted our attempts to determine directly the set (4.3). However, we can still bound this set by enlarging the joint values set \( \mathbb{T} \times \mathbb{N} \) to \( \mathbb{T} \times \mathbb{Z} \), that is, studying instead of (4.3) the set \( \mathcal{M}(\mathbb{T} \times \mathbb{Z}) \). This case reduces to the case of position \( Q \) and momentum \( P \) (or angle and \( (\mathbb{Z} \cdot) \)-number) on \( \mathcal{H} = L^2(\mathbb{T}) \) studied in great detail in [14].

Let \( G^\sigma : B(\mathbb{T} \times \mathbb{Z}) \to \mathcal{L}(\mathcal{H}) \) be the covariant phase space observable generated by a state \( \sigma \in \mathcal{S}(\mathcal{H}) \) so that its margins are the smeared position and momentum observables \( Q_\sigma \ast Q \) and \( P_\sigma \ast P \), smeared by the position and momentum distributions \( Q_\sigma \) and \( P_\sigma \) in state \( \sigma \), respectively [14, 30]. The observable \( E^\sigma : B(\mathbb{T} \times \mathbb{Z}) \to \mathcal{L}(\mathcal{H}) \), defined as

\[
E^\sigma(X \times Y) = V^* G^\sigma(X \times Y) V,
\]

has then the smeared phase \( E^\sigma_\tau = Q_\sigma \ast \Phi \) and smeared number \( E^\sigma_\tau = P_\sigma \ast N \) as its margins. By remark 3, the errors now reduce to the preparation uncertainties of \( Q \) and \( P \) in state \( \sigma \)

\[
d(E^\sigma_\tau, \Phi) = \sqrt{Q_\sigma[2]} \quad \text{and} \quad d(E^\sigma_\tau, N) = \sqrt{P_\sigma[2]}.
\]

The following proposition bounds the error set \( \mathcal{M}(\mathbb{T} \times \mathbb{N}) \) by the bounds of the larger set \( \mathcal{M}(\mathbb{T} \times \mathbb{Z}) \).

**Proposition 2.** Let \( F : B(\mathbb{T} \times \mathbb{Z}) \to \mathcal{L}(\mathcal{H}) \) be an observable such that \( d(F_2, N) < \infty \). Then there exists a state operator \( \sigma \) on \( \mathcal{H} \), such that

\[
d(E^\sigma_\tau, \Phi) \leq d(F_1, \Phi) \quad \text{and} \quad d(E^\sigma_\tau, N) \leq d(F_2, N),
\]

where \( E^\sigma \) is given by (4.4). In particular, the boundary curve for the error set \( \mathcal{M}(\mathbb{T} \times \mathbb{Z}) \), which includes the set \( \mathcal{M}(\mathbb{T} \times \mathbb{N}) \), is the same as for \( Q \) and \( P \) on \( \mathcal{H} \), as characterised in [14].

The idea behind the proof is the following:

1. Starting from \( F \), construct an observable \( \mathcal{M} \) on \( \mathcal{H} \) in such a way that the errors of its margins with respect to \( Q \) and \( P \) reflect the original errors.
2. Average \( \mathcal{M} \) with respect to phase space translations so that the errors (actually, the state dependent errors) do not increase.
3. Project the averaged observable \( \mathcal{M} \) back to \( \mathcal{H} \) to get the desired result.

**Proof.** Let \( F : B(\mathbb{T} \times \mathbb{Z}) \to \mathcal{L}(\mathcal{H}) \) be an observable with \( d(F_2, N) < \infty \). Define an observable \( \mathcal{M} : B(\mathbb{T} \times \mathbb{Z}) \to \mathcal{L}(\mathcal{H}) \) via

\[
\mathcal{M}(X \times Y) = VF(X \times Y)V^* + \sum_{n=1}^\infty \ell(X)\langle \gamma|F_2(-Y)|\gamma\rangle|e_{-\gamma}\rangle\langle e_{-\gamma}|. \quad (4.5)
\]

We now proceed by calculating the error \( d(M_2, P) \) for the second margin \( M_2 \). By remark 3, it is sufficient to take the supremum over the eigenstates \( |e_k\rangle \) of \( P \), and we have the probabilities

\[
p^{\mathcal{M}_k}(Y) = \langle e_k|\mathcal{M}(X \times Y)e_k\rangle = \begin{cases} 
\langle k|F_2(Y)|k\rangle & \text{for } k \geq 0, \\
\langle -k|F_2(-Y)|-k\rangle - k & \text{for } k < 0.
\end{cases}
\]
Since $p_{k_0}^P = \delta_k$, we have
\[
d(p_{M^2_{k_0}}^o, p_{k_0}^P) = \left( \sum_{l=-\infty}^{\infty} |l - k|^2 p^M_{k_0}(|l|) \right)^{1/2}
\]
so that for $k \geq 0$,
\[
d(p_{M^2_{k_0}}^o, p_{k_0}^P) = \left( \sum_{l=-\infty}^{\infty} |l - k|^2 \langle k|F_2(|I|)|k \rangle \right)^{1/2} = d(p_{F^2_{k_0}}^o, p_{k_0}^N)
\]
whereas for $k < 0$ we have
\[
d(p_{M^2_{k_0}}^o, p_{k_0}^P) = \left( \sum_{l=-\infty}^{\infty} |l - k|^2 \langle -k|F_2(|I|)|k \rangle \right)^{1/2} = d(p_{F^2_{-k_0}}^o, p_{-k_0}^N)
\]
Since $d(F_2, N)$ is also obtained by calculating the supremum over the number states $|k\rangle$, we have that
\[
d(M_2, P) = \sup_{k \in Z} d(p_{M^2_{k_0}}^o, p_{k_0}^P) = \sup_{k \in N} d(p_{F^2_{k_0}}^o, p_{k_0}^N) = d(F_2, N).
\]
(4.6)
For the first margin, we do not get such an equality due to the trivial term coming from the last term in equation (4.5). However, we may restrict to the states
\[
S_+ (\mathcal{H}) = \{ \rho \in S(\mathcal{H}) | \langle e_k | \rho e_l \rangle = 0 \text{ for all } k < 0 \text{ or } l < 0 \}
\]
so that $V^* S_+ (\mathcal{H}) V = S(\mathcal{H})$. Since for any $\rho \in S_+ (\mathcal{H})$ we have $\text{tr} [\rho M_1(Y)] = \text{tr} [V^* \rho V F_1(X)]$ and $\text{tr} [\rho Q(X)] = \text{tr} [V^* \rho V \Phi(X)]$, we have, in particular, that
\[
d(F_1, \Phi) = \sup_{\rho \in S(\mathcal{H})} d(p_{F_1}^o, p_{\Phi}^o) = \sup_{\rho \in S_+ (\mathcal{H})} d(p_{F_1}^o, p_{\Phi}^o) = \sup_{\rho \in S_+ (\mathcal{H})} d(M_1^o, p_{\Phi}^o).
\]
(4.7)
The next step is to average the observable $M$ with respect to phase space translations, and to show that the averaged observable $\overline{M}$ satisfies
\[
\sup_{\rho \in S_+ (\mathcal{H})} d(p_{M}^o, p_{\Phi}^o) = \sup_{\rho \in S_+ (\mathcal{H})} d(p_{M}^o, p_{\Phi}^o) \quad \text{and} \quad d(\overline{M}_2, P) = d(M_2, P)
\]
(4.8)
We perform the averaging by using an invariant mean $m$ on $T \times Z$, see, for instance, [29]. For any trace class operator $T \in T(\mathcal{H})$ and any bounded continuous function $f : T \times Z \rightarrow \mathbb{C}$, define
\[
\Theta[T, f](\theta, k) = \text{tr} \left[ T W(\theta, k)^* M(f(\theta, k)) W(\theta, k) \right]
\]
where \( W(\theta, k) \) are the Weyl operators and \( f^{(\theta, k)} \) denotes the translate of \( f \). Then \( \Theta[T, f] : \mathbb{T} \times \mathbb{Z} \rightarrow \mathbb{C} \) is a bounded continuous function, and by standard arguments the formula

\[
\text{tr} [\mathcal{T}\overline{\mathcal{M}}(f)] = m(\Theta[T, f])
\]
determines a covariant phase space observable \( \overline{\mathcal{M}} : \mathcal{B}(\mathbb{T} \times \mathbb{Z}) \rightarrow \mathcal{L}(\tilde{\mathcal{H}}) \) (since \( d(M_2, P) = d(F_2, N) < \infty \) and \( d(M_1, \Phi) < \infty \) trivially by the compactness of \( \mathbb{T} \), the normalization of \( \overline{\mathcal{M}} \) is guaranteed [11]).

Let \( \rho \in \mathcal{S}(\tilde{\mathcal{H}}) \). Then by the Kantorovich duality, for any bounded continuous functions \( f, g : \mathbb{T} \rightarrow \mathbb{R} \) such that \( f(\theta) - g(\theta') \leq \delta(\theta, \theta')^2 \) we have

\[
\text{tr} [\rho(M_1(f) - Q(g))] \leq d(p^M_\rho, p^\mathcal{O}_\rho).
\]

Since the above class of functions is invariant with respect to translations, we have

\[
\text{tr} \left[ W(\theta, k)\rho W(\theta, k)^* (M_1(f^{(\theta)}) - Q(g^{(\theta)})) \right] = \text{tr} \left[ \rho W(\theta, k)^* M_1(f^{(\theta)}) W(\theta, k) \right] - \text{tr} [\rho Q(g)] 
\]

or equivalently,

\[
\text{tr} \left[ \rho W(\theta, k)^* M_1(f^{(\theta, k)}) W(\theta, k) \right] \leq \text{tr} [\rho Q(g)] + d(p^M_\rho, p^\mathcal{O}_\rho)
\]

where \( f_1(\alpha, l) = f(\alpha) \). By applying the invariant mean, we obtain

\[
\text{tr} \left[ \rho \mathcal{M}_1(f) \right] - \text{tr} [\rho Q(g)] \leq d(p^M_\rho, p^\mathcal{O}_\rho)
\]

for all \( f, g \). By taking the supremum over such functions we get

\[
d(p^M_\rho, p^\mathcal{O}_\rho) \leq d(p^M_\rho, p^\mathcal{O}_\rho)
\]

for all \( \rho \in \mathcal{S}(\tilde{\mathcal{H}}) \). The same holds also for the second margin. Hence, we conclude that equation (4.8) holds.

Since \( \overline{\mathcal{M}} \) is a covariant phase space observable, we know that \( \overline{\mathcal{M}} = G^\sigma \) for some \( \sigma \in \mathcal{S}(\tilde{\mathcal{H}}) \). We now set \( E^\sigma = V^* G^\sigma V = V^* \overline{\mathcal{M}} V \), so that

\[
d(E^\sigma_1, \Phi) = d(V^* \overline{\mathcal{M}} V, V^* Q V) = \sup_{\rho \in \mathcal{S}(\mathcal{L}(\mathcal{B}^d(\tilde{\mathcal{H}})))} d(p^\overline{\mathcal{M}}_{\rho V}, p^\mathcal{O}_{\rho V}) = \sup_{\rho \in \mathcal{S}_+(\tilde{\mathcal{H}})} d(p^M_\rho, p^\mathcal{O}_\rho)
\]

\[
\leq \sup_{\rho \in \mathcal{S}_+(\tilde{\mathcal{H}})} d(p^M_\rho, p^\mathcal{O}_\rho) = d(F_1, \Phi)
\]

and similarly \( d(E^\sigma_2, N) \leq d(F_2, N) \). \( \square \)

For any \( F \) for which \( d(F_2, N) \) is finite there is thus an \( E^\sigma \) such that \( d(E^\sigma_1, \Phi) \leq d(F_1, \Phi) \) and \( d(E^\sigma_2, N) \leq d(F_2, N) \), so that\(^7\)

\[
\text{Recall that due to the arc distance on } \mathbb{T}, \text{ the error } Q_\rho[2] = \int_{-\pi}^\pi \arccos dQ_\rho(\theta) \text{ so that also the operator } Q^2 = \int_{-\pi}^\pi \theta^2 dQ(\theta).\]

\(^7\)
\[
d(F_1, \Phi)^2 + d(F_2, N)^2 \geq d(E^\sigma_1, \Phi)^2 + d(E^\sigma_2, N)^2 = Q_2[2] + P_2[2] = \text{tr} \left[ \sigma(Q^2 + P^2) \right] \geq \tilde{E}_0,\
\]
where \( \tilde{E}_0 > 0 \) is the smallest eigenvalue of the oscillator energy operator \( Q^2 + P^2 \) in \( \tilde{H} \).

Though the existence of \( \tilde{E}_0 \) is known, we can only give its approximate value \( \tilde{E}_0 \approx 0.9996 \) (see appendix). If \( \psi \in \tilde{H} \) is a corresponding eigenvector then \( E^{\psi}|\psi\rangle \langle \psi| \) is an optimal joint measurement of \( \Phi \) and \( N \) with the value space \( T \times Z \). For a detailed analysis of the boundary curve of the convex hull of the monotone hull of the error sets \( \text{MU}(T \times Z) \) we refer to [14], in particular, its sections 4, 5 and 6.

**Remark 4.** By extending the value space of the approximate joint measurements from \( T \times N \) to \( T \times Z \), we are potentially enlarging also the initial error set. This leaves us with a question if the inclusion \( \text{MU}(T \times N) \subseteq \text{MU}(T \times Z) \) is a proper one. Natural candidates for optimal joint observables on \( T \times N \) are the observables \( E^\sigma \) whose support is contained in \( T \times N \). This amounts to the requirement that the generating operator \( \sigma \in S(\tilde{H}) \) is supported on the positive number states, that is, \( \langle e_k|\sigma e_l\rangle = 0 \) wherever \( k < 0 \) or \( l < 0 \). Optimizing over such states is equivalent to optimizing the preparation uncertainties for \( \Phi \) and \( N \) over all states \( \rho \in S(H) \). Based on numerical calculations, the uncertainties lead to a strict subset of \( \text{MU}(T \times Z) \) giving evidence that this inclusion could be a proper one. However, we are lacking an argument which would show that these are indeed optimal \( T \times N \) valued approximate joint observables. We are thus also left with the problem of proving or disproving that the optimal \( T \times N \) valued approximate joint observables for \( \Phi \) and \( N \) are given by those \( E^\sigma \) whose support is contained in \( T \times N \).

**5. Conclusions**

We have proved that number and canonical phase are complementary observables in the sense of definition (3.1) (theorem 1). Although it is known that the greatest lower bound of the effects \( \Phi(X) \) and \( N(Y) \) exists for arbitrary sets, it remains an open question to determine this operator for unbounded (not finite) \( Y \). We also bound the joint predictability of number and canonical phase beyond their probabilistic complementarity (proposition 1). Finally, proposition 2 bounds the possible measurement uncertainties appearing in an approximate joint measurement of number and canonical phase. As a result, we obtain a measurement uncertainty relation (4.9) for these observables. However, as pointed out in remark 4, it remains an open question if the given lower bound is optimal.

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**Appendix. Proof of the existence of the lower bound in (4.9)**

In this appendix we give a simple proof of the well-known fact that the operator \( P^2 + Q^2 \) in \( \tilde{H} \), as well as the operator \( N^2 + \Phi[2] \) in \( H \), has a discrete spectrum with a strictly positive lowest eigenvalue. For that end, we fix a separable Hilbert space (with the identity \( I \)) and assume that all operators (bounded or not) act in this space. We let \( B \) denote the unit ball of the Hilbert space.
Lemma A.3. Let $E$ and $F$ be bounded operators such that $0 \leq E \leq F \leq I$ and $\|I - E\| < 1$. Then $E$ and $F$ are invertible and $E^{-1} \geq F^{-1} \geq I$.

Proof. Since $\|I - E\| < 1$ it follows that $\lim_{s \to \infty} \|I - E\|^s = 0$, and $I + \sum_{k=1}^{\infty} (I - E)^k$ converges in the operator norm to a bounded operator. Moreover,

$$E^{k} \left[I - (I - E)^k\right] = I - (I - E)^k \to I$$

when $s \to \infty$, so that

$$E^{-1} = I + \sum_{k=1}^{\infty} (I - E)^k \geq I.$$

Indeed, $(I - E)^k = \int_0^{\|I - E\|} x^k \,d\mathcal{M}(x) \geq 0$, for all $k = 1, 2, \ldots$, where $\mathcal{M}$ is the spectral measure of $I - E \geq 0$. Since $0 \leq I - E \leq E$ it follows that $\|I - F\| = \sup_{\psi \in \mathcal{B}} \langle \psi | (I - F)\psi \rangle \leq \|I - E\| < 1$, and (similarly as above) one sees that $F$ is invertible. Let $F^{-1/2}$ (resp. $F^{-1/2}$) be the square root operators of $F$ (resp. $E^{-1} \geq I$). Now $G = F^{-1/2}E^{-1/2} \geq 0$ is invertible with the inverse $G^{-1} = F^{1/2}E^{-1}F^{1/2}$ and the condition $E \leq F$ is equivalent to $G \leq I$. Now $\|I - G\| < 1$ since otherwise (i.e. if $\|I - G\| = 1$) there would exist a sequence $\{\psi_n\}_{n=1}^{\infty} \subset \mathcal{B}$ of unit vectors such that

$$\lim_{n \to \infty} (\psi_n | (I - G)\psi_n) = 1,$$

that is, $\|G^{1/2}\psi_n\|^2 = (\psi_n | G\psi_n) \to 0$. $n \to \infty$, and thus $1 = \|\psi_n\| = \|G^{-1/2}G^{1/2}\psi_n\| \leq \|G^{-1/2}\| \|G^{1/2}\psi_n\| \to 0$ when $n \to \infty$. Hence, by the above calculation, $G^{-1} \geq I$ so that $E^{-1} = F^{-1/2}G^{-1/2}F^{-1/2} \geq F^{-1}$.

Proposition A.3. Let $T$ be a positive (possibly unbounded) selfadjoint operator with a purely discrete non-degenerate spectrum. Assume that its eigenvalues $0 \leq p_0 < p_1 < p_2 < \ldots$ are such that $\sum_n (1 + p_n)^{-1} < \infty$. Let $V$ be a positive bounded operator. Then the spectrum of $H = T + V$ is discrete. The lowest eigenvalue of $H$ is zero if and only if $p_0 = 0$ and $V\phi_0 = 0$ where $\phi_0 \neq 0$ is an eigenvector of $T$ related to the eigenvalue $p_0$

Proof. If the Hilbert space is finite dimensional then the proof is trivial so we consider only an infinite dimensional case. By assumption, $T = \sum_{n=0}^{\infty} p_n |\phi_n\rangle \langle \phi_n|$ for an orthonormal basis $\{\phi_n\}$. The domain of $T$ is $\mathcal{D} = \left\{\sum_{n=0}^{\infty} c_n |\phi_n\rangle | \sum_{n=0}^{\infty} p_n^2 |c_n|^2 < \infty\right\}$. Now $(T + I)^{-1} = \sum_{n=0}^{\infty} p_n' |\phi_n\rangle \langle \phi_n|$, with $p_n' = (1 + p_n)^{-1} \in (0, 1]$, is a positive trace class operator. Define $W = T + \|V\| I + I$ on $\mathcal{D}$ so that

$$W^{-1/2} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{p_n + \|V\| + 1}} |\phi_n\rangle \langle \phi_n|$$

is a bounded operator with the norm $\|W^{-1/2}\| = \sup_n (p_n + \|V\| + 1)^{-1/2} = (p_0 + \|V\| + 1)^{-1/2}$. Let $A = T + I$ and $B = T + V + I$ be positive operators defined on $\mathcal{D}$. Since $V \leq \|V\| I$ one gets $0 \leq \langle \psi | A\psi \rangle \leq \langle \psi | B\psi \rangle \leq \langle \psi | W\psi \rangle$, $\psi \in \mathcal{V} = \text{lin} \{\phi_n\} \subset \mathcal{D}$, or, since $W^{-1/2} \mathcal{V} \subset \mathcal{V}$,

$$0 \leq W^{-1/2} AW^{-1/2} \leq W^{-1/2} BW^{-1/2} \leq I$$

where, e.g. $W^{-1/2} BW^{-1/2}$ is a bounded operator determined uniquely by the corresponding bounded sesquilinear form $\mathcal{V} \times \mathcal{V} \ni (\varphi, \psi) \mapsto \langle W^{-1/2}\varphi | BW^{-1/2}\psi \rangle \in \mathbb{C}$. 

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Since $\|I - W^{-1/2}AW^{-1/2}\| = \sup_n \left( \frac{\|V\|}{\rho_n + \|V\| + 1} \right) = \frac{\|V\|}{\rho_1 + \|V\| + 1} < 1$, from lemma A.3, one sees that

$$\left( W^{-1/2}AW^{-1/2} \right)^{-1} = \left( W^{-1/2}BW^{-1/2} \right)^{-1} \geq I,$$

that is, $p_n' = \langle \phi_n | (T + I)^{-1} \phi_n \rangle \geq \langle \phi_n | (T + V + I)^{-1} \phi_n \rangle \geq (p_n + \|V\| + 1)^{-1} > 0$ and

$$\sum_{n=0}^{\infty} \langle \phi_n | W^{-1} \phi_n \rangle \leq \sum_{n=0}^{\infty} \langle \phi_n | (T + V + I)^{-1} \phi_n \rangle \leq \sum_{n=0}^{\infty} p_n' < \infty$$

showing that $(T + V + I)^{-1} \geq W^{-1}$ is a (positive) trace-class operator. Let

$$(T + V + I)^{-1} = \sum_{l=0}^{\infty} \lambda_l |\varphi_l\rangle \langle \varphi_l|$$

where $\{\varphi_l\}$ is an orthonormal basis and $\lambda_l \in (0, 1]$, $\sum_{l=0}^{\infty} \lambda_l < \infty$. Hence,

$$H = T + V = \sum_{l=0}^{\infty} q_l |\varphi_l\rangle \langle \varphi_l|$$

where $q_l = \lambda_l^{-1} - 1 \geq 0$. Finally, let $\phi \in \mathcal{D}$. Then, $H\phi = 0$ if and only if $0 = \langle \phi | H\phi \rangle = \langle \phi | T\phi \rangle + \langle \phi | V\phi \rangle$ if and only if $\langle \phi | T\phi \rangle = 0 = \langle \phi | V\phi \rangle$ if and only if $T\phi = 0 = V\phi$. \[\square\]

Note that, in the context of the above proposition, all operators $T + cV, c > 0$, have discrete spectra, and their spectra have non-zero smallest eigenvalues (i.e. positive spectra) if $T + V$ has a positive spectrum.

In either case, $\hat{H} = P^2 + Q^2 = P^2 + \int_{-\pi}^{\pi} \theta^2 dQ(\theta)$ (in $\tilde{\mathcal{H}}$) or $H = N^2 + \Phi[2] = N^2 + \int_{-\pi}^{\pi} \theta^2 d\Phi(\theta)$ (in $\mathcal{H}$), the assumptions of proposition A.3 are satisfied; in particular, both of the positive operators $Q^2$ or $\Phi[2]$ have a purely continuous spectrum (with no eigenvalues): $\sigma(Q^2) = \sigma(\Phi[2]) = [0, \pi^2]$. Hence both operators $\hat{H}, H$ have strictly positive lowest eigenvalues $E_0, E_0$, respectively. Also, this follows directly from proposition A.3 by noting that $\langle \psi_0 | Q^2 \psi_0 \rangle = \langle \psi_0 | \Phi[2] \psi_0 \rangle = \int_{-\pi}^{\pi} \theta^2 d\Phi(\theta)/(2\pi) > 0$, i.e. $P^2 \psi_0 \neq 0$ but $Q^2 \psi_0 \neq 0$ and $N^2 \psi_0 \neq 0$ but $\Phi[2] \psi_0 \neq 0$. Numerically, $E_0 \approx 0.9996...$ associated with the (normalized) eigenvector $\psi_{\text{min}} = \sum_{l=-\infty}^{\infty} c_l \psi_l$ where $c_0 \approx 0.7518, c_{\pm 1} \approx 0.4550, c_{\pm 2} \approx 0.1017, c_{\pm 3} \approx 0.0083, c_{\pm 4} \approx 0.0002, \ldots, c_{\pm N} \approx 0.0002$. Moreover, $E_0 \approx 1.5818...$, with the eigenvector $\psi_{\text{min}} \approx 0.7276\{0 | + 0.6632\{1 | + 0.1745\{2 | + 0.0167\{3 | + 0.0002\{4 | + \ldots$. To conclude, if $M : B(\mathbb{T} \times \mathbb{Z}) \to \mathcal{L}(\mathcal{H})$ is any approximate joint measurement of $\Phi$ and $N$, with $d(M_2, N) < \infty$, then

$$d(M_1, \Phi) + d(M_2, N) \geq \tilde{E}_0 \approx 1.$$

It remains, however, an open question if the eigenvalue $E_0$ of $N^2 + \Phi[2]$ bounds the error sum $d(M_1, \Phi) + d(M_2, N)$ for the $T \times N$-valued approximate joint measurements of phase and number.

**Remark A.5.** The above numerical results for the smallest eigenvalues and the corresponding eigenvectors is based on the following facts: let $H = T + V, T = \sum_{n=0}^{\infty} p_n |\phi_n\rangle \langle \phi_n|$ be as in proposition A.3 (we assume that the Hilbert space is infinite-dimensional). Let $C_{\psi_{\text{min}}} \geq 0$ be the lowest eigenvalue of $H$ with the (normalized) eigenvector $\psi_{\text{min}}$. Let $P_h = \sum_{n=0}^{\infty} |\phi_n\rangle \langle \phi_n|$
so that $P_k \to I$, $k \to \infty$, with respect to the strong (and weak) operator topology. Denote $H_k = P_k H P_k \geq 0$ and let $\alpha_k$ be the smallest eigenvalue of the ‘finite positive matrix’ $H_k$. Let $\eta_k \in \mathcal{B}$, $P_k \eta_k = \eta_k$, be the corresponding eigenvector of $H_k$ that is, $H_k \eta_k = \alpha_k \eta_k$. Since $\alpha_k = \inf\{\langle \psi | H_k \psi \rangle \mid \psi \in \mathcal{B}, P_k \psi = \psi \}$ and $P_{k+1} P_k = P_k$ one gets

$$C_{\psi_{\min}} \leq \langle \eta_{k+1} | H_{k+1} \eta_{k+1} \rangle = \alpha_{k+1} \leq \langle P_k \psi_{\min} | H_k P_k \psi_{\min} \rangle \left\| P_k \psi_{\min} \right\|^2.$$

Since $\lim_{k \to \infty} \| P_k \psi_{\min} \| = 1$, to get $\lim_{k \to \infty} \alpha_k = C_{\psi_{\min}}$, one is left to show that (when $k \to \infty$)

$$\langle P_k \psi_{\min} | H_k P_k \psi_{\min} \rangle = \langle P_k \psi_{\min} | H P_k \psi_{\min} \rangle \to \langle \psi_{\min} | H \psi_{\min} \rangle = C_{\psi_{\max}}$$

or$^8$ that $H P_k \psi_{\min} \to H \psi_{\min} = C_{\psi_{\max}} \psi_{\max}$. But this is obvious (see the end of the proof of the proposition):

$$\left\| H \psi_{\min} - H P_k \psi_{\min} \right\|^2 = \sum_{l=1}^{\infty} (q_l)^2 \left( |\langle \varphi_l | (I - P_k) \psi_{\min} \rangle| \right)^2 \to 0 \quad (k \to \infty).$$

We have proved that $\lim_{k \to \infty} \alpha_k = C_{\psi_{\max}}$, i.e. $\lim_{k \to \infty} \langle \eta_k | H \eta_k \rangle = \langle \psi_{\min} | H \psi_{\min} \rangle$. Hence, one can numerically solve the smallest eigenvalues $\alpha_k$ of the finite matrices $H_k$. When $k$ is large enough one gets $C_{\psi_{\max}} \approx \alpha_k$.

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$^8$ $(\psi \cdots \psi) = (P_k \psi \cdots P_k \psi) + (P_k^+ \psi \cdots P_k \psi) + (P_k \psi \cdots P_k^+ \psi) + (P_k^+ \psi \cdots P_k^+ \psi)$ where $P_k^+ = I - P_k$. 
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