Unconditionally energy-stable schemes based on the SAV approach for the
inductionless MHD equations

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Abstract

In this paper, we consider numerical approximations for solving the inductionless magnetohydrodynamic (MHD) equations. By utilizing the scalar auxiliary variable (SAV) approach for dealing with the convective and coupling terms, we propose some first- and second-order schemes for this system. These schemes are linear, decoupled, unconditionally energy stable, and only require solving a sequence of differential equations with constant coefficients at each time step. We further derive a rigorous error analysis for the first-order scheme, establishing optimal convergence rates for the velocity, pressure, current density and electric potential in the two-dimensional case. Numerical examples are presented to verify the theoretical findings and show the performances of the schemes.

Keywords: inductionless MHD equations, SAV, energy stable, decoupled, error estimates

1. Introduction

The incompressible MHD describes the dynamic behavior of an electrically conducting fluid under the influence of a magnetic field. It has been widely used in many science and engineering applications, such as liquid metal cooling for nuclear reactors, and sustained plasma confinement for controlled thermonuclear fusion, see \cite{1, 2, 3}. Mathematically, the most frequently used model is obtained by coupling the Navier-Stokes equations for hydrodynamics with the Maxwell equations for electromagnetism. However, in most terrestrial applications, the magnetic Reynolds number of MHD flows is small. Consequently, the magnetic induction can usually be negligible compared with the external magnetic field and the electric field is considered to be quasi-static. The MHD equations with this simplification are referred as the inductionless MHD equations. Much effort has been spent on theoretical analysis and mathematical modeling of the inductionless MHD equations, see \cite{4, 5} and the references therein.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded domain with Lipschitz-continuous boundary $\Gamma := \partial \Omega$. In this paper, we consider the incompressible inductionless MHD equations as follows,

\begin{align}
    u_t - R_e^{-1} \Delta u + u \cdot \nabla u + \nabla p - \kappa J \times B &= 0 \quad \text{in } \Omega \times (0, T], \\
    \text{div} u &= 0 \quad \text{in } \Omega \times (0, T], \\
    J + \nabla \phi - u \times B &= 0 \quad \text{in } \Omega \times (0, T], \\
    \text{div} J &= 0 \quad \text{in } \Omega \times (0, T],
\end{align}

where $T > 0$ is the terminal time, $u$ is the fluid velocity, $p$ the hydrodynamic pressure, $J$ the current density, $\phi$ the electric potential, $R_e$ is the fluid Reynolds number, $\kappa$ is the coupling number. Function $B$ is the applied magnetic field which is assumed to be given. The system is considered in conjunction with the following initial and boundary conditions,

\begin{align}
    u(x, 0) &= u^0(x) \text{ in } \Omega, \\
    u &= 0, \quad J \cdot n = 0 \text{ on } \Gamma \times (0, T],
\end{align}

where the initial value satisfies $\nabla \cdot u_0 = 0$ and $n$ is the unit outer normal vector on $\Gamma$.

Numerical solving the inductionless MHD model has drawn a considerable amount of attention. Since it replaces Maxwell’s equations with Poisson’s equation for the electric potential, numerical solution is more economic compared with the full MHD model. In particular, the inductionless MHD model is further simplified as the reduced MHD model by eliminating the current density $J$. In \cite{6, 7}, Yuksel et al. studied the error analysis for both semi-discrete and fully discrete finite element approximate of the reduced MHD model. In \cite{8, 9, 10}, Ni and his collaborators studied a class
of consistent and charge-conservative finite volume scheme for the inductionless MHD equations on both structured and unstructured meshes. In [11], Planas et al. proposed a stabilized finite element method to solve the inductionless MHD problem, in which the time discretization is based on the Backward-Euler method. In 2019, Li et al. [12] proposed a fully discrete and charge-conservative finite element method and provided a plain convergence analysis. Later, Long further presented optimal error estimates for both the semi-discrete and full-discrete scheme [13]. Most of the schemes mentioned above are coupled-type where they need to assemble and solve a multi-physics and large system at each time step. Thus, it could be computationally expensive in numerical computation, especially in three dimensions.

To address this issue, some decoupled methods have been investigated attractive in the literature. Since decoupled methods usually solve the coupled problem by successively solving the sub-physics problems at each step and many efficient solvers can be used for each of them. However, decoupled methods are more computationally economical but may lose some stability. Thus, it is desirable to design decoupled methods while preserving the energy stability, in the sense that the discrete energy dissipation laws hold. For the reduced MHD model, Layton et al. [14] introduced two partitioned methods and studied the stability, where the first order method is shown to be unconditionally stable and the second order method is shown to be conditionally stable. In 2021, Zhang et al. [15] proposed and analyzed a decoupled, unconditionally energy stable and charge-conservative finite element method. By adding a first-order stabilized term to the magnetic problem and making some subtle implicit-explicit treatments for coupling terms, their scheme can decouple the computation of the magnetic problem from the fluid problem while ensuring the energy stability. In these works, the nonlinear terms are treated either implicitly or semi-implicitly, so that one needs to solve a nonlinear system or some linear systems with variable coefficients at each time step. Thus, it is desirable to be able to treat the nonlinear term explicitly while maintaining energy stability. With such treatment, the schemes only require the solution of linear systems with constant coefficients upon discretization, and thus are very efficient and popular for dynamical simulations.

Recently, SAV based schemes have gained much attention recently due to their efficiency, flexibility and accuracy. The SAV approach was first studied in [16, 17] to construct efficient schemes for gradient flows. Nowadays, it has been a powerful approach to develop energy stable numerical schemes for general dissipative systems, such as Navier-Stokes equations [18, 19], magnetohydrodynamic equations [20, 21] and Cahn-Hilliard-Navier-Stokes equations [22, 23]. Based on an auxiliary variable associated with the total system energy, \( q(t) = \sqrt{E_{\text{NS}}(t)} \) with \( E_{\text{NS}}(t) = \frac{1}{2} \| u \|^2 \), Dong et al. [18] constructed a numerical scheme for the NS equations. Within each time step, the scheme involves the computations of two generalized Stokes equations with constant coefficient matrices, together with a nonlinear algebraic equation for the auxiliary variable. To address the theoretical and practical issues form the nonlinear algebraic equation, Li et al. [19] proposed and analyzed some first- and second-order pressure correction schemes using the SAV approach for the NS equations, where \( q(t) = \exp(-t/T) \). Later, they extend the proposed approach and theoretical findings to the MHD equations in [20] and the Cahn-Hilliard-Navier-Stokes system in [23] for dealing with the nonlinear and coupling terms that satisfy the “zero-energy-contribution” feature. Meanwhile, Yang [22, 24] designed a series of linear and energy stable schemes for the flow-coupled phase-field models. In these works, the auxiliary variable for the NS equations is simpler, \( q(t) = 1 \). Following the same idea, this approach was later extended to devise fully decoupled finite element schemes for the MHD equations in [21, 25].

The purpose of this paper is to propose and analyze some SAV schemes for the inductionless MHD equations. By utilizing the SAV approach for dealing with the convective term and coupling terms, some first- and second-order schemes are constructed for this system. These schemes are linear, decoupled, unconditionally energy stable, and only require solving a sequence of differential equations with constant coefficients at each time step. Thus, they are very efficient and easy to implement. We further establish rigorous unconditional energy stability and error analysis for the first-order scheme in the two-dimensional case. Some numerical experiments are provided to confirm the predictions of the theory and demonstrate the efficiency of the proposed schemes.

While the construction of the SAV schemes for the inductionless MHD equations is quite straightforward, it is much more difficult to carry out the error analysis as we have to deal with the issues due to the non-local coupling between the SAV and other variables, and the explicit treatment of the convective terms and coupling terms. It is also remarkable that while the error analysis is somewhat similar to the ones in [19, 20], the extension of error analysis for the inductionless MHD equations is still non-trivial. On the one hand, compared to the Navier-Stokes equations, the error analysis for the inductionless MHD equations is much more involved due to the coupling terms. On the other hand, compared to the full MHD equations, the inductionless MHD equations is a hybrid system. More precisely, the fluid problem is unsteady while the electromagnetic problem is steady. The lack of the derivative term to time in electromagnetic problem makes the error analysis more complicated and tough. Therefore, more delicate analyses are needed for the error estimates.

The paper is organized as follows. In Section 2, we introduce some notations and present the energy estimate for the inductionless MHD equations. In Section 3, we propose the SAV schemes and prove the unconditional stability. In Section 4, we carry out a rigorous error analysis for the first-order scheme in the two-dimensional case. In Section 5, we present some numerical experiments. In Section 6, we conclude with a few remarks.
2. Preliminaries

We begin with introducing some notations and Sobolev spaces. For all $1 \leq q \leq \infty$, $L^q(\Omega)$ for the $q$-integrable function space with the norm $\| \cdot \|_{0,q}$. Particularly, $L^2(\Omega)$ is equipped with the inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$. The subspace of $L^2(\Omega)$ with zero mean value over $\Omega$ is further denoted as $L^2_0(\Omega)$. For all $m \in \mathbb{N}^+, 1 \leq q \leq \infty$, let $W^{m,q}(\Omega)$ denote the standard Sobolev space equipped with the standard Sobolev norm $\| \cdot \|_{m,q}$. For $q = 2$, we write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and its corresponding norm is $\| \cdot \|_m$. Let $H(\text{div}, \Omega)$ be the subspace of $L^2(\Omega)$ with square integrable divergence, the norm is defined by $\| \cdot \|_{\text{div}}$. Spaces $H^1_0(\Omega)$ and $H_0(\text{div}, \Omega)$ denote their subspaces with vanishing traces and vanishing normal traces, respectively. For a given Sobolev space $X$, we write $L^q(0, T; X)$ for the Bochner space and its norm is written by $\| \cdot \|_{L^q(0, T; X)}$. Here and what follows, we use $C$ to denote generic positive constants independent of the discretization parameters, which may take different values at different places.

For convenience, we introduce some notations for function spaces

$$X := H^1_0(\Omega), \quad V := \{ v \in X : \nabla \cdot v = 0 \}, \quad Y = L^2_0(\Omega), \quad D := H_0(\text{div}, \Omega), \quad S = L^2_0(\Omega).$$

We will use the following trilinear form,

$$b(u, v, w) = (u \cdot \nabla v, w) \quad \forall u, v, w \in X.$$ 

It is easy to see that the trilinear form $b(\cdot, \cdot, \cdot)$ is a skew-symmetric with respect to its last two arguments,

$$b(u, v, w) = -b(u, w, v) \quad \forall u \in V, \ v, w \in X,$$

and

$$b(u, v, v) = 0 \quad \forall u \in V, \ v \in X. \tag{5}$$

Now, we are in a position to establish the energy estimate for the inductionless MHD system. By taking the $L^2$-inner product of $u$ with (1a) and using the integration by parts and (1b), we get

$$\frac{1}{2} \frac{d}{dt} \| u \|^2 + R_e^{-1} \| \nabla u \|^2 + \kappa(J \times B, u) = 0. \tag{6}$$

By taking the $L^2$-inner product of $\kappa J$ with (1c) and using the integration by parts and (1d), we have

$$\kappa \| J \|^2 - \kappa(u \times B, J) = 0. \tag{7}$$

By combining (6)-(7) and using (5), we obtain the law of energy dissipation that reads as,

$$\frac{d}{dt} E(t) = -R_e^{-1} \| \nabla u \|^2 - \kappa \| J \|^2 \quad \text{with} \quad E(t) = \frac{1}{2} \| u \|^2. \tag{8}$$

The energy law describes the variation of the total energy caused by energy conversion. Since the induced magnetic field is neglected and the electric field is considered to be quasi-static, the total energy $E$ only consists of the fluid kinetic energy $\frac{1}{2} \| u \|^2$. The dissipation of $E$ stems from the friction losses $R_e^{-1} \| \nabla u \|^2$ and the Ohmic losses $\kappa \| J \|^2$. The above proof to obtain the law of energy dissipation (8) lies on the following two identities,

$$(u \cdot \nabla u, u) = 0, \quad \kappa(J \times B, u) - \kappa(u \times B, J) = 0. \tag{9}$$

These two equities can be regarded as the contribution of two types of coupling terms to the total free energy of the system is zero. These unique “zero-energy-contribution” property will be used to design decoupling type numerical schemes.

3. The SAV schemes

In this section, we first reformulate the inductionless MHD model into an equivalent system with SAV. Then, we construct first- and second-order SAV schemes and prove that they are unconditionally energy stable.

Let $\{ t^n = n \tau : n = 0, 1, \ldots, N \}$, $\tau = T/N$, be an equidistant partition of the time interval $[0, T]$. We denote $(\cdot)^n$ as the variable $(\cdot)$ at time step $n$. For any function $v(x, t)$, define

$$\delta_tv^{n+1} = \frac{v^{n+1} - v^n}{\tau}, \quad \delta_t^2v^{n+1} = \frac{3v^{n+1} - 4v^n + v^{n-1}}{2\tau}, \quad \dot{v}^{n+1} = 2v^n - v^{n-1}$$

In particular, when $n = 0$, we denote

$$\delta_t^2v^1 = \frac{v^1 - v^0}{\tau}, \quad \dot{v}^1 = v^0. \tag{10}$$
3.1. Reformulated system

Inspired by [19], we introduce a scalar auxiliary variable \( q(t) \),

\[
q(t) = \exp\left( -\frac{t}{T} \right).
\] (11)

Note that \( q(t) \) is a scalar-valued number, not a field function. This function will serve as the scalar auxiliary variable. By taking the derivative of (11) with respect to \( t \), we obtain

\[
\frac{dq}{dt} = -\frac{q}{T}.
\]

Observed that this a linear and dissipative ordinary differential equation for the scalar auxiliary variable. This feature is vital for designing unconditionally energy-stable and linear schemes.

In light of equation \( q \exp\left( \frac{t}{T} \right) = 1 \) and (9), we rewrite the system (1) into the following form

\[
\begin{align*}
\mathbf{u}_t - \mathbf{R}^{-1}_e \Delta \mathbf{u} + \nabla p + q \exp\left( \frac{t}{T} \right) \mathbf{u} \cdot \nabla \mathbf{u} - \kappa \mathbf{J} \times \mathbf{B} &= 0, \\
\text{div} \mathbf{u} &= 0, \\
\mathbf{J} + \nabla \phi - q \exp\left( \frac{t}{T} \right) \mathbf{u} \times \mathbf{B} &= 0, \\
\text{div} \mathbf{J} &= 0,
\end{align*}
\] (12a)-(12d)

\[
\frac{dq}{dt} + \frac{q}{T} - \exp\left( \frac{t}{T} \right) \left( \mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{u} \right) - \kappa \left( \mathbf{u} \times \mathbf{B}, \mathbf{J} \right) - \kappa \left( \mathbf{J} \times \mathbf{B}, \mathbf{u} \right) = 0,
\] (12e)

Note that in the reformulated system, \( q(t) \) is treated as an approximation of \( \exp\left( -\frac{t}{T} \right) \) and is computed by solving this system of equations, not by using equation (11). The initial condition for \( q(t) \) is set as \( q(0) = 1 \). The last term in the equation for \( q \) of (12e) is added to balance the nonlinear term and coupling term in (12) in the discretized case. We next focus on this reformulated system, and present unconditionally energy-stable schemes for this system.

Theorem 3.1. The reformulated system (12) admits the following law of energy dissipation,

\[
\frac{d}{dt} E_{\text{SAV}}(t) = -R^{-1}_e \|\nabla \mathbf{u}\|^2 - \kappa \|\mathbf{J}\|^2 - \frac{1}{T} |q|^2 \quad \text{with} \quad E_{\text{SAV}}(t) = \frac{1}{2} \|\mathbf{u}\|^2 + \frac{1}{2} |q|^2.
\] (13)

Proof. Taking the \( L^2 \)-inner product of \( \mathbf{u} \) with (12a), using the integration by parts and (12b), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|^2 + R^{-1}_e \|\nabla \mathbf{u}\|^2 + q \exp\left( \frac{t}{T} \right) \left( \mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{u} \right) + q(t) \exp\left( \frac{t}{T} \right) \kappa \left( \mathbf{J} \times \mathbf{B}, \mathbf{u} \right) = 0.
\] (14)

Taking the \( L^2 \)-inner product of \( \kappa \mathbf{J} \) with (12c), using the integration by parts and (12d), we obtain

\[
\kappa \|\mathbf{J}\|^2 - q \exp\left( \frac{t}{T} \right) \kappa \left( \mathbf{u} \times \mathbf{B}, \mathbf{J} \right) = 0.
\] (15)

Multiplying \( q \) with (12e) leads to

\[
\frac{1}{2} \frac{d}{dt} |q|^2 + \frac{1}{T} |q|^2 - q \exp\left( \frac{t}{T} \right) \left( \mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{u} \right) - \kappa \left( \mathbf{u} \times \mathbf{B}, \mathbf{J} \right) - \kappa \left( \mathbf{J} \times \mathbf{B}, \mathbf{u} \right) = 0.
\] (16)

By combining (14)-(16), we derive (13).

Remark 3.1. In this paper, the scalar auxiliary variable is only a time-dependent function \( q(t) = \exp(-t/T) \) not a energy-related function. With this treatment, the algebraic equation for the scalar auxiliary variable is linear and uni-solvent. Moreover, the scalar auxiliary variable of this type admits a general form, \( q(t) = C_{q,0} \exp(-C_{q,1} t/T) \) with \( C_{q,0} \neq 0 \) and \( C_{q,1} \geq 0 \). We refer to [26] for more details about this extension.
3.2. First-order scheme

A first-order scheme for solving the system (12) can be readily derived by the backward Euler method. For all \( n \geq 0 \), we compute \((u^{n+1}, p^{n+1}, J^{n+1}, \phi^{n+1})\) by solving

\[
\begin{align*}
\delta_t u^{n+1} - R^{-1}_e \Delta u^{n+1} + \nabla p^{n+1} + q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) (u^n \cdot \nabla u^n - \kappa J^n \times B^{n+1}) &= 0, \tag{17a} \\
\text{div} u^{n+1} &= 0, \tag{17b} \\
J^{n+1} + \nabla \phi^{n+1} - q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) (u^n \times B^{n+1}) &= 0, \tag{17c} \\
\text{div} J^{n+1} &= 0, \tag{17d} \\
\delta_t q^{n+1} + \frac{q^{n+1}}{T} - \exp\left( \frac{t^{n+1}}{T} \right) \left( (u^n \cdot \nabla u^n, u^{n+1}) - \kappa (u^n \times B^{n+1}, J^{n+1}) - \kappa (J^n \times B^{n+1}, u^{n+1}) \right) &= 0. \tag{17e}
\end{align*}
\]

**Remark 3.2.** The initial data \( J^0 \) is obtained by solving (1c)-(1d) at \( t = 0 \). Namely, \( J^0 \) is a part of the solution to

\[
\begin{align*}
J^0 + \nabla \phi^0 - u^0 \times B^0 &= 0, \tag{18a} \\
\text{div} J^0 &= 0. \tag{18b}
\end{align*}
\]

A more reliable way is to compute the system at \( t = t_1 \) using a coupled scheme [12] or decoupled scheme [15] to obtain \((u^1, p^1, J^1, \phi^1)\). Then the proposed scheme is implemented from \( t = t_2 \) and initialized by the solution at \( t = t_1 \).

**Remark 3.3.** It is worth noting that the function \( B \) is a given external magnetic field in this paper. In the numerical scheme, the symbol \( B^{n+1} \) used only indicates that its value may change with time, not that it must be computed within. This remark also applies to the second-order scheme (28).

First of all, we prove the unconditionally energy stability of the scheme as follows.

**Theorem 3.2.** The scheme (17) is unconditionally energy stable in the sense that the following energy estimate

\[
\delta_t E_{EL}^{n+1} \leq -R^{-1}_e \left\| \nabla u^{n+1} \right\|^2 - \kappa \left\| J^{n+1} \right\|^2 - \frac{1}{T} \left\| q^{n+1} \right\|^2 \quad \forall n \geq 0,
\]

holds, where \( E_{EL}^{n+1} := \frac{1}{2} \left\| u^{n+1} \right\|^2 + \frac{1}{2} \left\| q^{n+1} \right\|^2 \).

**Proof.** Taking the inner product of (17a) with \( u^{n+1} \), and using the identity 2 \((a - b, a) = a^2 - b^2 + (a - b)^2\), it yields,

\[
\begin{align*}
\left\| u^{n+1} \right\|^2 - \left\| u^n \right\|^2 + \left\| u^{n+1} - u^n \right\|^2 + R^{-1}_e \left\| \nabla u^{n+1} \right\|^2 \\
+ q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) (u^n \cdot \nabla u^n, u^{n+1}) - q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) \kappa (J^n \times B^{n+1}, u^{n+1}) &= 0.
\end{align*}
\]

Taking the inner product of (17c) with \( \kappa J^{n+1} \), we obtain

\[
\kappa \left\| J^{n+1} \right\|^2 - q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) (u^n \times B^{n+1}, J^{n+1}) = 0. \tag{21}
\]

Multiplying (17e) by \( q^{n+1} \) leads to

\[
\begin{align*}
\left\| u^{n+1} \right\|^2 - \left\| u^n \right\|^2 + \left\| u^{n+1} - u^n \right\|^2 + R^{-1}_e \left\| \nabla u^{n+1} \right\|^2 \\
- q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) \left( (u^n \cdot \nabla u^n, u^{n+1}) - \kappa (u^n \times B^{n+1}, J^{n+1}) - \kappa (J^n \times B^{n+1}, u^{n+1}) \right) &= 0 
\end{align*}
\]

By taking the summations of (20), (21) and (22), we obtain

\[
\begin{align*}
\frac{\left\| u^{n+1} \right\|^2 - \left\| u^n \right\|^2 + \left\| u^{n+1} - u^n \right\|^2}{2T} + R^{-1}_e \left\| \nabla u^{n+1} \right\|^2 \\
+ \kappa \left\| J^{n+1} \right\|^2 + \left\| q^{n+1} \right\|^2 - \left\| q^n \right\|^2 + \left\| q^{n+1} - q^n \right\|^2 + \frac{1}{T} \left\| q^{n+1} \right\|^2 &= 0
\end{align*}
\]

This yields (19).
Thus, we arrive at the final solution algorithm. It involves the following steps:

1. Solve equations (24) and (25) for \( u_{i}^{n+1}, p_{i}^{n+1} \) and \( J_{i}^{n+1}, \phi_{i}^{n+1} \), \( i = 1, 2 \).
2. Compute \( q^{n+1} \) by (26).
3. Compute \( u_{i}^{n+1}, p_{i}^{n+1}, J_{i}^{n+1}, \phi_{i}^{n+1} \) by (26), compute \( q^{n+1} \) by (27) and (23).

In summary, at each time step, we only need to solve two generalized Stokes equations in (24) and one Darcy equations in (25) with constant coefficients plus a linear algebraic equation (27) at each time step. Hence, the scheme is very efficient in practical calculations.

Remark 3.4. The non-homogeneous boundary conditions \( u = u_b, J \cdot n = J_b \) on \( \Gamma \) instead of (3) can be handled by making several modifications. We only need to slightly modify the SAV variable \( q(t) \) to include the boundary integration as follows,

\[
\frac{d q}{dt} + \frac{q}{T} - \exp \left( \frac{t}{T} \right) \left( u \cdot \nabla u, u \right) - \frac{1}{2} \int_{\Gamma} (u_b \cdot n) |u_b|^2 \, ds - \kappa (u \times B^{n+1}, J) - \kappa (J \times B^{n+1}, n) + \frac{1}{2} \int_{\Gamma} (u_b \cdot n) |u_b|^2 \, ds - \kappa (u \times B^{n+1}, J) - \kappa (J \times B^{n+1}, n) = 0.
\]

In the decoupled procedures of (24)-(25), we need to impose the following boundary conditions on \( \Gamma, u_{i}^{n+1} = u_b, J_{i}^{n+1} \cdot n = J_b \) and \( u_{i}^{n+1} = 0, J_{i}^{n+1} \cdot n = 0 \).

Finally, we state that the scheme (17) is well-defined. That is to say, it is uniquely solvable at each time step.
The scheme (28) is unconditionally energy stable in the sense that the following energy estimate holds, where

\[ -\left( u^n \cdot \nabla u^n, u_2^{n+1} \right) + \kappa \left( J^n \times B^{n+1}, u_2^{n+1} \right) = \frac{\|u_2^{n+1}\|^2}{\tau} + R_e^{-1} \|\nabla u_2^{n+1}\|^2. \]

By taking the \( L^2 \)-inner product of the third equation in (24) with \( u_2^{n+1} \) and using the last equation in (24), we get

\[ \kappa \left( u^n \times B^{n+1}, J_2^{n+1} \right) = \kappa \|J_2^{n+1}\|^2. \]

Combining all the estimates above, we conclude

\[ \frac{T + \tau}{T} - \exp \left( \frac{2n+1}{T} \right) A_2 = \frac{T + \tau}{T} + \exp \left( \frac{2n+1}{T} \right) \left( \frac{\|u_2^{n+1}\|^2}{\tau} + R_e^{-1} \|\nabla u_2^{n+1}\|^2 + \kappa \|J_2^{n+1}\|^2 \right) > 0. \]

This yields the existence and uniqueness of \( S_2^{n+1} \). Hence, the scheme (17) admits a unique solution. The proof is finished.

\[ \square \]

3.3. Second-order scheme

A second-order scheme based on backward differential formula (BDF) for (12) is constructed as follows. For all \( n \geq 1 \), we compute \( (u^{n+1}, p^{n+1}, J^{n+1}, \phi^{n+1}) \) by solving

\[ \delta^2 u^{n+1} - \nu \Delta u^{n+1} + \nabla p^{n+1} + q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) \left( \hat{u}^{n+1} \cdot \nabla \hat{u}^{n+1} - \kappa \hat{J}^{n+1} \times B^{n+1} \right) = 0, \]

\[ \text{div} u^{n+1} = 0, \]

\[ J^{n+1} + \nabla \phi^{n+1} - q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) (\hat{u}^{n+1} \times B^{n+1}) = 0, \]

\[ \text{div} J^{n+1} = 0, \]

\[ \delta t q^{n+1} + \frac{q^{n+1}}{T} \]

\[ - \exp \left( \frac{t^{n+1}}{T} \right) \left( \hat{u}^{n+1} \cdot \hat{u}^{n+1}, u^{n+1} \right) - \kappa \left( \hat{u}^{n+1} \times B^{n+1}, J^{n+1} \right) - \kappa \left( J^{n+1} \times B^{n+1}, u^{n+1} \right) = 0. \]

From (10), it is easy to see that when \( n = 0 \), \( (u^1, p^1, J^1, \phi^1, q^1) \) is computed by the first-order scheme described in (17). Now we derive the energy stability of the above scheme.

Theorem 3.4. The scheme (28) is unconditionally energy stable in the sense that the following energy estimate holds, where

\[ \delta t E_{BDP}^{n+1} \leq -R_e^{-1} \|\nabla u^{n+1}\|^2 - \kappa \|J^{n+1}\|^2 - \frac{1}{T} |q^{n+1}|^2 \quad \forall n \geq 0, \]

\[ E_{BDP}^{n+1} := \frac{1}{2} \left( \|u^{n+1}\|^2 + \|2u^{n+1} - u^n\|^2 \right) + \frac{1}{2} \left( \|q^{n+1}\|^2 + \|2q^{n+1} - q^n\|^2 \right). \]

Proof. Taking the inner product of (28a) with \( u^{n+1} \), and using the identity

\[ 2(3a - 4b + c, a) = |a|^2 - |b|^2 + |2a - b|^2 - |2b - c|^2 + |a - 2b + c|^2, \]

it yields,

\[ \frac{1}{4T} \left( \|u^{n+1}\|^2 + \|2u^{n+1} - u^n\|^2 - \|u^n\|^2 - \|2u^n - u^{n-1}\|^2 + \|u^{n+1} - 2u^n + u^{n-1}\|^2 \right) \]

\[ + R_e^{-1} \|\nabla u^{n+1}\|^2 + q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) \left( \hat{u}^{n+1} \cdot \hat{u}^{n+1}, u^{n+1} \right) - q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) \kappa \left( J^{n+1} \times B^{n+1}, u^{n+1} \right) = 0. \]
Taking the inner product of (28c) with $\kappa \mathbf{J}^{n+1}$, we have

$$\kappa \left\| \mathbf{J}^{n+1} \right\|^2 - q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) (\hat{\mathbf{u}}^{n+1} \times \mathbf{B}^{n+1}, \mathbf{J}^{n+1}) = 0. \quad (31)$$

Multiplying (28e) by $q^{n+1}$ leads to

$$
\frac{1}{4T} \left( \left\| q^{n+1} \right\|^2 + \left\| 2q^{n+1} - q^n \right\|^2 - \left\| q^n \right\|^2 \right) + \frac{1}{T} \left( \left\| q^{n+1} \right\|^2 - q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) (\hat{\mathbf{u}}^{n+1} \cdot \nabla \hat{\mathbf{u}}^{n+1}, \mathbf{u}^{n+1}) \right)
\quad + \frac{1}{4T} \left( \left\| q^{n+1} \right\|^2 + \left\| 2q^{n+1} - q^n \right\|^2 - \left\| q^n \right\|^2 \right)
\quad + \frac{1}{4T} \left( \left\| 2q^{n+1} - q^n \right\|^2 - \left\| q^n \right\|^2 \right)
\quad + R_e^{-1} \left\| \nabla \mathbf{u}^{n+1} \right\|^2 + \kappa \left\| \mathbf{J}^{n+1} \right\|^2 + \frac{1}{T} \left( \left\| q^{n+1} \right\|^2 \right) = 0,
$$

which completes the proof.

The second-order scheme (28) can be implemented efficiently in the same way as the first-scheme (17). For the convenience of the readers, we present the final solution algorithm. For $n = 0$, we have discussed the implementation in the previous subsection. For $n \geq 1$, it involves the following steps:

1. Get the solutions $(\mathbf{u}_i^{n+1}, p_i^{n+1}, \mathbf{J}_i^{n+1}, \phi_i^{n+1})$, $i = 1, 2$.

   (a) Get the solutions $(\mathbf{u}_i^{n+1}, p_i^{n+1})$, $i = 1, 2$, by solving

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{3}{2} \mathbf{u}_i^{n+1} - 4q^{n+1} \mathbf{u}_i^{n+1} - R_e^{-1} \Delta \mathbf{u}_i^{n+1} + \nabla p_i^{n+1} = 0, \\
\text{div} \mathbf{u}_i^{n+1} = 0,
\end{array} \right.
\end{align*}
\]

and

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{3}{2} \mathbf{u}_i^{n+1} - 4q^{n+1} \mathbf{u}_i^{n+1} - R_e^{-1} \Delta \mathbf{u}_i^{n+1} + \nabla p_i^{n+1} = \kappa \mathbf{J}_i^{n+1} \times \mathbf{B}^{n+1} - \hat{\mathbf{u}}^{n+1} \cdot \nabla \hat{\mathbf{u}}^{n+1}, \\
\text{div} \mathbf{u}_i^{n+1} = 0.
\end{array} \right.
\end{align*}
\]

(b) Get the solutions $(\mathbf{J}_i^{n+1}, \phi_i^{n+1})$, $i = 1, 2$, by solving

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathbf{J}_1^{n+1} + \nabla \phi_1^{n+1} = 0, \\
\text{div} \mathbf{J}_1^{n+1} = 0,
\end{array} \right.
\end{align*}
\]

and

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathbf{J}_2^{n+1} + \nabla \phi_2^{n+1} = \hat{\mathbf{u}}^{n+1} \times \mathbf{B}^{n+1}, \\
\text{div} \mathbf{J}_2^{n+1} = 0.
\end{array} \right.
\end{align*}
\]

2. Get the solution $S^{n+1}$ by solving

\[
\left( \frac{3T + 2\tau}{2\tau T} - \exp \left( \frac{2t^{n+1}}{T} \right) A_2 \right) \exp \left( -\frac{t^{n+1}}{T} \right) S^{n+1} = \exp \left( \frac{t^{n+1}}{T} \right) A_1 + \frac{1}{2\tau} (4q^n - q^{n-1}),
\]

where $A_i$, $i = 1, 2$ is defined by

\[
A_i = (\hat{\mathbf{u}}^{n+1} \cdot \nabla \mathbf{u}_i^{n+1}, \mathbf{u}_i^{n+1}) - \kappa (\mathbf{J}_i^{n+1} \times \mathbf{B}^{n+1}, \mathbf{u}_i^{n+1}) - \kappa (\hat{\mathbf{u}}^{n+1} \times \mathbf{B}^{n+1}, \mathbf{J}_i^{n+1}).
\]

3. Compute $(\mathbf{u}^{n+1}, p^{n+1}, \mathbf{J}^{n+1}, \phi^{n+1})$ by

\[
(\mathbf{u}^{n+1}, p^{n+1}, \mathbf{J}^{n+1}, \phi^{n+1}) = (\mathbf{u}_1^{n+1}, p_1^{n+1}, \mathbf{J}_1^{n+1}, \phi_1^{n+1}) + S^{n+1} (\mathbf{u}_2^{n+1}, p_2^{n+1}, \mathbf{J}_2^{n+1}, \phi_2^{n+1}),
\]

and compute $q^{n+1}$ by $q^{n+1} = S^{n+1} \exp (-t^{n+1}/T)$.
Similar to the implementation of the first-order scheme, we find that the solutions to (35) are $J_{1}^{n+1} = 0$ and $\phi_{1}^{n+1} = 0$, which means that they do not need to be solved veritably. Therefore, the second-order scheme can be efficiently implemented as the first-order scheme by solving a sequence of linear systems with constant coefficients.

By using exactly the same procedure as Theorem 3.3 for the first-order scheme (17), we can show that the second-order scheme (28) is uniquely solvable.

**Theorem 3.5.** The scheme (28) admits a unique solution at each time step.

**Proof.** Since the proof are similar to the one for Theorem 3.3, we omit the details. □

4. Error analysis

In this section, we derive the error estimates for the first-order scheme. Similar analysis can also be carried out for the second-order scheme by combing the procedures below but the detail is much more tedious. We also emphasize that while both schemes can be used in the three-dimensional case, the error analysis can not be easily extended to the three-dimensional case due to some technical issues. Hence, we set $d = 2$ in this section.

To do this, we denote the error functions as

$$e_{u}^{n} = u^{n} - u(t^{n}), e_{p}^{n} = p^{n} - p(t^{n}), e_{J}^{n} = J^{n} - J(t^{n}), e_{\phi}^{n} = \phi^{n} - \phi(t^{n}), e_{q}^{n} = q^{n} - q(t^{n}).$$

Subtracting (12a) at $t^{n+1}$ from (17), we obtain the following error equations,

$$\delta t e_{u}^{n+1} - R_{e}^{-1} \Delta e_{u}^{n+1} + \nabla e_{u}^{n+1} - \exp\left(\frac{t^{n+1}}{T}\right) (q(t^{n+1}) u(t^{n+1}) \cdot \nabla u(t^{n+1}) - q^{n+1} u^{n} \cdot \nabla u^{n}) = R_{u}^{n+1} + \kappa \exp\left(\frac{t^{n+1}}{T}\right) \left( q(t^{n+1}) J^{n+1} \times B^{n+1} - q^{n+1} J(t^{n+1}) \times B^{n+1} \right), \quad (38)$$

$$\nabla \cdot e_{u}^{n+1} = 0, \quad (39)$$

$$e_{J}^{n+1} + \nabla e_{\phi}^{n+1} = \exp\left(\frac{t^{n+1}}{T}\right) \left( q(t^{n+1}) u^{n+1} \times B^{n+1} - q(t^{n+1}) u(t^{n+1}) \times B^{n+1} \right), \quad (40)$$

$$\delta t e_{q}^{n+1} + \frac{1}{T} e_{q}^{n+1} - \exp\left(\frac{t^{n+1}}{T}\right) \left( (u^{n} \cdot \nabla u^{n}, u^{n+1}) - (u(t^{n+1}) \cdot \nabla u(t^{n+1}), u(t^{n+1})) \right) = R_{q}^{n+1} - \kappa \exp\left(\frac{t^{n+1}}{T}\right) \left( (J^{n} \times B^{n+1}, J^{n+1}) - (J(t^{n+1}) \times B^{n+1}, J(t^{n+1})) \right), \quad (41)$$

where the truncation errors are defined by

$$R_{u}^{n+1} = \frac{1}{\tau} \int_{t^{n}}^{t^{n+1}} (t^{n} - t) u_{tt} dt, \quad R_{q}^{n+1} = \frac{1}{\tau} \int_{t^{n}}^{t^{n+1}} (t^{n} - t) q_{tt} dt.$$

**Lemma 4.1** (Stability). Let $(u^{n}, p^{n}, J^{n}, \phi^{n}, q^{n}), n \geq 0$ solve (17). Then it satisfies the following stability estimate for any $m \geq 0$,

$$\|u^{m}\|^2 + |q^{m}|^2 \leq k_{1}, \quad (42)$$

$$\tau \sum_{n=0}^{m} \left( R_{e}^{-1} \|\nabla u^{n}\|^2 + \kappa \|J^{n}\|^2 + \frac{1}{T} |q^{n}|^2 \right) \leq k_{2}, \quad (43)$$

where the constants $k_{i} (i = 1, 2)$ are independent of $\tau$.

**Proof.** By Theorem 3.2, summing up inequality (19) from $n = 0$ to $m - 1$, we obtain

$$\|u^{m}\|^2 + |q^{m}|^2 + 2\tau \sum_{n=0}^{m} \left( R_{e}^{-1} \|\nabla u^{n+1}\|^2 + \kappa \|J^{n+1}\|^2 + \frac{1}{T} |q^{n+1}|^2 \right) \leq \|u^{0}\|^2 + |q^{0}|^2.$$

This implies the desired result. □
Let $P$ be the orthogonal projector in $L^2(\Omega)$ onto $V$, we define the Stokes operator $A$ by

$$Au = -P \Delta u, \quad \forall u \in D(A) = H^2(\Omega) \cap X.$$

The following estimates for the trilinear form holds for $s > 1/2$,

$$b(u, v, w) \leq C_{b,0} \|\nabla u\| \|\nabla v\| \|\nabla w\| \quad \forall u, v, w \in X,$$

(44)

Moreover, for $d = 2$, we have

$$b(u, v, w) \leq C_{b,0} \|\nabla u\|^{1/2} \|u\|^{1/2} \|\nabla w\| \quad \forall u \in V, \quad v, w \in X,$$

(50)

$$b(u, v, w) \leq C_{b,7} \|\nabla u\|^{1/2} \|u\|^{1/2} \|\nabla v\| \| A v \| \| w \| \quad \forall v \in X \cap H^2(\Omega), \quad u, w \in X,$$

(51)

$$b(u, v, w) \leq C_{b,8} \|u\|^{1/2} \|\nabla u\| \| w \| \quad \forall u \in X \cap H^2(\Omega), \quad v, w \in X.$$

(52)

Lemma 4.2. The following estimates of the trilinear form holds for $s > 1/2$,

$$b(u, v, w) \leq C_{b,0} \|\nabla u\| \|\nabla v\| \|\nabla w\| \quad \forall u, v, w \in X,$$

(45)

Theorem 4.1. Assuming $u \in H^2(0,T;H^{-1}(\Omega)) \cap H^1(0,T;H^{1+s}(\Omega)) \cap L^\infty(0,T;H^{1+s}(\Omega)), \quad J \in L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and $B \in L^\infty(0,T;L^2(\Omega))$, then for the scheme (17), the following error estimate holds for $m \geq 0$,

$$\|e_{u}^{m+1}\|^2 + \|e_{v}^{m+1}\|^2 + \nu \tau \sum_{n=0}^{m} \|\nabla e_{u}^{n+1}\|^2 + \kappa \tau \sum_{n=0}^{m} \|\nabla e_{v}^{n+1}\|^2 + \tau \sum_{n=0}^{m} |e_{v}^{n+1}|^2$$

$$+ \sum_{n=0}^{m} \|e_{u}^{n+1} - e_{u}^{n}\|^2 + \sum_{n=0}^{m} |e_{v}^{n+1} - e_{v}^{n}|^2 \leq C \tau^2.$$

(53)

Remark 4.1. In Theorem 4.1, the regularity assumption for the exact solutions are needed to deliver the error estimates. Compared with the existing works, we have lowered the regularity index $s = 1$ to $s > 1/2$, which is a small improvement. The regularity for the exact solutions in is relative weak and may be a reasonable hypothesis in such a setting.

The proof of the above theorem will be carried out with a sequence of lemmas below.

Lemma 4.4. Under the assumptions of Theorem 4.1, the following error estimate holds for $0 \leq n \leq N - 1$,

$$\|e_{u}^{n+1}\|^2 - \|e_{u}^{n}\|^2 + |e_{u}^{n+1} - e_{u}^{n}|^2 \leq \frac{R_{e}^{-1} \tau}{2} \|\nabla e_{u}^{n+1}\|^2$$

$$+ \exp \left( \frac{\nu \tau}{2} \right) e_{q}^{n+1} \left( \kappa (J^n \times B^{n+1}, e_{u}^{n+1} - (u^n \cdot \nabla u^n, e_{u}^{n+1})) \right).$$
\begin{align*}
+ C \left( \| u^n \|_{1+s}^2 + \| u^{n+1} \|_{1+s}^2 + \| \nabla e^n_u \|^2 + \| e^n_u \|^2 + C \| B^{n+1} \|_{0.3}^2 \| e^n_j \|^2 \right) \\
+ C \tau \int_{t^n}^{t_{n+1}} \| u(t) \|^2_{1+s} \, dt + C \tau \| u(t^n) \|^2_{1+s} \int_{t^n}^{t_{n+1}} \| u(t) \|^2 \, dt \\
+ C \tau \| u^n \| \int_{t^n}^{t_{n+1}} \| u(t) \|^2_{1+s} \, dt + C \tau \| B^{n+1} \|^2_{0.3} \int_{t^n}^{t_{n+1}} \| J_i \|^2 \, dt.
\end{align*}

(54)

Proof. Taking the $L^2$-inner product of (38) with $e^{n+1}_u$ and using (39), we get

\begin{align*}
&\| e^{n+1}_u \|^2 - \| e^n_u \|^2 + \| e^n_u - e^n_{u^l} \|^2  \\
&= (R^{n+1}_u, e^{n+1}_u) + \exp \left( \frac{t_{n+1}^q}{T} \right) (q(t_{n+1}^n) u(t_{n+1}^n) \cdot \nabla u(t_{n+1}^n) - q^n u^n \cdot \nabla u^n, e^{n+1}_u) \\
&+ \kappa \exp \left( \frac{t_{n+1}^q}{T} \right) \left( q^{n+1} J \times B^{n+1} - q(t_{n+1}^n) J(t_{n+1}^n) \times B^{n+1} \right) \cdot \nabla u^n, e^{n+1}_u \\
&:= \sum_{i=1}^3 I_i
\end{align*}

(55)

For term $I_1$, we use the Young inequality to have

\begin{align*}
I_1 \leq \frac{R^{-1}_e}{8} \| \nabla e^{n+1}_u \|^2 + C \tau \int_{t^n}^{t_{n+1}} \| u(t) \|^2_{-1} \, dt.
\end{align*}

(56)

For term $I_2$, we rearrange it as follows,

\begin{align*}
I_2 &= \left( (u(t_{n+1}^n) - u^n) \cdot \nabla u(t_{n+1}^n), e^{n+1}_u \right) + \left( u^n \cdot \nabla u(t_{n+1}^n) - u^n \cdot \nabla u^n, e^{n+1}_u \right) \\
&- \exp \left( \frac{t_{n+1}^q}{T} \right) e^{n+1}_q \left( u^n \cdot \nabla u^n, e^{n+1}_u \right) \\
&= I_{2,1} + I_{2,2} - \exp \left( \frac{t_{n+1}^q}{T} \right) e^{n+1}_q \left( u^n \cdot \nabla u^n, e^{n+1}_u \right).
\end{align*}

(57)

For term $I_{2,1}$, it can be bounded by using (45) and Young inequality,

\begin{align*}
I_{2,1} &\leq C_{b_1} \| u(t_{n+1}^n) - u^n \| \| u(t_{n+1}^n) \|_{1+s} \| \nabla e^{n+1}_u \| \\
&\leq C_{b_1} \| e^n_u \| \| u(t_{n+1}^n) \|_{1+s} \| \nabla e^{n+1}_u \| + C_{b_1} \left\| \int_{t^n}^{t_{n+1}} u(t) \, dt \right\| \| u(t_{n+1}^n) \|_{1+s} \| \nabla e^{n+1}_u \| \\
&\leq R^{-1}_e \| \nabla e^{n+1}_u \|^2 + C \| u(t_{n+1}^n) \|^2_{1+s} \| e^n_u \|^2 + C \tau \| u(t_{n+1}^n) \|^2_{1+s} \int_{t^n}^{t_{n+1}} \| u(t) \|^2 \, dt.
\end{align*}

(58)

Similarly, term $I_{2,2}$ can be estimated by using (45)-(46) and Young inequality,

\begin{align*}
I_{2,2} &= \left( u^n \cdot \nabla (u(t_{n+1}^n) - u(t^n)), e^{n+1}_u \right) - \left( e^n_u \cdot \nabla (u(t_{n+1}^n) - u(t^n)), e^{n+1}_u \right) - \left( u(t^n) \cdot \nabla e^n_u, e^{n+1}_u \right) \\
&= \left( u^n \cdot \nabla (u(t_{n+1}^n) - u(t^n)), e^{n+1}_u \right) + \left( e^n_u \cdot \nabla (u(t_{n+1}^n) - u(t^n)), e^{n+1}_u \right) + \left( u(t^n) \cdot \nabla e^n_u, e^{n+1}_u \right) \\
&\leq C_{b_1} \| u^n \| \| u(t_{n+1}^n) - u(t^n) \|_{1+s} \| \nabla e^{n+1}_u \| + C_{b_6} \| e^n_u \| \| \nabla e^{n+1}_u \| \| \nabla e^{n+1}_u \| + C_{b_{2,1}} \| u(t^n) \|_{1+s} \| \nabla e^{n+1}_u \| \| e^n_u \| \\
&\leq C_{b_1} \| u^n \| \left\| \int_{t^n}^{t_{n+1}} u(t) \, dt \right\| \| \nabla e^{n+1}_u \| + C_{b_6} \| e^n_u \| \| \nabla e^{n+1}_u \| + C_{b_{2,1}} \| u(t^n) \|_{1+s} \| \nabla e^{n+1}_u \| \| e^n_u \| \\
&\leq R^{-1}_e \| \nabla e^{n+1}_u \|^2 + C \left( \| u(t^n) \|^2_{1+s} + \| \nabla e^n_u \|^2 \right) \| e^n_u \|^2 + C \tau \| u^n \|^2 \int_{t^n}^{t_{n+1}} \| u(t) \|^2 \, dt.
\end{align*}

(59)

For term $I_3$, we invoke with Hölder inequality and Young inequality to deduce that

\begin{align*}
I_3 &= \kappa \exp \left( \frac{t_{n+1}^q}{T} \right) e^{n+1}_q \left( J^n \times B^{n+1} + e^{n+1}_u \right) + \left( (J^n - J(t_{n+1}^n)) \times B^{n+1} \right) \cdot e^{n+1}_u \\
&= \kappa \exp \left( \frac{t_{n+1}^q}{T} \right) e^{n+1}_q \left( J^n \times B^{n+1} + e^{n+1}_u \right) + \kappa \left( J^n - J(t_{n+1}^n) \right) \| B^{n+1} \|_{0.3} \| e^{n+1}_u \|_{0.6}
\end{align*}
\begin{align*}
\leq \kappa \exp \left( \frac{t^{n+1}}{T} \right) e_q^{n+1} (J^n \times B^{n+1}, e_{u}^{n+1}) + C_p \kappa \| e_q^n + J^n \|_{0,3} \| \nabla e_{u}^{n+1} \| \\
\leq \kappa \exp \left( \frac{t^{n+1}}{T} \right) e_q^{n+1} (J^n \times B^{n+1}, e_{u}^{n+1}) + \frac{R e^{-1}}{8} \| \nabla e_{u}^{n+1} \|^2 \\
+ C \left\| B^{n+1} \right\|_{0,3}^2 \| e_J \|^2 + C \tau \left\| B^{n+1} \right\|_{0,3}^2 \int_{t^n}^{t^{n+1}} \| J \|^2 dt. \tag{60}
\end{align*}

Combining (55) with (56)–(60) leads to the desired result.

Next, we derive a bound for the errors of the current density.

**Lemma 4.5.** Under the assumptions of Theorem 4.1, the following error estimate holds for $0 \leq n \leq N - 1$,

$$\frac{\kappa}{2} \| e_J^{n+1} \|^2 \leq \kappa \exp \left( \frac{t^{n+1}}{T} \right) e_q^{n+1} (u^n \times B^{n+1}, e_J^{n+1}) + C \left\| B^{n+1} \right\|_{0,3}^2 \| \nabla e_{u}^{n+1} \|^2 + C \tau \left\| B^{n+1} \right\|_{0,3}^2 \int_{t^n}^{t^{n+1}} \| \nabla u_t \|^2 dt. \tag{61}$$

**Proof.** Taking the $L^2$-inner product of (40) with $\kappa e_J^{n+1}$ and utilizing Hölder inequality and Young inequality, we obtain

\begin{align*}
\kappa \| e_J^{n+1} \|^2 &= \kappa \exp \left( \frac{t^{n+1}}{T} \right) \left( \langle q^{n+1} u^n \times B^{n+1} - q (t^{n+1}) u (t^{n+1}) \times B^{n+1} \rangle, e_J^{n+1} \right) \\
&= \kappa \exp \left( \frac{t^{n+1}}{T} \right) e_q^{n+1} (u^n \times B^{n+1}, e_J^{n+1}) - \kappa \left( \langle u (t^{n+1}) - u^n \rangle \times B^{n+1}, e_J^{n+1} \right) \\
&\leq \kappa \exp \left( \frac{t^{n+1}}{T} \right) e_q^{n+1} (u^n \times B^{n+1}, e_J^{n+1}) + C_p \kappa \| \nabla (u (t^{n+1}) - u^n) \| \| e_J^{n+1} \| \| B^{n+1} \|_{0,3} \\
&\leq \kappa \exp \left( \frac{t^{n+1}}{T} \right) e_q^{n+1} (u^n \times B^{n+1}, e_J^{n+1}) + \frac{\kappa}{2} \| e_J^{n+1} \|^2 \\
+ C \left\| B^{n+1} \right\|_{0,3}^2 \| \nabla e_{u}^{n+1} \|^2 + C \tau \left\| B^{n+1} \right\|_{0,3}^2 \int_{t^n}^{t^{n+1}} \| \nabla u_t \|^2 dt. \tag{62}
\end{align*}

This leads to the desired result.

In the next lemma, we derive a bound for the errors with respect to $q$.

**Lemma 4.6.** Under the assumptions of Theorem 4.1, the following error estimate holds for $0 \leq n \leq N - 1$,

\begin{align*}
\frac{|e_q^{n+1}|^2 - 2 |e_q^n|^2 + |e_q^{n+1} - e_q^n|^2}{2} + \frac{1}{2T} \| e_q^{n+1} \|^2 \\
= \exp \left( \frac{t^{n+1}}{T} \right) e_q^{n+1} (u^n \cdot \nabla u^n, e_{u}^{n+1}) - \kappa \exp \left( \frac{t^{n+1}}{T} \right) e_q^{n+1} (J^n \times B^{n+1}, e_{u}^{n+1}) \\
- \kappa \exp \left( \frac{t^{n+1}}{T} \right) e_q^{n+1} (u^n \times B^{n+1}, e_J^{n+1}) + \frac{1}{4k_2} \| \nabla u^n \|^2 \| e_q^{n+1} \|^2 \\
+ C \left( \| u (t^{n+1}) \|^2_{1+s} + \| \nabla u (t^{n+1}) \|^2 \| u (t^{n+1}) \|^2_{1+s} + \| B^{n+1} \|^2_{0,3} \| J (t^{n+1}) \|^2 \right) \| e_{u}^{n+1} \|^2 \\
+ C \left\| B^{n+1} \right\|_{0,3}^2 \| \nabla u (t^{n+1}) \|^2 \| e_{J}^{n+1} \|^2 + C \tau \left\| B^{n+1} \right\|_{0,3}^2 \| \nabla u (t^{n+1}) \|^2 \int_{t^n}^{t^{n+1}} \| J \|^2 dt + C \tau \int_{t^n}^{t^{n+1}} |q_u (s)|^2 ds \\
+ C \tau \left( \| u (t^{n+1}) \|^2_{1+s} + \| \nabla u (t^{n+1}) \|^2 \| u (t^{n+1}) \|^2_{1+s} + \| B^{n+1} \|^2_{0,3} \| J (t^{n+1}) \|^2 \right) \int_{t^n}^{t^{n+1}} \| \nabla u_t (s) \|^2 ds. \tag{63}
\end{align*}

**Proof.** Multiplying both sides of (41) by $e_q^{n+1}$ yields

\begin{align*}
\frac{|e_q^{n+1}|^2 - 2 |e_q^n|^2 + |e_q^{n+1} - e_q^n|^2}{2T} + \frac{1}{T} \| e_q^{n+1} \|^2 \\
= R e^{n+1} e_q^{n+1} + \exp \left( \frac{t^{n+1}}{T} \right) e_q^{n+1} \left( \langle u^n \cdot \nabla u^n, u^{n+1} \rangle - \langle u (t^{n+1}) \times \nabla u (t^{n+1}), u (t^{n+1}) \rangle \right) \\
- \kappa \exp \left( \frac{t^{n+1}}{T} \right) e_q^{n+1} \left( \langle J^n \times B^{n+1}, u^{n+1} \rangle - \langle J (t^{n+1}) \times B^{n+1}, u (t^{n+1}) \rangle \right)
\end{align*}
\[-\kappa \exp \left(\frac{t^{n+1}}{T}\right) e^{q+1}_q \left((u^n \times B^{n+1}, J^{n+1}) - (u(t^{n+1}) \times B^{n+1}, J(t^{n+1}))\right)\]

\[\text{where} \quad I_i = \frac{1}{8T} e^{q+1}_q \left|e^{q+1}_q\right| + C \tau \int_{t^n}^{t^{n+1}} \left|q^n(s)\right|^2 ds. \]

Term \(I_2\) can be recast as

\[I_2 = \exp \left(\frac{t^{n+1}}{T}\right) e^{q+1}_q (u^n \cdot \nabla u^n, e^{q+1}_q) + \exp \left(\frac{t^{n+1}}{T}\right) e^{q+1}_q (u^n \cdot \nabla (u^n - u(t^{n+1})), u(t^{n+1}))\]

\[+ \exp \left(\frac{t^{n+1}}{T}\right) e^{q+1}_q ((u^n - u(t^{n+1})) \cdot \nabla u(t^{n+1}), u(t^{n+1}))\]

\[\text{where} \quad k_2 \text{ is given by} \ (43). \text{ In a same manner, term } I_{2,2} \text{ can be bounded by using} \ (49) \text{ and Young inequality,}\]

\[I_{2,2} = \exp \left(\frac{t^{n+1}}{T}\right) e^{q+1}_q ((u^n - u(t^{n+1})) \cdot \nabla u(t^{n+1}), u(t^{n+1}))\]

\[\leq \exp(1) C_{b,4} \left|\nabla u^n\right| \left\|u(t^{n+1})\right\|_{1+s,2} \left\|u^n - u(t^{n+1})\right\|_0 \left|e^{q+1}_q\right|\]

\[\leq C \left|\nabla u^n\right| \left\|u(t^{n+1})\right\|_{1+s,2} \left\|u(t^{n+1}) - u(t^n) - e^{q+1}_q\right\|_0 \left|e^{q+1}_q\right|\]

\[\leq \frac{1}{4k_2} \left|\nabla u^n\right|^2 + C \left|e^{q+1}_q\right|^2 \left\|u(t^{n+1})\right\|^2_{1+s,2} + C \tau \left\|\nabla u(t^{n+1})\right\|^2 \left\|u(t^{n+1})\right\|^2_{1+s,2} \int_{t^n}^{t^{n+1}} \left\|u_t\right\|^2 dt. \]

Using Hölder inequality and Young inequality, term \(I_3\) can be estimated by

\[I_3 = \kappa \exp \left(\frac{t^{n+1}}{T}\right) e^{q+1}_q ((J(t^{n+1}) - J^n) \times B^{n+1}, u(t^{n+1})) - \kappa \exp \left(\frac{t^{n+1}}{T}\right) e^{q+1}_q (J^n \times B^{n+1}, e^{q+1}_u)\]

\[\leq \kappa \exp(1) \left|e^{q+1}_q\right| \left\|J(t^{n+1}) - J^n\right\| \left\|B^{n+1}\right\|_{0,3} \left\|u(t^{n+1})\right\|_{0,6} - \kappa \exp \left(\frac{t^{n+1}}{T}\right) e^{q+1}_q (J^n \times B^{n+1}, e^{q+1}_u)\]

\[\leq C \left|e^{q+1}_q\right| \left\|J(t^{n+1}) - J^n\right\| \left\|B^{n+1}\right\|_{0,3} \left\|\nabla u(t^{n+1})\right\| - \kappa \exp \left(\frac{t^{n+1}}{T}\right) e^{q+1}_q (J^n \times B^{n+1}, e^{q+1}_u)\]

\[\leq \frac{1}{8T} \left|e^{q+1}_q\right|^2 + C \left\|B^{n+1}\right\|^2_{0,3} \left\|\nabla u(t^{n+1})\right\|^2 \left|e^{q+1}_q\right|^2 + C \tau \left\|B^{n+1}\right\|^2_{0,3} \left\|\nabla u(t^{n+1})\right\|^2 \int_{t^n}^{t^{n+1}} \left\|J_t\right\|^2 dt\]

\[-\kappa \exp \left(\frac{t^{n+1}}{T}\right) e^{q+1}_q (J^n \times B^{n+1}, e^{q+1}_u). \]

Using the similar procedure, term \(I_4\) can be bounded by

\[I_4 = -\kappa \exp \left(\frac{t^{n+1}}{T}\right) e^{q+1}_q (B^{n+1} \times u^n, e^{q+1}_j) - \kappa \exp \left(\frac{t^{n+1}}{T}\right) e^{q+1}_q (B^{n+1} \times (u^n - u(t^{n+1})), J(t^{n+1}))\]
Combining (64) with (65)-(70) leads to the desired result.

Proof of Theorem 4.1. Summing up (54), (61) and (63) leads to

\[
\begin{align*}
\|e_n^{n+1}\|^2 - \|e_n^n\|^2 + \|e_n^{n+1} - e_n^n\|^2 &+ \frac{R_{e^{-1}}}{2} \|\nabla e_{n+1}^n\|^2 + \frac{\kappa}{2} \|e_n^{n+1}\|^2 \\
+ \frac{\|e_q^n\|^2 - \|e_q^0\|^2}{2\tau} + \|e_q^{n+1} - e_q^n\|^2 &+ \frac{1}{2\tau} \|e_q^{n+1}\|^2 \\
&\leq \frac{1}{4k_2} \|\nabla u_n^n\|^2 \|e_q^{n+1}\|^2 + C \left(1 + \|\nabla e_u^0\|^2\right) \|e_u^n\|^2 + C \|\nabla e_u^n\|^2 + C \|e_J^n\|^2 \\
+ C \tau \int_{\tau_n}^{\tau_{n+1}} \left(||u_t||_{1+\tau} + ||u_{tt}||_{-1+\tau} + |q_t|^2 + ||J_t||^2 \right) dt.
\end{align*}
\]

(71)

We will prove the error estimate by mathematical induction argument. For the case of \(k = m = 0\), by using (71), we have

\[
\begin{align*}
\|e_u^0\|^2 - \|e_u^n\|^2 + \|e_u^1 - e_u^0\|^2 &+ \frac{\|e_q^1\|^2 - \|e_q^0\|^2}{2\tau} + \frac{\|e_q^1 - e_q^0\|^2}{2\tau} \\
&\leq \frac{1}{4k_2} \|\nabla u_0^0\|^2 \|e_q^1\|^2 + C \|e_u^0\|^2 + C \|\nabla e_u^0\|^2 + C \|e_J^0\|^2 + C \tau.
\end{align*}
\]

Invoking with \(e_u^0 = e_J^0 = 0\) and (43), we can easily get

\[
\begin{align*}
\|e_u^1\|^2 + \|e_q^1\|^2 + \|e_u^0 - e_u^1\|^2 &+ \|e_q^1 - e_q^0\|^2 \\
&\leq C \tau^2.
\end{align*}
\]

This means that the error estimate (53) holds for \(m = 0\). For \(0 < k \leq m - 1\), we assume that the error estimate (53) is valid. We will show it is also valid for \(k = m\). We first deduce a bound for \(e_{q, m}^{n+1}\), where \(m^*\) is the time step such that

\[
|e_{q, m}^{n+1}| = \max_{0 \leq n \leq m} |e_{q, n}^{n+1}|.
\]

(72)

Multiplying (71) by \(2\tau\), summing up over \(n\) from 0 to \(m^*\), using (72) and the recursive hypothesis, we get

\[
\begin{align*}
\|e_{u, m}^{m+1}\|^2 + \tau \sum_{n=0}^{m^*} \left(R_{e^{-1}} \|\nabla e_{u, n}^{n+1}\|^2 + \|e_{u, n}^{n+1}\|^2\right) + \|e_{q, m}^{m+1}\|^2 + \frac{\tau}{T} \sum_{n=0}^{m^*} |e_q^{n+1}|^2 \\
+ \sum_{n=0}^{m^*} \|e_{u, n}^{n+1} - e_{u, n}^n\|^2 + \sum_{n=0}^{m^*} |e_q^{n+1} - e_q^n|^2 \\
&\leq \frac{\tau}{2k_2} \sum_{n=0}^{m^*} \|\nabla u_n^n\|^2 \|e_q^{n+1}\|^2 + C \tau \sum_{n=0}^{m^*} \left(1 + \|\nabla e_u^n\|^2\right) \|e_u^n\|^2 + C \tau \sum_{n=0}^{m^*} \|\nabla e_u^n\|^2
\end{align*}
\]
Now we turn to (71), multiply it by \( \tau \int_0^{t^{m+1}} \left( \| u_t \|^2 + \| u_{tt} \|^2_1 + \| q_{tt} \|^2 \right) dt, \)

\[
\leq \frac{\tau}{2k_2} \sum_{n=0}^{m^*} \| \nabla u_n \|^2 |e_q^{m+1}|^2 + C\tau \sum_{n=0}^{m^*} \left( 1 + \| \nabla e_u^n \|^2 \right) \| e_q^n \|^2 + C\tau^2 \\
+ C\tau \int_0^{t^{m+1}} \left( \| u_t \|^2 + \| u_{tt} \|^2_1 + \| q_{tt} \|^2 \right) dt.
\]

It follows from (43) that

\[
\frac{\tau}{2k_2} \sum_{n=0}^{m^*} \| \nabla u_n \|^2 |e_q^{m+1}|^2 \leq \frac{1}{2} |e_q^{m+1}|^2, \quad \tau \sum_{n=0}^{m^*} \left( 1 + \| \nabla e_u^n \|^2 \right) \leq C.
\]

Invoking with the discrete Gronwall inequality in Lemma 4.3, we obtain

\[
\| e_{u_t}^{m+1} \|^2 + \tau \sum_{n=0}^{m} \left( R_c^{-1} \| \nabla e_u^{n+1} \|^2 + \kappa \| e_J^{n+1} \|^2 \right) + |e_{q_t}^{m+1}|^2 + \frac{\tau}{2} \sum_{n=0}^{m} |e_{q_t}^{n+1}|^2 \\
\leq C\tau^2 \int_0^{t^{m+1}} \left( \| u_t \|^2 + \| u_{tt} \|^2_1 + \| q_{tt} \|^2 \right) dt.
\]

Now we turn to (71), multiply it by \( 2\tau \) and sum up over \( n \) from 0 to \( m \), and using (72),

\[
\| e_{u_t}^{m+1} \|^2 + |e_{q_t}^{m+1}|^2 + \tau R_c^{-1} \sum_{n=0}^{m} \| \nabla e_u^{n+1} \|^2 + \tau \kappa \sum_{n=0}^{m} |e_{J_t}^{n+1}|^2 \\
+ \sum_{n=0}^{m} \| e_{u_t}^{n+1} - e_{u_t}^{n} \|^2 + \sum_{n=0}^{m} |e_{q_t}^{n+1} - e_{q_t}^{n}|^2 \\
\leq \frac{\tau}{2k_2} \sum_{n=0}^{m} \| \nabla u_n \|^2 |e_q^{m+1}|^2 + C\tau \sum_{n=0}^{m} \left( 1 + \| \nabla e_u^n \|^2 \right) \| e_u^n \|^2 + C\tau \sum_{n=0}^{m} \| \nabla e_u^n \|^2 \\
+ C\tau \sum_{n=0}^{m} \| e_{q_t}^{n} \|^2 + C\tau \int_0^{t^{m+1}} \left( \| u_t \|^2 + \| u_{tt} \|^2_1 + \| q_{tt} \|^2 \right) dt.
\]

From (43) again, we have that

\[
\tau \sum_{n=0}^{m} \left( 1 + \| \nabla e_u^n \|^2 \right) \leq C.
\]

Using (74) and the discrete Gronwall inequality in Lemma 4.3 to (75), we complete the proof. \( \square \)

4.2. Error estimates for the pressure and electric potential

This section is devoted to presenting error estimates for the pressure and electric potential. We first deduce the error estimates for the electric potential.

**Theorem 4.2.** Under the assumption of Theorem 4.1, the following error estimate holds for \( 0 \leq m \leq N - 1 \),

\[
\tau \sum_{n=0}^{m} \| e_{q_t}^{n+1} \|^2 \leq C\tau^2.
\]

**Proof.** By taking the inner product of (40) with \( K \in D \) and using the similar arguments in (62), we obtain

\[
\left( e_{q_t}^{n+1}, \text{div} K \right) = \left( e_{J_t}^{n+1}, K \right) - \exp \left( \frac{t^{n+1}}{T} \right) \left( q^{n+1} u^n \times B^{n+1} - q \left( t^{n+1} \right) u \left( t^{n+1} \right) \times B^{n+1} \right) K
\]
\[ \leq \|e_{j}^{n+1}\|_{K} + \exp(1) |e_{q}^{n+1}|_{u} \|B_{n+1}\|_{0,3} \|K\|_3 + \|u (t^{n+1}) - u^n\|_{0,6} B_{n+1}\|_{0,3} \|K\|_3 \]

\[ \leq \|e_{j}^{n+1}\|_{K} \text{div} + C|e_{q}^{n+1}| \|\nabla u^n\| B_{n+1}\|_{0,3} \|K\|_3 + C \|\nabla (u (t^{n+1}) - u^n)\| B_{n+1}\|_{0,3} \|K\|_3 \]

\[ \leq \|e_{j}^{n+1}\|_{K} \text{div} + C|e_{q}^{n+1}| \|\nabla u^n\| B_{n+1}\|_{0,3} \|K\|_3 + C \|\nabla e_{u}^{n}\| B_{n+1}\|_{0,3} \|K\|_3 \]

\[ + C \tau^2 \left( \int_{t^n}^{t^{n+1}} \|\nabla u_t\| \, dt \right)^{1/2} B_{n+1}\|_{0,3} \|K\|_3. \]

Using Theorem 4.1, Lemma 4.7 and the inf-sup condition,

\[ \beta_m \|e_{\phi}^{n+1}\|_3 \leq \sup_{K \in D} \left( e_{\phi}^{n+1}, \text{div}K \right), \]

we have

\[ \tau \sum_{n=0}^{m} \|e_{\phi}^{n+1}\|^2 \leq C \left( \tau \sum_{n=0}^{m} \|e_{j}^{n+1}\|^2 + \tau \sum_{n=0}^{m} \|e_{q}^{n+1}\|^2 + \tau \sum_{n=0}^{m} \|\nabla e_{u}^{n}\|^2 + C \tau \right) \]

\[ \leq C \tau^2. \]

This completes the proof.

To derive error estimate for the pressure, we need to establish the estimate for \( \delta e_{u}^{n+1} \).

**Lemma 4.7.** Assuming \( u \in H^2(0, T; H^2(\Omega)) \cap H^1(0, T; \mathbb{H}^2(\Omega)) \cap L^\infty(0, T; H^2(\Omega)), J \in L^\infty(0, T; L^2(\Omega)) \cap H^1(0, T; L^2(\Omega)) \) and \( B \in L^\infty(0, T; L^\infty(\Omega)) \), then we have the following error estimate for \( 0 \leq n \leq N - 1 \),

\[ \|\nabla e_{u}^{n+1}\|^2 + \tau \sum_{n=0}^{m} \|\delta e_{u}^{n+1}\|^2 + \nu \tau \sum_{n=0}^{m} \|Ae_{u}^{n+1}\|^2 \leq C \tau^2. \] (77)

**Proof.** First of all, in virtue of (53), we have

\[ \|\nabla e_{u}^{n+1}\|^2 \leq \tau^{-1} \left( \tau \sum_{k=0}^{m} \|\nabla e_{u}^{k+1}\|^2 \right) \leq C \tau, \quad \|e_{j}^{n+1}\|^2 \leq \tau^{-1} \left( \tau \sum_{k=0}^{m} \|\nabla e_{j}^{k+1}\|^2 \right) \leq C \tau. \]

Hence, there holds that

\[ \|\nabla u^{n+1}\| \leq \|\nabla e_{u}^{n+1}\| + \|\nabla (u (t_n^{n+1}))\| \leq C \left( \tau^{1/2} + \|\nabla (u (t_n^{n+1}))\| \right), \] (78)

\[ \|J^{n+1}\| \leq \|e_{j}^{n+1}\| + \|J (t_n^{n+1})\| \leq C \left( \tau^{1/2} + \|J (t_n^{n+1})\| \right). \] (79)

Taking the inner product of (38) with \( Ae_{u}^{n+1} + \delta e_{u}^{n+1} \), we obtain

\[ (1 + R_c^{-1}) \|\nabla e_{u}^{n+1}\|^2 - \|\nabla e_{u}^{n}\|^2 + \|\nabla e_{u}^{n+1} - \nabla e_{u}^{n}\|^2 + \|\delta e_{u}^{n+1}\|^2 + R_c^{-1} \|Ae_{u}^{n+1}\|^2 \]

\[ = (R_u^{n+1}, Ae_{u}^{n+1} + \delta e_{u}^{n+1}) + \kappa \exp \left( \frac{t_n^{n+1}}{T} \right) \left( q_n^{n+1} J^{n} \times B^{n+1} - q (t_n^{n+1}) J (t_n^{n+1}) \times B^{n+1}, Ae_{u}^{n+1} + \delta e_{u}^{n+1} \right) \]

\[ + \exp \left( \frac{t_n^{n+1}}{T} \right) (q (t_n^{n+1}) u (t_n^{n+1}) \cdot \nabla u (t_n^{n+1}) - q_n^{n+1} u^n \cdot \nabla u^n, Ae_{u}^{n+1} + \delta e_{u}^{n+1} \right) \]

\[ := \sum_{i=1}^{3} I_i. \] (80)

For term I_1, we use Cauchy-Schwarz and Young inequality to estimate it as

\[ (R_u^{n+1}, Ae_{u}^{n+1} + \delta e_{u}^{n+1}) \leq \frac{1}{12} \|\delta e_{u}^{n+1}\|^2 + \frac{\nu}{24} \|Ae_{u}^{n+1}\|^2 + C \tau \int_{t^n}^{t^{n+1}} \|u_{tt}\|^2 \, dt. \] (81)

For term I_2, using Hölder inequality and Young inequality, we obtain

\[ I_2 = \exp \left( \frac{t_n^{n+1}}{T} \right) q_n^{n+1} (J^{n} \times B^{n+1}, Ae_{u}^{n+1} + \delta e_{u}^{n+1}) + (J^{n} - J (t_n^{n+1})) \times B^{n+1}, Ae_{u}^{n+1} + \delta e_{u}^{n+1} \]
\[
= \exp \left( \frac{t_n^{n+1}}{T} \right) e_q^n + \left( J \times B^{n+1}, A_{e_u}^{n+1} + \delta e_u^{n+1} \right) + (e_j \times B^{n+1}, A_{e_u}^{n+1} + \delta e_u^{n+1}) \\
+ \left( J \left( t^{n+1} \right) - J \left( t^{n+1} \right) \right) \times B^{n+1}, A_{e_u}^{n+1} + \delta e_u^{n+1} \\
\leq \exp(1) \left\| e_q^n \right\| \left\| J \right\| \left\| B^{n+1} \right\| _{0, \infty} \left\| A_{e_u}^{n+1} + \delta e_u^{n+1} \right\| \left\| e_j^n \right\| \left\| B^{n+1} \right\| _{0, \infty} \left\| A_{e_u}^{n+1} + \delta e_u^{n+1} \right\| \\
+ \left\| \int_{t_n}^{t_{n+1}} J_i dt \right\| \left\| B^{n+1} \right\| _{0, \infty} \left\| A_{e_u}^{n+1} + \delta e_u^{n+1} \right\|
\]
\[
\leq \frac{1}{6} \left\| \delta e_u^{n+1} \right\|^2 + \frac{\nu}{12} \left\| A_{e_u}^{n+1} \right\|^2 + C \left( \tau + \left\| J \left( t^{n+1} \right) \right\|^2 \right) \left\| B^{n+1} \right\|^2 _{0, \infty} \left\| e_q^n \right\|^2 \\
+ C \left\| B^{n+1} \right\|^2 _{0, \infty} \left\| e_j^n \right\|^2 + C \tau \left\| B^{n+1} \right\|^2 _{0, \infty} \int_{t_n}^{t_{n+1}} \left\| J_i \right\|^2 dt.
\]
(82)

For term $I_3$, we rearrange it as follows

\[
I_3 = - \exp \left( \frac{t_n^{n+1}}{T} \right) e_q^n \left( u^n \cdot \nabla u^n, A_{e_u}^{n+1} + \delta e_u^{n+1} \right) + (u \left( t^{n+1} \right) - u^n) \cdot \nabla u \left( t^{n+1} \right), A_{e_u}^{n+1} + \delta e_u^{n+1}) \\
+ (u^n \cdot \nabla (u \left( t^{n+1} \right) - u^n), A_{e_u}^{n+1} + \delta e_u^{n+1}) \\
\leq 3 \sum_{i=1}^{3} I_{3,i}.
\]

Term $I_{3.1}$ can be bounded by using (51), (49) and (78), the first term on the right hand side of (83) can be bounded by

\[
I_{3.1} = - \exp \left( \frac{t_n^{n+1}}{T} \right) e_q^n \left( u^n \cdot \nabla u^n, A_{e_u}^{n+1} + \delta e_u^{n+1} \right) - \exp \left( \frac{t_n^{n+1}}{T} \right) e_q^n \left( (u^n \cdot \nabla u(t^n), A_{e_u}^{n+1} + \delta e_u^{n+1}) \\
\leq \exp(1) C_{b,7} \left\| e_q^n \right\| \left\| u^n \right\|^2 _{1/2} \left\| \nabla u^n \right\|^2 _{1/2} \left\| \nabla e_u^n \right\|^2 _{1/2} \left\| A_{e_u}^{n+1} \right\|^2 _{1/2} \left\| A_{e_u}^{n+1} + \delta e_u^{n+1} \right\| \\
+ \exp(1) C_{b,5} \left\| e_q^n \right\| \left\| \nabla u^n \right\| _{1/2} \left\| u(t^n) \right\| _{1/2} \left\| A_{e_u}^{n+1} + \delta e_u^{n+1} \right\|
\]
\[
\leq \frac{1}{12} \left\| \delta e_u^{n+1} \right\|^2 + \frac{\nu}{24} \left\| A_{e_u}^{n+1} \right\|^2 + \frac{\nu}{8} \left\| A_{e_u}^{n+1} \right\|^2 \\
+ C \left( \tau + \left\| \nabla u \left( t^n \right) \right\|^2 \right) \left\| \nabla e_u^n \right\|^2 + C \left( \tau + \left\| \nabla u \left( t^n \right) \right\|^2 \right) \left\| u \left( t^n \right) \right\| _{2} \left\| e_q^n \right\|^2.
\]
(84)

Similarly, term $I_{3.2}$ can be estimated by

\[
I_{3.2} \leq C_{b,5} \left\| \nabla u \left( t^{n+1} \right) - \nabla u^n \right\| \left\| u \left( t^{n+1} \right) \right\| _{2} \left\| A_{e_u}^{n+1} + \delta e_u^{n+1} \right\|
\]
\[
\leq \frac{1}{12} \left\| \delta e_u^{n+1} \right\|^2 + \frac{\nu}{24} \left\| A_{e_u}^{n+1} \right\|^2 + C \left\| u \left( t^{n+1} \right) \right\| _{2} \left\| \nabla e_u^n \right\|^2 + C \left\| u \left( t^{n+1} \right) \right\| _{2} \tau \int_{t_n}^{t_{n+1}} \left\| \nabla u(t) \right\|^2 dt.
\]
(85)

For term $I_{3.3}$, we deduce that

\[
I_{3.3} = \left( u^n \cdot \nabla (u \left( t^{n+1} \right) - u(t^n)), A_{e_u}^{n+1} + \delta e_u^{n+1} \right) - \left( u^n \cdot \nabla e_u^n, A_{e_u}^{n+1} + \delta e_u^{n+1} \right)
\]
\[
\leq C_{b,5} \left\| \nabla u^n \right\| \left\| u \left( t^{n+1} \right) - u(t^n) \right\| _{2} \left\| A_{e_u}^{n+1} + \delta e_u^{n+1} \right\|
\]
\[
+ C_{b,6} \left\| u^n \right\| ^{1/2} \left\| \nabla u^n \right\| ^{1/2} \left\| \nabla e_u^n \right\| ^{1/2} \left\| A_{e_u}^{n+1} \right\| ^{1/2} \left\| A_{e_u}^{n+1} + \delta e_u^{n+1} \right\|
\]
\[
\leq \frac{1}{12} \left\| \delta e_u^{n+1} \right\|^2 + \frac{\nu}{24} \left\| A_{e_u}^{n+1} \right\|^2 + C \left( \tau + \left\| \nabla u \left( t^n \right) \right\|^2 \right) \left\| \nabla e_u^n \right\|^2 \\
+ \frac{\nu}{8} \left\| A_{e_u}^{n+1} \right\|^2 + C \tau \left( \tau + \left\| \nabla u \left( t^n \right) \right\|^2 \right) \int_{t_n}^{t_{n+1}} \left\| u(t) \right\|^2 dt.
\]
(86)

Combining (80) with (81)-(83), we have

\[
\left( 1 + R_n \right) \left\| \nabla e_u^{n+1} \right\|^2 - \left\| \nabla e_u^n \right\|^2 + \left\| \nabla e_u^{n+1} - \nabla e_u^n \right\|^2 + \left\| \delta e_u^{n+1} \right\|^2 + R_n \left\| A_{e_u}^{n+1} \right\|^2 \\
\leq R_n \left\| A_{e_u}^{n+1} \right\|^2 + C \left( 1 + \tau + \left\| \nabla u \left( t^n \right) \right\|^2 \right) \left( \left\| \nabla e_u^n \right\|^2 + \left\| e_u^{n+1} \right\|^2 \right) \\
+ C \tau \int_{t_n}^{t_{n+1}} \left( \left\| u(t) \right\|^2 + \left\| \nabla u(t) \right\|^2 + \left\| u(t) \right\|^2 \right) dt.
\]
(87)
Multiplying (87) by $2\tau$ and summing over $n$ from 0 to $m$, and applying the discrete Gronwall inequality in Lemma 4.3, we obtain
\[
\|\nabla e_u^{n+1}\|^2 + \tau \sum_{n=0}^{m} \|\delta_t e_u^{n+1}\|^2 + \tau R_e^{-1} \sum_{n=0}^{m} \|A e_u^{n+1}\|^2 \\
\leq C \left(1 + \tau + \|\nabla u(t^n)\|^2\right) \tau \sum_{n=0}^{m} \left(\|\nabla e_u^n\|^2 + |e_q^{n+1}|^2\right) + C\tau^2. \tag{88}
\]
Combining the above estimate with Theorem 53, we obtain the desired result.

We are now in position to prove the pressure estimate.

**Theorem 4.3.** Assuming $u \in H^2(0, T; H^2(\Omega)) \cap H^1(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^2(\Omega))$, $J \in L^\infty(0, T; L^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$ and $B \in L^\infty(0, T; L^\infty(\Omega))$, then we have the following error estimate for $0 \leq m \leq N - 1$,
\[
\tau \sum_{n=0}^{m} \|e_p^{n+1}\|^2 \leq C\tau^2. \tag{89}
\]

**Proof.** Taking the inner product of (38) with $v \in X$, we obtain
\[
\langle \nabla e_u^{n+1}, v \rangle = -\langle \delta_t e_u^{n+1}, v \rangle + R_e^{-1} \langle \nabla e_u^{n+1}, v \rangle + \langle R_u^{n+1}, v \rangle \\
+ \exp\left(\frac{t^{n+1}}{T}\right) \left(q(t^{n+1}) \langle u(t^{n+1}) \cdot \nabla \rangle u(t^{n+1}) - q^{n+1} \langle u^n \cdot \nabla \rangle u^n, v\right) \\
+ \kappa \exp\left(\frac{t^{n+1}}{T}\right) \left(q^{n+1} J^n \times B^{n+1} - q(t^{n+1}) J(t^{n+1}) \times B^{n+1}, v\right) \tag{90}
\]
It is easy to see that the first three terms can be easily bounded by
\[
-\langle \delta_t e_u^{n+1}, v \rangle + R_e^{-1} \langle \nabla e_u^{n+1}, v \rangle + \langle R_u^{n+1}, v \rangle \leq C \left(\|\delta_t e_u^{n+1}\| + \|\nabla e_u^{n+1}\| + \|R_u^{n+1}\|\right) \|\nabla v\|. \tag{91}
\]
For the forth term, by using (44)-(50) and (78), we have that for all $v \in X$,
\[
\exp\left(\frac{t^{n+1}}{T}\right) \left(q(t^{n+1})u(t^{n+1}) \cdot \nabla u(t^{n+1}) - q^{n+1} u^n \cdot \nabla u^n, v\right) \\
= \langle (u(t^{n+1}) - u^n) \cdot \nabla u(t^{n+1}), v \rangle - e_q^{n+1} \exp\left(\frac{t^{n+1}}{T}\right) \langle u^n \cdot \nabla u^n, v \rangle + \langle u^n \cdot \nabla (u(t^{n+1}) - u^n), v \rangle \\
\leq C \|\nabla u(t^{n+1}) - \nabla u^n\| \|u(t^{n+1})\|_2 \|\nabla v\| + C \|e_q^{n+1}\| \|\nabla u^n\| \|\nabla u^n\| \|\nabla v\| \\
+ C \|\nabla u^n\| \|\nabla u(t^{n+1}) - \nabla u^n\| \|\nabla v\| \\
\leq C \left(\|\nabla e_u^n\| + \int_{t^n}^{t^{n+1}} \nabla u_t \cdot dt \right) \|\nabla v\|. \tag{92}
\]
For the last term, we invoke to estimate it as we have that for all $v \in X$,
\[
\kappa \exp\left(\frac{t^{n+1}}{T}\right) \left(q^{n+1} J^n \times B^{n+1} - q(t^{n+1}) J(t^{n+1}) \times B^{n+1}, v\right) \\
= \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left(J^n \times B^{n+1}, v\right) + \left((J^n - J(t^{n+1})) \times B^{n+1}, v\right) \\
= \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left(J^n \times B^{n+1}, v\right) + \left(e_q^n \times B^{n+1}, v\right) + \left((J(t^n) - J(t^{n+1})) \times B^{n+1}, v\right) \\
\leq \exp(1) \|e_q^{n+1}\| \|J^n\| \|B^{n+1}\|_0,\infty \|v\| + \|e_q^n\| \|B^{n+1}\|_0,\infty \|v\| + \int_{t^n}^{t^{n+1}} J_t dt \|B^{n+1}\|_0,\infty \|v\| \\
\leq C \left(\|e_q^{n+1}\| + \|e_q^n\| + \int_{t^n}^{t^{n+1}} J_t dt\right) \|\nabla v\|. \tag{93}
\]
Using Theorem 4.1, Lemma 4.7 and the inf-sup condition,

$$\beta_s \|e^{n+1}_p\| \leq \sup_{v \in X} \frac{(\nabla e^{n+1}_p, v)}{\|\nabla v\|}, \quad (94)$$

we get

$$\tau \sum_{n=0}^{m} \|e^{n+1}_p\|^2 \leq C \tau \sum_{n=0}^{m} \left( \|\delta_t e^{n+1}_u\|^2 + R^{-1} \|\nabla e^{n+1}_u\|^2 + \|\nabla e^{n}_u\|^2 + \|e^{n+1}_q\|^2 + \|e^{n+1}_J\|^2 \right)$$

$$+ C \tau^2 \int_{0}^{m+1} \left( \|\nabla u_t\|^2 + \|u_t\|_{-1}^2 + \|J_t\|^2 \right) dt \leq C \tau^2.$$

The proof is complete. \qed

**Remark 4.3.** In this paper, we only focus on designing unconditionally energy-stable and linear SAV schemes for the inductionless MHD equations. The velocity and pressure can be further decoupled by using the classical pressure correction scheme [19, 21, 30], and we leave it to the interested readers.

5. Numerical experiments

In this section, we present a series of numerical experiments to verify the theoretical results of the proposed schemes. The numerical experiments are implemented on the finite element software FreeFEM [31].

5.1. The fully-discrete schemes

Although we only discussed semi-discretization in time in the previous sections, the SAV schemes can be coupled with any compatible spatial discretization. In this work, the spatial discretization is based on mixed finite element method.

Let $\mathcal{T}_h$ be a quasi-uniform and shape-regular tetrahedral mesh of $\Omega$. As usual, we introduce the local mesh size $h_K = \text{diam}(K)$ and the global mesh size $h := \max_{K \in \mathcal{T}_h} h_K$. For any integer $k \geq 0$, let $P_k(K)$ be the space of polynomials of degree $k$ on element $K$ and define $P_h(K) = P_k(K)^3$. Following [5, 12, 15], we employ the Mini-element to approximate the velocity and pressure

$$X_h = P^{h}_{1, h} \times Y_h = \left\{ r_h \in H^1(\Omega) : r_h|_{K} \in P_1(K), \forall K \in \mathcal{T}_h \right\} \cap Y,$$

where $P^{h}_{1, h} = \left\{ v_h \in C^0(\Omega) : v_h|_{K} \in P_1(K) \oplus \text{span}\{\delta\}, \forall K \in \mathcal{T}_h \right\}$, $\delta$ is a bubble function on $K$. We choose the lowest-order Raviart-Thomas element space given by

$$D_h = \left\{ K_h \in D : K_h|_{K} \in P_0(K) + xP_0(K), \forall K \in \mathcal{T}_h \right\},$$

combined with the discontinuous and piece-wise constant finite element space

$$S_h = \left\{ \psi_h \in L^2(\Omega) : \psi_h|_{K} \in P_0(K), \forall K \in \mathcal{T}_h \right\} \cap S.$$

From [32, 33, 34], the two finite element pairs, $(X_h, Y_h)$ and $(D_h, S_h)$, satisfy the following uniform inf-sup conditions,

$$\inf_{0 \neq q_h \in Q_h} \sup_{0 \neq v_h \in V_h} \frac{b_s(q_h, v_h)}{\|\nabla v_h\|_0 \|q_h\|} \geq \beta_s, \quad \inf_{0 \neq \psi_h \in S_h} \sup_{0 \neq K_h \in D_h} \frac{b_m(\psi_h, K_h)}{\|K_h\|_{\text{div}} \|\psi_h\|} \geq \beta_m,$$  \quad (95)

where $\beta_s$ and $\beta_m$ are constants independent of the mesh size.

Based on (17), a fully discrete first-order SAV scheme is as follows. For all $n \geq 0$, we find $(u^{n+1}_h, p^{n+1}_h, J^{n+1}_h, \phi^{n+1}_h) \in X_h \times Y_h \times D_h \times S_h$ and $q_h \in \mathbb{R}$ such that for all $(v_h, r_h, K_h, \psi_h) \in X_h \times Y_h \times D_h \times S_h$,

$$\begin{align*}
(\delta_t u^{n+1}_h, v_h) + R^{-1} (\nabla u^{n+1}_h, \nabla v_h) - (p^{n+1}_h, \nabla \cdot v_h) + q_h^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) ((u^n_h, \nabla u^n_h, v_h) - \kappa (J^n_h \times B^{n+1}_h, v_h)) &= 0, \\
(\nabla \cdot u^{n+1}_h, r_h) &= 0, \\
(J^{n+1}_h, K_h) - (\phi^{n+1}_h, \nabla \cdot K_h) - q_h^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) (u^n_h \times B^{n+1}_h, K_h) &= 0,
\end{align*} \quad (96a)$$

where $\kappa$ is the ratio of the magnetic Reynolds number to the magnetic Prandtl number.
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where \( \mathbf{J} = (\sin t, \cos t) \), \( \phi = \cos t \),

\begin{align*}
\nabla \cdot \mathbf{J}^{n+1}_h, \psi_h &= 0, \\
\delta q^{n+1}_h + \frac{q^{n+1}_h}{T} \exp \left( \frac{t^{n+1}}{T} \right) (u^n_h \cdot \nabla u^n_h, u^n_h) - \kappa (u^n_h \times B^{n+1}_h, J^{n+1}_h) - \kappa (J^n_h \times B^{n+1}_h, u^{n+1}_h) = 0. 
\end{align*}

For convenience, the init data \( q^0 = q^0 = 1, u^0 \) is taken as the standard interpolation of \( u^0 \) onto \( X_h, J^0 \) is obtained by solving discrete problem, for all \( (K_h, \psi_h) \in D_h \times S_h \),

\begin{equation}
(J^0_h, K_h) - (\phi^0_h, \nabla \cdot K_h) - (u^0 \times B^0, K_h) = 0, \\
\nabla \cdot J^0_h, \psi_h = 0.
\end{equation}

Similarly, a fully discrete version of the second-order scheme (28) is as follows. For all \( n \geq 0 \), we find \( (u^{n+1}_h, p^{n+1}_h, J^{n+1}_h, \phi^{n+1}_h) \in X_h \times Y_h \times D_h \times S_h \) and \( q_h \in \mathbb{R} \) such that for all \( (\psi_h, r_h, K_h, \psi_h) \in X_h \times Y_h \times D_h \times S_h \),

\begin{align*}
\delta^2 u^{n+1}_h + R^{-1}_c (\nabla u^{n+1}_h, \nabla v_h) + q^{n+1}_h \exp \left( \frac{t^{n+1}}{T} \right) \left( u^{n+1}_h \cdot \nabla u^{n+1}_h, v_h \right) - \kappa (J^{n+1}_h \times B^{n+1}_h, v_h) &= 0, \\
(\nabla \cdot u^{n+1}_h, r_h) &= 0, \\
(J^{n+1}_h, K_h) - (\phi^{n+1}_h, \nabla \cdot K_h) - q^{n+1}_h \exp \left( \frac{t^{n+1}}{T} \right) (u^{n+1}_h \times B^{n+1}_h, K_h) &= 0, \\
(\nabla \cdot J^{n+1}_h, \psi_h) &= 0, \\
\delta^2 q^{n+1}_h + \frac{q^{n+1}_h}{T} \exp \left( \frac{t^{n+1}}{T} \right) (u^{n+1}_h \cdot \nabla u^{n+1}_h, u^{n+1}_h) - \kappa (u^{n+1}_h \times B^{n+1}_h, J^{n+1}_h) - \kappa (J^{n+1}_h \times B^{n+1}_h, u^{n+1}_h) &= 0.
\end{align*}

The init data is set by the similar way as the first-order scheme (96).

Following the similar procedure as in the proof of Theorems 3.2 and 3.4, we can obtain the following stability result.

**Theorem 5.1.** The schemes (96) and (97) are unconditionally energy stable in the sense that the following energy estimate holds for \( n \geq 0 \),

\begin{align*}
\delta E^{n+1}_{EL,h} &\leq -R^{-1}_c \| \nabla u^{n+1}_h \|^2 - \kappa \| J^{n+1}_h \|^2 - \frac{1}{T} \| q^{n+1}_h \|^2, \\
\delta E^{n+1}_{BDF,h} &\leq -R^{-1}_c \| \nabla u^{n+1}_h \|^2 - \kappa \| J^{n+1}_h \|^2 - \frac{1}{T} \| q^{n+1}_h \|^2,
\end{align*}

where

\begin{align*}
E^{n+1}_{EL,h} := &\frac{1}{2} \| u^{n+1}_h \|^2 + \frac{1}{2} \| q^{n+1}_h \|^2, \\
E^{n+1}_{BDF,h} := &\frac{1}{4} \left( \| u^{n+1}_h \|^2 + \| 2u^{n+1}_h - \hat{u}_h \|^2 \right) + \frac{1}{4} \left( \| q^{n+1}_h \|^2 + \| 2q^{n+1}_h - \hat{q}_h \|^2 \right).
\end{align*}

**Proof.** Setting \( (\psi_h, r_h, K_h, \psi_h) = (u^{n+1}_h, p^{n+1}_h, \kappa J^{n+1}_h, \phi^{n+1}_h) \) in (96), multiplying (17e) by \( q^{n+1} \), and adding the resulting equations yields (98). The estimate (99) can be proved in a similar way and we omit it.

By using the non-local and scalar property of the auxiliary variable \( q^{n+1}_h \), we can carry out the fully-discrete schemes (96) and (97) efficiently as the semi-discrete schemes (17) and (28) in Section 3. We leave the detailed procedures to the interested readers.

5.2. Accuracy test

We first verify the first- and second-order accuracy of the proposed numerical schemes. The computational domain is set as \( \Omega = (0, 1)^d \), \( d = 2, 3 \), and the external magnetic field is \( \mathbf{B} = (0, 0, 1)^T \). The physical parameters are given by \( R = \kappa = 1 \) and the terminal time \( T = 1 \). The right-hand sides, the initial condition and the Dirichlet boundary conditions are chosen so that so that the exact solution is given by

\begin{align*}
\mathbf{u} &= (\exp (-t), x \cos (t)), \quad p = \sin (t), \quad \mathbf{J} = (\sin (t), \cos (t)), \quad \phi = \cos (t),
\end{align*}
for $d = 2$ and

$$u = (z \sin(t), x, y \exp(-t)), \quad p = 0, \quad J = (\cos(t), t^2, 0), \quad \phi = 0.$$

for $d = 3$.

Note that the exact solutions are linear or constant in space, the only error comes from the discretization of the time variable. We fix a mesh size with $h = 1/6$ and test the convergence rate with respect to the time step. The errors and convergence orders are displayed in Tables 1-2 for $d = 2$ and Tables 3-4 for $d = 3$, respectively. From these tables, we observe that the errors of all variable become smaller and smaller as the time step is refined. Besides, the corresponding convergence rates are of the order of $O(\tau)$ for the first-order scheme and $O(\tau^2)$ for the second-order scheme asymptotically. This accord well with our theoretical analysis.

Table 1: Errors and convergence rates for the first-order scheme (17) in 2D.

| $\tau$   | $\|e^N_u\|$ | $\|\nabla e^N_u\|$ | $\|e^N_p\|$ | $\|e^N_J\|$  | $\|e^N_q\|$ |
|----------|--------------|---------------------|-------------|-------------|-------------|
| 0.2      | 2.13e-04     | 1.74e-03            | 3.53e-02    | 7.88e-06    | 1.29e-05    |
| 0.1      | 1.03e-04     | 8.35e-04            | 3.17e-02    | 3.51e-06    | 1.19e-02    |
| 0.05     | 5.02e-05     | 4.09e-04            | 1.64e-06    | 6.16e-03    | 2.01e-03    |
| 0.025    | 2.49e-05     | 2.02e-04            | 7.88e-07    | 3.14e-03    | 8.93e-04    |
| 0.0125   | 1.24e-05     | 1.01e-04            | 2.47e-03    | 1.58e-03    | 4.17e-04    |

Table 2: Errors and convergence rates for the second-order scheme (28) in 2D.

| $\tau$   | $\|e^N_u\|$ | $\|\nabla e^N_u\|$ | $\|e^N_p\|$ | $\|e^N_J\|$  | $\|e^N_q\|$ |
|----------|--------------|---------------------|-------------|-------------|-------------|
| 0.2      | 3.33e-05     | 2.71e-04            | 7.23e-03    | 9.06e-07    | 4.54e-03    |
| 0.1      | 8.42e-06     | 6.86e-05            | 1.68e-03    | 2.55e-07    | 1.02e-03    |
| 0.05     | 2.12e-06     | 1.73e-05            | 4.31e-04    | 6.42e-08    | 2.40e-04    |
| 0.025    | 5.32e-07     | 4.33e-06            | 1.11e-04    | 5.82e-05    | 5.42e-05    |
| 0.0125   | 1.33e-07     | 1.08e-06            | 2.81e-05    | 4.03e-09    | 1.43e-05    |

Table 3: Errors and convergence rates for the first-order scheme (17) in 3D.

| $\tau$   | $\|e^N_u\|$ | $\|\nabla e^N_u\|$ | $\|e^N_p\|$ | $\|e^N_J\|$  | $\|e^N_q\|$ |
|----------|--------------|---------------------|-------------|-------------|-------------|
| 0.2      | 4.21e-04     | 3.90e-03            | 1.56e-01    | 2.43e-02    | 5.66e-02    |
| 0.1      | 1.99e-04     | 1.85e-03            | 7.67e-02    | 1.62e-02    | 3.08e-02    |
| 0.05     | 9.93e-05     | 9.26e-04            | 3.78e-02    | 9.06e-03    | 1.60e-02    |
| 0.025    | 4.99e-05     | 4.67e-04            | 1.88e-02    | 4.76e-03    | 8.13e-03    |
| 0.0125   | 2.51e-05     | 2.35e-04            | 9.34e-03    | 2.44e-03    | 4.10e-03    |

Table 4: Errors and convergence rates for the second-order scheme (28) in 3D.

| $\tau$   | $\|e^N_u\|$ | $\|\nabla e^N_u\|$ | $\|e^N_p\|$ | $\|e^N_J\|$  | $\|e^N_q\|$ |
|----------|--------------|---------------------|-------------|-------------|-------------|
| 0.2      | 2.38e-04     | 2.24e-03            | 6.03e-02    | 3.04e-02    | 3.43e-02    |
| 0.1      | 6.26e-05     | 5.89e-04            | 1.54e-01    | 7.86e-03    | 8.93e-03    |
| 0.05     | 1.06e-05     | 1.30e-04            | 3.90e-03    | 2.00e-03    | 2.28e-03    |
| 0.025    | 4.03e-06     | 3.80e-05            | 9.81e-04    | 5.05e-04    | 5.74e-04    |
| 0.0125   | 1.01e-06     | 9.54e-06            | 2.46e-02    | 1.27e-04    | 1.44e-04    |

5.3. Stability test

This example is devoted to test the stability of the of the proposed SAV schemes. We set the computed domain to be $\Omega = (0, 1)^d$, $d = 2, 3$ and the external magnetic field to be $B = (0, 0, 1)^T$. The physical parameters are given by $R_e = \kappa = 20, 100$ and the terminal time $T = 3$. The initial condition is chosen as

$$u^0 = (\sin(\pi x) \cos(\pi y), -\cos(\pi x) \sin(\pi y)).$$

for $d = 2$ and

$$u^0 = -\pi/2 \sin(\pi x) \sin(\pi y) \Psi,$$

$$\Psi = (\sin(\pi x) \cos(\pi y) \cos(\pi z), -2 \cos(\pi x) \sin(\pi y) \cos(\pi z), \cos(\pi x) \cos(\pi y) \sin(\pi z))^T.$$

for $d = 3$.

With the prescribed data, we test the energy stability of the SAV schemes on the fixed mesh size with $h = 1/150$ for 2D and $h = 1/16$ for 3D. Figures 1-2 present the time evolution of the energy for different time steps. We observe that all energy curves decay monotonically for all time step sizes in both 2D and 3D. This confirms that the SAV schemes are unconditionally energy stable.
(a) First-order SAV scheme with $R_e = \kappa = 20$.

(b) First-order SAV scheme with $R_e = \kappa = 100$.

(c) Second-order SAV scheme with $R_e = \kappa = 20$.

(d) Second-order SAV scheme with $R_e = \kappa = 100$.

Figure 1: Time evolution of the energy for different time step sizes in 2D.
(a) First-order SAV scheme with $R_e = \kappa = 20$.

(b) First-order SAV scheme with $R_e = \kappa = 100$.

(c) Second-order SAV scheme with $R_e = \kappa = 20$.

(d) Second-order SAV scheme with $R_e = \kappa = 100$.

Figure 2: Time evolution of the energy for different time step sizes in 3D.
6. Concluding remarks

In this paper, we propose and analyze some SAV schemes for inductionless MHD equations. The attractive points of these schemes are they are decoupled, linear, unconditionally energy stable and easy to implement. We further derive

5.4. Lid driven cavity

In this example, we consider a well-known benchmark problem in fluid dynamics, known as lid-cavity flow. For this end, we assume that the cavity is a unit cubic in 3D. The physical parameters are set by $Re = 200, \kappa = 10$. The applied magnetic field is $B = (1, 0, 0)^T$ and the initial values are given by $u_0 = (g_1, 0, 0)^T$, where $g_1 = g_1(z)$ is a continuous function and satisfies

$$g_1(x, y, 1) = 1,$$

$$g_1(x, y, z) = 0 \quad \forall z \in [0, 1 - h],$$

where $h$ is the mesh size. The boundary conditions are set by

$$u = u_0, \quad \phi = 0 \quad \text{on} \quad \Gamma.$$

Note that our schemes apply equally to the above boundary conditions.

For this problem, we want to see how the fluid flows under the influence of the magnetic field. We perform the numerical tests by using the second-order SAV scheme with the mesh size $h = 1/32$ and the time step $\tau = 0.01$. For convenience, the terminal time $T$ is set by $T = 10$. Figure 3 displays the streamlines of $u_h$ and the distributions of $|J|_h$ on the cross-section $y = 0.5$ at the terminal time. It can be seen that the structure of vortex is similar to those reported in [5, 35] where the steady inductionless MHD equations are considered. To investigate the formation of the final vortex, we show some snapshots of the streamlines of $u_h$ on the cross-section $y = 0.5$ in Figure 4. Our numerical results indicate that the fluid yields more large vertices and tends to be stratified as time evolves until the physical fields reach steady states.

Figure 3: Streamlines of $u_h$ and distributions of $|J|_h$ on the cross-section $y = 0.5$ (Right).

Figure 4: Time-evolution of streamlines of $u_h$ on the cross-section $y = 0.5$. Left: $t = 0.1$, Middle: $t = 1$, Right: $t = 2$. 
rigorous error estimates for the first-order scheme in the two-dimensional case without any condition on the time step. A series of numerical experiments are given to confirm the theoretical findings and show the performances of the schemes.

Remarkably, we only present the error analysis for the first-order scheme in the two-dimensional case. We believe that the error estimates can also be established for the second-order scheme in the two-dimensional case, although the process will surely be much more tedious. However, it appears that the error estimates cannot be easily extended to the three-dimensional case, as our proof uses essentially some inequalities which are only valid in the two-dimensional case. In the further, the error estimates in three dimensions will be considered. Moreover, we have only considered time discretization in this work. Error analysis for full discretization will be left as a subject of future endeavors.

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