Quark Helicity and Transversity Distributions

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Abstract

The quark transversity distribution inside nucleon is less understood than the quark unpolarized and helicity distributions inside nucleon. In particular, it is important to know clearly why the quark helicity and transversity distributions are different. We investigate the origin of their discrepancy.

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1 Introduction

The unpolarized distribution $f_1(x)$ and the helicity distribution $g_1(x)$ of quarks inside nucleon have been extensively investigated. However, the transversity distribution $h_1(x)$ is less known since it can not be measured in fully deep inelastic scattering since it is chiral-odd. The transversity distribution $h_1(x)$ can be extracted by measuring the double-spin asymmetry in the Drell-Yan process $A^\uparrow B^\uparrow \rightarrow l^+ l^- X$, where $A^\uparrow$ and $B^\uparrow$ are two transversely polarized protons or antiprotons, $l^+ l^-$ are lepton pairs and $X$ is the undetected hadronic system [1]. There is also an approach which applies the Collins mechanism [2] to the single-spin asymmetry in the process $lp^\uparrow \rightarrow l\pi X$ of semi-inclusive deep inelastic scattering. By these methods important experimental and theoretical progresses have been made in investigating the quark transversity distribution inside nucleon [3 4 5 6].

The transversity distribution was first introduced in Ref. [7], and then there have been extensive studies on this subject [1 8 9 10 11]. However, it is desirable to have a better understanding of the origin of the difference between $g_1(x)$ and $h_1(x)$. For example, we can find the following sentence in Ref. [11]: “It would be very useful to have a better idea of the dynamical and relativistic effects which generate differences between $g_1$ and $h_1$.” In this paper we show that the discrepancy between the helicity and transversity distributions is rooted in the difference between the Bjorken-Drell spinors and the light-front spinors. As a result, the quark helicity and transversity distributions are equal if the quark has no transverse momentum, since the Bjorken-Drell spinors and the light-front spinors are the same when the transverse momentum is zero.

In this paper we show that the precise description of $f_1(x)$, $g_1(x)$ and $h_1(x)$ is as follows: In a hadron $f_1(x)$ is the probability of finding a quark with momentum fraction $x$ of the longitudinal momentum of the hadron. In a longitudinally polarized hadron $g_1(x)$ is the number density of quarks with momentum fraction $x$ and in the spin state of $u_1^{LF}(k)$ minus the number density of quarks with the same momentum
fraction and in the spin state of $u_2^{LF}(k)$. In a hadron transversely polarized to the positive $x$ direction $h_1(x)$ is the number density of quarks with momentum fraction $x$ and in the spin state of $\frac{1}{\sqrt{2}}(u_1^{LF}(k) + u_2^{LF}(k))$ minus the number density of quarks with the same momentum fraction and in the spin state of $\frac{1}{\sqrt{2}}(u_1^{LF}(k) - u_2^{LF}(k))$, where $u_1^{LF}(k)$ and $u_2^{LF}(k)$ are the light-front spinors defined in Eq. (12).

In the literature the transversity distribution $h_1(x)$ is commonly described by an expression like “In a transversely polarised hadron $h_1(x)$ is the number density of quarks with momentum fraction $x$ and polarization parallel to that of the hadron minus the number density of quarks with the same momentum fraction and antiparallel polarization.” In this description it is not clear what is meant by “polarization parallel to that of the hadron”. We described $h_1(x)$ in the later part of the previous paragraph without such ambiguity. In reality the quark spin state of $\frac{1}{\sqrt{2}}(u_1^{LF}(k) + u_2^{LF}(k))$ is not the spin state polarized along the positive $x$ direction. In addition it is not accurate to call $g_1(x)$ the helicity distribution, since the helicity eigenstate spinors and the light-front spinors are the same only when the quark mass is zero. In this paper we explain these properties and show why $g_1(x)$ and $h_1(x)$ are different and when they are equal.

2 Parton Distributions

2.1 Definitions

The transverse momentum dependent parton distributions are defined through the vector, axial-vector and tensor currents:

$$\int \frac{dy^-d^2\vec{y}_\perp}{16(\pi)^3} e^{ixP^+ - ik\cdot\vec{y}_\perp} \langle P, \lambda' | \bar{\psi}(0) \gamma^+ \psi(y) | P, \lambda \rangle |_{y^+ = 0}$$

$$= \frac{1}{2P^+} \mathcal{U}(P, \lambda') f_1(x, \vec{k}_\perp) \gamma^+ U(P, \lambda),$$

$$\int \frac{dy^-d^2\vec{y}_\perp}{16(\pi)^3} e^{ixP^+ - ik\cdot\vec{y}_\perp} \langle P, \lambda' | \bar{\psi}(0) \gamma^+ \gamma_5 \psi(y) | P, \lambda \rangle |_{y^+ = 0}$$

$$= \frac{1}{2P^+} \mathcal{U}(P, \lambda') g_1(x, \vec{k}_\perp) \gamma^+ \gamma_5 U(P, \lambda),$$
where the parton distributions over $f$ are expressed in terms of the light-front wavefunctions as

\[ f_1(x) = \int [d^2k_\perp] f_1(x, k_\perp), \]
\[ g_1(x) = \int [d^2k_\perp] g_1(x, k_\perp), \]
\[ h_1(x) = \int [d^2k_\perp] h_1(x, k_\perp), \]

(2)

where $[d^2k_\perp]$ is $d^2k_\perp$ times a common overall constant which normalizes $f_1(x)$ to satisfy $\int_0^1 f_1(x) \, dx = 1$.

### 2.2 Wavefunction Representations

The state of proton is represented by the light-front Fock expansion \[12, 13\]:

\[
\left| \psi_p(P^+, \vec{P}_\perp; \lambda) \right> = \sum_n \prod_{i=1}^n \frac{dx_i \, d^2k_{\perp i}}{\sqrt{x_i} \, 16\pi^3} \, 16\pi^3 \delta \left( 1 - \sum_{i=1}^n x_i \right) \delta^{(2)} \left( \sum_{i=1}^n k_{\perp i} \right) \times \psi_n(x_i, k_{\perp i}, \lambda_i) \left| n; x_i P^+, x_i \vec{P}_\perp + k_{\perp i}, \lambda_i \right>,
\]

(3)

where $x_i = k_i^+ / P^+$ and $k_{\perp i}$ is the relative transverse momentum of constituent. From (1) and (3) we find that the transverse momentum dependent parton distributions are expressed in terms of the light-front wavefunctions as

\[ f_1(x, k_\perp) = \mathcal{A} \psi_n^{\uparrow \ast}(x_i, k_{\perp i}, \lambda_i) \psi_n^{\uparrow}(x_i, k_{\perp i}, \lambda_i), \]
\[ g_1(x, k_\perp) = \mathcal{A} \lambda_1 \psi_n^{\uparrow \ast}(x_i, k_{\perp i}, \lambda_i) \psi_n^{\uparrow}(x_i, k_{\perp i}, \lambda_i), \]
\[ h_1(x, k_\perp) = \mathcal{A} (-\lambda_1) \psi_n^{\uparrow \ast}(x_i, k_{\perp i}, \lambda_i) \psi_n^{\uparrow}(x_i, k_{\perp i}, \lambda_i), \]

(4)
where
\[
\mathcal{A} = \sum_{n,\lambda_i} \int \frac{d^4 x_i q_i^2 k_{i\perp}}{16\pi^3} \int (1 - \sum_{j=1}^{n} x_j) \delta^{(2)}(x - x_1) \delta^{(2)}(k_{\perp} - k_{\perp 1}) .
\]  

(5)

The formulas given in (4) can be used to find the transverse momentum dependent distributions in an explicit model. These formulas can also be applied in getting model independent relations. For example, we can show the Soffer’s inequality \[14\]. After some calculations, from (4) we get
\[
\left[ (f_1(x, k_{\perp}) + g_1(x, k_{\perp})) \pm 2 h_1(x, k_{\perp}) \right] = \mathcal{A}
\times \left[ \psi^\dagger_{(n)}(x_i, k_{\perp i}, \lambda_1 = \uparrow, \lambda_i \neq 1) \pm \psi^\dagger_{(n)}(x_i, k_{\perp i}, \lambda_1 = \downarrow, \lambda_i \neq 1) \right]
\times \left[ \psi_{(n)}(x_i, k_{\perp i}, \lambda_1 = \uparrow, \lambda_i \neq 1) \pm \psi_{(n)}(x_i, k_{\perp i}, \lambda_1 = \downarrow, \lambda_i \neq 1) \right] ,
\]  
which shows the Soffer’s inequality as
\[
(f_1(x, k_{\perp}) + g_1(x, k_{\perp})) \pm 2 h_1(x, k_{\perp}) \geq 0 ,
\]  
where the equality holds when
\[
\psi^\dagger_{(n)}(x_i, k_{\perp i}, \lambda_1 = \uparrow, \lambda_i \neq 1) \pm \psi^\dagger_{(n)}(x_i, k_{\perp i}, \lambda_1 = \downarrow, \lambda_i \neq 1) = 0 .
\]  

(8)

From the formulas for \( f_1(x) \), \( g_1(x) \) and \( h_1(x) \) given by Eqs. (2) and (4), we can show the following: In a hadron \( f_1(x) \) is the probability of finding a quark with momentum fraction \( x \) of the longitudinal momentum of the hadron. In a longitudinally polarized hadron \( g_1(x) \) is the number density of quarks with momentum fraction \( x \) and in the spin state of \( u_{1\perp}^L(k) \) minus the number density of quarks with the same momentum fraction and in the spin state of \( u_{2\perp}^L(k) \). In a hadron transversely polarized to the positive \( x \) direction \( h_1(x) \) is the number density of quarks with momentum fraction \( x \) and in the spin state of \( \frac{1}{\sqrt{2}}(u_{1\perp}^L(k) + u_{2\perp}^L(k)) \) minus the number density of quarks with the same momentum fraction and in the spin state of \( \frac{1}{\sqrt{2}}(u_{1\perp}^L(k) - u_{2\perp}^L(k)) \), where \( u_{1\perp}^L(k) \) and \( u_{2\perp}^L(k) \) are the light-front spinors given in Eq. (12). We emphasize that the quark spin states of \( u_{1\perp}^L(k) \) and \( u_{2\perp}^L(k) \) are not the quark helicity eigenstates
when quark mass is not zero as we can see in Eqs. (44) and (48) in Appendix A, and the quark spin states of $\frac{1}{\sqrt{2}}(u_1^{LF}(k) \pm u_2^{LF}(k))$ are not the angular momentum eigenstates polarized along the $\pm x$ directions. Those eigenstates are the quark spin states of $\frac{1}{\sqrt{2}}(u_1^{BD}(k) \pm u_2^{BD}(k))$, where $u_1^{BD}(k)$ and $u_2^{BD}(k)$ are the Bjorken-Drell spinors given in Eq. (11). We show this property in Appendix B.

3 The reason why $g_1(x) \neq h_1(x)$

We use the notations $k^R = k_1 + ik^2$, $k^L = k_1 - ik^2$, $k^\pm = k^0 \pm k^3$, and the $\gamma$ matrices in the Dirac representation:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\sigma^i$ are the Pauli matrices given in Appendix A.

We consider two sets of the positive energy solutions of the Dirac equation

$$(k - m) u(k) = 0. \quad (10)$$

The Bjorken-Drell spinors are two linearly independent solutions of (10) \cite{15} given by

$$u_1^{BD}(k) = \frac{1}{\sqrt{k^0 + m}} \begin{pmatrix} k^0 + m \\ 0 \\ k^3 \\ k^R \end{pmatrix}, \quad u_2^{BD}(k) = \frac{1}{\sqrt{k^0 + m}} \begin{pmatrix} 0 \\ k^0 + m \\ k^L \\ -k^3 \end{pmatrix}. \quad (11)$$

The light-front spinors are another set of linear combinations of the solutions of (10) \cite{16, 17, 18} given, in the convention of Ref. \cite{17}, by

$$u_1^{LF}(k) = \frac{1}{\sqrt{2k^+}} \begin{pmatrix} k^+ + m \\ k^R \\ k^+ - m \\ k^R \end{pmatrix}, \quad u_2^{LF}(k) = \frac{1}{\sqrt{2k^+}} \begin{pmatrix} -k^L \\ k^+ + m \\ k^L \\ -k^+ + m \end{pmatrix}. \quad (12)$$
The two sets $u^{BD}(k)$ in (11) and $u^{LF}(k)$ in (12) are related as:

$$
u_{BD1}^{(k)} = \frac{1}{\sqrt{(k^+ + m)^2 + k^2_\perp}} \left( (k^+ + m) u_{1}^{LF}(k) - k^R u_{2}^{LF}(k) \right),$$

$$
u_{BD2}^{(k)} = \frac{1}{\sqrt{(k^+ + m)^2 + k^2_\perp}} \left( k^L u_{1}^{LF}(k) + (k^+ + m) u_{2}^{LF}(k) \right).$$

(13)

We organize the relations among the Bjorken-Drell spinors, the light-front spinors and the helicity eigenstate spinors in Appendix A.

The Bjorken-Drell spinors $u_{1,2}^{BD}(k)$ given in (11) satisfy

$$j^+ u_{1}^{BD}(k) = 0, \quad j^- u_{2}^{BD}(k) = 0, \quad j^- u_{1}^{BD}(k) = u_{2}^{BD}(k), \quad j^+ u_{2}^{BD}(k) = u_{1}^{BD}(k),$$

(14)

where $j^\pm = j^1 \pm j^2$ and

$$j^i = s^i + l^i, \quad s^i = \frac{1}{2} \Sigma^i = \frac{1}{2} \left( \begin{array}{cc} \sigma^i & 0 \\ 0 & \sigma^i \end{array} \right), \quad l^i = -i \epsilon^{ijk} k^j \frac{\partial}{\partial k^k}, \quad \epsilon^{123} = 1.$$ 

(15)

We explain in Appendix B the reason why Bjorken-Drell spinors satisfy (14).

Since $u_{1,2}^{BD}(k)$ satisfy the transformation properties (14), they are spin-half states which are eigenstates of $j^3$ with eigenvalues $\pm \frac{1}{2}$ and the $\frac{1}{\sqrt{2}}(u_1^{BD}(k) \pm u_2^{BD}(k))$ state are spin-half states which are eigenstates of $j^2$ with the eigenvalues $\pm \frac{1}{2}$. Then we can construct the proton spin states $|P; \lambda = \pm \frac{1}{2} >$ by using the Clebsch-Gordan coefficients with $u_{1,2}^{BD}(k)$ for the quark state, which we will do in the next section. The proton states $|P; \lambda = \pm \frac{1}{2} >$ satisfy

$$J^+ |P; \lambda = \frac{1}{2} >= 0, \quad J^- |P; \lambda = -\frac{1}{2} >= 0$$

$$J^- |P; \lambda = \frac{1}{2} >= |P; \lambda = -\frac{1}{2} >, \quad J^+ |P; \lambda = -\frac{1}{2} >= |P; \lambda = \frac{1}{2} >,$$

(16)

where $J^i$ is the total angular momentum operator for the proton given by $J^i = \sum_a \mathbf{j}^i = \sum_a (s^i_a + l^i_a)$, which is the sum over the constituents $a$, and $J^\pm = J^1 \pm J^2$.

We define $g_1^{BD}(x)$ as the probability of the quark’s being in the $u_1^{BD}(k)$ state minus that of being in the $u_2^{BD}(k)$ state when the proton’s state is $|\lambda = +\frac{1}{2} >$, where the $u_1^{BD}(k)$, $u_2^{BD}(k)$ and $|\lambda = +\frac{1}{2} >$ states are all angular momentum eigenstates of
spin-half with the eigenvalues of $j^3$ (or $J^3$) as $+\frac{1}{2}$ or $-\frac{1}{2}$. We define $h_1^{\text{BD}}(x)$ as the probability of the quark’s being in the $\frac{1}{\sqrt{2}}(u_1^{\text{BD}}(k) + u_2^{\text{BD}}(k))$ state minus that of being in the $\frac{1}{\sqrt{2}}(u_1^{\text{BD}}(k) - u_2^{\text{BD}}(k))$ state when the proton’s state is $\frac{1}{\sqrt{2}}(|\lambda = +\frac{1}{2} > + |\lambda = -\frac{1}{2} >)$. Here, $|\lambda = \pm \frac{1}{2} >$ means $|P; \lambda = \pm \frac{1}{2} >$. When the proton spin states $|P; \lambda >$ satisfy (16), $\frac{1}{\sqrt{2}}(|\lambda = +\frac{1}{2} > + |\lambda = -\frac{1}{2} >)$ is an angular momentum eigenstate of spin-half with the eigenvalue of $j^2$ as $+\frac{1}{2}$, whereas the $\frac{1}{\sqrt{2}}(u_1^{\text{BD}}(k) + u_2^{\text{BD}}(k))$ and $\frac{1}{\sqrt{2}}(u_1^{\text{BD}}(k) - u_2^{\text{BD}}(k))$ states are angular momentum eigenstates of spin-half with the eigenvalues of $j^2$ as $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Therefore, the former and latter three states are equivalent and only their quantization axes are different. Then, the relation $g_1^{\text{BD}}(x) = h_1^{\text{BD}}(x)$ is satisfied.

On the other hand, the helicity and transversity distributions $g_1(x)$ and $h_1(x)$ given by Eqs. (2) and (4) can be interpreted as: $g_1(x)$ is the probability of the quark’s being in the $u_1^{\text{LF}}(k)$ state minus that of being in the $u_2^{\text{LF}}(k)$ state when the proton’s state is $|\lambda = +\frac{1}{2} >$, and $h_1(x)$ is the probability of the quark’s being in the $\frac{1}{\sqrt{2}}(u_1^{\text{LF}}(k) + u_2^{\text{LF}}(k))$ state minus that of being in the $\frac{1}{\sqrt{2}}(u_1^{\text{LF}}(k) - u_2^{\text{LF}}(k))$ state when the proton’s state is $\frac{1}{\sqrt{2}}(|\lambda = +\frac{1}{2} > + |\lambda = -\frac{1}{2} >)$. That is, $g_1(x)$ and $h_1(x)$ are related with the light-front spinors. On the contrary to the case of the Bjorken-Drell spinors, the light-front spinors given in (12) do not satisfy the property (14) and then the $\frac{1}{\sqrt{2}}(u_1^{\text{LF}}(k) \pm u_2^{\text{LF}}(k))$ states are not eigenstates of $j^2$, whereas the $\frac{1}{\sqrt{2}}(|\lambda = +\frac{1}{2} > + |\lambda = -\frac{1}{2} >)$ proton state is an eigenstate of $J^2$ with eigenvalue $+\frac{1}{2}$. Therefore, the situation for $g_1(x)$ and that for $h_1(x)$ are not equivalent, and then $g_1(x)$ and $h_1(x)$ are different. We will see these properties in explicit examples in the next section. When the transverse momentum is zero, the Bjorken-Drell spinors and the light-front spinors are the same as we can see in (13). Therefore, when the quark transverse momentum is zero, $g_1(x)$ and $h_1(x)$ are equal.
4 Explicit Calculations in Diquark Models

In this section we perform explicit calculations in diquark models in order to see by examples what we found in previous sections. We use the Bjorken-Drell spinors for the quark spin states when we construct the nucleon spin states by using the Clebsch-Gordan coefficients, and then the resulting nucleon spin states become eigenstates of the total angular momentum.

4.1 S-wave Scalar Diquark Model

The nucleon state composed of a scalar diquark and an S-wave quark is represented as

\[ |\uparrow\rangle_N = |\uparrow\rangle_q \sqrt{\frac{1}{4\pi}} R_0(|\vec{k}|), \]

\[ |\downarrow\rangle_N = |\downarrow\rangle_q \sqrt{\frac{1}{4\pi}} R_0(|\vec{k}|). \]  \hspace{1cm} (17)

The nucleon state represented by (17) have the distribution functions given by

\[ f_1(x) = \int \left[ d^2\vec{k}_\perp \right] \frac{1}{4\pi} \left( R_0(|\vec{k}|) \right)^2, \]

\[ g_1(x) = \int \left[ d^2\vec{k}_\perp \right] \frac{1}{4\pi} \left( R_0(|\vec{k}|) \right)^2 \frac{1}{(k^+ + m)^2 + \vec{k}_\perp^2} \left[ (k^+ + m)^2 - \vec{k}_\perp^2 \right], \]

\[ h_1(x) = \int \left[ d^2\vec{k}_\perp \right] \frac{1}{4\pi} \left( R_0(|\vec{k}|) \right)^2 \frac{1}{(k^+ + m)^2 + \vec{k}_\perp^2} \left[ (k^+ + m)^2 \right], \]

\[ g_1^{BD}(x) = \int \left[ d^2\vec{k}_\perp \right] \frac{1}{4\pi} \left( R_0(|\vec{k}|) \right)^2. \]  \hspace{1cm} (18)

We checked by explicit calculation that \( h_1^{BD}(x, \vec{k}_\perp) \) is the same as \( g_1^{BD}(x, \vec{k}_\perp) \) given in (18). From the results in (18), we see that the Soffer’s inequality is satisfied with equality in this model: \( f_1(x) + g_1(x) = 2h_1(x). \)
4.2 S-wave Axial-vector Diquark Model

The nucleon state composed of an axial-vector diquark and an S-wave quark is represented as

\[
| \uparrow_N \rangle = \left( -\sqrt{\frac{1}{3}} | \uparrow_q \rangle | 10 >_{av} + \sqrt{\frac{2}{3}} | \downarrow_q \rangle | 1 + 1 >_{av} \right) \sqrt{\frac{1}{4\pi}} R_0(|\vec{k}|),
\]

\[
| \downarrow_N \rangle = \left( -\sqrt{\frac{2}{3}} | \uparrow_q \rangle | 1 - 1 >_{av} + \sqrt{\frac{1}{3}} | \downarrow_q \rangle | 10 >_{av} \right) \sqrt{\frac{1}{4\pi}} R_0(|\vec{k}|).
\]

The nucleon state represented by (19) have the distribution functions given by

\[
f_1(x) = \int \frac{d^2\vec{k}_\perp}{4\pi} \left( R_0(|\vec{k}|) \right)^2,
\]

\[
g_1(x) = -\frac{1}{3} \int \frac{d^2\vec{k}_\perp}{4\pi} \left( R_0(|\vec{k}|) \right)^2 \frac{1}{(k^2 + m^2 + k^2_\perp)} \left[ (k^2 + m^2 - k^2_\perp) \right],
\]

\[
h_1(x) = -\frac{1}{3} \int \frac{d^2\vec{k}_\perp}{4\pi} \left( R_0(|\vec{k}|) \right)^2 \frac{1}{(k^2 + m^2 + k^2_\perp)} \left[ (k^2 + m^2) \right],
\]

\[
g_1^{BD}(x) = -\frac{1}{3} \int \frac{d^2\vec{k}_\perp}{4\pi} \left( R_0(|\vec{k}|) \right)^2.
\]

We checked by explicit calculation that \( h_1^{BD}(x, \vec{k}_\perp) \) is the same as \( g_1^{BD}(x, \vec{k}_\perp) \) given in (20). From the results in (20), we see that the Soffer’s inequality \( f_1(x) + g_1(x) > 2|h_1(x)| \) is satisfied.

4.3 P-wave Scalar Diquark Model

In this section we consider the P-wave scalar diquark model, in which the orbital angular momentum of the quark is incorporated in the spin contents of nucleon. We consider here the scalar diquark to be a pseudo-scalar one in order that the parity of the nucleon is even. Following the usual construction, the nucleon state composed of a scalar diquark and a P-wave quark is represented as [19]

\[
| \uparrow_N \rangle = \left( -\sqrt{\frac{1}{3}} | \uparrow_q \rangle | Y_{10}(\hat{k}) \rangle + \sqrt{\frac{2}{3}} | \downarrow_q \rangle | Y_{1+1}(\hat{k}) \rangle \right) R_1(|\vec{k}|),
\]

\[
| \downarrow_N \rangle = \left( -\sqrt{\frac{2}{3}} | \uparrow_q \rangle | Y_{1-1}(\hat{k}) \rangle + \sqrt{\frac{1}{3}} | \downarrow_q \rangle | Y_{10}(\hat{k}) \rangle \right) R_1(|\vec{k}|),
\]

(21)
where
\[ Y_{10}(k) = \sqrt{\frac{3}{4\pi}} \frac{k^3}{|k|}, \quad Y_{1\pm 1}(k) = \mp \sqrt{\frac{3}{8\pi}} \frac{k^1 \pm ik^2}{|k|}. \] (22)

The nucleon state represented by (21) have the distribution functions given by
\[
\begin{align*}
    f_1(x) &= \int \left[ d^2 k_\perp \frac{1}{4\pi} (R_1(|\mathbf{k}|))^2 \frac{(k^0 + m)^2}{(k^+ + m)^2 + k_\perp^2} \frac{1}{|\mathbf{k}|^2} \frac{1}{|\mathbf{k}|^2} \right], \\
    g_1(x) &= \int \left[ d^2 k_\perp \frac{1}{4\pi} (R_1(|\mathbf{k}|))^2 \frac{(k^0 + m)^2}{(k^+ + m)^2 + k_\perp^2} \frac{1}{|\mathbf{k}|^2} \frac{1}{|\mathbf{k}|^2} \right], \\
    h_1(x) &= \int \left[ d^2 k_\perp \frac{1}{4\pi} (R_1(|\mathbf{k}|))^2 \frac{(k^0 + m)^2}{(k^+ + m)^2 + k_\perp^2} \frac{1}{|\mathbf{k}|^2} \frac{1}{|\mathbf{k}|^2} \right], \\
    g_1^{BD}(x) &= \int \left[ d^2 k_\perp \frac{1}{4\pi} (R_1(|\mathbf{k}|))^2 \left[ -\frac{1}{3} \right] \right]. \quad (23)
\end{align*}
\]

We checked by explicit calculation that \( h_1^{BD}(x, \tilde{k}_\perp) \) is the same as \( g_1^{BD}(x, \tilde{k}_\perp) \) given in (23). From the results in (23), we see that the Soffer’s inequality is satisfied with equality in this model: \( f_1(x) + g_1(x) = 2|h_1(x)| \).

5 Conclusion

When we define \( g_1^{BD}(x) \) as the probability of the quark’s being in the \( u_1^{BD}(k) \) state minus that of being in the \( u_2^{BD}(k) \) state when the proton’s state is \( |\lambda = +\frac{1}{2} > \), and define \( h_1^{BD}(x) \) as the probability of the quark’s being in the \( \frac{1}{\sqrt{2}}(u_1^{BD}(k) + u_2^{BD}(k)) \) state minus that of being in the \( \frac{1}{\sqrt{2}}(u_1^{BD}(k) - u_2^{BD}(k)) \) state when the proton’s state is \( \frac{1}{\sqrt{2}}(|\lambda = +\frac{1}{2} > +|\lambda = -\frac{1}{2} >) \), the relation \( g_1^{BD}(x) = h_1^{BD}(x) \) is satisfied. The reason for the above is the following: \( u_1^{BD}(k) \), \( u_2^{BD}(k) \) and \( |\lambda = +\frac{1}{2} > \) are all angular momentum eigenstates of spin-half with the eigenvalues of \( J^3 \) (or \( J_z \)) as \( +\frac{1}{2} \) or \( -\frac{1}{2} \), and \( \frac{1}{\sqrt{2}}(u_1^{BD}(k) + u_2^{BD}(k)) \), \( \frac{1}{\sqrt{2}}(u_1^{BD}(k) - u_2^{BD}(k)) \) and \( \frac{1}{\sqrt{2}}(|\lambda = +\frac{1}{2} > +|\lambda = -\frac{1}{2} >) \) are all angular momentum eigenstates of spin-half with the eigenvalues of \( J^2 \) (or \( J_z^2 \)) as \( +\frac{1}{2} \) or \( -\frac{1}{2} \). Therefore, the former and latter three states are equivalent and only their quantization axes are different. Then, the relation \( g_1^{BD}(x) = h_1^{BD}(x) \) is satisfied.

However, the situation concerning the relation between the helicity and transversity distributions \( g_1(x) \) and \( h_1(x) \) is different. The states given by \( \frac{1}{\sqrt{2}}(u_1^{LF}(k) \pm u_2^{LF}(k)) \)
are not angular momentum eigenstates of spin-half with the eigenvalue of $j^2$ as $\pm \frac{1}{2}$, and there is no equivalence which existed in the previous paragraph for $g^{\text{BD}}_1(x)$ and $h^{\text{BD}}_1(x)$. Then, $g_1(x)$ and $h_1(x)$ are not equal. The condition of $g_1(x)$ and $h_1(x)$ being equal is that the quark transverse momentum is zero. We explained these properties and also showed that $g_1(x)$ is a helicity distribution only when the quark mass is zero.

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Appendix A

A1  Relation between $u^{BD}(p)$ and $u^{LF}(p)$

We use the $\gamma$ matrices in the Dirac representation:

$$
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \\
\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha^i = \gamma^0 \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \sigma^{12} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}, \\
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

(24)

We use the following notations:

$$
p^R = p^1 + ip^2, \quad p^L = p^1 - ip^2, \quad p^+ = p^0 + p^3, \quad p^- = p^0 - p^3.
$$

(25)

Let us study the equation

$$(\not{p} - m) \ u(p) = 0.
$$

(26)

The following $u(p)$ satisfies (26):

$$
u(p) = \frac{1}{\sqrt{N}} (\not{p} + m) \ \gamma^0 \ \chi.
$$

(27)

A1.1  $u^{BD}(p)$

When we take $\chi$ and $N$ as

$$
\chi_1^{BD} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \chi_2^{BD} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad N^{BD} = p^0 + m,
$$

(28)
we have two linearly independent solutions:

\[ u_{iBD}(p) = \frac{\hat{\phi} + m}{\sqrt{p^0 + m}} \chi_{iBD} = \left( \frac{\sqrt{p^0 + m}}{\sqrt{p^0 + m}} \phi_{iBD}^{BD} \right) \text{,} \]  

where

\[ \phi_{1BD} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{,} \quad \phi_{2BD} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]  

When we write (29) explicitly, we have

\[ u_{1BD}(p) = \frac{1}{\sqrt{p^0 + m}} \begin{pmatrix} p^0 + m \\ 0 \\ p^3 \\ p^R \end{pmatrix}, \quad u_{2BD}(p) = \frac{1}{\sqrt{p^0 + m}} \begin{pmatrix} 0 \\ p^0 + m \\ p^L \\ -p^3 \end{pmatrix}. \]  

**A1.2 \( u_{LF}(p) \)**

When we take \( \chi \) and \( N \) as

\[ \chi_{1LF} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \chi_{2LF} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad N_{LF} = p^+, \]  

we have another set of two linearly independent solutions of (26):

\[ u_{iLF}(p) = \frac{1}{\sqrt{p^0 + m}} (p^+ + \beta m + \vec{\alpha}_\perp \cdot \vec{p}_\perp) \chi_{iLF}. \]  

We adopt the convention of Ref. [17] in (33). When we write (33) explicitly, we have

\[ u_{1LF}(p) = \frac{1}{\sqrt{2p^+}} \begin{pmatrix} p^+ + m \\ p^R \\ p^+ - m \\ p^R \end{pmatrix}, \quad u_{2LF}(p) = \frac{1}{\sqrt{2p^+}} \begin{pmatrix} -p^L \\ p^+ + m \\ p^L \\ -p^+ + m \end{pmatrix}. \]
The two different sets $u^{BD}(p)$ in (31) and $u^{LF}(p)$ in (34) are related as:

\[
\begin{pmatrix}
  u^{BD}_1(p) \\
  u^{BD}_2(p)
\end{pmatrix} = \frac{1}{\sqrt{2p^+(p^0 + m)}} \begin{pmatrix}
  (p^+ + m) & -p^R \\
  p^L & (p^+ + m)
\end{pmatrix} \begin{pmatrix}
  u^{LF}_1(p) \\
  u^{LF}_2(p)
\end{pmatrix}, \tag{35}
\]

\[
\begin{pmatrix}
  u^{LF}_1(p) \\
  u^{LF}_2(p)
\end{pmatrix} = \frac{1}{\sqrt{2p^+(p^0 + m)}} \begin{pmatrix}
  (p^+ + m) & p^R \\
  -p^L & (p^+ + m)
\end{pmatrix} \begin{pmatrix}
  u^{BD}_1(p) \\
  u^{BD}_2(p)
\end{pmatrix}. \tag{36}
\]

A2 \quad \frac{\vec{p} \cdot \Sigma}{|\vec{p}|}(p)

The spin matrix is given by

\[
\Sigma^i = \begin{pmatrix}
  \sigma^i & 0 \\
  0 & \sigma^i
\end{pmatrix}. \tag{37}
\]

When we write $\vec{p} \cdot \Sigma$ and $\hat{p}$ matrices explicitly, we have

\[
\vec{p} \cdot \Sigma = \begin{pmatrix}
  p^3 & p^L & 0 & 0 \\
  p^R & -p^3 & 0 & 0 \\
  0 & 0 & p^3 & p^L \\
  0 & 0 & p^R & -p^3
\end{pmatrix}, \quad \hat{p} = \begin{pmatrix}
  p^0 & 0 & -p^3 & -p^L \\
  0 & p^0 & -p^R & p^3 \\
  p^3 & p^L & -p^0 & 0 \\
  p^R & -p^3 & 0 & -p^0
\end{pmatrix}. \tag{38}
\]

We can check by explicit matrix multiplications of $\vec{p} \cdot \Sigma$ and $\hat{p}$ matrices in (38) that

\[
\hat{p} (\vec{p} \cdot \Sigma) = (\vec{p} \cdot \Sigma) \hat{p}. \tag{39}
\]

Let us find the eigenstates of $\vec{p} \cdot \Sigma$ which satisfy

\[
\vec{p} \cdot \Sigma u(p) = \lambda u(p). \tag{40}
\]

From $|\vec{p} \cdot \Sigma - \lambda I| = 0$ we have $\lambda = +|\vec{p}|$ and $\lambda = -|\vec{p}|$.

For $\lambda = +|\vec{p}|$, the solution of (40) is given by

\[
u_{+1}^{\vec{p} \cdot \Sigma}(p) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}| + p^3)}} \left( (|\vec{p}| + p^3) u^{BD}_1(p) + p^R u^{BD}_2(p) \right), \tag{41}\]
which is given explicitly as

\[
\hat{u}^{\Sigma}_{+1}(p) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}| + p^3)}} \frac{1}{\sqrt{p^0 + m}} \left( \begin{array}{c} (p^0 + m) \\ |\vec{p}| + p^3 \\ -|\vec{p}| \end{array} \right) .
\]

(42)

Using (35), (41) can also be written as

\[
\hat{u}^{\Sigma}_{+1}(p) = \frac{1}{\sqrt{2p^+(p^0 + m)}} \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}| + p^3)}} \times \left( (|\vec{p}| + p^3) \left( (p^+ + m) - (|\vec{p}| + p^3) \right) u^{LF}_{1}(p) + p^R \left( (p^+ + m) - (|\vec{p}| + p^3) \right) u^{LF}_2(p) \right) .
\]

(43)

For reference, if we consider the case of \( m = 0 \), (43) becomes

\[
\hat{u}^{\Sigma}_{+1}(p ; m = 0) = u^{LF}_{1}(p) .
\]

(44)

We chose the phase of \( \hat{u}^{\Sigma}_{+1}(p) \) so that (44) is satisfied with identity.

For \( \lambda = -|\vec{p}| \), the solution of (40) is given by

\[
\hat{u}^{\Sigma}_{-1}(p) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}| + p^3)}} \left( -p^L u^{BD}_{1}(p) + (|\vec{p}| + p^3) u^{BD}_{2}(p) \right) .
\]

(45)

which is given explicitly as

\[
\hat{u}^{\Sigma}_{-1}(p) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}| + p^3)}} \frac{1}{\sqrt{p^0 + m}} \left( \begin{array}{c} (p^0 + m) \\ -p^L \\ -|\vec{p}| \end{array} \right) .
\]

(46)

Using (35), (45) can also be written as

\[
\hat{u}^{\Sigma}_{-1}(p) = \frac{1}{\sqrt{2p^+(p^0 + m)}} \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}| + p^3)}} \times \left( -p^L \left( (p^+ + m) - (|\vec{p}| + p^3) \right) u^{LF}_{1}(p) + (|\vec{p}| + p^3) \left( (p^+ + m) - (|\vec{p}| - p^3) \right) u^{LF}_2(p) \right) .
\]

(47)
For reference, if we consider the case of $m = 0$, (47) becomes
\[ u_{-1}^L(p ; m = 0) = u_{2}^L(p) . \] (48)

From (41) and (45) we get
\[ u_{1}^B(p) = \frac{1}{\sqrt{2|p|(|p| + p^3)}} \left( \left( |p| + p^3 \right) u_{+1}^L(p) - p^R u_{-1}^L(p) \right) , \] (49)
\[ u_{2}^B(p) = \frac{1}{\sqrt{2|p|(|p| + p^3)}} \left( p^L u_{+1}^L(p) + (|p| + p^3) u_{-1}^L(p) \right) , \]
and from (43) and (47) we get
\[ u_{1}^L(p) = \frac{1}{\sqrt{2|p^+|(|p^+| + m)}} \left( \left( |p^+| + m \right) - (|p^-| + p^3) \right) u_{+1}^L(p), \] (50)
\[ u_{2}^L(p) = \frac{1}{\sqrt{2|p^+|(|p^+| + m)}} \left( p^L \left( (|p^+| + m) - (|p^-| + p^3) \right) u_{+1}^L(p) + (|p^-| + p^3) \left( (|p^+| + m) - (|p^-| + p^3) \right) u_{-1}^L(p) \right) . \]

### A3 Unitary Matrices

We can write the relations among $u_B^D(p)$, $u_L^F(p)$ and $u_{\Sigma}^{\pm\Sigma}(p)$ by unitary matrices as follows:
\[ \begin{pmatrix} u_{1}^B(p) \\ u_{2}^B(p) \end{pmatrix} = U^{-1} \begin{pmatrix} u_{1}^L(p) \\ u_{2}^L(p) \end{pmatrix} , \] (51)
where
\[ U^{-1} = \frac{1}{\sqrt{2p^+(p^0 + m)}} \begin{pmatrix} p^+ + m & -p^R \\ p^L & p^+ + m \end{pmatrix} , \] (52)
\[ U = \frac{1}{\sqrt{2p^+(p^0 + m)}} \begin{pmatrix} p^+ + m & p^R \\ -p^L & p^+ + m \end{pmatrix} . \] (53)
where
\[ \mathcal{V} = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}| + p^3)}} \begin{pmatrix} |\vec{p}| + p^3 & p^R \\ -p^L & |\vec{p}| + p^3 \end{pmatrix}, \] (55)
and
\[ \mathcal{W} = \frac{1}{\sqrt{2p^+(p^0 + m)}} \begin{pmatrix} |\vec{p}| + p^3 & (p^+ + m) + (|\vec{p}| - p^3) \\ -p^L & (p^+ + m) - (|\vec{p}| + p^3) \end{pmatrix} \begin{pmatrix} p^R (p^+ + m) - (|\vec{p}| + p^3) \\ (|\vec{p}| + p^3)((p^+ + m) + (|\vec{p}| - p^3)) \end{pmatrix}. \]

We can check that the relation \( \mathcal{W} = \mathcal{V} \mathcal{U}^{-1} \) is satisfied.

We can express the relations in the above as follows:
\[ \begin{pmatrix} u_1^{BD} \\ u_2^{BD} \end{pmatrix} = \mathcal{U}^{-1} \begin{pmatrix} u_1^{LF} \\ u_2^{LF} \end{pmatrix}, \quad \begin{pmatrix} \hat{u}_1^{\Sigma} \\ \hat{u}_2^{\Sigma} \end{pmatrix} = \mathcal{V} \begin{pmatrix} u_1^{BD} \\ u_2^{BD} \end{pmatrix}, \quad \begin{pmatrix} \hat{u}_1^{\Sigma} \\ \hat{u}_2^{\Sigma} \end{pmatrix} = \mathcal{W} \begin{pmatrix} u_1^{LF} \\ u_2^{LF} \end{pmatrix}, \] (58)
where
\[ \mathcal{U} = \frac{1}{\sqrt{2p^+(p^0 + m)}} \begin{pmatrix} p^+ + m & p^R \\ -p^L & p^+ + m \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta e^{i\phi} \\ -\sin\theta e^{-i\phi} & \cos\theta \end{pmatrix}, \] (59)
\[ \cos\theta = \frac{p^+ + m}{\sqrt{2p^+(p^0 + m)}}, \quad \sin\theta = \frac{|\vec{p}_\perp|}{\sqrt{2p^+(p^0 + m)}}, \quad |\vec{p}_\perp| = \sqrt{(p^1)^2 + (p^2)^2}, \] (60)
\[ e^{i\phi} = \frac{p^R}{|\vec{p}_\perp|}, \quad e^{-i\phi} = \frac{p^L}{|\vec{p}_\perp|}, \] (61)
\[ \mathcal{V} = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}| + p^3)}} \begin{pmatrix} |\vec{p}| + p^3 & p^R \\ -p^L & |\vec{p}| + p^3 \end{pmatrix} = \begin{pmatrix} \cos\chi & \sin\chi e^{i\phi} \\ -\sin\chi e^{-i\phi} & \cos\chi \end{pmatrix}, \] (62)
\[\cos \chi = \frac{|\vec{p}| + p^3}{\sqrt{2|\vec{p}|(|\vec{p}| + p^3)}}, \quad \sin \chi = \frac{|\vec{p}_\perp|}{\sqrt{2|\vec{p}|(|\vec{p}| + p^3)}}, \quad (63)\]

and

\[W = \frac{1}{\sqrt{2p^+(p^0 + m)}} \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}| + p^3)}} \times \begin{pmatrix}
(p^+ + m) + (|\vec{p}| - p^3) & p^R (p^+ + m) - (|\vec{p}| + p^3) \\
-p^L (p^+ + m) - (|\vec{p}| + p^3) & (|\vec{p}| + p^3)(p^+ + m) + (|\vec{p}| - p^3)
\end{pmatrix}, \quad (64)\]

\[= \begin{pmatrix}
\cos(\chi - \theta) & \sin(\chi - \theta) e^{i\phi} \\
-\sin(\chi - \theta) e^{-i\phi} & \cos(\chi - \theta)
\end{pmatrix} = \begin{pmatrix}
\cos \psi & \sin \psi e^{i\phi} \\
-\sin \psi e^{-i\phi} & \cos \psi
\end{pmatrix}, \quad (65)\]

\[\cos \psi = \frac{(|\vec{p}| + p^3)(p^+ + m) + (|\vec{p}| - p^3)}{\sqrt{2p^+(p^0 + m)} \sqrt{2|\vec{p}|(|\vec{p}| + p^3)}}, \quad (66)\]

\[\sin \psi = \frac{|\vec{p}_\perp| (p^+ + m) - (|\vec{p}| + p^3)}{\sqrt{2p^+(p^0 + m)} \sqrt{2|\vec{p}|(|\vec{p}| + p^3)}}. \quad (67)\]

We can write the relations in the above more compactly as:

\[U = \begin{pmatrix}
\cos \theta & \sin \theta e^{i\phi} \\
-\sin \theta e^{-i\phi} & \cos \theta
\end{pmatrix} = I \cos \theta + i \sin \theta \vec{\sigma} \cdot \hat{n} = e^{i \vec{\sigma} \cdot \hat{n} \theta}, \quad (68)\]

where

\[\hat{n} = (\sin \phi, \cos \phi, 0), \quad (69)\]

\[V = \begin{pmatrix}
\cos \chi & \sin \chi e^{i\phi} \\
-\sin \chi e^{-i\phi} & \cos \chi
\end{pmatrix} = e^{i \vec{\sigma} \cdot \hat{n} \chi}, \quad (70)\]

\[W = \begin{pmatrix}
\cos \psi & \sin \psi e^{i\phi} \\
-\sin \psi e^{-i\phi} & \cos \psi
\end{pmatrix} = e^{i \vec{\sigma} \cdot \hat{n} \psi}. \quad (71)\]

The above expressions given in (66), (68) and (69) are useful. For example, we can understand the relations written in the last line of (64) easily as

\[W = V U^{-1} = e^{i \vec{\sigma} \cdot \hat{n} \chi} e^{i \vec{\sigma} \cdot \hat{n} (-\theta)} = e^{i \vec{\sigma} \cdot \hat{n} (\chi - \theta)} \quad (72)\]
Appendix B

The Bjorken-Drell spinors $u_i^{BD}(p)$ given in (29) can be obtained by applying the Lorentz boost operator $S(p)$ to the spinors $\chi_i^{BD}$ given in (28), which is the positive energy eigenstates of the Dirac equation in the quark rest frame, where $S(p)$ is given by

$$S(p) = \sqrt{p^0 + m} \begin{pmatrix} 1 & \bar{p}^i \sigma_i \\ p^0 + m & 1 \end{pmatrix}.$$  \hfill (71)

The total angular momentum operator $j^i$ of quark is given by the sum of the spin operator $s^i$ and the orbital angular momentum operator $l^i$ as:

$$j^i = s^i + l^i, \quad s^i = \frac{1}{2} \Sigma^i = \frac{1}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \quad l^i = -i \epsilon^{ijk} k_j \frac{\partial}{\partial k^k}, \quad \epsilon^{123} = 1.$$  \hfill (72)

We can show that $j^i$ in (72) and $S(p)$ in (71) commute:

$$[j^i, S(p)] = 0.$$  \hfill (73)

Then, if a quark spinor is an eigenstate of the operator $j^i$ in the quark rest frame (in this frame $j^i = s^i$), the Bjorken-Drell spinor which is made by Lorentz boosting that quark spinor by multiplying $S(p)$ given in (71) is also an eigenstate of the same operator $j^i$. We illustrate this by considering an example in which the quark spinor is an eigenstate of $s^2$ in the quark rest frame:

$$j^2 \frac{1}{\sqrt{2}} (u_1^{BD}(p) + u_2^{BD}(p)) = j^2 S(p) \frac{1}{\sqrt{2}} (\chi_1^{BD} + \chi_2^{BD}) = S(p) j^2 \frac{1}{\sqrt{2}} (\chi_1^{BD} + \chi_2^{BD})$$

$$= S(p) s^2 \frac{1}{\sqrt{2}} (\chi_1^{BD} + \chi_2^{BD}) = S(p) \frac{1}{2} \frac{1}{\sqrt{2}} (\chi_1^{BD} + \chi_2^{BD}) = \frac{1}{2} \frac{1}{\sqrt{2}} (u_1^{BD}(p) + u_2^{BD}(p)),$$  \hfill (74)

where $\chi_i^{BD}$ and $u_i^{BD}(p)$ are given in (28) and (29). Eq. (74) shows that $\frac{1}{\sqrt{2}} (\chi_1^{BD} + \chi_2^{BD})$ is the eigenstate of $s^2$ with eigenvalue $+\frac{1}{2}$ in the quark rest frame, and then $\frac{1}{\sqrt{2}} (u_1^{BD}(p) + u_2^{BD}(p))$ is the eigenstate of $j^2$ with eigenvalue $+\frac{1}{2}$.
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