Summonable Supermoney: virtual tokens for a relativistic economy

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We propose definitions and implementations of “supermoney” – virtual tokens designed for high value fast transactions on networks with relativistic or other trusted signalling constraints. Supermoney is more flexible than standard quantum or classical money in the sense that it can solve deterministic summoning tasks that they cannot. It requires networks of agents with classical data storage and communication, but no long term quantum state storage, and is feasible with current technology. User privacy can be incorporated by secure bit commitment and zero knowledge proof protocols. The level of privacy feasible in given scenarios depends on efficiency and composable security questions that remain to be systematically addressed.

INTRODUCTION

Money, share certificates, passwords and other tokens are familiar quantities whose definitions nonetheless continue to evolve. For example, our conceptions of the function and physical form of money were expanded by the inventions of cryptocurrencies (see e.g. Ref. [1]) and Wiesner’s quantum money [2] respectively.

Here we reconsider money and tokens from another perspective, in which they are used on a network of points in space-time with a causal structure enforced by trusted constraints. Our primary motivation is that relativistic signalling constraints imply a causal structure that plays a significant and growing role in the global economy. Since general relativistic corrections in weak gravitational fields are small, the background space-time around Earth is well approximated by Minkowski space, and we use this as our main illustration. Our discussion applies to any fixed background space-time, so general relativistic corrections can also be included where significant.

Our discussion also applies to network causal structures defined by additional constraints arising from trusted technological limitations. For example, one or both parties might accept that the other cannot practically communicate at or near light speed through the interior of the Earth, even though physics gives several ways to do this in principle. For parties who cannot communicate through the Earth’s interior at speeds greater than \( \frac{c}{2\pi} \), the fastest signals between points on the Earth’s surface travel at light speed on a great circle. Parties might additionally trust lower than light speed signalling bounds, enforced by specific features of a given network.

The light speed bound on communications plays a significant role in the global financial system and some of its implications for arbitrage are well known. However, the broader financial implications of special relativity have received little attention, with the notably amusing exception of Krugman’s analysis of interstellar interest rates [3].

We identify two problems that are intrinsic to trades on networks with trusted signalling constraints when transaction speed is critical. The first is preventing illegitimate duplication without incurring the transaction delays required in order to cross-check across the network. The second is allowing the location of token presentation to depend on incoming data across the network, so that the token user can effectively allow the token to be “summoned” to a particular point by the incoming data, in a way that is as flexible as possible. We propose a solution in the form of “supermoney” schemes defining virtual tokens. We show that supermoney is more flexible than standard quantum or classical money, in the sense that it can solve any deterministic summoning task that they can, and many that they cannot. If unencrypted, supermoney does not protect the user’s privacy: the issuer has access to the same token data as the user. However, we show that supermoney may be encrypted so as to retain user privacy.

The two problems supermoney addresses are not new in principle. Indeed, in one sense they were larger in past, given the low technological bounds on signalling speeds before the development of the telegraph and radio [27]. The relatively undeveloped state of the pre-1830s global financial system, computing technology and cryptography presumably help to explain the lack of historical precedents.

NETWORKS AND AGENTS

We are interested in token schemes that allow some form of access – for example to goods, services, data, physical objects or environments. Depending on the context, such tokens might for example play the role of money, passwords, keys or passcards. They may be used by parties with many possible relationships. The parties may be individuals or may be networks of collaborating agents who trust one another.
For illustration, we suppose that the scheme involves money issued by one agency, $B$ (Bob, the Bank, or issuer) to another $A$ (Alice, the Acquirer, or user). We suppose that $A$ and $B$ have pre-equipped themselves to carry out communications and transactions at a pre-agreed finite set of space-time points defining a network.

A network point $P$ here represents a small pre-agreed local space-time region around the point $P$, which can contain separate secure sites for user and issuer agents, and authenticated classical and/or quantum channels between these sites. All communications associated with a given network point must take place within the associated space-time region. Typically, when there is a natural fixed frame choice to describe the network, we assume that the spatial diameter of these regions is significantly smaller than the spatial separation between any pair of network points.

Unless otherwise stated, we assume below that the only signalling constraint is the impossibility of superluminal communication, and that both $A$ and $B$ can send secure classical signals (for instance using one-time pads maintained by quantum key distribution) from any network point to any other network point within its future light cone.

As usual in relativistic cryptography [4], we assume that each agency may be represented by a set of agents who are completely trusted and whose actions are coordinated, so that they can be identified as effectively a single party. The parties may have unequal technological resources. We suppose that $B$ plays the role of a universal bank, and has an agent available at each network point; the same agent may be available at many timelike separated points. $A$ may be a single user or small organisation, with only one or a few agents, moving between network points, perhaps in response to locally acquired data. Alternatively, she may be an organisation similar to $B$, with agents available at every network point.

**Authentication**

We assume that at each network site user and issuer agents are able to authenticate each other’s communications. This may be by physical means or by using some standard cryptographic authentication scheme, or a combination. We also assume that all user agents have authenticated communication channels with one another and that the same applies to all issuer agents.

**THE PROBLEM OF DUPLICATION**

**Classical data money in space-time**

Consider first a simple money scheme based on classical data strings. To issue a money token, $B$ generates a secure random string $S$ at some point $P_0$ and transmits it securely to $A$ at some network point $P \succeq P_0$. The string has an agreed monetary value $V(S)$. $B$’s agent at some point $P_1$, where $P_0 \preceq P_1 \preceq P$, securely communicates the values $(S, V(S))$ to all $B$’s agents in the future light cone of $P$. $A$’s agent at $P$ may securely carry the string $S$ to some network point $Q \succ P$, or securely transmit it to another agent of $A$ at such a point. Here we write $Q \succ P$ if and only if $Q$ is in the causal future of $P$. After some sequence of such actions, $A$’s agent at a network point $Z \succ \ldots \succ Q \succ P$ may present the string to $B$’s agent at $Z$, who checks against his records, validates it, and issues resources to the value $V(S)$. Having done so, he communicates the fact to all $B$’s agents in the future light cone of $Z$, ensuring that they will not accept the string if it is re-presented to them.

Among other problems, this scheme is vulnerable to illegitimate multiple presentations of the same token. Once $A$ receives the string $S$, she may copy it and transmit it to agents at two or more pairwise space-like separated network points $Z, Z', \ldots$, who may present it to $B$’s agents at each of these points. None of $B$’s agents at these points can be aware that the string is also being presented at the other points. If they follow the scheme as defined, they thus each issue resources to the value $V(S)$, allowing $A$ illegitimate access to multiple resources.

**Detection of fraud**

$B$’s agents will communicate their acceptance to one another, so that $B$ will eventually become aware of $A$’s fraud. Legal and/or other sanctions may follow. These may be a sufficient deterrent in some scenarios, but not all. Fraudulent transactions may be difficult to reverse, particularly if $A$’s agents exploit, consume, abscond with or trade on resources. The legal situation may be complex; each agent of $A$ may claim that they behaved honestly, unaware of the others’ actions. If $A$’s motive is, say, sabotage of a financial system, she may not be overly concerned about detection.
Cross-checking and delays

For these reasons, among others, B may want to make the scheme secure against multiple presentation. One possible security mechanism is for any agent of B who receives the money token to cross-check with all other agents of B (either individually or via a central server) and make sure they are aware he has received the token, and have not themselves previously received it, before he issues the resources. If access delays of the order of twice the network diameter are acceptable, this may be a viable solution. For global markets with superfast local trading responding to local data, let alone future interplanetary and interstellar markets, it will not be.

PREVENTING DUPLICATION WITHOUT CROSS-CHECKING

We now consider a simple alternative proposal, fixed path classical data money. This can be thought of as a special case of the supermoney schemes we define below. Like general supermoney schemes, it prevents illegitimate duplication. However it lacks the flexibility of general supermoney schemes.

Fixed Path Classical Data Money

We define fixed path classical data money to be classical data money with an extra condition on the user, A, that ensures that the money token follows a single path through the network. When A’s agent at P receives the string S, she is required to tell B’s agent there at which point Q ≺ P she wants the token to be valid. B’s agent at P communicates this to B’s agent at Q, who will only accept the token as valid if he has received this confirmation.

If A’s agent at Q wants to propagate the money token further, then she does not present the string to B’s agent there, but tells him at which point R ≺ Q she wants it to be valid. B’s agent at Q communicates this to B’s agent at R, along with confirmation that he received a communication from the agent at P confirming that the token propagated from P to Q. The agent at R will only accept the token as valid if he has received these confirmations.

This continues until the token is presented by A’s agent at Z to B’s agent at Z. He checks that S is the correct string and that B’s agents at P, Q, R . . . have confirmed that the token was validly propagated from P → Q, Q → R, and so on. A thus cannot validate the token at two spacelike separated points Z, Z′, since there cannot be two distinct valid propagation paths P → . . . → Z and P → . . . → Z′.

Drawbacks of Fixed Path Classical Data Money

Clearly, in this scheme, B knows ahead of time where the money token will be valid, so that the user’s privacy is compromised. This can be addressed. We describe later ways in which fixed path classical data money, and the more general supermoney schemes we consider below, can be extended to ensure A’s privacy below.

Another concern may be that the scheme reduces A’s flexibility. In a general network, A’s agent at P may not know at that point which site the token should go to next, and may wish to decide later. In principle, this concern can be significantly mitigated by refining the network to include more decision points. If the token were a physical object (which may be massless and travel at light speed) being propagated by A, then at any given point P the user A would know the token’s current planned trajectory and hence, to good approximation, a near future space-time point Q it will reach. A sufficiently refined network would allow her to propagate the token from P to Q, reevaluate the trajectory there, and so continue. If one thinks of the user as an individual moving about the planet carrying a physical token, for example, then in principle our scheme can emulate this if the user carries a mobile device exchanging signals with issuer’s local stations, which can calculate her velocity at any given point and propagate the token to an appropriate future network point.[28] Similarly, our scheme can emulate a data token sent at light speed over a user’s network, by taking switching points to be points of the scheme’s network and propagating our token between switching points according to the scheme.

A less obvious drawback is that, as the name implies, fixed path classical data money tokens follow a definite path through space-time. We now argue that relativistic or other signalling constraints motivate a more general concept of money token, which does not require definite token paths.
MONEY AS A SOLUTION TO TASKS IN SPACE-TIME

Money and information

We start by proposing a way of thinking about money and other tokens in general, whether or not signalling constraints apply. This is to consider money and tokens as solutions to problems that arise because the information users receive over time is significant for their decisions and generally unpredictable in advance. Money and tokens allow users (qua consumers) to acquire or access resources, and (qua producers) to sell or allow access to resources, when they receive information that tells them this is possible and advantageous. If we had perfect advance information, we could pre-arrange every transaction that we will wish to carry out during every trip, and leave our wallets at home. We could even pre-arrange a set of contracts at birth, committing ourselves to a lifetime of balanced production and consumption, signing off on the payment for the choir at our funeral before exploring our crib. In fact, we don’t know how we or the world will evolve, and so we don’t know what we will want, nor what the world will want from or offer us. So we generally carry some form of money, in order to buy or sell when relevant information arrives.

Relativistic and other signalling constraints impose limits on when and where specific information can arrive. For everyday decisions, relativistic constraints are currently rarely relevant. Lemon prices in Australia rarely fluctuate fast enough to affect whether a consumer in the U.K. should buy lemonade, however precise her decision calculus and however avidly she tracks the global lemon market. For high-frequency financial market trades and other trades involving relativistic arbitrage, though, they are crucial.

Even when relevant, relativistic constraints do not fundamentally change the role of money for individuals, so long as they act only as individuals. If someone follows a definite path in space-time, and makes all their financial decisions themselves, during that path, their decisions at any point on the path are made only on the basis of information that has reached them up to that point on the path. This is true in either Galilean or Minkowski space-time.

However, this no longer holds true in Minkowski space-time when individuals act as agents of a larger enterprise. A team of separated collaborating agents each acquire information on their own paths. The team thus may receive information at various points distributed throughout space-time, which may be shared as fast as signalling constraints allow. This shared information may make a persuasive case for buying or selling at various other points in space-time. For money to enable buying or selling, it needs to be available at the relevant points. So a money token scheme needs to solve the problem of responding to information distributed throughout space-time, by making money available at appropriate points, while preventing illegitimate duplication.

A team of user agents who have only one money token, which they propagate along a definite causal path, only has the option of using it at points on that path. The present state of such a token is defined at any given time in an agreed frame (or more generally on any given hypersurface) by its location. The evolution of its future state is determined by its velocity and higher derivatives. These can only be affected by information in the past light cone of its present location. For this reason, fixed path classical data money and other schemes that require money tokens to follow definite paths in space-time do not allow optimally flexible responses to incoming information distributed over space-time, as the examples below illustrate.

A model environment for money token schemes

We define a model environment for money tokens in which local user agents may acquire new information at specific points in space-time and agents collaborate on decisions to acquire or access resources. We will assume here that the relevant information and resources are classical. User agents at some set of input points \( \{P_i\}_{i=1}^m \) receive incoming classical information, which may for example be integers \( n_i \), real numbers \( x_i \), or real vectors \( x_i \). We also allow the possibility that the user agents at some or all of the \( P_i \) receive no information.

The user agents collaborate in order to optimize their decisions about acquiring a local resource at some presentation point \( Q \). There is a finite set \( \{Q_i\}_{i=1}^n \) of allowed presentation points. The user agents’ collective decisions may either select one of the \( Q_i \) or select none of them, in which case they effectively choose not to acquire the resource. We suppose that the sets \( \{P_i\} \) and \( \{Q_i\} \) are known well in advance. They may overlap: the user agents at some or all of the \( Q_i \) may receive incoming classical information relevant to their choice of \( Q \).

For example, the global financial network defines market prices at many different market locations at any given time, and these may be available to local user agents in real time. These prices in turn are determined in response to locally acquired information, which may for example include revealed actions of market participants or independent actors, physical events, and the outcomes of local computations. Users may wish to use information from many points in space-time to inform their decisions about acquiring a local resource at some given point \( Q \).
DEFINING SUPERMONEY

Supermoney schemes define virtual tokens that allow users to solve summoning tasks defined by incoming information distributed over a space-time network. User privacy is not essential to solve such tasks, and so we may define unencrypted supermoney schemes, in which the user keeps nothing private from the issuer. In these schemes, the issuer’s agents learn the user’s agents inputs immediately, and the issuer may be able to predict in advance where the virtual token will be presented. We later discuss encrypted versions of the schemes that preserve user privacy.

Supermoney with free user choice of the presentation point

We are interested in schemes that allow virtual tokens to be generated and presented with some or all of the following properties:

(i) *(free user choice)* the user may present a valid token at one of a number of pre-agreed space-time points, some or all of which may be space-like separated. The identity of the point at which the token is valid depends solely on data that the user’s agents input into the token scheme.

(ii) *(instant verification by issuer)* the issuer may verify at any such pre-agreed point \( Q \) whether the token is valid at \( Q \).

(iii) *(instant verification by user)* the user may verify at any such pre-agreed point \( Q \) whether the token is valid at \( Q \).

(iv) *(security against duplication)* the scheme guarantees to the issuer that valid tokens cannot be presented at two or more points.

We now give general definitions of schemes for supermoney tokens. They can be used on a very broad class of networks. In general they define tokens that are not required to follow definite paths in space-time.

The first definition below describes the simplest versions of these schemes, which have free user choice, instant verification by issuer and user and security against duplication. It is illustrated in our discussions below of classical emulation of summoning quantum money and of Examples 1 and 2.

Supermoney (with free user choice and optimal issuer and user communication)

(i) user agents at each network point \( P_i \) in an agreed set \( \{P_1, \ldots, P_n\} \) communicate to the corresponding issuer agent at \( P_i \) classical information encoding statements of some pre-agreed form. These statements collectively constrain the network points \( Q \) at which the token may potentially be valid.

(ii) The issuer agents at each \( P_i \) communicate the statements they received to all issuer agents at points \( Q \succ P_i \).

(iii) The user agents at each \( P_i \) communicate the statements they transmitted to all user agents at points \( Q \succ P_i \).

(iv) The token is valid at a network point \( Q \), and should be accepted by the issuer’s agent at \( Q \) if presented there, if and only if the statements communicated to the issuer and user agents at \( Q \) imply that (i) the token is potentially valid at \( Q \), (ii) the statements received by issuer and user agents at any network point \( R \) spacelike separated from \( Q \) exclude the possibility that the token is also potentially valid at \( R \).

(v) if a valid token is presented and accepted at \( Q \), \( B \)’s agent there communicates this to all \( B \)’s agents at points \( Q' \succ Q \), so it will not be accepted again at any such \( Q' \). Similarly \( A \)’s agent communicates this to all \( A \)’s agents at point \( Q' \succ Q \), so that they are aware that it will not be accepted again at any such \( Q' \).

Note that conditions (i), (ii), (iii) and (v) all describe communications that are relevant in applying condition (iv). Conditions (i) and (iv) give separate definitions of potentially valid and actually valid tokens, which are convenient in understanding some of the examples we discuss. These definitions can alternatively be combined, so that a token is valid at \( Q \) if the set of statements received at \( Q \) belong to a list of valid sets. In this case, every valid set of statements must have the property that they exclude the possibility of a valid set of statements being received at any point \( R \) spacelike separated from \( Q \). The analogous definitions for the other types of supermoney discussed below can similarly be combined.

Conditions (ii), (iii) and (v) ensure that each issuer (respectively user) agent communicates all relevant information to all issuer (respectively user) agents in his (her) causal future. These are useful but not essential simplifying features. We may instead allow the issuer and user agents to follow agreed weaker rules, with the tradeoffs that the criteria for token validity are more stringent and that instant user verification may not be possible, as follows.
**Supermoney (with free user choice, without optimal issuer and user communication):**

(i) user agents at each network point $P_i$ in an agreed set $\{P_1, \ldots, P_n\}$ communicate to the corresponding issuer agent at $P_i$ classical information encoding statements of some pre-agreed form. These statements collectively constrain the network points $Q$ at which the token may potentially be valid.

(ii) The issuer agents at each $P_i$ communicate the statements they receive to some subset $S_i$ of the issuer agents at points $Q \succ P_i$. The subsets $S_i$ are known to both parties in advance of the protocol.

(iii) The user agents at each $P_i$ communicate the statements they transmitted to some subset $T_i$ of the user agents at points $Q \succ P_i$.

(iv) The token is valid, and should be accepted by the issuer’s agent at a network point $Q$ if presented there, if and only if the statements communicated to $Q$ imply that (i) the token is potentially valid at $Q$, (ii) the statements received by the issuer agent at every network point $R \neq Q$ exclude the possibility that the token is also potentially valid at $R$.

(v) If a valid token is presented and accepted at $Q$, B’s agent there communicates this to some subset $S_Q$ of B’s agents at points $Q’ \succ Q$. The subsets $S_Q$ are known to both parties in advance of the protocol. Similarly A’s agent there communicates this to some subset $T_Q$ of A’s agents at points $Q’ \succ Q$.

Note that conditions (i), (ii), (iii) and (v) all describe communications that are relevant in applying condition (iv).

The subsets in condition (ii) and (v) are allowed to be the empty set or the full set of all points in the causal future of $P_i$ and $Q$ respectively. Of course, if every subset is the full set of all points in the causal future of the relevant point, this becomes a supermoney scheme with optimal issuer and user communication. Schemes without optimal issuer and user communication are defined to allow instant verification by the issuer. A sufficient condition to ensure instant verification by the user is that the subsets satisfy $S_i \subseteq T_i$ for all $i$ and $S_Q \subseteq T_Q$ for all $Q$. In schemes without instant verification by the user, it may sometimes be the case that the issuer would accept the token as valid at some presentation points, but the user does not know this at the relevant points.

For simplicity, we focus on schemes with optimal issuer and user communication in the examples below.

**Supermoney with valid presentation points jointly determined by user and issuer**

We are also interested in supermoney token schemes in which both the user and issuer input data into the scheme, such that the identity of the point at which the token is valid is a joint function of user and issuer inputs. By definition, such schemes do not have free user choice. Examples 3 and 4 illustrate such schemes, in which the user and issuer jointly determine the point at which the token is valid. We give a general definition of these schemes, assuming optimal issuer and user communication.

**Supermoney (without free user choice, with optimal issuer and user communication)**

(i) user agents at each network point $P_i$ in an agreed set $\{P_1, \ldots, P_n\}$ communicate to the corresponding issuer agent at $P_i$ classical information encoding statements of some pre-agreed form. Issuer agents at each $P_i$ also communicate to the corresponding user agent at $P_i$ classical information encoding statements of some pre-agreed form. The issuer may be required to respect pre-agreed constraints relating their statements. These statements collectively constrain the network points $Q$ at which the token may potentially be valid.

(ii) The issuer agents at each $P_i$ communicate the statements they sent and received to all issuer agents at points $Q \succ P_i$. The user agents at each $P_i$ likewise communicate the statements they sent and received to all user agents at points $Q \succ P_i$.

(iii) The token is valid at a network point $Q$, and should be accepted by the issuer’s agent at $Q$ if presented there, if and only if the statements communicated to the issuer and user agents at $Q$ imply that (i) the token is potentially valid at $Q$, (ii) the statements received by issuer and user agents at any network point $R$ spacelike separated from $Q$ exclude the possibility that the token is also potentially valid at $R$, assuming that the issuer has respected the pre-agreed constraints relating their statements.

(iv) If a valid token is presented and accepted at $Q$, B’s agent there communicates this to all B’s agents at points $Q’ \succ Q$, so it will not be accepted again at any such $Q’$. Similarly, A’s agent there communicates this to all A’s agents at points $Q’ \succ Q$. 


Note that conditions (i), (ii), and (iv) all describe communications that are relevant in applying condition (iii).

As before, the assumption of optimal communications is made to simplify the scheme, but is not essential. We may instead allow the issuer and/or user agents to follow agreed weaker rules, with the tradeoffs that the criteria for token validity are more stringent and that instant user verification may not be possible. The definition of schemes without optimal user and issuer communication parallels that given above for schemes with free user choice.

The definition allows the possibility of pre-agreed constraints relating the issuer’s inputs, which a dishonest issuer may violate, but does not allow a similar possibility for the user. This makes sense in asymmetric scenarios in which the reputational costs to a cheating issuer are higher than those to a cheating user, and the potential consequences of user cheating are more serious. Of course, if users perceive the risks or costs of issuer cheating as too high, they may prefer schemes in which the issuer’s inputs are not required to be related. We discuss these points further in Examples 3 and 4 below.

EXAMPLE: EMULATING QUANTUM MONEY WITH SUPERMONEY

Advantages of quantum money in relativistic scenarios

Wiesner’s quantum money [2] uses quantum information embedded within a physical token as a security mechanism. One well understood property of quantum money is its effective unforgeability [2, 5]; this also holds for other versions of quantum money [4, 8] that require specific quantum information to be held locally in order to validate the quantum token.

What has perhaps not been sufficiently emphasized to date is that the advantages of using quantum states rather than classical states for money tokens are much clearer in a relativistic context than in a non-relativistic context, for at least two reasons.

1. The advantage given by the unforgeability of quantum tokens is effectively reproducible in principle by cross-checking in non-relativistic contexts, but not in relativistic contexts.

Unforgeability implies that a quantum token cannot be presented at two space-like separated points, and this gives an advantage that cannot be replicated using standard classical tokens.

In contrast, in effectively non-relativistic scenarios, in which communication times are negligible, effectively instantaneous cross-checking gives another theoretical solution to the problem of duplication. Admittedly, cross-checking requires potentially large communication and data storage resources. However, quantum memory technology may also require costly resources, if and when such technology becomes available. It seems plausible that, in many or even possibly all scenarios, classical data money with cross-checking may always be cheaper than using quantum money. If so, unforgeability may be a good reason for preferring quantum money mainly or even only in scenarios where relativistic (or other) signalling constraints are relevant and cross-checking delays are a significant issue.

2. Quantum tokens may be delocalized in space-time and used to solve summoning and other intrinsically relativistic tasks that standard classical tokens cannot.

Unknown quantum states, such as those stored in quantum money tokens, can be delocalized and effectively propagated along multiple paths via teleportation and secret sharing. This allows users to respond to distributed data in ways that are impossible if they use standard classical tokens that must follow a fixed path in space-time [9–12]. We discuss this further below.

Quantum money and summoning

Summoning

Summoning was originally [9, 13] defined as a task between two mistrustful parties, Alice and Bob, who each have networks of collaborating agents. Bob creates a quantum state, keeping its classical description secret, and gives it to Alice at some starting point. Alice is required to return it at some later point that depends on communications received from Bob at other points. Various versions of the task have been studied [6–8].

We will consider here a form of summoning described in Ref. [12]. Alice receives an unknown quantum state at the start point $P$. Alice must either produce the state at some presentation point $Q \supset P$, where $Q \in \{Q_1, \ldots, Q_N\}$ or, if there is no valid presentation point, she should not return the state anywhere [12]. We write $Q = \emptyset$ in the latter case. In our examples we will assume that there is at most one valid presentation point, although our discussion also applies to the case where there may be any number of valid presentation points if there is a quantum algorithm that always chooses the same valid presentation point for a given set of inputs [12]. So we have $Q \in \emptyset, Q_1, \ldots, Q_N$. The identity of $Q$ depends on classical information received at some set of input points $\{P_i\}_{i=1}^M$. The point $P$ and the sets $\{P_i\}$ and $\{Q_i\}$ are known to both parties in advance of the task. The sets may overlap, and may include $P$. We assume that $Q_i \supset P$ for all $i$. We take the sets of input and presentation points to be finite. The classical information Alice receives at $P_i$ is an integer $m_i$ in the range $0 \leq m_i \leq (n_i - 1)$. Thus if
If $Q(m_1, \ldots, m_M) \in \{Q_1, \ldots, Q_N\}$ then Alice should return the state to $Q(m_1, \ldots, m_N)$, and if $Q(m_1, \ldots, m_M) = \emptyset$ then she should not return the state. Alice knows the functional dependence of $Q$ and the set $\{m_i\}_{i=1}^M$ in advance of the task; she learns the values of the $m_i$ only at the points $P_i$.

**Scenarios for summoning quantum money**

Quantum money schemes, in their simplest form, also involve two parties, who may have networks of collaborating agents, and who play the roles of issuer (Bob, the Bank) and user (Alice, the Acquirer), as in the classical money schemes discussed above. As in the case of summoning, Bob creates a quantum state, keeping its classical description secret, and gives it to Alice at some starting point, in this case as part of a quantum money token.

A natural scenario which combines summoning and quantum money is that Alice is the acquirer of a quantum money token and wishes to get its quantum state to some point $Q$ in space-time, in order to spend the quantum money there. The point $Q$ – the best place to spend the money – is determined by classical information she receives at other points in space-time. In this case we can identify the parties called Alice in the definitions of summoning and of quantum money: there is one party, Alice, who has quantum money and needs to satisfy a summoning task in order to get its quantum information to the right point.

In some scenarios the parties called Bob might also be identified as a single party: the issuer of the quantum money might, for some reason, be supplying information that effectively guides Alice as to where to redeem it. In general, though, it is more natural to separate the roles of the two Bobs. In the context of summoning quantum money, and in Examples 1 and 2 below of supermoney schemes, we will think of the classical inputs to the summoning task as given to Alice by “nature”, or by some other agency or agencies not necessarily associated with the quantum money issuer (Bob, the Bank). The role of the issuer is restricted to participating in the token or money scheme and verifying that a previously issued token is validly presented according to the scheme rules. Alice wishes to satisfy the summoning task not because Bob requires her to – he may not care where she redeems her money, so long as it is valid – but because she knows, given the classical information she received in the summoning task, that it is most advantageous to her to redeem the money token at a particular point.

For example, the classical inputs in the summoning task may be local prices of some commodity, or some more general function of local market data, or local temperatures or wind speeds, at the given space-time points. The summoning task here is chosen to indicate to Alice where it is most likely to be advantageous for her to present her money or token in order to carry out a transaction.

We will assume that Alice does not need to prove to Bob that the classical inputs took the values $m_i$ that she learned at the points $P_i$. Indeed she may not necessarily need even to report these values to Bob at all, unless the form of the token scheme uses these values as part of the definition of the token. If they do, then, for example, her local agents at $P_i$ may be required to give the classical values $m_i$ to Bob’s local agents at $P_i$. However the token scheme allows Bob to accept these reported values without independent verification. If Alice misreports in this scenario, then the token will either be valid at a less advantageous point or not valid anywhere. Alice thus has no motive to misreport; if she does, it is her problem, not Bob’s.

In Examples 3 and 4, Alice receives inputs both from nature and from Bob, the Bank. These jointly define her summoning task. In these examples we again assume that Alice does not need to prove to Bob that the classical inputs took the values $m_i$ that she learned at the points $P_i$.

**Quantum money and deterministic summoning tasks**

Suppose then that Alice is given a quantum money token containing an unknown quantum state $\psi$ at a point $P$ in a network, and wishes to propagate the quantum state over the network, exploiting the power of general quantum operations, with the goal of satisfying a summoning task that allows her to present the money at a desired space-time point identified as a function of incoming data at earlier points. Suppose further that this summoning task is possible and that she follows a deterministic algorithm that implements it. By “deterministic”, we mean here that the algorithm always propagates the state to the same presentation point for a given set of incoming data.

Following this algorithm, her local agent at any network point $Q \succ P$ may receive a quantum state produced by the previous actions of local agents. The local agent at $Q$ may also introduce one or more new quantum states, which may be entangled with states that may be introduced elsewhere in the network – for example parts of singlets to be used for teleportation. The local agent at $Q$ may then apply a general unitary $U_Q$ which acts on these states collectively and propagates outputs along secure channels $C_i$ to further network points $R_i \succ Q$. The states introduced at $Q$ and unitaries $U_Q$ may depend on local information made available to the local agent at (or before) $Q$. Local agents may also apply general measurement operations. We can either treat these either as projective measurements (with appropriate ancillae) in the standard projection postulate formalism, or enlarge the Hilbert space and consider measurements as unitary quantum operations. We take the latter course, since it simplifies
the discussion. So, although the user’s algorithm may involve operations such as teleportation that, if we apply the projection postulate, produce random classical data, we treat these as deterministic unitary quantum operations.

By assumption, Alice’s local agents follow an algorithm which determines all these local operations as a function of local classical information inputs \( m \) at the input points \( P_i \), in a manner consistent with the causal structure (so the operations at \( Q \) depend only on inputs at points \( P_i \) such that \( Q \succ P_i \)). We suppose that this algorithm guarantees that the quantum money token state \( \psi \) will always be recreated at a valid presentation point \( Q_j \) if there is one, and that for any given set of data inputs the chosen valid presentation point is always the same. We also suppose that the algorithm gives Alice’s agent at \( Q_j \) the information that she has the token state and that \( Q_j \) is a valid presentation point. The agent then, having reconstructed \( \psi \), may return the quantum money token. That is, Alice’s algorithm guarantees successfully summoning the money to the required point, which is uniquely determined by the data inputs.

Classical emulation of summoning quantum money

To emulate a successful quantum summoning algorithm in a classical supermoney token scheme of type \( S \), the user may proceed as follows. At each point \( Q \), \( A \)’s local agent communicates to \( B \)’s local agent a classical description of the choice of \( U_Q \) and the nature of any new quantum states introduced. For example, at \( Q \) they may say “we introduce here singlet \( n \), whose other subsystem is available for introduction at point \( Q’ \)”. If they do, then at \( Q’ \) they may say “we introduce here singlet \( n \), whose other subsystem is available for introduction at point \( Q’ \), but they may instead say they are introducing some other state, or none. Whether their actions are coordinated or not depends on their decision algorithm and on the information available to them at the relevant points.

To present a supermoney token at \( Z \), \( A \)’s local agent describes the operation \( U_Z \) and then states that she wishes to present the token. \( B \)’s local agent at \( Z \) verifies that the token is potentially valid by checking that the set of quantum operations and introduced states described in the past light cone of \( Z \) would indeed have produced the state \( \psi \) at \( Z \). If so, the no-cloning theorem assures \( B \) that no potentially valid token can be presented at any point space-like separated from \( Z \). The token is thus valid, and may be accepted.

Any deterministic quantum money summoning algorithm may thus be emulated by a classical supermoney token scheme. In this sense, supermoney can solve any deterministic summoning task that quantum money can.

Efficient supermoney schemes for summoning tasks

We have shown that supermoney can classically emulate summoning quantum money tokens. This gives one way in which supermoney can solve a deterministic summoning task that is solvable by quantum money summoning. However, there are often far more efficient supermoney solutions.

**Example 1** Consider a task in which incoming data arrives at a set of pre-agreed input points \( \{ P_i \}_{i=1}^N \) and the supermoney token may be presented at one of a set of presentation points \( \{ Q_i \}_{i=1}^2 \), with the spatial locations of all \( P_i \) and \( Q_i \) lying on the surface of a sphere (for example, the Earth) in a given frame. Here the \( P_i \) are all located on the Equator, and have time coordinate \( t = 0 \); \( Q_1 \) and \( Q_2 \) are located at the two poles, and have time coordinate \( t = T \), where \( T \) is the time taken to send a light signal from a point on the Equator to one of the poles. (We may either assume it is possible for light speed signals to be sent through the interior of the sphere, or that both parties agree this is technologically infeasible, in which case they must go around the surface. The time \( T \) is calculated accordingly.) Suppose that at \( t = 0 \) each point \( P_i \) supplies data in the form of a subset \( S_i \) of the set \( \{ Q_1, Q_2 \} \). That is, each \( P_i \) “chooses” neither presentation point, or one of them, or both of them.

Suppose that it is pre-agreed that if any subset of \( \{ P_i \}_{i=1}^N \) belonging to some pre-agreed list \( \{ S_1(1), \ldots, S_1(N_1) \} \) chooses \( Q_1 \), the token is potentially valid at \( Q_1 \). Similarly, it is pre-agreed that if any subset of \( \{ P_i \}_{i=1}^N \) belonging to some pre-agreed list \( \{ S_2(1), \ldots, S_2(N_2) \} \) of input points chooses \( Q_2 \), the token is potentially valid at \( Q_2 \).

We can define a supermoney scheme for this task as follows. As soon as Alice’s agents at \( P_i \) receive the set \( S_i \), they send the set \( S_i \) to Bob’s agents at \( P_j \). They also send the information to Alice’s agents at light speed to both presentation points. Bob’s agents at \( P_j \) also send the information they receive at light speed to Bob’s agents at both presentation points. The token may be presented as soon as the signals have reached the presentation points. To check that the token is valid and may be presented and accepted at \( Q_1 \), Alice and Bob’s local agents need to check:

(a) that some valid subset \( S_1(i) \) of input points have included \( Q_1 \), so that the token is potentially valid at \( Q_1 \);
(b) that no valid subset \( S_2(j) \) of input points have included \( Q_2 \), so that the token cannot also be potentially valid at \( Q_2 \).

Validity at \( Q_2 \) is similarly determined.
Example 2  Consider input points whose spatial locations are distributed on a spatial network over the surface of a sphere. We may take this network to be relatively dense. For instance, if the sphere is radius 1 we could suppose that every circle on the surface of radius $\epsilon$ contains at least one network point, for some $\epsilon \ll 1$.

In this example, the spatial location of every input point is also the spatial location of a presentation point, and the time coordinate of the input points is $t = 0$ in an agreed frame. Thus the input points take the form $P_i = (x_i,0)$, where the 3-vector $x_i$ represents the $i$-th spatial location. The presentation points are $Q_j = (x_j, T)$, where $T$ is the time required for a light signal to go between antipodal points. (Again, the parties may agree that communication at light speed by shortest path is possible, or alternatively that no communication through the interior of the sphere is possible; $T$ is calculated accordingly.)

The data presented to Alice’s agents at each input point $P_i$ is the value of some function $f(P_i)$ calculated at that point. This function might, for example, be the price of a stock at the relevant node of the financial network at $t = 0$. It might also be some data generated from market data available at $P_i$ by a complicated algorithm that Alice does not necessarily want to reveal to Bob. Alice’s local agent at each $P_i$ gives the value $f(P_i)$ to Bob’s co-located agent, who broadcasts them to all Bob’s agents located at the presentation points. Alice’s local agents also broadcast the values $f(P_i)$ to all Alice’s agents at the presentation points. At time $T$, Alice’s and Bob’s agents at each presentation point will have received all the data from all the input points. The token is potentially valid at a presentation point $Q_j = (x_j, T)$, if the data show that the function $f$ takes its maximum value at $P_j = (x_j,0)$. The token is valid at $Q_j$, if it is also potentially valid at any other presentation point, i.e. if the data show that the maximum at $P_j$ is unique. (Thus, in this scheme, if there are $N > 1$ global maxima then there is no valid presentation point for the token.)

Solutions with quantum money tokens

Standard quantum money tokens can also solve the summoning tasks given in the first two examples. One method, applicable to both examples, is to begin with the token at one input point and carry out a sequence of teleportation operations through all the input points, implementing a non-local quantum computation that results in a quantum state at the correct presentation point which can be converted to the token state using the accumulated incoming teleportation data [14, 15].

This requires entanglement resources that depend exponentially on the number of sites, and near-perfect teleportation. It thus may be far from an optimal solution in terms of resource use. It would be very interesting to understand how efficiently these tasks and others of their type can be solved and to prove lower bounds on the resources required. We leave these questions for future work.

Solving constrained summoning tasks with supermoney

Example 3  Consider a variation of Example 2, with the same spatial locations for the network points, but different supermoney token rules. In this version, at $t = 0$, the protocol requires Bob’s agents to make coordinated announcements to Alice’s co-located agents at each input point $P_i$, each identifying the same subset $S \subset \{P_i\}$ that defines the subset of Bob’s agents that will participate in the scheme. (Bob might, for example, know in advance that some of his agents will be out of action at any given time, but not wish to reveal which ahead of time.) Thus, both Bob-the-bank and “nature” provide input data for Alice’s summoning task.

In this variation, the supermoney token may become potentially valid after time $T'$, where $T'$ is the maximum time required for a light signal to travel between a pair $(P_i, P_j)$ where $\{P_i, P_j\} \subset S$. It is potentially valid at $P_i \in S$ if $f(P_i) = \max\{f(P_j) : P_j \in S\}$. It is valid at $P_i \in S$ if there is no other presentation point at which it is potentially valid, i.e. if $P_i$ is its unique maximum in $S$. (Again, this scheme allows no valid presentation point if there are $N > 1$ global maxima in $S$.)

The supermoney scheme guarantees that, if Bob behaves honestly by declaring the same subset $S$ at each input point, then both Alice and Bob can verify that the supermoney token is valid at $T'$ at the unique maximum point $P_i \in S$, when there is a unique maximum. Alice can check there that Bob’s agents at every $P_i \in S$ identified the same set $S$ of participants; Bob can check there that it is indeed the unique maximum of $f$ in $S$.

The no-summoning theorem [13] shows that there are configurations of input and presentation points for which Alice cannot generally solve the task defining this scheme by summoning a quantum money token. For example, suppose there are two disjoint subsets $S, S' \subset \{P_i\}$ such that the corresponding sets of presentation points $R = \{Q_i : P_i \in S\}$ and $R' = \{Q_i : P_i \in S'\}$ have the properties that $R$ is space-like separated from $S'$ and $R'$ from $S$. Suppose that Bob’s agents at points $P_i \in S$ all identify $S$, while Bob’s agents at points $P_i \in S'$ all identify $S'$. Then Alice would be required to propagate the quantum token state to two space-like separated points, which is impossible. Now, her ability to solve the task if Bob identifies $S$ at all points in $S$ is
located agents. If they do, there may be no point in space-time at which Alice is persuaded the supermoney token is even separated points. This would be the case, for example, if the are disjoint sets $R = \{Q_i : P_i \in S\}$ and $R' = \{Q_i : P_i \in S'\}$ have the property that $R$ is spacelike separated from $S'$ and $R'$ is spacelike separated from $S$, and Bob’s agents at points $P_i \in S$ all identify $S$, while Bob’s agents at points $P_i \in S'$ all identify $S'$. If so, Alice may present and try to spend the token at two or more points; Bob then either has to allow the token to be spent multiply or acknowledge his cheating at one or more of the relevant points.

Another possibility is that two or more of Alice’s may be persuaded the token is actually valid at two or more space-like separated points. This would be the case, for example, if there are disjoint sets $R \subseteq S$ and $R' \subseteq S'$ such that the corresponding sets of presentation points $R = \{Q_i : P_i \in S\}$ and $R' = \{Q_i : P_i \in S'\}$ have the property that $R$ is spacelike separated from $S'$ and $R'$ is spacelike separated from $S$, and Bob’s agents at points $P_i \in S$ all identify $S$, while Bob’s agents at points $P_i \in S'$ all identify $S'$. If so, Alice may present and try to spend the token at two or more points; Bob then either has to allow the token to be spent multiply or acknowledge his cheating at one or more of the relevant points.

While the possibility of presenting inconsistent information in this scheme may be advantageous to Bob in some scenarios, Alice retains the decision whether to actually spend the token. Also, a bank that cheats by making inconsistent declarations is likely to suffer reputationally. Alice may thus be willing to participate in a scheme of this type, despite the possibility that Bob may cheat, since she may see the risks and drawbacks as acceptably limited.

An interesting variant of this task prevents Bob from sending inconsistent inputs while preventing Alice from learning his inputs before the time at which she may present a valid token. This is done by requiring Bob’s agent at some pre-agreed point $P_0 \in \{P_i\}$ to make a cryptographic commitment of Bob’s input to Alice’s co-located agent, using a secure bit string commitment scheme. In this variant, the token is valid after time $T$ (where $T$ is defined as in Example 2), so that Alice’s agent at $P_0$ may broadcast the commitment data received from Bob’s agent there to Alice’s agents at all the presentation points at $Q_i$. Similarly, Bob’s agent at $P_0$ broadcasts all the data he received or retained from the commitment to all Bob’s agents at the $Q_i$. Bob’s agents at each presentation point $Q_i$ may then simultaneously unveil the commitment, and Alice’s located agents may each instantaneously verify the validity of the unveiling. The encrypted commitment prevents Alice from using the information in Bob’s input until it is unveiled, and means that the task cannot be solved by summoning a quantum money token. (This remains true even if the presentation points $Q_i$ are at time later than $T$.) We discuss this further in Example 4 below.

Supermoney with variable value tokens and sub-tokens

Variable value tokens

Supermoney may be extended by including rules that make the value of a valid supermoney token depend on the information defining the token. For instance, in Example 2 above, the token value could depend on some function of the global distribution of the values of the function that are reported at the presentation point and that validate the token there. One simple possibility would be for the token value to be the difference between the global maximum and global minimum.

Variable value subtokens with specified validity constraints

A further interesting class of extensions allows the scheme to define one or more valid sub-tokens that may be presented at different (perhaps space-like separated) points, and whose values may also depend on the scheme data communicated at or to those points. In this case, the issuer defines the security of the scheme by a set of constraints on the values and locations of the sub-tokens. The scheme data communicated at or to any sub-token presentation point must guarantee that no set of sub-tokens may be presented at locations that violate these constraints.

Such constraints may be quite general: a sub-token need not have value given by a fixed fraction of a token. For example, the scheme might allow either 10 credits at a single market site, or sub-tokens to a sum value of no more than 7 credits at up to three sites on the same continent, or sub-tokens with an individual value of up to 3 credits on sites at up to 5 distinct continents.

These features are not generally replicable by any standard quantum money token scheme. For instance, in Example 2, a quantum token may be delocalized and propagated so that it is reconstructed at the unique maximum point. However, for the user and issuer to agree on its value there requires the user to prove to the issuer the value of a function of the global distribution of function values chosen at the input points. In a supermoney solution, these chosen values are communicated directly by local user agents to local issuer agents. Absent such communications, no set of operations by the user on the quantum token can encode this value (for general functions) in a way that allows the issuer to deduce it with certainty when the output state is returned to him.
TOKEN SCHEMES AND USER PRIVACY

Privacy for tokens that follow definite paths

Token schemes may be designed to guarantee the user various types of privacy. For tokens that are defined to follow definite paths in space-time we can sensibly describe these in terms of the inferences that can or cannot be made about points or segments of the token’s path.

Suppose that such a token is issued at point \( P \) and is defined for a network with points \( V = \{ P_i \}_{i=1}^{N} \), so that it may be transmitted causally between some pairs of points on the network, given by a set \( E \subset \{(P_i, P_j) : 1 \leq i \leq N, 1 \leq j \leq N, P_i \prec P_j \} \). Thus a token following the scheme rules will follow a path \( P \rightarrow P_1 \rightarrow \ldots \rightarrow P_k \), where \( (P, P_1), (P_1, P_2), \ldots, (P_{k-1}, P_k) \in E \), and may or may not be presented at the final point \( P_k \). We can assign it a worldline by assigning it a linear path segment between any successive pair of points. Thus any spacelike hypersurface \( S \) contains a finite subset of points \( \text{Poss}(S) = \{ S_i \}_{i=1}^{l} \) through which a valid token’s path may possibly pass.

We say such a scheme offers present privacy if it does not require the user to give away any information about the token’s location on any spacelike hypersurface, unless she chooses to present the token there, beyond the information that can be inferred from the constraints on possible paths implied by \( E \). Thus, if the token has not been presented in the causal past of \( S \), and if the user follows the scheme, the collective information available to all issuer agents at points \( Q \prec S \) or \( Q \in S \) implies only that the token may be at any \( S_j \in \text{Poss}(S) \).

We say the scheme offers future privacy if the information it requires the user to give at any locations \( Q \prec S \) or \( Q \in S \) does not imply any information about the token’s locations at points in the causal future of \( S \), beyond the constraints implied by \( E \).

We say it offers past privacy if the information it requires the user to give in propagating the token by any valid means from \( P \) to some network point \( Q \succ P \) and in presenting it there implies no information about any intermediate locations, beyond the constraints implied by \( E \). That is, a presentation at \( Q \) implies only that the token followed some path \( P \rightarrow P_1 \rightarrow \ldots \rightarrow P_k \rightarrow Q \), where each successive pair belongs to \( E \).

Quantum money token schemes and user privacy

Consider now a quantum money token scheme in which the user is given a token quantum state at a start point \( P \), may carry out arbitrary quantum operations at network points \( P_i \), and may present the token quantum state at some network point \( Q \succ P \). For general networks, this allows the user to delocalize the quantum state, relocalizing it before or at the point where it is presented. The token thus need not follow a definite path.

Consider a standard cryptographic scenario in which the user’s agents are able to carry out quantum operations of their choice at secure sites in the vicinity of each network point, and are able to communicate securely from network point \( P_i \) to network point \( P_j \) whenever \( P_j \succ P_i \). In this scenario, the issuer obtains no information about the token’s presentation point, prior to presentation, beyond that implied by signalling constraints: it may be presented at any network point \( Q \succ P \). The issuer also obtains no information, after presentation, beyond that implied by signalling constraints and quantum theory: the token may have been propagated from \( P \) to \( Q \) by any algorithm allowed by relativistic quantum theory.

In this sense, the standard cryptographic scenario for quantum money tokens offers the user as complete a form of privacy as relativistic quantum theory permits. This includes past, present and future privacy as defined above in the case where the user is restricted to propagating the token along definite paths.

We may extend the above definitions of past, present and future privacy to schemes, such as quantum money token schemes, in which the token has a quantum state \( \psi_S \) defined on any spacelike hypersurface \( S \), with the property that the possible user actions (including actions on the token and presentation of the token) in the future of \( S \) depend only on \( \psi_S \).

We say such a scheme offers present privacy if it does not require the user to give away any information about the token state \( \psi_S \) on any spacelike hypersurface \( S \), unless she chooses to present the token there, beyond the information that can be inferred from signalling constraints.

We say the scheme offers future privacy if the information it requires the user to give at any locations \( Q \prec S \) or \( Q \in S \) does not imply any information about the token state \( \psi_{S'} \) on any hypersurfaces \( S' \) in the causal future of \( S \), beyond those implied by signalling constraints.

We say it offers past privacy if the information it requires the user to give in propagating the token by any valid means from \( P \) to some network point \( Q \succ P \) and in presenting it there implies no information about its state \( \psi_S \) on any hypersurface \( S \) lying between \( P \) and \( Q \), beyond those implied by signalling constraints.

The standard cryptographic scenario for quantum money tokens offers the user past, present and future privacy, by these definitions.
User privacy for encrypted supermoney schemes

We now consider user privacy for virtual token schemes such as supermoney. In unencrypted supermoney schemes such as those described in Examples 1 and 2, the virtual token is defined by unencrypted communications – user token inputs – from the user’s agents to the issuer’s co-located agents at input points, and the valid presentation point is determined by these communications. In Example 3, the virtual token is defined by a combination of the user token inputs and unencrypted communications – issuer token inputs – from the issuer’s agents to the user’s co-located agents, at the same input points.

We want to consider encrypted versions of supermoney schemes such as these, which allow past, present and future privacy, in appropriate senses. Specifically, we consider encryptions with the following features:

1. each user token input may be introduced, at the relevant input point, as the committed string in a bit string commitment protocol initiated between the user’s and issuer’s local agents.
2. data obtained from the commitment phase of the protocol by the user and issuer’s local agents are classical, and thus may be broadcast to all presentation points in the causal future of the relevant input point (the commitment point).
3. data obtained from any subsequent sustaining phases by user and issuer agents at appropriate points are also classical and may similarly be broadcast.
4. the user’s local agent may unveil the committed string at any/all presentation points in the causal future of the commitment point, and the issuer’s local agent may verify the unveiled commitment at any presentation point where it is unveiled. (For some types of protocol (e.g. [4]), the verification protocol may require a pre-agreed time interval. If so, this interval should be short compared to typical network scales.)

We say an encrypted supermoney scheme of this type offers the user present and future privacy if the issuer obtains no information about the user token inputs prior to the presentation of the token, beyond that implied by the scheme rules that constrain allowed inputs. In particular this means that, for any spacelike hypersurface $S$, if the token has not been presented at any point $Q \prec S$ or $Q \in S$, the user’s inputs at all such points give the issuer no information about the points $Q' \succ S$ at which the token may validly be presented, beyond that implied by signalling constraints.

We say the scheme offers the user past privacy if the issuer obtains no information about the user token inputs even after presentation of the token, beyond that implied by the scheme rules. This means that, if the token is presented at $Q$, the issuer learns only that the user gave some valid set of inputs consistent with presentation at $Q$: he learns no information about which set of inputs, if the scheme rules allow more than one.

Motivations for user privacy

Each type of privacy may be valued for a variety of reasons. For token types that could be stolen if located, present and future privacy enhance security. For tokens carried by individuals, whose location thus closely correlates with the token’s, past, present or future token privacy are necessary for the corresponding forms of personal locational privacy.

Since supermoney schemes assume that user and issuer communications are authenticated, they protect the parties against token theft, even in the absence of privacy. However, in some scenarios, the user may only have one or a few agents, who may value their personal locational privacy. Even if the user is a financial institution represented by a network of collaborating agents operating at known secure sites, present and future privacy of the token scheme may be highly desirable for other reasons. For example, if the user’s token represents a large sum of money to be used somewhere on the global financial network, with the location and time decided by a complex trading strategy, she may wish to conceal the token’s present and future location from other agencies, so that they cannot exploit advance information about her possible trades. She may perhaps also value past privacy, so as not to reveal any more information about her trading strategy than necessary, even after a trade.

Encrypted supermoney with present and future privacy

Encrypted supermoney schemes add an extra layer of security, by allowing the user to communicate all token data, up to the point of presentation, in an encrypted form. In their simplest form, the user’s local agent at each input point $P_i$ introduces her user token input as the committed string in a single (i.e. non-redundant) bit string commitment protocol. The protocol commits her to the token input but does not reveal it to the issuer’s local agent.

Bit string commitments may be made using protocols involving the short distance transmission of classical or quantum data, with security based on relativistic signalling constraints [4,16-20]. Such schemes need two or more user agents communicating
with adjacent issuer agents; many configurations of agents are possible. Depending on the agent configurations and the choice of spacetime point at which the commitment may be unveiled, the schemes may require multiple rounds of communication between each pair of agents. The bit string commitments could also be made by schemes that are not information theoretically secure but are presently believed to be computationally secure (e.g. [21]) or by “post-quantum” bit string commitment schemes designed to be computationally secure against quantum computers.

We require protocols in which the data obtained from the commitment phase by the user and issuer’s local agents are classical, and thus may be broadcast to all presentation points in the causal future of the relevant input point (the commitment point), and the data obtained from any subsequent sustaining phases by user and issuer agents at appropriate points are also classical and may similarly be broadcast. This is true of the schemes of Refs. [4, 16]. However the security of multi-round versions of these schemes against general quantum adversaries is not proven. It is also true of classical schemes that are believed to be computationally secure against classical or quantum computers, although of course these schemes are not information-theoretically secure. For definiteness, we focus our present discussion on a variant of the scheme of Ref. [17], for which the appropriate information theoretic security is provable [19, 22]. As we discuss next, this variant actually does not necessarily require the user to commit herself to a given bit string at the point where the communications defining the commitment phase take place. It is thus more accurately described as a protocol for a closely related relativistic task, which we term bit string coordination. As we explain below, this is adequate, since the security criteria for encrypted supermoney also require secure bit string coordination rather than secure bit string commitment.

**Bit string coordination and bit string commitment**

In our main scenario for supermoney, the user inputs data at various space-time points, and these inputs collectively determine the space-time point at which the supermoney token will be valid. Effectively, the user freely controls the destiny of the supermoney token, and the scheme is designed to give as much freedom to the user as possible while preventing fraudulent presentation of duplicate tokens. In this scenario, it is not an additional security requirement that the user necessarily must be committed to a specific destination by choices made at specific pre-agreed points in space-time.

So, if the user’s inputs are encrypted, the necessary security requirement for supermoney is that the user should not be able to produce different apparently valid decrypts of the same input at different space-time points (and in particular at space-like separated points). This does not necessarily imply that the user was committed to a specific decrypt value at the input point. It is thus possible in principle to implement secure supermoney schemes, and to maintain present and future user privacy, while using subprotocols that guarantee bit string coordination, meaning that any two apparently valid decrypts of a bit string must (with suitably high probability) be identical. This can be done without necessarily guaranteeing bit string commitment at some specified input point.

The weaker security requirement is significant even if protocols that guarantee secure bit commitment, and not only secure bit coordination, are used in a supermoney scheme. There are several reasons for this. One is that the security parameters for the relevant schemes need be chosen only so as ensure appropriately secure bit coordination rather than bit commitment. Another is that, for schemes with incomplete security proofs against general attacks, we only need to prove that the scheme securely implements bit coordination rather than bit commitment. Moreover, the composable security of bit commitment protocols in a supermoney scheme need only be analysed with respect to composablely secure coordination rather than composablely secure commitment.

**Definition of bit coordination**

A bit coordination scheme in space-time is a protocol defined in the standard scenario for mistrustful relativistic cryptography. That is, Alice is represented by a network of collaborating and mutually trusting agents distributed throughout space-time, occupying secure sites; Bob is represented by a similar network. The secure sites occupied by Alice’s agents are disjoint from those occupied by Bob’s. The protocol may (and in the cases we focus on does) prescribe a sequence of coordinated communications between adjacent agents of Alice and Bob taking place within prescribed time windows. However, we assume the secure sites of adjacent agents are so close in space and the communications so close in time, compared to network point separations and other protocol time parameters, that we may sensibly consider them as effectively taking place at a single space-time point.

At some finite set of initial points \( \{P_i\}_{i=1}^m \), Alice’s and Bob’s co-located agents exchange classical and/or quantum information, following a prescribed protocol, which may require either or both to make random choices from prescribed distributions.

At another set of final points \( \{Q_j\}_{j=1}^n \), Alice’s and Bob’s co-located agents make further exchanges of classical and/or quantum information, again following a prescribed protocol. These include a declaration of a bit value \( b_j \in \{0, 1\} \) by each of
Alice’s agents. At each $Q_j$, the protocol stipulates an algorithm in which $B_j$ inputs all the information he received or retained as part of the protocol, including the value $b_j$, and outputs $O_j$ taking the value accept or reject. If there are no constraints on Bob’s signalling other than those implied by the causal structure of space-time, we may take the final points $Q_j$ to be mutually space-like separated, since if Alice declares a bit value $b_j$ at $Q_j$ then Bob’s agent there may send the declared value $b_j$ and his output $O_j$ to any point $Q \succ Q_j$.

Alice’s agents may carry out any classical or quantum operations on any information they generate or receive and may transmit such information to other agents of Alice. Such operations and communications may be required as part of the protocol and may be required to be implemented securely; we assume secure devices and channels are available as required. The same holds true for Bob. The protocol stipulates non-empty sets $S_b = \{A_i^j : i \in I\}$ of coordinated sets of actions, $A_i^j$, for Alice’s agents, where the same indexing set $I$ labels $S_0$ and $S_1$. Alice may choose any set of actions $A_i^j \in S_b$ if she wishes all her agents to unveil bit value $b$. Each $A_i^j$ includes the prescription that each $A_i^j$ should declare bit value $b$ at $Q_j$.

We say the protocol is $\epsilon$-sound if, when Bob honestly follows the protocol and Alice follows any $\epsilon$-sound, $\epsilon$-secure and $\epsilon$'-concealing if each of its components is. None of Alice’s agents know the value of $b$ until the point $Q_j$. Nonetheless, this does not per se violate the security criteria. A secure protocol guarantees to Bob that he will almost certainly reject if any two of Alice’s agents unveil different bit values; it is not required to guarantee to him that the agents knew these values in advance of unveiling.

Note too that Alice may have strategies that almost surely result in her agents’ all unveiling the same bit and Bob’s agents all accepting these unveilings, even though some or all of her agents do not know the value of the bit to be unveiled in advance. She may, for example, prepare a state of the form

$$\sum_{b=0,1} a_b |b\rangle_{A_1} \ldots |b\rangle_{A_n} |b\rangle_{I_1} \ldots |b\rangle_{I_N},$$

where $\sum_{b=0,1} |a_b|^2 = 1$, $a_0 \neq 0 \neq a_1$, the state $|\rangle_{A_i}$ is an ancilla state sent to Alice’s agent at $Q_i$ and the $|\rangle_{I_j}$ states are input into the protocol as required, where input $|b\rangle$ corresponds to Alice choosing bit value $b$. Alice’s agents at $Q_i$ may each measure their $|\rangle_{A_i}$ state, each obtaining the same bit value $b$, which they declare along with the unveiling data prescribed by the protocol. This superposition strategy is $\epsilon$-sound, $\epsilon'$-secure and $\epsilon''$-concealing if each of its components is. None of Alice’s agents know the value of $b$ until the point $Q_i$. Nonetheless, this does not per se violate the security criteria. A secure protocol guarantees to Bob that he will almost certainly reject if any two of Alice’s agents unveil different bit values; it is not required to guarantee to him that the agents knew these values in advance of unveiling.

\section*{Definition of bit string coordination}

We define a bit string coordination scheme in space-time similarly. As before, at some finite set of initial points $\{P_i\}_{i=1}^m$, Alice’s and Bob’s co-located agents exchange classical and/or quantum information, following a prescribed protocol, which may require either or both to make random choices from prescribed distributions.

At another set of final points $\{Q_j\}_{j=1}^n$, Alice’s and Bob’s co-located agents make further exchanges of classical and/or quantum information, again following a prescribed protocol. These include a declaration of a bit string value

$$s_j = b_{j,0} \ldots b_{j,t-1} \in \{0,1\}^t$$

by each of Alice’s agents. At each $Q_j$, the protocol stipulates an algorithm in which $B_j$ inputs all the information he received or retained as part of the protocol, including the value $s_j$, and outputs $O_j$ taking the value accept or reject. Again, if there are
no constraints on Bob’s signalling other than those implied by the causal structure of space-time, we may take the final points to be mutually space-like separated.

As before, Alice’s agents may carry out any classical or quantum operations on any information they generate or receive and may transmit such information to other agents of Alice. Such operations and communications may be required as part of the protocol and may be required to be implemented securely; we assume secure devices and channels are available as required. The same holds true for Bob. The protocol stipulates non-empty sets $S_i = \{A_{i}^s : i \in I\}$ of coordinated sets of actions, $A_{i}^s$, for Alice’s agents, where the same indexing set $I$ labels each $S_i$. Alice may choose any $A_{i}^s \in S_i$ if she wishes all her agents to unveil string value $s$. Each $A_{i}^s$ includes the prescription that each $A_i$ should declare string value $s$ at $Q_j$.

We say the protocol is $\epsilon$-sound if, when Bob honestly follows the protocol and Alice follows any $A_{i}^s \in S_i$, the probability that all of Bob’s agents accept obeys

$$\operatorname{Prob}(O_1 = \ldots = O_n = \text{accept}) \geq 1 - \epsilon.$$  \hspace{1cm} (4)

We say the protocol is $\epsilon$-secure against Alice if, whatever strategy she uses, including strategies not in any $S_i$,

$$\operatorname{Prob}(\exists j \neq j' \text{ such that } s_j \neq s_{j'} \& O_j = O_{j'} = \text{accept}) \leq \epsilon.$$  \hspace{1cm} (5)

We say the protocol is $\epsilon$-concealing if the following condition holds. Suppose it is given that Alice will follow prescription $A_{i}^s$ for some given $i \in I$ known to Bob and a string value $s \in \{0, 1\}^4$ of her choice, everywhere up to the unveiling points $Q_j$, where Alice’s agents do not send any information. Suppose that Bob’s agents then share all the information received during the protocol, communicating it all to a single agent $B'$ at some point $Q'$, where $Q' \geq Q_j$ for all $j$. Then $B'$ can learn no more than $\epsilon$ bits of information about Alice’s chosen string $s$.

As in the case of bit coordination, the protocol may allow Alice other sound and concealing strategies, including strategies in which her agents coordinate to reveal a string that remains in a superposition state until unveiling.

**A secure bit string coordination protocol**

To define a secure bit coordination protocol, consider the following variant of the scheme of Ref. [18]. A secure bit string coordination protocol may be defined simply by composing a number of these secure bit coordination protocols. Alice and Bob agree on a space-time point $P$ and $n$ space-like separated points $\{Q_i\}_{i=1}^n$ in the causal future of $P$. They each have agents, separated in secure laboratories, adjacent to each of the points $P, Q_1, \ldots, Q_n$. To simplify for the moment, we take the distances from the labs to the relevant points as negligible. Alice may have other agents at other locations. We assume that Alice’s agents have secure quantum and classical communication channels as required. Alice’s secure classical channels could, for example, be created by pre-sharing one-time pads between her agent at $P$ and those at $Q_0$ and $Q_1$ and sending pad-encrypted classical signals. If necessary or desired, these pads could be periodically replenished by quantum key distribution links between the relevant agents. Her secure quantum channels could, for example, be teleportation using pre-shared entangled states and (not necessarily secure, but authenticated) classical communication. We assume that Bob’s agents either have secure classical communication channels or pre-share the classical description of the qubit string used by Bob in the protocol.

Bob securely prepares a set of qubits $|\psi\rangle_{i=1}^N =$ independently randomly chosen from the BB84 states $\{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$ (where $|\pm\rangle = 1/\sqrt{2}(|0\rangle \pm |1\rangle)$) and sends them to Alice to arrive (essentially) at $P$.

We define a strategy $S_{P'}$ for Alice to produce a coordinated unveiling of the bit value of her choice, for any point $P'$ such that $P \leq P' \leq Q_i$ for all $i$. To be able to unveil 0 in strategy $S_{P'}$, an agent of Alice’s agent at $P'$ measures each state in the $\{|0\rangle, |1\rangle\}$ basis, and sends the outcomes over secure classical channels to her agents at each of the $Q_i$. To be able to unveil 1, Alice’s agent at $P'$ measures each state in the $\{|+\rangle, |-\rangle\}$ basis, and sends the outcomes as above.

To unveil her coordinated bit, Alice’s agents at each of the $Q_i$ reveal her measurement choices and outcomes to Bob’s agents there. Each of Bob’s agents checks that the choices and outcomes revealed to them are statistically consistent with the qubit string initially sent and that the choices correspond to the declared bit value. If so, they accept the revelation; if not, they reject.

For Alice to be able to successfully cheat on any coordinated bit, at least two of her agents must be able to produce measurement choice and outcome data statistically consistent with measurements in the two BB84 bases on the same set of $N$ qubits. There are $n(n-1)/2$ pairs of agents, and the probability of any pair successfully doing so [22] decreases exponentially with $N$.

The protocol is thus $\epsilon(N)$-secure against Alice, where $\epsilon(N) \to 0$ as the security parameter $N \to \infty$.

Alice gives Bob no information until unveiling, so the protocol is perfectly (i.e. 0-) concealing.

In the ideal case of perfect preparation, transmission and measurement, Alice’s results are always statistically consistent with her declared bit value and measurement choices. The protocols is thus perfectly (i.e. 0-) sound in this case.

The protocol can also be shown to be secure and sound in the presence of non-zero errors and losses [19,23].
Notice that, if any pair \( Q_i \) and \( Q_j \) are space-like separated, Alice’s successful unveiling at \( Q_i \) and \( Q_j \) guarantees that she was committed to the unveiled bit, in the standard sense \([17]\), at some point in the past of \( Q_i \) and \( Q_j \). However, it does not identify the point at which she was committed. In particular, it does not guarantee that she was committed at \( P \).

**Privacy for issuer inputs in supermoney schemes**

We have defined present and future privacy as security desiderata for the user, without considering the issuer’s privacy. Indeed, in many scenarios the issuer either does not input any information during the scheme or may input only information that he is happy to be made public. In either case, he has no privacy concerns.

However, in some scenarios, when participating in supermoney schemes such as Example 3, in which he inputs data that affect when and where the token may be valid, the issuer may wish to ensure that these data are kept temporarily private and/or that they are made public only if and when a token is presented. He may ensure this by using bit string commitment or coordination schemes to commit to the data without immediately revealing them. Precisely what form of security the user may demand from the issuer – bit string commitment or bit string coordination – depends on the scenario. The parties also need to agree how to coordinate the unveilings of their respective bit string commitments (for example by agreeing that one of them should unveil first, or by agreeing to unveil at adjacent spacelike separated locations) in protocols of this type.

**Example 4** Consider the following variation of Example 3, with the same network as Example 2 but different supermoney token rules. In this version, at \( t = 0 \), Bob’s agent at some pre-agreed point \( P \in \{ P_i \} \) makes a commitment to Alice’s co-located agent, using a secure bit string commitment or coordination scheme that guarantees coordinated unveilings at all the presentation points \( Q_j \). (Recall that these have the same spatial locations as the input points, and time coordinate \( T \), the time required for a signal to travel between antipodal points.) Bob’s commitment identifies a subset \( S \subset \{ P_i \} \) that defines the subset of Bob’s agents that will participate in the scheme. Alice’s local agent at each \( P_i \) make coordinated commitments of the value \( f(P_i) \) to Bob’s co-located agents, using a secure bit string commitment or coordination scheme that guarantees coordinated unveilings at all the presentation points \( Q_j \).

At time \( T \), Bob’s agents at each presentation point unveil the bit string commitment identifying the subset \( S \) of input/presentation points on which the token is defined. Alice’s agents unveil their coordinated commitments of the values of \( f(P_i) \) for the input points belonging to that subset.

In this variation, the supermoney token is potentially valid after time \( T \) at a location \( P_i \in S \) if \( f(P_i) = \max \{ f(P_j) : P_j \in S \} \). It is valid at \( P_i \) if there is no other point at which it could be potentially valid, i.e. if \( P_i \) is the unique maximum point in \( S \). (Thus, again, this scheme allows no valid presentation point if there are \( N > 1 \) global maxima in \( S \).)

The supermoney scheme guarantees that both Alice and Bob can verify that the supermoney token is valid at \( T \) at the unique maximum point \( P_i \in S \), when there is a unique maximum. Bob can check there that it is indeed the unique maximum of \( f \) in \( S \). Bob’s bit string commitment or coordination scheme guarantees to Alice that Bob’s agents at all of the presentation points \( Q_j \) identify the same set \( S \) of participants; it guarantees to Bob that Alice cannot identify this set \( S \) until he unveils at the presentation points. Alice’s bit string commitment or coordination scheme guarantees to Bob that Alice’s agents at each presentation point \( Q_j \) unveil the same set of values \( f(P_i) \) of the function at the input points.

The no-summoning theorem \([13]\) shows that Alice cannot generally solve the task defining this scheme by summoning a quantum money token. If \( S \) and \( S' \) are space-like separated sets of presentation points, she may be required to present the quantum token either at a presentation point in \( S \) or at a presentation point in \( S' \), and has no advance information as to which.

**Encrypted supermoney and past privacy**

In the embodiments discussed above, \( A \) presents to \( B \) at her final chosen point \( Q \) a classical string of bits defining commitment data for the various commitments that encrypt the data defining the supermoney token. \( B \) can use these commitment data, together with his validation data, to unveil the commitments, and thus obtain the unencrypted supermoney token input data. This allows \( B \) to validate the supermoney token at \( Q \), but also gives \( B \) all the token input data.

To prevent this and ensure past privacy, \( A \) and \( B \) may proceed as follows. \( B \) may present to \( A \) the requirements for a token to be valid at \( Q \) in the form of a testing algorithm, which may for example include statistical tests that allow for some level of errors and for statistical variance in the outcomes of any quantum measurements \( A \) may have made in the course of the commitment protocols. This algorithm may conveniently be agreed in advance, although in principle it could be specified by \( B \) at \( Q \). Instead of sending commitment data directly, \( A \) may encrypt it using any standard cryptographically or unconditionally secure bit commitment scheme, in a redundant form that allows \( A \) and \( B \) to follow a zero-knowledge proof protocol (e.g. Ref. \([23]\)) that simultaneously guarantees to \( B \) that the commitment data correspond to a token valid at \( Q \) and guarantees to \( A \) that
$B$ learns no information about the supermoney token data other than the information implied by its initial state, its validity at $Q$, the rules of the given supermoney scheme, and the no-superluminal signalling principle.

SECURITY AND PRACTICALITY ISSUES

We have suggested a conception of secure supermoney tokens in space-time implemented by local communications between user and issuer agents, and characterised by security properties of the token scheme. The key feature of our schemes is a guarantee against multiple presentation of apparently valid supermoney tokens. This is ensured by the logical structure of the schemes. The schemes require communications to be completed in small regions around each network point. In their simplest versions, they require the issuer agents to transmit information received from user agents to all issuer agents in their causal future, who must store it until it can no longer be relevant. Depending on the details of the token scheme, this may require storing it until they learn that the token has been validly presented. These communication and storage requirements imply technology-dependent bounds on the network density and on token lifetimes. However, the schemes can clearly be implemented securely for many interesting and realistic examples of networks and token lifetimes.

Extensions of the scheme to allow present, future and past privacy raise new questions. Unconditionally secure bit commitment schemes can be proven secure, against both classical and quantum adversaries, assuming only the validity of quantum theory and special relativity, when used to commit a single bit, at least for a single round $[4, 17–20]$. These schemes are also unconditionally and compositely secure against the receiver, Bob, since from his perspective the information he receives $[34]$ for each committed bit is random, and is uncorrelated between commitments. Specific zero knowledge protocols based on relativistic bit commitments have also been proved secure $[24]$. However, security and efficiency questions for general zero knowledge protocols based on relativistic bit commitment schemes remain open. A fortiori, this is also true for protocols that incorporate bit commitments and zero knowledge proofs as subprotocols. Supermoney token schemes add further motivation for exploring these intriguing questions.

As we have noted, secure supermoney schemes need to ensure that the decrypts of encrypted data are securely coordinated, not necessarily that the user was securely committed to these data at some specified input point. Still, the composable security of such schemes against a committer equipped with unlimited quantum technology, who may be unveiling many commitments, each at many sites, remains to be fully analysed. It also remains to be understood how to optimize such protocols for efficiency, so that a given level or type of security can be guaranteed with minimal (classical and/or quantum) communication, storage, entanglement or information processing resources.

As noted earlier, supermoney token schemes could alternatively be based on classical non-relativistic bit commitment schemes that are believed to be computationally secure against classical or quantum computers. Composability issues are a much less serious concern (albeit not proven irrelevant) if such schemes are used, and we believe present technology should allow our these versions of our schemes to be implemented in some interesting and practically relevant examples with good security.

CONCLUSIONS

We have proposed defining secure supermoney token schemes that (i) prevent space-like duplication, (ii) allow the solution of generalised summoning tasks, including tasks that cannot be solved by standard quantum money schemes. We have also proposed extensions of these schemes involving general types of subtoken that also satisfy (i-ii) and have very flexible properties. These token and sub-token schemes have implementations with current technology for many types of practically relevant network.

We have also proposed refinements of these schemes that offer past, present and future privacy, modulo questions about composability security that remain to be fully addressed. Their optimal implementations remain to be characterised, as does the full range of tasks for which they can be practically implemented with present technology.

Quantum money schemes offer other ways of satisfying (i) and of solving some generalised summoning tasks, and guarantee past, present and future privacy to the extent quantum theory allows. While supermoney token schemes can in principle solve any deterministic summoning task that can be solved with quantum money, the converse is not true. Quantum money schemes with long token lifetimes are not feasible with current technology.

Intriguing theoretical and practical questions about the relative advantages, implementability and resource costs of supermoney and quantum money token schemes thus remain. We hope this will motivate further theoretical and experimental work.

Comment Refs. $[29]$ contain some further examples and discussions.

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[27] These certainly slowed the responsivity of the global financial system and led to significant arbitrage opportunities. For example, it has been suggested that Nathan Rothschild was able to profitably exploit early information about the outcome of the Battle of Waterloo. It should be noted, though, that historical evidence conflicts with the popular anecdotes, which may largely be inventions by adversaries of the Rothschilds: some discussion can be found in Ref. [26]. The anecdotes nonetheless at least show that the existence of signalling arbitrage opportunities was widely recognised in the 19th century. It would be very surprising if such opportunities were not sometimes exploited.

[28] We suppose here that the network of local stations is sufficiently dense for the applications envisaged.

[29] Users might also acquire quantum information, which might be delocalized. They might also want to acquire delocalized quantum resources, such as singlets shared between specified sites. Our token schemes can allow for the possibility that users may acquire quantum information, if users generate classical token data from acquired quantum information. Our schemes could also be generalized to allow acquired quantum information to be part of the token data, although the propagation of such data is obviously constrained by the no-cloning and no-broadcasting theorems. Our generalized schemes involving subtokens may be used to allow users to acquire delocalized quantum resources. In such scenarios, the onus is on the user to ensure that their requests are appropriately coordinated as well as valid. For example, if the user requests a component of singlet s at point P and a component of singlet $s’ \neq s$ at a spacelike separated point Q, she will not acquire a singlet shared between P and Q, even if each request is valid according to the scheme.

[30] Here and in later definitions, this includes the possibility of the null statement, corresponding to no communication.

[31] For example, they may be required to make the same statement at each $P_i$, as in Example 3 below.

[32] In Ref. [19] and elsewhere in the summoning literature, the $Q_i$ are referred to as return points. We use the term presentation point here so that the same term can be used in the case of quantum money tokens, emulations of quantum money tokens by supermoney, and tokens in
general supermoney schemes. We prefer to refer to supermoney tokens being presented rather than returned, since the latter term suggests restoring some classical or quantum physical object to its originator.

[33] We assume here for simplicity that \( P \) is in the causal past of all network points at which the user may wish to introduce states or carry out operations. However, the discussion extends to the more general case in which some network points are not in the causal future of \( P \).

[34] If any – in some of these schemes he receives no information until unveiling.