MONODROMY APPROACH TO THE SCALING LIMITS IN THE ISOMONODROMY SYSTEMS

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Abstract. The isomonodromy deformation method is applied to the scaling limits in the linear $N \times N$ matrix equations with rational coefficients to obtain the deformation equations for the algebraic curves which describe the local behavior of the reduced versions for the relevant isomonodromy deformation equations. The approach is illustrated by the study of the algebraic curve associated to the $n$-large asymptotics in the sequence of the bi-orthogonal polynomials with cubic potentials.

1. Introduction

It is well known that, in certain asymptotic limits, the classical Painlevé equations [1] reduce to elliptic ones. For instance, the Painlevé sixth equation,

$$y_{xx} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) y_x^2 - \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) y_x + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left( c_1 + c_2 \frac{x}{y^2} + c_3 \frac{x-1}{(y-1)^2} + c_4 \frac{x(x-1)}{(y-x)^2} \right),$$

(PVI)

with the large parameters $c_j, j = 1, \ldots, 4$, after the changes $x = t_0 + \delta \tau$, $c_j = \delta^{-2} a_j + \delta^{-1} b_j$, where $t_0, a_j, b_j = const$, turns at $\delta = 0$ into an autonomous equation which has the first integral

$$D_0 = \frac{t_0^2(t_0-1)^2}{2y(y-1)(y-t_0)} y_x^2 - a_1 y + a_2 \frac{t_0}{y} + a_3 \frac{t_0-1}{y-1} + a_4 \frac{t_0(t_0-1)}{y-t_0}.$$  (1.1)

The asymptotics of the classical Painlevé transcendents w.r.t. parameters or initial data were studied in numerous works, see [2] for extended but not exhaustive bibliography. The limit transitions of such kind are called below the scaling limits.

The most effective to this date approach to the scaling limits in the Painlevé transcendents is based on the monodromy representation for the latter [3, 4]. In [2], the isomonodromy deformation technique of [5] was adapted to the study of the scaling limits in the equations of the isomonodromy deformations for the $2 \times 2$ matrix linear first order ODEs.
with rational coefficients. In particular, the results of [2] imply that
the modular parameters determining the limiting (hyper)elliptic curve
like $D_0$ in (1.1) are not arbitrary constants but certain functions of all
the deformation parameters. These functions are uniquely determined
by the system of transcendent modulation equations
\[ \text{Re} \int_{\ell} \mu(\lambda) \, d\lambda = \text{const}, \tag{1.2} \]
where $\ell$ is an arbitrary closed path on the Riemann surface of the
relevant spectral curve.

Below, this result will be extended to the scaling limits in the isomonodromy
deformation systems for $N \times N$ matrix linear ODEs with rational
coefficients. The work is motivated by recent developments in the
theory of coupled random matrices. Indeed, while the statistic prop-
erties of the ensembles of the single random matrices can be given in
terms of the asymptotics of the semi-classical orthogonal polynomials,
see [6, 7], which give rise to the linear first order $2 \times 2$ matrix ODEs
with rational coefficients [8], the ensembles of the coupled random
matrices in the very similar way give rise to the bi-orthogonal polynomials
[9, 10, 11] and the linear $N \times N$ matrix ODEs [10].

The paper is organized as follows. In Section 2, we recall the basic
facts in the isomonodromy deformations of the linear matrix equations
with rational coefficients, introduce the notion of the scaling limits
in such systems and describe the WKB approach to their asymptotic
solutions. In Section 3, we find the modulation equations for the non-
singular asymptotic spectral curve and prove their unique solvability.
In Section 4, we illustrate our approach using a particular $3 \times 3$ ma-
trix equation satisfied by the bi-orthogonal polynomials for the cubic
potentials.

2. THE ISOMONODROMY SYSTEMS WITH A LARGE PARAMETER

In this section, following [12, 13, 3], we recall the basic notions of the
theory of the linear matrix differential equations. Consider an $N \times N$
matrix first order ODE,
\[ \frac{d\Psi}{d\lambda} = A(\lambda)\Psi, \quad A(\lambda) = \sum_{\nu=1}^{n} \sum_{k=0}^{r_{\nu}} \frac{A_{\nu,-k}}{(\lambda - a^{(\nu)})^{k+1}} - \sum_{k=1}^{r_{\infty}} A_{\infty,-k} \lambda^{k-1}. \tag{2.1} \]
We call equation (2.1) generic if the eigenvalues of $A_{\nu,-r_{\nu}}$ are distinct
for $r_{\nu} \neq 0$ and if they are distinct modulo integers for $r_{\nu} = 0$. Without
loss of generality, $\lambda = \infty$ is the singular point of the highest Poincaré
rank, i.e. $r_{\infty} \geq r_{\nu}$, $\nu = 1, \ldots, n$. Assume that (2.1) is generic and
$A_{\infty,-r_{\infty}}$ is diagonal (for the non-generic situations, see [13]). Then, near the singularity $a^{(\nu)}$, equation (2.1) has the formal solution
\[ \Psi^{(\nu)}(\lambda) = W^{(\nu)}(\lambda)e^{\theta^{(\nu)}(\lambda)}, \quad \nu = 1, \ldots, n, \infty, \]
\[ \hat{\Psi}^{(\nu)}(\lambda) = I + \sum_{j=1}^{\infty} \psi_j^{(\nu)}(\lambda) \xi_j, \quad \theta^{(\nu)}(\lambda) = \sum_{j=1}^{r^{(\nu)}} x_j^{(\nu)} \frac{\xi_j}{(-j)} + x_0^{(\nu)} \ln \xi, \quad (2.2) \]
where $\xi = \lambda - a^{(\nu)}$ for a finite singularity $a^{(\nu)}$ and $\xi = 1/\lambda$ for infinity.

The matrix coefficients $\psi_j^{(\nu)}$ and the diagonal matrix coefficients $x_j^{(\nu)}$, $j \neq 0$, for $\theta^{(\nu)}(\lambda)$ in (2.2) are determined uniquely by the eigenvector matrix $W^{(\nu)}$ of $A_{\nu,-r_{\nu}}$.

The ratio $\tilde{\Psi}^{-1}(\lambda)\Psi(\lambda)$ of any two solutions $\Psi$ and $\tilde{\Psi}$ of (2.1) does not depend on $\lambda$. The ratios of the fundamental solutions normalized by (2.2) are called the monodromy data.

The set of deformation parameters is specified for generic equation (2.1) in [3] (for non-generic equations in [14]). These parameters include the positions $a^{(\nu)}$ of the singular points and the entries of the diagonal matrices $x_j^{(\nu)}$, $j \neq 0$, for $\theta^{(\nu)}(\lambda)$ in (2.2). All these quantities form together the vector $x$ of the deformation parameters. Remaining parameters $x_0^{(\nu)}$, $\nu = 1, \ldots, \infty$, are called the formal monodromy exponents. The latter satisfy the Fuchs’ identity,
\[ \sum_{\nu=1}^{n} \text{Tr} x_0^{(\nu)} + \text{Tr} x_0^{(\infty)} = 0. \quad (2.3) \]

**Remark 2.1.** Using the linear transformations of the complex $\lambda$-plane, one can fix the positions of two of the finite singular points or to fix two of the entries $(x_j^{(\nu)})_k$. Using the scalar gauge transformation, one can get $\text{Tr} A(\lambda) \equiv 0$ and therefore $\text{Tr} x_j^{(\nu)} = 0$. The orbits of the above transformations are called the essential deformation parameters.

Let $d$ denote the exterior differentiation w.r.t. entries of $x$. In accord with [3], the monodromy data of the generic equation (2.1) do not depend on $x$ if and only if there exist 1-forms $\Omega$ and $\Theta^{(\nu)}$ such that the fundamental solutions above additionally satisfy equations
\[ d\Psi = \Omega \Psi, \quad dW^{(\nu)} = \Theta^{(\nu)} W^{(\nu)}, \quad \nu = 1, \ldots, n, \infty. \quad (2.4) \]
Then the compatibility condition of (2.1), (2.4),
\[ dA = \frac{\partial \Omega}{\partial \lambda} + [\Omega, A], \quad d\Omega = \Omega \wedge \Omega, \quad (2.5) \]
is the completely integrable differential system whose fixed singularities are the planes \( a^{(\nu)} = a^{(\rho)} \), \( \nu \neq \rho \), \((x_j^{(\nu)})_{ii} = \infty\), \((x_{-r\nu})_{ii} = 0\), \((x_{-r\nu})_{jj} = (x_{-r\nu})_{jj} \) if \( i \neq j\), \( r\nu \neq 0\). The generic \( 2 \times 2 \) system (2.5) admitting the only one deformation parameter is equivalent to one of the classical Painlevé equations.

2.1. **Scaling limits in the isomonodromy systems.** Let \( A(\lambda) \) depend on an additional parameter \( \delta \),

\[
a^{(\nu)} = \delta^\varphi (b^{(\nu)} + \delta c^{(\nu)}), \quad \varphi = \text{const}, \quad \nu = 1, \ldots, n,
\]

\[
A_{\nu,-k} = \delta^{k\varphi} B_{\nu,-k}, \quad A_{\infty,-k} = \delta^{-k\varphi} B_{\infty,-k}, \quad k = 0, \ldots, r\nu,
\]

where \( B_{\nu,-k}, k = 0, \ldots, r\nu, \nu = 1, \ldots, n, \infty\), are ascending Erdelyi series in \( \delta \) such that, for generic equation (2.1),

\[
x^{(\nu)}_{-k} = \delta^{k\varphi} (t^{(\nu)}_{-k} + \delta t^{(\nu)}_{-k}), \quad x^{(\infty)}_{-k} = \delta^{-k\varphi} (t^{(\infty)}_{-k} + \delta t^{(\infty)}_{-k}).
\]

The constants \( b^{(\nu)} \) and the entries of \( t^{(\nu)}_{-k}, k \neq 0\), form the vector \( t \) for the center of the asymptotic domain in the parameter space as \( \delta \to +0\).

Inequalities \(|c^{(\nu)}|, ||\tau^{(\nu)}|| < \text{const}\), or simply \(||\tau|| < \text{const}\), yield the range of the “local” deformations. The entries of \( t \) are usually called the “slow” variables, while the entries of \( \tau \) are called the “quick” or “fast” variables. The entries of the vectors \( t^{(\nu)}_{0} \) and \( \tau^{(\nu)}_{0} \) for the formal monodromy exponents form the vectors \( \alpha \) and \( \beta \), respectively.

Substituting (2.6) and \( \lambda = \delta^\varphi \zeta \) into (2.1), (2.4), we find

\[
\frac{d\Psi}{d\zeta} = \delta^{-1} B(\zeta) \Psi, \quad d\Psi = \omega \Psi,
\]

\[
B(\zeta) = \sum_{\nu=1}^{n} \sum_{k=0}^{r\nu} \frac{B_{\nu,-k}}{\zeta - b^{(\nu)} - \delta c^{(\nu)} k + 1} - \sum_{k=1}^{r\infty} B_{\infty,-k} \zeta^{-k-1},
\]

whose compatibility reads

\[
\frac{dB}{d\zeta} = [\omega, B] + \delta \frac{\partial \omega}{\partial \zeta}, \quad d\omega = \omega \wedge \omega.
\]

**Remark 2.2.** If \( r\infty \neq 0 \) and \( (x^{(\infty)}_{-r\infty})_{11} \) does not depend on \( \delta \) then \( \varphi = -1/r\infty \). If all the singularities are Fuchsian, i.e. \( r\nu = 0, \nu = 1, \ldots, n, \infty \), one may put for simplicity \( \varphi = 0 \).

**Remark 2.3.** Following [15], it is possible to construct a Schlesinger transformation chain yielding \( x^{(\nu)}_{0} = \rho^{(\nu)} + l^{(\nu)} \) with constant \( \rho^{(\nu)} \in \mathbb{C}^N \) and \( l^{(\nu)} \in \mathbb{Z}^N \). Taking \(||l^{(\nu)}||\) and \( \delta^{-1} \) large and comparable,
The scaling parameter $\delta$ is defined up to a positive factor which gives rise to a scaling freedom in the set of the “slow” variables. Below, we assume that this scaling freedom is eliminated by a normalization of one of the non-trivial “slow” deformation parameters.

2.2. Complex WKB method. Here we recall the idea of the complex WKB method following in principal [12, 13]. Consider (2.8) as $\delta \to 0$ assuming that the coefficients of $B(\zeta)$ remain bounded. Let $T$ and $\Lambda_0$ be the eigenvector and eigenvalue matrices for $B(\zeta)$,

\[ T^{-1}BT = \Lambda_0 = \text{diag}(\mu_1, \ldots, \mu_N). \]  

(2.11)

Let $\mu_j \neq \mu_k$, $j \neq k$. Then the formal expression

\[ \Psi(\zeta) = T \sum_{n=0}^{\infty} \delta^n T_n \exp\left\{ \delta^{-1} \int_{\zeta_0}^{\zeta} \sum_{m=0}^{\infty} \delta^m \Lambda_m(\xi) d\xi \right\}, \quad T_0 = I, \]  

(2.12)

satisfies (2.8) provided the diagonal matrices $\Lambda_n$ and the off-diagonal matrices $T_n$, $n \geq 1$, solve the recursion

\[ [\Lambda_0, T_1] - \Lambda_1 = T^{-1} \frac{dT}{d\lambda}, \]

\[ [\Lambda_0, T_n] - \Lambda_n = \sum_{m=1}^{n-1} T_{n-m} \Lambda_m + T^{-1} \frac{d(T T_{n-1})}{d\lambda}, \quad n \geq 2. \]  

(2.13)

Let $\Gamma$ be the Riemann surface of the algebraic curve

\[ F(\zeta, \mu) := \det(B(\zeta) - \mu I) = 0. \]  

(2.14)

The branch points of (2.14) are called the turning points for (2.8). Let $\mathcal{L}_0$ be an open simply connected domain which is the complex $\zeta$-plane punctured at the singularities of $B(\zeta)$, at the turning points and cut along the segments connecting all the singularities and turning points. Let $\mathcal{L} \subset \mathcal{L}_0$ be a closed simply connected domain. By construction, $B(\zeta)$ is holomorphic in $\mathcal{L}$, and equation (2.8) has no turning point in $\mathcal{L}$. Thus all the roots $\mu_j(\lambda)$ of the characteristic equation (2.14) are distinct from each other and are holomorphic in $\mathcal{L}$. Therefore there exists a holomorphic non-special in $\mathcal{L}$ matrix $T(\zeta)$ which diagonalizes
the matrix $B(\zeta)$, $T^{-1}BT = \Lambda_0 = \text{diag}(\mu_1, \ldots, \mu_N)$. The matrices $T$, $T^{-1}$, $\Lambda_n$, $T_n$, are holomorphic and bounded in $\mathcal{L}$ [12].

Consider the reduced gauge matrix

$$\mathcal{T}^{(m)} = T \sum_{n=0}^m \delta^n T_n, \quad T_0 = I,$$

(2.15)

where $T_n$ are defined by (2.11)–(2.13), and $\zeta \in \mathcal{L}$. The matrix function $\Phi^{(m)} = (\mathcal{T}^{(m)})^{-1}\Psi$ solves the “almost diagonal” equation

$$\Phi^{(m)}(\zeta) = \delta^{-1} B^{(m)}(\zeta), \quad B^{(m)}(\zeta) = \sum_{n=0}^m \delta^n \Lambda_n + \delta^{m+1} R^{(m)}(\zeta),$$

(2.16)

where $R^{(m)}(\zeta)$ is holomorphic and bounded for $\zeta \in \mathcal{L}$ provided $\delta$ is small enough. Define the WKB approximation to (2.16),

$$\Phi_{\text{WKB}}^{(m)}(\zeta) = e^{\theta(\zeta_0, \zeta)} \delta^{-1} \int_{\zeta_0}^\zeta \Lambda^{(m)}(\zeta) d\zeta, \quad \Lambda^{(m)} = \sum_{n=0}^m \delta^n \Lambda_n,$$

(2.17)

and introduce the correction function $\chi^{(m)}$,

$$\Phi^{(m)}(\zeta) = \chi^{(m)}(\zeta) \Phi_{\text{WKB}}^{(m)}(\zeta),$$

(2.18)

and notations

$$\mu_{ij} = \mu_i - \mu_j, \quad \theta_{ij}(\xi, \zeta) = \left(\theta(\xi, \zeta)_{ii} - \theta(\xi, \zeta)_{jj}\right),$$

$$\hat{\theta}_{ij}(\xi, \zeta) = \theta_{ij}(\xi, \zeta) - \delta^{-1} \int_{\xi}^\zeta \mu_{ij}(s) ds.$$

(2.19)

The contour $\gamma_{ij}(\zeta) \subset \mathcal{L}$ connecting the finite or infinite point $\zeta_{ij}$ with $\zeta$ is called the $(i, j)$-canonical path if

$$\text{Re} \int_{\xi}^\zeta \mu_{ij}(s) ds \leq 0 \quad \forall \xi \in \gamma_{ij}(\zeta).$$

(2.20)

A closed simply connected domain $C_{ij} \subset \mathcal{L}$ is called $(i, j)$-canonical if there exists such a point $\zeta_{ij} \in C_{ij}$ that the contour $\gamma_{ij}(\zeta) \subset C_{ij}$ connecting $\zeta_{ij}$ with any given point $\zeta \in C_{ij}$ is homotopy equivalent to a canonical path. A closed simply connected domain $C$ is called canonical if it is $(i, j)$-canonical $\forall i, j = 1, \ldots, N, i \neq j$.

Using the arguments of [12], we thus obtain the following

**Theorem 2.1.** Let $C \subset \mathcal{L}$ be a canonical domain. If

$$\int_{\gamma_{ij}(\zeta)} e^{\text{Re} \hat{\theta}_{ij}(\xi, \zeta)} |(R^{(m)}(\xi))_{ij}| \cdot |d\xi| < \infty, \quad \forall i, j = 1, \ldots, N, \quad \forall \zeta \in C,$$
then there exist such positive constants $C$ and $\delta_0$ that
\[
\|\chi^{(m-1)}(\zeta) - I\| \leq C|\delta|^m \quad \forall \zeta \in \mathcal{C}, \quad \forall \delta \in (0, \delta_0].
\tag{2.21}
\]

In particular, $\chi^{(0)}(\zeta) = I + O(\delta)$.

To construct a canonical domain containing a given point $\zeta_0 \in \mathcal{L}$, consider a pair of $i, j \in \{1, \ldots, N\}$, $i \neq j$, and introduce the segment $\ell_{ij} \subset \mathcal{L}$ of the $(i, j)$-anti-Stokes level curve-line passing through $\zeta_0$:
\[
\ell_{ij} = \{ \zeta \in \mathcal{L} : \text{Im} \int_{\zeta_0}^\zeta \mu_{ij}(s) \, ds = 0 \}.
\tag{2.22}
\]

Choose two points $\zeta_{ij}, \zeta_{ji} \in \ell_{ij}$ in such a way that: a) $\zeta_0 \in [\zeta_{ij}, \zeta_{ji}]$; b) for any $\zeta \in \ell_{ij}$ separating $\zeta_{ij}$ from $\zeta_{ji}$, the curve-line segment $[\zeta_{ij}, \zeta] \subset \ell_{ij}$ is the $(i, j)$-canonical path, while $[\zeta_{ji}, \zeta] \subset \ell_{ij}$ is the $(j, i)$-canonical path. Introduce the segment $\ell^*_{ij} \subset \mathcal{L}$ of the $(i, j)$-Stokes level curve-line passing through the point $\zeta^* \in [\zeta_{ij}, \zeta_{ji}] \subset \ell_{ij}$:
\[
\ell^*_{ij} = \{ \zeta \in \mathcal{L} : \text{Re} \int_{\zeta^*}^\zeta \mu_{ij}(s) \, ds = 0 \}.
\tag{2.23}
\]

By construction, the union $\mathcal{C}_{ij}$ of all the curve-line segments $\ell^*_{ij}$,
\[
\mathcal{C}_{ij} = \bigcup_{\zeta^* \in [\zeta_{ij}, \zeta_{ji}]} \ell^*_{ij},
\tag{2.24}
\]

is the $(i, j)$- and $(j, i)$-canonical domain. The boundary of the constructed $(i, j)$-canonical domain $\mathcal{C}_{ij}$ is formed by the $(i, j)$-Stokes level curve-lines passing through the points $\zeta_{ij}$ and $\zeta_{ji}$ and partially by the boundary of $\mathcal{L}$. It is worth to note that, near the irregular singularities of (2.8), the $(i, j)$-canonical domain $\mathcal{C}_{ij}$ can be extended beyond the boundary of $\mathcal{L}$ to fill out certain sector in the complex $\zeta$-plane called the $(i, j)$-Stokes sector.

The canonical domain $\mathcal{C} \ni \zeta_0$ of validity of Theorem 2.1 is the intersection of the above $(i, j)$-canonical domains $\mathcal{C}_{ij}$,
\[
\mathcal{C} = \bigcap_{\substack{i, j \in \{1, \ldots, N\} \atop i \neq j}} \mathcal{C}_{ij}.
\tag{2.25}
\]

The above construction implies that any closed simply connected domain $\mathcal{L} \subset \mathcal{L}_0$ can be covered by a finite number of the overlapping canonical domains $\mathcal{C}^{(k)}$, $k = 1, \ldots, K$, since the opposite assumption can be easily brought to a contradiction.
3. Modulation of the spectral curve

Solutions of the Lax equation $dB = [\omega, B]$ are routinely interpreted as the approximate solutions for (2.9) as $\delta \to 0$ [16, 17]. Supplemented by the eigenvalue problem (2.11), the Lax equation constitutes the basis for the algebro-geometric integration of the “soliton” equations. However, in the theory of the “soliton” PDEs, the spectral curve (2.14) is determined by the initial data, while in the isomonodromy deformation context it is determined by the original $\lambda$-equation (2.1). Moreover the spectral curve for the typical “soliton” equation is an exact integral of motion while, in the isomonodromy case, the curve varies,

$$d(\ln \det(B - \mu I)) = \delta \text{Tr}\left(\frac{\partial \omega}{\partial \zeta}(B - \mu I)^{-1}\right) \neq 0.$$  

In what follows, we precisely describe the dependence of the algebraic curve (2.14) on the “slow” variables at $\delta = 0$. Below, the subscript $\text{as}$ denotes the relevant object at $\delta = 0$.

We call the curve (2.14) singular if its topological properties for all small enough $\delta \neq 0$ differ from those at $\delta = 0$. Given a parameterization of the curve, we define the discriminant set $\mathcal{S}$ in the total parameter space $\mathcal{P}$ as the set determining the singular curve. Also, let $\mathcal{F}$ be the union of the hyper-planes $b^{(\nu)} = b^{(\rho)}$, $\nu \neq \rho$, $(t^{(\nu)}_{-r_{\nu}})_{ii} = (t^{(\nu)}_{-r_{\nu}})_{jj}$, $i \neq j$, $(t^{(\nu)}_{-r_{\nu}})_{kk} = 0$ corresponding to the fixed singularities of (2.9) at $\delta = 0$. Below, we always assume that our deformation parameters are apart from the fixed singularities $\mathcal{F}$.

The differential $\mu_{as}(\zeta) d\zeta$ as well as its derivatives w.r.t. $b^{(\nu)}$ and entries of $t^{(\nu)}_{-k}$, $k = 0, \ldots, r_{\nu}$, $\nu = 1, \ldots, n, \infty$, are meromorphic on the Riemann surface $\Gamma_{as}$ of the curve. All the parameters $b^{(\nu)}, t^{(\nu)}_{-k}$ together completely determine the singular part of $\mu_{as}(\zeta) d\zeta$.

**Definition 3.1.** The parameter $D_j$ is called modular iff the differential $\frac{\partial}{\partial D_j} \mu_{as}(\zeta) d\zeta$ is holomorphic on the Riemann surface $\Gamma_{as}$.

Thus $\mathcal{P} = \mathcal{T} \otimes \mathcal{D}$, where $\mathcal{T}$ is the subspace of the deformation parameters $t = (t, \text{Re} \alpha)$ (see Remark 2.3 and constraint (2.3)) while $\mathcal{D}$ is the subspace of the remaining parameters $D = (D, \text{Im} \alpha)$.

**Theorem 3.1.** Let $(t_0, D_0) \in \mathcal{P} \setminus \mathcal{S}$. Then there exists an open neighborhood $\mathcal{U} \subset \mathcal{T} \setminus \mathcal{F}$ of $t_0$ such that, for any closed path $\ell$ on the Riemann surface $\Gamma_{as}$ punctured over the points $b^{(\nu)}$, $\nu = 1, \ldots, n, \infty$,

$$J_\ell(t, D) := \text{Re} \oint_{\ell} \mu_{as}(\zeta) d\zeta = h_\ell \quad \forall t \in \mathcal{U},$$  

where $h_\ell = J_\ell(t_0, D_0) = \text{const}$. 


Proof. Let $\ell_\zeta = \pi(\ell)$ be a projection of the closed path $\ell \subset \Gamma$ with the base point $(\zeta_0, \mu_0)$ on the punctured complex $\zeta$-plane. Let the integer $m_0$ be chosen in such a way that the lift $\hat{\ell}$ of $m_0\ell_\zeta$ on $\Gamma$ be closed for all branches of $\mu(\zeta_0)$. Consider the analytic continuation of the WKB approximation (2.12) along $\hat{\ell}$. The projection $\pi(\hat{\ell}) = m_0\ell_\zeta$ is covered by a finite number of the overlapping canonical domains $C_k$, $k = 1, \ldots, s$, $C_{s+1} = C_1$, in each of which (2.12) approximates uniformly in $\zeta$ an exact solution $\Psi_k(\zeta)$ of (2.8). Because $\Psi_{k+1}(\zeta) = \Psi_k(\zeta)G_k$ where $G_k$ is independent from both $\zeta$ and $t \in \mathcal{T} \setminus \mathcal{F}$, we obtain

$$M_\hat{\ell}(\Psi_{s+1}(\zeta)) = \Psi_1(\zeta)M_\hat{\ell}G_1 \cdots G_s,$$

where $M_\hat{\ell}$ is the operator of analytic continuation along $m_0\ell_\zeta$, and $M_\hat{\ell}$ is the monodromy matrix for $\Psi_1(\zeta)$ along $m_0\ell_\zeta$. Using for $\Psi_1(\zeta)$ and $\Psi_{s+1}(\zeta)$ our WKB approximation, we find

$$\exp\left\{\delta^{-1} \oint_\hat{\ell} \Lambda(\zeta) d\zeta\right\} = (I + O(\delta))G(\delta).$$

(3.2)

Since the curve is non-singular, the r.h.s. of (3.2) preserves while $t$ remains in a neighborhood of $t_0$. Equating the leading orders of the l.h.s. for (3.2) at $t_0$ and nearby points $t$, we arrive at (3.1). $\square$

Theorem 3.1 immediately provides us with the following assertion:

**Corollary 3.2.** Let $U \subset \mathcal{T} \setminus \mathcal{F}$ be an open domain and let (3.1) holds true. If the curve $F(\zeta, \mu) = 0$ remains non-singular at the boundary point $t_1 \in \partial U$, then there exists an open domain $W \subset \mathcal{T} \setminus \mathcal{F}$ such that $U \subset W$, $t_1 \in W$, and (3.1) is valid $\forall \, t \in W$.

For the subsequent discussion, the following assertion is useful:

**Proposition 3.3.** For any closed path $\ell$ on $\Gamma_{as}$ punctured over $b^{(\nu)}$, $\nu = 1, \ldots, n, \infty$, the integral $J_{\ell}(t, \mathcal{D})$ is continuous in $(t, \mathcal{D})$ outside the fixed singularities $\mathcal{F}$ of (2.9) and is differentiable in $(t, \mathcal{D})$ outside the discriminant set $\mathcal{S}$.

**Proof.** Since the contour $\ell$ is finite, the continuity of $J_{\ell}(t, \mathcal{D})$ follows from the continuity of $\mu_{as}(\zeta)$. If the point $(t_0, \mathcal{D}_0)$ is located apart from the discriminant set $\mathcal{S}$, then there exists an open neighborhood $\mathcal{V}$ of $(t_0, \mathcal{D}_0)$ such that the spectral curve (2.14) does not degenerate $\forall (t, \mathcal{D}) \in \mathcal{V}$. Then the differentiability of $J_{\ell}(t, \mathcal{D})$ at $(t_0, \mathcal{D}_0)$ follows from the continuous differentiability of $\mu_{as}(\zeta)$. $\square$

Theorem 3.1 and Proposition 3.3 imply...
Corollary 3.4. Let $U, \hat{U} \subset \mathcal{F} \setminus \mathfrak{F}$ be adjacent open domains and let $(\partial U \cap \partial \hat{U}) \subset \mathcal{F} \setminus \mathfrak{F}$ be not empty. If $J_\ell(t, D) = h_\ell \ \forall t \in U$ and $J_\ell(t, D) = \hat{h}_\ell \ \forall t \in \hat{U}$, then $h_\ell = \hat{h}_\ell$.

In accord with Corollaries 3.2 and 3.4, if the spectral curve (2.14) is non-singular at the initial point $(t_0, D_0)$, then the modulation equation (3.1) is valid in a domain $U$ bounded by the points where the spectral curve becomes singular. Applicability of (3.1) to a particular solution of (2.9) beyond this boundary depends on some subtle details in the initial data or, equivalently, in the relevant monodromy data of the isomonodromy system (2.8), see [2] and Section 4 below.

Let us discuss now the existence of the function $D(t)$ such that $J_\ell = \text{const}$. Varying the contour $\ell$ in (3.1), we obtain the system of equations $J_{\ell_j} = h_j$, where the set of contours $\ell_j$, $j = 1, \ldots, 2g$, form a homology basis of the Riemann surface of $\Gamma_{as}$ punctured over $b^{(\nu)}$. For instance, taking for $\ell$ a small circle $c_\nu$ around $(b^{(\nu)}, \mu_j^{(b^{(\nu)})})$, we find

$$\text{Im} \left( t_0^{(\nu)} \right)_{j\jmath} = -\frac{1}{2\pi} h_{c_\nu}^{(j\jmath)} = \text{const}, \quad (3.3)$$

which contains (2.10) as the particular case $h_{c_\nu}^{(j\jmath)} = 0$.

To discuss (3.1) further, it is convenient to impose the conditions (3.3) and to remove $(N-1)(n+1)$ small circles from the “sufficient” set of contours. The remaining cycles $\ell_j$, $j = 1, \ldots, 2g$, form a homology basis of $\Gamma_{as}$. Also, using (3.3), we exclude the constant parameters $\text{Im} t_0^{(\nu)}$ from the set of unknowns and assume below that the space $\mathcal{D}$ is $g$-dimensional complex space of the modular parameters.

Theorem 3.5. Let $(t_0, D_0) \in \mathfrak{P} \setminus \mathfrak{S}$. Then, in an open neighborhood $U \subset \mathcal{F} \setminus \mathfrak{F}$ of the point $t_0$, the system (3.1), where $\ell$ runs over the homology basis $\{\ell_j\}^{2g}_{j=1}$ of $\Gamma_{as}$, determines the unique differentiable in the real sense complex vector function $D(t, \bar{t})$ such that $D(t_0, \bar{t}_0) = D_0$.

Proof. Theorem 3.1 and Proposition 3.3 imply that $J_{\ell_j}(t, D)$ are the first integrals of the completely integrable Pfaffian system $dJ = 0$,

$$\omega \begin{pmatrix} dD \\ d\bar{D} \end{pmatrix} = -\Omega \begin{pmatrix} dt \\ d\bar{t} \end{pmatrix}, \quad (3.4)$$

where $\omega$ and $\Omega$ are the matrices of the partial derivatives of $J_{\ell_j}(t, D)$ w.r.t. the entries of the vectors $D, \bar{D}$ and $t, \bar{t}$, respectively. Here, the bar means the complex conjugation. Let constant $c_1$ be the determinant of the transformation of the natural basis $\left\{ \frac{\partial}{\partial D_j} \mu_{as} d\zeta \right\}^{g}_{j=1}$ into the basis of the normalized holomorphic differentials, and let $B$ be the matrix of the $B$-periods of the normalized holomorphic differentials. Since $\omega =
\( (A \bar{A} \bar{B} B) \) is the matrix of \( A \) - and \( B \) -periods of the holomorphic differentials and their complex conjugate, \( \det \omega = (-2i)^g |c_1|^2 \det(\text{Im} \hat{B}) \neq 0 \). Thus the matrix \( \omega \) is invertible until \( F(\zeta, \mu) = 0 \) remains non-singular, and therefore the integral manifold for (3.4) is well parameterized by the deformation parameters \( (t, \bar{t}, \text{Re} \alpha) \). \( \square \)

**Remark 3.1.** If \( h_{\ell_j} = J_{\ell_j}(t_0, D_0) \neq 0 \) then the cycle \( \ell_j \) can not collapse, and the encircled by \( \ell_j \) branch points can not coalesce. Thus, along the integral manifold for (3.4), the spectral curve remains non-singular provided \( h_{\ell_j} \neq 0 \) \( \forall j = 1, \ldots, 2g \). This observation ensures the applicability of (3.1), (3.4) and the existence of \( D(t, \bar{t}) \) in any connected domain \( U \subset T \setminus S \) containing the initial point \( t_0 \).

Let \( J_{\ell}(t, D) \) in (3.1) vanish for all closed paths,

\[
\text{Re} \oint_{\ell} \mu_{as}(\zeta) \, d\zeta = 0, \quad \forall \ell \subset \Gamma_{as}. \tag{3.5}
\]

This system does make sense regardless the choice of the initial point since there is no need to fix a homology basis. Traditionally, it is called the *Boutroux system*. We recall that (3.5) may be not applicable to a particular solution of (2.9) in certain sectors of \( \mathfrak{P} \setminus \mathfrak{S} \) in spite of the Boutroux system itself does make sense in the whole parameter space \( \mathfrak{P} \) (using Proposition 3.3, the system (3.5) is interpreted at the points of the discriminant set \( \mathfrak{S} \) as a continuation from \( \mathfrak{P} \setminus \mathfrak{S} \)).

**Remark 3.2.** As the integral manifold for (3.5) meets the discriminant set \( \mathfrak{S} \), at least one of the cycles \( \ell_j \) collapses, and the corresponding real equation in (3.5) becomes trivial as being replaced by the complex condition of coalescence of two branch points. Thus, generically, the intersection of the integral manifold for (3.5) with \( \mathfrak{S} \) has codim_\(R\) = 1 in the space of the deformation parameters.

**Theorem 3.6.** There exists the unique solution \( D(t, \bar{t}) \) of the Boutroux system (3.5).

*Proof.* Here, we give the sketch proof.

**Uniqueness.** Given \( t \), two solutions \( D \) and \( D' \) determine two differentials \( \mu_{as} d\zeta \) and \( \mu'_{as} d\zeta \) meromorphic on the respective Riemann surfaces \( \Gamma_{as} \) and \( \Gamma'_{as} \). The difference \( \phi = (\mu_{as} - \mu'_{as}) d\zeta \) is holomorphic on the covering Riemann surface \( \mathcal{G}_{as} \), and \( \text{Re} \oint_{\ell} \phi = 0 \) for all closed paths \( \ell \subset \mathcal{G}_{as} \). However, there is no differential \( \phi \) with such properties.

**Existence.** Choose a point \( (t_0, D_0) \in \mathfrak{P} \setminus \mathfrak{S} \) in such a way that \( h_{\ell_j}^{(0)} = J_{\ell_j}(t_0, D_0) \neq 0 \) for all contours \( \ell_j \) of a homology basis \( \{\ell_j\}_{1}^{2g} \). Consider
the extension of (3.4) where $2g$ real parameters $h = (h_1, \ldots, h_{2g})^T$ are added to the set of the independent variables,

$$\omega \left( \frac{dD}{d\bar{D}} \right) = -\Omega \left( \frac{dt}{d\bar{t}} \right) + dh.$$ 

Applying the arguments used in the proof of Theorem 3.5 and taking into account Remark 3.1, we establish the existence of the function $D(t, \bar{t}, h) \forall t \in U \subset \mathcal{X} \setminus \mathcal{F}$ and $\forall h: h_j \text{ sgn} (h_j^{(0)}) > 0, \ j = 1, \ldots, 2g$. The assumption that $D(t, \bar{t}, h)$ is unbounded as $h \to 0$ leads to a contradiction. From a bounded sequence $D_k(t, \bar{t}, h_k) \to D^*$, we extract a convergent subsequence, $\lim_{m \to \infty} D_k^{(m)} = D^*$. Then the continuity of the integrals $J_{\ell_j}(t, D)$ w.r.t. $D$ yields $J_{\ell_j}(t, D^*) = 0$. □

Remark 3.3. Assuming that the monodromy data of the system (2.8) are generic and do not depend on the scaling parameter $\delta^{-1}$, or this dependence is weak enough, it is possible to prove that the relevant spectral curve satisfies the Boutroux system (3.5). Here, we do not prove this assertion (look for more details and for the proof of this statement in the case $N = 2$ in [2]).

4. Modulation equations and the asymptotics of the bi-orthogonal polynomials

Modulation equations (3.1), (3.5) for the classical Painlevé equations were studied in [18, 2]. Here we note that the linear ODEs associated to the classical Painlevé equations [3, 4, 5] as well as the similar equations associated to the semi-classical orthogonal polynomials have matrix dimension $2 \times 2$ [8], thus all the relevant spectral curves are (hyper)elliptic. In this section, we discuss the spectral curve for the $3 \times 3$ matrix linear ODE which appears in the theory of the bi-orthogonal polynomials [9, 10] with the cubic potentials [19],

$$\frac{\partial \Psi_n}{\partial \lambda} (\lambda) = A_n(\lambda) \Psi_n(\lambda), \quad \Psi_{n+1}(\lambda) = R_n(\lambda) \Psi_n(\lambda),$$
$$\frac{\partial \Psi_n}{\partial t} = U_n(\lambda) \Psi_n, \quad \frac{\partial \Psi_n}{\partial y} = V_n(\lambda) \Psi_n, \quad \frac{\partial \Psi_n}{\partial \bar{t}} = W_n(\lambda) \Psi_n,$$

where

$$R_n(\lambda) = \begin{pmatrix} \lambda - a_{n,n} & \frac{a_{n,n-1}}{a_{n,n+1}} & \frac{a_{n,n-2}}{a_{n,n+1}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_n(\lambda) = -D_{n,n+2} R_{n+1}(\lambda) R_n(\lambda) - D_{n,n+1} R_n(\lambda) - D_{n,n} - D_{n,n-1} R_{n-1}(\lambda),$$

$$D_{n,m} = t \text{ diag}(b_{n,m}, b_{n-1,m-1}, b_{n-2,m-2}).$$
More details can be found in [19] where the Riemann-Hilbert problem for $\Psi_n(\lambda)$ is formulated. The fixed singularities of the relevant completely integrable system correspond to the infinite values of the deformation parameters $x, y, t$ as well as to $t = 0$. In the case we are interested here, the formal monodromy exponents at $\lambda = \infty$, which is the only singular point for the $\lambda$-equation in (4.1), are equal to $n, -n/2, -n/2$.

The asymptotics of $\Psi_n(\lambda)$ as $n \to \infty$ is of particular importance for the theory of coupled random matrices. The scaling changes (2.6) with $x = -1/r_\infty = -1/3$ and Remark 2.4 imply

$$\lambda = \delta^{-1/3} \zeta, \quad x = \delta^{-2/3} (x_0 + \delta x_1),$$
$$y = \delta^{-2/3} (y_0 + \delta y_1), \quad t = \delta^{-1/3} (t_0 + \delta t_1), \quad n = \delta^{-1}, \quad (4.2)$$

and yield the system (2.8) with the spectral curve

$$F(\zeta, \mu) = \mu^3 - t_0^3 \zeta^3 + \mu^2 \zeta^2 + x_0 \mu^2 + y_0 t_0^2 \zeta^2 - (t_0^3 - 1) \mu \zeta -$$
$$- \mu D_1 - \zeta D_2 - D_3 + O(\delta) = 0. \quad (4.3)$$

Generically, this curve has 10 first order branch points and therefore, via the Riemann-Hurwitz formula, has genus $g = 3$. Because the monodromy data for $\Psi_n(\lambda)$ are independent from $n, x, y, t$, the curve (4.3) for a generic solution of (4.1) satisfies (3.5), see Remark 3.3. By Theorem 3.6, given $x_0, y_0, t_0$, system (3.5) uniquely determines the modular parameters $D_j, j = 1, 2, 3$.

The analysis of (3.5) is significantly more involved then the similar analysis of the elliptic curves associated to the classical Painlevé equations, see [18]. Here, we present the results in the numeric study of the integral sub-manifold for (3.5) parameterized by $x_0 \in \mathbb{C}$ as $y_0 = \bar{x}_0$ and $t_0 = 1$ based on the use of MATLAB 6.1 package.

The graph on the complex $x_0$-plane shown in Figure 1 separates the regions with different topological properties of the relevant Stokes graphs. Namely, some of the cycles $\ell_j$ existing in the neighboring regions collapse at the points of their common boundary. Our numeric study suggests that, at the points of the very central triangular domain in Figure 1, the curve (4.3) subject to (3.5) has genus $g = 0$, and the relevant Stokes graph is consistent with the Riemann-Hilbert problem data of [19]. The latter observation implies the applicability of (3.5) to the asymptotic study of the $\Psi$-function for the bi-orthogonal polynomials. In particular, the typical configuration of the branch points implied by (3.5) suggests that, for $x_0 \neq 0$, the asymptotics of $\Psi_n(\lambda)$ involves, in certain domains of the complex $\lambda$-plane, the exponential, Airy and parabolic cylinder functions. For $x_0$ at the boundary of the central triangular domain in Figure 1, the asymptotics involves also the
Figure 1. The projection of the integral sub-manifold 
\( y_0 = \bar{x}_0, \ t_0 = 1 \) for (3.5) on the \( x_0 \)-plane

\( \Psi \)-function associated to the Painlevé first transcendent. For \( x_0 = 0 \), besides exponential and Airy functions, the asymptotic description requires also a third order special function. For the values of \( x_0 \) beyond this triangular domain, the relevant Stokes graphs seem not consistent with the Riemann-Hilbert problem data of [19]. Therefore it is unlikely that, for \( x_0 \) beyond the central triangular domain, the system (3.5) can be applied to the asymptotics of the bi-orthogonal polynomials. The detailed description of the asymptotics of the bi-orthogonal polynomials, however, is out of the scope of the present paper and will be published later elsewhere.

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