KAUFFMAN-JONES POLYNOMIAL OF A CURVE ON A SURFACE

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ABSTRACT. We introduce a Kauffman-Jones type polynomial $L_\gamma(A)$ for a curve $\gamma$ on an oriented surface, whose endpoints are on the boundary of the surface. The polynomial $L_\gamma(A)$ is a Laurent polynomial in one variable $A$ and is an invariant of the homotopy class of $\gamma$. As an application, we obtain an estimate in terms of the span of $L_\gamma(A)$ for the minimum self-intersection number of a curve within its homotopy class. We then give a chord diagrammatic description of $L_\gamma(A)$ and show some computational results on the span of $L_\gamma(A)$.

1. INTRODUCTION

Let $S$ be an oriented $C^\infty$-surface with non-empty boundary $\partial S$. By a curve on $S$, we mean a $C^\infty$-immersion $\gamma$ from the unit interval $I = [0, 1]$ to $S$, which has only transverse double points as its singularities and satisfies $\gamma^{-1}(\partial S) = \{0, 1\}$ with $\gamma(0) \neq \gamma(1)$.

In this article, we consider curves on $S$ from the viewpoint of virtual knots [6] or equivalently, abstract link diagrams [4], with emphasis on their invariants coming from the Kauffman bracket [5]. More concretely, we introduce Laurent polynomials $\langle D_\gamma \rangle$ and $L_\gamma(A)$ in one variable $A$. We show that the span of these polynomials can be used for estimating the number of double points of $\gamma$. In fact, the polynomials $\langle D_\gamma \rangle$ and $L_\gamma(A)$ depend only on combinatorics of the image of the curve $\gamma$ in its regular neighborhood in $S$. Based on this fact, we then give a chord diagrammatic description of these polynomials. An advantage of being free from the ambient surface $S$ is that it becomes easy to provide and compute examples. In §4, we show some computational results on the span of $\langle D_\gamma \rangle$ from this point of view.

In the rest of this section, we describe main constructions and results. Some proofs will be postponed to §2.

We begin with terminology. Let $X$ be a compact 1-manifold. Namely, $X$ is a disjoint union of finitely many $I$’s and circles:

$$X = I \sqcup \cdots \sqcup I \sqcup S^1 \sqcup \cdots \sqcup S^1.$$  

A $C^\infty$-immersion $f : X \to S$ is called generic if it has only transverse double points as its singularities, $f^{-1}(\partial S) = \partial X$, and $f|_{\partial X}$ is injective. A generalized link diagram on $S$ is a subset of $S$ of the form $D = f(X)$ for some generic immersion $f : X \to S$, endowed with a choice of crossing to each double point of $D$. See Figures 1 and 2.

Two generalized link diagrams $D$ and $D'$ are called equivalent (resp. regularly equivalent) if $D$ is transformed into $D'$ by a finite sequence of ambient isotopies of $S$ relative to $\partial S$, and the three Reidemeister moves $R_1$, $R_2$, and $R_3$ (resp. $R_2$ and $R_3$) shown in Figure 3. We write $D \sim D'$ (resp. $D \sim_r D'$) when $D$ is equivalent (resp. regularly equivalent) to $D'$.

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Let \( \gamma \) be a curve on \( S \). For each double point \( p \) of \( \gamma \), there is a neighborhood \( U \) of \( p \) such that \( U \cap \gamma(I) \) consists of two arcs \( J_1 \) and \( J_2 \) intersecting at \( p \), and \( J_1 \) is traversed first when we go along \( \gamma \) from \( \gamma(0) \). Then we replace \( p \) with a crossing with \( J_1 \) being overcrossing (see Figure 4). Let \( D_\gamma \) denote the generalized link diagram on \( S \) obtained in this way. In other words, \( D_\gamma \) is the projection diagram in the usual sense of the embedding \( I \to S \times I, t \mapsto (\gamma(t), 1 - t) \) by the projection \( S \times I \to S \times \{0\} \sim S, (x, t) \mapsto x \).

The following fact is crucial in our argument:

**Theorem 1.1.** Suppose that two curves \( \gamma \) and \( \gamma' \) on \( S \) are homotopic (resp. regularly homotopic) relative to \( \partial S \). Then \( D_\gamma \sim D_{\gamma'} \) (resp. \( D_\gamma \sim_r D_{\gamma'} \)).

The *Kauffman bracket* \([5]\) is extended to link diagrams on surfaces \([2]\). This extension is straightforward and applies to our generalized link diagrams also. For the sake of definiteness, let us recall the construction. Let \( D \) be a generalized link diagram on \( S \). We can split \( D \) at each crossing in two ways. We will distinguish these splittings as a type A splitting and a type B splitting, respectively (see Figures 5 and 6 according to the orientation of \( S \)). A *state* of \( D \) is a choice of splitting type for each crossing of \( D \). For a state \( s \) of
$F_{\text{IGURE} \ 5. \ \text{splitting with an orientation}}$

$F_{\text{IGURE} \ 6. \ \text{splitting with the other orientation}}$

$F_{\text{IGURE} \ 7. \ \text{three diagrams}}$

Let $D(s)$ be the compact 1-submanifold of $S$ obtained by splitting $D$ by $s$. If $D$ has $n$ crossings, there are $2^n$ states of $D$.

To each state $s$ of $D$, we assign the following three numbers:

- $\alpha(s) := $ the number of type A splittings,
- $\beta(s) := $ the number of type B splittings,
- $\mu(s) := $ the number of connected components of $D(s)$.

Then we define the bracket polynomial of $D$ by

$$\langle D \rangle := \sum_s A^{\alpha(s) - \beta(s)}(-A^2 - A^{-2})^{\mu(s)-1},$$

where $s$ runs over all states of $D$.

A basic property of the bracket polynomial is the following skein relation, whose proof is the same as that of the classical case [5].

**Lemma 1.2.** Let $D$ be a generalized link diagram on $S$.

1. Pick a crossing of $D$ and consider the two splittings of it as shown in Figure 7. Then

$$\langle D \rangle = A\langle D_A \rangle + A^{-1}\langle D_{A^{-1}} \rangle.$$

2. Let $T$ be a generalized link diagram which is connected and has no crossing.
   (a) We have $\langle T \rangle = 1$.
   (b) If $D$ and $T$ are disjoint, then $\langle D \sqcup T \rangle = (-A^2 - A^{-2})\langle D \rangle$.

Assume that a generalized link diagram $D = f(X)$ is oriented. That is, $X$ is oriented and $D$ inherits this orientation. For instance, if $\gamma$ is a curve on $S$, then $D_\gamma$ can be oriented from the natural orientation of $I$. Let $w(D)$ denote the writhe number of $D$. That is,

$$w(D) := \sum_p \varepsilon_p,$$
where \( p \) runs over all crossings of \( D \) and \( \varepsilon_p \in \{\pm 1\} \) is the sign of the crossing at \( p \) (see Figure 8). Then we define the Kauffman-Jones polynomial of \( D \) by

\[
\mathcal{L}_D(A) := (-A)^{-3w(D)} \langle D \rangle.
\]

The following result is an analogy of the result for ordinary link diagrams given by Kauffman [5], where Lemma 1.2 played a central role. His argument can also be applied to the case of generalized link diagrams, so we omit the proof.

**Theorem 1.3.** Let \( D \) and \( D' \) be generalized link diagrams on \( S \).

1. If \( D \) and \( D' \) are regularly equivalent, \( \langle D \rangle = \langle D' \rangle \).
2. Assume further that \( D \) and \( D' \) are oriented. If \( D \) and \( D' \) are equivalent, \( \mathcal{L}_D(A) = \mathcal{L}_{D'}(A) \).

To simplify notation, we denote \( \mathcal{L}_{\gamma}(A) := \mathcal{L}_{D_{\gamma}}(A) \) for a curve \( \gamma \). Combining Theorems 1.1 and 1.3, we obtain

**Theorem 1.4.** Let \( \gamma \) and \( \gamma' \) be curves on \( S \).

1. If \( \gamma \) and \( \gamma' \) are regularly homotopic relative to \( \partial S \), then \( \langle D_{\gamma} \rangle = \langle D_{\gamma'} \rangle \).
2. If \( \gamma \) and \( \gamma' \) are homotopic relative to \( \partial S \), then \( \mathcal{L}_{\gamma}(A) = \mathcal{L}_{\gamma'}(A) \).

For a Laurent polynomial \( f(A) \in \mathbb{Z}[A, A^{-1}] \), the span of \( f \), denoted by \( \text{span} f \), is defined to be the difference of the maximal and the minimal degrees of \( f \). Note that \( \text{span} \langle D_{\gamma} \rangle = \text{span} \mathcal{L}_{\gamma}(A) \) for any curve \( \gamma \). We denote by \( d(\gamma) \) the number of double points of a curve \( \gamma \). Then we have the following estimate for \( d(\gamma) \), which is analogous to [8] and [9].

**Theorem 1.5.** For a curve \( \gamma \) on \( S \), it holds that

\[
\text{span} \langle D_{\gamma} \rangle \leq 4d(\gamma).
\]

We define the minimum self-intersection number \( c(\gamma) \) of a curve \( \gamma \) by

\[
c(\gamma) := \min \{ d(\gamma') \mid \gamma' \text{ is a curve on } S \text{ homotopic to } \gamma \text{ relative to } \partial S \}.
\]

**Corollary 1.6.** For any curve \( \gamma \) on \( S \), it holds that

\[
\frac{\text{span} \langle D_{\gamma} \rangle}{4} \leq c(\gamma).
\]

We give examples of using Corollary 1.6 for estimating \( c(\gamma) \).

**Example 1.7.** Let \( \gamma_1 \) be the curve shown in Figure 9. The bracket polynomial of \( \gamma_1 \) is

\[
\langle D_{\gamma_1} \rangle = A - A^{-3} - A^{-5}.
\]

We see that \( \text{span} \langle D_{\gamma_1} \rangle = 6 \) and \( 6/4 \leq c(\gamma_1) \). Hence we obtain \( 2 \leq c(\gamma_1) \leq 3 \).

**Example 1.8.** Let \( \gamma_2 \) be the curve shown in Figure 10. The bracket polynomial of \( \gamma_2 \) is

\[
\langle D_{\gamma_2} \rangle = -A^5 + A + A^{-1} - A^{-3} - A^{-5}.
\]

Since \( \text{span} \langle D_{\gamma_2} \rangle = 10 \), we have \( 10/4 \leq c(\gamma_2) \). Therefore, \( c(\gamma_2) = 3 \).
In this section, we prove Theorems 1.1 and 1.5.

Proof of Theorem 1.1 If two curves $\gamma$ and $\gamma'$ are homotopic relative to $\partial S$, then $\gamma$ is transformed into $\gamma'$ by using a finite sequence of ambient isotopies of $S$ relative to $\partial S$ and the three local moves $\omega_1, \omega_2, \omega_3$, shown in Figure 11. See e.g., [1] Lemma 5.6.

It is easily seen that if $\gamma$ is transformed into $\gamma'$ by $\omega_i$ ($i = 1, 2, 3$), then $D_\gamma$ can be transformed into $D_{\gamma'}$ by $R_i$ ($i = 1, 2, 3$) respectively (see Figure 12). This completes the proof.

Next, we prove Theorem 1.5 Recall that a generalized link diagram $D$ has the form $D = f(X)$ for a generic immersion $f : X \rightarrow S$, endowed with a choice of crossing to each double point. We say that $D$ is connected if it is connected as a subset of $S$. Let $d(D)$ be the number of crossings of $D$.

Let us consider the following condition for a generalized link diagram $D = f(X)$:

(2.1) the number of connected components of $X$ homeomorphic to $I$ is at most one.

Since $D_\gamma$ is connected for any curve $\gamma$, Theorem 1.5 is a special case of the following: 

2. Proofs of Theorems 1.1 and 1.5
Proposition 2.1. Let $D$ be a connected generalized link diagram satisfying condition \(\text{(2.1)}\). Then it holds that

$$\text{span} \langle D \rangle \leq 4d(D).$$

**Proof.** The bracket polynomial of $D$ is written as

$$\langle D \rangle = \sum_s \langle D|s \rangle \delta^{\mu(s)-1},$$

where $s$ runs over all states of $D$ and we set $\langle D|s \rangle := A^{\alpha(s)} - \beta(s)$, $\delta := -A^2 - A^{-2}$.

Let $s$ be a state of $D$ having a type A splitting, and let $s'$ denote the state of $D$ obtained from $s$ by replacing the type A splitting with a type B splitting. Then we have

$$\langle D|s' \rangle = \langle D|s \rangle A^{-2} - \mu(s') \leq \mu(s) + 1, \quad \mu(s) \leq \mu(s') + 1.$$ 

Hence we have

$$\text{max} \deg \langle D|s' \rangle \delta^{\mu(s')-1} \leq \text{max} \deg \langle D|s \rangle \delta^{\mu(s)-1},$$

$$\text{min} \deg \langle D|s' \rangle \delta^{\mu(s')-1} \leq \text{min} \deg \langle D|s \rangle \delta^{\mu(s)-1}.$$ 

Let $s_A$ (resp. $s_B$) denote the state of $D$ whose splitting at each crossing is of type A (resp. of type B). Then we have

$$\text{max} \deg \langle D \rangle \leq \text{max} \deg \langle D|s_A \rangle \delta^{\mu(s_A)-1} = d(D) + 2(\mu(s_A) - 1),$$

$$\text{min} \deg \langle D \rangle \geq \text{min} \deg \langle D|s_B \rangle \delta^{\mu(s_B)-1} = -d(D) - 2(\mu(s_B) - 1).$$

From these inequalities, we have

$$\text{span} \langle D \rangle \leq 2d(D) + 2(\mu(s_A) + \mu(s_B) - 2).$$

**Lemma 2.2.** We have $\mu(s_A) + \mu(s_B) \leq d(D) + 2$.

**Proof.** If $d(D) = 0$, the inequality is obvious. Let $d(D) > 0$ and choose a crossing of $D$ and consider the two splittings of it as shown in Figure 7. Then, at least one of them is connected and satisfies condition \(\text{(2.1)}\) by virtue of the assumption \(\text{(2.1)}\) on $D$. Let $D'$ be such a generalized link diagram and assume that $D'$ is obtained from the type A splitting (the other case is treated similarly). Let $s_A'$ and $s_B'$ be the states of $D'$ defined by the same way as we introduce $s_A$ and $s_B$ to $D$. Then $\mu(s_A) = \mu(s_A')$ and $\mu(s_B) \leq \mu(s_B') + 1$, hence $\mu(s_A) + \mu(s_B) \leq \mu(s_A') + \mu(s_B') + 1$. Then the assertion is proved by induction on $d(D)$. \(\square\)

By Lemma 2.2 we conclude

$$\text{span} \langle D \rangle \leq 2d(D) + 2(\mu(s_A) + \mu(s_B) - 2) \leq 4d(D).$$

This completes the proof of Proposition 2.1. \(\square\)

3. Chord diagrammatic description

For a curve $\gamma$ on $S$, the bracket polynomial $\langle D_\gamma \rangle$ is actually determined by a regular neighborhood of $\gamma(f)$ in $S$. In this section, we study $\langle D_\gamma \rangle$ from this point of view.

Let $d$ be a positive integer. An oriented linear chord diagram of $d$ chords is a set $C = \{(i_1,j_1), \ldots, (i_d,j_d)\}$ of $d$ ordered pairs of integers such that $\{i_k\}_k \cup \{j_k\}_k = \{1, \ldots, 2d\}$. Each element of $C$ is called a chord of $C$. A chord $(i,j)$ is called positive if $i < j$, and negative otherwise. Finally, a state of $C$ is a map $s: C \to \{A,B\}$, where $A$ and $B$ are fixed symbols.
Let \( \gamma \) be a curve with \( d(\gamma) = d \). Then the inverse image of the double points of \( \gamma \) are \( 2d \) points on \( I \). We name them \( \{p_i\} \), so that \( 0 < p_1 < p_2 < \cdots < p_{2d} < 1 \). The oriented linear chord diagram \( C_\gamma \) is defined by the condition that an ordered pair \((i, j)\) is a chord of \( C_\gamma \) if and only if \( \gamma(p_i) = \gamma(p_j) \) and the pair \((d\gamma/dt(p_i), d\gamma/dt(p_j))\) of tangent vectors matches the orientation of \( S \).

**Remark 3.1.** Conversely, for any oriented linear chord diagram \( C \), there is a curve \( \gamma \) on some oriented surface \( S \) such that \( C = C_\gamma \).

Let \( C \) be an oriented linear chord diagram of \( d \) chords and \( s \) a state of \( C \). For each chord \( c = (i, j) \in C \), we define a subset \( R_c \subset S_{2d+1} \) of permutations of \( 2d + 1 \) letters \( \{0, 1, \ldots, 2d\} \) in the following way.

- If \( s(c) = A \) and \( c \) is positive, or \( s(c) = B \) and \( c \) is negative, then we set \( R_c = \{(i, j-1), (i-1, j)\} \).
- If \( s(c) = A \) and \( c \) is negative, or \( s(c) = B \) and \( c \) is positive, then we set \( R_c = \{(i, j), (i-1, j-1)\} \).

Consider the subgroup of \( S_{2d+1} \) generated by \( \bigcup_{c \in C} R_c \), and let \( \Gamma_s \) be the number of orbits of the action of this group on \( \{0, 1, \ldots, 2d\} \).

We set
\[
\langle C|s \rangle := A^{s^{-1}(A)}|s^{-1}(B)|(-A^2 - A^{-2})^{\Gamma_s - 1},
\]
where \( |s^{-1}(A)| \) denotes the cardinality of the set \( s^{-1}(A) \), and we define
\[
\langle C \rangle := \sum_s \langle C|s \rangle,
\]
where the sum runs over all states of \( C \). We also define
\[
L_C(A) := (-A)^{-3w(C)}\langle C \rangle,
\]
where \( w(C) \) is the number of positive chords minus the number of negative chords of \( C \).

**Proposition 3.2.** Let \( \gamma \) be a curve on \( S \). Then \( \langle D_\gamma \rangle = \langle C_\gamma \rangle \) and \( L_\gamma(A) = L_{C_\gamma}(A) \).

**Proof.** First of all, the second formula follows from the first, since \( w(D_\gamma) = w(C_\gamma) \).

Now introduce \( 2d + 1 \) points \( q_i, 0 \leq i \leq 2d \). With respect to the parametrization of \( \gamma \), these points have the following interpretation: \( q_0 = 0 \), \( q_i = (p_i + p_{i+1})/2 \) for \( 1 \leq i \leq 2d - 1 \), and \( q_{2d} = 1 \). For a state \( s \) of \( \gamma \), let \( \Gamma(C_\gamma, s) \) be the graph with the set of vertices being \( \{q_i\} \), and the set of edges determined by the condition that \( q_k \) and \( q_l \) are connected by an edge if and only if \( (k, l) \in \bigcup_{c \in C_\gamma} R_c \). Then \( \Gamma(C_\gamma, s) \) is homeomorphic to the splice of \( D_\gamma \) by \( s \). See Figure [13] (Here and in what follows, we assume the counter-clockwise orientation in any figure.) The first formula follows from this observation. \( \square \)

In the below, we record elementary properties of \( \langle D_\gamma \rangle \) in terms of chord diagrams.

Let \( C = \{(i_1, j_1), \ldots, (i_d, j_d)\} \) be an oriented linear chord diagram. Fix \( 0 \leq \ell \leq 2d \). For \( i \in \{1, \ldots, 2d\} \), we set
\[
i' := \begin{cases} i & \text{if } i \leq \ell, \\ i + 2 & \text{if } i > \ell. \end{cases}
\]

We define
\[
C_+^\ell := \{(i'_k, j'_k)\}_k \cup \{\ell, \ell + 1\},
\]
\[
C_-^\ell := \{(i'_k, j'_k)\}_k \cup \{\ell + 1, \ell\}.
\]
Also, we define

\[ C_{+}^{\gamma} := \{(i_k + 1, j_k + 1)\}_{k} \cup \{(1, 2d + 2)\}; \]
\[ C_{-}^{\gamma} := \{(i_k + 1, j_k + 1)\}_{k} \cup \{(2d + 2, 1)\}. \]

**Proposition 3.3** (Birth/death of monogons). We have

\[ \langle C_{+}^{\ell} \rangle = \langle C_{+}^{\gamma} \rangle = (-A^3)\langle C \rangle, \]
\[ \langle C_{-}^{\ell} \rangle = \langle C_{-}^{\gamma} \rangle = (-A^{-3})\langle C \rangle. \]

**Proof.** If \( C = C_{\gamma} \) for some curve \( \gamma \), then \( C_{+}^{\gamma} \) corresponds to a suitable insertion of a negative monogon to \( \gamma \). Therefore, from the behavior of the bracket polynomial under the Reidemeister move \( R_1 \), we obtain \( \langle C_{+}^{\gamma} \rangle = (-A^3)\langle C \rangle \). The other cases are treated similarly. \( \Box \)

Let

\[ C = \{(i_1, j_1), \ldots, (i_d, j_d)\} \quad \text{and} \quad D = \{(k_1, \ell_1), \ldots, (k_e, \ell_e)\} \]

be oriented linear chord diagrams. We define the stacking of \( C \) and \( D \) by

\[ C^*D := \{(i_a, j_a)\}_{a} \cup \{(k_b + 2d, \ell_b + 2d)\}_{b}. \]

**Proposition 3.4** (Stacking formula). We have \( \langle C^*D \rangle = \langle C \rangle \langle D \rangle \). In particular, \( \text{span} \langle C^*D \rangle = \text{span} \langle C \rangle + \text{span} \langle D \rangle \).

**Proof.** Since the chords of \( C^*D \) are in one-to-one correspondence with the disjoint union of the chords of \( C \) and \( D \), any state of \( C^*D \) is of the form \( s^*t \), where \( s \) is a state of \( C \) and \( t \) is a state of \( D \). The assertion follows from the observation that \( |\Gamma(C^*D, s^*t)| = |\Gamma(C, s)| + |\Gamma(D, t)| - 1. \) \( \Box \)

**Proposition 3.5.** Let \( C \) be an oriented linear chord diagram of \( d \) chords.

- If \( d \) is even, then \( \langle C \rangle \) has only terms of even degree.
- If \( d \) is odd, then \( \langle C \rangle \) has only terms of odd degree.

**Proof.** By definition, \( \langle C|s \rangle \) has this property, so does \( \langle C \rangle \). \( \Box \)

**Proposition 3.6** (Reversing all the chords). Let \( C = \{(i_1, j_1), \ldots, (i_d, j_d)\} \) be an oriented linear chord diagram and set \( \overline{C} := \{(j_1, i_1), \ldots, (j_d, i_d)\} \). Then \( \langle \overline{C} \rangle = \langle C \rangle|_{A \rightarrow A^{-1}}. \)
Proof. There is a natural bijection \( \iota \) from the set of chords of \( C \) to that of \( \overline{C} \) given by \( (i_k, j_k) \mapsto (j_k, i_k) \). This maps positive (resp. negative) chords to negative (resp. positive) chords. Moreover, it induces a bijection from the set of states of \( C \) to that of \( \overline{C} \) given by \( s \mapsto \overline{\iota(s)} \), determined by the condition that \( \{s(c), \overline{\iota}(c)\} = \{A, B\} \) for any chord \( c \) of \( C \). Then, it holds that \( \langle C | s \rangle = \langle \overline{C} | \overline{\iota}(s) \rangle |_{A \mapsto A^{-1}} \) for any state \( s \) of \( C \). This proves the formula. \( \square \)

4. THE RANGE OF THE SPAN

In this section, we study the range of span \( \langle C \rangle \). By Theorem 1.5, \( \text{span} \langle C \rangle \leq 4d \) if \( C \) has \( d \) chords. Also, by Proposition 3.5, \( \text{span} \langle C \rangle \) is always an even integer. Fixing \( d \), let us consider which even integers not greater than \( 4d \) are realized as \( \text{span} \langle C \rangle \) for some \( C \) with \( d \) chords.

We say that an even integer \( l \) is \( d \)-realizable if there exists an oriented linear chord diagram \( C \) of \( d \) chords such that \( \text{span} \langle C \rangle = l \).

If \( d = 1 \), \( C = C_1 := \{(1, 2)\} \) or \( C = \overline{C_1} \). Thus \( \langle C \rangle = -A^{\pm 3} \) and \( \text{span} \langle C \rangle = 0 \).

If \( d = 2 \), by a direct computation, we see that 0 and 6 are 2-realizable, while 2, 4, and 8 are not. For example, \( C_2 = \{(1, 3), (2, 4)\} \) satisfies \( \langle C_2 \rangle = A^2 + 1 - A^{-4} \), so that \( \text{span} \langle C_2 \rangle = 6 \).

If \( d = 3 \), we see that 0, 6, 10, and 12 are 3-realizable, while 2, 4, and 8 are not. For example, the stacking \( C_1 \sharp C_2 \) satisfies \( \text{span} \langle C_1 \sharp C_2 \rangle = 6 \); \( C_3 = \{(1, 5), (2, 4), (6, 3)\} \) satisfies \( \langle C_3 \rangle = -A^5 - A^3 + A + A^{-1} - A^{-5} \), so that \( \text{span} \langle C_3 \rangle = 10 \); the chord diagram \( C(3) \) in Example 4.1 below satisfies \( \text{span} \langle C(3) \rangle = 12 \).

To see the case \( d \geq 4 \), we consider the following two examples.

Example 4.1. Let \( d \geq 1 \) be an odd integer, and set \( C(d) := \{(i, i + d)\}_{i=1}^d \). Then

\[
\langle C(d) \rangle = \sum_{i=1}^{d-1} (-1)^{i-1} A^{-3d-2+4i} - A^{d+2}.
\]

In particular, if \( d \geq 3 \), then \( \text{span} \langle C \rangle = (d + 2) - (-3d + 2) = 4d \).
Proof. We have $C(d) = C_{\gamma_d}$, where $\gamma_d$ is the curve as shown in Figure 14. Let $d \geq 1$ be an odd integer. Then

$$\langle \gamma_{d+2} \rangle = A \langle \gamma_{d+2} \rangle + A^{-1} \langle \gamma_{d+2} \rangle + A^2 \langle \gamma_{d+2} \rangle + A^{-1} \cdot (-A^{-3})^{d+1} \langle \gamma_{d} \rangle = A^2 \langle \gamma_d \rangle + (-A^{-3})^d + A^{-3d-4}.$$ 

Now it is easy to see that $\langle \gamma_1 \rangle = -A^3$, and the formula is proved by an inductive argument.

Example 4.2. Let $d \geq 4$ be an even integer, and set

$$C(d) := \{(1, d), (d + 1, 2d)\} \cup \{(2d - i, i + 1)\}_{i=1}^{d-2}.$$ 

Then

$$\langle C(d) \rangle = A^{-3d+4} - A^{-3d+8} + 2 \left( \sum_{i=1}^{d-4} (-1)^{i-1} A^{-3d+8+4i} \right) + A^{d-4} - A^d + A^{d+4}.$$ 

In particular, $\text{span} \langle C(d) \rangle = (d + 4) - (-3d + 4) = 4d$. 

Figure 15. the curve $\gamma_d$ in Example 4.2
Proof. We have $C(d) = C_{\gamma_d}$, where $\gamma_d$ is the curve as shown in Figure 15. Let $d \geq 4$ be an even integer. Then

$$\langle \gamma_{d+2} \rangle = A \langle \gamma_d \rangle + A^{-1} \cdot (\bar{A}^{-3})^{d-2}$$

(4.1)

The diagram in the first term of (4.1) can be expanded as

$$A \langle \gamma_d \rangle$$

$$+ A^{-1} \langle \gamma_d \rangle$$

(4.1)

On the other hand, the second term of (4.1) is equal to

$$A^{-1} \cdot (\bar{A}^{-3})^{d-1}$$

Moreover, we compute

$$\langle \gamma_d \rangle = A \langle \gamma_d \rangle + A^{-1} \langle \gamma_d \rangle$$

Therefore, we obtain

$$\langle \gamma_{d+2} \rangle = A^2 \langle \gamma_d \rangle + (A^{-3})^{d-2} + A^{-1} \cdot (\bar{A}^{-3})^{d-1}(\bar{A}^{-4} - A^4)$$

$$= A^2 \langle \gamma_d \rangle + A^{-3d-2} - A^{-3d+2} + A^{-3d+6} - A^{-3d+10}.$$

Now, by a direct computation, we see that $\langle \gamma_4 \rangle = A^{-8} - A^{-4} + 1 - A^4 + A^8$, and the formula is proved by an inductive argument. \qed
Table 1. linear chord diagrams with $\text{span} \langle C \rangle = 4d$

| $d$ | 0  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9 |
|-----|----|----|----|----|----|----|----|----|---|
| 2   | 2  | 0  | 2  | 4  | 12 | 84 | 338| 1588| 8588|

**Theorem 4.3.** Let $d \geq 4$. Any even integer $l \in \{0, 6, 8, \ldots, 4d \}$ is $d$-realizable.

**Proof.** Let $d = 4$. First, 0, 6, 10, and 12 are 4-realizable. To see this, for any $l \in \{0, 6, 10, 12\}$ pick an oriented chord diagram $D$ of 3 chords with $\text{span} D = l$; then the stacking $C = C_1 \uparrow \downarrow D$ satisfies $\text{span} \langle C \rangle = l$. Next, 8 is 4-realizable, since $C_4 = \{(1, 4), (2, 7), (3, 5), (6, 8)\}$ satisfies $\langle C_4 \rangle = A^4 + A^2 + 1 - A^{-2} - A^{-4}$, so that $\text{span} \langle C_4 \rangle = 8$; also 14 is 4-realizable, since $C'_4 = \{(1, 5), (2, 4), (3, 7), (6, 8)\}$ satisfies $\langle C'_4 \rangle = A^8 + A^6 - A^4 - A^2 + 1 + A^{-2} - A^{-6}$, so that $\text{span} \langle C'_4 \rangle = 14$. Finally, the element $C(4)$ in Example 4.4 satisfies $\text{span} \langle C(4) \rangle = 16$.

Now let $d \geq 5$ and assume that any even integer $l \in \{0, 6, 8, \ldots, 4(d - 1)\}$ is $(d - 1)$-realizable. Then by considering the stacking of $C_1$ and oriented linear chord diagrams of $d - 1$ chords, we see that any $l \in \{0, 6, 8, \ldots, 4(d - 1)\}$ is $d$-realizable. Next, the element $C(d)$ in Examples 4.4 satisfies $\text{span} \langle C(d) \rangle = 4d$. Finally, the stacking $C = C_2 \uparrow \downarrow C(d - 2)$ satisfies $\text{span} \langle C \rangle = 6 + 4(d - 2) = 4d - 2$.

By induction on $d$, we obtain the assertion. \qed

It can be checked that 2 and 4 are not $d$-realizable for $d \leq 6$.

**Question 4.4.** Is there an oriented linear chord diagram $C$ such that $\text{span} \langle C \rangle = 2$ or 4?

Finally, we study the case where the equality $\text{span} \langle C \rangle = 4d$ holds. This class of linear chord diagrams might be of interest since it is closed under stacking by Proposition 3.4. Table 1 shows the number of linear chord diagrams with $\text{span} \langle C \rangle = 4d$ for a fixed integer $d \leq 9$. To give another motivation, let us recall the following classical result on characterization of alternating knots.

**Theorem 4.5 ([5] [8] [9]).** Let $K$ be an oriented knot in $S^3$ and assume that the span of the Jones polynomial $V_K(t) = \mathcal{L}_K(t^{-1/4})$ is equal to the minimum number of double points among all projection diagrams of $K$. Then $K$ is alternating.

Let $C$ be an oriented linear chord diagram of $d$ chords, and let $\gamma$ be a curve on an oriented surface $S$ such that $C_\gamma = C$. Let $s_A$ and $s_B$ be the states of $D_\gamma$ which appeared in the proof of Proposition 2.1. Suppose that $\text{span} \langle C \rangle = 4d$. Then, as we see from the proof of Proposition 2.1, we have

$$\mu(s_A) + \mu(s_B) = d + 2.$$

**Remark 4.6.** The condition (4.2) does not imply $\text{span}(C) = 4d$. For example, let $C = \{(1, 8), (2, 5), (3, 6), (4, 7)\}$. Then $\mu(s_A) + \mu(s_B) = 6$. However, $\langle C \rangle = A^5 + 1 - A^{-4}$ and $\text{span}(C) = 12 \neq 16$.

Let us consider the following condition for $C$.

$$\text{(4.3)} \quad \text{for any chord } (i, j) \text{ of } C, \text{ the parity of } i \text{ and } j \text{ are different.}$$

**Theorem 4.7.** Keep the notation as above. Then condition (4.2) implies condition (4.3). In particular, if $\text{span} \langle C \rangle = 4d$, then condition (4.3) holds.

To prove this, let us consider the following preliminary construction. Let $N$ be a regular neighborhood of $\gamma$ in $S$. We modify $N$ in a neighborhood of every double point of $\gamma$ by
inserting two half-twisted bands as illustrated in Figure 16. The result is denoted by $N'$, in which the curve $\gamma$ embeds naturally. Next, we give a labelling $A$ or $B$ to each boundary component of a neighborhood in $N'$ of each double point as shown in Figure 16. Then, this labelling extends naturally to a locally constant function $\phi: \partial N' \setminus \partial S \to \{A, B\}$. From the construction, we see that the inverse image $\phi^{-1}(A)$ (resp. $\phi^{-1}(B)$) is homeomorphic to the splice of $D_{\gamma}$ by $s_A$ (resp. $s_B$). Therefore, if we denote by $r'$ the number of boundary components of $N'$, we have

$$ r' = \mu(s_A) + \mu(s_B) - 1. \quad (4.4) $$

**Lemma 4.8.** The surface $N'$ is orientable if and only if condition $(4.3)$ holds.

**Proof.** Let $\{p_k\}_{k=1}^{d}$ be the set of double points of $\gamma$, and fix a parametrization $\gamma: I \to S$. For each $k$, write $\gamma^{-1}(p_k) = \{t^1_k, t^2_k\}$ so that $t^1_k < t^2_k$, and let $c_k \in H_1(N'; \mathbb{Z})$ be the homology class of the loop defined as the restriction of $\gamma$ to $[t^1_k, t^2_k]$. Then, the set $\{c_k\}_{k=1}^{d}$ constitutes a $\mathbb{Z}$-basis for $H_1(N'; \mathbb{Z})$.

Now, let $w_1 \in H^1(N'; \mathbb{Z}_2) \cong \text{Hom}(H_1(N'; \mathbb{Z}), \mathbb{Z}_2)$ be the first Stiefel-Whitney class of the tangent bundle of $N'$. Let $(i_k, j_k)$ be the chord of $C$ corresponding to $p_k$. Then, $w_1(c_k)$ is just the number of inserted half-twisted bands along the representative of $c_k$, and this is equal to $|i_k - j_k| + 1$. Since $N'$ is orientable if and only if $w_1 = 0$, the assertion follows. \hfill $\Box$

**Proof of Theorem 4.7.** Since $N'$ is homotopy equivalent to the bouquet of $d$ circles, the Euler characteristic of $N'$ is $\chi(N') = 1 - d$.

Assume that $N'$ is unorientable. Since $r' = d + 1$ from $(4.2)$ and $(4.4)$, there exists an integer $g > 0$ and $N'$ is homeomorphic to a connected sum of $g$ copies of $\mathbb{R}P^2$ minus the interior of $d + 1$ disjoint union of closed disks. Hence

$$ \chi(N') = 2 - g - (d + 1) = 1 - g - d. $$

Thus $g = 0$, a contradiction. Therefore $N'$ is orientable, and the conclusion follows by Lemma 4.8. \hfill $\Box$
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