BLOWUP ON AN ARBITRARY COMPACT SET FOR A SCHRÖDINGER EQUATION WITH NONLINEAR SOURCE TERM

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Abstract. We consider the nonlinear Schrödinger equation on $\mathbb{R}^N$, $N \geq 1$,

$$\partial_t u = i\Delta u + \lambda |u|^\alpha u$$

on $\mathbb{R}^N$, $\alpha > 0$, with $\lambda \in \mathbb{C}$ and $\Re \lambda > 0$, for $H^1$-subcritical nonlinearities, i.e. $\alpha > 0$ and $(N-2)\alpha < 4$. Given a compact set $K \subset \mathbb{R}^N$, we construct $H^1$ solutions that are defined on $(-T,0)$ for some $T > 0$, and blow up on $K$ at $t = 0$. The construction is based on an appropriate ansatz. The initial ansatz is simply $U_0(t,x) = (\Re \lambda)^{-\frac{1}{\alpha}}(-\alpha t + A(x))^{-\frac{1}{\alpha} - i\Im \lambda \alpha \Re \lambda}$, where $A \geq 0$ vanishes exactly on $K$, which is a solution of the ODE $u' = \lambda |u|^\alpha u$. We refine this ansatz inductively, using ODE techniques. We complete the proof by energy estimates and a compactness argument. This strategy is reminiscent of [3, 4].

1. Introduction

We consider the nonlinear Schrödinger equation

$$\partial_t u = i\Delta u + \lambda |u|^\alpha u$$

(1.1)

on $\mathbb{R}^N$, where

$$\alpha > 0, \quad (N-2)\alpha < 4.$$  \hfill (1.2)

and $\Re \lambda > 0$. Note that by scaling invariance of (1.1), we may assume that $\Re \lambda = 1$, so we write

$$\lambda = 1 + i\mu, \quad \mu \in \mathbb{R}. \hfill (1.3)$$

Under assumption (1.2), equation (1.1) is $H^1$-subcritical, so that the corresponding Cauchy problem is locally well posed in $H^1(\mathbb{R}^N)$.

Concerning blowup, it is proved in [1, Theorem 1.1] that, for $\alpha < 2/N$, equation (1.1) has no global in time $H^1$ solution that remains bounded in $H^1$. In other words, every $H^1$ solution blows up, in finite or infinite time. Moreover, it is proved in [3] that under assumption (1.2) with the restriction $\alpha \geq 2$, and for $\lambda = 1$, finite-time blowup occurs. This result is extended in [8] to the case $\alpha > 1$ and $\lambda \in \mathbb{C}$ with $(\alpha + 2)\Re \lambda \geq \alpha|\lambda|$.

In this paper, we extend the previous blow-up results to the whole range of $H^1$ subcritical powers and arbitrary $\lambda \in \mathbb{C}$ with $\Re \lambda > 0$. Moreover, we prove blowup on any prescribed compact subset of $\mathbb{R}^N$. Our result is the following.

Theorem 1.1. Let $N \geq 1$, let $\alpha > 0$ satisfy (1.2), let $\mu \in \mathbb{R}$ and $\lambda$ given by (1.3), and let $K$ be a nonempty compact subset of $\mathbb{R}^N$. It follows that there exist $T > 0$ and a solution $u \in C((-T,0), H^1(\mathbb{R}^N))$ of (1.1) which blows up at time 0 exactly on $K$ in the following sense.

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(i) If $x_0 \in K$ then for any $r > 0$,
\[
\lim_{t \to 0} \|u(t)\|_{L^2((|x-x_0|<r)} = \infty.
\] (1.4)

(ii) If $U$ is an open subset of $\mathbb{R}^N$ such that $K \subset U$, then
\[
\lim_{t \to 0} \|\nabla u(t)\|_{L^2(U)} = \infty.
\] (1.5)

(iii) If $\Omega$ is an open subset of $\mathbb{R}^N$ such that $\Omega \cap K = \emptyset$, then
\[
\sup_{t \in (-T,0)} \|u(t)\|_{H^1(\Omega)} < \infty.
\] (1.6)

**Remark 1.2.** Here are some comments on Theorem 1.1.

(i) Estimate (1.4) can be refined. More precisely, it follows from (5.12) that
\[
(-t)^{-\frac{\alpha}{2}+\varepsilon} \leq \|u(t)\|_{L^2(|x-x_0|<r)} \leq (-t)^{-\frac{\alpha}{2}}
\] where $k > \frac{N\alpha}{2}$ is given by (4.1). (4.4).

(ii) From the proof of Theorem 1.1, the blow-up mechanism for $u$ is described as follows. If $U_0(t,x) = (-\alpha t + A(x))^{-\frac{\alpha}{2}+\frac{\alpha}{4}}$ where $A \geq 0$ (which vanishes exactly on $K$) is given by (5.1), then $u(t) = U_0(t) + V(t) + \varepsilon(t)$, where $\|V(t)\|_{H^1} \lesssim (-t)^\nu \|U_0(t)\|_{L^2}$ and $\|\varepsilon(t)\|_{H^1} \lesssim (-t)^\nu$ for some $\nu > 0$, see (3.28) and (5.6).

To prove Theorem 1.1, we follow the strategy, introduced in [3] (see also the references there), of defining the ansatz $U_0$, blowing-up solution of the ODE $\partial_t U_0 = \lambda |U_0|^{\alpha} U_0$, and then using energy estimates and compactness arguments. In [3], restricted to $\alpha \geq 2$ and $\lambda = 1$, $U_0$ is a sufficiently good approximation and blowup is proved at any finite number of points. To treat any subcritical $\alpha$ and any $\lambda \in \mathbb{C}$ with $\Re \lambda > 0$, we need to refine the ansatz following the technique developed in [4] for the semilinear wave equation. We emphasize that this technique only uses ODE arguments. See the beginning of Sections 3 and 4 for more details. See also Remark 5.1 below for comments on the restriction (1.2) to $H^1$-subcritical powers.

The rest of this paper is organized as follows. In Section 2 we introduce some notation that we use throughout the paper and we recall some useful estimates. In Section 3 we construct the appropriate blow-up ansatz. Section 4 is devoted to the construction of a sequence of solutions of (1.1) close to the blow-up ansatz and to estimates of this sequence. Finally, we complete the proof of Theorem 1.1 in Section 5.

2. Notation and preliminary estimates

2.1. Some Taylor’s inequalities. Let
\[
f(u) = |u|^{\alpha} u.
\] (2.1)

In general, $f$ is not $C^1$ as a function $\mathbb{C} \to \mathbb{C}$ (except for $\alpha \in 2\mathbb{N}$, when $f$ is analytic). However, $f$ is $C^1$ as a function $\mathbb{R}^2 \to \mathbb{R}^2$. We denote by $df$ the derivative of $f$ is this sense, and we have
\[
df(u)v = \frac{\alpha + 2}{2} |u|^{\alpha} v + \frac{\alpha}{2} |u|^{\alpha-2} u^2 \overline{v}.
\] (2.2)

We also have
\[
\nabla f(u) = \frac{\alpha + 2}{2} |u|^{\alpha} \nabla u + \frac{\alpha}{2} |u|^{\alpha-2} u^2 \nabla \overline{u},
\]
so that
\[
\Re(\nabla f(u) \cdot \nabla \overline{u}) = \frac{\alpha + 2}{2} |u|^{\alpha} |\nabla u|^2 + \frac{\alpha}{2} \Re(|u|^{\alpha-2} u^2 (\nabla \overline{u})^2) \geq |u|^{\alpha} |\nabla u|^2.
\] (2.3)

In addition, we have the following estimates.
Lemma 2.1. Set
\[ C_\alpha = \begin{cases} 0 & 0 < \alpha \leq 1 \\ 1 & \alpha > 1. \end{cases} \] (2.4)

There exists a constant \( M \geq 1 \) such that
\[ |df(u)| \leq M |u|^\alpha \] (2.5)
\[ |f(u + v) - f(u)| \leq M (|u| + |v|) |v| \] (2.6)
\[ |df(u + v) - df(u)| \leq M (|v| + C_\alpha |u|^{\alpha-1}) |v| \] (2.7)
\[ |df(u + v) - df(u)| \leq M (|v| + C_\alpha |u|^{\alpha-1}) |v|^2 \] (2.8)
for all \( u, v \in \mathbb{C} \). Moreover,
\[ |\nabla [f(u + v) - f(u) - f(v)]| \leq M (|u| + |v|) (|\nabla u| + |\nabla v|) \] (2.10)
a.e. for all \( u, v \in H^1(\mathbb{R}^N) \).

Proof. Estimate (2.5) is an immediate consequence of (2.2), and (2.6) follows, using
\[ f(u + v) - f(u) = \int_0^1 df(u + tv) dt = \int_0^1 df(u + tv) v dt. \]

Estimate (2.7) is classical, see e.g. [2, formulas (2.26)-(2.27)]; and (2.8) follows from (2.7) and the elementary estimate
\[ |v|^\alpha + C_\alpha |u|^{\alpha-1} |v| \leq 2(|u|^\alpha + |v|) \].

Writing
\[ f(u + v) - f(u) - df(u) v = \int_0^1 [df(u + \theta v) v - df(u) v] d\theta \]
and using (2.7), we obtain (2.9). Finally, given \( u, v \in H^1(\mathbb{R}^N) \),
\[ \partial_{x_j} [f(u + v) - f(u) - f(v)] = [df(u + v) - df(u)] \partial_{x_j} u + [df(u + v) - df(v)] \partial_{x_j} v \]
so that (2.10) follows from (2.8). \( \Box \)

2.2. A Sobolev inequality. We have
\[ |u|^\alpha |\nabla u|^2 \geq |u|^\alpha |\nabla |u||^2 = \frac{4}{(\alpha + 2)^2} |\nabla (|u|^{\frac{2\alpha + 2}{\alpha + 2}})|^2. \] (2.11)

Note that by (1.2)
\[ \theta := \frac{N\alpha}{4(\alpha + 1)} \in (0, 1). \]

Since \( \frac{\alpha + 2}{3(\alpha + 1)} = \frac{1}{2} - \frac{\theta}{N} \), we deduce from Gagliardo-Nirenberg’s inequality that
\[ \|v\|_{L^{\frac{4(\alpha + 1)}{N(\alpha + 2)}}} \lesssim \|v\|_{L^2}^{1-\theta} \|
abla v\|^\theta_{L^2}, \]
so that, letting \( v = |u|^{\frac{2\alpha + 2}{\alpha + 2}} \) and using (2.11),
\[ \int |u|^{2\alpha + 2} \lesssim \left( \int |u|^{\alpha + 2} \right)^{\frac{4}{4-N(\alpha + 1)}} \left( \int |u|^\alpha |\nabla u|^2 \right)^{\frac{N\alpha}{2(\alpha + 2)}}. \]

Since \( \frac{N\alpha}{2(\alpha + 2)} < 1 \) by (1.2), we see that for every \( \eta > 0 \), there exists \( C_\eta > 0 \) such that
\[ \int |u|^{2\alpha + 2} \leq \eta \int |u|^{\alpha} |\nabla u|^2 + C_\eta \int |u|^{\alpha + 2}. \] (2.12)
2.3. Faa di Bruno’s formula. We recall that by the Faa di Bruno formula (see Corollary 2.10 in [5]), if $\beta$ is a multi-index, $|\beta| \geq 1$, if $a \in C^{|\beta|}(O, \mathbb{R})$ where $O$ is an open subset of $\mathbb{R}^N$, and if $\varphi \in C^{|\beta|}(U, \mathbb{R})$ where $U$ is a neighborhood of $a(O)$, then on $O$, $\partial_x^{|\beta|}[\varphi(a(\cdot))]$ is a sum of terms of the form
\[
\varphi^{(\nu)}(a(\cdot)) \prod_{\ell=1}^{|\beta|} (\partial_x^\ell a(\cdot))^\nu_{\ell},
\]
with appropriate coefficients, where $\nu \in \{1, \ldots, |\beta|\}$, $\nu_{\ell} \geq 0$, $\sum_{\ell=1}^{|\beta|} \nu_{\ell} = \nu$, $\sum_{\ell=1}^{|\beta|} \nu_{\ell} \beta_{\ell} = \beta$.

3. The blow-up ansatz

In this section, we construct inductively an appropriate blow-up ansatz. The first ansatz is $U_0$ defined by (3.3) below. $U_0$ is a natural candidate, since it is an explicit blowing-up solution of the ODE $\partial_t U_0 = \lambda f(U_0)$. Moreover, the error term $i \Delta U_0$ is of lower order than both $\partial_t U_0$ and $f(U_0)$. (See Lemma (3.1) below.) However, we need at least the error term to be integrable in time near the singularity. Since $\Delta U_0$ is of order $(-t)^{-\frac{N}{2}} |U_0| \lesssim t^{-\frac{N-2}{2}}$, this is not the case for any choice of $k$ if $\alpha \leq 1$. In Section 3.2, we introduce a procedure to reduce the singularity of the error term at any order of $(-t)$ by refining the approximate solution. This is important, not only to obtain blowup for arbitrarily small powers $\alpha$, but also to avoid any condition between $\alpha$ and $3\lambda$. We also point out that in this section, there is no condition on the power $\alpha$ other than $\alpha > 0$.

Throughout this section, we assume
\[
k > \max\{6, N\alpha\}. \tag{3.1}
\]
Let $A \in C^{k-1}(\mathbb{R}^N, \mathbb{R})$ be piecewise of class $C^k$ and satisfy
\[
\begin{aligned}
A \geq 0 \text{ and } |\partial_\beta A| &\lesssim A^{1-\frac{|\beta|}{4}} \text{ on } \mathbb{R}^N \text{ for } |\beta| \leq k - 1, \\
A(x) &= |x|^k \text{ for } x \in \mathbb{R}^N, |x| \geq 2.
\end{aligned} \tag{3.2}
\]
Assuming (1.3), (3.1) and (3.2), and set
\[
U_0(t, x) = (-\alpha t + A(x))^{1-\frac{i\lambda}{4}} \quad t < 0, x \in \mathbb{R}^N. \tag{3.3}
\]
It follows that
\[
U_0 \text{ is } C^\infty \text{ in } t < 0 \text{ and } C^{k-1} \text{ in } x \in \mathbb{R}^N, \tag{3.4}
\]
\[
\partial_t U_0 = \lambda f(U_0), \tag{3.5}
\]
\[
|U_0| = (-\alpha t + A(x))^{-\frac{i\lambda}{4}} \leq (-\alpha t)^{-\frac{i\lambda}{4}}. \tag{3.6}
\]
Moreover,
\[
\partial_t |U_0| = f(|U_0|) > 0 \tag{3.7}
\]
and
\[
A(x) \leq |U_0|^{-\alpha}. \tag{3.8}
\]

3.1. Estimates of $U_0$.

**Lemma 3.1.** Assume (1.3), (3.1), (3.2) and let $U_0$ be given by (3.3). If $p \geq 1$ then
\[
\|U_0(t)\|_{L^p} \lesssim (-t)^{-\frac{1}{2}}, \tag{3.9}
\]
for $-1 \leq t < 0$. In addition, for every $\rho \in \mathbb{R}$, $\ell \in \mathbb{N}$ and $|\beta| \leq k - 1$,

$$|\partial_2^\beta \partial_0^\rho U_0| \lesssim |U_0|^{1 + |\rho + \frac{\beta}{2}| \leq (t - \frac{\ell}{\rho})^+ |U_0|, \quad (3.10)$$

$$|\partial_2^\beta (|U_0|\rho)| \lesssim |U_0|^{\rho + \frac{|\beta|}{2}} \lesssim (t - \frac{\rho}{\rho})^+ |U_0|, \quad (3.11)$$

$$|\partial_2^\beta (|U_0|\rho - 1)U_0| \lesssim |U_0|^{\rho + \frac{|\beta|}{2}} \lesssim (t - \frac{\rho}{\rho})^+ |U_0|, \quad (3.12)$$

for $x \in \mathbb{R}^N$, $t < 0$, and

$$U_0 \in C^\infty((-\infty, 0), H^{k-1}(\mathbb{R}^N)). \quad (3.13)$$

Furthermore, for any $x_0 \in \mathbb{R}^N$ such that $A(x_0) = 0$, for any $r > 0$, $-1 \leq t < 0$ and $1 \leq p \leq \infty$,

$$(-t)^{-\frac{2}{n} + \frac{|\alpha|}{2}} \lesssim \|U_0(t)||_{L^p(|x - x_0| < r)}, \quad (3.14)$$

where the implicit constant in (3.14) depend on $r$ and $p$.

**Proof.** Estimate (3.9) is a consequence of (3.6), (3.2) and (3.1). Indeed,

$$\int_{\mathbb{R}^N} |U_0|^p \leq \int_{|x| < 2} |U_0|^p + \int_{|x| > 2} |U_0|^p \lesssim (t - \frac{\rho}{\rho})^+ + \int_{|x| > 2} |x|^{-\frac{2}{n} + \frac{|\alpha|}{2}} \lesssim (t - \frac{\rho}{\rho}).$$

We now set

$$U_0 = W^{-\frac{2}{n} - \frac{\rho}{\rho}}$$

where $W = -at + A(x) > 0$, and we prove (3.11)-(3.12). We let $|\beta| \geq 1$, and we write $|U_0|^p = \varphi(W)$ with $\varphi(s) = s^{-\frac{2}{n} - \nu}$. We see that $|\varphi^{(\nu)}(s)| \lesssim s^{-\frac{2}{n} - \nu}$ for $s > 0$. Moreover, if $1 \leq |\gamma| \leq |\beta|$, then

$$|\partial_2^\beta \varphi(W)| \lesssim |\partial_2^\beta A| \lesssim A^1 - \frac{|\beta|}{2} \lesssim W^{-\frac{|\beta|}{2}}, \quad (3.15)$$

where we used (3.2), $A \leq W$, and $1 - \frac{|\beta|}{2} \geq 1 - \frac{|\alpha|}{2} \geq \frac{1}{\rho} > 0$. By the Faa di Bruno formula (see Section 2.3), we deduce that $|\partial_2^\beta (|U_0|^p)|$ is estimated by a sum of terms of the form

$$B = W^{-\frac{2}{n} - \nu} \prod_{\ell = 1}^{\beta} W^{(1 - \frac{|\beta|}{2})\nu_\ell},$$

with appropriate coefficients, where $\nu \in \{1, \cdots, |\beta|\}$, $\nu_\ell \geq 0$, $\sum_{\ell = 1}^{\beta} \nu_\ell = \nu$, $\sum_{\ell = 1}^{\beta} \nu_\ell \beta_\ell = \beta$. It follows that

$$B = W^{-\frac{2}{n} - \nu} W^{\nu - \frac{|\beta|}{2}} = W^{-\frac{2}{n} - \frac{|\beta|}{2}} = |U_0|^{\rho + \frac{\beta}{2}|\beta|},$$

and (3.11) follows.

Now we claim that if $1 \leq |\beta| \leq k - 1$, then

$$|\partial_2^\beta U_0| \lesssim |U_0|^{1 + \frac{|\beta|}{2}}. \quad (3.16)$$

Indeed, we write

$$U_0 = |U_0| \left( \cos \left( \frac{\mu}{\alpha} \log W \right) + i \sin \left( \frac{\mu}{\alpha} \log W \right) \right) \quad (3.17)$$

Since $(\log s)^{(\nu)} \lesssim s^{-\nu}$ for $\nu \geq 1$, it follows easily from Faa di Bruno’s formula and (3.15) that

$$|\partial_2^\gamma (\log W)| \lesssim W^{-\frac{|\gamma|}{2}} \quad (3.18)$$

if $1 \leq |\gamma| \leq |\beta|$. Using again Faa di Bruno’s formula together with (3.18), we obtain

$$\left| \partial_2^\gamma \left( \cos \left( \frac{\mu}{\alpha} \log W \right) \right) \right| + \left| \partial_2^\gamma \left( \sin \left( \frac{\mu}{\alpha} \log W \right) \right) \right| \lesssim W^{-\frac{|\gamma|}{2}} \quad (3.19)$$

if $1 \leq |\gamma| \leq |\beta|$. Estimate (3.16) follows from (3.17), (3.11) with $\rho = 1$, (3.19), and Leibnitz’s formula.

Estimate (3.12) follows from (3.11), (3.16), and Leibnitz’s formula.
To prove (3.10), we observe that \( \partial_t^j \tilde{U}_0 = cW^{\frac{1}{\alpha}} - t - i \tilde{\Phi} \) for some constant \( c \). Therefore, \( \partial_t^j \tilde{U}_0 = c \tilde{U}_0 \), where \( \tilde{U}_0 = W^{-1/\alpha} - i \tilde{\mu} / \tilde{\alpha} \) with \( \tilde{\alpha} = \frac{\alpha}{1 + \mu} \) and \( \tilde{\mu} = \frac{\mu}{1 + \mu} \). In particular, \( |\tilde{U}_0| = |U_0|^{1 + t \alpha} \). Applying formula (3.16) with \( \alpha \) and \( \mu \) replaced by \( \tilde{\alpha} \) and \( \tilde{\mu} \), we obtain
\[
|\partial_t^j \tilde{U}_0| \lesssim |\tilde{U}_0|^{1 + \tilde{\beta}}|\tilde{\beta}| = |U_0|^{1 + t \alpha + \tilde{\beta}}|\tilde{\beta}|
\]
from which (3.10) follows.

Property (3.13) is an immediate consequence of (3.4), (3.10) and (3.9).

Finally, we prove (3.14). Let \( x_0 \in \mathbb{R}^N \) be such that \( A(x_0) = 0 \) and \( r > 0 \). We have \( |U_0(t, x_0)| = (\alpha t)^{- \frac{1}{\alpha}} \), and (3.14) follows in the case \( p = \infty \). We now assume \( p < \infty \). Since \( A \) is piecewise \( C^k \), it follows easily from (3.2) that for any \( x \) such that \( |x - x_0| < r \), we have \( |A(x)| \leq C(r)|x - x_0|^k \); and so
\[
|U_0(t, x)|^p \geq (t - |x - x_0|^k)^{- \frac{p}{k}}
\]

Estimate (3.14) then follows from
\[
\int_{|y| < r} (t - |y|^k)^{- \frac{p}{k}} dy = (t - |y|^k)^{- \frac{p}{k}} \frac{1}{\alpha} (1 + |z|^k)^{- \frac{p}{k}} dz \geq (t - |y|^k)^{- \frac{p}{k}} \frac{1}{\alpha} (1 + |z|^k)^{- \frac{p}{k}} dz \geq (t - |y|^k)^{- \frac{p}{k}} \frac{1}{\alpha}.
\]

This completes the proof.

\[\square\]

3.2. The refined blow-up ansatz. We consider the linearization of equation (3.5)
\[
\partial_t w = \lambda \frac{\alpha + 2}{2} |U_0|^\alpha w + \lambda \frac{\alpha}{2} |U_0|^\alpha U_0^2 w = \lambda df(U_0)w,
\]
where \( df \) is defined by (2.2). Equation (3.21) has the two solutions \( w = iU_0 \) and \( w = \partial_t U_0 \), i.e. \( w = \lambda |U_0|^\alpha U_0 \). Moreover, \( \partial_t (|U_0|^\alpha U_0) = (\lambda + \alpha)|U_0|^{2\alpha} U_0 \). Elementary calculations show that for suitable \( G \), the corresponding nonhomogeneous equation
\[
\partial_t w = \lambda df(U_0)w + G
\]

has the solution \( w = P(G) \), where
\[
P(G) = |U_0|^\alpha U_0 \int_0^t [\|U_0\|^{\alpha - 2} \Re(U_0G)](s) ds + iU_0 \int_0^t [\|U_0\|^{-2} \Im(U_0G)](s) ds + i\alpha \mu U_0 \int_0^t |U_0(s)|^{2\alpha} \int_0^s [\|U_0\|^{-\alpha - 2} \Re(U_0G)](\sigma) d\sigma ds.
\]

We define \( U_j, w_j, E_j \) by
\[
w_0 = iU_0, \quad E_0 = -\partial_t U_0 + i\Delta U_0 + \lambda f(U_0) = i\Delta U_0,
\]
and then recursively
\[
w_j = P(E_{j-1}), \quad U_j = U_{j-1} + w_j, \quad E_j = -\partial_t U_j + i\Delta U_j + \lambda f(U_j),
\]
for \( j \geq 1 \), as long as this makes sense. We will see that for \( j \leq \frac{k-1}{2} \), \( P(E_{j-1}) \) is well defined at each step, on a sufficiently small time interval. We have the following estimates.

**Lemma 3.2.** Assume (1.3), (3.1), (3.2), and let \( U_j, w_j, E_j \) be given by (3.3), (3.24) and (3.25). There exists \(-1 \leq T < 0\) such that the following estimates hold for all \( 0 \leq j \leq \frac{k-1}{2} \).

(i) If \( 0 \leq |\beta| \leq k - 1 - 2j \), then
\[
|\partial_t^\beta w_j| \lesssim (t_j^{1 - \frac{p}{k}})^{\frac{|\beta|}{p}} |U_0|, \quad T \leq t < 0, x \in \mathbb{R}^N.
\]
(ii) If $0 \leq |\beta| \leq k - 3 - 2j$, then

$$|\partial^2_x E_j| \lesssim (-t)^{j(1 - \frac{2}{k}) - \frac{|\beta|}{k}} |U_0|, \quad T \leq t < 0, x \in \mathbb{R}^N. \quad (3.27)$$

(iii) If $0 \leq |\beta| \leq k - 1 - 2j$, then

$$|\partial^2_x (U_j - U_0)| \lesssim (-t)^{j - \frac{|\beta|}{k}} |U_0|, \quad T \leq t < 0, x \in \mathbb{R}^N. \quad (3.28)$$

Moreover

$$\frac{1}{2} |U_0| \leq |U_j| \leq 2 |U_0|, \quad T \leq t < 0, x \in \mathbb{R}^N. \quad (3.29)$$

In addition,

$$U_j \in C^4((T, 0), H^{k-1-2j}(\mathbb{R}^N)). \quad (3.30)$$

**Proof.** For $j = 0$, estimates (3.26) and (3.27) are immediate consequences of (3.12) and (3.6) (for $T = -\infty$), and estimates (3.28) and (3.29) are trivial. We now proceed by induction on $j$. We assume that for some $1 \leq j \leq \frac{k-3}{2}$, estimates (3.26)–(3.29) hold for $0, \cdots, j - 1$, and we prove estimates (3.26)–(3.29) for $j$, by possibly assuming $T$ smaller.

**Proof of (3.26).** Let $0 \leq |\beta| \leq k - 1 - 2j$. Given $\rho \in \mathbb{R}$, it follows from Leibnitz’s formula, (3.12), and (3.27) for $j - 1$ that

$$|\partial^3_x ([U_0]^\rho E_{j-1})| \lesssim \sum_{\beta_1 + \beta_2 = \beta} \left| \partial_{x_1}^\beta ([U_0]^\rho U_0) \right| \left| \partial_{x_2}^\beta E_{j-1} \right| \lesssim \sum_{\beta_1 + \beta_2 = \beta} (-t)^{-\frac{|\beta_1|}{\rho}} |U_0|^\rho (-t)^{j - 1 - \frac{k-3}{2}} |U_0| \lesssim (-t)^{j(1 - \frac{2}{k}) - \frac{|\beta|}{\rho}} |U_0|^\rho + 2.$$ Integrating on $(t, 0)$ for $t \in (T, 0)$, and using that $|U_0|^{-1}$ is a decreasing function of $t$ by (3.7), we see that if $\rho + 2 \leq 0$, then

$$\left| \partial^3_x \int_0^t [U_0]^\rho E_{j-1} ds \right| \lesssim |U_0(t)|^{\rho+2} \int_t^0 (-s)^{j(1 - \frac{2}{k}) - \frac{|\beta|}{\rho}} ds.$$ Note that $j \geq 1$ and $\frac{|\beta|}{\rho} \leq 1 - \frac{2j+1}{k}$, so that $j(1 - \frac{2}{k}) - \frac{|\beta|}{k} \geq 1 - \frac{2}{k} - \frac{3}{k} > 0$; and so,

$$\left| \partial^3_x \int_0^t [U_0]^\rho E_{j-1} ds \right| \lesssim (-t)^{j(1 - \frac{2}{k}) - \frac{|\beta|}{k}} |U_0(t)|^{\rho+2}. \quad (3.31)$$

It follows from Leibnitz’s formula, (3.12), (3.31), (3.11) and (3.6) that

$$|\partial^3_x w_j| \lesssim \sum_{\beta_1 + \beta_2 = \beta} (-t)^{-\frac{|\beta_1|}{\rho}} |U_0|^{\rho+1} (-t)^{j(1 - \frac{2}{k}) - \frac{|\beta|}{k}} |U_0|^{-\alpha} + \sum_{\beta_1 + \beta_2 = \beta} (-t)^{-\frac{|\beta_1|}{\rho}} |U_0| (-t)^{j(1 - \frac{2}{k}) - \frac{|\beta|}{k}} \int_t^0 (-s)^{\rho - 1 - \frac{2j+1}{k}} |U_0|^\rho |U_0|^{-\alpha} ds$$

$$\lesssim (-t)^{j(1 - \frac{2}{k}) - \frac{|\beta|}{k}} |U_0| + \sum_{\beta_1 + \beta_2 = \beta} (-t)^{-\frac{|\beta_1|}{\rho}} |U_0| \int_t^0 (-s)^{\rho - 1 - \frac{2j+1}{k}} |U_0|^\rho (-s)^{j(1 - \frac{2}{k}) - \frac{|\beta_2|}{k}} |U_0|^{-\alpha} ds$$

This proves (3.26).
Proof of (3.28) and (3.29). Since \( U_j - U_0 = w_j + U_{j-1} - U_0 \), estimate (3.28) for \( j \) follows from (3.26) for \( j \) and (3.28) for \( j - 1 \). Estimate (3.29) follows from (3.28) by possibly choosing \( T > 0 \) smaller.

Proof of (3.27). Since \( U_j - U_{j-1} = w_j \), it follows from (3.25) and the definition of \( P \) that
\[
\mathcal{E}_j - \mathcal{E}_{j-1} = -\partial_t w_j + i\Delta w_j + \lambda f(U_j) - \lambda f(U_{j-1})
\]
so that
\[
\mathcal{E}_j = i\Delta w_j + \lambda [f(U_{j-1} + w_j) - f(U_{j-1}) - df(U_0)w_j]
\]
\[
= : A_1 + \lambda A_2.
\]
It follows from (3.26) (for \( j \)) that if \(|\beta| \leq k - 3 - 2j \) (so that \(|\beta| + 2 \leq k - 1 - 2j \)), then
\[
|\partial_t^2 A_1| \lesssim (-\tau)^{[(1 - \frac{3}{2}) - \frac{|\beta| + 2}{2}]} |U_0|.
\]
(3.32)
We now estimate \( A_2 \), and we write
\[
f(U_{j-1} + w_j) - f(U_{j-1}) = \int_0^1 \frac{d}{d\theta} f(U_0 + g_j(\theta)) \, d\theta = \int_0^1 df(U_0 + g_j(\theta))w_j \, d\theta,
\]
where
\[
g_j(\theta) = U_{j-1} - U_0 + \theta w_j,
\]
so that
\[
A_2 = \int_0^1 [df(U_0 + g_j(\theta))w_j - df(U_0)w_j] \, d\theta
\]
\[
= \frac{\alpha + 2}{2} \int_0^1 (|U_0 + g_j(\theta)|^\alpha - |U_0|^\alpha)w_j \, d\theta
\]
\[
+ \frac{\alpha}{2} \int_0^1 (|U_0 + g_j(\theta)|^{\alpha - 2}(U_0 + g_j(\theta))^2 - |U_0|^{\alpha - 2}U_0^2)w_j \, d\theta.
\]
We write
\[
|U_0 + g_j(\theta)|^\alpha - |U_0|^\alpha = \int_0^1 \frac{d}{d\tau}(\tau|U_0 + g_j(\theta)|^2 + (1 - \tau)|U_0|^2)^{\frac{\alpha}{2}} \, d\tau
\]
\[
= \frac{\alpha}{2} \eta (|U_0 + g_j(\theta)|^2 - |U_0|^2)
\]
\[
= \frac{\alpha}{2} \eta (2\Re(U_0g_j(\theta)) + |g_j(\theta)|^2)
\]
where
\[
\eta = \int_0^1 (\tau|U_0 + g_j(\theta)|^2 + (1 - \tau)|U_0|^2)^{\frac{\alpha - 1}{2}} \, d\tau.
\]
Similarly,
\[
|U_0 + g_j(\theta)|^{\alpha - 2}(U_0 + g_j(\theta))^2 - |U_0|^{\alpha - 2}U_0^2
\]
\[
= (|U_0 + g_j(\theta)|^{\alpha - 2} - |U_0|^{\alpha - 2})(U_0 + g_j(\theta))^2 + |U_0|^{\alpha - 2}((U_0 + g_j(\theta))^2 - U_0^2)
\]
\[
= \frac{\alpha - 2}{2} \eta (2\Re(U_0g_j(\theta)) + |g_j(\theta)|^2 + |U_0|^{\alpha - 2}(2U_0 + g_j(\theta))g_j(\theta),
\]
where
\[
\tilde{\eta} = (U_0 + g_j(\theta))^2 \int_0^1 (\tau|U_0 + g_j(\theta)|^2 + (1 - \tau)|U_0|^2)^{\frac{\alpha - 2}{2}} \, d\tau.
\]
Thus we may write
\[
A_2 = \frac{\alpha(\alpha + 2)}{4} \int_0^1 \eta[2\Re(U_0g_j(\theta))] + |g_j(\theta)|^2 |\omega_j| d\theta
+ \frac{\alpha(\alpha - 2)}{4} \int_0^1 \eta[2\Re(U_0g_j(\theta))] + |g_j(\theta)|^2 |\omega_j| d\theta
+ \frac{\alpha}{2} \int_0^1 |U_0|^{\alpha - 2}(2U_0 + g_j(\theta))g_j(\theta) |\omega_j| d\theta
= \frac{\alpha(\alpha + 2)}{4} B_1 + \frac{\alpha(\alpha - 2)}{4} B_2 + \frac{\alpha}{2} B_3.
\]
(3.33)

Using (3.26), we obtain by choosing $T$ possibly smaller
\[
|g_j(\theta)| \leq \sum_{i=1}^j |\omega_i| \leq C(-t)^{1 + \frac{\alpha}{2}} |U_0| \leq \frac{1}{2} |U_0|,
\]
so that
\[
\frac{1}{4} |U_0|^2 \leq \tau |U_0 + g_j(\theta)|^2 + (1 - \tau)|U_0|^2 \leq 3|U_0|^2
\]
(3.34)
for all $0 \leq \tau, \theta \leq 1$. Applying (3.11), (3.12), (3.26), and Leibnitz’s formula, it is not difficult to show that if $|\beta| \leq k - 3 - 2j$ then
\[
|\partial^\beta_x [(U_0 + g_j(\theta))^2 + (1 - \tau)|U_0|^2]| \lesssim (-t)^{-|\partial^\beta_x| |\omega_0|^{\alpha - 2}}. \tag{3.35}
\]

Using now (3.34), (3.35), and the Faà di Bruno formula, we deduce that
\[
|\partial^\beta_x \eta| \lesssim (-t)^{-|\partial^\beta_x| |\omega_0|^{\alpha - 2}}. \tag{3.36}
\]

Similarly (using in addition Leibnitz’s formula), we see that
\[
|\partial^\beta_x \eta| \lesssim (-t)^{-|\partial^\beta_x| |\omega_0|^{\alpha - 2}}. \tag{3.37}
\]

Next, we deduce from (3.12), (3.26), (3.36), (3.37), (3.6), and Leibnitz’s formula that if $|\beta| \leq k - 3 - 2j$ then
\[
|\partial^\beta_x B_1| + |\partial^\beta_x B_2| \lesssim (-t)^{(1 - \frac{\alpha}{2}) - \frac{|\beta| + 2}{2}} |U_0|.
\]
(3.38)

Using (3.11) with $\rho = \alpha - 2$, we obtain similarly
\[
|\partial^\beta_x B_3| \lesssim (-t)^{(1 - \frac{\alpha}{2}) - \frac{|\beta| + 2}{2}} |U_0|.
\]
(3.39)

Estimate (3.27) follows from (3.32), (3.33), (3.38) and (3.39).

Finally, we prove (3.30). For this, we prove by induction on $j$ that
\[
U_j, w_j \in C^1((T, 0), H^{k-1-2j}(\mathbb{R}^N)) \text{ and } \mathcal{E}_j \subset C((T, 0), H^{k-3-2j}(\mathbb{R}^N)).
\]
(3.40)

For $j = 0$, (3.40) holds, by (3.13). We assume that for some $1 \leq j \leq \frac{k - 4}{2}$, property (3.40) holds for $j - 1$, and we prove it for $j$. Let $t_0 \in (T, 0)$. It follows from (3.26) and (3.9) that $w_j(t_0) \in H^{k-1-2j}(\mathbb{R}^N)$. Moreover, $\mathcal{E}_{j-1} \subset C((T, 0), H^{k-1-2j}(\mathbb{R}^N))$ by the induction assumption. Since $\partial_t w_j = \lambda df(U_0) w_j + \mathcal{E}_{j-1}$, it is not difficult to prove (using Lemma 3.1 for the relevant estimates of $U_0$) that $w_j \in C^1((T, 0), H^{k-1-2j}(\mathbb{R}^N))$. Hence $U_j \in C^1((T, 0), H^{k-1-2j}(\mathbb{R}^N))$, and by definition of $\mathcal{E}_j$, we deduce that $\mathcal{E}_j \subset C((T, 0), H^{k-3-2j}(\mathbb{R}^N))$. This proves (3.40). \qed
4. Construction and estimates of approximate solutions

In this section, we construct a sequence $u_n$ of solutions of (1.1), close to the ansatz $U_J$ of Section 3, which will eventually converge to the blowing-up solution of Theorem 1.1. We estimate $\varepsilon_n = u_n - U_J$ by an energy method. More precisely, we estimate $(-t)^{-\sigma} \|\varepsilon_n\|_{L^2} + (-t)^{-1-\theta}\sigma \|\nabla \varepsilon_n\|_{L^2}$ for some appropriate parameters $\sigma \geq 0$ and $0 \leq \theta \leq 1$. This parameter $\sigma$ is taken large enough to avoid unnecessary condition on $\lambda$, see (4.18) and (4.33). Moreover, the parameter $J$ of the ansatz $U_J$ is chosen sufficiently large to absorb the singularity $(-t)^{-\sigma}$, see (4.17) and (4.25).

We now go into details. We define $\sigma, \theta > 0$ by

$$\sigma = \max \{2^{\alpha+1}\alpha^{-1}|\lambda|M, 2\alpha|\lambda|(8M|\lambda|^\alpha\right\}, \tag{4.1}$$

$$\theta = \min \left\{\frac{1}{N}, \frac{1}{N\alpha + 1}, \frac{1}{3\alpha}\right\}, \tag{4.2}$$

where $M$ is given by Lemma 2.1, and we set

$$J = \left[\frac{2}{\alpha} + 4\sigma\right] + 1, \tag{4.3}$$

$$k = \max \{2J + 4, \frac{4}{\theta\sigma}, 2N\alpha\}. \tag{4.4}$$

In particular, $k$ satisfies (3.1). We let $A \in C^{k-1}(\mathbb{R}^N, \mathbb{R})$ be piecewise of class $C^k$ and satisfy (3.2), and we consider the ansatz $U_J$ constructed in Section 3, and $T < 0$ given by Lemma 3.2. (This is possible since $2J \leq k - 4$ by (4.4).) For $n > -\frac{1}{J}$, we set

$$T_n = -\frac{1}{n} \in (T, 0).$$

Since $U_J(T_n) \in H^2(\mathbb{R}^N)$ (by (3.30) and (4.4)) it follows that there exist $s_n < T_n$ and a unique solution $u_n \in C((s_n, T_n], H^2(\mathbb{R}^N)) \cap C^1((s_n, T_n], L^2(\mathbb{R}^N))$ of

$$\begin{cases}
\partial_t u_n = i\Delta u_n + \lambda f(u_n) \\
u_n(T_n) = U_J(T_n),
\end{cases} \tag{4.5}$$

defined on the maximal interval $(s_n, T_n]$, with the blow-up alternative that if $s_n > -\infty$, then

$$\|\nabla u_n(t)\|_{L^2} \xrightarrow{t \downarrow s_n} \infty. \tag{4.6}$$

See [6].

We let $\varepsilon_n \in C((\max\{s_n, T\}, T_n], H^2(\mathbb{R}^N)) \cap C^1((\max\{s_n, T\}, T_n], L^2(\mathbb{R}^N))$ be defined by

$$u_n = U_J + \varepsilon_n \tag{4.7}$$

and we have the following estimate.

**Proposition 4.1.** Assume (4.1), (4.2), (4.3) and (4.4). If $\varepsilon_n$ is given by (4.7), then there exist $T \leq S < 0$ and $n_0 > -\frac{1}{J}$ such that

$$s_n < S, \quad \text{for all } n \geq n_0. \tag{4.8}$$

Moreover,

$$\|\varepsilon_n(t)\|_{L^2} \leq (-t)^{\sigma} \tag{4.9}$$

$$\|\nabla \varepsilon_n(t)\|_{L^2} \leq (-t)^{(1-\theta)\sigma} \tag{4.10}$$

for all $n \geq n_0$ and $S \leq t \leq T_n$, and

$$\int_S^{T_n} \int_{\mathbb{R}^N} |\varepsilon_n|^\alpha |\nabla \varepsilon_n|^2 \leq 1. \tag{4.11}$$
Proof. Throughout the proof, we write $\varepsilon$ instead of $\varepsilon_n$. Moreover, $C$ denotes a constant that may change from line to line, but that is independent of $n$ and $t$. Unless otherwise specified, all integrals are over $\mathbb{R}^N$. Using (4.5) and (3.25), we have
\[
\begin{aligned}
\partial_t \varepsilon &= i\Delta \varepsilon + \lambda (f(U_J + \varepsilon) - f(U_J)) + \mathcal{E}_J \\
\varepsilon(T_n) &= 0.
\end{aligned}
\] (4.12)
We control $\varepsilon$ by energy estimates. Let
\[
\tau_n = \inf \{ t \in [\max\{T, s_n\}, T_n]; \| \varepsilon(s) \|_{L^2} \leq (-s)\sigma \text{ and } \| \nabla \varepsilon(s) \|_{L^2} \leq (-s)^{(1-\theta)\sigma} \text{ for } t \leq T_n, \text{ and } \int_t^{T_n} \int_{\mathbb{R}^N} |\varepsilon|^{\alpha} |\nabla \varepsilon|^2 \leq 1 \}.
\] (4.13)
Since $\varepsilon(T_n) = 0$, we see that $T \leq \tau_n < T_n$. Moreover, it follows from the blowup alternative (4.6) that
\[
s_n < \tau_n.
\] (4.14)
In addition, by Gagliardo-Nirenberg’s inequality, (4.13) and (4.2),
\[
\| \varepsilon(t) \|_{L^{\alpha+2}}^2 \leq C(-t)^{(\alpha+2-\frac{2N}{\alpha})\sigma} = C(-t)^{2\sigma + \alpha(1-\frac{2}{\alpha})\sigma} \leq C(-t)^{2\sigma},
\] (4.15)
for $\tau_n \leq t \leq T_n$. Moreover, it follows from (3.29), (3.6), (3.28) and (3.10) that
\[
|U_J| \leq 2|U_0| \leq 2\alpha^{-\frac{1}{3}}(-t)^{-\frac{1}{3}}, \quad |\nabla U_J| \leq C(-t)^{-\frac{1}{3}}|U_0| \leq C(-t)^{-\frac{1}{3}+\frac{2}{3}}
\] (4.16)
for all $T < t < 0$.

We first estimate $\| \varepsilon \|_{L^2}$. Multiplying (4.12) by $\overline{\varepsilon}$ and taking the real part, we obtain
\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \| \varepsilon(t) \|_{L^2}^2 &= \Re \left( \int [df(U_J) \varepsilon] \overline{\varepsilon} \right) \\
&\quad + \Re \left( \int [\overline{f(U_J + \varepsilon) - f(U_J) - df(U_J)\varepsilon}] \overline{\varepsilon} \right) + \Re \int \mathcal{E}_J \overline{\varepsilon}.
\end{aligned}
\]
Using (2.5) and (2.9), we deduce that
\[
\left| \frac{d}{dt} \| \varepsilon(t) \|_{L^2}^2 \right| \leq 2|\lambda|M \int |U_J|^\alpha |\varepsilon|^2 + 2|\lambda|M \int |\varepsilon|^{\alpha+2} + C_\alpha |U_J|^{\alpha-1} |\varepsilon|^3 \right| + \| \mathcal{E}_J \|_{L^2} \| \varepsilon \|_{L^2}.
\]
By (4.16),
\[
2|\lambda|M \int |U_J|^\alpha |\varepsilon|^2 \leq 2^{\alpha+1} \alpha^{-1} |\lambda|M(-t)^{-1} \| \varepsilon \|_{L^2}^2.
\]
The term $\int |\varepsilon|^{\alpha+2}$ is estimated by (4.15). Note that $C_\alpha \neq 0$ only if $\alpha > 1$. In this case $\alpha < 1$ and $2 < 3 < \alpha + 2$, so we deduce from (4.16), Hölder’s inequality, (4.13), (4.15) and (4.2) that
\[
C_\alpha \int |U_J|^\alpha |\varepsilon|^3 \leq CC_\alpha(-t)^{-1+\frac{2}{3}} \int |\varepsilon|^3 \leq C(-t)^{-1+\frac{2}{3}+2\sigma}.
\]
Next, by (3.27), (3.9) and (4.13),
\[
\| \mathcal{E}_J \|_{L^2} \| \varepsilon \|_{L^2} \leq C(-t)^{J(1-\frac{2}{3})+\frac{5}{3}+\sigma} = C(-t)^{-1+J+\frac{5}{3}+\sigma}.
\]
Note that by (4.3) and (4.4),
\[
(J + 1)(1-\frac{2}{k}) - \frac{1}{\alpha} + \sigma \geq \frac{1}{2}(J + 1) - \frac{1}{\alpha} + \sigma \geq 3\sigma,
\]
so that
\[
\| \mathcal{E}_J \|_{L^2} \| \varepsilon \|_{L^2} \leq C(-t)^{-1+3\sigma}.
\] (4.17)
It follows from the above inequalities that
\[
\frac{d}{dt} \| \varepsilon(t) \|_{L^2}^2 \geq -2^{\alpha+1} \alpha^{-1} |\lambda|M(-t)^{-1} \| \varepsilon \|_{L^2}^2 - C(-t)^{-1+2\sigma+\nu},
\]
where \( \nu = \min\{1, \frac{1}{\alpha}, \sigma\} \); and so

\[
\frac{d}{dt} \left( (-t)^{-\sigma} \| \varepsilon(t) \|^2_{L^2} \right) = \sigma (-t)^{-\sigma-1} \| \varepsilon(t) \|^2_{L^2} + (-t)^{-\sigma} \frac{d}{dt} \| \varepsilon(t) \|^2_{L^2} \\
\geq \left[ \sigma - 2^{\alpha+1} \alpha^{-1} |\lambda| M \right] (-t)^{-\sigma-1} \| \varepsilon(t) \|^2_{L^2} - C (-t)^{-1+\sigma+\nu}.
\] (4.18)

Using (4.1), we obtain

\[
\frac{d}{dt} \left( (-t)^{-\sigma} \| \varepsilon(t) \|^2_{L^2} \right) \geq -C (-t)^{-1+\sigma+\nu}.
\]

Integrating on \((t, T)\) and using \(\varepsilon(T) = 0\), we deduce that

\[
(-t)^{-\sigma} \| \varepsilon(t) \|^2_{L^2} \leq C (-t)^{\sigma+\nu},
\]

hence

\[
\| \varepsilon(t) \|_{L^2} \leq C (-t)^{\sigma+\frac{\nu}{2}},
\] (4.19)

for all \(t \in (\tau_n, T)\).

We now define the energy

\[
E(t) = \frac{1}{2} \int |\nabla \varepsilon(t)|^2 - \frac{\mu}{\alpha + 2} \int | \varepsilon(t) |^{\alpha+2}.
\] (4.20)

Multiplying equation (4.12) by \(-\Delta \varepsilon - \mu \varepsilon(t)\) and taking the real part, we obtain after integrating by parts

\[
\frac{d}{dt} E(t) = \Re \int \nabla f(\varepsilon) \cdot \nabla \varepsilon + \Re \int \lambda \nabla (f(U_j + \varepsilon) - f(U_j) - f(\varepsilon)) \cdot \nabla \varepsilon \\
- \mu \Re \int \lambda (f(U_j + \varepsilon) - f(U_j)) f(\varepsilon) + \Re \int \nabla E_j \cdot \nabla \varepsilon \\
- \mu \Re \int \varepsilon f(\varepsilon) =: A_1 + A_2 + A_3 + A_4 + A_5.
\] (4.21)

Using (2.3), we have

\[
A_1 \geq \int |\varepsilon|^\alpha |\nabla \varepsilon|^2.
\] (4.22)

Moreover, it follows from (2.6), (4.16), (4.15), (2.12) that

\[
|A_3| \leq |\lambda|^2 M \int (|U_j|^\alpha |\varepsilon|^{\alpha+2} + |\varepsilon|^{2\alpha+2}) \\
\leq C (-t)^{-1+2\sigma} + \left( \frac{1}{8} + C(-t)^{\frac{\alpha}{\nu}} \right) \int |\varepsilon|^{\alpha} |\nabla \varepsilon|^2.
\]

We let \( \tilde{\kappa} < 0 \) be defined by

\[
C(-\tilde{\kappa})^{\frac{2}{\nu}} = \frac{1}{8},
\] (4.23)

and we deduce that

\[
|A_3| \leq C (-t)^{-1+2\sigma} + \frac{1}{4} \int |\varepsilon|^{\alpha} |\nabla \varepsilon|^2,
\] (4.24)

for all \( \alpha \geq -\frac{1}{2} \) and all \( \tau_n < t \leq T_n \) such that \( t \geq \tilde{\kappa} \).

Next by (3.27), (3.9) and (4.13),

\[
|A_4| \leq \| \nabla E_j \|_{L^2} \| \nabla \varepsilon \|_{L^2} \leq C (-t)^{J(1-\tilde{\kappa})^{-\frac{1}{2}}} \| U_0 \|_{L^2} \| \nabla \varepsilon \|_{L^2} \\
\leq C (-t)^{J(1-\tilde{\kappa})^{-\frac{1}{2}} + \frac{1}{4} + (1-\theta)\sigma}.
\]

Moreover, using (3.27), (3.9) and (4.15),

\[
|A_5| \leq C \| \mathcal{E}_j \|_{L^{\alpha+2}} \| \varepsilon \|_{L^{\alpha+2}}^{\alpha+1} \leq C (-t)^{J(1-\tilde{\kappa})^{-\frac{1}{2}} - \frac{1}{2} + \frac{\alpha+1}{2} (\alpha+2-\frac{4}{\nu}) \sigma}.
\]
Using \( \min\{1 - \theta, \frac{\alpha + 1}{\alpha + 2}(\alpha + 2 - \frac{N\alpha}{2}\theta)\} \geq 1 - (N\alpha + 1)\theta \geq 0 \) \( \text{(by (4.2))} \), we conclude that
\[
|A_4 + A_5| \leq C(-t)^{(J(1-\frac{2}{J}) - \frac{1}{\alpha} + \frac{1}{k}J)} \leq C(-t)^{-1+(J+1)(1-\frac{2}{J}) + \frac{1}{\alpha} + \frac{1}{k}J}.
\]
Note that by (4.3) and (4.4)
\[
(J+1)(1 - \frac{3}{k}) - \frac{1}{\alpha} + \frac{1}{k}J \geq (J+1)(1 - \frac{3}{k}) - \frac{1}{\alpha} \geq \frac{1}{2}(J+1) - \frac{1}{\alpha} \geq 2\sigma,
\]
so that
\[
|A_4 + A_5| \leq C(-t)^{-1+2\sigma}. \tag{4.25}
\]
We now estimate \( A_2 \). We write
\[
\nabla(f(U_J + \epsilon) - f(U_J)) = (df(U_J + \epsilon) - df(\epsilon))\nabla \epsilon + (df(U_J + \epsilon) - df(U_J))\nabla U_J,
\]
so that
\[
|A_2| \leq |\lambda| \left( \int B_1 + \int B_2 \right) \tag{4.26}
\]
with
\[
B_1 = |df(U_J + \epsilon) - df(\epsilon)| |\nabla \epsilon|^2, \quad B_2 = |df(U_J + \epsilon) - df(U_J)| |\nabla U_J| |\nabla \epsilon|.
\]
It follows from (2.7) and (3.29) that
\[
|df(U_J + \epsilon) - df(\epsilon)| \leq 2^\alpha M|U_0|^\alpha + 2MC_\alpha|\epsilon|^{\alpha-1}|U_0|.
\]
If \( \alpha > 1 \), then \( |\epsilon|^{\alpha-1}|U_0| \leq (8M|\lambda|)^{-1}|\epsilon|^{\alpha} + (8M|\lambda|)^{\alpha-1}|U_0|^\alpha \), so that (recall \( |\lambda| \geq 1 \))
\[
|df(U_J + \epsilon) - df(\epsilon)| \leq (2^\alpha M + 2M(8M|\lambda|)^{\alpha-1})|U_0|^\alpha + \frac{1}{4|\lambda|}|\epsilon|^{\alpha}
\]
\[
\leq (8M|\lambda|)^\alpha|U_0|^\alpha + \frac{1}{4|\lambda|}|\epsilon|^{\alpha}.
\]
Using (3.6), we deduce that
\[
\int B_1 \leq \alpha(8M|\lambda|)^\alpha(-t)^{-1}||\nabla \epsilon||^2_{L^2} + \frac{1}{4|\lambda|} \int |\epsilon|^{\alpha}||\nabla \epsilon||^2. \tag{4.27}
\]
To estimate \( B_2 \), we consider separately the cases \( \alpha \leq 1 \), \( 1 < \alpha \leq 3 \), and \( \alpha > 3 \).

Suppose first \( \alpha \leq 1 \). Using (4.16), we see that
\[
|U_J|^{\alpha-1} |\nabla U_J| \leq C(-t)^{-\frac{\alpha}{J}} |U_0|^\alpha \leq (-t)^{-1+\frac{1}{J}}. \tag{4.28}
\]
Now if \( |v| \leq \frac{1}{2} |u| \), then \( |u + sv| \geq \frac{1}{2} |u| \) for all \( 0 \leq s \leq 1 \). Writing \( df(u + v) - df(u) = \int_0^1 \frac{dv}{dt} df(u + sv) \), it follows easily that
\[
|df(u + v) - df(u)| \leq C \left( \min_{0 \leq s \leq 1} |u + sv| \right)^{\alpha-1} |v| \leq C |u|^{\alpha-1} |v|, \quad \text{if } |v| \leq \frac{1}{2} |u|.
\]
It follows that
\[
\int_{|\epsilon| \leq \frac{1}{2} |U_J|} B_2 \leq C \int |U_J|^{\alpha-1} |\epsilon| |\nabla U_J| |\nabla \epsilon| \leq C(-t)^{-1+\frac{1}{J}} ||\epsilon||_{L^2} ||\nabla \epsilon||_{L^2}
\]
\[
\leq C(-t)^{-1+\frac{1}{J}+(2-\theta)\sigma},
\]
where we used (4.28) and (4.13). Moreover, using (2.8) and (4.28),
\[
\int_{|\epsilon| > \frac{1}{2} |U_J|} B_2 \leq C \int |\epsilon|^{\alpha-1} |\nabla U_J| |\nabla \epsilon| \leq C \int |\epsilon|^{\alpha-1} |\nabla U_J| |\epsilon| |\nabla \epsilon|
\]
\[
\leq C \int |U_J|^{\alpha-1} |\nabla U_J| |\epsilon| |\nabla \epsilon| \leq C(-t)^{-1+\frac{1}{J}+(2-\theta)\sigma}.
\]
Thus we see that
\[
\int B_2 \leq C(-t)^{-1+\frac{1}{J}+(2-\theta)\sigma}. \tag{4.29}
\]
When $\alpha > 1$, we deduce from (2.7), (4.28) and (4.13) that
\[
\int B_2 \leq M \int |U_j|^\alpha \nabla U_j |\nabla \varepsilon| + M \int |\nabla U_j| |\nabla \varepsilon| \\
\leq C(-t)^{-1 - \frac{1}{\theta} + (2 - \theta)\sigma} + M \int |\nabla U_j| |\nabla \varepsilon|.
\]

Suppose $1 < \alpha \leq 3$. By (1.2) we have $0 \leq N - (N - 2)\alpha < 2\alpha$, and by Gagliardo-Nirenberg’s inequality
\[
\|\varepsilon\|_{L^{2\alpha}} \leq C\|\nabla \varepsilon\|^{\frac{N(\alpha - 1)}{2\alpha}}_{L^2} \|\varepsilon\|^{\frac{N - (N - 2)\alpha}{2\alpha}}_{L^2}.
\]
Using (4.16) and (4.13),
\[
\int |\nabla U_j| |\nabla \varepsilon| \leq C(-t)^{-\frac{1}{\theta} - \frac{1}{\theta}} \|\varepsilon\|_{L^{2\alpha}} \|\nabla \varepsilon\|_{L^2} \leq C(-t)^{-\frac{1}{\theta} - \frac{1}{\theta} + (\alpha + 1 - \frac{N\alpha^2}{2(\alpha + 2)} - N\theta)\sigma} \\
\leq C(-t)^{-\frac{1}{\theta} - \frac{1}{\theta} + (\alpha + 1 - \frac{N\alpha^2}{2(\alpha + 2)} - N\theta)\sigma} = C(-t)^{-1 - \frac{1}{\theta} + (2 - \theta)\sigma + (\alpha + 1 - \frac{N\alpha^2}{2(\alpha + 2)} - N\theta)\sigma}.
\]
Using (4.1), we conclude that in this case
\[
\int B_2 \leq C(-t)^{-1 - \frac{1}{\theta} + (2 - \theta)\sigma}. \tag{4.30}
\]
If $\alpha > 3$, then by (4.16),
\[
M |\nabla U_j| |\nabla \varepsilon| \leq \frac{1}{4|\lambda|} |\nabla \varepsilon|^2 + C |\nabla U_j|^2 \\
\leq \frac{1}{4|\lambda|} |\nabla \varepsilon|^2 + C(-t)^{-\frac{1}{\theta} - \frac{1}{\theta}} |\nabla \varepsilon| |U_0|^2.
\]
Applying (4.15) and (3.9), we obtain
\[
\int |\nabla \varepsilon| |U_0|^2 \leq \|\nabla \varepsilon\|_{L^{2\alpha + 2}}^2 \|U_0\|_{L^{2\alpha + 2}}^2 \leq C(-t)^{-\frac{1}{\theta} + (\alpha - \frac{N\alpha^2}{2(\alpha + 2)} - N\theta)\sigma} \\
\leq C(-t)^{-1 + (\alpha - \frac{N\alpha^2}{2(\alpha + 2)} - N\theta)\sigma}.
\]

Since $\alpha \geq 3$ and $N \leq 3$,
\[
(\alpha - \frac{N\alpha^2}{2(\alpha + 2)} - \theta)\sigma = (2 - \theta)\sigma + (\alpha - 2 - \frac{N\alpha^2}{2(\alpha + 2)} - \theta)\sigma \\
\geq (2 - \theta)\sigma + (1 - 3\alpha^2\theta)\sigma
\]
Using (4.2), we deduce that
\[
\int |\nabla \varepsilon| |U_0|^2 \leq C(-t)^{-1 + (2 - \theta)\sigma};
\]
and so,
\[
M \int |\nabla U_j| |\nabla \varepsilon| \leq \frac{1}{4|\lambda|} \int |\nabla \varepsilon|^2 + C(-t)^{-1 - \frac{1}{\theta} + (2 - \theta)\sigma}.
\]
so that in this case
\[
\int B_2 \leq \frac{1}{4|\lambda|} \int |\nabla \varepsilon|^2 + C(-t)^{-1 - \frac{1}{\theta} + (2 - \theta)\sigma}. \tag{4.31}
\]
Estimates (4.26), (4.27), (4.29), (4.30) and (4.31) imply
\[
|A_2| \leq |\lambda|(8M|\lambda|)^{\alpha - 1} \|\nabla \varepsilon\|_{L^2}^2 + \frac{1}{2} \int |\nabla \varepsilon|^2 + C(-t)^{-1 - \frac{1}{\theta} + (2 - \theta)\sigma}.
\]
Using (4.4), we see that $-\frac{1}{\theta} + (2 - \theta)\sigma \geq 2(1 - \theta) + \frac{2\theta}{\alpha}$, hence
\[
|A_2| \leq |\lambda|(8M|\lambda|)^{\alpha - 1} \|\nabla \varepsilon\|_{L^2}^2 + \frac{1}{2} \int |\nabla \varepsilon|^2 + C(-t)^{-1 + 2(1 - \theta)\sigma + \frac{2\theta}{\alpha}}. \tag{4.32}
\]
Combining (4.21), (4.22), (4.24), (4.25) and (4.32), we obtain
\[
\frac{d}{dt} E(t) \geq \frac{1}{4} \int |\varepsilon|^\alpha |\nabla \varepsilon|^2 - \alpha |\lambda|^2|\nabla \varepsilon|^2 - C(-t)^{-1+2(1-\theta)\sigma + \frac{d}{2}}.
\]
Using (4.15) and (4.1), we deduce that
\[
\frac{d}{dt} \left( (-t)^{-\sigma} E(t) \right) = \sigma (-t)^{-\sigma} E(t) + (-t)^{-\sigma} \frac{d}{dt} E(t)
\geq \frac{1}{4} (-t)^{-\sigma} \int |\varepsilon|^\alpha |\nabla \varepsilon|^2 - C(-t)^{-1+2(1-\theta)\sigma + \frac{d}{2}}.
\]
(4.33)
It follows from (4.2) that \((1 - 2\theta)\sigma \geq 0\), so that the power of \(-t\) on the right-hand side of the above inequality are (strictly) larger than \(-1\). Integrating on \((t, T_0)\), using \(\varepsilon(T_0) = 0\), and multiplying by \((-t)^\sigma\), we obtain
\[
\frac{1}{4} (-t)^{-\sigma} \int_0^{T_0} (-s)^{-\sigma} \int |\varepsilon|^\alpha |\nabla \varepsilon|^2 + E(t) \leq C(-t)^{2(1-\theta)\sigma + \frac{d}{2}}.
\]
Using (4.15), we deduce that
\[
(-t)^\sigma \int_0^{T_0} \int |\varepsilon|^\alpha |\nabla \varepsilon|^2 + \|\nabla \varepsilon\|^2_{L^2} \leq C(-t)^{2(1-\theta)\sigma + \frac{d}{2}}.
\]
(4.34)
for all \(n \geq -\frac{1}{4}\) and all \(\tau_n < t \leq T_0\) such that \(t \geq \tilde{s}\).

We now conclude as follows. By (4.19) and (4.34) (and since \(2(1 - \theta)\sigma \geq \sigma\)), there exists \(S < 0\) such that for \(n\) sufficiently large (so that \(S < T_0\)),
\[
\|\varepsilon(t)\|_{L^2} \leq (-t)^\sigma, \quad \|\nabla \varepsilon\|^2_{L^2} \leq (-t)^{2(1-\theta)\sigma}, \quad \int_t^{T_0} \int |\varepsilon|^\alpha |\nabla \varepsilon|^2 \leq 1,
\]
(4.35)
for all \(\tau_n < t < T_0\) such that \(t \geq S\). By the definition (4.13) of \(\tau_n\), this implies \(\tau_n \leq S\). Using property (4.14), we conclude that \(s_n < S\) and that (4.9), (4.10) and (4.11) hold.

\[\Box\]

5. Proof of Theorem 1.1

Let \(K\) be any compact set of \(\mathbb{R}^N\) included in the ball of center 0 and radius 1 (by the scaling invariance of equation (1.1), this assumption does not restrict the generality). It is well-known that there exists a smooth function \(Z: \mathbb{R}^N \to [0, \infty)\) which vanishes exactly on \(K\) (see e.g. Lemma 1.4, page 20 of [9]). For \(\alpha\) satisfying (1.2), let \(\sigma, \theta, J, k\) be defined by (4.1), (4.2), (4.3) and (4.4). Define the function \(A: \mathbb{R}^N \to [0, \infty)\) by
\[
A(x) = (Z(x)\chi(|x|) + (1 - \chi(|x|))|x|)^k,
\]
(5.1)
where
\[
\begin{cases}
\chi \in C^\infty(\mathbb{R}, \mathbb{R}) \\
\chi(s) = \begin{cases}
1 & 0 \leq s \leq 1 \\
0 & s \geq 2
\end{cases}
\end{cases}
\]
\[
\chi'(s) \leq 0 \leq \chi(s) \leq 1, \quad s \geq 0.
\]
It follows that the function \(A\) satisfies (3.2) and vanishes exactly on \(K\).

We consider the solution \(u_n\) of equation (4.5), \(\varepsilon_n\) defined by (4.7), and \(n_0 \geq 1\) and \(S < 0\) given by Proposition 4.1. Using the estimate (2.6) and the embeddings \(H^1(\mathbb{R}^N) \hookrightarrow L^{n+2}(\mathbb{R}^N), L^\frac{n+2}{n}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N)\), we deduce from equation (4.12) that
\[
\|\partial_t \varepsilon_n\|_{H^{-1}} \lesssim \|\varepsilon_n\|_{H^1} + \|U_J\|_{H^1} \|\varepsilon_n\|_{H^1} + \|\varepsilon_n\|_{H^{1+\frac{d}{2}}} + \|\mathcal{E}_J\|_{L^2}
\]
so that, applying (4.9), (4.10), (3.27), (3.28), (3.10) and (3.9), there exists $\kappa > 0$ such that
\[ \| \partial_t \varepsilon_n \|_{H^{-1}} \leq C(-t)^{-\kappa}, \quad S \leq t \leq T_n. \] (5.2)

Given $\tau \in (S, 0)$, it follows from (4.9), (4.10) and (5.2) that $\varepsilon_n$ is bounded in $L^\infty((S, \tau), H^1(\mathbb{R}^N)) \cap W^{1, \infty}((S, \tau), H^{-1}(\mathbb{R}^N))$. Therefore, after possibly extracting a subsequence, there exists $\varepsilon \in L^\infty((S, \tau), H^1(\mathbb{R}^N)) \cap W^{1, \infty}((S, \tau), H^{-1}(\mathbb{R}^N))$ such that
\[ \varepsilon_n \xrightarrow{n \to \infty} \varepsilon \text{ in } L^\infty((S, \tau), H^1(\mathbb{R}^N)) \text{ weak}^*, \] (5.3)
\[ \partial_t \varepsilon_n \xrightarrow{n \to \infty} \partial_t \varepsilon \text{ in } L^\infty((S, \tau), H^{-1}(\mathbb{R}^N)) \text{ weak}^*, \] (5.4)
\[ \varepsilon_n(t) \xrightarrow{n \to \infty} \varepsilon(t) \text{ weakly in } H^1(\mathbb{R}^N), \text{ for all } S \leq t \leq \tau \] (5.5)

Since $\tau \in (S, 0)$ is arbitrary, a standard argument of diagonal extraction shows that there exists $\varepsilon \in L^\infty((S, 0), H^1(\mathbb{R}^N)) \cap W^{1, \infty}_{loc}((S, 0), H^{-1}(\mathbb{R}^N))$ such that (after extraction of a subsequence) (5.3), (5.4) and (5.5) hold for all $S < \tau < 0$. Moreover, (4.9), (4.10) and (5.5) imply that
\[ \| \varepsilon(t) \|_{L^2} \leq (-t)^{\sigma}, \quad \| \nabla \varepsilon(t) \|_{L^2} \leq (-t)^{(1-\theta)\sigma}, \] (5.6)
for $S \leq t < 0$, and (5.2) and (5.4) imply that
\[ \| \partial_t \varepsilon \|_{L^\infty((S, \tau), H^{-1})} \leq C(-\tau)^{-\kappa} \] (5.7)
for all $S < \tau < 0$. In addition, it follows easily from (4.12) and the convergence properties (5.3)–(5.5) that
\[ \partial_t \varepsilon = i \Delta \varepsilon + \lambda (f(U_j + \varepsilon) - f(U_j)) + \mathcal{E}_J \] (5.8)
in $L^\infty_{loc}((S, 0), H^{-1}(\mathbb{R}^N))$. Therefore, setting
\[ u(t) = U_j(t) + \varepsilon(t) \quad S \leq t \leq 0 \] (5.9)
we see that $u \in L^\infty_{loc}((S, 0), H^1(\mathbb{R}^N)) \cap W^{1, \infty}_{loc}((S, 0), H^{-1}(\mathbb{R}^N))$ and, using (3.25), that
\[ \partial_t u = i \Delta u + \lambda f(u) \] (5.10)
in $L^\infty_{loc}((S, 0), H^{-1}(\mathbb{R}^N))$. By local existence in $H^1(\mathbb{R}^N)$ and uniqueness in $L^\infty_t H^1_x$, we conclude that $u \in C((S, 0), H^1(\mathbb{R}^N)) \cap C^1((S, 0), H^{-1}(\mathbb{R}^N))$.

We now prove properties (i), (ii) and (iii). Let $\Omega$ be an open subset of $\mathbb{R}^N$ such that $\overline{\Omega} \cap K = \emptyset$. It follows from (5.1) that $A > 0$ on $\Omega$, and so there exists a constant $c > 0$ such that $A(x) \geq c(1 + |x|)^k$ on $\Omega$. Using (3.6), we deduce that $|U_0| \leq C(1 + |x|)^{-\frac{k}{2}}$ on $\Omega$. Since $(1 + |x|)^{-\frac{k}{2}} \in L^2(\mathbb{R}^N)$ by (4.4), we conclude, applying (3.28) and (3.10), that
\[ \limsup_{t \uparrow 0} \| U_j(t) \|_{H^1(\Omega)} < \infty. \] (5.11)

Property (iii) follows, using (5.6). Let now $x_0 \in K$ and $r > 0$, and set $\omega = \{|x - x_0| < r\}$. Let $p \geq 2$ satisfy $(N - 2)p \leq 2N$, so that $H^1(\omega) \hookrightarrow L^p(\omega)$. It follows from (3.29), (3.9) and (3.14) that
\[ (-t)^{-\frac{1}{2} + \frac{N}{2p}} \lesssim \| U_j(t) \|_{L^p(\omega)} \lesssim \| U_j(t) \|_{H^1(\mathbb{R}^N)} \lesssim (-t)^{-\frac{1}{2}}. \]

Using (5.6) and the embedding $H^1(\omega) \hookrightarrow L^p(\omega)$ we deduce that
\[ (-t)^{-\frac{1}{2} + \frac{N}{2p}} \lesssim \| u(t) \|_{L^p(\omega)} \lesssim \| u(t) \|_{H^1(\mathbb{R}^N)} \lesssim (-t)^{-\frac{1}{2}}. \] (5.12)

Property (i) follows by letting $p = 2$. Next, we prove that
\[ \lim_{t \uparrow 0} \| \nabla u(t) \|_{L^2(\mathbb{R}^N)} = \infty. \] (5.13)
If $N \geq 3$, this follows from (5.12) with $p = \frac{2N}{N-2}$ and Sobolev’s inequality
\[ \|u(t)\|_{L^{2N/(N-2)}(\mathbb{R}^N)} \lesssim \|\nabla u(t)\|_{L^2(\mathbb{R}^N)}. \]
If $N = 1$, we apply (5.12) with $p = \infty$ and obtain using Gagliardo-Nirenberg’s inequality
\[ (-t)^{-\frac{2}{N}} \lesssim \|u(t)\|_{L^{2N}(\mathbb{R})}^{\frac{2}{N}} \lesssim \|\nabla u(t)\|_{L^2(\mathbb{R})} \lesssim (-t)^{-\frac{1}{2}} \|\nabla u(t)\|_{L^2(\mathbb{R})}, \]
so that $\|\nabla u(t)\|_{L^2(\mathbb{R})} \gtrsim (-t)^{\frac{1}{2}}$. If $N = 2$, we apply (5.12) and Gagliardo-Nirenberg to obtain
\[ (-t)^{-\frac{1}{2} + \frac{N}{2p}} \lesssim \|u(t)\|_{L^p(\mathbb{R}^2)} \lesssim \|\nabla u(t)\|_{L^2(\mathbb{R}^2)}^{\frac{p}{2}} \|u(t)\|_{L^2(\mathbb{R}^2)}^{\frac{p-2}{2}} \lesssim (-t)^{-\frac{3}{2p}} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)}. \]
For $p > 2 + \frac{N_0}{2}$, we deduce that $\|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \gtrsim (-t)^{-\nu}$ with $\nu > 0$. This completes the proof of (5.13). Property (ii) is an immediate consequence of (5.13) and (1.6).

The proof of Theorem 1.1 is now complete.

**Remark 5.1.** As observed at the beginning of Section 3, the construction of the blow-up ansatz does not require any upper bound on the power $\alpha$. Theorem 1.1 is restricted to $H^1$-subcritical powers because the energy estimates of Section 4 only provide $H^1$ bounds. It is not too difficult to see that a similar result holds in the $H^1$-critical case $N \geq 3$ and $\alpha = \frac{1}{N-2}$. Indeed, in this case, the blow-up alternative is not that $\|u(t)\|_{H^1}$ blows up, but that certain Strichartz norms blow up, for instance $\|u\|_{L_t^\infty L_x^{2N/(N-2)}}$. Control of this norm is given by estimate (4.35) and the inequality
\[ \|u\|_{L_t^\infty L_x^{2N/(N-2)}} = \|u\|_{L_t^{2N/(N-2)} L_x^{2N/(N-2)}} \lesssim \|\nabla (|u|^{\frac{N-2}{2}})\|_{L_x^2} \lesssim \int |u|^3 |\nabla u|^2. \]
For $H^1$ supercritical powers, higher order estimates would be required. It is not unlikely that a result similar to Theorem 1.1 can be proved in the $H^2$-subcritical case $(N-4)\alpha < 4$, by establishing $H^2$ estimates through $L^2$ estimates of $\partial_t u$, in the spirit of [7].

**References**

[1] Cazenave T., Correia S., Dickstein F. and Weissler F.B.: A Fujita-type blowup result and low energy scattering for a nonlinear Schrödinger equation. São Paulo J. Math. Sci. 9 (2015), no. 2, 146–161. (MR3457455) (doi: 10.1007/s40863-015-0020-6)
[2] Cazenave T., Fang D. and Han Z.: Continuous dependence for NLS in fractional order spaces. Ann. Inst. H. Poincaré Anal. Non Linéaire 28 (2011), no. 1, 135–147. (MR2765515) (doi: 10.1016/j.anihpc.2010.11.005)
[3] Cazenave T., Martel Y. and Zhao L.: Finite-time blowup for a Schrödinger equation with nonlinear source term. Discrete Contin. Dynam. Systems 39 (2019), no. 2, 1171–1183. (doi: 10.3934/dcds.2019050)
[4] Cazenave T., Martel Y. and Zhao L.: Solutions blowing up on any given compact set for the energy subcritical wave equation. arXiv 1812.03949 (link: https://arxiv.org/abs/1812.03949)
[5] Constantine G. M. and Savits T. H.: A multivariate Faa di Bruno formula with applications. Trans. Amer. Math. Soc. 348 (1996), no.2, 503–520. (MR1325915) (doi: 10.1090/S0002-9947-96-01501-2)
[6] Kato T.: On nonlinear Schrödinger equations. Ann. Inst. H. Poincaré Phys. Théor. 46 (1987), no. 1, 113–129. (MR877998) (link: http://www.numdam.org/item?id=AIHPA_1987__46_1_113_0)
[7] Kato T.: Nonlinear Schrödinger equations, in *Schrödinger Operators* (Sønderborg, 1988). Lecture Notes in Phys. 345, Springer, Berlin, 1989, 218–263. (MR1037322) (doi: 10.1007/3-540-51783-9_22)
[8] Kawakami S. and Machihara S.: Blowup solutions for the nonlinear Schrödinger equation with complex coefficient. Preprint, 2019. arXiv:1905.13037 (link: https://arxiv.org/abs/1905.13037)
[9] Moerdijk I. and Reyes G.: *Models for smooth infinitesimal analysis*. Springer-Verlag, New York, 1991. (MR1083355) (doi: 10.1007/978-1-4757-4143-8)

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