New Topological Gauss-Bonnet Black Holes in Five Dimensions

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We investigate vacuum static black hole solutions of Einstein-Gauss-Bonnet gravity with a negative cosmological constant in five dimensions. These are solutions with horizons of nontrivial topologies. The first one possesses a horizon with the topology \( S^1 \times H^2 \), and a varying Gauss-Bonnet coupling constant \( \alpha \). By looking into its thermodynamic properties, we find that its specific heat capacity with fixed volume is negative, therefore it is thermodynamically unstable. The second one is equipped with a so-called “Sol-manifold” as its horizon, and interestingly, the product of the Gauss-Bonnet coupling constant \( \alpha \) and the cosmological constant \( \Lambda \) is fixed. For the second solution, the total energy and entropy vanish. These results enlarge our knowledge of both topological black holes in higher dimensions and the property of higher curvature corrections of gravitational theories.

I. INTRODUCTION

Black holes are simple and meanwhile complicated objects in nature. The smooth event horizon of a stationary black hole can only have spherical topology, therefore they are simple. However, this simplicity is ensured only when the spacetime dimension is 4 and the dominant energy condition (DEC) is satisfied — this is the topology theorem by Hawking in 1972 [1]. In the year 1994 Chruściel and Wald [2] proposed a topology theorem without the condition of smoothness. Black hole horizons can be complicated while those two conditions are violated, i.e. one can consider higher dimensions or break DEC.

In 5 dimensions, there are not only the famous “Myers-Perry” black holes [3] with spherical topology \( S^3 \), also “black rings” found by Emparan and Reall [4] with topology \( S^1 \times S^2 \). Actually, these two topologies are the only possibilities for 5 dimensional asymptotically flat stationary blackhole horizons according to the generalization [5] to higher dimensions of Hawking’s topology theorem. In dimensions \( D > 5 \) there are much more complicated black hole horizons, including \( S^{D-2} \) and \( S^1 \times S^{D-3} \) topologies [6, 7].

DEC can be broken by introducing the negative cosmological constant \( \Lambda \). The asymptotically anti-de Sitter (AdS, the maximally symmetric space with negative curvature) black holes are solutions with negative \( \Lambda \) and they can have horizons of 3 types: sphere with positive curvature, torus with flat geometry and hyperbolic space with negative curvature [8–10]. These 3 types appear in arbitrary dimensions. If one considers generic black holes that are not asymptotically AdS, with negative \( \Lambda \), there are even black hole horizons with arbitrary genus \( g > 1 \) [11]. These are spacetimes locally equal to pure AdS spacetimes. Moreover, some static, plane symmetric solutions and cylindrically symmetric solutions of Einstein-Maxwell equations with a negative cosmological constant are investigated in the paper [12] by Rong-Gen Cai and Yuan-Zhong Zhang.

It is interesting to explore the possibility of certain nontrivial horizon types in the presence of a negative \( \Lambda \). If certain types were proven to be forbidden, this might help establishing new topology theorems; if certain types turned out to exist, these would be novel discoveries worth noticing. There is an important example, Ref. [13], which obtained nontrivial black holes in 5 dimensions. These black holes have 3-dimensional horizons of different types. These types all belong to the eight-dimensional “model-geometries” classified by Thurston [14]. They are: Euclidean \( E^3 \), spherical \( S^3 \), hyperbolic \( H^3 \), \( S^1 \times S^2 \), \( S^1 \times H^2 \) and the so-called “Sol”, “Nil” and “\( SL_2 R \)” geometries. They all admit homogeneous metrics. For more details of these eight manifolds one can refer to [13–15]. These manifolds should be admit if one uses them to construct black hole horizons, and the topologies and the compactification of these eight manifolds are also described in [13]. Its main results are Einsteinian solutions of black holes with “Sol” and “Nil” horizons1. For recent developments of this

What do we do is to explore the effect of higher curvature corrections on horizon topology. To be exact, we consider 5-dimensional Einstein-Gauss-Bonnet (EGB for short) gravity theory, a special case of the Lovelock gravity theory [16], the general second-order covariant gravity theory in dimensions higher than four, with a negative \( \Lambda \). Moreover, the Gauss-Bonnet term can be regarded as corrections from the heterotic string theory [17, 18]. There had already been spherical, Euclidean and hyperbolic black hole horizons in this theory, shown by the famous paper [19] by Rong-Gen Cai and Yuan-Zhong Zhang.

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1 According to [13], there had already been black hole horizons for the first five geometries except \( S^1 \times S^2 \), while “\( SL_2 R \)” horizons are still unknown.
Gen Cai. What we found are a black hole with a \( S^1 \times H^2 \) horizon and one with a Sol-manifold horizon. These are static vacuum black holes. We then discuss the thermodynamic properties of these solutions, and interestingly, the Sol-manifold solution has zero entropy and mass. This may be caused by the fact that in the Sol-manifold solution the Gauss-Bonnet coupling constant has a fixed value. We make some discussions at the end of this paper. We hope our results may help further understanding black holes in higher curvature gravity theories and help exploring possible topology theorems in generic cases.

II. THE TOPOLOGICAL BLACK HOLE SOLUTIONS

The action of EGB gravity theory with a negative cosmological constant

\[
S = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{-g} \left[ R - 2\Lambda + \alpha (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) \right],
\]

where \( G \) is the Newton constant and the cosmological constant \( \Lambda \), the Gauss-Bonnet coupling denoted by \( \alpha \). The equations of motion are

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} - \alpha \left( 4R_{\mu\rho}R^\rho_\nu - 2RR_{\mu\nu} + 4R^{\rho\sigma}R_{\mu\rho\nu\sigma} - 2R_{\mu}^{\rho\sigma\tau}R_{\rho\sigma\tau\nu} + \frac{1}{2} g_{\mu\nu}(R^2 - 4R_{\rho\sigma}R^{\rho\sigma} + R_{\rho\sigma\tau\pi}R^{\rho\sigma\tau\pi}) \right) = 0.
\]

A. The first solution with a constant curvature subspace

As we will see, the horizon topology of the first black hole solution is \( S^1 \times H^2 \). The metric of the first solution is assumed to be a warped product of a 3-dimensional static black hole \( BH_3 \) and a 2-dimensional hyperbolic space \( \Sigma_2 \) with constant negative curvature:

\[
ds^2 = -V(r)dt^2 + \frac{1}{V(r)}dr^2 + r^2d\xi^2 + ad\Sigma_2^2
\]

with \( V(r) \) an unknown function of coordinate \( r \), \( a \) a positive constant, and \( d\Sigma_2^2 \) the line element of the 2-d hyperbolic space,

\[
d\Sigma_2^2 = d\theta^2 + \sinh^2\theta d\phi^2.
\]

The new black hole solution in 5 dimensions has

\[
V(r) = -\frac{\Lambda r^2}{3} - M, \quad a = \frac{3}{2\Lambda} - 2\alpha,
\]

where \( \alpha \) is the Gauss-Bonnet parameter, \( \Lambda \) is the negative cosmological constant. The constant \( M \) is related to the total energy of the black hole, and is related to the horizon radius \( r_h \) by the relation

\[
M = -\frac{\Lambda r_h^2}{3}.
\]

The solution above with \( \alpha \to 0 \) gives the Eq.(II.18) in the paper [13].

B. The second solution with nontrivial horizon topology

The horizon of the second black hole solution is the so-called Sol-manifold[13–15]. The Sol-manifold is described by

\[
ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,
\]

The ansatz metric of the whole spacetime is

\[
ds^2 = -V(r)dt^2 + \frac{1}{V(r)}dr^2 + f(r)e^{2z}dx^2 + g(r)e^{-2z}dy^2 + h(r)dz^2
\]
which gives the metric on a constant \{t, r\} surface up to some constant. Here \(V(r), f(r), g(r)\) and \(h(r)\) are functions of \(r\) to be determined.

After putting the ansatz into the equations of motion, we obtain a solution as follows:

\[
V(r) = -\frac{\Lambda r^2}{3} - M, \quad g(r) = r^2, \quad h(r) = \frac{r^2}{M}. \tag{2.9}
\]

Here the coupling constant \(\alpha\) is fixed to satisfy

\[
\alpha = -\frac{3}{4\Lambda} \tag{2.10}
\]

and \(M\) is an integration constant and \(f(r)\) an unfixed function\(^2\) even when the equations of motion are satisfied. The horizon radius \(r_h\) also satisfies the equation (2.6).

This solution is quite different from the Sol-manifold solution of Einstein gravity in \([13]\). Interestingly, when \(f(r)\) and \(g(r)\) are both set to be \(r^2\), the horizon manifold can be arbitrary as shown in the paper \([20]\).

### III. THERMODYNAMICS OF THE NONTRIVIAL SOLUTIONS

In this section we study the thermodynamics of the black hole solutions given above.

#### A. The first solution

For the first solution (2.5), we can identify the cosmological constant with the thermodynamic pressure

\[
P = -\frac{\Lambda}{8\pi G}. \tag{2.11}
\]

This identification had been applied to explore the extended phase space of asymptotically AdS black holes\([21]\).

The temperature can be obtained by the semi-classical method of removing the conical singularity of the near-horizon geometry

\[
T = \frac{V'(r)}{4\pi} \bigg|_{r \to r_h} = -\frac{\Lambda r_h}{6\pi} = \frac{4GP r_h}{3}, \tag{3.1}
\]

and the entropy can be obtained by applying the Wald entropy formula \([22, 23]\)

\[
S = -2\pi \int_{\text{horizon}} \sqrt{\hat{g}} \, d^{d-1}x \left( \frac{\partial L}{\partial R_{abcd}} \right) \epsilon_{ab} \epsilon_{cd} = -\frac{3r_h \Omega(1 + 4\alpha \Lambda)}{8G} = \frac{3r_h \Omega(1 - 32\pi \alpha G P)}{64\pi G^2 P}. \tag{3.2}
\]

where all the hatted quantities are intrinsic quantities on the horizon cross section on which the integral is defined and \(\epsilon_{ab}\) is the natural volume element on the tangent space orthogonal to the cross section. The constant \(\Omega\) is the volume of the hyperbolic subspace.

If we consider the first law of thermodynamics \(dE = TdS + \Phi dP\) without including \(\Lambda\) or \(P\) as a variable, we will obtain the expression for the total energy

\[
E = \frac{r_h^2 \Omega(1 + 4\alpha \Lambda)}{32\pi G}. \tag{3.3}
\]

However, when \(\Lambda\) or \(P\) is a thermodynamic variable, this quantity should be interpreted as the total enthalpy

\[
H = \frac{r_h^2 \Omega(1 + 4\alpha \Lambda)}{32\pi G} = \frac{3r_h \Omega(1 - 32\pi \alpha G P)}{64\pi G^2 P}. \tag{3.4}
\]

After rewriting \(H(r_h, P)\) as the function of \(S\) and \(P\), we arrive at

\[
H(S, P) = \frac{128\pi G^3 P^2 S^2}{9\Omega - 288\pi \alpha G P \Omega}, \tag{3.5}
\]

and the first law of thermodynamics becomes

\[
dH = TdS + VdP \tag{3.6}
\]

\(^2\) One can also set \(f(r) = r^2\), and leave \(g(r)\) undetermined. These are the same solutions with different signs of \(z\).
with $V$ being the thermodynamic volume
\[ V = \frac{256\pi G^3 P S^2 (1 - 16\pi \alpha GP)}{9\Omega (1 - 32\pi \alpha GP)^2} = \frac{1}{16} r_h^2 \Omega \left( \frac{1}{\pi GP} - 16\alpha \right) = \frac{9T^2 \Omega (1 - 16\pi \alpha GP)}{256\pi G^3 P^3}. \] (3.7)

The last equation above gives the equation of state of the black hole. One can see that the derivative of $V$ with respect to $P$ is always negative
\[ \left( \frac{\partial V}{\partial P} \right)_T = \frac{9T^2 \Omega (32\pi \alpha GP - 3)}{256\pi G^3 P^3} < 0 \] (3.8)
for all $T$, since $1 - 32\pi \alpha GP > 0$ must be satisfied in Eq.(3.2).

The specific heat capacity $C_p$ with fixed pressure is
\[ C_p = T \left( \frac{\partial S}{\partial T} \right)_P = \frac{9T \Omega (1 - 32\pi \alpha GP)}{256\pi G^3 P^2} > 0 \] (3.9)
and the specific heat capacity $C_v$ with volume fixed is
\[ C_v = C_p - T \left( \frac{\partial P}{\partial T} \right)_V \left( \frac{\partial V}{\partial T} \right)_P = \frac{9T \Omega}{256\pi G^3 P^2 (32\pi \alpha GP - 3)} < 0 \] (3.10)
since $1 - 32\pi \alpha GP > 0$. So this system is thermodynamically unstable in the sense of specific heat capacity with fixed volume, $C_v$.

**B. The second solution**

After applying the Wald entropy (3.2) formula to the second solution (2.9), we found that the entropy vanishes, since the integrand
\[ \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab} \epsilon_{cd} \propto \frac{(3M + \Lambda r^2) f'(r)}{2\Lambda f(r)} \] (3.11)
vanishes on the horizon, i.e. at the point $r = r_h$. According to the first law of thermodynamics $dE = TdS$, the total energy\(^3\) of this solution also vanishes as the entropy,
\[ S = 0, \quad E = 0. \] (3.12)
This implies that the integration constant $M$ in this solution does not stand for the total energy or mass.

**IV. CONCLUSION AND DISCUSSION**

In this paper we present two novel 5-dimensional black hole solutions in Einstein-Gauss-Bonnet theory with a negative cosmological constant. They are simple vacuum static black holes with horizons with nontrivial horizons. Their horizons corresponds to two of the 8 types of 3-dimensional “Thurston model geometries” in the literature[14, 15].

The horizon topology of the first black hole solution is $S^1 \times H^2$, and it allows an extended phase interpretation of thermodynamics by including the cosmological constant as the thermodynamic pressure. We derive the heat capacities of this solution and find that it is thermodynamically unstable. This solution has the form of the so-called “warped product”, the perturbations of which can be studied in the formalism provided by the paper [24] by Rong-Gen Cai and Li-Ming Cao. In this formalism, the hyperbolicity and causality of this solution can be investigated in the same way as the paper [25] by Li-Ming Cao and Liang-Bi Wu. In that paper they analyzed various spacetimes, including dynamical spacetimes.

The second black hole has the “Sol-manifold” as its horizon. The Gauss-Bonnet coupling constant $\alpha$ of this solution is fixed as $\alpha = -3/(4\Lambda)$. Curiously, there is an unfixed function $f(r)$ in the metric in Eq.(2.8). It seems that there may be redundant degrees of freedom in this solution. Moreover, by applying the Wald entropy formula we find that

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\(^3\) For the second solution we do not address the extended phase space, so we only worry about the total energy instead of the enthalpy.
its entropy vanishes, so does its total energy due to the first law of thermodynamics. This phenomenon is thought-provoking. In the paper [26] by Rong-Gen Cai, Li-Ming Cao and Nobuyoshi Ohta, some black hole solutions with zero mass and entropy had been presented in the context of Lovelock gravity theory. The authors fix the coupling constant to such a critical value that the kinetic fluctuations around the background spacetime vanish, so there are no excitations of the background. Since the entropy is related to the quantum degrees of freedom of the black hole, if the fluctuations are forbidden, the entropy is expected to vanish. This is why the entropy vanishes in their case. This is closely related to the fact that the effective gravitational constant $G_{\text{eff}} \to \infty$ [27]. Our case is similar: the coupling constant $\alpha$ is fixed, and the entropy is zero. However, our coefficient choice is different from that paper. This is an intriguing fact, implying that there might be more than one critical value. Our coefficient choice (2.10) is the same as the paper on dimensionally continued gravity[28] and the paper [29] by Zhong-Ying Fan, Bin Chen and Hong Lü, where they found the critical value of the coupling constant actually forbids kinetic fluctuations around the AdS backgrounds. The connection of the entropy and the kinetic fluctuations is quite interesting and deserves further investigation.

Among the 8 model geometries[14, 15] in 3 dimensions, the black holes with $S^3$, $H^3$ and $R^3$ horizons already exist in both Einstein and Gauss-Bonnet gravity. However, except the $S^1 \times H^2$ and the Sol-geometry horizons, we have not found static black holes with $S^1 \times S^2$, Nil-geometry or $SL_2R$ horizons in Gauss-Bonnet gravity yet, although such black holes (except $SL_2R$) have been found in Einstein gravity. To try to find or to rule out these solutions are important future directions. To generalize our results to charged cases is also interesting.

Another interesting direction is about the connection between quantum information and gravity. The concept quantum complexity denotes the computation cost of reaching a certain quantum state, its holographic dual [30–33] as well as its precise definition (still unclear) on the quantum theory side had received much attention in recent years[34–57]. The late-time growth rate of complexity $C$ is proportional to the product of the temperature and entropy $\dot{C} = TS$. It will be interesting to look into the solutions with vanishing entropy and see the behavior of the complexity.

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