On one generalization of finite nilpotent groups

Zhang Chi *
Department of Mathematics, University of Science and Technology of China,
Hefei 230026, P. R. China
E-mail: zcqxj32@mail.ustc.edu.cn

Alexander N. Skiba
Department of Mathematics and Technologies of Programming, Francisk Skorina Gomel State University,
Gomel 246019, Belarus
E-mail: alexander.skiba49@gmail.com

Abstract

Let \( \sigma = \{ \sigma_i | i \in I \} \) be a partition of the set \( \mathbb{P} \) of all primes and \( G \) a finite group. A chief factor \( H/K \) of \( G \) is said to be \( \sigma \)-central if the semidirect product \( (H/K) \rtimes (G/C_G(H/K)) \) is a \( \sigma_i \)-group for some \( i = i(H/K) \). \( G \) is called \( \sigma \)-nilpotent if every chief factor of \( G \) is \( \sigma \)-central. We say that \( G \) is semi-\( \sigma \)-nilpotent (respectively weakly semi-\( \sigma \)-nilpotent) if the normalizer \( N_G(A) \) of every non-normal (respectively every non-subnormal) \( \sigma \)-nilpotent subgroup \( A \) of \( G \) is \( \sigma \)-nilpotent.

In this paper we determine the structure of finite semi-\( \sigma \)-nilpotent and weakly semi-\( \sigma \)-nilpotent groups.

1 Introduction

Throughout this paper, all groups are finite and \( G \) always denotes a finite group. Moreover, \( \mathbb{P} \) is the set of all primes, \( \pi \subseteq \mathbb{P} \) and \( \pi' = \mathbb{P} \setminus \pi \). If \( n \) is an integer, the symbol \( \pi(n) \) denotes the set of all primes dividing \( n \); as usual, \( \pi(G) = \pi(|G|) \), the set of all primes dividing the order of \( G \).

In what follows, \( \sigma = \{ \sigma_i | i \in I \} \) is some partition of \( \mathbb{P} \), that is, \( \mathbb{P} = \bigcup_{i \in I} \sigma_i \) and \( \sigma_i \cap \sigma_j = \emptyset \) for all \( i \neq j \). By the analogy with the notation \( \pi(n) \), we write \( \sigma(n) \) to denote the set \( \{ \sigma_i | \sigma_i \cap \pi(n) \neq \emptyset \} \); \( \sigma(G) = \sigma(|G|) \). A group is said to be \( \sigma \)-primary if it is a \( \sigma_i \)-group for some \( i \).

A chief factor \( H/K \) of \( G \) is said to be \( \sigma \)-central (in \( G \)) if the semidirect product \( (H/K) \rtimes (G/C_G(H/K)) \) is \( \sigma \)-primary. The normal subgroup \( E \) of \( G \) is called \( \sigma \)-hypercentral in \( G \) if either \( E = 1 \) or every chief factor of \( G \) below \( E \) is \( \sigma \)-central.

Recall also that \( G \) is called \( \sigma \)-nilpotent if every chief factor of \( G \) is \( \sigma \)-central.

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An arbitrary group $G$ has two canonical $\sigma$-nilpotent subgroups of particular importance in this context. The first of these is the $\sigma$-Fitting subgroup $F_\sigma(G)$, that is, the product of all normal $\sigma$-nilpotent subgroups of $G$. The other useful subgroup is the $\sigma$-hypercentre $Z_\sigma(G)$ of $G$, that is, the product of all $\sigma$-hypercentral subgroups of $G$.

Note that in the classical case, when $\sigma = \sigma^1 = \{\{2\}, \{3\}, \ldots\}$ (we use here the notation in [2]), $F_\sigma(G) = F(G)$ is the Fitting subgroup and $Z_\sigma(G) = Z_\infty(G)$ is the hypercentre of $G$.

In fact, the $\sigma$-nilpotent groups are exactly the groups $G$ which can be written in the form $G = G_1 \times \cdots \times G_t$ for some $\sigma$-primary groups $G_1, \ldots, G_t$ [1], and such groups have proved to be very useful in the formation theory (see, in particular, the papers [3, 4] and the books [5, Ch. IV], [6, Ch. 6]). In the recent years, the $\sigma$-nilpotent groups have found new and to some extent unexpected applications in the theories of permutable and generalized subnormal subgroups (see, in particular, [1, 2], [7]–[18] and the survey [19]).

In view of the results in the paper [20], the $\sigma$-nilpotent groups can be characterized as the groups in which the normalizer of any $\sigma$-nilpotent subgroup is $\sigma$-nilpotent. Groups in which normalizers of all non-normal $\sigma$-nilpotent subgroups are $\sigma$-nilpotent may be non-$\sigma$-nilpotent (see Example 1.3 below), and in the case when $\sigma = \sigma^1$ such groups have been described in [21, Ch. 4, Section 7] (see also [22]). In this paper, we determine the structure of such groups $G$ for the case arbitrary $\sigma$.

**Definition 1.1.** We say that $G$ is (i) **semi-$\sigma$-nilpotent** if the normalizer of every non-normal $\sigma$-nilpotent subgroup of $G$ is $\sigma$-nilpotent;

(ii) **weakly semi-$\sigma$-nilpotent** if the normalizer of every non-subnormal $\sigma$-nilpotent subgroup of $G$ is $\sigma$-nilpotent;

(iii) **weakly semi-nilpotent** if $G$ is weakly semi-$\sigma^1$-nilpotent.

**Remark 1.2.** (i) Every $\sigma$-nilpotent group is semi-$\sigma$-nilpotent, and every semi-$\sigma$-nilpotent group is weakly semi-$\sigma$-nilpotent.

(ii) The semi-$\sigma^1$-nilpotent groups are exactly the **semi-nilpotent groups** studied in [21, Ch. 4, Section 7] (see also [22]).

(iii) We show that $G$ is (weakly) semi-$\sigma$-nilpotent if and only if the normalizer of every non-normal (respectively non-subnormal) $\sigma$-primary subgroup of $G$ is $\sigma$-nilpotent. Since every $\sigma$-primary group is $\sigma$-nilpotent, it is enough to show that if the normalizer of every non-normal (respectively non-subnormal) $\sigma$-primary subgroup $A$ of $G$ is $\sigma$-nilpotent, then $G$ is $\sigma$-semi-nilpotent (respectively weakly semi-$\sigma$-nilpotent). First note that $A \neq 1$ and $A = A_1 \times \cdots \times A_n$, where $\{A_1, \ldots, A_n\}$ is a complete Hall $\sigma$-set of $A$. The subgroups $A_i$ are characteristic in $A$, so $N_G(A) = N_G(A_1) \cap \cdots \cap N_G(A_n)$, where either $N_G(A_n) = G$ or $N_G(A_n)$ is $\sigma$-nilpotent. Since $A$ is non-normal (respectively non-subnormal) in $G$, there is $i$ such that $N_G(A_i)$ is $\sigma$-nilpotent. Therefore $N_G(A)$ is $\sigma$-nilpotent by Lemma 2.2(i) below. Hence $G$ is semi-$\sigma$-nilpotent (respectively weakly semi-$\sigma$-nilpotent).

**Example 1.3.** Let $p > q > r > t > 2$ be primes, where $q$ divides $p - 1$ and $t$ divides $r - 1$, and
let $\sigma = \{\{p\}, \{q\}, \{p,q\}\}$. Let $R$ be the quaternion group of order 8, $A$ a group of order $p$, and let $B = C_p \times C_q$ be a non-nilpotent group of order $pq$ and $C$ a non-nilpotent group of order $rt$. Then $B \times R$ is a non-$\sigma$-nilpotent semi-$\sigma$-nilpotent group and $B \times C$ is not semi-$\sigma$-nilpotent.

Now let $G = A \times (Q \times R)$, where $Q$ is a simple $\mathbb{F}_q R$-module which is faithful for $R$. Then for every subgroup $V$ of $R$ we have $N_G(V) = A \times Q$, so $G$ is weakly semi-$\sigma$-nilpotent. On the other hand, $QV$ is supersoluble for every subgroup $V$ of $R$ of order 2 and so for some subgroup $L$ of $Q$ with $1 < L < Q$ we have $V \leq N_G(L)$ and $[L,V] \neq 1$. Hence $G$ is not semi-$\sigma$-nilpotent.

Recall that $G^{\sigma_0}$ is the $\sigma$-nilpotent residual of $G$, that is, the intersection of all normal subgroups $N$ of $G$ with $\sigma$-nilpotent quotient $G/N$.

Our goal here is to determine the structure of weakly semi-$\sigma$-nilpotent and semi-$\sigma$-nilpotent groups. In fact, the following concept is an important tool to achieve such a goal.

**Definition 1.4.** Let $H$ be a $\sigma$-nilpotent subgroup of $G$. Then we say that $H$ is $\sigma$-Carter subgroup of $G$ if $H$ is an $\mathfrak{S}_\sigma$-covering subgroup of $G$ [6, p. 101], that is, $U_{\mathfrak{S}_\sigma} H = U$ for every subgroup $U$ of $G$ containing $H$.

Note that in Example 1.3, the subgroup $C_q C$ is a $\sigma$-Carter subgroup of the group $B \times C$. It is clear also that a group $H$ of a soluble group $G$ is a Carter subgroup of $G$ if and only if it is a $\sigma^1$-Carter subgroup of $G$.

A complete set of Sylow subgroups of $G$ contains exactly one Sylow $p$-subgroup for each prime $p$ dividing $|G|$. In general, we say that a set $\mathcal{H}$ of subgroups of $G$ is a complete Hall $\sigma$-set of $G$ if every member $H$ of $\mathcal{H}$ is a $\sigma_i$-Carter subgroup of $G$ for some $i$ and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $\sigma_i \in \sigma(G)$.

Our first result is the following

**Theorem A.** If $G$ is weakly semi-$\sigma$-nilpotent, then:

(i) $G$ has a complete Hall $\sigma$-set $\{H_1, \ldots, H_t\}$ such that for some $1 \leq r \leq t$ the subgroups $H_1, \ldots, H_r$ are normal in $G$, $H_i$ is not normal in $G$ for all $i > r$, and

\[
\langle H_{r+1}, \ldots, H_t \rangle = H_{r+1} \times \cdots \times H_t.
\]

(ii) If $G$ is not $\sigma$-nilpotent, then $N_G(H_i)$ is a $\sigma$-Carter subgroup of $G$ for all $i > r$.

(iii) $F_{\sigma}(G)$ is a maximal $\sigma$-nilpotent subgroup of $G$ and $F_{\sigma}(G) = F_{0\sigma}(G)Z_{\sigma}(G)$, where $F_{0\sigma}(G) = H_1 \cdots H_r$.

(iv) $V_G = Z_{\sigma}(G)$ for every maximal $\sigma$-nilpotent subgroup $V$ of $G$ such that $G = F_{\sigma}(G)V$.

(v) $G/F(G)$ is $\sigma$-nilpotent.

On the basis of Theorem A we prove also the following

**Theorem B.** Suppose that $G$ is semi-$\sigma$-nilpotent, and let $\{H_1, \ldots, H_t\}$ be a complete Hall $\sigma$-set
of $G$, where $H_1, \ldots, H_r$ are normal in $G$ and $H_i$ is not normal in $G$ for all $i > r$. Suppose also that non-normal Sylow subgroups of any Schmidt subgroup $A \leq H_i$ have prime order for all $i > r$. Then:

(i) $G/F_\sigma(G)$ is abelian.

(ii) If $U$ is any maximal $\sigma$-nilpotent non-normal subgroup of $G$, then $U$ is a $\sigma$-Carter subgroup of $G$ and $U_G = Z_\sigma(G)$.

(iii) If the subgroups $H_1, \ldots, H_r$ are nilpotent, then $G/F_\sigma(G)$ is cyclic.

(iv) Every quotient and every subgroup of $G$ are semi-$\sigma$-nilpotent.

Now we consider some of corollaries of Theorems A and B in the three classical cases. First of all note that in the case when $\sigma = \sigma^1$, Theorems A and B not only cover the main results in [21, Ch. 5 Section 7] but they also give the alternative proofs of them. Moreover, in this case we get from the theorems the following results.

**Corollary 1.4.** If $G$ is weakly semi-nilpotent, then:

(i) $G$ has a complete set of Sylow subgroups $\{P_1, \ldots, P_t\}$ such that for some $1 \leq r \leq t$ the subgroups $P_1, \ldots, P_r$ are normal in $G$, $P_i$ is not normal in $G$ for all $i > r$, and $(P_{r+1}, \ldots, P_t) = P_{r+1} \times \cdots \times P_t$.

(ii) $F(G)$ is a maximal nilpotent subgroup of $G$ and $F(G) = F_{0\sigma}(G)Z_\infty(G)$, where $F_{0\sigma}(G) = P_1 \cdots P_r$.

(iii) If $G$ is not nilpotent, then $N_G(P_i)$ is a Carter subgroup of $G$ for all $i > r$.

**Corollary 1.5 (See Theorem 7.6 in [21, Ch. 4]).** If $G$ is semi-nilpotent and $F_0(G)$ denotes the product of its normal Sylow subgroups, then $G/F_0(G)$ is nilpotent.

**Corollary 1.6 (See Theorem 7.8 in [21, Ch. 4]).** If $G$ is semi-nilpotent, then:

(a) $F(G)$ is a maximal nilpotent subgroup of $G$.

(b) If $U$ is a maximal nilpotent subgroup of $G$ and $U$ is not normal in $G$, then $U_G = Z_\infty(G)$.

**Corollary 1.7 (See Theorem 7.10 in [21, Ch. 4]).** The class of all semi-nilpotent groups is closed under taking subgroups and homomorphic images.

In the other classical case when $\sigma = \sigma^\pi = \{\pi, \pi'\}$, $G$ is $\sigma^\pi$-nilpotent if and only if $G$ is $\pi$-decomposable, that is, $G = O_\pi(G) \times O_{\pi'}(G)$.

Thus $G$ is semi-$\sigma^\pi$-nilpotent if and only if the normalizer of every $\pi$-decomposable non-normal subgroup of $G$ is $\pi$-decomposable; $G$ is weakly semi-$\sigma^\pi$-nilpotent if and only if the normalizer of every $\pi$-decomposable non-subnormal subgroup of $G$ is $\pi$-decomposable. Therefore in this case we get from Theorems A and B the following results.

**Corollary 1.8.** Suppose that $G$ is not $\pi$-decomposable. If the normalizer of every $\pi$-decomposable non-subnormal subgroup of $G$ is $\pi$-decomposable, then:

(i) $G$ has a Hall $\pi$-subgroup $H_1$ and a Hall $\pi'$-subgroup $H_2$, and exactly one of these subgroups,
$H_1$ say, is normal in $G$.

(ii) $G/F(G)$ is $\pi$-decomposable.

(iii) $N_G(H_2)$ is an $\mathfrak{F}$-covering subgroup of $G$, where $\mathfrak{F}$ is the class of all $\pi$-decomposable groups.

(iv) $O_\pi(G) \times O_{\pi'}(G) = H_1 \times O_{\pi'}(G)$ is a maximal $\pi$-decomposable subgroup of $G$ and every element of $G$ induces a $\pi'$-automorphism on every chief factor of $G$ below $O_{\pi'}(G)$.

**Corollary 1.9.** Suppose that $G$ is not $\pi'$-closed and the normalizer of every $\pi$-decomposable non-normal subgroup of $G$ is $\pi$-decomposable. Then $G = H_1 \times H_2$, where $H_1$ is a Hall $\pi$-subgroup and $H_2$ is a Hall $\pi'$-subgroup of $G$. Moreover, if non-normal Sylow subgroups of any Schmidt subgroup $A \leq H_2$ have prime order, then:

(i) $G/O_\pi(G) \times O_{\pi'}(G)$ is abelian.

(ii) Every maximal $\pi$-decomposable non-normal subgroup of $G$ is an $\mathfrak{F}$-covering subgroup of $G$, where $\mathfrak{F}$ is the class of all $\pi$-decomposable groups.

(iii) If $H_1$ is nilpotent, then $G/O_\pi(G) \times O_{\pi'}(G)$ is cyclic.

In fact, in the theory of $\pi$-soluble groups ($\pi = \{p_1, \ldots, p_n\}$) we deal with the partition $\sigma = \sigma^{1_\pi} = \{\{p_1\}, \ldots, \{p_n\}, \pi'\}$. Moreover, $G$ is $\sigma^{1_\pi}$-nilpotent if and only if $G$ is $\pi$-special \[23\], that is, $G = O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\pi'}(G)$.

Thus $G$ is semi-$\sigma^{1_\pi}$-nilpotent if and only if the normalizer of every $\pi$-special non-normal subgroup of $G$ is $\pi$-special; $G$ is weakly semi-$\sigma^{1_\pi}$-nilpotent if and only if the normalizer of every $\pi$-special non-subnormal subgroup of $G$ is $\pi$-special. Therefore in this case we get from Theorems A and B the following results.

**Corollary 1.10.** Let $P_i$ be a Sylow $p_i$-subgroup of $G$ for all $p \in \pi = \{p_1, \ldots, p_n\}$. If the normalizer of every $\pi$-special non-subnormal subgroup of $G$ is $\pi$-special, then:

(i) $G$ has a Hall $\pi'$-subgroup $H$ and at least one of subgroups $P_1, \ldots, P_n, H$ is normal in $G$.

(ii) $O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\pi'}(G)$ is a maximal $\pi$-special subgroup of $G$.

(iii) $G/F(G)$ is $\pi$-special.

**Corollary 1.11.** Suppose that the normalizer of every $\pi$-special non-normal subgroup of $G$ is $\pi$-special. If non-normal Sylow subgroups of any Schmidt $\pi'$-subgroup of $G$ have prime order, then:

(i) $G/(O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\pi'}(G))$ is abelian.

(ii) Every maximal $\pi$-special non-normal subgroup of $G$ is an $\mathfrak{F}$-covering subgroup of $G$, where $\mathfrak{F}$ is the class of all $\pi$-special groups.

(iii) If every normal in $G$ subgroup $A \in \{P_1, \ldots, P_n, H\}$ is nilpotent, then $G/(O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\pi'}(G))$ is cyclic.
2 Preliminaries

Recall that \( G \) is said to be: a \( D_\pi \)-group if \( G \) possesses a Hall \( \pi \)-subgroup \( E \) and every \( \pi \)-subgroup of \( G \) is contained in some conjugate of \( E \); a \( \sigma \)-full group of Sylow type \( \Pi \)-section 2.2 if every subgroup \( E \) of \( G \) is a \( D_\sigma \)-group for every \( \sigma_i \in \sigma(E) \); \( \sigma \)-soluble \( \Pi \) if every chief factor of \( G \) is \( \sigma \)-primary.

**Lemma 2.1** (See Theorem A and B in \([13]\)). If \( G \) is \( \sigma \)-soluble, then \( G \) is a \( \sigma \)-full group of Sylow type and, for every \( i \), \( G \) has a Hall \( \sigma_i \)-subgroup and every two Hall \( \sigma_i \)-subgroups of \( G \) are conjugate.

A subgroup \( A \) of \( G \) is said to be \( \sigma \)-subnormal in \( G \) \([1]\) if there is a subgroup chain \( A = A_0 \leq A_1 \leq \cdots \leq A_n = G \) such that either \( A_{i-1} \leq A_i \) or \( A_i/(A_{i-1}) \) is \( \sigma \)-primary for all \( i = 1, \ldots, n \). Note that a subgroup \( A \) of \( G \) is subnormal in \( G \) if and only if \( A \) is \( \sigma^1 \)-subnormal in \( G \) (where \( \sigma^1 = \{\{2\}, \{3\}, \ldots\} \)).

**Lemma 2.2.**

(i) The class of all \( \sigma \)-nilpotent groups \( \Pi_\sigma \) is closed under taking direct products, homomorphic images and subgroups. Moreover, if \( H \) is a normal subgroup of \( G \) and \( H/H \cap \Phi(G) \) is \( \sigma \)-nilpotent, then \( H \) is \( \sigma \)-nilpotent (See Lemma 2.5 in \([13]\)).

(ii) \( G \) is \( \sigma \)-nilpotent if and only if every subgroup of \( G \) is \( \sigma \)-subnormal in \( G \) (See \([18\) Proposition 3.4]).

(iii) \( G \) is \( \sigma \)-nilpotent if and only if \( G = G_1 \times \cdots \times G_n \) for some \( \sigma \)-primary groups \( G_1, \ldots, G_n \) (See \([18\) Proposition 3.4]).

**Lemma 2.3** (See Lemma 2.6 in \([1]\)). Let \( A, K \) and \( N \) be subgroups of \( G \). Suppose that \( A \) is \( \sigma \)-subnormal in \( G \) and \( N \) is normal in \( G \).

1. If \( N \leq K \) and \( K/N \) is \( \sigma \)-subnormal in \( G/N \), then \( K \) is \( \sigma \)-subnormal in \( G \).
2. \( A \cap K \) is \( \sigma \)-subnormal in \( K \).
3. If \( A \) is \( \sigma \)-nilpotent, then \( A \leq F_\sigma(G) \).
4. \( AN/N \) is \( \sigma \)-subnormal in \( G/N \).
5. If \( A \) is a Hall \( \sigma_i \)-subgroup of \( G \) for some \( i \), then \( A \) is normal in \( G \).

In view of Proposition 2.2.8 in \([6\), we get from Lemma 2.2 the following

**Lemma 2.4.** If \( N \) is a normal subgroup of \( G \), then \((G/N)^{\Omega_{\sigma}} = G^{\Omega_{\sigma}} N/N \).

**Lemma 2.5.** If \( G \) is \( \sigma \)-soluble and, for some \( i \) and some Hall \( \sigma_i \)-subgroup \( H \) of \( G \), \( N_G(H) \) is \( \sigma \)-nilpotent, then \( N_G(H) \) is a \( \sigma \)-Carter subgroup of \( G \).

**Proof.** Let \( N = N_G(H) \) and \( N \leq U \leq G \). Suppose that \( U^{\Omega_{\sigma}} N \neq U \) and let \( M \) be a maximal subgroup of \( U \) such that \( U^{\Omega_{\sigma}} N \leq M \). Then \( M \) is \( \sigma \)-subnormal in \( U \) by Lemmas 2.2(i, ii) and 2.3(1), so \( U/M_U \) is a \( \sigma_j \)-group for some \( j \) since \( U \) is clearly \( \sigma \)-soluble. Therefore \(|U : M| \) is a \( \sigma_j \)-number, so \( j \neq i \) and hence \( H \leq M_U \). But then \( U = M_U N_U(H) \leq M < U \) by Lemma 2.1 and the Frattini argument. This contradiction completes the proof of the lemma.

It is clear that if \( A \) is \( \sigma \)-Carter subgroup of \( G \), then \( A \) is a \( \sigma \)-Carter subgroup in every subgroup
of $G$ containing $A$. Moreover, in view of Proposition 2.3.14 in [6], the following useful facts are true.

**Lemma 2.6.** Let $H$ and $R$ be subgroups of $G$, where $R$ is normal in $G$.

(i) If $H$ is a $\sigma$-Carter subgroup of $G$, then $HR/R$ is a $\sigma$-Carter subgroup of $G/R$.

(ii) If $U/R$ is a $\sigma$-Carter subgroup of $G/R$ and $H$ is a $\sigma$-Carter subgroup of $U$, then $H$ is a $\sigma$-Carter subgroup of $G$.

**Lemma 2.7.** Suppose that $G$ possesses a $\sigma$-Carter subgroup. If $G$ is $\sigma$-soluble, then any two of its $\sigma$-Carter subgroups are conjugate.

**Proof.** Assume that this lemma is false and let $G$ be a counterexample of minimal order. Then $|\sigma(G)| > 1$.

Let $A$ and $B$ be $\sigma$-Carter subgroups of $G$, and let $R$ be a minimal normal subgroup of $G$. Then $AR/R$ and $BR/R$ are $\sigma$-Carter subgroups of $G/R$ by Lemma 2.6(i). Therefore for some $x \in G$ we have $AR/R = B^xR/R$ by the choice of $G$. If $AR \neq G$, then $A$ and $B^x$ are conjugate in $AR$ by the choice of $G$ and so $A$ and $B$ are conjugate.

Now assume that $AR = G = B^xR = BR$. If $R \leq A$, then $A = G$ is $\sigma$-nilpotent and so $A = B$. Therefore we can assume that $A_G = 1 = B_G$.

Since $G$ is $\sigma$-soluble, $R$ is a $\sigma_i$-group for some $i$. Let $H$ be a Hall $\sigma_i'$-subgroup of $A$. Since $|\sigma(G)| > 1$, it follows that $H \neq 1$ and so $N = N_G(H) \neq 1$. Since $A$ and $B$ be $\sigma$-Carter subgroups of $G$, both these subgroups are $\sigma$-nilpotent. Hence $A \leq N$ and, for some $x \in G$, $B^x \leq N$ by Lemma 2.1. But then the choice of $G$ implies that $A$ and $B^x$ are conjugate in $N$. So we again get that $A$ and $B$ are conjugate. The lemma is proved.

If $G \notin \mathfrak{M}_\sigma$ but every proper subgroup of $G$ belongs to $\mathfrak{M}_\sigma$, then $G$ is called an $\mathfrak{M}_\sigma$-critical or a minimal non-$\sigma$-nilpotent group. If $G$ is an $\mathfrak{M}_{\sigma_i}$-critical group, that is, $G$ is not nilpotent but every proper subgroup of $G$ is nilpotent, then $G$ is said to be a Schmidt group.

**Lemma 2.8** (See [5] Ch. V, Theorem 26.1). If $G$ is a Schmidt group, then $G = P \rtimes Q$, where $P = G^{\sigma_1} = G'$ is a Sylow $p$-subgroup of $G$ and $Q = \langle x \rangle$ is a cyclic Sylow $q$-subgroup of $G$ with $\langle x^0 \rangle \leq Z(G) \cap \Phi(G)$. Hence $Q^G = G$.

**Lemma 2.9.** If $G$ is an $\mathfrak{M}_{\sigma_i}$-critical group, then $G$ is a Schmidt group.

**Proof.** For some $i$, $G$ is an $\mathfrak{M}_{\sigma_0}$-critical group, where $\sigma_0 = \{\sigma_i, \sigma_i'\}$. Hence $G$ is a Schmidt group by [20].

**Lemma 2.10.** Let $Z = Z_\sigma(G)$. Let $A$, $B$ and $N$ be subgroups of $G$, where $N$ is normal in $G$.

(i) $Z$ is $\sigma$-hypercentral in $G$.

(ii) If $N \leq Z$, then $Z/N = Z_\sigma(G/N)$.

(iii) $Z_\sigma(B) \cap A \leq Z_\sigma(B \cap A)$.

(iv) If $A$ is $\sigma$-nilpotent, then $ZA$ is also $\sigma$-nilpotent. Hence $Z$ is contained in each maximal
\(\sigma\)-nilpotent subgroup of \(G\).

(v) If \(G/Z\) is \(\sigma\)-nilpotent, then \(G\) is also \(\sigma\)-nilpotent.

**Proof.** (i) It is enough to consider the case when \(Z = A_1A_2\), where \(A_1\) and \(A_2\) are normal \(\sigma\)-hypercentral subgroups of \(G\). Moreover, in view of the Jordan-Hölder theorem for the chief series, it is enough to show that if \(A_1 \leq K < H \leq A_1A_2\), then \(H/K\) is \(\sigma\)-central. But in this case we have \(H = A_1(H \cap A_2)\), where \(H \cap A_2 \not\supset K\) and so from the \(G\)-isomorphism \((H \cap A_2)/(K \cap A_2) \cong (H \cap A_2)K/K = H/K\) we get that \(C_G(H/K) = C_G((H \cap A_2)/(K \cap A_2))\) and hence \(H/K\) is \(\sigma\)-central in \(G\).

(ii) This assertion is a corollary of Part (i) and the Jordan-Hölder theorem for the chief series.

(iii) First assume that \(B = G\), and let \(1 = Z_0 < Z_1 < \cdots < Z_t = Z\) be a chief series of \(G\) below \(Z\) and \(C_i = C_G(Z_i/Z_{i-1})\). Now consider the series

\[1 = Z_0 \cap A \leq Z_1 \cap A \leq \cdots \leq Z_t \cap A = Z \cap A.\]

We can assume without loss of generality that this series is a chief series of \(A\) below \(Z \cap A\).

Let \(i \in \{1, \ldots, t\}\). Then, by Part (i), \(Z_i/Z_{i-1}\) is \(\sigma\)-central in \(G\), \((Z_i/Z_{i-1}) \times (G/C_i)\) is a \(\sigma_k\)-group say. Hence \((Z_i \cap A)/(Z_{i-1} \cap A)\) is a \(\sigma_k\)-group. On the other hand, \(A/A \cap C_i \cong C_i A/C_i\) is a \(\sigma_k\)-group and

\[A \cap C_i \leq C_A((Z_i \cap A)/(Z_{i-1} \cap A)).\]

Thus \((Z_i \cap A)/(Z_{i-1} \cap A)\) is \(\sigma\)-central in \(A\). Therefore, in view of the Jordan-Hölder theorem for the chief series, we have \(Z \cap A \leq Z_{\sigma}(A)\).

Now assume that \(B\) is any subgroup of \(G\). Then, in view of the preceding paragraph, we have

\[Z_{\sigma}(B) \cap A = Z_{\sigma}(B) \cap (B \cap A) \leq Z_{\sigma}(B \cap A).\]

(iv) Since \(A\) is \(\sigma\)-nilpotent, \(ZA/Z \cong A/A \cap Z\) is \(\sigma\)-nilpotent by Lemma 2.2(i). On the other hand, \(Z \leq Z_{\sigma}(ZA)\) by Part (iii). Hence \(ZA\) is \(\sigma\)-nilpotent by Part (i).

(v) This assertion follows from Part (i).

The lemma is proved.

The following lemma is a corollary of Lemmas 2.2(i) and 2.10(v).

**Lemma 2.11.** \(F_\sigma(G)/\Phi(G) = F_\sigma(G/\Phi(G))\) and \(F_\sigma(G)/Z_{\sigma}(G) = F_\sigma(G/Z_{\sigma}(G))\).

### 3 Proofs of the main results

**Proof of Theorem A.** Assume that this theorem is false and let \(G\) be a counterexample of minimal order. Then \(G\) is not \(\sigma\)-nilpotent.
(1) Every proper subgroup \( E \) of \( G \) is weakly semi-\( \sigma \)-nilpotent. Hence the conclusion of the theorem holds for \( E \).

Let \( V \) be a non-subnormal \( \sigma \)-nilpotent subgroup of \( E \). Then \( V \) is not subnormal in \( G \) by Lemma 2.3(2), so \( N_G(V) \) is \( \sigma \)-nilpotent by hypothesis. Hence \( N_E(V) = N_G(V) \cap E \) is \( \sigma \)-nilpotent by Lemma 2.2(i).

(2) Every proper quotient \( G/N \) of \( G \) (that is, \( N \neq 1 \)) is weakly semi-\( \sigma \)-nilpotent. Hence the conclusion of the theorem holds for \( G/N \).

In view of Remark 1.2(iii) and the choice of \( G \), it is enough to show that if \( U/N \) is any non-subnormal \( \sigma \)-primary subgroup of \( G/N \), then \( N_{G/N}(U/N) \) is \( \sigma \)-nilpotent. We can assume without loss of generality that \( N \) is a minimal normal subgroup of \( G \).

Since \( U/N \) is not subnormal in \( G/N \), \( U/N < G/N \) and \( U \) is not subnormal in \( G \). Hence \( U \) is a proper subgroup of \( G \), which implies that \( U \) is \( \sigma \)-soluble by Claim (1). Hence \( N \) is a \( \sigma \)-group for some \( i \).

If \( U/N \) is a \( \sigma \)-group, then \( U \) is \( \sigma \)-primary and so \( N_G(U) \) is \( \sigma \)-nilpotent. Hence \( N_{G/N}(U/N) = N_G(U)/N \) is \( \sigma \)-nilpotent by Lemma 2.2(i). Now suppose that \( U/N \) is a \( \sigma \)-group for some \( j \neq i \). Then \( N \) has a complement \( V \) in \( U \) by the Schur-Zassenhaus theorem. Moreover, from the Feit-Thompson theorem it follows that at least one of the groups \( N \) or \( U/N \) is soluble and so every two complements to \( N \) in \( U \) are conjugate in \( U \). Therefore \( N_G(U) = N_G(NV) = NN_G(V) \). Since \( U = NV \) is not subnormal in \( G \), \( V \) is not subnormal in \( G \) by Lemma 2.3(1, 4) and so \( N_G(V) \) is \( \sigma \)-nilpotent. Hence \( N_{G/N}(U/N) = N_G(U)/N \) is \( \sigma \)-nilpotent.

(3) If \( A \) is an \( \mathfrak{N}_\sigma \)-critical subgroup of \( G \), then \( A = P \times Q \), where \( P = A^R = A' \) is a Sylow \( p \)-subgroup of \( A \) and \( Q \) is a Sylow \( q \)-subgroup of \( A \) for some different primes \( p \) and \( q \). Moreover, \( P \) is subnormal in \( G \) and so \( P \leq O_p(G) \).

The first assertion of the claim directly follows from Lemmas 2.8 and 2.9. Since \( A \) is not \( \sigma \)-nilpotent, \( P \) is subnormal in \( G \) by hypothesis. Therefore \( P \leq O_p(G) \) by Lemma 2.3(3).

(4) \( G \) is \( \sigma \)-soluble.

Suppose that this is false. Then \( G \) is a non-abelian simple group since every proper section of \( G \) is \( \sigma \)-soluble by Claims (1) and (2). Moreover, \( G \) is not \( \sigma \)-nilpotent and so it has an \( \mathfrak{N}_\sigma \)-critical subgroup \( A \). Claim (3) implies that for some Sylow subgroup \( P \) of \( A \) we have \( 1 < P \leq O_p(G) < G \). This contradiction shows that we have (4).

(5) Statements (i) and (ii) hold for \( G \).

Since \( G \) is \( \sigma \)-soluble by Claim (4), it is a \( \sigma \)-full group of Sylow type by Lemma 2.1. In particular, \( G \) possesses a complete Hall \( \sigma \)-set \( \{ H_1, \ldots, H_t \} \). Then there is an index \( k \) such that \( H_k \) is not subnormal in \( G \) by Lemma 2.3(5) since \( G \) is not \( \sigma \)-nilpotent. Then \( N_G(H_k) \) is \( \sigma \)-nilpotent by hypothesis, so \( N_G(H_i) \) is a \( \sigma \)-Carter subgroup of \( G \) by Lemma 2.5 for all \( i > r \).
If for some $j \neq k$ the subgroup $H_j$ is not subnormal in $G$, then $N_G(H_j)$ is also a $\sigma$-Carter subgroup of $G$. But then $N_G(H_k)$ and $N_G(H_j)$ are conjugate in $G$ by Lemma 2.7. Hence for some $x \in G$ we have $H^x_k \leq N_G(H_j)$. Therefore, since $G$ is not $\sigma$-nilpotent, there is a complete Hall $\sigma$-set \{L_1, \ldots, L_t\} of $G$ such that for some $1 \leq r < t$ the subgroups $L_1, \ldots, L_r$ are normal in $G$, $L_i$ is not normal in $G$ for all $i > r$, and $\langle L_{r+1}, \ldots, L_t \rangle = L_{r+1} \times \cdots \times L_t$.

(6) Every subgroup $V$ of $G$ containing $F_\sigma(G)$ is $\sigma$-subnormal in $G$, so $F_\sigma(V) = F_\sigma(G)$.

From Claim (5) it follows that $H_1, \ldots, H_r \leq F_\sigma(G)$ and

$$G/F_\sigma(G) = F_\sigma(G)(H_{r+1} \cdots H_t)/F_\sigma(G) \simeq (H_{r+1} \cdots H_t)/(H_{r+1} \cdots H_t \cap F_\sigma(G))$$

is $\sigma$-nilpotent. Hence every subgroup of $G/F_\sigma(G)$ is $\sigma$-subnormal in $G/F_\sigma(G)$ by Lemma 2.2(ii). Therefore $V$ is $\sigma$-subnormal in $G$ by Lemma 2.3(1), so $F_\sigma(V) \leq F_\sigma(G) \leq F_\sigma(V)$ by Lemma 2.3(3). Hence we have (6).

(7) Statement (iii) holds for $G$.

First note that $F_\sigma(G)$ is a maximal $\sigma$-nilpotent subgroup of $G$ by Claim (6). In fact, $F_\sigma(G) = F_{0\sigma}(G) \times O_{\sigma_{i_1}}(G) \times \cdots \times O_{\sigma_{i_m}}(G)$ for some $i_1, \ldots, i_m \subseteq \{r + 1, \ldots, t\}$. Moreover, in view of Claim (5), we get clearly that $G/C_G(O_{\sigma_{i_k}}(G))$ is a $\sigma_{i_k}$-group and so $O_{\sigma_{i_k}}(G) \subseteq Z_\sigma(G)$. Hence $F_\sigma(G) = F_{0\sigma}(G)Z_\sigma(G)$.

(8) Statement (iv) holds for $G$.

First we show that $U_G \leq Z_\sigma(G)$ for every $\sigma$-nilpotent subgroup $U$ of $G$ such that $G = F_\sigma(G)U$. Suppose that this is false. Then $U_G \neq 1$. Let $R$ be a minimal normal subgroup of $G$ contained in $U$ and $C = C_G(R)$. Then

$$G/R = (F_\sigma(G)R/R)(U/R) = F_\sigma(G/R)(U/R),$$

so

$$U_G/R = (U/R)G/R \leq Z_\sigma(G/R)$$

by Claim (2). Since $G$ is $\sigma$-soluble, $R$ is a $\sigma_i$-group for some $i$. Moreover, from $G = F_\sigma(G)U$ and Lemma 2.1 we get that for some Hall $\sigma_i'$-subgroups $E$, $V$ and $W$ of $G$, of $F_\sigma(G)$ and of $U$, respectively, we have $E = VW$. But $R \leq F_\sigma(G) \cap U$, where $F_\sigma(G)$ and $U$ are $\sigma$-nilpotent. Therefore $E \leq C$, so $R/1$ is $\sigma$-central in $G$. Hence $R \leq Z_\sigma(G)$ and so $Z_\sigma(G/R) = Z_\sigma(G)/R$ by Lemma 2.10(ii). But then $U_G \leq Z_\sigma(G)$. Finally, if $V$ is any maximal $\sigma$-nilpotent subgroup of $G$ with $G = F_\sigma(G)V$, then $Z_\sigma(G) \leq V$ by Lemma 2.11(iv) and so $V_G = Z_\sigma(G)$.

(9) Statement (v) holds for $G$.

In view of Lemma 2.2(i), it is enough to show that $D = G^{\pi_\sigma}$ is nilpotent. Assume that this is false. Then $D \neq 1$, and for any minimal normal subgroup $R$ of $G$ we have that $(G/R)^{\pi_\sigma} = RD/R \simeq D/D \cap R$ is nilpotent by Claim (2) and Lemmas 2.2(i) and 2.4. Moreover, Lemma 2.2(i) implies that $R$ is a unique minimal normal subgroup of $G$, $R \leq D$ and $R \notin \Phi(G)$. Since $G$ is not $\sigma$-nilpotent,
Claim (3) and [24] Ch. A, 15.6 imply that $R = C_G(R) = O_p(G) = F(G)$ for some prime $p$. Then $R < D$ and $G = R \rtimes M$, where $M$ is not $\sigma$-nilpotent, and so $M$ has an $\mathfrak{H}_\sigma$-critical subgroup $A$. Claim (3) implies that for some prime $q$ dividing $|A|$ and for a Sylow $q$-subgroup $Q$ of $A$ we have $1 < Q \leq F(G) \cap M = R \cap M = 1$. This contradiction completes the proof of (9).

From Claims (5), (7), (8) and (9) it follows that the conclusion of the theorem is true for $G$, contrary to the choice of $G$. The theorem is proved.

**Proof of Theorem B.** Assume that this theorem is false and let $G$ be a counterexample of minimal order. Then $G$ is not $\sigma$-nilpotent. Nevertheless, $G$ is $\sigma$-soluble by Theorem A. Let $F_0(\sigma)(G) = H_1 \cdots H_r$ and $E = H_{r+1} \cdots H_t$. Then $E$ is $\sigma$-nilpotent by Theorem A(ii).

1. Every proper subgroup $E$ of $G$ is semi-$\sigma$-nilpotent. Hence Statements (i) and (ii) hold for $E$ (See Claim (1) in the proof of Theorem A).

2. The hypothesis holds for every proper quotient $G/N$ of $G$. Hence Statements (i), (ii) and (iv) hold for $G/N$.

It is not difficult to show that $G/N$ is semi-$\sigma$-nilpotent (see Claim (2) in the proof of Theorem A).

Now let $U/N$ be any Schmidt $\sigma_i$-subgroup of $G/N$ such that $U/N \leq W/N$ for some non-normal in $G/N$ Hall $\sigma_i$-subgroup $W/N$ of $G/N$. In view of Lemma 2.1, we can assume without loss of generality that $W/N = H_iN/N$. Let $L$ be any minimal supplement to $N$ in $U$. Then $L \cap N \leq \Phi(L)$ and, by Lemma 2.8, $U/N = LN/N \simeq L/L \cap N$ is a $\sigma_i$-group and $L/L \cap N = (P/L \cap N) \times (Q/L \cap N)$, where $P/L \cap N = (L/L \cap N)^R = (L/L \cap N)^q$ is a Sylow $p$-subgroup of $L/L \cap N$ and $Q/L \cap N = \langle x \rangle$ is a cyclic Sylow $q$-subgroup of $L/L \cap N$ with $V/L \cap N = \langle x^q \rangle = \Phi(Q/L \cap N) \leq \Phi(L/L \cap N) \cap Z(L/L \cap N)$ and $p, q \in \sigma_i$. Suppose that $|Q/L \cap N| > q$. Then $L \cap N < V$.

In view of Lemma 2.2(i), a Sylow $p$-subgroup of $L$ is normal in $L$. Hence, in view of Lemma 2.8, for any Schmidt subgroup $A$ of $L$ we have $A = A_p \rtimes A_q$, where $A_p$ is a Sylow $p$-subgroup of $A$, $A_q$ is a Sylow $q$-subgroup of $A$ and $(A_q)^A = A$. We can assume without loss of generality that $A_q(L \cap N)/(L \cap N) \leq Q/L \cap N$. Then $A_q(L \cap N)/(L \cap N) \leq V/L \cap N$ since $V \leq \Phi(L)$. It follows that $A_q \not\leq N$. Since $W/N = H_iN/N$ is not normal in $G/N$, $H_i$ is not normal in $G$. But for some $x \in G$ we have $A^x \leq H_i$, so $|A^x_q| = |A_q| = q$ by hypothesis.

Note that $|Q/V| = q$ since $Q/L \cap N$ is cyclic and $V/L \cap N = \Phi(Q/L \cap N)$. Hence

$$(V/L \cap N)(A_q(L \cap N)/(L \cap N)) = (V/L \cap N) \times (A_q(L \cap N)/(L \cap N)) = Q/(L \cap N),$$

which implies that $Q/(L \cap N)$ is not cyclic. This contradiction shows that $|Q/L \cap N| = q$, so for a Sylow $q$-subgroup $S$ of $U/N$ we have $|S| = q$. Therefore the hypothesis holds for $G/N$. Hence we have (2) by the choice of $G$.

3. If $A$ is an $\mathfrak{H}_\sigma$-critical subgroup of $G$, then $A = P \rtimes Q$, where $P = A^R = A'$ is a Sylow $p$-subgroup of $A$ and $Q$ is a Sylow $q$-subgroup of $A$ for some different primes $p$ and $q$. Moreover, the
subgroup $P$ is normal in $G$. Hence $G$ has an abelian minimal normal subgroup $R$ (See Claim (3) in the proof of Theorem A).

(4) Statement (i) holds for $G$.

In view of Lemma 2.2(i), it is enough to show that $G'$ is $\sigma$-nilpotent. Suppose that this is false.

(a) $R = C_G(R) = O_p(G) = F(G) \not\in \Phi(G)$ for some prime $p$ and $|R| > p$.

From Claim (2) it follows that for every minimal normal subgroup $N$ of $G$, $(G/N)' = G'/G' \cap N$ is $\sigma$-nilpotent. If $R \neq N$, it follows that $G'/((G' \cap N) \cap (G' \cap R)) = G'/1$ is $\sigma$-nilpotent by Lemma 2.2(i). Therefore $R$ is a unique minimal normal subgroup of $G$, $R \leq D$ and $R \not\in \Phi(G)$ by Lemma 2.2(i). Hence $R = C_G(R) = O_p(G) = F(G)$ by Theorem 15.6 in [24 Ch. A], so $|R| > p$ since otherwise $G/R = G/C_G(R)$ is cyclic, which implies that $G' = R$ is $\sigma$-nilpotent.

(b) $F_\sigma(V) = F_\sigma(G)$ for every subgroup $V$ of $G$ containing $F_\sigma(G)$ (See Claim (6) in the proof of Theorem A).

(c) $G = H_1 \rtimes H_2$, where $R \leq H_1 = F_\sigma(G)$ and $H_2$ is a minimal non-abelian group.

From Theorem A and Claim (a) it follows that $r = 1$ and $R \leq H_1 = F_\sigma(G)$.

Now let $W = F_\sigma(G)V$, where $V$ is a maximal subgroup of $E$. Then $F_\sigma(G) = F_\sigma(W)$ by Claim (b), so $W/F_\sigma(W) = W/F_\sigma(G) \simeq V$ is abelian by Claim (1). Therefore $E$ is not abelian but every proper subgroup of $E$ is abelian, so $E = H_2$ since $E$ is $\sigma$-nilpotent. Hence we have (c).

(d) $H_1 = R$ is a Sylow $p$-subgroup of $G$ and every subgroup $H \neq 1$ of $H_2$ acts irreducibly on $R$. Hence every proper subgroup $H$ of $H_2$ is cyclic.

Suppose that $|\pi(H_1)| > 1$. There is a Sylow $p$-subgroup $P$ of $H_1$ such that $H_2 \leq N_G(P)$ by Claim (c) and the Frattini argument. Let $K = PH_2$. Then $K < G$ and $P = H_1 \cap K$ is normal in $K$, so $R \leq P = F_\sigma(K)$ since $C_G(R) = R$ by Claim (a). Then $K/F_\sigma(K) = K/P \simeq H_2$ is abelian by Claim (1), a contradiction. Hence $H_1$ is a normal Sylow $p$-subgroup of $G$. Hence $H_1 \leq F(G) \leq C_G(R) = R$ by [24 Ch. A, 13.8(b)], so $H_1 = R$.

Now let $S = RH$. By the Maschke theorem, $R = R_1 \times \cdots \times R_n$, where $R_i$ is a minimal normal subgroup of $S$ for all $i$. Then $R = C_S(R) = C_S(R_1) \cap \cdots \cap C_S(R_n)$. Hence, for some $i$, the subgroup $R_iH$ is not $\sigma$-nilpotent and so it has an $\forall \sigma$-critical subgroup $A$ such that $1 < A'$ is normal in $G$ by Claim (3). But then $R \leq A$. Therefore $i = 1$, so we have (d) since $H$ is abelian by Claim (c).

(e) $H_2$ is not nilpotent. Hence $|\pi(H_2)| > 1$.

Suppose that $H_2 = Q \times H$ is nilpotent, where $Q \neq 1$ is a Sylow $q$-subgroup of $H_2$. If $H \neq 1$, then $Q$ and $H$ are proper subgroups of $H_2$ and so the groups $Q$, $H$ and $H_2$ are abelian by Claim (c). Therefore $H_2 = Q$ is a $q$-group. Then, since every maximal subgroup of $H_2$ is cyclic by Claim (d), $q = 2$ by [25 Ch. 5, Theorems 4.3, 4.4]. Therefore $|R| = p$, contrary to Claim (a). Hence we have (e).

(f) $H_2 = A \rtimes B$, where $A = C_{H_2}(A)$ is a group of prime order $q \neq p$ and $B = \langle a \rangle$ is a group of
order \( r \) for some prime \( r \not\in \{p, q\} \).

From Claims (d) and (e) it follows that \( H_2 \) is a Schmidt group with cyclic Sylow subgroups. Therefore Claim (f) follows from the hypothesis and Lemma 2.8.

Final contradiction for (4). Suppose that for some \( x = yz \in RA \), where \( y \in R \) and \( z \in A \), we have \( xa = ax \). Then \( x \in N_G(B) \), so \( R \cap \langle x \rangle = 1 \) since \( B \) acts irreducible on \( R \) by Claim (d). Hence \( \langle x \rangle \) is a \( q \)-group and \( V = \langle x \rangle B \) is abelian group such that \( B \cap R = 1 \). Hence from the isomorphism \( G/R \cong H_2 \) we get that \( x = 1 \). Therefore \( a \) induces a fixed-point-free automorphism on \( RA \) and hence \( RA \) is nilpotent by the Thompson theorem [25, Ch. 10, Theorem 2.1]. But then \( A \leq C_G(R) = R \). This contradiction completes the proof of (4).

(5) Statement (ii) holds for \( G \).

Suppose that this is false. By Lemma 2.10(iv), \( \sigma(G) \leq U \). On the other, \( U/Z_\sigma(G) \) is a maximal \( \sigma \)-nilpotent non-normal subgroup of \( G/Z_\sigma(G) \) by Lemma 2.10(v). Hence in the case \( \sigma(G) \neq 1 \) Claim (2) implies that \( U/Z_\sigma(G) \) is a \( \sigma \)-Carter subgroup \( G/Z_\sigma(G) \), so \( U \) is a \( \sigma \)-Carter subgroup of \( G \) by Lemma 2.6(ii). Hence \( \sigma(G) = 1 \), so Theorem A(iii) implies that \( F_\sigma(G) = F_{0\sigma}(G) = H_1 \cdots H_r \). Hence \( E \cong G/F_{0\sigma}(G) \) is abelian by Claim (4).

Let \( V = F_\sigma(G)U \). If \( V = G \), then for some \( x \) we have \( H_{r+1}^x \leq U \) by Lemma 2.1. Hence \( U \leq N_G(H_{r+1}^x) \) and so \( U = N_G(H_{r+1}^x) \) is a \( \sigma \)-Carter subgroup of \( G \) by Theorem A(ii). Therefore \( V = F_\sigma(G)U \) is a normal proper subgroup of \( G \). Let \( x \in G \). If the subgroup \( U^x \) is normal in \( V \), then \( U^x \) is subnormal in \( G \) and so \( U^x, U \leq F_\sigma(G) \) by Lemma 2.3(3), which implies that \( U = F_\sigma(G) \) is normal in \( G \) since \( F_\sigma(G) \) and \( U \) are maximal \( \sigma \)-nilpotent subgroups of \( G \) by Theorem A(iii). This contradiction shows that \( U^x \) and \( U \) are non-normal maximal \( \sigma \)-nilpotent subgroups of \( V \). Since \( V < G \), Claim (1) implies that \( U^x \) and \( U \) are \( \sigma \)-Carter subgroups of \( V \). Since \( V \) is \( \sigma \)-soluble, \( U \) and \( U^x \) are conjugate in \( V \) by Lemma 2.7. Therefore \( G = V N_G(U) \) by the Frattini argument. Since \( U \) is a maximal \( \sigma \)-nilpotent non-normal subgroup of \( G \), \( U = N_G(U) \). Hence \( G = V U = (F_\sigma(G)U)U = F_\sigma(G)U < G \). This contradiction completes the proof of the fact that every maximal \( \sigma \)-nilpotent non-normal subgroup \( U \) of \( G \) is a \( \sigma \)-Carter subgroup of \( G \). But then \( G = F_\sigma(G)U \) since \( G/F_\sigma(G) \) is \( \sigma \)-nilpotent by Claim (4) and so \( U_G = Z_\sigma(G) \) by Theorem A(iv). Hence we have (5).

(6) If \( F_{0\sigma}(G) \leq F(G) \), then \( G/F_\sigma(G) \) is cyclic.

Assume that this is false.

(i) \( \Phi(F_{0\sigma}(G)) = 1 \). Hence \( F_{0\sigma}(G) \) is the direct product of some minimal normal subgroups \( R_1, \ldots, R_k \) of \( G \).

Suppose that \( \Phi(F_{0\sigma}(G)) \neq 1 \) and let \( N \) be a minimal normal subgroup of \( G \) contained in \( \Phi(F_{0\sigma}(G)) \leq \Phi(G) \). Then \( N \) is a \( p \)-group for some prime \( p \).

We show that the hypothesis holds for \( G/N \). First note that \( G/N \) is semi-\( \sigma \)-nilpotent by Claim (2). Now let \( V/N \) be a normal Hall \( \sigma_i \)-subgroup of \( G/N \) for some \( \sigma_i \in \sigma(G/N) \). If \( p \in \sigma_i \), then \( V \) is normal Hall \( \sigma_i \)-subgroup of \( G \), so \( V \leq F(G) \) by hypothesis and hence \( V/N \leq F(G)N/N \leq F(G/N) \).
Now assume that $p \notin \sigma_i$ and let $W$ be a Hall $\sigma_i$-subgroup of $V$. Then $W$ is a Hall $\sigma_i$-subgroup of $G$. Moreover, every two Hall $\sigma_i$-subgroups of $V$ are conjugate in $V$ by Lemma 2.1, so $G = VN_G(W) = NW_NG(W) = N_NG(W) = N_G(W)$ by the Frattini argument. Therefore $W \leq F(G)$, so $V/N = WN/N \leq F(G/N)$. Hence $F_{0\sigma}(G/N) \leq F(G/N)$, so the hypothesis holds for $G/N$. The choice of $G$ and Lemma 2.11 imply that $(G/N)/F_{\sigma}(G/N) = (G/N)/(F_{\sigma}(G)/F_{\sigma}(G)) \simeq G/F_{\sigma}(G)$ is cyclic, a contradiction. Hence $\Phi(F_{0\sigma}(G)) = 1$, so we have (i) by [24, Ch. A, Theorem 10.6(c)].

(ii) $Z_{\sigma}(G) = 1$. Hence $F_{0\sigma}(G) = F_{\sigma}(G) = F(G)$.

Since $Z_{\sigma}(G/Z_{\sigma}(G)) = 1$ by Lemma 2.10(ii), Lemma 2.11 and Theorem A(iii) imply that

\[ F_{0\sigma}(G/Z_{\sigma}(G)) = F_{\sigma}(G/Z_{\sigma}(G)) = F_{\sigma}(G)/Z_{\sigma}(G) = F_{0\sigma}(G)Z_{\sigma}(G)/Z_{\sigma}(G), \]

where $F_{0\sigma}(G) \leq F(G)$ and so $F_{0\sigma}(G/Z_{\sigma}(G)) \leq F(G/Z_{\sigma}(G))$. Therefore the hypothesis holds for $G/Z_{\sigma}(G)$ and hence, in the case when $Z_{\sigma}(G) \neq 1$, $G/F_{\sigma}(G) \simeq (G/Z_{\sigma}(G))/F_{\sigma}(G/Z_{\sigma}(G))$ is cyclic by the choice of $G$. Hence we have (ii).

**Final contradiction for (6).** Since $E \simeq G/F(G)$ is abelian by Claims (4) and (ii) and $G$ is not nilpotent, there is an index $i$ such that $V = R_i \rtimes E$ is not nilpotent. Then $C_{R_i}(E) \neq R_i$. By the Maschke theorem, $R_i = L_1 \times \cdots \times L_m$ for some minimal normal subgroups $L_1, \ldots, L_m$ of $V$. Then, since $C_{R_i}(E) \neq R_i$, for some $j$ we have $L_j \times E \neq L_j \times E$. Hence $L_j E$ contains a Schmidt subgroup $A_p \rtimes A_q$ such that $A_p = R_i$, so $m = 1$. But then $E$ acts irreducible on $R_i$ and hence $G/F(G) \simeq E$ is cyclic. This contradiction completes the proof of (6).

From Claims (1), (2), (4), (5) and (6) it follows that the conclusion of the theorem is true for $G$, contrary to the choice of $G$. The theorem is proved.

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