Islands in Generalized Dilaton Theories

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ABSTRACT: In this work we systematically study the island formula in the general asymptotically flat eternal black holes in generalized dilaton gravity theories or in higher dimensional spherical black holes. Under some reasonable and mild assumptions we prove that the island always appears barely outside of the horizon in the late time of Hawking radiation so that the information paradox is resolved. In particular, we find proper island in Liouville black hole which solves the puzzle of [1].
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1 Introduction

Over the last five years there is a dramatic acceleration of progress on quantum aspects of black holes [2]. The most exciting achievement is the successful derivation of Page curve [3] which resolves the long-standing information paradox [4] by proposing a new rule, the island formula [5–7], for computing the entanglement entropy of the Hawking radiations. The island formula mimics the quantum extremal surface (QES) prescription [8] of the generalized entanglement entropy:

\[
S_R = \min \left\{ \text{ext} \left\{ \frac{A(\partial I)}{4G_N} + S_{\text{semi-cl}}(\text{Rad} \cup I) \right\} \right\},
\]

where \( I \) is the island which is a codimension-one region and \( A(\partial I) \) is the area of its boundary \( \partial I \), the QES. The surprising fact about this formula is that its right-hand side only depends on semi-classical physics and importantly the island is not added by hand but its existence can be justified by the replica tricks of the gravitational Euclidean path integral [9, 10]. Another way to understand the island formula is by combining the AdS/BCFT correspondence and the brane world holography [11–14].

The island formula has been successfully applied to black holes in various of gravitational theories [19–39]. However there is also a notable counterexample [1] where it was claimed that the island formula can not save the information paradox of Liouville black hole. More puzzlingly this claim seems to be inconsistent with the systematic analysis of QES in general D-dimensional asymptotically flat (or AdS) eternal black hole performed in [39]. In this work, we solve this puzzle by finding a new Liouville black hole solution with the help of the general construction of classical solutions of generalized dilaton theories. We also conduct a systematic analysis of islands in asymptotically flat eternal black holes. Our general results agree with the ones in [39].

2 2D Generalized dilaton Gravity Theory

Because of their remarkable solvability, 2D GDTs as toy models of quantum gravity serve as a laboratory for studying properties of black holes. The general action of 2D GDTs is

\[
S = \frac{1}{2\pi} \int d^2 x \sqrt{-g} \left[ XR - U(X)(\nabla X)^2 - 2V(X) \right],
\]

which depends on the metric \( g_{\mu\nu} \) and the scalar field \( X \). The scalar field \( X \) is usually related to the dilaton \( \phi \) via \( X = e^{-2\phi} \). In terms of \( \Phi \), the action is in a more familiar form

\[
S_{\text{dil}} = \frac{1}{2\pi} \int d^2 x \sqrt{-g} e^{-2\phi} \left[ R - \tilde{U}(\Phi)(\nabla \Phi)^2 - 2\tilde{V}(\Phi) \right]
\]
with the identifications

\[ \tilde{U} = 4e^{-2\Phi}U, \quad \tilde{V} = e^{2\Phi}V. \]  

(2.3)

All the classical solutions of (2.1) can be found in a closed form with the help of the first-order formalism of GDT. We will show our convention and review the construction of classic solutions in the Appendix A. The general solution of (2.1) is given by\(^1\) [15, 16]

\[ ds^2 = 2e^Q d v(d X + (w(X) - C_0) d v) \]

(2.4)

\[ = 2 d v d \tilde{X} + \xi(\tilde{X}) d v^2, \]

(2.5)

where we have introduced

\[ d \tilde{X} = d X e^Q, \quad \xi(\tilde{X}) = 2e^Q(w - C_0), \]

(2.6)

\[ Q = \int^X U(y) dy, \]

(2.7)

\[ w = \int^X e^Q V(y) dy. \]

(2.8)

The solution is parameterized by a constant \(C_0\) which is usually related to the mass of the black hole. We can transform the metric to the diagonal gauge by introducing the coordinates \(x^0 = r, x^1 = r\), thus \(d \tilde{X} = \tilde{X} dt + \tilde{X}' dr, d v = \dot{v} dt + v' dr\) and setting

\[ \tilde{X} = \tilde{X}(r), \quad \tilde{X}' + \xi' = 0. \]

(2.9)

The resulting metric is

\[ ds^2 = \xi(\dot{v}^2 dt^2 - v'^2 dr^2). \]

(2.10)

If we further set \(\tilde{X} = r\) and \(\dot{v} = 1\), we can get

\[ ds^2 = \xi \frac{1}{\xi} dt^2 - \frac{1}{\xi} dr^2, \]

(2.11)

which is in the Schwarzschild gauge if the solution describes a black hole. In this paper we will focus on asymptotically flat solutions so we take the ansatz that that \(\xi\) approaches to some constant

\[ \lim_{r \to \infty} \xi = -\xi_0^2, \]

(2.12)

\(^1\)here we only consider the interesting linear dilaton vacua.
where $\xi_0^2$ should be positive to ensure the correct signature. Then we rescale the coordinates as

$$ t = \frac{t'}{\xi_0}, \quad r = \xi_0 r' $$

(2.13)

such that

$$ ds^2 = \frac{\xi^2}{\xi_0^2} dt'^2 - \frac{\xi_0^2}{\xi} dr'^2, \quad \lim_{r' \to \infty} ds^2 = -dt^2 + dr^2. $$

(2.14)

The horizon of the black hole is at $r'_H$, $\xi(\xi'_H) = 0$. The temperature and the entropy of the black hole are

$$ T_{BH} = \frac{\partial r}{\partial \xi} \frac{\xi_0}{\xi'_H} \bigg|_{r'} = 2 X \left( r'_H \right), \quad S_{BH} = 2 X \left( r'_H \right). $$

(2.15)

3 General results

3.1 Setting up the calculation

Our goal is to compute the entanglement entropy of the Hawking radiation of the eternal black hole with the island formula (1.1). For 2d GDT (2.1), the first term is given by the value of dilaton field at the position of the island boundary $2X(\partial I)$ [17]. $S_{\text{semi-cl}}[\text{Rad} \cup I]$ is semi-classical entanglement entropy of Hawking radiation in the region Rad $\cup I$. For eternal black holes, we first introduce Kruskal coordinates

$$ ds^2 = -e^{2\rho(y^+, y^-)} dy^+ dy^- $$

(3.1)

to cover the whole regions then we choose Rad to be two symmetric intervals $[y_{L\infty}, y_{-a}] \cup [y_a, y_{R\infty}]$, where the coordinates of the end points are

$$ y_{L\infty} = (y_a^0, -\infty), \quad y_{R\infty} = (y_a^0, \infty), \quad y_{-a} = (y_a^0, -y_a^1), \quad y_a = (y_a^0, y_a^1). $$

(3.2)

With this symmetric choice, it is reasonable to expect that the island $[y_{-a}, y_d]$ also enjoys this symmetry

$$ y_{-d} = (y_d^0, -y_d^1), \quad y_d = (y_d^0, y_d^1). $$

(3.3)

The Kruskal coordinates are related to two copies of Schwarzschild coordinates via

$$ e^{lx_R} = ly_R^+, \quad e^{-lx_R} = -ly_R^-, \quad e^{-lx_L} = -ly_L^+, \quad e^{lx_L} = ly_L^-, $$

$$ y_R^+ \geq 0, \quad y_R^- \leq 0, \quad y_L^+ \leq 0, \quad y_L^- \geq 0, $$

(3.4)

We assume that the one-interval island configuration dominates.
where \( y_{R(L)}^\pm = y_{R(L)}^0 \pm \xi_{R(L)}^1 \) labels the position in the right (left) patch of Kruskal spacetime and \( l \) is some convenient constant and

\[
    x^\pm = t' \pm x^*, \quad x^* = \int \frac{dr'}{-\xi/\xi_0^2}.
\]

(3.6)

From the transformations (about \( x_R \)) we find that

\[
    l y^+ \frac{dx^+}{d y^+} = \frac{d y^+}{d y^-}, \quad -l y^- \frac{dx^-}{d y^-} = \frac{d y^-}{d y^-}
\]

(3.7)

\[
    e^{2\rho} = \frac{\xi(y^+ y^-)}{\xi_0^2 l^2 y^+ y^-}.
\]

(3.8)

We will model the Hawking radiation with a probe conformal field theory with a central charge \( c \ll \frac{1}{G_N} \) so can use the semi-classical formula to compute the entanglement entropy \( S_{\text{semi-cl}}(\text{Rad} \cup I) \).

### 3.2 Entanglement entropy without islands

In the early time of Hawking radiation, there are very few Hawking quanta in the interior of black holes so the island configuration can not be supported. So as more and more Hawking quanta escape to the infinity the asymptotic (Schwarzschild) observer should see a growing entanglement entropy. Assuming that the initial state of the quantum field is pure then

\[
    S_{\text{semi-cl}}[\text{Rad}] = \frac{c}{6} \log \left( |(y_a - y_{-a})^+ (y_a - y_{-a})^-| e^{\rho(y_a)} e^{\rho(y_{-a})} \right)
\]

(3.9)

\[
    = \frac{c}{6} \log \left( |2 y_a^1|^2 e^{\rho(y_a^+ y_{-a}^-)} e^{\rho(y_{-a}^+ y_a^-)} \right),
\]

\[
    = \frac{c}{3} \log (2 \cosh(l t'_a)) + \frac{c}{6} \log \left( -\frac{\xi}{\xi_0^2 l^2} \right),
\]

(3.10)

thus indeed as shown in the Page curve the growth of early entanglement entropy is general. Without introducing islands, the entanglement entropy will exceed the Bekenstein-Hawking entropy before the black hole completely evaporates. However the Bekenstein-Hawking entropy should be the upper bound of entanglement entropy. This contradiction is the well known information paradox.

### 3.3 Entanglement entropy with islands

In the late time of black hole evaporation, as we have seen that we have to include the islands. But as stressed in [18], the island region is not included by hand but its appearance is a result of evaluating gravitational path integral around non-trivial saddles. As a result, the black hole evaporation is unitary such that the entanglement
entropy will vanish in the end as the Page curve shows. This implies the position of the
\( \partial I \) is very close to the horizon i.e. \( y_d^\alpha = y_d^0 - y_d^\alpha \approx 0 \) which means that \( y_d^\alpha \) is very large. Therefore we can approximate the entanglement entropy \( S_{\text{Rad}}([y_a^-, y_d^+] \cup [y_d^+, y_a^+]) \) with
\[
2S_{\text{Rad}}([y_d^+, y_a])
\]
thus
\[
S_{\text{island}} = \frac{4}{G_N} X(y_d) + \frac{c}{3} \log \left( |(y_a - y_d^+)(y_a - y_d^-)| e^{\rho(y_a)} e^{\rho(y_d)} \right), \tag{3.11}
\]
where we have added back the Newton’s constant \( G_N \) to indicate that in the semi-
classical limit, the first term should be much larger than the second term. Taking
this approximation as an ansatz and then solving \( y_d \) by extremizing the generalized
entanglement entropy (3.11) we can show indeed this approximation is correct.

Differentiating (3.11) with respect to \( y_d^\pm \) gives two extremal conditions
\[
\frac{4}{G_N} \frac{d}{dy_d^+} X + \frac{c}{3} \left( \frac{1}{y_d^+ - y_a^+} + \frac{d \rho}{d y_d^+} \right) = 0, \tag{3.12}
\]
\[
\frac{4}{G_N} \frac{d}{dy_d^-} X + \frac{c}{3} \left( \frac{1}{y_d^- - y_a^-} + \frac{d \rho}{d y_d^-} \right) = 0. \tag{3.13}
\]
Recalling that \( \tilde{X} = r \) first term can be evaluated as
\[
\frac{d X}{d^2 \tilde{r}} = \frac{d X}{d \tilde{r}} \frac{d r}{d y_d^\pm} = e^{-Q} \xi_0 \frac{d r}{d y_d^\pm} = -e^{-Q} \frac{\xi_0}{\xi} \frac{d x^*}{d y_d^\pm} = -e^{-Q} \frac{\xi}{2l \xi_0 y_d^\pm} \tag{3.14}
\]
where \( \xi(z) \) should be understood as function of \( y_d^+ y_d^- \equiv z \). Using the expression (3.8)
the last term can be computed as
\[
\frac{d \rho}{d y_d^\pm} = \frac{1}{2y_d^\mp} \left( \frac{\xi'}{\xi} \frac{d z}{y_d^\pm} - \frac{1}{y_d^\pm} \right) = \frac{1}{2y_d^\mp} \left( \frac{\xi'}{\xi} z - 1 \right) \tag{3.15}
\]
Then the two equations (3.12) and (3.13) can be written as
\[
\frac{1}{3} \frac{1}{y_a^+ - y_d^+} - \frac{2 \xi e^{-Q}}{c G_N l \xi_0} \frac{1}{y_d^+} + \frac{1}{3} \frac{1 - z \xi'}{2 \xi} \frac{1}{y_d^+} = 0, \tag{3.16}
\]
from which we can obtain the relation
\[
\frac{y_d^+}{y_a^+ - y_d^+} = \frac{y_d^-}{y_a^- - y_d^-}, \quad \therefore \quad y_a^+ y_d^- = y_a^- y_d^+, \quad \text{or} \quad \frac{y_a^+}{y_a^-} = \frac{y_d^+}{y_d^-}. \tag{3.17}
\]
In the semi-classical limit, \( c G_N \ll 1 \) thus the last term in (3.16) can be neglected so we only need consider the equations
\[
\frac{z}{y_d^+ y_a^+ - z} = -\frac{6 \xi e^{-Q}}{c G_N l \xi_0}, \quad \frac{z}{y_d^- y_a^-} = -\frac{6 \xi e^{-Q}}{c G_N l \xi_0} \tag{3.18}
\]
from which we can derive a single equation for $z$
\[ z \left( 1 - \frac{\epsilon}{\xi e^{-Q}} \right)^2 = y_a^+ y_a^- , \quad \epsilon \equiv \frac{cG_N l \xi_0}{6}, \quad (3.19) \]
where we will also assume that the "effective" coupling $\epsilon$ is small. Solving it and using the relation (3.17) we can solve $y_a^\pm$.

In the end we need to transfer to the Schwarzschild coordinate because it describes the asymptotic observer and it turns out it is more convenient to do the computation with the Schwarzschild coordinates. Let the positions $y_a$ and $y_d$ are $(t_a,a)$ and $(t_d,d)$ in the Schwarzschild coordinates such that
\[
\frac{1}{l} Y e^{lt_a} = y_a^+ , \quad \frac{1}{l} Y e^{-lt_a} = -y_a^- , \quad Y = e^{ly^*} , \quad y^* = \int_{t_d}^{t_a} \frac{dr'}{-\xi/\xi_0^2} , \quad (3.20)
\]
\[
\frac{1}{l} D e^{lt_d} = y_d^+ , \quad \frac{1}{l} D e^{-lt_d} = -y_d^- , \quad D = e^{ld^*} , \quad d^* = \int_{d}^{d} \frac{dr'}{-\xi/\xi_0^2} , \quad (3.21)
\]
thus the entanglement entropy (3.11) can be written as
\[
S_{\text{island}} = \frac{4X(d)}{G_N} + \frac{c}{3} \log \frac{1}{l^2} (Y^2 + D^2 - 2YD \cosh l(t_a - t_d)) + \frac{c}{3} (\rho(d) + \rho(a)) , \quad (3.22)
\]
Varying with respect to $t_d$ implies $t_d = t_a$ therefore the extremal of $S_{\text{island}}$ is time-independent as expected. Assuming the temperature of the black hole (2.14) is not zero so it can evaporate then $\xi$ has a single zero at $r' = r'_H$. It implies that $d^*$ has a logarithm singularity at $d = r'_H$ so we can rewrite
\[
d^* = f(d) + \frac{1}{2l} \log(d - r'_H) , \quad (3.23)
\]
where $f(d)$ is regular at $r'_H$. Since we have assumed that the island is very close to the horizon $d \approx r'_H$ or equivalently $z \approx 0$ thus we have the approximation
\[
e^{2ld^*} = e^{2lf(d)}(d - r'_H) = -\frac{1}{l^2} z , \quad \rightarrow \quad (3.24)
\]
\[
d = r'_H - e^{-2lf(r'_H)} \frac{1}{l^2} z + \mathcal{O}(z^2) . \quad (3.25)
\]
If we also assume that $e^{-Q}$ is regular and not vanishing at $d = r'_H$ then
\[
\xi e^{-Q} = (d - r'_H) g(d) , \quad (3.26)
\]
where $g(d)$ is a regular at $r'_H$. Thus (3.19) becomes
\[
z(1 - \frac{\epsilon^2 e^{2lf(r'_H)}}{zg(d)})^2 \approx z + \frac{1}{z} \left( \frac{\epsilon^2 e^{2lf(r'_H)}}{g(r'_H)} \right)^2 - 2 \frac{\epsilon^2 e^{2lf(r'_H)}}{g(r'_H)} = y_a^+ y_a^- \equiv y^2 . \quad (3.27)
\]
There is indeed one solution which satisfies our ansatz \( z \approx 0 \):
\[
\begin{align*}
\beta^2 y^2 + \mathcal{O}(\beta^3), & \quad \beta = \frac{\epsilon f^2 e^{2f(r_H')}}{g(r_H')}, \\
y_d^+ = -\frac{\beta}{y_a^-}, & \quad y_d^- = -\frac{\beta}{y_a^+},
\end{align*}
\]

which leads to
\[
\begin{align*}
S_{\text{island}} &= \frac{4X(r_H')}{G_N^0} + \frac{c}{3} \log \left( |y_a^+ y_a^-| e^{\rho(y_d^\pm)} \right), \\
&= 2S_{\text{BH}} + S_{\text{matter}}
\end{align*}
\]

where \( S_{\text{matter}} \) is the quantum correction of order \( \mathcal{O}(G_N^0) \) due to the presence of matter fields. This is the main result of this paper: under some reasonable assumptions for a general asymptotically flat eternal black hole in GDT we can find island such that (generalized) entanglement entropy of Hawking radiation follows Page curve which resolves the information paradox.

**Examples**

### 4 CGHS model

The most well-studied GDT which admits an asymptotically flat black hole solution is the Callan-Giddings-Harvery-Strominger model (CGHS model) [40]. Islands in this model have been found in [23, 38]. In this section, we will rederive the island with our general procedures to confirm the validity of our general analysis. The action of the CGHS model is
\[
S = \frac{1}{2\pi} \int d^2 x \sqrt{-g} \left[ e^{-2\phi} (R + 4(\nabla \phi)^2 + 4\lambda^2) \right]
\]

#### 4.1 The geometry

Comparing (4.1) with (2.2) we recognize that
\[
U = -\frac{1}{X}, \quad V = -2\lambda^2 X.
\]

Therefore according to our general discussion, we can compute the following data
\[
\begin{align*}
e^Q &= \frac{1}{X}, \quad X = \exp \tilde{X}, \quad w = -2\lambda^2 X = -2\lambda^2 e^{\tilde{X}} \\
\xi &= -2C_0 e^{-\tilde{X}} - 4\lambda^2, \quad \xi_0 = 2\lambda.
\end{align*}
\]
So the Schwarzschild metric is
\[ ds^2 = -\left( \frac{C_0}{2\lambda^2} e^{-2\lambda r'} + 1 \right) dt^2 + \frac{dr'^2}{\left( \frac{C_0}{2\lambda^2} e^{-2\lambda r'} + 1 \right)}, \tag{4.5} \]

It is easy to find that the horizon and curvature singularity are located at
\[ r'_H = -\frac{1}{2\lambda} \log \left( -\frac{2\lambda^2}{C_0} \right), \quad r'_s = -\infty. \tag{4.6} \]

Therefore we should take \( \lambda^2 > 0 \) and \( C_0 \equiv -C < 0 \) to get the black hole geometry. To summarize, the classical solution describes a asymptotically flat black hole with metric and dilaton
\[ ds^2 = -(1 - \frac{C}{2\lambda^2} e^{-2\lambda r'}) dt^2 + \frac{1}{(1 - \frac{C}{2\lambda^2} e^{-2\lambda r'})} dr'^2, \tag{4.7} \]
\[ X = \exp (2\lambda r'). \tag{4.8} \]

The temperature of this black hole can be found using the equation
\[ T = -\frac{1}{4\pi} \partial_{r'} \sqrt{-g_{r'r'}} \bigg|_{r'=r_H} = \frac{\lambda}{2\pi}, \tag{4.9} \]
and the entropy is given by the Wald formula
\[ S = 2X|_{r'=r_H} = 2e^{2\lambda r'}|_{r'=r_H} = \frac{C}{\lambda^2}. \tag{4.10} \]

Introducing the new variable (3.6)
\[ x^* = \int \frac{d r'}{1 - \frac{C}{2\lambda^2} e^{-2\lambda r'}} = \frac{\log \left( 2\lambda e^{2\lambda r'} - C \right)}{2\lambda}, \tag{4.11} \]
we can obtain the Kruskal coordinates \( y^\pm \) as for example through
\[ e^{\lambda x^+} = \lambda y^+, \quad e^{-\lambda x^-} = -\lambda y^- \tag{4.12} \]
thus the metric and dilaton become
\[ ds^2 = -\frac{d y^+ d y^-}{C - \lambda^2 y^+ y^-}, \quad e^{2\rho} = \frac{1}{C - \lambda^2 y^+ y^-}, \tag{4.13} \]
\[ X = \frac{1}{2\lambda^2} \left( C - \lambda^2 y^+ y^- \right). \tag{4.14} \]

In the Kruskal coordinates, the horizon is located at \( y^+ y^- = 0 \) and the singularity is located at \( y^+ y^- = C \). As we have shown in the general discussion, without including the island the entanglement entropy is given by the general formula (3.10). Let us focus on the derivation of the island.
4.2 The derivation of island

In the late time, the entanglement entropy with island is given by:

\[ S_{\text{island}} = \frac{1}{G_N} \frac{2}{2\lambda^2} \left( C - \lambda^2 y^+_d y^-_d \right) + \frac{c}{3} \log \left( \frac{|(y_a - y_d)^+ (y_a - y_d)^-|}{\sqrt{C - \lambda^2 y^+_d y^-_a \sqrt{C - \lambda^2 y^+_a y^-_d}}} \right). \] (4.15)

Taking derivative with respect to \( y^-_d \) and \( y^+_d \) we obtain the equations

\[
\frac{\lambda^2 y^+_d}{6(C - \lambda^2 y^+_d y^-_d)} + \frac{1}{3(y^-_d - y^+_a)} - \frac{2y^+_d}{cG_N} = 0, \] (4.16)

\[
\frac{\lambda^2 y^-_d}{6(C - \lambda^2 y^+_d y^-_a)} + \frac{1}{3(y^-_a - y^+_d)} - \frac{2y^-_d}{cG_N} = 0, \] (4.17)

The exact solutions can be straightforwardly obtained but the exact solutions are very complicated. To exact the useful information we again take the semi-classical limit \( G_N \to 0 \). In this limit, we find that the non-trivial solutions are

\[ y^+_d = -\frac{cG_N}{6y^-_a}, \quad y^-_d = -\frac{cG_N}{6y^+_a}, \] (4.18)

and corresponding extremal entanglement entropy in the Schwarzschild coordinate is

\[ S_{\text{island}} = 2 \frac{1}{G_N} \frac{C}{\lambda^2} + \frac{c}{3} \log \left( \frac{|y^+_a y^-_a|}{\sqrt{C} (C - \lambda^2 y^+_a y^-_a)} \right) \] (4.19)

which is time-independent and coincides with the results in [23, 38].

Alternatively, we can apply our general result (3.19):

\[ z(1 - \frac{\epsilon}{\xi e^{-Q}})^2 = y^2 \quad \rightarrow \quad z(1 - \frac{\epsilon}{2\lambda^2 z})^2 = y^2, \quad \epsilon = \frac{cG_N \lambda^2}{3} \] (4.20)

which has two solutions

\[ z_1 = y^2 + \frac{\epsilon}{\lambda^2} - \frac{\epsilon^2}{4\lambda^4 y^2} + \mathcal{O}(\epsilon^3), \quad z_2 = \frac{\epsilon^2}{4y^2 \lambda^4} + \mathcal{O}(\epsilon^3). \] (4.21)

The solution \( z_1 \) leads to the trivial solution while the solution \( z_2 \) leads to (4.18).

5 Liouville gravity

A particular generalization of CGHS model is the one with exponential potential. The action is

\[
S = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[ RX + \sum_i 4\alpha_i^2 e^{\beta_i X} \right]. \] (5.1)
There are some interesting reasons to consider such exponential potentials. It is shown in \[42\] that this kind of model admits extra (conformal) symmetries. If we add \(2X\) in the potential, this kind of models as deformations of JT gravity is shown to have a matrix model dual \[41\]. So it means that

\[
U(X) = 0, \quad V(X) = -2 \sum_i \alpha_i^2 e^{\beta_i X}.
\]  

(5.2)

For simplicity, let us take \(k = 1\) and the model is called the Liouville gravity. Surprisingly, it is claimed in \[1\] that island formula can not resolve the information paradox of Liouville gravity based on the black hole solution which is found in \[42, 43\]. In this section, we will use the general solution of GDT to derive a different solution such that the island formula successfully resolves the information paradox.

### 5.1 The geometry

Given the potentials (5.2) we can compute the following data

\[
Q = 0, \quad X = \tilde{X} = r, \quad w = -\frac{2\alpha^2 e^{\beta X}}{\beta},
\]

(5.3)

\[
\xi = 2(w - C_0^L) = -2\left(\frac{2\alpha^2 e^{\beta X}}{\beta} + C_0^L\right), \quad \xi_0 = \sqrt{2C_0^L},
\]

(5.4)

where we choose \(\beta < 0\). The corresponding Schwarzschild metric is

\[
ds^2 = -(1 + \frac{2\alpha^2 e^{\beta \sqrt{2C_0^L}r}}{C_0^L \beta}) dt^2 + \frac{1}{1 + \frac{2\alpha^2 e^{\beta \sqrt{2C_0^L}r}}{C_0^L \beta}} dr^2.
\]

(5.5)

Therefore the horizon is at

\[
r'_H = \frac{1}{\beta} \log \left(\frac{-C_0^L \beta}{2\alpha^2}\right),
\]

(5.6)

and the Ricci scalar is

\[
R = -4\beta e^{\beta \sqrt{2C_0^L}} \alpha^2.
\]

(5.7)

So to ensure asymptotic flatness we can set

\[
\beta < 0, \quad C_0^L > 0, \quad \alpha^2 > 0,
\]

(5.8)

Note that the metric (5.5) is same as (4.7) if we identity

\[
\beta = -\frac{2\lambda}{\sqrt{2C_0^L}}, \quad \alpha^2 = \frac{C}{2\lambda \sqrt{\frac{C_0^L}{2}}},
\]

(5.9)
If we also set \( C_0^L = 2\lambda^2 \) such that \( \alpha^2 = \frac{C}{2}, \beta = -1 \) then the dilaton is given by

\[
X = \tilde{X} = 2\lambda y',
\]

thus the dilaton of the Liouville black hole is the Logarithm of the one of the CGHS black hole. Below we will keep \( C_0^L \) general. Similarly the geometry in the Kruskal coordinates are

\[
d^2 s = -\frac{d y^+ \, d y^-}{C - \lambda^2 y^+ y^-},
\]

\[
X = -\frac{1}{\beta} \log \left[ \frac{1}{2\lambda^2} \left( C - \lambda^2 y^+ y^- \right) \right].
\]

Since the geometry is same the entanglement entropy without island is also same as one in CGHS black hole. Let us focus on the entanglement entropy of the Hawking radiation in the presence of possible islands.

### 5.2 The derivation of island

In the late time, the entanglement entropy with island is given by:

\[
S_{\text{island}} = \frac{4}{G_N} \frac{\sqrt{2C_0^L}}{2\lambda} \log \left[ \frac{1}{2\lambda^2} \left( C - \lambda^2 y_d^+ y_d^- \right) \right] + \frac{c}{3} \log \left( \frac{(y_a - y_d)^+ (y_a - y_d)^-}{\sqrt{C' - \lambda^2 y_a^+ y_a^- \sqrt{C' - \lambda^2 y_d^+ y_d^-}}} \right).
\]

The extremal conditions are

\[
-\frac{4}{G_N} \frac{\sqrt{2C_0^L}}{2\lambda} \frac{\lambda^2 y_d^+}{C - \lambda^2 y_d^+ y_d^-} + \frac{c}{3(y_d^- - y_a^-)} + \frac{c y_d^+}{6(C - \lambda^2 y_d^+ y_d^-)} = 0,
\]

\[
-\frac{4}{G_N} \frac{\sqrt{2C_0^L}}{2\lambda} \frac{\lambda^2 y_d^-}{C - \lambda^2 y_d^+ y_d^-} + \frac{c}{3(y_d^+ - y_a^+)} + \frac{c y_d^-}{6(C - \lambda^2 y_d^+ y_d^-)} = 0.
\]

These equations are quadratic so can be easily solved. In the limit \( G_N \to 0 \) the two solutions behave as

\[
y_d^- = \frac{cG_N C \beta}{12\lambda^2 y_a^+} = -\frac{cG_N C}{6 \sqrt{2C_0^L y_a^+ \lambda}}, \quad y_d^+ = \frac{cG_N C \beta}{12\lambda^2 y_a^-} = -\frac{cG_N C}{6 \sqrt{2C_0^L y_a^- \lambda}},
\]

\[
y_d^- = y_a^- + \mathcal{O}(G_N), \quad y_d^+ = y_a^+ + \mathcal{O}(G_N).
\]

Let us try to derive these solutions directly from our general result (3.19):

\[
z(1 - \frac{e}{\xi e^{-\xi}})^2 = y_d^- \quad \rightarrow \quad z \left( 1 + \frac{e}{2C_0^L} \right) - \frac{eC}{2C_0^L \lambda^2 z} \right)^2 = y_d^-,
\]

\[3\text{We have double checked that the solutions indeed solve the equations of motion in the second order formalism.}\]
with

\[ \epsilon = \frac{c G_N \lambda \sqrt{2C_0'}}{6}. \]  

(5.19)

The equation (5.18) is also quadratic with solutions to be

\[ z_1 = \frac{C^2 \epsilon^2}{4C_0'^2 \lambda^4 y^2} + \mathcal{O}(\epsilon^3), \quad z_2 = y^2 - \frac{\epsilon(\lambda^2 y^2 - C)}{C_0'^2 \lambda^2} + \mathcal{O}(\epsilon^2) \]

(5.20)

which will correspond to (5.16) and (5.17), respectively. The first solution (5.16) is the non-trivial one which gives the generalized entanglement entropy

\[ S_{\text{island}} = 2 \frac{1}{G_N} \frac{\sqrt{2C_0'}}{\lambda} \log \left( \frac{C}{2\lambda^2} \right) + \frac{c}{3} \log \left( \frac{|y_+ y_-|}{\sqrt{C (C - \lambda^2 y_+ y_-)}} \right). \]

(5.21)

Thus we have derived the Page curve for the Liouville black hole. The reason why we succeed is that we have derived another black hole solution whose parameters are opposite to those in the solutions used in [1] or derived in [42, 43]. Let us revisit the black geometry which is used in [1].

### 5.3 The other black geometry

To get that solution we start from (5.5) and reverse the radial coordinate

\[ r' \to -r', \]

(5.22)

such that the metric becomes

\[ ds^2 = -(1 + \frac{2\alpha^2 e^{-\beta \sqrt{2C_0'} r'}}{C_0'^2 \beta}) dt'^2 + \frac{1}{1 + \frac{2\alpha^2 e^{-\beta \sqrt{2C_0'} r'}}{C_0'^2 \beta}} dr'^2. \]

(5.23)

The position of the event horizon and the Ricci scalar are

\[ r'_H = \frac{1}{\sqrt{2C_0' \beta}} \log \left( -\frac{2\alpha^2}{C_0' \beta} \right), \quad R = -4\beta \alpha^2 e^{-\sqrt{2C_0'} \beta r'}. \]

(5.24)

So requiring the asymptotic flatness at \( r' \to \infty \) forces the choice

\[ \beta > 0, \]

(5.25)

and having a well-defined horizon forces the choice

\[ \alpha^2 < 0. \]

(5.26)
With these choices, the solution in the Kruskal coordinates are still given by (5.11)

\[ d^2 s = - \frac{d y^+ d y^-}{C - \lambda^2 y^+ y^-}, \] (5.27)

\[ X = - \frac{1}{\beta} \log \left[ \frac{1}{2 \lambda^2} (C - \lambda^2 y^+ y^-) \right]. \] (5.28)

but with different identification

\[ \beta = \frac{2 \lambda}{\sqrt{2C_0}}, \quad \alpha^2 = - \frac{C}{2\lambda} \sqrt{\frac{C_0}{2}}, \] (5.29)

But in this black hole solution the position of the island is at

\[ y_d = \frac{c G_N C \beta}{12 \lambda^2 y_a^+} = \frac{c G_N C}{6 \sqrt{2C_0 y_a^+} \lambda}, \quad d^+ = \frac{c G_N C \beta}{12 \lambda^2 y_a^-} = \frac{c G_N C}{6 \sqrt{2C_0 y_a^-} \lambda}, \] (5.30)

which is in the left Kruskal patch. This contradicts the assumption that \( y_d \) is in the right patch and this is why [1] claims the failure of island formula. However we have shown this is only because a “wrong” solution is used.

To summarize, island can save the information paradox of Liouville gravity.

6 ab-family

In this section, we consider a large family of dilaton gravity theories which has the following potentials

\[ U(X) = - \frac{a}{X}, \quad V(X) = - \frac{B}{2} X^{a+b}. \] (6.1)

In general, there are two free parameters and sometimes this family is called the ab-family [44]. From our general discussion, the classical solution is

\[ d^2 s = 2X^{-a} d X d v - X^{-2} (2C_0 + \frac{BX^{b+1}}{b+1}) d^2 v. \] (6.2)

Because we are interested in the asymptotically flat black hole solutions \( \lim_{X \to \infty} R = 0 \) we will choose [44]

\[ b = a - 1, \quad a \in (0, 1). \] (6.3)

This choice can be understood form the behavior of Ricci scalar

\[ R = -2a C_0 X^{a-2} + \frac{bB(a - b - 1)}{b+1} X^{a+b-1}, \] (6.4)

\[ \text{here we consider the case } b \neq 1. \]
by noticing that $C_0$ is related to mass of black hole therefore the solution with $C_0 = 0$ should be the Minkowski spacetime. Following the general discussion we compute

$$Q = -a \log X, \quad X = (1 - a)\frac{1}{1-a} X^{1-a}, \quad w = -\frac{B}{2a} (1 - a)\frac{a}{1-a} X^{1-a}, \quad (6.5)$$

$$\xi = -\frac{B}{a} - \frac{2C_0^f}{(1 - a)\frac{a}{1-a} X^{1-a}}, \quad \xi_0 = \sqrt{\frac{B}{a}}. \quad (6.6)$$

Thus the corresponding Schwarzschild metric is

$$ds^2 = -(1 - \frac{1}{2\lambda^2 r'^{1-a}}) dt^2 + \frac{1}{1 - \frac{1}{2\lambda^2 r'^{1-a}}} dr'^2, \quad (6.7)$$

where

$$\frac{1}{2\lambda} = -\frac{2C_0^f a}{B} \left( \sqrt{\frac{B}{a}} (1 - a) \right)^{\frac{a}{a-1}}, \quad C_0^f < 0, \quad \lambda > 0. \quad (6.8)$$

So the horizon is located at

$$r_H' = (2\lambda)^{\frac{a-1}{a}}. \quad (6.9)$$

Next we can transform it the conformal gauge by introducing

$$x^* = \int \frac{dr'}{1 - \frac{1}{2\lambda^2 r'^{1-a}}} = 2(a - 1)\lambda r'^{\frac{1}{1-a}} _2F_1 \left( 1, \frac{1}{a}; 1 + \frac{1}{a}; 2r'^{\frac{a}{1-a}} \lambda \right) + c_1, \quad (6.10)$$

where $c_1$ is a constant which can be chosen for our convenience. The hypergeometric function generally can not be inverted to write $r'(x^*)$ as a function of $x^*$ explicitly. However for the special case of $a = 1/2$, we can invert the function with product logarithm:

$$x^* = r' + \frac{\log(2\lambda r' - 1)}{2\lambda} + c_1, \quad \rightarrow \quad (6.11)$$

$$r' = \frac{1}{2\lambda} + \frac{W_0(e^{2\lambda(x^*-c_1)-1})}{2\lambda}, \quad (6.12)$$

where $W_0$ is the principle branch of the Lambert $W$ function or product logarithm. Thus it is natural to introduce the Kruskal coordinates as

$$e^{\pm\lambda x^\pm} = \pm \lambda y^\pm, \quad c_1 = -\frac{1}{2\lambda}, \quad (6.13)$$

$$ds^2 = -e^{2\rho} dy^+ dy^-, \quad e^{2\rho} = \frac{1}{e^{W_0(-\lambda^2 y^2)} - \lambda^2 y^2}, \quad (6.14)$$

$$X = \sqrt{\frac{B}{4\lambda}} \frac{1}{2} \left( 1 + W_0(-\lambda^2 y^2) \right), \quad y^2 = y^+ y^- \quad (6.15)$$
The equation for determining the position of island becomes
\[
z \left( \frac{e^{1/4} \sqrt{\lambda} W_0 (\lambda^2(-z)) + 1}{B^{5/4} W_0 (\lambda^2(-z))} + 1 \right)^2 = y^2, \quad \epsilon = \frac{c G_N \lambda}{6} \sqrt{\frac{B}{a}}. \quad (6.16)
\]
Assuming \( \epsilon \to 0 \) we can expand the left-hand side to the first order of \( z \) then the we
will obtain two solutions
\[
z_1 = y^2 + \frac{2^{1/4} \alpha (2 - 3y^2 \lambda^2)}{B^{5/4} \lambda^{3/2}} + \mathcal{O} (\epsilon^2), \quad z_2 = \frac{\beta^2}{y^2}, \quad \beta^2 = \frac{\sqrt{2} \epsilon^2}{B^{5/2} \lambda^3}. \quad (6.17)
\]
Thus the physical solution is
\[
y^+_d = \frac{\beta}{y_a}, \quad y^-_d = \frac{\beta}{y^+_a}, \quad (6.18)
\]
which leads to
\[
S_{\text{island}} = \sqrt{\frac{B}{2G_N \lambda}} + \frac{c}{3} \log \frac{|y^+_a y^-_a|}{\sqrt{e^{W_0(-\lambda^2 y^2)} - \lambda^2 y^2}} + \mathcal{O}(G_N). \quad (6.19)
\]
For generic \( a \), we observe that \( x^* \) can be always decomposed into
\[
x^* = f(r') + \frac{1 - a}{a} (2\lambda) \frac{a - 1}{a} \log(r' - r'_H), \quad (6.20)
\]
where \( f(r') \) is regular at \( r' = r'_H \) as we expect in (3.23). Here we omit the further analysis since it is very similar to the general result.

## 7 Reissner-Nordstrom

Islands in charged black hole have been studied in [31–34]. Even though they considered 4-dimensional black holes, effectively and technically the model is still 2-dimensional after a dimensional reduction of the two sphere. Therefore we can also study them with our general procedure. To support a Reissner-Nordstrom black hole, the simplest choice of potentials are
\[
U(X) = -\frac{1}{2X}, \quad V(X) = -\lambda^2 + \frac{A}{X} \quad (7.1)
\]
which lead to the following data
\[
e^Q = \frac{1}{\sqrt{X}}, \quad X = \frac{\tilde{X}^2}{4}, \quad w = -\frac{2(A + \lambda^2 X)}{\sqrt{X}} = -\lambda^2 \tilde{X} - \frac{4A}{\tilde{X}}, \quad (7.2)
\]
\[
\xi = -4\lambda^2 - \frac{4C_0 R}{\tilde{X}} - \frac{16A}{X^2}, \quad \xi_0 = 2\lambda. \quad (7.3)
\]
Thus the metric and dilaton are
\[
d s^2 = -(1 + \frac{C_0^R}{2r^3\lambda^3} + \frac{A}{r^r\lambda^4})dt^2 + \frac{1}{1 + \frac{C_0^R}{2r^3\lambda^3} + \frac{A}{r^r\lambda^4}}dr^2, \tag{7.4}
\]
\[X = \lambda^2 r'^2. \tag{7.5}\]

Comparing with the standard Reissner-Nordstrom we can identify the following parameters
\[
C_0^R = -4M\lambda^3, \quad A = \lambda^4 Q_c^2, \tag{7.6}
\]
\[M = \frac{r_+ + r_-}{2}, \quad Q_c = \sqrt{r_+ r_-}, \tag{7.7}\]
where $M$ and $Q_c$ are the mass and charge of the black hole and $r_\pm$ are positions of the outer (+) and inner (−) horizons. Next we introduce new variable
\[
x^* = \int \frac{dr'}{1 - \frac{2M}{r'} + \frac{Q_c^2}{r'^2}} = r' + \frac{r_+^2 \log(r' - r_+) - r_-^2 \log(r' - r_-)}{r_+ - r_-}, \tag{7.8}
\]
\[\exp(2x^*) = e^{2x'}(r' - r_+)^{x_+ - x_-}(r' - r_-)^{-x_+ + x_-}, \tag{7.9}\]
to get the conformal metric
\[
d s^2 = -H(r') dx^+ dx^-, \quad H(r') = 1 - \frac{2M}{r'} + \frac{Q_c^2}{r'^2}. \tag{7.10}\]
The Kruskal coordinates can be defined as
\[
e^{lx^+} = ly^+, \quad e^{-lx^-} = -ly^-, \quad l = \frac{r_+ - r_-}{2r_+^2}. \tag{7.11}\]
In order to use (3.19) to solve the position of the island, we first express $d$ in terms of $z$. Using (7.9) we can directly get
\[-l^2 z = e^{2d}(d - r_+)(d - r_-) \frac{r_+^2}{r_-^2}, \tag{7.12}\]
which leads to
\[d = r_+ - e^{-2l}r_+^2(r_+ - r_-) \frac{r_-^2}{r_+^2} z + O(z^2). \tag{7.13}\]
Recall that we expect that $z \to 0$. Substituting (7.13) into (3.19) with help of (7.2) and (7.3) we end up an equation of $z$. The non-trivial solution is
\[z = \frac{\beta^2}{y^2}, \quad \beta^2 = \frac{e^2 e^{4l}r_+^2(r_+ - r_-) \frac{r_-^2}{r_+^2} - 2y^2}{16l^4\lambda^6y^2}, \quad \epsilon = \frac{cG_N l \lambda}{3}, \tag{7.14}\]
\[\dot{y}^+_a = -\frac{\beta}{y_a^+}, \quad \dot{y}^-_a = -\frac{\beta}{y_a^-}. \tag{7.15}\]
7.1 Other Charged dilaton Black Hole I

We can also consider island in other charged dilaton black hole. In [33], the charged dilaton black hole has the metric

\[ ds^2 = -r^2 \left( 1 - \frac{2M}{r^2} + \frac{Q_c^2}{4r^4} \right) dt^2 + \left( 1 - \frac{2M}{r^2} + \frac{Q_c^2}{4r^4} \right)^{-1} dr^2 + r^2(dx^2 + dy^2) \] (7.16)

The effective 2D model is

\[ ds^2 = -H(r) dt^2 + r^2 H(r)^{-1} dr^2, \quad X = r^2, \] (7.17)

where

\[ H(r) = r^2 \left( 1 - \frac{2M}{r^2} + \frac{Q_c^2}{4r^4} \right). \] (7.18)

To transform to the Schwarzschild metric let us introduce

\[ dr' = 2r \, dr, \quad \rightarrow, \quad r' = r^2, \] (7.19)

such that

\[ ds^2 = -H(r') dt'^2 + H(r')^{-1} dr'^2, \quad t' = t/2, \] (7.20)

\[ H(r') = 4r' - 8M - \frac{Q_c^2}{r'^2} = \frac{4(r - r_+)(r - r_-)}{r'}, \quad r_\pm = M \pm \sqrt{4M^2 - Q_c^2}. \] (7.21)

The geometry is asymptotically flat. The outer event horizon and curvature singularity are located at \( r' = r_+ \) and \( r' = 0 \) respectively. The solution can be embedded into dilaton gravity by choosing the possible potentials to be

\[ U(X) = 0, \quad V(X) = -2 + \frac{Q_c^2}{2X^2}, \quad C_0 = 4M. \] (7.22)

When \( Q_c = 0 \), the geometry (7.20) reduces to Rindler patch. From the potential we obtain

\[ Q = 0, \quad X = \tilde{X} = r', \quad w = -2X - \frac{Q_c^2}{2X}, \quad \xi = 8M - \frac{Q_c^2}{X} - 4X. \] (7.23)

Next we introduce new variable

\[ x^* = \int \frac{dr'}{H(r')} = \frac{r_+}{4(r_+ - r_-)} \log(r' - r_+) - \frac{r_-}{4(r_+ - r_-)} \log(r' - r_-), \] (7.24)

\[ e^{2x^*} = (r' - r_-)^{-\frac{r_-}{r_+}} (r' - r_+), \quad \frac{1}{2l} = \frac{r_+}{4(r_+ - r_-)}. \] (7.25)
Following the general procedure we find that in the late time the position of the island is
\[ d = r_+ - l^2 (r_+ - r_-)^{\frac{2}{1 + n}} z, \]  
\[ z = \frac{\beta^2}{y^2}, \quad \beta^2 = \frac{\epsilon^2 r_+^2 (r_+ - r_-) 2(r_+ + r_-)}{16 l^4}, \quad \epsilon = \frac{c G_N l}{6}, \]  
\[ y_d^+ = -\frac{\beta}{y_a^-}, \quad y_d^- = -\frac{\beta}{y_a^+}, \]  
which coincides with the results in [33].

7.2 Other Charged dilaton Black Hole II

In [32], the charged dilaton black hole has the metric
\[ ds^2 = -W(r) dt^2 + W^{-1} dr^2 + R(r)^2 d\Omega^2, \]  
with the function
\[ W(r) = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^n, \quad R^2 = r^2 \left(1 - \frac{r_-}{r}\right)^{1-n}, \quad n \in [0, 1). \]  
The effective 2D model is
\[ ds^2 = -W(r) dt^2 + W^{-1} dr^2, \quad X = r^2 \left(1 - \frac{r_-}{r}\right)^{1-n} \equiv f(r). \]  
The corresponding 2d dilaton potentials can be
\[ e^{Q} = \frac{d f^{-1}(X)}{d X} \equiv f^{-1'}, \quad U(X) = \frac{d \ln(f^{-1'})}{d X}, \]  
\[ V(X) = -\frac{1}{2} e^{-Q(x)} \frac{d (e^{-Q(x)} W(f^{-1}(X)))}{d X}. \]  
In general, \( f(r) \) is hard to invert but to solve the island the explicit expressions of the potentials are not needed. We only need the following quantity
\[ \xi e^{-Q} = -W \frac{d X}{d r} = -\frac{(2r - (1 + n)r_-)(r - r_+)}{r}, \]  
which appears in (3.19) and the relation between \( r \) and \( z \):
\[ x^* = \int \frac{dr}{W(r)} = \frac{r^n}{(r - r_-)^{n-1}} + (nr_- + r_+) B_{1-n}(1 - n, 0) \]  
\[- \left( \frac{r_+}{r_+ - r_-} \right)^n B_n(1 - n, 0), \quad t = \frac{r_+}{r_+ - r_-} \left(1 - \frac{r_-}{r}\right), \]  
\[ r_+ = \frac{1 + r_-}{r_-}, \quad \beta_+ = \frac{1}{1 + n} \beta^- . \]
where $B_\alpha(a, b)$ is the incomplete beta function. Note that

$$\lim_{t \to 1} B_t(1 - n, 0) = -\log(t - 1) \quad (7.36)$$

thus let us denote $x^*$ as

$$x^* = R + \left( \frac{r_+}{r_+ - r_-} \right)^n \log(r - r_+), \quad (7.37)$$

$$e^{2lx^*} = e^{2lR} (r - r_+), \quad \frac{1}{2l} = \left( \frac{r_+}{r_+ - r_-} \right)^n. \quad (7.38)$$

It implies that

$$d = r_+ - t^2 e^{-2lR(r_+)} z + O(z^2). \quad (7.39)$$

Substituting into (3.19) we can solve

$$z = \frac{\beta^2}{y^2}, \quad \beta = \frac{cr_+}{e^{-2lR(r_+)} (2r_+ - r_- - nr_-)}, \quad \epsilon = \frac{cG_N}{6}, \quad (7.40)$$

$$y_d^+ = -\frac{\beta}{y_a^+}, \quad y_d^- = -\frac{\beta}{y_a^+}, \quad (7.41)$$

which is consistent with the result in [32] while our method is much simpler.

### 8 Kaluza-Klein black holes

Our last example is the 4-dimensional Kaluza-Klein black hole. The island of this black hole is studied in [35]. The metric of a non-rotating KK black hole in 4d asymptotically flat spacetime is

$$\mathrm{d}s^2 = -W(r) \, \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{W(r)} + H^{1/2} r^2 \, \mathrm{d} \Omega^2, \quad (8.1)$$

where

$$W(r) = \frac{f(r)}{\sqrt{H(r)}}, \quad f(r) = 1 - \frac{r_h}{r}, \quad H(r) = 1 + \frac{Q_c}{r}. \quad (8.2)$$

When $Q = 0$, (8.1) is just the 4 dimensional Schwarzschild black hole and the corresponding dilaton gravity model has the potentials

$$U(X) = -\frac{1}{2X}, \quad V(X) = -\lambda^2. \quad (8.3)$$
To embed the solution (8.1) into dilaton gravity we can first identify
\[ X = H^{1/2}r^2 \equiv f(\tilde{X}), \quad r = \tilde{X}. \quad (8.4) \]
Supposing that \( f(\tilde{X}) \) is invertible, we can solve \( \tilde{X} = f^{-1}(X) \). The potentials of the corresponding GDT can be found with (7.32) and (7.33). For the solution (8.1), the results are very involved. The results in leading order of \( Q \) are
\[ U(X) = -\frac{1}{2X} + \frac{3Q^2}{32X^2}, \quad V(X) = -1 + \frac{r_h Q}{4X} + \frac{Q^2(2\sqrt{X} - 3r_h)}{8X^{3/2}}. \quad (8.5) \]
To compute the generalized entanglement entropy we can directly use the 2d dilaton gravity solutions
\[ ds^2 = -W(r) dt^2 + \frac{dr^2}{W(r)}, \quad (8.6) \]
\[ X = \sqrt{1 + \frac{Q_c}{r} r^2}, \quad W(r) = (1 - \frac{r_h}{r})(1 + \frac{Q_c}{r})^{-\frac{1}{2}}, \quad (8.7) \]
which are already in the Schwarzschild coordinates. To derive the island first we compute
\[ \xi e^{-Q} = -W \frac{dX}{dr} = -\frac{(4r + 3Q_c)}{2(r + Q_c)}(r - r_h), \quad (8.8) \]
and
\[ x^* = \int \frac{dr}{W(r)} = \sqrt{r(Q_c + r)} + (Q_c + r_h) \sinh^{-1} \left( \frac{r}{\sqrt{Q_c}} \right) \]
\[ - \sqrt{r_h(Q_c + r)} \log \left( 1 + \frac{r(Q_c + r)}{r_h(Q_c + r)} \right) + \sqrt{r_h(Q_c + r)} \log \left( 1 - \sqrt{r_h(Q_c + r)} \right) \]
\[ = R + \sqrt{r_h(Q_c + r)} \log(r - r_h), \quad (8.9) \]
where \( R \) is again regular at \( r = r_h \). Then we can solve \( d \) in terms of \( z \):
\[ e^{2x^*} = e^{2lR}(r - r_h), \quad \frac{1}{2l} = \sqrt{r_h(Q_c + r)}, \quad (8.11) \]
\[ d = r_h - l^2 e^{-2lR(r_h)} z. \quad (8.12) \]
Substituting into (3.19) we find
\[ z = \frac{\beta^2}{y^2}, \quad \beta = \frac{2e(r_h + Q_c)}{e^{-2lR(r_h)}(3Q_c + 4r)} \quad \epsilon = \frac{cG_N}{6}, \quad (8.13) \]
\[ y^+_d = -\frac{\beta}{y^-}, \quad y^-_d = -\frac{\beta}{y^+_d}. \quad (8.14) \]
9 Conclusion and Discussion

In this work we have studied island formula (1.1) in the general asymptotically flat eternal black holes in GDT. Under some reasonable and mild assumptions we prove that the island always appears barely outside of the horizon in the late time of Hawking radiation so that the information paradox is resolve, in particular, in the Liouville gravity theory in which it was reported in [1] that the island proposal failed. We find that failure is due to the use of a “wrong” black hole solution. With the help of general construction of classical solutions of GDT we find a different black hole solution where the island appears as expected. We further apply our general analysis to a large family of GDT and several 4-dimensional black holes including different charged dilaton black holes and the KK black hole. It turns out that our procedure for finding island is much simpler.

There are some possible generalizations of our analysis.

• Our general analysis should be simply generalized to the asymptotically AdS black holes in GDT by gluing a flat bath. Since after gluing the flat bath, the whole spacetime is similar to the asymptotically flat black hole and cut-off surface $y_a$ can be chosen to be boundary of the AdS space.

• In this work we only consider the classical solutions of GDT. It is also possible to include the quantum effect which comes from the conformal anomaly following for example [24].

• It is also possible to generalize our results to single-sided black hole and consider a truly evaporating black hole. Some examples are [24, 28].

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A Review of 2D GDT

Conventions

The local Lorentz metric and the Lorentz transformation invariant tensor are chosen to be

$$\eta_{ab} = \eta^{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \epsilon^a_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (A.1)$$
Thus the Levi-Civita tensors are
\[ \epsilon_{ac} = \epsilon_{cb} \eta^{bc} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_{ab} = \eta_{ac} \epsilon^c_b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \] (A.2)

The volume form is related to the local Lorentz basis \( e^a \) via
\[ \epsilon = \frac{1}{2} \epsilon_{ab} e^a \wedge e^b = \frac{1}{2} \epsilon_{ab} e^a_{\mu} dx^\mu \wedge dx^\nu = \frac{1}{2} \epsilon_{ab} (e^a_1 e^b_0 - e^a_0 e^b_1) dx^1 \wedge dx^0 \] (A.3)
\[ = (e^1 e^0 - e^0 e^1) dx^1 \wedge dx^0 = \sqrt{-g} dx^1 \wedge dx^0 \rightarrow \sqrt{-g} d^2 x. \] (A.4)

In 2d the spin connection should be proportional to \( \epsilon \):
\[ \omega^a_{\ b} = \epsilon^a_b \] and Ricci tensor
\[ R_{ab} = d \omega_{ab}, \quad R_{ab} = \epsilon_{ab} (\partial_\mu \omega^\mu_{ab} - \partial_b \omega_{\mu a}). \] (A.5)
So the Ricci scalar is
\[ (R_{\mu \nu})_{ab} e^a_\mu e^b_\nu = 2 \epsilon_{\mu \nu} (\partial_\mu \omega^\mu_{ab} - \partial_b \omega^\mu_{\mu}) = 2 \epsilon_{\mu \nu} \partial_\mu \omega^\mu_{ab} \] (A.6)
\[ = 2 |\epsilon|^{-1} (\partial_0 \omega_1 - \partial_1 \omega_0) \] (A.7)

On the other hand we have
\[ d \omega = \partial_\mu \omega_{\nu ab} dx^\mu \wedge dx^\nu = (\partial_1 \omega_0 - \partial_0 \omega_1) dx^1 \wedge dx^0 = -\frac{1}{2} R \sqrt{-g} d^2 x. \] (A.8)
The torsion two-form is given by
\[ T^a = (D)^a_{\ b} e^b = (\delta^a_{\ b} d + \omega^a_{\ b}) e^b = d e^a + \omega^a_{\ b} \wedge e^b, \] (A.9)
with its components are
\[ T_{\mu \nu}^a = \partial_\mu e^a_{\nu} - \partial_\nu e^a_{\mu} + (\omega^a_{\ b})_{\ b} e^b_{\nu} - (\omega^a_{\ b})_{\ b} e^b_{\mu} = D_\mu e^a_{\nu} - D_\nu e^a_{\mu}. \] (A.10)

It is convenient to use the Light-cone gauge:
\[ x^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^1), \quad x^0 = \frac{1}{\sqrt{2}} (x^+ + x^-), \quad x^1 = \frac{1}{\sqrt{2}} (x^+ - x^-). \] (A.12)
The Lorentz transformation connecting these gauges is
\[ \Lambda_{\ a}^a = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}. \] (A.13)

Thus we can find that
\[ \eta_{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^a_{\ b} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] (A.14)
such that the torsion form \( (A.10) \) can be expressed as
\[ T^\pm = (d \pm \omega) e^\pm. \] (A.15)
A.1 The first order formalism of GDT

The action \( (2.1) \) is equivalent to

\[
I_{\text{gen}}[e_a, \omega, X, X^a] = \int \left( X \, d\omega + X_a(d \, e^a + e^a_b \omega \wedge e^b) + \frac{1}{2} e^{ab} e_a \wedge e_b V(X, X^c X^c) \right). 
\]

(A.16)

We will first solve all its classical solution and then prove the equivalence. Varying with respect to \( \omega \) gives

\[
X \, d\delta\omega + X^a e_{ab} \delta\omega \wedge e^b = -d X \wedge \delta\omega - X^a e_{ab} e^b \wedge \delta\omega \rightarrow 
\]

\[
d X + X^a e_{ab} e^b = 0. 
\]

(A.17)

Varying with respect to \( e \) we get

\[
X^a d \delta e_a + X^a e^b_a e^b \delta e_b + \frac{1}{2} e^{ab} (\delta e_a \wedge e_b - e_a \wedge \delta e_b) V \rightarrow 
\]

\[
-d X^a \wedge \delta e_a + X^b e^b_a \omega \wedge \delta e_a - e^{ab} e_b \wedge \delta e_a V \rightarrow 
\]

\[
dX^a - X^b e^b_a \omega + e^{ab} e_b V = dX^a + X^b e^b_a \omega + e^{ab} e_b V = 0, 
\]

(A.18)

where in the last line we used \( e^a_b = \eta_{bc} e^c_d \eta^{da} = -e^a_b \). The other two equations of motion are

\[
d\omega + \frac{1}{2} e^{ab} e_a \wedge e_b \frac{\partial V}{\partial X} = 0, 
\]

(A.19)

\[
d e_a + e^b_a \omega \wedge e_b + \frac{1}{2} e^{ab} e_a \wedge e_b \frac{\partial V}{\partial X^a}. 
\]

(A.20)

In the light-cone gauge the equations of motion become

\[
d X + X^+ e^- - X^- e^+ = 0, 
\]

(A.21)

\[
(d \pm \omega) X^\pm \pm \nu e^\pm = 0, 
\]

(A.22)

\[
d \omega + \frac{\partial V}{\partial X} = 0, 
\]

(A.23)

\[
(d \pm \omega) e^\pm + \epsilon \frac{\partial V}{\partial X^\pm} = (d \pm \omega) e^\pm - \epsilon \frac{\partial V}{\partial X^\mp} = 0, 
\]

(A.24)

where the volume form is \( \epsilon = e^+ \wedge e^- \) and in the last line we have used \( X_\pm = -X^\mp \). From (A.22) we get

\[
X^- d X^+ + X^+ d X^- + \nu (X^- e^+ - X^+ e^-) = 0, 
\]

(A.25)
then using (A.21) we get

$$d(X^-X^+) + \mathcal{V}(X^-X^+,X) \, dX = 0.$$  \hfill (A.26)

This equation indicates that there exists a conserved quantity defined by integrating (A.26).

If $X^+ \neq 0$, from (A.22) we can get

$$\omega = -\frac{dX^+}{X^+} - Z \mathcal{V}, \quad Z \equiv \frac{e^+}{X^+},$$  \hfill (A.27)

and from (A.21) we can get

$$e^- = -\frac{dX^+}{X^+} + X^- Z.$$  \hfill (A.28)

Substituting the expression of volume form

$$\epsilon = \frac{1}{2} \epsilon_{ab} e^a \wedge e^b = e^+ \wedge e^- = dX \wedge Z$$  \hfill (A.29)

into (A.24) gives

$$d e^+ + \omega \wedge e^+ - dX \wedge Z \frac{\partial \mathcal{V}}{\partial X^-} = 0$$  \hfill (A.30)

$$= X^+ dZ + dX^+ \wedge Z - dX^+ \wedge Z - dX \wedge Z \frac{\partial \mathcal{V}}{\partial X^-} = 0.$$  \hfill (A.31)

Therefore we end up with

$$dZ = -Z \wedge \frac{dX}{X^+} \frac{\partial \mathcal{V}}{\partial X^-}.$$  \hfill (A.32)

Taking the ansatz of $Z$ as

$$Z = d e^{Q(X)} , \quad dZ = e^{Q(X)} \frac{dQ}{dX} \, dX \wedge dv ,$$  \hfill (A.33)

and substituting into (A.32) gives

$$\frac{dQ}{dX} = \frac{1}{X^+} \frac{\partial \mathcal{V}}{\partial X^-} , \rightarrow$$  \hfill (A.34)

$$Q = \int^{X} X^+ \frac{\partial \mathcal{V}}{\partial X^-} .$$  \hfill (A.35)

Recall that the metric is

$$d s^2 = \eta_{ab} e^a \cdot e^b = -2 e^+ e^- = 2(Z \, dX - X^+ X^- Z^2) = 2e^Q(dv \, dX - e^Q Y \, d^2 v)(A.36)$$
where \( Y \equiv X^+X^- \). So all solutions\(^5\) for all generalized dilaton gravity models obey a generalized Birkhoff theorem, in the sense that all solutions exhibit a Killing vector \( \partial_v \).

The solution space is parameterized by two constants of integration. The one coming from the integration of (A.26) is non-trivial, while the one coming from (A.35) is trivial and can be fixed by a choice of units.

### A.2 Back to Second order formalism

First we separate out the torsion-free part of the spin-connection. To do that we notice

\[
\ast T_a = \ast(d e_a + \epsilon_a^b \omega \wedge e_b) = \ast d e_a + \epsilon_a^b \omega^c \ast (e_c \wedge e_b). \tag{A.37}
\]

Using

\[
e_a \wedge e_b = e_{a \mu} e_{b \nu} d x^\mu \wedge d x^\nu, \tag{A.38}
\]

\[
\ast e_a \wedge e_b = e_{a \mu} e_{b \nu} \ast d x^\mu \wedge d x^\nu = e_{a \mu} e_{b \nu} \epsilon^{\mu \nu} = \epsilon_{ab} \tag{A.39}
\]

we get

\[
\ast T_a = \ast d e_a + \epsilon_a^b \omega^c \epsilon_{cb} = \ast d e_a - \omega_a. \tag{A.40}
\]

So we can rewrite the spin-connection as

\[
\omega = \omega^a e_a = (\ast d e_a - \ast T_a) e^a = e^a \ast d e_a - \epsilon^a \ast T_a. \tag{A.41}
\]

Then \( \tilde{\omega} = e^a \ast d e_a \) is the torsion-free part which in terms of components is given by

\[
\ast d e_a = \partial_{\mu}(e_{\nu})_a e^{\mu \nu}, \quad \tilde{\omega} = e^a \partial_{\mu}(e_{\nu})_a e^{\mu \nu}. \tag{A.42}
\]

Recall that the action in the first formalism is

\[
I_{\text{gen}} \sim \int X d \omega + e^c + X^a T_a. \tag{A.43}
\]

The first term can be manipulated as

\[
X d \omega = -d X \wedge \omega = -d X \wedge (\tilde{\omega} - \epsilon^a \ast T_a) = X d \tilde{\omega} + d X \wedge \epsilon^a \ast T_a. \tag{A.44}
\]

Note that

\[
d \tilde{\omega} = \partial_{\mu} \omega_{\nu} d x^\mu \wedge d x^\nu \rightarrow \frac{R}{2} \sqrt{-g} d^2 x, \tag{A.45}
\]

\(^5\)in the linear dilaton vacua
which is exactly the first term in the action (2.1). It is obvious that
\[ \epsilon \mathcal{V}(X, X^a X_a) \rightarrow \sqrt{-g} \mathcal{V}(X, X^a X_a) \, d^2 x. \] (A.46)

So the last thing to do is to remove \( X^a \) with the help of equation of motion (A.20):
\[ T_a = -\frac{1}{2} \epsilon^{bc} e_b \wedge e_c \frac{\partial \mathcal{V}}{\partial X^a} \rightarrow \] (A.47)
\[ \star T_a = -\frac{1}{2} \epsilon^{bc} \epsilon_{bc} \frac{\partial \mathcal{V}}{\partial X^a} = \frac{\partial \mathcal{V}}{\partial X^a} \] (A.48)
and (A.17):
\[ \partial_{\mu} X + X^b \epsilon_a \epsilon_{b\mu} = 0 \rightarrow \] (A.49)
\[ \partial_{\mu} X + X^a \epsilon_\mu \epsilon_{\nu} = 0 \rightarrow \] (A.50)
\[ X^a = -\epsilon_a \epsilon^\mu \partial_{\mu} X. \] (A.51)

So terms involved with \( T^a \) are cancelled to each other:
\[ X^a T_a = \frac{1}{2} \epsilon^a_\mu \epsilon_{\nu \mu} \partial_{\nu} X \epsilon^{eb} \wedge e_c \frac{\partial \mathcal{V}}{\partial X^a} \rightarrow \epsilon^a_\mu \epsilon_{\nu \mu} \partial_{\nu} X \frac{\partial \mathcal{V}}{\partial X^a} \sqrt{-g} \, d^2 x, \] (A.52)
\[ d X \wedge \epsilon^a \star T_a = \partial_{\mu} X \epsilon^a_\nu \frac{\partial \mathcal{V}}{\partial X^a} \, d x^\mu \wedge d x^\nu \rightarrow -\epsilon^a_\mu \epsilon_{\nu \mu} \partial_{\nu} X \frac{\partial \mathcal{V}}{\partial X^a} \sqrt{-g} \, d^2 x. \] (A.53)

Then we arrive at the action in second order formalism
\[ -\frac{1}{2} \int \sqrt{-g} \left( X R - 2 \mathcal{V}(X, -(\partial X)^2) \right), \] (A.54)
where we have used
\[ X^a X_a = -((\partial X)^2). \] (A.55)

Therefore we find that the \( Q \) function is given by
\[ Q = \int^X U(y) \, dy, \] (A.56)
and the conserved quantity (A.26) is given by
\[ C_0 = e^Q Y + w, \quad Y = X^+ X^-, \quad w = \int^X e^Q V(y) \, dy \] (A.57)
\[ dC_0 = e^Q dY + Ye^Q U(X) dX + e^Q V dX = 0. \] (A.58)

Using this we can rewrite the metric as
\[ d s^2 = 2e^Q d v (d X + (w(X) - C_0) \, d v) \] (A.59)
\[ = 2d v d \bar{X} + \xi(\bar{X}) \, d v^2, \] (A.60)
where we have introduced
\[ d \bar{X} = d X e^Q, \quad \xi(\bar{X}) = 2e^Q (w - C_0). \] (A.61)
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