Superrenormalizable quantum gravity with complex ghosts

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A B S T R A C T
We suggest and briefly review a new sort of superrenormalizable models of higher derivative quantum gravity. The higher derivative terms in the action can be introduced in such a way that all the unphysical massive states have complex poles. According to the literature on Lee–Wick quantization, in this case the theory can be formulated as unitary, since all massive ghosts-like degrees of freedom are unstable.

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1. Introduction

One of the main theoretical problems concerning quantum gravity is a well-known conflict between renormalizability and unitarity. Quantum gravity theory based on general relativity is not renormalizable by power counting, since the loop expansion parameter has inverse-mass dimension. Renormalizability can be achieved by introducing fourth derivative terms into the action, because in such a theory the main coupling constant (parameter of the loop expansion in the UV) is dimensionless [1] (see also [2] for an introduction). However, in the case the particle spectrum of the theory includes unphysical massive ghosts which can not be removed without violating unitarity of the S-matrix. The possibility to solve the problem by using the imaginary poles in the dressed propagator of gravitons was discussed in a number of remarkable papers [3–5], but the final conclusion was that a definite solution requires full information about these complex poles at the non-perturbative level [6], that is far away from the state of art in quantum gravity (regardless of an interesting attempt in this direction [7]).

Introducing into the starting action some extra terms with more than four derivatives of the metric and with real massive poles provides a superrenormalizable theory, since the loop expansion parameter in this case has positive mass dimension. However, such modification does not change situation with ghosts, since they remain in the spectrum of the theory [8]. An interesting possibility which should be mentioned is to introduce a specially tuned terms which are non-polynomial in derivatives1 that can provide a ghost-free structure of the theory at the tree level [12] (see also [13]). However, one can prove that taking loop corrections into account the dressed propagator in such a theory gains infinitely many ghost-like poles [14], all of them with complex squares of “masses”. This situation shows that the “ghost-free” model of [9] and [12] is a non-local generalization of the historically first model of superrenormalizable theory of quantum gravity [8]. The ghosts-like states are indeed present in both local polynomial and non-local and non-polynomial versions of the theory. The difference is that in the non-polynomial case ghosts show up only after quantum corrections are taken into account, and that the number of such ghosts is infinite.

In the mentioned conflict between renormalizability and unitarity there is an unexplored possibility which we start to consider here. Namely, in the present work we discuss a new sort of superrenormalizable quantum gravity theory, when all massive states correspond to the complex poles. According to existing literature (see, e.g., [17–20]), in this case the theory can be formulated as

1 This was originally done for the gravitational theories in by Tseytlin [9] (see also [10]) in order to provide the singularity-free modified Newtonian limit. Recently the non-singular potential was “rediscovered” in [11] without the use of auxiliary fields.

2 More recently this theory has been generalized to any dimension [15] and explicitly showed to be finite at any order in the loop expansion in both odd and even dimensions when some extra local operators (only two of them are necessary in D = 4) are included [16].
unitary, since all ghosts are unstable. Due to the existence of these works, we mainly need to review them and discuss possible applications to higher derivative gravity models. The organization of the manuscript is as follows. In Sect. 2 the superrenormalizable models of quantum gravity are briefly reviewed. After this we consider the simplest such model with six derivatives and obtain the conditions for the complex massive poles. In Sect. 3 there is a discussion of existing works on the Quantum Field Theories (QFT) with complex poles and the application of complex poles to gravity with higher derivatives. We also present an example of how it works on a toy model with higher derivative insertion (polynomial in the case, but in principle it can be generalized to the general version). In Sect. 4 the stability of Lee–Wick unitarity under the radiative corrections is discussed, for both superrenormalizable and finite versions of higher-derivative quantum gravity. Finally, in Sect. 5 we draw our conclusions and present some discussions.

2. Superrenormalizable gravity with complex poles

The action of the general superrenormalizable polynomial model [8] can be written as

\[
S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R + 2\Lambda \right) + \int d^4x \sqrt{-g} \left[ c_1 R_{\mu\nu\alpha\beta} + c_2 R_{\mu\nu}^2 + c_3 R^2 + d_1 R_{\mu\nu\alpha\beta}^2 R^{\mu\nu\alpha\beta} + 2d_2 R_{\mu\nu} R^{\mu\nu} + d_3 R R^2 + d_4 R^2 + d_5 R R_{\mu\nu}^2 R_{\mu\nu} + \ldots + f_1 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + f_2 R_{\mu\nu} R_{\mu\nu} R^{\mu\nu} + f_3 R^2 R + \ldots + f_{4,5,\ldots} k^{k+2} \right].
\]

Here the first integral is the Einstein–Hilbert action with cosmological constant and the second includes higher derivative terms. We assume that \( k = 1, 2, \ldots \). The terms indicated by dots in (1) and the terms \( f_{4,5,\ldots} k^{k+2} \) denote the set of all covariant local terms with the derivatives up to the order \( 2k + 4 \). All surface terms are omitted for brevity. \( c_{1,2,3}, d_{1,2,3}, \ldots f_{1,2,3,\ldots} \) are arbitrary coefficients.

The discussion of unitarity and renormalization is much simpler for the flat background, hence in what follows we assume that \( \Lambda = 0 \). As it was explained already in [1], the results are not affected by this assumption.

The evaluation of the superficial degree of divergence \( D \) of the Feynman diagrams in the theory (1) leads to the following result [8]:

\[ D + d = 4 + k(1 - p), \]

where \( d \) is the number of metric derivatives in the counterterms at the \( p \)-loop level. For the logarithmically divergent diagrams with \( D = 0 \) the relation (2) indicates that the models with \( k = 1 \) have divergences only up to the three-loop order, for \( k = 2 \) divergences show up only up to the two-loop order. Finally, the models with \( k \geq 3 \) may have only one-loop divergences with restricted number of derivatives of the metric, \( d = 4 \), \( d = 2 \) and \( d = 0 \). Moreover, in all cases only the parameters \( \Lambda, G, c_{1,2,3} \) gain divergent contributions, and therefore the coefficients of the higher derivative terms do not require infinite renormalization. This means also that the terms with derivatives higher than four are not running. At the same time, the coefficients of the highest derivative terms define the running of the cosmological and Newton constants and of the coefficients \( c_{1,2,3} \). For \( k \geq 3 \) the corresponding one-loop beta-functions are exact.

In order to complete the story, let us note that one can choose the terms with highest derivatives and terms \( O(\kappa^4) \) in such a way that the divergences cancel and the theory becomes finite [13] (see also [14] for an alternative consideration). Furthermore, if there are divergences, they do not depend on the choice of the gauge-fixing parameters, hence the \( \beta \)-functions in this theory are unambiguous.

Indeed, the renormalizability or superrenormalizability of the theories (1) has a price, and this price is not small. The physical spectrum of the theory includes not only a usual massless graviton, but also a set of massive tensor and scalar modes, and part of these extra degrees of freedom are ghosts. In the first paper [8] it was shown that the models with real mass spectrum always have both ghosts and physically massive fields, with alternating signs of the masses and residues. In the present work we shall elaborate on the case of complex masses of the ghost modes.

In what follows we shall discuss general \( k \), but will mainly concern describing the simplest situation with complex poles for the model with \( k = 1 \). The situation for \( k > 1 \) is qualitatively similar and hence the simplest \( k = 1 \) case gives sufficiently clear general understanding. The structure of poles in the propagator on a flat background is defined by the terms which are at most quadratic in curvature tensor. One can make further simplification if remember the relation

\[
R_{\mu\nu\alpha\beta} C^{\mu\nu} C^{\alpha\beta} - 4R_{\mu\nu} C^{\mu\nu} R + R C^{\mu\nu} = \nabla_\mu C^{\mu\nu} + O(R^3),
\]

for any \( l \).

As a result the analysis of the propagator can be done for \( c_1 = d_1 = \ldots = f_1 = 0 \). It proves useful to introduce another basis and notations for the relevant terms in the action

\[
S_{\text{red}} = -\frac{2}{k^2} \int d^4x \sqrt{-g} R \left[ -\alpha \int d^4x \sqrt{-g} \left( \frac{1}{2} C_{\mu\nu\alpha\beta} \Pi_2(\Xi) C^{\mu\nu\alpha\beta} + \omega R \Pi_0(\Xi) R \right) \right],
\]

where \( C_{\mu\nu\alpha\beta} \) is Weyl tensor, \( \kappa = (32\pi G)^{-1/2} = 1/M_P \) is the inverse of reduced Planck mass, \( \alpha \) and \( \omega \) are arbitrary numerical parameters and \( \Pi_{2,0}(x) = 1 + \ldots \) are some polynomials of order \( k \). With these notations we can use directly the results of [23] to arrive at the part of the action (4) which is quadratic in the perturbations,

\[ \kappa h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}, \]

\[
S_{\text{red}}^{(2)} = -\int d^4x \left( \frac{1}{2} h^{\mu\nu} \left[ \frac{\alpha k^2}{2} \Pi_2(\alpha) \nabla^2 \right] \nabla^2 - 1 \right) \nabla^2 P^{(2)}_{\mu\nu, \rho\sigma} h^{\rho\sigma} + \nabla^{\mu\nu} \left[ \alpha \omega k^2 \Pi_0(\alpha^2) \nabla^2 - 1 \right] \nabla^2 P_{\mu\nu, \rho\sigma}^{(0)} h^{\rho\sigma} \right].
\]

An obvious difference between (4) and (5) is that the last is based on the flat-space metric and on the partial derivatives (e.g., \( \partial^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu \)), while the first is a covariant expression. The projectors to tensor \( P^{(2)}_{\mu\nu, \rho\sigma} \) and scalar \( P_{\mu\nu, \rho\sigma}^{(0)} \) states on the flat background are defined in a standard way (see, e.g., [1] or [2]).

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3 This expression corresponds to the four spacetime dimensions. Generalization to an arbitrary dimension is possible, but it will be considered elsewhere [22].

4 Let us note that there is a gauge fixing dependence in the scalar sector of this expression, which was discussed in [2]. However, the expression (5) which we take from [23] corresponds to the gauge-independent interaction between two sources and hence can be considered as well-defined at the tree-level.
\[ p^{(0-s)}_{\mu \nu, \rho \sigma} = \frac{1}{3} \theta_{\mu \nu} \theta_{\rho \sigma}, \]

\[ p^{(2)}_{\mu \nu, \rho \sigma} = \frac{1}{2} \left( \theta_{\mu \rho} \theta_{\nu \sigma} + \theta_{\nu \rho} \theta_{\mu \sigma} \right) - p^{(0-s)}_{\mu \nu, \rho \sigma}, \]

where \( \theta_{\mu \nu} = \eta_{\mu \nu} - \frac{\partial_{\mu} \partial_{\nu}}{2a^2} \).

After Wick rotation to Euclidean space, the equations for the poles have the form

\[ \alpha \Pi_2(p^2) = 2M_p^2, \quad \alpha \omega \Pi_0(p^2) = M_p^2. \]

In the simplest case of the fourth-derivative theory [1,23], \( \Pi_2 = \Pi_0 = 1 \), hence the solutions for the poles, in the tensor and scalar sectors, are

\[ p^2 = m_2^2 = \frac{2M_p^2}{\alpha} \quad \text{and} \quad p^2 = m_0^2 = \frac{M_p^2}{\alpha \omega}. \]

The positive signs of the masses correspond to the negative sign of the higher-derivative terms in (4).

Let us consider the next order and choose

\[ \Pi_2(p^2) = 1 + \frac{p^2}{2A_2}, \quad \Pi_0(p^2) = 1 + \frac{p^2}{2A_0}. \]

where \( A_0 = A_2 \) are some constants with the dimension of the square of mass. Since the two equations (7) are similar, let us present the solution only for the tensor part,

\[ p^2 = m_2^2 = -A_2 \pm \sqrt{A_2^2 + \frac{4A_2 M_p^2}{\alpha}}. \]

This expression shows that, in principle, one can have the following types of solutions:

- Two real positive solutions \( 0 < m_{2p} < m_{2k} \) for the poles. This is the case discussed in [8] and we will not consider it again here.
- Two pairs of complex conjugate solutions for the mass, one with positive and one with negative imaginary parts. In the rest of the paper we concentrate on this case, assuming that the poles of the propagator include one massless state (graviton) and that all other poles are massive and complex.

The quantization of QFT with complex poles was pioneered by Lee and Wick in [17]. The subject attracted a great deal of attention, see, e.g., [18–20] and more recent [21]. The net result is that such theories may be formulated as unitary, since the optical theorem (see, e.g., [24–26]) is satisfied.\(^5\) The complex conjugate poles do not appear on shell since this would mean that one meets physical observables with imaginary component. The physically relevant part of the propagator is composed only by the states corresponding to real poles. In our case this means that the physically relevant part of the propagator is the same as in Einstein gravity.

The application of these ideas to the fourth-derivative Quantum Gravity also has a long history, starting from the works of Stelle [1], Tomboulis [3] and Salam and Stratdhee [4], where the condition of unitarity at the quantum level has been formulated in form of the Froissart Bound which must be satisfied by a dressed propagator with quantum corrections taken into account. The one-loop corrections typically split the real massive pole into a couple of complex conjugate poles [3,28,5,29]. The explicit (rather complicated, technically) calculations of the one-loop corrections in four-derivative gravity were done in [30–32] and carefully checked in [33], including the hypothesis of the relevant role of the Gauss–Bonnet term [34]. We know that the one-loop \( \beta \)-functions [31, 32] have “correct” signs, exactly as the contributions of matter fields, which can be also used in the framework of the large-\( N \) approximation [3]. Unfortunately, it was shown in Ref. [6] that the one-loop calculations are not completely conclusive, the same also concerns the large-\( N \) approximation, which does not provide reliable non-perturbative information about quantum gravity. Hence the question of whether the dressed propagator of metric perturbations has a form which satisfies the Froissart Bound remains open, until we will be able to get a full non-perturbative form of quantum corrections.

From our point of view the great advantage of the superrenormalizable models [1] is that in this case one can provide the desirable complex structure of the propagator already at the tree-level. In this case the conditions of Ref. [4] can be easily satisfied. Moreover, in the superrenormalizable theory the form of loop corrections can be easily set under control, hence the desirable form holds also for the dressed propagator with full quantum corrections. In the next section we present a brief description of the unitarity in the theory with complex poles at the tree level and after that discuss the loop corrections.

3. Unitarity in the theory with complex poles

In QFT unitarity of the \( S \)-matrix means

\[ S|S = 1. \]

In terms of the \( T \)-matrix defined by

\[ S = 1 + iT, \]

the unitarity condition (11) turns out to be

\[ -i(T - T^\dagger) = T^\dagger T. \]

One has to consider the matrix element of the above equation between the initial state \( |i\) and the final state \( |f\),

\[ -i \left( \langle f|T|i\rangle - \langle f|T^\dagger|i\rangle \right) = \langle f|T^\dagger \left( \sum_k |k\rangle \langle k| \right) T|i\rangle. \]

By defining the scattering amplitude as

\[ \langle f|T|i\rangle = (2\pi)^D \delta^D(p_i - p_f) T_{fi}, \]

we arrive at

\[ -i \left( T_{fi} - T_{if}^\dagger \right) = \sum_k T_{ik} T_{ki}. \]

Assuming that for the forward scattering amplitude \( i = f \), previous equation simplifies to

\[ 2 \text{Im} T_{ii} = \sum_k T_{ik}^\dagger T_{ik} > 0. \]

We now present a systematic consideration of the tree-level unitarity, partially following the work by Accioly et al. Ref. [35]. A general theory is certainly well-defined if “tachyons” and “ghosts” are absent, in which case the propagator has only first poles at \( k^2 - M^2 = 0 \) with real masses (no tachyons) and with positive residues (no ghosts). In order to test the tree-level unitarity of a superrenormalizable higher derivative gravity one can introduce an external conserved stress-energy tensor, \( \Theta^{\mu \nu} \), and examine the amplitude at the pole. When we introduce such a general source, the linearized action including the gauge-fixing term reads

\[ An interesting attempt to implement this scheme in the four-derivative quantum gravity can be found in [27], but there is no concrete mechanism which provides an absence of physical poles in this case.
\[ \mathcal{L}_{\text{Heo}} = \frac{1}{2} h^{\mu \nu} \mathbb{O}_{\mu \nu, \rho \sigma} h_{\rho \sigma} - g h_{\mu \nu} \Theta^{\mu \nu}. \]  

(17)

The transition amplitude in momentum space is defined by the expression

\[ i T = (-i)^2 g^2 \Theta^{\mu \nu} i \Delta_{F_{\mu \nu, \rho \sigma}} \Theta^{\mu \nu}, \]

where \( \langle 0 | T \{ h_{\mu \nu}(x) h_{\rho (\sigma}(x) \} | 0 \rangle = i \Delta_{F_{\mu \nu, \rho \sigma}}(k) \equiv i \Theta^{-1}_{\mu \nu, \rho \sigma}(k). \]

(18)

It proves useful to expand the sources using independent vectors in the momentum space,

\[ k^\mu = (k^0, \mathbf{k}), \quad \tilde{k}^\mu = (k^0, -\mathbf{k}), \quad \epsilon_i^\mu = (0, \epsilon_i), \quad i = 1, 2, \]  

(19)

where \( \epsilon_i \) are unit vectors orthogonal to each other and to \( \mathbf{k} \). The symmetric stress-energy tensor reads

\[ \Theta^{\mu \nu} = ak^{\mu} \tilde{k}^\nu + b k^{\mu} \tilde{k}^\nu + c_{ij} \epsilon_i^{(\mu} \epsilon_j^{\nu)} + d k^{(\mu} \tilde{k}^{\nu)}, \]

\[ + e(k^\mu \epsilon^\nu + f \tilde{k}^{(\mu} \epsilon^{\nu)}), \]

(20)

where \( a, b, c, d, e, f \) are some coefficients, which can be partially constrained by the conservation law conditions \( k^\mu \Theta^{\mu \nu} = 0 \).

In the presence of the usual graviton pole and a finite sequence of complex conjugate poles the Feynman propagator reads

\[ i \Delta_F(k) = i \left[ \frac{1}{k^2 + i \epsilon} + \sum_n \left( \frac{c_n}{k^2 - \eta_n^2} + \frac{\eta_n^2}{k^2 - \eta_n^2} \right) \right] \times \left( p^{(2)} - \frac{1}{2} p^{(0)} \right), \]

(21)

where the projectors \( p^{(2)} \) and \( p^{(0)} \) can be consulted in Eq. (6). In the last formula the spacetime indices in \( \Delta_F \) and in the projectors are omitted. Replacing the last expression into (18) we arrive at the result,

\[ i T = (2\pi)^4 \delta(P - P_f) i T_{ff} \]

\[ = -g^2 \Theta^{\mu \nu} i \Delta_{F_{\mu \nu, \rho \sigma}} \Theta^{\mu \nu} = (2\pi)^4 \delta(P - P_f) i T_{ff}, \]

(22)

where

\[ T_{ff} = (-i)^2 \Theta^{\mu \nu} \left[ \frac{1}{k^2 + i \epsilon} + \sum_n \left( \frac{c_n}{k^2 - \eta_n^2} + \frac{\eta_n^2}{k^2 - \eta_n^2} \right) \right] \times \left( p^{(2)} - \frac{1}{2} p^{(0)} \right)_{\mu \nu, \rho \sigma} \Theta^{\rho \sigma}. \]

(23)

Using projectors (6) and the conservation low \( k^\mu \Theta^{\mu \nu} = 0 \) in (23), the imaginary part of \( T_{ff} \) reads

\[ \text{Im} \left[ T_{ff} \right] = \text{Im} \left[ (-i)^2 \left( \Theta_{\mu \nu} \Theta^{\mu \nu} - \frac{1}{2} \Theta^{\mu \nu} \right) \right] \times \left[ \frac{k^2 - i \epsilon}{k^2 + i \epsilon} + \sum_n \left( \frac{c_n}{k^2 - \eta_n^2} + \frac{\eta_n^2}{k^2 - \eta_n^2} \right) \right] \]

\[ = \frac{g^2}{k^2 + i \epsilon} \left( \Theta_{\mu \nu} \Theta^{\mu \nu} - \frac{1}{2} \Theta^{\mu \nu} \right) \]

\[ \rightarrow \pi g^2 \left( \Theta_{\mu \nu} \Theta^{\mu \nu} - \frac{1}{2} \Theta^{\mu \nu} \right) \delta(k^2). \]

(24)

where the usual cut rule has been assumed at the limit \( \epsilon \to 0 \).

From (16) and (24) the tree-level unitarity requirement simplifies to

\[ \text{Im} \left\{ \Theta(k) \Theta^{-1}_{\mu \nu, \rho \sigma} \Theta(k) \right\} \]

\[ = \pi \text{Res} \left\{ \Theta(k) \Theta^{-1}_{\mu \nu, \rho \sigma} \Theta(k) \rho \sigma \right\} |_{k^2 = 0} > 0 \]

(25)

and (24) can be recast into

\[ \text{Res} \left\{ A \right\} |_{k^2 = 0} = g^2 \left( c_0^2 - \frac{1}{2} c_i^2 \right). \]

(26)

with the coefficients \( c_i \) defined in (20).

In the Lee–Wick theory the propagator shows extra complex poles and at the moment it is not obvious how to derive, if any, the usual Largest Time Equation. However, we can still analyze Eq. (16) for the case of individual graphs by cutting the diagrams (19) (see also (26) for the introduction). Energy–momentum conservation must be satisfied by both sides of (16). Therefore, if we cut through normal particle propagators (in our case this means the massless graviton) we have to replace the propagator with \( \overline{\delta(k^2)} \). If we cut through the Lee–Wick propagators, these just correspond to take the imaginary part of the sum in (21), and the imaginary part of the sum of complex conjugate poles vanish. In particular, in \( IT \) we only have to sum over intermediate normal particle states. Therefore, the unitary is unitary in the subspace of the real normal and stable particles as a consequence of the energy–momentum conservation and the presence of extra poles in the propagator that always come in complex conjugate pairs.

In order to illustrate the general arguments, let us consider, as a toy model, the case of a relatively simple Lee–Wick interacting theory to explicitly show the perturbative unitarity of theories with complex conjugate poles. Consider a theory of scalar field with cubic interaction \( \lambda \phi^3 \). The one-loop self-energy diagram provides the contribution to the S-matrix as follows:

\[ S^{(2)} = i T = (-i)^2 \lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k^2 + i \epsilon)\Pi(k^2)} \]

\[ \cdot \left[ \frac{1}{(k + p)^2 + i \epsilon} \Pi((k + p)^2) \right]. \]

(27)

where \( \Pi(k^2) \) has only complex conjugate zeros. For further simplicity, one can take an example with one pair of complex conjugate poles, \( \Pi(k^2) = 1 + k^2/\Lambda^2 \). To verify the unitarity condition (15) we can extract the imaginary part of the integral above without evaluating it explicitly.

\[ \text{Im} T = \text{Im} \left\{ -i\lambda^2 \int \frac{d^4k}{(2\pi)^4} \left[ \frac{1}{(k^2 + i \epsilon)\Pi((k + p)^2)} \right] \right\} \]

\[ = \text{Im} \left\{ -i\lambda^2 \int \frac{d^4k}{(2\pi)^4} \left[ \frac{k^2(k + p)^2 - \epsilon^2 - i\epsilon k^2 - i\epsilon(k + p)^2}{(k^4 + \epsilon^2)\Pi(k^2) \cdot [(k + p)^4 + \epsilon^2)\Pi((k + p)^2)} \right] \right\} \]

\[ = \text{Im} \left\{ -i\lambda^2 \int \frac{d^4k}{(2\pi)^4} \left[ \frac{k^2}{(k^4 + \epsilon^2)\Pi(k^2)} \right] \right\} \]

\[ \cdot \left[ \frac{1}{(k + p)^2 + i \epsilon} \Pi((k + p)^2) \right] \]

\[ \cdot \left[ \frac{\pi \delta(k^2)}{\Pi(k^2)} \cdot \frac{\pi \delta((k + p)^2)}{\Pi((k + p)^2)} \right] \]

\[ = \text{Im} \left\{ -i\lambda^2 \int \frac{d^4k}{(2\pi)^4} \left[ \frac{k^2}{(k^4 + \epsilon^2)\Pi(k^2)} \right] \right\} \]

\[ \cdot \left[ \frac{1}{(k + p)^2 + i \epsilon} \Pi((k + p)^2) \right] \]

\[ \cdot \left[ \frac{-i\pi \delta(k^2)}{\Pi(k^2)} \cdot \frac{\pi \delta((k + p)^2)}{\Pi((k + p)^2)} \right] \]

\[ \cdot \left[ \frac{(k + p)^2}{\Pi((k + p)^2)} \right] \]

\[ \cdot \left[ \frac{1}{(k + p)^4 + \epsilon^2)\Pi((k + p)^2)} \right] \]
\[
\text{Im} \left\{ -i \lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^2}{(k^2 + \epsilon^2)\Pi(k^2)} \cdot \frac{(k + p)^2}{[(k + p)^4 + \epsilon^2\Pi((k + p)^2)]} \right. \\
- \pi^2 \delta(k^2) \delta((k + p)^2) + \frac{k^2}{(k^4 + \epsilon^2)\Pi(k^2)} (-i)\pi \delta(k^2) + \frac{(k + p)^2}{[(k + p)^4 + \epsilon^2\Pi((k + p)^2)]} \right\}.
\]

(28)

The imaginary parts of the last two integrals are zero, while the second integral is identical to the right hand side of equation (15). Then we still have to prove that the imaginary part of the first integral is zero. The expression is

\[
\text{Im} \left\{ -i \lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^2}{(k^4 + \epsilon^2)\Pi(k^2)} \cdot \frac{(k + p)^2}{[(k + p)^4 + \epsilon^2\Pi((k + p)^2)]} \right\} = 0,
\]

(29)

and it is zero because of the extra factor of \(i\). One can find the same result using the Feynman \(\epsilon\) prescription. We end up with the following amplitude,

\[
\text{Im} \mathcal{T} = \pi^2 \lambda^2 \int \frac{d^4kE}{(2\pi)^4} \frac{1}{k^2E} \frac{1}{(k + pE)^2} \delta((k + p)^2),
\]

(31)

which agrees with the optical theorem.

Notice that if \(\Pi(k^2)\) has a pole of complex mass square (without complex conjugate partner) then the integral (30) acquires an imaginary part. If we have derivative self-adjoint interactions only even powers of the momenta \(k\) will give a non-zero contribution to the loop integrals, and the proof of this section remains basically unchanged. In other words unitarity in a theory with complex conjugate poles works like in a two derivative theory.

The generalization to the case of higher number of loops is straightforward. Taking the imaginary part of any amplitude, we get the right product of Dirac delta, namely the generalization of (31) to the case of a given loop order, plus extra integrals whose contribution to the imaginary part is identically zero.

4. Quantum corrections and complex poles

At quantum level the theory can be superrenormalizable or it can be even finite, if terms of third- and fourth-order in curvature are properly introduced (see details in [14]). In both cases there will be a non-local form factor in the propagator of gravitational field at the quantum level. The difference is that for the finite theories the beta functions are zero, there is no need to introduce counterterms, and the propagator does not gain logarithmic form factors, that means the non-local form factors are weaker that logarithmic. In the finite theories of massless fields this should mean that there will not be any quantum corrections to the propagator. However in the superrenormalizable quantum gravity there are mass parameters, hence this scenario is not possible.

From the general perspective it is somehow simpler to evaluate the effect of quantum correction in the case where divergences and logarithmic form factors occur, but when this happens only at the one-loop level. For the sake of simplicity, let’s consider the six-derivative version with one-loop divergences only.

According to the power counting in Eq. (2), in this case the modification in the first of equations of (7) and (9) are as follows

\[
\alpha \left[ 1 + \beta_W \ln \left( \frac{p^2}{\mu^2} \right) + \frac{p^2}{2A_2} \right] \beta^2 = 2M_2^2 \left[ 1 + \beta_\kappa \ln \left( \frac{p^2}{\mu^2} \right) \right].
\]

(32)

One can see that the six-derivative term does not gain logarithmic quantum correction, while such contributions are present for the four-derivative and Einstein–Hilbert terms. The values of the beta-functions \(\beta_\kappa\), \(\beta_\omega\) depend on the relations between highest-derivative terms and terms which are cubic and fourth-order in curvature. One can provide desirable values to \(\beta_\kappa\), \(\beta_\omega\) without much effort. For the sake of simplicity we assume here that the beta-function for the cosmological constant is identically zero. This is also easy to provide, just looking at the expression which was derived explicitly in [8].

The properties and positions of the poles depend on the of the solutions of Eq. (32). From the mathematical side it is a complicated transcendental equation which can not be solved analytically. However, this equation enables one to easily make a qualitative analysis, which can be also confirmed by numerical study with some given values of the parameters. In order to understand the role of the quantum corrections in Eq. (32), let us first note that the logarithmic form of the form factors actually holds only in the far UV, when the energy scale is much higher that the masses of the fields. In the present case, when the masses are defined by the Planck scale, this limit means a far transplanckian energies. On the contrary, at the lower energies, comparable to and below the Planck mass, the effect of masses starts to be essential. At the sub-Planck energies we are probably going to observe a kind of quantum decoupling, such as it was obtained for the semiclassical corrections to gravity from massive fields [36]. The concrete form of the form-factors is relatively complicated, but the general structure is similar to the simple replacement (using Euclidean signature)

\[
\ln \left( \frac{p^2}{\mu^2} \right) \rightarrow \ln \left( \frac{p^2 + m^2}{\mu^2} \right)
\]

(33)

in (32). At this point we can make the following consideration. The logarithmic functions in (32) are slowly varying everywhere except the IR regime, where they must be replaced to other even more slowly varying functions qualitatively similar to (33). This means that the non-local logarithmic insertions in the last equation do not increase the number of poles in the propagator, they can only lead to a certain shift in the positions of existing complex poles. The situation is opposite to the one in the theory with real massive poles, since in this case logarithmic corrections lead to the splitting of massive real poles into a couple of complex conjugate poles [3,4,28].

After all, the Lee–Wick – type unitarity is save in the finite or in general superrenormalizable theories of gravity which are based on the polynomial actions (1).

5. Conclusions and discussions

The semiclassical or quantum treatment of gravity in four-dimensional spacetime always leads to higher derivatives in the

\footnote{7 The consideration for the scalar part is very similar and we will not bother the readers with repetition.}

\footnote{8 Situation is going to be very similar for the finite theories, where we also have only weaker than logarithmic corrections.}
action. In the minimal four-derivative version this means either massive unphysical ghost or tachyon. Including more derivatives with extra massive real poles in the propagator does not change this fact and does not make ghosts more massive [8], since ghost is always a lightest massive spin-2 excitation, after the massless graviton.

In this paper we introduced a new type of the superrenormalizable theories of gravity which are based on the polynomial actions [1]. Different from the models considered before in [8], this new type of theories does not have a hierarchy of massive ghost and massive normal particles with growing masses. Instead, the massive poles appear in a complex conjugate pairs. The quantization of the theories with complex poles is well-known [17–20], but has been never applied to the important case of higher derivative gravity. Our analysis shows that there is a chance to formulate the theory of gravity with complex poles as unitary in the Lee–Wick approach.

The introduction of six- and higher-derivative terms into the action can be seen as an UV completion of the theory with four derivatives, which is useful to remove physical real massive poles from the spectrum. We do not pretend to say the final word on this subject here. But, in our opinion the theories with complex poles at the classical level deserve complete study. In case of a completely consistent formulation, the theory (1) with complex massive poles may be an ideal starting point to construct a successful theory of quantum gravity.

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