On the cardinality of $\pi(\delta)$

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March 26, 1999

Abstract

We prove that the cardinality of transitive quasi-uniformities in a quasi-proximity class is at least $2^{2^{\aleph_0}}$ if there exist at least two transitive quasi-uniformities in the class. The transitive elements of $\pi(\delta)$ are characterized if $\mathcal{V}_A$ is transitive, and in this case we give a condition when there exists a unique transitive quasi-uniformity in $\pi(\delta)$.

Keywords: compatible quasi-proximity, quasi-proximity class, compatible quasi-uniformity, transitive quasi-uniformity, totally bounded quasi-uniformity, l-base, p-filter

AMS Subject Classification: 54E15; 54A25

1 Introduction

The purpose of this paper is the generalization of a result of [12] which states that there exist either a unique or at least $2^{\aleph_0}$ compatible transitive quasi-uniformities on a topological space. Moreover we proved there that $|\pi(\delta^1)| = 1$ or $\geq 2^{2^{\aleph_0}}$ where $\delta^1$ denotes the finest compatible quasi-proximity on $X$, and $\pi(\delta^1)$ is its quasi-proximity class, namely $\pi(\delta^1) = \{V : V \supset P\}$ where $P$ is the Pervin quasi-uniformity of $X$. The natural question arises: is something similar true for $\pi(\delta)$ if $\delta$ is an arbitrary compatible quasi-proximity? In this paper we partly answer this question by proving that $|\pi(\delta)| = 1$ or $\geq 2^{2^{\aleph_0}}$ if the coarsest element of $\pi(\delta)$ is transitive.

In [13] we prove that in the class of infinite locally compact $T_2$ spaces $|\pi(\delta^0)| = 1$ if and only if $X$ is compact or non-Lindelöf ($\delta^0$ denotes the coarsest compatible quasi-proximity) and if $X$ is non-compact and Lindelöf, then $|\pi(\delta^0)| \geq 2^{2^{\aleph_0}}$.

We give two elementary definitions.

Definition 1.1 Let $(X, \tau)$ be a topological space. Then $N(X)$ or $N(\tau)$ ($T(X)$ or $T(\tau)$) denotes the set of all compatible (transitive) quasi-uniformities on $X$ respectively.
Definition 1.2 A base \( B \) of a topological space is called an l-base or a lattice-base if it is closed under finite union and finite intersection and \( \emptyset, X \in B \).

We enumerate some results connected with this notion. For the proofs of these results the reader may wish to consult \[11\].

We say that \( \alpha \) is an interior preserving open cover if it is an open cover and for every \( x \in X \) the set \( \bigcap \{ N \in \alpha : x \in N \} \) is open or equivalently \( \alpha' \subseteq \alpha \) implies \( \bigcap \alpha' \) is open. We can assign a transitive neighbournet \( U_\alpha \) to \( \alpha \) in the following way: \( U_\alpha(x) = \bigcap \{ N \in \alpha : x \in N \} \ (x \in X) \). If \( \alpha' = \{ U_\alpha(x) : x \in X \} \) then \( \alpha' \) is also an interior preserving open cover and \( U_\alpha' = U_\alpha \). An important remark is that if \( \alpha \) is finite then \( \{ U_\alpha(x) : x \in X \} \) is finite too and if \( U_\alpha \in \mathcal{P} \) then \( \alpha \) is finite where \( \mathcal{P} \) denotes the Pervin quasi-uniformity of \( X \) namely \( \mathcal{P} = \text{fil}_{X \times X} \{ U_\alpha : \alpha \text{ is finite} \} \). It is known that a quasi-uniformity is totally bounded if and only if it is contained in \( \mathcal{P} \) (see \[2\]).

In \[11\] we showed that there is a one-to-one correspondence between the set of compatible totally bounded transitive quasi-uniformities and the set of l-bases. Namely if \( \mathcal{V} \) is a quasi-uniformity of the mentioned type then \( \mathcal{B}(\mathcal{V}) = \{ N \in \tau : U_N \in \mathcal{V} \} \) is an l-base, and if \( \mathcal{B} \) is an l-base then \( \mathcal{V}(\mathcal{B}) = \text{fil}_{X \times X} \{ U_N : N \in \mathcal{B} \} \) is a totally bounded transitive quasi-uniformity, where \( U_N = (N \times N) \cup ((X - N) \times X) = U_{(N, X)} \) for the cover \( \{ N, X \} \).

If we say that \( \delta \) is a quasi-proximity we use it in the sense of \[2\]. If \( \mathcal{V} \) is a quasi-uniformity, then \( \delta(\mathcal{V}) \) and \( \tau(\mathcal{V}) \) will always denote the quasi-proximity and the topology induced by \( \mathcal{V} \) respectively. \( \tau(\delta) \) will denote the topology induced by \( \delta \). Let \( \pi(\delta) = \{ \mathcal{V} : \delta(\mathcal{V}) = \delta \} \). We know from \[2\] that for every \( \delta, \pi(\delta) \neq \emptyset \). Moreover there exists a coarsest element of \( \pi(\delta) \), it is denoted by \( \mathcal{V}^{\delta} \). This is totally bounded and the only totally bounded member of \( \pi(\delta) \). If a quasi-uniformity \( \mathcal{V} \) is given then \( \mathcal{V}_\omega \) denotes the coarsest element of \( \pi(\delta(\mathcal{V})) \).

2 The main results

First we want to characterize the transitive elements of \( \pi(\delta) \) where \( \delta \) is a compatible quasi-proximity such that \( \mathcal{V}_\delta \) is transitive. To this aim we will need some lemmas.

Lemma 2.1 Let \( \delta \) be a quasi-proximity on \( X \) compatible with \( \tau \) such that \( \mathcal{V}_\delta \) is transitive. Let \( \mathcal{V} \in T(X) \) such that \( \mathcal{V}_\delta \subseteq \mathcal{V} \). In this case \( \mathcal{V} \in \pi(\delta) \) if and only if \( N \in \tau, U_N \in \mathcal{V} \) imply that \( N \in \mathcal{B}(\delta) = \mathcal{B}(\mathcal{V}_\delta) \).

Proof: By \[10\] we know that \( (\mathcal{V}_\omega)_t = (\mathcal{P})_t \cap (\mathcal{V})_t \) where for a quasi-uniformity \( \mathcal{U}, (\mathcal{U})_t = \{ U \in \mathcal{U} : U \text{ is transitive} \} \).

If \( \mathcal{V} \in \pi(\delta) \) then \( \mathcal{V}_\omega = \mathcal{V}_\delta \). By the previous observation if \( U_N \in \mathcal{V} \) then \( U_N \in \mathcal{V}_\delta \) hence \( N \in \mathcal{B}(\delta) \). Let us prove the sufficiency. We get \( (\mathcal{V})_t \cap (\mathcal{P})_t \subseteq \mathcal{V}_\delta \) and \( \mathcal{V}_\omega \subseteq \mathcal{V}_\delta \) which yields that \( \mathcal{V} \in \pi(\delta) \).
Lemma 2.2 Let $V \in T(\tau)$, $B = B(V)$ and $U \in V$ be transitive. In this case $U(A) \in B$ if $A \subseteq X$.

Proof: Let $A \subseteq X$ and $N = U(A)$. It is easy to check that $U \subseteq U_N$ so $U_N \in V$, and $U_N \in V$. $ullet$

Corollary 2.3 Let $V \in T(\tau)$, $B = B(V)$ and $A_\alpha \in V$ where $\alpha$ is an interior preserving open cover. Then $A \in \alpha$ implies that $A \in B$.

Proof: $A = A_\alpha(A)$. $ullet$

Proposition 2.4 Let $\delta$ be a compatible quasi-proximity on $X$ such that $V_\delta$ is transitive and let $B$ be the l-base associated with $V_\delta$. Let $V = \text{fil}_{X \times X}\{V_\delta, U_i : i \in I\}$ where $U_i$ is transitive ($i \in I$) and the system $\{U_i : i \in I\}$ is closed under finite intersection. Then $V \in \pi(\delta)$ if and only if $\forall i \in I, \forall A \subseteq X U_i(A) \in B$.

Proof: If $V \in \pi(\delta)$ then $V = V_\delta$ and we get the statement by the previous lemma (2.2).

To prove the opposite case let $N = \tau = \tau(\delta)$, $U_N \in V$. We need that $N \in B$ by 2.1. We know that there are $M_i \in \tau$, $U_{M_i} \in V_\delta$ and $j \in I$ such that

$$\cap_{i=1}^n U_{M_i} \cap U_j \subseteq U_N$$

or in other words $U_{\{M_1, \ldots, M_n, X\}} \cap U_j \subseteq U_N$ for the cover $\{M_1, \ldots, M_N, X\}$. Obviously $M_i \in B$ and we can assume the system $\{M_1, \ldots, M_n, X\}$ is closed for union and intersection and $M_n = X$. Let $U' = U_{\{M_1, \ldots, M_n\}}$. If $x \in N$ then there is $M_i$ such that $x \in M_i \cap U_j(x) \subseteq U_N(x) = N$. Now

$$N = \bigcup_{x \in N} (U_{\{M_1, \ldots, M_n\}} \cap U_j)(x) =$$

$$\bigcup_{x: U'(x) = M_1} (M_1 \cap U_j(x)) \bigcup_{x: U'(x) = M_2} (M_2 \cap U_j(x)) \bigcup \ldots \bigcup_{x: U'(x) = M_n} (M_n \cap U_j(x)) =$$

$$\bigcup_{x \in U'(x) = M_1} (M_1 \cap U_j(x)) \bigcup \ldots \bigcup_{x \in U'(x) = M_n} (M_n \cap U_j(x)) \in B$$

since $U_j(x \in N : U'(x) = M_k) \in B$ by assumption and $B$ is an l-base. $ullet$

Corollary 2.5 Let $\delta$ be a quasi-proximity, $V_\delta$ be transitive, $B$ be the l-base associated with it and $V$ be a transitive quasi-uniformity on $X$. Then $V \in \pi(\delta)$ if and only if $V \subseteq V$ and for every transitive $U \in V$ and for every $A \subseteq X$, $U(A) \in B$.
Proof: This is an obvious consequence of 2.4. •

Now we can give condition for $|\pi(\delta) \cap T(X)| > 1$.

**Theorem 2.6** Let $\delta$ be a compatible quasi-proximity, $\mathcal{V}_\delta$ be transitive and $\mathcal{B}$ be the $i$-base associated with $\mathcal{V}_\delta$. In this case there exists $\mathcal{V} \in \pi(\delta) \cap T(\tau)$, $\mathcal{V} \neq \mathcal{V}_\delta$ if and only if either

1. there exists a system of sets $\{N_i : i \in \mathbb{N}\}$ such that $N_i \in \mathcal{B}, N_i \subseteq N_{i+1}, N_i \neq N_{i+1}$ and $\bigcup_1^\infty N_i \in \mathcal{B}$ or
2. there exists a system of sets $\{N_i : i \in \mathbb{N}\}$ such that $N_i \in \mathcal{B}, N_{i+1} \subseteq N_i, N_i \neq N_{i+1}$ and $\bigcap_1^\infty N_i \in \mathcal{B}$.

Proof: Let us prove first the sufficiency and suppose that condition 1 holds. Let $\alpha = \{X; N_i : i \in \mathbb{N}\}$ and $\mathcal{V} = \text{fil}_{X \times X}\{\mathcal{V}_\delta, U_\alpha\}$. By 2.4 $\mathcal{V} \in \pi(\delta) \cap T(\tau)$ since if $A \subseteq X$ then $U_\alpha(A)$ is either a finite union of $N_i$’s or if it is infinite then it equals to $\bigcup_1^\infty N_i \in \mathcal{B}$. If 2 holds then let $\alpha = \{X; \bigcap_1^\infty N_i\} \cup \{N_i : i \in \mathbb{N}\}$ and $\mathcal{V} = \text{fil}_{X \times X}\{\mathcal{V}_\delta, U_\alpha\}$. Then $U_\alpha(A)$ is always a finite union so it is in $\mathcal{B}$. Finally by [1][2.7] $U_\alpha \neq \mathcal{V}_\delta$ in both cases.

Let us prove the necessity. If $\mathcal{V} \in \pi(\delta)$ is transitive, $\mathcal{V} \neq \mathcal{V}_\delta$ then there is a transitive $U \in \mathcal{V}$ such that $\{U(x) : x \in X\}$ is infinite. If we suppose that there is $A = \{x_i \in X : i \in \mathbb{N}\}$ such that $U(x_n) \not\subseteq \bigcup_{n=1}^{n-1} U(x_i)$ ($\forall n \in \mathbb{N}$) then condition 1 holds since by 2.5 $U(x_i) \in \mathcal{B}, U(A) \in \mathcal{B}$ and the system $\{\bigcup_{n=1}^{n-1} U(x_j) : n \in \mathbb{N}\}$ will work.

In case there exists no such system then there is a finite set $A_1 \subseteq X$ such that $X = U(A_1)$. Since $\{U(x) : x \in X\}$ is infinite then there exists $x_1 \in A_1$ such that $\{U(y) : y \in U(x_1)\}$ is infinite by $U$ being transitive. Then there is a finite $A_2 \subseteq X$ such that $U(A_2) = U(x_1)$ and there exists $x_2 \in A_2$ such that $\{U(y) : y \in U(x_2)\}$ is infinite. Let us continue this process. We get a system $\{U(x_i) : i \in \mathbb{N}\}$ such that $U(x_i) \supset U(x_{i+1})$. $U(x_i) \neq U(x_{i+1})$ can also be assumed. By 2.3 $\bigcap_1^\infty U(x_i) \in \mathcal{B}$. •

**Corollary 2.7** If $\delta_1$ is coarser than $\delta_2$, $\mathcal{V}_{\delta_1}, \mathcal{V}_{\delta_2}$ are transitive, $|\pi(\delta_1) \cap T(\tau)| > 1$ then $|\pi(\delta_2) \cap T(\tau)| > 1$.

Proof: Obviously $B(\delta_1) \subseteq B(\delta_2)$ and apply 2.6. •

We will need definitions and a theorem from [12] dealing with p-filters.

**Definition 2.8** ([12][2.2]) Let $N \subseteq \mathbb{N}$. We call a subset $H$ of $N$ a pile of $N$ or an $N$-pile or simply pile (if there is no misunderstanding) if

1. $i,j \in H$ and $i < k < j$, $k \in \mathbb{N}$ implies that $k \in H$ and
2. $H$ is maximal for the previous property.

**Definition 2.9** ([12][2.3]) Let again $N \subseteq \mathbb{N}$. We say that $Z \subseteq \mathbb{N}$ is admissible for $N$ if there exists $k \in \mathbb{N}$ such that $H$ being an $N$-pile implies that $|H \cap Z| \leq k$.

**Definition 2.10** ([12][2.4]) We call a filter $\sigma$ on $\mathbb{N}$ a p-filter if, whenever $N \in \sigma$ and $Z$ is admissible for $N$, then $N - Z \in \sigma$. 
Theorem 2.11 \( |\{\sigma : \sigma \text{ is a } p\text{-filter}\}| = 2^{\aleph_0} \).

Now we are ready to prove the main theorem.

Theorem 2.12 Let \( \delta \) be a quasi-proximity such that \( \mathcal{V}_\delta \) is transitive. In this case \( |\pi(\delta)| = 1 \) or \( |\pi(\delta)| \geq 2^{\aleph_0} \), moreover \( |\pi(\delta) \cap T(\tau)| = 1 \) or \( \geq 2^{\aleph_0} \).

Proof: Let \( B \) be the 1-base associated with \( \mathcal{V}_\delta \). If \( |\pi(\delta)| > 1 \) then condition 1 or 2 holds in theorem 2.6.

1. Suppose that there exists a system of sets \( \{N_i : i \in \mathbb{N}\} \) such that \( N_i \in \mathcal{B}, N_i \subseteq N_{i+1}, N_i \neq N_{i+1} \) and \( \bigcup_{i=0}^{\infty} N_i \in \mathcal{B}. \)

Let \( A \subseteq \mathbb{N} \) then let

\[ \alpha_A = \{X, N_i : i \notin A\}. \]

Then \( \alpha_A \) is an interior preserving open cover of \( X \). Let \( U_A(x) = U_{\alpha_A}(x) = \cap \{M \in \alpha_A : x \in M\} \). If \( \sigma \) is a p-filter on \( \mathbb{N} \) then let

\[ \mathcal{V}_\sigma = \text{fil}_{X \times X} \{\mathcal{V}_\delta, U_A : A \in \sigma\}. \]

It is obvious that \( \mathcal{V}_\sigma \in T(\tau) \). It is also straightforward that \( U_{A_1} \cap U_{A_2} = U_{A_1 \cap A_2} \).

By 2.4, \( \mathcal{V}_\sigma \in \pi(\delta) \).

We show that if \( \sigma_1 \neq \sigma_2 \) then \( \mathcal{V}_{\sigma_1} \neq \mathcal{V}_{\sigma_2} \). This yields immediately that the cardinality of the set of all p-filters on \( \mathbb{N} \) is less than or equal to the cardinality of \( |\pi(\delta) \cap T(\tau)| \), hence \( |\pi(\delta) \cap T(\tau)| \geq 2^{\aleph_0} \).

Now let \( \mathcal{V}_{\sigma_1} \subseteq \mathcal{V}_{\sigma_2} \) then we will show that \( \sigma_1 \subseteq \sigma_2 \), which is enough to prove since with the opposite case it implies the required statement.

Let \( A_1 \in \sigma_1 \). Then \( U_{A_1} \in \mathcal{V}_{\sigma_1} \) hence there are \( A_2 \in \sigma_2 \) and \( P = U_\beta \in \mathcal{V}_\delta \) such that \( U_\beta \cap U_{A_2} \subseteq U_{A_1} \) where \( \beta \) is a finite subset of \( \mathcal{B} \). Let \( \{\{P(x) : x \in X\} : k \in \mathbb{N}\} \). Let \( H \subseteq A_2 \) be an \( A_2 \)-pile such that \( H = \{r \in \mathbb{N} : p < r < q\} \). Suppose that \( i \in H \setminus A_1 \) and let \( x_i \in N_i \setminus N_{i-1} \). Then \( U_{A_1}(x_i) = N_i \) and \( U_{A_2}(x_i) = N_q \) and we get that \( N_q \cap P(x_i) \subseteq N_i \) which implies that \( P(x_i) \cap (N_q \setminus N_i) = \emptyset \). If \( j \in H \setminus A_1 \) such that \( i < j \) then \( x_j \in P(x_j) \cap (N_q \setminus N_i) \neq \emptyset \) so \( P(x_i) \neq P(x_j) \) which verifies that the function \( f \) defined by \( f(i) = P(x_i) \) \( (i \in H \setminus A_1) \) is injective. A similar argument applies if \( q = \infty \).

Hence \( |H \setminus A_1| \leq k \) and there is \( Z = A_2 - A_1 \subseteq \mathbb{N} \) such that \( A_2 - Z \subseteq A_1 \) where \( Z \) is admissible for \( A_2 \). Since \( \sigma_2 \) is a p-filter then \( A_1 \in \sigma_2 \) and \( \sigma_1 \subseteq \sigma_2 \).

2. Suppose that there exists a system of sets \( \{N_i : i \in \mathbb{N}\} \) such that \( N_i \in \mathcal{B}, N_{i+1} \subseteq N_i, N_i \neq N_{i+1} \) and \( \bigcap_{i=1}^{\infty} N_i \in \mathcal{B}. \)

Let \( N_0 = X \). If \( A \subseteq \mathbb{N} \) then let

\[ \alpha_A = \{X, \bigcap_{j=1}^{\infty} N_j\} \cup \{N_i : i \notin A\}. \]

Then \( \alpha_A \) is an interior preserving open cover of \( X \). Let \( U_A(x) = \cap \{M \in \alpha_A : x \in M\} \). If \( \sigma \) is a p-filter on \( \mathbb{N} \) then let

\[ \mathcal{V}_\sigma = \text{fil}_{X \times X} \{\mathcal{V}_\delta, U_A : A \in \sigma\}. \]
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0-dimensional is interesting.

We show that if σ₁ ≠ σ₂ then V_{σ₁} ≠ V_{σ₂}. This yields immediately that |

Finally we present some open problems which seem to be interesting.

Corollary 2.13 Let X be locally compact, T₂, non-compact, Lindelöf and 0- dimensional. If δ is a compatible quasi-proximity such that V_δ is transitive then |

Proof: By 2.7 it is enough to verify the statement for δ⁰. In this case \( B = \{\text{compact-open sets}\} \cup \{\emptyset, X\} \), and there is a strictly increasing sequence \( \{N_i\} \) of compact-open sets, such that \( \bigcup_{i=0}^{\infty} N_i = X \) and 2.6 and 2.12 are applicable.

Finally we present some open problems which seem to be interesting.

Problem 1 What can we say about \(|\pi(δ)|\) if V_δ is not transitive?

Remark: We note that in the meantime this problem have been solved by Künzi in [8].

Problem 2 Let V_δ be transitive. What can be said about \(|\pi(δ) \cap (N(X) - T(X))|\)? Can it occur that there exists no non-transitive quasi-uniformity in \(\pi(δ)\) if \(|\pi(δ)| > 1\)?

Problem 3 Is there any connection between \(|\pi(δ_1)|\) and \(|\pi(δ_2)|\) if δ₁ is coarser than δ₂? Is it true that \(|\pi(δ_1)| ≤ |\pi(δ_2)|\) in this case?

Problem 4 Is there always a non-transitive quasi-uniformity that is finer than \(\mathcal{P}\) (assuming that \(|\pi(δ^1)| > 1\))? (see also [7]Remark 1)
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