On $\delta^k$-colouring of Powers of Paths and Cycles

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Recommended Citation
Ellumkalayil, Merlin Thomas Ms and Naduvath, Sudev (2021) "On $\delta^k$-colouring of Powers of Paths and Cycles," Theory and Applications of Graphs: Vol. 8 : Iss. 2 , Article 3.
DOI: 10.20429/tag.2021.080203
Available at: https://digitalcommons.georgiasouthern.edu/tag/vol8/iss2/3

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On $\delta^k$-colouring of Powers of Paths and Cycles

Cover Page Footnote
Authors of the paper would like to thank Dr Johan Kok, Visiting Professor, Department of Mathematics, CHRIST (Deemed to be University), Bangalore, India., for his valuable comments and inputs which significantly improved the content and the style of the presentation.

This article is available in Theory and Applications of Graphs: https://digitalcommons.georgiasouthern.edu/tag/vol8/iss2/3
Abstract

In an improper vertex colouring of a graph, adjacent vertices are permitted to receive same colours. An edge of an improperly coloured graph is said to be a bad edge if its end vertices have the same colour. A near-proper colouring of a graph is a colouring which minimises the number of bad edges by restricting the number of colour classes that can have adjacency among their own elements. The $\delta^{(k)}$-colouring is a near-proper colouring of $G$ consisting of $k$ given colours, where $1 \leq k \leq \chi(G) - 1$, which minimises the number of bad edges by permitting at most one colour class to have adjacency among the vertices in it. In this paper, we discuss the number of bad edges of powers of paths and cycles.

Keywords: Improper colouring, near-proper colouring, $\delta^{(k)}$-colouring, bad edges.

MSC 2020: 05C15, 05C38.

1 Introduction

Unless mentioned otherwise, all graphs considered here are undirected, simple, finite and connected. The $r$-th power of an undirected graph $G$ is another graph $G^r$ that has the same set of vertices and two vertices are adjacent when their distance in $G$ is at most $r$ (see [6]). The power graph $G^r$ is a complete graph when $r = d$ where $d$ is the diameter of the graph $G$.

A vertex colouring of a graph $G$ is a mapping $c : V(G) \to \mathcal{C}$, a set of colours. When no two adjacent vertices are assigned the same colour, it is called a proper colouring of $G$ (see [2, 1]). The minimum number of colours required to colour a graph properly is called the chromatic number of $G$ and is denoted by $\chi(G)$. A set $\{c_i \in \mathcal{C} : c(v) = c_i \text{ for at least one } v \in V(G)\}$ is called a colour class of the colouring of $G$.

Since its inception, the graph colouring methods have been widely used to model many practical problems. In many problems, availability of colours may be restricted. This led to the study on certain colouring techniques in which the colouring is not proper. In these types of colouring, some pairs of adjacent vertices may have the same colour. Such edges are called bad edges.

A near-proper colouring is a colouring which minimises the number of bad edges by restricting the number of colour classes that can have adjacency among their own elements [4]. From the terminology of near-proper colouring, the notion of $\delta^{(k)}$-colouring has been introduced in [4] as follows:

Definition 1.1. [4] For $1 \leq k < \chi(G)$, the $\delta^{(k)}$-colouring of a graph $G$ is a near-proper colouring of $G$ consisting of $k$ given colours, which minimises the number of bad edges by permitting at most one colour class to have adjacency among the vertices in it.

The number of bad edges in a graph $G$ obtained from a $\delta^{(k)}$-colouring of $G$ is denoted by $b_k(G)$.

Unless mentioned otherwise, the colour class $C_1$ will be the relaxed colour class (that is, the colour class which will have adjacency between the vertices in it), throughout the discussion. In this paper, the bad edges of the graphs are illustrated by dotted lines.
Different types of improper colouring of graphs in general and $\delta^{(k)}$-colouring in particular have their own significance in real-life situations or practical problems. Since graph colouring methods are widely used to model many of such problems, the constraints, limitations and/or restrictions applicable to those problems will lead to different types of improper colouring of the corresponding graphical models. The $\delta^{(k)}$-colouring and variants play a vital role, when we seek for the most economical or affordable solutions to those problems. Some interesting studies in this direction can be seen in [4, 5, 3].

Motivated by the above-mentioned studies, we investigate the $\delta^{(k)}$-colouring of powers of paths and cycles in this paper.

2 $\delta^{(k)}$-Colouring of Some Graph Powers

Among different classes of connected graphs, paths and cycles are the most elementary graph classes. In this section, we discuss the $\delta^{(k)}$-colouring and the optimum number of bad edges corresponding to this colouring of the powers of paths and cycles. Because of their structural characteristics, the powers of paths and cycles possess much importance in our study on $\delta^{(k)}$-colouring and the corresponding bad edges.

2.1 $\delta^{(k)}$-Colouring of Powers of Paths

Theorem 2.1. For $n \equiv m \pmod{r+1}$, the number of bad edges obtained from an $\delta^{(k)}$-colouring in $P_r^n$, where $2 \leq r < n - 1$ and $k \leq r$ is,

\[
b_k(P_r^n) = \begin{cases} 
\frac{(r-k+2)(r-k+1)}{2}(2\lceil \frac{n}{r+1} \rceil - 1), & \text{if } 0 \leq m < k, \\
\frac{(r-k+2)(r-k+1)}{2}(2\lceil \frac{n}{r+1} \rceil - 1) + \\
\frac{(m-k+1)(m-k)}{2} + \frac{m-k+1}{2}(2r - m - k + 2), & \text{if } m \geq k.
\end{cases}
\]

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of a path and let $k$ where $k \leq r$, be the number of available colours. We consider the colour class $C_1$ to have adjacency between the vertices in it. In all the possible cases of powers of paths, we colour the vertices in the following pattern:

Assign the colour $c_k$ to the vertex $v_1$. The vertices $v_2, v_3, \ldots, v_{r+1}$ cannot be given the colour $c_k$ as they are adjacent to the vertex $v_1$. Hence we assign the colour $c_{k-1}$ to the vertex $v_2$. Continuing this colouring pattern to all the vertices, the vertex $v_{k-1}$ is assigned the colour $c_2$ (note that, the colours are assigned in the descending order for the ease of explanation). Now, all the vertices $v_1, v_2, \ldots, v_{k-1}$ are at least adjacent to the vertex $v_{r+1}$ and hence the vertices from $v_k$ to $v_{r+1}$ are assigned the colour $c_1$ to satisfy the requirements of $\delta^{(k)}$-colouring. Thus, using the $k - 1$ colours, excluding the colour $c_1$, the vertices $v_1, v_2, \ldots, v_{k-1}$ are properly coloured and the remaining $r + k - 2$ vertices namely, $v_k, v_{k+1}, \ldots, v_{r+1}$ are assigned the colour $c_1$. Since the vertex $v_1$ is not adjacent to the vertex $v_{r+2}$ and since the vertices $v_2, v_3, \ldots, v_{r+1}$ are at least adjacent to the vertex $v_{r+2}$, the vertex $v_{r+2}$ can be given no colours other than $c_k$ or $c_1$. Here, we assign the vertex the colour $c_k$. In this manner, we assign the vertices $v_{r+3}, v_{r+4}, \ldots, v_{r+k}$ the colours $c_{k-1}, c_{k-2}, \ldots, c_2$ respectively. The remaining $r - k + 2$ vertices namely, $v_{r+k+2}, v_{r+k+3}, \ldots, v_{2(r+1)}$ are again assigned the
colour $c_1$ to maintain the requirements of $\delta^{(k)}$-colouring. We continue this colouring pattern to the rest of the vertices and determine the minimum number of bad edges resulting from this $\delta^{(k)}$-colouring.

Now, consider $n \equiv m \pmod{r + 1}$, where $m \leq r$. Assigning the $k$ available colours to the $n$ vertices using the above mentioned colouring pattern, we observe that there are $\left\lfloor \frac{n}{r+1} \right\rfloor$ vertex disjoint complete subgraphs and a vertex disjoint complete subgraph of order $m$ where $m \leq r$. And each vertex disjoint complete subgraph lead to $\left\lfloor \frac{(r-k+2)(r-k+1)}{2} \right\rfloor$ bad edges. Also, it can be noted that there are $(\left\lfloor \frac{n}{r+1} \right\rfloor - 1)\left\lfloor \frac{(r-k+2)(r-k+1)}{2} \right\rfloor$ bad edges between consecutive vertex disjoint complete subgraphs.

Now, when $0 \leq m < k$, there are no bad edges in the vertex disjoint subgraph of order $m$. Thus, the total number of bad edges obtained from this $\delta^{(k)}$-colouring in $P_n^r$ when $n \equiv m \pmod{r + 1}$ where $0 \leq m < k$ is $\left\lfloor \frac{n}{r+1} \right\rfloor \left( \frac{(r-k+2)(r-k+1)}{2} \right) + (\left\lfloor \frac{n}{r+1} \right\rfloor - 1)\left\lfloor \frac{(r-k+2)(r-k+1)}{2} \right\rfloor = \left\lfloor \frac{n}{r+1} \right\rfloor (r-k+2)(r-k+1) - \frac{(r-k+2)(r-k+1)}{2}$ (see the Figure 1 for illustration).

![Figure 1 $\delta^{(2)}$-colouring of $P_{10}^2$](image)

However, when $n \equiv m \pmod{r + 1}$ where $m \geq k$, the vertex disjoint complete subgraph of order $m$ will have $m-k+1$ vertices coloured only with the colour $c_1$ which are adjacent to few vertices among the $r-k+2$ vertices assigned with the colour $c_1$ in the previous section. The last vertex of this section is adjacent to $r+1-m$ vertices. Similarly, the second last, third last and so on the first vertex of this section is adjacent to $r+2-m, r+3-m, \ldots, r+(m-(k+1))$ vertices of the previous section respectively. Therefore, $S_{m-k+1} = \frac{m-k+1}{2} (r+1-m+r+1-k) = \frac{m-k+1}{2} (2r-m-k+2)$ (note that, $S_{m-k+1}$ is the sum of the arithmetic progression $r+2-m, r+3-m, \ldots, r+1-k$ with $m-k+1$ terms, first and last terms to be $r+2-m$ and $r+1-k$ respectively and the common difference to be 1). Thus, the number of bad edges between these two sections of vertices having coloured with only the colour $c_1$ is $\frac{m-k+1}{2} (2r-m-k+2)$. Hence, the total number of bad edges obtained from $\delta^{(k)}$-colouring in $P_n^r$ when $n \equiv m \pmod{r + 1}$ and $m \geq k$ is, $b_k(P_n^r) = \frac{(r-k+2)(r-k+1)}{2} (\left\lfloor \frac{n}{r+1} \right\rfloor - 1) + \frac{(m-k+1)(m-k)}{2} + \frac{m-k+1}{2} (2r-m-k+2)$ (see the Figure 2 for illustration).

![Figure 2 $\delta^{(2)}$-colouring of $P_{11}^3$](image)

We call the above mentioned $\delta^{(k)}$-colouring to be the optimum colouring and the bad edges obtained from this $\delta^{(k)}$-colouring to be the minimum. The explanation for the same is given below.
Since, in $P^n_r$ every vertex is adjacent to the vertices that are of at most distance $r$, the graph $P^n_r$ induces complete subgraphs of order $r + 1$. As each vertex is in a complete subgraph, we consider the vertex disjoint complete subgraphs each of order $r + 1$. Now, there are $\left[ \frac{n}{r+1} \right]$ vertex disjoint complete subgraphs in $P^n_r$ which can be properly coloured using $r + 1$ colours and hence we consider the $k$ to be less than or equal to $r$. Now, the minimum number of bad edges in a complete graph is already proved in ([4]) as $b_k(K_n) = \frac{x(x+1)}{2}$, where $x = 1, 2, 3, \ldots, n - 2$ and $k = n - x$. Substituting the values of $n$ as $r + 1$ and $x = n - k = r + 1 - k$, we have $\frac{(r-k+1)(r-k+2)}{2}$ number of bad edges in each vertex disjoint complete subgraph of order $r + 1$. Thus, there are a total of $\left[ \frac{n}{r+1} \right] \frac{(r-k+1)(r-k+2)}{2}$ number of bad edges in the vertex disjoint complete subgraphs. Since, the underlined graph is a path graph, there will always be adjacency between two consecutive vertex disjoint complete subgraphs. So, to determine the minimum number of bad edges between any two vertex disjoint complete subgraphs, it is enough to find the minimum number of bad edges between any two consecutive vertex disjoint complete subgraphs. Since, $r - k + 2 < r + 1$, every vertex among the $r - k + 2$ vertices is adjacent to at most $r + 1$ vertices of the other vertex disjoint complete subgraph. To minimise the adjacency between the vertices assigned the colour $c_1$ in a complete subgraph to that of the vertices coloured with $c_1$ in the consecutive complete subgraph, there can be three ways as explained below.

Case 1: Consider the first two vertex disjoint subgraphs. Assign the first $r - k + 2$ vertices of both the complete subgraphs the colour $c_1$ and the rest of the $k - 1$ vertices of the two complete subgraphs the colours other than $c_1$. Among the $r - k + 2$ vertices that are assigned the colour $c_1$, the vertex $v_1$ is not adjacent to any vertex of second complete subgraph and hence $v_1$ will not contribute to any bad edge between the complete subgraphs. Now, the vertex $v_2$, assigned the colour $c_1$, is adjacent to the vertex $v_{r+2}$ of the second complete subgraph which is assigned the colour $c_1$. This leads to one bad edge. The third vertex $v_3$, assigned the colour $c_1$ is at most adjacent to the vertices $v_{r+2}$ and $v_{r+3}$ of the second complete subgraph which are assigned the colour $c_1$. This leads to two bad edges between the two vertex disjoint complete subgraphs. In this way, the vertex $v_{r-k+2}$ of the first complete subgraph is adjacent to the $r - k + 1$-th and the previous vertices (i.e the vertices $v_{r+2}, v_{r+3}, \ldots, v_{2r-k+2}$) that are assigned the colour $c_1$ of the second complete subgraph and thus this lead to $r - k + 1$ bad edges. Thus, the total number of bad edges between each consecutive vertex disjoint complete subgraph is $\frac{(r-k+1)(r-k+2)}{2}$.

Case 2: Assign the colours $c_k, c_{k-1}, \ldots, c_2$ properly to the first $k - 1$ vertices of both the vertex disjoint complete subgraphs and the rest $r - k + 2$ vertices the colour $c_1$ (as mentioned in our $\delta(k)$-colouring). Now, the vertex $v_k$, assigned the colour $c_1$, is at most adjacent to the vertex $v_{r+k}$ which is assigned the colour $c_2$. Hence, this vertex lead to no bad edges between the two vertex disjoint complete subgraphs. Now as explained in Case 1, the remaining $r - k + 1$ vertices lead to bad edges between the complete subgraphs. Thus, the total number of bad edges between each consecutive vertex disjoint complete subgraph is $\frac{(r-k+1)(r-k+2)}{2}$.

Case 3: Assign the first vertex of a vertex disjoint complete subgraph the colour $c_1$, then the second with $c_2$, the third with $c_1$, the fourth with $c_3$ and so on. Now, if $r - k + 1 < k - 1$, there is a possibility of getting the last few vertices coloured only with $c_1$. Now, the first vertex is not adjacent to the consecutive vertex disjoint complete subgraphs and this will not lead to any bad edges. Thus, as explained in the above two cases the remaining $r - k + 1$
vertices in each complete subgraphs will lead to bad edges between any two vertex disjoint
consecutive complete subgraphs, or if the colour \( c_1 \) is assigned to a vertex other than the
first vertex, that particular vertex will always be adjacent to a vertex assigned the colour
other than \( c_1 \). Thus, in either of the cases only \( r - k + 1 \) vertices of any two consecutive
vertex disjoint complete subgraphs leads to bad edges between any two consecutive vertex
disjoint complete subgraphs and the total number is \( \frac{(r-k+1)(r-k+2)}{2} \).

Thus, from all of the above three cases it can be observed that the minimum num-
ber of bad edges between any two consecutive vertex disjoint complete subgraphs in \( P_n^r \) is
\( \frac{(r-k+1)(r-k+2)}{2} \). Since there are \( \left\lfloor \frac{n}{r+1} \right\rfloor \) vertex disjoint complete subgraphs of order \( r + 1 \), the
minimum number of bad edges between the subgraphs is \( \left( \left\lfloor \frac{n}{r+1} \right\rfloor - 1 \right) \frac{(r-k+1)(r-k+2)}{2} \). Now,
among the \( n \) vertices there are \( m \) vertices, where \( m \leq r \), which form a vertex disjoint sub-
graph of order \( m \). These \( m \) vertices and the previous vertex disjoint complete subgraph
of order \( r + 1 \) can have lesser number of bad edges between them. Thus, taking this into
consideration the following two cases are addressed.

**Case 4:** Let \( n \equiv m \mod r + 1 \) and \( m < k \), then the remaining \( m \) vertices can be
assigned the colours other than \( c_1 \). This type of colouring where the last \( m \) vertices are
given distinct colours other than \( c_1 \) can be obtained if and only if each \( m \) vertices are not
adjacent to the same colour as that of itself and this is only possible if all the first \( k - 1 \)
vertices of each vertex disjoint complete subgraph of order \( r + 1 \) is coloured with \( k - 1 \)
distinct colours and the remaining \( r - k + 2 \) with the colour \( c_1 \). This will lead to have no bad
edges among the last \( m \) vertices which induces a complete subgraph on \( m \) vertices and also
no bad edges between the \( \left\lfloor \frac{n}{r+1} \right\rfloor \)-th complete subgraph and the \( m \) vertices. Thus, the total
number of bad edges resulting from a \( \delta(k) \)-colouring when \( n \equiv m \mod (r + 1) \) and \( m < k \)
is
\[ \left\lfloor \frac{n}{r+1} \right\rfloor \frac{(r-k+2)(r-k+1)}{2} + \left( \left\lfloor \frac{n}{r+1} \right\rfloor - 1 \right) \frac{(r-k+1)(r-k+2)}{2} = \left\lfloor \frac{n}{r+1} \right\rfloor (r-k+2)(r-k+1) - \frac{(r-k+1)(r-k+2)}{2}. \]

**Case 5:** Let \( n \equiv m \mod (r + 1) \) and \( m \geq k \). Since \( m \geq k \), not every \( m \) vertices can be
coloured with colours other than \( c_1 \). But we try to maximise the use of the colours other
than \( c_1 \) to colour the last \( m \) vertices to minimise the number of bad edges between the last
vertex disjoint complete subgraph and the complete subgraph induced by the \( m \) vertices. As
explained in the above case, this can be obtained only if every first \( k - 1 \) vertices of each
complete subgraph of order \( r + 1 \) is assigned the \( k - 1 \) colours distinctly and the remaining
\( r - k + 2 \) vertices the colour \( c_1 \). This colouring will lead in properly assigning the \( k - 1 \) vertices
among the \( m \) vertices the colours \( c_k, c_{k-1}, \ldots, c_2 \) and the remaining \( m - k + 1 \) vertices the
colour \( c_1 \), to attain the requirements of \( \delta(k) \)-colouring. Thus, this lead to \( \frac{(m-k+1)(m-k)}{2} \) bad
edges in the vertex disjoint complete subgraph on \( m \) vertices. Now, between the last vertex
disjoint complete subgraph and the complete subgraph induced by the \( m \) vertices the number
of bad edges as explained above is \( \frac{m-k+1}{2}(2r - m - k + 2) \). Hence, the total number of bad
dges when \( n \equiv m \mod (r + 1) \) and \( m \geq k \) is \( \frac{(r-k+2)(r-k+1)}{2} \left( \left\lfloor \frac{n}{r+1} \right\rfloor - 1 \right) + \frac{(m-k+1)(m-k)}{2} + \frac{m-k+1}{2}(2r - m - k + 2) \).

The path graph \( P_n \) is the subgraph of the cycle graph \( C_n \) and so it is obvious that
few points discussed about the powers of paths in the above theorem will be similar when
considered the powers of cycles. Now, as discussed in the 2.1, the cycle graph when taken
the power also induces \( \left\lfloor \frac{n}{r+1} \right\rfloor \) complete subgraphs of order \( r + 1 \) and the number of bad edges
in each of the vertex disjoint complete subgraphs is same as that of the bad edges in each of
the vertex disjoint complete subgraphs of powers of paths which is equal to \( \frac{(r-k+2)(r-k+1)}{2} \). However, it can be noted that, unlike in the powers of paths, the first and the last vertex disjoint complete subgraph \( \left( \frac{n}{r+1} \right) \)th of order \( r+1 \), have edges between them and so there are a total of \( \frac{n}{r+1} \) \( (r-k+2)(r-k+1) \) bad edges between the vertex disjoint complete subgraphs in powers of cycle. For different values of \( n, r \) and \( k \), the bad edges between the first and last vertex disjoint complete subgraphs vary. The \( \delta(k) \)-colouring of powers of cycles will be discussed in the following Theorems.

### 2.2 \( \delta(k) \)-Colouring of Powers of Cycles

First, recall that the diameter of a cycle graph of order \( n \) is \( \left\lfloor \frac{n}{2} \right\rfloor \). The cases when \( r = 1 \) and \( r = \left\lfloor \frac{n}{2} \right\rfloor \) have already been discussed in [4]. Hence, we discuss the \( \delta(k) \)-colouring of the \( r \)-th power of a cycle \( C_n \) for \( 1 < r < \left\lfloor \frac{n}{2} \right\rfloor \) in the following theorems.

**Theorem 2.2.** For \( C_n^r \), where \( 2 \leq r < \left\lfloor \frac{n}{2} \right\rfloor \) and \( 2 \leq k < \chi(G) \) and \( k \leq r \), the number of bad edges obtained from \( \delta(k) \)-colouring is given by,

\[
b_k(C_n^r) = \begin{cases} \frac{(n}{r+1})(r-k+2)(r-k+1), & \text{if } n \equiv 0 \pmod{r+1}, \\ \frac{(r-k+1)(r-k+2)}{2}(2\left\lfloor \frac{n}{r+1} \right\rfloor - 1) + \frac{(r-k+3)(r-k+2)}{2}, & \text{if } n \equiv 1 \pmod{r+1}. \end{cases}
\]

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be the vertices of \( C_n^r \) taken in order. Let \( c_1, c_2, \ldots, c_k \) be the \( k \) available colours and \( C_1, C_2, \ldots, C_k \) be their respective colour classes. By Definition 1.1, only the vertices of one colour class are permitted to have adjacency among themselves. Hence, we assume that the vertices in the colour class \( C_1 \) to have adjacency among themselves. As explained above, the number of bad edges in each vertex disjoint complete subgraph is \( \frac{(r-k+2)(r-k+1)}{2} \). Now, between any two consecutive vertex disjoint subgraphs there are a minimum of \( \frac{(r-k+1)(r-k+2)}{2} \) bad edges. Thus, the total number of bad edges between any two consecutive vertex disjoint complete subgraphs is \( \frac{n}{r+1} \) \( \frac{(r-k+1)(r-k+2)}{2} \). Now, we need to address the following cases.

**Case 1:** Let \( n \equiv 0 \pmod{r+1} \). Since \( n \equiv 0 \pmod{r+1} \), there are \( \frac{n}{r+1} \) vertex disjoint complete subgraphs. Thus, the total number of bad edges between any two consecutive vertex disjoint complete subgraphs is \( \frac{n}{r+1} \) \( \frac{(r-k+1)(r-k+2)}{2} \). The total number of bad edges in each vertex disjoint complete subgraph is \( \frac{(r-k+1)(r-k+2)}{2} \). Thus, the total number of minimum bad edges from a \( \delta(k) \)-colouring is \( \frac{n}{r+1} \) \( (r-k+2)(r-k+1) \).

Below given is a \( \delta(k) \)-colouring explaining the proof. Let \( v_1, v_2, \ldots, v_n \) be the vertices of \( C_n^r \) taken in order (for the ease of explanation we colour the graph in clockwise direction). We follow the following colouring pattern to assign colours to the first \( r+1 \) vertices. Colour the vertices \( v_1, v_2, \ldots, v_k \) with the colours \( c_1, c_2, \ldots, c_k \) respectively and the vertices \( v_{k+1}, v_{k+2}, \ldots, v_{r+1} \) with the colour \( c_1 \) to maintain the requirements of \( \delta(k) \)-colouring. Since, the vertex \( v_2 \) and the vertices \( v_3, \ldots, v_{r+1} \) is at least adjacent to vertex \( v_{r+2} \), this vertex cannot be assigned any colour other than \( c_1 \). Now, since vertex \( v_3 \) and the vertices \( v_4, v_5, \ldots, v_{r+1} \) are at least adjacent to \( v_{r+3} \), this vertex can be assigned the colour \( c_1 \) or \( c_2 \). To minimise the number of bad edges we assign the vertex the colour \( c_2 \). Similarly, vertex \( v_{r+4} \) is assigned the colour \( c_3 \), \( v_{r+5} \) the colour \( c_4 \) and so on, the vertex \( v_{r+k+1} \) the colour \( c_k \). Now, the remaining
r - k + 1 vertices from the \( v_{r+k+1} \)-th vertex are given the colour \( c_1 \). This colouring pattern is followed on each vertex disjoint complete subgraph of order \( r + 1 \). Thus, each vertex disjoint complete subgraph will have \( \frac{(r-k)(r-k+1)}{2} \) bad edges. Now, between any two consecutive vertex disjoint complete subgraphs there are \( \frac{(r-k+1)(r-k+2)}{2} \) number of bad edges (see the Theorem 2.1). Thus, the total number of bad edges is \( \frac{n(r-k+2)(r-k+1)}{2} \) (see the Figure 3 for illustration).

![Figure 3](image-url)  

**Figure 3**  The \( \delta^{(2)} \)-colouring of \( C_6^2 \)

**Case 2:** Let \( n \equiv 1 \mod (r+1) \). Since, paths are the subgraphs of cycles and as proved in the Theorem 2.1, there will always be \( \frac{(r-k)(r-k+1)}{2} \) bad edges in each of the vertex disjoint complete subgraphs and between any two consecutive complete subgraphs there will always be a minimum of \( \frac{(r-k+2)(r-k+1)}{2} \) bad edges. Now, since \( n \equiv 1 \mod (r+1) \), there is a vertex say \( v_n \) which does not belong to any of the vertex disjoint complete subgraphs. Now, this vertex can be assigned the colour \( c_1 \) or any colour other than \( c_1 \). The two possibilities are discussed below.

**Subcase 2.1:** If \( c(v_n) = c_1 \). In this case, any \( \delta^{(k)} \)-colouring that colours each vertex disjoint complete subgraph with \( k \) colours will lead to \( \frac{(r-k)(r-k+1)}{2} \) bad edges in each of the vertex disjoint complete subgraphs and \( \frac{(r-k+2)(r-k+1)}{2} \) bad edges between any two vertex disjoint complete subgraphs. But, due to the vertex \( v_n \), the number of bad edges between the last and the first vertex disjoint complete subgraphs (the one to the left of the vertex \( v_n \) (the last) and the one to right of the vertex \( v_n \) (the first)) as proved in Case 1, can be decreased by \( r - k + 1 \). Hence, the minimum number of bad edges between the last and the first vertex disjoint complete subgraphs is \( \frac{(r-k)(r-k+1)}{2} \) and the vertex \( v_n \) is at most adjacent to \( v_{n-r} \)-th and \( v_{n-1} \)-th vertices of the last and first vertex disjoint complete subgraphs. Thus, the minimum number of bad edges arising from the vertex \( v_n \) are \( 2(r-k+1) \). A \( \delta^{(k)} \)-colouring that explains this case is discussed below.

We follow the same colouring pattern as that of the Case 1 of the Theorem 2.4 and see that there are \( \frac{(r-k+2)(r-k+1)}{2} \) bad edges in each of the vertex disjoint complete subgraphs and between each consecutive vertex disjoint complete subgraphs there are \( \frac{(r-k+1)(r-k+2)}{2} \) bad edges. The vertex \( v_n \) is assigned the colour \( c_1 \). Now, the vertices \( v_{r+1} \) and \( v_{n-(r+1)} \) of the first and last vertex disjoint complete subgraphs respectively are assigned the colour \( c_1 \). Thus, the
vertex $v_n$ will be adjacent to $r-k+1$ vertices of both the complete subgraphs that are assigned the colour $c_1$, leading to $2(r - k + 1)$ bad edges all together, between the vertex $v_n$ and the vertex disjoint complete subgraphs. Now, we discuss the number of bad edges between the last and first vertex disjoint complete subgraphs. The vertices from $v_{n-1}$ to $v_{n-(r-k+1)}$ that are assigned the colour $c_1$ are the only vertices that lead to bad edges between the last and the first vertex disjoint complete subgraphs and hence we consider only these vertices. Since, there is a vertex $v_n$, unlike in the Case 1, the vertex $v_{n-1}$ is at most adjacent to the vertex $v_{r-1}$. Thus, among the $r+1$ vertices of the first vertex disjoint complete subgraph, the vertex $v_{n-1}$ is not adjacent to the remaining two vertices that are assigned the colour $c_1$. Thus, among the $r-k+2$ vertices that are assigned the colour $c_1$, $v_{n-1}$ is adjacent to $r-k$ vertices coloured with $c_1$ leading to $r-k$ bad edges. Similarly, the vertices $v_{n-(r-k)}$ and $v_{n-(r-k+1)}$ are at most adjacent to $v_k$ and $v_{k-1}$ vertices respectively (i.e. $v_{r-(r-k)}$ and $v_{r-(r-k+1)}$) and hence both the vertices will be adjacent to only one vertex coloured with the colour $c_1$ i.e the vertex $v_1$ and this thereby lead to one bad edge each respectively. Thus, there are $r-k+r-k+1+r-k-2+\ldots+1$ bad edges from vertex $v_{n-1}$ to $v_{n-(r-k)}$ and a bad edges lead by the vertex $v_{n-(r-k+1)}$. Hence, the total number of bad edges between these two vertex disjoint complete subgraphs are $\frac{(r-k)(r-k+1)}{2} + 1$. Thus, the total number of bad edges resulting from this $\delta^{(k)}$-colouring is $\left\lfloor \frac{n}{r+1} \right\rfloor (r-k+2) (r-k+1) - \frac{(r-k)(r-k+1)}{2} + 2 (r-k+1) + \frac{(r-k)(r-k+1)}{2} + 1 = \frac{(r-k+1)(r-k+2)}{2} \left( \frac{n}{r+1} - 1 \right) + 2 (r-k+1) + \frac{(r-k)(r-k+1)}{2} + 1.$

**Subcase 2.2:** If $c(v_n) = c_k$, where $k = 2, 3, \ldots, \chi(G) - 1$. Then, only the $r+1$-th vertex of both the vertex disjoint complete subgraphs from $v_n$ can be assigned the colour $c_k$. Thus, in this case the vertex $v_n$ does not lead to any bad edges. Each vertex disjoint complete subgraph will have a minimum of $\frac{(r-k+2)(r-k)}{2}$ bad edges and between each complete subgraphs there are $\frac{(r-k+1)(r-k+2)}{2}$ bad edges. But in this case, as mentioned earlier the $r+1$-th vertex of the first (i.e. $v_{r+1}$-th vertex) and last (i.e. $v_{n-(r+1)}$-th vertex) vertex disjoint complete subgraphs are assigned the colour $c_k$ and following this pattern from either of the directions, every $r+1$-th vertex of every complete subgraph is assigned the colour $c_k$ in order to maintain the requirements of $\delta^{(k)}$-colouring. Thus, either of one vertex disjoint complete subgraph that is adjacent to first or the last complete subgraph cannot be given the colour $c_k$ (this is because each colour other than $c_1$ can only be repeated $\left\lfloor \frac{n}{r+1} \right\rfloor$ times in the graph and the colour $c_k$ is already been assigned to the vertex $v_n$ and so there will be a vertex disjoint complete subgraph that cannot be assigned the colour $c_k$) and so this particular vertex disjoint complete subgraph will have $r-k+3$ vertices assigned the colour $c_1$. Hence, this will lead to $\frac{(r-k+3)(r-k+2)}{2}$ bad edges in one vertex disjoint complete subgraph and the rest $\frac{(r-k+2)(r-k+1)}{2}$ bad edges. Now, between the last and first vertex disjoint complete subgraphs there are $\frac{(r-k+1)(r-k+2)}{2}$ bad edges and this is because to determine the number of bad edges between these vertex disjoint complete subgraphs it is enough to consider the $r+1$ vertices from the vertex $v_{n-r}$ to $v_n$ (instead of the original vertex disjoint complete subgraph that has the vertices $v_{n-(r+1)}$ to $v_{n-1}$). This is possible as the graph is regular and the vertex $v_n$ is assigned a colour other than $c_1$. Thus, the number of bad edges between these two vertex disjoint complete subgraphs is $\frac{(r-k+1)(r-k+2)}{2}$ (See the Theorem 2.1). Thus, the total number of bad edges resulting from this case is given as $\left\lfloor \frac{n}{r+1} \right\rfloor (r-k+2)(r-k+1) - \frac{(r-k+1)(r-k+2)}{2} + \frac{(r-k+3)(r-k+2)}{2}$. A $\delta^{(k)}$-colouring explaining the above discussed case is given as follows:
To maintain the minimum number of bad edges between any two consecutive complete subgraphs which is \( \frac{(r-k+1)(r-k+2)}{2} \) (See the Theorem 2.1), we assign the first \( k-2 \) vertices (from the vertex \( v_1 \)) of either of the complete subgraphs, say, the last one with \( k-2 \) distinct colours say \( c_3, c_3, \ldots, c_{k-2} \). Now since, the \( v_{n-1} \)-th vertex of the last complete subgraph is at most adjacent to the \( r-1 \)-th vertex of the first complete subgraph, the colour \( c_2 \) can be assigned to the vertex \( v_r \). Similarly, the vertex \( v_{n-2} \) is at most adjacent to the vertex \( v_{r-2} \), and hence the vertex \( v_{r-1} \) can be assigned the colour \( c_3 \). The vertex \( v_{n-(k-2)} \) is at most adjacent to the vertex \( v_{r-k+2} \) of the first complete subgraph and so the vertex \( v_{r-k+3} \) can be assigned the colour \( c_{k-2} \). Thus, the remaining \( r-k+2 \) vertices of the last and the first complete subgraphs are assigned the colour \( c_1 \) respectively. Now, to find the number of bad edges between the last and first vertex disjoint complete subgraphs, it is enough to consider the vertices from \( v_{n-r-1} \) to \( v_{n-k+1} \) from the last vertex disjoint complete subgraph and the vertices from \( v_1 \) to \( v_{r-k+1} \) from the first vertex disjoint complete subgraph, as these are the vertices which lead to bad edges between the last and the first vertex disjoint complete subgraphs. The vertex \( v_{n-r-1} \) (assigned the colour \( c_1 \)) is at most adjacent to the vertex \( v_1 \), (assigned the colour \( c_1 \)), leading to one bad edge. Similarly, \( v_{n-r} \) is at most adjacent to the vertex \( v_2 \), leading to two bad edges. Continuing in this manner, the vertices \( v_{n-k} \) and \( v_{n-k+1} \) are at most adjacent to the vertices \( v_{r-k} \) and \( v_{r-k+1} \), leading to \( r-k \) and \( r-k+1 \) bad edges respectively. Thus, the minimum number of bad edges between the last and the first vertex disjoint complete subgraphs is \( \frac{(r-k+1)(r-k+2)}{2} \).

We use the same colouring pattern to colour the rest of the vertex disjoint complete subgraphs, say we begin colouring the vertex disjoint complete subgraph that is adjacent to the first vertex disjoint complete subgraph and follow the pattern likewise in clockwise direction. Since, the \( r+1 \)-th vertex of every vertex disjoint complete subgraph is coloured with the colour \( c_k \), the \( r+1 \)-th vertex of the vertex disjoint complete subgraph that is adjacent to the last complete subgraph cannot be given the colour \( c_k \) and so this particular vertex disjoint complete subgraph will have \( r-k+3 \) vertices assigned the colour \( c_1 \). Hence, this particular vertex disjoint complete subgraph will have \( \frac{(r-k+3)(r-k+2)}{2} \) and the rest \( \frac{(r-k+2)(r-k+1)}{2} \) bad edges. This vertex disjoint complete subgraph is coloured in such a way that the minimum number of bad edges between any two consecutive vertex disjoint complete subgraphs (see the Theorem 2.1) is maintained. The first \( r-k+3 \) vertices among the \( r+1 \) vertices of this complete subgraph is assigned the colour \( c_1 \) and the remaining \( k-2 \) the colours \( c_{k-2}, c_{k-3}, \ldots, c_2 \) in this order respectively, if not coloured in this order there would be adjacency between the elements of colour classes other than \( C_1 \). This colouring pattern will maintain the minimum number of bad edges between the second last vertex disjoint complete subgraph and its consecutive vertex disjoint complete subgraphs i.e. the third last and the last. Since there are \( r-k+2 \) vertices assigned the colour \( c_1 \) in the third last vertex disjoint complete subgraph and since the first vertex of this vertex disjoint complete subgraph coloured with \( c_1 \) is not adjacent to the second last vertex disjoint complete subgraph, this vertex will not lead to any bad edges between the second last vertex disjoint complete subgraph and itself. Now, the second vertex assigned the colour \( c_1 \), is at most adjacent to the first vertex of the second last vertex disjoint complete subgraph which is assigned the colour \( c_1 \) and this vertex will lead to one bad edge. Again, the third vertex of the third vertex disjoint complete subgraph (assigned the colour \( c_1 \)) is at most adjacent to the second vertex of the second vertex disjoint complete subgraph coloured with \( c_1 \), leading to two bad edges.
Similarly, the $r - k + 2$-th vertex of the third vertex disjoint complete subgraph (assigned the colour $c_1$) is at most adjacent to the $r - k + 1$-th vertex of the second vertex disjoint complete subgraph coloured with $c_1$, leading to $r - k + 1$ bad edges. Thus, between these two vertex disjoint complete subgraphs there are $\frac{(r-k+1)(r-k+2)}{2}$ bad edges. Now, between the second last and the last vertex disjoint complete subgraph there will also be $\frac{(r-k+1)(r-k+2)}{2}$ bad edges as explained below. The first vertex of second last vertex disjoint complete subgraph coloured with $c_1$ is not adjacent to the last vertex disjoint complete subgraph, this vertex thereby will not lead to any bad edges between the last complete subgraph and itself. Now, the second vertex assigned the colour $c_1$, is at most adjacent to the vertex $v_{n-(r+1)}$ of the last vertex disjoint complete subgraph which is assigned the colour $c_k$ and this leads to no bad edges. The remaining $r - k + 1$ vertices assigned the colour $c_1$, will lead to $\frac{(r-k+1)(r-k+2)}{2}$ bad edges between the last vertex disjoint complete subgraph and the second last vertex disjoint complete subgraph. Thus, the minimum total number of bad edges obtained from this $\delta^{(k)}$-colouring is 
\[ \frac{(r-k+1)(r-k+2)}{2} \left( \lfloor \frac{n}{r+1} \rfloor - 1 \right) + \frac{(r-k+3)(r-k+2)}{2} \left( \lfloor \frac{n}{r+3} \rfloor - 1 \right) = \frac{(r-k+1)(r-k+2)}{2} \left( 2 \lfloor \frac{n}{r+1} \rfloor - 1 \right) + \frac{(r-k+3)(r-k+2)}{2} \left( 2 \lfloor \frac{n}{r+3} \rfloor - 1 \right). \]

Now, when both the $\delta^{(k)}$-colourings are compared, the bad edges in both the cases are the same (See Figure 4a and Figure 4b for illustration).

Now, from the Theorem 2.4 it is clear that since $C_n^r$ is a regular graph and since it induces $\lfloor \frac{n}{r+1} \rfloor$ vertex disjoint complete subgraphs, any $\delta^{(k)}$-colouring with $k$ colours will give same number of bad edges if the colours other than the colour $c_1$ is assigned to the maximum number of vertices in the graph. Now, every colour other than $c_1$ will have only one possibility of colouring a single vertex of each vertex disjoint complete subgraph. As, there are $k - 1$ colours (excluding the colour $c_1$) to colour each vertex disjoint complete subgraph, there are $(k-1)\lfloor \frac{n}{r+1} \rfloor$ possibilities of using the colours other than $c_1$ in the optimal $\delta^{(k)}$-colouring of powers of cycles and the remaining vertices are assigned the colour $c_1$ to maintain the requirements of $\delta^{(k)}$-colouring.
The below theorem discusses the number of bad edges obtained from the $\delta^{(k)}$-colouring of $C_n^r$ when $n \equiv m \pmod{r+1}$ where $m \geq 2$.

**Theorem 2.3.** For the powers of cycle $C_n^r$, where $2 \leq r < \left\lfloor \frac{n}{2} \right\rfloor$, the number of bad edges obtained from $\delta^{(k)}$-colouring when $n \equiv m \pmod{r+1}$ where $m \geq 2$ is given by,

$$b_k(C_n^r) = ab + a(b - 1) + c + d + e,$$

where,

$$a = \frac{(r - k + 2)(r - k + 1)}{2},$$

$$b = \left\lfloor \frac{n}{r+1} \right\rfloor,$$

$$c = \begin{cases} r - m + \frac{(r-m-k)(r-m-k+1)}{2}, & \text{if } r - m < r - k + 1 \text{ when } k > m, \\ r - k + 1 + \frac{(r-m-k)(r-m-k+1)}{2}, & \text{if } r - m \geq r - k + 1 \text{ when } k \leq m. \end{cases}$$

$$d = \begin{cases} \frac{m}{2}(2(r - k) - m + 3), & \text{if } m < r - k, \\ m - r + k, & \text{if } m \leq k \text{ and } m = k = r. \end{cases}$$

$$e = \begin{cases} m(r - k + 1), & \text{if } m \leq k, \\ k(r - k + 1) + \frac{m-k}{2}(2r - m - k + 1), & \text{if } m > k. \end{cases}$$

**Proof.** Let $c_1, c_2, \ldots, c_k$ be the $k$ available colours and let $C_1, C_2, \ldots, C_k$ be their respective colour classes. Now, as explained above, since, $C_n^r$ is a $2r$ regular graph every vertex is in complete subgraph of order $r + 1$. Now, any vertex can be assigned any colour only once so we consider the vertex disjoint complete subgraphs of $C_n^r$ each of order $r+1$. There are $\left\lfloor \frac{n}{r+1} \right\rfloor$ vertex disjoint complete subgraphs of order $r+1$ and the remaining $m$ vertices do not belong to any of the vertex disjoint complete subgraphs, also since, $2 \leq m \leq r$, these $m$ vertices will not form a complete subgraph of order $r+1$. However, these $m$ vertices form a vertex disjoint complete subgraph of order $m$. Now, every colour other than $c_1$ will have only one possibility of colouring a single vertex of each vertex disjoint complete subgraph. Since there are $k - 1$ colours (excluding the colour $c_1$) to colour each vertex disjoint complete subgraph, there are $(k - 1)\left\lfloor \frac{m}{r+1} \right\rfloor$ possibilities of using the colours other than $c_1$ and the remaining vertices are assigned the colour $c_1$ to maintain the requirements of $\delta^{(k)}$-colouring. A $\delta^{(k)}$-colouring that explains this is same as that of the colouring mentioned for the case where $n \equiv 1 \pmod{r+1}$ and $k \leq r$, explained in the Theorem 2.4. In the case where $n \equiv 1 \pmod{r+1}$ and $k \leq r$, since $m = 1$, the vertex $v_n$ receives the colour $c_1$ and in this particular case all the $m$ vertices $m \geq 2$ will receive the colour $c_1$. Thus, every vertex disjoint complete subgraph will have \(\frac{(r-k+2)(r-k+1)}{2}\) bad edges and between any two vertex disjoint complete subgraphs of order $r+1$, there are \(\frac{(r-k+1)(r-k+2)}{2}\) bad edges (see the Theorem 2.4 for detailed explanation). The $m$ vertices, assigned the colour $c_1$, to maintain the prerequisites of $\delta^{(k)}$-colouring, will lead to \(\frac{m(m-1)}{2}\) bad edges. Now, since $m \leq r$, there can be bad edges between the last and the
first vertex disjoint subgraphs each of order \( r + 1 \), but the number can be less than that of the number of bad edges between the other vertex disjoint complete subgraphs of order \( r + 1 \). Also, there can be bad edges between the \( m \) vertices and the first and the last vertex disjoint complete subgraphs which are as discussed below.

**Case 1:** First we discuss the bad edges between the first and the last vertex disjoint complete subgraphs. The number of bad edges between these two vertex disjoint complete subgraphs can vary with different values of \( r, k \) and \( m \). There are two cases for the same as explained below.

**Case 1a:** Let \( r - m < r - k + 1 \) when \( k > m \). The vertex \( v_1 \) of the first vertex disjoint complete subgraph is adjacent to \( r \) vertices of the last vertex disjoint complete subgraph among which \( m \) vertices are not taken into consideration. Now, there are \( r - k + 2 \) vertices in each of the vertex disjoint complete subgraphs that are assigned the colour \( c_1 \) and the vertex \( v_1 \) is adjacent to a maximum of \( r - k + 1 \) vertices whose colour is \( c_1 \). Since, \( r - m < r - k + 1 \), the vertex \( v_1 \) is adjacent to \( r - m \) vertices coloured only with the colour \( c_1 \), leading to \( r - m \) bad edges. Further, the remaining \( k - 1 \) vertices \( v_2, v_3, \ldots, v_k \), assigned the colour \( c_2, c_3, \ldots, c_k \) respectively, are adjacent to \( r - m - 1, r - m - 2, \ldots, r - m - (k - 1) \) number of vertices respectively of the last vertex disjoint complete subgraph which lead to no bad edges. Now, the remaining vertices \( v_{k+1} \) to \( v_{r+1} \), assigned the colour \( c_1 \), adjacent to the last vertex disjoint complete subgraph (if any), lead in \( r - m - k + r - m - (k + 1) + \ldots + 1 = \frac{(r-m-k)(r-m-k+1)}{2} \) bad edges. Thus, the total number of bad edges resulting from this case is \( r - m + \frac{(r-m-k)(r-m-k+1)}{2} \).

**Case 1b:** Let \( r - m \geq r - k + 1 \) when \( k \leq m \). Then, the vertex \( v_1 \) is adjacent to \( r - k + 1 \) vertices that are solely assigned the colour \( c_1 \). The remaining vertices that are assigned the colour \( c_1 \) in the first vertex disjoint complete subgraph will lead in \( \frac{(r-m-k)(r-m-k+1)}{2} \) bad edges as explained in the case 1a. Thus, the total number of bad edges resulting from this case is \( r - k + 1 + \frac{(r-m-k)(r-m-k+1)}{2} \).

Now, we discuss the number of bad edges with respect to the \( m \) vertices and the first and last vertex disjoint complete subgraphs. First we focus on the bad edges between the former and then explain the latter. Each of the two cases have subcases which are as detailed below.

**Case 2:** In this case, we discuss the number of bad edges with respect to the \( m \) vertices and the first vertex disjoint complete subgraph.

**Case 2a:** Let \( m \geq r - k \). Here, the vertex \( v_n \) among the \( m \) vertices is adjacent to the vertices up to the vertex \( v_r \). Now, since there are \( r - k + 2 \) vertices that receive the colour \( c_1 \) in the vertex disjoint complete subgraphs and since \( v_n \) is not adjacent to the vertex \( v_{r+1} \) which is also assigned the colour \( c_1 \), it is adjacent to \( r - k + 1 \) vertices that are assigned the colour \( c_1 \). This will lead in \( r - k + 1 \) bad edges. The next vertex \( v_{n-1} \) is adjacent to the vertices up to \( v_{r-1} \) and this vertex will lead in \( r - k \) bad edges. Continuing in this manner, the vertex \( v_{n-r+k+1} \) is adjacent to the vertices up to the vertex \( v_{r-(r-k-1)} = v_{k+1} \), among which only two vertices, \( v_{k+1} \) and \( v_1 \) are assigned the colour \( c_1 \) and this leads to two bad edges. Now, since \( m \geq r - k \), there will be \( m - r + k \) vertices among the \( m \) vertices that are adjacent to the vertices coloured with colours other than \( c_1 \) and a single vertex \( v_1 \) that is assigned the colour \( c_1 \). Thus, all these vertices will lead in a total of \( m - r + k \) bad edges. Hence, the total number of bad edges resulting from this case is \( r - k + 1 + r - k + \ldots + 2 + m - r + k = \frac{(r-k)(r-k+1)+2m}{2} \).
Case 2b: Let $m < r - k$. Similar to the Case 2a, the vertex $v_n$ will lead in $r - k + 1$ bad edges, the vertex $v_{n-1}$ will lead in $r - k$ bad edges and so on. Now, since $m < r - k$, the $m$-th vertex $v_{n-m+1}$ is adjacent to the vertices up to the vertex $v_{r-m+1}$ of the first vertex disjoint complete subgraph and this leads to $r - k + 1 - m + 1 = r - k - m + 2$ bad edges. Thus, there are a total of $r - k + 1 + r - k + \ldots + r - k - m + 2 = \frac{m}{2}(2(r - k) - m + 3)$ bad edges resulting from this particular case.

Case 2c: Now, there are two special cases from the case 2a where the value of $r - k + 1 + r - k + \ldots + 2 = 0$ and the bad edges resulting from the special cases will be $m - r + k$. The cases arise when $m < k$ and $k = r$ and when $m = r = k$. In both the cases, there are only two vertices in the first vertex disjoint complete subgraphs that receive the colour $c_1$ i.e., the vertex $v_1$ and $v_{r+1}$. Now, none of the $m$ vertices are adjacent to the vertex $v_{r+1}$, thus, all the $m$ vertices will lead to $m - r + k$ or simply $m$ bad edges in both the cases.

Case 3: Next, we discuss the number of bad edges between the $m$ vertices and the last vertex disjoint complete subgraph of order $r + 1$. There are two cases for this which are explained below.

Case 3a: Let $k \geq m$. Now, none of the $m$ vertices are adjacent to the vertex $v_{n-m-r}$ which is assigned the colour $c_1$, thus the bad edges obtained in this case will be from the $r - k + 1$ vertices assigned the colour $c_1$ in the last vertex disjoint complete subgraph. Now, the first vertex among the $m$ vertices i.e., the vertex $v_{n-m+1}$ is adjacent to the $r$ vertices i.e., up to the vertex $v_{n-m-r+1}$. This will lead in $r - k + 1$ bad edges. The next vertex $v_{n-m+2}$ is adjacent to the vertices up to the vertex $v_{n-m-r+2}$ which is assigned the colour $c_2$, thus this vertex $v_{n-m+2}$ will also lead in $r - k + 1$ bad edges. Similarly, the vertex $v_{n-m+3}$ is adjacent to the vertices up to the vertex $v_{n-m-r+3}$ which is assigned the colour $c_3$ leading to again $r - k + 1$ bad edges. Similarly, the $m$-th vertex $v_n$ is adjacent to vertices up to $v_{n-m-r+m} = v_{n-r}$ which is assigned the colour $c_{m+1}$ leading to again $r - k + 1$ bad edges. Thus, the total number of bad edges arising from this case is $m(r - k + 1)$.

Case 3b: Let $k < m$. As explained in the case 3a and since $k < m$, the $k$ vertices among the $m$ vertices will lead in $r - k + 1$ bad edges. Now, there are $m - k$ vertices remaining. The first vertex among the $m - k$ vertices will lead in $r - k$ bad edges, the second $r - k - 1$ and so on the $m-k$-th vertex will lead in $r-k-(m-k-1) = r-m+1$ bad edges. Thus, the total number of bad edges resulting from this case is $k(r-k+1)+r-k+r-k-1+\ldots+r-m+1$. Hence, the total number of bad edges for $C_n^r$ resulting from the $\delta^{(k)}$-colouring when $n \equiv m \pmod{r+1}$ where $m \leq r$ and $k \leq r$ is given as $ab + a(b - 1) + c + d + e$, where,

$$a = \frac{(r - k + 2)(r - k + 1)}{2}$$
$$b = \left\lfloor \frac{n}{r+1} \right\rfloor$$
$$c = \begin{cases} 
  r - m + \frac{(r-m-k)(r-m-k+1)}{2}, & \text{if } r - m < r - k + 1 \text{ when } k > m, \\
  r - k + 1 + \frac{(r-m-k)(r-m-k+1)}{2}, & \text{if } r - m \geq r - k + 1 \text{ when } k \leq m.
\end{cases}$$
The below theorem discusses the number of bad edges for two cases of powers of cycles resulting from $\delta^{(k)}$-colouring when $k > r$.

**Theorem 2.4.** For $C_r^n$, where $2 \leq r < \lfloor \frac{n}{2} \rfloor$ and $3 \leq k < \chi(G)$ and $k > r$, the number of bad edges obtained from $\delta^{(k)}$-colouring is given by,

$$b_k(C_r^n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{r+1}, \\ 1, & \text{if } n \equiv 1 \pmod{r+1}. \end{cases}$$

**Proof.** Case 1: When $n \equiv 0 \pmod{r+1}$, we know that there are exactly $\frac{n}{r+1}$ vertex disjoint complete subgraphs which includes all the $n$ vertices. Since, there are complete subgraphs of order $r + 1$ in $C_r^n$, $\chi(C_r^n) \geq r + 1$. Now, since there are exactly $\frac{n}{r+1}$ vertex disjoint complete subgraphs, every vertex disjoint complete subgraph can be properly coloured using $k = r + 1$ colours leading to no bad edges in the graph. Thus, there are no bad edges resulting from a $\delta^{(k)}$-colouring in $C_r^n$ when $k > r$.

Case 2: Let $n \equiv 1 \pmod{r+1}$. There will be $\lfloor \frac{n}{r+1} \rfloor$ vertex disjoint complete subgraphs including all the $n - 1$ vertices among the $n$ vertices and hence as explained in the above case, each vertex disjoint complete subgraph can be properly coloured with $k = r + 1$ colours $c_1, c_2, \ldots, c_{r+1}$. The remaining one vertex can be assigned a colour say $c_{r+2}$. Thus, when $n \equiv 1 \pmod{r+1}$, the $\chi(C_r^n) = r + 2$. Now, if $k = r + 1$, then we colour the vertices in the following manner. Assign the vertices $v_1, v_2, \ldots, v_{r+1}$ the colours $c_1, c_2, \ldots, c_{r+1}$ respectively. Thus, each vertex disjoint complete subgraph is properly coloured with the $r + 1$ colours leading to no bad edges in each of the vertex disjoint complete subgraphs and also no bad edges between any two vertex disjoint complete subgraphs. Now, after assigning colours to the $n - 1$ vertices that belong to $\lfloor \frac{n}{r+1} \rfloor$ vertex disjoint complete subgraphs, there is a vertex say $v_n$ which has to be assigned the colour $c_1$, to maintain the requirements of $\delta^{(k)}$-colouring and this vertex is adjacent to the vertex $v_1$ which is assigned the colour $c_1$ leading to one bad edge in the graph. Thus, $b_k(C_r^n) = 1$, when $n \equiv 1 \pmod{r+1}$ and $k = r + 1$. Note that, in this case, any $\delta^{(k)}$-colouring other than the above mentioned will always give one bad edge when $k = r + 1$. \qed

3 Conclusion

In this paper, we have discussed the $\delta^{(k)}$-colouring of the powers of paths and cycles. Every case for different parities of $n$ and for different values of $k$ and $r$ are keenly studied and an
optimal $\delta^{(k)}$-colouring and the minimum number of bad edges obtained from the same is determined.

The study can be extended to many graph classes, graph powers and graph products. In this discussion, we have restricted the number of colour classes, which can have relaxation in proper colouring requirements, to one. This restriction may also be relaxed ensuring the minimum possible number of bad edges in the given graph.

**Acknowledgements**

Authors of the paper would like to thank Dr Johan Kok, Visiting Professor, Department of Mathematics, CHRIST (Deemed to be University), Bangalore, India., for his valuable comments and inputs which improved the content and the style of the presentation. The authors also acknowledge the critical and creative comments and suggestions of the referees which improved the quality of the paper significantly.

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