Jet Methods in Time–Dependent Lagrangian Biomechanics

Tijana T. Ivancevic
Society for Nonlinear Dynamics in Human Factors, Adelaide, Australia
and
CITECH Research IP Pty Ltd, Adelaide, Australia
e-mail: tijana.ivancevic@alumni.adelaide.edu.au

Abstract

In this paper we propose the time-dependent generalization of an ‘ordinary’ autonomous human biomechanics, in which total mechanical + biochemical energy is not conserved. We introduce a general framework for time-dependent biomechanics in terms of jet manifolds associated to the extended musculo-skeletal configuration manifold, called the configuration bundle. We start with an ordinary configuration manifold of human body motion, given as a set of its all active degrees of freedom (DOF) for a particular movement. This is a Riemannian manifold with a material metric tensor given by the total mass-inertia matrix of the human body segments. This is the base manifold for standard autonomous biomechanics. To make its time-dependent generalization, we need to extend it with a real time axis. By this extension, using techniques from fibre bundles, we defined the biomechanical configuration bundle. On the biomechanical bundle we define vector-fields, differential forms and affine connections, as well as the associated jet manifolds. Using the formalism of jet manifolds of velocities and accelerations, we develop the time-dependent Lagrangian biomechanics. Its underlying geometric evolution is given by the Ricci flow equation.

Keywords: Human time-dependent biomechanics, configuration bundle, jet spaces, Ricci flow

1 Introduction

It is a well-known fact that most of dynamics in both classical and quantum physics is based on assumption of a total energy conservation (see, e.g. [1]). Dynamics based on this assumption of time-independent energy, usually given by Lagrangian or Hamiltonian energy function, is called autonomous. This basic assumption is naturally inherited in human biomechanics, formally developed using Newton–Euler, Lagrangian or Hamiltonian formalisms (see [2] [3] [4] [5] [6] [7] [8] [9]).

And this works fine for most individual movement simulations and predictions, in which the total human energy dissipations are insignificant. However, if we analyze a 100 m-dash sprinting motion, which is in case of top athletes finished under 10 s, we can recognize a significant slow-down after about 70 m in all athletes – despite of their strong intention to finish and win the race, which is an obvious sign of the total energy dissipation. This can be seen, for example, in a current record-braking speeddistance curve of Usain Bolt, the world-record holder with 9.69 s [10], or in a former record-braking speeddistance curve of Carl Lewis, the former world-record holder (and 9 time Olympic gold medalist) with 9.86 s (see Figure 3.7 in [9]). In other words, the total mechanical
biochemical energy of a sprinter cannot be conserved even for 10 s. So, if we want to develop a realistic model of intensive human motion that is longer than 7–8 s (not to speak for instance of a 4 hour tennis match), we necessarily need to use the more advanced formalism of time-dependent mechanics.

Similarly, if we analyze individual movements of gymnasts, we can clearly see that the high speed of these movements is based on quickly-varying mass-inertia distribution of various body segments (mostly arms and legs). Similar is the case of pirouettes in ice skating. As the total mass-inertia matrix $M_{ij}$ of a biomechanical system corresponds to the Riemannian metric tensor $g_{ij}$ of its configuration manifold, we can formulate this problem in terms of time-dependent Riemannian geometry [4, 1].

The purpose of this paper is to introduce a general framework for time-dependent biomechanics, consisting of the following four steps:

1. Human biomechanical configuration manifold and its (co)tangent bundles;
2. Biomechanical configuration bundle, as a time–extension of the configuration manifold;
3. Biomechanical jet spaces and prolongation of locomotion vector-fields developed on the configuration bundle; and
4. Time–dependent Lagrangian dynamics using biomechanical jet spaces.

In addition, we will show that Riemannian geometrical basis of this framework is defined by the Ricci flow. In particular, we will show that the exponential–like decay of total biomechanical energy (due to exhaustion of biochemical resources [9]) is closely related to the Ricci flow on the configuration manifold of human motion.

2 Configuration Manifold for Autonomous Biomechanics

Recall from [6] that representation of an ideal humanoid–robot motion is rigorously defined in terms of rotational constrained $SO(3)$–groups in all main robot joints. Therefore, the configuration manifold $Q_{rob}$ for humanoid dynamics is defined as a topological product of all included $SO(3)$ groups, $Q_{rob} = \prod_i SO(3)^i$. Consequently, the natural stage for autonomous Lagrangian dynamics of robot motion is the tangent bundle $TQ_{rob}$, defined as follows. To each $n$–dimensional (nD) configuration manifold $Q$ there is associated its $2n$D velocity phase–space manifold, denoted by $TQ$ and called the tangent bundle of $Q$. The original smooth manifold $Q$ is called the base of $TQ$. There is an onto map $\pi : TQ \to Q$, called the projection. Above each point $x \in Q$ there is a tangent space $T_xQ = \pi^{-1}(x)$ to $Q$ at $x$, which is called a fibre. The fibre $T_xQ \subset TQ$ is the subset of $TQ$, such that the total tangent bundle, $TQ = \bigsqcup_{x \in Q} T_xQ$, is a disjoint union of tangent spaces $T_xQ$ to $Q$ for all points $x \in Q$. From dynamical perspective, the most important quantity in the tangent bundle concept is the smooth map $v : Q \to TQ$, which is an inverse to the projection $\pi$, i.e., $\pi \circ v = \text{Id}_Q$, $\pi(v(x)) = x$. It is called the velocity vector–field. Its graph $(x, v(x))$ represents the cross–section of the tangent bundle $TQ$. This explains the dynamical term velocity phase–space, given to the tangent bundle $TQ$ of the manifold $Q$. The tangent bundle is where tangent vectors
live, and is itself a smooth manifold. Vector-fields are cross-sections of the tangent bundle. Robot’s Lagrangian (energy function) is a natural energy function on the tangent bundle $TQ$. On the other hand, human joints are more flexible than robot joints. Namely, every rotation in all synovial human joints is followed by the corresponding micro-translation, which occurs after the rotational amplitude is reached [6]. So, representation of human motion is rigorously defined in terms of Euclidean $SE(3)$–groups of full rigid–body motion [2, 4, 11] in all main human joints (see Figure[11]). Therefore, the configuration manifold $Q$ for human dynamics is defined as a topological product of all included constrained $SE(3)$–groups[2]. Consequently, the natural stage for autonomous Lagrangian dynamics of human motion is the tangent bundle $TQ$ (and for the corresponding autonomous Hamiltonian dynamics is the cotangent bundle $T^*Q$).

Therefore, the configuration manifold $Q$ for human musculo-skeletal dynamics is defined as a Cartesian product of all included constrained $SE(3)$ groups, $Q = \prod_j SE(3)^j$ where $j$ labels the active joints. The configuration manifold $Q$ is coordinated by local joint coordinates $x^i(t)$, $i = 1, ..., n = total\ number\ of\ active\ DOF$. The corresponding joint velocities $\dot{x}^i(t)$ live in the velocity phase space $TQ$, which is the tangent bundle of the configuration manifold $Q$.

The velocity phase space $TQ$ has the Riemannian geometry with the local metric form:

$$\langle g \rangle \equiv ds^2 = g_{ij} dx^i dx^j,$$

(Einstein’s summation convention is in always use)

where $g_{ij}(x)$ is the material metric tensor defined by the biomechanical system’s mass-inertia matrix and $dx^i$ are differentials of the local joint coordinates $x^i$ on $Q$. Besides giving the local distances between the points on the manifold $Q$, the Riemannian metric form $\langle g \rangle$ defines the system’s kinetic energy:

$$T = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j,$$

---

1. The corresponding autonomous Hamiltonian robot dynamics takes place in the cotangent bundle $T^*Q_{rob}$, defined as follows. A dual notion to the tangent space $T_m Q$ to a smooth manifold $Q$ at a point $m$ is its cotangent space $T^*_m Q$ at the same point $m$. Similarly to the tangent bundle, for a smooth manifold $Q$ of dimension $n$, its cotangent bundle $T^*Q$ is the disjoint union of all its cotangent spaces $T^*_m Q$ at all points $m \in Q$, i.e., $T^*Q = \bigsqcup_{m \in Q} T^*_m Q$. Therefore, the cotangent bundle of an $n$–manifold $Q$ is the vector bundle $T^*Q = (TQ)^*$, the (real) dual of the tangent bundle $TQ$. The cotangent bundle is where 1–forms live, and is itself a smooth manifold. Covector–fields (1-forms) are cross-sections of the cotangent bundle. Robot’s Hamiltonian is a natural energy function on the cotangent bundle.

2 Briefly, the Euclidean $SE(3)$–group is defined as a semidirect (noncommutative) product (denoted by $\triangleright$) of 3D rotations and 3D translations: $SE(3) := SO(3) \triangleright \mathbb{R}^3$. Its most important subgroups are the following (for technical details see [4,11]):

| Subgroup | Definition |
|----------|------------|
| $SO(3)$, group of rotations in 3D (a spherical joint) | Set of all proper orthogonal rotational matrices |
| $SE(2)$, special Euclidean group in 2D (all planar motions) | Set of all $3 \times 3$ matrices: $\begin{bmatrix} \cos \theta & \sin \theta & r_x \\ -\sin \theta & \cos \theta & r_y \\ 0 & 0 & 1 \end{bmatrix}$ |
| $SO(2)$, group of rotations in 2D subgroup of $SE(2)$–group (a revolute joint) | Set of all proper orthogonal $2 \times 2$ rotational matrices included in $SE(2)$ – group |
| $\mathbb{R}^3$, group of translations in 3D (all spatial displacements) | Euclidean 3D vector space |
Figure 1: The configuration manifold $Q$ of the human musculo-skeletal dynamics is defined as a topological product of constrained $SE(3)$ groups acting in all major (synovial) human joints, $Q = \prod_i SE(3)^i$.

giving the Lagrangian equations of the conservative skeleton motion with kinetic-minus-potential energy Lagrangian $L = T - V$, with the corresponding geodesic form [8]

$$\frac{d}{dt} L_{\dot{x}^i} - L_{x^i} = 0 \quad \text{or} \quad \ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0,$$

where subscripts denote partial derivatives, while $\Gamma^i_{jk}$ are the Christoffel symbols of the affine Levi-Civita connection of the biomechanical manifold $Q$.

This is the basic geometrical structure for autonomous Lagrangian biomechanics. In the next sections we will extend this basic structure to embrace the time-dependent Lagrangian biomechanics.

3 Biomechanical Bundle, Sections and Connections

While standard autonomous Lagrangian biomechanics is developed on the configuration manifold $X$, the time-dependent biomechanics necessarily includes also the real time axis $\mathbb{R}$, so we have an extended configuration manifold $\mathbb{R} \times X$. Slightly more generally, the fundamental geometrical structure is the so-called configuration bundle $\pi : X \rightarrow \mathbb{R}$. Time-dependent biomechanics is thus formally developed either on the extended configuration manifold $\mathbb{R} \times X$, or on the configuration bundle $\pi : X \rightarrow \mathbb{R}$, using the concept of jets, which are based on the idea of higher-order tangency, or higher-order contact, at some designated point (i.e., certain joint) on a biomechanical configuration manifold $X$.

In general, tangent and cotangent bundles, $TM$ and $T^*M$, of a smooth manifold $M$, are special cases of a more general geometrical object called fibre bundle, denoted $\pi : Y \rightarrow X$, where the word
fiber $V$ of a map $\pi : Y \rightarrow X$ is the \textit{preimage} $\pi^{-1}(x)$ of an element $x \in X$. It is a space which \textit{locally} looks like a product of two spaces (similarly as a manifold locally looks like Euclidean space), but may possess a different \textit{global} structure. To get a visual intuition behind this fundamental geometrical concept, we can say that a fibre bundle $Y$ is a \textit{homeomorphic generalization} of a product space $X \times V$ (see Figure 2), where $X$ and $V$ are called the \textit{base} and the \textit{fibre}, respectively. $\pi : Y \rightarrow X$ is called the \textit{projection}, $Y_x = \pi^{-1}(x)$ denotes a fibre over a point $x$ of the base $X$, while the map $f = \pi^{-1} : X \rightarrow Y$ defines the \textit{cross-section}, producing the graph $(x, f(x))$ in the bundle $Y$ (e.g., in case of a tangent bundle, $f = \dot{x}$ represents a velocity vector-field).

Figure 2: A sketch of a locally trivial fibre bundle $Y \approx X \times V$ as a generalization of a product space $X \times V$; left – main components; right – a few details (see text for explanation).

More generally, a biomechanical configuration bundle, $\pi : Y \rightarrow X$, is a locally trivial fibred (or, projection) manifold over the base $X$. It is endowed with an atlas of fibred bundle coordinates $(x^\lambda, y^i)$, where $(x^\lambda)$ are coordinates of $X$.

All dynamical objects in time–dependent biomechanics (including vectors, tensors, differential forms and gauge potentials) are \textit{cross-sections} of biomechanical bundles, representing generalizations of graphs of continuous functions.

An \textit{exterior differential form} $\alpha$ of order $p$ (or, a $p$–form $\alpha$) on a base manifold $X$ is a section of the bundle $\bigwedge^p T^* X \rightarrow X$ \cite{19}. It has the following expression in local coordinates on $X$

$$\alpha = \alpha_{\lambda_1...\lambda_p} dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_p} \quad \text{(such that } |\alpha| = p),$$

where summation is performed over all ordered collections $(\lambda_1, ..., \lambda_p)$. $\Omega^p(X)$ is the vector space of $p$–forms on a biomechanical manifold $X$. In particular, the 1–forms are called the Pfaffian forms.

The \textit{contraction} $\lfloor$ of any vector-field $u = u^\mu \partial_\mu$ and a $p$–form $\alpha = \alpha_{\lambda_1...\lambda_p} dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_p}$ on a biomechanical manifold $X$ is given in local coordinates on $X$ by

$$u \lfloor \alpha = u^\mu \alpha_{\mu \lambda_1...\lambda_{p-1}} dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_{p-1}}.$$ 

It satisfies the following relation

$$u \lfloor (\alpha \wedge \beta) = u \lfloor \alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge u \lfloor \beta.$$

The \textit{Lie derivative} $L_u \alpha$ of $p$–form $\alpha$ along a vector-field $u$ is defined by Cartan’s ‘magic’ formula (see \cite{4, 11}):

$$L_u \alpha = u \lfloor d \alpha + d(u \lfloor \alpha).$$
It satisfies the Leibnitz relation

\[ \mathcal{L}_u(\alpha \wedge \beta) = \mathcal{L}_u \alpha \wedge \beta + \alpha \wedge \mathcal{L}_u \beta. \]

A linear connection \( \bar{\Gamma} \) on a biomechanical bundle \( Y \to X \) is given in local coordinates on \( Y \) by

\[ \bar{\Gamma} = dx^\lambda \otimes [\partial_\lambda - \Gamma^i_{j\lambda}(x)y^j\partial_i]. \] (2)

An affine connection \( \Gamma \) on a biomechanical bundle \( Y \to X \) is given in local coordinates on \( Y \) by

\[ \Gamma = dx^\lambda \otimes [\partial_\lambda + (-\Gamma^i_{j\lambda}(x)y^j + \Gamma^i_{\lambda}(x))\partial_i]. \]

Clearly, a linear connection \( \bar{\Gamma} \) is a special case of an affine connection \( \Gamma \).

4 Biomechanical Jets

A pair of smooth manifold maps, \( f_1, f_2 : M \to N \) (see Figure 3), are said to be \( k \)-tangent (or tangent of order \( k \), or have a \( k \)th order contact) at a point \( x \) on a domain manifold \( M \), denoted by \( f_1 \sim f_2 \), iff

\[
\begin{align*}
    f_1(x) &= f_2(x) \quad \text{called 0–tangent}, \\
    \partial_x f_1(x) &= \partial_x f_2(x), \quad \text{called 1–tangent}, \\
    \partial_{xx} f_1(x) &= \partial_{xx} f_2(x), \quad \text{called 2–tangent}, \\
    &\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{etc. to the order } k
\end{align*}
\]

In this way defined \( k \)-tangency is an equivalence relation.

\[ j^k_x f : Q \to N = \{ f' : f' \text{ is } k \text{–tangent to } f \text{ at } x \}. \]
For example, consider a simple function \( f : X \to Y, x \mapsto y = f(x) \), mapping the \( X \)–axis into the \( Y \)–axis in \( \mathbb{R}^2 \). At a chosen point \( x \in X \) we have:

- a 0–jet is a graph: \((x, f(x))\);
- a 1–jet is a triple: \((x, f(x), f'(x))\);
- a 2–jet is a quadruple: \((x, f(x), f'(x), f''(x))\),

and so on, up to the order \( k \) by the coordinate conditions

\[ a_2 \frac{\partial}{\partial x^2} \in \mathfrak{X}(X) \]

is endowed with the adapted coordinates \((x, y, \dot{y}, \ddot{y})\) of \( \gamma(t) \), that is by \((t, q^i, \dot{q}^i, \ddot{q}^i)\), that is by \((time, \text{coordinates and velocities})\) at every active human joint, so the 1–jets are local joint coordinate maps

\[ j_1^i: \mathbb{R} \to Q, \quad t \mapsto (t, q^i, \dot{q}^i). \]

The repeated jet manifold \( J^1 J^1 Y \) is defined to be the jet manifold of the bundle \( J^1 Y \to X \). It is endowed with the adapted coordinates \((x^\lambda, y^i, y^i_\lambda, y^i_\lambda\mu)\).

The second order jet manifold \( J^2 Y \) of a bundle \( Y \to X \) is the subbundle of \( J^2 Y \to J^1 Y \) defined by the coordinate conditions \( y^i_\lambda\mu = y^j_\mu\lambda \). It has the local coordinates \((x^\lambda, y^i, y^i_\lambda, y^i_\lambda\mu, y^i_\lambda\mu\nu)\) together with the transition functions

\[ y^i_\lambda \mu = \frac{\partial x^\alpha}{\partial x'^\beta}(\partial_\alpha + y^j_\mu \partial_j + y^j_\mu\alpha \partial_j') y^i_\beta. \]

The second order jet manifold \( J^2 Y \) of \( Y \to X \) comprises the equivalence classes \( j^2_1 s \) of sections \( s \) of \( Y \to X \) such that

\[ y^i_\lambda(j^2_1 s) = \partial_\lambda s^i(x), \quad y^i_\lambda\mu(j^2_1 s) = \partial_\mu \partial_\lambda s^i(x). \]

In other words, two sections \( s, s' \in j^2_1 s \) are identified by their values and the values of their first and second order derivatives at the point \( x \in X \).
In particular, given the biomechanical configuration bundle \( Q \rightarrow \mathbb{R} \) over the time axis \( \mathbb{R} \), the 2−jet space \( J^2(\mathbb{R}, Q) \) is the set of equivalence classes \( j^2_s \) of sections \( s^i : \mathbb{R} \rightarrow Q \) of the configuration bundle \( \pi : Q \rightarrow \mathbb{R} \), which are identified by their values \( s^i(t) \), as well as the values of their first and second partial derivatives, \( \partial_t s^i = \partial_t s^i(t) \) and \( \partial_{tt} s^i = \partial_{tt} s^i(t) \), respectively, at time points \( t \in \mathbb{R} \).

The 2−jet manifold \( J^2(\mathbb{R}, Q) \) is coordinated by \((t, q^i, \dot{q}^i, \ddot{q}^i)\), that is by (time, coordinates, velocities and accelerations) at every active human joint, so the 2−jets are local joint coordinate maps:

\[
j^2_s : \mathbb{R} \rightarrow Q, \quad t \mapsto (t, q^i, \dot{q}^i, \ddot{q}^i).
\]

5 Lagrangian Time–Dependent Biomechanics

5.1 Jet Dynamics and Quadratic Equations

The general form of time-dependent Lagrangian biomechanics with time-dependent Lagrangian function \( L(t; q^i; \dot{q}^i) \) defined on the jet space \( X = J^1(\mathbb{R}, Q) \cong \mathbb{R} \times TQ \), with local canonical coordinates: \((t; q^i; \dot{q}^i) = (\text{time}, \text{coordinates} \text{and} \text{velocities}) \) in active local joints, can be formulated as \([4, 1]\)

\[
\frac{d}{dt} L_{\dot{q}^i} - L_{q^i} = \mathcal{F}_i(t, q, \dot{q}), \quad (i = 1, \ldots, n),
\]

(3)

where the coordinate and velocity partial derivatives of the Lagrangian are respectively denoted by \( L_{q^i} \) and \( L_{\dot{q}^i} \).

The most interesting instances of (3) are quadratic biomechanical equations, of the general form

\[
\xi^i \equiv \ddot{q}^i = a^i_{jk}(q^\nu)\dot{q}^j \dot{q}^k + b^i_{jk}(q^\nu)\dot{q}^j \dot{q}^k + f^i(q^\nu).
\]

(4)

They are coordinate–independent due to the affine transformation law of coordinates \( \dot{q}^i \). Then, it is clear that the corresponding dynamical connection \( \Gamma_{\xi} \) is affine \([4, 1]\):

\[
\Gamma_{\xi} = dq^\alpha \otimes [\partial_\alpha + (\Gamma_{\lambda 0}^\alpha(q^\nu) + \Gamma_{\lambda j}^\alpha(q^\nu)\dot{q}^j)\partial_t^i].
\]

This connection is symmetric iff \( \Gamma_{\lambda 0}^\mu = \Gamma_{\mu 0}^\lambda \).

There is 1–1 correspondence between the affine connections \( \Gamma \) on the affine jet bundle \( J^1(\mathbb{R}, Q) \rightarrow Q \) and the linear connections \( K \) on the tangent bundle \( TQ \rightarrow Q \) of the autonomous biomechanical manifold \( Q \). This correspondence is given by the relation

\[
\Gamma_{\mu}^i = \Gamma_{\lambda 0}^i + \Gamma_{\mu j}^i \dot{q}^j, \quad \Gamma_{\mu \lambda}^i = K_{\mu \alpha}^i.
\]

Any quadratic biomechanical equation \([4]\) is equivalent to the geodesic equation \([19]\)

\[
\ddot{t} = 1, \quad \dddot{t} = 0, \quad \dddot{q}^i = a^i_{jk}(q^\nu)\dddot{q}^j \dddot{q}^k + b^i_{jk}(q^\nu)\dddot{q}^j \dddot{q}^k + f^i(q^\nu)\ddot{t},
\]

for the symmetric linear connection

\[
K = dq^\alpha \otimes (\partial_\alpha + K_{\alpha \nu}^\mu(t, q^i)\dot{q}^\nu \partial_\mu).
\]

Footnote 3: For more technical details on jet spaces with their physical applications, see \([17, 18, 19, 20]\).
on the tangent bundle $TQ \rightarrow Q$, given by the components

$$K^0_{αν} = 0, \quad K^i_0 = f^i, \quad K^i_j = K^j_0 = \frac{1}{2}b^i_j, \quad K^i_j k = a^i_{jk}.$$  

Conversely, any linear connection $K$ on the tangent bundle $TQ \rightarrow Q$ defines the quadratic dynamical equation

$$\ddot{q}^i = K^j_k q^j \dot{q}^k + (K^0_j + K^i_0) \dot{q}^i + K^0_0,$$

written with respect to a given reference frame $(t, q^i) \equiv q^μ$ (see [19] for technical details).

### 5.2 Local Muscle–Joint Mechanics

The right-hand side terms $F_i(t, q, \dot{q})$ of (3) denote any type of external torques and forces, including excitation and contraction dynamics of muscular–actuators and rotational dynamics of hybrid robot actuators, as well as (nonlinear) dissipative joint torques and forces and external stochastic perturbation torques and forces. In particular, we have [2, 3]):

1. **Synovial joint mechanics**, giving the first stabilizing effect to the conservative skeleton dynamics, is described by the $(q, \dot{q})$–form of the Rayleigh–Van der Pol’s dissipation function

$$R = \frac{1}{2} \sum_{i=1}^{n} (\dot{q}^i)^2 [α_i + β_i(\dot{q}^i)^2],$$

where $α_i$ and $β_i$ denote dissipation parameters. Its partial derivatives give rise to the viscous–damping torques and forces in the joints

$$F^\text{joint}_i = \partial R/\partial \dot{q}^i,$$

which are linear in $\dot{q}^i$ and quadratic in $q^i$.

2. **Muscular mechanics**, giving the driving torques and forces $F^\text{muscle}_i = F^\text{muscle}_i(t, q, \dot{q})$ with $(i = 1, \ldots, n)$ for human biomechanics, describes the internal excitation and contraction dynamics of equivalent muscular actuators [12].

   (a) **Excitation dynamics** can be described by an impulse force–time relation

$$F^\text{imp}_i = F^0_i(1 - e^{-t/τ_i}) \quad \text{if stimulation} > 0$$

$$F^\text{imp}_i = F^0_i e^{-t/τ_i} \quad \text{if stimulation} = 0,$$

where $F^0_i$ denote the maximal isometric muscular torques and forces, while $τ_i$ denote the associated time characteristics of particular muscular actuators. This relation represents a solution of the Wilkie’s muscular active–state element equation [13]

$$\dot{μ} + Γ μ = Γ S A, \quad μ(0) = 0, \quad 0 < S < 1,$$

where $μ = μ(t)$ represents the active state of the muscle, $Γ$ denotes the element gain, $A$ corresponds to the maximum tension the element can develop, and $S = S(r)$ is the ‘desired’ active state as a function of the motor unit stimulus rate $r$. This is the basis for biomechanical force controller.

   (b) **Contraction dynamics** has classically been described by the Hill’s hyperbolic force–velocity relation [14]

$$F^\text{Hill}_i = \frac{(F^0_i b_k - δ_{ij} a_i \dot{q}^j)}{(δ_{ij} \dot{q}^j + b_k)},$$
where $a_i$ and $b_i$ denote the Hill’s parameters, corresponding to the energy dissipated during the contraction and the phosphagenic energy conversion rate, respectively, while $\delta_{ij}$ is the Kronecker’s $\delta$–tensor.

In this way, human biomechanics describes the excitation/contraction dynamics for the $i$th equivalent muscle–joint actuator, using the simple impulse–hyperbolic product relation

$$F_{i}^{\text{muscle}}(t, q, \dot{q}) = F_{i}^{\text{imp}} \times F_{i}^{\text{Hill}}.$$  

Now, for the purpose of biomedical engineering and rehabilitation, human biomechanics has developed the so–called hybrid rotational actuator. It includes, along with muscular and viscous forces, the D.C. motor drives, as used in robotics

$$F_{i}^{\text{robo}} = i_k(t) - J_k \dot{q}_k(t) - B_k \ddot{q}_k(t),$$

where $k = 1, \ldots, n$, $i_k(t)$ and $u_k(t)$ denote currents and voltages in the rotors of the drives, $R_k, l_k$ and $C_k$ are resistances, inductances and capacitances in the rotors, respectively, while $J_k$ and $B_k$ correspond to inertia moments and viscous dampings of the drives, respectively.

Finally, to make the model more realistic, we need to add some stochastic torques and forces

$$F_{i}^{\text{stoch}} = B_{ij} \left[ q_i(t), t \right] dW_j(t),$$

where $B_{ij}[q(t), t]$ represents continuous stochastic diffusion fluctuations, and $W_j(t)$ is an $N$–variable Wiener process (i.e., generalized Brownian motion), with

$$dW_j(t) = W_j(t + dt) - W_j(t), \quad (\text{for } j = 1, \ldots, N).$$

6 Time–Dependent Riemannian Geometry of Biomechanics

As illustrated in the introduction, the mass-inertia matrix of human body, defining the Riemannian metric tensor $g_{ij}(q)$ need not be time-constant, as in case of fast gymnastic movements and pirouettes in ice skating, which are based on quick variations of inertia moments and products constituting the material metric tensor $g_{ij}(q)$. In particular, in the geodesic framework, the (in)stability of the biomechanical joint and center-of-mass trajectories is the (in)stability of the geodesics, and it is completely determined by the curvature properties of the underlying manifold according to the Jacobi equation of geodesic deviation

$$\frac{D^2 J^i}{ds^2} + R^i_{jkm} \frac{dq^j}{ds} \frac{dq^m}{ds} = 0,$$

whose solution $J$, usually called Jacobi variation field, locally measures the distance between nearby geodesics; $D/ds$ stands for the covariant derivative along a geodesic and $R^i_{jkm}$ are the components of the Riemann curvature tensor.

In general, the biomechanical metric tensor $g_{ij}$ is both time and joint dependent, $g_{ij} = g_{ij}(t, q)$. This time-dependent Riemannian geometry can be formalized in terms of the Ricci flow, the nonlinear heat–like evolution metric equation:

$$\partial_t g_{ij} = -R_{ij}, \quad (5)$$

10
for a time–dependent Riemannian metric \( g = g_{ij}(t) \) on a smooth \( n \)-manifold \( Q \) with the Ricci curvature tensor \( R_{ij} \). This equation roughly says that we can deform any metric on the configuration manifold \( Q \) by the negative of its curvature; after normalization, the final state of such deformation will be a metric with constant curvature. The negative sign in \( \text{(5)} \) insures a kind of global \textit{volume exponential decay}\footnote{This complex geometric process is globally similar to a generic exponential decay ODE: \( \dot{q} = -\lambda f(q) \), for a positive function \( f(q) \). We can get some insight into its solution from the simple exponential decay ODE, \( \dot{q} = -\lambda q \) with the solution \( q(t) = q_0 e^{-\lambda t} \).} since the Ricci flow equation \( \text{(5)} \) is a kind of nonlinear geometric generalization of the standard linear heat equation

\[
\partial_t u = \Delta u.
\]

In a suitable local coordinate system, the Ricci flow equation \( \text{(5)} \) on a biomechanical configuration manifold \( Q \) has a nonlinear heat–type form, as follows. At any time \( t \), we can choose local harmonic coordinates so that the coordinate functions are locally defined harmonic functions in the metric \( g(t) \). Then the Ricci flow takes the general form \( \text{(6)} \)

\[
\frac{\partial g_{ij}}{\partial t} = \Delta_Q g_{ij} + G_{ij}(g, \partial g), \quad \text{where}
\]

\[
\Delta_Q \equiv \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial q^i} \left( \sqrt{\det(g)} g^{ij} \frac{\partial}{\partial q^j} \right)
\]

is the \textit{Laplace–Beltrami operator} of the configuration manifold \( Q \) and \( G_{ij}(g, \partial g) \) is a lower–order term quadratic in \( g \) and its first order partial derivatives \( \partial g \). From the analysis of nonlinear heat PDEs, one obtains existence and uniqueness of forward–time solutions to the Ricci flow on some time interval, starting at any smooth initial metric \( g_0 \) on \( Q \).

The exponentially-decaying geometrical diffusion \( \text{(6)} \) is a formal description for pirouettes in ice skating and fast rotational movements in gymnastics.

\section{Conclusion}

We have presented the time-dependent generalization of an ‘ordinary’ autonomous human biomechanics, in which total mechanical + biochemical energy is not conserved. We have introduced a general framework for time-dependent biomechanics in terms of jet manifolds associated to the extended musculo-skeletal configuration manifold, called the configuration bundle. We start with an ordinary, autonomous configuration manifold of human body motion, given as a set of its all active DOF for a particular movement. This is a Riemannian manifold with a material metric tensor given by the total mass-inertia matrix of the human body segments. This is the base manifold for standard autonomous biomechanics. To make its time-dependent generalization, we had to extend it with a real time axis. By this extension, using techniques from fibre bundles, we defined the biomechanical configuration bundle. On the biomechanical bundle we defined vector-fields, differential forms and affine connections, as well as first and second biomechanical jet manifolds of velocities and accelerations and prolongations of locomotion vector-fields. Using the formalism
of jet manifolds of velocities and accelerations, we have developed the time-dependent Lagrangian biomechanics. Finally, we have shown that the underlying geometric evolution is given by the parabolic Ricci flow equation.

References

[1] Ivancevic, V., Ivancevic, T., Applied Differential Geometry: A Modern Introduction. World Scientific, Singapore, (2007)

[2] Ivancevic, V., Ivancevic, T., Human–Like Biomechanics: A Unified Mathematical Approach to Human Biomechanics and Humanoid Robotics. Springer, Dordrecht, (2006)

[3] Ivancevic, V., Ivancevic, T., Natural Biodynamics. World Scientific, Singapore (2006)

[4] Ivancevic, V., Ivancevic, T., Geometrical Dynamics of Complex Systems: A Unified Modelling Approach to Physics, Control, Biomechanics, Neurodynamics and Psycho-Socio-Economical Dynamics. Springer, Dordrecht, (2006)

[5] Ivancevic, V., Ivancevic, T., High-Dimensional Chaotic and Attractor Systems. Springer, Berlin, (2006)

[6] Ivancevic, V., Ivancevic, T., Human versus humanoid robot biodynamics. Int. J. Hum. Rob. 5(4), 699-713, (2008)

[7] Ivancevic, T., Jovanovic, B., Djukic, M., Markovic, S., Djukic, N., Biomechanical Analysis of Shots and Ball Motion in Tennis and the Analogy with Handball Throws, J. Facta Universitatis, Series: Sport, 6(1), 51–66, (2008)

[8] Ivancevic, T., Jain, L., Pattison, J., Hariz, A., Nonlinear Dynamics and Chaos Methods in Neurodynamics and Complex Data Analysis, Nonl. Dyn. (Springer), 56(1-2), 23–44, (2009)

[9] Ivancevic, T., Jovanovic, B., Djukic, S., Djukic, M., Markovic, S., Complex Sports Biodynamics: With Practical Applications in Tennis, Springer, Berlin, (2009)

[10] Tucker, R., Dugas, J.: Beijing 2008: Men 100m race analysis. Bolt’s 9.69s. Analysis of speed during the world record. The Science of Sport, http://www.sportsscientists.com/2008/08/beijing-2008-men-100m-race-analysis.html, (2008)

[11] Park, J., Chung, W.-K., Geometric Integration on Euclidean Group With Application to Articulated Multibody Systems. IEEE Trans. Rob. 21(5), 850–863 (2005)

[12] Hatze, H., A general myocybernetic control model of skeletal muscle. Biol. Cyber. 28, 143–157, (1978)

[13] Wilkie, D.R., The mechanical properties of muscle. Brit. Med. Bull. 12, 177–182, (1956)

[14] Hill, A.V., The heat of shortening and the dynamic constants of muscle. Proc. Roy. Soc. B76, 136–195, (1938)
[15] Vukobratovic, M., Borovac, B., Surla, D., Stokic, D., Biped Locomotion: Dynamics, Stability, Control, and Applications. Springer, Berlin, (1990)

[16] Ivancevic, V., Ivancevic, T., Neuro–Fuzzy Associative Machinery for Comprehensive Brain and Cognition Modelling. Springer, Berlin, (2007)

[17] Saunders, D.J.: The Geometry of Jet Bundles. Lond. Math. Soc. Lect. Notes Ser. 142, Cambr. Univ. Pr., (1989)

[18] Massa, E., Pagani, E., Jet bundle geometry, dynamical connections and the inverse problem of Lagrangian mechanics. Ann. Inst. Henri Poincaré 61, 17, (1994)

[19] Giachetta, G., Mangiarotti, L., Sardanashvily, G., New Lagrangian and Hamiltonian Methods in Field Theory, World Scientific, Singapore, (1997)

[20] Sardanashvily, G.: Hamiltonian time-dependent mechanics. J. Math. Phys. 39, 2714, (1998)

[21] R.S. Hamilton, Three-manifolds with positive Ricci curvature, J. Diff. Geom. 17, 255-306, (1982)