ABSENCE OF SINGULAR CONTINUOUS SPECTRUM FOR PERTURBED DISCRETE SCHRÖDINGER OPERATORS

WENCAY LIU

ABSTRACT. We show that the spectral measure of discrete Schrödinger operators \((Hu)(n) = u(n + 1) + u(n - 1) + V(n)u(n)\) does not have singular continuous component if the potential \(V(n) = O(n^{-1})\).

1. Introduction and main results

We consider the discrete Schrödinger operator on \(\ell^2(\mathbb{Z}^+)\),
\[
(Hu)(n) = u(n + 1) + u(n - 1) + V(n)u(n),
\]
where \(V(n)\) is the potential.

Denote by \(H_0\) the free discrete Schrödinger operator on \(\ell^2(\mathbb{Z}^+)\). Without loss of generality, we assume the operator given by (1) satisfies the Dirichlet boundary condition at zero.

In this paper, we are interested in the spectral theory of \(H_0 + V\) with power-decaying potentials:
\[
|V(n)| \leq O\left(\frac{1}{1 + n^\alpha}\right)
\]
for some \(\alpha > 0\).

We also introduce the continuous Schrödinger operator on \(L^2(\mathbb{R}^+)\), namely,
\[
Hu = -u'' + Vu
\]
with \(|V(x)| \leq O\left(\frac{1}{1 + x^\alpha}\right)\).

Schrödinger operators with power decay potentials have attracted a lot of attentions and achieved a remarkable progress. Roughly speaking, \(\alpha = \frac{1}{2}\) is the sharp transition for \(\sigma_{ac}(H_0 + V) = \sigma_{ac}(H_0)\) and \(\alpha = 0\) is the sharp transition for absence of (singular continuous spectrum) embedded eigenvalues. We refer readers to a survey paper [3] for the progress in this area.

Let us go back to the discrete model. If \(V(n) = \frac{a(1)}{n}\), \(\sigma_{pp}(H_0 + V) \cap (-2, 2) = \emptyset\). Wigner-von Neumann type functions \(V(n) = \frac{a(1)}{n + \sin(kn + \varphi)}\) show that \(H_0 + V\) may have eigenvalues in \((-2, 2)\) if we allow \(V(n) = O\left(\frac{1}{1 + n}\right)\). See [11] for the quantitative results. For the singular continuous spectrum, Remling proved that \(\sigma_{sc}(H_0 + V) = \emptyset\) if \(V(n) = \frac{a(1)}{n}\) [12]. In this paper, we obtain

\textbf{Theorem 1.1.} Suppose the potential \(V(n)\) satisfies \(\limsup_{n \to \infty} n|V(n)| < \infty\). Then the operator \(H_0 + V\) does not have singular continuous spectrum.

One of our motivations is from the continuous Schrödinger operator. For the continuous case, Kiselev proved that \(\sigma_{ac}(H_0 + V) = \emptyset\) if \(V(x) = \frac{O(1)}{1 + x}\) and for any given any positive function \(h(x)\) tending to infinity as \(x \to \infty\), there exist potentials \(V(x)\) such that \(|V(x)| \leq \frac{h(x)}{1 + x}\) and the singular continuous spectrum of the operator \(H_0 + V\) is non-empty [6]. It is natural to ask whether such sharp spectral transitions hold for discrete cases or not. In this note, we...
prove that the absence of the singular continuous spectrum is still true for discrete cases. We
conjecture here that $|V(n)| = O(1) \frac{1}{1+n}$ is the sharp transition for absence for singular continuous
spectra. In the forthcoming paper, the author will study the same topic of perturbed periodic
operators [8]. Comparing to continuous cases, the spectral properties of discrete cases strongly
depend on the arithmetic properties of the quasimomentum [10] and the Prüfer angle is evolved
in a singular way (there is a cot function involved). Because of those difficulties, the spectral
features of discrete operators are usual much more delicate than those of continuous cases.
For example, the sharp transition for single embedded eigenvalues for the continuous case was
known forty years ago [1]. However, the sharp transition for single embedded eigenvalues for
the discrete case was partially solved by the author only a short time ago [10]. The construction
of potentials with dense embedded eigenvalues for perturbed periodic operator was known for
around 20 years [8]. However, similar results for the discrete case were only done in very recent
papers [5, 11]. Although the proof of this paper follows the strategy for the continuous case
[6], the extension is not completely straightforward.

In the following, we always assume that

$$|V(n)| \leq B \frac{1}{1+n},$$

for some $B > 0$.

2. Preliminaries

For $z \in \mathbb{C} \setminus \mathbb{R}$, denote by $\tilde{v}(n, z)$ ($\tilde{u}(n, z)$) the solution of (1) with boundary condition
$\tilde{v}(0, z) = 1$ and $\tilde{v}(1, z) = 0$ ($\tilde{u}(0, z) = 0$ and $\tilde{u}(1, z) = 1$). The Weyl $m$-function $m(z)$ (well de-
dined on $z \in \mathbb{C} \setminus \mathbb{R}$) is given by the unique complex number $m(z)$ so that $\tilde{v}(n, z) + m(z)\tilde{u}(n, z) \in \ell^2(\mathbb{Z}^+)$. The spectral measure $\mu$ on $\mathbb{R}$, is given by the follow formula, for $z \in \mathbb{C} \setminus \mathbb{R}$

$$m(z) = \int \frac{d\mu(x)}{x - z}.$$

Denote $\mu_{sc}$ by the singular continuous component of $\mu$. It is well known that $\sigma_{sc}(H_0 + V) = \emptyset$
if and only if $\mu_{sc} = 0$.

By Weyl law, $\sigma_{ess}(H) = (-2, 2)$. In order to prove Theorem 1.1 it suffices to show $\mu_{sc}(-2, 2) = 0$.

For any $E \in (-2, 2)$, let $E = 2\cos \pi k$ with $k \in (0, 1)$. We mention that $k$ depends on $E$.
However, we omit the dependence for simplicity. By symmetry, we only need to show there
is no sc component in $(0, 2)$. Fix any closed interval $I$ in $(0, 2)$, define $\hat{I} = \{ k(E) : E = 2\cos \pi k(E) \in I \}$ so that $\hat{I}$ is a closed interval in $(0, \frac{1}{2})$. In the following, we always assume $E \in I$ ($k \in \hat{I}$).

Let us introduce the Prüfer transformation first (cf. [8, 12]). Suppose $u(n, E)$ (sometimes we also use $u(n, k)$) is a solution of (1) with $u(0, E) = 0$ and $u(1, E) = 1$.

Let

$$Y(n, k) = \frac{1}{\sin \pi k} \begin{pmatrix} \sin \pi k & 0 \\ -\cos \pi k & 1 \end{pmatrix} \begin{pmatrix} u(n-1, k) \\ u(n, k) \end{pmatrix}.$$

Define the Prüfer variables $R(n, k)$ and $\theta(n, k)$ as

$$Y(n, k) = R(n, k) \begin{pmatrix} \sin(\pi \theta(n, k) - \pi k) \\ \cos(\pi \theta(n, k) - \pi k) \end{pmatrix}.$$
It is well known that $R$ and $\theta$ obey the equations

\begin{equation}
R(n + 1, k)^2 = 1 - \frac{V(n)}{\sin \pi k} \sin 2\pi \theta(n, k) + \frac{V(n)^2}{\sin^2 \pi k} \sin^2 \pi \theta(n, k)
\end{equation}

and

\begin{equation}
\cot(\pi \theta(n + 1, k) - \pi k) = \cot \pi \theta(n, k) - \frac{V(n)}{\sin \pi k}.
\end{equation}

By the Dirichlet boundary condition, we have the initial conditions

\[ R(0, k) = \frac{1}{\sin \pi k}, \theta(0, k) = k. \]

We will give several Lemmas, which will be used in the following sections.

**Lemma 2.1.** \[7, Prop. 2.4\] Suppose $\theta(n, k)$ satisfies (6) and $|\frac{V(n)}{\sin \pi k}| < \frac{1}{2}$. Then we have

\begin{equation}
|\theta(n + 1, k) - k - \theta(n, k)| \leq \left| \frac{V(n)}{\sin \pi k} \right|.
\end{equation}

**Lemma 2.2.** \[7, Lemma 4.4\] Let $\{e_i\}_{i=1}^N$ be a set of unit vector in a Hilbert space $\mathcal{H}$ so that

\[ \alpha = N \sup_{i \neq j} |\langle e_i, e_j \rangle| < 1. \]

Then

\begin{equation}
\sum_{i=1}^N |\langle g, e_i \rangle|^2 \leq (1 + \alpha)||g||^2.
\end{equation}

For $L \in \mathbb{Z}^+$, let $V_L$ be the cut off $V$ up to $L$. Namely, $V_L(n) = V(n)$ for $0 \leq n \leq L$ and $V_L(n) = 0$ for $n > L$. Let $\mu_L$ be the spectral measure corresponding to the operator with potential $V_L$.

**Lemma 2.3.** \[7\] Fix some compact interval $I \in (-2, 2)$ and arbitrary $M, \sigma > 0$. Then for any $\epsilon > L^{-1+\sigma}, we have

\begin{equation}
\mu(E - \epsilon, E + \epsilon) \geq \mu_L(E - \frac{\epsilon}{2}, E + \frac{\epsilon}{2}) - C(I, \sigma, B, M)e^M
\end{equation}

for any $(E - \epsilon, E + \epsilon) \subset (-2, 2)$.

**Lemma 2.4.** \[7\] Under the assumption of \[7\], the spectral measure $\mu$ of $H = H_0 + V$ is zero dimensional.

3. **Technical Lemmas**

**Lemma 3.1.** For any $k \in \hat{I}$, we have

\begin{equation}
\left| \sum_{n=1}^L \frac{\cos 4\theta(n, k)}{n} \right| \leq C(I, B).
\end{equation}

For any $k_1, k_2 \in \hat{I}$ and $k_1 \neq k_2$, we have

\begin{equation}
\left| \sum_{n=1}^L \frac{\sin 2\theta(n, k_1) \sin 2\theta(n, k_2)}{n} \right| \leq C(I, B) \log(|k_1 - k_2|^{-1}) + C(I, B).
\end{equation}
Proof. We start with the proof of (10). It suffices to show
\[ \left| \sum_{n=1}^{L} \frac{e^{4i\theta(n,k)}}{n} \right| \leq C(I,B). \]

Straightforwardly,
\[
\left| (e^{4\pi i k} - 1) \sum_{n=1}^{L} \frac{e^{4i\theta(n,k)}}{n} \right| = \left| \sum_{n=1}^{L} \frac{e^{4i\theta(n,k) + k}}{n} - \sum_{n=1}^{L} \frac{e^{4i\theta(n,k)}}{n} \right|
\]
\[
= \left| \sum_{n=1}^{L} \frac{e^{4i\theta(n+1,k)}}{n} - \sum_{n=1}^{L} \frac{e^{4i\theta(n,k)}}{n} + \sum_{n=1}^{L} \frac{e^{4i\theta(n,k)+k}}{n} - \sum_{n=1}^{L} \frac{e^{4i\theta(n+1,k)}}{n} \right|
\]
\[
\leq \left| \sum_{n=1}^{L} \frac{e^{4i\theta(n+1,k)}}{n} - \sum_{n=1}^{L} \frac{e^{4i\theta(n,k)}}{n} \right| + \left| \sum_{n=1}^{L} \frac{e^{4i\theta(n,k)+k}}{n} - \sum_{n=1}^{L} \frac{e^{4i\theta(n+1,k)}}{n} \right|
\]
\[
(12)
\]
\[
\leq 2 + \left| \sum_{n=1}^{L-1} \frac{1}{n} - \frac{1}{n+1} \right| e^{4i\theta(n+1,k)}.
\]
By (7), (12) and \( \left| e^{4\pi i k} - 1 \right| = 2\sin 2\pi k \), we have
\[
\left| \sum_{n=1}^{L} \frac{e^{4i\theta(n,k)}}{n} \right| \leq C(I,B).
\]

Now we are in the position to prove (11). Trigonometric identity implies
\[
(13) \quad 2\sin 2\theta(n,k_1)\sin 2\theta(n,k_2) = \cos 2(\theta(n,k_1) - \theta(n,k_2)) - \cos 2(\theta(n,k_1) + \theta(n,k_2)).
\]

By the same proof of (10), one has
\[
\left| \sum_{n=1}^{L} \frac{\cos 2(\theta(n,k_1) + \theta(n,k_2))}{n} \right| \leq C(I,B).
\]

It suffices to show
\[
\left| \sum_{n=1}^{L} \frac{e^{2i\theta(n,k_1) - 2i\theta(n,k_2)}}{n} \right| \leq C(I,B) + C(I,B) \log(|k_1 - k_2|^{-1}).
\]

Since
\[
\left| \sum_{n=1}^{L} \frac{\left| e^{2i\theta(n,k_1) - 2i\theta(n,k_2)} \right|}{n} \right| \leq C(I,B) + C(I,B) \log(|k_1 - k_2|^{-1}),
\]
we only need to prove
\[
\left| \sum_{n=\lfloor k_1 - k_2 \rfloor^{-1}}^{L} \frac{e^{2i\theta(n,k_1) - 2i\theta(n,k_2)}}{n} \right| \leq C(I,B).
\]

By the proof of (12), we have
\[
\left| (e^{2\pi i (k_1-k_2)} - 1) \sum_{n=\lfloor k_1 - k_2 \rfloor^{-1}}^{L} \frac{e^{2i\theta(n,k_1) - 2i\theta(n,k_2)}}{n} \right| \leq C(I,B) \sum_{n=\lfloor k_1 - k_2 \rfloor^{-1}}^{L} \frac{1}{n^2}
\]
\[
\leq C(I,B)|k_1 - k_2|.
\]
It leads to
\[
| \sum_{n=|k_1-k_2|}^{L} \frac{e^{2i\theta(n,k_1)-2i\theta(n,k_2)}}{n} | \leq C(I,B).
\]

We finish the proof. \[\square\]

**Lemma 3.2.** The following formula hold,

\begin{equation}
\frac{d\mu_L(E)}{dE} = \frac{1}{\pi \sin \pi k} \frac{1}{R^2(L+1, E)}
\end{equation}

for \( E \in (-2,2) \).

**Proof.** Let \( z = E + i\varepsilon \) for \( E \in (-2,2) \) and \( \varepsilon > 0 \). Let \( k(z) + i\gamma(z) \) be such that \( 2 \cos \pi (k(z) + i\gamma(z)) = z \) with \( k(z) \in \mathbb{R} \) and \( \gamma(z) \in \mathbb{R} \). Thus
\[
(e^{-\pi\gamma} + e^{\pi\gamma}) \cos \pi k = E; (e^{-\pi\gamma} - e^{\pi\gamma}) \sin \pi k = \varepsilon.
\]

Let us choose the branch so that \( k(z) \in (0,1) \) and \( \gamma(z) < 0 \). It is easy to see
\[
\lim_{\varepsilon \to 0^+} k(E + i\varepsilon) = k(E), \quad \lim_{\varepsilon \to 0^+} \gamma(E + i\varepsilon) = 0
\]
where \( 2 \cos \pi k(E) = E \) with \( k(E) \in (0,1) \).

Define \( \tilde{u}(n,z) = e^{-i\pi(k+i\gamma)n} \) for \( n \geq L \) and extend \( \tilde{u}(n,z) \) to \( 0 \leq n \leq L \) by solving equation
\[
\tilde{u}(n+1,z) + \tilde{u}(n-1,z) + (V_L(n) - z)\tilde{u}(n,z) = 0
\]
for \( 0 \leq n \leq L - 1 \). Since \( \gamma(z) < 0 \), one has \( \tilde{u}(n,z) \in \ell^2(\mathbb{Z}^+) \). By spectral theory (we refer the readers to [13] and references therein for details), we have
\[
m(z) = -\frac{\tilde{u}(1,z)}{\tilde{u}(0,z)}
\]
and

\begin{equation}
\frac{d\mu_L}{dE} = \frac{1}{\pi \varepsilon \to 0^+} \Im m(E + i\varepsilon).
\end{equation}

Let \( T(z) \) be the transfer matrix of \( H_0 + V_L \) from \( 0 \) to \( L \), that is
\[
T(z) \begin{pmatrix} \phi(0) \\ \phi(1) \end{pmatrix} = \begin{pmatrix} \phi(L) \\ \phi(L+1) \end{pmatrix}
\]
for any solution \( \phi \) of \( (H_0 + V_L)\phi = z\phi \).

Let
\[
T(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}.
\]

Thus
\[
\begin{pmatrix} \tilde{u}(0,z) \\ \tilde{u}(1,z) \end{pmatrix} = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}^{-1} \begin{pmatrix} \tilde{u}(L,z) \\ \tilde{u}(L+1,z) \end{pmatrix} = \begin{pmatrix} d(z) & -b(z) \\ -c(z) & a(z) \end{pmatrix} \begin{pmatrix} \tilde{u}(L,z) \\ \tilde{u}(L+1,z) \end{pmatrix}.
\]

Direct computation implies that
\[
\lim_{\varepsilon \to 0^+} \Im m(E + i\varepsilon) = -\Im \frac{ae^{-i\pi k} - c}{d - be^{-i\pi k}} = \frac{\sin \pi k}{(d - b \cos \pi k)^2 + b^2 \sin^2 \pi k}.
\]

\begin{equation}
\text{Lemma 3.2. The following formula hold,}
\end{equation}
It is easy to see that
\[
\begin{pmatrix} u(L) \\ u(L+1) \end{pmatrix} = T(E) \begin{pmatrix} u(0) \\ u(1) \end{pmatrix} = T(E) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}.
\]
By (3) and (4), one has
\[
\sin^2 \pi k = \frac{1}{R^2(L+1,E)} = \frac{\sin^2 \pi k}{(d-b \cos \pi k)^2 + b^2 \sin^2 \pi k}.
\]
Now the Lemma follows from (15), (16) and (17). □

4. Proof of Theorem 1.1

Once we have Lemmas 3.1 and 3.2 at hand, Theorem 1.1 can be proved in a similar way as the argument in [6]. For convenience, we give all the details here.

Fix 0 < β < 1, M = 1 + β and σ > 0. We will choose small enough ǫ > 0 (depends on B, β, M > 1 and σ > 0). Let
\[
L = \left\lfloor \frac{1}{\epsilon} - 1 \right\rfloor - \sigma \right\rfloor,
\]
where \( \lfloor x \rfloor \) is the integer part of x. Let
\[
C_1 = C_1(B,I),
\]
which will be determined later.

We say a subset \( S \subset I \) is ǫ-N separate, if the following two conditions hold:

For any \( k \in S \),
\[
|\sum_{n=1}^{L} V(n) \sin 2\theta(n,k)| \geq (1 - \beta)C_1(B,I) \log \epsilon^{-1}.
\]

For any \( k_1, k_2 \in S \) and \( k_1 \neq k_2 \),
\[
|k_1 - k_2| \geq \epsilon^{1/N^2}.
\]

Theorem 4.1. There exists \( \epsilon_1(B,I,\sigma,\beta) > 0 \) and \( C(B,I,\sigma,\beta) \) such that for any \( \epsilon < \epsilon_1 \) and \( N \geq C(B,I,\sigma,\beta) \), the \( \epsilon-N \) separate set \( S \) satisfies \#S ≤ N.

Proof. We consider the Hilbert space
\[
\mathcal{H} = \{ u \in \mathbb{R}^L : \sum_{n=1}^{L} n|u(n)|^2 < \infty \}
\]
with the inner product
\[
\langle u, v \rangle = \sum_{n=1}^{L} u(n)v(n)n.
\]

In \( \mathcal{H} \), by (2) we have
\[
||V||_{\mathcal{H}}^2 \leq B^2 \log L.
\]

Let
\[
e_i(n) = \frac{1}{\sqrt{A_i}} \sin 2\theta(n,k_i) \chi_{[1,L]}(n),
\]
where \( A_i \) is chosen so that \( e_i \) is a unit vector in \( \mathcal{H} \). We have the following estimate,
\[
A_i = \sum_{n=1}^{L} \frac{\sin^2 2\theta(n,k_i)}{n} = \sum_{n=1}^{L} \frac{1}{2n} - \sum_{n=1}^{L} \frac{\cos 4\theta(n,k_i)}{2n}.
\]
By (18), one has
\[(21)\quad |A_i - \frac{1}{2} \log L| \leq C(I, B)\]

By (11) and (21), we have
\[(22)\quad |\langle e_i, e_j \rangle| \leq \frac{C(B, I)}{1 + \sigma} N^{-2} + \frac{C(B, I)}{(1 + \sigma) \log \epsilon^{-1}}.\]

The first condition (18) implies
\[(23)\quad |\langle V, e_i \rangle|^2 \geq \frac{(1 - \beta)^2 C_I^2}{1 + \sigma} \log \epsilon^{-1}.\]

By (20), (23) and (24), we have
\[
\begin{align*}
N \left( \frac{C_I^2 (1 - \beta)^2}{1 + \sigma} \log \epsilon^{-1} \right) & \leq \left( 1 + \frac{C(B, I)}{1 + \sigma} N^{-1} + \frac{NC(B, I)}{(1 + \sigma) \log \epsilon^{-1}} \right) B^2 (1 + \sigma) \log \epsilon^{-1}.
\end{align*}
\]

This implies the Lemma. \(\square\)

Assume that the singular continuous spectrum is not empty. As the analysis in the beginning of [2] there exists \(\delta > 0\) such that \(\mu_{sc}(I) = \delta\). Fix a small number \(\epsilon\) and a large number \(N\) such that Theorem 4.1 holds. By making \(\epsilon\) smaller and the continuity of \(\mu_{sc}\), we assume \(\mu_{sc}(J) < \frac{\epsilon}{\pi} \delta N^{-3}\) for any interval \(J \subset I\) such that \(|J| \leq \epsilon N^{-2}\).

Let \(m \in \mathbb{Z}^+\). We say that an interval \(J \subset I\) belongs to the scale \(m\) if \(|J| \leq \epsilon_m \equiv \epsilon_m^m\). We call an interval \(J\) of scale \(m\) singular if \(\mu_{sc}(J) \geq \epsilon_m^m\). We call two intervals of the scale \(m\) separated if the distance between their centers exceeds \(2 \epsilon_m^m\).

**Lemma 4.2.** There can be no more than \(N\) separated singular intervals at each scale.

**Proof.** Assume that \(J_i^m, i = 1, \cdots, N\) are separated singular intervals of scale \(m\). Let \(L_m = \lfloor \epsilon_m^{1 - \sigma} \rfloor\) and denote by \(\mu_m\) the spectral measure corresponding to the potential being cut off at \(L_m\). Denote by \(2J_i^m\) the interval with the same center as \(J_i^m\) but twice its size. Then by (14) we have
\[(25)\quad \mu_m(2J_i^m) \geq \mu(J_i^m) - C(B, I, \sigma, \beta) \epsilon_m^M \geq \frac{1}{2} \epsilon_m^m,\]

provided \(\epsilon\) is small enough. Combining (14) with (25), we see that there exist \(k_i^m \in 2J_i^m\) such that
\[(26)\quad R^2(L_m, k_i^m) \leq C(I) \epsilon_m^{1 - \beta}.\]

We will show that \(k_j^m, j = 1, 2, \cdots, N\) is \(\epsilon_m - N\) separate. By (5), one has
\[(27)\quad \ln R(L, k)^2 - \ln R(1, k)^2 = - \sum_{n=1}^L \frac{V(n)}{\sin \pi k} \sin 2\pi \theta(n, k) + O(1).\]

By (26) and (27), we have that the assumption (18) holds for suitable \(C_1\). Moreover, by the separation assumption of scale \(m\), \(|k_i^m - k_j^m| > \epsilon_{m-2}^m\), which implies (19). Now the Lemma follows from Theorem 4.1. \(\square\)
Proof of Theorem 1.1. Define the set $S_m$ as a union of all singular intervals $J$ at scale $m$. By Lemma 4.2, it is easy to see that the set $S_m$ can be covered by at most $8N$ intervals of size $\epsilon_m^{N^{-2}}$. We denote them by $\tilde{J}_m^l$. By the smallness choice of $\epsilon$, we have for any $m \in \mathbb{Z}^+$,

$$\mu_{sc}(S_m) \leq 8N \frac{1}{32} \delta N^{-3} = \frac{1}{4} N^{-2} \delta.$$ 

It yields that

$$\sum_{m=1}^{N^2} \mu_{sc}(S_m) \leq \frac{1}{4} \delta.$$ 

Denote by $\tilde{m} = \lfloor mN^{-2} \rfloor$. Then any interval $\tilde{J}_m^l$ satisfying $\mu_{sc}(\tilde{J}_m^l) \geq \epsilon_\tilde{m}^\beta$ already belongs to $S_{\tilde{m}}$ since $|\tilde{J}_m^l| = \epsilon_m^{N^{-2}} \leq \epsilon_{\tilde{m}}$. Therefore, for any $m \geq N^2$, we have

$$\mu_{sc}(S_m \setminus \bigcup_{l<m} S_l) \leq 8N \epsilon_{\tilde{m}}^\beta.$$ 

By (29) and the fact that each $\tilde{m}$ has at most $N^2$ corresponding $m$, we have

$$\sum_{m=N^2}^{\infty} \mu_{sc}(S_m \setminus \bigcup_{l<m} S_l) \leq \sum_{\tilde{m}=1}^{\infty} 8N^3 \epsilon_{\tilde{m}}^\beta \leq 16N^3 \epsilon^\beta.$$ 

By (28) and (30), we finally obtain

$$\mu_{sc}(\cup_{m} S_m) \leq \frac{\delta}{4} + 16N^3 \epsilon^\beta \leq \frac{\delta}{2},$$

if $\epsilon$ is small enough ($\epsilon^\beta \leq \frac{1}{16N^3}$). On the other hand, by Lemma 2.4, the spectral measure can only be zero-dimensional. Thus, $\mu_{sc}$ is supported on a set $S$ such that for any $E \in S$ and any $\alpha > 0$ (see [2, Corollary 2.2] for example),

$$D^\alpha \mu_{sc}(E) = \limsup_{\epsilon \to 0} \frac{\mu_{sc}(E - \epsilon, E + \epsilon)}{2^n \epsilon^\alpha} = \infty.$$ 

In particular, $S \subset \cup S_m$. It implies

$$\delta = \mu_{sc}(S) = \mu_{sc}(\cup S_m) \leq \frac{1}{2} \delta.$$ 

This is impossible. 

\[\square\]

Acknowledgments

W.L. was supported by NSF DMS-1700314. This research was also supported by NSF DMS-1401204.

References

[1] F. V. Atkinson and W. N. Everitt. Bounds for the point spectrum for a Sturm-Liouville equation. Proc. Roy. Soc. Edinburgh Sect. A, 80(1-2):57–66, 1978.

[2] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon. Operators with singular continuous spectrum. IV. Hausdorff dimensions, rank one perturbations, and localization. J. Anal. Math., 69:153–200, 1996.

[3] S. A. Denisov and A. Kiselev. Spectral properties of Schrödinger operators with decaying potentials. In Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon’s 60th birthday, volume 76 of Proc. Sympos. Pure Math., pages 565–589. Amer. Math. Soc., Providence, RI, 2007.

[4] F. Germinet, A. Kiselev, and S. Tcheremchantsev. Transfer matrices and transport for Schrödinger operators. Ann. Inst. Fourier (Grenoble), 54(3):787–830, 2004.
E. Judge, S. Naboko, and I. Wood. Spectral results for perturbed periodic Jacobi matrices using the discrete Levinson technique. *Studia Math.*, 242(2):179–215, 2018.

A. Kiselev. Imbedded singular continuous spectrum for Schrödinger operators. *J. Amer. Math. Soc.*, 18(3):571–603, 2005.

A. Kiselev, Y. Last, and B. Simon. Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators. *Comm. Math. Phys.*, 194(1):1–45, 1998.

A. Kiselev, C. Remling, and B. Simon. Effective perturbation methods for one-dimensional Schrödinger operators. *J. Differential Equations*, 151(2):290–312, 1999.

W. Liu. WKB and absence of singular continuous spectrum for perturbed periodic Schrödinger operators. *Preprint*.

W. Liu. Criteria for embedded eigenvalues for discrete Schrödinger operators. *arXiv preprint arXiv:1805.02817* 2018.

W. Liu and D. C. Ong. Sharp spectral transition for eigenvalues embedded into the spectral bands of perturbed periodic operators. *arXiv preprint arXiv:1805.01569* 2018.

C. Remling. The absolutely continuous spectrum of one-dimensional Schrödinger operators with decaying potentials. *Comm. Math. Phys.*, 193(1):151–170, 1998.

B. Simon. Analogs of the m-function in the theory of orthogonal polynomials on the unit circle. *J. Comput. Appl. Math.*, 171(1-2):411–424, 2004.

(Wencai Liu) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, IRVINE, CALIFORNIA 92697-3875, USA

E-mail address: liuwencai1226@gmail.com