Wilson polynomials/functions and intertwining operators for the generic quantum superintegrable system on the 2-sphere

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Abstract. The Wilson and Racah polynomials can be characterized as basis functions for irreducible representations of the quadratic symmetry algebra of the quantum superintegrable system on the 2-sphere, \( H \Psi = E \Psi \), with generic 3-parameter potential. Clearly, the polynomials are expansion coefficients for one eigenbasis of a symmetry operator \( L_2 \) of \( H \) in terms of an eigenbasis of another symmetry operator \( L_1 \), but the exact relationship appears not to have been made explicit. We work out the details of the expansion to show, explicitly, how the polynomials arise and how the principal properties of these functions: the measure, 3-term recurrence relation, 2nd order difference equation, duality of these relations, permutation symmetry, intertwining operators and an alternate derivation of Wilson functions – follow from the symmetry of this quantum system. This paper is an exercise to show that quantum mechanical concepts and recurrence relations for Gaussian hypergeometric functions alone suffice to explain these properties; we make no assumptions about the structure of Wilson polynomial/functions, but derive them from quantum principles. There is active interest in the relation between multivariable Wilson polynomials and the quantum superintegrable system on the \( n \)-sphere with generic potential, and these results should aid in the generalization. Contracting function space realizations of irreducible representations of this quadratic algebra to the other superintegrable systems one can obtain the full Askey scheme of orthogonal hypergeometric polynomials. All of these contractions of superintegrable systems with potential are uniquely induced by Wigner Lie algebra contractions of \( \mathfrak{so}(3, \mathbb{C}) \) and \( \mathfrak{e}(2, \mathbb{C}) \). All of the polynomials produced are interpretable as quantum expansion coefficients. It is important to extend this process to higher dimensions.

1. Introduction
We define a quantum superintegrable system as an integrable Hamiltonian system on an \( n \)-dimensional pseudo-Riemannian manifold with potential: \( H = \Delta_n + V \) that admits \( 2n - 1 \) algebraically independent partial differential operators commuting with \( H \), the maximum possible,[1]. Thus \( [H, L_j] = 0, \quad n = 1, 2, \cdots, 2n - 1 \) where \( \Delta_n \) is the Laplace-Beltrami operator on the manifold and we choose the generators \( L_j \) such that the sum of their orders is a small as possible. Superintegrability captures the properties of quantum Hamiltonian systems that allow the Schrödinger eigenvalue problem \( H \Psi = E \Psi \) to be solved exactly, analytically and algebraically. Typically, the basis symmetries \( L_j \) generate an algebra under commutation, not usually a Lie algebra that closes at finite order. It is this algebra that is responsible for the solvability of the quantum system.
The generic superintegrable system on the 2-sphere, system $S9$ in our listing [2], is

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}, \quad a_j = \frac{1}{4} - k_j^2,$$

where $J_3 = s_1 \partial_{s_1} - s_2 \partial_{s_2}$ and $J_2, J_3$ are obtained by cyclic permutations of indices. Here, $s_1^2 + s_2^2 + s_3^2 = 1$. The operators $J_k$ preserve the order of homogeneity when acting on functions in $\mathcal{R}^3$ so they act on $C^\infty$ functions on the sphere. The basis symmetries can be chosen as

$$L_1 = J_1^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \quad L_2 = J_2^2 + \frac{a_1 s_3^2}{s_1^2} + \frac{a_3 s_1^2}{s_2^2}, \quad L_3 = J_3^2 + \frac{a_2 s_1^2}{s_2^2} + \frac{a_1 s_2^2}{s_1^2},$$

where $J_1, J_2, J_3$ are rotation generators: $J_3 = s_2 \partial_{s_1} - s_1 \partial_{s_2}, \cdots$. Note the discrete symmetry of $H = L_1 + L_2 + L_3 + a_1 + a_2 + a_3$ with respect to permutations of the indices 1, 2, 3. With the commutator $[L_i, R] = [L_2, L_3] = [L_3, L_1]$, the algebra generated by these symmetries has the structure

$$[L_i, R] = 4\{L_i, L_k\} - 4\{L_j, L_k\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k), \quad (1)$$

where $i, j, k = 1, 2, 3$, pairwise distinct. Here, $\{A, B\} = AB + BA$, and $\{A, B, C\}$ is the symmetrizer of 3 operators. $R^2$ is contained in the algebra and expressed by

$$R^2 = 8 - \frac{52}{3}\{L_1, L_2, L_3\} + (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 +$$

$$+ \frac{32}{3}(a_1 + a_2 + a_3) + 48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3. \quad (2)$$

This algebra can be matched with $QR(3)$, the structure algebra of the Racah and Wilson polynomials, [3, 4, 5, 6, 7, 8]. The significance of these special functions is that they are the expansion coefficients of a basis of eigenfunctions of one of the spherical coordinate generators $L_i$ in terms of an eigenbasis of $L_j, i \neq j$. Long before, Dunkl in the remarkable paper [9], (see also [10, 11] and references contained therein) had computed these coefficients as expansions of polynomial bases (not eigenbases) and shown them to be Racah/Wilson polynomials. However, to our knowledge no one has worked out the details of this relationship between the functions and the expansion coefficients to understand how important properties of Wilson polynomials can be interpreted from a quantum mechanical viewpoint. Here we show the critical role of intertwining operators for the quantum system in determining parameter changing recurrences for Wilson/Racah polynomials. In particular, the $4F_3$ expressions for Wilson polynomials/functions follow from intertwining operators alone.

We recall the definition of $4F_3$ hypergeometric functions:

$$4F_3\left(\begin{array}{cccc}
  a_1, & a_2, & a_3, & a_4 \\
  b_1, & b_2, & b_3, & \end{array}; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k(a_3)_k(a_4)_k}{(b_1)_k(b_2)_k(b_3)_k k!} x^k$$

where $(a)_0 = 1, \quad (a)_k = a(a+1)(a+2)\cdots(a+k-1)$ if $k \geq 1$. If $a_1 = -n$ for $n$ a nonnegative integer then the sum is finite with $n + 1$ terms. The Wilson polynomials of order $n$ in $t^2$ are

$$\Phi_n(\alpha, \beta, \gamma; t) = 4F_3\left(\begin{array}{cccc}
  -n, & \alpha + \beta + \gamma + \delta + n - 1, & \alpha + t, & \alpha + t \\
  \alpha + \beta, & \alpha + \gamma, & \alpha + \delta & \end{array}; 1\right).$$

If $\alpha + \beta = -m, m$ a nonnegative integer we have the finite set $\Phi_0, \cdots, \Phi_m$ of Racah polynomials.
2. Structure algebra and interbasis expansion coefficients

The basis simultaneous eigenfunctions of \( L_1 \) and \( H \): take the form

\[
\Psi_{N-n,n} = (s_1^2 + s_2^2)^{(2n+k_1+k_2+1)}(1 - s_1^2 - s_2^2)^{(2k_3+\frac{1}{2})}\left(\frac{s_2^2}{s_1^2 + s_2^2}\right)^{(k_2+\frac{1}{2})}\left(\frac{s_1^2}{s_1^2 + s_2^2}\right)^{(k_1+\frac{1}{2})}P_{n}(k_2, k_1)\left(\frac{s_1^2 - s_2^2}{s_1^2 + s_2^2}\right)P_{n}(2n+k_1+k_2+1, k_3)(1 - 2s_1^2 - 2s_2^2),
\]

\[
\frac{1}{4}(L_1 + 2k_1k_2 + 2k_1 + 2k_2 + 3)\Psi_{N-n,n} = -n(n + k_1 + k_2 + 1)\Psi_{N-n,n}, \quad n = 0, 1, \ldots, N,
\]

\[
H \Psi_{N-n,n} = E_N \Psi_{N-n,n}, \quad E_N = -(2N + k_1 + k_2 + k_3 + 2)^2 + \frac{1}{4}, \quad N = 0, 1, \ldots,
\]

separable in spherical coordinates, \( x = \cos(2\varphi) \), \( y = \cos(2\theta) \) where \( s_1 = \sin \theta \cos \varphi \), \( s_2 = \sin \theta \sin \varphi \), \( s_3 = \cos \varphi \), orthogonal with respect to area measure on the 1st octant of the 2-sphere.

The dimension of eigenspace \( E_N \) is \( N + 1 \).

\[
P_n^{(\alpha, \beta)}(y) = \binom{n + \alpha}{n} \binom{\alpha + \beta + n + 1}{\alpha + 1} \left(\frac{1 - y}{2}\right), \quad \text{Jacobi polynomials}
\]

Define the functions \( \Lambda_{N-n,n} \) by the permutation \( 1 \leftrightarrow 3 \), in \( \Psi_{N-n,n} \). These are eigenfunctions of \( L_2, H \):

\[
\frac{1}{4}(L_2 + 2k_2k_3 + 2k_2 + 2k_3 + 3)\Lambda_{N-n,n} = -q(q + k_2 + k_3 + 1)\Lambda_{N-n,n}, \quad q = 0, 1, \ldots, N,
\]

\[
H \Lambda_{N-n,n} = E_N \Lambda_{N-n,n}, \quad E_N = -(2N + k_1 + k_2 + k_3 + 2)^2 + \frac{1}{4}, \quad N = 0, 1, \ldots.
\]

They are separable in a different set of spherical coordinates \( X, Y \) expressible in terms of \( x, y \) by

\[
X = \frac{1 + x + 3y - xy}{x + y + 3y^2}, \quad Y = \frac{x - y - 1 - xy}{x + y + 3y^2},
\]

and orthogonal with respect to area measure on the 1st octant of the 2-sphere. Due to the \( S_3 \) permutation symmetry there is also a basis of eigenfunctions of \( L_3 \) and \( H \), but we will not consider it here. Using the method described in [12] we can derive the structure algebra of \( S_9 \), just from the 1st order Gaussian differential recurrences for

\[
2F_1 \left( \begin{array}{c} a, \ b \ \\
\ c, \ z \end{array} \right) \rightarrow 2F_1 \left( \begin{array}{c} a \pm 1, \ b \ \\
\ c, \ z \end{array} \right), \quad 2F_1 \left( \begin{array}{c} a, \ b \ \\
\ c, \ z \end{array} \right) \rightarrow 2F_1 \left( \begin{array}{c} a \pm 1, \ b \mp 1 \ \\
\ c \mp 1, \ z \end{array} \right)
\]

and a limiting process. One consequence from Section 5.1 of that paper, is:

\[
L_2 \Psi_{m,n} = A_n \Psi_{m-1,n+1} + B_n \Psi_{m,n} + C_n \Psi_{m+1,n-1} =
\]

\[
\frac{4(N + k_3 - n)(N + n + k_1 + k_2 + k_3 + 2)(n + 1)(n + k_1 + k_2 + 1)}{(2n + k_1 + k_2 + 2)(2n + k_1 + k_2 + 1)} \Psi_{m-1,n+1}
\]

\[
\left[\frac{(k_2^2 - k_3^2)}{2(2n + k_1 + k_2 + 2)(2n + k_1 + k_2)} + \frac{1}{2} \frac{(2n + k_1 + k_2 + 1)^2}{(2n + k_1 + k_2 + 2)(2n + k_1 + k_2 + 1)} + \frac{1}{4} - k_1^2 - \frac{1}{2} \frac{(2N + 2 + k_1 + k_2 + k_3)^2}{(2n + k_1 + k_2)(2n + k_1 + k_2 + 1)} \right] \Psi_{m,n} -
\]

\[
\frac{4(N - n + 1)(N + n + k_1 + k_2 + 1)(n + k_1)(n + k_2)}{(2n + k_1 + k_2)(2n + k_1 + k_2 + 1)} \Psi_{m+1,n-1}.
\]

The action of \( L_1 \) on the \( L_2 \) eigenbasis follows immediately from permutation symmetry. Now we expand the \( L_2 \) eigenbasis in terms of the \( L_1 \) eigenbasis:

\[
\Lambda_{N-n,n}^{(k_1, k_2, k_3)} = \sum_{n=0}^{N} R_q^{(k_1, k_2, k_3)} \Psi_{N-n,n}^{(k_1, k_2, k_3)}, \quad q = 0, \ldots, N
\]
Using the self-adjoint properties of $L_1$, $L_2$ we can find recurrences to compute the norm:

$$||\Psi_{N-n,n}||^2 = \frac{1}{4n!\Gamma(N-n+1)\Gamma(n+k_1+1)\Gamma(n+k_2+1)\Gamma(N+n+k_3+1)\Gamma(N+n+k_1+k_2+2)} \Gamma(n+k_1+1)\Gamma(n+k_2+1)\Gamma(N-n+k_3+1)\Gamma(N+n+k_1+k_2+2).$$

The squared norm of $\Lambda_{N-n,n}$ follows by applying the permutation $1 \leftrightarrow 3$ to the above expression. To better exploit the symmetry of our system we rescale the bases and expansion:

$$\Psi'(k_1,k_2,k_3)_{N-n,n} = \frac{(-1)^n\Gamma(N-n+1)}{\Gamma(N-n+k_3+1)\Gamma(n+k_2+1)} \Psi(k_1,k_2,k_3)_{N-n,n}, \quad \Lambda'(k_1,k_2,k_3) = \sum_{n=0}^{N} R_{n}^m(k_1,k_2,k_3) \Psi'(k_1,k_2,k_3)_{N-n,n}, \quad q = 0, \cdots N$$

The two sets of $N + 1$ basis vectors $\{\Psi_{m,n}||\Psi_{m,n}'||\}$, $\{R_{n}^m||R_{n}^m'||\}$ are each orthonormal, implying that the $(N + 1) \times (N + 1)$ matrix $\begin{pmatrix} \Psi_{m,n}||\Psi_{m,n}'|| \\ R_{n}^m||R_{n}^m'|| \end{pmatrix}$, $0 \leq n, q \leq N$, is orthogonal. We have identities

$$\sum_{n=0}^{N} R_{n}^{(m_1)} R_{n}^{(m_2)} ||\Lambda'_{N-n,\ell,\mu}||^2 = \delta_{n_1,n_2} \sum_{n=0}^{N} R_{n}^{(m_1)} R_{n}^{(m_2)} ||\Psi'_{N-n_1,n_1}||^2, \quad \sum_{n=0}^{N} R_{n}^{(m_1)} R_{n}^{(m_2)} ||\Lambda'_{N-n_1,n_1}||^2 = \delta_{n_1,q_2} ||\Psi'_{N-n_1,n_1}||^2.$$ 

By permutation symmetry: $\Lambda'(k_1,k_2,k_3) = \Psi'(k_3,k_2,k_1).$ We set

$$R_{n}^m(k_1,k_2,k_3)_N \cdot ||\Psi'(k_1,k_2,k_3)_{N-n,n}||^2 = \Xi'(k_1 n, k_2 N, k_3 q).$$

Note that $\Xi'$ is invariant under the transposition of its 1st and 3rd columns, and satisfies

$$\sum_{q=0}^{N} \frac{\Xi'(k_1 n_1, k_2 N, k_3 q) \Xi'(k_1 n_2, k_2 N, k_3 q)}{||\Lambda'(k_1,k_2,k_3)||^2} = ||\Psi'(k_1,k_2,k_3)_{N-n,n}||^2 \delta_{n_1,n_2},$$

$$\sum_{\ell=0}^{N} \frac{\Xi'(k_1 \ell, k_2 N, k_3 q_1) \Xi'(k_1 \ell, N q_2, k_3)}{||\Lambda'(k_1,k_2,k_3)||^2} = ||\Lambda'(k_1,k_2,k_3)_{N-n_1,n_1}||^2 \delta_{q_1,q_2}.$$ 

Applying $L_1, L_2$ to both sides of the expansion we have $L_1 \Lambda'_{N-n,q} = \sum_{n=0}^{N} \mu_n R_{n}^m(\Psi'_{N-n,n}$ and $L_2 \Lambda'_{q,p} = \sum_{n=0}^{N} \lambda_n R_{n}^m(\Psi'_{N-n,n}$. We find

$$L_2 \Lambda'_{q,p} = \sum_{n=0}^{N} \mu_n R_{n}^m(\Psi'_{N-n,n} = \sum_{n=0}^{N} \left( A_{n-1} R_{n-1}^m + B_{n} R_{n}^m + C_{n+1} R_{n+1}^{m+1} \right) \Psi'_{N-n,n},$$

$$\mu_n = -(2n+1)^2 - 2(2n+1)(k_1 + k_2) + 2k_1 k_2 - \frac{1}{2}, \quad \lambda_n = -2q + 1)^2 - 2(2q+1)(k_2 + k_3) + 2k_2 k_3 - \frac{1}{2}.$$ 

By equating coefficients of $\Psi'_{N-n,n}$, we find a 3-term recurrence formula for $R_{n}^m$, hence for $\Xi'$. If we make the identifications

$$k_1 = \delta + \beta - 1, \quad k_2 = \alpha + \gamma - 1, \quad k_3 = \alpha - \gamma, \quad N = -\alpha - \beta, \quad t = q + \frac{k_2 + k_3 + 1}{2},$$

this formula for $\Xi'$ agrees exactly with the 3-term recurrence formula for the Racah polynomials, so, by symmetry, $\Xi'(k_1 n, k_2 N, k_3 q) = K(N,k_1,k_2,k_3) R_n(k_1,k_2,k_3,q)$ where $R_n$ is proportional
Moreover, the measure defined by $\left| |A^{(k_1,k_2,k_3)}_{N-q,q}\right|^2 \sim \frac{\Gamma(t-\alpha+1)\Gamma(t-\beta+1)\Gamma(t-\gamma+1)\Gamma(t-\delta+1)\Gamma(t)}{\Gamma(t+\alpha)\Gamma(t+\beta)\Gamma(t+\gamma)\Gamma(t+\delta)\Gamma(t+1)}$, so the measure is symmetric with respect to all permutations of $\alpha, \beta, \gamma, \delta$. This implies that the family of orthogonal polynomials determined by this measure must admit this symmetry up to a multiplicative factor. Further, the left hand side of the orthogonality relation is proportional to

$$\sum_{q=0}^{N} \frac{(2\alpha)_q(\alpha+1)_q(\alpha+\beta)_q(\alpha+\gamma)_q(\alpha+\delta)_q}{(\alpha)_q(\alpha-\beta+1)_q(\alpha-\gamma+1)_q(\alpha-\delta+1)_q} q! \equiv (k_1 \ k_2 \ k_3) \ n \ q \ \Xi(n,q),$$

precisely the measure for orthogonality of the Racah polynomials $\Phi_n^{(\alpha,\beta,\gamma,\delta)}(t^2)$ where $t = q + \alpha$.

**Duality:** By making the transpositions $k_1 \leftrightarrow k_3$, $n \leftrightarrow q$ we obtain the result of applying $L_1$ to the expansion of $\Psi^\prime_{n,n,n}$ in an $L_2$ eigenbasis. This gives a 3-term recurrence relation for $\Xi$, defining a family of orthogonal polynomials $\psi_n(n)$ in the variable $n$. It is a 2nd order difference equation in $q$, hence $t$, for the Racah polynomials as eigenfunctions. This action induces a model of an irreducible representation of the structure algebra of $S_9$ in which the basis functions are Racah polynomials in $t^2$ and the symmetry operators map to difference operators.

### 3. Intertwining operators

Let $W_{k_1,k_2,k_3}$ be the space of functions on the first octant of the 2-sphere and with Hamiltonian $H^{(k_1,k_2,k_3)}$, symmetry operators $L_j^{(k_1,k_2,k_3)}$ and inner product $\langle \Phi, \Psi \rangle_{k_1,k_2,k_3}$. Let $W_{k_1',k_2',k_3'}$ be another such space. An **intertwining operator** is a mapping $X^{(k_1,k_2,k_3)}: W_{k_1,k_2,k_3} \rightarrow W_{k_1',k_2',k_3'}$ such that

$$X^{(k_1,k_2,k_3)}H^{(k_1,k_2,k_3)} = H^{(k_1',k_2',k_3')}X^{(k_1,k_2,k_3)}.$$

Note that $X^{(k_1,k_2,k_3)}$ maps eigenfunctions of $H^{(k_1,k_2,k_3)}$ to eigenfunctions of $H^{(k_1',k_2',k_3')}$ and its adjoint $X^{(k_1,k_2,k_3)}$ reverses the action.

Such energy shifting transformations are induced by the basic differential recurrence relations obeyed by Gaussian hypergeometric functions. For example the standard recurrence

$$\left[ z(1-z) \frac{d}{dz} \right. \left. - (b + a - 1)z + c - 1 \right] {}_2F_1 \left( \frac{a}{c}, b; z \right) = \left( c - 1 \right) {}_2F_1 \left( \frac{a-1}{c-1}, b-1; z \right),$$

induces a 1st order differential operator $T^{(k_1,k_2,k_3)}: W_{k_1,k_2,k_3} \rightarrow W_{k_1-1,k_2-1,k_3}$ such that, in terms of the $x, y$ variables,

$$T^{(k_1,k_2,k_3)} = \sqrt{1-x^2} \partial_x - \frac{1}{2}(k_2 - \frac{1}{2}) \sqrt{\frac{1+x}{1-x}} \partial_x \sqrt{\frac{1-x}{1+x}} - (n+1)\psi_{m,n+1}^{(k_1-1,k_2-1,k_3)}.$$

The adjoint is induced by

$$\frac{d}{dz} {}_2F_1 \left( \frac{a}{c}, b; z \right) = \frac{ab}{c} {}_2F_1 \left( \frac{a+1}{c+1}, b+1; z \right),$$

$$T^{*(k_1,k_2,k_3)}: W_{k_1,k_2,k_3} \rightarrow W_{k_1+1,k_2+1,k_3}, \psi_{m,n}^{(k_1,k_2,k_3)} \rightarrow -(k_1 + k_2 + n + 1)\psi_{m,n}^{(k_1+1,k_2+1,k_3)},$$

$$T^{*(k_1,k_2,k_3)} = -\sqrt{1-x^2} \partial_x + \frac{1}{2}(k_2 + \frac{1}{2}) \sqrt{\frac{1+x}{1-x}} \partial_x \sqrt{\frac{1-x}{1+x}},$$

Note that these intertwining operators are defined independent of basis. The action of $T$ and $T^*$ on the $\Lambda$-basis can again be computed from 1st order relations obeyed by Gaussian
hypergeometric functions. To find these we transform to $X,Y$ coordinates and again make use of first order hypergeometric differential recurrences. We obtain

$$T(k_1,k_2,k_3)A^{(k_1,k_2,k_3)}_{p,q} = -\frac{1}{2q+k_2+k_3+1}\Lambda^{(k_1-1,k_2-1,k_3)}_{p,q+1} + \frac{1}{2q+k_2+k_3+1}\Lambda^{(k_1-1,k_2-1,k_3)}_{p+1,q},$$

$$\tau^{(a-\frac{1}{2},b-\frac{1}{2},\gamma-\frac{1}{2},\delta-\frac{1}{2})} z^2 \left( \begin{array}{c} k_1 - 1 \\ n \end{array} \right) \left( \begin{array}{c} k_2 - 1 \\ N + 1 \end{array} \right) = n(k_1 + k_2 + n - 1) \Xi' \left( \begin{array}{c} k_1 \\ n - 1 \end{array} \right) \left( \begin{array}{c} k_2 \\ N \end{array} \right).$$

$$\tau^{(a-\frac{1}{2},b-\frac{1}{2},\gamma-\frac{1}{2},\delta-\frac{1}{2})} f(t) = \frac{1}{2t} \left[ f(t + \frac{1}{2}) - f(t - \frac{1}{2}) \right], \quad t = q + \frac{k_2 + k_3 + 1}{2}.$$

$$\tau^{(a+\frac{1}{2},b+\frac{1}{2},\gamma+\frac{1}{2},\delta+\frac{1}{2})} \Xi' \left( \begin{array}{c} k_1 + 1 \\ n \end{array} \right) \left( \begin{array}{c} k_2 + 1 \\ N - 1 \end{array} \right) = \Xi' \left( \begin{array}{c} k_1 \\ n + 1 \end{array} \right) \left( \begin{array}{c} k_2 \\ N \end{array} \right).$$

Note that (with $t = q + \frac{k_2 + k_3 + 1}{2}$),

$$\tau^{(a+\frac{1}{2},b+\frac{1}{2},\gamma+\frac{1}{2},\delta+\frac{1}{2})} \tau^{(a,\beta,\gamma,\delta)} \Xi' \left( \begin{array}{c} k_1 \\ n \end{array} \right) \left( \begin{array}{c} k_2 \\ N \end{array} \right) = n(k_1 + k_2 + n + 1) \Xi' \left( \begin{array}{c} k_1 \\ n + 1 \end{array} \right) \left( \begin{array}{c} k_2 \\ N \end{array} \right),$$

a 2nd order difference equation for $\Xi'$ as a polynomial in $t^2$. We will solve this equation.

3.1. Calculation of Racah polynomials

Note that $\Xi_n'(t)$ can be written as $\Xi' = G(n,N,k_1,k_2,k_3) \Phi_n(t)$ where $\Phi_n(t)$ is a polynomial in $t^2$ such that $\Phi_n(0) = 1$. Thus we can write $\Phi_n(t) = \sum_{k=0}^{n} w_k P_k(\alpha,t)$, where $P_k(\alpha,t) = (\alpha + t)k(\alpha - t)k$, $w_0 = 1$. Applying $\tau$ to $P_k(\alpha,t)$, we get,

$$\tau P_k(\alpha,t) = -kP_{k-1}(\alpha + \frac{1}{2},t)$$

(3)

Applying $\tau^*$ to the shifted basis, we get

$$\tau^* P_k(\alpha + \frac{1}{2},t) = -(\alpha + \beta + \gamma + \delta + k)P_{k+1}(\alpha,t) + (\alpha + \beta + k)(\alpha + \gamma + k)(\alpha + \delta + k)P_k(\alpha,t).$$

(4)

Thus,

$$\tau^* P_k(\alpha,t) = k(\alpha + \beta + \gamma + \delta + k - 1)P_k(\alpha,t) - k(\alpha + \beta + k - 1)(\alpha + \gamma + k - 1)(\alpha + \delta + k - 1)P_{k-1}(\alpha,t),$$

a 2-term recurrence relation, which implies $w_{k+1} = \frac{(-n+k)(n+\alpha+\beta+\gamma+\delta-k-1)}{(k+1)(\alpha+\beta+k)(\alpha+\gamma+k)(\alpha+\delta+k)} w_k$, $w_0 = 1$. It is easy to solve this recurrence to obtain

$$w_k = \frac{(-n)_k(n+\alpha+\beta+\gamma+\delta-1)_k}{k!(\alpha+\beta)_k(\alpha+\gamma)_k(\alpha+\delta)_k}, \quad k = 0,1,\ldots.$$

(5)

Hence, unique up to a scalar multiple,

$$\Phi_n(t) = \binom{-n}{\alpha+\beta+\gamma+\delta-1} \binom{\alpha+\gamma+\delta-1}{\alpha+\beta+\gamma+\delta+1}.$$
3.3. More intertwining operators

The recurrence
\[
\left( z \frac{d}{dz} + c - 1 \right) _2 F_1 \left( \begin{array}{cc} a & b \\ c & \end{array} ; z \right) = (c - 1) _2 F_1 \left( \begin{array}{cc} a & b \\ c - 1 & \end{array} ; z \right)
\]

leads to an intertwining operator \( U^{(k_1,k_2,k_3)}_{(+,+,+)} : \mathcal{W}_{k_1,k_2,k_3} \to \mathcal{W}_{k_1+1,k_2+1,k_3}, \)

\[
\mu^{(\beta,\delta)} \Phi_n^{(a,\beta,\gamma,\delta)}(t) = \frac{(n + \beta + \delta)(n + \alpha + \gamma - 1)}{(\alpha + \gamma - 1)} \Phi_n^{(a,\beta,\gamma,\delta)}(t)
\]

Its action on the \( \Lambda' \) basis induces the recurrence

\[
\mu^{(\beta,\delta)} f(t) = \frac{1}{2\alpha} \left[ (\beta + t)(\delta + t)f(t) - (\beta - t)(\delta - t)f(t) - \frac{\alpha}{2} \right]
\]

\[
\mu^{(\beta,\delta)} \Xi \left( \begin{array}{ccc} k_1 & k_2 & k_3 \\ n & N & q \end{array} \right) = (n + k_1 + 1)(n + k_2 + 2) \Xi \left( \begin{array}{ccc} k_1 & k_2 & k_3 \\ n & N & q \end{array} \right).
\]

The permutation invariance of the Racah polynomials leads to a family of recurrences in the \( \mu \) such that any pair of \( \alpha, \beta, \gamma, \delta \) can be raised by \( \frac{1}{2} \) and the other pair lowered by \( \frac{1}{2} \). These also follow from intertwining operators induced by Gaussian hypergeometric differential recurrences.

In particular, the hypergeometric recurrence

\[
\left( z \frac{d}{dz} + a \right) _2 F_1 \left( \begin{array}{cc} a & b \\ c & \end{array} ; z \right) = a \ _2 F_1 \left( \begin{array}{cc} a + 1 & b \\ c & \end{array} ; z \right)
\]

induces the operator

\[
U^{(k_1,k_2,k_3)}_{(+,+,+)} = \sqrt{\frac{1 + y}{2}} \left[ - (1 - y) \partial_y - N - \frac{k_1}{2} - \frac{k_2}{2} - \frac{k_3}{2} + \frac{1}{2} \right] \Xi \left( \begin{array}{ccc} k_1 & k_2 & k_3 \\ n & N & q \end{array} \right)
\]

\[
U^{(k_1,k_2,k_3)}_{(+,+,+)} \Psi_{m,n}^{(k_1,k_2,k_3)} = -(n + N + k_1 + k_2 + 1) \Psi_{m-1,n}^{(k_1,k_2,k_3+1)}, \quad m \geq 1, n \geq 0.
\]

The action on the \( L_2 \) eigenbasis is

\[
\mu^{(\alpha,\beta)} \Xi \left( \begin{array}{ccc} k_1 & k_2 & k_3 + 1 \\ n & N - 1 & q \end{array} \right) = - \frac{2N + k_1 + k_2 + k_3 + 2}{2N + k_1 + k_2 + k_3 + 1} \Xi \left( \begin{array}{ccc} k_1 & k_2 & k_3 \\ n & N & q \end{array} \right),
\]

\[
\mu^{(\alpha,\beta)} \Phi_n^{(a,\beta,\gamma,\delta)} = (\alpha + \beta) \Phi_n^{(a,\beta,\gamma,\delta)}, \quad \alpha + \beta = -N.
\]

The recurrence
\[
\left( z \frac{d}{dz} + b \right) _2 F_1 \left( \begin{array}{cc} a & b \\ c & \end{array} ; z \right) = b \ _2 F_1 \left( \begin{array}{cc} a & b + 1 \\ c & \end{array} ; z \right)
\]

induces the operator

\[
U^{(k_1,k_2,k_3)}_{(+,+,+)} = \sqrt{\frac{1 + y}{2}} \left[ (y - 1) \partial_y + N + \frac{k_1}{2} + \frac{k_2}{2} + \frac{k_3}{2} + \frac{3}{2} \right] \Xi \left( \begin{array}{ccc} k_1 & k_2 & k_3 \\ n & N & q \end{array} \right)
\]

\[
U^{(k_1,k_2,k_3)}_{(+,+,+)} \Psi_{m,n}^{(k_1,k_2,k_3)} = (n + N + k_1 + k_2 + k_3 + 2) \Psi_{m,n}^{(k_1,k_2,k_3+1)}, \quad m \geq 0, n \geq 0.
\]

In terms of \( \Xi' \) and \( \Phi_n \) the action is

\[
\mu^{(\alpha,\beta)} \Xi' \left( \begin{array}{ccc} k_1 & k_2 & k_3 + 1 \\ n & N & q \end{array} \right) = \frac{2N + k_1 + k_2 + k_3 + 2}{2N + k_1 + k_2 + k_3 + 1} \Xi' \left( \begin{array}{ccc} k_1 & k_2 & k_3 \\ n & N & q \end{array} \right),
\]

\[
\mu^{(\alpha,\beta)} \Phi_n^{(a,\beta,\gamma,\delta)} = (\alpha + \delta) \Phi_n^{(a,\beta,\gamma,\delta)}, \quad \alpha + \delta = N + k_1 + k_2 + k_3.
\]

Note that the operators (7,10) are not basis independent, since they depend on \( N \). However, the intertwining operator \( V^{(k_1,k_2,k_3)} : \mathcal{W}_{k_1,k_2,k_3} \to \mathcal{W}_{k_1,k_2,k_3+1}, \)

\[
V^{(k_1,k_2,k_3)} = U^{(k_1,k_2,k_3)}_{(+,+,+)} + U^{(k_1,k_2,k_3)}_{(+,+,+)} = \sqrt{\frac{1 + y}{2}} \left[ 2(y - 1) \partial_y + k_3 + 1 \right] .
\]
3.4. The expansion coefficients
Solving all of these recurrences for Ξ', we find

\[ R_q'(k_1, k_2, k_3)_N \cdot ||\Psi_{N-n,n}(k_1, k_2, k_3)||^2 = \Xi'(k_1 \quad k_2 \quad k_3 \quad N \quad q) = \]

\[ \frac{4c(2N + k_1 + k_2 + k_3 + 2)\Gamma(N + 1)}{\Gamma(\alpha + \beta, \gamma, \delta)(t)^2} \Phi_n^{(\alpha, \beta, \gamma, \delta)}(t^2), \quad t = q + \frac{k_2 + k_3 + 1}{2}, \]

\[ \alpha = \frac{k_2 + k_3 + 1}{2}, \quad \beta = -N - \frac{k_2 + k_3 + 1}{2}, \quad \gamma = \frac{k_2 - k_3 + 1}{2}, \quad \delta = N + k_1 + \frac{k_2 + k_3 + 3}{2}, \]

\[ \Phi_n = 4F_3 \left( \begin{array}{ccc} -n, & k_1 + k_2 + n + 1, & -q, \\ -N, & k_2 + 1, & N + k_1 + k_3 + 2 \end{array} \right) \frac{k_3 + k_2 + q + 1}{1} \]

The overall scaling factor \( c(k_1, k_2, k_3) \) can be determined by evaluating the double integral for \( \Xi' \) in the simplest case \( n = q = N = 0 \) where it factors into a product of beta integrals.

4. Extension to Wilson polynomials
Racah polynomials are \( S^9 \) expansion coefficients for finite dimensional representations on the real 2-sphere. Wilson polynomials are expansion coefficients related to infinite dimensional representations for the Schrödinger eigenvalue equation of the generic potential on the upper sheet of the 2d hyperboloid. (Much earlier, Koornwinder [10, 11] pointed out the connection between expansion coefficients on higher dimensional hyperboloids but not specifically to the 2d case.) A Hilbert space structure is imposed on the eigenspace corresponding to a single continuous spectrum eigenvalue, where \( N \) is a negative real number, not an integer.

We expand the \( L_2 \) basis vectors in terms of the \( L_1 \) basis:

\[ \Lambda_q = \sum_{n=0}^{\infty} R_q^n \Psi_n, \quad \sum_{\ell=0}^{\infty} \frac{||\Psi_n||^2}{||\Lambda_q||^2} R_{q1}^n R_{q2}^n = \delta_{n1,n2}, \]

Applying \( L_2 \) to both sides of the expansion we can show that the \( R_q^n \) satisfy a three term recurrence relation and a difference equation as before, and the orthogonality relation can be rewritten in the form

\[ \sum_{q=0}^{\infty} \frac{(2\alpha)\alpha + 1)q(\alpha + \beta)q(\alpha + \gamma)q(\alpha + \delta)_q}{(\alpha)_q(\alpha - 1 + 1)q(\alpha - \gamma + 1)q(\alpha - \delta + 1)q} R_{q1}^n R_{q2}^n = \delta_{n1,n2}h_{n1}, \]

where \( R_{q1}^n \sim \Phi_n(t) \) is a Wilson polynomial. This is equivalent to a \( 5F_4 \) identity and can all be made rigorous. Wilson recast the orthogonality into the form of a contour integral which greatly extended its domain of validity. All of the Racah-intertwining operators extend to this case.

The quantum problem on the 2d hyperboloid has mixed spectrum, both bound states and continuous spectra, [13]. An interesting task for future research is to work out the interbasis expansion associated with the spectral decomposition in this case and to relate it explicitly to partially discrete, partially continuous orthogonality relations for Wilson polynomials.

5. Extension to Wilson functions
If \( n \) is not an integer a formal calculation using (3) and (4) that ignores series convergence still gives (6) as a solution of the eigenvalue equation \( \tau^* \tau \Phi_n = n(\alpha + \beta + \gamma + \delta - 1)\Phi_n \). However, a careful calculation gives

\[ \left[ \tau^* - n(\alpha + \beta + \gamma + \delta) \right] \sum_{k=0}^{K} \omega_k P(t, \alpha)_k = \frac{(-n)_{K+1}(\alpha + \beta + \gamma + \delta - 1)_{K+1}(\alpha + t)K(\alpha - t)K}{K! (\alpha + \beta)_{K}(\alpha + \gamma)_{K}(\alpha + \delta)_{K}}. \]
Taking the limit as \( K \to +\infty \), and making use of the Stirling formula, we obtain
\[
\left[ \tau^* \tau - n(n + \alpha + \beta + \gamma + \delta) \right] \Phi_n(t) = \frac{\Gamma(\alpha + \beta)\Gamma(\alpha + \gamma)\Gamma(\alpha + \delta)}{\Gamma(-n)\Gamma(n + \alpha + \beta + \gamma + \delta - 1)\Gamma(\alpha + t)\Gamma(\alpha - t)}.
\] (13)

Since the \( \Gamma \) function has a pole at the negative integers, we see that \( \Phi_n(t) \) satisfies the eigenvalue equation for \( n \) a nonnegative integer, but otherwise it does not, except for isolated choices of the parameters. Now consider the functions with relations
\[
Q(t, \alpha, \beta)_k = \frac{\Gamma(1 - \beta + t)\Gamma(1 - \beta - t)}{\Gamma(\alpha + t)\Gamma(\alpha - t)}(1 - \beta + t)_k(1 - \beta - t)_k, \quad \tau^* Q(t, \alpha, \beta)_k = (\alpha + \beta - k - 1) Q(t, \alpha + \frac{1}{2}, \beta + \frac{1}{2})_k, \quad \tau^* Q(t, \alpha + \frac{1}{2}, \beta + \frac{1}{2})_k = - (\gamma + \delta + k) Q(t, \alpha, \beta)_k + k(\beta - \delta)(\beta + \gamma) Q(t, \alpha, \beta)_{k-1}.
\]

Then computing formally without regard to series convergence, we obtain the nonpolynomial solution of the \( \tau^*\tau \) eigenvalue equation:
\[
\Psi_n(t) = \frac{\Gamma(1 - \beta + t)\Gamma(1 - \beta - t)}{\Gamma(\alpha + t)\Gamma(\alpha - t)} {_4F_3} \left( \begin{array}{c} 1 - n - \alpha - \beta \, n + \gamma + \delta \, 1 - \beta + t \, 1 - \beta - t \\ 2 - \alpha - \beta \, 1 - \beta + \gamma \, 1 - \beta + \delta \end{array} ; 1 \right).
\] (14)

However, a careful computation, exactly analogous to that for \( \Phi_n(t) \), yields the result
\[
(\tau^* \tau - n(n + \alpha + \beta + \gamma + \delta)) \Psi_n(t) = \frac{\Gamma(2 - \alpha - \beta)\Gamma(1 - \beta + \gamma)\Gamma(1 - \beta + \delta)}{\Gamma(1 - n - \alpha - \beta)\Gamma(n + \gamma + \delta)\Gamma(\alpha + t)\Gamma(\alpha - t)},
\] (15)

so \( \Psi_n(t) \) doesn’t satisfy the eigenvalue equation. Now, comparing (13), (15), we see that functions
\[
\tilde{\Phi}_n^{(\alpha, \beta, \gamma, \delta)}(t) = \Phi_n(t) - \frac{\Gamma(\alpha + \beta)\Gamma(\alpha + \gamma)\Gamma(\alpha + \delta)}{\Gamma(-n)\Gamma(n + \alpha + \beta + \gamma + \delta - 1)\Gamma(2 - \alpha - \beta)\Gamma(1 - \beta + \gamma)\Gamma(1 - \beta + \delta)} \Psi_n(t)
\] (16)
do satisfy the eigenvalue equation for general \( n = n' + c \) where \( n' \) runs over the integers and \( c \) is a fixed noninteger. Furthermore it is straightforward to verify that \( \tilde{\Phi}_n^{(\alpha, \beta, \gamma, \delta)}(t) \) satisfies all of the recurrence formulas induced by the intertwining operators \( \tau, \mu, \tau^*, \mu^* \) for general \( n \), such as (9), that are satisfied by \( \Phi_n \) for nonnegative integer values. The functions \( \Phi_n(t) \) have the duality property, hence since they satisfy the 2nd order difference eigenvalue equation for general \( n \) they must also satisfy the 3-term recurrence formula. Thus for fixed \( c \) these basis functions define an infinite-dimensional irreducible representation of the quadratic algebra of \( S9 \) in which the eigenvalues of \( L_1 \) are indexed by an arbitrary integer \( n' \). In the original quantum problem this representation is easy to construct: all of the recurrence relations that we have derived using hypergeometric functions remain valid for nonpolynomial hypergeometric functions. These nonpolynomial bases are no longer normalizable, but the recurrence relations remain valid. Not clear was how such representations could be realized in terms of difference operators. The solution is that the analytic continuation of the basis functions is (16), the Wilson functions which lead to associated Wilson polynomials \([14, 15]\), and to the Wilson transform \([16]\), which corresponds to infinite dimensional irreducible representations of the quadratic algebra of \( S9 \) in which the spectrum of \( L_1 \) is (partly) continuous. In the original quantum mechanical system the analytic continuation is evident, but the integral over the sphere, i.e. in \( x, y \), giving the interbasis expansion coefficients is deformed into a Pochhammer contour integral on a Riemann surface over the \( x \)-plane with branch points at \( \pm 1 \) and a similar surface over the \( y \)-plane. All of the intertwining operator recurrences can be verified by integration by parts.
6. Discussion and Conclusions

- We showed explicitly how Racah and Wilson polynomials, and the Wilson functions, arise as expansion coefficients for the generic superintegrable system on the complex 2-sphere, relating two different sets of spherical coordinate bases. We employed only techniques from quantum theory and facts about the differential recurrence relations of ordinary $_2F_1$ hypergeometric series. We did not use results from the established theory of Wilson polynomials and functions, although we did identify instances where our results correspond to known facts about the functions.

- We showed how the principal properties of these functions: the measure, 3-term recurrence relation, the orthogonality measure, 2nd order difference equation, duality, permutation symmetry, and intertwining operators – follow from the symmetry of this quantum system.

- The parameter changing difference relations for the polynomials follow from intertwining operators for the quantum system. All of the properties of this system are induced by the fundamental differential recurrence relations of the Gaussian hypergeometric functions.

- There is active interest in the relation between multivariable Wilson polynomials and the quantum superintegrable system on the $n$-sphere with generic potential, e.g. [17], and these results should aid in the generalization.

- By contracting function space realizations of irreducible representations of the $S^9$ quadratic algebra to the other superintegrable systems one obtains the full Askey scheme of orthogonal hypergeometric polynomials, uniquely induced by Lie algebra contractions of $so(3, C)$ and $e(2, C)$, [18, 19]. This work should be extended to multivariable orthogonal polynomials.

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