Multi-parametric R-matrix for the $\mathfrak{sl}(2|1)$ Yangian

Andrei Babichenko$^a$ and Alessandro Torrielli$^b$

$^a$Department of Particle Physics, Weizmann Institute, Rehovot 76100, Israel
$^b$Department of Mathematics, University of Surrey, Guildford, Surrey, GU2 7XH, United Kingdom

E-mail: babichen@weizmann.ac.il, a.torrielli@surrey.ac.uk

Abstract: We study the Yangian of the $\mathfrak{sl}(2|1)$ Lie superalgebra in a multi-parametric four-dimensional representation. We use Drinfeld’s second realization to derive the R-matrix, the antiparticle representation, the crossing and unitarity condition. We consistently apply the Yangian antipode and its inverse to the individual particles involved in the scattering. We explicitly find a scalar factor solving the crossing and unitarity conditions, and study the analytic structure of the resulting dressed R-matrix. The formulas we obtain bear some similarities with those familiar from the study of integrable structures in the AdS/CFT correspondence, although they present obvious crucial differences.
1 Introduction

In recent years there has been an increasing interest in integrable models based on superalgebra symmetries, both in the continuum and in their lattice versions. Examples of this sort include spin chains on one hand, and integrable and conformal field theories on the other hand, most notably two-dimensional sigma models on supergroup manifolds. The spectrum of spin chains with superalgebra symmetries turns out to be interesting, in particular, in view of their conformal limit \[1\text{–}5\]. This limit is expected to reproduce the data obtained by elaborated methods of logarithmic conformal field theories (for introductory reviews see e.g. \[6, 7\]). Integrability makes it in principle possible to determine the spin chain spectrum exactly, and therefore can allow to derive exact predictions for the CFT spectrum and partition function in the conformal limit.

The interest in two-dimensional sigma models on supergroup manifolds emerged both from string theory \[8\text{–}10\] and in context of disordered two-dimensional condensed matter systems \[11, 12\]. Later on, sigma models on a variety of supergroup manifolds, and their Gross-Neveu like analogs, were successfully investigated by integrability methods. Especially interesting is the relation of integrable structures to the CFT ones, when the sigma model is not only integrable but also conformal with a non chiral conformal symmetry \[8, 13\].

One instance where the integrability based on superalgebras revealed itself particularly powerful is in the investigation of AdS/CFT correspondence for maximally supersymmetric backgrounds. In the case of integrable backgrounds the same R-matrix appears in their
sigma model and spin chain incarnations, on the AdS and CFT side respectively, and enables an exact comparison of the quantities on both sides of correspondence. For a review see e.g. [14] and references therein. In the case of the $AdS_5 \times S^5$ background the R-matrix intertwines two fundamental representations of the centrally extended $\mathfrak{psl}(2|2)$ algebra [15], whereas in the case of $AdS_4 \times CP(3)$ the R-matrix acts in tensor product of fundamental and anti-fundamental representation of $\mathfrak{osp}(4|6)$ [16]. Another case of an integrable background where alternating spin chains seems to be relevant is $AdS_3 \times S^3 \times S^3 \times S^1$ [17, 18]. In all these cases the relevant superalgebra representations depend on some continuum parameters, which enter the non-relativistic dispersion relation of the excitations in the system. The dependence of the transfer-matrix spectrum on such additional parameters in the spin chain case, or, equivalently, its dependence on the ‘particle’ mass spectrum in the sigma model case, raises the important physical question of their interpretation in the framework of integrability. This question has been raised earlier in the literature [19].

In this paper we consider another example of rational R-matrix based on the superalgebra $\mathfrak{sl}(2|1)$, taken in a four dimensional representation (and its conjugated). These representations may be considered as fundamental and anti-fundamental for an $\mathfrak{osp}(2|2)$ algebra, which is isomorphic to $\mathfrak{sl}(2|1)$. R- and S- matrices in this representations and with this symmetry were considered earlier in the literature [1, 3, 11, 20–22], see also the recent paper [4], but in a different setup and not in full generality (namely, keeping all the independent representation parameters unspecified). We construct these R-matrices explicitly from the requirement of their commutation with the $\mathfrak{sl}(2|1)$ Yangian comultiplication, and show that they satisfy the Yang-Baxter equation. Each four dimensional representation of $\mathfrak{sl}(2|1)$ corresponds to a point in moduli space which depends on five parameters related by two constraints, such that the R-matrix depends on three parameters for each of the two representations it intertwines. The R-matrices we find depend on the difference of the Yangian spectral parameters carried by each representation, and in a non-difference form on all remaining parameters. The conjugation rule of representation involves a non trivial change of representation parameters. We succeed in finding a relativistic interpretation to these conjugation transformations as antiparticles, such that the R-matrix we find is crossing invariant and unitary. In order to achieve this, the R-matrix needs to be multiplied by a crossing-unitarizing scalar factor, which we find explicitly. An interesting effect is revealed concerning the role of the inverse of the antipode in the crossing relation.

Our multi-parametric R-matrix\(^1\) is therefore a good candidate for describing an integrable two-dimensional sigma model based on the $\mathfrak{sl}(2|1)$ superalgebra. One of the possible candidates is the ‘supersymmetric sign Gordon’ (SSSG) model on the super manifold $\mathfrak{osp}(3|2)/\mathfrak{osp}(2|2)$. For some recent developments on the SSSG see [23]. We also produce a formal interpretation of the parameters characterizing our representations in terms of the variables used in the context of integrability of the AdS/CFT correspondence. The R-matrix we find resembles very closely Beisert’s R-matrix [15], being however different. We nevertheless believe that our findings might be instrumental in resolving certain issues, related for instance to the possibility of a Drinfeld’s second realization of the AdS/CFT correspondence.

\(^1\)For related work, see [25, 26].
Yangian in the distinguished basis [24]. In this paper, we derive such a realization for a similar four-dimensional representation (although of a different superalgebra), and show how the supercharges get modified at the Yangian level by the presence of the multi-parametric deformation. This turns out to be quite similar to how certain so-called ‘secret’ charges, found in [27], appear in the AdS/CFT context. Such charges might therefore be related to an alternative choice of a Dynkin diagram with respect to the one in [29], and connected to it by commutation with the secret ‘automorphism’ generator $\hat{B}$.

The plan of the paper is the following. In section 2, we study the Yangian of $\mathfrak{sl}(2|1)$ in the distinguished basis and in the four-dimensional representation relevant to our interests. We utilize Drinfeld’s second realization of the Yangian, and derive the R-matrix in this representation. We also check the Yang-Baxter equation and the unitarity condition, and expand the R-matrix in terms of projectors onto irreducible components of the tensor product of two four-dimensional $\mathfrak{sl}(2|1)$ representation. In section 3, we derive the conjugate representation and its R-matrix, and then derive the antiparticle representation. We also comment on the similarities with the AdS/CFT case. In section 4, we derive the antiparticle R-matrix and the crossing symmetry condition, which we explicitly solve obtaining a crossing symmetric and unitary scalar factor. We consistently apply the Yangian antipode and its inverse separately on the two factors of the tensor product, and study its effect on the particle-antiparticle transformation. We finish with some conclusions, an appendix with formulas for the conjugate R-matrix, and another appendix with the analysis of the poles of the direct R-matrix (dressed with the scalar factor).

## 2 R matrix from the Yangian

Let us denote with $E_{ij}$ the matrix with all zeroes, but 1 in row $i$, column $j$. We will work with a so-called *distinguished* Dynkin diagram, *i.e.* with the lowest number of fermionic nodes (in this case, one). The representation we are interested in is the following:

\[
\begin{align*}
E_1 &= E_{43}, & F_1 &= E_{34}, & H_1 &= -E_{33} + E_{44}, \\
E_2 &= -a E_{14} + b E_{32}, & F_2 &= -d E_{23} + e E_{41}, & H_2 &= -c \mathbb{1} - E_{11} - E_{44},
\end{align*}
\]

where the parameters are constrained as

\[
ac = c + 1, \quad bd = c.
\]

The vector space on which this representation acts is generated by two bosons $|a\rangle$ (indices $a = 1, 2$) and two fermions $|\alpha\rangle$ (indices $|\alpha\rangle = 3, 4$). Notice that in the limit $c \to -1$ the above representation becomes reducible but still indecomposable. In fact, if we choose for instance $e \to 0$, the state $|1\rangle$ gets annihilated by all generators, but the state $|4\rangle$ is still sent to $|1\rangle$ by $E_2$. The Cartan matrix, whose entries we denote with $a_{ij}$, is a two by
two matrix with entries equal to 2 and 0 respectively on the diagonal, and −1’s on the anti-diagonal. The following assignment\(^2\):

\[
\begin{align*}
\xi_{i,0}^+ &= E_i, & \xi_{i,0}^- &= F_i, & \kappa_{i,0} &= H_i, \\
\xi_{i,1}^+ &= u E_i, & \xi_{i,1}^- &= u F_i, & \kappa_{i,1} &= u H_i, \\
\xi_{2,1}^+ &= b \left( u + \frac{1}{2} \right) E_{32} - a \left( u - \frac{1}{2} \right) E_{14}, & \xi_{2,1}^- &= e \left( u - \frac{1}{2} \right) E_{41} - d \left( u + \frac{1}{2} \right) E_{23}, \\
\kappa_{2,1} &= -c \left( u + \frac{1}{2} \right) I + \left( c - u + \frac{1}{2} \right) (E_{11} + E_{14}),
\end{align*}
\]

with the rest of the generators \(\xi_{i,n}^\pm, \kappa_{i,n}, n > 1\), consistently obtained by subsequent application of the relations (2.4) below to the above generating elements (2.3), defines a representation of the Yangian in Drinfeld’s second realization [28, 29]:

\[
\begin{align*}
[\kappa_{i,m}, \kappa_{j,n}] &= 0, & [\kappa_{i,0}, \xi_{j,m}^\pm] &= \pm a_{ij} \xi_{j,m}^\pm, \\
[\xi_{j,m}^+, \xi_{j,m}^-] &= \delta_{i,j} \kappa_{j,m}, \\
[\kappa_{i,m+1}, \xi_{j,n}^-] - [\kappa_{i,m}, \xi_{j,n+1}^+] &= \pm \frac{1}{2} a_{ij} \{\kappa_{i,m}, \xi_{j,n}^\pm\}, \\
[\xi_{i,m}^+, \xi_{j,n}^-] - [\xi_{i,m}^+, \xi_{j,n+1}^+] &= \pm \frac{1}{2} a_{ij} \{\xi_{i,m}^+ \xi_{j,n}^\pm\}, \\
i \neq j, & n_{ij} = 1 + |a_{ij}|, & \text{Sym}_{\{\kappa\}} [\xi_{i,k_1}^\pm, \ldots, [\xi_{i,k_{n_{ij}}}^\pm, \ldots]] &= 0.
\end{align*}
\]

One can actually go further, and prove that the all-level representation corresponding to (2.3) and which solves all the relations (2.4) is given by

\[
\begin{align*}
\xi_{1,n}^+ &= u^n E_1, & \xi_{1,n}^- &= u^n F_1, & \kappa_{1,n} &= u^n H_1, \\
\xi_{2,n}^+ &= b \left( u + \frac{1}{2} \right)^n E_{32} - a \left( u - \frac{1}{2} \right)^n E_{14}, & \xi_{2,n}^- &= e \left( u - \frac{1}{2} \right)^n E_{41} - d \left( u + \frac{1}{2} \right)^n E_{23}, \\
\kappa_{2,n} &= -ae \left( u - \frac{1}{2} \right)^n E_{11} - bd \left( u + \frac{1}{2} \right)^n E_{22} - bd \left( u + \frac{1}{2} \right)^n E_{33} - ae \left( u - \frac{1}{2} \right)^n E_{41}.
\end{align*}
\]

The \(R\)-matrix related to this Yangian representation must have very specific properties. Since it must satisfy

\[
\Delta^{op}(3) R = R \Delta(3)
\]

for any generator \(\mathfrak{g}\) of the Yangian, we can obtain strong constraints on its entries by focusing for instance on the Cartan subalgebra \(\{\kappa_{i,0}, i = 1, 2\}\). The coproduct in this subalgebra is trivial (as it is trivial on the entire level \(n = 0\) Lie subalgebra of the Yangian), namely

\(^2\)We will always denote with \([A, B]\) the graded commutator \(AB - (-)^{\text{deg}(A)\text{deg}(B)} BA\), and with \(\{A, B\}\) the combination \(AB + (-)^{\text{deg}(A)\text{deg}(B)} BA\). The grading is 0 for the bosonic indices 1, 2 and 1 for the fermionic indices 3, 4, so that \(\text{deg}(E_{ij}) = \text{deg}(i) + \text{deg}(j)\).
\[ \Delta(\kappa, 0) = [\kappa, 0]_{\text{rep} 1} \otimes 1 + 1 \otimes [\kappa, 0]_{\text{rep} 2} = \Delta^{op}(\kappa, 0), \] (2.7)

with \( \otimes \) being the \textit{graded} tensor product, such that \((X \otimes Z)(Y \otimes W) = (-)^{\text{deg}(Z)\text{deg}(Y)} XY \otimes ZW\) among operators and \((X \otimes Z)(v_1 \otimes v_2) = (-)^{\text{deg}(Z)\text{deg}(v_1)} X v_1 \otimes Z v_2\) when acting on states.

By looking at (2.1), and recalling that the total number of particles is conserved, we immediately obtain for example the conservation of the following numbers:

- ‘Total number of bosons of type 1’ \textit{minus} ‘Total number of bosons of type 2’
- ‘Total number of fermions of type 3’ \textit{minus} ‘Total number of fermions of type 4’

Notice that the \( \pm c \mathbb{1} \) term in the Cartan generators \( H_i \) simply drops out of the relation (2.6). The conservation of the above quantum numbers is enough to single out the structure of the \( R \)-matrix entries, which must be as follows (we denote by \( ij \) the state \(|i\rangle \otimes |j\rangle\) for simplicity\(^3\), and we choose a specific overall normalization):

\[
\begin{align*}
R_{11} &= 11, \\
R_{12} &= B_{12} + C_{21} + D_{34} + E_{43}, \\
R_{21} &= F_{12} + G_{21} + H_{34} + I_{43}, \\
R_{22} &= L_{22}, \\
R_{33} &= \Gamma_{33}, \\
R_{34} &= P_{12} + Q_{21} + N_{34} + \Theta_{43}, \\
R_{43} &= T_{12} + U_{21} + \Psi_{34} + \Xi_{43}, \\
R_{44} &= V_{44},
\end{align*}
\] (2.8)

and

\[
\begin{align*}
R_{13} &= \alpha_{13} + \alpha_{31}, \\
R_{14} &= \alpha_{34} + \alpha_{41}, \\
R_{23} &= \alpha_{52} + \alpha_{63}, \\
R_{24} &= \alpha_{72} + \alpha_{84}, \\
R_{31} &= \beta_{13} + \beta_{21}, \\
R_{41} &= \beta_{34} + \beta_{41}, \\
R_{32} &= \beta_{53} + \beta_{62}, \\
R_{42} &= \beta_{74} + \beta_{82}.
\end{align*}
\] (2.9)

\(^3\)We remind that the grading of the states is then \( \text{deg}(1) = \text{deg}(1) = 0, \text{deg}(2) = \text{deg}(2) = 0, \text{deg}(3) = \text{deg}(3) = 1, \text{deg}(4) = \text{deg}(4) = 1. \)
Incidentally, these are the same non-zero entries of Beisert’s $R$-matrix [15].

One can relate some of these entries to one another by imposing invariance under the generators $E_1$ and $F_1$ (which we call the ‘fermionic’ $\mathfrak{sl}(2)$ subalgebra). However, the algebra being $\mathfrak{sl}(1|2)$, no $\mathfrak{sl}(2)$ subalgebra is available for the bosonic states, therefore many of the coefficients of the $R$-matrix still remain unconstrained.

The comultiplication becomes non-trivial as soon as we move to the level one Yangian generators. One has

\[
\Delta(\kappa_{2,1}) = \kappa_{2,1} \otimes 1 + 1 \otimes \kappa_{2,1} + H_2 \otimes H_2 + F_1 \otimes E_1 - F_3 \otimes E_3,
\]
\[
\Delta(\kappa_{1,1}) = \kappa_{1,1} \otimes 1 + 1 \otimes \kappa_{1,1} + H_1 \otimes H_1 - 2F_1 \otimes E_1 + F_2 \otimes E_2 + F_3 \otimes E_3,
\]
\[
\Delta(\xi_{2,1}^+) = \xi_{2,1}^+ \otimes 1 + 1 \otimes \xi_{2,1}^+ + H_2 \otimes E_2 + F_1 \otimes E_3,
\]
\[
\Delta(\xi_{2,1}^-) = \xi_{2,1}^- \otimes 1 + 1 \otimes \xi_{2,1}^- + F_2 \otimes H_2 - F_3 \otimes E_1,
\]
\[
\Delta(\xi_{1,1}^+) = \xi_{1,1}^+ \otimes 1 + 1 \otimes \xi_{1,1}^+ + H_1 \otimes E_1 - F_2 \otimes E_3,
\]
\[
\Delta(\xi_{1,1}^-) = \xi_{1,1}^- \otimes 1 + 1 \otimes \xi_{1,1}^- + F_1 \otimes H_1 + F_3 \otimes E_2,
\]

where we denote the generators associated to the non-simple roots as

\[
E_3 = [E_1, E_2], \quad F_3 = [F_1, F_2].
\]

We have checked that these coproducts provide a homomorphism of the Yangian, namely, they respect the relations (2.4).

By imposing the condition (2.6) and using formulas (2.10), together with the remaining level zero coproducts

\[
\Delta(\xi_{i,0}^\pm) = \xi_{i,0}^\pm \otimes 1 + 1 \otimes \xi_{i,0}^\pm = \Delta^{op}(\xi_{i,0}^\pm),
\]

one is able to fix the $R$-matrix entries uniquely up to an overall scalar factor. If we define

\[
\delta u = u_1 - u_2,
\]

then one finds
\begin{align}
B &= \frac{(\delta u + c_1 - c_2)(1 + \delta u + c_1 - c_2)}{(-1 + \delta u - c_2)(\delta u - c_2)}, \\
D &= -E = -\frac{b_2(\delta u + c_1 - c_2)e_1}{(-1 + \delta u - c_2)(\delta u - c_2)}, \\
G &= \frac{\delta u(1 + \delta u)}{(-1 + \delta u - c_2)(\delta u - c_2)}, \\
H &= -I = -\frac{a_2 b_1 (1 + c_2)}{a_2 (-1 + \delta u - c_2)(\delta u - c_2)}, \\
L &= \frac{(\delta u + c_1)(1 + \delta u + c_1)}{(-1 + \delta u - c_2)(\delta u - c_2)}, \\
N &= \frac{\delta u(\delta u + c_1 - c_2)}{(-1 + \delta u - c_2)(\delta u - c_2)}, \\
P &= -\frac{a_1 d_2 (\delta u + c_1 - c_2)}{(-1 + \delta u - c_2)(\delta u - c_2)}, \\
V &= \Gamma, \quad T = -P, \\
\alpha_1 &= \alpha_3 = \frac{\delta u + c_1 - c_2}{-1 + \delta u - c_2}, \quad \alpha_2 = \alpha_4 = \frac{a_2 (1 + c_1)}{a_1 (1 - \delta u + c_2)}, \\
\alpha_5 &= \alpha_7 = \frac{\delta u(1 + \delta u + c_1)}{(-1 + \delta u - c_2)(\delta u - c_2)}, \quad \alpha_6 = \alpha_8 = \frac{b_1 (1 + \delta u + c_1)d_2}{(-1 + \delta u - c_2)(\delta u - c_2)}, \\
\beta_1 &= \beta_3 = \frac{(1 + c_1)e_2}{(1 - \delta u + c_2)e_1}, \quad \beta_2 = \beta_4 = \frac{\delta u}{-1 + \delta u - c_2}, \\
\beta_5 &= \beta_7 = \frac{b_2 (1 + \delta u + c_1)d_1}{(-1 + \delta u - c_2)(\delta u - c_2)}, \quad \beta_6 = \beta_8 = \frac{(1 + \delta u + c_1)(\delta u + c_1 - c_2)}{(-1 + \delta u - c_2)(\delta u - c_2)},
\end{align}

We have checked that the R-matrix satisfies the graded Yang-Baxter equation

\begin{align}
R_{12,12,12}(x_1, x_2) R_{13,13,13}(x_1, x_3) R_{12,12,13}(x_2, x_3) \equiv & \Psi = \Theta, \quad \Xi = N,
\end{align}

where all indices run from 1 to 4 and repeated indices are summed over. We have defined

\begin{align}
R_{ij} = R_{ijmn}(x_1, x_2)^m_n
\end{align}

using the notation of (2.8) for the states, and collectively indicating the representation parameters in representation \( i \) as

\begin{align}
x_i \equiv \{ a_i, b_i, c_i, d_i, e_i, u_i \},
\end{align}

constrained by (2.2).

We have also checked that the above R-matrix satisfy the unitarity condition

\begin{align}
(-)^{\phi + ab} R_{b_{\alpha \delta}}(x_2, x_1) R_{d_{\phi q}}(x_1, x_2) = \delta_{a \alpha} \delta_{b \phi}.
\end{align}
This implies that any overall scalar factor multiplying this R-matrix will have to satisfy unitarity on its own, namely.

\[ \Phi_{12} \Phi_{21} = 1. \quad (2.20) \]

Notice that the tensor Casimir of the algebra is given by:

\[ C_{12} = -\frac{1}{2}E_3 \otimes F_3 + \frac{1}{2}E_2 \otimes F_2 + \frac{1}{2}F_3 \otimes E_3 - \frac{1}{2}F_2 \otimes E_2 - \frac{1}{2}F_1 \otimes E_1 - \frac{1}{2}E_1 \otimes F_1 \]
\[ + H_2 \otimes H_2 + \frac{1}{2}(H_1 \otimes H_2 + H_2 \otimes H_1). \quad (2.21) \]

It satisfies \([C_{12}, \Delta(3)] = 0\) for any level zero generator 3. Since the level zero of the Yangian has a trivial coproduct, the R-matrix can be decomposed into a linear combination of projectors onto irreducible representations of the tensor product of representations 1 and 2. The irreducible components correspond to the eigenspaces of the Casimir operator, and there are three such eigenspaces, corresponding to the three distinct eigenvalues of \(C_{12}\) \([15, 32]\)

\[ \lambda_1 = c_1 c_2, \quad \lambda_2 = (1 + c_1)(1 + c_2), \quad \lambda_3 = \frac{1}{2}(c_1 + c_2 + 2 c_1 c_2). \quad (2.22) \]

The projectors onto the three eigenspaces are given by

\[ P_i = \frac{(C_{12} - \lambda_j)(C_{12} - \lambda_k)}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)} \quad (2.23) \]

with \((i, j, k) = (1, 2, 3), (2, 1, 3), (3, 1, 2)\), respectively. The R-matrix (without the overall scalar factor \(\Phi_{12}\)) can then be written as

\[ R = \frac{(u_1 - u_2 + c_1)(1 + u_1 - u_2 + c_1)}{(-1 + u_1 - u_2 - c_2)(u_1 - u_2 - c_2)} P_1 + P_2 + \frac{1 + u_1 - u_2 + c_1}{-1 + u_1 - u_2 - c_2} P_3. \quad (2.24) \]

The various coefficients in the above spectral decomposition correspond to the diagonal action of the R-matrix on the highest weight states in each irreducible component. Notice that all the functions multiplying the projectors depend only on the parameters \(c_{1,2}\) of the representations, while all the other parameters are hidden in the projectors.

### 3 Conjugate representation and antiparticles

The representation we consider in this section is the conjugate, i.e. the supertranspose, representation of the one studied in the previous section for the distinguished Dynkin diagram. One can show that such representation is generated by
\begin{align}
E_1 &= E_{34}, \quad F_1 = E_{43}, \quad H_1 = [E_1, F_1], \quad (3.1) \\
E_2 &= -b E_{23} - a E_{41}, \quad F_2 = -e E_{14} - d E_{32}, \quad H_2 = [E_2, F_2],
\end{align}

where the parameters are constrained as
\begin{align}
ae &= c + 1, \quad bd = c, \quad (3.2)
\end{align}

The vector space on which this representation acts is again generated by two bosons (indices 1 and 2) and two fermions (indices 3 and 4). The Cartan matrix is the same as in the previous section. The following assignment
\begin{align}
\xi^+_1 &= E_1, \quad \xi^-_1 = F_1, \quad \kappa_{1,0} = H_1, \\
\xi^+_1 &= u E_1, \quad \xi^-_1 = u F_1, \quad \kappa_{1,1} = u H_1, \\
\xi^+_2 &= -b \left( u - \frac{1}{2} \right) E_{23} - a \left( u + \frac{1}{2} \right) E_{41}, \quad \xi^-_2 = -e \left( u + \frac{1}{2} \right) E_{14} - d \left( u - \frac{1}{2} \right) E_{32}, \\
\kappa_{2,1} &= [\xi^+_2, \xi^-_1], \quad (3.3)
\end{align}

with the rest of the generators \( \xi^\pm_{i,n}, \kappa_{i,n}, n > 1 \), consistently obtained by iteration of (2.4), defines another representation of the same Yangian in Drinfeld’s second realization. One can promote to arbitrary levels the representation (3.3) simply by assigning
\begin{align}
\xi^+_1 &= u^n E_1, \quad \xi^-_1 = u^n F_1, \quad \kappa_{1,n} = u^n H_1, \\
\xi^+_2 &= -b \left( u - \frac{1}{2} \right)^n E_{23} - a \left( u + \frac{1}{2} \right)^n E_{41}, \quad \xi^-_2 = -e \left( u + \frac{1}{2} \right)^n E_{14} - d \left( u - \frac{1}{2} \right)^n E_{32}, \\
\kappa_{2,n} &= [\xi^+_2, \xi^-_1], \quad (3.4)
\end{align}

The R-matrix related to this Yangian representation must again satisfy
\begin{align}
\Delta^\text{op}(\hat{3}) \, R = R \, \Delta(\hat{3}) \quad (3.5)
\end{align}

for any generator \( \hat{3} \) of the Yangian. The coproduct is trivial on the entire level \( n = 0 \) Lie subalgebra of the Yangian.

We consider in this section both representations 1 and 2 to be the conjugate representation (3.3). By looking at (3.1) and recalling that the total number of particles is conserved, we immediately obtain the conservation law for the differences of the numbers of bosons and fermions, exactly as in the previous section. The R-matrix (choosing the same overall normalization as in the previous section) can therefore again be again parametrized by the same equations (2.8), (2.9).

The comultiplication becomes non-trivial as soon as we move to the level one Yangian generators. If we define once again
then we can directly use the formulas (2.10), which are universal for any representation\(^4\). The result of imposing the invariance of the R-matrix (3.5) is given in the appendix.

Let us now construct the antiparticle representation of the Yangian representation (2.1), (2.3). Such antiparticle representation is defined as

\[
E_1 = E_{43}, \quad F_1 = E_{34}, \quad H_1 = -E_{33} + E_{44},
\]

\[
E_2 = -\bar{a} E_{14} + \bar{b} E_{32}, \quad F_2 = -\bar{d} E_{23} + \bar{e} E_{41}, \quad H_2 = -\bar{c} \mathbb{1} - E_{11} - E_{44},
\]

where the parameters are constrained as

\[
\bar{a} \bar{e} = \bar{c} + 1, \quad \bar{b} \bar{d} = \bar{c}
\]  

(3.8)

and such that

\[
S(\mathfrak{J}) = \mathcal{C}^{-1} \mathfrak{J}^{\text{st}} \mathcal{C}.
\]  

(3.9)

In (3.9), \(\mathfrak{J}\) is any generator in the representation (2.3), \(\mathfrak{J}\) is any generator in the representation (3.7) and corresponding Yangian, \(\mathcal{C}\) is a suitable charge conjugation matrix, and \(S\) is the Yangian Hopf algebra antipode\(^5\).

In order to find a solution to the condition (3.9), we also need to allow the Yangian related to the representation (3.7) to have a spectral parameter \(\bar{u}\) different from \(u\) in (2.3). If we do that, we can find a consistent solution which reads

\[
\mathcal{C} = \frac{1}{\bar{a}} b E_{12} + \frac{1}{\bar{a}} \bar{b} E_{21} - E_{34} + E_{43},
\]

\[
\bar{c} = -c - 1, \quad \bar{u} = u + c.
\]  

(3.10)

(3.11)

Notice that the combination

\[
\vartheta = -2\pi i (u + \frac{c}{2}),
\]

(3.12)

\(^4\)We have checked that also in this representation the coproducts provide a homomorphism of the Yangian, namely, they respect the relations (2.4).

\(^5\)We remind that the antipode is defined on the whole Hopf algebra by the relation \(\mu (S \otimes 1) \Delta = \eta \epsilon\) (and \(\mu (1 \otimes S^{-1}) \Delta = \eta \epsilon\) for invertible antipodes) involving the multiplication \(\mu\), the coproduct \(\Delta\), the counit \(\epsilon\) and the unit \(\eta\). The counit \(\epsilon\) turns out to act as zero on all generators of the Yangian, while one has \(\epsilon(1) = 1\).
transforms as a relativistic rapidity under the crossing transformation (3.11), i.e.

$$\tilde{\vartheta} = \vartheta + i\pi.$$  \hspace{1cm} (3.13)

The R-matrix depends on the variables $\vartheta_1$ and $\vartheta_2$ only through their difference, consistently with the existence of a shift automorphism of the Yangian. The representations we find in this paper are all of the so-called evaluation type \cite{30,31}. In such representations, the shift automorphism simply transforms the spectral parameter as $u \to u + q$, where $q$ is a constant independent on the representation.

We can also introduce a set of parameters which are reminiscent of AdS/CFT \cite{15}. In fact, let us make the following choice:

$$a = -1, \quad b = -\alpha \left(1 - \frac{x^-}{x^+}\right), \quad d = \frac{i\beta}{x^-}, \quad e = i(x^+ - x^-),$$  \hspace{1cm} (3.14)

with

$$\alpha \beta = \frac{g^2}{2}, \quad x^+ + \frac{\alpha \beta}{x^+} - x^- - \frac{\alpha \beta}{x^-} = i.$$  \hspace{1cm} (3.15)

One can check that the constraint (2.2) is satisfied, with

$$c = -1 - i(x^+ - x^-).$$  \hspace{1cm} (3.16)

In terms of these new variables, the antiparticle transformation (3.11) for the variable $c$ amounts to the same map found by Janik \cite{32} in the AdS/CFT context\footnote{It is interesting to notice how the same map arises in the AdS/CFT context from imposing the condition $\mu (S \otimes 1)\Delta = \eta \epsilon$ on a nontrivial level zero coproduct, as showed in \cite{33}.}, namely

$$x^\pm = \frac{\alpha \beta}{x^\pm}.$$  \hspace{1cm} (3.17)

This map can be then expressed in terms of a generalized rapidity by means of Weierstrass functions (see \cite{32}). We also notice that, with the assignment (3.14), the representation (3.1) becomes precisely the AdS/CFT representation used in \cite{32}.

4 Crossing symmetry and S matrix

The mixed R-matrix which intertwines a representation of the type (3.7) with a representation of the type (2.1) has to satisfy the following crossing-symmetry condition:

$$(\mathcal{C}^{-1} \otimes 1) \Phi_{12} R_{12}^{st} (\mathcal{C} \otimes 1) \Phi_{12} R_{12} = 1 \otimes 1,$$  \hspace{1cm} (4.1)
derived from the following condition one imposes on the universal R-matrix (which is assumed to be invertible):

$$(S \otimes 1) R = R^{-1}. \tag{4.2}$$

In (4.1), $s_{t_1}$ means taking the supertranspose in the space 1 of the tensor product, the charge conjugation matrix is given by (3.10). The R-matrix $R_{12}$ coincides with the one we have obtained in section 2, while the mixed R-matrix $\bar{R}_{12}$ is given by straightforward substitution of the representation 1 with its associated antiparticle representation. We report the result here below for the convenience of the reader:

$$
\begin{align*}
R_{12} 11 &= 11, \\
R_{12} 12 &= B' 12 + C' 21 + D' 34 + E' 43, \\
R_{12} 21 &= F' 12 + G' 21 + H' 34 + I' 43, \\
R_{12} 22 &= L' 22, \\
R_{12} 33 &= T' 33, \\
R_{12} 34 &= P' 12 + Q' 21 + N' 34 + \Theta' 43, \\
R_{12} 43 &= T' 12 + U' 21 + \Psi' 34 + \Xi' 43, \\
R_{12} 44 &= V' 44, \tag{4.3}
\end{align*}
$$

$$
\begin{align*}
R_{12} 13 &= \alpha'_1 13 + \alpha'_2 31, \\
R_{12} 14 &= \alpha'_3 14 + \alpha'_4 41, \\
R_{12} 23 &= \alpha'_5 23 + \alpha'_6 32, \\
R_{12} 24 &= \alpha'_7 24 + \alpha'_8 42, \\
R_{12} 31 &= \beta'_1 13 + \beta'_2 31, \\
R_{12} 34 &= \beta'_3 14 + \beta'_4 41, \\
R_{12} 43 &= \beta'_5 23 + \beta'_6 32, \\
R_{12} 42 &= \beta'_7 24 + \beta'_8 42, \tag{4.4}
\end{align*}
$$

$$
\delta u = u_1 - u_2. \tag{4.5}
$$
\[ B' = \frac{(\delta u - 1 - c_2)(\delta u - c_2)}{(-1 + \delta u + c_1 - c_2)(\delta u + c_1 - c_2)}, \quad C' = \frac{b_2(1 + c_2)d_1\bar{e}_1}{(-1 + \delta u + c_1 - c_2)(\delta u + c_1 - c_2)e_2}, \]
\[ D' = -E' = -\frac{b_2(\delta u - 1 - c_2)e_1}{(-1 + \delta u + c_1 - c_2)(\delta u + c_1 - c_2)}, \]
\[ F' = \frac{\bar{a}_1\bar{b}_1(1 + c_2)d_2}{a_2(-1 + \delta u + c_1 - c_2)(\delta u + c_1 - c_2)}, \]
\[ G' = \frac{(\delta u + c_1)(1 + \delta u + c_1)}{(-1 + \delta u + c_1 - c_2)(\delta u + c_1 - c_2)}, \quad V' = \Gamma', \quad T' = -P', \]
\[ H' = -I' = -\frac{(\delta u + c_1)\bar{b}_1(1 + c_2)}{a_2(-1 + \delta u + c_1 - c_2)(\delta u + c_1 - c_2)}, \quad \Xi' = N', \]
\[ L' = \frac{(\delta u + c_1)(\delta u - 1 - c_2)}{(-1 + \delta u + c_1 - c_2)(\delta u + c_1 - c_2)}, \quad \Gamma' = \frac{\delta u}{(-1 + \delta u + c_1 - c_2)}, \quad \Psi' = \Theta', \]
\[ N' = \frac{b_2d_1d_2}{a_2(1 + c_2)} \frac{(\delta u + c_1)(1 + c_2)}{(-1 + \delta u + c_1 - c_2)(\delta u + c_1 - c_2)}, \]
\[ P' = -\frac{(-1 + \delta u + c_1 - c_2)(\delta u + c_1 - c_2)}{(1 + c_2)} \frac{1}{\bar{d}_1(\delta u + c_1 + 1 + c_2)} \frac{(\delta u + c_1 + 1 + c_2)}{(-1 + \delta u + c_1 - c_2)(\delta u + c_1 - c_2)}, \]
\[ \alpha' = \frac{\delta u - 1 - c_2}{1 + \delta u + c_1 - c_2}, \quad \alpha' = \frac{a_2c_2}{a_1(1 - \delta u - c_1 + c_2)}, \]
\[ \alpha' = \frac{\delta u(\delta u + c_1)}{(1 + \delta u + c_1 - c_2)(\delta u + c_1 - c_2)}, \quad \beta' = \frac{\delta u + c_1}{-1 + \delta u + c_1 - c_2}, \]
\[ \beta' = \frac{\delta u(\delta u - 1 - c_2)}{(-1 + \delta u + c_1 - c_2)(\delta u + c_1 - c_2)}, \quad \beta' = \frac{\delta u + c_1}{-1 + \delta u + c_1 - c_2}, \] (4.6)

Moreover, the overall scalar factor multiplying the R-matrix is

\[ \Phi_{12} = \Phi(a_1, b_1, c_1, d_1, e_1, u_1, a_2, b_2, c_2, d_2, e_2, u_2), \] (4.8)

and

\[ \Phi_{12} = \Phi(\bar{a}_1, b_1, -c_1 - 1, \bar{d}_1, \bar{e}_1, u_1 + c_1, a_2, b_2, c_2, d_2, e_2, u_2), \] (4.9)

where the scalar function $\Phi$ appearing in (4.8), (4.9) is fixed by requiring crossing symmetry and unitarity to hold. Notice also that, in the parametrization given by (3.14), one has
\[ a_m = -1, \quad \bar{b}_m = -\alpha \left(1 - \frac{x_m^+}{x_m} \right), \quad \tilde{a}_m = \frac{ix_m^-}{\alpha}, \quad \tilde{e}_m = -1 - i(x_m^+ - x_m^-). \quad (4.10) \]

with \( m = 1, 2 \).

By making use of the above expressions, one can show that the crossing condition (4.1) reduces to the following equation for the scalar factor \( \Phi \):

\[ \Phi_{12} \Phi_{12} = \frac{(c_2 + u_2 - u_1 - c_1)(1 + c_2 + u_2 - u_1 - c_1)}{(-u_2 + u_1 + c_1)(1 - u_2 + u_1 + c_1)} \equiv f(c_1, c_2, u_1, u_2). \quad (4.11) \]

In terms of the parameters \( x^\pm \) (3.14) and (4.10) this reads

\[ \Phi(x_1^+, x_2^+, u_1, u_2) \Phi \left( \frac{\alpha^2}{x_1^+}, x_2^+, u_1 - 1 - i(x_1^+ - x_1^-), u_2 \right) \]
\[ = \frac{(u_2 - u_1 - i(x_1^- - x_2^- - x_1^+ + x_2^+))(u_2 - u_1 - i(i + x_1^- - x_2^- - x_1^+ + x_2^+))}{(u_2 - u_1 - i(x_1^- - x_1^+))(1 + u_2 - u_1 - ix_1^- + ix_1^+)} \quad (4.12) \]

It is quite interesting to notice what happens when considering antiparticles in the second factor of the tensor product. In fact, the condition on the universal R-matrix complementary to (4.2) is (for an invertible antipode map)

\[ (\mathbb{1} \otimes S^{-1}) R = R, \quad (4.13) \]

which means that in the second factor of the tensor product we have to analyze the equation

\[ S^{-1}(\bar{\mathfrak{J}}) = \tilde{\mathfrak{C}}^{-1} \tilde{\mathfrak{C}}, \quad (4.14) \]

complementary to (3.9). In order to do this, we notice that \( S^2 = \mathbb{1} \) on the level zero of the Yangian (since at level zero the antipode just changes the sign to any generator). This means that the inverse of the antipode equals the antipode itself at level zero, and the condition (4.14) coincides with (3.9) at level zero, which fixes

\[ \tilde{\mathfrak{C}} = \mathfrak{C}, \quad \tilde{c} = c = -c - 1. \quad (4.15) \]

At Yangian-level instead, one has the following:

\[ S^2(\xi_{i,1}^\pm) = \xi_{i,1}^\pm - \xi_{i,0}^\pm \quad (4.16) \]

This means

\[ S^{-1}(\xi_{i,1}^\pm) = S(\xi_{i,1}^\pm) - \xi_{i,0}^\pm \quad (4.17) \]
which in turn implies that (4.14) is solved for all generators by the following requirement:

\[ \bar{u} = \bar{u} + 1 = u + c + 1. \]  

(4.18)

We then consider the R-matrix, and indeed we find that it satisfies the analog of Eq. (4.1), this time for the inverse-antipodal representation we just found, namely

\[ (\bar{1} \otimes C^{-1}) \Phi_{12} R_{12}^{st} (\bar{1} \otimes C) \Phi_{12} R_{12} = \bar{1} \otimes 1. \]  

(4.19)

The R-matrix \( R_{12} \) is obtained by substituting the antiparticle representation (4.15), (4.18) in the second factor of the tensor product. One can show that the relation (4.19) amounts to the following requirement for the overall scalar factor \( \Phi \) of (4.8):

\[ \Phi_{12} \Phi_{12} = \left[ 1 + \frac{c_1(1 + c_1)}{u_2 - u_1} - \frac{(2 + c_1)(1 + c_1)}{u_2 - u_1 + 1} \right]^{-1} \equiv g(c_1, c_2, u_1, u_2). \]  

(4.20)

with

\[ \Phi_{12} = \Phi(a_1, b_1, c_1, d_1, e_1, u_1, \tilde{a}_2, \tilde{b}_2, -c_2 - 1, \tilde{d}_2, \tilde{e}_2, u_2 + c_2 + 1). \]

Let us comment on consistency of the crossing relation, double crossing and unitarity. First, if we apply the crossing transformation (3.11) on particle 1 one more time to (4.11), we schematically obtain (explicitly displaying only the variables affected by the transformation)

\[ \Phi(-c_1 - 1, u_1 + c_1) \Phi(c_1, u_1 - 1) = f(-c_1 - 1, c_2, u_1 + c_1, u_2), \]  

(4.21)

with no apparent contradiction with (4.11). Similarly, applying one more time the crossing (4.18) on particle 2 to (4.20) results in

\[ \Phi(-c_2 - 1, u_2 + c_2 + 1) \Phi(c_2, u_2 + 1) = g(c_1, -c_2 - 1, u_1, u_2 + c_2 + 1), \]  

(4.22)

with no apparent contradiction with (4.20). Finally, if we consider the unitarity relation (2.20) and the two crossing relations (4.11) and (4.20), we can deduce both

\[ \Phi_{12}^{-1} \Phi_{12}^{-1} = \Phi_{21} \Phi_{21} = f^{-1}(c_1, c_2, u_1, u_2) \]  

(4.23)

and at the same time

\[ \Phi_{21} \Phi_{21} = g(c_2, -c_1 - 1, u_2, u_1 + c_1). \]  

(4.24)

The latter formula is obtained by exchanging 1 and 2 in (4.20), and subsequently sending \( c_1 \to -c_1 - 1 \) and \( u_1 \to u_1 + c_1 \), in such a way that \( \Phi_{21} \Phi_{21} = \Phi(c_1, u_1) \Phi(-c_1 - 1, u_1 + c_1 + 1) \).
precisely becomes $\Phi_{21} \Phi_{21} = \Phi(c_1, u_1) \Phi(-c_1 - 1, u_1 + c_1)$. By taking into account the explicit form of the functions $f$ and $g$, one can check that (4.23) and (4.24) are consistent with each other.

Notice that we can find a solution to (4.11) and (4.20) simultaneously, namely

$$
\Phi_{12}^{(0)} = \frac{\Gamma(1 + c_2 - u_1 + u_2) \Gamma(2 + c_2 - u_1 + u_2) \Gamma(-1 - c_1 - u_1 + u_2) \Gamma(-c_1 - u_1 + u_2)}{\Gamma(-u_1 + u_2) \Gamma(1 - u_1 + u_2) \Gamma(c_2 - c_1 - u_1 + u_2) \Gamma(1 + c_2 - c_1 - u_1 + u_2)}.
$$

However, the above factor is not unitary. In fact, solving (4.11) and (4.20) simultaneously only implies for instance

$$
\Phi_{12}^{(0)} \Phi_{21}^{(0)} \Phi_{12}^{(0)} = 1,
$$

which is not equivalent to the relation (2.20) (although it is compatible with it). A formal solution of (4.11) and (4.20) which is also unitary is then obtained as

$$
\Phi_{12} = \sqrt{\frac{\Phi_{12}^{(0)}}{\Phi_{21}^{(0)}}}.
$$

As a further check, we have computed the scalar factor as it comes from evaluating the universal R-matrix [34, 35] on the all-level Yangian representation (2.5), and found that, after unitarization, it precisely coincides with (4.26). More precisely, the universal R-matrix reads

$$
R = R_E R_H R_F,
$$

where

$$
R_E = \prod_{n \geq 0} \exp(-e_n \otimes f_{-n-1}),
$$

$$
R_F = \prod_{n \geq 0} \exp(-f_n \otimes e_{-n-1}),
$$

$$
R_H = \prod_{n \geq 0} \exp \left\{ \text{Res}_{u=v} \left[ \frac{d}{du} (\log H^+(u)) \otimes \log H^-(v + 2n + 1) \right] \right\}.
$$

One defines

$$
\text{Res}_{u=v} (A(u) \otimes B(v)) = \sum_k a_k \otimes b_{-k-1}
$$

for $A(u) = \sum_k a_k u^{-k-1}$ and $B(u) = \sum_k b_k u^{-k-1}$, and the so-called Drinfeld’s currents are given by

$$
E^\pm(u) = \pm \sum_{n \geq 0} e_n u^{-n-1}, \quad F^\pm(u) = \pm \sum_{n \geq 0} f_n u^{-n-1},
$$

$$
H^\pm(u) = 1 \pm \sum_{n < 0} h_n u^{-n-1}.
$$
The arrows on the products indicate the ordering one has to follow in the multiplication, and are a consequence of the normal ordering prescription for the root factors in the universal R-matrix (see Khoroshkin-Tolstoy \cite{34}). In order to determine the scalar factor, we can simply act on the state $1\bar{1}$. The root factors $R_E$ and $R_F$ act as identity, and all one is left with is calculating the contribution from the Cartan part $R_H$. This gives

$$R_H \ 1\bar{1} = \frac{\Gamma(u_1 - u_2)\Gamma(1 + u_1 - u_2)\Gamma(c_1 - c_2 + u_1 - u_2)\Gamma(1 + c_1 - c_2 + u_1 - u_2)}{(\Gamma(1 + c_1 + u_1 - u_2)\Gamma(2 + c_1 + u_1 - u_2)\Gamma(-c_2 + u_1 - u_2)\Gamma(-c_2 + u_1 - u_2)}.$$  

Unitarizing this result in the fashion (4.26) produces a scalar factor which coincides with what is obtained by unitarizing $\Phi^{(0)}_1$.  

We finish by noticing that the R-matrix $R_{1\bar{2}}$ can easily be obtained by substituting the appropriate representations in the two tensor product factors. Furthermore, in order to obtain the physical S-matrix one needs to apply the graded permutation operator to any R-matrix from this paper, i.e. $S = PR$.  

It is convenient to write down the R-matrix $R_{1\bar{2}}$, including the crossing-unitarity factor, in terms of the variable $\vartheta$ (3.12). We define $x = (\vartheta_2 - \vartheta_1)/\pi i$, in terms of which the unitary and crossing symmetric S-matrix reads

$$R_{1\bar{2}}(x) = \left\{ P_2 + \frac{(x/2 + \bar{c})(x/2 + \bar{c} + 1)}{(x/2 - \bar{c})(x/2 - \bar{c} - 1)}P_1 + \frac{x/2 + \bar{c} + 1}{x/2 - \bar{c} - 1}P_3 \right\}. \quad (4.33)$$

Here we have defined

$$\bar{c} = \frac{c_1 + c_2}{2}, \quad \delta c = \frac{c_2 - c_1}{2}.$$  

In appendix B we report the structure of poles of the R-matrix in the physical sheet $0 < x < 1$.  

5 Conclusions

In this paper we have constructed a rational R-matrix with \textit{sl}(2|1) Yangian symmetry in a four dimensional representation and its conjugate (antiparticle). Each of these representations depend on three additional continuum parameters. The calculation was done by working out of explicit form of the Yangian representation in the so-called Drinfeld’s second realization, and by making use of the associated Hopf-algebra coproducts. We also succeeded to interpret the found R-matrix as a relativistic scattering S-matrix. We have derived the unitarity and crossing relations and we have solved them, determining in this way the overall scalar factor of the R-matrix (apart from possible CDD factors). The scalar factor we single out corresponds to the unitarization of the scalar factor coming from the universal R-matrix.
Let us point out further steps of investigation. The first necessary step in the procedure of exactly solving an integrable model is working out the Bethe equations, for instance by means of the algebraic or coordinate Bethe ansatz. We plan to derive these equations and to utilize them as a starting point for the investigation of the spectrum of integrable alternating spin chains with $\mathfrak{sl}(2|1)$ symmetry in four dimensional representations, their thermodynamics and their conformal limit spectrum. Especially interesting is the question about the dependence of thermodynamic and conformal properties of the spin chain on the continuum parameters of the four dimensional representation.

One of the possible physical interpretations of the obtained S-matrix can be found in the context of SSSG model with the $\mathfrak{osp}(3|2)/\mathfrak{osp}(2|2)$ symmetry. A check of such correspondence can be attempted by using thermodynamic Bethe ansatz techniques based on the Bethe equations. Another important question is the investigation of the S-matrix poles structure in the physical strip and bound state spectrum encoded in these poles.

Finally, it is very interesting to notice how the R-matrices and representations we have obtained are very similar to the ones one encounters in context of AdS/CFT integrability, and the comparison can be very fruitful in terms of a better understanding of the features of the AdS/CFT Yangian in Drinfeld’s second realization for various choices of the Dynkin diagram.

We hope to return to this and other questions in further publications.

6 Acknowledgements

A. Babichenko is thankful to the Einstein center of Weizmann Institute for support. A. Torrielli thanks the UK EPSRC for funding under grant EP/H000054/1 during the initial stage of this work, and Nordita-Stockholm for hospitality during a subsequent stage of this work.

7 Appendix A

We report here below the R-matrix intertwining two conjugate representations of section 3. Defining

$$\delta u = u_1 - u_2,$$  \hspace{1cm} (7.1)

one finds
Below we analyze the structure of singularities of the R-matrix \((4.33)\) in the physical strip \(0 < x < 1\). First, we start by listing the poles and zeroes of the the scalar factor. One can see that most of the poles and zeroes under the square root are double poles and zeroes, and those poles which are not actually cancel out, such that the remaining poles and zeroes after taking the square root are all simple ones, and we are not left with square root branch cuts.

The set of potential poles of the scalar factor\(^7\) is

\[
\{2\delta c - 2n\} \cup \{-2\delta c - 2n\} \cup \{2\tilde{c} + 2n + 2\} \cup \{-2\tilde{c} + 2n - 2\}, \quad n = 1, 2, \ldots \tag{8.1}
\]

and the set of zeroes is

\[
\{-2\delta c + 2n\} \cup \{2\delta c + 2n\} \cup \{-2\tilde{c} - 2n - 2\} \cup \{2\tilde{c} - 2n + 2\}, \quad n = 1, 2, \ldots \tag{8.2}
\]

The analysis of potential poles and their cancellation with zeroes in the physical strip leads to the following result, which we describe in each scattering channel \(P_{12,3}\) separately.

\(^7\)A potential pole is such that it becomes a physical pole if it belongs to the physical strip.
In what follows, by \( [a] \) (respectively, \( \{a\} \)) we mean the integer (respectively, fractional) part of \( a \). For convenience we also define the following functions:

\[
\begin{align*}
m_1(a) &= \min(-[a], [a] + 2, 0), \quad m_2(a) = \max(-[a] - 3, [a] - 1, 1), \\
m_3(a) &= \min(-[a], [a] + 2, -1), \quad m_4(a) = \max(-[a] - 3, [a] - 1, 0), \\
m_5(a) &= \max(-[a] - 5, [a] - 1, 0), \quad m_6(a) = \min(-[a], [a] + 2, -2) \\
\end{align*}
\]  

(8.3)

**P\(_2\) channel**

The poles of the \( P_2 \) channel are defined purely by the scalar factor. For generic values of \( c_1 \) and \( c_2 \), i.e. when neither of them is integer, nor their sum or difference, there are poles in the physical strip

- at \( \{c_2 - c_1\} \) if \( c_2 - c_1 > 2 \) and \( |c_2 - c_1| \) is even
- at 1 – \( \{c_2 - c_1\} \) if \( c_2 - c_1 < -2 \) and \( |c_2 - c_1| \) is odd
- at \( \{c_2 + c_1\} \) if \( c_2 + c_1 < -3 \) and \( |c_2 + c_1| \) is even
- at 1 – \( \{c_2 + c_1\} \) if \( c_2 + c_1 > -1 \) and \( |c_2 + c_1| \) is odd

Notice that there are no poles if \(-5/2 \leq c_1, c_2 \leq 1/2\). As soon as \( \{c_2 - c_1\} \) (respectively, \( \{c_2 + c_1\} \)) becomes integer, the poles at \( \{c_2 - c_1\} \) and 1 – \( \{c_2 - c_1\} \) (respectively, \( \{c_2 + c_1\} \) and 1 – \( \{c_2 + c_1\} \)) fall out of the physical sheet (see below for a remark about the special cases \( \{c_2 + c_1\} = 0, -1, -2\)).

The picture becomes more complicated in the case when either \( c_1 \) or \( c_2 \) is integer, but not both simultaneously. If \( c_1 \) is integer but \( c_2 \) is not, the picture of poles in the physical strip will be modified as follows.

- the pole at \( \{c_2 - c_1\} \) can exist if \( [c_2 - c_1] \) is even and \( c_2 - c_1 > 2 \), and it is a double pole coinciding with the pole at \( \{c_2 + c_1\} \) if \( c_1 < m_6(c_2) \), it is a simple pole if \( m_6(c_2) \leq c_1 < -[c_2] \), and it is cancelled by a zero if \( -[c_2] \leq c_1 \)
- the pole at 1 – \( \{c_2 - c_1\} \) can exist if \( [c_2 - c_1] \) is odd and \( c_2 - c_1 < -2 \), and it is a double pole coinciding with the pole at 1 – \( \{c_2 + c_1\} \) if \( m_5(c_2) < c_1 \), it is a simple pole if \( -[c_2] - 5 < c_1 \leq m_5(c_2) \), and it is cancelled by a zero if \( c_1 \leq -[c_2] - 5 \)
- the pole at \( \{c_2 + c_1\} \) can exist if \( [c_2 + c_1] \) is even and \( c_2 + c_1 < -3 \), it is a double pole coinciding, as we said above, with the pole at \( \{c_2 - c_1\} \) if \( c_1 < m_6(c_2) \), it is a simple pole if \( m_6(c_2) \leq c_1 \leq -[c_2] + 2 \), and it is cancelled by a zero if \( [c_2] + 2 \leq c_1 \)
- the pole at 1 – \( \{c_2 + c_1\} \) can exist if \( [c_2 + c_1] \) is odd and \( c_2 + c_1 > -1 \), it is a double pole coinciding, as we said above, with the pole at 1 – \( \{c_2 - c_1\} \) if \( m_5(c_2) < c_1 \), it is a simple pole if \( c_2 - 1 < c_1 \leq m_5(c_2) \), and it is cancelled by a zero if \( c_1 \leq -[c_2] - 1 \)

If \( c_2 \) is integer but \( c_1 \) is not:
• the pole at \( \{c_2 - c_1\} \) can exist if \([c_2 - c_1]\) is even and \(c_2 - c_1 > 2\), it is a double pole coinciding with the pole at \(1 - \{c_2 + c_1\}\) if \(m_5(c_1) < c_2\), it is a simple pole if \(-5 - [c_1] < c_2 \leq m_5(c_1)\), and it is cancelled by a zero if \(c_2 \leq -5 - [c_1]\)

• the pole at \(1 - \{c_2 - c_1\}\) can exist if \([c_2 - c_1]\) is odd and \(c_2 - c_1 < -2\), it is a double pole coinciding with the pole at \(\{c_2 + c_1\}\) if \(c_2 < m_6(c_1)\), it is a simple pole if \(m_6(c_1) \leq c_2 < -[c_1]\), and it is cancelled by a zero if \(-[c_1] \leq c_2\)

• the pole at \(\{c_2 + c_1\}\) can exist if \([c_2 + c_1]\) is even and \(c_2 + c_1 < -3\), as we said, it is a double pole coinciding with the pole at \(1 - \{c_2 - c_1\}\) if \(c_2 < m_6(c_1)\), it is a simple pole if \(m_6(c_1) \leq c_2 < [c_1] + 2\), and it is cancelled by a zero if \([c_1] + 2 \leq c_2\)

• the pole at \(1 - \{c_2 + c_1\}\) can exist if \([c_2 + c_1]\) is odd and \(c_2 + c_1 > -1\), it becomes a double pole coinciding, as we said above, with the pole at \(\{c_2 - c_1\}\) if \(m_5(c_1) < c_2\), it is a simple pole if \([c_1] - 1 < c_2 \leq m_5(c_1)\), and it is cancelled by a zero if \(c_2 \leq [c_1] - 1\)

The spectral decomposition of channels \(P_1\) and \(P_3\) is only slightly different from the one of of the \(P_2\) channel.

\(P_1\) channel

The factor before the projector \(P_1\) cancels one pole in the set of poles of the scalar factor, and adds to the set two additional poles and one zero. For generic values of \(c_1\) and \(c_2\) (see remarks above), in the \(P_1\) channel there are physical strip poles

• at \(\{c_2 - c_1\}\) if \(c_2 - c_1 > 2\) and \([c_2 - c_1]\) is even

• at \(1 - \{c_2 - c_1\}\) if \(c_2 - c_1 < -2\) and \([c_2 - c_1]\) is odd

• at \(\{c_2 + c_1\}\) if \(c_2 + c_1 < 1\) and \([c_2 + c_1]\) is even

• at \(1 - \{c_2 + c_1\}\) if \(c_2 + c_1 > 1\) and \([c_2 + c_1]\) is odd

If \(c_1\) is integer, but not \(c_2\), the picture of poles in the \(P_1\) channel in the physical strip is the following:

• the pole at \(\{c_2 - c_1\}\) can exist if \([c_2 - c_1]\) is even and \(c_2 - c_1 > 2\), it is a double pole coinciding with the pole at \(\{c_2 + c_1\}\) if \(c_1 < m_1(c_2)\), it is a simple pole if \(m_1(c_2) \leq c_1 < -[c_2]\), and it is cancelled by a zero if \(-[c_2] \leq c_1\)

• the pole at \(1 - \{c_2 - c_1\}\) can exist if \([c_2 - c_1]\) is odd and \(c_2 - c_1 < -2\), it is a double pole coinciding with the pole at \(1 - \{c_2 + c_1\}\) if \(m_2(c_2) < c_1\), it is a simple pole if \(-[c_2] - 3 < c_1 \leq m_2(c_2)\), and it is cancelled by a zero if \(c_1 \leq -[c_2] - 3\)

• the pole at \(\{c_2 + c_1\}\) can exist if \([c_2 + c_1]\) is even and \(c_2 + c_1 < 1\), it is a double pole coinciding, as we said, with the pole at \(\{c_2 - c_1\}\) if \(c_1 < m_1(c_2)\), it is a simple pole if \(m_1(c_2) \leq c_1 < [c_2] + 2\), and it is cancelled by a zero if \([c_2] + 2 \leq c_1\)
• the pole at \(1 - \{c_2 + c_1\}\) can exist if \([c_2 + c_1]\) is odd and \(c_2 + c_1 > 1\), it is a double pole coinciding, as we said, with the pole at \(1 - \{c_2 - c_1\}\) if \(m_2(c_2) < c_1\), it is a simple pole if \([c_2] - 1 < c_1 \leq m_2(c_2)\), and it is cancelled by a zero if \(c_1 \leq [c_2] - 1\).

Conversely, if \(c_2\) is integer, but \(c_1\) is not, the situation is the following:

• the pole at \(\{c_2 - c_1\}\) can exist if \([c_2 - c_1]\) is even and \(c_2 - c_1 > 2\), it is a double pole coinciding with the pole at \(1 - \{c_2 + c_1\}\) if \(m_2(c_1) < c_2\), it is a simple pole if \(-[c_1] - 3 < c_2 \leq m_2(c_1)\), and it is cancelled by a zero if \(c_2 \leq -[c_1] - 3\).

• the pole at \(1 - \{c_2 - c_1\}\) can exist if \([c_2 - c_1]\) is odd and \(c_2 - c_1 < -2\), it is a double pole coinciding with the pole at \(\{c_2 + c_1\}\) if \(c_2 < m_1(c_1)\), it is a simple pole if \(m_1(c_1) \leq c_2 \leq -[c_1]\), and it is cancelled by a zero if \(-[c_1] \leq c_2\).

• the pole at \(\{c_2 + c_1\}\) can exist if \([c_2 + c_1]\) is even and \(c_2 + c_1 < 1\), it is a double pole coinciding, as we said in the previous item, with the pole at \(1 - \{c_2 - c_1\}\) if \(c_2 < m_1(c_1)\), it is a simple pole if \(m_1(c_1) \leq c_2 < [c_1] + 2\), and it is cancelled by a zero if \([c_1] + 2 \leq c_2\).

• the pole at \(1 - \{c_2 + c_1\}\) can exist if \([c_2 + c_1]\) is odd and \(c_2 + c_1 > 1\), it is a double pole coinciding with the pole at \(\{c_2 - c_1\}\) if \(m_2(c_1) < c_2\), it is a simple pole if \([c_1] - 1 < c_2 \leq m_2(c_1)\), and it is cancelled by a zero if \(c_2 \leq [c_1] - 1\).

**P\(_3\)** **channel**

For generic values of \(c_1\) and \(c_2\), the physical strip poles are:

• at \(\{c_2 - c_1\}\) if \(c_2 - c_1 > 2\) and \([c_2 - c_1]\) is even

• at \(1 - \{c_2 - c_1\}\) if \(c_2 - c_1 < -2\) and \([c_2 - c_1]\) is odd

• at \(\{c_2 + c_1\}\) if \(c_2 + c_1 < -1\) and \([c_2 + c_1]\) is even

• at \(1 - \{c_2 + c_1\}\) if \(c_2 + c_1 > -1\) and \([c_2 + c_1]\) is odd

For integer \(c_1\) and not integer \(c_2\) their structure in the physical strip is modified as follows:

• the pole at \(\{c_2 - c_1\}\) can exist if \([c_2 - c_1]\) is even and \(c_2 - c_1 > 2\), and it is a double pole coinciding with the pole at \(\{c_2 + c_1\}\) if \(c_1 < m_3(c_2)\), it is a simple pole if \(m_3(c_2) \leq c_1 < -[c_2]\), and it is cancelled by a zero if \(-[c_2] \leq c_1\).

• the pole at \(1 - \{c_2 - c_1\}\) can exist if \([c_2 - c_1]\) is odd and \(c_2 - c_1 < -2\). The situation here is different for \([c_2] < -3\) and for \([c_2] \geq -3\). If \([c_2] \geq -3\) it is a double pole coinciding with the pole at \(1 - \{c_2 + c_1\}\) if \(m_5(c_2) < c_1\), and it is a simple pole if \(-[c_2] - 3 < c_1 \leq m_5(c_2)\). If \([c_2] < -3\) it is a double pole coinciding with the pole at \(1 - \{c_2 + c_1\}\) if \(-[c_2] - 3 < c_1\), and it is a simple pole if \(m_5(c_2) < c_1 \leq -[c_2] - 3\).
the projectors \( P \) on the physical sheet. A separate analysis is however required for the poles at \( \{c_2 - c_1\} \) if \( c_1 < m_3(c_2) \), it is a simple pole if \( m_3(c_2) \leq c_1 < |c_2| + 2 \), and it is cancelled by a zero if \( |c_2| + 2 \leq c_1 \).

- the pole at \( 1 - \{c_2 + c_1\} \) can exist if \( c_2 + c_1 \) is odd and \( c_2 + c_1 > -1 \), it is a double pole coinciding with the pole at \( 1 - \{c_2 - c_1\} \) if \( m_5(c_2) < c_1 \), it is a simple pole if \( |c_2| - 1 < c_1 \leq m_5(c_2) \), and it is cancelled by a zero if \( c_1 \leq |c_2| - 1 \).

Conversely, if \( c_2 \) is integer and \( c_1 \) is not

- the pole at \( \{c_2 - c_1\} \) can exist if \( c_2 - c_1 \) is even and \( c_2 - c_1 > 2 \). In this case there are two subcases: \( |c_1| - 3 \) and \( |c_1| \geq 3 \). If \( |c_1| \geq 3 \) it is a double pole coinciding with the pole at \( 1 - \{c_2 + c_1\} \) if \( m_5(c_1) < c_2 \), it is a simple pole if \( -|c_1| - 3 < c_2 \leq m_5(c_1) \). If \( |c_1| < 3 \) it is a double pole coinciding with the pole at \( 1 - \{c_2 + c_1\} \) if \( -|c_1| - 3 < c_2 \), and it is a simple pole if \( m_3(c_1) < c_2 \leq -|c_1| - 3 \).

- the pole at \( 1 - \{c_2 - c_1\} \) can exist if \( c_2 - c_1 \) is odd and \( c_2 - c_1 > -2 \), it is a double pole coinciding with the pole at \( \{c_2 + c_1\} \) if \( c_2 < m_3(c_1) \), it is a simple pole if \( m_3(c_1) \leq c_2 < -|c_1| \), and it is cancelled by a zero if \( -|c_1| \leq c_2 \).

- the pole at \( \{c_2 + c_1\} \) can exist if \( c_2 + c_1 \) is even and \( c_2 + c_1 < -1 \), it is a double pole coinciding with the pole at \( 1 - \{c_2 - c_1\} \) if \( c_2 < m_3(c_1) \), it is a simple pole if \( m_3(c_1) \leq c_2 < |c_1| + 2 \), and it is cancelled by a zero if \( |c_1| + 2 \leq c_2 \).

- the pole at \( 1 - \{c_2 + c_1\} \) can exist if \( c_2 + c_1 \) is odd and \( c_2 + c_1 > -1 \), it is a double pole coinciding with the pole at \( \{c_2 - c_1\} \) if \( m_5(c_1) < c_2 \), it is a simple pole if \( |c_1| - 1 < c_2 \leq m_5(c_1) \), and it is cancelled by a zero if \( c_2 \leq |c_1| - 1 \).

For all the channels, as soon as \( \{c_2 - c_1\} \) (respectively, \( \{c_2 + c_1\} \)) becomes integer, the poles at \( \{c_2 - c_1\} \) and \( 1 - \{c_2 - c_1\} \) (respectively, \( \{c_2 + c_1\} \) and \( 1 - \{c_2 + c_1\} \)) fall out of the physical sheet. A separate analysis is however required for \( c_1 + c_2 = 0, -1, -2 \), since the projectors \( P_1, P_2, P_3 \) are singular in this case.

Let us notice that the presence of double poles in the physical strip for specific values of the representation parameters might be an indication of the Coleman-Weinberg mechanism [36]. One can also expect our R-matrix, which we directly obtained form the Yangian construction, to be a bootstrap R-matrix for particles in the fundamental three-dimensional representation of \( \mathfrak{sl}(2|1) \). The double poles we observe should then be subject to a consistent multi-scattering interpretation in the related bootstrap approach [37, 38]. We reserve this point for a further investigation.

References

[1] Z. Maassarani, “\( U_q(\mathfrak{osp}(2,2)) \) Lattice Models,” J. Phys. A28 (1995) 1305 [arXiv:hep-th/9407032].

[2] F.H.S. Essler, H. Frahm, H. Saleur, “Continuum Limit of the Integrable \( \mathfrak{sl}(2|1) \) 3 \( \rightarrow \) 3 Superspin Chain,” Nucl. Phys. B712 (2005) 513 [arXiv:cond-mat/0501197].
3] W. Galleas, M.J. Martins, “Exact solution and finite size properties of the $U_q[osp(2|2m)]$ vertex model,” Nucl. Phys. B768 (2007) 219, [arXiv:hep-th/0612281].

4] H. Frahm, M.J. Martins, “Phase Diagram of an Integrable Alternating $U_q[sl(2|1)]$ Superspin Chain”, [arXiv:1202.4676].

5] C. Candu, “Continuum Limit of $gl(M|N)$ Spin Chains,” JHEP 1107, 069 (2011) [arXiv:1012.0050].

6] M. R. Gaberdiel, “An algebraic approach to logarithmic conformal field theory,” Int. J. Mod. Phys. A18 (2003) 4593 [arXiv:hep-th/0111260].

7] M. Flohr, “Bits and Pieces in Logarithmic Conformal Field Theory,” Int. J. Mod. Phys. A18 (2003) 4497 [arXiv:hep-th/0111228].

8] M. Bershadsky, S. Zhukov, A. Vaintrob, “PSL(n|n) Sigma Model as a Conformal Field Theory,” Nucl. Phys. B559 (1999) 205 [arXiv:hep-th/9902180].

9] N. Berkovits, C. Vafa, E. Witten, “Conformal Field Theory of AdS Background with Ramond-Ramond Flux,” JHEP 9903, 018, (1999) [arXiv:hep-th/9902098].

10] I. Bena, J. Polchinski and R. Roiban, “Hidden symmetries of the AdS(5) x S**5 superstring,” Phys. Rev. D 69 (2004) 046002 [hep-th/0305116].

11] Z.S. Bassi, A. LeClair, “The Exact S-Matrix for an $osp(2|2)$ Disordered System,” Nucl. Phys. B578 (2000) 577 [arXiv:hep-th/9911105].

12] S. Guruswamy, A. LeClair, A.W.W. Ludwig, “$gl(N|N)$ Super-Current Algebras for Disordered Dirac Fermions in Two Dimensions” Nucl. Phys. B583 (2000) 475 [arXiv:cond-mat/9909143].

13] A. Babichenko, “Conformal invariance and quantum integrability of sigma models on symmetric superspaces,” Phys. Lett. B648 2007 254 [arXiv:hep-th/0611214].

14] N. Beisert et al., “Review of AdS/CFT Integrability: An Overview,” [arXiv:1012.3982].

15] N. Beisert, “The $su(2|2)$ dynamic S-matrix,” Adv. Theor. Math. Phys. 12 (2008) 945 [hep-th/0511082].

16] J.A. Minahan, K. Zarembo, “The Bethe Ansatz for superconformal Chern-Simons,” JHEP 0809, 040, (2008) [arXiv:0806.3951]

17] O.O. Sax, B. Stefanski, jr., “Integrability, spin-chains and the $AdS_3/CFT_2$ correspondence,” [arXiv:1106.2558]

18] A. Babichenko, B. Stefanski, jr., K. Zarembo, “Integrability and the $AdS(3)/CFT(2)$ correspondence,” JHEP 1003, 058, (2010) [arXiv:0912.1723]

19] V. Bazhanov, Talk at the Conference on Integrability in Gauge and String Theory, Max Planck Institute for Gravitational Physics (Albert-Einstein Institute), Potsdam, 29 June - 3 July 2009, [http://int09.aei.mpg.de/];

20] J. Gruneberg, “On the construction and solution of $U_q(\hat{gl}(2,1;\mathbb{C}))$ - symmetric models,” Nucl. Phys. B568 (2000) 594.

21] J. Gruneberg, “On exact solution of models based on non-standard representations,” Commun. Math. Phys. 206 (1999) 383.

22] H. Saleur, B. Pozsgay, “Scattering and duality in the two dimensional OSP(2|2) Gross Neveu and sigma models,” JHEP 1002, 008, (2010) [arXiv:0910.0637].
[23] B. Hoare, T.J. Hollowood, J. L. Miramontes, “A Relativistic Relative of the Magnon S-Matrix,” JHEP 1111, 048, (2011) [arXiv:1107.0628].
[24] F. Spill, “Yangians in Integrable Field Theories, Spin Chains and Gauge-String Dualities,” arXiv:1201.1884 [hep-th].
[25] A. J. Bracken, M. D. Gould, Y. -Z. Zhang and G. W. Delius, “Solutions of the quantum Yang-Baxter equation with extra nonadditive parameters,” J. Phys. A A 27 (1994) 6551 [hep-th/9405138].
[26] G. W. Delius, M. D. Gould, J. R. Links and Y. -Z. Zhang, “Solutions of the Yang-Baxter equation with extra nonadditive parameters. II: $U_q(\mathfrak{gl}(m|n))$,” J. Phys. A A 28 (1995) 6203 [hep-th/9411241].
[27] T. Matsumoto, S. Moriyama and A. Torrielli, “A Secret Symmetry of the AdS/CFT S-matrix,” JHEP 0709, 099 (2007) [arXiv:0708.1285 [hep-th]].
[28] Drinfeld, V. G., “A new realization of Yangians and quantized affine algebras,” Soviet. Math. Dokl. 36 (1988) 212
[29] F. Spill and A. Torrielli, “On Drinfeld’s second realization of the AdS/CFT $\mathfrak{su}(2|2)$ Yangian,” J. Geom. Phys. 59 (2009) 489 [arXiv:0803.3194 [hep-th]].
[30] A. Torrielli, “Review of AdS/CFT Integrability, Chapter VI.2: Yangian Algebra,” arXiv:1012.4005 [hep-th].
[31] A. Torrielli, “Yangians, S-matrices and AdS/CFT,” J. Phys. A A 44 (2011) 263001 [arXiv:1104.2474 [hep-th]].
[32] R. A. Janik, “The $AdS(5) \times S^5$ superstring worldsheet S-matrix and crossing symmetry,” Phys. Rev. D 73 (2006) 086006 [hep-th/0603038].
[33] J. Plefka, F. Spill and A. Torrielli, “On the Hopf algebra structure of the AdS/CFT S-matrix,” Phys. Rev. D 74 (2006) 066008 [hep-th/0608038].
[34] Khoroshkin, S. M. and Tolstoy, V. N. “Yangian double,” Lett. Math. Phys. 36 (1996) 373 [hep-th/9406194]
[35] A. Rej and F. Spill, The Yangian of $\mathfrak{sl}(n|m)$ and the universal R-matrix, JHEP 1105 (2011) 012, [arXiv:1008.0872].
[36] S. R. Coleman and H. J. Thun, “On The Prosaic Origin Of The Double Poles In The Sine-gordon S Matrix,” Commun. Math. Phys. 61 (1978) 31.
[37] G. Mussardo, “Off-Critical Statistical Models: Factorized Scattering Theories and Bootstrap Program,” Phys. Rep. 218 (1992) 215.
[38] P. Dorey, “Exact S matrices,” hep-th/9810026.