Special Grand Unification

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Abstract

We discuss new-type grand unified theories based on grand unified groups broken to their special subgroups as well as their regular subgroups. In the framework, when we construct four-dimensional (4D) chiral gauge theories, i.e., the Standard Model (SM), 4D gauge anomaly cancellation restricts the minimal number of generations of the 4D SM Weyl fermions. We show that in a six-dimensional (6D) $SU(16)$ gauge theory on $M^4 \times T^2/\mathbb{Z}_2$ one generation of the SM fermions can be embedded into a 6D bulk Weyl fermion. For the model including three chiral generations of the SM fermions, the 6D and 4D gauge anomalies on the bulk and fixed points are canceled out without exotic 4D chiral fermions.

1 Introduction

One of the most attractive ideas to construct unified theories beyond the standard model (SM) is grand unification since it appeared [1]. It is known in e.g., Ref. [2,3] that the candidates for grand unified theory (GUT) gauge groups in four-dimensional (4D) theories are only $SU(n)(n \geq 5)$, $SO(4n+2)(n \geq 2)$, and $E_6$ because of rank and type of representations. For examples, there are GUTs based on $SU(5)$ [1], $SU(6)$ [4], $SO(10)$ [5], $SO(14)$ [6], $SO(18)$ [7], $E_6$ [8] gauge groups. Recently, in Ref. [3], the author showed that the candidates for GUT gauge groups in higher dimensional theories are $SU(n)(n \geq 5)$, $SO(n)(n \geq 9)$, $USp(2n)(n \geq 4)$, $E_n(n = 6,7,8)$, $F_4$. Five-dimensional (5D) gauge theories based on e.g., $SU(5)$ [9,10], $SU(6)$ [11,12], $SU(7)$ [13], $SU(8)$ [13], $SO(10)$ [14,15], $SO(11)$ [16–20], $E_6$ [17,21] gauge groups and six-dimensional (6D) ones based on e.g., $SO(11)$ [22] are discussed.

More than half a century ago, E. Dynkin showed in Refs. [23,24] that simple Lie algebras has not only regular subalgebras but also special (or irregular) subalgebras. All regular subalgebras are obtained by deleting dots from Dynkin diagrams, while special subalgebras are not. For example, for an embedding $SU(3) \supset SU(2) \times U(1)$, its $SU(2) \times U(1)$ is a maximal regular subalgebra of $SU(3)$, while for an embedding $SU(3) \supset SU(2)$, its $SU(2)$ is a maximal special subalgebra. One of the branching rules of the regular subalgebra $SU(3) \supset SU(2) \times U(1)$ is $3 = (2)(1) \oplus (1)(-2)$, where $3$ in the left-hand side stands for an $SU(3)$ representation, and $(2)(1)$ and $1(-2)$ in the right-hand side show $SU(2) \times U(1)$ representations. (The same rule will be applied below.) One of the branching rules of the special subalgebra $SU(3) \supset SU(2)$ is $3 = (3)$. For an embedding $G \supset H$, a maximal subalgebra $H$ is a subalgebra of $G$ if there is no larger subalgebra containing it except $G$ itself. (See e.g., Refs. [3,25] in detail.)

In principle, we can construct GUTs based on Lie groups broken to not only their regular subgroups but also their special subgroups. However, at present, there seem to be no GUTs by using special embeddings, which is referred as special grand unification or special GUTs. In the following, we consider large GUT gauge groups $G$ broken to smaller GUT groups $H$, called small

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GUT gauge groups. The groups $H$ is regarded as usual GUT groups e.g., $SU(5)$, $SO(10)$, $E_6$. When we apply them for a special GUT model building, the rank of its large GUT gauge groups is usually much larger than small GUT gauge groups. Even a famous GUT review paper written by R. Slansky does not contain enough information to construct special GUTs. Recently, simple Lie algebras with up to rank-15 and $D_{16}$ and their regular and special subalgebras are given in Ref. 3, where the information in its tables is not enough to survey special GUTs.

We can find candidates for large GUT gauge groups in special GUTs by using usual methods about branching rules of Lie algebras under their maximal subalgebras discussed in Refs. 3,25. Under usual requirements for their small GUT gauge groups that have complex representations and contain the SM gauge group $G_{SM} := SU(3)_C \times SU(2)_L \times U(1)_Y$ the candidates for large GUT gauge groups broken to maximal special subgroups in 4D theories are e.g., $SU(\frac{n(n+1)}{2}) \supset SU(n)$, $SU(\frac{n(n-1)}{2}) \supset SU(n)$, $SU(\frac{n(n-3)}{2}) \supset SU(n)$, $SU(\frac{n(n+1)}{2}) \supset SU(n)$, $SU(n)$ $(n \geq 5)$, $SU(2^{2k}) \supset SO(4k+2)$ $(k \geq 2)$ and $SU(27) \supset E_6$. In five- or higher-dimensional theories, the requirement for the small GUT gauge groups that contain the SM gauge group $G_{SM}$ leads to additional candidates e.g., $SO(n^2-1) \supset SU(n)$ $(n \geq 5)$, $USp(2^{4\ell+1}) \supset SO(8\ell+4)$ $(\ell \geq 1)$, $USp(56) \supset E_7$, $SO(248) \supset E_8$, $SO(26) \supset F_4$, where almost the above pairs are explicitly shown by E. Dynkin in Ref. 23.

To construct unified theories beyond the SM based on a GUT gauge group $G_{GUT} \supset G_{SM}$, we must take into account for not only how to realize the SM gauge group at a vacuum but also how to embed the SM matter content into the GUT matter content. To do so, we need the information about the branching rules of Lie algebras under their subalgebras. It is known that their branching rules can be calculated by using their corresponding projection matrices introduced in Ref. 26. Many examples of the branching rules of simple Lie algebras with up to rank 8 under their regular and special semi-simple subalgebras are listed in Ref. 24. Recently, more examples up to rank-15 and $SO(32)$ including non-semi-simple subalgebras are given in Ref. 3. Other cases can be calculated by using appropriate computer programs, such as Susyno program 28 and LieART 29, and appropriate projection matrices, which can be calculated by using a usual weight diagram discussed in Refs. 3,25.

It is also known that one generation of the five SM left-handed Weyl fermions, which consist of a quark doublet, up- and down-type quarks, a lepton doublet and a charged lepton, can be embedded into an $SU(5)$ reducible representation $(10 \oplus 5)$, where we will omit left-handed if it is not needed. One generation of the SM Weyl fermions and vectorlike fermions can be embedded into an $SU(n)$ $(n \geq 6)$ reducible representation $\left(\frac{n(n-1)}{2} \oplus (n-4)\overline{3}\right)$. (Note that their 4D $SU(n)$ gauge anomaly coefficients from the 4D Weyl fermions is zero. It can be checked by using tables in Ref. 3.) One generation of the SM fermions and a SM singlet fermion can be embedded into an $SO(10)$ spinor representation 16. For the other candidates for rank-4 and -5 GUT gauge groups $SO(9)$, $USp(8)$, $F_4$, $SO(11)$, and $USp(10)$, it is examined which representations of them can contain the SM fermions in Ref. 3.

There are several good features of special GUTs. First, we can eliminate almost all unnecessary $U(1)$s even if we consider very large GUT gauge groups. For example, the rank of a gauge group $SO(32)$ is sixteen. We have the SM gauge group and twelve extra $U(1)$s if we use only regular embeddings, while we have only the SM gauge group and two extra $U(1)$s if we use a special embedding $SU(16) \supset SO(10)$. Second, by using only regular embeddings, we cannot embed the representations of the SM fermions into an $SO(32)$ vector representation 32, while by using regular and special embeddings $SO(32) \supset SU(16) \times U(1) \supset SO(10) \times U(1)$, the representations of the SM fermions can be embedded into an $SO(32)$ vector representation because an $SO(32)$ vector representation 32 is decomposed into $SO(10)$ spinor representations 16 and 10 3. Third, asymptotic freedom may be realized in special GUTs. This is because the larger contribution comes from large GUT gauge fields while the smaller contribution comes from the GUT fermions.

In this paper, we discuss special GUTs based on a large GUT gauge group $G$ broken to its maximal special subgroup $H(\subset G)$, where $H$ is regarded as a usual GUT gauge group. In 4D...
framework, to realize a 4D chiral gauge theory, the large GUT groups \( G \) and their maximal special subgroups \( H \) must contain complex representations. In the following discussion, we focus on \( G = SU(n) \) with less than rank-30 and \( H = SU(k), SO(4k + 2), E_6 \). Also, the special groups \( H \) must have rank four or greater because the rank of the SM gauge group \( G_{SM} \) is four. The conditions are satisfied by the following nine pairs of Lie groups and their maximal special subgroups: \( SU\left( \frac{n^2}{2} \right) \supset SU(n) \) (\( n = 5, 6, 7, 8 \)); \( SU\left( \frac{n^2+1}{2} \right) \supset SU(n) \) (\( n = 5, 6, 7 \)); \( SU(16) \supset SO(10) \); and \( SU(27) \supset E_6 \).

We find the following facts from Tables in Ref. [3] and its calculating methods. We consider 4D \( SU(n) \) (\( n \leq 30 \)) special GUTs whose fermions are a 4D Weyl fermion in an \( SU(n) \) reducible representation \( (n-4)n + \frac{n(n-1)}{2} \), where the representation is 4D anomaly-free. Only for the \( SU(16) \supset SO(10) \) and \( SU(27) \supset E_6 \) cases, their matter contents contain the SM chiral fermions. For the \( SU(16) \supset SO(10) \) case, the fermion is a 4D \( SU(16) \) \( (12 \times 16 \oplus 120) \) Weyl fermion, which stands for a 4D left-handed Weyl fermion in an \( SU(16) \) reducible representation \( (12 \times 16 \oplus 120) \), where the similar notation is used below. As we will see later, for 4D framework there are twelve generations of the SM Weyl fermions but fortunately there is no exotic non-SM chiral fermions due to a special property of the \( SU(16) \) complex representation \( 120 \). For the \( SU(27) \supset E_6 \) case, its matter content contains non-SM chiral fermions.

The main purpose of this paper is to show that in a 6D \( SU(16) \) special GUT on \( M^4 \times T^2/\mathbb{Z}_2 \) we can realize three generations of the 4D SM Weyl fermions from twelve 6D \( SU(16) \) 16 bulk Weyl fermions.

This paper is organized as follows. In Sec. 2 we first discuss basic properties of a special embedding \( SU(16) \supset SO(10) \) in 4D framework. After that, by using the special embedding, we construct a 6D \( SU(16) \) special GUT on \( M^4 \times T^2/\mathbb{Z}_2 \). Section 3 is devoted to a summary and discussion.

2 Special grand unification

Before we start to construct 6D special GUTs, we examine how to embed the SM Weyl fermions into \( SU(16) \) GUT multiplets in purely 4D framework. For a special embedding \( SU(16) \supset SO(10) \), an \( SU(16) \) fundamental representation \( 16 \) \( (16) \) is identified as an \( SO(10) \) spinor representation \( 16 \) \( (16) \) [3]:

\[
16 = 16 \quad (16) = (16).
\]  

Further, the \( SO(10) \) spinor representation \( 16 \) is decomposed as usual into \( G_{SM} \times U(1)_X = SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X \) representations

\[
16 = (3, 2)(-1)(1) \oplus (\overline{3}, 1)(-2)(-3) \oplus (\overline{3}, 1)(4)(1) \\
\oplus (1, 2)(3)(-3) \oplus (1, 1)(-6)(1) \oplus (1, 1)(0)(5),
\]

where the convention of their \( U(1) \) normalization is the same as one in Ref. [3]. Obviously, we can identify an \( SU(16) \) 16 fermion as one generation of the SM fermions plus a right-handed neutrino. The 4D anomaly coefficient of an \( SU(16) \) 16 fermion is non-zero, while the 4D anomaly coefficient of any \( SO(10) \) fermion is zero.

For \( SU(16) \) models instead of \( SO(10) \) ones, we cannot construct 4D anomaly-free theories with a chiral matter content by using only 4D \( SU(16) \) 16 and/or \( 16 \) Weyl fermions. We need to introduce an additional Weyl fermion to cancel out 4D \( SU(16) \) gauge anomaly. A primary candidate is a 4D \( SU(16) \) \( 120 \) Weyl fermion, where \( 120 \) is the second lowest dimensional complex representation of \( SU(16) \) and has the second smallest value of the 4D anomaly coefficient in \( SU(16) \) complex representations with at least up to \( 10^8 \) dimension. Its anomaly coefficient of \( SU(16) \) is \(-12\). By using \( SU(16) \) complex representations \( 16 \) and \( 120 \), we can realize a 4D anomaly-free chiral gauge theory whose fermion content is a 4D \( SU(16) \) \( (12 \times 16 \oplus 120) \) Weyl fermion.
From Tables in Ref. [3] for the branching rules of $SU(16) \supset SO(10)$, we find that an $SU(16)$ complex representation $\mathbf{120}$ ($\mathbf{120}$) is just identified as an $SO(10)$ real representation $\mathbf{120}$:

\[ \mathbf{120} = \mathbf{120} \quad (\mathbf{120} = \mathbf{120}). \] (2.3)

We consider a 4D $SU(16)$ special GUT whose fermion content is an $SU(16)$ ($12 \times \mathbf{16} \oplus \mathbf{120}$) Weyl fermion. When $SU(16)$ is broken to $SO(10)$ via a non-vanishing VEV of a GUT breaking Higgs scalar boson, there are twelve generations of $SO(10)$ $16$ Weyl fermions. One may wonder what kind of GUT breaking Higgs can reduce $SU(16)$ to $SO(10)$, properly. The lowest dimensional $SU(16)$ representation is $\mathbf{5440}$ ($\mathbf{5440}$) that contains singlet under the $SO(10)$ special subgroup. Its $SO(10)$ decomposition is shown in Ref. [3]:

\[ \mathbf{5440} = 4125 \oplus \mathbf{1050} \oplus \mathbf{210} \oplus \mathbf{54} \oplus \mathbf{1} \quad (\mathbf{5440} = 4125 \oplus 1050 \oplus 210 \oplus 54 \oplus 1). \] (2.4)

An additional question is how to break $SU(16)$ to $G_{\text{SM}}$. One way of achieving it is to use several GUT breaking Higgses, where we assume their proper components get non-vanishing VEVs. One example is to introduce three $SU(16)$ $\mathbf{5440}$, $\mathbf{255}$, $\mathbf{16}$ scalar fields. In this case, the non-vanishing VEV of the $SU(16)$ $\mathbf{5440}$ scalar field is responsible for breaking $SU(16)$ to $SO(10)$; the additional VEV of the $SU(16)$ $\mathbf{16}$ scalar breaks $SU(16) \supset SO(10)$ to $SU(5)$; the last VEV of the $SU(16)$ $\mathbf{255}$ scalar reduces $SU(16) \supset SO(10) \supset SU(5)$ to $G_{\text{SM}}$, where the $SU(16)$ adjoint representation $\mathbf{255}$ is decomposed into $SO(10)$ representations

\[ \mathbf{255} = \mathbf{210} \oplus \mathbf{45}. \] (2.5)

It may be possible to reduce the number of the scalar fields e.g., by using orbifold BCs in extra dimension [30,31] or the Hosotani mechanism [32,33].

We start to discuss an $SU(16)$ special GUT on 6D orbifold spacetime $M^4 \times T^2/\mathbb{Z}_2$ with the Randall-Sundrum (RS) type metric [22,34,35] given by

\[ ds^2 = e^{-2\sigma(|y|)}(\eta_{\mu\nu}dx^\mu dx^\nu + dv^2) + dy^2, \] (2.6)

where $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$, $y$ is the fifth coordinate of RS warped space, $v$ is the sixth coordinate of $S^1$ ($v \sim v + 2\pi R_6$) and $\sigma(|y|) = \sigma(-y) = \sigma(y + 2\pi R_5)$, $\sigma(y) = k|y|$ for $|y| \leq \pi R_5$. The space has four fixed points on $T^2/\mathbb{Z}_2$ at $(y_0, v_0) = (0, 0)$, $(y_1, v_1) = (\pi R_5, 0)$, $(y_2, v_2) = (0, \pi R_6)$, and $(y_3, v_3) = (\pi R_5, \pi R_6)$. The $\mathbb{Z}_2$ parity reflection around each fixed point is given by

\[ P_j : (x_\mu, y_j + y, v_j + v) \to (x_\mu, y_j - y, v_j - v), \] (2.7)

where $j = 0, 1, 2, 3$. Note that only three of them is independent because of $P_3 = P_1P_0P_2 = P_2P_0P_1$. Also, 5th and 6th dimensional translation $U_5 : (x_\mu, y, v) \to (x_\mu, y + 2\pi R_5, v)$ and $U_6 : (x_\mu, y, v) \to (x_\mu, y, v + 2\pi R_6)$ can be described as $U_5 = P_1P_0$ and $U_6 = P_2P_0$, respectively.

The orbifold BCs of a 6D $SU(16)$ $\mathbf{16}$ positive or negative Weyl bulk fermion can be written by

\[ \Psi_{16\pm}(x, y_j - y, v_j - v) = \eta_j(-i\Gamma^5\Gamma^6)P_{j\mathbf{16}}\Psi_{16\pm}(x, y_j + y, v_j + v), \] (2.8)

where the subscript of $\Psi$ ± stands for 6D chirality, $\eta_j$ is a positive or negative sign, $\Gamma^M (M = 1, 2, \cdots, 7)$ is a 6D gamma matrix, and $P_{j\mathbf{16}}$ is a projection matrix acting on the $SU(16)$ representation $\mathbf{16}$. In the following discussion, we take $P_{j\mathbf{16}} = 1$, but still we need to choose $\eta_j$, where $\prod_{j=0}^{3}\eta_j = 1$ must be satisfied because of consistency of orbifold BCs [22].

We introduce three copies of the sets of two 6D $SU(16)$ $\mathbf{16}$ positive and negative Weyl fermion pairs. They must have appropriate orbifold BCs to realize 6D bulk and 4D brane gauge anomaly-free, and three chiral generations of quarks and leptons. More explicitly, each set of 6D Weyl fermions consists of a 6D $SU(16)$ $\mathbf{16}$ positive Weyl fermion with orbifold BCs ($\eta_0, \eta_1, \eta_2, \eta_3) = (-, -, -, -)$; a 6D positive one with orbifold BCs ($-, +, +$); a 6D negative
one with orbifold BCs \((-,+,+,-)\); and a 6D negative one with orbifold BCs \((-,+,−,+)\).

In this case, only the first 6D positive Weyl fermion with the orbifold BCs \((-,+,−,-)\) has zero modes for its 4D (left-handed) Weyl fermion components because its 4D left-handed Weyl fermion components have Neumann BCs at all the fixed points, while its 4D right-handed Weyl fermion components have Dirichlet BCs at all the fixed points. For the other three fermions, all their components have two Neumann and two Dirichlet BCs at four fixed points \((y_j, v_j)\).

From the above discussion, we find that in 4D framework the 4D anomaly-free SU(16) chiral gauge theories cannot realize the SM matter content because the 4D SU(16) anomaly cancellation does not allow three chiral generation of quarks and leptons. The 4D anomaly situation is exactly the same as each 4D fixed point in 6D theories with two dimensional orbifold extra dimension \(T^2/\mathbb{Z}_2\). However, as we will see below, even when effectively twelve 4D SU(16) 16 Weyl fermions from six 6D SU(16) 16 positive and negative Weyl fermions exist at an 4D fixed point, three 6D positive Weyl fermions have zero modes and their zero modes are only 4D SU(16) 16 Weyl fermion. When we take into account appropriate symmetry breaking effects, the zero modes become three generations of the SM fermions.

Here we check the contribution to 6D bulk and 4D brane anomalies from the above four 6D Weyl fermion set. Obviously, the fermion set does not contribute to 6D gauge anomaly because the number of 6D SU(16) 16 positive and negative Weyl fermions are the same. In other words, the fermion set forms 6D vectorlike fermions in the bulk. We calculate 4D gauge anomaly numbers at four fixed points \((y_j, v_j)(j = 0, 1, 2, 3)\) by using 4D SU(16) anomaly coefficients listed in Ref. [3], where for our convention the anomaly coefficient of a 4D SU(16) 16 Weyl fermion is +1. The above first 4D positive Weyl fermion contributes +1 to the SU(16) anomaly number for all the four fixed points \((y_j, v_j)(j = 0, 1, 2, 3)\); the second 6D positive Weyl fermion contributes +1 for the fixed points \((y_0, v_0)\) and \((y_1, v_1)\) and −1 for the fixed points \((y_2, v_2)\) and \((y_3, v_3)\); the third 6D negative Weyl fermion contributes +1 for the fixed points \((y_0, v_0)\) and \((y_3, v_3)\) and −1 for the fixed points \((y_1, v_1)\) and \((y_2, v_2)\); the last 6D negative Weyl fermion contributes +1 for the fixed points \((y_0, v_0)\) and \((y_2, v_2)\) and −1 for the fixed points \((y_1, v_1)\) and \((y_3, v_3)\). Therefore, one set of the four 6D SU(16) 16 Weyl fermions contributes +4 for the fixed point \((y_0, v_0)\) and 0 for the other three fixed points \((y_1, v_1)\), \((y_2, v_2)\), and \((y_3, v_3)\). Since the SM has three chiral generations of quarks and leptons, we need to introduce three sets of the four 6D Weyl fermions. Its total anomaly number from three set of the four 6D Weyl fermions are +12 for the fixed point \((y_0, v_0)\) and 0 for the other fixed points \((y_1, v_1)\), \((y_2, v_2)\), and \((y_3, v_3)\).

To cancel the 4D SU(16) anomaly on all the fixed points, we need to introduce at least one 4D Weyl fermion of a SU(16) non-anomaly-free complex representation on the fixed point \((y_0, v_0)\). As we saw before, a 4D SU(16) \(\mathbf{T}_{20}\) Weyl fermion contributes −12 to the SU(16) anomaly number. Therefore, by introducing a 4D SU(16) \(\mathbf{T}_{20}\) Weyl brane fermion at the fixed point \((y_0, v_0)\), all the 4D SU(16) anomalies at the fixed points are canceled out.

In the 6D framework, we introduce three 5D SU(16) 5440, 255, 16 brane scalar fields \(\Phi_{5440}, \Phi_{255}, \Phi_{16}\) on the 5D brane \((y = 0)\). Their orbifold BCs are given by

\[
\Phi_\ell(x, v_\ell - v) = \eta_\ell x P_{\ell x} \Phi_\ell(x, v_\ell + v),
\]

where \(\ell = 0, 2, x = 5440, 255, 16\), \(\eta_\ell x\) is a positive or negative sign, and \(P_{\ell x}\) is a projection matrix. In our orbifold BCs, \(P_{\ell x} = 1\). The scalar fields are responsible for breaking SU(16) to \(G_{\text{SM}}\). When we take \(\eta_\ell x = 1\), they have zero modes and can get nonvanishing VEVs. We assume that the nonvanishing VEVs of the scalars break SU(16) to \(G_{\text{SM}}\).

To summarize, the matter content in the SU(16) special GUT consists of a 6D SU(16) bulk gauge boson \(A_M\); three 6D SU(16) 16 positive Weyl fermions with the orbifold BCs \((\eta_0, \eta_1, \eta_2, \eta_3) = (−, −, −, −)\); \(\Psi_1^{(a)}\) \((a = 1, 2, 3)\); three 6D positive one with \((−, −, +, +)\); \(\Psi_2^{(b)}\) \((b = 1, 2, 3)\); three 6D negative one with \((−, +, +, −)\); \(\Psi_3^{(c)}\) \((c = 1, 2, 3)\); three 6D negative one with \((−, +, −, +)\); \(\Psi_4^{(d)}\) \((d = 1, 2, 3)\); three 5D SU(16) 5440, 255, 16 brane scalar bosons at \(y = 0\) \(\Phi_{5440}, \Phi_{255}, \Phi_{16}\); one 4D SU(16) \(\mathbf{T}_{20}\) Weyl brane fermion at the fixed point \((y_0, v_0) = (0, 0)\). The matter content of the SU(16) special GUT is summarized in Table [4].
Table 1: The table shows the matter content in the $SU(16)$ special GUT on $M^4 \times T^2/\mathbb{Z}_2$. The representations of $SU(16)$ and 6D, 5D, 4D Lorentz group, the orbifold BCs of 6D bulk fields and 5D brane fields, and the spacetime location of 5D and 4D brane fields are shown.

### 3 Summary and discussion

In this paper, we constructed $SU(16)$ special GUTs by using a special embedding $SU(16) \supset SO(10)$. In this framework, we found that the 4D $SU(16)$ 16 Weyl fermion can be identified with one generation of quarks and leptons; the 4D $SU(16)$ gauge anomaly on the fixed points restricts the minimal number of generations; we cannot construct $SU(16)$ special GUTs with three generations of quarks and leptons in 4D framework, while we can construct an $SU(16)$ special GUT in 6D framework; also exotic chiral fermions do not exist due to a special feature of the $SU(16)$ complex representation $\mathbf{120}$ once we take into account the symmetry breaking of the large GUT gauge group $SU(16)$ to the small GUT gauge group $SO(10)$.

We comment on the SM fermion masses in the 6D $SU(16)$ special GUT. Since one generation of the SM fermions is unified into a 6D $SU(16)$ 16 Weyl fermion, the masses of quarks and leptons for each generation are degenerate without $SU(16)$ breaking effects. We introduced a 5D $SU(16)$ 16 brane fermion and a 4D $SU(16)$ $\mathbf{120}$ brane fermion at a fixed point $(y_0, v_0) = (0, 0)$. The $SO(10)$ 16 and 120 representations are decomposed into $SU(5) \times U(1)$ representations

\begin{align}
\mathbf{16} &= \mathbf{(10)}(1) \oplus \mathbf{(5)}(-3) \oplus \mathbf{(1)}(5), \quad (3.1) \\
\mathbf{120} &= \mathbf{(45)}(2) \oplus \mathbf{(45)}(-2) \oplus \mathbf{(5)}(-2) \oplus \mathbf{(10)}(6) \oplus \mathbf{(10)}(-6) \oplus \mathbf{(5)}(2). \quad (3.2)
\end{align}

Since an $SO(10)$ tensor product $\mathbf{16} \otimes \mathbf{16} \otimes \mathbf{120}$ contains singlet, the 6D $SU(16)$ 16 bulk and 4D $SU(16)$ $\mathbf{120}$ brane fermions can be mixed via the VEV of the 5D $SU(16)$ 16 brane scalar once its corresponding brane interaction term is generated. The effective mass term divides up-type quark mass with down-type quark and charged lepton masses. Next, we introduced a 5D $SU(16)$ 255 brane scalar field on the brane $y = 0$ and its VEV is responsible for breaking $SU(5)(\subset SU(16))$ to $G_{SM}$. The VEV of the brane scalar field can mix three 6D $SU(16)$ 16 bulk Weyl fermions containing zero modes with the other nine 6D $SU(16)$ 16 ones. Therefore, the $SU(16)$ model seems to realize the SM fermion masses, but seems to give us no prediction about quark and lepton masses. We will leave the detail analysis in future studies.

In the 6D $SU(16)$ special GUT, we use the fact that an $SU(16)$ complex representation $\mathbf{120}$ is identified with an $SO(10)$ real representation $\mathbf{120}$ for the special embedding $SU(16) \supset SO(10)$. It is important to eliminate non-SM chiral fermions at low-energy physics. This type of representations, which is complex under a Lie group but is real under its maximal special subgroup, cannot be found in the other special embeddings $G \supset H$ with up to the rank-30 of $G$ except $G = SU(27) \supset H = E_6$. (It is checked at least for up to $10^4$ dimensional representations of $SU(10) \supset SU(5)$ and $SU(15) \supset SU(5)$; up to $10^5$ dimensional representations of $SU(15) \supset SU(10)$.\supset$
SU(6), SU(21) ⊃ SU(6), SU(21) ⊃ SU(7), SU(28) ⊃ SU(7), SU(28) ⊃ SU(8).) For a special embedding \(SU(27) \supset E_6\), an \(SU(27)\) complex representation \(2925\) \((2925)\) becomes an \(E_6\) real representation \(2925\):

\[
2925 = 2925 \quad (2925 = 2925).
\] (3.3)

Many branching rules of \(SU(10) \supset SU(5), SU(15) \supset SU(5), \) and \(SU(15) \supset SU(6)\) are listed in Ref. [3]; for branching rules of \(SU(21) \supset SU(6), SU(21) \supset SU(7), SU(28) \supset SU(7), SU(28) \supset SU(8), SU(27) \supset E_6, \) there are no references, but their branching rules can be calculated by using methods in Ref. [3].

We consider a special GUT based on a small GUT gauge group \(E_6\). As the same as \(SO(10)\) gauge theories, there is no 4D \(E_6\) gauge anomaly, while there can be 4D \(SU(27)\) gauge anomaly in 4D chiral theories. We have to check how to cancel out the 4D \(SU(27)\) gauge anomaly. When we take the 4D anomaly coefficient of a 4D \(SU(27)\) \(27\) Weyl fermion as +1, the 4D anomaly coefficient of a 4D \(SU(27)\) \(2925\) Weyl fermion is \(-252\), which can be calculated by using the general formula of 4D anomaly coefficients of \(SU(n)\) representations discussed in Refs. [3, 36].

Thus, if we consider a model whose fermion matter content is a 4D \(SU(27)\) \((252 \times 27 \oplus 2925\) Weyl fermion, its matter content does not contain exotic non-SM chiral fermions. However, there are 252 generations of the SM Weyl fermions. It seems to be impossible to realize three chiral generations of quarks and leptons.

In our 6D \(SU(16)\) special GUT, we assumed that 5D brane scalar fields on the 5D brane on 5th dimensional space \(y = 0\) are responsible for breaking the large GUT gauge group \(SU(16)\) to \(G_{SM}\). However, a part of its symmetry breaking may be replaced by orbifold BCs and/or the Hosotani mechanism breaking. In the above discussion, we took the projection matrix \(P_{16j}(j = 0, 1, 2, 3)\) for the \(SU(16)\) representation \(16\) as an identity matrix, but we can choose different ways. For example, when we take \(P_{016} = P_{116} = I\) and \(P_{216} = P_{316} = \text{diag}(I_{15}, -1)\), the orbifold BCs break \(SU(16)\) to \(SU(15) \times U(1)\), and we assume that the non-vanishing VEV of the \(SU(16)\) \(5440\) GUT Higgs breaks the large GUT gauge group \(SU(16)\) to the small GUT gauge group \(SO(10)\). The two symmetry breaking combination leaves only \(SU(5)\), where it can be recognized by using the correspondence of the root vectors between \(SU(16)\) and its special subgroup \(SO(10)\).

Also, the \(SU(16)\) adjoint representation \(255\) is decomposed into \(SO(10)\) representations \(210\) and \(45\). The \(SO(10)\) representation \(210\) includes not only the \(SU(5)\) adjoint representation \(24\) and but also all the \(SU(5)\) fundamental representations \(5, 10, 10, 5\). It can be used as GUT or the electro-weak symmetry breaking Higgs in gauge-Higgs GUT scenarios.

The use of the special embedding \(SU(16) \supset SO(10)\) may be interesting for string inspired GUT model builders because one of the regular embeddings of \(SO(32)\) is \(SU(16) \times U(1)\). The special embedding may be incorporated with model buildings in an \(SO(32)\) heterotic string theory. For one of the amazing features, \(SO(10)\) spinor representations \(16\) and \(\overline{16}\) are embedded into an \(SO(32)\) vector representation \(32\).

Another motivation may come from string GUT model buildings to realize higher-level gauge groups by lower “costs” compared with e.g., so-called diagonal embeddings [9, 37, 38]. For example, by using a diagonal embedding, we need four \(SO(10)_1\) to obtain \(SO(10)_4\), while by using a special embedding \(SU(16)_1 \supset SO(10)_4\), we need one \(SU(16)_1\), where the subscript is the index of embedding, which stands for the ratio of the second order Dynkin indices of corresponding representations between a Lie algebra and its subalgebra. The rank in the former is twenty, while one in the latter is fifteen [37].

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