Metric convexity in the symplectic category

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Abstract

We introduce an extension of the standard Local-to-Global Principle used in the proof of the convexity theorems for the momentum map to handle closed maps that take values in a length metric space. This extension is used to study the convexity properties of the cylinder valued momentum map introduced by Condevaux, Dazord, and Molino in [8] and allows us to obtain the most general convexity statement available in the literature for momentum maps associated to a symplectic Lie group action.

Keywords: symplectic geometry, momentum maps, convexity, length metric spaces.

1 Introduction

The study of the convexity properties of the image of momentum maps has been a very active field of research for the last twenty years ever since Atiyah [2] and, independently, Guillemin and Sternberg [12], realized that this is a natural way to encode very important problems in linear algebra and representation theory. One of them is the classical result of Schur [30] and Horn [16] that states that the set of diagonals of an isospectral set of $n \times n$ Hermitian matrices equals the convex hull of the $n!$ points obtained by permuting all the eigenvalues. A second linear algebra problem that can be described in the momentum map setup is the characterization of all possible eigenvalues of the sum $A + B$ be sorted in decreasing order, this problem amounts to describing the intersection of the image of a coadjoint equivariant momentum map $J : M \to u(n)^*$ with one of the Weyl chambers of $u(n)^*$. More explicitly, in this case $M$ is the symplectic manifold obtained by taking the Cartesian product of the two $U(n)$ coadjoint orbits $O_\mu \times O_\lambda$ and $J : M \to u(n)^*$ is the momentum map corresponding to the diagonal $U(n)$-action on that set.

This problem is a particular case of a more general situation which consists of describing the image of a coadjoint equivariant momentum map $J : M \to g^*$ associated to the action of a compact Lie group $G$ on a compact symplectic manifold $M$. Guillemin and Sternberg [13] proved that $J(M) \cap t^*_+ \mu$ is a union of compact convex polytopes ($t$ is a maximal toral algebra of $g$ and $t^*_+$ the positive Weyl chamber of...
g^*) and Kirwan showed that this set is connected thereby concluding that $J(M) \cap t^*_+ \cap \mathfrak{g}^*$ is a compact convex polytope.

These convexity results have been generalized to compact group actions on noncompact manifolds with proper momentum maps by Condevaux, Dazord, and Molino who designed a very ingenious topological proof of the convexity theorems based on the local properties of the momentum map. This method is usually referred to as Local-to-Global Principle. The main tool for capturing the local properties of the momentum map was the so-called Marle-Guillemin-Sternberg normal form. Using this approach they proved that a momentum map of a torus Hamiltonian action is locally fiber connected, locally open onto its image, and has local convexity data. This method was further analyzed and applied to many interesting situations by Hilgert, Neeb, and Plank. Later on Sjamaar and Knop proved that a momentum map in the non-Abelian case has the same local topological properties mentioned above.

In spite of its generality, the Local-to-Global Principle cannot be applied to obtain a convexity result which would include other interesting examples like, for example, those introduced by Prato since the momentum map in this case is not proper. This inconvenience has been fixed by the authors in where the properness condition has been replaced by just closedness.

All the above convexity results concern maps with values in vector spaces. Consequently, they address the situation in which the group action in question has an associated standard momentum map. When such a map does not exist one still can define a momentum map that captures most of the properties of the standard object but which, in general, takes values in an Abelian group isomorphic to a cylinder. This momentum map that will be referred to as the cylinder valued momentum map was introduced in Condevaux, Dazord, and Molino and carefully studied in Ortega and Ratiu in the context of reduction. Additionally, its local properties are as well known as those for the standard momentum map. Hence, it is very natural to ask if one could extend to the cylinder valued momentum map the knowledge that we have about the convexity properties of the classical momentum map by using an appropriate generalization of the Local-to-Global Principle. This question is actually posed as an open problem in the original article. An affirmative answer to this twenty year old question is the main achievement of this paper.

The main results of the present work are divided into two parts. First, in Section we extend the Local-to-Global Principle to the category of maps that take values in length spaces. Our approach is based on the extension of the classical Hopf-Rinow theorem to length metric spaces due to Cohn-Vossen. This circle of ideas also appears for compact spaces in. The results on length metric spaces necessary for the comprehension of the paper are recalled in an appendix at the end. The metric approach seems to be the best adapted generalization of the classical setup to our problem since, under certain hypotheses related to the topological nature of the Hamiltonian holonomy of the problem (a concept defined carefully later on), the target space of the cylinder valued momentum map has an associated canonical length space structure. The extension presented here recovers the previous Local-to-Global Principle proved in by the authors for the case of closed maps with values in vector spaces. It is expected that this general result can be applied to many other situations going beyond symplectic geometry. Second, in Section we apply this generalized Local-to-Global Principle to the cylinder valued momentum map and we obtain a convexity result similar to the classical one for the standard momentum map.
2 Image convexity for maps with values in length spaces

One of the main goals in this paper is the study of the convexity properties of the image of a natural generalization of the momentum map. The notion of convexity is usually associated with vector spaces. However, the map considered in this paper has values in a manifold that is, in general, diffeomorphic to a cylinder. Thus, one is forced to work in a more general setting. As reviewed in the appendix §4 most of the concepts pertaining to convexity can be extended to the context of the so called length spaces. It turns out that the target space of the map that we are going to study can be naturally endowed with a length space structure and hence convexity will be used in this context. We give in §4 a self-contained brief summary of all the definitions and results on length spaces necessary in this paper.

The convexity program has been successfully carried out for the standard momentum map by several means. One possible approach consists in determining certain local properties of the map that guarantee that it has a globally convex image. This strategy relies on a fundamental result called the Local-to-Global Principle which has been introduced in [8, 15] for maps whose target space is an Euclidean vector space. Since the extension of the standard momentum map with which we will be working does not map into a vector space but into a length space, a generalization of the Local-to-Global Principle is needed to handle this situation. This is the main goal of the present section.

Let \( f : X \to Y \) be a continuous map between two connected Hausdorff topological spaces. Define the following equivalence relation on the topological space \( X \): declare two points \( x_1, x_2 \in X \) to be equivalent if and only if \( f(x_1) = f(x_2) = y \) and they belong to the same connected component of \( f^{-1}(y) \). The topological quotient space, whose elements are the connected components of the fibers of \( f \), will be denoted by \( X_f \), the projection map by \( \pi_f : X \to X_f \), and the induced map on \( X_f \) by \( \tilde{f} : X_f \to Y \). Thus, \( \tilde{f} \circ \pi_f = f \) uniquely characterizes \( \tilde{f} \). The map \( \tilde{f} \) is continuous and if the fibers of \( f \) are connected then it is also injective.

\[ \text{Definition 2.1} \quad \text{Let } X \text{ and } Y \text{ be two topological spaces and } f : X \to Y \text{ a continuous map. The subset } A \subset X \text{ satisfies the locally fiber connected condition (LFC) if } A \text{ does not intersect two different connected components of the fiber } f^{-1}(f(x)), \text{ for any } x \in A. \]

Let \( X \) be an arcwise connected Hausdorff topological space. The continuous map \( f : X \to Y \) is said to be locally fiber connected if for each \( x \in X \), any open neighborhood of \( x \) contains a connected neighborhood \( U_x \) of \( x \) such that \( U_x \) satisfies the (LFC) condition.

The following consequences of the definition will be useful later on. A subset of a set that satisfies (LFC) also satisfies (LFC). If \( A \subset X \) satisfies the (LFC) property, then its saturation \( \pi_f^{-1}(\pi_f(A)) \) also satisfies (LFC). If \( f \) is locally fiber connected, then any open neighborhood of \( x \in X \) contains an open neighborhood \( U_x \) of \( x \) such that the restriction of \( \tilde{f} \) to \( \pi_f(U_x) \) is injective.

\[ \text{Definition 2.2} \quad \text{A continuous map } f : X \to Y \text{ is said to be locally open onto its image if for any } x \in X \text{ there exists an open neighborhood } U_x \text{ of } x \text{ such that the restriction } f|_{U_x} : U_x \to f(U_x) \text{ is an open map, where } f(U_x) \text{ has the topology induced by } Y. \text{ We say that such a neighborhood satisfies the (LOI) condition.} \]

Benoist proved in Lemma 3.7 of [3] the following result that will be used later on.
**Lemma 2.3** Suppose $f : X \to Y$ is a continuous map between two topological spaces. If $f$ is locally fiber connected and locally open onto its image then $\pi_f$ is an open map.

The following characterization of closed maps will be needed in what follows (see [9], Theorems 1.4.12 and 1.4.13).

**Theorem 2.4** Let $f : X \to Y$ be a continuous mapping.

(i) $f$ is closed if and only if for every $B \subset Y$ and every open set $A \subset X$ which contains $f^{-1}(B)$, there exists an open set $C \subset Y$ containing $B$ and such that $f^{-1}(C) \subset A$.

(ii) $f$ is closed if and only if for every point $y \in Y$ and every open set $U \subset X$ which contains $f^{-1}(y)$, there exists a neighborhood $V_y$ of the point $y$ in $Y$ such that $f^{-1}(V_y) \subset U$.

For the next results we recall the following standard definitions (see, e.g., Engelking [9]). A topological space $X$ is called a $T_1$-space if for every pair of distinct points $x_1, x_2 \in X$, there exists an open set $U \subset X$ such that $x_1 \in U$ and $x_2 \notin U$. The topological space $X$ is normal if it is a $T_1$-space and for any closed disjoint subsets $A, B \subset X$ there exist open subsets $U, V \subset X$ such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$. The topological space $X$ is first countable if every point admits a countable base of open neighborhoods.

**Lemma 2.5** Let $X$ be a normal, first countable, arcwise connected, and Hausdorff topological space and $Y$ a Hausdorff topological space. Let $f : X \to Y$ be a continuous map that is locally open onto its image and is locally fiber connected. If $f$ is a closed map, then

(i) the projection $\pi_f : X \to X_f$ is also a closed map,

(ii) the quotient $X_f$ is a Hausdorff topological space.

**Proof.** (i) Let $[x]$ be an arbitrary point in $X_f$ and $U \subset X$ an arbitrary open set that includes $E_x := \pi_f^{-1}([x])$, the connected component of $f^{-1}(f(x))$ that contains $x$. Denote by $F := f^{-1}(f(x)) \setminus E_x$ the union of all (closed) connected components of $f^{-1}(f(x))$ different from $E_x$. We claim that $F$ is a closed subset of $X$. Indeed, if $z \in F$, by first countability of $X$, there exists a sequence $\{z_n\}_{n \in \mathbb{N}}$ in $F$ which is convergent to $z$. Since $f(z_n) = f(x)$, by continuity of $f$ we conclude that $f(x) = f(z_n) \to f(z)$ and hence $z \in f^{-1}(f(x))$. If $z \in E_x$ then any neighborhood of $z$ intersects at least one other connected component of the fiber $f^{-1}(f(x))$ since $z \in F$. This, however, contradicts the (LFC) condition. Therefore, $z \in F$ and hence $F$ is closed. The same argument as above shows that the (LFC) condition implies that $E_x$ is also closed in $X$.

Using the normality of $X$ there exist two open sets $U_{E_x}$ and $W$ such that $E_x \subset U_{E_x}$, $F \subset W$, and $U_{E_x} \cap W = \emptyset$. After shrinking, if necessary, we can assume that $U_{E_x} \subset U$. Applying Theorem 2.4(ii), the closedness of $f$ ensures the existence of an open neighborhood $V_{f(x)}$ of $f(x)$ in $V$ such that $E_x \subset f^{-1}(f(x)) \subset f^{-1}(V_{f(x)}) \subset U_{E_x} \cup W$.

The set $A := U_{E_x} \cap f^{-1}(V_{f(x)})$ is a nonempty open subset of $X$ and is also saturated with respect to the equivalence relation that defines $\pi_f$ or, equivalently, $\pi_f^{-1}(\pi_f(A)) = A$. Indeed, if a connected component of a fiber of $f$ from $f^{-1}(V_{f(x)})$ intersects $U_{E_x}$, respectively $W$, then it is entirely contained either in $U_{E_x}$ or in $W$ since $U_{E_x} \cap W = \emptyset$. 


Since \( A \) is open in \( X \), by the definition of the quotient topology of \( X_f \), it follows that \( \pi_f(A) \) is an open neighborhood of \([x]\). Note that \( \pi_f^{-1}(\pi_f(A)) \subset U \), which shows via Theorem 2.6 ii that \( \pi_f \) is a closed map.

(ii) We shall prove that \( X_f \) is Hausdorff by showing that the projection \( \pi_f \) is an open map (which holds by Lemma 2.3) and that the graph of the equivalence relation that defines \( X_f \) is closed.

To show that the graph is closed, we need some preliminary considerations. For every \( x \in X \) there exists a neighborhood \( U_x \) that satisfies (LOI) and (LFC). By normality of \( X \) there exists also a neighborhood \( U'_x \) of \( x \) with \( \overline{U'_x} \subset U_x \). We shall prove that \( \pi_f^{-1}(\pi_f(U'_x)) \subset \pi_f^{-1}(\pi_f(U_x)) \) which shows that for every connected component \( E_0 \) of a fiber there exists a saturated neighborhood of it which contains a smaller saturated neighborhood whose closure still satisfies (LFC). In order to prove the above inclusion observe that since \( \pi_f \) is continuous and closed we have that \( \overline{\pi_f(U'_x)} = \pi_f(\overline{U'_x}) \subset \pi_f(U_x) \). By the continuity of \( \pi_f \), we obtain the inclusion \( \pi_f^{-1}(\pi_f(U'_x)) \subset \pi_f^{-1}(\pi_f(U_x)) \), and thus \( X_f \) is Hausdorff by Lemma 2.5. This implies that \( \pi_f^{-1}(\pi_f(U'_x)) \subset \pi_f^{-1}(\pi_f(U_x)) \) is saturated. Therefore, \( x, y \in \pi_f^{-1}(\pi_f(U'_x)) \). But \( \pi_f^{-1}(\pi_f(U'_x)) \) satisfies (LFC) and thus \( x \) and \( y \) belong to the same connected component of the fiber \( f^{-1}(f(x)) \). This shows that the graph of the equivalence relation is closed, as required. ■

On \( X_f \) we define the function \( \overline{d} : X_f \times X_f \to [0, \infty) \) in the following way: for \([x] \) and \([y] \) in \( X_f \) let \( \overline{d}([x], [y]) \) be the infimum of all the lengths \( l_d(\overline{f} \circ \gamma) \) where \( \gamma \) is a continuous curve in \( X_f \) that connects \([x] \) and \([y] \). The length \( l_d \) is computed with respect to the distance \( d \) on \( Y \). From the definition it follows that \( \overline{d}(\overline{f([x])}, \overline{f([y])}) \leq \overline{d}([x], [y]) \).

**Proposition 2.6** Let \( X \) be a normal, first countable, arcwise connected, and Hausdorff topological space and \((Y, d)\) a metric space. Assume that \( f : X \to Y \) is a continuous closed map that is also locally open onto its image and locally fiber connected. Then the function on \( \overline{d} : X_f \times X_f \to [0, \infty) \) introduced above defines a metric on \( X_f \).

**Proof.** The positivity, symmetry, and the triangle inequality of \( \overline{d} \) are obvious from the definition of \( \overline{d} \). It remains to be proved that if \( \overline{d}([x], [y]) = 0 \) then \([x] = [y] \). Suppose that there exist \([x] \neq [y] \) with \( \overline{d}([x], [y]) = 0 \). Then \( d(\overline{f([x])}, \overline{f([y])}) = 0 \) and hence \( f([x]) = f([x]) = f([y]) = f([y]) \). This implies that \([x] \) and \([y] \) are images under the the projection \( \pi_f \) of two different connected components of the same fiber.

By the (LOI) property of \( f \) and the openness of \( \pi_f \) (see Lemma 2.3), there exist two open neighborhoods \( U_{[x]} \) and \( U_{[y]} \) in the quotient topology of \( X_f \) around \([x] \), respectively \([y] \), such that \( \overline{f}_{[x]} \) and \( \overline{f}_{[y]} \) are open onto their images. Consequently, there exist two open neighborhoods \( U'_{[x]} \subset U_{[x]} \) and \( U'_{[y]} \subset U_{[y]} \) such that \( \overline{f}(U'_{[x]}) \supset B(\overline{f}([x]), r) \cap f(U_{[x]}) \) and \( \overline{f}(U'_{[y]}) \supset B(\overline{f}([y]), r') \cap f(U_{[y]}) \) for some small enough constants \( r, r' > 0 \). Moreover, we can choose the above neighborhoods \( U'_{[x]} \) and \( U'_{[y]} \) such that \( U'_{[x]} \cap U'_{[y]} = \emptyset \) since the quotient topology of \( X_f \) is Hausdorff by Lemma 2.5.
Any curve $\gamma$ in $X_f$ that connects $[x]$ and $[y]$ is mapped by $\tilde{f}$ into a loop in $Y$ based at $\tilde{f}([x]) = \tilde{f}([y])$ and has to exit $U'_x$ and enter $U'_y$ (since $U'_x \cap U'_y = \emptyset$) in order to connect $[x]$ and $[y]$. Consequently, the curve $f \circ \gamma$ in $Y$ has to exit $B_d(f([x]), R)$ and reenter $B_d(f([y]), R')$ and hence $l_d(f \circ \gamma) > R + R'$. This is in contradiction with the hypothesis that $d([x], [y]) = 0$ for $[x] \neq [y]$. ■

In order to put the following definition in context, the reader is encouraged to look at the appendix \[\text{[4]}\] where the concepts of length metric and geodesic metric space are discussed.

**Definition 2.7** A subset $C$ in a metric space $(X, d)$ is said to be **convex** if the restriction of $d$ to $C$ is a finite length metric (see Definition \[\text{[1, 4]}\]).

**Definition 2.8** Let $X$ be a connected Hausdorff space and $(Y, d)$ a length space (see Definition \[\text{[4, 4]}\]). A continuous mapping $f : X \rightarrow Y$ is said to have **local convexity data** if for each $x \in X$ and every sufficiently small neighborhood $U_x$ of $x$ the set $f(U_x)$ is a convex subset of $Y$. We say that $U_x$ satisfies the (LCD) condition.

**Proposition 2.9** Let $X$ be a normal, first countable, arcwise connected, and Hausdorff topological space and $(Y, d)$ a geodesic metric space (see Definition \[\text{[4, 8]}\]). Assume that $f : X \rightarrow Y$ is a continuous closed map that is also locally open onto its image, locally fiber connected, and has local convexity data. Then $(X_f, \tilde{d})$ is a length space and the topology induced by $\tilde{d}$ coincides with the quotient topology of $X_f$.

**Proof.** First we will prove that for close enough $[x]$ and $[y]$ we have $d(\tilde{f}([x]), \tilde{f}([y])) = \tilde{d}([x], [y])$. Indeed, let $[x], [y] \in X_f$ be such that they are contained in an open set $\tilde{U} \subset X_f$ (open in the quotient topology of $X_f$) such that $\tilde{U} = \pi_f(U)$, where $U$ open in $X$ and satisfies the (LCD), (LOI), and (LFC) conditions. Let $\Omega := f(U) = \tilde{f}(\tilde{U})$. Since $\Omega$ is convex (because (LCD) holds for $U$), by Lemma \[\text{[4, 12]}\] there exists a rectifiable shortest path $\gamma_0$ entirely contained in $\Omega$ that connects $\tilde{f}([x])$ and $\tilde{f}([y])$, that is, $l_d(\gamma_0) = d(\tilde{f}([x]), \tilde{f}([y]))$. Note that $\tilde{f}|_{\tilde{U}} : \tilde{U} \rightarrow \Omega$ is a homeomorphism $\tilde{U}$ endowed with the quotient topology of $X_f$ because $\tilde{f}|_{\tilde{U}}$ is open, since $U$ satisfies (LOI), and is injective, because $U$ satisfies (LFC). The curve $c_0 := \tilde{f}^{-1}(\gamma_0)$ is continuous and connects $[x]$ with $[y]$. From the definition of $\tilde{d}$ we have that $\tilde{d}([x], [y]) \leq l_d(f \circ c_0) = l_d(\gamma_0) = d(\tilde{f}([x]), \tilde{f}([y]))$. As the inequality $d(\tilde{f}([x]), \tilde{f}([y])) \leq \tilde{d}([x], [y])$ always holds, we obtained the desired equality $d(\tilde{f}([x]), \tilde{f}([y])) = \tilde{d}([x], [y])$. Consequently, the homeomorphism $\tilde{f}|_{\tilde{U}} : \tilde{U} \rightarrow \Omega$ is also an isometry from $(\tilde{U}, \tilde{d}|_{\tilde{U}})$ to $(\Omega, d|_{\Omega})$ and thus the quotient topology of $X_f$ coincides with the metric topology induced by $\tilde{d}$.

Next we will prove that $(X_f, \tilde{d})$ is a length space. Let $c : [a, b] \rightarrow X_f$ be a continuous curve connecting two arbitrary points $[x]$ and $[y]$ in $X_f$. For two partitions $\Delta_n$ and $\Delta_{n+1}$ of the interval $[a, b]$ with $\Delta_{n+1}$ finer than $\Delta_n$ we have that $\sum_{i=1}^{n} d(c(t_i), c(t_{i+1})) \leq \sum_{i=1}^{n+1} d(c(t_i), c(t_{i+1}))$ due to the triangle inequality. Therefore, in order to compute $l_{\tilde{d}}(c)$ it suffices to work with partitions fine enough such that two consecutive points $c(t_i), c(t_{i+1})$, corresponding to a partition $\Delta_n$, are close enough as above. Therefore, by
what was just proved, we have $d(\tilde{f}(c(t_i)), \tilde{f}(c(t_{i+1}))) = \tilde{d}(c(t_i), c(t_{i+1}))$ and we conclude

$$l_\tilde{d}(c) = \sup_{\Delta_n} \sum_{1}^{n} \tilde{d}(c(t_i), c(t_{i+1}))$$

$$= \sup_{\Delta_n} \sum_{1}^{n} d(\tilde{f}(c(t_i)), \tilde{f}(c(t_{i+1})))$$

$$= l_{\tilde{d}}(\tilde{f} \circ c).$$

Consequently, $\tilde{f} \circ c$ is a rectifiable curve in $(Y, d)$ if and only if $c$ is rectifiable in $(X_f, \tilde{d})$. The equality

$$\tilde{d}([x], [y]) = \inf l_{\tilde{d}}(\tilde{f} \circ c) = \inf l_{\tilde{d}}(c) = \tilde{d}([x], [y]),$$

shows that $(X_f, \tilde{d})$ is a length space. ■

The proof of the following lemma can be found as Proposition 4.4.16 in [9].

Lemma 2.10 (Vainšteĭn) If $f : X \to Y$ is a closed mapping from a metrizable space $X$ onto a metrizable space $Y$, then for every $y \in Y$ the boundary $\text{bd}(f^{-1}(y)) := f^{-1}(y) \cap (X \setminus f^{-1}(y))$ is compact.

Definition 2.11 Let $X$ be a Hausdorff topological space and $f : X \to Y$ a continuous map. We call $f$ a proper map if it is closed and all fibers $f^{-1}(y)$ are compact subsets of $X$.

The proof of the following theorem can be found in Engelking [9], Proposition 3.7.2.

Theorem 2.12 If $f : X \to Y$ is a proper map, then for every compact subset $Z \subset Y$ the inverse image $f^{-1}(Z)$ is compact.

A converse of this theorem is available when $Y$ is a $k$-space (i.e. $Y$ is a Hausdorff topological space that is the image of a locally compact space under a quotient mapping). For example every first countable Hausdorff space is a $k$-space (see Theorem 3.3.20 of [9]).

Proposition 2.13 Let $X$ be a normal, first countable, arcwise connected, and Hausdorff topological space and $(Y, d)$ a complete locally compact length space (and thus, by Hopf-Rinow-Cohn-Vossen a geodesic metric space; see Theorem 4.9). Assume that $f : X \to Y$ is a continuous closed map that is also locally open onto its image, locally fiber connected, and has local convexity data. Then $(X_f, \tilde{d})$ is a complete locally compact length space and hence a geodesic metric space.

Proof. First we will prove that $\tilde{f}$ is a proper map. It is a closed map since $f$ is a closed map. Local injectivity of $\tilde{f}$ implies that $\text{bd}(\tilde{f}^{-1}(y)) = \tilde{f}^{-1}(y)$. By Vainšteĭn’s Lemma 2.10 we conclude that the fibers of $\tilde{f}$ are all compact and consequently $\tilde{f}$ is a proper map.

Because local compactness is an inverse invariant for proper maps we obtain also that $(X_f, \tilde{d})$ is locally compact since $\tilde{f} : X_f \to Y$ is a proper map.

By the Hopf-Rinow-Cohn-Vossen Theorem 4.9 it suffices to show that every closed metric ball in $X_f$ is compact in order to conclude that $(X_f, \tilde{d})$ is a complete metric space. The set $B([x], r)$ is
closed in $X_f$ and, by definition of the metric $\tilde{d}$, the inclusion $\tilde{f}(\mathcal{B}([x], r)) \subset \mathcal{B}(\tilde{f}(x), r)$ holds. By the Hopf-Rinow-Cohn-Vossen Theorem it follows that $\mathcal{B}(\tilde{f}(x), r)$ is a compact set in $Y$ and, consequently, $\tilde{f}^{-1}(\mathcal{B}(\tilde{f}(x), r))$ is compact in $X_f$ due to properness of $\tilde{f}$. Since $\mathcal{B}([x], r)$ is a closed subset of $\tilde{f}^{-1}(\mathcal{B}(\tilde{f}(x), r))$, $\tilde{f}$ it is necessarily compact in $X_f$.

As a consequence of the above proposition, $(X_f, \tilde{d})$ satisfies all the conditions of the Hopf-Rinow-Cohn-Vossen Theorem which implies that for any two points $[x], [y] \in X_f$ there exists a shortest geodesic connecting them.

**Definition 2.14** Let $(X, d)$ be a geodesic metric space. We say that $C$ is weakly convex if for any two points $x, y \in C$ there exists a geodesic between $x$ and $y$ entirely contained in $C$.

Note that weak convexity does not require that the geodesic to be a shortest one. Now we can present the main result of this section.

**Theorem 2.15 (Local-to-Global Principle)** Let $X$ be a normal, first countable, arcwise connected, and Hausdorff topological space and $(Y, d)$ a complete, locally compact length space. Assume that $f : X \to Y$ is a continuous closed map that is also locally open onto its image, locally fiber connected, and has local convexity data. Then the following hold:

(i) $f(X) \subset (Y, d)$ is a weakly convex subset of $Y$.

(ii) If, in addition, $(Y, d)$ is uniquely geodesic (that is, any two points can be joined by a unique geodesic) and $d(y_1, y_2) < \infty$ for all $y_1, y_2 \in Y$, then $f(X)$ is a convex subset of $(Y, d)$, $f$ has connected fibers, and it is open onto its image.

**Proof.** (i) We have to prove that for any $y_1, y_2 \in f(X)$ there exists a geodesic (not necessary shortest) in $(Y, d)$ completely included in $f(X)$. Indeed, take $[x_1], [x_2] \in X_f$ such that $f([x_1]) = y_1$ and $f([x_2]) = y_2$. As was explained above, there exists a shortest geodesic $c : [a, b] \to X_f$ with the properties $c([a]) = [x_1]$, $c([b]) = [x_2]$, and $\tilde{d}([x_1], [x_2]) = l_d(c)$. We will show that $\tilde{f} \circ c \subset f(X)$ is a geodesic that connects $y_1$ with $y_2$. Since $f$ is locally open onto its image, locally fiber connected, and has local convexity data, each $[x] \in X_f$ admits an open neighborhood $U_{[x]}$ such that $\tilde{f}|_{U_{[x]} : U_{[x]} \to \tilde{f}(U_{[x]})}$ is injective, open onto its image, and $\tilde{f}(U_{[x]})$ is convex in $Y$. Choose now $[x]$ in the image of $c$ and let $t_0$ be such that $c(t_0) = [x]$. Then the intersection of the image of $c$ with $U_{[x]}$ is the image of a curve $c' : I \to U_{[x]}$ with $I \subset [a, b]$ a subinterval. If we take $U_{[x]}$ small enough then for any subinterval $[t_1, t_2] \subset I$ with $t_0 \in [t_1, t_2]$ we have $d(c(t_1), c(t_2)) = d(\tilde{f}(c(t_1)), \tilde{f}(c(t_2)))$ as was explained in the proof of Proposition 2.9. Because $c$ is a shortest geodesic we have that $d(c(t_1), c(t_2)) = l_d(c|_{[t_1, t_2]})$ and since $l_d(c|_{[t_1, t_2]}) = l_d(f \circ c|_{[t_1, t_2]})$ (as was shown in the proof of Proposition 2.9) we obtain the desired equality $d(\tilde{f}(c(t_1)), \tilde{f}(c(t_2))) = l_d(\tilde{f} \circ c|_{[t_1, t_2]})$. This proves that $\tilde{f} \circ c$ is a geodesic in $(Y, d)$ since it is a local distance minimizer.

(ii) From (i) we already know that any two points in $f(X)$ can be connected by a geodesic included in $f(X)$. Since $Y$ is uniquely geodesic this chosen geodesic must be the shortest one. Also, the restriction of $d$ to $f(X)$ is finite and therefore $f(X)$ is a convex subset of $(Y, d)$.

Now we will prove that $f$ has connected fibers. Suppose the contrary, that is, there exist $[x] \neq [y]$ with $\tilde{f}([x]) = \tilde{f}([y])$. Then there exists a shortest geodesic $c : [a, b] \to X_f$ that links $[x]$ and $[y]$ which is
mapped by \( \tilde{f} \) to a loop based at \( \tilde{f}(x) = \tilde{f}(y) \). As was proved in (i), \( \tilde{f} \circ c \) is a geodesic in \( Y \). Since
\( (Y, d) \) is uniquely geodesic we obtain that \( f \circ c \) is the constant loop. Consequently, \( \tilde{f}(c(t)) = \tilde{f}(x) \)
for all \( t \in [a, b] \). This implies that \( c(t) \) and \( x \) belong to the same fiber of \( f \) for all \( t \in [a, b] \) which
contradicts the local injectivity of \( \tilde{f} \) implied by the (LFC) property of \( f \).

Since \( \tilde{f} \) is a closed injective map, it is also open onto its image and, consequently, \( f \) is open onto its image. \( \blacksquare \)

\textbf{Remark 2.16} Unlike the situation encountered in the classical Local-to-Global Principle \cite{8,15} in
which the target space of the map is a Euclidean vector space and hence uniquely geodesic, \( f \) could
have, in general, a weakly convex image but not connected fibers. See Section 3.5 for an example.

\textbf{Remark 2.17} If \( Y \) is an Euclidean vector space and \( C \) is a convex subset of \( Y \) then Theorem 2.15
applied to the map \( f : X \to C \) yields the generalization of the classical Local-to-Global Principle introduced in
Theorem 2.28 of \cite{4}.

\section{Metric convexity for cylinder valued momentum maps}

The goal of this section is to apply the general results obtained in \S 2 to study the convexity properties of
the image of the \textit{cylinder valued momentum map}. This object, introduced by Condevaux, Dazord, and
Molino \cite{8}, naturally generalizes the standard momentum map definition due to Kostant and Souriau.
The standard momentum map is associated to a canonical Lie algebra action on a symplectic manifold.
Its convexity properties have been extensively studied \cite{2,12,13,15,31}.

Unlike the standard momentum map, the cylinder valued momentum map always exists for any
canonical Lie algebra action. However, the convexity properties of the standard momentum map cannot
be trivially extended to this object because it does not map into a vector space but into a manifold that is,
in general, diffeomorphic to a cylinder. Thus, in order to study the convexity properties of the cylinder
valued momentum map the notion of convexity introduced and studied in \S 2 is necessary.

\subsection{The cylinder valued momentum map}

In the following paragraphs we quickly review the elementary properties of the cylinder valued
momentum map. For more information and for detailed proofs the reader is encouraged to check with \cite{8}
or with Chapter 5 of \cite{26}.

Let \((M, \omega)\) be a connected paracompact symplectic manifold and let \( g \) be a Lie algebra that acts
canonically on \( M \). Take the Cartesian product \( M \times g^* \) and let \( \pi : M \times g^* \to M \) be the projection onto \( M \).
Consider \( \pi \) as the bundle map of the trivial principal fiber bundle \((M \times g^*, M, \pi, g^*)\) that has \((g^*, +)\)
as Abelian structure group. The group \((g^*, +)\) acts on \( M \times g^* \) by \( v \cdot (m, \mu) := (m, \mu + v) \), with \( m \in M \) and
\( \mu, v \in g^* \). Let \( \alpha \in \Omega^1(M \times g^*; g^*) \) be the connection one-form defined by
\[
\langle \alpha(m, \mu)(v_m, v), \xi \rangle := (i_{\xi_m} \omega)(m) (v_m) - \langle v, \xi \rangle,
\]
where \((m, \mu) \in M \times g^*, (v_m, v) \in T_m M \times g^*, \langle \cdot, \cdot \rangle\) denotes the natural pairing between \( g^* \) and \( g \), and \( \xi_M \)
is the infinitesimal generator vector field associated to \( \xi \in g \). The connection \( \alpha \) is flat. For \((z, \mu) \in \g^* \times \m \)
$M \times g^*$, let $(M \times g^*)(z,\mu)$ be the holonomy bundle through $(z,\mu)$ and let $\mathcal{H}(z,\mu)$ be the holonomy group of $\alpha$ with reference point $(z,\mu)$ (which is an Abelian zero dimensional Lie subgroup of $g^*$ by the flatness of $\alpha$). The principal bundle $((M \times g^*)(z,\mu),\mathcal{H}(z,\mu))$ is a reduction of the principal bundle $(M \times g^*,\pi,M)$ (commutative: group of the flatness of $\alpha$). To simplify notation, we will write $(\tilde{M},\tilde{\varphi})$ instead of $((M \times g^*)(z,\mu),\mathcal{H}(z,\mu))$. Let $\tilde{K} : \tilde{M} \rightarrow \pi_c \times M$ be the projection into the $\pi_c$-factor.

Let $\tilde{\mathcal{H}}$ be the closure of $\mathcal{H}$ in $g^*$. Since $\tilde{\mathcal{H}}$ is a closed subgroup of $(g^*,+)$, the quotient $C := g^*/\tilde{\mathcal{H}}$ is a cylinder (that is, it is isomorphic to the Abelian Lie group $\mathbb{R}^a \times \mathbb{T}^b$ for some $a,b \in \mathbb{N}$). Let $\pi_c : g^* \rightarrow g^*/\tilde{\mathcal{H}}$ be the projection. Define $K : M \rightarrow C$ to be the map that makes the following diagram commutative:

$$
\begin{array}{c}
\tilde{M} \\
\downarrow \tilde{\varphi} \\
\tilde{\mathcal{H}}
\end{array}
\quad
\begin{array}{c}
g^* \\
\downarrow \pi_c \\
g^*/\tilde{\mathcal{H}}
\end{array}
\quad
\begin{array}{c}
M \\
\downarrow K \\
g^*/\tilde{\mathcal{H}}
\end{array}
(3.2)

In other words, $K$ is defined by $K(m) = \pi_c(v)$, where $v \in g^*$ is any element such that $(m,v) \in \tilde{M}$.

We will refer to $K : M \rightarrow g^*/\tilde{\mathcal{H}}$ as a cylinder valued momentum map associated to the canonical $g$-action on $(M,\omega)$ and to $\tilde{\mathcal{H}}$ as the Hamiltonian holonomy of the $g$-action on $(M,\omega)$.

**Elementary properties.** The cylinder valued momentum map is a strict generalization of the standard (Kostant-Souriau) momentum map since the $g$-action has a standard momentum map if and only if the holonomy group $\mathcal{H}$ is trivial. In such a case the cylinder valued momentum map is a standard momentum map. The cylinder valued momentum map satisfies Noether’s Theorem, that is, for any $g$-invariant function $h \in C^\infty(M)^g := \{ f \in C^\infty(M) \mid df(\xi_m) = 0 \text{ for all } \xi \in g \}$, the flow $F_t$ of its associated Hamiltonian vector field $X_h$ satisfies the identity $K \circ F_t = K_{\text{Dom}(F_t)}$. Additionally, for any $v_m \in T_m M$, $m \in M$, $T_mK(v_m) = T_m\pi_c(v)$, where $\mu \in g^*$ is any element such that $K(m) = \pi_c(\mu)$ and $v \in g^*$ is uniquely determined by $\langle v,\xi \rangle = (K_{\xi_m}(\omega))(m)(v_m)$, for any $\xi \in g$. Also, $\ker(T_mK) = \left(\left(\text{Lie}(\tilde{\mathcal{H}})\right)^\circ \cdot m\right)^\circ$, where $\text{Lie}(\tilde{\mathcal{H}}) \subset g^*$ is the Lie algebra of $\tilde{\mathcal{H}}$, and range $(T_mK) = T_m\pi_c((g_m)^\circ)$ (Bifurcation Lemma). In the first statement we use the fact that the annihilator $\left(\text{Lie}(\tilde{\mathcal{H}})\right)^\circ$ is a Lie subalgebra of $g$. The notation $\mathfrak{k} \cdot m$ for any Lie subalgebra $\mathfrak{k} \subset g$ means the vector subspace of $T_m M$ formed by evaluating all infinitesimal generators $\eta_{\mathfrak{k} m}$ at the point $m \in M$ for all $\eta \in \mathfrak{k}$. The upper index $\circ$ denotes the $\omega$-orthogonal complement of the set in question.

**Equivariance properties of the cylinder valued momentum map.** Suppose now that the $g$-Lie algebra action on $(M,\omega)$ is obtained from a canonical action of the Lie group $G$ on $(M,\omega)$ by taking the infinitesimal generators of all elements in $g$. There is a $G$-action on the target space of the cylinder valued momentum map $K : M \rightarrow g^*/\tilde{\mathcal{H}}$ with respect to which it is $G$-equivariant. This action is constructed by noticing first that the Hamiltonian holonomy $\mathcal{H}$ is invariant under the coadjoint action, that is, $\text{Ad}_g^{\star-1}\mathcal{H} \subset \mathcal{H}$, for any $g \in G$. Actually, if $G$ is connected, then $\mathcal{H}$ is pointwise fixed by the coadjoint action. Hence, there is a unique group action $\mathcal{Ad}_g^* : G \times g^*/\tilde{\mathcal{H}} \rightarrow g^*/\tilde{\mathcal{H}}$ such that for any $g \in G$, $\mathcal{Ad}_g^* \circ \pi_c = \pi_c \circ \text{Ad}_g^{\star-1}$. With this in mind, we define $\mathcal{S} : G \times M \rightarrow g^*/\tilde{\mathcal{H}}$ by

$$
\mathcal{S}(g,m) := K(\Phi_g(m)) = \mathcal{Ad}_g^{\star-1}K(m).
$$
Since $M$ is connected by hypothesis, it can be shown that $\Theta$ does not depend on the points $m \in M$ and hence it defines a map $\sigma : G \to \mathfrak{g}^*/\mathcal{H}$ which is a group valued one-cocycle: for any $g, h \in G$, it satisfies the equality $\sigma(gh) = \sigma(g) + \mathcal{A}d^*_{\sigma(g)} \sigma(h)$. This guarantees that the map

$$
\Theta : \ G \times \mathfrak{g}^*/\mathcal{H} \longrightarrow \mathfrak{g}^*/\mathcal{H},
\quad (g, \pi_{C}(\mu)) \longmapsto \mathcal{A}d^*_{\sigma(g)}(\pi_{C}(\mu)) + \sigma(g),
$$

defines a $G$-action on $\mathfrak{g}^*/\mathcal{H}$ with respect to which the cylinder valued momentum map $K$ is $G$-equivariant, that is, for any $g \in G, m \in M$, we have

$$
K(\Phi_{g}(m)) = \Theta_{g}(K(m)).
$$

We will refer to $\sigma : G \to \mathfrak{g}^*/\mathcal{H}$ as the non-equivariance one-cocycle of the cylinder valued momentum map $K : M \to \mathfrak{g}^*/\mathcal{H}$ and to $\Theta$ as the affine $G$-action on $\mathfrak{g}^*/\mathcal{H}$ induced by $\sigma$. The infinitesimal generators of the affine $G$-action on $\mathfrak{g}^*/\mathcal{H}$ are given by the expression

$$
\xi_{g}^\pi_{C}(\mu) = -T_{\mu} \pi_{C}(\Psi(m)(\xi, \cdot)),
$$

for any $\xi \in \mathfrak{g}$, $(m, \mu) \in \tilde{M}$, where $\Psi : M \to Z^{2}(\mathfrak{g})$ is the Chu map defined by $\Psi(\xi, \eta) := \omega(\xi_{M}, \eta_{M})$, for any $\xi, \eta \in \mathfrak{g}$.

**The Poisson structure on $\mathfrak{g}^*/\mathcal{H}$**. The bracket $\{\cdot, \cdot\}_{\mathfrak{g}^*/\mathcal{H}} : \mathcal{C}^{\infty}(\mathfrak{g}^*/\mathcal{H}) \times \mathcal{C}^{\infty}(\mathfrak{g}^*/\mathcal{H}) \to \mathbb{R}$ defined by

$$
\{f, g\}_{\mathfrak{g}^*/\mathcal{H}}(\pi_{C}(\mu)) = \Psi(m) \left( \frac{\delta(f \circ \pi_{C})}{\delta \mu}, \frac{\delta(g \circ \pi_{C})}{\delta \mu} \right),
$$

where $f, g \in \mathcal{C}^{\infty}(\mathfrak{g}^*/\mathcal{H}), (m, \mu) \in \tilde{M}$, $\pi_{C} : \mathfrak{g}^* \to \mathfrak{g}^*/\mathcal{H}$ is the projection, and $\Psi : M \to Z^{2}(\mathfrak{g})$ is the Chu map, defines a Poisson structure on $\mathfrak{g}^*/\mathcal{H}$ such that $K : M \to \mathfrak{g}^*/\mathcal{H}$ is a Poisson map.

### 3.2 A normal form for the cylinder valued momentum map

A major technical tool in some proofs of the classical convexity theorems for the standard momentum map is a normal form obtained by Marle [21] and by Guillemin and Sternberg [14]. This normal form is a version of the classical Slice Theorem for proper group actions adapted to the symplectic symmetric setup that provides a semi-global set of coordinates (global only in the direction of the group orbits) in which the standard momentum map takes a particularly convenient and simple form and in which the conditions of the Local-to-Global Principle can be verified. This normal form has been generalized to the context of the cylinder valued momentum map in [23] [28]. We briefly review this generalization in the following paragraphs.

In this section we will work on a connected and paracompact symplectic manifold $(M, \omega)$ acted properly and symplectically upon by the Lie group $G$ with Lie algebra $\mathfrak{g}$. The first step in the construction of the symplectic slice theorem is the splitting of the Lie algebra $\mathfrak{g}$ of $G$ into three parts. The first summand is defined by

$$
\mathfrak{e} := \left\{ \xi \in \mathfrak{g} \mid \xi_{M}(m) \in (\mathfrak{g} \cdot m)^{\omega(m)} \right\},
$$

(3.4)
where \( m \in M \) is the point around whose \( G \)-orbit we want to construct the symplectic slice. The set \( \mathfrak{t} \) is clearly a vector subspace of \( \mathfrak{g} \) that contains the Lie algebra \( \mathfrak{g}_m \) of the isotropy subgroup \( G_m \) of the point \( m \in M \). In fact, \( \mathfrak{t} \) is a Lie subalgebra of \( \mathfrak{g} \). Since the \( G \)-action is by hypothesis proper, the isotropy subgroup \( G_m \) is compact and hence there is an \( Ad_{G_m} \)-invariant inner product \( \langle \cdot, \cdot \rangle_\mathfrak{g} \) on \( \mathfrak{g} \). We decompose

\[
\mathfrak{t} = \mathfrak{g}_m \oplus \mathfrak{m} \quad \text{and} \quad \mathfrak{g} = \mathfrak{g}_m \oplus \mathfrak{m} \oplus \mathfrak{q},
\]

where \( \mathfrak{m} \) is the \( \langle \cdot, \cdot \rangle_\mathfrak{g} \)-orthogonal complement of \( \mathfrak{g}_m \) in \( \mathfrak{t} \) and \( \mathfrak{q} \) is the \( \langle \cdot, \cdot \rangle_\mathfrak{g} \)-orthogonal complement of \( \mathfrak{t} \) in \( \mathfrak{g} \). The splittings in (3.5) induce similar ones on the duals

\[
\mathfrak{t}^* = \mathfrak{g}_m^* \oplus \mathfrak{m}^* \quad \text{and} \quad \mathfrak{g}^* = \mathfrak{g}_m^* \oplus \mathfrak{m}^* \oplus \mathfrak{q}^*.
\]

Each of the spaces in this decomposition should be understood as the set of covectors in \( \mathfrak{g}^* \) that can be written as \( \langle \xi, \cdot \rangle_\mathfrak{g} \), with \( \xi \) in the corresponding subspace. For example, \( \mathfrak{q}^* = \{ \langle \xi, \cdot \rangle_\mathfrak{g} \mid \xi \in \mathfrak{q} \} \).

The subspace \( \mathfrak{q} \cdot m \) is a symplectic subspace of \((T_m M, \omega(m))\).

Let now \( \ll \cdot, \cdot \gg \) be a \( G_m \)-invariant inner product in \( T_m M \) (available again by the compactness of \( G_m \)). Define \( V \) as the orthogonal complement to \( \mathfrak{q} \cdot m \cap (\mathfrak{g} \cdot m)^{\omega(m)} = \mathfrak{t} \cdot m \) in \((\mathfrak{g} \cdot m)^{\omega(m)} \) with respect to \( \ll \cdot, \cdot \gg \), that is:

\[
(\mathfrak{g} \cdot m)^{\omega(m)} = \mathfrak{g} \cdot m \cap (\mathfrak{g} \cdot m)^{\omega(m)} \oplus V = \mathfrak{t} \cdot m \oplus V.
\]

The subspace \( V \) is a symplectic \( G_m \)-invariant subspace of \((T_m M, \omega(m))\) such that \( V \cap \mathfrak{q} \cdot m = \{0\} \).

Any such space \( V \) is called a \textbf{symplectic normal space} at \( m \). Since the \( G_m \)-action on \((V, \omega(m)|_V)\) is linear and symplectic it has a standard equivariant associated momentum map \( J_V : V \to \mathfrak{g}_m^* \) given by \( \langle J_V(v), \eta \rangle = \frac{1}{2} \omega(m)(\xi_v(v), v) \). The proof of the following two results can be found in [25,26].

**Proposition 3.1 (The symplectic tube)** Let \((M, \omega)\) be a connected paracompact symplectic manifold and \( G \) a Lie group acting properly and canonically on it. Let \( m \in M \), \( V \) be a symplectic normal space at \( m \), and \( \mathfrak{m} \subset \mathfrak{g} \) the subspace introduced in the splitting (3.5). Then there exist \( G_m \)-invariant neighborhoods \( \mathfrak{m}_1^* \) and \( V_r \) of the origin in \( \mathfrak{m}^* \) and \( V \), respectively, such that the twisted product

\[
Y_r := G \times_{\mathfrak{g}_m} (\mathfrak{m}_1^* \times V_r)
\]

is a symplectic manifold with the two–form \( \omega_{Y_r} \) defined by:

\[
\omega_{Y_r}([g, \rho, v])(T_{(g, \rho, v)} g \cdot m, \alpha_1, u_1) = T_{(g, \rho, v)} \pi(T_{(g, \rho, v)} g \cdot \xi_1, \alpha_1, u_1)
\]

\[
:= \langle \alpha_2 + T_J(V(u_2), \xi_2), -\alpha_1 + T_J(V(u_1), \xi_1) \rangle + \langle \rho + J(V(\xi_2), \xi_1), \xi_2 \rangle
\]

\[
+ \Psi(m)(\xi_1, \xi_2) + \omega(m)(u_1, u_2),
\]

where \( \Psi : M \to Z^2(\mathfrak{g}) \) is the Chu map associated to the \( G \)-action on \((M, \omega)\), \( \pi : G \times (\mathfrak{m}_1^* \times V_r) \to G \times_{\mathfrak{g}_m} (\mathfrak{m}_1^* \times V_r) \) is the projection, \([g, \rho, v] \in Y_r, \xi_1, \xi_2 \in \mathfrak{g}, \alpha_1, \alpha_2 \in \mathfrak{m}^*, \) and \( u_1, u_2 \in V \).

The Lie group \( G \) acts canonically on \((Y_r, \omega_{Y_r})\) by \( g \cdot [h, \eta, v] := [gh, \eta, v], \) for any \( g \in G \) and any \([h, \eta, v] \in Y_r\).

In the sequel will refer to the symplectic manifold \((Y_r, \omega_{Y_r})\) as a \textbf{symplectic tube} of \((M, \omega)\) at the point \( m \).
Theorem 3.2 (Symplectic Slice Theorem) Let \((M, \omega)\) be a symplectic manifold and let \(G\) be a Lie group acting properly and canonically on \(M\). Let \(m \in M\) and let \((Y_r, \omega_Y)\) be the \(G\)-symplectic tube at that point constructed in Proposition 3.1. Then there is a \(G\)-invariant neighborhood \(U\) of \(m\) in \(M\) and a \(G\)-equivariant symplectomorphism \(\phi : U \rightarrow Y_r\) satisfying \(\phi(m) = [e, 0, 0]\).

We now provide an expression in the symplectic tube for the cylinder valued momentum map. This is what we call the normal form for the cylinder valued momentum map. The proof of the following theorem can be found in [28].

Theorem 3.3 (Normal form for the cylinder valued momentum map) Let \((M, \omega)\) be a connected paracompact symplectic manifold acted properly and canonically upon by a connected Lie group \(G\). Let \(m \in M\) and \((Y_r, \omega_Y)\) be a symplectic tube at \(m\) that models a \(G\)-invariant neighborhood \(U\) of the orbit \(G \cdot m\) via the \(G\)-equivariant symplectomorphism \(\phi : (Y_r, \omega_Y) \rightarrow (U, \omega|_U)\). Let \(K : M \rightarrow g^* / \mathcal{H}\) be a cylinder valued momentum map associated to the \(G\)-action on \(M\) with non-equivariance one-cocycle \(\sigma : G \rightarrow g^* / \mathcal{H}\). Then for any \([g, \rho, v] \in Y_r\) we have

\[
K(\phi[g, \rho, v]) = \Theta_g(K(m)) + \pi_C(\rho + J_v(v)) + \pi_C(\text{Ad}_{\gamma^{-1}}(\rho + J_v(v)))
\]

where \(\pi_C : g^* \rightarrow g^* / \mathcal{H}\) is the projection and \(\Theta : G \times g^* / \mathcal{H} \rightarrow g^* / \mathcal{H}\) is the affine action associated to the non-equivariance one-cocycle \(\sigma\).

3.3 Closed Hamiltonian holonomies and covering spaces

We will use in our study of the convexity properties of the cylinder valued momentum map a hypothesis that allows us to naturally endow the target space of this map with all the necessary metric properties. More specifically, we will assume in all that follows that the Hamiltonian holonomy \(\mathcal{H}\) is a closed subgroup of \((g^*, +)\).

In order to spell out the implications of this hypothesis we introduce some terminology. Let \(G\) be a group that acts on a topological space \(X\). This action is called totally discontinuous if every point \(x \in X\) has a neighborhood \(U\) such that \(g \cdot U \cap U = \emptyset\) for all \(g \in G\) satisfying \(g \cdot x \neq x\).

Let \(X\) and \(Y\) be two topological spaces and \(f : X \rightarrow Y\) a continuous map. An open set \(V \subset Y\) is said to be evenly covered if its inverse image \(f^{-1}(V)\) is a disjoint union of open sets \(U_i \subset X\) such that the restrictions \(f|_{U_i} : U_i \rightarrow V\) are homeomorphisms. The map \(f\) is a covering map if every point \(y \in Y\) has an evenly covered neighborhood.

A continuous map \(f : X \rightarrow Y\) between two topological spaces induces a homomorphism of fundamental groups \(f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)\) which maps the class of a loop \(\gamma\) in \(X\) to the class of the loop \(f \circ \gamma\) in \(Y\). A covering map \(f : X \rightarrow Y\) is said to be regular if the following equivalent properties hold:

(i) \(f_* (\pi_1(X, x_0))\) is a normal subgroup of \(\pi_1(Y, y_0)\).

(ii) \(f_* (\pi_1(X, x_0))\) does not depend on \(x_0 \in f^{-1}(x_0)\).
A deck transformation of a covering $f : X \to Y$ is a homeomorphism which permutes the sheets of the cover or equivalently permutes the points of $f^{-1}(y)$ for any $y \in Y$. The proof of the following two results can be found in [7], Propositions 3.4.15 and 3.4.16.

**Proposition 3.4** Let $G$ be a group acting on a topological space $X$ freely and totally discontinuously. Then, the projection $\pi_G : X \to X/G$ onto the orbit space is a regular covering map. Moreover, the group of its deck transformations coincides with $G$.

**Theorem 3.5** Let $f : X \to Y$ be a regular covering and $G$ its group of deck transformations. Then, the length metrics on $Y$ are in one-to-one correspondence with the $G$-invariant length metrics on $X$ so that for corresponding metrics $d_X$ on $X$ and $d_Y$ on $Y$, $f$ is a local isometry.

**Lemma 3.6** Let $X$ and $Y$ be two metric spaces and $f : X \to Y$ a covering map that is also a local isometry. If $X$ is complete then $Y$ is complete.

**Proof.** Let $\{y_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $(Y, d_Y)$. Given that the map $f$ is a covering map and a local isometry there exists $\epsilon_0 > 0$ small enough so that $f^{-1}(B_Y(y, \epsilon_0)) = \bigcup_{x \in f^{-1}(y)} B_X(x, \epsilon_0)$ is a disjoint union. As $\{y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence, there exists $N_{\epsilon_0} \in \mathbb{N}$ such that $d_Y(y_n, y_m) < N_{\epsilon_0}$ for all $n, m \geq N_{\epsilon_0}$. Consequently, for $m \geq N_{\epsilon_0}$ we have that $y_m \in B_Y(y_{N_{\epsilon_0}}, \epsilon_0)$. Select an arbitrary ball $B_X(x_0, \epsilon_0)$ in the disjoint union $\bigcup_{x \in f^{-1}(y_{N_{\epsilon_0}})} B_X(x, \epsilon_0)$. Define the sequence $\{x_m\}_{m \geq N_{\epsilon_0}}$, $x_m := f^{-1}(y_m) \cap B_X(x_0, \epsilon_0)$.

Since $f$ is a local isometry, the sequence $\{x_m\}_{m \geq N_{\epsilon_0}}$ is clearly Cauchy in $X$ and hence the completeness of $X$ and the continuity of $f$ ensures that $\{y_n\}_{n \in \mathbb{N}}$ has a convergent subsequence. ■

We now come back to the implications of the closedness hypothesis on the Hamiltonian holonomy $\mathcal{H}$.

**Proposition 3.7** Let $(M, \omega)$ be a connected paracompact symplectic manifold acted canonically upon by the Lie algebra $\mathfrak{g}$ with Hamiltonian holonomy $\mathcal{H} \subset \mathfrak{g}^*$. If $\mathcal{H}$ is a closed subset of $\mathfrak{g}^*$ then:

(i) The projection $\pi_C : \mathfrak{g}^* \to \mathfrak{g}^*/\mathcal{H}$ is a regular covering smooth map and hence the Euclidean metric in $\mathfrak{g}^*$ projects naturally to a length metric on $\mathfrak{g}^*/\mathcal{H}$ with respect to which this space is complete and locally compact and the projection $\pi_C$ is a local isometry.

(ii) Suppose that there exists a compact connected Lie group $G$ whose Lie algebra is $\mathfrak{g}$. Identify the positive Weyl chamber $\mathfrak{t}^*_+ \subset \mathfrak{g}^*$ with the orbit space of the coadjoint action of $G$ on $\mathfrak{g}^*$. Then, $\mathfrak{t}^*_+$ is a closed convex subset of $\mathfrak{g}^*$ and hence it has a natural length space structure with respect to which it is complete and locally compact. $\mathcal{H}$ acts on $\mathfrak{t}^*_+$ in a totally discontinuous fashion and hence the length metric in $\mathfrak{t}^*_+$ projects naturally to a length metric on $\mathfrak{t}^*_+ / \mathcal{H}$ with respect to which this space is complete and locally compact and the orbit space projection $\pi^*_C : \mathfrak{t}^*_+ \to \mathfrak{t}^*_+ / \mathcal{H}$ is a local isometry.

(iii) In the hypotheses of part (ii), the natural identification of $\mathfrak{t}^*_+$ with the orbit space $\mathfrak{g}^*/G$ of the coadjoint action induces an identification of $\mathfrak{t}^*_+ / \mathcal{H}$ with the orbit space $(\mathfrak{g}^*/\mathcal{H})/G$ of the $\mathcal{A}d^*$
action of $G$ on $g^*/\mathcal{H}$ with respect to which the following diagram commutes:

$$
g^* \xrightarrow{\pi_C} \frac{g^*}{\mathcal{H}} \xrightarrow{\pi_G} \frac{g^*}{G} \xrightarrow{\pi_C^+} \frac{t^*_+}{\mathcal{H}} \xrightarrow{\pi_G} \frac{t^*_+/G}{(g^*/\mathcal{H})/G}. \quad (3.11)
$$

**Proof.** Since $\mathcal{H}$ acts on $(g^*, +)$ by translations, the Euclidean metric in $g^*$ is $\mathcal{H}$-invariant. Additionally, as this action is free and proper, the Slice Theorem guarantees that any point $\mu \in g^*$ has a $\mathcal{H}$-invariant neighborhood that is equivariantly diffeomorphic to the product $\mathcal{H} \times U$, with $U \subset g^*$ an open neighborhood of zero in $g^*$. In this semiglobal model the point $\mu$ is represented by the element $(0, 0)$. Since $\mathcal{H}$ is a closed zero dimensional submanifold of $g^*$, it follows that the set $\{0\} \times U$ is a open neighborhood of $(0, 0) \equiv \mu$. Moreover, for any $\nu \in \mathcal{H}$ different from zero, we have $\nu \cdot (\{0\} \times U) = \{\nu\} \times U$ and since $(\{\nu\} \times U) \cap (\{0\} \times U) = \emptyset$ we conclude that $\mathcal{H}$ acts totally discontinuously on $g^*$. The statement in part (i) follows then by Proposition 3.4, Theorem 3.5, and Lemma 3.6. The local compactness of $g^*/\mathcal{H}$ is a consequence of the open character of the orbit projection $\pi_C$ and the local compactness of $g^*$ (we recall that every orbit projection is an open map and that if $f : X \to Y$ is an arbitrary open map from a locally compact topological space onto a Hausdorff space $Y$, then $Y$ is locally compact).

As to part (ii) we recall that the positive Weyl chamber $t_+^*$ is a closed convex subset of $g^*$ and hence a complete and locally compact length metric space (see Definition 4.11 and Lemma 4.12). We now recall that if $G$ is connected, then $\mathcal{H}$ is pointwise fixed by the coadjoint action $[27]$ and hence the $G$-coadjoint action and the $\mathcal{H}$-action on $g^*$ commute, which guarantees that the $\mathcal{H}$-action on $g^*$ drops to an $\mathcal{H}$-action on $g^*/G \simeq t_+^*$. Since this action can be viewed as the restriction to $t_+^*$ of the totally discontinuous $\mathcal{H}$-action on $g^*$, we conclude that it is also totally discontinuous and hence Proposition 3.4, Theorem 3.5, and Lemma 3.6 apply, which establishes the statement.

Regarding part (iii), the identification $t_+^*/\mathcal{H} \simeq (g^*/\mathcal{H})/G$ is a consequence of the fact that the $G$-coadjoint action and the $\mathcal{H}$-action on $g^*$ commute because $G$ is connected. The rest of the statement is a straightforward diagram chasing exercise.

### 3.4 Convexity properties of the cylinder valued momentum map. The Abelian case.

In this subsection it will be shown that the image of the cylinder valued momentum map associated to a proper Abelian Lie group action is weakly convex. The approach taken to prove this statement consists in using the Symplectic Slice Theorem [2.12] to show that this map satisfies the local hypotheses needed to apply the generalization of the Local-to-Global Principle for length spaces (Theorem 2.15). The main step in that direction is taken in the following proposition.

**Proposition 3.8** Let $(M, \omega)$ be a connected paracompact symplectic manifold and let $G$ be a connected Abelian Lie group acting properly and canonically on $M$ with closed Hamiltonian holonomy $\mathcal{H}$. Let $K : M \to g^*/\mathcal{H}$ be a cylinder valued momentum map for this action and $m \in M$ arbitrary such that $K(m) = [\mu] \in g^*/\mathcal{H}$. Then there exists an open neighborhood $U$ of $m$ in $M$ and open neighborhoods $W$ and $V$ of $\mu \in g^*$ and $[\mu] \in g^*/\mathcal{H}$, respectively, such that $K(U) \subset V$, $\pi_C|_W : W \to V$ is a diffeomorphism, and

$$
\pi_C|_W^{-1} \circ K|_U = J_U + c, \quad (3.12)
$$

where $J_U$ is the canonical momentum map on $U$.
with \( c \in \mathfrak{g}^* \) a constant and \( J_U : U \rightarrow \mathfrak{g}^* \) a map that in symplectic slice coordinates around the point \( m \) has the expression
\[
J_U([g, p, v]) = p + J_V(v) - \langle \mathbb{P}_q(\exp^{-1}(s([g])))\rangle_q.
\]
(3.13)
The neighborhood \( U \) has been chosen so that it can be written in slice coordinates as \( U \equiv U_e \times G_m(m^* \times V_r) \), with \( U_e \) an open \( G_m \)-invariant neighborhood of \( e \) in \( G \) small enough so that there exists a local section \( s : U_e/G_m \rightarrow V_e \) for the projection \( G \to G/G_m \). \( V_e \) is an open neighborhood of \( e \) in \( G \) such that \( \exp : U_0 \to V_e \) is a diffeomorphism, for some open neighborhood \( U_0 \) of \( 0 \in \mathfrak{g}^* \). \( \mathbb{P}_q : \mathfrak{g} = \mathfrak{g}_m \oplus \mathfrak{m} \oplus \mathfrak{q} \rightarrow \mathfrak{q} \) is the projection onto \( \mathfrak{q} \) constructed using the splitting in \( \mathfrak{g} \) and \( \langle \cdot, \cdot \rangle_q \) is the non-degenerate bilinear form on \( \mathfrak{q} \) induced by the Chu map at \( m \), that is, for any \( \xi, \eta \in \mathfrak{q}, \langle \xi, \eta \rangle_q := \omega(m)(\xi_M(m), \eta_M(m)) \).

**Proof.** Due to the closedness hypothesis on the Hamiltonian holonomy \( \mathcal{H} \), the projection \( \pi_C \) is a local diffeomorphism (see Proposition 3.7) and hence there exists an open neighborhood \( \mathbb{P}_q \circ \exp \) of some element in the fiber \( \pi_C^{-1}(K(m)) \subset \mathfrak{g}^* \) such that \( \pi_C \) is the bundle projection. Let \( U \) be the connected component containing \( m \), that is, for any \( \xi, \eta \in \mathfrak{q}, \langle \xi, \eta \rangle_q := \omega(m)(\xi_M(m), \eta_M(m)) \).

We start by noticing that (3.13) is well defined because for any \( h \in G_m \) and any \( \xi, \eta, \zeta \in \mathfrak{g} \), the map \( J_U^\xi := \langle J_U, \xi \rangle \) satisfies
\[
X_{J_U^\xi}(z) = \xi_M(z),
\]
(3.14)
with \( X_{J_U^\xi} \) the Hamiltonian vector field associated to the function \( J_U^\xi \in C^\omega(M) \). Since this is a local statement, it suffices to show that
\[
i_{\omega_Y} \omega_Y([\exp \zeta, p, v]) = dJ_U^\xi([\exp \zeta, p, v]),
\]
(3.15)
where \([\exp \zeta, p, v] \) is the expression of \( z \) in slice coordinates and \( \zeta \in \mathfrak{g} \) chosen so that \( s([\exp \zeta]) = \exp \zeta \).

We prove (3.15) by using the expression of \( \omega_Y \) in Proposition 3.1. First of all, notice that
\[
\xi_M([\exp \zeta, p, v]) = T_{(\exp \zeta, p, v)}(T_e \exp \zeta(\xi_M), 0, 0),
\]
where \( \pi : G \times (m_m^* \times V_r) \to G \times G_m(m^*_m \times V_r) \) is the orbit projection. If \( w := T_{(\exp \zeta, p, v)}(T_e \exp \zeta(\eta), \alpha, u) \in T_{[\exp \zeta, p, v]}(G \times G_m(m^*_m \times V_r)) \) we have
\[
\omega_Y([\exp \zeta, p, v])(\xi_M([\exp \zeta, p, v]), w) = \langle \alpha + T_e J_u(\alpha), \xi_M(m), \eta_M(m) \rangle
\]
(3.16)
On the other hand, by \((3.13)\), we have
\[
\mathbf{d}J^5_U([\exp \zeta, p, v]) \cdot w = \langle \alpha + T_c J_U (u), \xi \rangle - \frac{d}{dt} \bigg|_{t=0} \langle \mathbb{P}_q \exp^{-1} s([\exp \zeta \exp \eta]), \mathbb{P}_q \xi \rangle. \tag{3.17}
\]

In order to compute the second summand of the right hand side, notice that
\[
s([\exp \zeta \exp \eta]) = s([\exp (\zeta + \eta)]) = \exp (\zeta + \eta) h(t),
\]
with \(h(t)\) a curve in \(G_m\) such that \(h(0) = e\) and \(h'(0) = \lambda \in \mathfrak{g}_m\). Consequently,
\[
\frac{d}{dt} \bigg|_{t=0} s([\exp \zeta \exp \eta]) = T_c L_{\exp \zeta} (\eta + \lambda) = \frac{d}{dt} \bigg|_{t=0} \exp (\zeta + t(\eta + \lambda))
\]
and hence, since \(\mathbb{P}_q \lambda = 0\), we have
\[
\frac{d}{dt} \bigg|_{t=0} \langle \mathbb{P}_q \exp^{-1} s([\exp \zeta \exp \eta]), \mathbb{P}_q \xi \rangle = \frac{d}{dt} \bigg|_{t=0} \langle \mathbb{P}_q (\zeta + t(\eta + \lambda)), \mathbb{P}_q \xi \rangle = \langle \mathbb{P}_q \eta, \mathbb{P}_q \xi \rangle. \tag{3.18}
\]

The equalities \((3.16)\), \((3.17)\), and \((3.18)\) show that \((3.14)\) holds.

With this in mind we will now show that for any \(z \in U\)
\[
T_c (\pi_C|_W^{-1} \circ K|_U) = T_c J_U. \tag{3.19}
\]

Indeed, for any \(v \in T_z M\) and \(p \in \pi_C^{-1}(V)\) such that \(K(z) = \pi_C(p)\),
\[
T_c (\pi_C|_W^{-1} \circ K|_U)(v) = \left( T_{K(z)} \pi_C|_W^{-1} \circ T_c K \right)(v) = \left( T_{K(z)} \pi_C|_W^{-1} \circ T_p \pi_C \right)(v) = v,
\]
where \(v \in \mathfrak{g}^*\) is uniquely determined by the expression
\[
\langle v, \xi \rangle = \langle i_{\xi_M} \omega \rangle (z) \langle v \rangle, \quad \text{for all } \xi \in \mathfrak{g}. \tag{3.20}
\]

On the other hand, by \((3.14)\), we can write
\[
\langle T_c J_U (v), \xi \rangle = \mathbf{d}J^5_U (z)(v) = (i_{\xi_M} \omega) (z) \langle v \rangle.
\]

This, together with \((3.20)\), shows that \((3.19)\) holds.

Let \(c(t)\) be a smooth curve such that \(c(0) = m\) and \(c(1) = z\), available by the connectedness of \(U\). Then by \((3.19)\)
\[
\left( \pi_C|_W^{-1} \circ K \right)(z) = \int_0^1 \frac{d}{dt} \left( \pi_C|_W^{-1} \circ K \right)(c(t)) dt
\]
and
\[
\int_0^1 T_{c(t)} \left( \pi_C|_W^{-1} \circ K \right)(\dot{c}(t)) dt = \int_0^1 T_{c(t)} J_U (\dot{c}(t)) dt = J_U (z) - J_U (m).
\]

Since \(z \in M\) is arbitrary and \(m \in M\) is fixed, the previous equality shows that \((3.12)\) holds by setting \(c = \left( \pi_C|_W^{-1} \circ K \right)(m) - J_U (m)\). \[\square\]
Theorem 3.9 Let \((M,\omega)\) be a connected paracompact symplectic manifold and let \(G\) a connected Abelian Lie group acting properly and canonically on \(M\) with closed Hamiltonian holonomy \(\mathcal{H}\). Let \(K : M \to g^*/\mathcal{H}\) be a cylinder valued momentum map for this action. If \(K\) is a closed map then the image \(K(M) \subset g^*/\mathcal{H}\) is a weakly convex subset of \(g^*/\mathcal{H}\). We think of \(g^*/\mathcal{H}\) as a length metric space with the length metric naturally inherited from \(g^*/H\) (see Proposition 3.7). If, in addition, \(g^*/\mathcal{H}\) is uniquely geodesic then \(K(M)\) is convex, \(K\) has connected fibers, and it is open onto its image.

Proof. We will establish this result by using the Local-to-Global Principle for length spaces (Theorem 2.15). First of all, notice that the closedness of the Hamiltonian holonomy implies, by Proposition 3.7, that \(g^*/\mathcal{H}\) is a complete and locally compact length space and that the projection \(\pi_C\) is a local isometry. Therefore, in order to apply Theorem 2.15 we need to show that \(K\) is locally open onto its image, locally fiber connected, and has local convexity data. Now, by Proposition 3.8 (more specifically by (3.12)) and Lemma 4.13, it suffices to prove that those three local properties are satisfied by the map \(J_U : U \to g^*\).

We start the proof of this fact by recalling that since the \(G\)-action is proper the isotropy subgroup \(G_m\) is compact and hence its connected component containing the identity is isomorphic to a torus. Consequently, the map \(J_V : V \to g^*_m\) is the momentum map of the symplectic representation of a torus on the symplectic vector space \(V\) and hence it automatically has (see for instance [15] for a proof) local convexity data and it is locally fiber connected and locally open onto its image. Additionally, if we split \(g^*\) as \(g^*/g^*_m \oplus m^* \oplus q^*\) then the map \(J_U - c\) can be decomposed as
\[
J_U([g,\rho,v]) - c = (J_V(v),\rho, -\langle \rho_q(\exp^{-1}(\lambda([g]))),\cdot\rangle_q).
\]
Each of the three components of the map has local convexity data, is locally open onto its image, and is locally fiber connected. Thus \(J_U\) also has these properties. ■

3.5 Convexity for Abelian Lie group valued momentum maps

We now discuss the convexity properties of the Lie group valued momentum maps introduced in [22, 10, 18, 17, 11]. We give the definition of these objects only for Abelian symmetry groups because in the non-Abelian case these momentum maps are defined on spaces that are not symplectic (they are called quasi Hamiltonian spaces) thereby leaving the category on which we focus in this paper.

Definition 3.10 Let \((M,\omega)\) be a symplectic manifold and \(T^k\) a torus acting canonically on \((M,\omega)\). The map \(\mu : M \to T^k\) is called a Lie group valued momentum map if for any \(\xi \in t\),
\[
\mathbf{i}_{\xi_m}\omega = \mu^*(\theta, \xi),
\]
where \(\theta \in \Omega^1(T^k, t)\) is the bi-invariant Maurer-Cartan form.

A typical example for this momentum map is provided by the following situation. Take the symplectic manifold \(T^2 = S^1 \times S^1\) with symplectic form the standard area form and consider the action of the circle on the first circle of \(T^2\). The \(S^1\)-valued momentum map associated to this action is the projection on the second circle of \(T^2\), namely, \(\mu(e^{i\theta_1}, e^{i\theta_2}) = e^{i\theta_2}\).
**Proposition 3.11** The image of any Abelian Lie group valued momentum map \( \mu : M \to T^k \) is a weakly convex subset of \( T^k \).

**Proof.** Alekseev, Malkin, and Meinrenken [1] (see Proposition 3.4 and Remark 3.3 of this paper) prove that for each point in \( M \) there is an open simply connected neighborhood \( U \subset M \) and a standard momentum map \( \Phi : U \to t \) (\( t \) and \( t^* \) are identified here) such that the restriction \( \exp|_{\Phi(U)} : \Phi(U) \to T^k \) is a diffeomorphism onto its image and one has \( \mu|_U = \exp \circ \Phi \). This immediately implies that \( \mu \) has the local properties (LFC), (LOI), and (LCD). Thus the hypotheses of Theorem 2.15 hold and the statement is a direct consequence of this theorem. \( \blacksquare \)

For a map with values in a metric spaces that is not uniquely geodesic, the convexity property of its image is not related to the connectedness of its fibers. This is in sharp contrast to the situation encountered for standard momentum maps. For instance, if in the above example we multiply the symplectic form by two then the associated \( S^1 \)-valued momentum map is \( \mu(e^{i\phi_1}, e^{i\phi_2}) = e^{2i\phi_2} \) which satisfies the hypothesis of Theorem 2.15 and hence has a convex image but does not have connected fibers.

### 3.6 Convexity properties of the cylinder valued momentum map. The Non-Abelian case.

The study of the convexity properties of the image of the cylinder valued momentum map for non-Abelian groups presents two main complications with respect to its Abelian analog. First, unless we assume additional hypotheses, we do not have available a result similar to Proposition 3.8 that provides a convenient local representation for \( \mathbf{K} \) out of which one can easily conclude the necessary local properties to ensure convexity out of the Local-to-Global Principle. Second, the entire image is not likely to be convex since, already in the standard momentum map case one has to take a convex piece of the dual of the Lie algebra to obtain convexity. We will take care of the first problem by working with special actions, namely those that are **tubewise Hamiltonian**, whose definition will be recalled below. As to the second question we will look, not at the image of the cylinder valued momentum map but, as expected from the classical case, at the intersection of this image with \( t^* / \mathcal{H} \), taking advantage at the same time of the good behavior of the projection \( \pi_C^+ : t^*_+ \to t^*_+ / \mathcal{H} \) introduced and discussed in the second part of Proposition 3.7.

**Definition 3.12** Let \((M, \omega)\) be a symplectic manifold acted canonically upon by a Lie group \( G \). For any point \( m \in M \), we say that the \( G \)-action on \( M \) is **tubewise Hamiltonian at** \( m \) if there exists a \( G \)-invariant open neighborhood of the orbit \( G \cdot m \) such that the restriction of the action to the symplectic manifold \((U, \omega)_{U} \) has an associated standard momentum map. The \( G \)-action on \( M \) is called **tubewise Hamiltonian** if it is tubewise Hamiltonian at any point of \( M \).

Sufficient conditions ensuring that a canonical action is tubewise Hamiltonian have been given in [25, 24]. For example, here are two useful results.

**Proposition 3.13** Let \((M, \omega)\) be a symplectic manifold and let \( G \) a Lie group with Lie algebra \( \mathfrak{g} \) acting properly and canonically on \( M \). For \( m \in M \) let \( Y_r := G \times_{G_m} (m^*_r \times V_r) \) be the slice model around the
orbit $G \cdot m$ introduced in Proposition 3.1 If the $G$-equivariant $\mathfrak{g}^*$-valued one form $\gamma \in \Omega^1(G; \mathfrak{g}^*)$ defined by
\[
(\gamma(g)(T_{eL_g}(\eta)), \xi) := -\omega(m)(\Ad_{e^{-\rho}}^{-1}(\xi), \eta_M(m))
\]
for any $g \in G$ and $\xi, \eta \in \mathfrak{g}$ is exact, then the $G$-action on $Y_f$ given by $g \cdot [h, \eta, v] := [gh, \eta, v]$, for any $g \in G$ and any $[h, \eta, v] \in Y_f$, has an associated standard momentum map and thus the $G$-action on $(M, \omega)$ is tubewise Hamiltonian at $m$.

**Corollary 3.14** Let $(M, \omega)$ be a symplectic manifold and let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ acting properly and canonically on $M$. If either

(i) $H^1(G) = 0$, or

(ii) the orbit $G \cdot m$ is isotropic

then the $G$-action on $(M, \omega)$ is tubewise Hamiltonian at $m$.

The following result is the analog of Proposition 3.8 in the non-Abelian setup.

**Proposition 3.15** Let $(M, \omega)$ be a connected paracompact symplectic manifold and let $G$ be a compact connected Lie group acting canonically on $M$ in a tubewise Hamiltonian fashion with closed Hamiltonian holonomy $\mathcal{H}$. Let $K : M \to \mathfrak{g}^*/\mathcal{H}$ be a cylinder valued momentum map for this action and $m \in M$ arbitrary such that $K(m) = [\mu] \in \mathfrak{g}^*/\mathcal{H}$. Then, there exists an open neighborhood $U$ of $m$ in $M$ and open neighborhoods $W$ and $V$ of $\mu \in \mathfrak{g}^*$ and $[\mu] \in \mathfrak{g}^*/\mathcal{H}$, respectively, such that $K(U) \subset V$, $\pi_C|_W : W \to V$ is a diffeomorphism, and
\[
\pi_C|_W^{-1} \circ K|_U = J_U + c,
\]
where $c \in \mathfrak{g}^*$ is a constant and $J_U : U \to \mathfrak{g}^*$ is a map that in symplectic slice coordinates around the point $m$ has the expression
\[
J_U([g, \rho, v]) = \Ad_{g^{-1}}(v + \rho + J_V(v)),
\]
with $v \in \mathfrak{g}^*$ a constant.

**Proof.** Due to the closedness hypothesis on the Hamiltonian holonomy $\mathcal{H}$, the projection $\pi_C$ is a local diffeomorphism (see Proposition 3.7) and hence there exists an open neighborhood $V$ of $K(m)$ in $\mathfrak{g}^*/\mathcal{H}$ and a neighborhood $W$ of some element in the fiber $\pi_C^{-1}(K(m)) \subset \mathfrak{g}^*$ such that $\pi_C|_W : W \to V$ is a diffeomorphism. Let $U$ be the connected component containing $m$ of the intersection of $K^{-1}(V)$ with the domain of a symplectic slice chart around $m$. Given that the $G$-action is by hypothesis tubewise Hamiltonian, the symplectic slice chart can be chosen so that the restriction of the $G$-action to that chart has a standard associated momentum map $J_Y$, that in slice coordinates has, by the Marle-Guillemin-Sternberg normal form [21] [14], the expression
\[
J_Y([g, \rho, v]) = \Ad_g^{-1}(v + \rho + J_V(v)) + \sigma(g),
\]
with $v \in \mathfrak{g}^*$ a constant and $\sigma : G \to \mathfrak{g}^*$ the non-equivariance one-cocycle of $J_Y$. Since the group $G$ is compact, $J_Y$ can be chosen equivariant and hence with trivial non-equivariance cocycle $\sigma$ (see [23] for the original source of this result, or [26], Proposition 4.5.19). Let $J_U$ be the restriction of that
equivariant momentum map to $U$. By the definition of the standard momentum map we have that for any $z \in U$ and any $\xi \in \mathfrak{g}$, the map $J^z_U := (J_U, \xi)$ satisfies $X^g_{J^z_U}(z) = \xi M(z)$, with $X^g_{J^z_U}$ the Hamiltonian vector field associated to the function $J^z_U \in C^\infty(M)$. With this in mind, it suffices to mimic the proof of Proposition 3.8 starting from expression (3.19) to establish the statement of the proposition. □

**Theorem 3.16** Let $(M, \omega)$ be a connected paracompact symplectic manifold and let $G$ a compact connected Lie group acting canonically on $M$ in a tubewise Hamiltonian fashion with closed Hamiltonian holonomy $\mathcal{H}$. Let $K : M \to \mathfrak{g}^*/\mathcal{H}$ be a cylinder valued momentum map for this action. If $K$ is a closed map then the intersection of the image $K(M) \subset \mathfrak{g}^*/\mathcal{H}$ with $t^+_* \mathcal{H}$ is a weakly convex subset of $\mathfrak{g}^*/\mathcal{H}$. We think of $\mathfrak{g}^*/\mathcal{H}$ and $t^+_* \mathcal{H}$ as length metric spaces with the length metric naturally inherited from $\mathfrak{g}^*$ (see Proposition 3.7). If, in addition, $t^+_* \mathcal{H}$ is uniquely geodesic then $K(M) \cap (t^+_* \mathcal{H})$ is convex, $K$ has connected fibers, and it is open onto its image.

**Proof.** First of all, notice that the closedness of the Hamiltonian holonomy implies, by Proposition 3.7, that $\mathfrak{g}^*/\mathcal{H}$ and $t^+_* \mathcal{H}$ are complete and locally compact length spaces and that the projections $\pi_C$ and $\pi_C^+$ are local isometries. Moreover, the identification $t^+_* \mathcal{H} \simeq (\mathfrak{g}^*/\mathcal{H})/G$, introduced in part (iii) of Proposition 3.7 and the diagram (3.11) allow us to think of $\pi_C^+$ as the restriction of $\pi_C$ to $t^+_* \mathcal{H}$. Consequently, if $V \subset \mathfrak{g}^*/\mathcal{H}$ is an open set such that $\pi_C^+|_W : W \cap t^+_* \mathcal{H} \rightarrow \pi_C^+(V) \simeq (W \cap t^+_* \mathcal{H})/\mathcal{H}$ is an isometry. Notice that $\pi_C(W) \simeq W \cap t^+_* \mathcal{H}$ is an open set in $t^+_* \mathcal{H}$, since $\pi_C$ is an open map.

Using the identification $K(M) \cap (t^+_* \mathcal{H}) \simeq (\mathfrak{g}^*/\mathcal{H})/G$ we can study the convexity properties of the intersection $K(M) \cap (t^+_* \mathcal{H})$ by looking at the convexity properties of the image of the map $k := \pi_C^+ \circ K : M \to t^+_* \mathcal{H}$. We will do so by applying the Local-to-Global Principle for length spaces (Theorem 2.15) to $k$, that is, by showing that $k$ is locally open onto its image, locally fiber connected, and has local convexity data. By Proposition 3.13 there exists an open neighborhood $U$ of $m$ in $M$ and an open neighborhood $V$ of $[\mu]$ in $\mathfrak{g}^*/\mathcal{H}$ such that $K(U) \subset V$, $\pi_C|_W : W \rightarrow V$ is a diffeomorphism, and $\pi_C^{-1}|_W \circ K|_U = J_U + \epsilon$, with $\epsilon \in \mathfrak{g}^*$ a constant and $J_U : U \to \mathfrak{g}^*$ a map that has the expression (3.23). If we apply $\pi_C$ to both sides of this equality and we use the commutativity of diagram (3.11) and the remarks above we obtain that

$$\left(\pi_C^+|_{\pi_C(W)}\right)^{-1} \circ k|_U = j_U + \pi_C(c),$$

with $j_U := \pi_C \circ J_U$. Two results due to Sjamaar [31, Theorem 6.5] and Knop [20, Theorem 5.1] show that $j_U$ is locally open onto its image, locally fiber connected, and has local convexity data. Consequently, since $\pi_C^+|_{\pi_C(W)}$ is an isometry, Lemma 4.13 guarantees that $k|_U$ also shares those three local properties. The statement of the theorem follows then as a consequence of Theorem 2.15. □

**Remark 3.17** The classical convexity theorem of Kirwan [19] states that if $G$ is a compact connected Lie group acting canonically on the compact connected symplectic manifold $(M, \omega)$ and this action has an associated standard coadjoint equivariant momentum map $J : M \to \mathfrak{g}^*$, then $J(M) \cap t^*_G$ is a compact convex polytope. The convexity part of this theorem can be obtained from Theorem 3.16 since the global existence of a standard momentum map implies that the action is, in particular, tubewise Hamiltonian and that a cylinder valued momentum map for it is the momentum map $J : M \to \mathfrak{g}^*$ ($\mathcal{H} = \{0\}$ in this case). Since the manifold $M$ is by hypothesis compact then $J$ is necessarily a closed map and,
moreover, in this case $t_+^* / \mathcal{H} = t_+^*$ is uniquely geodesic. Consequently, by Theorem 3.16 $\mathbf{J}(M) \cap t_+^*$ is convex, $\mathbf{J}$ has connected fibers, and it is a $G$-open map onto its image. See also [4] where the same result was obtained in a different manner.

4 Appendix: metric and length spaces

In this appendix we collect the standard results and we fix the notations that we use when dealing with metric and length spaces. Most of the quoted statements below can be found in Bridson and Haefliger [5] and Burago et al. [7].

Definition 4.1 Let $X$ be an arbitrary set. A function $d : X \times X \to \mathbb{R} \cup \{\infty\}$ is called a metric on $X$ if the following conditions are satisfied for all $x, y, z \in X$:

(i) Positiveness: $d(x, y) > 0$ if $x \neq y$ and $d(x, x) = 0$;

(ii) Symmetry: $d(x, y) = d(y, x)$;

(iii) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

A metric space is a pair $(X, d)$ with $d$ a metric on the set $X$.

Let $(X, d)$ and $(X', d')$ be two metric spaces. A map $f : X \to X'$ is called distance-preserving if $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. A bijective distance-preserving map is called an isometry. Note that a distance-preserving map is always injective.

Given $x \in X$ and $r > 0$ the open ball of radius $r$ and center $x$ is defined to be the set $B(x, r) := \{y \in X \mid d(x, y) < r\}$. Similarly, the closed ball of radius $r$ and center $x$ is the set $\overline{B}(x, r) := \{y \in X \mid d(x, y) \leq r\}$. On $(X, d)$ the topology given by the metric $d$ is, by definition, the topology whose basis of neighborhoods at every point $x \in X$ is the collection of all open balls $B(x, r)$ for $r > 0$. Thus, a set $U$ in the metric space $(X, d)$ is open if and only if for every point $x \in U$ there exists $r > 0$ such that $B(x, r) \subseteq U$. It is easy to see that $\overline{B}(x, r)$ are closed sets in the metric topology but, in general, they are strictly larger that the closure $\overline{B}(x, r)$ of the open balls in the same topology.

Definition 4.2 Let $(X, d)$ be a metric space. A sequence $\{x_n\}$ is called Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$. A metric space is called complete if every Cauchy sequence has a convergent subsequence.

Let $(X, d)$ be a metric space and $Y \subset X$ a subset. Then the restriction $d_Y$ of the metric $d$ to $Y \times Y$ defines a metric on $Y$. Thus, $(Y, d_Y)$ is a metric space whose metric topology coincides with the relative topology induced from the metric topology of $X$. Also, if $Y$ is a complete metric space relative to the induced metric $d_Y$, then $Y$ is closed in $X$ and if $(X, d)$ is a complete metric space and $Y$ is closed in $X$, then $Y$ is complete.

Let $(X, d)$ be a metric space. A curve or a path in $X$ is a continuous map $c : I \to X$ with $I$ a connected interval of $\mathbb{R}$. If $c_1 : [a_1, b_1] \to X$ and $c_2 : [a_2, b_2] \to X$ are two paths such that $c_1(b_1) = c_2(a_2)$, their concatenation is the path $c : [a_1, b_1 + b_2 - a_2] \to X$ defined by $c(t) = c_1(t)$ if $t \in [a_1, b_1]$ and $c(t) = c_2(t + a_2 - b_1)$ if $t \in [b_1, b_1 + b_2 - a_2]$.
Definition 4.3 The length \( l_d(c) \) of a curve \( c : [a, b] \to X \) induced by the metric \( d \) is

\[
l_d(c) := \sup_{\Delta_n} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})),
\]

where the supremum is taken over all possible partitions \( \Delta_n : a = t_0 \leq t_1 \leq \cdots \leq t_n = b \) of the interval \([a, b] \subset \mathbb{R}\).

Consequently, the length of a curve is a non-negative number or it is infinite. The curve \( c \) is said to be rectifiable if its length is finite.

Next, we recall several properties of the length of a curve in a metric space.

(i) \( l_d(c) \geq d(c(a), d(c(b))) \), for any path \( c : [a, b] \to X \).

(ii) If \( \phi : [a', b'] \to [a, b] \) is an onto monotone map, then \( l_d(c) = l_d(c \circ \phi) \).

(iii) Additivity: if \( c \) is the concatenation of two paths \( c_1 \) and \( c_2 \) then \( l_d(c) = l_d(c_1) + l_d(c_2) \).

(iv) If \( c \) is rectifiable of length \( l \), then the function \( \lambda : [a, b] \to [0, l] \) defined by \( \lambda(t) = l_d(c_{[a, t]} \) is a continuous weakly monotone function.

(v) Reparametrization by arc length: if \( c \) and \( \lambda \) are as in as in the previous point, then there is a unique path \( \tilde{c} : [0, l] \to X \) such that \( \tilde{c} \circ \lambda = c \) and \( l_d(\tilde{c}_{[0, t]} = t \).

(vi) Lower semicontinuity: let \( (c_n) \) be a sequence of paths \( [a, b] \to X \) converging uniformly to a path \( c \). If \( c \) rectifiable, then for every \( \varepsilon > 0 \), there exists an integer \( N_\varepsilon \) such that

\[
l_d(c) \leq l_d(c_n) + \varepsilon
\]

whenever \( n > N_\varepsilon \).

Next we will introduce the notion of length metric or inner metric. It is well known that every Riemannian metric on a manifold induces a length. Unfortunately, Riemannian metrics can be defined only in the differentiable setting. As we will see, length metrics share many properties with Riemannian metrics but they can be defined in more general settings.

Definition 4.4 Let \((X, d)\) be a metric space. The distance \( d \) is said to be a length metric or an inner metric if the distance between every pair of points \( x, y \in X \) is equal to the infimum of the length of rectifiable curves joining them. If there are no such curves then, by definition, \( d(x, y) = \infty \). If \( d \) is a length metric then \((X, d)\) is called a length space or an inner space.

Other authors refer to length spaces as path metric spaces (see, e.g., Gromov [11]). For various properties and characterizations of length metrics see Bridson and Haefliger [8], Burago et al. [7], and Gromov [11].
Having a metric space \((X, d)\) we can always construct a length metric \(\overline{d}\) induced by the initial metric \(d\) in the following way,

\[
\overline{d}(x, y) := \inf_{\gamma \in R_{x,y}} \int_0^1 \sqrt{g_{ij}(t)c_i(t)c_j(t)} \, dt,
\]

where \(R_{x,y} := \{\text{all rectifiable curves connecting } x \text{ and } y\}\). If there are no such curves then we set \(\overline{d}(x, y) = \infty\).

**Proposition 4.5 (Bridson and Haefliger [5])** The induced length metric has the following properties:

(i) \(\overline{d}\) is a metric.

(ii) \(\overline{d}(x, y) \geq d(x, y)\) for all \(x, y \in X\).

(iii) If \(c : [a, b] \to X\) is continuous with respect to the topology induced by \(\overline{d}\), then it is continuous with respect to the topology induced by \(d\). (The converse is false, in general).

(iv) If a map \(c : [a, b] \to X\) is a rectifiable curve in \((X, d)\), then it is a continuous and rectifiable curve in \((X, \overline{d})\).

(v) The length of a curve \(c : [a, b] \to X\) in \((X, \overline{d})\) is the same as its length in \((X, d)\).

(vi) \(\overline{d} = \overline{d}\).

The assertion in point (iii) of the above proposition is a consequence of the fact that the topology induced by the metric \(d\) is coarser than the topology induced by the metric \(\overline{d}\). Note that \((X, d)\) is a length space if and only if \(\overline{d} = d\).

Classical examples of a length spaces are Riemannian manifolds. Let \((X, g)\) be a Riemannian manifold and \(c : [a, b] \to X\) a piecewise differentiable path. The Riemannian length \(l_g(c)\) is defined as

\[
l_g(c) := \int_a^b \sqrt{g_{ij}(t)c_i(t)c_j(t)} \, dt.
\]

**Proposition 4.6** Let \(X\) be a connected Riemannian manifold. Given \(x, y \in X\), let \(d(x, y)\) be the infimum of the Riemannian length of piecewise continuously differentiable paths \(c : [0, 1] \to X\) such that \(c(0) = x\) and \(c(1) = y\). Then

(i) \(d\) is a metric on \(X\).

(ii) The topology on \(X\) defined by this distance is the same as the given manifold topology on \(X\).

(iii) \((X, d)\) is a length space.

**Definition 4.7** A curve \(c : [a, b] \to (X, d)\) is called a **shortest path** if its length is minimal among all the curves with the same endpoints. Shortest paths in length spaces are also called **distance minimizers**.
If \((X,d)\) is a length space then a curve \(c : [a,b] \to X\) is a shortest path if and only if its length is equal with the distance between endpoints, that is, \(l_d(c) = d(c(a), c(b))\).

Next we will introduce the notion of geodesic in length spaces that generalizes the one in Riemannian geometry.

**Definition 4.8** Let \((X,d)\) be a length space. A curve \(c : I \subset \mathbb{R} \to X\) is called **geodesic** if for every \(t \in I\) there exist a subinterval \(J\) containing a neighborhood of \(t\) in \(I\) such that \(c|_J\) is a shortest path. In other words, a geodesic is a curve which is locally a distance minimizer. A length space \((X,d)\) is called a **geodesic metric space** if for any two points \(x,y \in X\) there exists a shortest path between \(x\) and \(y\).

Clearly in a length space a shortest path is a geodesic. The extension of the Hopf-Rinow theorem from Riemannian geometry to the case of length metric spaces is due Cohn-Vossen and is a key result in this paper. Its proof can be found in Bridson and Haefliger [5] or Burago et al. [7].

**Theorem 4.9** (Hopf-Rinow-Cohn-Vossen) For a locally compact length space \((X,d)\), the following assertions are equivalent:

(i) \(X\) is complete,

(ii) every closed metric ball in \(X\) is compact.

If one of the above assertions holds, then for any two points \(x,y \in X\) there exists a shortest path connecting them. In other words, \((X,d)\) is a geodesic metric space.

**Corollary 4.10** Every complete, connected, Riemannian manifold is a geodesic metric space.

Having introduced the notion of shortest path one can define the key concept of metric convexity.

**Definition 4.11** A subset \(C\) in a metric space \((X,d)\) is said to be **convex** if the restriction of \(d\) to \(C\) is a finite length metric.

For the case of geodesic metric spaces we have the following characterization of convex sets.

**Lemma 4.12** Let \((X,d)\) be a geodesic metric space. Then a subset \(C \subset X\) is convex if and only if for any two points \(x,y \in C\) there exists a rectifiable shortest path \(\gamma\) connecting \(x\) and \(y\) which is entirely contained in \(C\).

**Proof.** Let \(d_C\) be the restriction of \(d\) to the subset \(C\) and assume \(C\) is a convex subset of \((X,d)\). Since \(d_C = d_C\) and \(d_C < \infty\) we have the following equality

\[
\inf_{\gamma \in R_{x,y}^C} l_d(\gamma) = d_C(x,y) = d_C(x,y) = \inf_{\gamma \in R_{x,y}^C} l_d(\gamma),
\]

where \(R_{x,y}^C = \{\gamma : [a,b] \to C \mid \gamma\) rectifiable curve entirely contained in \(C\) with \(\gamma(a) = x\) and \(\gamma(b) = y\}\). Observe that \(R_{x,y}^C \neq \emptyset\) because \(C\) is a length space whose metric \(d_C\) is finite. This shows that taking
the infimum over \( R_{x,y}^C \), completely determines \( d_C(x,y) \). Because \( (X,d) \) is a geodesic metric space this infimum is attained for a curve entirely contained in \( C \).

Conversely, suppose that for any two points \( x,y \in C \) there exists a rectifiable shortest path between \( x \) and \( y \) which is entirely contained in \( C \). We have to prove that \( d_C < \infty \) and that \( d_C = \overline{d_C} \). To prove the first claim note that, by the working hypothesis, there exists a rectifiable path \( \gamma_0 \) entirely belonging to \( C \) connecting \( x \) and \( y \) such that \( d_C(x,y) = l_d(\gamma_0) < \infty \). Since \( \inf_{R_{x,y}} l_d(\gamma) \) is attained for paths entirely belonging to \( C \) we obtain the equality \( d_C = \overline{d_C} \), which proves the second claim.

**Lemma 4.13** Let \( (X,d_X) \) and \( (Y,d_Y) \) be two geodesic metric spaces and \( f : X \rightarrow Y \) a distance-preserving map. If \( d_X \) is finite then the image \( \text{Im}(f) := f(X) \) is a convex subset of \( Y \).

**Proof.** Since \( f \) is a distance preserving map it is injective. Let \( y_1, y_2 \in f(X) \) and \( x_1, x_2 \in X \) be the corresponding preimages. Then there exists a shortest geodesic between \( x_1 \) and \( x_2 \), namely a curve \( c : [a,b] \rightarrow X \) satisfying \( c(a) = x_1 \), \( c(b) = x_2 \) and \( l_{d_X}(c) = d_X(x_1,x_2) < \infty \). Since \( f \) is distance preserving we have that \( l_{d_Y}(f \circ c) = l_{d_Y}(f \circ c) \). Consequently \( l_{d_Y}(f \circ c) = d_Y(y_1,y_2) < \infty \) which proves that \( f \circ c \) is a shortest geodesic connecting \( y_1 \) and \( y_2 \) entirely contained in \( f(X) \). Consequently, \( f(X) \) is a convex subset of \( Y \).

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