Massless particles in five and higher dimensions

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Abstract

We describe a five-dimensional analogue of Wigner’s operator equation $\mathcal{W}_a = \lambda P_a$, where $\mathcal{W}_a$ is the Pauli-Lubanski vector, $P_a$ the energy-momentum operator, and $\lambda$ the helicity of a massless particle. Higher dimensional generalisations are also given.
1 Introduction

The unitary representations of the Poincaré group in four dimensions were classified by Wigner in 1939 [1], see [2] for a recent review. Our modern understanding of elementary particles is based on this classification.

Unitary representations of the Poincaré group $\text{ISO}_0(d-1,1)$ in higher dimensions, $d > 4$, have been studied in the literature, see, e.g., [3]. However, there still remain some aspects that are not fully understood, see, e.g., [4] for a recent discussion. In this note we analyse the irreducible massless representations of $\text{ISO}_0(4,1)$ with a finite (discrete) spin.

We recall that the Poincaré algebra $\text{iso}(d-1,1)$ in $d$ dimensions is characterised by the commutation relations\footnote{We make use of the mostly plus Minkowski metric $\eta_{ab}$ and normalise the Levi-Civita tensor $\varepsilon_{a_1...a_d}$ by $\varepsilon_{01...d-1} = 1.$}

\begin{align}
\left[ P_a, P_b \right] &= 0 \ , \\
\left[ J_{ab}, P_c \right] &= i\eta_{ac} P_b - i\eta_{bc} P_a \\
\left[ J_{ab}, J_{cd} \right] &= i\eta_{ac} J_{bd} - i\eta_{bd} J_{ac} - i\eta_{bc} J_{ad} .
\end{align}

In any unitary representation of (the universal covering group of) the Poincaré group, the energy-momentum operator $P_a$ and the Lorentz generators $J_{ab}$ are Hermitian. For every dimension $d$, the operator $P^a P_a$ is a Casimir operator. Other Casimir operators are dimension dependent.

In four dimensions, the second Casimir operator is $\mathbb{W}^a \mathbb{W}_a$, where

\begin{equation}
\mathbb{W}^a = \frac{1}{2} \varepsilon^{abcd} J_{bc} P_d
\end{equation}

is the Pauli-Lubanski vector. Using the commutation relations (1.1), it follows that the Pauli-Lubanski vector is translationally invariant,

\begin{equation}
\left[ P_a, \mathbb{W}_b \right] = 0
\end{equation}

and possesses the following properties:

\begin{align}
\mathbb{W}^a P_a &= 0 \\
\left[ J_{ab}, \mathbb{W}_c \right] &= i\eta_{ac} \mathbb{W}_b - i\eta_{bc} \mathbb{W}_a \\
\left[ \mathbb{W}_a, \mathbb{W}_b \right] &= i\varepsilon_{abcd} \mathbb{W}^c P^d .
\end{align}
The irreducible massive representations are characterised by the conditions

\[ P^a P_a = -m^2 \mathbb{1}, \quad m^2 > 0, \quad \text{sign} P^0 > 0, \quad (1.4a) \]

\[ \mathbb{W}^a \mathbb{W}_a = m^2 s(s + 1) \mathbb{1}, \quad (1.4b) \]

where the quantum number \( s \) is called spin. Its possible values in different representations are \( s = 0, 1/2, 1, 3/2, \ldots \). The massless representations are characterised by the condition \( P^a P_a = 0 \). For the physically interesting massless representations, it holds that

\[ \mathbb{W}_a = \lambda P_a, \quad (1.5) \]

where the parameter \( \lambda \) determines the representation and is called the helicity. Its possible values are \( 0, \pm \frac{1}{2}, \pm 1, \) and so on. The parameter \( |\lambda| \) is called the spin of a massless particle.

In this paper we present a generalisation of Wigner’s equation \( (1.5) \) to five and higher dimensions.

2 Unitary representations of \( \text{ISO}_0(4, 1) \)

The five-dimensional analogue of \( (1.2) \) is the Pauli-Lubanski tensor

\[ \mathbb{W}^{ab} = \frac{1}{2} \varepsilon^{abcde} J_{cd} P_e. \quad (2.1) \]

It is translationally invariant,

\[ [\mathbb{W}_{ab}, P_c] = 0, \quad (2.2) \]

and possesses the following properties:

\[ \mathbb{W}_{ab} P^b = 0, \quad (2.3a) \]

\[ [\mathbb{W}_{ab}, J_{cd}] = i \eta_{ac} \mathbb{W}_{bd} - i \eta_{ad} \mathbb{W}_{bc} - i \eta_{bd} \mathbb{W}_{ac} + i \eta_{bc} \mathbb{W}_{ad}, \quad (2.3b) \]

\[ [\mathbb{W}_{ab}, \mathbb{W}_{cd}] = i \varepsilon_{acdef} \mathbb{W}_{b} f P^g - i \varepsilon_{bcdef} \mathbb{W}_{a} f P^g. \quad (2.3c) \]

Making use of \( \mathbb{W}_{ab} \) allows one to construct two Casimir operators, which are

\[ \mathbb{W}_{ab} \mathbb{W}^{ab}, \quad \mathbb{H} := \mathbb{W}^{ab} J_{ab}. \quad (2.4) \]
2.1 Irreducible massive representations

The irreducible massive representations of the Poincaré group ISO\(_0(4, 1)\) are characterised by two conditions

\[
\frac{1}{8} \left( W_{ab} W_{ab} + m^2 \right) = m^2 s_1 (s_1 + 1) \mathbb{1}, \tag{2.5a}
\]

\[
\frac{1}{8} \left( W_{ab} W_{ab} - m^2 \right) = m^2 s_2 (s_2 + 1) \mathbb{1}, \tag{2.5b}
\]

in addition to (1.4a). Here \(s_1\) and \(s_2\) are two spin values corresponding to the two SU(2) subgroups of the universal covering group Spin(4) \(\cong SU(2) \times SU(2)\) of the little group.\(^2\)

2.2 Irreducible massless representations

It turns out that all irreducible massless representations of ISO\(_0(4, 1)\) with a finite spin are characterised by the condition

\[
\varepsilon_{abcde} P^c W^{de} = 0 \iff P^{[a} W^{bc]} = 0 . \tag{2.6}
\]

Both Casimir operators (2.4) are equal to zero in these representations, \(W_{ab} W^{ab} = 0\) and \(W^{ab} J_{ab} = 0\).

Let \(|p, \sigma\rangle\) be an orthonormal basis in the Hilbert space of one-particle states, where \(p^a\) denotes the momentum of a particle, \(P^a |p, \sigma\rangle = p^a |p, \sigma\rangle\), and \(\sigma\) stands for the spin degrees of freedom. For a massless particle, we choose as our standard 5-momentum \(k^a = (E, 0, 0, 0, E)\). On this eigenstate:

\[
W^{ab} |k, \sigma\rangle = \frac{1}{2} \varepsilon_{abcde} J_{cd} P_e |k, \sigma\rangle = \frac{E}{2} \left( \varepsilon^{abcd4} J_{cd} - \varepsilon^{abcd0} J_{cd} \right) |k, \sigma\rangle . \tag{2.7}
\]

Running through the elements of \(W^{ab}\), one finds:

\[
W^{01} = W^{41} = -E J_{23} , \quad W^{12} = E (J_{30} + J_{34}) , \quad W^{02} = W^{42} = -E J_{31} , \quad W^{23} = E (J_{10} + J_{14}) ,
\]

\[
W^{03} = W^{43} = -E J_{12} , \quad W^{31} = E (J_{20} + J_{24}) , \quad W^{04} = 0 . \tag{2.8}
\]

\(^2\)The equations (2.5) were independently derived during the academic year 1992-93 by Arkady Segal and David Zinger, who were undergraduates at Tomsk State University at the time.
If we rescale these generators and define:

\[ R_1 \equiv \frac{1}{E} \mathbb{W}^{2i}, \quad R_2 \equiv \frac{1}{E} \mathbb{W}^{3i}, \quad R_3 \equiv \frac{1}{E} \mathbb{W}^{12}, \]

\[ J_i \equiv -\frac{1}{E} \mathbb{W}^{0i}, \]

then these new operators satisfy:

\[ [J_i, J_j] = i \epsilon^{ijk} J_k, \quad [J_i, R_j] = i \epsilon^{ijk} R_k, \quad [R_i, R_j] = 0. \]

These are the commutation relations for the three-dimensional Euclidean algebra, \( \mathfrak{iso}(3) \).

The irreducible unitary representations of \( \mathfrak{iso}(3) \) are labelled by a continuous parameter \( \mu^2 \), corresponding to the value the Casimir operator \( R_i R_i \) takes. Since \( R_i \) commute among themselves the operators can be simultaneously diagonalised, and the eigenvectors \( |r_i\rangle \) taken as a basis. However the only restriction on these is that \( r_i r_i = \mu^2 \), which for non-zero \( \mu^2 \) permits a continuous basis and is thus an infinite dimensional representation. Because we want only finite-dimensional representations, we must take:

\[ \mu^2 = 0 \implies R_i = 0 \iff J_0 = -J_4. \]

We are therefore restricted to those representations in which the translation component is trivial, and so only the generators \( J_i \) remain, which generate the algebra \( \mathfrak{so}(3) \). The algebra of the little group on massless representations is thus \( \mathfrak{so}(3) \) which is isomorphic to \( \mathfrak{su}(2) \).

As stated previously, the irreducible representations of \( \mathfrak{su}(2) \) are labelled by a non-negative (half) integer \( s \) and have a single Casimir operator \( J^i J_i \) which takes the value \( s(s + 1) \).

This analysis leads to (2.6).

The spin value of a massless representation can still be found using a ‘spin’ operator. The following relation holds on massless representations:

\[ S_a := -\frac{1}{4} \varepsilon_{abcd} J^{bc} \mathbb{W}^{de} = J^2 P_a = s(s + 1) P_a, \]

where \( J^2 = J^i J_i \) is the Casimir operator for the \( \mathfrak{so}(3) \) generators in (2.9). The parameter \( s \) is the spin of a massless particle. Its possible values in different representations are \( s = 0, 1/2, 1, \) and so on. Equation (2.12) naturally holds for massless spinor and vector fields [5].

In general, the operator \( S_a \) is not translationally invariant,

\[ [S_b, P_a] = \frac{i}{2} \varepsilon_{abcd} P^c \mathbb{W}^{de}. \]

It is only for the massless representations with finite spin that the quantity on the right vanishes so that the spin operator commutes with the momentum operators. Equation (2.12)
is the five-dimensional analogue of the operator equation (1.5). Its consistency condition is (2.6).3

3 Generalisations

The results of section 2.2 can be generalised to $d > 5$ dimensions. The Pauli-Lubanski tensor (2.1) turns into

$$W^{a_1...a_{d-3}} = \frac{1}{2} \varepsilon^{a_1...a_{d-3}bce} J_{bc} P_e.$$ (3.1)

The condition (2.6) is replaced with

$$P^{[a} W^{b_1...b_{d-3}]} = 0.$$ (3.2)

This equation is very similar to another that has appeared in the literature using the considerations of conformal invariance [6–9]. One readily checks that (3.2) is equivalent to

$$J_{ab} P^2 + 2 J_{[a|P_b|} P^c = 0 \quad \Rightarrow \quad J_{[c|P_b|} P^c = 0.$$ (3.3)

The latter is solved on the momentum eigenstates by $J_{ab} P^b \propto p_a$, which is of the form considered in [6–9].4

Equation (3.2) characterises all irreducible massless representations of ISO$_0(d-1, 1)$ with a finite (discrete) spin. Finally, the spin equation (2.12) turns into

$$S_a := \frac{(-1)^d}{2(d-3)!} \varepsilon_{abce_1...e_{d-3}} J^{bc} W^{e_1...e_{d-3}} = J^2 P_a,$$ (3.4)

where $J^2 = \frac{1}{2} \mathcal{J}^{ij} J_{ij}$ is the quadratic Casimir operator of the algebra so$(d-2)$, with $i, j = 1, \ldots, d-2$. For every irreducible massless representation of ISO$_0(d-1, 1)$ with a finite spin, it holds that $J^2 \propto 1$.

We can extend this further to higher-order Casimir operators of so$(d-2)$. As a generalisation of (3.1), we introduce the $n^{th}$ Pauli-Lubanski tensor

$$W^{(n)}_{a_1...a_{d-2n-1}} = \frac{1}{2^n \varepsilon_{a_1...a_d} J^{a_{d-2n}a_{d-2n+1}} \cdots J^{a_{d-2}a_{d-1}}} P^{a_d}, \quad 1 \leq n \leq \left\lfloor \frac{d-2}{2} \right\rfloor$$ (3.5)

3The consistency condition for (1.5) is $P^{[a} \mathcal{W}^{b]} = 0$, which is the four-dimensional counterpart of (2.6).

4We are grateful to Warren Siegel for useful comments.

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which is order $n$ in the Lorentz generators (the operator (3.1) coincides with $\mathbb{W}(1)$). Then higher-order spin operators can be defined as

$$S^{(n)}_{a_1} = \frac{(-1)^d}{2(d-2n-1)!} \varepsilon_{a_1...a_d} J^{a_2 a_3} \ldots J^{a_{2n} a_{2n+1}} \mathbb{W}^{(n)}(a_{2n+2}...a_d),$$  \hspace{1cm} (3.6)

which are order $2n$ in the Lorentz generators. Using the fact that $J^{a_0} = J^{a_0} = 1$ in the frame with a standard $d$-momentum $k^a = (E, 0, \ldots, 0, E)$, one can show that

$$S^{(n)}_{a_1} = C^{(n)} P_a$$  \hspace{1cm} (3.7)

where $C^{(n)}$ is an order $2n$ Casimir operator for $\mathfrak{so}(d-2)$ defined by

$$C^{(n)} = \frac{-1}{2^{n+1}(d-2n-2)!} \varepsilon_{0i_1...i_{d-2}} d-1 \varepsilon_{0j_1...j_{2n} i_{2n+1}...i_{d-2}} d-1 \times J^{i_1 i_2} \ldots J^{i_{2n-1} i_{2n}} J_{j_1 j_2} \ldots J_{j_{2n-1} j_{2n}}$$  \hspace{1cm} (3.8)

If $d$ is odd, the operators (3.8) can be constructed up to $n = \frac{d-3}{2}$ (the order $n = \frac{d-1}{2}$ Pauli-Lubanski tensor is a scalar). If $d$ is even, it suffices to restrict $n$ to run from 1 to $n = \frac{d-4}{2}$, since the Pauli-Lubanski tensor of order $\frac{d-2}{2}$,

$$\mathbb{W}^{(\frac{d-1}{2})}_{a_1} = \frac{1}{2^{\frac{d-1}{2}}} \varepsilon_{a_1...a_d} J^{a_2 a_3} \ldots J^{a_{d-2} a_{d-1}} P^{a_d},$$  \hspace{1cm} (3.9)

is itself a ‘spin operator’ with the property

$$\mathbb{W}^{(\frac{d-1}{2})}_{a_1} = \Lambda^{(\frac{d-1}{2})} P_a,$$  \hspace{1cm} (3.10)

where

$$\Lambda^{(\frac{d-1}{2})} = -\frac{1}{2^{\frac{d-1}{2}}} \varepsilon_{0i_1...i_{d-2}} d-1 J^{i_1 i_2} \ldots J^{i_{d-2} i_{d-2}}$$  \hspace{1cm} (3.11)

Note that the $d = 4$ case corresponds to (1.5).

For every irreducible massless representation of $\text{ISO}_0(d-1, 1)$ with a finite spin, the operator $C^{(n)}$ in (3.7) is a multiple of the identity operator, $C^{(n)} \propto 1$. Then the translational invariance of the equations (3.7) implies (3.2) and the relation

$$\mathbb{W}^{(n-1)}_{a_1 a_2 b_1...b_{d-2n-1}} \mathbb{W}^{(n)}(b_1...b_{d-2n-1}) = 0 .$$  \hspace{1cm} (3.12)

\textsuperscript{5}In the massless case, all Casimir operators of the Poincaré group $(\mathbb{W}^{(n)}_{a_1...a_{d-2n-1}})^2$ vanish, and so does the scalar operator $\mathbb{W}^{(\frac{d-1}{2})}$, which is defined when $d$ is odd.
It is possible to derive a five-dimensional analogue of the operator equation defining the \( \mathcal{N} = 1 \) superhelicity \( \kappa \) in four dimensions [10]. The latter has the form\(^6\)

\[
\mathbb{L}_a = \left( \kappa + \frac{1}{4} \right) P_a ,
\]

(3.13)

where the operator \( \mathbb{L}_a \) is defined by

\[
\mathbb{L}_a = \mathbb{W}_a - \frac{1}{16} ( \tilde{\sigma}_a ) ^{\hat{\alpha} \alpha} [ Q_\alpha , \bar{Q}^{\hat{\alpha}} ] .
\]

(3.14)

The fundamental properties of the operator \( \mathbb{L}_a \) (the latter differs from the supersymmetric Pauli-Lubanski vector [11]) are that it is translationally invariant and commutes with the supercharges \( Q_\alpha \) and \( \bar{Q}^{\hat{\alpha}} \) in the massless representations of the \( \mathcal{N} = 1 \) super-Poincaré group.\(^7\) The superhelicity operator (3.14) was generalised to higher dimensions in [12,13]. Generalisations of (3.13) to five and higher dimensions will be discussed elsewhere.

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\(^6\)In the supersymmetric case, the conventions of [10] are used, in particular the Levi-Civita tensor \( \varepsilon_{abcd} \) is normalised by \( \varepsilon_{0123} = -1 \).

\(^7\)The irreducible massless representation of superhelicity \( \kappa \) is the direct sum of two irreducible massless Poincaré representations corresponding to the helicity values \( \kappa \) and \( \kappa + \frac{1}{2} \).
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