Abstract. For a general set transformation \( R \) between two measure spaces, we define the rearrangement of a measurable function by means of the Layer’s cake formula. We study some functional properties of the Lorentz spaces defined in terms of \( R \), giving a unified approach to the classical rearrangement, Steiner’s symmetrization, the multidimensional case, and the discrete setting of trees.

1 Introduction

Given two measure spaces \((X, \Sigma_X, \mu)\) and \((Y, \Sigma_Y, \nu)\) we consider a general set transformation \( R : \Sigma_X \to \Sigma_Y \). We denote by \( f_R^* \) the rearrangement of a \( \mu \)-measurable function \( f \) with respect to the transformation \( R \) by means of the “Layer cake formula” (see [12]):

\[
f_R^*(y) = \int_0^\infty \chi_{R\{x \in X: |f(x)| > t\}}(y) \, dt,
\]

whenever it defines a \( \nu \)-measurable function on \( Y \).

For \( Y = (0, \infty) \) and \( R \) the transformation defined by \( R(E) = (0, |E|) \), where \(|E|\) denotes the Lebesgue measure of a set \( E \in \Sigma_X \), we have that \( f_R^* = f^* \), the usual decreasing rearrangement of a measurable function \( f \) defined on \( X \), which we will refer as the classical case (see [4] for more information).

Formula (1) has been used recently to define the rearrangement of functions with respect to some order in very different contexts: in [9, 10] a new decreasing rearrangement is defined for functions on homogeneous trees and in [3] a multidimensional rearrangement is considered for functions on \( \mathbb{R}^n \).

The work is organized as follows: in Section 2 we develop the main results concerning general rearrangements from a measure theoretical point of view. In Section 3 we introduce the weighted Lorentz spaces associated to this general kind of transformations and also we review some functional properties for these spaces in two known contexts: the multidimensional rearrangement and the rearrangement on homogeneous trees, completing the characterization of normability already proved in [3, 9]. The theory that we develop allows us to unify and extend these kind of results to two kind of preserving measures rearrangements that appear very frequently in applications: Steiner’s symmetrization and spherical rearrangements.

\[\text{Keywords: Rearrangement, Lorentz space, symmetrization.}\]

\[\text{MSC2000: 46E30, 46B42.}\]

Both authors have been partially supported by Grants MTM2004-02299 and 2005SGR00556.


## 2 General rearrangement transformations

In this section, we review how the basic results, well-known in the classical theory, actually imply some a priori assumptions on \( \mathcal{R} \) which, in many cases, turn out to be equivalent statements. From (1), we observe that the rearrangement of a function is a non degenerate transformation; that is, \( f \neq 0 \) implies \( f^*_\mathcal{R} \neq 0 \), if there exists \( F \in \Sigma_Y \) with \( \nu(F) > 0 \) such that a.e. \( y \in F \), there exists \( A_y \in (0, \infty) \), with positive Lebesgue measure, such that \( y \in \cap_{k \in A_y} \mathcal{R}\{|f| > t\} \). It is clear from the definitions that having a non degenerate transformation implies that \( \nu(\mathcal{R}(\emptyset)) > 0 \), or \( \nu(\mathcal{R}(E)) > 0 \) if \( \mu(E) > 0 \). The reverse property is also true if \( \mathcal{R} \) is a monotone transformation, in the sense that \( E \subset F \) implies \( \mathcal{R}(E) \subset \mathcal{R}(F) \).

To show that more conditions, like monotonicity, are necessary to have a non degenerate transformation, let us consider the following counterexample: assume that \( X \) is a subset of \( \mathbb{R}^n \) of finite measure and \( Y = (0, \infty) \), with \( \mathcal{R}(E) = (|E|, 2|E|) \) (here \( |E| \) denotes the Lebesgue measure of the set \( E \subset X \)), which is not a monotone rearrangement. An easy application of the Layer’s cake formula shows that \( f^*_\mathcal{R}(t) = f^*(t/2) - f^*(t) \), where \( f^* \) denotes the usual rearrangement of \( f \) with respect to the Lebesgue measure. We deduce then that any constant function \( f \) has \( f^*_\mathcal{R}(y) \equiv 0 \), and so the transformation \( \mathcal{R} \) is degenerate, although in this case, \( \nu(\mathcal{R}(E)) > 0 \) if \( \mu(E) > 0 \).

**Remark 2.1** A simple non-negative function \( f \) can be written as \( f(x) = \sum_{k=1}^{N} b_k \chi_{F_k}(x) \), with \( (b_k)_k > 0 \) and \( (F_k)_{1 \leq k \leq N} \) an increasing sequence of sets. In this case, (1) gives us that

\[
f^*_\mathcal{R}(y) = \sum_{k=1}^{N} b_k \chi_{\mathcal{R}(F_k)}(y), \quad \text{a.e. } y \in Y,
\]

provided that the transformation \( \mathcal{R} \) satisfies \( \nu(\mathcal{R}(\emptyset)) = 0 \). In fact, this condition is necessary for (2) to hold. Thus, from now on, we will always assume that \( \nu(\mathcal{R}(\emptyset)) = 0 \), and hence \( (\chi_A)_{\mathcal{R}} = \chi_{\mathcal{R}(A)} \), a.e., for all \( A \in \Sigma_X \).

Also, assuming in addition that \( \mathcal{R} \) is a monotone transformation on \( \Sigma_X \), for \( f(x) = \sum_{j=1}^{N} a_j \chi_{E_j}(x) \), with \( a_1 > a_2 > \cdots > a_N > 0 \) and \( E_i \cap E_j = \emptyset, \, i \neq j \), then

\[
f^*_\mathcal{R}(y) = \sum_{j=1}^{N} a_j \chi_{\mathcal{R}(F_j) \setminus \mathcal{R}(F_{j-1})}(y), \quad \text{a.e. } y \in Y,
\]

where \( F_j = \bigcup_{k=1}^{j} E_k \), and \( F_0 = \emptyset \).

**Lemma 2.2** Suppose \( f, g \) are measurable functions on \( X \), and \( \mathcal{R} \) is a monotone transformation.

(a) If \(|g| \leq |f| \) a.e., then \( g^*_\mathcal{R} \leq f^*_\mathcal{R} \).

(b) \( \{f^*_\mathcal{R} > t\} \subseteq \mathcal{R}\{|f| > t\} \subseteq \{f^*_\mathcal{R} \geq t\} \).
(c) \((|f|^p)_R = (f_R^*)^p, 0 < p < \infty\).

**Proof:** (a) is a consequence of the monotonicity property on \(\mathcal{R}\), because if \(|g| \leq |f|\) then \(\chi_{\mathcal{R}(|g| > t)} \leq \chi_{\mathcal{R}(|f| > t)}\) and hence \(g_R^* \leq f_R^*\).

To show (b), fix \(y \in Y\) such that \(f_R^*(y) > t\), which is equivalent to 
\[
\int_0^\infty \chi_{\mathcal{R}(|f| > s)}(y) \, ds > t.
\]
Then, \(|\{s \in (0, \infty) : y \in \mathcal{R}(|f| > s)\}| > t\); i.e., there is \(\varepsilon_y > 0\) such that
\[
(0, t + \varepsilon_y) \subset \{s \in (0, \infty) : y \in \mathcal{R}(|f| > s)\},
\]
and hence \(y \in \mathcal{R}(|f| > t)\). Therefore, we have proved that:
\[
\{f_R^* > t\} \subseteq \mathcal{R}\{|f| > t\}. \tag{3}
\]
On the other hand, since \(y \in \mathcal{R}(|f| > t)\) then \(y \in \mathcal{R}(|f| > s)\), \(\forall s \in [0, t]\) and hence \(f_R^*(y) \geq t\). Therefore,
\[
\mathcal{R}\{|f| > t\} \subseteq \{f_R^* \geq t\}. \tag{4}
\]
To see (c), we use (3) and obtain
\[
(|f|^p)_R^*(x) = \int_0^\infty \chi_{\mathcal{R}(|f|^p > t)}(x) \, dt = p \int_0^\infty t^{p-1} \chi_{\mathcal{R}(|f| > t)}(x) \, dt
\]
\[
\geq p \int_0^\infty t^{p-1} \chi_{\mathcal{R}(|f_R^*| > t)}(x) \, dt = p \int_0^{f_R^*(x)} t^{p-1} \, dt = (f_R^*)^p(x).
\]
On the other hand, the inclusion (4) establishes the reverse inequality
\[
(|f|^p)_R^*(x) = p \int_0^\infty t^{p-1} \chi_{\mathcal{R}(|f|^p > t)}(x) \, dt
\]
\[
\leq p \int_0^\infty t^{p-1} \chi_{\mathcal{R}(|f_R^*| \geq t)}(x) \, dt = p \int_0^\infty t^{p-1} \, dt = (f_R^*)^p(x).
\]

\[\square\]

**Definition 2.3** We will say that the transformation \(\mathcal{R}\) satisfies the Fatou property if for every increasing sequence of positive measurable functions \(f_n\) converging to \(f\), \(\mu\)-a.e., we have that also \((f_n)_R^*\) converges increasingly to \((f)_R^*\), \(\nu\)-a.e.

The following lemma proves that the Fatou property is equivalent to the fact that the transformation \(\mathcal{R}\) preserves increasing sequences of sets. The notation \((A_j)_j \uparrow A\) used below means that \(A_j \subset A_{j+1}\), and \(A = \cup_j A_j\).

3
Proposition 2.4 Let $(X, \Sigma_X, \mu)$ and $(Y, \Sigma_Y, \nu)$ be two measure spaces and $\mathcal{R} : \Sigma_X \to \Sigma_Y$. Then, the following statements are equivalent:

(a) $\mathcal{R}$ satisfies the Fatou property.

(b) For every increasing sequence of sets $(A_j)_j \uparrow A$, $(\mathcal{R}(A_j))_j \uparrow \mathcal{R}(A)$. In particular, $\mathcal{R}$ is monotone.

(c) For $f$ and $f_n$, $n \geq 1$, measurable functions on $X$,

$$|f| \leq \liminf_{n \to \infty} |f_n| \Rightarrow f^*_R \leq \liminf_{n \to \infty} (f_n)^*_R.$$

Proof:

First, assume that $\mathcal{R}$ satisfies the Fatou property. The condition $(A_j)_j \uparrow A$ is equivalent to $\chi_{(A_j)}(x) \uparrow \chi_A(x)$ for every $x \in X$ and hence, using (2)

$$(\chi_{A_j})^*_R(y) = \chi_{\mathcal{R}(A_j)}(y) \uparrow \chi_{\mathcal{R}(A)}(y).$$

To prove that condition (b) implies (c), we define for a fixed $\lambda \geq 0$,

$$E := \{x : |f(x)| > \lambda\}, \quad E_n := \{x : |f_n(x)| > \lambda\} \quad (n = 1, 2, \cdots).$$

Clearly, $E \subset \bigcup_{m \geq 1} \bigcap_{n \geq m} E_n$ and hence, using (b),

$$\mathcal{R}(E) \subseteq \mathcal{R} \left( \bigcup_{m \geq 1} \left( \bigcap_{n \geq m} E_n \right) \right) = \bigcup_{m \geq 1} \mathcal{R} \left( \bigcap_{n \geq m} E_n \right) \subseteq \bigcup_{m \geq 1} \bigcap_{n \geq m} \mathcal{R}(E_n) = \liminf_n \mathcal{R}(E_n).$$

This inclusion implies that, for $y \in Y$,

$$\chi_{\mathcal{R}(E)}(y) \leq \liminf_{n \to \infty} \chi_{\mathcal{R}(E_n)}(y),$$

and then, using (1) and Fatou’s lemma

$$f^*_R \leq \liminf_{n \to \infty} (f_n)^*_R.$$

Finally, it is easy to see that (c) implies that $\mathcal{R}$ is monotone. Now consider $f_j \uparrow f$. On the one hand, by Lemma 2.2 $(f_j)^*_R \leq f^*_R$ so that $\limsup_j (f_j)^*_R \leq f^*_R$, and by hypothesis, $f^*_R \leq \liminf_j (f_j)^*_R$, which proves (a).

Theorem 2.5 Let $\mathcal{R}$ be a set transformation between two measure spaces $(X, \Sigma_X, \mu)$ and $(Y, \Sigma_Y, \nu)$. Assume that $\mathcal{R}$ satisfies the Fatou property. Then, the following are equivalent conditions:
(a) \( \mu(A \cap B) \leq \nu(\mathcal{R}(A) \cap \mathcal{R}(B)) \), for every \( A, B \in \Sigma_X \).

(b) \( \int_A f \, d\mu \leq \int_{\mathcal{R}(A)} f^*_\mathcal{R} \, d\nu \), for every non-negative measurable function \( f \) on \( X \), and \( A \in \Sigma_X \).

(c) \( \int_X fg \, d\mu \leq \int_Y f^*_\mathcal{R} g^*_\mathcal{R} \, d\nu \), for every non-negative measurable functions \( f \) and \( g \) on \( X \).

**Proof:** Let us assume (a). Using the Fatou property, it is enough to prove (b) just for a simple function of the form

\[
f(x) = \sum_{j=1}^N b_j \chi_{E_j}(x), \quad \text{with } (b_j)_j > 0 \text{ and } E_j \text{ an increasing sequence of sets,}
\]

since we can always find a sequence \((s'_k)_k\) of simple functions such that \(0 \leq (s'_1)_R \leq \cdots \leq (s'_k)_R \leq f^*_R\) and \((s'_k)_R(y) \to f^*_R(y)\), as \(k \to \infty\), a.e. \( y \in Y \).

Then,

\[
\int_A g \, d\mu = \sum_{j=1}^N b_j \mu(A \cap E_j) \leq \sum_{j=1}^N b_j \nu(\mathcal{R}(A) \cap \mathcal{R}(E_j)) = \sum_{j=1}^N b_j \int_{\mathcal{R}(A)} \chi_{\mathcal{R}(E_j)}(y) \, d\nu(y) = \int_{\mathcal{R}(A)} g^*_\mathcal{R} \, d\nu.
\]

To prove (c) assuming (b), we can also suppose \( f(x) = \sum_{j=1}^N a_j \chi_{E_j}(x) \), \( a_j > 0 \) and \( E_j \) an increasing sequence of sets. Then, by (b),

\[
\int_X fg \, d\mu = \sum_{j=1}^N a_j \int_{E_j} g \, d\mu \leq \sum_{j=1}^N a_j \int_{\mathcal{R}(E_j)} g^*_\mathcal{R} \, d\nu = \int_{\mathcal{R}(E_j)} g^*_\mathcal{R} \, d\nu = \int_{\mathcal{R}(E_j)} f^*_\mathcal{R}(y)g^*_\mathcal{R}(y) \, d\nu.
\]

Finally if we take \( f = \chi_A \) and \( g = \chi_B \) in condition (c) we obtain (a). \( \square \)

**Definition 2.6** We will say that \( \mathcal{R} \) is a measure preserving transformation from \( \Sigma_X \) into \( \Sigma_Y \), if \( \mu(E) = \nu(\mathcal{R}(E)) \), for every \( E \in \Sigma_X \).

**Proposition 2.7** Let us suppose that \( \mathcal{R} \) is a monotone transformation. Then, the following statements are equivalent:

(a) \( \mathcal{R} \) is a measure preserving transformation.

(b) If \( s(x) = \sum_{j=1}^N a_j \chi_{E_j}(x) \), with \( a_1 > a_2 > \cdots > a_N > 0 \), \( E_i \cap E_j = \emptyset \), \( i \neq j \) and \( p > 0 \), then

\[
\int_X s^p \, d\mu(x) = \int_Y (s^*_\mathcal{R})^p(y) \, d\nu(y).
\]
Proof: If \( R \) is a monotone measure preserving transformation, and \( E \cap F = \emptyset \), with \( \mu(E) < \infty \), then
\[
\nu(R(E \cup F) \setminus R(E)) = \nu(R(E \cup F)) - \nu(R(E)) = \mu(E \cup F) - \mu(E) = \mu(F).
\]
Thus, (b) follows by Remark 2.1, since
\[
(s_R^*)^p(y) = \sum_{j=1}^N a_j^p \chi_{R(F_j \setminus R(F_{j-1})}(y),
\]
with \( F_j = \bigcup_{k=1}^j E_k \), and \( F_0 = \emptyset \). (a) follows from (b) by taking \( s = \chi_A \).

3 Lorentz spaces and symmetrization

In this section we prove some properties of a new type of Lorentz spaces, defined using the general transformations \( R \). Let \( v \) be a weight on \( Y \) (i.e., \( v \in L^1_{\text{loc}}(Y, d\nu) \), \( v \geq 0 \) and satisfies the following non-cancellation property: if \( \mu(A) > 0 \), then \( \int_{R(A)} v(y) \, d\nu(y) > 0 \), and \( 0 < p < \infty \). We will say that a \( \mu \)-measurable function on \( X \) belongs to the Lorentz space \( \Lambda^p_R(v) \), provided
\[
\|f\|_{\Lambda^p_R(v)} := \left( \int_Y (f_R(y))^p \, v(y) \, d\nu(y) \right)^{1/p},
\]
is finite. The case \( X = \mathbb{R}^n \), \( Y = \mathbb{R}^+ \), \( R(E) = (0, |E|) \), and \( v(y) = y^{p/q-1} \) gives the classical Lorentz space: \( \Lambda^p_R(v) = L^{q,p}(\mathbb{R}^n) \).

The question whether the functional defined in (5) is a norm was answered by Lorentz in the euclidean case (see [13] for a proof and [8, 14] for related questions). Also, M.J. Carro and J. Soria ([7]) characterized the weights \( v \) such that it becomes a quasi-norm, if \( X \) is no atomic. Later, in [6], the quasi-normability was completed for all \( X \). The analogous characterization was established in [3] for the multidimensional rearrangement and in [9] for the case of homogeneous trees. In this section we give partial answers to this question in the context of a general transformation \( R \), satisfying the Fatou property, between \( \sigma \)-finite measure spaces \( X \) and \( Y \) (from now on, we will always assume these two conditions).

We adopt the notation \( V(E) = \int_E v(y) \, d\nu(y) \), for every measurable set \( E \subset Y \) and every weight \( v \) in \( Y \). Then, the functional (5) has the following description:

**Lemma 3.1** Let \( 0 < p < \infty \). Then, for all \( f \in \Lambda^p_R(v) \), we have
\[
\|f\|_{\Lambda^p_R(v)} = \left( \int_0^\infty p \lambda^{p-1} V(R(\{|f| > \lambda\}) \, d\lambda \right)^{1/p}.
\]
**Proof:** Using Lemma 2.2 (c) we have:

\[
\|f\|_{\mathcal{L}^p_{\mathcal{R}(v)}} = \left( \int_Y (|f|^p \mathcal{R}(y))^p \, v(y) \, d\nu(y) \right)^{1/p}.
\]

Then, by \((1)\) and Fubini’s Theorem,

\[
\|f\|_{\mathcal{L}^p_{\mathcal{R}(v)}} = \left( \int_Y \left( \int_0^\infty \chi_{\mathcal{R}(|f|^p > \lambda)}(y) \, d\lambda \right) v(y) \, d\nu(y) \right)^{1/p}
\]

\[
= \left( \int_Y \left( \int_0^\infty p\xi^{p-1} \chi_{\mathcal{R}(|f| > \xi)}(y) \, d\xi \right) v(y) \, d\nu(y) \right)^{1/p}
\]

\[
= \left( \int_0^\infty p\xi^{p-1} \left( \int_{\mathcal{R}(|f| > \xi)} v(y) \, d\nu(y) \right) \, d\xi \right)^{1/p}.
\]

\[\Box\]

Our first result gives a characterization of the quasi-normability of the functional defined in \((5)\).

**Theorem 3.2** The functional \(\| \cdot \|_{\mathcal{L}^p_{\mathcal{R}(v)}}\) is a quasi-norm if and only if there exists a constant \(C > 0\) such that

\[
V(\mathcal{R}(A \cup B)) \leq C(V(\mathcal{R}(A)) + V(\mathcal{R}(B))),
\]

for all sets \(A, B \in \Sigma_X\).

**Proof:** Assume first \((7)\): by Lemma 3.1, if \(\|f\|_{\mathcal{L}^p_{\mathcal{R}(v)}} = 0\), then

\[
V(\mathcal{R}\{|f| > \lambda\}) = 0,
\]

for all \(\lambda > 0\), and by hypothesis \(\mu(\{|f| > \lambda\}) = 0\), for all \(\lambda\); that is \(f \equiv 0\). Also by Lemma 3.1 the hypothesis and the monotonicity in \(\mathcal{R}\), we have:

\[
\|f + g\|^p_{\mathcal{L}^p_{\mathcal{R}(v)}} = \int_0^\infty p\lambda^{p-1}V(\mathcal{R}(|f + g| > \lambda)) \, d\lambda
\]

\[
\leq \int_0^\infty p\lambda^{p-1}V(\mathcal{R}(\{|f| > \lambda/2\} \cup \{|g| > \lambda/2\})) \, d\lambda
\]

\[
\leq C \left( \int_0^\infty p\lambda^{p-1}V(\mathcal{R}(|f| > \lambda/2)) \, d\lambda + \int_0^\infty p\lambda^{p-1}V(\mathcal{R}(|g| > \lambda/2)) \, d\lambda \right)
\]

\[
= 2C \left( \int_0^\infty p\lambda^{p-1}V(\mathcal{R}(|f| > \lambda)) \, d\lambda + \int_0^\infty p\lambda^{p-1}V(\mathcal{R}(|g| > \lambda)) \, d\lambda \right)
\]

\[
= 2C(\|f\|^p_{\mathcal{L}^p_{\mathcal{R}(v)}} + \|g\|^p_{\mathcal{L}^p_{\mathcal{R}(v)}}) \leq C_p(\|f\|^p_{\mathcal{L}^p_{\mathcal{R}(v)}} + \|g\|^p_{\mathcal{L}^p_{\mathcal{R}(v)}})^p.
\]

Conversely, suppose that the functional is a quasi-norm and take \(A\) and \(B\). Then,

\[
V(\mathcal{R}(A \cup B))^{1/p} = \|\chi_{A \cup B}\|_{\mathcal{L}^p_{\mathcal{R}(v)}} \leq C(\|\chi_A\|_{\mathcal{L}^p_{\mathcal{R}(v)}} + \|\chi_B\|_{\mathcal{L}^p_{\mathcal{R}(v)}})
\]

\[
= C(V(\mathcal{R}(A))^{1/p} + V(\mathcal{R}(B))^{1/p}) \leq C(V(\mathcal{R}(A)) + V(\mathcal{R}(B)))^{1/p}.
\]

\[\Box\]
Concerning the normability of $\Lambda^p_{\mathcal{R}}(\nu)$, we can establish the following partial results:

**Theorem 3.3** Let $1 \leq p < \infty$, and $\nu$ be a weight on $Y$. If $\| \cdot \|_{\Lambda^p_{\mathcal{R}}(\nu)}$ is a norm then, for all $A, B \in \Sigma_X$,

$$V(\mathcal{R}(A \cup B)) + V(\mathcal{R}(A \cap B)) \leq V(\mathcal{R}(A)) + V(\mathcal{R}(B)).$$

(8)

**Proof:** If $\| \cdot \|_{\Lambda^p_{\mathcal{R}}(\nu)}$ is a norm, take $A, B \subset X$, $\delta > 0$ and define the functions

$$f(x) = (1 + \delta)\chi_A(x) + \chi_{(A \cup B) \setminus A}(x)$$

and

$$g(x) = (1 + \delta)\chi_A(x) + \chi_{(A \cup B) \setminus B}(x).$$

Then,

$$f^*_\mathcal{R}(y) = (1 + \delta)\chi_{\mathcal{R}(A)}(y) + \chi_{\mathcal{R}(A \cup B) \setminus \mathcal{R}(A)}(y),$$

$$g^*_\mathcal{R}(y) = (1 + \delta)\chi_{\mathcal{R}(B)}(y) + \chi_{\mathcal{R}(A \cup B) \setminus \mathcal{R}(B)}(y),$$

$$(f + g)^*_\mathcal{R}(y) = (2 + 2\delta)\chi_{\mathcal{R}(A \cap B)}(y) + (2 + \delta)\chi_{\mathcal{R}(A \cup B) \setminus \mathcal{R}(A \cap B)}(y).$$

The triangle inequality and the fact that $1/p \leq 1$ imply

$$\| f + g \|_{\Lambda^p_{\mathcal{R}(\nu)}} = ((2 + 2\delta)^p V(\mathcal{R}(A \cap B)) + (2 + \delta)^p V(\mathcal{R}(A \cup B) \setminus \mathcal{R}(A))^{1/p} \leq \| f \|_{\Lambda^p_{\mathcal{R}(\nu)}} + \| g \|_{\Lambda^p_{\mathcal{R}(\nu)}} = ((1 + \delta)^p V(\mathcal{R}(A)) + V(\mathcal{R}(A \cup B) \setminus \mathcal{R}(A)))^{1/p} + (1 + \delta)^p V(\mathcal{R}(B)) + V(\mathcal{R}(A \cup B) \setminus \mathcal{R}(B))^{1/p} \leq 2^{1-1/p}((1 + \delta)^p V(\mathcal{R}(A)) + V(\mathcal{R}(A \cup B) \setminus \mathcal{R}(A)) + (1 + \delta)^p V(\mathcal{R}(B)) + V(\mathcal{R}(A \cup B) \setminus \mathcal{R}(B))^{1/p}.$$

Collecting terms, dividing both sides by $2^{p-1}((1 + \delta)^p - 1)$ and letting $\delta \to 0$, we finally obtain

$$V(\mathcal{R}(A \cup B)) + V(\mathcal{R}(A \cap B)) \leq V(\mathcal{R}(A)) + V(\mathcal{R}(B)).$$

\[ \square \]

Condition (3), in the classical case, implies that $V$ is a concave function, and we will refer to it as the **Concavity Condition**. A sufficient condition in a general setting to ensure that the functional $\| \cdot \|_{\Lambda^p_{\mathcal{R}}(\nu)}$ defines a norm is the following **Saturation Property**:

**Theorem 3.4** Let $1 \leq p < \infty$, and $\nu$ be a weight on $Y$ such that $\nu$ coincides with $h^*_R$ for some $h$ defined on $X$. Then, $\| \cdot \|_{\Lambda^p_{\mathcal{R}}(\nu)}$ is a norm if for all measurable functions $f$ in $X$, the equality

$$\sup_{\{h: h^*_R = \nu\}} \int_X |f(x) h(x)| \, d\mu(x) = \int_Y f^*_\mathcal{R}(y) \, v(y) \, d\nu(y)$$

(9)

holds.
Proof: We apply Lemma 2.2 (c) and the hypothesis:

\[ \|f + g\|_{\Lambda^p_Y(v)} = \left( \int_Y (f + g)_R^p(y) \, v(y) \, d\nu(y) \right)^{1/p} = \left( \int_Y (|f + g|^p)_R(y) \, v(y) \, d\nu(y) \right)^{1/p} \]

\[ = \sup_{\{h:s_R^h=v\}} \left( \int_X |f(x) + g(x)|^p \, h(x) \, d\mu(x) \right)^{1/p} \]

\[ \leq \sup_{\{h:s_R^h=v\}} \left( \int_X |f(x)|^p \, h(x) \, d\mu(x) \right)^{1/p} + \sup_{\{h:s_R^h=v\}} \left( \int_X |g(x)|^p \, h(x) \, d\mu(x) \right)^{1/p} \]

\[ = \left( \int_Y (f_R)_R^p(y) \, v(y) \, d\nu(y) \right)^{1/p} + \left( \int_Y (g_R)_R^p(y) \, v(y) \, d\nu(y) \right)^{1/p} \]

\[ = \|f\|_{\Lambda^p_Y(v)} + \|g\|_{\Lambda^p_Y(v)}. \]

\[ \square \]

Remark 3.5 We observe that in order for \( \| \cdot \|_{\Lambda^p_Y(v)} \) to be a norm is not enough that the weight \( v \) be the rearrangement of some function \( h \) defined on \( X \). The conditions are, in general, more restrictive: see [9] in the case of trees or [3] in the multidimensional setting. In the next section we will deal with these examples.

Even though normability can fail, completeness of \( \Lambda^p_Y(v) \) always holds:

**Proposition 3.6** Assume that \( v \) is a weight on \( Y \), such that the Lorentz space \( \Lambda := \Lambda^p_Y(v) \) is continuously embedded in the space \( L^1_{\text{loc}}(X) \), and \( \| \cdot \|_{\Lambda} \) is a quasi-norm. If \( (f_n) \) is a Cauchy sequence in \( \Lambda \) then, there exists a measurable function \( f \in \Lambda \) such that \( \lim_{n \to \infty} \|f - f_n\|_{\Lambda} = 0. \)

**Proof:** Since \( \| \cdot \| \) is a quasi-norm and \( (f_n) \) is Cauchy, there exists a constant \( C > 0 \) such that \( \|f_n\|_{\Lambda} \leq C < \infty \), for all \( n \in \mathbb{N} \).

Since \( X \) is \( \sigma \)-finite, let us write \( X = \bigcup_{k \geq 1} A_k \), with \( \mu(A_k) < \infty \) and \( A_k \) an increasing sequence of sets.

It is clear that \( f_n\chi_{A_k} \) is a Cauchy sequence in \( L^1(A_k) \) and hence the sequence \( f_n\chi_{A_k} \) converges to a function \( g_k \) in \( L^1(A_k) \), for each \( k \). Let us define \( f := g_k \) in \( A_k \), which is well-defined by the monotonicity of \( A_k \). We have to prove that \( f_n \to f \) in \( \Lambda \). By standard arguments, we can find a subsequence \( f_{j_k} \to f \) a.e. \( x \in X \). Then, by Proposition 2.4 (c) and Fatou’s lemma, we have that \( f \in \Lambda \), and

\[ \int_Y (f_R)_R^p(y) \, v(y) \, d\nu(y) \leq \int_Y \liminf_k (f_{j_k})_R^p \, v(y) \, d\nu(y) \]

\[ \leq \liminf_k \int_Y (f_{j_k})_R^p \, v(y) \, d\nu(y) = \liminf_k \|f_{j_k}\|_\Lambda^p \leq C^p. \]

Using Fatou’s lemma again and the fact that \( (f_k) \) is a Cauchy sequence, we finally get

\[ \|f - f_n\|_{\Lambda} \leq C(\|f - f_{j_k}\|_{\Lambda} + \|f_{j_k} - f_n\|_{\Lambda}) \to 0, \]

as \( n, k \to \infty \). \[ \square \]
Definition 3.7 A weight \( v \) defined on the space \( Y \) is called \( \mathcal{R} \)-admissible if for every \( A \in \Sigma_X \) and all \( 0 < \varepsilon < \int_{\mathcal{R}(A)} v(y) \, d\nu(y) \), there exists \( \mathcal{R}(B) \subset \mathcal{R}(A) \), such that \( \int_{\mathcal{R}(B)} v(y) \, d\nu(y) = \varepsilon \).

Now, we can show the following necessary condition on \( p \) for which the functional \( \| \cdot \|_{\Lambda^p_R(v)} \) defines a norm:

Theorem 3.8 Let \( v \) be an \( \mathcal{R} \)-admissible weight and \( 0 < p < \infty \). If \( \Lambda^p_R(v) \) is a Banach space, then \( p \geq 1 \).

Proof: Since \( \Lambda^p_R(v) \) is a Banach space, there exists \( \| \cdot \| \), a norm on \( \Lambda^p_R(v) \), such that \( \| f \|_{\Lambda^p_R(v)} \simeq \| f \| \). Hence,

\[
\left\| \sum_{k=1}^{N} f_k \right\|_{\Lambda^p_R(v)} \leq C \sum_{k=1}^{N} \| f_k \| \leq \tilde{C} \sum_{k=1}^{N} \| f_k \|_{\Lambda^p_R(v)},
\]

for all \( N \in \mathbb{N} \). Suppose \( 0 < p < 1 \). Due to the hypothesis assumed on \( v \), we can take a decreasing sequence of subsets \( A_{k+1} \subset A_k \cdots \subset X \),

such that \( \int_{\mathcal{R}(A_k)} v(y) \, d\nu(y) = 2^{-kp} \). If \( f_k = 2^k \chi_{A_k} \), then \( \| f_k \|_{\Lambda^p_R(v)} = 1 \). But for a fixed \( N \), we have

\[
\frac{1}{N} \left\| \sum_{k=1}^{N} f_k \right\|_{\Lambda^p_R(v)} \leq \tilde{C} < \infty.
\]

On the other hand, since by Remark 2.1 \((\sum_{k=1}^{N} 2^k \chi_{A_k})_R^* = \sum_{k=1}^{N} 2^k \chi_{\mathcal{R}(A_k)} \) and \( \mathcal{R}(A_{k+1}) \subset \mathcal{R}(A_k) \subset \cdots \subset Y \), we have (taking \( \mathcal{R}(A_{N+1}) = \emptyset \)),

\[
\frac{1}{N} \left\| \sum_{k=1}^{N} f_k \right\|_{\Lambda^p_R(v)} = \frac{1}{N} \left\| \sum_{k=1}^{N} 2^k \chi_{A_k} \right\|_{\Lambda^p_R(v)}
\]

\[
= \frac{1}{N} \left( \int_{Y} \left( \sum_{k=1}^{N} 2^k \chi_{\mathcal{R}(A_k)} \right)^p \right) \varepsilon(y) \, d\nu(y) \frac{1}{p}
\]

\[
= \frac{1}{N} \left( \int_{Y} \left( \sum_{k=1}^{N} \left( \sum_{j=1}^{k} 2^j \right)^p \chi_{\mathcal{R}(A_k) \setminus \mathcal{R}(A_{k+1})} \right) \varepsilon(y) \, d\nu(y) \right) \frac{1}{p}
\]

\[
= \frac{1}{N} \left( \sum_{k=1}^{N} \left( \sum_{j=1}^{k} 2^j \right)^p \left( \int_{\mathcal{R}(A_k)} \varepsilon(y) \, d\nu(y) - \int_{\mathcal{R}(A_{k+1})} \varepsilon(y) \, d\nu(y) \right) \right) \frac{1}{p}
\]

\[
\geq \frac{C}{N} \left( \sum_{k=1}^{N} (1 - 2^{-k})^p \right)^{1/p} \geq \frac{C}{N} \left( \sum_{k=1}^{N} 2^{-p} \right)^{1/p} = \frac{C \cdot N^{1/p}}{N} \to \infty, \text{ as } N \to \infty,
\]

which is a contradiction. Hence \( p \geq 1 \).
Remark 3.9 We observe that in Theorem 3.8 the hypothesis assumed on $Y$ is not compatible with the fact that $Y$ is a completely atomic measure space. In the case $0 < p < 1$, if $Y$ is completely atomic, we observe that the functional $\| \cdot \|_{\Lambda_p^R(v)}$ is a norm if and only if $\text{supp } v$ is contained in some atom $R(A)$ such that, for every measurable set $R(B)$ in $Y$, $R(A) \subset R(B)$. Observe that this is the case of the discrete setting (see [9] for a proof in the context of homogeneous trees).

The classical Lorentz spaces are generalizations of the Lebesgue spaces, since $\Lambda_p^X(1) = L^p(X)$. The next proposition shows that for a general transformation $R$, the corresponding Lorentz space also satisfies this property provided that $R$ is a measure preserving transformation.

Proposition 3.10 Let $0 < p < \infty$. Then, $R$ is a measure preserving transformation if and only if $\Lambda_p^R(1) = L^p(X)$, with equality of norms.

Proof: If $R$ is a measure preserving transformation, by Fubini’s theorem and Lemma 2.2, we have:

$$\| f \|_{L^p(X)}^p = \int_X |f(x)|^p \, d\mu(x) = \int_0^\infty \int_{\{|f|^p > t\}} d\mu(x) \, dt = \int_0^\infty \int_{R(|f|^p > t)} d\nu(y) \, dt$$

$$= \int_Y \int_0^\infty \chi_{R(|f|^p > t)}(y) \, dt \, d\nu(y) = \int_Y (|f|^p)_R(y) \, d\nu(y)$$

$$= \int_Y (|f|)_R^p(y) \, d\nu(y) = \| f \|_{\Lambda_p^R(1)}^p.$$

The converse follows by taking $f = \chi_A$. 

In the general context of a monotone transformation $R$ between measure spaces, Theorems 3.3 and 3.4 give two conditions (one necessary and the other sufficient) to ensure that the functional given by (5) defines a norm. Both conditions are known as the concavity condition and the saturation property, respectively, and are equivalent in the classical setting. Moreover, they are also equivalent to the fact that the weight $v$ must be decreasing (see [13]).

In the case of the two-dimensional rearrangement it has been proved that (5) is a norm if and only if the concavity condition holds and the weight $v$ defined on $\mathbb{R}^2_+$ is a decreasing function that only depends on one variable (see [3, Theorem 3.7]). On the other hand, in the case of rearrangement defined on homogeneous trees it has been shown (see [9, Theorem 4.9]) that the saturation property holds for linear decreasing weights (see [10] for the definition) and both conditions are equivalent to the fact that (5) defines a norm.

We can briefly resume these conditions in the following list:

1. **(Norm)**: (5) defines a norm.
2. **(CC)**: Concavity Condition (8).
3. **(SP)**: Saturation Property (9).
4. **(MP)**: Monotonicity properties on the weight.

11
Then, in the classical setting:

\[(\text{Norm}) \iff (\text{CC}) \iff (\text{SP}) \iff (\text{MP}),\]

in the multidimensional setting:

\[(\text{Norm}) \iff (\text{CC}) \iff (\text{MP}) \iff (\text{SP}),\]

and in the case of trees:

\[(\text{Norm}) \iff (\text{SP}) \iff (\text{MP}) \implies (\text{CC}).\]

We will now complete the missing results in the above list, and extend the equivalences to two more rearrangements (spherical and Steiner’s symmetrization).

In the case of the multidimensional rearrangement, to simplify the notation, we will restrict ourselves to the two-dimensional case. We can establish the following saturation property which completes the characterization of normability of Lorentz spaces in this context (see [2, 3]). Also, it is proved in [3] that given a function \(f(x, y)\) defined on \(\mathbb{R}^2\), its two dimensional rearrangement, \(f^*_2(s, t)\), \(s, t > 0\), can be understood as an iterative procedure with respect to the usual rearrangement in each variable. More precisely, \(f^*_2(s, t) = (f^*_y(\cdot, t))^*_x(s)\). That is, first we rearrange with respect to \(y\) and after with respect to \(x\). In this case (MP) is given by the fact that the weight \(v(s, t) = v(t)\), where \(v\) is a decreasing function.

**Proposition 3.11** For any measurable function in \(\mathbb{R}^2\),

\[
\sup_{h^*_2 = v} \int \int_{\mathbb{R}^2} f(x, y) h(x, y) \, dx \, dy = \int \int_{\mathbb{R}^2} f^*_2(s, t) \, v(t) \, ds \, dt,
\]

where \(v(t)\) is a decreasing function with respect to the variable \(t \in \mathbb{R}^+\).

**Proof:** Applying Hardy-Littlewood inequality with respect the one dimensional decreasing rearrangement we have that,

\[
\int_{\mathbb{R}^2} f(x, y) h(x, y) \, dy \, dx = \int_{\mathbb{R}^2} f_x(y) h(x, y) \, dy \, dx \leq \int_{\mathbb{R}^2} \int_0^\infty f^*_y(x, t) h^*_y(x, t) \, dt \, dx
\]

\[
\leq \int_0^\infty \int_0^\infty (f^*_y(\cdot, t))^*_x(s) (h^*_y(\cdot, t))^*_x(s) \, dt \, ds
\]

\[
= \int_0^\infty \int_0^\infty f^*_2(s, t) \, v(t) \, dt \, ds.
\]

To prove the converse, we use that for \(u\) a decreasing function (see [11]),

\[
\sup_{\sigma} \int_{\mathbb{R}^n} |f(x)| u(\sigma(x)) \, dx = \int_0^\infty f^*(t) u(t) \, dt, \tag{10}
\]
where the supremum is taken over all measure preserving transformations \( \sigma : \mathbb{R}^n \rightarrow \mathbb{R}^+ \).

Let us show that
\[
\int_0^\infty \int_0^\infty (f_y^*(\cdot, t))_{x}^*(s) v(t) \, dt \, ds \leq \sup_{\sigma} \int_{\mathbb{R}^2} f(x, y) \, v(\sigma_x(y)) \, dy \, dx.
\] (11)

For a given \( \varepsilon > 0 \) and \( x \in \mathbb{R} \), using (10), there exists \( \sigma_x : \mathbb{R} \rightarrow \mathbb{R}^+ \) such that
\[
\int_{\mathbb{R}} f_y^*(x, t) v(t) \, dt \leq \int_{\mathbb{R}} f(x, y) \, v(\sigma_x(y)) \, dy.
\]

We integrate over \( x \in \mathbb{R} \) and obtain
\[
\frac{1}{1 + \varepsilon} \int_{\mathbb{R}} \left( \int_0^\infty f_y^*(x, t) \, v(t) \, dt \right) \, dx = \frac{1}{1 + \varepsilon} \int_{\mathbb{R}} \left( \int_0^\infty (f_y^*(\cdot, t))_{x}^*(s) \, v(t) \, dt \, ds \right) \leq \sup_{\sigma} \int_{\mathbb{R}^2} f(x, y) \, v(\sigma_x(y)) \, dy \, dx,
\]
which gives us (11).

In the case of a tree, we can complete the set of equivalences by showing that also the concavity condition for a function \( v \), defined on the tree, implies that \( v \) is a linear decreasing weight, which is (MP) (see [9, 10]):

**Definition 3.12** For two given disjoint sets \( A \) and \( B \) in the boundary \( \partial T \) of a homogeneous tree \( T \), we write \( A \leq B \), if \( \alpha \leq \beta \) for all \( \alpha \in A \) and all \( \beta \in B \). Then, given two vertices \( x \) and \( y \) in \( T \), we define

\[ x \preceq y \]

if and only if

\[ x \preceq y \text{ or } I(x) \geq I(y), \]

where \( I(x) \) is the set of all geodesics passing through \( x \). We say that the function \( f \) is linearly decreasing if \( f(x) \geq f(y) \) whenever \( x \preceq y \).

**Proposition 3.13** Let \( v \) be a weight in \( T \). If \( v \) satisfies the concavity condition (CC), then \( v \) is linearly decreasing.

**Proof:** Let us consider two vertices \( x \prec y \). It is enough to consider the case \( I(x) \geq I(y) \).

Set \( A = [o, x] \) and \( B = [1, y] \cup [1, x] \). If we denote by \( x - 1 \) the vertex in the geodesic \([o, x]\) with distance to \( x \) equal to \( 1 \) then, \( A = A^*, B^* = [o, y-1] \cup [1, x], A \cup B = (A \cup B)^* = [o, y] \cup [1, x] \) and \((A \cap B)^* = [o, x - 1]\).

Applying (CC) for these sets we easily obtain that \( v(y) \leq v(x) \); that is, \( v \) is linearly decreasing.

\[ \square \]
We will now consider two more well-known rearrangements. Let \( A \) be a measurable set in \( \mathbb{R}^n, \ n \geq 2 \). The spherical symmetrization of a set \( A \) is \( \mathcal{R}(A) = A^* = B(0, (\sigma_n^{-1}|A|)^{1/n}) \), where \( \sigma_n \) is the volume of the \( n \)-dimensional ball (see [1] for further information). To define Steiner symmetrization (see [12, 5]) of order \( k \geq 1 \), we write points in \( x \in \mathbb{R}^n \) as pairs \((\bar{x}, y)\) with \( \bar{x} \in \mathbb{R}^{n-k} \) and \( y \in \mathbb{R}^k \). The Steiner symmetrization of order \( k \) of \( A \) is \( \mathcal{R}(A) = \mathcal{S}_k(A) \), the set whose \( k \)-dimensional cross sections parallel to the hyperplane \( \bar{x} = 0 \) are balls with measure equal to the corresponding cross sections of \( A \). This symmetrization method shows up in applications to PDE’s, like the isoperimetric inequality (see [12, 11]).

For \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) a measurable function, we define the spherical symmetrization \( f^*_{Sp} \) and the Steiner symmetrization \( (\mathcal{S}_k f)^* \) of \( f \), using [11]:

\[
f^*_{Sp}(x) = \int_0^\infty \chi_{\{f > s\}}(x) \, ds,
\]

\[
(\mathcal{S}_k f)^*(x) = \int_0^\infty \chi_{\{\mathcal{S}_k(f > s)\}}(x) \, ds.
\]

First, we observe that, by an easy change of variables in (12), we obtain that if \( f^* \) denotes the classical decreasing rearrangement of \( f \),

\[
f^*_{Sp}(x) = f^*(\sigma_n|x|^n), \ x \in \mathbb{R}^n.
\]

In particular, the spherical rearrangement \( f^*_{Sp} \), of a measurable function \( f \) in \( \mathbb{R}^n \), is a radial decreasing function.

By a change of variables into spherical coordinates in \( \mathbb{R}^k \), we can write the last \( k \)-coordinates of \( x \in \mathbb{R}^n \), \((x_{n-k+1}, \ldots, x_n)\) as \( \rho(\theta_{k-1}) \), with \( \rho > 0 \) and \( \theta_{k-1} \in \Sigma_{k-1} \) (the unit sphere in \( \mathbb{R}^k \)). Thus, using (14), we have that

\[
(\mathcal{S}_k f)^*(x) = (f_{\bar{x}})^*(\sigma_k \rho k),
\]

where \((f_{\bar{x}})^*\) is the classical decreasing rearrangement of the function defined on \( \mathbb{R}^k \) as follows: \( f_{\bar{x}}(\cdot) := f(\bar{x}, \cdot) \), with respect to the coordinates \((x_{n-k+1}, \ldots, x_n) \in \mathbb{R}^k \).

Taking into account these considerations, given a weight \( v \) defined on \( \mathbb{R}^n \), we can establish the following formula for the functional defining the Lorentz spaces \( \Lambda^p_{\mathcal{S}_k}(v) \), with respect to the rearrangement given by the Steiner symmetrization of order \( k \).

**Proposition 3.14** Let \( 0 < p < \infty \). Given a weight \( v \) defined on \( \mathbb{R}^n \), there exists another weight \( \bar{v} \) on \( \mathbb{R}^{n-k} \times \mathbb{R}^+ \) such that, for all measurable functions \( f \) on \( \mathbb{R}^n \),

\[
\|f\|_{\Lambda^p_{\mathcal{S}_k}(v)}^p = \frac{1}{k \sigma_k} \int_{\mathbb{R}^{n-k}} \int_0^\infty (f_{\bar{x}})^{sp}(s) \, \bar{v}(\bar{x}, s) \, ds \, d\bar{x}.
\]

**Proof:** This is just a consequence of (15) after a change into spherical coordinates \( \rho(\theta_{k-1}) \), with \( \rho > 0 \) and \( \theta_{k-1} \in \Sigma_{k-1} \). Then calling \( s = \sigma_k \rho k \), the weight \( \bar{v} \) associated to \( v \) is given by

\[
\bar{v}(\bar{x}, s) := \int_{\Sigma_{k-1}} v(\bar{x}, (s/\sigma_k)^{1/k} \theta_{k-1}) \, d\theta_{k-1}, \ (\bar{x}, s) \in \mathbb{R}^{n-k} \times \mathbb{R}^+.
\]

\( \square \)

14
Remark 3.15 We remark that in the case of Steiner symmetrization of order $k = 1$ (the corresponding to one dimensional cross sections), the associated weight to $v$ is just $\bar{v}(\bar{x}, y) = v(\bar{x}, y) + v(\bar{x}, -y), \ (\bar{x}, y) \in \mathbb{R}^{n-1} \times \mathbb{R}^+.$

Looking at the formula (10), by means of a change into spherical coordinates and the use of (14), we can deduce that, if $u$ is a decreasing function in $\mathbb{R}^+$, then the following saturation formula for the spherical rearrangement hold:

$$\sup_{\sigma} \int_{\mathbb{R}^n} |f(x)| u(\sigma(x)) \, dx = \int_{\mathbb{R}^n} f_{Sp}^*(x) u(\sigma_n |x|^{n}) \, dx,$$

where the supremum is taken over all measure preserving transformations $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^+$.

All these facts lead us to establish the following characterization of the normability of Lorentz spaces with respect the spherical rearrangement with essentially the same proof as in the classical case (see [13]).

Theorem 3.16 Let $v$ be a weight in $\mathbb{R}^n$ and $p \geq 1$. The following facts are equivalent:

(a) The functional $\| \cdot \|_{L^p_{Sp}(v)}$ is a norm.

(b) For every $A, B \in \mathbb{R}^n$, $V((A \cup B)^*) + V((A \cap B)^*) \leq V(A^*) + V(B^*)$.

(c) The weight $\bar{v}(s) := \int_{\Sigma_{n-1}} v((s/\sigma_n)^{1/n} \theta_{n-1}) \, d\theta_{n-1}, \ s \in \mathbb{R}^+$, is a decreasing function.

(d) For all measurable functions $f$ in $\mathbb{R}^n$, the equality

$$\sup_{h_{Sp}^* = h} \int_{\mathbb{R}^n} |f(x)h(x)| \, dx = \int_{\mathbb{R}^n} f_{Sp}^*(x) \bar{v}(\sigma_n |x|^{n}) \, dx$$

holds.

Proof: Theorem 3.3 gives us that (a) implies (b).

Assume that (b) holds, and consider $0 < \varepsilon < a \leq b$ and the following sets in $\mathbb{R}^n$,

$$A = B(0, a), B = B(0, b) \setminus B(0, \varepsilon).$$

Then,

$$A = A^*, \ B^* = B(0, (b^n - \varepsilon^n)^{1/n}), \ (A \cup B)^* = B(0, b), \ (A \cap B)^* = B(0, (a^n - \varepsilon^n)^{1/n}).$$

Condition (b) implies that

$$V(B(0, b)) - V(B(0, (b^n - \varepsilon^n)^{1/n})) \leq V(B(0, a)) - V(B(0, (a^n - \varepsilon^n)^{1/n})).$$
After a change into spherical coordinates \((\rho, \theta_{n-1}) \in \mathbb{R}^+ \times \Sigma_{n-1}\) and calling \(s = \sigma_n \rho^n\), we obtain that the above condition can be written as
\[
\int_{\sigma_n(b^n - \varepsilon^n)}^{\sigma_n(b^n)} \bar{v}(s) \, ds \leq \int_{\sigma_n(a^n - \varepsilon^n)}^{\sigma_n(a^n)} \bar{v}(s) \, ds.
\]

Dividing both sides by \(\sigma_n \varepsilon^n\) and letting \(\varepsilon \to 0\), we obtain \(\bar{v}(\sigma_n b^n) \leq \bar{v}(\sigma_n a^n)\); that is, \(\bar{v}\) is a decreasing function of \(s\).

That condition (c) implies (d) is equality (17). Finally, we observe that Theorem 3.4 proves that condition (d) implies (a).

Similarly, in order to study when the functional \(\|\cdot\|_{\Lambda^k(v)}\) is a norm, we observe that, due to Proposition 3.14, the condition is reduced to the fact that the associated weight \(\bar{v}(\bar{x}, s)\), defined in (16), must be a decreasing function in \(s\), and, also with essentially the same proof, we can establish the following characterization.

**Theorem 3.17** Let \(v\) be a weight in \(\mathbb{R}^n\), \(p \geq 1\) and \(k \geq 1\) an integer. The following facts are equivalent:

(a) The functional \(\|\cdot\|_{\Lambda^k(v)}\) is a norm.

(b) For every \(A, B \subset \mathbb{R}^n\), \(V(S_k(A \cup B)) + V(S_k(A \cap B)) \leq V(S_k(A)) + V(S_k(B))\).

(c) The weight \(\bar{v}(\bar{x}, s)\) defined in (17) is a decreasing function in the variable \(s\).

(d) For all measurable functions \(f\) in \(\mathbb{R}^n\), the equality
\[
\sup_{(S_k f)^* = \hat{\theta}} \int_{\mathbb{R}^n} |f(x) h(x)| \, dx = \int_{\mathbb{R}^n-k \times R^+} (S_k f)^*(\bar{x}, s) \, \bar{v}(\bar{x}, s) \, d\bar{x} \, ds
\]
holds.

**References**

[1] F.J. Almgren and E.H. Lieb, *Symmetric decreasing rearrangement can be discontinuous*, Bull. Amer. Math. Soc. (N.S.) 20 (1989), 177–180.

[2] S. Barza, L.E. Persson, and J. Soria, *Sharp weighted multidimensional integral inequalities for monotone functions*, Math. Nachr. 210 (2000), 43–58.

[3] S. Barza, L.E. Persson, and J. Soria, *Multidimensional rearrangement and Lorentz spaces*, Acta Math. Hungar. 104 (2004), 203–224.

[4] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics, 129. Academic Press, Inc., Boston, MA, 1988.
[5] A. Burchard, *Steiner symmetrization is continuous in $W^{1,p}$*, Geom. Funct. Anal. 7 (1997), 823–860.

[6] M.J. Carro, J.A. Raposo, and J. Soria, *Recent Developments in the Theory of Lorentz Spaces and Weighted Inequalities*, Mem. Amer. Math. Soc. 187. Providence, RI, 2007.

[7] M.J. Carro and J. Soria, *Weighted Lorentz spaces and the Hardy operator*, J. Funct. Anal. 112 (1993), 480–494.

[8] M.J. Carro and J. Soria, *The Hardy–Littlewood maximal function and weighted Lorentz spaces*, J. London Math. Soc. 55 (1997), 146–158.

[9] J.L. Garcia-Domingo and J. Soria, *Lorentz spaces for decreasing rearrangements of functions on trees*, Math. Inequal. Appl. 7 (2004), 471–490.

[10] J.L. Garcia-Domingo and J. Soria, *A decreasing rearrangement for functions on homogeneous trees*, European J. Combin. 26 (2005), 201–225.

[11] B. Kawohl, *Rearrangements and Convexity of Level Sets in PDE*, Lecture Notes in Mathematics, 1150. Springer-Verlag, Berlin, 1985.

[12] E.H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 1997.

[13] G. G. Lorentz, *On the theory of spaces $\Lambda$*, Pacific J. Math. 1 (1951), 411–429.

[14] E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. 96 (1990), 145–158.

Santiago Boza  
Dept. Appl. Math. IV  
EPSEVG  
Polytechnical University of Catalonia  
E-08880 Vilanova i Geltrú, SPAIN  
E-mail: boza@ma4.upc.edu

Javier Soria  
Dept. Appl. Math. and Analysis  
University of Barcelona  
E-08007 Barcelona, SPAIN  
E-mail: soria@ub.edu