A Pedagogical Model of Static Friction

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Abstract

While dry Coulombic friction is an elementary topic in any standard introductory course in mechanics, the critical distinction between the kinetic and static friction forces is something that is both hard to teach and to learn. In this paper, I describe a geometric model of static friction that may help introductory students to both understand and apply the Coulomb static friction approximation.
I. INTRODUCTION

One of the key disputes that had to be settled when the modern formulation of mechanics was being constructed was whether friction had a primary or secondary role. A hard-won step forward was made with the recognition that the inertia principle (momentum in a system can only be changed through an interaction with its surroundings) is fundamental, and not the “friction principle” (that systems naturally and continuously slow down unless acted upon by their surroundings). That the “friction principle” seems to be so naturally confirmed by everyday experience is one of the big challenges in teaching the inertia principle, and is the source of many naive pre- and mis-conceptions on the part of students.

So, it is no small victory when a student builds sufficient mastery with Newtonian ideas to successfully solve the usual spectrum of problems set in a first-semester mechanics course. The pedagogical problem I wish to explore here is a common one in these courses (both at the high school and university level). The Coulomb model of dry friction is simple enough to state, and while the Coulombic kinetic friction is fraught with its own set of pitfalls for students\(^1,2\), the static Coulomb friction approximation is (in my experience) even more tangled with students’ conceptual difficulties.

Indeed, the static Coulomb friction approximation puts an upper limit on a quantity that students have to work extraordinarily hard to convince themselves even exists. This paper is (yet one more\(^3,5\)) attempt to give students a sense of what this maximum force means, and why the angle of repose is a good measure of the static coefficient of friction, and to put this pedagogical tool into the hands of instructors.

II. ELEMENTARY PROBLEM

An elementary problem will illustrate the conceptual problem that I encounter all too often in my own first-semester mechanics course. The situation is shown in Figure I. A uniform cylinder of radius R is placed on a ramp inclined from the horizontal at an angle \(\theta\). The coefficient of static friction and kinetic friction, \(\mu_s\) and \(\mu_k\) are given. Assuming that the object rolls without slipping, what is its acceleration down the ramp?

Taking the \(x\)-axis as going down the ramp, and the \(y\) axis as normal to the ramp, a good
FIG. 1. A disc of moment of inertia $I$ rolls without slipping on a surface with static coefficient of friction $\mu_s$.

A student will generate the following from Newton's second law:

$$mg \sin \theta - f = ma,$$

(1)

and

$$- mg \cos \theta + N = 0.$$  

(2)

The first is the resolution of the weight and friction force in the $x$-direction as being the explanation for the $x$-component of the acceleration of the disc, and the second is a really quite sophisticated statement that the constraint normal force $N$ will take whatever value is required so that the observed motion ($a_y = 0$) is maintained. At this point, there are three unknowns in the problem, $f$, $a$ and $N$, and it is all too tempting to close the set of equations with the statement of Coulomb kinetic friction - because the disc is moving:

$$f = \mu_k N.$$  

(3)

This is, of course, an excellent solution to a completely different problem, not the one at hand. For, the resulting (incorrect) acceleration is just that of an object sliding down a incline:

$$a = g(\sin \theta - \mu_k \cos \theta).$$

(4)

But, this is completely unsatisfactory, and I attempt to guide students to discover why this is a bad solution from a skeptical review of the result. For instance, we could easily choose $\tan \theta = \mu_k$ so that the "acceleration" would vanish. Thus, at this "magic" angle, a disc placed at rest on such a surface would remain in equilibrium - defying every experience
every student has had with objects rolling down hill. We can make $a$ small, but it is very
hard to imagine how it can be made to vanish at finite $\theta$ ... or even become negative!

The resolution is trivial for an instructor, of course. We have not even used the “rolling
without slipping” constraint, and the critical role the moment of inertia of the disc plays in
the physics has not been established. In fact, the friction force is determined by the rolling
without slipping constraint: if $f$ were too big (small), the disc would over (under) rotate,
and the point of contact between the disc and the ramp would slide. The correct analysis
eventually leads to

$$a = g \sin \theta \frac{mR^2}{I + mR^2}, \quad (5)$$

where $I$ is the moment of inertia of the disc and

$$\tan \theta < \mu_s, \quad (6)$$

ensuring that the friction force stays below its maximal value.

And, while this is indeed a trivial problem that we expect any well-trained student to
be able to solve, there is still the slightly (to me) unsatisfactory and qualitative difference
between the kinetic friction force (which has a given magnitude proportional to the loading
of the surfaces) and the static friction force (which has the character of a constraint force)
with its mysterious maximal value. As a student, the idea that the kinetic friction force was
the result of asperities on each surface interacting was convincing and suggestive. What,
however, is being “broken” in the static friction force? Dry welding of asperities, or adhesion
of the surfaces, or any other number of phenomenon are called to mind as at least a metaphor
for Coulomb dry static friction. While dry friction is far too complex a phenomenon to
explain with a simple geometric theory, I can explain its general features in a pedagogical
and geometric toy model.

III. $\mu_s$ AS A GEOMETRIC PARAMETER

Before anyone mucks around with modeling surface interactions with friction, we intro-
duce the physicist’s toy model of the “frictionless” surface, and we train students quite well
in how to analyze systems built from these imaginary surfaces. Consider the symmetric
wedge of mass $m$ and bottom angle $2\phi$ resting on a conforming frictionless surface, as in
Figure 2.
and
\[ f_2 = \frac{1}{2} \left( \csc \phi \, mg + \sec \phi \, F \right). \] (10)

Not surprisingly, \( f_2 \) has a larger magnitude than \( f_1 \), because \( F \) is dragging the wedge off to the right. Much more enlightening is to consider the maximal value of \( F \) consistent with equilibrium. Here, because the left surface is providing \( f_1 \) by pressing away from itself, the maximal value of \( F \) will occur when \( f_1 = 0 \):
\[ F = \cot \phi \, mg = \tan \theta \, mg. \] (11)

Here, we have replaced the half-opening angle of the wedge, \( \phi \) with the incline angel of the surface \( \theta = \pi/2 - \phi \). Thus, if the lateral force \( F \) satisfies:
\[ F < \mu_s N \text{ with } \mu_s = \tan \theta \text{ and } N = mg, \] (12)
then the wedge will remain in equilibrium. Should \( F \) exceed this value, the wedge will start to accelerate. This is a simple, straightforward model for Coulomb static friction.
IV. DISCUSSION

Very well, a single wedge resting in a conforming trough behaves similarly to an object resting on a flat surface characterized by static friction \( \mu_s \). If we conceptualize such a flat, frictioned surface as consisting of two surfaces as in Figure 3 we have a useful toy model for static friction. Here, the upper and lower surfaces are frictionless in the usual sense, but consist of a uniform sawtooth profile with a “roughness” angle \( \theta \). If a lateral force \( F \) is applied to the upper object, the argument above implies that each wedge-shaped subregion will remain in equilibrium only so long as

\[
F < \tan \theta n_o m_o g \equiv \tan \theta N_{tot},
\]

where \( n_o \) is the total number of wedges in the upper surface, each consisting of a mass \( m_o \).

If we were to tilt the lower surface at an angle \( \alpha \), as long as \( \alpha < \theta \), there will be a nonzero reaction force between the two surfaces, and the object will remain motionless. When \( \alpha > \theta \) the frictionless surfaces making up the system are free to slide past each other. Thus, the angle of repose matches the usual Euler condition:

\[
\tan \alpha = \mu_s \equiv \tan \theta.
\]
This toy model could be made very explicit by adding low-friction caster wheels to the wedge and trough in Figure 2, and could be used as a lecture demonstration or as a stand-alone experiment in equilibrium leading to a study of static friction.

The thing that I enjoy so much about this model is that it places kinetic and static friction upon the same footing. That the kinetic coefficient of friction between two surfaces is related (in some hard to analyze but definite manner) to the shapes or the surfaces, as characterized by a RMS tangent angle ⟨θ⟩ is a fixture in elementary mechanics texts. This model makes it clear that static friction arises from the same basic physics - two rough surfaces interacting.

An interesting implication of the model is that it may be possible to create surfaces with anisotropic friction but in more than just the sense of their being low friction axes on a surface. Indeed, if microscopically corrugated surfaces with a different forward roughness, θ_f ≠ the backward roughness θ_b, different threshold forces in the forward and reverse directions could be engineered. Thus, not just an axis but a specific direction on a surface - a tribological diode could be engineered.

V. CONCLUSION

I have constructed a simple mechanical toy model for dry static Coulombic friction, and naturally interpreted the roughness of a surface as the source of all of the important static friction properties introduced in a first-semester mechanics course. Surfaces are modeled as frictionless but impenetrable, and it is the tangent angle of the sawtooth contacts surfaces make with each other that gives rise to μ_s.

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