ON THE COMMON TRANSVERSAL PROBABILITY IN FINITE GROUPS

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Abstract. Let $G$ be a finite group, and let $H$ be a subgroup of $G$. We compute the probability, denoted by $P_G(H)$, that a left transversal of $H$ in $G$ is also a right transversal, thus a two-sided one. Moreover, we define, and denote by $tp(G)$, the common transversal probability of $G$ to be the minimum, taken over all subgroups $H$ of $G$, of $P_G(H)$. We prove a number of results regarding the invariant $tp(G)$, like lower and upper bounds, and possible values it can attain. We also show that $tp(G)$ determines structural properties of $G$. Finally, several open problems are formulated and discussed.

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1. Introduction

We begin with some general remarks and a little bit of history. It is often the case that probability theory interacts with group theory. The area that we now call “probabilistic group theory” may reasonably be said to begin with a series of papers by Erdős and Turán—stretching from 1965 to 1972 and beginning with [7]—where statistical properties of the symmetric group are examined in detail. For survey articles of this area we refer the interested reader to [24], [25], [6], [3], as well as the

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Recent concepts that probabilistic group theory concerns itself with include the so-called “probabilistic generation” and the “commuting probability” as well as variants of the latter. The former is about questions of the type:

Given a family of groups \( \{G_n\}_{n} \), what is the probability that a \( d \)-tuple of elements from \( G_n \), chosen uniformly at random, generates \( G_n \) as \( n \to \infty \)?

Recent work in this direction has mainly focussed on the so-called \((2,3)\)-generation problem for finite simple groups. The question here is whether all finite simple groups can be generated by an involution and an element of order 3. The commuting probability of \( G \), often denoted by \( \text{cp}(G) \), is the probability that two elements of \( G \), chosen uniformly at random, commute. It was popularised by Gustafson who, in turn, traces its origin to Erdős and Turán’s series of papers on the statistics of the symmetric group. Using the class equation one can prove that \( \text{cp}(G) = \frac{k(G)}{|G|} \), where \( k(G) \) is the number of conjugacy classes of \( G \). Gustafson also established the gap result stating that, if \( G \) is non-abelian then \( \text{cp}(G) \leq \frac{5}{8} \), with equality if, and only if, \( G/Z(G) \cong C_2 \times C_2 \). Of course, \( G \) is abelian if, and only if, \( \text{cp}(G) = 1 \) and, despite the \( (5/8,1) \) gap, the commuting probability may be viewed as an arithmetic quantification of the “abelianness” of a group. Certain other variants of the commuting probability have also received attention. We take this opportunity to mention only a couple of those: Tărnăuceanu’s concept of the “subgroup permutability degree” as the probability that two subgroups of \( G \) permute. It turns out that this probability is an arithmetic measure of how close \( G \) is to being an Iwasawa group or, equivalently, a nilpotent modular group. A second interesting variant is due to Blackburn, Britnell, and Wildon. In the authors introduce the probability that two elements, chosen independently and at random from a (finite) group \( G \), are conjugate and study many of the fundamental properties that this probability enjoys.

We now focus our attention on the contents of the paper at hand. Let \( G \) be a finite group, and let \( H \) be a subgroup of \( G \). By Hall’s Marriage Theorem (see [14]) there exists a common set of representatives for the left and the right cosets of \( H \) in \( G \); such a set is called a double or two-sided transversal of \( H \) in \( G \). Even though the existence of such a double transversal is guaranteed, it seems unlikely that (in general) a left transversal of a non-normal subgroup will also be a right transversal. One of the main objectives of this paper is to compute the probability that a randomly chosen (with the uniform distribution) left transversal of \( H \) in \( G \) is also a right transversal. If the index of \( H \) in \( G \) is \( n \) and we denote by \( \text{DT}_G(H) \) the set of all double transversals of \( H \) in \( G \), then the quotient

\[
P_G(H) := \frac{|	ext{DT}_G(H)|}{|H|^n}
\]

is precisely the (Laplacian) probability in question. Actually, what we compute is the above probability in a more general setting, where two subgroups \( H, K \) of the same index are given, and we are seeking left transversals of \( H \) that are also right transversals for \( K \); see Proposition 3.2.

In the present article, and in the spirit of \( \text{cp}(G) \), a novel invariant is introduced and studied: the transversal probability \( \text{tp}(G) \) of \( G \) defined as

\[
\text{tp}(G) := \min_{H \leq G} P_G(H).
\]

Our objective here is two-fold: firstly, to compute \( P_G(H) \) and to discuss its various numerical properties. In this direction, for example, we prove that if \( n = (G : H) \),
where $H$ is not normal in $G$, then (see Corollary 3.10)

\[
\frac{(n-1)!}{(n-1)^{n-1}} \leq P_G(H) \leq \frac{1}{2},
\]

and (see Theorem 3.11)

\[
\lim_{n \to \infty} P_G(H) = 0;
\]

secondly, to investigate how $\text{tp}(G)$ influences the structure of the group $G$. Results in this direction are for example the following:

- If $\text{tp}(G) > (1/2)^{40}$ then either $G$ is soluble or has a section isomorphic to $A_5$. (Proposition 4.3)
- If $\text{tp}(G) > (1/2)^8$ then either $G$ is supersoluble or has a section isomorphic to $A_4$. (Theorem 4.4)
- If $\text{tp}(G) > (2/9)^2$ then either $G$ is nilpotent or has a section isomorphic to one of the groups $\{A_4, D_3, D_5, D_7\}$. (Theorem 4.5)
- If $\text{tp}(G) > 4/81$ and $G$ is non-abelian then it has derived length 2. (Proposition 4.14)

Furthermore, the results above are all sharp. In addition, the upper bound of $1/2$ obtained for $P_G(H)$ which holds for every non-normal $H \leq G$ carries over to $\text{tp}(G)$ for every non-Dedekind group $G$; that is, $\text{tp}(G) \leq 1/2$ for every such group. We are able to fully characterise groups $G$ with $\text{tp}(G) = 1/2$, as well as those with $\text{tp}(G) = 1/4$ (cf. Corollary 4.11 and Theorem 4.13).

Finally, while investigating the possible values of $\text{tp}(\cdot)$, the following number theoretic result, of independent interest, was obtained (see Theorem 4.6):

- If $\prod_{i=1}^n t_i! = \prod_{i=1}^k p_i!$ where the numbers $t_i > 1$ are positive integers and the $p_i$ are distinct primes, then $k = n$ and, after appropriate rearrangement, $t_i = p_i$ for all $i = 1, \ldots, n$.

The article is organised as follows. In Section 2 the coset intersection graph is introduced and its properties needed for the computation of $P_G(\cdot, \cdot)$ are analyzed. In Section 3 the function $P_G(\cdot, \cdot)$ is introduced and its precise value is calculated. A connection with the permanent of a matrix is established and several bounds on its values are given. In addition, the behaviour of $P_G(\cdot)$ with respect to subgroups and quotients is determined. In Section 4 we work with the function $\text{tp}(G)$, discussing the values it can and cannot attain, and the way it interacts with the structure of the group $G$. Finally, in Section 5 some open problems and questions are posed and discussed.

2. Coset intersection graphs

In this section we introduce the concept of a coset intersection graph, which is the required background for the computation of $P_G(\cdot, \cdot)$. We follow the definition in [2].

**Definition 2.1.** Let $G$ be a group, and let $H, K \leq G$ be subgroups. The **coset intersection graph** $\Gamma = \Gamma^G_{H,K}$ has vertex set $V(\Gamma) = G/H \cup K\backslash G$, two vertices being connected by an (undirected) edge if, and only if, the corresponding cosets have non-empty intersection. If $H = K$, we set $\Gamma^G_H := \Gamma^G_{H,H}$.
By definition, $\Gamma_{H,K}^G$ is a bipartite graph, split between $\{l_i H\}_{i \in I}$ and $\{K r_j\}_{j \in J}$, where $\{l_i\}_{i \in I}$ is a left transversal for $H$ in $G$ and $\{r_j\}_{j \in J}$ is a right transversal for $K$ in $G$. In this paper, $G$ will always be a finite group; thus, the coset intersection graphs considered in what follows will all be finite. We also denote by $K_{a,b}$ the complete bipartite graph on $a$ and $b$ vertices, for any positive integers $a$ and $b$.

**Lemma 2.2.** Let $\Delta \leq \Gamma_{H,K}^G$ be a connected component. Then

(i) $\bigcup_{gH \in V(\Delta)} gH = \bigcup_{Kg' \in V(\Delta)} Kg'$;

(ii) $\Delta$ is a complete bipartite graph.

**Proof.** (i) For given $g \in G$ with $gH \in V(\Delta)$, we have

$$gH \subseteq \bigcup_{Kg' \in K \setminus G} Kg' \subseteq \bigcup_{gH \cap Kg' \neq \emptyset} Kg',$$

so that the left-hand side of (i) is contained in the right-hand side, and a similar argument establishes the reverse inclusion.

(ii) This is [2, Thm. 3].

Our next result, which (in somewhat different language and with a less direct proof) goes back to Ore [22], computes the number of connected components of a coset intersection graph.

**Lemma 2.3.** The number of connected components of the graph $\Gamma_{H,K}^G$ equals the number $|K \setminus G/H|$ of $(K,H)$-double cosets of $G$.

**Proof.** Denote by $C(\Gamma)$ the collection of all connected components of $\Gamma = \Gamma_{H,K}^G$. If $aH, Kb \in V(\Delta)$ for some connected component $\Delta$ of $\Gamma$ then, by Part (ii) of Lemma 2.2, there exist elements $h \in H$ and $k \in K$, such that $ah = kb$, or $b = k^{-1}ah$. Hence, $KbH = KaH$. Keeping $aH \in V(\Delta)$ fixed while running through all right vertices $Kb$ of $\Delta$, we thus see that

$$KbH = KaH, \quad (Kb \in V(\Delta)).$$

A symmetric argument, this time keeping a right vertex $Kb \in V(\Delta)$ fixed, and running through the left vertices $aH$ of $\Delta$ shows that

$$KaH = KbH, \quad (aH \in V(\Delta)).$$

It follows that the whole component $\Delta$ (meaning every vertex of $\Delta$) is contained in one and the same $(K,H)$-double coset $Kg_\Delta H$, and sending $\Delta$ to $Kg_\Delta H$ yields a well-defined map

$$c : C(\Gamma_{H,K}^G) \longrightarrow K \setminus G/H, \quad c(\Delta) = Kg_\Delta H, \quad (gH \in V(\Delta)).$$

Conversely, let $KgH \in K \setminus G/H$ be a given $(K,H)$-double coset. Then each left $H$-coset $kgH$ and every right $K$-coset $Kgh$ contained in $KgH$ intersect non-trivially, as $kgh \in kgH \cap Kgh$. An argument analogous to the one given above now shows that all left $H$-cosets and all right $K$-cosets contained in $KgH$ lie in one and the same connected component $\Delta_g$ of $\Gamma$, and sending $KgH$ to $\Delta_g$ gives a well-defined map

$$d : K \setminus G/H \longrightarrow C(\Gamma_{H,K}^G), \quad d(KgH) = \Delta, \quad (gH \in V(\Delta)).$$

The fact that $d(c(\Delta)) = \Delta$ for $\Delta \in C(\Gamma)$ is now obvious (pin down $\Delta$ by means of a vertex $gH \in V(\Delta)$) and, similarly, we find that $c(d(KgH)) = KgH$, finishing the proof.
With every edge of a coset intersection graph we associate a weight in the following natural way.

**Definition 2.4.** Given a coset intersection graph $\Gamma = \Gamma_{H,K}^G$, we associate to an edge $e = l_iH - Kr_j$ in $\Gamma$ the weight $w(e) := |l_iH \cap Kr_j|$.

If $g \in G$ and if $\Delta \leq \Gamma$ is a connected component of the coset intersection graph $\Gamma = \Gamma_{H,K}^G$, we shall write $g \in \Delta$ to mean $gH \in V(\Delta)$. Observe that, as $g \in gH \cap Kg$, we clearly have $gH \in V(\Delta)$ if and only if $Kg \in V(\Delta)$.

**Lemma 2.5.** Let $\Delta \leq \Gamma_{H,K}^G$ be a connected component. Then all edges of $\Delta$ carry the same weight $w$, and we have

$$w = |gHg^{-1} \cap K| = |H \cap g^{-1}Kg|, \quad (g \in \Delta).$$

**Proof.** Let $aH, Kb \in V(\Delta)$ and $g \in aH \cap Kb$. Then $aH = gH$ and $Kb = Kg$, and thus

$$w_{a,b} := w(aH - Kb) = |aH \cap Kb| = |gH \cap Kg| = |gHg^{-1} \cap K|.$$ Keeping $aH$ fixed, consider an arbitrary right coset $Kc \in V(\Delta)$, and set $w_{a,c} := w(aH - Kc)$. Then, by the previous computation, $w_{a,c} = |xHx^{-1} \cap K|$ for any $x \in aH \cap Kc$. Since $x \in aH = gH$, we have $x = gh'$ for some $h' \in H$, so that

$$w_{a,c} = |gh'H(h')^{-1}g^{-1} \cap K| = |gHg^{-1} \cap K| = w_{a,b}. $$

Next, consider an arbitrary left coset $dH \in V(\Delta)$, noting that, at this stage, $dH - Kc$ is an arbitrary edge in $\Delta$. Changing the roles of $H$ and $K$ in the previous argument we also get $w_{a,c} = w_{d,c}$. Hence

$$w_{d,c} = w_{a,c} = w_{a,b} = w,$$

and our result follows. $\blacksquare$

**Definition 2.6.** If $\Delta \leq \Gamma_{H,K}^G$ is a connected component, then we call the common weight of the edges in $\Delta$ the weight of $\Delta$, denoted by $w(\Delta)$.

**Lemma 2.7.**

(a) If $\Delta_\sigma \equiv K_{s_\sigma,t_\sigma}$ is a connected component of $\Gamma_{H,K}^G$, then $|H| = w_\sigma t_\sigma$ and $|K| = w_\sigma s_\sigma$, where $w_\sigma = w(\Delta_\sigma)$ is the weight of $\Delta_\sigma$. In particular, $s_\sigma/t_\sigma = |K|/|H|$, and we have $s_\sigma = t_\sigma$ provided that $|H| = |K|$. In the latter case, the number $m$ of components of type $K_{1,1}$ is given by

$$m = \begin{cases} (N_C(H) : H), & \text{if } H \text{ is conjugate to } K \text{ in } G, \\ 0, & \text{otherwise}, \end{cases}$$

and the weight of such a component equals $|H|$.

(b) If $G = \bigcup_{\sigma = 1}^s K_{s_\sigma,t_\sigma}H$, then $t_\sigma = (H : H \cap K_{s_\sigma})$, where $\Delta_\sigma \equiv K_{t_\sigma,t_\sigma}$ is the connected component of $\Gamma_{H,K}^G$ containing $s_\sigma$.

**Proof.** (a) Let $\Delta_\sigma \equiv K_{s_\sigma,t_\sigma}$ be a connected component of the coset intersection graph $\Gamma_{H,K}^G$ of weight $w(\Delta_\sigma) = w_\sigma$. By Lemma 2.5, each given left coset $L_i$ of $\Delta_\sigma$ intersects every right coset $R_j \in V(\Delta_\sigma)$ in exactly $w_\sigma$ elements and, as the right cosets of $\Delta_\sigma$ are pairwise disjoint, we have

$$|L_i \cap \bigcup_{R_j \in V(\Delta_\sigma)} R_j| = w_\sigma t_\sigma, \quad (L_i \in V(\Delta_\sigma)).$$
However, $L_i \subseteq \bigcup_{R_j \in \mathcal{V}(\Delta)} R_j$ by Lemma 2.2(i), so that in fact

$$L_i \cap \bigcup_{R_j \in \mathcal{V}(\Delta)} R_j = L_i,$$

and we find that

$$|H| = |L_i| = w_{0, t_0},$$

as claimed. A symmetric argument yields that $|K| = w_{0, s_0}$. As concerns the assertion about the number $m$ of trivial components, we note that $1 = t_0 = s_0$ precisely when $w_\sigma = |H| = |K|$, which in turn implies (by Lemma 2.5) that $K = gHg^{-1} \cap K$ and $H = H \cap g^{-1}Kg$ for $g \in \Delta_\sigma$. Hence $H, K$ are conjugate and

$$m = \left| \left\{ g \in G : H^g = K \right\} \right|/|H|,$$

from which the given formula follows. The remaining assertions are now clear. 

(b) This is a special case of [2, Prop. 6].

We record upper and lower bounds for the number $s = |K\setminus G/H|$ of connected components of a symmetric coset intersection graph $\Gamma_{H,K}^G$.

**Lemma 2.8.** Let $H, K \leq G$ be such that $(G : H) = n = (G : K)$, and let $s = |K\setminus G/H|$. Then we have

$$\frac{n-m}{|H|} + m \leq s \leq \frac{n-m}{p} + m,$$

where $p$ is the smallest prime divisor of $|H|$, and $m$ is given by (2.1).

**Proof.** Let

$$G = \bigcup_{\sigma=1}^{s} K_{g_\sigma}H,$$

where $g_\sigma \in \Delta_\sigma$ for $1 \leq \sigma \leq s$. We observe that

$$|K_{g_\sigma}H| = \frac{|H| \cdot |K|}{|H \cap K^{g_\sigma}|} = \frac{|H|^2}{|H \cap K^{g_\sigma}|},$$

and that

$$|H| \leq \frac{|H|^2}{|H \cap K^{g_\sigma}|} \leq |H|^2.$$

Here, the left-hand side is assumed if, and only if, $g_\sigma \in \{ g \in G : K^g = H \}$, while the right-hand side is assumed if, and only if, $H \cap K^{g_\sigma} = 1$. It follows that

$$n|H| = |G| = \sum_{\sigma=1}^{s} |K_{g_\sigma}H| = m|H| + \sum_{\sigma=m+1}^{s} |K_{g_\sigma}H| \leq m|H| + (s-m)|H|^2.$$

Dividing both sides by $|H|$ yields $n \leq m + (s-m)|H|$ or equivalently

$$\frac{n-m}{|H|} + m \leq s.$$
This proves the first half of the assertion. The upper bound for \( s \) is obtained in a similar way:

\[
|H| = |G| = \sum_{a=1}^{s} |K_{g_a}H| \\
= m|H| + \sum_{a=m+1}^{s} |H| \cdot (H : H \cap K^{\langle a \rangle}) \\
\geq m|H| + (s - m)p|H|.
\]

Dividing both sides by \(|H|\) yields \( n \geq m + (s - m)p \), or equivalently

\[
\frac{n-m}{p} + m \geq s,
\]

completing the proof.

With the help of Lemma 2.8, we can characterise groups \( G \) having a non-normal subgroup \( H \) satisfying \(|H \setminus G/H| = 2\), as the next proposition shows.

**Proposition 2.9.** Let \( G \) be a finite group and let \( H \leq G \) be non-normal and of index \( n \) in \( G \). If \( s = |H \setminus G/H| = 2 \) and \(|H| = n - 1\), then \( G \) is a Frobenius group and \( H \) is a Frobenius complement. Conversely, suppose that \( G \) is a Frobenius group and that \( H \) is a Frobenius complement. Then \( s = 2 \) if, and only if, \(|H| = n - 1\).

**Proof.** By Lemma 2.8 with \( H = K \), we have

\[
2 \geq \frac{n-m}{|H|} + m,
\]

which implies that \( m = 1 \), since \( H \) is not normal in \( G \) (i.e., \( m < n \)); thus, \( N_G(H) = H \).

Now let \( g \in G - H \). Since \( s = 2 \), we have \( G = H \cup HgH \), and therefore

\[
|G| = |H| + \frac{|H|^2}{|H \cap Hg|}.
\]

Consequently, for \( s = 2 \) and \(|H| = n - 1\),

\[
n = 1 + \frac{|H|}{|H \cap Hg|} = 1 + \frac{n - 1}{|H \cap Hg|}.
\]

It follows that \(|H \cap H^g| = 1\), thus \( H \cap H^g = 1 \). Since \( g \) was arbitrary subject to lying outside \( H \), we deduce that \( G \) is a Frobenius group with Frobenius complement \( H \).

Suppose now that \( G \) is a Frobenius group, and that \( H \) is a Frobenius complement. Then we have \( s = 2 \) if, and only if, \( G = H \cup HgH \) for every \( g \in G - H \), or, equivalently, if, and only if,

\[
|G| = |H| + |H|^2;
\]

that is, if, and only if, \(|H| = n - 1\), which is as desired.

3. **The common transversal probability** \( P_G(-,-) \) of given subgroups.

Let \( G \) be a finite group, and let \( H \leq G \) be a subgroup. As we have noted before, a given left transversal for \( H \) in \( G \) may or may not be a two-sided one. The aim of this section is to compute the probability \( P_G(H) \) that this happens. We actually give a generalized definition of the above probability, where two groups \( H, K \) of the same index \( n \) are concerned, and compute its value. This way we are able to show that the bigger the index \( n \) is, the smaller the probability of a left transversal of \( H \) to be
a right transversal of $K$ is (see Theorem 3.11 below). In addition, we give bounds for
the values of $P_C(H,K)$ and associate $P_C(H,K)$ with a permanent of a doubly stochastic
matrix. Finally in the last subsection we show that $P_C(-)$ behaves well with respect
to subgroups and homomorphic images.

3.1. **Definition and computation of $P_C(-,-)$.**

We start with a definition of the main actors of this paper.

**Definition 3.1.** Let $G$ be a finite group, and let $H,K \leq G$ be subgroups such that
$(G : H) = n = (G : K)$.

(i) We denote by $DT_C(H,K)$ the set of all left transversals for $H$ in $G$, which are
also right transversals for $K$ in $G$.

(ii) We let

$$P_C(H,K) := \frac{|DT_C(H,K)|}{|H|^n},$$

so that $P_C(H,K)$ is the (Laplacian) probability that a left transversal for $H$ in
$G$ is also a right transversal for $K$ in $G$. If $H=K$, we let $P_C(H) := P_C(H,H)$ be
the probability that a left transversal for $H$ in $G$ is a two-sided one. We call
$P_C(H,K)$ the *common transversal probability* for the subgroups $H,K \leq G$.

(iii) We set

$$tp(G) := \min_{H \leq G} P_C(H),$$

which is an invariant of $G$ alone, termed the *common transversal probability*
of the group $G$.

The first result of this section computes the probability $P_C(H,K)$ in terms of the sizes
of the connected components of the coset intersection graph $\Gamma_{H,K}^G$.

**Proposition 3.2.** Let $G$ be a finite group, and let $H,K \leq G$ be subgroups such that

$$(G : H) = (G : K) = n.$$

Assume further that $|K \setminus G/H| = s$ and let $\Delta_\sigma = K_{t_\sigma,t_\sigma}$ for $\sigma \in [s]$ denote the connected
components of $\Gamma = \Gamma_{H,K}^G$. Then

$$(3.1) \quad P_C(H,K) = \prod_{\sigma=1}^s \frac{t_\sigma!}{t_\sigma} \cdot \frac{t_\sigma!}{t_\sigma} \cdot \frac{t_\sigma!}{t_\sigma} \cdot |H|^n . \prod_{\sigma=1}^s \frac{t_\sigma!}{t_\sigma}$$

where $t_\sigma \mid |H|$ and $\sum_{\sigma=1}^s t_\sigma = n$.

**Proof.** It is clear from the definition of $\Gamma$ that the number of common transversals
for the pair $(H,K)$ in $G$ equals the product over the number of common transversals
for the $t_\sigma$ left-$H$ and right-$K$ cosets in each component $\Delta_\sigma$ of $\Gamma$. Moreover, the fact that
each component $\Delta_\sigma$ is a symmetric complete bipartite graph $K_{t_\sigma,t_\sigma}$, while the weight
$w_\sigma$ on every edge of $\Delta_\sigma$ is the size of the intersection of any left-$H$ with with any
right-$K$ coset of $\Delta_\sigma$, implies that the number of common $(H,K)$-double transversals
for the $t_\sigma$ left-$H$ and $t_\sigma$ right-$K$ cosets of $\Delta_\sigma$ equals $t_\sigma! \cdot w_\sigma^{t_\sigma}$. By Lemma 2.7, we have
$w_\sigma \mid |H|/t_\sigma$, in particular, $t_\sigma \mid |H|$ for each $\sigma \in [s]$. Since $\sum_{\sigma=1}^s t_\sigma = n$, it follows that

$$|DT_G(H,K)| = \prod_{\sigma=1}^s t_\sigma! \cdot w_\sigma^{t_\sigma} = \prod_{\sigma=1}^s \left( \frac{t_\sigma!}{t_\sigma} \cdot |H|^{|H|/t_\sigma} \right) = |H|^{\sum_{\sigma=1}^s t_\sigma} \cdot \prod_{\sigma=1}^s \frac{t_\sigma!}{t_\sigma}$$
Hence,

\[ P_G(H, K) = \frac{|D(T_G(H, K))|}{|H|^n} = \prod_{\sigma=1}^{s} \frac{t_{\sigma}!}{t_{\sigma}^{t_{\sigma}}}, \]

as claimed.  

**Definition 3.3.** Let \( H \leq G \) and let \( \{\Delta_{t_{\sigma}}\}_{\sigma} \) be the family of connected components of \( \Gamma^G_{H'} \), where the family of positive integers \( \{t_{\sigma}\} \) is defined as above. We list the integers \( t_{\sigma} \) in decreasing order, say \( (t_1, \cdots, t_s) \), and call the resulting vector the \( t \)-vector of \( H \) in \( G \).

**Corollary 3.4.** We have \( P_G(H, K) = 1 \) if, and only if, \( H = K \leq G \). Hence \( tp(G) = 1 \) if, and only if, \( G \) is a Dedekind group; that is, if, and only if, every subgroup in \( G \) is normal.

**Proof.** If \( H = K \leq G \), then \( P_G(H, K) = P_G(H) = 1 \). Conversely, suppose that \( H, K \leq G \) are such that \( P_G(H, K) = 1 \). By Proposition 3.2,

\[ P_G(H, K) = \prod_{\sigma=1}^{s} \frac{t_{\sigma}!}{t_{\sigma}^{t_{\sigma}}} = 1, \]

where \( s = |K \setminus G| \), so that \( t_{\sigma} = 1 \) for all \( \sigma \in [s] \). Since \( \sum_{\sigma} t_{\sigma} = n \), it follows that \( m = s = n \), thus, by (2.1), \( H \) and \( K \) are conjugate, and we have \( N_G(H) = G \), so that \( H \leq G \) and \( H = K \). The rest of the Corollary follows directly from the definition of \( tp(G) \).

**Corollary 3.5.** Suppose that \( H, K \leq G \) are such that \( (G : H) = n = (G : K) \), and that \( H \cap K^g = 1 \) for \( g \notin KH \). Then

\[ P_G(H, K) = \left( \frac{|H|!}{|H \cap K^g|} \right)^{\frac{n-1}{n}}. \]

**Proof.** Apart from the double coset \( HK \) for every other \( g_{\sigma} \in \Delta_{\sigma} \) we have

\[ t_{\sigma} = \frac{|H|}{|H \cap K^{g_{\sigma}}|} = |H|, \quad g_{\sigma} \notin KH \]

by Lemma 2.7(b). Hence

\[ P_G(H, K) = \left( \frac{|H|!}{|H|} \right)^{s-1} = \left( \frac{|H|!}{|H||H|} \right)^{\frac{n-1}{n}} \]

by Proposition 3.2, plus the fact that \( n = \sum_{\sigma=1}^{s} t_{\sigma} = 1 + |H|(s-1) \).

**Corollary 3.6.** Let \( G \) be a finite Frobenius group with Frobenius complement \( H \), and let \( (G : H) = n \). Then

\[ P_G(H) = \left( \frac{|H|!}{|H||H|} \right)^{\frac{n-1}{n}}. \]

**Proof.** Here \( H = K, H \cap H^g = 1 \) for \( g \notin H \), and \( N_G(H) = H \), whence the result by Corollary 3.5.
Corollary 3.7. Let $H \leq G$ be a subgroup of order $|H| = p$ a prime, and suppose that $(G : H) = n$. Then

$$P_G(H) = \left( \frac{p!}{p^n} \right)^{\frac{n-m}{p}},$$

where $m = (N_G(H) : H)$, and the exponent on the right-hand side is an integer.

Proof. Apart from $m$ trivial components in $\Gamma^G_H := \Gamma^G_{H,H}$, we have $\frac{n-m}{p}$ components $\Delta_v \equiv K_{p,p}$. In particular, this quotient is an integer. The result now follows from Proposition 3.2. ■

3.2. The function $P_G$ and permanents.

At this point, we wish to briefly discuss a connection between the probability $P_G(H,K)$, where $H$ and $K$ are subgroups of the finite group $G$ such that $(G : H) = n = (G : K)$, and the permanent of a certain associated matrix (for properties of permanents used in what follows, the reader is referred to the standard reference [21]). The matrix we associate to the triple $(G,H,K)$ is the $n \times n$ weight matrix $W = W^G_{H,K} = (w_{i,j})$, whose $(i,j)$ entry is

$$w_{i,j} = |l_i H \cap K r_j|.$$ 

That is, $w_{i,j}$ is the weight assigned to the edge $e = l_i H - K r_j$ of the coset intersection graph $\Gamma = \Gamma^G_{H,K}$. A simple combinatorial argument shows that

$$|\text{DT}_G(H,K)| = \sum_{\sigma \in S_n} \prod_{i=1}^n w_{i,\sigma(i)}.$$ 

As the right hand side in the above equation is exactly the permanent of the matrix $W$, we get

$$|\text{DT}_G(H,K)| = \text{per}(W^G_{H,K}).$$

Observe now that all row and column sums of $W$ are equal to $|H| = |K|$; that is,

$$\sum_{i=1}^n w_{i,r} = |H| = \sum_{j=1}^n w_{t,j}, \quad (1 \leq r, t \leq n).$$

Thus, the matrix

$$M = M^G_{H,K} := \frac{1}{|H|} \cdot W^G_{H,K}$$

is an $n \times n$ doubly stochastic matrix (all row and column sums equal to 1); furthermore, its permanent is exactly the common transversal probability for the subgroups $H,K \leq G$, as

$$\text{per}(M) = \frac{1}{|H|^n} \cdot \text{per}(W^G_{H,K}) = P_G(H,K).$$

In addition, after appropriate row and column permutations, we can group together cosets (left for $H$ and right for $K$) which intersect. In this way, $M$ is seen to be equivalent to a block diagonal matrix: there exist $n \times n$ permutation matrices $P$ and $Q$, such that

$$PMQ = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_s \end{bmatrix},$$
where, for $\sigma = 1, \ldots, s$, the matrix $M_{\sigma}$ is the $t_\sigma \times t_\sigma$ rational matrix whose entries all equal $\frac{1}{t_\sigma}$. As usual, $t_\sigma = (H : H \cap K^{G_\sigma})$, where $\{t_\sigma\}_{\sigma=1}^s$ is the family of sizes of the connected components of the coset intersection graph $\Gamma_{H,K}^G$. Collecting together these observations, we get most of the following.

**Proposition 3.8.** The doubly stochastic matrix $M = M^G_{H,K}$ satisfies

\[
(3.2) \quad \text{per}(M) = P_G(H,K) = \prod_{\sigma=1}^s \frac{(t_\sigma)!}{t_\sigma^s}.
\]

The eigenvalues of $M$ are 0 with multiplicity $n-s$, and 1 with multiplicity $s$. Moreover, $M$ is positive semi-definite.

**Proof.** We have already seen that $\text{per}(M) = P_G(H,K)$ and the right-hand side equals the desired product according to Proposition 3.2. We remark here that an alternative way to see that $\text{per}(M^G_{H,K}) = \prod_{\sigma=1}^s \frac{(t_\sigma)!}{t_\sigma^s}$ is the following: Since $PMQ = \text{diag}(M_1, \ldots, M_s)$, we have $\text{per}(PMQ) = \text{per}(M) = \prod_{\sigma=1}^s \text{per}(M_\sigma)$. On the other hand, working directly from the definition, the permanent of a square matrix $M = (m_{ij})$ of size $c$ with all entries equal to some fixed number $a$ is seen to be

\[
\text{per}(M) = \sum_{\sigma \in S_c} \prod_{i=1}^c m_{i,\sigma(i)} = \sum_{\sigma \in S_c} a^c = c! a^c.
\]

Specifying $a = 1/t_\sigma$ and $c = t_\sigma$, for each $\sigma \in [s]$, yields

\[
\text{per}(M_\sigma) = t_\sigma/t_\sigma^s,
\]

whence (3.2). The assertion concerning the eigenvalues of $M$ follows from the fact that the characteristic polynomial of the matrix $M_\sigma$ is

\[
\det(M_\sigma - \lambda I) = (-\lambda)^{t_\sigma-1}(1 - \lambda),
\]

which implies that the characteristic polynomial of the matrix $M$ is given by

\[
\det(M - \lambda I) = (-\lambda)^{n-s}(1 - \lambda)^s.
\]

Thus, $\text{tr}(M) = s$, while $\det(M) = 0$, which can, of course, also be seen directly. Also, $\text{rk}(M) = s$. Since $M$ is symmetric and its eigenvalues are non-negative, it follows that $M$ is positive semi-definite. \hfill \blacksquare

### 3.3. A limit theorem.

The aim in this subsection is twofold. Firstly, we will give sharp bounds for the value $P_G(H,K)$; so, for example, we will prove that for non-normal conjugate subgroups its value cannot exceed 1/2. Secondly, we will show that the bigger the index $n = (G : H) = (G : K)$ is the smaller the value $P_G(H,K)$ becomes. We start with the following purely arithmetic result.

**Lemma 3.9.** Let $s \geq 1$, and let $\{t_\sigma\}_{\sigma=1}^s$ be a family of positive integers with $\sum_{\sigma=1}^s t_\sigma = n$. Then

\[
(3.3) \quad \frac{n!}{n^n} \leq \prod_{\sigma=1}^s \frac{t_\sigma!}{t_\sigma^s} \leq \left(\frac{n+s}{2n}\right)^n.
\]

Furthermore if $m$ of the $t_\sigma$ are equal to 1, then

\[
(3.4) \quad \frac{(n-m)!}{(n-m)^{n-m}} \leq \prod_{\sigma=1}^s \frac{t_\sigma!}{t_\sigma^s} \leq \frac{1}{2^{s-m}}.
\]
Proof. We first prove the left-hand inequality in (3.3) by induction on \( s \). For \( s = 1 \), both sides are equal. Next, suppose that \( s = 2 \), so that \( t_1 + t_2 = n \). Then \( t_1! \cdot t_2! = \left( \frac{n!}{t_1! \cdot t_2!} \right)^{-1} n! \), while \( n!! = (t_1 + t_2)! \geq \left( \frac{n!}{t_1! \cdot t_2!} \right) \cdot t_1 \cdot t_2 \). Thus,

\[
\frac{n!}{n!!} = \frac{\left( \frac{n!}{t_1! \cdot t_2!} \right) \cdot t_1 \cdot t_2}{\frac{n!}{t_1! \cdot t_2!}} = \prod_{\sigma=1}^{n} \frac{t_{\sigma}!}{t_{\sigma}^2}
\]

as desired. For the induction step, suppose that the left-hand inequality in (3.3) holds for all \( s \leq u \) with some \( u > 2 \), and let \( n = t_1 + t_2 + \cdots + t_u + 1 \). Then \( n = t_1 + (n - t_1) \), and the inductive hypothesis for \( s = 2 \) and \( s = u \) gives

\[
\frac{n!}{n!!} \leq \frac{t_1!}{t_1^2} \cdot \frac{(n - t_1)!}{(n - t_1)^{n - t_1}} \leq \frac{t_1!}{t_1^2} \cdot \prod_{\sigma=2}^{u} \frac{t_{\sigma}!}{t_{\sigma}^2} = \prod_{\sigma=1}^{u} \frac{t_{\sigma}!}{t_{\sigma}^2}
\]

whence our result.

The right-hand inequality in (3.3) follows from the arithmetic-geometric mean inequality as follows:

\[
\prod_{\sigma=1}^{s} \frac{t_{\sigma}!}{t_{\sigma}^s} = \prod_{\sigma=1}^{s} \prod_{\tau=1}^{t_{\sigma}} \tau/t_{\sigma} \leq \left( \sum_{\sigma=1}^{s} \sum_{\tau=1}^{t_{\sigma}} \tau/t_{\sigma} \right)^{s} = \left( \frac{1}{n} \sum_{\sigma=1}^{s} \frac{1}{t_{\sigma}} \right) = \left( \frac{1}{n} \sum_{\sigma=1}^{s} \frac{(t_{\sigma} + 1)}{2n} \right)^n = \left( \frac{n + s}{2n} \right)^n.
\]

This establishes the first part of the lemma. Now suppose that

\[ t_1 = t_2 = \cdots = t_m = 1 \]

for some \( m \leq s \), while \( t_{\sigma} > 2 \) for \( m + 1 \leq \sigma \leq s \). Then \( n - m = t_{m+1} + \cdots + t_s \) thus, by the left-hand inequality in (3.3),

\[
\frac{(n - m)!}{(n - m)^{n-m}} \leq \prod_{\sigma=m+1}^{s} \frac{t_{\sigma}!}{t_{\sigma}^s} = \prod_{\sigma=1}^{s} \frac{t_{\sigma}!}{t_{\sigma}^s}.
\]

Furthermore, for each \( \sigma > m \) we have \( t_{\sigma} \geq 2 \), so that \( t_{\sigma}!/t_{\sigma}^s \leq 1/2 \), thus

\[
\prod_{\sigma=1}^{s} \frac{t_{\sigma}!}{t_{\sigma}^s} \leq 2^{-(s-m)}.
\]

This completes the proof of the lemma. \( \blacksquare \)

Corollary 3.10. Let \( G \) be a finite group, and let \( H, K \leq G \) be conjugate, non-normal subgroups of index \( n \) in \( G \). Then we have

\[
\frac{(n - 1)!}{(n - 1)^{n-1}} \leq P_G(H, K) \leq \frac{1}{2},
\]

and these bounds are best possible. In particular, for \( n = 3 \), we have \( P_G(H, K) = 1/2 \).

Proof. Let \( H, K \) be conjugate subgroups of \( G \). Then, according to Lemma 2.7(a), the number \( m \) of connected components \( \Delta_{\sigma} \) of \( \Gamma_G^{H, K} \) of type \( \Delta_{\sigma} \approx K_{t_1} \) is at least one, i.e., \( m \geq 1 \). On the other hand, we cannot have \( m = s = |K\setminus G/H| \), since otherwise \( P_G(H, K) = 1 \) by Proposition 3.2, so that \( H = K \leq G \) by Corollary 3.4, contradicting our hypothesis. Hence, we have \( 1 \leq m < s \). Since the left-hand side of (3.4) decreases with \( m \), while the right-hand side increases with \( m \), the result follows.
We next focus on the sharpness of the bounds in (3.5). Let $F$ be a field with $q$ elements, where $q$ is a power of a prime. Then the action of the multiplicative group $F^*$ on the additive group of $F$ is of Frobenius type. The resulting group $G = F \rtimes F^*$ is a Frobenius group with Frobenius complement $H \cong F^*$ of order $q - 1$ and index $q$. Thus, by Corollary 3.6,

$$P_G(H) = \left( \frac{(q-1)!}{(q-1)^{q-1}} \right)^{\frac{q-1}{q}} = \frac{(q-1)!}{(q-1)^{q-1}},$$

which shows that the lower bound in Equation (3.5) is attained.

Also, for $q = 3$, let $G \cong S_3$, and let $H$ be any Sylow 2-subgroup of $G$. In this case, $P_G(H) = 1/2$, so that the upper bound in (3.5) is attained as well. ■

We can now establish our first main result.

**Theorem 3.11.** Let $G$ be a finite group, and let $H, K \leq G$ be subgroups such that $(G : H) = n = (G : K)$, and such that at least one of $H, K$ is not normal in $G$. Then

$$\lim_{n \to \infty} P_G(H, K) = 0.$$

**Proof.** According to Proposition 3.2 and Lemma 3.9, we have

$$P_G(H, K) = \prod_{i=1}^{s} \frac{t_i!}{t_i^{t_i}} \leq \left( \frac{n + s}{2n} \right)^n,$$

where $s = |K \setminus G/H|$ equals the number of connected components of $\Gamma_{G, H}$. According to Lemma 2.7 exactly $m$ of those $t_\sigma$’s are equal to 1, while for the rest we get $t_\sigma \geq 2$, where $m = (N_G(H) : H)$ if $H, K$ are conjugate, or $m = 0$ if they are not. As $\sum_{\sigma=1}^{s} t_\sigma = n$, we conclude that

$$s \leq m + \frac{n - m}{2} = \frac{n + m}{2}.$$

Furthermore, by (2.1), $m$ is either zero, or divides $n$ properly (otherwise we would have $H = K \leq G$ contradicting our hypothesis); thus $m \leq \frac{n}{2}$. Hence, $s \leq \frac{3n}{4}$ in either case, and Equation (3.6) implies

$$P_G(H, K) \leq \left( \frac{n + s}{2n} \right)^n \leq \left( \frac{7}{8} \right)^n.$$

As $\lim_{n \to \infty} \left( \frac{7}{8} \right)^n = 0$, the theorem follows. ■

### 3.4. An improved bound for $P_G(H, K)$

We digress briefly to discuss an upper bound for the function $\prod_{\sigma=1}^{s} t_\sigma! / t_\sigma^{t_\sigma}$, which is stronger than (3.7). For $x > 0$, let

$$f(x) := \frac{\Gamma(x + 1)}{x^x},$$

where $\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} dt$ is the well-known $\Gamma$–function. Observe that $f(t_\sigma) = t_\sigma! / t_\sigma^{t_\sigma}$ and thus

$$\prod_{\sigma=1}^{s} \frac{t_\sigma!}{t_\sigma^{t_\sigma}} = \prod_{\sigma=1}^{s} f(t_\sigma).$$
We argue that $f$ is strictly log-concave; that is, the function $g(x) = \log f(x)$ is strictly concave. First, $g$ is infinitely differentiable. Second, 

$$g''(x) = \psi'(x+1) - \frac{1}{x},$$

where $\psi(x) = (\log \Gamma(x))' = \Gamma'(x)/\Gamma(x)$ is the so-called digamma function. By [10, Lemma 2], we have 

$$\psi'(x+1) = e^{\frac{1}{x+1}} - 1, \quad (x > 0),$$

while 

$$e^{\frac{1}{x+1}} - 1 < \frac{1}{x}, \quad (x > 0)$$

by virtue of the elementary inequality 

$$e < \left(1 + \frac{1}{x}\right)^{x+1},$$

which is valid for all $x > 0$. Thus 

$$g''(x) = \psi'(x+1) - \frac{1}{x} < 0, \quad (x > 0),$$

which implies that $g$ is strictly concave, as claimed.

Now let 

$$\Pi := \prod_{\sigma=1}^{s} f(t_\sigma).$$

Then 

$$\log \Pi = \sum_{\sigma=1}^{s} \log f(t_\sigma) \leq s \log f\left(\frac{\sum_{\sigma=1}^{s} t_\sigma}{s}\right) = s \log f\left(\frac{n}{s}\right)$$

where the inequality is a consequence of Jensen's inequality applied to the concave function $g(x) = \log f(x)$. We conclude that 

$$\prod_{\sigma=1}^{s} \frac{t_\sigma!}{t_\sigma} = \prod_{\sigma=1}^{s} f(t_\sigma) \leq f\left(\frac{n}{s}\right)^s.$$ 

At this point we are interested in comparing the two bounds 

$$f\left(\frac{n}{s}\right)^s \quad \text{and} \quad \left(\frac{n+s}{2n}\right)^n.$$ 

Specifically, we will show the following.

**Proposition 3.12.** For all $n \in \mathbb{N}$ and for all $s$ with $s \leq \frac{3}{4}n$, we have 

$$\tag{3.8} f\left(\frac{n}{s}\right)^s \leq c^n \left(\frac{n+s}{2n}\right)^n,$$

where 

$$c := \frac{8}{7} f(4/3)^{3/4} = 0.976986... .$$

**Proof.** Proving the above inequality is tantamount to proving that $f(y) \leq c^y \left(\frac{y+1}{2y}\right)^y$ for all $y \geq 4/3$, where we have made the substitution $y := n/s$. It will clearly suffice
to show that \( \log f(y) \leq y \left[ \log \left( \frac{y+1}{2y} \right) + \log c \right] \). So let

\[
H(y) := \log f(y) - y \left[ \log \left( \frac{y+1}{2y} \right) + \log c \right]
\]

\[
= \log \Gamma(y+1) - y \log y - y \left[ \log \left( \frac{y+1}{2y} \right) + \log c \right]
\]

\[
= \log \Gamma(y+1) - y \left[ \log \left( \frac{y+1}{2} \right) + \log c \right]
\]

for \( y \geq 4/3 \), and observe that \( H(4/3) = 0 \). Thus, the claim will have been established, provided that we can show that \( H'(y) \leq 0 \) for \( y \geq 4/3 \). Now observe that

\[
H'(y) = \psi(y+1) - \log \left( \frac{y+1}{2} \right) - \log c - \frac{y}{y+1}.
\]

By Lemma 1 in [10], we have

\[
\psi(y+1) < \log(y+1) - \frac{1}{2(y+1)} \quad (y > 0).
\]

Hence,

\[
H'(y) < \log(y+1) - \frac{1}{2(y+1)} - \log \left( \frac{y+1}{2} \right) - \log c - \frac{y}{y+1}
\]

\[
= \frac{y[2\log(2/c) - 2] + 2\log(2/c) - 1}{2(y+1)}.
\]

We note that the numerator \( N \) in the last expression is negative, since \( N \geq 0 \) would imply that

\[
y \leq \frac{2\log(2/c) - 1}{2 - 2\log(2/c)} \approx 0.763233 < \frac{4}{3},
\]

contradicting our assumption that \( y \geq 4/3 \), while the denominator is positive. Thus \( H'(y) \leq 0 \), which is what we wanted to prove.

\[
\text{Remark 3.13. It is clear that we have equality in (3.8) if and only if } n = (4/3)s \text{ since } H(4/3) = 0 \text{ and } H'(y) \text{ is in fact strictly negative for } y > 4/3. \text{ Moreover, if } s \text{ remains bounded and thus } y = n/s \text{ grows without bound, } f(n/s)^s \text{ becomes an even better bound than } \left( \frac{n+s}{2n} \right)^n \text{ since}
\]

\[
f(n/s)^s \sim \left( \frac{2\pi n^2}{e^n} \right)^{s/2} \quad \text{while} \quad \left( \frac{n+s}{2n} \right)^n \sim \frac{e^s}{2^n},
\]

where the first relation follows from Stirling’s approximation.

3.5. **Subgroups, homomorphic images, and the function** \( P_G(-) \).

From now on, we shall concentrate on the case where \( H = K \); that is, we shall focus on the functions of one variable \( P_G(-) \) and \( t \). Our aim in this subsection is to show that \( P_G(-) \) behaves well with respect to subgroups and homomorphic images.

**Proposition 3.14.** Let \( K \leq H \leq G \) be finite groups. Then every connected component of \( \Gamma^H_K \) is also a connected component of \( \Gamma^G_K \); in particular, \( \Gamma^H_K \) is an induced subgraph of \( \Gamma^G_K \) and we have

\[
P_G(K) \leq P_H(K).
\]
\textbf{Proof.} First, it is clear that $\Gamma^H_K$ is a subgraph of $\Gamma^G_K$. Second, we observe that

$$hK \cap Kg \neq \emptyset \implies g \in H, \quad (h \in H, g \in G),$$

since, by hypothesis, there have to exist elements $k_1, k_2 \in K$ such that $hk_1 = k_2g$, so that $g = k_2^{-1}hk_1 \in H$, as $K \subseteq H$. Therefore, an edge in $\Gamma^G_K$ with one vertex in the set $H/K$ necessarily has its other bounding vertex in $K\setminus H$, which implies that the connected components of $\Gamma^H_K$ are identical with certain components of $\Gamma^G_K$. Third, we obviously have

$$s_H := |K\setminus H/K| \leq |K\setminus G/K| =: s_G.$$ 

Thinking of the connected components of $\Gamma^H_K$ as being listed first among the components of $\Gamma^G_K$, we now find that

$$P_G(K) = \frac{\sum_{\sigma \in [n]} (t_\sigma)!}{t_\sigma^{|G\setminus \sigma|}} = \frac{\sum_{\sigma \in [n]} (t_\sigma)!}{t_\sigma^{|G\setminus \sigma|}} \times \frac{\sum_{\sigma \in [s_H]} (t_\sigma)!}{t_\sigma^{|G\setminus \sigma|}} \leq \frac{\sum_{\sigma \in [s_H]} (t_\sigma)!}{t_\sigma^{|G\setminus \sigma|}} = P_H(K),$$

whence (3.9). \hfill \blacksquare

Our next result concerns the connection of $P_G$ with homomorphic images.

\textbf{Theorem 3.15.} Let $G$ and $K$ be finite groups, $f : G \to K$ a group homomorphism, and let $H \subseteq G$ be a subgroup. Then, if $N := \ker(f) \subseteq H$, we have

$$P_G(H) = P_{f(G)}(f(H)).$$

\textbf{Proof.} Our aim is to define and analyse a certain map

$$\Phi : \text{DT}_G(H) \coloneqq \text{DT}_G(H,H) \longrightarrow \text{DT}_{f(G)}(f(H)).$$

Suppose that $(G : H) = n$, and let $T = \{t_1, t_2, \ldots, t_n\}$ be a two-sided transversal for $H$ in $G$. Applying the homomorphism $f$, we obtain what looks at first like a multiset $\overline{T} = \{f(t_1), f(t_2), \ldots, f(t_n)\} \subseteq K$, and we claim that $\overline{T}$ is in fact a two-sided transversal for $f(H)$ in $f(G)$; in particular, $\overline{T}$ is an $n$-set. Suppose first that

$$f(t_{v_1}, H) \cap f(t_{v_2}, H) = f(t_{v_1})f(H) \cap f(t_{v_2})f(H) \neq \emptyset, \quad (v_1, v_2 \in [n]).$$

Then there exist elements $h_1, h_2 \in H$, such that $f(t_{v_1}h_1) = f(t_{v_2}h_2)$, or $f(t_{v_1}h_1^{-1}h_2^{-1}) = 1$. Thus, there are elements $n,n' \in N$, such that

$$t_{v_2} = n^{-1}t_{v_1}h_1^{-1} = t_{v_1}n'h_2^{-1} \in t_{v_1}H,$$

as $N \subseteq H$ by hypothesis. This forces $v_1 = v_2$, since $T$ is a left transversal for $H$ in $G$. Since $(f(G) : f(H)) \leq n$, it follows that $\overline{T}$ is a complete set of pairwise inequivalent representatives for the left cosets of $f(H)$ in $f(G)$. Similarly, if

$$f(Ht_{v_1}) \cap f(Ht_{v_2}) = f(H)f(t_{v_1}) \cap f(H)f(t_{v_2}) \neq \emptyset, \quad (v_1, v_2 \in [n]),$$

we find that $f(h_1t_{v_1}^{-1}h_2^{-1}) = 1$ for some $h_1, h_2 \in H$, or

$$t_{v_1}t_{v_2}^{-1} = h_1^{-1}nh_2 \in H, \quad (n \in N),$$

again implying $v_1 = v_2$, so that $\overline{T}$ is also a right transversal for $f(H)$ in $f(G)$; therefore, $\overline{T} \in \text{DT}_{f(G)}(f(H))$. We now set $\Phi(T) := \overline{T}$, to obtain a well-defined map from $\text{DT}_G(H)$ to $\text{DT}_{f(G)}(f(H))$, and claim that $\Phi$ is surjective. In order to justify this claim, let $\{f(s_i)\}_{1 \leq i \leq n} \in \text{DT}_{f(G)}(f(H))$,
pick arbitrary pre-images \( \bar{s}_v \in f^{-1}(f(s_v)) \) for \( 1 \leq v \leq n \), and form the set \( \bar{S} = \{ \bar{s}_v : 1 \leq v \leq n \} \). Then \( \bar{S} \in DT_G(H) \). Indeed, if \( \bar{s}_v^{-1} \bar{s}_v \in H \), then

\[
f(\bar{s}_v^{-1} \bar{s}_v) = f(\bar{s}_v)^{-1} f(\bar{s}_v) = f(s_v)^{-1} f(s_v) \in f(H),
\]

so that \( v_1 = v_2 \). A similar argument works for right cosets and, by construction,

\[
\Phi(\bar{S}) = \{ f(\bar{s}_v) \}_{1 \leq v \leq n} = \{ f(s_v) \}_{1 \leq v \leq n},
\]

whence our claim.

We observe that, as \( N = \ker(f) \leq H \), if \( T = \{ t_v \}_{1 \leq v \leq n} \in DT_G(H) \), so is \( T' = \{ m_v t_v \}_{1 \leq v \leq n} \), where \( m_1, \ldots, m_n \in N \), as \( Hm_v t_v = Ht_v \) and \( m_v t_v H = t_v m'_v H = t_v H \), where \( m'_v \in N \). Moreover, we have \( \Phi(T) = \Phi(T') \), since \( f(m_v t_v) = f(t_v) \). Consequently, for each \( \bar{T} \in DT_{f(G)}(f(H)) \), there are at least \( |N|^n \) distinct pre-images under \( \Phi \). Actually, we have

\[
|\Phi^{-1}(\bar{T})| = |N|^n, \quad (\bar{T} \in DT_{f(G)}(f(H))).
\]

Namely, if \( S = \{ s_v \}_{1 \leq v \leq n} \in DT_G(H) \) with \( \Phi(S) = \Phi(T) \), where \( T \in DT_G(H) \) is a given two-sided transversal then, after appropriate rearrangement of indices, we have \( f(s_v) = f(t_v) \) for \( 1 \leq v \leq n \), thus

\[
s_v = m_v t_v, \quad (1 \leq v \leq n),
\]

where \( m_v \in N \), as claimed. Summarising our findings so far, we have shown that

\[
|DT_{f(G)}(f(H))| = \frac{|DT_G(H)|}{|N|^n}.
\]

It follows that

\[
P_{f(G)}(f(H)) = \frac{|DT_{f(G)}(f(H))|}{|f(H)|^n} = \frac{|DT_G(H)|}{|N|^n} = \frac{|DT_G(H)|}{|H|^n} = P_G(H),
\]

completing the proof.

\[\text{Corollary 3.16.} \text{ Let } G \text{ be a finite group, let } H \text{ be a subgroup of } G, \text{ and let } f : G \to K \text{ be an isomorphism. Then we have } P_G(H) = P_K(f(H)), \text{ in particular, } tp(G) = tp(K); \text{ that is, } tp(\cdot) \text{ is an isomorphism invariant.}\]

Taking \( f : G \to G/N \) the canonical projection, where \( N \) is a normal subgroup of \( G \), Theorem 3.15 gives the following.

\[\text{Corollary 3.17.} \text{ Let } G \text{ be a finite group, and let } N \leq H \text{ be subgroups of } G \text{ with } N \text{ normal in } G. \text{ Then } P_G(H) = P_{G/N}(H/N).\]

4. The common transversal probability \( tp(\cdot) \) of a group.

We now turn our attention to the invariant \( tp(\cdot) \), defined as

\[
\text{tp}(G) := \min_{H \leq G} P_C(H).
\]

The main objective of the section is to relate the values of \( tp(G) \) to key properties for the group \( G \). Roughly this means that the larger \( tp(G) \) is, the more normal structure \( G \) exhibits, in particular, we show that solubility, supersolubility and nilpotency are characterised by certain values of \( tp(G) \). We also compute the common transversal probability of certain groups and show that specific values of \( tp \) are achieved by specific groups while others are not attained at all. We also show that \( tp \) behaves well with respect to subgroups, quotients and group extensions.
4.1. The function $\text{tp}(-)$ and group structure.

Our first result records the behaviour of $\text{tp}(-)$ under taking subgroups, quotients, and sections.

**Proposition 4.1.** Let $G$ be a finite group.

(i) For every proper subgroup $H$ of $G$ we have $\text{tp}(G) \leq \text{tp}(H)$.

(ii) For every normal subgroup $N$ of $G$ we have $\text{tp}(G) \leq \text{tp}(G/N)$.

(iii) For every section $X$ of $G$ we have $\text{tp}(G) \leq \text{tp}(X)$.

(iv) If $G$ is a non-Dedekind $p$-group, then $\text{tp}(G) \leq p!/p^p$.

**Proof.**

(i) Let $K \leq H$ be such that $\text{tp}(H) = P_H(K)$. Then, according to Proposition 3.14, we have $P_G(K) \leq P_H(K)$. Hence,$$
\text{tp}(G) \leq P_G(K) \leq P_K(K) = \text{tp}(H).
$$

(ii) In view of Corollary 3.17 we have $\text{tp}(G/N) = \min_{N \leq H \leq G} \text{tp}(G/H) = \min_{N \leq H \leq G} P_G(H) \geq \text{tp}(G)$.

(iii) Let $X = H/N$, where $N \leq H \leq G$. Then, by Parts (i) and (ii),
$$
\text{tp}(G) \leq \text{tp}(H) \leq \text{tp}(H/N) = \text{tp}(X),
$$
whence our result.

(iv) Suppose that $H \leq G$ is not normal in $G$, and that $(G : H) = n$. Then the number $m$ of trivial connected components of the graph $\Gamma^G_H$ satisfies $m = (N_G(H) : H) < n$. Thus, there exists at least one non-trivial component $\Delta_\sigma \cong K_{t_\sigma, t_\sigma}$ in $\Gamma^G_H$, and we have $t_\sigma \geq p$, since $t_\sigma | |H|$ by Proposition 3.2. Hence, by (3.1),
$$
\text{tp}(G) \leq P_G(H) = \prod_{\sigma=1}^s t_\sigma! / t_\sigma^{t_\sigma} \leq p! / p^p.
$$

The proof is complete.

We next give the common transversal probabilities of various groups, which will be needed later on.

**Lemma 4.2.** For an odd prime $p$ and any integer $n \geq 1$, let

$$
C_p \rtimes C_{2n} = \langle a, b \mid a^p = b^{2n} = 1, a^b = a^{-1} \rangle,
$$

and let $D_n$ be the dihedral group of order $2n$. Then

(i) $\text{tp}(C_p \rtimes C_{2n}) = \left(\frac{1}{2}\right)^{p-1}$,

(ii) $\text{tp}(D_n) = \begin{cases} 
\left(\frac{1}{2}\right)^{n+1} & \text{if } n \text{ is odd}, \\
\left(\frac{1}{2}\right)^{n+2} & \text{if } n \text{ is even},
\end{cases}$

(iii) $\text{tp}(Q_{16}) = \text{tp}(C_4 \rtimes C_4) = 1/2$,

(iv) $\text{tp}(A_4) = 2/9$,

(v) $\text{tp}(A_5) = (1/2)^{14}$,
(vi) \(\text{tp}(\text{PSL}_3(2)) = (1/2)^{40}\),
(vii) \(\text{tp}(C_3 \rtimes C_7) = (1/2)^{12}\),
(viii) \(\text{tp}(C_3 \rtimes C_4) = (1/2)^8\),
(ix) \(\text{tp}(\text{SL}_2(3)) = (2/9)^2\).

**Proof.** (i) Let \(G_p = C_p \rtimes C_{2^n}\) be as in (4.1). Then the only subgroups of \(G_p\) that are not normal are the conjugates of \(\langle b \rangle \cong C_{2^n}\). To see this observe that \(a\) centralises \(b^2\). This is because \(a^b = a^{-1}\) implies that \(aba = b\), and thus \(a^k b a^k = b\) for every \(k \geq 1\). Hence

\[
ab^2a^{-1} = (aba^{-1})^2 = (ba^p - 2)^2 = ba^p - 2ba^p - 2 = b^2.
\]

Thus \(\langle b^2 \rangle\), which is the unique maximal subgroup of \(\langle b \rangle\), is normal in \(G\). Since \(\langle b^2 \rangle \cong C_{2^{n-1}}\) is cyclic, all of its subgroups (that is, all proper subgroups of \(\langle b \rangle\)) are also normal in \(G_p\). Now let \(H \leq G_p\) be an arbitrary subgroup, and let \(S\) be a 2-Sylow subgroup of \(H\). If \(S \leq \langle b^2 \rangle\) then, by the above, \(S\) is normal in \(G_p\), and we have \(H = S\) or \(H = S(a)\), so \(H \leq G_p\) in both cases. There remains the case that \(S = \langle b \rangle^x\) for some \(x \in G_p\). Then either \(H = S = \langle b \rangle^x\) is a conjugate of \(\langle b \rangle\), or we have \(H = G_p\), which is again a normal subgroup of \(G_p\). Our claim follows. Hence, by Corollary 3.16 we have \(\text{tp}(G_p) = P_{G_p}(\langle b \rangle)\).

Now note that \(N_{G_p}(\langle b \rangle) = \langle b \rangle\) while for any \(x \in G_p\) we get \(\langle b \rangle^x \cap \langle b \rangle = \langle b^2 \rangle\). Hence

\[
|\langle b \rangle : \langle b \rangle^x \cap \langle b \rangle| = 2 \text{ and the } t \text{-vector of } \langle b \rangle \text{ is } (2,2,\cdots,2,1).
\]

Hence \(P_{G_p}(\langle b \rangle) = (\frac{1}{2})^{n-1} \text{ and } (p-1)/2\)

the first part of the lemma follows.

(ii) We let

\[
G = \langle a, b \mid a^2 = b^n = 1, b^a = b^{-1} \rangle \cong D_n,
\]

and work by induction on \(n\). The claim is true for \(n = 3\) and \(n = 4\) according to a GAP computation [?]. Suppose that the claim has been established for all positive integers \(j\) such that \(3 \leq j < n\) and let \(K\) be a non-normal subgroup of \(G = D_n\). If \(|K| > 2\), then \(K\) is dihedral and thus \(K \cong D_k\) for some non-trivial proper divisor \(k\) of \(n\). Moreover, let

\[
N := \text{core}_G(K) = K \cap C,
\]

where \(C = \langle b \rangle\) is the normal cyclic subgroup of index 2 in \(D_n\) generated by a rotation by \(\frac{2\pi}{n}\) degrees. Since \(G/N = \langle b^{n/k}, a \rangle \cong D_{n/k}\), Corollary 3.17 plus the induction hypothesis yield

\[
P_G(K) = P_{G/N}(K/N) \geq \text{tp}(D_{n/k}) = \left(\frac{1}{2}\right)^{\frac{4\pi^2}{2\pi k}} \quad i = \begin{cases} 1, & n/k \equiv 1(2) \\ 2, & n/k \equiv 0(2) \end{cases}.
\]

On the other hand, \(\text{Aut}(D_n)\) acts transitively on the (non-normal) involutions of \(D_n\). To see this, recall that \(D_n\) is generated by a rotation \(r\) and a reflection \(s\) subject to the relation \(r^2 = s^{-1}\). Now consider the map \(\phi_i : D_n \to D_n\) fixing every rotation and mapping \(sr^i\) to \(sr^{i+1}\), where the exponents work modulo \(n\). Then \(\phi_i\) is a homomorphism (the reader is invited to check this detail) and since it is injective, it is an automorphism of \(D_n\). Finally, notice that if \(sr^k, sr^\ell\) are reflections, then \(\phi_i \circ \phi_k^{-1}\) sends \(sr^k\) to \(sr^\ell\).
Thus $P_G(H)$ has the same value for all non-normal subgroups $H$ of $G$ of order 2 by Corollary 3.16. Let $H$ be one such subgroup. Then

$$P_G(H) = \begin{cases} \left(\frac{1}{2}\right)^\frac{n-m}{2} & \text{if } n \text{ is odd,} \\ \left(\frac{1}{2}\right)^\frac{n-m}{2} & \text{if } n \text{ is even} \end{cases}$$

by Corollary 3.7, where the distinction between cases is explained by the fact that $H$ is self-normalising if $n$ is odd (and thus $m = 1$ in that case), while $m = (N_G(H) : H) = 2$ if $n$ is even. Since $tp(D_n) > P_G(H)$ in either case, $tp(G)$ is attained for a non-normal cyclic subgroup of order 2, and the induction is complete.

For Parts (iii)–(ix) of Lemma 4.2 we use a GAP-routine to compute $tp(G)$. 

Our next result connects the function $tp(-)$ to solubility of the corresponding finite group.

**Proposition 4.3.** Let $G$ be a finite group such that $tp(G) > (1/2)^{40}$. Then $G$ is soluble or has a section isomorphic to $A_5$. Moreover, this result is best possible.

**Proof.** We argue by induction on the order of $G$. Our claim is obviously true if $G$ is the trivial group, so that our induction begins. Assume now that $|G| \geq 2$, that $tp(G) > (1/2)^{40}$, that $G$ is non-abelian, and that our claim holds for groups of smaller order. Suppose first that $G$ is a non-abelian simple group, and let $H$ be a subgroup of order 2 in $G$ (existence of $H$ is guaranteed by the celebrated Feit-Thompson Odd Order Theorem [9]). Then, by Corollary 3.7 and our hypothesis,

$$P_G(H) = \left(\frac{1}{2}\right)^\frac{n-m}{2} > tp(G) > \left(\frac{1}{2}\right)^{40},$$

where $n = (G : H)$ and $m = (N_G(H) : H)$. Thus $\frac{n-m}{2} < 40$, or equivalently $n - m < 80$. It follows that

$$|G| - |N_G(H)| < 160. \tag{4.2}$$

Since $G$ is non-abelian simple, no proper subgroup can have index less than 5, so $|N_G(H)| \leq \frac{|G|}{5}$ and thus

$$|G| - |N_G(H)| \geq \frac{4|G|}{5}. \tag{4.3}$$

Combining (4.3) and (4.2) yields $\frac{4|G|}{5} < 160$, or equivalently $|G| < 200$. However, the only non-abelian simple group of order less than 200 are $A_5$ (of order 60) and $PSL_3(2)$ (of order 168). Now $A_5$ satisfies $tp(A_5) = (1/2)^{14} > (1/2)^{40}$ by Part (v) of Lemma 4.2, but trivially has a section isomorphic to $A_5$, thus is acceptable, while $tp(PSL_3(2)) = (1/2)^{40}$, so that $G \cong PSL_3(2)$ is ruled out by our hypothesis. Thus, $G \cong A_5$ for $G$ non-abelian simple, and the claim follows in that case.

Consequently, we may assume further that $G$ is not simple, thus $G$ has a proper non-trivial normal subgroup $N$. By Parts (i) and (ii) of Proposition 4.1, we have $tp(N) \geq tp(G) > 2^{-40}$ and $tp(G/N) \geq tp(G) > 2^{-40}$, thus, by the induction hypothesis applied to $N$ and $G/N$, each of $N$ and $G/N$ is either soluble or has a section isomorphic to $A_5$. If both are soluble, so is $G$. If $N$ has a section isomorphic to $A_5$, then so does $G$ while, if $G/N$ has a section isomorphic to $A_5$, this section lifts to give a corresponding section isomorphic to $A_5$ in $G$. This completes the induction. Finally, since

---

$^1$The web address https://fourier.math.uoc.gr/~marial/tp-Code contains our GAP-code for computing $tp(-)$. 

G = \text{PSL}_3(2) is not soluble (in fact, G is non-abelian simple), 5 \nmid |G| (so that G cannot contain a section isomorphic to \text{A}_5), and \text{tp}(G) = (1/2)^{40} by Part (vi) of Lemma 4.2, our result is indeed best possible.

We now turn our attention to the connection of \text{tp}(\cdot) and supersolubility. The proof of Theorem 4.4 below is considerably harder than that of Proposition 4.3, which is mainly due to the fact that the class of supersoluble groups is not closed under extensions (otherwise every metabelian group would be supersoluble, but \text{A}_4 is a counterexample).

**Theorem 4.4.** Let G be a finite group with \text{tp}(G) > (1/2)^8. Then either G is supersoluble, or G has a section isomorphic to \text{A}_4. Moreover, the bound (1/2)^8 is sharp.

**Proof.** We induce on the order of G. If G is trivial, then the theorem clearly holds, so that the induction starts. Suppose that |G| ≥ 2, that G is non-abelian, that \text{tp}(G) > 2^{-8}, and that the result holds for groups of smaller order. Since \text{tp}(G) > (1/2)^8 > (1/2)^{40}, our group G is soluble by Proposition 4.3. Let A be a minimal normal subgroup of G. Then A is an elementary abelian group of order \text{pr}^r for some prime \text{pr} and some positive integer r (see, for instance, Satz 9.13 in [15, Chap. I]). Since, by Part (ii) of Proposition 4.1 plus our hypothesis concerning G,

\[(1/2)^8 < \text{tp}(G) \leq \text{tp}(G/A),\]

the inductive hypothesis applies to the quotient G/A. Hence, either G/A is supersoluble or G/A has a section isomorphic to \text{A}_4. In the latter case, the section in question lifts isomorphically to a section in G; thus, we may suppose that G/A is supersoluble.

If A is cyclic, then G is also supersoluble (see, for instance, [23, 7.2.14]). Thus, we may suppose further that A is not cyclic; that is, r > 1 and A \nsubseteq Z(G). If A \nsubseteq \Phi(G) then G/A supersoluble would imply that G itself is supersoluble owing to the fact that supersoluble groups comprise a saturated formation (see, for instance, Satz 8.6(a) in [15, Chap. VI]). Consequently, we may also assume that A \nsubseteq \Phi(G), so that there exists a maximal subgroup M of G with A \nsubseteq M; in particular, AM = G. As A is normal in G, we have

\[(A \cap M)^g = A^g \cap M^g = A \cap M^g = A \cap M, \quad (g \in G),\]

where M = M^g, comes from the fact that \text{MM}^g \neq G by a theorem of Ore’s (see Satz 3.9 in [15, Chap. II]). Thus, A \cap M \nsubseteq G, and since A \cap M < A, minimality of A forces A \cap M = 1, so that G splits over A. Furthermore, G/A \cong M and thus M is supersoluble. Let T be a minimal normal subgroup of M. Then T is a cyclic group of order \text{qr} for some prime \text{qr}, while \text{N}(T) \supseteq M. Since M is maximal in G, either \text{N}(T) = M or T \subseteq G. In the second case, invoking Proposition 4.1(ii), our assumption that \text{tp}(G) > (1/2)^8, and the induction hypothesis, we deduce that either G/T (and thus G) is supersoluble, or G/T, and thus G, has a section isomorphic to \text{A}_4. We may therefore assume that \text{N}(T) = M. If (M : T) = t, then Corollary 3.7 implies that

\[(4.4) \quad \left(\frac{1}{2}\right)^8 < \text{tp}(G) \leq \text{P}(T) = \left(\frac{q^r - 1}{q - 1}\right)\quad ,\]

where we note that

\[G : M = (A \rtimes M : M) = |A| = \text{pr}^r,\]
and that the exponent $t(p^r - 1)/q$ is integral (see Corollary 3.7). Suppose first that $q > 5$. Then $q!/(q^q) \leq 5!/5^5 = 24/625$ and thus

$$\left(\frac{q!}{q^q}\right)^2 < \left(\frac{1}{2}\right)^8 < \text{tp}(G) \leq P_C(T) = \left(\frac{q!}{q^q}\right)^{\frac{t(p^r - 1)}{q}}.$$  

The above inequality forces the exponent of the right-hand-side to be equal to $1$ so that $q = t(p^r - 1)$. As $r > 1$ we conclude that $t = 1$ and so $p = 2$ and $q$ is a Mersenne prime (thus $r$ is prime as well). But the only Mersenne prime $q$ such that $q!/(q^q) > (1/2)^8$ is $q = 7$, which implies that $|G| = 56$. Using GAP we see that the only non-supersoluble group of order $56$ is the Frobenius group $C_2^3 \rtimes C_7$, whose transversal probability is $1/4096 < 2^{-8}$ by Part (vii) of Lemma 4.2.

Now suppose that $q = 3$. Then

$$\left(\frac{3!}{3^3}\right)^4 < \left(\frac{1}{2}\right)^8 < \text{tp}(G) \leq P_C(T) = \left(\frac{3!}{3^3}\right)^{\frac{t(p^r - 1)}{3}}.$$  

So $\frac{t(p^r - 1)}{3} \leq 3$. As $r > 1$ and $(p^r - 1) > 3$, we conclude that one of the following occurs:

- $t = 1$ and $p^r = 4, 8, 9$,
- $t = 2$ and $p^r = 4$,
- $t = 3$ and $p^r = 4$.

So in all cases $t \cdot p^r \leq 12$ and thus $|G| = 3 \cdot t \cdot p^r \leq 36$. Using GAP, we see that the non-supersoluble groups of order at most $36$ satisfying $\text{tp} > (1/2)^8$ are $A_4, \text{SL}_2(3), S_4, C_2 \rtimes A_4, C_3, A_4$, and $C_3 \times A_4$, all of which have a section isomorphic to $A_4$ (actually, all but $S_4$ have a quotient isomorphic to $A_4$).

We should mention here that among the non-supersoluble groups of order $36$ is the group $G = C_3^2 \rtimes C_4$ whose transversal probability is exactly $(1/2)^8$ and $G$ contains no section isomorphic to $A_4$; see Part (viii) of Lemma 4.2. This implies that our bound, when proved, will be sharp.

We are left with the case where $q = 2$. Clearly, for (4.4) to hold, we must have $\frac{t(p^r - 1)}{2} \leq 7$. We still have $r > 1$ and $p^r - 1 \geq 3$, thus we get one of the following cases:

- $t = 1$ and $p^r = 4, 8, 9$,
- $t = 2$ and $p^r = 4, 8$,
- $t = 3$ and $p^r = 4$,
- $t = 4$ and $p^r = 4$.

In all cases, we have $t \cdot p^r \leq 16$, thus $|G| = 2 \cdot t \cdot p^r \leq 32$. As we have already checked, there is no counterexample to our claim among the groups of order at most $36$, so the proof is complete.

The last theorem of this subsection relates $\text{tp}(-)$ and nilpotency. In order to express ourselves concisely and avoid repetition, we adopt a local convention. We will say that the group $G$ has a bad section if it has a section isomorphic to one of the groups in the following set: $\{A_4, D_3, D_5, D_7\}$.

**Theorem 4.5.** Let $G$ be a finite group with $\text{tp}(G) > 4/81 = (2/9)^2$. Then either $G$ is nilpotent or $G$ has a bad section. Furthermore, the bound is sharp.
We have

Thus (see Theorem 6.4 in [17]) $G$ is isomorphic to a Frobenius group with $A$ the Frobenius kernel and $M$ a Frobenius complement being cyclic of order $d$, where $d \mid p - 1$. By Corollary 3.6 we have

If $d = p - 1$, then the fact that $d! / d^d > 4 / 81$ forces $d \leq 4$ and thus $d \in \{1, 2, 4\}$. Then $G$ is isomorphic to $C_2$ or to $D_3$ or to the Frobenius group of order 20 respectively, so $G$ is nilpotent or $G$ has a bad section (in the last case because it has a subgroup isomorphic to $D_5$). We may therefore assume that $d < p - 1$, hence $(p - 1) / d \geq 2$. Thus

forcing $d! / d^d > 2 / 9$. That is only possible if $d = 2$ whence $G$ is itself a dihedral group. But the inequality

only holds for $p \leq 7$ in which case $G$ has a bad section. The proof is complete.
The semi-direct product \( C_7 \rtimes C_3 \) has common transversal probability equal to \( 4/81 \) and is neither nilpotent nor does it contain a bad section. Thus, our result is indeed sharp.

\[ 4.11 \]

\[ 4.13 \]

\[ 4.8 \]

below). On the other hand if \( \text{tp}(G) = p! / p^p \) for a prime \( p \) then \( p = 2 \) and the group \( G \) is one of a very specific list of groups (cf. Corollary 4.11 below). We will also show, that like the variant \( t \), it is natural to wonder if we can characterise all groups with transversal probability \( \text{tp}(G) = 1/2 \). We will actually show that if \( \text{tp}(G) = p! / p^p \) for a prime \( p \) then \( p = 2 \) and the group \( G \) is one of a very specific list of groups (cf. Corollary 4.11 below). We will also show, that like the value \( p! / p^p \) for \( p \) odd prime that can’t be attained, also \( \text{tp} \) can never equal \( p! / p^p \) for distinct primes \( p < q \) (see Theorem 4.8 below). On the other hand if \( p = q = 2 \) we fully characterise groups with transversal probability \( \text{tp}(G) = 1/4 \) (cf. Theorem 4.13).

Our first result is purely arithmetic and is of independent interest.

**Theorem 4.6.** Assume that

\begin{equation}
\prod_{i=1}^{n} \frac{t_i!}{t_i^{t_i}} = \prod_{i=1}^{k} \frac{p_i!}{p_i^{t_i}},
\end{equation}

where the numbers \( t_i > 1 \) are positive integers and the \( p_i \) are distinct primes. Then \( k = n \) and, after appropriate rearrangement, \( t_i = p_i \) for all \( i = 1, \ldots, n \).

**Proof.** We induce on \( k \). Assume \( k = 1 \), that is \( \prod_{i=1}^{n} \frac{t_i!}{t_i^{t_i}} = \frac{p!}{p^p} \). Then there exists a \( t_\sigma \) such that \( p | t_\sigma \) and without loss we may assume \( p | t_1 \). Now, the sequence \( a_n = \frac{n!}{n^n} \) is monotonically decreasing, as

\[ \frac{a_{n+1}}{a_n} = \left( \frac{n}{n+1} \right)^n = \left( \frac{1}{1+1/n} \right)^n < 1, \]

while clearly \( a_n < 1 \) for all \( n > 1 \). Hence

\[ \frac{p!}{p^p} = \prod_{i=1}^{n} \frac{t_i!}{t_i^{t_i}} \leq \frac{t_1!}{t_1^{t_1}} \leq \frac{p!}{p^p}. \]

Therefore \( t_1 = p \) and \( n = 1 \).

Assume now that the inductive hypothesis holds for all values less than \( k \) and we will prove it for \( k \). Thus we have \( \prod_{i=1}^{n} \frac{t_i!}{t_i^{t_i}} = \prod_{i=1}^{k} \frac{p_i!}{p_i^{t_i}} \) for distinct primes \( \{p_i\}_{i=1}^{k} \) and we assume that \( \{t_i\} \) and \( \{p_i\} \) are written in a decreasing order. So \( p := p_1 \) is the largest prime involved in the right hand side and \( t_1 > t_2 > \ldots > t_n \). Clearly \( p > 2 \) and equation (4.5) implies

\[ \prod_{i=2}^{p-1} \prod_{i=1}^{k} \frac{p_i^{t_i}}{t_i^{t_i}} \prod_{i=1}^{n} \frac{t_i!}{t_i^{t_i}} = (p-1)! \prod_{i=2}^{k} \frac{p_i!}{p_i^{t_i}} \prod_{i=1}^{n} \frac{t_i!}{t_i^{t_i}}. \]

Hence \( p^{p-1} \) divides \( \prod_{i=1}^{n} t_i^{t_i} \), and thus \( p \) divides at least one of the \( t_i \)’s. If \( p = t_\sigma \) for some \( \sigma = 1, \ldots, n \), then we are done by the inductive hypothesis. To see this, divide
both sides of (4.5) by $p! / p^\sigma$ to get
\[ \prod_{i=1, i\neq 0}^{n} \frac{t_i!}{t_i} = \prod_{i=2}^{k} \frac{p_i!}{p_i^\tau}, \]
where the inductive hypothesis now applies.

So we may assume that $t_i \neq p$ for all $i = 1, \ldots, n$, while the fact that $p$ divides some $t_i$ implies that $t_1 \geq 2p$.

For any $x = m/n \in \mathbb{Q}$ let $v_p(x)$ denote the $p$-adic valuation of $x = m/n$, that is, if $x \in \mathbb{Z}$ then $v_p(x)$ is the exponent of the largest power of $p$ that divides $x$ and $v_p(x) = 0$ if $p \nmid x$, while for $x = m/n$ we write $v_p(x) = v_p(m/n) = v_p(m) - v_p(n)$. Then observe that for any integer $t$ we have that
\[ v_p(t!/t^\sigma) < 0 \quad \text{if and only if} \quad p \mid t. \]

Now let $q$ be the biggest prime smaller than $t_1$. Then $q > p$ because $t_1 \geq 2p$ and according to Bertrand’s postulate there exists a prime in the interval $(p, 2p)$; cf., for instance, [16, Thm. 418]. Hence
\[ p < q \leq t_1. \]

Assume now that $t_1 = q$ and observe that $v_q(t_i!/t_i^\sigma) = 1 - q$ for all $i$ with $t_i = q$ (there is at least one such, namely $t_1$), while for the remaining we have $v_q(t_i!/t_i^\sigma) = 0$, since $t_1$ is the biggest of the $t_i$’s. Hence equation (4.5) implies
\[ \sum_{t_i=q} (1-q) = \sum_{i=1}^{n} v_q(t_i!/t_i^\sigma) = \sum_{i=1}^{k} v_q(\frac{p_i!}{p_i^\tau}) = 0 \]
which is clearly absurd. Hence $q < t_1$. In addition, applying Bertrand’s postulate again for the prime $q$, we conclude that $t_1 < 2q$ (or else a bigger prime than $q$ exists in $(q, 2q)$ and this prime would be smaller than $t_1$ contradicting the choice of $q$). Hence
\[ p < q < t_1 < 2q. \]

Therefore $q \nmid t_1$. So $v_q(t_1!/t_1^\sigma) = 1$, while $v_q(\prod_{i=1}^{k} \frac{p_i!}{p_i^\tau}) = 0$ as $q > p$ and $p$ is the largest of the $p_i$’s.

Thus, in view of (4.5), we get
\[ 1 + \sum_{i=2}^{n} v_q(t_i!/t_i^\sigma) = v_q(\prod_{i=1}^{k} \frac{p_i!}{p_i^\tau}) = 0. \]

We conclude that $\sum_{i=2}^{n} v_q(t_i!/t_i^\sigma) < 0$. Hence there exists $t_s$, for some $s = 2, \ldots, n$ so that $q \mid t_s$. But for every $i = 1, \ldots, n$ we have $t_i \leq t_1 < 2q$. Hence there exists some $s = 2, \ldots, n$ with $t_s = q$ and thus $v_q(t_s!/t_s^\sigma) = 1 - q$. We conclude that
\[ 2 - q + \sum_{i=2, i\neq s} v_q(t_i!/t_i^\sigma) = 0. \]

Hence there exist at least $q - 2$ among the $t_i$’s that are greater than $q$. These along with $t_1$ and $t_s$ provide at least $q$ elements among the $t_i$’s that are $\geq q$. Hence $t_i \geq q$ for all $i = 1, \ldots, q$ (as these are $\geq$ of the $q$-previously picked $t_i$’s) and therefore
\[ \sum_{i=1}^{q} t_i \geq q^2 > p^2 > \frac{p(p+1)}{2}. \]
while

$$\sum_{i=1}^{k} p_i \leq \sum_{k=1}^{p_1} k = \frac{p(p+1)}{2}$$

We conclude that the vector \((t_1, t_2, \ldots, t_q, \ldots, t_n)\) weakly majorises the vector \((p_1, p_2, \ldots, p_k)\), while all the hypothesis of Proposition A.6 (see appendix) are satisfied. Hence

$$\prod_{i=1}^{n} t_i^{l_i} < \prod_{i=1}^{k} p_i^{l_i}$$

contradicting the hypothesis of the theorem. This final contradiction implies that there does exist \(t\) with \(t_i = p\) and so, as we have seen, the inductive hypothesis completes the proof of the theorem. 

Combining Theorem 4.6 with Proposition 3.2 we get the following.

**Corollary 4.7.** Assume that \(P_G(H) = \frac{p!}{p^r} \cdot \frac{q!}{q^s}\) with distinct primes \(p < q\), where \(G\) is a finite group with subgroup \(H\). Then the \(t\)-vector of \(H\) in \(G\) is given by \((q, p, 1, \cdots, 1)\); in particular, \(pq | |H|\).

With the help of Corollary 4.7, we can now prove a result excluding certain rational numbers from the range of the function \(tp(-)\).

**Theorem 4.8.** There exists no finite group \(G\) such that \(tp(G) = \frac{p!}{p^r} \cdot \frac{q!}{q^s}\) for primes \(p < q\).

**Proof.** Let \(G\) be a counterexample of smallest possible order, and let \(H\) be a proper subgroup of \(G\) for which \(tp(G)\) is realised; that is,

$$P_G(H) = tp(G) = \frac{p!}{p^r} \cdot \frac{q!}{q^s}$$

for distinct primes \(p < q\). We argue that \(H\) is core-free. Suppose not, and let \(N := core_G(H)\). Then \(N > 1\) and \(P_G/H(H/N) = P_G(H)\) by Corollary 3.17. Moreover, by Part (ii) of Proposition 4.1, we have

$$P_G/H(H/N) > tp(G/N) > tp(G) = P_G(H).$$

Therefore, \(tp(G/N) = tp(G)\), contradicting the minimality of \(G\). Thus \(H\) is core-free, as claimed.

The prime \(q\) divides \(|H|\) according to Corollary 4.7. Let \(K < H\) be of order \(q\). Then \(K\) is not normal in \(G\), as \(H\) is core-free. Also

$$P_G(K) = \left(\frac{q!}{q^s}\right)^{n \mod \frac{n}{q}},$$

in view of Corollary 3.7, where \(n = (G : K), m = (N_G(K) : K), m | n, and m \neq n\). In addition, we have

$$\left(\frac{q!}{q^s}\right)^{n \mod \frac{n}{q}} = P_G(K) \geq tp(G) = \frac{p!}{p^r} \cdot \frac{q!}{q^s}.$$ 

But \(p < q\), and thus \(\frac{p!}{p^r} > \frac{q!}{q^s}\). So the above inequality holds if, and only if, \(\frac{n \mod \frac{n}{q}}{q} = 1\), which in turn, as \(m | n\), implies that \(m | q\). Hence either \(m = 1\) and \(n = q + 1\), or \(m = q\) and \(n = 2q\).
In the first case, we get $K = N_G(K)$, and thus $K = N_H(K)$, while $|G| = q(q + 1)$. Furthermore, by Proposition 3.14 and Corollary 3.7, we have

$$\frac{q^l}{q^0} = P_G(K) \leq P_H(K) = \left(\frac{q^l}{q^0}\right)^{(H:K) - 1}.$$  

Hence, $0 \leq (H:K) - 1 \leq 1$. As $H \neq K$, we necessarily have $(H : K) = q + 1 = (G : K)$ or, equivalently, $G = H$, contradicting the fact that we had chosen $H$ as a proper subgroup of $G$. So the first case, where $m = 1$, does not occur.

Consequently, we must have $m = q$, $n = 2q$, and $|G| = 2q^2$. This forces $p = 2$, the only other prime involved in the order of $G$. As $H < G$, while 2 and $q$ both divide $H$ by Corollary 4.7, we get $|H| = 2q$, thus $K$ is a normal subgroup of $H$. So $H \leq N_G(K)$, and thus

$$2 = (H : K) \mid (N_G(K) : K) = q.$$  

This final contradiction now implies the theorem. □

We can now show that if for a specific subgroup $H \leq G$ we know that $P_C(H) = \frac{p^l}{p^0}$ for some prime $p$, then $H$ has a very restricted place inside $G$.

**Lemma 4.9.** Let $G$ be a finite group, and let $H \leq G$ be a subgroup. If $P_C(H) = \frac{p^l}{p^0}$ for some prime $p$, then $p \mid |H|$ and one of the following occurs:

(i) $H = N_G(H)$ and $(G : H) = p + 1$, or  

(ii) $(N_G(H) : H) = p$, while $(G : H) = 2p$.

In particular, $P_C(H) = 1/2$ implies that $H$ is a subgroup of even order, whose index in $G$ is either 3 or 4.

**Proof.** Suppose that $P_C(H) = \frac{p^l}{p^0} = \prod_t t_t^l / t_t^0$ for some prime $p$, and let $m = (N_G(H) : H)$. Combining Proposition 3.2 with Theorem 4.6, we see that $\Gamma^C_H$ has precisely one non-trivial component $\Delta_{a_1} \cong K_{t_{a_1}, t_{a_1}}$, and that $t_{a_1} = p$. Hence, $p \mid |H|$, and we have $n = (G : H) = m + p$. Since $m \mid n$, we get $m \mid p$. Thus, either $m = 1$ (that is, $N_G(H) = H$) and $n = p + 1$, or $m = p$ and $n = 2p$, giving $(N_G(H) : H) = p$ in the second case. □

Our next result takes a major step towards classifying groups $G$ where $tp(G) = \frac{p^l}{p^0}$ for some prime $p$.

**Theorem 4.10.** Let $G$ be a finite group. Assume that, for every non-normal subgroup $H \leq G$, we have $P_C(H) = \frac{p^l}{p^0}$ with some fixed prime $p$. Then one of the following holds:

(i) $G$ is a Dedekind group. In this case, all subgroups are normal and $tp(G) = 1$.

(ii) $G \cong C_3 \rtimes C_{2n}$, for some integer $n \geq 1$.

(iii) $G \cong D_4$.

(iv) $G \cong Q_{16}$.

(v) $G \cong C_4 \rtimes C_4$.

In Cases (ii)–(v), we have $p = 2$ and $tp(G) = 1/2$.  

Proof. Clearly, $\text{tp}(G) = 1$ if, and only if, every subgroup of $G$ is normal or, equivalently, if, and only if, $G$ is a Dedekind group. Thus, discarding Case (i), we assume from now on that $\text{tp}(G) < 1$.

Fix some non-normal subgroup $H$ of $G$. Then, by hypothesis, $P_G(H) = P! / P^p$; thus, by Lemma 4.9, we have $(G : H) = p + 1$ or $(G : H) = 2p$, as well as $p \mid |H|$.

Claim: Every proper subgroup of $H$ is normal in $G$ and $H \cong C_p^k$, a cyclic group of order $p^k$ for some positive integer $k$.

Proof of Claim. If $(G : H) = p + 1$ and $K < H$, then clearly $(G : K) \neq p + 1$. Also, $(G : K) \neq 2p$, or else $2p = r(p + 1)$ for some integer $r \geq 2$. This implies that $p \mid r$, while $2 = r + r/p > 2$, a contradiction. If $(G : H) = 2p$ and $K < H$, then $(G : K) \geq 4p$; in particular, $(G : K) \neq p + 1, 2p$. Hence, in both cases, $K$ is normal, and the first part of the claim follows.

If $M_1, M_2$ are distinct maximal subgroups of $H$, then they are normal in $G$, and their product is $M_1M_2 = H$. This would imply that $H$ is normal in $G$, contrary to our assumption. Hence, $H$ has a unique maximal subgroup; thus, $H$ is cyclic of prime power order. Moreover, as $p \mid |H|$, we conclude that $H \cong C_p^k$ for some $k \geq 1$, and our claim follows. □

For the rest of the proof, we shall need to distinguish two cases.

Case 1: $p$ is odd. As we have seen, we either have $(G : H) = p + 1$ and $H = N_G(H)$, or $(G : H) = 2p$ and $|N_G(H)| = p^{k+1}$. Hence, either $|G| = p^k(p + 1)$, or $|G| = 2p^{k+1}$. Therefore, in either case, $2 \mid |G|$, so that there exists some $x \in G$ of order 2. Let $T = \langle x \rangle$. Clearly, $T$ is not a subgroup of $N_G(H)$ (in either case). This implies that $T$ cannot be normal in $G$. Indeed, if $T \subseteq G$, then $x \in Z(G)$, and so $T$ would be a subgroup of $N_G(H)$. It follows that $p \mid |T| = 2$, forcing $p = 2$, which contradicts our case assumption. Consequently, $p$ has to be even.

Case 2: $p = 2$. Here, there are two subcases:

(a) $N_G(H) = H$ and $(G : H) = 3$, or (b) $(N_G(H) : H) = 2 = (G : N_G(H))$.

Subcase (a). By our claim proved above, $H \cong C_2^n$ for some positive integer $n$. Since $(G : H) = 3$, so that $|G| = 3 \cdot 2^n$, it follows that, if $Q$ is a Sylow 3-subgroup of $G$, then $Q \cong C_3$. Moreover, $Q$ is normal in $G$, since otherwise $P_G(Q) = 1/2$, so that $Q$ would have to have even order by Lemma 4.9. By a well-known theorem of Zassenhaus (see, for instance, [15, Hauptsatz I.18.1]), we have

$$G \cong \langle a, b \mid a^3 = b^{2n} = 1, a^b = a^{-1} \rangle \cong C_3 \rtimes C_{2^n}, \quad n \geq 1,$$

since $\text{tp}(G) < 1$, whence Case (ii) of the theorem.

Subcase (b). Now suppose that

$$(G : N_G(H)) = 2 = (N_G(H) : H),$$

where $H \cong C_2^n$ for some $n \geq 1$, so that $G$ is a 2-group. At this point, we are tasked with determining the collection of all 2-groups $G$ of order $|G| = 8$, which enjoy the following property:

(†) All subgroups of $G$ are normal in $G$ except some subgroups of index 4.

Note that we are not allowing for any ambiguity in (†); there must exist non-normal subgroups of index 4. The only group of order 8 with this property is clearly $D_4$. 

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and we use GAP to find the groups of order 16: they are $Q_{16}$ and SmallGroup(16, 4), which is of type $C_4 \rtimes C_4$.

We now argue that those are the only 2-groups with this property. To prove the assertion, we shall show that, if $G$ satisfies (†), then $|G| < 16$. Indeed, suppose that $G$ is a minimal counterexample to our last claim, and let $H$ be a non-normal subgroup of index 4 in $G$. Let $N$ be a subgroup of order 2 in $H$, and observe that $N$ is normal in $G$, since $|G| \geq 32$, and $G$ satisfies (†). By the Correspondence Theorem, $G/N$ satisfies (†) and is not a counterexample. Thus, $|G/N| < 16$, and so $|G| < 32$ which implies that $|G| = 32$. Using GAP again, we check that no group of order 32 satisfies (†), and we have reached the desired contradiction. Hence, our claim holds, and the only 2 groups satisfying (†) are those exhibited above.

As a consequence of Theorem 4.10, we can now determine, up to isomorphism, all groups $G$ with $\text{tp}(G) = 1/2$.

**Corollary 4.11.** Let $G$ be a finite group. We have $\text{tp}(G) = 1/2$ if, and only if, one of the following occurs:

(i) $G = C_3 \rtimes C_2^n$, for some integer $n \geq 1$,

(ii) $G = D_4$,

(iii) $G = Q_{16}$,

(iv) $G = C_4 \rtimes C_4$.

**Proof.** This follows from Theorem 4.10 in conjunction with Corollary 3.10.

As a further application of Theorem 4.10, we have the following.

**Corollary 4.12.** If $G$ is a non-abelian group of odd order, then $\text{tp}(G) \leq 4/81$.

**Proof.** Let $H \leq G$ be a non-normal subgroup. As

$$m = (N_G(H): H) < (G: H) = n,$$

the graph $\Gamma_G^C$ has at least one non-trivial component $\Delta_{\sigma} \cong K_{t_{\sigma}:t_{\sigma'}}$, and since $1 < t_{\sigma}| |H||G|$ by Proposition 3.2, and $G$ has odd order, we have $t_{\sigma} \geq 3$. It follows that, for any such group $G$, either $\text{tp}(G) \leq 4/81$, or $P_G(H) = 2/9$ for all subgroups $H$ of $G$ which are not normal in $G$. In the second case, Theorem 4.10 applies, so that $G$ would have to be one of the groups listed in (i)–(v). However, by the well-known classification of Dedekind groups, Case (i) does not apply, while Cases (ii)–(v) are ruled out as $G$ has odd order by hypothesis. Hence, $\text{tp}(G) \leq 4/81$, as claimed.

Our next result determines all groups $G$ with $\text{tp}(G) = 1/4$.

**Theorem 4.13.** Let $G$ be a finite group. We have $\text{tp}(G) = 1/4$ if, and only if, one of the following occurs:

(i) $G = \langle a, b | a^5 = b^{2k} = 1, a^b = a^{-1} \rangle$ for some integer $k \geq 1$.

(ii) $G$ has a normal cyclic 2-subgroup $M = C_{2^k}$ for some integer $k \geq 0$, so that $G/M$ is one of the groups $D_6$, $M_4(2)$, $C_4 \circ D_4$, $C_2 \times D_4$, or $C_2^2 \rtimes C_4$. 
Proof. The proof is similar to that of Theorem 4.10 so we only sketch some steps. Assume first that \( P_G(H) = 1/4 \) for some subgroup \( H \) of \( G \), where \( 1/4 = \prod_{\alpha=1}^{s} \frac{t_{\alpha}!}{n_{\alpha}!} \). Then note that we necessarily have two values \( t_{\alpha_1} = t_{\alpha_2} = 2 \), while the rest are all 1. The reason is that if there exist \( t_{\alpha} \geq 3 \), then \( \frac{(t_{\alpha})!}{n_{\alpha}!} \leq 2/9 \) and thus \( \prod_{\alpha=1}^{s} \frac{t_{\alpha}!}{n_{\alpha}!} \leq 2/9 \). Hence, \( 2\|H\| \) and \( 4 + m = n \), where \( m,n \) are, as usual, the index of \( H \) in \( N_G(H) \) and in \( G \), respectively. Furthermore, we have \( m \mid n \), and thus \( m \mid 4 \). Hence \( m = 1,2, \) or 4. In conclusion, if \( P_G(H) = 1/4 \), then \( H \) is group of even order whose indices \( m, n \) equal one of the following:

\[
m = 1 \text{ and } n = 5, \quad m = 2 \text{ and } n = 6, \quad \text{or } m = 4 \text{ and } n = 8.
\]

Assume now that \( tp(G) = 1/4 \). Then for every non-normal subgroup \( T \) of \( G \) we should have \( P_G(T) = 1/4 \) or \( P_G(T) = 1/2 \). Hence, the only possible values for the index \( (G:T) \) are 3, 4, 5, 6, and 8. (The first two values occur in the case that \( P_G(T) = 1/2 \), according to Lemma 4.9.) As \( tp(G) = 1/4 \), there exists a non-normal subgroup \( H \leq G \) with \( P_G(H) = 1/4 \), and thus \( (G:H) \in \{5,6,8\} \), while \( H \) has even order. Observe that if \( M \) is any proper subgroup of \( H \), then \( (G:M) \notin \{3,4,5,6,8\} \). Therefore any proper subgroup of \( H \) is normal in \( G \). We conclude that \( H = C_{2^k} \) for some integer \( k \geq 1 \). We now distinguish three cases.

Case 1: \( (G:H) = 5 \) and \( H = N_G(H) \).

If \( C \) is a 5-Sylow subgroup of \( G \), then \( P_G(C) = 1 \), as \( |C| = 5 \) is odd. Hence \( C \) is a normal subgroup of \( G \). If \( M \) is the maximal subgroup of \( H \), then \( C \cdot M = CM \). So \( M \) is a central subgroup of \( G \). Let \( C = \langle a \mid a^5 = 1 \rangle \) and \( H = \langle b \mid b^{2^k} = 1 \rangle \). Then \( b \) acts as an automorphism of order 2 on \( C \). We conclude that

\[
G = \langle a,b \mid a^5 = b^{2^k} = 1, a^b = a^{-1} \rangle
\]

According to Lemma 4.2(i), the above group has transversal probability equal to 1/4 and Case 1 is completed.

Case 2: \( (G:H) = 6 \) and \( (N_G(H):H) = 2 \).

In this case the 3-Sylow subgroup \( C \) of \( G \) is a normal subgroup of \( G \), or else we would have \( P_G(C) \) being a power of 2/9. If \( |H| = 2 \) then \( G = D_6 \). So we may assume that \( H = C_{2^k} \) with \( k > 1 \). Let \( M = \langle b^2 \rangle \) be the maximal subgroup of \( H \). Then \( 1 \neq M \leq G \). In addition \( 1/4 = P_G(H) = P_{G/MM}(H/M) \), while \( tp(G) \leq tp(G/M) \). We conclude that \( G/M \) is a group of order 12 whose common transversal probability is 1/4. Using GAP, we find that \( G/M = D_6 \).

Case 3: \( (G:H) = 8 \) and \( (N_G(H):H) = 2 \).

In this case \( G \) is a 2-group. If \( |H| = 2 \) then \( G \) is a group of order 16. Using GAP again, we see that the only groups of order 16 that have transversal probability 1/4 are \( M_4(2), C_4 \circ D_4, C_2 \times D_4, \) and \( C_{2^2} \rtimes C_4 \). We may assume \( |H| > 2 \), and we write \( M \) for its unique maximal subgroup. Then \( M \) is normal in \( G \). As earlier, we get \( tp(G/M) = 1/4 \), while \( |G/M| = 16 \). Hence \( G/M \) is one of the previously mentioned groups.

This completes the proof in Case 3, and the theorem follows. \( \blacksquare \)

Proposition 4.3 might have left the reader (as it did us) with the nagging feeling that a refinement might perhaps be possible, which connects \( tp(\cdot) \) with the derived length of a soluble group. The final result of this section is a first indication that such a result might indeed exist, although we have been unable so far to find it. The
reader should also note that the bound obtained in the next result is the same one achieved in Theorem 4.5.

**Proposition 4.14.** If \( G \) is a non-abelian group with \( \text{tp}(G) > 4/81 \), then \( G \) has derived length 2. Furthermore, the bound is sharp.

**Proof.** We induct on the order of \( G \). Note that \( G \) cannot have odd order according to Corollary 4.12. Furthermore, if \( H \) is a subgroup of \( G \) of odd order, then \( H \) must be abelian; otherwise, combining Corollary 4.12 with Part (i) of Proposition 4.1, we would get

\[
4/81 < \text{tp}(G) \leq \text{tp}(H) \leq 4/81,
\]
a contradiction. Also, since

\[
\text{tp}(G) > 4/81 > (1/2)^{14} = \text{tp}(A_5)
\]

\( G \) is necessarily soluble by Proposition 4.3 and Proposition 4.1(iii).

Next, we argue that \( G \) has a unique minimal normal subgroup. Assume otherwise and let \( N_1, N_2 \) be distinct minimal normal subgroups of \( G \). Since \( G/N_i \) is either abelian, or of derived length 2 by the induction hypothesis plus Proposition 4.1(ii), we see that \( (G/N_1)' \) is abelian, as is \( (G/N_2)' \). We have

\[
(G/N_i)' = G'/N_i \cong G'/G' \cap N_i, \quad (i = 1, 2)
\]

and, since \( G' \cap N_i \leq G \), either \( G' \cap N_i = 1 \), or \( N_i \leq G' \), due to minimality of \( N_i \). If \( G' \cap N_1 = 1 \), then \( G' \) is abelian, and the induction is complete. Arguing similarly for \( N_2 \), we deduce that \( N_1, N_2 \leq G' \). Therefore both \( G'/N_1 \) and \( G'/N_2 \) are abelian groups. Now notice that \( N_1 \cap N_2 = 1 \), again due to minimality of the \( N_i \). Moreover, the group \( G'/\langle N_1 \cap N_2 \rangle \), which is isomorphic to a subgroup of \( G'/N_1 \times G'/N_2 \), is abelian as well. It follows that \( G' \) is abelian, thus \( G \) has derived length 2, as required.

We may therefore assume that \( G \) has a unique minimal normal subgroup. It follows that for every subgroup \( H \) of order 2 in \( G \), either \( H \) is normal and thus central in \( G \) (and moreover there can exist only one such subgroup by the previous observation) or, by Part (i) of Proposition 4.1 plus Corollary 3.7, we have

\[
2^{\frac{n-m}{2}} < 81/4,
\]

where \( n = |G|/2 \) and \( m = |N_G(H)|/2 \). This implies \( n - m \leq 8 \) and thus \( |G| - |N_G(H)| \leq 16 \). Since \( |N_G(H)| \leq |G|/2 \), the previous inequality implies that \( |G| \leq 32 \). A GAP check confirms that every group of order \( < 32 \) with \( \text{tp}(G) > 4/81 \) has derived length \( \leq 2 \) and thus the proof is complete in that case.

It therefore suffices to treat the case where a Sylow 2-subgroup of \( G \) has only one minimal subgroup which is moreover the unique minimal normal subgroup of \( G \). If \( G \) is a 2-group, then it is either cyclic or a generalised quaternion group (these being the only 2-groups with a unique minimal subgroup); the first case is ruled out by our hypothesis that \( G \) is non-abelian. In the second case, the derived length equals 2. We may thus assume that \( G \) is not a 2-group.

Let \( P \) be a non-trivial Sylow \( p \)-subgroup of \( G \) for some odd prime \( p \) and let \( T \leq P \) be of order \( p \). As the only minimal normal subgroup of \( G \) is a 2-group, the cyclic group \( T \) is not normal in \( G \), thus \( N_G(T) < G \) and, by Proposition 4.1(i) and Corollary 3.7,

\[
4/81 < \text{tp}(G) \leq \text{tp}(T) \leq \text{tp}(T) = \left( \frac{p!}{p^p} \right)^{\frac{n-m}{p}},
\]

where \( n = |G|/p \) and \( m = |N_G(T)|/p \leq n \). Since \( n-m \geq 1 \) and \( p!/p^p \leq 24/625 < 4/81 \) for \( p \geq 5 \), we must have \( p = 3 \) and \( n - m = 3 \). Moreover, since \( m \mid n \), we conclude that \( m \mid 3 \).
Thus, either \( m = 1 \) and \(|G| = 12\), or \( m = 3 \) and \(|G| = 18\). As we have already checked, no group of order 12 or 18 has derived length \( > 2 \) and transversal probability \( > 4/81\), and our proof is complete.

Finally, we note that the transversal probability of \( SL_2(3) \) equals \( 4/81 \) by Part (ix) of Lemma 4.2, while its derived length is 3; so that our bound is indeed sharp as claimed.

### 4.3. **Group extensions and the function** \( \text{tp}(\cdot) \).

The aim in this section is to see how the function \( \text{tp}(\cdot) \) behaves with respect to group extensions.

Let \( G \) and \( K \) be groups, and let \((V,W)\) be a factor system for \( G \) by \( K \). In detail, this means that \( V : K \to \text{Aut}(G) \) and \( W : K \times K \to G \) are maps, such that

\[
V(k_2) \circ V(k_1) = i_{W(k_1,k_2)} \circ V(k_1 k_2), \quad (k_1,k_2 \in K)
\]

\[
W(k_1,k_2,k_3)W(k_2,k_3) = W(k_1 k_2,k_3)V(k_3)(W(k_1,k_2)), \quad (k_1,k_2,k_3 \in K),
\]

where \( i_g(x) = g^{-1} x g \) for \( x, g \in G \), so that \( i_g \) is the inner automorphism of \( G \) associated with the element \( g \in G \). Let \( \widehat{G} \) be the extension of \( G \) by \( K \) associated with \((V,W)\); that is, \( \widehat{G} = K \times G \) as a set, with group law given by

\[
(k_1, g_1) \cdot (k_2, g_2) = (k_1 k_2, W(k_1,k_2)V(k_2)(g_1)g_2), \quad (k_1,k_2 \in K; g_1,g_2 \in G),
\]

and we have a short exact sequence

\[
1 \longrightarrow G \longrightarrow \widehat{G} \longrightarrow K \longrightarrow 1,
\]

where \( i(g) = (1, W(1,1)^{-1} g) \) and \( \pi(k,g) = k \), and the associated map \( \varphi_E : K \to \widehat{G} \) is given by \( \varphi_E(k) = (k,1) \); cf. [23, Sec. 9.4], in particular Statement 9.4.5. We shall need the following properties of a factor system.

**Lemma 4.15.** Let \((V,W)\) be a factor system for \( G \) by \( K \). Then

(i) \( V(1) = i_{W(1,1)} \),

(ii) \( W(k,1) = W(1,1), \quad (k \in K) \),

(iii) \( W(1,k) = V(k)(W(1,1)), \quad (k \in K) \).

**Proof.** (i) Setting \( k_1 = k_2 = 1 \) in (4.6) gives

\[
V(1) \circ V(1) = i_{W(1,1)} \circ V(1).
\]

Thus,

\[
V(1) = V(1) \circ V(1) \circ V(1)^{-1} = i_{W(1,1)} \circ V(1) \circ V(1)^{-1} = i_{W(1,1)},
\]

as desired.

(ii) Setting \( k_1 = k \) and \( k_2 = k_3 = 1 \) in (4.7), we get

\[
W(k,1)W(1,1) = W(k,1)V(1)(W(k,1)).
\]

Applying Part (i), this equation may be rewritten as

\[
W(k,1)W(1,1) = W(k,1)W(1,1)^{-1}W(k,1)W(1,1).
\]

Multiplying the last equation from the left by \( W(k,1)^{-1} \) and from the right by \( W(1,1)^{-1} \), the result follows.

(iii) Setting \( k_1 = k_2 = 1 \) and \( k_3 = k \) in (4.7) yields

\[
W(1,k)^2 = W(1,k)V(k)(W(1,1)),
\]
whence (iii).

\[ \textbf{Theorem 4.16.} \text{ Let } \hat{G} \text{ be an extension of the group } G \text{ by the group } K \text{ with associated factor system } (V, W). \text{ Then} 
\]

\[ tp(\hat{G}) \leq \min_{H \leq G} \prod_{k \in K} P_G(H, V(k)(H)). \]

\[ (4.9) \]

\[ \textbf{Proof.} \text{ Let } H \leq G \text{ be a subgroup (with } G \text{ considered as a subgroup of } \hat{G} \text{ via the embedding } \iota). \text{ Then, for } h \in H \text{ and } (k, g) \in \hat{G}, \text{ we have} 
\]

\[ (k, g)\iota(h) = (k, g)(1, W(1, 1)^{-1}h) \]

\[ = (k, W(k, 1)V(1)(g)W(1, 1)^{-1}h) \]

\[ = (k, W(k, 1)W(1, 1)^{-1}gW(1, 1)W(1, 1)^{-1}h) \]

\[ = (k, W(k, 1)W(1, 1)^{-1}gh) = (k, gh), \]

where we have applied the definition of \( \iota \) in the first step, \((4.8)\) in the second step, Lemma 4.15(i) in Step 3, and Lemma 4.15(ii) in the last step. Hence,

\[ (k, g) \cdot H = (k, gH), \quad ((k, g) \in \hat{G}, H \leq G). \]

\[ (4.10) \]

Similarly, making use of the definition of \( \iota \), the group law \((4.8)\), and Part (iii) of Lemma 4.15, we find that, for \( h \in H \) and \((k, g) \in \hat{G}, \)

\[ \iota(h)(k, g) = (1, W(1, 1)^{-1}h)(k, g) \]

\[ = (k, W(1, k)V(k)(W(1, 1)^{-1}h)g) \]

\[ = (k, W(1, k)V(k)(W(1, 1))^{-1}V(k)(h)g) \]

\[ = (k, V(k)(h)g), \]

implying

\[ (4.11) \quad H \cdot (k, g) = (k, V(k)(H)g), \quad ((k, g) \in \hat{G}, H \leq G). \]

Now let

\[ \hat{T} = \{(k_i, g_j) \in DT_{\hat{G}}(H) \}
\]

be any two-sided transversal for \( H \) in \( \hat{G} \). From either \((4.10)\) or \((4.11)\), when applied to \( H \leq G \) and \((k_i, g_j) \in \hat{T} \), it is clear that all elements of \( K \) must occur as first component of an element in \( \hat{T} \), since otherwise

\[ (k, 1) \notin \bigcup_{(i, j)}(k_i, g_j)H = \hat{G} \]

for some \( k \in K \), a contradiction. Given \( k \in K \), consider the set

\[ T(k) := \{ g \in G : (k, g) \in \hat{T} \} \subseteq G. \]

We claim that \( T(k) \in DT_G(H, V(k)(H)) \).

First, let \( k \in K \) be given, and let \( g \in G \) be arbitrary. Then

\[ (k, g) \in (k_i, g_j)H = (k_i, g_j; H) \]
for some \((i, j)\). This implies \(k_i = k\) and \(g \in g_j H\). Thus, \(g_j \in T(k)\) and \(g \in \bigcup_{g \ell \in T(k)} g \ell H\), so that

\[ G = \bigcup_{g \ell \in T(k)} g \ell H. \]

Similarly, given \(g \in G\), we have

\[ (k, g) \in H(k_i, g_j) = (k_i, V(k_i)(H) g_j) \]

for some \((i, j)\), thus \(k_i = k\), \(g \in V(k)(H) g_j\), and \(g_j \in T(k)\), so that

\[ G = \bigcup_{g \ell \in T(k)} V(k)(H) g \ell. \]

Next, suppose that \(g_j H \cap g \ell H \neq \emptyset\), where \(g_j, g \ell \in T(k)\). Then

\[ (k, g_j) H \cap (k, g \ell) H = (k, g_j H) \cap (k, g \ell H) = (k, g_j H \cap g \ell H) \neq \emptyset, \]

implying \((k, g_j) = (k, g \ell)\) by our hypothesis on \(\tilde{D}\), thus \(g_j = g \ell\). Hence, \(T(k)\) is a left transversal for \(H\) in \(G\). Similarly, suppose that

\[ V(k)(H) g_j \cap V(k)(H) g \ell \neq \emptyset \]

for some \(g_j, g \ell \in T(k)\). Then we have

\[ H(k, g_j) \cap H(k, g \ell) = (k, V(k)(H) g_j) \cap (k, V(k)(H) g \ell) = (k, V(k)(H) g_j \cap V(k)(H) g \ell) \neq \emptyset, \]

so that, again, \(g_j = g \ell\). Consequently, \(T(k)\) is also a right transversal for \(V(k)(H)\) in \(G\), therefore \(T(k) \in DT_G(H, V(k)(H))\), as claimed.

Mapping \(k \in K\) to \(T(k)\) for given \(\tilde{T}\) thus gives a choice function

\[ f(\tilde{T}) : K \rightarrow \bigcup_{k \in K} DT_G(H, V(k)(H)), \]

and, subsequently, sending \(\tilde{T} \in DT_G(H)\) to \(f(\tilde{T})\), defines a map

\[ \tilde{\Phi} : DT_G(H) \rightarrow CF(K, H, G), \]

where, unsurprisingly, \(CF(K, H, G)\) denotes the set of all choice functions

\[ f : K \rightarrow \bigcup_{k \in K} DT_G(H, V(k)(H)); \]

that is, functions \(f\) as above, such that \(f(k) \in DT_G(H, V(k)(H))\) for each \(k \in K\). For later use we observe that, obviously,

\[ |CF(K, H, G)| = \prod_{k \in K} |DT_G(H, V(k)(H))|. \]

Next, we note that, if \(\tilde{T}_1, \tilde{T}_2 \in DT_G(H)\) are such that \(\tilde{\Phi}(\tilde{T}_1) = f = \tilde{\Phi}(\tilde{T}_2)\) then, by definition of \(\tilde{\Phi}\),

\[ \tilde{T}_1 = \bigcup_{k \in K} (k, f(k)) = \tilde{T}_2, \]

so that \(\tilde{\Phi}\) is injective. We want to show that \(\tilde{\Phi}\) is surjective as well. Let \(f \in CF(K, H, G)\) be given, and set

\[ \tilde{T}_f := \bigcup_{k \in K} (k, f(k)) \subseteq \widehat{G}. \]
We claim that $\overline{T}_f \in DT_{\overline{G}}(H)$. By (4.10), we have

$$\bigcup_{t \in \overline{T}_f} tH = \bigcup_{k \in K} \bigcup_{g \in f(k)} (k, g)H = \bigcup_{k \in K} \bigcup_{g \in f(k)} (k, g) = \bigcup_{k \in K} (k, gH) = \bigcup_{k \in K} (k, G) = \overline{G}.$$ 

Similarly, by (4.11), we have

$$\bigcup_{t \in \overline{T}_f} HT = \bigcup_{k \in K} \bigcup_{g \in f(k)} H(k, g) = \bigcup_{k \in K} \bigcup_{g \in f(k)} (k, V(k)(H)g) = \bigcup_{k \in K} (k, V(k)(H)) = \bigcup_{k \in K} (k, G) = \overline{G}.$$ 

Moreover, for $\overline{t}_1 = (k_1, g_1), \overline{t}_2 = (k_2, g_2) \in \overline{T}_f$, where $k_1, k_2 \in K$, $g_1 \in f(k_1)$, and $g_2 \in f(k_2)$, the hypothesis

$$\overline{t}_1 H \cap \overline{t}_2 H = (k_1, g_1)H \cap (k_2, g_2)H = (k_1, g_2) \cap (k_2, g_2H) \neq \emptyset$$

first implies $k_1 = k_2 =: k$, thus $g_1, g_2 \in f(k)$, as well as $g_1H \cap g_2H \neq \emptyset$, which forces $g_1 = g_2$, since $f(k)$ forms a left transversal for $H$ in $G$. Hence, $\overline{t}_1 = \overline{t}_2$. Also, with $t_1, t_2, k_1, k_2, g_1, g_2$ as above, the assumption that

$$Ht_1 \cap Ht_2 = H(k_1, g_1) \cap H(k_2, g_2)$$

implies $k_1 = k_2 =: k$, thus $g_1, g_2 \in f(k)$, and $V(k)(H)g_1 \cap V(k)(H)g_2 \neq \emptyset$, forcing $g_1 = g_2$, as $f(k)$ also forms a right transversal for $V(k)(H)$ in $G$. Hence, again, $\overline{t}_1 = \overline{t}_2$, and it follows that $\overline{T}_f \in DT_{\overline{G}}(H)$. Since $\Phi(\overline{T}_f) = f$ by construction, we conclude that $\Phi$ is surjective, thus a bijection. Therefore,

$$|DT_{\overline{G}}(H)| = |CF(K, H, G)|,$$

and we conclude that

$$\text{tp} (\overline{G}) \leq \min_{H \in G} P_{\overline{G}}(H) = \min_{H \in G} \frac{|DT_{\overline{G}}(H)|}{|H|^{G:H}} = \min_{H \in G} \frac{|DT_{\overline{G}}(H)|}{|H|^{G:H}|K|} = \min_{H \in G} \prod_{k \in K} \frac{|DT_G(H, V(k)(H))|}{|H|^{G:H}} = \min_{H \in G} \prod_{k \in K} P_G(H, V(k)(H)),$$

whence (4.9). $\blacksquare$

**Corollary 4.17.** Let $\overline{G} = G \rtimes K$, viewed as an internal semidirect product. Then we have

$$\text{tp}(\overline{G}) \leq \min_{H \in G} \prod_{k \in K} P_G(H, H^k).$$
Proof. Suppose that the factor system \((V, W)\) in Theorem 4.16 splits, so that \((V, W)\) is equivalent to some factor system \((V', W')\) such that \(V' : K \to \text{Aut}(G)\) is a homomorphism, \(W'(k_1, k_2) = 1\) for all \(k_1, k_2 \in K\), and \(\varphi_E\) is a section to the projection \(\pi\). Identifying \(g \in G\) with \(\iota(g) = (1, g)\) and \(k \in K\) with \(\varphi_E(k) = (k, 1)\), we have

\[ V(k)(g) = g^k, \quad (k \in K, g \in G), \]

so that \(\text{DT}_G(H, V(k)(H)) = \text{DT}_G(H, H^k)\), and thus

\[ P_G(H, V(k)(H)) = P_G(H, H^k). \]

Our claim follows now from Theorem 4.16. \(\blacksquare\)

Corollary 4.18. Let \(\widehat{G} = G_1 \times \cdots \times G_r\), where the \(G_\rho\) are finite groups, and set \(m := |\widehat{G}|\). Then we have

\[ \text{tp} (\widehat{G}) \leq \min_{1 \leq \rho \leq r} \text{tp}(G_\rho)^{m/|G_\rho|}. \] (4.12)

Proof. An immediate induction on \(r\), starting with the trivial case where \(r = 1\), reduces us to the case where \(r = 2\); that is, to

\[ \text{tp}(G \times K) \leq \min \{ \text{tp}(G)^{|K|}, \text{tp}(K)^{|G|} \}. \] (4.13)

Moreover, as \(\text{tp}\) is an isomorphism invariant by Corollary 3.16, it suffices to show for (4.12) that

\[ \text{tp}(G \times K) \leq \text{tp}(G)^{|K|} \]

However, assuming that \([G, K] = 1\), Corollary 4.17 gives

\[ \text{tp}(G \times K) \leq \min_{H \leq G} \prod_{k \in K} P_G(H) = \prod_{k \in K} \min_{H \leq G} P_G(H) = \text{tp}(G)^{|K|}, \]

whence the result. \(\blacksquare\)

It is easy to construct examples where inequality (4.12) is sharp. For instance, let \(G = S_3 \times C_p\), where \(p \geq 5\) is a prime number. Then the minimum on the right-hand side of (4.13) equals \(\text{tp}(S_3)^p = 2^{-p}\). Since \(p \nmid |S_3|\), all subgroups of \(G\) are of the form \(H = U \times C\), where \(U \leq S_3\) and \(C \leq C_p\). Discarding normal subgroups, and making use of Corollary 3.16, we see that, as regards \(\text{tp}(G)\), we only need to check the subgroups \(H_1 = \langle (1, 2) \rangle \times 1\) and \(H_2 = \langle (1, 2) \rangle \times C_p\), whose transversal probabilities are given by \(P_G(H_1) = 2^{-p}\) and \(P_G(H_2) = 1/2\), respectively. Hence,

\[ \text{tp}(G) = \min \{1, 1/2, 2^{-p} \} = 2^{-p}, \]

as desired.

5. Some problems and questions

In this final section we outline several open problems, conjectures, and questions, hoping thereby to stimulate further research on this topic.

Problem 5.1. Assume a block diagonal doubly stochastic matrix

\[ T = \text{diag}(T_1, \ldots, T_s) \]

is given where, for \(i = 1, \ldots, s\), the matrix \(T_i\) has order \(t_i\), and its entries all equal \(1/t_i\). Do there exist a finite group \(G\) and subgroups \(H, K\) of \(G\) (of the same order), such that \(\text{per}(T) = P_G(H, K)\)? If, in addition, we know that the number of \(t_i\)'s equal to 1 is non-zero, do there exist a finite group \(G\) and a subgroup \(H\), such that \(\text{per}(T) = P_G(H)\)?
Problem 5.2. Suppose that $H \leq G$ and $N \trianglelefteq G$, where $G$ is finite. Is it then always true that $P_G(H) \leq P_G(HN)$ (GAP computations confirm this up to $|G| = 200$)? Such a result would have an interesting consequence: combining it with Theorem 3.15, it would follow that, for any group homomorphism $f : G \to K$, and with $N := \ker(f)$,

\begin{equation}
P_G(H) \leq P_G(HN) = P_{f(G)}(f(HN)) = P_{f(G)}(f(H)),
\end{equation}

so that we would have a general inequality relating to the function $P(\cdot)$ associated with each homomorphism $f$. Moreover, it should be possible to characterise equality in (5.1).

Problem 5.3. Characterise equality in (4.13). In particular, equality should hold, if $(|G|, |K|) = 1$ (this seems plausible, and is supported by massive computational evidence).

Problem 5.4. What is the appropriate setting for the quantity $P_G(H)$ to make sense when $G$ is an infinite group and $H$ is a finite index subgroup of $G$?

Recall that $cp(G)$ is the commuting probability of $G$, i.e. the probability that two randomly chosen elements of $G$ commute. Regarding the relation between $tp(G)$ and $cp(G)$ we propose the following conjecture.

Problem 5.5. Let $G$ be a finite group. Then $tp(G) \leq cp(G)$ except if $G$ is Dedekind non-abelian, in which case $cp(G)/tp(G) = 5/8$, or $G = Q_{16}$ in which case $cp(G)/tp(G) = 7/8$. Moreover, if $cp(G) = tp(G)$, then either $G$ is abelian or

$$G \cong \langle a, b : a^3 = b^{2^n} = 1, a^b = a^{-1} \rangle$$

for some positive integer $n$ and thus $tp(G) = cp(G) = 1/2$.

Problem 5.6. Does Theorem 4.6 generalise? In particular, is the following true: Assume that

\begin{equation}
\prod_{i=1}^{n} f(t_i) = \prod_{j=1}^{k} f(s_j),
\end{equation}

where $t_i, s_j$ are (distinct) integers greater than 1, for all appropriate $i, j$ and $f(x) = \Gamma(x+1)/x^x$. Can we then conclude that $n = k$ and $t_i = s_i$ for all $i$ after rearranging appropriately?

Problem 5.7. Let $G$ be a finite group and suppose that $K \leq H \leq G$. In Proposition 3.14 we saw that $P_G(K) \leq P_H(K)$. Are $P_G(K)$ and $P_G(H)$ connected in some way? If so, how?

Finally, as regards the quantity $tp(G)$, we ask the following which, in our opinion, is the most important relevant question.

Problem 5.8. Let $G$ be a finite group. Does there exist a prime divisor $p$ of $|G|$ and a cyclic $p$-subgroup $H$ such that $tp(G) = P_G(H)$ and moreover $H$ has the property $(H : N) \leq p$, where $N = \text{core}_G(H)$?

We think the answer is “yes”.
APPENDIX A. MAJORISATION

We begin by recalling basic definitions and concepts from the theory of majorisation and refer the reader to the canonical work on this topic \cite{20} for further information. Towards the end of this section we will prove a result that we have appealed to in the proof of Theorem 4.6.

Fix a positive integer \(s\) and let \(\mathbb{R}_+ := [0, +\infty)\). For any \(x = (x_1, \ldots, x_s) \in \mathbb{R}^s\), let 
\[x[1] \geq \cdots \geq x[s]\]
denote the components of \(x\) in decreasing order, and let 
\[x_↓ = (x[1], \ldots, x[s])\]
denote the decreasing rearrangement of \(x\).

**Definition A.1.** For \(x, y \in \mathbb{R}^s\) we write \(x \preceq y\) and say that \(x\) is majorised by \(y\) (or that \(y\) majorises \(x\)) if
\[\sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i], \quad k = 1, \ldots, s-1, \quad \text{and} \quad \sum_{i=1}^{s} x[i] = \sum_{i=1}^{s} y[i].\]

Inequality in the final equality in the definition above leads to the concept of weak majorisation.

**Definition A.2.** For \(x, y \in \mathbb{R}^n\) we write \(x \prec_w y\) and say that \(x\) is weakly majorised by \(y\) (or that \(y\) weakly majorises \(x\)) if 
\[\sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i]\]
for all \(k \in [s]\).

We present now the fundamental concept of Schur-convexity/concavity.

**Definition A.3.** A real-valued function \(\phi\) defined on a set \(\mathcal{A} \subset \mathbb{R}^n\) is said to be Schur-convex on \(\mathcal{A}\) if \(x \preceq y\) on \(\mathcal{A}\) implies that \(\phi(x) \leq \phi(y)\). If, in addition, \(\phi(x) < \phi(y)\) whenever \(x < y\) but \(x\) is not a permutation of \(y\), then \(\phi\) is said to be strictly Schur-convex on \(\mathcal{A}\). Similarly, \(\phi\) is said to be Schur-concave on \(\mathcal{A}\) if \(x < y\) on \(\mathcal{A}\) implies that \(\phi(x) \geq \phi(y)\) and \(\phi\) is strictly Schur-concave on \(\mathcal{A}\) if strict inequality \(\phi(x) > \phi(y)\) holds when \(x\) is not a permutation of \(y\).

Schur’s fundamental result asserts that if \(\phi\) is a convex real function, then the function \(\sum_{i=1}^{s} \phi(x_i)\) is Schur-convex. Since we are interested in products, the following result helps make the transition from sums to products.

**Theorem A.4** (\cite[Prop. E1, pp. 105–106]{20}). Let \(f\) be a continuous non-negative function defined on an interval \(I \subset \mathbb{R}\). Then
\[h(x) = \prod_{i=1}^{s} f(x_i), \quad x \in I^s\]
is Schur-convex/concave on \(I^s\) if, and only if, \(\log f\) is convex/concave on \(I\). Moreover, \(h\) is strictly Schur-convex/concave on \(I^s\) if, and only if, \(\log f\) is strictly convex/concave on \(I\).
Of course, it follows from the above result that $\phi(x) = \prod_{i=1}^{n} g(x_i)$ is (strictly) Schur-concave if, and only if, $\log g$ is (strictly) concave.

Finally, we mention the following crucial result.

**Theorem A.5** ([20, A.9.a, p. 177]). Let $x, y \in \mathbb{R}^s$. If $x \prec_w y$, then there exists a vector $v$ such that $x < v$ and $v \leq y$; that is, we have $v_i \leq y_i$ for all $i \in [s]$.

For $x \geq 0$, recall that

$$f(x) := \frac{\Gamma(x+1)}{x^x},$$

where $\Gamma(x) = \int_{0}^{\infty} t^{x-1}e^{-t}dt$ is the $\Gamma$–function. Note that $\log f$ is strictly concave on $I = [0, +\infty)$ owing to the fact that $(\log f)'' < 0$ which we have established in section 3.4.

It is a consequence of Theorem A.4 and the comment immediately following it that, since $\log f(x)$ is strictly concave on $I = [0, +\infty)$, the function $h : I^s \to \mathbb{R}$ with

$$h(x_1, \ldots, x_s) = \prod_{\sigma=1}^{s} f(x_\sigma)$$

is strictly Schur-concave on $I^s$.

**Proposition A.6.** Let $x, y \in I^s$ and suppose that $x = x_1, y = y_1$. Assume that $x \prec_w y$, that is, $x_1 + \ldots x_i \leq y_1 + \ldots y_i$ for all $i \in [s]$. Suppose that the coordinates of $x$ are pairwise distinct. Moreover, assume that there exists a $k \in [s]$ such that $x_1 + \ldots x_i < y_1 + \ldots y_i$ for all $i \in [k, s]$, so that strict inequality holds in the weak majorisation order between $x$ and $y$ from a certain point on. Then $h(x) > h(y)$, where $h$ is as in (A.1).

**Proof.** By Theorem A.5 there exists a vector $v$ such that $x \prec v$ and $v \leq y$. Since $x \prec v$ and $h$ is strictly Schur-concave, we have that $h(x) \succ h(v)$. We justify this claim. We certainly have $h(x) \succeq h(v)$. If we had $h(x) = h(v)$, then (by definition) $v$ would have to be a permutation of $x$. But $x$ has distinct coordinates so we would have $x = v$ and thus $x_i \leq y_i$ for all $i$. Now $f$ is strictly decreasing for $x \geq 1$ thus $f(x_i) \succeq f(y_i)$ and moreover at least one strict inequality holds by the initial assumption on $k$. Therefore

$$h(x) = h(x_1, \ldots, x_s) = \prod_{\sigma=1}^{s} f(x_\sigma) > \prod_{\sigma=1}^{s} f(y_\sigma) = h(y),$$

as wanted.

We may therefore assume that $h(x) > h(v)$. A similar argument as above shows that $v \prec y$ plus monotonicity of $f$ implies that $h(v) \prec h(y)$. In conclusion we have $h(x) > h(v) \succeq h(y)$, proving our assertion.

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