Deformed Heisenberg algebra: origin of q-calculus

P. Narayana Swamy

Professor Emeritus, Department of Physics, Southern Illinois University, Edwardsville, IL 62026-1654

Abstract

The intimate connection between q-deformed Heisenberg uncertainty relation and the Jackson derivative based on q-basic numbers has been noted in the literature. The purpose of this work is to establish this connection in a clear and self-consistent formulation and to show explicitly how the Jackson derivative arises naturally. We utilize a holomorphic representation to arrive at the correct algebra to describe q-deformed bosons. We investigate the algebra of q-fermions and point out how different it is from the theory of q-bosons. We show that the holomorphic representation for q-fermions is indeed feasible in the framework of the theory of generalized fermions. We also examine several different q-algebras in the context of the modified Heisenberg equation of motion.

PACS 02.20.Uw, 05.30.-d, 05.90.+m

Typeset using REVTeX
I. INTRODUCTION

Representations of the q-deformed Heisenberg algebra have received much attention in the literature [1] and its connection with the q-calculus based on basic numbers have been investigated by many authors [2–6]. However, the origin of q-calculus has not been established satisfactorily in a self-consistent formulation. Past investigation on this subject has not fully clarified what the assumptions are and what consequences follow from them. In particular, it has not been clearly established that the q-modified Heisenberg uncertainty relation directly leads to the q-algebra of bosons and to the Jackson Derivative (JD) [7]. Moreover, only a few of the authors employ the formulation which is symmetric under $q \leftrightarrow q^{-1}$. The consequences of q-deformed Heisenberg equation of motion, again, have not been established in a comprehensive manner. In the context of isomorphic or holomorphic representations of q-algebras, the role of q-calculus in the theory of q-fermions has been omitted in the literature. In this work we propose to remedy this situation.

In Sec.II, we shall develop \textit{ab initio} a self-consistent and self-contained formulation to establish the connection between basic numbers, the algebra of q-oscillators and the JD. We shall demonstrate that the q-boson algebra is intimately connected with the q-deformed Heisenberg relation and that the JD is intimately linked to the q-boson algebra. In Sec.III, we investigate the theory of q-fermions and show that the q-fermion algebra does not share these connections. We shall show that the theory of generalized fermions, extending beyond the exclusion principle, provides a convenient framework to introduce a holomorphic representation, even though it has unexpected classical limits. Sec.IV is devoted to a study of the consequences of the modified Heisenberg equation of motion. We shall investigate the important role played by the modified Heisenberg equation of motion in the algebra of both q-bosons and q-fermions. It will be shown that the formulation of q-fermions arises more naturally from the theory of generalized fermions which do not obey the familiar exclusion principle. Sec.V provides a brief summary.

First we begin with the dilatation operator

$$\theta \equiv x \frac{\partial}{\partial x}, \quad (1)$$

which evidently satisfies the properties

$$\theta x = x(\theta + 1), \quad \theta^r x = x(\theta + 1)^r, \quad f(\theta) x = xf(\theta + 1), \quad (2)$$

where $r$ is any number and $f$ is an arbitrary polynomial. It is easy to show that such properties extend to any monomial of a real variable $q$, thus

$$q^\theta x = xq^{\theta+1}. \quad (3)$$

This can be further extended to the basic numbers (bracket numbers) so that
where the basic numbers are defined by

\[
[\alpha] \equiv \frac{q^{\alpha} - q^{-\alpha}}{q - q^{-1}},
\]

in terms of \(q\), an arbitrary real number, \(0 < q < \infty\). In the symmetric formulation one can further restrict it to \(0 < q < 1\) or \(1 < q < \infty\). This formulation is designed to be symmetric under \(q \leftrightarrow q^{-1}\). The JD forms the basis [8] of what is referred to in the literature as q-calculus and is also intimately connected with the basic numbers which play a fundamental role. We shall now establish that the JD and the entire framework of q-calculus has its origin in the q-deformed Heisenberg uncertainty relation.

II. Q-HEISENBERG ALGEBRA AND JD: Q-BOSONS

Let us consider the q-deformed uncertainty principle, the q-Heisenberg relation,

\[
q \, x p - p x = i \hbar \Delta.
\]

There are at least two choices for \(\Delta\) that would preserve the correct limit (“classical limit”) when \(q \rightarrow 1\). The choice \(\Delta = 1\) has been studied by Finkelstein [2]. We shall choose \(\Delta = q^{-N}\), where \(N\) is to be specified later, in order to develop the formulation which is symmetric under \(q \leftrightarrow q^{-1}\). We accordingly introduce an ansatz or a hypothesis,

\[
q \, x p - p x = i \hbar q^{-N}.
\]

For \(q \neq 1\), the momentum \(p = -i \hbar \partial_x\) has to be replaced by the generalized operator \(p = -i \hbar D\), and accordingly,

\[
Dx - q \,xD = q^{-N}.
\]

We seek the solution of Eq.(8) for the operator \(D\). We can show that the solution [2] is

\[
D = \frac{1}{x} \frac{q^N - q^{-N}}{q - q^{-1}}.
\]

This also renders the deformation (7) unique. The basic number occurring here, defined in Eq.(5) forms the basis of the algebra of q-deformed oscillators. The proof is straightforward and utilizes the property \(x [N + 1] = [N] x\). An alternative proof which is instructive, consists of first showing
\[ [N] x = x [N + 1] = x q [N] + x q^{-N}, \]  

which leads to

\[ [N] x - q x [N] = x q^{-N}. \]  

From Eqs.(8,11) the solution, \( D = x^{-1} [N] \) follows immediately. This is the fundamental connection between the Jackson derivative and the q-basic number: the q-deformation of the uncertainty relation thus automatically leads to the JD via the basic number. In order to show that Eq.(9) is indeed the same as the familiar generalized derivative in the standard form, we proceed as follows.

Continuing from Eq.(3) we can establish the further property,

\[ \theta x^r = r x^r, \]  

for any number \( r \), which can be extended to a monomial, thus

\[ \theta^a x^r = r^a x^r. \]  

Upon using the series

\[ q^\theta = 1 + \theta \ln q + \frac{\theta^2}{2!} (\ln q)^2 + \cdots, \]  

we obtain, for a monomial,

\[ q^\theta x^r = x^r q^r = (q x)^r. \]  

This relation immediately generalizes to a polynomial and we obtain Eq.(12) for any polynomial function. We now make the observation [2,5] that there exists the holomorphic representation

\[ \theta \iff N, \quad \theta = x \partial_x \iff a^\dagger a = N, \quad \text{or} \quad \theta = x D \iff a^\dagger a = [N], \]  

where \( N \) is the number operator of q-deformed bosons and \( a, a^\dagger \) denote the annihilation and creation operators of q-boson oscillators. The property, Eq.(15), is also valid for an arbitrary polynomial function, where \( \theta \) can be either \( N \) or \( [N] \). If we choose \( \theta = N \), we derive the property

\[ q^N f(x) = f(q x), \]  

which can be further extended to the result
\[ q^{-N} f(x) = f(q^{-1}x). \] (18)

From this we may immediately obtain the important result
\[ Df(x) = \frac{1}{x} \frac{q^N - q^{-N}}{q - q^{-1}} f(x) = \frac{1}{x} \frac{f(qx) - f(q^{-1}x)}{q - q^{-1}}, \] (19)

which is recognized as the standard definition [8,9] of the JD in the symmetric formulation. It is clear that the JD reduces to the ordinary derivative in the limit \( q \to 1 \). Hence we have established the connection between the \( q \)-Heisenberg relation on one hand and the \( q \)-basic number and JD on the other hand in the symmetric formulation. Consequently, the introduction of JD arises naturally from the \( q \)-deformed Heisenberg relation via the \( q \)-basic number.

Let us now consider \( q \)-bosons and the JD which obeys the relation (8). If we choose the representation \( D \Rightarrow a, \ x \Rightarrow a^\dagger \) as in (16), then we immediately obtain
\[ aa^\dagger - qa^\dagger a = q^{-N}. \] (20)

Accordingly, the algebra of creation and annihilation operators of the \( q \)-deformed oscillators can be regarded as arising from a representation of the coordinate and the JD. Thus in this sense, the \( q \)-deformed algebra of bosons is an immediate consequence of the deformed Heisenberg algebra.

We conclude this section by recording some results for later use,
\[ q^N a = a q^{N-1}, \quad [N] a = a [N - 1], \quad [N + 1] = q [N] + q^{-N}. \] (21)

**III. Q-FERMIONS**

We shall now turn our attention to \( q \)-fermions. The statistical mechanics and thermodynamics of \( q \)-deformed fermions has been investigated thoroughly [10] on the basis of \( q \)-deformed algebra. Let us specifically consider the basic number as defined in Eq.(5) while the creation and annihilation operators satisfy [10] the algebra
\[ bb^\dagger + q^{-1} b^\dagger b = q^{-N}. \] (22)

This formulation which leads to the determination of many thermodynamic functions of \( q \)-bosons and \( q \)-fermions as well as to interesting predictions, displays a clearly desirable symmetry among fermions and bosons and moreover the basic number used for fermions is exactly the same as for bosons. When we examine the algebra described by Eqs.(5) and (22), together with the Heisenberg relation, we conclude that the modified Heisenberg uncertainty relation such as in Eq.(7), or one...
of its variants, does not provide a representation for the JD (9) to produce the q-fermion algebra (22). In other words, there does not exist any useful holomorphic representation for the JD, as in Eq.(16), such as \( x \leftrightarrow b^\dagger, \partial_x \leftrightarrow b \) that will produce the algebra in Eq.(22). On the other hand it should be noted that the JD in the case of q-fermions still arises from the q-deformed Heisenberg relation as was shown before. In the theory based on this algebra of q-fermions, the eigenvalues of the number operator \( N \) can take the values \( n = 0, 1 \) only. Thus the theory obeys Pauli exclusion principle just as in the case of undeformed fermions. Furthermore, it can be shown that algebras such as above can be transformed to the case of undeformed ordinary fermions [11,12] and thus it would appear that such algebras may not represent genuine deformations or generalizations. Such transformations themselves might, however, be of some interest [13].

There exists another interesting representation resulting in a different q-deformed fermion algebra which we shall now investigate. Let us introduce the definition of the fermion basic numbers [12] by

\[
[z]_F = \frac{q^{-z} - (-1)^z q^z}{q + q^{-1}},
\]

which has many useful and interesting properties. In this case we obtain the result that the equation analogous to Eq.(8) for the boson case,

\[
B x + q x B = q^{-N},
\]

has the solution given by \( B = x^{-1} [N]_F \). This can be verified as follows. First we confirm that the properties in Eqs.(1-3) and (12-15) are also valid in the fermion case when we use the generalized basic numbers, Eq.(23). We need an extension of these basic properties. By writing \((-q)^N\) as \((e^{i\pi q})^N\) and using the series form for the exponential, we derive

\[
(-q)^N x = x(-q)^{N+1}.
\]

It should be stressed that \( N \) is an operator and the above equation is in no conflict with Eq.(3). Consequently, we indeed have the holomorphic representation \((1/x)[N]_F \leftrightarrow b, x \leftrightarrow b^\dagger\) which leads to

\[
bb^\dagger + qb^\dagger b = q^{-N},
\]

as can be readily verified. We can also provide a direct proof for the solution of Eq.(24). The explicit proof begins with the observation that \([N]_F x = x [N + 1]_F\), together with the property \(q[N] + [N + 1] = q^{-N}\) and then showing

\[
q x \frac{1}{x} [N]_F + \frac{1}{x} x [N + 1]_F = q^{-N}.
\]
Eq.(26) follows immediately. Some observations are in order here. The algebra (26) which can be regarded as resulting from the representation stated above, arises from the special fermion basic number defined in Eq.(23), which is precisely the one introduced by Chaichian et al [12]. Secondly, it is remarkable that this fermion basic number readily provides a consistent holomorphic representation in terms of the creation and annihilation operators for the q-fermions, hence providing a fundamental justification for the definition (23) itself. Thirdly, we note that this q-fermion algebra is not based on the q-Heisenberg relation. The connection with the q-Heisenberg relation is valid for q-bosons only. In spite of appearances, the operator $B$ appearing in Eq.(24) has no relation to the JD, which has its connection only to the q-bosons; it does not have any resemblance to the ordinary derivative in the classical limit. Finally we observe that this leads to a generalization of fermions [12] in the sense that this q-deformed algebra goes beyond the Pauli exclusion principle and the eigenvalues of $N$ are arbitrary and not restricted to the values 0, 1. It must be stressed that the fermion basic number introduced above does not have the expected classical limit, \( \lim_{q \to 1} [N]_F \neq N \). Indeed we find that when \( q \to 1 \),

\[
[N]_F \to \frac{1 - (-1)^N}{q + q^{-1}} \to \frac{1}{2}(1 - (-1)^N),
\]

which reduces to unity when $N$ is odd and vanishes when $N$ is even. The limits in more general cases are described in ref. [12]. We reiterate that while the q-boson algebra, the q-deformed Heisenberg relation and the JD are intimately connected, the q-calculus for q-fermions has a different origin and is not related to the q-Heisenberg relation.

IV. HEISENBERG EQUATION OF MOTION AND Q-ALGEBRA

The Heisenberg equation of motion (H-equation)

\[
\dot{F} = \frac{\partial F}{\partial t} = [F, H] = FH - HF,
\]

determines the time evolution of an operator, and hence the dynamics, in standard quantum mechanics. In a q-deformed theory, we expect this to be modified according to a form such as

\[
\dot{F} = FH - qHF,
\]

which is not unique since many variations of this form might have the same classical \( q \to 1 \) limit. If we consider the boson oscillator Hamiltonian, \( H = \frac{1}{2}\hbar \omega (a^\dagger a + a a^\dagger) \) in the above equation, together with the q-boson algebra \( aa^\dagger - qa^\dagger a = 1 \) in the nonsymmetric formulation, as in ref. [2] we obtain

\[
\dot{a} = aH - qHa = \hbar \omega a,
\]

7
which then yields \( a \sim e^{-i\omega t} \) and everything is consistent. On the other hand, in the formulation symmetric under \( q \rightarrow q^{-1} \), we might work alternatively with the undeformed H-equation,

\[
\dot{F} = [F, H],
\]

as a variation of the above, together with the algebra \( aa^\dagger - qa^\dagger a = q^{-N} \) and the Hamiltonian of q-bosons, \( H = \frac{1}{2}\hbar \omega ([N] + [N + 1]) \). We then obtain in this case,

\[
\dot{a} = \frac{1}{\hbar} \omega a, \quad q^N + q^{-N-1} \quad e^{-i\omega t},
\]

which has the solution

\[
a = a_0 \frac{q^N + q^{-N}}{2} e^{-i\omega t},
\]

where the constant \( a_0 \) is independent of \( q \). On the other hand if we modify the H-equation as in Eq.(31), we obtain, for the same theory of q-bosons,

\[
a = a_0 \frac{q^{-N-1} + q^{-N}}{2} e^{-i\omega t}.
\]

Thus there is no significant difference from (34) in the time dependence or the \( q \)-dependence in these variants. While the deformed H-equation is thus non-unique, for the sake of definiteness, we shall henceforward make an arbitrary choice and adhere consistently to the following modified H-equation:

\[
\dot{F} = FH - q^{-1}HF.
\]

First let us consider the system of q-bosons described by

\[
H = \frac{1}{2}\hbar \omega (a^\dagger a + aa^\dagger) = \frac{1}{2}\hbar \omega ([N] + [N + 1]).
\]

If we employ the deformed H-equation, Eq.(36), we obtain

\[
\dot{a} = \frac{1}{\hbar} \omega \left\{ a ([N] + [N + 1]) - q^{-1}([N] + [N + 1]) a \right\}
= \frac{1}{2}\hbar \omega a \left( q^N + q^{-N} \right).
\]

Consequently the time dependence is given by
\[ a = a_0 \frac{q^N + q^{N-1}}{2} e^{-i\omega t}, \]

where \( a_0 \) is independent of \( q \).

We shall next consider the case of q-fermions. The system of “standard” q-fermions as in ref. [10] is described by the Hamiltonian,

\[ H = \frac{1}{2} \hbar \omega \left( a a^\dagger - a^\dagger a \right) = \frac{1}{2} \hbar \omega \left( [N] - [1 - N] \right), \]

which has the “correct” classical limit \( q \to 1, \lim H = \pm \frac{1}{2} \hbar \omega \), corresponding to the eigenvalues \( N = 0, 1 \). If we employ Eq.(36), we obtain in this case,

\[ i\hbar \omega \dot{a} = \frac{1}{2} \hbar \omega \left\{ a ([N] - [1 - N]) - q^{-1} ([N] - [1 - N]) a \right\} \]
\[ = \frac{1}{2} \hbar \omega \left\{ q^{N-1} + q^{N-2} \right\}. \]

Accordingly the time dependence is

\[ a = a_0 \frac{q^{N-1} + q^{N-2}}{2} e^{-i\omega t}, \]

in accordance with expectations.

Let us finally consider the Chaichian formulation [12] of generalized q-fermions which is described by

\[ a^\dagger a = [N]_F, \quad aa^\dagger = [N + 1]_F, \quad H = \frac{1}{2} \hbar \omega \left( [N]_F - [1 + N]_F \right). \]

We note the classical limit, \( q \to 1 \), of the Hamiltonian is \( H \to H_0 = \frac{1}{2} \hbar \omega \), in contrast to the case of the standard version where

\[ H_0 = \frac{1}{2} \hbar \omega (2N - 1) = \pm \frac{1}{2} \hbar \omega. \]

Consequently, this system not only describes generalized fermions which are not constrained by the exclusion principle, it also does not behave like the standard system of fermions in the classical limit. Upon employing the H-equation, Eq.(36), we obtain the result

\[ a = a_0 \frac{(-q)^N - (-q)^{N-1}}{2} e^{-i\omega t}, \]
which is significantly different from (42) in terms of the q-dependence. To see that it is so, we might examine the classical limit, \( q \to 1 \), and obtain \( a = a_0(-1)^N e^{-i\omega t} \). It looks similar to the expected classical limit but the phase alternates between 1 and \(-1\) for the allowed values \( N = 0, 1, 2, \ldots \infty \). However, the theory is self-consistent. Hence we conclude that in each of the above three cases, the deformed H-equation leads to acceptable time evolution of the creation and annihilation operators consistent with the corresponding q-algebra and the appropriate definition of the basic numbers. In particular, the time dependence of the annihilation and creation operators depend implicitly on \( q \) and \( N \), where the latter quantity in turn depends on \( q \).

V. SUMMARY

In this work we have shown explicitly that the q-deformed Heisenberg uncertainty relation, together with the q-deformed algebra of boson oscillators leads naturally to the Jackson Derivative. We have established this intimate connection in a self-contained and comprehensive formulation, symmetric under \( q \leftrightarrow q^{-1} \). We also showed that the q-deformed boson algebra itself can be regarded as arising from a holomorphic representation for the creation and annihilation operators. We have thus established the basis of q-calculus as a direct consequence of the basic numbers.

Upon examining the algebra of q-fermions we concluded that it has no direct link with the q-deformed Heisenberg relation. For the q-fermions, the fermion basic numbers are connected to the q-fermion algebra by a holomorphic representation in a self-consistently formulated theory of generalized fermions which go beyond the exclusion principle. Finally we have examined the Heisenberg equation of motion, deformed for the purpose of describing q-deformed systems. We have studied three different algebras in this context and we have shown that the q-bosons as well as the generalized fermions do possess desirable properties.

ACKNOWLEDGMENTS

I thank A. Lavagno of Politecnico di Torino, Italy, for fruitful discussions on the subject of the Jackson derivative and q-bosons and q-fermions.
REFERENCES

[1] L. Hellstrom and S. Silvestrov, *Commuting Elements in q-deformed Heisenberg Algebras*, World Scientific (2000), Singapore. This contains extensive references on the subject.

[2] R. Finkelstein, *Int. J. Mod. Phys. A* 13 (1998) 1795-1803; R. Finkelstein and E. Marcus, *J. Math. Phys.* 36 (1995) 2652-2672; arXiv:hep-th/9906137.

[3] A. Hebecker et al, *Z. Phys. C* 64, (1994) 355-359.

[4] J. Schwenk and J. Wess, *Phys. Lett. B* 291 (1992) 273.

[5] E. Floratos, *J. Phys. A: Math. Gen.* A 24, (1991) 4739-4750.

[6] B. Cerchiai et al, *Eur. Phys. J. C* 8, (1999) 547-558.

[7] F. Jackson, *Mess. Math.* 38, (1909) 57

[8] H. Exton, *q-Hypergeometric Functions and Applications*, Ellis Horwood Limited, 1983. The basic numbers and the JD are in the non-symmetric formulation in this work.

[9] A. Lavagno and P. Narayana Swamy *Phys. Rev. E* 61 (2000) 1218-1226

[10] A. Lavagno and P. Narayana Swamy, *Phys. Rev. E* (2002) 036101.

[11] K.S. Viswanathan, R. Parthasarathy and R. Jagannathan, *J. Phys.* A25, L335 (1992); S. Jing and J. Xu, *J. Phys. A: Math. Gen A* 24, (1991), L891.

[12] M. Chaichian et al, *J. Phys. A: Math. Gen.* A 27 (1994), 2045-2051.

[13] P. Narayana Swamy, *Mod. Phys. Lett. B* 15, (2001) 915-920.