Thermodynamics of the inhomogeneous LTB model: Generalized Bekenstein-Hawking System

Subhajit Saha
Subenoy Chakraborty

Department of Mathematics, Jadavpur University, Kolkata-700032, India.

Abstract

The present work deals with three alternative generalized Bekenstein-Hawking formulation of thermodynamical parameters namely entropy and temperature for the universal thermodynamical system bounded by a horizon in the frame work of inhomogeneous LTB model of the universe. The first law of thermodynamics holds for all the three choices while we need some restrictions for the validity of the generalized second law of thermodynamics. Finally, it is found that due to some thermal fluctuations there is a logarithmic correction to the Bekenstein entropy.

Keywords: Inhomogeneous LTB model, Thermodynamical laws, Generalized Hawking temperature, Modified Bekenstein entropy.

Pacs numbers: 98.80.Cq, 98.80.-k

I. INTRODUCTION

In 1975, Hawking [1] was able to show radiation from black holes using semi-classical description. Since then black hole(BH) is considered as a thermodynamical object, i.e., laws of BH physics and thermodynamical laws are equivalent [2,3]. Subsequently, there were a lot of works dealing with thermodynamical studies of the universe bounded by apparent horizon bounded by apparent horizon [4-11] or by the event horizon [12-15], specially for homogeneous and isotropic FRW model of the universe. Also for inhomogeneous LTB model [16], Einstein field equations were shown to be equivalent with the unified first law of thermodynamics on the apparent horizon and generalized second law of thermodynamics(GSLT) has been shown to be valid both on the apparent and on the event horizon with some restrictions.

In 2006, Wang etal [11], based on a comparative study of the thermodynamical laws for FRW model of the universe bounded by apparent and event horizons, concluded that universe bounded by apparent horizon is a perfect thermodynamical system(Bekenstein system) while both first and second law of thermodynamics break down on the event horizon, i.e., a unphysical system. Very recently, a generalized Hawking temperature(which coincides with the Hawking temperature on the apparent horizon) or a modified Bekenstein entropy has been introduced [17] on the event horizon and it is possible to show the validity of both the thermodynamical laws on the event horizon for any fluid system in FRW model. In the present work, we make an attempt to extend the idea of generalized Hawking temperature or modified Bekenstein entropy for inhomogeneous LTB model and examine the validity of both the thermodynamical laws on any horizon. Current supernova and some other data [18] provides a justification in support of the inhomogeneous LTB model of the universe. Also very recently, using perturbation analysis, Clarkson and Marteens [19] has given a justification for the

1subhajit1729@gmail.com
2schakraborty@math.jdvu.ac.in
inhomogeneous model.

II. BASIC EQUATIONS IN THE INHOMOGENEOUS LTB MODEL

The line element for inhomogeneous spherically symmetric Lemaitre-Tolman-Bondi (LTB) space-time in a co-moving frame is given by

\[ ds^2 = h_{ab}dx^a dx^b + R^2 d\Omega^2 \]

where a, b can take values 0 and 1, the two dimensional metric tensor \( h_{ab} \) (known as normal metric) is given by

\[ h_{ab} = \text{diag}[-1, \frac{R'^2}{1 + f(r)}] \]

with \( x^a \) being the associated co-ordinates \( (x^0 = t, x^1 = r) \). Here \( R = R(r,t) \) is the area radius of the spherical surface (a scalar field in the normal 2D space) and the curvature scalar \( f(r) \) classifies the space-time as follows:

a) bounded: \(-1 < f(r) < 0\)
b) marginally bounded: \( f(r) = 0 \)
c) unbounded: \( f(r) > 0 \).

Now by introducing the mass function \( F(r,t) \) (related to the mass within the co-moving radius \( 'r' \)) as

\[ F(r,t) = R(\dot{R}^2 - f(r)) \]

the Einstein field equations read

\[ 8\pi G\rho = \frac{F'(r,t)}{R^2 R'}, \quad 8\pi Gp = -\frac{\dot{F}(r,t)}{R^2 R} \]

and the evolution equation for \( R \) is

\[ 2R \ddot{R} + \dot{R}^2 + 8\pi GpR^2 = f(r). \]

In the above, \( \rho \) and \( p \) are the energy densities and thermodynamic pressure corresponding to perfect fluid having energy-momentum tensor

\[ T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \]

where the fluid 4-velocity \( u^\mu \) is normalized by \( u_\mu u^\mu = -1 \). The energy-momentum conservation relation \( T_{\mu;\nu} = 0 \) for the above space-time model results

\[ \dot{\rho} + 3H(\rho + p) = 0, \quad p' = 0 \]

with \( H = \frac{1}{3}(\dot{R}^2 + 2\ddot{R}) \) as the Hubble parameter. We now introduce another relevant scalar quantity on this normal space as

\[ \chi(x) = h^{ab}\partial_a R\partial_b R = 1 - \frac{F(r,t)}{R}. \]

Usually, the apparent horizon is defined at the vanishing of this scalar, i.e., \( \chi(x) = 0 \), which gives

\[ R_A = F(r,t). \]

So the surface gravity at the apparent horizon is given by

\[ \kappa_A = -\frac{1}{2} \frac{\partial \chi}{\partial R} \bigg|_{R=R_A} = -\frac{1}{2R_A} \]
and hence the Hawking temperature at the apparent horizon is

$$T_A = \frac{|\kappa_A|}{2\pi} = \frac{1}{4\pi R_A}. \quad (11)$$

In the literature, the Hawking temperature at any horizon (say $R = R_h$) is defined similar to that at the apparent horizon (i.e., equation (11)) as

$$T_h = \frac{1}{4\pi R_h}. \quad (12)$$

### III. VALIDITY OF THE FIRST AND THE SECOND LAWS OF THERMODYNAMICS

In the present work we start with the definition of surface gravity at the horizon as

$$\kappa_h = -\frac{1}{2} \frac{\partial \chi}{\partial R}\bigg|_{R=R_h} = -\frac{F}{2 R_h^2} \quad (13)$$

and the modified Hawking temperature on the horizon is

$$T_h^{(m)} = \frac{|\kappa_h|}{2\pi} = \frac{F}{4\pi R_h^2} = \frac{R_A}{4\pi R_h^2}. \quad (14)$$

It is to be noted that in contrary to FRW model [15], if $R_h > R_A$ then $T_h^{(m)} < T_A$.

To evaluate the energy flow across the horizon, we consider the unified first law on the horizon [16,21], i.e.,

$$dE_h = A\Psi + WdV, \quad (15)$$

where $A = 4\pi R_h^2$, $V = \frac{4}{3}\pi R_h^3$ are respectively the area and volume bounded by the horizon, $\Psi = \psi_a dx^a$, $\psi_a = T_a^b \partial_b R + W \partial_a R$ is the energy flux (or momentum density) and $W = -\frac{1}{2} \text{Trace}(T)$ is the work function, with trace over the 2D space normal to the spheres of symmetry. For the present model, $\Psi = -\frac{1}{2}(\rho + p)(R' dt - \dot{R} dr)$, $W = \frac{1}{2}(\rho - p)$ and hence

$$dE_h = 4\pi R^2 [\rho R' dr - p \dot{R} dt]|_{R=R_h} = \frac{1}{2G} dF, \quad (16)$$

where the last equality is obtained by using the Einstein field equations (4).

Now, if we use the Bekenstein’s entropy-area relation on the given horizon and the temperature on the horizon (known as generalized Hawking temperature) as [17]

$$T_h^{(g)} = \alpha T_h^{(m)} \quad (17)$$

where the parameter $\alpha$ has the value unity on the apparent horizon (so that entropy and temperature on the apparent horizon are that of Bekenstein’s entropy and Hawking temperature respectively). Then to satisfy the Clausius relation, $\alpha$ turns out to be

$$\alpha = \frac{dR_A/R_A}{dR_h/R_h}. \quad (18)$$

The above expression of $\alpha$ for the present inhomogeneous model of the universe is a generalization to that for homogeneous FRW model [17]. Further, the interpretation of $\alpha$ remains the same, i.e., $\alpha^{-1}$ gives the ratio of the relative growth rate of the radius of the given horizon to that of the apparent horizon [17]. Thus we have generalized the modified Hawking temperature, keeping the Bekenstein entropy-area relation unchanged and we are able to show the validity of the first law of thermodynamics on any horizon, irrespective of any fluid distribution.
Now, to examine the validity of the generalized second law of thermodynamics (GSLT) we start with the Gibb’s law [11,22], to find the entropy variation of the bounded fluid distribution:

\[ T_{fl} dS_{fl} = dE + pdV \]  

(19)

where \( T_{fl} \) and \( S_{fl} \) are respectively the temperature and entropy of the given fluid distribution, \( V = \frac{4\pi}{3} R_h^3 \) and \( E = \rho V \). The above equation explicitly takes the form

\[ T_{fl} dS_{fl} = 4\pi R_h^3 (\rho + p) \left\{ \frac{\dot{R}_h}{R_h} - H \right\} dt + \left\{ \frac{R_h'}{R_h} - \frac{1}{3} \frac{\omega'}{\omega (1 + \omega)} \right\} dr \]  

(20)

where we have assumed the barotropic equation of state \( p = \omega \rho \) for the bounding fluid.

Also using the first law (i.e., Clausius relation), \( T_h dS_h = dE_h \), where \( E_h \) is given by equation (16). We have,

\[ T_h dS_h = 4\pi R_h^3 \rho \left\{ \frac{\dot{R}_h}{R_h} dt + \frac{R_h'}{R_h} dr \right\} \]  

(21)

Now for equilibrium configuration, we assume \( T_{fl} = T_h \), i.e., the inside fluid has the same temperature as the bounding surface and we obtain

\[ T_h dS_T = 4\pi R_h^3 \rho \left\{ \frac{\dot{R}_h}{R_h} - (1 + \omega) H \right\} dt + \left\{ (2 + \omega) \frac{R_h'}{R_h} - \frac{1}{3} \frac{\omega'}{\omega} \right\} dr \]  

(22)

with \( S_T = S_{fl} + S_h \), the total entropy of the universal system. Thus for the validity of the GSLT, we must have,

i) \( \frac{\dot{R}_h}{R_h} > (1 + \omega) H \) and ii) \( (2 + \omega) \frac{R_h'}{R_h} > \frac{1}{3} \frac{\omega'}{\omega} \), according as \( dr > 0 \) or < 0,

i.e., GSLT depends on the evolution of the matter as well as on the evolution of the horizon radius.

**IV. MODIFIED BEKENSTEIN ENTROPY: LOGARITHMIC CORRECTION**

We shall now examine the following two possible modifications of entropy and temperature on the horizon so that Clausius relation holds on the horizon, i.e.,

a) \( S_h = S_h^{(m)} = \beta S_h^{(B)} , T_h = T_h^{(m)} \) and b) \( S_h = S_h^{(m)} = \delta S_h^{(B)} , T_h = \frac{1}{\delta} T_h^{(m)} \)

where as before \( \beta \) and \( \delta \) are chosen to be unity on the apparent horizon and \((S_h^{(B)}, S_h^{(m)}, T_h^{(m)})\) respectively denote the Bekenstein entropy, modified Bekenstein entropy and modified Hawking temperature (given by equation (14)) on the horizon. Then for the choice (a), in order to satisfy the Clausius relation, \( \beta \) turns out to be (in integral form) as

\[ \beta = \frac{2}{R_h^2} \int R_h^2 \frac{dR_A}{R_A} \]  

(23)

One may note that for the choice (a), if we consider generalized Hawking temperature (see equations (17) and (18)) instead of \( T_h^{(m)} \), then validity of Clausius relation results \( \beta \) to be unity, i.e, we get back to the original system (i.e, \( S_h = S_h^{(B)}, T_h = T_h^{(g)} \)) discussed above. Further, we can proceed exactly in the same way as above to examine the validity of the GSLT for this choice (i.e., choice (a)). As the result will be very similar to the previous case so we are not presenting it here.

For the choice (b) validity of the first law of thermodynamics demands the parameter \( \delta \) to be \( \frac{R_A^2}{R_h^2} \) and

\[ 4 \]
as a result the entropy and temperature on the horizon coincides with the Bekenstein entropy and Hawking temperature on the apparent horizon, i.e., entropy and temperature turn out to be constant. So this choice of entropy and temperature on the horizon is not of much physical interest and we are not considering this choice further.

We now consider an interesting phenomena related to choice (a): Suppose due to some thermal fluctuations the apparent horizon is slightly displaced so that in the new position the area of the horizon $A_f$ is deviated from the original area of the apparent horizon $A_A$ by an infinitesimal quantity $\epsilon$, i.e.,

$$A_f = A_A + \epsilon$$

(24)

or equivalently, $R_f^2 = R_A^2 + \frac{\epsilon}{4\pi}$.

Then $\beta$ simplifies (upto first order) to

$$\beta = 1 - \frac{\epsilon'}{\pi R_A^2} + \frac{2\epsilon'}{\pi R_A^2} \ln R_A, \quad \epsilon' = \frac{\epsilon}{4}$$

(25)

and we have

$$S_{Af} = S_A^{(B)} + \frac{2\epsilon'}{G} \ln R_A$$

(26)

and

$$T_A = \frac{R_A}{4\pi R_A^2} (1 - \frac{\epsilon}{4\pi R_A^2}) = \frac{1}{4\pi R_A} - \frac{\epsilon}{16\pi^2 R_A^3}$$

(27)

Thus we have a logarithmic correction to the Bekenstein entropy and correction to the Hawking temperature is proportional to $R_A^{-3}$. It is to be noted that if instead of choosing the deviation of the area by an infinitesimal quantity, the radius of the horizon deviates infinitesimally, i.e., $R_f = R_A + \epsilon$, then we have a power-law correction to the Bekenstein entropy.

V. CONCLUSIONS

Thus the present work deals with thermodynamical analysis of the universe bounded by a horizon for inhomogeneous LTB model. At first, we define the modified Hawking temperature on the horizon and then examine the validity of the first law of thermodynamics for three possible choices of entropy and temperature on the horizon namely,

i) $T_h = T_h^{(g)} = \alpha T_h^{(m)}, \quad S_h = S_h^{(m)} = S_h^{(B)}$;

ii) $S_h = S_h^{(m)} = \beta S_h^{(B)}, \quad T_h = T_h^{(m)}$;

iii) $T_h = \frac{1}{\delta} T_h^{(m)}, \quad S_h = S_h^{(m)} = \delta S_h^{(B)}$.

For all the three choices of temperature and entropy, first law of thermodynamics has been shown to be valid on the horizon irrespective of any fluid distribution and the value of the parameters can be determined uniquely. However, choice (iii) is not of much physical interest as temperature and entropy becomes constant (to that at the apparent horizon). Further, in all the three cases, to derive the first law of thermodynamics we have used the Einstein field equations (4) or on the other way, it is possible to derive the Einstein field equations assuming the first law of thermodynamics. Therefore, in analogy to the homogeneous FRW model, the inhomogeneous LTB model bounded by a horizon is a possible generalized Bekenstein-Hawking system and the first law of thermodynamics and Einstein field equations are equivalent on the horizon. Further, for choice (ii), we have seen that due to some thermal fluctuations, if the apparent horizon deviates infinitesimally, then there is a logarithmic correction to the Bekenstein entropy, i.e., it is possible to have classically a logarithmic correction to
entropy. For future work, some more generalization in the definition of temperature and entropy can be made so that GSLT holds on the horizon irrespective of any fluid distribution.

ACKNOWLEDGEMENT

One of the authors (SC) is thankful to IUCAA, Pune, India for their warm hospitality as a part of the work has been done during a visit. Also SC is thankful to DRS programme in the Department of Mathematics, Jadavpur University. The author SS is thankful to DST-PURSE Programme of Jadavpur University for awarding JRF.

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