ON JAEGER’S HOMFLY-PT EXPANSIONS, BRANCHING RULES AND LINK HOMOLOGY: A PROGRESS REPORT

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Abstract. We describe Jaeger’s HOMFLY-PT expansion of the Kauffman polynomial and how to generalize it to other quantum invariants using the so-called “branching rules” for Lie algebra representations. We present a program which aims to construct Jaeger expansions for link homology theories. This note is an updated write-up of a talk given by the author at the Meeting of the Sociedade Portuguesa de Matemática in July 2012.

1. LINK POLYNOMIALS AND JAEGER EXPANSIONS

This story starts with two celebrated invariant polynomials of links.

Definition 1.1. The Kauffman polynomial \( F = F(a, q) \) is the unique invariant of framed unoriented links satisfying

\[
F \left( \begin{array}{c} \bar{\text{x}} \\ \text{x} \end{array} \right) - F \left( \begin{array}{c} \text{x} \\ \bar{\text{x}} \end{array} \right) = (q - q^{-1}) \left( F \left( \begin{array}{c} \text{x} \\ \bar{\text{x}} \end{array} \right) - F \left( \begin{array}{c} \bar{\text{x}} \\ \text{x} \end{array} \right) \right)
\]

\[
F \left( \begin{array}{c} \bar{\text{x}} \\ \text{x} \end{array} \right) = a^{-2} q F \left( \begin{array}{c} \text{x} \\ \bar{\text{x}} \end{array} \right) \quad \text{and} \quad F \left( \begin{array}{c} \bar{\text{x}} \\ \text{x} \end{array} \right) = \frac{a^2 q^{-1} - a^{-2} q}{q - q^{-1}} + 1.
\]

Definition 1.2. The HOMFLY-PT polynomial \( P = P(a, q) \) is the unique invariant of framed oriented links satisfying

\[
a P \left( \begin{array}{c} \bar{\text{x}} \\ \text{x} \end{array} \right) - a^{-1} P \left( \begin{array}{c} \text{x} \\ \bar{\text{x}} \end{array} \right) = (q - q^{-1}) P \left( \begin{array}{c} \text{x} \\ \bar{\text{x}} \end{array} \right)
\]

\[
P \left( \begin{array}{c} \bar{\text{x}} \\ \text{x} \end{array} \right) = a^{-1} P \left( \begin{array}{c} \text{x} \\ \bar{\text{x}} \end{array} \right), \quad P \left( \begin{array}{c} \bar{\text{x}} \\ \text{x} \end{array} \right) = a P \left( \begin{array}{c} \text{x} \\ \bar{\text{x}} \end{array} \right) \quad \text{and} \quad P \left( \begin{array}{c} \bar{\text{x}} \\ \text{x} \end{array} \right) = \frac{a - a^{-1}}{q - q^{-1}}.
\]

Either the Kauffman and the HOMFLY-PT polynomials can be normalized to give ambient isotopy invariants of (oriented) links, but we will not proceed in this direction. In 1989 François Jaeger showed that the Kauffman polynomial of a link \( L \) can be obtained as a weighted sum of HOMFLY-PT polynomials on certain links associated to \( L \). Consider the following formalism

\[
\left[ \begin{array}{c} \text{x} \\ \bar{\text{x}} \end{array} \right] = (q - q^{-1}) \left( \left[ \begin{array}{c} \text{x} \\ \bar{\text{x}} \end{array} \right] - \left[ \begin{array}{c} \bar{\text{x}} \\ \text{x} \end{array} \right] \right) + \left[ \begin{array}{c} \text{x} \\ \bar{\text{x}} \end{array} \right] + \left[ \begin{array}{c} \bar{\text{x}} \\ \text{x} \end{array} \right] + \left[ \begin{array}{c} \text{x} \\ \bar{\text{x}} \end{array} \right] + \left[ \begin{array}{c} \bar{\text{x}} \\ \text{x} \end{array} \right]
\]

\[
\left[ \begin{array}{c} \bar{\text{x}} \\ \text{x} \end{array} \right] = \left[ \begin{array}{c} \text{x} \\ \bar{\text{x}} \end{array} \right] + \left[ \begin{array}{c} \bar{\text{x}} \\ \text{x} \end{array} \right]
\]

Key words and phrases. Quantum invariant, branching rules, link homology.
where the r.h.s. is evaluated to HOMFLY-PT polynomials completed with information about rotation numbers \( [\vec{D}] = (a^{-1}q)^{\text{rot}D} P(\vec{D}) \) (of course we only take the diagrams which are globally coherently oriented). The expression in (1) can be seen as a state expansion with the coefficients given as vertex weights. For a state \( \sigma \) we denote its weight by \( w(\sigma) \). The proof of the following can be found in [1].

**Theorem 1.3** (F. Jaeger, 1989). Let \( D \) be a diagram of a link \( L \) and \( \Sigma(D) \) denote the set of states of \( D \). The sum

\[
\sum_{\sigma \in \Sigma(D)} w(\sigma)[\sigma] = F(D)
\]

is a HOMFLY-PT expansion of the Kauffman polynomial of \( L \).

It is not hard to see how to extend this expansion to tangles (see [5]). To explain this expansion we look into the representation theory of quantum enveloping algebras of the simple Lie algebras (QEA). It is known that the HOMFLY-PT polynomial is related to the representation theory of the QEAs of type \( A_{n-1} \) (e.g. \( \mathfrak{sl}_n \)) and that in turn the Kauffman polynomial is related to the QEAs of types \( B_n, C_n \) and \( D_n \) (\( \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n} \) and \( \mathfrak{so}_{2n} \) respectively). Taking \( a = q^n \) in \( F(a, q) \) and \( P(a, q) \) we obtain the \( \mathfrak{so}_{2n} \) and the \( \mathfrak{sl}_n \) polynomials respectively.

Following N. Reshetikhin and V. Turaev there is a functor from the category of tangles whose arcs are colored by irreducible finite dimensional (f.d.) representations of a QEA \( g \) to the (tensor) category of f.d. representations of \( g \) (see [3, 4]). In other words, for each of these tangles there is a \( g \)-invariant map which depends only on the (regular) isotopy class of the tangle which gives a full isotopy invariant in the cases we are interested in. So what we really have in Theorem 1.3 is an \( \mathfrak{sl}_n \)-expansion of the \( \mathfrak{so}_{2n} \)-polynomial (the case where all strands are colored by the fundamental representation)!

There are other (2-variable) HOMFLY-PT expansions of \( F(a, q) \) resulting in \( \mathfrak{sl}_n \)-expansions of the \( \mathfrak{so}_{2n+1} \) and \( \mathfrak{sp}_{2n} \) polynomials after the specialization \( a = q^n \). For example the assignment (this was found by the author together with E. Wagner and will appear somewhere in the literature [6])

\[
\begin{align*}
[\begin{array}{c}
\end{array}]_{B_n} &= (q - q^{-1})\left( [\begin{array}{c}
\end{array}]_{B_n} + [\begin{array}{c}
\end{array}]_{B_n} - [\begin{array}{c}
\end{array}]_{B_n} - [\begin{array}{c}
\end{array}]_{B_n} \right) \\
&+ [\begin{array}{c}
\end{array}]_{B_n} + [\begin{array}{c}
\end{array}]_{B_n} + [\begin{array}{c}
\end{array}]_{B_n} + [\begin{array}{c}
\end{array}]_{B_n} - [\begin{array}{c}
\end{array}]_{B_n} - [\begin{array}{c}
\end{array}]_{B_n} \\
&- [\begin{array}{c}
\end{array}]_{B_n} - [\begin{array}{c}
\end{array}]_{B_n}
\end{align*}
\]

\[
[\begin{array}{c}
\end{array}]_{B_n} = [\begin{array}{c}
\end{array}]_{B_n} + [\begin{array}{c}
\end{array}]_{B_n} + [\begin{array}{c}
\end{array}]_{B_n}
\]

where \( [\vec{D}]_{B_n} = a^{-\text{rot}D} P(\vec{D}) \) and a dashed line means the corresponding strand is to be erased, gives an \( \mathfrak{sl}_n \)-expansion of the \( \mathfrak{so}_{2n+1} \)-polynomial.
2. Branching rules, link homology and categorification

Let us give an explanation for this phenomenon. An inclusion \( l \hookrightarrow g \) of Lie algebras (resp. QEAs) gives rise to functors (\( \text{Ind} \) and \( \text{Res} \)) between their categories of representations. In general \( \text{Res} \) does not send an irreducible \( M \) over \( g \) to an irreducible over \( l \) but if we restrict ourselves to finite-dimensional representations we know that \( \text{Res}(M) \) decomposes as a direct sum of irreducibles for \( l \). The branching rule tells us how to obtain such a decomposition i.e. how to express an irreducible for \( g \) as a direct sum of irreducibles for \( l \).

This is what we had before! For example, the expression \( \mathcal{D}_n \) \( \mathcal{D}_n \mathcal{D}_n \mathcal{D}_n \) which can be obtained from the extension of Theorem 1.3 to tangles, is a diagrammatic interpretation of the isomorphism \( V_{\text{fund}}(\mathfrak{so}_{2n}) \cong V_{\text{fund}}(\mathfrak{sl}_n) \oplus V_{\text{fund}}^*(\mathfrak{sl}_n) \) for \( \mathfrak{so}_{2n} \supset \mathfrak{sl}_n \), and \( \mathcal{I}_B \) \( \mathcal{I}_B \mathcal{I}_B \mathcal{I}_B \) correponds to \( V_{\text{fund}}(\mathfrak{so}_{2n+1}) \cong V_{\text{fund}}(\mathfrak{sl}_n) \oplus V_{\text{fund}}^*(\mathfrak{sl}_n) \oplus V_{\text{triv}}(\mathfrak{sl}_n) \) for \( \mathfrak{so}_{2n+1} \supset \mathfrak{sl}_n \).

The general picture of categorification of quantum link invariants, pioneered by M. Khovanov [2] lifts the representations \( W \) appearing in the RT picture to categories \( C_g(W) \) (which are required to satisfy certain properties) and the RT map \( f_{RT} \) to a (derived) functor \( \mathcal{F}_{RT} \) between (the derived categories of) these categories. Again, the isomorphism class of this functor depends only on the isotopy class of the tangle. The categorification of \( f_{RT} \) for general f.d. irreducible representations of QEAs was constructed by B. Webster in [7,8].

We can try to use Webster’s work to construct categorical \( l \)-expansions for the categorified \( g \)-RT invariants. The categories \( C_g \) appearing in [7] extend to linear combinations of (arbitrary) f.d. irreducibles of \( g \) which means that Webster’s functors extend to (formal) linear combinations of tangles.

**Definition 2.1.** A categorical Jaeger expansion consists of (i) categorified branching rules i.e. a functor \( C_g(V^g) \rightarrow C_l(\oplus_i V^l_i) \) for \( l \subset g \), which is full, bijective on objects and descends to a map between the respective Grothendieck groups giving an isomorphism of \( l \)-representations \( V^g \cong \oplus_i V^l_i \), and (ii) its extension to corresponding decompositions of the “tangle functors”. Here \( V^g \) and each of the \( V^l_i \) are irreducible f.d. representations of \( g \) and \( l \) respectively (resp. tensor products of such representations).

Although (ii) seems desirable from the topological point of view (work still in progress), the fulfillment of (i) (see [9,10]) is already very interesting, due to the potential applications to areas like representation theory and physics.
Theorem 2.2. There are functors $C_q(V^g) \rightarrow C_l(\oplus_i V^i)$ categorifying the branching rules for $\mathfrak{sl}_{n+1} \supset \mathfrak{sl}_n$, $\mathfrak{so}_{2n+1} \supset \mathfrak{so}_{2n-1}$, $\mathfrak{sp}_{2n} \supset \mathfrak{sp}_{2n-2}$, $\mathfrak{so}_{2n} \supset \mathfrak{so}_{2n-2}$, (for all finite dimensional representations and tensor products of minuscule representations), $\mathfrak{so}_{2n}$, $\mathfrak{so}_{2n+1} \supset \mathfrak{sl}_n$ (for fundamental representations and their tensor products).

RÉFÉRENCES

[1] L. Kauffman, Knots and Physics, World Scientific, Singapore, 1991.
[2] M. Khovanov, “A categorification of the Jones polynomial”, Duke. Math. J., Vol. 101, No. 3 (2000), pp. 359-426.
[3] N. Reshetikhin e V. Turaev, “Ribbon graphs and their invariants derived from quantum groups”, Commun. Math. Phys., Vol. 127, No. 1 (1990), pp. 1-26.
[4] V. Turaev, “The Yang-Baxter equation and invariants of links”, Invent. Math., Vol. 92, No. 3 (1988), pp. 527-553.
[5] P. Vaz e E. Wagner, “A remark on BMW algebra, $q$-Schur algebras and categorification”, arXiv:1203.4628v1 [math.QA] (2012).
[6] P. Vaz e E. Wagner, “(work in progress)”, (2012).
[7] B. Webster, “Knot invariants and higher representation theory I : diagrammatic and geometric categorification of tensor products”, arXiv:1001.2020v7 [math.GT] (2011).
[8] B. Webster, “Knot invariants and higher representation theory II : the categorification of quantum knot invariants”, arXiv:1005.4559v5 [math.GT] (2011).
[9] P. Vaz, “KLR algebras and the branching rule I : The Gelfand-Tsetlin basis in type $A_n$”, arXiv:1309.0330v1 [math.RT] (2013).
[10] P. Vaz, KLR algebras and the branching rule II : The Gelfand-Tsetlin basis in types $B$, $C$ and $D$, (2013) (in preparation).

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