$K \to \pi\pi$ decay amplitudes from the lattice

Changhoan Kim, Norman H. Christ

Department of Physics, Columbia University, New York, NY, 10027

In order to directly compute physical two-pion K-decay amplitudes using lattice methods we must prepare a two-pion state with non-zero relative momentum. Building upon a proposal of Lellouch and Lüscher, we describe a finite-volume method to realize such a state as the lowest energy state of two pions.

1. Introduction

The techniques of lattice gauge theory offer the possibility of accurate, non-perturbative calculation of the masses and matrix elements predicted by low-energy QCD. Of particular interest are the properties of the K-meson system and the low-energy, weak matrix elements that are required to understand Kaon decays, especially those which enter the $K \to \pi\pi$ amplitudes which violate CP symmetry. Presently well-developed methods permit the calculation of $K$ to vacuum and $K$ to $\pi$ weak matrix elements. When combined with chiral perturbation theory and the accurate chiral symmetry of the domain wall fermion formulation, lattice methods can be used to evaluate all the matrix elements that contribute to the weak $K \to \pi\pi$ amplitudes in the chiral limit. In this paper we describe a method which may provide a practical approach to the direct lattice calculation of the desired, on-shell, $K \to \pi\pi$ amplitudes without the use of chiral perturbation theory.

2. Finite volume decay amplitudes

We begin by reviewing the finite-volume approach developed by Lellouch and Lüscher. To compute the matrix element of an operator $O$ between two physical states $|A\rangle$ and $|B\rangle$, one typically evaluates the Euclidean Green’s function:

$$G(t_1,t_2,t_3) = \text{Tr} \left\{ B(t_1)O(t_2)A(t_3)e^{-HT} \right\}$$

in the limit $T \gg t_1 \gg t_2 \gg t_3$. Here the operators $A$ and $B$ are chosen to carry quantum numbers which insure that the desired states, $|A\rangle$ and $|B\rangle$ are each the lowest energy states that can be created from the vacuum by $A$ and $B$, respectively. As demonstrated by Maiani and Testa, this method fails when one of the desired states is a two-particle state above threshold. In this case the desired final state cannot be uniquely selected by its quantum numbers and the Euclidean time limit will give an unwanted, threshold amplitude.

In a recent paper, Lellouch and Lüscher propose a method to circumvent this difficulty by tuning the finite lattice volume so that the first excited two-particle state allowed in that volume has the energy of the initial decaying particle. They work out an explicit formula which relates the resulting finite-volume matrix element to the desired infinite volume decay amplitude. The same result is obtained by different methods by Lin et al., offering further insight. As proposed, this method faces serious practical difficulties given our present computational capabilities. First, a large spatial volume of $(6 \text{Fermi})^3$ is required if the first-excited $\pi - \pi$ state is to have an energy matching the Kaon mass. Second, the extraction of the next-leading exponential in the large-time limit is numerically difficult, especially if an accurate result is desired. Finally, while the matrix element of the $I = 2$ final state can be obtained from the coefficient of such a next-leading exponential, the more important $I = 0$ state also receives a vacuum contribution implying that the desired amplitude is the third term in an expansion in increasingly small exponentials.
We propose to overcome these difficulties by imposing an anti-periodic boundary condition on the pions, say in the $z$-direction, while using the usual periodic boundary conditions in the other two directions. With this choice the lowest energy state of a single pion has momentum $p_z = \frac{\pi}{L}$ instead of zero. With such a choice of boundary conditions, the lowest energy state with $I = 2$ will be two pions, each with momentum $p_z \approx \pm \frac{\pi}{L}$. Thus, the matrix element of such a state can be extracted from the leading exponential in the Euclidean Green’s function. Since the vacuum state is unaffected by these boundary conditions, our large-time behavior and a vacuum subtraction will be necessary for calculation of $\Delta I = \frac{1}{2}$ amplitudes.

The condition which insures that the two-pion state has the relative momentum required for a physical $K \to \pi\pi$ decay, now requires $L_z \approx 3$ Fermi, only one-half the lattice extent needed for the periodic case. The other two spatial dimensions are not constrained by these kinematics and might be chosen to be similar $L_x = L_y = L_z$ giving a reduced box size of $(3$ Fermi$)^3$. Thus, the proposed method decreases the required spatial size from $(6$ Fermi$)^3$ to $(3$ Fermi$)^3$ and promotes the amplitude of interest to one of leading $(I = 2)$ or next-leading $(I = 0)$ large-time behavior. Since only periodic and cubically symmetric periodic boundary conditions were discussed in the original finite volume analysis\cite{6} on which the Lellouch-Lüscher paper\cite{4} is based, we must recompute Lüscher’s function $\phi(k)$ for this new, less symmetrical case. This has been done without difficulty following the method presented in the original paper, Ref.\cite{6}.

3. G-parity boundary condition

It is not immediately obvious how to realize this boundary condition for the pion, because we have control only on the underlying quark fields. However, the G-parity operation gives the solution since under G-parity: $G|\pi> = -|\pi>$. From this, we can see that anti-periodic boundary conditions for the pion can be achieved by performing a G-parity operation at the boundary—an operation that also can be defined at the quark level: $u \to d$ and $d \to -\bar{u}$. (Recall that G is defined as the product of charge-conjugation and an isospin rotation, $G = C \exp i\pi I_3$.) These unusual boundary conditions were originally proposed and implemented by Wiese\cite{6} and similar boundary conditions have also been used in Ref.\cite{8}.

Since we must also treat the Kaon, we must impose a charge conjugate boundary condition on the strange quark for consistency. This is easily done by including a fictitious charmed quark, degenerate with the strange quark and extending the usual G-parity by treating the $(c, s)$ quark pair as an independent iso-doublet. Our initial finite volume eigenstate can then be constructed as a G-even mixture $(K^0 + D^0)/\sqrt{2}$. However, because our weak decay operator and our final states have definite strangeness and charm, this mixing only introduces an extra “finite-volume factor” of $1/\sqrt{2}$ which can easily be removed.

4. Lattice Action

4.1. Symmetry

By imposing this G-parity boundary condition, we break chiral symmetry in a manner similar to the chiral symmetry breaking of a mass term. This was studied for the continuum in Ref.\cite{7}. On the lattice, the boundary condition is embedded in the action itself, adding a boundary-specific term. For example, with G-parity boundary conditions we have a term in the action density of the form:

$$\pi(L_z - 1)U^3(L_z - 1)\gamma_3 C\bar{d}(0)^T,$$

for Dirac conventions in which charge conjugation is represented by $q \to Cq^T$. Here $\pi(L_z - 1)$ is evaluated at the site $z = L_z - 1$ and $\bar{d}(0)$ at the site $z = 0$ while $U^3(L_z - 1)$ is the link variable joining these two sites across the boundary. Under chiral $SU(2)_L \otimes SU(2)_R$ such a term transforms as $a \frac{1}{2}, \frac{1}{2}$. While such a boundary term may appear to break translational invariance, the fact that the lattice Lagrangian is invariant under the G-parity operation ensures us that the system has translational symmetry, if appropriately defined. (Note, when exploring unconventional boundary conditions it is important to preserve the conser-
vation of lattice momentum and hence to require translational invariance.)

4.2. Implementation

The actual implementation of these boundary conditions represents a fairly minor modification of the usual lattice Dirac operator. An easy way of visualizing these G-parity boundary conditions begins with doubling the lattice in the z-direction, with the complex conjugates of gauge links in the original volume copied onto the links translated by $L_z$ into this extended volume. One then easily defines a single-flavor Dirac operator on this $L_x \times L_y \times 2 \cdot L_z$ lattice in the usual way, but imposing anti-periodic boundary conditions between the $z = 0$ and $z = 2L_z - 1$ boundaries. If a quark field in the original volume is identified as a $u$ quark, then the quark field in the doubled, $L_z \leq z < 2L_z - 1$ volume would be viewed as a $d$ quark. Note, the computational load is twice that of a usual calculation on a $L_x \times L_y \times L_z$ volume—our boundary conditions have effectively required separate treatment for the $u$ and $d$ quarks. Such an approach works for both Wilson and domain wall fermions.

5. Conclusion

Since the G-parity boundary condition is imposed in only one direction, we lose cubic symmetry. In the cubically symmetric case the state of interest with orbital angular momenta $l = 0$ is mixed only with states having $l$ differing by multiples of 4. However, without this cubic symmetry only parity invariance remains which permits mixing with all even $l$. Thus, in the practical application of the Lellouch and Lüscher method where the mixed, higher-$l$ states are treated as free and their phase shifts ignored, we must neglect the $l = 2$ phase shift $\delta_2(m_K)$. This reduces the accuracy of this approximation, introducing errors of order $(R/L)^4$ rather than $(R/L)^8$ as is the case for cubic symmetry[1]. Here $R$ is a typical strong interaction distance, e.g. $R = 1/m_\rho$.

There is an additional complexity for the $I = 0$ final state that should be discussed. For simplicity, we consider a full QCD calculation which includes only dynamical $u$ and $d$ quarks. For such a system, in addition to the three pions, there will be a fourth $SU(2)$ singlet, quark-anti-quark state called here the $\eta'$. This $\eta'$ state is G-parity even in contrast to G-parity odd triplet of pions and hence will obey even boundary conditions. Thus an intended two-pion, $I = 0$ state may mix with an unwanted, threshold two-$\eta'$ state which will have the same quantum numbers. Fortunately, in the two-flavor calculation just described, axial anomaly effects are expected make the non-Goldstone $\eta'$ heavier than the physical Kaon so this two-$\eta'$ state will play no role. However, in a quenched calculation, this two-$\eta'$ state will be essentially degenerate with the desired $I = 0$ two-pion state and obstruct the calculation. In fact, the hairpin diagrams entering the quenched two-$\eta'$ amplitudes will give an extra power of $t$ or $t^2$ further obscuring the state of interest. Thus, this G-parity boundary condition can be used in the quenched approximation only for the evaluation of $\Delta I = 3/2$ amplitudes.

In summary, we propose imposing anti-periodic boundary conditions on the pion by implementing a G-parity operation at boundary. This reduces the required simulation volume to $\approx (3 \text{ Fermi})^3$ and, more importantly, allows the physical $\langle \pi \pi | K_{\text{weak}} | K \rangle$ amplitude to be extracted from the leading/next-leading large-time behavior of a Euclidean correlation function for the $I = 2/1 = 0$ final states. We hope to try a calculation according to this method in the coming year and thank our colleagues in the RBC collaboration for useful suggestions and criticisms.

REFERENCES

1. J. I. Noaki, et al., hep-lat/0108013.
2. T. Blum, et al., hep-lat/0110073.
3. L. Maiani and M. Testa, Phys. Lett. B245 (1990) 585–590.
4. L. Lellouch and M. Luscher, Comm. Math. Phys. 219 (2001) 31-44.
5. C. J. D. Lin, et al., Int. J. Mod. Phys. C 11, 637 (2000).