POINCARÉ DUALITY COMPLEXES IN DIMENSION FOUR

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Abstract. We describe an algebraic structure on chain complexes yielding algebraic models which classify homotopy types of PD$^4$-complexes. Generalizing Turaev’s fundamental triples of PD$^3$-complexes we introduce fundamental triples for PD$^n$-complexes and show that two PD$^n$-complexes are orientedly homotopy equivalent if and only if their fundamental triples are isomorphic. As applications we establish a conjecture of Turaev and obtain a criterion for the existence of degree 1 maps between $n$-dimensional manifolds.

Introduction

In order to study the homotopy types of closed manifolds, Browder and Wall introduced the notion of Poincaré duality complexes. A Poincaré duality complex, or PD$n$-complex, is a CW–complex, $X$, whose cohomology satisfies a certain algebraic condition. Equivalently, the chain complex, $\hat{C}(X)$, of the universal cover of $X$ must satisfy a corresponding algebraic condition. Thus Poincaré complexes form a mixture of topological and algebraic data and it is an old quest to provide purely algebraic data determining the homotopy type of PD$n$-complexes. This has been achieved for $n = 3$, but, for $n = 4$, only partial results are available in the literature.

Homotopy types of 3–manifolds and PD$^3$–complexes were considered by Thomas [18], Swarup [17] and Hendriks [9]. The homotopy type of a PD$^3$–complex, $X$, is determined by its fundamental triple, consisting of the fundamental group, $\pi_1(X)$, the orientation character, $\omega$, and the image in $H_3(\pi, \mathbb{Z})$ of the fundamental class, $[X]$. Turaev [20] provided an algebraic condition for a triple to be realizable by a PD$^3$–complex. Thus, in dimension 3, there are purely algebraic invariants which provide a complete classification.

Using primary cohomological invariants like the fundamental group, characteristic classes and intersection pairings, partial results were obtained for $n = 4$ by imposing conditions on the fundamental group. For example, Hambleton, Kreck and Teichner classified PD$^4$-complexes with finite fundamental group having periodic cohomology of dimension 4 (see [6], [19] and [2]). Cavicchioli, Hegenbarth and Piccarreta studied PD$^4$-complexes with free fundamental group (see [4] and [8]), as did Hillman [14], who also considered PD$^4$–complexes with fundamental group a PD$^3$–group [10]. Recently, Hillman [13] considered homotopy types of PD$^4$–complexes whose fundamental group has cohomological dimension 2 and one end.

It is doubtful whether primary invariants are sufficient for the homotopy classification of PD$^4$–complexes in general and we thus follow Ranicki’s approach ([14] and [15]) who assigned to each PD$n$–complex, $X$, an algebraic Poincaré duality complex given by the chain complex, $\hat{C}(X)$, together with a symmetric or quadratic
structure. However, Ranicki considered neither the realizability of such algebraic Poincaré duality complexes nor whether the homotopy type of a PD$^n$–complex is determined by the homotopy type of its algebraic Poincaré duality complex.

This paper presents a structure on chain complexes which completely classifies PD$^4$–complexes up to homotopy. The classification uses fundamental triples of PD$^4$–complexes, and, in fact, the chain complex model yields algebraic conditions for the realizability of fundamental triples.

Fundamental triples of formal dimension $n \geq 3$ comprise an $(n-2)$–type $T$, a homomorphism $\omega : \pi_1(T) \to \mathbb{Z}/2\mathbb{Z}$ and a homology class $t \in H_n(T,\mathbb{Z})$. There is a functor,

$$\tau_+ : \text{PD}^n_+ \to \text{Trp}^n_+,$$

from the category $\text{PD}^n_+$ of PD$^n$–complexes and maps of degree one to the category $\text{Trp}^n_+$ of triples and morphisms inducing surjections on fundamental groups. Our first main result is

**Theorem 3.1.** The functor $\tau_+$ reflects isomorphisms and is full for $n \geq 3$.

**Corollary 3.2.** Take $n \geq 3$. Two closed $n$–dimensional manifolds or two PD$^n$–complexes, respectively, are orientedly homotopy equivalent if and only if their fundamental triples are isomorphic.

Corollary 3.2 extends results of Thomas [18], Swarup [17] and Hendriks [9] for dimension 3 to arbitrary dimension and establishes Turaev’s conjecture [20] on PD$^n$–complexes whose $(n-2)$–type is an Eilenberg–Mac Lane space $K(\pi_1 X, 1)$. Corollary 4.2 is even of interest in the case of simply connected or highly connected manifolds.

Theorem 3.1 also yields a criterion for the existence of a map of degree one between PD$^n$–complexes, recovering Swarup’s result for maps between 3–manifolds and Hendriks’ result for maps between PD$^3$–complexes.

In the oriented case, special cases of Corollary 3.2 were proved by Hambleton and Kreck [6] and Cavicchioli and Spaggiari [5]. In fact, in [6], Corollary 3.2 is obtained under the condition that either the fundamental group is finite or the second rational homology of the 2–type is non–zero. Corresponding conditions were used in [5] for oriented PD$^{2n}$–complexes with $(n-1)$–connected universal covers, and Teichner extended the approach of [6] to the non–oriented case in his thesis [19]. Our result shows that the conditions on finiteness and rational homology used in these papers are not necessary.

It follows directly from Poincaré duality and Whitehead’s Theorem that the functor $\tau_+$ reflects isomorphisms. To show that $\tau_+$ is full requires work. Given PD$^n$–complexes $Y$ and $X$, $n \geq 3$, and a morphism $f : \tau_+ Y \to \tau_+ X$ in $\text{Trp}^n_+$, we first construct a chain map $\xi : \tilde{C}(Y) \to \tilde{C}(X)$ preserving fundamental classes, that is, $\xi_*[Y] = [X]$. Then we use the category $H^{k+1}_+\text{ of homotopy systems of order } (k+1)$ introduced in [11] to realize $\xi$ by a map $\overline{f} : Y \to X$ with $\tau_+(\overline{f}) = f$.

Our second main result describes algebraic models of homotopy types of PD$^4$–complexes. We introduce the notion of PD$^n$–chain complex and show that PD$^3$–chain complexes are equivalent to PD$^3$–complexes up to homotopy. In Section 5, we show that PD$^4$–chain complexes classify homotopy types of PD$^4$–complexes up to 2–torsion. In particular, we obtain
Theorem 5.3. The functor $\hat{C}$ induces a 1–1 correspondence between homotopy types of PD$^4$–complexes with finite fundamental group of odd order and homotopy types of PD$^4$–chain complexes with homotopy co–commutative diagonal and finite fundamental group of odd order.

To obtain a complete homotopy classification of PD$^4$–complexes, we study the chain complex of a 2–type in Section 6. We compute this chain complex up to fundamental group of odd order.

Corollary 7.4. The functor $\hat{C}$ induces a 1–1 correspondence between homotopy types of PD$^4$–complexes and homotopy types of β–PD$^4$–chain complexes.

Corollary 7.4 highlights the crucial rôle of Peiffer commutators for the homotopy classification of 4–manifolds.

The proofs of our results rely on the obstruction theory [1] for the realizability of chain maps which we recall in Section 8.

1. Chain complexes

Let $X^n$ denote the n–skeleton of the CW–complex $X$. We call $X$ reduced if $X^0 = \ast$ is the base point. The objects of the category $\mathbf{CW}_0$ are reduced CW–complexes $X$ with universal covering $p : \hat{X} \to X$, such that $p(\ast) = \ast$, where $\ast \in \hat{X}^0$ is the base point of $\hat{X}$. Here the n–skeleton of $\hat{X}$ is $\hat{X}^n = p^{-1}(X^n)$. Morphisms in $\mathbf{CW}_0$ are cellular maps $f : X \to Y$ and homotopies in $\mathbf{CW}_0$ are base point preserving. A map $f : X \to Y$ in $\mathbf{CW}_0$ induces a unique covering map $\hat{f} : \hat{X} \to \hat{Y}$ with $\hat{f}(\ast) = \ast$, which is equivariant with respect to $\varphi = \pi_1(f)$.

We consider pairs $(\pi, C)$, where $\pi$ is a group and $C$ a chain complex of left modules over the group ring $\mathbb{Z}[\pi]$. We write $\Lambda = \mathbb{Z}[\pi]$ and $C$ for $(\pi, C)$, whenever $\pi$ is understood. We call $(\pi, C)$ free if each $C_n, n \in \mathbb{Z}$, is a free $\Lambda$–module. Let aug : $\Lambda \to \mathbb{Z}$ be the augmentation homomorphism, defined by aug($g$) = 1 for all $g \in \pi$. Every group homomorphism, $\varphi : \pi \to \pi'$, induces a ring homomorphism $\varphi_\#: \Lambda \to \Lambda'$, where $\Lambda' = \mathbb{Z}[\pi']$. A chain map is a pair $(\varphi, F) : (\pi, C) \to (\pi', C')$, where $\varphi$ is a group homomorphism and $F : C \to C'$ a $\varphi$–equivariant chain map, that is a chain map of the underlying abelian chain complexes, such that $F(\lambda c) = \varphi_\#(\lambda)F(c)$ for $\lambda \in \Lambda$ and $c \in C$. Two such chain maps are homotopic, $(\varphi, F) \simeq (\psi, G)$ if $\varphi = \psi$ and if there is a $\varphi$–equivariant map $\alpha : C \to C'$ of degree +1 such that $G - F = da + \alpha d$.

A pair $(\pi, C)$ is a reduced chain complex if $C_0 = \Lambda$ with generator $\ast$, $C_i = 0$ for $i < 0$ and $H_0C = \mathbb{Z}$ such that $C_0 = \Lambda \to H_0C = \mathbb{Z}$ is the augmentation of $\Lambda$. A chain map, $(\varphi, f) : (\pi, C) \to (\pi', C')$, of reduced chain complexes, is reduced if $f_0$ is induced by $\varphi_\#$, and a chain homotopy $\alpha$ of reduced chain maps is reduced if $\alpha_0 = 0$. The objects of the category $\mathbf{H}_0$ are reduced chain complexes and the morphisms are reduced chain maps. Homotopies in $\mathbf{H}_0$ are reduced chain homotopies. Every chain complex $(\pi, C)$ in $\mathbf{H}_0$ is equipped with the augmentation $\varepsilon : C \to \mathbb{Z}$ in $\mathbf{H}_0$.

The ring homomorphism $\mathbb{Z} \to \Lambda$ yields the co–augmentation $\varepsilon : \mathbb{Z} \to C$, where we view $\mathbb{Z} = (0, \mathbb{Z})$ as chain complex with trivial group $\pi = 0$ concentrated in degree 0. Note that $\varepsilon = \text{id}_{\mathbb{Z}}$, and the composite $\varepsilon \varepsilon : C \to C'$ is the trivial map.

For an object $X$ in $\mathbf{CW}_0$, the cellular chain complex $C(\hat{X})$ of the universal cover $\hat{X}$ is given by $C_n(\hat{X}) = H_n(\hat{X}^n, \hat{X}^{n+1})$. The fundamental group $\pi = \pi_1(X)$ acts...
on \(C(\tilde{X})\), and viewing \(C(\tilde{X})\) as a complex of left \(\Lambda\)-modules, we obtain the object \(\tilde{C}(X) = (\pi, C(\tilde{X}))\) in \(H_0\). Moreover, a morphism \(f : X \to Y\) in \(CW_0\) induces the homomorphism \(\pi_1(f)\) on the fundamental groups and the \(\pi_1(f)\)-equivariant map \(\hat{f} : \tilde{X} \to \tilde{Y}\) which in turn induces the \(\pi_1(f)\)-equivariant chain map \(\hat{f}_* : C(\tilde{X}) \to C(\tilde{Y})\) in \(H_0\). As \(\hat{f}\) preserves base points, \(\tilde{C}(f) = (\pi_1(f), \hat{f}_*)\) is a reduced chain map. We obtain the functor

\[
\tilde{C} : CW_0 \longrightarrow H_0.
\]

The chain complex \(C\) in \(H_0\) is \(2\)-realizable if there is an object \(X\) in \(CW_0\) such that \(\tilde{C}(X^2) \cong C_{\leq 2}\), that is, \(\tilde{C}(X^2)\) is isomorphic to \(C\) in degree \(\leq 2\). Given two objects \(X\) and \(Y\) in \(CW_0\), their product again carries a cellular structure and we obtain the object \(X \times Y\) in \(CW_0\) with base point \((*, *)\) and universal cover \((X \times Y)^\circ = \tilde{X} \times \tilde{Y}\), so that

\[
\tilde{C}(X \times Y) = (\pi \times \pi, C(\tilde{X}) \otimes_Z C(\tilde{Y})). 
\]

For \(i = 1, 2\), let \(p_i : X \times X \to X\) be the projection onto the \(i\)-th factor. A diagonal \(\Delta : X \to X \times X\) in \(CW_0\) is a cellular map with \(p_i\Delta \simeq \text{id}_X\) in \(CW_0\) for \(i = 1, 2\). A diagonal on \((\pi, C)\) in \(H_0\) is a chain map \((\delta, \Delta) : (\pi, C) \to (\pi \times \pi, C \otimes_Z C)\) in \(H_0\) with \(\delta : \pi \to \pi \times \pi, g \mapsto (g, g)\), such that \(p_i\Delta \simeq \text{id}_C\) for \(i = 1, 2\), where \(p_1 = \text{id} \otimes \varepsilon\) and \(p_2 = \varepsilon \otimes \text{id}\).

The diagonal \((\delta, \Delta)\) in \(H_0\) is homotopy co–associative if the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes_Z C \\
\downarrow & & \downarrow \text{id} \otimes \Delta \\
C \otimes_Z C & \xrightarrow{\Delta \otimes \text{id}} & C \otimes_Z C \otimes_Z C
\end{array}
\]

commutes up to chain homotopy in \(H_0\). The diagonal \((\delta, \Delta)\) in \(H_0\) is homotopy co–commutative if the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes_Z C \\
\downarrow & & \downarrow T \\
C \otimes_Z C & \xrightarrow{\Delta} & C \otimes_Z C
\end{array}
\]

commutes up to chain homotopy in \(H_0\), where \(T\) is given by \(T(c \otimes d) = (-1)^{|c||d|} d \otimes c\).

By the cellular approximation theorem, there is a diagonal \(\Delta : X \to X \times X\) in \(CW_0\) for every object \(X\) in \(CW_0\). Applying the functor \(\tilde{C}\) to such a diagonal, we obtain the diagonal \(\tilde{C}(\Delta)\) in \(H_0\). This raises the question of realizability, that is, given a diagonal \((\delta, \Delta) : \tilde{C}(X) \to \tilde{C}(X) \otimes_Z \tilde{C}(X)\) in \(H_0\), is there a diagonal \(\Delta\) in \(CW_0\) with \(\tilde{C}(\Delta) = (\delta, \Delta)\)? As \(\tilde{C}(\Delta)\) is homotopy co–associative and homotopy co–commutative for any diagonal \(\Delta\) in \(CW_0\), homotopy co–associativity and homotopy co–commutativity of \((\delta, \Delta)\) are necessary conditions for realizability.

To discuss questions of realizability for a functor \(\lambda : A \to B\), we consider pairs \((A, b)\), where \(b : \lambda A \cong B\) is an equivalence in \(B\). Two such pairs are equivalent, \((A, b) \sim (A', b')\), if and only if there is an equivalence \(g : A' \cong A\) in \(A\) with \(\lambda g = b^{-1}b'\). The classes of this equivalence relation form the class of \(\lambda\)-realizations of \(B\),

\[
\text{Real}_\lambda(B) = \{(A, b) \mid b : \lambda A \cong B\}/\sim.
\]
We say that \( B \) is \( \lambda \)-realizable if \( \text{Real}_\lambda(B) \) is non-empty. The functor \( \lambda : \mathbf{A} \to \mathbf{B} \) is representative if all objects \( B \) in \( \mathbf{B} \) are \( \lambda \)-realizable. Further, we say that \( \lambda \) reflects isomorphisms, if a morphism \( f \) in \( \mathbf{A} \) is an equivalence whenever \( \lambda(f) \) is an equivalence in \( \mathbf{B} \). The functor \( \lambda \) is full if, for every morphism \( \mathbf{f} : \lambda(A) \to \lambda(A') \) in \( \mathbf{B} \), there is a morphism \( f : A \to A' \) in \( \mathbf{A} \), such that \( \lambda(f) = \mathbf{f} \), we then say \( \mathbf{f} \) is \( \lambda \)-realizable.

2. PD–chain complexes and PD–complexes

We start this section with a description of the cap product on chain complexes. We fix a homomorphism \( \omega : \pi \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \) which gives rise to the anticommutator \( \mathbb{Z} : \Lambda \to \Lambda \) of rings defined by \( \mathbb{Z} = (-1)^{\omega(g)}g^{-1} \) for \( g \in \pi \). To the left \( \Lambda \)-module \( M \) we associate the right \( \Lambda \)-module \( M^\omega \) with the same underlying abelian group and action given by \( \lambda.a = a.\overline{x} \) for \( a \in A \) and \( \lambda \in \Lambda \). Proceeding analogously for a right \( \Lambda \)-module \( N \), we obtain a left \( \Lambda \)-module \( \omega N \). We put
\[
H_n(C, M^\omega) = H_n(M^\omega \otimes_\Lambda C); \quad H^k(C, M) = H_{-k}(\text{Hom}_\Lambda(C, M)).
\]

To define the \( \omega \)-twisted cap product \( \cap \) for a chain complex \( C \) in \( \mathbf{H}_0 \) with diagonal \((\delta, \Delta)\), write \( \Delta(c) = \sum_{i+j=n, \alpha} c^i_\alpha \otimes c^j_\alpha \) for \( c \in C \). Then
\[
\cap : \text{Hom}_\Lambda(C, M) \otimes \mathbb{Z} (Z^\omega \otimes_\Lambda C)_n \to (M^\omega \otimes_\Lambda C)_{n-k}
\]

\[
\psi \otimes (z \otimes c) \mapsto \sum_{\alpha} z \psi(c^i_\alpha) \otimes c^j_{n-k, \alpha}
\]

for every left \( \Lambda \)-module \( M \). Passing to homology and composing with
\[
H^r(C, M) \otimes \mathbb{Z} H_* (C \otimes \mathbb{Z}, C, Z^\omega) \to H_* (\text{Hom}_\Lambda(C, M)) \otimes \mathbb{Z} (Z^\omega \otimes_\Lambda (C \otimes \mathbb{Z}, C))
\]

we obtain
\[
(2.1) \quad \cap : H^k(C, M) \otimes \mathbb{Z} H_n(C, Z^\omega) \to H_{n-k}(C, M^\omega).
\]

A PD\(^n\)–chain complex \( C = ((\pi, C), \omega, [C], \Delta) \) consists of a free chain complex \((\pi, C)\) in \( \mathbf{H}_0 \) with \( \pi \) finitely presented, a group homomorphism \( \omega : \pi \to \mathbb{Z}/2\mathbb{Z} \), a fundamental class \([C] \in H_n(C, Z^\omega)\) and a diagonal \( \Delta : C \to C \otimes C \) in \( \mathbf{H}_0 \), such that \( H_1(C) = 0 \) and
\[
(2.2) \quad \cap [C] : H^r(C, M) \to H_{n-r}(C, M^\omega); \quad \alpha \mapsto \alpha \cap [X]
\]

is an isomorphism of abelian groups for every \( r \in \mathbb{Z} \) and every left \( \Lambda \)-module \( M \). A morphism of PD\(^n\)–chain complexes \( f : ((\pi, C), \omega, [C], \Delta) \to ((\pi', C'), \omega', [C'], \Delta') \) is a morphism \( (\phi, f) : (\pi, C) \to (\pi', C') \) in \( \mathbf{H}_0 \) such that \( \omega = \omega' \varphi \) and \( (f \otimes f)\Delta \simeq \Delta' f \). The category PD\(^n\) is the category of PD\(^n\)–chain complexes and morphisms between them. Homotopies in PD\(^n\) are reduced chain homotopies. The subcategory PD\(^n\)\(^+\) of PD\(^n\) is the category consisting of PD\(^n\)–chain complexes and oriented or degree 1 morphisms of PD\(^n\)–chain complexes, that is, morphisms \( f : C \to D \) with \( f_*[C] = [D] \). Wall [21] showed that it is enough to demand that (2.2) is an isomorphism for \( M = \Lambda \). If \( 1 \otimes x \in Z^\omega \otimes_\Lambda C_n \) represents the fundamental class \([C]\), where \( C_i \) is finitely generated for \( i \in \mathbb{Z} \), then \( \cap [C] \) in (2.2) is an isomorphism if and only if
\[
(2.3) \quad \cap 1 \otimes x : C^* = \text{Hom}_\Lambda(C^*, \Lambda) \to \Lambda \otimes_\Lambda C = C
\]
is a homotopy equivalence of chain complexes of degree \( n \). Here finite generation implies that \( C^* \) is a free chain complex.
Lemma 2.1. Every PD\(^n\)–chain complex is homotopy equivalent in PD\(^n\) to a 2–realizable PD\(^n\)–chain complex.

Proof. This follows from Proposition III 2.13 and Theorem III 2.12 in \[1\]. □

A PD\(^n\)–complex \(X = (X, \omega, [X], \Delta)\) consists of an object \(X\) in \(\text{CW}_0\) with finitely presented fundamental group \(\pi_1(X)\), a group homomorphism \(\omega : \pi_1 X \to \mathbb{Z}/2\mathbb{Z}\), a fundamental class \([X] \in H_n(X, \mathbb{Z}^\omega)\) and a diagonal \(\Delta : X \to X \times X\) in \(\text{CW}_0\), such that \((\tilde{C}X, \omega, [X], \tilde{C}\Delta)\) is a PD\(^n\)–chain complex. A morphism of PD\(^n\)–complexes \(f : (X, \omega, [X], \Delta) \to (X', \omega', [X'], \Delta')\) is a morphism \(f : X \to X'\) in \(\text{CW}_0\) such that \(\omega = \omega' \pi_1(f)\). The category PD\(^n\) is the category of PD\(^n\)–complexes and morphisms between them. Homotopies in PD\(^n\) are homotopies in \(\text{CW}_0\). The subcategory PD\(^n\) of PD\(^n\) is the category consisting of PD\(^n\)–complexes and oriented or degree 1 morphisms of PD\(^n\)–complexes, that is, morphisms \(f : X \to Y\) with \(f_*[X] = [Y]\).

Remark 2.2. Our PD\(^n\)–complexes have finitely presented fundamental groups by definition and are thus finitely dominated by Proposition 1.1 in \[23\].

Let \(X\) be a PD\(^n\)–complex with \(n \geq 3\). We say that \(X\) is standard, if \(X\) is a CW–complex which is \(n\)–dimensional and has exactly one \(n\)–cell \(e^n\). We say that \(X\) is weakly standard, if \(X\) has a subcomplex \(X'\) with \(X = X' \cup e^n\), where \(X'\) is \(n\)–dimensional and satisfies \(H^i(X', B) = 0\) for all coefficient modules \(B\). In this sense \(X'\) is homologically \((n - 1)\)–dimensional. Of course standard implies weakly standard with \(X' = X^{n-1}\).

Remark. Every compact connected manifold \(M\) of dimension \(n\) has the homotopy type of a finite standard PD\(^n\)–complex.

Remark 2.3. Wall’s Theorem 2.4 in \[21\] and Theorem E in \[22\] imply that, for \(n \geq 4\), every PD\(^n\)–complex is homotopy equivalent to a standard PD\(^n\)–complex and, for \(n = 3\), every PD\(^3\)–complex is homotopy equivalent to a weakly standard PD\(^3\)–complex.

Let \(C\) be a PD\(^n\)–chain complex with \(n \geq 3\). We say that \(C\) is standard, if \(C\) is 2–realizable, \(C_i = 0\) for \(i > n\), and \(C_n = \Lambda[e_n]\), where \([e_n] \in C_n\). We say that \(C\) is weakly standard, if \(C\) is 2–realizable and has a subcomplex \(C'\) with \(C = C' \oplus \Lambda[e_n]\), where \(C'\) is \(n\)–dimensional and satisfies \(H^i(C', B) = 0\) for all coefficient modules \(B\).

Remark 2.4. A PD\(^n\)–complex, \(X\), is homotopy equivalent to a finite standard, standard or weakly standard PD\(^n\)–complex, respectively, if and only if the PD\(^n\)–chain complex \(\tilde{C}X\) is homotopy equivalent to a finite standard, standard or weakly standard PD\(^n\)–chain complex, respectively.

3. Fundamental Triples

Homotopy types of 3–manifolds and PD\(^3\)–complexes were considered by Thomas \[18\], Swarup \[17\] and Hendriks \[9\]. In particular, Hendriks and Swarup provided a criterion for the existence of degree 1 maps between 3–manifolds and PD\(^3\)–complexes, respectively. In this section we generalize these results to manifolds and Poincaré duality complexes of arbitrary dimension \(n\).

Let \(k\)–types be the full subcategory of \(\text{CW}_0 \simeq \text{CW}\) consisting of \(\text{CW}\)–complexes \(X\) in \(\text{CW}_0\) with \(\pi_i(X) = 0\) for \(i > k\). The \(k\)–th Postnikov functor

\[P_k : \text{CW}_0 \to \text{\(k\)–types}\]
is defined as follows. For $X$ in $\text{CW}_0$ we obtain $P_kX$ by “killing homotopy groups”, that is, we choose a CW–complex $P_kX$ with $(k+1)$–skeleton $(P_kX)^{k+1} = X^{k+1}$ and $\pi_i(P_kX) = 0$ for $i > k$. For a morphism $f : X \to Y$ in $\text{CW}_0$ we may choose a map $Pf : P_kX \to P_kY$ which extends the restriction $f^{k+1} : X^{k+1} \to Y^{k+1}$ as $\pi_i(Pf) = 0$ for $i > k$. Then the functor $P_k$ assigns $P_kX$ to $X$ and the homotopy class of $Pf$ to $f$. Different choices for $P_kX$ yield canonically isomorphic functors $P_k$. The CW–complex $P_1X = K(\pi_1X,1)$ is an Eilenberg–MacLane space and, as a functor, $P_1$ is equivalent to the functor $\pi_1$ of fundamental groups. There are natural maps

$$p_k : X \to P_kX$$

in $\text{CW}_0/ \simeq$ extending the inclusion $X^{k+1} \subseteq P_kX$.

For $n \geq 3$, a fundamental triple $T = (X, \omega, t)$ of formal dimension $n$ consists of an $(n-2)$–type $X$, a homomorphism $\omega : \pi_1X \to \mathbb{Z}/2\mathbb{Z}$ and an element $t \in H_n(X,\mathbb{Z}^\omega)$. A morphism $(X,\omega_X,t_X) \to (Y,\omega_Y,t_Y)$ between fundamental triples is a homotopy class of maps of the $(n-2)$–types, such that $\omega_X = \omega_Y \pi_1(f)$ and $f_*t_X = t_Y$. We obtain the category $\text{Trp}_n$ of fundamental triples $T$ of formal dimension $n$ and the functor

$$\tau : \text{PD}_n^+ / \simeq \to \text{Trp}_n^+, \quad X \mapsto (P_{n-2}X, \omega_X, p_{n-2}([X])).$$

Every degree 1 morphism $Y \to X$ in $\text{PD}_n^+$ induces a surjection $\pi_1Y \to \pi_1X$ on fundamental groups, see for example [3], and hence we introduce the subcategory $\text{Trp}_n^+ \subseteq \text{Trp}_n$ consisting of all morphisms inducing surjections on fundamental groups. Then the functor $\tau$ yields the functor

$$\tau_+ : \text{PD}_n^+ / \simeq \to \text{Trp}_n^+.$$

As a main result in this section we show

**Theorem 3.1.** The functor $\tau_+$ reflects isomorphisms and is full for $n \geq 3$.

As corollaries we mention

**Corollary 3.2.** Take $n \geq 3$. Two $n$–dimensional manifolds, respectively, two $\text{PD}_n^+$–complexes, are orientedly homotopy equivalent if and only if their fundamental triples are isomorphic.

**Remark.** For $n = 3$, Corollary 3.2 yields the results by Thomas [18], Swarup [17] and Hendriks [9]. Turaev reproves Hendriks’ result in the appendix of [20], although the proof needs further explanation. We reprove the result again in a more algebraic way.

**Remark.** Turaev conjectures in [20] that his proof for $n = 3$ has a generalization to $\text{PD}_n^+$–complexes whose $(n-2)$–type is an Eilenberg–MacLane space $K(\pi,1)$. Corollary 3.2 proves this conjecture.

Next consider $\text{PD}_n^+$–complexes $X$ and $Y$ and a diagram

$$\begin{array}{ccc}
Y & \xrightarrow{p_{n-2}} & P_{n-2}Y \\
\downarrow f & & \downarrow \\
X & \xrightarrow{p_{n-2}} & P_{n-2}X.
\end{array}$$

(3.3)
Corollary 3.3. For \( n \geq 3 \), there is a degree 1 map \( \overrightarrow{f} \) rendering Diagram (3.3) homotopy commutative if and only if \( f \) induces a surjection on fundamental groups, is compatible with the orientations \( \omega_X \) and \( \omega_Y \), that is, \( \omega_X \pi_1(f) = \omega_Y \), and

\[
f_*p_{n-2*}[Y] = p_{n-2*}[X].
\]

Remark. Swarup [17] and Hendriks [9] prove Corollary 3.3 for 3–manifolds and PD\(^3\)–complexes, respectively.

Remark. For oriented PD\(^3\)–complexes with finite fundamental group and \( f \) a homotopy equivalence, the map \( \overrightarrow{f} \) corresponds to the map \( h \) in Lemma 1.3 [6] of Hambleton and Kreck. The reader is invited to compare our proof to that of Lemma 1.3 [6] which shows the existence of \( h \) but not the fact that \( h \) is of degree 1.

By Remark 2.3, Theorem 3.1 is a consequence of the following Lemmata 3.4 and 3.5.

Lemma 3.4. The functor \( \tau_+ \) reflects isomorphisms.

Proof. This is a consequence of Poincaré duality and Whitehead’s Theorem. \( \square \)

Remark. For \( n \geq 3 \), let \( \lfloor \frac{n}{2} \rfloor \) be the largest integer \( \leq n \). Associating with a PD\(^n\)–complex, \( X \), the pre-fundamental triple \( (\hat{P}\lfloor \frac{n}{2}\rfloor X, \omega_X, p\lfloor \frac{n}{2}\rfloor e[X]) \), there is an analogue of Lemma 3.4, namely, an orientation preserving map between PD\(^n\)–complexes is a homotopy equivalence if and only if the induced map between pre-fundamental triples is an isomorphism. However, pre-fundamental triples do not determine the homotopy type of a PD\(^n\)–complex as in Corollary 3.2, which is demonstrated by the fake products \( X = (S^n \cup S^n) \cup_a e^{2n} \), where \( a \) is the sum of the Whitehead product \([\iota_1, \iota_2]\) and an element \( \iota_1 \beta \) with \( \beta \in \pi_{2n-1}(S^n) \) having trivial Hopf invariant. Pre-fundamental triples coincide with the fundamental triple for \( n = 3 \) and \( n = 4 \). It remains an open problem to enrich the structure of a pre-fundamental triple to obtain an analogue of Corollary 3.2.

Lemma 3.5. Let \( X \) and \( Y \) be standard PD\(^n\)–complexes for \( n \geq 4 \) and weakly standard for \( n = 3 \) and let \( f : \tau_+ Y \to \tau_+ X \) be a morphism in \( \text{Trp}_+^n \). Then \( f \) is \( \tau \)–realizable by a map \( \overrightarrow{f} : Y \to X \) in PD\(^n\)\(_+\) with \( \tau \overrightarrow{f} = f \).

For the proof of Lemma 3.5, we use

Lemma 3.6. Let \( X = X' \cup e^n \) be a weakly standard PD\(^n\)–complex. Then there is a generator \( [e] \in \hat{C}_n(X) \), with \( \hat{C}_n X = \hat{C}_n X' \oplus \Lambda [e] \), corresponding to the cell \( e^n \), such that \( 1 \otimes [e] \in \mathbb{Z}^\omega \otimes_\Lambda \hat{C}_n X \) is a cycle representing the fundamental class \( [X] \). Let \( \{e_m\}_{m \in M} \) be a basis of \( \hat{C}_{n-1} X = \hat{C}_{n-1} X' \). Then the coefficients \( \{a_m\}_{m \in M} \), \( a_m \in \Lambda \) for \( m \in M \), of the linear combination \( d_n[e] = \sum a_m[e_m] \) generate \( H(\pi_1 X) \) as a right \( \Lambda \)–module.

Proof. Poincaré duality implies \( H_n(X, \mathbb{Z}^\omega) \cong H^0(X, \mathbb{Z}) \cong \mathbb{Z} \). Hence \( 1 \otimes d \) maps a multiple of the generator \( 1 \otimes [e] \) of \( \mathbb{Z}^\omega \otimes_\Lambda \hat{C}_n(X) = \mathbb{Z}^\omega \otimes_\Lambda \Lambda [e] \cong \mathbb{Z} \) to zero, that is, there is an \( n \in \mathbb{N} \) such that

\[
0 = 1 \otimes d(n(1 \otimes [e])) = n(1 \otimes d[e]) = n(1 \otimes \sum_{m \in M} a_m[e_m])
\]

\[
= n \sum a_m \otimes [e_m] = n \sum_{m \in M} \text{aug}(\overline{a_m}) \otimes [e_m].
\]
Since $\mathbb{Z}^\omega \otimes_\Lambda D_{n-1} = \mathbb{Z}^\omega \otimes_\Lambda \bigoplus_{m \in M} \Lambda[e_m] \cong \bigoplus_{m \in M} \mathbb{Z}^\omega \otimes_\Lambda \Lambda[e_m] = \bigoplus_{m \in M} \mathbb{Z}$ is free as abelian group, this implies $\text{aug}(\mathbb{Z}_m) = 0$ and hence $a_m \in I$ for every $m \in M$. Therefore $1 \otimes d(1 \otimes [e]) = 0$ and $1 \otimes [e] \in \mathbb{Z}^\omega \otimes_\Lambda D_n$ is a cycle representing a generator of the group $H_n(X, \mathbb{Z}^\omega)$. Without loss of generality we may assume that $e$ is oriented such that $1 \otimes e$ represents the fundamental class $[X]$. Further, Poincaré duality implies $H^n(X, \mathbb{Z}) \cong \mathbb{Z}$ and hence $I(\pi) \cong \text{im}(d^* \pi)$. But, for every $\varphi \in \text{Hom}_\Lambda(C_{n-1, \ast} \Lambda)$,

$$(d^* \varphi)[e] = \varphi(d[e]) = \varphi\left(\sum a_m[e_m]\right) = \sum a_m \varphi[e_m] = \left(\sum \varphi[e_m] a_m[e]\right)[e],$$

where $[e] : \Lambda[e] \to \Lambda, [e] \mapsto 1$. Thus $I(\pi)$ is generated by $\{a_m\}_{m \in M}$ as a left $\Lambda$–module and hence $I(\pi)$ is generated by $\{a_m\}_{m \in M}$ as a right $\Lambda$–module \hfill $\square$

**Lemma 3.7.** Let $\overline{X} = X' \cup_f e^3$ be a weakly standard PD$^3$–complex. Then we can choose a homotopy $f \simeq g$ such that $X = X' \cup_g e^3$ admits a splitting $\tilde{C}_2X = S \oplus d_3(\tilde{C}_3X')$ as a direct sum of $\Lambda$–modules satisfying $d_3[e] \in S$.

**Proof.** As $X'$ is homologically 2–dimensional, $\tilde{C}(\overline{X})$ admits a splitting,

$$\tilde{C}_2(\overline{X}) = \text{im}d_3' \oplus S,$$

as a direct sum of $\Lambda$–modules, where $d_3' : \tilde{C}_3(X') \to \tilde{C}_2(X')$. Thus $d_3[e] \in \tilde{C}_2(\overline{X}) = \text{im}d_3' \oplus S$ decomposes as a sum $d_3[e] = \alpha + \beta$ with $\alpha \in \text{im}d_3'$ and $\beta \in S$. Since $\alpha$, viewed as a map $S^2 \to X'$, is homotopically trivial in $X'$, there is a homotopy $f \simeq g$, where $g$ represents $\beta$, such that $X = X' \cup_g e^3$ has the stated properties. \hfill $\square$

**Proof of Lemma 3.6** Certain aspects of the proof for the case $n = 3$ differ from that for the case $n \geq 4$. Those parts of the proof pertaining to the case $n = 3$ appear in square brackets [\ldots]. [For $n = 3$ we assume that $X = X' \cup_g e^3$ is chosen as in Lemma 3.7.] Given $X = X' \cup_g e^n$ and $Y = Y' \cup_{g'} e^n$ and a morphism $\varphi = \{f\} : \tau(Y) = (P, \omega_Y, t_Y) \to \tau(X) = (Q, \omega_X, t_X)$ in $\text{Trp}_n$, the diagram

$$
\begin{array}{ccc}
X^{n-1} \subset X' \subset X & P = P_{n-2} X \\
\uparrow & \uparrow f \\
Y^{n-1} \subset Y' \subset Y & Q = P_{n-2} Y,
\end{array}
$$

commutes in $\text{CW}_0$, where $p$ and $p'$ coincide with the identity morphisms on the $(n-1)$–skeleta, and where $f$ is the restriction of $f$. For $n \geq 4$, we have $X' = X^{n-1}$ and $Y' = Y^{n-1}$. We obtain the following commutative diagram of chain complexes in $H_0$

$$
\begin{array}{ccc}
\tilde{C}X^{n-1} \subset \tilde{C}X & \tilde{C}P \\
\uparrow \pi & \uparrow f \\
\tilde{C}Y^{n-1} \subset \tilde{C}Y & \tilde{C}Q,
\end{array}
$$

\hfill $\square$
Thus there are elements \( \eta \) is homologically 2–dimensional, there is a map \( \phi \) representing \( \varphi \) with \( p\eta' = f\eta' \).

We write \( \pi = \pi_1X, \pi' = \pi_1Y, \Lambda = \mathbb{Z}[\pi] \) and \( \Lambda' = \mathbb{Z}[\pi'] \) and let \( [e'] \in \hat{C}_nY \) and \( [e] \in \hat{C}_nX \) be the elements corresponding to the \( n \)-cells \( e_n \) and \( e'_n \), respectively, \( n \geq 3 \). Since \( \{f\} \) is a morphism in \( \text{Trp}^n \), we obtain \( f_*p'_n[Y] = p_*[X] \) in \( H_n(P, \mathbb{Z}^\omega) \) and hence

\[
f_*p'_n[e'] - p_*[e] \in \text{im}(d : \hat{C}_{n+1}P \to \hat{C}_nP) + \overline{I(\pi)}\hat{C}_nP.
\]

Thus there are elements \( x \in \hat{C}_{n+1}P \) and \( y \in \overline{I(\pi)}\hat{C}_nP \) with

\[
f_*p'_n[e'] - p_*[e] = dx + y.
\]

Let \( \{e'_m\}_{m \in M} \) be a basis of \( \hat{C}_{n-1}Y \). By Lemma 3.6

\[
d[e'] = \sum a_m[e'_m],
\]

for some \( a_m \in \Lambda', m \in M \), where \( \{a_m\}_{m \in M} \) generate \( \overline{I(\pi')} \) as right \( \Lambda' \)-module. Since \( \varphi = f_* \) is surjective, \( \overline{I(\pi)} \) is generated by \( \{\varphi(a_m)\}_{m \in M} \) as right \( \Lambda \)-module, and we may write

\[
y = \sum_{m \in M} \varphi(a_m)z_m,
\]

for some \( z_m \in \hat{C}_nP, m \in M \), since there is a surjection \( \bigoplus_{m \in M} \Lambda[m] \to \overline{I(\pi)} \) of right \( \Lambda \)-modules which maps the generator \( [m] \) to \( \varphi(a_m) \). Then (3.5) implies

\[
d(f_*p'_n[e'] - p_*[e]) = dy = \sum_{m \in M} \varphi(a_m)dz_m,
\]

and hence

\[
p_*d[e] = \sum_{m \in M} \varphi(a_m)f_*p'_n[e'_m] - \sum_{m \in M} \varphi(a_m)dz_m.
\]

We define the \( \varphi \)-equivariant homomorphism

\[
\overline{\alpha}_n : \hat{C}_{n-1}Y \to \hat{C}_nP \quad \text{by} \quad \overline{\alpha}_n([e'_m]) = -z_m.
\]

For \( n \geq 4 \), we define \( \xi : \hat{C}Y \to \hat{C}X \) by \( \xi[e'] = [e] \) and

\[
\xi_i = \begin{cases} \hat{C}_{n-1}(\eta) + d\overline{\alpha}_n & \text{for } i = n - 1, \\ \hat{C}_i(\eta) & \text{for } i < n - 1. \end{cases}
\]
For $n = 3$ we use the splitting $\tilde{C}_2Y = S \oplus d_3\tilde{C}_3Y'$ in Lemma 3.7 and define
\[\xi_i : \tilde{C}_1Y \to \tilde{C}_1X \text{ by } \xi_3[e'] = [e], \xi_3|\tilde{C}_3Y' = \tilde{C}_3\eta', \text{ and}\]
\[\xi_2|S = (\tilde{C}_2\eta' + d\tilde{C}_3)|S, \]
\[\xi_2|d_3\tilde{C}_3Y' = \tilde{C}_2\eta'|d_3\tilde{C}_3Y', \]
\[\xi_i = \tilde{C}_i\eta \text{ for } i < 2.\]

To ensure that $\xi$ is a chain map, it is now enough to show that $d\xi[e'] = \xi d[e']$.

But, for the injection $\tilde{C}(p) = p_*$, we obtain
\[p_*\xi d[e'] = p_*(\tilde{C}_{n-1}(\eta') + d\tilde{C}_n)(e') =\]
\[(p \circ \eta')_{n-1}d[e'] + p_*(d\tilde{C}_n(\sum a_m[e'_m])) =\]
\[(f \circ p')_{n-1}d[e'] + p_*(\sum \varphi(a_m)d\tilde{C}_n[e'_m]) =\]
\[\sum \varphi(a_m)p'_*d[e'_m] - p_*\sum \varphi(a_m)dz_m = p_*d[e'] = p_*d\xi[e'], \text{ by } (3.8).\]

[For $n = 3$, Theorem 3.8 now implies that there is a map $\tilde{f} : Y \to X$ such that $\tilde{C}(\tilde{f}) = \xi$. Then $\tau(\tilde{f}) = f$, $\tilde{f}$ is a degree 1 map and the proof is complete for $n = 3$.]

Now let $n \geq 4$. To check that $(\xi, \eta)$ is a morphism in $H^n_{n-1}$, note that the attaching map satisfies the cocycle condition and hence, by its definition, the map $\xi_{n-1}$ commutes with attaching maps in $r(X)$ and $r(Y)$, since $\tilde{C}_{n-1}\eta$ has this property. We must show that Diagram (3.4) is homotopy commutative. But $r(f) = (f_*, \eta)$ and $r(p) = (p_*, j), r(p') = (p'_*, j')$, where $j$ and $j'$ are the identity morphisms on $X^n = P^{n-2}$ and $Y^n = Q^{n-2}$, respectively. Hence we must find a homotopy
\[\alpha : (p_*, \xi, \eta) \simeq (f_*p'_*, \eta) \text{ in } H^n_{n-1}, \text{ that is, } \varphi\text{-equivariant maps }\]
\[\alpha_{i+1} : \tilde{C}_iY \to \tilde{C}_{i+1}P, \ i \geq n - 1,\]
such that
\[\{\eta\} + g_{n-1}\alpha_{n-1} = \{\eta\}, \]
\[(p_*(\xi)_i - (f \circ p')_i) = \alpha_id + d\alpha_{i+1} \text{ for } i \geq n - 1,\]
where $g_{n-1}$ is the attaching map of $(n-1)$-cells in $P$. Define $\alpha$ by $\alpha_{n+1}[e'] = -x$, see (3.10), and
\[\alpha_i = \begin{cases} \tau_n & \text{for } i = n, \\ 0 & \text{for } i < n. \end{cases}\]

Then $\alpha$ satisfies (3.11) trivially. For $i = n - 1$, we obtain
\[(p_*(\xi)_{n-1} - (f \circ p')_{n-1}) = \xi_{n-1} - \tilde{C}_{n-1}(f) = \xi_{n-1} - \tilde{C}_{n-1}(\tilde{f}) = d\alpha_n, \text{ by } (3.10) \text{ and } (3.13).\]

For $i = n$, we evaluate (3.12) on $[e']$. By (3.9),
\[(p_*(\xi - f_*p'_*)[e']) = p_*[e] - f_*p'_*[e'] = -dx - y.\]
On the other hand,
\[(d\alpha_n + \alpha_n d)[c'] = d\alpha_{n+1}[c'] + \alpha_n \sum_{m \in M} a_m [c'_m], \quad \text{by \(3.6\)},\]
\[= -dx - \sum_{m \in M} \varphi(a_m)z_m, \quad \text{by \(3.13\) and \(3.9\)},\]
\[= -dx - y \quad \text{by \(3.7\)}.\]
Hence \(\alpha\) satisfies \(3.12\) and Diagram \(3.3\) is homotopy commutative.

To construct a morphism \(\mathbf{f} : Y \to X\) in \(\text{PD}^n\) with \(\tau(\mathbf{f}) = f\), consider the obstruction \(O(\xi, \eta) \in H^n(Y, \Gamma_{n-1}X)\) (see Section 3 and note that \(p\) induces an isomorphism \(p_* : \Gamma_{n-1}X \to \Gamma_{n-1}P\), see II.4.8 \[1\]. Hence the obstruction for the composite \(r(p)(\xi, \eta)\) coincides with \(p_*O(\xi, \eta)\), where \(p_*\) is an isomorphism. On the other hand, the obstruction for \(r(f)r(p')\) vanishes, since this map is \(\lambda\)-realizable. Thus, by the homotopy commutativity of \(3.4\), \(p_*O(\xi, \eta) = O(r(f)r(p')) = 0\), so that \(O(\xi, \eta) = 0\) and there is a \(\lambda\)-realization \((\xi, \eta')\) of \((\xi, \eta)\) in \(H_n\). Since \(H^{n+1}(Y, \Gamma_nX) = 0\), there is a \(\lambda\)-realization \((\xi, \mathbf{f})\) of \((\xi, \eta')\) in \(H_{n+1}^r\). As \(Y = Y^n, X = X^n\) and \(\xi\) is compatible with fundamental classes by construction, \(\mathbf{f} : Y \to X\) is a degree 1 map in \(\text{PD}^n\) realizing the map \(f\) in \(\text{Trp}^n\).

\[\square\]

4. \(\text{PD}^3\)-complexes

The fundamental triple of a \(\text{PD}^3\)-complex consists of a group \(\pi\), an orientation \(\omega\), and an element \(t \in H_3(\pi, \mathbb{Z})\). Here we use the standard fact that the homology of a group \(\pi\) coincides with the homology of the corresponding Eilenberg–MacLane space \(K(\pi, 1)\). In general, it is a difficult problem to actually compute \(H_3(\pi, \mathbb{Z})\). The homotopy type of a \(\text{PD}^3\)-complex is characterized by its fundamental triple, but not every fundamental triple occurs as the fundamental triple of a \(\text{PD}^3\)-complex. Via the invariant \(\nu_C(t)\) Turaev \[20\] characterizes those fundamental triples which are realizable by a \(\text{PD}^3\)-complex. Let \(\text{Trp}^3_{+, \nu}\) be the full subcategory of \(\text{Trp}^3_{+}\) consisting of fundamental triples satisfying Turaev’s realization condition. Then Theorem \(3.1\) implies

**Theorem 4.1.** The functor
\[\tau_+ : \text{PD}^3_+ / \sim \to \text{Trp}^3_{+, \nu}\]
reflects isomorphisms and is representative and full.

**Remark.** Turaev does not mention that the functor \(\tau_+\) is actually full and thus only proves the first part of the following corollary which is one of the consequences of Theorem \(4.1\).

**Corollary 4.2.** The functor \(\tau_+\) yields a 1–1 correspondence between oriented homotopy types of \(\text{PD}^3\)-complexes and isomorphism types of fundamental triples satisfying Turaev’s realization condition. Moreover, for every \(\text{PD}^3\)-complex \(X\), there is a surjection of groups
\[\tau_+ : \text{Aut}_+(X) \to \text{Aut}(\tau(X)),\]
where \(\text{Aut}_+(X)\) is the group of oriented homotopy equivalences of \(X\) in \(\text{PD}^3_+ / \sim\) and \(\text{Aut}(\tau(X))\) is the group of automorphisms of the triple \(\tau(X)\) in \(\text{Trp}^3_{+}\) which is a subgroup of \(\text{Aut}(\pi_1 X)\).
As every 3–manifold has the homotopy type of a finite standard PD³–complex, the question arises which fundamental triples in \( \text{Trp}_+ ^{\text{PD}^3} \) correspond to finite standard PD³–complexes. While Turaev does not discuss this question, we use the concept of PD³–chain complexes (see Section 2) in the category PD³∗ to do so.

**Theorem 4.3.** The functor \( \hat{C} : \text{PD}^3/ \simeq \to \text{PD}^3/ \simeq \) reflects isomorphisms and is representative and full.

**Proof.** This follows from Theorems 10.1 and 10.2 in Section 10. □

**Corollary 4.4.** The functor \( \hat{C} \) yields a 1–1 correspondence between homotopy types of PD³–complexes and homotopy types of PD³–chain complexes. Moreover, for every PD³–complex \( X \) there is a surjection of groups

\[
\hat{C} : \text{Aut}(X) \to \text{Aut}(\hat{C}(X)).
\]

**Remark 4.5.** Corollary 4.4 implies that the diagonal of every PD³–chain complex is, in fact, homotopy co–associative and homotopy co–commutative.

Connecting the functor \( \hat{C} \) and the functor \( \tau_+ \), we obtain the diagram

\[
\begin{array}{ccc}
\text{PD}_+ ^{3/ \simeq} & \xrightarrow{\hat{C}_+} & \text{PD}_+ ^{3/ \simeq} \\
\downarrow{\tau_+} & & \downarrow{\tau_*} \\
\text{Trp}^{3, \nu}_+ & \xrightarrow{\Gamma} & \tau_*^{\text{PD}^3}_+ 
\end{array}
\]

where \( \tau_+ \) determines \( \tau_* \) together with a natural isomorphism \( \tau_* \hat{C} \cong \tau_+ \).

**Corollary 4.6.** All of the functors \( \hat{C}, \tau_+ \) and \( \tau_* \) reflect isomorphisms and are full and representative.

By Remark 2.4, the functor \( \hat{C} \) yields a 1–1 correspondence between homotopy types of finite standard PD³–complexes and finite standard PD³–chain complexes, respectively.

### 5. Realizability of PD⁴–chain complexes

Given a PD⁴–chain complex \( C \), we define an invariant \( \mathcal{O}(C) \) which vanishes if and only if \( C \) is realizable by a PD⁴–complex. To this end we recall the quadratic functor \( \Gamma \) (see also (4.1) p. 13 in [1]). A function \( f : A \to B \) between abelian groups is called a quadratic map if \( f(-a) = f(a) \), for \( a \in A \), and if the function \( A \times A \to B, (a,b) \mapsto f(a+b) - f(a) - f(b) \) is bilinear. There is a universal quadratic map

\[
\gamma : A \to \Gamma(A),
\]

such that for all quadratic maps \( f : A \to B \) there is a unique homomorphism \( f^\square : \Gamma(A) \to B \) satisfying \( f^\square \gamma = f \). Using the cross effect of \( \gamma \), we obtain the Whitehead product map

\[
P : A \otimes A \to \Gamma(A),
\]

\[
a \otimes b \mapsto [a,b] = \gamma(a+b) - \gamma(a) - \gamma(b).
\]

The exterior product \( \Lambda^2 A \) of the abelian group \( A \) is defined so that we obtain the natural exact sequence

\[
\Gamma(A) \xrightarrow{H} A \otimes A \to \Lambda^2 A \to 0,
\]
where $H$ maps $\gamma(a)$ to $a \otimes a$ for $a \in A$ (see also p.14 in [1]). The composite $PH : \Gamma(A) \to \Gamma(A)$ coincides with $2id_{\Gamma(A)}$, in fact, $PH$ maps $\gamma(a)$ to $[a,a] = 2\gamma(a)$. Given a CW–complex $X$, there is a natural isomorphism $\Gamma_3(X) \cong \Gamma(\pi_2X)$, by an old result of J.H.C. Whitehead [25], where $\Gamma_3$ is Whitehead’s functor in A Certain Exact Sequence [25].

**Theorem 5.1.** Let $C = ((\pi, C), \omega, [C], \Delta)$ be a PD$^4$–chain complex with homology module $H_2(C, \Lambda) = H_2$. Then there is an invariant

$$O(C) \in H_0(\pi, \Lambda^2H_2^\zeta)$$

with $O(C) = 0$ if and only if there is a PD$^4$–complex $X$ such that $\mathcal{C}(X)$ is isomorphic to $C$ in PD$^4_\zeta$. Moreover, if $O(C) = 0$, the group

$$\ker (H_\ast : H_0(\pi, \Gamma(H_2^\zeta) \to H_0(\pi, H_2^\zeta \otimes H_2^\zeta)))$$

acts transitively and effectively on the set Real$_{\mathcal{C}}(C)$ of realizations of $C$ in PD$^4_\zeta$. Here $\ker H_\ast$ is 2–torsion.

**Proof.** First note that

$$H^4(C, \Lambda^2H_2) \cong H_0(\pi, \Lambda^2H_2^\zeta) \cong H_0(\pi, \Lambda^2H_2^\zeta).$$

By Lemma [2.3] we may assume that $C$ is 2–realizable. By Proposition 5.3, there is thus a 4–dimensional CW–complex $X$ together with an isomorphism $\mathcal{C}X \cong (\pi, C)$. The CW–complex $X$ yields the homotopy systems $\lambda X$ in $H^3_\zeta$ and $\lambda X$ in $H^4$ with $\lambda X = r(X)$ and $\lambda X = \lambda X$. By Theorem [10.1] we may choose a diagonal $\Delta : X \to \lambda X \otimes \lambda X$ inducing $\Delta : C \to C \otimes C$, whose homotopy class is determined by $\Delta$. However, $\lambda X$ need not be $\lambda$–realizable. Lemma [6.1] shows that there is an obstruction

$$O' = O_{\lambda X \otimes \lambda X}(\Delta) \in H^4(C, \Gamma_3(\lambda X \otimes \lambda X))$$

which vanishes if and only if there is a diagonal $\Delta : \lambda X \to \lambda X \otimes \lambda X$ realizing $\lambda X$. Note that $O'$ is determined by the diagonal $\Delta$ on $C$, since the obstruction depends on the homotopy class of $\lambda X$ only. By Theorem [10.2] the existence of $\lambda X$ realizing $\lambda X$ also implies the existence of $\Delta : X \to X \times X$ realizing $\Delta$. But

$$\Gamma_3(\lambda X \otimes \lambda X) \cong \Gamma(\pi_2(\lambda X \otimes \lambda X))$$

$$\cong \Gamma(\pi_2(\lambda X \times \lambda X))$$

$$\cong \Gamma(\pi_2 \otimes \pi_2)$$

where $\pi_2 = \pi_2 X$.

Applying Lemma [6.2] (1), we see that

$$O' \in \ker p_i \ast \text{ for } i = 1, 2,$$

where $p_i : \pi_2 \ast \pi_2 \to \pi_2$ is the $i$–th projection. Now

$$\Gamma(\pi_2 \ast \pi_2) = \Gamma(\pi_2) \ast \pi_2 \ast \pi_2 \ast \Gamma(\pi_2)$$

and hence $O'$ yields $O'' \in H^4(C, \pi_2 \ast \pi_2)$. While the homotopy type of $\pi X$ is determined by $C$, the homotopy type of $\lambda X$ is an element of Real$\lambda(\lambda X)$ and the group $H^4(C, \Gamma(\pi_2))$ acts transitively and effectively on this set of realizations. To describe the behaviour of the obstruction under this action using Lemma [5.3] we first consider the homomorphism

$$\nabla = \Delta_* - \iota_{1*} - \iota_{2*} : \Gamma(\pi_2) \to \Gamma(\pi_2 + \pi_2),$$
where $\Delta : \pi_2 \to \pi_2 \oplus \pi_2$ maps $x \in \pi_2$ to $\iota_1(x) + \iota_2(x)$. We obtain, for $x \in \pi_2$,

$$\nabla(\gamma(x)) = \gamma(\iota_1(x) + \iota_2(x)) - \gamma(\iota_1(x)) - \gamma(\iota_2(x))$$

$$= [\iota_1(x), \iota_2(x)]$$

$$= x \otimes x \in \pi_2 \otimes \pi_2 \subset \Gamma(\pi_2 \oplus \pi_2),$$

showing that $\nabla$ coincides with $H : \Gamma(\pi_2) \to \pi_2 \otimes \pi_2$. Given $\alpha \in \mathbb{H}^4(C, \Gamma(\pi_2))$, the obstruction $O'' = \mathcal{O}_{\bigwedge^2 \pi_2 \otimes \pi_2}(\Delta)$ with $\bigwedge^2 \pi_2 = X + \alpha$ satisfies

$$O'' = O'' + H_\ast \alpha,$$

by Lemma 9.3. The exact sequence

$$0 \to H^4(\mathbb{C}, \Gamma(\pi_2)) \to H^4(\mathbb{C}, \pi_2 \otimes \pi_2) \to H^4(\mathbb{C}, \Lambda^2 \pi_2) \to 0$$

allows us to identify the coset of $\im H_\ast$ represented by $O''$ with an element

$$\mathcal{O} \in H^4(\mathbb{C}, \Lambda^2 H_2),$$

where $H_2 = H_2(\mathbb{C}, \Lambda) \cong \pi_2$. By the isomorphisms (5.2), this element yields the invariant

$$\mathcal{O} \in H_0(\pi, \Lambda^2 H_2)$$

with the properties stated. Given that $O''$ vanishes, the obstruction $O''$ vanishes if and only if $\alpha \in \ker H_\ast$, and Proposition 8.3 yields the result on Real$_\mathbb{C}(\mathbb{C})$. We observe that $\ker H_\ast$ is 2–torsion as $H_\ast(x) = 0$ implies $2x = P_\ast H_\ast x = 0$. □

**Theorem 5.2.** Let $C = ((\pi, \mathbb{C}), \omega, [C], \Delta)$ be a PD$^4$–chain complex for which $\Delta$ is homotopy co-commutative. Then the obstruction $\mathcal{O}(C)$ is 2–torsion, that is, $2\mathcal{O}(C) = 0$.

**Proof.** Lemma 9.2 (2) states

$$\mathcal{O} \in \ker(\id_\ast - T_\ast)\ast,$$

where $\id$ is the identity on $\pi_2 \oplus \pi_2$ and $T$ is the interchange map on $\pi_2 \oplus \pi_2$ with $T_{\iota_1} = \iota_2$ and $T_{\iota_2} = \iota_1$. Thus $T$ induces the map $-\id$ on $\Lambda^2 \pi_2$ and the result follows. □

**Remark.** Lemma 9.2 (3) concerning homotopy associativity of the diagonal does not yield a restriction of the invariant $\mathcal{O}(C)$.

**Theorem 5.3.** The functor $\hat{C}$ induces a 1–1 correspondence between homotopy types of PD$^4$–complexes with finite fundamental group of odd order and homotopy types of PD$^4$–chain complexes with homotopy co–commutative diagonal and finite fundamental group of odd order.

**Proof.** Since $\pi$ is of odd order, the cohomology $H^0(\pi, M)$ is odd torsion and the result follows from Theorem 5.1. □

**Remark.** By Theorem 5.3, every PD$^4$–chain complex with homotopy co–commutative diagonal and odd fundamental group has a homotopy co–associative diagonal.

Up to 2–torsion, Theorem 5.1 yields a correspondence between homotopy types of PD$^4$–complexes and homotopy types of PD$^3$–chain complexes. In Section 7 below we provide a precise condition for a PD$^4$–chain complex to be realizable by a PD$^4$–complex.
6. The chains of a 2–type

The fundamental triple of a PD$^4$–complex $X$ comprises its 2–type $T = P_2 X$ and an element of the homology $H_4(T, \mathbb{Z}^\omega)$. To compute $H_4(T, \mathbb{Z}^\omega)$, we construct a chain complex $P(T)$ which approximates the chain complex $\tilde{C}(T)$ up to dimension 4. Our construction uses a presentation of the fundamental group as well as the concepts of pre–crossed module and Peiffer commutator. To introduce these concepts, we work with right group actions as in [1], and define $P(T)$ as a chain complex of right $\Lambda$–modules. With any left $\Lambda$–module $M$ we associate a right $\Lambda$–module in the usual way by setting $x.\alpha = \alpha^{-1}.x$, for $\alpha \in \pi$ and $x \in M$, and vice versa.

A pre–crossed module is a group homomorphism $\partial : \rho_2 \to \rho_1$ together with a right action of $\rho_1$ on $\rho_2$, such that

$$\partial(x^\alpha) = -\alpha + \partial x + \alpha \quad \text{for} \quad x \in \rho_2, \alpha \in \rho_1,$$

where we use additive notation for the group law in $\rho_1$ and $\rho_2$, as in [1]. For $x, y \in \rho_2$, the Peiffer commutator is given by

$$\langle x, y \rangle = -x - y + x + y^{\partial x}.$$

A pre–crossed module is a crossed module, if all Peiffer commutators vanish. A map of pre–crossed modules, $(m, n) : \partial \to \partial'$ is given by a commutative diagram

$$\begin{array}{ccc}
\rho_2 & \xrightarrow{m} & \rho'_2 \\
\downarrow{\partial} & & \downarrow{\partial'} \\
\rho_1 & \xrightarrow{n} & \rho'_1
\end{array}$$

in the category of groups, where $m$ is $n$–equivariant. Let cross be the category of crossed modules and such morphisms. A weak equivalence in cross is a map $(m, n) : \partial \to \partial'$, which induces isomorphisms $\text{coker}\partial \cong \text{coker}\partial'$ and $\ker\partial \cong \ker\partial'$, and we denote the localization of cross with respect to weak equivalences by $\text{Ho}(\text{cross})$. By an old result of Whitehead–Mac Lane, there is an equivalence of categories

$$\pi : 2 \text{– types} \to \text{Ho}(\text{cross}),$$

compare Theorem III 8.2 in [1]. The functor $\pi$ carries a 2–type $T$ to the crossed module $\partial : \pi_2(T, T^1) \to \pi_1(T^1)$.

A pre–crossed module is totally free, if $\rho_1 = \langle E_1 \rangle$ is a free group generated by a set $E_1$ and $\rho_2 = \langle E_2 \times \rho_1 \rangle$ is a free group generated by a free $\rho_1$–set $E_2 \times \rho_1$ with the obvious right action of $\rho_1$. A function $f : E_2 \to \langle E_1 \rangle$ yields the associated totally free pre–crossed module $\partial_f : \rho_2 \to \rho_1$ with $\partial_f(x) = f(x)$ for $x \in E_2$. Let $\text{Pei}_n(\partial_f) \subset \rho_2$ be the subgroup generated by $n$–fold Peiffer commutators and put $\overline{\rho}_2 = \rho_2/\text{Pei}_2(\partial_f)$. Let cross$^=$ be the category whose objects are pairs $(\partial_f, B)$, where $\overline{\partial}_f$ is a totally free pre–crossed module $\overline{\partial}_f : \rho_2 \to \rho_1$ and $B$ is a submodule of $\ker(\partial : \overline{\rho}_2 \to \rho_1)$. Further, a morphism $m : (\partial_f, B) \to (\partial_{f'}, B')$ in cross$^=$ is a map $\overline{\partial}_f \to \overline{\partial}_{f'}$ which maps $B$ into $B'$. Then there is a functor

$$q : \text{cross}^= \to \text{cross} \to \text{Ho}(\text{cross}),$$

which assigns to $(\partial_f, B)$ the crossed module $\overline{\rho}_2/B \to \rho_1$, and one can check that $q$ is full and representative. Given any map $g : T \to T'$ between 2–types, we may choose a map $\overline{g} : (\partial_f, B) \to (\partial_{f'}, B')$ in cross$^=$ representing the homotopy class of $g$ via the functor $q$ and the equivalence $\pi$. We call $\overline{g}$ a map associated with $g$. 
Given an action of the group $\pi$ on the group $M$ and a group homomorphism $\varphi : N \to \pi$, a $\varphi$-crossed homomorphism $h : N \to M$ is a function satisfying
\[ h(x + y) = (h(x))^{\varphi(y)} + h(y) \quad \text{for } x, y \in N. \]
By an old result of Whitehead \cite{Whitehead}, the totally free crossed module $\mathfrak{m}_2 \to \mathfrak{m}_1$ enjoys the following properties.

**Lemma 6.1.** Let $X^2$ be a 2–dimensional CW–complex in $\mathbf{CW}_0$ with attaching map of 2–cells $f : E_2 \to (E_1) = \pi_1(X^1)$. Then there is a commutative diagram
\[
\begin{array}{ccc}
\pi_2(X^2, X^1) & \xrightarrow{\partial} & \pi_1(X^1) \\
\mathfrak{m}_2 & \xrightarrow{\mathfrak{m}_1} & \mathfrak{m}_1,
\end{array}
\]
identifying $\partial$ with the totally free crossed module $\mathfrak{m}_1$. Moreover, the abelianization of $\mathfrak{m}_2$ coincides with $\hat{C}_2(X^2)$, identifying the kernel of $\mathfrak{m}_1$ with the kernel of $d_2 : \hat{C}_2(X^2) \to \hat{C}_1(X^2)$, and $\mathfrak{m}_1$ determines the boundary $d_2$ via the commutative diagram
\[
\begin{array}{ccc}
\hat{C}_2(X^2) & \xrightarrow{\partial} & \hat{C}_1(X^2) \\
\mathfrak{m}_2 & \xrightarrow{\mathfrak{m}_1} & \mathfrak{m}_1,
\end{array}
\]
Here $h_2$ is the quotient map and $h_1$ is the $(q : \mathfrak{m}_1 \to \pi_1(X^2))$–crossed homomorphism which is the identity on the generating set $E_1$. Each map $\mathfrak{m}_1 \to \mathfrak{m}_1$ induces a chain map $\hat{C}_2(X^2) \to \hat{C}_2(X'^2)$ where $X^2$ and $X'^2$ are the 2–dimensional CW–complexes with attaching maps $f$ and $f'$, respectively.

In addition to Lemma 6.1 we need the following result on Peiffer commutators, which was originally proved in IV (1.8) of \cite{Conduche} and generalized in a paper with Conduché \cite{Conduche}.

**Lemma 6.2.** With the notation in Lemma 6.1, there is a short exact sequence
\[ 0 \to \Gamma(K) \to \hat{C}_2(X^2) \otimes \hat{C}_2(X^2) \xrightarrow{\omega} \text{Peier}(\mathfrak{m}_1)/\text{Peier}(\mathfrak{m}_1) \to 0, \]
where $K = \ker d_2 = \pi_2 X^2$ and $\omega$ maps $x \otimes y$ to the Peiffer commutator $[\xi, \eta]$ with $\xi, \eta \in \rho_2$ representing $x$ and $y$, respectively.

**Definition 6.3.** Given a 2–type $T$ in 2–types, we define the chain complex $P(T) = P(\mathfrak{m}_1, B)$ as follows. Let $f : E_2 \to (E_1)$ be the attaching map of 2–cells in $T$ and put $C_i = \hat{C}_i(T)$. Then the 2–skeleton of $P(T)$ coincides with $\hat{C}(T^2)$, that is, $P_i(T) = C_i$ for $i \leq 2$, and $P_i(T) = 0$ for $i > 4$. To define $P_4(T)$, let $H$ be the map in \cite{Conduche} and put $B = \text{im}(d : C_3 \to C_2)$ and $\nabla_B = B \otimes B + H[B, C_2]$ as a submodule of $C_2 \otimes C_2$. Then $P_4(T)$ is given by the quotient
\[ P_4(T) = C_2 \otimes C_2/\nabla_B. \]
To define $P_3(T)$, we use Lemma 6.1, Lemma 6.2 and the identification $\pi_2 T^2 = \ker(d : C_2 \to C_1)$ and put $\sigma_2 = \rho_2/\text{Peier}(\mathfrak{m}_1)$. Then $P_3(T)$ is given by the pull–back
The chain complex $P(T)$ is determined by the commutative diagram

\[
\begin{array}{ccccccc}
P_4(T) & \xrightarrow{d} & P_3(T) & \xrightarrow{\pi} & P_2(T) & \xrightarrow{\psi} & P_1(T) & \xrightarrow{\pi} & P_0(T) \\
C_2 \otimes C_2 / \nabla_B & \xrightarrow{-\omega} & \sigma_2 / \omega \nabla_B & B & \xrightarrow{\psi} & C_2 & \xrightarrow{\psi} & C_1 & \xrightarrow{\psi} & C_0.
\end{array}
\]

Clearly, $P(T) = P(\partial_f, B)$ depends on the pair $(\partial_f, B)$ only and yields a functor $P : \operatorname{cross} \rightarrow \mathbb{H}_0$.

The homology of $P(T)$ is given by

\[
H_i(P(T)) = \begin{cases} 
0 & \text{for } i = 1 \text{ and } i = 3, \\
H_2 C = \pi_2 T & \text{for } i = 2, \\
\Gamma(\pi_2(T)) & \text{for } i = 4.
\end{cases}
\]

**Lemma 6.4.** Given a 2–type $T$, there is a chain map

\[
\bar{\beta} : \tilde{C}(T) \rightarrow P(T)
\]

inducing isomorphisms in homology in degree $\leq 4$. The map $\bar{\beta}$ is natural in $T$ up to homotopy, that is, a map $g : T \rightarrow T'$ between 2–types yields a homotopy commutative diagram

\[
\begin{array}{cccccc}
\tilde{C}(T) & \xrightarrow{g^*} & \tilde{C}(T') \\
\xrightarrow{\bar{\beta}} & \xrightarrow{\bar{\beta}} & \xrightarrow{\bar{\beta}} \\
P(T) & \xrightarrow{g^*} & P(T'),
\end{array}
\]

where $g^*$ is induced by a map $\bar{\beta} : \partial_f \rightarrow \partial_f'$ associated with $g$.

For a proof of Lemma 6.4 we refer the reader to diagram (1.2) in Chapter V of [1]. In order to compute the fourth homology or cohomology of a 2–type $T$ with coefficients, choose a pair $(\partial_f, B)$ representing $T$ and a free chain complex $C$ together with a weak equivalence of chain complexes

\[
C \xrightarrow{\sim} P(\partial_f, B).
\]

Then, for right $\Lambda$–modules $M$ and left $\Lambda$–modules $N$,

\[
\begin{align*}
H_4(T, M) &= H_4(C \otimes M), \\
H^4(T, N) &= H^4(\text{Hom}_\Lambda(C, N)).
\end{align*}
\]

This allows for the computation of $H_4$ in terms of chain complexes only, as is the case for the computation of group homology in Section 4. Of course, it is also possible to compute the homology of $T$ in terms of a spectral sequence associated with the fibration

\[
K(\pi_2(T), 2) \rightarrow T \rightarrow K(\pi_1(T), 1).
\]
However, in general, this yields non–trivial differentials, which may be related to the properties of the chain complex \( P(\partial f, B) \).

7. Algebraic models of PD\(^4\)--complexes

Let \( X \) be a 4–dimensional CW–complex and let
\[
p_2 : X \longrightarrow P_2X = T
\]
be the map to the 2–type of \( X \), as in (3.1). Then \( p_2 \) yields the chain map
\[
\beta : \hat{C}(X) \xrightarrow{p_2 \ast} \hat{C}(T) \xrightarrow{\gamma} P(T) = P(\partial f, B),
\]
were \( \partial f \) is given by the attaching map of 2–cells in \( X \) and \( B = \text{im}(d_3 : \hat{C}_3(X) \rightarrow \hat{C}_2(X)) \). We call the chain map \( \beta \) the cellular boundary invariant of \( X \).

**Lemma 7.1.** Let \( X \) and \( X' \) be 4–dimensional CW–complexes. A chain map \( \varphi : \hat{C}(X) \rightarrow \hat{C}(X') \) is realizable by a map \( g : X \rightarrow X' \) in \( \mathbf{CW}_0 \), that is, \( \varphi = g_* \), if and only if the diagram
\[
\begin{array}{ccc}
\hat{C}(X) & \xrightarrow{\varphi} & \hat{C}(X') \\
\beta \downarrow & & \beta' \downarrow \\
P(\partial f, B) & \xrightarrow{\varphi} & P(\partial f', B')
\end{array}
\]
commutes up to homotopy. Here \( \varphi : \partial f \rightarrow \partial f' \) is a map in \( \text{cross}^= \) inducing \( \varphi_{\leq 2} : \hat{C}(X^2) \rightarrow \hat{C}(X'^2) \) as in Lemma 6.1.

**Proof.** By Lemma 6.4, the diagram
\[
\begin{array}{ccc}
\hat{C}(X) & \xrightarrow{\varphi} & \hat{C}(X') \\
\beta \downarrow & & \beta' \downarrow \\
P(\partial f, B) & \xrightarrow{\varphi} & P(\partial f', B')
\end{array}
\]
is homotopy commutative, where \( g \) is given by \( q(\beta) \) in \( \text{Ho}(\text{cross}) \). Since \( p_2 \ast \) and \( g_* \) are realizable, the obstruction \( O_{X,X'}(\varphi) \) vanishes. \( \square \)

**Definition 7.2.** A \( \beta \)-PD\(^4\)--chain complex is a PD\(^4\)--chain complex \(((\pi, C), \omega, |C|, \Delta)\) together with a totally free pre–crossed module \( \partial f \) inducing \( d_2 : C_2 \rightarrow C_1 \) and a chain map
\[
\beta : C \longrightarrow P(\partial f, B)
\]
which is the identity in degree \( \leq 2 \). Here \( B = \text{im}(d_3 : C_3 \rightarrow C_2) \), the diagram
\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\beta \downarrow & & \beta \otimes \Delta \downarrow \\
P(\partial f, B) & \xrightarrow{\beta \otimes} & P(\partial f \otimes f, B \otimes)
\end{array}
\]
commutes up to homotopy and \( \beta \) is the cellular boundary invariant \( \beta_* \) of a totally free quadratic chain complex \( \sigma \) defined in V(1.8) of [1]. Further, \( \beta \otimes \) is the cellular boundary invariant of the quadratic chain complex \( \sigma \otimes \sigma \) defined in Section IV 12 of [1], and there is an explicit formula expressing \( \beta \otimes \) in terms of \( \beta \), which we do
not recall here. The function \( f \otimes f \) is the attaching map of 2–cells in the product \( X^2 \times X^2 \), where \( X^2 \) is given by \( f \), and \( B^\otimes \) is the image of \( d_3 \) in \( C \otimes C \). The map \( \overline{\Delta} \) in \( \text{cross}^\equiv \) is chosen such that \( \overline{\Delta} \) induces \( \Delta \) in degree \( \leq 2 \) as in Lemma 7.1. Let \( \text{PD}_{4,\beta} \) be the category whose objects are \( \beta \)-PD\(^4\)–chain complexes and whose morphisms are maps \( \varphi \) in \( \text{PD}_4^\ast \) such that the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\varphi} & C' \\
\beta \downarrow & & \downarrow \beta' \\
P(\partial f, B) & \xrightarrow{\overline{\varphi}} & P(\partial f', B')
\end{array}
\]

is homotopy commutative, where \( \overline{\varphi} \) induces \( \varphi_{\leq 2} \) as in Lemma 7.1.

**Theorem 7.3.** The functor \( \widehat{\mathcal{C}} \) yields a functor

\[
\widehat{\mathcal{C}} : \text{PD}_4^\ast / \simeq \longrightarrow \text{PD}_{4,\beta}^\ast / \simeq
\]

which reflects isomorphisms and is representative and full.

**Proof.** Since \( C \) is 2–realizable, there is a 4–dimensional CW–complex \( X \) with \( \widehat{\mathcal{C}}(X) = C \) and cellular boundary invariant \( \beta \). By Lemma 7.1, the diagonal \( \Delta \) is realizable by a diagonal \( X \to X \times X \), showing that \( X \) is a PD\(^4\)–complex. By Lemma 7.1, a map \( \varphi \) is realizable by a map \( X \to X' \). \( \square \)

**Corollary 7.4.** The functor \( \widehat{\mathcal{C}} \) induces a 1–1 correspondence between homotopy types of PD\(^4\)–complexes and homotopy types of \( \beta \)-PD\(^4\)–chain complexes.

The functor \( \tau \) in Section 3 yields the diagram of functors

\[
\begin{array}{ccc}
\text{PD}_4^\ast / \simeq & \xrightarrow{\widehat{\mathcal{C}}} & \text{PD}_{4,\beta}^\ast / \simeq \\
\tau_+ & \downarrow & \tau_+ \\
\text{Trp}_4^\ast & \xrightarrow{\tau_*} & \text{Trp}_{4,\beta}^\ast
\end{array}
\]

where \( \tau_+ \) determines \( \tau_* \) together with a natural isomorphism \( \tau_* \widehat{\mathcal{C}} \cong \tau_+ \).

**Corollary 7.5.** The functor \( \tau_* \) in (7.1) reflects isomorphisms and is full.

8. Homotopy systems of order \((k+1)\)

To investigate questions of realizability, we work in the category \( \mathcal{H}_{k}^{k+1} \) of homotopy systems of order \((k+1)\). Let \( \text{CW}_0^k \) be the full subcategory of \( \text{CW}_0 \) consisting of \( k \)-dimensional CW–complexes. A 0–homotopy \( H \) in \( \text{CW}_0 \), denoted by \( \simeq^0 \), is a homotopy for which \( H_t \) is cellular for each \( t, 0 \leq t \leq 1 \).

Let \( k \geq 2 \). A homotopy system of order \((k+1)\) is a triple \( X = (C, f_{k+1}, X^k) \), where \( X^k \) is an object in \( \text{CW}_0^k \), \( C \) is a chain complex of free \( \pi_1(X^k) \)–modules, which coincides with \( \widehat{\mathcal{C}}(X^k) \) in degree \( \leq k \), and where \( f_{k+1} \) is a homomorphism of left \( \pi_1(X^k) \)–modules such that

\[
\begin{array}{ccc}
C_{k+1} & \xrightarrow{f_{k+1}} & \pi_k(X^k) \\
\downarrow d & & \downarrow j \\
C_k & \xrightarrow{h_k} & \pi_k(X^k, X^{k-1})
\end{array}
\]
commutes. Here $d$ is the boundary in $C$,

$$h_k : \pi_k(X^k, X^{k-1}) \xrightarrow{p_k^{-1}} \pi_k(\tilde{X}^k, \tilde{X}^{k-1}) \xrightarrow{h} \text{Hom}(\tilde{X}^k, \tilde{X}^{k-1}),$$

given by the Hurewicz isomorphism $h$ and the inverse of the isomorphism on the relative homotopy groups induced by the universal covering $p : \tilde{X} \to X$. Moreover, $f_{k+1}$ satisfies the cocycle condition

$$f_{k+1}d(C_{k+2}) = 0.$$ 

For an object $X$ in $\text{CW}^1_0$, the triple $r(X) = (\tilde{C}(X), f_{k+1}, X^k)$ is a homotopy system of order $(k+1)$, where $X^k$ is the $k$–skeleton of $X$, and

$$f_{k+1} : \tilde{C}_{k+1}(X) \xrightarrow{\pi_{k+1}(X^{k+1}, X^k)} \pi_k(X^k)$$

is the attaching map of $(k+1)$–cells in $X$. A morphism or map between homotopy systems of order $(k+1)$ is a pair

$$(\xi, \eta) : (C, f_{k+1}, X^k) \to (C', g_{k+1}, Y^k),$$

where $\eta : X^k \to Y^k$ is a morphism in $\text{CW}_{\mathbb{Z}/2}\eta \simeq 0$ and the $\pi_{k+1}(\eta)$–equivariant chain map $\xi : C \to C'$ coincides with $\tilde{C}_*(\eta)$ in degree $\leq k$ such that

$$\begin{array}{ccc}
C_{k+1} & \xrightarrow{\xi_{k+1}} & C'_{k+1} \\
\downarrow f_{k+1} & & \downarrow g_{k+1} \\
\pi_k(X^{k+1}) & \xrightarrow{\eta} & \pi_k(Y^k)
\end{array}$$

commutes. We also write $\pi_1 X = \pi_1(X^k)$ for an object $X = (C, f_{k+1}, X^k)$ in $\text{Hom}^1_{\mathbb{Z}/2}$.

To define the homotopy relation in $\text{Hom}^1_{\mathbb{Z}/2}$, we use the action (see ??? in [1])

$$(8.1) \quad [X^k, Y]_{\varphi} \times \tilde{H}^k(X^k, \varphi^* \pi_k Y) \to [X^k, Y]_{\varphi}, \quad (F, \{\alpha\}) \mapsto F + \{\alpha\},$$

where $[X^n, Y]_{\varphi}$ is the set of elements in $[X^n, Y]$ which induce $\varphi$ on the fundamental groups. Two morphisms

$$(\xi, \eta), (\xi', \eta') : (C, f_{k+1}, X^k) \to (C', g_{k+1}, Y^k)$$

are homotopy equivalent in $\text{Hom}^1_{\mathbb{Z}/2}$ if $\pi_1(\eta) = \pi_1(\eta') = \varphi$ and if there are $\varphi$–equivariant homomorphisms $\alpha_{j+1} : C_j \to C'_{j+1}$ for $j \geq k$ such that

$$\{\eta\} + g_{k+1} \alpha_{k+1} = \{\eta'\} \quad \text{and} \quad \xi'_{i} - \xi_i = \alpha_i \alpha + d \alpha_{i+1}, \quad i \geq k+1,$$

where $\{\eta\}$ denotes the homotopy class of $\eta$ in $[X^k, Y^k]$ and $+ \alpha$ is the action (8.1).

Given homotopy systems $X = (C, f_{k+1}, X^k)$ and $Y = (C', g_{k+1}, Y^k)$, consider

$$X \otimes Y = (C \otimes_{\mathbb{Z}} C', h_{k+1}, (X^k \times Y^k)^k),$$

where we choose CW–complexes $X^{k+1}$ and $Y^{k+1}$ with attaching maps $f_{k+1}$ and $g_{k+1}$, respectively, and $h_{k+1}$ is given by the attaching maps of $(k+1)$–cells in $X^{k+1} \times Y^{k+1}$. Then $X \otimes Y$ is a homotopy system of order $(k+1)$, and

$$\otimes : \text{Hom}^1_{\mathbb{Z}/2} \times \text{Hom}^1_{\mathbb{Z}/2} \to \text{Hom}^1_{\mathbb{Z}/2}$$

is a bi–functor, called the tensor product of homotopy systems. The projections $p_1 : X \otimes Y \to X$ and $p_2 : X \otimes Y \to Y$ in $\text{Hom}^1_{\mathbb{Z}/2}$ are given by the projections of
the tensor product and the product of CW–complexes. Similarly, we obtain the
inclusions \( i_1 : X \to X \otimes Y \) and \( i_2 : Y \to X \otimes Y \). Then \( p_1 i_1 = \text{id}_X \) and \( p_2 i_2 = \text{id}_Y \),
while \( p_1 i_2 \) and \( p_2 i_1 \) yield the trivial maps.

There are functors

\[
\begin{align*}
\mathcal{CW}_0 & \longrightarrow H_{k+1}^f & \longrightarrow H_k^f & \longrightarrow H_0
\end{align*}
\]

for \( k \geq 3 \), with \( r(X) = (\tilde{C}(X), f_{k+1}, X^k) \) such that \( r = \lambda r \). We write \( \lambda X = \overline{X} \) for
objects \( X \) in \( H_{k+1}^f \). As \( X \otimes Y = \lambda(X \otimes Y) = \overline{X} \otimes \overline{Y} \), the functor \( \lambda \), and also \( r \)
and \( C \), is a monoidal functor between monoidal categories. There is a homotopy
relation defined on the category \( H_{k+1}^f \), such that these functors induce functors
between homotopy categories

\[
\mathcal{CW}_0/ \simeq \frac{r}{H_{k+1}^f} \simeq \frac{\lambda}{H_k^f} \simeq \frac{C}{H_0}/ \simeq .
\]

For \( k \geq 3 \), Whitehead’s functor \( \Gamma_k \) factors through the functor \( r : \mathcal{CW} \to H_k^f \),
so that the cohomology \( H_m(\overline{X}, \varphi^* \Gamma_k(\overline{Y})) = H^m(C, \varphi^* \Gamma_k(\overline{Y})) \) is defined, where
\( \varphi : \pi_1 \overline{X} \to \pi_1 \overline{Y} \) and \( \overline{X} \) and \( \overline{Y} \) are objects in \( H_k^f \).

To describe the obstruction to realizing a map \( f = (\xi, \eta) : \overline{X} \to \overline{Y} \) in \( H_k^f \), where
\( \overline{X} = \lambda X \) and \( \overline{Y} = \lambda Y \), by a map \( X \to Y \) in \( H_{k+1}^f \) for objects \( X = (C, f_{k+1}, X^k) \)
and \( Y = (C', g_{k+1}, Y^k) \), choose a map \( F : X^k \to Y^k \) in \( \mathcal{CW}/\simeq_0 \) extending \( \eta : X^{k-1} \to Y^{k-1} \) and for which \( \tilde{C}_* F \) coincides with \( \xi \) in degree \( \leq n \). Then

\[
\begin{array}{ccc}
C_{k+1} & \longrightarrow & C'_{k+1} \\
\downarrow f_{k+1} & & \downarrow g_{k+1} \\
\pi_k(X^k) & \stackrel{F_*}{\longrightarrow} & \pi_k(Y^k)
\end{array}
\]

need not commute and the difference \( O(F) = -g_{k+1} \xi_{k+1} + F_* f_{k+1} \) is a cocycle in
\( \text{Hom}_\varphi(C_{k+1}, \Gamma_k(\overline{Y})) \). Theorem II 3.3 in \[1\] implies

**Proposition 8.1.** The map \( f = (\xi, \eta) : \overline{X} \to \overline{Y} \) in \( H_k^f \) can be realized by a map
\( f_0 = (\xi, \eta_0) : X \to Y \) in \( H_{k+1}^f \) if and only if \( O_{X,Y}(f) = \{ O(F) \} \in \tilde{H}^{k+1}(\overline{X}, \varphi^* \Gamma_k(\overline{Y})) \)
vanishes. The obstruction \( O \) is a derivation, that is, for \( f : \overline{X} \to \overline{Y} \) and \( g : \overline{Y} \to \overline{Z} \),

\[
O_{X,Z}(gf) = g_* O_{X,Y}(f) + f^* O_{Y,Z}(g),
\]

and \( O_{X,Y}(f) \) depends on the homotopy class of \( f \) only.

Denoting the set of morphisms \( X \to Y \) in \( H_{k+1}^f \) by \([X, Y]_\varphi\), and the subset of morphisms inducing \( \varphi \) on the fundamental groups by \([X, Y]_\varphi \subseteq [X, Y] \), there is a group action

\[
[X, Y]_\varphi \times \tilde{H}^k(\overline{X}, \varphi^* \Gamma_k(\overline{Y})) \text{ acts transitively on } [X, Y]_\varphi,
\]

where \( \overline{X} = \lambda X \) and \( \overline{Y} = \lambda Y \). Theorem II 3.3 in \[1\] implies

**Proposition 8.2.** Given morphisms \( f_0, f'_0 \in [X, Y]_\varphi \), then \( \lambda f_0 = \lambda f'_0 = f \) if
and only if there is an \( \alpha \in \tilde{H}^k(\overline{X}, \varphi^* \Gamma_k(\overline{Y})) \) with \( f'_0 = f_0 + \alpha \). In other words,
\( \tilde{H}^k(\overline{X}, \varphi^* \Gamma_k(\overline{Y})) \) acts transitively on the set of realizations of \( f \). Further, the action satisfies the linear distributivity law

\[
(f_0 + \alpha)(g_0 + \beta) = f_0 g_0 + f_* \beta + g^* \alpha.
\]

For the functor \( \lambda \) in \[2\], Theorem II 3.3 in \[1\] implies
Lemma 9.1. Let \( \Delta : X \to X \otimes X \) in \( \text{H}_{k+1}^c \) be a diagonal. Suppose \( \lambda \) realizes the same homotopy class of maps for all \( \lambda \in \hat{H}^{k+1}(X, \Gamma_k X) \), there is an object \( X' \) in \( \text{H}_{k+1}^c \) with \( \lambda(X') = \lambda(X) = X \) and \( O_{X,X}(\text{id}_X) = \alpha \). We then write \( X' = X + \alpha \).

Now let \( Y \) be an object in \( \text{H}_{k}^c \). Then the group \( \hat{H}^{k+1}(Y, \Gamma_k Y) \) acts transitively and effectively on \( \text{Real}_\lambda(Y) \) via +, provided \( \text{Real}_\lambda(Y) \) is non-empty. Moreover, \( \text{Real}_\lambda(Y) \) is non-empty if and only if an obstruction \( O(Y) \in \hat{H}^{k+2}(Y, \Gamma_k Y) \) vanishes.

For objects \( X \) and \( Y \) in \( \text{H}_{k+1}^c \) and a morphism \( f : X \to Y \) in \( \text{H}_{k}^c \), Propositions 8.1 and 8.3 yield

\[
\begin{align*}
O_{X+\alpha,Y+\beta}(f) &= O_{X,Y}(f) - f_\alpha + f^* \beta \\
O_{X \otimes Z,Y \otimes Z}(f \otimes \text{id}_Z) &= \tau_1 \cdot p_1^* O_{X,Y}(f), \\
O_{Z \otimes X,Z \otimes Y}(\text{id}_Z \otimes f) &= \tau_2 \cdot p_2^* O_{X,Y}(f),
\end{align*}
\]

for all \( \alpha \in \hat{H}^{k+1}(X, \Gamma_k X) \) and \( \beta \in \hat{H}^{k+1}(Y, \Gamma_k Y) \). Given another object \( Z \) in \( \text{H}_{k+1}^c \) with \( \lambda Z = \overline{Z} \),

\[
\begin{align*}
\tau_1 : X &\to X \otimes \overline{Z}, \\
p_1 : X \otimes \overline{Z} &\to X
\end{align*}
\]

are the inclusion of and projection onto the first factor and \( \tau_2 \) and \( p_2 \) are defined analogously. We obtain

\[
(X + \alpha) \otimes (Y + \beta) = (X \otimes Y) + \tau_1 \cdot p_1^* \alpha + \tau_2 \cdot p_2^* \beta.
\]

9. Obstructions to the diagonal

Let \( k \geq 2 \). A diagonal on \( X = (C, f_{k+1}, X^k) \) in \( \text{H}_{k+1}^c \) is a morphism, \( \Delta : X \to X \otimes X \), such that, for \( i = 1, 2 \), the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \otimes X \\
\downarrow{\text{id}} & & \downarrow{p_i} \\
X & & 
\end{array}
\]

commutes up to homotopy in \( \text{H}_{k+1}^c \). Applying the functor \( r : \text{CW}_0 \to \text{H}_{k}^c \) to a diagonal \( \Delta : X \to X \times X \) in \( \text{CW}_0 \), we obtain the diagonal \( r(\Delta) : r(X) \to r(X) \otimes r(X) \) in \( \text{H}_{k}^c \).

Lemma 9.1. Let \( X \) be an object in \( \text{H}_{k+1}^c \). Then every \( \lambda \)-realizable diagonal \( \overline{\Delta} : \overline{X} = \lambda X \to \overline{X} \otimes \overline{X} \) in \( \text{H}_{k}^c \) \( \approx \) has a \( \lambda \)-realization \( \Delta : X \to X \otimes X \) in \( \text{H}_{k+1}^c \) \( \approx \) which is a diagonal in \( \text{H}_{k+1}^c \).

Proof. Suppose \( \Delta' : X \to X \times X \) is a \( \lambda \)-realization of \( \overline{\Delta} \) in \( \text{H}_{k+1}^c \). The projection \( p_\ell : X \to X \otimes X \) realizes the projection \( p_\ell : \overline{X} \to \overline{X} \otimes \overline{X} \) and hence \( p_\ell \Delta' \) realizes \( p_\ell \overline{\Delta} \) for \( \ell = 1, 2 \). Now the identity on \( X \) realizes the identity on \( \overline{X} \) and \( p_\ell \Delta \) is homotopic to the identity on \( \overline{X} \) by assumption. Hence \( p_\ell \Delta' \) and the identity on \( X \) realize the same homotopy class of maps for \( \ell = 1, 2 \). As the group \( \hat{H}^k(\overline{X}, \Gamma_k \overline{X}) \) acts transitively on the set of realizations of this homotopy class by Proposition 8.2, there are elements \( \alpha_\ell \in \hat{H}^k(\overline{X}, \Gamma_k \overline{X}) \) such that

\[
\{p_\ell \Delta'\} + \alpha_\ell = \{\text{id}_X\} \quad \text{for} \quad \ell = 1, 2,
\]

where \( \{f\} \) denotes the homotopy class of the morphism \( f \) in \( \text{H}_{k+1}^c \). We put

\[
\{\Delta\} = \{\Delta'\} + \iota_1 \alpha_1 + \iota_2 \alpha_2.
\]
By Proposition 5.2

\[
\{p_\ell \Delta\} = \{p_\ell\} (\{\Delta'\} + \ell_1 \alpha_1 + \ell_2 \alpha_2)
\]

\[
= \{p_\ell \Delta'\} + p_\ell \cdot \ell_1 \alpha_1 + p_\ell \cdot \ell_2 \alpha_2
\]

\[
= \{p_\ell \Delta'\} + \alpha_\ell = \{\text{id}_X\}.
\]

\[\square\]

**Lemma 9.2.** For \(X\) in \(\text{H}^k_{\ell+1}\), let \(\Delta_X : X \to X \otimes X\) be a diagonal on \(\bar{X} = \lambda X\) in \(\text{H}^k_{\ell}\). Then we obtain, in \(H^{k+1}(\bar{X}, \Gamma_k(\bar{X} \otimes \bar{X}))\),

1. \(O_{X,X \otimes X}(\Delta_{\bar{X}}) \in \ker p_i \) for \(i = 1, 2\),
2. \(O_{X,X \otimes X}(\Delta_{\bar{X}}) \in \ker (id_{\bar{X}} - T_\lambda)_*\) if \(\Delta_{\bar{X}}\) is homotopy commutative and
3. \(O_{X,X \otimes X}(\Delta_{\bar{X}}) \in \ker (\tau_{1,2} + \tau_{2,3} + (\Delta_{\bar{X}} \otimes \text{id}_{\bar{X}})_* - (\text{id}_{\bar{X}} \otimes \Delta_{\bar{X}})_*)_\) if \(\Delta_{\bar{X}}\) is homotopy associative.

**Proof.** By definition, \(p_i \Delta_{\bar{X}} \approx id_{\bar{X}}\) for \(i = 1, 2\). As the identity on \(\bar{X}\) is realized by the identity on \(X\) and \(\bar{p}_i : X \otimes X \to \bar{X}\) is realized by \(p_i : X \otimes X \to X\), Proposition 8.1 implies \(O_{X,X \otimes X}(id_{\bar{X}}) = 0\) and \(O_{X \otimes X,X}(\bar{p}_i) = 0\). Since \(O\) is a derivation, we obtain

\[0 = O_{X,X}(p_i \Delta_{\bar{X}}) = p_i O_{X,X \otimes X}(\Delta_{\bar{X}}) + \Delta_{\bar{X}} O_{X \otimes X,X}(\bar{p}_i) = p_i O_{X,X \otimes X}(\Delta_{\bar{X}}),\]

and hence \(O_{X,X \otimes X}(\Delta_{\bar{X}}) \in \ker p_i \) for \(i = 1, 2\). If \(\Delta_{\bar{X}}\) is homotopy commutative, then

\[O_{X,X \otimes X}(\Delta_{\bar{X}}) = O_{X,X \otimes X}(T \Delta_{\bar{X}}) = T_* O_{X,X \otimes X}(\Delta_{\bar{X}}),\]

since \(O_{X \otimes X,X}(T) = 0\), as \(T\) is \(\lambda\)-realizable. Hence \(O_{X,X \otimes X}(\Delta_{\bar{X}}) \in \ker (id_{\bar{X}} - T_\lambda)_*\). For \(1 \leq k < \ell \leq 3\), let \(\iota_{k,\ell} : X \otimes X \to X \otimes X \otimes X\) denote the inclusion of the \(k\)-th and \(\ell\)-th factors and suppose \(\Delta_{\bar{X}}\) is a homotopy commutative diagonal in \(\text{H}^k_{\ell}\). Then \(O_{X \otimes X \otimes X}((\Delta_{\bar{X}} \otimes id_{\bar{X}}) \Delta_{\bar{X}}) = O_{X \otimes X \otimes X}((id_{\bar{X}} \otimes \Delta_{\bar{X}}) \Delta_{\bar{X}})\), as the obstruction depends on the homotopy class of a morphism only, and

\[O_{X,X \otimes X}(\Delta_{\bar{X}} \otimes id_{\bar{X}}) = \tau_{1,2} \bar{p}_1 O_{X,X \otimes X}(\Delta_{\bar{X}})\]

\[O_{X,X \otimes X}(id_{\bar{X}} \otimes \Delta_{\bar{X}}) = \tau_{2,3} \bar{p}_2 O_{X,X \otimes X}(\Delta_{\bar{X}}),\]

by 8.3 and 8.7. Omitting the objects in the notation for the obstruction, we obtain

\[O((\Delta_{\bar{X}} \otimes id_{\bar{X}}) \Delta_{\bar{X}}) = \Delta_{\bar{X}} O((\Delta_{\bar{X}} \otimes id_{\bar{X}}) \Delta_{\bar{X}}) + (\Delta_{\bar{X}} \otimes id_{\bar{X}})_* O((\Delta_{\bar{X}})\)

\[= \Delta_{\bar{X}} O_{\text{homotopy associative}} + (\Delta_{\bar{X}} \otimes id_{\bar{X}})_* O((\Delta_{\bar{X}})\)

\[= \tau_{1,2} \bar{p}_1 O_{\text{homotopy associative}} + (\Delta_{\bar{X}} \otimes id_{\bar{X}})_* O((\Delta_{\bar{X}})\)

\[= \tau_{1,2} \bar{p}_1 O_{\text{homotopy associative}} + (\Delta_{\bar{X}} \otimes id_{\bar{X}})_* O((\Delta_{\bar{X}})\).

Similarly, we obtain

\[O((id_{\bar{X}} \otimes \Delta_{\bar{X}}) \Delta_{\bar{X}}) = \tau_{2,3} \bar{p}_2 O_{\text{homotopy associative}} + (\Delta_{\bar{X}} \otimes id_{\bar{X}})_* O((\Delta_{\bar{X}})),\]

which proves \(3\).

\[\square\]

**Question.** Given a \(\lambda\)-realizable object \(\bar{X}\) with a diagonal \(\Delta_{\bar{X}} : X \to X \otimes X\) in \(\text{H}^k_{\ell}\), is there an object \(X\) with \(\lambda X = \bar{X}\) and a diagonal \(\Delta_X : X \to X \otimes X\) in \(\text{H}^k_{\ell+1}\) such that \(\lambda \Delta_X = \Delta_{\bar{X}}\)?
Lemma 9.1 guarantees the existence of a diagonal \( \Delta \) 2–realizable, that is, there is an object \( \Delta \in \text{PD}(10.1) \) such that \( \Delta \) is isomorphic, is representative and full.

Theorem 10.2. The functor \( C : \text{PD}^n / \cong \longrightarrow \text{PD}^n / \cong \) is an equivalence of categories for \( n \geq 3 \).

Proof. The functor \( C \) is full and faithful by Theorem III 2.9 and Theorem III 2.12 in [1]. By Lemma 2.1, every PD\(^n\)-chain complex, \( X = (D, \omega, [D], \Delta) \), in PD\(^n\) is 2–realizable, that is, there is an object \( X^2 \) in CW\(^3\) such that \( C(X^2) = D \leq 2 \), and we obtain the object \( X = (D, f_3, X^2) \) in PD\(^3\). As \( C \) is monoidal, full and faithful, the diagonal \( \Delta \) on \( X \) is realized by a diagonal \( \Delta \) on \( X \) and hence \( (X, \omega, [D], \Delta) \) is an object in PD\(^3\), with \( C(X) = \Delta \).

Theorem 10.3. For \( n \geq 3 \), the functor \( r : \text{PD}^n / \cong \longrightarrow \text{PD}^n / \cong \) reflects isomorphisms, is representative and full.

Proof. That \( r \) reflects isomorphisms follows from Whitehead’s Theorem.

Poincaré duality implies \( H^{n+1}(Y, \Gamma_n Y) = H^{n+2}(Y, \Gamma_n Y) = 0 \), for every object \( Y = (Y, \omega_Y, [Y], \Delta_Y) \) in PD\(^n\). Hence, by Proposition 8.3, \( Y = \lambda(X) \) for some object \( X \) in PD\(^n\), and, by Proposition 8.3, the diagonal \( \Delta_Y \) is \( \lambda \)-realizable. Thus Lemma 9.1 guarantees the existence of a diagonal \( \Delta_X : X 

\longrightarrow \quad X \otimes X \quad (\Delta_X) = (X \otimes X) + \tau_1, p_1^\alpha + \tau_2, p_2^\alpha \quad \text{and as the obstruction } \mathcal{O} \quad \text{is a derivation, we obtain}

\begin{align*}
\mathcal{O}_{X', X' \otimes X'}(\Delta_{X'}) &= \mathcal{O}_{X', X'(X + \alpha), X'(X + \alpha)}(\Delta_{X'}) \\
 &= \mathcal{O}_{X, X \otimes X}(\Delta_X) - (\Delta_X^* + \tau_1, p_1^\alpha + \tau_2, p_2^\alpha) \\
 &= \mathcal{O}_{X, X \otimes X}(\Delta_X) - (\Delta_X^* - \tau_1 - \tau_2)\alpha,
\end{align*}

since \( \Delta^* \tau_i \pi_i = \tau_i(\pi_i \Delta)^* = \tau_i \alpha \), for \( i = 1, 2 \).

Lemma 9.3. For \( X \in \text{PD}_{k+1} \), let \( \Delta_X : X \rightarrow \overline{X} \otimes X \) be a diagonal on \( \overline{X} = \lambda X \) in \( \text{PD}_{k+1} \) and let \( X' = X + \alpha \) for some \( \alpha \in \text{PD}^k \). Then we obtain, in \( \text{PD}^k \),

\begin{align*}
\mathcal{O}_{X', X' \otimes X'}(\Delta_{X'}) &= \mathcal{O}_{X, X \otimes X}(\Delta_X) - (\Delta_X^* - \tau_1 - \tau_2)\alpha.
\end{align*}

10. PD\(^n\)-homotopy systems

A PD\(^n\)-homotopy system \( X = (X, \omega_X, [X], \Delta_X) \) of order \((k + 1)\) consists of an object \( X = (C, f_{k+1}, X^k) \) in PD\(^k\), a group homomorphism \( \omega_X : \pi_1 X \rightarrow \mathbb{Z}/2\mathbb{Z} \), a fundamental class \( [X] \in H_n(C, \mathbb{Z}) \) and a diagonal \( \Delta : X \rightarrow X \otimes X \) in PD\(^k\) such that \((C, \omega_X, [X], \Delta_X)\) is a PD\(^n\)-chain complex. A map \( f : (X, \omega_X, [X], \Delta_X) \rightarrow (Y, \omega_Y, [Y], \Delta_Y) \) of PD\(^n\)-homotopy systems of order \((k + 1)\) is a morphism in PD\(^k\) such that \( \omega_X = \omega_Y \pi_1(f) \) and \((f \otimes f)\Delta_X \simeq \Delta_Y f\), and we thus obtain the category PD\(^n\) of PD\(^n\)-homotopy systems of order \((k + 1)\). Homotopies in PD\(^n\) are homotopies in PD\(^k\), and restricting the functors in PD\(^n\), we obtain, for \( k \geq 3 \), the functors

\[ \text{PD}^n \xrightarrow{r} \text{PD}^n_{[k+1]} \xrightarrow{\lambda} \text{PD}^n_{[k]} \xrightarrow{C} \text{PD}^n. \]

These functors induce functors between homotopy categories

\[ \text{PD}^n / \cong \xrightarrow{r} \text{PD}^n_{[k+1]}/ \cong \xrightarrow{\lambda} \text{PD}^n_{[k]}/ \cong \xrightarrow{C} \text{PD}^n / \cong. \]

Theorem 10.1. The functor \( C : \text{PD}^n_{[3]} / \cong \longrightarrow \text{PD}^n / \cong \) is an equivalence of categories for \( n \geq 3 \).

Proof. The functor \( C \) is full and faithful by Theorem III 2.9 and Theorem III 2.12 in [1]. By Lemma 2.1, every PD\(^n\)-chain complex, \( X = (D, \omega, [D], \Delta) \), in PD\(^n\) is 2–realizable, that is, there is an object \( X^2 \) in CW\(^3\) such that \( C(X^2) = D \leq 2 \), and we obtain the object \( X = (D, f_3, X^2) \) in PD\(^3\). As \( C \) is monoidal, full and faithful, the diagonal \( \Delta \) on \( X \) is realized by a diagonal \( \Delta \) on \( X \) and hence \((X, \omega, [D], \Delta)\) is an object in PD\(^3\), with \( C(X) = \Delta \).
\[ \lambda \Delta_X = \Delta_Y. \] The homomorphism \( \omega_Y \) and the fundamental class \([Y]\) determine a homomorphism \( \omega_X : \pi_1 X \to \mathbb{Z}/2\mathbb{Z} \) and a fundamental class \([X]\) \( \in H_0(C, \mathbb{Z}^\omega) \), such that \( X = (X, \omega_X, [X], \Delta_X) \) is an object in \( \text{PD}^n_{[n+1]} \). Inductively, we obtain an object \((X_k, \omega_{X_k}, [X_k], \Delta_{X_k})\) realizing \((Y, \omega_Y, [Y], \Delta_Y)\) in \( \text{PD}^n_k \) for \( k > n \), and in the limit an object \( X = (X, \omega_X, [X], \Delta_X) \) in \( \text{PD}^n \) with \( r(x) = Y \).

Proposition [31] together with the fact that, by Poincaré duality, \( \hat{H}^k(X, B) = 0 \) for \( k > n \) and every \( A \)-module \( B \), implies that \( r \) is full. \( \square \)

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