Clock Synchronisation in Inertial Frames and Gravitational Fields

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Abstract. The special relativistic test theory of Mansouri and Sexl is sketched. Theories based on different clock synchronisations are found to be equivalent to special relativity, as regards experimental results. The conventionality of clock synchronisation is shown not to hold, by means of an example, in a simple accelerated system and through the principle of equivalence in gravitational fields, especially when the metric is not static. Experimental implications on very precise clock synchronisation on earth are discussed.

1 The special relativistic test theory of Mansouri-Sexl

As early as 1911, Frank and Rothe showed that the relativity principle alone implies the existence of a constant velocity in all inertial frames. If we assume that this velocity is the velocity of light, we see that the constancy of the velocity of light is a consequence of the relativity principle. Following Mansouri and Sexl, we want to construct a set of theories, which enable to express departures from the relativity principle. This set of theories consists essentially of transformations between inertial frames, but in view of the goal, we should avoid using the relativity principle in the derivation of such transformations.

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We assume, that there is at least one inertial frame in which light behaves isotropically. We call it the privileged frame Σ and denote space and time coordinates in this frame by the capital letters: (X, Y, Z, T). In Σ, clocks are synchronised with Einstein’s procedure. We consider also another system S moving with uniform velocity v < c along the X-axis in the positive direction. In S, the coordinates are written with lower case letters (x, y, z, t).

Under rather general assumptions, symmetry conditions on the two systems, the assumption that the two-way velocity of light is c and furthermore that the time dilation factor has its relativistic value, one can derive the following transformation (see [2], [4]):

\[
\begin{align*}
    x &= \frac{1}{\sqrt{1 - \beta^2}} (X - vT) \\
    y &= Y \\
    z &= Z \\
    t &= s (X - vT) + \sqrt{1 - \beta^2} T,
\end{align*}
\]

(1.1)

where \( \beta = v/c \). The parameter s, which determines the synchronisation in the S frame remains unknown. Einstein's synchronisation in S involves: \( s = -v/(c^2 \sqrt{1 - \beta^2}) \) and (1.1) becomes a Lorentz boost. For a general s, the inverse one-way velocity of light is given by:

\[
\frac{1}{c_{\rightarrow}(\Theta)} = \frac{1}{c} + \left( \frac{\beta}{c} + s \sqrt{1 - \beta^2} \right) \cos \Theta,
\]

(1.2)

where Θ is the angle between the x-axis and the light ray in S. \( c_{\rightarrow}(\Theta) \) is in general dependent on the direction. A simple case is \( s = 0 \). This means from (1.1), that at \( T = 0 \) of Σ we set all clocks of S at \( t = 0 \) (external synchronisation), or that we synchronise the clocks by means of light rays with velocity \( c_{\rightarrow}(\Theta) = c/(1 + \beta \cos \Theta) \) (internal synchronisation). It should be stressed that, unlike to the parameters of length contraction and time dilation, the parameter s cannot be tested, but its value must be assigned in accordance with the synchronisation chosen in the experimental setup. For a recent and comprehensive discussion of this subject, see [3]. A striking consequence of (1.2) is that the negative result of the Michelson-Morley experiment does not rule out an ether. Only an ether with galilean transformations is excluded, because the galilean transformations do not lead to an invariant two-way velocity of light in a moving system.

2 One-way velocity of light on the rim of a disk

We now study synchronisation on the rim of a rotating disk. This problem has numerous applications, such as the Sagnac effect and the synchronisation of atomic clocks around the earth. Let consider a disk rotating anticlockwise with constant angular velocity \( \omega \) respective to the privileged frame Σ (for simplicity), around the Z-axis. Suppose light is constrained to move on the rim of the disk and that ideal clocks are put on that rim. Because of the Clock Hypothesis ideal clocks and rods on the rim behave exactly in the same way as in the instantaneously coinciding tangential inertial frame. Since it is obvious, at least tangentially to the rim, that the velocity of light is the same in the coinciding tangential inertial frame as
on the corresponding part of the rim (the gravitational field is perpendicular to the rim), \textit{we must choose the same synchronisation in these two frames.} Let us now construct a metric on the whole disk, using the transformations:

\[ T = t; \quad X = r \cos(\phi + \omega t); \quad Y = r \sin(\phi + \omega t); \quad Z = z \quad (2.1) \]

The coordinates \((t, r, \phi, z)\) give a possible description of the physical events for an observer at rest on the disk. In particular, the coordinate \(t\) is measured with a clock that runs \((1 - \omega^2 r^2/c^2)^{-1/2}\) faster than a clock at rest in \(\Sigma\), so that we have the right to write the first equation of \((2.1)\). From \((2.1)\) we obtain the line element:

\[ ds^2 = \left(1 - \frac{\omega^2 r^2}{c^2}\right) (cdt)^2 - 2 \frac{\omega r^2}{c} d\phi (cdt) - dr^2 - r^2 d\phi^2 - dz^2 \quad (2.2) \]

Let remark that the metric is not static. As is well known, the spatial part of the metric is not only given by the space-space coefficients of the four dimensional metric, but by:

\[ dl^2 = \left(-g_{\alpha\beta} + \frac{g_{0\alpha} g_{0\beta}}{g_{00}}\right) dx^\alpha dx^\beta = dr^2 + dz^2 + \frac{r^2 d\phi^2}{1 - \frac{\omega^2 r^2}{c^2}} \quad (2.3) \]

where 0 is the time index, and \(\alpha, \beta\) represent the space indices and can take the values 1, 2, 3. The right-hand-side of \((2.3)\), with \(dz = 0\), is the standard result, showing that the spatial part of the metric is not flat on the rotating disk.

We now use an Einstein’s synchronisation on the disk. Generally, if we send a light signal from point \(A\) with coordinates \(x^\alpha, \alpha = 1, 2, 3\) to an infinitesimally near point \(B\) with coordinates \(x^\alpha + dx^\alpha, \alpha = 1, 2, 3\) and back, the coordinate time difference \(dt_1, dt_2\) for the “there” (back) trip is obtained by solving the equation \(ds^2 = 0\). We obtain:

\[ dt_{1,2} = \frac{1}{cg_{00}} \left[ g_{0\alpha} dx^\alpha + \sqrt{(g_{0\alpha} g_{0\beta} - g_{\alpha\beta} g_{00}) dx^\alpha dx^\beta} \right] = \pm \frac{\omega r^2 d\phi}{c^2 \left(1 - \frac{\omega^2 r^2}{c^2}\right)} + \frac{dl_{AB}}{c \sqrt{1 - \frac{\omega^2 r^2}{c^2}}} \quad (2.4) \]

By definition the time \(t_A\) at \(A\) which is synchronous with the arrival time in \(t_B\) in \(B\) is the midtime of departure and arrival at \(A\). So, two Einstein-synchronous events are not coordinate-time- synchronous and have a difference \(\Delta t\) such that:

\[ t_B = t_A + \Delta t = t_A - \frac{1}{c} \frac{g_{0\alpha} dx^\alpha}{g_{00}} = t_A + \frac{\omega r^2 d\phi}{c^2 \left(1 - \frac{\omega^2 r^2}{c^2}\right)} \quad (2.5) \]

If we generalise this procedure, not only in an infinitesimal domain, but along a curve, we obtain that generally it is path dependant, because \(\Delta t\) is not a total differential. As consequence, time can not be defined globally on the rim with Einstein’s procedure. This is what in fact happens on earth: if one synchronises atomic clocks all around the earth with Einstein’s procedure and comes back to the point of departure after a whole round trip, a time lag will result. This means that a clock is not synchronisable with itself, which is clearly absurd. Hence the physicist Ashby from Boulder has said: “Thus one discards Einstein’s synchronisation in the rotating frame”\[6\]. On the other hand, we can easily define a global
time since the coordinate time $t$ is already global. Remembering that the coordinate time $t$ is measured with clocks that run faster than clocks at rest in $\Sigma$, we define a global time $t' = \sqrt{1 - \omega^2 r^2 / c^2} t$. The one-way velocity of light on the rim is now given by:

$$c_{\pm} = \frac{dl_{AB}}{dt'_{1,2}} = \frac{dl_{AB}}{dt_{1,2} \sqrt{1 - \omega^2 r^2 / c^2}} = \frac{c}{1 \pm \frac{\omega r}{c}}, \quad (2.6)$$

where the last step comes from (2.3) and (2.4) with $dr = dz = 0$ and $\pm$ stands for the anticlockwise (clockwise) propagation of light. It means that the velocity of light in the tangential inertial frame is also equal to: $c_{\pm}(\pm) = c/(1 \pm \beta)$, with $\beta = \omega r / c$, corresponding to a parameter $s = 0$ of (1.2) and an angle $\Theta$ of $0 (\pi)$.

3 Conclusion

In inertial systems, the synchronisation of clocks is conventional. When extended to accelerated systems, the synchronisation is no longer conventional [7]. A consistent theory must involve a one-way velocity of light which depends on the frame of reference. Transformations (1.2) with values of the synchronisation parameter other than $s = 0$ lead, from a physical point of view, to an inadequate description of time on the rim of a rotating disk.

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