Initial ideals of tangent cones to Richardson varieties in the symplectic Grassmannian

Papi Ray and Shyamashree Upadhyay

Department of Mathematics
Indian Institute of Technology, Guwahati
Assam-781039, INDIA
e-mail: popiroy93@iitg.ac.in

Department of Mathematics
Indian Institute of Technology, Guwahati
Assam-781039, INDIA
e-mail: shyamashree@iitg.ac.in

Abstract
We give an explicit Gröbner basis for the ideal of the tangent cone at any $T$-fixed point of a Richardson variety in the symplectic Grassmannian, thus generalizing a result of Ghorpade and Raghavan [4].

Keywords: Symplectic Grassmannian, Richardson variety, Initial ideal, Tangent cone, Gröbner basis.

2000 Mathematics Subject Classification: 05E10; 14M15.

Contents

1 Introduction 2

2 Notation and Preliminaries 3

2.1 Symplectic Grassmannian and Richardson varieties . . . 3
2.2 $(r,c)$ pairs and monomials . . . . . . . . . . . . . . . . . . 4
2.3 Notched and semistandard Young tableaux . . . . . . . . . 7
2.4 Schensted insertion and bounded insertion . . . . . . . . . 8
2.5 The bounded RSK correspondence . . . . . . . . . . . . . . 9
2.6 The Kodiyalam-Raghavan maps . . . . . . . . . . . . . . . 11

3 Statement of the main theorem 14

3.1 Extension of the Kodiyalam-Raghavan maps . . . . . . . . 14
3.2 Ideals of tangent cones to Richardson varieties . . . . . . . 16
3.3 Extended $\beta$-chains . . . . . . . . . . . . . . . . . . . . . . 18
1 Introduction

The study of Schubert varieties has a long and rich history. Richardson varieties are a natural generalization of Schubert varieties. We are interested in Richardson varieties in the symplectic Grassmannian. We consider initial ideals of tangent cones to Richardson varieties in the symplectic Grassmannian. In this paper, we give an explicit Gröbner basis for the ideal of the tangent cone at any $T$-fixed point of a Richardson variety in the symplectic Grassmannian.

In [7], Kodiyalam and Raghavan provide (with respect to certain conveniently chosen term orders) an explicit Gröbner basis for the ideal of the tangent cone at any $T$-fixed point of a Schubert variety in the ordinary Grassmannian, thereby proving the conjectures of Kreiman and Lakshmibai (made in [10]). Then in [4], Ghorpade and Raghavan do the analogous work for Schubert varieties in the symplectic Grassmannian. And finally in [13, 14], Raghavan and Upadhyay do the analogous work for Schubert varieties in the orthogonal Grassmannian.

The above results on Schubert varieties do not admit a straightforward generalization to Richardson varieties. The local properties of Schubert varieties at any $T$-fixed point determine the local properties at all other points, because of the $B$-action; but this does not extend to Richardson varieties, since Richardson varieties only have a $T$-action. However, in [9], Kreiman has extended the results of Kodiyalam and Raghavan to Richardson varieties in the ordinary Grassmannian. The analogous work for the orthogonal Grassmannian was done by Upadhyay in [17]. We address the analogous problem for Richardson varieties in the symplectic Grassmannian in this paper.

We are motivated by a work of Knutson, Woo and Yong [6], where they give a short proof of the fact that essentially all questions concerning singularities of Richardson varieties reduce to corresponding questions about Schubert varieties. Also, there is a second simplification to the work of [6] (One paper to this point is [20], due to Graham and Kreiman.) We are also motivated by the method used by Kreiman (in [9]) to compute an explicit Gröbner basis for the ideal of the tangent cone at any $T$-fixed point of a Richardson variety in the ordinary Grassmannian. Our motivation from [6] allows us to look at [4], where Ghorpade and Raghavan prove that in the case of Schubert varieties in the symplectic Grassmannian, certain objects called “good admissible pairs” give rise to a Gröbner basis for the ideal of the tangent cone at any $T$-fixed point.

In this paper, we have defined “good admissible pairs” as a natural extension of the “good admissible pairs” of [4]. Thereafter, we have followed the techniques used by Kreiman in [9] to obtain an explicit Gröbner basis in our case.
Sturmfels [16] and Herzog-Trung [5] proved results on a class of determinantal varieties which are equivalent to the results of [7, 8, 9] for the case of Schubert varieties at the \( T \)-fixed point \( e \). The key to their proofs was to use a version of the RSK correspondence (see [3] for the classical RSK) in order to establish a degree-preserving bijection between a set of monomials defined by an initial ideal and a ‘standard monomial basis’ (see [12] for a standard monomial basis).

In [7, 8], an explicit Gröbner basis for the ideal of the tangent cone of a Schubert variety in the ordinary Grassmannian at a torus-fixed point were obtained. In [9], Kreiman generalizes the results of [7, 8] to the case of Richardson varieties. In [9], Kreiman gives an explicit Gröbner basis for the ideal of the tangent cone at any \( T \)-fixed point of a Richardson variety in the ordinary Grassmannian, where \( T \) denotes a maximal torus in the general linear group. The proof given in [9] is based on a generalization of the Robinson-Schensted-Knuth (RSK) correspondence, which Kreiman calls the bounded RSK (BRSK). In [15], we had proved that the map \( BRSK \) of [9] and the map \( \tilde{\pi} \) of [7] are actually the same maps. In this paper, we use the map \( BRSK \) of [9] to obtain an explicit Gröbner basis for the ideal of the tangent cone at any \( T \)-fixed point of a Richardson variety in the symplectic Grassmannian. The way in which the map \( BRSK \) of [9] has been used here to obtain an explicit Gröbner basis has been explained in §3.5 of this paper.

In the study of singularities of Schubert varieties, Woo and Yong investigated Kazhdan-Lusztig ideals [18]. These ideals encode coordinates and equations for neighborhoods of type \( A \) Schubert varieties at torus fixed points. In [19], Woo and Yong provide a Gröbner basis for the Kazhdan-Lusztig ideals. Also in [1], the authors discuss three natural generalizations of Richardson varieties which they call projection varieties, intersection varieties, and rank varieties. In [1], they study the singularities of each type of generalization.

The organization of the paper is as follows. In §2.1, we define the main objects of interest, namely, the symplectic Grassmannian and Richardson varieties in it. In §2, we recall all the things necessary to state the main result of the paper. The main result of the paper comes as Theorem 3.4.6 and in §3.5, we provide a strategy to prove this theorem. In §4, we define the two sets needed to prove the main theorem and then we provide the main proof in §5. In a forthcoming paper, the results of this paper will be applied to give a combinatorial description of the multiplicity at any torus fixed point of a Richardson variety in the symplectic Grassmannian.

2 Notation and Preliminaries

2.1 Symplectic Grassmannian and Richardson varieties

The following definitions and notation are written in the same way as in [4]. Given any positive integer \( n \), we denote by \([n]\) the set \( \{1, 2, \ldots, n\} \). Given positive integers \( r \) and \( n \) with \( r \leq n \), we denote by \( I(r, n) \) the set of all \( r \)-element subsets of \([n] \). Let \( \alpha = (\alpha_1, \ldots, \alpha_r) \in I(r, n) \), where \( 1 \leq \alpha_1 < \ldots < \alpha_r \leq n \). If \( \beta = (\beta_1, \ldots, \beta_r) \in I(r, n) \) be such that \( 1 \leq \beta_1 < \ldots < \beta_r \leq n \), then we say that \( \alpha \leq \beta \) if \( \alpha_i \leq \beta_i \) for all \( i = 1, \ldots, r \). Clearly, \( \leq \) defines a partial order on
A positive integer \(d\) will be kept fixed throughout this paper. For \(j \in [2d]\), set \(j^* := 2d + 1 - j\). Let \(I(d)\) denote the set of all \(d\)-element subsets \(v\) of \([2d]\) with the property that exactly one of \(j, j^*\) belongs to \(v\) for every \(j \in [2d]\). Clearly \(I(d) \subseteq I(d, 2d)\). In particular, we have the partial order \(\leq\) on \(I(d)\) induced from \(I(d, 2d)\).

Fix a vector space \(V\) of dimension \(2d\) over an algebraically closed field of arbitrary characteristic. Fix a non-degenerate skew-symmetric bilinear form \(\langle \ ,\ \rangle\) on \(V\). Fix a basis \(e_1, \ldots, e_{2d}\) of \(V\) such that

\[
\langle e_i, e_j \rangle = \begin{cases} 
1 & \text{if } i = j^* \text{ and } i < j, \\
-1 & \text{if } i = j^* \text{ and } i > j, \\
0 & \text{otherwise.}
\end{cases}
\]

A linear subspace \(W\) of \(V\) is said to be isotropic if the form \(\langle \ ,\ \rangle\) vanishes identically on it. Let

\[
G_d(V) = \text{ the Grassmannian of all } d\text{-dimensional subspaces of } V
\]

and

\[
\mathfrak{M}_d(V) = \text{ the set of all maximal isotropic subspaces of } V.
\]

Then \(\mathfrak{M}_d(V)\) is a closed subvariety of \(G_d(V)\) and is called the symplectic Grassmannian.

Let \(Sp(V)\) denote the group of all linear automorphisms of \(V\) that preserve \(\langle \ ,\ \rangle\). The elements of \(Sp(V)\) that are diagonal with respect to the basis \(e_1, \ldots, e_{2d}\) form a maximal torus \(T\) of \(Sp(V)\). Similarly the elements of \(Sp(V)\) that are upper triangular with respect to \(e_1, \ldots, e_{2d}\) form a Borel subgroup \(B\) of \(Sp(V)\) and the elements of \(Sp(V)\) that are lower triangular with respect to \(e_1, \ldots, e_{2d}\) form a Borel subgroup opposite to \(B\) of \(Sp(V)\), it is denoted by \(B^\circ\).

The \(T\)-fixed points of \(\mathfrak{M}_d(V)\) are parametrized by \(I(d)\) (as explained in [4, §2]). The \(B\)-orbits (as well as \(B^\circ\)-orbits) of \(\mathfrak{M}_d(V)\) are naturally indexed by its \(T\)-fixed points: each \(B\)-orbit (as well as \(B^\circ\)-orbit) contains one and only one such point. Let \(\alpha \in I(d)\) be arbitrary and let \(e_\alpha\) denote the corresponding \(T\)-fixed point of \(\mathfrak{M}_d(V)\). The Zariski closure of the \(B\) (resp. \(B^\circ\)) orbit through \(e_\alpha\), with canonical reduced scheme structure, is called a Schubert variety (resp. opposite Schubert variety), and denoted by \(X^\alpha\) (resp. \(X_\alpha\)). For \(\alpha, \gamma \in I(d)\), the scheme-theoretic intersection \(X_\alpha^\gamma = X_\alpha \cap X_\gamma^\tau\) is called a Richardson variety. It can be seen easily that the set consisting of all pairs of elements of \(I(d)\) becomes an indexing set for Richardson varieties in \(\mathfrak{M}_d(V)\). It can also be shown that \(X_\gamma^\alpha\) is nonempty if and only if \(\alpha \leq \gamma\); and that for \(\beta \in I(d), e_\beta \in X_\alpha^\gamma\) if and only if \(\alpha \leq \beta \leq \gamma\).

For the rest of this paper, \(\alpha, \beta, \gamma\) are arbitrarily fixed elements of \(I(d)\) such that \(\alpha \leq \beta \leq \gamma\).

### 2.2 \((r, c)\) pairs and monomials

For this subsection, let us fix an arbitrary element \(v\) of \(I(d, 2d)\). We will be dealing extensively with ordered pairs \((r, c)\), \(1 \leq r, c \leq 2d\), such that \(r\) is not
and $c$ is an entry of $v$. Let

$R(v)$ denote the set of all such ordered pairs, that is, $R(v) = \{(r, c) | r \in \{1, \ldots, 2d\} \setminus v, c \in v\}$.

Set

$$
\begin{align*}
\mathcal{N}(v) &:= \{(r, c) \in R(v) | r > c\}, \\
\mathcal{ON}(v) &:= \{(r, c) \in R(v) | r \leq c^*\}, \\
\mathcal{OM}(v) &:= \{(r, c) \in R(v) | r > c, r \leq c^*\} = \mathcal{OM}(v) \cap \mathcal{N}(v), \\
\mathcal{OM}^v &:= \{(r, c) \in R(v) | r = c^*\}, \\
\mathcal{AR}(v) &:= \{(r, c) \in R(v) | r \geq c^*\}, \\
\mathcal{AN}(v) &:= \{(r, c) \in R(v) | r > c, r \geq c^*\}.
\end{align*}
$$

We will refer to $\mathcal{OM}^v$ as the diagonal. Let $\text{mon}\mathcal{N}(v)$ denote the set of all monomials in $\mathcal{N}(v)$.

Figure 2.2.1 in Example 2.2.1 below gives a pictorial look of the above sets.

**Example 2.2.1.** Let $d = 7$ and $v = (1, 3, 4, 7, 9, 10, 13)$.

![Figure 2.2.1: Illustration of the grid representing $\mathcal{N}(v)$](image)

The points (including the dark circles) of the above grid represent the set $\mathcal{N}(v)$ for $v = (1, 3, 4, 7, 9, 10, 13)$. The path sketched on the grid by some piecewise line segments denote the boundary of $\mathcal{N}(v)$. The points on this grid which lie on the boundary of $\mathcal{N}(v)$ or to the left of it belong to the set $\mathcal{OM}(v)$. The dark circles denote the diagonal elements. Points above and on the diagonal belong to the set $\mathcal{OM}(v)$. Again, points which are on and towards the left of the boundary of $\mathcal{N}(v)$, and which also lie on and above the diagonal, are the points of $\mathcal{OM}(v)$.

We will be considering monomials in some of these sets. A monomial, as usual, is a subset with each member being allowed a multiplicity (the multiplicity taking values in the non-negative integers). The degree of a monomial also has the usual meaning: consider the underlying set of the monomial and look at the multiplicity with which each element of this underlying set appears in the monomial, the degree of the monomial is the sum of these multiplicities.

For a monomial $S \in \mathcal{N}(v)$, let $S^\#$ denote the set $\{(c^*, r^*) | (r, c) \in S\}$.
Example 2.2.2. Let $d, v$ be as in Example 2.2.1 above. Let $\mathcal{S} = \{(2, 1), (6, 4)^2, (5, 10)\}$. Then $\mathcal{S}$ is a monomial in $\mathfrak{M}(v)$. The underlying set of the monomial $\mathcal{S}$ is \{(2, 1), (6, 4), (5, 10)\}. The degree of $\mathcal{S}$ is 4. The multiplicity of $(2, 1), (6, 4),$ and $(5, 10)$ are respectively 1, 2, and 1. Also for the monomial $\mathcal{S}$, $\mathcal{S}^\# = \{(14, 13), (11, 9)^2, (5, 10)\}$.

Let $S$ be any set. A multiset $E$ on $S$ is defined to be a function $E : S \to \{0, 1, 2, \ldots \}$. One should think of $E$ as consisting of the set $S$ of elements, but with each $s \in S$ occurring $E(s)$ times. Note that a set is a special type of multiset in which each element occurs exactly once. We call $E(s)$ the multiplicity of $s$ in $E$. Define the multiset $E \cup F$ as follows:

$$(E \cup F)(s) = E(s) + F(s), \quad s \in S$$

Let $\mathbb{N}$ denote the set of all positive integers. Let $A = \{a_1, a_2, \ldots \}$ and $B = \{b_1, b_2, \ldots \}$ be two multisets on $\mathbb{N}$ of the same degree, with $a_i \leq a_{i+1}$, $b_i \leq b_{i+1}$, for all $i$. We say that $A$ is less than or equal to $B$ in the termwise order if $a_i \leq b_i$ for all $i$. We denote this by $A \leq B$. We say that $A$ is less than $B$ in the strict termwise order if $a_i < b_i$ for all $i$. We denote this by $A < B$.

If $A$, $B$, $C$, and $D$ are multisets on $\mathbb{N}$ such that $|A \cup D| = |B \cup C|$, then we write

$$A - C \leq B - D$$

to indicate that $A \cup D \leq B \cup C$. \hfill (2.2.1)

Let $U = \{(e_1, f_1), (e_2, f_2), \ldots \}$ be a multiset on $\mathbb{N}^2$. Define $U_{(1)}$ and $U_{(2)}$ to be the multisets $\{e_1, e_2, \ldots \}$ and $\{f_1, f_2, \ldots \}$ respectively on $\mathbb{N}$. Define the non vanishing, negative, and positive parts of $U$ to be the following multisets:

$${U^\#}^0 = \{(e_i, f_i) \in U \mid e_i - f_i \neq 0\},$$

$${U^-} = \{(e_i, f_i) \in U \mid e_i - f_i < 0\},$$

$${U^+} = \{(e_i, f_i) \in U \mid e_i - f_i > 0\}.$$ 

We say that $U$ is non vanishing if $U \subset (\mathbb{N}^2)^0$, negative if $U \subset (\mathbb{N}^2)^-$, and positive if $U \subset (\mathbb{N}^2)^+$. Impose the following transitive relation on multisets on $\mathbb{N}^2$:

$$U \leq V \iff U_{(1)} - U_{(2)} \leq V_{(1)} - V_{(2)}.$$ \hfill (2.2.2)

A chain in $\mathbb{N}^2$ is a subset $C = \{(e_1, f_1), \ldots, (e_m, f_m)\}$ of $\mathbb{N}^2$ such that $e_1 < \cdots < e_m$ and $f_1 > \cdots > f_m$. Let $T$ and $W$ be negative and positive subsets of $\mathbb{N}^2$ respectively. A non vanishing multiset $U$ on $\mathbb{N}^2$ is said to be bounded by $T$, $W$ if for every chain $C$ which is contained in the underlying set of $U$, we have

$$T \leq C \leq W.$$ 

Let $\iota$ be the map on multisets on $\mathbb{N}^2$ defined by,

$$\iota((e_1, f_1), (e_2, f_2), \ldots ) := ((f_1, e_1), (f_2, e_2), \ldots )$$

Then $\iota$ is an involution, and it maps negative multisets on $\mathbb{N}^2$ to positive ones and visa-versa.
2.3 Notched and semistandard Young tableaux

In this subsection we are recalling the following things from [9].

A Young diagram (resp. notched diagram) is a collection of boxes arranged into a left and top justified array (resp. into left justified rows). The empty Young diagram is the Young diagram with no boxes. A notched diagram may contain rows with no boxes; however, a Young diagram may not, unless it is the empty Young diagram. A Young tableau (resp. notched tableau) is a filling of the boxes of a Young diagram (resp. notched diagram) with positive integers. The empty Young tableau is the Young tableau with no boxes. Let $P$ be either a notched tableau or a Young tableau. We say that $P$ is row strict if the entries of any row of $P$ strictly increase as one moves to the right. If $P$ is a Young tableau, then we say that $P$ is semistandard if it is row strict and the entries of any column weakly increase as one moves down.

Example 2.3.1 below illustrates a row strict notched tableau and a semistandard Young tableau.

Example 2.3.1. A row strict notched tableau $P$ and a semistandard Young tableau $R$.

\[
P = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 4 & & \\
3 & 6 & 7 & \\
4 & 5 & 7 & 8 \\
\end{array}
\quad \text{and} \quad R = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 4 & 5 & 6 \\
3 & 5 & & \\
4 & & & 6 \\
\end{array}
\]

Let $P$ be a row strict notched tableau and $b$ be a positive integer. Since $P$ is row strict, its entries which are greater than or equal to $b$ are right justified in each row. If we remove these entries (which are greater than or equal to $b$) from $P$, then we are left with a row strict notched tableau, which we denote by $P_{<b}$. We say that $P$ is semistandard on $b$ if $P_{<b}$ is a semistandard Young tableau.

Example 2.3.2 below illustrates $P_{<b}$, for fixed values of $P$ and $b$.

Example 2.3.2. For the row strict notched tableau $P$ in Example 2.3.1 and $b = 4$, we have

\[
P_{<4} = \begin{array}{ccc}
1 & 2 & 3 \\
2 & & \\
3 & 5 & \\
4 & 5 & 7 & 8 \\
\end{array}
\]

However, for the same $P$, if we take $b = 6$, then

\[
P_{<6} = \begin{array}{ccc}
1 & 2 & 3 & 4 \\
2 & 4 & & \\
3 & 5 & & \\
4 & 5 & & \\
\end{array}
\]

Hence $P$ is semistandard on 4, but not on 6.

A notched bitableau is a pair $(P,Q)$ of notched tableaux of the same shape (i.e., the same number of rows and the same number of boxes in each row). The degree of $(P,Q)$ is the number of boxes in $P$ (or $Q$). A notched bitableau $(P,Q)$ is said to be row strict if both $P$ and $Q$ are row strict. A row strict notched bitableau $(P,Q)$ is said to be semistandard if

\[P_1 - Q_1 \leq \cdots \leq P_r - Q_r, \quad (2.3.1)\]
where \( r \) is the total number of rows in \( P \) (or \( Q \)) and for each \( i \in \{1, \ldots, r\} \), \( P_i \) (resp. \( Q_i \)) denotes the \( i \)-th row (from the top) of \( P \) (resp. \( Q \)). A row strict notched bitableau \((P, Q)\) is said to be \textbf{negative} if \( P_i \preceq Q_i \), \( i = 1, \ldots, r \), \textbf{positive} if \( P_i \succeq Q_i \), \( i = 1, \ldots, r \), and \textbf{non vanishing} if either 
\begin{align}
P_i \preceq Q_i \quad \text{or} \quad P_i \succeq Q_i, 
\end{align}
for each \( i = 1, \ldots, r \).

Example 2.3.3 below gives an illustration of a row strict semistandard non vanishing bitableau.

\textbf{Example 2.3.3.} Consider the notched bitableau 
\[
(P, Q) = \left( \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 7 \\
6 & 8 & 9 \\
\end{array} \right), \quad \left( \begin{array}{ccc}
2 & 3 & 4 \\
5 & 6 & 7 \\
8 & 9 & 1 \\
\end{array} \right).
\]
We have that
1. \((P, Q)\) is row strict.
2. 
\[
P_1 \cup Q_2 = \{1, 2, 3, 3, 4, 5\} \preceq \{4, 5, 6, 7, 8, 9\} = P_2 \cup Q_1.
\]
Therefore, 
\[
P_1 - Q_1 \leq P_2 - Q_2.
\]
Thus \((P, Q)\) is semistandard.
3. \(P_1 \prec Q_1\) and \(P_2 \succ Q_2\). Thus \((P, Q)\) is non vanishing.

Let \((P, Q)\) be a semistandard notched bitableau. If for subsets \( T \) and \( W \) of \( \mathbb{N}^2 \),
\[
T_{(1)} - T_{(2)} \leq P_1 - Q_1 \quad \text{and} \quad P_r - Q_r \leq W_{(1)} - W_{(2)},
\]
then we say that \((P, Q)\) is \textbf{bounded by} \(T, W\).

If \((P, Q)\) is a non vanishing semistandard notched bitableau, then we define \(\iota(P, Q)\) to be the notched bitableau obtained by reversing the order of the rows of \((Q, P)\).

\section*{2.4 Schensted insertion and bounded insertion}

Let us now recall the \textbf{ordinary Schensted insertion} process from \cite{D}, §3. It is an algorithm which takes as input a semistandard Young tableau \(P\), a positive integer \(a\), and produces as output a new semistandard Young tableau with the same shape as \(P\) plus one extra box, and with the same entries as \(P\) (possibly in different locations) plus one additional entry, namely \(a\). To begin, insert \(a\) into the first row of \(P\), as follows. If \(a\) is strictly bigger than all entries in the first row of \(P\), then place \(a\) in a new box on the right end of the first row, and the insertion process terminates. Otherwise, find the smallest entry of the first row of \(P\) which is greater than or equal to \(a\), and replace that number with \(a\). We say that the number which was replaced was “bumped” from the first row. Insert the bumped number into the second row in precisely the same way as \(a\) was inserted into the first row. This process continues down the rows until, at some point, a number is placed in a new box on the right end of some row, at which point the process terminates.

We next describe the \textbf{bounded insertion algorithm}, which takes as input a positive integer \(b\), a notched tableau \(P\) which is semistandard on \(b\), and a positive integer \(a < b\), and produces as output a notched tableau which is semistandard on \(b\), which we denote by \(P \leftarrow_{b} a\).
Bounded Insertion:

**Step 1.** Remove all entries of $P$ which are greater than or equal to $b$ from $P$, resulting in the semistandard Young tableau $P^{<b}$.

**Step 2.** Insert $a$ into $P^{<b}$ using the ordinary Schensted insertion process (as described above).

**Step 3.** Place the entries of $P$ which were removed when forming $P^{<b}$ in Step 1 back into the Young tableau resulting from Step 2, in the same rows from which they were removed.

Example 2.4.1 below gives an illustration of the bounded insertion algorithm.

**Example 2.4.1.** Let $a = 3$ and $b = 4$. We compute $P \leftarrow^b a$, where

$$P = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 4 \\
5 & 6 & 7 \\
4 & 5 & 7 & 8
\end{array}.$$  

Observe that in Step 1, we obtain $P^{<4}$, where

$$P^{<4} = \begin{array}{cc}
1 & 2 & 3 \\
2
\end{array}.$$  

In Step 2, we insert $a = 3$ into $P^{<4}$ using the ordinary Schensted insertion process, to get

$$\begin{array}{cc}
1 & 2 & 3 \\
2 & 3
\end{array}.$$  

And finally in Step 3, we obtain

$$P \leftarrow^4 3 = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 4 \\
5 & 6 & 7 \\
4 & 5 & 7 & 8
\end{array}.$$  

### 2.5 The bounded RSK correspondence

We next define the **bounded RSK correspondence**, $BRSK$, a function which maps negative multisets on $\mathbb{N}^2$ to negative semistandard notched bitableaux. Let

$$U = \{(a_1, b_1), \ldots, (a_t, b_t)\}$$

be a negative multiset on $\mathbb{N}^2$, whose entries we assume are listed in lexicographic order: (i) $b_1 \geq \cdots \geq b_t$ and (ii) if for any $i \in \{1, \ldots, t - 1\}$, $b_i = b_{i+1}$, then $a_i \geq a_{i+1}$. We inductively form a sequence of notched bitableaux $(P^{(0)}, Q^{(0)}), (P^{(1)}, Q^{(1)}), \ldots, (P^{(t)}, Q^{(t)})$, such that $P^{(i)}$ is semistandard on $b_i$, $i = 1, \ldots, t$, as follows:

Let $(P^{(0)}, Q^{(0)}) = (\emptyset, \emptyset)$ and let $b_0 = b_1$. Assume inductively that we have formed $(P^{(i)}, Q^{(i)})$, such that $P^{(i)}$ is semistandard on $b_i$, and thus on $b_{i+1}$, since $b_{i+1} \leq b_i$. Define $P^{(i+1)} = P^{(i)} \leftarrow a_{i+1}$.
Since bounded insertion preserves semistandardness on \( b_{i+1} \), \( P^{(i+1)} \) is also semistandard on \( b_{i+1} \). Let \( j \) be the row number of the new box of this bounded insertion. Define \( Q^{(i+1)} \) to be the notched tableau obtained by placing \( b_{i+1} \) on the left end of row \( j \) of \( Q^{(i)} \) (and shifting all other entries of \( Q^{(i)} \) to the right one box). Clearly \( P^{(i+1)} \) and \( Q^{(i+1)} \) have the same shape.

Then \( BRSK(U) \) is defined to be \( (P^{(i)}, Q^{(i)}) \). In the process above, we write \( (P^{(i+1)}, Q^{(i+1)}) = (P^{(i)}, Q^{(i)}) \overset{b_{i+1}}{\leftarrow} a_{i+1} \).

In terms of this notation,

\[
BRSK(U) = ((\emptyset, \emptyset) \overset{b_1}{\leftarrow} a_1) \cdots \overset{b_t}{\leftarrow} a_t.
\]

If \( U \) is a positive multiset on \( \mathbb{N}^2 \), then we define \( BRSK(U) \) to be \( \iota(BRSK(\iota(U))) \).

Example 2.5.1 below gives an illustration of the map \( BRSK \).

**Example 2.5.1.** Let \( U = \{(2, 1), (5, 3), (6, 4), (6, 9), (8, 13), (11, 13)\} \) be a multiset on \( \mathbb{N}^2 \). Now

\[
\{(2, 1), (5, 3), (6, 4)\} \subset (\mathbb{N}^2)^+ \quad \text{and} \quad \{(6, 9), (8, 13), (11, 13)\} \subset (\mathbb{N}^2)^-. 
\]

Let \( U^+ = \{(2, 1), (5, 3), (6, 4)\} \) and \( U^- = \{(6, 9), (8, 13), (11, 13)\} \). So \( U = U^+ \cup U^- \). Now after arranging \( \iota(U^+) \) in lexicographic order, we have

\[
\iota(U^+) = \{(4, 6), (3, 5), (1, 2)\}.
\]

Let us first apply the map \( BRSK \) on \( \iota(U^+) \). Then

\[
\begin{align*}
P^{(0)} &= \emptyset & Q^{(0)} &= \emptyset \\
P^{(1)} &= \emptyset \overset{6}{\leftarrow} 4 = \begin{array}{c} \text{4} \end{array} & Q^{(1)} &= \begin{array}{c} \text{6} \end{array} \\
P^{(2)} &= \begin{array}{c} \text{4} \end{array} \overset{5}{\leftarrow} 3 = \begin{array}{c} \text{3} \\ \text{4} \end{array} & Q^{(2)} &= \begin{array}{c} \text{6} \\ \text{5} \end{array} \\
P^{(3)} &= \begin{array}{c} \text{3} \\ \text{4} \end{array} \overset{2}{\leftarrow} 1 = \begin{array}{c} \text{1} \\ \text{3} \\ \text{4} \end{array} & Q^{(3)} &= \begin{array}{c} \text{2} \\ \text{6} \\ \text{5} \end{array}
\end{align*}
\]

Therefore

\[
BRSK(U^+) = \iota(BRSK(\iota(U^+))) = \begin{pmatrix} 5 \\ 2 \\ 4 \\ 1 \\ 3 \\ 1 \\ 3 \end{pmatrix}.
\]

After arranging \( U^- \) in lexicographic order, we have \( U^- = \{(11, 13), (8, 13), (6, 9)\} \). Let us now apply the map \( BRSK \) on \( U^- \). Then
Remark 2.6.2. is distinguished.

Example 2.6.1. For \( v = (1, 2, 4, 7, 8, 11) \), the subset \( S \) of \( \mathcal{M}(v) \) given by

\[
S = \{(3, 2), (5, 1), (9, 8), (12, 7)\}
\]

is distinguished.

Remark 2.6.2. By [7] Proposition 4.3, there exists a bijection between elements \( w \) of \( I(d, 2d) \) satisfying \( w \geq v \) on the one hand, and distinguished subsets of \( \mathcal{N}(v) \) on the other hand. We denote this bijective correspondence by \( w \leftrightarrow S_w \).
Let $\mathcal{S}$ be a non-empty monomial in $\mathcal{R}(v)$. If $\beta_1 > \cdots > \beta_t$ is a $v$-chain in $\mathcal{S}$, then we call $\beta_1$ the head of the $v$-chain and $\beta_t$ its tail. We call $t$ to be the length of the $v$-chain. We say that an element $\beta$ of $\mathcal{S}$ is $t$-deep in $\mathcal{S}$ (where $t$ is a positive integer) if $\beta$ is the tail of a $v$-chain in $\mathcal{S}$ of length $t$. The depth of $\beta$ in $\mathcal{S}$ is defined to be $t$ if $\beta$ is $t$-deep in $\mathcal{S}$ but not $(t+1)$-deep in $\mathcal{S}$.

Example 2.6.3. Let $v = (1, 2, 4, 7, 8, 11, 14)$ and $\mathcal{S} = \{(9, 1), (6, 2), (5, 4), (13, 8), (12, 11)\}$ be a monomial in $\mathcal{R}(v)$. Let $\beta = (5, 4)$. Then it is easy to see that $\beta$ is 1-deep, 2-deep, and 3-deep in $\mathcal{S}$. But $\beta$ is not 4-deep in $\mathcal{S}$. In fact, $(9, 1) > (6, 2) > (5, 4)$ is a $v$-chain in $\mathcal{S}$, and this is a $v$-chain in $\mathcal{S}$ of maximum length having $\beta = (5, 4)$ as its tail. Hence the depth of $\beta$ in $\mathcal{S}$ is 3 here.

We will now recall the map $\pi$ of [7]. Let $\mathcal{S}$ be a non-empty monomial in the elements of $\mathcal{R}(v)$. We partition $\mathcal{S}$ in two stages. First we partition $\mathcal{S}$ into subsets $\mathcal{S}_1, \ldots, \mathcal{S}_k$, where $k$ is the largest length of a $v$-chain in $\mathcal{S}$: $\beta \in \mathcal{S}$ belongs to $\mathcal{S}_j$ if it is $j$-deep but not $(j+1)$-deep.

Now we partition each $\mathcal{S}_j$ into subsets called blocks as follows. We arrange the elements of $\mathcal{S}_j$ in non-decreasing order of their row numbers (all arrangements are from left to right; and elements occur with their respective multiplicities). Among those with the same row number, the arrangement is by non-decreasing order of column numbers. Two consecutive members ($r, c$), $(R, C)$ in this arrangement are said to be related if $r > C$. The blocks are the equivalence classes of the smallest equivalence relation containing the above relations.

Let $\mathcal{B}$ be a single block of some $\mathcal{S}_j$. Let

$$(r_1, c_1), \ldots, (r_p, c_p)$$

be the elements of $\mathcal{B}$ written in non-decreasing order of both row and column numbers (in such an arrangement, the elements occur with their respective multiplicities). We set $w(\mathcal{B}) := (r_p, c_1)$ and $\mathcal{B}'$ to be the monomial

$$\{(r_1, c_2), (r_2, c_3), \ldots, (r_{p-2}, c_{p-1}), (r_{p-1}, c_p)\}.$$

Set $\mathcal{S}_j^{(1)} := \bigcup_{\mathcal{B}'} \mathcal{B}'$ (where the index $\mathcal{B}$ runs over all blocks of $\mathcal{S}_j$) and $\mathcal{S}^{(1)} := \bigcup_{j=1}^k \mathcal{S}_j^{(1)}$. It follows from [7, Corollary 4.13] that the set

$$\{w(\mathcal{B})|\mathcal{B} \text{ is a block of } \mathcal{S}\}$$

is a distinguished subset of $\mathcal{R}(v)$. Let $w$ be the corresponding element of $I(d, 2d)$ (under the correspondence given in Remark 2.6.2). Set

$$\pi(\mathcal{S}) := (w, \mathcal{S}^{(1)}).$$

This finishes the description of the map $\pi$ of [7].

Example 2.6.4. Let $d = 7$ and $v = (1, 2, 4, 7, 8, 11, 14)$. The dark circles in the grid in Figure 2.6.1 represent a monomial $\mathcal{S}$ in $\mathcal{R}^d$, where $\mathcal{S} = \{(3, 2), (5, 4), (6, 2), (9, 1), (9, 1), (10, 7), (10, 7), (10, 7), (10, 8), (12, 1), (13, 4)\}$.
The numbers written near the dark circles denote the multiplicities of these elements in the monomial \( \mathcal{G} \). For this monomial \( \mathcal{G} \), we have

\[
\mathcal{G}_1 = \{(9, 1), (9, 1), (12, 1), (13, 4)\}
\]

\[
\mathcal{G}_2 = \{(3, 2), (6, 2), (10, 7), (10, 7), (10, 8)\}
\]

\[
\mathcal{G}_3 = \{(5, 4)\}
\]

Here \( \mathcal{G}_1 \) and \( \mathcal{G}_3 \) are single blocks. And \( \mathcal{G}_2 \) has two blocks given by

\[
\{(3, 2), (6, 2)\} \text{ and } \{(10, 7), (10, 7), (10, 8)\}.
\]

The dark line segments on the grid show the block decomposition of the monomial \( \mathcal{G} \). The set

\[
\{w(\mathcal{B})| \mathcal{B} \text{ is a block of } \mathcal{G}\} = \{(13, 1), (6, 2), (10, 7), (5, 4)\}.
\]

**Definition 2.6.5.** Let \( v \in I(d, 2d) \). A **standard sequence** in \( I(d, 2d) \) is a totally ordered sequence \( \theta_1 \geq \cdots \geq \theta_t \) of elements of \( I(d, 2d) \). Such a sequence is called \( v \)-**compatible** if each \( \theta_j \) is comparable to \( v \) but no \( \theta_j \) equals \( v \); it is called **anti-dominated** by \( v \) if \( \theta_t \geq v \). Let \( \hat{S}M^{v,v} \) denote the set of all \( v \)-compatible standard sequences in \( I(d, 2d) \) anti-dominated by \( v \).

**Example 2.6.6** below gives an illustration of \( \hat{S}M^{v,v} \).

Therefore

\[
w = (5, 6, 8, 10, 11, 13, 14) \text{ and } \mathcal{G}^{(1)} = \{(9, 1), (9, 1), (12, 4), (3, 2), (10, 7), (10, 7), (10, 8)\}.
\]
Example 2.6.6. Let $d = 4$ and $v = (1, 3, 5, 6)$. Then $(2, 5, 7, 8) \succeq (1, 4, 6, 8) \succeq (1, 4, 6, 7)$ is an element of $\tilde{SM}^{w,v}$.

Using $\pi$, we now recall the map $\tilde{\pi}$ of [7] from $\text{mon}\mathcal{R}(v)$ to $\tilde{SM}^{w,v}$. Proceed by induction on the degree of an element $\mathcal{S}$ of $\text{mon}\mathcal{R}(v)$. The image of the empty monomial under $\tilde{\pi}$ is taken to be the empty monomial. Let $\mathcal{S}$ be non-empty, and suppose that $\pi(\mathcal{S}) = (w, \mathcal{S}^{(1)})$. By (1) and (2) of [7] Proposition 4.1, the degree of $\mathcal{S}^{(1)}$ is strictly less than that of $\mathcal{S}$, and so by induction $\tilde{\pi}(\mathcal{S}^{(1)})$ is defined. Suppose that $\tilde{\pi}(\mathcal{S}^{(1)}) = w' \geq \ldots$. By induction we also know that the degree of $\mathcal{S}^{(1)}$ is the same as that of $w' \geq \ldots$ and that $w'$ is the least element of $I(d,2d)$ that dominates $\mathcal{S}^{(1)}$. By (3) of [7] Proposition 4.1, we have $w \geq w'$, and we set $\tilde{\pi}(\mathcal{S}) := w \geq \tilde{\pi}(\mathcal{S}^{(1)})$. This finishes the description of the map $\tilde{\pi}$ of [7].

Example 2.6.7 below gives an illustration of the map $\tilde{\pi}$ for a monomial in $\mathcal{R}(v)$.

Example 2.6.7. For the monomial $\mathcal{S}$ in Example 2.6.6 above, we have 

$$\tilde{\pi}(\mathcal{S}) = (5, 6, 8, 10, 11, 13, 14) \geq (3, 4, 8, 10, 11, 12, 14) \geq (2, 4, 7, 8, 10, 11, 14) \geq (1, 2, 7, 8, 10, 11, 14) \geq (1, 2, 4, 8, 9, 11, 14) \geq (1, 2, 4, 7, 9, 11, 14).$$

3 Statement of the main theorem

3.1 Extension of the Kodiyalam-Raghavan maps

In this subsection, we will extend the map $\tilde{\pi}$ of Kodiyalam-Raghavan [7] to the entire $\mathcal{R}(v)$.

Fix an element $v$ in $I(d, 2d)$. Let $v = (v_1, \ldots, v_d)$. Given $\beta_1 = (r_1, c_1)$ and $\beta_2 = (r_2, c_2)$ in $\mathcal{R}(v) \setminus \mathcal{R}(v)$, we say that $\beta_1 > \beta_2$ if $r_1 < r_2$ and $c_2 < c_1$. A sequence $\beta_1 > \cdots > \beta_t$ of elements of $\mathcal{R}(v) \setminus \mathcal{R}(v)$ is called an anti-$v$-chain. Given an anti-$v$-chain $\beta_1 > \cdots > \beta_t = (r_t, c_t)$, we define

$$s_{\beta_1} \cdots s_{\beta_t} v := (\{v_1, \ldots, v_d\} \setminus \{c_1, \ldots, c_t\}) \cup \{r_1, \ldots, r_t\}.$$

We say that an element $w$ of $I(d, 2d)$ anti-dominates the anti-$v$-chain $\beta_1 > \cdots > \beta_t$ if $w \leq s_{\beta_1} \cdots s_{\beta_t} v$. Let $\mathcal{S}$ be a monomial in $\mathcal{R}(v) \setminus \mathcal{R}(v)$. We say that $w$ anti-dominates $\mathcal{S}$ if $w$ anti-dominates every anti-$v$-chain in $\mathcal{S}$.

We call distinguished the subsets $\mathcal{S}$ of $\mathcal{R}(v) \setminus \mathcal{R}(v)$ satisfying the following conditions:

1. For $(r, c) \neq (r', c')$ in $\mathcal{S}$, we have $r \neq r'$ and $c \neq c'$.

2. If $\mathcal{S} = \{(r_1, c_1), \ldots, (r_p, c_p)\}$ with $r_1 > r_2 > \ldots > r_p$, then for $j$, $1 \leq j \leq p - 1$, we have either $c_j < c_{j+1}$ or $r_j > r_{j+1}$.

Remark 3.1.1. It can be proved similarly as in [7] Proposition 4.3 that there exists a bijection between elements $w$ of $I(d, 2d)$ satisfying $w \leq v$ on the one hand and distinguished subsets of $\mathcal{R}(v) \setminus \mathcal{R}(v)$ on the other hand. We denote this bijective correspondence by $w \leftrightarrow \mathcal{S}_w$.

Let $\mathcal{S}$ be a non-empty monomial in $\mathcal{R}(v) \setminus \mathcal{R}(v)$. If $\beta_1 > \cdots > \beta_t$ is an anti-$v$-chain in $\mathcal{S}$, then we call $\beta_1$ the head of the anti-$v$-chain and $\beta_t$ its tail. We
call $t$ to be the **length** of the anti-$v$-chain. We say that an element $\beta$ of $S$ is $t$-**deep** in $S$ (where $t$ is a positive integer) if $\beta$ is the tail of an anti-$v$-chain in $S$ of length $t$. The **depth** of $\beta$ in $S$ is defined to be $t$ if $\beta$ is $t$-deep in $S$ but not $(t + 1)$-deep in $S$.

We will now define the map $\pi$ on any monomial in $\mathcal{R}(v) \setminus \mathcal{N}(v)$. Let $S$ be a non-empty monomial in the elements of $\mathcal{R}(v) \setminus \mathcal{R}(v)$. We partition $S$ in two stages. First we partition $S$ into subsets $S_1, \ldots, S_k$, where $k$ is the largest length of an anti-$v$-chain in $S$: $\beta \in S$ belongs to $S_j$ if it is $j$-deep but not $(j + 1)$-deep.

Now we partition each $S_j$ into subsets called **blocks** as follows. We arrange the elements of $S_j$ in non-increasing order of their row numbers (where elements occur with their respective multiplicities). Among those with the same row number, the arrangement is by non-increasing order of column numbers. Two consecutive members $(r, c), (R, C)$ in this arrangement are said to be **related** if $r < C$. The blocks are the equivalence classes of the smallest equivalence relation containing the above relations.

Let $B$ be a single block of some $S_j$. Let

\[(r_1, c_1), \ldots, (r_p, c_p)\]

be the elements of $B$ written in non-increasing order of both row and column numbers (in such an arrangement, the elements occur with their respective multiplicities). We set $w(B) := (r_p, c_1)$ and $B'$ to be the monomial

\[\{(r_1, c_2), (r_2, c_3), \ldots, (r_{p-2}, c_{p-1}), (r_{p-1}, c_p)\} \cup B\]  

Set $S_j^{(1)} := \cup_B B'$ (where the index $B$ runs over all blocks of $S_j$) and $S^{(1)} := \cup_{j=1}^k S_j^{(1)}$. It follows (similarly as in [7, Corollary 4.13]) that the set

\[\{w(B) | B \text{ is a block of } S\}\]

is a distinguished subset of $\mathcal{R}(v) \setminus \mathcal{N}(v)$. Let $w$ be the corresponding element of $I(d, 2d)$ (under the correspondence given in Remark 3.1.1). Set

\[\pi(S) := (w, S^{(1)}).\]

This finishes the description of the map $\pi$.

A **standard sequence** in $I(d, 2d)$ is a totally ordered sequence $\theta_1 \geq \cdots \geq \theta_t$ of elements of $I(d, 2d)$. A standard sequence $\theta_1 \geq \cdots \geq \theta_t$ in $I(d, 2d)$ is called **dominated** by $v$ if $v \geq \theta_1$. Such a sequence is called $v$-**compatible** if each $\theta_j$ is comparable to $v$ but no $\theta_j$ equals $v$. Let $\mathcal{S}_v$ denote the set of all $v$-**compatible standard sequences** in $I(d, 2d)$ dominated by $v$.

Using $\pi$, we now define the map $\tilde{\pi}$ from the set of all monomials in $\mathcal{R}(v) \setminus \mathcal{N}(v)$ to $\mathcal{S}_v$. We proceed by induction on the degree of a monomial $S$ in $\mathcal{R}(v) \setminus \mathcal{R}(v)$. The image of the empty monomial under $\tilde{\pi}$ is taken to be the empty monomial. Let $S$ be non-empty, and suppose that $\pi(S) = (w, S^{(1)})$. It can be shown (similarly as in (1) and (2) of [7, Proposition 4.1]) that the degree of $S^{(1)}$ is strictly less than that of $S$, and so by induction $\tilde{\pi}(S^{(1)})$ is defined. Suppose that $\tilde{\pi}(S^{(1)}) = w' \leq \cdots$. It can be shown (similarly as in (3) of [7]
Proposition 4.1) that \( w \leq w' \). We set \( \tilde{\pi}(S) := w \leq \pi(S^{(1)}) \). This finishes the description of the map \( \tilde{\pi} \) on the set of all monomials in \( \mathcal{R}(v) \setminus \mathcal{R}(v) \).

Example 3.1.2 below gives an illustration of the map \( \tilde{\pi} \) for a monomial in \( \mathcal{R}(v) \setminus \mathcal{R}(v) \).

Example 3.1.2. Let \( d = 6 \) and \( v = (3, 6, 8, 10, 11, 12) \). Let

\[ S = \{(9, 11), (4, 11), (7, 10), (5, 10), (7, 8), (1, 8), (4, 6)\} \]

be a finite monomial in \( \mathcal{R}(v) \setminus \mathcal{R}(v) \). Figure 3.1.1 shows the monomial \( S \) and its block decomposition. The dark circles in the figure represent points in the monomial \( S \) with their respective multiplicities (the multiplicity of each point in the monomial \( S \) is 1 here, which is written near those points in the grid). The dark line segments (together with the point \( (7, 8) \)) denote the blocks of \( S \).

For this monomial \( S \), we have \( \pi(S) = (w_0, S^{(1)}) \), where

\[ w_0 = (1, 3, 4, 6, 7, 12) \text{ and } S^{(1)} = \{(9, 11), (4, 8), (7, 10), (5, 6)\} \]

Then \( \pi(S^{(1)}) = (w_1, S^{(2)}) \), where \( w_1 = (3, 4, 5, 8, 10, 12) \) and \( S^{(2)} = \{(9, 10), (7, 8)\} \). And finally, \( \pi(S^{(2)}) = (w_2, \emptyset) \), where \( w_2 = (3, 6, 7, 9, 11, 12) \).

Now for the above monomial \( S \) we have,

\[ \tilde{\pi}(S) = (1, 3, 4, 6, 7, 12) \leq (3, 4, 5, 8, 10, 12) \leq (3, 6, 7, 9, 11, 12). \]

The relation between the maps \( \tilde{\pi} \) and \( \text{BRSK} \) was given by [13, Corollary 2.3.2], which is stated here as the following Proposition.

**Proposition 3.1.3.** For any monomial \( U \) in \( \mathcal{R}(v) \), \( \tilde{\pi} = \text{BRSK}(U) \).

### 3.2 Ideals of tangent cones to Richardson varieties

Let \( \beta \) be the element of \( I(d) \), which was fixed at the beginning of this section. Consider the matrix of size \( 2d \times d \) whose columns are numbered by the entries of \( \beta \), the rows by \( \{1, \ldots, 2d\} \), the rows corresponding to the entries of \( \beta \) form
the $d \times d$ identity matrix, and the remaining $d$ rows form a matrix whose entries are $X_{(r,c)}$ such that $(r,c) \in \mathfrak{M}(\beta)$, where $X_{(r,c)} = -X_{(c',r')}$ if either $r > d$ and $c' < d$ or $r < d$ and $c' > d$, and $X_{(r,c)} = X_{(c',r')}$ otherwise.

For $d = 4$, $\beta = (1, 2, 5, 6)$, the $2d \times d$ matrix is given in below:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
x_{31} & x_{32} & x_{35} & x_{36} \\
x_{41} & x_{42} & x_{45} & x_{35} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
x_{71} & x_{72} & x_{42} & -x_{32} \\
x_{81} & x_{71} & x_{41} & -x_{31}
\end{pmatrix}
\]  

(3.2.1)

Let $\mathfrak{M}_d(V) \subseteq G_d(V) \hookrightarrow \mathbb{P}(\bigwedge^d V)$ be the Plücker embedding (where $G_d(V)$ denotes the Grassmannian of all $d$-dimensional subspaces of $V$). For $\theta$ in $I(d, 2d)$, let $p_\theta$ denote the corresponding Plücker coordinate. Consider the affine patch $\mathbb{A}$ of $\mathbb{P}(\bigwedge^d V)$ given by $p_\beta \neq 0$. The affine patch $\mathbb{A}^\beta := \mathfrak{M}_d(V) \cap \mathbb{A}$ of the symplectic Grassmannian $\mathfrak{M}_d(V)$ is an affine space whose coordinate ring can be taken to be the polynomial ring in variables of the form $X_{(r,c)}$ with $(r, c) \in \mathfrak{M}(\beta)$.

For $\theta \in I(d, 2d)$, consider the submatrix of the above mentioned matrix given by the rows numbered $\theta \setminus \beta$ and columns numbered $\beta \setminus \theta$. Let $f_{\theta, \beta}$ denote the determinant of this submatrix. Clearly, $f_{\theta, \beta}$ is a homogeneous polynomial in the variables $X_{(r,c)}$, where $(r, c) \in \mathfrak{M}(\beta)$.

Example 3.2.1 below gives an illustration of $f_{\theta, \beta}$.

Example 3.2.1. Let $d = 4$, $\beta = (1, 2, 5, 6)$, and $\theta = (1, 3, 4, 5)$. So $\theta \in I(4, 8)$.

Now $f_{\theta, \beta}$ is the determinant of the submatrix whose rows are numbered by $\theta \setminus \beta = \{3, 4\}$ and columns are numbered by $\beta \setminus \theta = \{2, 6\}$, that is

\[
f_{\theta, \beta} = x_{32} x_{36} - x_{32} x_{35} - x_{36} x_{42}.
\]

(3.2.2)

Clearly, $f_{\theta, \beta}$ is a homogeneous polynomial in the variables $X_{(r,c)}$, where $(r, c) \in \mathfrak{M}(\beta)$.

The $\epsilon$-degree of an element $x$ of $I(d)$ is defined as the cardinality of $x \setminus [d]$ or equivalently that of $[d] \setminus x$. An ordered pair $w = (x, y)$ of elements of $I(d)$ is called an admissible pair if $x \geq y$ and the $\epsilon$-degrees of $x$ and $y$ are equal.

We refer to $x$ and $y$ as the top and the bottom of $w$ and write top($w$) for $x$ and bot($w$) for $y$. Given any admissible pairs $w = (x, y)$ and $w' = (x', y')$, we say that $w \geq w'$ if $y \geq y'$, that is, if $x \geq y \geq x' \geq y'$. Let $w = (x, y)$ be an admissible pair. Let $\theta$ be the element $(x \cap [d]) \cup (y \cap [d]^c)$ of $I(d, 2d)$ (as mentioned in [H] Proposition 3.4]). For any admissible pair $w$, let us denote by $f_{w, \beta}$ the polynomial $f_{\theta, \beta}$.

Example 3.2.2 below gives an illustration of the admissible pairs.

Example 3.2.2. Let $d = 4$, so $\epsilon = (1, 2, 3, 4)$. Let $x = (1, 4, 6, 7)$. The $\epsilon$-degree of $x$ is 2. Let $y = (1, 3, 5, 7)$. Clearly, the $\epsilon$-degree of $y$ is also 2, and $x, y \in I(d)$ with $x \geq y$. Hence $w = (x, y)$ is an admissible pair. Also here
top(w) = (1, 4, 6, 7) and bot(w) = (1, 3, 5, 7). Again let x' = (1, 3, 5, 7) and
y' = (1, 2, 4, 6). Then the ε-degrees of both x' and y' are 1 and x', y' ∈ I(d)
with x' ≥ y'. So w' = (x', y') is also an admissible pair. As x ≥ y ≥ x' ≥ y',
so w ≥ w'. Again for the above w, θ = (1, 4, 5, 7). So for this θ and for
β = (1, 2, 5, 6), (using the 8 × 4 matrix of Statement 3.2.1) we have,
\[ f_{w, \beta} = f_{\theta, \beta} = -x_{32}x_{42} - x_{35}x_{72}. \]

Set \( Y^\alpha_\beta(\beta) := X^\alpha_\beta \cap A^\beta. \) From 11 we can deduce a set of generators for the
ideal \( I^\alpha_{n, \beta} \) of functions on \( A^\beta \) vanishing on \( Y^\alpha_\beta(\beta) \). The following equation gives the generators:
\[ I^\alpha_{n, \beta} = (f_{m, \beta} \mid w = (x, y) \text{ is an admissible pair, } \alpha \not\leq y \text{ or } x \not\leq \gamma) \quad (3.2.3) \]

We are interested in the tangent cone to \( X^\alpha_\beta \) at \( e_\beta \) or, what is the same, the
tangent cone to \( Y^\alpha_\beta(\beta) \) at the origin. Observe that \( f_{m, \beta} \) is a homogeneous polynomial. Because of this, \( Y^\alpha_\beta(\beta) \) itself is a cone and so equal to its tangent
cone at the origin. The ideal of the tangent cone to \( X^\alpha_\beta \) at \( e_\beta \) is therefore the ideal
\( I^\alpha_{n, \beta} \) in Equation (3.2.3).

### 3.3 Extended β-chains

Let \( \beta \) be the element of \( I(d) \), which was fixed at the end of 22. For elements \( \lambda = (R, C), \mu = (r, c) \) of \( \mathfrak{R}(\beta) \), we write \( \lambda > \mu \) if \( R > r \) and \( C < c \) (note that these are strict inequalities). A sequence \( \lambda_1 > \cdots > \lambda_k \) of elements of \( \mathfrak{R}(\beta) \)
is called an extended β-chain. Note that an extended β-chain can also be empty. Letting \( C \) to be an extended β-chain, we define \( C^+ := C \cap \mathfrak{R}(\beta) \) and
\( C^- := C \cap (\mathfrak{R}(\beta) \setminus \mathfrak{R}(\beta)) \). We call \( C^+ \) (resp. \( C^- \)) the positive (resp. negative) part of the extended β-chain \( C \). We call an extended β-chain \( C \) positive (resp. negative) if \( C = C^+ \) (resp. \( C = C^- \)). The extended β-chain \( C \) is called non-vanishing if at least one of its positive or negative part is non-empty. Clearly then, every non-empty extended β-chain is non-vanishing.

An extended β-chain that lies completely in \( \mathfrak{D}\mathfrak{R}(\beta) \) is called an extended upper β-chain. We similarly define extended upper positive and extended upper negative β-chains.

Example 3.3.1 below illustrates an extended β-chain and an extended upper
β-chain.

**Example 3.3.1.** Let \( d = 7 \) and \( \beta = (1, 3, 4, 7, 9, 10, 13) \). Let \( \lambda_1 = (14, 1), \lambda_2 = (12, 3), \lambda_3 = (6, 7), \) and \( \lambda_4 = (5, 13) \). Then clearly \( \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 \). So this is an extended β-chain in \( \mathfrak{R}(\beta) \). If we denote the above β-chain by \( C \), then
\( C^+ = \lambda_1 > \lambda_2 \) and \( C^- = \lambda_3 > \lambda_4 \). Again \( \lambda_1 > \lambda_2 > \lambda_3 \) is an extended upper
β-chain.

**Definition 3.3.2.** Let \( \beta \) be as fixed earlier. Let \( \beta \) denote the set \( \lceil 2d \rceil \setminus \beta \). We call \( \beta \) the complement of \( \beta \).

**Definition 3.3.3.** Let \( A \subset \beta \) and \( B \subset \beta \). We define \( A - B \) as the set \( A \cup (\beta \setminus B) \).

**Definition 3.3.4.** Let \( C \) be an extended upper β-chain. Let \( (P^C, Q^C) = BRSK(C \cup C^\#) \). \( (P^C_1, Q^C_1) \) denote the topmost row of \( (P^C, Q^C) \) and \( (P^C, Q^C) \) denote the bottom-most row of \( (P^C, Q^C) \). Let top\( (C^+) \) denote the element \( P^C_r - Q^C_r \) of \( I(d, 2d) \) and bot\( (C^-) \) denote the element \( P^C_1 - Q^C_1 \) of \( I(d, 2d) \),
respectively of the multiset

\[ \text{Let } (C \cup C^#)^+ \text{ and } (C \cup C^#)^- \text{ denote the positive and negative parts respectively of the multiset } C \cup C^#. \]

We know that \( BRSK(C \cup C^#) \) is equal to the notched bitableau obtained by placing the notched bitableau \( BRSK((C \cup C^#)^-) \) on top of the notched bitableau \( BRSK((C \cup C^#)^+) \).

Recall the map \( \tilde{\pi} \) from [7 §4]. We know from [15 Corollary 2.3.2] that \( BRSK((C \cup C^#)^+) = \tilde{\pi}((C \cup C^#)^+) \). Also \( (C \cup C^#)^+ = ((C \cup C^#)^+)^# \). It hence follows from [4 Proposition 5.6] that all the elements of \( I(d, 2d) \) corresponding to all the rows of \( BRSK((C \cup C^#)^+) \) in fact belong to \( I(d) \). In particular, the element \( P_1^C - Q_1^C = \text{top}(C^+) \) also belongs to \( I(d) \). The proof of the fact that \( \text{bot}(C^-) \) belongs to \( I(d) \) is similar [We omit the proof here because it involves proving that the maps \( BRSK \) and \( \tilde{\pi} \) are equal on negative multisets. And this proof is similar to that in [15]].

The example below illustrates Theorem 3.3.5.

**Example 3.3.6.** Let \( d = 7 \) and \( \beta = (1, 3, 4, 7, 9, 10, 13) \). Clearly, \( \beta \in I(d) \).

Now \( \beta = (2, 5, 6, 8, 11, 12, 14) \). Consider the upper extended \( \beta \)-chain

\[ C = \{(12, 1), (11, 3), (8, 4), (6, 7), (5, 9), (2, 10)\}. \]

According to [9],

\[ BRSK(C \cup C^#) = \begin{pmatrix} 2 & 5 & 6 & 9 & 10 & 13 \\ 5 & 6 & 8 & 7 & 9 & 10 \\ 8 & 11 & 12 & 3 & 4 & 7 \\ 11 & 12 & 13 & 14 \end{pmatrix} \quad (3.3.1) \]

Now \( \text{top}(C^+) = P_1^C - Q_1^C = \{11, 12, 14\} \cup (\beta \setminus \{1, 3, 4\}) = (7, 9, 10, 11, 12, 13, 14) \), and \( \text{bot}(C^-) = P_1^C - Q_1^C = \{2, 5, 6\} \cup (\beta \setminus \{9, 10, 13\}) = (1, 2, 3, 4, 5, 6, 7) \).

Clearly, both of \( \text{top}(C^+) \) and \( \text{bot}(C^-) \) belong to \( I(d) \).

### 3.4 Gröbner basis for ideals of tangent cones

We now specify the term order \( \gg \) on monomials in the coordinate functions \( \{X_{(r,c)}(r, c) \in DR(\beta)\} \) with respect to which the initial ideal of the ideal \( I_{\alpha, \beta}^\gamma \) of the tangent cone is to be taken.

**Definition 3.4.1.** Let \( \gg \) be the total order on \( DR(\beta) \) satisfying the following condition:

- \( X_{(r,c)} \gg X_{(r',c')} \) if either (a) \( r > r' \) or (b) \( r = r' \) and \( c < c' \).

Let \( \gg \) be the term order on monomials in \( DR(\beta) \) given by the homogeneous lexicographic order with respect to \( \gg \).

**Example 3.4.2.** Let \( d = 7 \) and \( \beta = (1, 3, 4, 7, 9, 10, 13) \). Now all of \( (14, 1), (12, 3), (11, 3), (11, 1) \) are elements in \( DR(\beta) \), and according to the above term order we have, \( X_{(14,1)} \gg X_{(12,3)} \gg X_{(11,1)} \gg X_{(11,3)} \). Let \( X_{\phi_1} = X_{(14,1)}^3 X_{(11,1)} X_{(11,3)}^2 \), \( X_{\phi_2} = X_{(14,1)} X_{(12,3)} X_{(11,3)} \).
and $X_{\Theta_3} = X_{(14,1)}X_{(11,1)}$. Clearly $\Theta_1$, $\Theta_2$, and $\Theta_3$ all are monomials in $\mathcal{O}(\beta)$. Now the degree of the polynomial $X_{\Theta_1}$ is greater than that of $X_{\Theta_2}$ and $X_{\Theta_3}$. So $X_{\Theta_1} \triangleright X_{\Theta_2}$ and $X_{\Theta_1} \triangleright X_{\Theta_3}$. Though the degree of $X_{\Theta_3}$ is equal to the degree of $X_{\Theta_3}$, but $X_{(12,3)} > X_{(11,1)}$ and in $X_{\Theta_2}$, the degree of $X_{(12,3)}$ is one and in $X_{\Theta_3}$, the degree of $X_{(12,3)}$ is zero. So according to the definition of homogeneous lexicographic order, we have $X_{\Theta_3} \triangleright X_{\Theta_2}$. Hence $X_{\Theta_1} \triangleright X_{\Theta_2} \triangleright X_{\Theta_3}$.

Now recall that the ideal of the tangent cone to $X_\alpha$ at $e_\beta$ is the ideal $I_{\alpha,\beta}$ given by Equation 3.2.3. Let $\triangleright$ be as in 3.3. For any element $f \in I_{\alpha,\beta}$, let $\text{in}_\triangleright f$ denote the initial term of $f$ with respect to the term order $\triangleright$. We define $\text{in}_\triangleright I_{\alpha,\beta}$ to be the ideal $(\text{in}_\triangleright f | f \in I_{\alpha,\beta})$ inside the polynomial ring $P := K[X(r,c) | (r,c) \in \mathcal{O}(\beta)]$.

**Definition 3.4.3.** An admissible pair $w = (t, u)$ (where $t \geq u$) is called a **good admissible pair** if it satisfies both of the following 2 properties:

1. $\alpha \not\in u$ or $t \not\in \gamma$.
2. Either $\text{in}_\triangleright f_{w,\beta}$ forms a positive upper extended $\beta$-chain $C^+$ such that $C^+_{(1)} - C^+_{(2)} \not\subseteq \gamma$ or $\text{in}_\triangleright f_{w,\beta}$ forms a negative upper extended $\beta$-chain $C^-$ such that $C^-_{(1)} - C^-_{(2)} \not\subseteq \alpha$.

Let $G_{\alpha,\beta}^\gamma$ denote the set $\{ f_{w,\beta} | w \text{ is good} \}$.

Example 3.4.4 below illustrates a good admissible pair.

**Example 3.4.4.** Let $d = 4$, $\alpha = (1,2,3,5)$, $\beta = (1,2,5,6)$, and $\gamma = (2,3,5,8)$. Let $w = (t, u)$ be an admissible pair, where $t = (3,4,7,8)$ and $u = (1,2,5,6)$. Clearly $t \not\in \gamma$. Now in § 3.2 we have already defined that $\theta = (t \cap [d]) \cup (u \cap [d]^c)$, and $f_{w,\beta} = f_{\theta,\beta}$. Hence in this example $\theta = (3,4,5,6)$ and from the matrix which is given in § 3.2 we have

\[
f_{w,\beta} = \begin{vmatrix} x_{31} & x_{32} \\ x_{41} & x_{42} \end{vmatrix}
\]  

(3.4.1)

Observe that $\text{in}_\triangleright f_{w,\beta} = -x_{41}x_{32}$. Clearly, $\text{in}_\triangleright f_{w,\beta}$ forms a positive upper extended $\beta$-chain $C^+$ such that $C^+_{(1)} - C^+_{(2)} = \{3,4\} \cup (\beta \setminus \{1,2\}) = (3,4,5,6) \not\subseteq \gamma$. Hence $w = (t, u)$ is a good admissible pair.

**Definition 3.4.5.** If $S$ is any nonempty subset of the polynomial ring $P := K[X(r,c) | (r,c) \in \mathcal{O}(\beta)]$ such that $S \neq \{0\}$. We define $\text{in}_\triangleright S$ to be the ideal $(\text{in}_\triangleright s | s \in S)$. The main result of this paper is the following:

**Theorem 3.4.6.** The set $G_{\alpha,\beta}^\gamma$ is a Gröbner basis for the ideal $I_{\alpha,\beta}$.

### 3.5 Strategy of the proof

To explain the strategy of the proof of Theorem 3.4.6 we need the following definition.

**Definition 3.5.1.** We call $f = f_{w_1,\beta} \cdots f_{w_r,\beta} \in P = K[X(r,c) | (r,c) \in \mathcal{O}(\beta)]$ a standard monomial if

\[
w_1 \leq \cdots \leq w_r,
\]

and for each $i \in \{1,\ldots,r\}$, we have

Either $\beta \geq \text{top}(w_i)$ or $\text{top}(w_i) \geq \beta$,

(3.5.2)
and either $\beta \geq \text{bot}(\text{w}_1)$ or $\text{bot}(\text{w}_1) \geq \beta$, \hspace{1cm} (3.5.3)
and $\text{w}_1 \not= (\beta, \beta)$, \hspace{1cm} (3.5.4)

If in addition, for $\alpha, \gamma \in I(d)$, we have

$$\alpha \leq \text{bot}(\text{w}_1) \text{ and } \text{top}(\text{w}_r) \leq \gamma,$$

then we say that $f$ is \textbf{standard on $Y_\alpha^\gamma(\beta)$}.

Example 3.5.2 below gives an illustration of a standard monomial on $Y_\alpha^\gamma(\beta)$.

\textbf{Example 3.5.2.} Let $d = 4$, $\alpha = (1, 2, 3, 5)$, $\beta = (1, 2, 5, 6)$, and $\gamma = (3, 4, 7, 8)$. For this $\beta$, the $8 \times 4$ matrix is given by 3.2.1. Let $\text{w}_1 = ((1, 2, 4, 6), (1, 2, 3, 5))$ and $\text{w}_2 = ((2, 4, 6, 8), (2, 3, 5, 8))$. Clearly, $\text{w}_1$ and $\text{w}_2$ both are admissible pairs.

Let $\theta_1$ and $\theta_2$ be the images of $\text{w}_1$ and $\text{w}_2$ respectively, under the correspondence given by $\text{w} = (x, y) \mapsto \theta = (x \cap [d]) \cup (y \cap [d]^c)$ as mentioned in [4, Proposition 3.4]. So we have, $\theta_1 = (1, 2, 4, 5)$ and $\theta_2 = (2, 4, 5, 8)$. Now using the $8 \times 4$ matrix of 3.2.1 we have,

$$f_{\text{w}_1, \beta} = f_{\theta_1, \beta} = x_{35} \in P$$
and
$$f_{\text{w}_2, \beta} = f_{\theta_2, \beta} = -x_{41}x_{31} - x_{35}x_{81} \in P.$$ 

Hence $f = f_{\text{w}_1, \beta}f_{\text{w}_2, \beta} \in P$. Clearly, $\text{w}_1 \leq \text{w}_2$. Again $\beta \geq \text{top}(\text{w}_1)$ and $\beta \geq \text{bot}(\text{w}_1)$. Also $\beta \leq \text{top}(\text{w}_2)$ and $\beta \leq \text{bot}(\text{w}_2)$, and $\text{w}_1 \not= (\beta, \beta)$ for all $i \in [1, 2]$. Again $\alpha, \gamma \in I(d)$ are such that $\alpha \leq \text{bot}(\text{w}_1)$ and $\beta \geq \text{top}(\text{w}_2)$. Hence $f$ is standard on $Y_\alpha^\gamma(\beta)$.

\textbf{Definition 3.5.3.} Let $f = f_{\text{w}_1, \beta} \cdots f_{\text{w}_r, \beta}$ be a standard monomial on $Y_\alpha^\gamma(\beta)$. We define the \textit{degree} of $f$ to be the sum of the $\beta$-degrees of $\text{w}_1, \ldots, \text{w}_r$, where given any admissible pair $\text{w} = (x, y)$, the $\beta$-degree of $\text{w}$ is defined to be $\frac{1}{2}(|x| - |\beta| + |y| - |\beta|)$.

We now briefly sketch the proof of Theorem 3.4.6 (the details are found in 35). Clearly, $G_\alpha^\gamma(\beta)$ is contained in the ideal $I_{\alpha, \beta}$. So $\text{in}_P G_\alpha^\gamma(\beta) \subseteq \text{in}_P I_{\alpha, \beta}$. Hence to prove Theorem 3.4.6 we only need to show that in any degree, the number of monomials of $\text{in}_P G_\alpha^\gamma(\beta)$ is at least as great as the number of monomials of $\text{in}_P I_{\alpha, \beta}$ (the other inequality being trivial). Equivalently, we need to prove that in any degree, the number of monomials of $P \setminus \text{in}_P G_\alpha^\gamma(\beta)$ is no greater than the number of monomials of $P \setminus \text{in}_P I_{\alpha, \beta}$. Both the monomials of $P \setminus \text{in}_P I_{\alpha, \beta}$ and the standard monomials on $Y_\alpha^\gamma(\beta)$ (the definition of a standard monomial on $Y_\alpha^\gamma(\beta)$ is given in Definition 3.5.1) form a basis for $P/I_{\alpha, \beta}$, and thus agree in cardinality in any degree. Therefore it suffices to prove that, in any degree, the number of monomials of $P \setminus \text{in}_P G_\alpha^\gamma(\beta)$ is less than or equal to the number of standard monomials on $Y_\alpha^\gamma(\beta)$. In this paper, we consider two sets, namely, the set of all “non-vanishing special semistandard notched bitableaux on $(\beta \times \beta)^*$ (bounded by $T_\alpha, W_\gamma$)”, and the set of all “non-vanishing semistandard notched bitableaux on $(\beta \times \beta)^*$ (bounded by $T_\alpha, W_\gamma$)”. The meaning attached to these two sets is given in 31 below. In 35 below, we will first show that there exists a degree doubling injection from the set of all monomials of $P \setminus \text{in}_P G_\alpha^\gamma(\beta)$ to the former set. Then we will show that, there exists a degree-halving injection from the later set (namely, the set of all “non-vanishing semistandard notched bitableaux on $(\beta \times \beta)^*$ (bounded by $T_\alpha, W_\gamma$)”) to the set of all standard monomials on $Y_\alpha^\gamma(\beta)$. And then we will prove that the map $BRSK$ of 39 is a degree preserving bijection from the former set to the later. This will complete the proof.

21
Example 3.5.4 below gives an illustration of a Gröbner basis.

**Example 3.5.4.** Let $d = 4$, $\alpha = (1, 2, 3, 5)$, $\beta = (1, 2, 5, 6)$, and $\gamma = (2, 3, 5, 8)$. Then from Example 3.4.4, we know that $w = (t, u)$ is an admissible pair, where $t = (3, 4, 7, 8)$ and $u = (1, 2, 5, 6)$. For the above $\alpha, \beta, \gamma$ if we consider all the admissible pairs which satisfying both the conditions of good admissible pair, then we will get the set $G_{\alpha, \beta}$ which is given by $\{f_{w, \beta} | w \in G\}$, where $G$ is the following set (of all good admissible pairs):

$$G = \{(1, 2, 3, 4), (1, 2, 3, 5), (1, 4, 6, 7), (1, 2, 5, 6), (2, 4, 6, 8), (1, 2, 5, 6), (3, 4, 7, 8), (1, 2, 5, 6), (1, 5, 6, 7), (2, 5, 6, 7), (1, 5, 6, 7), (3, 5, 7, 8), (1, 5, 6, 7), (4, 6, 7, 8), (1, 5, 6, 7), (2, 5, 6, 8), (3, 5, 7, 8), (2, 5, 6, 8), (4, 6, 7, 8), (2, 5, 6, 8), (5, 6, 7, 8), (5, 6, 7, 8)\}.$$  

As in Example 3.4.4, we can easily find the initial term of the above good admissible pairs. In this case, we have

$$\text{in}_{\beta} G_{\alpha, \beta} = \{(x_{11} x_{32}, x_{71}, x_{72}, x_{44} x_{32}, x_{41}, x_{42}, x_{45} x_{36}, x_{81} x_{72}, x_{81} x_{42}, x_{81} x_{32}, x_{81} x_{71} x_{42})\}.$$  

### 4 The two sets

As mentioned towards the end of the previous section, the two sets under consideration are “non-vanishing special multisets on $\bar{\beta} \times \bar{\beta}$ (bounded by $T_{\alpha}, W_{\gamma}$)” and “non-vanishing semistandard notched bitableaux on $(\bar{\beta} \times \bar{\beta})^*$ (bounded by $T_{\alpha}, W_{\gamma}$).” We will now explain the meaning of these two sets.

Let $\alpha, \beta, \gamma$ be as before (3.4.1). Let $I_{\beta}$ be the set of all pairs $(R, S)$ such that $R \subset \bar{\beta}$, $S \subset \bar{\beta}$, and $|R| = |S|$. Let $I_{\beta}^*$ be the subset of $I_{\beta}$ consisting of all pairs $(R, S)$ such that $R = S^*$. Clearly then, the map $(R, S) \mapsto R - S$ is a bijection from $I_{\beta}^*$ to $I(d)$. Indeed, the inverse map is given by $\theta \mapsto (\theta \setminus \beta \setminus \theta)$.

Let $(R_{\alpha}, S_{\alpha})$ and $(R_{\gamma}, S_{\gamma})$ be the preimages of $\alpha$ and $\gamma$ respectively under the bijection from $I_{\beta}^*$ to $I(d)$. Define $T_{\alpha}$ and $W_{\gamma}$ to be any subsets of $\bar{\beta} \times \bar{\beta}$ such that $(T_{\alpha})_{\bar{\beta}} = R_{\alpha}$, $(T_{\alpha})_{\overline{\beta}} = S_{\alpha}$, $(W_{\gamma})_{\bar{\beta}} = R_{\gamma}$, and $(W_{\gamma})_{\overline{\beta}} = S_{\gamma}$. Note that there always exist subsets $T_{\alpha}$ and $W_{\gamma}$ of $\bar{\beta} \times \bar{\beta}$ such that $T_{\alpha}$ is negative and $W_{\gamma}$ is positive. This is because $\beta \leq \gamma$. Apply the first half of the proof of [7] Proposition 4.3 to get a distinguished monomial corresponding to $\gamma$. This distinguished monomial will serve as a positive subset $W_{\gamma}$ of $\bar{\beta} \times \bar{\beta}$. Similarly, we can get a negative subset $T_{\alpha}$ of $\bar{\beta} \times \bar{\beta}$ corresponding to $\alpha$ (which is $\leq \beta$). Hence we can choose $T_{\alpha}$ and $W_{\gamma}$ in such a way that the former is negative and the latter is positive.

Example 4.0.1 below gives an illustration of the above paragraph.

**Example 4.0.1.** Let $d = 7$ and $\beta = (1, 3, 4, 7, 9, 10, 13)$. So $\bar{\beta} = (2, 5, 6, 8, 11, 12, 14)$. Let $\alpha = (1, 2, 3, 5, 7, 9, 11)$ and $\gamma = (4, 5, 6, 7, 12, 13, 14)$. So $\alpha \leq \beta \leq \gamma$. Now $I_{\beta}$ is the set of all pairs $(R, S)$ such that $R \subset (2, 5, 6, 8, 11, 12, 14)$, $S \subset (1, 3, 4, 7, 9, 10, 13)$, and $|R| = |S|$. Let $R = (2, 5, 6, 8, 11)$ and $S = (4, 7, 9, 13)$. Clearly, $|R| = |S|$ and $R = S^*$. So according to the definition of $I_{\beta}^*$, $(R, S) \in I_{\beta}^*$. Now $R - S = (1, 2, 3, 6, 8, 10, 11)$ is in $I(d)$. Again both of

$$(R_{\alpha}, S_{\alpha}) = ((2, 5, 11), (4, 10, 13)) \quad \text{and} \quad (R_{\gamma}, S_{\gamma}) = ((5, 6, 12, 14), (1, 3, 9, 10))$$

are in $I_{\beta}^*$. Let $T_{\alpha} = \{(2, 4), (5, 10), (11, 13)\}$ and $W_{\gamma} = \{(5, 1), (6, 3), (12, 9), (14, 10)\}$. Then clearly $T_{\alpha}$ is a negative and $W_{\gamma}$ is a positive subset of $\bar{\beta} \times \bar{\beta}$.  

22
4.1 The first set

A non-vanishing multiset on $\beta \times \beta$ (bounded by $T_\alpha$, $W_\gamma$) has the same meaning as in §2.3. Such a multiset $\mathcal{G}$ is called a non-vanishing special multiset on $\beta \times \beta$ (bounded by $T_\alpha$, $W_\gamma$) if moreover, the following two properties are satisfied:

1. $\mathcal{G} = \mathcal{G}^\#$.
2. The multiplicity of any diagonal element in $\mathcal{G}$ is even.

Example 4.1.1 below gives an illustration of the first set.

**Example 4.1.1.** Let $d = 7$ and $\beta = (1, 3, 4, 7, 9, 10, 13)$. Let $\alpha = (1, 2, 3, 5, 7, 9, 11)$ and $\gamma = (4, 5, 6, 7, 12, 13, 14)$. Let $\mathcal{G} = \{(2, 3), (12, 13), (5, 10), (5, 10)\}$. Clearly, $\mathcal{G} = \mathcal{G}^\#$ and the multiplicity of any diagonal element in $\mathcal{G}$ is even. The only $\beta$-chains in $\mathcal{G}$ are $C_1 = \{(2, 3)\}$, $C_2 = \{(12, 13)\}$, and $C_3 = \{(5, 10)\}$.

Let us take $T_\alpha = \{(2, 4), (5, 10), (11, 13)\}$ and $W_\gamma = \{(5, 1), (6, 3), (12, 9), (14, 10)\}$. Clearly, $(T_\alpha)_1 - (T_\alpha)_2 = \alpha$ and $(W_\gamma)_1 - (W_\gamma)_2 = \gamma$. We have to check that $T_\alpha \leq C_1 \leq W_\gamma$ for all $i \in \{1, 2, 3\}$. Now,

$$\{2\} - \{3\} = \{2\} \cup (\beta \setminus \{3\}) = \{1, 2, 4, 7, 9, 10, 13\}.$$  

So $T_\alpha \leq C_1 \leq W_\gamma$. Similarly one can check that $T_\alpha \leq C_2 \leq W_\gamma$ and $T_\alpha \leq C_3 \leq W_\gamma$. Hence $\mathcal{G}$ is a special multiset.

4.2 The second set

A non-vanishing semistandard notched bitableau on $\beta \times \beta$ bounded by $T_\alpha$, $W_\gamma$ has the same meaning as in §2.3. Such a notched bitableau $(P, Q)$ is said to be a non-vanishing semistandard notched bitableau on $(\beta \times \beta)^*$ (bounded by $T_\alpha$, $W_\gamma$) if moreover, the following 5 conditions are satisfied:

1. $P_i = Q_i^\ast$ for every row number $i$ of $(P, Q)$.
2. $(P, Q)$ doesn’t contain any empty rows.
3. The total number of rows in $P$ (or $Q$) is either even, or it is odd but

$$P_1 - Q_1 \leq \cdots \leq P_n - Q_n \leq \beta \leq P_{n+1} - Q_{n+1} \leq \cdots \leq P_{n+p} - Q_{n+p},$$

where $n + p$ is the total number of rows in $P$ (or $Q$), and $(P_i, Q_i)$ (for $1 \leq i \leq n$) is the negative part of $(P, Q)$, and $(P_{n+i}, Q_{n+i})$ (for $1 \leq i \leq p$) is the positive part of $(P, Q)$.

Let us denote by $\delta_1 \leq \cdots \leq \delta_{n+p+1}$ the sequence $P_1 - Q_1 \leq \cdots \leq P_n - Q_n \leq \beta \leq P_{n+1} - Q_{n+1} \leq \cdots \leq P_{n+p} - Q_{n+p}$, where $n + p$ is odd.

4. Either the total number of rows of $P$ (or $Q$) is even, and the $\epsilon$-degrees (where $\epsilon = (1, 2, \ldots, d) \in I(d)$) of $P_j - Q_j$ and $P_{j+1} - Q_{j+1}$ are equal for each $j$ odd, or the total number of rows in $P$ (or $Q$) is odd (say, $n + p$), and the $\epsilon$-degrees of $\delta_1$ and $\delta_{j+1}$ are equal for each $j$ odd, where the $\delta_j$’s are as mentioned in item (3) above.

5. The total number of boxes in $P$ (or $Q$) is even.

Example 4.2.1 below gives an illustration of the second set.
Example 4.2.1. Let $d = 7$ and $\beta = (1, 3, 4, 7, 9, 10, 13)$. Let

$$(P, Q) = \begin{pmatrix} 2 & 11 \\ 5 & 12 \\ 6 & 14 \end{pmatrix}, \begin{pmatrix} 4 & 13 \\ 3 & 10 \\ 1 & 9 \end{pmatrix}.$$  

Clearly, $(P, Q)$ is a notched bitableau on $\bar{\beta} \times \beta$. Let $\alpha = (1, 2, 3, 5, 7, 9, 11)$ and $\gamma = (4, 5, 6, 7, 12, 13, 14)$. Let us take $T_\alpha = \{(2, 4), (5, 10), (11, 13)\}$ and $W_\gamma = \{(5, 1), (6, 3), (12, 9), (14, 10)\}$. Observe that

$$P_1 - Q_1 = (1, 2, 3, 7, 9, 10, 11),$$
$$P_2 - Q_2 = (1, 4, 5, 7, 9, 12, 13),$$
and $P_3 - Q_3 = (3, 4, 6, 7, 10, 13, 14)$.

Since $P_1 < Q_1$, $P_2 > Q_2$, and $P_3 > Q_3$, we have $(P, Q)$ is non-vanishing. Also, $P_1 - Q_1 \leq P_2 - Q_2 \leq P_3 - Q_3$. Hence $(P, Q)$ is semistandard. Again,

$$(T_\alpha)_{(1)} - (T_\alpha)_{(2)} = \alpha \leq P_1 - Q_1$$
and
$$P_3 - Q_3 \leq \gamma = (W_\gamma)_{(1)} - (W_\gamma)_{(2)}.$$  

So $(P, Q)$ is bounded by $T_\alpha$ and $W_\gamma$. Observe now that

1. $P_i = Q_i^1$ for all $i \in \{1, 2, 3\}$.
2. $(P, Q)$ does not contain any empty rows.
3. The total number of rows in $P$ (or $Q$) is 3, which is odd, but

$$P_1 - Q_1 \leq \beta \leq P_2 - Q_2 \leq P_3 - Q_3.$$  

4. The $\epsilon$-degrees of $P_1 - Q_1$ and $\beta$ are the same (both are 3). Also, the $\epsilon$-degrees of $P_2 - Q_2$ and $P_3 - Q_3$ are the same (both are 3).
5. The total number of boxes in $P$ (or $Q$) is 6, which is even.

Hence $(P, Q)$ is on $(\bar{\beta} \times \beta)^*.$

5 The proof

The main result (Theorem 4.3.6) will be obtained as a consequence of Theorem 5.0.7, 5.0.10, and 5.0.13. For this section, we fix $\alpha, \beta$, and $\gamma$ in $I(d)$ such that $\alpha \leq \beta \leq \gamma$. Also we need some definitions for this section, which we will state first.

Definition 5.0.1. An ordered sequence $(w_1, \ldots, w_t)$ of admissible pairs is called a standard sequence of admissible pairs if $w_i \geq w_{i+1}$ for $1 \leq i < t$. We often write $w_1 \geq \cdots \geq w_t$ to denote the standard sequence $(w_1, \ldots, w_t)$ of admissible pairs. Given any $v \in I(d)$, we say that a standard sequence $w_1 \geq \cdots \geq w_t$ of admissible pairs is $v$-compatible if for each $w_i$, either $v \geq \text{top}(w_i)$ or $\text{bot}(w_i) \geq v$, and $w_i \neq (v, v)$. A standard sequence $w_1 \geq \cdots \geq w_t$ of admissible pairs is called anti-dominated by $v$ if $\text{bot}(w_t) \geq v$. Let $SM_v^w$ denote the set of all $v$-compatible standard sequences of admissible pairs that are anti-dominated by $v$. 

24
Example 5.0.2 below gives an illustration of $SM^{v,e}$.

**Example 5.0.2.** Let $d = 4$ and $v = (1, 2, 3, 5)$.

**Definition 5.0.3.** A monomial $\mathfrak{S}$ of $mon\mathfrak{R}(\beta)$ is special if

1. $\mathfrak{S} = \mathfrak{S}^\#$ and
2. the multiplicity of any diagonal element in $\mathfrak{S}$ is even.

Example 5.0.4 below gives an illustration of the special monomial.

**Example 5.0.4.** Let $d = 4$ and $\beta = (1, 2, 5, 6)$. Then the monomial

$$\mathfrak{S} = \{(8, 1), (8, 1), (7, 1), (8, 2)\}$$

is a special monomial of $mon\mathfrak{R}(\beta)$.

Now we recall [4] Proposition 4.1, which has been used in the proof of Theorem 5.0.7. [4] Proposition 4.1 is stated below as Proposition 5.0.5.

**Proposition 5.0.5.** There is a bijection between $SM^{3, \beta}$ and $mon\mathfrak{R}(\beta)$ that respects domination and degree.

Before we start the proof of the Theorem 5.0.7, let us recall the notation of $P$ and $in_\triangleright S$, which will be used in the proof of the Theorem 5.0.7.

**Definition 5.0.6.** If $S$ is any nonempty subset of the polynomial ring $P := K[X_{(r,c)} | (r,c) \in \mathfrak{D}\mathfrak{R}(\beta)]$, such that $S \neq \{0\}$. We define $in_\triangleright S$ to be the ideal $(in_\triangleright (s))_{s \in S}$, where $in_\triangleright$ is as in Definition 3.4.1.

**Theorem 5.0.7.** There exists a degree doubling injection from the set of all monomials of $P \setminus in_\triangleright G^{\gamma}_{\alpha, \beta}$ to the set of all non-vanishing special multisets on $\beta \times \beta$ (bounded by $T_\alpha$, $W_\gamma$).

**Proof.** Clearly,

$$in_\triangleright G^{\gamma}_{\alpha, \beta} = (in_\triangleright f_{m, \beta} : \text{w is good}) = (G^+ \cup G^-),$$

where

$$G^+ = \{x_{C^+} : C^+ \text{ a positive upper extended } \beta\text{-chain such that } C^+_{(1)} - C^+_{(2)} \not\leq \gamma\},$$

and

$$G^- = \{x_{C^-} : C^- \text{ a negative upper extended } \beta\text{-chain such that } \alpha \not\leq C^-_{(1)} - C^-_{(2)}\}.$$ 

Let

$$G'^+ := \{x_{C^+} : C^+ \text{ a positive upper extended } \beta\text{-chain such that } C^+ \not\leq W_\gamma\},$$

and

$$G'^- := \{x_{C^-} : C^- \text{ a negative upper extended } \beta\text{-chain such that } T_\alpha \not\leq C^-\}.$$ 

It is then easy to observe that $G'^+ = G'^+ \cup G'^-$ and $G'^- = G'^-$. Therefore

$$in_\triangleright G^{\gamma}_{\alpha, \beta} = (G'^+ \cup G'^-).$$

The definition of a generating set for an ideal will now imply that $x_U$ is a monomial in $in_\triangleright G^{\gamma}_{\alpha, \beta}$ if and only if $x_U$ is a multiple of some $x_{C^+}$ or some $x_{C^-}$, where $C^+$ is a positive upper extended $\beta$-chain such that $C^+ \not\leq W_\gamma$ and $C^-$ is a negative upper extended $\beta$-chain such that $T_\alpha \not\leq C^-$. Therefore

$$x_U$$

is a monomial in $P \setminus in_\triangleright G^{\gamma}_{\alpha, \beta}$.
\( x_U \) is not divisible by any \( x_{C^+} \) (where \( C^+ \) is a positive upper extended \( \beta \)-chain such that \( C^+ \not\leq W_\gamma \)) or by any \( x_{C^-} \) (where \( C^- \) is a negative upper extended \( \beta \)-chain such that \( T_\alpha \not\leq C^- \))

\[ \Rightarrow U \text{ contains no extended upper } \beta \text{-chains } C \text{ such that } T_\alpha \not\leq C^- \text{ or } C^+ \not\leq W_\gamma \]

Observe now that as the bijection of [4] Proposition 4.1 respects domination, and [15] Corollary 2.3.2 holds true, so \( C^+ \not\leq W_\gamma \) implies that \( \text{top}(C^+) \not\leq \gamma \).

A similar argument will show that \( T_\alpha \not\leq C^- \) implies \( \alpha \not\leq \bot(C^-) \). So we now have:

\[ U \text{ contains no extended upper } \beta \text{-chains } C \text{ such that } T_\alpha \not\leq C^- \text{ or } C^+ \not\leq W_\gamma \]

\[ \Rightarrow U \text{ contains no extended upper } \beta \text{-chains } C \text{ such that } \alpha \not\leq \bot(C^-) \text{ or } \text{top}(C^+) \not\leq \gamma \]

\[ \Rightarrow \alpha \leq \bot(C^-) \text{ and } \text{top}(C^+) \leq \gamma \text{ for any extended upper } \beta \text{-chain } C \text{ in } U \]

\[ \Rightarrow C \cup C^\# \text{ is bounded by } T_\alpha, W_\gamma \text{ for any extended upper } \beta \text{-chain } C \text{ in } U \]

where the last \( \Rightarrow \) follows because \( \bot(C^-) \) and \( \text{top}(C^+) \) are the two elements of \( I(d) \) (as mentioned in Definition 5.0.3) obtained by applying the map \( BRSK \) to the monomial \( C \cup C^\# \), and the map \( BRSK \) preserves domination.

Observe now that given any extended \( \beta \)-chain \( D \) in \( U \cup U^\# \), we can naturally get hold of an extended upper \( \beta \)-chain \( C \) (in \( U \)) from it in the following way:

If \( D = (r_1, c_1) > \cdots > (r_i, c_i) \) and \( (r_{i_1}, c_{i_1}), (r_{i_2}, c_{i_2}), \ldots, (r_{i_k}, c_{i_k}) \) (where \( i_1 < i_2 < \cdots < i_k \)) are such that \( r_{ij} > c_{ij} \) for all \( 1 \leq j \leq k \), then it is easy to check that the monomial formed by replacing all \( (r_{ij}, c_{ij}) \) \((1 \leq j \leq k)\) in \( D \) by \( (c_{ij}^*, r_{ij}^*) \) forms an extended upper \( \beta \)-chain in \( U \). Call this extended upper \( \beta \)-chain in \( U \) as \( C \).

Note that \( D \) is an extended \( \beta \)-chain in the monomial \( C \cup C^\# \). So if the monomial \( C \cup C^\# \) is bounded by \( T_\alpha, W_\gamma \), then \( T_\alpha \leq D \leq W_\gamma \).

Therefore

\[ C \cup C^\# \text{ is bounded by } T_\alpha, W_\gamma \text{ for any extended upper } \beta \text{-chain } C \text{ in } U \]

\[ \Rightarrow T_\alpha \leq D \leq W_\gamma \text{ for any extended } \beta \text{-chain } D \text{ in } U \cup U^\# \]

\[ \Leftrightarrow U \cup U^\# \text{ is bounded by } T_\alpha, W_\gamma. \]

The map \( U \mapsto U \cup U^\# \) from the set of all monomials of \( P \setminus \text{in}_{\beta} G_{\alpha, \beta}^\gamma \) to the set of all non-vanishing special multisets on \( \bar{\beta} \times \beta \) (bounded by \( T_\alpha, W_\gamma \)) is the required degree-doubling injection.

Example 5.0.8 below gives an illustration of Theorem 5.0.7.

Example 5.0.8. Let \( d = 4 \) and \( \alpha, \beta, \gamma \) be as given in Example 3.5.4. Here \( T_\alpha = \{(3, 6)\} \) and \( W_\gamma = \{(3, 1), (8, 6)\} \). Take the monomial \( U = \{(3, 1), (3, 2), (3, 5)\} \) in \( P \setminus \text{in}_{\beta} G_{\alpha, \beta}^\gamma \). It is now easy to verify that

\[ U \cup U^\# = \{(3, 1), (3, 2), (3, 5), (8, 6), (7, 6), (4, 6)\} \]

is a non-vanishing special multiset on \( \bar{\beta} \times \beta \) (bounded by \( T_\alpha, W_\gamma \)).
The following result follows easily from [2] Propositions 6 and 7 followed by a proof similar to the proof of [1] Proposition 3.9:

**Proposition 5.0.9.** The standard monomials on $Y_\alpha^* (\beta)$ form a basis for $K[Y_\alpha^* (\beta)]$.

**Theorem 5.0.10.** There exists a degree-halving injection from the set of all non-vanishing semistandard notched bitableaux on $(\beta \times \beta)^*$ (bounded by $T_\alpha, W_\gamma$) to the set of all standard monomials on $Y_\alpha^* (\beta)$.

**Proof.** Given any non-vanishing semistandard notched bitableau $(P, Q)$ on $(\beta \times \beta)^*$, let $P_1, \ldots, P_r$ (resp. $Q_1, \ldots, Q_r$) denote the rows of $P$ (resp. $Q$) from top to bottom. If $r$ is even (say, $r = 2s$), then let us denote by

$$
\mu_1 \leq \cdots \leq \mu_{2s}
$$

the sequence $P_1 - Q_1 \leq \cdots \leq P_{2s} - Q_{2s}$. If $r$ is odd (say, $r = 2s - 1$), then let us denote by

$$
\mu_1 \leq \cdots \leq \mu_{2s}
$$

the sequence $P_1 - Q_1 \leq \cdots \leq P_n - Q_n \leq \beta \leq P_{n+1} - Q_{n+1} \leq \cdots \leq P_{2s-1} - Q_{2s-1}$, where $(P_i, Q_i)$ (for $1 \leq i \leq n$) is the negative part of $(P, Q)$, and $(P_i, Q_j)$ (for $n + 1 \leq j \leq 2m - 1$) is the positive part of $(P, Q)$. We can then form the monomial

$$
f = f_{(\mu_2, \mu_1)} \beta f_{(\mu_4, \mu_3)} \beta \cdots f_{(\mu_{2s}, \mu_{2s-1})} \beta\gamma
$$

which belongs to $K[X_{(r, c)} | (r, c) \in \mathcal{B}(\beta)]$.

The notched bitableau $(P, Q)$ is non-vanishing which implies that, any $\mu_i$ $(1 \leq i \leq 2s)$ which is not equal to $\beta$ is such that either $\mu_i < \beta$ or $\mu_i > \beta$. And this in turn implies that (3.5.2), (3.5.3), (3.5.4) are satisfied by $f_{(\mu_2, \mu_1)} \beta \cdots f_{(\mu_{2s}, \mu_{2s-1})} \beta$. The notched bitableau $(P, Q)$ is semistandard $\Rightarrow$ $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{2s-1} \leq \mu_{2s}$. Further, $(P, Q)$ is a notched bitableau on $(\beta \times \beta)^*$ implies that the pairs $(\mu_2, \mu_1), \ldots, (\mu_{2s}, \mu_{2s-1})$ are all admissible pairs. These two facts together imply that (3.5.1) is satisfied by $f_{(\mu_2, \mu_1)} \beta \cdots f_{(\mu_{2s}, \mu_{2s-1})} \beta$. If in addition, $(P, Q)$ is bounded by $T_\alpha, W_\gamma$, then it is implied that (3.5.5) is satisfied by $f_{(\mu_2, \mu_1)} \beta \cdots f_{(\mu_{2s}, \mu_{2s-1})} \beta$. It is now easy to verify that $(P, Q)$ is a non-vanishing, semistandard, notched bitableau on $(\beta \times \beta)^*$ (bounded by $T_\alpha, W_\gamma$) $\Rightarrow f_{(\mu_2, \mu_1)} \beta f_{(\mu_4, \mu_3)} \beta \cdots f_{(\mu_{2s}, \mu_{2s-1})} \beta$ is standard on $Y_\alpha^* (\beta)$. Moreover, the degree of $(P, Q)$ equals the total number of boxes in $P$ (or $Q$). The total number of boxes in $P$ clearly equals $\Sigma_{i=1}^{r} (P_i - Q_i) \setminus \beta$, which in turn equals $\Sigma_{i=1}^{2m-1} |\mu_i \setminus \beta|$, which in turn equals twice the degree of $f_{(\mu_2, \mu_1)} \beta \cdots f_{(\mu_{2s}, \mu_{2s-1})} \beta$.

The map $(P, Q) \mapsto f_{(\mu_2, \mu_1)} \beta \cdots f_{(\mu_{2s}, \mu_{2s-1})} \beta$ is the required degree-halving injection.

**Example 5.0.11** below illustrates Theorem 5.0.10.

**Example 5.0.11.** Let $d = 4$ and $\alpha, \beta, \gamma$ be as given in Example 3.5.3. Here $T_\alpha = \{(3, 6)\}$ and $W_\gamma = \{(3, 1), (8, 6)\}$. Let

$$
(P, Q) = \begin{pmatrix}
3 & 4 & 6 \\
3 & 7 & 5 \\
3 & 8 & 2 \\
\end{pmatrix}
$$
Now \((T_\alpha)_{(1)} - (T_\alpha)_{(2)} = (1, 2, 3, 5), (W_\gamma)_{(1)} - (W_\gamma)_{(2)} = (2, 3, 5, 8), P_1 - Q_1 = (1, 2, 3, 5), P_2 - Q_2 = (1, 2, 4, 6), P_3 - Q_3 = (1, 3, 5, 7),\) and \(P_4 - Q_4 = (2, 3, 5, 8).\) Since \(P_i < Q_i\) for \(i = 1, 2\) and \(P_i > Q_i\) for \(i = 3, 4,\) so \((P, Q)\) is non-vanishing. Again \(P_1 - Q_1 < Q_2 < P_2 - Q_2 < P_3 - Q_3 < P_4 - Q_4,\) so \((P, Q)\) is semistandard. Also \((T_\alpha)_{(1)} - (T_\alpha)_{(2)} < P_1 - Q_1\) and \(P_4 - Q_4 < (W_\gamma)_{(1)} - (W_\gamma)_{(2)},\) so \((P, Q)\) is bounded by \(T_\alpha, W_\gamma.\)

Observe now that \((P, Q)\) is on \((\beta \times \beta)^*\) because:

1. \(P_i = Q_i\) for every row number \(i.\)
2. \((P, Q)\) does not contain any empty rows.
3. The total number of rows in \(P\) (or \(Q\)) is 4, which is even.
4. The \(\epsilon\)-degrees of both \(P_1 - Q_1\) and \(P_2 - Q_2\) is 1. Also the \(\epsilon\)-degrees of both \(P_3 - Q_3\) and \(P_4 - Q_4\) is 2. That is, the \(\epsilon\)-degrees of \(P_i - Q_i\) and \(P_{i+1} - Q_{i+1}\) are equal for every odd \(i.\)
5. The total number of box in \(P\) (or \(Q\)) is 6 that is even.

So \((P, Q)\) is on \((\beta \times \beta)^*\). Hence \((P, Q)\) is a non-vanishing semistandard notched bitableau on \((\beta \times \beta)^*\) (bounded by \(T_\alpha, W_\gamma\)).

Here \(\mu_1 = (1, 2, 3, 5), \mu_2 = (1, 2, 4, 6), \mu_3 = (1, 3, 5, 7), \mu_4 = (2, 3, 5, 8).\) Under the degree-halving injective map of Theorem 5.0.10 \((P, Q)\) maps to \(f(\mu_2, \mu_1, 3) f(\mu_4, \mu_3, 5, 8)\). which is a standard monomial on \(Y^*_\alpha(\beta)\).

Before we start with the last theorem of this section, we state [9, Lemma 6.3], which is going to be used in the proof of the theorem. [9, Lemma 6.3] is stated below as Lemma 5.0.12.

**Lemma 5.0.12.** The map \(\text{BRSK}\) is a degree-preserving bijection from the set of negative multisets on \(\mathbb{N}^2\) to the set of negative semistandard notched bitableaux.

**Theorem 5.0.13.** The map \(\text{BRSK}\) of [7] is a degree-preserving bijection from the set of all non-vanishing special multisets on \(\beta \times \beta\) (bounded by \(T_\alpha, W_\gamma\)) to the set of all non-vanishing semistandard notched bitableaux on \((\beta \times \beta)^*\) (bounded by \(T_\alpha, W_\gamma\)).

**Proof.** The fact that the map \(\text{BRSK}\) of [7] is degree-preserving is obvious from [9, Lemma 6.3] itself. For the rest of this proof, we will follow the notation and terminology of [7, §4.1] as well as the notation and terminology of [9].

There exists a natural injection from \(SM^\beta, \beta\) to \(SM^\beta, \beta\) as given in [7, §4.1]. Let \(\hat{SM}^\beta, \beta\) denote the image of \(SM^\beta, \beta\) in \(\hat{SM}^\beta, \beta\) under this injection. Let \(\mathcal{E}\) denote the set of all special monomials in \(\text{mon}\mathcal{R}(\beta)\). The map \(\pi\) of [7] is a degree and domination preserving bijection between the sets \(\mathcal{E}\) and \(\hat{SM}^\beta, \beta\). The set \(\mathcal{E}\) is the same as the set of all positive special multisets on \(\beta \times \beta\), and the set \(\hat{SM}^\beta, \beta\) is in a natural bijection (induced by the bijection map from \(I_\beta^*\) to \(I(\beta)\)) with the set of all positive semistandard notched bitableaux on \((\beta \times \beta)^*\). Also, we know from [15, Corollary 2.3.2] that the map \(\text{BRSK}\) of [9] and the map \(\pi\) of [7] are the same on positive multisets on \(\beta \times \beta\). Moreover, it follows from [9] (see [9, Lemma 7.2] and the fact that inside the proof of [7, Proposition 2.3]), the inequality \(b\) is actually an equality) that a positive multiset \(U\) on \(\beta \times \beta\) is bounded by \(\emptyset, W_\gamma\) if and only if \(\text{BRSK}(U)\) is bounded by \(\emptyset, W_\gamma\). Therefore, we can now conclude that the map \(\text{BRSK}\) of [9] is a degree-preserving bijection.
from the set of all positive special multisets on $\bar{\beta} \times \beta$ (bounded by $\emptyset$, $W_\gamma$) to the set of all positive semistandard notched bitableaux on $(\bar{\beta} \times \beta)^*$ (bounded by $\emptyset$, $W_\gamma$).

The proof for the negative part is similar. For the negative part, the multisets (as well as the notched bitableaux) will be bounded by $T_\alpha$, $\emptyset$ instead. □

Example 5.0.14 below illustrates Theorem 5.0.13.

Example 5.0.14. Let $d = 4$ and $\alpha, \beta, \gamma$ be as given in Example 3.5.4. Here $T_\alpha = \{(3, 6)\}$ and $W_\gamma = \{(3, 1), (8, 6)\}$. Let

$$\mathfrak{S} = \{(3, 1), (3, 2), (3, 5), (8, 6), (7, 6), (4, 6)\}.$$

Then $\mathfrak{S}$ is a non-vanishing special multiset on $\bar{\beta} \times \beta$ (bounded by $T_\alpha$, $W_\gamma$).

$$\text{BRSK}(\mathfrak{S}) = (P, Q) = \begin{pmatrix} 3 & 2 \\ 4 & 6 \\ 3 & 5 \\ 8 & 7 \\ 5 & 2 \\ 8 & 6 \end{pmatrix}.$$

By Example 5.0.11, $\text{BRSK}(\mathfrak{S}) = (P, Q)$ is a non-vanishing semistandard notched bitableau on $(\bar{\beta} \times \beta)^*$ (bounded by $T_\alpha$, $W_\gamma$).

The proof of Theorem 3.4.6 now follows easily from Theorems 5.0.7, 5.0.10, and 5.0.13.

References

[1] Billey, S., Coskun, I. (2012). Singularities of Generalized Richardson Varieties. Comm. Algebra. 40(4): 1466-1495.

[2] Brion, M., Lakshmibai, V. (2003). A Geometric Approach to Standard Monomial Theory. Represent. Theory. Volume 7: 651-680.

[3] Fulton, W. (1997). Young Tableaux. London Mathematical Society Student Texts. Volume 35. Cambridge, UK: Cambridge University Press.

[4] Ghorpade, S., Raghavan, K. N. (2006). Hilbert Functions of Points on Schubert Varieties in the Symplectic Grassmannians. Trans. Amer. Math. Soc. 358(12): 5401-5423.

[5] Herzog, J., Trung, N. V. (1992). Gröbner Bases and Multiplicity of Determinantal and Pfaffian Ideals. Adv. Math. 96(1): 1-37.

[6] Knutson, A., Woo, A., Yong, A. (2013). Singularities of Richardson Varieties. Math. Res. Lett. 20(2): 391-400.

[7] Kodiyalam, V., Raghavan, K. N. (2003). Hilbert Functions of Points on Schubert Varieties in Grassmannians. J. Algebra. 270(1): 28-54.

[8] Kreiman, V. (2003). Monomial Bases and Applications for Richardson and Schubert Varieties in Ordinary and Affine Grassmannians. PhD dissertation. Northeastern University, Boston, USA.

[9] Kreiman, V. (2008). Local Properties of Richardson Varieties in the Grassmannian via a Bounded Robinson-Schensted-Knuth Correspondence. J. Algebraic. Combin. 27(3): 351-382.
[10] Kreiman, V., Lakshmibai, V. (2004). Multiplicities of Singular Points in Schubert Varieties of Grassmannians. Algebra, arithmetic and geometry with applications. (West Lafayette, IN, 2000), Springer, Berlin: 553-563.

[11] Lakshmibai, V., Musili, C., Seshadri, C. S. (1979). Geometry of G/P - IV (Standard Monomial Theory for Classical Types). Proc. Indian. Acad. Sci. Sect. A. Math. Sci. 88(4): 279-362.

[12] Lakshmibai, V., Raghavan, K. N. (2008). Standard Monomial Theory, Invariant Theoretic Approach. Encyclopedia of Mathematical Sciences. Invariant Theory and Algebraic Transformation Groups VIII. Volume 137. New York, USA: Springer.

[13] Raghavan, K. N., Upadhyay, S. (2009). Initial Ideals of Tangent Cones to Schubert Varieties in Orthogonal Grassmannians. J. Combin. Theory. Ser. A. 116(3): 663-683.

[14] Raghavan, K. N., Upadhyay, S. (2010). Hilbert Functions of Points on Schubert Varieties in the Orthogonal Grassmannians. J. Algebraic. Combin. 31(3): 355-409.

[15] Ray, P., Upadhyay, S. (2022). Schubert Varieties in the Grassmannian and the Symplectic Grassmannian via a Bounded RSK Correspondence. Indian. J. Pure. Appl. Math: https://doi.org/10.1007/s13226-022-00334-6.

[16] Sturmfels, B. (1990). Gröbner Bases and Stanley Decompositions of Determinantal Rings. Math. Z. 205(1): 137-144.

[17] Upadhyay, S. (2013). Initial Ideals of Tangent Cones to the Richardson Varieties in the Orthogonal Grassmannian. Int. J. Comb. Volume 2013. Art. ID. 392437. 19 pages. DOI: http://dx.doi.org/10.1155/2013/392437.

[18] Woo, A., Yong, A. (2008). Governing Singularities of Schubert Varieties. J. Algebra. 320(2):495-520.

[19] Woo, A., Yong, A. (2012). A Gröbner Basis for Kazhdan-Lusztig Ideals. Amer. J. Math. 134(4):1089-1137.

[20] Graham, W., Kreiman, V. (2015). Excited Young Diagrams, Equivariant K-Theory, and Schubert Varieties. Trans. Amer. Math. Soc. 367(9):6597-6645.