Distance Magic Index One Graphs

A V Prajeesh*, Krishnan Paramasivam**

*Department of Mathematics  
National Institute of Technology Calicut  
Kozhikode 673601, India.

Abstract

Let \( S \) be a finite set of positive integers. A graph \( G = (V(G), E(G)) \) is said to be \( S \)-magic if there exists a bijection \( f : V(G) \to S \) such that for any vertex \( u \) of \( G \), \( \sum_{v \in N_G(u)} f(v) \) is a constant, where \( N_G(u) \) is the set of all vertices adjacent to \( u \). Let \( \alpha(S) = \max x \). Define \( i(G) = \min_{S} \alpha(S) \), where the minimum runs over all \( S \) for which the graph \( G \) is \( S \)-magic. Then \( i(G) - |V(G)| \) is called the distance magic index of a graph \( G \). In this paper, we compute the distance magic index of graphs \( G[\bar{K}_n] \), where \( G \) is any arbitrary regular graph, disjoint union of \( m \) copies of complete multipartite graph and disjoint union of \( m \) copies of graph \( C_p[\bar{K}_n] \), with \( m \geq 1 \). In addition to that, we also prove some necessary conditions for an regular graph to be of distance magic index one.

Keywords: Distance magic, \( S \)-magic graph, distance magic index, complete multi-partite graphs, lexicographic product.

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1. Introduction

In this paper, we consider only simple and finite graphs. We use \( V(G) \) for the vertex set and \( E(G) \) for the edge set of a graph \( G \). The neighborhood, \( N_G(v) \) or shortly \( N(v) \) of a vertex \( v \) of \( G \) is the set of all vertices adjacent to \( v \) in \( G \). For further graph theoretic terminology and notation, we refer Bondy and Murty [1] and Hammack et al. [2].

*Corresponding author

Email addresses: prajeesh_p150078ma@nitc.ac.in (A V Prajeesh),  
sivam@nitc.ac.in (Krishnan Paramasivam)

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A distance magic labeling of a graph $G$ is a bijection $f : V(G) \to \{1, \ldots, |V(G)|\}$, such that for any $u$ of $G$, the weight of $u$, $w_G(u) = \sum_{v \in N_G(u)} f(v)$ is a constant, say $c$. A graph $G$ that admits such a labeling is called a distance magic graph.

The motivation for distance magic labeling came from the concept of magic squares and tournament scheduling. An equalized incomplete tournament, denoted by $EIT(n, r)$, is a tournament, with $n$ teams and $r$ rounds, which satisfies the following conditions:

(i) every team plays against exactly $r$ opponents.

(ii) the total strength of the opponents, against which each team plays is a constant.

Therefore, finding a solution for an equalized incomplete tournament $EIT(n, r)$ is equivalent to establish a distance magic labeling of an $r$-regular graph of order $n$. For more details, one can refer [3, 4].

The following results provide some necessary condition for distance magicness of regular graphs.

**Theorem 1.** [5, 6, 7, 8] No $r$-regular graph with $r$-odd can be a distance magic graph.

**Theorem 2.** [4] Let $EIT(n, r)$ be an equalized tournament with an even number $n$ of teams and $r \equiv 2 \mod 4$. Then $n \equiv 0 \mod 4$.

In [6], Miller et al. discussed the distance magic labeling of the graph $H_{n, p}$, the complete multi-partite graph with $p$ partitions in which each partition has exactly $n$ vertices, $n \geq 1$ and $p \geq 1$. It is clear that $H_{n, 1}$ is a distance magic graph. From [6] it is observed that $K_n$ is distance magic if and only if $n = 1$ and hence, $H_{1, p} \cong K_p$ is not distance magic for all $p \neq 1$. The next result gives a characterization for the distance magicness of $H_{n, p}$.

**Theorem 3.** [6] Let $n > 1$ and $p > 1$. $H_{n, p}$ has a labeling if and only if either $n$ is even or both $n$ and $p$ are odd.

Recall a standard graph product (see [2]). Let $G$ and $H$ be two graphs. Then, the lexicographic product $G \circ H$ or $G[H]$ is a graph with the vertex set $V(G) \times V(H)$. Two vertices $(g, h)$ and $(g', h')$ are adjacent in $G[H]$ if and only if $g$ is adjacent to $g'$ in $G$, or $g = g'$ and $h$ is adjacent to $h'$ in $H$.

Miller et al. [6] proved the following.
Theorem 4. [6] Let $G$ be an arbitrary regular graph. Then $G[\overline{K}_n]$ is distance magic for any even $n$.

Later, Froncek et al. [4, 9] proved the following results.

Theorem 5. [4] For $n$ even an EIT$(n,r)$ exists if and only if $2 \leq r \leq n - 2; r \equiv 0 \mod 2$ and either $n \equiv 0 \mod 4$ or $n \equiv r + 2 \equiv 2 \mod 4$.

Theorem 6. [9] Let $n$ be odd, $p \equiv r \equiv 2 \mod 4$, and $G$ be an $r$-regular graph with $p$ vertices. Then $G[\overline{K}_n]$ is not distance magic.

Theorem 7. [9] Let $G$ be an arbitrary $r$-regular graph with an odd number of vertices and $n$ be an odd positive integer. Then $r$ is even and the graph $G[\overline{K}_n]$ is distance magic.

The following results by Shafiq et al. [10], discusses the distance magic labeling of disjoint union of $m$ copies of complete multi-partite graphs, $H_{n,p}$, and disjoint union of $m$ copies of product graphs, $C_p[\overline{K}_n]$.

Theorem 8. [10]

(i) If $n$ is even or $mnp$ is odd, $m \geq 1; n > 1$ and $p > 1$; then $mH_{n,p}$ has a distance magic labeling.

(ii) If $np$ is odd, $p \equiv 3 \mod 4$, and $m$ is even, then $mH_{n,p}$ does not have a distance magic labeling.

Theorem 9. [10] Let $m \geq 1, n > 1$ and $p \geq 3$. $mC_p[\overline{K}_n]$ has a distance magic labeling if and only if either $n$ is even or $mnp$ is odd or $n$ is odd and $p \equiv 0 \mod 4$.

In [10], Shafiq et al. posted a problem on the graph $mH_{n,p}$.

Problem 1. For the graph $mH_{n,p}$, where $m$ is even, $n$ is odd, $p \equiv 1 \mod 4$, and $p > 1$, determine if there is a distance magic labeling.

Later, Froncek et al. [9] proved the following necessary condition for $mH_{n,p}$.

Theorem 10. The graph $mH_{n,p}$, where $m$ is even, $n$ is odd, $p \equiv 1 \mod 4$, and $p > 1$, is not distance magic.
Figure 1: A graph $G$ with $c' = 13$ and $S = \{1, 3, 4, 5, 6, 7\}$.

For more details and results, one can refer Arumugam et al. [11].

From Theorem 1, one can observe that any odd-regular graph $G$ of order $n$ is not distance magic. But if we label the graph with respect to a different set $S$ of positive integers with $|S| = n$, then $G$ may admit a magic labeling with a magic constant $c'$. See Figure 1.

Motivated by this fact Godinho et al. [? ] defined the concept of $S$-magic labeling of a graph.

**Definition 1.** [? ] Let $G = (V(G), E(G))$ be a graph and let $S$ be a set of positive integers with $|V(G)| = |S|$. Then $G$ is said to be $S$-magic if there exists a bijection $f : V(G) \rightarrow S$ satisfying $\sum_{v \in N(u)} f(v) = c$ (a constant) for every $u \in V(G)$. The constant $c$ is called the $S$-magic constant.

**Definition 2.** [? ] Let $\alpha(S) = \max \{s : s \in S\}$. Let $i(G) = \min \alpha(S)$, where the minimum is taken over all sets $S$ for which the graph $G$ admits an $S$-magic labeling. Then $i(G) - |V(G)|$ is called the distance magic index of a graph $G$ and is denoted by $\theta(G)$.

From above definitions, one can observe that a graph $G$ is distance magic if and only if $\theta(G) = 0$ and if $G$ is not $S$-magic for any $S$ with $|V(G)| = |S|$, then $\theta(G) = \infty$.

Let $G$ be a graph for which $\theta(G)$ is finite (however so small) and non-zero. Now, a natural question arises that for all such graphs $G$, does there exist an $S$-magic labeling with $\theta(G) = 1$?

In the following section, we prove some necessary conditions for an $r$-regular $S$-magic graph $G$ to have $\theta(G) = 1$. Further, we compute the distance
magic index of disjoint union of $m$ copies of $H_{n,p}$ and disjoint union of $m$
copies of $C_p[K_n]$, where $m \geq 1$. Also, for any arbitrary regular graph $G$,
we compute the distance magic index of the graph $G[K_n]$. In addition to that,
we construct twin sets $S$ and $S'$ for the same graph $H_{n,p}$ with $\theta(G) = 1$,
for which $H_{n,p}$ is both $S$-magic and $S'$-magic with distinct magic constants.
We also discuss the maximum and minimum bounds attained by the magic
constant for the graph $H_{n,p}$.

2. Main results

If $G$ is a graph with $\theta(G) = 1$, then it is clear that $G$ is $S$-magic for
$S = \{1, ..., n + 1\} \setminus \{a\}$, for at least one $a \in \{1, ..., n\}$. We call $a$, the deleted
label of $S$.

The following results are similar to that of Theorem 1 and 2.

**Lemma 1.** If $G$ is an odd $r$-regular $S$-magic graph with $\theta(G) = 1$, then
$a \neq 1$.

**Proof.** Assume that $G$ is an $r$-regular graph with $\theta(G) = 1$, where $r$ is odd.
If $S = \{1, ..., n + 1\} \setminus \{a\}$ with the $S$-magic constant $c$, then,

\begin{align*}
nc &= r(1 + ... + n + 1) - ra \\
c &= \frac{rn + 3r}{2} + \frac{r - ra}{n}.
\end{align*}

Therefore, if $a = 1$, then $c$ is not an integer, a contradiction.

**Lemma 2.** If $G$ is an $r$-regular $S$-magic graph with $\theta(G) = 1$ and $r, n \equiv 2$
mod 4, then $a$ is an even integer, $a \neq 2, n$.

**Proof.** Assume that $G$ is an $r$-regular graph with $\theta(G) = 1$ and $r, n \equiv 2$
mod 4. Let $c$ be the $S$-magic constant of $G$, where $S = \{1, 2, ..., n + 1\} \setminus \{a\}$
and $a$ is an odd integer belonging to $\{1, 2, ..., n\}$. Let $r = 4k + 2$ and $n = 4k' + 2$, with $0 < k < k'$.

**Case 1:** when $a = 1$, from eq.(2), we have,

\[ c = (2k + 1)(4k' + 5). \]

Here $c$ is an odd integer and every vertex is adjacent to odd number of vertices
which are labeled with odd integers. Note that, here there are $2k' + 1$ such
vertices. Then the graph induced by the vertices having odd label has every vertex of odd degree, a contradiction.

Case 2: When $a = 2q + 1$, with $q > 0$. Then $rn + 3r \equiv 2 \mod 4$ and $r - ra \equiv 0 \mod 4$ and hence $c$ fails to be an integer.

Case 3: When $a = 2$ or $a = n$, $c$ is not an integer and hence the result follows.

The following theorem discusses the distance magic index of the graph, $H_{n,p}$, $n > 1$ and $p > 1$. We define the integer-valued function $\alpha$ given by

$$\alpha(j) = \begin{cases} 0 & \text{for } j \text{ even} \\ 1 & \text{for } j \text{ odd} \end{cases}$$

and the sets $\Omega_k = \{i : 5 \leq i \leq n-1 \text{ and } i \equiv k \mod 4\}$, where $k \in \{0, 1, 2, 3\}$. Both $\alpha$ and $\Omega_i$’s are used in the next theorem.

**Theorem 11.** If $G$ is a complete multi-partite graph $H_{n,p}$ with $p$ partitions having $n$ vertices in each partition, then

$$\theta(G) = \begin{cases} 0 & \text{for } n \text{ even or } n \text{ and } p \text{ both odd} \\ 1 & \text{for } n \text{ odd and } p \text{ even} \end{cases}$$

**Proof.** Let $G \cong H_{n,p}$ with $n > 1, p > 1$. From Theorem 3, it is clear that if $n$ is even or when $n$ and $p$ both are odd, then $\theta(G) = 0$.

Now, to construct an $(n \times p)$-rectangular matrix $A = (a_{i,j})$ with distinct entries from a set $S$ having column sum $b$ (a constant) is equivalent to find an $S$-magic labeling of $G$ with magic constant $(p-1)b$.

Note that $j^{th}$ column of $A$ can be used to label the vertices of $j^{th}$ partition of $G$ and hence $G$ admits a magic labeling with magic constant $(p-1)b$. In addition, if the entries of $A$ are all distinct and are from $S = \{1, ..., np+1\} \setminus \{a\}$, where $a \in \{1, ..., np\}$, then $G$ is $S$-magic with $\theta(G) = 1$.

Let $n$ be an odd and $p$ be an even integer.

Case 1: If $n = 3$ and $p = 2m, m > 0$, then construct $A$ as,

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & \cdots & 2m-3 & 2m-2 & 2m-1 & 2m \\
3m & 4m & 3m-1 & 4m-1 & \cdots & 2m+2 & 3m+2 & 2m+1 & 3m+1 \\
6m+1 & 5m & 6m & 5m-1 & \cdots & 5m+3 & 4m+2 & 5m+2 & 4m+1
\end{pmatrix}
$$

Note that, the deleted label is $5m+1$ here. One can observe that each column adds up to a constant $9m+2$ and thus, $\theta(H_{3,2m}) = 1$. 




Here, the deleted label is 9m + 1 and each column adds up to a constant 25m + 3. Therefore, \( \theta(H_{5,2m}) = 1 \).

Case 3: If \( n > 5 \) is odd and \( p = 2m, m > 0 \), then for each \( j \in \{1, ..., p\} \), construct \( A \) as follows.

\[
a_{i,j} = \begin{cases} 
  j & \text{for } i = 1 \\
  (2i-1)m - (\frac{i-1}{2}) + \alpha(j + 1)(m + \frac{1}{2}) & \text{for } i = 2, 4 \\
  2mi - (\frac{i-1}{2}) + \alpha(j + 1)(-m + \frac{1}{2}) & \text{for } i = 3 \\
  2mi - m + \frac{i}{2} + \alpha(j)(-m + \frac{1}{2}) & \text{for } i \equiv 1 \mod 4, i \in \{5, 6, ..., n-1\} \\
  2mi - m + \frac{i}{2} + \alpha(j)\frac{1}{2} - \alpha(j+1)m & \text{for } i \equiv 2 \mod 4, i \in \{5, 6, ..., n-1\} \\
  2mi - (\frac{i-1}{2}) + \alpha(j)(-m) + \alpha(j+1)\frac{1}{2} & \text{for } i \equiv 3 \mod 4, i \in \{5, 6, ..., n-1\} \\
  2mi - m + \frac{j}{2} + \alpha(j)\frac{3}{2} - \alpha(j+1)m & \text{for } i = n \equiv 1 \mod 4 \\
  2mi - \frac{j}{2} + \frac{1}{2} - \alpha(j+1)(m + \frac{1}{2}) & \text{for } i = n \equiv 3 \mod 4 
\end{cases}
\]

Therefore, \( m(2n - 1) + 1 \) is the deleted label in this case.

Subcase 1: If \( n \equiv 1 \mod 4 \), then \( n - 5 \equiv 0 \mod 4 \). Let \( n = 4q + 5 \), where \( q \geq 1 \).

Now for any fixed odd \( j \), the \( j^{th} \) column sum in \( A \) is,

\[
\sum_{i=1}^{n} a_{i,j} = \sum_{i=1}^{4} a_{i,j} + \sum_{i \in \Omega_1} a_{i,j} + \sum_{i \in \Omega_2} a_{i,j} + \sum_{i \in \Omega_3} a_{i,j} + \sum_{i \in \Omega_0} a_{i,j} + \left(2mn - m + \frac{j}{2} + \frac{3}{2}\right)
\]

\[
= j + 3m - (\frac{j-1}{2}) + 6m - (\frac{j-1}{2}) + 7m - (\frac{j-1}{2}) + \sum_{k=1}^{q} \left(2m(4k+1) - 2m + \frac{j+1}{2}\right) + \sum_{k=1}^{q} \left(2m(4k + 2) - m + \frac{j+1}{2}\right) + \sum_{k=1}^{q} \left(2m(4k + 3) - (\frac{j-1}{2})\right) + \sum_{k=1}^{q} \left(2m(4k + 4) - m - (\frac{j-1}{2})\right) + 2mn - m + 1 + \frac{j+1}{2}
\]
\[= 15m + 32mq + 16mq^2 + 2mn + 2q + 3 = \frac{n^2p+n+1}{2}.\]

Similarly, for any fixed even \(j\), the \(j^{th}\) column sum in \(A\) is,
\[
\sum_{i=1}^{n} a_{i,j} = \sum_{i=1}^{4} a_{i,j} + \sum_{i \in \Omega_1} a_{i,j} + \sum_{i \in \Omega_2} a_{i,j} + \sum_{i \in \Omega_3} a_{i,j} + \sum_{i \in \Omega_0} a_{i,j} + \left(2mn - 2m + \frac{j}{2}\right)
\]
\[
= j + 4m - \left(\frac{j-2}{2}\right) + 5m - \left(\frac{j-2}{2}\right) + 8m - \left(\frac{j-2}{2}\right) + \sum_{k=1}^{q} \left(2m(4k+1) - m + \frac{j}{2}\right) + \sum_{k=1}^{q} \left(2m(4k + 2) - 2m + \frac{j}{2}\right) + \sum_{k=1}^{q} \left(2m(4k + 4) - \left(\frac{j-2}{2}\right)\right) + 2mn - 2m + \frac{j}{2}
\]
\[
= 15m + 32mq + 16mq^2 + 2mn + 2q + 3 = \frac{n^2p+n+1}{2}.
\]

Subcase 2: if \(n \equiv 3 \pmod{4}\), then \(n - 5 \equiv 2 \pmod{4}\). Let \(n = 4q + 3\) where \(q \geq 0\).

Now, for any fixed odd \(j\), the \(j^{th}\) column sum in \(A\) is,
\[
\sum_{i=1}^{n} a_{i,j} = \sum_{i=1}^{4} a_{i,j} + \sum_{i \in \Omega_1} a_{i,j} + \sum_{i \in \Omega_2} a_{i,j} + \sum_{i \in \Omega_3} a_{i,j} + \sum_{i \in \Omega_0} a_{i,j} + \left(2mn - \frac{j}{2} + \frac{3}{2}\right)
\]
\[
= j + 3m - \left(\frac{j-1}{2}\right) + 6m - \left(\frac{j-1}{2}\right) + 7m - \left(\frac{j-1}{2}\right) + \sum_{k=1}^{q+1} \left(2m(4k+1) - 2m + \frac{j+1}{2}\right) + \sum_{k=1}^{q+1} \left(2m(4k + 2) - m + \frac{j+1}{2}\right) + \sum_{k=1}^{q+1} \left(2m(4k + 4) - \left(\frac{j-1}{2}\right)\right) + 2mn + 1 - \left(\frac{j-1}{2}\right)
\]
\[
= 35m + 48mq + 16mq^2 + 2mn + 2q + 4 = \frac{n^2p+n+1}{2}.
\]

Similarly, for any fixed even \(j\), the \(j^{th}\) column sum in \(A\) is,
\[
\sum_{i=1}^{n} a_{i,j} = \sum_{i=1}^{4} a_{i,j} + \sum_{i \in \Omega_1} a_{i,j} + \sum_{i \in \Omega_2} a_{i,j} + \sum_{i \in \Omega_3} a_{i,j} + \sum_{i \in \Omega_0} a_{i,j} + \left(2mn - m + \frac{j}{2} + 1\right)
\]
\[
= j + 4m - \left(\frac{j-2}{2}\right) + 5m - \left(\frac{j-2}{2}\right) + 8m - \left(\frac{j-2}{2}\right) + \sum_{k=1}^{q+1} \left(2m(4k+1) - m + \frac{j}{2}\right) + \sum_{k=1}^{q+1} \left(2m(4k + 2) - 2m + \frac{j}{2}\right) + \sum_{k=1}^{q} \left(2m(4k + 3) - \left(\frac{j-2}{2}\right)\right) + \sum_{k=1}^{q} \left(2m(4k)^2 - \left(\frac{j-2}{2}\right)\right) +
\]
\[
= 35m + 48mq + 16mq^2 + 2mn + 2q + 4 = \frac{n^2p+n+1}{2}.
\]
\[ \sum_{k=1}^{q} \left( 2m(4k + 4) - \left( \frac{j-2}{2} \right) \right) + 2mn - m - \left( \frac{j-2}{2} \right) \]

\[ = 35m + 48mq + 16mq^2 + 2mn + 2q + 4 = \frac{n^2p+n+1}{2}. \]

Since the sum of the entries in each column of \( A \) is \( \frac{n^2p+n+1}{2} \) for odd \( n > 5 \), \( H_{n,2m} \) is \( S \)-magic with magic constant \( \frac{n^2p+n+1}{2}(p-1) \) and \( \theta(H_{n,2m}) = 1. \)

**Theorem 12.** If \( G \cong H_{n,p} \) is an \( S \)-magic graph with \( \theta(G) = 1 \) and \( S \)-magic constant \( \frac{n^2p+n+1}{2}(p-1) \), then there exists a set \( S' \) such that \( G \) is an \( S' \)-magic graph with \( \theta(G) = 1 \) and \( S' \)-magic constant \( \frac{n^2p+3n-1}{2}(p-1) \).

**Proof.** For every \( S \)-magic graph \( G \cong H_{n,p} \) with \( \theta(G) = 1 \), one can obtain the corresponding rectangular matrix \( A = (a_{i,j}) \) associated with \( G \) by Theorem 11.

Define a new \((n \times p)\)-rectangular matrix \( A' = (a'_{i,j}) \) with entries,

\[ a'_{i,j} = (np + 2) - a_{i,j} \text{ for all } i \text{ and } j. \]

By Theorem 11, it is clear that the entries in \( A \) belong to the set \( \{1, ..., np + 1\} \setminus \{np + 1 - \frac{p}{2}\} \), which sum up to \( \frac{n^2p^2+p(n+1)}{2} \) and is divisible by \( p \). Hence the magic constant is \( \frac{n^2p+n+1}{2}(p-1) \).

Now using (3), define the new set \( S' = S \cup \{np + 1 - \frac{p}{2}\} \) and the sum of all the entries in \( A' = np(np + 2) - \left( \frac{n^2p^2+p(n+1)}{2} \right) \right) = \frac{n^2p^2+3np-p}{2} \), which is divisible by \( p \). Therefore, we obtain the magic constant as \( \frac{n^2p+3n-1}{2}(p-1) \).

The rectangular matrices \( A \) and \( A' \) associated with \( H_{5,6} \) are given below,

\[
A = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
9 & 12 & 8 & 11 & 7 & 10 \\
18 & 15 & 17 & 14 & 16 & 13 \\
21 & 24 & 20 & 23 & 19 & 22 \\
29 & 25 & 30 & 26 & 31 & 27
\end{pmatrix}
A' = \begin{pmatrix}
31 & 30 & 29 & 28 & 27 & 26 \\
23 & 20 & 24 & 21 & 25 & 22 \\
14 & 17 & 15 & 18 & 16 & 19 \\
11 & 8 & 12 & 9 & 13 & 10 \\
3 & 7 & 2 & 6 & 1 & 5
\end{pmatrix}
\]

Here, the sum of the entries in each column of \( A \) and \( A' \) are 78 and 82 respectively. Then, \( H_{5,6} \) is \( S \)-magic with magic constant 390 and \( S' \)-magic
with magic constant 410.

Now the following result is immediate.

**Lemma 3.** If $G$ is an $r$-regular graph with $\theta(G) = 1$ and with $S$-magic constant $c$, then

$$\frac{nr + r}{2} + \frac{r}{n} \leq c \leq \frac{nr + 3r}{2}.$$  

**Proof.** The proof is obtained from Lemma 1 by substituting $a = 1$ and $a = n$ for $c$. \qed

**Observation 1.** If $G \cong H_{n,p}$ is a graph with $\theta(G) = 1$ and $S$-magic constant $c$, then

$$\frac{n^2p + n + 1}{2} (p - 1) \leq c \leq \frac{n^2p + 3n - 1}{2} (p - 1).$$  

The lower and upper bounds in Observation 1 are tight when one compares with Lemma 3. It is noticed that if $S = \{1, ..., np + 1\} \setminus \{a\}$, which confirms that $H_{n,p}$ is $S$-magic, then the sum of all the entries in $S$ is divisible by $p$. Therefore, the highest $a$ that can be removed to get a multiple of $p$ is
and the lowest $a$ that can be removed to get a multiple of $p$ is $p^2 + 1$. Hence the result follows.

**Lemma 4.** Let $B$ be an $(n \times p)$-rectangular matrix with distinct entries from the set $\{1, 2, ..., np+1\} \setminus \{a\}$, where $a \in \{1, 2, ..., np\}$ having column sum $s$. If there exists an integer $m \geq 1$, $m|p$, then there exists $m$, $(n \times t)$-rectangular matrices, $B_m, (1 \leq m \leq t)$, having column sum $s$.

**Proof.** Consider the $(n \times mt)$-rectangular matrix $B$ with distinct entries from the set $\{1, 2, ..., np+1\} \setminus \{a\}$, where $a \in \{1, 2, ..., np\}$ and having column sum $s$.

Construct an $(n \times t)$-rectangular matrix, $B_1$ by choosing any $t$ distinct columns of $B$ and update the $B$ matrix by replacing all the entries in the newly chosen $t$ columns with $0$'s. Now the updated $B$ matrix will have exactly $(m - 1)t$ nonzero columns and $t$ columns having all zero entries.

Now, repeat the process to obtain the next matrix $B_2$ by choosing any $t$ non-zero columns from the remaining $(m - 1)t$ columns and update the $B$ matrix in the same manner as in first step. Now repeatedly apply the above technique to obtain the remaining $m - 2$ matrices, $B_i, (3 \leq i \leq m)$, until the matrix $B$ becomes an zero matrix. 

From Theorem 8, it is observed that in both the cases when $n$ is odd, $p$ is even and when $np$ is odd, $p \equiv 3 \mod 4$ and $m$ is even, $\theta(mH_{n,p}) \neq 0$. The following theorem computes the distance magic index of $mH_{n,p}$ for above cases.

**Theorem 13.** If $n > 1, p > 1, m \geq 1$, then

$$\theta(mH_{n,p}) = \begin{cases} 0 & \text{for } n \text{ even or } mnp \text{ is odd}, \\ 1 & \text{otherwise.} \end{cases}$$

**Proof.** Using Theorem 8, it is clear that $\theta(mH_{n,p}) = 0$, when either $n$ is even or $mnp$ is odd and $\theta(mH_{n,p}) \neq 0$, when either $np$ is odd, $p \equiv 3 \mod 4$, and $m$ is even. On the other hand, by Theorem 10, one can conclude that $\theta(mH_{n,p}) \neq 0$, when $m$ is even, $n$ is odd, $p \equiv 1 \mod 4$, and $p > 1$.

For all the remaining cases, use Theorem 11 to construct the rectangular matrix $A$ associated with the graph $H_{n,mp}$. Now using Lemma 4, construct the $(n \times p)$-matrices $B_k$, for $k \in \{1, ..., m\}$ Here, each $B_k$ forms the matrix associated with the $k^{th}$ copy of $H_{n,p}$ and hence we obtain an $S$-magic labeling of $mH_{n,p}$ with $c = \frac{n^2mp+n+1}{2} (p - 1)$. Therefore, $\theta(mH_{n,p}) = 1$. 

\[\]
Theorem 9 confirms that if \( n \) is even or \( mnp \) is odd or \( n \) is odd and \( p \equiv 0 \pmod{4} \), then \( \theta(mC_p[K_n]) = 0 \). Now the remaining cases are given below.

**Case 1:** \( n \) is odd, \( m \) is even, \( p \equiv 2 \pmod{4} \).

**Case 2:** \( n \) is odd, \( m \) is odd, \( p \equiv 2 \pmod{4} \).

**Case 3:** \( n \) is odd, \( m \) is even, \( p \) is odd.

The following theorem determines the distance magic index of the graph \( mC_p[K_n] \) for all the above mentioned three cases.

**Theorem 14.** Let \( m \geq 1, n > 1 \) and \( p \geq 3 \), then

\[
\theta(mC_p[K_n]) = \begin{cases} 
0 & \text{if } n \text{ is even or } mnp \text{ is odd or } n \text{ is odd, } p \equiv 0 \pmod{4}, \\
1 & \text{otherwise.}
\end{cases}
\]

**Proof.** Let \( G \cong mC_p[K_n] \). From Theorem 9 it is clear that \( \theta(G) = 0 \), when \( n \) even or \( mnp \) is odd or \( n \) is odd and \( p \equiv 0 \pmod{4} \).

Now, for all the remaining cases, using Theorem 11 construct the matrix \( A \) associated with the graph \( H_{n,mp} \) and use \( A \) in Lemma 4 to construct the \( m \) rectangular matrices associated with \( m \) copies of graph \( C_p[K_n] \). Hence, we obtain a \( S \)-magic labeling of \( G \) with \( c = n^2mp + n + 1 \) and hence \( \theta(G) = 1 \).

Let \( G \) be an \( r \)-regular graph on \( p \) vertices. From Theorem 5, for the graph \( G[K_n] \), if \( n \) is odd, \( r \) is even and \( p \) is even except when \( p \equiv r \equiv 2 \pmod{4} \), then \( \theta(G[K_n]) = 0 \). The following theorem computes the distance magic index of the graph \( G[K_n] \).

**Theorem 15.** Let \( G \) be an \( r \)-regular graph on \( p \) vertices. Then,

\[
\theta(G[K_n]) = \begin{cases} 
0 & \text{if } n \text{ is even or } n,p \text{ are odd, } r \text{ is even,} \\
1 & \text{if } n,r \text{ are odd or } n \text{ is odd, } r \equiv p \equiv 2 \pmod{4} \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** Let \( G \) be a graph on \( p \) vertices \( v_1, ..., v_p \) and let \( V_i = \{v_1, ..., v_n\} \) be set the vertices of \( G[K_n] \) that replace the vertex \( v_i \) of \( G \) for all \( i = 1, ..., p \).

Note that here \( V(G[K_n]) = \bigcup_{i=1}^{p} V_i \).

When \( n \) is even, by Theorem 4 \( \theta(G[K_n]) = 0 \) and when \( n \) is odd, \( p \) is odd and \( r \) is even, by Theorem 7 \( \theta(G[K_n]) = 0 \). Further, when \( n \) is odd and \( p \equiv r \equiv 2 \pmod{4} \), then by Theorem 6 \( \theta(G[K_n]) \neq 0 \). Also when \( n \) is odd and \( r \) is odd, by Theorem 1 \( \theta(G[K_n]) \neq 0 \). Further for all the other cases \( \theta(G[K_n]) = 0 \) by Theorem 5. Now for both the cases when \( \theta(G[K_n]) \neq 0 \), use Theorem 11 to construct the rectangular matrix \( A \) associated with the
graph $H_{n,p}$ and use the $i^{th}$ column of $A$ to label the set of vertices, $V_i$, for all $i = 1, 2, \ldots, p$. Hence, we obtain a $S$-magic labeling of $G[K_n]$, with $c = r\left(\frac{n^2p+n+1}{2}\right)$. Therefore we obtain that $\theta(C_p[K_n]) = 1$. \hfill \square

3. Conclusion

In this paper, the distance magic index of disjoint union of $m$ copies of $H_{n,p}$ and disjoint union of $m$ copies of $C_p[K_n]$ are computed and few necessary conditions are derived for a regular graph $G$ for which $\theta(G)$ is 1. The paper establishes a technique to construct a new set of labels from an existing one in such a way that both magic constants are distinct. Further, the lower and upper bounds of magic constant of a regular graph $G$ with $\theta(G) = 1$, are also determined.

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