FREIDLIN-WENTZELL’S LARGE DEVIATIONS FOR STOCHASTIC EVOLUTION EQUATIONS

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Abstract. We prove a Freidlin-Wentzell large deviation principle for general stochastic evolution equations with small perturbation multiplicative noises. In particular, our general result can be used to deal with a large class of quasi linear stochastic partial differential equations, such as stochastic porous medium equations and stochastic reaction diffusion equations with polynomial growth zero order term and $p$-Laplacian second order term.

1. Introduction

Since the work of Freidlin and Wentzell [14], the theory of small perturbation large deviations for stochastic differential equations(SDE) has been extensively developed(cf. [2, 30], etc.). In classical method, to establish such a large deviation principle(LDP) for SDEs, one needs to discretize the time variable and then prove various necessary exponential continuity and tightness for stochastic dynamical systems in different spaces by using comparison principle. However, such verifications would become rather complicated and even impossible in some cases for infinite stochastic partial differential equations with multiplicative noises.

Recently, Dupuis and Ellis [11] systematically developed a weak convergence approach to the theory of large deviation. The core idea is to prove some variational representation formula about the Laplace transform of bounded continuous functionals, which will lead to proving an equivalent Laplace principle with LDP. In particular, for Brownian functionals, an elegant variational representation formula has been established by Boué-Dupuis [3] and Budhiraja-Dupuis [5]. A simplified proof is given by the second named author [32]. This variational representation has been proved to be very effective for various finite dimensional stochastic dynamical systems with irregular coefficients(cf. [4, 23, 24], etc.). One of the main advantages of this argument is that one only needs to make some necessary moment estimates. This can be seen completely from the present paper that it also works very well for infinite dimensional stochastic dynamical systems.

In the past two decades, there are numerous results about the LDP for stochastic partial differential equations(SPDE) (cf. [29, 10, 20, 16, 7, 12, 6], etc.). All these results are concentrated on semi-linear SPDEs, i.e., the second order term is linear, and their proofs, except [12, 6], are mainly based on the classical exponential tightness method. In [12], the approach for LDP is based on nonlinear semigroup and infinite dimensional

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Hamilton-Jacobi equations. The approach in [6] is based on the variational representation. Recently, Röckner-Wang-Wu [26] proved an LDP for stochastic porous medium equation with additive noise by using the classical comparison principle. It should be pointed out that the equation of this type has a non-linear and degenerated second order term. Since additive noise was considered in [26], they can discretize time and prove some necessary estimates. It seems difficult to extend their result to the multiplicative noise case by using the classical method.

On the other hand, the existence and uniqueness of SPDEs have already been studied in various literatures prior to LDP for SPDEs(cf. [8, 18, 27, 10, 15, 31], etc.). In the theory of SPDEs, there exist two main tools: semigroup method and variation method(or monotone method). One of the merits of semigroup method is that the noise can take values in a larger space(cf. [10]). But, it can only deal with semi-linear SPDEs. The variation method combined with Galerkin’s approximation is usually used in the framework of evolution triple(cf. [18, 31]). Thus, as in the deterministic case(cf. [28]), it can tackle a large class of SPDEs. But, the diffusion coefficients need to be in the space of Hilbert-Schmidt operators.

Our aim in the present paper is to prove a Freidlin-Wentzell’s large deviation for stochastic evolution equations in the evolution triple case by using the weak convergence approach as done in [6]. Thus, the main point is to prove the tightness of some control stochastic evolution equations. This will be realized by making some moment estimates in suitable space(see Lemma 3.2 below) and then using the general tightness criterion for stochastic processes(see Lemma 3.4 below). Moreover, in order to treat the SPDEs with polynomial growth, we will work in the framework of [31], which is a little different from [18]. Compared with the well-known results, our proof is succinct, and we believe that our method can be adapted to some other non-linear stochastic equations such as stochastic Navier-Stokes equation.

This paper is organized as follows: In Section 2, we shall give our framework and recall an abstract criterion for Laplace principle due to Budhiraja-Dupuis [5], as well as an existence and uniqueness result for stochastic evolution equation essentially due to Krylov-Rozovski [18]. In Section 3, we first prove a Laplace principle for stochastic evolution equation(see Theorem 3.5 below) without any compact embedding requirement. In order to prove the corresponding rate function is good, we need an extra compact assumption (see Lemma 3.7 below). Lastly, in Section 4 we give three applications.

2. Framework and Preliminaries

Let $X$ be a reflexive and separable Banach space, which is densely and continuously injected in a separable Hilbert space $H$. Identifying $H$ with its dual we get

$$X \subset H \simeq H^* \subset X^*,$$

where the star ‘*’ denotes the dual spaces.

Assume that the norm in $X$ is given by

$$||x||_X := ||x||_{1,X} + ||x||_{2,X}, \quad x \in X.$$

Denote by $X_i$, $i = 1, 2$ the completions of $X$ with respect to the norms $|| \cdot ||_{i,X} := || \cdot ||_{X_i}$. Then $X = X_1 \cap X_2$. Let us also assume that both spaces are reflexive and embedded in $H$. Thus, we get two triples:

$$X_1 \subset H \simeq H^* \subset X_1^*, \quad X_2 \subset H \simeq H^* \subset X_2^*.$$

Noticing that $X_1^*$ and $X_2^*$ can be thought as subspaces of $X^*$, one may define a Banach space $Y := X_1^* + X_2^* \subset X^*$ as follows: $f \in Y$ if and only if $f = f_1 + f_2$, $f_i \in X_i^*, i = 1, 2$
and the norm of $f$ is defined by
\[ \|f\|_{Y} = \inf_{f=f_{1}+f_{2}} (\|f_{1}\|_{X_{1}} + \|f_{2}\|_{X_{2}}). \]

In the following, the dual pairs of $(X, X^{*})$ and $(X_{i}, X_{i}^{*})$, $i = 1, 2$ are denoted respectively by
\[ \langle \cdot, \cdot \rangle_{X}, \quad \langle \cdot, \cdot \rangle_{X_{i}}, \quad i = 1, 2. \]
Then, for any $x \in X$ and $f = f_{1} + f_{2} \in Y \subset X^{*},$
\[ [x, f]_{X} = [x, f_{1}]_{X_{1}} + [x, f_{2}]_{X_{2}}. \]

We remark that if $f \in \mathbb{H}$ and $x \in X,$ then
\[ [x, f]_{X} = [x, f]_{X_{1}} = [x, f]_{X_{2}} = \langle x, f \rangle_{\mathbb{H}}, \tag{1} \]

where $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ stands for the inner product in $\mathbb{H}$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_{t})_{t \geq 0}, P)$ be a complete separable filtration probability space, and $Q$ a nonnegative definite and symmetric trace operator defined on another separable Hilbert space $\mathbb{U}$. A $Q$-Wiener process $\{W(t), t \geq 0\}$ defined on $(\Omega, \mathcal{F}, P)$ is given and assumed to be adapted to $(\mathcal{F}_{t})_{t \geq 0}$ (cf. [10]). Set $\mathbb{U}_{Q} := Q^{1/2}(\mathbb{U})$ and let $L_{2}(\mathbb{U}_{Q}, \mathbb{H})$ denote the Hilbert space consisting of all Hilbert-Schmidt operators from $\mathbb{U}_{Q}$ to $\mathbb{H}$, where the inner product is denoted by $\langle \cdot, \cdot \rangle_{L_{2}(\mathbb{U}_{Q}, \mathbb{H})}$, and the norm by $\| \cdot \|_{L_{2}(\mathbb{U}_{Q}, \mathbb{H})}$.

In the following, we will work in the finite time interval $[0, T]$. For a Banach space $\mathbb{B}$ we shall denote by $\mathbb{C}_{T}(\mathbb{B})$ the continuous functions space from $[0, T]$ to $\mathbb{B}$, which is endowed with the uniform norm. Define
\[ \mathbb{L}_{Q} := \left\{ h = \int_{0}^{T} \dot{h}(s)ds : \dot{h} \in L^{2}(0, T; \mathbb{U}_{Q}) \right\} \]
with the norm
\[ \|h\|_{\mathbb{L}_{Q}} := \left( \int_{0}^{1} \|\dot{h}(s)\|_{L_{Q}}^{2}ds \right)^{1/2}, \]

where the dot denotes the generalized derivative. Let $\mu_{Q}$ be the law of the $Q$-Wiener process $W$ in $\mathbb{C}_{T}(\mathbb{U})$. Then
\[ (\mathbb{C}_{T}(\mathbb{U}), \mathbb{L}_{Q}, \mu_{Q}) \]
forms an abstract Wiener space.

For $N > 0$ we set $D_{N} := \{ h \in \mathbb{L}_{Q} : \|h\|_{\mathbb{L}_{Q}} \leq N \}$. Then $D_{N}$ is metrizable as a compact Polish space with respect to the weak topology in $\mathbb{L}_{Q}$. Let $\mathcal{A}_{N}$ denote all continuous and $\mathcal{F}_{t}$-adapted process $h$ from $[0, T]$ to $\mathbb{U}_{Q}$ such that for almost all $\omega$, $h(\cdot, \omega) \in D_{N}$, i.e.,
\[ \int_{0}^{T} \|\dot{h}(s, \omega)\|_{\mathbb{U}_{Q}}^{2}ds \leq N. \tag{2} \]

Let $\mathbb{S}$ be a Polish space. A function $I : \mathbb{S} \to [0, \infty]$ is given.

**Definition 2.1.** The function $I$ is called a rate function if $I$ is lower semicontinuous. The function $I$ is called a good rate function if for every $a < \infty$, $\{ f \in \mathbb{S} : I(f) \leq a \}$ is compact.

Let $Z^{\varepsilon} : \mathbb{C}_{T}(\mathbb{U}) \to \mathbb{S}$ be a family of measurable mappings. We assume that

**Hypothesis:** There is a measurable map $Z^{0} : \mathbb{L}_{Q} \hookrightarrow \mathbb{S}$ such that for any $N > 0$, if a family $\{ h^{\varepsilon} \} \subset \mathcal{A}_{N}$ (as random variables in $D_{N}$) converges in distribution to a $v \in \mathcal{A}_{N}$, then for some subsequence $\varepsilon_{k}$, $Z^{\varepsilon_{k}}(\cdot + \frac{h^{\varepsilon_{k}}(\cdot)}{\sqrt{\varepsilon_{k}}})$ converges in distribution to $Z^{0}(v)$ in $\mathbb{S}$. 







For each \( f \in \mathcal{S} \), define
\[
I(f) := \frac{1}{2} \inf_{h \in \mathcal{L}_q, f = Z^0(h)} \| h \|_{I_q}^2,
\]
where \( \inf \emptyset = \infty \) by convention.

We recall the following result due to [3, 5, see also 32, Theorem 4.4].

**Theorem 2.2.** \( \{ Z^\varepsilon, \varepsilon \in (0, 1) \} \) satisfies the Laplace principle with the rate function \( I(f) \) given by (3). That is, for each real bounded continuous function \( g \) on \( \mathcal{S} \):
\[
\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E} \left( \exp \left[ -\frac{g(Z^\varepsilon)}{\varepsilon} \right] \right) = -\inf_{f \in \mathcal{S}} \{ g(f) + I(f) \}.
\]

**Remark 2.3.** If \( I \) in (4) is not lower semicontinuous, then the regularization of \( I \)
\[
\tilde{I}(f) := \lim_{\varepsilon \to 0} \inf_{f' \in B_\varepsilon(f)} I(f)
\]
still satisfies (4), where \( B_\varepsilon(f) \) is the ball in \( \mathcal{S} \) with center \( f \) and radius \( \varepsilon \). Moreover, if \( I \) is a good rate function, then the Laplace principle is equivalent to the large deviation principle (cf. [3, Theorem 1.2.3]).

We now introduce three evolution operators used in the present paper (cf. [31]):
\[
A_i : [0, T] \times \mathfrak{X}_i \to \mathfrak{X}_i^* \subset \mathcal{B}(B([0, T]) \times \mathcal{B}(\mathfrak{X}_i) \big/ \mathcal{B}(\mathfrak{X}_i^*), \quad i = 1, 2,
\]
and
\[
B : [0, T] \times \mathbb{H} \to L_2(\mathbb{U}_Q, \mathbb{H}) \subset \mathcal{B}(B([0, T]) \times \mathcal{B}(\mathbb{H}) \big/ \mathcal{B}(L_2(\mathbb{U}_Q, \mathbb{H}))
\]
In the following, for the sake of simplicity, we write
\[
A = A_1 + A_2 \in \mathbb{Y} \subset \mathfrak{X}^*,
\]
and assume throughout this paper that

\[ \textbf{H1} \] (Hemicontinuity) For any \( t \in [0, T] \) and \( x, y, z \in \mathfrak{X} \), the mapping
\[
[0, 1] \ni \varepsilon \mapsto [x, A(t, y + \varepsilon z)]_\mathfrak{X}
\]
is continuous.

\[ \textbf{H2} \] (Weak coercivity) There exist \( q_1, q_2 \geq 2 \) and \( \lambda_1, \lambda_2, \lambda_3 > 0 \) such that for all \( x \in \mathfrak{X} \) and \( t \in [0, T] \)
\[
[x, A(t, x)]_\mathfrak{X} \leq -\lambda_1 \cdot \| x \|_{\mathfrak{X}_1}^{q_1} - \lambda_2 \cdot \| x \|_{\mathfrak{X}_2}^{q_2} + \lambda_3 \cdot (\| x \|_{\mathbb{H}} + 1).
\]

\[ \textbf{H3} \] (Weak monotonicity) There exist \( \lambda_0, \lambda'_1, \lambda'_2 \geq 0 \) such that for all \( x, y \in \mathfrak{X} \) and \( t \in [0, T] \)
\[
[x - y, A(t, x) - A(t, y)]_\mathfrak{X} \leq -\lambda'_1 \| x - y \|_{\mathfrak{X}_1}^{q_1} - \lambda'_2 \| x - y \|_{\mathfrak{X}_2}^{q_2} + \lambda_0 \cdot \| x - y \|_{\mathbb{H}}^2,
\]
where \( q_1 \) and \( q_2 \) are same as in (H2).

\[ \textbf{H4} \] (Boundedness) There exist \( c_A, c_{A_2} > 0 \) such that for all \( x \in \mathfrak{X} \) and \( t \in [0, T] \)
\[
\| A_i(t, x) \|_{\mathfrak{X}_i^*} \leq c_{A_i} \cdot (\| x \|_{\mathfrak{X}_i}^{q_i - 1} + 1), \quad i = 1, 2,
\]
where \( q_1 \) and \( q_2 \) are same as in (H2).

\[ \textbf{H5} \] There exists a \( \beta_1 > 0 \) such that for all \( x, y \in \mathbb{H} \) and \( t \in [0, T] \)
\[
\| B(t, x) - B(t, y) \|_{L_2(\mathbb{U}_Q, \mathbb{H})} \leq \beta_1 \| x - y \|_{\mathbb{H}}
\]
and
\[
\| B(t, x) \|_{L_2(\mathbb{U}_Q, \mathbb{H})} \leq \beta_1 (1 + \| x \|_{\mathbb{H}}).
\]
We take the polish space $\mathbb{S}$ in Theorem 3.7 as follows
\[ \mathbb{S} := C_T(\mathbb{H}) \cap L^{q_1}(0, T; \mathbb{X}_1) \cap L^{q_2}(0, T; \mathbb{X}_2) \] (5)
with the norm
\[ \|x\|_\mathbb{S} := \sup_{t \in [0, T]} \|x(t)\|_{\mathbb{H}} + \sum_{i=1, 2} \left( \int_0^T \|x(t)\|^2_{\mathbb{X}_i} dt \right)^{1/q_i}. \]

Consider the following stochastic evolution equation:
\[
\begin{align*}
\left\{ \begin{array}{l}
dX(t) = A(t, X(t))dt + B(t, X(t))dW(t), \\
X(0) = x_0 \in \mathbb{H}.
\end{array} \right.
\end{align*}
\] (6)
By $[18, 31]$ and $[25]$, we have the following existence of unique strong solution to Eq. (6).

**Theorem 2.4.** Assume that (H1)-(H5) hold. Then there exists a unique measurable functional $\Phi$ from $C_T(\mathbb{U})$ to $\mathbb{S}$ such that $X(t, \omega) = \Phi(W(\omega))(t)$ solves the following equation in $\mathbb{X}^*$
\[ X(t) = x_0 + \int_0^t A(s, X(s))ds + \int_0^t B(s, X(s))dW(s), \]
where the Itô stochastic integral is calculated in Hilbert space $\mathbb{H}$. Moreover, for any $h \in \mathcal{A}_N$, $X^h(t, \omega) = \Phi(W(\omega) + h(\omega))(t)$ solves the following equation in $\mathbb{X}^*$
\[ X(t) = x_0 + \int_0^t A(s, X(s))ds + \int_0^t B(s, X(s))dW(s) + \int_0^t B(s, X(s))\dot{h}(s)ds. \]

**Remark 2.5.** The second conclusion follows from the Girsanov theorem.

### 3. Laplace and Large Deviation Principle

Consider the following small perturbation to stochastic evolution equation (6):
\[
\left\{ \begin{array}{l}
dX_\varepsilon(t) = A(t, X_\varepsilon(t))dt + \sqrt{\varepsilon}B(t, X_\varepsilon(t))dW(t), \quad \varepsilon \in (0, 1), \\
X_\varepsilon(0) = x_0 \in \mathbb{H}.
\end{array} \right.
\] (7)
By Theorem 2.4 there exists a measurable mapping $\Phi_\varepsilon : C_T(\mathbb{U}) \to \mathbb{S}$ such that
\[ X_\varepsilon(t, \omega) = \Phi_\varepsilon(W(\omega))(t). \]
We now fix a family of processes $\{h_\varepsilon\}$ in $\mathcal{A}_N$, and put
\[ X^\varepsilon(t, \omega) := \Phi_\varepsilon(W(\omega) + h_\varepsilon(\omega)/\sqrt{\varepsilon})(t). \]
It should be noticed that we have used a little confused notations $X_\varepsilon$ and $X^\varepsilon$, but it is clearly different. Note that $X^\varepsilon(t)$ solves the following stochastic evolution equation:
\[
\left\{ \begin{array}{l}
dX^\varepsilon(t) = A(t, X^\varepsilon(t))dt + \sqrt{\varepsilon}B(t, X^\varepsilon(t))dW(t) + B(t, X^\varepsilon(t))\dot{h}(t)dt, \\
X^\varepsilon(0) = x_0 \in \mathbb{H}.
\end{array} \right.
\] (8)
Moreover, the following energy identity holds(cf. [18], also called Itô’s formula):
\[
\|X^\varepsilon(t)\|^2_{\mathbb{H}} = \|x_0\|^2_{\mathbb{H}} + 2 \int_0^t \langle X^\varepsilon(s), A(s, X^\varepsilon(s)) \rangle ds + M^\varepsilon(t)
+ 2 \int_0^t \langle X^\varepsilon(s), B(s, X^\varepsilon(s)) \dot{h}(s) \rangle ds
+ \varepsilon \int_0^t \|B(s, X^\varepsilon(s))\|^2_{L_2(\mathbb{U}, \mathbb{H})} ds,
\] (9)
where \( t \mapsto M^\varepsilon(t) \) is a real continuous martingale given by
\[
M^\varepsilon(t) := 2\sqrt{\varepsilon} \int_0^t \langle X^\varepsilon(s), B(s, X^\varepsilon(s)) \rangle dW(s).
\]
Note that the square variation process of \( M^\varepsilon(t) \) is given by
\[
<M^\varepsilon>_t = 4\varepsilon \sum_j \int_0^t \langle X^\varepsilon(s), B(s, X^\varepsilon(s))Q^{1/2}(e_j) \rangle^2 ds,
\]
where \( \{e_j\} \) is an orthonormal basis of \( U \).

**Convention:** The letter \( C \) below with or without subscripts will denote positive constants whose values may change in different occasions.

Our main task is to verify the above **(Hypothesis)**. We first prove some uniform estimates about \( X^\varepsilon(t) \).

**Lemma 3.1.** For any \( p \geq 2 \) and \( T > 0 \), there exists a constant \( C_{p,T} > 0 \) such that for all \( \varepsilon \in (0, 1] \)
\[
E \left( \sup_{t \in [0,T]} \| X^\varepsilon(t) \|^2_{p,H} \right) \leq C_{p,T}(\| x_0 \|^2_{p,H} + 1),
\]
and for \( i = 1, 2 \)
\[
E \left( \int_0^T \| X^\varepsilon(s) \|^2_{p,H} ds \right)^p \leq C_{p,T}(\| x_0 \|^2_{p,H} + 1).
\]

**Proof.** By (2) and Itô’s formula, we find that
\[
\| X^\varepsilon(t) \|^2_{p,H} = \| x_0 \|^2_{p,H} + 2p \int_0^t \| X^\varepsilon(s) \|^2_{p-2,H} [X^\varepsilon(s), A(s, X^\varepsilon(s))] ds
+ p \int_0^t \| X^\varepsilon(s) \|^2_{p-2,H} dM^\varepsilon(s) + \frac{p(p-1)}{2} \int_0^t \| X^\varepsilon(s) \|^2_{p-2,H} ds
+ 2p \int_0^t \| X^\varepsilon(s) \|^2_{p-2,H} \langle X^\varepsilon(s), B(s, X^\varepsilon(s)) \rangle_H ds
+ \varepsilon p \int_0^t \| X^\varepsilon(s) \|^2_{p-2,H} \| B(s, X^\varepsilon(s)) \|^2_{L^2(U_Q,H)} ds.
\]
By (H2) and (H5) we have
\[
\| X^\varepsilon(t) \|^2_{p,H} \leq \| x_0 \|^2_{p,H} + C \int_0^t \left( \| X^\varepsilon(s) \|^2_{p,H} (\| \dot{h}^\varepsilon(s) \|_{U_Q} + 1) + 1 \right) ds
+ p \int_0^t \| X^\varepsilon(s) \|^2_{p-2,H} dM^\varepsilon(s).
\]
Hence, by Gronwall’s inequality and (2) we get
\[
\| X^\varepsilon(t) \|^2_{p,H} \leq C_N \left( \| x_0 \|^2_{p,H} + 1 + \int_0^t \left( \left\| X^\varepsilon(r) \right\|^2_{p-2,H} dM^\varepsilon(r) \right) ds \right).
\]
Put
\[
f(t) := E \left( \sup_{r \in [0,t]} \| X^\varepsilon(r) \|^2_{p,H} \right).
\]
Then, by BDG’s inequality and Young’s inequality we obtain
\[
f(t) \leq C_N \cdot (\| x_0 \|^2_{p,H} + 1) + C_N \cdot T \cdot E \left( \int_0^t \| X^\varepsilon(r) \|^4_{p-1,H} d < M^\varepsilon>_r \right)^{1/2}.
\]
Hence note that the following equality holds in \( X^\varepsilon \) where
\[
\sup_{r \in [0,t]} \| X^\varepsilon (r) \|_{H^2}^2 \cdot \int_0^t \| X^\varepsilon (r) \|_{H^2}^2 + 1 \, dr \]
\[
\leq C_N \cdot (\| x_0 \|_{H^2}^2 + 1) + C_N \mathbb{E} \left( \sup_{r \in [0,t]} \| X^\varepsilon (r) \|_{H^2}^2 + 1 \right)^{1/2}
\leq C_N \cdot (\| x_0 \|_{H^2}^2 + 1) + \frac{1}{2} f(t) + C_N \int_0^t (\mathbb{E} \| X^\varepsilon (r) \|_{H^2}^2 + 1) \, dr.
\]

Therefore,
\[
f(t) \leq 2C_N \cdot (\| x_0 \|_{H^2}^2 + 1) + 2C_N \int_0^t (f(r) + 1) \, dr.
\]
By Gronwall’s inequality again, we obtain the first estimate.

As for the second estimate, from (9) and (H2), (H5) we also have
\[
\sum_{i=1,2} \int_0^T \| X^\varepsilon (s) \|_{X^*}^p \, ds \leq C \left( \| x_0 \|_{H^2}^2 + | M^\varepsilon (t) | + \int_0^t \| X^\varepsilon (s) \|_{H^2}^2 \| \dot{h}^\varepsilon (s) \|_{U_0} + 1 \right) \, ds.
\]

Using the first estimate, we immediately get the desired second estimate.

\[\square\]

**Lemma 3.2.** For any \( p \geq 2 \), there exists a constant \( C \) depending on \( p, T, N \) and \( x_0 \) such that for all \( t, r \in [0, T] \) and \( \varepsilon \in (0, 1) \)
\[
\mathbb{E} \| X^\varepsilon (t) - X^\varepsilon (r) \|^p_{X^*} \leq C |t - r|^{p \sqrt{\frac{p}{q_1 q_2}}}
\]

**Proof.** Note that the following equality holds in \( X^* \)
\[
X^\varepsilon (t) - X^\varepsilon (r) = \int_r^t A(s, X^\varepsilon (s)) \, ds + \sqrt{\varepsilon} \int_r^t B(s, X^\varepsilon (s)) \, dW(s)
\]
\[
+ \int_r^t B(s, X^\varepsilon (s)) \dot{h}^\varepsilon (s) \, ds.
\]

Hence
\[
\mathbb{E} \| X^\varepsilon (t) - X^\varepsilon (r) \|^p_{X^*} \leq 3^{p-1} (I_1 + I_2 + I_3),
\]
where
\[
I_1 := \mathbb{E} \left( \int_r^t \| A(s, X^\varepsilon (s)) \|_{X^*} \, ds \right)^p
\]
\[
I_2 := \mathbb{E} \left\| \sqrt{\varepsilon} \int_r^t B(s, X^\varepsilon (s)) \, dW(s) \right\|^p_{X^*}
\]
\[
I_3 := \mathbb{E} \left\| \int_r^t B(s, X^\varepsilon (s)) \dot{h}^\varepsilon (s) \, ds \right\|^p_{X^*}.
\]

For \( I_1 \), we have by (H4) and Hölder’s inequality
\[
I_1 \leq C \mathbb{E} \left( \int_r^t (\| A_1 (s, X^\varepsilon (s)) \|_{X^1} + \| A_2 (s, X^\varepsilon (s)) \|_{X^2}) \, ds \right)^p
\]
\[
\leq C \mathbb{E} \left( \int_r^t (\| X^\varepsilon (s) \|_{X^1}^{q_1-1} + \| X^\varepsilon (s) \|_{X^2}^{q_2-1} + 1) \, ds \right)^p
\]
\[
\leq C (t - r)^p + C \sum_{i=1,2} \left[ \mathbb{E} \left( \int_r^t \| X^\varepsilon (s) \|_{X^i}^{q_i} \, ds \right)^{\frac{p(q_i-1)}{q_i}} (t - r)^{\frac{q_i}{2}} \right].
\]

For \( I_2 \), we have by BDG’s inequality and (H5)
\[
I_2 \leq C \mathbb{E} \left\| \sqrt{\varepsilon} \int_r^t B(s, X^\varepsilon (s)) \, dW(s) \right\|^p_H
\]
Moreover, there exists a subsequence $\lambda$ and $X$. Assume that for almost all $t$, $\lambda$ solves the following equation

$$I_3 \leq C \mathbb{E} \left( \int_0^t \| B(s, X^\varepsilon(s)) \|_{L_2(U_Q, H)}^2 ds \right)^{p/2}$$

$$\leq C |t - r|^{p/2-1} \left( \int_r^t (\mathbb{E} \| X^\varepsilon(s) \|_H^p + 1) ds \right).$$

For $I_3$, we have by Hölder’s inequality, (2) and (H5)

$$I_3 \leq C \mathbb{E} \left( \int_0^t \| B(s, X^\varepsilon(s)) \|_{L_2(U_Q, H)} \cdot \| \dot{X}^\varepsilon(s) \|_{U_Q} ds \right)^p$$

$$\leq C \mathbb{E} \left( \int_0^t \| B(s, X^\varepsilon(s)) \|_{L_2(U_Q, H)}^2 ds \right)^{p/2} \left( \int_0^T \| \dot{X}^\varepsilon(s) \|_{U_Q}^2 ds \right)^{p/2}$$

$$\leq C |t - r|^{p/2-1} \left( \int_r^t (\mathbb{E} \| X^\varepsilon(s) \|_H^p + 1) ds \right). N^{p/2}. \quad (10)$$

The desired estimate now follows by combining the above estimates and Lemma 3.1. □

**Lemma 3.3.** Assume that for almost all $\omega$, $h^\varepsilon(\cdot, \omega)$ weakly converge to $h(\cdot, \omega)$ in $L_Q$, and $X^\varepsilon(\cdot, \omega)$ strongly converge to $X(\cdot, \omega)$ in $C_T(X^*)$. Then $X(\cdot, \omega)$ solves the following equation

$$X(t, \omega) = x_0 + \int_0^t A(s, X(s, \omega)) ds + \int_0^t B(s, X(s, \omega)) \dot{h}(s, \omega) ds.$$

Moreover, there exists a subsequence $\varepsilon_k$ such that as $k \to \infty$,

$$\mathbb{E} \left( \sup_{t \in [0, T]} \| X^\varepsilon_k(t) - X(t) \|_H^2 \right) \to 0, \quad (11)$$

and if $\lambda_1, \lambda_2 > 0$ in (H3), then for $i = 1, 2$

$$\int_0^T \mathbb{E} \| X^\varepsilon_k(t) - X(t) \|_{K_i}^q dt \to 0. \quad (12)$$

**Proof.** Set for $i = 1, 2$

$$K_{1,i} := L_{q_i/(q_i-1)}([0, T] \times \Omega, \mathcal{M}, dt \times dP; X_i^+)$$

and

$$K_{2,i} := L_q([0, T] \times \Omega, \mathcal{M}, dt \times dP; X_i),$$

where $\mathcal{M}$ denotes the progressively $\sigma$-algebra associated with $\mathcal{F}_t$. Then $K_{1,i}$ and $K_{2,i}$ are reflexive and separable Banach spaces.

We have by Lemma 3.1

$$\sup_{\varepsilon \in (0, 1]} \mathbb{E} \| X^\varepsilon(T) \|_H^2 + \sup_{\varepsilon \in (0, 1]} \sum_{i=1,2} \| X^\varepsilon \|_{K_{2,i}}^q < +\infty \quad (13)$$

and

$$\sup_{\varepsilon \in (0, 1]} \mathbb{E} \left( \sup_{t \in [0, T]} \| X^\varepsilon(t) \|_H^4 \right) < +\infty. \quad (14)$$

Hence, by the strong convergence of $X^\varepsilon(\cdot, \omega)$ to $X(\cdot, \omega)$ in $C_T(X^*)$ we have

$$\mathbb{E} \| X(T) \|_H^2 \leq \lim_{\varepsilon \downarrow 0} \mathbb{E} \| X^\varepsilon(T) \|_H^2 \leq C_{p,T}(\| x_0 \|_H^2 + 1) \quad (15)$$
\[
\int_0^T \mathbb{E}\|X(s)\|_{X_{\beta}}^2 ds \leq \lim_{\varepsilon \to 0} \int_0^T \mathbb{E}\|X^\varepsilon(s)\|_{X_{\beta}}^2 ds \leq C_{p,T}(\|x_0\|_{\mathbb{H}}^2 + 1),
\]
(16)
as well as by (13)
\[
\lim_{\varepsilon \to 0} \mathbb{E}\left(\sup_{t \in [0,T]} \|X^\varepsilon(s) - X(s)\|_{X_{\beta}}^2 \right) = 0.
\]
Thus, by (11) we have as \(\varepsilon \downarrow 0\)
\[
\mathbb{E}\left(\int_0^T \|X^\varepsilon(s) - X(s)\|_{X_{\beta}}^2 ds \right) = \mathbb{E}\left(\int_0^T |X^\varepsilon(s) - X(s), X^\varepsilon(s) - X(s)|_{X_{\beta}}|ds \right)
\leq \int_0^T \mathbb{E}\left(\|X^\varepsilon(s) - X(s)\|_{X_{\beta}} \cdot \|X^\varepsilon(s) - X(s)\|_{X_{\beta}} \right)ds
\leq \left( \int_0^T \mathbb{E}\|X^\varepsilon(s) - X(s)\|_{X_{\beta}}^2 ds \int_0^T \mathbb{E}\|X^\varepsilon(s) - X(s)\|_{X_{\beta}}^2 ds \right)^{1/2} \to 0.
\]
(17)
Notice also that by (H4) and (13)
\[
\sup_{\varepsilon \in (0,1]} \|A_i(\cdot, X^\varepsilon(\cdot))\|_{K_i} < +\infty, \quad i = 1, 2.
\]
By this and (13) and the weak compactness of \(K_{1,i}\) and \(K_{2,i}, \ i = 1, 2,\) there exist a subsequence \(\varepsilon_k\) (still denoted by \(\varepsilon\) for simplicity) and \(Y_i \in K_{1,i}, i = 1, 2,\) \(X \in \cap_{i=1,2} K_{2,i},\)
\(X_T \in L^2(\Omega)\) such that
\[
X^\varepsilon \to X \text{ weakly in } K_{2,i}, \ i = 1, 2,
\]
(18)
\[
X^\varepsilon(T) \to X_T \text{ weakly in } L^2(\Omega),
\]
(19)
and
\[
Y_i^\varepsilon := A_i(\cdot, X^\varepsilon(\cdot)) \to Y_i \text{ weakly in } K_{1,i}, \ i = 1, 2.
\]
(20)
Put \(Y = Y_1 + Y_2\) and define
\[
\tilde{X}(t) := x_0 + \int_0^t Y(s)ds + \int_0^t B(s, X(s))\dot{h}(s)ds.
\]
Note that
\[
X^\varepsilon(t) = x_0 + \int_0^t A(s, X^\varepsilon(s))ds + \sqrt{\varepsilon} \int_0^t B(s, X^\varepsilon(s))dW^\varepsilon(s)
\]
\[+ \int_0^t B(s, X^\varepsilon(s))\dot{h}(s)ds.
\]
By taking weak limits and (17), it is not hard to see that (see also the proof of (25) below)
\[
\tilde{X}(t, \omega) = X(t, \omega) = \hat{X}(t, \omega) \text{ for } dt \times dP-\text{almost all } (t, \omega)
\]
and
\[
\tilde{X}(T) = X(T) = X_T.
\]
(21)
In the following we use the unified notation \(X,\) and only need to prove by the usual monotonicity argument that
\[
Y(s, \omega) = A(s, X(s, \omega)) \text{ for } dt \times dP-\text{almost all } (t, \omega).
\]
(22)
Without loss of generality, we assume that \(\lambda_0 = 0\) in (H3) (cf. [21] [31]). It is clear that in (9)
\[
\mathbb{E}M^\varepsilon(T) = 0,
\]
(23)
and
\[ \lim_{\varepsilon \downarrow 0} \varepsilon \mathbb{E} \int_0^T \| B(s, X^\varepsilon(s)) \|^2_{L^2(U, H)} ds = 0. \] (24)

Let us prove the following limit:
\[ \lim_{\varepsilon \downarrow 0} \mathbb{E} \left| \int_0^T \left( \langle X^\varepsilon(s), B(s, X^\varepsilon(s)) \rangle_{H} - \langle X(s), B(s, X(s)) \rangle_{H} \right) ds \right| = 0. \] (25)

Since for almost all \( \omega, h^\varepsilon(\cdot, \omega) \) weakly converges to \( h(\cdot, \omega) \) in \( L^1_Q \), by the dominated convergence theorem we have
\[ \lim_{\varepsilon \downarrow 0} \mathbb{E} \left| \int_0^T \langle X(s), B(s, X(s)) \rangle \left( \hat{h}^\varepsilon(s) - \hat{h}(s) \right)_{H} ds \right| = 0. \]

By (2), Lemma 3.1 and (17) we also have
\[ \mathbb{E} \left| \int_0^T \langle X^\varepsilon(s) - X(s), B(s, X^\varepsilon(s)) \hat{h}^\varepsilon(s) \rangle_{H} ds \right| \]
\[ \leq \mathbb{E} \int_0^T \| X^\varepsilon(s) - X(s) \|_{H} \cdot (\| X^\varepsilon(s) \|_{H} + 1) \cdot \| \hat{h}^\varepsilon(s) \|_{U,Q} ds \]
\[ \leq C_N \mathbb{E} \left( \int_0^T \| X^\varepsilon(s) - X(s) \|_{H}^2 \cdot (\| X^\varepsilon(s) \|_{H} + 1)^2 ds \right)^{1/2} \]
\[ \leq C_N \mathbb{E} \left( \left( \sup_{s \in [0,T]} \| X^\varepsilon(s) \|_{H}^2 + 1 \right) \cdot \int_0^T \| X^\varepsilon(s) - X(s) \|_{H}^2 ds \right)^{1/2} \]
\[ \leq C_N \left( \int_0^T \mathbb{E} \| X^\varepsilon(s) - X(s) \|_{H}^2 ds \right)^{1/2} \to 0, \] (26)

and
\[ \mathbb{E} \left| \int_0^T \langle X(s), (B(s, X^\varepsilon(s)) - B(s, X(s))) \hat{h}^\varepsilon(s) \rangle_{H} ds \right| \]
\[ \leq \mathbb{E} \int_0^T \| X(s) \|_{H} \cdot \| X^\varepsilon(s) - X(s) \|_{H} \cdot \| \hat{h}^\varepsilon(s) \|_{U,Q} ds \]
\[ \leq C_N \mathbb{E} \left( \int_0^T \| X(s) \|_{H}^2 \cdot \| X^\varepsilon(s) - X(s) \|_{H}^2 ds \right)^{1/2} \to 0. \]

The limit (25) now follows.

Notice that for any \( \Phi \in K_{2,1} \cap K_{2,2} \)
\[ \mathbb{E} \int_0^T [X^\varepsilon(s), A(s, X^\varepsilon(s))]_{X} ds \leq \mathbb{E} \int_0^T [\Phi(s), A(s, X^\varepsilon(s)) - A(s, \Phi(s))]_{X} ds \]
\[ + \mathbb{E} \int_0^T [X^\varepsilon(s), A(s, \Phi(s))]_{X} ds \quad (\because \lambda_0 = 0) \]
\[ \to \mathbb{E} \int_0^T [\Phi(s), Y(s) - A(s, \Phi(s))]_{X} ds \]
\[ + \mathbb{E} \int_0^T [X(s), A(s, \Phi(s))]_{X} ds, \] (27)

as \( \varepsilon \downarrow 0 \). The limits are due to (18) and (20).
Combining (9), (15), (23)-(27) yields that
\[ \mathbb{E}\|X_T\|_{H}^2 \leq \lim_{\varepsilon \to 0} \mathbb{E}\|X^\varepsilon(T)\|_{H}^2 \]
\leq \|x_0\|_{H}^2 + 2\mathbb{E} \int_0^T [\Phi(s), Y(s) - A(s, \Phi(s))]_\mathcal{X} ds
+ 2\varepsilon \int_0^T \langle X(s), A(s, \Phi(s)) \rangle_\mathcal{X} ds
+ 2\mathbb{E} \int_0^T \langle X(s), B(s, X(s)) \hat{h}(s) \rangle_{H} ds.

On the other hand, by the energy equality (see (9)) we have
\[ \|\tilde{X}(T)\|_{H}^2 = \|x_0\|_{H}^2 + 2\int_0^T [X(s), Y(s)]_\mathcal{X} ds + 2\int_0^T \langle X(s), B(s, X(s)) \hat{h}(s) \rangle_{H} ds. \]
So by (21)
\[ \mathbb{E} \int_0^T [X(s) - \Phi(s), Y(s) - A(s, \Phi(s))]_\mathcal{X} ds \leq 0, \]
which then yields (22) by (H1) and Lemma 3.1 (see also 21).

Lastly, let us prove the limits (11) and (12). By Itô's formula, we have
\[ \|X^\varepsilon(t) - X(t)\|_{H}^2 = I_1^\varepsilon(t) + I_2^\varepsilon(t) + I_3^\varepsilon(t) + I_4^\varepsilon(t). \]
where
\[ I_1^\varepsilon(t) := 2 \int_0^t \langle X^\varepsilon(s) - X(s), A(s, X^\varepsilon(s)) - A(s, X(s)) \rangle_\mathcal{X} ds \]
\[ I_2^\varepsilon(t) := 2 \int_0^t \langle X^\varepsilon(s) - X(s), B(s, X^\varepsilon(s)) \hat{h}(s) \rangle_{H} ds \]
\[ I_3^\varepsilon(t) := -2 \int_0^t \langle X^\varepsilon(s) - X(s), B(s, X(s)) \hat{h}(s) \rangle_{H} ds \]
\[ I_4^\varepsilon(t) := 2\sqrt{\varepsilon} \int_0^t \langle X^\varepsilon(s) - X(s), B(s, X^\varepsilon(s)) dW(s) \rangle_{H} \]
\[ I_5^\varepsilon(t) := \varepsilon \int_0^t \|B(s, X^\varepsilon(s))\|_{L_2(U_Q, H)}^2 ds. \]

By BDG's inequality and Lemma 3.1 we obviously have
\[ \lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{t \in [0, T]} \left( |I_1^\varepsilon(t)| + |I_5^\varepsilon(t)| \right) \right) = 0. \]

For \( I_2^\varepsilon \), as in the proof of (26) we have
\[ \mathbb{E} \left( \sup_{t \in [0, T]} |I_2^\varepsilon(t)| \right) \leq C \mathbb{E} \int_0^T \|X^\varepsilon(s) - X(s)\|_H \cdot (\|X^\varepsilon(s)\|_H + 1) \cdot \|\hat{h}(s)\|_{U_Q} ds \]
\[ \leq C_N \mathbb{E} \left( \int_0^T \|X^\varepsilon(s) - X(s)\|_H^2 \cdot (\|X^\varepsilon(s)\|_H + 1)^2 ds \right)^{1/2} \]
\[ \to 0, \quad \text{as} \ \varepsilon \to 0. \]
Similarly
\[ \lim_{k \to \infty} \mathbb{E} \left( \sup_{t \in [0, T]} |I^k_f(t)| \right) = 0. \]

Assume \( \lambda_1', \lambda_2' > 0 \), then
\[ I^1_f(t) \leq -\sum_{i=1,2} \lambda'_i \int_0^t \|X^\varepsilon(s) - X(s)\|_{X_i}^q ds + \lambda_0 \int_0^t \|X^\varepsilon(s) - X(s)\|_H^2 ds. \tag{28} \]

If we put
\[ f(t) := \lim_{\varepsilon \to \infty} \mathbb{E} \left( \sup_{s \in [0, t]} \|X^\varepsilon(s) - X(s)\|_H^2 \right), \]
then
\[ f(t) \leq \lambda_0 \int_0^t f(s) = 0. \]

So
\[ f(T) = 0. \]

The limits (11) and (12) is straightforward by noting (28). \( \square \)

We may prove the following main lemma.

**Lemma 3.4.** There exists a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) and a sequence (still indexed by \(\varepsilon\) for simplicity) \(\{(\tilde{h}^\varepsilon, \tilde{X}^\varepsilon, \tilde{W}^\varepsilon)\}\) and \((h^\varepsilon, W)\) defined on this probability space and taking values in \(D_N \times C_T(X^*) \times C_T(U)\) such that
(a) \((\tilde{h}^\varepsilon, \tilde{X}^\varepsilon, \tilde{W}^\varepsilon)\) has the same law as \((h^\varepsilon, X^\varepsilon, W)\) for each \(\varepsilon\);
(b) \((\tilde{h}^\varepsilon, \tilde{X}^\varepsilon, \tilde{W}^\varepsilon) \to (h^\varepsilon, W)\) in \(D_N \times C_T(X^*) \times C_T(U)\), \(\tilde{P}\)-a.s. as \(\varepsilon \to 0\); 
(c) \((h, X^h)\) uniquely solves the following equation:
\[ X^h(t) = x_0 + \int_0^t A(s, X^h(s)) ds + \int_0^t B(s, X^h(s))\dot{h}(s) ds. \tag{29} \]

Moreover, there exists a subsequence \(\varepsilon_k\) such that as \(k \to \infty\),
\[ \mathbb{E}^{\tilde{P}} \left( \sup_{t \in [0, T]} \|\tilde{X}^{\varepsilon_k}(t) - X^h(t)\|_H^2 \right) \to 0, \tag{30} \]
and if \(\lambda_1', \lambda_2' > 0\) in (H3), then for \(i = 1, 2\)
\[ \int_0^T \mathbb{E}^{\tilde{P}} \|\tilde{X}^{\varepsilon_k}(t) - X^h(t)\|_{X_i}^q dt \to 0. \tag{31} \]

**Proof.** By Lemma [3.2](#) and [17, Corollary 14.9](#), the laws of \((h^\varepsilon, X^\varepsilon, W)\) in \(D_N \times C_T(X^*) \times C_T(U)\) is tight. By Skorohod’s embedding theorem, the conclusions (a) and (b) hold. Note that \(\tilde{X}^\varepsilon(0) = x_0\) \(\tilde{P}\)-a.s. and
\[ \tilde{X}^\varepsilon(t) = x_0 + \int_0^t A(s, \tilde{X}^\varepsilon(s)) ds + \sqrt{\varepsilon} \int_0^t B(s, \tilde{X}^\varepsilon(s)) d\tilde{W}^\varepsilon(s) + \int_0^t B(s, \tilde{X}^\varepsilon(s)) \dot{\tilde{h}}(s) ds. \]

The other conclusions follows from Lemma [3.3](#). \( \square \)

From this lemma, one sees that (Hypothesis) holds. Thus, by Theorem [3.7](#) we obtain
**Theorem 3.5.** Assume (H1)-(H5) hold, and $\lambda_1', \lambda_2' > 0$ in (H3). Then for all real bounded continuous functions $g$ on $\mathcal{S}$

$$
\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E} \left( \exp \left[ - \frac{g(X_\varepsilon)}{\varepsilon} \right] \right) = - \inf_{f \in \mathcal{S}} \{ g(f) + I(f) \},
$$

where $I(f)$ is defined by

$$
I(f) := \frac{1}{2} \inf_{h \in \mathcal{L}_Q \colon f = Xh} \| h \|^2_{\mathcal{L}_Q},
$$

and $X^h$ solves (29).

**Remark 3.6.** If $\lambda_1', \lambda_2' = 0$ in (H3), then the conclusion still holds if $\mathcal{S}$ is replaced by $\mathcal{C}_T(\mathbb{H})$.

In order to show the large deviation principle, we need to prove that $I(f)$ is a good rate function. For this aim, we need an extra assumption:

$$
\mathbb{X} \hookrightarrow \mathbb{H} \text{ compactly.}
$$

**Lemma 3.7.** In addition to (H1)-(H5) and $\lambda_1', \lambda_2' > 0$, we also assume that $\mathbb{X}$ is compactly embedded in $\mathbb{H}$. Then $I(f)$ is a good rate function, i.e., for any $a > 0$,

$$
\{ f \in \mathcal{S} : I(f) \leq a \}
$$

is compact.

**Proof.** It suffices to prove that if $h_n \in D_\mathbb{X}$ weakly converge to $h$ in $\mathbb{L}_Q$, then there exists a subsequence $n_k$ (still denoted by $n_k$) such that

$$
\lim_{n \to \infty} \| X^{h_n} - X^h \|_{\mathcal{S}} = 0.
$$

In fact, assume that $I(f_n) \leq a$. By the definition of $I(f_n)$, there exists a sequence $h_n \in \mathbb{L}_Q$ such that

$$
X^{h_n} = f_n \quad \text{and} \quad \frac{1}{2} \| h_n \|^2_{\mathcal{L}_Q} \leq a + \frac{1}{n}.
$$

By the weak compactness of $D_{2a+1}$, there exists a subsequence $n_k$ (still denoted by $n_k$) and $h \in \mathbb{L}_Q$ such that $h_n$ weakly converge to $h$ and

$$
\| h \|^2_{\mathcal{L}_Q} \leq \lim_{n \to \infty} \| h_n \|^2_{\mathcal{L}_Q} \leq 2a.
$$

Thus, by (33) we get the desired compactness.

We now prove (33). As in the proofs of Lemma 3.1 and Lemma 3.2, we may prove

$$
\| X^{h_n} \|_{\mathcal{S}} = \sup_{t \in [0, T]} \| X^{h_n}(t) \|_{\mathcal{H}} + \sum_{i=1,2} \left( \int_0^T \| X^{h_n}(t) \|_{\mathcal{H}}^q \, dt \right)^{1/q_i} \leq C
$$

and

$$
\| X^{h_n}(t) - X^{h_n}(r) \|_{\mathcal{H}^*} \leq C|t - r|^{1/n^{\sqrt{q_2}}},
$$

where $C$ is independent of $n$.

Since $\mathbb{X} \hookrightarrow \mathbb{H}$ is compact, by [13, Theorem 2.1] there exists a subsequence $n_k$ (still denoted by $n$) and an $X \in L^2(0, T; \mathbb{H})$ such that

$$
\int_0^T \| X^{h_n}(t) - X(t) \|^2_{\mathcal{H}} \, dt = 0.
$$

Basing on this convergence, as in the proof of Lemma 3.3, we in fact have $X = X^h$ and the desired limit (33) hold.

Using Theorem 3.5 and Lemma 3.7, we obtain the following large deviation principle.
Theorem 3.8. Assume (H1)-(H5) hold, and $\mathbb{X}$ is compactly embedded in $\mathbb{H}$, $\lambda_1', \lambda_2' > 0$ in (H3). Let the law of $X_\varepsilon$ in $\mathbb{S}$ be denoted by $\nu_\varepsilon$. Then for any $A \in \mathcal{B}(\mathbb{S})$

$$- \inf_{f \in A^c} I(f) \leq \liminf_{\varepsilon \to 0} \varepsilon \log \nu_\varepsilon(A) \leq \limsup_{\varepsilon \to 0} \varepsilon \log \nu_\varepsilon(A) \leq - \inf_{f \in A} I(f),$$

where the closure and the interior are taken in $\mathbb{S}$, and $I(f)$ is a good rate function defined by (32).

4. Applications

4.1. SDE with monotone drift. We consider the following small perturbation of stochastic ordinary differential equation with monotone drift:

$$dX_\varepsilon(t) = b(t, X_\varepsilon(t))dt + \sqrt{\varepsilon}\sigma(t, X_\varepsilon(t))dW(t), \quad X(0) = x_0 \in \mathbb{R}^d,$$

where $W$ is an $m$-dimensional Brownian motion, $\sigma$ and $b$ satisfy that

(Hσ) There exists a constant $C_\sigma > 0$ such that for all $t \in [0,T]$ and $x \in \mathbb{R}^d$

$$\|\sigma(t, x) - \sigma(t, y)\|_{\mathbb{R}^{d \times m}} \leq C_\sigma \|x - y\|_{\mathbb{R}^d}.$$

(Hb) There exists a constant $C_b > 0$ such that for all $t \in [0,T]$ and $x \in \mathbb{R}^d$

$$\langle x - y, b(t, x) - b(t, y) \rangle_{\mathbb{R}^d} \leq C_b \|x - y\|^2_{\mathbb{R}^d}.$$

Let $\nu_\varepsilon$ be the law of $X_\varepsilon(t)$ in the continuous functions space $C_T(\mathbb{R}^d)$. Then the conclusion of Theorem 3.8 holds.

The following two examples can also be found in [32].

4.2. Stochastic reaction diffusion equation. Let $\mathcal{O}$ be an open and bounded set in Euclidean space $\mathbb{R}^d$, where the boundary $\partial \mathcal{O}$ of $\mathcal{O}$ is assumed to be smooth. For $q \geq 2$, let $W_0^{1,q}(\mathcal{O})$ and $W^{-1,\frac{q}{q-1}}(\mathcal{O})$ be the usual Sobolev spaces(cf. [1]).

Suppose that for each integer $j = 1, \cdots, d$, we are given a function $a_j : \mathcal{O} \times \mathbb{R} \mapsto \mathbb{R}$ such that

$$r \mapsto a_j(\xi, r)$$

is continuous and non-decreasing for each $\xi \in \mathcal{O}$,

$$a_j(\xi, r) \xi \geq C_1 |r|^{q_1} - C_2, \quad (\xi, r) \in \mathcal{O} \times \mathbb{R},$$

$$|a_j(\xi, r)| \leq C_3 (|r|^{q_1-1} + 1), \quad (\xi, r) \in \mathcal{O} \times \mathbb{R},$$

where $q_1 \geq 2$ and $C_1, C_2, C_3 > 0$.

Let $b$ be another continuous function satisfying (34)-(36) and with a different constant $q_2 \geq 2$. Let $l^2$ be the usual Hilbert space of square summable real number sequences. Let $\sigma(\xi, r) : \mathcal{O} \times \mathbb{R} \mapsto l^2$ satisfy that $\sigma(\cdot, 0) \in L^2(\mathcal{O}; l^2)$ and for some $c_1 > 0$

$$\|\sigma(\xi, r) - \sigma(\xi, r')\|_{l^2} \leq c_1 \cdot |r - r'|, \quad \xi \in \mathcal{O}, r, r' \in \mathbb{R},$$

We consider the following small perturbation of stochastic reaction diffusion equation

$$\begin{cases}
    dX_\varepsilon(t, \xi) = \left[ \sum_{j=1}^d \partial_{a_j(\xi)} X_\varepsilon(t, \xi) - b(\xi, X_\varepsilon(t, \xi)) \right] dt \\
    \quad + \sqrt{\varepsilon} \sum_{j=1}^\infty \sigma_j(\xi, X_\varepsilon(t, \xi)) dW_j(t), \\
    X_\varepsilon(t, \xi) = 0, \forall \xi \in \partial \mathcal{O}, \\
    X_\varepsilon(0, \xi) = x(\xi) \in L^2(\mathcal{O}),
\end{cases}$$

where $W_j(t) = (W(t), \ell_j)_U$, and $\{\ell_j, j \in \mathbb{N}\}$ is an orthogonal basis of $U_Q$.

Set

$$X_1 := W_0^{1,q_1}(\mathcal{O}), \quad X_2 := L^{q_2}(\mathcal{O}), \quad H := L^2(\mathcal{O}),$$

and

$$X_1^* := W^{-1,\frac{q_1}{q_1-1}}(\mathcal{O}), \quad X_2^* := L^{\frac{q_2}{q_2-1}}(\mathcal{O}).$$
Then

\[ \mathcal{X} := \mathcal{X}_1 \cap \mathcal{X}_2 \subset \mathcal{H} \subset (\mathcal{X}_1^* + \mathcal{X}_2^*) \subset \mathcal{X}^* \]

forms an evolutional triple.

Now, define for \( u, v \in \mathcal{X}_1 \)

\[ [A_1(u), v]_{\mathcal{X}_1} := - \sum_{i=1}^{d} \int_{\mathcal{O}} a_i(\xi, \partial_i u(\xi)) \cdot \partial_i v(\xi) \, d\xi \]

and for \( u, v \in \mathcal{X}_2 \)

\[ [A_2(u), v]_{\mathcal{X}_2} := - \int_{\mathcal{O}} b(\xi, u(\xi)) \cdot v(\xi) \, d\xi. \]

Clearly, for each \( u \in \mathcal{X}_1 \), \( [A_1(u), \cdot]_{\mathcal{X}_1} \in \mathcal{X}_1^* \) and for each \( u \in \mathcal{X}_2 \), \( [A_2(u), \cdot]_{\mathcal{X}_2} \in \mathcal{X}_2^* \). Thus, \( A_1 : \mathcal{X}_1 \mapsto \mathcal{X}_1^* \), \( A_2 : \mathcal{X}_2 \mapsto \mathcal{X}_2^* \), and it is easy to verify that \( A := A_1 + A_2 \) satisfies \((H1)-(H4)\).

Moreover, if we define for \( x \in \mathcal{H} = L^2(\mathcal{O}) \)

\[ B(t, x) := \sum_{j=1}^{\infty} \sigma_j(\cdot, x(\cdot)) \ell_j \]

then for any \( x, y \in L^2(\mathcal{O}) \)

\[ \|B(t, x) - B(t, y)\|_{L^2(\mathcal{O}; \mathcal{H})}^2 = \sum_{j=1}^{\infty} \|\sigma_j(\cdot, x(\cdot)) - \sigma_j(\cdot, y(\cdot))\|_{L^2(\mathcal{O})}^2 \]

\[ \leq C\|x(\cdot) - y(\cdot)\|_{L^2(\mathcal{O})}^2. \]

Thus, \((H5)\) holds.

Let \( \nu_\varepsilon \) be the law of \( X_\varepsilon(t) \) in \( \mathcal{S} \), where \( \mathcal{S} \) is defined by \((5)\). Then the conclusion of Theorem 3.8 holds.

### 4.3. Stochastic Porous Medium Equation

As in the previous subsection, we consider the bounded domain \( \mathcal{O} \) in \( \mathbb{R}^d \) with smooth boundary.

For \( p \geq 2 \), set

\[ \mathcal{X} := L^p(\mathcal{O}), \quad \mathcal{H} := W^{-1,2}(\mathcal{O}), \quad \mathcal{X}^* := L^{p/(p-1)}(\mathcal{O}). \]

The inner product in \( \mathcal{H} \) is given by

\[ \langle x, y \rangle_{\mathcal{H}} := \int_{\mathcal{O}} (-\Delta)^{-1/2} x(\xi) \cdot (-\Delta)^{-1/2} y(\xi) \, d\xi, \quad x, y \in \mathcal{H} = W^{-1,2}(\mathcal{O}). \]

Note that \(-\Delta\) establishes an isomorphism between \( W_0^{1,2}(\mathcal{O}) \) and \( W^{-1,2}(\mathcal{O}) \). We shall identify \( W_0^{1,2}(\mathcal{O}) \) with the dual space \( \mathcal{H}^* \) of \( \mathcal{H} \), and hence \( \mathcal{H}^* = W_0^{1,2}(\mathcal{O}) \subset L^{p/(p-1)}(\mathcal{O}) \). Thus, we have the evolution triple

\[ \mathcal{X} \subset \mathcal{H} \simeq \mathcal{H}^* \subset \mathcal{X}^* \]

where \( \simeq \) is understood through \(-\Delta\).

Let \( \phi_p(r) := r^{|p-2/2|} \), and define for \( x \in \mathcal{X} = L^p(\mathcal{O}) \)

\[ A(x) := \Delta \phi_p(x). \]

Then \( A(x) \in \mathcal{X}^* \) and \((H1)-(H4)\) hold(cf. [28, 21]).
Let $B_1, \ldots, B_n \in L_2(U, \mathbb{H})$, and define
\begin{equation}
B(t, x) := \sum_{k=1}^{n} g_k([e_{n_1}, x]_{\mathbb{H}}, \ldots, [e_{n_k}, x]_{\mathbb{H}})B_k, \quad e_{n_j} \in \mathbb{H},
\end{equation}
where $g_k$ are Lipschitz continuous functions on $\mathbb{R}^{n_k}$. Then such $B(t, x)$ satisfies (H5). It should be noticed that if $\sigma \in C^\infty_b(\mathbb{R})$ is not linear, the mapping $x \mapsto \sigma(x)$ is in general not Lipschitz from $W^{-1,2}(O)$ to $W^{-1,2}(O)$.

Consider the following small perturbation of stochastic porous medium equation
\begin{equation}
\begin{cases}
dX_\varepsilon(t) = \Delta(\phi_p(X_\varepsilon(t)))dt + \sqrt{\varepsilon}B(t, X_\varepsilon(t))dW(t), \\
X_\varepsilon(t, \xi) = 0, \quad \forall \xi \in \partial O, \\
X_\varepsilon(0, \xi) = x(\xi) \in W^{-1,2}(O).
\end{cases}
\end{equation}
Let $\nu_\varepsilon$ be the law of $X_\varepsilon(t)$ in $C_T(\mathbb{H}) \cap L^p(0, T; \mathbb{X})$. Then the conclusion of Theorem 3.8 holds.

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**References**

[1] R.A. Adams: *Sobolev space*. Academic Press, 1975.
[2] R. Azencott: Grandes d´eviations et applications. *Ecole d’Été de Probabilités de Saint-Flour VIII, 1978, Lect. Notes. in Math.*, 779, 1-176. Springer, New York, 1980.
[3] M. Boué and P. Dupuis: A variational representation for certain functionals of Brownian motion. *Ann. of Prob.*, 1998, Vol. 26, No.4, 1641-1659.
[4] M. Boué, P. Dupuis and R.S. Ellis: Large deviations for small noise diffusions with discontinuous statistics. *Prob. Theory Relat.Fields.*, 116,125-149(2000).
[5] A. Budhiraja and P. Dupuis: A variational representation for positive functionals of infinite dimensional Brownian motion. *Probab. Math. Statist.* 20 (2000), no. 1, Acta Univ. Wratislav. No. 2246, 39–61.
[6] A. Budhiraja, P. Dupuis and V. Maroulas: Large deviations for infinite dimensional stochastic dynamical systems. to appear in *Ann. of Prob.*
[7] S. Cerrai and M. Röckner: Large deviations for stochastic reaction diffusion systems with multiplicative noise and non-Lipschitz reaction term. *The Annals of Probability*, 32:1100-1139, 1996.
[8] E. Pardoux: Stochastic partial differential equations and filtering of diffusion processes. *Stochastic*. 1979, 127-167.
[9] G. Da Prato and M. Röckner: Weak solutions to stochastic porous media equations, *J. Evolution Equ.* 4(2004), 249–271.
[10] G. Da Prato and J. Zabczyk: *Stochastic equations in infinite dimensions*. Cambridge: Cambridge University Press, 1992.
[11] P. Dupuis and R.S. Ellis: *A Weak Convergence Approach to the Theory of Large Deviations*. Wiley, New-York, 1997.
[12] J. Feng and T.G. Kurtz: *Large deviations for stochastic processes*. Math. Surveys and Mono., Vol. 131, AMS(2006).
[13] F. Flandoli and D. Gatarek: Martingale and stationary solutions for stochastic Navier-Stokes equations. *Probab. Theory Related Fields*, 102, 367-391(1995).
[14] M.I. Freidlin and A.D. Wentzell: On small random perturbations of dynamical system, *Russian Math. Surveys* 25 (1970), 1-55.
[15] I. Gyöngy and A. Millet: On Discretization Schemes for Stochastic Evolution Equations. *Potential Analysis*, (2005)23:99-134.
[16] G. Kallianpur, J. Xiong: Large deviations for a class of stochastic partial differential equations. *The Annals of Probability*, 24(1):320-345, 1996.
[17] O. Kallenberg: *Foundations of Modern Probability*. Springer-Verlag, Berlin, 1997.
[18] N.V. Krylov and B.L. Rozovskii: Stochastic evolution equations. *J. Soviet Math.* (Russian), 1979, pp. 71-147, Transl. 16(1981), 1233-1277.
[19] W. Liu: Large deviations for stochastic evolution equations with small multiplicative noise. Preprint.
[20] S. Peszat: Large deviation estimates for stochastic evolution equations. *Prob. Th. Rel. Fields*, 98:113-136, 1994.
[21] C. Prévôt and M. Röckner: A concise course on stochastic partial differential equations. Lecture Notes in Mathematics, 1905. Springer, Berlin, 2007. vi+144 pp
[22] J. Ren, M. Röckner and F. Wang: Stochastic Generalized Porous Media and Fast Diffusion Equations. Preprint.
[23] J. Ren and X. Zhang: Schilder theorem for the Brownian motion on the diffeomorphism group of the circle. *J. Func. Anal.* Vol. 224, I. 1, 107-133(2005).
[24] J. Ren and X. Zhang: Freidlin-Wentzell’s large deviations for homeomorphism flows of non-Lipschitz SDEs. *Bull. Sci. Math. 2 Serie*, Vol 129/8 pp 643-655(2005).
[25] M. Röckner, B. Schmuland and X. Zhang: Yamada-Watanabe Theorem for Stochastic Evolution Equations in Infinite Dimensions. Preprint.
[26] M. Röckner, F.Y. Wang and L. Wu: Large deviations for stochastic generalized porous media equations. *Stoch. Proc. and their Appl.*, 116(2006)1677-1689.
[27] B.L. Rozovskii: *Stochastic evolution systems. Linear theory and applications to nonlinear filtering*. Mathematics and its Applications (Soviet Series), 35, Kluwer Academic Publishers, 1990.
[28] R.E. Showalter: *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, AMS, Math. Surveys and Monographs, Vol.49, 1997.
[29] R.B. Sower: Large deviations for a reaction-diffusion equation with non-Gaussian perturbations. *The Annals of Probability*, 20(1): 504-537, 1992.
[30] D.W. Stroock: An Introduction to the Theory of Large Deviations, Springer-Verlag, New York, 1984.
[31] X. Zhang: On Stochastic Evolution Equations with non-Lipschitz Coefficients. Preprint.
[32] X. Zhang: A variational representation for random functionals on abstract Wiener spaces. Preprint.