Isospin excitations of a trapped 1D gas of attractively interacting fermions

M. Colomé-Tatché 1, G. V. Shlyapnikov1,2 and A. M. Tsvelik3

1 Laboratoire de Physique Théorique et Modèles Statistiques, Université. Paris Sud, CNRS, 91405 Orsay, France
2 Van der Waals-Zeeman Institute, University of Amsterdam, Valckenierstraat 65/67, 1018 XE Amsterdam, The Netherlands
3 Department of Condensed Matter Physics and Materials Science, Brookhaven National Laboratory, Upton, NY 11973-5000, USA

(Dated: February 2, 2008)

PACS numbers: 05.30. Jp, 03.75. Kk, 03.75. Nt, 05.60. Gg

We consider a gas of fermions with a short-range attractive intercomponent interaction in a parabolic external potential and derive the conditions of the local density approximation. The obtained spectrum of quasiparticle (isospin) excitations shows equidistant low-energy levels, which is equivalent to a linear momentum dependence and is fundamentally different from the ordinary Dirac spectrum in the spatially uniform case.

Fast progress in experiments with cold atoms has led to the creation of one-dimensional (1D) atomic gases by (tightly) confining the motion of atoms in two directions to zero point oscillations. One-dimensional quantum gases show a remarkable physics not encountered in higher dimensions. In particular, since the density of states in 1D increases towards the zone boundary, the effective strength of interactions in the 1D gas increases with decreasing density. Therefore, decreasing density in a gas of attractively interacting fermions one can crossover from the BCS-like regime of strongly overlapping pairs to the regime of compact pairs forming a Bose-Einstein condensate (BEC). For spatially uniform systems this problem is well understood due to availability of exact solutions obtained by the Bethe Ansatz techniques [4, 5, 6].

The 1D gases are usually obtained in an external harmonic potential, which introduces a finite size of the system and makes it spatially nonuniform. Trapped 1D Bose gases have been intensively studied in the last years (see [7] for review). Recently, the 1D regime has been achieved for atomic fermions [8], and the discussion of trapped 1D Fermi gases was focused on the occurrence of the BCS-BEC crossover, manifestation of spin-charge separation, role of the imbalance between atomic components (see [9, 10] for review). One of the key problems is revealing the influence of an external harmonic potential on the quasiparticle spectrum of this system.

In this paper we consider a 1D $N$-component Fermi gas with a point-like attraction in a parabolic trap. We first treat this problem in the limit of $N >> 1$ using the standard $1/N$-expansion [11]. In the limit of $N \to \infty$ the saddle point approximation becomes exact and one can deal with the spatially non-uniform distribution of particles in a controllable way. We show that the spatial inhomogeneity strongly affects the quantization rules for the quasiparticle spectrum. As a result, the system in a parabolic trap cannot be mapped onto an integrable system in a rectangular box. We then discuss the applicability of the large-$N$ results for the experimentally relevant case of a two-component Fermi gas ($N = 2$).

In terms of field operators $\psi_j^+$ and $\psi_j$, with the index $j$ labelling the fermionic species, the bare Hamiltonian is:

$$H = \sum_j \int dx \left[ -\frac{1}{2m} \psi_j^+ \frac{\partial^2}{\partial x^2} \psi_j \right] - \frac{g}{N} \sum_{j \neq j} \psi_j^+ \psi_j^+ \psi_j \psi_j,$$  \hspace{1cm} (1)

where $m$ is the atom mass, $\omega$ is the trap frequency, the coupling constant of the point-like attraction is written as $-g/N$ with $g > 0$, and we set $\hbar = 1$. We will work in the thermodynamic limit defined by the relations

$$E_F = \omega N/N = \text{const.}, \quad N \to \infty, \quad \omega \to 0;$$  \hspace{1cm} (2)

with $N$ being the total number of particles, and $E_F$ the Fermi energy.

We first briefly outline the results for the uniform case where the system is in a rectangular box with periodic boundary conditions. Then, in the thermodynamic limit the excitation spectrum consists of a gapless collective mode describing fluctuations of the total density (charge mode) and $N - 1$ branches of particles (isospin modes) [12, 13]. The latter have spectral gaps: the particle of the $q$-th branch transforms according to the fundamental representation of the SU($N$) group with the Young tableau consisting of one column of height $q$. In the limit of $\gamma << 1$ where

$$\gamma = \frac{mg}{\pi n}$$  \hspace{1cm} (3)

and $n$ is the mean density of each fermionic component, the spectrum of the gapless particles is approximately Lorentz invariant:

$$\epsilon_q(p) = \sqrt{(pv_F)^2 + M_q^2},$$  \hspace{1cm} (4)

$$M_q = M_0 \sin(2\pi q/N), \quad q = 1,...N - 1;$$  \hspace{1cm} (5)

where $C$ is a numerical constant. At a finite $N$ these particles interact and the ones with $q > 1$ are bound.
states of the fundamental particle with $q = 1$. However, in the limit of $N \to \infty$ the interaction vanishes and the only particles which remain are the particle with $q = 1$ and its antiparticle having $q = N - 1$. The case of $N \gg 1$ can be treated perturbatively using $1/N$ expansion [11].

This approach can be employed for a spatially nonuniform system as is done in this paper. To make calculations easier we assume that in the major part of the trap the coupling constant is small. This allows us to use the Thomas-Fermi density profile for non interacting fermions:

$$n(x) = n_0 \sqrt{1 - x^2/x_0^2}, \quad N = \frac{\pi}{2} n_0 x_0 N; \quad x_0 = \sqrt{\frac{2N}{mN\omega^2}}, \quad (6)$$

where $x_0$ is the Thomas-Fermi (half)size of the sample. The Thomas-Fermi approximation breaks down at distances near the Thomas-Fermi boundary, where $dn^{-1}(x)/dx \sim 1$ and, hence, $(1 - x^2/x_0^2)^{1/2} \sim N^{-1/3}$. The condition of weak coupling breaks down in a narrow range of distances near the boundary, where $\gamma(n(x)) \sim 1$ and $(1 - x^2/x_0^2)^{1/2} \sim \gamma(0) \equiv mg/\pi n_0 \ll 1$.

Assuming that the weak coupling approximation holds, we can linearize the spectrum introducing right and left moving fermions $R, L$:

$$\psi_j(x) = e^{-ik_Fx}\phi_j(x) + e^{ik_Fx}\phi_j(x).$$

The coordinate-dependent Fermi momentum is $k_F(x) = \pi n(x)$, and Hamiltonian [1] takes the form:

$$H = \sum_j \int_{\tau_0}^{\tau} dt \left\{ i(-R_j^\dagger \partial_t R_j + L_j^\dagger \partial_t L_j) \right\},$$

$$-\frac{\gamma(\tau)}{2N} \sum_{j \neq j'} (L_j^\dagger R_{j'}) \right\}, \quad (7)$$

where $\gamma$ is given by Eq. (3), and

$$\tau = \int_0^\tau dx' / v_F(x') = \omega^{-1} \arcsin(x/x_0), \quad (8)$$

with the Fermi velocity $v_F = k_F/m$, and $\tau_0 = \pi/2\omega$.

At $N \gg 1$ the interaction can be treated in the saddle-point approximation which becomes exact in the limit of $N \to \infty, \gamma = \text{const}$. To develop such an approximation one does the Hubbard-Stratonovich transformation introducing an auxiliary complex field $\Delta(\tau, t)$, and formally integrates over the fermions. The resulting effective action is

$$S = N \int dtd\tau \left\{ \frac{|\Delta(t, \tau)|^2}{2\gamma(\tau)} - \text{Tr} \ln \left[ i(\partial_t - \partial_\tau) \Delta \Delta^* i(\partial_t + \partial_\tau) \right] \right\},$$

where $t$ is the Matsubara time. Expanding the logarithm in gradients of $\Delta$ we obtain the Lagrangian density:

$$\mathcal{L} = \frac{N}{4\pi} \left\{ \left[ \frac{2\pi}{\gamma(\tau)} - \ln(\Lambda/|\Delta|) \right] |\Delta|^2 + \frac{1}{2} |\Delta|^2 \left[ |\partial_t \Delta|^2 + |\partial_\tau \Delta|^2 \right] \right\} + \ldots \quad (9)$$

$$+ \frac{1}{2} \left\{ \frac{(\partial_t \rho)^2}{\rho^2} + \frac{(\partial_\tau \rho)^2}{\rho^2} \right\} + \ldots \right\}, \quad (13)$$

There are two features of this expansion, with a different level of robustness under deviations from the condition of weak coupling, $\gamma \ll 1$. First of all, the phase $\phi$ is decoupled from the amplitude field, which is a simple consequence of $\phi$ being the Goldstone mode of the Charge Density Wave order parameter field $\langle |\Delta| \rangle$. As such it should be gapless and weakly coupled to other excitations. However, the second feature, namely the fact that the stiffness of the field $\phi$ is independent of the particle density $n(x)$, holds only in the limit of $\gamma \ll 1$.

As follows from the Bethe Ansatz calculations valid in the uniform case, the stiffness (or the Luttinger parameter $K_x$) starts to acquire the density dependence at $\gamma \sim 1$.

Since the action following from Eqs. (11)-(13) is proportional to large $N$, fluctuations are suppressed. In particular, one can find excitation energies by considering the Dirac Hamiltonian with the coordinate-dependent
mass determined by the minimum of the action $13$:

$$2\rho[2\pi/\gamma(\tau) - \ln(\Lambda/\rho)] - \rho^{-1}\partial_{\tau}\ln \rho + \ldots = 0 \quad (14)$$

The solution can be represented as

$$\rho(\tau) = \rho(0) \exp \left\{ 2\pi \int [n(0) - n(\tau)]/\gamma \right\} .$$

The second term in the exponent of this expression can be omitted if one satisfies the inequality

$$\Delta_0/\omega \gg 1/\gamma, \quad (15)$$

where $\gamma_0 = \gamma(0)$. One then has

$$\rho(\tau) = \Delta_0 \exp\{2\pi^2[n_0 - n(\tau)]/mg\} . \quad (16)$$

Now we find the excitation spectrum of quasiparticles. As we have said, in the limit of $N \to \infty$ their spectrum is decoupled from the phase excitations. Then the fermions are described by the effective Dirac Hamiltonian

$$H_F = \sum_j \int d\tau (R^+_j, L^+_j) \left( \begin{array}{c} -i \frac{d}{d\tau} \rho(\tau) \\ i \frac{d}{d\tau} \end{array} \right) \left( \begin{array}{c} R_j \\ L_j \end{array} \right) , \quad (17)$$

where $\rho(\tau)$ is given by Eq. $16$. A uniform system in a rectangular box with periodic boundary conditions, is described by the same Hamiltonian, with $\rho(\tau) = \Delta_0$. Equations of motion following from this Hamiltonian yield solutions in the form of plane waves with the spectrum

$$\epsilon_k = (\pi kv_F/\ell)^2 + \Delta_0^2, \quad k = 0, \pm 1, \pm 2, \ldots , \quad (18)$$

where $2\ell$ is the length of the box. This corresponds to the spectrum of $q = 1, N - 1$ excitation branches in Eq. $3$.

Actually, having derived Eq. $10$ for $\rho(\tau)$ entering the Hamiltonian $H_F$, we obtained the local density approximation for our nonuniform system. Our results show that in the Dirac Hamiltonian $17$ one can use the same expression for the mass (gap) as in the uniform case, but the exponent is coordinate dependent through the spatial dependence of the density (parameter $\gamma$) and the coordinate dependence of the preexponential factor is omitted. However, the quasiparticle spectrum is quite different from that of Eq. $13$.

Eigenvectors of Hamiltonian $17$ can be written as

$$R = \rho^{1/2} F, \quad L = (\epsilon R + iR) / \rho, \quad (19)$$

where the function $F(\tau)$ satisfies the equation

$$- F_{\tau\tau} + \left[ \rho^2(\tau) - \epsilon^2 + \left( \frac{\rho^2}{2\rho} - \left( \frac{\rho^2}{2\rho} \right)^2 \right) + i \frac{\rho^2}{\rho} \right] F = 0, \quad (20)$$

and the notations $F_\tau$ and $F_{\tau\tau}$ mean the first and second derivative with respect to $\tau$. For finding the lowest eigenstates one can approximate

$$\rho^2(\tau) \approx \Delta_0^2 \left[ 1 + \frac{2\pi \omega^2 \tau^2}{\gamma(0)} \right] \quad (21)$$

and check that under the condition $15$ the terms in the round brackets in Eq. $20$ are small at least as $(\omega/\sqrt{\gamma_0})^{1/2}$ compared to both terms on the rhs of Eq. $21$. Then, omitting the terms in the round brackets, for the energy eigenvalues we obtain:

$$\epsilon_k = \sqrt{\Delta_0^2 + \Delta_0 \omega (8\pi/\gamma_0)^{1/2} (k + 1/2)} , \quad (22)$$

where $k > 0$ is an integer. Identifying $k$ as a (rescaled) momentum of a particle confined in a rectangular box and comparing Eq. $22$ with Eq. $13$ one sees that the quasiparticle energy levels in the parabolic trap cannot be mapped onto the Dirac particle spectrum in the box.

For finding high-energy eigenstates one can use the WKB quantization rule. Omitting again the terms in the round brackets in Eq. $20$ we have

$$\int_{-\tau(\epsilon)}^{\tau(\epsilon)} d\tau \sqrt{\epsilon^2 - \rho^2(\tau)} = \pi k, \quad (23)$$

where $\tau(\epsilon)$ is determined by the condition $\epsilon = \rho(\tau)$. Rewriting Eq. $10$ as

$$\rho(\tau) = \Delta_0 \exp\{2\pi^2(1 - \cos \omega \tau)/\gamma_0\}$$

we notice that considering $\epsilon \ll E_F$ one can write the exponent of $\rho(\tau)$ as $\pi \omega^2 \tau^2 / \gamma_0$. This leads to

$$\tau(\epsilon) \approx \frac{1}{\omega} \sqrt{\gamma_0 \ln (\epsilon/\Delta_0)} . \quad (24)$$

Then performing the integration in Eq. $23$ we obtain:

$$\frac{\sqrt{\epsilon_k^2 - \Delta_0^2}}{\omega} \left( \frac{2\gamma_0}{\pi} \right)^{1/2} \sqrt{\ln \left( 1 + \beta_k^2 - \frac{\Delta_0^2}{\Delta_0^2} \right)} = \pi k, \quad (25)$$

where for eigenstates near the bottom of the gap, that is for $\epsilon_k - \Delta_0 \ll \Delta_0$, the coefficient $\beta$ is equal to $\pi^2/16$ and Eq. $25$ gives the same result as Eq. $22$ at large quantum numbers $k$: $\epsilon_k = \Delta_0 + \Delta_0 \omega (8\pi/\gamma_0)^{1/2} k$. For excitation energies $\epsilon_k \gtrsim \Delta_0$ one has $\beta$ close to unity, and in the limit of $\epsilon_k \gg \Delta_0$ corresponding to quantum numbers $k \gg \Delta_0 \gamma_0 / \omega$, Eq. $25$ yields:

$$\epsilon_k = \left( \frac{\pi^3}{4\gamma_0} \right)^{1/2} \sqrt{\frac{\omega k}{\ln((\pi^2/2\gamma_0)^{1/2} \omega k / \Delta_0)}} . \quad (26)$$

As we see, this result also differs significantly from the simple Dirac spectrum.

In Fig. 1 we compare the results of Eqs. $22$ and $25$ with numerical calculation of the spectrum from Eq. $20$. One sees a good agreement even for $\Delta_0/\omega$ about $20$, and for larger values of $\Delta_0/\omega$ the agreement is nearly perfect.

We believe that our main result, a pronounced difference between quantization rules for the quasiparticle
spectrum in harmonic and rectangular confining potentials, remains valid in the experimentally relevant case of a two-component Fermi gas ($N = 2$). It is likely that the local density approximation for the mass gap $\rho$ calculated along the lines of [14, 15], together with Eq. (23), remain robust. The main difference of small from large in the properties of the quantum gas.

For a trapped 1D atomic ultracold Fermi gas one can expect the number of particles $N \sim 10^4$. Then, recalling that in the two-species gas the Fermi energy is $E_F = N\omega/2$, for $\gamma_0 \approx 1$ we obtain $\Delta_0/\omega \approx 10$ and our results are applicable. The isospin modes can be excited optically, for example by using pulses of polarised $\sigma^-$ light acting on one of the atomic components (spin-up) and pulses of $\sigma^+$ light acting on the other component (spin-down). The $\sigma^-$ and $\sigma^+$ pulses provide periodic optical potentials shifted by a quarter of a wavelength with respect to each other, so that the minimum of the $\sigma^-$ potential corresponds to the maximum of the $\sigma^+$ potential and the sum of the two potentials is zero. Thus, the spin-up and spin-down particles get kicks in the opposite directions, which creates isospin modes. At the same time, charge (density) modes corresponding to in-phase oscillations of the two components are not excited.

In conclusion, we have found the quasiparticle (isospin) spectrum for attractively interacting fermions in a parabolic potential. The spectrum shows equidistant low-energy levels (linear momentum dependence) and is drastically different from the ordinary Dirac spectrum in the spatially uniform case. Experimental verification of this result will provide a clear demonstration of the fact that the parabolic confinement can fundamentally change the properties of the quantum gas.

We are grateful to B. L. Altshuler, I. L. Aleiner, J. Dalibard, F. Gerbier, D.L. Kovrizhin, and L.P. Pitaevskii for discussions and acknowledge support of Institut Henri Poincaré during the workshop "Quantum Gases" where part of this work has been done. The work was also supported by the IFIRA Institute, by ANR (grants 05-BLAN-0205 and 06-NANO-014-01), by the QUDEDIS program of ESF, and by the Dutch Foundation FOM. AMT was supported by US DOE under contract number DE-AC02 -98 CH 10886. LPTMS is a mixed research unit No. 8626 of CNRS and Université Paris Sud.

References:

[1] M. Gaudin, Phys. Lett. A24, 55 (1967).
[2] C. N. Yang, Phys. Rev. Lett. 19, 1312 (1967).
[3] F. H. L. Essler, H. Frahm, F. Goehmann, A. Klumper, and V. E. Korepin, The One-Dimensional Hubbard Model (Cambridge University Press, Cambridge, UK, 2005).
[4] A. O. Gogolin, A. A. Nersesyan and A. M. Tsvelik, Bosonization and Strongly Correlated Systems (Cambridge University Press, Cambridge 1999).
[5] A. M. Tsvelik Quantum Field Theory in Condensed Matter Systems (Cambridge University Press, Cambridge 2003).
[6] T. Giamarchi, Quantum Physics in One Dimension (Oxford University Press, Oxford, 2004).
[7] D.S. Petrov, D.M. Gangardt, and G.V. Shlyapnikov, J. Phys. IV (France) 116, 5 (2004); Y. Castin, ibid 116, 89 (2004).
[8] H. Moritz et al, Phys. Rev. Lett. 94, 210401 (2005).
[9] S. Giorgini, L.P. Pitaevskii, and S. Stringari, arXiv:0706.3360
[10] C. Kollath and U. Schollwock, New Journal of Physics, 8, 220 (2006).
[11] R. Koberle, V. Kurak and J. A. Swieca, Phys. Rev. D 20, 897 (1979).
[12] B. Schroer, T. T. Truong and P. Weisz, Phys. Lett. B 63, 422 (1976).
[13] N. Andrei, Phys. Lett. B 90, 106 (1980).
[14] V.Ya. Krivnov and A.A. Ovchinnikov, Zh. Eksp. Teor. Fiz. 67, 1568 (1974) [Sov. Phys. JETP 40, 781 (1975)]
[15] A.I. Larkin and J. Sak, Phys. Rev. Lett. 39, 1025 (1977).
[16] F. H. L. Essler and A. M. Tsvelik, Phys. Rev. Lett 90, 126401 (2003).
[17] This scheme lies in the basis of laser cooling below the

\[ \Delta_0/\omega = 95 \ (\gamma_0 = 0.5) \]

The solid and dashed curves show the results of numerical solution of Eq. (20).
Doppler limit: J. Dalibard and C. Gohen-Tannoudji, J. Opt. Soc. Am. 6, 2023 (1989). A similar scheme was used for controlled coherent transport in spin-dependent optical lattice potentials: O. Mandel et al, Phys. Rev. Lett. 91, 010407 (2003).