ON MONOIDS OF INJECTIVE PARTIAL SELFMAPS ALMOST EVERYWHERE THE IDENTITY

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ABSTRACT. In this paper we study the semigroup $J^\infty_\omega$ of injective partial selfmaps almost everywhere the identity of a set of infinite cardinality $\lambda$. We describe the Green relations on $J^\infty_\omega$, all (two-sided) ideals and all congruences of the semigroup $J^\infty_\omega$. We prove that every Hausdorff hereditary Baire topology $\tau$ on $J^\infty_\omega$ such that $(J^\infty_\omega, \tau)$ is a semitopological semigroup is discrete and describe the closure of the discrete semigroup $J^\infty_\omega$ in a topological semigroup. Also we show that for an infinite cardinal $\lambda$ the discrete semigroup $J^\infty_\omega$ does not embed into a compact topological semigroup and construct two non-discrete Hausdorff topologies turning $J^\infty_\omega$ into a topological inverse semigroup.

1. Introduction and preliminaries

In this paper all spaces are assumed to be Hausdorff. Furthermore we shall follow the terminology of [3, 4, 7, 9, 23]. By $\omega$ we shall denote the first infinite cardinal and by $|A|$ the cardinality of the set $A$. If $Y$ is a subspace of a topological space $X$ and $A \subseteq Y$, then by $cl_Y(A)$ and $Int_Y(A)$ we shall denote the topological closure and the interior of $A$ in $Y$, respectively.

For a semigroup $S$ we denote the semigroup $S$ with the adjoined unit by $S^1$ (see [9]).

An algebraic semigroup $S$ is called inverse if for any element $x \in S$ there exists a unique element $x^{-1} \in S$ (called the inverse of $x$) such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. If $S$ is an inverse semigroup, then the function $inv : S \to S$ which assigns to every element $x$ of $S$ its inverse element $x^{-1}$ is called inversion.

If $S$ is an inverse semigroup, then by $E(S)$ we shall denote the band (i.e., the subsemigroup of idempotents) of $S$. If the band $E(S)$ is a non-empty subset of $S$, then the semigroup operation on $S$ determines a partial order $\leq$ on $E(S)$: $e \leq f$ if and only if $ef = fe = e$. This order is called natural. A semilattice is a commutative semigroup of idempotents. A semilattice $E$ is called linearly ordered or a chain if the semilattice operation induces a linear natural order on $E$. A maximal chain of a semilattice $E$ is a chain which is properly contained in no other chain of $E$. The Axiom of Choice implies the existence of maximal chains in any partially ordered set. According to [21, Definition II.5.12] a chain $L$ is called an $\omega$-chain if $L$ is isomorphic to $\{0, -1, -2, -3, \ldots\}$ with the usual order $\leq$. Let $E$ be a semilattice and $e \in E$. We denote $\downarrow e = \{f \in E \mid f \leq e\}$ and $\uparrow e = \{f \in E \mid e \leq f\}$. By $(P_{<\omega}(\lambda), \cup)$ we shall denote the free semilattice with identity over a cardinal $\lambda \geq \omega$, i.e., $P_{<\omega}(\lambda)$ is the set of all finite subsets of $\lambda$ with the binary operation $a \cdot b = a \cup b$, for $a, b \in P_{<\omega}(\lambda)$.

If $S$ is a semigroup, then we shall denote by $R, L, J, D$ and $H$ the Green relations on $S$ (see [9]):

\[ aRb \text{ if and only if } aS^1 = bS^1; \]
\[ aLb \text{ if and only if } S^1a = S^1b; \]
\[ aJb \text{ if and only if } S^1aS^1 = S^1bS^1; \]
\[ D = L \cap R = R \cap L; \]
\[ H = L \cap R. \]

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The relation $\mathcal{J}$ induced a quasi-order $\leq_{\mathcal{J}}$ on $S$ as follows:

$$a \leq_{\mathcal{J}} b \quad \text{if and only if} \quad S^1aS^1 \subseteq S^1bS^1,$$

for $a, b \in S$. This implies that the inclusion order among two-sided ideals of $S$ induces a partial order among the $\mathcal{J}$-equivalence classes:

$$J_a \preceq J_b \quad \text{if and only if} \quad S^1aS^1 \subseteq S^1bS^1,$$

for $a, b \in S$, where by $J_a$ we denote the $\mathcal{J}$-class in $S$ which contains an element $a \in S$ (see [17 Section 2.1]). Then we may thus regard $S/\mathcal{J}$ with the relation $\leq_{\mathcal{J}}$ as a partially ordered set.

A semigroup $S$ is called simple if $S$ does not contain proper two-sided ideals.

A semitopological (resp. topological) semigroup is a topological space together with a separately (resp. jointly) continuous semigroup operation. An inverse topological semigroup with the continuous inversion is called a topological inverse semigroup.

In the remainder of the paper $\lambda$ denotes an infinite cardinal.

Let $\mathcal{I}_\lambda$ denote the set of all partial one-to-one transformations of an infinite cardinal $\lambda$ together with the following semigroup operation: $x(\alpha\beta) = (x\alpha)\beta$ if $x \in \text{dom}(\alpha\beta) = \{y \in \text{dom} \alpha \mid y\alpha \in \text{dom} \beta\}$, for $\alpha, \beta \in \mathcal{I}_\lambda$. The semigroup $\mathcal{I}_\lambda$ is called the symmetric inverse semigroup over the cardinal $\lambda$ (see [4]). The symmetric inverse semigroup was introduced by Wagner [25] and it plays a major role in the theory of semigroups.

A partial map $\alpha \in \mathcal{I}_\lambda$ is called almost everywhere the identity if the set $\lambda \setminus \text{dom} \alpha$ is finite and $(x)\alpha \neq x$ only for finitely many $x \in \lambda$. We denote

$$\mathcal{I}_\lambda^\infty = \{\alpha \in \mathcal{I}_\lambda \mid \alpha \text{ is almost everywhere the identity}\}.$$

Obviously, $\mathcal{I}_\lambda^\infty$ is an inverse subsemigroup of the semigroup $\mathcal{I}_\omega$. The semigroup $\mathcal{I}_\lambda^\infty$ is called the semigroup of injective partial selfmaps almost everywhere the identity of $\lambda$. We shall denote every element $\alpha$ of the semigroup $\mathcal{I}_\lambda^\infty$ by

$$\left(\begin{array}{cc}
  x_1 & \cdots & x_n \\
  y_1 & \cdots & y_n \\
\end{array}\right) A$$

and this means that the following conditions hold:

(i) $A$ is the maximal subset of $\lambda$ with the finite complement such that $\alpha|_A : A \to A$ is an identity map;
(ii) $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are finite (not necessary non-empty) subsets of $\lambda \setminus A$; and
(iii) $\alpha$ maps $x_i$ into $y_i$ for all $i = 1, \ldots, n$.

We denote the identity of the semigroup $\mathcal{I}_\lambda^\infty$ by $I$.

Many semigroup theorists have considered topological semigroups of (continuous) transformations of topological spaces. Beida [2], Orlov [19][20], and Subbiah [24] have considered semigroup and inverse semigroup topologies on semigroups of partial homeomorphisms of some classes of topological spaces.

Gutik and Pavlyk [12] considered the special case of the semigroup $\mathcal{I}_\lambda^\infty$: an infinite topological semigroup of $\lambda \times \lambda$-matrix units $B_\lambda$. They showed that an infinite topological semigroup of $\lambda \times \lambda$-matrix units $B_\lambda$ does not embed into a compact topological semigroup and that $B_\lambda$ is algebraically $h$-closed in the class of topological inverse semigroups. They also described the Bohr compactification of $B_\lambda$, minimal semigroup and minimal semigroup inverse topologies on $B_\lambda$.

Gutik, Lawson and Repovš [21] introduced the notion of a semigroup with a tight ideal series and investigated their closures in semitopological semigroups, in particular, in inverse semigroups with continuous inversion. As a corollary they showed that the symmetric inverse semigroup of finite transformations $\mathcal{I}_\lambda^n$ of infinite cardinal $\lambda$ is algebraically closed in the class of (semi)topological inverse semigroups with continuous inversion. They also derived related results about the nonexistence of (partial) compactifications of semigroups with a tight ideal series.

Gutik and Reiter [13] showed that the topological inverse semigroup $\mathcal{I}_\lambda^\infty$ is algebraically $h$-closed in the class of topological inverse semigroups. They also proved that a topological semigroup $S$ with
countably compact square $S \times S$ does not contain the semigroup $\mathcal{I}_\lambda^n$ for infinite cardinals $\lambda$ and showed that the Bohr compactification of an infinite topological semigroup $\mathcal{I}_\lambda^n$ is the trivial semigroup.

In [13] Gutik and Reiter showed that the symmetric inverse semigroup of finite transformations $\mathcal{I}_\lambda^n$ of infinite cardinal $\lambda$ is algebraically closed in the class of semitopological inverse semigroups with continuous inversion. Also there they described all congruences on the semigroup $\mathcal{I}_\lambda^n$ and all compact and countably compact topologies $\tau$ on $\mathcal{I}_\lambda^n$ such that $(\mathcal{I}_\lambda^n, \tau)$ is a semitopological semigroup.

Gutik, Pavlyk and Reiter [13] showed that a topological semigroup of finite partial bijections with pseudocompact square contain $\mathcal{F}_\lambda^n$ as a subsemigroup. They proved that every continuous homomorphism from a topological semigroup $\mathcal{I}_\lambda^n$ into a Hausdorff countably compact topological semigroup or Tychonoff topological semigroup with pseudocompact square is an annihilating. They also gave sufficient conditions for a topological semigroup $\mathcal{I}_\lambda^n$ to be non-$H$-closed and showed that the topological inverse semigroup $\mathcal{I}_\lambda^n$ is absolutely $H$-closed if and only if the band $E(\mathcal{I}_\lambda^n)$ is compact [13].

In [16] Gutik and Repovš studied the semigroup $\mathcal{F}_\lambda^n(\mathbb{N})$ of partial cofinite monotone bijective transformations of the set of positive integers $\mathbb{N}$. They showed that the semigroup $\mathcal{F}_\lambda^n(\mathbb{N})$ has algebraic properties similar to the bicyclic semigroup: it is bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. They proved that every locally compact topology $\tau$ on $\mathcal{F}_\lambda^n(\mathbb{N})$ such that $(\mathcal{F}_\lambda^n(\mathbb{N}), \tau)$ is a topological inverse semigroup, is discrete and described the closure of $(\mathcal{F}_\lambda^n(\mathbb{N}), \tau)$ in a topological semigroup.

In [4] Gutik and Chuchman studied the semigroup $\mathcal{F}_\lambda^n(\mathbb{N})$ of partial co-finite almost monotone bijective transformations of the set of positive integers $\mathbb{N}$. They showed that the semigroup $\mathcal{F}_\lambda^n(\mathbb{N})$ has algebraic properties similar to the bicyclic semigroup: it is bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. Also they proved that every Baire topology $\tau$ on $\mathcal{F}_\lambda^n(\mathbb{N})$ such that $(\mathcal{F}_\lambda^n(\mathbb{N}), \tau)$ is a semitopological semigroup is discrete, described the closure of $(\mathcal{F}_\lambda^n(\mathbb{N}), \tau)$ in a topological semigroup and constructed non-discrete Hausdorff semigroup topologies on the semigroup $\mathcal{F}_\lambda^n(\mathbb{N})$.

In this paper we study the semigroup $\mathcal{I}_\lambda^\infty$ of injective partial selfmaps almost everywhere the identity of a set of infinite cardinality $\lambda$. We describe the Green relations on $\mathcal{I}_\lambda^\infty$, all (two-sided) ideals and all congruences of the semigroup $\mathcal{I}_\lambda^\infty$. We prove that every Hausdorff hereditary Baire topology $\tau$ on $\mathcal{I}_\lambda^\infty$ such that $(\mathcal{I}_\lambda^\infty, \tau)$ is a semitopological semigroup is discrete and describe the closure of the discrete semigroup $\mathcal{I}_\lambda^\infty$ in a topological semigroup. Also we show that for an infinite cardinal $\lambda$ the discrete semigroup $\mathcal{I}_\lambda^\infty$ does not embed into a compact topological semigroup and construct two non-discrete Hausdorff topologies turning $\mathcal{I}_\lambda^\infty$ into a topological inverse semigroup.

### 2. Algebraic properties of the semigroup $\mathcal{I}_\lambda^\infty$

The definition of the semigroup $\mathcal{I}_\lambda^\infty$ implies the following proposition:

**Proposition 2.1.** A partial map $\alpha \in \mathcal{I}_\lambda$ is an element of the semigroup $\mathcal{I}_\lambda^\infty$ if and only if the following assertions hold:

(i) $|\lambda \setminus \text{dom } \alpha| = |\lambda \setminus \text{ran } \alpha|$; and 
(ii) there exists a subset $A \subseteq \text{dom } \alpha \cap \text{ran } \alpha$ such that $\lambda \setminus A$ is a finite subset of $\lambda$ and the restriction $\alpha|_A : A \to A$ is the identity map.

**Proposition 2.2.**

(i) An element $\alpha$ of the semigroup $\mathcal{I}_\lambda^\infty$ is an idempotent if and only if $(x)\alpha = x$ for every $x \in \text{dom } \alpha$.

(ii) If $\varepsilon, \iota \in E(\mathcal{I}_\lambda^\infty)$, then $\varepsilon \leq \iota$ if and only if $\text{dom } \varepsilon \subseteq \text{dom } \iota$.

(iii) The semilattice $E(\mathcal{I}_\lambda^\infty)$ is isomorphic to $(\mathcal{P}_\omega(\lambda), \cup)$ under the mapping $(\varepsilon)h = \lambda \setminus \text{dom } \varepsilon$.

(iv) Every maximal chain in $E(\mathcal{I}_\lambda^\infty)$ is an $\omega$-chain.

(v) $\alpha \mathcal{R} \beta$ in $\mathcal{I}_\lambda^\infty$ if and only if $\text{dom } \alpha = \text{dom } \beta$.

(vi) $\alpha \mathcal{L} \beta$ in $\mathcal{I}_\lambda^\infty$ if and only if $\text{ran } \alpha = \text{ran } \beta$.

(vii) $\alpha \mathcal{H} \beta$ in $\mathcal{I}_\lambda^\infty$ if and only if $\text{dom } \alpha = \text{dom } \beta$ and $\text{ran } \alpha = \text{ran } \beta$. 
(viii) \( \alpha \mathcal{D} \beta \) in \( \mathcal{I}_\lambda^\infty \) if and only if \( |\lambda \setminus \text{dom } \alpha| = |\lambda \setminus \text{dom } \beta| \).

(ix) If \( n \) is a non-negative integer, then for every \( \alpha, \beta \in \mathcal{I}_\lambda^\infty \) such that \( |\lambda \setminus \text{dom } \alpha| = |\lambda \setminus \text{dom } \beta| = n \) there exist \( \gamma, \delta \in \mathcal{I}_\lambda^\infty \) such that \( \alpha = \gamma \cdot \beta \cdot \delta \) and \( |\lambda \setminus \text{dom } \gamma| = |\lambda \setminus \text{dom } \delta| = n \).

(x) For every non-negative integer \( n \) the set \( I_n = \{ \alpha \in \mathcal{I}_\lambda^\infty \mid |\lambda \setminus \text{dom } \alpha| \geq n \} \) is an ideal in \( \mathcal{I}_\lambda^\infty \). Moreover, for every ideal \( I \) in \( \mathcal{I}_\lambda^\infty \) there exists an integer \( n \geq 0 \) such that \( I \) is equal to \( I_n \).

(xi) \( \mathcal{D} = \mathcal{I} \) in \( \mathcal{I}_\lambda^\infty \).

(xii) If \( \lambda_1 \) and \( \lambda_2 \) are infinite cardinals such that \( \lambda_1 \leq \lambda_2 \) then \( \mathcal{I}_{\lambda_1}^\infty \) is a subsemigroup of the semigroup \( \mathcal{I}_{\lambda_2}^\infty / \mathcal{I} , \leq \) is an \( \omega \)-chain for any infinite cardinal \( \lambda \).

Proof. Statements (i) – (iv) are trivial and they follow from the definition of the semigroup \( \mathcal{I}_\lambda^\infty \).

(v) Let be \( \alpha, \beta \in \mathcal{I}_\lambda^\infty \) such that \( \alpha \mathcal{D} \beta \). Since \( \alpha \mathcal{I}_\lambda^\infty = \beta \mathcal{I}_\lambda^\infty \) and \( \mathcal{I}_\lambda^\infty \) is an inverse semigroup, Theorem 1.17 [5] implies that \( \alpha \mathcal{I}_\lambda^\infty = \alpha \mathcal{L} \mathcal{I}_\lambda^\infty \) and \( \mathcal{I}_\lambda^\infty = \beta \mathcal{R} \mathcal{I}_\lambda^\infty \) and hence \( \alpha \mathcal{L} \mathcal{I}_\lambda^\infty = \beta \mathcal{R} \mathcal{I}_\lambda^\infty \).

The proof of statement (vi) is similar to (v).

Statement (vii) follows from (v) and (vi).

(viii) Let \( \alpha, \beta \in \mathcal{I}_\lambda^\infty \) be such that \( \alpha \mathcal{D} \beta \). Then there exists \( \gamma \in \mathcal{I}_\lambda^\infty \) such that \( \alpha \mathcal{L} \gamma \) and \( \gamma \mathcal{R} \beta \).

Then by statements (v) and (vi) we have that \( \text{ran } \alpha = \text{ran } \gamma \) and \( \text{ran } \gamma = \text{dom } \beta \). Then Proposition 2.1 implies that \( |\lambda \setminus \text{ran } \gamma| = |\lambda \setminus \text{dom } \gamma| \) and \( |\lambda \setminus \text{ran } \beta| = |\lambda \setminus \text{dom } \beta| \), and hence we get that \( |\lambda \setminus \text{dom } \alpha| = |\lambda \setminus \text{dom } \beta| \).

Let \( \alpha \) and \( \beta \) be elements of the semigroup \( \mathcal{I}_\lambda^\infty \) such that \( |\lambda \setminus \text{dom } \alpha| = |\lambda \setminus \text{dom } \beta| \). Then Proposition 2.1 implies that \( |\lambda \setminus \text{ran } \alpha| = |\lambda \setminus \text{dom } \alpha| \) and \( |\lambda \setminus \text{ran } \beta| = |\lambda \setminus \text{dom } \beta| \). Let \( A_\alpha \) and \( A_\beta \) be maximal subsets of \( \lambda \) such that the sets \( \lambda \setminus A_\alpha \) and \( \lambda \setminus A_\beta \) are finite and the restrictions \( \alpha|_{A_\alpha} : A_\alpha \rightarrow A_\alpha \) and \( \beta|_{A_\beta} : A_\beta \rightarrow A_\beta \) are identity maps. We put \( A = A_\alpha \cap A_\beta \). Since \( \lambda \setminus A_\alpha \) and \( \lambda \setminus A_\beta \) are finite subsets of \( \lambda \) we conclude that \( \lambda \setminus A \) is a finite subset of \( \lambda \). Since \( |\lambda \setminus \text{dom } \alpha| = |\lambda \setminus \text{dom } \beta| \) we have that \( |\lambda \setminus \text{dom } \alpha| = |\lambda \setminus \text{dom } \beta| \) and hence \( \alpha = \beta \).

Let \( \alpha \) and \( \beta \) be arbitrary elements of the semigroup \( \mathcal{I}_\lambda^\infty \) such that \( |\lambda \setminus \text{dom } \alpha| = |\lambda \setminus \text{dom } \beta| = n \) for some non-negative integer \( n \). If \( n = 0 \), then \( \alpha = \beta \). Suppose that \( n \geq 1 \). Let \( \{ x_1, \ldots, x_n \} = \text{ran } \alpha \setminus A \) and \( \{ y_1, \ldots, y_n \} = \text{dom } \beta \setminus A \). We define

\[
\gamma = \left( \begin{array}{c|c}
        y_1 & \cdots & y_n \\
        x_1 & \cdots & x_n \\
    \end{array} \right) .
\]

Then by statements (v) and (vi) we have that \( \alpha \mathcal{L} \gamma \) and \( \gamma \mathcal{R} \beta \) in \( \mathcal{I}_\lambda^\infty \). Hence \( \alpha \mathcal{D} \beta \) in \( \mathcal{I}_\lambda^\infty \).

(ix) Let \( \alpha \) and \( \beta \) be arbitrary elements of the semigroup \( \mathcal{I}_\lambda^\infty \) such that \( |\lambda \setminus \text{dom } \alpha| = |\lambda \setminus \text{dom } \beta| = n \) for some non-negative integer \( n \). Let \( A_\alpha \) and \( A_\beta \) be maximal subsets of \( \lambda \) such that the sets \( \lambda \setminus A_\alpha \) and \( \lambda \setminus A_\beta \) are finite and the restrictions \( \alpha|_{A_\alpha} : A_\alpha \rightarrow A_\alpha \) and \( \beta|_{A_\beta} : A_\beta \rightarrow A_\beta \) are identity maps. We put \( A = A_\alpha \cap A_\beta \). Since \( \lambda \setminus A_\alpha \) and \( \lambda \setminus A_\beta \) are finite subsets of \( \lambda \) we conclude that \( \lambda \setminus A \) is a finite subset of \( \lambda \). Since \( |\lambda \setminus \text{dom } \alpha| = |\lambda \setminus \text{dom } \beta| \) the definition of the semigroup \( \mathcal{I}_\lambda^\infty \) implies that \( |\lambda \setminus \text{dom } \alpha| = |\lambda \setminus \text{dom } \beta| < \omega \). If \( \alpha \mathcal{D} \beta \) then \( \alpha = \beta \) and hence \( \alpha = \gamma \cdot \beta \cdot \delta \) for \( \gamma = \delta = \mathbb{I} \). Otherwise we put \( \{ x_1, \ldots, x_k \} = \text{dom } \alpha \setminus A \), \( \{ y_1, \ldots, y_k \} = \text{dom } \beta \setminus A \), \( b_1 = (y_1)\beta, \ldots, b_k = (y_k)\beta \) and \( a_1 = (x_1)\alpha, \ldots, a_k = (x_k)\alpha \), for some positive integer \( k \). We define

\[
\gamma = \left( \begin{array}{c|c}
        x_1 & \cdots & x_k \\
        y_1 & \cdots & y_k \\
    \end{array} \right) \quad \text{and} \quad \delta = \left( \begin{array}{c|c}
        b_1 & \cdots & b_k \\
        a_1 & \cdots & a_k \\
    \end{array} \right) .
\]

Then \( \gamma, \delta \in \mathcal{I}_\lambda^\infty \), \( |\lambda \setminus \text{dom } \gamma| = |\lambda \setminus \text{dom } \delta| = n \) and \( \alpha = \gamma \cdot \beta \cdot \delta \).

(x) Let \( \alpha \) and \( \beta \) be arbitrary elements of the semigroup \( \mathcal{I}_\lambda^\infty \). Since \( \alpha \) and \( \beta \) are injective partial selfmaps almost everywhere the identity of the cardinal \( \lambda \) we conclude that

\[
|\lambda \setminus \text{dom } (\alpha \cdot \beta)| \geq \max \{|\lambda \setminus \text{dom } \alpha|, |\lambda \setminus \text{dom } \beta|\} .
\]

This implies the first assertion of statement (x).
Let $I$ be an ideal in $\mathcal{I}^\infty_\lambda$. Then the definition of the semigroup $\mathcal{I}^\infty_\lambda$ implies that there exists $\alpha \in I$ such that

$$|\lambda \setminus \text{dom } \alpha| = \min\{|\lambda \setminus \text{dom } \gamma| \mid \gamma \in I\}.$$ 

Then $|\lambda \setminus \text{dom } \alpha| = n$ for some integer $n \geq 0$. Hence $I \subseteq I_n$ and by statement (ix) we get that $I_n \subseteq I$. This implies the second assertion of the statement.

Statement (xi) follows from statement (ix).

(xii) Let $\alpha = \begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix} A$ be an arbitrary element of the semigroup $\mathcal{I}^\infty_{\lambda_1}$ and $B = \lambda_2 \setminus \lambda_1$. We put

$$\widetilde{\alpha} = \begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix} (A \cup B).$$

Obviously that $\widetilde{\alpha} \in \mathcal{I}^\infty_{\lambda_2}$. Simple verifications show that the map $h: \mathcal{I}^\infty_{\lambda_1} \to \mathcal{I}^\infty_{\lambda_2}$ defined by the formula $(\alpha)h = \widetilde{\alpha}$ is an isomorphic embedding of the semigroup $\mathcal{I}^\infty_{\lambda_1}$ into $\mathcal{I}^\infty_{\lambda_2}$.

Statement (xiii) follows from items (viii) and (xi).

Later we shall need the following proposition:

**Proposition 2.3.** Let $\lambda$ be an arbitrary infinite cardinal. Then for every finite subset $\{x_1, \ldots, x_n\}$ of $\lambda$ the semigroups $\mathcal{I}^\infty_\lambda$ and $\mathcal{I}^\infty_\eta$ are isomorphic for $\eta = \lambda \setminus \{x_1, \ldots, x_n\}$.

**Proof.** Since $\lambda$ is infinite we conclude that there exists a bijective map $f: \lambda \to \eta$. Then the bijection $f$ generates a map $h: \mathcal{I}^\infty_\lambda \to \mathcal{I}^\infty_\eta$ such that the following condition holds:

$$(\alpha_\lambda)h = (\alpha_\eta) \quad \text{if and only if} \quad ((x)f)\alpha_\eta = ((x)\alpha_\lambda) f \quad \text{for every } x \in \lambda,$$

where $\alpha_\lambda \in \mathcal{I}^\infty_\lambda$ and $\alpha_\eta \in \mathcal{I}^\infty_\eta$.

Now we shall show that so defined map $h$ is injective. Suppose to the contrary that there exist distinct elements $\alpha_\lambda, \beta_\lambda \in \mathcal{I}^\infty_\lambda$ such that $(\alpha_\lambda)h = (\beta_\lambda)h$. We denote $\alpha_\eta = (\alpha_\lambda)h$ and $\beta_\eta = (\beta_\lambda)h$. Then dom $\alpha_\eta = \text{dom } \beta_\eta$ and ran $\alpha_\eta = \text{ran } \beta_\eta$ and since $f: \lambda \to \eta$ is a bijective map we conclude that dom $\alpha_\lambda = \text{dom } \beta_\lambda$ and ran $\alpha_\lambda = \text{ran } \beta_\lambda$. Therefore there exists $x \in \text{ran } \alpha_\lambda$ such that $(x)\alpha_\lambda \neq (x)\beta_\lambda$. Since $(\alpha_\lambda)h = (\beta_\lambda)h$ we have that $((x)f)\alpha_\eta = ((x)f)\beta_\eta$. But $((x)f)\alpha_\eta = ((x)\alpha_\lambda)f$ and $((x)f)\beta_\eta = ((x)\beta_\lambda)f$ and since the map $f: \lambda \to \eta$ is bijective we conclude that $(x)\alpha_\lambda = (x)\beta_\lambda$, a contradiction.

The obtained contradiction implies that the map $h: \mathcal{I}^\infty_\lambda \to \mathcal{I}^\infty_\eta$ is injective.

Let $\alpha_\eta = \begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix} A$ be an arbitrary element of the semigroup $\mathcal{I}^\infty_\eta$, where $A \subseteq \eta$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in \eta$. Since the map $f: \lambda \to \eta$ is bijective we conclude that

$$\alpha_\lambda = \begin{pmatrix} (x_1)f^{-1} & \cdots & (x_n)f^{-1} \\ y_1 f^{-1} & \cdots & y_n f^{-1} \end{pmatrix} (A)f^{-1}$$

is a partial bijective map from $\lambda$ into $\lambda$ such that the sets $\lambda \setminus \text{dom } \alpha_\lambda$ and $\lambda \setminus \text{ran } \alpha_\lambda$ are finite. Therefore $\alpha_\lambda \in \mathcal{I}^\infty_\lambda$ and hence the map $h: \mathcal{I}^\infty_\lambda \to \mathcal{I}^\infty_\eta$ is bijective.

Now we prove that the map $h: \mathcal{I}^\infty_\lambda \to \mathcal{I}^\infty_\eta$ is a homomorphism. We fix arbitrary elements $\alpha_\lambda, \beta_\lambda \in \mathcal{I}^\infty_\lambda$ and denote $\alpha_\eta = (\alpha_\lambda)h$ and $\beta_\eta = (\beta_\lambda)h$. Then for every $x \in \text{ran } \alpha_\lambda$ we have that

$$((x)f)(\alpha_\eta \cdot \beta_\eta) = ((x)f)(\alpha_\eta) \beta_\eta = ((x)\alpha_\lambda \beta_\lambda)f = ((x)(\alpha_\lambda \cdot \beta_\lambda))f,$$

and hence $(\alpha_\lambda \cdot \beta_\lambda)h = (\alpha_\eta \cdot \beta_\eta) = (\alpha_\lambda)h \cdot (\beta_\lambda)h$.

Therefore $h$ is an isomorphism from the semigroup $\mathcal{I}^\infty_\lambda$ onto $\mathcal{I}^\infty_\eta$. □

**Proposition 2.4.** Let $\lambda$ be an arbitrary infinite cardinal. Then for every idempotent $\varepsilon$ of the semigroup $\mathcal{I}^\infty_\lambda$ the semigroups $\mathcal{I}^\infty_\lambda(\varepsilon) = \varepsilon \cdot \mathcal{I}^\infty_\lambda \cdot \varepsilon$ and $\mathcal{I}^\infty_\lambda$ are isomorphic.
Proof. Since
\[ \mathcal{I}_\lambda^\infty(\varepsilon) = \varepsilon \cdot \mathcal{I}_\lambda^\infty \cdot \varepsilon = \varepsilon \cdot \mathcal{I}_\lambda^\infty \cap \mathcal{I}_\lambda^\infty \cdot \varepsilon = \{ \alpha \in \mathcal{I}_\lambda^\infty \mid \text{dom } \alpha \subseteq \text{dom } \varepsilon \} \cap \{ \alpha \in \mathcal{I}_\lambda^\infty \mid \text{ran } \alpha \subseteq \text{ran } \varepsilon \} = \{ \alpha \in \mathcal{I}_\lambda^\infty \mid \text{dom } \alpha \subseteq \text{dom } \varepsilon \text{ and } \text{ran } \alpha \subseteq \text{ran } \varepsilon \}, \]
Proposition 2.3 implies the assertion of the proposition.

Proposition 2.5. For every \( \alpha, \beta \in \mathcal{I}_\lambda^\infty \), both sets \( \{ \chi \in \mathcal{I}_\lambda^\infty \mid \alpha \cdot \chi = \beta \} \) and \( \{ \chi \in \mathcal{I}_\lambda^\infty \mid \chi \cdot \alpha = \beta \} \) are finite. Consequently, every right translation and every left translation by an element of the semigroup \( \mathcal{I}_\lambda^\infty \) is a finite-to-one map.

Proof. We denote \( \lambda \) as a non-zero cardinal such that \( \lambda \cdot \chi = \beta \) implies the assertion of the proposition.

Proposition 2.6. Every maximal subchain of the semigroup \( \mathcal{I}_\lambda^\infty \) is isomorphic to \( S_\lambda \).

3. On congruences on the semigroup \( \mathcal{I}_\lambda^\infty \)

If \( \mathcal{R} \) is an arbitrary congruence on a semigroup \( S \), then we denote by \( \Phi_\mathcal{R} : S \to S/\mathcal{R} \) the natural homomorphisms, and \( \Omega_\mathcal{R} \) and \( \Delta_\mathcal{R} \) the universal and the identity congruences, respectively, on the semigroup \( S \), i.e., \( \Omega_\mathcal{R}(S) = S \times S \) and \( \Delta_\mathcal{R} = \{(s, s) \mid s \in S \} \).

The following lemma follows from the definition of a congruence on a semilattice:

Lemma 3.1. Let \( \mathcal{R} \) be an arbitrary congruence on a semilattice \( E \). Let \( a \) and \( b \) be elements of the semilattice \( E \) such that \( a \mathcal{R} b \). Then

(i) \( a \mathcal{R} (ab) \); and

(ii) if \( a \leq b \) then \( a \mathcal{R} c \) for all \( c \in E \) such that \( a \leq c \leq b \).

Proposition 3.2. Let \( \mathcal{R} \) be an arbitrary congruence on the semigroup \( \mathcal{I}_\lambda^\infty \). Let \( \varepsilon \) and \( \varphi \) be idempotents of \( \mathcal{I}_\lambda^\infty \) such that \( \varepsilon \mathcal{R} \varphi \) and \( \varepsilon \leq \varphi \). If \( |\text{dom } \varphi \setminus \text{dom } \varepsilon| = 1 \) then the following conditions hold:

(i) \( \varphi \mathcal{R} \iota \) for all idempotents \( \iota \in \downarrow \varphi \); and

(ii) \( \varphi \mathcal{R} \chi \) for all idempotents \( \chi \in \mathcal{I}_\lambda^\infty \) such that \( |\lambda \setminus \text{dom } \varphi| = |\lambda \setminus \text{dom } \chi| \).

Proof. (i) First we shall show that \( \varphi \mathcal{R} \psi \) for all idempotents \( \psi \in \downarrow \varepsilon \). By Proposition 2.2 (iv) there exists a maximal (not necessary unique) \( \omega \)-chain \( L \in E(\mathcal{I}_\lambda^\infty) \) which contains \( \varepsilon \) and \( \psi \). Let \( L_0 = \{\varepsilon_1, \ldots, \varepsilon_n\} \) be a maximal subchain in \( L \) such that \( \varepsilon = \varepsilon_1 < \ldots < \varepsilon_i = \varepsilon \), where \( n \) is some positive integer. The existence of the subchain \( L \) follows from Proposition 2.2 (iv) too. Let 

\[ x_n = \text{dom } \varepsilon_{n-1} \setminus \text{dom } \varepsilon_n, x_{n-1} = \text{dom } \varepsilon_{n-2} \setminus \text{dom } \varepsilon_{n-1}, \ldots, x_2 = \text{dom } \varepsilon_1 \setminus \text{dom } \varepsilon_2, x_1 = \varphi \setminus \text{dom } \varepsilon_1. \]

We put 

\[ \alpha_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid \text{dom } \varepsilon_2 \), \( \alpha_2 = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \mid \text{dom } \varepsilon_3 \), \( \ldots \), \( \alpha_{n-1} = \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix} \mid \text{dom } \varepsilon_n \). \]
Then we have that
\[
\alpha_1^{-1} \cdot \varphi \cdot \alpha_1 = \varepsilon_1 \quad \text{and} \quad \alpha_1^{-1} \cdot \varepsilon_1 \cdot \alpha_1 = \varepsilon_2;
\]
\[
\alpha_2^{-1} \cdot \varepsilon_1 \cdot \alpha_2 = \varepsilon_2 \quad \text{and} \quad \alpha_2^{-1} \cdot \varepsilon_2 \cdot \alpha_2 = \varepsilon_3;
\]
\[
\ldots \quad \ldots \quad \ldots \quad \ldots
\]
\[
\alpha_{n-1}^{-1} \cdot \varepsilon_{n-2} \cdot \alpha_{n-1} = \varepsilon_{n-1} \quad \text{and} \quad \alpha_{n-1}^{-1} \cdot \varepsilon_{n-1} \cdot \alpha_{n-1} = \varepsilon_n,
\]
and hence \( \varepsilon_1 \mathcal{R} \varepsilon_2, \varepsilon_2 \mathcal{R} \varepsilon_3, \ldots, \varepsilon_{n-1} \mathcal{R} \varepsilon_n \). Since \( \varphi \mathcal{R} \varepsilon \) we have that \( \varphi \mathcal{R} \varepsilon_n \). This completes the proof of the statement.

Let \( \iota \) be an arbitrary idempotent of the semigroup \( \mathcal{S}_{\lambda}^{\infty} \) such that \( \iota \in \downarrow \varphi \). We put \( \omega_0 = \varepsilon \cdot \iota \). Then by previous part of the proof we have that \( \omega_0 \mathcal{R} \varphi \) and hence by Lemma \[3.1\] we get \( \iota \mathcal{R} \varphi \).

(ii) Let \( \chi \) be an arbitrary idempotent of the semigroup \( \mathcal{S}_{\lambda}^{\infty} \) such that \( \varphi \neq \chi \) and \( |\lambda \setminus \text{dom} \varphi| = |\lambda \setminus \text{dom} \chi| \). Then \( \varepsilon \cdot \chi \leq \varphi \) and hence by statement (i) we get that \( (\varepsilon \cdot \chi) \mathcal{R} \varphi \). Since \( |\lambda \setminus \text{dom} \varphi| = |\lambda \setminus \text{dom} \chi| \) we conclude that \( |\text{dom} \varphi \setminus \text{dom}(\varepsilon \cdot \chi)| = |\text{dom} \chi \setminus \text{dom}(\varepsilon \cdot \chi)| \). Let be \( \{x_1, \ldots, x_k\} = \text{dom} \varphi \setminus \text{dom}(\varepsilon \cdot \chi) \) and \( \{y_1, \ldots, y_k\} = \text{dom} \chi \setminus \text{dom}(\varepsilon \cdot \chi) \). We put
\[
\alpha = \left( \begin{array}{ccc} x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \end{array} \right) \text{dom}(\varepsilon \cdot \chi),
\]
Then \( \alpha^{-1} \cdot \varepsilon \cdot \alpha = \chi \) and \( \alpha^{-1} \cdot (\varepsilon \cdot \chi) \cdot \alpha = \varepsilon \cdot \chi \). Therefore we get that \( (\varepsilon \cdot \chi) \mathcal{R} \chi \) and hence \( \varphi \mathcal{R} \chi \). This completes the proof of our statement.

**Theorem 3.3.** Let \( \mathcal{R} \) be an arbitrary congruence on the semigroup \( \mathcal{S}_{\lambda}^{\infty} \) and \( \varepsilon \) and \( \varphi \) be distinct \( \mathcal{R} \)-equivalent idempotents of \( \mathcal{S}_{\lambda}^{\infty} \). Then \( \alpha \mathcal{R} \varepsilon \) for every \( \alpha \in \mathcal{S}_{\lambda}^{\infty} \) such that \( |\lambda \setminus \text{dom} \alpha| \geq \min \{|\lambda \setminus \text{dom} \varphi|, |\lambda \setminus \text{dom} \varepsilon|\} \).

**Proof.** In the case when \( \alpha \) is an idempotent of the semigroup \( \mathcal{S}_{\lambda}^{\infty} \) the statement of the theorem follows from Lemma \[3.1\] and Proposition \[3.2\].

Suppose that \( \alpha \) is an arbitrary non-idempotent element of the semigroup \( \mathcal{S}_{\lambda}^{\infty} \) such that \( |\lambda \setminus \text{dom} \alpha| \geq \max \{|\lambda \setminus \text{dom} \varphi|, |\lambda \setminus \text{dom} \varepsilon|\} \). Since \( \mathcal{S}_{\lambda}^{\infty} \) is an inverse semigroup we have that \( \alpha \cdot \alpha^{-1} \cdot \alpha = \alpha \) and Propositions \[3.1\] and \[3.2\] imply that
\[
|\lambda \setminus \text{dom} \alpha| = |\lambda \setminus \text{dom} \alpha^{-1}| = |\lambda \setminus \text{dom}(\alpha \cdot \alpha^{-1})| = |\lambda \setminus \text{dom}(\alpha^{-1} \cdot \alpha)| \geq \min \{|\lambda \setminus \text{dom} \varphi|, |\lambda \setminus \text{dom} \varepsilon|\}.
\]
Hence \((\alpha \cdot \alpha^{-1}) \mathcal{R} \varepsilon \) and by Proposition \[3.2\] we have that \((\alpha \cdot \alpha^{-1}) \mathcal{R} \varepsilon \) for every idempotent \( \iota \) of the semigroup \( \mathcal{S}_{\lambda}^{\infty} \) such that \( \iota \in \downarrow \varepsilon \). Definition of the semigroup \( \mathcal{S}_{\lambda}^{\infty} \) implies that for every \( \alpha \in \mathcal{S}_{\lambda}^{\infty} \) there exists an idempotent \( \varsigma_{\alpha} \in \mathcal{S}_{\lambda}^{\infty} \) such that \( \alpha \cdot \varsigma = \varsigma \cdot \alpha = \varsigma \cdot (\alpha \cdot \alpha^{-1}) = \varsigma \) for all idempotents \( \varsigma \in \mathcal{S}_{\lambda}^{\infty} \) such that \( \varsigma \in \downarrow \varsigma_{\alpha} \). Let \( \nu = \varsigma_{\alpha} \cdot \varepsilon \). Then \((\alpha \cdot \alpha^{-1}) \mathcal{R} \nu \) and \( \alpha \cdot \nu = \nu \cdot \alpha = \nu \cdot (\alpha \cdot \alpha^{-1}) = \nu \). Therefore we get
\[
(\alpha) \Phi_{\mathcal{R}} = (\alpha \cdot \alpha^{-1}) \Phi_{\mathcal{R}} = (\alpha \cdot \alpha^{-1}) \Phi_{\mathcal{R}} \cdot (\alpha) \Phi_{\mathcal{R}} = (\nu) \Phi_{\mathcal{R}} \cdot (\alpha) \Phi_{\mathcal{R}} = (\nu \cdot \alpha) \Phi_{\mathcal{R}} = (\nu) \Phi_{\mathcal{R}}
\]
and \( \alpha \mathcal{R} \nu \). Hence we have that \( \alpha \mathcal{R} \varepsilon \).

**Proposition 3.4.** Let \( \mathcal{R} \) be an arbitrary congruence on the semigroup \( \mathcal{S}_{\lambda}^{\infty} \). Let \( \varepsilon \) be an idempotent of \( \mathcal{S}_{\lambda}^{\infty} \) such that \( |\lambda \setminus \text{dom} \varepsilon| \geq 1 \) and the following conditions hold:

(i) there exists an idempotent \( \varphi \in \mathcal{S}_{\lambda}^{\infty} \) such that \( \varepsilon \mathcal{R} \varphi \) and \( |\lambda \setminus \text{dom} \varphi| \geq |\lambda \setminus \text{dom} \varepsilon| \); and

(ii) does not exist an idempotent \( \psi \in \mathcal{S}_{\lambda}^{\infty} \) such that \( \varepsilon \mathcal{R} \psi \) and \( |\lambda \setminus \text{dom} \psi| < |\lambda \setminus \text{dom} \varepsilon| \).

Then there exists no element \( \alpha \) of the semigroup \( \mathcal{S}_{\lambda}^{\infty} \) such that \( \varepsilon \mathcal{R} \alpha \) and \( |\lambda \setminus \text{dom} \alpha| < |\lambda \setminus \text{dom} \varepsilon| \).

**Proof.** Suppose to the contrary that there exists \( \alpha \in \mathcal{S}_{\lambda}^{\infty} \) such that \( \varepsilon \mathcal{R} \alpha \) and \( |\lambda \setminus \text{dom} \alpha| < |\lambda \setminus \text{dom} \varepsilon| \). Since \( \mathcal{S}_{\lambda}^{\infty} \) is an inverse semigroup Lemma III.1.1 \[21\] implies that \( \varepsilon \mathcal{R} \alpha^{-1} \) and hence \( \varepsilon \mathcal{R} (\alpha \cdot \alpha^{-1}) \). But \( |\lambda \setminus \text{dom}(\alpha \cdot \alpha^{-1})| = |\lambda \setminus \text{dom} \alpha| < |\lambda \setminus \text{dom} \varepsilon| \), a contradiction. An obtained contradiction implies the statement of the proposition.

**Proposition 3.5.** Let \( \mathcal{R} \) be an arbitrary congruence on the semigroup \( \mathcal{S}_{\lambda}^{\infty} \). Let \( \alpha \) and \( \beta \) be non-\( \mathcal{H} \)-equivalent elements of \( \mathcal{S}_{\lambda}^{\infty} \) such that \( \alpha \mathcal{R} \beta \). Then \( \gamma \mathcal{R} \alpha \) for all \( \gamma \in \mathcal{S}_{\lambda}^{\infty} \) such that
\[
|\lambda \setminus \text{dom} \gamma| \geq \min \{|\lambda \setminus \text{dom} \alpha|, |\lambda \setminus \text{dom} \beta|\}.
\]
Proof. Since $\alpha$ and $\beta$ are non-$\mathcal{H}$-equivalent elements of the inverse semigroup $\mathcal{I}_\lambda^\infty$ we conclude that at least one of the following conditions holds:

(i) $\alpha \cdot \alpha^{-1} \neq \beta \cdot \beta^{-1}$;
(ii) $\alpha^{-1} \cdot \alpha \neq \beta^{-1} \cdot \beta$.

Suppose that the case $\alpha \cdot \alpha^{-1} \neq \beta \cdot \beta^{-1}$ holds. In the other case the proof is similar. Since $\mathcal{I}_\lambda^\infty$ is an inverse semigroup Lemma III.1.1 [21] implies that $\beta^{-1} \mathcal{R} \alpha^{-1}$ and hence $(\beta \cdot \beta^{-1}) \mathcal{R} (\alpha \cdot \alpha^{-1})$. Then we have that

$$|\lambda \setminus \text{dom } \alpha| = |\lambda \setminus \text{dom}(\alpha \cdot \alpha^{-1})| \quad \text{and} \quad |\lambda \setminus \text{dom } \beta| = |\lambda \setminus \text{dom}(\beta \cdot \beta^{-1})|$$

and hence the assumptions of the Theorem 3.3 hold. This completes the proof of the proposition. $\Box$

**Proposition 3.6.** Let $\mathcal{R}$ be an arbitrary congruence on the semigroup $\mathcal{I}_\lambda^\infty$. If $\alpha$ and $\beta$ are distinct $\mathcal{H}$-equivalent elements of $\mathcal{I}_\lambda^\infty$ such that $\alpha \mathcal{R} \beta$, then $\gamma \mathcal{R} \alpha$ for all $\gamma \in \mathcal{I}_\lambda^\infty$ such that

$$|\lambda \setminus \text{dom } \gamma| > |\lambda \setminus \text{dom } \alpha|.$$

Proof. Since $\mathcal{I}_\lambda^\infty$ is an inverse semigroup Theorem 2.20 [5] and Proposition 2.2 (viii) imply that without loss of generality we can assume that $\alpha$ and $\beta$ are elements of a maximal subgroup $H(\varepsilon)$ of $\mathcal{I}_\lambda^\infty$ with unity $\varepsilon$. Since $(\alpha \cdot \alpha^{-1}) \mathcal{R} (\beta \cdot \beta^{-1})$ we can assume that $\alpha$ is an identity of the subgroup $H(\varepsilon)$. Let $x \in \text{dom } \alpha$ such that $(\alpha \beta) x \neq x$. We put $\varepsilon_1 : \text{dom } \alpha \setminus \{x\} \to \text{dom } \alpha \setminus \{x\}$ be an identity map. Then $\varepsilon_1 \cdot \alpha = \varepsilon_1$ and $\text{ran}(\varepsilon_1 \cdot \beta) \neq \text{ran}(\varepsilon_1)$. Therefore by Proposition 2.2 (vii) we get that the elements $\varepsilon_1$ and $\varepsilon_1 \cdot \beta$ are not $\mathcal{H}$-equivalent. Since $|\lambda \setminus \text{dom } \varepsilon_1| = |\lambda \setminus \text{dom}(\varepsilon_1 \cdot \beta)|$ we have that the assumptions of Proposition 3.5 hold. This completes the proof of the proposition. $\Box$

Theorem 3.3 and Propositions 3.4, 3.5 and 3.6 imply the following proposition:

**Proposition 3.7.** Let $\mathcal{R}$ be an arbitrary congruence on the semigroup $\mathcal{I}_\lambda^\infty$. Let $\alpha$ and $\beta$ be distinct $\mathcal{H}$-equivalent elements of $\mathcal{I}_\lambda^\infty$ such that $\alpha \mathcal{R} \gamma$ and $|\lambda \setminus \text{dom } \gamma| < |\lambda \setminus \text{dom } \alpha|$. Then elements $\mu, \nu \in \mathcal{I}_\lambda^\infty$ with $|\lambda \setminus \text{dom } \mu| < |\lambda \setminus \text{dom } \alpha|$ and $|\lambda \setminus \text{dom } \nu| < |\lambda \setminus \text{dom } \alpha|$ are $\mathcal{R}$-equivalent if and only if $\mu = \nu$.

**Definition 3.8.** For every non-negative integer $n$ we denote by $\mathfrak{K}_n(I)$ the congruence on the semigroup $\mathcal{I}_\lambda^\infty$ generated by the ideal $I_n$, i.e., $\mathfrak{K}_n(I) = (I_n \times I_n) \cup \Delta(\mathcal{I}_\lambda^\infty)$. We observe that $\mathfrak{K}_0(I) = \Omega(\mathcal{I}_\lambda^\infty)$.

**Remark 3.9.** The group $S_\lambda^\infty(\lambda)$ has only one non-trivial normal subgroup: that is a group $A_\lambda^\infty(\lambda)$ of all even permutations of the set $\lambda$ (see [10], pp. 313–314, Example) or [18]. Therefore every non-trivial homomorphism of $S_\lambda^\infty(\lambda)$ is either an isomorphism or its image is a two-elements cyclic group.

**Definition 3.10.** Fix an arbitrary non-negative integer $n$. We shall say that elements $\alpha$ and $\beta$ of the semigroup $\mathcal{I}_\lambda^\infty$ are $n_{S_\lambda^\infty}$-equivalent if the following conditions hold:

(i) $\alpha \mathcal{H} \beta$; and
(ii) $|\lambda \setminus \text{dom } \alpha| = |\lambda \setminus \text{dom } \beta| = n$.

We define a relation $\mathfrak{K}_n(S_\lambda^\infty)$ on the semigroup $\mathcal{I}_\lambda^\infty$ as follows:

$$\mathfrak{K}_n(S_\lambda^\infty) = \{ (\alpha, \beta) \mid (\alpha, \beta) \in n_{S_\lambda^\infty} \} \cup (I_{n+1} \times I_{n+1}) \cup \Delta(\mathcal{I}_\lambda^\infty).$$

Simple verifications show that so defined relation $\mathfrak{K}_n(S_\lambda^\infty)$ on $\mathcal{I}_\lambda^\infty$ is an equivalence relation for every non-negative integer $n$.

**Proposition 3.11.** The relation $\mathfrak{K}_n(S_\lambda^\infty)$ is a congruence on the semigroup $\mathcal{I}_\lambda^\infty$.

Proof. First we consider the case when $n = 0$. If $\alpha$ and $\beta$ are distinct elements of the semigroup $\mathcal{I}_\lambda^\infty$ such that $\alpha \mathfrak{K}_0(S_\lambda^\infty) \beta$, then either $\alpha, \beta \in H(I)$ or $\alpha, \beta \in I_1$. Suppose that $\alpha, \beta \in H(I)$. Then for every $\gamma \in \mathcal{I}_\lambda^\infty$ we have that either $\alpha \cdot \gamma, \beta \cdot \gamma \in H(I)$ or $\alpha \cdot \gamma, \beta \cdot \gamma \in I_1$, and similarly we get that either $\gamma \cdot \alpha, \gamma \cdot \beta \in H(I)$ or $\gamma \cdot \alpha, \gamma \cdot \beta \in I_1$. If $\alpha, \beta \in I_1$, then for every $\gamma \in \mathcal{I}_\lambda^\infty$ we have that $\alpha \cdot \gamma, \beta \cdot \gamma, \alpha \cdot \gamma, \beta \cdot \gamma \in I_1$. Therefore $\mathfrak{K}_0(S_\lambda^\infty)$ is a congruence on the semigroup $\mathcal{I}_\lambda^\infty$.

Suppose that $n$ is an arbitrary positive integer. Let $\alpha$ and $\beta$ be distinct elements of the semigroup $\mathcal{I}_\lambda^\infty$ such that $\alpha \mathfrak{K}_n(S_\lambda^\infty) \beta$. The definition of the relation $\mathfrak{K}_n(S_\lambda^\infty)$ implies that only one of the following conditions holds:
First we consider the case when \( n \) is an arbitrary positive integer. Let \( \gamma \) be an arbitrary element of the semigroup \( \mathcal{I}_\lambda^\infty \). We consider two cases:

- \( a) \) dom \( \alpha \subseteq \text{ran} \gamma; \)
- \( b) \) dom \( \alpha \nsubseteq \text{ran} \gamma. \)

Since the elements \( \alpha \) and \( \beta \) are \( \mathcal{H} \)-equivalent in \( \mathcal{I}_\lambda^\infty \) Proposition 2.2 (viii) implies that in case \( a) \) we have that \( \text{dom}(\gamma \cdot \alpha) = \text{dom}(\gamma \cdot \beta) \) and \( \text{ran}(\gamma \cdot \alpha) = \text{ran}(\gamma \cdot \beta). \) Then again by Proposition 2.2 (vii) the elements \( \gamma \cdot \alpha \) and \( \gamma \cdot \beta \) are \( \mathcal{H} \)-equivalent in \( \mathcal{I}_\lambda^\infty \). Since \( \text{dom} \alpha \subseteq \text{ran} \gamma \) we get that \( |\lambda \setminus \text{dom}(\gamma \cdot \alpha)| = |\lambda \setminus \text{dom}(\gamma \cdot \beta)| = n. \) Hence we obtain that \((\gamma \cdot \alpha)\mathfrak{R}_n(S_\infty)(\gamma \cdot \beta). \) In case \( b) \) we have that \( \gamma \cdot \alpha, \gamma \cdot \beta \in I_{n+1} \) and hence \((\gamma \cdot \alpha)\mathfrak{R}_n(S_\infty)(\gamma \cdot \beta). \)

The proof the assertion that \( \alpha \mathfrak{R}_n(S_\infty) \beta \) implies \((\alpha \cdot \delta)\mathfrak{R}_n(S_\infty)(\beta \cdot \delta) \) for every \( \delta \in \mathcal{I}_\lambda^\infty \) is similar.

Suppose that \( \lambda \setminus \text{dom} \alpha > n \) and \( |\lambda \setminus \text{dom} \beta| > n. \) Then \( \alpha, \beta \in I_{n+1}. \) By Proposition 2.2 (x) we have that \( \gamma \cdot \alpha, \gamma \cdot \beta, \alpha \cdot \delta, \beta \cdot \delta \in I_{n+1} \) and hence \((\gamma \cdot \alpha)\mathfrak{R}_n(S_\infty)(\gamma \cdot \beta) \) and \((\alpha \cdot \delta)\mathfrak{R}_n(S_\infty)(\beta \cdot \delta) \) for all \( \gamma, \delta \in \mathcal{I}_\lambda^\infty. \) This completes the proof of the proposition.

**Definition 3.12.** Fix an arbitrary non-negative integer \( n. \) We shall say that elements \( \alpha \) and \( \beta \) of the semigroup \( \mathcal{I}_\lambda^\infty \) are \( n_\infty \)-equivalent if the following conditions hold:

(i) \( \alpha \mathcal{H} \beta; \)

(ii) \( \alpha \cdot \beta^{-1} \) is an even permutation of the set \( \text{dom} \alpha; \)

(iii) \( |\lambda \setminus \text{dom} \alpha| = |\lambda \setminus \text{dom} \beta| = n. \)

We define a relation \( \mathfrak{R}_n(A_\infty) \) on the semigroup \( \mathcal{I}_\lambda^\infty \) as follows:

\[ \mathfrak{R}_n(A_\infty) = \{(\alpha, \beta) \mid (\alpha, \beta) \in n_\infty \} \cup (I_{n+1} \times I_{n+1}) \cup \Delta(\mathcal{I}_\lambda^\infty). \]

Simple verifications show that so defined relation \( \mathfrak{R}_n(A_\infty) \) on \( \mathcal{I}_\lambda^\infty \) is an equivalence relation for every non-negative integer \( n. \)

**Proposition 3.13.** The relation \( \mathfrak{R}_n(A_\infty) \) is a congruence on the semigroup \( \mathcal{I}_\lambda^\infty. \)

**Proof.** First we consider the case when \( n = 0. \) If \( \alpha \) and \( \beta \) are distinct elements of the semigroup \( \mathcal{I}_\lambda^\infty \) such that \( \alpha \mathfrak{R}_0(S_\infty) \beta, \) then either \( \alpha, \beta \in H(I) \) or \( \alpha, \beta \in I_1. \) Suppose that \( \alpha, \beta \in H(I). \) Then for every \( \gamma \in H(I) \) we have that \( \alpha \cdot \gamma, \beta \cdot \gamma, \gamma \cdot \alpha, \gamma \cdot \beta \in H(I). \) Then \( (\alpha \cdot \gamma) \cdot (\beta \cdot \gamma)^{-1} = \alpha \cdot \gamma \cdot \gamma^{-1} \cdot \beta^{-1} = \alpha \cdot \beta^{-1} \) is an even permutation of the set \( \lambda. \) Also, since \( \alpha \cdot \beta^{-1} \) is an even permutation of the set \( \lambda \) we get that \((\gamma \cdot \alpha) \cdot (\gamma \cdot \beta)^{-1} = \gamma \cdot \alpha \cdot \beta^{-1} \cdot \gamma^{-1} \) is an even permutation of the set \( \lambda \) too. For every \( \gamma \in I_1 \) we have that \( \alpha \cdot \gamma, \beta \cdot \gamma, \gamma \cdot \alpha, \gamma \cdot \beta \in I_1. \) If \( \alpha, \beta \in I_1 \) then for every \( \gamma \in \mathcal{I}_\lambda^\infty \) we have that \( \alpha \cdot \gamma, \beta \cdot \gamma, \gamma \cdot \alpha, \gamma \cdot \beta \in I_1. \) Therefore \( \mathfrak{R}_0(A_\infty) \) is a congruence on the semigroup \( \mathcal{I}_\lambda^\infty. \)

Suppose that \( n \) is an arbitrary positive integer. Let \( \alpha \) and \( \beta \) be distinct elements of the semigroup \( \mathcal{I}_\lambda^\infty \) such that \( \alpha \mathfrak{R}_n(A_\infty) \beta. \) The definition of the relation \( \mathfrak{R}_n(A_\infty) \) implies that only one of the following conditions holds:

(i) \( |\lambda \setminus \text{dom} \alpha| = |\lambda \setminus \text{dom} \beta| = n; \) or

(ii) \( |\lambda \setminus \text{dom} \alpha| > n \) and \( |\lambda \setminus \text{dom} \beta| > n. \)

First we suppose that \( |\lambda \setminus \text{dom} \alpha| = |\lambda \setminus \text{dom} \beta| = n. \) Let \( \gamma \) be an arbitrary element of the semigroup \( \mathcal{I}_\lambda^\infty. \) We consider two cases:

- \( a) \) dom \( \alpha \subseteq \text{ran} \gamma; \)
- \( b) \) dom \( \alpha \nsubseteq \text{ran} \gamma. \)

Suppose case \( a) \) holds. Since the elements \( \alpha \) and \( \beta \) are \( \mathcal{H} \)-equivalent in \( \mathcal{I}_\lambda^\infty \) we have that Proposition 2.2 (viii) implies that \( \text{dom}(\gamma \cdot \alpha) = \text{dom}(\gamma \cdot \beta) \) and \( \text{ran}(\gamma \cdot \alpha) = \text{ran}(\gamma \cdot \beta). \) Then again by Proposition 2.2 (vii) the elements \( \gamma \cdot \alpha \) and \( \gamma \cdot \beta \) are \( \mathcal{H} \)-equivalent in \( \mathcal{I}_\lambda^\infty. \) Since \( \text{dom} \alpha \subseteq \text{ran} \gamma \) we get that \( |\lambda \setminus \text{dom}(\gamma \cdot \alpha)| = |\lambda \setminus \text{dom}(\gamma \cdot \beta)| = n. \) We define a partial map \( \gamma_1 : \lambda \to \lambda \) as follows \( \gamma_1 = \gamma_{(\text{dom} \alpha)}^{-1} : (\text{dom} \alpha)^{-1} \to \text{dom} \alpha. \) Then we get that \( |\lambda \setminus \text{dom} \gamma_1| = |\lambda \setminus \text{dom} \alpha| = |\lambda \setminus \text{dom} \beta| = n, \)

\( \gamma \cdot \alpha = \gamma_1 \cdot \alpha, \gamma \cdot \beta = \gamma_1 \cdot \beta \) and hence \( (\gamma \cdot \alpha) \cdot (\gamma \cdot \beta)^{-1} = (\gamma_1 \cdot \alpha) \cdot (\gamma_1 \cdot \beta)^{-1} = \gamma_1 \cdot \alpha \cdot \beta^{-1} \cdot \gamma_1^{-1}. \)

Since \( \alpha \cdot \beta^{-1} \) is an even permutation of the set \( \text{dom} \alpha \) we conclude that \( \gamma_1 \cdot \alpha \cdot \beta^{-1} \cdot \gamma_1^{-1} \) is an even
permutation of the set \( \text{dom} \gamma_1 = (\text{dom} \alpha) \gamma^{-1} \). Hence we obtain that \((\gamma \cdot \alpha)\mathcal{R}_n(A_{\infty})(\gamma \cdot \beta)\). In case \(b\) we have that \(\gamma \cdot \alpha, \gamma \cdot \beta \in I_{n+1}\) and hence \((\gamma \cdot \alpha)\mathcal{R}_n(A_{\infty})(\gamma \cdot \beta)\).

The proof the assertion that \(\alpha \mathcal{R}_n(A_{\infty}) \beta\) implies \((\alpha \cdot \delta)\mathcal{R}_n(A_{\infty})(\beta \cdot \delta)\) for every \(\delta \in \mathcal{I}_\lambda\) is similar.

Suppose that \(|\lambda \setminus \text{dom} \alpha| > n\) and \(|\lambda \setminus \text{dom} \beta| > n\). Then \(\alpha, \beta \in I_{n+1}\). By Proposition \(3.5\) \((x)\) we have that \(\gamma \cdot \alpha, \gamma \cdot \beta, \alpha \cdot \delta, \beta \cdot \delta \in I_{n+1}\) and hence \((\gamma \cdot \alpha)\mathcal{R}_n(A_{\infty})(\gamma \cdot \beta)\) and \((\alpha \cdot \delta)\mathcal{R}_n(A_{\infty})(\beta \cdot \delta)\), for all \(\gamma, \delta \in \mathcal{I}_\lambda\). This completes the proof of the proposition.

\[\square\]

**Theorem 3.14.** The family

\[
\text{Cong}(\mathcal{I}_\lambda) = \{\Delta(\mathcal{I}_\lambda), \Omega(\mathcal{I}_\lambda)\} \cup \{\mathcal{R}_n(S_{\infty}) \mid n = 0, 1, 2, \ldots\} \cup \{\mathcal{R}_n(A_{\infty}) \mid n = 0, 1, 2, \ldots\} \cup
\[
\cup \{\mathcal{R}_n(I_n) \mid n = 1, 2, \ldots\}
\]

determines all congruences on the semigroup \(\mathcal{I}_\lambda\).

**Proof.** Let \(\mathcal{R}\) be non-identity congruence on the semigroup \(\mathcal{I}_\lambda\). Since the set of all non-negative integers with respect to the usual order \(\leq\) is well ordered there exists a minimal non-negative integer \(n\) such that there are two distinct elements \(\alpha\) and \(\beta\) in \(\mathcal{I}_\lambda\) such that \(\alpha \mathcal{R} \beta\) and

\[
\min \{|\lambda \setminus \text{dom} \alpha|, |\lambda \setminus \text{dom} \beta|\} = n,
\]
i.e., for any non-negative integer \(m < n\) if for \(\alpha\) and \(\beta\) in \(\mathcal{I}_\lambda\) such that \(\alpha \mathcal{R} \beta\) and

\[
\min \{|\lambda \setminus \text{dom} \alpha|, |\lambda \setminus \text{dom} \beta|\} = m
\]
then \(\alpha = \beta\).

We consider two cases:

(i) \(|\lambda \setminus \text{dom} \alpha| \neq |\lambda \setminus \text{dom} \beta|\); and

(ii) \(|\lambda \setminus \text{dom} \alpha| = |\lambda \setminus \text{dom} \beta|\).

Suppose case (i) holds and \(|\lambda \setminus \text{dom} \alpha| = n < |\lambda \setminus \text{dom} \beta|\). Then \(\alpha\) and \(\beta\) are not \(\mathcal{H}\)-equivalent elements in \(\mathcal{I}_\lambda\) and hence by Proposition \(3.5\) we obtain that \(\alpha \mathcal{R} \gamma\) for all \(\gamma \in \mathcal{I}_\lambda\) with \(|\lambda \setminus \text{dom} \gamma| \geq n\). Then Proposition \(3.7\) implies that \(\mu \mathcal{R} \nu\) if and only if \(\mu = \nu\) for all elements \(\mu, \nu \in \mathcal{I}_\lambda\) such that \(|\lambda \setminus \text{dom} \mu| < n\) and \(|\lambda \setminus \text{dom} \nu| < n\). Hence we get that \(\mathcal{R} = \mathcal{R}_n(I)\). We observe if \(n = 0\) then \(\mathcal{R} = \Omega(\mathcal{I}_\lambda)\).

We henceforth assume that case (ii) holds.

If \(\alpha\) and \(\beta\) are not \(\mathcal{H}\)-equivalent elements in \(\mathcal{I}_\lambda\) and then by Proposition \(3.5\) we have that \(\alpha \mathcal{R} \gamma\) for all \(\gamma \in \mathcal{I}_\lambda\) such that \(|\lambda \setminus \text{dom} \gamma| \geq n\). Then Proposition \(3.7\) implies that \(\mu \mathcal{R} \nu\) if and only if \(\mu = \nu\) for all elements \(\mu, \nu \in \mathcal{I}_\lambda\) such that \(|\lambda \setminus \text{dom} \mu| < n\) and \(|\lambda \setminus \text{dom} \nu| < n\), and hence we have that \(\mathcal{R} = \mathcal{R}_n(I)\). Also in this case if \(n = 0\) then \(\mathcal{R} = \Omega(\mathcal{I}_\lambda)\).

Suppose that \(\alpha\) and \(\beta\) are \(\mathcal{H}\)-equivalent elements in \(\mathcal{I}_\lambda\) and there exists no non-\(\mathcal{H}\)-equivalent element \(\delta\) of the semigroup \(\mathcal{I}_\lambda\) such that \(\alpha \mathcal{R} \delta\). Otherwise by the previous part of the proof we have that \(\mathcal{R} = \mathcal{R}_n(I)\). Since \((\alpha \cdot \alpha^{-1})\mathcal{R}(\beta \cdot \alpha^{-1})\) we conclude that without loss of generality we can assume that \(\alpha\) is an identity element of \(\mathcal{H}\)-class \(H(\alpha)\) which contains \(\alpha\) and \(\beta \neq \alpha\). Since \(\alpha\) is an idempotent of the semigroup \(\mathcal{I}_\lambda\) we have that \(\text{dom} \alpha = \text{ran} \alpha\) and the restriction \(\alpha|_{\text{dom} \alpha}: \text{dom} \alpha \to \text{dom} \alpha\) is a permutation of the set \(\text{dom} \alpha\). Therefore without loss of generality we can consider \(\beta\) as a permutation of the set \(\text{dom} \alpha\).

We consider two cases:

(1) \(\beta\) is an odd permutation of the set \(\text{dom} \alpha\); and

(2) \(\beta\) is an even permutation of the set \(\text{dom} \alpha\).

Suppose that \(\beta\) is an odd permutation of the set \(\text{dom} \alpha\). Since \(H(\alpha)\) is a subgroup of the semigroup \(\mathcal{I}_\lambda\) we conclude that the image \((H(\alpha))\Phi_\mathcal{R}\) of \(H(\alpha)\) is a subgroup in \(\mathcal{I}_\lambda/\mathcal{R}\). Since the subgroup \(H(\alpha)\) is isomorphic to the group \(S_\infty(\lambda)\) and the group of all even permutations \(A_\infty(\lambda)\) of the set \(\lambda\) is a unique normal subgroup in \(S_\infty(\lambda)\) (see \([10]\), pp. 313–314, Example] or \([18]\)) we conclude that the image \((H(\alpha))\Phi_\mathcal{R}\) is singleton. Then by Theorem 2.20 \([5]\) and Proposition 2.22 \((viii)\) for every \(\gamma \in \mathcal{I}_\lambda\)
with $|\lambda \setminus \text{dom} \gamma| = |\lambda \setminus \text{dom} \alpha|$ the image $(H_\gamma)\Phi_R$ of the $\mathcal{H}$-class $H_\gamma$ which contains the element $\gamma$ is singleton and hence by Propositions 3.5, 3.6 and 3.7 we get that $R = \mathcal{R}_n(S_\infty)$.

Suppose that $\beta$ is an even permutation of the set $\text{dom} \alpha$. If the subgroup $H(\alpha)$ contains an odd permutation $\delta$ of the set $\text{dom} \alpha$ then by previous proof we get that $R = \mathcal{R}_n(S_\infty)$. Suppose the subgroup $H(\alpha)$ does not contain an odd permutation $\delta$ of the set $\text{dom} \alpha$. Since the subgroup $H(\alpha)$ is isomorphic to the group $S_\infty(\lambda)$ and the group of all even permutations $A_\infty(\lambda)$ of the set $\lambda$ is a unique normal subgroup in $S_\infty(\lambda)$ we conclude that the image $(H(\alpha))\Phi_R$ is a two-element subgroup in $\mathcal{F}_\lambda^{\infty}/R$. Then by Theorem 2.20 and Proposition 3.22 (viii) for every $\gamma \in \mathcal{F}_\lambda^{\infty}$ with $|\lambda \setminus \text{dom} \gamma| = |\lambda \setminus \text{dom} \alpha|$ the image $(H_\gamma)\Phi_R$ of the $\mathcal{H}$-class $H_\gamma$ which contains the element $\gamma$ is a two-element subset in $\mathcal{F}_\lambda^{\infty}/R$ and hence by Propositions 3.5, 3.6 and 3.7 we get that $R = \mathcal{R}_n(A_\infty)$. □

4. On topologizations of the free semilattice $(\mathcal{P}_{<\omega}(\lambda), \cup)$

Definition 4.1 ([7]). We shall say that a semigroup $S$ has the $F$-property if for every $a, b, c, d \in S^1$ the sets $\{x \in S \mid a \cdot x = b\}$ and $\{x \in S \mid x \cdot c = d\}$ are finite or empty.

Recall [7] an element $x$ of a semitopological semilattice $S$ is a local minimum if there exists an open neighbourhood $U(x)$ of $x$ such that $U(x) \cap \downarrow x = \{x\}$. This is equivalent to statement that $\uparrow x$ is an open subset in $S$.

A topological space $X$ is called Baire if for each sequence $A_1, A_2, \ldots, A_i, \ldots$ of nowhere dense subsets of $X$ the union $\bigcup_{i=1}^{\infty} A_i$ is a co-dense subset of $X$ [7]. A Tychonoff space $X$ is called Čech complete if for every compactification $cX$ of $X$ the remainder $cX \setminus c(X)$ is an $F_\sigma$-set in $cX$ [7].

A topological space $X$ is called hereditary Baire if every closed subset of $X$ is a Baire space [7]. Every Čech complete (and hence locally compact) space is hereditary Baire (see [7] Theorem 3.9.6). We shall say that a Hausdorff semitopological semigroup $S$ is an $I$-Baire space if for every $s \in S$ either $sS$ or $Ss$ is a Baire space [4].

Remark 4.2. We observe that every left ideal $SS$ and every right ideal $sS$ of a regular semigroup $S$ is generated by its idempotents. Therefore every principal left (right) ideal of a regular Hausdorff semitopological semigroup $S$ is a closed subset of $S$. Hence every regular Hausdorff hereditary Baire semitopological semigroup is a $I$-Baire space.

Theorem 4.3. Let $S$ be a semilattice with the $F$-property. Then every $I$-Baire topology $\tau$ on $S$ such that $(S, \tau)$ is a Hausdorff semitopological semilattice is discrete.

Proof. Let $x$ be an arbitrary element of the semilattice $S$. We need to show that $x$ is an isolated point in $(S, \tau)$.

Since $\tau$ is an $I$-Baire topology on $S$ we conclude that the subspace $\downarrow x$ is Baire. We denote $S_x = \downarrow x$.

For every positive integer $n$ we put

$$F_n = \{y \in S_x \mid |\uparrow y| = n\}.$$ 

Then we have that $S_x = \bigcup_{n=1}^{\infty} F_n$. Since the topological space $S_x$ is Baire we conclude that there exists $F_n \in \mathcal{F}$ such that $\text{Int}_{S_x}(F_n) \neq \emptyset$. We fix an arbitrary $y_0 \in \text{Int}_{S_x}(F_n)$. We observe that the definition of the family $\{F_n \mid n \in \mathbb{N}\}$ implies that for every non-empty subset $F_n$ and for any $s \in F_n$ the sets $\uparrow s \cap F_n$ and $\downarrow s \cap F_n$ are singleton. This implies that $y_0$ is a local minimum in $S_x$, i.e., $\uparrow y_0$ is an open subset of $S$. Since the semilattice $S_x$ has the $F$-property we conclude that the Hausdorffness of $S$ implies that $x$ is an isolated point in $S_x$. Then $x$ is a local minimum in $S$ and hence $\uparrow x$ is an open subset in $S$. Since the semilattice $S$ has the $F$-property we conclude that the Hausdorffness of $S$ implies that $x$ is an isolated point in $S$. □

Remark 4.4. We observe that the statement of Theorem 4.3 is true for a $T_1$-semitopological $I$-Baire semilattice with the $F$-property.

Since every Čech complete (and hence locally compact) space is hereditary Baire, Theorem 4.3 implies the following corollary:
Corollary 4.5. Let S be a semilattice with the F-property. Then every Čech complete (locally compact) topology \( \tau \) on S such that \((S, \tau)\) is a semitopological semilattice is discrete.

Since the free semilattice \((\mathcal{P}_{<\omega}(\lambda), \cup)\) has F-property, Theorem 4.3 implies the following corollary:

Corollary 4.6. Every Hausdorff I-Baire (Čech complete, locally compact) topology \( \tau \) on the free semilattice \( \mathcal{P}_{<\omega}(\lambda) \) such that \((\mathcal{P}_{<\omega}(\lambda), \tau)\) is a semitopological semilattice is discrete.

5. ON A TOPOLOGICAL SEMIGROUP \( \mathcal{I}_\infty \)

Theorem 5.1. Every hereditary Baire topology \( \tau \) on the semigroup \( \mathcal{I}_\infty \) such that \((\mathcal{I}_\infty, \tau)\) is a Hausdorff semitopological semigroup is discrete.

**Proof.** Let \( \alpha \) be an arbitrary element of the the semigroup \( \mathcal{I}_\infty \). We need to show that \( \alpha \) is an isolated point in \((\mathcal{I}_\infty, \tau)\).

For every non-negative integer \( n \) we denote \( C_n = \mathcal{I}_\infty \setminus I_{n+1} \).

By induction we shall prove that for every non-negative integer \( n \) the following statement holds: every \( \alpha \in C_n \) is an isolated point in \((\mathcal{I}_\infty, \tau)\).

First we shall show that our statement is true for \( n = 0 \). We define a family \( \mathcal{C} = \{\{\beta\} \mid \beta \in \mathcal{I}_\infty\} \).

Since the topological space \((\mathcal{I}_\infty, \tau)\) is Baire we have that the family \( \mathcal{C} \) has an element with non-empty interior and hence the topological space \((\mathcal{I}_\infty, \tau)\) has an isolated point \( \gamma \) in \((\mathcal{I}_\infty, \tau)\). Then \( |\omega \setminus \text{dom} \alpha| = 0 \) and hence statements (viii) – (xi) of Proposition 2.2 imply that there exist \( \mu, \nu \in \mathcal{I}_\infty \) such that \( \mu \cdot \alpha \cdot \nu = \gamma \). Since translations in \((\mathcal{I}_\infty, \tau)\) are continuous we conclude that Hausdorffness of the space \((\mathcal{I}_\infty, \tau)\) and Proposition 2.3 imply that \( \alpha \) is an isolated point in \((\mathcal{I}_\infty, \tau)\).

Suppose our statement is true for all \( n < k, \ k \in \mathbb{N} \). We shall show that its is true for \( n = k \). Our assumption implies that \( I_k \) is a closed subset of \((\mathcal{I}_\infty, \tau)\). Later we shall denote by \( \tau_k \) the topology induced from \((\mathcal{I}_\infty, \tau)\) onto \( I_k \). Then \((I_k, \tau_k)\) is a Baire space. We define a family \( \mathcal{C}_k = \{\{\beta\} \mid \beta \in I_k\} \).

Since the topological space \((I_k, \tau_k)\) is Baire we have that the family \( \mathcal{C}_k \) has an element with non-empty interior and hence the topological space \((I_k, \tau_k)\) has an isolated point \( \gamma \) in \((I_k, \tau_k)\). Let \( U(\gamma) \) be an open neighbourhood \( U(\gamma) \) of \( \gamma \) in \((\mathcal{I}_\infty, \tau)\) such that \( U(\gamma) \cap I_k = \{\gamma\} \). Since \((\mathcal{I}_\infty, \tau)\) is a semitopological semigroup we have that there exists an open neighbourhood \( V(\gamma) \) of \( \gamma \) in \((\mathcal{I}_\infty, \tau)\) such that \( V(\gamma) \subseteq U(\gamma) \) and \( \gamma \cdot \gamma^{-1} \cdot V(\gamma) \subseteq U(\gamma) \). We remark that \( \gamma \cdot \gamma^{-1} \cdot V(\gamma) \subseteq \{\gamma\} \). Hence by Proposition 2.3 the neighbourhood \( V(\gamma) \) is finite and Hausdorffness of the space \((\mathcal{I}_\infty, \tau)\) implies that \( \gamma \) is an isolated point in \((\mathcal{I}_\infty, \tau)\). Let \( \alpha \) be an arbitrary element of the set \( I_k \setminus I_{k+1} \). Then \( |\omega \setminus \text{dom} \alpha| = k \) and hence statements (viii) – (xi) of Proposition 2.2 imply that there exist \( \mu, \nu \in \mathcal{I}_\infty \) such that \( \mu \cdot \alpha \cdot \nu = \gamma \). Since translations in \((\mathcal{I}_\infty, \tau)\) are continuous we conclude that Hausdorffness of the space \((\mathcal{I}_\infty, \tau)\) and Proposition 2.3 imply that \( \alpha \) is an isolated point in \((\mathcal{I}_\infty, \tau)\). This completes the proof of our theorem.

**Remark 5.2.** We observe that the statement of Theorem 5.1 holds for every topology \( \tau \) on the semigroup \( \mathcal{I}_\infty \) such that \((\mathcal{I}_\infty, \tau)\) is a Hausdorff semitopological semigroup and every (two-sided) ideal in \((\mathcal{I}_\infty, \tau)\) is a Baire space.

Theorem 5.1 implies the following corollary:

Corollary 5.3. Every Čech complete (locally compact) topology \( \tau \) on the semigroup \( \mathcal{I}_\infty \) such that \((\mathcal{I}_\infty, \tau)\) is a Hausdorff semitopological semigroup is discrete.

Theorem 5.4. Let \( \lambda \) be an infinite cardinal and \( S \) be a topological semigroup which contains a dense discrete subsemigroup \( \mathcal{I}_\infty \). If \( I = S \setminus \mathcal{I}_\infty \neq \emptyset \) then \( I \) is an ideal of \( S \).

**Proof.** Suppose that \( I \) is not an ideal of \( S \). Then at least one of the following conditions holds:

1) \( I \cdot \mathcal{I}_\infty \notin I \), 2) \( \mathcal{I}_\infty \cdot I \notin I \), or 3) \( I \cdot I \notin I \).

Since \( \mathcal{I}_\infty \) is a discrete dense subspace of \( S \), Theorem 3.5.8 \[\] implies that \( \mathcal{I}_\infty \) is an open subspace of \( S \). Suppose there exist \( a \in \mathcal{I}_\infty \) and \( b \in I \) such that \( b \cdot a = c \notin I \). Since \( \mathcal{I}_\infty \) is a dense open discrete subspace of \( S \) the continuity of the semigroup operation in \( S \) implies that there exists an open
neighbourhood $U(b)$ of $b$ in $S$ such that $U(b) \cdot \{a\} = \{c\}$. But by Proposition 2.3 the equation $x \cdot a = c$ has finitely many solutions in $\mathcal{I}_\lambda^\infty$. This contradicts the assumption that $b \in S \setminus \mathcal{I}_\lambda^\infty$. Therefore $b \cdot a = c \in I$ and hence $I \cdot \mathcal{I}_\lambda^\infty \subseteq I$. The proof of the inclusion $\mathcal{I}_\lambda^\infty \cdot I \subseteq I$ is similar.

Suppose there exist $a, b \in I$ such that $a \cdot b = c \notin I$. Since $\mathcal{I}_\lambda^\infty$ is a dense open discrete subspace of $S$ the continuity of the semigroup operation in $S$ implies that there exist open neighbourhoods $U(a)$ and $U(b)$ of $a$ and $b$ in $S$, respectively, such that $U(a) \cdot U(b) = \{c\}$. But by Proposition 2.3 the equations $x \cdot b_0 = c$ and $a_0 \cdot y = c$ have finitely many solutions in $\mathcal{I}_\lambda^\infty$. This contradicts the assumption that $a, b \in S \setminus \mathcal{I}_\lambda^\infty$. Therefore $a \cdot b = c \in I$ and hence $I \cdot I \subseteq I$. □

**Proposition 5.5.** Let $S$ be a topological semigroup which contains a dense discrete subsemigroup $\mathcal{I}_\lambda^\infty$. Then for every $c \in \mathcal{I}_\lambda^\infty$ the set

$$D_c(\mathcal{I}_\lambda^\infty) = \{(x, y) \in \mathcal{I}_\lambda^\infty \times \mathcal{I}_\lambda^\infty \mid x \cdot y = c\}$$

is a closed-and-open subset of $S \times S$.

**Proof.** Since $\mathcal{I}_\lambda^\infty$ is a discrete subspace of $S$ we have that $D_c(\mathcal{I}_\lambda^\infty)$ is an open subset of $S \times S$.

Suppose that there exists $c \in \mathcal{I}_\lambda^\infty$ such that $D_c(\mathcal{I}_\lambda^\infty)$ is a non-closed subset of $S \times S$. Then there exists an accumulation point $(a, b) \in S \times S$ of the set $D_c(\mathcal{I}_\lambda^\infty)$. The continuity of the semigroup operation in $S$ implies that $a \cdot b = c$. But $\mathcal{I}_\lambda^\infty \times \mathcal{I}_\lambda^\infty$ is a discrete subspace of $S \times S$ and hence by Theorem 5.4 the points $a$ and $b$ belong to the ideal $I = S \setminus \mathcal{I}_\lambda^\infty$ and hence $a \cdot b \in S \setminus \mathcal{I}_\lambda^\infty$ cannot be equal to $c$. □

A topological space $X$ is defined to be pseudocompact if each locally finite open cover of $X$ is finite. According to [7, Theorem 3.10.22] a Tychonoff topological space $X$ is pseudocompact if and only if each continuous real-valued function on $X$ is bounded.

**Theorem 5.6.** If a topological semigroup $S$ contains $\mathcal{I}_\lambda^\infty$ as a dense discrete subsemigroup then the square $S \times S$ is not pseudocompact.

**Proof.** Since the square $S \times S$ contains an infinite closed-and-open discrete subspace $D_c(\mathcal{I}_\lambda^\infty)$, we conclude that $S \times S$ fails to be pseudocompact (see [7, Ex. 3.10.F(d)] or [6]). □

A topological space $X$ is called countably compact if any countable open cover of $X$ contains a finite subcover [7]. We observe that every Hausdorff countably compact space is pseudocompact.

Since the closure of an arbitrary subspace of a countably compact space is countably compact (see [7, Theorem 3.10.4]) Theorem 5.6 implies the following corollary:

**Corollary 5.7.** For every infinite cardinal $\lambda$ the discrete semigroup $\mathcal{I}_\lambda^\infty$ does not embed into a topological semigroup $S$ with the countably compact square $S \times S$.

Since every compact topological space is countably compact Theorem 3.24 [7] and Corollary 5.7 imply

**Corollary 5.8.** For every infinite cardinal $\lambda$ the discrete semigroup $\mathcal{I}_\lambda^\infty$ does not embed into a compact topological semigroup.

We recall that the Stone-Čech compactification of a Tychonoff space $X$ is a compact Hausdorff space $\beta X$ containing $X$ as a dense subspace so that each continuous map $f : X \to Y$ to a compact Hausdorff space $Y$ extends to a continuous map $\overline{f} : \beta X \to Y$ [7].

**Theorem 5.9.** For every infinite cardinal $\lambda$ the discrete semigroup $\mathcal{I}_\lambda^\infty$ does not embed into a Tychonoff topological semigroup $S$ with the pseudocompact square $S \times S$.

**Proof.** By Theorem 1.3 [1] for any topological semigroup $S$ with the pseudocompact square $S \times S$ the semigroup operation $\mu : S \times S \to S$ extends to a continuous semigroup operation $\beta \mu : \beta S \times \beta S \to \beta S$, so $S$ is a subsemigroup of the compact topological semigroup $\beta S$. Then Corollary 5.8 implies the statement of the theorem. □
The following example shows that there exists a non-discrete topology $\tau_F$ on the semigroup $\mathcal{I}_\infty^\times$ such that $(\mathcal{I}_\infty^\times, \tau_F)$ is a Tychonoff topological inverse semigroup.

**Example 5.10.** We define a topology $\tau_F$ on the semigroup $\mathcal{I}_\infty^\times$ as follows. For every $\alpha \in \mathcal{I}_\infty^\times$ we define a family

$$\mathcal{B}_F(\alpha) = \{ U_\alpha(F) \mid F \text{ is a finite subset of } \text{dom } \alpha \},$$

where

$$U_\alpha(F) = \{ \beta \in \mathcal{I}_\infty^\times \mid \text{dom } \alpha = \text{dom } \beta, \text{ran } \alpha = \text{ran } \beta \text{ and } (x)\beta = (x)\alpha \text{ for all } x \in F \}.$$ Since conditions (BP1)–(BP3) [7] hold for the family $\{ \mathcal{B}_F(\alpha) \}_{\alpha \in \mathcal{I}_\infty^\times}$ we conclude that the family $\{ \mathcal{B}_F(\alpha) \}_{\alpha \in \mathcal{I}_\infty^\times}$ is the base of the topology $\tau_F$ on the semigroup $\mathcal{I}_\infty^\times$.

**Proposition 5.11.** $(\mathcal{I}_\infty^\times, \tau_F)$ is a Tychonoff topological inverse semigroup.

**Proof.** Let $\alpha$ and $\beta$ be arbitrary elements of the semigroup $\mathcal{I}_\infty^\times$. We put $\gamma = \alpha \cdot \beta$ and let $F = \{ n_1, \ldots, n_i \}$ be a finite subset of dom $\gamma$. We denote $m_1 = (n_1)\alpha, \ldots, m_i = (n_i)\alpha$ and $k_1 = (n_1)\gamma, \ldots, k_i = (n_i)\gamma$. Then we get that $(m_1)\beta = k_1, \ldots, (m_i)\beta = k_i$. Hence we have that

$$U_\alpha(\{ n_1, \ldots, n_i \}) \cdot U_\beta(\{ m_1, \ldots, m_i \}) \subseteq U_\gamma(\{ n_1, \ldots, n_i \})$$

and

$$(U_\gamma(\{ n_1, \ldots, n_i \}))^{-1} \subseteq U_{\gamma^{-1}}(\{ k_1, \ldots, k_i \}).$$

Therefore the semigroup operation and the inversion are continuous in $(\mathcal{I}_\infty^\times, \tau_F)$.

We observe that the group of units $H(\mathbb{I})$ of the semigroup $\mathcal{I}_\infty^\times$ with the induced topology $\tau_F(H(\mathbb{I}))$ from $(\mathcal{I}_\infty^\times, \tau_F)$ is a topological group (see [10, pp. 313–314, Example] or [8, Theorem 8.4]) and the definition of the topology $\tau_F$ implies that every $\mathcal{H}$-class of the semigroup $\mathcal{I}_\infty^\times$ is an open-and-closed subset of the topological space $(\mathcal{I}_\infty^\times, \tau_F)$. Therefore Theorem 2.20 [5] implies that the topological space $(\mathcal{I}_\infty^\times, \tau_F)$ is homeomorphic to a countable topological sum of topological copies of $(H(\mathbb{I}), \tau_F(H(\mathbb{I})))$. Since every $T_0$-topological group is a Tychonoff topological space (see [22, Theorem 3.10] or [8, Theorem 8.4]) we conclude that the topological space $(\mathcal{I}_\infty^\times, \tau_F)$ is Tychonoff too. This completes the proof of the proposition. \(\square\)

**Remark 5.12.** We observe that the topology $\tau_F$ on $\mathcal{I}_\infty^\times$ induces the discrete topology on the band $E(\mathcal{I}_\infty^\times)$.

**Example 5.13.** We define a topology $\tau_{WF}$ on the semigroup $\mathcal{I}_\infty^\times$ as follows. For every $\alpha \in \mathcal{I}_\infty^\times$ we define a family

$$\mathcal{B}_{WF}(\alpha) = \{ U_\alpha(F) \mid F \text{ is a finite subset of } \text{dom } \alpha \},$$

where

$$U_\alpha(F) = \{ \beta \in \mathcal{I}_\infty^\times \mid \text{dom } \beta \subseteq \text{dom } \alpha \text{ and } (x)\beta = (x)\alpha \text{ for all } x \in F \}.$$ Since conditions (BP1)–(BP3) [7] hold for the family $\{ \mathcal{B}_{WF}(\alpha) \}_{\alpha \in \mathcal{I}_\infty^\times}$ we conclude that the family $\{ \mathcal{B}_{WF}(\alpha) \}_{\alpha \in \mathcal{I}_\infty^\times}$ is the base of the topology $\tau_{WF}$ on the semigroup $\mathcal{I}_\infty^\times$.

**Proposition 5.14.** $(\mathcal{I}_\infty^\times, \tau_{WF})$ is a Hausdorff topological inverse semigroup.

**Proof.** Let $\alpha$ and $\beta$ be arbitrary elements of the semigroup $\mathcal{I}_\infty^\times$. We put $\gamma = \alpha \cdot \beta$ and let $F = \{ n_1, \ldots, n_i \}$ be a finite subset of dom $\gamma$. We denote $m_1 = (n_1)\alpha, \ldots, m_i = (n_i)\alpha$ and $k_1 = (n_1)\gamma, \ldots, k_i = (n_i)\gamma$. Then we get that $(m_1)\beta = k_1, \ldots, (m_i)\beta = k_i$. Hence we have that

$$U_\alpha(\{ n_1, \ldots, n_i \}) \cdot U_\beta(\{ m_1, \ldots, m_i \}) \subseteq U_\gamma(\{ n_1, \ldots, n_i \})$$

and

$$(U_\gamma(\{ n_1, \ldots, n_i \}))^{-1} \subseteq U_{\gamma^{-1}}(\{ k_1, \ldots, k_i \}).$$

Therefore the semigroup operation and the inversion are continuous in $(\mathcal{I}_\infty^\times, \tau_{WF})$.

Later we shall show that the topology $\tau_{WF}$ is Hausdorff. Let $\alpha$ and $\beta$ be arbitrary distinct points of the space $(\mathcal{I}_\infty^\times, \tau_{WF})$. Then only one of the following conditions holds:
(i) \( \text{dom } \alpha = \text{dom } \beta \);
(ii) \( \text{dom } \alpha \neq \text{dom } \beta \).

In case \( \text{dom } \alpha = \text{dom } \beta \) we have that there exists \( x \in \text{dom } \alpha \) such that \( (x)\alpha \neq (x)\beta \). The definition of the topology \( \tau_{WF} \) implies that \( U_\alpha(\{x\}) \cap U_\beta(\{x\}) = \emptyset \).

If \( \text{dom } \alpha \neq \text{dom } \beta \), then only one of the following conditions holds:
(a) \( \text{dom } \alpha \subseteq \text{dom } \beta \);
(b) \( \text{dom } \beta \subseteq \text{dom } \alpha \);
(c) \( \text{dom } \alpha \setminus \text{dom } \beta \neq \emptyset \) and \( \text{dom } \beta \setminus \text{dom } \alpha \neq \emptyset \).

Suppose that case (a) holds. Let be \( x \in \text{dom } \beta \setminus \text{dom } \alpha \) and \( y \in \text{dom } \alpha \). The definition of the topology \( \tau_{WF} \) implies that \( U_\alpha(\{y\}) \cap U_\beta(\{x\}) = \emptyset \).

Case (b) is similar to (a).

Suppose that case (c) holds. Let be \( x \in \text{dom } \beta \setminus \text{dom } \alpha \) and \( y \in \text{dom } \alpha \setminus \text{dom } \beta \). The definition of the topology \( \tau_{WF} \) implies that \( U_\alpha(\{y\}) \cap U_\beta(\{x\}) = \emptyset \).

This completes the proof of the proposition. \( \square \)

**Remark 5.15.** We observe that the topology \( \tau_{WF} \) on \( \mathcal{I}_\infty \) induces a non-discrete topology (and hence a non-hereditary Baire topology) on the band \( E(\mathcal{I}_\infty) \). Moreover, \( \mathcal{H} \)-classes in \( (\mathcal{I}_\infty, \tau_{WF}) \) and \( (\mathcal{I}_\infty, \tau_F) \) are homeomorphic subspaces.

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