UPPER BOUNDS FOR HIGHER-ORDER POINCARÉ CONSTANTS

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Abstract. We introduce higher-order Poincaré constants for compact weighted manifolds and estimate them from above in terms of subsets. These estimates imply upper bounds for eigenvalues of the weighted Laplacian and the first nontrivial eigenvalue of the \( p \)-Laplacian. In the case of the closed eigenvalue problem and the Neumann eigenvalue problem these are related with the estimates obtained by Chung-Grigor’yau-Yau and Gozlan-Herry. We also obtain similar upper bounds for Dirichlet eigenvalues and multi-way isoperimetric constants. As an application, for manifolds with boundary of non-negative dimensional weighted Ricci curvature, we give upper bounds for inscribed radii in terms of dimension and the first Dirichlet Poincaré constant.

1. Introduction

Let \( M = (M, m) \) be a compact weighted Riemannian manifold, namely, \( M = (M, g) \) is a (connected) compact Riemannian manifold equipped with the Riemannian distance \( d \) determined by \( g \) and \( m \) is a (probability) weighted Riemannian volume measure defined as

\[
m := e^{-f} v_g
\]

for a smooth function \( f \in C^\infty(M) \) such that \( \int_M e^{-f} dv_g = 1 \), where \( v_g \) denotes the Riemannian volume measure induced from \( g \). The purpose of this article is to present a unifying way to give upper bounds for eigenvalues of the Laplacian and the \( p \)-Laplacian with or without a boundary condition.

1.1. Closed manifolds. We first consider the case where \( M \) is a closed manifold (i.e., its boundary \( \partial M \) is empty). We denote by \( \nabla \) and by \( \| \cdot \| \) the gradient operator and the canonical norm induced from \( g \)
respectively. For \( k \geq 1 \) and \( p \in [1, \infty) \), one of our main objects is the \( k \)-th (Neumann) \( p \)-Poincaré constant defined by

\[
\nu_{k,p}(M,m) := \inf_{L_k} \sup_{\phi \in L_k \setminus \{0\}} \frac{\int_M \|\nabla \phi\|^p \, dm}{\int_M |\phi - \int_M \phi \, dm|^p \, dm},
\]

where the infimum is taken over all \( k \)-dimensional subspaces \( L_k \) of the \( (1,p) \)-Sobolev space \( W^{1,p}(M,m) \) with \( L_k \cap C = \{0\} \) for the set \( C \) of all constant functions on \( M \). In the case of \( p = 2 \), this Poincaré constant is equal to the \( k \)-th non-trivial eigenvalue of the weighted Laplacian on \( M \) due to the min-max principle. In the case of \( p \in (1,\infty) \), the value \( \nu_{1,p}(M,m) \) is known to be equivalent to the so-called Cheeger isoperimetric constant (see Subsection 5.1).

We now assert one of our main results. For any finite sequence \( \{A_\alpha\}_{\alpha=0}^k \) of Borel subsets of \( M \) we set

\[
D(\{A_\alpha\}) := \min_{\alpha \neq \beta} d(A_\alpha, A_\beta),
\]

where \( d(A_\alpha, A_\beta) := \inf \{d(x,y) \mid x \in A_\alpha, y \in A_\beta\} \).

**Theorem 1.1.** Let \((M,m)\) be a closed weighted Riemannian manifold. For any sequence \( \{A_\alpha\}_{\alpha=0}^k \) of Borel subsets of \( M \) we have

\[
\nu_{k,p}(M,m) \leq \frac{2}{D(\{A_\alpha\})} \max_{\alpha=0,\ldots,k} \log \frac{e(1 - \sum_{\beta \neq \alpha} m(A_\beta))}{m(A_\alpha)}.
\]

**Remark 1.1.** In the case where \( p = 2 \), this type of inequality has been obtained by Chung-Grigor’yan-Yau. Under the same setting as in Theorem 1.1 they [8] have proven the following inequality (see Theorem 3.1 in [8], and see also [7]):

\[
\nu_{k,2}(M,m) \leq \frac{1}{D(\{A_\alpha\})} \max_{\alpha \neq \beta} \log \frac{e}{m(A_\alpha)m(A_\beta)}.
\]

When \( k = 1 \), this inequality (1.4) is due to Gromov-Milman [15]. The proof of (1.4) in [7] is based on the eigenfunction expansion of the heat kernel, and a heat kernel estimate. Recently, Gozlan-Herry [15] shown a similar inequality to (1.4) in a different way from that in [7] based on a simple geometric observation (see Proposition 2.2 in [15]). In Remark 3.1 we will compare our inequality (1.3) with the inequalities of Chung-Grigor’yan-Yau (1.4) and Gozlan-Herry.

Let us mention our method of the proof of Theorem 1.1. We prove the desired inequality by combining the following three principles, and a geometric observation: (1) domain monotonicity principle for a domain
Ω in M, which claims that the \((k+1)\)-th Dirichlet \(p\)-Poincaré constant on \(Ω\) is at least a modification of \(\nu_{k,p}(M,m)\) (see Lemma 3.1); (2) domain decomposing principle for a pairwise disjoint sequence \(\{\Omega_\alpha\}_{\alpha=0}^k\) of domains, which is a relation between the first Dirichlet \(p\)-Poincaré constants on the \(\Omega_\alpha\), for \(\alpha = 0,1,\ldots,k\), and the \((k+1)\)-th one on \(\bigcup_{\alpha=0}^k\Omega_\alpha\) (see Lemma 3.3); (3) boundary concentration inequality for a domain \(\Omega\), which states that the restricted weighted measure on \(Ω\) exponentially concentrates around its boundary when the first Dirichlet \(p\)-Poincaré constant on \(Ω\) is large (see Lemma 3.2). We notice that our method is quite different from that of (1.4) in [7].

1.2. Compact manifolds with boundary. We next consider the case where \(M\) is a compact manifold with boundary. In this case, our main object is the \(k\)-th Dirichlet \(p\)-Poincaré constant defined by

\[
\nu_{k,p}^D(M,m) := \inf_{L_k} \sup_{\phi \in L_k \setminus \{0\}} \frac{\int_M \|\nabla \phi\|^p \, dm}{\int_M |\phi|^p \, dm},
\]

where the infimum is taken over all \(k\)-dimensional subspaces \(L_k\) of the \((1,p)\)-Sobolev space \(W_0^{1,p}(M,m)\). When \(p = 2\), this constant is equal to the \(k\)-th Dirichlet eigenvalue of the weighted Laplacian. For \(p \in (1,\infty)\) the value \(\nu_{1,p}^D(M,m)\) coincides with the first Dirichlet eigenvalue of the weighted \(p\)-Laplacian (see Subsection 2.2), and \(\nu_{1,1}^D(M,m)\) the Dirichlet isoperimetric constant (see Subsection 5.2).

Our method of the proof of Theorem 1.1 also works in this case where \(\partial M\) is non-empty, and this yields the following analogue of Theorem 1.1 for the Dirichlet Poincaré constant. For any sequence \(\{A_\alpha\}_{\alpha=1}^k\) of Borel subsets of \(M\) we set

\[
D^\partial(\{A_\alpha\}) := \min \{ \min_{\alpha \neq \beta} d(A_\alpha, A_\beta), \min_{\alpha} d(A_\alpha, \partial M) \}.
\]

**Theorem 1.2.** Let \((M,m)\) be a compact weighted Riemannian manifold with boundary. For any sequence \(\{A_\alpha\}_{\alpha=1}^k\) of Borel subsets of \(M\),

\[
\nu_{k,p}^D(M,m)^\frac{1}{p} \leq \frac{2}{D^\partial(\{A_\alpha\})} \max_{\alpha=1,\ldots,k} \log \frac{e(1 - \sum_{\beta \neq \alpha} m(A_\beta))}{m(A_\alpha)}.
\]

The part \(D^\partial(\{A_\alpha\})\) in the right hand side of (1.7) concerns boundary concentration phenomena studied in [34]. Similarly to Theorem 1.1, this estimate (1.7) is new for \(p \neq 2\). When \(k \geq 2\), Theorem 1.2 is new even in the case of \(p = 2\) in view of the following two remarks.

**Remark 1.2.** Under the same setting as in Theorem 1.2, the second author [31] has obtained the following weaker estimate (see Lemma 4.1
\begin{align}
\tag{1.8}
\nu^D_{k,2}(M,m) \frac{1}{2} \leq \frac{2}{D^0(\{A_\alpha\})} \frac{1}{\sqrt{\min_{\alpha=1,\ldots,k} m(A_\alpha)}}.
\end{align}

After that the authors [14] improved (1.8), and proved Theorem 1.2 only when \( k = 1 \) (see Theorem 2.3 in [14]).

**Remark 1.3.** We now compare Theorem 1.2 for \( p = 2 \) with Theorem 1.1 in [7]. Chung-Grigor’yan-Yau [7] have obtained an upper estimate of the \( k \)-th Robin eigenvalues with same proof as (1.4). In our setting, their estimate holds in the following form: Under the same setting as in Theorem 1.2 we have

\begin{align}
(\nu^D_{k,2}(M,m) - \nu^D_{1,2}(M,m)) \frac{1}{2} \leq \frac{1}{\mathcal{D}(\{A_\alpha\})} \max_{\alpha \neq \beta} \log \frac{4}{\int_{A_\alpha} \phi^2_{1,2;m} \, dm \int_{A_\beta} \phi^2_{1,2;m} \, dm},
\end{align}

where \( \phi_{1,2;m} \) denotes an \( L^2 \)-normalized eigenfunction of the first Dirichlet eigenvalue of the weighted Laplacian. The advantage of Theorem 1.2 is that \( \nu^D_{1,2}(M,m) \) and the integral quantity of the eigenfunction do not appear in (1.7). We eliminate such values by considering \( \mathcal{D}^0(\{A_\alpha\}) \) instead of \( \mathcal{D}(\{A_\alpha\}) \). The form of (1.7) seems to be different from that of (1.9), and more close to that of (1.4).

1.3. **Organization.** In Section 2, we introduce the modified Poincaré constants and compare them with the Poincaré constants (1.2).

In Sections 3, 4, 5 we prove our main results. In Section 3 we prove Theorem 1.1. In Section 4 we formulate an analogue of Theorem 1.1 for Dirichlet eigenvalues of the weighted \( p \)-Laplacian on compact manifolds with boundary (see Theorem 1.2). In Section 5 we provide upper bounds for multi-way isoperimetric constant and also the multi-way Dirichlet isoperimetric constant (see Theorems 5.4 and 5.8).

In Section 6 we will discuss the sharpness of our main results.

Sections 7 and 8 are devoted to present some byproducts of the study in Sections 3, 4, 5. In Section 7 for compact manifolds with boundary of non-negative weighted Ricci curvature, we give an upper bound of its inscribed radius in terms of the Dirichlet Poincaré constant (1.5) by using the domain monotonicity principle (Lemma 4.1) and the boundary concentration inequality (Lemma 4.2) obtained in Section 4 (see Proposition 7.1). In Section 8 we discuss discrete cases. We show an analogue of Theorem 1.1 for weighted graphs (see Theorem 8.5).
2. Preliminaries

Hereafter, let \((M, m)\) denote a compact weighted Riemannian manifold defined as (1.1). Let \(k \geq 1\) and \(p \in [1, \infty)\).

2.1. Modified Poincaré constants. Let \(M\) be closed. We define the modified \(k\)-th \(p\)-Poincaré constant as

\[
\tilde{\nu}_{k,p}(M, m) := \inf_{L_{k+1}} \sup_{\phi \in L_{k+1} \backslash \{0\}} \frac{\int_M \|\nabla \phi\|^p \, dm}{\int_M |\phi|^p \, dm},
\]

where the infimum is taken over all \((k+1)\)-dimensional subspaces \(L_{k+1}\) of \(W^{1,p}(M, m)\). In stead of working with \(\nu_{k,p}(M, m)\) we will work with \(\tilde{\nu}_{k,p}(M, m)\). Let us study the relation between the Poincaré constant \(\nu_{k,p}(M, m)\) defined as (1.2), and the modified one \(\tilde{\nu}_{k,p}(M, m)\).

In the case of \(p = 2\) it is known and easy to show that \(\nu_{k,2}(M, m) = \tilde{\nu}_{k,2}(M, m)\), and they are also equal to the \(k\)-th non-trivial eigenvalue of the weighted Laplacian (see (2.2) below). We first recall the following elementary inequality (cf. Lemma 2.1 in [31]). For the completeness of this paper, we give its proof.

**Lemma 2.1.** For \(\phi \in W^{1,p}(M, m)\), we have

\[
\inf_{c \in \mathbb{R}} \|\phi - c\|_{L^p} \leq \left\| \phi - \int_M \phi \, dm \right\|_{L^p} \leq 2 \inf_{c \in \mathbb{R}} \|\phi - c\|_{L^p},
\]

where \(\cdot \|_{L^p}\) is the \(L^p\)-norm on \(M\) with respect to the measure \(m\).

**Proof.** Given any \(c \in \mathbb{R}\) we get

\[
\left\| \phi - \int_M \phi \, dm \right\|_{L^p} \leq \|\phi - c\|_{L^p} + \left\| \int_M \phi \, dm - c \right\|_{L^p} \\
= \|\phi - c\|_{L^p} + \left\| \int_M \phi \, dm - c \right\|_{L^1} \leq 2 \|\phi - c\|_{L^p}.
\]

This proves the lemma. \(\Box\)

Lemma 2.1 leads us to the following:

**Proposition 2.2.**

\[
\nu_{k,p}(M, m) \leq \tilde{\nu}_{k,p}(M, m) \leq 2^p \nu_{k,p}(M, m).
\]

**Proof.** We first prove \(\nu_{k,p}(M, m) \leq \tilde{\nu}_{k,p}(M, m)\). Fix a \((k+1)\)-dimensional subspace \(L_{k+1}\) of \(W^{1,p}(M, m)\), and also define a subspace

\[
\mathcal{L}_{k+1} := \left\{ \phi \in L_{k+1} \mid \int_M \phi \, dm = 0 \right\}.
\]
Note that the dimension of $T_{k+1}$ is at least $k$, and $T_{k+1} \cap C = \{0\}$ for the set $C$ of all constant functions on $M$. We now take a $k$-dimensional subspace $L_k \subset T_{k+1}$ of $W_{1,p}(M, m)$. Then it holds that

$$
\nu_{k,p}(M, m) \leq \sup_{\phi \in L_k \setminus \{0\}} \frac{\int_M \|\nabla \phi\|^p \, dm}{\int_M |\phi - \int_M \phi \, dm|^p \, dm} = \sup_{\phi \in L_k \setminus \{0\}} \frac{\int_M \|\nabla \phi\|^p \, dm}{\int_M |\phi|^p \, dm} \leq \sup_{\phi \in L_{k+1} \setminus \{0\}} \frac{\int_M \|\nabla \phi\|^p \, dm}{\int_M |\phi - \int_M \phi \, dm|^p \, dm}.
$$

This implies the desired inequality.

We next prove $\widehat{\nu}_{k,p}(M, m) \leq 2^p \nu_{k,p}(M, m)$. Let us fix a $k$-dimensional subspace $L_k$ of $W_{1,p}(M, m)$ with $L_k \cap C = \{0\}$. Lemma 2.1 yields that for every $\phi \in L_k$ we have

$$
\int_M \left| \phi - \int_M \phi \, dm \right|^p \, dm \leq 2^p \inf_{c \in \mathbb{R}} \int_M |\phi - c|^p \, dm.
$$

Define a $(k + 1)$-dimensional subspace $L_{k+1} := L_k \oplus C$. Then

$$
\widehat{\nu}_{k,p}(M, m) \leq \sup_{\phi \in L_{k+1} \setminus \{0\}} \frac{\int_M \|\nabla \phi\|^p \, dm}{\int_M |\phi|^p \, dm} \leq \sup_{\phi \in L_{k} \setminus \{0\}} \frac{\int_M \|\nabla \phi\|^p \, dm}{\int_M |\phi - \int_M \phi \, dm|^p \, dm} \leq 2^p \sup_{\phi \in L_k \setminus \{0\}} \frac{\int_M \|\nabla \phi\|^p \, dm}{\int_M |\phi - \int_M \phi \, dm|^p \, dm}.
$$

We complete the proof.

We further review the relation between $\nu_{k,p}(M, m)$, $\widehat{\nu}_{k,p}(M, m)$, and the spectrum of the weighted Laplacian

$$
\Delta_m := \Delta_g + g(\nabla f, \nabla \cdot),
$$

where $\Delta_g$ is the Laplacian defined as the minus of the trace of Hessian. We denote by

$$
0 = \lambda_0(M, m) < \lambda_1(M, m) \leq \cdots \leq \lambda_k(M, m) \leq \cdots \nearrow + \infty
$$

the all eigenvalues of $\Delta_m$, counting multiplicity. The min-max principle tells us that

$$
\lambda_k(M, m) = \nu_{k,2}(M, m) = \widehat{\nu}_{k,2}(M, m).
$$

For $p \in (1, \infty)$, the weighted $p$-Laplacian $\Delta_{m,p}$ is defined as

$$
\Delta_{m,p} := -e^f \text{div}_g \left( e^{-f} \|\nabla \cdot \|^p - \|\nabla \cdot \|^p \right),
$$

where $\text{div}_g$ is the divergence operator induced from $g$. Here $\Delta_{m,2} = \Delta_m$. Note that $\Delta_{m,p}$ is non-linear in the case of $p \neq 2$ (for its spectral theory, see e.g., [32] and the references therein). Let $\lambda_{1,p}(M, m)$ stand for the
smallest positive eigenvalue of $\Delta_{m,p}$, which is known to be variationally characterized as follows (see e.g., Corollary 2.11 in [19]):

$$\lambda_{1,p}(M, m) = \inf_{\phi \in W^{1,p}(M,m):\phi \in \mathcal{C}} \frac{\int_M \|\nabla \phi\|^p \, dm}{\inf_{c \in \mathbb{R}} \int_M |\phi - c|^p \, dm}$$

for the set $\mathcal{C}$ of all constant functions on $M$. In virtue of Lemma 2.1 and Proposition 2.2, we see

$$\lambda_{1,p}(M, m) \frac{1}{p} \simeq \nu_{1,p}(M, m) \frac{1}{p} \simeq \widehat{\nu}_{1,p}(M, m) \frac{1}{p}.$$  

Here $C_1 \simeq C_2$ means that $C_1$ and $C_2$ are equivalent up to universal explicit constants.

**Remark 2.1.** Let us compare $\widehat{\nu}_{k,p}(M, m)$ with an eigenvalue of the $p$-Laplacian in the case of $p \neq 2$. In this case, a similar min-max principle to (2.2) does not work in the higher-order case of $k \geq 2$. We refer to [32] for the summary of the higher-order eigenvalues of the weighted $p$-Laplacian. We here recall some construction of the higher-order eigenvalues. Let $B_p$ stand for the class of closed symmetric subsets of $\{ \phi \in W^{1,p}(M,m) | \|\phi\|_{L^p} = 1 \}$. For $B \in B_p$, its *genus* $\gamma^+(B)$ is defined as the supremum of $l \geq 1$ such that there is an odd continuous surjective map from the $l$-dimensional unit sphere $S^l$ in $\mathbb{R}^{l+1}$ to $B$. Its *cogenus* or *Krasnosel’skii genus* $\gamma^-(B)$ [20] is also defined as the infimum of $l$ such that there is an odd continuous map from $B$ to $S^l$. We now define

$$\lambda_{k,p}(M, m) := \inf_{B \in B_{k+1,p}^\pm} \sup_{\phi \in B} \int_M \|\nabla \phi\|^p \, dm,$$

where $B_{k+1,p}^\pm := \{ B \in B_p | \gamma^+(B) \geq k + 1 \}$. Then the two sequences $\{\lambda_{k,p}^\pm(M, m)\}_k$ are known to be increasing and unbounded sequences of eigenvalues of the weighted $p$-Laplacian and it is known that $\lambda_{1,p}(M, m)$ coincides with $\lambda_{1,p}^+(M, m)$ ([10]). $\lambda_{k,p}^+(M, m)$ was introduced by Drábek-Robinson [10]. For any $(k + 1)$-dimensional subspace $L_{k+1}$ of $W^{1,p}(M, m)$, it holds that $\gamma^+(L_{k+1} \cap \{\|\phi\|_{L^p} = 1\}) = k + 1$ (see e.g., Proposition 5.2 in [34] and Section 3 in [21]), and hence

$$\lambda_{k,p}^\pm(M, m) \leq \widehat{\nu}_{k,p}(M, m).$$  

In particular, upper bounds for $\widehat{\nu}_{k,p}(M, m)$ implies that for $\lambda_{k,p}^\pm(M, m)$.

### 2.2. Dirichlet Poincaré constants

Let $\partial M$ be non-empty. We next investigate the $k$-*th Dirichlet $p$-Poincaré constant* defined as (1.5). Let

$$0 < \lambda_1^D(M, m) < \lambda_2^D(M, m) \leq \cdots \leq \lambda_k^D(M, m) \leq \cdots \nearrow +\infty$$
stand for the all Dirichlet eigenvalues of the weighted Laplacian $\Delta_m$, counting multiplicity. The min-max principle states

$$\lambda_k^D(M, m) = \nu_{k, 2}^D(M, m).$$

For $p \in (1, \infty)$, let $\lambda_{1,p}^D(M, m)$ denote the smallest Dirichlet eigenvalue of the weighted $p$-Laplacian $\Delta_{m,p}$. It is well-known that $\lambda_{1,p}^D(M, m)$ is variationally characterized as

$$\lambda_{1,p}^D(M, m) = \nu_{1,p}^D(M, m).$$

We now present the boundary concentration inequality, which is a key ingredient of the proof of our main results. For $A \subset M$, let $B_r(A)$ denote its closed $r$-neighborhood in $M$.

**Proposition 2.3** ([14]). For every $r > 0$ we have

$$m(M \setminus B_r(\partial M)) \leq \exp \left(1 - \nu_{1,p}^D(M, m)^\frac{1}{p} r\right). \tag{2.4}$$

**Proof.** The inequality (2.4) has been obtained in [14] in the unweighted case of $f = \log v_g(M)$ and $p = 2$ (see Proposition 2.1 in [14]). It can be proved by the same argument as in [14]. We only outline the proof.

We begin with showing that

$$\left(1 + \epsilon \nu_{1,p}^D(M, m)\right) m(M \setminus B_{r+\epsilon}(\partial M)) \leq m(M \setminus B_r(\partial M)) \tag{2.5}$$

for all $\epsilon, r > 0$. Set $\Omega_1 := B_r(\partial M), \Omega_2 := M \setminus B_{r+\epsilon}(\partial M)$, and $v_\alpha := m(\Omega_\alpha)$ for any $\alpha \in \{1, 2\}$. Define a Lipschitz function $\varphi : M \to \mathbb{R}$ by

$$\varphi(x) := \min \left\{\frac{1}{\epsilon} d(x, \Omega_1), 1\right\}.$$

It holds that

$$\int_M |\varphi|^p dm \geq v_2, \quad \int_M \|\nabla \varphi\|^p dm \leq \frac{1}{\epsilon^p} (1 - v_1 - v_2),$$

and hence

$$\nu_{1,p}^D(M, m) \leq \frac{\int_M \|\nabla \varphi\|^p dm}{\int_M |\varphi|^p dm} \leq \frac{1}{\epsilon^p v_2} (1 - v_1 - v_2).$$

This proves (2.5).

One can derive (2.4) from (2.5). For $\epsilon_0 := \nu_{1,p}^D(M, m)^{-\frac{1}{p}}$, it suffices to consider the case where $r \in (0, \epsilon_0)$. Let $l$ be the integer determined by $\epsilon_0 r^{-1} \in [(l+1)^{-1}, l^{-1})$. Applying (2.5) to $m(M \setminus B_{l\epsilon_0}(\partial M))$ iteratively, we can conclude (2.4). \qed
3. Upper bounds for Poincaré constants

The present section is devoted to the proof of Theorem 1.1. Throughout this section, we always assume that $M$ is closed.

3.1. Key principles. In this subsection, we prepare three key principles for the proof of Theorem 1.1. For a domain $\Omega \subset M$, we introduce the following local version of the Dirichlet Poincaré constant:

$$\nu_{k,p}^D(\Omega, m) := \inf_{L_k} \sup_{\phi \in L_k \setminus \{0\}} \frac{\int_{\Omega} \|\nabla \phi\|^p dm}{\int_{\Omega} |\phi|^p dm},$$

where the infimum is taken over all $k$-dimensional subspaces $L_k$ of the $(1,p)$-Sobolev space $W^{1,p}_0(\Omega, m)$. This is same as (1.5) except for the difference of domain.

The following domain monotonicity principle follows from the above definition:

**Lemma 3.1.** For a domain $\Omega \subset M$,

$$\tilde{\nu}_{k,p}(M, m) \leq \nu_{k+1,p}^D(\Omega, m),$$

where $\tilde{\nu}_{k,p}(M, m)$ is defined as (2.1).

For a domain $\Omega \subset M$, let $m_\Omega$ be the normalized volume measure on $\Omega$ defined as

$$m_\Omega := \frac{1}{m(\Omega)} m|_{\Omega}.$$

By the same argument as in the proof of Proposition 2.3, we obtain the following boundary concentration inequality:

**Lemma 3.2.** Let $\Omega \subset M$ be a domain. Then for all $r > 0$,

$$m_\Omega(\Omega \setminus B_r(\partial \Omega)) \leq \exp \left( 1 - \nu_{1,p}^D(\Omega, m)^\frac{1}{p} r \right),$$

where $\partial \Omega$ denotes the boundary of $\Omega$.

We further observe the following domain decomposing principle:

**Lemma 3.3.** For any pairwise disjoint sequence $\{\Omega_\alpha\}_{\alpha=0}^k$ of domains in $M$, we have

$$\nu_{k+1,p}^D \left( \bigsqcup_{\alpha=0}^k \Omega_\alpha, m \right) \leq \max_{\alpha=0,\ldots,k} \nu_{1,p}^D(\Omega_\alpha, m).$$

**Proof.** Fix $\epsilon > 0$. For each $\alpha$, take a non-zero $\phi_\alpha \in W^{1,p}_0(\Omega_\alpha, m)$ with

$$\frac{\int_{\Omega_\alpha} \|\nabla \phi_\alpha\|^p dm}{\int_{\Omega_\alpha} |\phi_\alpha|^p dm} < \nu_{1,p}^D(\Omega_\alpha, m) + \epsilon.$$
We define \( \Omega := \bigsqcup_{k=0}^{k} \Omega_{\alpha} \), and regard \( \phi_{\alpha} \) as a function in \( W^{1,p}_{0}(\Omega, m) \) by setting it as zero outside \( \Omega_{\alpha} \). We note that \( \{\phi_{\alpha}\}_{\alpha=0}^{k} \) are independent in \( W^{1,p}_{0}(\Omega, m) \). Let \( L_{\epsilon} \) be the \((k+1)\)-dimensional subspace of \( W^{1,p}_{0}(\Omega, m) \) spanned by \( \phi_{0}, \ldots, \phi_{k} \). Fix \( \delta > 0 \), and take a non-zero \( \phi \in L_{\epsilon} \) with

\[
\sup_{\psi \in L_{\epsilon} \setminus \{0\}} \frac{\int_{\Omega} \|\nabla \psi\|^{p} dm}{\int_{\Omega} |\psi|^{p} dm} < \frac{\int_{\Omega} \|\nabla \phi\|^{p} dm}{\int_{\Omega} |\phi|^{p} dm} + \delta.
\]

Let \( c_{0}, \ldots, c_{k} \) be constants determined by \( \phi = \sum_{\alpha=0}^{k} c_{\alpha} \phi_{\alpha} \), and let \( I \) be the set of all \( \alpha \) such that \( c_{\alpha} \neq 0 \). Since \( \phi_{\alpha} \equiv 0 \) outside \( \Omega_{\alpha} \), we possess

\[
\int_{\Omega} |\phi_{\alpha}|^{p} dm = \sum_{\alpha \in I} |c_{\alpha}|^{p} \int_{\Omega_{\alpha}} |\phi_{\alpha}|^{p} dm,
\]

\[
\int_{\Omega} \|\nabla \phi_{\alpha}\|^{p} dm = \sum_{\alpha \in I} |c_{\alpha}|^{p} \int_{\Omega_{\alpha}} \|\nabla \phi_{\alpha}\|^{p} dm.
\]

We now recall the following elementary inequality: For \( \beta = 0, \ldots, l \), and for constants \( a_{\beta}, b_{\beta} \) with \( a_{\beta} \neq 0 \), we have

\[
\sum_{\beta=0}^{l} b_{\beta} \sum_{\beta=0}^{l} a_{\beta} \leq \max_{\beta=0, \ldots, l} a_{\beta}.
\]

The inequality (3.7) can be proven by induction on \( l \). From (3.4), (3.5), (3.6), (3.7), (3.3), we deduce

\[
\nu_{k+1,p}^{D}(\Omega, m) \leq \sup_{\psi \in L_{\epsilon} \setminus \{0\}} \frac{\int_{\Omega} \|\nabla \psi\|^{p} dm}{\int_{\Omega} |\psi|^{p} dm} < \frac{\int_{\Omega} \|\nabla \phi\|^{p} dm}{\int_{\Omega} |\phi|^{p} dm} + \delta
\]

\[
= \frac{\sum_{\alpha \in I} |c_{\alpha}|^{p} \int_{\Omega_{\alpha}} \|\nabla \phi_{\alpha}\|^{p} dm}{\sum_{\alpha \in I} |c_{\alpha}|^{p} \int_{\Omega_{\alpha}} |\phi_{\alpha}|^{p} dm} + \delta
\]

\[
\leq \max_{\alpha \in I} \frac{\int_{\Omega_{\alpha}} \|\nabla \phi_{\alpha}\|^{p} dm}{\int_{\Omega_{\alpha}} |\phi_{\alpha}|^{p} dm} + \delta < \nu_{1,p}^{D}(\Omega_{\alpha_{0}}, m) + \epsilon + \delta,
\]

where \( \alpha_{0} \in I \) is determined by

\[
\frac{\int_{\Omega_{\alpha_{0}}} \|\nabla \phi_{\alpha_{0}}\|^{p} dm}{\int_{\Omega_{\alpha_{0}}} |\phi_{\alpha_{0}}|^{p} dm} = \max_{\alpha \in I} \frac{\int_{\Omega_{\alpha}} \|\nabla \phi_{\alpha}\|^{p} dm}{\int_{\Omega_{\alpha}} |\phi_{\alpha}|^{p} dm}.
\]

We obtain

\[
\nu_{k+1,p}^{D}(\Omega, m) < \max_{\alpha=0, \ldots, k} \nu_{1,p}^{D}(\Omega_{\alpha}, m) + \epsilon + \delta.
\]

Letting \( \epsilon, \delta \to 0 \) leads us to the desired conclusion. \( \square \)
3.2. Proof of Theorem 1.1. Theorem 1.1 follows from the following:

**Theorem 3.4.** Let $(M, m)$ be a closed weighted Riemannian manifold. For any sequence $\{A_\alpha\}_{\alpha=0}^k$ of Borel subsets of $M$, we have

$$\hat{\nu}_{k,p}(M, m) \leq \frac{2}{\mathcal{D}(\{A_\alpha\})} \max_{\alpha=0, \ldots, k} \log \frac{e(1 - \sum_{\beta \neq \alpha} m(A_\beta))}{m(A_\alpha)}.$$

**Proof of Theorem 1.1.** In virtue of Theorem 3.4 and Proposition 2.2, we get (1.3). This completes the proof.

**Proof of Theorem 3.4.** Set $R := \mathcal{D}(\{A_\alpha\})$. For a fixed $r \in (0, R/2)$, we define a pairwise disjoint sequence $\{\Omega_\alpha\}_{\alpha=0}^k$ of domains $\Omega_\alpha \subset M$ as the open $r$-neighborhood of $A_\alpha$ in $M$. Lemmas 3.1 and 3.3 lead us to (3.8)

$$\hat{\nu}_{k,p}(M, m) \leq \nu_{k+1,p}^D \left( \bigcup_{\alpha=0}^k \Omega_\alpha, m \right) \leq \max_\alpha \nu_{1,p}^D(\Omega_\alpha, m).$$

For a fixed $\epsilon > 0$, we define

$$r_0 := \frac{1}{\nu_{1,p}^D(\Omega_{\alpha_0}, m)^{1/p}} \log \frac{e(1 - \sum_{\alpha \neq \alpha_0} m(A_\alpha))}{m(A_{\alpha_0})} + \epsilon,$$

where $\alpha_0$ is determined by $\nu_{1,p}^D(\Omega_{\alpha_0}, m) = \max_\alpha \nu_{1,p}^D(\Omega_\alpha, m)$.

Notice that

$$\frac{m(A_{\alpha_0})}{1 - \sum_{\alpha \neq \alpha_0} m(A_\alpha)} > \exp \left(1 - \nu_{1,p}^D(\Omega_{\alpha_0}, m)^{1/p} r_0\right).$$

Let us prove $r \leq r_0$. We have

$$m(\Omega_{\alpha_0}) \leq m \left( M \setminus \bigcup_{\alpha \neq \alpha_0} \Omega_\alpha \right) = 1 - \sum_{\alpha \neq \alpha_0} m(\Omega_\alpha) \leq 1 - \sum_{\alpha \neq \alpha_0} m(A_\alpha).$$

It follows that

$$m_{\Omega_{\alpha_0}}(A_{\alpha_0}) \geq \frac{m(A_{\alpha_0})}{m(\Omega_{\alpha_0})} = \frac{m(A_{\alpha_0})}{1 - \sum_{\alpha \neq \alpha_0} m(A_\alpha)}.$$

where $m_{\Omega_{\alpha_0}}$ is defined as (3.2). By combining (3.9) and (3.10), and by Lemma 3.2, we obtain

$$m_{\Omega_{\alpha_0}}(A_{\alpha_0}) \geq \exp \left(1 - \nu_{1,p}^D(\Omega_{\alpha_0}, m)^{1/p} r_0\right) \geq m_{\Omega_{\alpha_0}}(\Omega_{\alpha_0} \setminus B_{r_0}(\partial \Omega_{\alpha_0})).$$

This yields $B_{r_0}(\partial \Omega_{\alpha_0}) \cap A_{\alpha_0} \neq \emptyset$. Therefore, we see $r \leq r_0$. 

By letting $\epsilon \to 0$ and $r \to R/2$, and by (3.8), we arrive at

$$R \leq \frac{2}{\nu_{k,p}^{\Omega}(\Omega_{a_0},m)^{\frac{1}{p}}} \log \frac{e(1 - \sum_{\alpha \neq a_0} m(A_{\alpha}))}{m(A_{a_0})} \leq \frac{2}{\nu_{k,p}(M,m)^{\frac{1}{p}}} \max_{\alpha = 0, \ldots, k} \log \frac{e(1 - \sum_{\beta \neq \alpha} m(A_{\beta}))}{m(A_{\alpha})}.$$ 

This completes the proof of Theorem 3.4.

Combining Theorem 3.4 with (2.3) we obtain the following corollary. Refer to Remark 2.1 for the definition of $\lambda_{k,p}^{\pm}(M,m)$.

**Corollary 3.5.** Under the same assumption of Theorem 3.4, we have

$$\lambda_{k,p}^{\pm}(M,m)^{\frac{1}{p}} \leq \frac{2}{D(\{A_\alpha\})} \max_{\alpha = 0, \ldots, k} \log \frac{e(1 - \sum_{\beta \neq \alpha} m(A_{\beta}))}{m(A_{\alpha})}$$

and in particular for $k = 1$ we have

$$\lambda_{1,p}(M,m)^{\frac{1}{p}} \leq \frac{2}{d(A_0,A_1)} \max \left\{ \log \frac{e(1 - m(A_1))}{m(A_0)}, \log \frac{e(1 - m(A_0))}{m(A_1)} \right\}.$$ 

**Remark 3.1.** We shall compare our estimate (1.3) with the estimates obtained by Chung-Grigor’yan-Yau [7, 8] (1.4) and Gozlan-Herry [15].

Let $\{A_\alpha\}_{\alpha = 0}^k$ be a sequence of Borel subsets of $(M,m)$. By taking a permutation we may assume $m(A_0) \leq m(A_1) \leq \cdots \leq m(A_k)$. In this setting we have

$$\log \frac{e(1 - \sum_{\beta \neq 0} m(A_{\beta}))}{m(A_0)} = \max_{\alpha} \log \frac{e(1 - \sum_{\beta \neq \alpha} m(A_{\beta}))}{m(A_{\alpha})}$$

and

$$\log \frac{e}{m(A_0)m(A_1)} = \max_{\alpha \neq \beta} \frac{e}{m(A_\alpha)m(A_{\beta})}.$$ 

Thus our estimate (1.3) is better than Chung-Grigor’yan-Yau’s estimate (1.4) as long as

$$e m(A_1) \left( 1 - \sum_{\beta \neq 0} m(A_{\beta}) \right)^2 \leq m(A_0).$$

Otherwise Chung-Grigor’yan-Yau’s estimate (1.4) is better.

In Proposition 2.2 of [15] Gozlan-Herry imposed the following assumption for $k$ Borel subsets $A_1, A_2, \ldots, A_k$ of $M$;

$$m(A_{\alpha}) + \sum_{\beta = 1}^k m(A_{\beta}) \geq 1 \text{ for any } \alpha = 1, 2, \ldots, k.$$
Under the assumption setting $A_0 := M \setminus B_r(\bigcup_{\alpha=1}^k A_\alpha)$ they proved
\begin{equation}
\lambda_k(M, m) \leq \frac{2}{D(\{A_\alpha\})} \phi\left(\frac{1}{c} \log \frac{1 - \sum_{\beta \neq 0} m(A_\beta)}{m(A_0)}\right),
\end{equation}
where $\phi(x) := \max\{\sqrt{x}, x\}$ and $c > 0$ is a constant. In this setting observe that $m(A_0) \leq m(A_\alpha)$ for any $\alpha$ and thus we have (3.11). Note also that $\phi^{-1}(x) = \min\{x, x^2\}$ and
$$
\log \frac{e(1 - \sum_{\beta \neq 0} m(A_\beta))}{m(A_0)} \geq 1.
$$
Hence as long as
$$
\left(\frac{1 - \sum_{\beta \neq 0} m(A_\beta)}{m(A_0)}\right)^{\frac{1}{c} - 1} \geq e
$$
our estimate (1.3) is better than (3.12) and otherwise (3.12) is better.

We remark that Gozlan-Herry showed that $c = \log 5 - 1$ (proof of Theorem 2.1 of [15]) and so $\frac{1}{c} - 1 > 0$. Also our inequality holds without any restriction to $\{A_\alpha\}$.

4. Upper bounds for Dirichlet Poincaré constants

The aim of this section is to formulate an analogue of Theorem 1.1 for Dirichlet eigenvalues. In the present section, we always assume that $\partial M$ is non-empty.

We summarize some key lemmas to prove Theorem 1.2. We denote by $\text{Int} M$ the interior of $M$.

**Lemma 4.1.** For a domain $\Omega \subset \text{Int} M$, we have
$$
\nu_{k,p}^D(M, m) \leq \nu_{k,p}^D(\Omega, m),
$$
where $\nu_{k,p}^D(M, m)$ and $\nu_{k,p}^D(\Omega, m)$ are defined as (1.5) and (3.1), respectively.

**Lemma 4.2.** Let $\Omega \subset \text{Int} M$ be a domain. For all $r > 0$, we have
$$
m_\Omega(\Omega \setminus B_r(\partial \Omega)) \leq \exp \left(1 - \nu_{1,p}^D(\Omega, m)^\frac{1}{r} r\right),
$$
where $m_\Omega$ is defined as (3.2).

**Lemma 4.3.** For any pairwise disjoint sequence $\{\Omega_\alpha\}_{\alpha=1}^k$ of domains in $\text{Int} M$, we have
$$
\nu_{k,p}^D \left(\bigcup_{\alpha=1}^k \Omega_\alpha, m\right) \leq \max_{\alpha=1,\ldots,k} \nu_{1,p}^D(\Omega_\alpha, m).
$$
One can verify Lemmas 4.1, 4.2 and 4.3 by the same argument as in the proof of Lemmas 3.1, 3.2 and 3.3, respectively. We omit the proof.

**Proof of Theorem 1.2.** We complete the proof by setting $R := D^0(\{A_\alpha\})$, and using Lemmas 4.1, 4.2 and 4.3 instead of Lemmas 3.1, 3.2 and 3.3 along the line of the proof of Theorem 3.4, respectively. $\square$

5. **Upper bounds for multi-way isoperimetric constants**

In this section, we study multi-way isoperimetric constants which was introduced by Miclo [29] and studied in [9, 12, 25, 22, 30]. Multi-way isoperimetric constants are higher order version of isoperimetric constants (Cheeger constants) and it is expected that they possess similar properties of eigenvalues of the Laplacian.

### 5.1. Closed manifolds

Let $M$ be closed. For a Borel subset $A \subset M$,

$$m^+(A) := \liminf_{r \to 0} \frac{m(U_r(A)) - m(A)}{r},$$

where $U_r(A)$ is the open $r$-neighborhood of $A$. The *$k$-way isoperimetric constant* is defined as

$$\mathcal{I}_k(M, m) := \inf_{\{A_\alpha\}_{\alpha=0,\ldots,k}} \max_{\alpha} \frac{m^+(A_\alpha)}{m(A_\alpha)},$$

where the infimum is taken over all pairwise disjoint sequences $\{A_\alpha\}_{\alpha=0}^k$ of non-empty Borel subsets $A_\alpha \subset M$. When $k = 1$, this is nothing but the Cheeger constant. The following relation formally established by Federer-Fleming [11] (cf. Theorem 9.6 in [23], and Lemma 2.1):

$$\mathcal{I}_1(M, m) \simeq \nu_{1,1}(M, m). \quad (5.1)$$

For a domain $\Omega \subset M$, we consider the *local $k$-way Dirichlet isoperimetric constant*

$$\mathcal{I}_k^D(\Omega, m) := \inf_{\{A_\alpha\}_{\alpha=1,\ldots,k}} \max_{\alpha} \frac{m^+(A_\alpha)}{m(A_\alpha)},$$

where the infimum is taken over all pairwise disjoint sequences $\{A_\alpha\}_{\alpha=1}^k$ of non-empty Borel subsets of $\Omega$. Due to Federer-Fleming [11], we have the following (cf. Theorem 9.5 in [23]):

$$\mathcal{I}_1^D(\Omega, m) = \nu_{1,1}^D(\Omega, m), \quad (5.2)$$

where the right hand side is defined as (3.1).

One can verify the following domain monotonicity principle:

**Lemma 5.1.** For a domain $\Omega \subset M$, we have

$$\mathcal{I}_k(M, m) \leq \mathcal{I}_{k+1}^D(\Omega, m).$$
The following boundary concentration inequality follows from Lemma 3.2 with $p = 1$, and (5.2):

**Lemma 5.2.** Let $\Omega \subset M$ be a domain. For all $r > 0$, we have

$$m_\Omega(\Omega \setminus B_r(\partial \Omega)) \leq \exp\left(1 - I_{D_1}(\Omega, m) r\right),$$

where $m_\Omega$ is defined as (3.2).

By straightforward argument, we also have the following:

**Lemma 5.3.** For any pairwise disjoint sequence $\{\Omega_\alpha\}_{\alpha=0}^k$ of domains in $M$, we have

$$I_{D_{k+1}}\left(\bigcup_{\alpha=0}^k \Omega_\alpha, m\right) \leq \max_{\alpha=0,\ldots,k} I_{D_1}(\Omega_\alpha, m).$$

We can show the following assertion by using Lemmas 5.1, 5.2 and 5.3 along the line of the proof of Theorem 3.4 instead of Lemmas 3.1, 3.2 and 3.3, respectively. The proof is left to the reader.

**Theorem 5.4.** Let $M = (M, m)$ be a closed weighted Riemannian manifold. For any sequence $\{\Omega_\alpha\}_{\alpha=0}^k$ of Borel subsets of $M$, we have

$$I_k(M, m) \leq \frac{2}{\mathcal{D}(\{A_\alpha\})} \max_{\alpha=0,\ldots,k} \log \frac{e(1 - \sum_{\beta \neq \alpha} m(A_\beta))}{m(A_\alpha)}.$$

**Remark 5.1.** The first author [12] stated the following inequality to (5.3) (see Subsection 2.2 in [12]): Under the same setting as in Theorem 5.4, it holds that

$$I_k(M, m) \lesssim \frac{k^3}{\mathcal{D}(\{A_\alpha\})} \max_{\alpha \neq \beta} \log \frac{e}{m(A_\alpha)m(A_\beta)},$$

here $C_1 \lesssim C_2$ means that $C_1 \leq C_2$ for some universal explicit constant $C > 0$. In [12], he first pointed out that the higher-order Cheeger inequality in the graph setting established by Lee-Gharan-Trevisan [22] can be extended to the closed manifold setting by an appropriate modification of their proof (see Theorem 3.8 in [22], and Theorem 1.4 in [12]). He concluded (5.4) by combining the higher-order Cheeger inequality with the inequality (1.4) of Chung-Grigor’yan-Yau [8]. Note that $k^3$ does not appear in (5.3) and hence better than (5.4).

### 5.2. Manifolds with boundary

We next consider the case where $M$ is a compact manifold with boundary. The $k$-way Dirichlet isoperimetric constant is defined as

$$I_k^D(M, m) := \inf_{\{A_\alpha\}} \max_{\alpha=1,\ldots,k} \frac{m^+(A_\alpha)}{m(A_\alpha)}.$$
where the infimum is taken over all pairwise disjoint sequences \( \{ A_\alpha \}_{\alpha=1}^k \) of non-empty Borel subsets \( A_\alpha \subset \text{Int} \ M \). Due to Federer-Fleming [11],
\[
I^D_k(M, m) = \nu^D_{1,1}(M, m).
\]

Similarly to the case where \( M \) is closed, we have the following:

**Lemma 5.5.** For a domain \( \Omega \subset \text{Int} \ M \), we have
\[
I^D_k(M, m) \leq I^D_k(\Omega, m).
\]

**Lemma 5.6.** Let \( \Omega \subset \text{Int} \ M \) be a domain. For all \( r > 0 \), we have
\[
m_\Omega (\Omega \setminus B_r(\partial \Omega)) \leq \exp \left( 1 - I^D_1(\Omega, m) r \right).
\]

**Lemma 5.7.** For any pairwise disjoint sequence \( \{ \Omega_\alpha \}_{\alpha=1}^k \) of domains in \( \text{Int} \ M \), we have
\[
I^D_k \left( \bigcup_{\alpha=1}^k \Omega_\alpha, m \right) \leq \max_{\alpha=1,...,k} I^D_1(\Omega_\alpha, m).
\]

The above lemmas imply the following:

**Theorem 5.8.** Let \( M = (M, m) \) be a compact weighted Riemannian manifold with boundary. For any sequence \( \{ A_\alpha \}_{\alpha=1}^k \) of Borel subsets of \( M \), we have
\[
I^D_k(M, m) \leq \frac{2}{D^\beta(\{ A_\alpha \})} \max_{\alpha=1,...,k} \log \frac{e(1 - \sum_{\beta \neq \alpha} m(A_\beta))}{m(A_\alpha)}.
\]

6. Sharpness

Throughout this section, \( M \) is assumed to be closed.

6.1. Sharpness of Theorem 1.1. In this subsection, we prove that Theorem 1.1 is sharp with respect to the order of \( k \). We first prepare the following proposition, which is an extension of Proposition 3.1 of [13] originally proved by Buser [4] and Gromov [17] independently.

**Proposition 6.1.** Let \( \{ B_\alpha \}_{\alpha=0}^{k-1} \) be a sequence of compact subsets of \( M \) such that \( M = \bigcup_{\alpha=0}^{k-1} B_\alpha \) and \( m(B_\alpha \cap B_\beta) = 0 \) for \( \alpha \neq \beta \). Then we have
\[
\hat{\nu}_{k,p}(M, m)^{\frac{1}{p}} \gtrsim \frac{1}{p} \min_{\alpha} I_1(B_\alpha, m).
\]

**Proof.** We will follow the argument of Theorem 8.2.1 of [4]. By the definition of \( \hat{\nu}_{k,p}(M, m) \), for a given \( \epsilon > 0 \) there exists a \((k+1)\)-dimensional subspace \( L_{k+1} \) of \( W^{1,p}(M, m) \) such that

\[
\sup_{\phi \in L_{k+1} \setminus \{0\}} \frac{\int_M \| \nabla \phi \|^p \, dm}{\int_M |\phi|^p \, dm} < \hat{\nu}_{k,p}(M, m) + \epsilon.
\]
Since \( L_{k+1} \) is \((k+1)\)-dimensional, a standard argument of linear algebra implies the existence of \( \phi_0 \in L_{k+1} \) such that \( \int_M |\phi_0|^p \, dm = 1 \) and \( \phi_0 \) is orthogonal to the characteristic function \( 1_{B_\alpha} \) of \( B_\alpha \) for \( \alpha = 0, \ldots, k-1 \), i.e., \( \int_{B_\alpha} \phi_0 \, dm = 0 \). By the definition of \( \nu_{1,p}(B_\alpha, m) \) we get

\[
\nu_{1,p}(B_\alpha, m) \int_{B_\alpha} |\phi_0|^p \, dm \leq \int_{B_\alpha} \|\nabla \phi_0\|^p \, dm.
\]

From \( \int_M |\phi_0|^p \, dm = 1 \) it follows that

\[
\min_{\alpha} \nu_{1,p}(B_\alpha, m) \leq \sum_{\alpha=0}^{k-1} \nu_{1,p}(B_\alpha, m) \int_{B_\alpha} |\phi_0|^p \, dm \leq \sum_{\alpha=0}^{k-1} \int_{B_\alpha} \|\nabla \phi_0\|^p \, dm = \int_M \|\nabla \phi_0\|^p \, dm.
\]

Combining this with (6.1) and letting \( \epsilon \to 0 \) yield

\[
(6.2) \quad \min_{\alpha} \nu_{1,p}(B_\alpha, m) \leq \hat{\nu}_{k,p}(M, m).
\]

On the other hand, Proposition 2.5 in [31] tells us that

\[
(6.3) \quad \nu_{1,p}(B_\alpha, m)^{\frac{1}{p}} \gtrsim \frac{1}{p} \nu_{1,1}(B_\alpha, m).
\]

Therefore, (6.2), (6.3), and the equivalence of \( \nu_{1,1}(B_\alpha, m) \) and \( I_1(B_\alpha, m) \) lead us to the desired inequality. This completes the proof.

Let us show the sharpness of Theorem 1.1 with respect to the order of \( k \). We will show it by constructing an example, and estimating its Poincaré constant. We can construct an example of closed manifold, but its formulation will be slightly complicated. To avoid complexity, we will construct an example of compact manifold with boundary on which we consider the Neumann boundary condition, and conclude the sharpness for it. Here we notice that Theorem 1.1 and other results on closed manifolds can be applied to the Neumann eigenvalues on compact manifolds with boundary due to the min-max principle. In the end of this section, we will sketch an idea of the construction of the example of closed manifold (see Remark 6.2).

For \( a \in (0, 1) \), we work on a rectangle \([0, 1] \times [0, a]\) in \( \mathbb{R}^2 \). Given \( s, t \geq 0 \) setting

\[
\Omega_{s,t} := \left[ s, s + \frac{1}{2(k+1)} \right] \times [0, t],
\]

we define

\[
M_a := \bigcup_{\alpha=0, \ldots, k} \left( \Omega_{\frac{2\alpha}{2(k+1)}, a} \cup \Omega_{\frac{2\alpha+1}{2(k+1)}, \frac{a}{k+1}} \right).
\]
Let us consider a pairwise disjoint sequence \( \{A_\alpha\}_{\alpha=0}^{k+1} \) of subsets \( A_\alpha \subset M_a \) defined as follows: For \( \alpha = 0, 1, \ldots, k-1 \) we set
\[
A_\alpha := \Omega_{\frac{2\alpha}{2(k+1)}}.
\]
For \( \alpha = k, k+1 \) let us set
\[
A_k := \left[ \frac{6k}{6(k+1)}, \frac{6k+1}{6(k+1)} \right] \times [0, a]
\]
and
\[
A_{k+1} := \left[ \frac{6k+2}{6(k+1)}, \frac{6k+3}{6(k+1)} \right] \times [0, a].
\]
If \( d_{M_a} \) and \( m_{M_a} \) denote the Riemannian distance and the normalized uniform volume measure on \( M_a \) respectively, then we see
\[
m_{M_a}(A_\alpha) \geq \frac{1}{12(k+1)}, \quad d_{M_a}(A_\alpha, A_\beta) \geq \frac{1}{6(k+1)}
\]
and
\[
\sum_{\beta \neq \alpha} m_{M_a}(A_\beta) \geq 1 - \frac{2}{k+1} = \frac{k-1}{k+1}.
\]
Hence, Theorem 1.1 leads to
\[
\nu_{k+1,p}(M_a, m_{M_a}) \lesssim k+1 \sim k.
\]
On the other hand, one can show
\[
\nu_{k+1,p}(M_a, m_{M_a}) \gtrsim \frac{k}{p}
\]
provided that \( a \sim \frac{1}{k+1} \) as follows: For \( \alpha = 0, 1, \ldots, k \) we set
\[
B_\alpha := \Omega_{\frac{2\alpha}{2(k+1)}} \cup \Omega_{\frac{2\alpha+1}{2(k+1)}}.
\]
Then we have \( M_a = \cup_{\alpha=0}^k B_\alpha \) and \( m_{M_a}(B_\alpha \cap B_\beta) = 0 \) for \( \alpha \neq \beta \). Note that \( \mathcal{I}_1(s B_\alpha) = s^{-1} \mathcal{I}_1(B_\alpha) \) where \( s B_\alpha := \{sx \mid x \in B_\alpha\} \). Here we consider the Cheeger constant \( \mathcal{I}_1(s B_\alpha) \) in \( sM_a \) with the normalized uniform volume measure and the Cheeger constant \( \mathcal{I}_1(B_\alpha) \) in \( (M_a, m_{M_a}) \). Since \( a \sim \frac{1}{k+1} \) we have \( \mathcal{I}_1(B_\alpha) \gtrsim k \). Propositions 2.2 and 6.1 yield
\[
(6.5)
\]
Comparing (6.4) with (6.5), we can conclude that Theorem 1.1 is sharp with respect to the order of \( k \).

**Remark 6.1.** This example does not deny the possibility of
\[
\lambda_k(M, m)^{\frac{1}{2}} \lesssim \frac{1}{D(\{A_\alpha\})} \log k \max_{\alpha} \log \frac{1}{m(A_\alpha)}.
\]
This inequality was conjectured in \[13\] under the assumption of the non-negativity of Ricci curvature. One might not need the assumption.
6.2. **Note on Theorem 5.4.** Unlike Theorem 1.1, the authors do not know whether Theorem 5.4 is sharp with respect to the order of \( k \). In this subsection, we discuss its possibility.

For a measurable function \( \phi : M \to \mathbb{R} \), a real number \( \text{med}_\phi \) is called its **median** if it satisfies
\[
m(\{ x \in M \mid \phi(x) \geq \text{med}_\phi \}) \geq \frac{1}{2}, \quad m(\{ x \in M \mid \phi(x) \leq \text{med}_\phi \}) \geq \frac{1}{2}.
\]

Before we start the discussion, let us recall the following fact due to Maz’ya [28] and Federer-Fleming [11] (see e.g., Lemma 2.2 in [31]):

**Theorem 6.2.** The Cheeger constant \( \mathcal{I}_1(M, m) \) is the best constant for the following Poincaré inequality: For any \( \phi \in W^{1,1}(M, m) \),
\[
\mathcal{I}_1(M, m) \int_M |\phi - \text{med}_\phi| dm \leq \int_M \| \nabla \phi \| dm.
\]

Instead of the \( k \)-way isoperimetric constant \( \mathcal{I}_k(M, m) \), we consider the **modified \( k \)-way isoperimetric constant** \( \hat{\mathcal{I}}_k(M, m) \) defined by
\[
\hat{\mathcal{I}}_k(M, m) := \inf_{\{A_\alpha\}_{\alpha=0}^k} \max_{\alpha=0,\ldots,k} \frac{m^+(A_\alpha)}{m(A_\alpha)},
\]
where the infimum is taken over all pairwise disjoint sequences \( \{A_\alpha\}_{\alpha=0}^k \) of non-empty Borel subsets with \( M = \bigcup_{\alpha=0}^k A_\alpha \). We see that \( \mathcal{I}_k(M, m) \leq \hat{\mathcal{I}}_k(M, m) \); moreover the equality holds when \( k = 1 \), and \( \hat{\mathcal{I}}_1(M, m) \) is equivalent to \( \nu_{1,1}(M, m) \) (cf. (5.1)). We have the following assertion:

**Proposition 6.3.** Let \( \{B_\alpha\}_{\alpha=0}^{k-1} \) be a sequence of compact subsets of \( M \) such that \( M = \bigcup_{\alpha=0}^{k-1} B_\alpha \) and \( m(B_\alpha \cap B_\beta) = 0 \) for \( \alpha \neq \beta \). Then we have
\[
\hat{\mathcal{I}}_k(M, m) \geq \min_{\alpha} \mathcal{I}_1(B_\alpha, m).
\]

**Proof.** For any \( \epsilon > 0 \) there exists a pairwise disjoint sequences \( \{A_\beta\}_{\beta=0}^k \) of non-empty Borel subsets with \( M = \bigcup_{\beta=0}^k A_\beta \) such that
\[
\hat{\mathcal{I}}_k(M, m) + \epsilon > \max_{\beta} \frac{m^+(A_\beta)}{m(A_\beta)}.
\]
Using the Borsuk-Ulam theorem we can get constants \( c_0, c_1, \ldots, c_k \) such that \( \phi_0 := \sum_{\beta=0}^k c_\beta 1_{A_\beta} \) bisects each \( B_0, B_1, \ldots B_{k-1} \), i.e.,
\[
m(B_\alpha \cap \phi_0^{-1}[0, \infty)) \geq \frac{m(B_\alpha)}{2}, \quad m(B_\alpha \cap \phi_0^{-1}(-\infty, 0]) \geq \frac{m(B_\alpha)}{2}.
\]
In fact, according to Corollary of [36], in order to bisect \( k \) subsets by a finite combination of \( 1_{A_0}, 1_{A_1}, \ldots, 1_{A_k} \), it suffices to check that \( 1_{A_0}, 1_{A_1}, \ldots, 1_{A_k} \) are linearly independent modulo sets of measure zero (i.e., whenever \( a_0 1_{A_0} + a_1 1_{A_1} + \cdots + a_k 1_{A_k} = 0 \) over a Borel subset of
positive measure, we have \( a_0 = a_1 = \cdots = a_k = 0 \). This holds since \( M = \sqcup_{\beta} A_\beta \). We now apply Theorem 6.2 to Lipschitz functions which approximate, the characteristic function \( 1_{A_\beta} \), in an appropriate sense. Thus we see

\[
\min_{\alpha} I_1(B_\alpha, m) \int_M |\phi_0| dm \leq \sum_{\alpha=0}^{k-1} I_1(B_\alpha, m) \int_{B_\alpha} |\phi_0| dm \\
\leq \sum_{\alpha=0}^{k-1} \int_{B_\alpha} \|\nabla \phi_0\| dm = \int_M \|\nabla \phi_0\| dm \\
\leq \sum_{\beta=0}^{k} |c_\beta|m^+(A_\beta).
\]

From

\[
\int_M |\phi_0| dm = \sum_{\beta=0}^{k} |c_\beta|m(A_\beta),
\]

we derive

\[
\min_{\alpha} I_1(B_\alpha, m) \leq \max_{\beta} \frac{m^+(A_\beta)}{m(A_\beta)}.
\]

Therefore, this together with (6.6) completes the proof. \( \square \)

We now consider \( M_a \) introduced in the above subsection. By Theorem 5.4 we possess \( I_{k+1}(M_a, m_{M_a}) \lesssim k + 1 \). On the other hand, Proposition 6.3, and the same argument as in the above subsection yield \( \tilde{I}_{k+1}(M_a, m_{M_a}) \gtrsim k \) provided that \( a \sim \frac{1}{k+1} \). If one can replace \( \tilde{I}_{k+1}(M, m) \) with \( I_{k+1}(M, m) \) in Proposition 6.3, then we can show that \( I_{k+1}(M_a, m_{M_a}) \gtrsim k \), and conclude that Theorem 5.4 is also sharp.

Remark 6.2. We can discuss the sharpness of Theorems 1.1 and 5.4 by considering the following closed manifold instead of \( M_a \): We construct \((k+1)\) flat tori by identifying the edges of large rectangles in \( M_a \), and connect them by cylinders constructed by identifying the upper edges of small rectangles in \( M_a \) with their lower edges. We can discuss the sharpness by applying the same argument to this closed manifold.

### 7. Poincaré Constants and Inscribed Radii

In this section, we always assume that \( M \) is a compact manifold with boundary. Its inscribed radius is defined as

\[
\text{InRad} M := \sup_{x \in M} d(x, \partial M).
\]
We will discuss upper bounds for the inscribed radii under non-negativity of the weighted Ricci curvature.

We denote by $\dim M$ the dimension of $M$, and by $\text{Ric}_g$ the Ricci curvature induced from the Riemannian metric $g$. For $N \in (-\infty, \infty]$, the $N$-weighted Ricci curvature is defined as follows ([3], [24]):

$$\text{Ric}_N^m := \text{Ric}_g + \text{Hess} f - \frac{df \otimes df}{N - \dim M},$$

where $df$ and Hess $f$ are the differential and the Hessian of $f$, respectively. Let $\text{Ric}_{m,M}^N$ stand for the infimum of $\text{Ric}_N^m$ over the unit tangent bundle over $M$. We produce the following upper bound of the inscribed radius based on the fundamental principles established in Section 4.

**Proposition 7.1.** For $N \in [\dim M, \infty)$ we assume $\text{Ric}_{m,M}^N \geq 0$. Then

$$\text{InRad} M \leq \frac{2}{\nu_{1,p}^D(M,m)^{\frac{1}{p}}} (1 + N \log 2).$$

**Proof.** We set $R := \text{InRad} M$. Since $M$ is compact, we can find a point $x_0 \in M$ such that $R = d(x_0, \partial M)$. For $r > 0$, let $\Omega_r$ denote the the open ball of radius $r$ centered at $x_0$. By the definition of the inscribed radius, $\Omega_R$ is contained in $\text{Int} M$. This enables us to apply the volume comparison theorem of Bishop-Gromov type under $\text{Ric}_N^m \geq 0$ obtained by Qian [33] to $\Omega_R$, $\Omega_{R/2}$ (see Corollary 2 in [33]). Hence

$$2^{-N} \leq \frac{m(\Omega_{R/2})}{m(\Omega_R)} = m_{\Omega_R}(\Omega_{R/2}),$$

where $m_{\Omega_R}$ is defined as (3.2).

On the other hand, from Lemmas 4.1 and 4.2 we deduce

$$m_{\Omega_R}(\Omega_R \setminus B_r(\partial \Omega_R)) \leq \exp \left(1 - \nu_{1,p}^D(\Omega_R, m)^{\frac{1}{p}} r\right) \leq \exp \left(1 - \frac{1}{2} \nu_{1,p}^D(M,m)^{\frac{1}{p}} R \right),$$

for all $r > 0$. By letting $r \to R/2$ in (7.3), and combining it with (7.2), we arrive at

$$2^{-N} \leq m_{\Omega_R}(\Omega_{R/2}) \leq \exp \left(1 - \frac{1}{2} \nu_{1,p}^D(M,m)^{\frac{1}{p}} R \right).$$

This yields the desired inequality. \hfill $\square$

**Remark 7.1.** The authors wonder whether one can extend Proposition 7.1 to the higher-order case of $k \geq 2$ as follows: Under the same setting
as in Proposition 7.1
\[ \text{InRad} M \leq \frac{2k}{\nu_{k,p}^D(M,m)^{\frac{1}{2}}} (1 + N \log 2). \]

In the case of \( p = 2 \), a similar inequality was implicitly shown in [34].

The second author [34] has remarked that a classical method by Cheng [6] leads us to the following (see the inequality (2.16) in Remark 2.4 in [34]): If \( \text{Ric}_{m,M}^N \geq 0 \) for \( N \in [\text{dim} \, M, \infty) \), then
\[ \nu_{k,2}^D(M,m) \leq 2N(N + 4)k^2(\text{InRad} M)^{-2}. \]

The second author [34] has only considered the unweighted case where \( f = 0 \) and \( N = \text{dim} \, M \), but one can extend it to the weighted setting by using weighted comparison geometric results developed by Qian [33].

The inequality (7.4) can be rewritten as
\[ \text{InRad} M \leq \frac{\sqrt{2k}}{\nu_{k,2}^D(M,m)^{\frac{1}{2}}} \sqrt{N(N + 4)}. \]

**Remark 7.2.** The authors do not know whether Proposition 7.1 is optimal over \( p \) and \( N \). For example, for the Euclidean \( r \)-ball \( B_r \), its first Dirichlet eigenvalue \( \lambda_{1,p}^D(B_r) \) of the (unweighted) \( p \)-Laplacian satisfies
\[ \text{InRad} B_r = r \geq \frac{\text{dim} \, B_r}{p \lambda_{1,p}^D(B_r)^{\frac{1}{2}}}, \]
which is a consequence of the \( p \)-Cheeger inequality (see e.g., Theorem 2 in [37], Theorem 4.1 in [27], Theorem 4.3 in [26]). In particular, this simplest example does not show that Proposition 7.1 is optimal with respect to \( p \). Also, Mao [26] has obtained a more refined estimate than (7.1) in the case where \( M \) is a closed ball in a Riemannian manifold of non-negative Ricci curvature (see Theorem 4.3 in [26]). One might be able to extend his estimate to our general setting.

8. Discrete cases

The aim of this last section is to point out that we possess an discrete analogue of Theorem 1.1. We only state the setting, and the statement of the key lemmas and the main results. One can show them along the line of the argument in Section 3. The proof is left to the reader.

8.1. Weighted graphs and their Poincaré constants. We explain our discrete setting. We refer to Chapter 1 in [16] (see also Theorem 3.3 in [21], and Section 4 in [15]): Let \( G = (V,E) \) be a simple connected finite graph. A weighted graph is a pair \( (G, \mu) \) where \( \mu \) is a nonnegative
function on $V \times V$ such that (1) $\mu_{x,y} = \mu_{y,x}$; and (2) $\mu_{x,y} > 0$ iff $x$ and $y$ form an edge. We consider the graph distance on $V$.

For a point $x \in V$ and a subset $\Omega \subset V$ we set $\mu(x) := \sum_{y \in V} \mu_{x,y}$ and $\mu(\Omega) := \sum_{x \in \Omega} \mu(x)$. For a function $\phi : V \to \mathbb{R}$ we define

$$\overline{\phi} := \frac{1}{\mu(V)} \sum_{x \in V} \phi(x) \mu(x),$$

i.e., $\overline{\phi}$ is the mean of $\phi$ with respect to the weight $\mu$. Given $0 \leq k \leq \#V - 1$ we introduce the $k$-th discrete $p$-Poincaré constant of $(G, \mu)$ as

$$\nu_{k,p}(G, \mu) := \inf_{L_k} \sup_{\phi \in L_k \setminus \{0\}} \frac{1}{2} \sum_{x \in V} \left| \phi(y) - \phi(x) \right|^p \mu_{x,y},$$

where the infimum is taken over all $k$-dimensional subspace $L_k$ of the associated $L^p$-space $L^p(G, \mu)$ on $(G, \mu)$ which does not contain nontrivial constant functions. The (weighted) graph Laplacian $\Delta_\mu$ is defined as

$$\Delta_\mu \phi(x) := \phi(x) - \sum_{y \in V} \phi(y) \frac{\mu_{x,y}}{\mu(x)}.$$

Then the $k$-th nontrivial eigenvalue $\lambda_k(G, \mu)$ of $\Delta_\mu$ coincides with $\nu_{k,2}(G, \mu)$.

As in the Riemannian case we compare discrete Poincaré constants with its modified version and then we will work with the modified version. We introduce the $k$-th discrete modified $p$-Poincaré constant of $(G, \mu)$ as

$$\widehat{\nu}_{k,p}(G, \mu) := \inf_{L_{k+1}} \sup_{\phi \in L_{k+1} \setminus \{0\}} \frac{1}{2} \sum_{x \in V} \left| \phi(y) - \phi(x) \right|^p \mu_{x,y},$$

where the infimum is taken over all $(k + 1)$-dimensional subspace $L_{k+1}$ of $L^p(G, \mu)$. Then we have $\nu_{k,2}(G, \mu) = \widehat{\nu}_{k,2}(G, \mu)$ and the following:

**Lemma 8.1.**

$$\nu_{k,p}(G, \mu) \leq \widehat{\nu}_{k,p}(G, \mu) \leq 2^p \nu_{k,p}(G, \mu).$$

To analyze discrete modified Poincaré constants we prepare its local version. For $\Omega \subset V$, we denote by $\partial_\Omega \Omega$ its vertex boundary defined as the set of all $x \in \Omega$ satisfying $\mu_{x,y} > 0$ for some $y \in V \setminus \Omega$. Further, let $L^p_0(\Omega, \mu)$ denote a subspace of $L^p(\Omega, \mu)$ defined as the set of all functions $\phi \in L^p(\Omega, \mu)$ with $\phi|_{\partial_\Omega \Omega} = 0$. We now define the $k$-th discrete Dirichlet $p$-Poincaré constant by

$$\nu^D_{k,p}(\Omega, \mu) := \inf_{L_{k,0}} \sup_{\phi \in L_{k,0} \setminus \{0\}} \frac{1}{2} \sum_{x \in \Omega} \left| \phi(y) - \phi(x) \right|^p \mu_{x,y},$$

where the infimum is taken over all $(k + 1)$-dimensional subspace $L_{k+1}$ of $L^p(\Omega, \mu)$. Then we have $\nu_{k,2}(G, \mu) = \widehat{\nu}_{k,2}(G, \mu)$ and the following:

**Lemma 8.1.**

$$\nu_{k,p}(G, \mu) \leq \nu^D_{k,p}(\Omega, \mu) \leq 2^p \nu_{k,p}(G, \mu).$$

To analyze discrete modified Poincaré constants we prepare its local version. For $\Omega \subset V$, we denote by $\partial_\Omega \Omega$ its vertex boundary defined as the set of all $x \in \Omega$ satisfying $\mu_{x,y} > 0$ for some $y \in V \setminus \Omega$. Further, let $L^p_0(\Omega, \mu)$ denote a subspace of $L^p(\Omega, \mu)$ defined as the set of all functions $\phi \in L^p(\Omega, \mu)$ with $\phi|_{\partial_\Omega \Omega} = 0$. We now define the $k$-th discrete Dirichlet $p$-Poincaré constant by

$$\nu^D_{k,p}(\Omega, \mu) := \inf_{L_{k,0}} \sup_{\phi \in L_{k,0} \setminus \{0\}} \frac{1}{2} \sum_{x \in \Omega} \left| \phi(y) - \phi(x) \right|^p \mu_{x,y},$$

where the infimum is taken over all $(k + 1)$-dimensional subspace $L_{k+1}$ of $L^p(\Omega, \mu)$. Then we have $\nu_{k,2}(G, \mu) = \widehat{\nu}_{k,2}(G, \mu)$ and the following:
where the infimum is taken over all $k$-dimensional subspaces $L_{k,0}$ of $L_0^p(\Omega, \mu)$. We summarize key tools for the proof of our main result.

**Lemma 8.2.** For $\Omega \subset V$, we have
\[
\hat{\nu}_{k,p}(G, \mu) \leq \nu_{k+1,p}(\Omega, \mu).
\]

**Lemma 8.3.** Let $\Omega \subset V$. For all $r > 0$, we have
\[
\frac{\mu(\Omega \setminus B_r(\partial_v \Omega))}{\mu(\Omega)} \leq \exp \left(1 - \frac{\nu_{1,p}(\Omega, \mu)^{\frac{1}{p}} r}{p} \right).
\]

**Lemma 8.4.** For any pairwise disjoint sequence $\{\Omega_\alpha\}_{\alpha=0}^k$ of subsets in $V$, we have
\[
\nu_{k+1,p}(\bigcup_{\alpha=0}^k \Omega_\alpha, \mu) \leq \max_{\alpha=0, \ldots, k} \nu_{1,p}(\Omega_\alpha, \mu).
\]

We can conclude the following discrete analogue of Theorem 1.1:

**Theorem 8.5.** Let $(G, \mu)$ be a weighted graph. For any sequence $\{A_\alpha\}_{\alpha=0}^k$ of subsets of $V$ we have
\[
(8.1) \quad \nu_{k,p}(G, \mu)^{\frac{1}{p}} \leq \frac{2}{\mathcal{D}(\{A_\alpha\})} \max_{\alpha=0, \ldots, k} \log \frac{e(\mu(V) - \sum_{\beta \neq \alpha} \mu(A_\alpha))}{\mu(A_\alpha)}.
\]

**Remark 8.1.** The case where $k = 1$ and $p = 2$ were studied in ([1]). In this case Alon-Milman obtained a similar bound for $\lambda_1(G, \mu)$.

**Remark 8.2.** Let us compare Theorem 8.5 with Chung-Grigor’yan-Yau’s result ([7, 8], see Theorem 3.22 of [16]). Let $N := \#V$. Given $A_0, A_1, \ldots, A_k \subset V$ such that $\mathcal{D}(\{A_\alpha\}) > 1$ we set
\[
\delta(\{A_\alpha\}) := \max_{\alpha \neq \beta} \left(\frac{\mu(V \setminus A_\alpha)\mu(V \setminus A_\beta)}{\mu(A_\alpha)\mu(A_\beta)}\right)^{\frac{1}{\mathcal{D}(\{A_\alpha\})-1}}.
\]

We may assume that $\mu(A_0) \leq \mu(A_1) \leq \cdots \leq \mu(A_k)$, and thus $\mu(A_\alpha) \leq \frac{1}{2} \mu(V)$ for $\alpha = 0, \ldots, k - 1$; in particular, $\delta(\{A_\alpha\}) \geq 1$. In the case of $p = 2$ Chung-Grigor yan-Yau showed that
\[
(8.2) \quad \lambda_k(G, \mu) \leq \frac{\delta(\{A_\alpha\}) - 1}{\delta(\{A_\alpha\}) + 1} \lambda_{N-1}(G, \mu).
\]

Note that $\frac{N}{N-1} \leq \lambda_{N-1}(G, \mu) \leq 2$ (Theorem 2.11 of [16]) and hence the right-hand side of the above inequality (8.2) is at most 2 whereas our bound in (8.1) can be greater than 2. For example if we choose $\{A_\alpha\}_{\alpha=0}^k$ such that $\mu(A_\alpha)$ is small and $\mathcal{D}(\{A_\alpha\})$ is not large then the right-hand side of (8.1) is greater than 2, hence in this case the inequality (8.1) is not informative. It seems in many choices of $\{A_\alpha\}$ their inequality (8.2) is better than our inequality (8.1). Our inequality can be better when
\( (\mu(V) - \sum_{\beta \neq \alpha} \mu(A_\beta))/\mu(A_\alpha) \sim 1 \) and \( \mathcal{D}(\{A_\alpha\}) \gg 1 \). For example, let us consider a family \( \{(V_\alpha, E_\alpha)\}_{\alpha=0}^k \) of complete graphs such that \( \#V_\alpha = k \). For each \( \alpha = 0, 1, \ldots, k-1 \) we choose and fix a vertex \( x_\alpha \in V_\alpha \) and connect \( x_\alpha \) and \( x_{\alpha+1} \) by a path graph \( (V'_\alpha, E'_\alpha) \) of \( \#V'_\alpha \sim \log k \). We then set

\[
V := V_k \cup \bigcup_{\alpha=0}^{k-1} (V_\alpha \cup V'_\alpha) \quad \text{and} \quad E := E_k \cup \bigcup_{\alpha=0}^{k-1} (E_\alpha \cup E'_\alpha).
\]

Then \( G := (V, E) \) is a finite connected graph and we equip \( G \) the simple weight \( \mu \), i.e., \( \mu_{x,y} = 1 \) iff \( x, y \) form an edge in \( E \). We then normalize \( \mu \) so that \( \mu(V) = 1 \). For each \( \alpha = 0, 1, \ldots, k \) we set \( A_\alpha := V_\alpha \).

Then since \( \mu(A_\alpha) \geq \frac{1}{k+1} \) we have \( (1 - \sum_{\beta \neq \alpha} \mu(A_\beta))/\mu(A_\alpha) \sim 1 \) and \( \mathcal{D}(\{A_\alpha\}) \sim \log k \). Thus in this graph \( G \) our inequality (8.1) implies

\[
\lambda_k(G, \mu)^2 \lesssim \frac{2}{\log k}.
\]

(8.3)

On the other hand, since

\[
\frac{(1 - \mu(A_\alpha))(1 - \mu(A_\beta))}{\mu(A_\alpha)\mu(A_\beta)} \sim k^{1/\log k} = e,
\]

the inequality (8.2) implies

\[
\lambda_k(G, \mu) \lesssim \lambda_{N-1}(G, \mu) \sim 1.
\]

(8.4)

Comparing (8.3) with (8.4) our bound is better in this case.

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