Orientable smooth manifolds are essentially quasigroups

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Introduction

- In the mid-2010s Herman and Pakianathan introduced a functorial construction of closed surfaces from noncommutative finite groups.
- Semin Yoo and I decided to produce an $n$-dimensional generalization.
- The two main challenges in doing this were finding an appropriate analogue of noncommutative groups and in desingularizing the $n$-dimensional pseudomanifolds which arose at the first stage of our construction.
- Ultimately we found that every orientable triangulable manifold could be manufactured in the manner we described.
Talk outline

- Herman and Pakianathan’s construction
- Quasigroups
- The first functor: Open serenation
- The second functor: Serenation
Consider the quaternion group $G$ of order 8 whose universe is $G := \{\pm 1, \pm i, \pm j, \pm k\}$.

We begin by picking out all the pairs of elements $(x, y) \in G^2$ so that $xy \neq yx$. We call this collection $\text{NCT}(G)$.

We define $\text{In}(G)$ to be all the elements of $G$ which are entries in some pair $(x, y) \in \text{NCT}(G)$.

Similarly, $\text{Out}(G)$ is defined to be all the members of $G$ of the form $f(x, y)$ where $(x, y) \in \text{NCT}(G)$. 
Herman and Pakianathan’s construction

- In this case we have

\[ \text{NCT}(G) = \left\{ (\pm u, \pm v) \mid \{u, v\} \in \binom{\{i, j, k\}}{2} \right\} \]

so

\[ \text{In}(G) = \{\pm i, \pm j, \pm k\} \]

and

\[ \text{Out}(G) = \{\pm i, \pm j, \pm k\}. \]

- From this data we form a simplicial complex (actually a 2-pseudomanifold) whose facets are of the form \( \{x, y, f(x, y)\} \) where \((x, y) \in \text{NCT}(G)\).
Herman and Pakianathan’s construction

During the talk I drew a part of this complex here:
Herman and Pakianathan’s construction

- The three 4-cycles

\[(i, j, -i, -j), (i, k, -i, -k), \text{ and } (j, k, -j, -k).\]

each carry an octohedron.
This simplicial complex, which we call \( \text{Sim}(G) \) and Herman and Pakianathan called \( X(Q_8) \), consists of three 2-spheres, each pair of which is glued at two points.

Deleting these points to disjointize the spheres and filling the resulting holes yields the manifold we call \( \text{Ser}(G) \) and Herman and Pakianathan called \( Y(Q_8) \).

In this case \( \text{Ser}(G) \) is the disjoint union of three 2-spheres.
Quasigroups

Definition (Quasigroup)

A (binary) quasigroup is a magma $A := (A, f: A^2 \rightarrow A)$ such that if any two of the variables $x$, $y$, and $z$ are fixed the equation $f(x, y) = z$ has a unique solution.

That is, a quasigroup is a magma whose Cayley table is a Latin square, where each entry occurs once in each row and each column.

All groups are quasigroups, but quasigroups need not have identities or be associative.
The midpoint operation

\[ f(x, y) := \frac{1}{2}(x + y) \]

is a quasigroup operation on \( \mathbb{R}^n \).

The magma \((\mathbb{Z}, -)\) is a quasigroup.
Quasigroups

**Definition (Quasigroup)**

A *(binary)* quasigroup is an algebra \( A := (A, f, g_1, g_2) \) where for all \( x_1, x_2, y \in A \) we have

\[
\begin{align*}
  f(g_1(x_2, y), x_2) &= y, \\
  f(x_1, g_2(x_1, y)) &= y, \\
  g_1(x_2, f(x_1, x_2)) &= x_1,
\end{align*}
\]

and

\[
\begin{align*}
  g_2(x_1, f(x_1, x_2)) &= x_2.
\end{align*}
\]

- We think of \( g_1(x, y) \) as the division of \( y \) by \( x \) in the second coordinate.
The preceding definition shows that the class $\text{Quas}_2$ of all binary quasigroups can be defined by universally-quantified equations, or \textit{identities}.

This means that $\text{Quas}_2$ is a variety of algebras in the sense of universal algebra, and hence forms a category $\text{Quas}_2$ which is closed under taking quotients, subalgebras, and products.

Note that Herman and Pakianathan’s construction works with noncommutative quasigroups just as well as with groups.

We would then like an $n$-ary version of a quasigroup for our $n$-dimensional generalization.
Quasigroups

Definition (Quasigroup)

An \( n \)-quasigroup is an \( n \)-magma \( A := (A, f: A^n \to A) \) such that if any \( n - 1 \) of the variables \( x_1, \ldots, x_n, y \) are fixed the equation

\[
f(x_1, \ldots, x_n) = y
\]

has a unique solution.

- That is, an \( n \)-quasigroup is an \( n \)-magma whose Cayley table is a Latin \( n \)-cube.
- All \( n \)-ary groups are quasigroups, but quasigroups need not be associative.
Quasigroups

- Given any group $G$ the $n$-ary multiplication

\[
f(x_1, \ldots, x_n) := x_1 \cdots x_n
\]

is a quasigroup operation on $G$. 
Quasigroups

Definition (Quasigroup)

An $n$-quasigroup is an algebra

$$
A := (A, f, g_1, \ldots, g_n)
$$

where for all $x_1, \ldots, x_n, y \in A$ and each $i \in \{1, 2, \ldots, n\}$ we have

$$
f(x_1, \ldots, x_{i-1}, g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y), x_{i+1}, \ldots, x_n) = y
$$

and

$$
g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, f(x_1, \ldots, x_n)) \approx x_i.
$$

- We think of $g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y)$ as the division of $y$ simultaneously by $x_j$ in the $j^{th}$ coordinate for each $j \neq i$.  

Quasigroups

- We say that an \( n \)-quasigroup \( A \) is *commutative* when for all \( x_1, \ldots, x_n \in A \) and all \( \sigma \in \text{Perm}_n \) we have
  \[
  f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).
  \]

- We say that an \( n \)-quasigroup \( A \) is *alternating* when for all \( x_1, \ldots, x_n \in A \) and all \( \sigma \in \text{Alt}_n \) we have
  \[
  f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).
  \]

- Our “correct” analogue of the variety of groups will be the variety \( AQ_n \) of alternating \( n \)-ary quasigroups.
Quasigroups

- By general results in universal algebra there are nontrivial members of $AQ_n$ for each $n$, but the easiest examples are either commuting (take the $n$-ary multiplication for an abelian group) or infinite (the free alternating quasigroups, which we need later but are too much right now).

- We tediously found the following example by hand:
Quasigroups

Take \( S := (\mathbb{Z}/5\mathbb{Z})^3 \) and define \( h: \mathbb{Z}/5\mathbb{Z} \times \text{Alt}_3 \rightarrow \text{Perm}_S \) by

\[
(h(k, \sigma))(x_1, x_2, x_3) := (x_\sigma(1) + k, x_\sigma(2) + k, x_\sigma(3) + k).
\]

There are 7 members of \( \text{Orb}(h) \). One system of orbit representatives is:

\[
\{000, 011, 022, 012, 021, 013, 031\}.
\]
Quasigroups

Let $A := \mathbb{Z}/5\mathbb{Z}$ and define a ternary operation $f: A^3 \to A$ so that

$$f((h(k, \sigma))(x_1, x_2, x_3)) = f(x_1, x_2, x_3) + k$$

and $f$ is defined on the above set of orbit representatives as follows.

| $xyz$ | $f(x, y, z)$ |
|-------|--------------|
| 000   | 0            |
| 011   | 0            |
| 022   | 0            |
| 012   | 3            |
| 021   | 4            |
| 013   | 4            |
| 031   | 2            |
By taking products of $A := (A, f)$ this gives us infinitely many finite, noncommutative, alternating ternary quasigroups, but we only have one basic example.

We reached out to Jonathan Smith to see if anyone had studied the varieties of alternating $n$-quasigroups before, but it seemed that no one had.

He did, however, give us an example which we generalized into an alternating product construction which takes an $n$-ary commutative quasigroup and an $(n + 1)$-ary commutative quasigroup and yields an $n$-ary alternating quasigroup which is typically not commutative.
The first functor: Open serenation

- Our first construction gives a functor
  \[ \text{OSer}_n: \text{NCAQ}_n \to \text{SMfld}_n. \]

- We define
  \[ \text{Sim}_n: \text{NCAQ}_n \to \text{PMfld}_n \]
  similarly to our previous example for \( n = 2 \).

- We define \( \text{NCT}(A) \) to consist of all tuples \((a_1, \ldots, a_n) \in A^n\) such that \( f(a_1, \ldots, a_n) \neq f(a_2, a_1, \ldots, a_n) \).

- We define \( \text{In}(A) \) to consist of all entries in noncommuting tuples of \( A \) and \( \text{Out}(A) \) to consist of all \( f(a_1, \ldots, a_n) \) where \((a_1, \ldots, a_n) \in \text{NCT}(A)\).
The first functor: Open serenade

- We set

\[
\text{Sim}(A) := \{ a \mid a \in \text{ln}(A) \} \cup \{ \overline{a} \mid a \in \text{Out}(A) \}
\]

and

\[
\text{SimFace}(A) := \bigcup_{a \in \text{NCT}(A)} \text{Sb} \left( \left\{ a_1, \ldots, a_n, \overline{f(a)} \right\} \right).
\]

- We define

\[
\text{Sim}_n(A) := (\text{Sim}(A), \text{SimFace}(A)).
\]
The first functor: Open serenation

- We create $\text{OSer}_n(A)$ by taking the open geometric realization of $\text{Sim}_n(A)$ (basically all but the $(n - 2)$-skeleton of the open geometric realization) and then equipping it with a smooth atlas.

- The *standard open bipyramid* (or just *bipyramid*) in $\mathbb{R}^n$ is

$$\text{Bipy}_n := \text{OCvx} \left( \left\{ (0, \ldots, 0), \left( \frac{2}{n}, \ldots, \frac{2}{n} \right) \right\} \cup \{e_1, \ldots, e_n\} \right)$$

where $e_i$ is the $i^{\text{th}}$ standard basis vector of $\mathbb{R}^n$. 
The first functor: Open serenation

- Given an alternating $n$-quasigroup $\mathbf{A}$ and $a = (a_1, \ldots, a_n) \in \text{NCT}(\mathbf{A})$ the serene chart of input type for $a$ is

$$\phi_a : \text{Bipy}_{n} \rightarrow \text{OSer}_{n}(\mathbf{A}).$$

- We set

$$\phi_a(u_1, \ldots, u_n) := \sum_{i=1}^{n} u_i a_i + \left(1 - \sum_{i=1}^{n} u_i\right) \overline{f(a)}$$

when $\sum_{i=1}^{n} u_i \leq 1$.

- Otherwise,

$$\phi_a(u_1, \ldots, u_n) := \frac{2}{n} \sum_{i=1}^{n} \left(1 + \frac{n - 2}{2} u_i - \sum_{j \neq i} u_j\right) a_i + \left(-1 + \sum_{i=1}^{n} u_i\right) \overline{f(a')}.$$
The first functor: Open serenation

- There are also serene charts of output type, where are defined similarly.

- We set

  \[(O\text{Ser}_n(A), \tau) := (O\text{Geo}_n \circ \text{Sim}_n)(A)\].

- We then define

  \[O\text{Ser}_n(A) := (O\text{Ser}_n(A), \tau, \text{SerAt}_n(A))\]

  where

  \[\text{SerAt}_n(A) := \bigcup_{a \in NCT(A)} \{\phi_a, \bar{\phi}_a\} \].
The first functor: Open serenation

- The incidence graph of the facets of $\text{Sim}(A)$ for the ternary quasigroup $A$ from the previous example is pictured.
The first functor: Open serenade

The 1-skeleton of $\text{Sim}(A)$ for the ternary quasigroup $A$ from the previous example is pictured.
The first functor: Open serenation

- One may verify that $\text{OSer}(A)$ is a 3-sphere minus the graph pictured previously, which is homotopy equivalent to the join of 21 circles.
The second functor: Serenation

For any alternating quasigroup $A$ we may equip $O\text{Ser}(A)$ with a Riemannian metric in a functorial manner which makes $O\text{Ser}(A)$ flat.

We then define a *Euclidean metric completion functor*

$$\text{EuCmplt}: \text{Riem}_n \to \text{Mfld}_n$$

which assigns to a Riemannian manifold $(M, g)$ the topological manifold consisting of all points in the metric completion of $M$ which are locally Euclidean.
The second functor: Serenation

- The *serenation functor*

  \[ \text{Ser}_n : \text{NCAQ}_n \to \text{Mfld}_n \]

  is given by

  \[ \text{Ser}(A) := \text{EuCmplt}(\text{OSer}(A), g) \]

  where \( g \) is the standard metric on \( \text{OSer}(A) \).

- In the previous example of the ternary quasigroup \( A \) we find that \( \text{Ser}_3(A) \) is the 3-sphere.
The second functor: Serenation

Definition (Serene manifold)

We say that a connected orientable $n$-manifold $M$ is serene when there exists some alternating $n$-quasigroup $A$ such that $M$ is a component of $\text{Ser}(A)$. 
The second functor: Serenation

Theorem (A., Yoo (2021))

*Every connected orientable triangulable n-manifold is serene.*
The second functor: Serenation

Theorem (A., Yoo (2021))

Every connected orientable triangulable $n$-manifold is serene.

- Consider a triangulation of the given manifold $M$.
- Subdivide each facet in a manner I will draw off to the side.
- We have that $M$ is homeomorphic to a corresponding component of the serenation of a quotient of the free alternating $n$-quasigroup whose generators are the vertices of the subdivided triangulation.
References

- **Mark Herman and Jonathan Pakianathan.** “On a canonical construction of tessellated surfaces from finite groups”. *In: Topology Appl.* 228 (2017), pp. 158–207. [ISSN: 0166-8641](https://doi.org/10.1016/j.topol.2017.01.021)