J—holomorphic Curves, Legendre Submanifolds and Reeb Chords *

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Abstract
In this article, we prove that there exists at least one chord which is characteristic of Reeb vector field connecting a given Legendre submanifold in a closed contact manifold with any contact form.

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1 Introduction and results
Let Σ be a smooth closed oriented manifold of dimension 2n − 1. A contact form on Σ is a 1–form such that λ ∧ (dλ)n−1 is a volume form on Σ. Associated to λ there are two important structures. First of all the so-called Reeb vectorfield ħ = X defined by

\[i_X \lambda \equiv 1, \quad i_X d\lambda \equiv 0;\]

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and secondly the contact structure \( \xi = \xi_\lambda \mapsto \Sigma \) given by

\[
\xi_\lambda = \ker(\lambda) \subset T\Sigma.
\]

By a result of Gray, [7], the contact structure is very stable. In fact, if \((\lambda_t)_{t \in [0,1]}\) is a smooth arc of contact forms inducing the arc of contact structures \((\xi_t)_{t \in [0,1]}\), there exists a smooth arc \((\psi_t)_{t \in [0,1]}\) of diffeomorphisms with \(\psi_0 = Id\), such that

\[
T\psi_t(\xi_0) = \xi_t.
\]

(1.1)

here it is important that \(\Sigma\) is compact. From (1.1) and the fact that \(\psi_0 = Id\) it follows immediately that there exists a smooth family of maps \([0,1] \times \Sigma \mapsto (0,\infty) : (t,m) \mapsto f_t(m)\) such that

\[
\psi_t^*\lambda_t = f_t\lambda_0
\]

(1.2)

In contrast to the contact structure the dynamics of the Reeb vectorfield changes drastically under small perturbation and in general the flows associated to \(X_t\) and \(X_s\) for \(t \neq s\) will not be conjugated.

Concerning the dynamics of Reeb flow, there is a well-known conjecture raised by Arnold in [2] which concerned the Reeb orbit and Legendre submanifold in a contact manifold. If \((\Sigma, \lambda)\) is a contact manifold with contact form \(\lambda\) of dimension \(2n - 1\), then a Legendre submanifold is a submanifold \(L\) of \(\Sigma\), which is \((n-1)\)dimensional and everywhere tangent to the contact structure \(\ker \lambda\). Then a characteristic chord for \((\lambda, L)\) is a smooth path

\[
x : [0,T] \to M, T > 0
\]

with

\[
\dot{x}(t) = X_\lambda(x(t)) \text{ for } t \in (0,T),
\]

\[
x(0), x(T) \in L
\]

Arnold raised the following conjecture:

**Conjecture** (see [2]). Let \(\lambda_0\) be the standard tight contact form

\[
\lambda_0 = \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)
\]

on the three sphere

\[
S^3 = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 | x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\}.
\]
If \( f : S^3 \to (0, \infty) \) is a smooth function and \( \mathcal{L} \) is a Legendre knot in \( S^3 \), then there is a characteristic chord for \((f\lambda_0, \mathcal{L})\).

The main results of this paper as following:

**Theorem 1.1** Let \((\Sigma, \lambda)\) be a contact manifold with contact form \( \lambda \), \( X_\lambda \) its Reeb vector field, \( \mathcal{L} \) a closed Legendre submanifold, then there exists at least one characteristic chord for \((X_\lambda, \mathcal{L})\).

**Corollary 1.1** ([15, 16]) Let \((S^3, f\lambda_0)\) be a tight contact manifold with contact form \( f\lambda_0 \), \( X_{f\lambda_0} \) its Reeb vector field, \( \mathcal{L} \) a closed Legendre submanifold, then there exists at least one characteristic chord for \((X_{f\lambda_0}, \mathcal{L})\).

**Sketch of proofs:** We work in the framework as in [4, 8, 15]. In Section 2, we study the linear Cauchy-Riemann operator and sketch some basic properties. In section 3, first we construct a Lagrangian submanifold \( W \) under the assumption that there does not exists Reeb chord connecting the Legendre submanifold \( \mathcal{L} \); second, we study the space \( \mathcal{D}(V,W) \) of contractible disks in manifold \( V \) with boundary in Lagrangian submanifold \( W \) and construct a Fredholm section of tangent bundle of \( \mathcal{D}(V,W) \). In section 4, following [4, 8, 15], we prove that the Fredholm section is not proper by using a special anti-holomorphic section as in [4, 8, 15]. In section 5-6, we use a geometric argument to prove the boundaries of \( J^- \)-holomorphic curves remain in a finite part of Lagrangian submanifold \( W \). In the final section, we use nonlinear Fredholm trick in [4, 8, 15] to complete our proof.

## 2 Linear Fredholm Theory

For \( 100 < k < \infty \) consider the Hilbert space \( V_k \) consisting of all maps \( u \in H^{k,2}(D, C \times C^n) \), such that \( u(z) \in \{izR\} \times R^n \subset C \times C^n \) for almost all \( z \in \partial D \). \( L_{k-1} \) denotes the usual Sobolev space \( H_{k-1}(D, C \times C^n) \). We define an operator \( \bar{\partial} : V_k \mapsto L_{k-1} \) by

\[
\bar{\partial}u = u_s + iu_t
\]  

(2.1)

where the coordinates on \( D \) are \((s, t) = s + it, D = \{z||z| \leq 1\}\). The following result is well known (see[19]).
Proposition 2.1 \( \partial : V_k \mapsto L_{k-1} \) is a surjective real linear Fredholm operator of index \( n + 3 \). The kernel consists of \( (a_0 + i s z - \bar{a}_0 z^2, s_1, \ldots, s_n) \), \( a_0 \in C \), \( s, s_1, \ldots, s_n \in R \).

Let \( (\bar{C}^n, \sigma = -Im(\cdot, \cdot)) \) be the standard symplectic space. We consider a real \( n \)-dimensional plane \( R^n \subset C^n \). It is called Lagrangian if the skew-scalar product of any two vectors of \( R^n \) equals zero. For example, the plane \( \{(p, q) | p = 0\} \) and \( \{(p, q) | q = 0\} \) are two transversal Lagrangian subspaces. The manifold of all (nonoriented) Lagrangian subspaces of \( R^{2n} \) is called the Lagrangian-Grassmanian \( \Lambda(n) \). One can prove that the fundamental group of \( \Lambda(n) \) is free cyclic, i.e. \( \pi_1(\Lambda(n)) = Z \). Next assume \( (\Gamma(z))_{z \in \partial D} \) is a smooth map associating to a point \( z \in \partial D \) a Lagrangian subspace \( \Gamma(z) \) of \( C^n \), i.e. \( (\Gamma(z))_{z \in \partial D} \) defines a smooth curve \( \alpha \) in the Lagrangian-Grassmanian manifold \( \Lambda(n) \). Since \( \pi_1(\Lambda(n)) = Z \), one have \([\alpha] = k \epsilon \), we call integer \( k \) the Maslov index of curve \( \alpha \) and denote it by \( m(\Gamma) \), see([3, 19]).

Now let \( z : S^1 \mapsto \{R \times R^n \subset C \times C^n\} \in \Lambda(n + 1) \) be a constant curve. Then it defines a constant loop \( \alpha \) in Lagrangian-Grassmanian manifold \( \Lambda(n + 1) \). This loop defines the Maslov index \( m(\alpha) \) of the map \( z \) which is easily seen to be zero.

Now let \( (V, \omega) \) be a symplectic manifold, \( W \subset V \) a closed Lagrangian submanifold. Let \( (\bar{V}, \bar{\omega}) = (D \times V, \omega_0 + \omega) \) and \( \bar{W} = \partial D \times W \). Let \( \bar{u} = (id, u) : (D, \partial D) \mapsto (D \times V, \partial D \times W) \) be a smooth map homotopic to the map \( u_0 = (id, u_0) \), here \( u_0 : (D, \partial D) \mapsto p \in W \subset V \). Then \( \bar{u}^*TV \) is a symplectic vector bundle on \( D \) and \( (\bar{u}|_{\partial D})^*\bar{W} \) is a Lagrangian subbundle in \( \bar{u}^*TV|_{\partial D} \). Since \( \bar{u} : (D, \partial D) \mapsto (\bar{V}, \bar{W}) \) is homotopic to \( \bar{u}_0 \), i.e., there exists a homotopy \( h : [0, 1] \times (D, \partial D) \mapsto (\bar{V}, \bar{W}) \) such that \( h(0, z) = (z, p), h(1, z) = \bar{u}(z) \), we can take a trivialization of the symplectic vector bundle \( h^*\bar{V} \) on \([0, 1] \times (D, \partial D)\) as

\[
\Phi(h^*\bar{V}) = [0, 1] \times D \times C \times C^n
\]

and

\[
\Phi((h|_{[0,1] \times \partial D})^*\bar{W}) \subset [0, 1] \times S^1 \times C \times C^n
\]

Let

\[
\pi_2 : [0, 1] \times D \times C \times C^n \rightarrow C \times C^n
\]

then

\[
\bar{h} : (s, z) \in [0, 1] \times S^1 \rightarrow \pi_2\Phi(h|_{[0,1] \times \partial D})^*\bar{W}|(s, z) \in \Lambda(n + 1).
\]
Lemma 2.1 Let $\bar{u} : (D, \partial D) \to (V, W)$ be a $C^k$-map ($k \geq 1$) as above. Then,

$$m(\bar{u}) = 2.$$  

Proof. Since $\bar{u}$ is homotopic to $\bar{u}_0$ in $\bar{V}$ relative to $W$, by the above argument we have a homotopy $\Phi_s$ of trivializations such that

$$\Phi_s(\bar{u}^*TV) = D \times C \times C^n$$

and

$$\Phi_s((\bar{u}|_{\partial D})^*T\bar{W}) \subset S^1 \times C \times C^n$$

Moreover

$$\Phi_0((\bar{u}|_{\partial D})^*T\bar{W}) = S^1 \times izR \times R^n$$

So, the homotopy induces a homotopy $\tilde{h}$ in Lagrangian-Grassmanian manifold. Note that $m(\tilde{h}(0, \cdot)) = 0$. By the homotopy invariance of Maslov index, we know that $m(\tilde{u}|_{\partial D}) = 2$.

Consider the partial differential equation

$$\bar{\partial} \bar{u} + A(z)\bar{u} = 0 \text{ on } D$$

$$\bar{u}(z) \in \Gamma(z)(izR \times R^n) \text{ for } z \in \partial D$$

$$\Gamma(z) \in GL(2(n+1), R) \cap Sp(2(n+1))$$

$$m(\Gamma) = 2$$ (2.2)

For $100 < k < \infty$ consider the Banach space $\bar{V}_k$ consisting of all maps $u \in H^{k,2}(D, C^n)$ such that $u(z) \in \Gamma(z)$ for almost all $z \in \partial D$. Let $L_{k-1}$ the usual Sobolev space $H_{k-1}(D, C \times C^n)$

Proposition 2.2 $\bar{\partial} : \bar{V}_k \to L_{k-1}$ is a real linear Fredholm operator of index $n+3$.

3 Nonlinear Fredholm Theory

3.1 Constructions of Lagrangian submanifolds

Let $(\Sigma, \lambda)$ be a contact manifolds with contact form $\lambda$ and $X$ its Reeb vector field, then $X$ integrates to a Reeb flow $\eta_t$ for $t \in R^1$. Consider the form $d(e^{\theta} \lambda)$
on the manifold \((R \times \Sigma)\), then one can check that \(d(e^a \lambda)\) is a symplectic form on \(R \times \Sigma\). Moreover, one can check that

\[
i_X(e^a \lambda) = e^a \quad \text{(3.1)}
\]
\[
i_X(d(e^a \lambda)) = -de^a \quad \text{(3.2)}
\]

So, the symplectization of Reeb vector field \(X\) is the Hamilton vector field of \(e^a\) with respect to the symplectic form \(d(e^a \lambda)\). Therefore, the Reeb flow lifts to the Hamilton flow \(h_s\) on \(R \times \Sigma\) (see[3]).

Let \(\mathcal{L}\) be a closed Legendre submanifold in \((\Sigma, \lambda)\), i.e., there exists a smooth embedding \(Q : \mathcal{L} \to \Sigma\) such that \(Q^* \lambda |_{\mathcal{L}} = 0\), \(\lambda |_{Q(L)} = 0\). We also write \(\mathcal{L} = Q(\mathcal{L})\). Let

\[
(V', \omega') = (R \times \Sigma, d(e^a \lambda))
\]

and

\[
W' = \mathcal{L} \times R, \quad W'_s = \mathcal{L} \times \{s\}; \quad L' = (0, \cup_s \eta_s(Q(\mathcal{L}))), \quad L'_s = (0, \eta_s(Q(\mathcal{L})))
\]

(3.3)

define

\[
G' : W' \to V'
\]
\[
G'(w') = G'(l, s) = (0, \eta_s(Q(l)))
\]

(3.4)

**Lemma 3.1** There does not exist any Reeb chord connecting Legendre submanifold \(\mathcal{L}\) in \((\Sigma, \lambda)\) if and only if \(G'(W'_s) \cap G'(W'_{s'})\) is empty for \(s \neq s'\).

Proof. Obvious.

**Lemma 3.2** If there does not exist any Reeb chord for \((X_\lambda, \mathcal{L})\) in \((\Sigma, \lambda)\) then there exists a smooth embedding \(G' : W' \to V'\) with \(G'(l, s) = (0, \eta_s(Q(l)))\) such that

\[
G'_{K} : \mathcal{L} \times (-K, K) \to V'
\]

(3.5)

is a regular open Lagrangian embedding for any finite positive \(K\). We denote

\[
W'(-K, K) = G'_{K}(\mathcal{L} \times (-K, K))
\]

Proof. One check

\[
G'^*(d(e^a \lambda)) = \eta(\cdot, \cdot)^*d\lambda = (\eta^*_s d\lambda + i_X d\lambda \wedge ds) = 0
\]

(3.6)
This implies that $G'$ is a Lagrangian embedding, this proves Lemma 3.2.

In fact the above proof checks that

$$G'^*(\lambda) = \eta(\cdot, \cdot)\lambda = \eta_\nu^*\lambda + i_X\lambda ds = ds. \quad (3.7)$$

i.e., $W'$ is an exact Lagrangian submanifold.

Now we construct an isotopy of Lagrangian embeddings as follows:

$$F': \mathcal{L} \times R \times [0,1] \rightarrow R \times \Sigma$$

$$F'(l,s,t) = (a(s,t), G'(l,s)) = (a(s,t), \eta_s(Q(l)))$$

$$F'_t(l,s) = F'(l,s,t) \quad (3.8)$$

**Lemma 3.3** If there does not exist any Reeb chord for $(X_\lambda, \mathcal{L})$ in $(\Sigma, \lambda)$ and we choose the smooth $a(s,t)$ such that $\int_0^s a(\tau, t) d\tau$ and $\int_s^0 a(\tau, t) d\tau$ exists, then $F'$ is an exact isotopy of Lagrangian embeddings (not regular). Moreover if $a(s,0) \neq a(s,1)$, then $F'_0(\mathcal{L} \times R) \cap F'_1(\mathcal{L} \times R) = \emptyset$.

Proof. Let $F'_t = F'(\cdot, t) : \mathcal{L} \times R \rightarrow R \times \Sigma$. It is obvious that $F'_t$ is an embedding. We check

$$F'^*(d(e^a\lambda)) = d(F'^*(e^a\lambda))$$

$$= d(e^{a(s,t)} G'^*\lambda)$$

$$= d(e^{a(s,t)} ds)$$

$$= e^{a(s,t)} (a_s ds + a_t dt) \wedge ds$$

$$= e^{a(s,t)} a_t dt \wedge ds \quad (3.9)$$

which shows that $F'_t$ is a Lagrangian embedding for fixed $t$. Moreover for fixed $t$,

$$F'_t(e^a\lambda) = e^{a(s,t)} ds$$

$$= \left\{ \begin{array}{ll}
   d(\int_0^s e^{a(\tau,t)} d\tau) & \text{for } s \geq 0 \\
   d(- \int_s^0 e^{a(\tau,t)} d\tau) & \text{for } s \leq 0
\end{array} \right. \quad (3.10)$$

which shows that $F'_t$ is an exact Lagrangian embedding, this proves Lemma 3.3.

Now we take $a(s,t) = \frac{a_{0t}}{8} e^{-s^2}$ which satisfies the assumption in Lemma 3.3, then

$$F' : \mathcal{L} \times R \times [0,1] \rightarrow R \times \Sigma$$

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Let \( \psi_0(s, t) = se^{a(s, t)}a_s = -2\frac{a_0^s}{8}e^{\left(\frac{\partial}{\partial s}e^{-s^2}\right)}s^2 \) \hspace{1cm} (3.11)

\[ \psi_1(s, t) = \int_{-\infty}^{s} \psi_0(\tau, t)d\tau \] \hspace{1cm} (3.12)

\[ \psi = \frac{\partial \psi_1}{\partial t} - s e^{a(s, t)}a_t \] \hspace{1cm} (3.13)

and compute

\[ F'(\varepsilon^a) = e^{a(s, t)}ds \]
\[ = d(se^{a(s, t)}) - se^{a(s, t)}a_ds - se^{a(s, t)}a_t dt \]
\[ = d(se^{a(s, t)}) - ds\psi_1 - se^{a(s, t)}a_t dt \]
\[ = d((se^{a(s, t)} - \psi_1) + \frac{\partial \psi_1}{\partial t} dt - se^{a(s, t)}a_t dt \]
\[ = d\Psi' + \frac{\partial \psi_1}{\partial t} dt - se^{a(s, t)}a_t dt \]
\[ = d\Psi' - \psi(s, t)dt \]
\[ = d\Psi' - \bar{l} \] \hspace{1cm} (3.14)

Let \((V', \omega') = (R \times \Sigma, d(e^a\lambda)), W' = L \times R, \) and \((V, \omega) = (V' \times C, \omega' \oplus \omega_0)\). As in [8], we use figure eight trick invented by Gromov to construct a Lagrangian submanifold in \( V \) through the Lagrange isotopy \( F' \) in \( V' \). Fix a positive \( \delta < 1 \) and take a \( C^\infty \)-map \( \rho : S^1 \to [0, 1] \), where the circle \( S^1 \) via parametrized by \( \Theta \in [-1, 1] \), such that the \( \delta \)-neighbourhood \( I_0 \) of \( 0 \in S^1 \) goes to \( 0 \in [0, 1] \) and \( \delta \)-neighbourhood \( I_1 \) of \( \pm 1 \in S^1 \) goes \( 1 \in [0, 1] \). Let

\[ \bar{l} = -\psi(s, \rho(\Theta)) \rho'(\Theta)d\Theta \]
\[ = -\Phi d\Theta \] \hspace{1cm} (3.15)

be the pull-back of the form \( \bar{l} = -\psi(s, t)dt \) to \( W' \times S^1 \) under the map \((w', \Theta) \to (w', \rho(\Theta))\) and assume without loss of generality \( \Phi \) vanishes on \( W' \times (I_0 \cup I_1) \).

Next, consider a map \( \alpha \) of the annulus \( S^1 \times [5\Phi_-, 5\Phi_+] \) into \( R^2 \), where \( \Phi_- \) and \( \Phi_+ \) are the lower and the upper bound of the function \( \Phi \) correspondingly, such that
(i) The pull-back under $\alpha$ of the form $dx \wedge dy$ on $R^2$ equals $-d\Phi \wedge d\Theta$.

(ii) The map $\alpha$ is bijective on $I \times [5\Phi_-, 5\Phi_+]$ where $I \subset S^1$ is some closed subset, such that $I \cup I_0 \cup I_1 = S^1$; furthermore, the origin $0 \in R^2$ is a unique double point of the map $\alpha$ on $S^1 \times 0$, that is

$$0 = \alpha(0, 0) = \alpha(\pm 1, 0),$$

and $\alpha$ is injective on $S^1 = S^1 \times 0$ minus $\{0, \pm 1\}$.

(iii) The curve $S^1_0 = \alpha(S^1 \times 0) \subset R^2$ “bounds” zero area in $R^2$, that is

$$\int_{S^1_0} xdy = 0,$$

for the 1-form $xdy$ on $R^2$.

**Proposition 3.1** Let $V'$, $W'$ and $F'$ as above. Then there exists an exact Lagrangian embedding $F : W' \times S^1 \to V' \times R^2$ given by $F(w', \Theta) = (F'(w', \rho(\Theta)), \alpha(\Theta, \Phi))$.

Proof. We follow as in [8, 2.3]. Now let $F^* : W' \times S^1 \to V' \times R^2$ be given by $(w', \Theta) \to (F'(w', \rho(\Theta)), \alpha(\Theta, \Phi))$. Then

(i) The pull-back under $F^*$ of the form $\omega = \omega' + dx \wedge dy$ equals $dF^* - d\Phi \wedge d\Theta = 0$ on $W' \times S^1$.

(ii) The set of double points of $F^*$ is $W'_0 \cap W'_1 \subset V' = V' \times 0 \subset V' \times R^2$.

(iii) If $F^*$ has no double point then the Lagrangian submanifold $W = F^*(W' \times S^1) \subset (V' \times R^2, \omega' + dx \wedge dy)$ is exact if and only if $W'_0 \subset V'$ is such.

This completes the proof of Proposition 3.1.

### 3.2 Formulation of Hilbert bundles

Let $(\Sigma, \lambda)$ be a closed $(2n-1)$-dimensional manifold with a contact form $\lambda$. Let $SS = R \times \Sigma$ and put $\xi = \ker(\lambda)$. Let $J_\lambda'$ be an almost complex structure on $SS$ tamed by the symplectic form $d(e^a\lambda)$.

We define a metric $g_\lambda$ on $SS = R \times \Sigma$ by

$$g_\lambda = d(e^a\lambda)(\cdot, J_\lambda')$$

which is adapted to $J_\lambda$ and $d(e^a\lambda)$ but not complete.

In the following we denote by $(V', \omega') = ((R \times \Sigma), d(e^a\lambda))$ and $(V, \omega) = (V' \times R^2, \omega' + dx \wedge dy)$ with the metric $g = g' \oplus g_0$ induced by $\omega(\cdot, J)(J = J' \oplus i)$ and $W \subset V$ a Lagrangian submanifold which was constructed in section 3.1.
Let $\bar{V} = D \times V$, then $\pi_1 : \bar{V} \to D$ be a symplectic vector bundle. Let $J$ be an almost complex structure on $\bar{V}$ such that $\pi_1 : \bar{V} \to D$ is a holomorphic map and each fibre $\bar{V}_z = \pi_1(z)$ is a $J$ complex submanifold. Let $H^k(D)$ be the space of $H^k$-maps from $D$ to $\bar{V}$, here $H^k$ represents Sobolev derivatives up to order $k$. Let $\bar{W} = \partial D \times W$, $\bar{p} = \{1\} \times p$, $W^\pm = \{\pm i\} \times W$ and $\bar{u} \in H^k(D)|\bar{u}(x) \in \bar{W}$ a.e for $x \in \partial D$ and $\bar{u}(1) = \bar{p}, \bar{u}(\pm i) \in \{\pm i\} \times \bar{W}$

$D^k = \{\bar{u} \in H^k(D)|\bar{u}(x) \in \bar{W}$ a.e for $x \in \partial D$ and $\bar{u}(1) = \bar{p}, \bar{u}(\pm i) \in \{\pm i\} \times \bar{W}\}$

for $k \geq 100$.

**Lemma 3.4** Let $W$ be a closed Lagrangian submanifold in $V$. Then, $D^k$ is a pseudo-Hilbert manifold with the tangent bundle

$$TD^k = \bigcup_{\bar{u} \in D^k} \Lambda^{-1}$$

(3.17)

here

$$\Lambda^{-1} = \{\bar{w} \in H^{-1}(\bar{u}^*(T\bar{V})|\bar{w}(1) = 0, and \bar{w}(\pm i) \in T\bar{W}\}$$

**Note 3.1** Since $W$ is not regular we know that $D^k$ is in general complete, however it is enough for our purpose.

Proof: See [4, 13].

Now we consider a section from $D^k$ to $TD^k$ follows as in [4, 8], i.e., let $\bar{\partial} : D^k \to TD^k$ be the Cauchy-Riemann section

$$\bar{\partial} \bar{u} = \frac{\partial \bar{u}}{\partial s} + J \frac{\partial \bar{u}}{\partial t}$$

(3.18)

for $\bar{u} \in D^k$.

**Theorem 3.1** The Cauchy-Riemann section $\bar{\partial}$ defined in (3.18) is a Fredholm section of Index zero.

Proof. According to the definition of the Fredholm section, we need to prove that $\bar{u} \in D^k$, the linearization $D\bar{\partial}(\bar{u})$ of $\bar{\partial}$ at $\bar{u}$ is a linear Fredholm operator. Note that

$$D\bar{\partial}(\bar{u}) = D\bar{\partial}_{[\bar{u}]}$$

(3.19)
where
\[(D\bar{\partial}_{[\bar{u}]})v = \frac{\partial \bar{v}}{\partial s} + J\frac{\partial \bar{v}}{\partial t} + A(\bar{u})\bar{v}\]  
(3.20)

with
\[\bar{v}|_{\partial D} \in (\bar{u}|_{\partial D})^* T\bar{W}\]

here \(A(\bar{u})\) is \(2n \times 2n\) matrix induced by the torsion of almost complex structure, see [4, 8] for the computation.

Observe that the linearization \(D\bar{\partial}(\bar{u})\) of \(\bar{\partial}\) at \(\bar{u}\) is equivalent to the following Lagrangian boundary value problem
\[
\frac{\partial \bar{v}}{\partial s} + J\frac{\partial \bar{v}}{\partial t} + A(\bar{u})\bar{v} = \bar{f}, \quad \bar{v} \in \Lambda^k(\bar{u}^* TV)
\]
\[
\bar{v}(t) \in T_{\bar{u}(t)} W, \quad t \in \partial D
\]  
(3.21)

One can check that (3.21) defines a linear Fredholm operator. In fact, by proposition 2.2 and Lemma 2.1, since the operator \(A(\bar{u})\) is a compact, we know that the operator \(\bar{\partial}\) is a nonlinear Fredholm operator of the index zero.

**Definition 3.1** Let \(X\) be a Banach manifold and \(P : Y \rightarrow X\) the Banach vector bundle. A Fredholm section \(F : X \rightarrow Y\) is proper if \(F^{-1}(0)\) is a compact set and is called generic if \(F\) intersects the zero section transversally, see [4, 8].

**Definition 3.2** \(\text{deg}(F, y) = \sharp\{F^{-1}(0)\}\mod 2\) is called the Fredholm degree of a Fredholm section (see[4, 8]).

**Theorem 3.2** Assume that \(\bar{J} = i \oplus J\) on \(\bar{V}\) and \(i\) is complex structure on \(D\) and \(J\) the almost complex structure on \(V\) which is integrable near point \(p\). Then the Fredholm section \(F = \bar{\partial} : D^k \rightarrow T\bar{D}^k\) constructed in (3.18) has degree one, i.e.,
\[\text{deg}(F, 0) = 1\]

Proof: We assume that \(\bar{u} : D \rightarrow \bar{V}\) be a \(\bar{J}\)–holomorphic disk with boundary \(\bar{u}(\partial D) \subset W\) and by the assumption that \(\bar{u}\) is homotopic to the map \(\bar{u}_1 = (id, \bar{p})\). Since almost complex structure \(\bar{J}\) splits and is tamed by the symplectic form \(\bar{\omega}\), by stokes formula, we conclude the second component \(u : D \rightarrow V\) is a constant map. Because \(u(1) = p\), We know that \(F^{-1}(0) = (id, p)\). Next
we show that the linearization $DF_{(id,p)}$ of $F$ at $(id,p)$ is an isomorphism from $T_{(id,p)}D^k$ to $E$. This is equivalent to solve the equations

\[
\frac{\partial \bar{v}}{\partial s} + J \frac{\partial \bar{v}}{\partial t} = f
\]

(3.22)

\[
\bar{v}|_{\partial D} \subset T_{(id,p)}\bar{W}
\]

(3.23)

here $\bar{J} = i + J(p)$. By Lemma 2.1, we know that $DF((id,p))$ is an isomorphism. Therefore $\deg(F,0) = 1$.

4 Anti-holomorphic sections

In this section we construct a Fredholm section which is not proper as in [4, 8].

Let $(V', \omega') = (S\Sigma, d(e^\alpha \lambda))$ and $(V, \omega) = (V' \times C, \omega' \oplus \omega_0)$, $W$ as in section 3 and $J = J' \oplus i$, $g = g' \oplus g_0$, $g_0$ the standard metric on $C$.

Now let $c \in C$ be a non-zero vector. We consider $c$ as an anti-holomorphic homomorphism $c : TD \to TV' \oplus TC$, i.e., $c(\frac{\partial}{\partial z}) = (0, c \cdot \frac{\partial}{\partial z})$. Since the constant section $c$ is not a section of the Hilbert bundle in section 3 due to $c$ is not tangent to the Lagrangian submanifold $W$, we must modify it as follows:

Let $c$ as above, we define

\[
c_{\chi, \delta}(z, v) = \begin{cases} 
c & \text{if } |z| \leq 1 - 2\delta, \\
0 & \text{otherwise} \end{cases}
\]

(4.1)

Then by using the cut off function $\varphi_h(z)$ and its convolution with section $c_{\chi, \delta}$, we obtain a smooth section $c_{\delta}$ satisfying

\[
c_{\delta}(z, v) = \begin{cases} 
c & \text{if } |z| \leq 1 - 3\delta, \\
0 & \text{if } |z| \geq 1 - \delta. \end{cases}
\]

\[|c_{\delta}| \leq |c|\]

(4.2)

for $h$ small enough, for the convolution theory see [12, ch1.p16-17, Th1.3.1]. Then one can easily check that $\bar{c}_{\delta} = (0, 0, c_{\delta})$ is an anti-holomorphic section tangent to $\bar{W}$.

Now we put an almost complex structure $\bar{J} = i \oplus J$ on the symplectic fibration $D \times V \to D$ such that $\pi_1 : D \times V \to D$ is a holomorphic fibration.
and $\pi_1^{-1}(z)$ is an almost complex submanifold. Let $g = \bar{\omega}(\cdot, J\cdot)$ be the metric on $D \times V$.

Now we consider the equations

$$
\bar{v} = (id, v) = (id, v', f) : D \to D \times V' \times C
$$

$$
\bar{\partial}_J v = c_\delta \quad \text{or}
$$

$$
\bar{\partial}_{J'} v' = 0, \partial f = c_\delta \text{ on } D
$$

$$
v|_{\partial D} : \partial D \to W \tag{4.3}
$$

here $v$ homotopic to constant map $\{p\}$ relative to $W$. Note that $W \subset V \times B_{R}(0)$ for $\pi R^2 = 2\pi R(\varepsilon)^2$, here $R(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\varepsilon$ as in section 3.1.

**Lemma 4.1** Let $\bar{v} = (id, v)$ be the solutions of (4.3), then one has the following estimates

$$
E(v) = \left\{ \int_D (g'\left(\frac{\partial v}{\partial x}, J'\frac{\partial v'}{\partial x}\right) + g'\left(\frac{\partial v}{\partial y}, J'\frac{\partial v'}{\partial y}\right)
+ g_0\left(\frac{\partial f}{\partial x}, i\frac{\partial f}{\partial x}\right) + g_0\left(\frac{\partial f}{\partial y}, i\frac{\partial f}{\partial y}\right))d\sigma \right\} \leq 4\pi R(\varepsilon)^2. \tag{4.4}
$$

Proof: Since $v(z) = (v'(z), f(z))$ satisfy (4.3) and $v(z) = (v'(z), f(z)) \in V' \times C$ is homotopic to constant map $v_0 : D \to \{p\} \subset W$ in $(V, W)$, by the Stokes formula

$$
\int_D v^*(\omega' \oplus \omega_0) = 0 \tag{4.5}
$$

Note that the metric $g$ is adapted to the symplectic form $\omega$ and $J$, i.e.,

$$
g = \omega(\cdot, J\cdot) \tag{4.6}
$$

By the simple algebraic computation, we have

$$
\int_D v^*\omega = \frac{1}{4} \int_{D^2} (|\partial v|^2 - |\bar{\partial} v|^2) = 0 \tag{4.7}
$$

and

$$
|\nabla v| = \frac{1}{2} (|\partial v|^2 + |\bar{\partial} v|^2) \tag{4.8}
$$
Then
\[ E(v) = \int_D |\nabla v| \]
\[ = \int_D \left( \frac{1}{2} (|\partial v|^2 + |\bar{\partial} v|^2) \right) d\sigma \]
\[ = \int_D |c_\delta|^2 d\sigma \] (4.9)

By Cauchy integral formula,
\[ f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{\xi - z} d\xi + \frac{1}{2\pi i} \int_D \frac{\bar{\partial} f(\xi)}{\xi - z} d\xi \wedge d\bar{\xi} \] (4.10)

Since \( f \) is smooth up to the boundary, we integrate the two sides on \( D_r \) for \( r < 1 \), one get
\[ \int_{\partial D_r} f(z) dz = 0 + \frac{1}{2\pi i} \int_{\partial D_r} \frac{\bar{\partial} f(\xi)}{\xi - z} d\xi \wedge d\bar{\xi} \] (4.11)

Let \( r \to 1 \), we get
\[ \int_{\partial D} f(z) dz = \int_D \bar{\partial} f(\xi) d\xi \wedge d\bar{\xi} \] (4.12)

By the equations (4.3), one get
\[ \bar{\partial} f = c \text{ on } D_{1-2\delta} \] (4.13)

So, we have
\[ 2\pi i (1 - 2\delta) c = \int_{\partial D} f(z) dz - \int_{D - D_{1-2\delta}} \bar{\partial} f(\xi) d\xi \wedge d\bar{\xi} \] (4.14)

So,
\[ |c| \leq \frac{1}{2\pi(1 - 2\delta)} |\int_{\partial D} f(z) dz| + |\int_{D - D_{1-2\delta}} \bar{\partial} f(\xi) d\xi \wedge d\bar{\xi}| \]
\[
\leq \frac{1}{2\pi(1-2\delta)} 2\pi |\text{diam}(pr_2(W)) + c_1c_2|c|(\pi - \pi(1-2\delta)^2)) \quad (4.15)
\]

Therefore, one has
\[
|c| \leq c(\delta)R(\varepsilon) \quad (4.16)
\]

and
\[
E(v) = \pi \int_D |c_\delta|_g^2 = \pi c(\delta)^2 R(\varepsilon)^2. \quad (4.17)
\]

This finishes the proof of Lemma.

**Proposition 4.1** For \(|c| \geq 2c(\delta)R(\varepsilon)|, then the equations (4.3) has no solutions.

Proof. By 4.16, it is obvious.

**Theorem 4.1** The Fredholm section \(F_1 = \bar{\partial}j + c_\delta : D^k \to E\) is not proper.

Proof. By the Proposition 4.1 and Theorem 3.2, it is obvious (see [4, 8]).

5 **J–holomorphic section**

Recall that \(W(-K, K) \subset W \subset V' \times R^2\) as in section 3. The Riemann metric \(g\) on \(V' \times R^2\) induces a metric \(g|W\).

Now let \(c \in C\) be a non-zero vector and \(c_\delta\) the induced anti-holomorphic section. We consider the nonlinear inhomogeneous equations (4.3) and transform it into \(\bar{J}\)–holomorphic map by considering its graph as in [8, p319,1.4.C] or [4, p312, Lemma 5.2.3].

Denote by \(Y(1) \to D \times V\) the bundle of homomorphisms \(T_s(D) \to T_v(V)\). If \(D\) and \(V\) are given the disk and the almost Kähler manifold, then we distinguish the subbundle \(X(1) \subset Y(1)\) which consists of complex linear homomorphisms and we denote \(\bar{X}(1) \to D \times V\) the quotient bundle \(Y(1)/X(1)\).

Now, we assign to each \(C^1\)-map \(v : D \to V\) the section \(\bar{\partial}v\) of the bundle \(\bar{X}(1)\) over the graph \(\Gamma_v \subset D \times V\) by composing the differential of \(v\) with the quotient homomorphism \(Y(1) \to \bar{X}(1)\). If \(c_\delta : D \times V \to \bar{X}\) is a \(H^k\)–section we write \(\bar{\partial}v = c_\delta\) for the equation \(\bar{\partial}v = c_\delta|\Gamma_v\).
Lemma 5.1 (Gromov[8, 1.4.C]) There exists a unique almost complex structure $J_g$ on $D \times V$ (which also depends on the given structures in $D$ and in $V$), such that the (germs of) $J_g$-holomorphic sections $v : D \to D \times V$ are exactly and only the solutions of the equations $\delta v = c_5$. Furthermore, the fibres $z \times V \subset D \times V$ are $J_g$-holomorphic (i.e. the subbundles $T(z \times V) \subset T(D \times V)$ are $J_g$-complex) and the structure $J_g|z \times V$ equals the original structure on $V = z \times V$. Moreover $J_g$ is tamed by $k\omega \oplus \omega$ for $k$ large enough which is independent of $\delta$.

6 Gromov’s $C^0$–convergence theorem

6.1 Analysis of Gromov’s figure eight

Since $W' \subset S\Sigma$ is an exact Lagrangian submanifold and $F'_\rho$ is an exact Lagrangian isotopy (see section 3.1). Now we carefully check the Gromov’s construction of Lagrangian submanifold $W \subset V' \times R^2$ from the exact Lagrangian isotopy of $W'$ in section 3.

Let $S^1 \subset T^*S^1$ be a zero section and $S^1 = \bigcup_{i=1}^3 S_i$ be a partition of the zero section $S^1$ such that $S_1 = I_0$, $S_3 = I_1$. Write $S^1 \setminus \{I_0 \cup I_1\} = I_2 \cup I_3$ and $I_0 = (-\delta, -\frac{5\delta}{4}) \cup (-\frac{5\delta}{4}, \frac{5\delta}{4}) \cup \left[\frac{5\delta}{4}, \delta\right]$, similarly $I_1 = (1-\delta, 1-\frac{5\delta}{4}) \cup (1-\frac{5\delta}{4}, 1+\frac{5\delta}{4}) \cup [1+\frac{5\delta}{4}, 1+\delta) = I_1^- \cup I_1^+$. Let $S_2 = I_0^- \cup I_2 \cup I_1^-$, $S_4 = I_1^+ \cup I_3 \cup I_1^+$. Moreover, we can assume that the double points of map $\alpha$ in Gromov’s figure eight is contained in $(I_0 \cup I_1) \times \Phi_-, \Phi_+$, here $I_0 = (-\frac{5\delta}{12}, \frac{5\delta}{12})$ and $I_1 = (1-\frac{5\delta}{12}, 1+\frac{5\delta}{12})$. Recall that $\alpha : (S^1 \times [5\Phi_-, 5\Phi_+]) \to R^2$ is an exact symplectic immersion, i.e., $\alpha^*(-ydx) - \Psi d\Theta = dh$, $h : T^*S^1 \to R$. By the construction of figure eight, we can assume that $\alpha'_i = \alpha|((S^1 \setminus I_i') \times [5\Phi_-, 5\Phi_+])$ is an embedding for $i = 0, 1$. Let $Y = \alpha(S^1 \times [5\Phi_-, 5\Phi_+]) \subset R^2$ and $Y_i = \alpha(S_i \times [5\Phi_-, 5\Phi_+]) \subset R^2$. Let $\alpha_i = \alpha|Y_i(S^1 \times [5\Phi_-, 5\Phi_+])$. So, $\alpha_i$ puts the function $h$ to the function $h_{i0} = \alpha_{i0}^{-1}h$ on $Y_i$. We extend the function $h_{i0}$ to whole plane $R^2$. In the following we take the liouville form $\beta_{i0} = -ydx - dh_{i0}$ on $R^2$. This does not change the symplectic form $dx \wedge dy$ on $R^2$. But we have $\alpha_i^*\beta = \Phi d\Theta$ on $(S_i \times [5\Phi_-, 5\Phi_+])$ for $i = 1, 2, 3, 4$. Finally, note that

$$F : W' \times S^1 \to V' \times R^2;$$
$$F(w', \Theta) = (F'_\rho(\Theta)(w'), \alpha(\Theta, \Phi(w'), \rho(\Theta)).$$

(6.1)
Since $\rho(\Theta) = 0$ for $\Theta \in I_0$ and $\rho(\Theta) = 1$ for $\Theta \in I_1$, we know that $\Phi(w',\rho(\Theta)) = 0$ for $\Theta \in I_0 \cup I_1$. Therefore,

$$F(W' \times I_0) = W' \times \alpha(I_0); F(W' \times I_1) = W' \times \alpha(I_1).$$

(6.2)

### 6.2 Gromov’s Schwartz lemma

In our proof we need a crucial tool, i.e., Gromov’s Schwartz Lemma as in [8]. We first consider the case without boundary.

**Proposition 6.1** Let $(V, J, \mu)$ be as in section 4 and $V_K$ the compact part of $V$. There exist constants $\epsilon_0$ and $C$ (depending only on the $C^0$ norm of $\mu$ and on the $C^\alpha$ norm of $J$ and $A_0$) such that every $J$–holomorphic map of the unit disc to an $\epsilon_0$-ball of $V$ with center in $V_K$ and area less than $A_0$ has its derivatives up to order $k + 1 + \alpha$ on $D^\pm_1(0)$ bounded by $C$.

For a proof, see [8].

Now we consider the Gromov’s Schwartz Lemma for $J$–holomorphic map with boundary in a closed Lagrangian submanifold as in [8].

**Proposition 6.2** Let $(V, J, \mu)$ as above and $L \subset V$ be a closed Lagrangian submanifold and $V_K$ one compact part of $V$. There exist constants $\epsilon_0$ and $C$ (depending only on the $C^0$ norm of $\mu$ and on the $C^\alpha$ norm of $J$ and $K, A_0$) such that every $J$–holomorphic map of the half unit disc $D^+$ to a $\epsilon_0$-ball of $V$ with boundary in $L$ and area less than $A_0$ has its derivatives up to order $k + 1 + \alpha$ on $D^+_1(0)$ bounded by $C$.

For a proof see [8].

Since in our case $W$ is a non-compact Lagrangian submanifold, Proposition 6.2 can not be used directly but the proofs of Proposition 6.1-2 still holds in our case.

**Lemma 6.1** Recall that $V = V' \times R^2$. Let $(V, J, \mu)$ as above and $W \subset V$ be as above and $V_c$ the compact set in $V$. Let $\bar{V} = D \times V$, $\bar{W} = \partial D \times W$, and $\bar{V}_c = D \times \bar{V}_c$. Let $Y = \alpha(S^1 \times [5\Phi_-, 5\Phi_+]) \subset R^2$. Let $Y_j = \alpha(S_j \times [5\Phi_-, 5\Phi_+]) \subset R^2$. Let $\{X_j\}_{j=1}^q$ be a Darboux covering of $\Sigma$ and $V_j' = R \times X_j$. Let $\partial D = S^1_+ \cup S^1_-$. There exist constant $c_0$ such that every $J$–holomorphic map $v$
of the half unit disc $D^+$ to the $D \times V'_i \times R^2$ with its boundary $v((−1, 1)) \subset (S^{1±}) \times F(\mathcal{L} \times R \times S_i) \subset \tilde{W}, i = 1, .., 4$ has

$$\text{area}(v(D^+)) \leq c_0 l^2(v(\partial^D+)). \quad (6.3)$$

here $\partial^D+ = \partial D \setminus [−1, 1]$ and $l(v(\partial^D+)) = \text{length}(v(\partial^D+))$.

Proof. Let $\tilde{W}_i = S^{1±} \times F(W'_i \times S_i)$. Let $v = (v_1, v_2): D^+ \to \tilde{V} = D \times V$ be the $J$–holomorphic map with $v(\partial D^+) \subset \tilde{W}_i \subset \partial D \times W$, then

$$\text{area}(v) = \int_{D^+} v^*(\alpha_0 \oplus \alpha) = \int_{D^+} dv^*(\alpha_0 \oplus \alpha) = \int_{\partial D^+} v^*(\alpha_0 \oplus \alpha) = \int_{\partial D^+} v_1^*\alpha_0 + \int_{\partial D^+} v_2^*\alpha = \int_{\partial D^+ \cup [−1, +1]} v_1^*\alpha_0 + \int_{\partial D^+ \cup [−1, +1]} v_2^*(e^a\lambda - ydx - dh_{i0}) = \int_{\partial D^+ \cup [−1, +1]} v_1^*\alpha_0 + \int_{\partial D^+} v_2^*(e^a\lambda - ydx - dh_{i0}) + B_1, \quad (6.4)$$

here $B_1 = \int_{[−1, +1]} v_2^*(-d\Psi')$. Now take a zig-zag curve $C$ in $V'_j \times Y_i$ connecting $v_2(−1)$ and $v_2(+1)$ such that

$$\int_C (e^a\lambda + ydx) = B_1$$

$$\text{length}(C) \leq k_1 \text{length}(v_2(\partial^D+)) \quad (6.5)$$

Now take a minimal surface $M$ in $V'_j \times R^2$ bounded by $v_2(\partial^D+) \cup C$, then by the isoperimetric inequality(see[9, p283]), we get

$$\text{area}(M) \leq m_1 \text{length}(C + v_2(\partial^D+))^2 \leq m_2 \text{length}(v_2(\partial^D+))^2, \quad (6.6)$$

here we use the (6.5).

Since $\text{area}(M) \geq \int_M \omega$ and $\int_M \omega = \int_{D^+} v_2^*\omega = \text{area}(v)$, this proves the lemma.
Lemma 6.2 Let $v$ as in Lemma 6.1, then we have

$$\text{area}(v(D^+)) \geq c_0(\text{dist}(v(0), v(\partial^+D)))^2,$$  \hspace{1cm} (6.7)

here $c_0$ depends only on $\Sigma, J, \omega, \ldots$, etc, not on $v$.

Proof. By the standard argument as in [4, p79].

The following estimates is a crucial step in our proof.

Lemma 6.3 Recall that $V = V' \times \mathbb{R}^2$. Let $(V, J, \mu)$ as above and $W \subset V$ be as above and $V_c$ the compact set in $V$. Let $V = D \times V$, $W = \partial D \times W$, and $V_c = D \times V_c$. Let $Y = \alpha(S^1 \times [5\Phi_-, 5\Phi_+]) \subset \mathbb{R}^2$. Let $Y_i = \alpha(S_i \times [5\Phi_-, 5\Phi_+]) \subset \mathbb{R}^2$. Let $\partial D = S^{1+} \cup S^{1-}$. There exist constant $c_0$ such that every $J$–holomorphic map $v$ of the half unit disc $D^+$ to the $D \times V' \times \mathbb{R}^2$ with its boundary $v((-1, 1)) \subset (S^{1\pm}) \times F(L \times R \times S_i) \subset \bar{W}, i = 1, \ldots, 4$ has

$$\text{area}(v(D^+)) \leq c_0 l^2(v(\partial^+D^)).$$  \hspace{1cm} (6.8)

here $\partial^+D = \partial D \setminus [-1, 1]$ and $l(v(\partial^+D)) = \text{length}(v(\partial^+D))$.

Proof. We first assume that $\varepsilon$ in section 3.1 is small enough. Let $l_0$ is a constant small enough. If $\text{length}(\partial^+D^+) \geq l_0$, then Lemma 6.3 holds. If $\text{length}(\partial^+D^+) \leq l_0$ and $v(D^+) \subset D \times V_j' \times \mathbb{R}^2$, then Lemma6.3 reduces to Lemma6.1. If $\text{length}(\partial^+D^+) \leq l_0$ and $v(D^+) \subset D \times V_j' \times \mathbb{R}^2$, then Lemma6.2 implies $\text{area}(v) \geq \tau_0 > 100\pi R(\varepsilon)^2$, this is a contradiction. Therefore we proved the lemma.

Proposition 6.3 Let $(V, J, \mu)$ and $W \subset V$ be as in section 4 and $V_K$ the compact part of $V$. Let $\bar{V}, \bar{V}_K$ and $\bar{W}$ as section 5.1. There exist constants $\varepsilon_0$ (depending only on the $C^0$– norm of $\mu$ and on the $C^\alpha$ norm of $J$) and $C$ (depending only on the $C^0$ norm of $\mu$ and on the $C^{k+\alpha}$ norm of $J$) such that every $J$–holomorphic map of the half unit disc $D^+$ to the $D \times V' \times \mathbb{R}^2$ with its boundary $v((-1, 1)) \subset (S^{1\pm}) \times F(L \times R \times S_i) \subset \bar{W}, i = 1, \ldots, 4$ has its derivatives up to order $k + 1 + \alpha$ on $D^+_2(0)$ bounded by $C$.

Proof. One uses Lemma 6.3 and Gromov’s proof on Schwartz lemma to yield proposition 6.3.
6.3 Removal singularity of $J$–curves

In our proof we need another crucial tools, i.e., Gromov’s removal singularity theorem[8]. We first consider the case without boundary.

**Proposition 6.4** Let $(V, J, \mu)$ be as in section 4 and $V_K$ the compact part of $V$. If $v : D \setminus \{0\} \to V_K$ be a $J$–holomorphic disk with bounded energy and bounded image, then $v$ extends to a $J$–holomorphic map from the unit disc $D$ to $V_K$.

For a proof, see[8].

Now we consider the Gromov’s removal singularity theorem for $J$–holomorphic map with boundary in a closed Lagrangian submanifold as in [8].

**Proposition 6.5** Let $(V, J, \mu)$ as above and $L \subset V$ be a closed Lagrangian submanifold and $V_K$ one compact part of $V$. If $v : (D^+ \setminus \{0\}, \partial^+ D^+ \setminus \{0\}) \to (V_K, L)$ be a $J$–holomorphic half-disk with bounded energy and bounded image, then $v$ extends to a $J$–holomorphic map from the half unit disc $(D^+, \partial^+ D^+)$ to $(V_K, L)$.

For a proof see [8].

**Proposition 6.6** Let $(V, J, \mu)$ and $W \subset V$ be as in section 4 and $V_c$ the compact set in $V$. Let $\bar{V} = D \times V$, $\bar{W} = \partial D \times W$, and $\bar{V}_c = D \times V_c$. Then every $J$–holomorphic map $v$ of the half unit disc $D^+ \setminus \{0\}$ to the $\bar{V}$ with center in $\bar{V}_c$ and its boundary $v((-1, 1) \setminus \{0\}) \subset (S^{1\pm}) \times F(L \times [-K, K] \times S_i) \subset \bar{W}$ and

\[
\text{area}(v(D^+ \setminus \{0\})) \leq E
\]  

(6.9)

extends to a $J$–holomorphic map $\bar{v} : (D^+, \partial^+ D) \to (\bar{V}_c, \bar{W})$.

Proof. This is ordinary Gromov’s removal singularity theorem by $K$–assumption.

6.4 $C^0$–Convergence Theorem

We now recall that the well-known Gromov’s compactness theorem for cusp’s curves for the compact symplectic manifolds with closed Lagrangian submanifolds in it. For reader’s convenience, we first recall the “weak-convergence” for closed curves.
Cusp-curves. Take a system of disjoint simple closed curves $\gamma_i$ in a closed surface $S$ for $i = 1, \ldots, k$, and denote by $S^0$ the surface obtained from $S \setminus \bigcup_{i=1}^{k} \gamma_i$. Denote by $\bar{S}$ the space obtained from $S$ by shrinking every $\gamma_i$ to a single point and observe the obvious map $\alpha : S^0 \rightarrow \bar{S}$ gluing pairs of points $s'_i$ and $s''_i$ in $S^0$, such that $\bar{s}_i = \alpha(s'_i) = \alpha(s''_i) \in \bar{S}$ are singular (or cuspidal) points in $\bar{S}$ (see[8]).

An almost complex structure in $\bar{S}$ by definition is that in $S^0$. A continuous map $\beta : \bar{S} \rightarrow V$ is called a (parametrized $J$–holomorphic) cusp-curve in $V$ if the composed map $\beta \circ \alpha : S^0 \rightarrow V$ is holomorphic.

Weak convergence. A sequence of closed $J$–curves $C_j \subset V$ is said to weakly converge to a cusp-curve $\bar{C} \subset V$ if the following four conditions are satisfied

(i) all curves $C_j$ are parametrized by a fixed surface $S$ whose almost complex structure depends on $j$, say $C_j = f_j(S)$ for some holomorphic maps $f_j : (S, J_j) \rightarrow (V, J)$

(ii) There are disjoint simple closed curves $\gamma_i \in S$, $i = 1, \ldots, k$, such that $\bar{C} = \bar{f}(\bar{S})$ for a map $\bar{f} : \bar{S} \rightarrow V$ which is holomorphic for some almost complex structure $\bar{J}$ on $\bar{S}$.

(iii) The structures $J_j$ uniformly $C^\infty$–converge to $\bar{J}$ on compact subsets in $S \setminus \bigcup_{i=1}^{k} \gamma_i$.

(iv) The maps $f_j$ uniformly $C^\infty$–converge to $\bar{f}$ on compact subsets in $S \setminus \bigcup_{i=1}^{k} \gamma_i$. Moreover, $f_j$ uniformly $C^0$–converge on entire $S$ to the composed map $S \rightarrow \bar{S} \xrightarrow{\bar{f}} V$. Furthermore,

\[
\text{Area}_\mu f_j(S) \rightarrow \text{Area}_\mu \bar{f}(\bar{S}) \text{ for } j \rightarrow \infty,
\]

where $\mu$ is a Riemannian metric in $V$ and where the area is counted with the geometric multiplicity (see[8]).

Gromov’s Compactness theorem for closed curves. Let $C_j$ be a sequence of closed $J$–curves of a fixed genus in a compact manifold $(V, J, \mu)$. If the areas of $C_j$ are uniformly bounded,

\[
\text{Area}_\mu \leq A, \ j = 1, \ldots,
\]

then some subsequence weakly converges to a cusp-curve $\bar{C}$ in $V$. 

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Cusp-curves with boundary. Let $T$ be a compact complex manifold with boundary of dimension 1 (i.e., it has an atlas of holomorphic charts onto open subsets of $C$ or of a closed half plane). Its double is a compact Riemann surface $S$ with a natural anti-holomorphic involution $\tau$ which exchanges $T$ and $S \setminus T$ while fixing the boundary $\partial T$. If $f : T \to V$ is a continuous map, holomorphic in the interior of $T$, it is convenient to extend $f$ to $S$ by

$$f = f \circ \tau$$

Take a totally real submanifold $W \subset (V, J)$ and consider compact holomorphic curves $C \subset V$ with boundaries, $(\tilde{C}, \partial \tilde{C}) \subset (V, W)$, which are, topologically speaking, obtained by shrinking to points some (short) closed loops in $C$ and also some (short) segments in $C$ between boundary points. This is seen by looking on the double $C \cup_{\partial C} \tilde{C}$.

Gromov's Compactness theorem for curves with boundary. Let $V$ be a closed Riemannian manifold, $W$ a totally real closed submanifold of $V$. Let $C_j$ be a sequence of $J-$curves with boundary in $W$ of a fixed genus in a compact manifold $(V, J, \mu)$. If the areas of $C_j$ are uniformly bounded,

$$\text{Area}_{\mu}(C_j) \leq A, \ j = 1, \ldots,$$

then some subsequence weakly converges to a cusp-curve $\tilde{C}$ in $V$.

The proofs of Gromov’s compactness theorem can found in [4, 8]. In our case the Lagrangian submanifold $W$ is not compact, Gromov’s compactness theorem can not be applied directly but its proof is still effective since the $W$ has the special geometry. In the following we modify Gromov’s proof to prove the $C^0$-compactness theorem in our case.

Now we state the $C^0-$convergence theorem in our case.

**Theorem 6.1** Let $(V, J, \omega, \mu)$ and $W$ as in section 4. Let $C_j$ be a sequence of $J_\delta-$holomorphic section $v_j = (id, ((a_j, u_j), f_j)) : D \to D \times V$ with $v_j : \partial D \to \partial D \times W$ and $v_j(1) = (1, p) \in \partial D \times W$ constructed from section 4. Then the areas of $C_j$ are uniformly bounded, i.e.,

$$\text{Area}_{\mu}(C_j) \leq A, \ j = 1, \ldots,$$

and some subsequence weakly converges to a cusp-section $\tilde{C}$ in $V$ (see[4, 8]).
Proof. We follow the proofs in [8]. Write \( v_j = (id, (a_j, u_j), f_j) \) then \( |a_{ij}| \leq a_0 \) by the ordinary Monotone inequality of minimal surface without boundary, see following Proposition 7.1. Similarly \( |f_j| \leq R_1 \) by using the fact \( f_j(\partial D) \) is bounded in \( B_{R_1}(0) \) and \( I D |\nabla f_j| \leq 4\pi R^2 \) via monotone inequality for minimal surfaces. So, we assume that \( v_j(D) \subset V_c \) for a compact set \( V_c \).

1. **Removal of a net.**

1a. Let \( \bar{V} = D \times V \) and \( v_j \) be regular curves. First we study induced metrics \( \mu_j \) in \( v_j \). We apply the ordinary monotone inequality for minimal surfaces without boundary to small concentric balls \( B_{\varepsilon}(A_j, \mu_j) \) for \( 0 < \varepsilon \leq \varepsilon_0 \) and conclude by the standard argument to the inequality

\[
\text{Area}(B_\varepsilon) \geq \varepsilon^2, \text{ for } \varepsilon \leq \varepsilon_0;
\]

Using this we easily find a interior \( \varepsilon - \text{net} \) \( F_j \subset (v_j, \mu_j) \) containing \( N \) points for a fixed integer \( N = (\bar{V}, \bar{J}, \mu) \), such that every topological annulus \( A \subset v_j \backslash F_j \) satisfies

\[
\text{Diam}_\mu A \leq 10 \text{length}_\mu \partial A.
\]  

(6.10)

Furthermore, let \( A \) be conformally equivalent to the cylinder \( S^1 \times [0, l] \) where \( S^1 \) is the circle of the unit length, and let \( S^1_t \subset A \) be the curve in \( A \) corresponding to the circle \( S^1 \times t \) for \( t \in [0, l] \). Then obviously

\[
\int_a^b (\text{length}_{S^1}^2)dt \leq \text{Area}(A) \leq C_5.
\]

(6.11)

for all \([a, b] \subset [0, l]\). Hence, the annulus \( A_t \subset A \) between the curves \( S^1_t \) and \( S^1_{t-t} \) satisfies

\[
\text{diam}_\mu A_t \leq 20 \left( \frac{C_5}{t} \right)
\]  

(6.12)

for all \( t \in [0, l] \).

1b. We consider the sets \( \partial v_j \cap ((S^1) \times F(W' \times I^\pm_i)), i = 0, 1 \). By the construction of Gromov’s figure eight, there exists a finite components, denote it by

\[
\partial v_j \cap ((S^1) \times F(\mathcal{L} \times R \times I^\pm_i)) = \{ \gamma^k_{ij} \}, i = 0, 1.
\]

(6.13)

Let \( m_i^\pm \) be the middle point of \( I^\pm_i \). If

\[
\gamma^k_{ij} \cap ((S^1) \times F(\mathcal{L} \times R \times m_i^\pm)) \neq \emptyset, i = 0, 1,
\]

(6.14)
we choose one point in $\bar{\gamma}^k_{ij}$ as a boundary puncture point in $\partial v_j$. Consider the concentric $\varepsilon$ half-disks or quadrature $B_\varepsilon(p)$ with center $p$ on $\bar{\gamma}^k_{ij}$, then

$$\text{Area}(B_\varepsilon(p)) \geq \tau_0.$$  \hfill (6.15)

Since $\text{Area}(v_j) \leq E_0$, there exists a uniform finite puncture points.

Consider the concentric $\varepsilon$ half-disks or quadrature $B_\varepsilon(p)$ with center $p$ on $\partial v_j$ and

$$\text{Area}(B_\varepsilon(p)) \geq \tau_0,$$  \hfill (6.16)

we puncture one point on such half-disk or quadrature. Since $\text{Area}(v_j) \leq E_0$, there exists a uniform finite puncture points.

So, we find a boundary net $G_j \subset \partial v_j$ containing $N_1$ points for a fixed integer $N_1$ $(\bar{V}, \bar{J}, \mu)$, such that every topological quadrature or half annulus $B \subset v_j \setminus \{F_j, G_j\}$ satisfies

$$\partial' B = \partial B \cap \bar{W} \subset (S^{1\pm}) \times F(\mathcal{L} \times R \times S_i), i = 1, 2, 3, 4.$$  \hfill (6.17)

2. Poincare’s metrics. 2a. Now, let $\mu^*_j$ be a metric of constant curvature $-1$ in $v_j(D) \setminus F_j \cup G_j$ conformally equivalent to $\mu_j$. Then for every $\mu^*_j$--ball $B_\rho$ in $v_j \setminus F_j \cup G_j$ of radius $\rho \leq 0.1$, there exists an annulus $A$ contained in $v_j \setminus F_j \cup G_j$ such that $B_\rho \subset A_t$ for $t = 0.01|\log|(see Lemma 3.2.2in [4, chVIII]). This implies with (6.3) the uniform continuity of the (inclusion) maps $(v_j \setminus F_j, \mu^*_j) \rightarrow (\bar{V}, \bar{\mu})$, and hence a uniform bound on the $r^{th}$ order differentials for every $r = 0, 1, 2, \ldots$

2b. Similarly, for every $\mu^*_j$--half ball $B^*_\rho$ in $v_j \setminus F_j \cup G_j$ of radius $\rho \leq 0.1$, there exists a half annulus or quadrature $B$ contained in $v_j \setminus F_j \cup G_j$ such that $B^*_\rho \subset B$ with

$$\partial' B = \partial B \cap \bar{W} \subset (S^{1\pm}) \times F(\mathcal{L} \times R \times S_i), i = 1, 2, 3, 4.$$  \hfill (6.18)

Then, by Gromov’s Schwartz Lemma, i.e., Proposition 6.1-6.3 implies the uniform bound on the $r^{th}$ order differentials for every $r = 0, 1, 2, \ldots$

3. Convergence of metrics. Next, by the standard (and obvious ) properties of hyperbolic surfaces there is a subsequence(see[4]), which is still denoted by $v_j$, such that

(a). There exist $k$ closed geodesics or geodesic arcs with boundaries in $\partial v_j \setminus F_j$, say

$$\gamma^j_i \subset (v_j \setminus F_j, \mu^*_j), i = 1, \ldots, k; j = 1, 2, \ldots,$$
whose \( \mu_j^* \)-length converges to zero as \( j \to \infty \), where \( k \) is a fixed integer.

(b). There exist \( k \) closed curves or geodesic arcs with boundaries in \( \partial S \) of a fixed surface, say \( \gamma_j \) in \( S \), and an almost complex structure \( J \) on the corresponding (singular) surface \( \bar{S} \), such that the almost complex structure \( J_j \) on \( v_j \setminus F_j \) induced from \( (V, J) \) \( C^\infty \)-converge to \( \bar{J} \) outside \( \bigcup_{j=1}^k \gamma_j \). Namely, there exist continuous maps \( g^j : v_j \to \bar{S} \) which are homeomorphisms outside the geodesics \( \gamma_j^j \), which pinch these geodesics to the corresponding singular points of \( \bar{S} \)(that are the images of \( \gamma_j \)) and which send \( F_j \) to a fixed subset \( F \) in the non-singular locus of \( \bar{S} \). Now, the convergence \( J_j \to \bar{J} \) is understood as the uniform \( C^\infty \)-convergence \( g^j_j(J_j) \to \bar{J} \) on the compact subsets in the non-singular locus \( \bar{S}^* \) of \( \bar{S} \) which is identified with \( S \setminus \bigcup_{i=1}^k \gamma_i \).

4. \( C^0 \)-interior convergence. The limit cusp-curve \( \bar{v} : \bar{S}^* \to \bar{V} \), that is a holomorphic map which is constructed by first taking the maps

\[
\bar{v}_j = (g_j)^{-1} : S \setminus \bigcup_{i=1}^k \gamma_i \to \bar{V}
\]

Near the nodes of \( \bar{S} \) including interior nodes and boundary nodes, by the properties of hyperbolic metric \( \mu^* \) on \( \bar{S} \), the neighbourhoods of interior nodes are corresponding to the annuli of the geodesic cycles. By the reparametrization of \( v_j \), called \( \bar{v}_j \) which is defined on \( S \) and extends the maps \( \bar{v}_j : S \to S_j \to V \)(see[4, 8]). Now let \( \{z_i : i = 1, ..., n\} \) be the interior nodes of \( \bar{S} \). Then the arguments in [4, 8] yield the \( C^0 \)-interior converse near \( z_i \).

5. \( C^0 \)-boundary convergence. Now it is possible that the boundary of the cusp curve \( \bar{v} \) does not remain in \( \bar{W} \). Write \( \bar{v}(z) = (h(z), (a(z), u(z), f(z))) \), here \( h(z) = z \) or \( h(z) \equiv z_i, i = 1, ..., n, z_i \) is cusp-point or bubble point. We can assume that \( \bar{p} = (1, p) \in \bar{v}_n \) is a puncture boundary point. Let \( \bar{v}_1 \) be the component of \( \bar{v} \) which through the point \( \bar{p} \). Let \( D = \{z|z = re^{i\theta}, 0 \leq r < \theta < \delta \} \). We assume that \( \bar{v}_1 : D \setminus \{e^{i\theta_i}\}_{i=1}^k \to V_c \), here \( e^{i\theta_i} \) is node or puncture point. Near \( e^{i\theta_i} \), we take a small disk \( D_i \) in \( D \) containing only one puncture or node point \( e^{i\theta_i} \). By the reparametrization and the convergence procedure, we can assume that \( \bar{v}_{1i} = (\bar{v}_1|D_i) \) as a map from \( D^+ \setminus \{0\} \to V_\epsilon \) with \( \bar{v}_1([-1, 1] \setminus \{0\}) \subset S^1 \times F(W' \times S^1) \) and area\(\bar{v}_{1i}) \leq a_0, a_0 \) small enough. Since Area\(\bar{v}_{1i}) \leq a_0, \) there exist curves \( c_k \) near 0 such that \( l(\bar{v}_{1i}(c_k)) \leq \delta \). By the construction of convergence, we can assume that \( l(\bar{v}_{1i}(c_k)) \leq 2\delta \). If \( \bar{v}_{1i}(\partial c_k) \subset (S^1) \times F(S^1) \), we have \( \bar{v}_n(\partial c_k) \subset (S^1) \times F(S^1) \) for \( n \) large enough. Now \( \bar{v}_n(c_k) \) cuts \( \bar{v}_n(D) \) as two parts, one part corresponds to \( \bar{v}_{1i} \), say \( \bar{u}_n(D) \). Then area\(\bar{u}_n(D)) =
area(h_{n1}) + |Ψ'(u_{n2}(c_k^1)) - Ψ'(u_{n2}(c_k^2))|, here ∂c_k = \{c_k^1, c_k^2\}. Then by the proof of Lemma 6.1-6.3, we know that \bar{u}_n(∂D\setminus c_k) \subset (S^1) \times F(\mathcal{L} \times [-100N_0, 100N_0] \times S^1). So, \bar{v}_1([-1, 1] \setminus \{0\}) \subset S^1 \times F(\mathcal{L} \times [-100N_0, 100N_0] \times S^1). By proposition 6.6, one singularity of \bar{v}_1 is deleted. We repeat this procedure, we proved that \bar{v}_1 is extended to whole D. So, the boundary node or puncture points of \bar{v} are removed. Then by choosing the sub-sub-sequences of \mu_j and \bar{v}_j, we know that \bar{v}_j converges to \bar{v} in C^0 near the boundary node or puncture point. This proved the C^0-boundary convergence. Since \bar{v}_j(1) = \bar{p}, \bar{p} \in \bar{v}(∂D), \bar{v}(∂D) \subset \bar{W}.

6. Convergence of area. Finally by the C^0-convergence and \text{area}(v_j) = \int_D v_j^* \bar{ω}, one easily deduces

\text{area}(v(S)) = \lim_{j \to \infty} (v_j(S_j)).

6.5 Bounded image of J-holomorphic curves in W

Proposition 6.7 Let v be the solutions of equations (4.16), then

\[d_W(p, v(∂D^2)) = \max\{d_W(p, q)|q \in f(∂D^2)\} \leq d_0 < +\infty\]

Proof. It follows directly from Gromov’s C^0-convergence theorem.

7 Proof of Theorem 1.1

Proposition 7.1 If J-holomorphic curves C \subset \bar{V} with boundary

∂C \subset D^2 \times ([0, \varepsilon] \times \Sigma) \times \mathbb{R}^2

and

C \cap (D^2 \times \{-3\} \times \Sigma) \times \mathbb{R}^2 \neq \emptyset

Then

\text{area}(C) \geq 2l_0.

Proof. It is obvious by monotone inequality argument for minimal surfaces.
Note 7.1 we first observe that any $J$–holomorphic curves with boundary in $R^+ \times \Sigma$ meet the hypersurface $\{-3\} \times \Sigma$ has energy at least $2l_0$, so we take $\varepsilon$ small enough such that the Gromov’s figure eight contained in $B_R(\varepsilon) \subset C$ for $\varepsilon$ small enough and the energy of solutions in section 4 is smaller than $l_0$. we specify the constant $a_0, \varepsilon$ in section 3.1-3 such that the above conditions satisfied.

Theorem 7.1 There exists a non-constant $J$–holomorphic map $u : (D, \partial D) \rightarrow (V', C, W)$ with $E(u) \leq 4\pi R(\varepsilon)^2$ for $\varepsilon$ small enough such that $4\pi R(\varepsilon)^2 \leq l_0$. Proof. By Proposition 5.1, we know that the image $\bar{v}(D)$ of solutions of equations (4.3) remains a bounded or compact part of the non-compact Lagrangian submanifold $W$. Then, all arguments in [4, 8] for the case $W$ is closed in $S \Sigma \times R^2$ can be extended to our case, especially Gromov’s $C^0$–convergence theorem applies. But the results in section 4 shows the solutions of equations (4.3) must denegerate to a cusp curves, i.e., we obtain a Sacks-Uhlenbeck-Gromov’s bubble, i.e., $J$–holomorphic sphere or disk with boundary in $W$, the exactness of $\omega$ rules out the possibility of $J$–holomorphic sphere. For the more detail, see the proof of Theorem 2.3.B in [8].

Proof of Theorem 1.1. If $(\Sigma, \lambda)$ has no Reeb chord, then we can construct a Lagrangian submanifold $W$ in $V = V' \times C$, see section 3. Then as in [4, 8], we construct an anti-holomorphic section $c$ and for large vector $c \in C$ we know that the nonlinear Fredholm section or Cauchy-Riemann section has no solution, this implies that the section is non-proper, see section 4. The non-properness of the section and the Gromov’s compactness theorem in section 6 implies the existences of the cusp-curves. So, we must have the $J$–holomorphic sphere or $J$–holomorphic disk with boundary in $W$. Since the symplectic manifold $V$ is an exact symplectic mainifold and $W$ is an exact Lagrangian submanifold in $V$, by Stokes formula, we know that the possibility of $J$–holomorphic sphere or disk elimitated. So our priori assumption does not holds which implies the contact maifold $(\Sigma, \lambda)$ has at least Reeb chord. This finishes the proof of Theorem 1.1.

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