Nonlocal conservation laws for supersymmetric KdV equations

P. Dargis and P. Mathieu

Département de Physique, Université Laval, Québec, Canada G1K 7P4

Abstract

The nonlocal conservation laws for the N=1 supersymmetric KdV equation are shown to be related in a simple way to powers of the fourth root of its Lax operator. This provides a direct link between the supersymmetry invariance and the existence of nonlocal conservation laws. It is also shown that nonlocal conservation laws exist for the two integrable N=2 supersymmetric KdV equations whose recursion operator is known.

1/93 (hepth@xxx/9301080)

1 Work supported by NSERC (Canada).
2 Work supported by NSERC (Canada) and FCAR (Québec).
1. Introduction

The supersymmetric KdV (sKdV) equation [1,2] has been much studied in the last few years [3-11]. Its hamiltonian forms have been obtained in two steps. The second hamiltonian structure, related to the superconformal algebra [3,4], has been obtained some time ago [2]. However it is only recently that its first hamiltonian structure has been discovered [6,7] and it turns out to be nonlocal.\footnote{The adjectives ‘first’ and ‘second’ have thus no chronological origin. They are inherited from their KdV relatives.} Both structures were shown to be related to the Lax operator by the standard Gelfand-Dickey brackets after reduction [6,7]. With these two hamiltonian structures, one can construct a recursion operator from which the infinite sequence of conservation laws can be obtained once the simplest one is known. These conservation laws can also be obtained in the standard way, from the super-residue of fractional powers of the Lax operators [1,2].

There remains a fact that has not received a clear explanation, namely the occurrence of nonlocal conservation laws. They have been found in [11] from a tedious symmetry analysis and traced back to the mere existence of the supersymmetry invariance. Here we show that these nonlocal conservation laws are related to the Lax operator. We also point out that, with a natural extension of the variationnal calculus, they can be generated recursively. Finally these results are partly extended to N=2 supersymmetric KdV (SKdV - where capital S stands for N=2) equations. For the two SKdV equations whose Lax operator is known [12] we prove, in each case, that there are two infinite sequences of nonlocal conservation laws which can be generated recursively. This was an expected result given the existence of two supersymmetries. However their relation to the Lax operators still has to be clarified. On the other hand, there is a third SKdV equation expected to be integrable [13], for which no nontrivial nonlocal conservation laws have been found.

2. N=1 sKdV equation

Let us introduce the fermionic superfield (with an implicit time dependence)

\[ \phi(x, \theta) = \theta u(x) + \xi(x) \]  

(2.1)
where $\theta$ is a Grassmann space variable and $u(x)$ is the usual KdV field. In term of this superfield and the superderivative $D$ defined as

\[
D = \theta \partial + \partial \theta \\
D^2 = \partial \equiv \partial_x
\]  

(2.2)

the sKdV equation reads [2]

\[
\phi_t = -\phi_{xxx} + 3(\phi D\phi)_x
\]  

(2.3)

Its Lax operator is [1,2]

\[
L = \partial^2 - \phi D \quad \text{or} \quad \partial^2 + \phi D - (D\phi)
\]  

(2.4)

(Parentheses are used to delimit the action of the derivatives). The degeneracy of the Lax formalism has been explained Lie algebraically in [5]. For calculations one can use either form. In terms of $L$, the usual local conservation laws can be written as

\[
H_n = \int dX \text{Res } L^{n/2}
\]  

(2.5)

d$X$ stands for $dxd\theta$ ($\int d\theta = 0$ and $\int d\theta \theta = 1$) while the residue (Res) of a super-pseudodifferential operator is the coefficient of $D^{-1}$. The explicit form of the first few can be found in [2,3]. Given that $L^{n/2} = \partial^n + \ldots$ is bosonic, its residue is fermionic. The measure being also fermionic, these conservation laws are bosonic. The two hamiltonian operators are [2,6,7]

\[
P_1 = \partial[D^3 - \phi]^{-1}\partial
\]  

(2.6)

\[
P_2 = -D^5 + 3\phi \partial + (D\phi)D + 2\phi_x
\]  

(2.7)

from which one constructs the recursion operator

\[
R = P_1^{-1}P_2 = (D^{-1} - \partial^{-1}\phi\partial^{-1})P_2
\]  

(2.8)

Let us now turn to the nonlocal conservation laws. The first few of them can be easily obtained from (2.3) (i.e., one writes down the most general nonlocal differential polynomial
of appropriate degree, with $\deg(\phi) = 3/2$, $\deg(D) = 1/2$, and adjusts its coefficients such that its integral is time independent). These are

$$
J_{1/2} = \int dX (D^{-1}\phi) \\
J_{3/2} = \int dX (D^{-1}\phi)^2 \\
J_{5/2} = \int dX \left[ (D^{-1}\phi)^3 - 6(D^{-1}(\phi D\phi)) \right] \\
J_{7/2} = \int dX \left[ (D^{-1}\phi)^3 - 12(D\phi)^2 - 24(D^{-1}(D\phi))(D^{-1}(\phi D\phi)) \right]
$$ (2.9)

The subscripts indicate the degree. One first notices that they are all fermionic and that there is one conservation law at each half integer degree. (Notice that the $H_n$ are non-zero only for $n$ odd, hence there are twice as many nonlocal conservation laws than local ones.) The infinite sequence can be constructed from $J_{1/2}$ and $J_{3/2}$ by using the recursion operator:

$$
R \frac{\delta}{\delta \phi} J_{1/2+i+2n} = \frac{\delta}{\delta \phi} J_{1/2+i+2n+2} \quad n = 0, 1, 2, \ldots \quad i = 0, 1 \quad \text{(2.10)}
$$

For this, however, one has to extend the validity of the usual formula for the variational derivative (recall that here $\phi$ is odd) [1,10,14]

$$
\frac{\delta}{\delta \phi} \int dX \ h(\phi) = \sum_k (-)^{k+k(k+1)/2} \left( D^k \frac{\partial h}{\partial (D^k\phi)} \right) \quad \text{(2.11)}
$$

to negative values of $k$. For instance

$$
\frac{\delta}{\delta \phi} \int dX \ (D^{-1}\phi) = -(D^{-1}1) = -\theta \quad \text{(2.12)}
$$

(since $(D\theta) = 1$). Similarly, the validity of the formula for integration by parts,

$$
\int dX \ (D^k A) B = (-)^{k\tilde{A}+k(k+1)/2} \int dX \ A(D^kB) \quad \text{(2.13)}
$$

where $\tilde{A} = 0 \ (1)$ if $A$ is even (odd), is extended to the case where $k < 0$. In particular one has

$$
\int dX \ (D^{-1}\phi) = -\int dX \ \phi(D^{-1}1) = -\int dX \ \phi\theta \quad \text{(2.14)}
$$

from which one can recover the result (2.12). Notice that with this definition, $u(\partial^{-1}u)$ is a total derivative, i.e., $\int dx \ u(\partial^{-1}u) = -\int dx \ (\partial^{-1}u)u = 0$. 

3
**Example:** Here are few steps of a sample calculation leading from \( J_{3/2} \) to \( J_{7/2} \):

\[
R \frac{1}{2} \frac{\delta}{\delta \phi} J_{3/2} = R(- (\partial^{-1} \phi))
\]

\[
= - (D^{-1} - \partial^{-1} \phi \partial^{-1}) \partial (- (D \phi) + 2 \phi (\partial^{-1} \phi) + (D^{-1} (\phi (D^{-1} \phi))))
\]

\[
= \phi_x - 2(D \phi)(\partial^{-1} \phi) + \phi(D^{-1} \phi) - (\partial^{-1} (\phi (D \phi))) + \frac{1}{6} (D^{-1} (D^{-1} \phi)^3)
\]

\[
= - \frac{1}{24} \frac{\delta}{\delta \phi} \int dX \left[ (D^{-1} \phi)^4 - 12(D \phi)^2 - 24(D^{-1} \phi)D^{-1}(\phi D \phi) \right]
\]

(2.15)

To get the second equality, we used the relations

\[
(D \phi)(D^{-1} \phi) = (D(\phi D^{-1} \phi)) \quad (\phi_x \partial^{-1} \phi) = (\partial(\phi \partial^{-1} \phi))
\]

(2.16)

which holds because \( \phi^2 = 0 \), \( \phi \) being fermionic. To get the third equality we used

\[
(D^{-1} (\phi D^{-1} \phi)) = \frac{1}{2} (D^{-1} \phi)^2
\]

(2.17)

This is easily checked by setting \( \phi = DF \) (\( F \) being thus bosonic):

\[
(D^{-1} (\phi D^{-1} \phi)) = (D^{-1} (FDF)) = \frac{1}{2} (D^{-1} D(F^2)) = \frac{1}{2} (D^{-1} \phi)^2
\]

(2.18)

Notice that \( \frac{\delta}{\delta \phi} J_{1/2+2n} \) always contains explicit \( \theta \) terms while \( \frac{\delta}{\delta \phi} J_{3/2+2n} \) does not.

Thus, given \( J_{1/2} \), \( J_{3/2} \) and \( R \), we conclude that there exists an infinite sequence of odd nonlocal conservation laws. But is there a more direct way to probe their existence? The clue to the answer is to notice that \( L \) admits not only a square root but also an odd fourth root, of the form \( L^{1/4} = D + ... \). It is then clear that (up to a constant) the fermionic conservation laws must be related to \( L \) via

\[
J_{l/2} = \int dX \text{Res} L^{1/4} \quad (l \text{ odd})
\]

(2.19)

Their nonlocality is rooted in the nonlocality of \( L^{1/4} \) itself, i.e.

\[
L^{1/4} = D + (\partial^{-1} \phi) - \frac{1}{2} (D^{-1} \phi)D^{-1} - \frac{1}{2} \phi D^{-2} + \frac{1}{4} (D \phi)D^{-3} + \frac{1}{8} (D^{-1} \phi)^2 D^{-3} + ...
\]

(2.20)

Now all these odd nonlocal conservation laws commute with the usual local bosonic conservation laws. This follows trivially from the fact that the latter are possible hamiltonians and the former are time independent. However, what about the commutation of the fermionic conservation laws among themselves? From the Jacobi identity for the Poisson
brackets of \( J_{l/2}, J_{k/2} \) and \( H_n \), it follows that \( \{ J_{l/2}, J_{k/2} \} \), calculated with the second Poisson structure, is necessarily a conservation law, and its degree is \( \frac{l+k}{2} \). Therefore up to numerical factors, one must have

\[
\begin{align*}
\{ J_{(4n+1)/2}, J_{(4m+1)/2} \} &= H_{2(n+m)+1} \\
\{ J_{(4n+3)/2}, J_{(4m+3)/2} \} &= H_{2(n+m)+3} \\
\{ J_{(4n+1)/2}, J_{(4m+3)/2} \} &= 0
\end{align*}
\] (2.21)

The last result is a consequence of the absence of conservation laws with even integer degree. The first two are readily verified explicitly by using the first few conservation laws given in (2.9). One checks in particular that the proportionality constants are nonzero. This algebra shows that nonlocal conservation laws are some sort of square root, in a Poisson bracket sense, of local conservation laws.

Let us now prove these relations at the level of Lax equations. For this we introduce a new infinite sequence of odd flows

\[
\partial_{\tau_l} L = [G^l_+, L]
\] (2.22)

generated by the fourth root of \( L \):

\[
G \equiv L^{1/4}
\] (2.23)

\( l \) is restricted to be an odd integer so that the \( \tau_l \) are odd parameters. These imply directly that

\[
\partial_{\tau_l} G^k = [G^l_+, G^k]
\] (2.24)

where the commutator is understood as a graded commutator, i.e.

\[
[A, B] = AB - (-)^{A\cdot B} BA
\] (2.25)

A direct calculation yields

\[
(\partial_{\tau_l} \partial_{\tau_k} + \partial_{\tau_k} \partial_{\tau_l}) L = [B, L]
\] (2.26)

where \( B \) is given by

\[
B = [G^k_+, G^l_+]_+ + [G^l_+, G^k]_+ - [G^l_+, G^k_+]
\] (2.27)

To obtain this equality, we used the graded Jacobi identity

\[
[[[A, B] C] + [[[C, A] B](-)^{C(A+B)}] + [[B, C] A](-)^{B(C+A)} = 0
\] (2.28)
Now with $G^k = G^k_+ + G^k_-$ and $[G^k_-, G^l_+] = 0$, one can rewrite $B$ as

$$B = (G^k G^l + G^l G^k)_+ = 2L^k_+^2$$ (2.29)

This shows that

$$\partial_{\tau_k} \partial_{\tau_l} + \partial_{\tau_l} \partial_{\tau_k} = 2\partial t_{k+l}$$ (2.30)

where the flow $t_n$ is defined as

$$\partial_{t_n} L = [L^{n/2}_+, L]$$ (2.31)

The flow $t_{k+l}$ is trivial if $k+l$ is even, and generated by $H_{k+l}$ if $k+l$ is odd. This completes the proof of (2.21).

As a simple check of (2.30), let us verify that $\partial^2_{\tau_{3/2}} = \partial_{t_3}$. With

$$L^{3/4}_+ = D^3 + (\partial^{-1}\phi)\partial - \frac{1}{2}(D^{-1}\phi)D$$ (2.32)

one easily finds that

$$\partial_{\tau_{3/2}} \phi = D^3 \phi + (\partial^{-1}\phi)(\partial \phi) - \frac{1}{2}(D^{-1}\phi)(D \phi)$$ (2.33)

Applying $\partial_{\tau_{3/2}}$ on this equation gives

$$\partial^2_{\tau_{3/2}} \phi = +\frac{1}{4}[-\phi_{xxx} + 3(\phi D \phi)_x]$$ (2.34)

(2.33) is thus the square root of the sKdV equation. On the other hand, we have checked directly that the first three nonlocal conservation laws are indeed given by (2.19). In particular, deriving $J_{5/2}$ in this way is non-trivial (and very long) since it contains two terms, hence a precise relative coefficient.

Relations (2.21) and (2.30) can be regarded as the global translation of the graded commutation relations obtained in [11] from the symmetry algebra.

A final remark will close this section. The nonlocal conservation laws, being written as a superintegral of a density expressed in terms of the superfield $\phi$ and the superderivative $D$, appear to be manifestly supersymmetric invariant. However the nonlocality may induce a breaking of supersymmetry. In fact all $J_{(4n+1)/2}$ are not invariant under a supersymmetry transformation. This is most simply seen with the component formulation (2.1). The supersymmetric transformation of the fields is

$$\delta u = \eta \xi_x \quad \delta \xi = \eta u$$ (2.35)
where $\eta$ is a constant anticommuting parameter. In components, $J_{1/2}$ reads $\int dx \xi$ so that

$$\delta J_{1/2} = \eta \int dx \ u \neq 0$$

(2.36)

On the other hand, for $J_{3/2}$ one has

$$\delta J_{3/2} = \delta \int dx \ \xi (\partial^{-1} u) = \eta \int dx \ (u(\partial^{-1} u) - \xi \xi) = 0$$

(2.37)

(the first term in the integrand being a total derivative), that is $J_{3/2}$ is invariant under a supersymmetry transformation. The following observation shows that $\delta J_{(4n+3)/2} = 0$ simply because $H_{2n+2} = 0$. Indeed, up to a constant factor, the action of $\delta$ is equivalent to taking the Poisson bracket of the second hamiltonian structure with $J_{1/2}$, e.g.

$$\delta \cdot = -\eta \{J_{1/2}, \cdot\}$$

(2.38)

With this, the above results are seen to be simple transcriptions of the first and third relations in (2.21) for $n = 0$.

3. N=2 SKdV equations

We now introduce an even N=2 superfield $\Phi(x, \theta_1, \theta_2)$ whose components are

$$\Phi(x, \theta_1, \theta_2) = \theta_2 \theta_1 u(x) + \theta_1 \xi_1(x) + \theta_2 \xi_2(x) + w(x)$$

(3.1)

Here $\theta_{1,2}$ are two anticommuting variables, $\theta_1^2 = \theta_2^2 = 0$, $\theta_1 \theta_2 = -\theta_2 \theta_1$, $\xi_{1,2}$ are two anticommuting fields and $w$ is a new bosonic field. Notice that $\text{deg}(u, \xi_i, w) = (2, 3/2, 1)$. One also introduces two superderivatives $D_{1,2}$ defined as

$$D_i = \theta_i \partial + \partial \theta_i$$

$$D_1^2 = D_2^2 = \partial \quad D_1 D_2 = -D_2 D_1$$

(3.2)

The two integrable SKdV equations, whose Lax operators are known are [12]

$$\Phi_t = -\Phi_{xxx} + 3(\Phi D_1 D_2 \Phi)_x + \frac{1}{2} (\alpha - 1) (D_1 D_2 \Phi^2)_x + 3\alpha \Phi^2 \Phi_x$$

$$\alpha = -2, 4$$

(3.3)
and denoted by SKdV\(_{-2}\) and SKdV\(_{4}\). The Lie algebraic structure underlying the first system was unravelled in [15]. The corresponding Lax operators are

\[
L_{(-2)}: \partial^2 + 2\Phi D_1 D_2 - (D_2 \Phi) D_1 + (D_1 \Phi) D_2
\]
\[
L_{(4)}: \partial^2 - 2\Phi D_1 D_2 + (D_2 \Phi) D_1 - (D_1 \Phi) D_2 - (D_1 D_2 \Phi) - \Phi^2
\]
\[
= -(D_1 D_2 + \Phi)^2
\]

and the two are self-adjoint. Both systems are bi-hamiltonian. They share a common second hamiltonian structure [12]

\[
\hat{P}_2 = D_1 D_2 \partial + 2\Phi \partial - (D_1 \Phi) D_1 - (D_2 \Phi) D_2 + 2\Phi x
\]

and their first hamiltonian structure is given respectively by [6]

\[
\hat{P}_{1 (-2)} = (D_1 D_2 \partial^{-1} - D_1^{-1} \Phi D_1^{-1} - D_2^{-1} \Phi D_2^{-1})^{-1}
\]
\[
\hat{P}_{1 (4)} = \partial
\]

One then has a recursion operator \(\hat{R} = \hat{P}_{1 (-2)}^{-1} \hat{P}_2\) for each hierarchy. Notice however that \(\text{deg} R_{(4)} = 1\) while \(\text{deg} R_{(-2)} = 2\). This translates into the fact that there are twice as many local conservation laws for the SKdV\(_4\) equation as for the SKdV\(_{-2}\) one. There exists one local conservation law for each integer degree in the former case and one for each odd integer degree in the latter one. Conservation laws for odd degrees in both cases are related to the Lax operator by

\[
H_n = \int d\hat{X} \hat{Res} L^{n/2} \quad n \text{ odd}
\]

where the N=2 super residue is defined as

\[
\hat{Res} = \text{coeff of } D_1 D_2 \partial^{-1}
\]

\(d\hat{X} = dxd\theta_1 d\theta_2\) and \(L^{n/2} = \partial^n + \ldots\). The origin of the conservation laws of even degree for the case \(\alpha = 4\) is rooted in the existence of another square root for \(L_{(4)}\) (different from \(\partial + \ldots\)) namely [6]

\[
L' = D_1 D_2 + \Phi \quad (L')^2 = -L_{(4)}
\]

In term of \(L\) and \(L'\) one can introduce the new flows [6]

\[
\partial_{t_n} L = [(L^{n/2} L')_+, L]
\]
giving rise to another infinite sequence of conservation laws.

\[ H_{n+1} = \int d\hat{X} \ \text{Res} \ (L^{n/2} L') \quad n \ \text{odd} \quad (3.11) \]

For these N=2 supersymmetric KdV equations, there exists also nonlocal conservation laws. The first few of them are

\[ \alpha = -2 \]

\[ J_{1/2}^{(1)} = \int d\hat{X} \ (D_1^{-1} \Phi) \]

\[ J_{3/2}^{(1)} = \int d\hat{X} \ \Phi(D_1^{-1} \Phi) \]

\[ J_{5/2}^{(1)} = \int d\hat{X} \ \left[ \frac{2}{3} (D_1^{-1} \Phi^3) + \frac{1}{6} \Phi^2(D_1^{-1} \Phi) + (D_1^{-1} (\Phi D_1 D_2 \Phi)) - \frac{1}{6} (D_1^{-1} \Phi)(D_1 D_2 \partial^{-1} \Phi)^2 \right] \quad (3.12) \]

\[ \alpha = 4 \]

\[ J_{1/2}^{(1)} = \int d\hat{X} \ (D_1^{-1} \Phi) \]

\[ J_{3/2}^{(1)} = \int d\hat{X} \ \Phi(D_1^{-1} \Phi) - 2(D_1^{-1} \Phi^2) \]

\[ J_{5/2}^{(1)} = \int d\hat{X} \ \left[ \frac{4}{3} (D_1^{-1} \Phi^3) - \frac{5}{6} \Phi^2(D_1^{-1} \Phi) + (D_1^{-1} (\Phi D_1 D_2 \Phi)) - \frac{1}{6} (D_1^{-1} \Phi)(D_1 D_2 \partial^{-1} \Phi)^2 \right] \quad (3.13) \]

In each case there is another infinite series \( J_{l/2}^{(2)} \) obtained from \( J_{l/2}^{(1)} \) with \( D_1 \leftrightarrow D_2 \). The first two fermionic conservation laws for \( \alpha = -2 \) in (3.12) are easily obtained by inspection; from these two, all others can be generated by the action of the recursion operator. On the other hand for \( \alpha = 4 \), one only needs to know \( J_{1/2}^{(1)} \) to get all the \( J_{l/2}^{(1)} \)’s recursively.

Computing the Poisson brackets of these first few nonlocal conservation laws, we infer the algebraic structure

\[ \{ J_{l/2}^{(1)}, J_{k/2}^{(2)} \} = 0 \quad (3.14) \]

for any \( l \) and \( n \), and

\[ \{ J_{l/2}^{(i)}, J_{k/2}^{(i)} \} = \begin{cases} H_{l+k} & \text{for } \frac{l+k}{2} \ \text{odd} \\ \delta_{\alpha,4} H_{l+k} & \text{for } \frac{l+k}{2} \ \text{even} \end{cases} \quad (3.15) \]

with \( i = 1 \) or 2.
Unfortunately, the Lax origin of these nonlocal conservation laws has not been unraveled. The operator $L$ would seem to have two distinct fourth roots:

$$G_{(i)} = D^i + \text{lower order operators}$$
$$G_{(i)}^4 = L \quad i = 1, 2$$

whose residues would be natural candidates for the nonlocal conserved densities. But a simple calculation shows that such $G_{(i)}$ do not exist. A variant of this idea, where $L^{1/2}$ is factorized into a symmetric product of two odd operators, is not successful either.

On the other hand, it has been conjectured in [13] that the SKdV$_1$ equation is also integrable on the basis of the existence of $H_5$ and $H_7$ (yet two other conservation laws have been found [16]). It is natural to see whether it has nonlocal conservation laws. Of course $J_{1/2}^{(1,2)}$ is conserved. However it is not difficult to check that there are no nonlocal fermionic conservation laws of the general form

$$J_{3/2}^{(1)}(\alpha) = \int d\hat{X} \left[ a\Phi(D^{-1}_1\Phi) + b(D^{-1}_1\Phi^2) \right]$$

for values of $\alpha$ different from 4 and $-2$.

4. Concluding remarks

In this work we have shown that nonlocal conservation laws of supersymmetric KdV equations can be analysed with essentially the same methods used for local ones. Actually, they can be regarded as some sort of square root (in a Poisson brackets sense) of the usual bosonic local conservation laws, an interpretation which clearly reflects the supersymmetric invariance.

It is clear that the existence of nonlocal conservation laws will modify the algebraic structure of mastersymmetries obtained in [6] for N=1. Most probably it will change the centerless Virasoro algebra into its appropriate supersymmetric extension.

On the other hand, the local conservation laws for the quantum version of the system considered here appear in the appropriate perturbed superconformal minimal models [17]. It is natural to expect that the quantum form of the nonlocal conservation laws will also be present in these off-critical theories. Actually it is rather simple to check this for the first few nonlocal conservation laws of the system considered. But these are conservation
laws which contain a single term. In the quantum case it turns out to be difficult to
calculate the relative coefficients of the different terms of a nonlocal conservation law.
We expect to return to this elsewhere. In that context, it should be stressed that the
nonlocal conservation laws considered here have no relation with those found in [18] for
perturbed conformal field theories. At first the latter have a purely quantal origin, and
furthermore they cannot be expressed in terms of the energy-momentum tensor, except
for special values of the central charge, at which they become local.

References

1. Yu.I. Manin and A.O. Radul, *Comm. Math. Phys.* **98** (1985) 65
2. P. Mathieu, *J. Math. Phys.* **29** (1988) 2499
3. I. Yamanaka and R. Sasaki, *Prog. Theor. Phys.* **79** (1988) 1167
4. P. Mathieu, *Phys. Lett.* **B203** (1988) 287
5. T. Inami and H. Kanno, *Comm. Math. Phys.* **136** (1991) 519
6. W. Oevel and Z. Popowicz, *Comm. Math. Phys.* **139** (1991) 441
7. J.M. Figuera-O’Farrill, J. Mas, and E. Ramos, *Leuven preprint* KUL-TF-91-19
8. I.N. McArthur, *Comm. Math. Phys.* **148** (1992) 177
9. P. Mathieu, in “Integrable and Superintegrable Systems”, ed. B. Kupershmidt (World
Scientific, 1990) p352
10. E.D.van der Lende and H.G.J.Pijls, *Indag. Math. N.S.* **1:2**, (1990) 221; E.D.van der
Lende, PhD thesis
11. P.H.M. Kersten, *Phys. Lett.* **134 A** (1988) 25
12. C.A. Laberge and P. Mathieu, *Phys. Lett.* **B215** (1988) 718
13. P. Labelle and P. Mathieu, *J. Math. Phys.* **32** (1991) 923
14. P. Mathieu, *Lett. Math. Phys.* **16** (1988) 199
15. T. Inami and H. Kanno, *Nucl. Phys.* **B359** (1991) 201
16. C.M. Yung, *Tasmania preprint*, UTAS-PHYS-92-17
17. P. Mathieu, *Nucl. Phys.* **B336** (1990) 338; P. Mathieu and M. Walton, *Phys. Lett.*
**254** (1991) 106
18. D. Bernard and A. LeClair, *Comm. Math. Phys.* **142** (1989) 99; C. Ahn, D. Bernard
and A. LeClair, *Nucl. Phys.* **B346** (1990) 409