Reissner–Nordstrøm–de Sitter manifold: photon sphere and maximal analytic extension

Mokdad Mokdad

Department of Mathematics, University of Brest, 6 Avenue Victor Le Gorgeu, 29200 Brest, France
E-mail: mokdad.al.mokdad@gmail.com

Received 15 February 2017, revised 11 July 2017
Accepted for publication 18 July 2017
Published 10 August 2017

Abstract
This paper is devoted to the study of Reissner–Nordstrøm–de Sitter black holes and their maximal analytic extensions. Here, we find the necessary and sufficient conditions on the parameters of the Reissner–Nordstrøm–de Sitter metric—namely, the mass, the charge, and the cosmological constant—to have three horizons. Under these conditions, we prove that there is only one photon sphere and we locate it. We then give a detailed construction of the maximal analytic extension of the Reissner–Nordstrøm–de Sitter manifold in the case of three horizons. Studying these properties lays the groundwork for obtaining (in separate papers) decay results (Mokdad 2017 Decay of Maxwell fields on Reissner–Nordstrøm–de Sitter black holes (arXiv:1704.06441)) and constructing conformal scattering theories for test fields on such spacetimes (Mokdad 2017 Conformal scattering of Maxwell fields on Reissner–Nordstrøm–de Sitter black hole spacetimes (arXiv:1706.06993)).

Keywords: black holes, maximal extensions of solutions to Einstein’s equations, photon sphere and orbits, Reissner–Nordstrøm–de Sitter black hole spacetime, general relativity

(Some figures may appear in colour only in the online journal)

1. Introduction

The Reissner–Nordstrøm–de Sitter solution (RNdS), is one of the spherically symmetric solutions of the Einstein–Maxwell field equations in the presence of a positive cosmological constant \( \Lambda \). This solution models a non-rotating spherically symmetric charged black hole with mass \( M > 0 \) and a charge \( Q \neq 0 \), in a de Sitter background. To fix notations, the Reissner–Nordstrøm–de Sitter metric we are studying is given in spherical coordinates by
\[ g_M = f(r)dr^2 - \frac{1}{f(r)}dr^2 - r^2d\omega^2, \]

where
\[ f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \Lambda r^2, \]
and \(d\omega^2\) is the Euclidean metric on the 2-Sphere, \(S^2\), which in spherical coordinates is
\[ d\omega^2 = d\theta^2 + \sin(\theta)^2d\phi^2, \]
and \(g_M\) is defined on \(M = \mathbb{R}_r \times [0, +\infty]_r \times S^2_{\theta, \phi}\). Similar to the Schwarzschild metric, the RNdS metric in these coordinates appears to have singularities at \(r = 0\) and at the zeros of \(f\). Only the singularity at \(r = 0\) is a real geometric singularity at which the curvature blows up. The apparent singularities at the zeros of \(f\) are artificial and due to this particular choice of coordinates can be removed using Kruskal–Szekeres coordinates. Again as in the Schwarzschild case, the Kruskal–Szekeres coordinates can be used to define the maximal analytic extension of the RNdS whose construction is detailed in this work. The regions of spacetime where \(f\) vanishes are the event horizons, and \(f\) is called the horizon function. If \(f\) has three positive zeros and a negative one, then the zeros in the positive range corresponds in an increasing order respectively to the Cauchy horizon or inner horizon, the horizon of the black hole or the outer horizon, and the cosmological horizon. Here, we work with three horizons corresponding to \(r\) equals to \(r_1, r_2,\) and \(r_3\), the three positive zeros of \(f\). In this case, the region corresponding to \([0, r_1]\) is a static region in the interior of the black hole, and the one corresponding to \([r_2, r_3]\) is another static region in the exterior. An interior dynamic region separating the two static regions lies in \([r_1, r_2]\), and the region given by \(r > r_1\) is a dynamic region near infinity. One can refer to classical books such as [8, 12, 24] for more on exact solutions and on Einstein’s general theory of relativity. However, few works can be found on the RNdS spacetime. Some of the early works [4, 16, 17] shortly discuss the construction of maximally extended RNdS spacetimes. Works on global spacetime solutions and on their constructions that include RNdS cases can also be found for example in [3, 13, 14]. Thus, the general aspects of the construction of the maximal analytic extension of RNdS spacetime is in the literature, but up to our knowledge, it has never been explicitly carried out in details. In part of this paper, we try to fill the ‘gap’ by giving conditions on the free parameters of the RNdS metric and by constructing the maximal analytic extension of this spacetime in the most complete case (three horizons) with sufficient details and natural motivations for each step. The radial null geodesics play an important role in specifying the coordinates used and in obtaining the different extensions. We also discuss some of the geometrical and causal properties of these extensions (in the spirit of [12]).

The region with \(r > 2M\) of the Schwarzschild spacetime with black hole of radius \(2M\), contains a hypersurface at \(r = 3M\) that distinguish between different behaviours of null geodesics, and it is an important geometric feature of the spacetime. Any future endless null geodesic in the maximally extended Schwarzschild spacetime starting at some point with \(r > 3M\) and initially directed outwards will continue outwards and escape to infinity. While any null geodesic starting at some point between \(r = 2M\) and \(r = 3M\) and initially directed inward will continue inward and fall into the black hole. The hypersurface at \(r = 3M\) is called the photon sphere: any null geodesic starting at some point of the photon sphere and initially tangent to the photon sphere will remain in the photon sphere. The reader can refer to [8, 10, 11] for more on the behaviour of null geodesics in the Schwarzschild spacetime. A similar photon sphere is present in the RNdS spacetime, where the photon can orbit the black hole. We
show here that, in the case of three horizons, this photon sphere is located in the static exterior region given by $r \in [r_2, r_3]$.

The presence of a photon sphere has important physical and mathematical implications. On the one hand, the existence/non-existence of photon spheres, or more generally photon surfaces (see [9] for example), in a spacetime has important implications for gravitational lensing and plays an important role on observations of astrophysical objects. For example, in any spacetime containing a photon sphere, gravitational lensing will give rise to relativistic images [21–23]. Other discussions of photon orbits and their effects can also be found in [5–7, 15]. On the other hand, the geometrical aspects of this spacetime are important for us in other works [19, 20]. In these works, we are interested in the decay in time of Maxwell fields in the exterior static region of the RNdS black hole spacetime. The decay of test fields (such as the electromagnetic fields) plays an important role in studying the stability of solutions of Einstein’s equation. The RNdS black holes can be considered as spherically symmetric models of the more important Kerr family of black holes which are believed to best represent real black holes that may be existing now in our Universe. The effect of the photon sphere concerning decay of test fields can be seen in [1, 2, 20] among others. A priori, the existence of a photon sphere is an obstacle for the decay. Fields can still decay, as shown in [20] or [2] and other works, but the photon sphere slows the decay. This is because there will be null geodesics that rotate around the black hole near the photon sphere for an arbitrary amount of time. We are also interested in constructing a conformal scattering theory on the exterior static region [19]. For this, we need to have access to the boundary of the region corresponding to infinite t-values in the static interior region. This boundary is part of the maximal analytic extension of the spacetime whose construction we discuss in this paper.

This paper has two main sections:

Section 2 We start the section by presenting the necessary and sufficient conditions (5) on the parameters $M, Q$, and $\Lambda$ of the RNdS metric so that it has three horizons. We then verify our claim regarding these conditions along with the fact that there is a photon sphere only at one value of $r > 0$ and it is located in the exterior static region. This is proposition 1, and up to our knowledge, this is not in the literature.

Section 3 This section is a detailed discussion and construction of the maximal analytic extension of the RNdS manifold in the case of three horizons at $0 < r_1 < r_2 < r_3$. We start by exploring some properties of the black hole in the RNdS coordinates $(t, r, \omega)$. We then discuss the Regge–Wheeler $r_*$ coordinate and use it to obtain coordinate expressions of the radial null geodesics. Using the radial null geodesics we define the Eddington–Finkelstein advanced and retarded coordinates and extensions. The place where horizons of the same $r$-value ‘meet’ is asymptotic to all of the Eddington–Finkelstein charts, these are the bifurcation spheres. To cover these spheres we need the Kruskal–Szekeres extensions. Each of these new extensions now cover all the horizons at $r = r_i$ and the bifurcation sphere where they intersect. Finally, we use the Kruskal–Szekeres charts to cover the manifold of the maximal analytic extension. We discuss its causal structure, and some properties of its timelike singularity at $r = 0$.

2. Photon sphere

In this section we study the horizon function $f$ given in (2) of the RNdS metric (1). We put,

$$R = \frac{1}{\sqrt{6\Lambda}} ; \quad \Delta = 1 - 12Q^2\Lambda ; \quad m_1 = R\sqrt{1 - \sqrt{\Delta}} ; \quad m_2 = R\sqrt{1 + \sqrt{\Delta}} \quad (3)$$
and we consider the following conditions,

\[ Q \neq 0 \quad \text{and} \quad 0 < \Lambda < \frac{1}{12Q^2} \quad \text{and} \quad M_1 < M < M_2. \]  

The main result of this section is:

**Proposition 1 (Three positive zeros and one photon sphere).** The function \( f \) has exactly three positive distinct zeros if and only if (5) holds. In this case, there is exactly one photon sphere in the static exterior region of the black hole defined by the portion between the largest two zeros of \( f \).

The proof is divided into parts: First, we study the conditions on \( M, Q, \) and \( \Lambda \) for \( f \) to have three positive zeros, and then we show that in this case there is only one photon sphere.

### 2.1. The zeros of the horizon function

The zeros of the function \( f \) are the roots of the polynomial

\[ r^2 f(r) = P(r) = -4\Lambda r^3 + 2r - 2Mr + Q^2. \]  

Let us show that \( P \) has exactly three positive and one negative real roots if and only if (5) holds. We will proof this in two lemmata.

**Lemma 2.** The polynomial \( P \) has three positive roots if and only if

\[ P'(R) > 0 \quad \text{and} \quad P(s_1) < 0 \quad \text{and} \quad P(s_2) > 0, \]  

where \( 0 < s_1 < s_2 \) are the two positive roots of \( P' \).

**Proof.** The expressions of \( P' \) and \( P'' \) are \( P'(r) = -12\Lambda r^2 + 2r - 2M \), and so \( P''(r) = 2 \), \( P' \) is increasing on \([0,R]\) and decreasing on \([R, +\infty]\) with a local maximum at \( R \). If \( P'(R) \) is non positive, and since \( P'(0) = -2M < 0 \), then \( P' \) is everywhere non positive on \([0, +\infty]\). Thus, \( P \) is decreasing on \([0, +\infty]\), and has only one root there as it decreases from \( P(0) = Q^2 > 0 \) to \(-\infty\). Therefore, a necessary condition for \( P \) to have three positive roots is that \( P'(R) \) be positive. Clearly \( P'(R) > 0 \) is equivalent to \( M < \frac{1}{2}\Lambda \). As \( P'(0) < 0 \), and \( \lim_{r \to \infty} P'(r) = +\infty \), then having a positive local maximum at \( R \) implies that \( P' \) has exactly two roots \( 0 < s_1 < R < s_2 \) on the positive axis, and one on the negative axis. Also \( P' \) changes sign after passing through each of its roots \( s_1 \) and \( s_2 \), which means that \( P(s_1) \) and \( P(s_2) \) are respectively the local minimum and the local maximum of \( P \) over the interval \([0, +\infty]\). We can conclude the following:

- If \( P(s_1) > 0 \), then \( P \) has one positive root \( x \), with \( s_2 < x \).
- If \( P(s_1) = 0 \), then \( P \) has two positive roots \( s_1 \) and \( x \), with \( s_1 < s_2 < x \).
- If \( P(s_1) < 0 \), then:
  - If \( P(s_2) < 0 \), then \( P \) has one positive root \( x \), with \( x < s_1 \).
  - If \( P(s_2) = 0 \), then \( P \) has two positive roots \( x \) and \( s_2 \), with \( 0 < x < s_1 < s_2 \).
  - If \( P(s_2) > 0 \), then \( P \) has three positive roots \( r_1, r_2, \) and \( r_3 \), with \( 0 < r_1 < s_1 < r_2 < s_2 < r_3 \).

This concludes the proof. □
Instead of finding $s_1$ and $s_2$ explicitly, we will, using the next lemma, transform the conditions in (7) to those in (5) directly.

**Lemma 3.** If $P'(R) > 0$, with $s_1$ and $s_2$ the positive roots of $P$, then

$$P(s_1) < 0 \text{ if and only if } P'(m_1) < 0;$$

$$P(s_2) > 0 \text{ if and only if } P'(m_2) > 0,$$

where $m_1$ and $m_2$ are defined in (3).

**Proof.** We first note that $P(r) = -\Delta r^4 + 2Mr + Q^2 = rP'(r) + T(r)$, where $T$ is the polynomial $T(r) = 3\Delta r^2 - r^2 + Q^2$. So, $P(s_1) = T(s_1)$ and $P(s_2) = T(s_2)$. Therefore if we study the sign of $T$ we shall know the sign of $P(s_1)$ and $P(s_2)$. Let $T(r^2) = T(r)$, i.e. $T(r) = 3\Delta r^2 - r + Q^2$, which has discriminant $\Delta = 1 - 12\Lambda Q^2$. We investigate the different cases.

- If $\Delta < 0$ then $T$ has no real roots and is always positive, and hence so is $T$. In particular, this means that $T(s_1)$ and $T(s_2)$ are both positive, which is not the desired case.
- If $\Delta = 0$ then $R$ is a double root for $T$ and it is non negative. It follows that $T$ is also non negative, and the conditions of (7) cannot be satisfied.
- Finally, if $\Delta > 0$ which is $\Lambda < \frac{1}{12Q^2}$, then $T$ has two positive roots $m_1^2$, $m_2^2$ and hence $\pm m_1$, $\pm m_2$ are the roots of $T$, and $T$ is positive on $[0,m_1]$, negative on $[m_1,m_2]$, and positive on $[m_2, +\infty[$.

Thus, noting that $s_1$ and $m_1$ are strictly less than $R$, and $s_2$ and $m_2$ are strictly greater than $R$ when $\Lambda < \frac{1}{12Q^2}$ and assuming $P'(R) > 0$, we see that

$$P(s_1) = T(s_1) < 0 \text{ if and only if } m_1 < s_1$$

$$P(s_2) = T(s_2) > 0 \text{ if and only if } m_2 > s_2.$$  \(\square\)

Recalling $M_1$ and $M_2$ from (4) and noting that $P'(m_1) = -4\Lambda R^3(1 - \sqrt{\Delta})$, the only thing left to show is that $0 < M_1 < M_2 < \frac{2}{3}R$ whenever $0 < \Lambda < \frac{1}{12Q^2}$ and $Q \neq 0$. Consider the polynomial $A(x) = x - 2\Lambda x^3$. We have $\lim_{x \to \pm\infty} A(x) = \pm\infty$ when $\Lambda > 0$, and the roots of $A$ are zero and $\pm a$ where $a = \frac{1}{\sqrt{2\Lambda}}$. Also, $A$ is positive on $[0,a]$ with $R$ its local positive maximum on $x \geq 0$ and $A(R) = \frac{2}{3}R$. Moreover, $0 < m_1 < R < a$ and $0 < m_2 < R\sqrt{2} < a$, and since $A(m_1) = M_1$, if follows that $0 < M_1, M_2 < \frac{2}{3}R$. Finally, to see that $M_1 < M_2$ we note that

$$M_2 - M_1 = (m_2 - m_1)(1 - 2\Lambda(m_2^2 + m_1m_2 + m_1^2)) = (m_2 - m_1) \left(1 - \frac{2 - \sqrt{\Delta}}{3}\right) > 0.$$  \(\square\)

2.2. Photon sphere

Henceforth and unless otherwise specified, we will always assume that the conditions in (5) hold. Let us continue the proof of proposition 1 by showing that there is only one photon sphere, which is situated in the exterior static region.
In the coordinates \((t, r, \theta, \varphi) = (x^0, x^1, x^2, x^3)\), the non zero Christoffel symbols are:
\[
\begin{align*}
\Gamma^0_{01} &= -\Gamma^1_{11} = \frac{f'}{2f} ; \quad \Gamma^1_{00} = \frac{ff'}{2} ; \quad \Gamma^1_{22} = -rf ; \quad \Gamma^1_{33} = -rf \sin(\theta)^2 \\
\Gamma^2_{12} &= \Gamma^3_{13} = \frac{1}{r} ; \quad \Gamma^3_{23} = -\cos(\theta) \sin(\theta) ; \quad \Gamma^2_{33} = \cot(\theta).
\end{align*}
\] (12)

If we take a non zero purely rotational vector field along the angle \(\varphi\) it will be of the form
\[\mathcal{V} = \alpha \partial_t + \beta \partial_\varphi\text{ and it be a null vector if } f(r) = \alpha^{-2} \beta^2 r^2 \sin(\theta)^2.\]

Therefore, a photon sphere could only exist in the regions where \(f \geq 0\). Also, from the null condition, it is enough to examine the case where \(\alpha = 1\). In this case,
\[
\nabla_\mathcal{V} \mathcal{V} = f \left( \frac{f'}{2} - \frac{f}{r} \right) \partial_r - \frac{\cot(\theta)f}{r^2} \partial_\theta,
\]
and since we have \(f > 0\), we see that
\[
\nabla_\mathcal{V} \mathcal{V} = 0 \iff f \left( \frac{f'}{2} - \frac{f}{r} \right) = 0 \text{ and } \cot(\theta)f = 0
\]
\[
\iff rf' - 2f = 0 \text{ and } \theta = \frac{\pi}{2}.
\] (13)

One then can see that if we assume from the beginning that \(\theta = \frac{\pi}{2}\), the integral curves of \(\mathcal{V}\) at the zeros of \(rf'(r) - 2f(r)\) are geodesics. Hence, by the spherical symmetry, we get a full ‘sphere’ of null geodesics outside the black hole, this is referred to as the photon sphere around the black hole. Since \(rf'(r) - 2f(r) = \frac{6M}{r} - \frac{4Q^2}{r^2} - 2\), then by studying the polynomial \(S(r) = -r^2 + 3Mr - 2Q^2\) we can determine the zeros. The discriminant of \(S\) is
\[
\Delta_S = 9M^2 - 8Q^2 = (3M - 2\sqrt{2}Q)(3M + 2\sqrt{2}Q)\text{ which is positive if } M > \frac{2\sqrt{2}Q}{3}.
\]

The two roots, if they exist, have the expressions
\[
P_1 = \frac{3M - \sqrt{\Delta_S}}{2}\text{ and } P_2 = \frac{3M + \sqrt{\Delta_S}}{2}.
\] (14)

We can directly show that the last inequality holds when (5) is satisfied, however, by studying the sign of \(rf' - 2f\) near the zeros of \(f\), not only can one show that it has two zeros but also one can know their positions relative to the horizons, which is the important thing. This is proposition 1 and the argument is in its proof which we will present now.

**Continuation of the proof of proposition 1.** We showed that \(f\) has three positive zeros \(r_1, r_2, \text{ and } r_3\) if (5) holds. Note that \(f\) and \(P\), given in (6), are both smooth and have the same sign over \([0, +\infty[\), and we know the sign of \(P\) everywhere. In a small interval around \(r_i\), \(f\) is decreasing since it is positive to the left of \(r_i\) and negative to its right, thus \(f' < 0\) over this interval. Shrinking the interval if necessary, it follows that in the acceleration of the vector field \(\mathcal{V}\) (see (13)), the factor \(f(2^{-1}f' - r^{-1}f)\) is negative to the left of \(r_1\) and positive to its right. Using exactly the same logic, the last statement holds true for \(r_2\) and \(r_3\) also (figure 1). Since the acceleration vector field is continuous, it must vanish in order to change sign. And since its zeros are \(\{r_1, r_2, r_3, P_1, P_2\}\) (see (14)), then by the above argument the zeros are necessarily ordered as follows: \(r_1 < P_1 < r_2 < P_2 < r_3\), which is what we wanted to prove.

Note that \(\{r = P_1\}\) is not a photon sphere since \(f\) is negative on \([r_1, r_2]\) and so the rotational vector \(\mathcal{V}\) is necessarily spacelike. This means that there are no orbits inside the black hole horizon, which is consistent with the fact that this region is dynamic. We also note that in spite
of the covering of the photon sphere by null geodesics it is not a null hypersurface, as a matter of fact, the spacelike vector $\partial_r$ is normal to the photon sphere hypersurface, and therefore it is a timelike hypersurface. (Figure 2)

3. Maximal analytic extension

The two dimensional diagrams presented in this section are two dimensional cross-sections of the spacetime at fixed generic angular direction $\omega_0 = (\theta_0, \varphi_0)$, or equivalently, they are quotients of the spacetime by the action of the rotation group.
We start by reviewing some properties of the RNdS coordinates \((t, r, \omega)\). Consider the open sub-
sets \(U_i = \mathbb{R}_t \times I_i \times S^2_{\theta, \phi}\) of \(\mathcal{M}\), where \(I_i\) are the intervals \(I_1 = ]0, r_1[, \ I_2 = ]r_1, r_2[, \ I_3 = ]r_2, r_3[,\) and \(I_4 = ]r_3, +\infty[\). We also refer to these four regions by I, II, III, and IV.

We orient \(\mathcal{M}\) by requiring \((\partial_t, \partial_r, \partial_{\theta}, \partial_{\phi})\) to be a positively oriented frame. On the other
hand, because \(\mathcal{M}\) is not connected as a Lorentzian manifold when we remove the hypersurfaces
at \(r = r_i\), there is no canonical way of defining a continuous time-orientation on it \textit{a priori}. In
effect, each connected component has exactly two time orientations, \(\pm \partial_t\) for I and III, and \(\pm \partial_r\)
for II and IV. This amounts to a total of sixteen different configurations for time orienting \(\mathcal{M}\).

We shall see that each configuration is isometrically embedded in a time-orientation preserv-
ning way, in a connected part of the maximal extension. When we want to distinguish between
different time orientations, we shall designate \((U_1, +\partial_t)\) by I, and \((U_1, -\partial_t)\) by \(I'\), and the same
for \(U_3\). The time orientation on the other regions is indicated similarly, with II and IV for \(+\partial_r\),
and \(\text{II}'\) and \(\text{IV}'\) for \(-\partial_r\). We note that \(\mathcal{M}\) admits no global timelike Killing vector field. Only
regions \(U_1\) and \(U_3\) admit a timelike Killing vector field, and are in fact static.

### 3.2. Regge–Wheeler charts

Since \((\mathcal{M}, g)\) is a spherically symmetric spacetime, radial null geodesics are particularly
important. Consider a radial null geodesic \(\gamma\) of \(\mathcal{M}\). \(\gamma\) must be an integral curve of a vector
field of the form \(c(f^{-1}\partial_t \pm \partial_r)\) for some non zero constant \(c\). Hence, it is sufficient to study
the integral curves of the vector fields \(Y^\pm = f^{-1}\partial_t \pm \partial_r\) that generates the others (figure 3).

If \(\gamma\) is an integral curve of \(Y^- = f^{-1}\partial_t - \partial_r\), then \(r\) is an affine parameter of \(\gamma^-\), and
\[
\frac{d(t \circ \gamma^-)}{dr}(r) = \frac{1}{f(r)}.
\]

Thus, \(t(\gamma^- (r))\) is, up to an integration constant, nothing but the Regge–Wheeler coordinate func-
tion \(r_*(r)\) which we will presently define, and \(\gamma^-(r) = (r_*(r) + C, r, \omega_0)\) for some constant \(C\).

Similarly, an integral curve of \(Y^+ = f^{-1}\partial_t + \partial_r\) is of the form \(\gamma^+(s) = (C - r_*(-s), -s, \omega_0)\)
defined for $s < 0$. If we choose $\partial_t$ to be future-oriented on $U_3$, the null vector fields $Y^\pm = f^{-1} \partial_\pm \partial_r$, with the light cones shown where they meet, and the arrows on the geodesics show the increasing direction of their affine parameters.

The Regge–Wheeler radial coordinate function (also known as the tortoise coordinate), is defined by

$$\frac{dr}{dr_*} = f(r) \text{ and } r_* = 0 \text{ when } r = P_2,$$

where $P_2$ is the localization of the photon sphere outside the black hole given by (14). To get the explicit expression of $r_*$ in terms of $r$, let the four zeros of $f$ be $r_i$ with $r_0 < 0 < r_1 < r_2 < r_3$, and let us write $f$ as

$$f(r) = -\frac{\Lambda}{r^2} \prod_{i=0}^{3} (r - r_i).$$

We integrate,

$$r_*(r) = \int_{P_2}^{r} \frac{1}{f(s)} ds = \sum_{i=0}^{3} a_i \ln |r - r_i| + a,$$

where

$$a_i = -\frac{r^2}{\Lambda} \prod_{j \neq i} \frac{1}{(r_i - r_j)} ; a = -\sum_{i=0}^{3} a_i \ln |P_2 - r_i|.$$

We note that $a_0, a_2 > 0$ and $a_1, a_3 < 0$, $f'(r_i) = \frac{1}{a_i}$, and $dr = f dr_*$ on $I_i$ defined above. Since $f$ has a constant sign on each interval $I_i$, each $r_*(r) := r_*(r_i) = r_*(I_i)$ is a monotonically increasing function on $I_i$, and in fact, analytic. Thus, on each $U_i$, we define the Regge–Wheeler coordinates $(t, r_*, \omega) \in U_i = \mathbb{R} \times I^*_i \times S^2$, where $I^*_i = r_*(I_i)$. The metric in these coordinates is

$$g = f(r)(dt^2 - dr_*^2) - r^2 d\omega^2,$$

where $r = r(r_*)$ is the inverse function of $r_*(r)$.

We shall usually drop the $i$ in $r_*$ (and in other coordinates later) for clarity. The ordered basis $(\partial_t, \partial_\omega, \partial_\phi, \partial_r)$ is positively oriented on $U^*_1$ and $U^*_3$, and negatively oriented on the other

\footnote{This choice of the origin for $r_*$ is convenient when we deal with decay in [20].}
two domains. To determine the intervals \( I^*_r \) we calculate the limits of \( r_*(r) \) at the singularity \( r = 0 \), at the horizons \( r = r_\pm \), and at infinity \( r = +\infty \).

First,
\[
\lim_{r \to 0} r_*(r) = r_*(0) = b \in \mathbb{R},
\]
and from the signs of the coefficients \( a, s \), we have the two sided limits:
\[
\begin{align*}
\lim_{r \to -r_+} r_*(r) &= +\infty, \\
\lim_{r \to -r_-} r_*(r) &= -\infty, \\
\lim_{r \to +r_-} r_*(r) &= +\infty,
\end{align*}
\]
and,
\[
\lim_{r \to +\infty} r_*(r) = a.
\]
Therefore, \( I^*_1 = [-b, +\infty[ \), \( I^*_2 = \mathbb{R} \), and \( I^*_3 = ]-\infty, a[ \).

The time orientation of \( \partial_r = f \partial_t \) is the same as \( \partial_r \).

In \( (t, r, \omega) \)-coordinates, \( Y^\pm = f^{-1}(\partial_t \pm \partial_r) \) and we readily see (figure 4). We note that the boundary hypersurface \( r_\pm = a \) is conformally spacelike, and the spacelike nature of \( \mathcal{J} \) is revealed more in the Regge–Wheeler coordinates.

### 3.3. Eddington–Finkelstein extensions

Let us temporarily fix a time orientation on \( \mathcal{M} \). Let \( \partial_t \) be future oriented on \( U_1 \) and \( U_3 \), so, \( Y^\pm = f^{-1}(\partial_t \pm \partial_r) \) is future-oriented there, and we choose the orientation given by \( \partial_t \) on \( U_2 \) and \( U_4 \), then \( Y^- \) is future-oriented while \( Y^+ \) is past-oriented in regions \( U_2 \) and \( U_4 \). Since we defined incoming and outgoing geodesics to be future-directed, \( \gamma^- \) will be an outgoing radial null geodesic on \( \mathcal{M} \), while there are no incoming radial null geodesics in the dynamic regions for this particular time orientation. In the \( (t, r, \omega) \)-coordinates, the coordinate expression of \( \gamma^- \) has discontinuities at \( r = r_\pm \) since its \( t \)-coordinate blows up because \( r_* \) does, but this could be a mere bad choice of coordinate. To check this, we use the coordinates given by the flows of \( Y^\pm \), i.e. using the geodesics \( \gamma \) themselves as coordinate lines: For each point \( p = (t_p, r_p, \omega_0) \) of the spacetime in the plane \( \{\omega = \omega_0\} \), there is a unique \( C_p \in \mathbb{R} \) such that \( \gamma^-(r) = (r_*(r) + C_p, r, \omega_0) \) passes through \( p \), and \( p \) can be given the new coordinates \( (C_p, r_p, \omega_0) \), with \( t_p = r_*(r_p) + C_p \). We thus define a new coordinate \( u_- : = t - r_* \), this is Eddington–Finkelstein retarded null coordinate 2.

The Eddington–Finkelstein retarded coordinate chart on \( U_3 \) is a \( (u_, r, \omega) \in \mathbb{R} \times I_{r_+} \times S^2_\omega \), with \( u_+ = t - r_* \). In this chart the metric is:
\[
g = f(r)du_-^2 + 2du_+dr - r^2d\omega^2. \tag{18}
\]

This expression of the metric is analytic for all values \( (u_, r, \omega) \in \mathbb{R} \times [0, +\infty[ \times S^2_\omega \), including \( r = r_* \). The Lorentzian manifold \( \mathcal{M}^- = \mathbb{R} \times [0, +\infty[ \times S^2_\omega \) with the metric (18) is called the retarded Eddington–Finkelstein extension of the RNds manifold. Taking the orientation of \( \partial_\omega \), \( \partial_r, \partial_{\omega_0}, \partial_\omega \) is positively oriented on \( \mathcal{M}^- \), and when \( \partial_r \) is chosen to be future-oriented 3, we denote \( \mathcal{M}^- \) by \( \mathcal{M}_F^- \) and call it the future retarded extension (figure 5).

1 This is not the coordinate vector field \( \partial_\omega \) of the chart \( (t, r, \omega) \). If we denote the Eddington–Finkelstein retarded coordinates by \( (u_, r, \omega) \) then \( \partial_- = f^{-1}\partial_t + \partial_r = Y^- \).

2 A null, time, or space coordinate is one whose level surfaces are null, spacelike, or timelike hypersurfaces respectively.

3 A null, time, or space coordinate is one whose level surfaces are null, spacelike, or timelike hypersurfaces respectively.
$Y^-$ is a smooth null vector field and it is equal to $\partial_t$ (in the retarded coordinates). Its integral curves $\gamma^-$ are outgoing radial null geodesics, and they are just straight lines of constant $u_-$ and $\omega$: $\gamma^-(r) = (C, r, \omega_0)$. For an observer in III, light coming from the singularity and passing through the first two event horizons of the black hole is travelling forward in time and hence is from the past. Therefore the observer will consider the singularity to be in the past as well as the past inner horizon $\mathcal{H}^-_I = \mathbb{R}_{u_-} \times \{r = r_1\} \times S^2$, and the past outer horizon $\mathcal{H}^-_O = \mathbb{R}_{u_-} \times \{r = r_2\} \times S^2$, which are now regular null hypersurfaces. Similarly, the observer can only send but never receive any signal from the last horizon and $\mathcal{J}$. In this extension, we denote $\mathcal{J}$ by $\mathcal{J}^+$ since it lies in the future of the observer, and so does the future cosmological horizon $\mathcal{H}^+_C = \mathbb{R}_{u_+} \times \{r = r_3\} \times S^2$ which is a regular null hypersurface for the metric (18). The null horizons are generated by null geodesics each lying in a fixed angular plane (figure 6).

Although the outgoing geodesics $\gamma^-$ are inextendible in the extension $\mathcal{M}^+_F$, the incoming radial null geodesics (in the static regions I and III) $\gamma^+(s) = (C - 2r_s(-s), -s + c, \omega_0)$ are not. For this reason, $\mathcal{M}^+_F$ is also called the outgoing Eddington–Finkelstein extension. Nonetheless, if we reverse the time orientation, outgoing and incoming will be reversed, and the integral curves of $-Y^-$ will be the incoming geodesics crossing the horizons. We refer to $\mathcal{M}^-$ with this time orientation as $\mathcal{M}^-_F$, the past retarded extension. Of course, this is not the only extension in which the incoming geodesics are inextendible, had we chosen the opposite time orientation on $U_2$ and $U_4$ so that $\mathcal{M}$ is time-oriented by $\partial_t$ and $-\partial_t$ of the $(t, r, \omega)$-chart, the same procedure with $Y^+$ and $Y^-$ exchanging places would have lead instead to the advanced Eddington–Finkelstein null coordinate $u_+ = t + r_s$ and to a new extension $\mathcal{M}^+_F$ covered by the single chart $(u_+, r, \omega) \in \mathbb{R}_{u_+} \times [0, +\infty) \times S^2 = \mathcal{M}^+_F$ and endowed with the analytic metric $g = f(r)du_+^2 - 2du_+dr - r^2d\omega^2$, where $(\partial_{u_+}, \partial_r, \partial_\omega)$ is positively oriented and $-\partial_t$ is future-oriented. This is the future advanced Eddington–Finkelstein extension $\mathcal{M}^+_F$ (figure 7). Similarly, with $\partial_t$ future-oriented we get the past advanced Eddington–Finkelstein extension $\mathcal{M}^-_F$. The picture given by $\mathcal{M}^-_F$ and the one given by $\mathcal{M}^+_F$ are alike but not quiet the same. In both, the singularity and the horizons at $r = r_1$ and $r = r_2$ are in the future of region III, while the horizon at $r = r_3$ is in the past of the region where also past zero null infinity $\mathcal{J}^-$ is. In $\mathcal{M}^-_F$, we have the future inner horizon $\mathcal{H}^+_I = \mathbb{R}_{u_+} \times \{r = r_1\} \times S^2$, the future outer horizon $\mathcal{H}^+_O = \mathbb{R}_{u_+} \times \{r = r_2\} \times S^2$, and the past cosmological horizon $\mathcal{H}^-_C = \mathbb{R}_{u_-} \times \{r = r_3\} \times S^2$. For the past extensions, $\mathcal{M}^-_F$, we shall denote the horizon by a minus sign when we want to be specific: $-\mathcal{H}^\pm$. With these four extensions in hand we can see what the different regions of $\mathcal{M}$ represent. Although, as seen from the geodesics $\gamma^\pm$, none of the extensions are locally inextendible and

**Figure 4.** Oriented Regge–Wheeler charts defined on $U_i$. The null geodesics are integral curves of $Y^\pm = f^{-1}(\partial_t \pm \partial_\omega)$ (lines at $\pm 45^\circ$). The hypersurfaces $r = r_i$ (indicated in parenthesis) are off the chart since they are limits of $r_s$. 

---

Class. Quantum Grav. 34 (2017) 175014 M Mokdad
of course none are geodesically complete. Yet, when combined they give us an almost full picture: For almost any radial null geodesic in \( M \) there is an Eddington–Finkelstein extension for which the given geodesic is future-directed and maximal i.e. extending from the singularity to \( I^+ \). We say almost because the null generators of the horizons are not maximal in any Eddington–Finkelstein extension, even when combined.

From the convention of labelling the regions with primed and unprimed Roman numbers to indicate time orientation, we can identify the parts of different Eddington–Finkelstein extensions which are isometric in a time-orientation preserving manner: Different parts that carry the same label describe exactly the same geometry, the same orientation, and the same time orientation, so, the difference is merely a change of coordinates. Each labelled region will be covered by exactly two of these extensions. However, this is not the case for the horizons. For example, \( M_{+F} \) both agree on III, but in \( M_{-F} \) the null geodesic \( \gamma^- \) intersects the past outer horizon \( \mathcal{H}_{-2}^- \) and the future cosmological horizon \( \mathcal{H}_{+3}^+ \), whereas in \( M_{+F} \), \( \gamma^- \) never touches the hypersurfaces \( r = r_2 \) and \( r = r_3 \) which correspond to the future outer horizon \( \mathcal{H}_{+2}^+ \) and the past cosmological horizon \( \mathcal{H}_{-3}^- \). In fact, \( \mathcal{H}_{+2}^+ \) and \( \mathcal{H}_{-3}^- \) are asymptotic to \( M_{-F} \) where \( u^- = \pm \infty \). Thus, the horizons cannot be identified with each other, this is why we distinguish between future and past horizons.

3.4. Kruskal–Szekeres extensions

Even if we extend using the four Eddington–Finkelstein extensions at the same time, we still do not get a locally inextendible manifold. To see why this is the case, let us examine what do we mean by extending all Eddington–Finkelstein extensions at the same time. Each of the previous extensions is done basically by a change of coordinates on a region \( U_i \), then noticing that the metric in the new coordinates is analytic on a domain bigger than the original domain of the new chart, and then \( M \) is isometrically embedded in this bigger domain. We follow a similar strategy here, however we are not going to able cover the new extension, or even \( M \) for this matter, by a single coordinate chart. We need three charts, and this is related to having three horizons.

We start by defining on \( U_i \) the double null coordinates \( u_{-i} = t - r_i \) and \( u_{+i} = t + r_i \). We have \( (u_{-i}, u_{+i}, \omega) \in U_i \) with:

\[ u_{-i} = t - r_i \quad \text{and} \quad u_{+i} = t + r_i \]

In fact, even with the four combined we still do not have a geodesically complete spacetime because of the singularity at \( r = 0 \) beyond which the metric cannot be smoothly or even continuously extended. So we do not expect the maximal extension to be geodesically complete.
\[ \hat{U}_1 = \{ (u^{--}, u^{++}) \in \mathbb{R}^2 : u^{++} - u^{--} > 2b \} \times S^2_0 ; \]
\[ \hat{U}_2 = \mathbb{R}_{u^{--}} \times \mathbb{R}_{u^{++}} \times S^2_0 ; \]
\[ \hat{U}_3 = \mathbb{R}_{u^{--}} \times \mathbb{R}_{u^{--}} \times S^2_0 ; \]
\[ \hat{U}_4 = \{ (u^{--}, u^{++}) \in \mathbb{R}^2 : u^{++} - u^{--} < 2a \} \times S^2_0 \).

The frame \((\partial_{u^{--}}, \partial_{u^{++}}, \partial_{\theta}, \partial_{\phi})\) is positively oriented on \(\hat{U}_1\) and \(\hat{U}_3\), and negatively oriented on \(\hat{U}_2\) and \(\hat{U}_4\). The metric in these coordinates is \(g = f(r)du^{--}du^{++} - r^2d\omega^2\), where \(r\) is implicitly given by \(u^{++} - u^{--} = 2r_{\star i}(r)\).

To put these charts in the context of the Eddington–Finkelstein extensions, we need to choose orientations on the \(\hat{U}_i\)’s. We can then use the radial null geodesics and the extensions \(\mathcal{M}_F^\pm\) and \(\mathcal{M}_P^\pm\) to determine the asymptotics of the oriented double null coordinates charts. This is figure 8.

The next step is to ‘glue’ the charts along their common (asymptotic) horizons, and since we
want to understand these charts as oriented coordinate systems on Eddington–Finkelstein extensions, there is only one way of putting them together, which is shown in figure 9.

It is clear that we have left out where the four null hypersurfaces $\pm \mathcal{H}_i^\pm$ (for the same $i$) meet, this is a sphere $S_i$ called the bifurcation sphere since the hypersurface $r = r_i$ bifurcates into four horizons. To see that the missing spheres are regular and the metric can be analytically extended on them, we need to define new coordinates on $U_i$ for which we can identify (glue) the corresponding horizons as regular hypersurfaces and not just asymptotically. On the one hand, to bring the horizons back to finite coordinate values, one choice is exponential functions of the null coordinates $u_-$ and $u_+$ with the correct weights. This is the Kruskal–Szekeres choice of coordinates. On the other hand, the metric $g_{00}$, defined on the two dimensional space $\{\omega = \omega_0\}$ and locally given by $f(r)du_- du_+$, is locally conformally flat as we can see from the double null coordinates expression. The only coordinate transformation that leaves it in such a form, is if one of the new coordinates is function of $u_-$ only, and the other is function of $u_+$ only. The simplest of such transformations would be

$$U_{+i} = \beta_+ e^{\alpha_u u_+}, \quad U_{-i} = \beta_- e^{\alpha_u u_-},$$

with non zero constant weights $\alpha_\pm$ and $\beta_{\pm}$ (indexed by $i$ but we drop it for now). The metric in such coordinates would be

$$g = \frac{f(r)}{\alpha_+ \alpha_- U_{+i} U_{-i}} dU_{+i} dU_{-i} - r^2 d\omega^2,$$

but we need to express $r$ in terms of $U_{+i}$ and $U_{-i}$. In fact, $U_{+i} U_{-i} = \beta_+ \beta_- e^{(\alpha_- - \alpha_+) r_i}$, and in order to define $r$ as a function of $(U_{+i}, U_{-i})$ using this relation, we must eliminate the $r$ variable from the relation. So, we take $\alpha_+ = -\alpha_- =: \alpha_i$. This simplifies the above expression,

$$U_{+i} U_{-i} = \beta_+ \beta_- e^{2\alpha_i r_i} = \beta_+ \beta_- A_i \prod_{j=0}^{3} |r - r_j|^{2\alpha_i} =: h_i(r), \quad (19)$$

where $A_i = e^{2\alpha_i}$. The function $h_i$ is bijective on $I_i$ since $r_j$ is bijective. Thus, $r(U_{+i}, U_{-i}) = h_i^{-1}(U_{+i}, U_{-i})$ is a one-to-one function from $h_i(I_i)$ onto $I_i$. Actually, it is analytic since

$$\frac{dh_i}{dr}(r) = \beta_+ \beta_- \frac{2\alpha_i}{f(r)} e^{2\alpha_i r_i} (r) \neq 0.$$

It follows that

$$g = \frac{f(r)}{-\alpha^2 U_{+i} U_{-i}} dU_{+i} dU_{-i} - r^2 d\omega^2, \quad (20)$$

is also analytic on the domain $U'_i = \{(U_{+i}, U_{-i}) \in \mathbb{R}^2; \beta_+ U_{+i} > 0; \beta_- U_{-i} > 0; U_{+i}, U_{-i} \in h_i(I_i)\} \times S^2_\alpha$.

If we want $(\partial U_{+i}, \partial U_{-i}, \partial \theta, \partial \varphi)$ to be positively oriented everywhere, then we are bound to $\beta_+ \beta_- < 0$ on $U_2$ and $U_4$, and $\beta_+ \beta_- > 0$ on $U_1$ and $U_3$. This means that $h_1, h_3$ are negative and $h_2, h_4$ are positive. There is no serious restriction in assuming that $|\beta_{\pm}| = 1$, since it is their sign which is interesting to us. Accordingly, we have

$$h_i(r) = (-1)^i e^{2\alpha_i r_i} = (-1)^i A_i \prod_{j=0}^{3} |r - r_j|^{2\alpha_i}.$$
As before, we have defined a new coordinate system, we now try to extend its domain of definition, keeping in mind that we wish to assign finite double null coordinates for the horizons. We see from the expression of $h_i$ that for a good choice of $\alpha_i$, we can extend $h_i$ (and hence the domain of the chart) analytically to an interval containing a horizon at $r = r_j$ where $r_j$ is a boundary point of $I_i$ different than zero. Thus, the choice of $\alpha_i$ is self suggesting as $\frac{1}{aj}$ for some $j$. However, for the matter to be analytic we need to take $\alpha_i = \frac{1}{2ai}$.

Therefore, for $i = 1, 2, 3$, let $\alpha_i = \alpha_{i+1} = \frac{1}{2ai}$, then $A_i = A_{i+1}$ and the function

$$H_i(r) = (-1)^iA_i(r_i - r) \prod_{j \neq i, j = 0}^{3} |r - r_j|^{\frac{a_j}{2ai}} = \begin{cases} h_i(r) & r \in I_i \\ 0 & r = r_i \\ h_{i+1}(r) & r \in I_{i+1} \end{cases},$$

is continuous on $I_i \cup \{r_i\} \cup I_{i+1}$. Moreover, since $f$ has the opposite sign of $a_i$ over $I_i$ and the same sign over $I_{i+1}$, $H_i$ is monotonic on its domain:

$$\frac{dH_i}{dr}\bigg|_I = \frac{(-1)^iA_i}{a_j} e^{\frac{1}{a_j}r_j}r_i(r),$$

$$\frac{dH_i}{dr}(r_i) = (-1)^{i+1}A_j \prod_{j \neq i, j = 0}^{3} |r_i - r_j|^{\frac{a_j}{2ai}}$$

$$\frac{dH_i}{dr}\bigg|_{I_{i+1}} = \frac{(-1)^{i+1}}{a_j} e^{\frac{1}{a_j}r_{i+1}(r)},$$

so, $H_1$ and $H_3$ are increasing, while $H_2$ is decreasing. Thus, $H_i$ is an analytic bijection from $I_i \cup \{r_i\} \cup I_{i+1}$ onto its image, and its inverse is also analytic. To find the domain of the inverse functions we take the limits. From the limits of $r_i$ in (17) we have:

$$-\infty < \lim_{r \to 0} H_i(r) = H_i(0) = -e^{\frac{1}{a_i}} := B < 0,$$
and \( \lim_{r \to r_1} H_1(r) = +\infty \). Thus, \( H_1 : [0, r_2] \to [B, +\infty] \). Similarly, \( \lim_{r \to r_2} H_2(r) = +\infty \) and \( \lim_{r \to r_3} H_2(r) = -\infty \), so, \( H_2 : [r_1, r_3] \to [-\infty, +\infty] \). Also, \( \lim_{r \to +\infty} H_3(r) = e^A = A > 0 \), and \( H_3 : [r_2, +\infty] \to [-\infty, A] \). Using the \( H_s \) and the formal expression (20), we can define three Lorentzian manifolds \( (K_i, g_{K_i}) \) for \( i = 1, 2, 3 \), called the Kruskal–Szekeres extensions, as follows:

\[
K_i = \{(U^*_+, U^*_-) \in \mathbb{R}^2; U^*_+ U^*_- \in H_i(I_i \cup \{r_i\} \cup I_{i+1}) \} \times S^2,
\]
equipped with the analytic metric

\[
g_{K_i} = \frac{-4a_i^2 f(r)}{H_i(r)} dU^*_+ dU^*_- - r^2 d\omega^2,
\]
where \( r(U^*_+, U^*_-) = H_i^{-1}(U^*_+, U^*_-) \).

Now, to see these manifolds as local extensions of the Eddington–Finkelstein manifolds and of \( \mathcal{M} \), let us embed the \( U_i \)s in them via the transformation \( U^*_+ = \beta_+ e^{\frac{1}{2}u_+} \) and \( U^*_- = \beta_- e^{-\frac{1}{2}u_-} \), for \( j = i, i+1 \), where \( u_{\pm i} = t \pm r_{+i} \). If we want the transformation to be orientation preserving with \( U_i \) oriented by the positively oriented frame \( (\partial_t, \partial_r, \partial_\theta, \partial_\phi) \), then as we mentioned above, we must have \( \beta_+ \beta_- \) positive for \( i = 2, 4 \) and

Figure 9. The gluing of double null coordinates along their asymptotic horizons, covering the different Eddington–Finkelstein extensions, with the pattern repeating infinitely. See figure 8 for the legend.
negative for \( i = 1, 3 \). Then, form the definition of \( r(U^+_{+i}, U^+_i) \) and \( H_i(r) \), we see that two 'diagonally opposite quadrants' of \( K_i \) are each isometric to \( U_i \), and the other two 'quadrants' to \( U_{i+1} \), and the horizons at \( r = r_i \) corresponds to the 'axis' of \( K_i \). Of course each of these parts of \( K_i \) is a product with \( S^2 \). We note also that since \( H_i(r) \) and \( f(r) \) have opposite signs, \( \partial U_{+i}^+ + \partial U_{+i}^- \) is timelike on \( K \). The choice of this vector being future or past oriented is equivalent to fixing the sign of each \( \beta_{bj} \). These choices can be decided alternatively and equivalently by following the geodesics of \( Y^\pm \) guided by figure 9, where \( Y^\pm \) are now given in the Kruskal–Szekeres coordinates by \( Y^\pm = \frac{1}{a_f(r)} U^*_i \partial_{U^*_i} \). We note that since

\[
Y^- = \frac{1}{a_f(r)} U^*_i \partial_{U^*_i} = \frac{H_i(r)}{a_f(r)} U^*_i \partial_{U^*_i},
\]

\( Y^- \) is actually defined and smooth on \( K_i \setminus \{ U^*_i = 0 \} \). Similarly for \( Y^+ \). The geodesics along the horizons are given by \( Y^\pm_{\phi_i} = \pm \frac{1}{\pi} \partial_{\phi_i} \) on \( U^*_{\phi_i} = 0 \). Figure 10 summarizes all of this when \( \partial U^+_{+i} + \partial U^-_{+i} \) is future-oriented. We remark that using this coordinates change we can recover \( t \) as a function of \( (U^+_{+i}, U^+_{i}) \) through

\[
\frac{U^+_{+i}}{U^+_{i}} = \beta_{+i} \beta_{-j} e^{\pi}.
\]

(22)

3.5. The maximal extension

The maximal analytic extension of \( M \) is a Lorentzian manifold \( M^* \) covered by an atlas \( A^* \) consisting of coordinate charts given by the \( K_i \)'s, and is endowed with the metric \( g^* \), given locally as \( g_{K_i} \) (or simply \( g \)). Equipped with the usual topology, let

\[
M^* = \left( \mathbb{R}^2 \setminus \bigcup_{k,l \in \mathbb{Z}} S_{k,l} \right) \times S^2,
\]

where \( S_{k,l} \) is the square block \( S_{k,l} = \{ (x,y) \in \mathbb{R}^2; \frac{x}{2} \leq x \sqrt{2} - 2k\pi \leq \frac{3}{2} \pi; -\frac{x}{2} \leq y \sqrt{2} + 2l\pi \leq \frac{1}{2} \pi \} \), and let the atlas be \( A^* = \{ (A_{k,l}, \phi_{k,l}), (B_{k,l}, \chi_{k,l}), (C_{k,l}, \psi_{k,l}) \}; k,l \in \mathbb{Z} \} \), with the charts defined as follows: Let \( n = l - k \) and \( m = l + k \), and set \( X = \frac{1}{\sqrt{2}} (y + x) \) and \( Y = \frac{1}{\sqrt{2}} (y - x) \). The open \( A_{k,l}, B_{k,l}, \) and \( C_{k,l} \) are
Here, we ignore the fact that the 2-sphere needs multiple charts to cover it.
behaved at all points of wormhole to region II passing through the \( \prime \). This indicates that the singularity is timelike. However, null and timelike curves can avoid hitting the singularity and go from region II to region II passing through the ‘wormhole’. This indicates that the singularity is timelike.

The geodesic incompleteness comes from the singularity at \( r = 0 \): Radial null geodesics hit the singularity in a finite amount of their affine parameter, so, \( \mathcal{M}^* \) is geodesically incomplete. However, null and timelike curves can avoid hitting the singularity and go from region II to region II passing through the ‘wormhole’. This indicates that the singularity is timelike.

The metric on \( \mathcal{M}^* \) is \( g^* \) whose coordinate expression on each chart domain is given by (21). This extension is the maximal analytic extension of RNdS manifold. It is maximal in the sense that it is locally inextensible. It is also unique if the topology is not changed.

The structure of \( \mathcal{M}^* \) is shown in figure 11. First, we note that the metric is analytic and well behaved at all points of \( \mathcal{M}^* \), including the horizons (which are now given by \( U_\pm, U_{-1} = 0 \) and the bifurcation spheres \( (U_+, U_-) = (0, 0) \). The RNdS radius \( r \) is a scalar field on \( \mathcal{M}^* \), but the same cannot be said regarding the time parameter \( t \), which is given in the different regions through (22) as shown in the diagram, and is not defined on the horizons where it becomes infinite. \( \mathcal{M}^* \) contains infinitely many isometric copies of the original spacetime \( \mathcal{M} \). Each consists of four regions numbered from one to four in Roman, possibly primed or mixed. There are 16 (infinite) families of these copies, each family corresponds to one of the 16 different ways of time-orienting \( \mathcal{M} \). Four of the families correspond to the Eddington–Finkelstein extensions of \( \mathcal{M} \). Examples of the others along with these four are shown in figure 12. The causal structure of \( \mathcal{M}^* \) can also be seen from figure 9. Upon choosing a time orientation on \( \mathcal{M}^* \), say \( \partial_{U_-} + \partial_{U_+} \), then all future directed timelike causal curves in region IV end at \( \mathcal{I}^+ \), and all past directed causal curves end at \( \mathcal{I}^- \). Unlike Minkowski, Schwarzschild, Reissner–Nordström, or Kerr spacetimes, in RNdS, null infinity or \( \mathcal{I} \) is not a null ‘hypersurface’, instead it is spacelike due to the de Sitter nature of our spacetime. Using the conformal factor \( \sqrt{|\mathcal{I}|} \) one can define the metric on this hypersurface, and see that it is indeed spacelike for the conformal metric, but the conformal metric will not be analytic or even smooth on \( \mathcal{M}^* \). In coordinates, \( \mathcal{I} \) is given by \( U_+U_- = A \) which also corresponds to \( r = \infty \), and its spacelike nature produces a behaviour near infinity similar to that of a spacelike singularity. Near \( \mathcal{I}^- \), future-directed causal curves are bound to ‘go to infinity’ once they enter region IV. Of course, unlike the spacelike singularity in Schwarzschild, no observer or light ray can reach infinity in a finite amount of an affine parameter of these null and timelike geodesics, so no geodesic incompleteness is caused by the dynamics of region IV.

The geodesic incompleteness comes from the singularity at \( r = 0 \): Radial null geodesics hit the singularity in a finite amount of their affine parameter, so, \( \mathcal{M}^* \) is geodesically incomplete. However, null and timelike curves can avoid hitting the singularity and go from region II to region II passing through the ‘wormhole’. This indicates that the singularity is timelike.

![Figure 12. Examples of different time-orientations on \( \mathcal{M} \) as connected subsets of \( \mathcal{M}^* \).](image-url)
Despite geodesic incompleteness, the spacetime is timelike geodesically complete as the singularity is repulsive, due to the Reissner–Nordstrøm nature of our spacetime\(^6\). Therefore, no timelike geodesic can hit the singularity. The timelike nature of the singularity also means that there are points in the spacetime whose both future and past null cones meet the singularity inside the same region I. Another consequence of this nature is the absence of a (global) Cauchy hypersurface, as there are inextendible timelike curves of arbitrarily small length which start and end at the singularity. For instance, the spacelike hypersurface \(\mathcal{S}\) in figure 11 is a Cauchy hypersurface for regions covered by the domains \(B_{k,J-1}\) and \(C_{k,J-1}\) for all \(k \in \mathbb{Z}\). Yet, there are future-directed and past-directed inextendible timelike curves of \(\mathcal{M}\) which do not intersect \(\mathcal{S}\). Such curves hit the singularity inside region I and never cross the horizons at \(r = r_1\) towards \(\mathcal{S}\). Therefore, data in region I do not depend on data at \(\mathcal{S}\). The hypersurfaces \(-\mathcal{H}_1^- \cup \mathcal{H}_1^-\) and \(-\mathcal{H}_1^+ \cup \mathcal{H}_1^+\) bounding regions II and II’ in \(B_{k,J-1}\) (for all \(k\)) are said to be Cauchy horizons for the spacelike section \(\mathcal{S}\) (see [12]).

**Remark.** We end with a remark about the different number of horizons. The construction carried out in the paper can be easily modified to account for the cases with less number of horizons. The case we treated here is in some sense the most complete. In the case of three horizons, the maximal extension contains all the blocks that can appear in the maximal extensions of the cases with fewer horizons. That is, in the other cases, only some of the blocks I, II, III, IV are present. Also, the conditions on the mass, the charge, and the cosmological constant, for \(f\) to have less number of zeros, can be found using arguments similar to those in section 2.1.

**Acknowledgments**

The results of this paper and the mentioned decay and conformal scattering results [19, 20], were obtained during my PhD thesis [18], which was prepared at the Department of Mathematics of the University of Brest (LMBA - UBO). This research was funded by the University of Brest and was partly supported by the ANR funding ANR-12-BS01-012-01, ANR AARG project. I would like to thank my thesis advisor Prof Jean-Philippe Nicolas for his indispensable guidance during the thesis.

**ORCID iDs**

Mokdad Mokdad https://orcid.org/0000-0002-4857-5482

**References**

[1] Andersson L and Blue P 2015 Hidden symmetries and decay for the wave equation on the Kerr spacetime *Ann. Math.* 182 787–853
[2] Blue P 2008 Decay of the Maxwell field on the Schwarzschild manifold *J. Hyperbolic Differ. Equ.* 05 807–56
[3] Bolokhov S V, Bronnikov K A and Skvortsova M V 2012 Magnetic black universes and wormholes with a phantom scalar *Class. Quantum Grav.* 29 245006
[4] Bronnikov K A 1979 Inverted black holes and anisotropic collapse *Sov. Phys. J.* 22 594–600

\(^6\)This repulsive behaviour of the singularity is similar to that of the Reissner–Nordstrøm spacetime. See [8] for details.
[5] Chakraborty S 2015 Equilibrium configuration of perfect fluid orbiting around black holes in some classes of alternative gravity theories Class. Quantum Grav. 32 075007
[6] Chakraborty S 2015 Aspects of neutrino oscillation in alternative gravity theories J. Cosmol. Astropart. Phys. JCAP10(2015)019
[7] Chakraborty S and SenGupta S 2017 Strong gravitational lensing—a probe for extra dimensions and Kalb–Ramond field J. Cosmol. Astropart. Phys. JCAP07(2017)045
[8] Chandrasekhar S 1984 The Mathematical Theory of Black Holes (Oxford: Clarendon)
[9] Claudel C-M, Virbhadra K S and Ellis G F R 2001 The geometry of photon surfaces J. Math. Phys. 42 818–38
[10] Darwin C 1959 The gravity field of a particle Proc. R. Soc. Lond. A 249 180–94
[11] Darwin C 1961 The gravity field of a particle. II Proc. R. Soc. Lond. A 263 39–50
[12] Hawking S W and Ellis G F R 1973 The Large Scale Structure of Space-Time (Cambridge: Cambridge University Press)
[13] Katanaev M O, Kloesch T and Kummer W 1999 Global properties of warped solutions in general relativity Ann. Phys. 276 191–222
[14] Katanaev M O, Kummer W and Liebl H 1996 Geometric interpretation and classification of global solutions in generalized dilaton gravity Phys. Rev. D 53 5609–18
[15] Khoo F S and Ong Y C 2016 Lux in obscuro: photon orbits of extremal black holes revisited Class. Quantum Grav. 33 235002
[16] Lake K 1979 Reissner–Nordstrøm-de Sitter metric, the third law, and cosmic censorship Phys. Rev. D 19 421
[17] Laue H and Weiss M 1977 Maximally extended Reissner–Nordstrøm manifold with cosmological constant Phys. Rev. D 16 3376–9
[18] Mokdad M 2016 Maxwell field on the Reissner–Nordstrøm–de Sitter manifold: decay and conformal scattering. Phd Thesis Université de Bretagne Occidentale—Brest (Engl. http://theses.fr/2016BRES0060)
[19] Mokdad M 2017 Conformal scattering of Maxwell fields on Reissner–Nordstrøm–de Sitter black hole spacetimes (arXiv:1706.06993)
[20] Mokdad M 2017 Decay of Maxwell fields on Reissner–Nordstrøm–de Sitter black holes (arXiv:1704.06441)
[21] Virbhadra K S 2009 Relativistic images of Schwarzschild black hole lensing Phys. Rev. D 79 083004
[22] Virbhadra K S and Ellis G F R 2002 Gravitational lensing by naked singularities Phys. Rev. D 65 103004
[23] Virbhadra K S and Ellis G F R 2000 Schwarzschild black hole lensing Phys. Rev. D 62 084003
[24] Wald R M 2010 General Relativity (Chicago, IL: University of Chicago Press)