Closure structures parameterized by systems of isotone Galois connections

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Abstract

We study properties of classes of closure operators and closure systems parameterized by systems of isotone Galois connections. The parameterizations express stronger requirements on idempotency and monotony conditions of closure operators. The present approach extends previous approaches to fuzzy closure operators which appeared in analysis of object-attribute data with graded attributes and reasoning with if-then rules in graded setting and is also related to analogous results developed in linear temporal logic. In the paper, we present foundations of the operators and include examples of general problems in data analysis where such operators appear.

1 Introduction

In this paper we deal with closure structures which emerge in data-analytical applications such as formal concept analysis [30] of data with fuzzy attributes [14, 39] and approximate reasoning such as inference of fuzzy if-then rules from data [8, 9]. In particular, our paper generalizes and extends observations on fuzzy closure operators and related structures. Since their inception, fuzzy closure operators have been the subject of extensive research, the most influential

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early papers on the topic include \cite{3, 14, 17, 19, 20, 31, 45, 46}. As it is usual with graded (fuzzy) generalizations of classic notions, there are several sound ways to introduce closure operators in fuzzy setting. While most authors agree on the conditions of extensivity and idempotency, which take the same form as in the classic setting, the approaches differ in the treatment of the monotony condition. There are two major approaches:

1. Using the bivalent notion of inclusion of fuzzy sets, where the monotony condition can be written as “$A \subseteq B$ implies $c(A) \subseteq c(B)$” and means that “the closure of $A$ is fully contained in the closure of $B$ whenever $A$ is fully contained in $B$.” The full containment of fuzzy sets (here denoted “$\subseteq$”) is defined as a bivalent relation on fuzzy sets so that for any fuzzy sets $C$ and $D$ in the universe $X$, we put $C \subseteq D$ whenever for each element $x \in X$, the degree to which $x$ belongs to $D$ is at least as high as the degree to which it belongs to $C$.

2. The second option uses a graded notion of inclusion of fuzzy sets. In this case, the monotony condition can be written as $S(A, B) \leq S(c(A), c(B))$, where $\leq$ is the order on the set of truth degrees (the usual order of reals if the scale of degrees is the real unit interval) and $S$ is a suitable graded subsethood. Both $S(A, B)$ and $S(c(A), c(B))$ are general degrees of inclusion, i.e., $S(A, B)$ is the degree to which $A$ is included in $B$ and analogously for $S(c(A), c(B))$. Hence, the monotony condition can be read “the degree to which the closure of $A$ is included in the closure of $B$ is at least as high as the degree to which $A$ is included in $B$.” In other words, $S(A, B)$ gives a lower bound of the inclusion degree of the closure of $A$ in the closure of $B$. In the context of approximate inference, this is a desirable property because one can obtain a lower approximation of the degree $S(c(A), c(B))$ without the need to actually compute either of $c(A)$ and $c(B)$.

These two basic approaches can be seen as two borderline requirements on the monotony condition and for reasonable choices of $S$, which includes the residuum-based fuzzy set inclusion proposed by Goguen \cite{33}, the second ap-
proach constitutes a stronger requirement than the first one. Interestingly, both
the approaches can be handled by a single theory which leaves the approaches as
special cases. In fact, there are several results on fuzzy closure operators where
both the approaches result as special cases. The initial paper [4] by Belohlavek
uses a general monotony condition which is parameterized by an order-filter on
the set of truth degrees. Conceptually different approach has been introduced
in [6] where the authors employ linguistic hedges, again, as parameters of the
monotony condition. In both the approaches, the two basic notions of monotony
result by chosen parameterizations—either a special filter in case of [4] or a spe-
cial hedge in case of [6].

In our recent paper [53], we have developed a theory of graded if-then rules
with general semantics parameterized by systems of isotone Galois connections.
In this setting, general fuzzy closure operators with parameterized idempotency
conditions appeared. Interestingly, in the approach to attribute implications
with temporal semantics introduced first in [50] and developed further in [49],
we have utilized conceptually similar structures which utilize the notion of be-
ing closed under “time shifts.” In this paper, we present results showing that
most results related to closure structures in [53] and [49] can be handled by a
single theory of closure structures defined on complete lattices which are pa-
parameterized by systems of isotone Galois connections. In addition, we also show
that the approaches in [4] [6] result as special cases of the presented formalism.
Therefore, the present paper studies closure structures from the perspective of
a general class of parameterizations, makes conclusions on a general level, and
particular results like those in [4] [6] [49] [53] can be obtained by selecting concrete
parameterizations on complete lattices.

Our paper is organized as follows. In Section 2 we present a survey of no-
tions related to closure operators and closure systems and introduce notation
which is used further in the paper. In Section 3 we present the notions of
closure operators and closure systems parameterized by systems of isotone Ga-
lois connections and show their relationship to parameterized closure structures
studied in the past. In Section 4 we present details on two important fields
related to data analysis where the closure structures parameterized by systems of isotone Galois connections appear either in the general setting or as special cases. In Section 5, we investigate general properties of the closure structures and their parameterizations. We give conclusion and final remarks in Section 6.

2 Preliminaries

In the paper, we use the usual notions from the theory of ordered sets and lattices \([18, 23]\). Recall that a partial order \(\leq\) on a non-empty set \(L\) is a binary relation which is reflexive, antisymmetric, and transitive; the pair \(\langle L, \leq \rangle\) is called a partially ordered set. Furthermore, \(\langle L, \leq \rangle\) is called a complete lattice and denoted \(\mathbf{L} = \langle L, \leq \rangle\) whenever each \(K \subseteq L\) has its supremum and infimum in \(L\) which are denoted by \(\bigvee K\) and \(\bigwedge K\), respectively. Each complete lattice \(\mathbf{L}\) has its greatest and least elements \(1 = \bigvee L = \bigwedge \emptyset\) and \(0 = \bigwedge L = \bigvee \emptyset\).

A non-empty subset \(K \subseteq L\) is called an \(\leq\)-filter in \(\mathbf{L}\) if for every \(a, b \in L\) such that \(a \leq b\) we have \(b \in K\) whenever \(a \in K\).

A lattice element \(a \in L\) is called compact whenever \(a \leq \bigvee J\) for \(J \subseteq L\) implies there is a finite \(J' \subseteq J\) such that \(a \leq \bigvee J'\). A complete lattice \(\mathbf{L}\) is called algebraic (or compactly generated) whenever each \(a \in L\) can be expressed as \(a = \bigvee K\) where \(K\) is some subset of \(L\) consisting solely of compact elements.

2.1 Isotone Galois connections

Let \(\mathbf{L} = \langle L, \leq \rangle\) be a complete lattice. A pair \(\langle f, g \rangle\) of operators \(f : L \to L\) and \(g : L \to L\) is called an isotone Galois connection in \(\mathbf{L}\) whenever

\[
f(a) \leq b \iff a \leq g(b) \quad (1)
\]

for all \(a, b \in L\); \(f\) is called the lower adjoint of \(g\) and, dually, \(g\) is called the upper adjoint of \(f\). In an isotone Galois connection \(\langle f, g \rangle\), \(f\) uniquely determines \(g\).
and vice versa. In particular,
\[
f(a) = \bigwedge \{ b \in L; a \leq g(b) \}, \tag{2}
\]
\[
g(b) = \bigvee \{ a \in L; f(a) \leq b \}. \tag{3}
\]
In our paper, we utilize the following properties which are consequences of (1).

For any \(a, b \in L\) and \(a_i \in L\) \((i \in I)\), we have:
\[
a \leq g(f(a)), \tag{4}
\]
\[
f(g(b)) \leq b \tag{5}
\]
\[
a \leq b \text{ implies } f(a) \leq f(b), \tag{6}
\]
\[
a \leq b \text{ implies } g(a) \leq g(b), \tag{7}
\]
\[
f(\bigvee \{ a_i; i \in I \}) = \bigvee \{ f(a_i); i \in I \}, \tag{8}
\]
\[
g(\bigwedge \{ a_i; i \in I \}) = \bigwedge \{ g(a_i); i \in I \}. \tag{9}
\]

A \textit{composition} of isotone Galois connections is defined in terms of the ordinary composition of maps. That is, for isotone Galois connections \(\langle f_1, g_1 \rangle\) and \(\langle f_2, g_2 \rangle\) in \(L\), we put
\[
\langle f_1, g_1 \rangle \circ \langle f_2, g_2 \rangle = \langle f_1 f_2, g_2 g_1 \rangle, \tag{10}
\]
where \(f_1 f_2\) is a composed operator such that \(f_1 f_2 (a) = f_1(f_2(a))\) for all \(a \in L\) and analogously for \(g_2 g_1\). It is easy to see that the composition is again an isotone Galois connection in \(L\).

We denote by \(1_L\) the identity operator in \(L\), i.e., \(1_L(a) = a\) for all \(a \in L\). Obviously, \(\langle 1_L, 1_L \rangle\) is an isotone Galois connection in \(L\). If \(L\) is clear from the context, we write just \(1\) to denote \(1_L\). As a consequence of the fact that \(\circ\) is associative and \(\langle 1, 1 \rangle\) is neutral with respect to \(\circ\), it follows that all isotone Galois connections in \(L\) together with binary operation \(\circ\) defined by (10) and \(\langle 1, 1 \rangle\) form a monoid.

### 2.2 Residuated lattices and related structures

Residuated lattices \[54\] are structures based on lattices which are enriched by a couple of binary operations satisfying an additional condition. The struc-
tures are widely used in fuzzy and substructural logics \cite{21, 22, 28} and include structures of degrees based on left-continuous triangular norms \cite{38} which are popular in applications \cite{39}. From the point of view of isotone Galois connections, such structures can be seen as lattices endowed by particular systems of isotone Galois connections.

An ordered structure $L = (L, \leq, \otimes, \rightarrow, 0, 1)$ is called a \textit{(commutative integral) complete residuated lattice} whenever $(L, \leq)$ is a complete lattice with 0 and 1 being the least and greatest elements, respectively, $\otimes$ is a binary operation in $L$ (called a \textit{multiplication}) which is associative and commutative with 1 being its neutral element, and $\rightarrow$ is a binary operation in $L$ (called a \textit{residuum}) such that

\begin{equation}
a \otimes b \leq c \text{ iff } b \leq a \rightarrow c
\end{equation}

holds for all $a, b, c \in L$. In the terminology of residuated lattices, (11) is called the \textit{adjointness property}. In fuzzy logics \cite{23, 35}, $\otimes$ and $\rightarrow$ are used as truth functions of (fuzzy) logical connectives “conjunction” and “implication”, respectively. Alternatively, residuated lattices can be introduced in terms of isotone Galois connections as follows. Let $L = (L, \leq)$ be a complete lattice and let $\otimes$ and $\rightarrow$ be binary operations in $L$ such that $\otimes$ is associative, commutative, and neutral with respect to 1. In this setting, for any $a \in L$, we define maps $f_{a \otimes} : L \rightarrow L$ and $g_{a \rightarrow} : L \rightarrow L$ by

\begin{align}
f_{a \otimes} (b) &= a \otimes b, \\
g_{a \rightarrow} (b) &= a \rightarrow b
\end{align}

for all $b \in L$. Now, $L = (L, \leq, \otimes, \rightarrow, 0, 1)$ is a complete residuated lattice if and only if $(f_{a \otimes}, g_{a \rightarrow})$ is an isotone Galois connection for any $a \in L$. Indeed, $a \otimes b \leq c \text{ iff } f_{a \otimes} (b) \leq c \text{ iff } b \leq g_{a \rightarrow} (c) \text{ iff } b \leq a \rightarrow c$ provided that $(f_{a \otimes}, g_{a \rightarrow})$ is an isotone Galois connection and, conversely, $f_{a \otimes} (b) \leq c \text{ iff } a \otimes b \leq c \text{ iff } b \leq a \rightarrow c \text{ iff } b \leq g_{a \rightarrow} (c)$ provided that (11) holds.

In the paper, we are going to use general complete as well as concrete residuated lattices that are used in problem domains related to data analysis and
approximate inference. Namely, we are going to use residuated lattices of fuzzy sets. Such structures can be understood as direct powers of complete residuated lattices that serve as structures of truth degrees. In a more detail, let $\mathbf{L} = \langle L, \leq, \otimes, \to, 0, 1 \rangle$ be a complete residuated lattice and let $Y \neq \emptyset$ be a universe set. Then, the complete residuated lattice of $\mathbf{L}$-sets in (the universe) $Y$ is a structure $\mathbf{L}^Y = \langle L^Y, \subseteq, \otimes^Y, \to^Y, 0_Y, 1_Y \rangle$, where

- $L^Y$ is the set of all maps of the form $A : Y \to L$, each $A \in L^Y$ is called an $\mathbf{L}$-fuzzy set (shortly, an $\mathbf{L}$-set) in $Y$, see [32, 33];
- $\subseteq$ is a binary relation on $L^Y$ such that $A \subseteq B$ whenever $A(y) \leq B(y)$ for all $y \in Y$;
- $\otimes^Y$ and $\to^Y$ are defined componentwise using $\otimes$ and $\to$, i.e., for any $A, B \in L^Y$ and $y \in Y$, we have
  \begin{align}
  (A \otimes^Y B)(y) &= A(y) \otimes B(y), \\
  (A \to^Y B)(y) &= A(y) \to B(y);
  \end{align}

If there is no danger of confusion and both $\mathbf{L}$ and $Y$ are clear from context, we write just $\otimes$ and $\to$ to denote $\otimes^Y$ and $\to^Y$;

- $0_Y \in L^Y$ and $1_Y \in L^Y$ so that $0_Y(y) = 0$ and $1_Y(y) = 1$ for all $y \in Y$.

It follows from the basic properties of complete residuated lattices that $\mathbf{L}^Y$ is a complete residuated lattice (the class of complete lattices is closed under arbitrary direct products [18, 55]). In examples, we are going to use the usual notation for writing $\mathbf{L}$-sets, e.g., $\{y^a, z^b\}$ represents $A : \{y, z\} \to L$ such that $A(y) = a$ and $A(z) = b$.

The relation $\subseteq$ in $\mathbf{L}^Y$ is called a subsethood (or inclusion relation) of $\mathbf{L}$-sets and $A \subseteq B$ expresses the fact that $A$ is fully contained in $B$. If $A(y)$ and $B(y)$ are interpreted as degrees to which $y \in Y$ belongs to $A$ and $B$ respectively, it follows that $A \subseteq B$ means that, for each $y \in Y$, the degree to which $y$ belongs to $B$ is at least as high as the degree to which $y$ belongs to $A$. In addition to this type of “full subsethood”, it is reasonable to define its graded counterpart which
expresses general degrees [32, 33] to which one L-set is included in another one. For $A, B \in L^Y$, we put

$$S(A, B) = \bigwedge \{ A(y) \to B(y); \ y \in Y \} \quad (16)$$

and call $S(A, B) \in L$ the subsethood degree (of $A$ in $B$). That is, $S$ defined by (16) is a map of the form $S: L^Y \times L^Y \to L$. It can be easily seen that $A \subseteq B$ iff $A(y) \to B(y) = 1$ for all $y \in Y$ which is iff $S(A, B) = 1$.

In addition to our understanding of $\otimes$ and $\to$ as operations on $L$ and $L^Y$ (which are, in fact, $\otimes_{Y}$ and $\to_{Y}$), we also consider $\otimes$ and $\to$ as maps $\otimes: L \times L^Y \to L^Y$ and $\to: L \times L^Y \to L^Y$ given by

$$(a \otimes A)(y) = a \otimes A(y), \quad (a \to A)(y) = a \to A(y), \quad (17) \quad (18)$$

for all $A \in L^Y$ and $a \in L$. For fixed $a \in L$, $a \otimes A$, and $a \to A$ given by (17) and (18) are called the $a$-multiple and $a$-shift of $A$, respectively. Note that $\to$ is not commutative, i.e., one may also consider $A \to a$ but this operation is not relevant to our investigation.

Using (11), (16), (17), and (18), we derive the following property which is extensively used in our paper:

$$a \otimes B \subseteq C \iff B \subseteq a \to C \iff a \leq S(B, C) \quad (19)$$

for any $a \in L$ and any $B, C \in L^Y$. Indeed, $a \otimes B \subseteq C$ iff $a \otimes B(y) \leq C(y)$ for any $y \in Y$, i.e., by (11), we get $B(y) \leq a \to C(y)$ for any $y \in Y$, meaning $B \subseteq a \to C$. Furthermore, using the commutativity of $\otimes$ and (11) twice, it follows that $B(y) \leq a \to C(y)$ for any $y \in Y$ iff $a \leq B(y) \to C(y)$ for any $y \in Y$ which is iff $a \leq S(B, C)$ because $S(B, C)$ is the greatest lower bound of all $B(y) \to C(y)$ where $y \in Y$.

### 2.3 Closure structures

In this section, we recall two influential types of closure operators defined in complete residuated lattices of L-sets. The operators differ in definitions of the
isotony condition which is in both cases stronger than the ordinary isotony.

The approach in [6] introduced $L^*$-closure operators as operators on $L^Y$ whose isotony condition is parameterized by truth-stressing linguistic hedges [41, 57]. In our setting a truth-stressing linguistic hedge (shortly, a hedge) on $L$ is any map $*: L \to L$ such that

\begin{align}
1^* &= 1, \\
a^* &\leq a, \\
(a \rightarrow b)^* &\leq a^* \rightarrow b^*,
\end{align}

for all $a, b \in L$.

Remark 1. The conditions (20)–(22) are a subset of conditions of truth-stressing hedges as they were studied by Hájek in [36] and interpreted as truth functions of logical connectives “very true.” As it is argued in [36], (20)–(22) may be considered natural properties of (truth functions of) logical connectives “very true”: (20) says that “1 (degree denoting the full truth) is very true”; (21) reflects the fact that if a proposition is considered “very true”, then it is also considered “true” (i.e., being “very true” is at least as strong as being “true”) and (22) says that from a very true implication with a very true antecedent, one derives a very true consequent. Notice that (20) and (22) yield

\begin{align}
\text{if } a \leq b \text{ then } a^* &\leq b^*
\end{align}

for all $a, b \in L$. In fact, in this paper, we rely on (23), i.e., all the considerations can be made for hedges satisfying (20), (21), and (23).

Most of the applications in relational data analysis as well as other results where closure structures parameterized by hedges appear rely on idempotent truth-stressing hedges, i.e., $^*$ is in addition required to satisfy

\begin{align}
a^* &= a^{**}
\end{align}

for all $a \in L$. 

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Remark 2. The idempotency, as a property of hedges, is disputable. In case of the truth-stressing hedges, it is widely accepted that “very very true” is strictly stronger an emphasis than “very true” \[41, 57, 58, 59, 60\]. Indeed, \( x^* = x^2 \) is often taken as a truth function for a hedge if \( L \) is defined on the real unit interval using a left-continuous triangular norm acting as \( \otimes \) which is a typical choice in applications. In contrast, \[37\] argues that in case of truth-depressing (or truth-intensifying) hedges \[52\], the idempotency may seem natural. For instance, it is not so frequent that “more or less” is chained in order to further lessen the impact of an utterance. Nevertheless, major results in relational data analysis \[14\] and logics of if-then rules \[15, 7, 11, 12\] where the hedges were employed use \[24\] and the condition cannot be dropped without losing important properties of the studied concepts. We refer readers interested in treatment of hedges in fuzzy logics to recent papers \[26\] and \[22, Section VII–3\].

Fuzzy closure operators in \( L^Y \) parameterized by hedges have been introduced in \[6\] as follows:

**Definition 1.** Let \( L \) be a complete residuated lattice, \( * \) be a hedge satisfying \[20–22\], \( Y \) be a non-empty universe set. An operator \( c : L^Y \rightarrow L^Y \) is called an \( L^* \)-closure operator \[6\] in \( L^Y \) whenever

\[
A \subseteq c(A), \tag{25}
\]

\[
S(A, B)^* \leq S(c(A), c(B)), \tag{26}
\]

\[
c(c(A)) \subseteq c(A), \tag{27}
\]

for all \( A, B \in L^Y \).

Recall that \( S \) in \[20\] is the residuum-based graded subsesthod defined by \[11\]. Therefore, taking into account the interpretation of \( * \) as a truth function of connective “very true”, \[26\] can be read: “If it is very true that \( A \) is included in \( B \), then \( c(A) \) is included in \( c(B) \)”.

A finer reading, which involves explicit references to degrees is: “The degree to which \( c(A) \) is included in \( c(B) \) is at least as high as the degree to which it is very true that \( A \) is included in \( B \)”. 

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Let us stress that $L^*$-closure operators are indeed ordinary closure operators, i.e., (26) implies the ordinary isotony condition. Indeed, if $A \subseteq B$, then $S(A, B) = 1$, i.e., applying (20), we get that

$$\text{if } A \subseteq B \text{ then } c(A) \subseteq c(B).$$

(28)

In general, (26) is stronger than (28) at it is shown in [6, Remark 2.3]. The conditions become equivalent if $*$ is the so-called globalization [48], i.e., if for any $a \in L$, we have

$$a^* = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(29)

If $*$ is globalization, then $A \nsubseteq B$ gives $S(A, B^*) = 0$ because $S(A, B) < 1$. Therefore, (26) is indeed equivalent to (28).

The second important approach [4] to parameterized closure operators on complete residuated lattices of $L$-sets, which predates the approach by hedges, is based on the notion of an $\leq$-filter which has been recalled in the beginning of Section 2.

**Definition 2.** Let $L$ be a complete residuated lattice, $K \subseteq L$ be an $\leq$-filter in $L$, and $Y$ be a non-empty universe set. An operator $c: L^Y \to L^Y$ is called an $L_K$-closure operator [4] in $L^Y$ whenever (25), (27), and the following condition:

$$\text{if } S(A, B) \in K \text{ then } S(A, B) \leq S(c(A), c(B))$$

(30)

are satisfied for all $A, B \in L^Y$.

Analogously as in the case of $L^*$-closure operators, we can see that condition (30) implies (28) because $1 \in K$ and thus $A \subseteq B$ yields $S(A, B) = 1 \in K$ which gives $S(c(A), c(B)) = 1$, meaning $c(A) \subseteq c(B)$. In addition, (30) becomes (28) for $K = \{1\}$ which is trivially an $\leq$-filter in $L$. Hence, following the notation in Definition 2, we refer to the ordinary closure operators in $L^Y$ as to the $L_{\{1\}}$-closure operators, see [4].
3 S-closure operators

In this section, we introduce S-closure operators and related notions. Furthermore, we focus on their relationship to the established closure operators parameterized by hedges and \( \leq \)-filters which have been recalled in Section 2. We show that both the parameterizations may be seen as special cases of the parameterization by systems of isotone Galois connections. In addition, the studied operators can be seen as generalizations of those appearing in [50, 53].

**Definition 3.** Let \( L = \langle L, \leq \rangle \) be a complete lattice. Any set \( S \) of isotone Galois connections in \( L \) such that \( \langle 1, 1 \rangle \in S \) is called an \( L \)-parameterization. An operator \( c: \langle L, \leq \rangle \to \langle L, \leq \rangle \) is called an \( S \)-closure operator in \( \langle L, \leq \rangle \) whenever

\[
a \leq c(a), \\
a \leq b \implies c(a) \leq c(b), \\
c(g(c(a))) \leq g(c(a)),
\]

are satisfied for all \( a, b \in L \) and all \( \langle f, g \rangle \in S \). If \( S \) is closed under compositions, i.e., if \( S = \{ S, \circ, \langle 1, 1 \rangle \} \) is a monoid, we call it an \( L \)-parameterization and, in addition, \( c \) satisfying (31)–(33) is called an \( S \)-closure operator.

It is evident that for \( S = \{ \langle 1, 1 \rangle \} \), all \( S \)-closure operators in \( L \) are exactly the classic closure operators in \( L \). Indeed, for \( \langle f, g \rangle \) being \( \langle 1, 1 \rangle \), (33) becomes

\[
c(c(a)) \leq c(a)
\]

which together with (31) yields the classic idempotency condition. It is easy to see that without the requirement of \( \langle 1, 1 \rangle \in S \), the operator may not be idempotent in general.

We present examples of \( S \)-closure operators which appear in several fields of relational data analysis in Section 4. In the rest of this section, we investigate the relationship of \( S \)-closure operators to the operators in residuated lattices of \( L \)-sets summarized in Section 2.3.
Theorem 1. Let $L$ be a complete residuated lattice, $Y \neq \emptyset$, and let $*$ be an idempotent truth-stressing hedge. An operator $c : L^Y \to L^Y$ is an $L^*$-closure operator iff it is an $L_{\{1\}}$-closure operator and
\[ c(a^* \to c(A)) \subseteq a^* \to c(A) \] holds for all $a \in L$ and $A \in L^Y$.

Proof. Let $c$ be an $L^*$-closure operator. Obviously, (28) is a particular case of (26), see [6, Remark 2.3]. As a consequence, $c$ is an $L_{\{1\}}$-closure operator. Therefore, it remains to show that $c$ satisfies (34). We prove this fact using (24), (26), and (27). Take any $a \in L$ and $A \in L^Y$. Using (19), from
\[ a^* \to (a^* \to c(A)) \subseteq a^* \to c(A) \]
it follows that
\[ a^* \otimes (a^* \to c(A)) \subseteq c(A) \]
which, again by (19), gives
\[ a^* \leq S(a^* \to c(A), c(A)). \]
Now, using (23), (24), (26), and (27), the last inequality yields
\[ a^* = a^{**} \]
\[ \leq S(a^* \to c(A), c(A))^* \]
\[ \leq S(c(a^* \to c(A)), c(c(A))) \]
\[ \leq S(c(a^* \to c(A)), c(A)), \]
i.e., $a^* \leq S(c(a^* \to c(A)), c(A))$. Now, using (19) twice, we get
\[ a^* \otimes c(a^* \to c(A)) \subseteq c(A) \]
and finally
\[ c(a^* \to c(A)) \subseteq a^* \to c(A), \]
i.e., (35) is satisfied.

Conversely, suppose that $c$ is an $L_{\{1\}}$-closure operator such that (35) holds. We show that $c$ satisfies (26). Take any $A, B \in L^Y$. Observe that using (35) for $a = S(A, B)$, we get

$$c(S(A, B)^* \rightarrow c(B)) \subseteq S(A, B)^* \rightarrow c(B)$$

from which, owing to (19), it follows that

$$S(A, B)^* \leq S(c(S(A, B)^* \rightarrow c(B)), c(B)).$$

Since the graded subsethood is antitone in the first argument, we obtain (26) as a consequence of the previous inequality and the fact that $c(A) \subseteq c(S(A, B)^* \rightarrow c(B))$ which is indeed true: Using (21), (25), and the isotony of $S$ in the second argument, we get

$$S(A, B)^* \leq S(A, B) \leq S(A, c(B))$$

and thus $A \subseteq S(A, B)^* \rightarrow c(B)$ by (19). Using (28), we further get

$$c(A) \subseteq c(S(A, B)^* \rightarrow c(B))$$

As a consequence,

$$S(A, B)^* \leq S(c(S(A, B)^* \rightarrow c(B)), c(B)) \leq S(c(A), c(B))$$

which proves (26).
Remark 3. (a) Observe that in the only-if part of the proof of Theorem 1, we have utilized the isotony (23) of $\ast$ instead of the stronger condition (22). Also, in the only-if part of the proof, we have not utilized the subdiagonality condition (21). Neither (20) nor (23) nor (24) can be omitted because otherwise the only-if part of the assertion would not hold. This is obvious in the case of (20). In order to see that (23) is necessary, suppose that $L$ is defined on a four-element linearly ordered set $L = \{0, a, b, 1\}$ so that $0 < a < b < 1$. It can be easily checked that $\otimes$ and $\rightarrow$ defined by the tables in Figure 1 satisfy the adjointness property. Moreover, we can consider $\ast : L \to L$ such that $c^\ast = c$ for $c \in \{0, a, 1\}$ and $b^\ast = 0$. Obviously, $\ast$ satisfies (20), (24), and it does not satisfy (23). Now, take $Y = \{y\}$ and consider an operator $c : L^\{y\} \to L^\{y\}$ such that

$$c(A) = \begin{cases} 
\{y^0\}, & \text{for } A = \{y^0\}, \\
\{y^1\}, & \text{otherwise.}
\end{cases}$$

(36)

By a routine check, it follows that $c$ satisfies (25), (26), and (27). On the other hand, (35) is not satisfied. Indeed, for $A = \{y^0\}$, we have

$$c(a^\ast \to c(\{y^0\})) = c(a \to \{y^0\})$$

$$= c(\{y^{a^\ast - 0}\})$$

$$= c(\{y^b\})$$

$$= \{y^1\}$$

$$\not\subseteq \{y^b\} = a^\ast \to c(\{y^0\}).$$
Therefore, at least (23) is necessary in order to establish the assertion of Theorem 1.

(b) Analogously as in (a), we may proceed for an operator \( \ast \) which is not idempotent. Consider \( L \) with \( \otimes \) and \( \to \) defined as in Figure 2. Furthermore, consider \( \ast \) such that \( 0^\ast = a^\ast = 0, \ b^\ast = a, \) and \( 1^\ast = 1. \) Considering the same order of elements in \( L \) as before, \( L \) is a complete residuated lattice and \( \ast \) satisfies (20), (21), and (22). Thus, \( \ast \) is a truth-stressing hedge which is not idempotent because \( b^{\ast\ast} \neq b^\ast. \) Furthermore, \( c : L^{(y)} \to L^{(y)} \) defined by

\[
c(A) = \begin{cases} 
A, & \text{if } A = \{y^0\} \text{ or } A = \{y^a\}, \\
\{y^1\}, & \text{otherwise.}
\end{cases}
\]

is an \( L^\ast \)-closure operator. It is easy to see that

\[
c(b^\ast \to c(\{y^0\})) = c(a \to c(\{y^0\})) = c(a \to \{y^0\}) = c(\{y^b\}) = \{y^1\} \not\subseteq \{y^b\} = b^\ast \to c(\{y^0\}),
\]

i.e., (35) does not hold.

(c) In the if-part of Theorem 1 the subdiagonality condition (21) is also necessary. If \( L \) is the two-valued Boolean algebra with \( L = \{0, 1\} \) and \( 0 < 1, \) then for a map \( \ast : L \to L \) such that \( 0^\ast = 1^\ast = 1 \) it follows that the identity operator \( 1_{L(\leq)} \) satisfies (35). On the contrary, we have

\[
S(\{y^1\}, \{y^0\})^* = 0^* = 1 \not\subseteq 0 = S(\{y^1\}, \{y^0\}) = S(1_{L(\leq)}(\{y^1\}), 1_{L(\leq)}(\{y^0\})),
\]

i.e., \( 1_{L(\leq)} \) violates (26).

As a consequence of Theorem 1 we get a corollary presented below which states that \( L^\ast \)-closure operators are particular \( S \)-closure operators provided that \( \ast \) is idempotent. In the corollary, we use lower and upper adjoints written as
$f_{a\otimes}$ and $g_{a\rightarrow}$ for some $a \in L$ which are in fact defined analogously as (17) and (18), respectively. That is, by putting

$$f_{a\otimes}(B) = a \otimes B, \quad (37)$$

$$g_{a\rightarrow}(B) = a \rightarrow B, \quad (38)$$

for all $a \in L$ and $B \in L^Y$ and considering (19) together with Theorem 1, we have the following observation:

**Corollary 2.** Let $L$ be a complete residuated lattice, $Y \neq \emptyset$, and let $*$ be an idempotent truth-stressing hedge. An operator $c : L^Y \rightarrow L^Y$ is an $L^*$-closure operator iff it is an $S$-closure operator for $S = \{ (f_{a\otimes}, g_{a\rightarrow}) ; a \in L \}$. □

Let us note that in Corollary 2 the considered operator is indeed an $S$-closure operator and not just $S$-closure operator, i.e., $S$ is closed under the composition of isotone Galois connections. Indeed, for any $a, b \in L$ and $c = a^* \otimes b^*$, we have $f_{a^*\otimes}f_{b^*\otimes} = f_{c^*\otimes}$ and $g_{b^*\rightarrow}g_{a^*\rightarrow} = g_{c^*\rightarrow}$ which follows from the fact that $a^* \otimes b^* = (a^* \otimes b^*)^*$ provided that the hedge $*$ is idempotent, see [13, Lemma 2], cf. also [53, Example 1 (b)]

We now turn our attention to the relationship to $L_K$-closure operators. Analogously as in Theorem 1 we may establish the following characterization.

**Theorem 3.** An operator $c : L^Y \rightarrow L^Y$ is an $L_K$-closure operator iff it is an $L_{\{1\}}$-closure operator and

$$c(a \rightarrow c(A)) \subseteq a \rightarrow c(A) \quad (39)$$

holds for all $a \in K$ and $A \in L^Y$.

**Proof.** Most parts of the proof use similar arguments as in the proof of Theorem 1. Therefore, we only comment on the technical differences. Suppose that $c$ is an $L_K$-closure operator. Clearly, $c$ satisfies (28) because $1 \in K$. Take any $a \in K$ and $A \in L^Y$. By (19), we get

$$a \leq S(a \rightarrow c(A), c(A))$$
and thus $S(a \to c(A), c(A)) \in K$ because $K$ is an $\leq$-filter. Therefore, applying (29), (27), and the isotony of $S$ in the second argument, it follows that

$$a \leq S(a \to c(A), c(A)) \leq S(c(a \to c(A)), c(c(A))) \leq S(c(a \to c(A)), c(A)).$$

Thus, using (19), we obtain (39).

Conversely, assuming that $c$ is an $L\{1\}$-closure operator satisfying (39), we prove that (30) holds. Take $A, B \in L^Y$ such that $S(A, B) \in K$. As a particular case of (39), we get

$$c(S(A, B) \to c(B)) \subseteq S(A, B) \to c(B)$$

and thus, using (19), we get

$$S(A, B) \leq S(c(S(A, B) \to c(B)), c(B)).$$

In order to finish the proof, we show that $c(A) \subseteq c(S(A, B) \to c(B))$ which results by applying (25), (28) together with the isotony of $S$ in the second argument, and (19) in much the same way as in the proof of Theorem 1.

Remark 4. Analogously as in the case of Theorem 1, we can show that the condition of $K$ being an $\leq$-filter in Theorem 3 is essential. Indeed, take $L$ with $L = \{0, a, b, 1\}$ as in Remark 3 and let $\otimes$ and $\rightarrow$ be defined as in Figure 1. Furthermore, take $K = \{a, 1\}$. Obviously, $K$ is not an $\leq$-filter because $b \not\in K$.

Now, an operator $c : L^\{y\} \to L^\{y\}$ defined by (36) satisfies (25), (27), and (30). On the other hand, it does not satisfy (79). Indeed, observe that for $a \in K$ and $A = \{y^0\}$, we have

$$c(a \rightarrow c(\{y^0\})) = c(a \rightarrow \{y^0\}) = c(\{y^b\}) = \{y^1\} \not\subseteq \{y^b\} = a \rightarrow c(\{y^0\}),$$

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i.e., if $K$ is not an $\leq$-filter, then $c$ does not satisfy (39) in general. Note that in the if-part of Theorem 3, no assumptions on $K$ are needed.

**Corollary 4.** Let $L$ be a complete residuated lattice, $Y \neq \emptyset$, and let $K$ be an $\leq$-filter. An operator $c: L^Y \to L^Y$ is an $L_K$-closure operator iff it is an $S$-closure operator for $S = \{(f_{a\otimes}, g_{a\rightarrow}); a \in K\}$.

Unlike 2, $S$ in Corollary 4 is not closed under compositions in general. As a consequence, there are $L_K$-operators which are not $S$-closure operators for $S = \{(f_{a\otimes}, g_{a\rightarrow}); a \in K\}$. For instance, if $L$ is the standard Lukasiewicz algebra, i.e., if $L$ is the real unit interval with its natural ordering and $\otimes$ is given by $a \otimes b = \max(a + b - 1, 0)$, then $K = [0.5, 1]$ is obviously an $\leq$-filter but $S = \{(f_{0.5\otimes}, g_{0\rightarrow}); a \geq 0.5\}$ is not closed under $\circ$ because, e.g., $\langle f_{0.5\otimes}, g_{0.5\rightarrow} \rangle \circ \langle f_{0.5\otimes}, g_{0.5\rightarrow} \rangle = \langle f_{0\otimes}, g_{0\rightarrow} \rangle \notin K$.

4 Examples

In the following subsections, we show two important areas of reasoning with if-then rules and extracting information from relational data where $S$-closure operators naturally appear.

4.1 Enriched Armstrong-style inference systems

In this subsection, we show an important example of $S$-closure operators induced by enriched Armstrong-style inference systems [1] defined on algebraic lattices. We introduce general inference systems and consider operators which map each element of a lattice to its syntactic closure which is defined by the inference system. We prove the operator is indeed an $S$-closure operator and show that several inference systems that appeared in the past in different contexts have syntactic closures which may be viewed as special cases of the general one.

Recall the notions of compactness of lattice elements and (complete) algebraic lattices from Section 2 and assume that $L = (L, \leq)$ is a complete algebraic lattice with $K \subseteq L$ being the set of all its compact elements. We use this as-
sumption throughout the entire section and we are not going to repeat it. At this point, we may view $L$ as an abstract system of elements which can be used to form particular formulas. Namely, any ordered pair $(a, b) \in K \times K$ is called a (well-formed) formula and for better readability we are going to denote it $a \Rightarrow b$ and read it “if $a$ then $b$.” The intended meaning of $a \Rightarrow b$ is to express that if a lattice element is at least as high as $a$, then it is also at least as high as $b$. We introduce an inference system for the rules which resembles the famous Armstrong system for reasoning with functional dependencies [1,44]. In our setting, the system is enriched by new inference rules defined by lower adjoints coming from particular $L$-parameterizations.

**Definition 4.** Let $S$ be an $L$-parameterization. If for any $a \in K$ and any $(f,g) \in S$, we have $f(a) \in K$, then $S$ is called a compact $L$-parameterization. Under this assumption, an $S$-inference system for $L$ is a set of the following inference rules:

1. (from no assumptions) infer $a \vee b \Rightarrow b$,
2. from $a \Rightarrow b$ and $b \vee c \Rightarrow d$ infer $a \vee c \Rightarrow d$,
3. from $a \Rightarrow b$ infer $f(a) \Rightarrow f(b)$

for any $a, b, c, d \in K$ and any $(f,g) \in S$. Let $\Sigma$ be a set of formulas. An $S$-proof of $a \Rightarrow b$ by $\Sigma$ is any finite sequence $\delta_1, \ldots, \delta_n$ of formulas such that $\delta_n$ is $a \Rightarrow b$ and for each $i = 1, \ldots, n$, we have that

- $\delta_i \in \Sigma$, or
- $\delta_i$ results from some of the formulas $\delta_j$ ($j < i$) using a single application of one of the inference rules 1.–3.

We put $\Sigma \vdash a \Rightarrow b$ and say that $a \Rightarrow b$ is $S$-provable by $\Sigma$ whenever there is an $S$-proof of $a \Rightarrow b$ by $\Sigma$.

**Remark 5.** (a) Note that the inference rules of an $S$-inference system produce only well-formed formulas. Indeed, this is a consequence of the following facts. First, if $a, b \in K$, then $a \vee b \in K$, i.e., the supremum of compact elements in
is a compact element in \( \mathbf{L} \). Thus, the inference rules 1. and 2. produce well-formed formulas. Second, if \( a \in K \), then \( f(a) \in K \) because we assume that \( S \) is a compact \( L \)-parameterization. This shows that 3. always produces a well-formed formula. Also note that 1. is in fact an axiom schema saying that each formula of the form \( a \Rightarrow b \) where \( b \leq a \) is \( S \)-provable by any set of formulas.

(b) The classic system of Armstrong rules can be seen as an inference system which is a particular case of that in Definition 4. Take a finite \( R \) which is the set of attributes of a relation scheme and consider the finite complete lattice \( \mathbf{L} = \langle 2^R, \subseteq \rangle \) of all subsets of \( R \). Each element of \( \mathbf{L} \) is compact because \( 2^R \) is finite, i.e., \( K = L \). Then, formulas in our sense are expressions of the form \( A \Rightarrow B \), where \( A, B \in 2^R \) which agrees with the type of formulas used in the Armstrong system. Furthermore, the inference rules 1. and 2. become

- infer \( A \cup B \Rightarrow B \), and
- from \( A \Rightarrow B \) and \( B \cup C \Rightarrow D \) infer \( A \cup C \Rightarrow D \),

which are equivalent to the classic Armstrong rules and are called the axiom and pseudo-transitivity in [44]. Finally, for \( S = \{ \langle 1, 1 \rangle \} \), which is trivially a compact \( L \)-parameterization, the last inference rule becomes trivial: from \( A \Rightarrow B \) infer \( A \Rightarrow B \) and can be disregarded.

(c) As in the case of the classic Armstrong system, we can easily derive the following basic properties of \( S \)-provability [44]:

- weakening: if \( \Sigma \vdash a \Rightarrow c \), then \( \Sigma \vdash a \vee b \Rightarrow c \),
- transitivity: if \( \Sigma \vdash a \Rightarrow b \) and \( \Sigma \vdash b \Rightarrow c \), then \( \Sigma \vdash a \Rightarrow c \),
- addition: if \( \Sigma \vdash a \Rightarrow b \) and \( \Sigma \vdash a \Rightarrow c \), then \( \Sigma \vdash a \Rightarrow b \vee c \),

for any \( \Sigma \) and any \( a, b, c \in K \).

We now focus on particular \( S \)-closure operators which are induced by \( S \)-inference systems and given sets of formulas and show their basic properties.
Definition 5. Let $S$ be a compact $L$-parameterization of $L$. For any set $\Sigma$ of formulas, we define operators $\mathcal{C}_\Sigma : L \to 2^K$ and $\mathcal{C}_\Sigma : L \to L$ by

$$\mathcal{C}_\Sigma(a) = \{b \in K; \Sigma \vdash c \Rightarrow b \text{ for some } c \in K \text{ such that } c \leq a\}, \quad (40)$$

$$\mathcal{C}_\Sigma(a) = \bigvee \mathcal{C}_\Sigma(a) \quad (41)$$

for all $a \in L$; $\mathcal{C}_\Sigma(a)$ is called the syntactic $S$-closure of $a \in L$ (using $\Sigma$).

The following technical observation shows that (41) can be simplified provided that $a$ is a compact element of $L$.

Lemma 5. If $a \in K$, then $\mathcal{C}_\Sigma(a) = \bigvee \{b \in K; \Sigma \vdash a \Rightarrow b\}$.

Proof. Let $a \in K$ and let $b \in \mathcal{C}_\Sigma(a)$. That is, there is $c \in K$ such that $c \leq a$ and $\Sigma \vdash c \Rightarrow b$. Then, using the principle of weakening, see Remark 5(c), we get $\Sigma \vdash a \Rightarrow b$ and thus $b \leq \bigvee \{b \in K; \Sigma \vdash a \Rightarrow b\}$. Since $b \in \mathcal{C}_\Sigma(a)$ was taken arbitrarily, we get $\mathcal{C}_\Sigma(a) \leq \bigvee \{b \in K; \Sigma \vdash a \Rightarrow b\}$.

Conversely, take any $b \in K$ such that $\Sigma \vdash a \Rightarrow b$. Since $a \leq a$ and $a \in K$, it follows that $b \in \mathcal{C}_\Sigma(a)$ from which the claim readily follows.

The operator defined by (41) can be used to check whether a given $a \Rightarrow b$ ($a, b \in K$) is provable by a given $\Sigma$. Indeed, the next assertion shows that the problem of deciding $\Sigma \vdash a \Rightarrow b$ is equivalent to the problem of determining whether $\mathcal{C}_\Sigma(a)$ is at least as high as $b$ in terms of the underlying lattice order.

Theorem 6. Let $a, b \in K$. We have $\Sigma \vdash a \Rightarrow b$ iff $b \leq \mathcal{C}_\Sigma(a)$.

Proof. The only-if part follows directly by Lemma 5.

Conversely, suppose that $b \leq \mathcal{C}_\Sigma(a)$ holds. Using the fact that both $a$ and $b$ are compact and taking into account Lemma 5, we get that there is a finite $B = \{b_1, \ldots, b_n\} \subseteq K$ such that $b \leq \bigvee B \in K$ and $\Sigma \vdash b_i \Rightarrow b_i$ for all $i = 1, \ldots, n$.

Now, $b \leq \bigvee B \in K$ yields $\Sigma \vdash b_1 \lor \cdots \lor b_n \Rightarrow b$. Furthermore, using the addition principle multiple times, see Remark 5(c), it follows that $\Sigma \vdash a \Rightarrow b_1 \lor \cdots \lor b_n$. Thus, using the transitivity of $\vdash$, we get $\Sigma \vdash a \Rightarrow b$. 

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Note that similar characterizations of entailment as that in Theorem 6 appear in various approaches which involve if-then rules. For instance, in logic programming [43], an analogous role is played by least models of definite programs. The next assertion shows that \( c_{\Sigma} \) is indeed an \( S \)-closure operator which is in addition algebraic in the sense that the closure of any element \( a \in L \) can be obtained as the supremum of closures of all compact elements which are smaller than or equal to \( a \). In addition, we show that \( c_{\Sigma} \) is an \( S' \)-closure operator where \( S' \) is the uniquely given \( L \)-parameterization generated by \( S \). That is, \( S' \) is the least subset of isotone Galois connections in \( L \) such that \( S \subseteq S' \) and \( S' \) is closed under \( \circ \). In such a case, we say that \( S' = \langle S', \circ, \langle 1, 1 \rangle \rangle \) is a monoid (of isotone Galois connections) generated by \( S \).

**Theorem 7.** Let \( \Sigma \) be a theory and \( S \) be a compact \( L \)-parameterization. Then, \( c_{\Sigma} \) defined by (41) is an algebraic \( S' \)-closure operator where \( S' = \langle S', \circ, \langle 1, 1 \rangle \rangle \) is the monoid generated by \( S \).

**Proof.** First, we prove that \( c_{\Sigma} \) given by (41) is extensive, i.e., it satisfies (31). Since \( L \) is algebraic, any \( a \in L \) can be expressed as \( a = \bigvee B \) where \( B \subseteq K \). Take any \( b \in B \). It suffices to check that \( b \leq c_{\Sigma}(a) \). Observe that the facts that \( b \in K \), \( b \leq a \), and \( \Sigma \vdash b \Rightarrow b \) yield that \( b \in C_{\Sigma}(a) \) and thus \( b \leq c_{\Sigma}(a) \). Altogether, \( c_{\Sigma} \) is extensive.

Second, we check that \( c_{\Sigma} \) satisfies (32). This is easy to see. Observe that assuming \( a_1 \leq a_2 \), the fact \( b \in C_{\Sigma}(a_1) \) means there is \( c \in K \) such that \( c \leq a_1 \) and \( \Sigma \vdash c \Rightarrow b \). But in that case, \( c \leq a_1 \leq a_2 \) and so \( b \in C_{\Sigma}(a_2) \) from which we immediately get \( c_{\Sigma}(a_1) \leq c_{\Sigma}(a_2) \).

Third, we prove that \( c_{\Sigma} \) satisfies (33). Consider any \( \langle f, g \rangle \) which results as a composition of arbitrary finitely many isotone Galois connections in \( S \). We check that \( c_{\Sigma}(g(c_{\Sigma}(a))) \leq g(c_{\Sigma}(a)) \) for any \( a \in L \). Take \( b \in C_{\Sigma}(g(c_{\Sigma}(a))) \). It suffices to check that \( b \leq g(c_{\Sigma}(a)) \). Using (41), there is \( c \in K \) such that \( \Sigma \vdash c \Rightarrow b \) and \( c \leq g(c_{\Sigma}(a)) \). Using (41), it follows that \( f(c) \leq c_{\Sigma}(a) \) and, in addition, \( f(c) \) is a compact element because \( S \) is a compact \( L \)-parameterization. Therefore, from \( f(c) \leq c_{\Sigma}(a) \), it follows that there are \( \{b_1, \ldots, b_n\} \subseteq C_{\Sigma}(a) \) and
\{c_1, \ldots, c_n\} \subseteq K \text{ such that } f(c) \leq b_1 \lor \cdots \lor b_n \in K, c_i \leq a, \text{ and } \Sigma \vdash c_i \Rightarrow b_i \text{ for all } i = 1, \ldots, n. \text{ Hence, } \Sigma \vdash b_1 \lor \cdots \lor b_n \Rightarrow f(c) \text{ and } \Sigma \vdash c_1 \lor \cdots \lor c_n \Rightarrow b_1 \lor \cdots \lor b_n \text{ from which by the transitivity of } \vdash \text{ it follows that } \Sigma \vdash c_1 \lor \cdots \lor c_n \Rightarrow f(c). \text{ Moreover, we have observed that } \Sigma \vdash c \Rightarrow b, \text{ i.e., applying the third inference rule multiple times (recall that } f \text{ is in fact a composition of finitely many lower adjoints from } S), \text{ we get } \Sigma \vdash f(c) \Rightarrow f(b). \text{ Therefore, applying the transitivity of } \vdash \text{ once again, we get } \Sigma \vdash c_1 \lor \cdots \lor c_n \Rightarrow f(b). \text{ At this point, we have } c_1 \lor \cdots \lor c_n \in K \text{ such that } c_1 \lor \cdots \lor c_n \leq a \text{ and } \Sigma \vdash c_1 \lor \cdots \lor c_n \Rightarrow f(b) \text{ for } f(b) \in K, \text{ i.e., using (40), we get } f(b) \in C_{\Sigma}(a). \text{ Thus, } f(b) \leq c_{\Sigma}(a) \text{ and so } b \leq g(c_{\Sigma}(a)). \text{ Finally, it remains to prove that } c_{\Sigma} \text{ is algebraic. Take any } a \in L \text{ and } b \in C_{\Sigma}(a) \subseteq K, \text{ i.e., there is } c \in K \text{ such that } c \leq a \text{ and } \Sigma \vdash c \Rightarrow b. \text{ Hence, } b \in C_{\Sigma}(c), \text{ i.e., } b \leq c_{\Sigma}(c). \text{ As a consequence, } c_{\Sigma}(a) \leq \bigvee\{c_{\Sigma}(c); \ c \in K \text{ and } c \leq a\}. \]
\[ d \leq c(b \lor c). \] Therefore, using (32) and (34), it follows that
\[ d \leq c(b \lor c) \leq c(c(a) \lor c) \leq c(c(a \lor c)) \leq c(a \lor c). \]

Hence, \( a \lor c \Rightarrow d \in \Sigma_c \).

Take any \( (f, g) \in S \) and any \( a \Rightarrow b \in \Sigma_c \). We show that \( f(a) \Rightarrow f(b) \), which results by the application of the third inference rule in Definition 4 belongs to \( \Sigma_c \). Using (12), the assumption \( a \Rightarrow b \in \Sigma_c \) yields \( b \leq c(a) \) and thus, owing to (13), we get \( f(b) \leq f(c(a)) \). Thus, it suffices to check that \( f(c(a)) \leq c(f(a)) \) which is indeed the case. We postpone the proof of this general and important inequality to the next section, see Theorem 17.

Now, the claim of Lemma 8 follows directly by our observations using induction over the length of an \( S \)-proof by \( \Sigma_c \).

\[ \text{Theorem 9. Let } c \text{ be an algebraic } S\text{-closure operator where } S \text{ is a compact } L\text{-parameterization. Then, for } \Sigma_c \text{ given by (12), we have } c = c_{\Sigma_c}. \]

\[ \text{Proof. Owing to Theorem 7} \ c_{\Sigma_c} \text{ is an algebraic } S\text{-closure operator and thus it suffices to show that } c(a) = c_{\Sigma_c}(a) \text{ for each } a \in K. \]

First, take \( a \in K \) and let \( b \in K \) such that \( b \leq c(a) \). Using (12), we get \( a \Rightarrow b \in \Sigma_c \) and thus \( \Sigma_c \vdash a \Rightarrow b \). Therefore, using Theorem 6 we have \( b \leq c_{\Sigma_c}(a) \). Since \( L \) is an algebraic lattice and \( b \in K \) was taken arbitrarily, we get that \( c(a) \leq c_{\Sigma_c}(a) \) for \( a \in K \).

Conversely, for any \( a \in K \) and any \( b \in K \), we show that \( b \leq c(a) \) provided that \( b \leq c_{\Sigma_c}(a) \). Using Theorem 6 and the assumption \( b \leq c_{\Sigma_c}(a) \), it follows that \( \Sigma_c \vdash a \Rightarrow b \). Using Lemma 8, the last fact yields \( a \Rightarrow b \in \Sigma_c \). Now, directly from (12), we get \( b \leq c(a) \) which finishes the proof.

\[ \text{Remark 6. Let us note that the presented inference system for if-then formulas which is parameterized by a system of isotone Galois connections is not the only one possible. Several equivalent systems can be introduced which are analogous to the graph-based system in [51] and the system based on the simplification equivalence [5] which is suitable for automated provers. Also note that Theorem 7 and Theorem 9 are limited to algebraic } S\text{-closure operators on algebraic } \]
lattices. The assertions can be extended to unrestricted $S$-closure operators on arbitrary complete lattices by extending the inference system by an infinitary deduction rule in a similar way as in the approaches in \[10, 40\].

Finally, we show the relationship of the general approach presented in this section to the well-established approaches to reasoning with various types of if-then dependencies.

**Example 1.** (a) As we have mentioned in Remark \[5\] the classic Armstrong system for reasoning with functional dependencies is a particular case of the presented system. In addition, owing to the equivalence of the semantic entailment of functional dependencies and particular formulas in the classic propositional logic, see \[24, 27, 47\], the inference system can also be used for reasoning with attribute implications \[34\] which appear as basic formulas in formal concept analysis \[30\].

(b) Our approach generalizes the inference systems for fuzzy attribute implications \[8\] and similarity-based functional dependencies \[9\] parameterized by linguistic hedges, see also \[15, 7\] for detailed survey. In these approaches, $L$ is the residuated lattice of all $L$-sets in a finite universe $Y$. In this setting, formulas are considered as expressions $A \Rightarrow B$ where $A, B \in L^Y$ and are called fuzzy (graded) attribute implications. The inference system for fuzzy attribute implications consists of inference rules which correspond to 1. and 2. from Definition \[4\]. In addition, the inference system uses the following rule of multiplication: From $A \Rightarrow B$, infer $c^* \circ A \Rightarrow c^* \circ B$. Therefore, it can easily be seen that the inference system corresponds to the general system presented in this section for $S = \{(f_{c^* \circ}, g_{c^* \to}) ; c \in L\}$ with $f_{c^* \circ}$ and $g_{c^* \to}$ given as in \[12\] and \[12\], respectively. The approach to parameterization by linguistic hedges was further developed in \[53\] where we have presented fuzzy if-then rules with semantics parameterized by systems of isotone Galois connections; \[53\] is in fact the major motivation for the present paper which looks at the closure structures from a more general perspective because it is not concerned only with complete residuated lattices of fuzzy sets and extends the observations on the closure
structures studied initially in \[53\].

(c) The present approach also generalizes an inference system for attribute implications with temporal semantics. Attribute implications annotated by time points have been introduced in \[50\] and investigated in more detail in \[49\]. From the point of view of the present formalism, the formulas are expressions of the form \( A \Rightarrow B \) such that \( A, B \subseteq Y \times Z \), where \( Y \) is a finite set of attributes, both \( A \) and \( B \) are finite, and the integers from \( Z \) are used to denote relative time points. For instance, \( A = \{ \langle y, -1 \rangle, \langle z, 2 \rangle \} \subseteq Y \times Z \) is interpreted as expressing the fact that \( y \) holds in the time point prior to the present one (i.e., \(-1\) time units from now) and \( z \) holds in the time point after the next one (i.e., \(+2\) time units from now). A complete Armstrong-style inference system for such formulas is presented in \[49\]. It can be seen as a particular case of the general system presented in this section by considering \( L \) with \( L = 2^{Y \times Z} \) and \( \subseteq \) being the complete lattice order on \( L \). Obviously, all finite \( A \subseteq Y \times Z \) are exactly the compact elements of \( L \) and, in addition, \( L \) is an algebraic lattice. Furthermore, put \( s(A) = \{ \langle y, i + 1 \rangle \mid \langle y, i \rangle \in A \} \) for any \( A \subseteq Y \times Z \) and consider its inverse \( s^{-1} \). In this setting, the inference system from \[49\] becomes the general inference system for \( S = \{ \langle 1, 1 \rangle, \langle s, s^{-1} \rangle \} \).

4.2 Consequence operators induced by data

One of the basic problems in FCA \[30, 56\] is discovery of attribute dependencies described by sets of attribute implications which hold in given data. In this subsection, we approach the problem from the perspective of closure operators and general parameterizations. Analogously as in the previous subsection, we show that the notion of an if-then dependency being true in data can be defined on a general level and the basic criterion for a dependency being true in data can be expressed using fixed points of the parameterized closure operators studied in our paper.

Recall that in the classic FCA \[30, 54\], the input data is an object-attribute data table, called a (dyadic) formal context, which is formalized as a binary
relation \( I \subseteq X \times Y \) between the set \( X \) of objects and the set \( Y \) of attributes under consideration. An attribute implication \( A \Rightarrow B \) where \( A, B \subseteq Y \) is considered true in \( I \) whenever \( A \subseteq \{ x \}^I \) implies \( B \subseteq \{ x \}^I \) for all \( x \in X \) where \( \{ x \}^I \) is the standard concept-forming operator, i.e., \( \{ x \}^I = \{ y \in Y; \langle x, y \rangle \in I \} \).

It is well-known that \( A \Rightarrow B \) being true in \( I \) can equivalently be expressed by means of the inclusion of \( B \) in the intent of \( I \) generated by \( A \). We know show an analogous result on a more general level: First, instead of a dyadic formal context, we take a subset \( M \subseteq L \) where \( L = \langle L, \leq \rangle \) is a complete lattice; \( M \) can be seen as representing “input data.” Attribute implications will be considered in much the same way as in the previous subsection, and their interpretation in \( M \) is defined using an \( L \)-parameterization \( S \):

**Definition 6.** Let \( M \subseteq L \) and let \( S \) be an \( L \)-parameterization. For any \( a, b \in L \), we put \( M \models a \Rightarrow b \) and say that \( a \Rightarrow b \) is true in \( M \) whenever the following condition is satisfied:

\[
\text{if } f(a) \leq m \text{ then } f(b) \leq m
\]

for all \( m \in M \) and any \( \langle f, g \rangle \) such that

\[
\langle f, g \rangle = \langle f_1, g_1 \rangle \circ \cdots \circ \langle f_n, g_n \rangle
\]

for some \( \langle f_1, g_1 \rangle, \ldots, \langle f_n, g_n \rangle \in S \).

**Remark 7.** (a) The classic notion of an attribute implication being true in a formal context is generalized by Definition 6 as follows: \( L \) is the complete lattice of all subsets of \( Y \) ordered by the usual set inclusion and for each formal context \( I \subseteq X \times Y \) we take \( M = \{ \{ x \}^I; x \in X \} \subseteq L = 2^Y \). In this setting, for \( S = \{ \langle 1, 1 \rangle \}, \models \) from Definition 6 becomes the usual notion of satisfaction of an attribute implication in a formal context.

(b) It is easy to see that \( \models \) from Definition 6 can be seen as a relation of satisfaction (i.e., \( M \models a \Rightarrow b \) may be understood so that \( a \Rightarrow b \) is satisfied in \( M \)) which generalizes properties of the classic relation of satisfaction of attribute implications in formal contexts. For instance, \( M \models a \Rightarrow c \) implies \( M \models a \lor b \Rightarrow c \).
which can be seen as a form of the law of weakening; $M \models a \Rightarrow b \lor c$ implies $M \models a \Rightarrow c$; $M \models a \Rightarrow b$ and $M \models b \Rightarrow c$ imply $M \models a \Rightarrow c$ which can be seen as a form of the transitivity of $\models$.

With each $M \subseteq L$ and $a \in L$ which serves as an antecedent of a formula we associate a set of all consequents $b \in L$ such that $M |\models a \Rightarrow b$. Sets of this form are introduced in the following definition.

**Definition 7.** Let $S$ be an $L$-parameterization. For any $M \subseteq L$ we define operators $C_M : L \to 2^L$ and $c_M : L \to L$ by

\begin{align}
C_M(a) &= \{b \in L; M |\models a \Rightarrow b\}, \\
c_M(a) &= \bigvee C_M(a),
\end{align}

for all $a \in L$; $c_M(a)$ is called the semantic $S$-closure of $a \in L$ (using $M$).

The following assertion shows that $C_M(a)$ is closed under arbitrary suprema and thus $c_M(a)$ belongs to $C_M(a)$.

**Lemma 10.** Let $M \subseteq L$ and let $S$ be an $L$-parameterization. For any $a \in L$, we have $c_M(a) \in C_M(a)$.

**Proof.** Take any $\{b_i \in L; i \in I\} \subseteq C_M(a)$. We prove that the supremum of $\{b_i \in L; i \in I\}$ belongs to $C_M(a)$ and thus we obtain the proof of Lemma 10 as a particular case of this observation. Using (45), $b_i \in C_M(a)$ ($i \in I$) means that $M |\models a \Rightarrow b_i$ ($i \in I$). Thus, if $f(a) \leq m$ for some $m \in M$ and $(f, g)$ of the form (14), then $f(b_i) \leq m$ for all $i \in I$. Therefore, $\bigvee \{f(b_i); i \in I\} \leq m$. Now, we can apply (8) to get $f(\bigvee \{b_i; i \in I\}) \leq m$. Hence, $M |\models a \Rightarrow \bigvee \{b_i; i \in I\}$ and so $\bigvee \{b_i; i \in I\} \in C_M(a)$. \hfill \qed

As an immediate consequence of Lemma 10 we get that $c_M(a)$ is the greatest element of $C_M(a)$. Hence, using (45) together with Lemma 10 we get the following corollary which generalizes the standard criterion for classic attribute implications being true in dyadic formal contexts, cf. Remark 7 (b).

**Corollary 11.** For any $M \subseteq L$ and $a, b \in L$, $M \models a \Rightarrow b$ iff $b \leq c_M(a)$. \hfill \qed
In the following observations, we show that $c_M$ defined by $10$ is an $S$-closure operator. Actually, we prove more than that. The operator is an $S'$-closure operator for $S'$ being the $L$-parameterization generated by $S$.

**Lemma 12.** Let $M \subseteq L$, $S$ be an $L$-parameterization, and $a, b \in L$ such that $M \models a \Rightarrow b$. Then, $M \models f(a) \Rightarrow f(b)$ for all $(f, g)$ of the form $(11)$.

**Proof.** Take any $(f_1, g_1), (f_2, g_2)$ of the form $(11)$ and suppose that $f_1(f_2(a)) \leq m$ for some $m \in M$. Since $M \models a \Rightarrow b$ and $f_1 \circ f_2$ is also of the form $(11)$, it readily follows that $f_1(f_2(b)) \leq m$. Hence, $M \models f_2(a) \Rightarrow f_2(b)$ as a consequence of the fact that $(f_1, g_1)$ was taken arbitrarily. \hfill $\square$

**Theorem 13.** Let $M \subseteq L$ and let $S$ be an $L$-parameterization. Then, $c_M$ defined by $10$ is an $S'$-closure operator where $S' = \langle S', \circ, \langle 1, 1 \rangle \rangle$ is the monoid generated by $S$.

**Proof.** First, observe that the fact that $(f, g)$ is of the form $(11)$ is equivalent to stating that $(f, g)$ belongs to $S'$. Using this observation we prove that $c_M$ is indeed an $S'$-closure operator.

For any $a \in L$, we trivially have $M \models a \Rightarrow a$ and thus $a \in C_M(a)$, meaning that $a \leq c_M(a)$. Thus, $c_M$ satisfies $(31)$.

In order to prove $(32)$, take $a_1, a_2 \in L$ such that $a_1 \leq a_2$. Let $b \in C_M(a_1)$. By definition, we have $M \models a_1 \Rightarrow b$. Now, take any $m \in M$ and $(f, g) \in S'$ such that $f(a_2) \leq m$. Using $a_1 \leq a_2$ and $(9)$, we get $f(a_1) \leq f(a_2) \leq m$ and so $f(b) \leq m$. That is, $M \models a_2 \Rightarrow b$, i.e., $b \in C_M(a_2)$. Therefore, $C_M(a_1) \subseteq C_M(a_2)$ and thus $c_M(a_1) \leq c_M(a_2)$, proving $(32)$.

It remains to check that $(33)$ is satisfied. Using $(15)$ and $(16)$, it suffices to check that $b \leq g(c_M(a))$ provided that $b \in C_M(g(c_M(a)))$. Note that the fact that $b \in C_M(g(c_M(a)))$ means $M \models g(c_M(a)) \Rightarrow b$. Using Lemma 12 the last observation gives $M \models f(g(c_M(a))) \Rightarrow f(b)$. Furthermore, using $(5)$, it follows that $M \models c_M(a) \Rightarrow f(b)$, see Remark $(b)$. Applying $M \models a \Rightarrow c_M(a)$, which is a consequence of Lemma 10 together with the transitivity of $\leq$, we get $M \models a \Rightarrow f(b)$. Hence, $f(b) \leq c_M(a)$ and $b \leq g(c_M(a))$ owing to $(14)$. \hfill $\square$
Our last observation in this subsection shows that \( c_M \) can also be introduced without any explicit reference to \( |= \). The next assertion shows a way to compute \( c_M(a) \) for any \( a \in L \) using the data (i.e., the subset \( M \)) and the utilized parameterization.

**Theorem 14.** Let \( S \) be an \( L \)-parameterization and let \( S' = \langle S', \circ, \langle 1, 1 \rangle \rangle \) be the monoid generated by \( S \). Then,

\[
c_M(a) = \bigwedge \{ g(m); m \in M, (f, g) \in S', \text{ and } a \leq g(m) \}, \quad (47)
\]

for any \( M \subseteq L \) and any \( a \in L \).

**Proof.** Using Definition 6, (1), and the fact that infima in complete lattices can be expressed as suprema of lower cones, we get

\[
c_M(a) = \bigvee \{ b \in L; M |= a \Rightarrow b \} \\
= \bigvee \{ b \in L; \text{ for all } m \in M, (f, g) \in S', f(a) \leq m \Rightarrow f(b) \leq m \} \\
= \bigvee \{ b \in L; \text{ for all } m \in M, (f, g) \in S', a \leq g(m); b \leq g(m) \} \\
= \bigwedge \{ g(m); m \in M, (f, g) \in S', \text{ and } a \leq g(m) \},
\]

which proves the claim. \( \Box \)

**Remark 8.** Returning to Remark 7(a), it can be shown that (47) generalizes the operator which for any \( A \subseteq Y \) returns the intent generated by \( A \). Indeed, taking into account the special case described in Remark 7(a), applying (47), we get

\[
c_M(A) = \bigcap \{ \{ x \}^{\uparrow I}; x \in X \text{ and } A \subseteq \{ x \}^{\uparrow I} \} \\
= A^{\uparrow I},
\]

where \( \uparrow I \) is the composition of the usual concept-forming operators [29, 30]. Hence, Corollary 11 and Theorem 14 yield \( M |= A \Rightarrow B \text{ iff } B \subseteq A^{\uparrow I} \) which is the standard criterion for \( A \Rightarrow B \) being true in the formal context \( I \), see [30, p. 80] for more details.
Example 2. Our general definition of if-then formulas true in given $M \subseteq L$ under an $L$-parameterization $S$ encompasses several approaches which appeared earlier. We have already shown in Remark 7 that this is true for the classic attribute implications. In addition, the present approach generalizes the notions of formulas being true in data with graded attributes and temporal data.

(a) In the approaches to fuzzy attribute implications parameterized by hedges and similarity-based functional dependencies with the same type of parameterization [7], we can consider the same parameterization as in Example 1(b). In that case, $A \Rightarrow B$ is true in $I: X \times Y \rightarrow L$ to degree 1 iff $M \models A \Rightarrow B$ in the sense of Definition 6 for $M = \{I_x; x \in X\}$ where $I_x: Y \rightarrow L$ such that $I_x(y) = I(x, y)$ for all $x \in X$ and all $y \in Y$, see [8] for details. Analogous observation can be made for the approach to similarity-based functional dependencies in [9]. Let us add that both [8] and [9] deal with graded notion of truth of fuzzy attribute implications, i.e., in general the papers consider degrees (other than 0 and 1) to which formulas are true in given data. By the formalism presented in this section, we capture only the concept of being true to degree 1 which may be seen as a limitation, however, as it is shown in [53, Theorem 22], in the presence of general parameterizations, the graded notion of truth of fuzzy attribute implications can be expressed solely using the concept of full truth (i.e., truth to degree 1).

(b) From the point of view of attribute implications annotated by time points that have been introduced in [50], we can consider the same corresponding parameterization as in Example 1(c). According to [50], the input data can be seen as a triadic context [42] with conditions being time points in $Z$. Thus, for $I \subseteq X \times Y \times Z$ and $A, B \subseteq Y \times Z$ (where both $A$ and $B$ are finite), $A \Rightarrow B$ is considered true in $I$, see [50], whenever for every $i \in Z$ and $x \in X$, we have that

if for any $\langle y, z \rangle \in A$ we have $\langle x, y, z + i \rangle \in I$,

then for any $\langle y, z \rangle \in B$ we have $\langle x, y, z + i \rangle \in I$.

As in the previous cases, it can be easily seen that $A \Rightarrow B$ is true in $I$ in sense of [50] iff $M \models A \Rightarrow B$ according to Definition 6 for $M = \{I_x; x \in X\}$ where
5 Properties of S-closure operators

In this section, we present further properties of S-closure operators. We present equivalent ways to define such parameterized operators and present S-closure systems which are related to S-closure operators in a way which is analogous to the relationship of the classic closure systems and operators. In order to simplify notation, \( L = \langle L, \leq \rangle \) always stands for a complete lattice and \( \langle S, \circ, \langle 1, 1 \rangle \rangle \) always stands for an \( L \)-parameterization.

**Theorem 15.** An operator \( c : L \to L \) is an S-closure operator in \( L \) iff it satisfies (31) and

\[
a \leq g(c(b)) \text{ implies } c(a) \leq g(c(b)),
\]

holds for all \( a, b \in L \) and all \( \langle f, g \rangle \in S \).

**Proof.** First, we show that each S-closure operator satisfies (48). Suppose that for \( a, b \in L \) we have \( a \leq g(c(b)) \). Using (32) and (33), it readily follows that \( c(a) \leq c(g(c(b))) \leq g(c(b)) \), showing that (48) holds.

Conversely, we show that assuming (31) and (48), it follows that the operator \( c \) satisfies (32) and (33). First, assume that \( a \leq b \) and using (31) and the fact that \( \langle 1, 1 \rangle \in S \), we get

\[
a \leq b \leq c(b) = 1(c(b)).
\]

Therefore, applying (48) for \( g = 1 \), it follows that

\[
c(a) \leq 1(c(b)) = c(b)
\]

which proves that (32) is satisfied. In order to show that (33) holds, it suffices to consider \( g(c(a)) \leq g(c(a)) \) and apply (48) to get (33). \[ \square \]
Theorem 16. An operator $c: \mathbf{L} \to \mathbf{L}$ is an $S$-closure operator in $\mathbf{L}$ iff it satisfies (31), (34), and

$$a \leq g(b) \implies c(a) \leq g(c(b))$$

holds for all $a, b \in \mathbf{L}$ and all $\langle f, g \rangle \in S$.

Proof. The fact that each $S$-closure operator satisfies (49) follows by Theorem 15 using the fact that (49) is a weaker condition than (48). Indeed, if $c$ is an $S$-closure operator and $a \leq g(b)$, then we get $a \leq c(g(b))$ owing to (31) and (7).

Thus, $c(a) \leq g(c(b))$ is a consequence of (48). Conversely, let $c$ satisfy (31), (34), and (49). Observe that the isotony (32) follows directly by (49) for $g = 1$. Thus, it suffices to check that $c$ satisfies (33). Take any $a \in \mathbf{L}$ and $\langle f, g \rangle \in S$.

Using $g(c(a)) \leq g(c(a))$ and (49), we get

$$c(g(c(a))) \leq g(c(c(a))).$$

Therefore, (33) is a consequence of the previous inequality, (7), and (34). \qed

Remark 9. The requirement of (34) being satisfied in Theorem 16 cannot be dropped. For instance, if $S = \{\langle 1, 1 \rangle\}$ and $\mathbf{L} = \{0, a, 1\}$ such that $0 < a < 1$, then we can consider an isotone, extensive, and non-idempotent operator $c: \mathbf{L} \to \mathbf{L}$ defined by $c(0) = a$ and $c(a) = c(1) = 1$. Obviously, such an operator satisfies both (31) and (49) but it is not an $S$-closure operator.

Note that using (11), we can express (48) and (49) in terms of the lower adjoints in $S$ instead of the upper adjoints in $S$. The following assertions show further properties of $S$-closure operators related to lower and upper adjoints.

Theorem 17. Let $c: \mathbf{L} \to \mathbf{L}$ be an $S$-closure operator in $\mathbf{L}$. Then,

$$f(c(a)) \leq c(f(a)), \quad (50)$$

$$c(g(a)) \leq g(c(a)), \quad (51)$$

$$f(c(g(a))) \leq g(c(f(a))), \quad (52)$$

hold for all $a \in \mathbf{L}$ and any $\langle f, g \rangle \in S$. 34
Proof. In order to prove (50), we apply (31) to get \( f(a) \leq c(f(a)) \) and thus \( a \leq g(c(f(a))) \) owing to (1). Using (32), we further get \( c(a) \leq c(g(c(f(a)))) \).

Now, applying (33), it follows that \( c(a) \leq g(c(f(a))) \). Hence, \( f(c(a)) \leq c(f(a)) \) using (1).

To see that (51) holds, we start with \( a \leq c(a) \), which is an instance of (31), and apply (7) to get \( g(a) \leq g(c(a)) \). Thus, \( c(g(a)) \leq c(g(c(a))) \) as a consequence of (32). Hence, using (33), we obtain \( c(g(a)) \leq g(c(a)) \) which proves (51).

Now, (52) is a consequence of (50) and (51). Indeed, (50) and (1) yield \( a \leq b \) implies \( f(c(a)) \leq c(f(b)) \), analogously (51) and (1) give \( f(c(g(a))) \leq g(c(f(b))) \). Then, (52) follows using the transitivity of \( \leq \).

Since \( S \)-closure operators as well as all the upper and lower adjoints are isotone, the inequalities from the previous assertion can be generalized to if-then conditions which can be seen as generalized isotony conditions.

**Corollary 18.** Let \( c : L \to L \) be an \( S \)-closure operator in \( L \). Then,

\[
\begin{align*}
  a \leq b & \implies f(c(a)) \leq c(f(b)), \\
  a \leq b & \implies c(g(a)) \leq g(c(b)), \\
  a \leq b & \implies f(c(g(a))) \leq g(c(f(b))),
\end{align*}
\]

hold for all \( a, b \in L \) and any \( \langle f, g \rangle \in S \).

**Remark 10.** The converse inequalities to those in (50) and (51) do not hold in general. In case of (50), we can take \( L = \{0, 1\} \) with \( 0 < 1 \) and \( \langle f, g \rangle \) such that \( f(0) = f(1) = 0 \) and \( g(0) = g(1) = 1 \). In this setting, \( c : L \to L \) such that \( c(0) = c(1) = 1 \) is an \( S \)-closure operator for \( S = \{\langle f, g \rangle, \langle 1, 1 \rangle\} \). In addition, we have \( f(c(1)) = 0 \) and \( c(f(1)) = 1 \), i.e., \( c(f(a)) \not\leq f(c(a)) \) for \( a = 1 \).

In case of (51), consider \( L = \{0, a, 1\} \) such that \( 0 < a < 1 \). Furthermore, a pair \( \langle f, g \rangle \) of maps defined by \( f(0) = g(0) = 0, f(a) = f(1) = a, \) and \( g(a) = g(1) = 1 \) is an isotone Galois connection in \( L \). In addition, \( c \) defined by \( c(0) = c(a) = a \) and \( c(1) = 1 \) is an \( S \)-closure operator for \( S = \{\langle f, g \rangle, \langle 1, 1 \rangle\} \) such that \( g(c(0)) = 1 \not\leq a = c(g(0)) \).
Theorem 19. Let $c : L \to L$ be a closure operator in $L$. Then, the following conditions are equivalent:

(i) $c$ is an $S$-closure operator in $L$;

(ii) $c$ satisfies (50);

(iii) $c$ satisfies (51).

Proof. Using Theorem 17, it follows that (i) implies both (ii) and (iii). Therefore, it suffices to show that either of (ii) and (iii) implies (33).

Suppose that (ii) holds. First, $f(c(g(c(a)))) \leq c(f(g(c(a))))$ holds true because it is a particular case of (50). Furthermore, using (53) and (32), it follows that $f(g(c(a))) \leq c(a)$ and $c(f(g(c(a)))) \leq c(c(a))$. Applying (34) to the last inequality, we get $c(f(g(c(a)))) \leq c(a)$. Hence, the transitivity of $\leq$ together with (1) give $f(c(g(c(a)))) \leq c(a)$ and $c(g(c(a))) \leq g(c(a))$ which proves (33).

In case of (iii), the argument is straightforward: We have $c(g(c(a))) \leq g(c(c(a)))$ as a particular case of (51). Thus, using (54) together with (7), we get $c(g(c(a))) \leq g(c(a))$, i.e., (33) holds.

Let us recall that the $L^*$-closure operators and $L_K$-closure operators, which are important examples of $S$-closure operators on complete residuated lattices of fuzzy sets, were defined as closure operators with stronger isotony conditions. In contrast, $S$-closure operators have been introduced as operators with the classic isotony condition and with stronger form of the idempotency. Based on our observations, $S$-closure operators can also be seen as closure operators with stronger isotony conditions. Indeed, it is easy to see that (53) and (54) imply (50) and (51), respectively. In addition, (53) and (54) imply the ordinary isotony condition for $f = 1$ and $g = 1$, respectively. Therefore, using Theorem 19, we come to the following conclusion:

Corollary 20. Let $c : L \to L$ be an operator in $L$ which satisfies (31) and (34).

Then, the following conditions are equivalent:
(i) \( c \) is an \( S \)-closure operator in \( L \);

(ii) \( c \) satisfies (53);

(iii) \( c \) satisfies (54). \( \square \)

We now turn our attention to closure systems related to \( S \)-closure operators. The systems can be seen as systems closed under applications of upper adjoints of all isotone Galois connections in \( S \).

**Definition 8.** Let \( L = \langle L, \leq \rangle \) be a complete lattice and let \( S \) be an \( L \)-parameterization. A system \( S \subseteq L \) is called an \( S \)-closure system in \( L \) whenever it is a closure system in \( L \) such that \( g(a) \in S \) for any \( \langle f, g \rangle \in S \) and any \( a \in L \). In addition, if \( S = \langle S, \circ, \langle 1, 1 \rangle \rangle \) is an \( L \)-parameterization, then \( S \) is called an \( S \)-closure system.

The requirements of being closed under arbitrary infima and being closed under applications of upper adjoints can be replaced by a single condition as it is shown in the following assertion.

**Theorem 21.** A system \( S \subseteq L \) is an \( S \)-closure system in \( L \) iff

\[
\wedge \{g(b); a \leq g(b), b \in S, \text{ and } \langle f, g \rangle \in S\} \in S
\]  

(56)

for any \( a \in L \).

**Proof.** In order to show the only-if part, observe that for each \( b \in S \), we have that \( g(b) \in S \), i.e., (56) is a consequence of the fact that \( S \) is closed under \( \wedge \) and applications of \( g \) of any \( \langle f, g \rangle \in S \).

Conversely, let \( S \subseteq L \) satisfy (56) for any \( a \in L \). Take any \( c \in S \) and \( \langle f, g \rangle \in S \). Putting \( a = g(c) \), (56) yields

\[
g(c) \leq \wedge \{g(b); g(c) \leq g(b), b \in S, \text{ and } \langle f, g \rangle \in S\} \in S,
\]

because \( g(c) \leq g(b) \) for all \( b \in S \) and \( \langle f, g \rangle \in S \) and thus the infimum in the previous inequality is an infimum of elements which are greater than or equal to
Therefore, in order to prove that $g(c) \in S$, it suffices to check the converse inequality but this is easy to see since $c \in S$. Indeed, the fact that $c \in S$ yields

$$g(c) \in \{g(c); \langle f, g \rangle \in S\}$$

$$= \{g(c); g(c) \leq g(c) \text{ and } \langle f, g \rangle \in S\}$$

$$\subseteq \{g(b); g(c) \leq g(b), b \in S, \text{ and } \langle f, g \rangle \in S\}.$$

As a consequence, we have

$$g(c) = \bigwedge \{g(b); g(c) \leq g(b), b \in S, \text{ and } \langle f, g \rangle \in S\} \in S,$$

which shows that $S$ is closed under applications of all $g$’s. The fact that $S$ is closed under arbitrary infima can be proved by similar arguments: Take $R \subseteq S$ and observe that

$$\bigwedge R \leq \bigwedge \{g(b); \bigwedge R \leq g(b), b \in S, \text{ and } \langle f, g \rangle \in S\} \in S.$$

Moreover, for any $c \in R \subseteq S$, we have that

$$c \in \{1(b); \bigwedge R \leq 1(b), \text{ and } b \in S\}$$

$$\subseteq \{g(b); \bigwedge R \leq g(b), b \in S, \text{ and } \langle f, g \rangle \in S\}.$$

As a consequence, for any $c \in R \subseteq S$, it follows that

$$\bigwedge \{g(b); \bigwedge R \leq g(b), b \in S, \text{ and } \langle f, g \rangle \in S\} \leq c$$

Therefore, we have

$$\bigwedge \{g(b); \bigwedge R \leq g(b), b \in S, \text{ and } \langle f, g \rangle \in S\} \leq \bigwedge R,$$

Now, we use (56) for $a = \bigwedge R$ to conclude that $\bigwedge R \in S$. □

Remark 11. Let us note that $S$-closure operators and $S$-closure systems in $L$ are related in a way which is fully analogous to the relationship of the classic closure operators and closure systems. That is, given an $S$-closure operator $c$, the set $S_c = \{a \in L; c(a) = a\} = \{c(a); a \in L\}$ of all fixed points of $c$ is an $S$-closure system; given an $S$-closure system $S$, an operator $c_S : L \to L$ defined by
\[ c_S(a) = \bigwedge \{ b \in S; a \leq b \} \] for any \( a \in L \) is an \( S \)-closure operator. Furthermore, we have that \( S = S_{c_S} \) and \( c = c_{S_c} \). These claims are routine to check and use similar arguments as in the classic setting.

We conclude this section with notes on a partial order which can be introduced on the set of all isotone Galois connections in \( L \) and its relationship to \( L \)-parameterizations. For any isotone Galois connections \( \langle f_1, g_1 \rangle \) and \( \langle f_2, g_2 \rangle \) in \( L \), we put \( \langle f_1, g_1 \rangle \leq \langle f_2, g_2 \rangle \) whenever

\[ f_1(a) \leq f_2(a) \] (57)

for all \( a \in L \). Obviously, \( \leq \) defined by (57) is a partial order on the set of all isotone Galois connections in \( L \). Indeed, its reflexivity and transitivity follow directly from the properties of the complete lattice order \( \leq \) in \( L \). In addition, the antisymmetry of \( \leq \) follows by applying the antisymmetry of \( \leq \) together with the fact that each isotone Galois connection is uniquely given by its lower adjoint. It is easy to see that \( \leq \) can be defined in an equivalent way in terms of the upper adjoints in \( S \) instead of the lower ones:

**Lemma 22.** For \( \leq \) defined as in (57), we have \( \langle f_1, g_1 \rangle \leq \langle f_2, g_2 \rangle \) iff

\[ g_2(a) \leq g_1(a) \] (58)

for all \( a \in L \).

**Proof.** Let \( f_1(a) \leq f_2(a) \) for all \( a \in L \). In particular, for \( a = g_2(b) \) where \( b \in L \), using (6) and (5), it follows that

\[ f_1(g_2(b)) \leq f_2(g_2(b)) \leq b. \]

Therefore, by adjointness, we get \( g_2(b) \leq g_1(b) \). Since \( b \in L \) was taken arbitrarily, this proves that (58) holds. Analogously, we can check that assuming (58), we get that (57) holds. \( \square \)

The following assertion shows that \( \leq \) defined by (57) is compatible with composition of isotone Galois connections.
Lemma 23. Let \( \leq \) be defined as in (57). If \( \langle f_1, g_1 \rangle \leq \langle f_2, g_2 \rangle \) and \( \langle f_3, g_3 \rangle \leq \langle f_4, g_4 \rangle \), then
\[
\langle f_1, g_1 \rangle \circ \langle f_3, g_3 \rangle \leq \langle f_2, g_2 \rangle \circ \langle f_4, g_4 \rangle.
\]

Proof. Take any \( a \in L \). Since \( \langle f_3, g_3 \rangle \leq \langle f_4, g_4 \rangle \), we have \( f_3(a) \leq f_4(a) \). Furthermore, \( \langle f_1, g_1 \rangle \leq \langle f_2, g_2 \rangle \) applied together with (6) yields
\[
f_1(f_3(a)) \leq f_2(f_3(a)) \leq f_2(f_4(a)).
\]
The rest follows from the fact that \( a \in L \) was chosen arbitrarily. \( \Box \)

As a consequence of the previous lemma, in case of \( L \)-parameterizations, i.e., \( L \)-parameterizations which are closed under \( \circ \), we in fact deal with partially-ordered monoids, i.e., partially-ordered algebras whose operations are compatible with the underlying partial order:

Corollary 24. Let \( S = \langle S, \circ, \{1, 1\} \rangle \) be an \( L \)-parameterization and denote by \( \leq \) the relation on \( S \) defined as in (57). Then, \( S = \langle S, \leq, \circ, \{1, 1\} \rangle \) is a partially-ordered monoid. \( \Box \)

Remark 12. Depending on particular families of \( L \)-parameterizations, we can get even stronger properties of \( S = \langle S, \leq, \circ, \{1, 1\} \rangle \). In this remark, we comment on properties related to the existence of extremal elements with respect to \( \leq \).

(a) Put \( f_\perp(a) = 0 \) and \( g_\perp(a) = 1 \) for all \( a \in L \). Clearly, \( \langle f_\perp, g_\perp \rangle \) is an isotone Galois connection in \( L \). If \( \langle f_\perp, g_\perp \rangle \in S \), then it is the least element of \( S \) with respect to \( \leq \). Thus, each \( L \)-parameterization can be extended to an \( L \)-parameterization with a least element.

(b) Consider \( f_\top \) and \( g_\top \) defined by
\[
f_\top(a) = \begin{cases} 0, & \text{if } a = 0, \\ 1, & \text{otherwise}, \end{cases} \quad g_\top(a) = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise}. \end{cases}
\]
for all \( a \in L \). Recall that in the context of fuzzy sets, \( g_\top \) corresponds to the globalization \( 48 \), cf. (29) and \( f_\top \) is a dual hedge \( 52 \) which was also investigated on linear Gödel chains in \( 2 \). By moment’s reflection, we get that \( \langle f_\top, g_\top \rangle \)
is an isotone Galois connection in $L$. In addition, for any $a \in L$ and any isotone Galois connection $\langle f, g \rangle$ in $L$, we have $f(a) \leq f(\top)(a)$ (observe that (1) and $0 \leq g(0)$ give $f(0) = 0$). Thus, if $\langle f_\top, g_\top \rangle \in S$, then it is the greatest element of $S$ with respect to $\leq$. Recall that $\langle 1, 1 \rangle$ is neutral with respect to $\circ$ but as we have just shown, it may not be the greatest element of $S$ with respect to $\leq$. As a consequence, partially-ordered monoids of $L$-parameterizations described in Corollary 24 are not integral [28] in general. Note that $\langle 1, 1 \rangle$ is neutral with respect to $\circ$. In other words, if we consider any $L^*$-closure operator $c : L^Y \rightarrow L^Y$ (a fuzzy closure operator with the monotony condition parameterized by $^*$) and view it as a special case of an $S$-closure operator with $S$ being an $L$-parameterization as in Corollary 2, then $S$ is not only a partially-ordered monoid but in addition $\leq$ is a complete lattice order on $S$ and the composition $\circ$ of isotone Galois connections restricted to $S$ has its residuum satisfying the adjointness property (11):

**Theorem 25.** Let $L$ be a complete residuated lattice, $^*$ be an idempotent truth-stressing hedge, and let $S = \{\langle f_{a^* \otimes} g_{a^* \rightarrow} \rangle; a \in L \}$. Then,

$$S = \langle S, \leq, \circ, \rightarrow, \langle f_\bot, g_\bot \rangle, \langle 1, 1 \rangle \rangle,$$

(60)

where $\rightarrow$ is a binary operation on $S$ defined by

$$\langle f_{a^* \otimes} g_{a^* \rightarrow} \rangle \rightarrow \langle f_{b^* \otimes} g_{b^* \rightarrow} \rangle = \bigvee \{ \langle f_{c^* \otimes} g_{c^* \rightarrow} \rangle; a^* \otimes c^* \leq b^*, c \in L \},$$

(61)

for all $a, b \in L$, is a complete (commutative integral) residuated lattice.

**Proof.** First, observe that $\circ$ given by (10) and restricted to $S$ is obviously commutative which follows directly using the commutativity of $\otimes$ in $L$. Furthermore, $\langle f_\bot, g_\bot \rangle = \langle f_{0^* \otimes} g_{0^* \rightarrow} \rangle$ and $\langle 1, 1 \rangle = \langle f_{1^* \otimes} g_{1^* \rightarrow} \rangle$ are the least and the greatest elements of $S$, respectively, cf. Remark 12(b). Therefore, $S$ is a bounded
commutative integral partially-ordered monoid. Furthermore, it is also easy to see that for any \( a, b \in L \), we have

\[
\langle f_{a^* \otimes} g_{a^* \rightarrow} \rangle \leq \langle f_{b^* \otimes} g_{b^* \rightarrow} \rangle \text{ iff } a^* \leq b^*.
\] (62)

Moreover, \( S \) is closed under arbitrary suprema—this follows from the fact that a truth-stressing hedge on \( L \) satisfying (20)–(24) is an interior operator, i.e., \( \bigvee \{a_i^*; i \in I\} \) is a fixed point of \( ^* \). Hence,

\[
\bigvee \{\langle f_{a_i^* \otimes} g_{a_i^* \rightarrow} \rangle; i \in I\} = \langle f_{\bigvee \{a_i^*; i \in I\} \otimes} g_{\bigvee \{a_i^*; i \in I\} \rightarrow} \rangle \in S.
\]

In order to show that \( \circ \) and \( \hookrightarrow \) defined by (61) satisfy the adjointness property, we first show that \( \circ \) is distributive over \( \bigvee \) which is a condition equivalent to stating that \( \circ \) has a residuum satisfying the adjointness property. For any \( a \in L \) and any \( b_i \in L \) (\( i \in I \)), our previous observations together with (10) and the fact that \( \otimes \) is distributive over \( \bigvee \) in \( L \) yield

\[
\langle f_{a^* \otimes} g_{a^* \rightarrow} \rangle \circ \bigvee \{\langle f_{b_i^* \otimes} g_{b_i^* \rightarrow} \rangle; i \in I\} = \langle f_{\bigvee \{a^* \otimes b_i^*; i \in I\} \otimes} g_{\bigvee \{a^* \otimes b_i^*; i \in I\} \rightarrow} \rangle = \bigvee \{\langle f_{a^* \otimes b_i^*} g_{a^* \otimes b_i^* \rightarrow} \rangle; i \in I\}.
\]

Hence, there is uniquely given \( \rightarrow \) such that \( \circ \) and \( \hookrightarrow \) satisfy (11). Using standard properties of complete residuated lattices \( [28] \), we have

\[
\langle f_{a^* \otimes} g_{a^* \rightarrow} \rangle \rightarrow \langle f_{b^* \otimes} g_{b^* \rightarrow} \rangle = \bigvee \{\langle f_{c^* \otimes} g_{c^* \rightarrow} \rangle; \langle f_{a^* \otimes} g_{a^* \rightarrow} \rangle \circ \langle f_{c^* \otimes} g_{c^* \rightarrow} \rangle \leq \langle f_{b^* \otimes} g_{b^* \rightarrow} \rangle\}.
\]

Therefore, it remains to show that exactly \( \hookrightarrow \) defined by (61) is the residuum of \( \circ \) in \( L \), i.e., \( \hookrightarrow \) coincides with \( \rightarrow \). Using (62) together with the definition of
≤ and the latter observation, we get
\[
\langle f_{a^* \otimes}, g_{a^* \rightarrow} \rangle \rightarrow \langle f_{b^* \otimes}, g_{b^* \rightarrow} \rangle = \\
\bigvee \{ \langle f_{c^* \otimes}, g_{c^* \rightarrow} \rangle \in S; \langle f_{a^* \otimes}, g_{a^* \rightarrow} \rangle \circ \langle f_{c^* \otimes}, g_{c^* \rightarrow} \rangle \leq \langle f_{b^* \otimes}, g_{b^* \rightarrow} \rangle \} = \\
\bigvee \{ \langle f_{c^* \otimes}, g_{c^* \rightarrow} \rangle \in S; \langle f_{a^* \otimes c^* \otimes}, g_{(a^* \otimes c^*) \rightarrow} \rangle \leq \langle f_{b^* \otimes}, g_{b^* \rightarrow} \rangle \} = \\
\bigvee \{ \langle f_{c^* \otimes}, g_{c^* \rightarrow} \rangle \in S; a^* \otimes c^* \leq b^* \} = \\
\bigvee \{ \langle f_{c^* \otimes}, g_{c^* \rightarrow} \rangle; a^* \otimes c^* \leq b^* \text{ and } c \in \mathbb{L} \}
\]
which shows that \( \sim \) defined by (61) coincides with \( \rightarrow \). □

6 Conclusion

In this paper, we have extended and developed the theory of closure operators and corresponding closure systems parameterized by systems of isotone Galois connections. We have shown that the parameterizations may be viewed from two basic standpoints: First, as requirements on stronger isotony conditions of closure operators. Second, as requirements on stronger idempotency conditions of closure operators. From the point of view of the corresponding closure systems, the parameterizations represent additional requirements on relationship between elements in a closure system: The presence of an element in a parameterized closure system implies the presence of other elements obtained by applying upper adjoints of the utilized parameterization. In addition to the investigation of properties of the closure structures, we have presented two extended examples of areas where such operators naturally appear. As we have shown, the operators appear as operators of syntactic consequence in various types of non-classical logics of if-then rules which, as special cases, include logics of fuzzy and temporal if-then rules. We have also shown that such operators naturally emerge in the analysis of dependencies in data and, again, as special cases, several approaches in the analysis of graded and temporal data can be seen as special cases.

Future research in several directions is promising. First, it may be interesting to further investigate properties of structures of \( \mathbb{L} \)-parameterizations based
on ways in which the parameterizations can be induced. The first step in this
direction can be found in Section 5, most notably in Theorem 25 saying that for
parameterizations induced by hedges the structure is a complete residuated lat-
tice. Second, a thorough analysis of properties which are necessary and sufficient
in order to establish analogs of results known in further applications would be
desirable. Third, the class of parameterizations studied in this paper is much
more general than those studied earlier. Examples and applications in other
areas than fuzzy and temporal logics may be anticipated.

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