Analysis of the Transformation of Random Signals and Noise Using Poly-Gaussian Models

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Abstract. Analysis performed transformation of random signals and noise in linear and nonlinear systems based on the use of poly-Gaussian models and multidimensional PDF of the output paths of information-measuring and radio systems. The classification of elements of these systems, as well as expressions describing the input action and output response of the system are given. It is shown that the analysis of information-measuring and systems can be carried out using poly-Gaussian models. The analysis is carried out with a series connection of a linear system and a nonlinear element, a series connection of a nonlinear element and a linear system, as well as with a parallel connection of the named links. The output response in all cases will be a mixture of a poly-Gaussian distribution with a number of components. An example of the analysis of signal transmission through an intermediate frequency amplifier and a linear detector against a background of non-Gaussian noise is given. The resulting probability density distribution of the sum of the signal and non-Gaussian noise at the output of the detector will be poly-Rice. The multidimensional probability distribution density of the output processes of the nonlinear signal envelope detector is also obtained. The results of modeling the found distribution densities are presented. It is shown that the use of the poly-Gaussian representation of signals and noise, as well as the impulse response of the system, makes it possible to effectively analyze inertial systems in the time domain.

1. Introduction

The rapid development of information-measuring and radio engineering systems (EMRES), in particular, aerospace, solving a number of different tasks related to control, radio telemetry, radio communication, etc., put the agenda of increasing noise immunity among one of the main problems of radio engineering.

As you know, the equipment of aerospace systems is exposed to various kinds of influences associated with intense vibration, temperature changes, pressure, radioactive exposure, etc., which ultimately causes vibroacoustic noise in the operating equipment, as well as various types of additive-multiplicative noise. In addition, the operation of radio lines is greatly influenced by refraction, as well as the Earth's surface, causing, up to tens of decibels, noise, irregularity of the directional pattern of ground-based transceivers.

In EMRES the signals and the noises affecting them are random. Moreover, their probability density function (PDF), as a rule, is described by a non-Gaussian distribution law [1–4 etc.].

The non-Gaussian nature of signals and noises, the nonlinearity of their processing procedures and restrictions on the feasibility of these procedures lead to the inadequacy of using only Gaussian models.
and related linear processing algorithms that characterize the correlation theory. Therefore, it is necessary to apply new methods of analysis and synthesis of EMRES, one of which is based on the use of poly-Gaussian models to describe signals and noise [5-8 etc.].

The aim of this work is to analyze the transformation of random signals and noise in linear and nonlinear systems based on the use of poly-Gaussian models and multidimensional PDF of the output paths of EMRES.

2. Analysis of the transformation of random signals and noise into equipment of information-measuring and radio engineering systems

As a rule, when carrying out the task of analyzing the transformation of random processes in linear and nonlinear systems, it is assumed that the parameters of the system and the statistical characteristics of the input process $\alpha(t)$ are known. It is required to find the statistical characteristics of the process $\beta(t)$ resulting from the output of the system.

Two types of problems are usually investigated. This is either the definition of a system of moment functions at the output, or multidimensional distribution laws of the output process. The most complete statistical characteristics of a random process are multidimensional probability distribution laws. The solution of the first problem can be obtained from the solution of the second problem.

Let us consider and analyze multidimensional PDFs of the output processes of typical paths of these systems. Before moving on to them, let us consider the classification of the main elements that make up EMRES.

3. Classification of information-measuring and radio engineering system elements

As you know, the elements of EMRES are usually divided into two main groups: inertial and non-inertial. In turn, both of these systems can be both linear and non-linear.

Non-linear inertial elements. The characteristic of a nonlinear inertial element can be specified as a nonlinear transformation

\[ y(t) = f\left[ x(t) \right], \quad (1) \]

where $x$ refers to the input and $y$ refers to the output. Moreover, the value $y(t)$ at a given time $t$ is determined by the value only at that time $t$.

In many cases, characteristics of the form (1) quite accurately describe the operation of a large group of radio engineering elements: modulators, detectors, mixers, frequency converters, etc.

Inertial systems. Process values $y(t)$ at the output of the inertial system depend not only on the process value $x(t)$ acting at the output at the same time moment $t$, but also from its value at other times.

Linear inertial systems. These systems are characterized by the fact that the process $y(t)$ is obtained by superposition (addition) of all values $x(t)$, each of which is multiplied by a weighting factor $k(\tau)$ depending both on the moment of observation $t$ of the process at the output of the system and on the moment of application $\tau$ to the entrance.

Thus, at the output of the linear system, the process $y(t)$ can be expressed through the process at the entrance $x(t)$ using the integral

\[ y(t) = \int_0^t k(t, \tau)x(\tau)d\tau. \quad (2) \]

Function $k(t, \tau)$ received the name impulse transient function. It fully characterizes a linear system, the parameters of which change over time. Such systems and the processes occurring in them are called parametric.

For linear systems with constant parameters, the impulse transient function depends only on the difference $(t - \tau)$ moments of observation at the output and application of influence on the input of the system, that is $k(t, \tau) = k(t - \tau)$. 
In this case, expression (2) is transformed to the form
\[ y(t) = \int_0^t k(t-\tau) x(\tau) d\tau. \]

This formula can be used to calculate the output signals of differentiating and integrating circuits, filters, amplifiers, etc.

Linear systems with time-invariable parameters are subdivided into linear systems with lumped constant and distributed constant parameters.

Linear systems with lumped parameters include delay lines, differentiating and integrating chains, ladder filters, multichannel amplifiers, single and coupled circuits, etc. Long lines and waveguides are related to linear systems with distributed constant parameters.

Nonlinear inertial systems. Such systems can be characterized by some nonlinear integro-differential equation, multidimensional impulse responses, multidimensional transfer function.

In a nonlinear inertia system, the impact \( x(t) \) and response \( y(t) \) connects with each other the Volterra series, which is a generalization of the Duhamel integral (2)
\[ y(t) = Q_0 + \sum_{i=1}^{b} \prod_{j=1}^{v} Q_v(t, \tau_1, \ldots, \tau_v) \int_0^t x(\tau) d\tau, \]
where \( Q_v(t, \tau_1, \ldots, \tau_v) \), \( v = 1, 2, \ldots, b \) are the nuclei of the Volterra series.

If a \( Q_v = 0 \) for all \( j > 1 \), then we obtain a linear inertial system, and \( Q_v(t, \tau) \) is impulse transient function of a linear system. Adding members of series (3) at \( j > 1 \) means the introduction of nonlinearity. A collection of cores \( Q_0, Q_1, \ldots, Q_v \) characterizes a nonlinear filter of the \( v \)-th order.

4. Multidimensional probability distribution densities of the output processes of typical paths
As is known, many problems of transforming random processes in statistical radio engineering are reduced to the analysis of various compounds of linear inertial and inertialess nonlinear elements. In general, the analysis of such devices can be carried out on the basis of poly-Gaussian models of real disturbances [8, 9].

Let the analyzed device consist of a series-connected linear inertial system with an impulse transient response \( k(t, \tau) \) and a nonlinear inertia-free element with an amplitude characteristic \( z(t) = f[y(t)] \) (Fig. 1).

![Figure 1. Block diagram of the series connection of a linear system and a non-linear element.](image-url)

Let's find a multidimensional PDF at the output of a linear inertial system. When analyzing linear systems, it is convenient to use poly-Gaussian models of processes because of their invariance under linear transformations.

We assume that a random process, signal or noise, is poly-Gaussian. That is, their corresponding probability distribution law is \( F(.) \) and PDF \( W(.) \) in discrete form can be represented by Gaussian mixtures:
\[ F(.) = \sum_k p_k F_k(.); \quad W(.) = \sum_k p_k W_k(.) ; \quad \sum_k p_k = 1, \]
where $F_k(.)$ is Gaussian probability distribution, $W_k(.)$ is Gaussian PDF, $p_k$ is the likelihood of the presence of individual components.

Notice, that $\{p_k\}$ is «weighing» factor satisfying the normalization condition.

Applying the well-known expressions for the linear function of a random argument to (4), we obtain that the reaction of the linear system to the poly-Gaussian action is also poly-Gaussian for the same number $N$ and probabilities $p_n$, component, that is

$$W(y) = \sum_{n=1}^{N} p_n W_n(y, m_{n,\text{out}}, M_{n,\text{out}}); \quad \sum_{n} p_n = 1,$$

(5)

where $m_{n,\text{out}}$, $M_{n,\text{out}}$ are respectively, the vector of mathematical expectation and the correlation matrix of the Gaussian components of the output process $y$; $l$ is determines the dimension of the PDF (5) of the output process, $T$ is observation interval, $0 \leq t \leq T$.

Expectation vectors $m_{n,\text{out}}$ and correlation matrices $M_{n,\text{out}}$ Gaussian components of the output process $y$ and characteristics $k(t, \tau)$ of the analyzed system are connected by known expressions

$$m_{n,\text{out}}(t) = \int_{0}^{t} k(t, \tau) m_{n,\text{in}}(\tau) d\tau;$$

$$M_{n,\text{out}}(t_1,t_2) = \int_{0}^{t} \int_{0}^{t} k(t_1, \tau_1) k(t_2, \tau_2) M_{n,\text{in}}(\tau_1, \tau_2) d\tau_1 d\tau_2.$$

Note that in this case the dimension $l$ The PDF of the output process (5) can be written arbitrarily.

Multidimensional PDF $W(z)$ (see Fig. 1), the output process $z = \{z(t_1), z(t_2), \ldots, z(t_l)\}$ can be represented as

$$W(z) = \sum_{n} p_n (2\pi)^{-l/2} |M_{n,y}|^{-0.5} \sum_{k=1}^{K} I_{mk} \exp\left\{-0.5 \left(y_k(z) - m_{n,y}\right)^{\top} M_{n,y}^{-1} \left(y_k(z) - m_{n,y}\right)\right\},$$

(6)

where $y_k(z)$ is the k-th branch of the inverse function $y(z)$, $I_{mk}$ is the Jacobian of the transformation of the k-th branch from $y$ to $z$.

If the analyzed system consists of a series connection of an inertial nonlinear element and a linear inertial device (Fig. 2), then using the poly-Gaussian representation of each of the non-Gaussian distributions at the output of the non-linear element, for example, the term in the index $n$ in expression (6), we again obtain a mixture of the poly-Gaussian distribution with the total number of components $K = \sum_{n=1}^{N} K_n$:

$$W(z) = \sum_{n=1}^{N} \sum_{k=1}^{K_n} p_{nk} N_n[z, m_{n,y,z}, M_{n,y,z}].$$

With the parallel connection of the considered typical links, all implementations of the output processes are added algebraically. In this case, the resulting process, being an algebraic sum of poly-Gaussian processes, is also poly-Gaussian [9].

By combining the considered methods, it is possible to reduce to a correlation equation many problems of analysis of the transformation of non-Gaussian random processes in nonlinear inertial systems, divided into linear inertial devices and inertialess nonlinear elements.

At the same time, for a wide class of practically important tasks, with random influences, the determination of the output probabilistic characteristics of devices can be brought to engineering solutions.
5. Passing a signal through an intermediate frequency amplifier and detector against a background of non-Gaussian noise

Imagine the PDF $W(U_n)$ of a non-Gaussian noise $U_n = \{U_{n,1}, \ldots, U_{n,J}\}$ where $U_{n,i} = U_{s,i}(t_i)$, $i = 1, J$, an expression of the form (4).

After the parameters are determined $\{p_n\}$, $\{M_m(t_i,t_j)\}$, $\{m_m(t_i)\}$ of all Gaussian noise components, in the steady state, using expression (5), one can determine the PDF of the sum of the signal $s(t)$ and noise $U_n(t)$ at the output of the intermediate frequency amplifier (IFA).

To obtain the PDF of the voltage at the output of the detector, we will use the joint PDF $W(U_{c,i}, U_{s,i})$ of the quadrature components $U_{c,i} = U_{c,\text{out}}(t_i)$ and $U_{s,i} = U_{s,\text{out}}(t_i)$ output voltage of the intermediate frequency amplifier.

Using the technique described in [9], we introduce the notation for the determinants $D_n$ matrices $R_{n,k}$ formed from the quadrature components of the correlation functions of the Gaussian components at the output of the IF amplifier, as well as

$$a_{in} = s_i(t_i) + m_{in,\text{out}}(t_i); \quad b_{in} = s_i(t_i) + m_{in,\text{out}}(t_i),$$

where $s_i(t_i)$, $s_j(t_j)$ and $m_{in,\text{out}}(t_i)$, $m_{in,\text{out}}(t_i)$ is, respectively, the quadrature components of the signal, and the mathematical expectations of the Gaussian components at the output of the IF amplifier.

In this case, the joint PDF of the quadrature components will be determined by the expression

$$W(U_{c,i}, U_{s,1}, \ldots, U_{c,j}, U_{s,j}) = \sum_{n=1}^{N} p_n (2\pi)^{0.5J} D_n^{0.5} \exp \left\{ -\frac{1}{2\sigma_n^2} \sum_{i,j=1}^{J} \left[ \rho_{n}^{ij}(U_{c,i} - a_i)(U_{c,j} - a_j) + \rho_{n}^{i2}(U_{c,i} - b_i)(U_{c,j} - b_j) + \rho_{n}^{i2}(U_{c,i} - a_i)(U_{s,j} - b_j) - \rho_{n}^{i2}(U_{c,j} - a_j)(U_{s,i} - b_i) \right] \right\},$$

where $\rho_{n}^{ij}$ are the elements of the matrix inverse $\|R_{n,k}\|^{ij}, k = 1,2; \quad L_n = \frac{2i-1}{2j-1}; \quad L_2 = \frac{2i-1}{2j}.$

Passing, according to [9], to polar coordinates, we write:

$$U_{c,i} = U_i \cos \Theta_i; \quad a_{in} = U_{in} \cos \varphi_{in}; \quad \rho_{n}^{ii} = \beta_{n,ij} \cos \varphi_{n,ij};$$

$$U_{s,i} = U_i \sin \Theta_i; \quad b_{in} = U_{in} \sin \varphi_{in}; \quad \rho_{n}^{i2} = \beta_{n,ij} \sin \varphi_{n,ij};$$

$$\beta_{n,ij} = \beta_{n,ji}; \quad \varphi_{n,ij} = \varphi_{n,ji}; \quad \varphi_{n,ii} = 0.$$

Integrating (7) over all $\Theta_i$, $i = 1, J$ using the notation [9], we obtain the required PDF of the sum of the signal and non-Gaussian noise at the output of the linear detector in the form of a multidimensional Polarcz PDF:
\[ W(U_1, \ldots, U_J) = \sum_{n=1}^{N} p_n \sigma_n^{-2J} D_{2n} \prod_{i=1}^{J} U_i \times \]
\[ \times \exp \left\{ -\frac{1}{2\sigma_n^2} \left[ \sum_{i=1}^{J} \beta_{n,i} \left( U_i^2 - U_i^2 \right) + 2 \sum_{i=1}^{J-1} \sum_{j>i} \beta_{n,ij} U_i U_j \cos \varphi_{n,ij} \right] \right\} \]
\[ \times \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} I_m \left( \frac{\beta_{n,1} U_{1} U_{2}}{\sigma_n^2} \right) \] \[ \times \prod_{i=1}^{J} \exp \left\{ \sum_{k=1}^{J} \beta_{n,ik} I_i \left( \frac{\beta_{n,1} U_{1} U_{2}}{\sigma_n^2} \right) \right\} \]

where \( I_m(.) \) is the modified Bessel function of the first kind of the \( m \)-th order.

When \( J = 2 \) expression (8) takes the form:

\[ W(U_1, U_2) = \sum_{n=1}^{N} p_n \sigma_n^{-2J} D_{2n} \exp \left\{ -\frac{1}{2\sigma_n^2} \left[ \beta_{n,11} U_1^2 + \beta_{n,12} U_1 U_2 + \beta_{n,22} U_2^2 + 2 \beta_{n,12} U_1 U_2 \right] \right\} \]

\[ \times \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} I_m \left( \frac{\beta_{n,12} U_{1} U_{2}}{\sigma_n^2} \right) \] \[ \times \prod_{i=1}^{J} \exp \left\{ \sum_{k=1}^{J} \beta_{n,ik} I_i \left( \frac{\beta_{n,12} U_{1} U_{2}}{\sigma_n^2} \right) \right\} \]

Note that this expression of the multidimensional PDF of the linear envelope detector (8) allows one to obtain the multidimensional PDF of the output processes of nonlinear envelope detectors \( W_{nl,ed}(U_1, \ldots, U_J) \).

For this, at the beginning, it is necessary to determine the Jacobians and inverse functions of the corresponding nonlinear elements. Then multiply the Jacobians of the transformation with expression (8), having previously replaced its arguments \( U_i = \varphi[U_i] \), \( i = 1, J \):

\[ W_{nl,ed}(U_1, \ldots, U_J) = \sum_{k} W[k\varphi(U_1), \ldots, \varphi(U_J)] J_k \]

where \( \varphi(U_j) \) is the \( k \)-th branch of the inverse function, \( J_k \) is the Jacobian of the transformation of the \( k \)-th branch of the inverse function.

Note that expressions (8), (9) are very convenient for mathematical modeling on personal computers. They allow calculating the multidimensional PDF of the system from the IF amplifier with arbitrarily specified properties of the envelope detector for arbitrary input signals and interfering noise.

Figure 3 shows a block diagram of a typical radio engineering link consisting of a linear filter (LF) and an envelope detector (ED).

\[ \begin{align*}
W_{in}(U_{in}(t_1), U_{in}(t_2)) & \rightarrow \text{LF} & W_{out}(U_{out}(t_1), U_{out}(t_2)) & \rightarrow \text{ED} & W_{ED}(U(t_1), U(t_2)) \\
U_{in}(t) & \rightarrow & U_{out}(t) & \rightarrow & U_{ED}(t)
\end{align*} \]

Figure 4 shows the results of modeling the PDF of voltages at the input and output of the IF amplifier, as well as at the output of the linear DO, calculated by expression (9).
Figure 4. PDF counts: $a$ – filter input signal; $b$ – filter output signal; $c$ – the envelope of the output signal.
Note that in the simulation it was assumed that the impulse response of the LF is described by the expression:

\[ k(t) = 4 \exp\{-1.5t\} \cos 20t. \]

The components of the signal samples, at the input of the LF, are set by the functions of the mean and covariance, described, respectively:

\[ m_n(t) = \sum_{k=1}^{2} m_{n,k} \exp\{-0.8kt\} \cos[(\omega_{n,k} + k\Omega_{m})t]; \]
\[ R_{n,k}(\tau) = \sum_{l=1}^{2} R_{n,k,l} \exp\{-0.6k|\tau|\} \cos[(\omega_{n,k} + k\Omega_{m})\tau]. \]

6. Conclusions

Reviewed analysis of the transformation of random signals and noise in the EMRES. It is shown that, thanks to the use of the poly-Gaussian representation of the input processes and the impulse response of a linear system, it is possible to obtain a poly-Gaussian method that makes it possible to analyze a wide range of elements used in EMRES. It has been shown that the impulse response and poly-Gaussian representation of input processes, signals and noise are an effective tool for analyzing nonlinear inertial systems in the time domain.

The scientific novelty of the article lies in the use of the poly-Gaussian method for the analysis and synthesis of EMRES with non-Gaussian input influences and non-linear processing of signals and noise.

The practical significance lies in the fact that the possibility of using poly-Gaussian models of random signals and noise for describing linear and non-linear elements of EMRES is shown.

7. References

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