UNIVERSALLY DEFINED REPRESENTATIONS OF LIE CONFORMAL SUPERALGEBRAS

PAVEL KOLESNIKOV

Abstract. We distinguish a class of irreducible finite representations of conformal Lie (super)algebras. These representations (called universally defined) are the simplest ones from the computational point of view: a universally defined representation of a conformal Lie (super)algebra $L$ is completely determined by commutation relations of $L$ and by the requirement of associative locality of generators. We describe such representations for conformal superalgebras $W_n$, $n \geq 0$, with respect to a natural set of generators. We also consider the problem for superalgebras $K_n$. In particular, we find a universally defined representation for the Neveu–Schwartz conformal superalgebra $K_1$ and show that the analogues of this representation for $n \geq 2$ are not universally defined.

1. Introduction

Conformal Lie (super)algebras, as introduced in [17], provide a formal language to operate with certain infinite-dimensional Lie (super)algebras in conformal field theory and (super)string theory. From the algebraic point of view, a conformal algebra is an algebraic system based on a linear space $C$ over $\mathbb{C}$ endowed with a family of bilinear operations (conformal products) $(\cdot \circ_n \cdot)$, where $n$ ranges over the set $\mathbb{Z}_+$ of non-negative integers.

In conformal field theory, $C$ is assumed to be a space of pairwise mutually local fields; if $a, b \in C$ then the sequence $a \circ_n b$, $n \in \mathbb{Z}_+$, encodes the singular part of the operator product expansion (OPE) of $a$ and $b$. The properties of OPE give rise to the axioms of (Lie) conformal algebras [17, 18]. Roughly speaking, a Lie conformal algebra is a “singular part” of a vertex operator algebra [5, 16].

The problem of classification of simple and semisimple Lie conformal superalgebras of finite type (i.e., finitely generated modules over $\mathbb{C}[D]$) was solved in [8, 13, 14, 15], see also [19].

An important role in conformal field theory and representation theory belongs to vertex operator realizations of infinite-dimensional Lie (super)algebras. These constructions lead to the notion of a conformal module [9, 17] which is equivalent to the notion of a module over a conformal algebra. Irreducible conformal modules over Lie conformal superalgebras have been studied in a series of papers. In particular, the complete list of irreducible modules over several simple conformal Lie superalgebras of finite type was found in [9, 12]; extensions of such modules were described in [11, 10].

In this paper, we develop a combinatorial approach to representations of Lie conformal superalgebras. In the case of ordinary algebras, every representation of a Lie algebra $L$ gives rise to a representation of its universal associative enveloping
algebra $U(L)$. This is not the case for conformal algebras since there is no universal associative enveloping conformal algebra for a Lie conformal algebra.

However, given a conformal Lie (super)algebra $C$ generated by its subset $B$ one may consider a class of associative envelopes of $C$ with a restriction on the locality function on $B$ [26]. There exists the universal envelope in that class, so we obtain a lattice of universal envelopes of $C$. Every irreducible conformal $C$-module of finite type corresponds to a simple homomorphic image of a universal envelope $U$ of $C$, so the first points of interest are the minimal (non-trivial) elements of the lattice of universal envelopes, namely, simple universal envelopes of at most linear growth. Every universal envelope of this kind defines a representation which is called universally defined.

We describe all universally defined representations of Lie conformal superalgebras $W_n$, $n \geq 0$, with respect to a natural set of generators. It turns out that there exists only one universally defined representation of $W_0$ (the Virasoro conformal algebra) and two inequivalent representations of $W_n$, $n > 0$. We also show that the induced representations of $K_n \subset W_n$, $n \geq 1$, are irreducible for any $n \neq 2$, equivalent to a universally defined representation for $n = 1$ (i.e., for the Neveu–Schwartz conformal superalgebra), but for $n \geq 2$ neither of these representations is universally defined.

2. Main definitions

Definition 2.1 (Kac, 1997). A conformal algebra is a linear space $C$ over a field $\mathbb{k}$ (char $\mathbb{k} = 0$) endowed with a linear map $D : C \to C$ and a family of linear maps $\circ_n : C \otimes C \to C$ satisfying the following axioms:

(C1) for any $a, b \in C$ there exists $N = N(a, b)$ such that $a \circ_n b = 0$ for all $n \geq N$;

(C2) $Da \circ_n b = -na \circ_{n-1} b$;

(C3) $a \circ_n Db = D(a \circ_n b) + na \circ_{n-1} b$.

If $C$ is a finitely generated $\mathbb{k}[D]$-module then $C$ is said to be a conformal algebra of finite type (or finite conformal algebra).

Axiom (C1) allows to define the so-called locality function $N_C : C \times C \to \mathbb{Z}_+$,

$$N_C(x, y) = \min\{N \in \mathbb{Z}_+ \mid x \circ_n y = 0 \text{ for all } n \geq N\}.$$ 

A conformal algebra $C$ is said to be $\mathbb{Z}_2$-graded if $C = C_0 \oplus C_1$ as a $\mathbb{k}[D]$-module and $C_i \circ_n C_j \subseteq C_{(i+j) \text{mod} 2}$. By $p(a)$ we denote the parity of $a \in C$: $p(a) = i$ if $a \in C_i$, $i = 0, 1$.

For any conformal algebra $C$ there exists an ordinary (non-associative, in general) algebra $A$ such that $C$ can be embedded into the space of formal power series $A[[z, z^{-1}]]$, where $D = \partial_z$ and the $\circ_n$-products on $A[[z, z^{-1}]]$ are given by

$$a(z) \circ_n b(z) = \text{Res}_{w=0} a(w)b(z)(w - z)^n, \quad n \in \mathbb{Z}_+,$$

where $\text{Res}_{w=0} f(z, w)$ stands for the coefficient of $w^{-1}$ in $f(z, w)$. Such an algebra $A$ is not unique, but there exists a universal one denoted by $\text{Coeff} C$. Namely [18,25], $\text{Coeff} C = \mathbb{k}[[t, t^{-1}] \otimes \mathbb{k}[D]]$ as a right $\mathbb{k}[D]$-module thinking of $D$ as of $-\frac{d}{dt}$. Let us write $a(n)$ for $t^n \otimes a$, $a \in C$, $n \in \mathbb{Z}$. The multiplication on $\text{Coeff} C$ is well-defined by

$$a(n)b(m) = \sum_{s \geq 0} \left(\begin{array}{c} n \\ s \end{array}\right) (a \circ_{n-s} b)(m + s).$$
The algebra $\text{Coeff } C$ is called the coefficient algebra of $C$.

There is a correspondence between identities on $\text{Coeff } C$ and conformal identities on $C$. In particular, $\text{Coeff } C$ is associative if and only if $C$ satisfies

$$ (a \circ_n b) \circ_m c = \sum_{s \geq 0} (-1)^s \binom{n}{s} a \circ_{n-s} (b \circ_{m+s} c), \quad a, b, c \in C, \quad n, m \in \mathbb{Z}_+. $$

The system of relations (2.1) is equivalent to

$$ a \circ_n (b \circ_m c) = \sum_{s \geq 0} \binom{n}{s} (a \circ_{n-s} b) \circ_{m+s} c, \quad a, b, c \in C, \quad n, m \in \mathbb{Z}_+. $$

If $C$ is $\mathbb{Z}_2$-graded then $\text{Coeff } C$ inherits the grading; $p(a(n)) = p(a)$. Coefficient algebra $\text{Coeff } C$ is a Lie superalgebra if and only if $C$ satisfies

$$ a \circ_n + (-1)^{p(a)p(b)} \{a \circ_n b\} = 0, \quad (2.3) $$

$$ a \circ_n (b \circ_m c) - (-1)^{p(a)p(b)} b \circ_m (a \circ_n c) = \sum_{s \geq 0} \binom{n}{s} (a \circ_{n-s} b) \circ_{m+s} c, \quad (2.4) $$

where $\{b \circ_n a\} = \sum_{s \geq 0} (-1)^{n+s} D^s (b \circ_{n+s} a), \quad n, m \in \mathbb{Z}_+$.

A conformal algebra $C$ is called associative, if $\text{Coeff } C$ is associative, i.e., if $C$ satisfies (2.1) or (2.2). Analogously, $C$ is called Lie conformal superalgebra, if $\text{Coeff } C$ is a Lie superalgebra, i.e., if $C$ satisfies (2.3) and (2.4). In order to distinguish notations, we will denote conformal products in associative conformal algebras by $(\cdot \circ_n \cdot)$ and in Lie conformal algebras by $(\cdot \circ \cdot)$, $n \in \mathbb{Z}_+$.

**Proposition 2.2** (e.g. Kac, 1999). Let $C$ be an associative $\mathbb{Z}_2$-graded conformal algebra. Then the same $k[D]$-module $C$ endowed with new operations $a \sqcap_n b = [a \circ_n b]$, where

$$ [a \circ_n b] = a \circ_n b - (-1)^{p(a)p(b)} \{b \circ_n a\}, \quad a, b \in C, \quad n \in \mathbb{Z}_+, $$

is a Lie conformal superalgebra denoted by $C^{(-)}$.

**Definition 2.3** (Kac, 1997). Let $V$ be a left (unital) $k[D]$-module. A conformal endomorphism is a $k$-linear map $a : k[D] \rightarrow \text{End}_k V$ such that

(i) $\text{codim}\{ h \in k[D] \mid a(h)v = 0 \} < \infty$ for any $v \in V$;

(ii) $a(h)Dv = Da(h)v + a(h')v$, $h'$ is the ordinary derivative of $h$.

Let $\text{Cend}_V$ denotes the set of all conformal endomorphisms. One may define operations $D$ and $(\cdot \circ_n \cdot)$, $n \in \mathbb{Z}_+$, on $\text{Cend}_V$ as follows:

$$ (Da)(h) = -a(h'), \quad (a \circ_n b)(h) = \sum_{s \geq 0} (-1)^s \binom{n}{s} a(D^{n-s})b(D^s h). $$

Then (C2), (C3) and (2.1) hold. If $V$ is a finitely generated $k[D]$-module then (C1) also holds, so $\text{Cend}_V$ turns into an associative conformal algebra. If $V$ is a free $N$-generated $k[D]$-module then $\text{Cend}_V$ is denoted by $\text{Cend}_N$. The structure of this algebra was particulary considered in [6].

**Definition 2.4** (Cheng et al., 1997b; Kac, 1999). Let $C$ be a Lie conformal superalgebra. A representation of $C$ on a $k[D]$-module $V$ is a linear map $\rho : L \rightarrow \text{Cend}_V$
such that
\[ \rho(Da) = D\rho(a), \]
\[ \rho(a \circ_n b) = (\rho(a) \circ_n \rho(b)) - (-1)^{\rho(a)\rho(b)} \{\rho(b) \circ_n \rho(a)\} \]
for all \( a, b \in L, n \in \mathbb{Z}_+ \). If \( V \) is a finitely generated \( k[D] \)-module then the representation \( \rho \) is said to be finite.

3. **Free conformal algebras and the Composition-Diamond lemma**

The study of free conformal algebras was initiated in [25], where free associative and Lie conformal algebras were constructed.

For a set of generators \( B \) and a locality function \( N : B \times B \to \mathbb{Z}_+ \) there exists an associative (Lie) conformal algebra \( F_N(B) \) such that for any associative (resp., Lie) conformal algebra \( C \) and for any map \( \iota : B \to C \) such that \( N_C(\iota(a), \iota(b)) \leq N(a, b) \), \( a, b \in B \), there exists a unique homomorphism \( \varphi : F_N(B) \to C \) such that \( \varphi(a) = \iota(a), a \in B \).

Let us present the construction of the free associative conformal algebra with a constant locality function \( N \) (c.f. [2]). Consider the (ordinary) free associative algebra \( k\langle v, B \rangle \), where \( v \) is a formal variable, \( v \notin B \). Free \( k[D] \)-module \( F(B) = k[D] \otimes k\langle v, B \rangle \) can be endowed with conformal products by setting
\[ (1 \otimes f) \circ_n (1 \otimes g) = 1 \otimes f^\iota g, \]
for \( f, g \in k\langle v, B \rangle \), and then by making use of (C2), (C3). These operations turn \( F(B) \) into an associative conformal algebra. Conformal subalgebra of \( F(B) \) generated by \( \{v^{N-1}a \mid a \in B\} \) is isomorphic to \( F_N(B) \).

The following monomials (normal words)
\[ w = D^s(a_1 \circ_{n_1} (a_2 \circ_{n_2} \ldots \circ_{n_{k-1}} (a_k \circ_{n_k} a_{k+1}) \ldots)), \]
\[ s \geq 0, \quad a_i \in B, \quad 0 \leq n_i < N, \]
form a linear basis of \( F_N(B) \). Linear combinations of normal words are called conformal polynomials.

Assume \( w \) to be a normal word (3.1). By \( \text{wt}(w) \) we denote the following string:
\[ \text{wt}(w) = (n_1, n_2, \ldots, n_k, s). \]
If \( B \) is endowed with a linear order \( \leq \) such that \( (B, \leq) \) is a well-ordered set then we can expand this order to normal words by the rule
\[ v \leq w \iff \text{wt}(v) \leq \text{wt}(w), \]
comparing strings (3.2) by their length first and then lexicographically. For a conformal polynomial \( f \in F_N(B) \) denote by \( \bar{f} \) its principal word:
\[ f = \alpha \bar{f} + \sum u, \quad \alpha \neq 0, \quad u < \bar{f}. \]

Every associative conformal algebra \( C \) generated by \( B \) such that \( N_C(a, b) \leq N, \ a, b \in B \), is isomorphic to the quotient algebra \( F_N(B)/I \) for some ideal \( I \). As usual, a set \( S \subseteq F_N(B) \) generating \( I \) as an ideal of \( F_N(B) \) is called a set of defining relations of \( C \). There is a natural problem: given a set \( S \) of defining relations of \( C \), how to decide whether two conformal polynomials are equal in \( C \)? In general, this problem is algorithmically unsolvable [2], but there is a generalized (infinite) algorithm to treat it, somewhat similar to the one of [28, 4, 1, 20].
Let $S \subseteq F_N(B)$ be a set of conformal polynomials. A normal word $w$ is said to be $S$-reduced if $w$ cannot be presented as a principal part of

$$D^s(u \circ_n f \circ_m v) \text{ or } D^s(u \circ_n g),$$

where $f, g \in S$, $f$ is a $D$-free polynomial, $u$ and $v$ are normal words, $0 \leq n, m < N$, $s \geq 0$.

In [2] [4], the notion of a composition $(f, g)_w$ of conformal polynomials $f, g$ was introduced. In general, there are six types of compositions of such polynomials. A set $S \subseteq F_N(B)$ is called a Gröbner–Shirshov basis (GSB) if it is closed under all compositions.

**Theorem 3.1** ([4]). Let $S$ be a set of defining relations of an associative conformal algebra $C$. If $S$ is a GSB then $S$-reduced normal words form a linear basis of $C$. The converse is true if $S$ consists of $D$-free polynomials.

### 4. Associative envelopes of Lie conformal superalgebras

Let $L$ be a conformal Lie superalgebra with operations $D$ and $(\cdot \circ_n \cdot)$, $n \in \mathbb{Z}_+$. 

**Definition 4.1.** An associative envelope of $L$ is a pair $(A, \varphi)$, where $A$ is an associative conformal algebra, $\varphi : L \rightarrow A$ is a $D$-invariant linear map such that

$$\varphi(a \circ_n b) = \varphi(a) \circ_n \varphi(b) - (-1)^{p(a)p(b)}(\varphi(b) \circ_n \varphi(a)), \quad a, b \in L, \quad n \in \mathbb{Z}_+,$$

and $A$ is generated by $\varphi(L)$ as a conformal algebra.

Note that $\varphi$ is not necessarily injective.

Two associative envelopes $(A_1, \varphi_1), (A_2, \varphi_2)$ of $L$ are isomorphic if there exists an isomorphism $\psi : A_2 \rightarrow A_1$ of associative conformal algebras such that $\psi \varphi_2 = \varphi_1$.

The set $\mathcal{E}(L)$ of isomorphism classes of associative envelopes can be ordered in the usual way: $(A_1, \varphi_1) \leq (A_2, \varphi_2)$ if there exists a homomorphism $\psi : A_2 \rightarrow A_1$ such that $\psi \varphi_2 = \varphi_1$.

In contrast to the case of ordinary algebras, the partially ordered set $(\mathcal{E}(L), \leq)$ has no greatest element. The main reason is the requirement of locality of elements $\varphi(L)$ in an associative envelope $A$. However, there is a way to fix this problem [20].

Let $B$ be a set of generators of $L$, and let $N : B \times B \rightarrow \mathbb{Z}_+$ be a fixed function. Denote by $\mathcal{E}_N(L, B)$ the set of all associative envelopes $(A, \varphi)$ of $L$ such that $N_A(\varphi(a), \varphi(b)) \leq N(a, b)$ for all $a, b \in B$. Then $\mathcal{E}_N(L, B)$ has the greatest element (the universal associative envelope with respect to generators $B$ and locality $N$) denoted by $(U_N(L, B), \tau_N(L, B))$ or just $U_N(L, B)$, for short.

Let us state here the construction of $U_N(L, B)$. Consider the coefficient algebra $\mathcal{L} = \text{Coeff } L$, this is a Lie superalgebra generated by $\{b(n) \mid b \in B, n \in \mathbb{Z}\}$. Denote by $\mathcal{I}_N(B)$ the ideal of $U(\mathcal{L})$ generated by

$$\sum_{s \geq 0} (-1)^s \binom{N(a, b)}{s} a(n - s)b(m + s), \quad a, b \in B, \quad n, m \in \mathbb{Z}. \tag{4.1}$$

Then formal power series $\bar{a}(z) = \sum_{n \in \mathbb{Z}} (a(n) + \mathcal{I}_N(B))z^{-n-1} \in U(\mathcal{L})/\mathcal{I}_N(B)[[z, z^{-1}]]$ are pairwise mutually local, therefore, they generate an associative conformal algebra that is $U_N(L, B)$.

Another way to construct $U_N(L, B)$ is to use a presentation of $L$ by generators $B$ and defining relations $S_{\text{Lie}}$ [4]. Consider the ideal $I_N(B)$ of $F_N(B)$ generated by
$S$, where $S$ is obtained from $S_{L}$ by rewriting $(\cdot \circ_n \cdot) = [\cdot \circ_n \cdot]$ via (2.9). Then $U_{N}(L, B) \simeq F_{N}(B)/I_{N}(B)$.

Note that if $B$ consists of homogeneous elements of $L$ then $U_{N}(L, B)$ inherits the grading, so $\iota_{N}(L, B): L \to U_{N}(L, B)^{(-)}$ is a homomorphism of Lie conformal superalgebras.

A superinvolution of a $\mathbb{Z}_{2}$-graded conformal algebra $C$ is a $\mathbb{k}[D]$-linear map $\sigma: C \to C$ such that $p(\sigma(a)) = p(a)$, $\sigma^{2} = \mathrm{Id}_{C}$, $\sigma(a \circ_{n} b) = (-1)^{p(a)p(b)}\{\sigma(b) \circ_{n} \sigma(a)\}$, $a, b \in C$.

Lemma 4.2. Let $L$ be a Lie conformal superalgebra generated by a subset $B$ of homogeneous elements. Superinvolution $\sigma : L \to L$, $x \mapsto -x$, can be expanded to $U_{N}(L, B)$ if and only if $N(a, b) = N(b, a)$ for all $a, b \in B$.

Proof. Consider the canonical antipode $S : U(\mathcal{L}) \to U(\mathcal{L})$, $\mathcal{L} = \text{Coeff } L$, $S(x) = -x$ for $x \in \mathcal{L}$. If $N(a, b) = N(b, a)$ for any $a, b \in B$ then the relation (4.1) holds under $S$, so $S$ induces a superinvolution of $U(\mathcal{L})/I_{N}(B)$ that can be expanded to the conformal algebra $U_{N}(L, B)$. \hfill $\square$

Any associative envelope $(A, \varphi) \in \mathcal{E}_{N}(L, B)$ is a homomorphic image of $U_{N}(L, B)$. Therefore, it is interesting to explore the cases when $U_{N}(L, B)$ is a simple conformal algebra.

This case seems to be interesting by one more reason. For a fixed set of generators $B$ one may consider the lattice $\mathcal{U}\mathcal{E}(L, B)$ of universal envelopes $U_{N}(L, B)$ as a subset of $\mathcal{E}(L)$. Assume $L^{\mathbb{Z}} = L$ (e.g., $L$ is simple). Then the lowest point of the lattice $\mathcal{U}\mathcal{E}(L, B)$ is \{0\}. The set of minimal (nonzero) points of this lattice necessarily includes all simple universal envelopes. If we extend the set of generators, i.e., consider $B' \supset B$, then $\mathcal{U}\mathcal{E}(L, B) \subseteq \mathcal{U}\mathcal{E}(L, B')$. Some of minimal points of the lattice $\mathcal{U}\mathcal{E}(L, B)$ may not be minimal in $\mathcal{U}\mathcal{E}(L, B')$. But simple universal envelopes of $\mathcal{U}\mathcal{E}(L, B)$ are always minimal in $\mathcal{U}\mathcal{E}(L, B')$.

Let $L$ be a finitely generated conformal Lie superalgebra. Any simple associative envelope of $L$ of at most linear growth defines an irreducible finite representation of $L$. Indeed, it was shown in [24, 25] that every simple finitely generated associative conformal algebra of at most linear growth can be embedded into $\text{Cend } V$, rank $V < \infty$, as an irreducible subalgebra.

Conversely, let $\rho$ be a representation of a conformal superalgebra $L$ on a finite module $V$. Denote by $A_{V}(L)$ the associative conformal subalgebra generated by $\rho(L)$ in $\text{Cend } V$. The pair $(A_{V}(L), \rho)$ is an associative envelope of $L$; two representations are equivalent if and only if the corresponding envelopes are isomorphic. If $\rho$ is irreducible then the associative conformal algebra $A_{V}(L)$ generated in $\text{Cend } V$ by $\rho(L)$ acts irreducibly on $V$. Irreducible subalgebras of $\text{Cend } V$ were completely described in [24]. In particular, $A_{V}(L)$ is a simple conformal algebra of at most linear growth.

A finite representation $\rho$ induces a locality function $N$ on $L$, therefore, on a set of generators $B \subset L$. Namely, $N(x, y) = N_{A_{V}(L)}(\rho(x), \rho(y))$, $x, y \in B$. The envelope $A_{V}(L)$ is a homomorphic image of the corresponding universal envelope $U_{N}(L, B)$. Therefore, those finite irreducible representations that appear from $A_{V}(L) \simeq U_{N}(L, B)$ are in some sense simplest ones.

Definition 4.3. Let $L$ be a conformal Lie superalgebra generated by a subset $B$. An irreducible finite representation $\rho : L \to \text{Cend } V$ is called universally defined if
A = A_V(L) \simeq U_N(L, B), where N : B \times B \to \mathbb{Z}_+ is the locality function induced by \rho, i.e., N(a, b) := N_A(\rho(a), \rho(b)).

Note that this property depends on the choice of generating set B. However, if a representation \rho is universally defined with respect to B then so is \rho with respect to any \beta' \supseteq B.

**Example 4.4.** Consider \( L = \text{Cur} \, \text{sl}_2 \) (current conformal algebra over \text{sl}_2), and let \( B = \{1 \otimes e, 1 \otimes f, 1 \otimes h\} \), where \( e, f, h \) is the standard basis of \text{sl}_2. There are no universally defined representations of \( L \) with respect to \( B \).

**Example 4.5.** For \( L = \text{Vir} = \mathbb{k}[D]v \) (Virasoro conformal algebra), \( B = \{v\} \), a universally defined representation exists and unique. Namely, if \( N(v, v) = 2 \) then \( U_N(L, B) \simeq \text{Cend}_{1,v} \) as it was actually shown in [2].

5. Universally defined representations of \( W_n \)

In this section we describe universally defined representations of \( \mathbb{Z}_2 \)-graded extensions \( W_n \), \( n \geq 0 \), of the Virasoro conformal algebra. The explicit construction of these conformal Lie superalgebras via formal power series is stated, for example, in [14]. It is easy to show that one may present \( W_n \) by generators and defining relations as follows.

**Proposition 5.1** (Kolesnikov, 2004). Conformal superalgebra \( W_n \) is generated by the set \( B = \{v, \xi_i, \partial_i \mid i = 1, \ldots, n\} \) with defining relations

\[
\begin{align*}
\xi_i \circ_0 \xi_j &= -\xi_j \circ_0 \xi_i, \quad \partial_i \circ_n \partial_j = 0, \quad n \geq 0, \\
\partial_j \circ_0 v &= \delta_{ij} v, \quad v \circ_0 \xi_i &= \xi_i \circ_0 v = D\xi_i, \quad \xi_i \circ_1 v = 2\xi_i, \quad \partial_j \circ_0 v &= 0, \\
\partial_j \circ_1 v &= \partial_j, \quad \xi_i \circ_0 \xi_j &= \frac{1}{2}D(\xi_i \circ_1 \xi_j), \quad v \circ_0 v &= Dv, \quad v \circ_1 v &= 2v, \\
\xi_i \circ_n \xi_j &= v \circ_n \partial_j = \xi_i \circ_n \partial_j = v \circ_n \xi_i = v \circ_n v = 0, \quad n \geq 2,
\end{align*}
\]

(\text{here } p(v) = 0, \ p(\xi_i) = p(\partial_i) = 1).

Let \( A_n \) be the (ordinary) associative algebra with a unit generated by the set \( \{\xi_i, \partial_i \mid i = 1, \ldots, n\} \) with the following relations:

\[
\begin{align*}
\xi_i \xi_j + \xi_j \xi_i &= 0, \quad \partial_i \partial_j + \partial_j \partial_i = 0, \\
\partial_i \xi_j + \xi_j \partial_i &= \delta_{ij}.
\end{align*}
\]

We may consider \( \mathbb{k}[D] \otimes A_n[v] \) as an associative conformal algebra with operations

\[
(1 \otimes a(v)) \circ_n (1 \otimes b(v)) = 1 \otimes a(v)\frac{\partial^n b(v)}{\partial v^n},
\]

\( a(v), b(v) \in A_n[v] \). Since \( A_n \simeq \text{M}_{2n}(\mathbb{k}) \), the associative conformal algebra obtained is isomorphic to \( \text{Cend}_{1,v} \). This algebra is \( \mathbb{Z}_2 \)-graded with respect to the usual grading on \( A_n[v] \) \( (p(v) = 0, \ p(\xi_i) = p(\partial_i) = 1) \).

**Proposition 5.2.** The following maps define homomorphisms of conformal Lie superalgebras \( W_n \to (\mathbb{k}[D] \otimes A_n[v])^{(-1)} \):

\[
\begin{align*}
\varphi_1 : v &\mapsto v - D, \ \xi_i \mapsto (v - D)\xi_i, \ \partial_i \mapsto \partial_i; \\
\varphi_2 : v &\mapsto v, \ \xi_i \mapsto v\xi_i, \ \partial_i \mapsto \partial_i.
\end{align*}
\]

**Proof.** It is enough to check that the relations (5.1) hold under \( \varphi_k \), \( k = 1, 2 \). The computation is straightforward. \( \square \)
Denote by $C_k$ ($k = 1, 2$) the associative conformal subalgebra of $k[D] \otimes A_n[v]$ generated by $\varphi_k(W_n)$. To write down the explicit form of these algebras, let us fix an isomorphism $k[D] \otimes A_n[v] \to \text{Cend}_{2^n}$ as follows. If we identify the Grassman algebra $\Lambda_n(\xi_1, \ldots, \xi_n)$ with the $2^n$-dimensional vector space over $k$ then $A_n$ turns into the full algebra of linear transformations of this space. Let us fix a linear basis $(e_1, \ldots, e_{2^n})$ of $\Lambda_n$ in such a way that $e_1 = 1$, $e_{2^n} = \xi_1 \cdots \xi_n$ and identify $A_n \simeq \text{End} \Lambda_n$ with $M_{2^n}(k)$.

Then $C_1$ maps onto $\text{Cend}_{2^n, Q} \simeq M_{2^n}(k[D, v])Q(v - D)$, where $Q(v) = \text{diag}(v, 1, \ldots, 1)$. Analogously, conformal algebra $C_2$ can be identified with $\tilde{Q} = \text{diag}(1, \ldots, 1, v)$. Analogously, conformal algebra $C_2$ can be identified with $\text{Cend}_{\tilde{Q}, 2^n} \simeq \tilde{Q}(v)M_{2^n}(k[D, v])$, where $\tilde{Q} = \text{diag}(1, \ldots, 1, v)$.

Let us compute the locality functions $N_k : B \times B \to \mathbb{Z}_+$, $k = 1, 2$, where $N_k(x, y) = N_{C_k}(\varphi_k(x), \varphi_k(y))$, $x, y \in B$.

| $x$ | $y$ |
|-----|-----|
| $v$ | $\xi_1 \ldots \xi_n \partial_1 \ldots \partial_n$ |
| $2$ | $2$  |
| $2$ | $2$  |
| $2$ | $2$  |
| $2$ | $0$  |
| $1$ | $1$  |
| $1$ | $1$  |
| $0$ | $0$  |
| $\partial_n$ | $2$  |
| $\partial_n$ | $2$  |
| $\partial_n$ | $2$  |
| $\partial_n$ | $1$  |
| $\partial_n$ | $1$  |
| $\partial_n$ | $0$  |

| $x$ | $y$ |
|-----|-----|
| $v$ | $\xi_1 \ldots \xi_n \partial_1 \ldots \partial_n$ |
| $2$ | $2$  |
| $2$ | $2$  |
| $2$ | $2$  |
| $2$ | $0$  |
| $1$ | $1$  |
| $1$ | $1$  |
| $1$ | $1$  |
| $0$ | $0$  |
| $\partial_n$ | $2$  |
| $\partial_n$ | $2$  |
| $\partial_n$ | $2$  |
| $\partial_n$ | $1$  |
| $\partial_n$ | $1$  |
| $\partial_n$ | $0$  |

Although conformal algebras $C_1$ and $C_2$ are isomorphic, the associative envelopes $(C_1, \varphi_1)$ and $(C_2, \varphi_2)$ are not isomorphic for $n > 0$ (hence, the corresponding representations are not equivalent). The reason is that $N_1(v, \partial_1) = 2 \neq N_2(v, \partial_1) = 1$. For $n = 0$ these envelopes are isomorphic: they correspond to the universally defined representation of the Virasoro conformal algebra from Example 1.5.

**Theorem 5.3.** For $n > 0$ there exist exactly two universally defined representations of $W_n$ with respect to $B = \{v, \xi_i, \partial_i \mid i = 1, \ldots, n\}$. Namely, these representations correspond to the associative envelopes $(C_1, \varphi_1)$ and $(C_2, \varphi_2)$.

**Proof.** In [22], the Gröbner–Shirshov basis $S_1$ of $U_{N_1}(W_n, B)$ was found. The set of $S_1$-reduced conformal words consists of

\[ (5.4) \quad D^i((v \circ_0)^n \xi_{i_1} \circ_0 \cdots \circ_0 \xi_{i_s} \circ_0 \partial_{j_1} \circ_0 \cdots \circ_0 \partial_{j_t}), \quad n > 0, \quad s, q \geq 0, \]

\[ (5.5) \quad D^j((\xi_{i_1} \circ_0 \cdots \circ_0 \xi_{i_{s+1}} \circ_1 \cdots \circ_1 \xi_{i_s} \circ_1 \partial_{j_1} \circ_0 \cdots \circ_0 \partial_{j_t}), \quad 1 \leq r \leq s, \quad q \geq 0, \]

\[ (5.6) \quad D^k((\xi_{i_1} \circ_1 \cdots \circ_1 \xi_{i_{s+1}} \circ_0 \cdots \circ_0 \partial_{j_s} \circ_0 \cdots \circ_0 \partial_{j_t}), \quad s \geq 0, \quad q \geq 0, \]

where $1 \leq i_1 < \cdots < i_s \leq n$, $1 \leq j_1 < \cdots < j_q \leq n$, $t \geq 0$; by default, we assume the bracketing is right-justified. Here we use the following order on $B$: $v < \xi_1 < \cdots < \xi_n < \partial_1 < \cdots < \partial_n$.

The map $\varphi_1 : W_n \to C_1$ can be extended to a homomorphism $U_{N_1}(W_n, B) \to C_1$ because of the choice of $N_1$. Let us also denote this homomorphism by $\varphi_1$. It is easy to see that the images of (5.4)–(5.6) are linearly independent in $C_1$, so $(C_1, \varphi_1)$ is the universal associative envelope corresponding to the locality function $N_1$ on $B$. Therefore, $\varphi_1$ is a universally defined representation with respect to $B$. 

Now, let us show that $C_2 \simeq U_{N_2}(W_n, B)$. The initial set of defining relations of $U_{N_2}(W_n, B)$ appears from (5.1):

\[
\begin{align*}
(5.7) & \quad \partial_i \circ_0 \xi_j + \xi_j \circ_0 \partial_i = \delta_{ij} v, \\
(5.8) & \quad 2(\xi_i \circ_0 \xi_j + \xi_j \circ_0 \xi_i) = D(\xi_i \circ_1 \xi_j + \xi_j \circ_1 \xi_i), \quad i \neq j, \\
(5.9) & \quad \partial_i \circ_0 \partial_j + \partial_j \circ_0 \partial_i = 0, \quad i \neq j, \\
(5.10) & \quad v \circ_0 \xi_i - \xi_i \circ_0 v + D(\xi_i \circ_1 v) = D\xi_i, \\
(5.11) & \quad \xi_i \circ_0 v - v \circ_0 \xi_i + D(v \circ_1 \xi_i) = D\xi_i, \\
(5.12) & \quad \xi_i \circ_1 v + v \circ_1 \xi_i = 2\xi_i, \\
(5.13) & \quad \partial_i \circ_0 v - v \circ_0 \partial_i = 0, \\
(5.14) & \quad \partial_i \circ_1 v = \partial_i, \quad v \circ_1 v = v.
\end{align*}
\]

**Lemma 5.4.** The following relations hold on $U_{N_2}(W_n, B)$:

\[
\begin{align*}
(5.15) & \quad \xi_i \circ_0 \xi_j = -\xi_j \circ_0 \xi_i, \quad \xi_i \circ_1 \xi_j = -\xi_j \circ_1 \xi_i, \\
(5.16) & \quad v \circ_0 \xi_i = \xi_i \circ_0 v, \quad \xi_i \circ_1 v = \xi_i, \quad v \circ_1 \xi_i = \xi_i, \\
(5.17) & \quad \xi_i \circ_1 (\xi_j \circ_0 \xi_k) = 2\xi_j \circ_0 (\xi_j \circ_0 \xi_k), \quad i < j < k, \\
(5.18) & \quad \xi_i \circ_1 (\xi_j \circ_0 v) = 2\xi_j \circ_0 \xi_j, \quad i < j, \\
(5.19) & \quad \partial_i \circ_1 (\xi_j \circ_0 v) = 2\partial_i \circ_0 \xi_j, \\
(5.20) & \quad \partial_k \circ_1 \xi_j \circ_0 \xi_k = 2\partial_k (\xi_j \circ_1 \xi_k), \quad i < j.
\end{align*}
\]

*Proof.* To deduce the required relations, we are going to perform the Buchberger–Shirshov algorithm for conformal algebras starting with relations (5.7)–(5.14). Define the order of conformal monomials as in (3.3) assuming $v > \xi_n > \cdots > \xi_1 > \partial_n > \cdots > \partial_1$.

Consider (5.7) for $i = j$ and multiply with $\xi_j \circ_0$ and $\circ_0 \xi_j$ to obtain $\xi_j \circ_0 v = \xi_j \circ_0 \partial_j \circ_0 \xi_j = v \circ_0 \xi_j$.

Multiplying (5.12) with $\circ_1 v$ and applying (5.14) we obtain $v \circ_1 (\xi_j \circ_1 v) = v \circ_1 \xi_j$. The same relations allow to compute the left-hand side: $v \circ_1 (\xi_j \circ_1 v) = (v \circ_1 \xi_j \circ_1 v) = v \circ_1 \xi_j \circ_1 v + 2\xi_j \circ_0 v$. Therefore, $\xi_j \circ_1 v = v \circ_1 \xi_j = \xi_j$, and proved. To get the remaining relations (5.15), one can multiply (5.8) with $\circ_1 v$ and $\circ_2 v$.

Relation (5.18) appears as the composition of intersection $(f, g)_w$ [4], where $f = v \circ_0 \xi_j - \xi_j \circ_0 v, g = \xi_j \circ_1 v - \xi_j, w = \xi_i \circ_0 v \circ_0 \xi_j$.

To deduce (5.17), consider the composition of intersection $(f, g)_w$, where $f = \xi_i \circ_1 (\xi_j \circ_0 v) - 2\xi_i \circ_0 \xi_j - \xi_k, w = \xi_i \circ_1 \xi_j \circ_0 v \circ_1 \xi_k$.

Relations (5.19) and (5.20) can be obtained in a similar way. □

Let $S_2$ stands for the set of relations (5.7), (5.9), (5.12)–(5.20). We do not need (5.8), (5.10), (5.11) any more since these relations follow from (5.15), (5.16). The set of $S_2$-reduced normal words consists of

\[
D^n(\partial_j \circ_0 \cdots \partial_j \circ_0 \xi_i \circ_0 \cdots \circ_0 \xi_i (v \circ_0)^n \circ_0 v), \quad n \geq 0, \quad s, k \geq 0,
\]

\[
D^t(\partial_j \circ_0 \cdots \partial_j \circ_0 \xi_i \circ_0 \cdots \circ_0 \xi_i (v \circ_0 \cdots \circ_0 \xi_i), \quad 1 \leq r \leq s, \quad k \geq 0,
\]

\[
D^k(\partial_j \circ_0 \cdots \partial_j \circ_0 \xi_i \circ_0 \cdots \circ_0 \xi_i), \quad k > 0, \quad s, k \geq 0,
\]

where $t \geq 0, 1 \leq j_1 < \cdots < j_k \leq n, 1 \leq i_1 < \cdots < i_s \leq n$.

There exists a homomorphism $U_{N_2}(W_n, B) \to C_2$ extending $\varphi_2 : W_n \to C_2$. Let us denote it also by $\varphi_2$. It is easy to compute the images of (5.21) under $\varphi_2$: these
are
\[ D^t \otimes \partial_j \ldots \partial_j \xi_i \ldots \xi_i, v^{n+s}, \quad n > 0, \quad s, k \geq 0, \]
\[ D^t \otimes \partial_j \ldots \partial_j \xi_i \ldots \xi_i, v^r, \quad 1 \leq r \leq s, \quad k \geq 0, \]
\[ D^t \otimes \partial_j \ldots \partial_j \xi_i \ldots \xi_i, \quad k > 0, \quad s \geq 0, \]
respectively. The images obtained are linearly independent in \( k[D] \otimes A_n[v] \), hence, the homomorphism \( \varphi_2 : U_{N_2}(W_n, B) \to C_2 \) is an isomorphism of universal envelopes.

We have proved that the associative envelope \((C_2, \varphi_2)\) of \( W_n \) gives rise to a universally defined representation of \( W_n \) which is not equivalent (for \( n > 0 \)) to the representation coming from \((C_1, \varphi_1)\). Let us show that there are no other universally defined representations with respect to the set of generators \( B \) (as well as to any greater set of generators \( B' \supseteq B \)).

Assume that an associative envelope \((C, \varphi)\) of \( W_n \) corresponds to a universally defined representation with respect to \( B \), i.e., \( C \simeq U_N(W_n, B) \), where \( N(x, y) = N_C(\varphi(x), \varphi(y)) \), \( x, y \in B \).

If there exists \( k \in \{1, 2\} \) such that \( N_k(x, y) \leq N(x, y) \) for all \( x, y \in B \) then \( C_k \) is a homomorphic image of \( C \) (the homomorphism would preserve \( B \)). Since \( C \) is necessarily simple \([24]\), the associative envelope \((C, \varphi)\) should be isomorphic to \((C_k, \varphi_k)\), so the representations \( \varphi \) and \( \varphi_k \) are equivalent.

Hence, we have to assume that there exist \( x_1, y_1, x_2, y_2 \in B \) such that \( N(x_1, y_1) < N_1(x_1, y_1) \) and \( N(x_2, y_2) < N_2(x_2, y_2) \). But \( U_N(W_n, B) \) cannot be simple for such \( N \). To show that, it is sufficient to consider several cases. Let us focus on some of these cases to show the technique.

Suppose \( N(v, \xi_i) < 2 \) for some \( i \). Denote \( \xi = \xi_i \), \( \partial = \partial_i \) and proceed as follows. Defining relations \((5.1)\) imply \( \xi \circ_1 v = 2v, \xi \circ_0 v - v \circ_0 \xi = D \xi \). For any \( m \geq 0 \) we have
\[
\xi \circ_m \xi = \frac{1}{m+1} D \xi \circ_{m+1} \xi = -\frac{1}{m+1} (\xi \circ_0 v - v \circ_0 \xi) \circ_{m+1} \xi
= \frac{1}{m+1} v \circ_0 (\xi \circ_{m+1} \xi).
\]
Since \( \xi \circ_m \xi = 0 \) for a sufficiently large \( m \), we may conclude that \( N(\xi, \xi) = 0 \). Now, multiply the defining relation
\[
\partial \circ_0 \xi + \xi \circ_0 \partial - D(\xi \circ_1 \partial) + \cdots = v
\]
with \( \circ_0 \xi \) and \( \xi \circ_0 \). Then \( v \circ_0 \xi = \xi \circ_0 \partial \circ_0 \xi = \xi \circ_0 v \), so \( D \xi = 0 \) in \( U_N(W_n, B) \).

Hence, \( \varphi : W_n \to C \) is not injective and \( C = 0 \).

In the same way, one may get \( C = 0 \) assuming that \( N(v, \partial_i) < 1 \) for some \( i \in \{1, \ldots, n\} \).

If there exist \( i \neq j \) such that \( N(\partial_i, \partial_j) < 1 \) then \( N(\partial_i, \partial_j) = N(\partial_j, \partial_i) = 0 \). Consider the relation
\[
v = [\partial_i \circ_0 \xi_i] = \partial_i \circ_0 \xi_i + \{\xi_i \circ_0 \partial_i\}
\]
and multiply with \( \circ_m \partial_j, m \geq 0 \). Then \( v \circ_m \partial_j = \partial_i \circ_0 (\xi_i \circ_m \partial_j) = -\partial_i \circ_0 \{\partial_j \circ_m \xi_i\} = 0 \). Hence, \( N(v, \partial_j) = 0 \) but this case has already been explored.

Probably, the most difficult case is when \( N(\xi_i, \partial_j) < 2 \) and \( N(\partial_i, \xi_k) < 2 \) for some \( i, j, k \in \{1, \ldots, n\} \). Since \( \xi_i \circ_m \partial_j = 0 \) for \( m \geq 1 \), we have
\[
0 = \partial_i \circ_0 (\xi_i \circ_m \partial_j) = (v - \{\xi_i \circ_0 \partial_i\}) \circ_m \partial_j = v \circ_m \partial_j - \xi_i \circ_0 (\partial_i \circ_0 \partial_j).
\]
Note that \( \partial_i \circ_0 \partial_j + \{ \partial_j \circ_0 \partial_i \} = 0 \), so
\[
\xi_i \circ_m (\partial_i \circ_0 \partial_j) = -\xi_i \circ_m \{ \partial_j \circ_0 \partial_i \} = -\{ (\xi_i \circ_m \partial_j) \circ_0 \partial_i \} = 0,
\]
therefore, \( v \circ_m \partial_j = 0 \) for any \( m \geq 1 \).

Relation \( \partial_l \circ_m \xi_k = 0 \) \((m \geq 1)\) implies
\[
0 = \{(\partial_l \circ_m \xi_k) \circ_0 \partial_l \} = \partial_l \circ_m (v - \partial_k \circ_0 \xi_k).
\]
But
\[
\partial_l \circ_m (\partial_k \circ_0 \xi_k) = \{ \partial_l \circ_0 \partial_k \} \circ_m \xi_k = - (\partial_k \circ_0 \partial_l) \circ_m \xi_k = 0,
\]
so \( \partial_l \circ_m v = 0 \) for any \( m \geq 1 \).

We have obtained \( N(v, \partial_j) \leq 1, N(\partial_l, v) \leq 1 \). If \( j = l \) then the result is obvious; if \( j \neq l \) then \( \partial_j \circ_0 v = \partial_j, v \circ_0 \partial_l = \partial_l, N(\partial_j, v) = 2 \), and for any \( m \geq 0 \)
\[
\partial_l \circ_m \partial_j = \partial_l \circ_m (\partial_j \circ_0 v) = - \{ \partial_l \circ_m \partial_l \} \circ_0 v = (-1)^{m+1} (\partial_j \circ_{m+1} \partial_l) \circ_0 v.
\]

Hence, \( N(\partial_l, \partial_l) = 0 \), but this case has already been explored.

In the same way, all other choices of \( x_1, y_1, x_2, y_2 \in B \) also lead to \( C = 0 \). Therefore, there are no more universally defined representations of \( W_n \) with respect to \( B \). \( \Box \)

**Corollary 5.5.** The set \( S_2 \) of relations (5.7), (5.9), (5.13)–(5.20) is a Gröbner–Shirshov basis of \( U_{N_2}(W_n, B) \).

6. ON UNIVERSALLY DEFINED REPRESENTATIONS OF \( K_n \)

Consider the linear map \( \wedge_n \oplus \sum_{j=1}^n \wedge_n \partial_j \to W_n \) defined by
\[
\begin{align*}
1 & \mapsto v, \\
\partial_j & \mapsto \partial_j, \quad i = 1, \ldots, n, \\
\xi_I = \xi_{i_1} \cdots \xi_{i_r} & \mapsto \frac{1}{2^{r-1}} \xi_{i_1} \cdots \xi_{i_r}, \quad 1 \leq r \leq n, \\
\xi_I \partial_j = \xi_{i_1} \cdots \xi_{i_r} \partial_j & \mapsto \frac{1}{2^{r-1}} (\xi_{i_1} \cdots \xi_{i_r}) \partial_1 \partial_j, \quad 1 \leq r \leq n,
\end{align*}
\]
where \( I = \{i_1, \ldots, i_r\} \subseteq \mathcal{N} := \{1, \ldots, n\}, i_1 < \cdots < i_r \), the bracketing is assumed to be right-justified. Since the map is injective, we will identify elements of \( W_n \) with their preimages under this map.

Conformal superalgebra \( K_n \) is a subalgebra of \( W_n \) generated (as a \( k[D] \)-module) by the elements
\[
(6.1) \quad g_I = (2 - |I|) \xi_I + (-1)^{|I|} \sum_{i=1}^n (D_\xi_I \xi_I \partial_i + \partial_i (\xi_I) \partial_i), \quad I \subseteq \mathcal{N}.
\]

Universally defined representations of \( W_n \) induce finite representations of \( K_n \):
\[
\psi_k = \varphi_k|_{K_n} : K_n \to C_k \cong \begin{cases} 
\text{Cend}_{2^n, Q}, \quad Q = \text{diag}(v, 1, \ldots, 1), & \text{if } k = 1; \\
\text{Cend}_{Q, 2^n}, \quad Q = \text{diag}(1, \ldots, 1, v), & \text{if } k = 2.
\end{cases}
\]

Let \( B \) stands for the set \( \{ g_I \mid I \subseteq \mathcal{N} \} \).

**Theorem 6.1.**

1. \( (C_1, \psi_1), (C_2, \psi_2) \) are associative envelopes of \( K_n \) for \( n \neq 2 \). Therefore, the induced representations are irreducible.
(2) If \( n = 1 \) then the envelopes \((C_1, \psi_1)\) and \((C_2, \psi_2)\) are isomorphic. Moreover, the corresponding representation of the Neveu–Schwartz conformal superalgebra \(K_1\) is universally defined with respect to \(B\).

(3) If \( n > 2 \) then neither of \(\psi_1, \psi_2\) is a universally defined representation with respect to \(B\).

**Proof.** (1) Let us compute the images of \(g_I\) under \(\psi_k, \, k = 1, 2\), as elements of \(\mathbb{K}[D] \otimes A_n[v]\):

\[
(6.2) \quad \psi_1(g_I) = (2 - |I|)(v - D)\xi_I + (-1)^{|I|} \sum_{i=1}^{n} (D\xi_i \partial_i + \partial_i (\xi_i) \partial_i),
\]

\[
(6.3) \quad \psi_2(g_I) = (2 - |I|)v \xi_I - \sum_{i=1}^{n} (D\partial_i \xi_I + \partial_i \cdot \partial_i (\xi_i)).
\]

For a subset \(I \subseteq \mathcal{N}\) and an index \(i \in \mathcal{N}\), set

\[
\alpha(i, I) = \begin{cases} 0, & i \in I, \\ (-1)^{|I \setminus \{j\}|}, & i \notin I. \end{cases}
\]

Then \(\xi_I \partial_i \xi_J = \alpha(i, I)\xi_{I \cup \{i\}}\partial_i\xi_{I \cup \{i\}} = \alpha(i, I)\xi_I\partial_i\). It is also easy to observe that

\[
\xi_i \partial_i \xi_J = \begin{cases} \xi_J, & i \in J, \\ (-1)^{|J|} \alpha(i, J)\xi_{I \cup \{i\}}\partial_i, & i \notin J. \end{cases}
\]

Consider the associative conformal algebra \(B_n\) generated by \(\psi_1(K_n)\) in \(C_1\) for \(n \neq 2\). It is straightforward to compute that

\[
(6.4) \quad \psi_1(g_{\emptyset}) \circ \psi_1(g_{\emptyset \setminus \{i\}}) = (-1)^{n-i}(4 - 2n)\xi_{\emptyset} \partial_i,
\]

\[
(6.5) \quad \psi_1(g_{\{i\}}) \circ \xi_{I \cup \{j\}} \partial_k = -\alpha(i, I)\xi_{I \cup \{i\}} \partial_k - (-1)^{|I|+1} \xi_{I \cup \{i\}} \partial_i \partial_k.
\]

It follows from \((6.4)\) that \(\xi_{\emptyset} \partial_i \in B_n\) for any \(i \in \mathcal{N}\).

Let us show by induction on \(|I|\) that \(\xi_I \partial_j \in B_n\) for any \(I \subseteq \mathcal{N}, \, j \in \mathcal{N}\).

For \(|I| = \mathcal{N}\) we are done. For a smaller \(I\), assume that \(\xi_i \partial_j \in B_n\) for all \(i \in \mathcal{N}\) and for all \(J \subseteq \mathcal{N}\) such that \(|J| > |I|\). In order to show that \(\xi_I \partial_j \in B_n\) one has to consider two cases: \(j \notin I\) and \(j \in I\).

If \(j \notin I\) then \(\xi_{I \cup \{j\}} \partial_j \in B_n\) by the induction assumption, so by \((6.5)\)

\[
B_n \ni \psi_1(g_{\{j\}}) \circ \xi_{I \cup \{j\}} \partial_j = -\alpha(j, I)\xi_I \partial_j.
\]

Hence, \(\xi_I \partial_j \in B_n\).

If \(j \in I\) then denote \(I_j = I \setminus \{j\}, \, \, I^k_j = (I \setminus \{j\}) \cup \{k\}\), and consider

\[
a(I, j) := (-1)^{|I|-1} \psi_1(g_{\emptyset}) \circ \psi_1(g_{I_j}) = \psi_1(g_{\emptyset}) \circ \sum_{k=1}^{n} D(\xi_k \xi_{I_j} \partial_k).
\]
Since $a(I,j) \in B_n$ by the induction assumption, we have
\[
a(I,j) = \sum_{k=1}^{n} \left[ 4\alpha(k, I_j)\xi_{I_j}^k \partial_k - 2\alpha(k, I_j) \sum_{i=1}^{n} \xi_i \partial_i \xi_{I_j}^i \partial_k \right]
\]
\[
= \sum_{k=1}^{n} 2\alpha(k, I_j) \left[ 2\xi_{I_j}^k \partial_k - \sum_{i \in I_j^k} \xi_{I_j}^i \partial_k - \sum_{i \notin I_j^k} (-1)^{|I|} \xi_{I_j^k \cup \{i\}} \partial_i \partial_k \right]
\]
(6.6)
\[
= \sum_{k=1}^{n} 2\alpha(k, I_j) \left[ (2 - |I|) \xi_{I_j}^k \partial_k - \sum_{i \notin I_j^k} (-1)^{|I|} \xi_{I_j^k \cup \{i\}} \partial_i \partial_k \right].
\]

Recall that $\xi_{I_j^k \cup \{i\}} \partial_k \in B_n$ for any $i \notin I_j^k$. Now,
\[
B_n \ni g_{i(i)} \circ_0 \xi_{I_j^k \cup \{i\}} \partial_k = -\alpha(i, I_j^k) \xi_{I_j}^i \partial_k - (-1)^{|I|+1} \xi_{I_j^k \cup \{i\}} \partial_i \partial_k.
\]
Hence,
(6.7)
\[
\xi_{I_j^k \cup \{i\}} \partial_i \partial_k \equiv (-1)^{|I|} \alpha(i, I_j^k) \xi_{I_j}^i \partial_k \pmod{B_n}
\]
for any $i \notin I_j^k$. Substitute the last relation into (6.6) and obtain
\[
a(I,j) \equiv 2 \sum_{k=1}^{n} \alpha(k, I_j) (2 - n) \xi_{I_j}^k \partial_k \pmod{B_n},
\]
so
(6.8)
\[
\sum_{k=1}^{n} \alpha(k, I_j) \xi_{I_j}^k \partial_k \in B_n.
\]

Note that for any two different $k_1, k_2 \notin I_j$ (such a pair exists since $|I| < n$) we have $k_1 \notin I_j^{k_2}$ and $k_2 \notin I_j^{k_1}$. So by (6.4) we have
\[
\alpha(k_1, I_j^{k_2}) \xi_{I_j^{k_2}} \partial_{k_2} = (-1)^{|I|} \xi_{I_j \cup \{k_1, k_2\}} \partial_{k_2} \partial_{k_2} = (-1)^{|I|+1} \xi_{I_j \cup \{k_1, k_2\}} \partial_{k_2} \partial_{k_2} = \alpha(k_2, I_j^{k_1}) \xi_{I_j^{k_1}} \partial_{k_1} \pmod{B_n}.
\]

It is easy to observe that
\[
\alpha(k_1, I_j^{k_2}) = \begin{cases} \alpha(k_1, I_j), & k_1 < k_2, \\ -\alpha(k_1, I_j), & k_1 > k_2. \end{cases}
\]

Hence, all terms in (6.8) are equal modulo $B_n$, so for any $k \notin I_j$ we have
\[
(n - |I| + 1) \xi_{I_j} \partial_k \in B_n.
\]

In particular, for $k = j$ we have the required relation $\xi_i \partial_j \in B_n$.

We have proved that all elements of the form $\xi_i \partial_j$, $I \subseteq N \ni j$, belong to $B_n$. It remains to show that $(-D + v) \xi_I \in B_n$, $I \subseteq N$. It is enough to consider $I = \emptyset$ and $|I| = 1$.

Since $\xi_i \partial_i \in B_n$, we also have
\[
-D + v = \frac{1}{2} \left( \psi_1(g_0) - \sum_{i=1}^{n} D(\xi_i \partial_i) \right) \in B_n.
\]

Moreover, $\psi_1(g_{ik}) \equiv (-D + v) \xi_k \pmod{B_n}$, so $(-D + v) \xi_k \in B_n$ for any $k \in N$. 
Therefore, the image of $K_n$ under $\psi_1$ generates the entire algebra $C_1$. For the representation $\psi_2$ the proof is completely analogous.

(2) It was found in [21] that the universal envelope $U_{N_1}(K_1, B)$ (where $N_1$ is the locality function induced by $\psi_1$) is isomorphic to $C_1$. Hence, the associative envelope $(C_1, \psi_1)$ corresponds to a universally defined representation. Later we will show that this is not the case for $K_n$, $n \geq 2$.

Since $N_{C_2}(\psi_2(a), \psi_2(b)) = N_1(a, b)$ for any $a, b \in B$, the associative envelope $(C_2, \psi_2)$ has to be isomorphic to $(C_1, \psi_1)$.

(3) It is easy to note that for any $I \subseteq N$

\[
\psi_2(g_I) = \psi_1(g_I) - (n - 2)D\xi_I.
\]

Hence, for $n = 2$ the representations $\psi_1$ and $\psi_2$ of $K_2$ coincide, so the representation obtained is not an irreducible one.

It is not clear whether $(C_1, \psi_1) \simeq (C_2, \psi_2)$ for $n > 2$, but in any case neither of these envelopes is a universal one. Let us compute the induced locality functions $N_k(a, b) = N_{C_k}(\psi_k(a), \psi_k(b))$, $k = 1, 2$, $a, b \in B$. It is straightforward to check that $N_1 \equiv N_2 \equiv N$, where

\[
N(g_I, g_J) = \begin{cases}
3, & I \cap J = \emptyset, |I \cup J| \leq n - 1, \\
2, & |I \cap J| = 1 \text{ or } I \cap J = \emptyset, |I \cup J| = n, \\
1, & |I \cap J| = 2, \\
0, & |I \cap J| \geq 3.
\end{cases}
\]

In particular, $N(a, b) = N(b, a)$ for all $a, b \in B$.

**Lemma 6.2.** Conformal algebra $C = Cend_{2^n, Q}$, $Q = \text{diag}(v, 1, \ldots, 1)$, has no superinvolutions if $n > 1$.

*Proof.* Consider the $\mathbb{Z}_2$-grading $V = V_0 \oplus V_1$ on $V = k[D] \otimes k^{2^n}$ that induces the $\mathbb{Z}_2$-grading on $Cend_{2^n} \simeq k[D] \otimes A_n[v]$. This is exactly the canonical grading on $A_n$.

Denote

\[
\begin{align*}
C_{00} &= \{a \in C_0 \mid a \circ_m V_1 = 0 \forall m \geq 0\}, \\
C_{01} &= \{a \in C_0 \mid a \circ_m V_0 = 0 \forall m \geq 0\}.
\end{align*}
\]

It is clear that $C_{00} \simeq \text{Cend}_{2^{n-1}, Q}$, $C_{01} \simeq \text{Cend}_{2^{n-1}}$, $C_0 = C_{00} \oplus C_{01}$. Let $\pi$ stands for the projection of $C_0$ onto $C_{01}$.

Suppose $\sigma$ is a superinvolution of $C$. Note that $I = \pi(\sigma(C_{00}))$ is an ideal of $C_{01}$. Hence, either $I = 0$ or $I = C_{01}$. In the first case $\sigma|_{C_{00}}$ is an involution of $C_{00}$. In the last case $\sigma|_I$ is an isomorphism of $C_{00}$ and $C_{01}$. But it was shown in [8] that $\text{Cend}_{N, Q} \not\cong \text{Cend}_{N}$ if $Q$ is not invertible, and $\text{Cend}_{N, Q}$, $Q = \text{diag}(v, 1, \ldots, 1)$ has no involutions for $N > 1$.

It remains to apply Lemma 4.2 to show that neither of $\psi_k$, $k = 1, 2$, is a universally defined representation of $K_n$ with respect to $B$. ☐

**Acknowledgements**

This work is partially supported by RFBR 05–01–00230, Complex Integration Program SB RAS (2006–1.9) and SB RAS grant for young researchers (Presidium SB RAS, act N.29 of January 26, 2006). The author gratefully acknowledges the support of the Pierre Deligne fund based on his 2004 Balzan prize in mathematics.
The main results of this paper were presented on the Seventh Asian Symposium on Computer Mathematics in Seoul (December 8–10, 2005). The author is very grateful to Hyungju Park and Seok Jin Kang for their support in attending the Symposium.

References

[1] Bokut, L. A., 1976. Imbeddings into simple associative algebras [Russian], Algebra i Logika, 15, 117–142.
[2] Bokut, L. A., Fong, Y., Ke, W.-F., 2000. Gröbner–Shirshov bases and composition lemma for associative conformal algebras: an example, Contemporary Math. 264, 63–90.
[3] Bokut, L. A., Fong, Y., Ke, W.-F., Kolesnikov, P. S., 2000. Gröbner and Gröbner–Shirshov bases in algebra and conformal algebras [Russian], Fundam. Prikl. Mat. 6 (3), 669–706.
[4] Bokut, L. A., Fong, Y., Ke, W.-F., 2004. Composition-Diamond lemma for associative conformal algebras, J. Algebra 272 (2), 739–774.
[5] Borcherds, R. E., 1986. Vertex algebras, Kac-Moody algebras, and the Monster, Proc. Nat. Acad. Sci. U.S.A. 83, 3068–3071.
[6] Boyallian, C., Kac, V. G., Liberati, J. I., 2003. On the classification of subalgebras of Cend_N and gc_N, J. Algebra 260 (1), 32–63.
[7] Buchberger, B., 1970. An algorithmical criteria for the solvability of algebraic systems of equations [German]. Aequationes Math. 4, 374–383.
[8] Cheng, S.-J., Kac, V. G., 1997. A new N = 6 superconformal algebra, Commun. Math. Phys. 186, 219–231.
[9] Cheng, S.-J., Kac, V. G., 1997. Conformal modules, Asian J. Math. 1, 181–193.
[10] Cheng, S.-J., Kac, V. G., Wakimoto, M., 1998. Extensions of conformal modules, in: Topological field theory, primitive forms and related topics (Kyoto, 1996), 79–129, Progr. Math. 160, Birkhauser, Boston, MA.
[11] Cheng, S.-J., Kac, V. G., Wakimoto, M., 2000. Extensions of Neveu-Schwarz conformal modules, J. Math. Phys. 41 (4), 2271–2294.
[12] Cheng, S.-J., Lam, N., 2001. Finite conformal modules over N = 2, 3, 4 superconformal algebras, J. Math. Phys. 42 (2), 906–933.
[13] D’Andrea, A., Kac, V. G., 1998. Structure theory of finite conformal algebras, Sel. Math., New Ser. 4, 377–418.
[14] Fattori, D., Kac, V. G., 2002. Classification of finite simple Lie conformal superalgebras, J. Algebra 258 (1), 23–59.
[15] Fattori, D., Kac, V. G., Retakh, A., Structure theory of finite Lie conformal superalgebras, in: H.-D. Doebner and V.K.Dobrev (Eds.), Lie Theory and its Applications in Physics V, World Sci., 2004, 27–64.
[16] Frenkel, I. B., Lepowsky, J., Meurman, A., 1998. Vertex operator algebras and the Monster, Pure and Applied Math. V. 134, Academic Press.
[17] Kac, V.G., 1997. Vertex algebras for beginners, Univ. Lect. Series 10, AMS, Providence, RI.
[18] Kac, V.G., 1999. Formal distribution algebras and conformal algebras, in: XII-th International Congress in Mathematical Physics (ICMP’97), Brisbane, Internat. Press, Cambridge, MA, pp. 89–97.
[19] Kac, V. G., 2002. Classification of supersymmetries. Proc. of the International Congress of Mathematicians, Vol. I (Beijing, 2002), 319–344, Higher Ed. Press, Beijing.
[20] Kang, S.-J., Lee, K.-H., 2000. Gröbner–Shirshov bases for representation theory, J. Korean Math. Soc. 37 (1), 55–72.
[21] Kolesnikov, P. S., 2002. Universal representations of some Lie conformal superalgebras [Russian], Vestnik Quart. J. of Novosibirsk State Univ., Series: math., mech. and informatics 2 (3), 30–45.
[22] Kolesnikov, P. S., 2004. Gröbner–Shirshov bases of universal enveloping simple conformal Lie superalgebras of the series WN, Algebra Logic 43 (2), 109–122.
[23] Kolesnikov, P. S., 2006. Simple associative conformal algebras of linear growth, J. Algebra 295 (1), 247–268.
[24] Kolesnikov, P. S., 2006. Associative conformal algebras with finite faithful representation, Adv. Math. 202 (2), 602–637.
[25] Roitman, M., 1999. On free conformal and vertex algebras, J. Algebra 217 (2), 496–527.
[26] Roitman, M., 2000. Universal enveloping conformal algebras, Sel. Math., New Ser., 6 (3), 319–345.
[27] Retakh, A., 2001. Associative conformal algebras of linear growth, J. Algebra 237 (2), 769–788.
[28] Shirshov, A. I., 1962. Some algorithmic problems for Lie algebras [Russian], Sib. Mat. Z., 3, 292–296; English translation: ACM SIGSAM Bulletin, 33 (1999).

Sobolev Institute of Mathematics, Akad. Koptyug prospekt, 4, Novosibirsk, 630090, Russia

E-mail address: pavelsk@math.nsc.ru