Abstract. We show that solutions to the parabolic-elliptic Keller-Segel system on $\mathbb{S}^1$ with critical fractional diffusion $(-\Delta)^{\frac{1}{2}}$ remain smooth for any initial data and any positive time. This disproves, at least in the periodic setting, the large-data-blowup conjecture by Bournaveas and Calvez [10]. As a tool, we show smoothness of solutions to a modified critical Burgers equation via a generalization of the method of moduli of continuity by Kiselev, Nazarov and Shterenberg [26] over a setting where the considered equation has no scaling. This auxiliary result may be interesting by itself. Finally, we study the asymptotic behavior of global solutions, improving the existing results.

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1. Introduction

We study the following 1d parabolic-elliptic Keller-Segel equations\footnote{Known also as the Smoluchowski-Poisson equations. We will use both names interchangeably: the parabolic-elliptic Keller-Segel and the Smoluchowski-Poisson equations.} with the critical fractional diffusion $\Lambda = (-\Delta)^{\frac{1}{2}}$

\begin{equation}
\partial_t u = -\Lambda u - \chi \partial_x (u \partial_x v) \quad \text{in} \ (0,T) \times \mathbb{S}^1, \\
u(0) = u_0
\end{equation}
on the circle group $\mathbb{S}^1$. Alternatively, one can think of a 1d periodic torus $T = [-L, L]$. $\chi > 0$ is a parameter. Let us observe immediately that the formal integration of (1) in space implies

\begin{equation}
\int_{\mathbb{S}^1} u(x,t)dx = \int_{\mathbb{S}^1} u_0(x)dx =: m,
\end{equation}
i.e. conservation of the mean mass $m$. Unknown $v$ is governed by

\begin{equation}0 = \partial^2_x v + u - m \quad \text{in} \ (0,T) \times \mathbb{S}^1.
\end{equation}

Solutions to (3) are unique up to an additive constant. We choose it by requiring

\begin{equation}\int_{\mathbb{S}^1} v(x,t)dx = 0.
\end{equation}
In this paper, we deal with the global-in-time existence of classical solution to (1) - (3) and their asymptotic behavior. In particular, we will prove that every $L^2$-initial datum gives rise to a unique solution $(u,v)$ that remains smooth for any $T < \infty$. Moreover, if $\chi m$ is small enough, the solution $(u,v)$ tends to the homogeneous state $(m,0)$.

In order to deal with the introduced Keller-Segel system, we study the following modified fractional Burgers equation

$$
\partial_t Z = -\Delta Z + f Z \partial_x Z \quad \text{in } (0,T) \times S^1,
$$

where $f = f(t)$, $f : \mathbb{R} \rightarrow \mathbb{R}$ is a given smooth function.

In fact, we will use a simple yet powerful observation that solutions to our Keller-Segel system (1) - (3) are given as derivatives of solutions to the modified fractional Burgers equation (5).

1.1. Outline of the paper. The second part of introduction contains a comparison of our periodic case with the real-line case, motivations to study the problem as well as review of certain known results on both fractional Keller-Segel and Burgers equations. The introduction is concluded with necessary preliminaries, including the notion of the half-laplacian and definitions of a weak solution. Section 2 gathers our main results. There, Theorem 1 provides unique, large-data solutions to the Keller-Segel equations (1) - (3), globally in time. Its analogue for the modified Burgers (5) is Theorem 2. Theorem 3 provides steady-state asymptotics for (1) - (3) with initial datum $u_0$ having its mean mass of medium size$^2$. Next, in Section 3 we provide some auxiliary results: short-time solvability and a continuation criterion, with sketches of proofs or relevant references. Section 4 is devoted to proofs of our main theorems. In the concluding Section 5 we mention some open questions.

1.2. Comparison with equations on $\mathbb{R}$. Role of $m$. In the real-line case it is common to consider

$$
0 = \partial_x^2 v + u \quad \text{in } (0,T) \times \mathbb{R}
$$

as the equation for $v$, compare Escudero [20] and Bournaveas & Calvez [10]. In fact, the proper periodic counterpart of (6) is our (3) and not (6) on $S^1$. Let us explain this matter.

Firstly, when one considers a family of problems (1) - (3) on periodic tori $[-L,L]$, all with the same total initial mass $\int_{-L}^{L} u_0(x)dx$ and takes $L \rightarrow \infty$, then (3) goes (formally) to (6). Hence (3) is a legitimate periodic counterpart of (6).

Moreover, if we had dropped $m$ in (3) (in the periodic case), integration by parts in the resulting Poisson equation would force $\int_{S^1} u(x,t)dx = 0$. Since the applications call for nonnegative $u_0$ and $u$, the dynamics would become trivial. This is not the case on the real line, so we may consider the simplest possible (6) there.

Finally, equation (3) can be seen as the elliptic simplification of

$$
\varepsilon \partial_t v = \partial_x^2 v + u - m,
$$

appearing for instance in [11] by Burczak, Cieslak & Morales-Rodrigo. Observe that (7) formally yields $\int_{S^1} v(x,t)dx \equiv 0$ by the mass conservation. Taking limit $\varepsilon \rightarrow 0$ motivates our choice of (3) as well as our zero-mean disambiguation (4).

1.3. Motivation.

1.3.1. Applications. From the perspective of mathematical biology, the equations (1) - (3) are a 1d model of behavior of microorganisms (with density $u$) attracted by a chemical substance (with normalized density $v$). The parameter $\chi > 0$ quantifies the sensitivity of organisms to the chemical signal. In the original system by Keller & Segel [25] and its classical variations (compare for instance the survey [23] by Hillen & Painter) the natural motility of microorganisms is modeled by $-\Delta u$, instead of our $(-\Delta)^{\frac{3}{2}}u$. However, skippy movements of shrimps provide

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$^2$By medium size we mean that a smallness condition is needed, but, to the best of our knowledge, it is less restrictive than conditions currently available in the literature.
a good heuristic counterexample to the choice of laplacian. In fact, there is a strong evidence, both theoretical and empirical, that feeding strategies based on a Lévy process (generated by a fractional laplacian) are both closer to optimal ones and indeed used by certain organisms, especially in low-prey-density conditions. The interested reader can consult Lewandowsky, White & Schuster [29] for amoebas, Klafter, Lewandowsky & White [28] as well as Bartumeus et al. [3] for microzooplancton, Shlesinger & Klafter [36] for flying ants and Cole [19] in the context of fruit flies. Surprisingly, even for groups of large vertebrates, their feeding behavior is argued to follow Lévy motions, the fact referred sometimes as to the Lévy flight foraging hypothesis. See Atkinson, Rhodes, MacDonald & Anderson [2] for jackals, Viswanathan et al. [37] for albatrosses, Focardi, Marcellini & Montanaro [21] for deers and Raichlen et al. [35] for the Hadza tribe.

Thus, using $(-\Delta)^{\frac{\alpha}{2}}$ in the equation for the density $u$ is fully justified as a 1d model of behavior of certain organisms. Both our focus on $(-\Delta)^{\frac{\alpha}{2}}$ and choice of the equation (3) that models the chemoattractant diffusion will become clear in the next subsection.

At a qualitative first glance, it seems possible that organisms either aggregate in space (due to their tropism in accord with the gradient of density of the chemical substance) or they do not (due to their Lévy-type diffusion). This is the question that we address in this paper.

Let us finally remark that writing $v := -\phi$ in (1) - (3), we obtain a system for $(u, \phi)$ that is important in mathematical chemistry, cosmology and gravitation theory. It is very similar in spirit to the Zeldovich approximation used in cosmology to study the formation of large-scale structure in the primordial universe, see the works by Ascasibar, Granero-Belinchón & Moreno [1] and Biler [4]. It is also connected with the Chandrasekhar equation for the gravitational equilibrium of polytropic stars, statistical mechanics and the Debye system for electrolytes, see for example the work by Biler & Nadzieja [8].

1.3.2. Mathematical interest.

Results on the Smoluchowski-Poisson equation. We begin with explaining our focus on the half-laplacian in the equation for $u$. On $\mathbb{R}$, the problem

$$ \partial_t u = -\Lambda^\alpha u - \chi \partial_x (u \partial_x v), $$
$$ 0 = \partial_x^2 v + u, $$

where $\Lambda^\alpha = (-\Delta)^{\frac{\alpha}{2}}$, has global-in-time, regular solutions for $\alpha > 1$ and blowups for $\alpha < 1$. More precisely, Escudero in [20] shows the global regularity in the case $\alpha > 1$ for any initial datum $u_0 \in H^1$. Bourneaves & Calvez in [10] generalize this result by allowing any $u_0$ from $L^{1+\delta}$, $\delta > 0$. More importantly, for $\alpha < 1$ they provide a class of initial data that gives rise to the finite-time blowup, whereas for $\alpha \geq 1$ — a smallness condition $|u_0|_{L^\frac{2}{\alpha}} \leq K(\alpha)$ implying the global existence. Consequently, the case $\alpha = 1$ seems critical. To the best of our knowledge, presently the sharpest result is contained in Ascasibar, Granero-Belinchón & Moreno [1]. It says, for the periodic torus $[-\pi, \pi]$, that the condition $m \leq (2\pi)^{-2}$ implies global existence and convergence towards the homogeneous steady state. This bound on the initial data for the case (1) - (3) can be recovered from [13] by the authors and [22] by Granero-Belinchón & Orive-Illera, where more general systems were considered.

Hence, the remaining unresolved case is $\alpha = 1$ for initial data with large masses. For this case, Bourneaves & Calvez [10] conjecture a blowup of solutions, for certain large data, upon a numerical evidence. In this context Theorem 1 seems especially interesting, since it provides an analytical proof for the opposite.

Notice that in [14] we have obtained a series of regularity results for the doubly parabolic generalization of (1) - (3).

Let us recall some results for systems of type (1) - (3) in higher dimensions. They generally concern a system

$$ \partial_t u = -\mu \Lambda^\alpha u - \nabla \cdot (u \nabla v), $$
$$ v = K \ast u $$
in $\mathbb{R}^d$, $d \geq 2$, where $K$ stands for a (nonincreasing, i.e. attractive) interaction kernel. In [7] Biler, Karch & Laurençot, under rather general assumptions on $K$ (that allow for the Newtonian potential in particular), show blow up of solutions in the inviscid ($\mu = 0$) or $\alpha \in (0, 1)$ cases. For $d = 2$ this result was generalized over $\alpha \in (0, 2)$ by Li, Rodrigo & Zhang in [32]. A study for $K = e^{-|x|^2}$ was performed by Li & Rodrigo in [30], [31], [33]. In particular, Li & Rodrigo in [33] show regularity for $\alpha > 1$ and for $\alpha = 1$ with small data and in [30] — blowups for $\alpha \in (0, 1)$. Notice that this choice of $K$ implies that $v$ solves

$$c_d(1 - \Delta)^\frac{d+1}{2} v = u,$$

where $c_d(1 - \Delta)^\frac{d+1}{2}$ is the pseudo-differential operator given on the Fourier side by

$$c_d(1 - \Delta)^\frac{d+1}{2} v(\xi) = c_d(1 + |\xi|^2)^\frac{d+1}{2} v(\xi).$$

Let us mention here also a paper by Biler & Wu [9] that provides for $d = 2$ and $K = \frac{1}{\ln |x|}$ a local well-posedness result in critical Besov spaces for $\alpha \in (1, 2)$.

**Results on fractional Burgers equation.** The literature on the fractional dissipative Burgers equation is very extensive, so let us focus on the $1d$ case. Kiselev, Nazarov & Sheterenberg in [26] develop a method for proving the regularity in the critical case (5) with $f \equiv 1$. This provides a rather complete picture: one has global, regular solutions for $\alpha \geq 1$ and blowups for some initial data for $\alpha < 1$. Their method was used to solve the regularity problem of the $2d$ critical surface quasi-geostrophic equation, see the celebrated paper by Kiselev, Nazarov & Volberg [27].

Let us recall here also the earlier work [5] by Biler, Funaki & Woyczyński. Chae, Córdoba, Córdoba & Fontelos [17] (see also Castro & Córdoba [16] and Córdoba, Córdoba & Fontelos [18]) considered

$$\partial_t Z = -\mu \Lambda^{\alpha} Z - \delta \partial_x (Z H(Z)) - (1 - \delta) \partial_x Z H(Z),$$

where $H$ stands for the Hilbert transform and $\delta \in [0, 1]$. In the inviscid case $\mu = 0$, they show that (8) develops finite-time singularities in the whole range $0 \leq \delta \leq 1$. What’s more interesting for our considerations, in the case $\mu > 0$ and $\alpha = \delta = 1$, (8) develops singularities for large data and remains regular for small data. Let us also mention that Bae & Granero-Belinchón [12] proved global existence of weak solution to (8) for $\delta \geq 0.5$.

Li and Rodrigo in [34] considered

$$\partial_t Z = -\mu \Lambda^{\alpha} Z + \partial_x (Z H(Z))$$

and obtained blow up in the whole range of $0 \leq \alpha \leq 2$.

Equations (8) and (9) were proposed to get some insight into the behavior of $2d$ surface quasi-geostrophic equation.

Comparing the regularity result of Kiselev, Nazarov & Sheterenberg [26] and our Theorem 2 with the just-mentioned large-data blowup result for equations (8), (9), we realize how subtle matter is the exact form of the nonlinearity. More precisely, $\pm \partial_x (ZH(Z))$ at (8) with $\delta = 1$ and at (9) acts in fact as both a semilinear diffusion $\pm \Lambda Z$ and a nonlinear transport term $\pm H(Z) \partial_x Z$. Consequently, the sign of this nonlinearity turns out to be decisive, because it may produce either a forward or a backward-type nonlinear diffusion equation. On the other hand, the nonlinearity $\pm \partial_x (Z^2)$ of [26] or of our (5) is merely a nonlinear transport term. Hence it does not spoil the critical-case global regularity for any sign $\pm$ (compare Theorem 2 and observe that $f$ at (5) may be negative). Moreover, the $-$ sign has a regularizing effect. In this context, let us recall that Granero-Belinchón & Hunter [24] show for

$$\partial_t Z = (\Lambda^{\gamma} - \epsilon \Lambda^{1+\delta}) Z - Z \partial_x Z$$

with $1 > \delta > 0$, $\gamma \in [0, 1 + \delta)$, $0 < \epsilon < 1$, the global existence of solution. Furthermore, these solutions develop spatio-temporal chaos and remain close to the bounded attractor. Notice that the linearized version of (10) in Fourier space is

$$\frac{d}{dt} \hat{Z} = (|\xi|^\gamma - \epsilon |\xi|^{1+\delta}) \hat{Z}.$$
Since the linear part of (10) contains the backward diffusion $\Lambda^\gamma$, the linear term pumps energy into the lower Fourier modes. Consequently, the boundedness of the solutions to (10) is due only to the nonlinear term $-Z\partial_x Z$.

1.4. Definitions and analytic preliminaries. We use standard definitions for Hilbert spaces and write for short $| \cdot |_{H^k} = | \cdot |_k$, where $k = 0$ stands for the $L^2$ norm. We will sometimes suppress indication of the domain involved, when there is no danger of confusion.

1.4.1. Half-laplacian. Take $\alpha \in (0, 2)$. For an $f : \Omega \to \mathbb{R}$ we have by definition

$$\tilde{\Lambda}^\alpha f(\xi) = |\xi|^\alpha \hat{f}(\xi)$$

in Fourier variables, $\Omega$ being either $\mathbb{S}^1$ or $\mathbb{R}$. This operator is the infinitesimal generator of the isotropic stable Lévy process. Let us focus on the half-laplacian case $\alpha = 1$. It admits the following equivalent kernel representations for $\Omega = \mathbb{R}$

$$\Lambda f(x) = \frac{1}{\pi} \text{P.V.} \int_\mathbb{R} \frac{f(x) - f(y)}{|x - y|^2} dy,$$

(11)

$$\Lambda f(x) = \left[ \frac{d}{dh}(P_h * f) \right]_{h=0},$$

where $P_h = \frac{1}{\pi} \frac{h}{x+y} \pi$ is the Poisson kernel. The latter representation follows from the harmonic extension theorem, see the celebrated [15] by Caffarelli & Silvestre.

It is common to use on $\mathbb{S}^1$ the kernel representation

$$\Lambda f(x) = \frac{1}{\pi} \sum_{\gamma \in \mathbb{Z}} \text{P.V.} \int_\mathbb{S}^1 \frac{f(x) - f(y)}{|x - y - 2L\gamma|^2} dy,$$

(12)

that explicitly involves a chosen period $2L$, whereas the expressions (11) are standard for the real line case. In the particular case $L = \pi$, using complex analysis tools to add the previous series, we can write

$$\Lambda f(x) = \frac{1}{2\pi} \text{P.V.} \int_{-\pi}^{\pi} \frac{f(x) - f(x-y)}{\sin^2(y/2)} dy.$$

(13)

Nevertheless, we derive our main results using real-line expressions (11) for periodic function. This is admissible, since a $2L$-periodic, sufficiently smooth $u$ produces in (11) a $2L$-periodic $\Lambda f$, whereas the smoothness of $f$ gives the integrability at infinity in (11). Both principal values at (11) and at (12), then, concern in fact only the behavior around the origin. Moreover, the regularity of the involved functions will pose no problem, since a local-in-time smoothness is used in proofs of main results. For functions with little regularity, for instance in proofs of local-in-time existence, one can rely on the Fourier-side definition. The advantage of using (11) becomes clear in the proofs. Besides, it allows to justify our reasoning for an arbitrary period $2L$ at once.

For the proof of our result on the asymptotic behavior, we will utilize representation (13).

1.4.2. Weak solutions.

Definition 1 (Weak solutions to Keller-Segel system). Choose $u_0 \in L^2(\mathbb{S}^1)$. Fix an arbitrary $T \in (0, \infty)$. The couple

$$u \in C([0, T), L^2(\mathbb{S}^1)), \quad v \in C([0, T), H^2(\mathbb{S}^1))$$

is a weak solution of (1) - (3) if and only if

$$\int_0^T \int_{\mathbb{S}^1} -u \partial_t \varphi + u \Lambda \varphi - (\chi u \partial_x v) \partial_x \varphi = \int_{\mathbb{S}^1} u_0 \varphi(x, 0) dx,$$

$$\partial_x^2 v(x, t) = u(x, t) - m \quad \text{a.e. in } [0, T) \times \mathbb{S}^1,$$

where $\varphi$ is an arbitrary $C^\infty((-1, T) \times \mathbb{S}^1)$ function.

Similarly, we introduce
Definition 2 (Weak solutions to modified Burgers equation). Choose $u_0 \in L^2(S^1)$. Fix an arbitrary $T \in (0, \infty)$.

\[ Z \in C([0,T), L^2(S^1)) \]

is a weak solution of (5) if and only if

\[ \int_0^T \int_{S^1} -Z \partial_t \varphi + Z \Delta \varphi + \frac{f}{2} Z^2 \partial_x \varphi = \int_{S^1} Z_0 \varphi(x,0) dx, \]

where $\varphi$ is an arbitrary function from $C^\infty((−1, T) \times S^1)$.

2. Main results

Theorem 1 (Regularity of critical Keller-Segel). Fix any $T < \infty$ and a natural $s \geq 0$. For any initial datum $u_0 \in H^s(S^1)$ there exists a weak solution $(u,v)$ to (1) - (3) that satisfies

\[
\begin{align*}
    u &\in C([0,T]; H^s(S^1)) \cap C^\infty((0,T) \times S^1) \\
    v &\in C([0,T]; H^{s+2}(S^1)) \cap C^\infty((0,T) \times S^1).
\end{align*}
\]

Furthermore, this solution is unique among weak solutions.

Now we can state

Theorem 2 (Regularity of critical modified Burgers). Fix any $T < \infty$, a natural $s \geq 1$ and a smooth $f = f(t)$. For any initial datum $Z_0 \in H^s(S^1)$, problem (5) admits a smooth solution

\[ Z \in C([0,T]; H^s(S^1)) \cap C^\infty((0,T) \times S^1). \]

It is unique among weak solutions.

Finally, we study the asymptotic behavior of solutions to (1) - (3). Recall that, by (2), $m$ denotes the initial mean mass.

Theorem 3 (Steady state asymptotics of critical Keller-Segel). Let $u_0 \in L^2(S^1)$ be a $2\pi$—periodic initial datum for the system (1) - (3). Assume further that

\[ \chi m < 1. \]

Then, there exist numbers $\sigma, \Sigma > 0$ such that the solution couple $(u,v)$ to (1) - (3) verifies

\[ |u - m|_0 + |v|_2 \leq \Sigma e^{-\sigma t}. \]

Remark 1. Theorem 3 can also be proved for a general period $2L$ by analogous computations. Then the smallness condition takes the form $\chi m < C(L)$.

Let us recall that for the case $\chi = 1$ a similar condition, namely $m < (2\pi)^{-2}$, was proved in Ascasibar, Granero-Belinchón & Moreno [1] with a different method. Consequently, Theorem 3 improves the result in [1].

3. Auxiliary results

In order to proceed with our global regularity proofs, we need to make sure that we are equipped with sufficient smoothness. To this end, the easiest way is to provide the following short-time regularity results for the modified Burgers equation (5). Concerning their proofs, for brevity, we restrict ourselves to providing the appropriate reference and commenting on some minor changes needed. Let us emphasize that from now on, we fix an arbitrary $T < \infty$ and a particular smooth $f(t), f : \mathbb{R} \to \mathbb{R}$. Consequently, we do not write dependencies on $f$ and $T$ explicitly. Let us begin with the unique, local-in-time existence of weak solutions.

Lemma 1. Take $Z_0 \in H^s(S^1)$ with $s \geq 1$. Then, there exists $T_* = T_*(|Z_0|_s)$ such that problem (5) admits a weak solution such that

\[ Z \in L^2(0,T_*; H^{s+\frac{1}{2}}) \cap C([0,T_*]; H^s). \]

Moreover, this weak solution is unique in the class

\[ L^2(0,T_*; H^s) \cap C([0,T_*]; H^s). \]
For $f \equiv 1$, this is Theorem 2.5 (existence of a weak solution) and Theorem 2.8 with $\delta = 1$ (uniqueness) of [26]. Allowing for a non-constant $f(t)$ does not change these proofs, since smooth $f : \mathbb{R} \to \mathbb{R}$ and $T < \infty$ are prefixed.

Next, we state a simple continuation criterion

**Lemma 2.** Take a weak, local in time solution $Z$ to (5) with its existence time $T_* > 0$. If there exists $C$ such that $|Z(t)|_1 \leq C$ on $[0, T_*)$, then for an $\varepsilon > 0$, $Z$ can be continued beyond $T_*$ up to $T_* + \varepsilon$.

The proof comes down to restarting our evolution an instant before $T_*$ and use of Lemma 1. Finally, we arrive at a conditional smoothness result.

**Lemma 3.** If $|Z(t)|_1 \leq C$ on $[0, T)$, then the solution $Z$ to (5) given by Lemma 1 satisfies

$$Z \in C^\infty((0, T) \times \mathbb{T}).$$

Using $|Z(t)|_1 \leq C$ on $[0, T)$ and Lemma 2, we know that the weak solution to (5) can be continued up to $T$. Now, smoothness follows from Corollary 2.6. of [26], up to minor differences concerning $f(t)$, that we can deal with as before.

### 4. Proofs of main results

The main steps of proof of Theorem 1, presented in Section 4.1, are as follows

1. Restating considerations for Keller-Segel equations as considerations for a fractional Burgers-type equation (5). Roughly speaking, equation (5) serves as an equation for the primitive function of $u$ solving (1).

2. Using smoothness of solutions to (5) provided by Theorem 2.

In turn, in order to show Theorem 2 we will construct a family of moduli of continuity that are preserved over the evolution in time. This approach follows an ingenious methodology developed in [26] by Kiselev, Nazarov and Shterenberg for the fractional Burgers equation.

Our case differs from theirs in two aspects.

(i) Firstly, our equation (5) is slightly more complex, compared to pure fractional Burgers equation, since it involves $f$. Most importantly, it lacks the scaling of the original case $f \equiv 1$.

(ii) Secondly, our Theorem 2 allows for weak initial data.

It may be sufficient to say that we follow the lines of the respective part of [26], whereas we deal with (ii) via short-time regularity and with (i) by constructing at once an entire family of moduli of continuity, so that any (sufficiently smooth) initial datum enjoys one of them. Nevertheless, we provide rigorous proofs in Section 4.2. One motivation is completeness. A more important one reads as follows. Using a one parameter family of moduli of continuity, which is the most natural approach suggested by the scaling of the full-space case, turned out to be insufficient to deal with (i). We need to introduce another parameter $N$, that divides the middle and large arguments of moduli of continuity (see proof of Theorem 2).

#### 4.1. Proof of Theorem 1

To fix ideas, let us choose a reference point 0 on $\mathbb{S}^1$ and denote its half-length with $L$, i.e. we work with the periodic torus $\mathbb{T} = [-L, L]$. We assume here that we have Theorem 2 at our disposal.

A *primitive-type equation*. Take $u_0 \in H^s$, $s \geq 0$. Recall that by definition $m = \int_{-L}^{L} u_0(y)dy$. Let us choose

$$f(t) := e^{\chi mt},$$

$$Z(0, x) := \int_{-L}^{x} u_0(y)dy - m(x + L) + c,$$

where

$$c = Lm - \int_{-L}^{L} \int_{-L}^{x} u_0(y)dydx$$
is selected so that
\[ \int_{-L}^{L} Z(0, x)dx = 0. \]
Observe that \( Z(0, x) \) is periodic and belongs to \( H^{s+1} \). Hence, Theorem 2 implies that problem (5) with choice (14) admits a unique, smooth solution
\[ Z \in C([0, T]; H^{s+1}(S^1)) \cap C^\infty((0, T) \times S^1). \]
By integration of (5) in space we see that \( \int_{S^1} Z(x, t)dx \equiv 0 \).

**Change of variables.** Formula
\[ (15) \quad W(x, t) := e^{\nu \gamma t} Z(x, t) \]
provides a one-to-one correspondence between solutions of (5) with choice (14) and
\[ W \in C([0, T]; H^{s+1}(S^1)) \cap C^\infty((0, T) \times S^1) \]
solving
\[ \begin{align*}
\partial_t W &= -\Lambda W + \chi(\partial_x W)W + \chi m W \quad \text{in } (0, T) \times S^1, \\
W(0, x) &= \int_{-L}^{x} u_0(y)dy - mx - \int_{-L}^{L} \int_{-L}^{x} u_0(y)dydx.
\end{align*} \]

The definition (15) of \( W \) implies \( \int_{S^1} W(x, t)dx \equiv 0 \).

**Recovering solutions to Keller-Segel system.** Let us take
\[ (17) \quad u := \partial_x W + m. \]
Smoothness of \( W \) gives
\[ u \in C([0, T]; H^s(S^1)) \cap C^\infty((0, T) \times S^1). \]
It solves
\[ \begin{align*}
\partial_t u &= -\Lambda u + \chi \partial_x (uW) \quad \text{in } (0, T) \times S^1, \\
w(0, x) &= u_0(x),
\end{align*} \]

since \( W \) solves (16) and \( \partial_x \) and \( \Lambda \) commute. Observe that \( \int_{S^1} u(x, t)dx \equiv m \). Function \( u \) of (18) gives in fact solution to (1). It becomes fully clear after we recover \( v \) solving (3). To this end, let \( V \) be a solution to
\[ (19) \quad -\partial_x^2 V(x, t) = W(x, t) \quad \text{in } (0, T) \times S^1, \]
where \( W \) is an admissible right-hand side, because \( \int_{S^1} W(x, t)dx \equiv 0 \). Hence \( v := \partial_x V \) is the zero-mean solution of
\[ (20) \quad -\partial_x^2 v = u - m \quad \text{in } (0, T) \times S^1, \]
compare (17). By the definition \( v = \partial_x V \) and (19), we get \( \partial_x v = -W \). Plugging this in (18) and looking at (20) yields that
\[ \begin{align*}
u \in C([0, T]; H^s(S^1)) \cap C^\infty((0, T) \times S^1) \\
u \in C([0, T]; H^{s+2}(S^1)) \cap C^\infty((0, T) \times S^1)
\end{align*} \]
solve (1) - (3) with the initial condition \( u_0 \).

4.1.1. **Uniqueness.** Since Theorem 1 provides uniqueness of \( Z \) and (17) holds, uniqueness of \( u \) solving (1) follows. Since \( v \) is the zero-mean solution of (20), it is also unique.

4.2. **Proof of Theorem 2.**
4.2.1. Preliminaries. Moduli of continuity. As indicated, the approach of \cite{26} that we follow, relies on a construction of special moduli of continuity (in space). Hence, we first gather some preliminaries concerning these moduli.

**Definition 3.** We denote by $\mathcal{O}$ the class of moduli of continuity $\omega : \mathbb{R}_+ \to \mathbb{R}_+ \ (\text{non-decreasing, concave functions with } \omega(0) = 0)$, whose derivatives satisfy

$$\omega'(0) < \infty, \quad \lim_{\xi \to 0} \omega''(\xi) = -\infty.$$ 

and $\omega'(\xi)$ is continuous at 0.

Recall that a function $f$ has modulus of continuity $\omega$ if

$$|f(x + h) - f(x)| \leq \omega(|h|).$$

Using Taylor formula, we get

$$\left| \partial_x f(x)h + \partial^2_x f(\eta) \frac{h^2}{2} \right| \leq \omega'(|h|) + \omega''(|\eta_1|) \frac{h^2}{2},$$

with $\eta, \eta_1$ in-between 0 and $|h|$. For $\omega \in \mathcal{O}$, knowing that $\partial_x f$ and $\partial^2_x f$ are locally qualitatively bounded, we arrive at (21)

$$|\partial_x f|_{\infty} < \omega'(0).$$

In particular, it holds

**Proposition 1.** If a smooth, periodic, real function $f$ has a modulus of continuity $\omega \in \mathcal{O}$, then (21) holds.

A concrete family.

**Definition 4.** For parameters $K, B, \xi_0$ let us define the family

$$(22) \quad \omega_{K, B, \xi_0}(\xi) := \begin{cases} \frac{B\xi}{1 + K\sqrt{B\xi}} & \text{for } \xi \in [0, \xi_0) \\ C_{K, B, \xi_0} \ln(B\xi) & \text{for } \xi \geq \xi_0, \end{cases}$$

where

$$C_{K, B, \xi_0} = \frac{B\xi_0}{\ln(B\xi_0)(1 + K\sqrt{B\xi_0})}.$$ 

**Proposition 2.** For any positive $K, B, \xi_0$ such that

$$B\xi_0 \geq e^2,$$

we have that $\omega_{K, B, \xi_0}(\xi) \in \mathcal{O}$. Furthermore, it holds

$$\omega'_{K, B, \xi_0}(\xi) = \begin{cases} \frac{B}{2(1 + K\sqrt{B\xi})^2} & \text{for } \xi \in [0, \xi_0) \\ C_{K, B, \xi_0} \xi^{-1} & \text{for } \xi \geq \xi_0, \end{cases}$$

$$\omega''_{K, B, \xi_0}(\xi) = \begin{cases} -KB^2 \frac{3B\xi_0^{-2} + K}{4(1 + K\sqrt{B\xi_0})^3} & \text{for } \xi \in [0, \xi_0) \\ -C_{K, B, \xi_0} \xi^{-2} & \text{for } \xi \geq \xi_0. \end{cases}$$

**Proof.** The choice of $C_{K, B, \xi_0}$ in Definition 4 implies continuity of $\omega_{K, B, \xi_0}$. Formulas for derivatives hold by a computation. In particular, they give

$$\partial_\xi \omega_{K, B, \xi_0}(0) = B, \quad \lim_{\xi \to 0} \partial^2_\xi \omega_{K, B, \xi_0}(\xi) = -\infty.$$ 

To finish the proof, we need to show concavity of $\omega_B$. We have

$$\omega'_{K, B, \xi_0}(\xi_0^-) \geq \omega'_{K, B, \xi_0}(\xi_0^+) \iff 2(1 + K\sqrt{B\xi_0}) \leq \ln(B\xi_0)(2 + K\sqrt{B\xi_0})$$

and for the latter it is sufficient to assume $B\xi_0 \geq e^2$. \hfill $\square$

4.2.2. Smoothness.
Reduction of problem to keeping the modulus of continuity. First we show that if a solution of (5) keeps a modulus of continuity \( \omega \in \mathcal{O} \), then it remains smooth.

**Lemma 4.** Assume that a zero-mean solution \( Z(t) \) to (5) satisfies

\[
Z(t_0) \in C^\infty(\mathbb{S}^1),
\]

and there exists

\[
\omega \in \mathcal{O} \quad \text{that} \quad |Z(t,x) - Z(t,y)| \leq \omega(|x - y|) \quad \text{on} \quad [t_0,t_1] \times \mathbb{S}^1,
\]

where \( 0 \leq t_0 \leq t_1 < \infty \). Then, there exists \( \varepsilon > 0 \) such that \( Z(t) \in C^\infty(\mathbb{S}^1) \) on \([t_0,t_1 + \varepsilon]\).

Recall that \( |\cdot|_0 \) denotes the \( L^2 \) norm.

**Proof.** Once the \( f(t_0) \) holds the modulus \( \omega \), formula (21) gives

\[
|\partial_x Z(t_0)|_\infty < \omega'(0).
\]

Hence

\[
|\partial_x Z(t_0)|_0 < \omega'(0)|\mathbb{S}^1|^{\frac{1}{2}}.
\]

The zero mean assumption implies then

\[
|Z(t_0)|_1 < C\omega'(0)|\mathbb{S}^1|^{\frac{1}{2}} =: D.
\]

Theorem 1 for the initial datum \( Z(t_0) \) with Lemma 3 give that \( Z(t) \) is smooth on \([t_0,t_0 + T_s(D)]\). It keeps the zero-mean over its evolution. By assumption, \( Z(t) \) does not violate \( \omega \) as long as \( t_0 + T_s(D) \leq t_1 \). In such case, we restart our evolution at \( t_0^{(1)} := t_0 + T_s(D) \) and repeat our considerations. This gives smoothness of \( Z \) up to \( t_0^{(2)} := t_0 + 2T_s(D) \). We can continue this procedure up to \( t_0^{(n+1)} \), where \( t_0^{(n)} < t_1 \) and \( t_0^{(n+1)} > t_1 \). \( \square \)

Notice also that the assumption of smoothness of \( Z(t_0) \) is in fact not a restriction, since Lemma 3 implies that the Burgers problem (5) enjoys a local-in-time smooth solution for every initial data \( Z(0) \in H^1 \).

**Keeping the modulus of continuity.** Now we show that, given smooth \( Z(t_0) \), there exists such \( \omega \in \mathcal{O} \) that \( \omega \) is a modulus of continuity for \( Z(t_0) \). Furthermore, we will also prove that this modulus of continuity is kept for all times, over the evolution of a solution \( Z \) of (5) starting from \( Z(t_0) \). Consequently, via Lemma 4, we will obtain that \( Z \) is smooth for all times.

**Lemma 5.** Choose any \( T < \infty \). If for a \( t_0 \in [0,T] \)

\[
Z(t_0) \in C^\infty(\mathbb{S}^1),
\]

then \( Z \) remains smooth on \([t_0,T]\), i.e. \( Z \in C^\infty([t_0,T] \times \mathbb{S}^1) \).

The outline of the proof is as follows. We collect the requirements on \( K, B, \xi_0 \) needed to use Lemma 4 with \( \omega_{K,B,\xi_0} \) of Definition 4. Namely, in the first step, for an arbitrary smooth \( Z(t_0) \) we obtain a sufficient condition on \( K, B, \xi_0 \) so that \( \omega_{K,B,\xi_0} \) is kept by \( Z(t_0) \). Next, in the second step, we provide such conditions on \( K, B, \xi_0 \) that \( \omega_{K,B,\xi_0} \) is kept for all times \( t \geq t_0 \) over the evolution. Finally we choose such parameters \( K, B, \xi_0 \) that all the mentioned conditions are satisfied and we conclude by Lemma 4.

We could have chosen a shorter way of presenting the proof, namely list all the conditions on \( K, B, \xi_0 \) at the beginning and then check that \( \omega_{K,B,\xi_0} \) is kept over the evolution. Then, however, it would be less clear what all these conditions are needed for. Let us notice that we can work here with smooth solutions, which is not a priori estimate. Namely, for short times we have smoothness via Lemma 3. Next, as long as a modulus of continuity is conserved, we have smoothness by Lemma 4.
Proof. Step 1. (Requirements for compliance with $Z(t_0)$.) We collect the requirements on $K, B, \xi_0$ needed for the inequality
\begin{equation}
|Z(x, t_0) - Z(y, t_0)| \leq \omega_{K, B, \xi_0}(|x - y|).
\end{equation}
We have
\begin{equation}
|Z(x, t_0) - Z(y, t_0)| \leq |\partial_x Z(t_0)|_{\infty} |x - y|.
\end{equation}
Take $\xi := |x - y|$. Recalling Definition 4, we see that in the case $\xi \in [0, \xi_0)$, for (23) suffices
\begin{equation}
\frac{B}{1 + K\sqrt{B\xi_0}} \geq |\partial_x Z(t_0)|_{\infty}.
\end{equation}
We split the case $\xi \geq \xi_0$ into $\xi \in [\xi_0, N\xi_0]$ and $\xi > N\xi_0$, where $N \geq 1$ will be chosen later. Here for (23), it suffices to take
\begin{equation}
\frac{B}{1 + K\sqrt{B\xi_0}} \geq N|\partial_x Z(t_0)|_{\infty} \quad \text{in the case } \xi \in [\xi_0, N\xi_0]
\end{equation}
\begin{equation}
\frac{B\xi_0}{\ln(B\xi_0)} \geq 2|Z(t_0)|_{\infty} \quad \text{in the case } \xi > N\xi_0.
\end{equation}
It may happen that the half-period $L$ of the torus is smaller than $N\xi_0$ or even than $\xi_0$. Then, the corresponding conditions are not necessary.

Step 2. (Requirements for keeping the modulus) We collect the requirements on $K, B, \xi_0$ needed for
\begin{equation}
|Z(x, t) - Z(y, t)| \leq \omega_{K, B, \xi_0}(|x - y|)
\end{equation}
for $t > t_0$.

First, let us consider any $\omega \in \mathcal{O}$. Assume that
\begin{equation}
Z \text{ looses the modulus of continuity } \omega \text{ at a certain time } t \in (t_0, T].
\end{equation}
Let us take such $\tau$ that
\begin{equation}
\tau := \sup\{\tau \in [t_0, T] : |Z(x, t) - Z(y, t)| \leq \omega(|x - y|)\}.
\end{equation}
There are now two scenarios of what happens at time $\tau$. Either we have the two-point blowup scenario:
\begin{equation}
\exists y \neq x \text{ such that } |Z(x, \tau) - Z(y, \tau)| = \omega(|x - y|)
\end{equation}
or we have the single-point blowup scenario.

Roughly speaking, the latter option is realized when for $t_n \to \tau$, $x_n \to x$, $\xi_n \to 0$ we have
\begin{equation}
|Z(x_n, t_n) - Z(x_n + \xi_n, t_n)| \leq \omega(|\xi_n|),
\end{equation}
with the above inequality becoming equality in the limit $n \to \infty$. Equivalently,
\begin{equation}
|\partial_x Z(x_n + \tilde{\xi}_n, t_n)|_{\xi_n} \leq \omega'(|\tilde{\xi}_n|)\xi_n,
\end{equation}
where $\tilde{\xi}_n, \xi_n \in [0, \xi_n]$ and the inequality becomes equality in the limit $n \to \infty$. Hence it is indeed a single-point blowup, because in the limit we get
\begin{equation}
|\partial_x Z(x, \tau)| = \omega'(0).
\end{equation}
This is impossible, due to formula (21). Now we provide a rigorous argument for this impossibility of the single-point blowup scenario.

Substep 2.1. (Ruling out a single-point blowup scenario) This is Lemma 3.4 of [26]. For reader’s convenience we present here a slightly different argument. Our choice (28) of $\tau$ and Lemma 4 says that $Z(t)$ is smooth an $\varepsilon$ beyond $\tau$. As a consequence, we can use formula (21) to get $|\partial_x Z(t)|_{\infty} < \omega'(0)$ for all $t \in [\tau, \tau + \varepsilon]$. This sharp inequality and the fact that $\omega'(\xi)$ is continuous at 0 implies that there exists such $\delta > 0$ that $\sup_{t \in [\tau, \tau + \varepsilon]} |\partial_x Z(t)|_{\infty} < \omega'(\delta)$. Hence for $t \in [\tau, \tau + \varepsilon]$ and $|x - y| \leq \delta$

\begin{equation}
|Z(x, t) - Z(y, t)| \leq |\partial_x Z(t)|_{\infty} |x - y| < \omega'(\delta)|x - y| \leq \omega(|x - y|),
\end{equation}
where in the last inequality we have used the concavity and the mean value theorem. Consequently, the solution can lose the modulus of continuity only in the two-point blowup scenario (29).

Substep 2.2. (Ruling out a two-point blowup scenario) Without loss of generality we can assume $Z(x, \tau) \geq Z(y, \tau)$, drop the absolute value in (29), hence considering

$$Z(x, \tau) - Z(y, \tau) = \omega(|x - y|).$$

We have arrived at the heart of the regularity proof. To contradict (27), it suffices to show that

$$\partial_t (Z(x, t) - Z(y, t))\bigg|_{t=\tau} < 0,$$

because then we arrive at contradiction with the definition of $\tau$.

Since we work at the exact time instant $\tau$, let us suppress the time dependence below. We get the conditions on $K, B, \xi_0$ for (31) as follows.

Use the equation (5) to write

$$\partial_t (Z(x) - Z(y)) = [\Lambda Z(x) - \Lambda Z(y)] + f(\partial_z Z(x)Z(x) - \partial_z Z(y)Z(y)] =: I + fII$$

Due to translation-invariance of our problem, we can choose the reference point 0 on $S^1$ so that $x = \frac{\xi}{2}$, $y = \frac{\xi}{2}$. We write $\xi := x - y$. Via the differential formula in (11) and basic properties of the Poisson kernel, we arrive at

$$I \leq \frac{1}{\pi} \int_0^{\xi/2} \frac{\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^2} d\eta + \frac{1}{\pi} \int_0^{\xi/2} \frac{\omega(\xi + 2\eta) - \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^2} d\eta.$$

For more details of this computation, see [26], p. 222. Both terms above are nonpositive in view of concavity of $\omega$. Similarly we deal with $II$. Let us perform this computation. We have

$$\partial_z Z(x)Z(x) - \partial_z Z(y)Z(y) = \frac{d}{dh} (\frac{d}{dh} (Z(x + hZ(x)) - Z(y + hZ(y))))\bigg|_{h=0} = \frac{d}{dh} g(h)\bigg|_{h=0}.$$

Now, we use monotonicity of $\omega$ and (30) to compute $\frac{d}{dh} g(0)$. Indeed,

$$g(h) \leq \omega(\xi + h|Z(x) - Z(y)|) \leq \omega(\xi + h\omega(\xi)) \quad \text{and} \quad g(0) = \omega(\xi).$$

Consequently,

$$\frac{g(h) - g(0)}{h} \leq \frac{\omega(\xi + h\omega(\xi))}{h} - \frac{\omega(\xi)}{h} \quad \Rightarrow \quad \frac{d}{dh} g(0) \leq \omega(\xi)\omega'(\xi),$$

hence

$$II \leq \omega(\xi)\omega'(\xi).$$

Putting together estimates for $I$ and $II$ we obtain

$$\partial_t (Z(x) - Z(y)) \leq \frac{1}{\pi} \int_0^{\xi/2} \frac{\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^2} d\eta$$

$$+ \frac{1}{\pi} \int_0^{\xi/2} \frac{\omega(\xi + 2\eta) - \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^2} d\eta + f(\omega(\xi)\omega'(\xi))$$

$$=: I_1 + I_2 + I_3,$$

where $I_1, I_2$ are nonpositive.

Next, it remains to pinpoint such $\omega \in \mathcal{O}$ that

$$I_1 + I_2 + I_3 < 0,$$

since then (32) implies (31). To this end, it suffices to choose $\omega$ from our family $\omega_{K,B,\xi_0}$. Recalling that $f = f(t)$, let us take

$$\Gamma = \sup_{t \in [0,T]} f(t),$$

and
where \( T \) is a pre-fixed, arbitrary finite time. We have via Proposition 2

\[
I_3 = f w_{K, B, \xi_0}(\xi) \omega''_{K, B, \xi_0}(\xi) \leq \begin{cases} \Gamma B^2 \frac{\xi (2 + K \sqrt{B \xi})}{2(1 + K \sqrt{B \xi})^3} & \text{for } \xi \in [0, \xi_0), \\ \Gamma C_{K, B, \xi_0}^2 \xi^{-1} \ln(B \xi) & \text{for } \xi \geq \xi_0. \end{cases}
\]

Let us now consider the cases \( \xi \in [0, \xi_0) \) and \( \xi \geq \xi_0 \) separately.

(i) Case \( \xi \in [0, \xi_0) \). Proposition 2 gives that \( \omega_{K, B, \xi_0}(\xi) \) is decreasing, \( \omega''_{K, B, \xi_0}(\xi) \) is increasing on \([0, \xi_0]\) and negative. Hence for \( 2\eta \in [0, \xi_0]\)

\[
\begin{align*}
\omega_{K, B, \xi_0}(\xi + 2\eta) - \omega_{K, B, \xi_0}(\xi) &\leq \omega'_{B}(\xi)2\eta, \\
\omega_{K, B, \xi_0}(\xi - 2\eta) - \omega_{B}(\xi) &\leq -\omega'_{K, B, \xi_0}(\xi)2\eta + \omega''_{K, B, \xi_0}(\theta)2\eta^2 \\
&\leq -\omega'_{K, B, \xi_0}(\xi)2\eta + \omega''_{K, B, \xi_0}(\xi)2\eta^2.
\end{align*}
\]

Adding these inequalities yields

\[
I_1 \leq \frac{1}{\pi} \xi \omega''_{K, B, \xi_0}(\xi) = -\frac{1}{4\pi} KB^2 \xi^3 (B \xi)^{-\frac{1}{2}} + K \\
= \frac{3}{(1 + K \sqrt{B \xi})^3}.
\]

Since \( I_2 \leq 0 \), for \( I_1 + I_2 + I_3 < 0 \) suffices \( I_1 + I_3 < 0 \). This is equivalent to

\[
K(3(B \xi)^{-\frac{1}{2}} + K) > 2\pi \Gamma(2 + K \sqrt{B \xi}) = 2\pi \Gamma \sqrt{B \xi}(2(\frac{1}{2} + K),
\]

for which in turn the condition

\[
(34) \quad K > 2\pi \Gamma \sqrt{B \xi_0}
\]

is sufficient.

(ii) Case \( \xi \geq \xi_0 \). Due to the concavity of \( \omega_{K, B, \xi_0} \), we have

\[
\omega_{K, B, \xi_0}(2\eta + \xi) - \omega_{K, B, \xi_0}(2\eta - \xi) \leq \omega_{K, B, \xi_0}(2\xi).
\]

From the formula for \( w_{K, B, \xi_0} \) and assumption \( B \xi_0 \geq e^2 \) of Proposition 2, we obtain that

\[
\omega_{K, B, \xi_0}(2\xi) \leq \frac{3}{2} \omega_{K, B, \xi_0}(\xi).
\]

Consequently,

\[
I_2 \leq -\frac{\omega_{K, B, \xi_0}(\xi)}{\pi \xi} = \frac{C_{K, B, \xi_0}}{\pi} \xi^{-1} \ln(B \xi).
\]

Therefore

\[
I_2 + I_3 < 0 \iff f \pi < C_{K, B, \xi_0}^{-1}
\]

This condition is equivalent to

\[
(35) \quad \Gamma \pi B \xi_0 < \ln(B \xi_0)(1 + K \sqrt{B \xi_0}).
\]

Step 3. (Meeting all requirements) Finally, for \( \Gamma, |Z(t_0)|, |\partial x Z(t_0)| \) we need to choose such parameters \( K, B, \xi_0 \) and \( N \) (recall the splitting \( N \xi_0 \) in (25)) that requirements (24), (25), (34), (35) and \( B \xi_0 \geq e^2 \) (needed for Proposition 2) are satisfied. Let us first make the following choices

\[
B \xi_0 = e^2, \quad K = 4\pi \Gamma \sqrt{B \xi_0},
\]

so that (34) is fulfilled. Hence we are left with parameters \( B \) and \( N \) and we need to comply with (24), (25) and (35). Respectively, they take now the form

\[
\begin{align*}
\frac{B}{1 + 4\pi \Gamma e^2} &\geq |\partial x Z(t_0)|, \\
\frac{B}{1 + 4\pi \Gamma e^2} &\geq N|\partial x Z(t_0)| \quad \text{in the case } \xi \in [\xi_0, N \xi_0] \\
\frac{e^2}{(1 + 4\pi \Gamma e^2)} \ln(B \xi) &\geq 2|Z(t_0)| \quad \text{in the case } \xi > N \xi_0
\end{align*}
\]

and

\[
\Gamma \pi e^2 < 2(1 + 4\pi \Gamma e^2).
\]
The last one holds automatically. For the condition containing $|Z(t_0)|_\infty$, it suffices to choose $N \geq 1$ such that

$$\frac{e^2}{(1 + 4\pi e^2)} \ln(N e^2) = 2|Z(t_0)|_\infty.$$ 

This choice fixes $N$. Finally we choose

$$B = N|\partial_x Z(t_0)|_\infty(1 + 4\pi e^2)$$

which suffices for the two requirements involving $|\partial_x Z(t_0)|_\infty$. Consequently, we have met all the conditions. It means that for an arbitrary $Z(t_0) \in C^\infty(S^1)$ we have found a modulus $\omega \in \mathcal{O}$ such that it is kept by $Z(t_0) \in C^\infty(S^1)$ and is not violated for $t \geq t_0$ over the evolution. Hence Lemma 4 gives us the thesis. \hfill \Box

We have now all ingredients needed for our proof of Theorem 2. Lemmas 1, 3 imply that any initial datum in $H^s$, $s \geq 1$ gives rise to a locally-in-time smooth solution. Lemma 5 says that it can be continued for any $T < \infty$. Theorem 2 is proved. \hfill \Box

4.3. **Proof of Theorem 3.** Notice that the $W^{1, \infty}$-norm of $Z(t)$ is controlled by

$$C|\partial_x Z(t_0)|_\infty \Gamma \sim C|\partial_x Z(t_0)|_\infty e^{m \chi t},$$

compare inequality (21), Proposition 2 and and Step 3. of the proof of Theorem 2. Consequently, $u \sim e^{m \chi t} \partial_x Z$ may grow exponentially. On the other hand, a $L^2$-estimate for $Z$ solving the Burgers equation (5) implies an exponential decay. Hence, in the case when the exponential decay of Burgers controls (overweights) the exponential growth $e^{\chi m t}$, we obtain a global bound (an exponential decay, respectively) of solutions to (1) - (3).

Below we provide a rigorous proof of this result via a slightly different procedure. It allows for a wider range of parameters $\chi, m$ that admit an exponential decay, than the one sketched above.

Recall that we restricted ourselves here to the periodic torus $[-\pi, \pi]$. We denoted the full Hilbert norm as $|f|_{H^k} = |f|_k$, where $k = 0$ stands for the $L^2$ norm. Since we will work in what follows with homogeneous Hilbert norms, for clarity, we will write them as $|\Lambda^k f|_0$.

For zero mean functions, we will need the Poincaré inequality

$$|f|^2_0 \leq |\Lambda^{\frac{1}{2}} f|^2_0$$

and the interpolation inequality

$$|f|^4_{L^4} \leq C I|f|^2_0|\Lambda^{\frac{1}{2}} f|^2_0.$$ 

Recall that by assumption we have a fixed number $\chi m < 1$. $C$ is a generic constant that may vary between lines.

**Zero-order decays.** Testing equation (16) with $W$, we obtain

$$\frac{1}{2} \frac{d}{dt}|W(t)|^2_0 \leq \chi m |W(t)|^2_0 - |\Lambda^{\frac{1}{2}} W(t)|^2_0.$$  

Hence

$$|W(t)|_0 \leq e^{(-1+\chi m)t}|W(0)|_0,$$

where we have used the Poincaré inequality (36). Using the assumption $\chi m < 1$, we can conclude from (38) that

$$\int_0^\infty |\Lambda^{\frac{1}{2}} W(s)|^2_0 ds \leq C.$$ 

Let us denote by $f(x^*) = \max_{x \in S^1} f(x)$. Using the formula (13) for $\Lambda$ we obtain

$$\Lambda f(x^*) = \frac{1}{2\pi} \text{P.V.} \int_{-\pi}^{\pi} \frac{f(x) - f(x - y)}{\sin^2(y/2)} dy \geq f(x^*),$$
for a zero mean function $f$, since $\sin^2 1 \leq 1$. This inequality and tracking the spatial maximum of $W(t)$, i.e. $W(x, t)$ via (16), we obtain the $L^\infty$-decay

$$|W(t)|_{L^\infty} \leq e^{-(1+\chi m)t}|W(0)|_{L^\infty}.$$ 

The $L^2$ and $L^\infty$ decays allow us to choose $1 \ll T^*$ such that

$$\sup_{t \geq T^*} \chi (|W(t)|_0 + |W(t)|_{L^\infty}) \leq \min \left( 1 - \chi m, \frac{1}{2} \right)$$

for any $t \geq T^*$.

**Half-order decays.** Testing (16) with $\Lambda W$, we obtain

$$\frac{1}{2} \frac{d}{dt} |\Lambda^{1/2} W(t)|_0^2 \leq |\Lambda W(t)\|_{L^2}(1 + \chi |W(t)|_{L^\infty}) + \chi m|\Lambda^{1/2} W(t)|_0^2.$$ 

This estimate, together with the choice (39) and the Poincaré inequality (36) gives

$$\sup_{t \geq T^*} |\Lambda^{1/2} W(t)|_0^2 \leq |\Lambda^{1/2} W(T^*)|_0^2.$$ 

Consequently,

$$\int_0^\infty |\Lambda W(s)|_0^2 ds = \int_0^\infty |u(s) - m|^2 ds \leq C,$$

where the middle equality follows from $\partial_x W = u - m$, see (17), the fact that $\Lambda = \partial_x H$ and the $L^2$-isometry of the Hilbert transform $H$ for periodic, zero mean functions.

The $L^2$-estimate for the Keller-Segel problem (1) - (3) implies

$$\frac{1}{2} \frac{d}{dt} |u(t)|_0^2 \leq -|\Lambda^{1/2} u(t)|_0^2 + \frac{\chi}{2} |u(t)|_{L^4}^3$$

$$\leq -|\Lambda^{1/2} u(t)|_0^2 + \frac{\chi}{2} |u(t)|_{L^2} |u(t)|_{L^4}^2$$

$$\leq -|\Lambda^{1/2} u(t)|_0^2 + \frac{C}{2} |u(t)|_0^2 |\Lambda^{1/2} u(t)|_0,$$

where for the last inequality we used the interpolation (37). Hence Young’s and Gronwall’s inequalities, together with the bound (40), yield

$$\sup_{t \geq T^*} |u(t)|_0 \leq C,$$

and consequently

$$\int_0^\infty |\Lambda^{1/2} u(s)|_0^2 ds \leq C.$$ 

**First-order decays.** Testing equation (1) with $\Lambda u$, we obtain

$$\frac{1}{2} \frac{d}{dt} |\Lambda^{1/2} u(t)|_0^2 \leq -|\Lambda u(t)|_0^2 - \chi \int_{-\pi}^\pi (\partial_x u \partial_x v \Lambda u)(t) + \chi \int_{-\pi}^\pi (u - m) \Lambda u(t).$$

For the middle term on the right-hand side, we use the inequality

$$-\chi \int_{-\pi}^\pi (\partial_x u \partial_x v \Lambda u)(t) \leq \chi |\Lambda u(t)|_0^2 |\partial_x v|_{L^\infty}$$

in tandem with

$$|\partial_x v|_{L^\infty} = |W|_{L^\infty} \leq \frac{0.5}{\chi},$$

compare (39). For the last term we write

$$\chi \int_{-\pi}^\pi (u - m) \Lambda u = -\chi m |\Lambda^{1/2} u|_0^2 + \chi |\Lambda^{1/2} u|_0 |\Lambda^{1/2} u|_{L^2}^2 \leq C|\Lambda^{1/2} u|_0^2 (1 + |u|_0),$$

\footnote{For more details of this procedure, including its rigorousness, compare [1, 13, 22].}
where for the inequality we used
\[ |\Lambda^{\frac{1}{2}} u|_0^2 \leq C|\Lambda^{\frac{1}{2}} u|_0 u|_{L^\infty} \leq C|\Lambda^{\frac{1}{2}} u|_0 \Lambda u|_0. \]
Together, we obtain
\[ \frac{1}{2} \frac{d}{dt} |\Lambda^{\frac{1}{2}} u|_0^2 \leq -\frac{1}{4} |\Lambda u|_0^2 + C|\Lambda^{\frac{1}{2}} u|_0^4. \]
This, with aid of (41), implies
\[ \sup_{t \geq T^*} |\Lambda^{\frac{1}{2}} u(t)|_0 \leq C. \]
Therefore, using \( \partial_x W = u - m \) and Theorem 2 for \( t \leq T^* \), we arrive at
\[ (42) \quad \sup_{t \in \mathbb{R}} |W(t)|_1^2 \leq C. \]

**Asymptotics.** From (38) and (42), using interpolation in Sobolev spaces, we arrive at
\[ |(u - m)(t)|_0 \leq |W(t)|_1 \leq C|W(t)|_0^{1/3} |W(t)|_0^{2/3} \leq \Sigma e^{\frac{(t+\chi m)}{\sigma}}, \]
for certain, finite \( \Sigma, \sigma \).

\[ \square \]

### 5. Conclusion

We have showed that the critical fractal Keller-Segel problem in the periodic setting admits global-in-time, smooth solutions. This closes an open question, posed in [10] and present earlier in [20], at least in the periodic setting. Apart from this, we believe that both

- the observation that solutions to parabolic-elliptic Keller-Segel system are given as derivatives of the solutions to a Burgers-type equation, and
- the generalization of [26] methodology over equations with no scaling

may be useful for further studies.

Additionally, we obtained the steady-state asymptotics of solutions to (1) - (3), provided parameters \( \chi m \) are small enough.

Concerning the system (1) - (3), we have two questions in mind.

Firstly, it would be interesting to know whether the need for size restriction of \( \chi m \) in Theorem 3 is merely a technicality or there is an infinite-time blowup for large enough initial masses.

Secondly, how to generalize our result over wider class of Keller-Segel systems, especially over the doubly parabolic case.

### References

[1] Ascasibar, Y., Granero-Belinchón, Y., Moreno, J.M. (2013). An approximate treatment of gravitational collapse. *Physica D: Nonlinear Phenomena*, 262:71 – 82.

[2] Atkinson, R. P. D., Rhodes, C. J., MacDonald, D. W., Anderson, R. M. (2002). Scale-Free Dynamics in the Movement Patterns of Jackals. *Oikos*, vol. 98, Fasc. 1:134 – 140.

[3] Bartumeus, F., Peters, F., Pueyo, S., Marras, C., Catalan, J., (2003). Helical Lévy walks: Adjusting searching statistics to resource availability in microzooplankton. *Proc. Nat. Ac. Sci. USA (PNAS)*, vol. 100 no. 22:12771 – 12775, doi: 10.1073/pnas.2137243100.

[4] Biler., P. (1995). Growth and accretion of mass in an astrophysical model. *Appl. Math. (Warsaw)*, 23(2):179 – 189.

[5] Biler, P., Funaki, T., Woyczynski, W.A. (1998). Fractal Burgers equations. *J. Differential Equations* 148 (1998), no. 1.

[6] Biler, P., Karch, G., (2010). Blowup of solutions to generalized Keller-Segel model. *J. Evol. Equ.* 10.

[7] Biler, P., Karch, G., Laurençot, Ph. (2009). Blowup of solutions to a diffusive aggregation model. *Nonlinearity* 22 (2009), no. 7.

[8] Biler, P., Nadzieja, T. (1994). Existence and nonexistence of solutions for a model of gravitational interaction of particles, i. *Colloq. Math.*, vol. 6:319–334.

[9] Biler, P., Wu, G. (2009). Two-dimensional chemotaxis models with fractional diffusion. *Math. Methods Appl. Sci.* 32, no. 1:112 – 126.

[10] Bourbaveas, N., Calvez V. (2010). The one-dimensional Keller-Segel model with fractional diffusion. *Nonlinearity*, 23(4):923 – 935.

[11] Burczak, J., Cieślak, T., Morales-Rodrigo, C. (2012). Global existence vs. blowup in a fully parabolic quasi-linear 1D Keller-Segel system. *Nonlinear Anal.* 75, no. 13: 5215 – 5228.
[12] Bae, H., Granero-Belinchón, R. (2015). Global existence for some transport equations with nonlocal velocity. Adv. Math. 269:197 – 219.

[13] Burczak, J., Granero-Belinchón, R. Boundedness of large-time solutions to a chemotaxis model with nonlocal and semilinear flux. To appear in Topological Methods in Nonlinear Analysis. Arxiv Preprint arXiv:1409.8102 [math.AP].

[15] Caffarelli, L., Silvestre, L. (2007). An extension problem related to the fractional Laplacian. Comm. Partial Differential Equations 32, no. 7-9:1245 – 1260.

[16] Castro, A., Córdoba, D. (2008). Global existence, singularities and ill-posedness for a nonlocal and semilinear flux. Advances in Mathematics, 219(6):1916–1936.

[17] Chae, D., Córdoba, A., Córdoba, D., Fontelos, M.A. (2005). Finite time singularities in a 1D model of the quasi-geostrophic equation. Adv. Math. 194, no. 1:203 – 223

[18] Córdoba, A., Córdoba, D., Fontelos, M.A. (2005). Formation of singularities for a transport equation with nonlocal velocity. Ann. of Math. (2) 162, no. 3.

[19] Cole, B., J. (1995). Fractal time in animal behavior: the movement activity of Drosophila. An. Behav. 50:1317 – 1324.

[20] Escudero., C.. (2006). The fractional Keller-Segel model, Nonlinearity, 19(12):2909 – 2918.

[21] Focardi, S., Marcellini, P., Montanaro, P. (1996). Do Ungulates Exhibit a Food Density Threshold? A Field Study of Optimal Foraging and Movement Patterns, Journal of Animal Ecology Vol. 65, No. 5:606 – 620

[22] Granero-Belinchón, R., Orive-Illera., R. (2014). An aggregation equation with a nonlocal flux. Nonlinear Analysis: Theory, Methods & Applications, 108(0):260 – 274.

[23] Hillen, T., Painter, K. J., (2009). A user’s guide to PDE models for chemotaxis. J. Math. Biol., 58(1-2):183 – 217.

[24] Granero-Belinchón, R., Hunter, J. (2015). On a nonlocal analog of the Kuramoto-Sivashinsky equation Nonlinearity, 28(4):1103 – 1133.

[25] Keller, E.F., Segel, L. A. (1971). Model for chemotaxis. J. Theor. Biol. 30.

[26] Kiselev, A., Nazarov, F., Shterenberg, R. (2008). Blow up and regularity for fractal Burgers equation. (2008). Dyn. Partial Differ. Equ. 5, no. 3: 2011 – 240.

[27] Klafter, J., Lewandowsky, M., White, B. S. Microzooplankton feeding behavior and the Lévy walk in Biological Motion, editors: W. Alt et al., Springer 1990.

[28] Klafter, J., Lewandowsky, M., White, B. S. Microzooplankton feeding behavior and the Lévy walk in Biological Motion, editors: W. Alt et al., Springer 1990.

[29] Lewandowsky, M., White, B. S., Schuster, F. L. (1997). Random movements of soil amebas, Acta Protozoologica, 36 (4):237 – 248.

[30] Li, D., Rodrigo, J. L. (2009). Refined blowup criteria and nonsymmetric blowup of an aggregation equation, Adv. Math. 220, no. 6.

[31] Li, D., Rodrigo, J. L. (2009). Finite-time singularities of an aggregation equation in Rn with fractional dissipation, Comm. Math. Phys. 287, no. 2.

[32] Li, D., Rodrigo, J. L., Zhang X. (2010). Exploding solutions for a nonlocal quadratic evolution problem, Rev. Mat. Iberoam. 26, no. 1.

[33] Li, D., Rodrigo, J. L. (2010). Wellposedness and regularity of solutions of an aggregation equation, Rev. Mat. Iberoam. 26, no. 1.

[34] Raichlen, D., A., Wood, B., M., Gordon, A., D., Mabulla, A., Z., Marlowe F., W., Pontzer, H. (2011). On a one-dimensional nonlocal flux with fractional dissipation. SIAM J. Math. Anal. 43, no. 1.

[35] Shlesinger, M., Klafter, J. (1986). Lévy Walks Versus Lévy Flights, in: On Growth and Form. Fractal and Non-Fractal Patterns in Physics, editors: Stanley, H., E., Ostrowsky, N., NATO ASI Series, Volume 100.

[36] Viswanathan, G. M., Afanasyev, V., Buldyrev, S. V., Murphy, E. J., Prince, P. A., Stanley, H. E. (1996). Lévy flight search patterns of wandering albatrosses, Nature 381: 413 – 415, 30 May 1996, doi:10.1038/381413a0.

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