DEFORMATIONS OF QUANTUM FIELD THEORIES AND CONNES–MARCOLLI’S RENORMALIZATION GROUP IN EPSTEIN–GLASER SCHEME

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1. INTRODUCTION

Over the decades, physicists have developed a number of state of the art techniques to produce quantities of great physical relevance out of mathematically ill-defined quantum field theories (QFTs). The general strategy is to define physical quantities in families in such a way that they are well-defined away from the certain limits of the parameters, and then to extract finite limits of these quantities by using regularization techniques.

The momentum space renormalizations have been widely used in QFTs. The combinatoric of divergencies and regularizations are beautifully encoded by Hopf algebras of Feynmann diagrams and Birkhoff decomposition of loops in these Hopf algebras (see [4, 5]). The complexified dimension plays the role of deformation parameter in Connes–Kreimer picture. In [6, 7], Connes & Marcolli have constructed a Riemann–Hilbert correspondence associated to perturbative renormalization based on Connes–Kreimer’s approach. They have observed an action of a pro-unipotent affine group scheme $U^*$, universal with respect to the physical theories, and pointed out its connection to the motivic Galois group of the scheme of 4-cyclotomic integers $\mathbb{Z}[\sqrt{i}]$.

On the other hand, Epstein–Glaser renormalization distinguishes itself among others: It produces finite QFTs from the very definition by choosing the domain of physical parameters suitably. The Epstein–Glaser’s approach is based on Dyson series

\begin{equation}
S = 1 + \sum_{n=1}^{\infty} \frac{(i)^n}{n!} \int_{M^n} dx_1 \cdots dx_n \, T_n(\mathcal{L}_I(x_1), \ldots, \mathcal{L}_I(x_n))
\end{equation}

of scattering operator (S-matrix) for a given potential term $\mathcal{L}_I$ of a Lagrangian. The problem in Epstein–Glaser setting is formulated as the problem of extensions of distributions $T_n(\mathcal{L}_I(x_1), \ldots, \mathcal{L}_I(x_n))$ defined on the configuration space $M^n \setminus \Delta$ of points on the spacetime $M$ to the diagonals $\Delta$.

Contrary to the common perception, that points at divergencies as sources of ambiguities in QFTs, ambiguities are still present in finite QFTs and are determined by distributions supported on the diagonals in Epstein–Glaser setting. This short note aims to describe the deformations of QFTs in terms of the distributions supported on the diagonals and then give an action of...
the pro-algebraic group $U^*$ which appears in Connes–Marcolli’s setting, on the finite QFTs constructed by Epstein-Glaser renormalization scheme.

This paper is organized as follows: In Section 2, we review some basic facts on Epstein–Glaser constructions of time ordered products. In the following section, we describe the deformation space of QFTs. In Section 4, we give an action of the pro-algebraic group $U^*$ on the space of QFTs. Finally, in Section 5, we discuss a number of corollaries of our constructions in §3 and §4, and their connections to some other renormalization related problems.

2. Epstein-Glaser renormalization in a nutshell

Let spacetime $M$ be Euclidian space $\mathbb{R}^d$, and $D(M)$ be the space of test functions on $M$ with the usual topology. Let $\mathcal{H}$ denote the Hilbert space of the free fields, $D$ a suitable dense subspace and $\Omega$ be the vacuum state.

2.1. Time ordered products. Time ordered products form a collection of operator valued distributions

\[
\{ T_N : D(M^N) \to \text{End}(D) \mid N := \{1, \ldots, n\}\},
\]

and, in Epstein-Glaser renormalization scheme, they are expected to satisfy a set of basic properties:

2.1.1. Symmetry. $T_N$'s are symmetric under permutations of indices, i.e.,

\[
T_N(f_1 \otimes \cdots \otimes f_n) = T_N(f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)})
\]

for all $\sigma$ in the symmetric group of index set $N$.

2.1.2. Causality. $T_N$ factorizes casually, i.e., if $I, I^c \neq \emptyset$ is a partition of $N$, and if $\text{supp}(f_i) \cap \text{supp}(f_j) = \emptyset$ for all $i \in I$ and $j \in I^c$, then

\[
T_N(f_1 \otimes \cdots \otimes f_n) = T_I(\bigotimes_{i \in I} f_i) \cdot T_{N \setminus I}(\bigotimes_{j \in I^c} f_j).
\]

2.1.3. Translation invariance. $T_N$ is invariant under translations:

\[
T_N(f_1(x_1) \otimes \cdots \otimes f_n(x_n)) = T_N(f_1(x_1 - a) \otimes \cdots \otimes f_n(x_n - a)).
\]

Epstein & Glaser constructed time ordered products essentially by using the causality in [9].

**Theorem 2.1** (Epstein & Glaser, [9]). Time-ordered products exist.

2.2. Wick expansions of time ordered products. The extension problem of operator valued distributions is reduced to an extension problem for numerical distributions by expanding time ordered products in terms of the Wick expansions.
Theorem 2.2. Let \( \phi^{k_1}(x_1), \ldots, \phi^{k_n}(x_n) \): Wick monomials for non-coinciding points \( x_1, \ldots, x_n \) in \( M \). Then

\[
T_N(\phi^{k_1}(x_1), \cdots, \phi^{k_n}(x_n)) = \sum_{\mathbf{J}=(i_1,\ldots,i_n)=0}^{(k_1,\ldots,k_n)} t_{\mathbf{J}}(x_1,\ldots,x_n) \times \frac{\phi^{i_1} \cdots \phi^{i_n}}{i_1! \cdots i_n!}
\]

where the numerical distribution \( t_{\mathbf{J}}(x_1,\ldots,x_n) \) is

\[
\langle \Omega, T_N(\phi^{k_1-i_1}(x_1), \cdots, \phi^{k_n-i_n}(x_n)) \Omega \rangle
\]

(for instance, see Theorem 2.4 in [2]).

3. Deformations of QFTs in Epstein–Glaser scheme

One of the main consequences of Epstein–Glaser construction is that the space which parameterizes the collection of time ordered products can be observed explicitly:

Lemma 3.1. Let \( T_N, \hat{T}_N \) be two different time ordered products. Then, the difference \( T_N - \hat{T}_N \) is an operator valued distribution supported on the diagonals \( \Delta \subset M^N \).

The proof of this lemma is straightforward and can be found in [18] and [14] for Minkowski and Euclidean cases respectively.

3.1. The space of QFTs. We can reformulate Lemma 3.1 on the level of numerical distributions and give the space of QFTs as follows: First, we use the translation invariance and set one of the points, say \( x_1 \), to 0, so that \( t_{\mathbf{J}} \in D(M^{1|N|-1}) \) for \( n \geq 2 \). Due to Lemma 3.1, we obtain a new distribution by adding another numerical distribution supported on the union of diagonals \( \Delta = \bigcup_{I \subset N} \Delta_I \) where \( \Delta_I := \{(0,x_2,\ldots,x_n) \mid x_i = x_i \text{ iff } i,j \in I \} \subset M^{1|N|-1} \), i.e.,

\[
t_{\mathbf{J}} \mapsto t_{\mathbf{J}} + d_{\mathbf{J}}, \quad \text{where } d_{\mathbf{J}} = \sum_{I \subset N} d_{\mathbf{J},I},
\]

and \( d_{\mathbf{J},I} \)'s are numerical distributions supported on the corresponding diagonals \( \Delta_I \subset M^{1|N|-1} \).

Due to the well known fact that distributions supported at one point are finite linear combinations of the \( \delta \) distribution and its derivatives, the summand supported on the deepest diagonal \( \Delta_N = \{0\} \) in (3.1) is given by

\[
d_{\mathbf{J},N} = \sum_{\alpha=(\alpha_1,\ldots,\alpha_d \text{ s.t. } \sum \alpha_s \leq sd(t_{\mathbf{J}}))}^\infty b_{\mathbf{J},N}^\alpha \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} \delta_N
\]

where \( \delta_N \) is the delta function supported on \( \{0\} \subset M^{1|N|-1} \). The degree is bounded by the generalized degree of homogeneity, called scaling degree

\[
sd(t_{\mathbf{J}}) := \inf \{s \mid \lim_{\lambda \to 0} \lambda^s \cdot \int t_{\mathbf{J}}(\lambda x) \omega(x)dx \}.
\]
The other $d_{J,I}$’s in (3.1) can be given as above case;

\[(3.2) \quad d_{J,I} = \sum_{\alpha=(\alpha_1,\ldots,\alpha_{nd}) \atop \sum \alpha_* \leq sd(t_J)} b^\alpha_{J,I} \cdot \partial_2^\alpha \cdots \partial_{nd}^\alpha \delta_I\]

where $\delta_I$ is the delta function supported on $\Delta_I \subset M^{[N]-1}$.

Hence, we can rephrase Lemma 3.1 as a deformation theory for QFTs in Epstein–Glaser setting: Let $\text{Def}(Q)$ be the space of QFTs around a given QFT determined by the set of numerical distributions $Q = \{t_J\}$.

**Theorem–Definition 3.1.** $\text{Def}(Q)$ is an infinite dimensional Euclidean space whose coordinate ring $H$ is $\mathbb{C}[b^\alpha_{J,I}]$ where $|J| > 2, |\alpha| \leq sd(t_J)$ and $I \subset N$.

It is important to note that $\text{Def}(Q)$ is unobstructed since the coordinate ring is $\mathbb{C}[b^\alpha_{J,I}]$, and therefore all $k$-th order deformations extends to the next order for all $k$.

### 3.2. Filtration of $\text{Def}(Q)$

$\text{Def}(Q)$ is filtered

$$\emptyset \subset \text{Def}^{(1)}(Q) \subset \cdots \subset \text{Def}^{(n)}(Q) \subset \text{Def}^{(n+1)}(Q) \subset \cdots$$

according to the cardinality of index set $J = (j_1,\ldots,j_n)$:

$$\text{Def}^{(n)}(Q) = \{d_{J} \mid |J| = n + 1\}.$$

for all $n = 1, 2, 3, \ldots$ The inclusions $\iota : I \hookrightarrow N$ of index sets induce imbeddings $\iota_\#: \text{Def}^{(|I|-1)}(Q) \hookrightarrow \text{Def}^{(|N|-1)}(Q)$.

### 4. Renormalization group in Epstein–Glaser scheme

**4.1. Symmetries acting on the space of QFTs.** The most general form of symmetries of $\text{Def}(Q)$ are given by the pseudo-group of all formal (local) diffeomorphisms $\mu : \text{Def}(Q) \to \text{Def}(Q)$. More elaborate symmetries form a Lie pseudo-group. They are prescribed by systems of nonlinear equations on jet bundles that are satisfying formal integrability and local solvability conditions. The remarkable fact is that the Maurer–Cartan form produces an explicit form of the pseudo-group structure equations, see [17].

A version of such a symmetry group, called the group of diffeographisms, is introduced as the group of formal diffeomorphisms tangent to the identity of the space of coupling constants of the theory by Connes & Kreimer in [5].

**4.2. Connes–Marcolli’s renormalization group in Epstein–Glaser setting.** From physics perspective, the subgroup of symmetries of $\text{Def}(Q)$ generated by the scaling transformations is of particular interest since it essentially gives the renormalization group.

In this paragraph, we present the action of a subgroup of scalings on the space of QFTs. Namely, we consider a pro-algebraic group of the form $\mathbb{U}^* = \mathbb{U} \times \mathbb{G}_m$ whose unipotent part $\mathbb{U}$ is generated by scaling transformations and is associated to the free graded Lie algebra $\mathcal{F}(1,2,\cdots)$, with one generator.
$e_n$ in each degree $n > 0$. The semi-direct product is given by the grading of $\mathbb{U}$.

4.2.1. *Pro-unipotent part* $\mathbb{U}$. Consider the scaling transformations

\begin{equation}
 b_{\mathbf{J},I}^0 \mapsto \sum_{K \subseteq I} \epsilon(\alpha, K, \mathbf{J}) b_{\mathbf{J},K}^0
\end{equation}

where

\begin{equation}
 \epsilon(\alpha, K, \mathbf{J}) = \begin{cases} 
 \lambda |K| & \text{if } j_1 > j_2 \\
 1 & \text{if } j_1 = j_2 \\
 0 & \text{if } j_1 < j_2.
\end{cases}
\end{equation}

They act upon the degree $n$ piece $\text{Def}^{(n)}(\mathcal{Q})$ of $\text{Def}(\mathcal{Q})$. Note that, the definition of $\epsilon$ guarantees that the matrix in (4.1) is upper triangular and therefore the transformation in (4.1) is pro-unipotent.

The infinitesimal generator of (4.1) is given by the following vector field

\begin{equation}
 e_n = \sum_{\mathbf{J}: j_1 > j_2} \sum_{I \subset \mathbb{N}, \alpha} \left( \sum_{K \subseteq I} |K| b_{\mathbf{J},K}^0 \frac{\partial}{\partial b_{\mathbf{J},K}^0} \right)
\end{equation}

in $T^*\text{Def}(\mathcal{Q})$. The pro-unipotent part $\mathbb{U}$ is associated to free graded Lie algebra $\mathcal{F}(1,2,\cdots)$, which is generated by the elements $e_n$ at each positive degree $n$.

4.2.2. *Multiplicative group* $\mathbb{G}_m$ and semi-direct product. Consider the 1-parameter group of automorphisms

\[ \theta_z : d_{\mathbf{J}} \mapsto e^{nz} \cdot d_{\mathbf{J}}, \quad \forall z \in \mathbb{C} \]

implementing the grading. Its infinitesimal generator is given by the grading operator

\[ Y(d_{\mathbf{J}}) := \left. \frac{d}{dz} (\theta_z d_{\mathbf{J}}) \right|_{z=0} = n \cdot d_{\mathbf{J}}. \]

Finally, we define, for all $u \in \mathbb{G}_m$, an action $u^Y$ on $\mathbb{U}$ by

\[ u^Y(X) = u^n X, \quad \forall X \text{ of degree } n. \]

We can then form the semi-direct product

\[ \mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m \]

and this shows that

**Theorem 4.1.** The pro-algebraic group $\mathbb{U}^*$ acts upon the space $\text{Def}(\mathcal{Q})$.

$\mathbb{U}^*$ is universal with respect to the set of physical theories, in the sense that it is canonically defined and independent of the physical theory.
Remark 4.2. In their seminal paper [6], Connes & Marcolli considered the same $U^*$ as renormalization group. Their motivation is to identify the renormalization group as a motivic Galois group as Cartier suggested in [3]. In their approach, the pro-unipotent part is the graded dual of the universal enveloping algebra $U(\mathcal{F}(1, 2, \cdots)_{\bullet})^\vee$ as a Hopf algebra. Then, they showed that the Tannakian category of flat equisingular connections which they have obtained from the differential system of counterterms is equivalent to a category of representations of the affine groups scheme $U^*$.

5. Remarks and further directions

There are numerous connections between Epstein–Glaser and other approaches to QFTs. Below, we summarize a few direct corollaries of the discussions of the previous section and speculate on a few possible applications in related fields.

5.0.3. Spacetime other than Euclidean spaces. In §4.2, we have presented an action of pro-algebraic group $U^*$ on the space $\text{Def}(Q)$ of $S$-matrices on Euclidean spacetime. However, the basics of this approach can be directly adapted for any spacetime manifold $M$. Once the time ordered products are given in terms of numerical distributions (as in [2], for instance), one can define a representation of $U^*$ by considering scaling properties of distributions supported on the diagonals $\Delta \subset M^n$. A construction of time ordered products for curved space-time along with a discussion of renormalization group which is very close to our description here can be found in [12].

5.0.4. Causal treatment of gauge theories. In their papers [15] and [20, 21], Kreimer and van Suijlekom extended the results of Connes & Marcolli to gauge field theories by discussing the Slavnov-Taylor identities for the couplings at the Hopf algebra level. Van Suijlekom showed that the Slavnov-Taylor identities generate a Hopf ideal of Hopf algebra of Feynman diagrams. Hence, these identities are compatible with renormalization, and the affine group scheme $U^*$ remains as a part of the renormalization picture.

On the other hand, using Epstein–Glaser in gauge field theories is not new to physics literature and has been studied extensively in both abelian and non-abelian gauge theories (for instance, see [8, 11, 13, 10, 19]). The gauge invariance condition in the causal approach is expressed in every order of perturbation theory separately by a relation of the $n$-point distributions $T_N$ with the charge $Q$, the generator of the free operator gauge transformations $[Q, T_N] = d_Q T_N$,

and this equation essentially encodes Slavnov-Taylor identities. By using the gauge invariant distributions supported on the diagonals, one gives the role of symmetry group of perturbative gauge theories to same $U^*$. The essential tool for casual approach in gauge theories is time ordered products in
Grassmann variables and it can be found in [19]. A very detailed exposition for Yang-Mills theory can be found in [11].

5.0.5. Epstein-Glaser vs. dimensional regularization. Even though, our main theorem states an action of the same pro-algebraic group $U^*$ on perturbative QFT’s as in Connes–Marcolli’s case, the action is quite different in nature. The main distinction is that $U^*$ acts upon the counterterms in their case. By contrast, the representation in [12] is given by an action on the renormalized values.

Moreover, in Connes–Marcolli’s construction, the affine group scheme $U^*$ appears as a motivic Galois group. However, it is unclear to us whether $U^*$ has any direct motivic role in Epstein–Glaser renormalization in the form discussed above. This question simply arises from the fact that the integrals in [11] contain distributions not rational functions, and therefore they are not periods.

5.0.6. Feynmann motives and motives of configuration spaces. There is an ongoing search for motivic origins of Feynman amplitudes (for an extensive account, see [16] and reference therein). This is a program initiated by Kontsevich’s suggestion that these numbers should be related to mixed Tate motives. There are several positive and negative results in this direction.

Epstein–Glaser approach hints a motivic treatment of Feynman integrals: Feynman rules associates a distribution to each Feynman graph on a configuration space which is also determined by the same graph. The divergencies of these integrals can be treated by the techniques that we have used for time ordered products. Alternatively, one can use Fulton–MacPherson type of compactifications of these configuration spaces and try to obtain regularized integrals on them. Fulton–MacPherson compactifications of these configuration spaces are mixed Tate motives when the spacetime is itself mixed Tate. This observation is quite intriguing and we are planning to discuss this approach in a subsequent paper.

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