Approximation by multivariate quasi-projection operators
and Fourier multipliers

Yurii Kolomoitsev\textsuperscript{1,2} and Maria Skopina\textsuperscript{3,4}

\textsuperscript{1}Universität zu Lübeck, Lübeck, Germany; kolomoitsev@math.uni-luebeck.de
\textsuperscript{2}Institute of Applied Mathematics and Mechanics of NAS of Ukraine, Slov’yan’s’k, Ukraine
\textsuperscript{3}Saint Petersburg State University, St. Petersburg, Russia; skopina@ms1167.spb.edu
\textsuperscript{4}Regional Mathematical Center of Southern Federal University

Abstract

Multivariate quasi-projection operators $Q_j(f, \varphi, \tilde{\varphi})$, associated with a function $\varphi$ and a distribution/function $\tilde{\varphi}$, are considered. The function $\varphi$ is supposed to satisfy the Strang-Fix conditions and a compatibility condition with $\tilde{\varphi}$. Using technique based on the Fourier multipliers, we studied approximation properties of such operators for functions $f$ from anisotropic Besov spaces and $L^p$ spaces with $1 \leq p \leq \infty$. In particular, upper and lower estimates of the $L^p$-error of approximation in terms of anisotropic moduli of smoothness and anisotropic best approximations are obtained.

Keywords. Quasi-projection operator, Besov space, Error estimate, Anisotropic best approximation, Anisotropic moduli of smoothness, Fourier multipliers

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1 Introduction

The multivariate quasi-projection operator with a matrix dilation $M$ is defined as:

$$Q_j(f, \varphi, \tilde{\varphi}) = |\det M|^j \sum_{n \in \mathbb{Z}^d} \langle f, \tilde{\varphi}(M^j \cdot + n) \rangle \varphi(M^j \cdot + n),$$

where $\varphi$ is a function, $\tilde{\varphi}$ is a tempered distribution, and $\langle f, \tilde{\varphi}(M^j \cdot + n) \rangle$ is an appropriate functional.

The class of operators $Q_j(f, \varphi, \tilde{\varphi})$ is quite large. It includes the operators associated with a regular function $\tilde{\varphi}$, in particular, the so-called scaling expansions appearing in wavelet constructions (see, e.g., \cite{3, 11, 12, 20, 21, 28}) as well as the Kantorovich-Kotelnikov operators and their generalizations (see, e.g., \cite{8, 16, 18, 25, 33}). An essentially different class consists of the operators $Q_j(f, \varphi, \tilde{\varphi})$ associated with a tempered distribution $\tilde{\varphi}$ related to the Dirac delta-function (the so-called sampling-type operators). The model example of such operators is the following classical sampling expansion, appeared originally in the Kotelnikov formula,

$$\sum_{n \in \mathbb{Z}} f(-2^{-j}n) \frac{\sin \pi(2^j x + n)}{\pi(2^j x + k)} = 2^j \sum_{n \in \mathbb{Z}} \langle f, \delta(2^j \cdot + n) \rangle \text{sinc}(2^j x + n),$$

where $\delta$ is the Dirac delta-function and $\text{sinc} x := \frac{\sin \pi x}{\pi x}$. In recent years, many authors have studied approximation properties of the sampling-type operators for various functions $\varphi$ (see, e.g., \cite{4, 5, 13, 15, 18, 21, 27, 32}). Consideration of functions $\varphi$ with a good decay is very useful for different engineering applications. In particular, the operators associated with a linear combination of B-splines as $\varphi$, and the Dirac delta-function as $\tilde{\varphi}$, was studied, e.g., in \cite{2, 6, 27}. For a class of fast decaying functions $\varphi$,
the sampling-type quasi-projection operators were considered in [21], where the error estimates in the $L_p$-norm, $p \geq 2$, were obtained in terms of the Fourier transform of $f$, and the approximation order of the operators was found in the case of an isotropic matrix $M$. These results were extended to an essentially wider class of functions $\varphi$ in [7] (see Theorem A below). Next, in the paper [17], the results of [21] were improved in several directions. Namely, the error estimates were obtained also for the case $1 \leq p < 2$, the requirement on the approximated function $f$ were weakened, and the estimates were given in terms of anisotropic moduli of smoothness and best approximations.

The main goal of the present paper is to extend the results of [17] to band-limited functions $\varphi$ and to the case $p = \infty$. The scheme of the proofs of our results is similar to the one given in [17], but the technic is essentially refined by means of using Fourier multipliers. This development allows also to improve the results for the class of fast decaying functions $\varphi$ and to obtain lower estimates for the $L_p$-error of approximation by quasi-projection operators in some special cases. Similarly, the main result of [16] (see Theorem B below) is essentially extended in several directions (lower estimates, fractional smoothness, approximation in the uniform metric).

The paper is organized as follows. Notation and preliminary information are given in Sections 2 and 3, respectively. Section 4 contains auxiliary results. The main results are presented in Section 5. In particular, the $L_p$-error estimates for quasi-projection operators $Q_j(f, \varphi, \bar{\varphi})$ in the case of weak compatibility of $\varphi$ and $\bar{\varphi}$ are obtained in Subsection 5.2. In this subsection, we also consider lower estimates for the $L_p$-error and a generalization of compatibility conditions to the case of fractional smoothness. Subsection 5.3 is devoted to approximation by operators $Q_j(f, \varphi, \bar{\varphi})$ in the case of strict compatibility $\varphi$ and $\bar{\varphi}$. Two generalizations of the Whittaker–Nyquist–Kotelnikov–Shannon-type theorem are also proved in this subsection.

2 Notation

As usual, we denote by $\mathbb{N}$ the set of positive integers, $\mathbb{R}^d$ is the $d$-dimensional Euclidean space, $\mathbb{Z}^d$ is the integer lattice in $\mathbb{R}^d$, $\mathbb{Z}_+^d := \{x \in \mathbb{Z}^d : x_k \geq 0, k = 1, \ldots, d\}$, and $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is the $d$-dimensional torus. Let $x = (x_1, \ldots, x_d)^T$ and $\xi = (\xi_1, \ldots, \xi_d)^T$ be column vectors in $\mathbb{R}^d$, then $(x, \xi) := x_1\xi_1 + \cdots + x_d\xi_d$, $|x| := \sqrt{x(x)}$, $0 = (0, \ldots, 0)^T \in \mathbb{R}^d$, and $B_\delta = \{x \in \mathbb{R}^d : |x| < \delta\}$.

Given $a, b \in \mathbb{R}^d$ and $\alpha \in \mathbb{Z}_+^d$, we set

$$[\alpha] = \sum_{k=1}^d \alpha_k, \quad D^\alpha f = \frac{\partial^{[\alpha]} f}{\partial x^\alpha} = \frac{\partial^{[\alpha]} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}, \quad (ab)^j = \prod_{j=1}^d a_j b_j, \quad \alpha! = \prod_{j=1}^d \alpha_j!.$$ 

If $M$ is a $d \times d$ matrix, then $\|M\|$ denotes its operator norm in $\mathbb{R}^d$; $M^*$ denotes the conjugate matrix to $M$, $m = |\det M|$. By $I$ we denote the identity matrix, i.e., $I = M^0$.

A $d \times d$ matrix $M$ is called a dilation matrix if its eigenvalues are bigger than one in modulus. We denote the set of all dilation matrices by $\mathfrak{M}$. It is well known that $\lim_{j \to \infty} \|M^{-j}\| = 0$ for dilation matrices. For any $M \in \mathfrak{M}$, we set $\mu_0 := \min\{\mu \in \mathbb{N} : \mathbb{T}^d \subset \frac{1}{\mu} M^\nu \mathbb{T}^d \text{ for all } \nu \geq \mu - 1\}$.

Recall that a matrix $M$ is isotropic if it is similar to a multiple of an orthogonal matrix, its eigenvalues $\lambda_1, \ldots, \lambda_d$ are such that $|\lambda_1| = \cdots = |\lambda_d|$.

As usual, $L_p$ denotes the space $L_p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, with the norm $\| \cdot \|_p = \| \cdot \|_{L_p(\mathbb{R}^d)}$, $C$ denotes the space of all uniformly continuous bounded functions on $\mathbb{R}^d$, and

$$C_0 := \{f \in C : \lim_{|x| \to \infty} f(x) = 0\}.$$ 

We use $W^n_p$, $1 \leq p \leq \infty$, $n \in \mathbb{N}$, to denote the Sobolev space on $\mathbb{R}^d$, i.e. the set of functions whose derivatives up to order $n$ are in $L_p$, with usual Sobolev norm.

If $f$ and $g$ are functions defined on $\mathbb{R}^d$ and $f \mathcal{G} \in L_1$, then

$$\langle f, g \rangle := \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx.$$
The convolution of functions $f$ and $g$ is defined by

$$f * g(x) = \int_{\mathbb{R}^d} f(t)g(x-t)dt.$$  

The Fourier transform of $f \in L_1$ is given by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} dx.$$  

For any function $f$, we denote $f^{-}(x) = \overline{f(-x)}$.

The Schwartz class of functions defined on $\mathbb{R}^d$ is denoted by $\mathcal{S}$. The dual space of $\mathcal{S}$ is $\mathcal{S}'$, i.e. $\mathcal{S}'$ is the space of tempered distributions. Suppose $f \in \mathcal{S}$ and $\varphi \in \mathcal{S}'$, then $(f, \varphi) := \varphi(f)$. For any $\varphi \in \mathcal{S}'$, we define $\overline{\varphi}$ and $\varphi^-$ by $(f, \overline{\varphi}) := \overline{(f, \varphi)}$, $f \in \mathcal{S}$, and $(f, \varphi^-) := (f, \varphi)$, $f \in \mathcal{S}$, respectively.

The Fourier transform of $\overline{\varphi}$ is defined by $(\hat{f}, \overline{\varphi}) = (f, \varphi)$, $f \in \mathcal{S}$. The convolution of $\varphi \in \mathcal{S}'$ and $f \in \mathcal{S}$ is given by $f * \varphi(x) = (f, \varphi(x \cdot)) = (f, \varphi(\cdot - x))$. For suitable functions/distributions $f$ and $h$, we denote by $\Lambda_h(f)$ the following multiplier operator:

$$\Lambda_h(f) := \mathcal{F}^{-1}(h\hat{f}).$$  

Next, for a fixed matrix $M \in \mathfrak{M}$ and a function $\varphi$, we define $\varphi_{jk}$ by

$$\varphi_{jk}(x) := m^{j/2}\varphi(M^jx + k), \quad j \in \mathbb{Z}, \quad k \in \mathbb{R}^d.$$  

For $\overline{\varphi} \in \mathcal{S}'$, $j \in \mathbb{Z}$, and $k \in \mathbb{Z}^d$, we define $\overline{\varphi}_{jk}$ by

$$(f, \overline{\varphi}_{jk}) := (f_{-j,-M^{-j}k}, \overline{\varphi}), \quad f \in \mathcal{S}.$$  

By $\mathcal{S}'_N$, $N \geq 0$, we denote the set of all tempered distribution $\widehat{\varphi}$ whose Fourier transform $\widehat{\varphi}$ is a measurable function on $\mathbb{R}^d$ such that $|\widehat{\varphi}(\xi)| \leq c(\widehat{\varphi})(1 + |\xi|)^N$ for almost all $\xi \in \mathbb{R}^d$.

Let $1 \leq p \leq \infty$. We set

$$\mathcal{L}_p := \left\{ \varphi \in L_p : \|\varphi\|_{\mathcal{L}_p} := \left\| \sum_{l \in \mathbb{Z}^d} |\varphi(\cdot + l)| \right\|_{L_{p}(\mathbb{T}^d)} < \infty \right\}.$$  

It is not difficult to see that $\mathcal{L}_1 = L_1$, $\|\varphi\|_{p} \leq \|\varphi\|_{\mathcal{L}_p}$, and $\|\varphi\|_{\mathcal{L}_q} \leq \|\varphi\|_{\mathcal{L}_p}$ for $1 \leq q \leq p \leq \infty$.

For any $d \times d$ matrix $A$, we introduce the space

$$\mathcal{B}_{A,p} := \{ g \in L_p : \operatorname{supp} \hat{g} \subset A^* \mathbb{T}^d \}$$  

and the corresponding anisotropic best approximations

$$E_{A}(f)_p := \inf \{ \| f - P \|_p : P \in \mathcal{B}_{A,p} \}.$$  

Let $\alpha$ be a positive function defined on the set of all $d \times d$ matrices $A$. We consider the following anisotropic Besov-type space associated with a matrix $A$. We say that $f \in \mathbb{B}^{\alpha(\cdot)}_{p,A}$, $1 \leq p \leq \infty$, if $f \in L_p$ for $p < \infty$, $f \in C_0$ for $p = \infty$, and

$$\| f \|_{\mathbb{B}^{\alpha(\cdot)}_{p,A}} := \| f \|_p + \sum_{\nu = 1}^{\infty} |\det A|^\nu \alpha(A\nu)E_{A\nu}(f)_p < \infty.$$  

Note that in the case $A = 2I$ and $\alpha(\cdot) \equiv \alpha_0 \in \mathbb{R}$, the space $\mathbb{B}^{\alpha(\cdot)}_{p,A}$ coincides with the classical Besov space $B^{d/p+\alpha_0}_{p}(\mathbb{R}^d)$.

For any matrix $M \in \mathfrak{M}$, we denote by $\mathcal{A}_M$ the set of all positive functions $\alpha : \mathbb{R}^{d \times d} \to \mathbb{R}_+$ that satisfy the condition $\alpha(M^{\mu+1}) \leq c(M)\alpha(M\mu)$ for all $\mu \in \mathbb{Z}_+$. 

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For any \( d \times d \) matrix \( A \), we introduce the anisotropic fractional modulus of smoothness of order \( s, s > 0 \),
\[
\Omega_s(f, A)_p := \sup_{|A^{-1}h| < 1} \| \Delta^s_h f \|_p,
\]
where
\[
\Delta^s_h f(x) := \sum_{\nu=0}^{\infty} (-1)^\nu \binom{s}{\nu} f(x + h\nu).
\]
Recall that the standard fractional modulus of smoothness of order \( s, s > 0 \), is defined by
\[
\omega_s(f, t)_p := \sup_{|h| < t} \| \Delta^s_h f \|_p, \quad t > 0.
\]
We refer to [19] for the collection of basic properties of moduli of smoothness in \( L_p(\mathbb{R}^d) \).

For an appropriate function \( f \) and \( s > 0 \), the fractional power of Laplacian is given by
\[
(-\Delta)^{s/2} f(x) := \mathcal{F}^{-1} \left( |\xi|^s \hat{f}(\xi) \right)(x).
\]

As usual, \( \ell_p, 1 \leq p \leq \infty \), denotes the space of all sequences \( a = \{a_n\}_{n \in \mathbb{Z}^d} \subset \mathbb{C} \) equipped with the norm
\[
\|a\|_{\ell_p} := \left\{ \begin{array}{ll}
\left( \sum_{n \in \mathbb{Z}^d} |a_n|^p \right)^{1/p}, & \text{if } p < \infty, \\
\sup_{n \in \mathbb{Z}^d} |a_n|, & \text{if } p = \infty,
\end{array} \right.
\]
and \( c_0 \) denotes the subspace of \( \ell_\infty \) consisting of the sequences converging to zero.

By \( \eta \) we denote a real-valued function in \( C^\infty(\mathbb{R}^d) \) such that \( \eta(\xi) = 1 \) for \( \xi \in \mathbb{T}^d \) and \( \eta(\xi) = 0 \) for \( \xi \notin 2\mathbb{T}^d \). For any \( \delta > 0 \), we denote \( \eta_\delta = \eta(\delta^{-1}) \).

Finally, for any \( p \in [1, \infty] \), we define \( p' \) by \( 1/p' + 1/p = 1 \) and write \( c, c_1, c_2, \ldots \) to denote positive constants that depend on indicated parameters.

## 3 Preliminary information and main definitions.

In what follows, we discuss the operator
\[
Q_j(f, \varphi, \overline{\varphi}) := \sum_{k \in \mathbb{Z}^d} \langle f, \overline{\varphi}_{jk} \rangle \varphi_{jk},
\]
where the "inner product" \( \langle f, \overline{\varphi}_{jk} \rangle \) has meaning in some sense. This operator is associated with a matrix \( M \), which is a matrix dilation by default.

The operator \( Q_j(f, \varphi, \overline{\varphi}) \) is an element of the shift-invariant space generated by the function \( \varphi \). It is known that a function \( f \) may be approximated by the elements of such shift-invariant space only if \( \varphi \) satisfies the so-called Strang-Fix conditions.

**Definition 1** We say that a function \( \varphi \) satisfies the Strang-Fix conditions of order \( s \) if \( D^\beta \overline{\varphi}(k) = 0 \) for every \( \beta \in \mathbb{Z}^d_+, |\beta| < s \), and for all \( k \in \mathbb{Z}^d \setminus \{0\} \).

Certain compatibility conditions for a distribution \( \overline{\varphi} \) and a function \( \varphi \) is also required to provide good approximation properties of the operator \( Q_j(f, \varphi, \overline{\varphi}) \). For our purposes, we will use the following two conditions.

**Definition 2** A tempered distribution \( \overline{\varphi} \) and a function \( \varphi \) is said to be weakly compatible of order \( s \) if \( D^\beta (1 - \overline{\varphi}\varphi)(0) = 0 \) for every \( \beta \in \mathbb{Z}^d_+, |\beta| < s \).

**Definition 3** A tempered distribution \( \overline{\varphi} \) and a function \( \varphi \) is said to be strictly compatible if there exists \( \delta > 0 \) such that \( \overline{\varphi}(\xi)\varphi(\xi) = 1 \) a.e. on \( \delta\mathbb{T}^d \).
For $\tilde{\varphi} \in S'_N$ and different classes of functions $\varphi$, approximation properties of quasi-projection operators $Q_j(f, \varphi, \tilde{\varphi})$ were studied in [23], [21], [18], and [13]. Generally speaking, if $\tilde{\varphi} \in S'$, then the functional $\langle f, \tilde{\varphi}_{jk} \rangle$ has meaning only for functions $f$ belonging to $S$. Under some additional restrictions on the distribution $\tilde{\varphi}$, the class of functions $f$ can be essentially extended. To this end, the quantity $\langle f, \tilde{\varphi}_{jk} \rangle$ was replaced by the inner product $\langle \tilde{f}, \tilde{\varphi}_{jk} \rangle$ in the mentioned papers. The following result is a combination of Theorem 14 from [7] and Theorem 5 from [21].

**Theorem A.** Let $2 \leq p < \infty$, $s \in \mathbb{N}$, $N \geq 0$, $\delta \in (0, 1/2)$, $M$ be an isotropic matrix, $\psi \in L_p$, and $\tilde{\psi} \in S_N$. Suppose

1) $\tilde{\psi} \in L_{p'}$ and $\sum_{k \in \mathbb{Z}^d} |\tilde{\psi}(\xi + k)|^{p'} < c_1$ for all $\xi \in \mathbb{R}^d$;

2) $\tilde{\psi}(\cdot + l) \in C^s(B_{\delta})$ for all $l \in \mathbb{Z}^d \setminus \{0\}$ and $\sum_{l \neq 0} \sum_{\|\theta\|_1 = s} \sup_{|\xi| < \delta} |D^\beta \tilde{\psi}(\xi + l)|^{p'} < c_2$;

3) the Strang-Fix conditions of order $s$ are satisfied for $\psi$;

4) $\tilde{\psi} \in C^s(B_{\delta})$;

5) $\psi$ and $\tilde{\psi}$ are weakly compatible of order $s$.

If $\tilde{f} \in L_{p'}$, and $\tilde{f}(\xi) = O(\|\xi\|^{-N-d-\varepsilon})$, $\varepsilon > 0$, as $|\xi| \to \infty$, then

$$\left\| f - \lim_{N \to \infty} \sum_{\|k\| \leq N} \frac{\langle \tilde{f}, \tilde{\psi}_{jk} \rangle \psi_{jk}}{\|k\|^{N-d-\varepsilon}} \right\|_p \leq C \begin{cases} |\lambda|^{-j(N+d/p+\varepsilon)} & \text{if } s > N + d/p + \varepsilon \\ (j+1)^{1/p'} |\lambda|^{-js} & \text{if } s = N + d/p + \varepsilon \\ |\lambda|^{-js} & \text{if } s < N + d/p + \varepsilon \end{cases},$$

where $\lambda$ is an eigenvalue of $M$ and the constant $C$ is independent on $j$.

This result is obtained for a wide class of operators $Q_j(f, \psi, \tilde{\psi})$, but unfortunately, the error estimate is given only for $p \geq 2$. Another drawback of this theorem is the restriction on the decay of $\tilde{f}$. It is not difficult to see that it is redundant, for example, if $\tilde{\psi} \in L_{p'}$ and $f \in L_p$. Also, although Theorem A provides approximation order for $Q_j(f, \psi, \tilde{\psi})$, more accurate error estimates in terms of smoothness of $f$ were not obtained.

The mentioned drawbacks of Theorem A were avoided in [16], where a class of Kantorovich-type operators $Q_j(f, \varphi, \tilde{\varphi})$ associated with a regular function $\varphi$ and a bandlimited function $\varphi$ was considered. In particular, the next theorem was obtained in [16] Theorem 17. To formulate it, we introduce the space $B$, which consists of functions $\varphi$ given by $\varphi = F^{-1}\theta$, where $\supp \theta \subset [a,b] := \{a_1, b_1\} \times \cdots \times \{a_d, b_d\}$ and $|\theta|_{[a,b]} \in C^d([a,b])$.

**Theorem B.** Let $1 < p < \infty$, $s \in \mathbb{N}$, $\delta > 0$, and $\varepsilon \in (0, 1)$. Suppose

1) $\varphi \in B$, supp $\tilde{\varphi} \subset B_{1-\varepsilon}$, and $\tilde{\varphi} \in C^{s+d+1}(B_{\delta})$;

2) $\varphi \in B \cup L_{p'}$ and $\tilde{\varphi} \in C^{s+d+1}(B_{\delta})$;

3) $\varphi$ and $\tilde{\varphi}$ are weakly compatible of order $s$.

Then, for every $f \in L_p$, we have

$$\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\varphi}_{jk} \rangle \varphi_{jk} \right\|_p \leq c \omega_s (f, \|M^{-j}\|)_p,$$

where $c$ is independent on $f$ and $j$. 
In what follows, we will consider a class of quasi-projection operators \( Q_j(f, \varphi, \tilde{\varphi}) \) associated with a tempered distribution \( \tilde{\varphi} \) belonging to the class \( S'_p \), where \( 1 \leq p \leq \infty \), \( M \in \mathfrak{M} \), and \( \alpha \in \mathcal{A}_M \). We say that \( \tilde{\varphi} \in S'_p \) if \( \tilde{\varphi} \) is a measurable locally bounded function and
\[
\| \Lambda_{F(\tilde{\varphi})} P_\mu \|_p \leq \alpha(M^\mu) \| P_\mu \|_p \quad \text{for all} \quad P_\mu \in \mathcal{B}_{M^\mu} \cap L_2, \mu \in \mathbb{Z}_+.
\]  
Obviously, inequality (1) is satisfied with \( \alpha \equiv 1 \) if \( \tilde{\varphi} \) is the Dirac delta-function or \( \tilde{\varphi} \in L_1 \). If \( M = \text{diag}(m_1, \ldots, m_d) \) and \( \tilde{\varphi} \) is a distribution corresponding to the differential operator of the form \( \tilde{\varphi}(x) = D^\beta \delta(x) \), \( \beta \in \mathbb{Z}^d_+ \), then \( \tilde{\varphi} \) belongs to the class \( S'_p \) with \( \alpha(M) = m^{\beta_1} \cdots m^{\beta_d} \). If \( M \) is an isotropic matrix, then \( \alpha(M) = m^{1/d} \). This follows from the Bernstein inequality (see, e.g., [30, p. 252]) given by
\[
\| P^p \|_{L_p(\mathbb{R})} \leq c \| P \|_{L_p(\mathbb{R})}, \quad P \in L_p(\mathbb{R}), \quad \text{supp} \tilde{P} \subset [-\sigma, \sigma].
\]
Now we are going to extend the operator \( Q_j(f, \varphi, \tilde{\varphi}) \) with \( \tilde{\varphi} \in S'_p \) onto the Besov spaces \( \mathbb{B}^{\alpha(\cdot)}_{p,M} \) and the space \( C_0 \). For this, we need to define (extend) the functional \( \langle f, \tilde{\varphi}_{jk} \rangle \) in an appropriate way. In the case \( \alpha(M) = | \det M |^{1/d+1/p} \) and \( 1 \leq p < \infty \), a similar extension was given in [17].

**Definition 4** Let \( 1 \leq p \leq \infty \), \( M \in \mathfrak{M} \), \( \alpha \in \mathcal{A}_M \), and \( \delta \in (0,1) \). For \( \tilde{\varphi} \in S'_q \) and \( f \in \mathbb{B}^{\alpha(\cdot)}_{p,M} \) or \( \tilde{\varphi} \in S'_{\text{const, } \infty; M} \) and \( f \in C_0 \), we set
\[
\langle f, \tilde{\varphi}_{0k} \rangle := \lim_{\mu \to \infty} \langle \tilde{P}_\mu, \tilde{\varphi}_{0k} \rangle, \quad k \in \mathbb{Z}^d,
\]
where the functions \( \{ P_\mu \} \) are such that \( P_\mu \in \mathcal{B}_{kM^\mu} \cap L_2 \) and
\[
\| f - P_\mu \|_p \leq c(d, \mu) E_{\delta, M^\mu}(f)_p, \quad \delta = \begin{cases} \frac{\delta}{2} & \text{if } p = \infty, \\ \delta & \text{if } p < \infty. \end{cases}
\]
Set also
\[
\langle f, \tilde{\varphi}_{jk} \rangle := m^{-j/2} \langle f(M^{-1}), \tilde{\varphi}_{0k} \rangle, \quad j \in \mathbb{Z}_+.
\]
Some comments are needed to approve this definition. First, it will be proved in Lemma 9 that the limit in (2) exists and does not depend on a choice of \( P_\mu \) and \( \delta \). Second, in view of Lemmas 14 and 15 one can always find functions \( P_\mu \in \mathcal{B}_{kM^\mu} \cap L_2, 1 \leq p \leq \infty \), such that (3) holds. Third, we can write
\[
\langle \tilde{P}_\mu, \tilde{\varphi}_{0k} \rangle = \Lambda_{F(\tilde{\varphi})} P_\mu(-k).
\]
Finally, we mention that if \( \tilde{\varphi} \in L_{p'}, \) then \( \langle f, \tilde{\varphi}_{jk} \rangle \) is the standard inner product, which has meaning for any \( f \in L_p \).

**Remark 5** Note that if \( \tilde{\varphi} \in S'_N \) for some \( N \geq 0 \) and the Fourier transform of a function \( f \) has a sufficiently good decay such that the inner product \( \langle f, \tilde{\varphi}_{0k} \rangle \) has sense, then it is natural to define the operator \( Q_j(f, \varphi, \tilde{\varphi}) \) by setting \( \langle f, \tilde{\varphi}_{0k} \rangle := \langle \hat{f}, \tilde{\varphi}_{0k} \rangle \) (see, e.g., [21], [18], [15] as well as Theorem A). It is not difficult to see that such an operator \( Q_j(f, \varphi, \tilde{\varphi}) \) is the same as the corresponding operator defined by means of Definition 4 (see, e.g., [17]).

The main tools in this paper are Fourier multipliers. Let us recall their definition and basic properties.

**Definition 6** Let \( h \) be a bounded measurable function on \( \mathbb{R}^d \). Consider the linear transformation \( \Lambda_h \) defined by \( \Lambda_h(f) = F^{-1}(hf) \), \( f \in L_2 \). The function \( h : \mathbb{R}^d \to \mathbb{C} \) is called a Fourier multiplier in \( L_p \), \( 1 \leq p \leq \infty \), (we write \( h \in \mathcal{M}_p \)) if there exists a constant \( K \) such that
\[
\| \Lambda_h(f) \|_p \leq K \| f \|_p \quad \text{for any} \quad f \in L_2 \\
\| h \|_{\mathcal{M}_p}.
\]  
The smallest \( K \), for which inequality (5) holds, is called the norm of the multiplier \( h \). We denote this norm by \( \| h \|_{\mathcal{M}_p} \).

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Note that if \([\mathcal{L}]\) holds and \(1 \leq p < \infty\), then the operator \(\Lambda_h\) has a unique bounded extension to \(L_p\), which satisfies the same inequality. As usual, we denote this extension by \(\Lambda_h\).

Let us recall some basic properties of Fourier multipliers (see, e.g., [1] Ch. 6 and [24] Ch. 1):

(i) if \(1 < p < 2\), then \(M_1 \subset \mathcal{M}_p \subset M_2 = L_\infty\);
(ii) if \(1 \leq p \leq \infty\), then \(\mathcal{M}_p = \mathcal{M}_p'\) and \(\|h\|_{\mathcal{M}_p} = \|h\|_{\mathcal{M}_p'}\);
(iii) if \(h_1, h_2 \in \mathcal{M}_p\), then \(h_1 + h_2 \in \mathcal{M}_p\) and \(h_1 h_2 \in \mathcal{M}_p\);
(iv) if \(h \in \mathcal{M}_p\), then \(h(A) \in \mathcal{M}_p\) and \(\|h(A)\|_{\mathcal{M}_p} = \|h\|_{\mathcal{M}_p}\) for any non-singular matrix \(A\).

The classical sufficient condition for Fourier multipliers in \(L_p\), \(1 < p < \infty\), is Mikhlin’s condition (see, e.g., [10] p. 367), which states that if a function \(h\) is such that

\[
|D^\nu h(x)| \leq K|\xi|^{-|\nu|}, \quad \xi \in \mathbb{R}^d \setminus \{0\},
\]

for all \(\nu \in \mathbb{Z}^d, \|\nu\| \leq d/2 + 1\), then \(h \in \mathcal{M}_p\) for all \(1 < p < \infty\) and \(\|h\|_{\mathcal{M}_p} \leq c(p, d)(\|h\|_\infty + K)\).

Concerning the limiting cases \(p = 1\) and \(\infty\), we note that if \(h\) is a continuous function, then \(h \in \mathcal{M}_1\) if and only if \(h\) is the Fourier transform of a finite Borel (complex-valued) measure. The multiplier itself is a convolution of a function and this measure. Numerous efficient sufficient conditions for Fourier multipliers in \(L_1\) and \(L_\infty\) can be found in the survey [22]. Here, we only mention the Beurling-type condition, which states that if \(h \in \mathcal{W}_2^d\) with \(h > d/2\), then \(h \in \mathcal{M}_1\) (see, e.g., [22] Theorem 6.1).

Finally, we note that if \(\varphi \in \mathcal{B}\), then \(\hat{\varphi} \in \mathcal{M}_p\) for all \(1 < p < \infty\). Indeed, for any \(\varphi \in \mathcal{B}\), we have \(\hat{\varphi} = \chi_{\Pi} \cdot \theta\), where \(\theta\) belongs to \(C^d(\mathbb{R}^d)\) and has a compact support. It is well known that the characteristic function of \(\Pi\) is a Fourier multiplier in \(L_p\), \(1 < p < \infty\) (see, e.g., [29] p. 100). By Mikhlin’s condition the same holds for the function \(\theta\). Thus, it follows from (iii) that \(\hat{\varphi} \in \mathcal{M}_p\).

4 Auxiliary results

Lemma 7 ([31] Theorem 4.3.1]) Let \(g \in L_p\), \(1 \leq p < \infty\), and supp \(\hat{g} \subset [-\sigma_1, \sigma_1] \times \cdots \times [-\sigma_d, \sigma_d]\), \(\sigma_j > 0, j = 1, \ldots, d\). Then

\[
\frac{1}{\sigma_1 \cdots \sigma_d} \max_{k \in \mathbb{Z}^d} \max_{x \in Q_{k, \sigma}} |g(x)|^p \leq c\|g\|_p^p,
\]

where \(Q_{k, \sigma} = \left[\frac{2k_1 - 1}{2\sigma_1}, \frac{2k_1 + 1}{2\sigma_1}\right] \times \cdots \times \left[\frac{2k_d - 1}{2\sigma_d}, \frac{2k_d + 1}{2\sigma_d}\right]\) and \(c\) depends only on \(p\) and \(d\).

Lemma 8 Let \(1 \leq p \leq \infty\), \(g \in L_p\), \(h \in L_p\), and \(\hat{h} \in \mathcal{M}_p\). Then the operator \(T(g) := h \ast g\) is bounded in \(L_p\) and \(\|h \ast g\|_p \leq \|\hat{h}\|_{\mathcal{M}_p}\|g\|_p\).

Proof. In the case \(p = \infty\), the statement follows from Minkowski’s inequality (without assumption \(\hat{h} \in \mathcal{M}_p\)). Consider the case \(p < \infty\). Choose a sequence \(\{g_n\}_n \subset \mathcal{S}\) converging to \(g\) in \(L_p\)-norm. Since \(\hat{h} \in \mathcal{M}_p\), the functions \(\Lambda_{\hat{h}}(g_n)\) form a Cauchy sequence in \(L_p\). Hence, \(\Lambda_{\hat{g}}(g_n) \to G, G \in L_p\). On the other hand, \(\Lambda_{\hat{h}}(g_n) = h \ast g_n\), and the sequence \(h \ast g_n\) converges to \(h \ast g\) almost everywhere. It follows that \((h \ast g)(x) = G(x)\) for almost all \(x\). Thus, we derive

\[
\|h \ast g\|_p = \|G\|_p \leq \|\Lambda_{\hat{h}}(g_n)\|_p + \|G - \Lambda_{\hat{g}}(g_n)\|_p \leq \|\hat{h}\|_{\mathcal{M}_p}\|g_n\|_p + \|G - \Lambda_{\hat{g}}(g_n)\|_p
\]

\[
\leq \|\hat{h}\|_{\mathcal{M}_p}\|g\|_p + \|\hat{h}\|_{\mathcal{M}_p}\|g - g_n\|_p + \|G - \Lambda_{\hat{g}}(g_n)\|_p.
\]

Finally, passing to the limit as \(n \to \infty\), we complete the proof.

Lemma 9 Let \(1 \leq p \leq \infty\), \(M \in \mathfrak{M}, n \in \mathbb{N}, \delta \in (0, 1]\), and \(\alpha \in \mathcal{A}_M\). Suppose that \(\hat{\varphi}, f,\) and the functions \(P_\mu, \mu \in \mathbb{Z}_+,\) are as in Definition [7] Then the sequence \(\left\{\{\hat{P}_\mu(\hat{\varphi}_0k)\}_k\right\}_{k=1}^\infty\) converges in \(\ell_p\) as \(\mu \to \infty\) and its limit is independent of the choice of \(P_\mu\) and \(\delta\); a fortiori for every \(k \in \mathbb{Z}^d\) there
exists the limit \( \lim_{n \to \infty} (\overline{P}_\mu, \hat{\varphi}_{0k}) \) independent of the choice of \( P_\mu \) and \( \delta \). Moreover, for all \( f \in \mathbb{R}^{\mathbb{Z}^d} \), we have
\[
\sum_{\mu=n}^{\infty} \left\| \langle \overline{P}_{\mu+1}, \hat{\varphi}_{0k} \rangle - \langle \overline{P}_\mu, \hat{\varphi}_{0k} \rangle \right\|_{\ell_p} \leq c \sum_{\mu=n}^{\infty} m_{\mathbb{Z}^d} \alpha(M^\mu) E_{\delta_{\mu}, M^\mu}(f)_p,
\] (6)
where \( c \) depends only on \( d \), \( p \), and \( M \).

**Proof.** Consider the case \( p < \infty \). Setting
\[
F(x) := \int_{\mathbb{R}^d} \left( \overline{P}_{\mu+1}(M^{\mu+1}x) - \overline{P}_\mu(M^{\mu+1}x) \right) e^{2\pi i (\xi, x)}\,d\xi,
\]
we get
\[
\left\| \langle \overline{P}_{\mu+1}, \hat{\varphi}_{0k} \rangle - \langle \overline{P}_\mu, \hat{\varphi}_{0k} \rangle \right\|_{\ell_p}^p = m_{\mathbb{Z}^d}^{\mu+1} \sum_{k \in \mathbb{Z}^d} |F(M^{\mu+1}k)|^p.
\] (7)
Since \( \text{supp} \overline{P} \subset [-\sigma, \sigma]^d \), where \( \sigma = \sigma(M, d) > 1 \), using Lemma 7 and taking into account that each set \( Q_{k, \alpha} \) contains a finite number (depending only on \( M \) and \( d \)) points \( M^{\mu+1}k, k \in \mathbb{Z}^d \), we obtain
\[
\sum_{k \in \mathbb{Z}^d} |F(M^{\mu+1}k)|^p \leq c_1 \sigma^d \int_{R^d} |F(x)|^p \,dx.
\] (8)
Recall that \( \mu_0 = \min\{\mu \in \mathbb{N} : T^\delta \subset \frac{1}{\mu} M^\mu T^d \text{ for all } \nu \geq \mu - 1 \} \). Since \( M^{\mu+1} \delta T^d \subset M^{\mu+\mu_0} \delta T^d \) and \( M^{\mu} \delta T^d \subset M^{\mu+\mu_0} \delta T^d \), both the functions \( P_\mu \) and \( P_{\mu+1} \) are in \( B^{\delta_{\mu}, \mu_0}_{\mu, p} \cap L_2 \). Thus, combing (7) and (8) and using (1), we derive
\[
\left\| \langle \overline{P}_{\mu+1}, \hat{\varphi}_{0k} \rangle - \langle \overline{P}_\mu, \hat{\varphi}_{0k} \rangle \right\|_{\ell_p} \leq c_2 m_{\mathbb{Z}^d}^{\mu+1} \left\| F \right\|_p = c_2 m_{\mathbb{Z}^d}^{\mu+1} \left\| F(M^{\mu+1}.) \right\|_p
\]
\[
\leq c_2 m_{\mathbb{Z}^d}^{\mu+1} \alpha(M^\mu) \left\| P_{\mu+1} - P_\mu \right\|_p \leq c_2 m_{\mathbb{Z}^d}^{\mu+1} \alpha(M^\mu) \left\| P_{\mu+1} - P_\mu \right\|_p
\]
\[
\leq c_3 m_{\mathbb{Z}^d}^{\mu+1} \alpha(M^\mu) E_{\delta_{\mu}, M^{\mu+1}}(f)_p + \alpha(M^{\mu+1}) E_{\delta_{\mu}, M^{\mu+1}}(f)_p,
\] (9)
which implies (6) after the corresponding summation.

Let now \( p = \infty \). Taking into account (4), we can write
\[
\langle \overline{P}_{\mu+1}, \hat{\varphi}_{0k} \rangle - \langle \overline{P}_\mu, \hat{\varphi}_{0k} \rangle = \Lambda_f(-)(P_{\mu+1} - P_\mu)(-k).
\]
Then, using (1), we obtain
\[
\left\| \langle \overline{P}_{\mu+1}, \hat{\varphi}_{0k} \rangle - \langle \overline{P}_\mu, \hat{\varphi}_{0k} \rangle \right\|_{\ell_\infty} \leq \left\| \Lambda_f(-)(P_{\mu+1} - P_\mu) \right\|_\infty \leq \alpha(M^{\mu+\mu_0}) \left\| P_{\mu+1} - P_\mu \right\|_\infty \leq c_4 \alpha(M^\mu) E_{\delta_{\mu}, M^{\mu+1}}(f)_\infty + \alpha(M^{\mu+1}) E_{\delta_{\mu}, M^{\mu+1}}(f)_\infty,
\] (10)
which again implies (6).

Next, it is clear that there exists \( \nu(\delta) \in \mathbb{N} \) such that \( E_{\delta_{\mu}, M^{\mu}}(f)_p \leq E_{M^{\mu-\nu(\delta)}}(f)_p \) and \( \alpha(M^\mu) \leq c(\delta, M) \alpha(M^{\mu-\nu(\delta)}) \) for all big enough \( \mu \). Thus, if \( f \in \mathbb{R}^{\mathbb{Z}^d} \), then it follows from (6) that \( \{\langle \overline{P}_\mu, \hat{\varphi}_{0k} \rangle\}_{\mu=1}^\infty \) is a Cauchy sequence in \( \ell_p \). Fortiori, for every \( k \in \mathbb{Z}^d \), the sequence \( \{\langle \overline{P}_\mu, \hat{\varphi}_{0k} \rangle\}_{\mu=1}^\infty \) has a limit.

Let now \( p = \infty \), \( \alpha = \text{const} \), and \( f \in C_0 \). For every \( \mu', \mu'' \in \mathbb{N} \), there exists \( \nu \in \mathbb{N} \) such that both the functions \( \overline{P}_{\mu'} \) and \( \overline{P}_{\mu''} \) are supported in \( M^{\mu'} T^d \), and similarly to (11), we have
\[
\left\| \langle \overline{P}_{\mu'}, \hat{\varphi}_{0k} \rangle - \langle \overline{P}_{\mu''}, \hat{\varphi}_{0k} \rangle \right\|_{\ell_\infty} \leq c_5 \left( E_{\delta_{\mu'}, M^{\mu'}}(f)_\infty + E_{\delta_{\mu''}, M^{\mu''}}(f)_\infty \right).
\]
Thus, again \( \{\langle \overline{P}_\mu, \hat{\varphi}_{0k} \rangle\}_{\mu=1}^\infty \) is a Cauchy sequence in \( \ell_\infty \) and every sequence \( \{\langle \overline{P}_\mu, \hat{\varphi}_{0k} \rangle\}_{\mu=1}^\infty \) has a limit.
Let us check that the limit of \( \{ (\hat{P}_\mu, \hat{\varphi}_{0k}) \}_{\mu=1}^\infty \) in \( \ell_p \) does not depend on the choice of functions \( P_\mu \) and \( \delta \). Let \( \delta' \in (0,1) \) and \( P'_\mu \in B_{\delta'M'(p)} \cap L_2 \) be such that \( \| f - P'_\mu \|_p \leq c'(d,p)E_{\delta'_M}(f) \). Since both the functions \( P_\mu \) and \( P'_\mu \) are in \( B_{\delta'M}(p) \cap L_2 \), repeating the arguments of the proof of inequalities (9) and (10) with \( P'_\mu \) instead of \( P_{\mu+1} \) and 0 instead of \( \mu_0 \), we obtain
\[
\| (\hat{P}_\mu - \hat{P}_\nu)(\hat{\varphi}_{0k}) \|_{\ell_p} \leq c_0 m^{\frac{m+1}{2}}(M'^*)\| P'_\mu - P_\nu \|_p \\
\leq c_7 m^{\frac{m+1}{2}}(M'^*)(E_{\delta'_M}(f)p + E_{\delta'_M}(f)p).
\]
It follows that \( \| (\hat{P}_\mu - \hat{P}_\nu)(\hat{\varphi}_{0k}) \|_{\ell_p} \to 0 \) as \( \mu \to \infty \), which yields the independence from the choice of \( P_\mu \) and \( \delta \). ◯

**Lemma 10** Let \( \varphi \in L_p \), \( 1 \leq p \leq \infty \), be such that \( \text{supp} \hat{\varphi} \) is compact and \( \hat{\varphi} \in \mathcal{M}_p \). Then, for any sequence \( \{ a_k \}_{k \in \mathbb{Z}^d} \in \ell_p \) if \( p < \infty \) and \( \{ a_k \}_{k \in \mathbb{Z}^d} \in c_0 \) if \( p = \infty \), the series \( \sum_{k \in \mathbb{Z}^d} a_k \varphi_{0k}(x) \) converges unconditionally in \( L_p \) and
\[
\left\| \sum_{k \in \mathbb{Z}^d} a_k \varphi_{0k} \right\|_p \leq c \left\| \{ a_k \} \right\|_{\ell_p},
\]
where \( c \) does not depend on \( \{ a_k \}_{k \in \mathbb{Z}^d} \).

**Proof.** Let us fix an integer \( n \). By duality, we can find a function \( g \in L_{p'} \) such that \( \| g \|_{p'} \leq 1 \) and
\[
\left\| \sum_{\| k \| \leq n} a_k \varphi(-k) \right\|_p = \left\| \left( \sum_{\| k \| \leq n} a_k \varphi(-k), g \right) \right\| = \left\| \sum_{\| k \| \leq n} a_k \langle \varphi(-k), g \rangle \right\|.
\]
(11)
Consider the case \( p > 1 \). Applying Hölder’s inequality, using Lemmas 7 and 8 and taking into account that \( \mathcal{M}_p = \mathcal{M}_{p'} \), we obtain
\[
\sum_{\| k \| \leq n} |a_k \langle \varphi(-k), g \rangle| \leq \| \{ a_k \} \|_{\ell_p} \left( \sum_{k \in \mathbb{Z}^d} \| \varphi(-k), g \|_p \right)^{\frac{1}{p'}}
\]
(12)
\[
= \| \{ a_k \} \|_{\ell_p} \left( \sum_{k \in \mathbb{Z}^d} \| \varphi g^- \|_p \right)^{\frac{1}{p'}} \leq c_1 \| \{ a_k \} \|_{\ell_p} \| \varphi \|_{p'} \| g^- \|_{p'} \\
\leq c_2 \| \{ a_k \} \|_{\ell_p} \| g \|_{p'}.
\]
where \( c_2 \) does not depend on \( n \). Combining (11) and (12), we get that the cubic sums of the series \( \sum_{k \in \mathbb{Z}^d} a_k \varphi_{0k} \) are bounded in \( L_p \)-norm.

Similarly, the boundedness of the cubic sums in \( L_1 \)-norm follows from
\[
\sum_{k \in \mathbb{Z}^d} |a_k \langle \varphi_j(-k), g \rangle| \leq \| \{ a_k \} \|_{\ell_1} \sup_k |\langle \varphi(-k), g \rangle| \\
\leq c_3 \| \{ a_k \} \|_{\ell_1} \| \varphi \|_{\infty} \| g^- \|_{\infty} \leq c_4 \| \{ a_k \} \|_{\ell_1} \| g \|_{\infty}.
\]
Now it is clear that all statements hold. ◯

**Lemma 11 (11, THEOREM 2.1)** Let \( \varphi \in \mathcal{L}_p \), \( 1 \leq p \leq \infty \). Then, for any sequence \( \{ a_k \}_{k \in \mathbb{Z}^d} \in \ell_p \), we have
\[
\left\| \sum_{k \in \mathbb{Z}^d} a_k \varphi_{0k} \right\|_p \leq \| \varphi \|_{\mathcal{L}_p} \| \{ a_k \} \|_{\ell_p}.
\]

**Lemma 12 (23, PROPOSITION 5)** Let \( f \in L_p \), \( 1 \leq p \leq \infty \), and \( \tilde{\varphi} \in \mathcal{L}_{p'} \). Then
\[
\left\| \{ f, \tilde{\varphi}_{0k} \} \right\|_{\ell_p} \leq \| \tilde{\varphi} \|_{\mathcal{L}_{p'}} \| f \|_p.
\]
Lemma 13 Let $f \in C_0$ and $\widehat{\varphi} \in \mathcal{S}'_{\text{const,}\infty,\mathcal{M}}$ for some $M \in \mathcal{M}$. Then $\{\langle f, \widehat{\varphi}_{0k} \rangle\}_k \in c_0$ and
\[
\|\{\langle f, \widehat{\varphi}_{0k} \rangle\}_k\|_{\ell_\infty} \leq c \|f\|_\infty,
\]
where $c$ does not depend on $f$.

Proof. By Lemma 14 for any $\varepsilon > 0$, there exists a function $P_\mu \in \mathcal{B}_{M^p,\infty} \cap L_2$ such that $\|f - P_\mu\|_\infty \leq c_1 E_{M^p}(f)_\infty$ and
\[
\|\{\langle f, \widehat{\varphi}_{0k} \rangle - \langle \widehat{P}_\mu, \widehat{\varphi}_{0k} \rangle\}_k\|_{\ell_\infty} < \varepsilon.
\]
(13)
Moreover, due to (11) and (4), we have
\[
|\langle \widehat{P}_\mu, \widehat{\varphi}_{0k} \rangle| = |A_{\mathcal{F}(\widehat{\varphi})}(-k)| \leq \|A_{\mathcal{F}(\widehat{\varphi})}P_\mu\|_\infty \leq \text{const} \|P_\mu\|_\infty \leq c_1 |f|_\infty.
\]
Combining this with (13), we prove the lemma. ◊

Let us also recall some basic inequalities for the best approximation and moduli of smoothness.

Lemma 14 (Lemma 8) Let $f \in L_p$, $1 \leq p < \infty$, and let $A$ be a $d \times d$-matrix. Then
\[
\inf_{P \in \mathcal{B}_{A,\infty} \cap L_2} \|f - P\|_p \leq c E_A(f)_p,
\]
where $c$ depends only on $p$ and $d$.

Lemma 15 Let $f \in C_0$ and let $A$ be a $d \times d$-matrix. Then
\[
\inf_{P \in \mathcal{B}_{2A,\infty} \cap L_2} \|f - P\|_\infty \leq c E_A(f)_\infty,
\]
where $c$ depends only on $d$.

Proof. Since $f \in C_0$, there exists a compactly supported $g \in C$ satisfying
\[
\|f - g\|_\infty \leq E_A(f)_\infty.
\]
Let $Q \in \mathcal{B}_{A,\infty}$ be such that
\[
\|g - Q\|_\infty \leq 2 E_A(g)_\infty.
\]
Denote $N_A = \mathcal{F}^{-1}(\eta(A^{-1}))$. Obviously, $N_A * Q = Q$ and $N_A * g \in \mathcal{B}_{2A,\infty} \cap L_2$. This, together with the above two inequalities, yields
\[
\inf_{P \in \mathcal{B}_{2A,\infty} \cap L_2} \|f - P\|_\infty \leq \|f - N_A * g\|_\infty \leq \|f - g\|_\infty + \|g - Q\|_\infty + \|N_A * (g - Q)\|_\infty
\]
\[
\leq 3 E_A(f)_\infty + \|N_A\|_1 \|g - Q\|_\infty \leq (3 + \|N_A\|_1) E_A(f)_\infty. \quad \diamond
\]

Lemma 16 (See [24, 5.2.1 (7)] or [30, 5.3.3]) Let $f \in L_p$, $1 \leq p \leq \infty$, and $s \in \mathbb{N}$. Then
\[
E_1(f)_p \leq c \omega_s(f,1)_p,
\]
where $c$ depends only on $d$ and $s$.

Finally, the next two statements can be found, e.g., in [34], see also [19].

Lemma 17 Let $P \in \mathcal{B}_{I,p}$, $1 \leq p \leq \infty$, and $s \in \mathbb{N}$. Then
\[
\sum_{|\beta| = s} \|D^\beta P\|_p \leq c \omega_s(P,1)_p,
\]
where $c$ does not depend on $P$.

Lemma 18 Let $P \in \mathcal{B}_{I,p}$, $1 < p < \infty$, and $s > 0$. Then
\[
c_1 \omega_s(P,1)_p \leq \|(-\Delta)^{s/2} P\|_p \leq c_2 \omega_s(P,1)_p,
\]
where the constants $c_1$ and $c_2$ do not depend on $P$. 

10
5 Main results

5.1 Main lemma

Let $M \in \mathcal{M}$, $\alpha \in A_M$, and let $\varphi$ belong to $S'_{\alpha,p,M}$. In what follows, we understand $(f, \varphi_{jk})$ in the sense of Definition 3. Thus, the quasi-projection operators

$$Q_j(f, \varphi, \varphi) = \sum_{k \in \mathbb{Z}^d} (f, \varphi_{jk}) \varphi_{jk}$$

are defined for all $f \in B^{\alpha(\cdot)}_{p,M}$. By Lemmas 9 and 13 we have that $\{ (f, \varphi_{jk}) \}_k \in \ell_p$ and $\{ (f, \varphi_{jk}) \}_k \in c_0$ if $p = \infty$. This, together with Lemmas 10 and 12 implies that the series $\sum_{k \in \mathbb{Z}^d} (f, \varphi_{jk}) \varphi_{jk}$ converges unconditionally in $L_p$. Thus, the operator $Q_j(f, \varphi, \varphi)$ is well defined.

An analogue of the following lemma for the case $\varphi \in L_p$, $\alpha(M) = |\det M|^{\frac{1}{N}}$, and $p < \infty$ can be found in [17]. In the general case, the proof is similar, but for completeness, we present it in detail.

**Lemma 19** Let $1 \leq p \leq \infty$, $M \in \mathcal{M}$, $\delta \in (0,1]$, and $\nu \in \mathbb{Z}_+$. Suppose that $\varphi \in L_p$ or $\varphi \in L_p$ is band-limited with $\hat{\varphi} \in M_p$, and the functions $P_{\mu}$, $\mu \in \mathbb{Z}_+$, are as in Definition 4.

(i) If $\alpha \in A_M$, $\varphi \in S'_{\alpha,p,M}$, and $f \in B^{\alpha(\cdot)}_{p,M}$, then

$$\|f - Q_0(f, \varphi, \varphi)\|_p \leq \|P_\nu - Q_0(P_\nu, \varphi, \varphi)\|_p + c \sum_{\mu=\nu} \|Q_0(f - P_\mu, \varphi, \varphi)\|_p + \|Q_0(f - P_\nu, \varphi, \varphi)\|_p + c E_{\delta_p,M^\nu}(f)_p.$$  \(14\)

(ii) If $\varphi \in L_p$ and $f \in L_p$, $p < \infty$, or $\varphi \in S'_{\text{const} \cdot \infty, M}$ and $f \in C_0$, $p = \infty$, then

$$\|f - Q_0(f, \varphi, \varphi)\|_p \leq \|P_\nu - Q_0(P_\nu, \varphi, \varphi)\|_p + c E_{\delta_p,M^\nu}(f)_p.$$  \(15\)

In the above two inequalities, the constant $c$ does not depend on $f$ and $\nu$.

**Proof.** Obviously,

$$\|f - Q_0(f, \varphi, \varphi)\|_p \leq \|P_\nu - Q_0(P_\nu, \varphi, \varphi)\|_p + \|f - P_\nu\|_p + \|Q_0(f - P_\nu, \varphi, \varphi)\|_p.$$  \(16\)

Suppose that conditions of item (i) hold. Then, using Lemmas 10, 11, and 9 we obtain

$$\|Q_0(f - P_\nu, \varphi, \varphi)\|_p \leq c_1 \|\{(f - P_\nu, \varphi_{0k})\}_k\|_{\ell_p} \leq c_2 \sum_{\mu=\nu} \|\{(P_{\mu+1} - P_\mu, \varphi_{0k})\}_k\|_{\ell_p}$$

$$\leq c_3 \sum_{\mu=\nu} m \|\alpha(M^\nu)\| E_{\delta_p,M^\nu}(f)_p.$$  \(17\)

Combining (16), (17), and (3), we get (14).

Similarly, under assumptions of item (ii) in the case $p < \infty$, it follows from Lemmas 10, 11, and 12 that

$$\|Q_0(f - P_\nu, \varphi, \varphi)\|_p \leq c_4 \|\{(f - P_\nu, \varphi_{0k})\}_k\|_{\ell_p} \leq c_4 \|\varphi\|_{L_p^\nu} \|f - P_\nu\|_p \leq c_5 E_{\delta_p,M^\nu}(f)_p.$$  \(18\)

Thus, combining (16), (18), and (3), we obtain (15) for $p < \infty$. In the case $p = \infty$, using Lemma 13 together with Lemma 10, we get

$$\|Q_0(f - P_\nu, \varphi, \varphi)\|_\infty \leq c_6 \|\{(f - P_\nu, \varphi_{0k})\}_k\|_{\ell_\infty} \leq c_7 E_{\delta_p,M^\nu}(f)_\infty,$$

which completes the proof of the lemma. \(\diamond\)
5.2 The case of weak compatibility of $\varphi$ and $\tilde{\varphi}$

In this subsection, we give error estimates for the quasi-projection operators associated with weakly compatible $\varphi$ and $\tilde{\varphi}$.

**Theorem 20** Let $1 \leq p \leq \infty$, $M \in \mathcal{M}$, $\alpha \in \mathcal{A}_M$, $s \in \mathbb{N}$, and $\delta \in (0, 1]$. Suppose that $\tilde{\varphi} \in \mathcal{S}'_{\alpha,p;M}$ and $\varphi \in L_p$ satisfy the following conditions:

1) $\varphi$ is band-limited with $\tilde{\varphi} \in \mathcal{M}_p$, or $\varphi \in L_p$;
2) the Strang-Fix condition of order $s$ holds for $\varphi$;
3) $\varphi$ and $\tilde{\varphi}$ are weakly compatible of order $s$;
4) $\eta \delta^j \varphi, \tilde{\varphi} \in \mathcal{M}_p$ and $\eta \delta^j \varphi, \tilde{\varphi} \in \mathcal{M}_p$ for all $\beta \in \mathbb{Z}_+^d$, $[\beta] = s$, and $l \in \mathbb{Z}^d \setminus \{0\}$;
5) $\sum_{l \neq 0} \|\eta \delta^j \varphi, \tilde{\varphi} \|_{\mathcal{M}_p} < \infty$ for all $\beta \in \mathbb{Z}_+^d$, $[\beta] = s$.

Then, for any $f \in \mathbb{B}^{\alpha,(-)}_{p,m}$, we have

$$
\|f - Q_j(f, \varphi, \tilde{\varphi})\|_p \leq c \left( \Omega_s(f, M^{-j})_p + m^{-j} \sum_{\nu=j}^{\infty} m^{\nu} \alpha(M^{\nu-j}) E_{M^{\nu}}(f)_p \right). 
$$

(19)

Moreover, if $\tilde{\varphi} \in L_{p'}$ and $f \in L_p$, $p < \infty$, or $\tilde{\varphi} \in \mathcal{S}'_{\text{const},\infty;M}$ and $f \in C_0$, $p = \infty$, then

$$
\|f - Q_j(f, \varphi, \tilde{\varphi})\|_p \leq c \Omega_s(f, M^{-j})_p.
$$

(20)

In the above inequalities, the constant $c$ does not depend on $f$ and $j$.

**Proof.** First we note that it suffices to prove (19) and (20) for $j = 0$. Indeed,

$$
\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \varphi_{jk} \rangle \varphi_{jk} \right\|_p = \left\| m^{-j/p} \left( f(M^{-j} \cdot) - \sum_{k \in \mathbb{Z}^d} \langle f(M^{-j} \cdot), \varphi_{0k} \rangle \varphi_{0k} \right) \right\|_p.
$$

Obviously, $m^{-j/p} f(M^{-j} \cdot) \in \mathbb{B}^{\alpha,(-)}_{p,m}$ whenever $f \in \mathbb{B}^{\alpha,(-)}_{p,m}$. We have also that

$$
E_{M^{\nu}}(m^{-j/p} f(M^{-j} \cdot))_p = E_{M^{\nu+j}}(f)_p
$$

and

$$
\omega_s(f(M^{-j} \cdot), 1)_p = m^{j/p} \Omega_s(f, M^{-j})_p,
$$

which yields (19) and (20) whenever these relations hold true for $j = 0$.

Next, in view of Lemmas 19 and 16 to prove the theorem, it suffices to show that

$$
\left\| P - \sum_{k \in \mathbb{Z}^d} \langle P, \varphi_{0k} \rangle \varphi_{0k} \right\|_p \leq c_1 \sum_{[\beta]=s} \| D^\beta P \|_p,
$$

(21)

where the function $P$ is such that $P \in \mathcal{B}_{\delta^j L_p} \cap L_2$ and $\| f - P \|_p \leq c(d, p) E_{\delta^j L_p}(f)_p$ (remind that $\delta_p = \delta$ for $p < \infty$ and $\delta_p = \delta/2$ for $p = \infty$, and the function $P$ exists in view of Lemmas 14 and 15). Indeed, due to Lemmas 17 and 16 there holds

$$
\sum_{[\beta]=s} \| D^\beta P \|_p \leq c_2 \omega_s(P, 1)_p \leq c_2 \left( \omega_s(f, 1)_p + E_{\delta^j L_p}(f)_p \right) \leq c_3 \omega_s(f, 1)_p.
$$

(22)

Thus, combining (22) and (21) with Lemma 19 we obtain

$$
\| f - Q_0(f, \varphi, \tilde{\varphi}) \|_p \leq c_4 \left( \omega_s(f, 1)_p + \sum_{\nu=0}^{\infty} m^{\nu} \alpha(M^\nu) E_{\delta^\nu M^{\nu}}(f)_p \right).
$$
Since there exists \( \nu_0 = \nu(\delta) \in \mathbb{N} \) such that \( E_{\delta, M^\nu}(f)_p \leq E_{M^\nu - \nu_0}(f)_p \) and \( \alpha(M^\nu) \leq c(\delta, M) \alpha(M^{\nu - \nu_0}) \) for all \( \nu > \nu_0 \), applying Lemma 16 and the inequality \( \omega_s(f, \lambda)_p \leq (1 + \lambda)^s \omega_s(f, 1)_p \) (see, e.g., 19) to the first \( \nu_0 \) terms of the sum, we get 19 for \( j = 0 \). Similarly, taking into account Lemmas 16 we derive (20). Thus, it remains to verifying inequality (21). 

Set

\[
\Psi_0 = 1 - \widehat{\varphi^2} \quad \text{and} \quad \Psi_l = \widehat{\varphi(\cdot + l)\varphi}, \quad l \in \mathbb{Z}^d \setminus \{0\},
\]

and estimate \( \| \Lambda_{\Psi_l}(P) \|_p \) for all \( l \in \mathbb{Z}^d \).

Let \( l \in \mathbb{Z}^d \setminus \{0\} \). Using condition 2) and Taylor’s formula, we have

\[
\widehat{\varphi}(\xi + l) = \sum_{|\beta| = s} \frac{s}{\beta!} \int_0^1 (1 - t)^{s-1} D^\beta \widehat{\varphi}(t\xi + l) dt,
\]

which yields for \( p < \infty \) that

\[
\| \Lambda_{\Psi_l}(P) \|_p^p = \int_{\mathbb{R}^d} dx \left| \sum_{|\beta| = s} \frac{s}{\beta!} \int_0^1 dt (1 - t)^{s-1} \int_{\mathbb{R}^d} d\xi \left( D^\beta \widehat{\varphi}(t\xi + l) \xi^\beta \overline{P(\xi)} \right) e^{2\pi i (\xi, x)} \right|^p = \int_{\mathbb{R}^d} dx \left| \sum_{\beta \neq (2\pi l, p)} \frac{s}{\beta!} \int_0^1 dt (1 - t)^{s-1} \int_{\mathbb{R}^d} d\xi_\beta (t\xi) D^\beta \widehat{\varphi}(t\xi + l) \overline{\Theta_\beta(\xi)} e^{2\pi i (\xi, x)} \right|^p,
\]

where

\[
\Theta_\beta = F^{-1} \left( \widehat{D^\beta P \varphi} \right).
\]

Since \( \eta_\delta(t) D^\beta \widehat{\varphi}(t \cdot + l) \in \mathcal{M}_p \) for every \( t > 0 \) and \( \| \eta_\delta(t) D^\beta \widehat{\varphi}(t \cdot + l) \|_{\mathcal{M}_p} \) does not depend on \( t \) (see property (iv) of Fourier multipliers), it follows from condition 4) and inequality 11 that

\[
\| \Lambda_{\Psi_l}(P) \|_p \leq \sum_{|\beta| = s} \sup_{t \in (0, 1)} \left\| F^{-1} \left( \eta_\delta(t) D^\beta \widehat{\varphi}(t \cdot + l) \overline{\Theta_\beta} \right) \right\|_p \leq \sum_{|\beta| = s} \| \eta_\delta D^\beta \widehat{\varphi}(\cdot + l) \|_{\mathcal{M}_p} \| \Theta_\beta \|_p = \sum_{|\beta| = s} \| \eta_\delta D^\beta \widehat{\varphi}(\cdot + l) \|_{\mathcal{M}_p} \Lambda_{\mathcal{M}_p}(D^\beta P) \|_p \leq \alpha(\delta I) \sum_{|\beta| = s} \| \eta_\delta D^\beta \widehat{\varphi}(\cdot + l) \|_{\mathcal{M}_p} \| D^\beta P \|_p. \quad (23)
\]

Similarly,

\[
\| \Lambda_{\Psi_l}(P) \|_\infty \leq \sum_{|\beta| = s} \sup_{t \in (0, 1)} \left\| F^{-1} \left( \eta_\delta(t) D^\beta \widehat{\varphi}(t \cdot + l) \overline{\Theta_\beta} \right) \right\|_\infty \leq \sum_{|\beta| = s} \| \eta_\delta D^\beta \widehat{\varphi}(\cdot + l) \|_{\mathcal{M}_\infty} \| \Lambda_{\mathcal{M}_\infty}(D^\beta P) \|_\infty \leq \alpha(\delta I) \sum_{|\beta| = s} \| \eta_\delta D^\beta \widehat{\varphi}(\cdot + l) \|_{\mathcal{M}_\infty} \| D^\beta P \|_\infty. \quad (24)
\]

Combining relations (23) and (24) with condition 5), we get

\[
\sum_{l \neq 0} \| \Lambda_{\Psi_l}(P) \|_p \leq c_5 \sum_{|\beta| = s} \| D^\beta P \|_p \quad (25)
\]

To estimate \( \| \Lambda_{\Psi_0}(P) \|_p \), we note that by condition 4) and Taylor’s formula, there holds

\[
\varphi(\xi) \overline{\varphi(\xi)} - 1 = \sum_{|\beta| = s} \frac{s}{\beta!} \xi^\beta \int_0^1 (1 - t)^{s-1} D^\beta \varphi_{\overline{\varphi}}(t\xi) dt.
\]
As above, we obtain for $p < \infty$ that
\[
\|\Lambda \varphi_0(P)\|_p = 
\left( \int_{\mathbb{R}^d} dx \left| \sum_{|\beta| = s} \beta!(2\pi i)^{|eta|} \int_0^1 dt (1-t)^{s-1} \int_{\mathbb{R}^d} d\xi \, \eta_{\delta}(t\xi) D^\beta (\hat{\varphi}(\xi)) (t\xi) D^\beta P(\xi) e^{2\pi i \xi, x} \right|^p \right)^{1/p}.
\]
(26)

\[
\begin{align*}
&\leq \sum_{|\beta| = s} \sup_{t \in (0,1)} \left\| F^{-1} \left( \eta_{\delta}(t\cdot) D^\beta (\hat{\varphi}(\cdot)) (t\cdot) \hat{P} \right) \right\|_p 
&\leq c_6 \sum_{|\beta| = s} \| D^\beta P \|_p,
\end{align*}
\]
and, similarly,
\[
\|\Lambda \varphi_0(P)\|_\infty \leq \sum_{|\beta| = s} \sup_{t \in (0,1)} \left\| F^{-1} \left( \eta_{\delta}(t\cdot) D^\beta (\hat{\varphi}(\cdot)) (t\cdot) \hat{P} \right) \right\|_\infty \leq c_7 \sum_{|\beta| = s} \| D^\beta P \|_\infty.
\]

Next, we set $G(\xi) := \sum_{k \in \mathbb{Z}^d} \hat{P}(\xi + k) \hat{\varphi}(\xi + k)$ and prove that
\[
\sum_{k \in \mathbb{Z}^d} \langle P, \varphi_0(k) \varphi_0 \rangle = F^{-1}(G \varphi).
\]
(27)

First we consider the case $\varphi \in L_p$. Let $l \in \mathbb{Z}^d \setminus \{0\}$ and let $h_l$ denote the restriction of $\hat{\varphi}$ onto the set $\mathbb{T}^d \setminus l$. Then
\[
F^{-1}(Gh_l)(x) = \int_{\mathbb{T}^d} G(\xi) h_l(\xi) e^{2\pi i \xi, x} \, d\xi = \int_{\mathbb{T}^d} \hat{P}(\xi) \hat{\varphi}(\xi) \xi \hat{\varphi}(\xi + l) e^{2\pi i \xi, x + l} \, d\xi = \Lambda \varphi_0(P)(x).
\]

Denote by $\Omega$ and $\Omega_N$ respectively the sum and the $N$-th cubic partial sum of $\sum_{l \in \mathbb{Z}^d} F^{-1}(Gh_l)$, which converges in $L_p$ because of (24). Let us check that $\Omega = F^{-1}(G \hat{\varphi})$ in the distribution sense. Since $\hat{\varphi}$ is bounded, the function $Gh_l$ is in $L_2$, which yields that $\sum_{|l| \leq N} Gh_l = F \Omega_N$ almost everywhere. Hence, for every $g \in \mathcal{S}$, we have
\[
\langle F^{-1}(G \hat{\varphi}) - \Omega_N, g \rangle = \langle G \hat{\varphi} - \sum_{|l| \leq N} \hat{G}h_l, \hat{g} \rangle \rightarrow 0 \quad N \rightarrow \infty
\]
and, obviously,
\[
(\Omega - \Omega_N, g) \rightarrow 0 \quad N \rightarrow \infty.
\]
Thus, the tempered distribution $\Omega$ coincides with $F^{-1}(G \hat{\varphi})$. On the other hand, using Lemma 1 from [24] and cubic convergence of the Fourier series of $G$ in $L_2$-norm, we have the equality
\[
F \left( \sum_{k \in \mathbb{Z}^d} \langle P, \varphi_0(k) \varphi_0 \rangle \right) = G \hat{\varphi}
\]
(28)
in the distribution sense. Thus, the functions $\Omega$ and $Q_0(f, \varphi, \hat{\varphi})$ coincide as distributions. But both $\Omega$ and $Q_0(f, \varphi, \hat{\varphi})$ are locally summable, hence, due to the du Bois-Reymond lemma, these functions coincide almost everywhere, and so (27) is proved.

Now consider the case of bandlimited $\varphi$. Again (28) holds true in the distribution sense. Since $G$ is locally in $L_2$ and $\hat{\varphi}$ is bounded and compactly supported, we have $G \hat{\varphi} \in L$. Thus, again both the functions $F^{-1}(G \hat{\varphi})$ and $\sum_{k \in \mathbb{Z}^d} \langle P, \varphi_0(k) \varphi_0 \rangle$ are locally summable, which yields (27).

It follows from (27) that
\[
\sum_{k \in \mathbb{Z}^d} \langle P, \varphi_0(k) \varphi_0 \rangle(x) = \int_{\mathbb{R}^d} G(\xi) \hat{\varphi}(\xi) e^{2\pi i \xi, x} \, d\xi
\]
\[
= \sum_{l \in \mathbb{Z}^d} \int_{\mathbb{T}^d} G(\xi) \hat{\varphi}(\xi + l) e^{2\pi i \xi + l, x} \, d\xi = \sum_{l \in \mathbb{Z}^d} \int_{\mathbb{T}^d} \hat{P}(\xi) \hat{\varphi}(\xi) \hat{\varphi}(\xi + l) e^{2\pi i \xi, x} \, d\xi.
\]
(29)
From this, taking into account that $P = F^{-1}(\tilde{P})$, we obtain

$$\left\| P - \sum_{k \in \mathbb{Z}^d} \langle P, \tilde{\varphi}_{0k} \rangle \varphi_{0k} \right\|_p \leq \|\Lambda \psi_1(P)\|_p + \sum_{l \neq 0} \|\Lambda \psi_l(P)\|_p,$$

which together with (25) and (26) yields (21). This completes the proof of the theorem. \(\diamond\)

**Corollary 21** Let $1 \leq p \leq \infty$, $s \in \mathbb{N}$, $\delta \in (0, 1)$, $M \in \mathbb{N}$, $\alpha \in A_M$, and let $\varphi \in L_p$ and $\tilde{\varphi} \in S_{s,p,M}$. Suppose that conditions 2) and 3) of Theorem 20 are satisfied and, additionally,

a) if $1 < p < \infty$, we suppose that for some $k \in \mathbb{N}$, $k > \frac{1}{p} - \frac{1}{2}$,

$\bar{\varphi}_\beta \in C^{s+k}(2\delta \mathbb{T}^d)$, \(\bar{\varphi}(\cdot + l) \in C^{s+k}(2\delta \mathbb{T}^d)\) for all \(l \in \mathbb{Z}^d \setminus \{0\}\),

and

$$\sum_{l \neq 0} \sup_{\xi \in 2\delta \mathbb{T}^d} |D^\beta \bar{\varphi}(\xi + l)|^{1 - \frac{d}{p} - \frac{1}{2}} < \infty \quad \text{for all} \quad \beta \in \mathbb{Z}^d_+, \quad [\beta] = s;$$

b) if $p = 1$ or $p = \infty$, we suppose that for some $k \in \mathbb{N}$, $k > \frac{d}{4}$,

$\bar{\varphi}_\beta \in W^{s+k}_2(2\delta \mathbb{T}^d)$, \(\bar{\varphi}(\cdot + l) \in W^{s+k}_2(2\delta \mathbb{T}^d)\) for all \(l \in \mathbb{Z}^d \setminus \{0\}\),

and

$$\sum_{l \neq 0} \|D^\beta \bar{\varphi}(\cdot + l)\|_{L_{2}(2\delta \mathbb{T}^d)}^{1 - \frac{d}{p}} < \infty \quad \text{for all} \quad \beta \in \mathbb{Z}^d_+, \quad [\beta] = s.$$

Then inequalities (19) and (20) hold true.

For a band-limited function $\varphi$, the above statement remains valid if the condition $\varphi \in L_p$ is replaced by the assumption that $\varphi \in L_1$ in the case $p = 1$, and $\varphi = F^{-1}(\chi_U \psi)$, where $U$ is compact and $\psi \in C_0^k(\mathbb{R}^d)$ if $1 < p < \infty$.

The proof of Corollary 21 easily follows from sufficient conditions for Fourier multipliers given in [14, Corollaries 2 and 3]. Let us compare this result with Theorem 10 in [17], where the same estimates are obtained for the case $\alpha(M) = |\det M|^N$, but the proof is given without using Fourier multipliers. A higher order of smoothness near the integer points is required for functions $\tilde{\varphi}$ and $\bar{\varphi}$ as compared to $\psi$ in that theorem, namely, differentiability of order $s + d + 1$ is assumed. However, the requirement on the decay for the functions $D^\beta \tilde{\varphi}$ is less restrictive there. To provide the same decay for $D^\beta \bar{\varphi}$, we give another corollary based on Mikhlin’s condition (see Section 3) for Fourier multipliers.

**Corollary 22** Let $1 < p < \infty$, $s \in \mathbb{N}$, $M \in \mathbb{N}$, $\alpha \in A_M$, $\delta \in (0, 1)$, $k \in \mathbb{N}$, $k > \frac{d}{4}$, and let $\tilde{\varphi} \in S_{s,p,M}$ and $\varphi \in L_p$. Suppose that conditions 2) and 3) of Theorem 20 are satisfied. Additionally,

$\bar{\varphi}_\beta \in C^{s+k}(2\delta \mathbb{T}^d)$, \(\bar{\varphi}(\cdot + l) \in C^{s+k}(2\delta \mathbb{T}^d)\) for all \(l \in \mathbb{Z}^d \setminus \{0\}\),

and

$$\sum_{l \neq 0} \sup_{\xi \in 2\delta \mathbb{T}^d} |D^\beta \bar{\varphi}(\xi + l)| < \infty \quad \text{for all} \quad \beta \in \mathbb{Z}^d_+, \quad [\beta] = s.$$

Then inequalities (19) and (20) hold true.

For a band-limited function $\varphi$, the above statement remains valid if the condition $\varphi \in L_p$ is replaced by the assumption that $\varphi = F^{-1}(\chi_U \psi)$, where $U$ is compact and $\psi \in C_0^k(\mathbb{R}^d)$.

**Example 1.** Let $d = 2$, $p = \infty$, $s = 2$, and let $Q_j(f, \varphi, \tilde{\varphi})$ be a mixed sampling-Kantorovich quasi-projection operator associated with $\varphi(x) = \frac{1}{4} \sin^3(\frac{x}{4}) \sin^3(\frac{x}{4})$ and $\tilde{\varphi}(x) = \delta(x_1) \chi_{\mathbb{T}}(x_2)$, i.e.,

$$Q_0(f, \varphi, \tilde{\varphi})(x) = \sum_{k \in \mathbb{Z}^2} \int_{k_2 - 1/2}^{k_2 + 1/2} f(k_1, t) dt \varphi(x - k).$$
It is easy to see that all assumptions of Theorem 20 for the case \( p = \infty \) and \( \tilde{\varphi} \in S_{\text{const}, \infty}^p \) are satisfied, which implies
\[
\| f - Q_j(f, \varphi, \tilde{\varphi}) \|_{\infty} \leq c \Omega_2(f, M^{-j})_{\infty}.
\]

**Example 2.** Let \( d = 2, 1 < p < \infty, s \in \mathbb{N}, f \in L_p, \) and let \( Q_j(f, \varphi, \tilde{\varphi}) \) be a differential sampling expansion of the form
\[
Q_j(f, \varphi, \tilde{\varphi})(x) = \sum_{k \in \mathbb{Z}^2} \left( f(M^{-j}k) + \frac{\partial^s}{\partial x^s} f(M^{-j}k) \right) \text{sinc}(M^j x - k),
\]
i.e., \( \varphi(x) = \text{sinc} x \) and \( \tilde{\varphi}(x) = (I + \frac{\partial^s}{\partial x^s}) \delta(x) \). We have \( \tilde{\varphi} \in S_{\alpha, p}^p \), where \( \alpha(M) = m_1^s \) in the case of the diagonal dilation matrix \( M = \text{diag}\{m_1, m_2\} \). Thus, using Theorem 20 it is not difficult to see that
\[
\| f - Q_j(f, \varphi, \tilde{\varphi}) \|_{p} \leq c \left( \Omega_s(f, M^{-j}) + m_1 (s + \frac{1}{2}) m_2 \sum_{\nu = j}^{\infty} m_1^{\nu - s} m_2^\nu E_{M^\nu}(f) \right),
\]
where \( f \in B_{p, \nu}^s \) and \( c \) does not depend on \( f \) and \( j \).

**Fractional smoothness and lower estimates**

We have the following generalization of Theorem 20 in terms of fractional moduli of smoothness.

**Theorem 23** Let \( 1 < p < \infty, s > 0, \delta \in (0, 1/2), M \in \mathbb{N}, \) and \( \alpha \in A_M \). Suppose that \( \tilde{\varphi} \in S_{\alpha, p}^p \), \( \varphi \in L_p \) and the following is satisfied:

1) \( \text{supp} \tilde{\varphi} \subset T^d \) and \( \tilde{\varphi} \in M_p \);

2) \( \eta_\delta \frac{1 - \text{sinc}^2}{2} \in M_p \).

Then, for any \( f \in B_{p, \nu}^s \), we have
\[
\| f - Q_j(f, \varphi, \tilde{\varphi}) \|_{p} \leq c \left( \Omega_s(f, M^{-j}) + m_1 (s + \frac{1}{2}) m_2 \sum_{\nu = j}^{\infty} m_1^{\nu - s} m_2^\nu E_{M^\nu}(f) \right).
\]

Moreover, if \( \tilde{\varphi} \in L_p \), then for any \( f \in L_p \), we have
\[
\| f - Q_j(f, \varphi, \tilde{\varphi}) \|_{p} \leq c \Omega_s(f, M^{-j})_{p}.
\]

In the above inequalities, the constant \( c \) does not depend on \( f \) and \( j \).

**Proof.** Repeating the arguments of the proof of Theorem 20 for the case of bandlimited \( \varphi \), we see that it suffices to verify that for any \( P \in B_{d, \nu} \cap L_2 \) such that \( \| f - P \|_{p} \leq c(d, \nu) E_{d, \nu}(f)_{p} \), one has
\[
\| \Lambda_{\Psi_0}(P) \|_{p} \leq c_1 \omega_s(f, 1)_{p}, \tag{30}
\]
where \( \Psi_0 = 1 - \tilde{\varphi} \).

Using condition 2), we derive
\[
\| \Lambda_{\Psi_0}(P) \|_{p} = \left\| \mathcal{F}^{-1} \left( (1 - \tilde{\varphi}) | \cdot |^{-s} \eta_\delta | \cdot |^{s} \right) \right\|_{p} \leq c_2 \| (-\Delta)^{s/2} P \|_{p}.
\]

Thus, to get (30), it remains to note that, due to Lemmas 15 and 16 we have
\[
\| (-\Delta)^{s/2} P \|_{p} \leq c_3 \omega_s(P, 1)_{p} \leq c_4 \| f - P \|_{p} + \omega_s(f, 1)_{p} \leq c_5 \omega_s(f, 1)_{p},
\]
which proves the theorem. \( \diamond \)

In the next theorem, we obtain lower estimates for the \( L_p \)-error of approximation by the quasi-projection operators \( Q_j(f, \varphi, \tilde{\varphi}) \). Note that such type of estimates are also called strong converse inequalities, see, e.g., [1].
Theorem 24  Let $1 < p < \infty$, $s > 0$, $M \in \mathcal{M}$, and $\alpha \in A_M$. Suppose that $\bar{\varphi} \in S'_{\alpha,p,M}$ and $\varphi \in L_p$ satisfy the following conditions:

1) $\text{supp} \bar{\varphi} \subset \mathbb{T}^d$ and $\bar{\varphi} \in \mathcal{M}_p$;

2) $\eta_{1-\bar{\varphi}} \in \mathcal{M}_p$.

Then, for any $f \in B_{\alpha,c}^{\alpha,(-)}$, we have

$$\Omega_s(f,M^{-j})_p \leq c \|f - Q_j(f,\varphi,\bar{\varphi})\|_p + cm^{-\frac{s}{4}} \sum_{\nu=j}^{\infty} m^\nu \alpha(M^{-j}) E_M\nu(f)_p.$$  

Moreover, if $\bar{\varphi} \in \mathcal{L}_{p'}$, then for any $f \in L_p$, we have

$$\Omega_s(f,M^{-j})_p \leq c \|f - Q_j(f,\varphi,\bar{\varphi})\|_p.$$  

In the above inequalities, the constant $c$ does not depend on $f$ and $j$.

Proof. As in the proof of the previous theorems, it suffices to consider only the case $j = 0$. Let $P \in B_{1,p} \cap L_2$ be such that $\|f - P\|_p \leq c(d,p) E_1(f)_p$. Due to the same arguments as in the proof of Theorem 20 we have (29), which takes now the following form

$$P - \sum_{k \in \mathbb{Z}^d} \langle P, \bar{\varphi}_{0k} \rangle \varphi_{0k} = F^{-1} \left( \hat{P} (1 - \hat{\bar{\varphi}}) \right).$$

Using this equality, Lemma 18 and condition 2), we derive

$$\Omega_s(f,I)_p = \omega_s(f,1)_p \leq \omega_s(P,1)_p + c_1 \|f - P\|_p \leq c_2 \left( \|(-\Delta)^{s/2} P\|_p + E_1(f)_p \right)$$

$$= c_2 \left( \left\| F^{-1} \left( \frac{\eta \cdot |\cdot|^s}{1 - \bar{\varphi}^2} \hat{P} (1 - \bar{\varphi}^2) \right) \right\|_p + E_1(f)_p \right)$$

$$\leq c_3 \left( \left\| P - \sum_{k \in \mathbb{Z}^d} \langle P, \bar{\varphi}_{0k} \rangle \varphi_{0k} \right\|_p + E_1(f)_p \right)$$

$$\leq c_3 \left( \left\| f - Q_0(f,\varphi,\bar{\varphi}) \right\|_p + \left\| P - f \right\|_p + \left\| \sum_{k \in \mathbb{Z}^d} (P - f, \bar{\varphi}_{0k}) \varphi_{0k} \right\|_p + E_1(f)_p \right)$$

$$\leq c_4 \left( \left\| f - Q_0(f,\varphi,\bar{\varphi}) \right\|_p + E_1(f)_p + \left\| Q_0(f,\varphi,\bar{\varphi}) \right\|_p \right)$$

$$\leq c_5 \left( \left\| f - Q_0(f,\varphi,\bar{\varphi}) \right\|_p + E_1(f)_p ,$$

where the last inequality follows from Lemmas 10 and 12. Thus, to prove (32), it remains to note that in view of the inclusion $\text{supp} F(Q_0(f,\varphi,\bar{\varphi})) \subset \text{supp} \bar{\varphi} \subset \mathbb{T}^d$, we have $E_1(f)_p \leq \|f - Q_0(f,\varphi,\bar{\varphi})\|_p$.

Remark 25  Note that the conditions on functions/distributions $\varphi$ and $\bar{\varphi}$ in Theorems 23 and 24 can be also given in terms of smoothness of $\bar{\varphi}$ and $\bar{\varphi}$, similarly to those given in Corollaries 21 and 22. For this, one can use the sufficient conditions for Fourier multipliers mentioned in Section 3 as well as some results of papers 22 and 14.

Example 3. Let $1 < p < \infty$, $\varphi(x) = \text{sinc}(x) := \prod_{\nu=1}^{d} \frac{\sin(\pi x_\nu)}{\pi x_\nu}$, and $\bar{\varphi}(x) = \chi_{\mathbb{T}^d}(x)$ (the characteristic function of $\mathbb{T}^d$). Then all conditions of Theorems 23 and 24 are satisfied and, therefore, for any $f \in L_p$, we have

$$\left\| f - \sum_{k \in \mathbb{Z}^d} m^j \left( \int_{M^{-j}\mathbb{T}^d} f(M^{-j}k - t)dt \right) \text{sinc}(M^j \cdot -k) \right\|_p = \Omega_2(f,M^{-j})_p,$$  

(33)
where $\simeq$ is a two-sided inequality with constants independent of $f$ and $j$. Note that if we replace $	ext{sinc } x$ by $	ext{sinc}^2 x$, then the upper estimate in (33) via the modulus $\Omega_2(f, M^{-j})_p$ holds for all $f \in L_p$, $1 \leq p \leq \infty$. This can be easily verified using Theorem 20 see also [16].

Similarly, using Theorems 23 and 24 and some basic properties of Fourier multipliers (see Section 3 and also [26] for some special multipliers), one can prove the following $L_p$ error estimates for approximation by quasi-projection operators generated by the Bochner-Riesz kernel of fractional order.

Example 4. Let $1 < p < \infty$ and $\varphi(x) = R^*_\gamma(x) := \mathcal{F}^{-1}\left((1 - |3\xi|^\alpha)^\gamma\right)(x)$, $s > 0$, $\gamma > \frac{d-1}{2}$.

1) If $\tilde{\varphi}(x) = \delta(x)$, then for any $f \in \mathbb{B}^{1}_{p,M}$, we have
   \[ c_1\Omega_s(f, M^{-j})_p \leq \left\| f - m^j \sum_{k \in \mathbb{Z}^d} f(M^{-j}k)R^*_\gamma(M^j \cdot -k) \right\|_p \leq c_2 m^{-\frac{j}{p}} \sum_{v=j}^\infty m^\frac{v}{p}\Omega_s(f, M^{-v})_p, \]
   where $c_1$ and $c_2$ are some positive constants independent of $f$ and $j$

2) If $\tilde{\varphi}(x) = \chi_{\gamma\delta}(x)$, then for any $f \in L_p$ and $s \in (0, 2]$, we have
   \[ \left\| f - m^j \sum_{k \in \mathbb{Z}^d} \left( \int_{M^{-j}\mathbb{T}^d} f(M^{-j}k-t) dt \right) R^*_\gamma(M^j \cdot -k) \right\|_p \simeq \Omega_s(f, M^{-j})_p, \]
   where $\simeq$ is a two-sided inequality with positive constants independent of $f$ and $j$.

5.3 The case of strict compatibility for $\varphi$ and $\tilde{\varphi}$

Theorem 26 Let $1 \leq p \leq \infty$, $\delta \in (0, 1]$, $M \in \mathfrak{M}$, and $\alpha \in A_M$. Suppose that $\tilde{\varphi} \in \mathbf{S}_\alpha^{t,M} \text{ and } \varphi \in L_p$ satisfy the following conditions:

1) supp $\tilde{\varphi} \subset \mathbb{T}^d$ and $\tilde{\varphi} \in \mathcal{M}_p$;

2) $\varphi$ and $\tilde{\varphi}$ are strictly compatible with respect to $\delta$.

Then, for any $f \in \mathbb{B}^{\alpha(t)}_{p,M}$, we have
   \[ \|f - Q_j(f, \varphi, \tilde{\varphi})\|_p \leq c m^{-\frac{j}{p}} \sum_{v=j}^\infty m^\frac{v}{p}\alpha(M^{v-j})E_{b,M^v}(f)_p. \]  
   (34)

Moreover, if $\tilde{\varphi} \in L_p$, and $f \in L_p$, $p < \infty$, or $\tilde{\varphi} \in \mathbf{S}_\text{const,}^{t,\infty}_M$ and $f \in C_0$; $p = \infty$, then
   \[ \|f - Q_j(f, \varphi, \tilde{\varphi})\|_p \leq c E_{b,M^j}(f)_p. \]  
   (35)

In the above inequalities, the constant $c$ does not depend on $f$ and $j$.

Proof. As above, it suffices to consider only the case $j = 0$. Repeating the arguments of the proof of Theorem 20 we obtain from (29) that
   \[ P_0 - \sum_{k \in \mathbb{Z}^d} \langle P_0, \tilde{\varphi}_{0k} \rangle \varphi_{0k} = 0, \]
   Thus, applying Lemma [19] we prove both the statements of the theorem. ◇

Remark 27 In the case $p = \infty$, $\delta_p$ in estimates (34) and (35) can be replaced by $\rho \delta$, where $\rho \in (0, 1)$. 

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Example 5. If \( \tilde{\varphi}(x) = \chi_{\mathbb{T}^d}(x) \) and \( \varphi(x) = F^{-1}\left(\frac{\chi_{\mathbb{R}^d}(\xi)}{\sin(\xi)}\right)(x) \), then Theorem 28 provides the following estimate for the corresponding Kantorovich-type operator

\[
\left\| f - \sum_{k \in \mathbb{Z}^d} m^j \left( \int_{M^{-j} k - t} f(M^{-j}k - t)dt \right) \varphi(M^j \cdot -k) \right\|_p \leq c E_{M^j}(f)_p,
\]

where \( f \in L_p, 1 < p < \infty \), and \( c \) does not depend on \( f \) and \( j \). Note that the corresponding estimate in the case of Kotelnikov operators (the case \( \tilde{\varphi}(x) = \delta(x) \) and \( \varphi(x) = \text{sinc}(x) \)) has the following form:

\[
\left\| f - m^j \sum_{k \in \mathbb{Z}^d} f(M^{-j}k)\text{sinc}(M^j \cdot -k) \right\|_p \leq c m^{-\frac{1}{p}} \sum_{\nu = j}^{\infty} m^{\frac{\nu}{p}} E_{M^{\nu}}(f)_p, \quad f \in B_{p,M}^1.
\]

Whittaker–Nyquist–Kotelnikov–Shannon-type theorems

One can easily see that the right hand side of (34) is identically zero if \( \text{supp} \tilde{f} \subset \delta_p M^* \mathbb{T}^d \) and the matrix \( M \) is such that \( M \mathbb{T}^d \subset \mathbb{T}^d \). This leads to the following counterpart of the classical Kotelnikov formula

\[
f = Q_1(f, \varphi, \tilde{\varphi}) \text{ a.e.}
\]

The next two theorems provide results of this type under significantly milder conditions.

Theorem 28 Let \( M \) be a non-degenerate matrix and \( \delta \in (0,1] \). Suppose that

1) \( \text{supp} \tilde{\varphi} \subset \mathbb{T}^d \) and \( \tilde{\varphi} \in L_\infty; \)

2) \( \varphi \in S' \) and \( \hat{\varphi} \) is bounded on \( \delta \mathbb{T}^d; \)

3) \( \varphi \) and \( \tilde{\varphi} \) are strictly compatible with respect to \( \delta \).

If a function \( f \) is such that \( \text{supp} \tilde{f} \subset \delta M^* \mathbb{T}^d \) and \( \hat{f} \in L_q, q > 1 \), then

\[
f(x) = \lim_{n \to \infty} \sum_{\|k\|_\infty \leq n} \langle \tilde{f}, \varphi_{1k}\rangle \varphi_{1k}(x) \text{ for almost all } x \in \mathbb{R}^d.
\]

Proof. First let \( M = I \). Set \( G(\xi) := \sum_{k \in \mathbb{Z}^d} \tilde{\varphi}(\xi + k)\varphi_k(\xi + k) \). Since \( G \in L_q(\mathbb{T}^d) \), its Fourier series is cubic convergent to \( G \) in \( L_q(\mathbb{T}^d) \), i.e., \( |G - G_n|_{L_q(\mathbb{T}^d)} \to 0 \), where \( G_n \) is the \( n \)-th cubic partial Fourier sum and

\[
G_n(\xi)\tilde{\varphi}(\xi) = \sum_{\|k\|_\infty \leq n} \tilde{G}(k)e^{2\pi i k \cdot \xi}\varphi_k(\xi) = \sum_{\|k\|_\infty \leq n} \langle \tilde{f}, \varphi_{0k}\rangle \varphi_{ok}(\xi) =: H_n(\xi).
\]

Obviously, \( H_n \in L_q \). This, together with condition 1), yields

\[
\|H_m - H_n\|_q^q = \int_{\mathbb{R}^d} |(G_m(\xi) - G_n(\xi))\tilde{\varphi}(\xi)|^q d\xi
\]

\[
= \int_{\mathbb{T}^d} |(G_m(\xi) - G_n(\xi))\tilde{\varphi}(\xi)|^q d\xi \leq \|\tilde{\varphi}\|_\infty^q \|G_m - G_n\|_{L_q(\mathbb{T}^d)}^q.
\]

Thus, the sequence \( \{H_n\} \) converges in \( L_q \). Without loss of generality, we can suppose that \( q \leq 2 \), by the Hausdorff-Young inequality, \( F^{-1}H_n \) and \( F^{-1}H_m \) are in \( L_p \), where \( p = \frac{q}{q-1} \), and

\[
\|F^{-1}H_m - F^{-1}H_n\|_p \leq \|H_m - H_n\|_q \to 0 \text{ as } n, m \to \infty.
\]

It follows that the series \( \sum_{k \in \mathbb{Z}^d} \langle \tilde{f}, \varphi_{0k}\rangle \varphi_{ok} \) is cubic convergent in \( L_p \) and its sum is in \( L_p \). Again by the Hausdorff-Young inequality,

\[
\left\| f - \lim_{n \to \infty} \sum_{\|k\|_\infty \leq n} \langle \tilde{f}, \varphi_{0k}\rangle \varphi(\cdot + k) \right\|_p \leq \|\tilde{f} - G\tilde{\varphi}\|_q
\]

\[
\leq \|\tilde{f}(1 - \tilde{\varphi})\|_q + \|\tilde{\varphi}\| \sum_{k \neq 0} \|\tilde{f}(\cdot + k)\tilde{\varphi}(\cdot + k)\|_q := I_1 + I_2.
\]
Using condition 3) and taking into account that \( \text{supp} \hat{f} \subset \delta \mathbb{T}^d \), we obtain that \( I_1 = 0 \). At the same time, we have that \( I_2 = 0 \) because both the functions \( \varphi \) and \( f \) are band-limited to \( \mathbb{T}^d \). This yields (36) for the case \( M = I_d \).

Now let \( M \) be an arbitrary non-degenerate matrix. Setting \( g := f(M^{-1} \cdot) \), we have \( \text{supp} \hat{g} \subset \delta \mathbb{T}^d \). Hence, equality (36) holds for \( g \) and

\[
\hat{f}(x) = g(Mx) = \lim_{n \to \infty} \sum_{\|k\|_\infty \leq n} \langle \hat{g}, \varphi_{0k} \rangle \varphi_{0k}(Mx) \quad \text{for almost all} \quad x \in \mathbb{R}^d.
\]

Finally, after a suitable change of variable in the inner products, we get (36). \( \Diamond \)

There are two drawbacks in the latter theorem (as well as in the Kotelnikov-type formula extracted from Theorem [20]). First, the Fourier transform of \( f \) is assumed to be in \( L_q \), \( q > 1 \), and second, the Kotelnikov-type equality holds only at almost all points. Under an additional restriction on \( \varphi \), these drawbacks are avoided in the following statement.

**Theorem 29** Let \( M \) be a non-degenerate matrix and \( \delta \in (0, 1] \). Suppose that

1) \( \text{supp} \hat{\varphi} \subset \mathbb{T}^d \) and \( \hat{\varphi} \in L_\infty \);
2) \( \left| \frac{\partial^2 \varphi}{\partial x_l \partial x_l} (\xi) \right| \leq B \) for all \( \xi \in \mathbb{R}^d \) and \( l = 1, \ldots, d \);
3) \( \varphi \in S' \) and \( \hat{\varphi} \) is bounded on \( \delta \mathbb{T}^d \);
4) \( \varphi \) and \( \hat{\varphi} \) are strictly compatible with parameter \( \delta \).

If a function \( f \) is such that \( \text{supp} \hat{f} \subset \delta M^* \mathbb{T}^d \) and \( \hat{f} \in L_1 \), then

\[
f(x) = \lim_{n \to \infty} \sum_{\|k\|_\infty \leq n} \langle \hat{f}, \varphi_{1k} \rangle \varphi_{1k}(x) \quad \text{for all} \quad x \in \mathbb{R}^d.
\]

**Proof.** First let \( M = I \). Set \( \Theta_x(\xi) := \sum_{s \in \mathbb{Z}^d} \hat{\varphi}(\xi + s)e^{2\pi i(x, \xi + s)} \). By the Poisson summation formula, \( \Theta_x \) is a summable 1-periodic (with respect to each variable) function and its \( n \)-th Fourier coefficient is

\[
\hat{\Theta}_x(k) = \int_{\mathbb{R}^d} \hat{\varphi}(\xi)e^{-2\pi i(k-x, \xi)} \, d\xi = \hat{\varphi}(k-x) = \varphi(x-k).
\]

Since \( \Theta_x \) is a bounded function, its Fourier series cubic converges almost everywhere. Let us check that the cubic partial Fourier sums are uniformly bounded in \( L_\infty \)-norm. Set

\[
S_n(\Theta_x, \xi) := \sum_{\|k\|_\infty \leq n} \hat{\Theta}_x(k)e^{2\pi i(k, \xi)}.
\]

Using the Lebesgue inequality and the Jackson type inequality for the rectangular best approximations of periodic functions (see [30] Sec. 5.3.1]), we have

\[
\|S_n(\Theta_x, \cdot)\|_\infty \leq \|\Theta_x\|_\infty + \|\Theta_x - S_n(\Theta_x, \cdot)\|_\infty \\
\leq \|\Theta_x\|_\infty + c(d) \log^d(n + 1) \sum_{\nu=1}^d \omega_1(\nu) \left( \Theta_x, \frac{1}{n} \right)_\infty,
\]

where \( \omega_1(\nu) (g, h)_\infty = \sup_{\|l\|_\infty \leq k} \| \Delta_{\nu l}^1 g, h \|_\infty \) and \( \{e_\nu\}_{\nu=0}^d \) is the standard basis in \( \mathbb{R}^d \). Since the function \( \Theta_x \) and all its partial derivatives are bounded (uniformly with respect to \( x \)), there exists a constant \( c_1 \) such that

\[
\sum_{\nu=1}^d \omega_1(\nu) \left( \Theta_x, \frac{1}{n} \right)_\infty \leq \frac{c_1}{n} \quad \text{for all} \quad x \in \mathbb{R}^d,
\]

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which together with (5.3) implies the required boundedness. Using this with Lebesgue’s dominated convergence theorem and taking into account that both the functions $\varphi$ and $f$ are band-limited to $T^d$, we derive

$$
\lim_{n \to \infty} \sum_{\|k\|_{\infty} \leq n} \langle \hat{f}, \hat{\varphi}_0 \rangle \varphi(x + k) = \lim_{n \to \infty} \int_{\mathbb{R}^d} \sum_{\|k\|_{\infty} \leq n} \varphi(x + k) e^{-2\pi i (k, \xi)} \hat{\varphi}(\xi) \hat{f}(\xi) \, d\xi
$$

$$
= \int_{\mathbb{R}^d} \lim_{n \to \infty} \sum_{\|k\|_{\infty} \leq n} \varphi(x - k) e^{2\pi i (k, \xi)} \hat{\varphi}(\xi) \hat{f}(\xi) \, d\xi
$$

$$
= \int_{\mathbb{R}^d} e^{2\pi i (x, \xi)} \sum_{s \in \mathbb{Z}^d} \hat{\varphi}(\xi + s) e^{2\pi i (x, s)} \hat{\varphi}(\xi) \hat{f}(\xi) \, d\xi = \int_{\mathbb{T}^d} \hat{\varphi}(\xi) \hat{\varphi}(\xi) \hat{f}(\xi) e^{2\pi i (x, \xi)} \, d\xi.
$$

Since $\text{supp} \hat{f} \subset \delta \mathbb{T}^d$, it follows from condition 4) that

$$
f(x) - \lim_{n \to \infty} \sum_{\|k\|_{\infty} \leq n} \langle \hat{f}, \hat{\varphi}_0 \rangle \varphi(x + k) = \int_{\delta \mathbb{T}^d} (1 - \hat{\varphi}(\xi) \hat{\varphi}(\xi)) \hat{f}(\xi) e^{2\pi i (x, \xi)} \, d\xi = 0
$$

for every $x \in \mathbb{R}^d$.

So, the theorem is proved for the case $M = I$. For the general case, it remains to repeat the arguments at the end of the proof of Theorem 28.

**Remark 30** Condition 2) in Theorem 29 can be replaced by the assumption that $\hat{\varphi} \in \text{Lip}_\alpha, \alpha > 0$, with respect to each variable. In the case $d = 1$, condition 2) can be also replaced by the requirement of bounded variation for $\hat{\varphi}$.

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