Parallel Stochastic Mirror Descent for MDPs

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Abstract

We consider the problem of learning the optimal policy for infinite-horizon Markov decision processes (MDPs). For this purpose, some variant of Stochastic Mirror Descent is proposed for convex programming problems with Lipschitz-continuous functionals. An important detail is the ability to use inexact values of functional constraints. We analyze this algorithm in a general case and obtain an estimate of the convergence rate that does not accumulate errors during the operation of the method. Using this algorithm, we get the first parallel algorithm for average-reward MDPs with a generative model. One of the main features of the presented method is low communication costs in a distributed centralized setting.

1 Introduction

The role of reinforcement learning (RL) last years only increases among the modern machine learning community [16, 17]. In general, the goal of an RL instance is to learn from an unknown and (possible) stochastic environment the best sequence of decisions to maximize the reward associated with the decisions. The developed mathematical model for this decision-making process is called Markov Decision Process.

Problem statement. Let us give formal definitions. An instance of MDP is a tuple $\mathcal{M} = (\mathcal{S}, \mathcal{A}, P, r)$, where $\mathcal{S}$ is a finite set of states; $\mathcal{A} = \bigcup_{i \in \mathcal{S}} \mathcal{A}_i$ is a finite state of actions, each set $\mathcal{A}_i$ contains actions from the state $i$. $P$ is the collection of state-to-state transition probabilities given actions: $P = \{p_{ij}(a_i) \mid i, j \in \mathcal{S}, a_i \in \mathcal{A}\}$ where $p_{ij}(a_i)$ is a probability of transition from a state $i$ to a state $j$ given an action $a_i$. Also, we define $r \in [0,1]|\mathcal{A}|$ as the state-action reward vector, $r_{i,a_i}$ is the instant reward received when taking the action $a_i$ at the state $i \in \mathcal{S}$. For consistency of notation with [9], let $(i,a_i) \in \mathcal{A}$ denote an action $a_i$ at a state $i$. $A_{\text{tot}} = |\mathcal{A}| = \sum_{i \in \mathcal{S}} |\mathcal{A}_i|$ denotes the total number of state-action pairs. Also we denote by $P$ the action-state transition probability matrix of size $|\mathcal{A}| \times |\mathcal{S}|$, where $P_{(i,a_i),j} = p_{ij}(a_i)$ in terms of $P$.

The goal is to find a stationary (randomized) policy that specifies actions to choose in the fixed state. Formally, a policy $\pi$ is a block vector such that $i$-th block corresponds to a probability distribution over $\mathcal{A}_i$. Define as $P^\pi$, $r^\pi$ the transition matrix and the cost vector under the fixed policy $\pi$.

Now we are going to define optimality of the policy. In this paper we consider the infinite-horizon average-reward MDP with the following objective to maximize:

$$\hat{\nu}^\pi = \lim_{t \to \infty} \mathbb{E}^\pi \left[ \frac{1}{T} \sum_{t=1}^{T} r_{i_t,a_t} \mid i_1 \sim q \right].$$

Here $\{i_1,a_1, \ldots, i_t,a_t\}$ are state-actions transitions generated by MDP under a policy $\pi$, $q$ is an initial distribution, and an expectation $\mathbb{E}^\pi[\cdot]$ is taken over trajectories. In our case we interested in the case then the Markov chain generated by an AMDP under a fixed policy $\pi$ has a unique stationary...
distribution $\nu^\pi : \nu^\pi \cdot P^\pi = \nu^\pi$. Notice that in this case the value of $\bar{v}^\pi$ does not depend on an initial distribution $q$. Then the objective simplifies a lot:

$$\bar{v}^\pi = \langle \nu^\pi, r^\pi \rangle.$$

To consider AMDPs with this property, the mixing assumption was introduced in [19, 9] and we will use it too. Informally speaking, this assumption told us that the Markov chain corresponding to any policy $\pi$ is converges to a stationary distribution rapidly. Also, this assumption makes our objective independent on an initial distribution.

It is natural to compare an average-reward MDP with discounted one. Instead of a discounted case [14, 13, 1], average-reward MDPs are not so nice from the theoretical point of view [4]. However, the popularity of AMDPs in the RL community increases [7, 18]. One of the reasons to examine an average-reward case is that many cyclical problems that appear in robotics and games could not be naturally represented as discounted problems [11].

Additionally, we consider solving AMDP with a generative model: there is a stochastic oracle that generates a transition state from a given state-action pair according to probabilities $P$. It is a much more suitable framework than knowledge of full matrix $P$ from the point of view of reinforcement learning. In the full-knowledge setting, much has been done: we refer to [4, 12] for a survey.

**Our contribution.** However, to the authors’ best knowledge, there are only two works that consider the infinite-horizon AMDP with a generative model – [9, 19]. In both papers the same general convex optimization algorithm was used – Stochastic Mirror Descent for saddle-point problems, and both of the presented algorithms are not designed for parallel computations. In this paper, we present the first parallel algorithm for this problem with very low communication costs. The parallelism gives a possibility to handle very large setups of MDPs that cannot be stored in the memory of one machine. In the case of simultaneous working of $A_{\text{tot}}$ workers, our algorithm works in $\tilde{O}(t^{2}_{\text{mix}} |S| \varepsilon^{-2})$ and outperforms approach of [9] which works in $\tilde{O}(t^{2}_{\text{mix}} A_{\text{tot}} \varepsilon^{-2})$.

**Technique.** Essentially, the proposed algorithm is Stochastic Mirror Descent with functional constraints [2, 3, 15], applied to a linear program corresponding to the solution to AMDP. However, in our case we do not have an exact model of a MDP instance, thus, it is not possible to compute exactly constraint functions. This setting was developed in [10] but in our case techniques of this paper can reach sample complexity only of order $O(\varepsilon^{-4})$. Additionally, the problem that could be solved efficiently by this scheme does not give an approximate policy directly. We needed to compute the approximate dual solution (in sense of the Lagrange dual problem). The main technical novelties of our algorithm are possibilities to use a noisy version of constraints and to compute the dual solution. As far as we know, it is the first such result in the stochastic setting.

**Notation.** For a matrix $A \in \mathbb{R}^{n \times m}$ we define its $i$-th row as $A_{(i)}$. We denote by $1 = (1, \ldots, 1)^T$ the vector filled with ones. By $e_i$ we define a standard basis vector. Also we define $\Delta^v = \{ x \in \mathbb{R}^n \mid \forall i : x_i \geq 0, \sum_{i=1}^n x_i = 1 \}$ and $\mathbb{R}^n_{\geq c} = [-c, c]^n$. $I$ is an identity matrix of size deducible from the context. Inner product $\langle \cdot, \cdot \rangle : E^* \times E \to \mathbb{R}$ is defined on pairs of vectors from dual and primal spaces. In the case of Euclidean spaces, it coincides with the standard inner product. Also we define $[m] = \{1, \ldots, m\}$.

\section{Stochastic Mirror Descent with noisy constraints}

In this section we develop techniques of [2] and [3]. Firstly, we introduce a basic notation that will be used in further. Then we propose a new algorithm for the constrained convex optimization problem in the model of inexact computation of constraint functions. Finally, we prove convergence of the algorithm in terms of the duality gap between primal and (Lagrange) dual problems. The last part is crucial for an application on average-reward MDPs.
2.1 Notation

We consider the constrained convex optimization problem over a convex compact set $Q \subseteq E$, where $E$ is a finite-dimensional normed space:

$$\min_{x \in Q} f(x),$$

s.t. $g^{(l)}(x) \leq 0, \forall l \in [m], \tag{1}$$

and $f : Q \to \mathbb{R}, g^{(l)} : Q \to \mathbb{R}$ are convex functions. We assume that subgradients of these functions exist for each $x \in Q$ for simplicity. We call $\nabla f(x), \nabla g^{(l)}(x)$ any subgradients of corresponding functions. However, in our algorithm we have an access only to stochastic subgradient oracles $\nabla f(x, \xi), \nabla g^{(l)}(x, \xi_{(l)})$.

Now we are going to introduce definitions that will be useful in the algorithm description.

**Definition 1.** Function $h_{\delta} : Q \to \mathbb{R}$ is called a $\delta$-approximation of $h : Q \to \mathbb{R}$ if $|h_{\delta}(x) - h(x)| \leq \delta$ for all $x \in Q$.

For our problem, we suggest that we have an oracle not for computation of constraints $g^{(l)}$ but their $\delta$-approximations $g^{(l)}_{\delta}$. It is the important difference with the setup of [2]: in our assumption we cannot consider only one constraint of form $g(x) = \max_{i \in [m]} g^{(l)}(x)$ because it does not seem possible to compute a subgradient of $g$ given subgradients of $g^{(l)}$ when index on which the maximal value attains is unknown.

Now we are ready to list all our assumptions on problem setup (1) for our algorithm:

**A1** $f$ and all $g^{(l)}$ are Lipschitz continuous with constants $M_f$ for objective function and $M_g$ for all constraints;

**A2** Stochastic subgradients are unbiased: $\mathbb{E} \nabla f(x, \xi) = \nabla f(x), \mathbb{E} \nabla g^{(l)}(x, \xi_{(l)}) = \nabla g^{(l)}(x)$;

**A3** Stochastic subgradients are bounded: $\|\nabla f(x, \xi)\|_* \leq M_f, \|\nabla g^{(l)}(x, \xi_{(l)})\|_* \leq M_g$ a.s.;

Since our algorithm is Mirror-Descent based, the next step is to define the proximal step and basic properties of Mirror Descent.

Firstly, we define a prox-function $d : Q \to \mathbb{R}$ as a continuous 1-strongly convex function $d$ with respect to the norm $\| \cdot \|$ on $E$ that admits continuous selection of subgradients $\nabla d(x)$ where they exist. Bregman divergence that corresponds to a prox-function $d$ is a function $V(x, y) = d(y) - d(x) - \langle \nabla d(x), y - x \rangle$.

Given vectors $x \in E$ and $v \in E^*$, the mirror step is defined as follows:

$$x^+ = \text{Mirr}(x, v) = \arg \min_{y \in X} \{ \langle v, y \rangle + V(x, y) \}.$$

We assume that the mirror step can be easily computed.

The main interest of the mirror step is the following lemma.

**Lemma 1 ([2]).** Let $h$ be some convex function over a set $Q$, $\eta > 0$ is a stepsize, $x \in Q$. Let the point $x^+ = \text{Mirr}(x, \eta(\nabla h(x) + \Delta))$, where $\Delta$ is some vector from the dual space. Then, for any $y \in Q$:

$$\eta(h(x) - h(y) + \langle \Delta, x - y \rangle) \leq \eta(\nabla h(x) + \Delta, x - y)$$

$$\leq \frac{\eta^2}{2} \|\nabla h(x) + \Delta\|_*^2 + V(x, y) - V(x^+, y).$$

2.2 Primal Problem

In this subsection, we consider problem (1) in terms of convergence of the objective function in confidence region. Formally speaking, a vector $\hat{x}$ is called an $(\epsilon_f, \epsilon_g, \sigma)$-solution to the primal problem (1), if

$$f(\hat{x}) - f(x^*) \leq \epsilon_f, \tag{2}$$

$$g^{(l)}(\hat{x}) \leq \epsilon_g \forall l \in \{1, \ldots, m\} \text{ with probability } \geq 1 - \sigma,$$
where \( x^* \) is a true minimizer of the problem. We assume that an algorithm have access only to stochastic subgradient oracles of functions \( f, g^{(l)} \) and to \( \delta \)-approximations \( g^\delta_{(l)} \) of constraint functions.

**Algorithm 1:** Stochastic Mirror Descent with noisy constraints

**Input:** accuracy \( \epsilon > 0 \), number of steps \( N \), constants \( M_f, M_g \)

\[
\begin{align*}
\eta_f &= \epsilon / M_f^2, \eta_g = \epsilon / M_g^2; \\
x^0 &= \arg\min_{x \in Q} d(x); \\
I &= \emptyset, J = \emptyset; \\
\text{for } k = 0, 1, 2, \ldots, N - 1 \text{ do} \\
\text{if } g^\delta_{(l)}(x^k) \leq \epsilon + \delta \forall l \in [m] \text{ then} \\
x^{k+1} &= \operatorname{Merr}(x^k, \eta_f \nabla f(x^k, \xi^k)); \quad \text{// "productive" steps} \\
&\text{Add } k \text{ to } I; \\
\text{else} \\
l(k) &= \arg\max_{i \in [m]} g^\delta_{(l)}(x^k); \\
x^{k+1} &= \operatorname{Merr}(x^k, \eta_g \nabla g^{(l(k))}(x^k, \xi^{l(k)})); \quad \text{// "non-productive" steps} \\
&\text{Add } k \text{ to } J; \\
\text{return } \hat{x} = \frac{1}{M} \sum_{k \in I} x^k;
\end{align*}
\]

Denote \( \hat{\nabla} f = \nabla f(x^k, \xi^k), \hat{\nabla} g = \nabla f(x^k) \) and \( \hat{\nabla} k g^{(l)} = \nabla g^{(l)}(x^k, \xi^l) \). Additionally, we define \( M = \max\{M_f, M_g\}, M = \min\{M_f, M_g\} \) and, finally, \( \Theta_0^2 = d(x^*) - d(x^0) \).

To simplify the notation, we introduce the stochastic gradient noise function

\[
\gamma_k(y) = \begin{cases} 
\eta_f (\hat{\nabla} f(y) - \nabla f(y) - x^k), & k \in I; \\
\eta_g (\hat{\nabla} g^{(l(k))}(y) - \nabla g^{(l(k))}(y) - x^k), & k \in J.
\end{cases}
\]

The main property of this quantity is that it forms a martingale-difference sequence.

Now we are going to provide some useful properties of Algorithm 1 in terms of bounds of error in objective function \( f \) and constraint satisfiability. Afterwards, we combine all properties to a convergence analysis of the algorithm.

**Lemma 2.** For point \( \hat{x} \) produced by Algorithm 1 and any \( y \in Q \) the following holds:

\[
\eta_f |I| \cdot (f(\hat{x}) - f(y)) \leq \frac{\eta_f^2 M_f^2}{2} |I| + \frac{\eta_g^2 M_g^2}{2} |J| + [d(y) - d(x^0)] - |J| \eta_g \epsilon \\
+ \sum_{k=0}^{N-1} \gamma_k(y) + \eta_g \sum_{k \in J} g^{(l(k))}(y).
\]

**Proof.** By the construction of "productive" and "non-productive" steps in Algorithm 1 and Lemma 1 we have for all \( y \in Q \):

\[
\eta_f (f(x^k) - f(y)) \leq \frac{\eta_f^2 M_f^2}{2} + V(x^k, y) - V(x^{k+1}, y) \\
+ \eta_f (\hat{\nabla} f(y) - \nabla f(y) - x^k), \\
\eta_g (g^{(l(k))}(x^k) - g^{(l(k))}(y)) \leq \frac{\eta_g^2 M_g^2}{2} + V(x^k, y) - V(x^{k+1}, y) \\
+ \eta_g (\hat{\nabla} g^{(l(k))} - \nabla g^{(l(k))}, y - x^k).
\]
By definition of \( \delta \)-approximation, we have the following inequalities for "productive" and "non-productive" steps respectively:

\[
\eta_f (f(x^k) - f(y)) \leq \frac{\eta_f^2 M_f^2}{2} + V(x^k, y) - V(x^{k+1}, y) + \gamma_k(y),
\]

\[
\eta_g (g^{(l(k))}(x^k) - g^{(l(k))}(y)) \leq \frac{\eta_g^2 M_g^2}{2} + V(x^k, y) - V(x^{k+1}, y) + \gamma_k(y) + \eta_g \delta + \eta_g.
\]

Sum all these inequalities over all \( k \in I \) and \( k \in J \) and use the fact that \( I \cup J = \{0, \ldots, N - 1\} \):

\[
\sum_{k \in I} \eta_f (f(x^k) - f(y)) + \sum_{k \in J} \eta_g (g^{(l(k))}(x^k) - g^{(l(k))}(y)) \leq \frac{\eta_f^2 M_f^2}{2} |I| + \frac{\eta_g^2 M_g^2}{2} |J| + \sum_{k=0}^{N-1} [V(x^k, y) - V(x^{k+1}, y)] + \sum_{k=0}^{N-1} \gamma_k(y) + |J| \eta_g \delta. \tag{5}
\]

By the choice of \( x^0 = \arg\min_{x \in Q} d(x) \), we have

\[
\sum_{k=0}^{N-1} [V(x^k, y) - V(x^{k+1}, y)] \leq V(x^0, y) - d(y) - d(x^0) - \langle \nabla d(x^0), y - x^0 \rangle = d(y) - d(x^0).
\]

Using definition of "non-productive" steps \( g^{(l(k))}(x^k) > \varepsilon + \delta \) and convexity of \( f \):

\[
\sum_{k \in I} \eta_f (f(x^k) - f(y)) + \sum_{k \in J} \eta_g (g^{(l(k))}(x^k) - g^{(l(k))}(y)) \geq \eta_f |I| (f(\hat{x}) - f(y)) + \eta_g |J| (\varepsilon + \delta) - \eta_g \sum_{k \in J} g^{(l(k))}(y).
\]

By application of inequality (5) and simple regrouping of terms, we finish the proof.

During this section we set \( y = x^* \). In this case \( d(x^*) - d(x^0) \leq \Theta_0^2 \). Further we are going to derive bound on \( g^{(l)} \) at \( \hat{x} \) for each separate function \( g^{(l)} \):

**Lemma 3.** For \( \hat{x} \) produced by Algorithm 1 the following holds:

\[
g^{(l)}(\hat{x}) \leq \varepsilon + 2\delta. \tag{6}
\]

**Proof.** By convexity and definition of a \( \delta \)-approximation:

\[
g^{(l)}(\hat{x}) \leq \frac{1}{|I|} \sum_{k \in I} g^{(l)}(x^k) \leq \frac{1}{|I|} \sum_{k \in I} (g^{(l)}(x^k) + \delta).
\]

We finish the proof by definition of "productive" steps and a set \( I \).

Before obtaining final bounds on \( f \) we prove some technical fact:

**Lemma 4.** For any \( y \in Q \): \( \|y - x^0\|^2 \leq 2(d(y) - d(x^0)) \).

**Proof.** Follows from strongly convexity of \( d \) with respect to norm \( \| \cdot \| \) and the fact that \( x^0 \) is a minimum of \( d \).

Now we derive bounds on sums of our stochastic noise function with high probability.

**Lemma 5.** Define event \( \mathcal{E} \) such that the following inequalities holds:

\[
\sum_{k=0}^{N-1} \gamma_k(x^*) \leq \frac{2\Theta_0}{M} \cdot \sqrt{2N\varepsilon^2 \log(1/\sigma)}
\]

Then \( \Pr[\mathcal{E}] \geq 1 - \sigma \).
Theorem 1. Algorithm 1 outputs $(\varepsilon, \varepsilon + 2\delta, \sigma)$-solution in sense of (2) after $N \geq N_0$ iterations, where $N_0$ is defined in Lemma 6.
Proof. To guarantee that \( f(\hat{x}) - f(x^*) \leq \varepsilon \) with probability at least \( 1 - \sigma \), we use the first inequality in Corollary 1 and Lemma 6. To guarantee satisfaction of constraints, we simply use Corollary 1.

Additionally, if case \( M_f = M_g = M \) we can give simple description of \( N_0 \) in terms of \( O \)-notation:

\[
N_0 = O \left( \frac{\Theta_0^2 M^2 \log(1/\sigma)}{\varepsilon^2} \right).
\]

Remark. Notice that from theoretical point of view selection of the maximum in line 9 of Algorithm 1 could not be avoided. However, in case of non-stochastic constraint computation one of used heuristic is to set \( l(k) \) to any index of violated constraint and we suggest that such heuristic could work in noisy setting too.

Remark. Notice that a quantity \( \Theta_0^2 \) is not used in the pseudocode of Algorithm 1. Thus, it is possible to use another initial \( x^0 \) and have a «warm start» such that our algorithm converges faster. However, it could be considered only as a heuristic since Mirror Descent algorithms require \( x^0 \) be a minimum of prox-function.

### 2.3 Primal-Dual Convergence

In this subsection, we extend properties of the previous algorithm and prove its primal-duality. First of all, let us define the (Lagrange) dual optimization problem associated with the problem (1):

\[
\max_{\lambda \in \mathbb{R}_+^m} \left\{ \phi(\lambda) := \min_{x \in Q} \left\{ f(x) + \sum_{i=1}^n \lambda_i g_i(x) \right\} \right\}.
\]

Call \( \lambda^* \) a solution to this dual problem (if it exists). We refer to [6] for an additional background and examples.

It is well-known that for any \( x \in Q : g_i^{(l)}(x) \leq 0 \ \forall l \in \{1, \ldots, m\} \) and \( \lambda \in \mathbb{R}_+^m \) the weak duality holds: \( \Delta(x, \lambda) = f(x) - \phi(\lambda) \geq 0 \), where \( \Delta \) is so-called the duality gap. We assume that for our primal problem (1) the Slater’s condition holds, i.e. \( \exists x \in Q : \forall l \in \{1, \ldots, m\} : g_i^{(l)}(x) < 0 \). It implies that the dual problem has a solution and there is the strong duality: \( \Delta(x^*, \lambda^*) = 0 \) for any \( x^* \) and \( \lambda^* \) are solutions to the primal and the dual problems respectively.

It gives us a natural way to measure a quality of the pair \((\hat{x}, \hat{\lambda})\) by the value of the duality gap \( \Delta(\hat{x}, \hat{\lambda}) \). Let us call the pair \((\hat{x}, \hat{\lambda})\) a primal-dual \((\varepsilon_\Delta, \varepsilon_g, \sigma)\)-solution to (1) if the following holds with probability at least \( 1 - \sigma \):

\[
\Delta(\hat{x}, \hat{\lambda}) \leq \varepsilon_\Delta,
\]

\[
g_i^{(l)}(\hat{x}) \leq \varepsilon_g \ \forall l \in [m].
\]

Notice that since \( \hat{x} \) is not a feasible solution to the primal problem (1), we do not have the weak duality inequality \( \Delta(\hat{x}, \hat{\lambda}) \geq 0 \). However, the value of duality gap could be controlled from below because of controlled unfeasibility.

The most powerful property of Algorithm 1 is a possibility to generate a pair of primal-dual solutions in sense of (8). We could control the value of the duality gap without the explicit access to the constraint functions.

As a values of dual variables we choose \( \hat{\lambda} \in \mathbb{R}_+^m : \)

\[
\hat{\lambda}_l = \frac{1}{\eta_f |I|} \sum_{k \in I} \eta_g I \{l = l(k)\}
\]

in terms of Algorithm 1. Additionally, we define useful constant \( \Theta_0^2 = \sup_{y \in Q} (d(y) - d(x^0)) \).

Using \( \hat{\lambda} \), we could provide primal-dual properties of Algorithm 1.
Lemma 7. Suppose \( \hat{x} \) is an output of Algorithm 7 and \( \hat{\lambda} \) is defined as in (9) and \( \Theta_0^2 \) \( \sup_{y \in Q} (d(y) - d(x^0)) \).

Then the following holds:

\[
\eta_f |I| \Delta(\hat{x}, \hat{\lambda}) \leq \frac{\eta_f^2 M_1^2}{2} |I| + \frac{\eta_g^2 M_2^2}{2} |J| + \Theta_0^2 - |J| \eta_g \varepsilon \\
+ \sum_{k=0}^{N-1} \gamma_k (x^0) + \Theta_0 \sqrt{2} \cdot \left\| \sum_{k=0}^{N-1} \Delta_k \right\|_a,
\]

where \( \Delta_k \) is defined as follows:

\[
\Delta_k = \begin{cases} 
\eta_f \left( \nabla_k f - \nabla f \right), & k \in I \\
\eta_g \left( \nabla_k g^{(l(k))} - \nabla g^{(l(k))} \right), & k \in J.
\end{cases}
\]

Proof. We start from Lemma 2. Here we move all terms consist of \( y \) to the right-hand side and minimize over \( y \):

\[
\eta_f |I| f(\hat{x}) \leq \frac{\eta_f^2 M_1^2}{2} |I| + \frac{\eta_g^2 M_2^2}{2} |J| - |J| \eta_g \varepsilon \\
+ \min_{y \in Q} \left\{ d(y) - d(x^0) + \sum_{k=0}^{N-1} \gamma_k (y) + \eta_f |I| f(y) + \eta_g \sum_{k \in J} g^{(l(k))}(y) \right\}.
\]

Notice that by definition of \( \hat{\lambda} \) we have

\[
\eta_f |I| \phi(\hat{\lambda}) = \min_{y \in Q} \left\{ \eta_f |I| f(y) + \eta_g \sum_{k \in J} g^{(l(k))} \right\}.
\]

Thus, we have to upper bound \( d(y) - d(x^0) \) and \( \sum_{k=0}^{N-1} \gamma_k (y) \) without dependence on \( y \) to obtain required result. The first upper bound is trivial: \( d(y) - d(x^0) \leq \Theta_0^2 \) by definition of \( \Theta_0^2 \).

To analyse the second term, we use the definition of \( \gamma_k \) in terms of \( \Delta_k \) and Holder inequality:

\[
\sum_{k=0}^{N-1} \gamma_k (y) = \sum_{k=0}^{N-1} \langle \Delta_k, y - x^0 + x^0 - x^k \rangle \leq \| y - x^0 \| \left\| \sum_{k=0}^{N-1} \Delta_k \right\|_a + \sum_{k=0}^{N-1} \gamma_k (x^0).
\]

The last step is to apply Lemma 3 and definition of \( \Theta_0^2 \) to obtain uniform bound on \( \| y - x^0 \| \).

Our next goal is to derive bound on the right-hand side of (10). It is possible using concentration of measure techniques as in Lemma 5.

Lemma 8. Define event \( E' \) such that the following inequalities holds:

\[
\sum_{k=0}^{N-1} \gamma_k (x^0) < \frac{2 \Theta_0}{M} \cdot \sqrt{2N \varepsilon^2 \log(2/\sigma)} \\
\left\| \sum_{k=0}^{N-1} \Delta_k \right\|_a < \frac{\sqrt{2} \kappa(E^*) + \sqrt{4 \log(2/\sigma)}}{M} \sqrt{2N \varepsilon^2},
\]

where \( \Delta_k \) is defined in 7 and \( \kappa(E^*) \) is a constant of Nemirovski’s inequality 5 for the dual space. Then \( \Pr[E'] \geq 1 - \sigma \).

Remark. If \( E \) has a finite dimension \( d \), then we always have \( \kappa(E^*) \leq d \). Additionally, if \( E \) is endowed with \( \ell_p \) norm, then \( E^* \) is endowed with \( \ell_q \) norm, where \( 1/p + 1/q = 1 \), and there is a more precise bound, according to 8:

\[
\kappa(E^*) \leq K \left( \frac{p}{p-1}, d \right) = \begin{cases} 
d^{\frac{p}{2}-1}, & p \in [1,2] 
d^{1 - \frac{p}{2}}, & p \in (2, + \infty],
\end{cases}
\]

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In particular, if $E$ has $\ell_2$ norm, $\kappa(E^*) = 1$. For $p \in [2, +\infty]$ this bound is tight, however, in the case $p \in [1,2]$ and $d \geq 3$ it could be improved to, for instance, a logarithmic bound $\kappa(E^*) \leq 2e \log(d) - e$, that could be useful in the case of $\ell_1$-norm and an entropy prox-function.

Proof. The case of first inequality is identical to Lemma 5 by rescaling $\sigma$ to $\sigma/2$.

To ensure the last inequality, we apply bounded difference inequality [5]. This follows by observing that $Z = \| \sum_{k=0}^{N-1} \Delta_k \|$ satisfies bounded difference condition with a constant $4 \cdot \varepsilon M^{-1}$ that is greater than $2 \cdot \| \Delta_k \|$ a.s.. Thus

$$\Pr[Z - \mathbb{E}Z > t_2] \leq \exp\left(\frac{-t_2^2}{2 \cdot N \varepsilon^2 \cdot 4M^{-2}}\right).$$

Take $t_2 = 2M^{-1} \sqrt{2N \varepsilon^2 \cdot \log(4/\sigma)}$ and the last we have to do is to bound expectation of $Z$. Here we apply Nemirovski’s inequality:

$$\left(\mathbb{E} \left\| \sum_{k=0}^{N-1} \Delta_k \right\|_\ast \right)^2 \leq \mathbb{E} \left[ \left\| \sum_{k=0}^{N-1} \Delta_k \right\|_\ast^2 \right] \leq \kappa(E^*) \sum_{k=0}^{N-1} \mathbb{E} \left[ \| \Delta_k \|_\ast^2 \right] \leq \kappa(E^*) 4N \varepsilon^2 M^{-2}.$$

By application of the union bound we finish the proof. □

Corollary 2. Under event $\mathcal{E}'$ defined in Lemma 3 the following inequalities holds for all $l \in [m]$:

$$\eta_l |I| \Delta(\hat{x}_l, \hat{\lambda}) < \eta_l |I| \varepsilon - \frac{\varepsilon^2 N}{2M} + \Theta_0^2 + \sqrt{2N \varepsilon^2} \left( \frac{2\Theta_0 (4\sqrt{\log(2/\sigma)} + \sqrt{2\kappa(E^*)})}{M} \right);$$

$$g^{(l)}(\hat{x}) < \varepsilon + 2\delta.$$

Proof. Inequalities on $g^{(l)}(\hat{x})$ is equivalent to the same in 1. Inequality on $\Delta(\hat{x}_l, \hat{\lambda})$ follows from a combination of Lemma 7 and Lemma 8 □

Now we are ready to state a technical lemma that is similar to Lemma 6.

Lemma 9. If $\sigma \leq 1/2$ and

$$N \geq N_0 = \left( \frac{M}{M} \right)^4 \left( \frac{128\Theta_0^2 M^2 (17 \log(2/\sigma) + 2\kappa(E^*))}{\varepsilon^2} \right)$$

then

$$\varepsilon^2 N \geq 2M^2 - \Theta_0^2 - \sqrt{2N \varepsilon^2} \left( \frac{2\Theta_0 (4\sqrt{\log(2/\sigma)} + \sqrt{2\kappa(E^*)})}{M} \right) \geq 0.$$

Proof. Using inequality $M_g \geq M$ and the same reasoning about solving quadratic inequality in terms of $t = \sqrt{2N \varepsilon^2}$ as in Lemma 6 it is sufficient to show that

$$\sqrt{2N \varepsilon^2} \geq 4 \left( \frac{M}{M} \right)^2 \left( \frac{2\Theta_0 M (4\sqrt{\log(2/\sigma)} + \sqrt{2\kappa(E^*)})}{M} \right) + \frac{\Theta_0^2 M}{2\Theta_0 (4\sqrt{\log(2/\sigma)} + \sqrt{2\kappa(E^*)})}.$$ 

Since $\sigma \leq 1/2 \Rightarrow \log(2/\sigma) \geq 1$, $\sqrt{2\kappa(E^*)} \geq 1$ and $M/M \geq 1$, we have that it is sufficient to show that

$$\sqrt{2N \varepsilon^2} \geq 4 \left( \frac{M}{M} \right)^2 \left( \frac{2\Theta_0 M ((4 + 1/20)\sqrt{\log(2/\sigma)} + \sqrt{2\kappa(E^*)})}{M} \right).$$

By numeric inequality $2a^2 + 2b^2 \geq (a + b)^2$ and $(4 + 1/20)^2 \leq 17$, it is satisfied with our choice of $N \geq N_0$. □
Now we are ready to state our main result.

**Theorem 2.** Let us choose \( \hat{\lambda} \in \mathbb{R}^m_+ \) as defined in \((9)\) and \( \hat{x} \) is an output of the algorithm. Then the pair \((\hat{x}, \hat{\lambda})\) is an \((\varepsilon, \varepsilon + 2\delta, \sigma)\)-solution in sense of \((8)\) after \( N \geq N_0' \) iterations, where \( N_0' \) is defined in Lemma\(9\).

**Proof.** The inequality on \( g(\hat{x}) \) is satisfies by Corollary\(2\). We have to satisfy inequality on duality gap. The inequality on duality gap follows directly from Corollary\(2\) and Lemma\(2\) since \( N \geq N_0' \).

We can write bound on \( N \) in case \( M_f = M_g = M \) using \( O \)-notation as follows:

\[
N = O \left( \Theta_0^2 M^2 \left( \log(1/\sigma) + \kappa(E^*) \right) \right).
\]

The only difference between primal and dual case is connected to the constant in Nemirovski’s inequality.

**Remark.** As in the primal case, in the complexity bounds we have a constant \( \Theta_0^2 \) that does not appear in the algorithm description. This fact gives us a chance to work much better in practice than using worst-case constant. The same situation with constant \( \kappa(E^*) \).

### 3 Mixing AMDP

In this section, we discuss the application of the developed algorithm to the problem of approximate solving mixing average-time Markov Decision Processes (MDP). Firstly, we propose basic definitions connected to MDPs. Then, we discuss some technical nuances that will appeared in the algorithm. Finally, we outline the full algorithm and its parallel implementation.

#### 3.1 Markov Decision Process

First of all, we define Bellman equations for an AMDP \(4\): \( \bar{v}^* \) is the optimal average reward if and only if there exists a vector \( h^* \in \mathbb{R}^{|S|} \) satisfying the following:

\[
\bar{v}^* + h^*_i = \max_{a_i \in A_i} \left\{ \sum_{j \in S} p_{ij}(a_i) v^*_j + r_{i,a_i} \right\}, \forall i \in S.
\]

We focus on study of the primal LP which solution is equivalent to the solution to Bellman equation:

\[
\min_{\bar{v}, h} \bar{v}
\]

s.t. \( \bar{v}1 + (\hat{I} - P)h - r \geq 0, \)

where \( \hat{I}_{(i,a_i),j} = I_{i,j} \).

However, without additional assumptions it is hard to analyze problem. Following \(9\) \(19\), we introduce one important assumption on an AMDP instance:

**Assumption 1 (Mixing AMDP).** We call an AMDP instance mixing if its so-called mixing time (defined below) is bounded by \(1/2\):

\[
t_{mix} := \max_{\pi} \left[ \arg\min_{\tau \geq 1} \| (P^{\pi \top})^\tau q - \nu^\pi \|_1 \right] \leq \frac{1}{2}.
\]

The most powerful corollary of this result is a possibility to make the search space a compact convex set. Formally speaking, in \(9\) it was proven that the search space for primal variables could be reduced to \( X = [0,1] \times \mathbb{B}_{2M}^{|A|} = [0,1] \times [-2M, 2M]^{|A|} \), where \( M = 2t_{mix} \). We choose bounds of size \( 2M \) instead of \( M \) because of the same reasons as \(9\) that will be described in Section\(3.3\).
Overall, we have a (linear) optimization problem on a compact with a large number of constraints:

\[
\min_{\tilde{v}, h \in X} \tilde{v} \\
\text{s.t. } \tilde{v}1 + (\tilde{I} - P)h - r \geq 0.
\]  

(11)

If we knew the matrix \(P\), we could apply Stochastic Mirror Descent with constraints \(^2\) and got an approximate solution to this LP problem. However, it is not the case: we have only a sampling access to the transition probability matrix. The second problem that we face is to obtain an optimal policy by an approximate solution to this linear program. It is known that there is a strong connection between the optimal policy and the dual LP but not the primal one. To overcome this, we use primal-duality of Algorithm and construct a policy using dual variables.

### 3.2 Preprocessing

In this subsection we aim to describe complexity of the preprocessing connected to the estimate of the transition probability matrix. Firstly, we prove one technical lemma.

**Lemma 10 (Estimation of parameters of multinomial distribution).** Suppose that we have samples \(\{X_1, \ldots, X_N\}\) drawn from multinomial distribution with a parameter \(s \in \Delta^d\). Define an empirical estimate \(\hat{s} = \frac{1}{N} \sum_{i=1}^N e_{X_i} \in \Delta^d\), where \(e_j\) is an \(j\)-th standard basis vector.

Then, for any \(\delta', \sigma' > 0\), the inequality \(\|s - \hat{s}\|_1 \leq \delta'\) holds with a probability at least \(1 - \sigma'\) if

\[
N \geq \frac{8d + 4 \log(1/\sigma')}{\delta'^2}.
\]

**Proof.** First of all, notice that \(\mathbb{E} e_{X_i} = s\). Denote by \(\Delta\) a centred random variable \(\hat{s} - s = \frac{1}{N} \sum_{i=1}^N (e_{X_i} - s)\). Then, by Nemirovski’s inequality and bound on the \(\ell_1\) norm of elements of a simplex, we might estimate the mean of the square of \(\ell_1\) norm of this random variable: \(\mathbb{E}\|\Delta\|_1^2 \leq 4d/N\).

Let us define function \(f(X_1, \ldots, X_N) = \|\Delta\|_1\). It might be checked that this function satisfies conditions of the bounded difference inequality \(^5\) with constant \(2/N\). Thus, for all \(t > 2\sqrt{d/N} \geq \mathbb{E}\|\Delta\|_1\):

\[
\Pr[\|\Delta\|_1 > t] = \Pr[\|\Delta\|_1 - \mathbb{E}\|\Delta\|_1 > t - \mathbb{E}\|\Delta\|_1] \leq \exp\left(-\frac{(t - 2\sqrt{d/N})^2}{2N}\right).
\]

Taking \(N \geq (8d + 4 \log(1/\sigma')) \cdot (\delta')^{-2}\) and \(t = \delta\), we finish the proof.

Using this simple lemma, we can easily obtain required sample complexity for the preprocessing even in parallel sampling setting. Remember that we are going to use an algorithm \(^1\), hence, we want to approximate each constraint function.

**Proposition 1.** For each \(\delta', \sigma' > 0\), the estimate \(\tilde{P}\) of \(P\), such that for each \(a \in A, h \in \mathbb{B}_{2M}^{\Delta_{\tilde{P}a}} : \| (\tilde{P}_{(a)} - P_{(a)} , h) \| \leq \delta'\) with a probability at least \(1 - \sigma'\), could be computed in \(O\left(\sum_{t_{\text{mix}}} A_{\text{tot}} \cdot \frac{|S| + \log(A_{\text{tot}}/\sigma')}{\delta'^2}\right)\) total samples, \(O(1)\) parallel depth and \(O\left(\sum_{t_{\text{mix}}} \frac{|S| + \log(A_{\text{tot}}/\sigma')}{\delta'^2}\right)\) samples proceed by each single node. In the case of \(m \leq A_{\text{tot}}\) available workers, it works in \(O\left(\frac{A_{\text{tot}}}{m} \cdot \frac{t_{\text{mix}}^2 |S| + \log(A_{\text{tot}}/\sigma')}{\delta'^2}\right)\) real time.

**Proof.** Notice that each \(P_{(a)} \in \Delta^{|S|}\) is a parameters of a multinomial distribution we sampling from, and the estimator from the previous lemma could be applied.

By Holder’s inequality and the definition of the search space for \(h\) we have that \(\| (\tilde{P}_{(a)} - P_{(a)} , h) \| \leq 2M \cdot \| (P_{(a)} - \tilde{P}_{(a)} , h) \|\). Hence, to make it less than \(\delta'\), we required to make \(\ell_1\) norm of difference less than \(\delta'/(2M)\). To make all conditions work simultaneously, it is enough to make the probability in the terms of the previous lemma less than \(\sigma'/A_{\text{tot}}\) and apply the union bound over all \(A_{\text{tot}}\) conditions.

The last observation that finishes the proof is that sampling for an estimation of each \(\tilde{P}_a\) could be done separately and independent.
3.3 Rounding to Optimal Policy

In this subsection, we prove the result that give us a possibility to obtain an approximate optimal policy from the dual variables produced by (1).

**Proposition 2.** Suppose that primal \((\bar{\nu}, h\bar{\nu})\) and dual \(\mu\bar{\nu}\) variables are \((\varepsilon_f, \varepsilon_g, \sigma)\)-approximate solution to (11) in terms of (expected) primal-dual convergence (8). Define the policy \(\pi_i, a_i = \frac{\mu_i}{\lambda_i}\), where \(\lambda_i \in \mathbb{R}^+_i\) is defined as \(\lambda_i = \sum_{a_i \in A_i} \mu_i, a_i\). Then \(\pi\) is an \(4(\varepsilon_f + \varepsilon_g)\)-optimal policy with probability at least \(1 - \sigma\).

**Proof.** Conditions of the proposition give us the following guarantees in terms of the duality gap and constraint satisfaction with needed probability \(\geq 1 - \sigma\):

\[
\bar{v}^\nu - \min_{\bar{v}, h} \bar{v} + (\mu^\nu)^\top (\bar{v}1 - (\bar{I} - \bar{P})h + r) \leq \varepsilon_f, \tag{12}
\]

\[
\bar{v}^\nu 1 + (\bar{I} - \bar{P})h\nu - r \geq -\varepsilon_g 1.
\]

We can rewrite the first condition in more suitable terms of \(\lambda\): \(\forall \bar{v}, \forall h \in \mathcal{X}: \varepsilon_f \geq (\bar{v}^\nu - \bar{v}) + \bar{v} \cdot (\lambda^\nu) 1 + (\lambda^\nu)^\top ([I - P\nu]h - r\nu). \tag{13} \)

Now we have all required instruments and we can bound the expectation of an average value of our policy:

\[
\nu^\nu = (\nu^\nu)^\top r^\nu = (\nu^\nu - \lambda^\nu)^\top ([P\nu - I]h^\nu + r^\nu) + (\lambda^\nu)^\top ([P\nu - I]h^\nu + r^\nu).
\]

Here we used the stationary of our policy: \((\nu^\nu)^\top (P\nu - I) = 0\). For simplicity, we remind that \(\mu^\nu, \lambda^\nu \geq 0\), and, hence, \((\mu^\nu, 1) = (\lambda^\nu, 1) = \|\lambda^\nu\|_1\).

Firstly, bound the second term by (13): \((\lambda^\nu)^\top ([P\nu - I]h^\nu + r^\nu) \geq \bar{v}^\nu - \bar{v}(1 - \|\lambda^\nu\|_1) - \varepsilon_f.\)

To bound the first term we will use Lemma 7 from [9]:

\[
\|([I - P\nu] + 1(\nu^\nu)^\top)^{-1}\|_\infty \leq M.
\]

Also we have for all \(\mu \geq 0\) by primal feasibility: \(\mu^\top ([I - P\nu]h^\nu + r^\nu 1) \geq 0\). We can combine it with our condition (13) for arbitrary \(h\) and \(\bar{v} = \bar{v}^\nu\), using the fact that \(\bar{v}^\nu \geq \bar{v} - \varepsilon_g\):

\[
\varepsilon_f + \varepsilon_g \geq \bar{v}^\nu - \bar{v}^\nu + (\mu^\nu)^\top ([I - P\nu]h - r + \bar{v}^\nu 1)
\]

\[
\geq (\mu^\nu)^\top ([I - P\nu]h - r + \bar{v}^\nu 1) - (\mu^\nu)^\top ([I - P\nu]h^\nu + r - \bar{v}^\nu 1)
\]

\[
= (\mu^\nu)^\top ([I - P\nu](h - h^\nu)) = (\lambda^\nu)^\top ([I - P\nu](h - h^\nu)).
\]

Hence, we have

\[
2M\|([\lambda^\nu]^\top [I - P\nu])\|_1 = \max_{h \in \mathbb{E}^2_M} \|([\lambda^\nu]^\top [I - P\nu])h\]

\[
= \max_{h \in \mathbb{E}^2_M} \|([\lambda^\nu]^\top [I - P\nu])(h - h^\nu) + ([\lambda^\nu]^\top [I - P\nu])h^\nu\]

\[
\leq \varepsilon_f + \varepsilon_g + M\|([\lambda^\nu]^\top [I - P\nu])\|_1.
\]

Combining with the fact that \(\nu^\nu (I - P\nu) = 0\), we obtain \(\|([\nu^\nu - \lambda^\nu]^\top [I - P\nu])\|_1 \leq \frac{\varepsilon_f + \varepsilon_g}{M}\). By almost the same argument as in [9], we also have \(|E([\nu^\nu - \lambda^\nu, r\nu])| \leq \varepsilon_f + \varepsilon_g\). Hence, there is a bound on the required first term:

\[
(\nu^\nu - \lambda^\nu)^\top ([P\nu - I]h^\nu + r^\nu) \geq -2M \cdot \frac{(\varepsilon_f + \varepsilon_g)}{M} - (\varepsilon_f + \varepsilon_g).
\]

Overall, we obtain the required inequality by taking \(\bar{v} = 0\) and by feasibility of the pair \((\bar{v}^\nu + \varepsilon_g, h^\nu)\):

\[
\bar{v}^\nu \geq \bar{v}^\nu - \bar{v}(1 - \|\lambda^\nu\|_1) - \varepsilon_f - 3(\varepsilon_f + \varepsilon_g) \geq \bar{v}^\nu - 4(\varepsilon_f + \varepsilon_g).
\]

\(\square\)
3.4 Parallel Algorithm

In this subsection, we describe a final algorithm to approximate solving AMDP in parallel. In our setup, each single node of a centralized network corresponds to a single state-action pair.

First of all, we describe linear program corresponding to AMDP \( \text{(1)} \) in terms of \( \text{(1)} \).

\[
\begin{align*}
\min_{\hat{v}, h, v} & \quad f(\hat{v}, h) = \hat{v}, \\
\text{s.t.} & \quad g^{(i,a)}(\hat{v}, h) = r(i,a) - \bar{v} + \langle (P(i,a), h), \hat{v} - h_i \rangle \leq 0 \quad \forall (i,a) \in A.
\end{align*}
\]  

where \( \mathcal{X} = [0,1] \times \mathbb{R}^{A_{\text{max}}} \). We set standard Euclidean prox-structure on \( \mathcal{X} \): \( \ell_2 \)-norm \( \| \cdot \|_2 \) and a prox-function \( d(x) = \frac{1}{2} \| \| x \| \|_2^2 \). In these terms, we have the following constants: \( M_f = 1, M_g = 2, \mathcal{M} = 2, \mathcal{M}_1 = 1 \) and \( \mathcal{O}_0 = (4M)^2|S| + 1 = O(t^{2}_{\text{max}}|S|) \).

However, in our case we cannot compute constraints since there is no access to the true model. To overcome this we firstly run preprocessing described in Section 3.2 and obtain \( \delta \)-approximation of constraints \( g^{(i,a)}_\delta \).

Additionally, we are going to use stochastic subgradients for \( g \) by using samples of next state \( s \sim P(i,a) \):

\[
\nabla_v g^{(i,a)}(\hat{v}, h) = -1, \quad \nabla_h g^{(i,a)}(\hat{v}, h, s) = e_s - e_i,
\]

with constants \( M_f = 1, M_g = 2 \). In this case, we can use Algorithm 1 but in this form this algorithm is sequential. To make a parallel version of algorithm, we observe that on each step we have to compute maximum of \( A \) constraints and a sparsity of updates ensure a low communication and complexity costs.

**Theorem 3.** For any \( \varepsilon > 0 \) and \( \sigma \in (0, 1/2) \), Algorithm 2 after preprocessing described in Section 3.2 performed with an accuracy \( \delta' = \varepsilon/16 \) and a probability of failure \( \sigma' = \sigma/2 \) returns \( \tilde{\pi} \) is an \( \varepsilon \)-approximate optimal policy with probability at least \( 1 - \sigma \) in

\[
N = O\left( \frac{t^{2}_{\text{max}}|S| \log(1/\sigma)}{\varepsilon^2} \right)
\]

Mirror Descent iterations, \( O(1) \) parallel depth, \( O(A_{\text{tot}} \cdot N) \) sample and running time complexity. In the case of \( m \leq A_{\text{tot}} \) available workers, algorithm works in \( O\left( \frac{A_{\text{tot}}}{m} \cdot t^{2}_{\text{max}}|S| \log(1/\sigma) \cdot \varepsilon^{-2} \right) \) real time.

**Remark 1.** Notice that complexity of preprocessing and Algorithm 2 matches up to logarithmic factors.

**Remark 2.** Messages “Productive step” and “Non-productive step” ensures that nodes update their values to actual ones.

**Remark 3.** The communication costs on each round of communication are low: each text message could be send using \( O(1) \) bits and each message with a sample could be send using only \( O(\log |S|) \) bits.

**Remark 4.** Additional advantage of the algorithm is a sparsity of updates: there are at most 2 values in the vector \( h \) updated each iteration. From the point of view of external memory algorithms, it gives us a possibility to do only 2 requests to the memory if the state-space is too large to store a vector \( h \) in RAM.

**Proof.** First of all, we ensure that each \( (i,a) \) worker nodes computes \( c^k = g^{(i,a)}_\delta(\bar{v}^k, h^k) \) correctly.

Induction on \( k \), the basis \( c^0 = g^{(i,a)}_\delta(0,0) = r(i,a) \) is correct. Now we are going to provide the induction step. In the case of productive steps:

\[
c^k = c^{k-1} + \varepsilon/16 = r(i,a) - v^{k-1} + \langle (P(i,a), h^{k-1}) - h_i^{k-1} \rangle + \varepsilon/16.
\]

It is correct since \( v^k = v^{k-1} - \varepsilon/16 \) and \( h^k = h^{k-1} \). In the case of non-productive steps:

\[
c^k = c^{k-1} - \varepsilon/64(1 + \langle P(i,a), s - P(i,a), i \rangle + 1)
\]

\[
= r(i,a) - (\bar{v}^{k-1} + \varepsilon/64) + \langle P(i,a), h^{k-1} - \varepsilon/64(e_s - e_i) \rangle - (h_i^{k-1} + \varepsilon/64).
\]

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Algorithm 2: Parallel average-reward MDP

**Input:** accuracy $\varepsilon > 0$, number of steps $N$;

**Algorithm HeadNode():**

1. for $k = 0, 1, 2, \ldots, N - 1$ do
2.   Send message «Check constraints» to all worker nodes;
3.   Receive $g_{\varepsilon/16}^{(i,a_i)}(\bar{v}^k, h^k) \forall (i, a_i) \in A$;
4.   if $\max_{a \in A} g_{\varepsilon/16}^{(i,a_i)}(\bar{v}^k, h^k) \leq \varepsilon/8$ then
5.     $\bar{v}^{k+1} = \bar{v}^k + \varepsilon/16$, $h^{k+1} = h^k$;
6.     Send message «Productive step» to all worker nodes;
7.     Add $k$ to $I$;
8. else
9.     $(i,a_i)_k = \arg\max_{a \in A} g_{\varepsilon/16}^{(i,a_i)}(\bar{v}^k, h^k)$;
10.    Send message «Sample» to worker node $(i,a_i)_k$;
11.   Receive state $s$;
12.   $\bar{v}^{k+1} = \bar{v}^k + \varepsilon/64$, $h^{k+1} = h^k - \varepsilon/64 \cdot (e_s - e_i)$;
13.   Send message «Non-productive step» and $s$ to all worker nodes;
14.   Add $k$ to $J$;
15. end
16. end
17. $\hat{\mu}_{(i,a_i)} = \frac{1}{4|I|} \sum_{k \in J} I\{(i,a_i)_k = (i,a_i)\}$;
18. return $\pi_{i,a_i} = \mu_{i,a_i} / (\sum_{a \in A} \mu_{i,a_i})$;

**Algorithm WorkerNode($i, a_i$):**

1. Compute $\bar{P}_{(i,a_i)}$;
2. $\varepsilon^0 = r_{i,a_i}$;
3. $k = 0$;
4. while is not finished do
5.   Wait message;
6.   if message is «Compute constraints» then
7.     Send $\varepsilon^k$ as $g_{\varepsilon/16}^{(i,a_i)}(\bar{v}^k, h^k)$;
8.   end
9. else if message is «Productive step» then
10.   $k = k + 1$;
11.   $\varepsilon^k = \varepsilon^{k-1} + \varepsilon/16$;
12. end
13. else if message is «Sample» then
14.   Sample $s \sim \bar{P}_{(i,a_i)}$ and send it;
15. end
16. else if message is «Non-productive step» then
17.   Receive $s$;
18.   $k = k + 1$;
19.   $\varepsilon^k = \varepsilon^{k-1} - \varepsilon/64(2 + \bar{P}_{(i,a_i),s} - \bar{P}_{(i,a_i),i})$;
20. end
21. end
22. end
Since \( v^k = v^{k-1} + \varepsilon/64 \) and \( h^k = h^{k-1} - \varepsilon/64(e_s - e_i) \), the computation of constraints is correct.

Then, by Proposition 2, we can produce \( \varepsilon \)-optimal policy with probability at least \( 1 - \sigma/2 \) by running Algorithm 1 on problem (14) with an accuracy \( \varepsilon' = \varepsilon/8 \) because \( 4(\varepsilon \Delta + \varepsilon g) = 4(\varepsilon' + 2\delta) = 4\varepsilon' + 8\delta = 8\varepsilon' \). Performing union bound for probability of failure for a preprocessing and an algorithm, we have required probability of success of whole scheme.

Since from the point of view of the head node Algorithm 2 is essentially Algorithm 1, we have needed guarantees on number of Mirror Descent iterations by computed constants \( M_f = M_g = O(1), \Theta_0 = O(t_{\max}^2 |S|), \kappa(E^*) = 1 \) and Theorem 2. Notice that the parallel depth of the algorithm is equal to 1.

Total running time complexity forms by additional \( A_{\text{tot}} \) computations on constraints on each iterations and each computation spends \( O(1) \) time by using previous values of computed constraints. The last observation connected to the fact that update of \( \bar{v}^k \) and \( \bar{h}^k \) also spends \( O(1) \) time by sparsity. □

4 Conclusion

In this work, we proposed a parallel algorithm for solving an average-reward MDP. As far as we know, it is the first parallel algorithm in the generative model setting. The interesting properties of the provided method are very low communication costs between the head node and all other nodes and the sparsity of updates that offer a possibility to work with a very large state space.

Another contribution is the development of Mirror Descent with constraints algorithms. We provide an algorithm that works with inexact computation of constraints and proves its primal-dual properties. The setting of inexact computation of constraints was developed in [10] but results on primal-dual convergence of such algorithms appeared in known literature only in a deterministic exact case [3].

Turning to possible extensions, there arise natural questions.

Could a preprocessing step be avoided and make the algorithm model-free? In the current version, we needed to do a required number of preprocessing iterations to guarantee the condition on constraints. It seems possible to use an unbiased stochastic oracle for constraint evaluation.

Another question is connected to the total work complexity. The cost of a high level of parallelism is a worse total running time in comparison to algorithms based on saddle-point formulations. Could the total work time be reduced without losing the possibility to run in parallel?

The last questions is about the choice of step-sizes. It seems possible to use an adaptive step-sizes scheme as in [2,15], and it might increase the practical value of the algorithm.

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