A LECTURE ABOUT THE USE OF ORLICZ SPACES IN INFORMATION GEOMETRY

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Abstract. Non-parametric Information Geometry and Statistical bundle and Orlicz space and Gaussian Orlicz-Sobolev spaces

1. Introduction

This chapter is a revision of the lecture and the related hand-out which I presented to the École de Physique des Houches on July 26-31 2020. Due to its strictly tutorial character, I shall not give detailed primary references, but I shall mention some in a final section.

I aim to review the basics of a peculiar setting for Information Geometry (IG) with the following peculiarities.

- It is on-parametric and infinite-dimensional.
- It provides an affine manifold modelled on a Banach space, namely the exponential Orlicz space.
- It focusses on a particular expression of the tangent bundle, called Statistical Bundle (ST).
- It allows for the use of (weakly) differentiable densities.

A previous tutorial paper presents the non-parametric construction in the case of a finite state space. Here, we will focus on the preliminaries to the infinite state space case.

There are many other successful presentations of IG which are indeed non-parametric. Some references are given in the concluding section. I think that any useful presentation should explain and include the following elements of its historical development.

1. The starting point to consider is the work of Ronald Fisher. A regular statistical model is a mapping from a set of real parameters $\Theta$ to probability densities on a given measured sample space $\mathcal{P}(X, \mathcal{X}, \mu)$, $\theta \mapsto p(\theta)$, such that the following computation is feasible. If $f$ is a given random variable, one wants to compute the variation of the expectation (assuming its existence), as

$$
\frac{\partial}{\partial \theta_i} \mathbb{E}_p(\theta) [f] = \frac{\partial}{\partial \theta_i} \mathbb{E}_\mu [fp(\theta)] = \frac{\partial}{\partial \theta_i} \langle f, p(\theta) \rangle_\mu = \left\langle f, \frac{\partial}{\partial \theta_i} p(\theta) \right\rangle_\mu = \left\langle f, \frac{\partial}{\partial \theta_i} \log p(\theta) \right\rangle_{p(\theta)} = \frac{\partial}{\partial \theta_i} \log p(\theta) \langle f, \frac{\partial}{\partial \theta_i} p(\theta) \rangle_{p(\theta)}.
$$

The random vector with components $\frac{\partial}{\partial \theta_i} \log p(\theta)$ is the Fisher score of the model at $\theta$. It has expected value with respect to $p(\theta)$ is 0 and its variance matrix with respect to $p(\theta)$ is the Fisher information matrix,

$$
I(\theta) = \mathbb{E}_p(\theta) \left[ \frac{\partial}{\partial \theta_i} \log p(\theta) \frac{\partial}{\partial \theta_j} \log p(\theta) \right]_{ij} = \int \frac{\partial}{\partial \theta_i} p(\theta) \frac{\partial}{\partial \theta_j} p(\theta) \frac{d\mu}{p(\theta)}.
$$

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The computations above show many peculiar features:

- One computes the variation of the expectation as a function of the statistical model. Moreover, a moving inner product \( \langle \cdot , \cdot \rangle_{p(\theta)} \) appears naturally.
- The score \( \partial_i \log p(\theta) \) represents the velocity of variation of the statistical model, while \( f - \mathbb{E}_{p(\theta)} [f] \) represents the gradient of the expectation function.
- The velocity at \( \theta \) lives in the space of random variables centred at \( p(\theta) \).
- The information matrix provides squared norms and scalar products of the velocities in the moving inner product.

2. One explanation of the Fisher computations results from the assumption of an exponential model,

\[
p(\theta) = e^{\sum_i \theta_i u_i - \kappa(\theta)} ,
\]

where \( u_i \) are the sufficient statistics and \( \kappa(\theta) \) is the cumulant of \( \sum_i \theta_i u_i \). In such a case, the score is the centered sufficient statistics,

\[
\partial_i \log p(\theta) = u_i - \partial_i \kappa(\theta) = u_i - \mathbb{E}_{p(\theta)} [u_i] ,
\]

and the information matrix equals the Hessian of \( \kappa \).

3. C. R. Rao has been the first statistician to remark that the Fisher information matrix is positive definite and smooth in an adequately defined regular model, hence it defines a Riemannian metric on the space of parameters. Moreover, he provided an embedding argument for the resulting manifold. It is likely to assume that this second part of his view finds its roots in a similar construction from Physics. The mapping

\[
\Theta \ni \theta \mapsto 2\sqrt{p(\theta)} = P(\theta)
\]

takes the parameters’ space to the \( L^2(\mu) \)-sphere of radius 2. The vectors

\[
\partial_i P(\theta) = \partial_i 2\sqrt{p(\theta)} = \frac{\partial p(\theta)}{\sqrt{p(\theta)}}
\]

are in the tangent space at \( P(\theta) \) of the sphere, and the inner product between tangent vector is

\[
\int \partial_i P(\theta) \partial_j P(\theta) \, d\mu = \int \frac{\partial_i p(\theta) \partial_j (x, \theta)}{p(x; \theta)} \, d\mu ,
\]

that is, the \( (i, j) \) element of the Fisher information matrix.

The Rao’s computations above reproduce all the metric structure of the Fisher computations but in one point. That is, now the velocity is not expressed by the logarithmic derivative \( \partial_i \log p(\theta) = \frac{\partial p(\theta)}{p(\theta)} \), but it is expressed by \( \partial_i 2\sqrt{p(\theta)} = \frac{\partial p(\theta)}{p(\theta)} \).

It is possible to solve, at least formally, the apparent contradiction by considering that there are here three different realizations of the same object: the tangent bundle of the set of densities \( TP \) whose tangent vectors \( \dot{p} \) satisfy \( \int \dot{p} \, d\mu = 0 \); the tangent bundle of the space \( TS_2 \); the Fisher’s statistical bundle \( SP \) consisting of all couples \( (p,u) \) such that \( p \in \mathcal{P} \) and \( \mathbb{E}_p [u] = 0 \). The Rao’s embedding \( p \mapsto 2\sqrt{p} \) provides the identification of \( TP \) with \( TS_2 \). The identification of \( TS_2 \) with \( SP \) is provided by

\[
TS_2 \ni (P, \dot{P}) \mapsto \left( \frac{1}{4} P^2, 2 \frac{\dot{P}}{P} \right) = (p,u) \in SP .
\]

In fact,

\[
\int p \, d\mu = \frac{1}{4} \int P^2 \, d\mu = 1 ,
\]

\[
\int u \, p \, d\mu = \frac{1}{2} \int \frac{P}{P} \, P^2 \, d\mu = \frac{1}{2} \int P \, \dot{P} \, d\mu = 0 ,
\]

\[
\int u^1 u^2 \, p \, d\mu = \int \frac{\dot{P}_1 \, P_2 - P_1 \, \dot{P}_2}{P^2} \, d\mu = \int \dot{P}_1 P_2 \, d\mu .
\]
A large part of the literature in IG uses the expressions $TP$ and $TS_2$. Still, my own choice is to use $SP$ because it fits well with the statistical picture and the exponential representation of strictly positive densities.

The choice of the exponential expression and Fisher’s score might seem arbitrary, but the following argument shows is not. Assume $t \mapsto \mu(t)$ is a one-dimensional model of probability densities and assume the mapping is smooth in the total variation topology. Fix a value $\tilde{t}$ of the parameter $t$. If a measurable set $A$ is a zero set for $\mu(\tilde{t})$, then $t \mapsto \mu(A,t)$ is minimum at $t = \tilde{t}$ then the derivative is zero at $\tilde{t}$, $\dot{\mu}(\tilde{t}) = 0$. It follows that the measure $\dot{\mu}(\tilde{t})$ is absolutely continuous with respect to $\mu(\tilde{t})$. The resulting density $d\mu(\tilde{t})/d\mu(\tilde{t})$ is the generalisation of the Fisher’s score.

4. The Riemannian approach by C. R. Rao can lead to a more in-depth study of the second-order properties of the manifold, namely the Levi-Civita connection and the curvature. Several authors, notably S-I. Amari, B. Efron, Ph. Dawid, and S. Lauritzen, have later observed that it is fruitful to study the geometry of statistical models from a more general point of view. In modern terminology, a statistical manifold consists of a metric and a couple of flat connections which are in duality for the given inner product. This set-up nicely solves the divide between Fisher’s approach and Rao’s approach by producing a unified theory. Moreover, the specific type of affine manifold that is relevant for IG is a Hessian manifold, that is, all its structure depends on a master convex functional. The vague statements above have a non-parametric justification I am going to present in the following sections.

1.0.1. 5. Both theoretical and applied research have recently shown interest in a particular type of non-parametric statistical models. Namely, models where the real space $\mathbb{R}^n$ is a model for the sample space, the reference measure is either the Lebesgue measure or the Gaussian measure, and the densities are required to have some level of smoothness.

The first example is the statistical estimation method based on Hyvärinen’s divergence,

\begin{equation}
DH(P|Q) = \frac{1}{2} \int |\nabla \log P(x) - \nabla \log Q(x)|^2 \; P(x) \; dx ,
\end{equation}

where $P, Q$ are positive probability densities of the $n$-dimensional Lebesgue space. It is assumed that the log-densities are smooth and the integral exists.

The second, related, example is Otto’s calculus, which uses a inner product defined by

\begin{equation}
\langle f, g \rangle_P = \int \nabla f(x) \cdot \nabla g(x) \; P(x) \; dx ,
\end{equation}

where $p$ is a probability density and $f, g$ are smooth random variables such that $E_P[f] = E_P[g] = 0$. Notice that the development of the square in the Hyvärinen divergence produces the term

\[ \langle \log P - E_P[\log P], \log Q - E_P[\log Q] \rangle_P . \]

In this note, we shall focus on the exponential representation of positive densities $p = e^{\gamma - K(u)}$. The reference measure is specialised to be $\mu(dx) = \gamma(x) dx$ where $\gamma$ is the standard Gaussian density of $\mathbb{R}^n$. The sufficient statistics $u$ is assumed to belong to an exponential Orlicz space, to be defined next, and such that $\int u(x) \gamma(x) dx = 0$. The normalizing constant $u \mapsto K(u)$ is a convex functional of the exponential Orlicz space whose interior of proper domain of $K$ has the role of parameter set of a maximal exponential model of the Gaussian space. Here, maximal means that the model contains all possible finite dimensional exponential families. We will aim to sketch a theory in which all concerns of items 1 to 5 meet a solution of sort.

2. ORLICZ SPACES

First, we review below part of the theory of Orlicz spaces and fix a convenient notation. Notice that we will not aim to full generality.
If \( \phi \in C[0, +\infty[ \) satisfies: 1) \( \phi(0) = 0 \); 2) \( \phi \) is strictly increasing; 3) \( \lim_{u \to +\infty} \phi(u) = +\infty \), its primitive function
\[
\Phi(x) = \int_0^x \phi(u) \, du , \quad x \geq 0 ,
\]
is strictly convex. \( \Phi \) is extended to \( \mathbb{R} \) by symmetry and such an extension is called a \textit{Young function}.

The inverse function \( \psi = \phi^{-1} \) has the same properties 1) to 3) as \( \phi \), so that its primitive
\[
\Psi(y) = \int_0^y \psi(v) \, dv , \quad y \geq 0 ,
\]
is again a Young function. The couple \((\Phi, \Psi)\), is a couple of \textit{conjugate} Young functions. The relation is symmetric and we write both \( \Psi = \Phi_* \), and \( \Phi = \Psi_* \). The Young inequality holds true,
\[
\Phi(x) + \Psi(y) \geq xy , \quad x, y \geq 0 ,
\]
and the Legendre equality holds true,
\[
\Phi(x) + \Psi(\phi(x)) = x\phi(x) , \quad x \geq 0 .
\]

Here are the specific cases we are going to use. \( \text{name}_2 \) means 2nd Taylor remainder.

(3) \[
\Phi_\alpha(x) = \frac{x^\alpha}{\alpha} , \quad \Psi_\beta(y) = \frac{y^\beta}{\beta} , \quad \alpha, \beta > 1 , \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1 ;
\]
(4) \[
\exp_2(x) = e^x - 1 - x , \quad (\exp_2)_*(y) = (1 + y) \log(1 + y) - y ;
\]
(5) \[
\cosh_2(x) = \cosh x - 1 , \quad (\cosh_2)_*(y) = \int_0^y \sinh^{-1}(v) \, dv ;
\]
(6) \[
\text{gauss}_2(x) = \exp \left( \frac{1}{2} x^2 \right) - 1 .
\]

Given a Young function \( \Phi \) and a probability measure \( \mu \), the \textit{Orlicz space} \( L_\Phi(\mu) \) is the Banach space whose closed unit ball is \( \{ f \in L^0(\mu) \mid \int \Phi(|f|) \, d\mu \leq 1 \} \). This defines the \textit{Luxembury norm}, characterized by
\[
\|f\|_{L_\Phi(\mu)} \leq \rho \quad \text{if, and only if,} \quad \int \Phi(\rho^{-1} |f|) \, d\mu \leq 1 .
\]

Because of the Young inequality, it holds
\[
\int |uv| \, d\mu \leq \int \Phi(|u|) \, d\mu + \int \Phi_*(|v|) \, d\mu .
\]
This provides a separating duality \( \langle u, v \rangle_{\mu} = \int uv \, d\mu \) of \( L_\Phi(\mu) \) and \( L_{\Phi_*}(\mu) \) such that
\[
\langle u, v \rangle_{\mu} \leq 2 \|u\|_{L_\Phi(\mu)} \|v\|_{L_{\Phi_*}(\mu)} .
\]
From the conjugation between \( \Phi \) and \( \Psi \), an equivalent norm can be defined, namely, the \textit{Orlicz norm}
\[
\|f\|_{L_\Phi(\mu)^*} = \sup \left\{ \langle f, g \rangle_{\mu} \mid \|f\|_{L_{\Phi}(\mu)} \leq 1 \right\} .
\]

The domination relation between Young functions imply continuous injection properties for the corresponding Orlicz spaces. We say that \( \Phi_2 \) \textit{eventually dominates} \( \Phi_1 \), written \( \Phi_1 < \Phi_2 \), if there is a constant \( k \) such that \( \Phi_1(x) \leq \Phi_2(kx) \) for all \( x \) larger than some \( \bar{x} \). As, in our case, \( \mu \) is a probability measure, the continuous embedding \( L_{\Phi_2}(\mu) \to L_{\Phi_1}(\mu) \) holds if, and only if, \( \Phi_1 < \Phi_2 \). If \( \Phi_1 < \Phi_2 \), then \( \Phi_2^* < (\Phi_1)_* \).

A special case occurs when there exists a function \( C \) such that \( \Phi(ax) \leq C(a)\Phi(x) \) for all \( a \geq 0 \). This is true, for example, for a power function and in the case of the functions \( \exp_2 \) and \( (\cosh-1) \). In such a case, the dual couple is a couple of reflexive Banach spaces and bounded functions are a dense set. We will return to this important topic below.

The spaces corresponding to power case \( \text{name}_3 \) coincides with the ordinary Lebesgue spaces. The norm are related by
\[
\|f\|_{L_{\Phi_\alpha}(\mu)} = \alpha^{1/\alpha} \|f\|_{L^\alpha(\mu)} .
\]
With reference to our examples \( \text{name}_4 \) and \( \text{name}_5 \), we see that \( \exp_2 \) and \( (\cosh-1) \) are equivalent. They both are eventually dominated by \( \text{gauss}_2 \) \( \text{name}_6 \) and eventually dominate all powers \( \text{name}_7 \). The cases...
and \( \Phi \) provide isomorphic B-spaces \( L_{\text{cosh}^{-1}}(\mu) \leftrightarrow L_{\text{exp}}^{\ast}(\mu) \) which are of special interest for us as they provide the model spaces for our non-parametric version of IG, see section 4 below. Clearly, a function belongs to the space \( L_{\text{cosh}^{2}}(\mu) \) if, and only if, its moment generating function \( \lambda \mapsto \int e^{\lambda f} \, d\mu \) is finite in a neighborhood of 0. In turn, this implies that the moment generating function is analytic at 0. This property is also expressed in terms of a large deviation inequality. A function \( f \) belongs to \( L_{\text{cosh}^{-1}}(\mu) \) if, and only if, it is sub-exponential, that is, there exist constants \( C_{1}, C_{2} > 0 \) such that

\[
\mu(|f| \geq t) \leq C_{1} \exp(-C_{2}t) , \quad t \geq 0.
\]

Let us check the equivalence above. If \( \|f\|_{L_{\text{cosh}^{2}}(\mu)} = \rho \), then \( \int e^{\rho |f|} \, d\mu \leq 4 \). It follows that

\[
\mu(|f| > t) = \frac{1}{\rho} \int e^{\rho |f|} > e^{\rho t} \leq \left( \int e^{\rho |f|} \, d\mu \right) e^{-\rho t} \leq 4 e^{-\rho t}.
\]

The sub-exponential inequality holds with \( C_{1} = 4 \) and \( C_{2} = \|f\|_{L_{\text{cosh}^{2}}(\mu)}^{-1} \). Conversely, for all \( \lambda > 0 \),

\[
\int e^{\lambda f} \, d\mu = \int_{1}^{\infty} \mu \left( e^{\beta f} > t \right) \lambda dt \leq C_{1} \int_{0}^{\infty} e^{-(C_{2} \lambda^{-1} s)} \, ds.
\]

The right-hand side is finite if \( \lambda < C_{2} \) and the same inequality holds for \(-f\). Sub-exponential random variable are of special interest in applications because they admit an explicit exponential bounds in the Law of Large Numbers. Another class of interest consists of the sub-Gaussian random variables, that is, those random variables whose square is sub-exponential.

The theory of sub-exponential random variables provides an equivalent norm for the space \( L_{\text{cosh}^{2}}(\mu) \), namely the norm

\[
f \mapsto \sup_{k} \left( \frac{(2k)!^{-1}}{(2k)!} \int f^{2k} \, d\mu \right)^{1/2k} = \|f\|_{L_{\text{cosh}^{2}}}.\]

Let us prove the equivalence. If \( \|f\|_{L_{\text{cosh}^{2}}(\mu)} \leq 1 \), then

\[
1 \geq \int \cosh f \, d\mu \geq \frac{1}{(2k)!} \int f^{2k} \, d\mu \quad \text{for all } k = 1, 2, \ldots ,
\]

so that \( 1 \geq \|f\|_{L_{\text{cosh}^{2}}} \). Conversely, if the latter inequality holds, then

\[
\int \cosh^{2}(f/\sqrt{2}) \, d\mu = \sum_{k=1}^{\infty} \frac{1}{(2k)!} \int f^{2k} \, d\mu \left( \frac{1}{2} \right)^{k} \leq 1 ,
\]

so that \( \|f\|_{L_{\text{cosh}^{2}}(\mu)} \leq \sqrt{2} \).

We will be led to use a further notation. For each Young function \( \Phi \), the function \( \overline{\Phi}(x) = \Phi(x^{2}) \) is again a Young function such that \( \|f\|_{L_{\Phi}(\mu)} \leq \lambda \) if, and only if, \( \|f\|_{L_{\Phi}(\mu)}^{2} \leq \lambda^{2} \). We denote the resulting space by \( L_{\Phi}^{2}(\mu) \). For example, gauss2 and cosh^{-1} are \( \sim \)-equivalent, hence the isomorphism \( L_{\text{gauss}_{2}}(\mu) \leftrightarrow L_{\text{cosh}^{-1}}^{2}(\mu) \). As an application of this notation, consider that for each increasing convex \( \Phi \) it holds \( \Phi(fg) \leq \Phi((f^{2} + g^{2})/2) \leq (\Phi(f^{2}) + \Phi(g^{2}))/2 \). It follows that when the \( L_{\Phi}^{2}(\mu) \)-norm of \( f \) and of \( g \) is bounded by one, the \( L_{\Phi}(\mu) \)-norm of \( f, g, \) and \( fg \), are all bounded by one. The space \( L_{\text{cosh}^{2}}(\mu) \) has a continuous injection in the Fréchet space \( L_{\text{inf}}(\mu) = \cap_{\alpha \geq 1} L_{\alpha}(\mu) \), which is an algebra. When we need the product, we can either assume the factor are both sub-Gaussian, or, move up the functional framework to the intersection of the Lebesgue spaces.

Let us now discuss specific issues of the Gaussian exponential Orlicz spaces \( L_{\text{cosh}^{2}}(\gamma) \), \( \gamma \) the standard \( n \)-variate Gaussian density. Dominated convergence does not hold in this space. The squared-norm function \( f(x) = |x|^{2} \) belongs to the Gaussian exponential Orlicz space \( L_{\text{cosh}^{2}}(\gamma) \) because

\[
\int \cosh^{2}(\lambda f(x)) \, \gamma(x) dx < \infty \quad \text{for all } \lambda < 1/2 .
\]
However, the sequence \( f_N(x) = f(x)(|x| \leq N) \) converges to \( f \) point-wise and in all \( L^\alpha(\gamma) \), \( 1 \leq \alpha < \infty \). However, the convergence does not hold in the Gaussian exponential Orlicz space. In fact, for all \( \lambda \geq 1/2 \),

\[
\int \cosh_2(\lambda(f(x) - f_N(x))) \gamma(x) \, dx = \int_{|x| > N} \cosh_2(\lambda f(x)) \gamma(x) \, dx = \infty ,
\]

but convergence would imply

\[
\lim_{N \to \infty} \sup \int \cosh_2(\lambda(f(x) - f_N(x))) \gamma(x) \, dx \leq 1 \quad \text{for all } \lambda > 0 .
\]

The closure in \( L_{\cosh_2}(\gamma) \) of the vector space of bounded functions is called \textit{Orlicz class} and it is denoted by \( M_{\cosh_2}(\gamma) \). One can prove that \( f \in M_{\cosh_2}(\gamma) \) if, and only if, the moment generating function \( \lambda \mapsto \int e^{\lambda x} \gamma(x) \, dx \) is finite for all \( \lambda \). An example is \( f(x) = x \). Bounded convergence holds in the Orlicz class. Assume \( f \in M_{\cosh_2}(\gamma) \) and consider the sequence \( f_N(x) = (|x| \leq N) f(x) \). Now,

\[
\int \cosh_2(\lambda(f(x) - f_N(x))) \gamma(x) \, dx = \int_{|x| \geq N} \cosh_2(\lambda f(x)) \gamma(x) \, dx \to 0 \quad \text{as } N \to \infty .
\]

### 3. Calculus of the Gaussian Space

We will review here a few simple facts about the analysis of the Gaussian space, the so-called Malliavin’s calculus, see [12, Ch. V].

Let us denote by \( C^k_{\text{poly}}(\mathbb{R}^n) \), \( k = 0, 1, \ldots \), the vector space of functions which are differentiable up to order \( k \) and which are bounded, together with all derivatives, by a polynomial. This class of functions is dense in \( L^2(\gamma) \). For each couple \( f, g \in C^1_{\text{poly}}(\mathbb{R}^n) \), we have

\[
\int f(x) \, \delta_ig(x) \, \gamma(x) \, dx = \int \delta_i f(x) \, g(x) \, \gamma(x) \, dx ,
\]

where the divergence operator \( \delta_i \) is defined by \( \delta_i f(x) = x_i f(x) - \partial_i f(x) \). Multidimensional notations will be used, for example,

\[
\int \nabla f(x) \cdot \nabla g(x) \, \gamma(x) \, dx = \int f(x) \, \delta \cdot \nabla g(x) \, \gamma(x) \, dx , \quad f, g \in C^2_{\text{poly}}(\mathbb{R}^n) ,
\]

with \( \delta \cdot \nabla g(x) = x \cdot \nabla g(x) - \Delta g(x) \).

For example, in this notation, the Hyvärinen divergence eq. (11) with \( P = p \cdot \gamma, Q = q \cdot \gamma \), and \( p, q \in C^2_{\text{poly}}(\mathbb{R}^n) \), becomes

\[
\frac{1}{2} \int |\nabla \log p(x) - \nabla \log q(x)|^2 p(x) \gamma(x) \, dx ,
\]

while the Otto’s inner product eq. (2) becomes, with \( P = p \cdot \gamma \) and \( f, g, p \in C^2_{\text{poly}}(\mathbb{R}^n) \), gives

\[
\int \nabla f(x) \cdot \nabla g(x) \, p(x) \, \gamma(x) \, dx = \int f(x) \delta \cdot \nabla g(x)(p(x)) \, \gamma(x) \, dx .
\]

Hermite polynomials \( H_\alpha = \delta^\alpha 1 \) provide an orthogonal basis for \( L^2(\gamma) \) such that \( \partial_i H_\alpha = \alpha_i H_{\alpha-\varepsilon} \). In turn, this provides a way to prove that there is a closure of both operator \( \partial_i \) and \( \delta_i \) on a domain which is an Hilbert subspace of \( L^2(\gamma) \). Moreover, the closure of \( \partial_i \) is the infinitesimal generator of the translation operator.
4. Exponential statistical bundle

In this section, we very briefly review and slightly generalise the known construction of the statistical manifold as a Banach manifold modeled on the exponential Orlicz space \( L_{\cosh 2}(\gamma) \). See [18, 20] for all details that are missing here.

The support of the manifold is the maximal exponential model \( \mathcal{E}(\gamma) \) of probability densities on \((\mathbb{R}^p, \gamma)\) of the form

\[
q = \exp\left( u - K_1(u) \right) \quad u \in L_{\cosh 2}(\gamma) \quad \int u(x) \gamma(x) dx = 0,
\]

where \( K_1(u) = \log \int e^{u(x)\gamma(x)} dx \) is the unique normalising constant (or partition function). More precisely, the mapping \( K_1: L_{\cosh 2}(\gamma) \) is convex and the topological interior of the proper domain contains the open unit ball and we take all \( u \) in such domain. The mapping \( s_1: q \mapsto u \) provides a global chart to the manifold. The tangent space of the manifold is expressed by the statistical bundle \( \mathcal{S} \) consisting of all the couples \((q, v)\) such that \( q \) is a density of the maximal exponential model and \( v \) is a \( q \)-centered random variable in the exponential Orlicz space.

The following statement is crucial to prove the consistency of the existence of the Banach manifold structure in infinite dimension as it shows that the fibers of the statistical bundle are isomorphic as Banach spaces.

For all \( p, q \in \mathcal{E}(\gamma) \) it holds \( q = e^{v - K_p(u)} \cdot p \), where \( u \in L_{(\cosh - 1)}(\gamma) \), \( \mathcal{E}_p[u] = 0 \), and \( u \) belongs to the interior of the proper domain of the convex function \( K_p \). This property is equivalent to any of the following:

1. \( p \) and \( q \) are connected by an open exponential arc;
2. \( L_{(\cosh - 1)}(p) = L_{(\cosh - 1)}(q) \) and the norms are equivalent;
3. \( p/q \in \cup_{\alpha > 1} L^\alpha(q) \) and \( q/p \in \cup_{\alpha > 1} L^\alpha(p) \).

We give here the essential part of the proof. Let \( F \) be logarithmically convex on \( \mathbb{R} \) and such that \( \Phi = F - 1 \) is a Young function. For example, such an assumption holds for both \( F(x) = \cosh x \) and \( F(x) = e^{x^2/2} \). For all real \( A \) and \( B \), the function

\[
\mathbb{R}^2 \ni (\lambda, t) \mapsto F(\lambda A)e^{tB} = \exp\left( \log F(\lambda A) + tB \right)
\]

is convex and so it is

\[
C(\lambda, t) = \int F(\lambda f(x))e^{tu(x)\gamma(x)} dx,
\]

where \( f \in L_\Phi(\gamma) \) with and \( u \in L_{\cosh 2}(\gamma) \) with \( \int u(x) \gamma(x) dx = 0 \). Without restriction of generality, we assume \( \|f\|_{L_\Phi(\gamma)} = 1 \). Let us derive two marginal inequalities. First, for \( t = 0 \), the definition of Luxemburg norm gives

\[
C(\lambda, 0) = \int F(\lambda f) \gamma(x) dx \leq 2, \quad -1 \leq \lambda \leq 1.
\]

Second, for \( \lambda = 0 \), consider \( K_1(tu) = \log \int e^{tu(x)\gamma(x)} dx \), where \( t \) belongs to an an open interval \( I \) containing \([0, 1]\) and such that \( K_1(tu) < +\infty \). It follows that

\[
C(0, t) = \int e^{tu(x)} \gamma(x) dx = e^{K_1(tu)} < +\infty.
\]

Choose a \( t > 1 \) in \( I \) and consider the convex combination

\[
\left( \frac{t - 1}{t}, 1 \right) = \frac{t - 1}{t}(1, 0) + \frac{1}{t}(0, t)
\]

and the inequality

\[
C\left( \frac{t - 1}{t}, 1 \right) \leq \frac{t - 1}{t} C(1, 0) + \frac{1}{t} C(0, t) \leq 2\frac{t - 1}{t} + \frac{1}{t} e^{K_1(tu)}.
\]
Now
\[
\int \Phi \left( \frac{t-1}{t} f(x) \right) e^{u(x) - K_1(u)} \gamma(x) dx = \int F\left( \frac{t-1}{t} f(x) \right) e^{u(x) - K_1(u)} \gamma(x) dx - 1 = \\
e^{-K_1(u)} \left( \frac{t-1}{t}, 1 \right) - 1 \leq e^{-K_1(u)} \left( 2 \frac{t-1}{t} + \frac{1}{t} e^{K_1(u)} \right) - 1
\]

It follows that \( f \in L_\Phi(p) \). Conversely, a similar argument shows the other implication. We have proved that all Orlicz spaces \( L_\Phi(p), p \in \mathcal{E}(\gamma) \) are equal. This, in turn, implies the the norms are equivalent and it is actually possible to derive explicit bounds.

5. Gaussian Orlicz-Sobolev spaces

Let us define the functional spaces that will be the coordinate spaces of our non-parametric statistical bundle. We consider now sub-Gaussian random variables of the Gaussian space such that all the weak partial derivatives are sub-Gaussian. It is indeed a special instance of a general scheme. General references about the functional background are [1, 3, 12].

We define
\[
W^{1} L_{\text{cosh}^2}^2(\gamma) = \{ f \in L_{\text{cosh}^2}^1(\gamma) \mid \partial_i f \in L_{\text{cosh}^2}^2(\gamma), i = 1, \ldots, n \},
\]
where the weak partial derivative \( \partial_i f \) exists if
\[
\langle \partial_i f, \phi \rangle_\gamma = \langle f, \delta_i \phi \rangle_\gamma \quad \text{for all} \quad \phi \in C_0^1(\mathbb{R}^n).
\]

The above definition of weak derivative coincides this the usual definition of derivative in the sense of Schwartz distributions because \( \phi \leftrightarrow \phi \cdot \gamma \) is a bijection of \( C_0^\infty(\mathbb{R}^n) \) and
\[
\langle f, \delta_i \phi \rangle_\gamma = - \int f(x) \frac{\partial}{\partial x_i} (\phi(x) \gamma(x)) \; dx.
\]

We recall the relation between translation and weak derivative, that is, the weak version of the Fundamental Theorem of Calculus. Let be given a locally integrable real mapping \( G \in L_\text{loc}^1(\mathbb{R}) \), and assume there exists a locally integrable function \( G' \) which is the weak derivative of \( G \), that is,
\[
\int G(x) \phi'(x) \; dx = - \int G'(x) \phi(x) \; dx, \quad \phi \in C_0^\infty(\mathbb{R}).
\]

Define the translation \( \tau_h G, h \in \mathbb{R} \), by \( t_h G(x) = G(x - h), h \in \mathbb{R} \). It follows immediately \( \tau_h G \in L_\text{loc}^1(\mathbb{R}) \) and
\[
\int (\tau_h G(x) - G(x)) \phi(x) \; dx = \int G(x + h) \phi(x) \; dx - \int G(x) \phi(x) \; dx
\]
\[
= \int G(x) (\phi(x - h) - \phi(x)) \; dx
\]
\[
= \int G(x) \phi(x - sh) \big|_{s=1}^{s=0} \; dx
\]
\[
= -h \int G(x) \int_0^1 \phi'(x - sh) \; ds \; dx
\]
\[
= -h \int_0^1 \int G(x) \phi'(x - sh) \; dx \; ds
\]
\[
= h \int_0^1 \int G'(x) \phi'(x - sh) \; dx \; ds
\]
\[
= h \int_0^1 \int G'(x + sh) \phi'(x) \; dx \; ds
\]
\[
= \int \left( h \int_0^1 G'(x + sh) \; ds \right) \phi(x) \; dx
\]
As $\phi$ is any function in $C^\infty_0(\mathbb{R})$, we have proved that

\begin{equation}
\tau_h G - G = h \int_0^1 \tau_{-sh} G' \, ds = hG' + h \int_0^1 (\tau_{-sh} G' - G') \, ds
\end{equation}

in $L^1_{\text{loc}}(\mathbb{R})$. In particular, if $G'$ is bounded by a constant $K$, then $G$ is almost surely $K$-Lipschitz, $|G(x - h) - G(x)| \leq K|h|$.

Conversely, if eq. (9) holds, then eq. (8) holds, see, for example, [3, Lemma 8.1-2]. The argument above extends to $n$-variate functions $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$, that is, $f, \partial_i f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $i = 1, \ldots, n$, by considering, for each $h \in \mathbb{R}^n$, the univariate function $t \mapsto \tau_{th} f$ defined by $\tau_{th} f(x) = f(x - th)$. We obtain

\begin{equation}
\tau_{th} f - f = t \nabla f \cdot h + t \int_0^1 (\tau_{-sth} \nabla f - \nabla f) \cdot h \, ds ,
\end{equation}

where the equality holds in $L^1_{\text{loc}}(\mathbb{R}^n)$. The same equality holds in all function space whose elements are locally integrable.

The result about the functional differentiability of translations is the following. For all $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ the following first increment equation holds,

\begin{equation}
(\tau_{th} f - f) - t \nabla f \cdot h = t \int_0^1 (\tau_{-sth} \nabla f - \nabla f) \cdot h \, ds ,
\end{equation}

and differentiability holds in $L^{\infty,0}(\mathbb{R}) = \cap_{a>1} L^a(\mathbb{R})$. In fact, the translations are continuous in all $L^a(\mathbb{R})$.

It is possible to extend the previous result to a property of the derivative of the composite function $G \circ f$. The increment of the composition expands as

\begin{equation}
G(f(x + th)) - G(f(x)) = G(f(x) + (f(x + th) - f(x))) = (f(x + th) - f(x))G'(f(x)) + (f(x + th) - f(x)) \int_0^1 (G'(f(x)) + s(f(x + th) - f(x))) - G'(f(x))) \, ds ,
\end{equation}

and the weak derivative of the composite function exists if $G'$ is bounded and $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$. However, differentiability holds in $L^{\infty,0}(\mathbb{R})$.

An interesting example of application is the Neuron of Deep Learning. If $f_1, \ldots, f_k \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$, $G$ is an activation function, for example, $G(x) = x^+$, and $a_i, w_{ij}, b_i$ are given constants, then

\begin{equation}
\sum_{i=1}^h a_i G\left( \sum_{j=1}^k w_{ij} f_j - b_i \right) \in W^{1,1}_{\text{loc}}(\mathbb{R}^n) .
\end{equation}

An interesting feature of the space defined in eq. (7) is the fact that each element has a continuous version. The embedding

\begin{equation}
L^1_{\text{cos}}(\mathbb{R}) , L^2_{\text{cos}}(\mathbb{R}) \subset \cap_{a>1} L^a(\mathbb{R})
\end{equation}

will allow to use the standard Sobolev inequalities to our case. $W^{1,a}_{\text{loc}}(\mathbb{R}^n)$ denotes the space of functions whose restriction to each open ball $B_\rho = \{ x \in \mathbb{R}^n : |x| < \rho \}$ is $a$-integrable, together with all weak partial derivatives. $C^\lambda(\overline{B}_\rho)$ denotes $\lambda$-Hölder functions on the closed ball.

1. The following restriction and imbedding hold true and are continuous,

\[ W^{1,1}_{\text{loc}}(\mathbb{R}^n) \to W^{1,a}(\overline{B}_\rho) \subset C^\lambda(\overline{B}_\rho) , \quad \rho > 0 , \quad 0 < \lambda < 1 . \]

2. The following inclusions hold true and are continuous:

\[ W^{1,1}_{\text{loc}}(\mathbb{R}^n) \subset \cap_{a>1} W^{1,a} \cap L^1_{\text{cos}} \subset L^1_{\text{cos}} , \]

where the space of continuous functions $C(\mathbb{R})$ is endowed with the uniform convergence on compact sets.
The embedding are easily verified. If \( f \in L^1_{\cosh^{-1}}(\gamma) \), then, for all \( k \in \mathbb{N} \), the inequalities \( x^{2k}/(2k)! \leq \cosh^2(x) \) and \( (2\pi)^{-n/2}e^{-\rho^2/2} \leq \gamma(x) \) for \( x \in B_{\rho} \) imply the inequality

\[
\frac{(2\pi)^{-n/2}e^{-\rho^2/2}}{(2k)!} \int_{B_{\rho}} \left( \frac{f(x)}{\|f\|_{L^1_{\cosh^2}(\gamma)}} \right)^{2k} dx \leq \int \cosh^2 \left( \frac{f(x)}{\|f\|_{L^1_{\cosh^2}(\gamma)}} \right) \gamma(x) \, dx \leq 1 ,
\]

so that

\[
\|f\|_{L^{2k}(B_{\rho})} \leq (2\pi)^{n/2}(2k)!e^{\rho^2/2} \|f\|_{L^1_{\cosh^2}(\gamma)} .
\]

A similar argument applies to the weak partial derivatives. Now we can use the Sobolev embedding theorem, see [1, Th. 4.12].

Let us conclude by explicitly reviewing the main properties of our space.

1. The space \( W^1L^{1,2}_{\cosh^2}(\gamma) \) contains the constants and all polynomial up to order 2.
2. Each element has a continuous version.
3. If \( G : \mathbb{R} \to \mathbb{R} \) is the primitive of a bounded function, then \( G \circ f \in W^1L^{1,2}_{\cosh^2}(\gamma) \).
4. \( \min(f,g), \max(f,g) \in W^1L^{1,2}_{\cosh^2}(\gamma) \).

Moreover, our space is a Banach space: The mapping

\[
W^1L^{1,2}_{\Phi}(\gamma) \ni f \mapsto \|f\|_{L^1_{\Phi}(\gamma)} + \sum_{i=1}^n \|\partial_i f\|_{L^2_{\Phi}(\gamma)}
\]

is a complete norm and thus defines a Banach space. The argument is a standard one in Functional Analysis. The weak gradient \( \nabla \) is a closed operator from \( L^1_{\Phi}(\gamma) \to (L^2_{\Phi}(\gamma))^n \), that is, the graph of \( \nabla \) is closed in \( L^1_{\Phi}(\gamma) \times (L^2_{\Phi}(\gamma))^n \). In fact, given a converging sequence in the graph, say \( f_n \to f \) and \( \partial_i f \to f_i \), it holds

\[
\langle \partial_i f, \phi \rangle \gamma = \langle f, \delta_i \phi \rangle \gamma = \lim_n \langle f_n, \delta_i \phi \rangle \gamma = \lim_n \langle \partial_i f, \phi \rangle \gamma = \langle f^i, \phi \rangle \gamma .
\]

The identification of the space \( W^1L^{1,2}_{\Phi}(\gamma) \) with the graph of \( \nabla \) provides a complete norm.

Finally, we remark that all exponential Orlicz space are isomorphic to a maximal exponential model. This, in turn implies the isomorphism

\[
W^1L^{1,2}_{\cosh^2}(\gamma) \leftrightarrow W^1L^{1,2}_{\cosh^2}(\cdot, \gamma) \quad \forall \gamma \in \mathcal{E}(\gamma) .
\]

It follows that this space is suitable as a model of the fibers of a statistical bundle.

6. Selected Bibliography

In a series of papers [17, 10, 19, 20] we have explored a version of the non-parametric Information Geometry (IG) for smooth densities on \( \mathbb{R}^n \). Especially, we have considered the IG associated to Orlicz spaces on the Gaussian space. The analysis of the Gaussian space is discussed, for example, in [13, 15]. This set-up provides a simple way to construct a statistical manifold modelled on Banach spaces of smooth densities. Other modelling options are in fact available, for example the global analysis methods of [8], but we prefer to work with assumptions that allow for the use of classical infinite dimensional differential geometry modelled on B-spaces as in [9].

see in [7, 10]. The second one is the Otto’s inner product [16, 11].

We want a restricted type of Young functions as defined below. Cf. the more general cases in [14, Ch. II] and [1, Ch. VII]. The same references provide the proofs that are not reproduced here.

There is a slight difference in notation from [22]. Here, See a proof in [1, Th. 8.2].
There is a large literature on this subject, see, for example, [1] [22] [23] [24].

The need to control the product of two random variables in $L_{(\cosh^{-1})(\mu)}$ appears, for example, in the study of the covariant derivatives of the statistical bundle, see [6] [11] [21] [4].

See the full theory in [13] [2] and the applications to IG in [10] [20].

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