RATIONALITY OF THE QUOTIENT OF $\mathbb{P}^2$ BY FINITE GROUP OF AUTOMORPHISMS OVER ARBITRARY FIELD OF CHARACTERISTIC ZERO

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Abstract. Let $k$ be a field, chark = 0 and $G$ be a finite group of automorphisms of $\mathbb{P}^2$. Castelnuovo’s Theorem implies that the quotient variety $\mathbb{P}^2_k/G$ is rational if the field $k$ is algebraically closed. In this paper we prove that the quotient $\mathbb{P}^2_k/G$ is rational for an arbitrary field $k$ of characteristic zero.

1. Introduction

Let $G$ be a finite group and $k$ be a field. Consider a pure transcendental extension $K/k$ of transcendental degree $n = \text{ord}G$. We may assume that $K = k\{(x_g)\}$, where $g$ runs through all the elements of group $G$. The group $G$ naturally acts on $K$ as $h(x_g) = x_{hg}$. Noether’s problem asks whether the field of invariants $K^G$ is rational (i.e. pure transcendental) over $k$ or not. On the language of algebraic geometry, this is a question about the rationality of the quotient variety $\mathbb{A}^n/G$.

The most complete answer to this question is known for abelian groups, but even in this case quotient variety can be non-rational (see [Swa69], [Vos-foi] [EM73], [Len74]).

Noether’s problem can be generalized as follows. Let $G$ be a finite group, let $V$ be finite-dimensional vector space over an arbitrary field $k$ and let $\rho : G \to GL(V)$ be a representation. The question is if the quotient variety $V/G$ is k-rational?

Note that $V/G$ has a natural birational structure of a $\mathbb{P}^1$-fibration over $\mathbb{P}(V)/G$, which is locally trivial in Zarisky topology. So rationality of $V/G$ follows from rationality of $\mathbb{P}(V)/G$.

In this generalization it is natural to start with a low-dimensional case.

The most general result is known for dimension 1 and 2.
Theorem 1.1 (Lüroth). Let $k$ be an arbitrary field and let $G \subset PGL_2(k)$ be a finite subgroup. Then $\mathbb{P}^1_k/G$ is $k$-rational.

The next theorem is a consequence of Castelnuovo’s Theorem [Cast].

Theorem 1.2. Let $k$ be an algebraically closed field of characteristic zero and let $G \subset PGL_3(k)$ be a finite subgroup. Then $\mathbb{P}^2_k/G$ is $k$-rational.

The main result of this paper is the following.

Theorem 1.3. Let $k$ be an arbitrary field of characteristic zero and let $G \subset PGL_3(k)$ be a finite subgroup. Then $\mathbb{P}^2_k/G$ is $k$-rational.

Corollary 1.4. Let $k$ be an arbitrary field of characteristic zero and let $G \subset GL_3(k)$ be a finite subgroup. The field of invariants $k(x_1, x_2, x_3)^G$ is $k$-rational.

To prove this statement we consider algebraical closure of the field $k$. We have two groups acting on $\mathbb{P}^2_k$: the geometrical group $G$ and the Galois group $\Gamma = \text{Gal}(\overline{k}/k)$. Then we consider the quotient variety $\mathbb{P}^2_k/N$ where $N$ is a normal subgroup of $G$ (if such a subgroup $N$ exists). Next, we resolve the singularities of $\mathbb{P}^2_k/N$, run the $G/N \times \Gamma$-equivariant minimal model program and get a surface $X$. Then we repeat the above procedure applying this method to the surface $X$ and the group $G/N$.

In the section 2 we describe notions and results of minimal model program which are used in this work. In the section 3 we sketch the classification of finite subgroups in $PGL_3(\overline{k})$ where $\overline{k}$ is an algebraically closed field of characteristic zero. In the section 4 we prove Theorem 1.3.

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We use the following notation.

- $k$ denotes an arbitrary field of characteristic zero.
- $\overline{k}$ denotes the algebraic closure of a field $k$.
- $C_n$ denotes the cyclic group of order $n$.
- $D_{2n}$ denotes the dihedral group of order $2n$.
- $S_n$ denotes the symmetric group of degree $n$.
- $A_n$ denotes the alternating group of degree $n$.
- $\omega = e^{2\pi i/3}$.
- $I_n$ denotes the identity matrix of dimension $n$.
- $\text{diag}(\alpha, \beta, \gamma) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$.
$K_X$ denotes the canonical divisor of a variety $X$.
Pic$(X)$ (resp. Pic$(X)^G$) denotes the (invariant) Picard group of a variety $X$.
$\rho(X)$ (resp. $\rho(X)^G$) denotes the (invariant) Picard number of a variety $X$.
$\mathbb{F}_n$ denotes the minimal rational ruled (Hirzebruch) surface $\mathbb{P}^{21}(\mathcal{O} \oplus \mathcal{O}(n))$.
$X \approx Y$ denotes birationally equivalence between varieties $X$ and $Y$.

2. $G$-equivariant minimal model program

In this section we follow papers [DI-fs], [DI-po], [Isk79].

Definition 2.1. A rational surface $X$ is a smooth projective surface over $k$ such that $\overline{X} = X \otimes \overline{k}$ is birationally isomorphic to $\mathbb{P}^2_k$.

Definition 2.2. A $G$-surface is a pair $(X, \rho)$ where $X$ is a smooth projective surface and $\rho$ is a monomorphism $G \rightarrow \text{Aut}(X)$. A morphism of surfaces $f : X \rightarrow X'$ is called a morphism of $G$-surfaces $(X, \rho) \rightarrow (X', \rho')$ if $\rho'(G) = f \circ \rho(G) \circ f^{-1}$.

Definition 2.3. A $G$-surface $(X, \rho)$ is called minimal if any birational morphism of $G$-surfaces $(X, \rho) \rightarrow (X', \rho')$ is an isomorphism.

Note that in our case there are two groups acting on $\overline{X}$: the geometrical group $G$ and the Galois group $\Gamma = \text{Gal}(\overline{k}/k)$ and the action of $G$ is $\Gamma$-equivariant. It means that for each $g \in G$, $\gamma \in \Gamma$ and $x \in \overline{X}$ one has $\gamma \rho(g)x = \rho(g)\gamma x$. Throughout this paper by minimal surface we mean $(G \times \Gamma)$-minimal surface.

The classification of minimal rational surfaces is well-known due to S. Mori. We introduce some important notions before surveying it.

Definition 2.4. A rational $G$-surface $(X, \rho)$ admits a structure of a conic bundle if there exists a $G$-equivariant morphism $\phi : X \rightarrow \mathbb{P}^1$ such that any fibre is isomorphic to a reduced conic in $\mathbb{P}^2$.

Note that a general fibre of $\phi$ is isomorphic to $\mathbb{P}^1_k$ and its self-intersection equals 0. At the same time there may be singular fibres each of which being a pair of intersecting $(-1)$-curves. It is clear that if a conic bundle is $G$-minimal then two components of each singular fibre are permuted by the group $G$.

We will use the next theorem to work with conic bundles.

Theorem 2.5. [Isk79, Theorem 4] Let $X \rightarrow \mathbb{P}^1$ be a conic bundle. Then $X$ is not minimal if $K_X^2 \in \{3, 5, 6, 7\}$.
Definition 2.6. A Del Pezzo surface is a smooth projective surface $X$ such that the anticanonical divisor $-K_X$ is ample.

Theorem 2.7. [Isk79, Theorem 1] Let $X$ be a minimal rational $G$-surface. Then either $X$ admits a structure of conic bundle with $\text{Pic}(X)^G \cong \mathbb{Z}^2$, or $X$ is isomorphic to a Del Pezzo surface with $\text{Pic}(X)^G \cong \mathbb{Z}$.

The next theorem is an important criterion for proving k-rationality over an arbitrary perfect field $k$ (see [Isk96]).

Theorem 2.8. A minimal rational surface $X$ over a perfect field $k$ is $k$-rational if and only if the following two conditions are satisfied:

(i) $X(k) \neq \emptyset$;

(ii) $d = K_X^2 \geq 5$.

Note that all surfaces in this work are rational by Theorem 1.2. Taking a quotient, resolving singularities and running a minimal model program don’t affect the existence of k-points, so there exists a k-point on each considered surface. Therefore in this work if $X$ is a smooth surface and $K_X^2 \geq 5$ then $X$ is k-rational.

The important type of rational surfaces is toric surfaces.

Definition 2.9. Toric variety is a normal variety containing an algebraic torus as a dense subset.

Remark 2.10. A minimal rational surface $X$ is toric if and only if $K_X^2 \geq 6$.

A minimal rational surface $X$ with $K_X^2 \geq 6$ is $\mathbb{P}_k^2$, $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, del Pezzo surface of degree 6 or a minimal rational ruled surface $\mathbb{F}_n$ ($n \geq 2$).

3. Finite subgroups in $PGL_3(\mathbb{C})$

Definition 3.1. Any finite subgroup of $GL_n(k)$ is called a linear group in $n$ variables.

We will use detailed classification of finite linear subgroups in 3 variables over an algebraically closed field of characteristic zero.

Let a linear group $G$ act on the space $V = k^3$ where $k$ is algebraically closed.

Definition 3.2. If the action of the group $G$ on $V$ is reducible the group $G$ is called intransitive. Otherwise the group $G$ is called transitive.

Definition 3.3. Let $G$ be a transitive group. If there exists a decomposition $V = V_1 \oplus \cdots \oplus V_l$ to subspaces such that for any element $g \in G$ one has $gV_i = V_j$ then the group $G$ is called imprimitive. Otherwise the group $G$ is called primitive.
Lemma 3.4. Let $k$ be an algebraically closed field of characteristic zero. Then any representation of a finite group $G$ in $GL_n(k)$ is conjugate to a representation of the group $G$ in $GL_n(\mathbb{Q})$.

According to Lemma 3.4, for our purpose, it is sufficient to know the classification of finite subgroups of $GL_3(\mathbb{C})$. In the classification, we do not distinguish between groups which are equivalent modulo scalar multiplications because they define the same subgroup in $PGL_3(\mathbb{C})$. Thus, we need to know the classification of finite subgroups of $SL_3(\mathbb{C})$ modulo scalar multiplications.

We use the following notation: $S = \text{diag}(1, \omega, \omega^2), \ T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, U = \text{diag}(\varepsilon, \varepsilon, \varepsilon \omega), \varepsilon^3 = \omega^2$.

The finite subgroups of $SL_3(\mathbb{C})$ were completely classified in [Bl17, Chapter V] and [MBD16, Chapter XII].

Theorem 3.5. Any finite subgroup in $SL_3(\mathbb{C})$ (modulo scalar multiplications) is conjugate to one of the following.

Intransitive group:

(A) A diagonal abelian group.

(B) A group having a unique invariant subspace of dimension 2.

Imprimitive group:

(C) A group having a normal abelian subgroup $N$ such that $G/N \cong C_3$.

(D) A group having a normal abelian subgroup $N$ such that $G/N \cong S_3$.

Primitive groups having normal subgroups:

(E) Group of order 108 generated by $S$, $T$ and $V$.

(F) Group of order 216 generated by (E) and $P = UVU^{-1}$.

(G) Group of order 648 generated by (E) and $U$.

Simple groups:

(H) Group of order 60 isomorphic to the group $\mathfrak{A}_5$.

(I) Klein group of order 168 isomorphic to the group $PSL_2(\mathbb{F}_7)$.

(K) Valentiner group $G$ of order 1080 i.e., its quotient $G/F$ is isomorphic to the group $\mathfrak{A}_6$, where $F$ is the group generated by $\omega I_3$.
4. Rationality of the quotient variety

In this section for each finite group $G \subset PGL_3(\mathbb{C})$ (see Theorem 3.5) we prove that the quotient variety $\mathbb{P}^2_k/G$ is $k$-rational. The main method is the following. We consider the algebraic closure $\overline{k}$ of the field $k$. Two groups act on $\mathbb{P}^2_{\overline{k}}$: the geometrical group $G$ and the Galois group $\Gamma = \text{Gal}(\overline{k}/k)$. In addition the action of the group $G$ is $\Gamma$-equivariant.

If the group $G$ is cyclic or simple, $k$-rationality of $\mathbb{P}^2_k/G$ is proved in the first and the last cases of this section.

Otherwise there is a normal subgroup $N$ in the group $G$. We consider the quotient variety $\mathbb{P}^2_k/N$, resolve singularities, run the $G/N \times \Gamma$-equivariant minimal model program and get a $G/N \times \Gamma$-minimal surface $X$. One has $\mathbb{P}^2_k/G = (\mathbb{P}^2_k/N)/(G/N) \approx X/(G/N)$, therefore it is sufficient to prove that $X/(G/N)$ is $k$-rational. If there is a normal group $M$ in the group $G/N$ we can repeat this method.

We will use the following definition for convenience.

**Definition 4.1.** Let $S$ be a $G \times \Gamma$-surface, $\tilde{S} \to S$ be its minimal resolution of singularities and $Y$ be a $G \times \Gamma$-equivariant minimal model of $\tilde{S}$. We denote the surface $Y$ by $G \times \Gamma$-MMP-reduction of $S$.

For short we will write an $\text{MMP-reduction}$ instead of a $G \times \Gamma\text{-MMP-reduction}$.

4.1. **Diagonal abelian groups.** Each abelian subgroup $G \subset SL_3(\overline{k})$ is conjugate to a diagonal subgroup, so its action on $\mathbb{P}^2_{\overline{k}}$ can be considered as the action of finite subgroup in an open torus in $\mathbb{P}^2_{\overline{k}}$.

**Lemma 4.2.** Let $X$ be a toric variety over an algebraic closed field $\overline{k}$, $\text{char} \overline{k} = 0$ and let $G$ be a finite subgroup of a torus. Then the quotient $X/G$ is a toric variety.

**Proof.** Let $\mathbb{T}^n$ be an open torus in $X$. The regular function’s algebra of $\mathbb{T}^n$ is $\overline{k}[x_1, \ldots, x_n, \frac{1}{x_1}, \ldots, \frac{1}{x_n}]$ and its monoms form a lattice $\mathbb{Z}^n$. The action of the group $G$ on this algebra is monomial so monoms of the algebra of $G$-invariants form a sublattice in this lattice. It means that $\mathbb{T}^n/G$ is a torus in $X/G$, so $X/G$ is a toric variety. \hfill \Box

**Remark 4.3.** Note that the resolution of singularities and minimal model programm don’t affect toric structure on a surface. So for a toric surface $X$ and finite subgroup $G$ of a torus one has an $\text{MMP-reduction}$ of $X/G$ is a minimal toric surface. Therefore if there exists a $k$-point on $X/G$ then it is $k$-rational by Theorem 2.8.
Let $G$ be a finite abelian subgroup in $PGL_3(k)$. Then the MMP-reduction of $\mathbb{P}^2_k/G$ is a minimal toric surface by Lemma 4.2, it is $k$-rational by Theorem 2.8.

4.2. Groups having a unique fixed point. In this case the group $G$ acts on $\mathbb{A}^3_k$ and there exists decomposition $\mathbb{A}^3_k = \mathbb{A}^2_k \oplus k$ into $G$-invariant linear spaces. Moreover, the one-dimensional subspace $k$ is $\Gamma$-invariant because the action of the group $G$ is $\Gamma$-equivariant and there are no other one-dimensional $G$-invariant subspaces. Therefore there exists decomposition $\mathbb{A}^3_k = \mathbb{A}^2_k \oplus k$ into $G$-invariant linear subspaces.

It means that there is a unique $G$-fixed point $p \in \mathbb{P}^2_k$ and a unique $G$-invariant line $l$. Let $F_1$ be the blowup of $\mathbb{P}^2_k$ at the point $p$. The surface $F_1$ admits a $G$-equivariant $\mathbb{P}^1_k$-bundle structure $F_1 \to \mathbb{P}^1_k$ which fibres are proper transforms of lines passing through the point $p$. Obviously this $\mathbb{P}^1_k$-bundle has $G$-invariant sections: the exceptional divisor of the blowup at $p$ and the proper transform of $l$. So one has $F_1/G \approx \mathbb{P}^1_k \times \mathbb{P}^1_k/G$. Therefore our problem is reduced to a one-dimensional case.

4.3. Imprimitive groups. Each imprimitive group $G$ contains a normal abelian subgroup $N$ conjugate to a diagonal abelian subgroup in $GL_3(k)$. The quotient group $G/N$ is isomorphic to $C_3$ or $S_3$ (it corresponds to cases (C) and (D) of Theorem 3.5). Moreover a MMP-reduction of $\mathbb{P}^2_k/N$ is a $k$-rational minimal toric surface by Lemma 4.2.

In this subsection we prove the following proposition:

**Proposition 4.4.** Let $X$ be a $k$-rational minimal toric surface and let $G$ be a group $C_3$ or $S_3$ $\Gamma$-equivariantly acting on $X$. Then $X/G$ is $k$-rational.

In the proof of this proposition quotient singularities of definite types play important role. The following remark is useful to work with them.

**Remark 4.5.** Let a group $G$ act on a surface $X$ and fix a point $p \in X$. Let $f : X \to X/G$ be the quotient map. Then one has:

For the action of the group $G = C_2$ on a tangent space at the point $p$ as $-I_2$ the point $f(p)$ is a du Val singularity of type $A_1$.

For the action of the group $G = C_3$ on a tangent space at the point $p$ as $\text{diag}(\omega, \omega^2)$ the point $f(p)$ is a du Val singularity of type $A_2$.

Moreover these singularities have the following properties:

(a) The minimal resolution $\pi : Y \to X/C_2$ of a singularity $A_1$ gives the exceptional divisor which is a $(-2)$-curve; one has $K^2_Y = K^2_{X/G}$.

For nonsingular curve $C_{X/C_2}$ passing through the singularity and $C_Y = \pi^*C_{X/C_2}$ we have $C^2_Y = C^2_{X/C_2} - \frac{1}{2}$. 


(b) The minimal resolution $\pi : Y \to X/C_3$ of a singularity $A_2$ gives the exceptional divisor which consists of two components. Each of them is a $(-2)$-curve and they intersect transversally at one point; one has $K_Y^2 = K_{X/C_3}^2$.

Let $C_X$ be a $C_3$-invariant nonsingular irreducible curve on $X$ passing through the point $p$, $C_{X/C_3} = f(C_X)$ and $C_Y = \pi^* C_{X/C_3}$. Then $C_Y^2 = C_{X/C_3}^2 - \frac{2}{3}$.

Let $C_X$ and $D_X$ be two $C_3$-invariant nonsingular irreducible curves on $X$ which intersect transversally at the point $p$. Then the curves $C_Y = \pi^* f(C_X)$ and $D_Y = \pi^* f(D_X)$ intersect with different components of the exceptional divisor.

Now we come to the proof of Proposition 4.4.

**Lemma 4.6.** Let $X$ be a k-rational minimal toric surface, $p$ be prime and the group $C_3$-equivariantly act on $X$. Then an MMP-reduction of $X/C_3$ is a k-rational minimal toric surface.

The Proposition 4.4 follows from this Lemma. If the group $G = C_3$ it directly follows from Lemma 4.6. If the group $G = S_3$ then a surface $Y$, which is $C_3 \times \Gamma$-MMP reduction of $X/C_3$, is a k-rational minimal toric surface by Lemma 4.6 and MMP-reduction of $Y/C_2$ is a k-rational minimal toric surface by Lemma 4.6.

To prove Lemma 4.6 we case-by-case consider $\mathbb{P}^2_k$, $\mathbb{P}^1_k \times \mathbb{P}^1_k$ del Pezzo surface of degree 6 and minimal rational ruled surfaces $F_n$ $(n \geq 2)$ with an action of the group $C_p$.

**4.3.1. Case 1: $\mathbb{P}^2_k$.** Each cyclic group $C_n$ is a finite subgroup of an open torus in $\mathbb{P}^2_k$. Therefore an MMP-reduction of $\mathbb{P}^2_k/C_n$ is a k-rational minimal toric surface by Lemma 4.2.

**4.3.2. Case 2: $\mathbb{P}^1_k \times \mathbb{P}^1_k$.** The automorphism group of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ is isomorphic to the group $(PGL_2(\bar{k}) \times PGL_2(\bar{k})) \rtimes C_2$. We have the exact sequence:

$$1 \to PGL_2(\bar{k}) \times PGL_2(\bar{k}) \hookrightarrow (PGL_2(\bar{k}) \times PGL_2(\bar{k})) \rtimes C_2 \twoheadrightarrow C_2 \to 1.$$

Let $p$ be a prime and $C_p$ be a cyclic subgroup of the group $\text{Aut}(\mathbb{P}^1_k \times \mathbb{P}^1_k) = (PGL_2(\bar{k}) \times PGL_2(\bar{k})) \rtimes C_2$. The composition of maps

$$C_p \hookrightarrow (PGL_2(\bar{k}) \times PGL_2(\bar{k})) \rtimes C_2 \twoheadrightarrow C_2$$

takes $C_p$ to identity if $p \neq 2$. Thus one has $C_p \hookrightarrow PGL_2(\bar{k}) \times PGL_2(\bar{k})$. So the group $C_p$ is a finite subgroup of an open torus in $\mathbb{P}^1_k \times \mathbb{P}^1_k$. An MMP-reduction of $(\mathbb{P}^1_k \times \mathbb{P}^1_k)/C_p$ is a k-rational minimal toric surface by Lemma 4.2.
If \( p = 2 \) and \( C_2 \) is not a subgroup of \( PGL_2(\mathbb{k}) \times PGL_2(\mathbb{k}) \) then the action of the group \( C_2 \) is conjugate to
\[
(x_1 : x_0; y_1 : y_0) \mapsto (y_1 : y_0; x_1 : x_0)
\]
where \((x_1 : x_0; y_1 : y_0)\) are homogeneous coordinates on \( \mathbb{P}^1_\mathbb{k} \times \mathbb{P}^1_\mathbb{k} \). There is a fixed curve \( \frac{\alpha}{x_0} = \frac{\alpha}{y_0} \) on \( \mathbb{P}^1_\mathbb{k} \times \mathbb{P}^1_\mathbb{k} \) which class in \( \text{Pic}(\mathbb{P}^1_\mathbb{k} \times \mathbb{P}^1_\mathbb{k}) \) equals \(-\frac{1}{2}K_{\mathbb{P}^1_\mathbb{k} \times \mathbb{P}^1_\mathbb{k}}\). The quotient variety \( Y = (\mathbb{P}^1_\mathbb{k} \times \mathbb{P}^1_\mathbb{k})/C_2 \) is nonsingular. By Hurwitz formula one has \( K_Y^2 = \frac{1}{2}(\frac{3}{2}K_X)^2 = 9 \). Thus the surface \( Y \) is isomorphic to \( \mathbb{P}^2_\mathbb{k} \) and toric.

4.3.3. Case 3: Minimal ruled surfaces \( \mathbb{F}_n \) (\( n \geq 2 \)). Let the group \( G \) act on \( \mathbb{F}_n \). Then the conic bundle structure \( \mathbb{F}_n \rightarrow \mathbb{P}^1_\mathbb{k} \) is \( G \)-equivariant. It means that there exists the exact sequence:
\[
1 \rightarrow G_F \hookrightarrow G \twoheadrightarrow G_B \rightarrow 1
\]
where \( G_F \) is a group of automorphisms of general fibre and \( G_B \) is a group of automorphisms of the base \( B = \mathbb{P}^1_\mathbb{k} \). Therefore for a prime \( p \) the action of the group \( C_p \) on the base is either faithful or trivial.

In the first case there are two fixed points on the base \( \mathbb{P}^1_\mathbb{k} \). The corresponding fibres of \( \mathbb{F}_n \rightarrow \mathbb{P}^1_\mathbb{k} \) are \( C_p \)-invariant. In the second case all fibres of \( \mathbb{F}_n \rightarrow \mathbb{P}^1_\mathbb{k} \) are \( C_p \)-invariant. So in both cases we can choose \( C_p \)-invariant fibre \( F_1 \).

The action of the group \( C_p \) on \( F_1 \) is either faithful or trivial. Therefore there are at least two \( C_p \)-fixed points. So we can choose a fixed point \( p \) which don’t lay on the \((-n)\)-section. Let us blow up the point \( p \) and contract the proper transform of \( F_1 \). We get a surface \( \mathbb{F}_{n-1} \) with the action of the group \( C_p \). By repeating of this procedure \( n \) times we can obtain a \( C_p \)-equivariant birational map \( f : \mathbb{F}_n \dashrightarrow \mathbb{P}^1_\mathbb{k} \times \mathbb{P}^1_\mathbb{k} \).

Note that \( \rho(\mathbb{P}^1_\mathbb{k} \times \mathbb{P}^1_\mathbb{k})^{C_p} = \rho(\mathbb{F}_n)^{C_p} = 2 \). Therefore if \( p = 2 \) the group \( C_2 \) can’t act as a permutation of the rulings of \( \mathbb{P}^1_\mathbb{k} \times \mathbb{P}^1_\mathbb{k} \). So the group \( C_p \) is a finite subgroup of an open torus in \( \mathbb{P}^1_\mathbb{k} \times \mathbb{P}^1_\mathbb{k} \). The birational map \( f^{-1} : \mathbb{P}^1_\mathbb{k} \times \mathbb{P}^1_\mathbb{k} \rightarrow \mathbb{F}_n \) preserves this torus. Therefore an MMP-reduction of \( \mathbb{F}_n/C_p \) is a \( k \)-rational minimal toric surface by Lemma [1,2].

4.3.4. Case 4: Del Pezzo surface of degree 6. A Del Pezzo surface of degree 6 \( X_6 \) is isomorphic over algebraically closed field \( \mathbb{k} \) to blowup \( \mathbb{P}^2_\mathbb{k} \) at three points in general position. It can be assumed that these points have homogenous coordinates \((1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\).

The automorphism group of \( X_6 \) is isomorphic to \( \mathbb{T}^2 \rtimes D_{12} \) where \( \mathbb{T}^2 \) is two-dimensional torus over \( \mathbb{k} \) and \( D_{12} \) acts on the set of \((-1)\)-curves (the exceptional divisors of the blowup and the proper transforms of lines passing through a pair of points of blowup).
Remark 4.7. To work with a Del Pezzo of degree 6 we will use coordinates on $\mathbb{P}^2_k$. An equation in these coordinates defines a curve on the open set, which is the Del Pezzo surface of degree 6 without 6 $(-1)$-curves. At the same time it is clear how the curve intersects each of $(-1)$-curves because the Del Pezzo surface of degree 6 is isomorphic to blowup $\mathbb{P}^2_k$ at three points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$.

We have the exact sequence:

$$1 \to \mathbb{T}^2 \hookrightarrow \mathbb{T}^2 \times D_{12} \twoheadrightarrow D_{12} \to 1.$$  

Let $p$ be a prime and $C_p$ be a cyclic subgroup of the group $\text{Aut}(X_6) = \mathbb{T}^2 \rtimes D_{12}$. The composition of maps

$$C_p \hookrightarrow \mathbb{T}^2 \times D_{12} \twoheadrightarrow D_{12}$$

takes $C_p$ to identity if $p > 3$. Thus one has $C_p \hookrightarrow \mathbb{T}^2$. So the group $C_p$ is a finite subgroup of an open torus in $X_6$. An MMP-reduction of $X_6/C_p$ is a $k$-rational minimal toric surface by Lemma 4.2.

If $p = 2$ or $p = 3$ and $C_p$ is not a subgroup of $\mathbb{T}^2$ then the group $C_p$ acts on the set of $(-1)$-curves forming a hexagon. There are four nonconjugate cyclic subgroups of prime order in the group $D_{12}$. The actions on the hexagon, which sides correspond to $(-1)$-curves and vertices correspond to their intersection points, are the following:

(a) A reflection along a line passing through middles of opposite sides of the hexagon;
(b) A reflection along a line passing through two opposite vertices of the hexagon;
(c) The central symmetry;
(d) The rotation by an angle of $\frac{\pi}{3}$.

In the case (a) we have two invariant disjointed $(-1)$-curves and the others are not invariant. Therefore the action of $C_2$ is not minimal and we can contract this pair and get $\mathbb{P}^1_k \times \mathbb{P}^1_k$ considered in the case 2.

In the case (b) we have the invariant pair of disjointed $(-1)$-curves (two other pairs of $(-1)$-curves are not disjointed). Therefore the action of $C_2$ is not minimal and we can contract this pair and get $\mathbb{P}^1_k \times \mathbb{P}^1_k$ considered in the case 2.
In the case (c) the action of $C_2$ is conjugate to $(x : y : z) \mapsto \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$. There are four fixed points:

$t_1 = (1 : 1 : 1), t_2 = (-1 : 1 : 1), t_3 = (-1 : -1 : 1), t_4 = (1 : -1 : 1)$.

For each pair of these points there is an unique invariant curve with zero selfintersection passing through this pair. Their equations are the following:

$m_{12} = \{y = z\}, m_{13} = \{x = y\}, m_{14} = \{x = z\},$ \[ m_{23} = \{x = -z\}, m_{24} = \{x = -y\}, m_{34} = \{y = -z\}. \]

These curves don’t intersect at any points on $X_6$ except $t_i$.

Let $f : X_6 \to X_6/C_2$ be the quotient map. The four points $f(t_i)$ are singularities of type $A_1$ and one has $(f(m_{ij}))^2 = 0$, $K^2_{X_6/C_2} = 3$.

Let $\pi : \tilde{X}_6/C_2 \to X_6/C_2$ be the minimal resolution of the singularities. By Remark 4.5 one has:

$\left(\pi^{-1}f(t_i)\right)^2 = -2, \ (\pi^*f(m_{ij}))^2 = -1, \ (\pi^*f(m_{ij})) \cdot (\pi^*f(m_{kl})) = 0$

if the pair $ij$ differs from the pair $kl$, $K^2_{\tilde{X}_6/C_2} = 3$. Therefore we can equivariantly contract six curves $\pi^*f(m_{ij})$ and get a surface $Y$ with $K^2_Y = 9$. $Y$ is rational so it is isomorphic to $\mathbb{P}^2$. So $Y$ is a toric surface and it is k-rational by Theorem [2,8].
In the case (d) the action of $C_3$ is conjugate to $(x : y : z) \mapsto (y : z : x)$. There are three fixed points:

$$t_1 = (1 : 1 : 1), t_2 = (\omega : \omega^2 : 1), t_3 = (\omega^2 : \omega : 1).$$

For each pair of these points there are exactly two invariant curves with selfintersection one passing through this pair. Their equations are the following:

$$m_{12} = \{\omega x + \omega^2 y + z = 0\}, m_{13} = \{\omega^2 x + \omega y + z = 0\}, m_{23} = \{x + y + z = 0\},$$

$$u_{12} = \{\omega xy + \omega^2 xz + yz = 0\}, u_{13} = \{\omega^2 xy + \omega xz + yz = 0\},$$

$$u_{23} = \{xy + xz + yz = 0\}.$$

The curves $m_{ij}$ and $m_{jk}$ with $i \neq k$ intersect at the point $t_j$, the curves $m_{ij}$ and $u_{ij}$ intersect at the points $t_i$ and $t_j$, the curves $m_{ij}$ and $u_{jk}$ with $i \neq k$ intersect with multiplicity 2 at the point $t_j$, the curves $u_{ij}$ and $u_{jk}$ with $i \neq k$ intersect at the point $t_j$.

Let $f : X_6 \to X_6/C_3$ be the quotient map. The three points $f(t_i)$ are singularities of type $A_2$ and one has $(f(m_{ij}))^2 = (f(u_{ij}))^2 = \frac{1}{3}$, $K_{X_6/C_3}^2 = 2$.

Let $\pi : \widetilde{X_6/C_3} \to X_6/C_3$ be the minimal resolution of the singularities. By Remark 4.5 the resolution of each $A_2$ singularity is a pair of $(-2)$-curves intersecting at a point, $(\pi^* f(m_{ij}))^2 = (\pi^* f(u_{ij}))^2 = -1$, $K_{\widetilde{X_6/C_2}}^2 = 2$. By direct computation it is easy to check that six curves $(\pi^* f(m_{ij}))^2$ and $(\pi^* f(u_{ij}))^2$ are disjointed. Therefore we can equivariantly contract these six curves and get a surface $Y$ with $K_Y^2 = 8$ (one can check that $Y$ is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$). So $Y$ is a toric surface and it is $k$-rational by Theorem 2.8.
Corollary 4.8. Let $X_6$ be a $k$-rational del Pezzo surface of degree $6$ and $G$ be a finite subgroup of $\Gamma$-equivariant authomorphisms of $X_6$. The quotient variety $X_6/G$ is $k$-rational.

Proof. The group $G$ is a subgroup of $T^2 \rtimes D_{12}$. Therefore there is a normal subgroup $N = G \cap T^2$. An MMP-reduction of $X_6/N$ is a $k$-rational minimal toric surface $Y$ by Lemma 4.2. The group $G/N$ is a subgroup of $D_{12}$. The center of $D_{12}$ is $C_2$. Let $M = G/N \cap C_2$ then an MMP-reduction of $Y/M$ is a $k$-rational minimal toric surface $Z$ by Lemma 4.6 and the group $(G/N)/M$ is a subgroup of $S_3$. Let $L = (G/N)/M \cap C_3$ then an MMP-reduction of $Z/L$ is a $k$-rational minimal toric surface $W$ by Lemma 4.6 and the group $((G/N)/M)/L$ is a subgroup of $C_2$. An MMP-reduction of $W/((((G/N)/M)/L)$ is $k$-rational toric surface by Lemma 4.6. □

4.4. Primitive groups having normal subgroups. Primitive groups having normal subgroups are groups of type (E), (F), (G) from
Theorem 3.5. Note that \((E) \subset (F) \subset (G)\). Moreover they have common subgroup \(N\) generated by \(S = \text{diag}(1, \omega, \omega^2)\), \(T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\) and \(V^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\).

Note that the group generated by \(\omega I_3\) is a subgroup of all these groups. Therefore for the map \(f : SL_3(\mathbb{F}) \to PGL_3(\mathbb{F})\) one has \(\text{ord}(f(N)) = 18\), \(\text{ord}(f((E))) = 36\), \(\text{ord}(f((F))) = 72\), \(\text{ord}(f((G))) = 216\).

Let us show that \(N\) is normal subgroup in all of these groups. We should check up equalities

\[ VSV^{-1} = T, VTV^{-1} = V^2SV^2, USU^{-1} = S, UTU^{-1} = S^2T, UV^2U^{-1} = S^2V^2 \]

that can be made by direct computation (\(U\) is an element from the classification 3.5). Therefore \(\mathbb{P}^2_k/G = (\mathbb{P}^2_k/N)/(G/N)\) where \(G\) is one from the groups \((E), (F), (G)\). The quotients of these groups by the subgroup \(N\) are the following: \((E)/N = C_2\), \((F)/N = C_2^2\), \((G)/N = A_4\).

Let us consider the quotient variety \(\mathbb{P}^2_k/N\). The group \(N\) consists of 18 elements and 9 of them have order 2 and fix different lines. The other elements (excepting identity) have order three and isolated fixed points (three per element). By Hurvitz formula

\[ K_{\mathbb{P}^2_k/N} = \frac{1}{18}(K_{\mathbb{P}^2_k} - 9l)^2 = 8. \]

Note that fixed points of elements of order 3 don’t give us singularities because in their tangent space the group acts as \(S_3\). At the same time for each element of order 2 there is an isolated fixed point. These 9 points are permuted by elements of order 3. So there is one \(A_1\) singular point on the quotient variety \(\mathbb{P}^2_k/N\). Its resolution is \((-2)\)-curve \(C\) and the received surface is isomorphic to \(\mathbb{F}_2\).

In the case of the action of the group \((E)\) the rationality of the quotient variety \(\mathbb{P}^2_k/C_2\) directly follows from Lemma 4.2. Note that, in the case of the action of the group \((G)\) on \(\mathbb{P}^2_k\) the group \(A_4\) acts on \(\mathbb{F}_2\). The action of the group \(A_4\) on the base is either trivial, or factors through the homomorphism \(A_4 \to C_3\), or is a faithful action of \(A_4\). Therefore the action on the base of the normal subgroup \(C_2^2\) of \(A_4\), corresponding to the action of the group \((F)\) on \(\mathbb{P}^2_k\), is either trivial or faithful. Let us prove that in the both cases an MMP-reduction of \(\mathbb{F}_2/C_2^2\) is a k-rational minimal toric surface. The rationality of \(\mathbb{P}^2_k/(F)\)
directly follows from this fact and the rationality of $\mathbb{P}^2_k/(G)$ follows from this fact and Lemma 4.2 because $\mathfrak{a}_4/C_2^2 = C_3$.

The action of $C_2^2$ on $\mathbb{P}^1_k$ is conjugate to the action of the subgroup in $PGL_2(\mathbb{R})$ generated by $I = \text{diag}(i, -i)$ and $J = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$. Each of element $I$, $J$, $K = IJ$ have a pair of fixed points on $\mathbb{P}^1_k$. Let denote these points by $p_1$, $p_2$, . . . , $p_6$. Note that $Jp_1 = Kp_1 = p_2$ etc.

If the action of the group $C_2^2$ is trivial then each fibre of $\mathbb{F}_2 \rightarrow \mathbb{P}^1_k$ is invariant. Each fibre intersects with $(−2)$-section $C$ at a fixed point. But the faithful action of the group $C_2^2$ on fibre which is isomorphic to $\mathbb{P}^1_k$ don’t have a fixed point. So this case can’t be achieved.

If the group $C_2^2$ acts on the base faithfully let us consider six fibres $F_i$ over points $p_i$. Note that these fibres are permuted by the group $\mathfrak{a}_4 = (G)/N$. The action of the element $I$ on $F_i$ is either trivial or faithful.

In the first case each $F_i$ is fixed by one of the elements $I$, $J$, $K$. Therefore by Hurwitz formula

$$K^2_{\mathbb{F}_2/C_2^2} = \frac{1}{4}(-2C - 4F - 6F)^2 = 8,$$

where $F$ is the class of fibre in Pic($\mathbb{F}_2$). So $\mathbb{F}_2/C_2^2$ is a $k$-rational toric surface.

If the action of the element $I$ on $F_1$ is faithful then there are two $I$-fixed points on $F_1$. One of them is the intersection of $F_1$ and $(−2)$-section $C$, we denote this point by $p_{−1}$ and the other $I$-fixed point on $F_1$ by $p_{+1}$. In the same way we can define points $p_{±2}$, $p_{±3}$, . . . , $p_{±6}$.

Let $f : \mathbb{F}_2 \rightarrow \mathbb{F}_2/C_2^2$ be the quotient map. The six points $f(p_{±i})$ are singularities of type $A_1$ and one has $(f(F_i))^2 = 0$, $(f(C))^2 = -\frac{1}{2}$, $K^2_{\mathbb{F}_2/C_2^2} = 2$.

Let $\pi : \overline{\mathbb{F}_2/C_2^2} \rightarrow \mathbb{F}_2/C_2^2$ be the minimal resolution of the singularities. By Remark 4.5 one has

$$(\pi^{-1}f(p_{±i}))^2 = -2, (\pi^*f(F_i))^2 = -1,$$

$$(\pi^*f(C))^2 = -2, (\pi^*f(F_i)) \cdot (\pi^*f(C)) = 0.$$

Let $g : \overline{\mathbb{F}_2/C_2^2} \rightarrow Y$ be the contraction of three curves $\pi^*f(F_i)$. One has

$$(g\pi^{-1}f(p_{±i}))^2 = -1, (g\pi^*f(C))^2 = -2, (g\pi^{-1}f(p_{±i})) \cdot (g\pi^*f(C)) = 1,$$

$$(g\pi^{-1}f(p_{±i})) \cdot (g\pi^*f(C)) = 0, K^2_Y = 5.$$
Therefore we can equivariantly contract three curves $g\pi^{-1}f(p_{-i})$ and get a surface $Z$ with $K_Z^2 = 8$ (one can check that $Z$ is isomorphic to $\mathbb{F}_1$). So $Z$ is a toric surface and it is $k$-rational by Theorem 2.8.

4.5. Simple groups. Note that each simple group $G$ is generated by elements of order 2 because its order is even and elements of order 2 generate a normal subgroup. The normal form of an element of order 2 in $\text{SL}_3(k)$ is $\text{diag}(-1, -1, 1)$. This element fixes a line on $\mathbb{P}^2_k$ so it is a reflection. It is well-known (a consequence from Shevalley-Shephard-Todd Theorem) that the quotient $\mathbb{P}^2_k$ by group generated by reflections is a weighted projective space. It is a toric surface so an MMP-reduction of $\mathbb{P}^2_k/G$ is $k$-rational by Theorem 2.8.

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