Optimization of synchronizability in multiplex networks by rewiring one layer

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The mathematical framework of multiplex networks has been increasingly realized as a more suitable framework for modelling real-world complex systems. In this work, we investigate the optimization of synchronizability in multiplex networks by evolving only one layer while keeping other layers fixed. Our main finding is to show the conditions under which the efficiency of convergence to the most optimal structure is almost as good as the case where both layers are rewired during an optimization process. In particular, inter-layer coupling strength responsible for the integration between the layers turns out to be a crucial factor governing the efficiency of optimization even for the cases when the layer going through the evolution has nodes interacting much weakly than those in the fixed layer. Additionally, we investigate the dependency of synchronizability on the rewiring probability which governs the network structure from a regular lattice to the random networks. The efficiency of the optimization process preceding evolution driven by the optimization process is maximum when the fixed layer has regular architecture, whereas the optimized network is more synchronizable for the fixed layer having the rewiring probability lying between the small-world transition and the random structure.

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Introduction: The framework for a single network has been extremely successful for predicting and understanding behaviour of complex systems [1]. However, recent studies of multiplex networks are providing new insights to the research in real-world complex systems by incorporating in the analysis the fact that they are composed by several types of networks (layers) and more than one type of interactions exist among the layers. Thus, multiplex networks are expected to provide better understanding about the underlying structural and dynamical properties of real-world systems as compared to the traditional isolated networks approach [2]. For instance, diffusion processes taking place on multiplex networks has been shown to exhibit abrupt transitional behaviour guided by inter-layer coupling strength [3]. Entropy rates and information transmission was shown to be strongly regulated by the ratio between inter-connectivity and the size of the single layer [4]. Similarly, cluster synchronization of a layer in multiplex networks has been demonstrated to be strongly affected by the network parameters of other layer [5]. Furthermore, endemic states in multiplex networks has been shown to crucially depend on the interconnectivity of the layers, not emerging in isolated layers considered in isolation [6]. The multiplex network framework has allowed to incorporate various new interconnected processes into the modelling of complex systems, such as the work in Ref. [7] that studies the spreading of an epidemic in individuals contributing to the understanding of how disease spreading can be controlled.

Further, synchronization phenomena or collective behaviour of coupled dynamical units has been a topic of intensive research [8]. Dynamical behaviour of interacting units depends on the structural properties of interactions. One such relation between the structural property of a network and the synchronous dynamical behaviour of units interacting via diffusive coupling is measured by the synchronizability of the network, defined by the ratio between the first nonzero and the largest eigenvalues of the corresponding Laplacian matrix [9, 10]. Larger (smaller) the $R$ values, the smaller (larger) coupling strength interval for which synchronization is observed.

The most optimized network in terms of synchronizability has been shown to exhibit homogeneity in its degree distribution and in the betweenness centrality of the nodes [11]. Optimization of synchronizability in networks with nodes connected by weighted strengths is a problem with an extra dimension of complexity. However, it has been shown that such networks can be successfully evolved to become optimally synchronizable [12, 13]. Even more challenging is the optimization of multiplex networks, which would require optimization strategies involving several network parameters and larger dimensional systems. Take the brain as an example, it learns by rewiring its synaptic connections. If the brain were to adapt (optimize behavior) based on all its possible scenarios, that would be a fantastic complex optimization process. Rather, it is plausible to think that optimization in the brain (such as those driven by Hebbian learning rules) is driven by evolution rules applied locally. This paper shows that indeed synchronizability of a whole multiplex network can be achieved by rewiring only one layer, thus showing that the computational complexity of optimization in multiplex networks can be drastically reduced.

More specifically, we study optimization of a layer in multiplex network such that the entire network becomes more synchronizable. During the evolution, only one layer is rewired while keeping the other layer(s)’s topology fixed. Changing the network architecture of one layer...
affects the dynamical evolution of the other layers because of the interactions mediated by the inter-layer couplings. We therefore investigate the efficiency of the optimization in terms of the interplay between the intra-layer coupling strengths of the layer going through the evolution process and inter-layer couplings. Furthermore, we investigate the impact of the network architecture of the fixed layer on the optimization efficiency. Our investigation reveals that the inter-layer coupling strength plays a crucial role in determining the impact of the optimization process on the synchronization of the entire network. Interestingly, even if the layer going through the evolution has much weaker intra-layer coupling strength as compared to that of the fixed layer, efficiency of optimization is high if there is a strong interaction between the layers. Moreover, the optimization leads to the best synchronizable multiplex network when the network architecture of the fixed layer lies between a complete random architecture and the one observed at the small-world transition arising due to the combined impact of the degree homogeneity and the diameter.

Optimization of complex networks is behind the success of technological as well as natural adaptive processes. The brain learns by rewiring its synaptic connections. Deep learning machines changes internal structures of its neural network to optimize its logical outputs. It is a current scientific challenge to understand natural optimization processes in order to reproduce it. The difficulty lies on the fact that optimization complexity increases exponentially by the size of the system. This paper shows that synchronizability of a whole multiplex network, the ability of the network to synchronize, can be optimized by only rewiring a single network layer. Thus, this paper opens up a new avenue of research, by showing that optimization complexity can be drastically optimized.

**Theoretical Framework:** Let $A$ and $B$ be two adjacency matrices with dimension $N \times N$ corresponding to network configurations representing the initial structure of two layers of a multiplex network. The elements in the adjacency matrices $[a_{ij}$ and $b_{ij}$ take value 1 and 0 depending upon whether there exists a connection between the $i$ and $j$ nodes or not. We perform optimization for individual layers with several architectures. The weighted adjacency matrix of the multiplex networks can be written as,

$$M = \begin{bmatrix} A & D_x I \\ D_x I^T & E_y B \end{bmatrix}$$

where $E_y$ is the intra-layer coupling strength of the layer, $D_x$ represents the inter-layer coupling strength, and $I$ is the inter-layer adjacency matrix representing the connections from $B$ to $A$, and $I^T$ (the transpose of $I$) represents the connections from layer $A$ to $B$.

We optimize the eigenvalue ratio $R = \frac{\lambda_1}{\lambda_2}$, inverse of synchronizability, where $\lambda_{\text{max}}$ and $\lambda_2$ are the largest and the first non-zero eigenvalue of the Laplacian matrix of the multiplex network constructed from $\sum_{j=1}^{2N} M_{ij} I - M$, where $I$ represents the identity matrix.

We use the simulated annealing technique [14] to perform the optimization of $R$. Our optimization aims at minimizing $R$, and thus, maximizing synchronizability. This optimization technique has several variations depending upon the problem in hand. For the current work, the method is explained as follows. We take an initial multiplex network with a given set of parameters. Next, we calculate the eigenvalue ratio $R_1$ of the corresponding Laplacian matrix of the initial multiplex network. Rewiring is performed only in one layer by keeping the second layer’s architecture fixed throughout the evolution. We calculate the eigenvalue ratio $R_2$ of the multiplex network after performing a single rewiring. The initial multiplex network is replaced by the rewired multiplex network if the latter is more synchronizable and $R_2 \leq R_1$ otherwise replaced with the probability $p = \exp((R_1 - R_2)/T)$. Whereas, the initial network is selected with the probability $1 - p$. $T$ is a constant taken initially 1.000. It is updated to the end of each generation by 0.999T.

During the optimization process, the fixed layer introduces a limit to the synchronizability of the entire multiplex network. Nevertheless, the effect of the fixed layer varies depending upon inter and intra-layer coupling strengths of both the layers. Naturally, if the layer going through the rewiring during evolution has stronger intra-layer couplings as compared to that of the fixed layer, the optimization should be more efficient. Interestingly, we find that the inter-layer coupling strength $D_x$
FIG. 2: (a) Clustering coefficient (circles), characteristic path length (squares) and normalized eigenvalue ratio (star) as a function of small-world rewiring probability (p_r) of an initial multiplex network having one layer represented by a small-world network with p_r rewiring probability and other layer represented by ER network. (b) Depicts impact of p_r on the optimized R value for the average degree of each layer taken as (k) = 10 (circles) and (k)=20 (square). The fixed layer is represented by a small-world network with p_r probability and the layer represented by ER network is evolved through the optimization mechanism. (c) Shows the impact of p_r on the optimization efficiency (R_{norm}) for (k) = 10 (circles) and (k) = 20 (squares). Each layer of the multiplex networks has N_1 = N_2 = 500 and value of D_x = 1.

has more profound impact on the optimization. To observe the impact of E_y and D_x on the efficiency of the optimization process, we systematically investigate the following cases. In case (I), inter-layer coupling strength is weak, i.e. D_x takes the value 1 and the layer with weaker intra-layer coupling strengths (i.e. layer A) is rewired resulting in evolution of this layer, whereas the architecture of the layer with stronger intra-layer coupling strengths (layer B) is maintained throughout the evolution process. In case (II), inter-layer coupling strength is strong (D_x is large), and other parameters are the same as for the case (I). In case III, D_x is large and the layer with smaller intra-layer coupling (layer A) is preserved during the evolution. The rewiring is performed only in the layer having larger intra-layer coupling strength (layer B). To compare the results about the impact of change in only one layer on the synchronizability of the entire multiplex network with those obtained for changes in both the layers, we consider two more cases. In case (IV) and (V), evolution is allowed in both the layers with case (IV) considering D_x > 1 and case (V) considering D_x = 1. In case (VI), D_x = 1 and the layer with weaker intra-layer coupling strengths (layer A) is preserved, and the layer with stronger intra-layer couplings is evolved. Further, we measure efficiency of synchronizability by R_{norm} = \frac{R_{opt}}{R_{ini}} where R_{opt} and R_{ini} represent value of R for the final optimized and the initial multiplex network, respectively. As the eigenvalue ratio (R) and the synchronizability of a network are inversely related, the lower the R_{norm} value, the better is the efficiency of the synchronizability.

Results: As evolution progress, the optimization attempts to bring the layer going through the rewiring to a structure which is favourable for synchronization, whereas the fixed layer imposes a limit to the synchronizability or on the efficiency of the synchronization. Fig. (1) demonstrates that for the case (I), optimization does not succeed in producing a synchronizable networks for any value of E_y we have considered. Whereas in the case (II), the optimization succeeds into finding synchronizable networks for all the values of E_y considered here. Though, the maximum efficiency corresponds to a value of E_y for which R_{norm} is minimal, the exact value of E_y for which efficiency is maximal depends on the size and average degree of the network. Further, a low value of D_x typically produces a low value of \lambda_{max}, whereas high values of D_x lead to high value of \lambda_{max}. Both these factors contribute to an increase in the R values and for the model considered here R can be determined as following: For D_x being smaller with respect to E_y, referred as weaker D_x case, one can understand the behaviour of R using the following approximation:

\[ R \approx \frac{\max_{\alpha} \left[ \lambda_{max}(L^\alpha) + D_x \right]}{2D_x} \]  \hspace{1cm} (2)

where L^\alpha is Laplacian of the \alpha^{th} layer, \lambda_{max}(L^\alpha) is maximum eigenvalue of the Laplacian of the \alpha^{th} layer. For the model considered in Eq. (1), the \alpha index represents the matrix A or matrix E_yB, and therefore L^A = \sum_j A_{ij}I - A, and L^B = \sum_j E_yB_{ij}I - E_yB.

When D_x > 1, i.e., inter-layer being stronger than the intra-layer;

\[ R \approx \frac{2D_x + \sqrt{2} \lambda_{max}(L^{AV})}{\lambda_2(L^{AV})} \]  \hspace{1cm} (3)

where L^{AV} is the average Laplacian of two layers.

For small D_x values, R is governed by Eq. (2). Since \lambda_{max} of the fixed layer having stronger intra-layer coupling strength governs the numerator of Eq. (2) which leads to the same value of R throughout the optimization resulting in R_{norm} \approx 1. For larger D_x values, Eq. (3) starts to dominate over Eq. (2). The layer going through the evolution, even though having smaller intra-layer couplings as compared to those of the fixed layer, contributes to R as because of the average of the Laplacians of both the layers appearing in the denominator of Eq. (3). Further, structural changes caused by the evolution process are capable of steering \lambda_2 of the evolved layer towards larger values, resulting in the smaller R values (Eq. (3)) and therefore, optimization is successful. For a further increase in D_x, Eq. (3) holds even better for the R values, and suddenly there is an increase in the efficiency of the optimization. However, the larger the values of D_x and E_y are, the stronger the contribution of the fixed layer coupling strength in L^{AV} of Eq. (3) is. As a result, the efficiency again decreases for the case (II). Efficiency for the cases (V) and (VI), i.e., for smaller values of D_x, can be explained by Eq. (2) where \lambda_{max} comes from the rewired layer, which has stronger intra-layer couplings and hence always dominates the numerator of Eq. (3). Interestingly, for smaller D_x values, rewiring in both the layers (case (V)) does not lead to
an increase in the efficiency as compared to the rewiring in a single layer having stronger coupling strength (case VI) as illustrated in (Fig. 1(b)). For larger values of $D_x$ and $E_y$, Eq. (3) controls the values of $R$ where structural properties of both the layers are crucial to determine the spectral properties of the $L^{AV}$ matrices. As a result, the efficiency is higher for the case (IV) corresponding to rewiring performed in both the layers as compared to that of the case (III), which corresponds to rewiring performed in only one layer. However, further increments in $D_x$ as well as in $E_y$ (as $E_y = D_x$ for $D_x > 1$) values lead to a domination of the contribution of stronger couplings in $L^{AV}$ and as a result, the efficiency for case (IV) converges towards that of the case (III).

Figure 3(b) depicts that efficiency of the optimization is same for the cases (V) and (VI), although there are huge differences in the computational cost for the optimization process. Case (V) considers rewiring performed in both layers and case (VI) has only one layer being rewired. Equation (2) explains this behaviour since for both cases the $R$ values depend on $\lambda_{max}$ which is only determined by the layer having the stronger intra-layer coupling strength going through rewiring for both the cases. Finally, we find that the results about the efficiency for cases (III) and (IV) are valid for the denser networks as well (Fig. 3(c)). The one difference as compared to the sparser networks is that the efficiency is equal for both cases having larger values of $D_x$. Again, this behaviour arises due to the nature of Eq. (3), an equation that becomes more accurate for larger values of $D_x$ i.e., for the multiplex networks having stronger inter-layer couplings.

To study the dependence of the optimization process on the topology of one fixed layer, we consider the initial fixed layer constructed by the Watts-Strogatz model with various rewiring probabilities $p_r$. The small-world transition (Fig. 2(a)) for the Watts-Strogatz model is characterised by a clustering coefficient as high as that of the regular network and the characteristics path length being as small as that of the random networks. For an ER network representing the layer going through the optimization during the optimization process, and for small values of $p_r$, typically smaller than the SW transition, the initial and the optimized multiplex networks have both the same synchronizability (Fig. 2(b)). For $p_r$ larger than the value for the SW transition, synchronizability of both the initial and the optimized multiplex networks start increasing and attains its maximum value (the lowest $R$ value) at a rewiring probability which is much higher than the critical parameter for the SW transition $p_c$, but much smaller than $p_c = 1$. Such a dependence of synchronizability on $p_r$ is the result of an interplay between the degree homogeneity of the fixed layer and the layer going through the optimization. Initially for a $p_r$ being smaller than the value for the SW transition, the diameter of the fixed layer is large resulting in a poor synchronizability of the entire multiplex network. For $p_r$ being greater than the value for the SW transition, as long as the fixed layer has still small degree heterogeneity, the optimized multiplex networks possess the following topological characteristics contributing to better synchronizability; (1) degree homogeneity for both the fixed layer and the layer experiencing the rewiring (i.e., the distribution of degrees is not broad), (2) small values of both the average path length and the diameter of the entire multiplex networks. For the fixed layer generated with $p_r = 1$ or close to 1, though the diameter and the average path length of the entire network are still small, the degree heterogeneity of the fixed layer is high enough which does not get balanced by the rewiring of another layer during the optimization process, resulting in a smaller synchronizability of the optimized network. The value of $p_r$, corresponding to the maximally synchronizable network achieved through the optimization process, decreases as the average degree of the initial networks increases. This shift in $p_r$ towards the lower values arises due to the fact that for denser networks, even very small rewiring probability values are sufficient to destroy the degree homogeneity of the initial fixed layer, having a similar impact on the synchronizability of the final evolved network.

Moreover, optimization of denser networks leads to a less synchronizable evolved networks than those achieved by optimizing sparser networks, since denser networks possess a larger amount of mismatch in the inter and the intra-layer connections. For the sparser networks, the efficiency of synchronizability is high for a very large range of $p_r$. However, denser networks reflect comparatively a lesser efficiency of the optimization, i.e., smaller values of $R_{norm}$ (Fig. 2(c)), as the fixed layer restricts the value of $R$ to decrease beyond a limit even though the second layer is rewired to enhance the synchronizability of
the entire multiplex network.

Further, to study the impact of change in the structural properties of the fixed layer on the efficiency of optimization, we consider the fixed layer being represented by ER random and scale-free networks. Fig. 3(a) depicts that there is a decrease in $R$ with an initial increase in $D_x$. With a further increase in $D_x$, $R$ starts increasing for the case of ER representing the fixed layer. For the fixed layer being represented by a scale-free network, $R$ first decreases with an initial increase in the value of $D_x$, and after attaining a minimum value it remains almost constant for a further increase in $D_x$ or for larger $D_x$ values. As $D_x$ increases further, $R$ finally starts increasing. Again, similar to the previous case of fixed layer represented by ER network, the networks with lower $D_x$ values are not optimizable (Fig. 3(b)). This result is in contrast to the behaviour exhibited for the un-restricted rewiring case. When both the layers are rewired, the networks are optimizable for all the $D_x$ values (Fig. 3(c)). Fig. 3(d) reflects that for the unrestricted rewiring, i.e. for rewiring taking place in both the layers, the efficiency of optimization is maximum for a certain value of $D_x$ after which it again decreases. Interestingly, $D_x$ for which efficiency is maximum is shifted towards a larger value for the case of fixed layer being represented by ER random networks which also corresponds to the maximum efficiency. There is more shift towards a larger value for the case of fixed layer represented by the SF networks. The reason behind this shift is that the local minima of $R$ gets shifted towards a higher value of $D_x$ for the layer having the scale-free architecture [12].

Conclusion: Our results show that there are several pathways to improve synchronizability of multiplex networks, either by altering parameters such as those that promote integration of the layers (increasing the inter-layer coupling strength), or by evolving the network topology by rewiring edges within layers, under an optimization process. The surprising result is however that optimization of a single layer can achieve networks that are roughly as capable to synchronize as networks where all the layers are evolved under similar optimization criteria. This result is particularly relevant to works intended to improve synchronization of systems where only one layer is accessible or when one wants to optimize a system in a very cost effective fashion. Having in mind that real-world systems are very large, complex, and composed by many layers, our work points that optimization in such systems can indeed be carried out.

We have also studied the effectiveness of the optimization process, measured by the network synchronizability achieved through the evolution process, when the initial pre-evolved networks have different initial topologies. We found that the optimization leads to the maximum synchronizable multiplex networks when the fixed non-evolved layer has a topology lying in between a network with incipient small-world and fully random topologies.

Networks theory has proven its aptness in providing insights into controllability at a fundamental level. The controllability is desirable for dynamical behavior associated with the functionality of real-world systems. In traditional approaches, external inputs are imposed to affect the dynamics of few nodes which further causes a control of the entire system [18]. Our work might refine the concept of controllability by addition of a new system (one layer) that changes the dynamical evolution of the entire system (multiplex) to a desired behavior. Further, our work might complements works on controllability by creating more synchronous evolved networks that could be more controllable.

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