Theories with maximal acceleration

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Abstract

Maximal accelerations are related to the existence of a minimal time for a given physical system. Such a minimal time can be either an intrinsic time scale of the system or connected to a quantum gravity induced ultraviolet cut off. In this paper we pedagogically introduce the four major formulations for kinematics accounting for a maximal acceleration. Some phenomenological repercussion are offered as hints for future investigations.

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1 Introduction

The search of a consistent unified framework incorporating quantum theory and gravity has been a fundamental issue for the development of theoretical physics. After many discussions and research, it seems natural to think that such a theory should come together with a modification of fundamental notions of physics. One of such notions could be the spacetime arena, the dynamical framework where physical description are setup.

In this context, the existence of a maximal acceleration has been demonstrated in various dynamical theories of quantum gravity, as in covariant loop quantum gravity [1] and in string theory [2]. Generally, the existence of a minimal scale in quantum gravitational models is related with the existence of an universal maximal acceleration [3, 4]. Then one could wonder if these dynamical effects can be implemented in a modification of the spacetime structure itself, namely, in the form of a new kinematical geometry of maximal acceleration.

The notion of maximal acceleration appeared also in non-linear theories of classical electrodynamics [5], in relation with the generalized uncertainty principle [6] and in theories addressing the problem of radiation-reaction [7, 8], just to mention some examples. This suggests that there must exist a general kinematical formalism to accommodate a maximal acceleration in different theories.

Indeed, during the last decades there has been steady interest on the hypothesis of a maximal acceleration in Nature. By this we mean the general idea that the
proper accelerations of test particles are bounded with respect to a given spacetime structure. The origin of this idea can be traced back to the foundational work of E. Caianiello [9, 10] and H. E. Brandt [11]. In particular, Caianiello and his collaborators showed many interesting consequences of the existence of a maximal acceleration in several areas of theoretical physics. Among the suggested consequences, there is the modification of the behaviour of singularities in cosmological models [12], in black hole solutions [13], showing the absence of absolute collapse (collapse to a point) of an extended gravitational body and the possibility of regularization of perturbative quantum field theory, specially for the effects on the structure of the propagators that maximal acceleration has [14]. These examples illustrate the relevance of the kinematical approach to maximal acceleration and provide additional motivation for a thorough investigation of the idea.

1.1 Arguments in favour of maximal proper acceleration

There is a simple heuristic argument in favour of the existence of maximal proper accelerations. Let us consider a physical system with a minimal time \( \delta \tau \). The latter can be a characteristic time scale of the system or due to a natural ultraviolet cut off emerging from the background spacetime at Planckian energies [15, 16]. If the system is relativistic, the lapse of time is associated to a proper time. Then in this case, the change in speed that the system can have is bounded by the speed of light divided by \( \delta \tau \),

\[
|h(a, a)| \leq \frac{c^2}{(\delta \tau)^2},
\]

where \( h \) is a Lorentzian metric background with signature \((1, -1, -1, -1)\) and \( a \) is the 4-acceleration. This argument is a direct generalization of Caldirola’s argument [17] from his theory of maximal acceleration for the extended electron [18].

Note that the above argument is substantiated on the existence of the fundamental time lapse \( \delta \tau \) and such lapse could depend on the specific interaction producing the acceleration or the characteristic of the accelerated system. Paraphrasing A. Feoli [19], there could be several maximal accelerations. In the case that such maximal accelerations are associated not to the specific system, but to the interactions, we can speak of maximal proper accelerations in electrodynamics, in quantum gravity or for extended objects dynamics, to put some examples. Indeed, the maximal acceleration could depend on the mass of the system [5, 7, 9] or it could be universally defined [1, 2, 11], for instance when associated with the Planck scale.

From the above discussion in turns out that the existence of an universal maximal acceleration is only consistent if there is an universal hierarchy in the accel-
erations and an adequate scaling of the maximal acceleration with the mass of the system.

Apart from this argument, the existence of an universal maximal acceleration has the following consequence. If the weak equivalence principle holds good, the existence of such scale implies the existence of a maximal gravitational field. Thus if the invariant expression of the gravitational field is the Ricci curvature, then the existence of the maximal acceleration implies bounded Ricci curvature.

1.2 Examples of theories with maximal acceleration: a quick overview

The first instance where the idea of proper maximal acceleration appeared was in the work of E. Caianiello [9]. In his theory, each quantum particle of mass $m$ has heuristically associated a proper maximal acceleration whose modulus is given by

$$a_{\text{max}} = \frac{\mu c^2}{m \lambda},$$

where $\lambda$ is a quantity with dimension of length and $\mu$ a quantity with dimensions of mass, the interpretation of which is provided by Caianiello’s theory [9]. A similar expression is also derivable by direct application of Heisenberg’s uncertainty principle [10].

A different class of relations are obtained in models of quantum gravity. Thus for universal acceleration associated with the Planck scale, the expressions of the maximal acceleration\(^1\) are of the form [1, 2, 11]

$$A_{\text{max}} = 2 \pi \alpha \left( \frac{c^7}{\hbar G} \right)^{1/2},$$

where $\alpha$ is a constant of order 1. This maximal acceleration is of order $A_{\text{max}} \sim 10^{52} \text{m/s}^2$.

In classical charged particle electrodynamics, some modifications of the Lorentz force equation imply that the maximal acceleration is bounded by an expression of the form [8]

$$a_{\text{max}} \sim m/q^2,$$

where $m \neq 0$ is the mass of the particle and $q \neq 0$ its charge. In [8] this bound is obtained from the modified Lorentz-Dirac equation,

$$m \ddot{x} = q F_{\nu}^\mu \dot{x}^\nu - \frac{2}{3} q^2 h_{\rho \sigma} \dot{x}^\rho \dot{x}^\sigma \dot{x}^\mu$$

\(^1\)If the maximal acceleration in question depends on the characteristics of the systems, like mass or charge, it will be denoted by $a_{\text{max}}$; if the maximal acceleration is an universal constant, independent of the system, it will be denoted by $A_{\text{max}}$.\]
in the regime where the modulus of the proper acceleration is much smaller than the maximal acceleration, where \( h \) is a Lorentzian metric. The non-linear terms of the modification implies the condition

\[
m^2 |a|^2 = |F_L|^2 \left( 1 - \left( \frac{2}{3} \left( \frac{q}{m} \right)^2 \right)^2 |F_L|^2 \right),
\]

where \( |F_L| \) is the modulus of the external Lorentz force \( F_L^\mu \). Then the assumption \( F_L = ma + \) higher order terms and that \( F_L^\mu \) must be spacelike, implies the condition (1.4) in the regime where the acceleration is very small compared with \( a_{\text{max}} \). The bound (1.4) coincides with the expression for the maximal acceleration found in Caldirola’s theory of the electron [17].

Another example where maximal acceleration appears is in Born-Infeld non-linear theory of electrodynamics. In this case, the maximal acceleration is given by the expression [5]

\[
a_{\text{max}} = \frac{q}{m} b^{-1}, \quad (1.6)
\]

where \( b \) is the coupling of the Born-Infeld theory, a constant independent of the particle [20].

Note the consistency of the limit \( q \to 0 \) in the expression (1.5): when the charge is neutral, there is no acceleration under an external electromagnetic field. Thus the maximal acceleration in such situations must be zero. Similarly, there is consistency with the limit \( b \to 0 \) in (1.6).

We also observe that the different expressions above for the maximal accelerations are different and more importantly, have a different dependence on the mass-charge \((m, q)\) pair in different ways. This observation can be the basis for possible test of different models of point electrodynamics [8].

1.3 Aim and scope of the present work

The present article has the purpose to give a critical and short overview of several relevant theories of maximal acceleration. It is well known that Lorentzian geometry and general relativity do not contain a maximal proper acceleration in their kinematical formalism. Thus if proper acceleration is bounded by a dynamical mechanism and such mechanism is of universal character, it could imply a modification of the kinematical theory itself [5, 7, 9, 11]. We will discuss these kinematical theories of maximal acceleration in this paper.

It is not the aim of this work to offer an comprehensive overview of the investigations in the last decades concerning maximal acceleration. We focus the attention on a particular argument line, namely, the attempts to find a complete
and consistent kinematical theory with maximal proper acceleration. For other accounts on maximal acceleration, the reader can have a look at [19, 21], for instance. We will also not discuss the current status for the experimental search of maximal acceleration. Several proposals explore the transverse doppler effect and the corresponding bounds on the value of maximal acceleration obtained by application of Mössbauer spectroscopy. The interested reader could find further information and developments in [22–24]. Other proposal to test the existence of maximal proper acceleration(s) include deviations for the relativistic Thomas’ precession law [25].

1.4 Notation

In the next sections, we will adopt the following notation and symbols. The four dimensional manifold will be indicated by \( M \), while \( M_4 \) stands for a manifold diffeomorphic to \( \mathbb{R}^4 \). The symbol \( TM \) indicates the tangent bundle of \( M \) and \( T^*M \) the co-tangent bundle. A generic Lorentzian structure signature \((1, -1, -1, -1)\) will be \((M, h)\), while the particular case of the Minkowski metric will be denoted by \( \eta \). Greek indices run from 0 to 3. Equal up and down indices will be understood as contracted. If a local coordinate system for \( M_4 \) is given by the local coordinate functions \( \{x^\mu, \mu = 0, 1, 2, 3\} \), then the associated local coordinates on \( TM \) are given by the functions metric in local coordinates on \( TM \) given by \( x^A \equiv (x^\mu, \frac{\hbar}{mc^2} \dot{x}^\mu). \) The acceleration with respect to \( h \) will be denoted by \( a^2 \), namely, \( a^2 = h(D\dot{x}, D\dot{x}) \), where \( D \) is the covariant derivative of the Levi-Civita connection of \( h \).

2 Kinematic theories with maximal acceleration

In this section we describe some of the kinematical theories of maximal acceleration, including some critical remarks on them.

2.1 Caianiello’s theory of maximal acceleration

E. Caianiello introduced the idea of a maximal proper acceleration [9] as a natural consequence of his geometric formulation of quantum mechanics. The geometrization of quantum mechanics proposed by Caianiello is based on a metric structure defined on the co-tangent space \( T^*M_4 \). If \( \eta \) is the Minkowski metric on \( M_4 \), then there is a natural metric on \( TM_4 \),

\[
g_s = \eta \oplus_\alpha \eta^*. \tag{2.1}
\]

Here \( \alpha \) is a constant related with the value of the maximal acceleration, \( \eta^* \) is the metric acting on the fiber space \( \pi^{-1}(x) \) and \( \oplus_\alpha \) means the weighted direct sum
operation. In local coordinates, the metric from Caianiello on $T^*M_4$ is given by the line element \[ c^2 ds^2 = c^2 dt^2 - d\vec{x}^2 + \frac{\hbar^2}{\mu^4 c^4} \left[ \frac{1}{c^2} dE^2 - d\vec{p}^2 \right], \] where $\mu$ is a constant indicating a characteristic mass of the system.

The argument for the maximal acceleration from Caianiello follows from the criteria that for a massive particle the proper acceleration must be a spacelike four vector with respect to the metric (2.2),

\[ c^2 - \vec{v}^2 + \frac{\hbar^2}{\mu^4 c^4} \left[ \frac{1}{c^2} \left( \frac{dE}{dt} \right)^2 - \left( \frac{d\vec{p}}{dt} \right)^2 \right] \geq 0. \] (2.3)

The evaluation of the left hand side using special relativity implies the bound

\[ c^2 - \vec{v}^2 + \frac{\hbar^2}{\mu^4 c^4} \left[ 1 - \frac{m^2 c^2}{(c^2 - \vec{v}^2)^3} \right] \geq 0. \]

Therefore, the proper acceleration $\vec{a}$ is bounded by the expression (1.2).

Several direct implications of the theory were discussed by Caianiello. Perhaps, the most relevant consequence is the fact that the existence of a maximal acceleration avoids the total collapse of a black hole [9].

There is another argument, also proposed by Caianiello (see also [11]) that shows the need of a maximal proper acceleration. This second argument is based on Heisenberg uncertainty principle [10]. The starting point is the relation

\[ \Delta E \Delta f(t) \geq \frac{\hbar}{2} \left| \frac{df}{dt} \right|, \]

where here $t$ is an arbitrary time parameter. If $\Delta E \leq E$, $f = v$ and the relativistic constraint $\Delta v \leq c$ holds, then from the relativistic relation $E = mc^2$ and applied to the coordinate system where the particle is at rest, we have that the proper acceleration $a$ must be bounded by a maximal value given by

\[ a_{\text{max}} = 2 \frac{me^3}{\hbar}. \] (2.4)

Compared with the expression (1.2), this value of the maximal acceleration depends on the mass $m$ of the particle, instead of the characteristic mass $\mu$. This derivation of the value of maximal proper acceleration do not depend upon the existence of a quantum of time $\delta \tau$ or characteristic length $\lambda$. 

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A different formulation of the theory of maximal acceleration of Caianiello is found in [26]. There, the flat spacetime manifold $M_4$ is embedded in the tangent space $TM_4$. The tangent space has a natural Sasaki-type metric of the form

$$g_S = \eta \oplus_\alpha \eta.$$  \hspace{1cm} (2.5)

In this framework, it is postulated the proper time measure by physical clocks is given by the line element

$$ds^2 = g_{AB} dx^A dx^B = \eta_{\mu\nu} dx^\mu dx^\nu + \frac{\hbar^2}{4m^2c^6} \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu. \hspace{1cm} (2.6)$$

Physical particles have associated a physical vector velocity field. Then for a massive particle, the induced proper time element in $M_4$ is given by [26]

$$d\tau^2 = ds^2 \left( 1 - \frac{\hbar^2}{4m^2c^6} |\eta_{\mu\nu} \dddot{x}^\mu \dddot{x}^\nu| \right). \hspace{1cm} (2.7)$$

In this expression, derivatives are taken with respect to $ds$, the proper time of the Minkowski metric $\eta$. It follows that the value for the maximal acceleration given by Caianiello’s formula (2.4). The requirement that the proper time of a massive particle is positive or zero is translated now to the condition

$$1 - \frac{|\eta_{\mu\nu} \dddot{x}^\mu \dddot{x}^\nu|}{m^2} \geq 0,$$

which implies the existence of a bound for the proper acceleration $a^2 = \eta_{\mu\nu} \dddot{x}^\mu \dddot{x}^\nu$ given by the maximal acceleration (1.2).

Let us remark that the maximal accelerations (2.1) and (2.5) are different and henceforth, provide different theories of maximal acceleration.

An interesting example discussed in [26] is the case of the modification of the Rindler metric. In standard coordinates, the Rindler $1 + 1$ spacetime metric is

$$ds^2 = \chi^2 d\kappa^2 - d\chi^2,$$

with $-\infty < \kappa < +\infty$, $0 < \chi < +\infty$. This metric is modified in Caianiello’s theory to the form

$$ds^2 = (\chi^2 - m^{-2}) d\kappa^2 - d\chi^2. \hspace{1cm} (2.8)$$

This is not a flat metric; the scalar curvature is

$$R = -\frac{2}{m^2} (\chi^2 - m^{-2})^{-2}, \hspace{1cm} (2.9)$$
which has a singularity at the modified Rindler horizon $\chi = m^{-1}$. This is an example of how the existence of a maximal acceleration modifies the causal structure of the spacetime and also prevents the system from collapsing to a single point. It was shown that the existence of the singularity at the horizon implies the emergence of an effective repulsive force, that avoids any test particle to penetrate the interior and the system to collapse [26].

Caianiello’s theory has many other significant consequences. Among them there is the appearance of a deflection mechanism in extended object cosmology, that prevents the formation of singularities at the cosmological level [12], the formation of a shell out of the horizon in the quantum geometry modifications Schwarzschild black body solution [13] and the Kerr spacetime type solution [27], violations of the weak equivalence principle [28]. All these modifications vanish in the limit $\hbar \to 0$. In this sense, these corrections have a quantum origin.

Another type of consequence comes from the application to the regularization of quantum field theories [14]. This is because the Lagrangian of particle moving with maximal proper acceleration has associated a Green function with a higher order momentum in the denominator. It is plausible that such quantum field theories could be ultra-violet finite and at the same time, being compatible with special relativity.

2.2 A critical view on Caianiello’s theory

Despite its far reaching consequences, the formulation of Caianiello’s theory has several dramatic limitations. Let us start by considering the theory of maximal proper acceleration geometry developed in [26], in particular the expression in local coordinates for the metric of maximal acceleration (2.7). This expression is not general covariant, which could be problematic for any theory aimed to embrace also gravity. The root of the problem is found in the definition of the Sasaki type metric (2.6).

In order to have a covariant Sasaki metric, it is necessary to introduce a non-linear connection. Let us consider the element of the form [29, 30] for a generic Lorentzian metric $h$ on $M$, generalizing the construction starting from the Minkowski metric $\eta$,

$$g_s = g_{AB} dx^A \otimes dx^B = h_{\mu\nu} dx^\mu \otimes dx^\nu + h_{\mu\nu} \left( \delta x^\mu \otimes \delta x^\nu \right).$$

(2.10)

In this expression the 1-forms $\delta x^\mu$ contain corrections due to the non-linear connection that makes the expression properly covariant under local coordinate changes. An introduction to these geometric notions can be found in [30] and a comprehensible treatments in [31, 32]. The notion of non-linear connection is similar to the
usual notion of affine connection, that allows for covariant differential of sections of vector bundles\(^2\).

The induced proper time element along congruences of world lines on \(M\) of the metric (2.10) is of the form

\[
d\tau^2 = \left( 1 - \frac{|(D_x \dot{x})^\sigma (D_x \dot{x})_\sigma|}{a_{\text{max}}^2} \right) ds^2.
\]

(2.11)

Note that for \(|(D_x \dot{x})^\sigma (D_x \dot{x})_\sigma| < a_{\text{max}}^2\), this expression is well defined. The covariant derivative here is the one associated with the Levi-Civita connection of the metric \(h\), although it could be in principle associated to any affine connection on \(M\). This construction is valid for an arbitrary Lorentzian manifold \((M,h)\), not only for Minkowski spacetime. Also, we did not specify the value of the maximal acceleration parameter \(a_{\text{max}}\).

The second main problem that we find with the above construction, perhaps deeper than the previous one, is that Caianiello’s maximal acceleration geometry (2.6) and the Sasaki type metric (2.10) need of an underlying Lorentzian structure \((M,h)\). The proper time can be either constructed from the underlying Lorentzian spacetime \((M,h)\) or from the metric of maximal acceleration (2.6). Therefore, the existence of more than one metric structure implies a dichotomy. If \((M,g)\) is the physical structure (by assumption, it determines the physical proper time), then it is unclear how one can obtain the Lorentzian structure \((M,h)\) by an operational method.

### 2.3 Brandt’s differential geometric approach to maximal acceleration

In 1983, H. E. Brandt provided several heuristic arguments for the existence of a maximal universal acceleration [11]. One of these arguments started considering Sakharov maximal temperature [33]. The maximal temperature that a system can have in equilibrium with black body radiation turns to be given by

\[
T_{\text{max}} = \frac{\alpha}{k} \sqrt{\frac{\epsilon \hbar}{G}},
\]

(2.12)

where \(k\) is the Boltzmann’s constant and \(\alpha\) a dimensional number of order unity. On the other hand, for an accelerated frame with proper acceleration \(a\), the co-moving observers experiment a thermal bath in vacuum with temperature [34, 35]

\[
T = \frac{\hbar a}{2 \pi k c}.
\]

(2.13)

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\(^2\)Given a vector bundle \(\mathcal{E}\) with canonical projection \(\pi : \mathcal{E} \to M\) on the base manifold \(M\), a section \(S\) is a map \(S : M \to \mathcal{E}\) such that \(\pi \circ S\) is the identity map \(Id : M \to M\).
Therefore, the maximal acceleration that a system can have is given by the expression (1.3). Remarkably, Caianiello and Landi argued how from the expression of maximal acceleration in Caianiello’s theory one can re-derive Sakharov’s maximal temperature [36].

Motivated by the above argument, Brandt developed a geometric approach to systems where the proper acceleration is bounded by a maximal acceleration $A_{\text{max}}$. If for a world line of a physical particle $a \leq A_{\text{max}}$, then

$$|(D_\dot{x})^\sigma (D_\dot{x})_\sigma| \leq A_{\text{max}}^2.$$ 

This relation was interpreted in [29] in terms of the positiveness of the bilinear form

$$h_{\mu\nu} \, dx^\mu \, dx^\nu + \frac{1}{A_{\text{max}}^2} \, h_{\mu\nu} \left( d\dot{x}^\mu + \Gamma^\alpha_{\alpha\beta} \, \dot{x}^\alpha \, dx^\beta \right) \left( d\dot{x}^\nu + \Gamma^\gamma_{\nu\delta} \, \dot{x}^\delta \, dx^\gamma \right), \quad (2.14)$$

where here $\Gamma^\alpha_{\beta\delta}$ are the connection coefficients of the affine connection $D$, usually taken the Levi-Civita connection of $h$ and it is assumed that $A_{\text{max}} > 0$, that is, $|a| < A_{\text{max}}$. The expression (2.14) is the bilinear, symmetric form defined on $TM$. Thus if one also assumes that this form is non-degenerate, it defines a metric on $TM$,

$$G = G_{AB} \, du^A \, du^B, \quad (2.15)$$

where $u^A = (x^\mu, \dot{x}^\mu)$ are natural coordinates in $TM$. In natural coordinates, the metric $G$ has the following matrix components,

$$G_{AB} = \begin{pmatrix} h_{\mu\nu} + \frac{1}{A_{\text{max}}^2} \, \Gamma^\alpha_{\lambda\mu} \, h_{\alpha\beta} \, \Gamma^\beta_{\delta\nu} \, \dot{x}^\lambda \, \dot{x}^\delta & - \frac{1}{A_{\text{max}}} \, h_{\alpha\nu} \, \Gamma^\alpha_{\delta\mu} \, \dot{x}^\delta \\ - \frac{1}{A_{\text{max}}} \, h_{\alpha\mu} \, \Gamma^\alpha_{\delta\nu} \, \dot{x}^\delta & h_{\mu\nu} \end{pmatrix}. \quad (2.16)$$

Except for the value of the proposed maximal acceleration, the metric (2.15) coincides with the so-called Sasaki type metric discussed in reference [30]. Also note that in the case the metric $h$ is the Minkowski metric $\eta$, then (2.15) metric coincides with Caianiello’s metric (2.5). However, Brandt’s theory is a manifestly general covariant theory.

Brandt’s theory is based upon the assumption that the physical metric, the one that is testable by experiments performed by macroscopic observers, is given by (2.15). Brandt also considered the geometric theory for the metric (2.15) and formulated the corresponding field equations, generalizations from Einstein equations of general relativity. Also, an interpretation of the metric (2.15) as a Kaluza-Klein type metric was explored [29]. Brandt formulated field theories in this framework
during the 90’s, showing that many of the fundamental concepts of modern field
theory could be formulated in his theory (see for instance [37] and references there).

However, the starting point in the construction of the metric (2.15) is the space-
time metric $h$ defined on $M$. By the assumptions of the theory, $h$ is not the metric
that should be reconstructed by operational measurements. Indeed, one needs
this metric $h$ as a back-ground structure defined on $M$, prior to the construction
of $G_{AB}$. Thus, analogously as in Caianiello’s theory, one has in Brandt’s theory
a dichotomy between $(M, h)$ and $(TM, G)$, since both structures can be used to
define observables, for instance, proper time for a generic world line $x : I \to M$.
It is unclear what is the physical meaning of $h$ in Brandt’s theory.

2.4 Schuller’s algebraic formulation of Born-Infeld theory

F. P. Schuller formulated a kinematical theory of maximal acceleration in [5],
motivated by Born-Infeld theory [20]. It is based on a pseudo-complex extension of
Lorentzian geometry as follows. The commutative ring of pseudo-complex numbers
is defined by the set

$$\mathbb{P} = \{ a + Ib \mid a, b \in \mathbb{R} \}$$

equipped with addition and multiplication laws induced by those on $\mathbb{R}$ and such
that $I$ is a pseudo-complex structure, namely, the relation $I^2 = 1$ holds. There
is a matrix representation of $\mathbb{P}$. Thus if $u = a + Ib \in \mathbb{P}$, then the matrix
representation is such that

$$1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where the operations on $\mathbb{P}$ correspond to the matrix operations.

A $\mathbb{P}$-module is an algebraic structure analogous to a vector space, but where the
coefficients in the linear combinations are taken with respect to a ring $\mathbb{P}$, instead
than to a numerical field, for instance the real numbers $\mathbb{R}$. It can be shown
that fundamental constructions, like $\mathbb{P}$-extension of a Lie algebra\(^3\), exponential
map, etc... carries over the $\mathbb{P}$-module representations in a closer form to the real
Lie algebra theory over real vector spaces [5]. In particular, the pseudo-complex
Lorentz group

$$O_p(1, 3) \equiv \{ \Lambda \in \text{Mat}(n, \mathbb{P}) \text{ s.t. } \Lambda^\top \eta \Lambda = \eta, \det \Lambda = 1 \}$$

\(^3\)If $L$ is a real Lie algebra, a $\mathbb{P}$-extension is an algebra of the form $\alpha_1 + \alpha_2 I$, where $\alpha_1, \alpha_2 \in L$
and the bracket operation is extended by linearity.
will play a relevant role in the generalized kinematics. At the level of the Lie algebra, one has the identity

$$so_P \cong so_\mathbb{R}(1,3) \oplus so_\mathbb{R}(1,3).$$  \hspace{1cm} (2.18)

The Lie group $O_P(1,3)$ leaves invariant the bilinear form

$$\eta_P : TV_P \otimes TV_P \to \mathbb{P},$$  \hspace{1cm} (2.19)

induced from the Minkowski metric defined on $M_4$, where here $V_P$ is the pseudo-complexification of the vector space $M_4 \cong \mathbb{R}^4$ and $TV_P$ is its tangent space. Thus one can see that $V_P \cong TM_4$ by comparing the corresponding real dimensions. In terms of the representation of $V_P \cong TM_4$, a point $v \in V_P$ is identified with $(x, \dot{x})$. Therefore, a generic point of $TV_P$ is identified with $(\dot{x}, \ddot{x}) \in TT_{(x,\dot{x})}M_4$.

The bilinear form $\eta_P$ defines the metric of maximal acceleration through two additional postulates:

- For physical orbits on $TTM_4$, the metric $\eta_P(\dot{X}, \dot{X}) \geq 0$,
- The proper time of a physical orbit $X : [a, b] \to TM_4$ is given by the proper time of $\eta_P$.

It can be shown that this theory implies a consistent bound of the proper maximal acceleration with respect to $\eta$ [5]. Indeed, the line element associated to this "metric of maximal acceleration is of the form

$$dw^2 = \left(1 - \frac{a^2}{a_{\text{max}}^2}\right) ds^2.$$  \hspace{1cm} (2.20)

We observe that the structures (2.19) and (2.20) in Schuller’s theory are equivalent to the corresponding structures (2.5) and (2.7) in Caianiello’s theory. However, note that in Schuller’s theory $a_{\text{max}}$ is not fixed by Caianiello’s maximal acceleration (2.4). Instead $a_{\text{max}}$ is a free parameter with dimension of proper acceleration. In particular and since it was motivated by Born-Infeld theory, $a_{\text{max}}$ could be given by the expression (1.6).

Schuller’s theory can be implemented in curved spacetimes $(M, h)$, by a pointwise generalization on $M$ of the above algebraic construction. It is also consistent with a canonical quantization procedure. This contrasts with previous formulations of Caianiello’s theory based on complex structures compatible with the connection associated with the metric (2.5). Indeed, it was proved that under reasonable assumptions, Caianiello’s metric (2.5) is not consistent with quantization, except if the metric $h$ is flat [5]. Related with this, it could be of relevance to investigate whether the pseudo complex General Relativity of Hess and Greiner [38] provides a framework for curve spacetimes consistent with maximal acceleration.
Schuller’s theory is general covariant, independent of the local coordinate used to be formulated. However, the second problem discussed for Caianiello’s theory is also present in Schuller’s construction, since it depends upon an underlying Lorentzian structure \((M, h)\), an apparently ad hoc structure in the framework of metrics with maximal acceleration.

2.5 An effective theory of metric geometries with maximal acceleration and jet geometry

The notion of proper maximal acceleration is not necessarily linked with the quantum description of physical systems. Indeed, since the proper acceleration is defined as the derivative of the four-velocity with respect to the proper time along a given world line, it seems more appropriate to think the concept of maximal proper acceleration from a classical point of view.

It also seems clear that there are different notions of maximal proper acceleration and that such notions depend upon the system and the dynamics involved. Therefore, it is natural to leave un-specified the value of the maximal acceleration \(a_{\text{max}}\) in the search of classical geometric frameworks for maximal proper acceleration.

One of the fundamental features of Caianiello’s, Brandt’s and Schuller’s theories is that the proper time depends not only on the instantaneous value of the speed with respect an inertial or free falling coordinate system, but also on the instantaneous four-acceleration. This is a violation of the so-called clock hypothesis in the theory of relativity [39], when the problem of the description physical phenomena in an accelerated coordinate system is considered.

On the other hand, violations of the clock hypothesis are expected to happen in physical situations where radiation reaction effects are of relevance [40, 41]. These situations are of particular relevance in electrodynamics, where the standard equation of motion of a point charged particle [42] present serious theoretical difficulties.

It is in the above context that the theory developed in [7] must be interpreted. The fundamental idea is that in the treatment of certain physical processes, the clock hypothesis could be violated, according to B. Mashhoon [40, 41]. Therefore, the natural generalization of the proper time must depend upon the acceleration. The metric of maximal acceleration must be a fundamental physical element. The fundamental idea of the theory is that the mathematical description of a metric of maximal acceleration of the type (2.7) is a geometric structure whose components live on the second jet bundle over \(M\). This is the bundle determined by the set

\[ J^2_0(M^4) \equiv \{(x, \dot{x}, \ddot{x}), x : I \rightarrow M^4 \text{ smooth}, 0 \in I \subset \mathbb{R}\}, \]

14
where the coordinates of a given point \( u \in J_0^2(M_4) \) are of the general form

\[
(x, \dot{x}, \ddot{x}) = \left( x^\mu(s), \frac{dx^\mu(s)}{ds}, \frac{d^2x^\mu(s)}{ds^2} \right)
\]

and by the canonical projection \( \pi_2 : J_0^2 \rightarrow M_4, (x, \dot{x}, \ddot{x}) \mapsto x \).

We recall here that jet theory is a framework to systematically deal with Taylor expansions of functions on manifolds and sections. Roughly speaking, the \( k \)-jet of a function or section at a given point corresponds to a Taylor expansion up to order \( k \), without considering the remaining term of order \( k+1 \). For instance, the \( k \)-jet of a curve

\[
\gamma : (-a, a) \rightarrow M
\]
at the point \( \gamma(0) \in M \) is

\[
(\gamma^\mu(0), \dot{\gamma}^\mu(0), \ddot{\gamma}^\mu(0), \ldots, \gamma^{\mu(k)}(0)),
\]

where

\[
\dot{\gamma}^\mu = \frac{d\gamma^\mu}{ds}, \quad \gamma^{\mu(k)} = \frac{d^k\gamma^\mu}{ds^k},
\]
evaluated at the point \( \gamma(0) \) and \( s \) is the parameter of the curve. Different parameters determine different jets, which are, nevertheless related by strict transformation rules. The collection of possible jets, for instance, the collection of \( k \)-Taylor expansions of curves on a manifold, can be full of differentiable structure, which makes them manifolds. Furthermore, there are natural projections that provide further structure to such manifolds, such as fiber bundles. Therefore, one can speak of jet bundles [43].

Let us consider the general case where the spacetime manifold is \( M_4 \). Then the maximal acceleration metric it is postulated to be determined by a map that associates to each physical world line \( x : I \rightarrow M_4 \) a smooth family of scalar products, one at each point of the curve,

\[
\{ g(2x(t)) : T_{x(t)}M \times T_{x(t)}M \rightarrow \mathbb{R}, \ t \in I \}
\]
along the world line \( x : I \rightarrow M \) whose components live on the second jet lift \( ^2x : I \rightarrow J_0^2(M) \). This family of scalar products can formally be expressed in a general way as

\[
g(2x) = g^0(x, \dot{x}, \ddot{x}) + g^1(x, \dot{x}, \ddot{x})\xi(x, \dot{x}, \ddot{x}, a_{\max})^2,
\]
where dot derivatives \( \dot{x} \) are meant with respect to the proper parameter of the metric \( g_0 \).
We can consider limits when \( a_{\text{max}} \to +\infty \) in the family of metrics \( g(a_{\text{max}}) \). Then we require that metric obtained by the limit

\[
\lim_{a_{\text{max}}^2 \to +\infty} g(2x)
\]

is compatible with the clock hypothesis. We also assume that \( \xi(x, \dot{x}, \ddot{x}, a_{\text{max}}^2) \) is analytical in \( 1/a_{\text{max}}^2 \) and has the form

\[
\xi(x, \dot{x}, \ddot{x}, a_{\text{max}}^2) = \sum_{n=1}^{+\infty} \xi_n(x, \dot{x}, \ddot{x}) \left( \frac{1}{a_{\text{max}}^2} \right)^n.
\]

Then one can argue that

\[
\lim_{a_{\text{max}}^2 \to +\infty} g(2x) = g^0(x, \dot{x}, \ddot{x})
\]

and by compatibility with the clock hypothesis,

\[
g^0(x, \dot{x}, \ddot{x}) \equiv g^0(x, \dot{x}).
\]

The form \( g^0(x, \dot{x}) \) is non-degenerate, since \( g \) is non-degenerate. \( g^0 \) is also symmetric and bilinear. Therefore, \( g^0 \) is indeed a generalized Finsler metric [44]. If we make the further assumption that \( g^0 = h \) is Lorentzian and we assume the identifications

\[
g^0 = h, \quad g^1 = h, \quad \xi_1 = h(\ddot{x}, \ddot{x})
\]

then we have a generalization of the line element (2.7). In this case, the metric of maximal acceleration (2.21) can be expressed as

\[
g_{\mu\nu}(x) := \left[ 1 - \frac{\left| h(D\ddot{\dot{x}}(t), D\ddot{\dot{x}}(t)) \right|}{g^0(\dot{x}, \dot{x}) a_{\text{max}}^2} \right] h_{\mu\nu}
\]

where the curve \( x : I \to M \) is parameterized by the proper time parameter of \( g \) in the limit when \( a_{\text{max}} \) tends to zero. In practical examples, this limit metric coincides with \( h \). One example where this procedure is consistent is the model for point charged particles discussed in [8].

The main advantage of this formalism with respect to others kinematical theories of maximal acceleration rests on the fact that from the beginning, the theory is formulated in terms of the metric of maximal acceleration; the theory just discussed is based on the notion of maximal acceleration given by a family of metrics defied on the second jet. Thus, apparently, the above mentioned dichotomy in Caianiello, Brandt and Schuller’s theories is resolved. The metric \( h \) has only a formal definition and it is attached to \( g \) and not the other way around.

However, the theory discussed above is in principle an effective theory, since in principle the theory as it is formulated is limited to a range of validity when \( a << a_{\text{max}} \). A deeper treatment is missing.
3 Conclusions

The idea of maximal acceleration and the associated modification of the spacetime geometry has been around for long time. Being a modification of the fundamental ideas of the theory of relativity, the theoretical consequences of maximal acceleration are sound. However, the current state of the art shows important deficiencies, probably the reason of why maximal acceleration has been not yet attracted enough attention. The theories of maximal acceleration are such that either they lack of a general covariant formulation or if such formulation exits [7], then it has the limitation to be perturbative and with limited domain of validity, far from the region of maximal acceleration. Also, other formulations [5, 9, 29, 30] are rooted on a pre-existent Lorentzian metric back-ground, with the conceptual consequences that this entails. These problems, however, are likely to be technical problems.

There is also some relation between the idea of maximal acceleration in physics and the conjecture of maximal tension or force [45]. This conjecture asserts that in general relativity, there is a limit to the tension or force that a physical system can feel, a limit given by the expression

$$F \leq F_{\text{max}} = \frac{c^4}{4G},$$

where $G$ is Newton’s gravitational constant. The factor 1/4 in this expression can vary, in the sense that it could depend on the details of the dynamical system. For instance, in presence of a cosmological term in some models of black-holes, this factor is instead given by 1/9 [46].

A general argument supporting this conjecture is based upon considerations on black hole merging in general relativity and a form of cosmic censorship [45, 46]. It is in any case, a classical argument.

The reader should note that the notions of maximal universal force and of maximal acceleration are not equivalent in general relativity. In fact, the mathematical structure of general relativity does not contain a notion of universal maximal acceleration, since in such a framework, there is no a minimal universal length scale. Thus if the mathematical framework is kept Lorentzian or pseudo-Riemannian, the interplay between maximal acceleration and maximal force these implies the introduction of new principles, like the aforementioned existence of a minimal length in quantum gravity [45]. Maximal acceleration is, however, compatible also with classical models in the case of generalized geometric frameworks, like maximal acceleration geometry [7].

In a theory where the maximal acceleration could depend upon the system itself, like in Born-Infeld dynamics, it is not necessary to introduce quantum mechanics to make compatible maximal acceleration with maximal universal force. To the maximal force $c^4/4G$ corresponds the maximal acceleration $c^4/4Gm$. 
Similar related conjectures, like maximal power and maximal angular momentum [46, 47] could be related with maximal acceleration in analogous ways. All these conjectures are likely to be seen also in a kinematical version, where maximal acceleration is involved.

From the phenomenological point of view, it has been investigated possible phenomenological signatures for maximal acceleration [23–25]. However, these different results only provide lower bounds for the maximal acceleration. Another possibility has been discussed in [8], linked with a new theory of classical electrodynamics. It was predicted the strict decrease of the maximal acceleration with the size (charge and mass) of the charge particle.

Interestingly the existence of a maximal acceleration could have repercussions also on the physics of evaporating black holes [48] and the related Unruh effect that has been at the center of a recent debate in the literature [49].

We have discussed only some aspects of the main idea of having a kinematical theory with maximal acceleration. It is to be expected that such a profound and simple idea of maximal acceleration could have further interesting consequences, both at the phenomenological and theoretical levels of physical description.

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