Characterizing finitary functions over
non-archimedean RCFs via a topological definition
of OVF-integrality

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Abstract

When $R$ is a non-archimedean real closed field we say that a function $f \in R(\bar{X})$ is finitary at a point $\bar{b} \in R^n$ if on some neighborhood of $\bar{b}$ the defined values of $f$ are in the finite part of $R$. In this note we give a characterization of rational functions which are finitary on a set defined by positivity and finiteness conditions. The main novel ingredient is a proof that OVF-integrality has a natural topological definition, which allows us to apply a known Ganzstellensatz for the relevant valuation. We also give some information about the Kochen geometry associated with OVF-integrality.

1 Introduction

Let $R$ be a non-archimedean real closed field, and recall that the finite part of $R$ is the convex hull of $\mathbb{Z} \subseteq R$.

Definition 1.1 We say that a function $f \in R(\bar{X})$ is finitary at $\bar{b} \in R^n$ if on some neighborhood of $\bar{b}$ its values $f(\bar{x})$ are in the finite part of $R$ whenever defined.

The above definition is quoted in [2] (Definition 4.4 there) with the term ‘bounded’, which the author proposed originally. However in retrospect ‘finitary’ is more appropriate, e.g. note that the collection of functions which are finitary at a given point is not closed under multiplication by an infinite constant.
Let $\bar{p} = (p_1, \ldots, p_m)$ be polynomials in $R[\bar{X}]$, let $\bar{g} = (g_1, \ldots, g_l)$ be rational functions in $R(\bar{X})$, and define

$$T = T_{\bar{p}, \bar{g}} = \{ \bar{b} \in R^n \mid p_1(\bar{b}), \ldots, p_m(\bar{b}) > 0 \wedge g_1, \ldots, g_l \text{ are finitary at } \bar{b}\}$$

There is a canonical valuation $v$ on $R$ such that its ring of integers $O_R = \{ x \in R \mid v(x) \geq 0 \}$ equals the finite part of $R$. We wish to use a Ganzstellensatz on $(R, v)$ (see [3], Theorem 7.4) to get a characterization of functions which are finitary on $T$ (i.e. at every point of $T$). To this end we will need to show that a function is finitary at a point $\bar{b}$ if and only if it is OVF-integral there. Indeed we’ll prove below (Proposition 2.6) that for any RCVF the analogous topological definition is equivalent to OVF-integrality, this result being the main contribution of the present paper.

Note that the proof given in [2] (Lemma 3.6 there) of the harder direction of Proposition 2.6 seems to be erroneous - clearly the bounded sequence $f(b_n)$ need not have a convergent sub-sequence, but even if it does, the value of the limit $\ell$ might not equal the formula given there. For example consider the function $f(X, Y) = \frac{X}{X-Y}$, which is not OVF-integral at $(0,0)$. Let $b_n = ([b_n]_X, [b_n]_Y)$, and note that $v([b_n]_X) - \min\{v([b_n]_X), v([b_n]_Y)\} \geq 0$, hence the limit can not equal the negative value $v(\ell)$, contrary to the claim in [2]. The issue here is that any sequence $b_n$ satisfying $v(f(b_n)) < 0$ has to converge to $(0,0)$ from a certain direction (here the line $X = Y$), so one can’t assume its points are ‘generic’ relative to the numerator and denominator of $f$, and at any rate the required OVF-valuation near $(0,0)$ has to ‘point’ in the same direction. See the second part of Remark 4.2 for a finer view of the concept of ‘direction’ that we need to consider.

The last step is to note that the set $T$ is defined by OVF-integrality conditions, so is still not first-order definable, while the mentioned Ganzstellensatz deals with the following variant, which is (first-order) defined by naive integrality conditions:

$$S = S_{\bar{p}, \bar{g}} = \{ \bar{b} \in R^n \mid p_1(\bar{b}), \ldots, p_m(\bar{b}) > 0 \wedge v(g_1(\bar{b})), \ldots, v(g_l(\bar{b})) \geq 0\}$$

(where $v(g(\bar{x})) \geq 0$ is understood to also mean that $g$ is defined at $\bar{x}$).

Since being finitary is equivalent to OVF-integrality, the set $T$ is actually contained in the Kochen closure of $S$ (see [1], Remark 3.3), therefore this gap will not pose a real problem. However by Remark 4.2 the set $T$ need not equal this Kochen closure, contrary to what is claimed in [2] (the proof of Corollary 4.6 there).

We also discuss briefly the ‘Kochen geometry’ associated with OVF-integrality (see Definition 2.4), and demonstrate in Remark 4.2 its dependence on the ambient variety with some nice examples.
2 OVF-integrality and the Kochen geometry

Given a valued field \((K, v)\) we denote its valuation group by \(\Gamma_K\), its valuation ring by \(O_K = \{a \in K : v(a) \geq 0\}\), and its ideal of ‘infinitesimals’ by \(M_K = \{a \in K : v(a) > 0\}\).

Recall that an ordered valued field (or OVF) is a ordered field with a convex non-trivial valuation ring. The following definitions by Haskell and the author (see [1], Subsection 2.2) intend to refine the naive notion of integrality at a point (consider for example whether \(f(X, Y) = X^2 + Y^2\) should be integral at \((0, 0)\), as opposed to \(f(X, Y) = \frac{X}{Y}\) at the same point).

We begin with naming the valuations which interest us in the context of the OVF category. Let \(L\) be a field extension of \(K\). A valuation \(\tilde{v}\) on \(L\) which extends \(v\) is called an OVF-valuation if there exists some order \(\leq L\) on \(L\) such that \((L, \tilde{v}, \leq L) \models OVF\).

**Definition 2.1** [1] Let \((K, v, \leq K)\) be an OVF, \(\bar{b} \in K^n\). We will say that an OVF-valuation \(\tilde{v}\) on \(L = K(\bar{X})\) is near \(\bar{b}\) if for every \(f \in L\) such that \(f(\bar{b}) = 0\) and every \(\gamma \in \Gamma_K\) we have \(\tilde{v}(f) > \gamma\).

**Definition 2.2** [1] Let \((K, v, \leq K)\) be an OVF. Given \(f \in L = K(\bar{X})\) and \(\bar{b} \in K^n\) we say that \(f\) is OVF-integrable at \(\bar{b}\) if for any OVF-valuation \(\tilde{v}\) on \(L\) which is near \(\bar{b}\) we have \(\tilde{v}(f) \geq 0\).

For \(T \subseteq K^n\) we say that \(f\) is OVF-integrable on \(T\) if \(f\) is OVF-integrable at \(\bar{b}\) for every \(\bar{b} \in T\).

**Remark 2.3** By existence of OVF-valuations near \(\bar{b}\) (see e.g. [1], Proposition 4.2) it is easy to conclude that OVF-integrality is equivalent to naive integrality whenever \(f\) is defined at \(\bar{b}\).

We now give a few remarks about the ‘Kochen geometry’ associated with OVF-integrality (see [1], Remark 3.3).

**Definition 2.4** [1] For any \(Q \subseteq K^n\) we define the Kochen closure of \(Q\) to be the set of points \(\bar{x} \in K^n\) such that every function \(f \in K(\bar{X})\) which is OVF-integrable on \(Q\) is also OVF-integrable at \(\bar{x}\).

Note that the Kochen closure operation is not a ‘geometry’ in the strict sense of the term, i.e. it is not a matroid, since it does not have finite character. It is not hard to show that any closed set is also Kochen-closed, however the converse is false - see Remark 4.2 for some nice examples. On the line \(K^1\) the converse is true, i.e. a subset of the line is Kochen-closed exactly
when it is closed. Finally note that in dimensions higher than one there is no ‘Kochen topology’, i.e. it is trivial: in \( K^2 \) for example the union of \( V^{\text{int}}(\bar{X}) \) and \( V^{\text{int}}(\bar{Y}) \) (using the notation of [1], Remark 3.3) equals \( K^2 \setminus \{(0,0)\} \).

**Remark 2.5** We note in passing that in the context of ACVF the Kochen closure is not always contained in the closure, e.g. the Kochen closure of \( K^2 \setminus (O_K)^2 \) is \( K^2 \), since in this context the complement of a Kochen-closed set is never bounded.

### 2.1 Topological statement of OVF-integrality for RCVFs

Recall that a real closed valued field (or RCVF) is an OVF which is real closed. The theory RCVF is the model companion of the theory OVF. In this section we give a topological property which is equivalent to OVF-integrality over an RCVF.

**Proposition 2.6** Let \((K,v,\leq_K)\) be a RCVF, let \( f \in K(\bar{X}) \), and let \( \bar{b} \in K^n \). Then \( f \) is OVF-integral at \( \bar{b} \) if and only if there is some neighborhood \( U \) of \( \bar{b} \) such that for \( \bar{x} \in U \), if \( f(\bar{x}) \) is defined then it is in \( O_K \).

**Proof** First assume that \( f \) is not OVF-integral at \( \bar{b} \), i.e. there exists some OVF-valuation \( \bar{v} \) near \( \bar{b} \) such that \( \bar{v}(f) < 0 \). Let \( \gamma \in \Gamma_K \), and denote the \( \gamma \)-neighborhood of \( \bar{b} \) by \( U_\gamma = \{ \bar{x} \in K^n \mid \bigwedge_{i=1}^n v(x_i - b_i) > \gamma \} \). We need to find some \( \bar{x} \in U_\gamma \) such that \( f(\bar{x}) \in K \setminus O_K \) (and in particular is defined). However since \( \bar{v} \) is an OVF-valuation there is some order \( \leq_L \) on \( L = K(\bar{X}) \) such that \((L,\bar{v},\leq_L)\) is an OVF, and the tuple \( X \in L^n \) satisfies \( \bigwedge_{i=1}^n (v(X_i - b_i) > \gamma) \land v(f(X)) < 0 \land q(X) \neq 0 \) (where \( q \) is the denominator of \( f \)). This is a first-order formula over \((K,v,\leq_K)\), which is an existentially closed OVF by virtue of being a RCVF (see for example [3], Section 3). Hence there is some \( \bar{x} \in K^n \) satisfying the same formula, as required.

For the other (and harder) direction, assume that for every \( \gamma \in \Gamma_K \) there is some \( \bar{x}_\gamma \in U_\gamma(\bar{b}) \) such that \( f(\bar{x}_\gamma) \in K \setminus O_K \). Now let \( P = P(\bar{X}) \) be the partial type over the valued field \((K,v)\) which says that the tuple \( \bar{X} \): (i) is contained in \( U_\gamma(\bar{b}) \) for every \( \gamma \in \Gamma_K \); (ii) satisfies \( v(f(\bar{X})) < 0 \); (iii) is transcendental over the field \( K \); and (iv) satisfies \( v(g(\bar{X})) \geq 0 \) for every \( g \in \mathcal{I}_{\text{ord}} = \{ \frac{1}{1+r(\bar{x}_\gamma)} : r(\bar{X}) \in K(\bar{X}) \text{ is a sum of squares} \} \) (see [1], the sequel to Lemma 4.1).

We may assume without loss that \( \bar{x}_\gamma \) is transcendental over \( K \), by continuity of \( f \) (which is defined at \( \bar{x}_\gamma \)). It then clearly follows that \( \frac{1}{1+r(\bar{x}_\gamma)} \in [0,1] \subseteq O_K \). Therefore \( P \) is indeed a partial type, i.e. it is consistent, hence
$P$ has a realization $\bar{X}$ in some valued field $(\bar{L}, \bar{v})$ extending $(K, v)$. Now the restriction of $\bar{v}$ to $L = K(\bar{X})$ is an OVF-valuation (since $\mathcal{I}_\text{ord}$ has the extension property - see [1]) which is near $\bar{b}$, and such that $v(f(\bar{X})) < 0$. It follows that $f$ is not OVF-integral at $\bar{b}$, as required. ◦

An interesting conclusion from Proposition 2.6 is that a basic Kochen-closed set (i.e. the OVF-integrality locus $V^{\text{int}}(f)$ of a single function $f$) is open.

3 The relevant ganzstellensatz

Let $(K, v)$ be any valued field, let $L$ be a field extension of $K$, and assume $A \subseteq L$ is an $O_K$-algebra such that $A \cap K = O_K$. The set $T = \{1 + ma : m \in \mathcal{M}_K, a \in A\}$ is multiplicative. The integral radical of $A$ in $L$ is the integral closure (in $L$) of the localization $A_T$, and is denoted by $\sqrt[\text{int}]{A}$ (see [1], Section 2).

Let $\bar{p} = (p_1, \ldots, p_m)$ be polynomials from $K[\bar{X}]$, let $\bar{g} = (g_1, \ldots, g_l)$ be rational functions from $K(\bar{X})$, and define:

$$S_{\bar{p}, \bar{g}} = \{\bar{b} \in K^n | p_1(\bar{b}), \ldots, p_m(\bar{b}) > 0 \land v(g_1(\bar{b})), \ldots, v(g_l(\bar{b})) \geq 0\}$$

The following result by Lavi and the author (see [3], Theorem 7.4) gives a characterization of rational functions which are OVF-integral on $S_{\bar{p}, \bar{g}}$: first, let $\text{Cone}(\bar{p})$ denote the positive cone generated by the polynomials $\{p_i | i\}$. We may assume $S_{\bar{p}, \bar{g}} \neq \emptyset$, hence $-1 \notin \text{Cone}(\bar{p})$, and we may define $I_{\bar{p}} = \{\frac{1}{f} | f \in \text{Cone}(\bar{p})\}$. Finally let $A_{\bar{p}, \bar{g}}$ be the $O_K$-algebra generated by $I_{\bar{p}} \cup \{g_1, \ldots, g_l\}$ in $L = K(\bar{X})$.

**Theorem 3.1** [3] Assume that $(K, v, \leq_K)$ is a real closed valued field.

Then for any $h \in L$, $h$ is OVF-integral on $S_{\bar{p}, \bar{g}}$ if and only if $h \in \sqrt[\text{int}]{A_{\bar{p}, \bar{g}}}$. 

4 Finitary functions over non-archimedean RCFs

Let $R$ be a non-archimedean RCF, and let $v$ be the canonical valuation whose ring of integers equals the finite part of $R$. Let $\bar{p} = (p_1, \ldots, p_m)$ be polynomials in $R[\bar{x}]$, let $\bar{g} = (g_1, \ldots, g_l)$ be rational functions in $L = R(\bar{x})$, and define

$$T = T_{\bar{p}, \bar{g}} = \{\bar{b} \in R^n | p_1(\bar{b}), \ldots, p_m(\bar{b}) > 0 \land g_1, \ldots, g_l \text{ are finitary at } \bar{b}\}$$
Define $I_p = \{1 + f \mid f \in Cone(\bar{p})\}$, and let $A_{p,\bar{g}}$ be the $O_R$-algebra generated by $I_p \cup \{g_1, \ldots, g_l\}$ in $L$.

The pieces are now in place for the following:

**Theorem 4.1** For any $h \in L$, $h$ is finitary on $T_{p,\bar{g}}$ if and only if $h \in \sqrt{A_{p,\bar{g}}}$.

**Proof** By Proposition 2.6 being finitary at a point is equivalent to OVF-integrality, hence we may apply Theorem 3.1 to the set $S = S_{p,\bar{g}} = \{\bar{b} \in R^n \mid p_1(\bar{b}), \ldots, p_m(\bar{b}) > 0 \land v(g_1(\bar{b})), \ldots, v(g_l(\bar{b})) \geq 0\}$ and conclude that $h$ is finitary on $S$ if and only if $h \in \sqrt{A_{p,\bar{g}}}$. Clearly $S \subseteq T$, hence it is now sufficient to show that if $h$ is finitary (or equivalently, OVF-integral) on $S$ then it has this property on $T$ as well.

But every generator of $A_{p,\bar{g}}$ is OVF-integral on $T_{p,\bar{g}}$ (for elements of $I_p$ by Proposition 4.7 of [3], for the $g_i$ by definition), and the collection of functions which are OVF-integral on some set is closed under passing to the generated $O_R$-algebra and taking the integral radical. Therefore by using the above characterization of functions which are OVF-integral on $S$ we are done. ⋄

**Remark 4.2** The set $T$ above is contained in the Kochen closure of $S$ (see Definition 2.4), however it need not equal this Kochen closure: consider for example $p(X) = X$, and note that for any RCVF $K$ the Kochen closure of $S_p = \{x \in K \mid x > 0\}$ equals $\{x \in K \mid x \geq 0\}$.

It is instructive to note here that for RCVFs the Kochen closure of any set $Q$ is contained in the usual closure of $Q$, however they need not be equal. For example the Kochen closure of $Q = \{(x,0) \in K^2 \mid x > 0\}$ does not contain the point $(0,0)$ (thanks to $f = X$). The reason that the Kochen closure is sensitive to the ambient variety is that the latter determines the set of possible directions.

More dramatically, although the Kochen closure of $\{(x,y) \in K^2 \mid x \neq 0\}$ equals $K^2$, the Kochen closure of the half-plane $H = \{(x,y) \in K^2 \mid x > 0\}$ does not contain the point $(0,0)$ (thanks to $f = \frac{X}{X+Y^2}$ for example. This last example gives a better appreciation of the set of possible ‘directions’ that we have in mind.

Note that the set $T$ defined in Theorem 4.1 is actually open, hence one can show more directly that being finitary on $T$ is equivalent to OVF-integrality on $T$: all one needs is the easier direction of Proposition 2.6 and
Remark 2.3 (as done in [2], Corollary 3.8). A similar remark applies to the open set $S$, of course.

However there are sets $T'$ which are not open for which there is a Ganzstellensatz, and if one wishes to generalize Theorem 4.1 to such sets the full force of Proposition 2.6 seems to be required. For example, in an unpublished work of Haskell and the author we prove a Ganzstellensatz for sets defined by equalities, therefore one can also obtain a Ganzstellensatz for sets defined by weak inequalities (a function is OVF-integral on the union $\{x \mid p(x) \geq 0\} = \{x \mid p(x) > 0\} \cup \{x \mid p(x) = 0\}$ exactly when it is in the intersection of the relevant integral radicals). If we wish to characterize functions which are finitary on the non-open set

$$T' = \{ \bar{b} \in \mathbb{R}^n \mid p_1(\bar{b}), \ldots, p_m(\bar{b}) \geq 0 \land g_1, \ldots, g_l \text{ are finitary at } \bar{b} \}$$

then we would need to produce suitable OVF-valuations near points on the boundary of $T'$, as done in the proof of Proposition 2.6.

References

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