From the Sinai’s walk to the Brox diffusion using bilinear forms. *

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Abstract
Using the generators, we establish a connection between the Sinai’s random walk and the so-called Brox process. We first find the Dirichlet form of the Brox diffusion, and then prove that it is the limit of the Dirichlet form of the Sinai’s random walk. This also gives a natural way to connect between the Brox diffusion and the Brownian motion.

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1 Introduction
Sinai [17] studied the limiting behaviour of a random walk in random environment, and four years later Brox [2] asked the same question for a diffusion with random coefficients. In both cases the same long time behaviour was found. In Seignourel [16] it was shown the convergence in distribution of the Sinai’s walk to the Brox diffusion. Here we propose a modification of the scaling and a different method to connect both processes. In particular, we what we do is to see that the generator of the Brox diffusion is the limit of the generator of the Sinai’s walk. This provides an straightforward way to establish the connection between

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these processes. Moreover, this way to proceed helps to see the Brownian motion and the Brox process as models arising from the same microscopic system but with different local conditions.

An account regarding the connection of both models can be found in [18]. Other places where one finds invariance principles in the context of random media are [9], where the random environment has a large drift; see [6] to find work involving the so-called Brownian motion in random scenery; [11], regarding random walks with random conductances in dimensions 2 or more; and [12], regarding branching particle systems.

Let us briefly introduce an example of Sinai’s walks (abbreviated in general RWRE), taken from Révész [15]. Let $E := \{p(z), z \in \mathbb{Z}\}$ be a family of i.i.d. random variables (called the environment) such that $P(p(z) = 3/4) = P(p(z) = 1/4) = 1/2$. Then, the RWRE $R := \{R_k, k = 1, 2, \ldots\}$ has the following dynamics,

$$P(R_{k+1} = y|R_k = z, E) = \begin{cases} 
p(z) & \text{if } y = z + 1 \\
1 - p(z) & \text{if } y = z - 1.
\end{cases}$$

We may write $p(z) = 1/2 + q(z)$, with the following Bernoulli random variable

$$q(z) := \begin{cases} 
1/4 & 1/2 \\
-1/4 & 1/2.
\end{cases}$$

Another way to present $R_n$ is the following:

$$R_n = \sum_{i=1}^{n} \xi_i$$

where $\xi_1, \xi_2, \ldots$ are a sequence of random variables specified by

$$\xi_i := \begin{cases} 
1 & p(R_{i-1}) \\
-1 & 1 - p(R_{i-1}).
\end{cases}$$

This form goes more in hand with the way it is generally presented the classical random walk, and it turns out to help in our proofs.

When there is no random environment in the random walk we know that, after some rescaling, the central limit theorem gives rise to the normal random variable. On the other hand, Sinai [17] proved a limit theorem for the random walk where the environment is not being modified by any type of scaling. In the current paper we concern with the situation where the environment also suffers some rescaling. Indeed, we consider random variables of the form

$$\xi_i^{(n)} := \begin{cases} 
1 & p_n(S_{i-1}) \\
-1 & 1 - p_n(S_{i-1}).
\end{cases}$$
where

\[ p_n(x) := \frac{1}{2} + \frac{q(x)}{n^{1/4}}. \]

Now set

\[ S_0 := 0 \text{ and } S_k^{(n)} := \sum_{i=1}^{k} \xi_i^{(n)}. \]

Then, in particular, we study limit behaviour, as \( n \to \infty \), of

\[ S_n^{(n)} / \sqrt{n}. \]

The limit is not Gaussian as perhaps one might try to conjecture. One can put this in contrast with the random variables

\[ R_n / (\log(n))^2, \]

which converges in distribution to non-trivial random variable, see [17].

Now, let us present the Brox diffusion, which strictly speaking is not even Markov, but one defines it first by conditioning to the environment, which indeed gives a diffusion. Define a continuous time stochastic process \( \{X_t, \ t \geq 0\} \) with continuous trajectories proposed through the expression

\[ dX_t = dB_t - \frac{1}{2} W'(X_t) dt, \ X_0 = 0, \]

where \( B \) and \( W \) are independent Brownian motions. This expression is meant to be stochastic differential equation with a random coefficient \( W' \). One way to see that process \( X \) exists is by noticing that its generator would take the form

\[ \frac{1}{2} e^{W(x)} \frac{d}{dx} e^{-W(x)} \frac{d}{dx}. \]

Once one defines the conditioned process \( X \) given an environment \( W \), using the law of total probability, one defines what really the process \( X \) is.

The aim of this paper is to prove that the Dirichlet form associated to the Sinai’s walk converges to the corresponding one of the Brox diffusion. To do that we need first to find the Dirichlet form of the Brox diffusion which is done in the following 2 sections. In Section 4, we present the sequence of Sinai’s random walks that would help to approximate the Brox diffusion. Then, in Section 5, we show the desired convergence. There is an appendix at the end with a result we use at some point.
2 The Brox diffusion

Informally speaking, we call the stochastic process $X := \{X_t, t \geq 0\}$ the Brox diffusion, if it is solution of the equation

$$dX_t = dB_t - \frac{1}{2} W'(X_t)dt, \quad X_0 = 0.$$ 

Here $B := \{B_t, t \geq 0\}$ is a standard BM and $W := \{W(x), x \in \mathbb{R}\}$ is a two-sided BM, they both independent from each other. This expression can be interpreted as a stochastic differential equation with a random coefficient $W'$.

Based on standard theory of diffusions, in Brox [2] it is properly defined $X$. This is done arguing that

$$L := \frac{1}{2} e^{W(x)} \frac{d}{dx} \left( e^{-W(x)} \frac{d}{dx} \right)$$

is the infinitesimal generator associated to the equation displayed above. Thus, leaving fixed a trajectory of $W$, one is able to see that process $X$ is Markov process with generator denoted by $L$. Moreover, this way of thinking corresponds to considering the scale function

$$A(x) := \int_0^x e^{W(y)} dy, \quad x \in \mathbb{R},$$

and the speed measure

$$m(C) := \int_C 2e^{-W(x)} dx, \quad \text{for Borel sets } C \subset \mathbb{R}.$$ 

Then, using the scale function and a time-change transformation, Brox [2] proposed the following explicit construction of $X$:

$$X_t = A^{-1}(B(T^{-1}(t, B)))), \quad t \geq 0,$$

where

$$T(u, B) := \int_0^u e^{-2W(A^{-1}(B(s)))} ds, \quad x \in \mathbb{R}, \quad u \geq 0.$$ 

We can see that there are two sources of randomness, one coming from $B$ and the other from $W$. We will say that $B$ is the intrinsic randomness of $X$, and that $W$ is the external source of randomness, i.e. the environment. We may leave fixed either $B$ or $W$ and study the random dynamics of the process $X$. For instance, let $F$ the function in (4) that defines $X$ given $W$, i.e.

$$X = F(W).$$
We could one step further and think of $F$ as a function of $B$ as well. If we leave fixed a trajectory $W$, one could write $X(W) := F(W)$ to emphasize that the process $X$ is conditioned to the environment $W$. The corresponding probability measure of $X(W)$ over $C[0, \infty)$ is denoted $P_W$ and it is called the quenched measure. Whereas the probability measure on $C[0, \infty)$ coming from $X$ without leaving fixed $W$ is called the annealed probability, and it is denoted by $P$. In other words, if $\mu$ is the probability measure over $C(\mathbb{R})$ associated to the two-sided Brownian motion $W : \Omega \to C(\mathbb{R})$, then

$$P(C) = \int_{\Omega} P_{W(\omega)}(C) \mu(d\omega),$$

for any measurable set $C$ in $C[0, \infty)$.

To simplify notation, instead of writing $X(W)$, with no risk of confusion we only write $X$, because we will in general be working with the stochastic process $X$ after leaving fixed an environment $W$, i.e. the quenched case.

Now, since $X$ is a Markov process for each fixed environment $W$, there is a semigroup $\{H_t, t \geq 0\}$ defined as

$$H_t f(x) := E[f(X_t)|X_0 = x],$$

with $f \in C_0$, the space of real-valued continuous functions vanishing when $|x| \to \infty$. Symbolically, the generator of such semigroup is $L$ in (1), and let $D$ be the domain of $L$. Observe that the domain $D$ depends on the environment $W$.

Before we continue, let us mention that in the rest of the paper we denote by $C_0$ the space $\mathbb{R} \to \mathbb{R}$ of continuous functions that vanish as $|x| \to \infty$, and by $C_0^k$ the subspace of $k$-times differentiable functions.

**Remark 1** Notice that at first glance $Lf$ seems to be undefined, because $dW/dx$ does not exist, but since $X(W)$ is indeed a diffusion, there is indeed a generator $L$ with domain $D$, and such domain is known to be dense in $C_0$ (see e.g. [10, Lemma 3, p. 23]). Therefore, $Lf$ is well defined for any $f \in D$.

The following result will be useful for our purposes.

**Proposition 2** For any environment $W$, the domain $D$ is contained in the space of differentiable functions $C^1(\mathbb{R})$.

**Proof.** According to Lemma 2 in [10, p. 22], if $g(x) := [L f](x)$ for $f \in D$ then

$$f(x) = \int_0^x \int_0^y g(s) dm(s) dA(y) + f(0) + A(x) \frac{df}{dA}(0),$$

for any $x \geq 0$. Therefore, $Lf$ is well defined for any $f \in D$. 

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where
\[
\frac{df}{dA}(0) := \lim_{y \to 0} \frac{f(y) - f(0)}{A(y) - A(0)} = \lim_{y \to 0} \frac{f(y) - f(0)}{\int_0^y e^{W(s)} ds} \frac{y}{A(y)} = \frac{f'(0)}{e^{W(0)}} = f'(0).
\] (8)

Thus, plugging the scale function (2) and the speed measure (3), we have that
\[
f(x) = 2 \int_0^x \int_0^y g(s) e^{-W(s)} e^{W(y)} ds dy + f(0) + \frac{df}{dA}(0) \int_0^x e^{W(y)} dy.
\]

This tells us that \(f\) is indeed differentiable at every \(x \in \mathbb{R}\).

3 Approximations and the Dirichlet form

Let \(\{W_n\}\) be a sequence of stochastic processes that converge weakly to \(W\). Then, by considering the map \(F\) in (6), one can study the processes \(X^{(n)} := F(W_n)\). If the trajectories of \(W_n\) are differentiable functions, \(X^{(n)}\) can be analyzed using standard tools. In particular, we can approximate the Dirichlet form of \(X\) by considering the Dirichlet form of \(X^{(n)}\).

Let us start. Let \(W_n, n = 1, 2, \ldots\) be a sequence of processes with piecewise differentiable paths and such that \(W_n\) converges weakly to \(W\) as \(n \to \infty\). As it was done for (2)-(5), define
\[
X^{(n)}_t(B) := A^{-1}_n(B(T^{−1}_n(t, B))),
\]
where
\[
A_n(x) := \int_0^x e^{W_n(y)} dy, \ x \in \mathbb{R}
\]
and
\[
T_n(u, B) := \int_0^u e^{-2W_n(A^{-1}_n(B(s)))} ds, \ u \geq 0.
\]

For each \(n = 1, 2, \ldots\) we denote by \(H^{(n)}_t\) the associated semigroup of \(X^{(n)}\) after leaving frozen the environment \(W_n\). After finishing our paper, we came across with the manuscript [7], where approximation of this type are also used to study the Brox diffusion.

The generator of \(X^{(n)}\) is given by
\[
[L^{(n)} f](x) := \frac{1}{2} e^{W_n(x)} \frac{d}{dx} \left( e^{-W_n(x)} \frac{d}{dx} \left( \frac{d}{dx} f \right) \right) = \frac{1}{2} \frac{d^2 f}{dx^2} - \frac{W'_n(x)}{2} \frac{df}{dx}.
\] (10)

Then we can describe \(L^{(n)}\) using the bilinear form
\[
\langle L^{(n)} f, g \rangle = -\frac{1}{2} \int_{-\infty}^{\infty} f'g' - \frac{1}{2} \int_{-\infty}^{\infty} f'g dW_n,
\]
for any \(f, g \in C^1_0\), the space of real-valued differentiable functions that vanishes as \(|x| \to \infty\).
Our aim is to analyze $\langle L^{(n)}f, g \rangle$ as $n \to \infty$.

Recall that a trajectory of $W$ is being fixed. In addition, one can go one step further and consider $X$ to be a function of each trajectory $B \in C[0, \infty)$. Indeed, consider the map

$$(t, B) \mapsto X_t(B),$$

and the same for $X^{(n)}$. As usual, we take $C[0, \infty)$ with the topology of uniform convergence in compact sets. The first thing we need to prove is that $X^{(n)}$ converges uniformly to $X$, in compact sets of the arguments $(t, B)$.

**Lemma 3** We have that

$$X^{(n)}_t(B) \to X_t(B), \text{ as } n \to \infty,$$

uniformly in compact sets of $(t, B) \in [0, \infty) \times C[0, \infty)$.  

**Proof.** To prove convergence uniformly in compact sets, it is a topological fact (see e.g. Chapter XII, section 7, p.267 of [3]) that it is enough to prove that

$$X^{(n)}_{t_n}(B_n) \to X_t(B), \text{ as } n \to \infty,$$

whenever $t_n \to t$ and $B_n \to B$.

First of all, since $W_n \to W$ uniformly in compact sets, then $A_n \to A$ pointwise, and by Theorem [11] in the Appendix, the convergence is also uniform in compact sets. Therefore, $A_n^{-1} \to A^{-1}$ uniformly in compact sets, see Lemma 6 in [16]. From this, we can see that the composition

$$W_n \circ A_n^{-1} \circ B_n$$

also converges to $W \circ A^{-1} \circ B$ uniformly in compact sets of the domain $[0, \infty)$. This implies that

$$T_n(t_n, B_n) = \int_0^{t_n} e^{-2W_n(A_n^{-1}(B_n(s)))} ds \to T(t, B) = \int_0^t e^{-2W(A^{-1}(B(s)))} ds, \text{ as } n \to \infty.$$ (12)

With all these we can see that

$$A_n^{-1} \circ B_n \circ T_n^{-1}(t_n, B_n) \to A^{-1} \circ B \circ T^{-1}(t, B), \text{ as } n \to \infty,$$

which is (11) \hfill \blacksquare

**Corollary 4** For almost every $B$, the map

$$F : C(\mathbb{R}) \to C[0, \infty)$$

described in (6) is continuous.
Proof. First, fix \(B\). We need to show that \(F(W_n) \to F(W)\) whenever \(W_n \to W\) as \(n \to \infty\). Let \(F_t(W_n) := X_t^{(n)}\) and \(F_t(W) := X_t\). As mentioned in the proof of Lemma 3, it is enough to prove that \(F_t(W_n) \to F_t(W)\) whenever \(t_n \to t\), statement which is contained in previous lemma. ■

Lemma 5 For each \(t \geq 0\) and \(f \in C_0\),

\[
\|H_t^{(n)} f - H_t f\| \to 0, \quad n \to \infty.
\]

Proof. From Lemma 3 we have that \(X_t^{(n)}(B) \to X_t(B)\) for each path \(B\), we are then saying that \(X_t^{(n)} \to X_t\) for almost every path \(B\). Then,

\[
X_t^{(n)} + x_n \to X_t + x
\]

almost surely whenever \(x_n \to x\). Therefore, because any \(f\) is bounded,

\[
E \left[ f(X_t^{(n)} + x_n) \right] \to E \left[ f(X_t + x) \right], \quad n \to \infty.
\]

So, we are actually saying that \(H_t^{(n)} f\) converges to \(H_t f\) uniformly in compact sets of \(\mathbb{R}\). However, since \(f\) vanishes at the infinity, given \(\epsilon > 0\) we can give a compact set \(K \subset \mathbb{R}\) such that

\[
\left| E[f(X_t^{(n)} + x)] - E[f(X_t + x)] \right| \leq \left| E[f(X_t^{(n)} + x)] - E[f(X_t + x)] \right| I_K(x) + \epsilon, \quad (13)
\]

where \(I_K(x) = 0\) if \(x \notin K\) and \(I_K(x) = 1\) if \(x \in K\). Therefore, we have that \(H_t^{(n)} f\) converges to \(H_t f\) uniformly in the whole real line, i.e. with the supremum norm. ■

We can now prove that \(L f_n \to L f\) in the supremum norm for those functions \(f\) where \(L f\) is well defined, i.e. if \(f \in D\).

Corollary 6 It holds that

\[
\|L_n f - L f\| \to 0
\]

for any \(f \in D\).

Proof. It follows from Lemma 5 and Theorem 6.1 in [4, p.28] that

\[
\|L_n f - L f\| \to 0 \quad \text{for any } f \in D,
\]

in particular for any \(f \in D\). ■

We now present the so-called Dirichlet form associated to the Markov process.
Theorem 7  We have that the generator \( L \) of the Brox diffusion satisfies
\[
\langle Lf, g \rangle = -\frac{1}{2} \int_{-\infty}^{\infty} f'g' - \frac{1}{2} \int_{-\infty}^{\infty} f'gdW,
\]
for functions \( f, g \in C^1_0 \).

Notice that since \( f \) and \( g \) are deterministic functions, the last integral of previous display can be considered an Itô’s stochastic integral.

Proof. From Corollary 6
\[
(\forall f \in C^2_0 \cap D) \, L_nf \to Lf
\]
with the supremum norm. Thus,
\[
(\forall f \in C^2_0 \cap D)(\forall g \in C_0) \langle L_nf, g \rangle \to \langle Lf, g \rangle.
\]

On the other hand, from the theory of stochastic integrals (by taking a subsequence from the \( L_2 \)-convergence) we can take the convergence
\[
\int f'gdW_n \to \int f'gdW
\]
to be almost surely, i.e. for almost every \( W \). Thus, we can say that
\[
\langle L_nf, g \rangle \to -\frac{1}{2} \int_{-\infty}^{\infty} f'g' - \frac{1}{2} \int_{-\infty}^{\infty} f'gdW, \, n \to \infty,
\]
which in fact holds for functions \( f, g \in C^1_0 \). However, according to Proposition 2, the domain of \( L \) is contained in \( C^1_0 \). Therefore, the bilinear form is the associated Dirichlet form of the generator \( L \). □

In previous result, one realizes that, when describing \( L \) through the bilinear form, the domain is considered to be \( C^1_0 \), which is a bigger set of functions and independent of the trajectory \( W \). This kind of phenomenon is part of the theory of Dirichlet forms. The interested reader on Dirichlet forms and Markov process can consult [5].

4 The Sinai’s walk

In this section we want to introduce the sequence of Sinai’s random walks that approximate the Brox diffusion. We will see that apart from rescaling time and space of the corresponding random walk, we also need to rescale the the transition probabilities, i.e. the random environment.
Consider a partition of $\mathbb{R}$ into equally spaced intervals of size $\Delta_n > 0$, and denote by $Z_n \subset \mathbb{R}$ the lattice given by the set of extreme points of the intervals. Do now the same with the open interval $[0, \infty)$ by considering the discrete time $T_n \subset [0, \infty)$ of equidistant point of size $h_n > 0$. In this paper we will focus in a rescaling such that $\Delta_n := 1/\sqrt{n}$ and $h_n := 1/n$.

We must warn the reader not to be confused by expressions such as $\Delta_n x$, which does not mean an increment on $x$ but it is just the multiplication $\Delta_n \times x$.

Now we present the sequence of approximating random walks, where the notation is such that the model fits into the description of the introduction.

For each $n = 1, 2, \ldots$ consider the following continuous time stochastic process

$$S_t^{(n)} := \sum_{i=0}^{\lfloor t/h_n \rfloor} \xi_i^{(n)}, \ t \geq 0,$$

where $[x]$ is the integer part of $x$ and

$$\xi_i^{(n)} := \begin{cases} +1 & p_n \left( S_{i-1}^{(n)} \right) \\ -1 & 1 - p_n \left( S_{i-1}^{(n)} \right), \end{cases}$$

with $p_n(k) := 1/2 + \sqrt{\Delta_n} q(k)$, with

$$q(z) := \begin{cases} 1/4 & 1/2 \\ -1/4 & 1/2. \end{cases}$$

Notice that $E \left[ \xi_i^{(n)} \xi_j^{(n)} \right] = 0$ as long as $S_{i-1}^{(n)} \neq S_j^{(n)}$, and $E \left[ \xi_i^{(n)} \xi_j^{(n)} \right] = n^{-1/2}$ otherwise. This information will be useful to know later.

Now, let

$$X_t^{(n)} := \frac{1}{\sqrt{n}} S_t^{(n)}.$$

For any $n = 1, 2, \ldots$, one can see that the random environment $E_n := \{ p_n(z), \ z \in \mathbb{Z} \}$ is known once the sequence $E := \{ q(z), \ z \in \mathbb{Z} \}$ is specified. That is, given $E$ we know all the values in $E_n$. Moreover, from the Donsker invariance principle we know that

$$\left\{ \sqrt{\Delta_n} \sum_{j=0}^{k/\Delta_n} q(j), \ k \in \mathbb{Z} \right\} \overset{d}{\longrightarrow} \left\{ \frac{1}{4} W(t), \ t \in \mathbb{R} \right\}, \ n \to \infty, \quad (14)$$
where the factor 1/4 was factorized from $q$. This tells us that we can associate to each sample of $E$ a trajectory of $W$, or the other way round. This fact will be used in later in Theorem 8.

Given the environment $E$, when we consider the Sinai’s walk $X_t^{(n)}$ only at the jumps, i.e. for $t \in T_n$. This gives a discrete time Markov chain. In this case, the generator $L^{(n)}$ of $\{X_t^{(n)}, t \in T_n\}$ acting on a function $f : \mathbb{Z}_n \to \mathbb{R}$ is given by

$$L^{(n)} f(x) = \frac{1}{h_n} (f(x + \Delta_n) p_n(x/\Delta_n) + f(x - \Delta_n)(1 - p_n(x/\Delta_n)) - f(x)), \ x \in \mathbb{Z}_n. \ (15)$$

5 Convergence of the Dirichlet form

Due to previous sections, we are now in position to prove our main result, which is the convergence of the Dirichlet forms.

Remember that we are dealing with RWREs that takes values in $\mathbb{Z}_n \subset \mathbb{R}$. From (15), we can think that $L^{(n)} f$ is a step valued function from $\mathbb{R}$ to $\mathbb{R}$. This corresponds to consider $L^{(n)}$ as the composition of two operators, one is a projection to a vector and the other is multiplying such projection with a matrix; this way of thinking was imported from Pacheco [13].

Then, for every pair $f, g \in C_0^1$,

$$\langle L^{(n)} f, g \rangle = \int_{-\infty}^{\infty} L^{(n)} f(y) g(y) dy$$

$$= \sum_{x \in \mathbb{Z}_n} \left\{ \frac{1}{h_n} (f(x + \Delta_n) p_n(x/\Delta_n) + f(x - \Delta_n)(1 - p_n(x/\Delta_n)) - f(x)) \int_x^{x+\Delta_n} g(y) dy \right\}.
$$

Now, we can prove that when leaving fixed an environment $W$, $\langle L^{(n)} f, g \rangle \to \langle L f, g \rangle$ as $n \to \infty$.

Theorem 8 There is a subsequence $\{n_k\}_{k \geq 1}$ such that

$$\langle L^{(n_k)} f, g \rangle \to \frac{1}{2} \int_{-\infty}^{\infty} f'' g - \frac{1}{2} \int_{-\infty}^{\infty} f' g dW, \ k \to \infty,$$

for almost every trajectory $W$ and for functions $f \in C_0^2$ and $g \in C_0$.

Proof. To simplify notation let us omit subscript $n$ in $\Delta_n$ and in $p_n$. Using the mean valued theorem for integrals (see e.g. Bartle[1]), there are values $y_x \in [x, x + \Delta)$ for each $x \in \mathbb{Z}_n$ such that

$$\int_x^{x+\Delta} g = g(y_x) \Delta.$$

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Define $q_n(x) := \sqrt{\Delta} q(x/\Delta)$. Then we have

$$
\langle L^{(n)} f, g \rangle = \sum_{x \in \mathbb{Z}_n} \left\{ \frac{1}{\Delta^2} \left( f(x + \Delta)p(x/\Delta) + f(x - \Delta)(1 - p(x/\Delta)) - f(x) \right) g(y_x) \Delta \right\}
$$

$$
= \sum_{x \in \mathbb{Z}_n} \left\{ \frac{1}{\Delta^2} \left( \left\{ \frac{1}{2} + q_n(x) \right\} \{ f(x + \Delta) - f(x - \Delta) \} + f(x - \Delta) - f(x) \right) g(y_x) \Delta \right\}
$$

$$
= \frac{1}{2} \sum_{x \in \mathbb{Z}_n} \frac{f(x + \Delta) - 2f(x) + f(x - \Delta)}{\Delta^2} g(y_x) \Delta
$$

$$
+ 2 \sum_{x \in \mathbb{Z}_n} \frac{f(x + \Delta) - f(x - \Delta)}{2\Delta} g(y_x) q_n(x).
$$

Notice that the variance of the random variables $q_n(x)$ is of order $\Delta$ and they from an independent sequence. Hence, using theory of stochastic integrals, and using the fact (14), one can check that as $n \to \infty$ for any pair $f, g \in C_0^2$

$$
\langle L^{(n)} f, g \rangle \xrightarrow{L^2} \frac{1}{2} \int_{-\infty}^{\infty} f'' g - \frac{1}{2} \int_{-\infty}^{\infty} f' g dW,
$$

where $\xrightarrow{L^2}$ stands for convergence in mean square of random variables. In previous limit we have factorized the constant $1/4$ from the Bernoulli random variable $q_n(x)$. Thus, we can suppose that there is a subsequence where we have convergence almost surely, that is for almost every fixed trajectory $W$. ■

One can also check from the proof the following.

**Remark 9** In the proof of previous theorem, suppose that $q_n$ are random variables with $\text{var}(q_n(x)) \leq c\Delta^\gamma$ for some constants $c > 0$ and $\gamma \in [0, 1)$. Then, in the limit the second term vanishes and one recovers the generator of the Brownian motion. In this way, we can think of the Brownian motion and the Brox diffusion coming from the same type of model with different local specifications.

**Remark 10** By taking $q_n(x)$ such that

$$
q_n(x) = \left\{ \begin{array}{ll}
(\sqrt{\Delta} + \kappa \Delta_n)/4 & 1/2 \\
(-\sqrt{\Delta} + \kappa \Delta_n)/4 & 1/2,
\end{array} \right.
$$

we can have the convergence when the environment $W(x)$ is of the form $\beta(x) + \kappa x$, where $\beta$ is another BM. This model is considered for instance in [8, 19].
6 Appendix

The following result turns out to be very useful to us. It can be found as an exercise 127 in [14, p. 81].

**Theorem 11** Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be a sequence of continuous functions that converge pointwise to another continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$. If in addition all these functions are monotone, then the convergence is uniform in compact sets.

To prove this result one could use the following idea. We need to prove that $f_n(x_n) \to f(x)$ whenever $x_n \to x$. Since

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)|,$$

we concentrate on proving that $|f_n(x_n) - f_n(x)| \to 0$.

If the convergence were not uniformly, given $\epsilon > 0$ we could take $\{n_k\} \subset \{n\}$ such that

$$|f_n(x_{n_k}) - f_{n_k}(x)| > \epsilon, \ k = 1, 2, \ldots.$$  \hspace{1cm} (16)

Now, since each $f_{n_k}$ is continuous, there are open sets $V_k$ such that

$$(\forall y \in V_k)|f_{n_k}(y) - f_{n_k}(x)| < \epsilon.$$  

Therefore, each $x_{n_k}$ cannot be inside $V_k$.

From the monotonicity assumption, one knows that the sets $V_k$ need to be convex intervals. But the fact $x_{n_k} \to x$, as $k \to \infty$, would imply that the length of $V_k$ goes to 0. This would contradict (16).

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