Asymptotic Properties of the $p$-Adic Fractional Integration Operator

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To the blessed memory of M. L. Gorbachuk

Abstract

We study asymptotic properties of the $p$-adic version of a fractional integration operator introduced in the paper by A. N. Kochubei, Radial solutions of non-Archimedean pseudo-differential equations, *Pacif. J. Math.* 269 (2014), 355–369.

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1 Introduction

1.1. In analysis of complex-valued functions on the field $\mathbb{Q}_p$ of $p$-adic numbers (or, more generally, on a non-Archimedean local field), the basic operator is Vladimirov’s fractional differentiation operator $D^\alpha$, $\alpha > 0$, defined via the Fourier transform or, for wider classes of functions, as a hypersingular integral operator [1, 5]. Properties of this $p$-adic pseudo-differential operator were studied by Vladimirov (see [5]) and found to be more complicated than those of its classical counterparts. For example, as an operator on $L^2(\mathbb{Q}_p)$, it has a point spectrum of infinite multiplicity. However, it was shown in [2] to behave much simpler on radial functions $x \to f(|x|_p)$.

In particular, in [2] the first author introduced a right inverse $I^\alpha$ to the operator $D^\alpha$ on radial functions, which can be seen as a $p$-adic analog of the Riemann-Liouville fractional integral of real analysis (including the case $\alpha = 1$ of the usual antiderivative). Just as the Riemann-Liouville fractional integral is a source of many problems of analysis, that must be true for the operator $I^\alpha$.

In this paper we study asymptotic properties of the function $I^\alpha f$ for a given asymptotic expansion of $f$; for the asymptotic properties of Riemann-Liouville fractional integral see [3, 4, 7].

1.2. Let us recall the main definitions and notation used below.

Let $p$ be a prime number. The field of $p$-adic numbers is the completion $\mathbb{Q}_p$ of the field $\mathbb{Q}$ of rational numbers, with respect to the absolute value $|x|_p$ defined by setting

$$|0|_p = 0,$$

$$|x|_p = p^{-\nu} \text{ if } x = p^\nu m / n,$$

where $\nu, m, n \in \mathbb{Z}$, and $m, n$ are prime to $p$. It is well known that $\mathbb{Q}_p$ is a locally compact topological field with the topology determined by the metric $|x - y|_p$, and that there are no absolute values on $\mathbb{Q}$, which are not equivalent to the “Euclidean” one, or one of $| \cdot |_p$. We will denote by $dx$ the Haar measure on the additive group of $\mathbb{Q}_p$ normalized by the condition

$$\int_{|x|_p \leq 1} dx = 1.$$

The absolute value $|x|_p$, $x \in \mathbb{Q}_p$, has the following properties:

$$|x|_p = 0 \text{ if and only if } x = 0;$$

$$|xy|_p = |x|_p \cdot |y|_p;$$

$$|x + y|_p \leq \max(|x|_p, |y|_p).$$

The latter property called the ultrametric inequality (or the non-Archimedean property) implies the total disconnectedness of $\mathbb{Q}_p$ and unusual geometric properties. Note also the following consequence of the ultrametric inequality:

$$|x + y|_p = \max(|x|_p, |y|_p) \quad \text{if } |x|_p \neq |y|_p.$$

We will often use the integration formulas (see [1, 5, 6]):

$$\int_{|x|_p \leq r} |x|_p^{-\alpha} dx = \frac{1 - r^{-1}}{1 - r^{-\alpha}} p^\alpha; \text{ here and below } n \in \mathbb{Z}, \alpha > 0;$$

$$\int_{|x|_p \leq p^n} |x|_p^{-\alpha} dx = \frac{1 - p^{-1}}{1 - p^{-\alpha}} p^{\alpha n}.$$
in particular,
\[
\int_{|x|_p \leq p^n} dx = p^n; \\
\int_{|x|_p = p^n} dx = (1 - \frac{1}{p}) p^n; \\
\int_{|x|_p = 1} |1 - x|_{p}^{\alpha-1} = \frac{p - 2 + p^{-\alpha}}{p(1 - p^{-\alpha})}.
\]

See [1, 5] for further details of analysis of complex-valued functions on \( \mathbb{Q}_p \).

From now on, we consider the case \( \alpha > 1 \). The integral operator \( I^\alpha \) introduced in [2] has the form
\[
(I^\alpha f)(x) = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} \int_{|y|_p \leq |x|_p} (|x - y|_{p}^{\alpha-1} - |y|_{p}^{\alpha-1}) f(y) \, dy,
\]
where \( f \) is a locally integrable function on \( \mathbb{Q}_p \). See [2] for its connection to the Vladimirov operator \( D^\alpha \) and applications to non-Archimedean counterparts of ordinary differential equations. Note that our results can be generalized easily to the case of general non-Archimedean local fields.

### 2 Asymptotics at the origin

Let \( 0 < M_0 < M_1 < M_2 < \ldots, M_n \to \infty \). Then the sequence \( f_n(x) = |x|_{p}^{M_n} \) is an asymptotic scale for \( x \to 0 \) (see, for example, §16 of [4] for the main notions regarding asymptotic expansions).

**Theorem 1.** Suppose that a function \( f \) admits an asymptotic series expansion
\[
f \sim \sum_{n=0}^{\infty} a_n |x|_{p}^{M_n}, \quad |x|_p \to 0, a_n \in \mathbb{C}.
\]

Then
\[
(I^\alpha f)(x) \sim \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} \sum_{n=0}^{\infty} a_n b_n |x|_{p}^{M_n+\alpha}, \quad |x|_p \to 0,
\]
where
\[
b_n = \frac{p^{-\alpha+1} - 1}{(1 - p^{-\alpha})p} + (1 - p^{-1}) \sum_{k=1}^{\infty} (1 - p^{-k(\alpha-1)}) p^{-k(M_n+1)}.
\]

**Proof.** We have
\[
f = \sum_{n=0}^{N} a_n |x|_{p}^{M_n} + R_N(x), \quad R_N(x) = o(|x|_{p}^{M_N}), \quad |x|_p \to 0.
\]
Then $I^\alpha f = I^\alpha_{(1)} + I^\alpha_{(2)}$,

$$I^\alpha_{(1)} = \frac{1 - p^{-\alpha}}{1 - p^{-\alpha_1}} \int_{|y|_p \leq |x|_p} (|x|_p^\alpha - |y|_p^\alpha) \left( \sum_{n=0}^{N} a_n |y|_p^M \right) dy,$$

$$I^\alpha_{(2)} = \frac{1 - p^{-\alpha}}{1 - p^{-\alpha_1}} \int_{|y|_p \leq |x|_p} (|x|_p^\alpha - |y|_p^\alpha) R_N(y) dy.$$

After the change of variables $y = sx$ we get

$$I^\alpha_{(1)} = \frac{1 - p^{-\alpha}}{1 - p^{-\alpha_1}} |x|_p^{\alpha_1} \int_{|s|_p \leq 1} (1 - \alpha s_1 - |s|_1^{\alpha}) \left( \sum_{n=0}^{N} a_n |x|_p^M \right) ds = \frac{1 - p^{-\alpha}}{1 - p^{-\alpha_1}} |x|_p^{\alpha}(A + B)$$

where

$$A = \int_{|s|_p < 1} (1 - \alpha s_1 - |s|_1^{\alpha}) \left( \sum_{n=0}^{N} a_n |x|_p^M \right) ds$$

$$= \sum_{n=0}^{N} a_n |x|_p^M \sum_{k=1}^{\infty} (1 - p^{-k(\alpha - 1)}) \int_{|s|_p = p^{-k}}^d ds$$

$$= (1 - p^{-1}) \sum_{n=0}^{N} a_n |x|_p^M \sum_{k=1}^{\infty} (1 - p^{-k(\alpha - 1)}) p^{-k(M_n+1)},$$

$$B = \sum_{n=0}^{N} a_n |x|_p^M \int_{|s|_p = 1} (1 - |s|_p^{\alpha - 1} - |s|_1^{\alpha}) ds = \frac{p^{-\alpha+1} - 1}{(1 - p^{-\alpha})p} \sum_{n=0}^{N} a_n |x|_p^M.$$

On the other hand, since $|R_N(x)| \leq C |x|_p^{M+1}$, we find that for some constant $C_1 > 0$,

$$|I^\alpha_{(2)}| \leq C_1 |x|_p^{a+M+1} \int_{|s|_p \leq 1} (|1 - |s|_p^{\alpha - 1} - |s|_1^{\alpha}) |s|_p^{M+1} ds = O(|x|_p^{a+M+1}).$$

The above calculations result in the asymptotic relation (2). $\blacksquare$

### 3 Asymptotics at infinity

For positive functions $\varphi, \psi$, we write $\varphi(x) \prec \psi(x), |x|_p \to \infty$, if $c\psi(x) \leq \varphi(x) \leq d\psi(x)$, for large values of $|x|_p, x \in \mathbb{R}_p$, for some positive constants $c, d$.

**Theorem 2.** Suppose that $a \leq f(x) \leq b \ (a, b > 0)$ for $|x|_p < 1, \ |f(x)| \leq C |x|_p^{-M}, \ M > 1, \ C > 0, \ for \ |x|_p \geq 1$. Then

$$(I^\alpha f)(x) \sim |x|_p^{\alpha-1}, \ |x|_p \to \infty.$$  \hspace{1cm} (3)
Proof. Let us rewrite (1) with $|x|_p \geq 1$ in the form $I^\alpha f = J^\alpha(1) f + J^\alpha(2) f$ where

$$ (J^\alpha(1) f) (x) = \frac{1 - p^{-\alpha}}{1 - p^{-1}} \int_{|y|_p < 1} (|x - y|_p^{\alpha-1} - |y|_p^{\alpha-1}) f(y) dy, $$

$$ (J^\alpha(2) f) (x) = \frac{1 - p^{-\alpha}}{1 - p^{-1}} \int_{1 \leq |y|_p \leq |x|_p} (|x - y|_p^{\alpha-1} - |y|_p^{\alpha-1}) f(y) dy. $$

Then

$$ (J^\alpha(1) f) (x) \asymp \int_{|y|_p < 1} (|x - y|_p^{\alpha-1} - |y|_p^{\alpha-1}) dy \asymp |x|_p^{\alpha-1}. $$

Next, if $|x|_p = p^N$, $N \geq 0$, then

$$ |(J^\alpha(2) f) (x)| \leq C \int_{1 \leq |y|_p \leq |x|_p} (|x - y|_p^{\alpha-1} - |y|_p^{\alpha-1}) |y|_p^{-M} dy $$

$$ = C \left\{ \sum_{j=0}^{N-1} \int_{|y|_p = p^j} (|y|_p^{\alpha-1} - |y|_p^{\alpha-1}) |y|_p^{-M} dy + \int_{|y|_p = p^N} (|x - y|_p^{\alpha-1} - p^{N(\alpha-1)}) p^{-MN} dy \right\} $$

$$ = C \left\{ (1 - \frac{1}{p}) \sum_{j=0}^{N-1} p^j (p^{N(\alpha-1)} - p^{j(\alpha-1)}) p^{-Mj} $$

$$ + p^{-MN} \int_{|y|_p = p^N} |x - y|_p^{\alpha-1} dy - (1 - \frac{1}{p}) p^{\alpha N - MN} \right\}. $$

Calculating the integral as above and finding the sums of geometric progressions we see that $|(J^\alpha(2) f) (x)| \leq \text{const} \cdot |x|_p^{\alpha-1}$, which proves (3).

4 Logarithmic asymptotics

If a function $f$ decays slower than it did under the assumptions of Theorem 2, then a richer asymptotic behavior is possible. Let us consider the case where $f(t) \geq 0$,

$$ f(x) \sim |x|_p^{-\beta} \sum_{n=0}^{\infty} a_n (\log |x|_p)_{\gamma}^{n}, \quad |x|_p \to \infty, \quad (4) $$

where $0 \leq \beta < 1$, $\gamma \geq 0$, $a_n \in \mathbb{R}$. First we need some auxiliary results.

Lemma 1. Let $0 \leq f(x) = o\left(|x|_p^{-\lambda}\right)$, $|x|_p \to \infty$, where $0 < \lambda < 1$. Then

$$ G_1(r) \overset{def}{=} \int_{|y|_p \leq r} f(y) dy = o(r^{1-\lambda}), \quad r \to \infty. \quad (5) $$
Proof. Let \( n_0 = [\log_p r] \). Then \( p^{n_0} \leq r \leq p^{n_0 + 1} \). It is known (see Section 1) that
\[
\int_{|y|_p \leq p^\nu} |y|_p^{-\lambda} \, dy = \frac{1 - p^{-\lambda}}{1 - p^{(1-\lambda)\nu}}, \quad \nu \in \mathbb{Z},
\]
so that
\[
G_2(r) \overset{\text{def}}{=} \int_{|y|_p \leq r} |y|_p^{-\lambda} \, dy = O(r^{1-\lambda}), \quad r \to \infty. \tag{7}
\]

By our assumption, for any \( n \in \mathbb{N} \), there exists such \( r_0 = r_0(n) \) that \( f(x) < \frac{1}{n} |x|_p^{-\lambda} \) for \( |x|_p > r_0 \). Then we can write
\[
\frac{G_1(r)}{G_2(r)} = \frac{G_1(r_0(n)) + (G_1(r) - G_1(r_0(n)))}{G_2(r_0(n)) + (G_2(r) - G_2(r_0(n)))} \leq \frac{G_1(r_0(n))}{G_2(r_0(n))} + \frac{1}{n} G_3(n, r)
\]
where
\[
G_3(n, r) = \int_{r_0 \leq |y|_p \leq r} |y|_p^{-\lambda} \, dy.
\]

It follows from (6) that \( G_3(n, r) \to \infty \), so that
\[
0 \leq \limsup_{r \to \infty} \frac{G_1(r)}{G_2(r)} \leq \frac{1}{n}
\]
where \( n \) is arbitrary. Therefore
\[
\lim_{r \to \infty} \frac{G_1(r)}{G_2(r)} = 0,
\]
which gives, together with (7), the required asymptotic relation (5). \( \blacksquare \)

Lemma 2. Let \( 0 \leq \beta < 1 \), \( k \in \mathbb{N} \). For any \( \varepsilon > 0 \), such that \( \beta + \varepsilon < 1 \),
\[
K_r \overset{\text{def}}{=} \int_{|t|_p \leq r^{-1}} ((1 - t|_p^{\alpha - 1} - |t|_p^\alpha - 1 - |t|_p^{\alpha - 1}) \log |t|_p|t|_p^{-\beta} \, dt = O(r^{\beta + \varepsilon - 1}), \quad r \to \infty. \tag{8}
\]

Proof. Assuming that \( r > 2 \), we have \( |t|_p < \frac{1}{2} \), so that \( |1 - t|_p^{\alpha - 1} - |t|_p^\alpha - 1 - |t|_p^{\alpha - 1} \leq 1 \), and we find that
\[
K_r \leq \int_{|t|_p \leq r^{-1}} |t|_p^{-\beta} \log |t|_p|t|_p^{-\beta} \, dt \leq \int_{|t|_p \leq r^{-1}} |t|_p^{-\beta - \varepsilon} \, dt,
\]
if \( r \) is large enough, and the relation (8) follows from the integration formula (6). \( \blacksquare \)

Now we are ready to consider the asymptotics of \( I_\alpha f \) for a function \( f \) satisfying (4). Below we use the notation
\[
\binom{\gamma}{n} = \frac{\gamma (\gamma - 1) \cdots (\gamma - n + 1)}{n!}
\]
for any real positive number \( \gamma \) and \( n \in \mathbb{N} \).
Theorem 3. If a function $f \geq 0$ satisfies the asymptotic relation (4), then

$$(I^a f)(x) \sim \frac{1 - p^{-\alpha}}{1 - p^{-\beta}} |x|^\alpha |x|^\beta \sum_{n=0}^{\infty} B_n (\log |x|)^{\gamma - n}, \quad |x|_p \to \infty,$$

where

$$B_n = \sum_{k=0}^{n} a_{n-k} \left( \gamma + k - n \right) \Omega(k, \alpha, \beta),$$

$$\Omega(k, \alpha, \beta) = \int_{|t|_p \leq 1} (|1 - t|^\alpha - |t|^\alpha) |t|^{-\beta} (\log |t|)^k dt.$$ 

Proof. Let us write $(I^a f)(x)$ for $|x|_p \geq 1$ as the sum of two integrals $I_1$ and $I_2$, with the integration over $\{y : |y|_p < |x|_p^{1/2}\}$ and $\{y : |x|_p^{1/2} \leq |y|_p \leq |x|_p\}$ respectively.

Denote $\mathcal{K}(x, y) = |x - y|_p^{-\alpha} - |y|_p^{-\alpha}$. Considering $I_1$, for $|y|_p \leq |x|_p$, we have

$$|\mathcal{K}(x, y)| \leq |x|_p^{\alpha - 1}. \quad (10)$$

Indeed, if $|x|_p > 1$, then $|y|_p < |x|_p$, $\mathcal{K}(x, y) = |x|_p^{-\alpha} - |y|_p^{-\alpha}$, and we get (10). If $|x|_p = 1$, $|y|_p < 1$, then $0 < \mathcal{K}(x, y) = 1 - |y|_p^{-\alpha} < |x|_p^{-\alpha}$.

It follows from (10) that

$$0 \leq I_1 \leq C |x|_p^{\alpha - 1} \int_{|y|_p < |x|_p^{1/2}} f(y) dy,$$

and by (4) and Lemma 1, for any small $\varepsilon > 0$,

$$I_1 = O \left( |x|_p^{\alpha - \beta + \frac{2\varepsilon - 1}{2}} \right), \quad |x|_p \to \infty. \quad (11)$$

Considering $I_2$ we write

$$f(t) = |t|^{-\beta} \sum_{n=0}^{N} a_n (\log |t|)^{\gamma - n} + R_N(t), \quad R_N(t) = O(|t|^{-\beta} (\log |t|)^{\gamma - N - 1}) \quad |t|_p \to \infty.$$

Denote

$$L(\alpha, \beta, \gamma, x) = \int_{|x|_p^{1/2} \leq |y|_p \leq |x|_p} (|x - y|_p^{\alpha - 1} - |y|_p^{\alpha - 1}) |y|_p^{-\beta} (\log |y|)^{\gamma} dy$$

$$= |x|_p^{\alpha - \beta} (\log |x|)^{\gamma} \int_{|x|_p^{1/2} \leq |t|_p \leq |x|_p} (|1 - t|_p^{\alpha - 1} - |t|_p^{\alpha - 1}) |t|_p^{-\beta} \left( 1 + \frac{\log |t|_p}{\log |x|_p} \right)^{\gamma} dt$$

where on the domain of integration,

$$\frac{\log |t|_p}{\log |x|_p} \leq \frac{1}{2}.$$
and we may write, for a non-integer $\gamma$, the convergent binomial series

$$(1 + \frac{\log |t|_p}{\log |x|_p})^\gamma = \sum_{k=0}^{\infty} \binom{\gamma}{k} \left( \frac{\log |t|_p}{\log |x|_p} \right)^k.$$ 

Note that we can use the Taylor formula with the integral form of the remainder

$$(1 + s)^\gamma = \sum_{k=0}^{N} \binom{\gamma}{k} s^k + \frac{\gamma(\gamma - 1) \cdots (\gamma - N)}{N!} \int_0^s (1 + \sigma)^{\gamma-N-1}(s - \sigma)^N \, d\sigma$$

where

$$\int_0^s (1 + \sigma)^{\gamma-N-1}(s - \sigma)^N \, d\sigma = s^{N+1} \int_0^1 (1 + s\tau)^{\gamma-N-1}(1 - \tau)^N \, d\tau = s^{N+1} \int_0^1 (1 + (s - 1)\tau)^{\gamma-N-1}\tau^N \, d\tau.$$ 

If $-\frac{1}{2} < s < \frac{1}{2}$, $0 < \tau < 1$, then $\frac{1}{2} \leq 1 + s(1 - \tau) \leq \frac{3}{2}$. Therefore

$$(1 + \frac{\log |t|_p}{\log |x|_p})^\gamma = \sum_{k=0}^{N} \binom{\gamma}{k} \left( \frac{\log |t|_p}{\log |x|_p} \right)^k + S_N(t, x),$$

$$S_N(t, x) = O \left( \left( \frac{\log |t|_p}{\log |x|_p} \right)^{N+1} \right), \quad |x|_p \to \infty,$$

and this asymptotics is uniform with respect to $t$, $|t|_p \in |x|_p^{-1/2}, 1]$.

Substituting and using Lemma 2 we obtain the expansion

$$L(\alpha, \beta, \gamma, x) = |x|_{p}^{\alpha - \beta} \sum_{k=0}^{N} \binom{\gamma}{k} \Omega(k, \alpha, \beta)(\log |x|_p)^{\gamma-k}$$

$$+ o(|x|_{p}^{\alpha - \beta}(\log |x|_p)^{\gamma-N}), \quad |x|_p \to \infty. \quad (12)$$

We have

$$I_2 = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} \sum_{n=0}^{N} a_n L(\alpha, \beta, \gamma - n, x) + \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} \int_{|y|_p^{1/2} \leq |y|_p \leq |x|_p} (|x - y|_{p}^{\alpha-1} - |y|_{p}^{\alpha-1}) R_N(y) \, dy$$

where

$$\int_{|x|_p^{1/2} \leq |y|_p \leq |x|_p} (|x - y|_{p}^{\alpha-1} - |y|_{p}^{\alpha-1}) R_N(y) \, dy \leq C L(\alpha, \beta, \gamma - N - 1, x) = O \left( |x|_{p}^{\alpha - \beta}(\log |x|_p)^{\gamma-N-1} \right), \quad |x|_p \to \infty.$$ 

The last estimate is a consequence of (12).

Now the asymptotic relations (11) and (12) imply the required relation (9). \[\square\]

In our final result, we give a modification of Theorem 3 for the case where $\beta = 1$. 

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Theorem 4. Suppose that $f$ is nonnegative,

$$f(x) \sim |x|^{-1} \sum_{n=0}^{\infty} a_n (\log |x|)^{\gamma - n}, \quad |x| \to \infty.$$ 

Then

$$(I^\alpha f)(x) \sim \frac{1 - p^{-\alpha}}{1 - p^{-\alpha - 1}} \left[ |x|^{\alpha - 1} \int_{|y| \leq |x|} f(y) dy + \sum_{n=0}^{\infty} \tilde{B}_n (\log |x|)^{\gamma - n} \right], \quad |x| \to \infty \quad (13)$$ 

where

$$\tilde{B}_n = \sum_{k=0}^{n} a_{n-k} \binom{\gamma + k - n}{k} \tilde{\Omega}(k, \alpha),$$ 

$$\tilde{\Omega}(k, \alpha) = \int_{|t| \leq 1} \left( |1-t|^{\alpha - 1} - |t|^{\alpha - 1} \right) |t|^{-1} (\log |t|)^k dt.$$ 

Proof. Let us write $I^\alpha f = \frac{1 - p^{-\alpha}}{1 - p^{-\alpha - 1}} (J_1 + J_2 + J_3)$ where

$$J_1 = \int_{|y| \leq |x|^{1/2}} \left( |x - y|^{\alpha - 1} - |y|^{\alpha - 1} - |x|^{\alpha - 1} \right) f(y) dy,$$

$$J_2 = \int_{|x|^{1/2} \leq |y| \leq |x|} \left( |x - y|^{\alpha - 1} - |y|^{\alpha - 1} - |x|^{\alpha - 1} \right) f(y) dy,$$

$$J_3 = |x|^{\alpha - 1} \int_{|y| \leq |x|} f(y) dy.$$ 

Choosing $\varepsilon > 0$, such that $1 + \varepsilon < \alpha$, we see that $f(x) = o\left(|x|^{-1+\varepsilon}\right)$, $|x| \to \infty$. By Lemma 1,

$$\int_{|y| \leq |x|^{1/2}} f(y) dy = o\left(|x|^\gamma\right), \quad |x| \to \infty.$$ 

For the kernel of the above integral operator we get, considering various cases, the estimate

$$||x - y|^{\alpha - 1} - |y|^{\alpha - 1} - |x|^{\alpha - 1}| \leq 2|y|^{\alpha - 1}$$

It follows from Lemma 1 that

$$|J_1| \leq 2 \int_{|y| \leq |x|^{1/2}} |y|^{\alpha - 1} f(y) dy = o\left(|x|^\gamma\right), \quad |x| \to \infty. \quad (14)$$

By our assumption,

$$f(t) = |t|^{-1} \sum_{n=0}^{N} a_n (\log |t|)^{\gamma - n} + R_N(t), \quad R_N(t) = O\left(|t|^{-1} (\log |t|)^{\gamma - N - 1}\right), \quad |t| \to \infty.$$
Let us consider the expression

$$\widetilde{L}(\alpha, \gamma, x) = \int_{|x|^{1/2} \leq |y| \leq |x|} \left( |x - y|^{\alpha - 1} - |y|^{\alpha - 1} - |x|^{\alpha - 1} \right) |y|^{-1} (\log |y| \gamma) dy$$

$$= |x|^{\alpha - 1} \int_{|x|^{-1/2} \leq |t| \leq 1} \left( |1 - t|^{\alpha - 1} - |t|^{\alpha - 1} - 1 \right) |t|^{-1} (\log |x| + \log |t| \gamma) dt.$$ 

It follows from the first integration formula from Section 1 that

$$\int_{|t|^{1/2} \leq |t| \leq |x|^{-1/2}} \left( |1 - t|^{\alpha - 1} - |t|^{\alpha - 1} - 1 \right) |t|^{-1} (\log |t| \gamma)^k dt = o\left(|x|^{\frac{1-\alpha + k}{2}}\right), \quad |x| \rightarrow \infty.$$ 

This implies (just as in the proof of Theorem 3) the expansion

$$\widetilde{L}(\alpha, \gamma, x) \sim |x|^{\alpha - 1} \sum_{k=0}^{\infty} \binom{\gamma}{k} (\log |x| \gamma)^{-\kappa} \Omega(k, \alpha), \quad |x| \rightarrow \infty.$$ 

Taking into account (14), we come to (13). ■

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