The Signless Laplacian Coefficients and the Incidence Energy of Graphs with a Given Bipartition

Lei Zhong\textsuperscript{a,b}, Wen-Huan Wang\textsuperscript{a}

\textsuperscript{a}Department of Mathematics, Shanghai University, Shanghai 200444, China
\textsuperscript{b}Department of Mathematics, University of Mississippi, Oxford, MS 38677, USA

Abstract. We consider two classes of the graphs with a given bipartition. One is trees and the other is unicyclic graphs. The signless Laplacian coefficients and the incidence energy are investigated for the sets of trees/unicyclic graphs with \( n \) vertices in which each tree/unicyclic graph has an \((n_1, n_2)\)-bipartition, where \( n_1 \) and \( n_2 \) are positive integers not less than 2 and \( n_1 + n_2 = n \). Four new graph transformations are proposed for studying the signless Laplacian coefficients. Among the sets of trees/unicyclic graphs considered, we obtain exactly, for each, the minimal element with respect to the quasi-ordering according to their signless Laplacian coefficients and the element with the minimal incidence energies.

1. Introduction

Let \( G = (V(G), E(G)) \) be a simple graph, where \( V(G) = \{v_1, \ldots, v_n\} \) and \( E(G) = \{e_1, \ldots, e_m\} \) are the vertex set and the edge set of \( G \), respectively. The adjacency matrix of \( G \) is denoted by \( A(G) \). The energy of \( G \), as introduced by Gutman [6], is defined as the sum of the absolute values of all the eigenvalues of \( A(G) \). Let \( B \) be a matrix with real entries. The singular values of \( B \) are the positive square roots of the eigenvalues of \( BB^t \), where \( B^t \) is the transpose of \( B \). Moreover, if \( B \) is a symmetric matrix, then its singular values are the absolute values of its eigenvalues. Nikiforov [18] extended the concept of energy to all matrices, defining the energy of a matrix as the sum of the singular values of the matrix.

We denote by \( I(G) \) the vertex-edge incidence matrix of \( G \), where \( I(G) \) is an \((n \times m)\)-matrix whose \((i, j)\)th entry is 1 if the vertex \( v_i \) is incident with the edge \( e_j \), and 0 otherwise. In 2009, Jooyandeh et al. [11] defined the incidence energy (IE) of a graph \( G \) as

\[
IE(G) = \sum_{i=1}^{n} \sigma_i
\]

where \( \sigma_1, \sigma_2, \ldots, \sigma_n \) are the singular values of \( I(G) \).

Let \( D(G) = \text{diag}(d_C(v_1), d_C(v_2), \ldots, d_C(v_n)) \) be the degree diagonal matrix of \( G \), where \( d_C(v_i) \) is the degree of vertex \( v_i \) of \( G \). We refer to \( L(G) = D(G) - A(G) \) and \( Q(G) = D(G) + A(G) \) as the Laplacian
matrix and the signless Laplacian matrix, respectively. Since \( I(G) = D(G) + A(G) = Q(G) \), we get [7]

\[
IE(G) = \sum_{i=1}^{n} \sqrt{q_i},
\]

where \( q_1, q_2, \cdots, q_n \) are the eigenvalues of the signless Laplacian matrix \( Q(G) \). It is noted that \( q_1, q_2, \cdots, q_n \) are real and non-negative.

The IE of \( G \), which originates from chemical graph theory, can help explain some phenomena of chemical molecule. The graphs having the extremal IEs are derived on basis of (5) and other methods. For the graphs with the extremal IEs and the upper and lower bounds of IE, one can refer to Refs. [5, 7, 10, 15, 20, 21, 24, 26, 27]. Kaya and Maden got some bounds for the generalized version of incidence energy [12].

The Laplacian and signless Laplacian characteristic polynomials of \( G \) are respectively defined as

\[
L(G; x) = \det[xI - L(G)] = \sum_{i=0}^{n} (-1)^{i} c_i(G)x^{n-i},
\]

\[
Q(G; x) = \det[xI - Q(G)] = \sum_{i=0}^{n} (-1)^{i} \varphi_i(G)x^{n-i},
\]

where \( I \) is the identity matrix of order \( n \), and \( c_i(G) \) and \( \varphi_i(G) \) are coefficients of corresponding characteristic polynomials. It is known that \( Q(G) \) and \( L(G) \) are similar if and only if (iff) \( G \) is bipartite. Therefore, the Laplacian coefficients are the same as the signless Laplacian coefficients (SLCs) iff \( G \) is bipartite.

Let \( G_n \) be the set of all the simple graphs of order \( n \). For \( G, H \in G_n \), we write \( G \leq H \) if \( c_i(G) \leq c_i(H) \) with \( 0 \leq i \leq n \). Similarly, we denote \( G \preceq H \) if \( \varphi_i(G) \leq \varphi_i(H) \) for \( 0 \leq i \leq n \). We write \( G < H \) if \( G \preceq H \) with an integer \( k \) in such a way that \( \varphi_k(G) < \varphi_k(H) \). Then we refer to this symbol \( \preceq \) as the quasi-ordering. Mirzakah and Kiani [16] obtained

\[
G \preceq H \iff IE(G) \leq IE(H),
\]

\[
G < H \iff IE(G) < IE(H).
\]

The Laplacian matrix has been studied extensively. Among various classes of graphs, some results have been derived about the partial ordering according to \( \leq \). For example, Laplacian-cospectral trees [17], trees with a fixed matching number [9], unicyclic graphs [19], and bicyclic graphs [8], etc.

The signless Laplacian matrix of \( G \) has attracted more and more attention due to it can be used to discover more structural characterization of graphs than the Laplacian matrix in some ways [24]. For the partial ordering according to \( \preceq \), there are many interesting results. Mirzakah and Kiani [16] studied the coefficients of the signless Laplacian matrix of unicyclic graphs. Li et al. [13] determined two maximal elements and two minimal elements among unicyclic graphs. Zhang and Zhang [24] got two minimal elements in bicyclic graphs. Among the unicyclic graphs having a fixed matching number, Zhang and Zhang [25] characterized all the minimal elements. In the connected graphs of \( n \) vertices and \( m \) edges without even cycles, Wang et al. [23] obtained the minimal element which has the minimum SLCs and the minimum IE. Among the unicyclic graphs with \( n \) vertices and \( r \) pendant vertices, where \( n \geq 4 \) and \( r \geq 1 \), Wang and Zhong [22] characterized a unique extremal graph which has the minimum SLCs and the minimum IE. For further information on the signless Laplacian matrix, one can refer to three surveys [2–4].

Let \( G \) be a connected bipartite graph with \( n \) vertices. Then \( V(G) \) can be partitioned into two subsets \( V_1(G) \) and \( V_2(G) \) in such a way that each edge in \( E(G) \) joins a vertex in \( V_1(G) \) with a vertex in \( V_2(G) \). Let \( |V_1(G)| = n_1 \) and \( |V_2(G)| = n_2 \) with \( n_1 + n_2 = n \). We say that \( G \) has an \((n_1, n_2)\)-bipartition. Let \( T_{n_1, n_2} / \mathcal{U}_{n_1, n_2} \) be the set of trees/unicyclic graphs with \( n \) vertices in which each tree/unicyclic graph has an \((n_1, n_2)\)-bipartition, where \( n_1 \) and \( n_2 \) are positive integers not less than 2 and \( n_1 + n_2 = n \).

Motivated by all the above-mentioned work, we will characterize, in the present study, the minimal graphs in terms of \( \preceq \) according to their SLCs, and then deduce the graphs with the minimal IEs in \( T_{n_1, n_2} \) and \( \mathcal{U}_{n_1, n_2} \).
The subdivision graph $S(G)$ of a graph $G$ is a graph obtained by inserting a new vertex on each edge of $G$. Among $T_{n,t}$, by comparing the number of $k$-matchings of the subdivision graphs of the graphs considered, Lin and Yan [14] characterized the trees having the minimal and the second minimal Laplacian coefficients. In this paper, we will use the $x$-transformation (presented in Lemma 3.3 in Subsection 3.1) to obtain the graph with the minimal SLCs among $T_{n,t}$. Since the graphs among $T_{n,t}$ and $U_{n,t}$ are bipartite, their Laplacian coefficients are the same as their SLCs. Thus, in this paper, another straightforward and simpler method is acquired to obtain the graph with the minimal Laplacian coefficients among $T_{n,t}$ (presented in Theorem 3.13 in Subsection 3.2), and the graph with the minimal Laplacian coefficients among $U_{n,t}$ is deduced (presented in Theorem 3.17 in Subsection 3.2).

The paper is organized as follows. In Subsection 3.1, four new transformations (see Lemmas 3.1–3.11) which keep the bipartition unchanged are derived. In Subsection 3.2, by the four transformations proposed in this paper, we obtain exactly, among $T_{n,t}$ and $U_{n,t}$, one minimal element with respect to the quasi-ordering $<$ according to their SLCs and we get the graph with the minimal IEs.

2. Preliminaries

Let $G$ be a graph of order $n$. A connected graph of order $n$ is an odd unicyclic graph if it has only one cycle with an odd length. A spanning subgraph of $G$ whose connected components are trees or odd unicyclic graphs is called a TU-subgraph of $G$. Let $H$ be a TU-subgraph of $G$ consisting of $s$ odd unicyclic graphs and $t$ trees $T_1, T_2, \ldots, T_t$ of orders $n_1, n_2, \ldots, n_t$, respectively. Then the weight of $H$ is denoted by

$$W(H) = 4^t \prod_{i=1}^{t} n_i. \quad (7)$$

If $H$ contains no trees, then $W(H) = 4^t$. If $H$ contains no cycles, then $W(H) = \prod_{i=1}^{t} n_i$. Note that isolated vertices in $H$ may be ignored since they do not contribute to $W(H)$.

To obtain the main results of this paper, Lemma 2.1 is introduced as follows.

**Lemma 2.1.** [1] Let $Q(G, x) = \det[xI - Q(G)] = \sum_{i=0}^{n} (-1)^i \varphi_i(G)x^{n-i}$ be the characteristic polynomial of the signless Laplacian matrix of a graph $G$ with order $n$. Then

$$\varphi_i(G) = \sum_{H_{i}} W(H_{i}), \quad (i = 0, 1, 2, \ldots, n), \quad (8)$$

where the summation runs over all TU-subgraphs $H_i$ of $G$ with $i$ edges.

In particular, $\varphi_0(G) = 1$, $\varphi_1(G) = 2m$ and $\varphi_2(G) = 2m^2 - m - \frac{1}{2} \sum_{v_i} d_i^2(v_i)$.

By Lemma 2.1, we get the following property, which is used to obtain our transformations in Subsection 3.1. Let $G_1$ and $G_2$ be two connected graphs with $n$ vertices. Let $i$ be a fixed number with $2 \leq i \leq n$. Let $\mathcal{H}_1 = \{H_1, H_2, \ldots, H_i\}$ and $\mathcal{H}_2 = \{\overline{H_1}, \overline{H_2}, \ldots, \overline{H_i}\}$ be the sets of all the TU-subgraphs of $G_1$ and of $G_2$ with $i$ edges exactly, respectively, where $s \leq t$. Then $\varphi_i(G_1) = \sum_{H_i} W(H_i)$ and $\varphi_i(G_2) = \sum_{\overline{H_i}} W(\overline{H_i})$. If there exists a mapping $f$ from $\mathcal{H}_1$ to $\mathcal{H}_2$ satisfying $W(H_k) \leq W(\overline{H_k})$, where $1 \leq k \leq s$, then we have $\varphi_i(G_1) = W(H_1) + \ldots + W(H_i) \leq W(\overline{H_1}) + \ldots + W(\overline{H_i}) = \varphi_i(G_2)$.

3. Main results

3.1. Four transformations for studying the SLCs of graphs considered

In this subsection, we will introduce four new transformations for studying the SLCs of the graphs with a given bipartition, which are shown in Lemmas 3.1–3.11. The bipartition for the graphs among $T_{n,t}$ and $U_{n,t}$ keeps unchanged within the framework of the four transformations.
For a subset $M$ of $E(G)$, $G - M$ denotes the graph obtained from $G$ by deleting all the edges in $M$. For an edge set $M'$ satisfying $M' \cap E(G) = \emptyset$, $G + M'$ denotes the graph obtained from $G$ by adding all the edges in $M'$. If $M = \{e\}$ and $M' = \{e\}$, then $G - M$ and $G + M'$ are rewritten as $G - e$ and $G + e$, respectively. For a subgraph $H$ of $G$, $G - H$ denotes the subgraph of $G$ induced by the vertices not in $H$.

Let $G_1$, $G_2$ and $G_3$ be three mutually disjoint graphs in which $v_i$ is a vertex of $G_i$ ($1 \leq i \leq 3$). We denote by $G_1 + u_1u_2 + G_2$ the graph obtained from $G_1$ and $G_2$ by adding an edge $u_1u_2$ between $u_1$ of $G_1$ with $u_2$ of $G_2$. Similarly, $G_1 + u_1u_2 + G_2 + u_2u_3 + G_3$ is the graph obtained from $G_1$, $G_2$ and $G_3$ by adding an edge $u_1u_2$ between $u_1$ of $G_1$ with $u_2$ of $G_2$ and adding an edge $u_2u_3$ between $u_2$ of $G_2$ with $u_3$ of $G_3$.

We denote by $N_C(v)$ the neighbors of $v$ in the graph $G$.

Let $Q$ be a connected graph with a vertex $x$, and $T_v$ and $T_w$ two trees with $v \in V(T_v)$ and $w \in V(T_w)$. Let $P_3 = uvw$ be a path of length 2. Let $A_n$ be the graph with $n$ vertices obtained from $Q$ by first identifying $x$ of $Q$ with $u$ of $P_3$, then identifying $v$ of $T_v$ with $v$ of $P_3$ and identifying $w$ of $T_w$ with $w$ of $P_3$. Let $A_n^*$ be the graph obtained from $A_n$ by replanting $T_w$ from $w$ to $u$. $A_n$ and $A_n^*$ are shown in Figs. 1(a) and 1(b), respectively. In other words,

$$A_n^* = A_n - \{wy \mid y \in N_{T_v}(w)\} + \{uy \mid y \in N_{T_v}(w)\}. \tag{9}$$

The transformation from $A_n$ to $A_n^*$ in (9) is called $\alpha$-transformation.

**Lemma 3.1.** Let $A_n$ and $A_n^*$ be the two graphs as defined in Fig. 1. If $Q$ is a connected unicyclic graph, then we have $\varphi_i(A_n) \geq \varphi_i(A_n^*)$ for $0 \leq i \leq n$, where the equalities do not hold for all $i$.

**Proof.** It follows from Lemma 2.1 that $\varphi_i(A_n) = \varphi_i(A_n^*)$ for $i = 0, 1$. Next, let $2 \leq i \leq n$. For a fixed $i$, let $\mathcal{H}$ and $\mathcal{H}'$ be the sets of all the TU-subgraphs of $A_n^*$ and of $A_n$ with $i$ edges exactly, respectively.

For an arbitrary TU-subgraph $H' \in \mathcal{H}'$, let

$$f_i : \mathcal{H}' \rightarrow \mathcal{H}, H' \rightarrow H = f_i(H'), \tag{10}$$

with $V(H) = V(H')$ and

$$E(H) = E(H') - \{ux \mid x \in N_{T_v}(u) \cap V(H')\} + \{wx \mid x \in N_{T_v}(u) \cap V(H')\}.$$ 

Obviously, $f_i$ is a bijection from $\mathcal{H}'$ to $\mathcal{H}$.

For the sake of conciseness, a tree component, an odd unicyclic component, an arbitrary component, and the same component are abbreviated as a TC, an OUC, an AC, and the SC, respectively. Let $N$ be the weight of all the components of $H'$ not containing $u$, $v$ or $w$. In $A_n^*$, let $uw = e_1$ and $vw = e_2$. Three cases are considered as follows.

**Case (I) $e_1, e_2 \notin E(H')$.**

In this case, for an arbitrary TU-subgraph $H'$ in $\mathcal{H}'$, we denote by $R_{nu}$, $R_{nv}$ and $R_{nw}$ the connected components of $H'$ containing $u$, $v$ and $w$, respectively. Since $e_1, e_2 \notin E(H')$, $R_{nu}$, $R_{nv}$ and $R_{nw} = \{w\}$ are mutually

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**Figure 1:** $\alpha$-transformation from $A_n$ to $A_n^*$
disjoint; and $R_u^*$ and $R_v^*$ are TCs. Let $|V(Q) \cap V(R_u^*) \setminus \{u\}| = a$, $|V(T_v) \cap V(R_v^*) \setminus \{v\}| = b$ and $|V(T_w) \cap V(R_w^*) \setminus \{w\}| = c$. Thus, we get

$$|V(R_u^*)| = a + b + 1, \quad |V(R_v^*)| = c + 1, \quad |V(R_w^*)| = 1. \quad (11)$$

By the bijection $f_1$, in $H$, there exist three components, denoted by $R_u$, $R_v$ and $R_w$, which correspond to $R_u^*$, $R_v^*$ and $R_w^*$, respectively. It is noted that $R_u$, $R_v$ and $R_w$ contain $u$, $v$ and $w$ in $H$, respectively; $R_v$ and $R_w$ are TCs; and $R_u$, $R_v$ and $R_w$ are mutually disjoint. Obviously, we have

$$|V(R_u)| = a + 1, \quad |V(R_v)| = c + 1, \quad |V(R_w)| = b + 1. \quad (12)$$

Furthermore, we have the following statement:

**Fact 3.2.** Except for the component(s) containing $u$, $v$ and $w$ in $H^*$, an AC of $H^*$ corresponds to the SC of $H$.

Two subcases are considered according to the fact $R_u^*$ is a TC or an OUC.

**Subcase (I.i) $R_u^*$ is a TC.**

In this subcase, $R_u$ is a TC. By Fact 3.2, (7), (11), and (12), we obtain

$$W(f_1(H^*)) - W(H^*) = (a + 1)(b + 1)(c + 1)N - (a + b + 1)(c + 1)N = Na(b + 1) \geq 0,$$

with the third equality iff $a = 0$ or $b = 0$.

**Subcase (I.ii) $R_u^*$ is an OUC.**

In this subcase, $R_u$ is an OUC. By Fact 3.2, (7), (11), and (12), we get

$$W(f_1(H^*)) - W(H^*) = 4(b + 1)(c + 1)N - 4(c + 1)N = 4Nb(c + 1) \geq 0,$$

with the third equality iff $b = 0$.

**Case (II) $e_1 \in E(H^*)$ and $e_2 \notin E(H^*)$.**

Let $R_{u,v}^* = R_u^* + vw + R_v^*$ and $R_{u,v} = R_u + vw + R_v$. Obviously, by (11) and (12), $R_{u,v}^*$ is a component of order $a + b + c + 2$ containing $u$, $v$ in $H^*$ and $R_{u,v}$ is a component of order $a + c + 2$ containing $u$, $v$ of $H$. Since $e_1 \in E(H^*)$ and $e_2 \notin E(H^*)$, by the bijection $f_1$, $R_{u,v}^*$ and $[w]$ in $H^*$ correspond to $R_{u,v}$ and $R_w$ in $H$, respectively. Two subcases are considered according to the fact $R_{u,v}^*$ is a TC or an OUC.

**Subcase (II.i) $R_{u,v}^*$ is a TC.**

In this subcase, $R_{u,v}$ is a TC. By Fact 3.2, (7), (11), and (12), we get

$$W(f_1(H^*)) - W(H^*) = (a + c + 2)(b + 1)N - (a + b + c + 2)N = Nb(a + c + 1) \geq 0,$$

with the third equality iff $b = 0$. We denote

$$H_1' = \{H' \in H^* | e_1 \in E(H'), \ e_2 \notin E(H') \text{ and } R_{u,v}^* \text{ is a TC}.\}$$

**Subcase (II.ii) $R_{u,v}^*$ is an OUC.**

In this subcase, $R_{u,v}$ is an OUC. It follows from Fact 3.2, (7) and (12) that

$$W(f_1(H^*)) - W(H^*) = 4(b + 1)N - 4N = 4Nb \geq 0,$$

with the third equality iff $b = 0$.

**Case (III) $e_1 \notin E(H^*)$ and $e_2 \in E(H^*)$.**

Since $e_1 \notin E(H^*)$ and $e_2 \in E(H^*)$, by the bijection $f_1$, $R_u^*$ and $R_{w,v}^* = R_v^* + vw + R_w^*$ in $H^*$ correspond to $R_u$ and $R_{w,v}$, respectively. Obviously, by (11) and (12), $R_{w,v}^*$ is a TC of order $c + 2$ containing $v$ and $w$ in $H^*$ and $R_{w,v}$ is a TC of order $b + c + 2$ containing $v$ and $w$ in $H$. Two subcases are considered according to the fact $R_{w,v}^*$ is a TC or an OUC.
Subcase (II.ii) $R_u^*$ is a TC.
In this subcase, $R_u$ is a TC. By Fact 3.2, (7), (11), and (12), we have
\[
W(f_1(H')) - W(H') = (a + 1)(b + c + 2)N - (a + b + 1)(c + 2)N - Nb(a - c - 1).
\]
\[
(17)
\]
We denote
\[
\mathcal{H}_2^* = \{H' \in \mathcal{H}^* | e_1 \notin E(H'), e_2 \in E(H') \text{ and } R_u^* \text{ is a TC}. \]
We construct a mapping $\xi_1$ from $\mathcal{H}_2^*$ to $\mathcal{H}_1^*$ as follows. For $H' \in \mathcal{H}_2^*$, let
\[
\xi_1 : H' \rightarrow (H^* - e_2) + e_1.
\]
Obviously, $\xi_1$ is bijective. Thus, there exists a one-to-one relationship between $\mathcal{H}_2^*$ and $\mathcal{H}_1^*$. Namely, for an arbitrary $H' \in \mathcal{H}_2^*$, we can find, by $\xi_1$, a unique element $\xi_1(H') \in \mathcal{H}_1^*$ corresponding to it, and vice versa. For $H' \in \mathcal{H}_2^*$ by (17) and (15), we obtain
\[
[W(f_1(H')) - W(H')] + [W(f_1(\xi_1(H'))) - W(\xi_1(H'))] = 2Nb \geq 0.
\]
Furthermore, by (19), we get
\[
\sum_{H' \in \mathcal{H}_2^*} [W(f_1(H')) - W(H')] + \sum_{H' \in \mathcal{H}_1^*} [W(f_1(H')) - W(H')] = \sum_{H' \in \mathcal{H}_2^*} [W(f_1(H')) - W(H') + W(f_1(\xi_1(H'))) - W(\xi_1(H'))] \geq 0.
\]
\[
(20)
\]
Subcase (III.ii) $R_u^*$ is an OUC.
In this subcase, $R_u$ is an OUC. By Fact 3.2 and (7), we obtain
\[
W(f_1(H')) - W(H') = 4(b + c + 2)N - 4(c + 2)N = 4Nb \geq 0,
\]
with the third equality iff $b = 0$.

Case (IV) $e_1, e_2 \in E(H)$.
We have three facts: (i) $u, v$ and $w$ of $H'$ are contained in a component of $H'$ (denoted by $R_{u,v,w}^*$); (ii) $R_{u,v,w}^*$ corresponds to a component (denoted by $R_{a,b,c}^{u,v,w}$) of $H$ containing $u, v$ and $w$; and (iii) $R_{a,b,c}^{u,v,w}$ and $R_{a,b,c}^{u,v,w}$ are TCs or OUCs simultaneously and have the same order. Therefore, it follows from Fact 3.2 and (7) that
\[
W(f_1(H')) = W(H').
\]
\[
(22)
\]
By (13), (14), (16), and (20)–(22), for a fixed $i (2 \leq i \leq n)$, we finally get
\[
\sum_{H' \in \mathcal{H}} W(f(H')) \geq \sum_{H' \in \mathcal{H}} W(H').
\]
\[
(23)
\]
The inequality in (23) holds when at least one of the inequalities in (14), (16) and (21) holds for $b \geq 1$. Therefore, by Lemma 2.1, for $0 \leq i \leq n$, we obtain $\varphi_i(A_n) \geq \varphi_i(A_n^*)$ and the equality holds iff $i = 0, 1$. Thus, we obtain Lemma 3.1. \hfill \Box

In $A_n$ and $A_n^*$, if $Q$ is a tree, then by deleting the proofs for Subcases (I.ii), (II.ii) and (III.ii) in Lemma 3.1, we can easily get Lemma 3.3 as follows.

**Lemma 3.3.** Let $A_n$ and $A_n^*$ be the two graphs as defined in Fig. 1. If $Q$ is a tree, then we have $\varphi_i(A_n) \geq \varphi_i(A_n^*)$ for $0 \leq i \leq n$ and the equalities do not hold for all $i$. 

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Remark 3.4. After performing the α-transformation once from $A_n$ to $A'_n$ in Lemma 3.1, $A_n$ and $A'_n$ have the same bipartition, and the number of pendant vertices of $A'_n$ is one more than that of $A_n$.

Let $B_n$ be the graph shown in Fig. 2(a), where $B_n$ satisfies the following conditions: (i) $v$ and $w$ are two adjacent vertices at $C_1$ of $B_n$; (ii) $u$ is not at $C_1$ and $u$ is adjacent to $v$; (iii) $u$, $v$ and $w$ are identified with $u'$ of a tree $T_3$, $v'$ of a tree $T_1$ and $w'$ of a tree $T_2$, respectively; and (iv) the other vertices at $C_1$, $C_2$ and $C_l$ may be or may not attached by trees. Let $B'_n$ be the graph obtained from $B_n$ by replanting $T_3$ from $u$ to $w$, where $B'_n$ is shown in Fig. 2(b). In other words,

$$B'_n = B_n - \{uy \mid y \in N_{T_3}(u')\} + \{wy \mid y \in N_{T_3}(u')\}. \quad (24)$$

The transformation from $B_n$ to $B'_n$ in (24) is called β-transformation.

Lemma 3.5. For $0 \leq i \leq n$, we have $q_i(B_n) \geq q_i(B'_n)$ where the equality does not hold for all $i$.

Proof. By Lemma 2.1, $q_i(B_n) = q_i(B'_n)$ for $i = 0, 1$. Next, we assume $2 \leq i \leq n$.

For a fixed $i$, we denote by $\mathcal{H}'$ and $\mathcal{H}$ the sets of all the TU-subgraphs of $B'_n$ and of $B_n$ with exactly $i$ edges, respectively. For an arbitrary TU-subgraph $H' \in \mathcal{H}'$, let

$$f_2 : \mathcal{H}' \to \mathcal{H}, H' \to H = f_2(H'), \quad (25)$$

with $V(H) = V(H')$ and

$$E(H) = E(H') - \{ux \mid x \in N_{T_3}(u') \cap V(H')\} + \{ux \mid x \in N_{T_3}(u') \cap V(H')\}.$$

Obviously, $f_2$ is bijective from $\mathcal{H}'$ to $\mathcal{H}$.

Let $N$ be the weight of all the components of $H'$ not containing $u$, $v$ or $w$. Next, four cases are considered as follows.

**Case (I) uv, vw \not\in E(H').**

Two subcases are considered as follows.

**Subcase (I.i) v and w of $H'$ are not contained in a SC.**

In this subcase, for an arbitrary TU-subgraph $H'$ in $\mathcal{H}'$, we denote by $\overline{R}_u$, $\overline{R}_v$ and $\overline{R}_w$ the connected components of $H'$ containing $u$, $v$ and $w$, respectively. Since $uv, vw \not\in E(H')$, $\overline{R}_u = \{u\}$, $\overline{R}_v$ and $\overline{R}_w$ are TCs and they are mutually disjoint. Let $|V(\overline{R}_u - v)| = a$, $|V(\overline{R}_w - T_3 - w)| = b$ and $|V(T_3) \cap V(\overline{R}_w)\setminus \{w\}| = c$. Thus, we get

$$|V(\overline{R}_u)| = 1, \quad |V(\overline{R}_v)| = a + 1, \quad |V(\overline{R}_w)| = b + c + 1. \quad (26)$$

By the bijection $f_2$, in $H$, there exist three components, denoted by $R'_u$, $R'_v$ and $R'_w$, which correspond to $\overline{R}_u$, $\overline{R}_v$ and $\overline{R}_w$, respectively. Obviously, we have: (i) $R'_u$, $R'_v$ and $R'_w$ contain $u$, $v$ and $w$ in $H$, respectively; (ii) $R'_u$, $R'_v$ and $R'_w$ are TCs and they are mutually disjoint; and (iii) $R'_u$ is $\overline{R}_v$. Furthermore, we have

$$|V(R'_u)| = c + 1, \quad |V(\overline{R}_{v})| = a + 1, \quad |V(R'_w)| = b + 1. \quad (27)$$

We have the following statement:
Fact 3.6. Except for the component(s) containing \( u, v \) and \( w \) in \( H' \), an AC of \( H' \) corresponds to the SC of \( H \).

Therefore, by Fact 3.6, (7), (26), and (27), we obtain
\[
W(f_2(H')) - W(H') = (a + 1)(b + 1)(c + 1)N - (a + 1)(b + c + 1)N \\
= N(a + 1)bc \geq 0,
\]
with the third equality iff \( b = 0 \) or \( c = 0 \).

Subcase (III) \( u \) and \( w \) of \( H' \) are contained in a SC.

In this subcase, for an arbitrary TU-subgraph \( H' \) in \( H' \), we denote by \( \overline{R}_1 \) the connected component of \( H' \) containing \( v \) and \( w \). Since \( uv \not\in E(H') \), \( \overline{R}_1 \) is a TC. Since \( uv \not\in E(H') \), \( u \) of \( H' \) is contained in \( \overline{R}_u = \{ u \} \).

Let \( |V(\overline{R}_1 - T_3 - v - w)| = h \) and \( |V(T_3) \cap V(\overline{R}_1)\{w\}| = c \). Thus, we get
\[
|V(\overline{R}_1)| = h + c + 2, \quad |V(\overline{R}_u)| = 1.
\]

By the bijection \( f_2 \), we obtain that \( \overline{R}_1 \) and \( \overline{R}_u \) in \( H' \) correspond to a TC (denoted by \( R'_1 \)) containing \( v \) and \( w \) and \( R'_u \) containing \( u \) in \( H \), respectively. Obviously, we have
\[
|V(R'_1)| = h + 2, \quad |V(R'_u)| = c + 1.
\]

Thus, by Fact 3.6, (7), (29), and (30), we have
\[
W(f_2(H')) - W(H') = (h + 2)(c + 1)N - (h + c + 2)N = N(h + 1)c \geq 0,
\]
with the third equality iff \( c = 0 \).

Case (II) \( uv \in E(H') \) and \( vw \not\in E(H') \).

Two subcases are considered as follows.

Subcase (II.i) \( v \) and \( w \) of \( H' \) are not contained in a SC.

Since \( uv \in E(H') \) and \( vw \not\in E(H') \), by the bijection \( f_2 \), we obtain that a TC (denoted by \( R_{u,v} \)) of order \( a + 2 \) containing \( u \) and \( v \) and \( R_w \) containing \( w \) in \( H' \) correspond respectively to a TC (denoted by \( R'_{u,v} \)) of order \( a + c + 2 \) containing \( u \) and \( v \) and \( R'_w \) containing \( w \) in \( H \), where \( \overline{R}_{u,v} = \overline{R}_u + uv + \overline{R}_v \) and \( \overline{R}'_{u,v} = \overline{R}'_u + uv + \overline{R}'_v \).

Thus, by Fact 3.6, (7), (26), and (27), we have
\[
W(f_2(H')) - W(H') = (a + c + 2)(b + 1)N - (a + 2)(b + c + 1)N \\
= Nc(b - a - 1).
\]

We denote
\[
H'_s = \{ H' \in H' \mid uv \in E(H'), \ vw \not\in E(H'), \ v \text{ and } w \text{ of } H' \text{ are not contained in a SC} \}.
\]

Subcase (II.ii) \( u \) and \( w \) of \( H' \) are contained in a SC.

In this subcase, \( u, v \) and \( w \) of \( H' \) are contained in a TC of order \( h + c + 3 \), which corresponds to a TC of order \( h + c + 3 \) containing \( u, v \) and \( w \) in \( H \) (by the bijection \( f_2 \)). Thus, by Fact 3.6 and (7), we obtain
\[
W(f_2(H')) - W(H') = 0.
\]

Case (III) \( uv \not\in E(H') \) and \( vw \in E(H') \).

Subcase (III.i) \( v \) and \( w \) of \( H' \) are contained in a TC (namely, \( \overline{R}_{v,w} = \overline{R}_v + vw + \overline{R}_w \) of \( H' \)).

By the bijection \( f_2 \), we obtain that \( \overline{R}_{v,w} \) of order \( a + b + c + 2 \) containing \( v \) and \( w \) and \( \overline{R}_u = \{ u \} \) in \( H' \) correspond respectively to a TC (namely, \( R'_{v,w} = R'_v + vw + R'_w \)) of order \( a + b + 2 \) containing \( v \) and \( w \) and \( R'_u \) in \( H \).

Thus, by Fact 3.6, (7) and (27), we have
\[
W(f_2(H')) - W(H') = (a + b + 2)(c + 1)N - (a + b + c + 2)N \\
= Nc(a + b + 1) \geq 0,
\]
with the third equality iff \( c = 0 \).

We denote
\[
\mathcal{H}'_i = \{ H' \in \mathcal{H} \mid uv \notin E(H'), \ vw \in E(H'), \ v \text{ and } w \text{ of } H' \text{ are contained in a TC of } H' \}.
\]

We construct a mapping \( \xi_2 \) from \( \mathcal{H}'_3 \) to \( \mathcal{H}'_4 \) as follows. For \( H' \in \mathcal{H}'_3 \), let
\[
\xi_2 : H' \rightarrow \xi_2(H') = H' - uv + vw.
\]

Obviously, \( \xi_2 \) is bijective. Therefore, there exists a one-to-one relationship between \( \mathcal{H}'_3 \) and \( \mathcal{H}'_4 \). Namely, for an arbitrary \( H' \in \mathcal{H}'_3 \), we can find, by \( \xi_2 \), a unique element \( \xi_2(H') \in \mathcal{H}'_4 \) corresponding to it, and vice versa. For \( H' \in \mathcal{H}'_4 \), by (32) and (34), we obtain
\[
|W_n(f_2(H')) - W(H')| + |W(f_2(\xi_2(H'))) - W(\xi_2(H'))| = 2Nbc \geq 0,
\]
with the second equality iff \( b = 0 \) or \( c = 0 \). Therefore, by (36), we get
\[
\sum_{H' \in \mathcal{H}'_4} |W(f_2(H')) - W(H')| + \sum_{H' \in \mathcal{H}'_4} |W(f_2(\xi_2(H'))) - W(\xi_2(H'))| \geq 0.
\]

**Subcase (III.ii)** \( v \) and \( w \) of \( H' \) are contained in an OUC (namely \( \tilde{R}_1 + vw \)).

By the bijection \( f_2 \), we obtain that \( \tilde{R}_1 + vw \) and \( \tilde{R}_{uv} = \{ u \} \) in \( H' \) correspond respectively to an OUC (namely \( R'_1 + vw \)) containing \( v \) and \( w \) and \( R'_v \) in \( H \). Thus, by Fact 3.6, (7) and (30), we have
\[
W(f_2(H')) - W(H') = 4(c + 1)N - 4N = 4Nc \geq 0,
\]
with the third equality iff \( c = 0 \).

**Case (IV)** \( uv, vw \in E(H') \).

We have three facts: (i) \( u, v \) and \( w \) of \( H' \) are contained in a component of \( H' \) (denoted by \( \tilde{R}_{u,v,uv} \)); (ii) \( \tilde{R}_{u,v,uv} \) corresponds to a component (denoted by \( f_2(\tilde{R}_{u,v,uv}) \)) of \( H \) containing \( u, v \) and \( w \); and (iii) \( \tilde{R}_{u,v,uv} \) and \( f_2(\tilde{R}_{u,v,uv}) \) are TCs or OUCs simultaneously and have the same order. Therefore, by Fact 3.6 and (7), we obtain
\[
W(f_2(H')) = W(H').
\]

By (28), (31), (33), and (37)–(39), for a fixed \( i \), \( 2 \leq i \leq n \), we finally get
\[
\sum_{H' \in \mathcal{H}_i} W(f_2(H')) \geq \sum_{H' \in \mathcal{H}_i} W(H').
\]

The inequality in (40) holds when at least one of the inequalities in (31) and (38) holds for \( c \geq 1 \). By Lemma 2.1, we get \( \varphi_i(B_n) \geq \varphi_i(B'_n) \) for \( 0 \leq i \leq n \) and the equalities do not hold for all \( i \). Therefore, we obtain Lemma 3.5. \( \square \)

**Remark 3.7.** If \( B_n \) is a bipartite unicyclic graph, then after performing the \( \beta \)-transformation once from \( B_n \) to \( B'_n \) in Lemma 3.5, we have three properties: (i) \( B_n \) and \( B'_n \) have the same girth; (ii) \( B_n \) and \( B'_n \) have the same bipartition; and (iii) the number of pendant vertices of \( B'_n \) is one more than that of \( B_n \).

Let \( F_n \) be the graph obtained from \( C_1 = v_1w_2 \cdots v_l \) by identifying \( w_i \) of \( C_1 \) with \( w'_i \) of \( T_i \), where \( T_i \) is a tree, \( w'_i \) is a vertex of \( T_i \), and \( 1 \leq i \leq l \). It is noted that \( T_i \) may be an empty graph, where \( 1 \leq i \leq l \). \( F_n \) is shown in Fig. 3(a). Let \( F'_n \) be the graph obtained from \( F_n \) through the following steps: (i) Replanting the tree \( T_2 \) from \( w_2 \) to \( w'_2 \); (ii) Replanting the tree \( T_3 \) from \( w_3 \) to \( w'_3 \); (ii) Deleting the edge \( w_3w_3 \); and (iv) adding a new edge \( w_1w_4 \). \( F'_n \) is shown in Fig. 3(b). In other words, we have
\[
F'_n = F_n - \{ w_2y : y \in N_{T_i}(w'_2) \} - \{ w_3y : y \in N_{T_i}(w'_3) \} - \{ w_2w_3 \} + \{ w_1y : y \in N_{T_i}(w'_2) \} + \{ w_1y : y \in N_{T_i}(w'_3) \} + \{ w_1w_4 \}.
\]

The transformation from \( F_n \) to \( F'_n \) in (41) is called \( \gamma \)-transformation.
Lemma 3.8. We have \( \varphi_i(F_n) \geq \varphi_i(F'_n) \) for \( 0 \leq i \leq n \) and the equalities do not hold for all \( i \).

Proof. It follows from Lemma 2.1 that \( \varphi_i(F_n) = \varphi_i(F'_n) \) when \( i = 0, 1 \). Next, let \( 2 \leq i \leq n \).

For a fixed \( i \), let \( \mathcal{H}' \) and \( \mathcal{H} \) be the sets of all the TU-subgraphs of \( F'_n \) and of \( F_n \) with exactly \( i \) edges, respectively. For an arbitrary TU-subgraph \( H' \in \mathcal{H}' \), let

\[
f_3 : \mathcal{H}' \to \mathcal{H}, H' \to H = f_3(H'),
\]

with \( V(H) = V(H') \) and

\[
E(H) = E(H') - \{w_4y \mid y \in A\} - \{w_1y \mid y \in B\} - \{w_1w_4\}
+ \{w_2y \mid y \in A\} + \{w_2y \mid y \in B\} + \{w_2w_3\},
\]

where \( A = N_{T_1}(w'_2) \cap V(H') \) and \( B = N_{T_1}(w'_2) \cap V(H') \). Obviously, \( f_3 \) is injective from \( \mathcal{H}' \) to \( \mathcal{H} \).

Let \( N \) be the weight of all the components of \( H' \) not containing \( w_1, w_2, w_3 \) or \( w_4 \).

If all of \( w_1w_2, w_1w_3 \) and \( w_3w_4 \) are contained in \( E(H') \), then we have three facts: (i) \( w_1, w_2, w_3 \) and \( w_4 \) are contained in a component of \( H' \) (denoted by \( R'_{1234} \)); (ii) \( R'_{1234} \) corresponds to a component \( f_3(R'_{1234}) \) of \( H \) containing \( w_1, w_2, w_3, \) and \( w_4 \); and (iii) \( R'_{1234} \) and \( f_3(R'_{1234}) \) are TCs or OUCs simultaneously and have the same order. Furthermore, we have the following statement.

Fact 3.9. Except for the component(s) containing \( w_1, w_2, w_3, \) and \( w_4 \) in \( H' \), an AC of \( H' \) corresponds to the SC of \( H \).

Therefore, by Fact 3.9 and (7), we obtain

\[
W(f_3(H')) = W(H').
\]  

Next, we assume that at least one of \( w_1w_2, w_1w_3 \) and \( w_3w_4 \) does not belong to \( E(H') \). Seven cases are considered as follows.

Case (I) \( w_1w_2, w_1w_4, w_3w_4 \notin E(H') \).

Two subcases are considered as follows.

Subcase (I.i) \( w_1 \) and \( w_0 \) of \( H' \) are not contained in a SC.

In this subcase, for an arbitrary TU-subgraph \( H' \) in \( \mathcal{H}' \), we denote by \( R'_1, R'_2, R'_3, \) and \( R'_4 \) the connected components of \( H' \) containing \( w_1, w_2, w_3, \) and \( w_4 \), respectively. Obviously, \( R'_2 = \{w_2\} \) and \( R'_3 = \{w_3\} \). It is noted that \( R'_1, R'_2, R'_3, \) and \( R'_4 \) are mutually disjoint and they are TCs since \( w_1w_2, w_1w_4, w_3w_4 \notin E(H') \). Let \( |V(R'_1 - T_3 - w_1)| = a, |V(T_2) \cap V(R'_1)\{|w_4|\} = b, |V(T_3) \cap V(R'_1)\{|w_1|\} = c, \) and \( |V(R'_1 - T_2 - w_4)| = d \). Thus, we get

\[
|V(R'_1)| = a + c + 1, \quad |V(R'_2)| = 1, \quad |V(R'_3)| = 1, \quad |V(R'_4)| = b + d + 1.
\]

By the bijection \( f_3 \), in \( H \), there exist four components, denoted by \( R_1, R_2, R_3, \) and \( R_4 \), which correspond to \( R'_1, R'_2, R'_3, \) and \( R'_4 \), respectively. It is noted that \( R_1, R_2, R_3, \) and \( R_4 \) contain respectively \( w_1, w_2, w_3, \) and \( w_4 \) in

Figure 3: \( \gamma \)-transformation from \( F_n \) to \( F'_n \)
We have
\[ |V(R_1)| = a + 1, \quad |V(R_2)| = b + 1, \quad |V(R_3)| = c + 1, \quad |V(R_4)| = d + 1. \] (45)

Therefore, by Fact 3.9, (7), (44), and (45), we obtain
\[ W(f_3(H')) - W(H') = (a + 1)(b + 1)(c + 1)(d + 1)N - (a + c + 1)(b + d + 1)N \]
\[ = N[abcd + ac(b + d + 1) + bd(a + c + 1)] \geq 0. \] (46)

**Subcase (I.ii)** \( w_1 \) and \( w_4 \) of \( H' \) are contained in a SC.

In this subcase, for an arbitrary TU-subgraph \( H' \) in \( H' \), we denote by \( R'_5 \) the connected component of \( H' \) containing \( w_1 \) and \( w_4 \). Since \( w_1 w_4 \notin E(H') \), \( R'_5 \) is a TC. Obviously, \( R'_5 = \{w_2\} \) and \( R'_3 = \{w_3\} \) are the components containing \( w_2 \) and \( w_3 \) in \( H' \), respectively. Let \( |V(R'_5 - T_2 - T_3 - w_1 - w_4)| = h, |V(T_2) \cap V(R'_3)\}|w_4| = b \) and \( |V(T_3) \cap V(R'_3)\}|w_1| = c. \) Thus, we get
\[ |V(R'_3)| = h + b + c + 2, \quad |V(R'_5)| = 1, \quad |V(R'_3)| = 1. \] (47)

By the bijection \( f_3 \), we obtain that \( R'_5, R'_2 \) and \( R'_1 \) in \( H' \) correspond respectively to a TC (denoted by \( R_5 \)) containing \( w_1 \) and \( w_4 \), \( R_2 \) containing \( w_2 \) and \( R_3 \) containing \( w_3 \) in \( H \). It is noted that \( R_5, R_2 \) and \( R_3 \) are mutually disjoint. Obviously, we have
\[ |V(R_3)| = h + 2, \quad |V(R_2)| = b + 1, \quad |V(R_3)| = c + 1. \] (48)

Thus, by Fact 3.9, (7), (47), and (48), we get
\[ W(f_3(H')) - W(H') = (h + 2)(b + 1)(c + 2)N - (h + b + c + 2)N \]
\[ = N[(h + 2)bc + (h + 1)(b + c)] \geq 0. \] (49)

**Case (II)** \( w_1 w_2, w_1 w_4 \in E(H') \) and \( w_3 w_4 \notin E(H') \).

In this case, \( w_3 \) of \( H' \) is contained in \( R'_{2,1,4} \) and \( w_1, w_2 \) and \( w_4 \) of \( H' \) are contained in a component denoted by \( R'_{2,1,4} \). Here \( R'_{i,j,k} = R'_i + w_i w_j + R'_j + w_j w_k + R'_k \) with \( 1 \leq i, j, k \leq 4 \). Obviously, \( |V(R'_{i,j,k})| = |V(R'_i)| + |V(R'_j)| + |V(R'_k)| \) and \( R'_{i,j,k} \) contains \( w_i, w_j, \) and \( w_k \) of \( H' \). Let \( R_{i,j,k} = R_i + w_i w_j + R_j + w_j w_k + R_k \) with \( 1 \leq i, j, k \leq 4 \). Obviously, \( |V(R_{i,j,k})| = |V(R_i)| + |V(R_j)| + |V(R_k)| \) and \( R_{i,j,k} \) contains \( w_i, w_j, \) and \( w_k \) of \( H \). Two subcases are considered as follows.

**Subcase (II.i)** \( R'_{2,1,4} \) is a TC.

By the bijection \( f_3 \), we obtain that \( R'_{2,1,4} \) of order \( a + b + c + d + 3 \) and \( R'_3 = \{w_3\} \) in \( H' \) correspond respectively to a TC (namely \( R_{2,1,3} \)) of order \( a + b + c + 3 \) and \( R_4 \) of order \( d + 1 \) in \( H \). Thus, by Fact 3.9 and (7), we get
\[ W(f_3(H')) - W(H') = (a + b + c + 3)(d + 1)N - (a + b + c + d + 3)N \]
\[ = Nd(a + b + c + 2) \geq 0. \] (50)

We denote
\[ H'_c = \{H' \in H' \mid w_1 w_2, w_1 w_4 \in E(H'), w_3 w_4 \notin E(H'), \text{ and } R'_{2,1,4} \text{ is a TC}. \}

**Subcase (II.ii)** \( R'_{2,1,4} \) is an OUC.

By the bijection \( f_3 \), we obtain that \( R'_{2,1,4} \) and \( R'_3 = \{w_3\} \) in \( H' \) correspond to a TC (namely \( R_5 + w_1 w_2 + R_2 + w_2 w_3 + R_3 \)) of order \( h + b + c + 4 \) containing \( w_1, w_2, w_3, \) and \( w_4 \) in \( H \). Thus, by Fact 3.9 and (7), we have
\[ W(f_3(H')) - W(H') = (h + b + c + 4)N - 4N = N(h + b + c) \geq 0. \] (51)

**Case (III)** \( w_1 w_2 \notin E(H') \) and \( w_1 w_4, w_3 w_4 \in E(H') \).
In this case, \( w_1, w_3 \) and \( w_4 \) of \( H' \) are contained in a component denoted by \( R_{1,4,3}' \). Two subcases are considered as follows.

**Subcase (III.i)** \( R_{1,4,3}' \) is a TC.

By the bijection \( f_3 \), we obtain that \( R_{1,4,3}' \) of order \( a + b + c + d + 3 \) and \( R_2' = \{ w_2 \} \) in \( H' \) correspond respectively to a TC (namely, \( R_{2,3,4} \)) of order \( b + c + d + 3 \) and \( R_1 \) of order \( a + 1 \) in \( H \). Thus, by Fact 3.9 and (7), we get

\[
W(f_3(H')) - W(H') = (a + 1)(b + c + d + 3)N - (a + b + c + d + 3)N
= Na(b + c + d + 2) \geq 0.
\]

We denote

\( \mathcal{H}_0 = \{ H' \in \mathcal{H} | w_1w_2 \notin E(H'), \ w_1w_4, w_3w_4 \in E(H'), \text{ and } R_{1,4,3}' \text{ is a TC} \}. \)

**Subcase (III.ii)** \( R_{1,4,3}' \) is an OUC.

By the bijection \( f_3 \), we obtain that \( R_{1,4,3}' \) and \( R_2' = \{ w_2 \} \) in \( H' \) correspond to a TC (namely, \( R_2 + w_2w_3 + R_3 + w_3w_4 + R_4 \)) of order \( b + c + 4 \) containing \( w_1, w_2, w_3, \) and \( w_4 \) in \( H \). Thus, by Fact 3.9 and (7), we get

\[
W(f_3(H')) - W(H') = (h + b + c + 4)N - 4N = N(h + b + c) \geq 0.
\]

**Case (IV)** \( w_1w_2, w_3w_4 \in E(H') \) and \( w_1w_4 \notin E(H') \).

Two subcases are considered as follows.

**Subcase (IV.i)** \( w_1 \) and \( w_4 \) of \( H' \) are not contained in a SC.

In this subcase, \( w_1 \) and \( w_2 \) of \( H' \) are contained in \( R_{1,2}' \) and \( w_3 \) and \( w_4 \) of \( H' \) are contained in \( R_{3,4}' \), where \( R_{1,2}' = R_1 + w_1w_2 + R_3 \) and \( R_{3,4}' = R_3 + w_3w_4 + R_4 \). By the bijection \( f_3 \), \( R_{1,2}' \) of order \( a + c + 2 \) and \( R_{3,4}' \) of order \( b + d + 2 \) correspond respectively to a TC (namely, \( R_{1,2} \)) of order \( a + b + 2 \) containing \( w_1 \) and \( w_2 \) and a TC (namely, \( R_{3,4} \)) of order \( c + d + 2 \) containing \( w_3 \) and \( w_4 \) in \( H \), where \( R_{1,2} = R_1 + w_1w_2 + R_2 \) and \( R_{3,4} = R_3 + w_3w_4 + R_4 \). Thus, by Fact 3.9 and (7), we get

\[
W(f_3(H')) - W(H') = (a + b + 2)(c + d + 2)N - (a + c + 2)(b + d + 2)N
= N(b - c)(d - a).
\]

We denote

\( \mathcal{H}_5 = \{ H' \in \mathcal{H} | w_1w_2, w_3w_4 \in E(H'), \ w_1w_4 \notin E(H'), \ w_1 \) and \( w_4 \) are not contained in a SC \}. \)

We construct a mapping \( \xi_3 \) from \( \mathcal{H}_5 \) to \( \mathcal{H}_5' \) and a mapping \( \xi_4 \) from \( \mathcal{H}_5 \) to \( \mathcal{H}_6' \) as follow. For \( H \in \mathcal{H}_5 \), let

\[
\xi_3 : H' \rightarrow \xi_3(H') = H' + w_1w_4 - w_3w_4,

\xi_4 : H' \rightarrow \xi_4(H') = H' + w_1w_4 - w_1w_2.
\]

For an arbitrary \( H' \in \mathcal{H}_5' \), we can find, by \( \xi_3 \) and \( \xi_4 \), a unique \( \xi_3(H') \in \mathcal{H}_5' \) and a unique \( \xi_4(H') \in \mathcal{H}_6' \) corresponding to it, respectively. For \( H \in \mathcal{H}_5 \), by (54), (50) and (52), we get

\[
[W(f_3(H')) - W(H')] + [W(f_3(\xi_3(H'))) - W(\xi_3(H'))]
+ [W(f_3(\xi_4(H'))) - W(\xi_4(H'))] = N[d(a + 2b + 2) + a(2c + d + 2)] \geq 0.
\]

Since \( \xi_3 \) and \( \xi_4 \) are bijective, by (55), we have

\[
\sum_{H' \in \mathcal{H}_5 \cup \mathcal{H}_6} [W(f(H')) - W(H')]
= \sum_{H' \in \mathcal{H}_5} [W(f_3(H')) - W(H') + W(f_3(\xi_3(H'))) - W(\xi_3(H'))]
+ [W(f_3(\xi_4(H'))) - W(\xi_4(H'))] \geq 0.
\]
Subcase (IV.ii) $w_1$ and $w_4$ are contained in a SC.
In this subcase, $w_1, w_2, w_3,$ and $w_4$ of $H'$ are contained in a TC of order $h + b + c + 4$, which corresponds to a TC of order $h + b + c + 4$ containing $w_1, w_2, w_3,$ and $w_4$ in $H$ (by the bijection $f_3$). Thus, by Fact 3.9 and (7), we get
\[
W(f_3(H')) - W(H') = 0.
\]
(57)

Case (V) $w_1w_2, w_1w_4 \notin E(H')$ and $w_3w_4 \in E(H')$.
Two subcases are considered as follows.

Subcase (V.i) $w_1$ and $w_4$ of $H'$ are not contained in a SC.
By the bijection $f_3$, we obtain that $R'_1, R'_2 = \{w_2\}$ and $R''_3, R''_4$ with order $b + d + 2$ in $H'$. Therefore, by Fact 3.9, (7), (44), and (45), we get
\[
W(f_3(H')) - W(H') = (a + 1)(b + 1)(c + d + 2)N - (a + c + 1)(b + d + 2)N
= N(ab(c + d + 1) + b(d + 1) + ac - c(d + 1)).
\]
(58)

We denote
\[
\mathcal{H}'_6 = \{H' \in \mathcal{H}' \mid w_1w_2, w_2w_4 \notin E(H'), \ w_3w_4 \in E(H')\}, \ w_1 \text{ and } w_4 \text{ of } H' \text{ are not contained in a SC}.
\]

Subcase (V.ii) $w_1$ and $w_4$ of $H'$ are contained in SC.
By the bijection $f_3$, a TC (namely, $R'_1 + w_3w_4 + R''_2$) with order $h + b + c + 3$ containing $w_1, w_3$ and $w_4$ and $R''_3 = \{w_2\}$ in $H'$ correspond respectively to a TC (namely, $R_3 + w_3w_4 + R_5$) with order $h + c + 3$ containing $w_1, w_3$ and $w_4$ and $R_4$ of order $b + 1$ in $H$. Therefore, by Fact 3.9 and (7), we get
\[
W(f_3(H')) - W(H') = (b + 1)(h + c + 3)N - (b + c + 3)N
= Nbh(h + c + 2) \geq 0.
\]
(59)

Case (VI) $w_1w_2 \in E(H')$ and $w_1w_4, w_3w_4 \notin E(H')$.
Two subcases are considered as follows.

Subcase (VI.i) $w_1$ and $w_4$ of $H'$ are not contained in SC.
By the bijection $f_3$, we obtain that $R''_1, R''_2 = \{w_3\}$ and $R''_4$ in $H'$ correspond respectively to a TC (namely, $R_{1,2}$) of order $a + c + 2$. Therefore, by Fact 3.9, (7), (44), and (45), we obtain
\[
W(f_3(H')) - W(H') = (a + b + 2)(c + 1)(d + 1)N - (a + c + 2)(b + d + 1)N
= N[cd(a + b + 1) + bd + c(a + 1) - b(a + 1)].
\]
(60)

We denote
\[
\mathcal{H}'_7 = \{H' \in \mathcal{H}' \mid w_1w_2 \in E(H'), \ w_1w_4, w_3w_4 \notin E(H')\}, \ w_1 \text{ and } w_4 \text{ of } H' \text{ are not contained in SC}.
\]

We construct a mapping $\xi_5$ from $\mathcal{H}'_6$ to $\mathcal{H}'_7$ as follows. For $H \in \mathcal{H}'_6$, let
\[
\xi_5 : H' \rightarrow \xi_5(H') = H' + w_1w_2 - w_3w_4.
\]

Obviously, $\xi_5$ is bijective. Thus, there exists a one-to-one relationship between $\mathcal{H}'_6$ and $\mathcal{H}'_7$. By (58) and (60), we have
\[
\sum_{H' \in \mathcal{H}'_7 \cup \mathcal{H}'_6} [W(f_3(H')) - W(H')]
= \sum_{H' \in \mathcal{H}'_7} [W(f_3(H')) - W(H') + W(f_3(\xi_5(H')))] - W(\xi_5(H'))] \geq 0.
\]
(61)

Subcase (VI.ii) $w_1$ and $w_4$ of $H'$ are contained in SC.
By the bijection $f_3$, we obtain that a TC (namely, $R_2^* + w_2w_1 + R_3^*$) of order $h + b + c + 3$ containing $w_1$, $w_2$ and $w_4$ and $R'_3 = \{w_3\}$ in $H^*$ correspond respectively to a TC (namely, $R_2 + w_2w_1 + R_3$) of order $h + b + 3$ containing $w_1$, $w_2$ and $w_3$ and $R_3$ of order $c + 1$ in $H$. Therefore, by Fact 3.9 and (7), we obtain

$$W(f_3(H^*)) - W(H^*) = (c + 1)(h + b + 3)N - (h + b + c + 3)N = Nc(h + b + 2) \geq 0.$$  

\section{Case (VII)} $w_1w_2, w_3w_4 \notin E(H^*)$ and $w_1w_4 \in E(H^*)$.

In this case, $w_1$ and $w_4$ of $H^*$ are contained in a SC. Two subcases are considered as follows.

\subsection{Subcase (VII.i)} $w_1$ and $w_4$ of $H^*$ are contained in a TC (namely $R_{14}^* = R_1^* + w_1w_4 + R'_1$).

By the bijection $f_3$, we get that $R_{14}^*$ with order $a + b + c + d + 2$, $R_2^* = \{w_2\}$ and $R'_3 = \{w_3\}$ in $H^*$ correspond respectively to a TC (namely, $R_{23} = R_2 + w_2w_3 + R_3$) of order $b + c + 2$ containing $w_2$ and $w_3$, $R_1$ and $R_4$ in $H$. Therefore, by Fact 3.9, (7) and (45), we get

$$W(f_3(H^*)) - W(H^*) = (a + 1)(b + c + 2)(d + 1)N - (a + b + c + d + 2)N = N[ad + a + d](b + c + 1 + ad) \geq 0.$$  

\subsection{Subcase (VII.ii)} $w_1$ and $w_3$ of $H^*$ are contained in an OUC (namely $R_5^* + w_1w_4$).

By the bijection $f_3$, we get that $R_5^*$ in $H^*$ corresponds to $R_5$ of order $h + 2$ containing $w_1$ and $w_4$ in $H$, and $R_2^* = \{w_2\}$ and $R'_3 = \{w_3\}$ in $H^*$ correspond to a TC (namely, $R_{23} = R_2 + w_2w_3 + R_3$) of order $b + c + 2$ containing $w_2$ and $w_3$ in $H$. Therefore, by Fact 3.9 and (7), we have

$$W(f_3(H^*)) - W(H^*) = (h + 2)(b + c + 2)N - 4N = Nh(b + c + 2(b + c + f)) \geq 0.$$  

By combining the proofs of Cases (I)–(VII), for a fixed $i (2 \leq i \leq n)$, it follows from (43), (46), (49), (51), (53), (56), (57), (59), and (61)–(64) that

$$\sum_{i \in E^*} W(f_3(H^*)) \geq \sum_{i \in E^*} W(H^*).$$  

The inequality in (65) holds when the inequalities in (51) and (53) hold for $b \geq 1$ or $c \geq 1$ or $h \geq 1$. Furthermore, by Lemma 2.1, we have $\varphi_i(F_n) \geq \varphi_i(F_n^*)$ for $0 \leq i \leq n$ and the equalities do not hold for all $i$. Thus, we get Lemma 3.8. \qed

\section{Remark 3.10} After performing the $\gamma$-transformation once from $F_n$ to $F_n^*$ in Lemma 3.8, we have three facts: (i) The girth of $F_n^*$ is two smaller than that of $F_n$; (ii) $F_n$ and $F_n^*$ have the same bipartition; and (iii) The number of pendant vertices of $F_n^*$ is two more than that of $F_n$.

Let $X_{n+1}$ be a star with $n + 1$ vertices and $w_0$ the center vertex of $X_{n+1}$. Let $n'_1 + n'_2 + n'_3 + n'_4 = n - 4, n'_1 + n'_2 + 2 = n_1$ and $n'_2 + n'_3 + 2 = n_2$, where $0 \leq n'_i \leq n - 4$ for $1 \leq i \leq 4$. Let $D_n$ be the graph obtained from $C_4 = w_1w_2w_3w_4$ by identifying $w_i$ with the center vertex of $X_{n'_i+1}$, where $1 \leq i \leq 4$. Let $M_n$ be the graph obtained from $C_4 = w_1w_2w_3w_4$ by identifying $w_1$ with the center vertex of $X_{n'_1+1}$ and by identifying $w_2$ with the center vertex of $X_{n'_2+1}$. In other words,

$$M_n = D_n - \{w_0y\mid y \in A\} - \{w_1y\mid y \in B\} + \{w_1y\mid y \in A\} + \{w_2y\mid y \in B\},$$

where $A = N_{X_{n'_1+1}}(w_0)$ and $B = N_{X_{n'_2+1}}(w_0)$. The transformation from $D_n$ to $M_n$ in (66) is called $\xi$-transformation. $D_n$ and $M_n$ are shown in Figs. 4(a) and (4b), respectively. Obviously, $D_n, M_n \in \mathcal{U}_{m_1,m_2}$.

\section{Lemma 3.11} For $0 \leq i \leq n$, we have $\varphi_i(D_n) \geq \varphi_i(M_n)$ and the equalities do not hold for all $i$. 

By Lemma 2.1, $\varphi_i(D_n) = \varphi_i(D_n')$ for $i = 0, 1$. Next, we assume $2 \leq i \leq n$.

For a fixed $i$, we denote by $H'$ and $H$ the sets of all the TU-subgraphs of $M_n$ and of $D_n$ with exactly $i$ edges, respectively. For an arbitrary TU-subgraph $H' \in H'$, let

$$f_i : H' \rightarrow H, H' \rightarrow H = f_i(H'),$$

with $V(H) = V(H')$ and

$$E(H) = E(H') - |w_1y| y \in A \cap V(H') - |w_2y| y \in B \cap V(H')$$

$$+ |w_3y| y \in A \cap V(H') + |w_4y| y \in B \cap V(H'),$$

where $A = N_{X_{n'}}(w_0)$ and $B = N_{X_{n'-1}}(w_0)$. Obviously, $f_i$ is a bijection from $H'$ to $H$.

Let $N$ be the weight of all the components of $H'$ not containing $w_1$, $w_2$, $w_3$, or $w_4$. In $M_n$, let $w_1w_2 = e_1$, $w_2w_3 = e_2$, $w_3w_4 = e_3$, and $w_1w_4 = e_4$.

If all of $e_1$, $e_2$, $e_3$, and $e_4$ are contained in $E(H')$, then the component containing $w_1$, $w_2$, $w_3$, and $w_4$ in $M_n$ has a cycle with even girth. This is contrary to the definition of TU-subgraph. Therefore, we get that at most three of $e_1$, $e_2$, $e_3$, and $e_4$ are contained in $E(H')$. Three cases are considered as follows.

**Case (i)** None of $e_1$, $e_2$, $e_3$, and $e_4$ is contained in $E(H')$.

In this case, for an arbitrary TU-subgraph $H'$ in $H'$, we denote by $R_{w_1}^*$, $R_{w_2}^*$, $R_{w_3}^*$, and $R_{w_4}^*$ the connected components of $H'$ containing $w_1$, $w_2$, $w_3$, and $w_4$, respectively. Obviously, $R_{w_1}^* = \{w_1\}$ and $R_{w_2}^* = \{w_2\}$. It is noted that $R_{w_1}^*$, $R_{w_2}^*$, $R_{w_3}^*$, and $R_{w_4}^*$ are mutually disjoint and they are TCs. Let $|V(R_{w_1}^* \cap V(X_{n'+1}))\{w_1\}| = s$, $|V(R_{w_2}^* \cap V(X_{n'+1}))\{w_2\}| = t$, and $|V(R_{w_3}^* \cap V(X_{n'+1}))\{w_3\}| = p$. Thus, we get

$$|V(R_{w_1}^*)| = s + q + 1, \quad |V(R_{w_2}^*)| = t + p + 1, \quad |V(R_{w_3}^*)| = 1, \quad |V(R_{w_4}^*)| = 1.$$  

By the bijection $f_i$ in $H$, there exist four components, denoted by $R_{w_1}'$, $R_{w_2}'$, $R_{w_3}'$, and $R_{w_4}'$, which correspond to $R_{w_1}^*$, $R_{w_2}^*$, $R_{w_3}^*$, and $R_{w_4}^*$, respectively. It is noted that $R_{w_1}'$, $R_{w_2}'$, $R_{w_3}'$, and $R_{w_4}'$ contain respectively $w_1$, $w_2$, $w_3$, and $w_4$ in $H$ and they are mutually disjoint. Obviously, $R_{w_1}'$, $R_{w_2}'$, $R_{w_3}'$, and $R_{w_4}'$ are TCs since $e_1, e_2, e_3, e_4 \notin E(H')$.

Thus, we have the following statement:

**Fact 3.12.** Except for the component(s) containing $w_1, w_2, w_3, and w_4 in H'$, an AC of $H'$ corresponds to the SC of $H$.
Therefore, by Fact 3.12, (7), (68), and (69), we get
\[ W(f_4(H')) - W(H') = (s + 1)(t + 1)(p + 1)(q + 1) - (s + q + 1)(t + p + 1) = stpq + sq(t + p + 1) + tp(s + q + 1) \geq 0. \] (70)

**Case (ii)** Only one of \( c_1, c_2, c_3, \) and \( c_4 \) is contained in \( E(H') \).

If \( c_1 \in E(H') \) and \( c_2, c_3, c_4 \notin E(H') \), then by the bijection \( f_4 \), we obtain that a TC (namely, \( R^*_{w_1} + w_1w_2 + R^*_{w_2} \) with order \( s + t + p + q + 2 \) containing \( w_1 \) and \( w_2 \), \( R^*_{w_3} = \{w_4\} \) and \( R^*_{w_4} = \{w_5\} \) in \( H' \) correspond to a TC (namely, \( R'_{w_1} + w_1w_2 + R'_{w_2} \) of order \( s + t + 2 \) containing \( w_1 \) and \( w_2 \), \( R'_{w_3} \) and \( R'_{w_4} \) in \( H \), respectively. Therefore, by Fact 3.12, (7) and (69), we obtain
\[ W(f_4(H')) - W(H') = (s + t + 2)(p + 1)(q + 1) - (s + t + p + q + 2). \] (71)

By the methods similar to (71), we get (72)–(74) as follows.

If \( c_2 \in E(H') \) and \( c_1, c_3, c_4 \notin E(H') \), then
\[ W(f_4(H')) - W(H') = (s + 1)(t + q + 2)(p + 1) - (s + q + 1)(t + p + 2). \] (72)

If \( c_3 \in E(H') \) and \( c_1, c_2, c_4 \notin E(H') \), then
\[ W(f_4(H')) - W(H') = (s + 1)(t + 1)(p + q + 2) - 2(s + q + 1)(t + p + 1). \] (73)

If \( c_4 \in E(H') \) and \( c_1, c_2, c_3 \notin E(H') \), then
\[ W(f_4(H')) - W(H') = (s + p + 2)(q + 1)(t + 1) - (s + q + 2)(t + p + 1). \] (74)

Therefore, in Case (ii), after adding (71)–(74) together, we get
\[ W(f_4(H')) - W(H') = 4p(t + s) + 2(ops + psq + pst + qst) \geq 0. \] (75)

**Case (iii)** Two of \( c_1, c_2, c_3, \) and \( c_4 \) are contained in \( E(H') \).

In this case, there exist six kinds of classification. By the same analysis as those for (75), we get
\[ W(f_4(H')) - W(H') = [(s + t + p + 3)(q + 1) - (s + t + p + q + 3)] + [(s + t + q + 3)(p + 1) - (s + t + p + q + 3)] + [(s + t + q + 3)(p + 1) - (s + t + p + q + 3)] + [(s + t + p + q + 3)(p + 1) - (s + t + p + q + 3)] + [(s + p + q + 2)(t + q + 2) - (s + q + 2)(t + p + 2)] + [(s + p + q + 3)(t + 1) - (s + q + 3)(t + 1) - (s + q + 3)(t + 1) - (s + q + 3)(t + 1)] + [(s + 1)(t + p + q + 3) - (s + q + 1)(t + p + 3)] = 4p(t + s) \geq 0. \] (76)

**Case (iv)** Three of \( c_1, c_2, c_3, \) and \( c_4 \) are contained in \( E(H') \).

In this case, there exist four kinds of classification. By the same analysis as those for (75), we obtain
\[ W(f_4(H)) = W(H'). \] (77)

By combining (70), (75)–(77), for a fixed \( i \) \((2 \leq i \leq n)\), we obtain
\[ \sum_{i \in H'} W(f_4(H')) \geq \sum_{i \in H} W(H'). \] (78)

The inequality in (78) holds when the inequalities in (75) and (76) hold for \( p, s, t \geq 1 \). Furthermore, by Lemma 2.1, we get \( \varphi_i(D_n) \geq \varphi_i(M_n) \) for \( 0 \leq i \leq n \) and the equalities do not hold for all \( i \). □
3.2. The graphs with the minimal SLCs and the minimal IEs among $T_{n_1, n_2}$ and $U_{n_1, n_2}$

In this subsection, we will use the $\alpha$-transformation (presented in Lemma 3.3 in Subsection 3.1) to obtain the graph with the minimal SLCs among $T_{n_1, n_2}$, which is shown in Theorem 3.13. The $\beta_\gamma$- and $\xi$-transformations, as presented in Lemmas 3.5, 3.8 and 3.11 in Subsection 3.1, respectively, will be used to obtain the graph with the minimal SLCs among $U_{n_1, n_2}$, which is shown in Theorem 3.17. Furthermore, by Theorems 3.13 and 3.17, we obtain the graphs with the minimal IEs among $T_{n_1, n_2}$ and $U_{n_1, n_2}$, respectively.

Let $S(n_1, n_2)$ be a tree obtained from $X_m$ and $X_{n_2}$ by adding an edge between the center vertices of $X_m$ and of $X_{n_2}$, where $n_1, n_2 \geq 2$ and $n_1 + n_2 = n$.

**Theorem 3.13.** Let $T \in T_{n_1, n_2}$ with $n_1, n_2 \geq 2$ and $n_1 + n_2 = n$. For $0 \leq i \leq n$, we have $\varphi_i(T) \geq \varphi_i(S(n_1, n_2))$ with all the equalities iff $T = S(n_1, n_2)$.

**Proof.** Let $T_0$ be the graph with the minimal SLCs in $T_{n_1, n_2}$, where $n_1, n_2 \geq 2$ and $n_1 + n_2 = n$. Let $\text{dia}(T_0)$ be the diameter of $T_0$. We suppose $\text{dia}(T_0) \geq 4$. Thus, in $T_0$, there exists a path $P$ of length at least 4. Let $u$, $v$ and $w$ be three vertices lying on $P$ in such a way that $v$ is adjacent to both $u$ and $w$ and the vertex degrees of $u$ and of $w$ are greater than 1. Therefore, $T_0$ can be viewed as the graph $A_n$ (as shown in Fig. 1(a)), where $T_e$ in $A_n$ may be an empty graph. By Lemma 3.3, we can find another graph $A^*_n$ (as shown in Fig. 1(b)) satisfying $\varphi_i(T_0) \geq \varphi_i(A^*_n)$ for $0 \leq i \leq n$ and the equalities do not hold for all $i$. This contradicts the minimality of $T_0$. Therefore, we obtain $\text{dia}(T_0) = 3$ or $\text{dia}(T_0) = 2$. If $\text{dia}(T_0) = 2$, then $T_0 = X_{n+1}$. Since $X_{n+1} \notin T_{n_1, n_2}$, as $n_1, n_2 \geq 2$, we get $\text{dia}(T_0) = 3$. As $T_0 \in T_{n_1, n_2}$ and $\text{dia}(T_0) = 3$, $T_0$ must be $S(n_1, n_2)$. Theorem 3.13 is thus proved.

From Theorem 3.13, we obtain the graph with the minimal IE in $T_{n_1, n_2}$, as shown in Theorem 3.14.

**Theorem 3.14.** Let $T \in T_{n_1, n_2}$ with $n_1, n_2 \geq 2$ and $n_1 + n_2 = n$. We have $\text{IE}(T) \geq \text{IE}(S(n_1, n_2))$ with the equality iff $T = S(n_1, n_2)$.

By Lemmas 3.5–3.11, we get the graph with the minimal SLCs among $U_{n_1, n_2}$, as shown in Theorem 3.17. To obtain Theorem 3.17, we introduce Lemmas 3.15 and 3.16 as follows.

**Lemma 3.15.** If $G_0$ has the minimum SLCs in $U_{n_1, n_2}$, then a cut-edge of $G_0$ must be a pendent edge.

**Proof.** Suppose that $G_0$ has a cut-edge $e = uv$ which is not a pendent edge. Hence $u$ and $v$ are two vertices of degree at least 2 with $N_G(u) \cap N_G(v) = \emptyset$. Without loss of generality, we assume that $G_0$ is $B_n$ (as shown in Fig. 2(a)). By employing the $\beta$-transformation and by Lemma 3.5, there is a graph $B^*_n$ (as shown in Fig. 2(b)) such that $\varphi_i(G_0) \geq \varphi_i(B^*_n)$ for $0 \leq i \leq n$, where $B^*_n$ satisfies these three properties as shown in Remark 3.7. This contradicts the minimality of $G_0$. Therefore, a cut-edge of $G_0$ must be a pendent edge.

**Lemma 3.16.** If $G_0$ has the minimum SLCs in $U_{n_1, n_2}$ and $C_i$ is the cycle of $G_0$, then $l = 4$.

**Proof.** We assume that $G_0$ is $F_i$ (as shown in Fig. 3(a)) and $l \geq 6$. By applying the $\gamma$-transformation and by Lemma 3.8, we obtain a new graph $F^*_i$ (as shown in Fig. 3(b)) having a cycle $C_{l-2}$ such that $\varphi_i(G_0) \geq \varphi_i(F^*_i)$ for $0 \leq i \leq n$ and the equalities do not hold for all $i$, where $F^*_i$ satisfies these three properties as shown in Remark 3.10. This contradicts the minimality of $G_0$. Therefore, $l = 4$.

**Theorem 3.17.** Let $G \in U_{n_1, n_2}$ with $n_1, n_2 \geq 2$ and $n_1 + n_2 = n$. We have $\varphi_i(G) \geq \varphi_i(M_n)$ for $0 \leq i \leq n$ and the equalities do not hold for all $i$.

**Proof.** Let $G_0$ be the graph with the minimum SLCs in $U_{n_1, n_2}$ and $C_i$ the cycle of $G_0$. By Lemmas 3.15 and 3.16, we get that a cut-edge of $G_0$ must be a pendent edge and $l = 4$, respectively. Therefore, we suppose $G_0 = D_n$, where $D_n$ is shown in Fig. 4(a). By applying the $\xi$-transformation and by Lemma 3.11, we have $\varphi_i(D_n) \geq \varphi_i(M_n)$ for $0 \leq i \leq n$ and the equalities do not hold for all $i$, where $M_n$ is shown in Fig. 4(b) and $M_n \in U_{n_1, n_2}$. This contradicts the minimality of $G_0$. Therefore, we finally get $G_0 = M_n$. Theorem 3.17 is thus proved.

By Theorem 3.17, we get the graph with the minimal IE among $U_{n_1, n_2}$, which is shown in Theorem 3.18.

**Theorem 3.18.** Let $G \in U_{n_1, n_2}$ with $n_1, n_2 \geq 2$ and $n_1 + n_2 = n$. We have $\text{IE}(G) \geq \text{IE}(M_n)$ with the equality iff $G = M_n$. 
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