The Roper resonance in a finite volume

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1. Introduction

In recent years much work has been done to understand the hadron spectrum as it emerges from quantum chromodynamics (QCD). However, the excited baryon spectrum of QCD is still not very well understood, and requires further theoretical investigations. At low energies, chiral perturbation theory (ChPT) has proved to be an important tool to describe hadrons and their interactions, particularly in the Goldstone boson sector. The inclusion of baryon states in ChPT is also possible, and baryon chiral perturbation theory (BChPT) is widely and successfully used today. BChPT requires a more sophisticated approach because of the breakdown of power counting due to the inclusion of their heavy degrees of freedom, and their large masses. This issue can, however, be resolved either by using the so-called heavy-baryon approach, or a suitably chosen renormalization scheme within an explicitly Lorentz-invariant formalism, such as infrared regularization (IR) or extended-on-shell (EOMS) approaches (for an example of this, see review [1]). This allows one to investigate the properties of a few low-lying excited states. Unitarization methods allow more meson and baryon resonances to be addressed, though at the cost of introducing some model-dependence, as various unitarization schemes can be employed. To access a larger part of the spectrum, a different approach is required. Lattice QCD is a first principles method that allows one to calculate the hadron spectrum from underlying fundamental quark and gluon fields. The calculations do an outstanding job in describing the low-est-lying hadron states. With ever-increasing computational power, improved algorithms, and refined finite volume methods, hadron ground states in particular are simulated more and more precisely on the lattice. Excited states are more difficult to access, though there has been some visible progress in recent years. The present state of the art is reviewed in [2].

One excited state of particular interest is the Roper resonance, which was discovered in 1964 by means of a partial wave analysis of pion-nucleon scattering data [3]. It is a spin-1/2 state with positive parity (like the nucleon), and with a mass of around $m_R = 1.44$ GeV$^5$ it lies slightly above the delta resonance. The most remarkable feature of this low-lying baryon resonance lies in its decays. In addition to decay into a pion and a nucleon, it also decays into a nucleon and two pions (via the $\Delta N$ and $N\pi$ intermediate states) with a branching fraction comparable to the $N\pi$ mode. This three-particle final state becomes important in lattice simulations involving three or more hadrons; see [5] for a recent review. It is also worth noting that experimental programs are currently being conducted to map out the electromagnetic structure of the Roper resonance, in particular through electro-excititation and related theoretical studies, see e.g. [6–9].
A dedicated lattice QCD study of Roper using both quark and hadron interpolators was performed in [10]; see also [11]. In [10] a number of three-quark interpolating fields was supplemented by operators for $N\pi$ in P-wave and $N\sigma$ in S-wave. In the center-of-momentum frame, three eigenstates below 1.65 GeV were found. No eigenstate corresponding to Roper at $m_R = 1.44$ GeV is found, which indicates that $N\pi$ elastic scattering alone does not yield a low-lying Roper. Coupling with other channels, most notably with $N\pi\pi$, seems to be important for generating Roper resonance. The study of coupled-channel scattering, including a three-particle decay $N\pi\pi$, remains a challenge.

Here, we follow another path. An effective field theory treatment of the Roper resonance has already been established. In order to improve the investigation of the Roper resonance on the lattice, a finite volume and finite volume, i.e. the finite volume corrections of the Roper resonance. Due to the presence of a narrow resonance, the energy levels in the box show a very characteristic behavior near the resonance energy. The energy levels shift when box size $L$ is altered, but they do not cross each other. This is the so-called ‘avoided level crossing’ [12].

In this work we aim to discover whether this behavior can also be seen in the energy levels of the Roper system. To do so, we study the finite volume corrections of the self-energy of the Roper up to the third chiral order $O(p^3)$, and perform a fit of the energy levels. A similar study has already been undertaken for delta resonance in [13], and we treat the Roper resonance accordingly.

Our paper is organized as follows: In section 2, we display the effective chiral two-flavor Lagrangian of pions and baryons (nucleon, delta, and Roper resonance) underlying our calculations. In section 3, we calculate the self-energy of the Roper resonance in the continuum volume. The calculation of Roper self-energy in the infinite volume is given in section 4. The results for the energy levels of the Roper resonance, and further pertinent discussion, are given in section 5. We conclude with a short summary and outlook in section 6. Some technicalities are relegated to the appendices.

2. Effective Lagrangian

Firstly, we discuss the chiral effective Lagrangian required for our calculations. It is taken from [14] (for earlier related work, see e.g. [15–18]) and is given by

$$L_{\text{eff}} = L_{\pi\pi} + L_{\pi\eta} + L_{\pi\sigma} + L_{\pi\Delta} + L_{\pi\Lambda} + L_{\pi\Sigma} + L_{\pi\Xi}.$$  

The dynamical degrees of freedom are pions ($\pi$), nucleons ($N$), the delta ($\Delta$) and the Roper resonance ($R$). We restrict ourselves to flavor SU(2), and work in the isospin limit ($m_u = m_d \equiv m$). In the following, we work to leading one-loop order, $O(p^3)$, where $p$ denotes a small momentum or mass. We count the pion mass as well as the mass differences $m_R - m_N$, $m_{\Delta} - m_N$ and $m_R - m_{\Delta}$ as of order $p$. With reference to higher orders, this naive counting requires modification, as detailed in [14]. Now let us enumerate the contributions required for the $O(p^3)$ calculation of Roper self-energy. The relevant terms from the mesonic Lagrangian are

$$L_{\pi\pi}^{(2)} = \frac{F^2}{4} \text{Tr}(\partial_{\mu}U\partial^{\mu}U^\dagger) + \frac{F^2}{4} \text{Tr}(U\chi^T + \chi U^\dagger),$$

where $U$ is a $2 \times 2$-matrix containing the pion fields, $F$ is the pion decay constant in the chiral limit, which will later be identified with the physical pion decay constant $F_\pi$. Further, $\chi$ is the external scalar source which is given by the diagonal matrix

$$\chi = \begin{pmatrix} M_+^2 & 0 \\ 0 & M_-^2 \end{pmatrix},$$

with the pion mass $M_\pi$ (we have already identified the leading term in the quark mass expansion of the pion mass with its physical value). The leading order (LO) terms of chiral dimension one, and one required next-to-leading order (NLO) term of chiral dimension two, containing pion fields and the spin-1/2 baryons, are as follows:

$$L_{\pi N}^{(1)} = \Psi_N \left( i\not{\partial} - m_{N0} + \frac{1}{2} g_A \not{\gamma} \gamma_5 \right) \Psi_N,$$

$$L_{\pi R}^{(1)} = \Psi_R \left( i\not{\partial} - m_{R0} + \frac{1}{2} g_R \not{\gamma} \gamma_5 \right) \Psi_R,$$

$$L_{\pi R}^{(2)} = c_i \tilde{\Psi}_R \text{Tr}(\chi_{\gamma_5}) \Psi_R,$$

here, $\Psi_N$ and $\Psi_R$ are the isospin doublet fields with chiral limit masses $m_{N0}$ and $m_{R0}$ of the nucleon, and the Roper resonance, respectively. The interaction of these fields with the pion field is characterized by the axial couplings $g_A$ and $g_R$ and the chiral vielbein

$$u_\mu = i(u^\dagger \partial_\mu u - u\partial_\mu u^\dagger),$$

where $u = \sqrt{U}$. The final equation in (4) denotes a term of the second order pion-Roper Lagrangian with the low-energy constant (LEC) $c_i$ and $\chi_+ = u^\dagger u^\dagger + uu$. This term is required for the calculation of the Roper self-energy to be discussed below. Since we are only interested in strong interaction processes we omit all other external sources. The covariant derivative is then given by

$$D_\mu \Psi_{N/R} = (\partial_\mu + \Gamma_\mu) \Psi_{N/R},$$

with

$$\Gamma_\mu = \frac{1}{2} \left[ u^\dagger \partial_\mu u - u\partial_\mu u^\dagger \right].$$

The spin-3/2 delta resonances are introduced as usual in terms of Rarita–Schwinger fields $\Psi_{i\mu}$, $i \in \{1, 2, 3\}$ [19]. The
LO Lagrangian reads
\[
\mathcal{L}^{(1)}_{\pi\Delta} = -\bar{\psi}^i\gamma^\mu \gamma^\nu \left( i\mathcal{D}^\mu - m_{\Delta0} \delta^\mu_\nu \right) g^{\nu\nu} \\
- i(\gamma^\nu D^{\mu} - \gamma^\mu D^{\nu}) + i(\gamma^\mu \mathcal{D}^\nu) \\
+ m_{\Delta0} \delta^\nu_\mu \gamma^\nu \\
+ \frac{g_1}{2} \mathcal{D}^\mu \gamma_{\Sigma S} g^{\nu\nu} + \frac{g_2}{2} (\gamma^\mu u^{\nu\nu} + u^{\nu\nu\nu}) \gamma_5 \\
+ \frac{g_3}{2} \gamma^\mu \mathcal{D}^\nu \gamma_{\Sigma S} g^{\nu\nu} \right) \xi_{ij} / \bar{\psi}_i, \tag{8}
\]
where \( m_{\Delta0} \) is the chiral limit mass of the delta, \( g_1, g_2, \) and \( g_3 \) are coupling constants; these, however, are not independent [20]. Further, \( \xi_{ij} \) is the isospin-3/2 projector
\[
\xi_{ij} = \delta_{ij} \frac{1}{3} \tau_i \tau_j, \tag{9}
\]
in terms of the Pauli-matrices \( \tau_i \). The propagator \( G_{\rho\nu}(k) \) of the spin-3/2 Rarita–Schwinger propagator in \( D \) space-time dimensions is given by
\[
G_{\rho\nu}(k) = \frac{-i(k^\mu + m_{\Delta0})}{k^2 - m_{\Delta0}^2 + i\epsilon} \left( g_{\rho\nu} - \frac{1}{D - 1} \gamma_{\rho\nu} \right) \\
+ \frac{k^\rho k^\mu - \gamma_{\rho\nu} k^\nu}{(D - 1)m_{\Delta0}} - \frac{D - 2}{(D - 1)m_{\Delta0}} k^\rho k^\nu \right). \tag{10}
\]
where we use the physical delta mass \( m_{\Delta0} \), which is legitimate for our calculation. The LO interactions between pions, nucleons, deltas, and Roper resonances are completed by
\[
\mathcal{L}^{(1)}_{\pi\Delta} = \overline{\Psi}_R \left( \frac{g_{\pi NR}}{2} M_{\Delta} \right) \Psi_N + \text{h.c.}, \tag{11}
\]
\[
\mathcal{L}^{(3)}_{\pi\Delta} = h \overline{\Psi}_R \frac{\xi_3}{2} \Theta^{\mu\nu}(z_1) \omega^\mu_R \Psi_N + \text{h.c.}, \tag{12}
\]
\[
\mathcal{L}^{(3)}_{\pi\Delta} = h \overline{\Psi}_R \frac{\xi_3}{2} \Theta^{\mu\nu}(z_2) \omega^\mu_R \Psi_R + \text{h.c.}, \tag{13}
\]
where \( g_{\pi NR} \), \( h \), and \( h_R \) are coupling constants and
\[
\omega^\mu_R = \frac{1}{2} \text{Tr}(\tau^a u^\mu_a), \quad \Theta^{\mu\nu}(z) = g^{\mu\nu} + z^{\mu\rho} \gamma^\rho,
\]
where \( z_1 \) and \( z_2 \) are off-shell parameters. Throughout this text we follow [14] and set \( g_1 = -g_2 = g_3 = 0 \); see also [21, 22]. Further terms must be taken into account if one is interested in performing calculations for higher chiral orders.

3. Self-energy of the Roper resonance

To calculate the mass of the Roper resonance in the infinite (and also finite) volume we need to determine the poles of the dressed propagator
\[
\check{\Sigma}_{\pi\rho}(p) = -\frac{i}{p - m_{\pi} - \Sigma_{\pi\rho}(p)}, \tag{13}
\]
here, \( \Sigma_{\pi\rho} \) denotes the self-energy of the Roper, which can be calculated from all one-particle-irreducible contributions to the two-point function of the Roper resonance field \( \Psi_R \). The poles are obtained by solving the equation
\[
[p - m_{\pi} - \Sigma_{\pi\rho}(p)]\big|_{p = Z} = 0, \tag{14}
\]
where in the infinite volume \( Z \) is parametrized by
\[
Z = m_{\pi} - i \frac{\Gamma_{\pi\rho}}{2}, \tag{15}
\]
in terms of the physical Roper mass \( m_{\pi} \) and its width \( \Gamma_{\pi\rho} \). This implies that the real part of the self-energy corresponds to corrections of \( m_{\pi} \), whereas the imaginary part corresponds to corrections of \( \Gamma_{\pi\rho} \).

At third chiral order, three one-loop diagrams contribute to the self-energy of the Roper resonance, as depicted in figure 1.

The diagrams differ in terms of the internal baryon state, which can be a Roper, a nucleon or a delta baryon. Additionally, there is a contact interaction coming from the second order Lagrangian of the Roper in equation (4). The self-energy up to order \( O(p^4) \) then reads
\[
\Sigma_{\pi\rho}(p) = \sum_{\text{contact int.}}^{(2)}(p) + \sum_{\text{loops}}^{(2)}(p) + \sum_{\text{contact int.}}^{(3)}(p) + O(p^4), \tag{16}
\]
Using the effective Lagrangians from section 2, it is a simple process to set down the expressions for self-energy. For the second order contact interaction we find
\[
\Sigma_{\pi\rho}^{(2)} = -4c_R M_\pi^2, \tag{17}
\]
and the three loop contributions are given by
\[
\Sigma_{\pi\rho}^{(3)}(p) = \frac{3g_{\pi NR}^2}{4F_\pi^2} \left[ \frac{\int d^4k}{(2\pi)^2} \frac{i\gamma_5(p - k + m_{\pi}) \gamma_5}{[(p - k)^2 - m_{\pi}^2 + i\epsilon]} \right], \tag{18}
\]
\[
\Sigma_{\pi\rho}^{(3)}(p) = \frac{3g_{\pi NR}^2}{4F_\pi^2} \left[ \frac{\int d^4k}{(2\pi)^2} \frac{i\gamma_5(p - k + m_{\pi}) \gamma_5}{[(p - k)^2 - m_{\pi}^2 + i\epsilon]} \right]. \tag{19}
\]
\[
\Sigma_{\pi\Delta}(p) = \frac{2\hbar^2}{F_\pi^2} \int \frac{d^4k}{(2\pi)^4} \frac{i\delta \gamma_5(p - k)G^{\pi\pi}(k)(p - k)_\nu}{((p - k)^2 - M_\pi^2 + i\epsilon)},
\]

where \(G^{\pi\pi}(k)\) is given in equation (10). The three one-loop contributions to the Roper mass can be expanded in terms of the scalar Passarino-Veltman integrals (PV integrals). This expansion is accomplished using the Mathematica package FeynCalc [23, 24]. The definitions and solutions of the PV integrals can be found in appendix A. This results in

\[
\Sigma_{\pi\Delta}(p = m_R) = \frac{3g_{\pi NN}^2}{32F_\pi^2} \left\{ m_R^2 B_0(m_R, m_R, M_\pi^2) + A_0(m_R^2) \right\},
\]

\[
\Sigma^{(2)}_{\pi\Delta}(p = m_R) = \frac{3g_{\pi NN}^2}{32F_\pi^2} \left\{ m_R^2 B_0(m_R, m_R, M_\pi^2) + A_0(m_R^2) \right\},
\]

\[
\Sigma^{(3)}_{\pi\Delta}(p = m_R) = \frac{3g_{\pi NN}^2}{32F_\pi^2} \left\{ m_R^2 B_0(m_R, m_R, M_\pi^2) + A_0(m_R^2) \right\},
\]

or

\[
\Sigma^{(3)}_{\pi\Delta}(p = m_R) = \frac{3g_{\pi NN}^2}{32F_\pi^2} \left\{ m_R^2 B_0(m_R, m_R, M_\pi^2) + A_0(m_R^2) \right\}.
\]

Evaluating these scalar integrals in the infinite volume leads to well-known infinities requiring to be tamed within the framework of renormalization. Procedures such as the \(\overline{\text{MS}}\) scheme use redefinitions of the bare parameters in the Lagrangian to subtract the infinities. Additionally, in baryonic ChPT one will encounter terms in the expansion of the loop diagrams which breach the terms of power counting. These so-called power counting violating terms may be addressed using various techniques, such as the heavy-baryon approach, IR, or the EOMS scheme. Within this EOMS scheme one performs additional finite subtractions in order to cancel out those which violate the power counting terms. Ultimately one obtains a finite result consistent with power counting. Further details relevant to our calculations can be found in [25, 26].

4. Finite volume formalism

Next, we consider the Roper resonance in a finite volume. We place our system in a cubic box of length \(L\) and calculate the difference between finite and infinite volume [27]. In the case of finite volume, the (Euclidean) loop integral is replaced by an infinite sum of the spatial momenta while the integration over the time component remains unchanged (in actual lattice QCD calculations, the time direction is also discrete, but we keep it continuous for simplicity):

\[
\int \frac{d^4k_E}{(2\pi)^4} \ldots \mapsto \int_{-\infty}^{\infty} \frac{dk_E}{2\pi} \frac{1}{L} \sum_{\vec{k}} \ldots.
\]

In a finite volume the spatial momenta are discretized and can only take values that are integer multiples of \(2\pi/L\) (for a general discussion of theories with spontaneous symmetry breaking in a finite volume, see e.g. [28]), thus:

\[
\vec{k} = \frac{2\pi}{L} \vec{n}, \quad \vec{n} \in \mathbb{Z}^3.
\]

This change obviously influences the self-energy of the Roper resonance. The poles of the propagator are now given by

\[
p = m_R - \Sigma^R(p) = 0,
\]

where \(\Sigma^R(p)\) denotes the self-energy of the Roper in the finite box. The difference between the self-energy in the box and in the infinite volume is defined as the finite volume correction (FV correction) of the system [29, 30]

\[
\Sigma^F(p) = \Sigma^R(p) - \text{Re} \left\{ \Sigma^R(p) \right\}.
\]

Note that in the case of finite volume, the self-energy can only yield real values due to the summation over real momenta \(\vec{k}\). Therefore we must restrict the infinite volume self-energy to its real part in order to ensure a non-imaginary FV correction.

Using equation (27) we can reformulate equation (26). We choose the center-of-mass frame \(p_L = (E, 0)\) and use the

We used Källén’s triangle function \(\lambda(x, y, z) = (x - y - z)^2 - 4yz\) to simplify the lengthy expression of \(\Sigma_{\pi\Delta}\).
on-shell condition $p = E$ to obtain

$$0 = E - m_R = |\Sigma(E) + \text{Re}\{\Sigma(E)\}|,$$

$$E = \frac{[m_R + \text{Re}\{\Sigma(E)\}]}{m_R}$$

\[ \Rightarrow m_R = E = -\Sigma^L(E), \]

where we used the definition of the physical Roper mass $m_R$, i.e. the real part of the pole, in the final step. At third chiral order, the contributions to self-energy are as shown in figure 1. We get

$$m_R - E = -\{\Sigma^{L(3)}(E) + \Sigma^{L(3)}(E) + \Sigma^{L(3)}_{\text{NR}}(E)\}, \quad (28)$$

and our goal will be the numerical evaluation of this equation. The three one-loop contributions to the Roper mass have been expanded in terms of the PV integrals in the last section. Now we must replace the infinite volume quantities with their finite volume expressions. We thereby obtain

$$\Sigma^{L(3)}_{\text{NR}}(E) = -\frac{3g^2_{\text{NR}}}{12\pi^2 F^2_E} \left(E + m_N\right) \left(E + m_N\right) - \chi \left(E, m_N^2, M^2\right) - A_0^L(m_N^2) + (m_N - E) A_0^L(M^2)\right\}, \quad (29)$$

$$\Sigma^{L(3)}_{\text{NR}}(E) = -\frac{3g^2_{\text{NR}}}{12\pi^2 F^2_E} \left(E + m_N\right) \left(E + m_N\right) - \chi \left(E, m_N^2, M^2\right) - A_0^L(m_N^2) + (m_N - E) A_0^L(M^2)\right\}, \quad (30)$$

$$\Sigma^{L(3)}_{\text{NR}}(E) = \frac{h^2}{96\pi^2 M^2_E} \left[-(m_{\Delta} + E)^2 - M^2\right] \times \left(\chi(E, m_N^2, M^2) B_0^L(E^2, M^2) + [m_N^2 + 2m_N E + (M^2 - E^2)^2]

- m_N^2(M^2 + 2M^2) + 2m_N E(E^2 - M^2) \right) A_0^L(m_N^2) + [m_N^2 + 2m_N E - 2m_N E^3 - E^4 - 2m_N M^2 + 6m_N EM^2]

+ 5E^2M^2 + M^4 A_0^L(M^2)), \quad (31)$$

where $A_0^L$ and $B_0^L$ are the finite volume corrections of the PV integrals, which will be determined in the following section.

### 4.1. Calculation of loop integrals in the finite volume

Let us consider as an example

$$A_0(m^2) = -16\pi^2 \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k^2 + m^2}. \quad (32)$$

in four-dimensional Minkowski space. We will follow the procedure described in [13] here. First of all we perform the Wick rotation $k_0 \to ik_4$ to Euclidean space, so that the integral can be rewritten as

$$A_0(m^2) = -16\pi^2 \int d^4k_E \frac{1}{(2\pi)^4} \frac{1}{k^2 + m^2}. \quad (33)$$

Now we can define the finite volume PV integral by replacing the Euclidean spatial integral with a discrete sum

$$\int d^4k_E \frac{1}{k^2 + m^2} \Rightarrow \sum_{\vec{n}} \frac{1}{k^2 + |\vec{k}|^2 + m^2}. \quad (34)$$

where the momenta $\vec{k}$ are restricted according to equation (25). The evaluation of this sum is the next task. Firstly, we note that the function inside the sum is regular, i.e. it does not possess a pole on the real axis for all values of $k_0$ and $\vec{k}$. We can therefore employ the so-called sum trick to simplify the calculation. We insert the Dirac Poisson delta into the equation

$$A_0^L(m^2) = -16\pi^2 \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k^2 + m^2} \sum_{\vec{n}} \exp(i\vec{L} \vec{n} \cdot \vec{k}). \quad (35)$$

Plugging this result into our finite volume PV integral we obtain

$$A_0^L(m^2) = -16\pi^2 \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k^2 + m^2} \sum_{\vec{n}} \exp(i\vec{L} \vec{n} \cdot \vec{k}). \quad (37)$$

where we have regained a four-dimensional integral over momenta multiplied by the sum of exponential functions. We observe that the term $\vec{n} = \vec{0}$ in the sum reproduces the finite volume PV integral from equation (33)

$$A_0^L(m^2) = -16\pi^2 \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k^2 + m^2} \sum_{\vec{n} \neq \vec{0}} \exp(i\vec{L} \vec{n} \cdot \vec{k}). \quad (38)$$

Thus, the finite volume correction is given by

$$\tilde{A}_0^L(m^2) = A_0^L(m^2) - A_0(m^2) = -16\pi^2 \sum_{\vec{n} \neq \vec{0}} \int \frac{d^4k_E}{(2\pi)^4} \frac{\exp(i\vec{L} \vec{n} \cdot \vec{k})}{k^2 + m^2}. \quad (39)$$

The remaining integral is finite and may be solved using standard methods. After integrating the spatial part we are left
with
\[ \tilde{A}_0^L (m^2) = -4 \sum_{j=0}^{\infty} \frac{1}{L} \int_0^\infty dk_4 e^{-L \sqrt{k_4^2 + m^2}} \]
\[ = -4m^2 \sum_{j=0}^{\infty} \frac{K_0(mL)}{mL}, \]  
(40)
where \( j = |m| = \sqrt{n_x^2 + n_y^2 + n_z^2} \) and \( K_0(z) \) is the modified Bessel function of the second kind. For large values of box length \( L \) and summation index \( j \), the finite volume correction decreases exponentially, so that it becomes negligible for large volumes (this is generally to be expected for \( M, L > 4 \)).

A similar calculation can be done for \( \tilde{B}_0^l \). Having performed the Wick rotation in the infinite volume, the integral has the form
\[ B_0^l (E^2, m_\chi^2, M_\chi^2) = 16 \pi^2 \]
\[ \times \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + M_\chi^2)((P - k)^2 + m^2)}, \]
and the finite volume expression is given by
\[ B_0^l (E^2, m_\chi^2, M_\chi^2) = 16 \pi^2 \int_{-\infty}^0 \frac{dk_4}{2\pi L} \]
\[ \times \sum_k \frac{1}{[k_x^2 + k_y^2 + k_z^2][(k - k)^2 + m^2]}. \]  
(42)
The next step is to use Feynman parameterization (see appendix B for further details) to combine the two denominators, before performing a shift in the non-discret momentum component \( k_x \). The resulting expression reads thus:
\[ B_0^l (E^2, m_\chi^2, M_\chi^2) = 16 \pi^2 \int_{-\infty}^0 dy \int \frac{dk_4}{2\pi L} \]
\[ \times \sum_k \frac{1}{[k_x^2 + k_y^2 + g_\chi (y, E^2)]^2}, \]  
(43)
with
\[ g_\chi (y, E^2) = y(y-1)E^2 + ym_\chi^2 + (1-y)M_\chi^2. \]  
(44)
Depending on the values for \( E, m_\chi \) and \( M_\chi \), the function \( g_\chi (y, E^2) \) can be positive, negative or zero for some values of \( y \). If \( g_\chi (y, E^2) > 0 \) for all \( y \in [0, 1] \), then the function inside the sum is regular, and we can again use the Poisson formula analogously to \( \tilde{A}_0^L \). The finite volume correction is then as follows:
\[ \tilde{B}_0^l (E^2, m_\chi^2, M_\chi^2) = 2 \int_{-\infty}^0 dy \sum_{j=0}^{\infty} K_0(Lj \sqrt{g_\chi (y, E^2)}). \]  
(45)
In addition, the correction here drops exponentially for large \( L \) and \( j \). Note that the parameter integral over \( y \) must be evaluated numerically.

However, if the function \( g_\chi \) is negative or equal to zero for some values of \( y \), the Poisson formula is no longer applicable. For example, if in our calculation \( m_N = m_\chi \) then the difference between the pole position \( E \) and \( m_\chi \) is small and \( g_\chi \) remains positive. If \( m_N = m_N, M_\chi \), then the function can become negative, and it is necessary to find an alternative method for evaluating the finite volume contribution. To do so we again follow [13], and introduce a scale \( \mu \), which will be used to subtract ultraviolet divergences from the infinite sum. Also the scale can be chosen in such a way that the function \( g_\chi (y, \mu^2) \) stays positive. We expand the finite volume correction as follows:
\[ \tilde{B}_0^l (E^2, m_\chi^2, M_\chi^2) = B_0^l (E^2, m_\chi^2, M_\chi^2) - \text{Re} \{ B_0^l (E^2, m_\chi^2, M_\chi^2) \}
\]
\[ = B_0^l (E^2, m_\chi^2, M_\chi^2) - \text{Re} \{ B_0^l (E^2, m_\chi^2, M_\chi^2) \}
\]
\[ + \tilde{B}_0^l (\mu^2, m_\chi^2, M_\chi^2) - B_0^l (\mu^2, m_\chi^2, M_\chi^2)
\]
\[ + B_0 (\mu^2, m_\chi^2, M_\chi^2)
\]
\[ + (E^2 - \mu^2) \frac{d}{dE^2} \tilde{B}_0^l (E^2, m_\chi^2, M_\chi^2)
\]
\[ - B_0 (E^2, m_\chi^2, M_\chi^2) + B_0 (E^2, m_\chi^2, M_\chi^2) \}
\]
\[ = 16 \pi^2 \{ H_0^l (E^2) + H_1^l (E^2) + H_2^l (E^2) \}, \]  
(46)
and separate it into three terms, which are given by
\[ 16 \pi^2 H_0^l (E^2) = \{ B_0^l (E^2, m_\chi^2, M_\chi^2) - B_0 (\mu^2, m_\chi^2, M_\chi^2)
\]
\[ - (E^2 - \mu^2) \frac{d}{dE^2} B_0^l
\]
\[ \times (E^2, m_\chi^2, M_\chi^2) \}
\]
\[ = 16 \pi^2 \{ H_0^l (E^2) + H_1^l (E^2) + H_2^l (E^2) \}, \]
(47)
\[ 16 \pi^2 H_1^l (E^2) = \{ \tilde{B}_0^l (\mu^2, m_\chi^2, M_\chi^2)
\]
\[ + (E^2 - \mu^2) \frac{d}{dE^2} \tilde{B}_0^l (E^2, m_\chi^2, M_\chi^2) \}
\]
(48)
\[ 16 \pi^2 H_2^l (E^2) = - \{ \text{Re} \{ B_0 (E^2, m_\chi^2, M_\chi^2) \}
\]
\[ - B_0 (\mu^2, m_\chi^2, M_\chi^2)
\]
\[ - (E^2 - \mu^2) \frac{d}{dE^2} B_0
\]
\[ \times (E^2, m_\chi^2, M_\chi^2) \}
\]
(49)
The first subtraction term, with the newly introduced scale \( \mu \), ensures that the correction converges, with the derivative terms leading to a faster convergence. The first term, \( H_0^l \), contains only terms with momentum sums. Having completed integration over \( k_4 \), one obtains
\[ H_0^l (E^2) = \{ \text{Re} \{ B_0 (E^2, m_\chi^2, M_\chi^2) \}
\]
\[ - B_0 (\mu^2, m_\chi^2, M_\chi^2)
\]
\[ - (E^2 - \mu^2) \frac{d}{dE^2} B_0
\]
\[ \times (E^2, m_\chi^2, M_\chi^2) \}
\]
\[ \times \frac{1}{(E^2 + E_\chi)^2 - E^2 ((E^2 + E_\chi)^2 - \mu^2)^2}. \]  
(50)
with
\[ E_\chi = \sqrt{m_\chi^2 + \left( \frac{2}{T} \right)^2 \frac{m^2}{m^2}} \]  
and
\[ E_\chi = \sqrt{M_\chi^2 + \left( \frac{2}{T} \right)^2 \frac{m^2}{m^2}}. \]
In the second term, one finds finite volume corrections that can be calculated by means of the Poisson summation formula, since the functions are regular, and obtain
\[ H_2^l (E^2) = \frac{1}{8 \pi^2} \int_0^1 dy \sum_{j=0}^{\infty} \left\{ K_0(Lj \sqrt{g_\chi (y, \mu^2)}
\]
\[ - (E^2 - \mu^2) \frac{y(y-1)Lj}{2 \sqrt{g_\chi (y, \mu^2)}} K_0(Lj \sqrt{g_\chi (y, \mu^2)}) \right\}, \]
(51)
The final term only contains infinite volume quantities, which can be calculated using standard methods:

\[ H^L_k(E^2) = \frac{-B_k}{32\pi^2E^2} \left\{ \int \left( \frac{E^2 + m_N^2 - M^2_e + B_k}{E^2 + m_N^2 - M^2_e - B_k} \right) \right. \\
+ \ln \left( \frac{E^2 - m_N^2 - M^2_e + B_k}{E^2 - m_N^2 - M^2_e - B_k} \right) \right. \\
+ \frac{B_p}{16\pi^2E^2} \left\{ \arctan \left( \frac{\mu^2 + m_N^2 - M^2_e}{B_p} \right) \right. \\
+ \left. \arctan \left( \frac{\mu^2 - m_N^2 + M^2_e}{B_p} \right) \right\} \right\}, \]

where we again used the triangle function to define

\[ B_k = \lambda^{\prime \prime /2}(E^2, m_N^2, M^2_e) \]  
\[ = \sqrt{(E^2 - m_N^2 - M^2_e)^2 - 4m_N^2M^2_e}, \]  
\[ B_p = i\lambda^{\prime \prime /2}(\mu^2, m_N^2, M^2_e) \]  
\[ = \sqrt{-(\mu^2 - m_N^2 - M^2_e)^2 + 4m_N^2M^2_e}. \]

We have now evaluated all the required PV integrals for the finite volume. We note that the issue of using the PV reduction in the finite volume has already been discussed in [13], and we refer to that paper for details. Thus we return to the main task, the numerical calculation of equation (28).

5. Results

5.1. Calculation of energy levels

We now want to determine the energy spectrum of the Roper resonance system. To obtain this we take a look at equation (28), and attempt to find numerical solutions for the energy \( E \) for a range of box sizes \( L \). Due to the presence of Roper resonance, we expect to see the so-called avoided level crossing of the energy levels.

Before we start to solve equation (28) by numerical methods, let us again consider the results of the finite volume PV integrals. We have seen that all regular functions in the self-energy of Roper resonance decrease exponentially for large \( L \). This includes all tadpoles, i.e. all \( \Lambda_0 \) functions, as well as \( \tilde{B}_0(E^2, m_N^2, M^2_e) \) from the \( \pi N \) loop in figure 1. Selecting a large value for \( L \) allows us to neglect the contributions from these functions, leaving only the FV correction from the \( \pi N \) and the \( \pi \Delta \) loop (see also [13, 31]). This significantly expediates the numerical computation of the energy levels. The simplified equation is as follows:

\[ m_R - E = \frac{3g_{\pi NR}^2}{128\pi^2F^2_P} (E + m_N)^2 \]

\[ \times [(E - m_N)^2 - M^2_e] \tilde{B}_0^L(E^2, m_N^2, M^2_e) \]

\[ + \frac{h_R^2}{96\pi^2F^2_P m_N^2 E} [(m_N + E)^2 - M^2_e] \]

\[ \times \lambda(E^2, m_N^2, M^2_e) \tilde{B}_0^L(E^2, m_N^2, M^2_e). \]  

(55)

where only non-regular functions, together with two LECs \( g_{\pi NR} \) and \( h_R \) remain. Omitting these contributions also simplifies the treatment of terms which breach power counting, and which would normally appear in such a calculation. The remaining expression, however, does not contain any such terms, so that an additional renormalization scheme, such as EOMS, is redundant. Additional remarks on this issue are given in [13]. Further numerical studies of the energy levels are performed by means of this equation. Values of the parameters used are given in the next subsection.

5.2. Numerical results

For hadron masses and constants we use the numerical values from [14]. The baryion masses are \( m_N = 939 \text{ MeV} \), \( m_R = 1365 \text{ MeV} \), and \( m_\pi = 1210 \text{ MeV} \). For the pion mass we use \( M_\pi = 139 \text{ MeV} \), and for the pion decay constant \( F_\pi = 92.2 \text{ MeV} \). The two coupling constants are also taken from [14] and are \( g_{\pi NR} = \pm 0.47 \), and \( h_R = h = 1.42 \), with the assumption that coupling \( h_R \) is equal to the pion-nucleon-delta-coupling \( h \) (so-called maximal mixing) [15]. Note that the sign of \( g_{\pi NR} \) does not matter, as this coupling appears squared in our analysis. We must also select the scale \( \mu \) for the calculation, set \( \mu = m_N \) for the nucleon, and \( \mu = m_\Delta \) for the delta case.

We now have everything we need in order to find the numerical values of \( E \). We evaluate the sums in the finite volume corrections from \( |\bar{\mu}|^2 = 1 \) up to and including \( |\bar{\mu}|^2 = 4 \). We then solve equation (55) based on the respective energy levels for different box sizes. To make things easier we first look at the Roper resonance without the delta, i.e. we set \( h_R = 0 \) and take only the interaction between Roper, nucleon, and pion into account. The results are displayed in figure 2.

Energy is depicted in units of the nucleon mass \( m_N \), and the box size \( L \) is multiplied by the pion mass \( M_\pi \). The red solid lines denote the numerical results of \( E \) for the respective energy levels, while the blue dashed lines denote the free energy levels of the pion-nucleon final states, i.e.

\[ E_{\pi N}^{\text{free}}(\bar{\mu}) = \sqrt{m_N^2 + \left(\frac{2\pi}{L}\right)^2 |\bar{\mu}|^2} + \sqrt{M_\pi^2 + \left(\frac{2\pi}{L}\right)^2 |\bar{\mu}|^2}, \]  

(56)

\[ \text{This is the more reliable pole mass [4]}. \]
for $|\vec{n}|^2 = 1, 2, 3, 4$. We can clearly see signs of avoided level crossing at small box sizes, whereas the curves asymptotically approach free energy levels at larger box sizes. Moreover, the curves seem to switch between free energy levels, which is also a typical behavior for a resonance (see [13]). This phenomenon can be observed in particular in the upmost curve between the $|\vec{n}|^2 = 3$ and $|\vec{n}|^2 = 4$ levels. This is precisely the energy region where the Roper resonance is to be found, i.e. $1365 \text{ MeV} / m_N \approx 1.45$ (which is called the ‘critical value’ from here on) and the curves increasingly approximate the free energy levels at energies below the critical value.

Now we will do the opposite and set $g_{\pi NR} = 0$. The calculation is performed as above, and is displayed in figure 3.

The free energy levels of the pion and delta in the final state, i.e.

$$E_{n \Delta}^{\text{free}}(\vec{n}) = \sqrt{m_{\Delta}^2 + \left(2\pi L \right)^2 |\vec{n}|^2} + \sqrt{M_r^2 + \left(2\pi L \right)^2 |\vec{n}|^2},$$

are denoted by the gray dashed lines. This time we see no clear evidence for an avoided level crossing. One reason for this is the fact that we are now in an energy region which is mostly above the critical value. Only the two lowest-lying energy levels come close to this energy. Another reason is the relatively large coupling $h_R$, which tends to ‘wash out’ the typical signature of avoided level crossing. This effect has been also observed in the energy levels for delta resonance in a box [13]. It is important to note that although the delta baryon is itself a resonance, we treat it as a stable particle here. This holds as a first approximation with the argument that the Roper resonance first decays in a pion and a delta baryon (or pion and nucleon), and that subsequently the delta can decay further. For future investigations we should take the unstable nature of the delta into account. One possible method of achieving this might be the replacement of the delta propagator in our calculations with a modified propagator, including the decay width of the delta. This will be attempted in a forthcoming work.

Now we take a look at the full system, including pions, nucleons, and deltas. Our results are given in figure 4, which now includes both possible interactions. The avoided level crossing is again visible at small box sizes, and most pronounced between the free $|\vec{n}|^2 = 3$ level of the pion and the nucleon, and at the free $|\vec{n}|^2 = 2$ level of the pion and the delta. Switching between different free energy levels can clearly be observed in the vicinity of the critical value. We also depict the results without free energy levels in figure 5, in order to better display the shape of the curves in this energy region. In regions further from the critical value, and for larger box sizes, the energy levels behave similarly to free energy levels. Regarding the part of the fit with small box sizes, one may wish to inquire as to what would happen at $M_r L$ values smaller than those depicted. Smaller box sizes are problematic for two reasons: one is the fact that for small box

Figure 2. Energy levels for various box sizes $L$, considering only pion and nucleon as intermediate states. Red solid lines display the numerical results, and blue dashed lines the free energy levels of the pion and nucleon for $|\vec{n}|^2 = 1, 2, 3, 4$ (lowest to highest curve).

Figure 3. Energy levels for different box sizes $L$ considering only pion and delta baryon as intermediate states. Red solid lines display the numerical results, and gray dashed lines the free energy levels of the pion and delta for $|\vec{n}|^2 = 1, 2, 3, 4$ (lowest to highest curve).

Figure 4. Energy levels for the full system for different box sizes $L$. Red solid lines display the numerical results and blue dashed lines, gray dashed lines display the free energy levels of the pion and nucleon, pion, and delta, respectively.
sizes around $M_N L \simeq 3$ the numerical calculation is already quite unstable due to the overlapping energy levels. At smaller $M_N L$ it will be extremely difficult to distinguish between the different levels. The other reason is that at smaller box sizes, exponentially suppressed contributions from the tadpoles and the $\pi R$ loop can no longer be neglected, and must be considered explicitly.

All in all the energy levels behave according to our expectations, and we see the typical signature of avoided level crossing due to a resonance. A next possible step would be the investigation of the energy levels with the inclusion of an unstable delta resonance propagator. Furthermore, a calculation beyond chiral order $O(p^3)$ should be considered.

6. Summary and conclusions

In this paper, we have analyzed Roper resonance in a finite volume. The calculation of the Roper self-energy has been repeated up to the third chiral order in the infinite volume, and the extension to the finite volume case has been achieved in order to find the finite volume corrections for the system. We have seen that the FV correction of self-energy contains exponentially suppressed contributions for large $L$, which we neglected, as well as contributions with poles requiring to be regularized. The calculation of energy levels has been performed using physical baryon and pion masses, and only two LECs had to be taken into account, as determined in [14]. The main results are as follows:

- In the delta-free case ($h_R = 0$), avoided level crossing can clearly be observed in the vicinity of the Roper resonance energy. For large box sizes, the energy levels approach those of free energy.
- In the nucleon-free case ($\langle \xi_{NR} \rangle = 0$) there are no clear signs for avoided level crossing. This is due to the large value of $h_R$ and the fact that the energy region lies mostly above that of Roper. The approach to free energy levels for large $L$ is not as explicit as that of delta free energy.
- Looking at the full system with nucleons and deltas, the avoided level crossing is observed again. Also in this case the approach to the free energy levels for large $L$ can be seen.

Note that all the calculations discussed here may also be performed at non-physical pion masses. The remaining question is the treatment of delta resonance in the finite volume. Assuming the delta to be a stable particle is a reasonable first approximation, but in any further calculations, its resonance characteristic must be included. Further, a calculation to fourth order (or higher) in the chiral expansion may be considered. However, this will increase the number of LECs to be taken into account.

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Appendix A. Passarino-Veltman Integrals

The Passarino-Veltman Integrals [32] (see also [33]) are a specific representation of loop integrals, which we use here. The infinities emerging from the evaluation of the loop integrals in dimensional regularization are contained in $R$, which is given by

$$ R = \frac{2}{D - 4} - \left[ \log(4\pi) + \Gamma'(1) + 1 \right], $$

where $D$ denotes the space-time dimension and $\Gamma$ is the Gamma function. This term will be canceled in the $\overline{\text{MS}}$ renormalization scheme commonly used in ChPT calculations.

The following list contains the loop functions that appear in our calculations, and gives their respective results in the infinite volume:

- Integral with one propagator:

$$ A_0(m^2) = -16\pi^2i\mu^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 - m^2 + i\epsilon} \bigg|_{k^2 = \mu^2} = -m^2 \left[ R + \log \left( \frac{m^2}{\mu^2} \right) \right]. $$

- Integral with two propagators:

$$ A_1(m^2) = \frac{1}{\mu^2} \left[ \frac{16\pi^2}{m^2} \right] \left[ R + \log \left( \frac{m^2}{\mu^2} \right) \right]. $$

- Integral with three propagators:

$$ A_2(m^2) = \frac{1}{\mu^2} \left[ \frac{16\pi^2}{m^2} \right] \left[ \frac{16\pi^2}{m^2} \right] \left[ R + \log \left( \frac{m^2}{\mu^2} \right) \right]. $$
Integral with two propagators:

\[
B_0(p^2, m^2, M^2) = -16\pi^2 i\mu^{4-D} \\
\times \int \frac{d^Dk}{(2\pi)^D} \frac{1}{[k^2 - m^2 + i\epsilon][(k - p)^2 - M^2 + i\epsilon]} \\
= (-1) \left[ R + 1 + \log \left( \frac{m^2}{\mu^2} \right) + \frac{p^2 - m^2 + M^2}{p^2} \log \left( \frac{M}{m} \right) \right] \\
+ \frac{2mM}{p^2} F(\Omega),
\]

where

\[
F(\Omega) = \left\{ \begin{array}{ll}
\sqrt{\Omega^2 - 1} \log(-\Omega - \sqrt{\Omega^2 - 1}), & \Omega \leq -1 \\
\sqrt{1 - \Omega^2} \arccos(-\Omega), & -1 \leq \Omega \leq 1 \\
\sqrt{\Omega^2 - 1} \log(\Omega + \sqrt{\Omega^2 - 1}) - i\pi \sqrt{\Omega^2 - 1}, & 1 \leq \Omega
\end{array} \right.
\]

and

\[
\Omega = \frac{p^2 - m^2 - M^2}{2mM}.
\]

Tensor integrals with two propagators:

\[
B^{\nu}(p^2, m^2, M^2) = -16\pi^2 i\mu^{4-D} \\
\times \int \frac{d^Dk}{(2\pi)^D} \frac{k^\nu}{[k^2 - m^2 + i\epsilon][(k - p)^2 - M^2 + i\epsilon]} \\
:= p^\nu B_1(p^2, m^2, M^2),
\]

where

\[
B_1(p^2, m^2, M^2) = \frac{1}{2p^2} \left[ (p^2 + m^2 - M^2) \right] \\
\times B_0(p^2, m^2, M^2) - A_0(m^2) + A_0(M^2),
\]

and

\[
B^{\mu\nu}(p^2, m^2, M^2) = -16\pi^2 i\mu^{4-D} \\
\times \int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu k^\nu}{[k^2 - m^2 + i\epsilon][(k - p)^2 - M^2 + i\epsilon]} \\
:= g^{\mu\nu} B_{00}(p^2, m^2, M^2) + p^{\mu} p^{\nu} B_1(p^2, m^2, M^2)
\]

with

\[
B_{00}(p^2, m^2, M^2) = \frac{1}{2(D - 1)} \left[ 2m^2 B_0(p^2, m^2, M^2) + A_0(M^2) \right] \\
- \{p^2 + m^2 - M^2\} B_1(p^2, m^2, M^2),
\]

and

\[
B_1(p^2, m^2, M^2) = \frac{1}{2p^2} \left[ (p^2 + m^2 - M^2) B_1(p^2, m^2, M^2) \right] \\
+ A_0(M^2) - 2B_{00}(p^2, m^2, M^2).
\]

Integral with three propagators:

\[
The integrals with three propagators are:
\]

\[
C_0(p^2, m^2, M^2) = i\mu^{4-D} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{[k^2 - m^2 + i\epsilon][(k - p)^2 - M^2 + i\epsilon]}
\]

\[
= \frac{1}{2m} \frac{\partial}{\partial m} B_0(p^2, m^2, M^2),
\]

and

\[
C_1(p^2, m^2, M^2) = \frac{1}{4p^2} [B_0(p^2, m^2, M^2) - B_0(0, m^2, m^2) + (m^2 - p^2 - M^2) C_0(p^2, m^2, M^2)].
\]

Some special cases also arise, which are listed below:

\[
B_0(0, m^2, m^2) = (-1) \left[ R + 1 + \log \left( \frac{m^2}{\mu^2} \right) \right],
\]

\[
B_0(m^2, 0, m^2) = (-1) \left[ R + 1 + \log \left( \frac{m^2}{\mu^2} \right) \right].
\]

These relations can be shown with the explicit form of $B_0$ and it follows that

\[
B_0(0, m^2, m^2) + 2 = B_0(m^2, 0, m^2).
\]

Appendix B. Useful Formulas

This section contains a handful of useful formulae used in our calculations.

- Feynman parameter:

\[
\frac{1}{AB} = \int_0^1 \frac{dy}{yA + (1 - y)B}.
\]

- Modified Bessel functions of the second kind (see, e.g., [34]):

\[
K_\nu(z) := \int_0^\infty dt \cosh(\nu t) e^{-z \cosh(t)}, \quad \text{for } z > 0.
\]

Special case:

\[
K_\nu(z) = \int_0^\infty dt \frac{\cos(zt)}{\sqrt{t^2 + 1}}, \quad \text{for } z > 0.
\]

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