Standing waves for 6-superlinear Chern-Simons-Schrödinger systems with indefinite potentials

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Abstract. In this paper we consider 6-superlinear Chern-Simons-Schrödinger systems. In contrast to most studies, we consider the case where the potential \( V \) is indefinite so that the Schrödinger operator \( -\Delta + V \) possesses a finite-dimensional negative space. We obtain nontrivial solutions for the problem via Morse theory.

Keywords: Chern-Simons-Schrödinger system; Palais-Smale condition; Local linking; Morse theory

1. Introduction

In this paper, we consider the following Chern-Simons-Schrödinger system (CCS system) in \( H^1(\mathbb{R}^2) \):

\[
\begin{aligned}
-\Delta u + V(x)u + A_0 u + \sum_{j=1}^{2} A_j^2 u &= f(x, u), \\
\partial_1 A_0 &= A_2 |u|^2, \\
\partial_2 A_0 &= -A_1 |u|^2, \\
\partial_1 A_2 - \partial_2 A_1 &= -\frac{1}{2} u^2, \\
\partial_1 A_1 + \partial_2 A_2 &= 0.
\end{aligned}
\]

where \( V \in C(\mathbb{R}^2) \) is potential and \( f \in C(\mathbb{R}^2, \mathbb{R}) \) is the nonlinearity. The (CCS) system describes the nonrelativistic thermodynamic behavior of large number of particles in an electromagnetic field.

The (CCS) system (1.1) arises when we are looking for standing waves for the following nonlinear Schrödinger system

\[
\begin{aligned}
iD_0 \phi + (D_1 D_1 + D_2 D_2) \phi + f(x, \phi) &= 0, \\
\partial_0 A_1 - \partial_1 A_0 &= -\text{Im} (\bar{\phi} D_2 \phi), \\
\partial_0 A_2 - \partial_2 A_0 &= \text{Im} (\bar{\phi} D_1 \phi), \\
\partial_1 A_2 - \partial_2 A_1 &= -\frac{1}{2} |\phi|^2,
\end{aligned}
\]

where \( i \) denotes the imaginary unit, \( \partial_0 = \frac{\partial}{\partial x_1}, \partial_1 = \frac{\partial}{\partial x_1}, \partial_2 = \frac{\partial}{\partial x_2} \), for \((t, x_1, x_2) \in \mathbb{R}^{1+2}, \phi : \mathbb{R}^{1+2} \to \mathbb{C} \) is the complex scalar field, \( A_\mu : \mathbb{R}^{1+2} \to \mathbb{R} \) is the gauge field. The associated covariant differential operators are given by

\[
D_\mu := \partial_\mu + i A_\mu, \quad \mu = 0, 1, 2.
\]

System (1.2) proposed in \([1, 2]\) consists of the Schrödinger equation augmented by the gauge field \( A_\mu \). This feature of the model is important for the study of the high-temperature superconductor, fractional quantum Hall effect and Aharonov-Bohm scattering.

We suppose that the gauge field satisfies the Coulomb gauge condition \( \partial_0 A_0 + \partial_1 A_1 + \partial_2 A_2 = 0 \), and \( A_\mu(x, t) = A_\mu(x), \mu = 0, 1, 2 \). Then we deduce that \( \partial_1 A_1 + \partial_2 A_2 = 0 \). Moreover, standing

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waves for (1.2) are obtained through the ansatz $\phi = u(x)e^{i\omega t}$, $f(x, u) = f(x, u)e^{i\omega t}$, $\omega > 0$, resulting in
\[
\begin{align*}
-\Delta u + \omega u + A_0 u + \sum_{j=1}^{2} A_j u = f(x, u), \\
\partial_1 A_0 = A_2 |u|^2, \\
\partial_1 A_1 = -2|u|^2, \\
\partial_1 A_1 + \partial_2 A_2 = 0.
\end{align*}
\] (1.3)

Here the components $A_1$ and $A_2$ in system (1.3) can be represented by solving the elliptic equation
\[\Delta A_1 = \partial_2 \left( \frac{|u|^2}{2} \right) \quad \text{and} \quad \Delta A_2 = -\partial_2 \left( \frac{|u|^2}{2} \right),\]
which provide
\[
\begin{align*}
A_1 &= A_1[u](x) = \frac{x_2}{2\pi |x|^2} * \left( \frac{|u|^2}{2} \right) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x_2 - y_2 |u(y)|^2}{|x - y|^2} \, dy, \\
A_2 &= A_2[u](x) = \frac{x_1}{2\pi |x|^2} * \left( \frac{|u|^2}{2} \right) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x_1 - y_1 |u(y)|^2}{|x - y|^2} \, dy,
\end{align*}\]
where $*$ denotes the convolution.

Similarly,
\[\Delta A_0 = \partial_1 (A_2 |u|^2) - \partial_2 (A_1 |u|^2),\]
which gives the following representation of the component $A_0$:
\[A_0 = A_0[u](x) = \frac{x_1}{2\pi |x|^2} * (A_2 |u|^2) - \frac{x_2}{2\pi |x|^2} * (A_1 |u|^2).\]

The (CCS) system (1.1) has attracted considerable attention in recent decades, which can be seen in [3, 4, 5, 6, 7, 8, 9] and the references therein. We emphasize that in all these papers, the authors only considered the case where the Schrödinger operator $-\Delta + V$ is positive definite. In this case, the quadratic part of the variational functional $\Phi$ given in (2.1) is positively definite, the zero function $u = 0$ is a local minimizer of $\Phi$ and the mountain pass theorem [10] can be applied. However, when the potential $V$ is negative somewhere so that the quadratic part of $\Phi$ is indefinite, the zero function $u = 0$ is no longer a local minimizer of $\Phi$, the mountain pass theorem is not applicable anymore. For stationary NLS equations
\[-\Delta u + V(x)u = f(x, u)\]
with indefinite Schrödinger operator $-\Delta + V$, one usually applies the linking theorem to get solution, see e.g. [11, 12]. For system (1.1), it seems hard to verify the linking geometry due to the nonnegative terms involving $A_j^2 u^2$ in (2.1), which prevent the functional $\Phi$ to be nonpositive on the negative space of the Schrödinger operator. Hence, the classical linking theorem [13, Lemma 2.12] is also not applicable.

For this reason, there are very few results about (1.1) with indefinite potential. It seems that [14] is the only work devoted to this situation. To overcome these difficulties, and the difficulty that the Sobolev embedding
\[H^1(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)\]
is not compact, it is assumed in [14] that, roughly speaking, $V$ is coercive so that the related Sobolev space is compactly embedded into $L^2(\mathbb{R}^2)$. Then the local linking theory of Li and Willem [15] is applied to get critical point of $\Phi$.

Motivated by the above observation and [16] on Schrödinger-Poisson systems (see also [17]), in this paper, we will consider the case that $V$ is bounded, so that the above-mentioned compact embedding may not be true. From now on all integrals are taken over $\mathbb{R}^2$ except stated explicitly.

Now we are ready to state our assumptions on $V$ and $f$. 

(V) $V \in C(\mathbb{R}^2)$ is a bounded function such that the quadratic form

$$Q(u) = \frac{1}{2} \int (|\nabla u|^2 + V(x)u^2)$$

is non-degenerate and the negative space of $Q$ is finite-dimensional.

(f$_1$) $f \in C(\mathbb{R}^2 \times \mathbb{R})$ satisfies

$$\lim_{|t| \to 0} \frac{f(x,t)}{t} = 0, \quad \lim_{|t| \to \infty} \frac{f(x,t)}{e^{\mu t^2}} = 0$$

for any $\mu > 0$, where $F(x,t) = \int_0^t f(x,\tau) d\tau$.

(f$_2$) For $(x,t) \in \mathbb{R}^2 \times \mathbb{R} \setminus \{0\}$ we have $0 < 6F(x,t) \leq tf(x,t)$, moreover, for almost all $x \in \mathbb{R}^2$

$$\lim_{|t| \to \infty} \frac{F(x,t)}{t^6} = +\infty. \quad (1.4)$$

(f$_3$) One of the following conditions is satisfied:

(f$_{31}$) there exist $C_0 > 0$ and $\nu \in (0, 6)$ satisfying $F(x,t) \geq C_0|t|^{\nu}$ for all $t \in \mathbb{R}$;

(f$_{32}$) for some $\delta > 0$, $F(x,t) \leq 0$ for all $|t| \leq \delta$.

(f$_4$) For any $r > 0$, we have

$$\lim_{|x| \to \infty} \sup_{0 < |t| \leq r} \frac{|f(x,t)|}{t} = 0.$$

(f$_5$) For some $s > 2$, $p, q > 1$ we have $a \in L^\infty(\mathbb{R}^2) \cap L^p(\mathbb{R}^2), b \in L^\infty(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ such that

$$|f(x,t)| \leq a(|x|)|t|^{s-1} + b(x)|t|^{q-1}. \quad (1.5)$$

We emphasize that in (f$_5$), the exponents $p$ and $q$ can be chosen arbitrarily from $(1, \infty)$, see Remark 4.2.

Now we are ready to state the main results of this paper.

**Theorem 1.1.** Suppose that (V), (f$_1$), (f$_2$), (f$_3$) and (f$_4$) are satisfied. Then system (1.1) has a nontrivial solution.

**Theorem 1.2.** Suppose that (V), (f$_1$), (f$_2$), (f$_3$) and (f$_5$) are satisfied. Then system (1.1) has a nontrivial solution.

**Remark 1.3.** As we have mentioned before, neither the mountain pass lemma nor the linking theorem can be applied to our functional $\Phi$. It turns out that $\Phi$ has a local linking at the origin. Unfortunately, at present all critical point theorems involving local linking require the functional to satisfy global compactness condition. The role of our condition (f$_4$) or (f$_5$) is to ensure such compactness.

The paper is organized as follows. In Section 2 we prove that the (PS) sequences of $\Phi$ are bounded and $\Phi$ satisfies the (PS) condition. In Section 3 we will prove Theorem 1.1 by applying Morse theory. For this purpose after recalling some concepts and results in infinite-dimensional Morse theory [18], we will compute the critical group of $\Phi$ at infinity and then give the proof of Theorem 1.1. Finally, after investigating the compactness of the operator $K'$ (see Lemma 4.1), we use Morse theory again to prove Theorem 1.2 in Section 4.

### 2. Palais-Smale Condition

Throughout this paper we always denote $X = H^1(\mathbb{R}^2)$. Under the assumptions (V) and (f$_1$), similar to Kang and Tang [14] we can show that the functional $\Phi : X \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2} \int (|\nabla u|^2 + V(x)u^2 + A_1^2(u)u^2 + A_2^2(u)u^2) - \int F(x,u). \quad (2.1)$$

is well defined and of class $C^1$. The derivative of $\Phi$ is given by

$$\langle \Phi'(u), v \rangle = \int \nabla u \cdot \nabla v + \int V(x)uv + \int \left[\left(A_1^2(u) + A_2^2(u)\right)uv + A_0uv\right] - \int f(x,u)v.$$
for \( u, v \in X \). Consequently, critical points of \( \Phi \) are weak solutions of system (1.1).

To study the functional \( \Phi \), it will be convenient to rewrite the quadratic part \( Q \) in a simpler form. It is well known that, if (V) holds, then there exists an equivalent norm \( \| \cdot \| \) on \( X \) such that

\[
Q(u) = \frac{1}{2} \left( \| u^+ \|^2 - \| u^- \|^2 \right),
\]

where \( u^+ \) is the orthogonal projection of \( u \) on \( X^+ \) being \( X^\pm \) the positive/negative space of \( Q \). Using this new norm, \( \Phi \) can be rewritten as

\[
\Phi(u) = \frac{1}{2} \left( \| u^+ \|^2 - \| u^- \|^2 \right) + \frac{1}{2} \int \left( A_1^2(u)u^2 + A_2^2(u)u^2 \right) - \int F(x, u).
\]

By simple calculation (also see [5, 9]), we obtain, for any \( u \in X \),

\[
\langle \Phi'(u), u \rangle = \| u^+ \|^2 - \| u^- \|^2 + \int \left( A_1^2(u)u^2 + A_2^2(u)u^2 \right) - \int f(x, u)u.
\]

Next, we recall the following properties of the terms involving \( A_0, A_1, A_2 \).

**Lemma 2.1 ([14]).** There is a constant \( a_1 > 0 \) such that for all \( u \in X \) we have

\[
0 \leq \int \left( A_1^2(u) + A_2^2(u) \right) u^2 \leq a_1 \| u \|^6.
\]

**Lemma 2.2 ([5, Proposition 2.1]).** Let \( 1 < r < 2 \) and \( \frac{1}{r} - \frac{1}{r} = \frac{1}{2} \). If \( u \in X \), then

\[
|A_0(u)| \leq C |u|_{2,r}^r, \quad |A_1(u)| \leq C |u|_{2,r}^r,
\]

where \( i = 1, 2 \) and \( | \cdot | \) is the \( L^r \)-norm.

**Lemma 2.3.** If (V), (f1) and (f2) hold, then all (PS) sequences of \( \Phi \) are bounded.

**Proof.** Let \( \{ u_n \} \) be a (PS) sequence of \( \Phi \), that is,

\[
\sup_n |\Phi(u_n)| < \infty, \quad \Phi'(u_n) \to 0.
\]

It suffices to show that \( \{ u_n \} \) is bounded. Suppose \( \{ u_n \} \) is unbounded, we may assume \( \| u_n \| \to \infty \). Let \( v_n = \| u_n \|^{-1} u_n \). Then

\[
v_n = v^+_n + v^-_n \to v = v^+ + v^- \in X, \quad v^+_n, v^-_n \in X^\pm.
\]

If \( v = 0 \), then \( v^- \to v^- = 0 \) because \( \dim X^- < \infty \). Since

\[
\| v^+_n \|^2 + \| v^-_n \|^2 = 1,
\]

for \( n \) large enough we have

\[
\| v^+_n \|^2 - \| v^-_n \|^2 \geq \frac{1}{2}.
\]

Therefore, by assumption (f2), we deduce that for \( n \) large enough,

\[
1 + \sup_n |\Phi(u_n)| + \| u_n \| \geq \Phi(u_n) - \frac{1}{6} \langle \Phi'(u_n), u_n \rangle
\]

\[
= \frac{1}{3} \| u_n \|^2 \left( \| v^+_n \|^2 - \| v^-_n \|^2 \right) + \int \left( \frac{1}{6} f(x, u_n)u_n - F(x, u_n) \right)
\]

\[
\geq \frac{1}{6} \| u_n \|^2,
\]

contradicting \( \| u_n \| \to \infty \).

If \( v \neq 0 \), then the set \( \Theta = \{ v \neq 0 \} \) has positive Lebesgue measure. For \( x \in \Theta \) we have \( |u_n(x)| \to \infty \) and

\[
\frac{F(x, u_n(x))}{\| u_n \|^6} = \frac{F(x, u_n(x))}{u_n^6(x)} v^6_n(x) \to +\infty,
\]

(2.4)
thanks to (1.4). By Fatou lemma we deduce from (2.4) that
\[
\int F(x, u_n) \leq \int_{\|u_n\|_0^6} F(x, u_n) \geq \frac{1}{\|u_n\|_0^6} \int F(x, u_n) \rightarrow +\infty. \tag{2.5}
\]
It follows from Lemma 2.1 that
\[
\frac{1}{\|u_n\|_0^6} \int F(x, u_n) = \frac{\|u_n^*\|^2 - \|u_n\|^2}{2\|u_n\|_0^6} + \frac{1}{2\|u_n\|_0^6} \int (A^2_1(u_n)u_n^2 + A^2_2(u_n)u_n^2) - \frac{\Phi(u_n)}{\|u_n\|_0^6} \leq \frac{a_1}{2} + 1,
\]
a contradiction to (2.5). Therefore \(\{u_n\}\) is bounded in \(X\).

To get a convergent subsequence of the (PS) sequence, we need some compact properties of operators involving \(A_j, j = 0,1,2\). Firstly, we need to investigate the \(C^1\)-functional \(N : X \rightarrow \mathbb{R}\),
\[
N(u) = \frac{1}{2} \int (A^2_1(u)u^2 + A^2_2(u)u^2).
\]
It is known that the derivative of \(N\) is given by
\[
\langle N'(u), v \rangle = \int \left( (A^2_1(u) + A^2_2(u))uv + A_0(u)uv \right), \quad u, v \in X.
\]

**Lemma 2.4.** The functional \(N\) is weakly lower semi-continuous, its derivative \(N' : X \rightarrow X^\ast\) is weakly sequentially continuous, where \(X^\ast = H^{-1}(\mathbb{R}^2)\) is the dual space of \(X = H^1(\mathbb{R}^2)\).

**Proof.** Let \(\{u_n\}\) be a sequence in \(X\) such that \(u_n \rightharpoonup u\) in \(X\), we need to show
\[
N(u) \leq \liminf N(u_n), \quad \langle N'(u_n), \phi \rangle \rightarrow \langle N'(u), \phi \rangle,
\]
for all \(\phi \in X\).
Since \(u_n \rightharpoonup u\) in \(X\), up to a subsequence, by the compactness of the embedding \(X \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^2)\), we have
\[
u_n \rightarrow u \quad \text{in} \ L^2_{\text{loc}}(\mathbb{R}^2), \quad u_n(x) \rightarrow u(x) \quad \text{a.e. in} \ \mathbb{R}^2.
\]
According to Wan and Tan [5, Proposition 2.2], for \(j = 1,2\) we have \(A^2_j(u_n) \rightarrow A^2_j(u)\) a.e. in \(\mathbb{R}^2\). Moreover, by the Fatou lemma,
\[
N(u) = \frac{1}{2} \int (A^2_1(u)u^2 + A^2_2(u)u^2) \leq \frac{1}{2} \liminf \int (A^2_1(u_n)u_n^2 + A^2_2(u_n)u_n^2) = \liminf N(u_n).
\]
Hence \(N\) is weakly lower semi-continuous.
To prove the weak continuity of \(N'\), we observe that
\[
\langle N'(u_n) - N'(u), \phi \rangle = \int \left( A^2_1(u_n)u_n\phi - A^2_1(u)u\phi \right) + \int \left( A^2_2(u_n)u_n\phi - A^2_2(u)u\phi \right)
\]
\[
+ \int (A_0(u_n)u_n\phi - A_0(u)u\phi). \tag{2.6}
\]
Since
\[
\int \left( A^2_1(u_n)u_n\phi - A^2_1(u)u\phi \right) = \int A^2_j(u_n)(u_n - u)\phi + \int (A^2_j(u_n) - A^2_j(u))u\phi, \quad j = 1,2. \tag{2.7}
\]
By Lemma 2.2, Hölder inequality and continuous embedding yield that
\[
\int |A^2_j(u_n)(u_n - u)|^2 \leq |A^2_j(u_n)|^2 \|u_n - u\|^2 \leq C\|u_n\|^4 \|u_n - u\|^2 \leq C.
\]
Combining $u_n \to u$ a.e. in $\mathbb{R}^2$ and $A_j^2(u_n) \to A_j^2(u)$ a.e. in $\mathbb{R}^2$, we have $A_j^2(u_n)(u_n - u) \to 0$ in $L^2(\mathbb{R}^2)$. Thus
\[
\int A_j^2(u_n)(u_n - u)\phi \to 0.
\] (2.8)

Similarly,
\[
\int \left( A_j^2(u_n) - A_j^2(u) \right) u\phi \to 0.
\] (2.9)

On the other hand
\[
\int (A_0(u_n)u_n\phi - A_0(u)u\phi) = \int A_0(u_n)(u_n - u)\phi + \int (A_0(u_n) - A_0(u)) u\phi.
\] (2.10)

By Lemma 2.2, Hölder inequality and continuous embedding yield that
\[
\int |A_0(u_n)(u_n - u)|^2 \leq |A_0(u_n)|^2 |u_n - u|^2
\]
\[
\leq C|u_n|^2|u_n - u|^2
\]
\[
\leq C||u_n||^4||u_n - u||^2 \leq C.
\]

Combining $u_n \to u$ a.e. in $\mathbb{R}^2$, we have $A_0(u_n)(u_n - u) \to 0$ in $L^2(\mathbb{R}^2)$. Thus
\[
\int A_0(u_n)(u_n - u)\phi \to 0.
\] (2.11)

Similarly,
\[
\int (A_0(u_n) - A_0(u)) u\phi \to 0.
\] (2.12)

From (2.6)-(2.12), for $\phi \in X$ we have
\[
\langle \mathcal{N}'(u_n), \phi \rangle \to \langle \mathcal{N}'(u), \phi \rangle.
\]

Therefore, we have proved that $\mathcal{N}'$ is weakly sequentially continuous. \qed

**Lemma 2.5.** Let $u_n \rightharpoonup u$ in $X$. Then
\[
\lim \int \left[ \left( A_1^2(u_n) + A_2^2(u_n) \right) u_n(u_n - u) + A_0(u_n)u_n(u_n - u) \right] \geq 0.
\]

**Proof.** Applying Lemma 2.4, we have
\[
\lim N(u_n) \geq N(u), \quad \lim \langle \mathcal{N}'(u_n), u \rangle = \langle \mathcal{N}'(u), u \rangle.
\]

Therefore,
\[
\lim \int \left[ \left( A_1^2(u_n) + A_2^2(u_n) \right) u_n(u_n - u) + A_0(u_n)u_n(u_n - u) \right]
\]
\[
= \lim \int \left[ 3 \left( A_1^2(u_n) + A_2^2(u_n) \right) u_n^2 - \left( A_1^2(u_n) + A_2^2(u_n) \right) u_n u - A_0(u_n)u_n u \right]
\]
\[
= \lim \left( 6N(u_n) - \langle \mathcal{N}'(u_n), u \rangle \right)
\]
\[
\geq 6N(u) - \langle \mathcal{N}'(u), u \rangle = 0.
\] \qed

**Lemma 2.6.** If $(V)$, $(f_1)$, $(f_2)$ and $(f_4)$ hold, then the functional $\Phi$ satisfies the $(PS)$ condition, that is, any $(PS)$ sequence $\{u_n\} \subset X$ possesses a convergent subsequence.

**Proof.** Let $\{u_n\}$ be a $(PS)$ sequence. We know from Lemma 2.3 that $\{u_n\}$ is bounded in $X$. Up to a subsequence we may assume $u_n \rightharpoonup u$ in $X$. We have
\[
\int (\nabla u_n \cdot \nabla v + V(x)u_n u) \to \int (|\nabla u|^2 + V(x)u^2) = ||u^+||^2 - ||u^-||^2.
\]

Consequently
\[
\phi(1) = \langle \Phi'(u_n), u_n - u \rangle
\]
We have \( u_n^{-} \to u^{-} \) and \( \|u_n^{-}\| = \|u^{-}\| \) because \( \dim X^{-} < \infty \). Collecting all infinitesimal terms, we obtain
\[
\|u_n^+\|^2 - \|u^+\|^2 = o(1) + \int f(x, u_n)(u_n - u) - \int \left( [A_1^2(u_n) + A_2^2(u_n)] u_n(u_n - u) + A_0(u_n) u_n(u_n - u) \right).
\]
Using the condition \((f_4)\), according to [19, p.29] we have
\[
\overline{\lim} \int f(x, u_n)(u_n - u) \leq 0.
\]
We deduce from Lemma 2.5 and (2.13) that
\[
\overline{\lim} \left( \|u_n^+\|^2 - \|u^+\|^2 \right) = \lim \left( \int f(x, u_n)(u_n - u) - \int \left( [A_1^2(u_n) + A_2^2(u_n)] u_n(u_n - u) + A_0(u_n) u_n(u_n - u) \right) \right)
\]
\[
= \lim \int f(x, u_n)(u_n - u) - \lim \int \left( [A_1^2(u_n) + A_2^2(u_n)] u_n(u_n - u) + A_0(u_n) u_n(u_n - u) \right)
\]
\[
\leq \lim \int f(x, u_n)(u_n - u) \leq 0.
\]
Combining this with the weakly lower semi-continuity of the norm functional \( u \mapsto \|u\| \), we obtain
\[
\|u^+\| \leq \lim \|u_n^+\| \leq \overline{\lim} \|u_n^+\| \leq \|u^+\|.
\]
Therefore \( \|u_n^+\| \to \|u^+\| \). Remembering \( \|u_n^{-}\| \to \|u^-\| \), we get \( \|u_n\| \to \|u\| \). Thus \( u_n \to u \) in \( X \). \( \square \)

3. Critical groups and the proof of Theorem 1.1

Having established the \((PS)\) condition for \( \Phi \), we are now ready to present the proof of Theorem 1.1. We start by recalling some concepts and results from infinite-dimensional Morse theory (see e.g., Chang [18] and Mawhin and Willem [20, Chapter 8]).

Let \( X \) be a Banach space, \( \varphi : X \to \mathbb{R} \) be a \( C^1 \) functional, \( u \) be an isolated critical point of \( \varphi \) and \( \varphi(u) = 0 \). Then
\[
C_q(\varphi, u) := H_q(\varphi, \varphi^{-1}(0)), \quad q \in \mathbb{N} = \{0, 1, 2, \ldots\},
\]
is called the \( q \)-th critical group of \( \varphi \) at \( u \), where \( \varphi^{-1}(\infty, c) \) and \( H_q \) stands for the singular homology with coefficients in \( \mathbb{Z} \).

If \( \varphi \) satisfies the \((PS)\) condition and the critical values of \( \varphi \) are bounded from below by \( \alpha \), then following Bartsch and Li [21], we define the \( q \)-th critical group of \( \varphi \) at infinity by
\[
C_q(\varphi, \infty) := H_q(X, \varphi_{\alpha}), \quad q \in \mathbb{N}.
\]
Due to the deformation lemma, it is well known that the homology on the right hand side does not depend on the choice of \( \alpha \).

**Proposition 3.1** ([21, Proposition 3.6]). If \( \varphi \in C^1(X, \mathbb{R}) \) satisfies the (PS) condition and \( C_\ell(\varphi, 0) \neq C_\ell(\varphi, \infty) \) for some \( \ell \in \mathbb{N} \), then \( \varphi \) has a nonzero critical point.

**Proposition 3.2** ([22, Theorem 2.1]). Suppose \( \varphi \in C^1(X, \mathbb{R}) \) has a local linking at 0 with respect to the decomposition \( X = Y \oplus Z \), i.e., for some \( \varepsilon > 0 \),

\[
\varphi(u) \leq 0 \quad \text{for} \quad u \in Y \cap B_\varepsilon,
\]

\[
\varphi(u) > 0 \quad \text{for} \quad u \in (Z \setminus \{0\}) \cap B_\varepsilon,
\]

where \( B_\varepsilon = \{ u \in X \mid \|u\| \leq \varepsilon \} \). If \( \ell = \dim Y < \infty \), then \( C_\ell(\varphi, 0) \neq 0 \)

To investigate \( C_\ell(\Phi, \infty) \), using the idea of [16, Lemma 3.3] we will prove the following lemma.

**Lemma 3.3.** If \((V), (f_1)\) and \((f_2)\) hold, there exists \( A > 0 \) such that, if \( \Phi(u) \leq -A \), then

\[
\frac{d}{dt} \bigg|_{t=1} \Phi(tu) < 0.
\]

**Proof.** Otherwise, there exists a sequence \( \{u_n\} \subset X \) such that \( \Phi(u_n) \leq -n \) but

\[
\langle \Phi'(u_n), u_n \rangle = \frac{d}{dt} \bigg|_{t=1} \Phi(tu_n) \geq 0.
\]

Consequently,

\[
2 \left( \|u_n^+\|^2 - \|u_n^-\|^2 \right) \leq 2 \left( \|u_n^+\|^2 - \|u_n^-\|^2 \right) + \int \left[ f(x, u_n)u_n - 6F(x, u_n) \right]
\]

\[
= 6\Phi(u_n) - \langle \Phi'(u_n), u_n \rangle \leq -6n
\]

Let \( v_n = \|u_n\|^{-1}u_n \) and \( v_n^\pm \) be the orthogonal projection of \( v_n \) on \( X^\pm \). Then up to a subsequence \( v_n^- \to v^- \) for some \( v^- \in X^- \), because \( \dim X^- < \infty \).

If \( v^- \neq 0 \), then \( v_n \to v \) in \( X \) for some \( v \in X \setminus \{0\} \). By assumption \((f_2)\) we have

\[
\frac{f(x, t)}{t^6} \geq \frac{6F(x, t)}{t^6} \to +\infty, \quad \text{as} \quad t \to +\infty.
\]

Thus, similar to the proof of (2.5), we obtain

\[
\frac{1}{\|u_n\|^6} \int f(x, u_n)u_n \to +\infty.
\]

Now, using Lemma 2.1 we have a contradiction

\[
0 \leq \frac{\langle \Phi'(u_n), u_n \rangle}{\|u_n\|^6} = \frac{1}{\|u_n\|^6} \left( \left( \|u_n^+\|^2 - \|u_n^-\|^2 \right) + 3 \int \left( A_1^2(u_n) + A_2^2(u_n) \right) u_n^2 - \int f(x, u)u \right)
\]

\[
\leq 1 + 3a_1 - \frac{1}{\|u_n\|^6} \int f(x, u_n)u_n \to -\infty.
\]

Hence, we must have \( v^- = 0 \). But \( \|v_n^+\|^2 + \|v_n^-\|^2 = 1 \), we deduce \( \|v_n^+\| \to 1 \). Now for large \( n \) we have

\[
\|u_n^+\| = \|u_n\||v_n^+| \geq \|u_n\||v_n^-| = \|u_n^-\|
\]

a contradiction to (3.2).

**Lemma 3.4.** \( C_q(\Phi, \infty) = 0 \) for all \( q \in \mathbb{N} \).
Proof. Let \( B = \{ v \in X \mid ||v|| \leq 1 \} \), \( S = \partial B \) be the unit sphere in \( X \), and \( A > 0 \) be the number given in Lemma 3.3. Without loss of generality, we may assume that 
\[-A < \inf_{0 < ||v|| \leq 2} \Phi(v) .\]
Using (1.4), it is easy to see that for any \( v \in S \)
\[
\Phi(sv) = \frac{s^2}{2} \left( ||v^+||^2 - ||v^-||^2 \right) + 3s^2 \int (\Lambda^1 sv + \Lambda^2 sv) v^2 - \int F(x, sv) \\
= s^6 \left\{ \frac{||v^+||^2 - ||v^-||^2}{2s^4} + \frac{3}{s^4} \right\} \left( \Lambda^1 sv + \Lambda^2 sv \right) v^2 - \int \frac{F(x, sv)}{s^6} \rightarrow -\infty ,
\]
as \( s \to +\infty \). Therefore, for \( v \in S \) there is \( s_v > 0 \) such that \( \Phi(s_v, v) = -A \).

Using Lemma 3.3 and the implicit function theorem, as in the proof of [16, Lemma 3.4] it can be shown that such \( s_v \) is uniquely determined by \( v \) and \( T : v \mapsto s_v \) is continuous on \( S \). Using the continuous function \( T \) it is standard (see [23]) to construct a deformation from \( X \setminus B \) to the level set \( \Phi = \Phi^{-1}(-\infty, -A) \), and deduce

\[
C_q(\Phi, \infty) = H_q(X, \Phi^{-1}) \equiv H_q(X, X \setminus B) = 0, \quad \text{for all } q \in \mathbb{N} .
\]

Proof of Theorem 1.1. From the assumption \((f_1)\), there exists \( \varepsilon > 0 \) and \( C_\varepsilon > 0 \) such that, for every \( \mu > 0 \),
\[
|F(x, u)| \leq \varepsilon |u|^2 + C_\varepsilon |u|^\ell (e^{\mu|u|} - 1) \quad \text{for all } \xi > 2 .
\]
Using the conditions \((V)\), \((f_1)\) and \((f_3)\), similar to [14, Lemmas 3.4 and 3.5], it is easy to see that
\[
\Phi(u) = \frac{1}{2} \left( ||u^+||^2 - ||u^-||^2 \right) + o(||u||^2) \quad \text{as } ||u|| \to 0 .
\]
Hence, there exists \( \varepsilon > 0 \) such that \( \Phi \) is positive on \((X^+ \setminus \{0\}) \cap B_\varepsilon\), and negative on \((X^- \setminus \{0\}) \cap B_\varepsilon\). That is, \( \Phi \) has a local linking with respect to the decomposition \( X = X^- \oplus X^+ \). Therefore Proposition 3.2 yields
\[
C_\varepsilon(\varphi, 0) \neq 0 ,
\]
where \( \ell = \dim X^- \). By Lemma 3.4, \( C_\varepsilon(\varphi, \infty) = 0 \). Applying Proposition 3.1, we see that \( \Phi \) has a nonzero critical point. The proof of Theorem 1.1 is completed.

4. Proof of Theorem 1.2

To prove Theorem 1.2, we need to recover the \((PS)\) condition with the help of condition \((f_3)\). Therefore, we need the following lemma.

Lemma 4.1. Suppose that \( f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \) is continuous and satisfies \((f_3)\). For the functional \( K : X \to \mathbb{R} \),
\[
K(u) = \int F(x, u)
\]
is well defined and of class \( C^1 \) with
\[
\langle K'(u), \phi \rangle = \int f(x, u) \phi, \quad \forall \phi \in X .
\]
Moreover, \( K' \) is compact.

Proof. From (1.5) we have
\[
|f(x, t)| \leq |a|\infty |t| + |b|\infty |t|^{s-1} ,
\]
so it is well known that \( K \) is well defined and of class \( C^1 \).

To show that \( K' : X \to X^* \) is compact, let \( u_n \to u \) in \( X \). Because \( a \in L^p(\mathbb{R}^2) \), \( b \in L^q(\mathbb{R}^2) \), for any \( \varepsilon > 0 \), there exists \( R > 0 \) such that
\[
\int_{B_R^c} |a|^p < \varepsilon^p, \quad \int_{B_R^c} |b|^q < \varepsilon^q , \quad (4.1)
\]
where \( B_R \) is the ball in \( \mathbb{R}^2 \) with radius \( R > 0 \) centering at the origin and \( B_R^c = \mathbb{R}^2 \setminus B_R \).

For \( \phi \in X, ||\phi|| = 1 \), using (1.5), (4.1) and the Hölder inequality and noting that \( u, \phi \in L^r(B_R^c) \) for any \( \gamma > 2 \), we have

\[
\int_{B_R^c} |f(x, u)||\phi| \leq \int_{B_R^c} |a||u||\phi| + \int_{B_R^c} |b||u|^{\gamma-1}|\phi|
\]

\[
\leq |a|_p|u|_{2p/(p-1)}|\phi|_{2p/(p-1)} + |b|_q|u|^{\gamma-1}_{2q/(q-1)}|\phi|_{2q/(q-1)}
\]

\[
\leq M \varepsilon,
\]

where \([\cdot]_p\) is the standard \( L^p(B_R^c) \) norm, \( M \) is a constant depending on \( \sup_n ||u_n|| \) but not on \( \phi \). A similar inequality for \( \int_{B_R^c} |f(x, u_n)||\phi| \) is also true, therefore

\[
\int_{B_R^c} |f(x, u_n) - f(x, u)||\phi| \leq \int_{B_R^c} |f(x, u_n)||\phi| + \int_{B_R^c} |f(x, u)||\phi| \leq 2M \varepsilon.
\] (4.3)

By the compactness of the embedding \( X \hookrightarrow L^2_{loc}(\mathbb{R}^2) \) we have

\[
\sup_{||\phi||=1} \int_{B_R} |f(x, u_n) - f(x, u)||\phi| \to 0.
\] (4.4)

It follows from (4.3) and (4.4) that

\[
\lim_{n \to \infty} ||K'(u_n) - K'(u)|| = \lim_{n \to \infty} \sup_{||\phi||=1} \int |f(x, u_n) - f(x, u)||\phi|
\]

\[
\leq \lim_{n \to \infty} \sup_{||\phi||=1} \int |f(x, u_n) - f(x, u)||\phi| + \lim_{n \to \infty} \sup_{||\phi||=1} \int |f(x, u_n) - f(x, u)||\phi|
\]

\[
\leq 2M \varepsilon.
\]

Let \( \varepsilon \to 0 \), we deduce \( K'(u_n) \to K'(u) \) in \( X \). Hence \( K' \) is compact. \( \Box \)

Remark 4.2. Lemma 4.1 is motivated by [24, Lemma 1]. In that paper, the space dimension \( N > 2 \) and the working space is \( D^{1,2}(\mathbb{R}^N) \), \( a \) and \( b \) have to be taken from \( L^\infty(\mathbb{R}^N) \cap L^r(\mathbb{R}^N) \) for certain \( \gamma \) depending on the power of \( |x| \) on the left hand side of (1.5). Here our space dimension is \( N = 2 \) and our working space is \( X = H^1(\mathbb{R}^2) \). Unlike \( D^{1,2}(\mathbb{R}^N) \) the Sobolev space \( X = H^1(\mathbb{R}^2) \) can be continuously embedded into \( L^r(\mathbb{R}^2) \) for any \( \gamma \geq 2 \). For this reason, in our assumption (\( f_5 \)) the integrability of the weight functions \( a \) and \( b \) can be quit flexible.

Proof of Theorem 1.2. As we have pointed out in Remark 1.3, our condition (\( f_5 \)) is to ensure the global compactness of our functional \( \Phi \). According to the proof of Lemma 2.6, it suffices to derive

\[
\lim \int f(x, u_n)(u_n - u) = 0.
\]

from \( u_n \to u \) in \( X \).

Indeed, if \( u_n \to u \) in \( X \), by the Lemma 4.1 we have \( K'(u_n) \to K'(u) \) and

\[
\left| \int f(x, u_n)(u_n - u) \right| = |\langle K'(u_n), u_n - u \rangle|
\]

\[
\leq |\langle K'(u_n) - K'(u), u_n - u \rangle| + |\langle K'(u), u_n - u \rangle|
\]

\[
\leq ||K'(u_n) - K'(u)|| ||u_n - u|| + o(1) \to 0.
\]

Therefore, similar to the proof of Lemma 2.6 we can show that the functional \( \Phi \) satisfies the (PS) condition. Furthermore, applying Lemma 3.3 and 3.4, using the same proof we deduce that under the assumptions of Theorem 1.2, \( \Phi \) has a nonzero critical point. This completes the proof of Theorem 1.2. \( \Box \)
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