CARRYING SIMPLEX IN THE LOTKA-VOLterra
COMPETITION MODEL WITH SEASONAL SUCCESSION
WITH APPLICATIONS

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Abstract. We investigate the dynamics of the Poincaré-map for an n-dimensional Lotka-Volterra competitive model with seasonal succession. It is proved that there exists an (n−1)-dimensional carrying simplex Σ which attracts every nontrivial orbit in \( \mathbb{R}_+^n \). By using the theory of the carrying simplex, we simplify the approach for the complete classification of global dynamics for the two-dimensional Lotka-Volterra competitive model with seasonal succession proposed in [Hsu and Zhao, J. Math. Biology 64(2012), 109-130]. Our approach avoids the complicated estimates for the Floquet multipliers of the positive periodic solutions.

1. Introduction. Seasonal succession is a natural phenomenon, which has an important influence on the growth and survival of species. With the changes in temperature, rainfall, wind and humidity according to the season, populations experience a periodic external environment. One impressive example occurs in phytoplankton and zooplankton of the temperate lakes, where the species grow during the warmer months and die off or form resting stages in the winter. Such phenomenon is called seasonal succession in Sommer et al. [26]. Seasonal forcing leads to a regular succession of species over the seasons, which is a major cause of nonequilibrium dynamics.

A two species competition model under seasonally fluctuating light in the chemostat was first studied by Hsu [7]. Stone et al. [19] gained further insights into the nonlinear dynamics of recurrent diseases through the analysis of the classical seasonally forced SIR epidemic model. Huppert et al. [6] analysed a generic bottom up nutrient phytoplankton model to comprehend the dynamics of seasonally recurring algae blooms. In [14], Klausmeier applied a novel approach (called successional state

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dynamics—SSD) to three additional models of two-species interactions: resource competition, facilitation, and flip-flop competition, and obtained some analytic results. Recently, Hsu and Zhao [8] explored the global dynamics of a Lotka-Volterra two species competition model with seasonal succession via the theory of monotone dynamical systems. By estimating the Floquet multipliers of the positive periodic solution, they gave a complete classification for the global dynamics and the effects of season succession on the competition outcomes. Based on this, Zhang and Zhao [30] incorporated reaction and diffusion into the model and obtained the existence and global stability of bistable travelling waves for such a new system under appropriate conditions. There are other works on seasonal succession, such as [2, 25, 15] and references therein.

In this paper, it is interesting for us to introduce the seasonal succession into the \( n \)-dimensional Lotka-Volterra competition model and consider the dynamics of such time-periodic system. Enlightened by the modelling methods in Klausmeier [14] and Hsu and Zhao [8], we propose an \( n \)-dimensional Lotka-Volterra competition model with seasonal succession as follows:

\[
\begin{align*}
\frac{dx_i}{dt} & = -\lambda_i x_i, & m\omega \leq t \leq m\omega + (1 - \varphi)\omega, \ i = 1, \ldots, n, \\
\frac{dx_i}{dt} & = x_i \left( b_i - \sum_{j=1}^{n} a_{ij} x_j \right), & m\omega + (1 - \varphi)\omega \leq t \leq (m + 1)\omega, \ i = 1, \ldots, n,
\end{align*}
\]

\[ (x_1(0), \ldots, x_n(0)) = x^0 \in \mathbb{R}_+^n, \quad (1) \]

where \( m \in \mathbb{Z}_+, \varphi \in [0, 1], \mathbb{R}_+^n = \{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0 \} \) and \( \omega, \lambda_i, b_i, \) and \( a_{ij} (i, j = 1, \ldots, n) \) are all positive constants.

Clearly, if \( \varphi = 0 \), then the system (1.1) becomes the following decoupling form

\[
\begin{align*}
\frac{dx_i}{dt} & = -\lambda_i x_i, & i = 1, \ldots, n, \\
(x_1(0), \ldots, x_n(0)) & = x^0 \in \mathbb{R}_+^n.
\end{align*}
\]

(2)

While, if \( \varphi = 1 \), system (1) turns out to be the classical Lotka-Volterra model

\[
\begin{align*}
\frac{dx_i}{dt} & = x_i \left( b_i - \sum_{j=1}^{n} a_{ij} x_j \right), & i = 1, \ldots, n, \\
(x_1(0), \ldots, x_n(0)) & = x^0 \in \mathbb{R}_+^n.
\end{align*}
\]

(3)

One can see that system (1) is a time-periodic system in a seasonal succession environment. Overall period is \( \omega \), and \( \varphi \) stands for the switching proportion of a period between two subsystems (2) and (3). Biologically, \( \varphi \) is used to describe the proportion of the period in the good season in which the species follow system (3), while \( (1 - \varphi) \) represents the proportion of the period in the bad season in which the species die exponentially according to system (2). This setup caricatures the annual forcing of pelagic food webs, where many species live only during the warmer months and die off in the winter.

Let

\[
b_i(t) = \begin{cases} 
-\lambda_i, & [m\omega, m\omega + (1 - \varphi)\omega), \\
b_i, & [m\omega + (1 - \varphi)\omega, (m + 1)\omega],
\end{cases}
\]

\[
a_{ij}(t) = \begin{cases} 
0, & [m\omega, m\omega + (1 - \varphi)\omega), \\
a_{ij}, & [m\omega + (1 - \varphi)\omega, (m + 1)\omega],
\end{cases}
\]

where \( i, j = 1, \ldots, n \). Then system (1) can be rewritten as an \( n \)-dimensional time \( \omega \)-periodic Lotka-Volterra system with discontinuous \( \omega \)-periodic coefficients \( b_i(t) \) and
\[ a_{ij}(t). \]

\[
\begin{align*}
\frac{dx_i(t)}{dt} &= x_i(t) \left( b_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) \right), \\
(x_1(0), \ldots, x_n(0)) &= x^0 \in \mathbb{R}^n_+.
\end{align*}
\]

It is not difficult to see that system (1) (or (4)) admits a unique nonnegative global solution \( x(t, x^0) \) on \( [0, +\infty) \) for any \( x^0 \in \mathbb{R}^n_+ \). Since the system is \( \omega \)-periodic, we only consider the Poincaré map \( S \) on \( \mathbb{R}^n_+ \), that is, \( S(x^0) = x(\omega, x^0) \) for any \( x^0 \in \mathbb{R}^n_+ \).

Let us first define a linear map \( L \) by

\[ L(x_1, x_2, \ldots, x_n) = (e^{-\lambda_1(1-\varphi)x_1}e^{-\lambda_2(1-\varphi)x_2}, \ldots, e^{-\lambda_n(1-\varphi)x_n}), \quad \forall (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+. \]

We also let \( \{Q_i\}_{i \geq 0} \) represent the solution flow associated with the Lotka-Volterra competition system (3). Then, we have

\[ S(x^0) = Q_{\varphi \omega}(Lx^0), \quad \forall x^0 \in \mathbb{R}^n_+, \quad \text{i.e.,} \quad S = Q_{\varphi \omega} \circ L. \]

In the following, we will focus on the dynamics of the discrete-time system \( \{S^n\}_{n \geq 0} \).

Smith [21] considered the dynamics of the Poincaré-map \( T \) associated with the so called time-periodic Kolmogorov systems of differential equations. Based on the special structure of the competitive Kolmogorov systems, he introduced the following six hypotheses (all the notations will be given in Section 2):

1. \((H_1)\) \( T \) is an injective, \( C^2 \)-diffeomorphism onto its image;
2. \((H_2)\) For each nonempty \( I \subset \mathbb{N} := \{1, 2, \ldots, n\} \), the sets \( H_I, H_I^+ \) and \( \hat{H}_I^+ \) are all positively invariant under both \( T \) and \( T^{-1} \), where \( H_I = \{x \in \mathbb{R}^n_+ : x_i = 0 \text{ for } j \notin I\} \), \( H_I^+ = \mathbb{R}^n_+ \cap H_I \) and \( \hat{H}_I^+ = \{x \in H_I^+ : x_i > 0 \text{ for } i \in I\} \);
3. \((H_3)\) For each nonempty subset \( I \subset \mathbb{N} \), the Jacobian matrix \( D(T|_{H_I^+})(x)^{-1} = (DT(x)^{-1})_I \gg 0 \) at every \( x \in H_I^+ \), where \( T|_{H_I^+} \) is the restriction of \( T \) on \( H_I^+ \);
4. \((H_4)\) If \( x \in \mathbb{R}^n_+ \) and \( y = Tx \) then \( [0, y] \subset T[0, x] \);
5. \((H_5)\) \( T|_{H_{\{i\}}^+} \) has a unique fixed point \( u_i > 0 \) with \( 0 < \frac{1}{\mu_i} (T|_{H_{\{i\}}^+})(u_i) < 1 \) for each \( i \in \mathbb{N} \);
6. \((H_6)\) If \( x \) is a nontrivial \( m \)-periodic point of \( T \) and \( I \subset \mathbb{N} \) is such that \( x \in \hat{H}_I^+ \), then \( \mu_{I, m}(x) < 1 \), where \( \mu_{I, m}(x) \) is the eigenvalue of the mapping \( D(T|_{H_I^+})^m(x) \) with the smallest modulus.

Under these hypotheses, he conjectured that, generically, there exists a carrying simplex (named after Zeeman [29]) attracting all the nontrivial orbits in \( \mathbb{R}^n_+ \). Wang and Jiang [28] introduced an additional hypothesis:

1. \((H_7)\) For each nonempty subset \( I \subset \mathbb{N} \) and \( x, y \in \hat{H}_I^+ \), if \( 0 < T_i x < T_i y \) for all \( i \in I \), then \( \frac{T_i x}{T_i y} \geq \frac{x_i}{y_i} \) for all \( i \in I \), and proved the conjecture. Moreover, they showed \((H_7)\) is indeed generically satisfied by the Poincaré-map associated with the competitive Kolmogorov systems.

Diekmann, Wang and Yan [3, Theorem 1.1] later improved the existence of carrying simplices by removing the not easily checkable assumption \((H_6)\) (this is the situation for system (1)), and modifying the assumptions \((H_1)\) and \((H_7)\) to

1. \((H'_1)\) \( T \) is a \( C^1 \)-diffeomorphism onto its image;
2. \((H'_7)\) For nonempty subsets \( I \subseteq J \subseteq \mathbb{N} \), \( x \in \hat{H}_J^+ \) and \( y \in \hat{H}_J^+ \), if \( T_i x < T_i y \) for all \( i \in \}
Meanwhile, they succeeded in applying the carrying simplex theory to the age-structured semelparous population model and Atkinson/Allen model (see Hirsch [5] and Ruiz-Herrera [17] for more discussion on the mappings). Based on this, the dynamics of two-dimensional and three-dimensional competition Atkinson/Allen model were analyzed by Jiang and Niu [10]. By introducing the index formula on the carrying simplex, they presented all the equivalence classes relative to the boundary of the carrying simplex for these two systems, which corresponds to Zee- man’s classification for three-dimensional competitive Lotka-Volterra model (see [29]). Later, Jiang and Niu [11] analyze a three-dimensional Leslie/Gower competitive map and obtained the amenable conditions that guarantee the existence of the carrying simplex, as well as the 33 stable equivalence classes. For a survey of related types of discrete-time competitive models, we refer to [10, 11, 16, 4]. On the other hand, it is a pity that, up to now, the carrying simplex theory has not yet been successful to be applied to time-periodic models of Kolmogorov differential equations. One of the main obstacle is that, compared to the concrete discrete-time competitive maps discussed in [10, 11, 17, 5], there is no explicit expression of the Poincaré-map for the time-periodic differential equations, even for the simplest form as system (1). This makes the research much more difficult and complicated on the dynamics of the Poincaré-map of the time-periodic Kolmogorov competitive systems.

The aim of this paper is to show the existence of the carrying simplex for system (1), as well as its application on the two-dimensional case. More precisely, we show that system (1) admits a carrying simplex, which attracts every nontrivial orbits in \( \mathbb{R}^n_+ \) (see Theorem 2.3). By applying the carrying simplex theory to the two-dimensional Lotka-Volterra competitive model with seasonal succession, we simplify the approach by Hsu and Zhao [8] to obtain the complete classification of global dynamics (see Theorem 3.6). Our novel approach avoids the complicated estimates in [8] for the Floquet multipliers of the positive periodic solution for system (1). The corresponding phase portraits of global dynamics are presented in Figure 3.1.

This paper is organized as follows. Section 2 is devoted to show that system (1) admits a carrying simplex. In Section 3, we present the complete classification of global dynamics of two-dimensional case by the theory of carrying simplex. The paper ends with a discussion in Section 4.

2. Carrying simplex. In this section, we first introduce some definitions and some useful lemmas; and then, present the existence theorem of the carrying simplex for system (1).

Let us first define \( \mathbb{N} := \{1, 2, \ldots, n\} \) and \( \mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0, \forall i \in \mathbb{N}\} \). Let \( X \subseteq \mathbb{R}^n_+ \) and \( S : X \to X \) be the Poincaré (period) map of system (1). The orbit of a state \( x \) for \( S \) is \( \gamma(x) = \{S^n(x), n \in \mathbb{Z}_+\} \). A fixed point \( x \) of \( S \) is a point \( x \in X \) such that \( S(x) = x \). A point \( y \in X \) is called a \( k \)-periodic point of \( S \) if there exists some positive integer \( k > 1 \), such that \( S^k(y) = y \) and \( S^m(y) \neq y \) for every positive integer \( m < k \). The \( k \)-periodic orbit of the \( k \)-periodic point \( y \), \( \gamma(y) = \{y, S(y), S^2(y), \ldots, S^{k-1}(y)\} \), is often called a periodic orbit for short. A set \( V \subseteq X \) is positively invariant under \( S \) if \( SV \subset V \), and invariant if \( SV = V \). If \( S \) is a differentiable map, we write \( DS(x) \) as the Jacobian matrix of \( S \) at the point \( x \). For an \( n \times n \) matrix \( A \), we write \( A \geq 0 \) iff \( A \) is a nonnegative matrix (i.e., all the
entries are nonnegative) and $A \gg 0$ iff $A$ is a positive matrix (i.e., all the entries are positive).

Given $\emptyset \neq I \subset \mathbb{N}$, let $H_I := \{ x \in \mathbb{R}^n : x_j = 0 \text{ for } j \notin I \}$, $H_I^+ = \mathbb{R}^n_+ \cap H_I$ and $H_I^+ = \{ x \in H_I^+ : x_i > 0 \text{ for } i \in I \}$. For two vectors $x, y \in \mathbb{R}^n$, we write $x \leq y$ if $x_i \leq y_i$ for all $i \in \mathbb{N}$, and $x < y$ if $x_i < y_i$ for all $i \in \mathbb{N}$. If $x \leq y$ but $x \neq y$, we write $x < y$. For $x, y \in \mathbb{R}^n$, let $[x, y] := \{ z \in \mathbb{R}^n : x \leq z \leq y \}$ be a closed order interval, and $([x, y]) := \{ z \in \mathbb{R}^n : x < z < y \}$ an open order interval.

A map $S$ is said to be competitive in $\mathbb{R}^n_+$, if $x, y \in \mathbb{R}^n_+$ and $S(x) < S(y)$, then $x < y$. Furthermore, $S$ is called strongly competitive in $\mathbb{R}^n_+$, if $S(x) < S(y)$ then $x \ll y$.

A carrying simplex for the periodic map $S$ is a subset $\Sigma \subset \mathbb{R}^n_+ \setminus \{0\}$ with the following properties (see [28, Theorem 11]):

1. $\Sigma$ is compact, invariant and unordered;
2. $\Sigma$ is homeomorphic via radial projection to the $(n-1)$-dim standard probability simplex $\Delta^{n-1} := \{ x \in \mathbb{R}^n_+ : \Sigma, x_1 = 1 \}$;
3. $\forall x \in \mathbb{R}^n_+ \setminus \{0\}$, there exists some $y \in \Sigma$ such that $\lim_{n \to \infty} |S^n x - S^n y| = 0$.

Lemma 2.1. (Hsu and Zhao [8, Lemma 2.1]) Let $x(t, x_0)$ be the unique solution of the following system

$$
\begin{align*}
\frac{dx}{dt} &= -\lambda x, & m\omega \leq t \leq m\omega + (1 - \varphi)\omega, & m = 0, 1, 2, \ldots, \\
\frac{dx}{dt} &= x(b - ax), & m\omega + (1 - \varphi)\omega \leq t \leq (m + 1)\omega, \\
x(0) &= x_0 \in \mathbb{R}_+.
\end{align*}
$$

Then the following two statements are valid:

(i) If $b\varphi - \lambda(1 - \varphi) \leq 0$, then $\lim_{t \to \infty} x(t, x_0) = 0$ for all $x_0 \in \mathbb{R}_+$;

(ii) If $b\varphi - \lambda(1 - \varphi) > 0$, then system (5) admits a unique positive $\omega$-periodic solution $x^*(t)$, and $\lim_{t \to \infty} (x(t, x_0) - x^*(t)) = 0$ for all $x_0 \in \mathbb{R}_+ \setminus \{0\}$.

Remark 1. By $(H_3)$, we have $DT(x)^{-1} > 0$ for any $x \in \text{Int}\mathbb{R}^n_+$. This implies that if $x, y \in \mathbb{R}^n_+$ and $S(x) < S(y)$, then $x < y$. In particular, if $x, y \in \text{Int}\mathbb{R}^n_+$ and $S(x) < S(y)$, then $x \ll y$ (See Smith [21, Proposition 2.1]).

Lemma 2.2. Assume $(H_1'), (H_2)$ and $(H_3)$, then $(H_4)$ holds automatically.

Proof. See Jiang et al. [13, Proposition 2.1].

In the following, we will present the existence theorem of the carrying simplex for system (1).

Theorem 2.3. (The existence of the carrying simplex) Assume that $b_i\varphi - \lambda_i(1 - \varphi) > 0, i = 1, 2, \ldots, n$. Let $S : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be the Poincaré-map induced by system (1). Then, system (1) admits a carrying simplex $\Sigma$ which attracts every nontrivial orbit in $\mathbb{R}^n_+$.

Proof. According to the Theorem 1.1 in Diekmann et al. [3], one needs to verify that the hypotheses (H$^*_1$), (H$^*_2$)-(H$^*_5$) and (H$^*_7$) hold for Poincaré-map $S$ associated to the system (1). Firstly, by the expression of the equation (4), we can see that all coordinate axes and coordinate planes are invariant under the periodic map $S$, which implies that (H$^*_2$) holds. In view of Lemma 2.1, it follows that the corresponding one-dimensional map possesses the convergence of all nontrivial orbits towards the positive fixed point, that is, (H$^*_3$) holds. Thirdly, we can obtain that
The corresponding adjoint equation is
\[
F = \text{the smoothness of the flow } S \Rightarrow \{S_t\}_{t \geq 0}.
\]
Then the statement (H_1) is satisfied. Next, let \( F(x) = (F_1, F_2, \ldots, F_n)^T \), where
\[
F_i = x_i \left( b_i - \sum_{j=1}^n a_{ij} x_j \right), \quad i = 1, 2, \ldots, n.
\]
For simplicity, we denote \( u(t, x) := Q_t(x) \) and \( V(t, x) := D_xu(t, x) = D_xQ_t(x) \).
Then \( S(x) = Q_{\varphi \omega}(Lx) = u(\varphi \omega, Lx) \), and hence,
\[
DS(x) = D(Q_{\varphi \omega}(Lx)) \cdot D(Lx)
= V(\varphi \omega, Lx) \cdot \text{diag} \left( e^{-\lambda_1(1-\varphi)\omega}, e^{-\lambda_2(1-\varphi)\omega}, \ldots, e^{-\lambda_n(1-\varphi)\omega} \right).
\]
Note that \( V(t, x) \) satisfies
\[
\frac{dV(t)}{dt} = DF(u(t, x))V(t), \quad V(0) = I.
\]
The corresponding adjoint equation is
\[
\frac{dY(t)}{dt} = -(DF(u(t, x)))^T Y(t), \quad Y(0) = I.
\]
Then \( Y(t) = (V(t)^T)^{-1} \). Clearly, \(-(DF(u(t, x)))^T \) is a cooperative and irreducible matrix (see Smith [20, Chapter 4]). By Smith [22, Theorem D(ii)], we obtain \( Y(t) = (V(t)^T)^{-1} \gg 0 \) for all \( t > 0 \), that is, \( V(t)^{-1} \gg 0 \) for all \( t > 0 \). Taking \( t = \varphi \omega \), we have \( DS(x)^{-1} = (DQ_{\varphi \omega}(Lx))^{-1} = V(\varphi \omega)^{-1} \gg 0 \). If \( x \in \hat{H}_I^+ \), then \( D(S|_{\hat{H}_I^+})(x) \) is a matrix obtained from \( DS(x) \) by deleting rows and columns indexed by the complement of \( I \) in \( N \). So, it follows that \( (DS|_{\hat{H}_I^+})^{-1} \gg 0 \).
Thus, the statement (H_3) is true for system (1). Again the condition (H_4) is a result of (H_1)-(H_3) by Lemma 2.2.

Finally, it remains to show that (H_2') holds. For this purpose, we simply rewrite system (3) as
\[
\dot{x}_i = x_i f_i(x), \quad i = 1, 2, \ldots, n,
\]
where \( f_i(x) = b_i - \sum_{j=1}^n a_{ij} x_j \). Let \( W(t) = \frac{u_i(t, Lx)}{u_i(t, Ly)} \). Since \( Q_t(x) = u(t, x) \) for \( t \in [0, \varphi \omega] \), it follows that \( \dot{u}_i(t, Lx) = u_i(t, Lx) f_i(u(t, Lx)) \), \( i = 1, 2, \ldots, n \) for \( t \in [0, \varphi \omega] \).
Then,
\[
W(t) = \frac{\dot{u}_i(t, Lx) u_i(t, Ly) - u_i(t, Lx) \dot{u}_i(t, Ly)}{(u_i(t, Ly))^2}
= \frac{u_i(t, Lx) f_i(u(t, Lx)) - u_i(t, Lx) u_i(t, Ly) f_i(u(t, Ly))}{(u_i(t, Ly))^2}
= \frac{u_i(t, Lx) (f_i(u(t, Lx)) - f_i(u(t, Ly)))}{u_i(t, Ly)}
= W(t) (f_i(u(t, Lx)) - f_i(u(t, Ly)))
= W(t) \sum_{j=1}^n \int_0^1 \frac{\partial f_i}{\partial x_j} (t, \lambda(s)) \cdot (u_j(t, Lx) - u_j(t, Ly)) ds,
\]
where \( \lambda(s) = su(t, Lx) + (1-s)u(t, Ly), \) \( s \in [0, 1] \). Noticing that the expression of system (1.3), we can get \( \frac{\partial f_i}{\partial x_j} < 0 \). If \( S_i x < S_i y \), then \( u_i(t, Lx) < u_i(t, Ly) \),
Let $t \in [0, \varphi \omega]$, $i = 1, 2, \ldots, n$. Thus, $W(t) > 0$, i.e., $W(\varphi \omega) > W(0)$. Therefore,
\[
\frac{u_i(\varphi \omega, Lx)}{u_i(\varphi \omega, Ly)} > \frac{u_i(0, Lx)}{u_i(0, Ly)} = \frac{(Lx)_i}{(Ly)_i} = e^{-\lambda_i(1-\varphi)\omega x_i} = \frac{x_i}{y_i}.
\]
This entails that $\frac{x_i}{y_i} > \frac{a_i}{b_i}$, that is, (H$_2$) holds. We have completed the proof. \hfill $\square$

3. Two-dimensional system. In this section, we will utilize the carrying simplex (Theorem 2.3) to investigate the global dynamics for the following 2-D Lotka-Volterra competitive model with seasonal succession:
\[
\begin{align*}
\frac{dx_i}{dt} &= -\lambda_i x_i, & m\omega \leq t \leq m\omega + (1-\varphi)\omega, & i = 1, 2, \\
\frac{dx_1}{dt} &= x_1(b_1 - a_{11} x_1 - a_{12} x_2), & m\omega + (1-\varphi)\omega \leq t \leq (m + 1)\omega, \\
\frac{dx_2}{dt} &= x_2(b_2 - a_{21} x_1 - a_{22} x_2), & m\omega + (1-\varphi)\omega \leq t \leq (m + 1)\omega, \\
(x(0), x(0)) &= x^0 \in \mathbb{R}_+^2, & m = 0, 1, 2, \ldots
\end{align*}
\]
(9)

In [8], Hsu and Zhao analyzed the stability of trivial fixed point $O$, axial fixed points $R_i$, $i = 1, 2$ and the positive fixed point $E$, and obtained a complete classification for the global dynamics of system (9). In this section, we will simplify their approach by using the theory of carrying simplex to obtain the complete classification of global dynamics. Our novel approach avoids the complicated estimates in [8] for the Floquet multipliers of the positive periodic solution for system (9).

Let $h_i = (h_i - \lambda_i(1-\varphi)\omega$, $i = 1, 2$. By Lemma 2.1, if $h_i \leq 0$, then the species $i$ will go to extinction, and hence the dynamics of the system are trivial. Throughout this paper, we only consider the case: $h_i > 0$, $i = 1, 2$.

Let $K := \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 \geq 0, x_2 \leq 0\}$. Then we write $x \leq_K y$ if $y - x \in K$; $x <_K y$ if $y - x \in K \setminus \{0\}$; and $x \ll_K y$ if $y - x \in \text{Int} K$. Notations such as $x \geq_K (>_K, \gg_K) y$ have the natural meanings. Let $C_1 := \{(x_1, 0) : x_1 \in R_+\}$ and $C_2 := \{(0, x_2) : x_2 \in R_+\}$. For any $a, b \in \mathbb{R}_+^2$ and $a \ll_K b$, we define the closed $K$-order interval $[a, b]_K := \{x \in \mathbb{R}_+^2 : a \leq_K x \leq_K b\}$ and the open $K$-order internal $]a, b]_K := \{x \in \mathbb{R}_+^2 : a <_K x <_K b\}$.

Hereafter, we always let $S$ be the Poincaré-map associated with 2-D time-periodic system (9).

**Lemma 3.1.** The following statements are valid for the map $S$:

(i) If $x \leq_K y$, then $S(x) \leq_K S(y)$. Moreover, if $x <_K y$ and $y \in \text{Int} \mathbb{R}_+^2$, then $S(x) \ll_K S(y)$.

(ii) For any $x^0 \in \mathbb{R}_+^2$, the sequence of points $S^n(x^0)$ converges to a fixed point of $S$ as $n \to \infty$.

(iii) $S$ admits an one-dimensional carrying simplex $\Sigma$ such that every nontrivial orbit is asymptotic to a fixed point on $\Sigma$.

**Proof.** For (i)-(ii), see Hsu and Zhao [8, Lemma 2.2]. For the strong monotonicity with respect to “$\leq_K$” of $S$, see also Smith and Waltman [23, Theorem B.6]. The statement (iii) is a result of (ii) and Theorem 2.3 for $n = 2$. \hfill $\square$

**Lemma 3.2.** Given system (9), then

(i) The trivial fixed point $O$ is a hyperbolic repeller.

(ii) If $\frac{b_2}{a_{21}} > \frac{h_1}{a_{11}}$, then $R_1$ is an unstable (saddle type) fixed point of $S$, and hence, $R_1$ repels along $\Sigma$; if $\frac{b_2}{a_{21}} < \frac{h_1}{a_{11}}$, then $R_1$ is an asymptotically stable (node type) fixed point of $S$, and hence, $R_1$ attracts along $\Sigma$.\hfill $\square$
(iii) If \( \frac{h_1}{a_{12}} > \frac{h_2}{a_{22}} \), then \( R_2 \) is an unstable (saddle type) fixed point of \( S \), and hence, \( R_2 \) repels along \( \Sigma \); if \( \frac{h_1}{a_{12}} < \frac{h_2}{a_{22}} \), then \( R_2 \) is an asymptotically stable (node type) fixed point of \( S \), and hence, \( R_2 \) attracts along \( \Sigma \).

Proof. For (i), see the proof of Theorem 2.1(i) in Hsu and Zhao\[8\]. Compared with Hsu and Zhao\[8, Lemma 2.3\] and Lemma 3.1(iii), it is easy to see that the statements (ii) and (iii) holds. \( \square \)

**Lemma 3.3.** Given system (9), then

(i) If \( S \) has a positive fixed point \( \bar{x} \), then \( y := \int_0^{\varphi} Q_t(L\bar{x})dt \) must be a positive solution of the following linear algebraic system

\[
\begin{cases}
a_{11}y_1 + a_{12}y_2 = h_1, \\
a_{21}y_1 + a_{22}y_2 = h_2.
\end{cases}
\]

(ii) If \( a_{11}a_{22} \neq a_{12}a_{21} \), then \( S \) has at most a positive fixed point.

Proof. See Hsu and Zhao \[8, Lemma 2.5\]. \( \square \)

**Lemma 3.4.** Let \((X, X^+)\) be an ordered Banach space with a solid cone \( X^+ \). Let \( S : U \to U \) be a continuous and monotone map, where \( U \) is a bounded open set of \( \text{Int} X^+ \). Assume that \( S \) has a fixed point \( x^* \in U \) such that

(a) The Fréchet derivative \( DS(x^*) : X \to X \) exists, \( DS(x^*) \) is strongly positive and bounded, and the spectral radius \( r(DS(x^*)) = 1 \);

(b) \( x^* \) does not attract any point \( x \in U \) with either \( x > x^* \) or \( x < x^* \).

Then there is a \( \delta > 0 \) such that the locally stable set \( W^s_\delta := \{ x \in U : \| x - x^* \| < \delta, \lim_{n \to \infty} S^n(x) = x^* \} \) is a locally strongly stable manifold of \( S \) at \( x^* \).

Proof. See Smith and Thieme \[24, Proof of Theorem 3.4\] or Jiang et al. \[12, Lemma 2.3\]. \( \square \)

**Lemma 3.5.** Let \( E \) be a positive fixed point of \( S \). Then the Jacobian matrix \( DS(E) \) is strongly \( K \)-positive and \( \det DS(E) < 1 \).

Proof. Let \( u(t) = (u_1(t), u_2(t)) := Q_t(LE) \) and \( V(t) = (V_1(t), V_2(t)) := D_2Q_t(LE) \). In view of the equation (7),

\[
\frac{dV(t)}{dt} = \begin{pmatrix}
b_1 - 2a_{11}u_1(t) - a_{12}u_2(t) & -a_{12}u_1(t) \\
-a_{21}u_2(t) & b_2 - 2a_{22}u_2(t) - a_{21}u_1(t)
\end{pmatrix}V(t), \quad V(0) = I.
\]

From the coefficient matrix of the above equation, it is easy to see that \( V(t) \) is strongly monotone with respect to \( \preceq_K \) in \( \text{Int} \mathbb{R}^2_+ \). For any \( \xi \gg_K 0 \), one has

\[
\text{diag}(e^{-\lambda_1(1-\varphi)\omega}, e^{-\lambda_2(1-\varphi)\omega}) \cdot \xi \gg_K 0.
\]

So,

\[
DS(E)\xi = V(\varphi \omega) \cdot \text{diag}(e^{-\lambda_1(1-\varphi)\omega}, e^{-\lambda_2(1-\varphi)\omega}) \cdot \xi \gg_K 0,
\]

which implies that \( DS(E) \) is strongly \( K \)-positive.

By the expression (6) of \( DS(x) \) in the proof of Theorem 2.3, it follows that

\[
\det DS(x) = \det V(\varphi \omega, LX) \cdot \exp \left( - (\lambda_1 + \lambda_2)(1 - \varphi) \omega \right).
\]

Due to (7) and Liouville’s formula, we have

\[
\det V(\varphi \omega, LE) = \exp \left( \int_0^{\varphi} \text{tr}(DF(u(t)))dt \right) = \exp \left( (\lambda_1 + \lambda_2)(1 - \varphi) \omega - \int_0^{\varphi} (a_{11}u_1(t) + a_{22}u_2(t))dt \right).
\]
where \( u(t) = (u_1(t), u_2(t)) := Q_t(LE) \). So,
\[
\det DS(E) = \exp \left( - \int_0^\omega (a_{11}u_1(t) + a_{22}u_2(t)) dt \right) < 1.
\]
We have proved the Lemma. \( \Box \)

Now, we are ready to present the global dynamics of \( S \) as below.

**Theorem 3.6.** For system (9),

1. (Competitive Exclusion) If \( \frac{h_2}{a_{21}} > \frac{h_1}{a_{11}}, \frac{h_1}{a_{12}} < \frac{h_2}{a_{22}} \), then \( S \) has no positive fixed point, and \( R_2 \) is globally asymptotically stable in \( \mathbb{R}_+^2 \setminus C_1 \).
2. (Competitive Exclusion) If \( \frac{h_2}{a_{21}} < \frac{h_1}{a_{11}}, \frac{h_1}{a_{12}} > \frac{h_2}{a_{22}} \), then \( S \) has no positive fixed point, and \( R_1 \) is globally asymptotically stable in \( \mathbb{R}_+^2 \setminus C_2 \).
3. (Coexistence) If \( \frac{h_2}{a_{21}} > \frac{h_1}{a_{11}}, \frac{h_1}{a_{12}} > \frac{h_2}{a_{22}} \), then \( S \) has a unique positive fixed point \( E \), which is globally asymptotically stable in \( \text{Int}\mathbb{R}_+^2 \).
4. (Bistability) If \( \frac{h_2}{a_{21}} < \frac{h_1}{a_{11}}, \frac{h_1}{a_{12}} < \frac{h_2}{a_{22}} \), \( R_1 \) and \( R_2 \) are locally asymptotically stable and \( S \) has a unique positive fixed point \( E \). Furthermore, there is a continuous, unbounded, invariant and unordered (with respect to “\( \leq \)”) curve \( \Gamma \subset \mathbb{R}_+^2 \) which separates the basins of attraction of \( R_1 \) and \( R_2 \).

Moreover, the corresponding phase portraits are demonstrated in Figure 3.1.

**Proof.** For (1)-(2), we only prove (1), because the other is similar. The linear algebraic system in Lemma 3.3(i) has no positive solution. So the positive fixed point does not exist. From Lemma 3.2(ii)-(iii), it follows that \( R_1 \) is a saddle point and \( R_2 \) is asymptotically stable. In view of Lemma 3.1(iii), we obtain that \( R_2 \) is globally asymptotically stable in \( \mathbb{R}_+^2 \setminus C_1 \).

(3) Since \( \frac{h_2}{a_{21}} > \frac{h_1}{a_{11}}, \frac{h_1}{a_{12}} > \frac{h_2}{a_{22}} \), both \( R_1 \) and \( R_2 \) are saddle points. By Lemma 3.1(ii), there exists at least one positive fixed point. From Lemma 3.3(ii), the positive fixed point is unique (written as \( E \)). Meanwhile, Lemma 3.1(iii) ensures that \( E \) is globally attracting in \( \text{Int}\mathbb{R}_+^2 \); and moreover, we have \( E \in [\{ R_2, R_1 \}]_K \). For any neighborhood \( U_1 \) of \( E \), one can find two points \( p, q \in U_1 \cap \Sigma \) such that \( p \ll_K E \ll_K q \) and \( [p, q]_K \subset U_1 \). Since the carrying simplex \( \Sigma \) is invariant, unordered (related to “\( \leq \)” and globally attracting. One may assume without loss of generality that \( p \ll_K S p < K S q \ll_K q \). Then Lemma 3.1(i) implies that \( p \ll_K S p < K S q < K \cdots < K S^n p < K S^n q \ll_K q \) for any \( n \geq 1 \). Then, for any \( x \in [p, q]_K \), we have \( S^n p < K S^n x < K S^n q \) for \( n \geq 0 \). Hence, \( S^n x \in [p, q]_K \subset U_1 \) for \( n \geq 0 \). This means that \( E \) is Lyapunov stable. Noticing that \( \lim_{n \to \infty} S^n p = \lim_{n \to \infty} S^n q = E \), we further obtain that \( E \) is locally asymptotically stable. Together with the global attractivity of \( E \), we have obtained that \( E \) is globally asymptotically stable.

(4) Let \( U := [R_2, R_1]_K = \{ x \in \mathbb{R}_+^2 : R_2 \leq_K x \leq_K R_1 \} \). Since \( \frac{h_2}{a_{21}} < \frac{h_1}{a_{11}}, \frac{h_1}{a_{12}} < \frac{h_2}{a_{22}} \), both \( R_1 \) and \( R_2 \) are locally stable by Lemma 3.2(ii)-(iii). By Proposition 2.1 in Hsu, Smith and Waltman [9], \( S \) has a positive fixed point \( E \) in \( U \), and \( R_2 \ll_K E \ll_K R_1 \). Moreover, Lemma 3.3(ii) implies that \( E \) is unique. Now we claim that \( E \) is unstable. Otherwise, \( E \) and \( R_3 \) are stable. Again, by Proposition 2.1 in [9], \( S \) has another positive fixed point \( E' \) in \( [E, R_1]_K \), which is a contradiction to the uniqueness of the positive fixed point. So, \( E \) is unstable.

Let \( B_1 \) and \( B_2 \) be the basins of attraction of \( R_1 \) and \( R_2 \) for \( S \) in \( \mathbb{R}_+^2 \), respectively. Then, \( B_1 \) and \( B_2 \) are relatively open in \( \mathbb{R}_+^2 \). Moreover, due to the connecting orbits
Theorem (see, e.g. Dancer and Hess [1, Proposition 1]), it is not difficult to see that \( \{ \omega \in \mathbb{R}_+^2 : E <_K \omega \leq_K R_1 \} \subset B_1 \) and \( \{ \omega \in \mathbb{R}_+^2 : R_2 \leq_K \omega <_K E \} \subset B_2 \).

Define \( \Gamma := \mathbb{R}_+^2 \setminus (B_1 \cup B_2) \). Clearly, \( O, E \in \Gamma \) and \( \Gamma \) is invariant. We first show that \( \Gamma \) is unbounded. If not, there exists a bounded open ball \( B \subset \mathbb{R}_+^2 \) with its closure \( \bar{B} \) such that \( \Gamma \subset \bar{B} \). Then, \( \mathbb{R}_+^2 \setminus \bar{B} = (B_1 \setminus B) \cup (B_2 \setminus B) \), which contradicts the connectedness of \( \mathbb{R}_+^2 \setminus \bar{B} \).

Next, we show that any two distinct points in \( \Gamma \) are unordered with respect to \( _{\leq_K} \). Suppose that there are two points \( x, y \in \Gamma \) with \( x <_K y \). Then, together with the fact that \( \Gamma \) is unbounded, \( \Gamma \) is unordered with respect to \( _{\leq_K} \). Notice that \( x, y \notin B_1 \cup B_2 \), it follows from Lemma 3.1(ii) that \( \lim_{n \to \infty} S^n x = S^n y = E \). This implies that \( E \) attracts an open subset \( \left[ [S^{n_1} x, S^{n_1} y] \right]_K \). By Lemma 3.5, the smallest positive eigenvalue of \( DS(E) \) is less than 1. Let \( \rho \) be the largest eigenvalue of \( DS(E) \). Since \( E \) is unstable, we have \( \rho \geq 1 \). If \( \rho > 1 \), \( E \) is a saddle point, which can not attract the open set \( \left[ [S^{n_1} x, S^{n_1} y] \right]_K \). This leads to a contradiction.

If \( \rho = 1 \), then Lemma 3.1(i) and Lemma 3.5 implies that \( S \) is a continuous and monotone map (with respect to \( _{\leq_K} \)) and \( DS(E) \) is strongly positive. Note also that \( \{ \omega \in \mathbb{R}_+^2 : E <_K \omega \leq_K R_1 \} \subset B_1 \) and \( \{ \omega \in \mathbb{R}_+^2 : R_2 \leq_K \omega <_K E \} \subset B_2 \), it follows from Lemma 3.4 that there is some \( \delta > 0 \) such that the locally stable set \( W^s_\delta(E) := \{ x \in \text{Int} \mathbb{R}_+^2 : \| x - E \| < \delta, \lim_{n \to \infty} S^n(x) = E \} \) is a local strongly stable manifold \( W^{ss}_{loc}(E) \) of \( E \). Since \( \lim_{n \to \infty} S^n x = \lim_{n \to \infty} S^n y = E \), one can choose some \( n_2 \in \mathbb{N} \) sufficiently large, such that \( z \in W^s_\delta(E) \) for any \( z \in \left[ [S^{n_2} x, S^{n_2} y] \right]_K \), which entails that \( \left[ [S^{n_2} x, S^{n_2} y] \right]_K \subset W^s_\delta(E) \). So, by Lemma 3.4, one can obtain \( \left[ [S^{n_2} x, S^{n_2} y] \right]_K \subset W^{ss}_{loc}(E) \). On the other hand, by Shub [18, Theorem III.8], one has \( W^{ss}_{loc}(E) \) is a Lipschitz graph; and hence, it can not contain the open set \( \left[ [S^{n_2} x, S^{n_2} y] \right]_K \), a contradiction. Therefore, \( \Gamma \) is unordered with respect to \( _{\leq_K} \).

Finally, we will show that \( \Gamma \) is a continuous curve. For this purpose, we note that \( S \) is strongly monotone (with respect to \( _{\leq_K} \)) in \( \text{Int} \mathbb{R}_+^2 \). Let \( \tilde{\Gamma} = \Gamma \setminus \{ O \} \). Then the pair \( (B_1 \cup \tilde{\Gamma}) \cap \text{Int} \mathbb{R}_+^2, (B_2 \cup \tilde{\Gamma}) \cap \text{Int} \mathbb{R}_+^2 \) is an order decomposition of \( \text{Int} \mathbb{R}_+^2 \) (see Takáč [27, Definition 1.1]) with \( \tilde{\Gamma} \) being the boundary (also called \( d \)-hypersurface of the order-decomposition). By Takáč [27, Proposition 1.3], we obtain that \( \tilde{\Gamma} \) is a one-dimensional Lip-continuous graph. More precisely, we can write \( \tilde{\Gamma} \) as a graph \( \{ (x_1, h(x_1)) \in \mathbb{R}_+^2 : x_1 > 0 \} \), where \( h : (0, +\infty) \to \mathbb{R}_+ \) is a strictly increasing Lip-continuous function. So, let \( P = \lim_{x_1 \to 0^+} (x_1, h(x_1)) \). Then, clearly, \( P \in \partial \mathbb{R}_+^2 \cap \tilde{\Gamma} \).

This implies that \( P = O \). Thus, \( \Gamma = \tilde{\Gamma} \cup \{ O \} \) and \( \Gamma \) is a continuous curve. We have completed the proof.

According to the above analysis, we present four corresponding phase portraits as follows.

\[ \text{Figure 3.1} \]
Remark 2. In Theorem 3.6, we obtain that the complete classification of global dynamics for system (9) via an alternative approach of those in Hsu and Zhao [8]. Our approach is based on the theory of carrying simplex, and avoid the estimates for the Floquet multipliers of the positive periodic solutions.

4. Discussion. In this paper, we focus on an \( n \)-species competitive Lotka-Volterra model with time-periodic coefficients (called Seasonal Succession). By the theory of carrying simplex for competitive map developed by Diekmann et al. in [3], we obtain that system (1) admits an \( (n-1) \)-dimensional carrying simplex which attracts every nontrivial orbit in \( \mathbb{R}^n_+ \) under the condition \( h_i > 0, i = 1, 2, \ldots, n \). In particular, when \( n = 2 \), the system admits a one-dimensional carrying simplex \( \Sigma \). Because every bounded solution of a competitive planar periodic system asymptotically approaches a periodic solution, every nontrivial orbit of \( S \) in \( \mathbb{R}^2_+ \) converges to a fixed point on \( \Sigma \). We apply the theory of carrying simplex to the two-dimensional model proposed by Hsu and Zhao [8] and obtain the same complete classification of global dynamics. More precisely, we only need to analyze the stability of axial fixed points and the uniqueness of the positive fixed point, instead of the estimate for the Floquet multipliers of the positive fixed point.

In addition, for three-dimensional competition case, it follows from Theorem 2.3 that there will exist a two-dimensional carrying simplex which attracts every nontrivial orbit in \( \mathbb{R}^3_+ \). In a future work, we will try to give a complete stable equivalence classes relative to the boundary of the carrying simplex for the Poincaré-map associated with the time-periodic 3-D competitive Lotka-Volterra model with seasonal succession. Among others, one of the challenging problems is the uniqueness of the positive periodic solution in such 3-D system.

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