REPRESENTATIONS OF \textit{asl}_2  

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Abstract. We study representations of the simple Lie antialgebra \textit{asl}_2 introduced in \cite{5}. We show that representations of \textit{asl}_2 are closely related to the famous ghost Casimir element of the universal enveloping algebra osp(1|2). We prove that \textit{asl}_2 has no non-trivial finite-dimensional representations; we define and classify some particular infinite-dimensional representations that we call weighted representations.

Introduction

"Lie antialgebras" is a new class of algebras introduced by V. Ovsienko \cite{5}. These algebras naturally appear in the context of symplectic and contact geometry of \(\mathbb{Z}_2\)-graded spaces, their algebraic properties are not yet well understood. Lie antialgebras is a surprising "mixture" of commutative algebras and Lie algebras.

The first example of Ovsienko's algebras is a simple Lie antialgebra \textit{asl}_2(\mathbb{K}), over \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{C}\). This algebra is of dimension 3, it has linear basis \(\{\varepsilon; a, b\}\) subject to the following relations:

\begin{align*}
\varepsilon \cdot \varepsilon &= \varepsilon, \\
\varepsilon \cdot a &= a \cdot \varepsilon = \frac{1}{2} a, \\
\varepsilon \cdot b &= b \cdot \varepsilon = \frac{1}{2} b, \\
a \cdot b &= -b \cdot a = \frac{1}{2} \varepsilon.
\end{align*}

The notion of representation of a Lie antialgebra was also defined in \cite{5}, and the problem of classification of representations of simple Lie antialgebras was formulated. In this paper, we study representations of \textit{asl}_2(\mathbb{K}).

The Lie antialgebra \textit{asl}_2(\mathbb{K}) is closely related to the simple classical Lie superalgebra osp(1|2). For instance, osp(1|2) = Der(\textit{asl}_2(\mathbb{K})). It was shown in \cite{5}, that every representation of \textit{asl}_2(\mathbb{K}) is naturally a representation of osp(1|2). The first problem is thus to determine the corresponding class of osp(1|2)-representations. It turns out that this class can be characterized by so-called ghost Casimir element of the universal enveloping algebra \(U(\text{osp}(1|2))\). This element first appeared in \cite{6}, see also \cite{1, 2, 3}.

\begin{theorem}
There is a one-to-one correspondence between representations of \textit{asl}_2(\mathbb{K}) and representations of osp(1|2) such that
\begin{align*}
\Gamma^2 &= \frac{1}{4} \text{Id},
\end{align*}
where \(\Gamma\) is the action of the ghost Casimir element.
\end{theorem}

We will see that in the case of a \(\mathbb{Z}_2\)-graded irreducible representation \(V = V_0 \oplus V_1\) of \textit{asl}_2(\mathbb{K}), one has a more complete information on the action of the ghost Casimir element, namely, the \(\mathbb{Z}_2\)-grading can be chosen in such a way that

\(\Gamma|_{V_0} = -\frac{1}{2} \text{Id}, \quad \Gamma|_{V_1} = \frac{1}{2} \text{Id}\).

We believe that this relation with the ghost Casimir element provides a better understanding for the nature of \textit{asl}_2(\mathbb{K}) itself.
In the finite-dimensional case, we prove the following

**Theorem 2.** The Lie antialgebra \( \mathfrak{asl}_2(\mathbb{K}) \) has no non-trivial finite-dimensional representations.

Let us emphasize that the algebra \( \mathfrak{asl}_2(\mathbb{K}) \) was defined in [5] as analog of the classical simple Lie algebra \( \mathfrak{sl}_2(\mathbb{K}) \), but is also has a certain similarity with the 3-dimensional Heisenberg algebra \( \mathfrak{h}_1 \). The above result is similar to the classical result that \( \mathfrak{h}_1 \) has no non-trivial irreducible representations.

In the infinite-dimensional case, we restrict our study to the case of *weighted representations*. A weighted representation is a representation containing an eigenvector for the action of the Cartan element \( H \in \mathfrak{osp}(1|2) \). We classify the irreducible weighted representations of \( \mathfrak{asl}_2(\mathbb{K}) \). We introduce a family of weighted representations \( V(\ell) \), for \( \ell \in \mathbb{K} \) (see Section 3.2 for the construction). Considering the set of parameters \( \mathcal{P} = [-1, 1] \) in the real case, or \( \mathcal{P} = [-1, 1] \cup \{ \ell \in \mathbb{C} \mid -1 \leq \Re(\ell) < 1 \} \) in the complex case, we obtain the complete classification of irreducible weighted representations.

**Theorem 3.** Any irreducible weighted representation is isomorphic to a \( V(\ell) \) for a unique \( \ell \in \mathcal{P} \).

The paper is organized as follows. In Section 1, we recall the general definitions of Lie antialgebras and their representations introduced in [5]. In Section 2, we obtain preliminary results about the representations of \( \mathfrak{asl}_2(\mathbb{K}) \). A link with the Lie algebra \( \mathfrak{osp}(1|2) \) and with the Casimir elements is established. We complete the proof of Theorem 1 in subsection 2.3. In Section 3, we introduce the notion of weighted representations and give the construction of the family of irreducible weighted representations \( V(\ell) \), \( \ell \in \mathbb{K} \). In Section 4, we formulate our results concerning the representations \( V(\ell) \) and complete the proofs of Theorem 2 and Theorem 3. In the end of the paper we discuss some general aspects of representation theory of \( \mathfrak{asl}_2(\mathbb{K}) \), such as the tensor product of two representations.

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1. **Lie antialgebras and their representations**

Let us give the definition of a Lie antialgebra equivalent to the original definition of [5]. Throughout the paper the ground vector field is \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \).

**Definition 1.1.** A Lie antialgebra is a \( \mathbb{Z}_2 \)-graded vector space \( \mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1 \), equipped with a bilinear product \( \cdot \) satisfying the following conditions.

1. it is even: \( \mathfrak{a}_i \cdot \mathfrak{a}_j \subset \mathfrak{a}_{i+j} \);
2. it is supercommutative, i.e., for all homogeneous elements \( x, y \in \mathfrak{a} \),
   \[ x \cdot y = (-1)^{p(x)p(y)} y \cdot x \]
   where \( p \) is the parity function defined by \( p(x) = i \) for \( x \in \mathfrak{a}_i \);
3. the subspace \( \mathfrak{a}_0 \) is a commutative associative algebra;
4. for all \( x_1, x_2 \in \mathfrak{a}_0 \) and \( y \in \mathfrak{a}_1 \), one has
   \[ x_1 \cdot (x_2 \cdot y) = \frac{1}{2} (x_1 \cdot x_2) \cdot y, \]
   in other words, the subspace \( \mathfrak{a}_1 \) is a module over \( \mathfrak{a}_0 \), homomorphism \( \varrho : \mathfrak{a}_0 \to \text{End}(\mathfrak{a}_1) \) being given by \( \varrho_x y = 2x \cdot y \) for all \( x \in \mathfrak{a}_0 \) and \( y \in \mathfrak{a}_1 \);
(5) for all \( x \in a_0 \) and \( y_1, y_2 \in a_1 \), the following Leibniz identity
\[
x \cdot (y_1 \cdot y_2) = (x \cdot y_1) \cdot y_2 + y_1 \cdot (x \cdot y_2)
\]
is satisfied;
(6) for all \( y_1, y_2, y_3 \in a_1 \), the following Jacobi-type identity
\[
y_1 \cdot (y_2 \cdot y_3) + y_2 \cdot (y_3 \cdot y_1) + y_3 \cdot (y_1 \cdot y_2) = 0
\]
is satisfied.

**Example 1.2.** It is easy to see that the above axioms are satisfied for \( \text{asl}_2(\mathbb{K}) \). In this case, the element \( \varepsilon \) spans the even part, \( \text{asl}_2(\mathbb{K})_0 \), while the elements \( a, b \) span the odd part, \( \text{asl}_2(\mathbb{K})_1 \).

Consider a \( \mathbb{Z}_2 \)-graded vector space \( V = V_0 \oplus V_1 \), the space \( \text{End}(V) \) of linear endomorphisms of \( V \) is a \( \mathbb{Z}_2 \)-graded associative algebra:
\[
\text{End}(V)_0 = \text{End}(V_0) \oplus \text{End}(V_1), \quad \text{End}(V)_1 = \text{Hom}(V_0, V_1) \oplus \text{Hom}(V_1, V_0).
\]
Following [5], we define the following “anticommutator” on \( \text{End}(V) \):
\[
\left[ X, Y \right] := X Y + (-1)^{p(X)p(Y)} Y X,
\]
where \( p \) is the parity function on \( \text{End}(V) \) and \( X, Y \in \text{End}(V) \) are homogeneous (purely even or purely odd) elements. Note that the sign rule in (1.3) is opposite to that of the usual commutator.

**Remark 1.3.** Let us stress that the operation (1.3) does not define a Lie antialgebra structure on the full space \( \text{End}(V) \) and it is not known what are the subspaces of \( \text{End}(V) \) for which this is the case. This operation provides, however, a definition of the notion of representation of a Lie antialgebra.

**Definition 1.4.** (a) We call representation of the Lie antialgebra \( a \), the data of \( (V, \chi) \) where \( V = V_0 \oplus V_1 \) is a \( \mathbb{Z}_2 \)-graded vector space and \( \chi : a \to \text{End}(V) \) is an even linear map such that
\[
\left[ \chi_x, \chi_y \right] = \chi_{x \cdot y},
\]
for all \( x, y \in a \).
(b) A subrepresentation is a \( \mathbb{Z}_2 \)-graded subspace \( V' \subset V \) stable under \( \chi_x \) for all \( x \in a \).
(c) A representation is called irreducible if it does not have proper subrepresentations.
(d) Two representations \( (V, \chi) \) and \( (V', \chi') \) are called equivalent if there exists a linear map \( \Phi : V \to V' \) such that \( \Phi \circ \chi_x = \chi'_x \circ \Phi \) for every \( x \in a \).

**Remark 1.5.** It is not clear a priori, that a given Lie antialgebra has at least one non-trivial representation. In the case of \( \text{asl}_2(\mathbb{K}) \), however, an example of representation was given in [5] in the context of geometry of the supercircle.

Let us finally mention that there is a notion of module over a Lie antialgebra which is different from that of representation. For instance, the “adjoint action” defined as usual by \( \text{ad}_x y = x \cdot y \) is not a representation, but it defines a structure of \( a \)-module on \( a \).

Given a Lie antialgebra \( a \), it was shown in [5] that there exists a Lie superalgebra, \( \mathfrak{g}_a \), canonically associated to \( a \). Every representation of \( a \) extends to a representation of \( \mathfrak{g}_a \). In the case of \( \text{asl}_2(\mathbb{K}) \), the corresponding Lie superalgebra is the classical
simple Lie antialgebra osp(1|2). We will give the explicit construction of this Lie superalgebra in the next section and use it as the main tool for our study.

2. REPRESENTATIONS OF asl₂(\mathbb{K}) AND THE GHOST CASIMIR OF osp(1|2)

In this section we collect the general information about the representations of asl₂(\mathbb{K}). We also introduce the action of osp(1|2) and prove Theorem 1.

2.1. Generators of asl₂(\mathbb{K}) and the Z₂-grading. Consider an asl₂(\mathbb{K})-representation \( V = V₀ ⊕ V₁ \) with \( χ : asl₂(\mathbb{K}) → End(V) \). The homomorphism condition (1.4) can be written explicitly in terms of the basis elements:

\[
\begin{align*}
χ_αχ_β - χ_βχ_α &= \frac{1}{2}χ_ε \\
χ_αχ_ε + χ_εχ_α &= \frac{1}{2}χ_a \\
χ_βχ_ε + χ_εχ_β &= \frac{1}{2}χ_b \\
χ_εχ_ε &= \frac{1}{2}χ_ε.
\end{align*}
\]

Let us simplify the notations by fixing the following elements of \( End(V) \):

\[
A = 2χ_a, \quad B = 2χ_b, \quad E = 2χ_ε.
\]

The above relations read:

\[
\begin{align*}
AB - BA &= E \\
AE + EA &= A \\
BE + EB &= B \\
E² &= E.
\end{align*}
\] (2.5)

The element \( E \) is a projector in \( V \). This leads to a decomposition of \( V \) into eigenspaces \( V = V^{(0)} ⊕ V^{(1)} \) defined by

\( V^{(λ)} = \{ v ∈ V | Ev = λv \}, \quad λ = 0, 1. \)

This decomposition is not necessarily the same as the initial one, \( V = V₀ ⊕ V₁ \). Since \( V_i \), where \( i = 0, 1 \), is stable under the action of \( E \), this gives a refinement:

\( V = V₀^{(0)} ⊕ V₀^{(1)} ⊕ V₁^{(0)} ⊕ V₁^{(1)} \)

where

\( V_i^{(λ)} = \{ v ∈ V_i | Ev = λv \}, \quad λ = 0, 1, \quad i = 0, 1. \)

**Proposition 2.1.** Any asl₂(\mathbb{K})-representation \( (V = V₀ ⊕ V₁, χ) \) is equivalent to a representation \( (V' = V₀' ⊕ V₁', χ') \) such that

\( E | V₀' = 0, \quad E | V₁' = Id. \)

**Proof.** Using the relations (2.5) it is easy to see that \( A \) and \( B \) send the spaces \( V_i^{(λ)} \) into \( V_i^{(1-λ)} \), where \( λ = 0, 1, \ i = 0, 1 \). Thus, changing the \( Z₂ \)-grading of \( V \) to \( V' = V₀' ⊕ V₁' \) where

\[
V₀' = V₀^{(0)} ⊕ V₁^{(0)} , \\
V₁' = V₀^{(1)} ⊕ V₁^{(1)} ,
\]

does not change the parity of the operators \( A, B \) and \( E \) viewed as elements of the \( Z₂ \)-graded space \( End(V') \). In other words, the map \( χ' : a → End(V') \) defined by \( χ'_x = χ_x \) for all \( x ∈ a \), is still an even map satisfying the condition (1.4).
By consequent, \((V', \chi')\) is also a representation. It is then clear that \((V', \chi')\) is equivalent (in the sense of Definition 1.4 (d)) to \((V, \chi)\).

As a consequence of the above proposition we will always assume in the sequel that the \(Z_2\)-grading of the representations \(V\) is given by the eigenspaces of the action \(E\).

2.2. Action of \(osp(1|2)\). This section provides a special case of the general construction of [5]. For the sake of completeness, we give here the details of the computations.

Given an \(asl_2(K)\)-representation \(V\), we define the operators \(E\), \(F\) and \(H\) by

\[
E = A^2, \quad F = -B^2, \quad H = -(AB + BA).
\]

These three elements define a structure of \(sl_2(K)\)-module on \(V\).

**Lemma 2.2.** One has:

\[
[H, E] = 2E \\
[H, F] = -2F \\
[E, F] = H
\]

**Proof.** These relations follow from relations (2.5). Indeed,

\[
[H, E] = -(AB + BA)A^2 + A^2(AB + BA) \\
= -ABA^2 - BA^3 + A^3B + A^2BA \\
= -2ABA^2 + (ABA^2 - BA^3) + (A^3B - A^2BA) + 2A^2BA \\
= -2ABA^2 + (AB - BA)A^2 + A^2(AB - BA) + 2A^2BA \\
= 2(A(AB - BA)A + (AB - BA)A^2 + A^2(AB - BA) \\
= 2AE + E^2A + A^2E \\
= (AEA + EA^2) + (AEA + A^2E) \\
= (AE + EA)A + A(EA + AE) \\
= A^2 + A^2 \\
= 2E.
\]

In the same way we obtain \([H, F] = -2F\). Finally,

\[
[E, F] = -A^2B^2 + B^2A^2 \\
= -A^2B^2 + ABAB - ABAB + BA^2B - BA^2B + BABABA - BABABA + B^2A^2 \\
= -A(AB - BA)B - (AB - BA)AB - BA(AB - BA) - B(AB - BA)A \\
= -AEB - EAB - BAE - BEEA \\
= -(AE + EA)B - B(AE + EA) \\
= -(AB - BA \\
= H
\]
With similar computations one can establish the following additional relations:

\[
\begin{align*}
[H, A] & = A & [E, A] & = 0 & [F, A] & = B, \\
[H, B] & = -B & [E, B] & = A & [F, B] & = 0, \\
[H, E] & = 0 & [E, E] & = 0 & [F, E] & = 0,
\end{align*}
\]

that can be summarized as follows.

**Proposition 2.3.** Every representation of \( \mathfrak{asl}_2(\mathbb{K}) \) has a structure of a module over the Lie superalgebra \( \mathfrak{osp}(1|2) \). The even part, \( \mathfrak{osp}(2|1)_0 \), is spanned by \( E, F, H \), while the odd part, \( \mathfrak{osp}(2|1)_1 \), is spanned by \( A, B \).

The relation (2.7) and (2.8) play an essential role in all our computations.

### 2.3. The ghost Casimir element

The notion of twisted adjoint action was introduced for a certain class of Lie superalgebras in [2]. We recall here the definition in the \( \mathfrak{osp}(1|2) \)-case.

Let \( X \) be an element of \( \mathfrak{osp}(1|2) \) and \( Y \) be an element of the universal enveloping algebra \( U(\mathfrak{osp}(1|2)) \). Define \( \tilde{\text{ad}} : \mathfrak{osp}(1|2) \to \text{End}(U(\mathfrak{osp}(1|2))) \) by

\[
\tilde{\text{ad}}_X Y := XY - (-1)^{\rho(X)(\rho(Y)+1)} YX.
\]

In other words,

\[
\tilde{\text{ad}}_X = \begin{cases} 
\text{ad}_X & \text{if } X \text{ is even} \\
-\text{ad}_X & \text{if } X \text{ is odd}.
\end{cases}
\]

Remarkably enough, \( \tilde{\text{ad}} \) defines an \( \mathfrak{osp}(1|2) \)-action on \( U(\mathfrak{osp}(1|2)) \).

The ghost Casimir elements are the invariants of the twisted adjoint action, see [2], and also [4]. In the case of \( \mathfrak{osp}(1|2) \), the ghost Casimir element is particularly simple:

\[
\Gamma = AB - BA - \frac{1}{2} \text{Id},
\]

The ghost Casimir satisfies \( \tilde{\text{ad}}_X \Gamma = 0 \) for all \( X \in \mathfrak{osp}(1|2) \).

The relation between the above twisted adjoint action and our situation is the following. Consider a representation \( \chi : \mathfrak{asl}_2(\mathbb{K}) \to \text{End}(V) \). Denote by \( U \) the subalgebra of \( \text{End}(V) \) generated by the image of \( \mathfrak{asl}_2(\mathbb{K}) \) under \( \chi \). The algebra \( U \) can be viewed as a quotient of \( U(\mathfrak{osp}(1|2)) \). A simple comparison of (2.9) and (1.3) shows that, if \( x \) is an odd element of \( \mathfrak{asl}_2(\mathbb{K}) \), then

\[
[\chi_x, Y] = \tilde{\text{ad}}_{\chi_x}(Y),
\]

for all \( Y \) in \( U \). The operator \( \mathcal{E} \) and the ghost Casimir \( \Gamma \) are obviously related by

\[
\Gamma = \mathcal{E} - \frac{1}{2} \text{Id}.
\]

It follows that the second and third relations in (2.5) are equivalent to \( \tilde{\text{ad}}_A \Gamma = 0 \) and \( \tilde{\text{ad}}_B \Gamma = 0 \), respectively, while the relation \( \mathcal{E}^2 = \mathcal{E} \) reads \( \Gamma^2 = \frac{1}{4} \text{Id} \).

This completes the proof of Theorem 1.
2.4. **Usual Casimir elements.** The operator $E$ is also related to the usual Casimir elements $C$, resp. $C_0$ of $osp(1|2)$, resp. $sl_2(\mathbb{K})$. Recall

$$C = EF + FE + \frac{1}{2}(H^2 + AB - BA),$$

$$C_0 = EF + FE + \frac{1}{2}H^2.$$  

We easily see

$$E = 2(C - C_0).$$

This implies that, if $V$ is an irreducible representation of $asl_2(\mathbb{K})$, then $E|_{V_0}$ and $E|_{V_1}$ are proportional to $Id$.

Moreover, straightforward computation in $U(osp(1|2))$ gives the following relation

$$4(C - C_0)^2 = 4C - 2C_0.$$  

It follows the condition $E^2 = E$ implies that $C$ acts trivially.

3. **Weighted representations of $asl_2(\mathbb{K})$.**

In this section, we introduce the notion of weighted representation of the Lie antialgebra $asl_2(\mathbb{K})$. This class of representation is characterized by the property that the action of the Cartan element $H$ of $osp(1|2)$ has at least one eigenvector. We do not require a priori the eigenspaces to be finite dimensional.

3.1. **The definition.** Let $V$ be a representation of $asl_2(\mathbb{K})$. We introduce the subspaces

$$V_\ell = \{ v \in V | Hv = \ell v \}, \quad \ell \in \mathbb{K}.$$  

Whenever $V_\ell \neq \{0\}$, we call this subspace a weight space of $V$ with weight $\ell$. We denote by $\Pi_H(V)$ the set of weights of representation $V$.

**Lemma 3.1.** With the above notations:

(i) The element $A$ (resp. $B$) maps $V_\ell$ into $V_{\ell + 1}$ (resp. $V_{\ell - 1}$).

(ii) The sum $\sum_{\ell \in \Pi_H(V)} V_\ell$ is direct in $V$.

(iii) The space

$$Wt(V) := \bigoplus_{\ell \in \Pi_H(V)} V_\ell$$  

is a subrepresentation of $V$.

**Proof.** Let $v$ be a vector in $V_\ell$. Using the relations (2.8) we obtain:

$$HAV = [H, A]v + AHv = Av + \ell Av = (\ell + 1)v$$

and

$$HBv = [H, B]v + BHv = -Bv + \ell Bv = (\ell - 1)v.$$  

Part (i) then follows.

Part (ii) is clear since the weight spaces are eigenspaces for $H$.

It follows from (i) that $Wt(V)$ is stable with respect to the action of $A$ and $B$ and, therefore, it is also stable under $E = AB - BA$. Hence (iii). \qed

**Corollary 3.2.** If $V$ is an irreducible representation then either

$$Wt(V) = \{0\} \quad \text{or} \quad Wt(V) = V.$$  

**Definition 3.3.** We call weighted representation any representation $V$ of $asl_2(\mathbb{K})$ such that $Wt(V) \neq \{0\}$. 


3.2. **The family of weighted representations** $V(\ell)$. For every $\ell \in \mathbb{K}$, we construct an irreducible weighted representation of $\mathfrak{asl}_2(\mathbb{K})$ that we denote $V(\ell)$. This representation contains an odd vector $e_1$ such that $He_1 = \ell e_1$ and, by irreducibility, every element of $V(\ell)$ is a result of the (iterated) $\mathfrak{asl}_2(\mathbb{K})$-action on $e_1$.

(a) **The case where $\ell$ is not an odd integer.** We start the construction with the generic weight $\ell$.

Consider a family of linearly independent vectors $\{e_k\}_{k \in \mathbb{Z}}$. We set $V(\ell) = \bigoplus_{k \in \mathbb{Z}} \mathbb{K} e_k$ and we define the operators $A$ and $B$ on $V(\ell)$ by

$$Ae_k = e_{k+1}, \quad \forall k \in \mathbb{Z}$$

$$Be_k = ((1 - \ell)/2 - [k/2])e_{k-1}, \quad \forall k \in \mathbb{Z},$$

The operator $\mathcal{E}$ is determined by $\mathcal{E} = AB - BA$. Introduce the following $\mathbb{Z}_2$-grading on $V(\ell)$:

$$V(\ell)_0 = \bigoplus_{k \text{ even}} \mathbb{K} e_k,$$

$$V(\ell)_1 = \bigoplus_{k \text{ odd}} \mathbb{K} e_k.$$

It is easy to see the operators $A$ and $B$ are odd operators with respect to this grading whereas $\mathcal{E}$ is even.

**Proposition 3.4.** The space $V(\ell)$ together with the operators $A, B, \mathcal{E}$ is an $\mathfrak{asl}_2(\mathbb{K})$-representation.

**Proof.** By simple straightforward computations we obtain:

$$A\mathcal{E} + \mathcal{E}A = A, \quad B\mathcal{E} + \mathcal{E}B = B.$$ 

Moreover, on the basis elements $e_k$ of $V(\ell)$

$$\mathcal{E} e_k = \begin{cases} e_k, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even} \end{cases}$$

so that $\mathcal{E}^2 = \mathcal{E}$. 

It is easy to see the basis elements $e_k$'s are weight vectors. Indeed, one checks

$$He_k = (\ell + k - 1) e_k, \quad \forall k \in \mathbb{Z}.$$ 

In particular, the element $e_1$ is a weight vector of weight $\ell$ and generates the representation $V(\ell)$.

The actions on the basis elements of $\mathfrak{osp}(1|2)$ can be pictured as follows:
The entire space $V(\ell)$ can be pictured as an infinite chain of the above diagrams.

(b) **Construction of $V(\ell)$ for $\ell$ a positive odd integer.** Consider a family of linearly independent vectors $\{e_k\}_{k \in \mathbb{Z}, k \geq 2 - \ell}$. We set $V(\ell) = \bigoplus_{k \geq 2-\ell} \mathbb{K}e_k$ and we define the operators $A$ and $B$ on $V(\ell)$ by a similar formula:

\[
Ae_k = e_{k+1}, \ \forall k \geq 2 - \ell \\
Be_k = ((1 - \ell)/2 - [k/2])e_{k-1}, \ \forall k > 2 - \ell, \\
Be_{2-\ell} = 0.
\]

The operator $E$ is again determined by $E = AB - BA$. The $\mathbb{Z}_2$-grading on $V(\ell)$ is defined by the same formula (3.11). The result of Lemma 3.4 holds true.

The element $e_1$ is an odd weight vector of weight $\ell$ and generates the representation $V(\ell)$. However, the vector $e_{2-\ell}$ is more interesting.

**Definition 3.5.** We call a representation $V$ a lowest weight (resp. highest weight) representation if it contains a vector $v$, such that $Bv = 0$ and the vectors $A^n v$ span $V$ (resp. $Av = 0$ and $B^n$ span $V$); the vector $v$ is called a lowest weight (resp. highest weight) vector.

Clearly, the vector $e_{2-\ell}$ is a lowest weight vector of the representation $V(\ell)$, if $\ell$ a positive odd integer. One obtains the following diagram.

\[V_1: \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}\]

\[V_0: \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}\]

Viewed as a representation of $\text{osp}(1|2)$, $V(\ell)$ is a Verma module.

(c) **Construction of $V(\ell)$ for $\ell$ a negative odd integer.** Consider a family of linearly independent vectors $\{e_k\}_{k \in \mathbb{Z}, k \leq -\ell}$. We set $V(\ell) = \bigoplus_{k \leq -\ell} \mathbb{K}e_k$ and we define the operators $A$ and $B$ on $V(\ell)$ by

\[
Ae_k = ((1 + \ell)/2 + [k/2])e_{k+1}, \ \forall k < -\ell, \\
Ae_{-\ell} = 0, \\
Be_k = e_{k-1}, \ \forall k \leq -\ell.
\]

As previously these operators define an $\mathfrak{asl}_2(\mathbb{K})$-representation. The vector $e_{-\ell}$ is a highest weight vector of $V(\ell)$. 

3.3. Geometric realization. It was shown in [5] that asl$_2$(K) has a representation in terms of vector fields on the 1|1-dimensional space. More precisely, consider \( \mathcal{F} = C^\infty_c(\mathbb{R}) \) the set of \( \mathbb{K} \)-valued \( C^\infty \)-functions of one real variable \( x \). Introduce \( \mathcal{A} = \mathcal{F}[\xi]/(\xi^2) \) with the \( \mathbb{Z}_2 \)-grading \( \mathcal{A}_0 = \mathcal{F}, \mathcal{A}_1 = \mathcal{F}(\xi) \). Define the vector field

\[
\mathcal{D} = \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial x}.
\]

It is very easy to check that the following vector fields:

\[
\begin{align*}
A &= \mathcal{D}, & B &= x \mathcal{D}, & E &= \xi \mathcal{D}
\end{align*}
\]

satisfy the relations (2.5). Therefore this defines an asl$_2$(K)-action on \( \mathcal{A} \).

Let us choose the following function:

\[
e_1 = x^\lambda \xi,
\]

where \( \lambda \in \mathbb{K} \). It turns out this function generates a weighted representation of asl$_2$(K), isomorphic to \( V(\ell) \), with the weight \( \ell = -2\lambda - 1 \). Note that the case \( \lambda \) is an integer gives the highest or lowest irreducible representation.

Remark 3.6. The vector field \( \mathcal{D} \) defines the standard contact structure on \( \mathbb{K}^{1|1} \). The vector fields (3.13) are therefore tangent to the contact structure. These vector fields do not span a Lie (super)algebra with respect to the usual Lie bracket. Remarkable enough, the corresponding generators of the osp(1|2)-action:

\[
\begin{align*}
E &= \frac{\partial}{\partial x}, & F &= -x^2 \frac{\partial}{\partial x} - \xi \mathcal{D}, & H &= -2x \frac{\partial}{\partial x} - \xi \mathcal{D}
\end{align*}
\]

are contact vector fields, while the above vector fields \( A \) and \( B \) are the only vector fields that are contact and tangent at the same time. The Lie superalgebra osp(1|2) thus preserves the contact structure in the usual way.

4. Classification results

In this section we prove Theorem 3 which is our main classification result.

4.1. Classification of weighted representations. The following statement shows that the representations \( V(\ell) \) are, indeed, irreducible; it also classifies all the isomorphisms between these representations.

Proposition 4.1. (i) For every \( \ell \in \mathbb{K} \), the representation \( V(\ell) \) is an irreducible infinite-dimensional representation.

(ii) Two weighted representations \( V(\ell) \) and \( V(\ell') \), where \( \ell \) and \( \ell' \) are not odd integers, are isomorphic if and only if \( \ell' - \ell = 2m \) for some \( m \in \mathbb{Z} \).

(iii) Two weighted representations \( V(\ell) \) and \( V(\ell') \), where \( \ell \) and \( \ell' \) are odd integers, are isomorphic if and only if \( \ell \) and \( \ell' \) have same sign.
Proof. (i) Irreducibility of $V(\ell)$:

Suppose there exists a subrepresentation $V'$ of $V(\ell)$. Consider a nonzero vector $v \in V'$ and write

$$v = \sum_{1 \leq i \leq N} \alpha_i e_{k_i}$$

with $\alpha_i \neq 0$, for all $1 \leq i \leq N$. Using (3.12), we obtain,

$$H^p v = \sum_{1 \leq i \leq N} \alpha_i (\ell + k_i - 1)^p e_{k_i},$$

where $0 \leq p \leq N - 1$. We may assume the coefficients $(\ell + k_i - 1)$ are nonzero. In other words, we assume the basis elements occurring in the decomposition of $v$ are not of weight zero. If this is not the case, we change $v$ to $A^k v$ or $B^k v$ for sufficiently large $k$.

The above equations form a linear system of type Vandermonde with distinct nonzero coefficients, and so it is solvable. We can express the $e_{k_i}$'s as linear combinations of $H^p v$'s. It follows that the vectors $e_{k_i}$ are in $V'$. Applying $A$ and $B$ to $e_{k_i}$, we produce all the vectors $e_k$, $k \in \mathbb{Z}$. Hence, the vectors $e_k$, of the basis are in $V'$ for all $k \in \mathbb{Z}$. This implies $V' = V(\ell)$. Therefore there is no proper subrepresentation of $V(\ell)$.

(ii) Let $\ell$ and $\ell'$ be two scalars in $\mathbb{K}$ that are not odd integers.

Denote by $\{e_k\}_k$ the standard basis of $V(\ell)$ and $\{e'_k\}_k$ the standard basis of $V(\ell')$. Suppose there exists an isomorphism of representation $\Phi : V(\ell') \to V(\ell)$.

The vector $\Phi(e'_1)$ is a vector of weight $\ell'$ in $V(\ell)$. The weights in $V(\ell')$ are of the form $\ell + k$ for some $k \in \mathbb{Z}$ and the corresponding weight space is $\mathbb{K}e_{k+1}$. We thus have $\Phi(e'_1) = \alpha e_{k+1}$ for some $\alpha \in \mathbb{K} \setminus \{0\}$, $k \in \mathbb{Z}$, and therefore $\ell' = \ell + k$.

Moreover, $\mathcal{E} \Phi(e'_1) = \Phi(e'_1)$, so $e_{k+1}$ has to be an odd vector, i.e., $k$ has to be even.

We proved that a necessary condition to have $V(\ell)$ isomorphic to $V(\ell')$ is

$$\ell' = \ell + 2m,$$

where $m \in \mathbb{Z}$.

Conversely, suppose $\ell' = \ell + 2m$, for some $m \in \mathbb{Z}$. It is easy to check that the linear map $\Phi : V(\ell') \to V(\ell)$ defined by

$$\Phi(e'_k) = e_{k+2m},$$

for all $k \in \mathbb{Z}$, is an isomorphism of representation.

(iii) Let $\ell$ and $\ell'$ be two odd integers. If $\ell$ and $\ell'$ have opposite sign then $V(\ell)$ and $V(\ell')$ cannot be isomorphic, since on one of the space $A$ acts injectively and on the other space $A$ has a non-trivial kernel.

Conversely, if $\ell$ and $\ell'$ have same sign, let us construct an explicit isomorphism between $V(\ell)$ and $V(\ell')$. One has: $\ell' = \ell - 2m$, for some $m \in \mathbb{N}$ and we define $\Phi : V(\ell') \to V(\ell)$ by

$$\Phi(e'_k) = e_{k+2m}, \quad \forall k \geq 2 - \ell'$$

in the case where $\ell$ and $\ell'$ are positive, or by

$$\Phi(e'_k) = e_{k+2m}, \quad \forall k \leq -\ell'$$

in the case where $\ell$ and $\ell'$ are negative. □
4.2. Structure of weighted representations. In this section we establish two lemmas crucial for the proofs of Theorem 2 and 3.

Let us consider an irreducible representation $V$ of $\mathfrak{asl}_2(\mathbb{K})$. Recall that the $\mathbb{Z}_2$-grading $V = V_0 \oplus V_1$ corresponds to the decomposition with respect to the eigenvalues of $E$, see Lemma 2.3.

**Lemma 4.2.** If there exists a nonzero element $v \in V$ such that $H v = \ell v$, $\ell \in \mathbb{K}$, then there exist nonzero elements $v_i \in V_i$ and $\ell_i \in \mathbb{K}$, $i = 0, 1$, such that $H v_i = \ell_i v_i$.

**Proof.** In the case where $v$ is a pure homogeneous element, i.e., $v \in V_i$, we can choose for $v_{1-i}$ the vector $Av$ or $Bv$. Indeed, one has

\begin{align}
HA v &= [H, A] v + AH v = Av + \ell Av = (\ell + 1)v,
HB v &= [H, B] v + BH v = -Bv + \ell Av = (\ell - 1)v.
\end{align}

If the two vectors $Av$ and $Bv$ are zero then $V = \mathbb{K}v$ is the trivial representation and, by consequent, the statement of the theorem is true. Otherwise, if $Av$ and $Bv$ are not zero, one obtains weight vectors of weight $\ell_{1-i} = \ell \pm 1$.

If $v$ is not a homogeneous element, we can write $v = v_0 + v_1$ with $v_0 \neq 0 \in V_0$ and $v_1 \neq 0 \in V_1$. One then has:

$$H v = H v_0 + H v_1$$
and $H v = \ell v = \ell v_0 + \ell v_1$.

Furthermore, $H v_0$ is an element of $V_0$ and $H v_1$ is an element of $V_1$, since the operator $H = -(AB + BA)$ is even. Therefore, by uniqueness of the writing in $V_0 \oplus V_1$, one has:

$$H v_0 = \ell v_0 	ext{ and } H v_1 = \ell v_1.$$ 

□

**Lemma 4.3.** If $v \in V_i$ is such that $H v = \ell v$ then:

(i) for all $k \geq 1$,

$$AB^k v = \lambda_k B^{k-1} v$$
with $\lambda_k = \left[\frac{k-i}{2}\right] + \frac{i-\ell}{2}$, where $\left[\frac{k-i}{2}\right]$ is the integral part of $(k-i)/2$;

(ii) for all $k \geq 1$,

$$BA^k v = \mu_k A^{k-1} v$$
with $\mu_k = -\left[\frac{k-i}{2}\right] - \frac{i+\ell}{2}$.

**Proof.** We first establish the formulas for $k = 1$. On the one hand one has:

$$AB v = -H v - BA v = -\ell v - BA v$$
and on the other hand,

$$AB v = E v + BA v = iv + BA v.$$

By adding or subtracting these two identities we deduce:

$$2AB v = (i-\ell) v \text{ and } 2BA v = (-i-\ell) v.$$ 

We get $\lambda_1 = (i-\ell)/2$ and $\mu_1 = (-i-\ell)/2$, so that the formulas are established at the order 1.
By induction on $k$,

\[
AB^{k-1}v = -HB^{k-1}v - BAB^{k-1}v
\]
\[
= -(\ell - (k-1))B^kv - \lambda_{k-1}B^{k-1}v
\]
\[
= (-\ell + k - 1 - \lambda_{k-1})B^{k-1}v
\]

We deduce the relations:

\[
\lambda_k = -\ell + k - 1 - \lambda_{k-1}
\]
\[
= -\ell + k - 1 - (-\ell + (k - 2) - \lambda_{k-2})
\]
\[
= 1 + \lambda_{k-2}.
\]

Knowing $\lambda_1 = \frac{i - \ell}{2}$ we can now obtain the explicit expression of $\lambda_k$:

\[
\lambda_k = \left[ \frac{k-i}{2} \right] + \frac{i - \ell}{2}.
\]

Hence part (i).

Part (ii) can be proved in a similar way. \[\square\]

4.3. **Proof of Theorem 2**

In this section we prove that $asl_2(\mathbb{K})$ has no non-trivial representations of finite dimension.

Let $V$ be an irreducible finite dimensional representation of $asl_2(\mathbb{K})$. Considering the actions of the elements $E = A^2$, $F = -B^2$ and $H$, the space $V$ has a structure of $sl_2$-module. Therefore, there exists a weight vector $v$ such that $Hv = \ell v$ for some $\ell \in \mathbb{Z}$.

By lemma 4.2, we can assume that $v$ is a homogeneous element, namely $v \in V_i$, $i = 0, 1$. Let us consider the family of vectors

\[
\mathcal{F} = \{ \cdots, B^k v, v, A^k v, \cdots, A^k v, \cdots \}.
\]

From formula (4.14) we know that all the nonzero vectors of $\mathcal{F}$ are eigenvectors of $H$ with distinct eigenvalues, $\ell \pm k$, $k \in \mathbb{N}$. Therefore, all the non-zero vectors of $\mathcal{F}$ are linearly independent.

Hence, there exists $N \geq 1$ such that

\[
B^{N-1}v \neq 0, \quad B^kv = 0, \quad \forall k \geq N
\]

and $M \geq 1$ such that

\[
A^{M-1}v \neq 0, \quad A^kv = 0, \quad \forall k \geq M.
\]

Using Lemma [4.3] we deduce

\[
\lambda_N = 0,
\]
\[
\mu_M = 0.
\]

This leads to

\[
\left[ \frac{N-i}{2} \right] + \frac{i - \ell}{2} = 0,
\]
\[
- \left[ \frac{M-i}{2} \right] - \frac{i + \ell}{2} = 0.
\]

By subtracting these two equations we obtain:

\[
\left[ \frac{N-i}{2} \right] + \left[ \frac{M-i}{2} \right] + i = 0.
\]
But one has: \( N, M \geq 1 \) and \( i = 0, 1 \). So, necessarily,

\[
N = M = 1, \quad i = 0.
\]

In conclusion, \( v \) is an even vector such that \( Av = Bv = 0 \) and \( V \) is nothing but the trivial representation.

Finally, if \( V \) is an arbitrary finite-dimensional representation, then \( V \) is completely reducible. This immediately follows from the classical theorem in the \( \text{osp}(1|2) \)-case.

Theorem 2 is proved.

4.4. **Proof of Theorem 3.** Let us consider an infinite-dimensional irreducible weighted representation \( V \).

We start by studying the cases of highest weight representations and lowest weight representations.

**Lemma 4.4.** Every irreducible highest weight representation is isomorphic to \( V(-1) \).

**Proof.** Consider an irreducible representation \( V \) containing a weight vector \( v \) of weight \( \ell \) such that \( Av = 0 \). We write \( v = v_0 + v_1 \) with \( v_i \in V_i, i = 0, 1 \). We also have \( Av_i = 0 \) and \( Hv_i = \ell v_i \) for \( i = 0, 1 \).

We first show that \( v_0 = 0 \). Consider the action of \( \mathfrak{sl}_2(K) \) on \( V_0 \), the vector \( v_0 \) is a highest weight vector for this action. Since \( Av_0 = 0 \), we get \( H v_0 = ABv_0 = -E v_0 = 0 \).

The vector \( v_0 \) is a highest weight vector of weight 0 for the action of \( \mathfrak{sl}_2(K) \). By consequence, \( v_0 \) is also a lowest weight vector, i.e., \( B^2 v_0 = 0 \). Thus, the space \( \text{Span} (v_0, B v_0) \) is stable under the action of \( A \) and \( B \). Since \( V \) is an infinite-dimensional irreducible representation one necessarily has \( v_0 = 0 \).

We can assume now that \( v \) belongs to \( V_1 \). Let us use Lemma 4.3 part (ii). From the relation \( B A v = 0 \), we deduce \( \mu_1 = 0 \) and thus \( \ell = -1 \). This implies that all the constants \( \lambda_k \) from Lemma 4.3 part (i) are non-zero. By induction we deduce, using Lemma 4.3 part (i), that all the vectors \( B^k v, k \in \mathbb{N} \) are non-zero. Moreover, these vectors are linearly independent since they are eigenvectors for \( H \) associated to distinct eigenvalues. By setting

\[
e_k = B^{1-k} v, \quad k \in \mathbb{Z}, \, k \leq 1,
\]

we obtain \( V(-1) \) as a subrepresentation of \( V \). We then deduce from the irreducibility assumption that \( V \simeq V(-1) \). \( \square \)

**Lemma 4.5.** Every irreducible lowest weight representation is isomorphic to \( V(1) \).

**Proof.** Similar to the proof of Lemma 4.4 \( \square \)

Now we are ready to prove Theorem 3.

Fix a weight vector \( v \in V_1 \) (such a vector exists by Lemma 4.2) of some weight \( \ell \in K \). Consider the family

\[
\mathcal{F} := \{ A^k v, B^k v, k \in \mathbb{N} \}.
\]
(a) Suppose that \( \ell \) is not an odd integer. It is easy to see that the constants \( \lambda_k \) and \( \mu_k \), \( k \in \mathbb{N} \), from Lemma 4.3 never vanish. Indeed,

\[
\begin{align*}
\lambda_k &= 0 \Rightarrow \ell = 2 \left\lfloor \frac{k-1}{2} \right\rfloor + 1, \\
\mu_k &= 0 \Rightarrow \ell = -2 \left\lfloor \frac{k-1}{2} \right\rfloor - 1.
\end{align*}
\]

By induction, we deduce that the elements in \( \mathcal{F} \) are different from zero. Moreover, the elements of \( \mathcal{F} \) are eigenvectors for the operator \( H \) with distinct eigenvalue \( \ell \pm k \), where \( k \in \mathbb{N} \), so that, they are linearly independent.

By setting

\[ e_k = \begin{cases} 
A^{k-1}v, & k \geq 1 \\
B^{1-k}v, & k \leq 0 
\end{cases} \]

we see that \( V(\ell) \) as a subrepresentation of \( V \). Again, by irreducibility assumption, we deduce \( V \cong V(\ell) \).

Finally, using Proposition 4.1, part (ii), one has:

\[ V \cong V(\ell'), \]

where \( \ell' \) is the unique element of

\[ \mathcal{P}^+ = [-1, 1] \cup \{ \ell \in \mathbb{C} | -1 \leq \text{Re}(\ell) < 1 \}, \]

such that \( \ell' = \ell + 2m \) for some \( m \in \mathbb{Z} \).

(b) Suppose that \( \ell \) is a positive odd integer. From the first statement of (4.15), we deduce the existence of an integer \( N \geq 1 \), such that \( \lambda_N = 0 \) and \( \lambda_k \neq 0 \) for all \( k < N \). Hence,

\[ B^k v \neq 0, \quad \forall k < N, \quad AB^N v = 0. \]

If \( B^N v \neq 0 \) then this vector is a highest weight vector. By Lemma 4.4 we obtain \( V \cong V(-1) \). But, in the highest weight representation \( V(-1) \), the set of weights is the set of negative integers. We obtain a contradiction since \( v \) has a positive weight.

It follows that \( B^N v = 0 \) and this implies that \( B^{N-1} \) is a lowest weight vector. Using Lemma 4.5 we conclude

\[ V \cong V(1). \]

(c) Suppose finally that \( \ell \) is a negative odd integer. Then similar arguments show:

\[ V \cong V(-1). \]

Theorem 3 is proved.

Remark 4.6. We proved that any irreducible weighted representation is a Harish-Chandra irreducible representation (i.e. the weight spaces are all finite dimensional). A classification of Harish-Chandra irreducible representations of \( \text{osp}(1|2) \) over the field of complex numbers is given in [3]. The correspondence between the representations \( V(\ell) \) and the representations given in Theorem 5.13 of [3] is the following:

(a) If \( \ell \in \mathcal{P}^+ \) is not an odd integer then

\[ V(\ell) \cong \mathcal{P}(l, \lambda_0), \]

for the choice \( l = 0 \) and \( \lambda_0 = \ell/2 \).
(b) The lowest weight representation is
\[ V(1) \cong [\lambda_0] \downarrow, \]
for the unique choice \( \lambda_0 = -1/2. \)

(c) The highest weight representation is
\[ V(-1) \cong [\lambda_0] \uparrow, \]
for the unique choice \( \lambda_0 = 1/2. \)

**Appendix: Tensor Product of Two Representations**

Given two representations, \( V \) and \( W \) of \( \text{asl}_2(\mathbb{K}) \), to what extent their tensor product, \( V \otimes W \) is again an \( \text{asl}_2 \)-representation? This question is non-trivial since \( \text{asl}_2(\mathbb{K}) \) is not a Lie algebra. We will show that \( \text{asl}_2(\mathbb{K}) \) does not act on \( V \otimes W \).

An attempt to define such an action leads to a deformation of the \( \text{asl}_2 \)-relations by the Casimir element of \( \mathcal{U}(\text{osp}(1|2)) \). The algebraic meaning of this deformation is not yet clear.

The operators \( A \) and \( B \) have canonical lifts to \( V \otimes W \) according to the Leibniz rule:
\[ \tilde{A} = A \otimes \text{Id} + \text{Id} \otimes A, \quad \tilde{B} = B \otimes \text{Id} + \text{Id} \otimes B, \]
since they belong to the \( \text{osp}(1|2) \)-action. It is then natural to define the lift of operator \( E \) by \( \mathcal{E} := \tilde{A}\tilde{B} - \tilde{B}\tilde{A} \). One immediately obtains the explicit formula
\[ \mathcal{E} = E \otimes \text{Id} + \text{Id} \otimes E + 2 (A \otimes B - B \otimes A). \]

The following statement is straightforward.

**Proposition 4.7.** The operators \( \tilde{A}, \tilde{B} \) and \( \mathcal{E} \) satisfy the following relations:
\[
\begin{align*}
\tilde{A}\tilde{B} - \tilde{B}\tilde{A} &= \mathcal{E} \\
\tilde{A}\mathcal{E} + \mathcal{E}\tilde{A} &= \tilde{A} \\
\tilde{B}\mathcal{E} + \mathcal{E}\tilde{B} &= \tilde{B} \\
\mathcal{E}^2 &= \mathcal{E} + 4 \bar{C},
\end{align*}
\]

where
\[ \bar{C} = E \otimes F + F \otimes E + \frac{1}{2} (H \otimes H + A \otimes B - B \otimes A). \]

This means that two of the relations (2.5) are satisfied, but not the last \( \text{asl}_2 \)-relation \( \mathcal{E}^2 = \mathcal{E} \).

Let us recall that the element \( C \in \mathcal{U}(\text{osp}(1|2)) \) given by
\[ C = EF + FE + \frac{1}{2} (H^2 + AB - BA) \]
is nothing but the classical Casimir element. The operator \( \bar{C} \) is the diagonal part of the standard lift of \( C \) to \( V \otimes W \). In particular, the operator \( \bar{C} \) commutes with the action of \( \text{asl}_2(\mathbb{K}) \) and \( \text{osp}(1|2) \):
\[ [\bar{C}, \tilde{A}] = [\bar{C}, \tilde{B}] = [\bar{C}, \mathcal{E}] = 0. \]

This is how the Casimir operator of \( \text{osp}(1|2) \) appears in the context of representations of \( \text{asl}_2(\mathbb{K}) \).

The relations (1.10) look like a “deformation” of the \( \text{asl}_2 \)-relations (2.5) with one parameter that commutes with all the generators. It would be interesting to find a precise algebraic sense of this deformation.


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