On cubic-linear polynomial mappings

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Abstract. In the field of the Jacobian conjecture it is well-known after Drużkowski that from a polynomial “cubic-homogeneous” mapping we can build a higher-dimensional “cubic-linear” mapping and the other way round, so that one of them is invertible if and only if the other one is. We make this point clearer through the concept of “pairing” and apply it to the related conjugability problem: one of the two maps is conjugable if and only if the other one is; moreover, we find simple formulas expressing the inverse or the conjugations of one in terms of the inverse or conjugations of the other. Two nontrivial examples of conjugable cubic-linear mappings are provided as an application.
1. Introduction

The following conjecture was essentially originated by Keller [14] in 1939:

**Jacobian Conjecture.** *For all* $n \in \mathbb{N}$, *if* $f: \mathbb{C}^n \to \mathbb{C}^n$ *has polynomial components and the Jacobian determinant* $\det f'(x)$ *is a nonzero constant throughout* $\mathbb{C}^n$, *then* $f$ *is a polynomial automorphism of* $\mathbb{C}^n$, *that is, a bijective polynomial map with polynomial inverse.*

There is a huge literature on this topic and also some wrong proofs were published. A basic paper on the subject is [2] by Bass, Connell and Wright. The recent proceedings of conference [10], and in particular its first paper, by the editor van den Essen, are a good update on this research field, rich in questions of different nature.

In everything that follows $\mathbb{R}$ or $\mathbb{R}^n$ can be substituted for $\mathbb{C}$ and $\mathbb{C}^n$ with only tripling adjustments. Before proceeding it is convenient to establish some notations first: if $x, y \in \mathbb{C}^n$ we will write $x * y$ for the componentwise product of the two vectors: $x * y := (x_1y_1, x_2y_2, \ldots, x_ny_n) \in \mathbb{C}^n$. The powers with respect to this multiplication will be denoted by $x^{*2}, x^{*3}, \ldots$. The symbol $I_n$ will be identity mapping (or matrix) in $\mathbb{C}^n$. The minus signs in formulas (1.1) and (1.2) below may seem odd but will later simplify some expressions in Section 4.

**Definition 1.1.** A mapping $f: \mathbb{C}^n \to \mathbb{C}^n$ will be called “cubic-homogeneous” if there exists a trilinear symmetric function $g: \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ such that

$$f(x) = x - g(x, x, x) \quad \text{for all} \ x \in \mathbb{C}^n, \tag{1.1}$$

It will be called “cubic-linear” if there exists an $n \times n$ matrix $A$ such that

$$f(x) = x - (Ax)^{*3} \quad \text{for all} \ x \in \mathbb{C}^n. \tag{1.2}$$

Cubic-homogeneous and cubic-linear mappings with constant Jacobian determinant will be called “Yagzhev maps” and “Drużkowski maps” respectively.

Two classical “reduction” results bear in particular on the present paper. They restrict the class of polynomial functions over which it is sufficient to concentrate the attention in order to prove or disprove the full conjecture: the first reduction was to Yagzhev maps (Yagzhev [18] and independently Bass-Connell-Wright [2]) and the second was to the smaller class of the Drużkowski maps (Drużkowski [7]).

A side issue of the Jacobian conjecture was introduced in [5]: given a parameter $\lambda \in \mathbb{C} \setminus \{0, 1\}$ and a polynomial mapping $f: \mathbb{C}^n \to \mathbb{C}^n$ such that $f(0) = 0$, $f'(0) = I_n$, the problem is to find a global analytic conjugation, i.e., an invertible analytic function $k_\lambda: \mathbb{C}^n \to \mathbb{C}^n$ such that the following diagram commutes:

$$\begin{array}{cc}
\mathbb{C}^n & \leftarrow \mathbb{C}^n \\
\downarrow \lambda F & \quad \downarrow \lambda I_n \\
\mathbb{C}^n & \leftarrow \mathbb{C}^n
\end{array} \tag{1.3}$$
and with again the “normalizing” conditions \( k_\lambda(0) = 0, \ k'_\lambda(0) = I_n \). We will also be handling functions \( k_\lambda \) that are like conjugation, except for either being defined only on a neighbourhood of \( 0 \in \mathbb{C}^n \) or for possibly failing to be invertible: these will be called pre-conjugations.

If a mapping \( f: \mathbb{C}^n \to \mathbb{C}^n \) admits a conjugation for some \( \lambda \), then the discrete dynamical system of the backward and forward iterates of \( \lambda f \) is “trivial”. In particular, \( f \) is invertible, and this observation was in fact the original motivation for raising the problem. There was some hope to prove that the Jacobian conjecture was true by proving first that all polynomial maps in a suitable class are conjugable.

With the work that has been done in the past few years on conjugations, and specially after the examples found by A. van den Essen and E. Hubbers, the hope to possibly prove the Jacobian conjecture through conjugations has dimmed, although we cannot rule it out yet. What is still very well possible is that a counterexample to the Jacobian conjecture may be found as a by-product of research in conjugability. One unexpected and encouraging by-product, in a seemingly unrelated area, is already here: it is a very simple and elegant counterexample to Markus-Yamabe conjecture in dimension \( \geq 3 \), due to A. Cima, A. van den Essen, A. Gasull, E. Hubbers and F. Mañosas [4].

The state of the art in the conjugation business is as follows. There are normalized polynomial automorphisms that are not conjugable: the earliest one was given in [11], and it is of “quintic-homogeneous” form, but we have later realized that also the old and well-known map in two dimensions

\[
f: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + (y + x^2)^2 \\ y + x^2 \end{pmatrix}
\] (1.4)

is a counterexample (three fixed points for \( \lambda f \) outside the origin are quickly found when \( \lambda \in \mathbb{C} \setminus \{0,1\} \)). We do not know yet if all Yagzhev maps are necessarily conjugable, although a theorem in [12] seems to make it hard to find counterexamples. Yagzhev maps have been given for which global conjugations exist that are analytic but not polynomial themselves ([10, page 231] and [12]). These last examples have in turn taught us a lesson on the Jacobian conjecture, namely, on the structure of the local inverse of Yagzhev maps (see [13]).

The present paper was born out of the effort to find whether a particular Družkowski mapping (Example 6.1 below) was conjugable. We have discovered that there is a strong link between the conjugability of a Družkowski map, or, more generally, of a cubic-linear map \( F \), and the conjugability of a certain lower-dimensional cubic-homogeneous map \( f \).

The exact relation between \( F \) and \( f \) is described as follows:

**Definition 1.2.** Given a cubic-homogeneous mapping \( f: \mathbb{C}^n \to \mathbb{C}^n \) and a cubic-linear mapping \( F: \mathbb{C}^N \to \mathbb{C}^N \), \( F(X) := X - (AX)^3 \), with \( N > n \), we will say that \( f \) and \( F \) are “paired” through the matrices \( B \) and \( C \) (of dimensions \( n \times N \) and \( N \times n \) respectively) if \( \ker A = \ker B \) and the following diagrams commute:

\[
\begin{array}{ccc}
\mathbb{C}^N & \xrightarrow{B} & \mathbb{C}^n \\
\uparrow{C} & & \uparrow{I_n} \\
\mathbb{C}^n & \xrightarrow{I_n} & \mathbb{C}^n \\
\end{array}
\quad
\begin{array}{ccc}
\mathbb{C}^N & \xleftarrow{C} & \mathbb{C}^n \\
\downarrow{F} & & \downarrow{f} \\
\mathbb{C}^N & \xrightarrow{B} & \mathbb{C}^n \\
\end{array}
\] (1.5)
that is, $BC = I_n$ and $f(x) = BF(Cx)$ for all $x \in \mathbb{C}^n$.

We will prove that when two maps are paired, one has a conjugation if and only if the other has. But the symmetry extends to invertibility too: in fact it turns out that the pairing concept underlies and somewhat elucidates Drużkowski’s reduction theorem, in particular the way it is proved in [9], and [10, page 11]. The main results of this paper can be summed up in the following theorem, which shows that the problems of both invertibility and conjugation have the same answers if two maps are paired.

**Theorem 1.3.** Every cubic-homogeneous map can be paired to a cubic-linear map and vice versa. Moreover, if $f$ and $F$ are paired, each of the following properties for one of the two mappings implies the same property for the other, for a given $\lambda \in \mathbb{C} \setminus \{0\}$, $|\lambda| \neq 1$:
1. one-to-one,
2. onto,
3. invertible with polynomial inverse,
4. constant Jacobian determinant,
5. existence of a global pre-conjugation,
6. the global pre-conjugation is onto,
7. the global pre-conjugation is one-to-one,
8. the global pre-conjugation is a polynomial map,
9. the conjugation has a polynomial inverse.

The existence of global pre-conjugations is guaranteed when $|\lambda| > 1$, and also when $f$ and $F$ are invertible.

Here is an assortment of formulas connecting $F, f$ and the respective (globally defined) pre-conjugations $K_\lambda, k_\lambda$, that are true for all $x, y \in \mathbb{C}^n$, $X, Y \in \mathbb{C}^N$, whenever every single piece just makes sense:

\[
\begin{align*}
f(BX) &= BF(X), & \det f'(x) &= \det F'(Cx), & \det F'(X) &= \det f'(BX), \\
f^{-1}(y) &= BF^{-1}(Cy), & F^{-1}(Y) &= Y - F(Cf^{-1}(BY)) + Cf^{-1}(BY), \\
k_\lambda(x) &= BK_\lambda(Cx), & k_\lambda(BX) &= BK_\lambda(X), \\
k^{-1}_\lambda(y) &= BK^{-1}_\lambda(Cy), & K^{-1}_\lambda(Y) &= Y - K_\lambda(Ck^{-1}_\lambda(BY)) + Ck^{-1}_\lambda(BY).
\end{align*}
\] (1.6)

The first two rows (and in particular the formula for $F^{-1}$ in terms of $f^{-1}$) were somehow implicit in the treatment of [9] and [10, page 11], but they were hidden beneath layers of changes of variables.

The rest of the paper is organized as follows. Section 2 concerns the existence of pairing and some basic properties. Section 3 is about invertibility. Section 4 is an introduction to pre-conjugations, in the simpler cubic-homogeneous setting and with a much easier proof than in [5], and not relying on Poincaré’s theorem [1, Sec. 25] either. Section 5 shows how pairing behaves under conjugation. Section 6 illustrates two examples: the first is Drużkowski’s example 7.8 from [9] in dimension 15, which turns out to be conjugable through a polynomial automorphism; in the end we compute a Drużkowski pairing to van den Essen’s example from [10, page 231], thus producing a new Drużkowski map in dimension 16 for which global analytic conjugations exist which are not polynomial.
2. Pairing a cubic-homogeneous mapping to a cubic-linear one, and vice versa

**Proposition 2.1.** Let \( f: \mathbb{C}^n \to \mathbb{C}^n \) be a cubic-homogeneous mapping. Then there exist \( N > n \) and linear maps \( A: \mathbb{C}^N \to \mathbb{C}^N \), \( B: \mathbb{C}^N \to \mathbb{C}^n \), \( C: \mathbb{C}^n \to \mathbb{C}^N \) such that \( f \) is paired to the cubic-linear mapping \( F(X) := X - (AX)^3 \) through \( B \) and \( C \).

**Proof.** (To follow the steps of this proof it may help to look at the last example of Section 6, where they are carried out in some detail on a nontrivial mapping \( f \)). Thanks to the algebraic identities (see [9])

\[
ab^2 = \frac{(a+b)^3 + (a-b)^3 - 2a^3}{6},
\]

\[
abc = \frac{(a+b+c)^3 + (a-b-c)^3 - (a+b-c)^3 - (a-b+c)^3}{24},
\]

we can write every third-degree monomial appearing in the components of \( f \) as a linear combination of cubic powers of linear forms of \( x \). Build a matrix \( D_0 \) by piling up in some order all the 1-row matrices representing these linear forms. Do not forget to insert the projections corresponding to the monomials such as \( x_1^3 \), that are cubic powers from the start. Next, build the matrix \( B_0 \) that combines the cubes of those linear forms so that

\[
f(x) = x - B_0(D_0 x)^3 \quad \text{for all } x \in \mathbb{C}^n.
\]

The matrix \( B_0 \) has the same dimensions as the transpose of \( D_0 \). By adding null columns to \( B_0 \) and an equal number of null rows to \( D \) we can assume that the number of columns of \( B_0 \) is \( > n \). The matrix \( B_0 \) may not yet be the \( B \) of the statement, because it need not be of full rank. But this problem is easily remedied by adding a few columns to \( B_0 \) and the same numbers of null rows to \( D_0 \). For example we can add a \( n \times n \) identity matrix at the right end of \( B_0 \) and a \( n \times n \) null matrix to the bottom of \( D_0 \). In a similar manner we can arrange that \( D_0 \) has full rank too. Call \( B, D \) the resulting matrices and \( N \) the number of columns of \( B \). We have that \( N > n \) and

\[
f(x) = x - B(D_0 x)^3 \quad \text{for all } x \in \mathbb{C}^n.
\]

Let \( C \) be any right-inverse of \( B \), i.e., an \( N \times n \) matrix such that \( BC = I_n \). What we are still missing is an \( N \times N \) matrix \( A \) that shares the same kernel as \( B \) and such that \( f(x) = BF(Cx) \), where \( F \) is defined as \( F(X) := X - (AX)^3 \). Let \( M \) be a matrix whose columns form a basis of the kernel of \( B \). If we are content for the time being to relax the equality of the kernels into the inclusion \( \ker A \supset \ker B \), this weaker condition in terms of \( M \) translates as \( AM = 0 \). On the other hand, if we impose that \( AC = D \), we will be able to write \( f(x) = x - B(ACx)^3 = B(Cx - (ACx)^3) = BF(Cx) \). The two equations \( AM = 0 \) and \( AC = D \) can be combined as

\[
A(C \mid M) = (D \mid 0), \quad \text{which solves for } A \text{ as } \quad A = (D \mid 0)(C \mid M)^{-1},
\]

where \( (C \mid M) \) is the matrix formed by joining the two blocks of columns of \( C \) and of \( M \), and \( (D \mid 0) \) similarly. The matrix \( (C \mid M) \) is indeed invertible because the range of \( C \) is a complement to the kernel of \( B \), since \( BC = I_n \). The proof is complete if we notice that with this choice of \( A \) the kernel of \( B \) is equal to, and not merely contained in, the kernel of \( A \), because the rank of \( A \) is the same as the rank of \( D \). \( \square \)
The reverse procedure from a cubic-linear to a paired cubic-homogeneous mapping is much easier. Throughout the rest of this paper $A$ will be a fixed linear mapping $A: \mathbb{C}^N \to \mathbb{C}^N$, that we will as usual identify with the matrix that represents it with respect to the canonical basis of $\mathbb{C}^N$, and $F(X) := X - (AX)^*^3$ for $X \in \mathbb{C}^N$. The matrix $A$ will be assumed to be singular, both because this is the case when the Jacobian determinant is constant (that is, if we are dealing with what we called Drużkowski maps; see [7]), and because the following theory trivializes anyway when $A$ is invertible. Before proceeding, take note of the following fact, that we will be using over and over again.

**Proposition 2.2.** If $X \in \mathbb{C}^N$ and $X_0 \in \ker A$, then $F(X + X_0) = F(X) + X_0$. In particular the differential satisfies $F'(X)X_0 = X_0$ and $F'(X + X_0) = F'(X)$.

**Proof.** Obvious: $F(X + X_0) = X + X_0 - (AX + AX_0)^*^3 = F(X) + X_0$. □

Let $n$ be the rank of $A$ and $B: \mathbb{C}^N \to \mathbb{C}^n$ be a linear mapping with the same kernel as $A$. In particular $B$ has full rank, coinciding with the rank of $A$. Let $C: \mathbb{C}^n \to \mathbb{C}^N$ be a right-inverse of $B$, that is, a linear mapping such that $BC = I_n$. A mapping $f$ that is paired to $F$ through $B$ and $C$ is trivial to define:

$$f(x) := BF(Cx) = x - B(ACx)^*^3$$  \hspace{1cm} (2.5)

A property of $B$ and $C$ that we will also be using all the time without explicit reference is that

$$CBX - X \in \ker A = \ker B$$  \hspace{1cm} (2.6)  

for all $X \in \mathbb{C}^N$.

The formula is true, because $B(CBX - X) = (BC)BX - BX = BX - BX = 0$.

**Proposition 2.3.** Once $B$ is given, the paired mapping $f$ defined in (2.5) is independent of the choice of the right-inverse $C$, and it makes the following diagram commute:

$$\begin{array}{ccc}
\mathbb{C}^N & \xrightarrow{B} & \mathbb{C}^n \\
\downarrow F & & \downarrow f \\
\mathbb{C}^N & \xrightarrow{B} & \mathbb{C}^n
\end{array}$$  \hspace{1cm} (2.7)

**Proof.** Let $C, \tilde{C}$ be two right inverses of $B$. Then $Cx = \tilde{C}x \in \ker A = \ker B$ for all $x \in \mathbb{C}^n$, because $B(Cx - \tilde{C}x) = BCx - B\tilde{C}x = x - x = 0$. The paired mapping $f$ does not depend on the choice of $C$ because

$$BF(\tilde{C}x) = B\left[F(\tilde{C}x) + \underbrace{Cx - \tilde{C}x}_{\in \ker B = \ker A}\right] = BF(\tilde{C}x + Cx - \tilde{C}x) = BF(Cx).$$  \hspace{1cm} (2.8)

As for diagram (2.7), noticing that $ACB = A$,

$$f(BX) = BF(CBX) = B\left[F(CBX) + \underbrace{X - CBX}_{\in \ker B = \ker A}\right] =$$  \hspace{1cm} (2.9)

$$BF(CBX + X - CBX) = BF(X).$$

□
Proposition 2.4. For all $x \in \mathbb{C}^n$, $X \in \mathbb{C}^N$ we have $\det f'(x) = \det F'(Cx)$ and $\det F'(X) = \det f'(BX)$. In particular $f$ has constant Jacobian determinant if and only if $F$ has.

Proof. To study the Jacobian determinants of $F$ and $f$ it is convenient to decompose first $\mathbb{C}^N = (\text{range } C) \oplus (\text{ker } A)$ and to choose a basis of $\mathbb{C}^N$ whose first $n$ vectors are the image through $C$ of the canonical basis of $\mathbb{C}^n$ (forming in particular a basis of the range of $C$) and the remaining ones are a basis of ker $A$. If on $\mathbb{C}^n$ we keep the canonical basis, the matrices representing $F'(X), C, B$ take the following forms, thanks also to Proposition 2.2,

$$F'(X) = \begin{pmatrix} R(X) & S(X) \\ 0 & I_n \end{pmatrix}, \quad C = \begin{pmatrix} I_n \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} I_n \\ 0 \end{pmatrix}, \quad (2.10)$$

for matrices $R(X), S(X)$ of suitable dimensions. Now

$$\det f'(x) = \det BF'(Cx)C =$$

$$= \det \left( I_n \begin{pmatrix} 0 & R(Cx) \\ \hline 0 & N(Cx) \end{pmatrix} \begin{pmatrix} I_n \\ 0 \end{pmatrix} \right) =$$

$$= \det R(Cx) = \det F'(Cx). \quad (2.11)$$

Conversely,

$$\det F'(X) = \det F'(CBX + X - CBX) = \det F'(CBX) = \det f'(BX). \quad (2.12)$$

\qed
3. Inverses of paired mappings

**Proposition 3.1.** If \( F \) is one-to-one, so is \( f \). If \( F \) is onto, so is \( f \). If \( F \) is a bijection, then so is \( f \), and \( f^{-1}(y) = BF^{-1}(Cy) \) for all \( y \in \mathbb{C}^n \), that is, the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{C}^N & \xrightarrow{B} & \mathbb{C}^n \\
\uparrow f^{-1} & & \uparrow f^{-1} \\
\mathbb{C}^N & \xleftarrow{C} & \mathbb{C}^n \\
\end{array}
\]

(3.1)

In particular, if \( F^{-1} \) is a polynomial mapping, so is \( f^{-1} \), and the degree of \( f^{-1} \) is not higher than the degree of \( F^{-1} \).

**Proof.** Suppose that \( F \) is one-to-one. Let \( x_0, x_1 \in \mathbb{C}^n \). Then

\[
f(x_1) = f(x_2) \implies BF(Cx_1) = BF(Cx_2) \implies F(Cx_1) = F(Cx_2) \implies Cx_1 - X_0 = Cx_2 \implies \text{range } C \ni C(x_1 - x_2) = X_0 \implies x_1 = x_2.
\]

Suppose that \( F \) is onto. For a given \( y \in \mathbb{C}^n \), we have to prove that \( y \) is in the range of \( f \). Let \( X \in \mathbb{C}^N \) be such that \( F(X) = Cy \). Then

\[
\text{range } f \ni f(BX) = BF(CBX) = B\left(F(CBX) + X - CBX\right) = B\left(F(CBX) + X - CBX\right) \in \ker B = \ker A = \ker B
\]

(3.3)

Finally, when \( F \) is a bijection, the vector \( X \) in (3.3) is simply \( F^{-1}(Cy) \), which proves the first formula for the inverse. \( \square \)

**Proposition 3.2.** If \( f \) is one-to-one, so is \( F \). If \( f \) is onto, so is \( F \). If \( f \) is a bijection, then so is \( F \), and for all \( Y \in \mathbb{C}^N \)

\[
F^{-1}(Y) = Y + \left(ACf^{-1}(BY)\right)^3 = Y - F(Cf^{-1}(BY)) + Cf^{-1}(BY)
\]

(3.4)

In particular, if \( f^{-1} \) is a polynomial mapping, so is \( F^{-1} \), and the degree of \( F^{-1} \) is at most three times the degree of \( f^{-1} \).
Proof. Suppose that \( f \) is one-to-one and let \( X_1, X_2 \in \mathbb{C}^N \). Then
\[
F(X_1) = F(X_2) \implies F\left( CBX_1 + X_1 - CBX_1 \right) = F\left( CBX_2 + X_2 - CBX_2 \right) \in \ker B = \ker A
\]
\[
\implies F(CBX_1) + X_1 - CBX_1 = F(CBX_2) + X_2 - CBX_2 \quad (*) \quad (3.5)
\]
\[
\implies BF(CBX_1) = BF(CBX_2)
\]
\[
\implies f(BX_1) = f(BX_2)
\]
\[
\implies BX_1 = BX_2 \quad \text{(using formula * above)}
\]
\[
\implies X_1 = X_2 .
\]

Suppose that \( f \) is onto and let \( Y \in \mathbb{C}^N \). Let \( x \in \mathbb{C}^n \) be such that \( f(x) = BY \). Then
\[
\text{range } F \ni F(Y + (ACx)^3) =
\]
\[
= F\left( Y - CBY + CBY + (ACx)^3 \right) =
\]
\[
= Y - Cf(x) + F\left( Cf(x) - F(Cx) + F(Cx) + (ACx)^3 \right) =
\]
\[
= Y - Cf(x) + Cf(x) - F(Cx) + F(Cx) =
\]
\[
= Y .
\]

Assume finally that \( f \) is a bijection. Then we can write \( x = f^{-1}(BX) \) in (3.6) and get the first formula for the inverse. The second expression is a simple consequence:
\[
F^{-1}(Y) = Y + \left( ACF^{-1}(BY) \right)^3 =
\]
\[
= Y - \left( Cf^{-1}(BY) - \left( ACF^{-1}(BY) \right)^3 \right) + Cf^{-1}(BY) =
\]
\[
= Y - F(Cf^{-1}(BY)) + Cf^{-1}(BY) .
\]

\( \square \)
4. Pre-conjugations for cubic-homogeneous mappings

**Proposition 4.1.** Let $\mathbb{X}$ be a complex Banach space, $\gamma : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ be a continuous trilinear symmetric form, and define the function $\varphi : \mathbb{X} \to \mathbb{X}$ as $\varphi(x) := x - \gamma(x, x, x)$. Then for any $\lambda \in \mathbb{C} \setminus \{0\}$, with $|\lambda| \neq 1$, there exists an analytic function $\kappa_\lambda$ defined in a neighbourhood of $0 \in \mathbb{X}$ and with values in $\mathbb{X}$, such that

$$
\kappa_\lambda(0) = 0, \quad \kappa'_\lambda(0) = I_{\mathbb{X}} \quad \text{(the identity operator on } \mathbb{X}) \quad \text{and}
$$

$$
\lambda \varphi(\kappa_\lambda(y)) = \kappa_\lambda(\lambda y) \quad \text{for all } y \in \mathbb{X} \text{ such that } y, \lambda y \in \text{dom } \kappa_\lambda.
$$

The function $\kappa_\lambda$ is unique, in the sense that any two functions with the same property must agree in a neighbourhood of the origin. If we denote by $\Psi_m$ the homogeneous term of degree $m$ in the Taylor series $\kappa_\lambda = \sum_{m \geq 0} \Psi_m$ of $\kappa_\lambda$ centered in the origin (ignoring the dependence on $\lambda$), the following recursive formulas hold:

$$
\begin{align*}
\Psi_0(y) := 0, \quad &\Psi_1(y) := y, \\
\Psi_m = \frac{1}{1 - \lambda^{m-1}} \sum_{\substack{p+q+r=m \\leq p, q, r < m}} \gamma(\Psi_p, \Psi_q, \Psi_r), & \quad \text{for } m \geq 2.
\end{align*}
$$

If either $|\lambda| > 1$ or $\varphi$ is invertible, then the function $\kappa_\lambda$ is defined and analytic on the whole of $\mathbb{X}$. Finally, if $\mathbb{X}$ is finite-dimensional and the Jacobian determinant of $\varphi$ is constant, then the same happens to $\kappa_\lambda$ on any connected open neighbourhood of the origin (both constants must be 1, of course, because $\varphi'(0) = \kappa'_\lambda(0) = I_{\mathbb{X}}$).

**Proof.** Uniqueness of $\kappa_\lambda$ and the recursive relations (4.2) are obtained as in [13] simply by substitution of $\kappa_\lambda = \sum_m \Psi_m$ into the conjugation formula $\lambda \varphi(\kappa_\lambda(y)) = \kappa_\lambda(\lambda y)$, using the multilinearity of $\gamma$ and the homogeneity of $\Psi_k$:

$$
\lambda \sum_{m \geq 0} \Psi_k - \lambda \sum_{p, q, r \geq 0} \gamma(\Psi_p, \Psi_q, \Psi_r) = \sum_{m \geq 0} \lambda^m \Psi_m,
$$

and then by grouping together the terms which are homogeneous of the same degree. The initial conditions on $\Psi_0, \Psi_1$ cannot be derived from the conjugation relation, and are simply the transcriptions of the normalizing conditions on $\kappa_\lambda(0), \kappa'_\lambda(0)$. The summation in (4.2) can be restricted to the $p, q, r$ strictly less than $m$ because $\Psi_0 = 0$. Observe that $\Psi_m = 0$ when $m$ is even, a fact that we have chosen not to highlight here, but that speeds up computations sometimes.

We have to prove that the series $\sum_k \Psi_k(y)$ converges when $\|y\|$ is small enough. Write

$$
\begin{align*}
a_m := \sup_{\|y\| \leq 1} \|\Psi_m(y)\|, \\
\|\gamma\| := \sup \left\{ \|\gamma(x, y, z)\| : \|x\| \leq 1, \|y\| \leq 1, \|z\| \leq 1 \right\}.
\end{align*}
$$
The series $\sum \Psi_m(y)$ will converge whenever $\sum a_m \|y\|^m < +\infty$. The following inequalities hold:

$$a_0 = 0, \quad a_1 = 1, \quad a_m \leq \frac{\|\gamma\|}{|1 - \lambda^{m-1}|} \sum_{p+q+r=m} a_p a_q a_r \leq \frac{\|\gamma\|}{|1 - |\lambda||} \sum_{p+q+r=m} a_p a_q a_r. \quad (4.5)$$

Then we see that $0 \leq a_m \leq b_m$ for all $m$, where $b_m$ is the sequence defined by recursion as $b_0 := 0, \quad b_1 := 1, \quad b_m := \sum_{p+q+r=m} b_p b_q b_r$ for $m \geq 2$, where $\alpha := \frac{\|\gamma\|}{|1 - |\lambda||}$. \quad (4.6)

If we define the one-variable (formal) power series $\mu(t) := \sum b_m t^m$, we see that the function $\mu$ should verify the relation

$$\mu(0) = 0, \quad \mu'(0) = 1, \quad \mu(t) - \alpha \mu(t)^3 = t \quad (4.7)$$

for all $t$ where $\mu(t)$ exists. This means that $\mu$ must be a local inverse of the complex variable function $u \mapsto u - \alpha u^3$, around the origin, mapping $0$ to $0$. But we very well know that such a local inverse exists and it is a power series with a positive radius $R$ of convergence. We could estimate $R$, if we wish, using Cardano’s formula for cubic equations. We conclude that the power series $\sum b_m t^m$ has positive radius $R$ of convergence. If $\|y\| < R$ we have that $\sum \|\Psi_m(y)\| \leq \sum a_m \|y\|^m \leq \sum b_m \|y\|^m = \mu(\|y\|) < +\infty$. The local existence of $\kappa_\lambda$ is established.

The fact that $\kappa_\lambda$ exists on the whole of $X$ if $|\lambda| > 1$ follows from the same simple argument used in [5]: the conjugation relation $\lambda \varphi(\kappa_\lambda(y)) = \kappa_\lambda(\lambda y)$ allows us to define $\kappa_\lambda(\lambda y)$ whenever we know $\kappa_\lambda(y)$, and the extensions that we obtain this way are analytical.

Similarly, when $\varphi$ is invertible, the conjugation relation can be rewritten as $\kappa_\lambda(y) = \varphi^{-1}(\kappa_\lambda(\lambda y)/\lambda)$, which allows us to extend analytically the definition of $\kappa_\lambda$ to the whole space if $0 < |\lambda| < 1$.

The derivative of the conjugation identity $\lambda \varphi(\kappa_\lambda(y)) = \kappa_\lambda(\lambda y)$ with respect to $y$ is $\lambda \varphi'(\kappa_\lambda(y)) \kappa_\lambda'(y) = \kappa_\lambda'(\lambda y)$. If $X = \mathbb{C}^n$ and $\varphi$ has constant Jacobian determinant, then this constant is $1$ because $\varphi'(0) = I_n$, and we deduce that $\det \kappa_\lambda'(\lambda y) = \det \kappa_\lambda'(y)$. If $y \in \mathbb{C}^n \setminus \{0\}$ is close enough to the origin then $\lambda^r y \in \text{dom } \kappa_\lambda$ either for all $r \geq 0$ or for all $r \leq 0$, depending on whether $|\lambda| > 1$ or $|\lambda| < 1$. In either case $\det \kappa_\lambda'$ has the same value along a sequence of points containing $y$ and with the origin as a cluster point. Then $\det \kappa_\lambda'(y) = \det k'(0) = 1$ because $\kappa_\lambda'$ is continuous. \quad \square

**Remark 4.2.** If we consider the local inverse of $\varphi$ around the origin, the terms of its Taylor expansion $\varphi^{-1} = \sum_m \Phi_m$ satisfy the same recursive relations as the $\Psi_m$, only with $\lambda = 0$ (see [8]). It follows from this with simple calculations that the $\Psi_m$ are scalar multiples of the corresponding $\Phi_m$ up to degree 5:

$$\Psi_1 = \Phi_1, \quad \Psi_3 = \frac{1}{1 - \lambda^2} \Phi_3, \quad \Psi_5 = \frac{1}{(1 - \lambda^2)(1 - \lambda^4)} \Phi_5. \quad (4.8)$$

However the property fails from degree 7 onward. For example

$$\Psi_7(x) = \frac{1}{(1 - \lambda^2)(1 - \lambda^4)(1 - \lambda^6)} \left(\Phi_7(x) + 3\lambda^2 \gamma(\gamma(x, x, x), \gamma(x, x, x))\right). \quad (4.9)$$
5. Conjugations of paired mappings

In this section we will use the letters $F, f, A, B, C$ with the same meaning as in Section 2. The function $F$ can be expressed as $F(X) = X - G(X, X, X)$, where $G$ is defined as

$$G(X, Y, Z) := (AX) \ast (AY) \ast (AZ) \text{ for } X, Y, Z \in \mathbb{C}^N.$$  \hspace{1cm} (5.1)

This $G$ is trilinear and symmetric from $\mathbb{C}^N \times \mathbb{C}^N \times \mathbb{C}^N$ into $\mathbb{C}^N$, and we can apply Proposition 4.1 to $F$: for $\lambda \in \mathbb{C} \setminus \{0\}$, $|\lambda| \neq 1$, there exists a unique analytic $K_\lambda$, defined as a convergent Taylor series in a neighbourhood $\text{dom } K_\lambda$ of $0 \in \mathbb{C}^N$ and with values in $\mathbb{C}^N$ such that $K_\lambda(0) = 0$, $K'_\lambda(0) = I_N$ and such that $\lambda F(K_\lambda(X)) = K_\lambda(\lambda X)$ for all $X$ such that $X, \lambda X \in \text{dom } K_\lambda$.

**Proposition 5.1.** If $X \in \text{dom } K_\lambda$ and $X_0 \in \ker A$ then $X + X_0 \in \text{dom } K_\lambda$ and $K_\lambda(X + X_0) = K_\lambda(X) + X_0$.

**Proof.** Consider the recursive formulas (4.2): to start with

$$\Psi_0(X + X_0) = 0 = \Psi_0(X), \quad \Psi_1(X + X_0) = X + X_0 = \Psi_1(X) + X_0. \hspace{1cm} (5.2)$$

If $\Psi_r(X + X_0)$ equals either $\Psi_r(X)$ or $\Psi_r(X) + X_0$ for all $r < m$, then $A \Psi_r(X + X_0) = A \Psi_r(X)$ and

$$\Psi_m(X + X_0) = \frac{1}{1 - \lambda^{m-1}} \sum_{p+q+r=m \atop 0 \leq p, q, r < m} (A \Psi_p(X + X_0)) \ast (A \Psi_q(X + X_0)) \ast (A \Psi_r(X + X_0)) = \frac{1}{1 - \lambda^{m-1}} \sum_{p+q+r=m \atop 0 \leq p, q, r < m} (A \Psi_p(X)) \ast (A \Psi_q(X)) \ast (A \Psi_r(X)) = \Psi_m(X) \text{ for } m \geq 2. \hspace{1cm} (5.3)$$

The paired function $f$ can be written in the form $f(x) = x - g(x, x, x)$, where $g$ is the trilinear symmetric form defined by

$$g(x, y, z) := B((ACx) \ast (ACY) \ast (ACz)). \hspace{1cm} (5.4)$$

Hence Proposition 4.1 can be applied to $f$ too: for $\lambda \in \mathbb{C} \setminus \{0\}$, $|\lambda| \neq 1$, there exists a unique analytic $k_\lambda$, defined as a convergent Taylor series in a neighbourhood of $0 \in \mathbb{C}^n$ and with values in $\mathbb{C}^n$ such that $k_\lambda(0) = 0$, $k'_\lambda(0) = I_n$ and such that $\lambda f(k_\lambda(x)) = k_\lambda(\lambda x)$ for all $x$ such that $x, \lambda x \in \text{dom } k_\lambda$.

The next two Propositions teach us that whenever either $k_\lambda$ or $K_\lambda$ is globally defined, then the other one is too, so that the following commutative diagrams always travel together:

$$\begin{array}{cccccc}
\mathbb{C}^n & \xleftarrow{k_\lambda} & \mathbb{C}^n & \xrightarrow{K_\lambda} & \mathbb{C}^N & \xleftarrow{\lambda} & \mathbb{C}^N \\
\downarrow{\lambda f} & & \downarrow{\lambda I_n} & & \downarrow{\lambda F} & & \downarrow{\lambda I_N} \\
\mathbb{C}^n & \xleftarrow{k_\lambda} & \mathbb{C}^n & \xrightarrow{K_\lambda} & \mathbb{C}^N & \xleftarrow{\lambda} & \mathbb{C}^N 
\end{array} \hspace{1cm} (5.5)$$
Proposition 5.2. For small \( x \in \mathbb{C}^n, X \in \mathbb{C}^N \) we have that \( k_\lambda(x) = BK_\lambda(Cx) \) and \( k_\lambda(BX) = BK_\lambda(X) \). Moreover, if \( K_\lambda \) is globally defined on \( \mathbb{C}^N \), then the function \( k_\lambda \) is globally defined on \( \mathbb{C}^n \) too, and the following diagrams commute:

\[
\begin{array}{ccc}
\mathbb{C}^N & \xrightarrow{B} & \mathbb{C}^n \\
\uparrow K_\lambda & & \uparrow k_\lambda \\
\mathbb{C}^N & \xleftarrow{C} & \mathbb{C}^n \\
\end{array}
\]

(5.6)

In particular, if \( K_\lambda \) is a polynomial mapping, so is \( k_\lambda \), and the degree of \( k_\lambda \) is not higher than the degree of \( K_\lambda \).

Proof. Let \( p(x) := BK_\lambda(Cx) \) for small \( x \in \mathbb{C}^n \). We have that \( p(0) = BK_\lambda(0) = 0 \), \( p'(0) = BK'_\lambda(0)C = BC = I_n \), and

\[
\lambda f(p(x)) = \lambda BF(Cp(x)) = \lambda BF(CBK_\lambda(Cx)) = \\
\quad = \lambda B \left( F(CBK_\lambda(Cx)) + K_\lambda(Cx) - CBK_\lambda(Cx) \right) = \\
\quad \in \ker A = \ker B \\
\quad = \lambda BF(CBK_\lambda(Cx) + K_\lambda(Cx) - CBK_\lambda(Cx)) = \\
\quad = \lambda BF(K_\lambda(Cx)) = BK_\lambda(\lambdaCx) = \\
\quad = p(\lambda x).
\]

The function \( p \) is obviously analytic and it satisfies the same relations that define \( k_\lambda \) uniquely by Proposition 4.1. Hence \( p = k_\lambda \) near the origin and the conjugation relation \( \lambda f(k_\lambda(x)) = k_\lambda(\lambda x) \) holds for small \( x \in \mathbb{C}^n \). Next, let \( X \in \mathbb{C}^N \) be small. From Proposition 5.1 we have that

\[
K_\lambda(X) = K_\lambda(CBX + X - CBX) = K_\lambda(CBX) + X - CBX, \quad (5.8)
\]

whence, applying \( B \) we get that \( BK_\lambda(X) = BK_\lambda(CBX) = k_\lambda(BX) \). If \( K_\lambda \) is globally defined, the identities extend to the whole spaces and define \( k_\lambda \) everywhere on \( \mathbb{C}^n \). The first one shows also that if \( K_\lambda \) is polynomial so is \( k_\lambda \), with no greater degree. \( \square \)

Proposition 5.3. For \( X \) in a neighbourhood of \( 0 \in \mathbb{C}^N \) we can write \( K_\lambda(X) = Ck_\lambda(BX) + Q(X) \), where \( Q \) is the unique analytic function such that \( Q'(0) = I_n - CB \) and such that

\[
Q(\lambda X) - \lambda Q(X) = \lambda(I_n - CB)F(Ck_\lambda(BX)) \quad \text{for small } X \in \mathbb{C}^N. \quad (5.9)
\]

If \( k_\lambda \) is globally defined on \( \mathbb{C}^n \), then \( Q \) and \( K_\lambda \) are also globally defined on \( \mathbb{C}^N \). Moreover, if \( k_\lambda \) is a polynomial mapping, so is \( K_\lambda \), and the degree of \( K_\lambda \) is at most three times the degree of \( k_\lambda \).
Proof. Let $Q$ be defined as $Q(X) := K_\lambda(X) - Ck_\lambda(BX)$ for small $X$. This function $Q$ is obviously analytic near the origin and $Q'(0) = K_\lambda'(0) - Ck_\lambda'(0)B = I_n - CB$. Using Proposition 5.2 we have that $BQ(X) = BK_\lambda(X) - BCk_\lambda(BX) = k_\lambda(BX) - k_\lambda(BX) = 0$, so that $Q(X) \in \ker B = \ker A$. Let us write the conjugation relation $\lambda F(K_\lambda(X)) = K_\lambda(\lambda X)$ in terms of $Q$: the left-hand side becomes

$$\lambda F(Ck_\lambda(BX) + Q(X)) = \lambda F(Ck_\lambda(BX)) + \lambda Q(X),$$

while the right-hand side is, using the conjugation relation for $f$, $k_\lambda$ and the definition of $f$,

$$Ck_\lambda(\lambda BX) + Q(\lambda X) = \lambda CF(k_\lambda(BX)) + Q(\lambda X) = \lambda CBF(Ck_\lambda(BX)) + Q(\lambda X).$$

Formula (5.9) is simply the rearranged combination of (5.10) and (5.11). Let $\sum \Phi_m(X)$ be the Taylor expansion of $X \mapsto (I_n - CB)F(Ck_\lambda(BX))$ centered in the origin (notice that this function has values in $\ker A$), and $\sum \varphi_m(X)$ the one of $Q(X)$. Relation (5.9) is equivalent to

$$(\lambda^{m-1} - 1)\varphi_m(X) = \Phi_m(X),$$

which determines uniquely all the terms $\varphi_m$ except the one with $m = 1$.

If we assume that $k_\lambda$ is globally defined and $0 < |\lambda| < 1$, then formula (5.9) can be used to extend analytically the definition of $Q$ from any ball $\{X : |X| < r\}$ to the larger ball $\{X : |\lambda X| < r\}$. This means that $Q$ is global, and hence $K_\lambda$ too. When $|\lambda| > 1$ both conjugation are global to begin with, because of Proposition 4.1.

If $k_\lambda$ is a polynomial mapping, then all the $\Phi_m$ vanish identically for $m$ beyond three times its degree, so the same happens for $\varphi_m$ too. \[\square\]

In the remaining part of this Section we will deduce the invertibility of each of $K_\lambda, k_\lambda$ from the invertibility of the other. For this we will assume that $k_\lambda$ and $K_\lambda$ are both globally defined, as it is always the case when either $|\lambda| > 1$ or $F$, $A$ are invertible.

**Proposition 5.4.** If $K_\lambda$ is one-to-one, so $k_\lambda$ is. If $K_\lambda$ is onto, so $k_\lambda$ is. If $K_\lambda$ is bijective, so is $k_\lambda$, and $k_\lambda^{-1}(y) = BK_\lambda^{-1}(Cy)$ for all $y \in \mathbb{C}^n$, i.e., the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{C}^N & \overset{C}{\leftarrow} & \mathbb{C}^n \\
\downarrow K_\lambda^{-1} & & \downarrow k_\lambda^{-1} \\
\mathbb{C}^N & \overset{B}{\rightarrow} & \mathbb{C}^n
\end{array}$$

In particular, if $K_\lambda^{-1}$ is a polynomial mapping, then so is $k_\lambda^{-1}$.

**Proof.** Suppose that $K_\lambda$ is one-to-one. Then for all $x_1, x_2 \in \mathbb{C}^n$

$$k_\lambda(x_1) = k_\lambda(x_2) \implies BK_\lambda(Cx_1) = BK_\lambda(Cx_2) \implies K_\lambda(Cx_1) - K_\lambda(Cx_2) = X_0 \in \ker A = \ker B \implies K_\lambda(Cx_1) = K_\lambda(Cx_2 + X_0) \implies Cx_1 = Cx_2 + X_0 \implies \text{range} C \ni C(x_1 - x_2) = X_0 \in \ker A = \ker B \implies C(x_1 - x_2) = X_0 = 0 \implies x_1 = x_2.$$
Suppose that $K_\lambda$ is onto. Let $y \in \mathbb{C}^n$ be arbitrary. There exists $Y \in \mathbb{C}^N$ such that $K_\lambda(Y) = Cy$. Then

$$\text{range } k_\lambda \ni k_\lambda(BY) = BK_\lambda(CBY) = BK_\lambda\left(CBY + \underbrace{Y - CBY}_{\in \ker A}\right) = BK_\lambda(Y) = BCy = y.$$ (5.15)

The inversion formula comes by writing $Y = K_\lambda^{-1}(Cy)$ in (5.15). □

**Proposition 5.5.** If $k_\lambda$ is one-to-one, so is $K_\lambda$. If $k_\lambda$ is onto, so is $K_\lambda$. If $k_\lambda$ is bijective, so is $K_\lambda$, and

$$K_\lambda^{-1}(Y) = Y - K_\lambda\left(Ck_\lambda^{-1}(BY)\right) + Ck_\lambda^{-1}(BY) \quad \text{for all } Y \in \mathbb{C}^N. \quad (5.16)$$

In particular, if $k_\lambda^{-1}$ is a polynomial mapping, so is $K_\lambda^{-1}$, and the degree of $K_\lambda^{-1}$ is not larger than the product of the degrees of $K_\lambda$ and $k_\lambda^{-1}$.

**Proof.** Suppose that $k_\lambda$ is one-to-one and let $X_1, X_2 \in \mathbb{C}^N$. Then

$$K_\lambda(X_1) = K_\lambda(X_2) \implies \implies K_\lambda\left(CBX_1 + \underbrace{X_1 - CBX_1}_{\in \ker B = \ker A}\right) = K_\lambda\left(CBX_2 + \underbrace{X_2 - CBX_2}_{\in \ker B = \ker A}\right) \implies K_\lambda(CBX_1) + X_1 - CBX_1 = K_\lambda(CBX_2) + X_2 - CBX_2 \quad (*) \quad (5.17)$$

$$\implies BK_\lambda(CBX_1) = BK_\lambda(CBX_2) \implies k_\lambda(BX_1) = k_\lambda(BX_2) \implies BX_1 = BX_2 \quad \text{ (using * above)} \implies X_1 = X_2.$$

Suppose that $k_\lambda$ is onto. Let $Y \in \mathbb{C}^N$. There exists $x \in \mathbb{C}^n$ such that $BY = k_\lambda(x) = BK_\lambda(Cx)$. In particular $Y - K_\lambda(Cx) \in \ker A = \ker B$. Then

$$\text{range } K_\lambda \ni K_\lambda\left(Cx + \underbrace{Y - K_\lambda(Cx)}_{\in \ker B = \ker A}\right) = K_\lambda(Cx) + Y - K_\lambda(Cx) = Y.$$ (5.18)

In particular, if $k_\lambda$ is bijective just write $x = k_\lambda^{-1}(BY)$ to get the inversion formula. Finally, if $k_\lambda^{-1}$ is a polynomial map, then also $k_\lambda$ must be polynomial by a well-known result (see e.g. [16]), and then $K_\lambda$ too by Proposition 5.3. □

If we weakened Proposition 5.5 by saying “if $k_\lambda^{-1}$ and $k_\lambda$ are polynomial mapping, so is $K_\lambda^{-1}$”, then we would not need to resort to the advanced complex analysis result of [16], and the result would extend to the real case too.

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6. Examples

**Example 6.1.** Consider the $15 \times 15$ matrix

\[
A = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & -2 & 2 & 2 & 2 & 0 & 0 & -2 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & -2 & 0 & -2 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\
0 & 2 & -4 & 0 & 0 & 0 & 2 & -2 & -2 & -2 & 0 & 0 & 0 & 2 \\
0 & 2 & 2 & -4 & 0 & 0 & 2 & -2 & -2 & -2 & 0 & 0 & 0 & 2 \\
2 & 0 & -2 & 0 & -2 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\
2 & 0 & 0 & -4 & -2 & 2 & 2 & 2 & 0 & 0 & -2 & 0 & 0 & -2 & 0 \\
0 & 2 & 0 & -4 & -2 & 2 & 2 & 2 & 0 & 0 & -2 & 0 & 0 & -2 & 0 \\
2 & 0 & 2 & 0 & 2 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\
2 & 0 & -2 & 4 & 0 & 0 & 0 & -2 & 2 & 2 & 2 & 0 & 0 & 0 & -2 \\
0 & 2 & 0 & 4 & 2 & -2 & -2 & -2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\
2 & 2 & 2 & -4 & 0 & 0 & 2 & -2 & -2 & -2 & 0 & 0 & 0 & 2 \\
2 & 2 & 0 & -4 & -2 & 2 & 2 & 2 & 0 & 0 & -2 & 0 & 0 & -2 & 0 \\
\end{pmatrix} \tag{6.1}
\]

The function $F: \mathbb{C}^{15} \to \mathbb{C}^{15}$ defined by $F(X) = X - (AX)^3$ was introduced by Drużkowski in [9] as a simpler alternative to an example by Rusek [17], concerning some geometric condition proposed by Yagzhev.

It can be verified that $A$ has rank equal to 5 and that $A^2 = 0$. A linear mapping (or matrix) $B: \mathbb{C}^{15} \to \mathbb{C}^5$ with the same kernel as $A$ is the following:

\[
B = \frac{1}{2} \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -4 & -2 & 2 & 2 & 2 & 0 & 0 & -2 & 0 & 0 & -2 & 0 \\
0 & 0 & 2 & -4 & 0 & 0 & 0 & 0 & 2 & -2 & -2 & -2 & 0 & 0 & 2 \\
0 & 0 & -2 & 0 & -2 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\
\end{pmatrix} \tag{6.2}
\]

Notice that, if we ignore the first couple of columns, the set of the rows of $B$ coincides with the set of the rows of $A$. It can be verified that the rows of $B$ are in fact a basis for the orthogonal to the kernel of $A$, with respect to the canonical scalar product. Anyway, a simple right inverse $C$ of $B$ is given by

\[
C^T := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
\end{pmatrix} \tag{6.3}
\]

The mapping $f: \mathbb{C}^5 \to \mathbb{C}^5$ paired to $F$ through $B$ and $C$ ($f(x) := BF(Cx)$) is calculated as

\[
f(x) = x + 3 \begin{pmatrix}
0 \\
0 \\
x_1^2 x_2 + x_1 x_2^2 + 2 x_1 x_2 x_4 - 2 x_2^2 x_5 \\
-x_1^2 x_2 - x_1 x_2^2 - 2 x_1 x_2 x_3 - 2 x_1^2 x_5 \\
-x_2^2 x_3 - x_2^2 x_4
\end{pmatrix} \tag{6.4}
\]
The inverse of $f$ is easily found by computer and it is a polynomial mapping of degree 7:

$$f^{-1}(y) = y + 3 \begin{pmatrix} 0 \\ 0 \\ -y_1^2 y_2 - y_1 y_2^2 - 2 y_1 y_2 y_4 + 2 y_2^2 y_5 \\ y_1^2 y_2 + y_1 y_2^2 + 2 y_1 y_2 y_3 + 2 y_2^2 y_4 \\ y_2^2 y_3 + y_2^2 y_4 \end{pmatrix} +$$

$$+ 18 \begin{pmatrix} 0 \\ 0 \\ -y_1^2 y_2 - y_1 y_2^2 - y_1^2 y_2^2 y_3 + y_1^2 y_2 y_3 - y_1^3 y_2 y_4 + 2 y_2^3 y_5 \\ y_1^3 y_2 y_4 - y_1^2 y_2^2 y_4 \end{pmatrix} + (6.5)$$

$$+ 108 \begin{pmatrix} 0 \\ 0 \\ y_1^3 y_2^2 + y_1^3 y_2^4 \\ -y_1^3 y_2^3 - y_1^3 y_2^4 \\ -y_1^3 y_2^4 - y_1^2 y_2^5 \end{pmatrix} .$$

Proposition 3.2 predicts now that the inverse of $F$ is a polynomial mapping of degree at most 21 and that it is given by the formula

$$F^{-1}(Y) = Y + \left(ACf^{-1}(BY)\right)^* = 2Y - F(Cf^{-1}(BY)) .$$

The pre-conjugation $k_\lambda$ of the paired mapping $f$ can be computed through the recursive formula (4.2) and turns out to be a polynomial mapping of degree 7:

$$k_\lambda(x) = x + \frac{3}{1 - \lambda^2} \begin{pmatrix} 0 \\ 0 \\ -x_1^2 x_2 - x_1 x_2^2 - 2x_1 x_2 x_4 + 2x_2^2 x_5 \\ x_1^2 x_2 + x_1 x_2^2 + 2x_1 x_2 x_3 + 2x_2^2 x_4 \end{pmatrix} +$$

$$+ \frac{18}{(1 - \lambda^2)(1 - \lambda^4)} \begin{pmatrix} 0 \\ 0 \\ -x_1^2 x_2 - x_1 x_2^2 - x_1^2 x_2 x_3 + x_1^2 x_2^2 x_4 - 2x_1^3 x_2 x_5 \\ -x_1^2 x_2 - x_1 x_2^2 + x_1^2 x_2^3 - x_1^2 x_2 x_3 + 2x_1^2 x_2 x_5 \end{pmatrix} + (6.7)$$

$$+ \frac{108}{(1 - \lambda^2)(1 - \lambda^4)(1 - \lambda^6)} \begin{pmatrix} 0 \\ 0 \\ x_1^3 x_2 + x_1^3 x_2^4 \\ -x_1^3 x_2^3 - x_1^3 x_2^4 \\ -x_1^3 x_2^3 - x_1^2 x_2^5 \end{pmatrix} .$$

Each homogeneous terms of $k_\lambda$ is a scalar multiple of the corresponding term in $f^{-1}$. This is because the trilinear form $g$ associated with $f$ happens to satisfy $g(g(x, x, x), g(x, x, x), x) \equiv 0$ (see Remark 4.2).
On cubic-linear polynomial mappings

Gianluca Gorni and Gaetano Zampieri

Using Proposition 5.3 we can predict that the pre-conjugation $K_\lambda$ for the cubic-linear mapping $F$ is a polynomial transformation of degree at most 21. The inverse of $k_\lambda$ can be computed easily enough, exploiting the fact that $k_\lambda$ is affine in the last three components:

$$k_\lambda^{-1}(y) = y + \frac{3}{1 - \lambda^2} \begin{pmatrix} 0 \\ y_1^2y_2 + y_1y_2^2 + 2y_1y_2y_4 - 2y_1^2y_5 \\ -y_1^2y_2 - y_1y_2^2 - 2y_1y_2y_3 - 2y_1^2y_5 \\ -y_2^2y_3 - y_2^2y_4 \\ \end{pmatrix} + \frac{18\lambda^2}{(1 - \lambda^2)(1 - \lambda^4)} \begin{pmatrix} 0 \\ -y_1^3y_2 - y_3^2y_2 - y_1^2y_2^2y_3 + y_1y_2^2y_4 - 2y_1^3y_2y_5 \\ -y_1^3y_2^2 - y_2^2y_3 + y_1^2y_2y_4 - y_1^2y_2^2y_5 + 2y_1^3y_2y_5 \\ y_1y_2^3 - y_1y_2^2y_4 + 2y_1^2y_2^2y_5 \\ \end{pmatrix} + \frac{108\lambda^6}{(1 - \lambda^2)(1 - \lambda^4)(1 - \lambda^6)} \begin{pmatrix} 0 \\ -y_1^4y_2^3 - y_1^3y_2^4 \\ y_1^4y_2^3 + y_1^3y_2^4 \\ y_1^3y_2^4 + y_1^2y_2^5 \\ \end{pmatrix}.$$  \quad (6.8)

From Proposition 5.5 we can draw that $K_\lambda$ is invertible and that $K_\lambda^{-1}$ is a polynomial transformation of degree at most $21 \cdot 7 = 147$.

**Example 6.2.** The following polynomial mapping of $\mathbb{C}^4$

$$f(x) := x + \begin{pmatrix} (x_3x_1 + x_4x_2)x_4 \\ -(x_3x_1 + x_4x_2)x_3 \\ x_3^3 \\ 0 \end{pmatrix}$$

for $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{C}^4$, \quad (6.9)

was introduced by van den Essen in [10, page 231]. It is a cubic-homogeneous mapping with polynomial inverse (of degree 7):

$$f^{-1}(y) = y + \begin{pmatrix} -y_1y_3y_4 - y_2^2y_4 \\ y_1y_3^2 + y_2y_3y_4 \\ -y_3^3 \\ 0 \end{pmatrix} + \begin{pmatrix} y_1^4y_4^4 \\ -2y_1y_3y_4^3 - y_2^4y_4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y_1^6y_4^6 \\ 0 \\ 0 \end{pmatrix}.$$  \quad (6.10)
It was shown in [10, page 231] with a very simple degree argument that the pre-conjugations \( k_\lambda \) could not possibly be themselves polynomial automorphisms. The Taylor series of \( k_\lambda \) truncated at the degree 7 is

\[
k_\lambda(x) = x + \frac{1}{1 - \lambda^2} \left( \begin{array}{c}
-x_1 x_3 x_4 - x_2 x_4^2 \\
x_1 x_3^2 + x_2 x_3 x_4 \\
-x_4^3 \\
0
\end{array} \right) +
\]

\[
+ \frac{1}{(1 - \lambda^2)(1 - \lambda^4)} \left( \begin{array}{c}
x_1 x_4^4 \\
-2 x_1 x_3 x_4 x_4 - x_2 x_4^4 \\
0 \\
0
\end{array} \right) +
\]

\[
+ \frac{1}{(1 - \lambda^2)(1 - \lambda^4)(1 - \lambda^6)} \left( \lambda^2 \left( \begin{array}{c}
x_1 x_3 x_4^5 - x_2 x_4^6 \\
x_1 x_5^2 x_4^3 + x_2 x_3 x_4^5 + x_1 x_4^6 \\
0 \\
0
\end{array} \right) + \left( \begin{array}{c}
0 \\
x_1 x_4^5 \\
0 \\
0
\end{array} \right) \right) + \ldots
\]

The paper [10, page 231] left the question open whether \( k_\lambda \) was globally defined for \(|\lambda| < 1\), and whether it was globally invertible for \(|\lambda| \neq 1\). The problem was later studied in detail in [12], and it was found that the pre-conjugations \( k_\lambda \) are in fact analytic automorphisms of \( \mathbb{C}^4 \) for \(|\lambda| \neq 1\), and the coefficients of the power series were also explicitly calculated.

Through the procedure delineated in Proposition 2.1 it is possible to pair \( f \) to a cubic-linear map \( F: \mathbb{C}^{16} \rightarrow \mathbb{C}^{16} \). The first step is to write the third-degree part of \( f(x) \) as a sum of cubes of linear forms, using formulas (2.1):

\[
f(x) - x = \frac{1}{24} \left( \begin{array}{c}
-8 x_2^3 + 4 ( x_2 - x_4 )^3 \\
+ ( x_1 - x_3 - x_4 )^3 \\
- ( x_1 + x_3 - x_4 )^3 \\
+ 4 ( x_2 + x_4 )^3 \\
- ( x_1 - x_3 + x_4 )^3 \\
+ ( x_1 + x_3 + x_4 )^3 \\
8 x_1^2 - 4 ( x_1 - x_3 )^3 \\
- 4 ( x_1 + x_3 )^3 \\
- ( x_2 - x_3 - x_4 )^3 \\
+ ( x_2 + x_3 - x_4 )^3 \\
+ ( x_2 - x_3 + x_4 )^3 \\
- ( x_2 + x_3 + x_4 )^3 \\
8 x_4^3 \\
0
\end{array} \right) = - \frac{1}{24} \left( \begin{array}{c}
8 \\
0 \\
0 \\
0 \\
-4 \\
1 \\
-1 \\
-4 \\
1 \\
-1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array} \right) T \left( \begin{array}{c}
x_2 \\
x_2 - x_4 \\
x_1 - x_3 - x_4 \\
x_1 + x_3 - x_4 \\
x_2 + x_4 \\
x_1 - x_3 + x_4 \\
x_1 + x_3 + x_4 \\
x_1 \\
x_1 - x_3 \\
x_1 + x_3 \\
x_2 - x_3 - x_4 \\
x_2 + x_3 - x_4 \\
x_2 - x_3 + x_4 \\
x_2 + x_3 + x_4 \\
x_4
\end{array} \right)^3
\]

(6.12)
Suitable matrices $B, D, C$ have 16 as the larger size and are given by

$$B = \frac{1}{24} \begin{pmatrix} 8 & -4 & -1 & 1 & -4 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 4 & 4 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 \end{pmatrix},$$

$$D^T = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \end{pmatrix}, \quad (6.13)$$

$$C^T = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(the last column of $B$ and the last row of $D$ have been added to make $B$ of full rank). We will skip writing down a basis of ker $B$ (although it has been used for the computation), and proceed to the final matrix $A$:

$$A = \frac{1}{24} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 4 & 4 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 4 & 4 & 1 & -1 & -1 & 1 & 0 & -24 \\ 8 & -4 & -1 & 1 & -4 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & -24 \\ 8 & -4 & -1 & 1 & -4 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & -24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 4 & 4 & 1 & -1 & -1 & 1 & 0 & 24 \\ 8 & -4 & -1 & 1 & -4 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & 24 \\ 8 & -4 & -1 & 1 & -4 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & 24 \\ 8 & -4 & -1 & 1 & -4 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & 0 \\ 8 & -4 & -1 & 1 & -4 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 4 & 4 & 1 & -1 & -1 & 1 & 24 & -24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 4 & 4 & 1 & -1 & -1 & 1 & 24 & -24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 4 & 4 & 1 & -1 & -1 & 1 & 24 & -24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & 24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 \end{pmatrix}. \quad (6.14)$$

It is possible to check that $A^2 \neq 0$, $A^3 = 0$. Through the results of Sections 3 and 5, the cubic-linear polynomial mapping

$$F(X) := X - (AX)^*^3 \quad \text{for } X \in \mathbb{C}^{16}$$

is a polynomial automorphism of $\mathbb{C}^{16}$, and its inverse is of degree at most 21. The conjugations $K_\lambda$ of $F$ are analytic but not polynomial automorphisms of $\mathbb{C}^{16}$, for all $\lambda \in \mathbb{C}\backslash \{0\}$, $|\lambda| \neq 1$. 

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