COHOMOLOGY OF HIERARCHICAL TILINGS

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Algebraic invariants, such as homotopy groups, homology groups, and cohomology groups, are used to study topological spaces and maps between them. The best known of these is homology. In a homology theory, we associate Abelian groups $C_k(X)$ of chains to a topological space $X$, and define a boundary operator $\partial_k : C_k(X) \to C_{k-1}(X)$ such that $\partial_{k-1} \circ \partial_k = 0$. Chains in the kernel of $\partial_k$ are called closed, and chains in the image of $\partial_{k+1}$ are called boundaries. The $k$-th homology of $X$ in this setup is $H_k(X) = \text{Ker}(\partial_k)/\text{Im}(\partial_{k+1})$. A continuous map $f : X \to Y$ induces push-forward maps $f_* : C_k(X) \to C_k(Y)$. (Strictly speaking there is one such map for each integer $k$, but they are all denoted $f_*$). This in turn induces a map (also denoted $f_*$) from $H_k(X)$ to $H_k(Y)$. Homotopic maps induce the same map on homology. Homology groups can then help us classify spaces, and the pushforward $f_* : H_k(X) \to H_k(Y)$ helps classify maps up to homotopy, and hence the relation between $X$ and $Y$. There are many different homology theories, including simplicial, singular and cellular. For CW complexes they all yield isomorphic groups, so we often get lazy and speak of the homology of a space $X$ without specifying the theory.

In a cohomology theory, we associate Abelian groups $C^k(X)$ of cochains to $X$ and define a coboundary operator $\delta_k : C^k(X) \to C^{k+1}(X)$ such that $\delta_{k+1} \circ \delta_k = 0$. Given such a setup, the $k$-th cohomology group of $X$ is $H^k(X) = \text{Ker}(\delta_k)/\text{Im}(\delta_{k+1})$. A continuous map $f : X \to Y$ induces pullback maps $f^* : C^k(Y) \to C^k(X)$ and $f^* : H^k(Y) \to H^k(X)$.

One way to get a cohomology theory is to start with a homology theory and dualize everything. We can define $C^k(X)$ to be the dual space of $C_k(X)$, and $\delta_k$ to be the transpose of $\partial_k$. That is, if $\alpha$ is a $k$-cochain and $c$ is a $(k+1)$-chain, then

$$\left(\delta_k \alpha\right)(c) := \alpha(\partial_{k+1} c),$$

since the boundary $\partial c$ of $c$ is a $k$-chain. This is how simplicial, singular, and cellular cohomology are defined. However, there are also cohomology theories that are defined intrinsically rather than via homology. In de Rham cohomology, if $X$ is a smooth manifold, then $C^k(X)$ is the set of $k$-forms on $X$, and $\delta_k$ is the exterior derivative. In Čech cohomology, the cochains are defined via open covers of $X$. Regardless of the setup, we call elements of $\text{Ker } \delta_k$ co-closed and elements of $\text{Im } \delta_{k-1}$ co-exact, and define

$$H^k(X) = (\text{Ker } \delta_k)/(\text{Im } \delta_{k-1}).$$

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1This is where the prefix “co” for objects related to cohomology comes from.

2When the dimension of a chain or cochain is clear, we often omit the subscript from $\partial_k$ or $\delta_k$. 
Since the 1990s, Čech cohomology has been used to study tiling spaces. This began with work of Kellendonk [Kel1], and really took off after the seminal work of Anderson and Putnam [AP]. This chapter will address three essential questions, all of which have generated a host of papers: (1) What is tiling cohomology? (2) How do you compute it? (3) What is it good for? Most of this chapter is review material, but the content of Sections 3.1.1 and 3.1.2 is new and is joint work with John Hunton.

1. WHAT IS TILING COHOMOLOGY?

Many algebraic invariants that are used to classify topological spaces do not work very well with tiling spaces. Tiling spaces (with finite local complexity) are “matchbox manifolds”; foliated spaces that locally look like the product of Euclidean space and a Cantor set. Tiling spaces have uncountably many path components. Most of the standard algebraic invariants are then useless, since they look at each path component separately, without regard to how the path components approximate one another. For instance, in singular homology, $H_0$ of a tiling space is a free group with uncountably many generators, while all higher homotopy groups vanish. The fundamental group and all higher homotopy groups also vanish.

To get around these difficulties, we need to employ less familiar cohomology theories, especially Čech cohomology, which is well adapted to tiling theory. In Subsection 1.1 we describe how to view tiling spaces as inverse limits. In Subsection 1.2 we describe Čech cohomology and explain how to view the cohomology of an inverse limit space. In Subsection 1.3 we go over pattern-equivariant cohomology. This is a theory, isomorphic to Čech cohomology, in which the cochains and cocycles can be viewed as functions on a single tiling. PV cohomology, described in subsection 1.4 is another reformulation of the Čech complex, only now the cochains are functions on Cantor sets. Finally, in subsection 1.5 we describe quotient cohomology, an analogue of relative cohomology that is very useful in computations.

1.1. INVERSE LIMIT SPACES. Let $\Gamma^0, \Gamma^1, \Gamma^2, \ldots$ be a sequence of topological spaces, and for each $n > 0$ let $\rho_n : \Gamma^n \to \Gamma^{n-1}$ be a continuous map. The inverse limit $\lim_{\leftarrow} (\Gamma^n, \rho_n)$ is a subset of the product space $\prod_n \Gamma_n$. It is the set of all sequences $(x_0, x_1, x_2, \ldots) \in \prod_n \Gamma^n$ such that for each $n > 0$, $\rho_n(x_n) = x_{n-1}$. The spaces $\Gamma^n$ are called approximants to the inverse limit, since knowing $x_n \in \Gamma^n$ determines the first $n + 1$ terms $(x_0, \ldots, x_n)$ in the sequence, and thus approximates the entire sequence in the product topology.

A simple example is the dyadic solenoid $Sol_2$. Each $\Gamma_n$ is the circle $\mathbb{R}/\mathbb{Z}$, and each $\rho_n$ is the doubling map. A point in $Sol_2 = \lim_{\leftarrow} (S^1, \times 2)$ is a point $x_0$ on the unit circle, together with a choice between two possible preimages $x_1$, another choice between possible preimages

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3 All tilings in this chapter will be assumed to have finite local complexity, and in particular to have tiles that meet full-edge to full-edge. Cohomology can also be used to study tiling spaces of infinite local complexity, but both the calculations and the interpretations are more complicated.
x_2$ of $x_1$, another choice of $x_3$, etc. Infinitely many discrete choices make a Cantor set, and $Sol_2$ is a Cantor set bundle over the circle.

There are many descriptions of tiling spaces as inverse limits, and we will present a few of the constructions in Section 2. If the tilings have finite local complexity, then the approximants are branched manifolds or branched orbifolds [AP, BGG, Sa1]. Even if the tilings do not have finite local complexity, it is usually possible to construct reasonable approximants. The approximants $\Gamma^n$ parametrize the possible restrictions of a tiling to a ball of radius $r_n$, with $\lim_{n \to \infty} r_n = \infty$, and the maps $\rho_n$ are obtained by restricting the tiling to a smaller region. A point in the inverse limit is a set of consistent instructions for tiling bigger and bigger balls around the origin, which is tantamount to a tiling of the entire plane.

1.2. Čech cohomology. The precise definition of the Čech cohomology $\check{H}^*(\Omega)$ of a topological space $\Omega$ involves the combinatorics of open covers of $\Omega$, and how the combinatorics change with refinements of the open covers. The (complicated!) details can be found in an algebraic topology text [BT, Hat, Sa3] and need not concern us here. What do concern us are some standard properties of Čech cohomology.

**Theorem 1.1.** If $X$ is a CW complex, then the Čech cohomology $\check{H}^*(X)$ is naturally isomorphic, as a ring, to the singular cohomology $H^*(X)$, and also to the cellular cohomology. If $X$ is a manifold, then the Čech cohomology with real coefficients is isomorphic to the de Rham cohomology $H^*_dR(X)$.

Recall that if we have a sequence $G_0, G_1, \ldots$ of groups, and a collection of homomorphisms $\eta_n^*: G_n \to G_{n+1}$, then the direct limit $\varinjlim (G_n, \eta_n)$ is the disjoint union of the $G_n$’s, modulo the relation that $x_n \in G_n$ is identified with $\eta_n(x_n) \in G_{n+1}$. Every element $x \in \varinjlim (G_n, \eta_n)$ is the equivalence class of an element of an approximating group $G_n$; there are no additional elements “at infinity”. For instance, $\mathbb{Z}[\frac{1}{2}] := \varinjlim (\mathbb{Z}, \times 2)$ is isomorphic to the set of dyadic rational numbers whose denominators are powers of 2. The element $k \in G_n$ is associated with the rational number $k/2^n$, and $k \in G_n$ equals $2k \in G_{n+1}$ (as it must). The rational number $5/16$ can be represented as $5 \in G_4$, $10 \in G_5$, or $20 \in G_6$, etc., but has no representative in $G_0, G_1, G_2$ or $G_3$.

**Theorem 1.2.** If $\Omega$ is the inverse limit $\varprojlim (\Gamma^n, \rho_n)$ of a sequence of spaces $\Gamma^n$ under a sequence of maps $\rho_n : \Gamma^n \to \Gamma^{n-1}$, then $\check{H}^*(\Omega)$ is isomorphic to the direct limit $\varinjlim (\check{H}^*(\Gamma^n), \rho^*_n)$. In other words, all cohomology theories on a nice space are the same, and the Čech cohomology of an inverse limit is the direct limit of the Čech cohomologies of the approximants.

This is how tiling cohomology is most frequently viewed in practice. Every element of $\check{H}^*(\Omega)$ can be represented by a class in $\check{H}^k(\Gamma^n)$ on some approximant $\Gamma^n$, and hence by a singular or cellular cochain on $\Gamma^n$. Instead of working with arbitrary open covers of the tiling space itself, we write everything in terms of the cells that compose the approximants.
As an example, consider the dyadic solenoid. \( H^0(S^1) = H^1(S^1) = \mathbb{Z} \). Since \( \rho_n \) wraps the circle twice around itself, \( \rho_n^* \) is the identity on \( H^0 \) and multiplication by 2 on \( H^1 \). Thus \( \check{H}^0(\text{Sol}_2) = \lim \mathbb{Z}, \times 1 = \mathbb{Z} \) and \( \check{H}^1(\text{Sol}_2) = \lim \mathbb{Z}, \times 2 = \mathbb{Z}[1/2] \). If we view \( S^1 \) as consisting of one 0-cell and one 1-cell, then for each \( m \geq n \), the element \( 2^{-n} \in \check{H}^1(\text{Sol}_2) \) can be represented by a cochain on \( \Gamma^m \) that evaluates to \( 2^{m-n} \) on the 1-cell.

1.3. Pattern-Equivariant Cohomology. Tiling cohomology can also be understood in terms of the properties of a single tiling of \( T \in \Omega \). This approach, called pattern-equivariant (PE) cohomology, was developed by Kellendonk and Putnam \[\text{Kel2, KP}\] using differential forms, and extended to integer-valued cohomology in \[\text{Sa2}\].

Suppose that \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is a smooth function. We say that \( f \) is pattern-equivariant (or PE) with radius \( R \) if the value of \( f(x) \) depends only on what the tiling \( T \) looks like in a ball of radius \( R \) around \( x \). That is, if \( x, y \in \mathbb{R}^d \), and if \( T - x \) and \( T - y \) agree exactly on a ball of radius \( R \) around the origin, then \( f(x) \) must equal \( f(y) \). A function is called strongly PE if it is PE with some finite radius \( R \). A function is weakly PE if it and all of its derivatives are uniform limits of strongly PE functions.

PE forms are defined similarly. Let \( \Lambda^k_{PE}(T) \) denote the \( k \)-forms on \( \mathbb{R}^d \) that are strongly PE with respect to the tiling \( T \). It is easy to see that the exterior derivative \( d_k \) maps \( \Lambda^k_{PE}(T) \) to \( \Lambda^{k+1}_{PE}(T) \), and we define

\[
H^k_{PE}(T, \mathbb{R}) = (\text{Ker} \; d_k)/(\text{Im} \; d_{k-1}).
\]

**Theorem 1.3 (KP).** If \( T \) is a tiling with finite local complexity with respect to translations, and if \( \Omega \) is the continuous hull of \( T \), then \( H^k_{PE}(T, \mathbb{R}) \) is naturally isomorphic to the Čech cohomology of \( \Omega \) with real coefficients, denoted \( \check{H}^k(\Omega, \mathbb{R}) \).

To get a PE interpretation of integer-valued cohomology, we use the fact that a tiling \( T \) is itself a decomposition of \( \mathbb{R}^d \) into 0-cells (vertices), 1-cells (edges), etc. A PE \( k \)-cochain \( \alpha \) is a function that assigns an integer to each oriented \( k \)-cell in a PE way. More precisely, there must be a radius \( R \) such that, if \( c_1 \) and \( c_2 \) are two \( k \)-cells with centers of mass \( x \) and \( y \), and if \( T - x \) and \( T - y \) agree on a ball of radius \( R \) around the origin, then \( \alpha(c_1) = \alpha(c_2) \). (For integer-valued functions, there is no distinction between strong and weak pattern-equivariance.) Let \( C^k_{PE}(T) \) denote the set of PE \( k \)-cochains. Instead of the exterior derivative, we consider the cellular coboundary map \( \delta_k \) that maps \( C^k_{PE}(T) \) to \( C^{k+1}_{PE}(T) \), and define

\[
H^k_{PE}(T) = (\text{Ker} \; \delta_k)/(\text{Im} \; \delta_{k-1}).
\]

**Theorem 1.4 (Sa2).** If \( T \) is a tiling with finite local complexity with respect to translations, and if \( \Omega \) is the continuous hull of \( T \), then \( H^k_{PE}(T) \) is naturally isomorphic to the Čech cohomology of \( \Omega \) with integer coefficients.

**Sketch of proof.** \( T \) induces a map \( \pi \) from \( \mathbb{R}^d \) to \( \Omega \), sending \( x \in \mathbb{R}^d \) to the tiling \( T - x \). Composing with the natural projection from \( \Omega \) to each approximant \( \Gamma^m \), we obtain a sequence...
of maps $\pi_n : \mathbb{R}^d \to \Gamma^n$. The orbit of $T$ is dense in $\Omega$, so these maps are surjective. Since $\Gamma^n$ parametrizes the central patch of a tiling, a function on $\mathbb{R}^d$ is (strongly) pattern-equivariant if and only if it is the pullback of a function on one of the approximants $\Gamma^n$, and the same goes for cochains. Studying $PE$ cochains of arbitrary radius is equivalent to studying cochains on $\Gamma^n$ and taking a limit as $n \to \infty$. In other words, $H^k_{PE}(T) = \lim_{n \to \infty} H^k(\Gamma^n) \cong \check{H}^k(\Omega)$. □

Example 1. Let $T$ be a Fibonacci tiling $\ldots babaabaa \ldots$ of $\mathbb{R}$ by long (a) and short (b) tiles. Let $i_a$ be a 1-cochain that evaluates to 1 on each a tile and 0 on each b tile, and let $i_b$ evaluate to 1 on each b and to 0 on each a. Since there are no 2-cells, $\delta i_a = \delta i_b = 0$, so $i_a$ and $i_b$ define classes in $H^1_{PE}(T)$. Once we develop the machinery of Barge-Diamond collaring, we will see that these classes correspond to the generators of $\check{H}^1(\Omega) = \mathbb{Z}^2$.

Example 2. If $T$ is a Thue-Morse tiling $\ldots abbabaabbaababba \ldots$, obtained from the substitution $a \to ab, b \to ba$, one can similarly define indicator 1-cochains $i_a$ and $i_b$ that count a and b tiles. However, these cochains are cohomologous. To see this, divide the tiling $T$ into 1-supertiles with each being either $ab$ or $ba$. Let $\gamma$ be a PE 0-cochain that evaluates to zero on the vertices that mark the beginning or end of such a supertile, to 1 on the vertex in the middle of an $ab$ supertile, and to $-1$ on the vertex in the middle of a $ab$ supertile. Then $\delta \gamma$ evaluates to 1 on every a tile (since the boundary of an a tile is either the middle vertex of an $ab$ supertile minus the beginning of that supertile, or the end vertex of a $ba$ supertile minus the middle vertex) and $-1$ on every b tile, so $\delta \gamma = i_a - i_b$.

The first Čech cohomology of the Thue-Morse tiling space is known to be $\mathbb{Z}[1/2] \oplus \mathbb{Z}$. The generators can be chosen as follows. Let $\alpha_n$ be a 1-cochain that evaluates to 1 on the first tile of each $n$-supertile and to 0 on the other $2^n - 1$ tiles. The cochain $\alpha_n$ basically counts $n$-supertiles. Since there are two $n$-supertiles in each $(n + 1)$-supertile, $\alpha_n$ is cohomologous to $2\alpha_{n+1}$. Let $\beta$ be a 1-cochain that evaluates to 1 on each a tile that is followed by a b tile, and to zero on b tiles or on a tiles that are followed by a tiles. This is not cohomologous to any combination of the $\alpha_n$ tiles since, on average, $\beta$ applied to a long interval yields a third of the length of the interval, something that no finite linear combination of the $\alpha_n$’s can do. The $\alpha_n$ cochains and $\beta$ generate all of $\check{H}^1$. In this example, the cochains $i_a$ and $i_b$ are both cohomologous to $\alpha_1$.

1.4. PV cohomology. Another cohomology theory, called PV cohomology after the Pimsner-Voiculescu exact sequence, was developed by Savinien and Bellissard [SB]. This theory is based on the structure of the transversal to the tiling space. Since the $C^*$ algebra associated with a tiling space is constructed from the transversal and the associated groupoid, this

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$^4$Recall that if a substitution tiling is non-periodic, then it can be decomposed into supertiles in a unique way, and that this decomposition is a local operation. In the Thue-Morse tiling, every patch of size 5 or greater contains either the sub-word $aa$ or the sub-word $bb$. The boundaries between 1-supertiles sit in the middle of these sub-words, and at all points at even distance from these middles.
provides a more intuitive link between the cohomology of a tiling space and the K-theory of the $C^*$ algebra.

We associate a distinguished point, called a **puncture**, to each type of tile. Usually these are chosen in the interior of the tile, say at the center of mass, but the precise choice of puncture is unimportant. The **canonical transversal** $\Xi$ of a tiling space is the set of tilings for which there is a puncture at the origin. This is a Cantor set, and we can study the ring of continuous integer-valued functions on $\Xi$, denoted $C(\Xi, \mathbb{Z})$. If $\alpha$ is a $d$-cochain, we define an associated function $f_\alpha$ on $\Xi$ as follows: if $T \in \Xi$, then $f_\alpha(T)$ equals $\alpha$ applied to the tile of $T$ that lies at the origin. This map induces an isomorphism (as an additive group) between $C(\Xi, \mathbb{Z})$ and $C^d_{PE}(T)$.

Similarly, we can define punctures for all of the lower-dimensional faces and edges and vertices of different tiles, with the condition that if (say) an edge is on the boundary of two tiles, then its puncture viewed as the boundary of the first tile is the same as its puncture viewed as the boundary of the second tile. For $k$ ranging from 0 to $d$, let $\Xi^k_\Delta$ be the set of tilings where the origin sits at a puncture of an $k$-cell. As with $\Xi = \Xi^d_\Delta$, $C(\Xi^k_\Delta, \mathbb{Z})$ is isomorphic to $C^k_{PE}(T)$.

In PV cohomology, the group of $k$-cochains is $C(\Xi^k_\Delta, \mathbb{Z})$ and the coboundary maps are built from the geometry of the specific tiles. After untangling the definitions, these coboundary maps turn out to be identical to the coboundary maps in PE-cohomology. Thus, PV theory and PE theory not only have the same cohomologies, but have isomorphic cochain complexes. For details of this argument, see [BK].

### 1.5. Quotient cohomology

So far we have been discussing the absolute cohomology of each tiling space. However, cohomology is also a functor that concerns maps between spaces. Inclusions give rise to relative cohomology (see [Hat]), while surjections give rise to a less-known construction called **quotient cohomology**.

Let $f : \Omega_X \to \Omega_Y$ be a factor map of tiling spaces. As long as the tilings have finite local complexity with respect to translations, the pullback map $f^*$ is an injection on cochains. (This argument applies both to Čech cochains and to pattern-equivariant cochains.) We then define the quotient cochain complex $C^k_Q(\Omega_X, \Omega_Y)$ to be $C^k(\Omega_X)/f^*(C^k(\Omega_Y))$, and the quotient cohomology $H^k_Q(\Omega_X, \Omega_Y)$ to be the cohomology of this complex. The short exact sequence of cochain complexes:

$$0 \to C^k(\Omega_Y) \xrightarrow{f^*} C^k(\Omega_X) \to C^k_Q(\Omega_X, \Omega_Y) \to 0$$

induces a long exact sequence of cohomology groups

$$\cdots \to \tilde{H}^k(\Omega_Y) \xrightarrow{f^*} \tilde{H}^k(\Omega_X) \to H^k_Q(\Omega_X, \Omega_Y) \to \tilde{H}^{k+1}(\Omega_Y) \to \cdots$$

As with ordinary relative (co)homology, there is an excision principle:
Theorem 1.5. [BSa] Let \( f : X \to Y \) be a quotient map such that \( f^* \) is an injection on cochains. If \( Z \subset X \) is an open set such that \( f \) is injective on the closure of \( Z \), then \( H^k_Q(X, Y) \) is isomorphic to \( H^k_Q(X - Z, Y - f(Z)) \).

For factor maps between tiling spaces, excision cannot be used directly. Every orbit is dense, so there are no open sets where \( f \) is injective on the closure. However, it is often the case that a factor map \( \Omega_X \to \Omega_Y \) is injective apart from a small set of tilings. In such circumstances, one can use homotopy to convert the tiling spaces into spaces where excision does apply.

Example 3. Let \( \Omega_X \) be the 1-dimensional tiling space obtained from the period-doubling substitution \( a \to ab, b \to aa \), and let \( \Omega_Y \) be the dyadic solenoid \( Sol_2 \) which can be viewed formally as coming from a substitution \( c \to cc \). (The dyadic solenoid is not actually a tiling space, but it has similar topological properties, being an inverse limit space, allowing us to apply the machinery of quotient cohomology.) There is a factor map \( f : \Omega_X \to \Omega_Y \) that identifies two translational orbits but is otherwise injective. This map sends a tiling \( T \) to the sequence \((x_0, x_1, \ldots)\), where \( x_k \) is the location of the endpoints of the \( k \)-supertiles (mod \( 2^k \)). In other words, \( f(T) \) gives the locations of the supertiles of all order in \( T \), but does not indicate which supertiles are of type \( a \) or type \( b \). However, in a period-doubling tiling the \( n \)-th order supertiles are identical except on the very last entry. Unless the tiling \( T \) consists of two infinite-order supertiles, \( f(T) \) determines \( T \). If the tiling \( T \) does consist of two infinite-order supertiles, then there is exactly one other tiling \( T' \), differing from \( T \) only at a single letter, such that \( f(T') = f(T) \).

In that last instance, we say that \( T \) and \( T' \) have a 0-dimensional feature, namely the boundary of an infinite-order supertile, and agree away from that feature. Let \( \Omega_{X_0} = \{T, T'\} \), and let \( \Omega_{Y_0} = f(\Omega_{X_0}) \). The map \( f \) is basically a quotient map, identifying the orbit of \( T \) with the orbit of \( T' \). This identification is the suspension of a map from the 2-point set \( \Omega_{X_0} \) to the 1-point set \( \Omega_{Y_0} \).

The situation of this example is quite common. There are many situations where a factor map \( f : \Omega_X \to \Omega_Y \) between tiling spaces (or solenoids) is injective except on the translational orbits of a set \( \Omega_{X_0} \) of tilings. Furthermore, \( \Omega_{X_0} \) has the structure of a \( d - \ell \)-dimensional tiling space, admitting an \( \mathbb{R}^{d-\ell} \) action and locally being the product of \( \mathbb{R}^{d-\ell} \) and a totally disconnected set. Defining \( \Omega_{Y_0} \) to be \( f(\Omega_{X_0}) \), the following theorem relates the quotient cohomologies of \( (\Omega_X, \Omega_Y) \) and \( (\Omega_{X_0}, \Omega_{Y_0}) \).

Theorem 1.6. [BSa] Let \( f : \Omega_X \to \Omega_Y \) be a quotient map of tiling spaces such that \( f^* \) is injective on cochains. Suppose that \( f \) is injective aside from the translational orbits of a codimension-\( \ell \) set \( \Omega_{X_0} \subset \Omega_X \) of tilings. Let \( \Omega_{Y_0} = f(\Omega_{X_0}) \). Then \( H^k_Q(\Omega_X, \Omega_Y) = H^{k-\ell}_Q(\Omega_{X_0}, \Omega_{Y_0}) \).
In our example, $\ell = 1$, $\Omega_{X_0}$ consists of two points, $\Omega_{Y_0}$ is a single point, $H^0_Q(\Omega_{X_0}, \Omega_{Y_0}) = \mathbb{Z}$, and so $H^1_Q(\Omega_X, \Omega_Y) = \mathbb{Z}$. Since $\hat{H}^1(\Omega_Y) = \mathbb{Z}[1/2]$, the long exact sequence \((\natural)\) shows that $\hat{H}^1(\Omega_X) = \mathbb{Z}[1/2] \oplus \mathbb{Z}$. This is in fact the first cohomology of the period-doubling space.

An extension of Theorem 1.6 relates the generators of $H^k_Q(\Omega_{X_0}, \Omega_{Y_0})$ to the generators of $H^k_Q(\Omega_X, \Omega_Y)$. This allows us to construct generators for $H^k(\Omega_X)$ from generators of $H^k(\Omega_Y)$ and from generators of $H^k_Q(\Omega_{X_0}, \Omega_{Y_0})$.

2. How do you compute tiling cohomology?

As with other topological spaces, there is no single “best” method for computing the cohomology of a tiling space. Different tiling spaces are best addressed with different methods.

Cut-and-project tiling spaces are measurably conjugate to Kronecker flows on higher-dimensional tori. As topological spaces, they are obtained from the tori by removing some hyperplanes and gluing them back in multiple times. Forrest, Hunton and Kellendonk [FHK], and later Kalugin [Kal] developed ways to compute the cohomology of $\Omega$ from the geometry of the “window” used in the cut-and-project scheme.

Substitution tilings can easily be expressed as inverse limits spaces in which all the approximants $\Gamma^n$ are homeomorphic to a single space $\Gamma^0$, and where the substitution $\sigma$ can be viewed as a map from $\Gamma^0$ to itself. For these spaces, computing the cohomology boils down to understanding the cohomology of $\Gamma^0$ and tracking how the classes evolve under the pullback map $\sigma^*$. There are many ways to do this, and each inverse limit scheme gives rise to a calculational method. In this section we develop several such schemes, beginning with the original ideas of Anderson and Putnam, and working our way through Gähler’s construction and the more recent ideas of Barge, Diamond, Hunton and Sadun. Variants of the Anderson-Putnam and Barge-Diamond methods are then applied to tilings with rotational symmetry and to hierarchical tilings that are not substitutions (e.g., the “generalized substitutions” of \[F\] [AFHI]).

Tilings that come from local matching rules are harder to understand. However, they can sometimes be related to substitution tilings \([MoZ], [GS], [Rad]\). When a substitution tiling space $\Omega_Y$ is the quotient of a local matching rules tiling space $\Omega_X$, we can study the cohomology of $\Omega_X$ via the cohomology of $\Omega_Y$ and the quotient cohomology $H^k_Q(\Omega_X, \Omega_Y)$.

2.1. The Anderson-Putnam complex. Suppose that we have a substitution tiling whose tiles are polygons that meet full-edge to full-edge. We construct an inverse limit space whose approximants $\Gamma^n$ describe partial tilings. Specifically, a point in $\Gamma^n$ describes where the origin sits within an $n$-supertile. Since this also determines where the origin sits within an $(n - 1)$-supertile, we have a natural map $\sigma : \Gamma^n \to \Gamma^{n-1}$ and can consider the inverse limit space $\Omega^0 = \lim (\Gamma^n, \sigma)$.

Since the origin can sit anywhere in a supertile of any type, $\Gamma^n$ consists of one copy of each type of supertile. However, there is an ambiguity when the origin sits on the boundary
of a supertile. If the origin sits on the boundary between supertile $A$ and supertile $B$, do we consider it as part of $A$ or $B$? The answer is to identify the two edges.

Specifically, $\Gamma^n$ is obtained by taking the disjoint union of one copy of each kind of (closed) $n$-supertile, and then applying the relation that, if somewhere in an admissible tiling an edge $e_1$ of supertile $A$ coincides with an edge $e_2$ of supertile $B$, then $e_1$ and $e_2$ are identified.

These identifications do not just come in pairs. It may happen that the right edge of $A$ is identified with the left edges of both $B$ and $C$, and that the left edge of $C$ is identified with the right edges of both $A$ and $D$. In that case, the left edges of $B$ and $C$ and the right edges of $A$ and $D$ would all be identified. The information contained in that point in $\Gamma^n$ would indicate that the origin either sits at a particular spot on the right edge of $A$, or at that spot on the right edge of $D$, and also that it sits at the corresponding spot on the left edge of either $B$ or $C$.

The set of possible $n$-supertiles looks just like the set of possible tiles, only scaled up by a factor of $\lambda^n$. As a result, $\Gamma^n$ is just a scaled-up version of $\Gamma^0$. $\Gamma^0$ is called the (uncollared) Anderson-Putnam complex of the substitution $\sigma$, and is denoted $\Gamma_{AP}$. Furthermore, the decomposition of $n$-supertiles into constituent $(n-1)$-supertiles is combinatorially the same for all $n$. After rescaling, there is a single map (which we again call $\sigma$) from $\Gamma_{AP}$ to itself. This map involves stretching each tile in $\Gamma_{AP}$ by a factor of $\lambda$, dividing it into tiles via the substitution rule, and then identifying pieces. We then define $\Omega^0 = \lim \leftarrow (\Gamma_{AP}, \sigma)$.

2.1.1. Forcing the border. Forcing the border was defined by Johannes Kellendonk in his study [Kel1] of the Penrose tiling. As we shall see, if a substitution forces the border, then $\Omega^0$ is homeomorphic to the tiling space $\Omega$, allowing for an easy computation of the cohomology of $\Omega$. If a substitution doesn’t force the border, then there are a variety of collaring techniques for describing the tiling space via a slightly different substitution that does force the border. By combining collaring with the Anderson-Putnam construction, we can compute the cohomology of arbitrary substitution tiling spaces.

Suppose we have a non-periodic substitution tiling space, so that $\sigma : \Omega \to \Omega$ is a homeomorphism [Mos, Sol]. This means that we can decompose each tiling $T$ uniquely into a collection of non-overlapping 1-supertiles, and by extension we can decompose $T$ uniquely into non-overlapping $k$-supertiles for every $k$. The substitution is said to force the border at level $k$ if any two $k$-supertiles of the same type not only have the same decomposition into tiles, but also have the same pattern of ordinary tiles surrounding them (i.e., the pattern of tiles that touch the supertiles at 1 or more points). Moreover, any two $n$-supertiles with $n > k$ have the same pattern of $(n-k)$-supertiles surrounding them.

The half-hex substitution is shown in Figure 1. The solid lines indicate the tiles within a supertile, and the dotted lines indicate the neighboring tiles that must also appear. This substitution forces the border at level 2, since the 2-supertile is completely surrounded by determined tiles, but does not force the border at level 1, since some of the tiles that touch
Figure 1. In bold face, a half-hex tile, an order-1 supertile, and an order-2 supertile. In dotted lines, the nearby tiles that these determine.

the four vertices of the 1-supertile are undetermined. By contrast, the chair tiling does not force the border at all, since tiles near the southwest corner of a chair supertile of arbitrary order can appear in either of the patterns shown in Figure 2.

Figure 2. There are two ways to extend a high-order chair supertile around the southwest corner.

If a substitution forces the border at level \( k \), then a point in \( \Gamma^n \) not only determines where the origin sits in a supertile of level \( n \), but it determines all of the \((n - k)\)-supertiles surrounding the supertile that contains the origin. If the origin sits on the boundary between two or more \( n \)-supertiles, then there is some ambiguity on the nature of the \( n \)-supertiles that surround the origin. However, there is no ambiguity about the \((n - k)\)-supertiles that surround the origin.

The inverse limit \( \Omega^0 = \lim \left( \Gamma^n_{AP}, \sigma \right) \) is a sequence of consistent instructions for placing higher and higher-order supertiles in a growing region containing the origin. The union of these regions is all of \( \mathbb{R}^d \). This is tantamount to

**Theorem 2.1.** If \( \sigma \) is a substitution that forces the border and has finite local complexity with respect to translations, then the corresponding tiling space \( \Omega \) is homeomorphic to \( \Omega^0 \).

2.1.2. **Anderson-Putnam collaring.** If the substitution \( \sigma \) does not force the border, then \( \Omega^0 \) is typically not homeomorphic to \( \Omega \). There is still a map \( \Omega \rightarrow \Omega^0 \), whose \( n \)-th coordinate is a description of the \( n \)-supertile containing the origin. Furthermore, this map is surjective. However, it is typically not injective. Even if the origin is not on a boundary, knowing the supertiles to all orders containing the origin may not describe the entire tiling, since
the union of these supertiles may be a quarter-plane or a half-plane. If there is more than one extension of this infinite partial-tiling to the entire plane, then there is more than one preimage in $\Omega$.

To remedy this, we construct a new substitution using collared tiles. Take a tiling $T$, and identify tiles that are (a) of the same type and (b) whose nearest neighbors are all of the same type. That is, tiles $t_1$ and $t_2$ are identified if, for some points $x \in t_1$ and $y \in t_2$, the tilings $T-x$ and $T-y$ agree exactly on the tile containing the origin and on all tiles touching this central tile. A collared tile is an equivalence class of tiles under this identification. Note that a collared tile has the same size and shape as an ordinary uncollared tile. The difference is that the label of the collared tile carries extra information about its surroundings.

Example 4. In the Fibonacci tiling, every $b$ tile is preceded and followed by an $a$ tile, while an $a$ tile has three possibilities for its neighbors. There are thus four collared tiles, which we denote $A_1 = (a)a(b)$, $A_2 = (b)a(a)$, $A_3 = (b)a(b)$ and $B = (a)b(a)$, where the notation $(x)y(z)$ means a $y$ tile that is preceded by an $x$ and followed by a $z$. Under substitution, $A_1 \rightarrow (ab)ab(a) = A_3B$, $A_2 \rightarrow (a)ab(ab) = A_1B$, $A_3 \rightarrow (a)ab(a) = A_1B$, and $B \rightarrow (ab)a(ab) = A_2$.

We can relabel all of our tiles according to their neighbors to obtain a new tiling by collared tiles. For instance, in the Fibonacci tiling the pattern $\ldots babaabaababa \ldots$ becomes $\ldots BA_3BA_2A_1BA_1BA_2A_1BA_3 \ldots$.

Theorem 2.2. [AP] Rewriting a substitution in terms of collared tiles always yields a system that forces the border.

Sketch of proof. A collared tile is a tile together with a pattern of nearest neighbors, thereby determining all the tiles in at least an $\epsilon$-neighborhood. After substituting $n$ times, we obtain an $n$-supertile together with a pattern of neighboring $n$-supertiles, thereby determining all the tiles within a distance $\lambda^n\epsilon$. Pick $n$ big enough that $\lambda^n\epsilon$ is more than twice the diameter of the largest tile. The $n$-times substituted (collared) tile then determines its neighboring uncollared tiles and the neighbors of these neighbors, and hence determines its neighboring collared tiles. □

For instance, in the Fibonacci example, $\sigma^2(A_1) = (aba)aba(ab) = (BA_2)A_1BA_2(A_1)$, $\sigma^2(A_2) = (ab)aba(aba) = (B)A_3BA_2(A_1B)$, $\sigma^2(A_3) = (ab)aba(ab) = (B)A_3BA_2(A_1)$ and $\sigma^2(B) = (aba)ab(aba) = (BA_2)A_1B(A_3B)$. In each case, substituting a collared tile twice determines at least two extra tiles on each side of the 2-supertile, and so determines the collared tile on each side of the supertile. Combining this theorem with the first Anderson-Putnam construction yields the following

Theorem 2.3. [AP] Let $\Omega$ be a tiling space derived from a substitution $\sigma$. Assume that there are only finitely many tile types, up to translation, and that the tiles are polygons (or
polyhedra) that meet full edge to full edge (or full face to full face). Then $\Omega$ is homeomorphic to $\lim\limits_{\leftarrow} (\tilde{\Gamma}_{AP}, \sigma)$, where $\tilde{\Gamma}_{AP}$ is constructed using once-collared tiles.

2.2. Gähler’s construction. One can iterate the collaring construction, rewriting an arbitrary tiling space $\Omega$ in terms of collared tiles, then in terms of collared collared tiles (i.e., two tiles of the same type are identified only if they have the same pattern of nearest and second-nearest neighbors), and more generally $n$-times collared tiles. Let $\Gamma^n_G$ be the Anderson-Putnam complex constructed from the $n$-times collared tiles. There is a natural quotient map $q_n : \Gamma^n_G \rightarrow \Gamma^{n-1}_G$ that merely forgets about the $n$-th nearest neighbors.

**Theorem 2.4.** Let $\Omega$ be any space of tilings that have finite local complexity with respect to translation. Then $\Omega$ is homeomorphic to the inverse limit of the approximants $\Gamma^n_G$ under the forgetful maps $q_n$.

**Sketch of proof.** (see [Gah, Sa1]) A point $p_n \in \Gamma^n_G$ is either a point in an $n$-collared tile, or is the identification of several possible points on the boundary of an $n$-collared tile. Either way, at least $n-1$ rings of tiles around $p_n$ are specified. The point $p_n$ can then be viewed as instructions for building a patch around the origin. A sequence $p_0, p_1, \ldots$ is then a consistent set of instructions for building larger and larger patches around the origin, whose union is $\mathbb{R}^d$. Hence $\lim\limits_{\leftarrow} (\Gamma^n_G, q_n)$ parametrizes tilings in $\Omega$.

Gähler’s construction is extremely useful for theoretical arguments, as it applies to all tiling spaces, not just to substitution tiling spaces. For instance, the identification of integer-valued pattern-equivariant cohomology with Čech cohomology [Sa2] is based on this construction. Unfortunately, it has not proven effective in computing cohomology. $\hat{H}^k(\Omega)$ does equal $\lim\limits_{\leftarrow} H^k(\Gamma^n_G)$, but there is no general procedure for computing $H^k(\Gamma^n_G)$. The number of cells in $\Gamma^n_G$ grows with $n$, and it is difficult to do computations that apply simultaneously to all values of $n$.

2.3. Barge-Diamond collaring. The Anderson-Putnam and Gähler constructions are based on collared tiles. The Barge-Diamond construction [BD2, BDHS] is based on collared points.

Let $T \in \Omega$ be a non-periodic substitution tiling. Recall that non-periodicity implies that the substitution $\sigma$ has an inverse on $\Omega$. Pick a radius $r$ and consider the equivalence relation on $\mathbb{R}^d$: $x \sim y$ if the tilings $T-x$ and $T-y$ agree out to distance $r$ around the origin. Likewise, let $x \sim_n y$ if the tilings $\sigma^{-n}(T-x)$ and $\sigma^{-n}(T-y)$ agree out to distance $r$. That is, if $T-x$ and $T-y$ have the same structure of $n$-supertiles out to distance $\lambda^n r$. (In particular, they also have the same structure of ordinary tiles out to distance $\lambda^n r$.) Let $\Gamma^n_{BD}$ be the quotient of $\mathbb{R}^d$ by $\sim_n$. A priori this would seem to depend on the tiling $T$, but for minimal tiling spaces all tilings have the same patterns and give rise to identical approximants. Since $x \sim_n y$ implies $x \sim_{n-1} y$, there is a natural quotient map $q_n : \Gamma^n_{BD} \rightarrow \Gamma^{n-1}_{BD}$. Furthermore, the complexes $\Gamma^n_{BD}$ are all homeomorphic. Indeed, if $T$ is a self-similar tiling with $\sigma(T) = T$,
then $x \sim y$ if and only if $\lambda^n x \sim_n \lambda^n y$, so $\Gamma^n_{BD}$ is just an enlarged copy of a single space $\Gamma_{BD}$ and the quotient maps $q_n$ are all induced from the substitution $\sigma$.

The radius $r$ is arbitrary, but for many applications it is convenient to take $r$ extremely small. The complex $\Gamma_{BD}$ is then a CW complex comprised of pieces of tiles. For instance, suppose that $T$ is a 1-dimensional tiling. Points $x$ and $y$ are identified if either (1) they are in corresponding places in tiles of the same type, and are farther than $r$ from the nearest vertex, or (2) they are in corresponding places in tiles of the same type, within distance $r$ of a vertex, and the tiles on the other side of the vertices are the same. If the tiles all have length 1, then the equivalence classes of the first type form 1-cells of length $1 - 2r$, one for each tile type. We call these tile cells. The equivalence classes of the second type form 1-cells of length $2r$, called vertex flaps, one for each possible transition from one tile to another. For instance, in the Fibonacci tiling, the possible 2-tile patches are $aa$, $ab$, and $ba$, so $\Gamma_{BD}$ consists of two tile cells ($a$ and $b$) and three vertex flaps, arranged as in Figure 3. In the Thue-Morse tiling, all four transitions $\{aa, ab, ba, bb\}$ are possible, so we have two edge cells and four vertex flaps, also shown in Figure 3.

In a 2-dimensional tiling, there are three kinds of 2-cells. Tile cells correspond to the interiors of tiles, edge flaps correspond to points that are within $r$ of an edge, and contain information about what tile is on the other side of the edge, and vertex polygons describe what is happening near a vertex, and have information about all of the tiles touching the vertex. If the tiles are unit squares meeting edge-to-edge, then the tile cells are $(1 - 2r) \times (1 - 2r)$ squares, the edge flaps are $2r \times (1 - 2r)$ rectangles, and the vertex polygons are $2r \times 2r$ squares. (Strictly speaking, this requires using the $L^\infty$ metric on $\mathbb{R}^2$ rather than the Euclidean metric, to avoid having arcs of circles on the boundaries of cells.)

**Theorem 2.5.** [BD2, BDHS] For any positive radius $r$, $\Omega$ is homeomorphic to the inverse limit $\lim_{\leftarrow}(\Gamma_{BD}, \sigma)$.

**Proof.** As with the Anderson-Putnam construction, a point in the inverse limit is a sequence of instructions for tiling larger and larger regions of the plane, insofar as the $n$-th approximant determines the structure of a tiling out to distance $\lambda^n r$. $\square$
The complexes $\Gamma^n_{BD}$ are all the same (up to scale), so it is relatively easy to compute $H^*(\Gamma^n_{BD}) = H^*(\Gamma_{BD})$. Unfortunately, the map $\sigma : \Gamma_{BD} \to \Gamma_{BD}$ is typically not a cellular map. For instance, for a square tiling $\sigma$ takes a $2r \times 2r$ vertex polygon to a $2\lambda r \times 2\lambda r$ square, which is a vertex polygon plus a small piece of the adjacent edge flaps and tile cells. To do our computations we need to use a map $\tilde{\sigma}$ that is cellular and homotopic to $\sigma$. (One way to get such a map $\tilde{\sigma}$ is to compose $\sigma$ with a flow that expands tile cells slightly at the expense of the edge cells and vertex polygons. The details are not important.) The map $\tilde{\sigma}$ sends vertex polygons to vertex polygons, edge flaps to a union of edge flaps and vertex polygons, and tile cells to a union of all three kinds of cells. Let $\tilde{\Omega} = \varprojlim_{N} (\Gamma_{BD}, \tilde{\sigma})$.

**Theorem 2.6.** The Čech cohomology of $\tilde{\Omega}$ is isomorphic to the Čech cohomology of $\Omega$.

**Proof.** Since $\sigma$ and $\tilde{\sigma}$ are homotopic, $\tilde{\sigma}^* = \sigma^*$ as operators on $H^*(\Gamma_{BD})$. Then $\tilde{H}^*(\tilde{\Omega}) = \tilde{H}^*(\varprojlim_{N} (\Gamma_{BD}, \tilde{\sigma})) = \varprojlim_{N} \tilde{H}^*(\Gamma_{BD}, \tilde{\sigma}^*) = \varprojlim_{N} \tilde{H}^*(\Gamma_{BD}, \sigma^*) = \tilde{H}^*(\varprojlim_{N} (\Gamma_{BD}, \sigma)) = \tilde{H}^*(\Omega). \quad \square$

This theorem does not say that $\tilde{\Omega}$ and $\Omega$ are homeomorphic. In many cases they are not. However, their cohomologies are the same, so we can always use the inverse limit structure of $\tilde{\Omega}$ to compute the cohomology of $\Omega$.

2.3.1. **One dimensional results.** [BD2]

Let $S_0 \subset \Gamma_{BD}$ be the sub-complex of vertex flaps, and let $S_1 = \Gamma_{BD}$. Since $\tilde{\sigma}$ maps $S_0$ to $S_0$ and $S_1$ to $S_1$, we can consider the inverse limit space $S_i = \varprojlim_{N} (S_i, \tilde{\sigma})$. Since $S_0 \subset S_1$, we can compute $\tilde{H}^*(\Omega) = \tilde{H}^*(S_1)$ by computing $\tilde{H}^*(S_0)$ and the relative cohomology $\tilde{H}^*(S_1, S_0)$ and then combining them with the long exact sequence

$$0 \to \tilde{H}^0(S_1, S_0) \to \tilde{H}^0(S_1) \to \tilde{H}^0(S_0) \to \tilde{H}^1(S_1, S_0) \to \tilde{H}^1(S_1) \to \tilde{H}^1(S_0) \to 0 \quad (7)$$

We examine each of these terms. $\tilde{H}^0(S_1, S_0)$ is the direct limit (under $\tilde{\sigma}^*$) of $H^0(S_1, S_0)$. Since $S_1$ is connected, this is zero. Likewise, $\tilde{H}^0(S_1) = \varinjlim H^0(S_1) = \mathbb{Z}$. Since $\tilde{\sigma}$ maps each cell of $S_0$ to a single cell, $\tilde{\sigma}$ merely permutes the cells of the eventual range $S_0^{ER}$. Thus $\varinjlim H^*(S_0) = H^*(S_0^{ER})$. If $S_0^{ER}$ has $k$ connected components and has $\ell$ loops, then $\tilde{H}^0(S_0) = \mathbb{Z}^k$ and $\tilde{H}^1(S_0) = \mathbb{Z}^\ell$. Meanwhile the quotient space $S_1/S_0$ is a wedge of circles, one for each tile type. $H^1(S_1, S_0) = \mathbb{Z}^N$, where $N$ is the number of tile types, and $\tilde{H}^1(S_1, S_0) = \varinjlim (\mathbb{Z}^N, A^T)$, where $A$ is the substitution matrix. Combining these observations, we have the long exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}^k \to \varinjlim (\mathbb{Z}^N, A^T) \to \tilde{H}^1(\Omega) \to \mathbb{Z}^\ell \to 0. \quad (8)$$

Using reduced cohomology, this can be further simplified to

$$0 \to \mathbb{Z}^{k-1} \to \varinjlim (\mathbb{Z}^N, A^T) \to \tilde{H}^1(\Omega) \to \mathbb{Z}^\ell \to 0. \quad (9)$$

In the Fibonacci tiling, $A = \left( \begin{array}{c} 1 & 1 \\ 1 & 0 \end{array} \right)$ and $S_0$ consists of three vertex flaps: $aa$, $ab$, and $ba$. These form a contractible set, so $k - 1 = \ell = 0$, and $\tilde{H}^1(\Omega) = \varinjlim (\mathbb{Z}^2, A^T) = \mathbb{Z}^2$. In fact,
whenever $S_{0}^{ER}$ is contractible, $\tilde{H}^{0}(S_{0}^{ER})$ and $H^{1}(S_{0}^{ER})$ vanish and $H^{1}(\Omega)$ is isomorphic to $\lim \left( \mathbb{Z}^{N}, A^{T} \right)$.

We can also describe the Fibonacci tiling using collared tiles $A_{1} = (a)a(b)$, $A_{2} = (b)a(a)$, $A_{3} = (b)a(b)$, and $B = (a)b(a)$. Collaring the Fibonacci tiles and then applying the Barge-Diamond construction is overkill, but this example shows the interplay of the substitution matrix and the cohomology of $S_{0}^{ER}$. Our complex $\Gamma_{BD}$ has four tile cells and five vertex flaps, namely $A_{1}B$, $A_{2}A_{1}$, $A_{3}B$, $BA_{2}$, and $BA_{3}$. However, $A_{1}B$ and $A_{3}B$ are not in $S_{0}^{ER}$, since all supertiles start with $A_{1}$, $A_{2}$, or $A_{3}$. $S_{0}^{ER}$ consists of just the flaps $A_{2}A_{1}$, $BA_{2}$, and $BA_{3}$, yielding $k = 2$ and $\ell = 0$. The substitution matrix is

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}
$$

This matrix has rank 3, with eigenvalues $(1 \pm \sqrt{5})/2$, $-1$, and 0, and $\lim \left( \mathbb{Z}^{3}, A^{T} \right) = \mathbb{Z}^{3}$. We then have $0 \to \mathbb{Z} \to \mathbb{Z}^{3} \to \tilde{H}^{1}(\Omega) \to 0$. After checking that the quotient of $\mathbb{Z}^{3}$ by $\mathbb{Z}$ is $\mathbb{Z}^{2}$ (with no torsion terms), we again obtain $\tilde{H}^{1}(\Omega) = \mathbb{Z}^{2}$.

In the Thue-Morse tiling, $M = (1 \ 1)$ and $S_{0}$ consists of four vertex flaps that form a loop. Now $\lim \left( \mathbb{Z}^{2}, A^{T} \right) = \mathbb{Z}[1/2]$ and $k = \ell = 1$, so we have $0 \to \mathbb{Z}[1/2] \to \tilde{H}^{1}(\Omega) \to \mathbb{Z} \to 0$. Since $\mathbb{Z}$ is free, this sequence splits, so $\tilde{H}^{1}(\Omega) = \mathbb{Z}[1/2] \oplus \mathbb{Z}$.

### 2.3.2. Higher dimensions. \cite{BDHS}

In higher dimensions the procedure is similar, but the results cannot be expressed in a single exact sequence such as (9). In two dimensions, we consider the complex $S_{0}$ of vertex polygons, $S_{1}$ of vertex polygons and edge flaps, and $S_{2} = \Gamma_{BD}$. We also consider the inverse limits $S_{i} = \lim(S_{i}, \tilde{\sigma})$. As in one dimension, $\tilde{\sigma}$ maps each vertex polygon to a single vertex polygon, so $\tilde{H}^{*}(S_{0}) = H^{*}(S_{0}^{ER})$. However, $S_{0}$ is a 2-dimensional complex, so computing the cohomology of $S_{0}^{ER}$ is more than just counting connected components and loops.

The next step is to consider $\tilde{H}^{*}(S_{1}, S_{0}) = \lim(\tilde{H}^{*}(S_{1}/S_{0}), \tilde{\sigma}^{*})$. This involves only the eventual range of $S_{1}$, but is typically a complicated calculation. The quotient space $S_{1}/S_{0}$ breaks into several pieces, one for each direction that an edge can point. In general, the pieces are not particularly simple, and it takes work to understand how $\tilde{\sigma}^{*}$ acts on $\tilde{H}^{*}(S_{1}/S_{0})$. Once $\tilde{H}^{*}(S_{1}, S_{0})$ is computed, we combine it with $\tilde{H}^{*}(S_{0})$ via the long exact sequence

$$
\cdots \to \tilde{H}^{k}(S_{1}, S_{0}) \to \tilde{H}^{k}(S_{1}) \to \tilde{H}^{k}(S_{0}) \to \tilde{H}^{k+1}(S_{1}, S_{0}) \to \cdots
$$

(10)

to compute $\tilde{H}^{*}(S_{1})$.

The relative cohomology $\tilde{H}^{*}(S_{2}, S_{1})$ is simpler. The quotient space $S_{2}/S_{1}$ is a wedge of spheres, so $\tilde{H}^{0} = \tilde{H}^{1} = 0$ and $\tilde{H}^{2} = \mathbb{Z}^{N}$. $\tilde{H}^{k}(S_{2}, S_{1})$ equals $\lim \left( \mathbb{Z}^{n}, A^{T} \right)$ when $k = 2$, and vanishes when $k = 0$ or 1. The final stage is combining $\tilde{H}^{*}(S_{1})$ and $\tilde{H}^{*}(S_{2}, S_{1})$ with the long exact sequence

$$
\cdots \to \tilde{H}^{k}(S_{2}, S_{1}) \to \tilde{H}^{k}(S_{2}) \to \tilde{H}^{k}(S_{1}) \to \tilde{H}^{k+1}(S_{2}, S_{1}) \to \cdots
$$

(11)
Example 5. Consider a tiling of $\mathbb{R}^2$ featuring three square tiles $A$, $B$, and $C$, and generated by the substitution $\text{[ ]} \to \begin{array}{cc} \text{A} & \ast \\ \text{B} & \text{C} \end{array}$ where “$\ast$” is shorthand for $A$, $B$ or $C$. This substitution does not force the border, so collaring is needed to compute its cohomology. $S_0$ involves many vertex polygons, but each of these maps to a vertex polygon of the form $\begin{array}{cc} \text{C} & \text{B} \\ \ast & \text{A} \end{array}$. $S_0$ is contractible, consisting of three squares glued together at their north and east edges, so $\tilde{H}^0(S_0) = \mathbb{Z}$ and $\tilde{H}^1(S_0) = \tilde{H}^2(S_0) = 0$.

$S_1/S_0$ consists of vertical and horizontal edge flaps. The vertical edge flaps are $B|C$, $C|B$, $A|\ast$ and $\ast|A$, but only $C|B$ and $\ast|A$ survive to the eventual range. This portion of $S_1/S_0$ retracts to the wedge of two circles, and $\tilde{\sigma}^*$ acts on its first cohomology by the matrix $\left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right)$, yielding a direct limit of $\mathbb{Z}[1/2]$. The horizontal edge flaps are similar, giving another factor of $\mathbb{Z}[1/2]$, so $\tilde{H}^1(S_1, S_0) = \mathbb{Z}[1/2]^2$ and $\tilde{H}^0(S_1, S_0) = \tilde{H}^2(S_1, S_0) = 0$.

$S_2/S_0$ is a wedge of three spheres, and the only nontrivial cohomology is $H^2 = \mathbb{Z}^3$. This transforms via $A^T = \left( \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right)$, so $\tilde{H}^2(S_2, S_1) = \lim_\longrightarrow(\mathbb{Z}^3, A^T)$ and $\tilde{H}^0(S_2, S_1) = \tilde{H}^1(S_2, S_1) = 0$.

We combine these relative cohomologies using the long exact sequences (10) and (11). The first of these yields:

$$0 \to 0 \to \tilde{H}^0(S_1) \to Z \to Z[1/2]^2 \to \tilde{H}^1(S_1) \to 0,$$

so $\tilde{H}^0(S_1) = \mathbb{Z}$ and $\tilde{H}^1(S_1) = Z[1/2]^2$ (and $\tilde{H}^2(S_1) = 0$). The second yields:

$$0 \to \tilde{H}^1(S_2) \to Z[1/2]^2 \to \lim_\longrightarrow(\mathbb{Z}^3, A^T) \to \tilde{H}^2(S_2) \to 0.$$ 

All maps commute with $\tilde{\sigma}^*$. Since the $Z[1/2]^2$ terms double with substitution, and since the eigenvalues of $A^T$ are 1, 1, and 4, the map from $Z[1/2]^2$ to $\lim_\longrightarrow(\mathbb{R}^3, A^T)$ must be zero. We then have

$$\tilde{H}^0(S_2) = \mathbb{Z}, \quad \tilde{H}^1(S_2) = Z[1/2]^2, \quad \tilde{H}^2(S_2) = \lim_\longrightarrow(\mathbb{Z}^3, A^T) = Z[1/4] \oplus \mathbb{Z}^2.$$ 

(We write $Z[1/4]$ rather than $Z[1/2]$ in $\tilde{H}^2$ to emphasize that this term scales by 4 under substitution.) This is the same cohomology as the half-hex substitution. In fact, this tiling space is homeomorphic to the half-hex tiling space.

2.4. Rotations and other symmetries. A natural question about any pattern is “what are its symmetries?” An aperiodic tiling cannot have any translational symmetries, but it can have rotational or reflectional symmetries. We consider actions of reflection and rotation (and translation, of course) on the tiling space $\Omega$, and examine how various quantities transform under that group action.

2.4.1. Decomposing by representation. Rotating a tile and then taking its boundary is the same as taking the boundary and then rotating. Likewise, rotations commute with coboundaries, and in most cases rotations commute with substitution, so it makes sense to decompose our
cochain complexes, and the cohomology of our tiling space, into representations of whatever rotation group \( G \) acts on our tiling space. By Schur’s Lemma, neither the coboundary nor substitution can mix different representations, and our calculations can proceed one representation at a time.

The trouble with this approach is that representations are vector spaces, and our cochain complexes take values in \( \mathbb{Z} \). We therefore consider the cohomology of tiling spaces with values in \( \mathbb{R} \) rather than \( \mathbb{Z} \). In the process we lose information about torsion and divisibility, but that’s the price we have to pay.

For example, the tiles and substitution rules for the Penrose tilings are shown in Figures 4 and 5. There are four types of tiles, each in 10 orientations, and four types of edges, each in 10 orientations. There are only four kinds of vertices \( a, b, c, d \) each of which can sit in the center of a pattern with 5-fold rotational symmetry. This means that \( a = t^2a, b = t^2b, c = t^2c \) and \( d = t^2d \), where \( t \) is a rotation by \( \pi/5 \). In fact, \( a = tb \) and \( b = ta \), as can be seen from the fact that the \( \alpha \) edge of \( A \) runs from \( b \) to \( a \) while the \( t\alpha \) edge of \( B \) runs from \( a \) to \( b \). Likewise, \( c = td \) and \( d = tc \). This tiling forces the border, so we do not need to collar.

The group \( G = \mathbb{Z}_{10} \) acts on the Anderson-Putnam complex \( \Gamma \) by permuting the tiles, and the eigenvalues of the generator \( t \) are the 10th roots of unity. Each tile type, and each edge type, can be described by the module \( \mathbb{R}[t]/(t^{10} - 1) \). The polynomial \( t^{10} - 1 \) factors as \((t - 1)(t + 1)(t^4 + t^3 + t^2 + t + 1)(t^4 - t^3 + t^2 - t + 1)\), with the factors corresponding to the primitive first, second, 5th and 10th roots, respectively. Each factor also corresponds to a representation. Since \( t^2 \) acts trivially on the vertices, only the representations with \( t = \pm 1 \) appear in \( C_0 \). More specifically, when working with the Anderson-Putnam complex, our
chains complexes are:
\[
C_0(\Gamma) = \left[ \mathbb{R}[t]/(t^2 - 1) \right]^2 = \left[ \mathbb{R}[t]/(t - 1) \right]^2 \oplus \left[ \mathbb{R}[t]/(t + 1) \right]^2
\]
\[
C_1(\Gamma) = \left[ \mathbb{R}[t]/(t^{10} - 1) \right]^4 = \left[ \mathbb{R}[t]/(t - 1) \oplus \mathbb{R}[t]/(t + 1) \oplus \mathbb{R}[t]/(t^4 + t^3 + t^2 + t + 1) \oplus \mathbb{R}[t]/(t^4 - t^3 + t^2 - t + 1) \right]^4
\]
\[
C_2(\Gamma) = \left[ \mathbb{R}[t]/(t^{10} - 1) \right]^4 = \left[ \mathbb{R}[t]/(t - 1) \oplus \mathbb{R}[t]/(t + 1) \oplus \mathbb{R}[t]/(t^4 + t^3 + t^2 + t + 1) \oplus \mathbb{R}[t]/(t^4 - t^3 + t^2 - t + 1) \right]^4
\]

The complexes \( C^k(\Gamma) \) are the dual spaces of \( C_k(\Gamma) \).

The matrices for the boundary maps \( \partial_1 : C_1 \to C_0 \) and \( \partial_2 : C_2 \to C_1 \) are:
\[
(16) \quad \partial_1 = \begin{pmatrix}
1 - t & -1 & -t & -1 \\
0 & 1 & 1 & t
\end{pmatrix}; \quad \partial_2 = \begin{pmatrix}
-1 & t^4 & -t^7 \\
-1 & t^9 & -t & t^8 \\
1 & -t^5 & 0 & 0 \\
0 & 0 & 1 & -t^5
\end{pmatrix}
\]
ine the representations \( t = \pm 1 \). In the other representations \( \partial_2 \) is the same, but \( \partial_1 \) is identically zero (since \( C_0 = 0 \)). The coboundary maps \( \delta_0 \) and \( \delta_1 \) are the transposes of \( \partial_1 \) and \( \partial_2 \), only with \( t \) replaced by \( t^{-1} \).

In the \( t = 1 \) representation, \( \delta_1 \) has rank 2 and \( \delta_0 \) has rank 1, and we get \( H^0 = H^1 = \mathbb{R} \) and \( H^2 = \mathbb{R}^2 \). These are the elements of cohomology that are invariant under rotation. We say that this portion of the cohomology rotates like a scalar.

In the \( t = -1 \) representation, \( \delta_0 \) and \( \delta_1 \) are each rank 2, and we get \( H^2 = \mathbb{R}^2 \) and \( H^1 = H^0 = 0 \). This portion of the cohomology rotates like a pseudoscalar, flipping sign with every 36 degree rotation.

In the representation with \( t^5 = 1 \) but \( t \neq 1 \) (that is, with \( t^4 + t^3 + t^2 + t + 1 = 0 \)), \( \delta_1 \) is a rank-4 isomorphism, so all cohomologies vanish. In the representations with \( t^5 = -1 \) (but \( t \neq -1 \)), \( \delta_1 \) has rank 3, so \( H^1 = H^2 = \mathbb{R}[t]/(t^4 - t^3 + t^2 - t + 1) \). This portion of the cohomology rotates like a vector, flipping sign after a 180 degree rotation.

Substitution acts on 2-cells by the matrix \( \begin{pmatrix}
t^7 & 0 & 0 & t^4 \\
0 & t^3 & t^6 & 0 \\
t^4 & 0 & t^1 & t^6
\end{pmatrix} \) and on 1-cells by \( \begin{pmatrix}
0 & 0 & 0 & t^8 \\
t^3 & t^7 & 0 & 0 \\
0 & 0 & t^3 & 0
\end{pmatrix} \).

Both of these matrices are invertible for all representations (in fact, both have determinant 1), so \( \tilde{H}^k(\Omega) = H^k(\Gamma) \) for \( k = 0, 1, 2 \). In summary:
\[
\tilde{H}^0(\Omega) = H^0(\Gamma) = \mathbb{R}[t]/(t - 1)
\]
\[
\tilde{H}^1(\Omega) = H^1(\Gamma) = \mathbb{R}[t]/(t - 1) \oplus \mathbb{R}[t]/(t^4 - t^3 + t^2 - t + 1)
\]
\[
\tilde{H}^2(\Omega) = H^2(\Gamma) = (\mathbb{R}[t]/(t - 1))^2 \oplus (\mathbb{R}[t]/(t + 1))^2 \oplus \mathbb{R}[t]/(t^4 - t^3 + t^2 - t + 1)
\]

The upshot is that \( \tilde{H}^0(\Omega) = \mathbb{R} \) and is rotationally invariant, which is no surprise, since the generator is the constant function. \( \tilde{H}^1(\Omega) = \mathbb{R}^5 \), of which 4 dimensions rotate like vectors, with \( t^5 = -1 \), and one is rotationally invariant. \( \tilde{H}^2(\Omega) = \mathbb{R}^8 \), consisting of a rotationally
invariant $\mathbb{R}^2$, a piece $\mathbb{R}^2$ that rotates like a pseudoscalar, and a piece $\mathbb{R}^4$ that rotates like a vector.

2.4.2. Three tiling spaces. For 2-dimensional substitution like the Penrose tiling and the chair tiling, there are actually three tiling spaces to be considered. We have been considering the space $\Omega$ that is the (translational) orbit closure of a single tiling. This would be, for instance, the set of all chair tilings where the edges are parallel to the coordinate axes. We can also consider a larger space $\Omega_{rot}$ of all rotations of tilings in $\Omega$. Finally we can consider the quotient space $\Omega_0$ of tilings modulo rotations. $\Omega_0$ can either be viewed as $\Omega_{rot}/S^1$ or as the quotient of $\Omega$ by the discrete group of rotations that acts on $\Omega$. For the Penrose space, we would have $\Omega_0 = \Omega/\mathbb{Z}_{10}$, while for the chair tiling we would have $\Omega_0 = \Omega/\mathbb{Z}_4$.

The cohomologies of the three spaces are related as follows [ORS, BDHS]:

**Theorem 2.7.** Working with real or complex coefficients, the cohomology of $\Omega_0$ is isomorphic to the rotationally invariant part of the cohomology of $\Omega$. The cohomology of $\Omega_{rot}$ is isomorphic to the cohomology of $\Omega_0 \times S^1$.

The upshot of this theorem is that $\Omega$ is the space with the richest cohomology, while $\Omega_{rot}$ and $\Omega_0$ have less cohomological structure. This is because all rotations on $\Omega_{rot}$ are homotopic to the trivial rotation, and so act trivially on $\check{H}^*(\Omega_{rot})$. Thus, only the rotationally invariant parts of the cohomology of $\Omega$ can manifest themselves in the cohomology of $\Omega_{rot}$.

In general, $\Omega_{rot}$ is not homeomorphic to $\Omega_0 \times S^1$, since the action of $S^1$ on $\Omega_{rot}$ is typically not free. There are some tilings in $\Omega_{rot}$ that have discrete $k$-fold rotational symmetry. For these tilings, rotation by $2\pi/k$ brings us back to the same tiling. $\Omega_{rot}$ has the structure of a circle bundle over $\Omega_0$ with some exceptional fibers corresponding to these symmetric tilings. (Seifert fibered 3-manifolds have a very similar structure.) When working with integer coefficients, these exceptional fibers can give rise to torsion in $\check{H}^2(\Omega_{rot})$.[BDHS]

These relations can also be used to compute the cohomology of the pinwheel tiling space. When there are tiles that point in all directions, the only two well-defined spaces are $\Omega_{rot}$ and $\Omega_0$. For the pinwheel, the cohomology of $\Omega_0$ can be computed with Barge-Diamond collaring, with the result that $\check{H}^0(\Omega_0) = \mathbb{Z}$, $\check{H}^1(\Omega_0) = \mathbb{Z}$ and $\check{H}^2(\Omega_0) = \mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3] \oplus \mathbb{Z}^5 \oplus \mathbb{Z}_2$. This then determines the real cohomology of $\Omega_0$, and, by Theorem 2.7, the real cohomology of $\Omega_{rot}$. To compute the integer cohomology of $\Omega_{rot}$, we have to consider the exceptional fibers. There are 6 pinwheel tilings with 2-fold rotational symmetry, as shown in Figure 9 below;
these give rise to a $\mathbb{Z}_2^5$ term in a spectral sequence

\[
\begin{array}{ccc}
\mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z}_2^5 & \mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3]^2 \oplus \mathbb{Z}_5^5 \oplus \mathbb{Z}_2 \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3]^2 \oplus \mathbb{Z}_5^5 \oplus \mathbb{Z}_2 \\
\end{array}
\]

that computes the cohomology of $\Omega_{rot}$. Furthermore, the $d_2$ map in the spectral sequence involves the torsion elements in a non-trivial way. The end result is that $\check{H}^1(\Omega_{rot}) = \mathbb{Z}_2^5$, $\check{H}^2(\Omega_{rot}) = \mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3]^2 \oplus \mathbb{Z}_5^5$ and $\check{H}^3(\Omega_{rot}) = \mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3]^2 \oplus \mathbb{Z}_5^5 \oplus \mathbb{Z}_2$. For details of this calculation, see [BDHS] or [Sa3].

3. What is cohomology good for?

3.1. Distinguishing spaces. The most obvious use of topological invariants such as Čech cohomology is to distinguish spaces. If tiling spaces $\Omega$ and $\Omega'$ have cohomologies that are not isomorphic (as rings), then $\Omega$ and $\Omega'$ cannot be homeomorphic. If a group $G$ of isometries of $\mathbb{R}^d$ (such as $\mathbb{Z}_{10}$ or $\mathbb{Z}_4$) acts on $\Omega$ and $\Omega'$, then we can decompose each cohomology group into representations of $G$. For each irreducible representation $\rho$ of $G$, let $\check{H}_\rho^k(\Omega)$ be the part of $\check{H}^k(\Omega)$ that transforms under $\rho$. If $\Omega$ and $\Omega'$ are homeomorphic via a map that intertwines the action of $G$, then for each representation $\rho$ we must have $\check{H}_\rho^k(\Omega) = \check{H}_\rho^k(\Omega')$. In particular, if a tiling space $\Omega'$ is related to the Penrose tiling $\Omega$ by a $\mathbb{Z}_{10}$-equivariant homeomorphism, then not only must $\check{H}^1(\Omega', \mathbb{R})$ equal $\mathbb{R}_5$, but $\check{H}^1(\Omega', \mathbb{R})$ must consist of a 1-dimensional rotationally invariant piece and a 4-dimensional piece that rotates like a vector.

For each subgroup $H < G$, we can also consider the topology of the set $\Omega_H$ of tilings in $\Omega$ that are fixed by $H$. If $\Omega$ and $\Omega'$ are tiling spaces with the same rotation group $G$, and if there exists an isomorphism that commutes with the action of $G$, then $\Omega_H$ and $\Omega'_H$ must be homeomorphic. In this sense, the structure of $\Omega_H$ is a topological invariant of the tiling space $\Omega$. If $H_1 < H_2$, then $\Omega_{H_2} \subset \Omega_{H_1}$. The way that these different spaces nest within one another is also manifestly invariant.

If $d = 2$ and $G$ is a subgroup of $SO(2)$, then $\Omega_H$ is not especially interesting. If $N$ is the normalizer of $H$ in $G$, then $N$ acts on $\Omega_H$, and there are typically only finitely many orbits. Understanding $\Omega_H$ boils down to counting these orbits and identifying how much symmetry a point in each orbit has.

Things get more interesting if $d > 2$, or if $H$ involves reflections. In that case, there may be a subspace $V \subset \mathbb{R}^d$ whose vectors are fixed by the action of $H$. $\Omega_H$ is invariant under translation by elements of $V$, and can often be realized as a space of tilings of $V$, or
as a disjoint union of several such lower-dimensional tiling spaces. In such cases, the Čech cohomology of $\Omega_H$ yields an interesting invariant.

We present two worked examples. We first consider the chair tiling of the plane, with $G = D_8 = O(2, \mathbb{Z})$, the group generated by rotation by 90 degrees and by reflection about the $x$ axis. We compute the structure of $\Omega_H$ for every nontrivial subgroup $H < G$. We then consider the pinwheel tilings, with $G = O(2)$.

3.1.1. *The chair tiling.* We work with the “arrow” version of the chair tiling. This is a 2-dimensional substitution tiling in which the tiles are all unit squares that meet full-edge to full-edge. Each square is decorated with an arrow pointing northeast, southeast, northwest or southwest. Rotation and reflection act naturally on arrows, so a counterclockwise rotation by 90 degrees would send a northeast arrow to a northwest, a northwest to a southwest, a southwest to a southeast, and a southeast to a northeast. Likewise, reflection about the $x$ axis interchanges northeast and southeast arrows, and interchanges northwest and southwest arrows. The substitution on northeast arrows is

\[
\begin{array}{c}
\text{↑} \\
\text{→} \\
\text{↑} \\
\end{array}
\rightarrow
\begin{array}{c}
\text{↑} \\
\text{→} \\
\text{↑} \\
\end{array}
\]

and the substitution on all other arrow tiles is obtained by rotating or reflecting this picture.

There are nine nontrivial subgroups of $G = D_8$. These include $H_1 = G$ itself, the rotation groups $H_2 = \mathbb{Z}_4$ and $H_3 = \mathbb{Z}_2$, the dihedral group $H_4 = D_4$ generated by reflections about the $x$ and $y$ axes, and the 2-element groups $H_5$ generated by reflection about the $x$ axis, $H_6$ generated by reflection about the $y$ axis, $H_7$ generated by reflection about the line $y = x$, and $H_8$ generated by reflection about the line $y = -x$. Finally, there is the dihedral group $H_9$ generated by $H_7$ and $H_8$.

There is only one tiling that is invariant under all of $G$, namely the fixed point of the substitution whose central patch involves four arrows pointing out from the origin:

\[
\begin{array}{c}
\text{↑} \\
\text{→} \\
\text{↑} \\
\text{→} \\
\end{array}
\]

This is also the only tiling that is invariant under $H_2$ or $H_3$ or $H_4$ or $H_9$. $H_5$ and $H_6$ are conjugate, so $\Omega_{H_5}$ and $\Omega_{H_6}$ are homeomorphic, with rotation by 90 degrees taking each set to the other. Likewise, $\Omega_{H_7}$ and $\Omega_{H_8}$ are homeomorphic. We therefore restrict our attention to $\Omega_{H_5}$ and $\Omega_{H_7}$.

We begin with $\Omega_{H_5}$. Since all vertices with incoming and outgoing arrows have either 3 or 0 incoming arrows, and since vertices with 3 incoming arrows cannot be symmetric under $H_5$, the tiles along the $x$ axis must alternate between the form $\begin{array}{c}
\text{↑} \\
\text{→} \\
\end{array}$ and $\begin{array}{c}
\text{→} \\
\text{↑} \\
\end{array}$. Although the pattern along the $x$ axis is periodic, there is a hierarchy from the way that tiles along the $x$ axis join with tiles once removed from the $x$ axis to form clusters of four tiles, which join tiles even farther away to form clusters of 16, and so on. $\Omega_{H_5}$ is connected and is homeomorphic to the dyadic solenoid $Sol_2$, so $\tilde{H}^1(\Omega_{H_5}) = \mathbb{Z}[1/2]$. 
To understand $\Omega_{H_7}$, we look at symmetric configurations of tiles along the line $y = x$. All vertices take one of three forms:

and the second of these patterns can occur at most once. In other words, either all of the arrows along the line $x = y$ point northeast, or all point southwest, or all point outwards from a special point where four infinite-order supertiles meet.

Reading from southwest to northeast along the line $x = y$, there are four kinds of collared tiles that appear, which we label $a$, $b$, $c$ and $d$. The label $a$ means (SW)SW(SW), while $b$ means (SW)SW(NE), $c$ means (SW)NE(NE) and $d$ means (NE)NE(NE). An $a$ can be followed by an $a$ or a $b$, a $b$ is always followed by a $c$, a $c$ is always followed by a $c$, and a $d$ is always followed by a $d$. The Anderson-Putnam complex is then given by the “eyeglasses” graph shown in Figure 6.

Substitution sends edge $a$ to $aa$, edge $b$ to $ab$, edge $c$ to $cd$ and edge $d$ to $dd$. The graph has $H^0 = \mathbb{Z}$ and $H^1 = \mathbb{Z}^2$, and substitution acts trivially on $H^0$ and by multiplication by 2 on $H^1$, so $\tilde{H}^0(\Omega_{H_7}) = \mathbb{Z}$ and $\tilde{H}^1(\Omega_{H_7}) = \mathbb{Z}[1/2]^2$.

One can apply a similar analysis to chair tilings in higher dimensions. For the 3-dimensional chair tiling, the relevant group is the 24-element group $G$ of symmetries of the cube, and there are significant subgroups of order 2, 3, 4, 6, and 12. Each cyclic subgroup $H$ gives rise to a space $\Omega_H$ with non-trivial $\tilde{H}^1$, while the non-Abelian subgroups $H < G$ have $\Omega_H$ finite. For each non-Abelian $H$, the only invariant is $\tilde{H}^0(\Omega_H) = \mathbb{Z}^{\Omega_H}$.

3.1.2. The pinwheel tilings. The pinwheel tilings are based on a single tile, up to reflection, rotation and translation. It is a 1-2-$\sqrt{5}$ right triangle with substitution rule shown in Figure 7.

The maximal symmetry group for any pinwheel tiling is $H_1 = D_4$ (say, invariance under reflection about both the $x$ and $y$ axes). There are four such tilings, all closely related. Each
There are two subgroups (up to conjugacy) of $H_1$, namely $H_2$ generated by 180 degree rotation, and $H_3$ generated by reflection about the $x$ axis. There are six $H_2$-invariant tilings (plus rotations of the same), whose central patches are shown in Figure 9. All are periodic points of the substitution, the first four of period four and the last two of period two. In other words, $\Omega_{H_2}$ consists of six disjoint circles, so $\tilde{H}^1(\Omega_{\mathbb{Z}^2}) = \tilde{H}^0(\Omega_{\mathbb{Z}^2}) = \mathbb{Z}^6$. These six circles are the same exceptional fibers in the fibration $\Omega_{rot} \to \Omega_0$ that gave rise to torsion in $\tilde{H}^k(\Omega_{rot})$.

Finally, we consider tilings that are $H_3$-invariant. Since no tiles are themselves reflection-symmetric, there must be edges along the $x$ axis, and these edges are either all hypotenuses or all of integer length.

For the symmetric tilings with hypotenuses along the $x$ axis, we get a 1-dimensional tiling space that comes from the substitution $a \to aabba, b \to baabb$, where $a$ and $b$ represent hypotenuses pointing in the two obvious directions. This edge substitution actually comes from the square of the pinwheel substitution, since the pinwheel substitution swaps hypotenuses and integer legs. A 1-dimensional Barge-Diamond calculation shows that this set of tilings has $\tilde{H}^1 = \mathbb{Z}^2 \oplus \mathbb{Z}[1/5]$. Pinwheel substitution, applied only once, swaps this space with the space of symmetric tilings involving integer edges along the $x$ axis, which therefore has the same cohomology. The upshot is that $H^1(\Omega_{H_3}) = \mathbb{Z}^4 \oplus \mathbb{Z}[1/5]^2$.

3.1.3. *Asymptotic structures*. A key difference between solenoids and spaces of 1-dimensional non-periodic tilings is that tilings may be forward or backwards asymptotic. Suppose that $T_1$ and $T_2$ are tilings in the same tiling space $\Omega$, but that the restrictions of $T_1$ and $T_2$ to the half-line $[0, \infty)$ are identical. Then $T_1 - t$ and $T_2 - t$ agree on a larger half-line $[-t, \infty)$, and $\lim_{t \to \infty} d(T_1 - t, T_2 - t) = 0$, where $d$ is the metric on $\Omega$. We say that $T_1$ and $T_2$ are *forward asymptotic*. Likewise, two tilings can be *backwards asymptotic*. The orbits
of \(T_1\) and \(T_2\) are called \textit{asymptotic composants}. Every substitution tiling space has a finite number of asymptotic composants, and the structure of these composants is reflected in the cohomology of \(\Omega\).

For instance, in the Thue-Morse tiling space there are four periodic points of the substitution of the form \(T_1 = \ldots a.a\ldots\), \(T_2 = \ldots a.b\ldots\), \(T_3 = \ldots b.a\ldots\) and \(T_4 = \ldots b.b\ldots\), where the central dot indicates the location of the origin. The tilings \(T_1\) and \(T_2\) are backwards asymptotic, as are \(T_3\) and \(T_4\). Likewise, \(T_1\) and \(T_3\) are forward asymptotic, as are \(T_2\) and \(T_4\). If we imagine asymptotic composants to be “joined at infinity”, then the orbits of these four tilings form an \textit{asymptotic cycle}. This asymptotic cycle manifests itself as the closed loop on \(\Gamma_{BD}\) that generated a \(\mathbb{Z}\) term in \(\check{H}^1(\Omega)\).

By studying asymptotic structures, Barge and Diamond [BD1] were able to construct a complete homeomorphism invariant of one-dimensional substitution tilings. Unfortunately, this invariant is extremely difficult to compute in practice. As a practical alternative, Barge and Smith [BSm] constructed an \textit{augmented cohomology} of one-dimensional substitution tilings. The precise definition involves the inverse limit of a variant of the Anderson-Putnam complex, but the basic idea is to identify all forward asymptotic tilings that are periodic points of the substitution, and separately to identify all backwards asymptotic periodic points. The cohomology of the resulting space, while not a complete invariant, yields finer information than the ordinary Čech cohomology.

In higher dimensions, asymptotic structures are more subtle, since there are (potentially) infinitely many directions to check. In 2 dimensions (with results that generalize somewhat to still higher dimensions), Barge and Olimb [BO] examined the \textit{periodic branch locus} of a substitution, namely the set of pairs of tilings, each periodic under the substitution, that agree on at least a half-plane. From this locus, and from translates of these pairs along certain special directions, they construct a larger \textit{branch locus} that can have a structure similar to that of a one-dimensional tiling space. The cohomology of the branch locus is a homeomorphism invariant of a tiling space.

With the chair tiling, as with a number of other examples, the branch locus seems to be closely related to the tilings that are symmetric under certain reflections, and the calculation of the cohomology of the branch locus resembles the computations of \(\check{H}^1(\Omega_{H_5})\) and \(\check{H}^1(\Omega_{H_7})\). These in turn are related to the quotient cohomology of the chair tiling space relative to the 2-dimensional dyadic solenoid. While it might be a coincidence, all three computations seem to be telling the same story! Unfortunately, the general relation between cohomology of branch loci, cohomology of tilings with symmetry and quotient cohomology is not yet understood.

3.2. \textbf{Gap labeling}. For tilings of \(\mathbb{R}^d\) with finite local complexity (with respect to translations), there is a natural trace map from the highest cohomology \(\check{H}^d(\Omega)\) to \(\mathbb{R}\). Each class \(\alpha \in \check{H}^d(\Omega)\) can be represented by a pattern-equivariant \(d\)-cochain \(i_\alpha\). Pick any bounded
region $R$ of a tiling $T$, let $i_\alpha(R)$ be the sum of the values of $i_\alpha$ on all of the tiles in $R$. If $i_\alpha$ and $i'_\alpha$ are cohomologous, then $i_\alpha - i'_\alpha = \delta i_\beta$ for some pattern-equivariant cochain $i_\beta$, and $i_\alpha(R) - i'_\alpha(R) = i_\beta(\partial R)$. Define

$$Tr(\alpha) = \lim_{r \to \infty} \frac{i_\alpha(B_r)}{Vol(B_r)},$$

where $B_r$ is the ball of radius $r$ around the origin in a fixed tiling $T$. Since $Vol(\partial B_r)/Vol(B_r) \to 0$ as $r \to \infty$, different representatives for the class $\alpha$ yield the same limit. Likewise, if $\Omega$ is uniquely ergodic, then all tilings $T$ yield the same limit.

For instance, in a Fibonacci tiling where the $a$ tiles have length $\phi = (1 + \sqrt{5})/2$ and the $b$ tiles have length 1, there are on average $\phi$ $a$ tiles for every $b$ tile, so the indicator cochain $i_a$ has trace $\phi/(\phi^2 + 1)$ and the cochain $i_b$ has trace $1/(\phi^2 + 1)$. Since $i_a$ and $i_b$ generate $\tilde{H}^1$, the image of the trace map is $(\phi^2 + 1)^{-1}\mathbb{Z}[\phi]$.

The image of the trace map is called the frequency module of the tiling space. The frequency module is isomorphic to the gap-labeling group, which in K-theory is the image of a trace map in $K^0$. Besides being an invariant of topological conjugacies, the gap-labeling group is used (as the name implies) to label gaps in the spectra of Schrödinger operators associated with a tiling. The key theorem is due to Bellissard ([Bel], see also [BBG, BHZ, BKL]):

**Theorem 3.1.** Let $T$ be a tiling in a minimal and uniquely ergodic tiling space $X$, and let $V : \mathbb{R}^d \to \mathbb{R}$ be a strongly pattern-equivariant function. Consider the Schrödinger operator

$$H = -\frac{\hbar^2}{2m} \Delta + V.$$

Let $E_0$ be a point that is not in the spectrum of $H$. (That is, $E_0$ lies in a gap in the spectrum.) Then the integrated density of states up to energy $E_0$ is an element of the gap-labeling group of $\Omega$.

Elements of the frequency module (or gap-labeling group) should not be viewed as pure numbers. Rather, they have units of (Volume)$^{-1}$, being the ratio of $i_\alpha(B_r)$ (a pure number) and $Vol(B_r)$. Likewise, the integrated density of states gives the number of eigenstates of $H$ up to energy $E_0$ (a pure number) per unit volume.

Traces of cohomologies in all dimensions were studied in [KP], and are known as Ruelle-Sullivan maps. These traces give a ring homomorphism from $\tilde{H}^*(\Omega)$ to the exterior algebra of $\mathbb{R}^d$.

### 3.3. Tiling deformations

Some properties of a tiling are consequences of the geometry of the tiles, while others follow from the combinatorics of how tiles fit together. To distinguish between the two, we consider different tiling spaces that have the same combinatorics, and parametrize the possible tile shapes.

Let $X$ be a tiling space. To specify the shapes of the tiles involved, we must indicate the displacement associated to every edge of every possible tile. Furthermore, if two tiles share
an edge, then those two edges must be described by the same vector, and the vectors for all the edges around a tile must sum to zero.

In other words, the shapes of all the tiles is described by a co-closed vector-valued 1-cochain on a space obtained by taking one copy of each tile type and identifying edges that can meet. That is, a cochain on the Anderson-Putnam complex $\Gamma_{AP}$! Different geometric versions of the same combinatorial tiling space are described by different shape covectors on the same Anderson-Putnam complex.

**Theorem 3.2.** [CS] Let $\Omega$ be a tiling space with shape cochain $\alpha_0$. There is a neighborhood $U$ of $\alpha_0$ in $C^1(\Gamma_{AP}, \mathbb{R}^d)$ such that, for any two co-closed shape cochains $\alpha_1, \alpha_2 \in U$, the tiling spaces $\Omega_{\alpha_1}$ and $\Omega_{\alpha_2}$ obtained from $\alpha_1$ and $\alpha_2$ are mutually locally derivable (MLD) if and only if $\alpha_1$ and $\alpha_2$ are cohomologous.

This theorem says that the first cohomology of $\Gamma_{AP}$, with values in $\mathbb{R}^d$, parametrizes local deformations of the shapes and sizes of the tiles, up to local equivalence. By considering changes in the shapes and sizes of collared tiles and taking a limit of repeated collaring (as in Gähler’s construction), we obtain

**Theorem 3.3.** [CS] Infinitesimal shape deformations of tiling spaces, modulo local equivalence, are parametrized by the vector-valued cohomology $\tilde{H}^1(\Omega, \mathbb{R}^d)$.

Among all shape changes, there are some that yield topological conjugacies. We call such a shape change a shape conjugacy. Shape conjugacies correspond to a subgroup $\tilde{H}^1_{an}(\Omega, \mathbb{R}^d)$ of $\tilde{H}^1(\Omega, \mathbb{R}^d)$ called the asymptotically negligible classes. These classes are neatly described in terms of pattern-equivariant functions:

**Theorem 3.4.** [Kel3] A class in $\tilde{H}^1(\Omega, \mathbb{R}^d)$ is asymptotically negligible if and only if it can be represented as a strongly pattern-equivariant vector-valued 1-form that is the differential of a weakly pattern-equivariant vector-valued function.

Asymptotically negligible classes don’t just describe shape conjugacies. They essentially describe all topological conjugacies, thanks to

**Theorem 3.5.** [KS1] If $f : \Omega_X \to \Omega_Y$ is a topological conjugacy of tiling spaces, then we can write $f$ as the composition $f_1 \circ f_2$ of two maps, such that $f_1$ is a shape conjugacy and $f_2$ is an MLD equivalence.

The importance of this theorem is that it allows us to check when a property of a tiling (e.g. having its vertices form a Meyer set) is invariant under topological conjugacies. One merely has to check whether the property is preserved by MLD maps (a local computation) and whether it is preserved by shape conjugacies. (The Meyer property turns out to be preserved by MLD maps but not by shape conjugacies [KS1].)

For substitution tilings, the asymptotically negligible classes are easy to characterize:
Theorem 3.6. [CS] Let $\Omega$ be a substitution tiling space generated from a substitution $\sigma$. Let $\sigma^*$ denote the action of $\sigma$ on the vector space $\breve{H}^1(\Omega, \mathbb{R}^d)$. The asymptotically negligible classes are the span of the generalized eigenvectors of $\sigma^*$ with eigenvalues strictly inside the unit circle.

For example, for the Fibonacci tiling we have $\breve{H}^1(\Omega) = \mathbb{Z}^2$, so $\breve{H}^1(\Omega, \mathbb{R}) = \mathbb{R}^2$. The substitution acts via the matrix $\left( \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right)$, with eigenvalues $\lambda_1 = \phi$ and $\lambda_2 = 1 - \phi$ and eigenvectors $\left( \begin{array}{c} \lambda_1 \\ 1 \end{array} \right)$. Deformations proportional to the second eigenvector are asymptotically negligible, so all deformations are locally equivalent to an overall rescaling followed by an asymptotically negligible deformation. In particular, any two Fibonacci tiling spaces are topologically conjugate, up to an overall rescaling.

Similar arguments apply to any 1-dimensional substitution tiling space where $\sigma^*$ acts on $\breve{H}^1$ via a Pisot matrix. The asymptotically negligible classes have codimension 1, so all deformations yield spaces that are topologically conjugate up to scale. In particular, for these substitutions, suspensions of subshifts (with all tiles having size 1) have the same qualitative properties as self-similar tilings.

For the Penrose tiling, $\breve{H}^1(\Omega, \mathbb{R}^2) = \mathbb{R}^5 \otimes \mathbb{R}^2 = \mathbb{R}^{10}$. The eigenvalues of $\sigma^*$ are $\phi$ and $1 - \phi$, each with multiplicity 4, and $-1$ with multiplicity 2. The multiplicity 4 for the large eigenvalue corresponds to the 4-dimensional family of linear transformations that can be applied to $\mathbb{R}^2$. (For self-similar tilings of $\mathbb{R}^d$, the leading eigenvalue of $\sigma^*$ always has multiplicity $d^2$.) Meanwhile, the two deformations with eigenvalue -1 break the 180-degree rotational symmetry of the tiling space. Thus, any combinatorial Penrose tiling space that maintains 180-degree rotational symmetry must be topologically conjugate to a linear combination applied to the “standard” Penrose tiling space.

For cut-and-project tilings, the asymptotically negligible classes depend on the shape of the “window”. When the window isn’t too complicated, there is an explicit description of these classes. This theorem applies even when $\breve{H}^1(\Omega, \mathbb{R}^d)$ is infinite-dimensional.

Theorem 3.7. [KS2] If the window of a cut-and-project scheme of codimension $n$ is a polytope, or a finite union of polytopes, then $\dim(\breve{H}^1_{an}(\Omega, \mathbb{R}^d)) = nd$. The elements of $\breve{H}^1_{an}(\Omega, \mathbb{R}^d)$ correspond to projections from $\mathbb{R}^{n+d}$ to $\mathbb{R}^d$, and all shape conjugacies amount to simply changing the projection by which points in the acceptance strip are sent to $\mathbb{R}^d$.

Besides the cohomology of strongly PE functions and forms, we can consider the weak PE cohomology of weakly PE functions and forms, and the mixed cohomology [Kel3]. We call a strongly PE form weakly exact if it can be written as $d$ of a weakly PE $(k-1)$-form. The $k$th mixed cohomology $H^k_{PE,m}(\mathbf{T})$ of a tiling $\mathbf{T}$ is the quotient of the closed strongly PE $k$-forms by the weakly exact $k$-forms. This should not be viewed as a subgroup of $H^k_{PE}(\mathbf{T}) \equiv \breve{H}^k(\Omega, \mathbb{R})$. Rather, it is a quotient of $H^k_{PE}(\mathbf{T})$ by those classes that can be represented by weakly exact
forms. This can be identified with a quotient of \( \tilde{H}^k(\Omega, \mathbb{R}) \). In dimension 1,
\[
H^1_{PE,m}(T) \equiv \tilde{H}^1(\Omega, \mathbb{R})/H^1_{an}(\Omega, \mathbb{R}).
\]
Since \( H^1_{an}(\Omega, \mathbb{R}^d) \) parametrizes shape conjugacies, this means that \( H^1_{PE,m}(T) \) parametrized deformations of a tiling space \( \Omega_T \) up to topological conjugacy rather than up to MLD equivalence [Kel3].

3.4. **Exact regularity.** A measure on a tiling space is equivalent to specifying the frequencies of all possible patches. Specifically, let \( P \) be a patch in a specific location (say, centered at the origin). Let \( U \) be an open set in \( \mathbb{R}^d \). Let \( \Omega_{P,U} \) be the set of all tilings \( T \) such that, for some \( x \in U \), \( T - x \) contains the patch \( P \). As long as \( U \) is chosen small enough, there is at most one \( x \in U \) that works. For any tiling \( T \), let \( freq_T(P) \) be the number of occurrences of \( P \), per unit area, in \( T \). That is, restrict \( T \) to a large ball, divide by the volume of the ball, and take a limit as the radius goes to infinity. The ergodic theorem says that this limit exists for \( \mu \)-almost every \( T \), with \( freq_T(P) = \mu(\Omega_{P,U})/Vol(U) \). If the tiling space is uniquely ergodic, then this statement applies to every \( T \), not just to almost every \( T \).

There are two natural questions. First, what are the possible values of \( freq_T(P) \)? Second, as we consider larger and larger balls, how quickly does the number of occurrences of \( P \) per unit area approach \( freq_T(P) \)? Both questions have cohomological answers.

**Theorem 3.8.** For each patch \( P \) and each sufficiently small open subset \( U \) of \( \mathbb{R}^d \), \( \mu(\Omega_{P,U})/Vol(U) \) takes values in the frequency module of \( X \).

**Proof.** Let \( i_P \) be a pattern-equivariant \( d \)-cochain that equals 1 on one of the tiles of \( P \) and is zero on all other tiles. Being of the top dimension, \( i_P \) is co-closed, and so represents a cohomology class. For any region \( R \), \( i_P(R) \) is just the number of occurrences of \( P \) in \( R \). The limiting number per unit area \( freq_T(P) \) is then the trace of the class of \( i_P \). \( \square \)

**Theorem 3.9.** [Sa4] Suppose that \( \tilde{H}^d(\Omega, \mathbb{Q}) = \mathbb{Q}^k \) for some integer \( k \). Then there exist patches \( P_1, \ldots, P_k \) with the following property: for any other patch \( P \), there exist rational numbers \( c_1(P), \ldots, c_k(P) \) such that, for any region \( R \) in any tiling \( T \in X \), the number of appearances of \( P \) in \( R \) equals \( \sum_{i=1}^k c_i(P)n_i(R) + e(P, R) \), where \( n_i(R) \) is the number of appearances of \( P_i \) in \( R \), and \( e(P, R) \) is an error term computable from the patterns that appear on the boundary of \( R \). In particular, the magnitude of \( e(P, R) \) is bounded by a constant (that may depend on \( P \)) times the measure of the boundary of \( R \). Furthermore, if \( \tilde{H}^d(\Omega) = \mathbb{Z}^k \) is finitely generated over the integers, then we can pick the coefficients \( c_i \) to be integers.

**Corollary 3.10.** If the patches \( P_1, \ldots, P_k \) have well-defined frequencies, then \( \Omega \) is uniquely ergodic and there exist uniform bounds for the convergence of all patch frequencies to their...
ergodic averages. If the regions $R$ are chosen to be balls, whose radii we denote $r$, then the number of $P$’s per unit area approaches $\sum c_i \text{freq}(P_i)$ at least as fast as one the frequency of one of the $P_i$’s approaches $\text{freq}(P_i)$, or as fast as $r^{-1}$, whichever is slower.

Note that this theorem and its corollary apply to all tiling spaces, and not just to substitution tiling spaces.

**Proof.** Using the isomorphism between Čech and pattern-equivariant cohomology, pick patches $P_1, \ldots, P_k$ such that the cohomology classes of $i_{P_i}$ are linearly independent. These classes then form a basis for $H^d_{PE}(X) = \mathbb{Q}^k$, and we can write $[i_{P_i}] = \sum c_i [i_{P_i}]$, where $[\alpha]$ denotes the cohomology class of the cochain $\alpha$. This means that there is a $(d-1)$-cochain $\beta$ such that $i_P = \sum c_i i_{P_i} + \delta \beta$. Then

$$\text{number of } P \text{ in } R = i_P(R) = \sum c_i i_{P_i}(R) + \delta \beta(R) = \sum c_i n_i(R) + \beta(\partial R).$$

Since $\beta$ is pattern-equivariant there is a maximum value that it takes on any $(d-1)$-cell, so the error term $\beta(\partial R)$ is bounded by a constant times the area of the boundary of $R$. Dividing by the volume of $R$, the deviation of the left-hand-side from $\text{freq}(P)$ is bounded by the deviation of $n_i(R)/\text{Vol}(R)$ from $\text{freq}(P_i)$ or by $|\partial R|/\text{Vol}(R) \sim r^{-1}$.

If $\tilde{H}^d(\Omega) = \mathbb{Z}^k$, then the same argument applies with the patches chosen such that $[i_{P_i}]$ are generators of $\mathbb{Z}^k$ and with integral coefficients $c_i$. \hfill \Box

When $\Omega$ is a 1-dimensional tiling space, it is possible to pick $R$ such that $\partial R$ is homologically trivial. Let $\beta$ be pattern-equivariant with radius $r_0$, and let $W$ be a word of length at least $2r_0$. Pick $R$ to be an interval that starts in the middle of one occurrence of $W$ and ends in the corresponding spot of another occurrence. Then $\delta \beta(R) = \beta(\partial R)$ vanishes, and the number of $P$’s in $R$ is exactly $\sum c_i(P) n_i(R)$. This is called exact regularity \cite{BBJS, SM}.

**3.5. Invariant measures and homology.** Exact regularity is dual to an earlier description \cite{BG} of invariant measures in terms of the real-valued homology $H_d(\Gamma^n, \mathbb{R})$ of the approximants. Recall that measures do not pull back, but instead push forward like homology classes: Given a measure $\mu$ on a space $X$ and a continuous map $f : X \to Y$, there is a measure $f_* \mu$ on $Y$. For any measurable set $S \subset Y$, $f_* \mu(S) := \mu(f^{-1}(S))$. Thus, a measure $\mu$ on a tiling space gives rise to a sequence of measures $\mu_n$ on the approximants $\Gamma^n$, with $(\rho_n)_* \mu_n = \mu_{n-1}$.

Integration gives a pairing between indicator $d$-cochains and measures, $\langle \mu, i_P \rangle = \text{freq}(P)$. This extends to a pairing between measures and cohomology, both for the tiling space and for each approximant. By the universal coefficients theorem, the dual space to the top cohomology group $H^d(\Gamma^n, \mathbb{R})$ is the top homology group $H_d(\Gamma^n, \mathbb{R})$, so we can view $\mu_n$ as living in $H_d(\Gamma^n, \mathbb{R})$, and $\mu$ as living in the inverse limit space $\limleftarrow H_d(\Gamma^n, \mathbb{R}), (\rho_n)_*$. (The identification of $\mu_d$ as an element of $H_d(\Gamma^n, \mathbb{R})$ can also be seen more directly. A measure
can be viewed as a chain satisfying certain “switching rules”, or “Kirchoff-like laws”. These rules are equivalent to saying that the boundary operator applied to $\mu_d$ is zero, i.e. that $\mu_d$ defines a homology class.)

The measure of any cylinder set is non-negative, so each $\mu_d$ must lie in the positive cone of $H_d(\Gamma^n, \mathbb{R})$. Not only is $\mu$ constrained to lie in the inverse limit of the top homologies of the approximants, but $\mu$ must lie in the inverse limit of the positive cones. All of the invariant measures on a tiling space can be determined from the transition matrices $(\rho_n)_*$, and in particular we can tell whether the tiling space is uniquely ergodic.

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