On edge-group choosability of graphs

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Abstract

In this paper, we study the concept of edge-group choosability of graphs. We say that $G$ is edge-$k$-group choosable if its line graph is $k$-group choosable. An edge-group choosability version of Vizing conjecture is given. The evidence of our claim are graphs with maximum degree less than 4, planar graphs with maximum degree at least 11, planar graphs without small cycles, outerplanar graphs and near-outerplanar graphs.

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1 Introduction

We consider only simple graphs in this paper unless otherwise stated. For a graph $G$, we denote its vertex set, edge set, minimum degree, and maximum degree by $V(G)$, $E(G)$, $\delta(G)$, and $\Delta(G)$, respectively. A plane graph is a particular drawing of a planar graph in the Euclidean plane. We denote the set of faces of a plane graph $G$ by $F(G)$. For a plane graph $G$ and $f \in F(G)$, we write $f = [u_1u_2\ldots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices on the boundary walk of $f$ enumerated clockwise. The degree of a face is the number of edge-steps in the boundary walk. Let $d_G(x)$, or simply $d(x)$, denote the degree of a vertex (or face) $x$ in $G$. A vertex (or face) of degree $k$ is called a $k$-vertex (or $k$-face). A 3-face $f = [u_1u_2u_3]$ is called a $(i, j, k)$-face if $(d(u_1), d(u_2), d(u_3)) = (i, j, k)$. For $v \in V(G)$, $N_G(v)$ is the set of all vertices of $G$ that are adjacent to $v$ in $G$. A 2-alternating cycle in a graph $G$ is a cycle of even length in which alternate vertices have degree 2.

A $k$-coloring of a graph $G$ is a mapping $\phi$ from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for every edge $xy$. A graph $G$ is $k$-colorable

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if it has a $k$-coloring. The chromatic number $\chi(G)$ is the smallest integer $k$ such that $G$ is $k$-colorable. A mapping $L$ is said to be a list assignment for $G$ if it supplies a list $L(v)$ of possible colors to each vertex $v$. A $k$-list assignment of $G$ is a list assignment $L$ with $|L(v)| = k$ for each vertex $v \in V(G)$. If $G$ has some $k$-coloring $\phi$ such that $\phi(v) \in L(v)$ for each vertex $v$, then $G$ is $L$-colorable or $\phi$ is an $L$-coloring of $G$. We say that $G$ is $k$-choosable if it is $L$-colorable for every $k$-list assignment $L$. The choice number or list chromatic number $\chi_l(G)$ is the smallest $k$ such that $G$ is $k$-choosable. To distinguish the objects by different notions we denote the line graph of a graph $G$ by $\ell(G)$. By considering colorings for $E(G)$, we can define analogous notions such as edge-$k$-colorability, edge-$k$-choosability, the chromatic index $\chi'(G)$, the choice index $\chi'_l(G)$, etc. Clearly, we have $\chi'(G) = \chi(\ell(G))$ and $\chi'_l(G) = \chi_l(\ell(G))$. The notion of list coloring of graphs has been introduced by Erdős, Rubin, and Taylor [9] and Vizing [20]. The following conjecture, which first appeared in [1], is well-known as the List Edge Coloring Conjecture.

**Conjecture 1.** If $G$ is a multi-graph, then $\chi'_l(G) = \chi'(G)$.

Although Conjecture 1 has been proved for a few special cases such as bipartite multigraphs [10], complete graphs of odd order [11], multicircuits [20], graphs with $\Delta(G) \geq 12$ that can be embedded in a surface of non-negative characteristic [2], and outerplanar graphs [21], it is regarded as very difficult. Vizing proposed the following weaker conjecture (see [15]).

**Conjecture 2.** Every graph $G$ is edge-$(\Delta(G) + 1)$-choosable.

Assume $A$ is an Abelian group and $F(G, A)$ denotes the set of all functions $f : E(G) \longrightarrow A$. Consider an arbitrary orientation of $G$. The graph $G$ is $A$-colorable if for every $f \in F(G, A)$, there is a vertex coloring $c : V(G) \longrightarrow A$ such that $c(x) - c(y) \neq f(xy)$ for each directed edge from $x$ to $y$. The group chromatic number of $G$, $\chi_g(G)$, is the minimum $k$ such that $G$ is $A$-colorable for any Abelian group $A$ of order at least $k$. The notion of group coloring of graphs was first introduced by Jaeger et al. [14].

The concept of group choosability is introduced by Král and Nejedlý [17]. Let $A$ be an Abelian group of order at least $k$ and $L : V(G) \longrightarrow 2^A$ be a list assignment of $G$. For $f \in F(G, A)$, an $(A, L, f)$-coloring under an orientation $D$ of $G$ is an $L$-coloring $c : V(G) \longrightarrow A$ such that $c(x) - c(y) \neq f(xy)$ for every edge $e = xy$, $e$ is directed from $x$ to $y$. If for each $f \in F(G, A)$ there exists an $(A, L, f)$-coloring for $G$, then we say that $G$ is $(A, L)$-colorable. The graph $G$ is $k$-group choosable if $G$ is $(A, L)$-colorable for each Abelian group $A$ of order at least $k$ and any $k$-list assignment $L : V(G) \longrightarrow \binom{A}{k}$. The minimum $k$ for which $G$ is $k$-group choosable is called the group choice number of $G$ and is denoted by $\chi_g(G)$. It is clear that the concept of group choosability is independent of the orientation on $G$. 
Graph $G$ is called edge-$k$-group choosable if its line graph is $k$-group choosable. The group-choice index of $G$, $\chi_{gl}'(G)$, is the smallest $k$ such that $G$ is edge-$k$-group choosable, i.e. $\chi_{gl}'(G) = \chi_{gl}((\ell(G)))$. It is easily seen that an even cycle is not edge-2-group choosable. This example shows that $\chi_{gl}'(G)$ is not generally equal to $\chi'(G)$. But we can extend the Vizing Conjecture as follows.

**Conjecture 3.** If $G$ is a multi-graph, then $\chi_{gl}'(G) \leq \Delta(G) + 1$.

Since $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$, as a sufficient condition, we have the following weaker conjecture.

**Conjecture 4.** If $G$ is a multi-graph, then $\chi_{gl}'(G) \leq \chi'(G) + 1$.

In this paper, we prove that Conjecture 3 (and consequently Conjecture 4) holds for certain classes of graphs. We shall see that Conjecture 3 is true for graphs with $\Delta(G) \leq 3$, planar graphs with $\Delta(G) \geq 11$, planar graphs without small cycles, outerplanar graphs, and near-outerplanar graphs.

## 2 Edge-group choosability of graphs with bounded degree

In this section, we aim to prove Conjecture 3 holds for a graph $G$ satisfying either $\Delta(G) \leq 3$ or planar with $\Delta(G) \geq 11$.

**Lemma 2.1** Let $l$ be a natural number, $v$ be a vertex of degree at most 2 of $G$ and $e$ be an edge adjacent to $v$. If $\chi_{gl}'(G-e) \leq \Delta(G)+l$, then $\chi_{gl}'(G) \leq \Delta(G)+l$.

**Proof.** Let $\Delta = \Delta(G)$, $D$ be an orientation of $\ell(G)$, $A$ be an Abelian group of order at least $\Delta + l$, $L : V(\ell(G)) \rightarrow \binom{A}{\Delta + l}$ be a $(\Delta + l)$-list assignment and $f \in F(\ell(G), A)$. Suppose that $G' = G - e$. Then $\ell(G') = \ell(G) - e$ and since $\chi_{gl}'(G') \leq \Delta + l$, there exists an $(A, L, f)$-coloring $c : V(\ell(G')) \rightarrow A$. For each $e' \in N_{\ell(G)}(e)$ we can consider, without loss of generality, $ee'$ to be directed from $e$ to $e'$. Then, since $|L(e)| = \Delta + l$ and $d_{\ell(G)}(e) \leq \Delta$, $|L(e) - \{c(e') + f(ee') : e' \in N_{\ell(G)}(e)\}| \geq 1$ and so there is now at least one color available for $e$. Thus we can color all edges of $G$. This completes the proof of lemma. \hfill \Box

An argument similar to the proof of Lemma 2.1 gives the following lemma.

**Lemma 2.2** Let $G$ be a graph with $\chi_{gl}'(G-e) < \chi_{gl}'(G)$ for each $e \in E(G)$. Then $\delta(\ell(G)) \geq \chi_{gl}'(G) - 1$.

**Lemma 2.3** Let $P_n$ and $C_n$ denote a path and a cycle of length $n$, respectively. Then

1. $\chi_{gl}(P_n) = 2$ and $\chi_{gl}(C_n) = 3$,
For any connected simple graph $G$, we have $\chi_{gl}(G) \leq \Delta(G) + 1$, with equality holds iff $G$ is either a cycle or a complete graph.

Immediately from Lemma 2.3 we have the following corollary.

**Corollary 2.4** $\chi'_{gl}(P_n) = \Delta(P_n) = 2$ and $\chi'_{gl}(C_n) = \Delta(C_n) + 1 = 3$.

**Theorem 2.5** Let $G$ be a graph with maximum degree $\Delta(G)$. If $\Delta(G) \leq 3$, then $\chi'_{gl}(G) \leq \Delta(G) + 1$ and if $\Delta(G) = 4$, then $\chi'_{gl}(G) \leq 6$.

**Proof.** It is clear that we can assume $G$ is connected. If $\Delta(G) = 1$, then $G = P_2$ and this theorem trivially holds. If $\Delta(G) = 2$, then $G = P_n$ or $G = C_n$ and the assertion holds by Corollary 2.4. It is clear that $\Delta(\ell(G)) \leq 4$ if $\Delta(G) \leq 3$ and $\Delta(\ell(G)) \leq 6$ if $\Delta(G) \leq 4$. The proof is completed by Lemma 2.3. ⊓⊔

The remainder of this paper consists of an investigation of Conjecture 3 for certain classes of planar graphs. Our first result shows that this conjecture is true for planar graphs with maximum degree at least 11.

**Lemma 2.6** [16] For every planar graph $G$ with minimum degree at least 3 there is an edge $e = uv$ with $d(u) + d(v) \leq 13$.

**Lemma 2.7** If $G$ is a planar graph with maximum degree $\Delta$, then

$$\chi'_{gl}(G) \leq \max\{\Delta + 1, 12\}.$$  

**Proof.** Let $G$ be a minimal counterexample to Lemma 2.7. By Lemmas 2.1 and 2.6, there exists $e \in V(\ell(G))$ with $d_{\ell(G)}(e) \leq 11$, which is a contradiction by Lemma 2.2. ⊓⊔

The truth of Conjecture 3 for planar graphs with maximum degree at least 11 immediately follows from Lemma 2.7. In fact we have the following.

**Corollary 2.8** Let $G$ be a planar graph with maximum degree $\Delta$.

1. If $\Delta \geq 11$, then $\chi'_{gl}(G) \leq \Delta + 1$,

2. If $\Delta \geq 10$, then $\chi'_{gl}(G) \leq \Delta + 2$.

3. **Edge-group choosability of planar graphs without small cycles**

In this section, we show that Conjecture 3 holds for planar graphs that contain no certain configurations. For planar graphs without 4-cycles we need the following structural lemma.

**Lemma 3.1** [19] If a connected plane graph $G$ with $\delta(G) \geq 3$ has no 4-cycles, then $G$ contains one of the following configurations.
(1) An edge \(xy\) with \(d(x) + d(y) \leq 7\),

(2) The subgraph \(G_1\) consisting of a 4-vertex \(v\) that is incident to two non-adjacent 3-faces \(f_1 = [vv_1v_2]\) and \(f_2 = [vv_3v_4]\), such that \(d(v_i) \geq 4\) for \(i = 1, 2, 3, 4\), and in \(\{v_1, v_2, v_3, v_4\}\) there is at most one vertex with degree greater than 4 (see Figure 1).

For the graph \(G_1\) of Lemma 3.1 set \(e_i = vv_i\), for \(i = 1, 2, 3, 4\), \(e_5 = v_3v_4\) and \(e_6 = v_1v_2\). Now we have the following result.

Lemma 3.2 Suppose that \(A\) is an Abelian group with \(|A| \geq 4\) and \(L : V(\ell(G_1)) \rightarrow 2^A\) an assignment for vertices of \(\ell(G_1)\) such that \(|L(e_5)| = 2\), \(|L(e_6)| = 1\), \(|L(e_1)| = 3\), and \(|L(e_i)| = 4\) for \(i = 2, 3, 4\). Then \(\ell(G_1)\) is \((A, L)\)-colorable. (see Figure 1).

![Figure 1: The graph \(G_1\) and its line graph.](image)

Proof. Let \(f \in F(\ell(G_1), A)\). Suppose that \(L = L(e_4) = \{a, b, c, d\}, L(e_2) = L', L(e_6) = \{h\}, f(e_4e_2) = f_1\) and \(f(e_6e_2) = f_2\). If \(L' \neq L - f_1\), say \(a - f_1 \notin L'\), we color \(e_4\) with \(a\). Then color \(e_6, e_1, e_5, e_3\) and \(e_2\) successively. So we may assume that \(L' = L - f_1\). If \(h - f_2 \in L'\), say \(h - f_2 = a - f_1\), then first color \(e_4\) with \(a\) and then color \(e_6, e_1, e_5, e_3\) and \(e_2\). So \(h - f_2 \notin L'\) and we color \(e_6, e_4, e_1, e_5, e_3, e_2\) successively.

Theorem 3.3 If \(G\) is a planar graph without 4-cycles with \(\Delta(G) \geq 5\), then \(G\) is edge-(\(\Delta(G) + 1\))-group choosable.

Proof. Let \(G\) be a minimal counterexample to Theorem 3.3 for some Abelian group \(A\) with \(|A| \geq \Delta(G) + 1\), a \((\Delta(G) + 1)\)-list assignment \(L : V(\ell(G)) \rightarrow \left(\frac{\Delta(G) + 1}{2}\right)\) and \(f \in F(\ell(G), A)\). By Lemma 2.1 and Lemma 2.5, we may assume \(\delta(G) \geq 3\). Using Lemma 3.1, suppose first that \(G\) contains an edge \(xy\) with \(d(x) + d(y) \leq 7\). Lemmas 2.2 and 2.5 yield the desired contradiction. Thus \(G\) has \(G_1\) as a subgraph. Remove the vertices of \(\ell(G_1)\) from \(\ell(G)\), and color the remaining vertices of \(\ell(G)\) from their lists, which is possible by the minimality of \(G\) as a counterexample and using Theorem 2.5. The numbers of colors available for vertices of \(\ell(G_1)\) are at least as stated in Lemma 3.2 and so \(\ell(G)\) is \((A, L, f)\)-colorable, which is impossible.

Theorem 3.3 together with Theorem 2.5 yields the following corollary.
Corollary 3.4 If $G$ is a planar graph without 4-cycles, then $\chi'_{gl}(G) \leq \Delta(G) + 2$.

Let $G$ be a planar graph. For $v \in V$ and $f \in F$, let $m_v(f)$ denote the number of times passing through $v$ of $f$ in clockwise order. Let $H_2$ be the subgraph induced by the edges incident with the 2-vertices of $G$. It is shown that $H_2$ contains a matching $M$ such that all 2-vertices in $H_2$ are saturated [24]. If $uv \in M$ and $d(u) = 2$, then $v$ is called the 2-master of $u$.

Lemma 3.5 If $G$ is a planar graph with maximum degree $\Delta(G) = 4$ such that $G$ has no cycles of length from 4 to 14, then $G$ contains one of the following configurations.

1. An edge $uv$ with $\min\{d(u), d(v)\} \leq 2$ and $d(u) + d(v) \leq 5$,

2. A 4-vertex $v$ incident with two 3-faces $vv_1v_2$ and $vv_3v_4$ with $d(v_1) = d(v_4) = 2$.

Proof. Let $G$ does not contain each of the mentioned configurations. We define the initial charge function $w(x) = d(x) - 4$ for each $x \in V(G) \cup F(G)$. It follows from Euler’s formula that $\sum_{x \in V(G) \cup F(G)} w(x) < 0$. We define the new charge function $\hat{w}(x)$ on $G$ as follows:

(R1) Each $r(\geq 15)$-face $f$ gives $(1 - \frac{4}{r})m_v(f)$ to its incident vertex $v$ if $v$ is a cut vertex and gives $1-4/r$, otherwise.

(R2) Each 2-vertex receives $19/24$ from its neighbors if it is incident with a 3-face and receives $8/15$ from its 2-master, otherwise.

(R3) Each 3-face receives $1/3$ from its incident vertices.

It is obvious that $\hat{w}(f) = 0$ for any face $f$. Let $v$ be an arbitrary vertex of $G$. First consider the case of $d(v) = 2$. If it is incident with a 3-face, then its other incident face $f$ must have degree at least 16. Since $G$ does not have configuration of (1) any neighbor of $v$ should be of degree 4. Hence, they can not be 2-vertices. It follows that $v$ receives at least $1 - \frac{4}{16} = 3/4$ from $f$ and $19/12$ from its neighbors, and gives $1/3$ to its incident 3-face. Otherwise, $v$ receives at least $22/15$ from its incident faces and $8/15$ from its 2-master. Hence, $\hat{w}(x) \geq w(x) + \min\{3/4 + 19/12 - 1/3, 22/15 + 8/15\} = 0$. Now consider the case of $d(v) = 3$. $v$ receives at least $22/15$ from its incident faces. Hence, $\hat{w}(v) \geq w(v) + 22/15 - 1/3 = 2/15 > 0$. If $d(v) = 4$ and it is incident with two 3-faces, then $v$ is adjacent to at most one 2-vertex since $G$ does not have configuration of (2). It follows that $\hat{w}(v) \geq w(v) + 22/15 - (2/3 + 19/24) = 1/120 > 0$. Otherwise, it receives at least $3 \times 11/15$ from its incident faces, and gives at most $1/3$ to its incident 3-face and $19/24 + 8/15$ to its adjacent 2-vertices. It follows that $\hat{w}(v) \geq w(v) + 33/15 - (1/3 + 19/24 + 8/15) = 13/24 > 0$. This implies that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} \hat{w}(x) > 0$, a contradiction. \(\square\)
Let $G$ be a planar graph with maximum degree $\Delta(G) = 4$ such that $G$ has no cycle of length $i$ for $4 \leq i \leq 14$. If $G$ is not edge-$\Delta(G)$-group choosable, then an argument similar to the proof of Theorem 3.3 shows that $G$ does not contain any of configurations mentioned in Lemma 3.5, this leads to a contradiction. So we have the following.

**Theorem 3.6** If $G$ is a planar graph with maximum degree $\Delta(G) = 4$ such that $G$ has no cycles of length from $4$ to $14$, then $\chi'_g(G) = \Delta(G)$.

In the sequel, we shall study edge-group choosability of plane graphs without 5-cycles, plane graphs without 5-cycles with a chord and plane graphs without induced 5-cycles. A cycle $C$ of length $k$ of a graph $G$ is called a $k$-hole (resp. $k$-net) if $C$ has no (resp. at least one) chord in $G$. The following is a structural lemma for plane graphs without 5-cycles.

**Lemma 3.7** If a plane graph $G$ with $\delta(G) \geq 3$ has no five cycles, then there exists an edge $xy$ of $G$ such that $d(x) = 3$ and $d(y) \leq 5$.

A subgraph $H$ of a plane graph $G$ is called a cluster if $H$ consists of a non-empty minimal set of 3-faces in $G$ such that no other 3-face is adjacent to a member of this set. A structural lemma for plane graphs with $\Delta(G) = 6$ and without 5-nets is as follows.

**Lemma 3.8** Let $G$ be a planar graph with $\Delta(G) = 6$ and without 5-nets. Then $G$ contains one of the following configurations.

1. An edge $xy$ with $d(x) + d(y) \leq 8$,
2. A 4-cycle $C = uwxv$ such that $d(u) = d(w) = 3$ and $d(v) = d(x) = 6$,
3. The subgraph $G_2$ consisting of a 6-vertex $x$ incident to four 3-faces $[xx_1x_2]$, $[xx_2x_3]$, $[xx_4x_5]$, $[xx_5x_6]$ such that $d(x_1) = d(x_5) = 3$, $d(x_3) = 4$ and $d(x_2) = d(x_4) = d(x_6) = 6$ (see Figure 2),
4. The graph $G_3$ consisting of a cluster with $\{u, x, y, z\}$ as its vertex set such that $d(u) = 3$, $d(x) = d(y) = d(z) = 6$, and satisfying the following properties:
   (a) $x$ is incident to $(3, 6, 6)$-faces $[uxy]$, $[uxz]$ and $[xx_1x_2]$ with $d(x_1) = 3$,
   (b) $y$ is incident to $(3, 6, 6)$-faces $[uxy]$, $[yuz]$, $[yy_1y_2]$ and $[yy_2y_3]$ with $d(y_2) = 3$.

For planar graphs without 5-nets, we need the following lemma.

**Lemma 3.9** Every planar graph $G$ with $\delta(G) \geq 3$ and without 5-nets contains an edge $xy$ such that $d(x) + d(y) \leq 9$.

For studying Conjecture 3 for planar graphs with maximum degree 5 and without 5-nets we need the following structural lemma.

**Lemma 3.10** Consider the following configurations.
(1) An edge $xy$ with $d(x) + d(y) \leq 7$,

(2) A 4-cycle $vwux$ such that $d(v) = d(w) = 3$ and $d(u) = d(x) = 5$,

(3) The subgraph $G_4$ consisting of a 5-vertex incident to three $(3, 5, 5)$-faces,

(4) The subgraph $G_5$ consisting of a 4-vertex $u$ adjacent to three 4-vertices $x, y, z$ and incident to a 3-face $[uxy]$.

Let $G$ be a planar graph with $\Delta(G) = 5$. Then

(a) If $G$ is without 5-nets and 6-nets, then it contains at least one of the graphs mentioned in (1), (2) and (3),

(b) If $G$ is without 4-nets and 5-nets, then it contains at least one of the graphs mentioned in (1), (2) and (4).

Lemma 3.11 [18] Let $G$ be a connected planar graph with $\delta(G) \geq 2$. If $G$ contains no 5-cycles or no 6-cycles, then $G$ contains either an edge $xy$ with $d(x) + d(y) \leq 9$ or a 2-alternating cycle.

Theorem 3.12 If $G$ is a planar graph without 5-cycles with maximum degree $\Delta$, then $G$ is edge-$(\Delta + 2)$-group choosable.

Proof. Let $G$ be a minimal counterexample to Theorem 3.12. Then by Lemma 2.1 and Theorem 2.5 we have $\delta(G) \geq 3$, $\Delta \geq 5$ and by Lemma 3.7, there exists a vertex $e \in V(\ell(G))$ with $d(\ell(G))(e) \leq 6$, which is impossible by Lemma 2.2. $\square$

For graphs with maximum degree at least 7, we have a stronger result as follows.

Theorem 3.13 If $G$ is a planar graph with maximum degree $\Delta \geq 7$ and without 5-cycles, then $\chi'_{gl}(G) \leq \Delta + 1$.

Proof. Let $G$ be a minimal counterexample to Theorem 3.13 for some Abelian group $A$ with $|A| \geq \Delta(G) + 1$, a $(\Delta(G) + 1)$-list assignment $L : V(\ell(G)) \rightarrow (A)_{\Delta(G)+1}$ and $f \in F(\ell(G), A)$. Then by Theorem 3.12 and Lemma 2.1 $\delta(G) \geq 3$ and we can apply Lemma 3.11. Hence there exists a vertex $e \in V(\ell(G))$ with $d(\ell(G))(e) \leq 7$, which is impossible by Theorem 3.12 and Lemma 2.2 $\square$

Theorem 3.14 Let $G$ be a planar graph without 5-nets,

(1) If $\Delta(G) \geq 6$, then $\chi'_{gl}(G) \leq \max\{8, \Delta(G) + 1\}$,

(2) If $\Delta(G) = 5$ and $G$ contains no 4-nets or 6-nets, then $\chi'_{gl}(G) \leq 7$.

Proof. (1) Let $k = \max\{8, \Delta(G) + 1\}$ and $G$ be a minimal counterexample to Theorem 3.14. Then there is an Abelian group $A$ with $|A| \geq k$, a $k$-assignment $L : V(\ell(G)) \rightarrow (A)_k$ and $f \in F(\ell(G), A)$ such that $\ell(G)$ is not $(A, L, f)$-colorable. Suppose first that $\Delta(G) = 6$. By Lemma 3.8, we consider
Figure 2: The graph $G_2$ and its line graph.

four cases as follows.

Cases 1, 2. $G$ contains an edge $xy$ with $d(x) + d(y) \leq 8$ or a 4-cycle $C = uvwx$ such that $d(u) = d(w) = 3$ and $d(v) = d(x) = 6$. Both cases lead to a contradiction by an argument similar to the proofs of Lemma 2.1 and Theorem 3.13.

Case 3. $G$ contains graph $G_2$ of Lemma 3.8 (see Figure 2). Remove the vertices of $\ell(G_2)$ from $\ell(G)$ and color the remaining vertices of $\ell(G)$ from their lists, which is possible by the minimality of $G$ as a counterexample. There are now $4$, $4$, $6$, $7$, $5$, $8$, $3$, $4$, $4$, $4$ colors available for the vertices $e_1, \ldots, e_{10}$, respectively where $e_1 = xx_4$, $e_2 = xx_6$, $e_3 = xx_3$, $e_4 = xx_1$, $e_5 = xx_2$, $e_6 = xx_5$, $e_7 = x_2x_3$, $e_8 = x_1x_2$, $e_9 = x_4x_5$, and $e_{10} = x_5x_6$. It is easily seen that we can color these vertices in their order. Thus $G$ is not a counterexample, which is a contradiction.

Case 4. $G$ contains graph $G_3$ of Lemma 3.8. The proof is similar to case 3.

If $\Delta(G) > 6$, by applying Lemmas 2.1 and 3.9 the assertion holds.

(2) If $\Delta(G) = 5$ and $G$ has no 4-nets, apply Lemma 3.10(b) and if $G$ has no 6-nets, apply Lemma 3.10(a).

The structure of planar graphs without non-induced 5-cycles is given in the following lemma.

**Lemma 3.15** [4] Let $G$ be a planar graph without non-induced 5-cycles. Then $G$ contains one of the following configurations.

1. An edge $uv$ with $d(u) + d(v) \leq \max\{8, \Delta(G) + 2\}$.

2. An even cycle $C : v_1v_2 \ldots v_{2n}$ with $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 3$ and $d(v_2) = d(v_4) = \cdots = d(v_{2n}) = \Delta(G)$.

**Theorem 3.16** If $G$ is a planar graph without non-induced 5-cycles, then $\chi_g'(G) \leq \max\{7, \Delta + 2\}$.

**Proof.** Applying Lemma 3.15 the proof is similar to the proof of Theorem 3.13.

$\Box$
If planar graph $G$ without non-induced 5-cycles in addition contains no even cycles, we can replace $\Delta + 2$ by $\Delta + 1$ in Theorem 3.16.

**Corollary 3.17** If $G$ is a planar graph without non-induced 5-cycles and without even cycle $C : v_1v_2 \ldots v_{2n}$ with $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 3$ and $d(v_2) = d(v_4) = \cdots = d(v_{2n}) = \Delta(G)$, then $\chi'_{gl}(G) \leq \max\{7, \Delta + 1\}$.

We finish this section by studying Conjecture 3 on planar graphs without 3-, 6- or 7-cycles. Because the proofs are similar to the proofs we did before, we omit them. The structural lemma for triangle free planar graphs is as follows.

**Lemma 3.18** [27] Let $G$ be a triangle-free plane graph with $\Delta(G) \geq 5$. If $d(x) + d(y) \geq \Delta(G) + 3$ for every edge $xy \in E(G)$, then $\Delta(G) = 5$ and $G$ contains a 4-face incident with two 3-vertices and two 5-vertices.

The following lemmas show the structure of planar graphs without adjacent triangles and without 6-cycles, respectively.

**Lemma 3.19** [13] Let $G$ be a plane graph without adjacent triangles. Then $G$ contains one of the following configurations.

1. An edge $uv$ with $d(u) + d(v) \leq \max\{8, \Delta + 2\}$,
2. A 4-cycle $vwxu$ such that $d(u) = d(w) = 3$ and $d(v) = d(x) = \Delta(G)$,
3. The subgraph $G_6$ consisting of a 6-vertex $v$ incident with two $(3, 6)$-faces and one $(4, 5, 6)$-face.

**Lemma 3.20** [22] If $G$ is a plane graph without 6-cycles and $\delta(G) \geq 3$, then there is an edge $xy \in E(G)$ such that $d(x) + d(y) \leq 8$.

**Lemma 3.21** [13] Every planar graph $G$ without 7-cycles contains one of the following configurations.

1. An edge $uv$ with $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$,
2. A 4-cycle $vwxu$ such that $d(u) = d(w) = 3$ and $d(v) = d(x) = \Delta(G)$.

Using Lemmas 3.18, 3.19, 3.20 and arguments similar to the proofs we did before, we can summarize all results on Conjecture 3 for planar graphs without 3-, 6- or 7-cycles in the following theorem.

**Theorem 3.22** Let $G$ be a planar graph,

1. If $G$ contains no 3-cycles and $\Delta(G) \geq 6$, then $\chi'_{gl}(G) \leq \Delta(G) + 1$,
2. If $G$ contains no 3-cycles and $\Delta(G) = 5$, then $\chi'_{gl}(G) \leq 7$,
3. If $G$ contains no adjacent triangles, then $\chi'_{gl}(G) \leq \max\{7, \Delta(G) + 2\}$,
(4) If $G$ contains no 6-cycles, then $\chi'_{gl}(G) \leq \max\{7, \Delta(G) + 1\}$,

(5) If $G$ contains no 7-cycles, then $\chi'_{gl}(G) \leq \max\{8, \Delta(G) + 2\}$.

As an immediate consequence of Theorem 3.22 and Theorem 2.5, we have the following.

**Corollary 3.23** If $G$ is a planar graph without 6-cycles or without 3-cycles, then $\chi'_{gl}(G) \leq \Delta(G) + 2$.

### 4 Edge-group choosability of outerplanar and near-outerplanar graphs

A planar graph is called outerplanar if it has a drawing in which each vertex lies on the boundary of the outer face. It is well-known that a graph is outerplanar iff it contains neither $K_4$ nor $K_{2,3}$ as a minor (see for example [7]). In this section, we see that Conjecture 3 is true for outerplanar graphs and near-outerplanar graphs, i.e. graphs that are either $K_4$-minor- free or $K_{2,3}$-minor-free.

**Lemma 4.1** If $G$ is an outerplanar graph, then at least one of the following cases holds.

1. $\delta(G) = 1$,
2. There exists an edge $uv$ such that $d(u) = d(v) = 2$,
3. There exists a 3-face $uxy$ such that $d(u) = 2, d(x) = 3$,
4. $G$ contains the graph $G_7$ consisting of two 3-faces $xu_1v_1$ and $xu_2v_2$ such that $d(u_1) = d(u_2) = 2$ and $d(x) = 4$ and these five vertices are all distinct.

This lemma lets us to prove that Conjecture 3 is true for outerplanar graphs. In fact, we prove a stronger result.

**Theorem 4.2** Let $G$ be an outerplanar graph with maximum degree $\Delta$. Then

1. If $\Delta < 5$, then $\chi'_{gl}(G) \leq \Delta + 1$,
2. If $\Delta \geq 5$, then $\chi'_{gl}(G) \leq \Delta$.

**Proof.** We prove the second part of the theorem, the proof of the first part is similar. Let $G$ be a minimal counterexample to the second part of Theorem 4.2. Then there is an Abelian group $A$ with $|A| \geq \Delta$, a $\Delta$-assignment $L : V(\ell(G)) \rightarrow \left(\frac{A}{\Delta}\right)$ and $f \in F(\ell(G), A)$ such that $\ell(G)$ is not $(A, L, f)$-colorable. By Lemma 4.1 we consider four cases as follows.

Case 1. $\delta(G) = 1$. Then $G$ contains an edge $uv$ such that $d(u) = 1$ and $d(v) \leq \Delta$. Hence $\ell(G)$ contains a vertex $e = uv$ with $d_{\ell(G)}(e) \leq \Delta - 1$, a contradiction by the first part of this theorem and Lemma 2.2.
Case 2. There exists an edge $uv$ such that $d(u) = d(v) = 2$. Then $\ell(G)$ contains a vertex $e$ with $d_{\ell(G)}(e) = 2$, a contradiction by the first part of this theorem and Lemma 2.2.

Case 3. There exists a 3-face $C: uxy$ such that $d(u) = 2$, $d(x) = 3$. Remove the vertices of $\ell(C)$ from $\ell(G)$ and color the remaining vertices of $\ell(G)$ from their lists. There are now at least $1, 2, 4$ colors available for the vertices of $\ell(C)$. It is easily seen that we can color these vertices. Thus $G$ is not a counterexample which is a contradiction.

Case 4. $G$ contains the subgraph $G_7$ of Lemma 4.1 (see Figure 3). Remove the vertices of $\ell(G_7)$ from $\ell(G)$ and color the remaining vertices of $\ell(G)$ from their lists, which is possible by the minimality of $G$ as a counterexample and using the second part of this theorem. There are now at least $2, 2, 2, 2, 5, 5$ colors available for the vertices of $\ell(G_7)$. It is easily seen that we can color these vertices in the above order. Thus $G$ is not a counterexample which is a contradiction. $\dashv$

A graph is called series-parallel if it has no subgraph isomorphic to a subdivision of $K_4$. It is well-known [8] that every simple series-parallel graph has a vertex of degree at most two. Thus using Lemma 2.1, we have the following corollary.

Corollary 4.3 If $G$ is a simple series-parallel graph, then $\chi'_{gl}(G) \leq \Delta(G) + 1$. In particular, every $K_4$-minor-free graph $G$ is edge-$(\Delta(G) + 1)$-group choosable.

In the following, we will prove that Conjecture 3 holds for $(K_2^c + (K_1 \cup K_2))$-minor-free graphs, where $K_2^c + (K_1 \cup K_2)$ is the graph obtained from the union of $K_1$ and $K_2$ and joining them by $K_2^c$, or, equivalently, it is the graph obtained from $K_{2,3}$ by adding an edge joining two vertices of degree 2. This implies that Conjecture 3 is true for $K_{2,3}$-minor-free graphs.

Lemma 4.4 [12] Let $G$ be a $(K_2^c + (K_1 \cup K_2))$-minor-free graph. Then each block of $G$ is either $K_4$-minor-free or isomorphic to $K_4$.

A graph $G$ is called $D$-group choosable if it is $(A, L)$-colorable for every Abelian group $A$ with $|A| \geq \Delta(G)$ and every list assignment $L : V(G) \rightarrow 2^A$ with $|L(v)| = d(v)$ for each vertex $v$. There is a characterization of all $D$-group choosable graphs in [6] as follows.
Theorem 4.5 A connected graph $G$ is not $D$-group choosable iff every block of $G$ is either complete or cycle.

By Theorem 4.5 it is easily seen that $\ell(K_4)$ is $D$-group choosable and so $\chi_{gl}'(K_4) \leq 4$. Moreover, it is well-known [25] that a $K_4$-minor-free graph $G$ with $|V(G)| \geq 4$ has at least two non-adjacent vertices with degree at most 2. By modifying some proofs in [12], we have the following result.

Theorem 4.6 If $G$ is a $(K_2^c + (K_1 \cup K_2))$-minor-free graph, then $\chi_{gl}'(G) \leq \Delta(G) + 1$. In particular, every $K_{2,3}$-minor-free graph $G$ is edge-$(\Delta(G) + 1)$-group choosable.

Proof. Let $G$ be a minimal counterexample to Theorem 4.6. Then there is an Abelian group $A$ with $|A| \geq \Delta(G) + 1$, a $(\Delta(G) + 1)$-assignment $L : V(\ell(G)) \rightarrow \left(\frac{A}{\Delta(G)+1}\right)$ and $f \in F(\ell(G), A)$ such that $\ell(G)$ is not $(A,L,f)$-colorable. Clearly $G$ is connected, $G \not\cong K_4$ and by Lemma 2.1 $\delta(G) \geq 3$. If $\Delta(G) = 3$, then $G$ is 3-regular and not complete, a contradiction follows since $\ell(G)$ is 4-regular and so it is $D$-group choosable by Theorem 4.5. So we may assume that $\Delta(G) \geq 4$. By Lemma 4.4 every block of $G$ is either $K_4$-minor-free or isomorphic to $K_4$. We first show that $G$ is not 2-connected. Suppose on the contrary that $G$ is 2-connected. Then $G$ is $K_4$-minor free since $\Delta(G) \geq 4$ and by Corollary 4.3 $\chi_{gl}'(G) \leq \Delta(G) + 1$, a contradiction. Thus $G$ is not 2-connected. Let $B$ be an end-block of $G$ with cut-vertex $z_0$. Clearly $B \not\cong K_2$. Moreover, if $B \cong K_4$, it is easily seen that an $(A,L,f)$-coloring of $\ell(G) - V(\ell(B))$ can be extended to an $(A,L,f)$-coloring of $\ell(G)$. This contradiction shows that $B \not\cong K_4$. Hence using Lemma 4.4 $B$ is $K_4$-minor-free, and so it contains at least two vertices of degree at most 2. Hence $G$ has at least one vertex of degree at most 2, this contradiction completes the proof of the theorem. $\square$

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