Quantum Mechanical Motion of Relativistic Particle in Non-Continuous Spacetime

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Abstract

The quantum mechanical motion of a relativistic particle in a non-continuous spacetime is investigated. The spacetime model is a dense, rationale subset of two-dimensional Minkowski spacetime. Solutions of the Dirac equation are calculated using a generalized version of Feynman’s checkerboard model. They turn out to be closely related to the continuum propagator.

Keywords: Dirac equation, Feynman checkerboard, discrete spacetime

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1 Introduction

Continuity and a metric of Lorentz signature are fundamental properties associated with spacetime in the framework of Special and General Relativity. Though intuitive, the assumption of continuity is debatable because of quantum mechanical arguments. Discrete models of spacetime have been investigated thus by various authors [1, 2, 3, 4, 5].

The modern idea that space could be discrete goes back to Riemann [6] who reflected about a natural measure of space. More recent attempts to consider spacetime as discrete either aim at resolving divergence problems in Quantum Field Theory at a fundamental level or seek to reconcile General Relativity and Quantum Theory to form a unified theory.

In this paper we investigate possible effects, which the hypothetical discreteness of spacetime could have on Quantum Theory. To this end,
Feynman’s path integral approach to Quantum Theory [7, ?] provides an ideal framework since it naturally accounts for spacetime properties. For continuous spacetime, the path integral formalism has been shown to be completely equivalent to ordinary wave mechanics. However, for a non-continuous spacetime one should not necessarily expect equivalent findings. To gain insight into the possible differences between quantum mechanics in the framework of common continuous spacetime and a non-continuous spacetime model, we seek solutions for the Dirac equation.

As has been pointed out by Feynman and Hibbs [7], the retarded propagator of the 1 + 1 dimensional Dirac equation

\[ i\frac{\partial \Psi}{\partial t} = -i\sigma_z \frac{\partial \Psi}{\partial x} - \sigma_x \Psi \]  

(with units \( c = \hbar/m = 1 \)) can be obtained from a random walk model in which the particle motion is restricted to movements either forward or backward at the speed of light. We consider this model in the framework of a particular non-continuous spacetime model [8] and demonstrate by an explicit calculation that the model yields common solutions of the Dirac equation.

The paper is organized as follows: Section 2 gives a short review of relevant properties of the non-continuous Minkowskian spacetime model described in [8]. In section 3 we calculate solutions of the Dirac equation. Section 4 summarizes and discusses the results.

2 Non-Continuous Minkowskian Spacetime

The two dimensional spacetime model considered here is defined by the subset \( M \) of \( \mathbb{R}^2 \)

\[ M = \{ t, x \} = \left\{ \frac{n}{m}(p^2 + q^2), \frac{n}{m}(p^2 - q^2) \right\} \subset \mathbb{R}^2, \quad n, m, p, q \in \mathbb{Z} \setminus \{0\} \]  

In the following, \( n, m, p, q \) may take any value of the indicated range and are not to be considered as fixed. Elements of \( M \) correspond to spacetime points with temporal and spatial coordinates \( (t, x) \). \( M \) is rational, hence countable, i.e. of cardinality \( \aleph_0 \) while \( \mathbb{R}^2 \) is of cardinality \( \aleph_1 \). With respect to this difference we denote \( M \) as non-continuous.
The set $M$ is invariant under the transformations

$$\phi : M \rightarrow M,$$  \hspace{1cm} (3)

and

$$\phi(s) = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} s$$

$$= \frac{1}{2pq} \begin{pmatrix} p^2 + q^2 & -(p^2 - q^2) \\ -(p^2 - q^2) & p^2 + q^2 \end{pmatrix} s,$$

where $p, q \in \mathbb{Z} \setminus \{0\}$.

The parameter

$$v = v(t, x) = \frac{x}{t} = \frac{p^2 - q^2}{p^2 + q^2}$$  \hspace{1cm} (4)

corresponds to the velocity defined as ratio of space $x$ and time $t$ displacements measured from the origin of the coordinate system. As illustrated by the second line of (4), the transformation $\phi$ maps rational spacetime coordinates $(t, x)$ onto rational ones $(t', x')$.

It is straightforward to verify that (i) the elements $\varphi \in \phi$ map onto $M$, that (ii) the group $\phi$ with respect to matrix multiplications $\circ$ possesses group structure and that (iii) $(\phi, \circ)$ is a subgroup of the common 1 + 1 dimensional Lorentz group (using natural units). According to these properties and the non-continuity of $M$ we denote $M$ as non-continuous Minkowskian spacetime. In [8] it has been shown that in the framework of the d-space formalism [9, 10, 11] the set $M$ corresponds to a spacetime model which is nowhere diffeomorphic to $\mathbb{R}^2$ but possesses the key properties of the common macrophysical spacetime. In the same context $M$ may be considered as a massless solution of the Einstein equations. Note that in contrast to other discrete spacetime models (e.g. [4, 5, 12] and references therein) which intrinsically violate relativistic covariance and introduce an observationally not supported minimal length, the spacetime model considered here exhibits a generalized form of covariance. Since the set $M$ is dense in $\mathbb{R}^2$, it doesn’t introduce a minimal length as well.

The following sections are based on the spacetime model (2). We will consider light-cone coordinates $(r, l) = \left(\frac{1}{2}(t+x), \frac{1}{2}(t-x)\right)$ in order to simplify calculations. In light-cone coordinates $M$ becomes

$$M^r = \{r, l\} = \left\{ \frac{n}{m} p^2, \frac{n}{m} q^2 \right\} \subset \mathbb{R}^2$$  \hspace{1cm} (5)

\[ m, n, p, q \in \mathbb{Z} \setminus \{0\}. \]
To end this section we note that since the set $M$ is dense, it is possible to formulate differential equations without referring to a calculus of finite differences.

### 3 Quadratic Checkerboard Model

In [7] Feynman and Hibbs described a model for the quantum mechanical motion of a relativistic electron. Solutions of the 1+1 dimensional Dirac equation are obtained by summing over all possible particle trajectories consisting of movements either forward or backward at the speed of light. Assuming natural units $c = \hbar/m = 1$, the motion of the particle corresponds to a sequence of straight path segments of slope $\pm 45^\circ$ in the $x$-$t$ plane. The retarded propagator $\psi_{\delta\gamma}(x,t)$ of the Dirac equation is obtained from the limiting process (e.g. [7,13])

$$
\psi_{\delta\gamma}(x,t) = \lim_{N \to \infty} A_{\delta\gamma}(\epsilon) \sum_{R \geq 0} N_{\delta\gamma}(R)(i\epsilon)^R
$$

$N$ is the number of segments with constant length $\epsilon = t/N$ between the start point (which is assumed to be the origin of the coordinate system) and the end point $(x,t)$ of the path. $R$ denotes the number of bends while $N_{\delta\gamma}(R)$ stands for the total number of paths consisting of $N$ segments with $R$ bends. The indices $\gamma$ and $\delta$ correspond to the directions forward or backward at the path’s start and end points, respectively, and refer to the components of $\psi$. $A_{\delta\gamma}(\epsilon)$ accounts for the appropriate normalization.

Since the early 80’ies, the checkerboard model of relativistic particle motion has been subject to anew interest (see e.g. [13,14,15,16,17,18,19]). In [19] it has been observed that paths with fixed start and end points have $R - 1$ degrees of freedom, i.e. the last bend of a path is fully specified by the location of the $R - 1$ preceding bends and the end point of the path. As well it has been shown that the normalization constant $A_{\delta\gamma}(\epsilon) \equiv 1$ if only the $R - 1$ bends enter the calculation which actually define the path of the particle. The general intuition behind this is the idea that the origin of the divergence problems of Quantum Field Theory could be related to an over-specification of the theory. Here, the corresponding interpretation of the checkerboard model and the calculation scheme described in [19] are adapted.

In the following we demonstrate by an explicit calculation that expression (6) when generalized to account for the key properties of $M$ yields solutions
of the Dirac equation. Before starting with the calculation, some remarks about the limiting process (3) and the terminology are in place. First observe the structure of (3). $\psi_\gamma(x,t)$ is defined in the limit $N \rightarrow \infty$ where $N$ is the number of path segments a path has. The length of each of these path segments is $\epsilon = t/N$. Bends of the particle trajectory occur only at boundaries of path segments. This is equivalent to consider a particle moving on a rectangular spacetime lattice with equal spacing length (or ‘resolution’) $\epsilon$. Consider now the expression

$$\sum_{R \geq 0} N_{\phi\gamma}(R)(i)R.$$ (7)

which is subject to the limit $N \rightarrow \infty$. For a given $N$, it is weighted sum of the number $N_{\phi\gamma}(R)$ of possible paths with $N$ segments and $R \geq 0$ bends linking the start and the end point under consideration. The complex weight (amplitude) each path is contributing is $(i\epsilon)^R$, i.e. each bend of a path contributes $(i\epsilon)$. As has been demonstrated (e.g. [7, 13]), taking the limit $N \rightarrow \infty$ yields solutions of the Dirac equation.

When adapting Feynman’s checkerboard model to the discrete spacetime $M$ we will follow as closely as possible the scheme outlined above. Two differences have to be taken into account, the first of which being a generalization. First, it is supposed that path segments can be of non-uniform length $\epsilon_n \neq \text{const.}$ and that the contribution of each bend to the overall amplitude of a path is proportional to the segment length $\epsilon_n$ immediately preceding the bend. This leads to the generalized form

$$\psi_\gamma(x,t) = \lim_{N \rightarrow \infty} A_{\phi\gamma}(\epsilon) \sum_{R \geq 0} N_{\phi\gamma}(R)(i)R \prod_{n=1}^{R} \epsilon_n.$$ (8)

of (3). Second, only particle paths on $M$ will be considered. These path are characterized by bends occurring at spacetime points $(t,x)$ satisfying

$$\frac{x}{t} = \frac{p^2 - q^2}{p^2 + q^2}.$$ (9)

In terms of $M^*$, the light-cone representation of $M$, this condition becomes

$$\frac{r}{l} = \frac{p^2}{q^2}.$$ (10)

where $(r,l) = \left(\frac{1}{2}(t+x), \frac{1}{2}(t-x)\right)$. Points of the spacetime lattice should account for restriction (10). We achieved this by considering quadratic spacetime lattices whose lattice points $(j_r, j_l) \in \mathbb{N}^2$ have, up to scaling, spacetime
lightcone coordinates \((r,l) = (j_r^2, j_l^2)\). The spacetime coordinates of lattice points of this kind naturally account for condition (1). Figure 1 shows an example for a path with \(N = 8\) segments. With respect to the modified properties of the spacetime lattice, the model is denoted as quadratic checkerboard model.

From the definition of the quadratic spacetime lattice (or directly from Figure 1) it is evident that path segments \(\epsilon_n\) are of non-uniform length and are depending on the lattice location \((j_r, j_l)\). We only will be interested in the length of paths segment preceding a bend and bends of a path to the right and to the left will be considered separately in the following. Observing that the length of a path segment preceding a bend to the right (left) is uniquely specifying by the lattice coordinate \(j_r\) \((j_l)\), the indexes \(r,l\) can be dropped. Then, the length of a path segment immediately following a bend can be written as

\[
\epsilon_j = (2j - 1)\epsilon_0 \tag{11}
\]

The constant \(\epsilon_0\) accounts for the scaling of the path. It depends on the end point \((t, x)\) and on the number of segments \(N\) a path has. For an explicit expression of \(\epsilon_0\) in terms of the number of segments to the left and right, see equation (24). Note that \(\epsilon_0\) scales not just the length of paths segment followed by a bend but the whole quadratic spacetime lattice. From this it follows immediately that for \(N \to \infty\) the spacetime lattice under consideration becomes a rectangular subset of \(M\).

In line with the generalization (8) of Feynman’s checkerboard model suppose now that each bend defining a path on \(M\) contributes an amplitude proportional to the length of the path segment immediately preceding the bend. Again, bends to the right and left are considered separately and the following notation is adopted: The index \(n\) enumerates the segments of a path followed by a bend to the right (left). \(j_n\) stands for the lattice coordinate \(j_r\) \((j_l)\) of the segment and the set \([j_n]\) indicates the path segments, after which bends to the right (left) occur. For a schematic example consider Figure 1. The length \(\epsilon_{jn}\) of the path segment followed by the \(n^{th}\) bend to the right (left) according to (11) becomes

\[
\epsilon_{jn} = (2j_n - 1)\epsilon_0 \tag{12}
\]

and the corresponding contribution of the \(n^{th}\) bend to the overall amplitude is

\[
\phi_{jn} = i\epsilon_{jn} = (2j_n - 1)\epsilon_0 \tag{13}
\]
The total amplitude of a path contributed by bends to the right (left) is given by the product

$$\phi = \prod_n (i \epsilon_j)$$  \hspace{1cm} (14)

Having established this notation, we start with the evaluation of (8). As in the 'linear' case (19) a path with \( R \) bends that starts with a positive velocity (i.e. to the right) and ends with a negative velocity (i.e. to the left) consists of exactly \((R - 1)/2 + 1\) bends to the left and \((R - 1)/2\) bends to the right. The \((R - 1)/2\) bends to the right can occur after an arbitrary path segment to the left. \((R - 1)/2\) of the \((R - 1)/2 + 1\) bends to the left occur in the same manner after path segments to the right while the additional bend to the left must occur after the last segment to the right. Let \( P \) denote the total number of path segments to the right (+) and \( Q \) those to the left (−). In total, the path has \( N = (P + Q) \) segments. The contribution of the \( R^+ = (R - 1)/2 \) bends to the right \( \psi_{-+} \) is

$$\psi_{-+}(R^+) = \sum_{j_1 < \ldots < j_{R^+}} (2j_1 - 1) \cdot \ldots \cdot (2j_n - 1) \cdot (i \epsilon_0)^{R^+}$$  \hspace{1cm} (15)

Next consider the situation where the path consists of a large number of segments to the right, i.e. \( P \gg 1 \) or equivalently \( N \to \infty \). This limiting process corresponds to the spacetime lattice becoming dense. For \( P \gg 1 \), \( \psi_{-+}(R^+) \) is approximated by

$$\psi_{-+}(R^+) \approx \frac{1}{R^+!} \sum_{j_1 \neq \ldots \neq j_{R^+}}^{P} (j_1 \cdot \ldots \cdot j_n) \cdot 2^{R^+} (i \epsilon_0)^{R^+}$$  \hspace{1cm} (16)

$$\approx \frac{2^{R^+} (i \epsilon_0)^{R^+}}{R^+!} \left( \sum_{j=1}^{P} j \right)^{R^+}$$  \hspace{1cm} (17)

$$\approx \frac{2^{R^+} (i \epsilon_0)^{R^+}}{R^+!} \left( \frac{P^2}{2} \right)^{R^+}$$  \hspace{1cm} (18)

$$= \frac{P^{2R^+} (i \epsilon_0)^{R^+}}{R^+!}$$  \hspace{1cm} (19)

The contribution of the \( R^- = (R - 1)/2 + 1 \) bends to the left is calculated similarly. The additional bend (occurring after the last segment to the right) does not enter the calculation since a path is fully determined by the \( R - 1 \)}
bends to the right and left, respectively. We find
\[
\psi_{-+}(R^-) \approx \frac{2(R^- - 1)(i\epsilon_0)(R^- - 1)}{(R^- - 1)!} \left(\frac{Q^2}{2}\right)^{(R^- - 1)}
\]
\[
= \frac{Q^{2(R^- - 1)}(i\epsilon_0)(R^- - 1)}{(R^- - 1)!}
\]
(20)
To get the total contribution of the \(R^-\) bends to the left and the \(R^+\) bends to the right, expressions \(\psi_{-+}(R^-)\) and \(\psi_{-+}(R^-)\) are multiplied yielding
\[
\psi_{-+}(R) = (i\epsilon_0)^{R-1} \frac{(PQ)^{R-1}}{[((R - 1)/2)!]^2}
\]
(22)
which represents the contribution of all paths with exactly \(R\) bends that start to the right and end to the left. Finally, the summation over all possible paths that start to the right and end to the left is performed by summing \(\psi_{-+}(R)\) over the total number of bends \(R\). This leads to
\[
\psi_{-+} = \sum_{\text{odd } R} (i\epsilon_0)^{R-1} \frac{(PQ)^{R-1}}{[((R - 1)/2)!]^2}
\]
(23)
where
\[
\epsilon_0 = \frac{t}{P^2 + Q^2}
\]
(24)
accounts for the proper scaling of the lattice. Notice that for \(N \to \infty\) the scaling factor \(\epsilon_0 \to 0\). Defining the classical velocity of the particle \(v = \Delta x/\Delta t = x/t = (P^2 - Q^2)/(P^2 + Q^2)\) we find \(PQ = (P^2 + Q^2)/2\gamma\) where \(\gamma = 1/\sqrt{1 - v^2}\). Finally \(\psi_{-+}\) becomes
\[
\psi_{-+} = \sum_{k=0}^{\infty} (-1)^k \frac{(t/2\gamma)^{2k}}{[(k)!]^2}
\]
(25)
\[
= J_0(t/\gamma)
\]
(26)
A similar calculation of \(\psi_{++}\) leads to the same result. This can be seen from interchanging the roles of \(P, Q, R^-\) and \(R^+\).

For \(\psi_{++}\), the number of bends to the right and to the left is \(R/2\) for each direction where \(R\) is even. However, the path again is defined by \(R^+ = R/2\) bends to the right and \(R^- = R/2 - 1\) to the left. Thus,
\[
\psi_{++} = \sum_{\text{even } R} (i\epsilon_0)^{R-1} \frac{P^RQ^{R-2}}{(R/2)!(R/2 - 1)!}
\]
(27)
\[ = \sum_{0,2,4,\ldots} (i\epsilon_0)^{R+1} \frac{P^2(PQ)^R}{(R/2 + 1)!(R/2)!} \]  
\[ = \frac{P}{Q} \sum_{k=0}^{\infty} (-1)^k \frac{(PQ\epsilon_0)^{2k+1}}{(k + 1)!(k)!} \]  
\[ = i\frac{P}{Q} \sum_{k=0}^{\infty} (-1)^k \frac{(t/2\gamma)^{2k+1}}{(k + 1)!(k)!} \]  
\[ = i\frac{P}{Q} J_1(t/\gamma) . \]  

With \( P/Q = (x + t)/(t^2 - x^2)^{1/2} \) the component \( \psi_{++} \) becomes
\[ \psi_{++} = i \frac{(t + x)}{t} \gamma J_1(t/\gamma) . \]  
A similar calculation for \( \psi_{--} \) leads to
\[ \psi_{--} = i \frac{(t - x)}{t} \gamma J_1(t/\gamma) . \]

which completes the calculation.

To relate the components \( \psi_{\delta \gamma} \) to the Dirac equation (1) consider its explicit representation with
\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \]  
In this representation, \( \psi_1 \) and \( \psi_2 \) defined as
\[ \psi_1 = \begin{pmatrix} \psi_{++} \\ \psi_{+-} \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} \psi_{+-} \\ \psi_{--} \end{pmatrix} \]  
are two common independent solutions of the Dirac equation (1). Since \( M \) is dense in \( \mathbb{R}^2 \), the differential form of (1) does not pose a problem even though \( M \) is non-continuous.

The four components \( \psi_{\delta \gamma} \) are closely related to the matrix elements of the retarded continuum propagator of the Dirac equation (see e.g. [13]). Despite this there is no full equivalence and a physical interpretation of \( \psi_{\delta \gamma} \) in terms of propagator components is not straightforward. In the last section we will come back to this and related questions in more detail.

As a side remark note that for the applied calculation scheme it is essential to consider only those bends which actually define the path of the
while it is straightforward to 'renormalize' Feynman’s original model if all bends are considered, this is not the case here. Taking into account all bends, the normalization ‘constant’ \( A_{\delta, \gamma}(\epsilon_0) \) for the checkerboard restricted to \( M \) turns out to not only depend on \( \epsilon_0 \) but also on \( \delta, \gamma \) and the classical velocity \( v \).

4 Discussion and Conclusions

It is counter-intuitive that Feynman’s spacetime based path integral approach to quantum theory yields in the case of the discussed example intriguing similarities between the common continuous and the discrete spacetime model \( M \). The solutions \( \psi \) obtained from the generalized checkerboard model and the common retarded continuum propagator of the Dirac equation are of the same form.

However, despite the same functional form, the components \( \psi_{\delta, \gamma} \) are not fully equivalent to the retarded continuum propagator. Reworking the outlined calculation with physical units \( \hbar/m \neq 1 \) shows the dimensionality of \( \psi_{\delta, \gamma} \) to be \( \text{length}^0 \) which is different from the dimensionality \( \text{length}^{-1} \) of the retarded continuum propagator. With respect to the original checkerboard model, the difference is related to two origins: First, taking into account only \( R - 1 \) bends means accounting for one factor \( m \) less. Second, Feynman’s original checkerboard model is 'renormalized' by dividing components like e.g. (23) and (32) by \( 2\epsilon \) in order to yield the retarded continuum propagator.

The dimensionality \( \text{length}^0 \) of \( \psi_{\delta, \gamma} \) may be seen as suggesting a picture consistent with the cardinality \( \aleph_0 \) of \( M \). Profoundly taking into account the cardinality \( \aleph_0 \), a continuum-like propagator is inconsistent with \( M \). This is due to the fact that because of its cardinality there exists no measure on \( M \). As a consequence, the concept of integration is not available. Instead, one can expect (infinite) sums to play the role of integration. This suggests that the components of a discrete counterpart of the continuum propagator (i.e. \( \psi_{\delta, \gamma} \)) should have dimensionality \( \text{length}^0 \). Fully clarifying the physical meaning of \( \psi_{\delta, \gamma} \) in the framework of the discrete spacetime model \( M \) is not straightforward. One is lead into number theoretical problems that are not easy to tackle.

Other open questions remain. For example: To what extend does the path integral approach capture the relation between spacetime and quantum mechanics? This question is raised not only by the present results but already by Feynman’s original checkerboard model. After all, according to
its formulation, bends occur only at spacetime points with rational coordinates. As well, the velocity spectrum of the particle is rational. In this sense, Feynman’s checkerboard model is of cardinality $\aleph_0$ and thus discrete in the same manner as $M$ is. To our knowledge this is a fact that the literature has not paid attention to.

In [20] it has been shown that the solution of the Dirac equation can be understood in terms of bit strings detached from any notion of spacetime. The present results seem to point into the same direction, namely, that there is (at least in the special case of the Dirac equation) only a loose connection between spacetime properties and quantum mechanics, except probably for the signature of the metric.

Other open issues are related to the spacetime model $M$ itself. The ratio $x/t$ interpreted as classical velocity of the particle (starting from the origin) implies a discrete velocity spectrum

$$\{v\} = \frac{p^2 - q^2}{p^2 + q^2}, \quad p, q \in \mathbb{Z} \setminus \{0\}. \quad (36)$$

What are the implications of the non-continuous spectrum (36)? Progress in this and other directions again is complicated by the fact that $M$ is of cardinality $\aleph_0$. Finally, a question not addressed here is where the bends of a path occur, that contribute most to the overall amplitude. Because of the structure of $M$ it is not evident that the locations are distributed as in the continuous case.

In summary, it has been shown that Feynman’s path integral approach restricted to the subset $M$ of $\mathbb{R}^2$ recovers common solutions of the Dirac equation. The particle path on $M$ is limited to bends at positions $i$ with coordinate ratios $x_i/t_i = (p^2 - q^2)/(p^2 + q^2)$ where $p, q \in \mathbb{Z} \setminus \{0\}$. The bends of paths on $M$ thus occur at a subset of the rational space and time coordinates only. It should be noted that both the characteristics of $M$ and the discrete velocity spectrum (36) are invariant under the Lorentz subgroup $\phi$. In particular, bends of paths occur only at rational space and time coordinates and the velocity remains rational in all coordinate systems. In this sense the model exhibits a generalized form of Lorentz invariance missing in Feynman’s original checkerboard model and most other lattice models of spacetime. This demonstrates that there is a discrete, two-dimensional spacetime model of cardinality $\aleph_0$ accounting for some of the key properties of special relativity and quantum mechanical features of relativistic particle motion.
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Figure 1: A path on $M$ between the origin of the coordinate system and a given end point $(t, x)$. The path has a total of $N = P + Q = 8$ segments, where $P = 5$ segments are going to the right and $Q = 3$ segments to the left. A total of $R = R^+ + R^- = 5$ bends occur, $R^+ = 2$ to the right and $R^- = 3$ to the left. Note that the last bend to the left is fully determined by the end point $(t, x)$ and the location of $R^+ - 1 = 4$ bends, i.e. by the $R^+ = 2$ bends to the right and the first $R^- - 1 = 2$ bends to the left.