This paper presents the first convergence result for random search algorithms to a subset of the Pareto set of given maximum size $k$ with bounds on the approximation quality $\epsilon$. The core of the algorithm is a new selection criterion based on a hypothetical multilevel grid on the objective space. It is shown that, when using this criterion for accepting new search points, the sequence of solution archives converges with probability one to a subset of the Pareto set that $\epsilon$-dominates the entire Pareto set. The obtained approximation quality $\epsilon$ is equal to the size of the grid cells on the finest level of resolution that allows an approximation with at most $k$ points in the family of grids considered. While the convergence result is of general theoretical interest, the archiving algorithm might be of high practical value for any type iterative multiobjective optimization method, such as evolutionary algorithms or other metaheuristics, which all rely on the usage of a finite on-line memory to store the best solutions found so far as the current approximation of the Pareto set.

1 Introduction

In multiobjective optimization, we are given $m \geq 2$ objective functions $f_i : X \rightarrow \mathbb{R}$, $i \in \{1, \ldots, m\}$, defined over some search space $X$, which might be implicitly defined by constraints. We assume the search space $X$ to be finite and that, w.l.o.g., all objective shall be maximized. We are therefore interested in solving

$$\max \left\{ f(x) = (f_1(x), \ldots, f_m(x))^T \mid x \in X \right\}$$

(1)

Here, maximization is understood with respect to the product order $\geq$ on $\mathbb{R}^m$, i.e., for any pair $(y, y') \in \mathbb{R}^m \times \mathbb{R}^m$, $y \geq y'$ if and only if $y_i \geq y'_i$ for all $i \in \{1, \ldots, m\}$. Hence
(\mathcal{Y}, \geq), where \( \mathcal{Y} = f(\mathcal{X}) \) is called the objective space, is a partially ordered set. This gives rise to the following order relation on the search space.

**Definition 1** (Pareto dominance). For any pair \((x, x') \in \mathcal{X} \times \mathcal{X}, x \) is said to weakly dominate \( x' \), denoted as \( x \geq x' \), if and only if \( f(x) \geq f(x') \). \( x \) said to dominate \( x' \), denoted as \( x \succ x' \), if and only if \( x \geq x' \) and \( x' \not\succ x \).

Note that \((\mathcal{X}, \geq)\) is a preordered set, while \((\mathcal{X}, \succ)\) is a strictly partially ordered set. A subset \( \mathcal{X}' \subseteq \mathcal{X} \) is called independent with respect to \( \geq \) if for all \((x, x') \in \mathcal{X}' \times \mathcal{X}' \) where \( x \neq x' \) it holds that \( x \not\geq x' \). Let the set of all such independent subsets of \( \mathcal{X} \) be denoted by \( \hat{\mathcal{X}} \), i.e.,

\[
\hat{\mathcal{X}} := \{ \mathcal{X}' \subseteq \mathcal{X} | \mathcal{X}' \text{ is independent with respect to } \geq \}.
\]

The set of minimal elements of the objective space,

\[
\mathcal{Y}^* := \max(\mathcal{Y}, \geq) = \{ y \in \mathcal{Y} | \bar{y} y' \neq y \text{ with } y' \geq y \},
\]

is called the Pareto front, and its preimage \( \mathcal{X}^* = f^{-1}(\mathcal{Y}) \) is called the Pareto set.

Ideally, when solving (1), one is interested in determining the Pareto front \( \mathcal{Y}^* \), together with an independent set \( \mathcal{X}' \) that should cover \( \mathcal{Y}^* \), i.e., \( f(\mathcal{X}') = \mathcal{Y}^* \). This means we are usually not interested in obtaining more than one preimage for each element of the Pareto front.

In many instances, the size of Pareto front might be immense, so we are interested in approximations. Here, our goal is for find some subset of \( \mathcal{X} \) of a given maximum size \( k \) that approximates \( \mathcal{Y}^* \) well in the following sense.

**Definition 2** (\( \epsilon \)-dominance). Let \( \epsilon \in \mathbb{R}^m, \epsilon \geq 0 \). For any pair \((x, x') \in \mathcal{X} \times \mathcal{X}, x \) is said to \( \epsilon \)-dominate \( x' \), denoted as \( x \succ^\epsilon x' \), if and only if \( f(x) + \epsilon \geq f(x') \).

A set \( \mathcal{A} \subseteq \hat{\mathcal{X}} \) is called an \( \epsilon \)-approximate Pareto set if for all \( x \in \mathcal{X} \) there is an \( x' \in \mathcal{A} \) with \( x' \succ^\epsilon x \). An \( \epsilon \)-approximate Pareto set that is a subset of \( \mathcal{X}^* \) is called an \( \epsilon \)-Pareto set. Thus, a reasonable task is to find an \( \epsilon \)-Pareto set of some given maximum cardinality \( k \). Note that for the special case of \( \epsilon = 0 \), the notions of \( \epsilon \)-approximate Pareto set, \( \epsilon \)-Pareto set, and covering independent set are equivalent.

We investigate in which sense simple random search is able to find an \( \epsilon \)-Pareto set of cardinality \( k \) of the multiobjective optimization problem (1), i.e., whether its solution set stochastically converges to such a set in the limit. For this, we consider Algorithm 1, which is pure random search where the set \( \mathcal{A} \) represents its archive of solutions, stored in a memory (array) of size at most \( k \).

### 2 Related work and previous results

Many different notions of Pareto set approximations have been proposed in the literature, see for instance the survey on concepts of \( \epsilon \)-efficiency in [7]. However, many of them deal with infinite sets and are therefore not practical as a solution concept for our purpose of producing and maintaining a representative subset of a fixed given size. The use
of discrete $\epsilon$-approximations of the Pareto set was proposed almost simultaneously by various authors [6, 4, 15, 18]. The general idea is that each Pareto-optimal point is approximately dominated by some point of the approximation set. The $\epsilon$-dominance given in Definition 2 is a typical instance of this approach, while it is also common to use a relative deviation instead of the absolute deviation employed here.

As relative deviation is essentially equivalent to absolute deviation on a logarithmically scaled objective space, this choice should not affect the convergence results obtained but rather depend on the actual application problem at hand. The nice property of relative deviation is that it allows to prove that, under very mild assumptions, there is always an $\epsilon$-Pareto set whose size is polynomial in the input length [13, 3]. Further approximation results for particular combinatorial multiobjective optimization problems are given in [2], where the question was how well a single solution can approximate the whole Pareto set, which is a special case of our question restricted to $k = 1$ and with focus on deterministic algorithms.

Despite the existence of suitable approximation concepts, investigations on the convergence of particular algorithms towards such approximation sets, that is, their ability to obtain a suitable Pareto set approximation in the limit, have remained rare. In [16, 17] the stochastic search procedure proposed by earlier by [14] was analyzed and proved to converge to an $\epsilon$-Pareto set with $\epsilon = 0$ in case of a finite search space. Obviously, the solution set maintained by this algorithm might in the worst case grow as large as the Pareto set $X^*$ itself. Thus, a different version with bounded memory of at most $k$ elements was proposed and shown to converge to some subset of $X^*$ of size at most $k$, but no guarantee about the approximation quality could be given. Similar results were obtained by [5] for continuous search spaces.

One option to control the approximation quality under size restrictions is to define a quality indicator which maps each possible solution set to a real value that can then be used to decide on the inclusion of a new search point. Several algorithms have been proposed that implement this concept [20, 1]. In case that such a quality indicator fulfils certain monotonicity conditions, it can be used as a potential function in the convergence analysis. As shown in [8, 9], this entails convergence to a subset of the Pareto set as a local optimum of the quality indicator, but it remained open how such a local optimum relates to a guarantee on the approximation quality $\epsilon$. [9] also analyzed an adaptive grid archiving method proposed in [10] and proved that after finite time, even though the solution set itself might permanently oscillate, it will always represent an $\epsilon$-
approximation whose approximation quality depends on the granularity of the adaptive grid and on the number of allowed solutions. The results depend on the additional assumption that the grid boundaries converge after finite time, which is fulfilled in certain special cases.

In [12], two archiving algorithms were proposed that provably maintain a finite-size \( \epsilon \)-approximation of all points ever generated during the search process. As an immediate corollary, these archiving strategies ensure convergence to a Pareto set approximation of given quality \( \epsilon \). While the desired \( \epsilon \) is an input to the algorithm that can be specified beforehand, the resulting size of the approximation can be bounded as a function of \( \epsilon \) and the ranges of the objective space \( \mathcal{Y} \). In [11] it was shown that in practise these bounds are sometimes not tight enough, which is a particular disadvantage when applied in a scenario where the maximum archive size \( k \) has to be specified beforehand. If information of the objective space ranges is available, the bounds can be used to find a valid value for \( \epsilon \), but this choice often turns out overly conservative so that far less solutions are attained than would be possible with a memory of size \( k \). In case where the objective ranges are unknown, a mechanism proposed in [12] can be used to systematically increase the value of \( \epsilon \) without losing the convergence properties, but this suffers from the same drawback of being overly conservative. Thus, it has remained open until now whether working with fixed size Pareto set approximations can guarantee convergence in the limit for arbitrary multiobjective optimization problems on finite search spaces, and at the same time guarantee a certain approximation quality.

In this paper we settle this question positively by presenting an archiving scheme that enables Algorithm 1 to produce a sequence of solution sets that converges with probability one to an \( \epsilon \)-Pareto set of a certain quality. The algorithm represents a combination of the two complementary algorithms discussed above [9, 12], thus combining the advantages of both: it can be seen as a variant of the adaptive grid archiving method where a multilevel, fixed grid is used instead of an adaptive grid with moving boundaries, which is crucial to obtain convergence. It can also be seen as a proper implementation of the adaptation mechanism for the \( \epsilon \) values proposed in [12], which is crucial to limit the size of the solution set to at most \( k \), but which is also able to reduce the value of \( \epsilon \) whenever possible. Finally, the algorithm can be seen as selection using a particular quality indicator [20], a notion that will be defined more precisely later on. However, instead of having to compute the actual indicator values, which might be computationally cumbersome, this indicator will only be used as a potential function in the analysis of the algorithm and never has to be computed. The actual comparison will be defined using very simple local rules that – and this is crucial – are in accordance with the quality indicator, which will be established via order homomorphisms.

3 Stochastic convergence analysis

The sequences of archives \( \{A_t, t \in \mathbb{N}_0\} \) generated by Algorithm 1 are realizations of a discrete-time stochastic process defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The sample space \( \Omega \) can be defined as the infinite product \( \mathcal{X}^\infty \) with the \( \sigma \)-algebra \( \mathcal{F} = 2^\Omega \) being
We show for all components
\[ A_t : \Omega \rightarrow \hat{X}, \quad A_t(\omega) = \omega_t \]
and \( A_t = A_t(\omega) \).

From Algorithm 1 it is clear that \( A_{t+1} \) only depends on \( A_t \), so the process is a homogenous finite Markov chain with state space \( \hat{X} \). Due to line 5 of the algorithm, all transition probabilities that change the state are equal to \( p = 1/|\hat{X}| \).

Let the transition graph of the Markov chain be denoted by \( G = (\hat{X}, E) \). Clearly, its arcs \( E \subseteq \hat{X} \times \hat{X} \) are determined by the update function given in Algorithm 2 as
\[ E = \{(A, A') \in \hat{X} \times \hat{X} \mid \exists x \in \hat{X} : A' = \text{update}(A, x)\}. \]

Our goal is to show that this transition graph, when ignoring loops, forms a directed acyclic graph, which immediately implies that absorption will take place with probability one in finite time. We then proceed to show that in all absorbing states the archives are \( \epsilon \)-Pareto sets, and finally give some guarantee of the approximation quality \( \epsilon \) obtained.

Instead of working on \( E \) directly, however, we define a potential function \( I \), according to which the set of possible archives \( \hat{X} \) can be linearly ordered. For this, some auxiliary notation is needed.

**Definition 3** (box index vector). The box index vector of a vector \( y \in \mathbb{R}^m \) at level \( b \in \mathbb{Z} \) is given by the value of the function
\[ \beta^{(b)} : \mathbb{R}^m \rightarrow \mathbb{Z}^m, \quad \beta^{(b)}(y) = \left( r^{(b)}(y_1), \ldots, r^{(b)}(y_m) \right)^T \]
where
\[ r^{(b)} : \mathbb{R} \rightarrow \mathbb{Z}, \quad r^{(b)}(z) = \left\lfloor z \cdot 2^{-b} + \frac{1}{2} \right\rfloor \]

**Definition 4** (box-dominance). Let \( b \in \mathbb{Z} \). For any pair \( (x, x') \in \mathcal{X} \times \mathcal{X} \), \( x \) is said to weakly box-dominate \( x' \) at level \( b \), denoted as \( x \geq_b x' \), if and only if \( \beta^{(b)}(f(x)) \geq \beta^{(b)}(f(x')) \). \( x \) is said to be box-equal to \( x' \), denoted as \( x \sim_b x' \), if \( x \geq_b x' \) and \( x' \geq_b x \). \( x \geq_b x' \) and \( x \not\sim_b x' \) then \( x \) is said to box-dominate \( x' \) at level \( b \), denoted as \( x \succ_b x' \).

Note that the relations \( \geq_b \) form a family of order extensions of the dominance relation \( \geq \). The accompanying equivalence relations \( \sim_b \) can be seen as a successive coarse-graining of approximate indifference between solutions.

**Lemma 1.** If \( x \geq_b x' \) then \( x \geq_c x' \) for all \( c \geq b \).

**Proof.** We show for all components \( i \in \{1, \ldots, m\} \) that if \( r^{(b)}(f_i(x)) \geq r^{(b)}(f_i(x')) \) then \( r^{(b+1)}(f_i(x)) \geq r^{(b+1)}(f_i(x')) \). The lemma follows then by induction. Let \( d := f_i(x) \cdot 2^{-b} + 1/2 \) and \( d' := f_i(x') \cdot 2^{-b} + 1/2 \). From the premise we can express \( d = p + 1/2 + g \) and \( d' = p + 1/2 + 1 - h \) for some \( p \in \mathbb{Z}, g \geq 0 \) and \( h > 0 \). If \( p = 2k \) for some \( k \in \mathbb{Z} \) then \( r^{(b+1)}(d) \geq [2k/2 + 1/2] = k \geq [2k/2 + 1 - d] = r^{(b+1)}(d') \). If \( p = 2k + 1 \) then \( r^{(b+1)}(d) \geq [(2k + 1)/2 + 1/2] = k + 1 \geq [(2k + 1)/2 + 1 - d] = r^{(b+1)}(d') \). \( \square \)
Algorithm 2 update(\(\mathcal{A}, x\))

1: \(\mathcal{A}' \leftarrow \mathcal{A} \cup \{x\}\)
2: if \(\exists a \in \mathcal{A} : a \succeq x\) then
3: return \(\mathcal{A}\)
4: else if \(\max(\mathcal{A}', \succ) \leq k\) then
5: return \(\max(\mathcal{A}', \succ)\)
6: else
7: \(\beta \leftarrow \min\{b \in \mathbb{Z} \mid \exists (a, a') \in \mathcal{A} \times \mathcal{A}' : a \succeq_b a'\ and\ a \neq a'\}\)
8: \(\mathcal{A}'' \leftarrow \{a \in \mathcal{A}' \mid \exists a' \in \mathcal{A}' : a' \succeq_b a\ and\ a' \neq a\}\)
9: if \(x \in \mathcal{A}''\) then
10: return \(\mathcal{A}\)
11: else
12: Draw \(a\) uniformly at random from \(\mathcal{A}''\)
13: return \(\mathcal{A} \setminus \{a\} \cup \{x\}\)
14: end if
15: end if

Definition 5 (potential function). Let

\[
I : \hat{\mathcal{X}} \rightarrow \mathbb{R}, \quad I(\mathcal{A}) := \sum_{i=-\bar{b}}^{\infty} |\mathcal{B}_{i}(\mathcal{A})| \cdot (|\mathcal{X}| + 1)^{-i}
\]

where

\[
\mathcal{B}_b(\mathcal{A}) = \{x \in \mathcal{X} \mid \exists a \in \mathcal{A} : a \succeq_b x\}
\]

and

\[
\bar{b} = \min\{b \in \mathbb{Z} \mid \forall (x, x') \in \mathcal{X} \times \hat{\mathcal{X}} : x \sim_b x'\}
\]

The power series defining \(I\) converges as the \(\mathcal{B}_b\) are subsets of \(\mathcal{X}\), which is finite. Moreover, \(\bar{b}\) exists since it is possible to enclose the whole objective space \(f(\mathcal{X})\) by one box by choosing \(b\) large enough.

The dominance relation on solutions can be used to define a natural preference relation on the set of independent sets \(\hat{\mathcal{X}}\).

Definition 6 (dominance of independent sets). Let \((\mathcal{A}, \mathcal{A}') \in \hat{\mathcal{A}} \times \hat{\mathcal{A}}\). The set \(\mathcal{A}\) is said to weakly dominate \(\mathcal{A}'\), denoted as \(\mathcal{A} \sqsupseteq \mathcal{A}'\), if \(\max(f(\mathcal{A} \cup \mathcal{A}'), \succeq) = \max(f(\mathcal{A}), \succeq)\).

Lemma 2. \(I\) is an order homomorphism of \((\hat{\mathcal{A}}, \sqsupseteq)\) into \((\mathbb{R}, \geq)\), i.e., if \(\mathcal{A} \sqsupseteq \mathcal{A}'\) then \(I(\mathcal{A}) \geq I(\mathcal{A}')\). \(I\) is also an order homomorphism of \((\hat{\mathcal{A}}, \sqsubset)\) into \((\mathbb{R}, >)\), i.e., if \(\mathcal{A} \sqsubset \mathcal{A}'\) then \(I(\mathcal{A}) > I(\mathcal{A}')\).

Proof. If \(\mathcal{A} \sqsupseteq \mathcal{A}'\) then the coefficients \(|\mathcal{B}_i(\mathcal{A})|\) in the power series of \(I(\mathcal{A})\) are uniformly not less than \(|\mathcal{B}_i(\mathcal{A}')|\) because \(\mathcal{B}_i(\mathcal{A}') \subseteq \mathcal{B}_i(\mathcal{A})\) for all \(i\). If additionally \(\mathcal{A}' \not\sqsupseteq \mathcal{A}\) then there exists an \(a \in \mathcal{A}\) and a \(b \in \mathbb{Z}\) such that there is no \(a' \in \mathcal{A}'\) with \(a' \succeq_b a\). Hence, \(\mathcal{B}_b(\mathcal{A}') \subseteq \mathcal{B}_b(\mathcal{A})\), which implies that \(I(\mathcal{A}) > I(\mathcal{A}')\). \(\square\)
The potential function $I$ can be seen as a quality indicator for independent sets. As an immediate corollary of Lemma 2, the $\succ$-relation on $I(\hat{A})$ represents a comparison method that is $\preceq$-compatible and $\succeq$-complete in the terminology of [21].

Lemma 3. If $(A_t, A_{t+1}) \in E$ and $A_t \neq A_{t+1}$ then $I(A_{t+1}) > I(A_t)$.

Proof. If $(A_t, A_{t+1}) \in E$ and $A_t \neq A_{t+1}$ then $x \in A_{t+1} = \text{update}(A_t, x)$. There are two cases that can cause $x$ to be included into the new archive (termination in line 5 or line 13). For termination to occur in line 5, $\text{max}(A', \succ) \leq k$ has to hold (line 4), and furthermore $x \in \text{max}(A_{t+1}, \succ)$ by contradiction to the condition in line 2. Since $A_t = \text{max}(A, \succ) \not\ni x$ it follows that $A_{t+1} = \text{max}(A', \succ) \supseteq A_t$ and thus, by Lemma 2, $I(A_{t+1}) > I(A_t)$. When termination occurs in line 13, $\text{max}(A', \succeq_{\beta})$ contains $x$ but not $a$. Hence $B_{\beta}(A_{t+1}) \supseteq B_{\beta}(A_t)$, which implies, by Lemma 1, that $B_{\beta}(A_{t+1}) \supseteq B_{\beta}(A_t)$ for all $b \geq \beta$. This implies $I(A_{t+1}) > I(A_t)$, which completes the proof.

Theorem 1. The Markov chain $\{A_t, t \in \mathbb{N}_0\}$ is absorbing.

Proof. Due to Lemma 3, the mapping $I$ is an order homomorphism of $G$ into a strict linear order. This implies that the transitive closure of $G$ is a partially ordered set, and hence $G$ is a directed acyclic graph. As any Markov chain on an directed acyclic graph is absorbing, the claim follows.

Theorem 2. For any absorbing state $A \in \text{max}(G)$ it holds that $A \subseteq X^*$.

Proof. Assume that $A \not\subseteq X^*$, so there is some $a \in A$ that is not in $X^*$. Then there exists some $x \in X$ with $x \succ a$. Let $\hat{A} = \text{update}(A, x)$. Since $x \succ A$, the condition $\text{max}(A', \succ) \leq k$ in line 4 holds because $a \not\in \text{max}(A', \succ)$. Thus, $\hat{A} = \text{max}(A', \succ) \neq A$, which is a contradiction to the assumption.

The next theorem shows that any absorbing state box-dominates the Pareto set at the lowest level possible with the least number of solutions necessary, while distributing the remaining solutions with maximum entropy over the nondominated boxes at the next lower level.

Theorem 3. Let

$$\delta = \min\{b \in \mathbb{Z} \mid \exists A \subseteq X^* \text{ with } |A| \leq k \text{ and } B_b(A) = X\},$$

(2)
denote the smallest level $b$ on which it is possible to box-dominate the Pareto set with at most $k$ solutions,

$$s = \min\{|A| : B_\delta(A) = X\},$$

(3)
denote the minimum size of such a set, and let

$$\mathcal{E} = \{X' \subseteq X^* \mid \forall (x, x') \in X' \times X' : x \sim_{\delta-1} x'\}$$

(4)
denote a partitioning of the Pareto set into the boxes of the next smaller level. Then for any absorbing state $A \in \text{max}(G)$ it holds that $B_\delta(A) = X$ and that

$$|\{X' \in \mathcal{E} : X' \cap A \neq \emptyset\}| = \min\{k, |\mathcal{E}|\}.$$
Proof. Assume that there is some \( x \in X^* \) with \( x \not\in B_\delta(A) \). If \(|A| < k\) then \( \max(A', \succ) \leq k \) in line 4, in which case \( A \) cannot be absorbing. If \(|A| = k\) then \( A'' \) cannot be empty, as this would contradict the definition of \( \beta \) in line 7. If \( x \in A'' \) then \( x \) will enter \( A \), so \( A \) cannot be absorbing. If \( x \not\in A'' \) then \( x \) will enter \( A \), so \( A \) cannot be absorbing. If \( \beta > \delta \) due to Lemma 1, since \( x \not\in B_\delta(A) \) but \( x \in B_\beta(A) \) by assumption. The definition of \( \beta \) in line 7 now implies that \( A' \) is an independent set of cardinality \( k + 1 \) with respect to \( \succeq_\delta \), hence \( A' \) serves as a witness for the fact that there is no \( A \) with \(|A| \leq k \) and \( B_\delta(A) = X \), which is a contradiction to (2) and completes the proof that \( B_\delta(A) = X \). To prove the second part of the proposition, assume that \( B_\delta(A) = X \) and \( d := |\{X' \in \mathcal{E} : X' \cap A \neq \emptyset \}| < \min\{k, |\mathcal{E}|\} \). Hence there is some \( x \in X' \in \mathcal{E} \) with \( X' \cap A = \emptyset \). If \(|A| < k\) then \( x \) will be accepted as there is no \( a \in A \) with \( a \succeq x \), so \( A \) cannot be absorbing. If \(|A| = k\) then \( \beta < \delta \) as otherwise \( d = k \). Now, as \( \beta a \in A \) with \( a \succeq \delta - 1 \) \( x \) and \( \beta \leq \delta - 1 \), it follows with Lemma 1 that \( x \not\in A'' \), thus \( x \) will be accepted, so \( A \) cannot be absorbing.

We have collected now all necessary ingredients to show the main result.

Theorem 4. The sequence \( \{A_t, t \in \mathbb{N}_0\} \) converges with probability one to some \( \epsilon \)-Pareto set with \( \epsilon = 2^\delta \), with \( \delta \) defined as in Theorem 3.

Proof. As a corollary to Theorems 1 and 2, \( A_t \) converges with probability one to some subset of the Pareto set. As a corollary of Theorem 3, for any absorbing state \( A \) there exists for all \( x \in X^* \) some \( a \in A \) such that \( a \succeq_\delta x \) and, hence, \( a \succeq_\epsilon x \) for all \( \epsilon \geq 2^\delta \).

4 Conclusions

In this paper, the first convergence result for random search algorithms to a subset of the Pareto set of given maximum size \( k \) and bounds on the approximation quality \( \epsilon \) was given. The convergence was enabled by a new selection scheme, given als Algorithm 2, that compares the new candidate solution to the current archive using a multi-level grid.

In many parts, the assumption of a finite search space was used. Even though this is a reasonable assumption for any implementation in computer arithmetic with finite precision, an extension to the continuous case would be desirable. Even though it might be a justified assumption that the results can be extended, recent experience [19] has shown that this might involve considerable effort.

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