Quantized vortices in dipolar supersolid Bose-Einstein condensed gases

A. Galemi, S. M. Roccuzzo, S. Stringari and A. Recati
INO-CNR BEC Center and Dipartimento di Fisica, Università degli Studi di Trento, 38123 Povo, Italy
Trento Institute for Fundamental Physics and Applications, INFN, 38123, Trento, Italy

We investigate the properties of quantized vortices in a dipolar Bose-Einstein condensed gas by means of a generalised Gross-Pitaevskii equation. The size of the vortex core hugely increases by increasing the weight of the dipolar interaction and approaching the transition to the supersolid phase. The critical angular velocity for the existence of an energetically stable vortex decreases in the supersolid, due to the reduced value of the density in the interdroplet region. The angular momentum per particle associated with the vortex line is shown to be smaller than $\hbar$, reflecting the reduction of the global superfluidity. The real-time vortex nucleation in a rotating trap is shown to be triggered, as for a standard condensate, by the softening of the quadrupole mode. For large angular velocities, when the distance between vortices becomes comparable to the interdroplet distance, the vortices are arranged into a honeycomb structure, which coexists with the triangular geometry of the supersolid lattice and persists during the free expansion of the atomic cloud.

I. INTRODUCTION

The recent realization of supersolidity in dipolar Bose-Einstein condensed gases [1–3] is stimulating novel experimental and theoretical work aimed at studying the superfluid properties of these intriguing systems, which exhibit the spontaneous breaking of both gauge and translational symmetry yielding superfluidity and crystal periodic order, respectively (see, e.g., [4–7]). Experimental evidence of phase coherence among the droplets forming the crystal structure [1–3], the occurrence of Goldstone modes associated with the spontaneous breaking of both symmetries [8–10] and the reduction of the moment of inertia with respect to the rigid value [11] have provided important signatures of the superfluid behavior of these systems. Conclusive proof of superfluidity is however given by the observation of quantized vortex lines, following the seminal papers of Refs. [12–15] and [16] in Bose-Einstein condensates and strongly interacting Fermi gases, respectively. The realization of quantized vortices, hosted by the crystal configuration of the supersolid, then represents a challenging task to pursue. This possibility, so far not yet experimentally realized, has been the object of first recent theoretical investigations [17]. Even the structure of quantized vortices in the fully-superfluid phase and in particular the effect of the long-range dipolar force on the size of the vortex core and on the value of the critical angular velocity needed to ensure the energetic stability of a single vortex line, represents an interesting topic, hopefully of near future experimental investigation. The purpose of this paper is to provide a first comprehensive theoretical investigation of the structure of quantized vortices in a Bose-Einstein condensate characterized by a long-range dipolar interaction with special focus on the supersolid phase.

Our investigation is based on the use of a suitable extension of the Gross-Pitaevskii equation to include the beyond mean-field term (see Section II) in the equation of state accounting for quantum fluctuations [18], which plays a crucial role in the emergence of supersolidity and the formation of self-bound droplets (see Fig. 1). In Section III we explore the properties of a single vortex line in both the superfluid and the supersolid case. In the superfluid phase the dipolar interaction hugely increases the value of the healing length when compared to Bose gases with only zero-range interaction. In the supersolid phase, vortices are hosted in the region separating the droplets forming the crystal structure and their shape is strongly deformed by the presence of the droplets. We show that the value of the critical angular velocity exhibits an important reduction by increasing the ratio between the dipolar and the zero-range strengths of the interatomic force. Furthermore we show that the angular momentum carried by a vortical line in an axi-symmetrically trapped supersolid is reduced with respect to the usual value $\hbar$ as a consequence of the reduced superfluidity of the system. By carrying out a time-dependent simulation we also point out that the nucleation process for the creation of a vortical line in the supersolid phase is favored

*Corresponding Author: alessio.recati@ino.it
by the softening of the quadrupole mode frequency. In Section [V] we study the case of higher angular velocities and show that the coexistence of the density modulation and vorticity yields a honeycomb vortex lattice in place of the usual triangular (Abrikosov) lattice, the coexistence persisting during the expansion following the sudden release of the trap. We sum up our conclusions in Section [VI].

II. DIPOLAR GROSS-PITAEVSKII EQUATION WITH LEE-HUANG-YANG CORRECTION

We consider a dipolar Bose gas of $^{164}$Dy atoms trapped by an in-plane isotropic harmonic potential $V_{ho}(r) = \frac{1}{2}m(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)$, with, if not otherwise stated, $\omega_x = \omega_y$ and $m$ the atomic mass. At zero temperature the gas can be characterized by a single macroscopic wave function $\Psi(r,t)$, whose temporal evolution is described by a generalized Gross-Pitaevskii equation. The latter takes into account the contact, the dipole-dipole interaction, and the quantum fluctuations [19], and can be written as

$$i\hbar \frac{\partial}{\partial t} \Psi(r,t) = \mathcal{H}(r) \Psi(r,t),$$

where the hamiltonian is

$$\mathcal{H}(r) = -\frac{\hbar^2}{2m} \nabla^2 + V_{ho}(r) + g|\Psi(r,t)|^2 + \gamma(\epsilon_{dd})|\Psi(r,t)|^3$$

$$+ \int dr V_{dd}(r-r')|\Psi(r',t)|^2,$$

with $g = 4\pi \alpha a/m$ the coupling constant fixed by the $s$-wave scattering length $a$ and $V_{dd}(r_1-r_2) = \frac{4\pi \alpha^2}{1 - 3\cos^2 \theta}|r_1-r_2|$, the dipole-dipole potential, being $\mu_0$ the magnetic permeability in vacuum, $\mu$ the magnetic dipole moment and $\theta$ the angle between the vector distance between dipoles and the polarization direction, which we choose as the $z$ axis. In the absence of trapping, the system can be fully characterised by the single parameter $\epsilon_{dd} = \mu_0 \mu^2/(3g) = a_{dd}/a$, i.e., the ratio between the strength of the dipolar and the contact interaction, eventually written in terms of the dipolar length $a_{dd}$ and the scattering length $a$. For the atom we are using, $a_{dd} = 131 a_B$, where $a_B$ is the Bohr radius. The third term of the Hamiltonian Eq. (2) corresponds to the local density approximation of the beyond-mean-field Lee-Huang-Yang (LHY) correction [18 [20], with

$$\gamma(\epsilon_{dd}) = \frac{16}{3\sqrt{\pi}} \frac{g\alpha^2}{\hbar^2} \text{Re} \left[ \int_0^\pi d\theta \sin \theta \left[ 1 + \epsilon_{dd}(3\cos^2 \theta - 1) \right] \right].$$

Experimental measurements and microscopic Monte Carlo calculations [21] have confirmed that the LHY term is an accurate correction to the mean-field theory given by the Gross-Pitaevskii equation both in dipolar gases and quantum mixtures [22 [29]. At the mean-field level, increasing the role of the dipolar interaction would lead to the collapse of the cloud because of the attractive component of the dipolar force. The collapse is prevented by a stronger confinement in the polarization direction $z$, $(\omega_z > \omega_x = \omega_y)$ causing the occurrence of a typical roton-like excitation spectrum [27], whose gap becomes smaller and smaller as one increases $\epsilon_{dd}$, and by the inclusion of the LHY term. Both effects are responsible for the emergence of new interesting phases. In particular a dipolar Bose gas confined in the polarization direction has been shown to be fully superfluid for a value of $\epsilon_{dd}$ lower than a certain critical value of the order of 1.3 (this value has a weak dependence on the trapping parameters and the total atom number $N$). Above this value – which in absence of the LHY effect would lead to collapse due to roton softening – the system presents supersolid properties characterized by density modulations and coherence between the density peaks. By further increasing $\epsilon_{dd}$, the system enters the crystal phase, where coherence between the density peaks is destroyed and global superfluidity is lost.

In Ref. [17] we have recently shown that supersolids are able to host quantized vortices in the low density region between the density peaks, with a deformed vortex core. In this work we determine the behaviour of the relevant properties of the vortices by changing $\epsilon_{dd}.$

In order to study vortices we add to the hamiltonian of the system the angular momentum constraint $-\Omega L_z$, where $L_z$ is the angular momentum operator and $\Omega$ is the angular rotation frequency, and solve the Gross-Pitaevskii equation either in imaginary or real time as explained in detail in the following. In particular, the ground state of a superfluid is insensitive to rotations for $\Omega$ lower than a critical value $\Omega_c$, above which the presence of a vortex in the system becomes energetically favourable. The new stable vortical configuration carries an angular momentum per particle equal to $\hbar$. Supersolids, instead, react to any value of the angular frequency due to the existence of a nonsuperfluid component [17] and for the same reason, as we explicitly show in the next section, the angular momentum carried by a vortex line is expected to be smaller than the usual value $\hbar$.

III. SINGLE VORTEX LINE

In this Section we analyze the properties of a single vortex line oriented along the $z$ direction and located at the center of the trap. We present the results for its core size, its energy and its angular momentum across the superfluid-to-supersolid transition. Furthermore we show that, similarly to the case of usual superfluids [28 [29], the nucleation of vortices in a rotating trap, is dictated by the quadrupole deformation of the superfluid component.
The structure of the vortex core in superfluids is deeply connected to a length called healing length. In condensates with only contact interactions, the healing length is computed as the half width at half maximum of the wave function. Keeping the same definition also the dipolar gases, seminal papers already studied in detail the dependence of the healing length on the scattering length for dipolar gases without the LHY correction \[30, 31\]. In these works, it has been shown that the healing length of the vortex increases by increasing \(\varepsilon\) until the gas collapses.

As mentioned above, quantum fluctuations prevent the collapse, and the supersolid phase emerges in the trapped geometry at higher values of \(\varepsilon_{dd}\). In Fig. 2 we report the healing length as the system goes from the superfluid to the supersolid regime, by solving in imaginary time the Gross-Pitaevskii equation in a rotating frame with angular frequency \(\Omega\). We find that the healing length keeps increasing till the transition point, after which a monotonic behaviour, but it remains roughly constant.

FIG. 2: Healing length defined as the vortex core size (see text) as a function of the scattering length \(a\) by crossing the superfluid to supersolid transition, both in the non-dipolar (solid line) and dipolar (points) case. The simulation’s parameters are: \(N = 40000\) atoms, \(\omega_{x,y,z} = 2\pi \times (60, 60, 120)\) Hz, and \(\Omega = 2\pi \times 12.7\) Hz.

A. Vortex core structure

The single vortex line is energetically stable only above a certain angular frequency \(\Omega_c\), which makes the energy in the rotating frame of the system with the vortex lower than the energy without the vortex. In Fig. 3(a) we show the numerical value of the critical rotation frequency for a stable vortex line in the trap center as a function of \(\varepsilon_{dd}\) across the whole phase diagram from the superfluid to the supersolid and to the droplet crystal regime. There are already several works \[30, 32\] accounting for the dependence of the critical rotation frequency as a function of \(\varepsilon_{dd}\) in dipolar condensates without including quantum fluctuations. In that case, \(\Omega_c\) increases with \(\varepsilon_{dd}\) reaching a maximum for \(\varepsilon_{dd} = 1\), and decreasing for larger values until the collapse is achieved. Thanks to the inclusion of the beyond mean-field term in the Gross-Pitaevskii equation one can go beyond the mean-field collapse, eventually entering the supersolid phase. We find that after the maximum is reached, the critical frequency keeps decreasing, showing a rather small jump at the transition to the supersolid regime, and continues decreasing until the crystal phase is reached.

B. Critical rotation frequency
C. Angular momentum carried by a vortex

In a fully superfluid system, the angular momentum per particle carried by a vortex line is \( \hbar \). In a partially superfluid system, this value should instead become smaller. The angular momentum carried by the vortex corresponds to the jump \( \Delta \hbar \) in the angular momentum per particle at \( \Omega = \Omega_c \) by increasing the angular velocity from below to above the critical value. We have determined such a jump across the whole zero-temperature phase diagram of the trapped dipolar gas and the value of \( \Delta \) is reported in Fig. 3(b). As intuitively expected, we find that \( \Delta = 1 \) in the superfluid phase, it decreases monotonically in the supersolid phase and eventually becomes zero in the droplet crystal phase. For completeness, the angular momentum per particle as a function of \( \Omega \) is reported in Figs. 3(c), (d) and (e), corresponding to the superfluid, supersolid and droplet crystal regime, respectively.

It is interesting to analyze the connection between \( \Delta \) and the superfluid fraction of the system, through the jump in the moment of inertia. Such a jump is only due to the presence of a superfluid part in a supersolid, and therefore \( \Delta \) itself is a natural quantity to evaluate the global superfluidity of the system. Another very relevant quantity to characterise the superfluidity of finite systems is the nonclassical rotation of inertia fraction that we studied in detail for the dipolar supersolid gas in Ref. [17].

The nonclassical rotation of inertia fraction is given by

\[
f_{\text{NCRI}} = 1 - \frac{\Theta}{\Theta_{\text{rig}}},
\]

where \( \Theta \) is the moment of inertia of the system and \( \Theta_{\text{rig}} \) is its rigid body value. As pointed out by Leggett [33] in cylindrical annulus, \( f_{\text{NCRI}} \) coincides with the superfluid fraction \( f_s \). In Fig. 1(b) we compare \( f_{\text{NCRI}} \) and \( \Delta \). In the superfluid phase, they are both equal to 1, i.e., the whole system is superfluid. In the supersolid region they start deviating from each other with \( f_{\text{NCRI}} > \Delta \). In the droplet crystal phase \( f_{\text{NCRI}} \) remains finite, while \( \Delta = 0 \). The reason is due to the fact that each density peak (droplet) in the crystal phase is superfluid by itself, increasing therefore the value of the nonclassical rotation of inertia. On the other hand \( \Delta \) only accounts for the superfluid component participating to the rotation due to the presence of a vortex phase in the order parameter.

D. Quadrupole Instability and Vortex Nucleation

In this Section we address the problem of vortex nucleation by rotating the harmonic trap, as implemented for the first time almost 20 years ago for a standard \(^{87}\)Rb BEC [29]. The dynamics of vortex nucleation in ordinary (non-dipolar) condensates in a rotating trap has been extensively studied. Vortex nucleation is induced by the introduction of a suitable rotating deformation of the trap, characterized by a rotation frequency \( \Omega \) and deformation parameter \( \epsilon = (\omega_x^2 - \omega_y^2)/(\omega_x^2 + \omega_y^2) \).

It turns out that there exists indeed a critical frequency for vortex nucleation \( \Omega_{vn} \) [13, 29], which is significantly higher than the one at which a vortex becomes energetically favourable. The reason is due to the presence of an energetic barrier [34] for the vortex to enter due to the need of creating a density depletion at the vortex position. In Refs. [28, 35] it was shown that for rotating harmonic traps, the mechanism of vortex nucleation is triggered by the dynamic instability of the quadrupole mode, according to the resonance condition

\[
\Omega_{vn} = \omega_q/2,
\]

with \( \omega_q = \sqrt{2} \omega_\perp \) the frequency of the quadrupole mode in the absence of rotation [36]. The dynamical instability leads to the spontaneous breaking of the cylindrical symmetry of the cloud creating the condition for vortices to be nucleated [28, 35].

Not surprisingly, considering the Gross-Pitaevskii equation without the LHY term, dipolar superfluids show the same kind of quadrupole instability [30, 37], which could therefore drive vortex nucleation, when the resonance condition is satisfied. Notice however that for a dipolar gas, the quadrupole frequency \( \omega_q \) is not simply given by \( \sqrt{2} \omega_\perp \), but it depends on the interaction strength and the trapping parameters of the dipolar gas (see, e.g. [37]). Quite remarkably, by direct numerical simulations, we have shown that the critical frequency for
A very small trap deformation ($\epsilon = 3.33 \times 10^{-3}$) to trigger the instability is also added. We observe that for any value of $\epsilon_{dd}$ there exists a critical angular frequency $\Omega_{vn}$, such that for $\Omega \geq \Omega_{vn}$ strong cloud deformations occur followed by the nucleation of a vortex. Typical snapshots during the time evolution are reported in Fig. 6, where lengths are in units of the geometric mean of the three harmonic oscillator lengths $a_{ho} = \sqrt{\hbar/m} \times (\omega_x \omega_y \omega_z)^{-1/6}$.

In Fig. 5 we report the calculated values of $\Omega_{vn}$ multiplied by the factor 2 in order to make the comparison with the lowest quadrupole frequency more direct and explicit. The resonance condition Eq. 5 considering the (superfluid) lower frequency mode is met.

A comment on the time scale for vortex nucleation is due here. Our simulation predicts rather long times (of the order of 1 second) for the vortex nucleation. However, in real experiments, noise and thermal effects are expected to trigger the instability on a much faster time scale. Larger trap deformations also help in speeding up the nucleation process.

IV. VORTEX LATTICES IN DIPOLAR SUPERSONDS

A superfluid can host more than a single vortex line, if the rotational frequency is large enough. Many vortices form, in the $z = 0$ plane, a 2D triangular lattice called Abrikosov lattice [14, 15]. The aim of this Section is to address the question of whether and how a dipolar superfluid can host many vortices. The presence of the regular (triangular lattice) density modulation can indeed interfere with the formation of the Abrikosov lattice, when the intervortex distance (which scales as $\Omega^{-1/2}$) becomes of the same order of the distance between the density peaks (the latter being fixed by the roton wave vector [27], which in supersolids is of the order of a few units of the axial oscillator length $\sqrt{\hbar/m \omega_z}$).

We have first checked that the triangular Abrikosov lattice persists in the whole superfluid regime, also in the presence of the dipolar interaction and LHY term, till the transition to the supersolid phase [27].

In the supersolid phase, it is energetically favourable to accommodate the vortices in the low density regions. Therefore when the vortex distance is of the same order of the distance between the density peaks, there is a competition between the natural tendency of vortices forming the triangular Abrikosov lattice and the vortices occupying the valleys of the supersolid density. In the panel (b) of Fig. 6 we show the numerical result of the imaginary-time evolution of the extended Gross-Pitaevskii equation in a rotating frame for large enough angular velocities. We obtain that the vortices are pinned by the minima of the supersolid density modulations forming – for the chosen value $\Omega = 2/3 \omega_x$ – a honeycomb lattice. It is instructive to compare it with the system for $\Omega = 0$, reported in panel (a). The increase of the cloud’s radius
(and the density reduction) due to the centrifugal potential allows a larger number of peaks to be hosted in the system.

It is important to notice that in the literature the pinning of Abrikosov vortex lattices in Bose gases has been addressed by a number of authors considering an underlying (square or triangular) rotating optical lattice [39–42], with a nice early experimental demonstration of the transition from the “natural” Abrikosov lattice to the pinned vortex lattice by Eric Cornell and coworkers [43]. We remind, however, that in our case, the structure of the density modulation is not imposed by an external potential, but is due to the spontaneous breaking of translational symmetry, yielding the supersolid phase.

The pinned honeycomb lattice persists as long as Ω is not too large for the intervortex distance to become smaller than the period of the supersolid lattice. By further increasing Ω we find that the vortices are hosted in the low density regions surrounding the droplets as shown in the panel (c) of Fig. 6 for Ω = 5/6 ωx.

V. EXPANDING A SUPERSOLID WITH VORTEX LINES

The previously reported results considered the possibility of addressing the system in-situ. Here we briefly discuss the effect of letting the cloud to expand, i.e., after switching off the trap in the transverse (z = 0) plane, in order to image the system with a better space resolution. We consider both the single and the many-vortex case.

Figure 7 shows the density profiles of a dipolar supersolid with and without a vortex line at two different times after the removal of the trap [43]. The ratio between the peak density and the central density, in the absence of the vortex, is less than 10 and it further decreases during the expansion, the minimum of the density remaining of the same order as that of an ordinary superfluid. Thus, with our choice of parameters, a good imaging system could easily identify the presence of the vortex in the center of the trap.

For the high angular frequency case, when many vortices appear, we consider the most interesting case, when the vortices form a honeycomb lattice, as in Figure 6 b). The expansion at two different times after switching off the transverse confinement is reported in Fig. 8 [19]. In

FIG. 6: Density plots of N = 110000 164 Dy atoms in the supersolid phase for εdd = 1.409, confined in an harmonic trap with frequencies ωx = 2ωx = 2π × 120 Hz. Panel a): non-rotating gas Ω = 0, Panel b) Ω = 2/5ωx and Panel c) Ω = 5/6ωx.

FIG. 7: In-situ density profiles along the z = 0 plane (panels (a) to (f)), and cuts along the x-axis (panels (g-i)) of an expanding dipolar supersolid in the absence (a-c) and in presence (d-f) of a vortex. The initially prepared ground state in the presence (d) of a vortex has been obtained for Ω = 2π × 20 Hz. Panel (g) shows the corresponding density cut along the x-axis. The red (black) line corresponds to the case with (without) vortex. Panels (h) and (i) show corresponding cuts along the x-axis. The other parameters are the same as in Fig. 6. A full movie of the expansion is available as Supplementary Material.
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the supersolid phase. We have found that the width
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Remarkably we could show that the nuclea-
ation of a vortex is always triggered by the softening of
the lowest quadrupole mode frequency (see Fig. 6), as in
standard Bose-Einstein condensates with contact inter-
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At large enough angular velocity we have addressed
the problem of the spatial arrangement of many vortices.
The supersolid phase forces the vortices to be pinned in
the density valleys. In particular we have shown that
vortices arrange in a regular honeycomb lattice when the
intervortex distance is of the same order of the solid peri-
docity (Fig. 6 (b)), the novel supersolid vortex structure
being preserved during the free expansion, following the
release of the trap.

VI. CONCLUSIONS

In this work we have provided a first comprehensive
study of quantized vortex lines in the supersolid phase of
an ultracold dipolar gas rotating in the plane orthogonal
to the polarization direction of the dipole moment of
the atoms. The analysis has been carried out by means of the
generalised Gross-Pitaevskii equation (Eqs. 1-2), includ-
ing LHY corrections for the stabilisation of the supersolid
phase.

We have studied in detail the stationary properties and
the nucleation dynamics of a single vortex line hosted in

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