Abstract

A graph $G$ is list $(b : a)$-colorable if for every assignment of lists of size $b$ to vertices of $G$, there exists a choice of an $a$-element subset of the list at each vertex such that the subsets chosen at adjacent vertices are disjoint. We prove that for every positive integer $a$, the family of minimal obstructions of girth at least five to list $(3a : a)$-colorability is strongly hyperbolic, in the sense of the hyperbolicity theory developed by Postle and Thomas. This has a number of consequences, e.g., that if a graph of girth at least five and Euler genus $g$ is not list $(3a : a)$-colorable, then $G$ contains a subgraph with $O(g)$ vertices which is not list $(3a : a)$-colorable.

While it is NP-hard to decide whether a planar graph is 3-colorable [9], Grötzsch [10] proved that every planar triangle-free graph is 3-colorable. These facts motivated the development of a rich theory of 3-colorability of triangle-free embedded graphs. Thomassen [17] proved that there are only finitely many 4-critical graphs (minimal obstructions to 3-colorability) of girth at least five drawn in any fixed surface. For triangle-free graphs, the situation is more complicated, as there are infinitely many minimal triangle-free non-3-colorable graphs which can be drawn in any surface other than the sphere. Nevertheless, Dvořák, Král’ and Thomas [5] gave a rough structural characterization of 4-critical triangle-free embedded graphs, sufficient to design a linear-time algorithm to test 3-colorability of triangle-free graphs drawn in a fixed surface [6]. A more detailed description of such 4-critical graphs was given by Dvořák and Lidický [7].

Before discussing further questions related to 3-colorability of embedded triangle-free graphs, let us briefly introduce the theory of hyperbolicity developed by Postle and Thomas [15]. A class $G$ of graphs embedded in closed surfaces (which possibly can have a boundary) is hyperbolic if there exists a constant $c_G$ such that for each graph $G \in G$ embedded in a surface $\Sigma$ and each open disk $\Lambda \subset \Sigma$ whose boundary $\partial \Lambda$ intersects $G$ only in vertices, the number of vertices of $G$ in $\Lambda$ is at most $c_G(|\partial \Lambda \cap G| - 1)$. The class is strongly hyperbolic if the same holds for all sets $\Lambda \subset \Sigma$ homeomorphic to an open cylinder (sphere with two holes). The importance of these notions in the study of graph colorings stems from the following facts. Firstly, the classes of minimal obstructions to many
kinds of colorings form strongly hyperbolic families. This is the case e.g. for 6-critical graphs (minimal obstructions to 5-colorability) \[14, 12\] and for 4-critical graphs of girth at least five \[4\]. Secondly, hyperbolicity or strong hyperbolicity is often relatively easy to establish, as it only involves dealing with the planar subgraphs drawn in \(\Lambda\). Finally, hyperbolicity and strong hyperbolicity has a number of important consequences, such as the following. Recall the edge-width of a graph drawn in a surface is the length of a shortest non-contractible cycle of the graph.

**Theorem 1** (Postle and Thomas \[15\]). *If \(G\) is a hyperbolic class, then each graph in \(G\) drawn in a surface without boundary of Euler genus \(g\) has edge-width \(O(\log g)\).*

Thus, e.g., there exists a constant \(c_g = O(\log g)\) such that every graph of girth at least five drawn in a surface of Euler genus \(g\) with edge-width at least \(c_g\) is 3-colorable.

**Theorem 2** (Dvořák and Kawarabayashi \[4\], Postle and Thomas \[15\]). *For any hyperbolic class \(G\), a surface \(\Sigma\), and an integer \(k\), there exists a linear-time algorithm to decide whether a graph drawn in \(\Sigma\) with at most \(k\) vertices contained in the boundary of \(\Sigma\) has a subgraph belonging to \(G\).*

Thus, e.g., it is possible to test in linear time whether a graph of girth at least five drawn in a fixed surface is 3-colorable, or more generally, whether a precoloring of a bounded number of vertices in such a graph extends to a 3-coloring.

**Theorem 3** (Postle and Thomas \[15\]). *If \(G\) is a strongly hyperbolic class, then each graph in \(G\) drawn in a surface \(\Sigma\) of Euler genus \(g\) with boundary \(\partial \Sigma\) has \(O(g + |\partial \Sigma \cap V(G)|)\) vertices.*

Thus, e.g., for every integer \(g \geq 0\), there exists a constant \(s_g = O(g)\) such that every non-3-colorable graph of girth at least five and Euler genus \(g\) contains a non-3-colorable subgraph with at most \(s_g\) vertices.

With these results in mind, let us return to the discussion of other variants of graph coloring in the context of triangle-free embedded graphs, especially the list coloring and the fractional coloring. Let \(L\) be an assignment of lists of colors to vertices of a graph \(G\). An \(L\)-coloring of \(G\) is a proper coloring of \(G\) such that the color of each vertex \(v\) belongs to the list \(L(v)\). A graph \(G\) is list \(k\)-colorable if it is \(L\)-colorable for every assignment \(L\) of lists of size at least \(k\). Clearly, a list \(k\)-colorable graph is also \(k\)-colorable. The direct counterpart of Grötzsch' theorem \[10\] is false: Voigt \[18\] found a triangle-free planar graph which is not list 3-colorable. On the other hand, Thomassen \[10\] proved that planar graphs of girth at least five are list 3-colorable. Furthermore, Dvořák and Kawarabayashi \[3\] proved that minimal obstructions of girth at least 5 to list 3-colorability are hyperbolic, and finally Postle \[13\] proved they are strongly hyperbolic. Hence, all the aforementioned results for 3-coloring of embedded graphs of girth at least five also apply in the list coloring setting.

Let us now turn our attention to fractional coloring. A function that assigns sets to all vertices of a graph is a set coloring if the sets assigned to adjacent vertices are disjoint. For positive integers \(a\) and \(b\), a \((b : a)\)-coloring of a graph \(G\) is a set coloring which to each vertex assigns an \(a\)-element subset of \(\{1, \ldots, b\}\).
The concept of \((b : a)\)-coloring is a generalization of the conventional vertex coloring. In fact, a \((b : 1)\)-coloring is exactly an ordinary proper \(b\)-coloring. The fractional chromatic number of \(G\), denoted by \(\chi_f(G)\), is the infimum of the fractions \(b/a\) such that \(G\) admits a \((b : a)\)-coloring. Note that \(\chi_f(G) \leq \chi(G)\) for any graph \(G\), where \(\chi(G)\) is the chromatic number of \(G\).

Grötzsch’ theorem can be improved only very mildly in the fractional coloring setting. Jones [11] found for each integer \(n\) such that \(n \equiv 2 \pmod{3}\) an \(n\)-vertex triangle-free planar graph with fractional chromatic number exactly \(3 - \frac{3}{n+1}\). On the other hand, Dvořák, Sereni and Volec [8] proved that every triangle-free planar graph with \(n\) vertices is \((9n : 3n + 1)\)-colorable, and thus its fractional chromatic number is at most \(3 - \frac{3}{3n+1}\). The examples given by Jones [11] have many 4-cycles, leading Dvořák and Mnich [2] to conjecture the following.

**Conjecture 4.** There exists a constant \(c < 3\) such that every planar graph of girth at least five has fractional chromatic number at most \(c\).

While the conjecture is open in general, in a followup paper we will prove that it holds for graphs with bounded maximum degree (i.e., for any \(\Delta\), there exists \(c_\Delta < 3\) such that planar graphs of girth at least five and maximum degree at most \(\Delta\) have fractional chromatic number at most \(c_\Delta\)). As a key step in that argument, we need to show that the class of minimal obstructions of girth at least five to \((6 : 2)\)-colorability is strongly hyperbolic.

Since every 3-colorable graph is also \((6 : 2)\)-colorable, it might seem obvious that this can be done by some modification of the argument of Dvořák, Král’ and Thomas [4] for 3-colorings. Somewhat surprisingly, this turns out not to be the case (the argument is based on reducible configurations, and even the simplest one—a 5-cycle of vertices of degree three—does not work for \((6 : 2)\)-coloring). Fortunately, the list-coloring argument of Postle [13] does the trick, subject to extensive groundwork and some minor modifications. As an added benefit, we obtain the result for the list variant of fractional coloring. For an assignment \(L\) of lists to vertices of a graph \(G\) and a positive integer \(a\), a set coloring \(\varphi\) of \(G\) is an \((L : a)\)-coloring if \(\varphi(v)\) is an \(a\)-element subset of \(L(v)\) for every \(v \in V(G)\). Let \(S\) be a proper subgraph of \(G\). We say \(G\) is \((a, L, S)\)-critical if for every proper subgraph \(H\) of \(G\) containing \(S\), there exists an \((L : a)\)-coloring of \(S\) which extends to an \((L : a)\)-coloring of \(H\), but not to an \((L : a)\)-coloring of \(G\). In particular, denoting by \(\emptyset\) the null subgraph (with no vertices and edges), \(G\) is \((a, L, \emptyset)\)-critical if and only if \(G\) is a minimal non-\((L : a)\)-colorable graph. Our main result is the following.

**Theorem 5.** Let \(G\) be the class of graphs of girth at least five drawn in surfaces such that if \(G \in \mathcal{G}\) is drawn in a surface \(\Sigma\) and \(S\) is the subgraph of \(G\) drawn in the boundary of \(\Sigma\), then \(G\) is \((a, L, \emptyset)\)-critical for some positive integer \(a\) and an assignment \(L\) of lists of size at least \(3a\) to vertices of \(G\). Then \(\mathcal{G}\) is strongly hyperbolic.

Thus, for example, by Theorem [3] for every integer \(g \geq 0\), there exists a constant \(s_g = O(g)\) such that if \(G\) is a graph of girth at least five drawn in a surface of Euler genus \(g\), and \(G\) is not \((L : a)\)-colorable for some positive integer \(a\) and an assignment \(L\) of lists of size at least \(3a\) to vertices of \(G\), then \(G\) contains a subgraph with at most \(s_g\) vertices which also is not \((L : a)\)-colorable.
The main difficulty in proving Theorem 5 is the need to establish the following result. For a positive integer \( a \) and an assignment \( L \) of lists of size \( 2a \) or \( 3a \) to vertices of a graph \( G \), a flaw is an edge joining two vertices with lists of size \( 2a \).

**Theorem 6.** Let \( a \) be a positive integer. Let \( G \) be a plane graph of girth at least 5 and let \( f_1 \) and \( f_2 \) be faces of \( G \). Let \( L \) be a list assignment for \( G \) such that \( |L(v)| \in \{2a, 3a\} \) for all \( v \in V(G) \) and all vertices with list of size \( 2a \) are incident with \( f_1 \) or \( f_2 \). If each flaw is at distance at least three from any other vertex with list of size \( 2a \) and at least four from any other flaw, then \( G \) is \((L : a)\)-colorable.

A weaker form of Theorem 6 for list 3-coloring was proven by Thomassen [17]; in his formulation, the lists must be subsets of \( \{1, 2, 3\} \), and more assumptions are made on distances between flaws and other vertices with lists of size \( 2a \). Thomassen’s argument can essentially be modified to work for list \((3a : a)\)-colorings as well; however, the argument is quite long and complicated and as presented in [17], leaves quite a lot of details to the reader to work out. Hence, rather than trying to verify all the details in the fractional list coloring setting, we developed a simpler proof along the same lines, which takes the majority of this paper. First, in Section 1, we prove an auxiliary claim (Theorem 7) regarding graphs with a precolored path and at most two flaws. Using this result, we can relatively easily deal with the most technical parts of the proof of Theorem 6—the case where \( f_1 \) and \( f_2 \) are close to each other. Theorem 6 is established in Section 2. Finally, in Section 3, we discuss the modifications to the argument of Postle [13] needed to prove Theorem 5.

1 Graphs with two flaws

Let \( a \) be a positive integer, let \( G \) be a plane graph and let \( P \) be a path contained in the boundary of its outer face. A list assignment \( L \) for \( G \) is \((a, P)\)-valid if \( |L(v)| = 3a \) for every vertex \( v \in V(G) \) not incident with the outer face of \( G \), \( |L(v)| \in \{2a, 3a\} \) for every vertex \( v \in V(G) \setminus V(P) \) incident with the outer face of \( G \), and \( P \) is \((L : a)\)-colorable. A flaw is an edge \( uv \) of \( G \) with \( u, v \notin V(P) \) such that \( |L(u)| = |L(v)| = 2a \). The first vertex \( p \) of \( P \) is adjacent to the flaw \( uv \) if \( pu \in E(G) \), and it is connected to the flaw \( uv \) if either \( pu \in E(G) \), or \( G \) contains a path \( pxuvy \) with \( x, y \notin V(P) \) and \( |L(x)| = 2a \). If \( p \) is connected to a flaw in a unique way and \( P \) has length two, then let \( c(G, P, L) \) be the set defined as follows. If \( pu \in E(G) \), then let \( c(G, P, L) = L(u) \). Otherwise, let \( c' \) be an \( a \)-element subset of \( L(y) \setminus L(u) \) and let \( c(G, P, L) \) be an \( a \)-element subset of \( L(x) \setminus c' \). If \( P \) has length at most one or \( p \) is not connected to a flaw, then let \( c(G, P, L) = \emptyset \). For a set \( c \) of colors and a vertex \( p \), an \((L : a)\)-coloring is \((p, c)\)-disjoint if the color set of \( p \) is disjoint from \( c \). For a cycle \( C \) in \( G \), let \( \text{int}(C) \) denote the subgraph of \( G \) drawn in the closed disk bounded by \( C \), and let \( \text{ext}(C) \) denote the subgraph of \( G \) drawn in the complement of the open disk bounded by \( C \).

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1Let us remark out writeup is a bit longer than Thomassen’s (15 pages including the auxiliary results compared to 11 pages), but we go into substantially greater detail in its presentation.
Theorem 7. Let \( a \) be a positive integer, let \( G \) be a plane graph of girth at least 5 and let \( P = p_0 \ldots p_k \) be a path of length at most two contained in the boundary of its outer face. Let \( L \) be an \((a, P)\)-valid list assignment for \( G \) with at most two flaws, such that distance between flaws is at least three. Assume furthermore that if \( \ell = 2 \), then either \( p_0 \) is not connected to a flaw or \( p_0 \) is connected to a flaw in a unique way. Then every \((p_0, c(G, P, L))\)-disjoint \((L : a)\)-coloring of \( P \) extends to an \((L : a)\)-coloring of \( G \).

Proof. Suppose for a contradiction that \( G \) is a counterexample with \(|V(G)| + |E(G)|\) minimum, and subject to that with \( \ell \) maximum. Let \( c = c(G, P, L) \) and let \( \psi \) denote a \((p_0, c)\)-disjoint \((L : a)\)-coloring of \( P \) which does not extend to an \((L : a)\)-coloring of \( G \) (such a coloring exists since \( G \) is a counterexample).

Note that \(|L(v)| \leq \deg(v)a\) for every \( v \in V(G) \setminus V(P) \), as otherwise \(|L(v)| \geq (\deg(v) + 1)a\) by the assumption that \( L \) is \((a, P)\)-valid, \( \psi \) extends to an \((L : a)\)-coloring of \( G - v \) by the minimality of \( G \), and we can greedily color \( v \), obtaining an \((L : a)\)-coloring of \( G \) which extends \( \psi \), contradicting the choice of \( \psi \). In particular, all vertices not incident with the outer face have degree at least three. Furthermore, \( G \) is clearly connected.

Suppose that \( G \) is not 2-connected; then \( G = G_1 \cup G_2 \) for proper induced subgraphs \( G_1 \) and \( G_2 \) intersecting in a single vertex \( v \). Without loss of generality, \( P \not\subseteq G_2 \). Let \( P_1 = P \cap G_1 \). If \( P_1 \subseteq G_1 \), then let \( P_2 = v \), otherwise let \( P_2 = P \setminus G_2 \). By the minimality of \( G \), the restriction of \( \psi \) to \( P_1 \) extends to an \((L : a)\)-coloring \( \varphi_1 \) of \( G_1 \). Since \( P_2 \) has length at most one, the restriction of \( \psi \cup \varphi_1 \) to \( P_2 \) extends to an \((L_2 : a)\)-coloring \( \varphi_2 \) of \( G_2 \) by the minimality of \( G \).

Then \( \varphi_1 \cup \varphi_2 \) is an \((L : a)\)-coloring of \( G \) extending \( \psi \), which is a contradiction. Hence, \( G \) is 2-connected.

Let \( K \) be the cycle bounding the outer face of \( G \). Analogously to the previous paragraph, we see that if \( K \) has a chord, then \( P \) has length three and the chord is incident with \( p_1 \). Suppose \( p_1v \) is such a chord, and let \( G = G_1 \cup G_2 \) for proper induced subgraphs \( G_1 \) and \( G_2 \) intersecting in \( p_1v \), where \( p_0 \in V(G_1) \) and \( p_2 \in V(G_2) \). By the minimality of \( G \), the restriction of \( \psi \) to \( p_1p_2 \) extends to an \((L : a)\)-coloring \( \varphi_2 \) of \( G_2 \). Note that \( c(G_1, p_1p_1v, L) \subseteq c(G, P, L) \), and thus the restriction of \( \psi \cup \varphi_2 \) to \( p_0p_1v \) extends to an \((L : a)\)-coloring \( \varphi_1 \) of \( G_1 \) by the minimality of \( G \). Then \( \varphi_1 \cup \varphi_2 \) is an \((L : a)\)-coloring of \( G \) extending \( \psi \), which is a contradiction. Consequently, \( K \) is an induced cycle.

Claim 1. Let \( H \) be a proper subgraph of \( G \) and let \( Q = q_0 \ldots q_k \) be an induced path in \( H \) contained in the boundary of the outer face of \( H \). For \( 4 \leq k \leq 8 \), Suppose that \( P \cap H \subseteq Q \), every vertex \( v \in V(H) \setminus V(Q) \) satisfies \(|L(v)| = 3a\), and the distance between \( q_0 \) and \( q_k \) in \( H \) is at least \( 4 \). If no vertex of \( H \) has more than two neighbors in \( Q \), then every \((L : a)\)-coloring \( \psi \) of \( Q \) extends to an \((L : a)\)-coloring of \( H \).

Subproof. Suppose first \( k \leq 6 \). Let \( L' \) be the list assignment for \( H - \{q_0, q_k\} \) obtained from \( L \) by setting \( L'(q_1) = \psi(q_1) \cup \psi(q_2), L'(q_{k-1}) = \psi(q_{k-1}) \cup \psi(q_{k-2}) \), and by choosing \( L'(v) \) for each vertex \( v \not\in \{q_1, q_{k-1}\} \) with a neighbor \( q \in \{q_0, q_k\} \) as a \( 2a \)-element subset of \( L(v) \setminus \psi(q) \). Note that the neighbors of \( q_0 \) and \( q_k \) form an independent set, since \( G \) has girth at least 5 and the distance between \( q_0 \) and \( q_k \) is at least 4; and in particular \( H - \{q_0, q_k\} \) with list assignment \( L' \) has no flaws. By the minimality of \( G \), we conclude that the restriction of \( \psi \) to \( q_2 \ldots q_{k-2} \)
extends to an \((L':a)\)-coloring of \(H - \{q_0, q_k\}\), which gives an \((L:a)\)-coloring of \(H\) extending \(\psi\).

For \(k \geq 7\), we prove the claim by induction on the number of vertices of \(H\). Suppose \(H\) contains an induced cycle \(C\) of length at most 8 with \(\text{int}(C) \neq C\). By the induction hypothesis, \(\psi\) extends to an \((L:a)\)-coloring \(\varphi_1\) of \(\text{ext}(C)\). Let \(uv\) be an edge of \(C\); since \(G\) has girth at least 5, the distance between \(u\) and \(v\) in \(\text{int}(C) - uv\) is at least 4. Since \(G\) has girth at least 5, no vertex has more than two neighbors in \(C\). By the induction hypothesis, the restriction of \(\varphi_1\) to the path \(C - uv\) extends to an \((L:a)\)-coloring of \(\text{int}(C) - uv\), giving an \((L:a)\)-coloring of \(H\) extending \(\psi\). Hence, we can assume that \(\text{int}(C) = C\) for every induced cycle of length at most 8 in \(H\).

Suppose some vertex \(v \notin V(Q)\) has two neighbors \(q_i\) and \(q_j\) in \(Q\), with \(i < j\). Note that \(j - i \leq k - 2\), since the distance between \(q_0\) and \(q_k\) is at least 4. Then \(H = H_1 \cup H_2\), where \(H_1\) and \(H_2\) are proper induced subgraphs of \(H\) intersecting in \(q_iq_j\) and the outer face of \(H_1\) is bounded by the cycle \(C_1 = q_i \ldots q_jv\). Since \(v\) has at most two neighbors in \(Q\), the cycle \(C_1\) is induced, and since its length is \(j - i + 2 \leq k \leq 8\), we have \(H_1 = \text{int}(C) = C\). Let \(P_2 = q_0 \ldots q_iq_j \ldots q_k\). Note that no vertex of \(H_2\) can be adjacent to \(v\) and two other vertices of \(P_2\), since the distance between \(q_0\) and \(q_k\) is at least 4 \(\geq k - 4\). By induction hypothesis, we can extend the coloring of \(P_2\) given by \(\psi\) and by coloring \(v\) by an \(a\)-element subset of \(L(v) \setminus (\psi(q_i) \cup \psi(q_j))\) to an \((L:a)\)-coloring of \(H_2\). This gives an \((L:a)\)-coloring of \(H\) extending \(\psi\). Hence, we can assume that no vertex \(v \notin V(Q)\) has more than one neighbor in \(Q\).

If \(k = 8\), then note that by planarity and the assumption that \(G\) has girth at least 5, \(G\) cannot contain both a path of length three between \(q_0\) and \(q_i\) and between \(q_i\) and \(q_k\). By symmetry, we can assume \(G\) does not contain such a path between \(q_0\) and \(q_i\). In case \(k = 7\), this is true as well by the assumption that the distance between \(q_0\) and \(q_k\) is at least 4. We consider the graph \(H' = H - \{q_0, q_i, q_k\}\) with the list assignment \(L'\) obtained as follows: we set \(L'(q_i) = \psi(q_i)\) and \(\psi(q_i) = \psi(q_i)\) for \(i \in \{5,6\}\), and for each vertex \(v \notin V(Q)\) with a neighbor \(x \in \{q_i, q_7, q_k\}\), we choose \(L'(v)\) as a 2\(a\)-element subset of \(L(v) \setminus \psi(x)\). Since \(G\) has girth at least 5, since no vertex \(v \in V(H) \setminus V(Q)\) has more than one neighbor in \(Q\), and since the distance between \(q_0\) and \(\{q_i, q_k\}\) is at least 4, we conclude that \(q_iq_k\) is the only flaw in \(H'\), and that this flaw is not connected to \(q_2\). By the minimality of \(G\), the restriction of \(\psi\) to \(q_2q_iq_k\) extends to an \((L' : a)\)-coloring of \(H'\), which gives an extension of \(\psi\) to an \((L : a)\)-coloring of \(H\).

A cycle \(C\) in \(G\) is tame if \(\text{int}(C) = C\), or \(|C| \geq 8\) and \(\text{int}(C)\) consists of \(C\) and its chord, or \(|C| \geq 9\) and \(\text{int}(C)\) consists of \(C\) and a vertex with three neighbors in \(C\).

**Claim 2.** All \((\leq 9)\)-cycles in \(G\) are tame.

**Subproof.** Suppose that \(G\) has a non-tame \((\leq 9)\)-cycle, and choose such a cycle \(C\) with \(\text{int}(C)\) minimal. Note that \(C\) is an induced cycle in \(\text{int}(C)\) and no vertex of \(\text{int}(C)\) has three neighbors in \(C\), as otherwise by the assumption that \(C\) is not tame, there would exist a \((\leq 6)\)-cycle \(C' \neq C\) in \(\text{int}(C)\) with \(\text{int}(C') \neq C'\). But then \(C'\) is not tame, contradicting the choice of \(C\).

By the minimality of \(G\), \(\psi_C\) extends to an \((L:a)\)-coloring \(\varphi_1\) of \(\text{ext}(C)\). Let \(uv\) be any edge of \(C\). Since \(G\) has girth at least 5, the distance between \(u\) and \(v\)
Let coloring of $G$ be $\varphi$ such that neither $u$ is in a path connecting $u$ to a flaw in $G$. By the minimality of $G$, it follows that there exists a cycle $C = u$ in $G$. Consequently, the path connecting $u$ to a flaw in $G$ is at least three, this is only possible if $u$ is adjacent to $x_2$, $x_3$, and $x_9$, and $|L(x_1)| = |L(x_3)| = |L(x_9)| = 2a$. In either case, $\psi$ extends to an $(L : a)$-coloring of $G$ by the choice of $c = c(G, L, P)$, which is a contradiction. Since $K$ is an induced cycle, it follows that $K$ is not tame, and by Claim 2, we have $|K| \geq 10$.

Claim 4. Let $Q = u_1u_2u_3$ be a path in $G$ with $u_1, u_3 \in V(K)$ and $u_2 \notin V(K)$, such that neither $u_1$ nor $u_3$ is the middle vertex of $P$ when $P$ has length two. Let $G = G_1 \cup G_2$, where $G_1$ and $G_2$ are proper induced subgraphs of $G$ intersecting in $Q$ and $P \subset G_1$. Then for $i \in \{1, 3\}$, either $u_i \in V(P)$ or $|L(u_i)| = 3a$, and $u_i$ is connected to a flaw in $G_2$ in a unique way.

Subproof. By the minimality of $G$, $\psi$ extends to an $(L : a)$-coloring $\varphi_1$ of $G_1$. Let $\psi$ be the restriction of $\varphi_1$ to $u_1u_2u_3$. Since $\psi$ does not extend to an $(L : a)$-coloring of $G$, we conclude that $\psi$ does not extend to an $(L : a)$-coloring of $G_2$. By the minimality of $G$, it follows that for $i \in \{1, 3\}$, the vertex $u_i$ is connected to a flaw in $G_2$. Since the distance between flaws in $G$ is at least three, this excludes the case that $u_i \notin V(P)$ and $|L(u_i)| = 2a$. Furthermore, assume that $Q$ has an endvertex with list of size $2a$ not belonging to $V(P)$. Then $Q$ is a subpath of $K$.
Claim 6. Let $Q = u_1u_2u_3u_4$ be a path in $G$ with $u_1 \in V(K)$, $u_4 \in V(K) \setminus V(P)$, and $u_2, u_3 \notin V(K)$, such that $|L(u_4)| = 2a$ and $u_1$ is not the middle vertex of $P$ when $P$ has length two. Let $G = G_1 \cup G_2$, where $G_1$ and $G_2$ are proper induced subgraphs of $G$ intersecting in $Q$ and $P \subset G_1$. Then $u_1$ and $u_3$ are connected to flaws in $G_2$ in a unique way, and any $(u_1, e(G_2, u_1u_2u_3, L))$-disjoint $(L : a)$-coloring of $u_1u_2u_3u_4$ extends to an $(L : a)$-coloring of $G_2$.

Subproof. By the minimality of $G$, $\psi_c$ extends to an $(L : a)$-coloring $\varphi_1$ of $G_1$. Let $\psi$ be the restriction of $\varphi_1$ to $u_1u_2u_3$ and let $L'$ be the list assignment obtained from $L$ by setting $L'(u_4) = \varphi_1(u_4) \cup \varphi_1(u_4)$. Since $\psi_c$ does not extend to an $(L : a)$-coloring of $G$, the precoloring $\psi$ of $u_1u_2u_3$ does not extend to an $(L' : a)$-coloring of $G_2$. By the minimality of $G$, it follows that $u_1$ and $u_3$ are connected to flaws in $G_2$, and by Claim 3 the paths connecting them to flaws are subpaths of $K + u_3u_4$; hence, they are unique.

Consider now a $(u_1, e(G_2, u_1u_2u_3, L))$-disjoint $(L : a)$-coloring $\theta$ of $u_1u_2u_3u_4$. Let $L''$ be the list assignment obtained from $L$ by setting $L''(u_4) = \theta(u_3) \cup \theta(u_4)$. Note that $e(G_2, u_1u_2u_3, L) = e(G_2, u_1u_2u_3, L'')$, and thus the restriction of $\theta$ to $u_1u_2u_3$ extends to an $(L'' : a)$-coloring $\varphi_2$ of $G_2$ by the minimality of $G$. Observe that $\varphi_2$ is an $(L : a)$-coloring of $G_2$ which extends $\theta$.

Let $K = p_0 \ldots p_0v_1v_2 \ldots$, and let $v_0 = p_0$. Suppose first that $p_0$ is adjacent to a flaw, necessarily $v_1v_2$ since $K$ is an induced cycle. If $\ell \leq 1$, then let $\psi$ be obtained from $\psi_c$ by choosing $\psi(v_1)$ as an $a$-element subset of $L(v_1) \setminus \psi_c(p_0)$, and let $P' = p_0 \ldots p_0v_1$. Note that $v_1$ is not connected to a flaw, since the distance between flaws is at least three. Hence, $\psi$ (and thus also $\psi_c$) extends to an $(L : a)$-coloring of $G$ (recall we chose a counterexample with $\ell$ maximum). This is a contradiction, and thus $\ell = 2$. Then $\psi_c(p_0) \cap L(v_1) = \emptyset$, since $c = L(v_1)$, and $p_0$ is not connected to a flaw in $G - p_0v_1$. Hence, $\psi_c$ extends to an $(L : a)$-coloring of $G - p_0v_1$ by the minimality of $G$, and the resulting coloring is also proper in $G$. This is a contradiction, showing that $p_0$ is not adjacent to a flaw.

That is, the minimum index $b \geq 1$ such that $|L(v_b)| = 3a$ satisfies $b \in \{1, 2\}$.

Suppose that $|L(v_{b+1})| = 3a$. Consider the graph $G' = G - v_{b-1}v_b$. Let $\psi = \psi_c$ if $b = 1$ and let $\psi$ be an $(L : a)$-coloring extending $\psi_c$ to the path $p_0 \ldots p_0v_1$ if $b = 2$. Let $L'$ be the list assignment for $G'$ obtained from $L$ by choosing $L'(v_b)$ as a $2a$-element subset of $L(v_b) \setminus \psi(v_{b-1})$ and if $b = 2$, additionally setting $L'(v_1) = \psi(v_1) \cup \psi(p_0)$. Since $K$ is an induced cycle, $v_b$ has no neighbor with list of size $2a$ in $G'$. Hence, the distance between flaws of $G'$ is at least three. Furthermore, since $G$ has girth at least 5, $v_b$ is not adjacent to $p_0$; together with Claim 3, this implies that $p_0$ is not connected to a flaw in $G'$.

By the minimality of $G$, we conclude that $\psi_c$ extends to an $(L' : a)$-coloring of $G'$. This gives an $(L : a)$-coloring of $G$ extending $\psi_c$, which is a contradiction. Therefore, $|L(v_{b+1})| = 2a$.

Suppose now that $b = 2$ and $|L(v_b)| = 2a$, and thus $p_0$ is connected to the flaw $v_3v_4$. If $\ell \leq 1$, then let $\psi$ be obtained from $\psi_c$ by choosing $\psi(v_1)$ as an $a$-element subset of $L(v_1) \setminus \psi_c(p_0)$, and let $P' = p_0 \ldots p_0v_1$. Note that $v_1$ is not connected to a flaw, since it has no neighbor with list of size $2a$ not belonging to $P$. Hence, $\psi$ (and thus also $\psi_c$) extends to an $(L : a)$-coloring of $G$ (since we choose a counterexample with $\ell$ maximum). This is a contradiction, and thus $\ell = 2$. Then by the choice of $c$, $\psi_c$ extends to an $(L : a)$-coloring $\psi'$ of $p_0p_0v_1v_2$ such that $\psi'(v_2) \cap L(v_3) = \emptyset$. Let $L'$ be the list assignment for $G - v_2$ obtained
from \( L \) by setting \( L'(v_1) = \psi'(v_1) \cup \psi'(v_0) \) and for each vertex \( v \neq v_1 \) adjacent to \( v_2 \), choosing \( L'(v) \) as a 2a-element subset of \( L(v) \setminus \psi'(v_2) \). By Claim 5 and the assumption that \( G \) has girth at least 5, we conclude that \( p_0 \) is not connected to a flaw in \( G - v_2 \) with the list assignment \( L' \), and by the minimality of \( G \), there exists an \((L' : a)\)-coloring of \( G - v_2 \) extending \( \psi_c \). However, this implies that \( G \) has an \((L : a)\)-coloring extending \( \psi_c \), which is a contradiction. Hence, \( p_0 \) is not connected to a flaw.

**Claim 7.** Suppose \(|L(v_{b+3})| = 2a\) (and consequently \(|L(v_{b+2})| = 3a\)) and \( G \) contains a 5-face bounded by a cycle \( v_b v_{b+1} v_{b+2} x_2 x_0 \). Then \( x_2 \) does not have a neighbor in \( P \) and \( G \) does not contain a path \( x_i y z \) with \( i \in \{0, 2\}, y \notin \{v_b, v_{b+2}\} \), \( z \in V(K) \setminus V(P) \), and \(|L(z)| = 2a\).

Subproof. Note that \( x_0, x_2 \notin V(K) \), since \( K \) is an induced cycle and \( v_b \) and \( v_{b+2} \) have degree greater than two. Since \( x_0 \) has degree at least three, Claim 2 implies that \( x_2 \) has no neighbor in \( P \).

Suppose now that \( G \) contains a path \( x_i y z \) as described in the statement of the claim. Let \( G = G_1 \cup G_2 \), where \( G_1 \) and \( G_2 \) are proper induced subgraphs of \( G \), \( P \subset G_1 \), and \( G_1 \) intersects \( G_2 \) only in the path \( Q = v_{b+i} x_i y \) or \( Q = v_{b+i} x_i y z \), depending on whether \( y \in V(K) \) or not. Let \( c' = c(G_2, v_{b+i} x_i y, L) \).

If \( i = 0 \), then note that \( v_b \) is not adjacent to a flaw in \( G_2 \), and thus \(|c'| \leq a\). Hence, \( \psi \) extends to a \((v_b, c')\)-disjoint \((L : a)\)-coloring of the path \( p_t \ldots p_0 v_1 \ldots v_b \). By the minimality of \( G \) and Claim 4 any \((v_b, c')\)-disjoint \((L : a)\)-coloring of \( Q \) extends to an \((L : a)\)-coloring of \( G_2 \). Since \( \psi \) does not extend to an \((L : a)\)-coloring of \( G \), it follows that \( \psi \) does not extend to an \((L : a)\)-coloring of \( G_1 \). Consider the graph \( G_1 - v_b \) with a list assignment \( L_1 \) obtained from \( L \) by setting \( L_1(v_1) = \psi(v_1) \cup \psi(p_0) \) if \( b = 2 \), and by choosing \( L_1(v) \) as a 2a-element subset of \( L(v) \setminus \psi(v_b) \) for each neighbor \( v \) of \( v_b \) other than \( v_{b-1} \). By Claim 5 we conclude that such a neighbor \( v \) cannot be adjacent to another vertex with list of size two, and that \( p_0 \) is not connected to a flaw in \( G_1 - v_b \) with the list assignment \( L_1 \). By the minimality of \( G \), \( \psi_c \) extends to an \((L_1 : a)\)-coloring of \( G_1 - v_b \); however, this also implies that \( \psi \) extends to an \((L : a)\)-coloring of \( G \), which is a contradiction.

Hence, \( i = 2 \). Observe \( \psi_c \) extends to an \((L : a)\)-coloring \( \psi \) of the path \( p_t \ldots p_0 v_1 \ldots v_{b+2} \) such that \( \psi(v_{b+2}) \cap L(v_{b+3}) = \emptyset \). By the minimality of \( G \) and Claim 4 any \((v_b, c')\)-disjoint \((L : a)\)-coloring of \( Q \) extends to an \((L : a)\)-coloring of \( G_2 \), and thus \( \psi \) does not extend to an \((L : a)\)-coloring of \( G_1 \). Suppose first that \( \ell \leq 1 \), or \( b = 1 \), or \( \ell = 2 \) and \( p_2 \) is not connected to a flaw in \( G_1 \). Let \( L_1' \) be a list assignment for \( G_1' = G_1 - \{v_{b+1}, v_{b+2}\} \) obtained by setting \( L_1'(v_i) = \psi(v_{i-1}) \cup \psi(v_i) \) for \( i = 1, \ldots, b \) and choosing \( L_1'(x_2) \) as a 2a-element subset of \( L(x_2) \setminus \psi(v_{b+2}) \). By Claim 4 \( x_2 \) is not incident with a flaw in \( G_1' \), and the only vertices with list of size \( 2a \) (not belonging to \( P \)) at distance at most two from \( v_b \) are \( x_2 \) and possibly \( v_{b-1} \). We created at most one flaw \( (v_1 v_2) \) when \( b = 2 \), and a flaw of \( G \) belongs to \( G_2 \) by Claim 4 and Claim 5, hence, \( G_1' \) has at most two flaws, and the distance between them is at least three. If \( \ell = 2 \), then by the assumptions either \( b = 1 \) (and then \( p_0 \) is not connected to a flaw in \( G_1' \)), or \( b = 2 \) and \( p_2 \) is not connected to a flaw in \( G \) (and then \( p_2 \) is not connected to a flaw in \( G_1' \), either, since \( v_1 \), \( v_2 \), and \( x_2 \) are not adjacent to \( p_2 \)). By the minimality of \( G \), we conclude that \( \psi_c \) extends to an \((L_1' : a)\)-coloring of \( G_1' \). Consequently, \( \psi \) extends to an \((L : a)\)-coloring of \( G_1 \), which is a contradiction.
Hence, we can assume that \( \ell = 2, b = 2, \) and \( p_2 \) is connected to a flaw \( uv \) in \( G_1 \). Suppose now that the distance between \( \{u, v\} \) and \( \{x_0, x_2\} \) is at least three. Then let \( G''_1 = G_1 - \{v_2, v_3, v_4\} \) with list assignment \( L''_1 \) obtained from \( L_1 \) by setting \( L''_1(v_1) = \psi(p_0) \cup \psi(v_1) \) and by choosing \( L''_1(w) \) as a 2a-element subset of \( L(w) \setminus \psi(r) \) for each vertex \( w \neq v_1 \) with a neighbor \( r \in \{v_2, v_4\} \). By Claim 3, neither \( v_0 \) nor \( x_2 \) has a neighbor with list of size 2a not in \( V(P) \cup \{x_0, x_2\} \).

Since \( G \) has girth at least 5 and by Claim 2, \( v_1 \) is at distance at least three in \( G''_1 \) from the newly created flaw \( x_0x_2 \), and thus \( p_0 \) is not connected to a flaw in \( G''_1 \). Note that \( uv \) and \( x_0x_2 \) are the only flaws in \( G_1 \), since \( G_2 \) contains a flaw and \( G \) contains at most two flaws. By the assumption, the distance between the flaws \( uv \) and \( x_0x_2 \) is at least three. By the minimality of \( G \), we conclude that \( \psi \) extends to an \((L''_1 : a)\)-coloring of \( G''_1 \). Consequently, \( \psi \) extends to an \((L : a)\)-coloring of \( G_1 \), which is a contradiction.

Hence, by Claim 3, the distance between \( \{x_0, x_2\} \) and \( \{u, v\} \) is exactly two, and we can without loss of generality assume \( z = v \). Since \( y \) is connected to a flaw in \( G_2 \) by Claim 4 and Claim 6 and the distance between flaws in \( G \) is at least three, we conclude that \( y \in V(K) \) and \( y \) is connected but not adjacent to a flaw in \( G_2 \). Since \( G \) contains only two flaws, \( v_4 \) is by Claim 1 connected to the same flaw in \( G_2 \). By Claim 2 we conclude that \( G_2 \) consists of the 9-cycle \( v_4 \ldots v_1x_2 \) (where \( y = v_1 \)) and a vertex adjacent to \( x_2, v_6, \) and \( v_9 \).

The flaw \( uv = v_3v_4 \) is connected to \( p_2 \); hence, the outer face of \( G_1 \) has length 11 or 13. Note that there exists a \((y, c(G_2, yv_2v_4, L))\)-disjoint \((L : a)\)-coloring \( \psi_3 \) of the path \( P_3 = v_1v_2 \ldots p_2p_0v_1 \) of length 6 or 8. If \( G_3 = G_1 - \{v_3, v_4\} \) had an \((L : a)\)-coloring extending \( \psi_3 \), then we could extend it to \( G_1 \), greedily and then to \( G_2 \) by the minimality of \( G \) since \( \psi_3 \) is \((y, c(G_2, yv_2v_4, L))\)-disjoint, obtaining an \((L : a)\)-coloring of \( G \). This is not possible, and thus \( \psi_3 \) does not extend to an \((L : a)\)-coloring of \( G_3 \). Furthermore, no vertex in \( V(G_3) \setminus V(P_3) \) has list of size 2a, and by Claim 6 the distance between \( v_1 \) and \( v_{11} \) in \( G_3 \) is at least 4. Since \( \psi_3 \) does not extend to an \((L : a)\)-coloring of \( G_3 \), Claim 1 implies that \( P_3 \) has length 8 and \( G \) contains a vertex \( w \) adjacent to \( p_1, v_{11}, \) and \( v_{14} \). Since \( x_0 \) has degree at least three, Claim 2 implies that \( x_0 \) is adjacent to \( p_1 \), and by Claim 2 and Claim 3, we conclude this uniquely determines the whole graph \( G \). However, this contradicts Claim 3.

This is a contradiction, showing that no path \( v_{b+i}x_1yz \) as described in the statement of the claim exists.

**Claim 8.** Suppose \( b = 1, \) \( |L(v_3)| = 2a, |L(v_4)| = 3a, |L(v_5)| = 2a, \) and a cycle \( v_2v_3v_4x_4x_2 \) (where possibly \( x_2 = v_1 \)) bounds a 5-face. Then \( x_4 \) does not have a neighbor in \( P \) and \( G \) does not contain a path \( x_iyz \) with \( i \in \{2, 4\}, y \not\in \{v_2, v_4\}, \) \( z \in V(K) \setminus V(P) \), and \( |L(z)| = 2a. \)

**Subproof.** Suppose \( x_4 \) has a neighbor in \( P \). By Claim 2 and the assumption that \( G \) has girth at least 5, we conclude that \( \ell = 2, x_4p_2 \in E(G) \), and \( x_2 = v_1 \). Let \( G_2 = G - \{p_1, p_0, v_1, v_2, v_3\} \). By Claim 1 both \( p_2 \) and \( v_4 \) are connected to a flaw in \( G_2 \). Since \( G \) has at most two flaws and \( v_2v_4 \) is one of them, both \( p_2 \) and \( v_4 \) are connected to the same flaw, and thus the outer face of \( G_2 \) is bounded by a 7- or 9-cycle. By Claim 2 and the fact that vertices with list of size 3a not in \( P \) must have degree at least three, we conclude that \( G_2 \) is bounded by a 9-cycle and contains a vertex \( u \) adjacent to \( v_6, v_9, \) and \( x_4 \). However, this contradicts Claim 3. Hence, \( x_4 \) has no neighbor in \( P \).
Suppose now that $G$ contains a path $x_{i}yz$ as described in the statement of the claim. Let $G = G_{1} \cup G_{2}$, where $G_{1}$ and $G_{2}$ are proper induced subgraphs of $G$, $P \subset G_{1}$, and $G_{1}$ intersects $G_{2}$ only in the path $Q = v_{x_{i}y}$ or $Q = v_{x_{i}yz}$, depending on whether $y \in V(K)$ or not. If $i = 2$, then by Claim 4 we have $x_{2} \neq v_{1}$. By Claim 3 and Claim 5 $v_{2}$ is connected to a flaw in $G_{2}$. However, since $|L(v_{2})| = 2a$, this contradicts the assumption that the distance between flaws is at least three.

Hence, we have $i = 4$. By Claim 4 and Claim 5 $G_{2}$ contains a flaw, and since $G$ has at most two flaws, $v_{2}v_{3}$ is the only flaw in $G_{1}$. Let $e' = e(G_{2}, v_{4}x_{4}y, L)$; the flaw of $G_{2}$ is at distance at least three from $v_{2}v_{3}$, and thus it is not adjacent to $v_{1}$, and thus $|e'| = a$. Observe that there exists a $(v_{4}, e')$-disjoint $(L : a)$-coloring of $G_{1}$, as otherwise the resulting coloring would further extend to $G_{2}$ and give an $(L : a)$-coloring of $G$ extending $\psi$. If $x_{2} \neq v_{1}$, then let $G_{1}' = G_{1} - \{v_{2}, v_{3}\}$, otherwise let $G_{1}' = G_{1} - \{v_{4}, v_{5}\}$. Let $L_{1}'$ be the list assignment for $G_{1}'$ obtained from $L$ by setting $L_{1}'(v_{1}) = \psi(v_{1}) \cup \psi(p_{0})$, $L_{1}'(v_{2}) = \psi(v_{2}) \cup \psi(v_{1})$ when $x_{2} \neq v_{1}$, and choosing $L_{1}'(x_{4})$ as a $2a$-element subset of $L(x_{4}) \setminus \psi(v_{1})$. Let $e = v_{1}v_{2}$ if $x_{2} \neq v_{1}$ and $e = v_{1}v_{4}$ otherwise. Then $e$ is the only flaw in $G_{1}'$ and when $\ell = 2$, the vertex $p_{2}$ is not connected to $e$, by Claim 5 and the fact that $x_{4}$ is not adjacent to $p_{2}$. Hence, $\psi$ extends to an $(L_{1} : a)$-coloring of $G_{1}$ by the minimality of $G$, and thus $\psi$ extends to an $(L : a)$-coloring of $G_{1}$, which is a contradiction.

Recall that $L(u_{i}) = 2a$ for $1 \leq i \leq b - 1$, $|L(v_{b})| = 3a$, and $|L(v_{b+1})| = 2a$. Furthermore, since, $p_{b}$ is not connected to a flaw, if $b = 2$, then $|L(v_{b+2})| = 3a$. Let us now define a set $X$ of vertices of $G$ depending on the sizes of lists of $v_{b+2}$, $v_{b+3}$, and $v_{b+4}$ as follows.

(X1) If $b = 1$, $|L(v_{3})| = 2a$ and $|L(v_{4})| = |L(v_{5})| = 3a$, then $X = \{v_{2}, v_{3}\}$.

(X2) If $b = 1$, $|L(v_{3})| = 2a$, $|L(v_{4})| = 3a$, $|L(v_{5})| = 2a$, and $v_{2}v_{3}v_{4}$ is not a subpath of the boundary of a 5-face, then $X = \{v_{2}, v_{3}, v_{4}\}$.

(X2a) If $b = 1$, $|L(v_{3})| = 2a$, $|L(v_{4})| = 3a$, $|L(v_{5})| = 2a$, and a cycle $v_{2}v_{3}v_{4}rs$ (where possibly $v = v_{1}$) bounds a 5-face, then $X = \{v_{2}, v_{3}, v_{4}, r, s\}$.

(X3) If $|L(v_{b+2})| = |L(v_{b+3})| = 3a$, then $X = \{v_{b}, v_{b+1}\}$.

(X4) If $|L(v_{b+2})| = 3a$, $|L(v_{b+3})| = 2a$, and $v_{b}v_{b+1}v_{b+2}$ is not a subpath of the boundary of a 5-face, then $X = \{v_{b}, v_{b+1}, v_{b+2}\}$.

(X4a) If $|L(v_{b+2})| = 3a$, $|L(v_{b+3})| = 2a$, and a cycle $v_{b}v_{b+1}v_{b+2}rs$ bounds a 5-face, then $X = \{v_{b}, v_{b+1}, v_{b+2}, r, s\}$.

Let $p$ be the minimum index such that $v_{p} \in X$, and let $m$ be the maximum index such that $v_{m} \in X$; note that $m \leq 4$. Observe that there exists an $(L : a)$-coloring $\psi$ of $G[v(P) \cup X \cup \{v_{1}, \ldots, v_{p-1}\}]$ extending $\psi_{e}$ such that if $|L(v_{m+1})| = 2a$, then $\psi(v_{m}) \cap L(v_{m+1}) = \emptyset$; in cases (X2a) and (X4a), this holds since $r$ has no neighbor in $P$ by Claim 4 and Claim 5.

Note that since $G$ has girth at least 5, each vertex of $V(G) \setminus X$ has at most one neighbor in $X$. Furthermore, by Claim 5 $V(G) \setminus (X \cup V(P) \cup \{v_{p-1}, v_{m+1}\})$ contains no vertex $v$ such that $|L(v)| = 2a$ and $v$ has a neighbor in $X$. Let $G' = G \setminus X$ and let $L'$ be the list assignment defined as follows. If $v \in V(G) \setminus X$, then $L'(v)$ is the list of colors that are not adjacent to any of $v_{i}$, $i = 1, 2, 3, 4$.
\((X \cup V(P) \cup \{v_1, \ldots, v_{p-1}\})\) has a neighbor \(x \in X\), then let \(L'(v)\) be a 2a-element subset of \(L(v) \setminus \psi(x)\), otherwise let \(L'(v) = L(v)\). For \(i = 1, \ldots, p - 1\), let
\[
L'(v_i) = \psi(v_{i-1}) \cup \psi(v_i).
\]
The choice of \(X\) and \(\psi\) ensures that \(|L'(v)| \in \{2a, 3a\}\) for all \(v \in V(G') \setminus V(P)\), and thus \(L'\) is an \((a, P)\)-valid list assignment for \(G'\).

Note that any \((L': a)\)-coloring of \(G'\) extending \(\psi_c\) would combine with \(\psi\) to an \((L: a)\)-coloring of \(G\), and thus \(\psi_c\) does not extend to an \((L': a)\)-coloring of \(G'\).

Suppose \(uv\) is a flaw of \(G'\) which does not appear in \(G\), say \(|L(u)| = 3a\). If \(|L(v)| = 3a\), then both \(u\) and \(v\) would have a neighbor in \(X\); since \(G\) has girth at least 5 and satisfies Claim 2 (and noting the assumption of the non-existence of a 5-cycle containing the path \(v_m \rightarrow v_{m-1} \rightarrow v_{m-1}\) in cases (X2) and (X4)), this is not possible. The case \(|L(v)| = 2a\) is excluded by Claim 7, Claim 7, and Claim 8. Consequently, \(G'\) has no new flaws, and in particular it has at most two flaws and the distance between them is at least three. Furthermore, \(p_0\) is not adjacent to a flaw for the same reason. By the minimality of \(G\), since \(\psi_c\) does not extend to an \((L': a)\)-coloring of \(G'\), we conclude that \(\ell = 2\) and \(p_0\) is connected but not adjacent to a flaw. Let \(Q = p_0xyuv\) be a path in \(G'\) such that \(x, y, u, v \not\in V(P)\) and \(|L'(x)| = |L'(u)| = |L'(v)| = 2a\). As we argued before, \(uv\) is also a flaw in \(G\); in particular, \(|L(u)| = 2a\) and \(u \in V(K)\). Clearly \(p_0xyuv\) is not a subpath of \(K\), and by Claim 7 we conclude that \(x \not\in V(K)\). But then Claim 4 or Claim 6 applied to \(p_0xyuv\) would imply that \(p_0\) is connected to a flaw in \(G\), which is a contradiction.

Therefore, no counterexample to Theorem 7 exists.

Let us remark that it would be possible to exactly determine the minimal graphs \(G\) satisfying the assumptions of Theorem 7 such that some \((L: a)\)-coloring of \(P\) does not extend to an \((L: a)\)-coloring of \(G\) (there are only finitely many). However, the weaker statement we gave is a bit easier to prove and sufficient in the application we have in mind.

## 2 Graphs with two special faces

Let us note an easy consequence of a special case of Theorem 7 with no precolored path, obtained by an argument taken from [17].

**Corollary 8.** Let \(a\) be a positive integer. Let \(G\) be a plane graph of girth at least 5 and let \(f\) be a face of \(G\). Let \(L\) be a list assignment for \(G\) such that \(|L(v)| \in \{2a, 3a\}\) for all \(v \in V(G)\) and the vertices with list of size 2a are all incident with \(f\). If each flaw is at distance at least three from any other vertex with list of size 2a and at least four from any other flaw, then \(G\) is \((L: a)\)-colorable.

**Proof.** We prove the claim by the induction on \(|V(G)|\). If \(G\) contains a flaw \(xy\), then choose an \((L: a)\)-coloring \(\psi\) of \(G[\{x, y\}]\) arbitrarily, and let \(L'\) be the list assignment for \(G' = G - \{x, y\}\) obtained from \(L\) by choosing \(L'(v)\) as a 2a-element subset of \(L(v) \setminus \psi(z)\) for every vertex \(v\) with a neighbor \(z \in \{x, y\}\). Note that the neighbors of \(\{x, y\}\) form an independent set, are not adjacent to other vertices with list of size 2a, and their distance to other flaws is at least three, and thus \(G'\) with the list assignment \(L'\) satisfies the assumptions of Corollary 8. By the induction hypothesis, there exists an \((L': a)\)-coloring of \(G'\), which together with \(\psi\) forms an \((L: a)\)-coloring of \(G\).
Hence, we can assume that there are no flaws. Let $p$ be an arbitrary vertex incident with $f$. By Theorem 8, any $(L : a)$-coloring of $p$ extends to an $(L : a)$-coloring of $G$.

We now strengthen Corollary 8 to the case vertices with lists of size 2 are incident with two faces of $G$.

Proof of Theorem 9. Suppose for a contradiction that $G$ is a counterexample with the smallest number of vertices. Without loss of generality, we can assume $f_1$ is the outer face of $G$. Clearly $G$ is connected, has minimum degree at least 2, and vertices with list of size 3 have degree at least three. Note that $G$ has no flaws: otherwise, $(L : a)$-color the vertices of a flaw $xy$, delete $x$ and $y$, remove the colors of $x$ and $y$ from the lists of their neighbors, and reduce further the lists of these neighbors to size 2. The neighbors form an independent set, are not incident to other vertices with list of size 2, and their distance to other flaws is at least three, implying that the resulting graph would be a counterexample smaller than $G$ (or contradict Corollary 8 if both $f_1$ and $f_2$ are incident with $\{x, y\}$, so that all vertices with list of size 2 in $G - \{x, y\}$ are incident with one face).

Furthermore, the faces $f_1$ and $f_2$ are bounded by cycles: otherwise, if say $f_1$ is not bounded by a cycle, then $G = G_1 \cup G_2$ for proper induced subgraphs intersecting in a vertex $p$ incident with $f_1$, such that $f_2$ is a face of $G_1$ and the boundary of the outer face of $G_2$ is part of the boundary of $f_1$. By the minimality of $G$, the graph $G_1$ is $(L : a)$-colorable, and by Theorem 8 the corresponding coloring of $p$ extends to an $(L : a)$-coloring of $G_2$, together giving an $(L : a)$-coloring of $G$.

Let $C_1$ and $C_2$ be the cycles bounding $f_1$ and $f_2$, respectively. Analogously to the proof of Claim 2, we conclude that the following holds.

Claim 9. Every $(\leq 9)$-cycle $C$ in $G$ such that $f_2$ is not contained in $\text{int}(C)$ is tame.

Next, let us restrict short paths with both ends in $C_1$ or both ends in $C_2$.

Claim 10. Let $Q = q_0 \ldots q_k$ be a path of length $k \leq 4$ in $G$. If $k = 4$, furthermore assume that $|L(q_4)| = 2a$. If $q_0, q_k \in V(C_i)$ for some $i \in \{1, 2\}$, then $Q \subseteq C_i$, or $k \geq 3$ and $q_3 \in V(C_i)$ and $q_0$ and $q_3$ have a common neighbor in $C_i$ with list of size $2a$ (and in particular, $|L(q_0)| = |L(q_3)| = 3a$), or $k = 4$ and $q_0$ is adjacent to $q_4$ in $C_i$. In the last two cases, the cycle consisting of $q_0q_1q_2q_3$ and the common neighbor of $q_0$ and $q_3$ bounds a 5-face.

Subproof. We prove the claim by induction on $k$, the case $k = 0$ being trivial. By symmetry, we can assume that $i = 1$. Furthermore, we can assume that $q_1, \ldots, q_{k-1} \notin V(C_1)$, since otherwise the claim follows by the induction hypothesis applied to subpaths of $Q$ between vertices belonging to $C_1$. Let $G = G_1 \cup G_2$, where $G_1$ and $G_2$ are proper induced subgraphs of $G$ intersecting in $Q$ and $f_2$ is a face of $G_1$. By the minimality of $G$, there exists an $(L : a)$-coloring $\varphi_1$ of $G_1$. If $k \leq 2$, then the restriction of $\varphi_1$ to $Q$ extends to an $(L : a)$-coloring of $G_2$ by Theorem 2. This gives an $(L : a)$-coloring of $G$, which is a contradiction.

If $k = 3$, then let $L'$ be the list assignment for $G_2$ obtained from $L$ by setting $L'(q_3) = \varphi_1(q_3) \cup \varphi_1(q_2)$, and let $\psi$ be the restriction of $\varphi_1$ to $Q - q_3$. Using the induction hypothesis (the case $k = 1$), observe that $q_3$ has at most one neighbor
$u \in V(G_2) \setminus V(Q)$ with list of size $2a$, and thus the list assignment $L'$ has at most one flaw $q_3u$. Since $G$ is not $(L : a)$-colorable, $\psi$ does not extend to an $(L' : a)$-coloring of $G_2$, and thus Theorem 4 implies the flaw $q_3u$ is connected to $q_0$. By Claim 9 and the fact that vertices with list of size $3a$ have degree at least three, this is only possible when $u$ is adjacent to $q_0$, and thus $q_0$ and $q_3$ have a common neighbor $u$ with list of size $2a$. The path $q_0uq_3$ is a subpath of $C_1$ by the induction hypothesis (the case $k = 2$).

Suppose now that $k = 4$. Let $L'$ be the list assignment obtained from $L$ by setting $L'(q_4) = \varphi_1(q_4) \cup \varphi_2(q_4)$ and $L'(q_0) = \varphi_1(q_0) \cup \varphi_1(q_1)$, and let $\psi$ be the restriction of $\varphi$ to $Q - \{q_0, q_1\}$. Note that $G_2$ with list assignment $L'$ has at most one flaw $q_0u$. Since $G$ is not $(L : a)$-colorable, Theorem 4 implies the flaw $q_0u$ is connected to $q_3$. By Claim 9 and the fact that vertices with list of size $3a$ have degree at least three, this is only possible when $u$ is adjacent to $q_3$, and by the induction hypothesis (case $k = 2$), we conclude that $u = q_4$.

In the last two cases, the cycle consisting of $q_0q_1q_2q_3$ and the common neighbor of $q_0$ and $q_3$ bounds a 5-face by Claim 9.

We now show that $G$ cannot have a short path between $C_1$ and $C_2$, successively showing better and better bounds on the distance between $C_1$ and $C_2$.

Suppose $Q$ is an induced path with ends in $C_1$ and $C_2$ and otherwise disjoint from $C_1 \cup C_2$. Each edge $e \notin E(Q)$ incident with a vertex of $Q$ goes either to the left or to the right side of $Q$ (as seen from $f_2$). Let $R(Q)$ denote the set of edges $e \notin E(Q)$ incident with vertices of $Q$ from the right side.

Claim 11. The cycles $C_1$ and $C_2$ are disjoint.

Subproof. Suppose for a contradiction that there exists a vertex $w \in V(C_1 \cap C_2)$.

Let $C_1 = wv_1v_2 \ldots$ and $C_2 = wu_1u_2 \ldots$, labeled in the clockwise order. Without loss of generality, we can assume $u_1 \neq v_1$. By symmetry, we can assume that $|L(u_1)| \leq |L(v_1)|$. If $|L(v_1)| = 3a$, then let $G' = G - R(w)$, let $\psi$ be an $(L : a)$-coloring of $w$ such that if $|L(u_1)| = 2a$ then $\psi(w) \cap L(u_1) = \emptyset$, and let $L'$ be the list assignment obtained from $L$ by choosing $L'(x)$ as a 2a-element subset of $L(x) \setminus \psi(w)$ for every vertex $x \neq w$ incident with an edge in $R(w)$. By Claim 10 $G'$ has no flaws other than $v_1v_2$ and possibly $u_1u_2$ when $|L(u_1)| = 3a$. If $|L(u_1)| = 3a$, then observe that the distance between $u_1u_2$ and $v_1v_2$ is at least three; otherwise by Claim 9 and the assumption that $G$ has girth at least five, we would conclude that $u_1$ or $v_1$ has degree two, which is a contradiction since both have list of size $3a$. By Theorem 4, $\psi$ extends to an $(L' : a)$-coloring of $G'$, which also gives an $(L : a)$-coloring of $G$.

If $|L(v_1)| = 2a$ (and thus also $|L(u_1)| = 2a$), then let $G' = G - R(w) - v_1$, let $\psi$ be an $(L : a)$-coloring of $wv_1$ such that $\psi(w) \cap L(u_1) = \emptyset$, and let $L'$ be the list assignment obtained from $L$ by choosing $L'(x)$ as a 2a-element subset of $L(x) \setminus \psi(z)$ for every vertex $x \in V(G) \setminus \{w, v_1\}$ with a neighbor $z$ such that either $z = v_1$, or $z = w$ and $wx \in R(w)$. By Claim 10 $G'$ has no flaws other than $v_2v_3$, and by Theorem 4 the restriction of $\psi$ to $w$ extends to an $(L' : a)$-coloring of $G'$, which also gives an $(L : a)$-coloring of $G$.

In both cases, we obtain a contradiction.

Claim 12. The distance between $C_1$ and $C_2$ is at least 2.
Claim 13. Let $C_1 = v_1 v_2 \ldots$ and $C_2 = u_1 u_2 \ldots$, labeled in the clockwise order, and suppose for a contradiction $u_1 v_1 \in E(G)$. By symmetry, we can assume that $|L(u_2)| \leq |L(v_2)|$.

Suppose first $|L(v_2)| = 3a$. Let $G' = G - R(u_1 v_1)$, let $\psi$ be an $(L : a)$-coloring of $u_1 v_1$ such that if $|L(u_2)| = 2a$ then $\psi(u_2) \cap L(u_2) = \emptyset$, and let $L'$ be the list assignment obtained from $L$ by choosing $L'(x)$ as a $2a$-element subset of $L(x) \setminus \psi(z)$ whenever $xz \in R(u_1 v_1)$ for some $z \in \{u_1, v_1\}$. By Claim 10, $G'$ has no flaws other than $v_2 v_3$ and possibly $u_2 u_3$ when $|L(u_2)| = 3a$. If either $G'$ has at most one flaw, or the distance between $u_2 u_3$ and $v_2 v_3$ is at least three, then Theorem 7 implies that $\psi$ extends to an $(L' : a)$-coloring of $G'$, which also gives an $(L : a)$-coloring of $G$.

Hence, suppose that both $u_2 u_3$ and $v_2 v_3$ are flaws (and thus $|L(u_2)| = 3a$ and $|L(u_3)| = |L(v_3)| = 2a$) and the distance between them is at most two. Since $u_2$ and $v_2$ have degree at least three, Claim 9 implies that $u_2$ and $v_2$ have a common neighbor $w \not\in V(C_1 \cup C_2)$ and the 5-cycle $u_1 u_2 w v_2 v_1$ bounds a face. In that case, let $\psi'$ be an $(L : a)$-coloring of $u_2 w v_2$ such that $\psi'(u_2) \cap L(u_2) = \emptyset$ and $\psi'(v_2) \cap L(v_2) = \emptyset$. Let $G'' = G - R(u_2 w v_2)$, with list assignment $L''$ obtained from $L$ by choosing $L''(x)$ as a $2a$-element subset of $L(x) \setminus \psi(z)$ whenever $xz \in R(u_2 w v_2)$ for some $z \in \{u_2, w, v_2\}$. Since $w$ has degree at least three, Claim 9 implies that vertices incident with edges of $R(u_2 w v_2)$ form an independent set in $G''$. Furthermore, these vertices have no other neighbors with list of size 2, by Claim 10. Claim 9 and the fact that $w$ has degree at least three. Hence, $G''$ with the list assignment $L''$ has no flaws, and by Theorem 7, $\psi'$ extends to an $(L'' : a)$-coloring of $G''$. This gives an $(L : a)$-coloring of $G$.

Finally, suppose $|L(v_2)| = |L(u_2)| = 2a$. By symmetry, we can assume that at least half of the edges of $R(u_1 v_1)$ are incident with $v_1$, and if exactly half of them are incident with $v_1$, then $\deg(v_1) = \deg(u_2)$. Let $G' = G - R(u_1 v_1) - v_2$, let $\psi$ be an $(L : a)$-coloring of $u_1 v_1$ such that $\psi(u_1) \cap L(u_2) = \emptyset$, and let $L'$ be the list assignment obtained from $L$ by choosing $L'(x)$ as a $2a$-element subset of $L(x) \setminus \psi(z)$ for every vertex $x \in V(G) \setminus \{u_1, v_1, v_2\}$ with a neighbor $z$ such that either $z = v_2$ or $xz \in R(u_1 v_1)$. If $G'$ with the list assignment $L'$ has no flaw other than $v_3 v_4$, then the restriction of $\psi$ to $u_1 v_1$ extends to an $(L' : a)$-coloring of $G$ by Theorem 7, giving an $(L : a)$-coloring of $G$. Hence, assume that $G'$ has a flaw $uv$ other than $v_3 v_4$. By Claim 10, $u$ is adjacent to $u_1$ and $v$ is adjacent to $v_2$. Since at least half of the edges of $R(u_1 v_1)$ are incident with $v_1$, Claim 9 applied to the 5-cycle $u_1 v_1 v_2 u v$ implies that $u = u_2$. Consequently, exactly one edge of $R(u_1 v_1)$ is incident with each of $u_1$ and $v_1$, and thus $\deg(v_2) \geq \deg(u_2)$ by the choice made at the beginning of the paragraph. It follows that $v \neq v_3$. Since $\deg(v) \geq 3$ and $\deg(v_3) \geq 3$, Claim 9 implies that the distance between $uv$ and $v_3 v_4$ in $G'$ is at least three. Hence, the restriction of $\psi$ to $u_1 v_1$ extends to an $(L' : a)$-coloring of $G$ by Theorem 7, giving an $(L : a)$-coloring of $G$.

In all cases, we obtain a contradiction. ■

Claim 13. The distance between $C_1$ and $C_2$ is at least 3.

Subproof. Let $C_1 = v_1 v_2 \ldots v_s$ and $C_2 = u_1 u_2 \ldots u_t$, labeled in the clockwise order, and suppose for a contradiction $u_1$ and $v_1$ have a common neighbor $w$. We flip the graph if possible so that $|L(u_2)| = |L(v_2)| = 2a$. If not (i.e., $|L(u_2)| + |L(v_2)| \geq 5a$ and $|L(u_3)| + |L(v_3)| \geq 5a$), then instead flip the graph so that $w$ is incident with an edge of $R(u_1 v_1 v)$ (this is possible, since $\deg(w) \geq 3$).
Let $\psi$ be an $(L : a)$-coloring of $u_1wv_1$ such that $\psi(u_1) \cap L(u_2) = \emptyset$ if $|L(u_2)| = 2a$ and and $\psi(v_1) \cap L(v_2) = \emptyset$ if $|L(v_2)| = 2a$. Let $G' = G - R(u_1wv_1)$ and let $L'$ be the list assignment obtained from $L$ by choosing $L'(x)$ as a 2a-element subset of $L(x) \setminus \psi(z)$ whenever $xz \in R(u_1wv_1)$ for some $z \in \{u_1, v, v_1\}$. Suppose first that $|L(u_2)| = |L(v_2)| = 2a$. By Claim 10 and Claim 13, we conclude that $G'$ contains at most one flaw, with $u$ adjacent to $u_1$ and $v$ adjacent to $v_1$.

Claim 11 and the assumption that $G$ has girth at least 5 also implies that this flaw is not connected to $u_1$ and $v_1$. Hence, Theorem 7 implies that $\psi$ extends to an $(L' : a)$-coloring of $G'$, which also gives an $(L : a)$-coloring of $G$.

Therefore, we can assume that $|L(u_2)| + |L(v_2)| \geq 5a$, and by the choice made in the first paragraph, $w$ is incident with an edge of $R(u_1wv_1)$ and we can assume that $|L(v_2)| = 3a$. By Claim 10, Claim 9 and the assumption that $w$ is incident with an edge of $R(u_1wv_1)$, $G'$ has no flaws other than $u_2w_3$ and $v_2v_3$. If it has both, then $|L(u_2)| = |L(v_2)| = 3a$, $\deg(u_2), \deg(v_2) \geq 3$, and the distance between the flaws is at least three by Claim 9 and the assumption that $w$ is incident with an edge of $R(u_1wv_1)$. No flaw in $G'$ is connected to $v_1$, since $|L'(v_1)| = |L(v_1)| = 3a$. Hence, Theorem 7 implies that $\psi$ extends to an $(L' : a)$-coloring of $G'$, which also gives an $(L : a)$-coloring of $G$.

In both cases, we obtain a contradiction.

If $C_2$ contained three consecutive vertices with list of size 3a, we could reduce the list of the middle one to 2a without violating the assumptions. Let $v_1u_2v_3u_4v_5u_6$ be a subwalk of $C_2$ (where possibly $u_1 = u_6$ if $|C_2| = 5$) such that $|L(u_2)| = |L(v_3)| = 3a$, and thus $|L(u_1)| = |L(u_4)| = 2a$. If $|L(u_6)| = 3a$, then let $X = \{u_4\}$. If $|L(u_6)| = 2a$, then let $X = \{u_4, u_5\}$. Let $\psi$ be an $(L : a)$-coloring of $G[X]$ such that $\psi(u_3) \cap L(u_6) = \emptyset$ when $|L(u_6)| = 2a$. Let $L'$ be the list assignment obtained from $L$ by choosing $L'(v)$ as a 2a-element subset of $L(v) \setminus \psi(x)$ for every vertex $v \in V(G) \setminus X$ with a neighbor $x \in X$. By Claim 10 and Claim 13, $G - X$ with list assignment $L'$ has no flaws, and an $(L' : a)$-coloring of $G - X$ which exists by the minimality of $G$ together with $\psi$ gives an $(L : a)$-coloring of $G$, which is a contradiction. We conclude that in $C_2$, the vertices with lists of size 2a and 3a alternate (and in particular, $|C_2|$ is even).

Claim 14. Let $u_1u_2u_3xy$ be a cycle bounding a 5-face in $G$, where $u_1u_2u_3$ is a subpath of $C_2$, $|L(u_1)| = |L(u_2)| = 3a$ and $|L(u_2)| = 2a$. Then $G$ does not contain a path $u_1ywv_1$ with $|L(v_1)| = 2a$.

Subproof. By Claim 10, $v_1 \notin V(C_2)$, and thus $v_1 \in V(C_1)$. Let $C_1 = v_1v_2 \ldots v_s$ and $C_2 = u_1u_2 \ldots v_1$, labeled in the clockwise order. Let $G' = G - R(u_1ywv_1) - u_2$, let $\psi$ be an $(L : a)$-coloring of $u_1ywv_1$ such that $\psi(u_1) \cap L(u_2) = \emptyset$, and let $L'$ be the list assignment for $G'$ obtained from $L$ by setting $L'(u_1) = \psi(v_1) \cup \psi(u_2)$ and by choosing $L'(v)$ as a 2a-element subset of $L(v) \setminus \psi(z)$ whenever $z \in R(u_1ywv_1)$ for some $z \in \{y, w, v_1\}$. By Claim 11, Claim 10, and Claim 13, the only possible flaws in $G'$ are $v_2v_3$ and an edge $u'v'$ such that $v_1ywv'v'$ is a 5-cycle bounding a face of $G$. Note that $v_1$ is the only neighbor of $w$ with list of size 2a in $G'$, and since $G$ has girth at least 5, we conclude that $w$ is not connected to either of the flaws. Suppose that either $G'$ contains at most one flaw or $v' \neq v_2$. Since $\deg(v') \geq 3$ and $\deg(v_2) \geq 3$, Claim 11 implies in the latter case that the distance between the flaws is at least three. Consequently,
the restriction of \( \psi \) to \( u_1yu \) extends to an \((L' : a)\)-coloring of \( G' \) by Theorem 7 giving an \((L : a)\)-coloring of \( G \).

Hence, \( G' \) contains two flaws and \( v' = v_2 \). Let \( S \) be the set of edges incident with \( u_1 \) or \( y \) distinct from \( yu_1 \) and not belonging to \( R(u_1yu_1v_1) \), together with the edge \( v_1v_2 \). Let \( G'' = G - S \). Let \( \psi' \) be an \((L : a)\)-coloring of \( u_1yu_1v_2 \) such that \( \psi'(u_1) \cap L(u_1) = \emptyset \) and \( \psi'(v_2) \cap L(v_1) = \emptyset \). Let \( L'' \) be the list assignment obtained from \( L \) by setting \( L''(v_2) = \psi'(v_2) \cup \psi'(u') \) and by choosing \( L''(v) \) as a 2a-element subset of \( L(v) \setminus \psi'(x) \) whenever \( v \in S \) for some \( z \in \{u_1, y\} \). By Claim 13 and Claim 13, the only flaws in \( G'' \) are \( v_1w \) and \( v_2v_3 \), and since the girth of \( G \) is at least 5, Claim 14 implies that the distance between these two flaws in \( G'' \) is at least three. Furthermore, neither of them is connected to \( u_1 \) by Claim 13. Hence, the restriction of \( \psi' \) to \( u_1yu' \) extends to an \((L'', a)\)-coloring of \( G'' \) by Theorem 4 giving an \((L : a)\)-coloring of \( G \).

In both cases, we obtain a contradiction. ■

Claim 15. Every vertex in \( C_2 \) with list of size \( 2a \) has degree two and is incident with a 5-face.

Subproof. Suppose for a contradiction a vertex \( u_3 \in V(C_2) \) with list of size \( 2a \) has degree at least three or is not incident with a 5-face. Let \( u_1u_2u_3u_4u_5u_6u_7 \) be a subwalk of \( C_2 \) (where possibly \( u_1 = u_7 \), with odd-indexed vertices having list of size \( 2a \).

If either \(|C_2| > 6\), or \(|C_2| = 6 \) and \( u_1 = u_7 \) has degree greater than two, then let \( X = \{u_2, \ldots, u_6\} \), and let \( \psi \) be an \((L : a)\)-coloring of the path \( G[X] \) such that \( \psi(u_2) \cap L(u_1) = \emptyset \) and \( \psi(u_6) \cap L(u_7) = \emptyset \). Let \( G' = G - X \) and let \( L' \) be the list assignment for \( G' \) obtained from \( L \) by choosing \( L'(v) \) as a 2a-element subset of \( L(v) \setminus \psi(x) \) for every vertex \( v \in V(G') \) with a neighbor \( x \in X \) (note such a neighbor is unique by Claim 10). Suppose \( uv \) is a flaw in \( G' \), where \( u \notin V(C_2) \) and \( u \) has a neighbor \( u_i \) in \( X \). By Claim 10 and Claim 13 we conclude that \( v \notin V(C_1 \cup C_2) \) and \( v \) has a neighbor \( u_j \) in \( X \). By Claim 10 applied to \( u_iuvu_j \), we conclude that \( u_i \) and \( u_j \) have a common neighbor \( u_k \) in \( C_2 \) of degree two in \( G \), and \( u_iuvu_ju_k \) bounds a 5-face. Since \( \deg(u_2) = \deg(u_4) \) is at least 3, and \( \deg(u_3) = \deg(u_5) \) is not incident with a 5-face, and either \(|C_1| > 6 \) or \( \deg(u_1) \geq 3 \), we conclude that \( uv \) is contained in a 5-cycle \( u_1u_2u_3u_4u_5u_6u_7 \) bounding a 5-face. Consequently, \( G' \) has at most one flaw, and if it has such a flaw, then by Claim 14 this flaw is at distance at least three from any other vertex with list of size \( 2a \). Hence, by the minimality of \( G \), the graph \( G' \) is \((L : a)\)-colorable, which also gives an \((L : a)\)-coloring of \( G \).

If \(|C_2| = 6 \) and \( u_1 \) has degree two, then let \( X' = \{u_1, \ldots, u_5\} \), let \( \psi' \) be an \((L : a)\)-coloring of \( u_2u_3u_4u_5 \), such that \( \psi'(u_2) \cap L(u_1) = \emptyset \), and let \( L'' \) be the list assignment for \( G'' = G - X' \) obtained from \( L \) by choosing \( L''(v) \) as a 2a-element subset of \( L(v) \setminus \psi'(x) \) for every vertex \( v \in V(G'') \) with a neighbor \( x \in \{u_2, u_3, u_4, u_5\} \). Note that we do not color the vertex \( u_1 \); instead, observe that by the choice of \( \psi'(u_2) \) and the fact that \( \deg(u_1) = 2 \), any extension of \( \psi' \) to \( G - u_1 \) extends to an \((L : a)\)-coloring of \( G \). Since \( \deg(u_4) \geq 3 \) and either \( \deg(u_3) \geq 3 \) or \( u_3 \) is not incident with a 5-face, Claim 10 and Claim 13 imply that \( G'' \) has no flaws. By the minimality of \( G \), the graph \( G'' \) is \((L'' : a)\)-colorable, which also gives an \((L : a)\)-coloring of \( G \).

In both cases, we obtain a contradiction. ■

Claim 16. Every vertex in \( C_2 \) with list of size \( 3a \) has degree at least 4.
Subproof. Suppose for a contradiction a vertex \( u_4 \in V(C_2) \) with list size \( 3a \) has degree three, and let \( u_1u_2u_3u_4 \) be a subpath of \( C_2 \). By Claim 15, there exists a 5-face bounded by the cycle \( u_2u_3u_4u_1u \). Let \( X = \{u_2, u_3, v, u\} \), let \( G' = G - X - u_4 \), and let \( \psi \) be an \((L:a)\)-coloring of the path \( G[X]\) such that \( \psi(u_2) \cap L(u_1) = \emptyset \) and \( |L(u_4) \setminus (\psi(u_2) \cup \psi(u))| \geq 2a \); such a coloring is obtained by first choosing \( \psi(u_2) \) disjoint from \( L(u_1) \), and \( \psi(u_3) \) disjoint from \( \psi(u_2) \), then choosing \( \psi(u) \) with as small intersection with \( L(u_4) \setminus \psi(u_3) \) as possible, and finally coloring \( v \). Let \( L' \) be the list assignment for \( G' \) obtained from \( L \) by choosing \( L'(v) \) as a 2a-element subset of \( L(v) \setminus \psi(x) \) for every vertex \( v \in V(G') \) with a neighbor \( x \in X \). By Claim 14 and Claim 10, \( G' \) has no flaws. By the minimality of \( G \), the graph \( G' \) is \((L':a)\)-colorable, which together with \( \psi \) gives an \((L:a)\)-coloring of \( G - u_4 \). This coloring extends to an \((L:a)\)-coloring of \( G \) by the choice of \( \psi(u) \), which is a contradiction.

Claim 17. There exists a subpath \( u_iu_{i+1}u_{i+2} \) of \( C_2 \) such that \( |L(u_i)| = |L(u_{i+2})| = 3a \) and neither \( u_i \) nor \( u_{i+2} \) is an endpoint of any path of length 4 with both ends in \( C_2 \) and no internal vertices in \( C_2 \).

Subproof. Otherwise, choose a path \( Q_0 \) of length 4 with both ends in \( C_2 \) and no internal vertices in \( C_2 \) so that, letting \( G = G_1 \cup G_2 \) for proper induced subgraphs \( G_1 \) and \( G_2 \) intersecting in \( Q_0 \) and with \( f_1 \) being the outer face of \( G_1 \), the graph \( G_2 \) is minimal. Let \( u_1u_2 \ldots u_k \) be the path whose concatenation with \( Q \) forms the outer face of \( G_2 \), and let \( Q_0 = u_1x y z u_k \). By Claim 14, we have \( |L(u_1)| = \ldots = |L(u_k)| = 3a \), and in particular \( k \) is odd. Since \( u_3 \) is incident with a 5-face by Claim 15, Claim 8 implies \( k \geq 5 \). Since \( \deg(u_3) \geq 4 \), Claim 3 further implies \( k \geq 7 \). We claim we can choose \( i = 3 \). Indeed, suppose that there exists a path \( Q_1 = u_j x' y' z'u \) such that \( j \in \{3, 5\}, u \in V(C_2) \), and \( x', y', z' \not\in V(C_2) \).

The minimality of \( G_2 \) and the planarity implies that \( \{x, y, z\} \cap \{x', y', z'\} \neq \emptyset \). By Claim 10, we have \( x' \not\in V(Q) \) and \( y' \not\in \{x, z\} \). If \( z' = y \), then Claim 10 applied to \( u_{j-1}u_j u_{j+2}u_k \) implies \( \deg(u) = 3 \), contradicting Claim 10. By the minimality of \( G_2 \), we have \( y' \neq y \). Consequently, \( y' \in \{x, z\} \). If \( y' = z \), then Claim 10 implies \( k = 7 \) and the case is symmetric to the case \( y' = x \). Hence, suppose \( y' = x \); by Claim 10, \( j = 3 \). By Claim 10, applied to \( u_1y'z'u \), and by Claim 9, we conclude that \( u_3 \) has degree three, contradicting Claim 16.

Let \( C_2 = u_1 \ldots u_t \), where the odd-indexed vertices have list of size \( 2a \), and by Claim 17 we can assume that neither \( u_1 \) nor \( u_2 \) is an endpoint of any path of length 4 with both ends in \( C_2 \) and no internal vertices in \( C_2 \). Let \( X \) denote the set of vertices of \( C_2 \) whose index is 2 or 3 modulo 4, and let \( Y \) denote the set of vertices whose index is 1 modulo 4. Let \( \psi \) be an \((L:a)\)-coloring of \( G[X]\) such that \( \psi(u_i) \cap L(u_{i-1}) = \emptyset \) for every \( i \) such that \( i \equiv 2 \) (mod 4). Let \( L' \) be the list assignment for \( G - (X \cup Y) \) obtained from \( L \) by choosing \( L'(v) \) as a 2a-element subset of \( L(v) \setminus \psi(x) \) for every vertex \( v \in V(G') \) with a neighbor \( x \in X \). If \( t \) is divisible by 4, then \( G' \) has no flaw by Claim 13 and Claim 10. Otherwise \( t \equiv 2 \) (mod 4), and by Claim 13 and Claim 11, \( G' \) only has the flaw \( uv \) such that the cycle \( u_1u_2u_3u \) bounds a 5-face. This flaw is at distance at least three from any other vertex with list of size \( 2a \), by Claim 13 and the assumption that neither \( u_1 \) nor \( u_2 \) is an endpoint of any path of length 4 with both ends in \( C_2 \) and no internal vertices in \( C_2 \). In either case, the graph \( G' \) is \((L':a)\)-colorable by the minimality of \( G \). The \((L':a)\)-coloring of \( G' \) extends to an \((L:a)\)-coloring.
of $G$ by coloring $X$ according to $\psi$ and extending to $Y$ greedily, which is a contradiction.

3 Strong hyperbolicity

We are now ready to prove Theorem 5.

Proof of Theorem 5. We follow the proof of Theorem 1.8 in [13]. The only major changes (other than trivial modifications such as replacing all statements of form $|L(v)| \geq k$ by $|L(v)| \geq ka$, all $L$-colorings by $(L : a)$-colorings, etc.) are as follows. An analogue of [13, Lemma 2.9] is implied directly by Corollary 8 so we do not need to prove the analogue of [13, Theorem 1.17]; all the other applications of [13, Theorem 1.17] can be replaced by applications of Corollary 8 as well. Instead of [13, Theorem 2.12], Theorem 6 is used. In the statement of [13, Claim 4.11], we replace $A(v) \subseteq A(u)$ by $|A(v) \setminus A(u)| < a$. We ignore all the claims based on [13, Claim 4.11], which are actually not needed, except for the following one. In the proof of [13, Claim 4.17], we observe that by the modified version of [13, Claim 4.11], we can choose $\phi(p_5)$ as an $a$-element subset of $A(p_5) \cap A(p_4)$, and then since $|A(p_3)| = |A(p_4)| = 3a$, we can choose $\phi(p_3)$ as an $a$-element subset of $A(p_3) \setminus (A(p_4) \setminus \phi(p_5))$.

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