WIGNER DISTRIBUTION AND ASSOCIATED UNCERTAINTY PRINCIPLES IN
THE FRAMEWORK OF OCTONION LINEAR CANONICAL TRANSFORM

Aamir H. Dar¹, and M. Younus Bhat²,*

¹ Department of Mathematical Sciences, Islamic University of Science and Technology, Kashmir. E-mail: ahdkul740@gmail.com
²,∗ Department of Mathematical Sciences, Islamic University of Science and Technology, Kashmir. E-mail: ggyounusg@gmail.com

Abstract. The most recent generalization of octonion Fourier transform (OFT) is the octonion linear canonical transform (OLCT) that has become popular in present era due to its applications in color image and signal processing. On the other hand the applications of Wigner distribution (WD) in signal and image analysis cannot be excluded. In this paper, we introduce novel integral transform coined as the Wigner distribution in the octonion linear canonical transform domain (WDOL). We first propose the definition of the one dimensional WDOL (1D-WDOL), we extend its relationship with 1D-OLCT and 1D-OFT. Then explore several important properties of 1D-WDOL, such as reconstruction formula, Rayleigh’s theorem. Second, we introduce the definition of three dimensional WDOL (3D-WDOL) and establish its relationships with the WD associated with quaternion LCT (WD-QLCT) and 3D-WD in LCT domain (3D-WDLCT). Then we study properties like reconstruction formula, Rayleigh’s theorem and Riemann-Lebesgue Lemma associated with 3D-WDOL. The crux of this paper lies in developing well known uncertainty principles (UPs) including Heisenberg’s UP, Logarithmic UP and Hausdorff-Young inequality associated with WDOL∗

Keywords: Octonion linear canonical transform(OLCT); Wigner distribution (WD); Rayleigh’s theorem; Riemann-Lebesgue Lemma ; Uncertainty principle (UP).

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1. Introduction

Of all the time-frequency distributions, Wigner distribution (WD) [1]-[12] is regarded as the most important distribution. WD is considered as an important frequency analysis tool that is more suitable for the analysis of time-frequency characteristics of chirp-like signals, such as the linear-frequency-modulated (LFM) signals that are frequently used in wireless communications, medical imaging sonar, radar and many more. For any finite energy signals \( f \) and \( g \) the WD is defined as \[ W_{f,g}(t, w) = \int_{\mathbb{R}} f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) e^{-iwx} dx, \] where \( f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \) represents the instantaneous auto-correlation relation of the signal \( f(x) \). It is also viewed as a local spatial frequency spectrum of the signal, with tremendous applications in optics, matrix optics, wave optics, geometrical optics, Fourier ray optics and radiometry [13].

In the last few decades, the researcher’s community has shown greater interest in the study of multidimensional hyper-complex signals defined by means of Cayley–Dickson algebras and there applications in image filtering, watermarking, color and image processing, edge detection and pattern recognition [16]-[21]. The Cayley–Dickson algebra of order 4 is known as quaternions. In quaternionic analysis, the quaternion Fourier transform (QFT) is the most basic and important time-frequency analysis tool for multidimensional quaternionic signals. QFTs are frequently studied in present era because
of its wide range of applications in signal and image processing. QFTs are best studied in [22]-[25]. The QFT is regarded as the generalization of the real and complex FT to the quaternionic case. As in recent times the generalization of integral transforms to quaternion setting is popular, in this regard Yang and Kou [26] generalized linear canonical transform (LCT) to quaternion-valued signals, known as the the quaternion LCT (QLCT), that is better effective signal processing tool than the classical QFT amid its extra parameters, see [27, 28, 29, 30, 31, 32, 33]. Later, El Haoui and Hitzer generalizes the offset linear canonical transform (OLCT) to QOLCT [34]. Recently Dar and Bhat introduces new integral transform which generalizes the most neoteric quadratic-phase Fourier transform (QPFT) for the quaternion algebra [35]. Authors in [36-40] introduced WD associated with QLCT (WD-QLCT) and OLCT (WD-OLCT)and studied their application in detection of quaternion LFM signals.

Moving towards other side, Cayley-Dickson algebra of order 8 is known as octonion algebra. Recently Hahn and Snopek introduces the octonion Fourier transform (OFT) [41]. From then OFT is becoming the hot area of research in modern signal processing. Later, authors in [42, 43, 44, 45] developed and studied the theory of OFT extensively. Recently Gao and Li generalized the OFT to the octonion linear canonical transform (OLCT) [46] by substituting the Fourier kernel with the LCT kernel. Later Bhat and Dar [47] introduced the octonion version of offset linear canonical transform. Furthermore, authors in [48, 49] introduced octonion short-time Fourier transform and octonion spectrum of 3d short-time LCT signals where they established classical properties besides establishing Pitt’s, Lieb’s and uncertainty inequalities.

So motivated and inspired by the merits of Wigner distribution and octonion linear canonical transform, we in this paper propose the novel integral transform coined as the Wigner distribution in the octonion linear canonical transform domain (WDOL), that gives a unified treatment for already existing classes of signal processing tools. Therefore it is worthwhile to rigorously study the WDOL and associated UPs which can be productive for signal processing theory and applications.

1.1. Paper Contributions.

The contributions of this paper are summarized below:

- To introduce a novel integral transform coined as the Wigner distribution in the octonion linear canonical transform domain (WDOL)
- To study the fundamental properties of the 1D-WDOL and 3D-WDOL , including the Reconstrucion Formula, Rayleigh’s theorem and Riemann-Lebesgue lemma.
- To establish the relationship of 3D-WDOL with WD-QLCT and with 3D-WDLCT.
- To formulate several classes of uncertainty principles, such as the Heisenberg UP, logarithmic UP and the Hausdorff-Young inequality associated with WDOL.

1.2. Paper Outlines.

The paper is organized as follows: In Section 2 we gave a brief review to the octonion algebra and properties of OFT and OLCT. The definition and the properties of the 1D-WDOL and 3D-WDOL are studied in Section 3. In Section 4 we establish several classes
of uncertainty principles, such as the Heisenberg UP, logarithmic UP and the Hausdorff-Young inequality associated with the proposed transform. Finally, a conclusion is drawn in Section 5.

2. Preliminaries

In this section, we collect some basic facts on the octonion algebra and the octonion Fourier transform (OFT) and octonion linear canonical transform (OLCT), which will be needed throughout the paper.

2.1. Octonion algebra.

The octonion algebra \( \mathbb{O} \), \([52]\) is generated by the eighth-order Cayley-Dickson construction. In this construction, a hyper-complex number \( o \in \mathbb{O} \) is an ordered pair \( q_0, q_1 \in \mathbb{H} \)

\[
o = (q_0, q_1)
\]

\[
= ((\xi_0, \xi_1), (\xi_2, \xi_3))
\]

\[
= q_0 + q_1 \tau_4
\]

\[
= (\xi_0 + \xi_1 \tau_2) + (\xi_2 + \xi_3 \tau_2) \tau_4
\]

(2.1)

which has equivalent form

\[
o = s_o + \sum_{i=1}^{7} s_i \tau_i = s_0 + s_1 \tau_1 + s_2 \tau_2 + s_3 \tau_3 + s_4 \tau_4 + s_5 \tau_5 + s_6 \tau_6 + s_7 \tau_7
\]

(2.2)

thus \( o \) is a hyper-complex number defined by eight real numbers \( s_i, i = 0, 1, \ldots, 7 \) and seven imaginary units \( \tau_i \) where \( i = 1, 2, \ldots, 7 \). It should be noted that the \( \mathbb{O} \) is non-commutative and non-associative algebra. We present a table to represent the multiplication of imaginary units in the Cayley-Dickson algebra of octonions as \([43]\).

| \( \cdot \) | 1 | \( \tau_1 \) | \( \tau_2 \) | \( \tau_3 \) | \( \tau_4 \) | \( \tau_5 \) | \( \tau_6 \) | \( \tau_7 \) |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | \( \tau_1 \) | \( \tau_2 \) | \( \tau_3 \) | \( \tau_4 \) | \( \tau_5 \) | \( \tau_6 \) | \( \tau_7 \) |
| \( \tau_1 \) | \( \tau_1 \) | \( -1 \) | \( \tau_3 \) | \( -\tau_2 \) | \( \tau_5 \) | \( -\tau_4 \) | \( -\tau_7 \) | \( \tau_6 \) |
| \( \tau_2 \) | \( \tau_2 \) | \( -\tau_3 \) | \( -1 \) | \( \tau_1 \) | \( \tau_6 \) | \( \tau_7 \) | \( -\tau_4 \) | \( -\tau_5 \) |
| \( \tau_3 \) | \( \tau_3 \) | \( \tau_2 \) | \( -\tau_1 \) | \( -1 \) | \( \tau_7 \) | \( -\tau_6 \) | \( \tau_5 \) | \( -\tau_4 \) |
| \( \tau_4 \) | \( \tau_4 \) | \( -\tau_5 \) | \( -\tau_6 \) | \( -\tau_7 \) | \( -1 \) | \( \tau_1 \) | \( \tau_2 \) | \( \tau_3 \) |
| \( \tau_5 \) | \( \tau_5 \) | \( \tau_4 \) | \( -\tau_7 \) | \( \tau_6 \) | \( -\tau_1 \) | \( -1 \) | \( -\tau_3 \) | \( \tau_2 \) |
| \( \tau_6 \) | \( \tau_6 \) | \( \tau_7 \) | \( \tau_4 \) | \( -\tau_5 \) | \( -\tau_2 \) | \( \tau_3 \) | \( -1 \) | \( -\tau_1 \) |
| \( \tau_7 \) | \( \tau_7 \) | \( -\tau_6 \) | \( \tau_5 \) | \( \tau_4 \) | \( -\tau_3 \) | \( -\tau_2 \) | \( \tau_1 \) | \( -1 \) |

Table I

Multiplication Rules in Octonion Algebra.

The conjugate of an octonion is defined as

\[
\overline{o} = s_0 - s_1 \tau_1 - s_2 \tau_2 - s_3 \tau_3 - s_4 \tau_4 - s_5 \tau_5 - s_6 \tau_6 - s_7 \tau_7
\]

(2.3)

Therefore norm is defined by \( |o| = \sqrt{\overline{o}o} \) and \( |o|^2 = \sum_{i=0}^{7} s_i \). Also \( |o_1o_2| = |o_1||o_2|, \forall o_1, o_2 \in \mathbb{O} \).

From (2.1) it is clear that every \( o \in \mathbb{O} \) can be reshaped in quaternion form like

\[
o = a + b \tau_4
\]

(2.4)

where \( a = s_0 + s_1 \tau_1 + s_2 \tau_2 + s_3 \tau_3 \) and \( b = s_4 + s_5 \tau_1 + s_6 \tau_2 + s_7 \tau_3 \) are both quaternions. Evidently, we have the following lemma.
Lemma 2.1. [43] Let $a, b \in \mathbb{H}$, then

1. $\tau_4 a = \overline{a} \tau_4$;  
2. $\tau_4 (a \tau_4) = -\overline{a}$;  
3. $(a \tau_4) \tau_4 = -a$;  
4. $a(b \tau_4) = (ba) \tau_4$;  
5. $(a \tau_4) b = (ab) \tau_4$;  
6. $(a \tau_4)(b \tau_4) = -\overline{ba}$.

It is evident from the above result that, for an octonion $a + b \tau_4, a, b \in \mathbb{H}$, we have

$$\overline{a + b \tau_4} = \overline{a} - b \tau_4$$

and

$$|a + b \tau_4|^2 = |a|^2 + |b|^2.$$  

Lemma 2.2. [46] Let $\tilde{\omega}, \tilde{\phi} \in \mathbb{O}$. Then $e^{\tilde{\omega}} e^{\tilde{\phi}} = e^{\tilde{\omega} + \tilde{\phi}}$ iff $\tilde{\omega} \tilde{\phi} = \tilde{\phi} \tilde{\omega}$.

An octonion-valued function $f : \mathbb{R}^3 \rightarrow \mathbb{O}$ has following explicit form

$$f(x) = f_0 + f_1(x) \tau_1 + f_2(x) \tau_2 + f_3(x) \tau_3 + f_4(x) \tau_4 + f_5(x) \tau_5 + f_6(x) \tau_6 + f_7(x) \tau_7$$

$$= f_0 + f_1 \tau_1 + (f_2 + f_3 \tau_1) \tau_2 + [f_4 + f_5 \tau_1 + (f_6 + f_7 \tau_1) \tau_2] \tau_4$$

$$= \tilde{f}(x) + \tilde{f}(x) \tau_4$$

where each $f_i(x)$ is a real valued functions, $\tilde{f}, \tilde{f} \in \mathbb{H}$ are as in (2.1) and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

### 2.2. Octonion Fourier Transform and Octonion Linear Canonical Transform

Let $f : \mathbb{R} \rightarrow \mathbb{O}$ be an octonion-valued function, then 1D octonion Fourier transform (OFT) [46][53] is given by

$$\mathcal{F}_{\tau_4}[f](w) = \int_{\mathbb{R}} f(x) e^{-\tau_4 2\pi x w} dx,$$  

(2.8)

and its inverse is given by

$$f(x) = \mathcal{F}_{\tau_4}^{-1} \{\mathcal{F}_{\tau_4}[f]\}(x) = \int_{\mathbb{R}} \mathcal{F}_{\tau_4}[f](w) e^{\tau_4 2\pi x w} dw.$$  

(2.9)

See [53] for the properties of 1D OFT.

Later, authors in [43][45] introduced 3D OFT of an octonion-valued signal $f \in L^1(\mathbb{R}^3, \mathbb{O}) \cap L^2(\mathbb{R}^3, \mathbb{O})$ as

$$\mathcal{F}_{\tau_1, \tau_2, \tau_4}[f](w) = \int_{\mathbb{R}} f(x) e^{-\tau_1 2\pi x_1 w_1} e^{-\tau_2 2\pi x_2 w_2} e^{-\tau_4 2\pi x_3 w_3} dx,$$  

(2.10)

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $w = (w_1, w_2, w_3) \in \mathbb{R}^3$ and multiplication in above integral is done from left to right. Also the order of imaginary units in (2.10) is fixed because octonions are neither commutative nor associative.

The inverse 3D OFT is given as

$$f(x) = \mathcal{F}_{\tau_1, \tau_2, \tau_4}^{-1} \{\mathcal{F}_{\tau_1, \tau_2, \tau_4}[f]\}(x) = \int_{\mathbb{R}} f(x) e^{\tau_1 2\pi x_3 w_1} e^{\tau_2 2\pi x_2 w_2} e^{\tau_4 2\pi x_1 w_3} dw,$$  

(2.11)

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $w = (w_1, w_2, w_3) \in \mathbb{R}^3$.

Authors in [46] introduced octonion linear canonical transform (OLCT) as:
Lemma 2.3. The 1D OLCT of any octonion-valued signal \( f \in L^1(\mathbb{R}^3, \mathbb{O}) \), with respect to the uni-modular matrix \( \Lambda = (a, b, c, d) \) is given by
\[
\mathcal{L}_{\tau_4}^\Lambda[f](w) = \int_{\mathbb{R}} f(x) K_{\Lambda}^{\tau_4}(x, w) dx,
\]
where
\[
K_{\Lambda}^{\tau_4}(x, w) = \frac{1}{\sqrt{2\pi|b|}} e^{\frac{\tau_4}{2\pi} \left[ a x^2 - 2xw + dw^2 - \frac{\pi}{4} \right]}, \quad b \neq 0
\]
with the inversion formula
\[
f(x) = \left\{ \mathcal{L}_{\tau_4}^\Lambda \right\}^{-1} \left[ \mathcal{L}_{\tau_4}^\Lambda[f] \right](x) = \int_{\mathbb{R}} \mathcal{L}_{\tau_4}^\Lambda[f](w) K_{\Lambda}^{-\tau_4}(x, w) dx,
\]
where \( K_{\Lambda}^{-\tau_4}(x, w) = K_{\Lambda^{-1}}^{\tau_4}(w, x) \) and \( \Lambda^{-1} = (d, -b, -c, a) \).

Lemma 2.3. [46] The 1D OLCT can be reduced to 1D OFT by following equation:
\[
\mathcal{L}_{\tau_4}^\Lambda[f](w) = \frac{1}{\sqrt{2\pi|b|}} \mathcal{F}_{\tau_4}[g] \left( \frac{b}{2\pi|b|} \right) e^{\tau_4 \left( \frac{b}{2\pi|b|} \right)} w^2 - \frac{\pi}{4},
\]
where \( g(x) = f(x) e^{\tau_4 \frac{b}{2\pi|b|} x^2} \).

Definition 2.2 (3D-OLCT [46]). For every octonion-valued signal \( f \in L^1(\mathbb{R}^3, \mathbb{O}) \cap L^2(\mathbb{R}^3, \mathbb{O}) \), the 3D-OLCT with respect to the matrix parameters \( \Lambda_k = (a_k, b_k, c_k, d_k) \), satisfying \( \det(\Lambda_k) = 1 \), \( k = 1, 2, 3 \) is defined as
\[
\mathcal{L}_{\tau_1, \tau_2, \tau_4}^{\Lambda_1, \Lambda_2, \Lambda_3} \{ f \}(w) = \int_{\mathbb{R}^3} f(x_1, x_2, x_3) K_{\Lambda_1}^{\tau_1}(x_1, w_1) K_{\Lambda_2}^{\tau_2}(x_2, w_2) K_{\Lambda_3}^{\tau_4}(x_3, w_3) dx
\]
where \( x = (x_1, x_2, x_3), w = (w_1, w_2, w_3) \), and multiplication in above integral is done from left to right and
\[
K_{\Lambda_1}^{\tau_1}(x_1, w_1) = \frac{1}{\sqrt{2\pi|b_1|}} e^{\frac{\tau_1}{2\pi} \left[ a_1 x_1^2 - 2x_1w_1 + dw_1^2 - \frac{\pi}{4} \right]}, \quad b_1 \neq 0
\]
\[
K_{\Lambda_2}^{\tau_2}(x_2, w_2) = \frac{1}{\sqrt{2\pi|b_2|}} e^{\frac{\tau_2}{2\pi} \left[ a_2 x_2^2 - 2x_2w_2 + dw_2^2 - \frac{\pi}{4} \right]}, \quad b_2 \neq 0
\]
and
\[
K_{\Lambda_3}^{\tau_4}(x_3, w_3) = \frac{1}{\sqrt{2\pi|b_3|}} e^{\frac{\tau_4}{2\pi} \left[ a_3 x_3^2 - 2x_3w_3 + dw_3^2 - \frac{\pi}{4} \right]}, \quad b_3 \neq 0.
\]
with the inversion formula
\[
f(x) = \left\{ \mathcal{L}_{\tau_1, \tau_2, \tau_4}^{\Lambda_1, \Lambda_2, \Lambda_3} \right\}^{-1} \left[ \mathcal{L}_{\tau_1, \tau_2, \tau_4}^{\Lambda_1, \Lambda_2, \Lambda_3} \{ f \} \right](x) = \int_{\mathbb{R}^3} \mathcal{L}_{\tau_1, \tau_2, \tau_4}^{\Lambda_1, \Lambda_2, \Lambda_3} \{ f \}(w) K_{\Lambda_3}^{-\tau_4}(w_3, x_3) K_{\Lambda_2}^{-\tau_2}(w_2, x_2) K_{\Lambda_1}^{-\tau_1}(w_1, x_1) dw,
\]
where \( \Lambda_k^{-1} = (d_k, -b_k, -c_k, a_k) \in \mathbb{R}^{2 \times 2} \), for \( k = 1, 2, 3 \).
Further authors in [46] expanded the kernel of 3D-OLCT as:

\[
K^{\tau_4}_{\Lambda_1}(x_1, w_1)K^{\tau_2}_{\Lambda_2}(x_2, w_2)K^{\tau_4}_{\Lambda_3}(x_3, w_3) = \frac{1}{2\pi \sqrt{2\pi |b_1b_2b_3|}} e^{\tau_1 \theta_1} e^{\tau_2 \theta_2} e^{\tau_4 \theta_4} = \frac{1}{2\pi \sqrt{2\pi |b_1b_2b_3|}} (c_1 + \tau_1 s_1)(c_2 + \tau_2 s_2)(c_3 + \tau_4 s_3) = \frac{1}{2\pi \sqrt{2\pi |b_1b_2b_3|}} (c_1 c_2 c_3 + s_1 c_2 c_3 \tau_1 + c_1 s_2 c_3 \tau_2 + s_1 s_2 c_3 \tau_3 + c_1 c_2 s_3 \tau_4 + s_1 c_2 s_3 \tau_5 + c_1 s_2 s_3 \tau_6 + s_1 s_2 s_3 \tau_7),
\]

where \( \theta_k = \frac{1}{2\theta_k} [a_k x_k^2 - 2x_k w_k + d_k w_k^2 - \frac{\pi}{2}] \), \( c_k = \cos \theta_k \) and \( s_k = \sin \theta_k \), \( k = 1, 2, 3 \).

3. Wigner Distribution In The Octonion Linear Canonical Domain

In this section we formally introduce the definition of Wigner distribution associated with octonion linear canonical transform (WDOL) and study its various properties.

According to 1D-OLCT, we can obtain the definition of 1D-WDOL as follows

**Definition 3.1 (1D-WDOL).** The 1D-WDOL of any octonion-valued signals \( f, g \in L^1(\mathbb{R}^3, \mathbb{O}) \), with respect to the uni-modular matrix \( \Lambda = (a, b, c, d) \) is given by

\[
\mathbb{W}^{\tau_4}_{f,g}(t, w) = \int_{\mathbb{R}} f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) K^{\tau_4}_{\Lambda}(x, w) dx,
\]

where kernel \( K^{\tau_4}_{\Lambda}(x, w) \) is given by (2.15).

Next we note that 1D-WDOL is an 1D-OLCT of a instantaneous correlation of octonion-valued functions \( f \) and \( g \) by means of remark as:

**Remark 3.1.** From definition of 1D-OLCT, it is clear that if we take

\[
h_t(x) = f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right),
\]

we have

\[
\mathbb{W}^{\tau_4}_{f,g}(t, w) = \mathcal{L}^{\tau_4}_\Lambda \left[ f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right] (w) = \mathcal{L}^{\tau_4}_\Lambda [h_t](w).
\]

And by using Lemma [2.3] we obtain the relation between 1D-WDOL and 1D-OFT as :

\[
\mathbb{W}^{\tau_4}_{f,g}(t, w) = \frac{1}{\sqrt{2\pi |b|}} \mathcal{F}_{\tau_4} [H_t] \left( \frac{w}{2\pi |b|} \right) e^{\tau_4 (w^2 - t^2)},
\]

where \( H_t(x) = h_t(x) e^{\tau_4 \frac{w^2}{2b} x^2} \).

**Theorem 3.2 (1D-WDOL Reconstruction Formula).** For \( f, g \in L^2(\mathbb{R}, \mathbb{O}) \) with \( g(0) \neq 0 \), we have the following inversion formula for 1D-WDOL

\[
f(\xi) = \frac{1}{g^*(0)} \int_{\mathbb{R}} \mathbb{W}^{\tau_4}_{f,g} \left( \frac{\xi}{2}, w \right) K^{\tau_4}_{\Lambda}(\xi, w) dw
\]
Proof. From (3.4), we obtain
\[
\sqrt{2\pi|b|}\mathbb{W}_{f,g}^{\tau_4}(t, w)e^{-\tau_4\left(\frac{d}{2\nu}w^2 - \frac{x}{2}\right)} = \mathcal{F}_{\tau_4}[H_t]\left(\frac{w}{2\pi|b|}\right),
\]
(3.6)
where \(H_t(x) = f\left(t + \frac{x}{2}\right)g^*\left(t - \frac{x}{2}\right)e^{\tau_4\frac{d}{2\nu}x^2}\).

Now by the application of (2.9), it follows that
\[
f\left(t + \frac{x}{2}\right)g^*\left(t - \frac{x}{2}\right) = \int_{\mathbb{R}} \sqrt{2\pi|b|}\mathbb{W}_{f,g}^{\tau_4}(t, w)e^{-\tau_4\left(\frac{d}{2\nu}w^2 - \frac{x}{2}\right)} e^{\tau_4\frac{d}{2\nu}x^2}dw.
\]
Which implies
\[
f\left(t + \frac{x}{2}\right)g^*\left(t - \frac{x}{2}\right) = \int_{\mathbb{R}} \mathbb{W}_{f,g}^{\tau_4}(t, w)\frac{1}{\sqrt{2\pi|b|}}e^{-\tau_4\left(\frac{d}{2\nu}w^2 - \frac{x}{2}\right)} e^{\tau_4\frac{d}{2\nu}x^2}dw = \int_{\mathbb{R}} \mathbb{W}_{f,g}^{\tau_4}(t, w)K^{-\tau_4}_\Lambda(x, w)dw.
\]
(3.7)
On setting \(\frac{x}{2} = t\), and applying change of variable \(\xi = 2t\), (3.7) yields
\[
f(\xi)g^*(0) = \int_{\mathbb{R}} \mathbb{W}_{f,g}^{\tau_4}\left(\frac{\xi}{2}, w\right)K^{-\tau_4}_\Lambda(\xi, w)dw,
\]
which completes the proof \(\Box\)

**Theorem 3.3** (Plancherel’s Theorem). Let \(f, g \in L^2(\mathbb{R}, \mathcal{O})\), then the 1D-WDOL satisfies:
\[
\|\mathbb{W}_{f,g}^{\tau_4}\|_{L^2(\mathbb{R}, \mathcal{O})}^2 = \|f\|_{L^2(\mathbb{R}, \mathcal{O})}^2\|g\|_{L^2(\mathbb{R}, \mathcal{O})}^2.
\]
(3.8)

**Proof.** We know that for \(f, g \in L^2(\mathbb{R}, \mathcal{O})\), we have
\[
\langle f, g \rangle_{L^2(\mathbb{R}, \mathcal{O})} = \int_{\mathbb{R}} f(x)g^*(x)dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{L}_{\tau_4}^\Lambda[f](w)K^{-\tau_4}_\Lambda(x, w)dw\right)g^*(x)dx = \int_{\mathbb{R}} \mathcal{L}_{\tau_4}^\Lambda[f](w)\left(\int_{\mathbb{R}} g(x)K^\tau_4(x, w)dx\right)^*dw = \int_{\mathbb{R}} \mathcal{L}_{\tau_4}^\Lambda[f](w)\mathcal{L}_{\tau_4}^\Lambda[g](w)^*dw = \langle \mathcal{L}_{\tau_4}^\Lambda[f](w), \mathcal{L}_{\tau_4}^\Lambda[g](w)\rangle_{L^2(\mathbb{R}, \mathcal{O})}.
\]
Thus for \(f = g\), above yields
\[
\|f\|_{L^2(\mathbb{R}, \mathcal{O})}^2 = \|\mathcal{L}_{\tau_4}^\Lambda[f]\|_{L^2(\mathbb{R}, \mathcal{O})}^2
\]
Replacing \(f(x)\) by \(h_t(x)\), we have
\[
\|h_t\|_{L^2(\mathbb{R}, \mathcal{O})}^2 = \|\mathcal{L}_{\tau_4}^\Lambda[h_t]\|_{L^2(\mathbb{R}, \mathcal{O})}^2
\]
Now applying (3.3), above eqn. yields

\[
\|W_{f,g}^{r_4}\|_{L^2(\mathbb{R},O)}^2 = \left\| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right\|_{L^2(\mathbb{R},O)}^2 \\
= \left( \int_{\mathbb{R}} \int_{\mathbb{R}} |f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right)|^2 dt ds \right) \\
= \int_{\mathbb{R}} |f(u)|^2 du \int_{\mathbb{R}} |g^*(v)|^2 dv \\
= \|f\|_{L^2(\mathbb{R},O)}^2 \|g\|_{L^2(\mathbb{R},O)}^2.
\]

Which completes the proof. □

By using the relationship between 1D-WDOL and 1D-OFT, we can prove the properties like \(\mathbb{R}\)-antilinearity, Scaling, Shift, Modulation of 1D-WDOLC following the procedure defined in [33].

Next, we will introduce the definition of the 3D-WDOL.

**Definition 3.2 (3D-WDOL).** Let \( f, g : \mathbb{R}^3 \rightarrow O \) be two octonion-valued functions, then 3D-WDOL with respect to the matrix parameters \( \Lambda_k = (a_k, b_k, c_k, d_k) \), satisfying \( \det(\Lambda_k) = 1 \), \( k = 1, 2, 3 \) is defined as

\[
W_{f,g}^{A_1,A_2,A_3}(t, w) = \int_{\mathbb{R}^3} f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) K_{A_1}^r(x_1, w_1) K_{A_2}^r(x_2, w_2) K_{A_3}^r(x_3, w_3) dx.
\]

where \( x = (x_1, x_2, x_3) \), \( w = (w_1, w_2, w_3) \), \( t = (t_1, t_2, t_3) \) and kernel signals \( K_{A_1}^r(x_1, w_1) \), \( K_{A_2}^r(x_2, w_2) \), and \( K_{A_3}^r(x_3, w_3) \) are given by (2.17), (2.18) and (2.19) respectively.

It should be noted that the multiplication in the above integrals is done from left to right as the octonion algebra is non-associative. Also we assume that the above signals \( f, g \) are continuous and both signals and there WDOL are integrable (in Lebesgue sense) in this paper.

Next we note that 3D-WDOL is an 3D-OLCT of a instantaneous correlation of 3D octonion-valued functions \( f \) and \( g \) by means of remark as:

**Remark 3.4.** From definition of 3D-OLCT, it is clear that if we take

\[
h_t(x) = f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right),
\]

we have

\[
W_{f,g}^{A_1,A_2,A_3}(t, w) = \mathcal{L}_{r_1,r_2,r_3}^{A_1,A_2,A_3}[h_t](w).
\]

The 3D-WDOL \( W_{f,g}^{A_1,A_2,A_3}(t, w) \) defined in (3.9) can be expressed as octonion sum of components of different parity by using (2.21) as:

\[
W_{f,g}^{A_1,A_2,A_3}(t, w) = W_{f,g}^{eie} + W_{f,g}^{eo} r_1 + W_{f,g}^{oe} r_2 + W_{f,g}^{oee} r_3 + G_{f,g}^{eoe} r_4 + W_{f,g}^{oee} r_5 + W_{f,g}^{oeo} r_6 + W_{f,g}^{ooo} r_7
\]

(3.12)
where

\[
W^{ree}_{f,g}(t, w) = \frac{1}{2\pi^2 |b_1 b_2 b_3|} \int_{\mathbb{R}^3} h^{ree}_t(x)c_1c_2c_3 dx,
\]

\[
W^{ree}_{t}(w) = \frac{1}{2\pi^2 |b_1 b_2 b_3|} \int_{\mathbb{R}^3} h^{ree}_t(x)s_1c_2c_3 dx,
\]

\[
W^{ree}_{f,g}(t, w) = \frac{1}{2\pi^2 |b_1 b_2 b_3|} \int_{\mathbb{R}^3} h^{ree}_t(x)c_1s_2c_3 dx,
\]

\[
W^{ree}_{f,g}(t, w) = \frac{1}{2\pi^2 |b_1 b_2 b_3|} \int_{\mathbb{R}^3} h^{ree}_t(x)c_1c_2s_3 dx,
\]

\[
W^{ree}_{f,g}(t, w) = \frac{1}{2\pi^2 |b_1 b_2 b_3|} \int_{\mathbb{R}^3} h^{ree}_t(x)c_1s_2s_3 dx,
\]

\[
W^{ree}_{t}(w) = \frac{1}{2\pi^2 |b_1 b_2 b_3|} \int_{\mathbb{R}^3} h^{ree}_t(x)c_1s_2s_3 dx,
\]

\[
W^{ree}_{f,g}(t, w) = \frac{1}{2\pi^2 |b_1 b_2 b_3|} \int_{\mathbb{R}^3} h^{ree}_t(x)s_1s_2s_3 dx.
\]

Where \( h_t(x) = f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \) and can be expressed as sum eight terms:

\[
h_t(x) = h^{reee}_t(x) + h^{rree}_t(x) + h^{rere}_t(x) + h^{eree}_t(x) + h^{rero}_t(x) + h^{eror}_t(x) + h^{oore}_t(x) + h^{ooreo}_t(x).
\]

\[
(3.21)
\]

where \( h^{lmm}_t(x), l, m, n \in \{ e, o \} \) are eight terms of different parity with relation to \( x_1, x_2 \) and \( x_3 \). Again using subscripts \( e \) and \( o \) to indicate that a function is either even (\( e \)) or odd (\( o \)) with respect to an appropriate variable, i.e. \( h^{rree}_t(x) \) is even with respect to \( x_1 \) and \( x_2 \) and odd with respect to \( x_3 \).

Now, we show that 3D-WDOL can be divided into four Wigner distributions in the QLCT domain (WDQLCT).

**Lemma 3.1 (Relation with WDQLCT).** For \( f, g \in L^2(\mathbb{R}^3, \mathcal{O}) \) and \( \mathcal{W}^{A_1,A_2,A_3}_{f,g}(t, w) \) be the 3D-WDOL, then

\[
\left| \mathcal{W}^{A_1,A_2,A_3}_{f,g}(t, w) \right|^2 = \frac{1}{2\pi b_3} \left( \left| \mathcal{W}^{A_1,A_2}_{(\bar{f},\bar{g})}(t, w) \right|^2 + \left| \mathcal{W}^{A_1,A_2}_{(\bar{f},\bar{g})^o}(t, w) \right|^2 \right.

\[
+ \left. \left| \mathcal{W}^{A_1,A_2}_{(f,g)}(t, w) \right|^2 + \left| \mathcal{W}^{A_1,A_2}_{(f,g)^o}(t, w) \right|^2 \right),
\]

\[
(3.22)
\]

where \( \bar{f}, \bar{g}, \bar{g} \in L^2(\mathbb{R}^2, \mathbb{H}) \).

**Proof.** Let \( h^r_t(x) = \frac{1}{2}[h_t(x_1, x_2, x_3) + h_t(x_1, x_2, -x_3)] \), and \( h^o_t(x) = \frac{1}{2}[h_t(x_1, x_2, x_3) - h_t(x_1, x_2, -x_3)] \), then \( h^r_t(x) \) and \( h^o_t(x) \) represents even and odd parts of \( h_t(x) \) but only in variable \( x_3 \).
Since every octonion function can be written in the quaternion form as \( h_t(x) = \tilde{h}_t + \hat{h}_t \tau_4 \), therefore from (3.11), we have

\[
\mathbb{W}^{\Lambda_1, \Lambda_2, \Lambda_3}_{f,g}(t, w) = \mathbf{L}^{\Lambda_1, \Lambda_2, \Lambda_3}_{\tau_1, \tau_2, \tau_4}[\tilde{h}_t + \hat{h}_t \tau_4](w)
\]

\[
= \int_{\mathbb{R}^3} (\tilde{h}_t + \hat{h}_t \tau_4) K_{\Lambda_1}^{\tau_1}(x_1, w_1) K_{\Lambda_2}^{\tau_2}(x_2, w_2) K_{\Lambda_3}^{\tau_4}(x_3, w_3) dx
\]

\[
+ \int_{\mathbb{R}^3} \tilde{h}_t(x) K_{\Lambda_1}^{\tau_1}(x_1, w_1) K_{\Lambda_2}^{\tau_2}(x_2, w_2) \tau_4 K_{\Lambda_3}^{\tau_4}(x_3, w_3) dx. \quad (3.23)
\]

Now taking even and odd parts of functions \( \tilde{h}_t \) and \( \hat{h}_t \), (3.23) yields

\[
\mathbb{W}^{\Lambda_1, \Lambda_2, \Lambda_3}_{f,g}(t, w) = \frac{1}{\sqrt{2\pi b_3}} \int_{\mathbb{R}^3} \tilde{h}_t^e(x) K_{\Lambda_1}^{\tau_1}(x_1, w_1) K_{\Lambda_2}^{\tau_2}(x_2, w_2) c_3 dx
\]

\[
+ \frac{1}{\sqrt{2\pi b_3}} \int_{\mathbb{R}^3} \tilde{h}_t^o(x) K_{\Lambda_1}^{\tau_1}(x_1, w_1) K_{\Lambda_2}^{\tau_2}(x_2, w_2) s_3 dx
\]

\[
+ \left( \frac{1}{\sqrt{2\pi b_3}} \int_{\mathbb{R}^3} \tilde{h}_t^e(x) K_{\Lambda_1}^{\tau_1}(x_1, w_1) K_{\Lambda_2}^{\tau_2}(x_2, w_2) c_3 dx \right) \tau_4, \quad (3.24)
\]

where \( \theta_k = \frac{1}{2b_k} [a_k x_k^2 - 2x_k w_k + d_k w_k^2 - \frac{u_k^2}{2}] \), \( c_k = \cos \theta_k \) and \( s_k = \sin \theta_k \), \( k = 1, 2, 3 \).

Thus

\[
\left| \mathbb{W}^{\Lambda_1, \Lambda_2, \Lambda_3}_{f,g}(t, w) \right|^2 = \frac{1}{2\pi b_3} \left( \left| \mathbf{L}^{\Lambda_1, \Lambda_2, \Lambda_3}_{\tau_1, \tau_2}[\tilde{h}_t^e](w) \right|^2 + \left| \mathbf{L}^{\Lambda_1, \Lambda_2, \Lambda_3}_{\tau_1, \tau_2}[\tilde{h}_t^o](w) \right|^2 \right.
\]

\[
+ \left| \mathbf{L}^{\Lambda_1, \Lambda_2, \Lambda_3}_{\tau_1, \tau_2}[\tilde{h}_t^e](w) \right|^2 + \left| \mathbf{L}^{\Lambda_1, \Lambda_2, \Lambda_3}_{\tau_1, \tau_2}[\tilde{h}_t^o](w) \right|^2 \right), \quad (3.25)
\]

where the terms in RHS denote the quaternion LCT of the respective correlation product. Applying the definition of WD in the QLCT domain, (3.25) gives

\[
\left| \mathbb{W}^{\Lambda_1, \Lambda_2, \Lambda_3}_{f,g}(t, w) \right|^2 = \frac{1}{2\pi b_3} \left( \left| \mathbb{W}^{\Lambda_1, \Lambda_2}_{(f, \theta)^e}(t, w) \right|^2 + \left| \mathbb{W}^{\Lambda_1, \Lambda_2}_{(f, \theta)^o}(t, w) \right|^2 \right.
\]

\[
+ \left| \mathbb{W}^{\Lambda_1, \Lambda_2}_{(f, \theta)^e}(t, w) \right|^2 + \left| \mathbb{W}^{\Lambda_1, \Lambda_2}_{(f, \theta)^o}(t, w) \right|^2 \right) .
\]

Which completes the proof. \( \square \)
**Theorem 3.5** (Reconstruction formula of the 3D-WDOL). If \( f, g \in L^2(\mathbb{R}^3, \mathcal{O}) \) and \( g(0) \neq 0 \), then \( f \) can be reconstructed by the inverse 3D-OLCT of 3D-WDOL \( \mathcal{W}_{f,g}^{\alpha_1,\alpha_2,\alpha_3} \), i.e.

\[
 f(x) = \frac{1}{g^*(0)} \int_{\mathbb{R}^3} \mathcal{W}_{f,g}^{\alpha_1,\alpha_2,\alpha_3} \left( \frac{x}{2}, w \right) \mathcal{K}_{\alpha_3}^{\tau_1}(w_3, x_3) \mathcal{K}_{\alpha_2}^{\tau_2}(w_2, x_2) \mathcal{K}_{\alpha_1}^{\tau_3}(w_1, x_1) \, dw.
\]

**Proof.** Applying 3D-OLCT inversion given in (2.20) to (3.11), we have

\[
 h_t(x) = \mathcal{L}_{\tau_1,\tau_2,\tau_3}^{-1} \left[ \mathcal{W}_{f,g}^{\alpha_1,\alpha_2,\alpha_3} \right](x) = \int_{\mathbb{R}^3} \mathcal{W}_{f,g}^{\alpha_1,\alpha_2,\alpha_3}(t, w) \mathcal{K}_{\alpha_3}^{\tau_1}(w_3, x_3) \mathcal{K}_{\alpha_2}^{\tau_2}(w_2, x_2) \mathcal{K}_{\alpha_1}^{\tau_3}(w_1, x_1) \, dw.
\]

Hence

\[
 f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) = \int_{\mathbb{R}^3} \mathcal{W}_{f,g}^{\alpha_1,\alpha_2,\alpha_3}(t, w) \mathcal{K}_{\alpha_3}^{\tau_1}(w_3, x_3) \mathcal{K}_{\alpha_2}^{\tau_2}(w_2, x_2) \mathcal{K}_{\alpha_1}^{\tau_3}(w_1, x_1) \, dw.
\]

Setting \( t = \frac{x}{2} \), above equation yields

\[
 f \left( 2t \right) g^* \left( 0 \right) = \int_{\mathbb{R}^3} \mathcal{W}_{f,g}^{\alpha_1,\alpha_2,\alpha_3}(t, w) \mathcal{K}_{\alpha_3}^{\tau_1}(w_3, 2t_3) \mathcal{K}_{\alpha_2}^{\tau_2}(w_2, 2t_2) \mathcal{K}_{\alpha_1}^{\tau_3}(w_1, 2t_1) \, dw.
\]

Applying the change of variable \( y = 2t \), we get

\[
 f(y) = \frac{1}{g^*(0)} \int_{\mathbb{R}^3} \mathcal{W}_{f,g}^{\alpha_1,\alpha_2,\alpha_3} \left( \frac{y}{2}, w \right) \mathcal{K}_{\alpha_3}^{\tau_1}(w_3, y_3) \mathcal{K}_{\alpha_2}^{\tau_2}(w_2, y_2) \mathcal{K}_{\alpha_1}^{\tau_3}(w_1, y_1) \, dw.
\]

Which completes the proof. \(\Box\)

**Theorem 3.6** (Rayleigh’s theorem for 3D-WDOL). Let \( f, g \in L^2(\mathbb{R}^3, \mathcal{O}) \), then we have

\[
 2\pi |b_3| \| \mathcal{W}_{f,g}^{\alpha_1,\alpha_2,\alpha_3}(t, w) \|_{L^2(\mathbb{R}^2, \mathcal{O})}^2 = \| f \|_{L^2(\mathbb{R}^2, \mathcal{O})}^2 \| g \|_{L^2(\mathbb{R}^2, \mathcal{O})}^2. \tag{3.26}
\]

**Proof.** The Rayleigh’s theorem is valid for the 3D-OLCT and reads (see Thm.4 [10])

\[
 2\pi |b_3| \mathcal{L}_{\tau_1,\tau_2,\tau_4}^{\alpha_1,\alpha_2,\alpha_3} \| f \|_{L^2(\mathbb{R}^2, \mathcal{O})}^2 = \| f \|_{L^2(\mathbb{R}^2, \mathcal{O})}^2
\]

Replacing \( f(x) \) by \( h_t(x) \) given in (3.11), above equation becomes

\[
 2\pi |b_3| \mathcal{L}_{\tau_1,\tau_2,\tau_4}^{\alpha_1,\alpha_2,\alpha_3} \| h_t \|_{L^2(\mathbb{R}^2, \mathcal{O})}^2 = \| h_t \|_{L^2(\mathbb{R}^2, \mathcal{O})}^2
\]

Applying (3.11), above equation yields

\[
 2\pi |b_3| \| \mathcal{W}_{f,g}^{\alpha_1,\alpha_2,\alpha_3}(t, w) \|_{L^2(\mathbb{R}^2, \mathcal{O})}^2 = \left\| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right\|_{L^2(\mathbb{R}^2, \mathcal{O})}^2
\]

Applying the Fubini theorem and using suitable change of variables (just like earlier), we obtain

\[
 2\pi |b_3| \| \mathcal{W}_{f,g}^{\alpha_1,\alpha_2,\alpha_3}(t, w) \|_{L^2(\mathbb{R}^2, \mathcal{O})}^2 = \| f \|_{L^2(\mathbb{R}^2, \mathcal{O})}^2 \| g \|_{L^2(\mathbb{R}^2, \mathcal{O})}^2.
\]

Which completes the proof. \(\Box\)
\textbf{Theorem 3.7.} Let \( f, g \in L^2(\mathbb{R}^3; \mathbb{O}) \), then the Riemann–Lebesgue lemma associated with 3D-WDOL holds with respect to \( w \), i.e.,
\[
\lim_{|w| \to 0} \left| \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) \right| \to 0, \quad w, x \in \mathbb{R}^3.
\] (3.27)

\textbf{Proof.} Since Riemann–Lebesgue lemma holds for 3D-OLCT (see \cite{46}), therefore using relation between 3D-OLCT and 3D-WDOL the proof of theorem follows. \( \square \)

Now, we shall establish the relation between 3D-WDOL and 3D-WDLCT.

\textbf{Theorem 3.8.} Let \( f, g : \mathbb{O} \to \mathbb{R}^3 \) be two octonion-valued signals and \( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) \) represents the 3D-WDLCT. Then the following equation holds
\[
\begin{align*}
\mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) &= \frac{1}{4} \left\{ \left( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) + \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda'_3}(t, w)(1 - \tau_3) \right) \\
&+ \left( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda'_2,\Lambda_3}(t, w) + \mathcal{W}_{f,g}^{\Lambda_1,\Lambda'_2,\Lambda'_3}(t, w)(1 + \tau_3) \right) \right\} \\
&+ \frac{1}{4} \left\{ \left( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) - \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda'_3}(t, w)(1 - \tau_3) \right) \\
&+ \left( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda'_2,\Lambda_3}(t, w) - \mathcal{W}_{f,g}^{\Lambda_1,\Lambda'_2,\Lambda'_3}(t, w)(1 + \tau_3) \right) \right\} \tau_3
\end{align*}
\] (3.28)

where \( \Lambda'_k = (a_k, -b_k, -c_k, d_k), \quad k = 2, 3. \)

\textbf{Proof.} For \( f, g \in L^2(\mathbb{R}^3) \) the 3D Wigner distribution associated with LCT (3D-WDLCT) is defined corresponding to the 3D-LCT \cite{46} as:
\[
\mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) = \frac{1}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} \int_{\mathbb{R}^3} f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) e^{\tau_1 \theta_1 e^{\tau_2 \theta_2} e^{\tau_3 \theta_3}} dx.
\] (3.29)

Now for \( \Lambda' \),
\[
\mathcal{W}_{f,g}^{\Lambda_1,\Lambda'_2,\Lambda_3}(t, w) = \frac{1}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} \int_{\mathbb{R}^3} f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) e^{\tau_1 \theta_1 e^{-\tau_2 \theta_2} e^{\tau_3 \theta_3}} dx.
\] (3.30)

By equivalent definition of sine and cosine functions, we obtain
\[
\frac{1}{2} \left( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) + \mathcal{W}_{f,g}^{\Lambda_1,\Lambda'_2,\Lambda_3}(t, w) \right) = \frac{1}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} \int_{\mathbb{R}^3} f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) e^{\tau_1 \theta_1 c_2 e^{\tau_3 \theta_3}} dx.
\] (3.31)

And
\[
\frac{1}{2} \left( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda'_2,\Lambda_3}(t, w) - \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) \right) = \frac{1}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} \int_{\mathbb{R}^3} f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) e^{\tau_1 \theta_1 (-\tau_1 s_2)} e^{\tau_3 \theta_3} dx.
\] (3.32)
Let us introduce new notation for simplification:

\[
\frac{1}{2} \left( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) - \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) \right) \tau_3
\]

Adding (3.31) and (3.33), we have

\[
\frac{1}{2} \left( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) + \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) \right) \\
+ \frac{1}{2} \left( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) - \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) \right) \tau_3
\]

Let us introduce new notation for simplification:

\[
\mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) = \frac{1}{2} \left( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) + \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) \right)
+ \frac{1}{2} \left( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) - \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) \right) \tau_3.
\]

Now for \( \Lambda'_3 = (a_3, -b_3, -c_3, d_3) \), then

\[
\mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda'_3}(t, w)
= \frac{1}{2} \left( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda'_3}(t, w) + \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda'_3}(t, w) \right)
+ \frac{1}{2} \left( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda'_3}(t, w) - \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda'_3}(t, w) \right) \tau_3.
\]

By following similar steps as before we get

\[
\frac{1}{2} \left( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda'_3}(t, w) - \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda'_3}(t, w) \right)
= \frac{1}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} \int_{\mathbb{R}^3} f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) e^{\tau_1 \theta_1 e^{\tau_2 \theta_2 e^{\tau_3 \theta_3}} dx.}
\]

And

\[
\frac{1}{2} \left( \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda'_3}(t, w) - \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda'_3}(t, w) \right)
= \frac{1}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} \int_{\mathbb{R}^3} f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) e^{\tau_1 \theta_1 e^{\tau_2 \theta_2 e^{\tau_3 \theta_3}} dx.}
\]
On multiplying (3.39) from right by \( \tau_5 \) and using multiplication rules from Table 2.1 we have

\[
\frac{1}{2} \left( W_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) - W_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) \right) \tau_5 \\
= \frac{1}{2 \pi \sqrt{2 \pi b_1 b_2 b_3}} \int_{\mathbb{R}^3} f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) e^{\tau_1} e^{\tau_2} e^{\tau_3} (x, s) dx.
\]

Adding (3.38) and (3.40), we get

\[
\frac{1}{2} \left( W_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) + W_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) \right) \\
+ \frac{1}{2} \left( W_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) - W_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) \right) \tau_5 \\
= \frac{1}{2 \pi \sqrt{2 \pi b_1 b_2 b_3}} \int_{\mathbb{R}^3} f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) e^{\tau_1} e^{\tau_2} e^{\tau_3} (x, s) dx.
\]

(3.41)

On substituting (3.35) and (3.36) in (3.41), we get the desired result. \( \square \)

4. Uncertainty principles of the WDOL

The uncertainty principles (UPs) lies in the heart of any integral transforms. In [46, 47] authors derived Heisenberg’s uncertainty principle, Hausdorff–Young inequality, logarithmic uncertainty inequality, Pitt’s inequality and local uncertainty inequality for the octonion linear canonical transform and octonion offset linear canonical transform. Recently, in [48, 49] extend these UPs to the short-time octonion Fourier transform and short-time octonion linear canonical transform. Considering the WDOL as an extension of WD-QLCT, so in this section we shall investigate some uncertainty inequalities for the STOLCT.

Let begin with the Heisenberg’s uncertainty principle for the WDOL.

**Theorem 4.1 (Heisenberg’s uncertainty principle for the WDOL).** Let \( f, g \in L^1(\mathbb{R}^3, \Omega) \cap L^2(\mathbb{R}^3, \Omega) \), then 3D-WDOL satisfies following inequality

\[
\left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} x^2 \left| f \left( t + \frac{x}{2} \right) \right| g^* \left( t - \frac{x}{2} \right) dx dt \right) \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \| W_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t, w) \|^2 dw dt \right) \\
\geq \frac{2}{\pi |b_1| b_2^2 b_3^2} \| f \|^2 \| g \|^2.
\]

(4.1)

**Proof.** For any signal \( f \in L^1(\mathbb{R}^3, O) \cap L^2(\mathbb{R}^3, O) \), the Heisenberg’s uncertainty principle associated with octonion linear canonical transform reads [46]

\[
\int_{\mathbb{R}^2} x^2 |f(x)|^2 dx \int_{\mathbb{R}^2} w^2 |\mathcal{L}_{\tau_1,\tau_2,\tau_3}[f](w)|^2 dw 
\geq \frac{2}{\pi |b_3| b_1^2 b_2^2} \| f \|^2. 
\]

(4.2)
Since \(f, g \in L^1(\mathbb{R}^3, O) \cap L^2(\mathbb{R}^3, O)\), which implies \(h_t(x)\) defined in (3.10) belongs to \(L^1(\mathbb{R}^3, O) \cap L^2(\mathbb{R}^3, O)\). Therefore on replacing \(f(x)\) by \(h_t(x)\), (4.2) yields
\[
\int_{\mathbb{R}^2} x^2 |h_t(x)|^2 dx \int_{\mathbb{R}^2} w^2 |\mathcal{L}^{A_1,A_2,A_3}_{\tau_1,\tau_2,t}| h_t(w)|^2 dw \geq \frac{2}{\pi |b_3|} b_1^2 b_2^2 \|h_t\|^2_2. \tag{4.3}
\]
Applying (3.11) to LHS of (4.3), we obtain
\[
\int_{\mathbb{R}^2} x^2 |h_t(x)|^2 dx \int_{\mathbb{R}^2} w^2 |\mathcal{W}_{f,g}^{A_1,A_2,A_3}(t, w)|^2 dw \geq \frac{2}{\pi |b_3|} b_1^2 b_2^2 \left( \int_{\mathbb{R}^2} |h_t(x)|^2 dx \right)^2. \tag{4.4}
\]
Then, we have
\[
\int_{\mathbb{R}^2} x^2 \left| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right|^2 dx \int_{\mathbb{R}^2} w^2 |\mathcal{W}_{f,g}^{A_1,A_2,A_3}(t, w)|^2 dw \\
\geq \frac{2}{\pi |b_3|} b_1^2 b_2^2 \left( \int_{\mathbb{R}^2} \left| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right|^2 dx \right)^2. \tag{4.5}
\]
To (4.3), we first take square root and then integrating it both sides with respect to \(dt\), we get
\[
\int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} x^2 \left| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^2} w^2 |\mathcal{W}_{f,g}^{A_1,A_2,A_3}(t, w)|^2 dw \right)^{1/2} \right) dt \\
\geq \sqrt{\frac{2}{\pi |b_3|}} b_1 b_2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right|^2 dx dt. \tag{4.6}
\]
Applying the Cauchy–Schwarz inequality to the LHS of (4.6), we have
\[
\left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} x^2 \left| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right|^2 dx dt \right)^{1/2} \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} w^2 |\mathcal{W}_{f,g}^{A_1,A_2,A_3}(t, w)|^2 dw dt \right)^{1/2} \\
\geq \sqrt{\frac{2}{\pi |b_3|}} b_1 b_2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right|^2 dx dt. \tag{4.7}
\]
Further applying the Fubini theorem and using suitable change of variables (just like earlier) to RHS of (4.7), we obtain
\[
\left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} x^2 \left| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right|^2 dx dt \right)^{1/2} \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} w^2 |\mathcal{W}_{f,g}^{A_1,A_2,A_3}(t, w)|^2 dw dt \right)^{1/2} \\
\geq \sqrt{\frac{2}{\pi |b_3|}} b_1 b_2 \|f\|_2 \|g\|_2. \tag{4.8}
\]
Which completes the proof.
Lemma 4.1 (Logarithmic Uncertainty Principle for the WD-QLCT). For \( f, g \in S(\mathbb{R}^2, \mathbb{H}) \), we have the following inequality
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left| f \left( t + \frac{X}{2} \right) g^* \left( t - \frac{X}{2} \right) \right|^2 dx dt + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| \left| W_{f,g}^{A_1,A_2}(t,w) \right|^2 dw dt \\
\geq (D + \ln |b|) \left| f \right|^2 \left| g \right|^2,
\]
where \( \phi(\frac{1}{2}) - \ln \pi, \phi(t) = \frac{\Gamma(t)}{\Gamma(0)} \) and \( \Gamma \) is a Gamma function.

Proof. Applying the procedure defined in proof of Theorem 1 in [50] to the Logarithmic Uncertainty Principle associated with QLCT [51], we get the desired result. \( \square \)

Theorem 4.2 (Logarithmic Uncertainty Principle for the WDOL). Let \( f, g \in S(\mathbb{R}^3, \mathbb{O}) \), then
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left| f \left( t + \frac{X}{2} \right) g^* \left( t - \frac{X}{2} \right) \right|^2 dx dt + 2\pi b_3 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| \left| W_{f,g}^{A_1,A_2,A_3}(t,w) \right|^2 dw dt \\
\geq (D + \ln |b|) \left| f \right|^2 \left| g \right|^2.
\]

Proof. From Lemma 3.1 3D-WDOL has been divided into four WD-QLCTs as
\[
\left| W_{f,g}^{A_1,A_2,A_3}(t,w) \right|^2 = \frac{1}{2\pi b_3} \left( \left| W_{(f,g)}^{A_1,A_2}(t,w) \right|^2 + \left| W_{(f,g)}^{A_2}(t,w) \right|^2 + \left| W_{(f,g)}^{A_2}(t,w) \right|^2 \right),
\]
where \( \tilde{f}, \tilde{g} \in L^2(\mathbb{R}^2, \mathbb{H}) \).
Thus
\[
2\pi b_3 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| \left| W_{f,g}^{A_1,A_2,A_3}(t,w) \right|^2 dw dt \\
= \left[ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| \left| W_{(f,g)}^{A_1,A_2}(t,w) \right|^2 dw dt + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| \left| W_{(f,g)}^{A_1,A_2}(t,w) \right|^2 dw dt \\
+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| \left| W_{(f,g)}^{A_1}(t,w) \right|^2 dw dt + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| \left| W_{(f,g)}^{A_1}(t,w) \right|^2 dw dt \right].
\]

(4.12)

Since every octonion function say \( h_t(x) \) defined in (3.10) can be written in the quaternion form as:
\[
|h_t(x)|^2 = |\tilde{h}_t^e(x)|^2 + |\tilde{h}_t^o(x)|^2 + |\tilde{h}_t^x(x)|^2 + |\tilde{h}_t^x(x)|^2.
\]

(4.13)

Therefore
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left| h_t(x) \right|^2 dx dt = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left| \tilde{h}_t^e(x) \right|^2 dx dt + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left| \tilde{h}_t^o(x) \right|^2 dx dt \\
+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left| \tilde{h}_t^x(x) \right|^2 dx dt + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left| \tilde{h}_t^x(x) \right|^2 dx dt.
\]

(4.14)
Now (4.14), implies

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right|^2 \, dx \, dt
\]

\[
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left\| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right\|^2 \, dx \, dt
\]

\[
+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left\| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right\|^2 \, dx \, dt
\]

\[
+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left\| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right\| \, dx \, dt
\]

\[
+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left\| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right\| \, dx \, dt.
\]

(4.15)

By the Logarithmic Uncertainty Principle for the WD-QLCT given in (??), we get

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| \left\| \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2}(t,w) \right\|^2 \, dw \, dt + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left\| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right\|^2 \, dx \, dt
\]

\[
\geq (D + \ln |b|) \left\| \hat{f} \right\|^2 \left\| \hat{g} \right\|^2.
\]

(4.16)

Similarly

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| \left\| \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2}(t,w) \right\|^2 \, dw \, dt + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left\| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right\|^2 \, dx \, dt
\]

\[
\geq (D + \ln |b|) \left\| \hat{f} \right\|^2 \left\| \hat{g} \right\|^2.
\]

(4.17)

And

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| \left\| \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2}(t,w) \right\|^2 \, dw \, dt + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left\| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right\|^2 \, dx \, dt
\]

\[
\geq (D + \ln |b|) \left\| \hat{f} \right\|^2 \left\| \hat{g} \right\|^2.
\]

(4.18)

And

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| \left\| \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2}(t,w) \right\|^2 \, dw \, dt + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left\| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right\|^2 \, dx \, dt
\]

\[
\geq (D + \ln |b|) \left\| \hat{f} \right\|^2 \left\| \hat{g} \right\|^2.
\]

Collecting equations (4.16), (4.17), (4.18) and (4.19), we obtain

\[
2\pi b_3 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |w| \left\| \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t,w) \right\|^2 \, dw \, dt + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x| \left| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right|^2 \, dx \, dt
\]

\[
\geq (D + \ln |b|) \left\| \hat{f} \right\|^2 \left\| \hat{g} \right\|^2.
\]

(4.19)

Which completes the proof.

\[\Box\]

**Theorem 4.3** (Hausdorff-Young inequality for the WDOL). *Let* \( 2 \leq p < \infty \) *and* \( f, g \in L^2(\mathbb{R}^2, \Omega) \), *we have*

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left\| \mathcal{W}_{f,g}^{\Lambda_1,\Lambda_2,\Lambda_3}(t,w) \right\|^q \, dw \, dt \leq E \frac{\| b_1 b_2 \|^{\frac{1}{2} + \frac{1}{q}}}{(2\pi)^{\frac{n+1}{2}}} \| f \|^2 \| g \|^2.
\]

(4.20)
Proof. For $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, the Hausdorff-Young inequality associated with OLCT \cite{40} is given by

\[
\| \mathcal{L}^{\Lambda_1, \Lambda_2, \Lambda_3}[f] \|_{q} \leq \frac{|b_1 b_2|^{-\frac{1}{2} + \frac{1}{q}}}{(2\pi)^{\frac{1}{2} + 1}|b_3|^{\frac{1}{2q}}}\| f \|_p.
\]  

(4.21)

Making use of (3.11), we can write

\[
\left( \int_{\mathbb{R}^2} \left| \mathcal{W}_{f,g}^{\Lambda_1, \Lambda_2, \Lambda_3}(t, w) \right|^q \, dw \right)^{\frac{1}{q}} = \left( \int_{\mathbb{R}^2} \left| \mathcal{L}^{\Lambda_1, \Lambda_2, \Lambda_3}[h_t] \right|^q \, dw \right)^{\frac{1}{q}},
\]  

(4.22)

where

\[ h_t(x) = f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right). \]

Therefore, on applying (4.21) to the right hand side of (4.22), we get

\[
\left( \int_{\mathbb{R}^2} \left| \mathcal{W}_{f,g}^{\Lambda_1, \Lambda_2, \Lambda_3}(t, w) \right|^q \, dw \right)^{\frac{1}{q}} \leq \frac{|b_1 b_2|^{-\frac{1}{2} + \frac{1}{q}}}{(2\pi)^{\frac{1}{2} + 1}|b_3|^{\frac{1}{2q}}} \| h_t \|_p.
\]

Further simplifying, we have

\[
\left( \int_{\mathbb{R}^2} \left| \mathcal{W}_{f,g}^{\Lambda_1, \Lambda_2, \Lambda_3}(t, w) \right|^q \, dw \right)^{\frac{1}{q}} \leq \frac{|b_1 b_2|^{-\frac{1}{2} + \frac{1}{q}}}{(2\pi)^{\frac{1}{2} + 1}|b_3|^{\frac{1}{2q}}} \left( \int_{\mathbb{R}^2} \left| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right|^p \, dx \right)^{\frac{1}{p}} dt.
\]

(4.23)

Integrating (4.24) both sides with respect to $dt$, it yields

\[
\int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \left| \mathcal{W}_{f,g}^{\Lambda_1, \Lambda_2, \Lambda_3}(t, w) \right|^q \, dw \right) dt \leq \frac{|b_1 b_2|^{-\frac{1}{2} + \frac{1}{q}}}{(2\pi)^{\frac{1}{2} + 1}|b_3|^{\frac{1}{2q}}} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \left| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right|^p \, dx \right)^{\frac{1}{p}} dt.
\]

(4.25)

Using procedure defined in \cite{40} (see theorem 1 relation (3.3)), we have

\[
\int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \left| f \left( t + \frac{x}{2} \right) g^* \left( t - \frac{x}{2} \right) \right|^p \, dx \right)^{\frac{2}{p}} dt \leq E \{ \| f \|_2 \| g \|_2 \}^q,
\]

(4.26)

where $E$ is positive constant.

Thus (4.25) and (4.26) together yields

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| \mathcal{W}_{f,g}^{\Lambda_1, \Lambda_2, \Lambda_3}(t, w) \right|^q \, dw \, dt \leq E \frac{|b_1 b_2|^{-\frac{1}{2} + \frac{1}{q}}}{(2\pi)^{\frac{1}{2} + 1}|b_3|^{\frac{1}{2q}}} \| f \|_2 \| g \|_2^q.
\]

Which completes the proof. □
5. Conclusion

In the present study, we examined three major objectives: first, we have studied the notion of 1D-WDOL in the framework of time-frequency analysis and examined all of its basic properties by means of the one dimensional octonion linear canonical transform transforms. Second, we propose the definition of 3D-WDOL and establish the fundamental properties associated with it, which includes the reconstruction formula, Rayleigh’s theorem and Riemann-Lebesgue Lemma. Moreover, we also establish the relation of WDOL with 3D-LCT and WD-QLCT. Third, well known uncertainty principles (UPs) including Heisenberg’s UP, Logarithmic UP and Hausdorff-Young inequality associated with WDOL are established.

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