Local $\nu$-Euler Derivations and Deligne’s Characteristic Class of Fedosov Star Products and Star Products of Special Type

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Dedicated to the memory of
Moshé Flato

Abstract

In this paper we explicitly construct local $\nu$-Euler derivations $E_\alpha = \nu \partial + L_{\xi_\alpha} + D_\alpha$, where the $\xi_\alpha$ are local, conformally symplectic vector fields and the $D_\alpha$ are formal series of locally defined differential operators, for Fedosov star products on a symplectic manifold $(M, \omega)$ by means of which we are able to compute Deligne’s characteristic class of these star products. We show that this class is given by $\frac{1}{\nu}[\omega] + \frac{1}{\nu}[\Omega]$ where $\Omega = \sum_{i=1}^{\infty} \nu^i \Omega_i$ is a formal series of closed two-forms on $M$ the cohomology class of which coincides with the one introduced by Fedosov to classify his star products. Moreover, our result implies that the normalisation condition used by Fedosov does not have any effect on the isomorphy class of the resulting star product. Finally we consider star products that have additional algebraic structures and compute the effect of these structures on the corresponding characteristic classes of these star products. Specifying the constituents of Fedosov’s construction we obtain star products with these special properties. Finally we investigate equivalence transformations between such special star products and prove existence of equivalence transformations being compatible with the considered algebraic structures.

Contents

1 Introduction 2
2 Fedosov Star Products and Deligne’s Characteristic Class 3

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1 Introduction

Since the very beginning of deformation quantisation in the pioneering articles [2] by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer there has been not only an immense interest in answering the question of existence of star products $\ast$ (i.e. formal, associative deformations of the classical Poisson algebra of complex-valued functions $C^\infty(M)$ on a symplectic or, more generally, Poisson manifold $M$, such that in the first order of the formal parameter $\nu$ the commutator of the star product yields the Poisson bracket) positively, but also in finding a classification of the star product algebras up to isomorphism of algebras. Therefore the proofs of existence given by DeWilde and Lecomte [12], Fedosov [13, 14] in the symplectic case and recently by Kontsevich [19] in the general case of a Poisson manifold always contained results on classification. Moreover, there have been several other results on classification up to equivalence by Nest and Tsygan [21, 22], Bertelson, Cahen and Gutt [3], Weinstein and Xu [23].

In the case of deformation quantisations with separation of variables on Kähler manifolds Karabegov proved existence and gave a classification using a deformation of the Kähler form in [17, 18]. Moreover, he has shown that the Fedosov approach to such star products considered by Bordemann and Waldmann in [8] corresponds to the trivial deformation in the sense of his classification. This observation gave the inspiration for our considerations concerning the Deligne class of Fedosov star products.

In his article [11] Deligne has introduced the notions of intrinsic derivation-related and characteristic class in order to compare the different constructions and classifications of DeWilde, Lecomte and Fedosov. In his paper Deligne uses the language of algebraic geometry to approach deformation theory and proves (cf. [11, Proposition 3.6.]) that the relative class $c(\ast) - c(\ast')$ of two Fedosov star products being the difference of the characteristic classes of two Fedosov star products $\ast, \ast'$ equals $\frac{1}{\nu}(F(\ast) - F(\ast'))$, where $F(\ast)$ denotes the cohomology class of the Weyl-curvature Fedosov introduced to classify his star products, that naturally arises when one constructs a star product using Fedosov’s method.

Recently, Gutt and Rawnsley [15] gave an alternative approach to Deligne’s various classes that avoids using methods of algebraic geometry. Using their methods we succeed in slightly generalising Deligne’s result in proving that Deligne’s characteristic class equals $\frac{1}{\nu}$ times the cohomology class of the Weyl-curvature. We should like to emphasise that our proof is purely algebraic and does not use any results on sheaf cohomology except for the de Rham isomorphism relating the second Čech cohomology with the second de Rham cohomology. Moreover, we follow Fedosov’s philosophy “Whatever you plan to do on a symplectic manifold $M$, do it first fibrewise on the tangent space, and pull it then down to $M$ by means of a compatible Fedosov derivative.”

The interest in the relation between the characteristic class and the Fedosov class is also motivated by the occurrence of the latter in formulas for canonical traces resp. trace densities obtained
by Halbout in [16] whose results are based on investigations of invariants in the cyclic cohomology of $M$ made by Connes, Flato and Sternheimer [17].

The paper is organised as follows: After a brief summary of Fedosov’s construction of star products on symplectic manifolds we close Section 2 by a short review of the definitions of Deligne’s various classes. Section 3 constitutes the main part of our work, where we give an explicit construction of local $\nu$-Euler derivations for an arbitrary Fedosov star product. After these preparations it is an easy task computing Deligne’s derivation-related and characteristic class in Section 4. As an application of the properties of Deligne’s characteristic class and its relation to Fedosov’s Weyl-curvature we study star products of special type in Section 5 that satisfy special algebraic identities with respect to complex conjugation and the mapping $\nu \mapsto -\nu$ changing the sign of the formal parameter and compute the influence on the corresponding characteristic classes. Moreover, we can show that there are always Fedosov star products satisfying these special algebraic identities the characteristic class of which coincides with a suitably given element of $\frac{\omega}{\nu} + H^2_{\text{dir}}(M)[[\nu]]$. Considering equivalent star products satisfying the same algebraic identities with respect to the mappings mentioned above we can show that there are always equivalence transformations between these star products commuting with these mappings. In Appendix A we give a short proof of the deformed Cartan formula that is of great value for our considerations in Section 4, but seems to be folklore. A further Appendix B is added for completeness giving the computation of the term of the characteristic class that cannot be determined from the algebraic considerations in Section 4.

Conventions: In what follows we shall use the Einstein summation convention where the sum over repeated coordinate indices is understood and the sum ranges from 1 to $\text{dim} M$. Moreover, we should notice that our convention for the occurrence of the formal parameter $\nu$ differs from the one used by Fedosov and in usual physics literature. In our considerations the star product commutator starts by $\nu$ times the Poisson bracket, whereas from the viewpoint of physics it is more natural to have $i\lambda$ times the Poisson bracket and the formal parameter $\lambda$ is assumed to be real and directly corresponds to Planck’s constant $\hbar$, so we consider $\nu$ as purely imaginary. The difference between these conventions is not of great importance as long as one does not consider star products that have an additional $\ast$-structure incorporated by complex conjugation (cf. [5, Theorem 7.3] for an example of this situation and Section 5). In order to distinguish Fedosov star products from star products in general they shall be denoted by $\ast$ whereas the latter ones are denoted by $\star$.

2 Fedosov Star Products and Deligne’s Characteristic Class

In this rather technical section we shall briefly recall Fedosov’s construction of a star product for a given symplectic manifold $(M, \omega)$. The notation is mainly the same as in Fedosov’s book [14] and in [13]. In addition we collect the definitions as they were introduced in [11] of Deligne’s intrinsic derivation-related class and Deligne’s characteristic class and the relations between them. For proofs and a detailed discussion of these topics the reader is referred to the exposition [15].

Let $(M, \omega)$ be a smooth symplectic manifold and define

$$W \otimes \Lambda(M) := (X_{s=0}^\infty \mathbb{C} (\Gamma^\infty ((V^s T^* M \otimes \wedge T^* M))) [[\nu]].$$

If there is no possibility for confusion we simply write $W \otimes \Lambda$ and denote by $W \otimes \Lambda^k$ the elements of anti-symmetric degree $k$ and set $W := W \otimes \Lambda^0$. For two elements $a, b \in W \otimes \Lambda$ we define their pointwise product denoted by $\mu(a \otimes b) = ab$ by the symmetric $\vee$-product in the first factor and the anti-symmetric $\wedge$-product in the second factor. Then the degree-maps $\deg_s$ and $\deg_a$ with respect to the symmetric and anti-symmetric degree are derivations of this product. Therefore we shall call $W \otimes \Lambda$ a formally $\mathbb{Z} \times \mathbb{Z}$-graded algebra with respect to the symmetric and anti-symmetric degree.
Moreover \((W \otimes \Lambda, \mu)\) is super-commutative with respect to the anti-symmetric degree. For a vector field \(X\) we define the symmetric substitution (insertion) \(i_s(X)\) and the anti-symmetric substitution \(i_a(X)\) which are super-derivations of symmetric degree \(-1\) resp. 0 and anti-symmetric degree 0 resp. \(-1\). Following Fedosov we define
\[
\delta := (1 \otimes dx^i)i_s(\partial_i) \quad \text{and} \quad \delta^* := (dx^i \otimes 1)i_a(\partial_i),
\]
(2)
where \(x^1, \ldots, x^n\) are local coordinates for \(M\) and \(\partial_i = \partial_{x^i}\) denotes the corresponding coordinate vector fields. For \(a \in W \otimes \Lambda\) with \(\text{deg}_s a = ka\) and \(\text{deg}_a a = la\) we define
\[
\delta^{-1} a := \begin{cases} 
\frac{1}{k+l} \delta^* a & \text{if } k + l \neq 0 \\
0 & \text{if } k + l = 0
\end{cases}
\]
(3)
and extend \(\delta^{-1}\) by linearity. Clearly \(\delta^2 = \delta^{*2} = 0\). Moreover, we denote by \(\sigma : W \otimes \Lambda \rightarrow C^\infty(M)[[\nu]]\) the projection onto the part of symmetric and anti-symmetric degree 0. Then one has the following ‘Hodge-decomposition’ for any \(a \in W \otimes \Lambda\) (see e. g. [3], eqn. (2.8)):
\[
a = \delta \delta^{-1} a + \delta^{-1} \delta a + \sigma(a).
\]
(4)
Now we consider the fibrewise associative deformation \(\circ\) of the pointwise product having the form
\[
a \circ b = \mu \circ \exp \left(\frac{\nu}{2} \Lambda^{ij}_{\nu} i_s(\partial_i) \otimes i_s(\partial_j)\right)(a \otimes b),
\]
(5)
where \(\Lambda^{ij}\) denotes the components of the Poisson tensor corresponding to the symplectic form \(\omega\) that are related to the ones of \(\omega\) by the equation \(\omega_{ij} = \delta^k_i\). Moreover, we define \(\text{deg}_s\)-graded super-commutators with respect to \(\circ\) and set \(\text{ad}(a)\) \(a\) := \([a, b]\). Now \(\text{deg}_s\) is no longer a derivation of the deformed product \(\circ\) but \(\text{Deg} := \text{deg}_s + 2\text{deg}_a\) is still a derivation and hence the algebra \((W \otimes \Lambda, \circ)\) is formally \(\text{Deg}\)-graded where \(\text{deg}_s\) := \(\nu\partial_\nu\). We shall refer to this degree as total degree.

According to Fedosov’s construction of a star product we consider a torsion free, symplectic connection \(\nabla\) on \(TM\) that extends in the usual way to a connection \(\nabla\) on \(T^*M\) and symmetric resp. anti-symmetric products thereof. Using this connection we define (using the same symbol as for the connection) the map \(\nabla : W \otimes \Lambda \rightarrow W \otimes \Lambda\) by
\[
\nabla := (1 \otimes dx^i)\nabla_{\partial_i},
\]
(6)
where \(\nabla_{\partial_i}\) denotes the covariant derivative with respect to \(\partial_i\). Then clearly \(\nabla\) is globally defined and due to the property of the connection being symplectic \(\nabla\) turns out to be a super-derivation of anti-symmetric degree 1 and symmetric and total degree 0 of the fibrewise product \(\circ\). Moreover \([\delta, \nabla] = 0\) since the connection is torsion free and \(\nabla^2 = \frac{1}{\nu}[\nabla, \nabla]\) turns out to be an inner super-derivation
\[
\nabla^2 = -\frac{1}{\nu} \text{ad}(R),
\]
(7)
where \(R := \frac{1}{4} \omega_{\nu} R_{ijkl}^\nu dx^i \otimes dx^j \otimes dx^k \wedge dx^l \in W \otimes \Lambda^2\) involves the curvature of the connection. Moreover one has \(\delta R = 0 = \nabla R\) as consequences of the Bianchi identities.

Now remember the following two theorems which are just restatements of Fedosov’s original theorems in [3], Theorem 3.2, 3.3] resp. [4] Theorem 5.3.3:

**Theorem 2.1** Let \(\nabla : W \otimes \Lambda \rightarrow W \otimes \Lambda\) be the super-derivation of \(\circ\) of anti-symmetric degree 1 and total degree 0 as defined in [4] satisfying \([\delta, \nabla] = 0\), and \(\nabla^2 = \frac{1}{\nu}[\nabla, \nabla] = -\frac{1}{\nu} \text{ad}(R)\) with \(R := \frac{1}{4} \omega_{\nu} R_{ijkl}^\nu dx^i \otimes dx^j \otimes dx^k \wedge dx^l \in W \otimes \Lambda^2\) of total degree 2, satisfying \(\delta R = 0 = \nabla R\). Moreover
let \( \Omega = \sum_{i=1}^{\infty} \nu^i \Omega_i \) denote a formal series of closed two-forms on \( M \) and let \( s = \sum_{k=1}^{\infty} s^{(k)} \in \mathcal{W} \) be given with \( \sigma(s) = 0 \) and \( \text{Deg} s^{(k)} = ks^{(k)} \). Then there exists a unique element \( r \in \mathcal{W} \otimes \Lambda^1 \) such that

\[
\delta r = R + \nabla r - \frac{1}{\nu} r \circ r + 1 \otimes \Omega \quad \text{and} \quad \delta^{-1} r = s.
\]

Moreover \( r = \sum_{k=2}^{\infty} r^{(k)} \) with \( \text{Deg} r^{(k)} = kr^{(k)} \) satisfies the formula

\[
r = \delta s + \delta^{-1} \left( R + 1 \otimes \Omega + \nabla r - \frac{1}{\nu} r \circ r \right)
\]

from which \( r \) can be determined recursively. In this case the Fedosov derivation

\[
\mathcal{D} := -\delta + \nabla - \frac{1}{\nu} \text{ad}(r)
\]

is a super-derivation of anti-symmetric degree 1 and has square zero: \( \mathcal{D}^2 = 0 \).

**Theorem 2.2** Let \( \mathcal{D} = -\delta + \nabla - \frac{1}{\nu} \text{ad}(r) : \mathcal{W} \otimes \Lambda \to \mathcal{W} \otimes \Lambda \) be given as in \([12]\) with \( r \) as in \((8)\).

i.) Then for any \( f \in C^\infty(M)[[\nu]] \) there exists a unique element \( \tau(f) \in \ker(\mathcal{D}) \cap \mathcal{W} \) such that

\[
\sigma(\tau(f)) = f
\]

and \( \tau : C^\infty(M)[[\nu]] : \ker(\mathcal{D}) \cap \mathcal{W} \subset \mathcal{W} \) is \( \mathbb{C}[[\nu]] \)-linear and referred to as the Fedosov-Taylor series corresponding to \( \mathcal{D} \).

ii.) In addition we have \( \tau(f) = \sum_{k=0}^{\infty} \tau(f)^{(k)} \) where \( \text{Deg} \tau(f)^{(k)} = k \tau(f)^{(k)} \) which can be obtained by the following recursion formula

\[
\tau(f)^{(0)} = f
\]

\[
\tau(f)^{(k+1)} = \delta^{-1} \left( \nabla \tau(f)^{(k)} - \frac{1}{\nu} \sum_{l=0}^{k-1} \text{ad}(r^{(l+2)}) \tau(f)^{(k-l)} \right).
\]

iii.) Since \( \mathcal{D} \) is a \( \circ \)-super-derivation of anti-symmetric degree 1 as constructed in Theorem \([2,7]\) \( \ker(\mathcal{D}) \cap \mathcal{W} \) is a \( \circ \)-sub-algebra and a new associative product \( \ast \) for \( C^\infty(M)[[\nu]] \) is defined by pull-back of \( \circ \) via \( \tau \), which turns out to be a star product.

Observe that in \([8]\) we allowed for an arbitrary element \( s \in \mathcal{W} \) with \( \sigma(s) = 0 \) that contains no terms of total degree lower than 3, as normalisation condition for \( r \), i.e. \( \delta^{-1} r = s \) instead of the usually used equation \( \delta^{-1} r = 0 \). In the sequel we shall especially show that this more general normalisation condition does not affect the isomorphy class of the resulting star product. In the following we shall refer to the associative product \( \ast \) defined above as the Fedosov star product. Moreover, we shall denote by \( F(\ast) \) Fedosov’s characteristic class of the star product \( \ast \) as discussed in \([14, \text{Section 5.3}]\) which is given by \( F(\ast) = [\omega] + [\Omega] \). There \( \omega + \Omega \) was introduced as the curvature of a certain connection being of importance for the Fedosov construction, the so-called Weyl-curvature.

Next we collect some basic concepts of characteristic classes for star products as they can be found in \([11, 15]\). Deligne’s characteristic class \( c(\ast) \) of a star product has been introduced in \([11]\) and classifies in a functorial way the isomorphy classes of star products on a symplectic manifold \((M, \omega)\). It lies in the affine space \([\omega] + H^2_{\text{dr}}(M)[[\nu]]\) and can be calculated by methods of Čech cohomology. Let us provide some details of the calculation as far as they are needed for our purposes. At this

5
instance we should mention that our conventions, that are as in [1], differ from those used in [15] by a sign in the Poisson bracket causing the positive sign in front of $\frac{\omega}{\nu}$ in $c(\ast)$.

If $\ast$ is a star product on the symplectic manifold $(M, \omega)$ there exists a good open cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of $M$ (i.e. all finite intersections of the $\mathcal{U}_\alpha$ are contractible) together with a family $\{E_\alpha\}_{\alpha \in I}$ of local $\nu$-Euler derivations of $(C^\infty(\mathcal{U}_\alpha)[[\nu]], \ast)$ i.e. a family of derivations $E_\alpha$ of $\ast$ over $\mathcal{U}_\alpha$ having the form

$$E_\alpha = \nu \partial_\nu + \mathcal{L}_{\xi_\alpha} + D_\alpha,$$

where $\xi_\alpha$ is conformally symplectic ($\mathcal{L}_{\xi_\alpha} \omega|_{\mathcal{U}_\alpha} = \omega|_{\mathcal{U}_\alpha}$) and $D_\alpha = \sum_{i=1}^{\infty} \nu^i D_{\alpha,i}$ is a formal series of differential operators over $\mathcal{U}_\alpha$. The existence of such $\nu$-Euler derivations has already been shown in [15] using cohomological methods, whereas in the case of a Fedosov star product we are going to give a very direct, purely algebraic proof of this fact in the next section since for our purposes we need a more concrete formula for the differential operators $D_\alpha$. As every $\nu$-linear derivation over a contractible, open set $\mathcal{U}$ is of the form $\frac{1}{\nu} \text{ad}_* (d)$ with $d \in C^\infty(\mathcal{U})[[\nu]]$ there exist formal functions $d_{\alpha\beta} \in C^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)[[\nu]]$ fulfilling

$$E_\alpha - E_\beta = \frac{1}{\nu} \text{ad}_* (d_{\alpha\beta})$$

over $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$. This fact can also be seen directly from the results of the following two sections. Now, whenever $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma \neq \emptyset$ the sums $d_{\alpha\beta\gamma} = d_{\beta\gamma} - d_{\alpha\gamma} + d_{\alpha\beta}$ lie in $\mathbb{C}[[\nu]]$ and define a 2-cocycle whose Čech class $[d_{\alpha\beta\gamma}] \in H^2(M, \mathbb{C})[[\nu]]$ does not depend on the choices made and the corresponding class $d(\ast) \in H^2_{\text{dr}}(M)[[\nu]]$ is called Deligne’s intrinsic derivation-related class.

**Definition 2.3** (cf. [13, Definition 6.3]) Deligne’s characteristic class $c(\ast)$ of a star product $\ast$ on $(M, \omega)$ is the element $c(\ast) = \frac{\omega}{\nu} + \sum_{i=0}^{\infty} \nu^i c(\ast)^i$ of the affine space $\frac{\omega}{\nu} + H^2_{\text{dr}}(M)[[\nu]]$ defined by

$$c(\ast)^0 = -2C_{2}^{-\frac{\ast}{2}}, \quad \partial_\nu c(\ast) = \frac{1}{\nu^2} d(\ast).$$

Hereby $C_{2}^{-\frac{\ast}{2}}$ is the projection onto the second part in the decomposition $H^2_{\text{Chev, nc}}(C^\infty(M), C^\infty(M)) = \mathbb{C} \oplus H^2_{\text{dr}}(M)$ of the second Chevalley cohomology (null on constants, on $(C^\infty(M), \{ , \})$ with respect to the adjoint representation) of the anti-symmetric part $C_2^{-\ast}(f, g) = \frac{1}{2} (C_2(f, g) - C_2(g, f))$ of the bidifferential operator $C_2$ in the expansion of $\ast$ which is a 2-cocycle with respect to this cohomology by the Jacobi-identity for star commutators.

**Remark 2.4** Notice that for Fedosov star products $\ast$ we have $C_1(f, g) = \frac{1}{2} \{f, g\}$ implying (cf. [13, Remark 6.1]) that $C_2^{-\ast}(f, g) = \rho_2(X_f, X_g)$ for a closed two-form $\rho_2$ on $M$, where $X_f$ denotes the Hamiltonian vector field with respect to $\omega$ that corresponds to $f \in C^\infty(M)$, and hence $C_{2}^{-\frac{\ast}{2}} = [\rho_2]$.

## 3 Explicit Construction of Local $\nu$-Euler Derivations

To simplify the notation we use the convention that whenever an equation contains indices $\alpha, \beta, \gamma$ this means that it is valid on the intersection of the members of the good open cover whose indices occur in it. As a first step in the construction of local $\nu$-Euler derivations we have to find local, conformally symplectic vector fields $\xi_\alpha$. Since $d\omega = 0$ we can find one-forms $\theta_\alpha$ on each $\mathcal{U}_\alpha$ such that $\omega = -d\theta_\alpha$ by the Poincaré lemma. Using these local one-forms we can define local vector fields $\xi_\alpha$ by $i_{\xi_\alpha} \omega = -\theta_\alpha$. These vector fields obviously satisfy $\mathcal{L}_{\xi_\alpha} \omega = \omega$ by Cartan’s formula for the Lie derivative. Using these vector fields we find the following lemma:
Lemma 3.1 Let $H_\alpha: W \otimes \Lambda(U_\alpha) \rightarrow W \otimes \Lambda(U_\alpha)$ be defined by
\[ H_\alpha := \nu \partial_\nu + L_{\xi_\alpha}, \] (16)
then $H_\alpha$ is a local (super-)derivation with respect to the fibrewise product $\circ$ of anti-symmetric and total degree 0, i. e.
\[ H_\alpha(a \circ b) = H_\alpha a \circ b + a \circ H_\alpha b \] for all $a, b \in W \otimes \Lambda(U_\alpha)$. Moreover we have $[L_{\xi_\alpha}, \delta] = [L_{\xi_\alpha}, \delta^*] = 0$ and $[H_\alpha, \delta] = [H_\alpha, \delta^*] = 0$.

PROOF: The proof is a straightforward computation using that $\nu \partial_\nu$ as well as $L_{\xi_\alpha}$ are derivations of the undeformed product $\mu$ and the equation $L_{\xi_\alpha} \Lambda = -\Lambda$ which follows from $L_{\xi_\alpha} \omega = \omega$. The commutation relations are obvious from the very definitions. \qed

At first sight it might be desirable to construct local derivations with respect to $\nabla$ by restricting $H_\alpha$ to $C^\infty(U_\alpha)[[\nu]]$. In fact this can be done in some special cases where the connection $\nabla$ is compatible with the Lie derivative with respect to the vector fields $\xi_\alpha$. An important example for this situation are homogeneous star products on cotangent bundles that have been discussed in [5, 6]. But this cannot be done in general since the failure of the connection to be compatible with the above Lie derivatives causes that the Fedosov derivation $D$ does not commute with $H_\alpha$ and hence $H_\alpha$ does not map elements of ker($D$) to elements of ker($D$). So we try to extend $H_\alpha$ to a (super-)derivation of anti-symmetric degree 0 that commutes with $D$. To this end we make the ansatz
\[ E_\alpha = H_\alpha + \frac{1}{\nu} \text{ad}(h_\alpha) = \nu \partial_\nu + L_{\xi_\alpha} + \frac{1}{\nu} \text{ad}(h_\alpha) \] (18)
with $h_\alpha \in W(U_\alpha)$ such that $\sigma(h_\alpha) = 0$ and compute $[D, E_\alpha]$.

Lemma 3.2 Let $E_\alpha$ be defined as above, then we have
\[ [D, E_\alpha] = \frac{1}{\nu} \text{ad}(Dh_\alpha) + [\nabla, L_{\xi_\alpha}] + \frac{1}{\nu} \text{ad}(H_\alpha r - r). \] (19)

PROOF: The proof of this formula relies on the fact that $D$ is a super-derivation of anti-symmetric degree 1 with respect to $\circ$ and that $H_\alpha$ is a (super-)derivation of anti-symmetric degree 0 with respect to $\circ$. Moreover we used $[\delta, H_\alpha] = 0$ and $[\nabla, H_\alpha] = [\nabla, L_{\xi_\alpha}]$. \qed

Now we consider the mapping $[\nabla, L_{\xi_\alpha}]$ more closely. The formulas we collect in the following two lemmas are essential for the whole construction of local $\nu$-Euler derivations.

Lemma 3.3 For the locally defined vector fields $\xi_\alpha$ the mapping $[\nabla, L_{\xi_\alpha}]$ enjoys the following properties:

i.) In local coordinates one has
\[ [\nabla, L_{\xi_\alpha}] = (dx^j \otimes dx^i)i_s((L_{\xi_\alpha} \nabla)_{i_\alpha j}) = (dx^j \otimes dx^i)i_s(S_\alpha(\partial_i, \partial_j)), \] (20)
where the local tensor field $S_\alpha \in \Gamma^\infty(T^2U_\alpha \otimes \nabla^2 U_\alpha \otimes T^2U_\alpha)$ is defined by
\[ S_\alpha(\partial_i, \partial_j) = (L_{\xi_\alpha} \nabla)_{i_\alpha j} := L_{\xi_\alpha} \nabla_{i_\alpha j} - \nabla_{i_\alpha} L_{\xi_\alpha} \partial_j - \nabla_{\xi_\alpha} \partial_i \partial_j = R(\xi_\alpha, \partial_i) \partial_j + \nabla^{(2)}(\partial_i, \partial_j) \xi_\alpha. \] (21)

ii.) $S_\alpha$ as defined above is symmetric, i. e. $S_\alpha \in \Gamma^\infty(\nabla^2 T^2U_\alpha \otimes T^2U_\alpha)$.

iii.) For all $X, Y, Z \in \Gamma^\infty(TU_\alpha)$ we have
\[ \omega(Z, S_\alpha(X, Y)) = -\omega(S_\alpha(X, Z), Y). \] (22)
Proof: The proof of the local expression for \([\nabla, \mathcal{L}_\xi]\) is a straightforward computation. The last equality in (22) follows from the torsion freeness of the connection \(\nabla\). The fact that \(S_\alpha\) is symmetric is a consequence from the first Bianchi identity for the connection \(\nabla\). Equation (22) follows from a direct computation essentially using \(\nabla \omega = 0\) and \(\mathcal{L}_{\xi}, \omega = \omega\).

Now the local tensor fields \(S_\alpha\) as defined above naturally give rise to elements \(T_\alpha\) of \(W \otimes \Lambda(U_\alpha)\) of symmetric degree 2 and anti-symmetric degree 1 by

\[
T_\alpha(Z, Y; X) := \omega(Z, \nabla_{S_\alpha}(X, Y)).
\]

In local coordinates this reads \(T_\alpha = \frac{1}{2} \omega_{ijk} S_{\alpha}^{ijk} dx^i \otimes dx^j \otimes dx^k\), where \(S_{\alpha}^{ijk} = dx^j(S_\alpha(\partial_k, \partial_l))\) denotes the components of \(S_\alpha\) in local coordinates.

**Lemma 3.4** The local tensor field \(T_\alpha\) as defined in (23) satisfies the following equations:

i.)

\[
\frac{1}{\nu} \text{ad}(T_\alpha) = [\nabla, \mathcal{L}_\xi],
\]

ii.)

\[
T_\alpha = i_\alpha(\xi_\alpha) R + \nabla \left( \frac{1}{2} D\theta_\alpha \otimes 1 \right),
\]

where the operator of symmetric covariant derivation \(D\) is defined by \(D := dx^i \vee \nabla_{\partial_i}\).

iii.)

\[
\delta T_\alpha = 0 \quad \text{and} \quad \nabla T_\alpha = \mathcal{L}_{\xi_\alpha} R - R.
\]

Proof: The first assertion easily follows from the properties of \(S_\alpha\) given in Lemma 3.3 by a direct computation. Part ii.) can be easily proven by direct computation using (21) and the definitions of \(R\) and \(T_\alpha\). The equations given in iii.) follow from the super-Jacobi-identity applied to the equations \([H_\alpha, [\delta, \nabla]] = 0\) and \([H_\alpha, \frac{1}{2}[\nabla, \nabla]] = -[H_\alpha, \frac{1}{2}\text{ad}(R)]\). For the second equation one has to observe that \(R\) does not depend on \(\nu\) and again that \(H_\alpha\) is a derivation with respect to \(\partial\). Moreover we used the fact that the only central elements of the Fedosov algebra \(W \otimes \Lambda\) with respect to \(\circ\) with symmetric degree 1 resp. 2 are zero.

Collecting our results we have shown that

\[
[D, \mathcal{E}_\alpha] = \frac{1}{\nu} \text{ad}(Dh_\alpha + T_\alpha + H_\alpha r - r).
\]

Our next aim is to prove that \(h_\alpha\) can be chosen such that \(Dh_\alpha + T_\alpha + H_\alpha r - r = 1 \otimes A_\alpha\) where \(A_\alpha\) is a formal series of locally defined one-forms that have to be chosen suitably, since then \([D, \mathcal{E}_\alpha] = 0\).

The necessary condition for this equation to be solvable is \(D(1 \otimes A_\alpha - T_\alpha - H_\alpha r + r) = 0\) since \(D^2 = 0\). But this is also sufficient since the \(D\)-cohomology on elements \(a\) with positive anti-symmetric degree is trivial since one has the following homotopy formula \(DD^{-1}a + D^{-1}Da = a\), where \(D^{-1}a := -\delta^{-1} \left( \frac{1}{\text{id} - \delta^{-1} \nabla - \frac{1}{2} \text{ad}(r)} \right) a\) (cf. [14, Theorem 5.2.5]).

**Lemma 3.5** Choosing local potentials \(\Theta_{i\alpha}\) for the closed two-forms \(\Omega_i\) on \(U_\alpha\), and defining

\[
A_\alpha := (\text{id} - H_\alpha) \Theta_\alpha = (\text{id} - H_\alpha) \sum_{i=1}^{\infty} \nu^i \Theta_{i\alpha}
\]

the equation \(D(1 \otimes A_\alpha - T_\alpha - H_\alpha r + r) = 0\) is fulfilled.
Proof: Using equation (29), \([H_\alpha, \delta] = 0\) and equation (17) as well as Lemma 3.4 i),iii.) one computes

\[ D(H_\alpha r - r) = 1 \otimes (\Omega - H_\alpha \Omega) + R - L_\xi_\alpha r + [\nabla, L_\xi_\alpha] r = 1 \otimes (\text{id} - H_\alpha)\Omega - \nabla T_\alpha + \frac{1}{\nu} \text{ad}(T_\alpha) r. \]

On the other hand we get from \(\delta T_\alpha = 0\) and \(dA_\alpha = d(\text{id} - H_\alpha)\Theta_\alpha = (\text{id} - H_\alpha)\Omega\) that

\[ D(1 \otimes A_\alpha - T_\alpha) = 1 \otimes (\text{id} - H_\alpha)\Omega - \nabla T_\alpha + \frac{1}{\nu} \text{ad}(r) T_\alpha, \]

proving the lemma.

This lemma enables us to prove the following important proposition.

**Proposition 3.6** There are uniquely determined elements \(h_\alpha \in W(U_\alpha)\) such that \(Dh_\alpha = 1 \otimes A_\alpha + r - H_\alpha r - T_\alpha\) and \(\sigma(h_\alpha) = 0\). Moreover \(h_\alpha\) is explicitly given by

\[ h_\alpha = D^{-1} (1 \otimes A_\alpha + r - H_\alpha r - T_\alpha), \quad (29) \]

where \(D^{-1} a = -\delta^{-1} \left( \frac{1}{\nu} \text{ad}(r) \right) a \). With these elements \(h_\alpha\) the fibrewise, local \(\nu\)-Euler derivations \(E_\alpha = \nu \partial_\nu + L_{\xi_\alpha} + \frac{1}{\nu} \text{ad}(h_\alpha)\) commute with the Fedosov derivation \(D\).

Proof: Using the homotopy formula \(a = DD^{-1}a + D^{-1}Da\) that is valid for elements \(a \in W \otimes \Lambda\) with positive anti-symmetric degree on \(1 \otimes A_\alpha + r - H_\alpha r - T_\alpha\) we get

\[ 1 \otimes A_\alpha + r - H_\alpha r - T_\alpha = DD^{-1}(1 \otimes A_\alpha + r - H_\alpha r - T_\alpha) \]

from the preceding lemma. Since we want the last expression to equal \(Dh_\alpha\) one gets \(h_\alpha = D^{-1}(1 \otimes A_\alpha + r - H_\alpha r - T_\alpha) + \tau(\varphi_\alpha)\) with arbitrary, locally defined formal functions \(\varphi_\alpha \in C^\infty(U_\alpha)[[\nu]]\). From the demand \(\sigma(h_\alpha) = 0\) we get \(\sigma(\tau(\varphi_\alpha)) = \varphi_\alpha = 0\) since \(D^{-1}\) raises the symmetric degree and the formula for \(h_\alpha\) is proven. The fact that \(E_\alpha\) commutes with \(D\) now follows from equation (27).

Using the fibrewise, local \(\nu\)-Euler derivations \(E_\alpha\) we constructed we are in the position to define local \(\nu\)-Euler derivations with respect to the Fedosov star product \(*\).

**Definition 3.7** Let \(h_\alpha \in W(U_\alpha)\) be given as in equation (29). Denoting by \(E_\alpha : W \otimes \Lambda(U_\alpha) \rightarrow W \otimes \Lambda(U_\alpha)\) the fibrewise local \(\nu\)-Euler derivations \(E_\alpha = \nu \partial_\nu + L_{\xi_\alpha} + \frac{1}{\nu} \text{ad}(h_\alpha)\) we define the mappings \(E_\alpha : C^\infty(U_\alpha)[[\nu]] \rightarrow C^\infty(U_\alpha)[[\nu]]\) by

\[ E_\alpha f := \sigma (E_\alpha \tau(f)) \quad (30) \]

for \(f \in C^\infty(U_\alpha)[[\nu]]\).

With this definition we get the main result of this section.

**Theorem 3.8** The mapping \(E_\alpha\) as defined in equation (30) is a local derivation with respect to the Fedosov star product \(*\). Moreover \(E_\alpha = \nu \partial_\nu + L_{\xi_\alpha} + D_\alpha\), where \(D_\alpha = \sum_{i=1}^\infty \nu^i D_{\alpha,i}\) is a formal series of differential operators over \(U_\alpha\).

Proof: The fact that \(E_\alpha\) is a local derivation with respect to \(*\) is obvious from the fact that \(E_\alpha\) is a local derivation with respect to \(\circ\) and the property of \(E_\alpha\) mapping elements in \(W(U_\alpha) \cap \ker D\) to elements in \(W(U_\alpha) \cap \ker D\) which was achieved by constructing \(E_\alpha\) such that \([D, E_\alpha] = 0\). The assertion about the shape of \(E_\alpha\) follows from the fact that \(\sigma\) commutes with \(\nu \partial_\nu\) and \(L_{\xi_\alpha}\) yielding \(E_\alpha f = \nu \partial_\nu f + L_{\xi_\alpha} f + \frac{1}{\nu} \sigma(\text{ad}(h_\alpha) \tau(f))\).

The fact that the last term involving \(h_\alpha\) and \(\tau\) defines a formal series of differential operators is obvious from the properties of the Fedosov-Taylor series. The only thing one has to observe is that this formal series starts at order one in the formal parameter. But this follows from the fact that \(h_\alpha\) only contains terms of total degree greater or equal to three, which is a consequence of \(D^{-1}\) raising the symmetric degree, not decreasing the \(\nu\)-degree and \(1 \otimes A_\alpha + r - H_\alpha r - T_\alpha\) only containing terms of total degree greater or equal to two. \(\Box\)
4 Computation of Deligne’s Characteristic Class

With the aid of the local ν-Euler’s derivations we constructed in the preceding section we are in the position to compute Deligne’s intrinsic derivation-related class \( d(*) \) and hence the characteristic class \( c(*) \) for every Fedosov star product \(*\) as defined in Section 2. To this end we have to find formal functions \( d_{\alpha\beta} \in C^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)[[\nu]] \) such that on \( \mathcal{U}_\alpha \cap \mathcal{U}_\beta \) we have \( E_\alpha - E_\beta = \frac{1}{\nu} \text{ad}_\nu(d_{\alpha\beta}). \) From the definition of the ν-Euler derivations \( E_\alpha \) and the deformed Cartan formula (cf. Appendix A) we have the following:

**Lemma 4.1** For \( g \in C^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)[[\nu]] \) we have

\[
(E_\alpha - E_\beta)(g) = \frac{1}{\nu} \sigma \left( \text{ad} \left( h_\alpha - h_\beta + f_{\alpha\beta} + df_{\alpha\beta} \otimes \Omega + \frac{1}{2} Ddf_{\alpha\beta} \otimes \Omega - i_a(X_{f_{\alpha\beta}}) \right) \tau(g) \right), \tag{31}
\]

where \( f_{\alpha\beta} \in C^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \) satisfies \( df_{\alpha\beta} = \theta_\alpha - \theta_\beta \) and the local one-forms \( \theta_\alpha \) satisfy \( d\theta_\alpha = -\omega. \)

**Proof:** By definition of the \( E_\alpha \) we have \( (E_\alpha - E_\beta)(g) = \sigma \left( (\mathcal{L}_{\xi_\alpha - \xi_\beta} + \frac{1}{\nu} \text{ad}(h_\alpha - h_\beta)) \tau(g) \right). \) Now on \( \mathcal{U}_\alpha \cap \mathcal{U}_\beta \) we have \( -d\theta_\alpha = \omega = -d\theta_\beta, \) and hence by the Poincaré lemma we can find locally defined functions \( f_{\alpha\beta} \) such that \( df_{\alpha\beta} = \theta_\alpha - \theta_\beta. \) Now by definition of the local vector fields \( \xi_\alpha \) we get \( d(-f_{\alpha\beta}) = i_{\xi_\alpha - \xi_\beta}\omega \) implying that \( \xi_\alpha - \xi_\beta = X_{-f_{\alpha\beta}} \) is the Hamiltonian vector field of the function \( -f_{\alpha\beta}. \) Thus we can apply the deformed Cartan formula (17) proven in Proposition 3.4 and immediately obtain the statement of the lemma since \( D\tau(g) = 0 \) and \( i_a(X_{-f_{\alpha\beta}})\tau(g) = 0 \) by Theorem 2.2 and the fact that \( \text{deg}_\alpha \tau(g) = 0. \)

Now we are to show that the term occurring in the argument of \( \text{ad} \) in equation (31) can be extended by adding a locally defined formal function \( a_{\alpha\beta} \in C^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)[[\nu]] \) (clearly satisfying \( \text{ad}(a_{\alpha\beta}) = 0 \)) such that the whole argument is the Fedosov-Taylor series \( \tau(f_{\alpha\beta} + a_{\alpha\beta}) \) of the local formal function \( a_{\alpha\beta} := f_{\alpha\beta} + a_{\alpha\beta}. \) If we succeed to find such a local function, equation (31) yields \( E_\alpha - E_\beta = \frac{1}{\nu} \text{ad}_\nu(a_{\alpha\beta}) \) enabling us to give an expression for Deligne’s intrinsic derivation-related class \( d(*) \) of \(*\). We thus have to show that \( a_{\alpha\beta} \) can be chosen such that

\[
D \left( h_\alpha - h_\beta + a_{\alpha\beta} + f_{\alpha\beta} + df_{\alpha\beta} \otimes \Omega + \frac{1}{2} Ddf_{\alpha\beta} \otimes \Omega - i_a(X_{f_{\alpha\beta}}) \right) = 0. \tag{32}
\]

**Lemma 4.2** With the notations from above we have

\[
D \left( h_\alpha - h_\beta + a_{\alpha\beta} + f_{\alpha\beta} + df_{\alpha\beta} \otimes \Omega + \frac{1}{2} Ddf_{\alpha\beta} \otimes \Omega - i_a(X_{f_{\alpha\beta}}) \right) = 1 \otimes ((i_\xi_\alpha \Omega + A_\alpha) - (i_\xi_\beta \Omega + A_\beta)), \tag{33}
\]

where \( A_\alpha \) is given as in Lemma 3.3.

**Proof:** From the construction of the elements \( h_\alpha \in W(\mathcal{U}_\alpha) \) we gave in the preceding section (cf. Proposition 3.4) we get \( D(h_\alpha - h_\beta) = 1 \otimes (A_\alpha - A_\beta) - T_\alpha + T_\beta - \mathcal{L}_{\xi_\alpha - \xi_\beta} r. \) Another straightforward calculation yields \( D \left( f_{\alpha\beta} + df_{\alpha\beta} \otimes 1 + \frac{1}{2} Ddf_{\alpha\beta} \otimes 1 \right) = \frac{1}{\nu} \text{ad}(r) \left( df_{\alpha\beta} \otimes 1 + \frac{1}{2} Ddf_{\alpha\beta} \otimes 1 \right) + \frac{1}{2} \nabla(Ddf_{\alpha\beta} \otimes 1). \) Using the deformed Cartan formula once again combined with equation (8) and the definition of \( D \) we get

\[
D(-i_a(X_{f_{\alpha\beta}}) r) = -(D\bar{i}_a(X_{f_{\alpha\beta}}) r + i_a(X_{f_{\alpha\beta}})Dr) + i_a(X_{f_{\alpha\beta}}) \left( -\delta r + \nabla r - \frac{1}{\nu} \text{ad}(r) r \right)
\]

\[
= \mathcal{L}_{\xi_\alpha - \xi_\beta} r + \frac{1}{\nu} \text{ad} \left( -df_{\alpha\beta} \otimes 1 - \frac{1}{2} Ddf_{\alpha\beta} \otimes 1 + i_a(X_{f_{\alpha\beta}}) r \right) r + i_a(\xi_\alpha - \xi_\beta) \left( R + 1 \otimes \Omega + \frac{1}{\nu} r \circ r \right)
\]

\[
= \mathcal{L}_{\xi_\alpha - \xi_\beta} r + \frac{1}{\nu} \text{ad} \left( -df_{\alpha\beta} \otimes 1 - \frac{1}{2} Ddf_{\alpha\beta} \otimes 1 \right) r + i_a(\xi_\alpha - \xi_\beta) R + 1 \otimes i_{\xi_\alpha - \xi_\beta} \Omega,
\]
since \( \frac{1}{2}i_\alpha(X_\xi\alpha - \xi_\beta)(r \circ r) = -\frac{1}{2}\text{ad}(i_\alpha(X_{f_{\alpha\beta}})r)r \). All these results together with equation (23) and \( df_{\alpha\beta} = \theta_\alpha - \theta_\beta \) prove the statement of the lemma.

After these preparations we are able to formulate the following proposition.

**Proposition 4.3** There are locally defined formal functions \( d_{\alpha\beta} \in C^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)[[\nu]] \) such that \( E_\alpha - E_\beta = {\frac{1}{\nu}}\text{ad}_\nu(d_{\alpha\beta}) \). Moreover, these formal functions satisfy \( dd_{\alpha\beta} = df_{\alpha\beta} + da_{\alpha\beta} = \theta_\alpha - \theta_\beta - ((A_\alpha + i\xi_\alpha \Omega) - (A_\beta + i\xi_\beta \Omega)) \). Thus they define a 2-cocycle and the image of the corresponding Čech class under the de Rham isomorphism, which is just Deligne’s intrinsic derivation-related class, is given by \( d(\ast) = -[\omega] - [\Omega - \nu \partial_\nu \Omega] \).

**Proof:** From Lemma 4.2 we get that \( a_{\alpha\beta} \) has to satisfy the equation \( da_{\alpha\beta} = -((A_\alpha + i\xi_\alpha \Omega) - (A_\beta + i\xi_\beta \Omega)) \) so that (22) is fulfilled. From the definition of \( A_\alpha \) we get that the right-hand side of this equation is closed since \( d(A_\alpha + i\xi_\alpha \Omega) = \Omega - \nu \partial_\nu \Omega \). Therefore the existence of \( a_{\alpha\beta} \in C^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)[[\nu]] \) as desired is guaranteed by the Poincaré lemma. Now we have \( \sigma(h_\alpha - h_\beta + a_{\alpha\beta} + f_{\alpha\beta} + df_{\alpha\beta} \otimes 1 + \frac{1}{2}Df_{\alpha\beta} \otimes 1 - i_\xi(X_\xi A_{\alpha\beta})r) = f_{\alpha\beta} + a_{\alpha\beta} \) and equation (23) is fulfilled implying \( (E_\alpha - E_\beta)(g) = \frac{1}{\nu} \sigma(\text{ad}(\nu f_{\alpha\beta} + a_{\alpha\beta})(\nu g)) = \frac{1}{\nu} \text{ad}_\nu(d_{\alpha\beta})g \) by Lemma 4.1. The assertion about the corresponding de Rham class is obvious from the properties of \( f_{\alpha\beta} \) and \( a_{\alpha\beta} \) we have already proven namely \( d(\theta_\alpha - (A_\alpha + i\xi_\alpha \Omega)) = -([\omega + \Omega - \nu \partial_\nu \Omega]) \).

From this proposition and from the computation of \( C_2^{-2} \) in Appendix 3 we obtain our final result.

**Theorem 4.4** Deligne’s characteristic class \( c(\ast) \) of a (slightly generalised) Fedosov star product \( \ast \) as constructed in Section 2 is given by

\[
 c(\ast) = \frac{1}{\nu} [\omega] + \frac{1}{\nu} [\Omega] = \frac{1}{\nu} F(\ast),
\]

where \( F(\ast) \) denotes Fedosov’s characteristic class of the star product \( \ast \).

**Proof:** From the differential equation \( \partial_\nu c(\ast) = \frac{1}{\nu} d(\ast) \) that relates the derivation-related class to the characteristic class and from the preceding proposition we get \( c(\ast) = \frac{1}{\nu} [\omega] + c(\ast) + \frac{1}{\nu} \sum_{i=2}^\infty [\Omega_i] \). By the result of Proposition 3.3 we get \( c(\ast)^0 = [\Omega] \) proving the theorem.

As an immediate corollary which originally is due to Fedosov (cf. [4, Corollary 5.5.4]) we find:

**Corollary 4.5** Two Fedosov star products \( \ast \) and \( \ast' \) for \((M, \omega)\) constructed as in Section 2 from the data \((\nabla, \Omega, s)\) and \((\nabla', \Omega', s')\) as in Theorem 2.4 are equivalent if and only if \([\Omega] = [\Omega']\).

**Remark 4.6** In the case of cotangent bundles \( T^* Q \) equipped with the symplectic form \( \omega_0 + \pi^* B_0 \), which is a sum of the canonical two-form and the pull-back of a closed two-form on \( Q \), Bordemann, Neumaier, Pflaum and Waldmann gave a ‘non-Fedosov’ approach to the construction of star products of a given characteristic class in [3, Section 4]. The main idea of this construction is to start with a star product whose characteristic class is \([0]\), to apply local equivalence transformations depending on local potentials of a formal series of closed two-forms \( \sum_{i=0}^\infty \nu^i B_i \) on \( Q \) and to show that the local star products obtained fit together to a globally defined star product.

## 5 Star Products of Special Type, their Characteristic Classes and Equivalence Transformations

In this section we consider star products that have additional algebraic properties and compute their characteristic classes showing that these properties give rise to restrictions on this class. Moreover,
we can show that for every characteristic class satisfying the necessary condition for a star product of this class to have the desired algebraic properties there are always Fedosov star products with suitably chosen data $\Omega, s$ having these properties. Although the following results might be known they do nevertheless not seem to have appeared in the literature except for the special case $\Omega = 0$ and $s = 0$ considered in [8, Lemma 3.3]. In this section $C : W \otimes \Lambda \to W \otimes \Lambda$ shall always denote the complex conjugation, where we define $C\nu := -\nu$ in view of our convention for the formal parameter being considered as purely imaginary. By $P : W \otimes \Lambda \to W \otimes \Lambda$ with $P := (-1)^{\deg_\nu}$ we denote the so-called $\nu$-parity operator. Using these maps fulfilling $C^2 = P^2 = \text{id}$ we can define special types of star products:

**Definition 5.1**

i.) For a given star product $\ast$ for $(M, \omega)$ we define the star products $\ast_{\text{flip}}, \ast_C, \ast_P$ for $(M, -\omega)$ by

\[
\begin{align*}
    f \ast_{\text{flip}} g & := g \ast f, \\
    f \ast_C g & := C ((Cf) \ast (Cg)), \\
    f \ast_P g & := P ((Pf) \ast (Pg)) = f \ast_{-\nu} g = \sum_{i=0}^{\infty} (-\nu)^i C_i(f, g),
\end{align*}
\]

where $f, g \in C^\infty(M)[[\nu]]$ and the bidifferential operators $C_i$ describe the star product $\ast$ by $f \ast g = \sum_{i=0}^{\infty} \nu^i C_i(f, g)$.

ii.) A star product $\ast$ is said to have the $\nu$-parity property if $P$ is an anti-automorphism of $\ast$, i. e.

\[
f \ast_P g = f \ast_{\text{flip}} g \quad \forall f, g \in C^\infty(M)[[\nu]].
\]

iii.) A star product $\ast$ is said to have a $\ast$-structure incorporated by complex conjugation if $C$ is an anti-automorphism of $\ast$, i. e.

\[
f \ast_C g = f \ast_{\text{flip}} g \quad \forall f, g \in C^\infty(M)[[\nu]].
\]

iv.) A star product $\ast$ is called of Weyl type if it has the $\nu$-parity property and has a $\ast$-structure incorporated by complex conjugation.

Using these definitions we find:

**Lemma 5.2**

i.) The characteristic classes of $\ast_{\text{flip}}, \ast_C, \ast_P$ are related to the characteristic class $c(\ast)$ of $\ast$ by the following equations:

\[
\begin{align*}
    c(\ast_{\text{flip}}) & = -c(\ast), \\
    c(\ast_C) & = Cc(\ast), \\
    c(\ast_P)(\nu) & = c(\ast_{-\nu})(\nu) = c(\ast)(-\nu) = P \left( c(\ast)(\nu) \right).
\end{align*}
\]

ii.) The characteristic class of a star product $\ast$ that has the $\nu$-parity property satisfies

\[
Pc(\ast) = -c(\ast),
\]

and hence $c(\ast) = \frac{[\omega]}{\nu} + \sum_{l=0}^{\infty} \nu^{2l+1} c(\ast)^{2l+1}$, i. e. $c(\ast)^{2l} = [0]$ for all $l \in \mathbb{N}$.

12
iii.) The characteristic class of a star product $\star$ that has $C$ as $\ast$-structure satisfies

$$Cc(\ast) = -c(\ast),$$

and hence $c(\ast)^{2l} = -Cc(\ast)^{2l}$ and $c(\ast)^{2l+1} = Cc(\ast)^{2l+1}$ for all $l \in \mathbb{N}$.

iv.) The characteristic class of a star product $\star$ that is of Weyl type satisfies

$$Pc(\ast) = -c(\ast) \quad \text{and} \quad Cc(\ast) = -c(\ast),$$

and hence $c(\ast)^{2l} = [0]$ and $c(\ast)^{2l+1} = Cc(\ast)^{2l+1}$ for all $l \in \mathbb{N}$.

**Proof:** The proof of part i.) relies on the observation, that local $\nu$-Euler derivations $E_\nu$ of $\ast$ yield such derivations for $\ast_{\text{flip}}$, $\ast_C$ and $\ast_P$ given by $E_\nu$, $CE_\nu C$ and $PCE_\nu P$. With these derivations one easily finds $d(\ast_{\text{flip}}) = -d(\ast)$, $d(\ast_C) = -Cd(\ast)$ and $d(\ast_P) = -Pd(\ast)$. From the definition of the characteristic class relating the derivation-related class $d$ with $c$ and the obvious observations that $c(\ast_{\text{flip}})^0 = -c(\ast)^0$, $c(\ast_C)^0 = Cc(\ast)^0$ and $c(\ast_P)^0 = c(\ast)^0$ one gets the asserted statements. The assertions ii.), iii.) and iv.) are obvious from part i.) and Definition 5.1 ii.), iii.) and iv.).

The preceding lemma states that in general there are equivalence classes of star products corresponding to the characteristic classes $c(\ast)$ that contain no representatives (i.e. star products $\ast$ with this characteristic class) satisfying the conditions (38) resp. (39), namely those whose characteristic classes do not satisfy the equations (43) resp. (44). Vice versa the following proposition states that for every class $c \in \frac{[\omega]}{\nu} + H^2_{\text{dr}}(M)[[\nu]]$ enjoying the properties $Cc = -c$ resp. $Pc = -c$ one can find even Fedosov star products having the characteristic class $c$ and satisfying the conditions (38) resp. (39).

**Proposition 5.3**

i.) For all $c \in \frac{[\omega]}{\nu} + H^2_{\text{dr}}(M)[[\nu]]$ with $Pc = -c$ there are Fedosov star products $\ast$ for $(M, \omega)$ with

$$c(\ast) = c \quad \text{and} \quad P((Pf) \ast (Pg)) = g \ast f \quad \text{for all } f, g \in C^\infty(M)[[\nu]].$$

ii.) For all $c \in \frac{[\omega]}{\nu} + H^2_{\text{dr}}(M)[[\nu]]$ with $Cc = -c$ there are Fedosov star products $\ast$ for $(M, \omega)$ with

$$c(\ast) = c \quad \text{and} \quad C((Cf) \ast (Cg)) = g \ast f \quad \text{for all } f, g \in C^\infty(M)[[\nu]].$$

iii.) For all $c \in \frac{[\omega]}{\nu} + H^2_{\text{dr}}(M)[[\nu]]$ with $Pc = -c = Cc$ there are Fedosov star products $\ast$ for $(M, \omega)$ with

$$c(\ast) = c \quad \text{and} \quad P((Pf) \ast (Pg)) = g \ast f = C((Cf) \ast (Cg)) \quad \text{for all } f, g \in C^\infty(M)[[\nu]].$$

**Proof:** For the proof we first observe that the fibrewise product $\circ$ satisfies $C((Ca) \circ (Cb)) = P((Pa) \circ (Pb)) = (-1)^{kl}b \circ a$ for all $a, b \in \mathcal{W} \otimes \Lambda$ with $\deg_a a = ka$ and $\deg_a b = lb$. Now let $c \in \frac{[\omega]}{\nu} + H^2_{\text{dr}}(M)[[\nu]]$ be written as $c = \frac{[\omega]}{\nu} + \sum_{i=0}^{\infty} \nu^i c^i$. For the proof of i.) we choose closed two-forms $\Omega_i$ such that $\Omega_{2l+1} = 0$ (to achieve $[\Omega_{2l+1}] = c^{2l} = [0]$) and $[\Omega_{2l+2}] = c^{2l+1}$ for all $l \in \mathbb{N}$ yielding $P\Omega = \Omega$. Moreover, we choose $s = \sum_{k=3}^{\infty} s^{(k)} \in \mathcal{W}$ with $\sigma(s) = 0$ and $Ps = s$. Under these preconditions one easily proves that $Pr$ satisfies the equations (8) implying $Pr = r$ by uniqueness of the solution of (8). With such an element $r \in \mathcal{W} \otimes \Lambda^1$ the Fedosov derivation $D$ obviously commutes with $P$ implying that $Pr(f) = \tau(Pf)$ for all $f \in C^\infty(M)[[\nu]]$ since $P$ obviously commutes with $\sigma$. Using this equation and the definition of $\ast$ together with $P((Pa) \circ (Pb)) = (-1)^{kl}b \circ a$ and observing that $\deg_{a\tau}(f) = 0$ one gets the asserted property of $\ast$ under the mapping $P$. From Theorem 4.3 we get $c(\ast) = \frac{[\omega]}{\nu} + \frac{1}{\nu}[\Omega] = c$. For ii.) one proceeds quite analogously.
Moreover, we choose $s \in W$ such that $Cs = s$. As in the proof of i.) one gets that $Cr = r$ yielding the desired behaviour of the corresponding star product $*$ under the mapping $C$ as in the proof of part i.). The fact that $c(s) = c$ again follows from Theorem 4.4 and the choice of $\Omega$. For the proof of part iii.) one just has to bring into line the choices made for i.) with the ones made for ii.) i.e. choose $s$ with $Cs = s = Ps$ and closed two-forms $\Omega_i$ with $C\Omega_2l+2 = \Omega_2l+2$ and $\Omega_2l+1 = 0$ such that $[\Omega_2l+2] = c^{2l+1}$ and $[\Omega_2l+1] = c^{2l} = [0]$ for all $l \in \mathbb{N}$. Then the argument as in i.) and ii.) yields the stated result.

**Remark 5.4** The interest in such special star products from the viewpoint of physics is based on the interpretation of the star product algebras $(C^\infty(M)[[\nu]], \ast)$ as the algebra of observables of the quantised system corresponding to the classical system described by the symplectic manifold $M$, and hence the existence of a $^*$-structure incorporated by complex conjugation (the $^*$-structure of the algebra of classical observables) is strongly recommended. Moreover, the Weyl-Moyal product on $T^*\mathbb{R}^n$ giving a correct description of the quantisation of observables that are polynomials in the coordinates is of Weyl type, motivating the general interest in such star products (cf. [5, 6] for further details). In addition there is the possibility of constructing $^*$-representations for star products with $C$ as $^*$-structure under the preconditions of having defined a formally positive functional on a suitable twosided ideal in $C^\infty(M)[[\nu]]$ that is stable under $C$ by a formal analogue of the GNS construction (cf. [3] for details).

To conclude this section we shall discuss the question of existence of special equivalence transformations between equivalent star products satisfying equations (38) and (39).

**Definition 5.5** Let $(C^\infty(M)[[\nu]], \ast_1)$ and $(C^\infty(M)[[\nu]], \ast_2)$ denote equivalent star product algebras.

i.) In case $(C^\infty(M)[[\nu]], \ast_1)$ and $(C^\infty(M)[[\nu]], \ast_2)$ have $C$ incorporated as $^*$-structure they are called equivalent as $^*$-algebras (resp. $C$-equivalent) if there is an equivalence transformation $S$ between them satisfying $SCS = S$.

ii.) In case $(C^\infty(M)[[\nu]], \ast_1)$ and $(C^\infty(M)[[\nu]], \ast_2)$ have the $\nu$-parity property they are called $P$-equivalent if there is an equivalence transformation $S$ between them satisfying $PSP = S$.

iii.) In case $(C^\infty(M)[[\nu]], \ast_1)$ and $(C^\infty(M)[[\nu]], \ast_2)$ are of Weyl-type they are called Weyl-equivalent if there is an equivalence transformation $S$ between them satisfying $CSC = S$ and $PSP = S$.

The following proposition states that for two equivalent star products enjoying the additional algebraic properties as in the preceding definition there are always equivalence transformations being compatible with the mappings $C$ and $P$.

**Proposition 5.6** Let $(C^\infty(M)[[\nu]], \ast_1)$ and $(C^\infty(M)[[\nu]], \ast_2)$ denote equivalent star product algebras.

i.) In case $C$ is a $^*$-structure for $\ast_1$ and $\ast_2$, then $(C^\infty(M)[[\nu]], \ast_1)$ and $(C^\infty(M)[[\nu]], \ast_2)$ are equivalent as $^*$-algebras.

ii.) In case $\ast_1$ and $\ast_2$ have the $\nu$-parity property then $(C^\infty(M)[[\nu]], \ast_1)$ and $(C^\infty(M)[[\nu]], \ast_2)$ are $P$-equivalent.

iii.) In case $\ast_1$ and $\ast_2$ are of Weyl-type then $(C^\infty(M)[[\nu]], \ast_1)$ and $(C^\infty(M)[[\nu]], \ast_2)$ are Weyl-equivalent.
Thus we find $P^\ast R = (T f) \ast_2 (T g)$ for all $f, g \in C^\infty(M)$. Obviously $CTC$ is also an equivalence transformation between $\ast_1$ and $\ast_2$ and hence there is an automorphism $A$ of $\ast_1$ such that $CTC = TA$. Conjugating this equation with $C$ and using $C^2 = \text{id}$ we obtain $T = CTCCAC = TACAC$ yielding $\text{ACAC} = \text{id}$. Since any automorphism of $\ast_1$ has the shape $A = \exp(\nu D)$ where $D$ is a derivation of $\ast_1$ we get $id = \exp(\nu D) \exp(-\nu CDC \text{D})$ implying $CDC = \text{D}$. For $t \in \mathbb{R}$ we consider the automorphisms $A_t := \exp(t \nu \text{D})$ of $\ast_1$ satisfying $CA_t^1 C = (A_t^1)^{-1} = A_t^{-1}$. Now $S_1 := T A_t^1$ obviously is an equivalence between $\ast_1$ and $\ast_2$ for all $t \in \mathbb{R}$ and we have $C S_1 C = TCA_t^1 - t = S_1 - t$. Therefore $S := S_{1/2}$ satisfies $S(f \ast_1 g) = (Sf) \ast_2 (Sg)$ and $CSC = S$ proving part i.) of the proposition. For the proof of part ii.) one proceeds completely analogously replacing $C$ by $P$ in the above argumentation. For the proof of part iii.) we consider some equivalence transformation $T$ between $\ast_1$ and $\ast_2$ and use the results of part i.) and part ii.) to obtain two further equivalence transformations $S_1 := T A_1^{1/2}$ and $S_2 := T A_2^{1/2}$ satisfying $C S_2 C = S_1$ and $PS_2 S_1 = S_2$, where $A_1$ and $A_2$ are automorphisms of $\ast_1$ given by $CTC = T A_1$ and $PTP = T A_2$. In general $S_1$ fails to satisfy $PS_1 S_1 = S_1$ as well as $S_2$ fails to commute with $C$, but by an analogous procedure as for the proofs of the statements i.) and ii.) $S_1$ and $S_2$ can be modified such that the resulting equivalence transformations have the desired properties. Since $P$ commutes with $C$ we have $C P T P C = P C T P C$ implying the crucial equation $A_1 (C A_2 C) = A_2 (P A_1 P)$ by the definitions of $A_1$ and $A_2$. Now we compute $C S_2 C = S_2 A_2^{1/2} A_1 C A_2^{1/2} C = S_2 F_2$, where $F_2 := A_2^{1/2} A_1 C A_2^{1/2} C$ is an automorphism of $\ast_1$ and hence $F_2 = \exp(\nu D_2)$ with a derivation $D_2$ of $\ast_1$. As in ii.) one gets $S_2 F_2 C = \text{id}$ and $R_2 := S_2 F_2^{1/2}$ with $F_2^{1/2} := \exp(\nu D_2)$ is an equivalence transformation between $\ast_1$ and $\ast_2$ satisfying $C R_2 C = R_2$. It remains to show that $R_2$ satisfies $P R_2 P = R_2$. To this end we compute $P F_2 P$ using $A_1 C = A_2 P A_1 P C A_2^{-1}$. Thus we find $P R_2 P = P S_2 P S_2^{1/2} = S_2 F_2^{1/2} = R_2$ proving part iii.). One can also modify $S_1$ to obtain another equivalence transformation $R_1 := S_1 F_1^{1/2}$ having the desired properties where $F_1 := A_1^{1/2} A_2 P A_1^{1/2} P$ again is an automorphism of $\ast_1$.

Remark 5.7 The assertion about the existence of equivalence transformations between equivalent star products with a $*$-structure incorporated by $C$ that commute with $C$ has an important consequence for the GNS representations one can construct for these star product algebras namely that such an equivalence transformation induces a unitary map between the GNS Hilbert spaces obtained by the GNS construction relating the corresponding GNS representations (cf. [4, Proposition 5.1]).

A The Deformed Cartan Formula

The aim of this section is to prove the deformed Cartan formula that was very useful for our computations in Section 3. This formula and the proof of it which we shall give already appeared in [4, Lemma 4.6]. A similar result has also been derived in [20, Proposition 4.3.] where the vector field with respect to which the Lie derivative is computed is assumed to be affine with respect to the symplectic connection $\nabla$.

Proposition A.1 For all vector fields $X \in \Gamma^\infty(TM)$ the Lie derivative $L_X : W \otimes \Lambda \to W \otimes \Lambda$ can be expressed in the following manner:

\[
L_X = D i_a(X) + i_a(X) D + i_a(X) dx^i \wedge i_a(\nabla_{\partial_i} X) + \frac{1}{\nu} \text{ad}(i_a(X) r).
\]  

(46)

In case $X = X_f$ is the Hamiltonian vector field of a function $f \in C^\infty(M)$, i.e. $i_{X_f} \omega = df$ this formula takes the following form:

\[
L_{X_f} = D i_a(X_f) + i_a(X_f) D - \frac{1}{\nu} \text{ad} \left( f + df \otimes 1 + \frac{1}{2} D df \otimes 1 - i_a(X_f) r \right),
\]  

(47)
where $D = dx^i \lor \nabla_{\partial_i}$ denotes the operator of symmetric covariant derivation.

**Proof:** The proof of formula (40) is obtained by collecting the following formulas the proofs of which are all straightforward computations just using the definitions of the involved mappings and applying them to factorised sections $a = A \otimes \alpha \in W \otimes A$:

\[
\delta i_a(X) + i_a(X) \delta = i_a(X) \quad (48)
\]

\[
\frac{1}{\nu} (ad(r)i_a(X) + i_a(X)ad(r)) = \frac{1}{\nu} ad(i_a(X)r) \quad (49)
\]

\[
(\nabla i_a(X) + i_a(X)\nabla)(A \otimes \alpha) = \nabla_X A \otimes \alpha + A \otimes \mathcal{L}_X \alpha. \quad (50)
\]

For a symmetric one-form $A$ it is easy to see that $\nabla_X A = \mathcal{L}_X A - dx^i \lor i_s(\nabla_{\partial_i} X)A$. Together with the observation that the operators on both sides of this equation are derivations with respect to the $\lor$-product this and (50) imply

\[
\nabla i_a(X) + i_a(X)\nabla = \mathcal{L}_X - dx^i \lor i_s(\nabla_{\partial_i} X). \quad (51)
\]

Combining (50), (49) and (51) we get the first statement of the proposition. For the second statement one just has to observe that $ad(f) = 0$ and that

\[
i_s(X_f) = \frac{1}{\nu} ad(df \otimes 1) \quad dx^i \lor i_s(\nabla_{\partial_i} X_f) = \frac{1}{\nu} ad \left( \frac{1}{2} Ddf \otimes 1 \right), \quad (52)
\]

which is again a straightforward computation in local coordinates using the explicit shape of the deformed product $\circ$. Using these equations combined with (16) finishes the proof of (17).

**B Computation of $C_2^{-s}$**

This section just gives a sketch of the computations that are necessary to determine the anti-symmetric part of the bidifferential operator $C_2$ that occurs in the expression of the Fedosov star product $f \ast g = fg + \nu C_1(f, g) + \nu^2 C_2(f, g) + \ldots$ of two functions $f, g \in C^\infty(M)$.

**Proposition B.1** The anti-symmetric part $C_2^-$ of the bidifferential operator $C_2$ is given by

\[
C_2^-(f, g) = \frac{1}{2} (C_2(f, g) - C_2(g, f)) = -\frac{1}{2} \left( \Omega_1 + ds_1^{(3)} \right)(X_f, X_g) = \rho_2(X_f, X_g), \quad (53)
\]

where $f, g \in C^\infty(M)$ and $X_f$ resp. $X_g$ denote the corresponding Hamiltonian vector fields with respect to $\omega$ and $s_1^{(3)} \in \Gamma\otimes(T^*M)$ denotes the one-form occurring in the first order of $\nu$ in $s^{(3)} = s_3^{(3)} + \nu s_1^{(3)}$, where $s_3^{(3)} \in \Gamma\otimes(V^3T^*M)$, that comes up from the normalisation condition $\delta^{-1}r = s$ (cf. equation (8)). Thus we have

\[
c(s)^0 = -2C_2^-= -2[\rho_2] = [\Omega_1]. \quad (54)
\]

**Proof:** Using the shape of the fibrewise product $\circ$ we obtain $f \ast g - g \ast f = \nu \sigma(\Lambda^{r^*}i_s(\partial_r)\tau(f)i_s(\partial_s)\tau(g)) + O(\nu^3)$. To compute the terms of order less or equal to two in $\nu$ we thus only have to know $\tau(f)$ and $\tau(g)$ except for terms of symmetric degree and $\nu$-degree greater than one. Hence it is enough to look at $\tau(f)^{(0)}, \ldots, \tau(f)^{(3)}$ since for $\tau(f)^{(k)}$ with $k \geq 4$ either the symmetric degree or the $\nu$-degree of the occurring terms are greater than one. Looking at the recursion formula (12) we thus see that the only terms of $\tau$ that are needed are given by $r^{(2)} = \delta s^{(3)}$ and $r^{(3)} = \delta s^{(4)} + \delta^{-1}(R + \nu \Omega_1 + \nabla r^{(2)} - \frac{1}{2} r^{(2)} \circ r^{(2)})$ what is obtained from (1) by writing down the terms of total degree 2 resp. 3. Writing $\approx$ for equations holding modulo terms
of symmetric degree resp. $\nu$-degree greater than one, one gets by lengthy but obvious computation that

$$
\tau(f)^{(0)} = f
$$

$$
\tau(f)^{(1)} = df \otimes 1
$$

$$
\tau(f)^{(2)} = \frac{1}{2}Ddf \otimes 1 - i_s(X_f)s_3^{(3)} \approx 0
$$

$$
\tau(f)^{(3)} \approx -\delta^{-1}(i_s(X_f)r^{(3)}) \approx -\nu \left( i_s(X_f)s_2^{(4)} + \frac{1}{2}i_{X_f} \left( \Omega_1 + ds_1^{(3)} \right) \right) \otimes 1,
$$

where we have written $s^{(4)} = s_4^{(4)} + \nu s_2^{(4)}$ with $\deg s_4^{(4)} = k s_4^{(4)}$. Inserting these results into $f \ast g - g \ast f$ as given above the terms involving $s_2^{(4)} \in \Gamma^\infty(V^2 T^*M)$ cancel because of their symmetry and one gets

$$
f \ast g - g \ast f = \nu \{f, g\} - \nu^2(\Omega_1 + ds_1^{(3)})(X_f, X_g) + O(\nu^3)
$$

proving the proposition. $\square$

One should observe, that this is the only instance of our proof of Theorem 4.4 where the modified normalisation condition on $r$ enters our considerations, whereas the other terms of $c(*)$ could be computed without making use of it.

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