Pseudoclassical neutrino in the external electromagnetic field

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Abstract

The problem of the passage of the neutral massless particle with anomalous magnetic moment through the external electromagnetic field is considered both in pseudoclassical and quantum mechanics. The quantum description uses the hamiltonian in the Foldy–Wouthuysen representation, obtained from the pseudoclassical hamiltonian of the massive charged particle with anomalous magnetic moment in interaction with the external electromagnetic field using Weyl quantization scheme.
1 Introduction

In the paper of the authors \[1\] in the pseudoclassical approach \[2\], when the spin degrees of freedom are described by Grassmann variables, the canonical quantization of the relativistic spinning particle with anomalous magnetic moment (AMM), interacting with the external electromagnetic field, was carried out in the space–time dimensions \(D = 2n\). The resulting quantum theory is the theory of the Dirac particle in the external electromagnetic field in the Foldy–Wouthuysen representation \[3, 4\]. The physical Hamiltonian of the theory was found, and in the dimension \(D = 4\) it is given by the expression

\[
H_{\text{phys}} = \Omega - g\kappa A_0(q, \tau) + \left[ \frac{i(g - 2mG)}{4\Omega} F_{ik}(q, \tau) - \frac{ig\kappa}{2\Omega(\Omega + m)} (F_{0k}(q, \tau)\pi_j - F_{0j}(q, \tau)\pi_k) + \frac{iG}{\Omega} \left( F_{0k}(q, \tau)\pi_j - F_{0j}(q, \tau)\pi_k - \frac{\kappa}{\Omega + m} F_{ij}(q, \tau)\pi_i\pi_k + \frac{\kappa}{\Omega + m} F_{ik}(q, \tau)\pi_i\pi_j \right) \right] (\psi_k \psi_j - \psi_j \psi_k).
\]

Here \(g, m, -G\) are the charge, mass and the AMM of the particle, correspondingly, \(\kappa\) is the parameter of the theory, which can be equal to \(\pm 1\) (\(\kappa = +1\) corresponds to the presence in the theory of the particle, while \(\kappa = -1\) corresponds to the presence of the antiparticle), \(\tau\) is a parameter along the trajectory of the particle; \(\Omega = \sqrt{p_i^2 + m^2}\), \(\pi_i = P_i - gA_i\); \(A_0, A_i, F_{0k}, F_{ik}\) are components of the vector-potential and of the stress-tensor of the electromagnetic field, correspondingly; \(p_i, q_j, \psi_k\) - are canonical (Newton-Wigner) variables of the theory. The first two variables correspond to physical momentum and coordinate , while the spin of the particle is described through \(\psi_k\) (\(\psi_k\), contrary to \(p_i, q_j\), are Grassmann odd variables).

The quantum Hamiltonian, corresponding to \(H_{\text{phys}}\), is obtained from (1) using Weyl prescription for operator ordering, adapted to the presence of Grassmann variables in the theory \[4\]. Then the operator \(\hat{\psi}_k\) in the \(D = 4\) space-time is realized through Pauli matrices: \(\hat{\psi}_k = \left( \frac{\hbar}{2} \right)^{1/2} \sigma_k\).
Note, that this method for the construction of the Hamiltonian operator in the FW-
representation differs from the usual method, when the operator in the FW – represen-
tation is obtained from the operator in the Dirac picture using the relation

\[ \hat{H}_{FW} = \exp(-iS)\hat{H}^D\exp(iS), \]  

(2)

where \( \hat{H}_{FW}, \hat{H}^D \) are Hamiltonian operators in FW and Dirac representations correspond-
ingly, \( S \) is the generator of Foldy-Wouthuysen transformation [5]. As is well known, for
the free particle the operator \( S \) can be construed exactly. However in the presence of the
external field the generator \( S \) (and hence \( \hat{H}_{FW} \)) is known only in a series of the param-
eter \( 1/m \) [5]. This makes the transition in the theory to the massless limit impossible.
Contrary to this, the expression for \( H_{phys} \), as it’s easy to see from (1), and the correspond-
ing quantum Hamiltonian operator \( \hat{H}_{phys} \) are analytic functions of the mass parameter
\( m \). Note, that the generalization of the Foldy – Wouthuysen hamiltonian to describe a
particle with AMM, which allows to set \( m = 0 \), was given by Sattorp and DeGroot [6].
However their hamiltonian was correct only to the first order of \( e \) and contained only
first derivatives of the electromagnetic potentials.

Note also, that in [7], where the Compton scattering by a proton was investigated,
the effective hamiltonian of the interaction of the particle with AMM with external
electromagnetic field in the FW-representation in the \((1/m)^3\) approximation was found.
Comparing that hamiltonian with the one obtained from \( \hat{H}_{phys} \) in the same approximation
we find that they coincide in the first order of \( \hbar \).

The analyticity of the expression for \( H_{phys} \), and hence of \( \hat{H}_{phys} \), with respect to the
parameter \( m \) allows to use (1) for \( m = 0 \) to describe the interaction of the massless
Dirac particle with AMM, interacting with the external electromagnetic field and this
problem is investigated in this paper. Namely, we will consider a massless neutral particle
\((m = 0, g = 0)\). This corresponds to the propagation of the four component neutrino
(with AMM equal to \(-G\)) in the external electromagnetic field. This problem attracted
attention recently in connection with the attempts to explain the anticorrelations between
the number of registered solar neutrinos and the magnetic activity of the sun. One of
the possible explanations was based on the assumption that neutrino has a anomalous magnetic moment: when the neutrino passes through the convective outer layers of the sun its spin precesses in the plane, perpendicular to the magnetic field of the sun. As a result some of the left neutrinos born in the core of the sun become right neutrinos, that are not registered in the solar neutrino experiment [8].

We’ll return to this question after solving the problem within the pseudoclassical approach using the expression (1) with \( m = 0, \ g = 0. \)

2 Pseudoclassical description

Now let us write down the expression for the pseudoclassical hamiltonian \( H \), which describes the interaction of the massless neutral particle with AMM with the external electromagnetic field. From (1) we find for \( g = 0, \ m = 0 \)

\[
H = \sqrt{\vec{p}^2 + 2G \left[ \vec{B} \vec{S} - (\vec{B} \vec{n})(\vec{S} \vec{n}) \right] - 2\kappa G \vec{S} \cdot \left[ \vec{E}, \vec{n} \right]} = |\vec{p}| + 2G \vec{S} \vec{D},
\]

(3)

where

\[
\vec{D} = \vec{B} - (\vec{B} \vec{n}) \vec{n} - \kappa \left[ \vec{E}, \vec{n} \right]
\]

(4)

In writing down the expressions (3) the following notations were used: \( E_k = F_{ok}, \ B_i = \frac{1}{2} \varepsilon_{ijk} F_{kj}; \ S_i \) is the pseudoclassical spin vector, defined as \( S_i = -\frac{i}{2} \varepsilon_{ikl} \psi_k \psi_l, \ \vec{n}, \ n_i = p_i/|\vec{p}|. \) Note that only transverse components of the fields \( \vec{E}, \vec{B} \) enter the interaction hamiltonian: \( B_i^\perp = B_i - (\vec{B} \vec{n}) n_i, \left[ \vec{E}, \vec{n} \right] = \left[ \vec{E}^\perp, \vec{n} \right]. \) Thus the \( \vec{D} \) lies in the plane, which is perpendicular to the momentum of the particle.

Consider now the equations of motion of canonical variables \( q_i, \ p_j \) and of the spin \( S_k \)

\[
\dot{q}_i = \{ q_i, H \}_D = \frac{\partial H}{\partial \dot{p}_i},
\]

(5)

\[
\dot{p}_j = \{ p_j, H \}_D = -\frac{\partial H}{\partial \dot{q}_j},
\]

(6)

\[
\dot{S}_k = \{ S_k, H \}_D.
\]

(7)
Here Dirac brackets are given by the relations
\[ \{ q_i, p_j \}_D = \delta_{ij}, \quad \{ \psi_i, \psi_j \} = -i\delta_{ij} \]
(all other brackets for the variables \( q_i, p_j, \psi_k \) are equal to zero). Using the explicit expression for the Hamiltonian (3) we find from (5)-(7)
\[ \dot{q}_i = n_i, \quad (8) \]
\[ \dot{p}_j = -2GS_k \frac{\partial D_k}{\partial q_j}, \quad (9) \]
\[ \dot{S}_k = 2G\varepsilon_{klm}D_lS_m. \quad (10) \]
The eq. (8) reflects the fact, that the modulus of the velocity of the particle doesn’t change and is equal to \( |\dot{q}_i| = 1 \) (in units \( c = 1 \)). As it follows from the equation (9) the momentum of the particle is constant in the presence of the constant external electromagnetic field (note that the vector \( \vec{D} \) lies in plane, perpendicular to the momentum vector). As for the equation (10), it describes the precession of the pseudoclassical spin vector around the external ”magnetic” field \( \vec{D} \).

In what follows we will consider the motion of the spinning particle in the constant electromagnetic field. Hence we have \( p_i = \text{const} \). Without loss of generality the axis \( q_3 \) can be chosen along the direction of the momentum vector: \( n_i = (0, 0, 1) \), and the axis \( q_1 \) along the \( \vec{D} \). In this coordinate system the expression (3) for the hamiltonian takes the form
\[ H = |p_3| + 2GS_1D_1, \quad D_1 = B_1 - \kappa E_2 \quad (11) \]
and the equations (10) are rewritten as follows:
\[ \dot{S}_1 = 0, \quad \dot{S}_2 = -2GD_1S_3, \quad \dot{S}_3 = 2GD_1S_2 \quad (12) \]
Evidently the solution of the first equation is \( S_1 = \text{const} \). We’ll write down the solution of the remaining set of equations in the form
\[ S_2 = A_2e^{2iGD_1t} + B_2e^{-2iGD_1t}, \quad S_3 = A_3e^{2iGD_1t} + B_3e^{-2iGD_1t} \quad (13) \]
where $A_2, B_2, A_3, B_3$ are some constants. Substituting (13) into (12) we find relations between these constants
\[ A_3 = -iA_2, \quad B_3 = iB_2. \]  
(14)

Suppose now, that for $t = 0$ the pseudoclassical spin vector had projections $S_2 = 0, S_3 = \frac{1}{2}\hbar$. In quantum case this corresponds to the particle with positive chirality. This initial condition allows to find all coefficients $A_2, B_2, A_3, B_3$:
\[ A_2 = \frac{i}{2}, \quad B_2 = -\frac{i}{2}; \quad A_3 = \frac{1}{2}, \quad B_3 = \frac{1}{2}. \]  
(15)

Hence from (13) we find
\[ S_2 = -\frac{1}{2}\hbar\sin(2GD_1t), \quad S_3 = \frac{1}{2}\hbar\cos(2GD_1t). \]  
(16)

Thus the motion of the spin is a precession in plane $(q_2q_3)$ ($S_1$- component is a constant of a motion)

### 3 Quantum description

Consider now the quantum mechanical problem of the propagation of the neutrino in the constant electromagnetic field. The choice of the coordinate system is as in sect.2: the momentum of the particle is in the direction of $q_3$, while the axis $q_1$ is directed along the vector $\vec{D}$. The neutrino, having the momentum $\vec{p} = (0, 0, p_3)$ is moving from the region $q_3 < 0$ towards the region of the constant electromagnetic field, confined between the planes $q_3 = 0$ and $q_3 = L$. The Hamilton operator of the system in the region $0 < q_3 < l$ is found from (11) and is given by
\[ \hat{H} = |\vec{p}_3| + G\hbar D_1\sigma_1, \]  
(17)

where the expression for the spin operator is used: $\hat{S}_i = (\hbar/2)\sigma_i, \sigma_i$ are the Pauli matrices. In the regions $q_3 < 0$ and $q_3 > L$ we have a free motion. Denote the wave functions in the regions $q_3 < 0, 0 < q_3 < l$ and $q_3 > L$ by $\psi^I, \psi^II, \psi^III$, correspondingly. Since the
hamiltonian doesn’t depend on time explicitly, the problem is reduced to finding solutions of the stationary states equation

\[ \hat{H}\psi_\alpha(q_3) = E\psi_\alpha(q_3) \]  

(18)

Consider first the region \(0 < q_3 < l\). The Hamilton operator is given by (17). To construct the \(\psi^{II}_\alpha\) we’ll find first the spin eigenfunctions \(\chi_\alpha\) from the equation

\[ (D_1\sigma_1)_{\alpha\beta}\chi_\beta = \lambda\chi_\alpha, \quad \chi_\alpha = (\chi_1, \chi_2), \]  

(19)

where \(\lambda\) the eigenvalues, corresponding to this functions. Nontrivial solutions of this equation exist under the condition

\[ \det |D_1\sigma_1 - \lambda| = 0, \]  

(20)

which together with the normalization condition \(\chi_1^2 + \chi_2^2 = 1\) leads to the following eigenvalues and eigenfunctions:

\[ \chi_\alpha^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_1 = D_1 \]

\[ \chi_\alpha^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda_2 = -D_1 \]  

(21)

We’ll look for the solution of (18) in the form of the decomposition through spin eigenfunctions \(\chi_\alpha^i = (\chi_\alpha^1, \chi_\alpha^2)\):

\[ \psi^{II}_\alpha(q_3) = \sum_i f_i(q_3)\chi_\alpha^i, \]  

(22)

where \(f_i(q_3)\) are scalar functions of the coordinates. Substituting (22) into (18) and taking into account the equation (19) and the completeness of the set of functions \(\chi_\alpha^i\), we come to the equation

\[ \left(|\hat{p}_3| + G\hbar\lambda(i) - E\right) f_i(q_3) = 0 \]  

(23)

The general solution of this equation is given by the expression

\[ f_i(q_3) = a_i e^{\frac{i}{\hbar}\hat{p}_3 q_3} + b_i e^{-\frac{i}{\hbar}\hat{p}_3 q_3} \]  

(24)
\[ p_3^{(i)} = E - G\hbar\lambda_{(i)} \]

where \( a_i, b_i \) are some constants. Substituting now (24) in (22) we come to the following expression for the wave function \( \psi^{II}_\alpha(q_3) \):

\[
\psi^{II}_\alpha(q_3) = \sum_{i=1,2} \left( a_i e^{i\hat{p}_3^{(i)} q_3} + b_i e^{-i\hat{p}_3^{(i)} q_3} \right) \chi^i_\alpha
\]

(25)

In the region \( q_3 < 0 \) both incident and reflected waves are present. The corresponding wave function \( \psi^I_\alpha(q_3) \), presented as a decomposition through spin eigenfunctions \( \chi^i_\alpha \) has the form

\[
\psi^I_\alpha(q_3) = \frac{1}{\sqrt{2}} \chi^1_\alpha e^{i\hat{p}_3 q_3} + \frac{1}{\sqrt{2}} \chi^2_\alpha e^{-i\hat{p}_3 q_3} + \chi^1_\alpha A_1 e^{-i\hat{p}_3 q_3} + \chi^2_\alpha A_2 e^{-i\hat{p}_3 q_3} =
\]

\[
e^{i\hat{p}_3 q_3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-i\hat{p}_3 q_3} \frac{1}{\sqrt{2}} \begin{pmatrix} A_1 + A_2 \\ A_1 - A_2 \end{pmatrix},
\]

(26)

where \( A_1, A_2 \) are some constants. The first summand in the second line of (26) corresponds to the wave of definite polarization (as it was mentioned earlier, the neutrinos born in the core of the sun are left handed).

In the region \( q_3 > L \) only the outgoing wave, which moves in the direction of the incident wave, is present:

\[
\psi^{III}_\alpha(q_3) = \chi^1_\alpha B_1 e^{i\hat{p}_3 q_3} + \chi^2_\alpha B_2 e^{i\hat{p}_3 q_3} =
\]

\[
e^{i\hat{p}_3 q_3} \frac{1}{\sqrt{2}} \begin{pmatrix} B_1 + B_2 \\ B_1 - B_2 \end{pmatrix} = e^{i\hat{p}_3 q_3} \chi^{III}_\alpha
\]

(27)

where \( B_1, B_2 \) are constants.

The constants \( a, b, A, B \) are defined from the condition of continuity of \( \psi \) and \( d\psi/dt \) at the planes \( q_3 = 0 \) and \( q_3 = L \). Because of the orthogonality of the functions \( \chi^1_\alpha, \chi^2_\alpha \), the corresponding set of equations is divided into two subsets of equations, each containing only one of the functions \( \chi^1_\alpha, \chi^2_\alpha \). Solving these equations we come to the following values for the constants:

\[
A_1 = -\frac{i\sqrt{2}G\hbar D_1 \sin(\frac{1}{\hbar}p_3^{(1)} L)}{(E + p_3^{(1)}) \left( e^{-i\hat{p}_3^{(1)} L} - \frac{(G\hbar D_1)^2}{(E + p_3^{(1)})^2} e^{i\hat{p}_3^{(1)} L} \right) - \frac{(G\hbar D_1)^2}{(E + p_3^{(1)})^2} e^{i\hat{p}_3^{(1)} L}}
\]

(28)
\[ A_2 = -\frac{i\sqrt{2}GhD_1 \sin(\frac{1}{2}p_3^{(1)}L)}{(E + p_d^{(2)})} \left[ e^{-\frac{i}{\hbar}p_3^{(2)}L} - \frac{(GhD_1)^2}{(E + p_d^{(2)})^2} e^{\frac{i}{\hbar}p_3^{(2)}L} \right] \] (29)

\[ B_1 = \frac{\sqrt{2}}{2} \frac{e^{-iGD_1L}}{1 - \frac{i(GhD_1)^2}{2Ep_3^{(1)}} \sin(\frac{1}{\hbar}p_3^{(1)}L) e^{\frac{i}{\hbar}p_3^{(1)}L}} \] (30)

\[ B_2 = \frac{\sqrt{2}}{2} \frac{e^{iGD_1L}}{1 - \frac{i(GhD_1)^2}{2Ep_3^{(2)}} \sin(\frac{1}{\hbar}p_3^{(2)}L) e^{\frac{i}{\hbar}p_3^{(2)}L}} \] (31)

As it was shown in [8], the value of the quantity $GhD_1/E$ is small for the real electromagnetic fields and medium energies of the neutrino. Hence the amplitudes of the reflected waves, as one can see from (28) and (29), are also small. As for the outgoing wave, taking into account the smallness of the named parameter, we find from (30) and (31) that

\[ B_1 = \frac{\sqrt{2}}{2} e^{-iGD_1L}, \quad B_2 = \frac{\sqrt{2}}{2} e^{iGD_1L}, \] (32)

which in their turn bring as to the following expression for $\psi_{\alpha}^{III}$ (see (27):

\[ \psi_{\alpha}^{III} = e^{\frac{i}{\hbar}p_3q_3} \begin{pmatrix} \cos(GD_1L) \\ -i \sin(GD_1L) \end{pmatrix} \] (33)

To compare this result with the pseudoclassical one we calculate the mean projection of the spin of the outgoing wave on the $q_2, q_3$ axes

\[ < S_2 > = \frac{\chi_{\alpha}^{III} h}{2} \sigma_2 \chi_{\alpha}^{III} = -\frac{\hbar}{2} \sin(2GD_1L), \quad < S_3 > = \frac{\chi_{\alpha}^{III} h}{2} \sigma_3 \chi_{\alpha}^{III} = \frac{\hbar}{2} \cos(2GD_1L) \] (34)

If we now insert in (16) $t = L$, we’ll find a coincidence of the results of the quantum and pseudoclassical description of the rotation of the spin of the massless neutral particle in the external electromagnetic field.

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