JOHN DISK AND K-QUASICONFORMAL HARMONIC MAPPINGS

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Abstract. The main aim of this article is to establish certain relationships between K-quasiconformal harmonic mappings and John disks. The results of this article are the generalizations of the corresponding results of Ch. Pommerenke [18].

1. Introduction and main results

For $a \in \mathbb{C}$ and $r > 0$, we let $D(a, r) = \{z : |z - a| < r\}$ so that $D_r := D(0, r)$ and thus, $\mathbb{D} := D_1$ denotes the open unit disk in the complex plane $\mathbb{C}$. This paper provides a necessary and sufficient condition for the image $\Omega = f(\mathbb{D})$ of univalent harmonic mappings $f$ defined on $\mathbb{D}$ to be a John disk (see Theorems 1 and 2). Some differential properties of $K$-quasiconformal harmonic mappings will also be characterized by using Pommerenke interior domains and John disks (see Theorem 4 and Corollary 1). In addition, we present a sufficient condition, in terms of harmonic analog of the pre-Schwarzian of $K$-quasiconformal harmonic mappings $f$ on $\mathbb{D}$, for $\Omega = f(\mathbb{D})$ to be a John disk (see Theorem 5). Similar results for analytic functions are proved earlier by Ahlfors and Weill [1], Becker and Pommerenke [2], and Pommerenke [18]. In order to state and prove our main results and related investigations, we need to recall some basic definitions, remarks and some results.

For a real $2 \times 2$ matrix $A$, we use the matrix norm $\|A\| = \sup\{|Az| : |z| = 1\}$ and the matrix function $l(A) = \inf\{|Az| : |z| = 1\}$. For $z = x + iy \in \mathbb{C}$, the formal derivative of the complex-valued functions $f = u + iv$ is given by

$$D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

so that

$$\|D_f\| = |f_z| + |f_{\bar{z}}| \quad \text{and} \quad l(D_f) = |f_z| - |f_{\bar{z}}|,$$

where $f_z = (1/2)(f_x - if_y)$ and $f_{\bar{z}} = (1/2)(f_x + if_y)$.

Let $\Omega$ be a domain in $\mathbb{C}$, with non-empty boundary. A sense-preserving homeomorphism $f$ from a domain $\Omega$ onto $\Omega'$, contained in the Sobolev class $W_{loc}^{1,2}(\Omega)$, is

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said to be a \( K\)-quasiconformal mapping if, for \( z \in \Omega \),
\[
\|D_{f}(z)\|^2 \leq K \operatorname{det}D_{f}(z), \text{ i.e., } \|D_{f}(z)\| \leq K\|I(D_{f}(z))\|,
\]
where \( K \geq 1 \) and \( \operatorname{det}D_{f} \) is the determinant of \( D_{f} \) (cf. \([12, 14, 22, 23]\)).

A complex-valued function \( f \) defined in a simply connected subdomain \( G \) of \( \mathbb{C} \) is called a harmonic mapping in \( G \) if and only if both the real and the imaginary parts of \( f \) are real harmonic in \( G \). It is indeed a simple fact that every harmonic mapping \( f \) in \( G \) admits a decomposition \( f = h + \overline{g} \), where \( h \) and \( g \) are analytic in \( G \). If we choose the additive constant such that \( g(0) = 0 \), then the decomposition is unique. Since the Jacobian \( \operatorname{det}D_{f} \) of \( f \) is given by
\[
\operatorname{det}D_{f} := |f_z|^2 - |f_{\overline{z}}|^2 = |h'|^2 - |g'|^2,
\]
\( f \) is locally univalent and sense-preserving in \( G \) if and only if \( |g'(z)| < |h'(z)| \) in \( G \); or equivalently if \( h'(z) \neq 0 \) and the dilatation \( \omega = g'/h' \) has the property that \( |\omega(z)| < 1 \) in \( G \) (see \([15]\) and also \([8]\)).

In the recent years, the family \( \mathcal{S}_{H} \) of all sense-preserving planar harmonic univalent mappings \( f = h + \overline{g} \) in \( \mathbb{D} \), with the normalization \( h(0) = g(0) = 0 \) and \( h'(0) = 1 \), attracted the attention of many function theorists. This class together with a few other geometric subclasses, originally investigated in details by \([6]\), became instrumental in the study of univalent harmonic mappings. See the monograph \([8]\) and the recent survey \([20]\) for the theory of these functions.

If the co-analytic part \( g \) is identically zero in the decomposition of \( f \), then the class \( \mathcal{S}_{H} \) reduces to the classical family \( \mathcal{S} \) of all normalized analytic univalent functions \( h(z) = z + \sum_{n=2}^{\infty} a_{n} z^{n} \) in \( \mathbb{D} \). If \( \mathcal{S}^{0}_{H} = \{ f = h + \overline{g} \in \mathcal{S}_{H} : g'(0) = 0 \} \), then the family \( \mathcal{S}^{0}_{H} \) is both normal and compact (see \([6, 8, 20]\)).

Let \( d_{\Omega}(z) \) be the Euclidean distance from \( z \) to the boundary \( \partial\Omega \) of \( \Omega \). In particular, we always use \( d(z) \) to denote the Euclidean distance from \( z \) to the boundary \( \partial\mathbb{D} \) of \( \mathbb{D} \).

**Definition 1.** A bounded simply connected plane domain \( G \) is called a \( c\)-John disk for \( c \geq 1 \) with John center \( w_{0} \in G \) if for each \( w_{1} \in G \) there is a rectifiable arc \( \gamma \), called a John curve, in \( G \) with end points \( w_{1} \) and \( w_{0} \) such that
\[
\sigma_{\ell}(w) \leq c d_{\Omega}(w)
\]
for all \( w \) on \( \gamma \), where \( \gamma[w_{1}, w] \) is the subarc of \( \gamma \) between \( w_{1} \) and \( w \), and \( \sigma_{\ell}(w) \) is the Euclidean length of \( \gamma[w_{1}, w] \) (see \([11, 9, 17, 19]\)).

**Remark 1.** If \( f \) is a complex-valued and univalent mapping in \( \mathbb{D}, G = f(\mathbb{D}) \) and, for \( z \in \mathbb{D}, \gamma = f([0, z]) \) in Definition 1, then we call \( c\)-John disk as a radial \( c\)-John disk, where \( w_{0} = f(0) \) and \( w = f(z) \). In particular, if \( f \) is a conformal mapping, then we call \( c\)-John disk as a hyperbolic \( c\)-John disk. It is well known that any point \( w_{0} \in G \) can be chosen as John center by modifying the constant \( c \) if necessary. Moreover, if we don’t emphasize the constant \( c \), we regard the \( c\)-John disk as the John disk (cf. \([11, 9, 17]\)).

In \([18]\) (see also \([19, p. 97]\)), Pommerenke proved that if \( f \) maps \( \mathbb{D} \) conformally onto a bounded domain \( G \), then \( G \) is a hyperbolic John disk if and only if there exist
constants $M > 0$ and $\delta \in (0, 1)$ such that for each $\zeta \in \partial \mathbb{D}$, and for $0 \leq r_1 \leq r_2 < 1$, we have
\[
|f'(r_2 \zeta)| \leq M |f'(r_1 \zeta)| \left( \frac{1 - r_2}{1 - r_1} \right)^{\delta - 1}.
\]
Later, in [9, Theorem 2.3], Kari Hag and Per Hag gave an alternate proof of this result. In this paper, our first aim is to extend this result to planar harmonic mappings.

**Theorem 1.** For $K \geq 1$, let $f \in \mathcal{S}_H^0$ be a $K$-quasiconformal harmonic mapping from $\mathbb{D}$ onto a bounded domain $\Omega$. Then $\Omega$ is a radial John disk if and only if there are constants $M(K) > 0$ and $\delta \in (0, 1)$ such that for each $\zeta \in \partial \mathbb{D}$ and for $0 \leq r \leq \rho < 1$,
\[
(1.1) \quad \|D_f(\rho \zeta)\| \leq M(K) \|D_f(r \zeta)\| \left( \frac{1 - \rho}{1 - r} \right)^{\delta - 1}.
\]

The following result is another characterization of radial John disk, which is also a generalization of [18, Theorem 1].

**Theorem 2.** For $K \geq 1$, let $f \in \mathcal{S}_H^0$ be a $K$-quasiconformal mapping and $\Omega = f(\mathbb{D})$ is a bounded domain. Then the following conditions are equivalent:

(a) $\Omega$ is a radial John disk;

(b) There is a positive constant $M_1$ such that, for all $z \in \mathbb{D}$,
\[
\text{diam}_f(B(z)) \leq M_1 d_\Omega(f(z)),
\]
where $B(z) = \{ \zeta : |z| \leq |\zeta| < 1, |\arg z - \arg \zeta| \leq \pi(1 - |z|) \}$;

(c) There is a positive constant $\delta \in (0, 1)$ such that, for all $z \in \mathbb{D}$ and $\zeta \in B(z)$,
\[
(1.2) \quad \|D_f(\zeta)\| \leq M_2 \|D_f(z)\| \left( \frac{1 - |\zeta|}{1 - |z|} \right)^{\delta - 1},
\]
where $M_2$ is a positive constant.

By using some distortion conditions in Theorem 2, we get a characterization of coefficients of $K$-quasiconformal harmonic mappings.

**Theorem 3.** For $K \geq 1$, let $f = h + \overline{g} \in \mathcal{S}_H^0$ be a $K$-quasiconformal harmonic mapping, where
\[
h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n.
\]

If $f$ satisfies the condition (b) or (c) in Theorem 2, then there is some $\beta_0 > 0$ such that
\[
\sum_{n=2}^{\infty} n^{1+\beta_0} (|a_n|^2 + |b_n|^2) < \infty.
\]

Using Theorems 2 and 3, it can be easily seen that the conclusion of Theorem 3 continues to hold if the assumption that "$f$ satisfies the condition (b) or (c) in Theorem 2" is replaced by "$\Omega = f(\mathbb{D})$ is a radial John disk".
For \( p \in (0, \infty] \), the generalized Hardy space \( H_p^g(D) \) consists of all those functions \( f : D \to \mathbb{C} \) such that \( f \) is measurable, \( M_p(r, f) \) exists for all \( r \in (0, 1) \) and \( \| f \|_p < \infty \), where

\[
\| f \|_p = \begin{cases} 
\sup_{0 < r < 1} M_p(r, f) & \text{if } p \in (0, \infty) \\
\sup_{z \in D} |f(z)| & \text{if } p = \infty
\end{cases}, \quad \text{and } M_p^g(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta.
\]

Let \( f \in \mathcal{S}_H \) be a \( K \)-quasiconformal harmonic mapping from \( D \) onto a domain \( G \). For \( 0 < r < 1 \) and \( w_1, w_2 \in f(\partial \mathbb{D}_r) \), let \( \gamma_r \) be the smaller subarc of \( f(\partial \mathbb{D}_r) \) between \( w_1 \) and \( w_2 \), and let

\[
d_{G_r}(w_1, w_2) = \inf_{\Gamma} \text{diam}\Gamma,
\]

where \( \Gamma \) runs through all arcs from \( w_1 \) to \( w_2 \) that lie in \( G_r = f(\mathbb{D}_r) \) except for their endpoints. If

\[
\sup_{0 < r < 1} \left\{ \sup_{w_1, w_2 \in \gamma_r} \frac{\ell(\gamma_r[w_1, w_2])}{d_{G_r}(w_1, w_2)} \right\} < \infty,
\]

then we call \( G \) as a Pommerenke interior domain (cf. [18]). In particular, if \( G \) is bounded, then we call \( G \) as a bounded Pommerenke interior domain. Our next theorem is an analogous result of [18, Theorem 3].

**Theorem 4.** Let \( f \in \mathcal{S}_H \) be a \( K \)-quasiconformal harmonic mapping from \( D \) onto a bounded Pommerenke interior domain \( G \). If there are constants \( M \) and \( \delta \in (0, 1) \) such that for each \( \zeta \in \partial \mathbb{D} \) and for \( 0 \leq r \leq \rho < 1 \),

\[
\| D_f(\rho \zeta) \| \leq M \| D_f(r \zeta) \| \left( \frac{1 - \rho}{1 - r} \right)^\delta,
\]

then \( \| D_f \| \in H_1^g(D) \).

The following result easily follows from Theorems 1 and 4.

**Corollary 1.** Let \( f \in \mathcal{S}_H^0 \) be a \( K \)-quasiconformal harmonic mapping from \( D \) onto a bounded Pommerenke interior domain \( G \). If \( G \) is a radial John disk, then \( \| D_f \| \in H_1^g(D) \).

In terms of the canonical representation of a sense-preserving harmonic mappings \( f = h + \overline{g} \) in \( \mathbb{D} \) with \( \omega = g'/h' \), as in the works of Hernández and Martín [10], the Pre-Schwarzian derivative \( P_f \) of \( f \) and the Schwarzian derivative \( S_f \) of \( f \) are defined by

\[
P_f = T_h - \frac{\overline{\omega''}\overline{\omega}}{1 - |\omega|^2}, \quad \text{and } S_f = Sh + \frac{\overline{\omega}}{1 - |\omega|^2} (T_h\omega' - \omega'') - \frac{3}{2} \left( \frac{\overline{\omega''}\overline{\omega}}{1 - |\omega|^2} \right)^2,
\]

respectively. Here

\[
T_h = \frac{h''}{h'} \quad \text{and} \quad Sh = T_h' - \frac{1}{2} T_h^2
\]

are referred to as the Pre-Schwarzian and Schwarzian (derivatives) of a locally univalent analytic function \( f \) in \( \mathbb{D} \), respectively. For the original definition of the Schwarzian derivative of harmonic mappings, see [4].
Ahlfors and Weill [1], Becker and Pommerenke [2] characterized the quasidisk by using the Pre-Schwarzian of analytic functions. On the basis of the works of Chuaqui, et al. [5], Kari Hag and Per Hag [9] discussed the relationships between the John disk and the Pre-Schwarzian of analytic functions. It is natural to ask whether a similar relationship is attainable (see [5, Theorem 4] and [9, Theorem 3.7]) with the help of Pre-Schwarzian of harmonic mappings. This is the content of our next result.

**Theorem 5.** Suppose that \( f \in S_0^H \) is a \( K \)-quasiconformal harmonic mapping of \( \mathbb{D} \) onto a bounded domain \( f(\mathbb{D}) \) for some \( K \geq 1 \) and such that

\[
\lim_{|z| \to 1^-} \sup \{(1 - |z|^2)\text{Re}(zP_f(z))\} < 1.
\]

If \( \ell(f([0, z])) < \infty \) for all \( z \in \mathbb{D} \), then \( f(\mathbb{D}) \) is a radial John disk.

The proofs of Theorems 1–5 will be given in Section 2.

### 2. The proofs of the main results

We begin the section by recalling the following results which play an important role in the proofs of Theorems 1–5.

**Theorem A.** ([13, Proposition 3.1] and [13, Theorem 3.2]) Let \( f \) be a \( K \)-quasiconformal harmonic mapping from \( \mathbb{D} \) onto itself. Then for all \( z \in \mathbb{D} \), we have

\[
\frac{1 + K}{2K} \left( \frac{1 - |f(z)|^2}{1 - |z|^2} \right) \leq |f_z(z)| \leq \frac{K + 1}{2} \left( \frac{1 - |f(z)|^2}{1 - |z|^2} \right).
\]

**Theorem B.** ([3, Theorem 3]) Let \( f \in S_0^H \). Then there is a positive constant \( c_1 < +\infty \) such that for \( \xi \in \partial \mathbb{D} \) and \( 0 \leq r_3 \leq r_4 < 1 \),

\[
\|D_f(r_4\xi)\| \geq \frac{1}{2^{1+c_1}} \|D_f(r_3\xi)\| \left( \frac{1 - r_4}{1 - r_3} \right)^{c_1 - 1}.
\]

**Proof of Theorem 1.** We first prove the sufficiency. Applying [16, Proposition 13], we obtain that

\[
(2.1) \quad \|D_f(z)\| \leq \frac{16Kd_\Omega(f(z))}{1 - |z|^2}.
\]
Also, by (1.1) and (2.1), for \(w = f(r\zeta)\) and \(w_1 = f(\rho\zeta)\), we have
\[
\sigma_\ell(w) = \int_r^\rho |df(t\zeta)| \leq \int_r^\rho \|Df(t\zeta)\| \, dt \\
\leq M(K)\|Df(r\zeta)\| \int_r^1 \left(\frac{1-t}{1-r}\right) \delta^{-1} \, dt, \text{ by (1.1)}, \\
= \frac{M(K)}{\delta}\|Df(r\zeta)\|(1-r) \\
\leq \frac{M(K)}{\delta}\|Df(r\zeta)\|(1-r^2) \\
\leq \frac{16KM(K)}{\delta}d_\Omega(w), \text{ by (2.1)},
\]
which implies that \(\Omega\) is a radial \((16KM(K)/\delta)\)-John disk with John center \(w_0 = f(0)\) and with \(\gamma = f([0, \rho\zeta])\) as the John curves, where \(r \in [0, 1)\), \(\rho \in [r, 1)\) and \(\zeta \in \partial \mathbb{D}\).

Now we prove the necessity. For \(z \in \mathbb{D}\), let
\[
\Delta = f^{-1}\left(\mathbb{D}(f(z), d_\Omega(f(z)))\right)
\]
and \(\phi\) be a conformal mapping of \(\mathbb{D}\) onto \(\Delta\) with \(\phi(0) = z\). Since \(\phi(\mathbb{D}) \subset \mathbb{D}\), we know that, for \(w \in \mathbb{D}\),
\[
|\phi'(w)| \leq \frac{1 - |\phi(w)|^2}{1 - |w|^2}.
\]
Then
\[
F(w) = \frac{1}{d_\Omega(f(z))}\left(f(\phi(w)) - f(z)\right)
\]
is a \(K\)-quasiconformal harmonic mapping of \(\mathbb{D}\) onto itself with \(F(0) = 0\). It is not difficult to know that
\[
\|D_F(w)\| = \frac{|\phi'(w)|\|D_f(\phi(w))\|}{d_\Omega(f(z))},
\]
which, together with (2.2) and Theorem A, give that
\[
\|D_f(z)\| = \|D_f(\phi(0))\| = \frac{d_\Omega(f(z))\|D_F(0)\|}{|\phi'(0)|} \\
\geq \frac{d_\Omega(f(z))\|D_F(0)\|}{1 - |z|^2} \\
\geq \frac{1 + K \, d_\Omega(f(z))}{2K} \frac{1}{1 - |z|^2}.
\]
(2.3)
Since \(\Omega\) is a radial John disk, we can choose \(w_0 = f(0)\) as the John center and \(\gamma = f([0, \rho\zeta])\) as the John curve; \(\Omega\) can be assumed to be a radial \(c\)-John disk with respect to this choice, where \(c \geq 1\). Hence for \(w = f(r\zeta)\) and \(w_1 = f(\rho\zeta)\), we have
\[
\sigma_\ell(w) \leq cd_\Omega(w) \text{ for all } \rho \in [r, 1).
\]
The boundedness of \( \Omega \) implies that \( d_{\Omega}(w) \) is finite for all \( w \in \Omega \). Hence the limit
\[
\lim_{\rho \to 1^{-}} \int_{r}^{\rho} |df(t\zeta)|
\]
does exist and is finite. By (2.4) and (2.5), we get
\[
\frac{1}{K} \int_{r}^{1} \|D_{f}(t\zeta)\|dt \leq \int_{r}^{1} l(D_{f}(t\zeta))dt \leq \int_{r}^{1} |df(t\zeta)| \leq cd_{\Omega}(w),
\]
where \( \zeta \in \partial \mathbb{D} \). By (2.3) and (2.6), we have
\[
\int_{r}^{1} \|D_{f}(t\zeta)\|dt \leq \frac{2cK^{2}}{1+K} (1-r^{2})\|D_{f}(r\zeta)\| \leq M_{0}(1-r)\|D_{f}(r\zeta)\|
\]
where \( M_{0} = \frac{4cK^{2}}{1+4K} \geq 2c \).

Next, we let \( \varphi(r) = (1-r) - \frac{1}{M_{0}} \int_{r}^{1} \|D_{f}(t\zeta)\|dt \).

By (2.7), we have
\[
\varphi'(r) = (1-r) - \frac{1}{M_{0}(1-r)} \left[ \frac{1}{M_{0}} \int_{r}^{1} \|D_{f}(t\zeta)\|dt - \|D_{f}(r\zeta)\| \right] \leq 0,
\]
which implies that \( \varphi(r) \) is decreasing on the unit interval \((0, 1)\).

By Theorem B, for \( \rho \leq t \leq 1+\rho^{2} \), there is a positive constant \( c_{1} \) such that
\[
\|D_{f}(\rho\zeta)\| \leq 4c_{1}\|D_{f}(t\zeta)\|,
\]
which gives
\[
\int_{r}^{1} \|D_{f}(t\zeta)\|dt \geq \int_{r}^{1} \|D_{f}(t\zeta)\|dt \geq 4^{-c_{1}}\|D_{f}(\rho\zeta)\| \int_{r}^{1} \|D_{f}(t\zeta)\|dt
\]
(2.8)
\[
= 2^{-2c_{1}-1}\|D_{f}(\rho\zeta)\|(1-\rho).
\]
For \( 0 \leq r \leq \rho < 1 \), by (2.7) and (2.8), we have
\[
(1-\rho)^{1-\frac{1}{M_{0}}} \|D_{f}(\rho\zeta)\| \leq 2^{1+2c_{1}} \varphi(\rho) \leq 2^{1+2c_{1}} \varphi(r)
\]
\[
\leq 2^{1+2c_{1}} M_{0}(1-r)^{1-\frac{1}{M_{0}}} \|D_{f}(r\zeta)\|,
\]
which yields
\[
\|D_{f}(\rho\zeta)\| \leq 2^{1+2c_{1}} M_{0}\|D_{f}(r\zeta)\| \left( \frac{1-r}{1-\rho} \right)^{1-\frac{1}{M_{0}}}
\]
\[
= 2^{1+2c_{1}} M_{0}\|D_{f}(r\zeta)\| \left( \frac{1-\rho}{1-r} \right)^{\frac{1}{M_{0}}-1}.
\]
The proof of the theorem is complete. \( \square \)

For \( z_{1}, z_{2} \in \mathbb{D} \), the hyperbolic metric (or Poincaré metric) is defined by
\[
\lambda_\mathbb{D}(z_1, z_2) = \min_\gamma \int_\gamma \frac{|dz|}{1 - |z|^2},
\]
where the minimum is taken over all curves \( \gamma \) in \( \mathbb{D} \) from \( z_1 \) and \( z_2 \). It is well-known that, for \( z_1, z_2 \in \mathbb{D} \),
\[
\lambda_\mathbb{D}(z_1, z_2) = \frac{1}{2} \log \frac{1 + |z_1 - z_2|/|1 - z_1 z_2|}{1 - |z_1 - z_2|/|1 - z_1 z_2|},
\]
which is equivalent to
\[
\left| \frac{z_1 - z_2}{1 - \overline{z}_1 z_2} \right| = \frac{e^{2\lambda_\mathbb{D}(z_1, z_2)} - 1}{e^{2\lambda_\mathbb{D}(z_1, z_2)} + 1} = \tanh \lambda_\mathbb{D}(z_1, z_2).
\]

In [21], Sheil-Small proved the following result.

**Lemma C.** Let \( f = h + \overline{g} \in \mathcal{S}_H \) and \( \alpha = \sup_{f \in \mathcal{S}_H} \frac{|h''(0)|}{2} \), where \( h \) and \( g \) are analytic in \( \mathbb{D} \). Then
\[
\frac{(1 - |z|)^{\alpha-1}}{(1 + |z|)^{\alpha+1}} \leq |h'(z)| \leq \frac{(1 + |z|)^{\alpha-1}}{(1 - |z|)^{\alpha+1}}.
\]

We remark that \( \alpha = \sup_{f \in \mathcal{S}_H} \frac{|h''(0)|}{2} \) is finite, but the sharp upper bound of \( \alpha \) is still unknown (see [8, 21]).

**Lemma 1.** Let \( f = h + \overline{g} \in \mathcal{S}_H \), where \( h \) and \( g \) are analytic in \( \mathbb{D} \). Then, for \( z_1, z_2 \in \mathbb{D} \),
\[
\frac{1}{2} \|D_f(z_1)\| e^{-(1+\alpha)\lambda_\mathbb{D}(z_1, z_2)} \leq \|D_f(z_2)\| \leq 2\|D_f(z_1)\| e^{(1+\alpha)\lambda_\mathbb{D}(z_1, z_2)},
\]
where \( \alpha \) is defined in Lemma C.

**Proof.** Let \( f = h + \overline{g} \in \mathcal{S}_H \) and \( z = \frac{z_2 - z_1}{1 - \overline{z}_1 z_2} \), where \( h, g \) are analytic in \( \mathbb{D} \) and \( z_1, z_2 \in \mathbb{D} \). Then
\[
F(z) = \frac{f(z_2) - f(z_1)}{(1 - |z|^2)h'(z_1)} \in \mathcal{S}_H,
\]
where \( z_2 = \frac{z + z_1}{1 + \overline{z}_1 z} \). By Lemma C, we get
\[
\frac{(1 - |z|)^{\alpha-1}}{(1 + |z|)^{\alpha+1}} \leq |F'(z)| = \frac{|h'(z_2)|}{|h'(z_1)||1 + \overline{z}_1 z|^2} \leq \frac{(1 + |z|)^{\alpha-1}}{(1 - |z|)^{\alpha+1}},
\]
which gives
\[
(1 - |z|)^{\alpha+1} |h'(z_1)| \leq |h'(z_2)| \leq (1 + |z|)^{\alpha+1} |h'(z_1)|.
\]
By (2.9), we obtain
\[
\frac{1}{2} \frac{(1 - |z|)^{\alpha+1}}{(1 + |z|)^{\alpha+1}} \|D_f(z_1)\| \leq \|D_f(z_2)\| \leq 2 \frac{(1 + |z|)^{\alpha+1}}{(1 - |z|)^{\alpha+1}} \|D_f(z_1)\|,
\]
which implies that
\[
\frac{1}{2} \|D_f(z_1)\| e^{-(1+\alpha)\lambda_\mathbb{D}(z_1, z_2)} \leq \|D_f(z_2)\| \leq 2 \|D_f(z_1)\| e^{(1+\alpha)\lambda_\mathbb{D}(z_1, z_2)}.\]
The proof of this lemma is complete. □

**Lemma 2.** Let $a_1, a_2$ and $a_3$ be positive constants and let $0 < |z_0| = 1 - \delta$, where $\delta \in (0, 1)$. If $f \in \mathcal{S}_H$, $0 \leq 1 - a_2 \delta \leq |z| \leq 1 - a_1 \delta$ and $|\arg z - \arg z_0| \leq a_3 \delta$, then

\[
\frac{1}{M(a_1, a_2, a_3)} \|D_f(z_0)\| \leq \|D_f(z)\| \leq M(a_1, a_2, a_3)\|D_f(z_0)\|,
\]

where $M(a_1, a_2, a_3) = 2e^{(1+\alpha)(a_3 + \frac{1}{2} \log \frac{2a_2 - a_1}{a_1})}$ and $\alpha$ is defined in Lemma C.

**Proof.** Let $\angle AOB = 2a_3 \delta$ and $z_1, z_2, z_3$ line in the line $OB$ with $|z_1| \leq |z_2| = |z_0| \leq |z_3|$, see Figure 1. Then the length of the circular arc from $z_0$ to $z_2$ is less than $a_3 \delta$.

![Figure 1](image)

By calculations, we have

\[
\lambda_B(z_0, z_2) < \frac{a_3 \delta}{1 - (1 - \delta)^2} = \frac{a_3}{2 - \delta} < a_3
\]

and

\[
\left| \frac{z_3 - z_1}{1 - z_1 \overline{z_3}} \right| = \frac{1 - a_1 \delta - (1 - a_2 \delta)}{1 - (1 - a_1 \delta)(1 - a_2 \delta)} = \frac{a_2 - a_1}{a_2 + a_1(1 - a_2 \delta)} \leq \frac{a_2 - a_1}{a_2}.
\]

Hence

\[
\lambda_B(z_0, z) \leq \lambda_B(z_0, z_2) + \lambda_B(z_2, z_1) \\
\leq \lambda_B(z_0, z_2) + \lambda_B(z_1, z_3) \\
\leq a_3 + \frac{1}{2} \log \frac{2a_2 - a_1}{a_1}.
\]

By Lemma 1, we see that

\[
\frac{1}{M(a_1, a_2, a_3)} \|D_f(z_0)\| \leq \|D_f(z)\| \leq M(a_1, a_2, a_3)\|D_f(z_0)\|,
\]

where $M(a_1, a_2, a_3)$ is defined as in the statement. □
Proof of Theorem 2. We first prove (c) ⇒ (b). Let \( z = r e^{i\theta} \in \mathbb{D} \) and \( r_1 e^{i\theta_1}, r_2 e^{i\theta_2} \in B(re^{i\theta}) \) with \( r_1 \leq r_2 \). Then, by (1.2), Lemma 2 and [16, Proposition 13], there is a positive constant \( M \) such that

\[
|f(r_2 e^{i\theta_2}) - f(r_1 e^{i\theta_1})| \leq |f(r_2 e^{i\theta_2}) - f(r e^{i\theta})| + |f(r_1 e^{i\theta_1}) - f(r e^{i\theta})| + |f(r_1 e^{i\theta_1}) - f(r_2 e^{i\theta_2})|
\]

\[
\leq \int_r^{r_2} \| D_f(r e^{i\theta}) \| \, dr + \int_r^{r_1} \| D_f(r e^{i\theta}) \| \, dr + r \int_{\gamma_0} \| D_f(r e^{i\theta}) \| \, dt \quad \text{(by Lemma 2)}
\]

\[
\leq M_2 \int_r^{r_2} \| D_f(r e^{i\theta}) \| \left( \frac{1 - \rho}{1 - r} \right) \delta^{-1} \, d\rho + M_2 \int_r^{r_1} \| D_f(r e^{i\theta}) \| \left( \frac{1 - \rho}{1 - r} \right) \delta^{-1} \, d\rho + Mr \int_{\gamma_0} \| D_f(r e^{i\theta}) \| \, dt
\]

\[
\leq \frac{2M_2}{\delta} \| D_f(r e^{i\theta}) \| (1 - r) + Mr \ell(\gamma_0) \| D_f(r e^{i\theta}) \| + \frac{2M_2}{\delta} \| D_f(r e^{i\theta}) \| (1 - r) + M|\theta_2 - \theta_1| \| D_f(r e^{i\theta}) \|
\]

\[
\leq \left( \frac{2M_2}{\delta} + 2\pi M \right) \| D_f(r e^{i\theta}) \| (1 - r)
\]

\[
\leq 16K \left( \frac{2M_2}{\delta} + 2\pi M \right) d_{\Omega}(f(z)), \quad \text{by [16, Proposition 13]},
\]

where \( \gamma_0 \) is the smaller subarc of \( \partial \mathbb{D} \) between \( re^{i\theta_1} \) and \( re^{i\theta_2} \). Hence there exists a positive constant \( M_1 \) such that, for all \( z \in \mathbb{D} \),

\[
diam f(B(z)) \leq M_1 d_{\Omega}(f(z)).
\]

Next we prove (b) ⇒ (c). For \( z = r e^{i\theta} \in \mathbb{D} \), let

\[
(2.10) \quad \phi(r) = \int_r^1 (1 - x) \| D_f(x e^{i\theta}) \|^2 \, dx
\]

and

\[
\Delta(r) = \{ \zeta = x + iy : r \leq x < 1, \ 0 \leq y \leq 1 - x \}.
\]

By Lemma 2, for \( \zeta = x + iy \in \Delta(r) \), there exists a positive constant \( M_3 \) such that

\[
\| D_f(x e^{i\theta}) \| \leq M_3 \| D_f(\zeta e^{i\theta}) \|,
\]
which implies that

\[(2.11) \quad \phi(r) \leq \int_r^1 \int_0^{1-x} \|D_f(xe^{i\theta})\|^2 dy dx \]

\[\leq M_3^2 \int_r^1 \int_0^{1-x} \|D_f(\zeta e^{i\theta})\|^2 dy dx \]

\[\leq KM_3^2 \int_r^1 \int_0^{1-x} f(\zeta e^{i\theta})dy dx \]

\[= KM_3^2 A(f(\Delta(re^{i\theta}))), \]

where

\[\Delta(re^{i\theta}) = \{\zeta e^{i\theta} = (x + iy)e^{i\theta} : r \leq x < 1, \ 0 \leq y \leq 1 - x\}.

It is not difficult to see that \(\Delta(re^{i\theta}) \subset B(re^{i\theta})\), which, together with (2.3) and (2.11), imply

\[(2.12) \quad \phi(r) \leq \frac{\pi KM_3^2}{4} \left(\frac{diam(f(B(re^{i\theta})))}{2}\right)^2 \]

\[\leq \frac{\pi KM_3^2 M_1^2}{4} (diam(f(z)))^2 \]

\[\leq \frac{\pi K^3 M_1^2 M_2^2}{(1 + K)^2} (1 - |z|^2)^2 \|D_f(z)\|^2. \]

By (2.10), for \(r \leq \rho < 1\), we get

\[(2.13) \quad \log \frac{\phi(\rho)}{\phi(r)} = \int_r^\rho \frac{\phi'(t)}{\phi(t)} dt \leq -\alpha \int_r^\rho \frac{dt}{1 - t} = \alpha \log \frac{1 - \rho}{1 - r}, \]

where \(\alpha = (1 + K)^2/[\pi K^3 M_1^2 M_2^2]\). For \(\rho \leq x \leq \frac{1 + \rho}{2}\), by Theorem B, there is a positive constant \(c_1^*\) such that

\[(2.14) \quad \|D_f(\rho e^{i\theta})\| \leq 4c_1^* \|D_f(xe^{i\theta})\|.

Applying (2.10), (2.13) and (2.14), we have

\[\frac{1 - \rho}{2c_1^*} \|D_f(\rho e^{i\theta})\|^2 = \frac{1 - \rho}{2c_1^*} \int_\rho^1 (1 - x)\|D_f(\rho e^{i\theta})\|^2 dx \]

\[\leq \int_\rho^1 (1 - x)\|D_f(xe^{i\theta})\|^2 dx = \phi(\rho) \]

\[\leq \phi(r) \left(\frac{1 - \rho}{1 - r}\right)^\alpha, \]

which, together with (2.12), yield that

\[\frac{1 - \rho}{2c_1^*} \|D_f(\rho e^{i\theta})\|^2 (1 - \rho)^2 \leq \phi(r) \left(\frac{1 - \rho}{1 - r}\right)^\alpha \leq \frac{1}{\alpha} (1 - \rho)^2 \|D_f(\rho e^{i\theta})\|^2 \left(\frac{1 - \rho}{1 - r}\right)^\alpha. \]
Then we conclude that

\[(2.15) \quad \|D_f(\rho e^{i\theta})\| \leq \sqrt{\frac{2^{1+4\gamma}}{\alpha}} \|D_f(z)\| \left(\frac{1-\rho}{1-r}\right)^{\frac{\alpha}{2}-1}.\]

By (2.15) and Lemma 2, for all \(\zeta = \rho e^{i\eta} \in B(z)\), there exists a positive constant \(M_4\) such that

\[
\|D_f(\rho e^{i\eta})\| \leq M_4 \|D_f(\rho e^{i\theta})\| \leq M_4 \sqrt{\frac{2^{1+4\gamma}}{\alpha}} \|D_f(z)\| \left(\frac{1-\rho}{1-r}\right)^{\frac{\alpha}{2}-1}.
\]

Now we prove (a)⇒(c). By Theorem 1, there are constants \(M\) and \(\delta \in (0,1)\) such that for each \(\zeta \in \partial \mathbb{D}\) and for \(0 \leq r \leq \rho < 1\),

\[(2.16) \quad \|D_f(\rho \zeta)\| \leq M \|D_f(r \zeta)\| \left(\frac{1-\rho}{1-r}\right)^{\delta-1}.
\]

For all \(\xi \in \partial \mathbb{D}\) with \(|\arg \xi - \arg \zeta| \leq \pi(1-r)\), by Lemma 2, there is a positive constant \(M'\) such that

\[(2.17) \quad \|D_f(r \zeta)\| \leq M' \|D_f(r \xi)\|.
\]

Hence (1.2) follows from (2.16) and (2.17).

At last, we prove (c)⇒(a). By [16, Proposition 13], (1.2), for \(w = f(r \zeta)\) and \(w_1 = f(\rho \zeta)\), we have

\[
\sigma_\ell(w) = \int_r^\rho |df(t \zeta)| dt \leq \int_r^\rho \|D_f(t \zeta)\| dt \leq M_2 \|D_f(r \zeta)\| \int_r^\rho \left(\frac{1-t}{1-r}\right)^{\delta-1} dt = \frac{M_2}{\delta} \|D_f(r \zeta)\| (1-r) \leq \frac{16KM_2}{\delta} d_\Omega(w),
\]

which implies that \(\Omega\) is a radial \((16KM_2/\delta)\)-John disk with John center 0 and with \(\gamma = f([0, \rho \zeta])\) as the John curves, where \(r \in [0,1)\), \(\rho \in [r,1)\) and \(\zeta \in \partial \mathbb{D}\). The proof is complete. \(\square\)

**Proof of Proposition 3.** Without loss of generality, we assume that there is a positive constant \(M^*_1\) such that, for all \(z \in \mathbb{D}\),

\[(2.18) \quad \text{diam} f(B(z)) \leq M^*_1 d_\Omega(f(z)),\]

where \(\Omega = f(\mathbb{D})\). For \(r \in [0,1)\), let

\[(2.19) \quad \varphi(r) = \frac{1}{2\pi} \int_0^{2\pi} \left( |f_z(re^{it})|^2 + |f_{\pi}(re^{it})|^2 \right) dt = 1 + \sum_{n=2}^\infty n^2 (|a_n|^2 + |b_n|^2) r^{2n-2}.\]
Then, by Theorem A and (2.18), we obtain
\begin{align}
\int_1^r \int_{-\pi(1-r)}^{\pi(1-r)} J_f(\rho e^{i(\theta + t)}) \rho d\theta d\rho &= A\left( f(B(re^{it})) \right) \\
&\leq \frac{\pi}{4} \text{diam}^2 \left( f(B(re^{it})) \right) \\
&\leq M^*(1 - r^2) \| D_f(re^{it}) \|^2,
\end{align}
where $M^* = \frac{\pi K^2 M_1^2}{(1 + K)^2}$.

By (2.20), for $r \in \left[ \frac{1}{2}, 1 \right)$, we obtain
\begin{align*}
\frac{1}{2K} \int_1^r \int_{-\pi(1-r)}^{\pi(1-r)} \varphi(\rho) d\theta d\rho &\leq \frac{1}{K} \int_1^r \rho \left( \int_0^{2\pi} \| D_f(\rho e^{i(t+\theta)}) \|^2 dt \right) d\theta d\rho \\
&\leq \int_r^1 \int_{-\pi(1-r)}^{\pi(1-r)} \rho \left( \int_0^{2\pi} J_f(\rho e^{i(t+\theta)}) \right) d\theta d\rho \\
&\leq 4M^*(1 - r)^2 \int_0^{2\pi} \| D_f(re^{it}) \|^2 dt \\
&\leq 16\pi M^*(1 - r)^2 \varphi(r),
\end{align*}
which gives that
\begin{equation}
\int_r^1 \varphi(\rho) d\rho \leq 16KM^*(1 - r) \varphi(r) = \beta(1 - r) \varphi(r),
\end{equation}
where $\beta = 16KM^*$. Applying (2.21), for $r \in \left[ \frac{1}{2}, 1 \right)$, we get
\begin{equation}
\frac{d}{dr} \left[ (1 - r)^{-2\beta_0} \int_r^1 \varphi(\rho) d\rho \right] = \frac{1}{2\beta_0} (1 - r)^{-2\beta_0 - 1} \int_r^1 \varphi(\rho) d\rho - (1 - r)^{-2\beta_0} \varphi(r) \leq 0,
\end{equation}
where $\beta_0 = 1/(2\beta)$. By (2.22), for $r \in \left[ \frac{1}{2}, 1 \right)$, we have
\begin{equation}
(1 - r)^{1 - 2\beta_0} \varphi(r) \leq (1 - r)^{-2\beta_0} \int_r^1 \varphi(\rho) d\rho \leq 2^{-2\beta_0} \int_{\frac{1}{2}}^1 \varphi(\rho) d\rho < \infty.
\end{equation}

It follows from (2.19) and (2.23) that there are two positive constants $M'_1$ and $M''_1$ such that
\begin{align*}
1 + \sum_{n=2}^{\infty} n^{1 + \beta_0} (\| a_n \|^2 + \| b_n \|^2) &\leq M'_1 \int_{\frac{1}{2}}^1 (1 - r)^{-\beta_0} \varphi(r) dr \\
&\leq M''_1 \int_{\frac{1}{2}}^1 (1 - r)^{\beta_0 - 1} dr < \infty.
\end{align*}
The proof of this proposition is complete. \qed
Lemma 3. Let \( f \in S_H \) be a \( K \)-quasiconformal harmonic mapping from \( \mathbb{D} \) onto a bounded domain \( G \). If there are constants \( M \) and \( \delta \in (0,1) \) such that for each \( \varsigma \in \partial \mathbb{D} \) and for \( 0 \leq r \leq \rho < 1 \),
\[
\| D_f(\rho \varsigma) \| \leq M \| D_f(r \varsigma) \| \left( \frac{1 - \rho}{1 - r} \right)^{\delta - 1},
\]
then, for \( a \in \mathbb{D} \), we have
\[
\text{diam}(f(I(a))) \leq M'_0 d_G(a),
\]
where
\[
I(a) = \{ z \in \partial \mathbb{D} : |\arg z - \arg a| \leq 1 - |a| \}
\]
and
\[
M'_0 = 32K \left( 2e^{(1+\alpha)} + \frac{M2e^{(1+\alpha)}}{\delta} + \frac{M}{\delta} \right).
\]

Proof. For \( a \in \mathbb{D} \), let \( a = \rho \zeta \) with \( \rho = |a| \). For \( z \in I(a) \), by (2.24) and Lemma 2, we have
\[
|f(z\rho) - f(\rho \zeta)| \leq \int_{\gamma'} \rho \| D_f(\rho \zeta) \| |d\xi|
\]
\[
\leq 2e^{(1+\alpha)} \rho \int_{\gamma'} \| D_f(\rho \zeta) \| |d\xi|, \quad \text{by Lemma 2},
\]
\[
= 2e^{(1+\alpha)} \rho \ell(\gamma') \| D_f(\rho \zeta) \|
\]
\[
= 2e^{(1+\alpha)} \rho^2 \| D_f(\rho \zeta) \| |\arg(\rho \zeta) - \arg z|
\]
\[
\leq 2e^{(1+\alpha)} \rho^2 (1 - \rho) \| D_f(\rho \zeta) \|
\]
\[
\leq 2e^{(1+\alpha)} (1 - \rho) \| D_f(\rho \zeta) \|
\]
(2.25)

\[
|f(z\rho) - f(z)| \leq \int_{\rho}^{1} \| D_f(tz) \| dt
\]
\[
\leq M \int_{\rho}^{1} \| D_f(\rho z) \| \left( \frac{1 - t}{1 - \rho} \right)^{\delta - 1} dt, \quad \text{by (2.24)},
\]
\[
= \frac{M}{\delta} (1 - \rho) \| D_f(\rho z) \|
\]
\[
\leq \frac{2Me^{(1+\alpha)}}{\delta} (1 - \rho) \| D_f(\rho \zeta) \|
\]
(2.26)

and

\[
|f(\zeta \rho) - f(\zeta)| \leq \int_{\rho}^{1} \| D_f(t\zeta) \| dt
\]
\[
\leq M \int_{\rho}^{1} \| D_f(\rho \zeta) \| \left( \frac{1 - t}{1 - \rho} \right)^{\delta - 1} dt, \quad \text{by (2.24)},
\]
\[
= \frac{M}{\delta} (1 - \rho) \| D_f(\rho \zeta) \|
\]
(2.27)
where $\gamma'$ is the smaller subarc of $\partial D_\rho$ between $\rho z$ and $\rho \zeta$.
Again, for $z \in I(a)$, by (2.1), (2.25), (2.26) and (2.27), we obtain
\[
|f(\zeta) - f(z)| \leq |f(\rho \zeta) - f(\rho z)| + |f(z) - f(\rho z)| + |f(\rho \zeta) - f(\zeta)| \\
\leq M_1^*(1 - \rho)\|Df(\rho \zeta)\| \\
\leq 16M_1^*Kd_G(a), \text{ by (2.1)},
\]
which in turn implies that $\text{diam} f(I(a)) \leq 32K M_1^*d_G(a)$, where
\[
(2.28) \quad M_1^* = 2e^{(1+\alpha)} + \frac{M 2e^{(1+\alpha)}}{\delta} + \frac{M}{\delta}.
\]
The proof of the lemma is complete. \hfill \Box

**Proof of Theorem 4.** Let $\frac{1}{2} < \nu < 1$ and
\[
(2.29) \quad \sup_{0 < r < 1} \left\{ \sup_{w_1, w_2 \in \gamma_r} \frac{\ell(\gamma_r[w_1, w_2])}{d_G(w_1, w_2)} \right\} = M_\gamma,
\]
where $\gamma_r$ is given by (1.3). Then, by (2.29), Lemma 3 and [7, Theorem 3], we have
\[
\nu \int_0^{2\pi} \|Df(\nu e^{i\theta})\|d\theta \leq \nu \int_0^{2\pi} l(Df(\nu e^{i\theta}))d\theta \\
\leq \int_0^{2\pi} |df(\nu e^{i\theta})| \\
\leq \sum_{k=1}^{7} \int_{I(z_k)} |df(\nu e^{i\theta})| \\
\leq M_\gamma \sum_{k=1}^{7} \text{diam} f(I(z_k)), \text{ by (2.29)},
\]
\[
\leq 32M_\gamma M_1^*K \sum_{k=1}^{7} d_G(f(z_k)), \text{ by Lemma 3},
\]
\[
\leq 64M_\gamma M_1^*K \sum_{k=1}^{7} \{(1 - |z_k|^2)\|Df(z_k)\|\} \\
\leq \frac{1792M_\gamma M_1^*K}{(1 + K)\pi}, \text{ by [7, Theorem 3]},
\]
which implies that $\|Df\| \in H^1_\nu(\mathbb{D})$, where $k \in \{1, 2, \ldots, 7\}$,
\[
z_k = \frac{1}{2}e^{i(k-1)}, \quad I(z_k) = \{z \in \partial \mathbb{D} : |\arg z - \arg z_k| \leq 1 - |z_k|\},
\]
and $M_1^*$ is given by (2.28). The proof of the theorem is complete. \hfill \Box
Proof of Theorem 5. By the assumption, we see that there is a \( \nu \in (0, 1) \) and \( r_0 \in (0, 1) \) such that, for \( r_0 \leq \eta < 1 \),

\[
\frac{\nu}{1 - \eta^2} \geq \text{Re} \left( \zeta P_f(\eta \zeta) \right) = \text{Re} \left( \frac{\zeta h''(\eta \zeta)}{h'(\eta \zeta)} \right) - \text{Re} \left( \frac{\zeta \omega'(\eta \zeta) \omega(\eta \zeta)}{1 - |\omega(\eta \zeta)|^2} \right),
\]

which shows that

\[
\text{Re} \left( \frac{\zeta h''(\eta \zeta)}{h'(\eta \zeta)} \right) \leq \text{Re} \left( \frac{\zeta \omega'(\eta \zeta) \omega(\eta \zeta)}{1 - |\omega(\eta \zeta)|^2} \right) + \frac{\nu}{1 - \eta^2},
\]

where \( \zeta \in \partial \mathbb{D} \). By Schwarz-Pick’s lemma, we obtain

\[
|\omega'(\eta \zeta)| \leq \frac{1 - |\omega(\eta \zeta)|^2}{1 - \eta^2}.
\]

By (2.30) and (2.31), we have

\[
\text{Re} \left( \frac{\zeta h''(\eta \zeta)}{h'(\eta \zeta)} \right) \leq 1 + \frac{\nu}{1 - \eta^2}.
\]

Choosing \( \lambda \in (0, 1 - \nu) \), there is an \( r_1 \in [r_0, 1) \) such that

\[
\text{Re} \left( \frac{\zeta h''(\eta \zeta)}{h'(\eta \zeta)} \right) < \frac{2\eta - 2\lambda}{1 - \eta^2} \quad \text{for all } \zeta \in \partial \mathbb{D},
\]

when \( \eta \in [r_1, 1) \). For \( 0 \leq r_1 \leq r < \rho < 1 \), by (2.32), we find that

\[
\log \left[ \frac{(1 - \rho^2)|h'(\rho \zeta)|}{(1 - r^2)|h'(r \zeta)|} \right] = \int_r^\rho \left[ \text{Re} \left( \frac{\zeta h''(\eta \zeta)}{h'(\eta \zeta)} \right) - \frac{2\eta}{1 - \eta^2} \right] d\eta
\]
\[
< -2\lambda \int_r^\rho \frac{d\eta}{1 - \eta^2}
\]
\[
= -\lambda \log \left( \frac{1 + \rho}{1 + r} \cdot \frac{1 - r}{1 - \rho} \right),
\]

which implies that

\[
\frac{|h'(\rho \zeta)|}{|h'(r \zeta)|} < \left( \frac{1 + r}{1 + \rho} \right)^{1+\lambda} \left( \frac{1 - \rho}{1 - r} \right)^{\lambda-1} \leq \left( \frac{1 - \rho}{1 - r} \right)^{\lambda-1}.
\]

By (2.33), we get

\[
\|D_f(\rho \zeta)\| \leq \frac{2K}{1 + K} |h'(\rho \zeta)| < \frac{2K}{1 + K} |h'(r \zeta)| \left( \frac{1 - \rho}{1 - r} \right)^{\lambda-1}
\]
\[
\leq \frac{2K}{1 + K} \|D_f(r \zeta)\| \left( \frac{1 - \rho}{1 - r} \right)^{\lambda-1}.
\]

In order to apply Theorem 1 and then to conclude that \( \Omega_1 = f(\mathbb{D}) \) is a radial John disk, we will use some proof techniques as in the proof of [9, Theorem 3.7] to remove
the restriction \( r \geq r_1 \) above. For \( 0 \leq r_1 \leq r \leq \rho < 1 \), by (2.1) and (2.34), we see that there is a constant \( c(\lambda) > 1 \) such that

\[
\sigma_r(w) \leq c(\lambda)d_{\Omega_1}(w),
\]

where \( w_1 = f(\rho \zeta), \ w = f(r \zeta) \) and \( \gamma = f([0, \rho \zeta]) \). It follows from (2.35) that

\[
\text{diam}(\gamma[w_1, w]) \leq c(\lambda)d_{\Omega_1}(w).
\]

Now we consider the case: \( 0 \leq r \leq r_1 \leq \rho < 1 \). Let \( \delta_0 = \text{dist}(f(\overline{D}_{r_1}), \partial \Omega_1) \) denote the Euclidean distance from \( f(\overline{D}_{r_1}) \) to the boundary \( \partial \Omega_1 \) of \( \Omega_1 \) and let \( \lambda_0 = \text{diam}(f(\overline{D}_{r_1})) \). Then

\[
\delta_0 > 0 \text{ and } \lambda_0 < \infty.
\]

For \( 0 \leq r \leq r_1 \leq \rho < 1 \), by the triangle inequality, (2.36) and (2.37), we get

\[
\text{diam}(\gamma[w, w_1]) \leq \text{diam}(\gamma[w, w_0]) + \text{diam}(\gamma[w_0, w_1])
\leq \lambda_0 + c(\lambda)d_{\Omega_1}(w_0)
\leq \lambda_0 + c(\lambda)(\lambda_0 + \delta_0)
\leq (c(\lambda) + c')\delta_0
\leq (c(\lambda) + c')d_{\Omega_1}(w),
\]

where \( w_1 = f(\rho \zeta), \ w = f(r \zeta), \ w_0 = f(r_1 \zeta) \) and \( c' = (1 + c(\lambda))\lambda_0/\delta_0 \).

The remaining case when \( 0 \leq r \leq \rho \leq r_1 < 1 \) is treated similarly. Therefore, for \( 0 \leq r \leq \rho < 1 \), there is a constant \( c_2 > 1 \) such that

\[
\text{diam}(\gamma[w, w_1]) \leq c_2d_{\Omega_1}(w),
\]

which implies that \( \text{car}_d(\gamma, c_2) \subset \Omega_1 \) (cf. [17]), where

\[
\text{car}_d(\gamma, c_2) = \bigcup \left\{ \mathbb{D}(w, \text{diam}(\gamma[w, w_1])/c_2) : w \in \gamma \setminus \{f(0), w_1\} \right\}.
\]

It follows from [17, Theorem 2.16] and [17, Part 2.26 in P.17] that \( \Omega_1 \) is a John disk. For the definition of the diameter of c-carrot, denoted by \( \text{car}_d(\gamma, c) \), we refer to [17]. The proof of the theorem is complete.

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