DERIVATION OF INDEX THEOREM BY SUPERSYMMETRY

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Abstract. The present paper gives calculations in detail to prove several special cases of Atiyah-Singer theorem through supersymmetric σ-models. Some technical tricks are employed to calculate the determinants of fluctuation operators of the path integrals. An intuitive and geometric argument is applied to overcome the complicated calculation on spin fields twisted by gauge fields.

1. INTRODUCTION

Topological invariants were defined in [1] for the theories in which supersymmetry is not spontaneously broken. It was suggested in that paper that the topological invariant $Tr(-1)^F$ plays an important role in the index theorem, which is an modern development of Riemann-Roch theorem. Since then, there had been a lot of intensive work on the relationship between supersymmetry and topological information of manifolds. Another influential paper [2] showed that Morse theory can be derived as conclusions of a supersymmetric theory.

On the other hand, it had been well known that anomalies in many situations are intimately connected to the index theorem. Fujikawa gave a very direct method to derive the gauge anomaly of a spin field, which explains the topological origin of anomalies most clearly [3],[4]. Gravitational anomaly was derived in [5] using Fujikawa’s method. [6] pointed out the relationship between gravitational anomalies and the index theorem. Gravitational anomalies of spin $\frac{1}{2}$ and spin $\frac{3}{2}$ fermions and self-dual gauge fields are considered in [6]. Then several papers were also concerned with this topic. [5] gave the method to calculate the index density for different complexes. Although the idea is simple, the calculations to realize the idea are complicated. [15] gave the calculation of Dirac genus by considering a $0 + \frac{1}{2}$ nonlinear σ model. [12],[13] gave the same result by using Green function which is essentially the same as the method of heat kernel. The present paper derives the Hirzebruch signature theorem and the Hirzebruch-Riemann-Roch theorem in detail. In the derivation of the Hirzebruch-Riemann-Roch theorem, we need to consider a supersymmetric theory coupled with gauge field. Some papers got the index of twisted spin field (twisted by gauge field) using very complicated calculation. However we will give a very simple idea, which can be applied to any gauge coupled index, to overcome this difficulty. In the first section of the present paper, we review the procedure of calculating index by supersymmetric theories. In the second section, we give the full calculation of the Hirzebruch signature theorem and Hirzebruch-Riemann-Roch theorem. In the last section the conclusion is drawn and further questions are asked.

Key words and phrases. supersymmetry, index theorem, nonlinear σ-model.
In this section, let’s review the basic procedure to calculate the index by a nonlinear supersymmetric $\sigma$ model. We will see how this model is connected with the Atiyah-Singer theorem. We follow the literature on this topic which are mentioned in the introduction part of this paper. Suppose we have supersymmetric theory and the supercharges are $Q_1, Q_2, ..., Q_k$ satisfying

$$Q_i^2 = Q_2^2 = ... = Q_k^2 = H,$$

$$Q_i Q_j + Q_j Q_i = 0(i \neq j).$$

(2.1)

We can choose one $Q_i$ and denote it by $Q$. An important fact is that $Q$ pairs a nonzero energy bosonic state with a fermionic state. Specifically, let $|b\rangle$ be a bosonic state with a nonzero energy $E$. Then define a fermionic state $|f\rangle = \frac{1}{\sqrt{E}}|b\rangle$. Since $Q^2 = H$, $|f\rangle$ also satisfies $Q|f\rangle = \sqrt{E}|b\rangle$ and $H|f\rangle = E|f\rangle$. On the other hand, obviously, if $E$ is zero, the construction above can’t hold which means zero energy states can’t be paired as bosons and fermions. We denote the number of zero-energy bosonic states by $n_B^0$ and the number of zero-energy fermionic states by $n_F^0$. As the parameters of the theory vary, a nonzero state maybe becomes a zero state. Its superpartner must also become a zero state. In the case, both $n_B^0$ and $n_F^0$ increase by 1. On the other hand, if a zero-energy state becomes a non-zero energy state, since every non-zero state must be paired with its superpartner, there must also exist another zero-energy state, which is the superpartner of the zero-energy state mentioned, becoming a non-zero energy state. In one word, $n_B^0 - n_F^0$ is an invariant of the parameters of the theory. We introduce the operator $(-1)^F$, where $F$ is the fermionic number operator. Then it is obvious that

$$n_B^0 - n_F^0 = Tr(-1)^F.$$  

(2.2)

In addition, the following important equality holds because non-zero energy states are paired.

$$Tr(-1)^F e^{-\beta H},$$

(2.3)

where $\beta \leq 0$. The above doesn’t depend on $\beta$.

The Hilbert space of the theory can be split into the bosonic subspace $H_B$ and the fermionic subspace $H_F$. Therefore $Q$ can be written in the form

$$Q = \begin{pmatrix} 0 & D^+ \\ D & 0 \end{pmatrix}.$$  

(2.4)

Now we can see that $n_B^0$ is the dimension of the space $D\phi = 0$ and the $n_F^0$ is the dimension of the space $D^+ \phi = 0$.

By above, we have

$$\text{ind} D = \ker D - \ker D^+ = Tr(-1)^F e^{-\beta H}.$$  

(2.5)

The right hand side of (2.5) can be calculated by path integrals if we change the time to negative imaginary time. Since the existence of $(-1)^F$ in the right hand side of (2.5), we need to impose periodic conditions on bosonic variables and fermionic variables.
Before we get into next section, let’s briefly go over the Atiyah-Singer theorem [7], [9], [8]: Let \((E, D)\) be an elliptic complex over an \(m\)-dimensional compact manifold \(M\) without a boundary. The index of this complex is given by

\[
\text{ind}(E, D) = (-1)^{m(m+1)/2} \int_M \text{ch}(\oplus_r(-1)^r E_r) \frac{\text{Td}(TM^C)}{e(TM)} |_{\text{vol}}.
\]

This theorem unifies all the underlying theorems. There have been different mathematical proofs for this theorem based on \(K\) theory, heat kernel and etc.

3. Calculations on different complexes

3.1. Hirzebruch signature theorem. Let us review the formula of Hirzebruch signature. Let \(M\) be a compact orientable manifold of even dimension which is \(m = 2l\). Then we can define a bilinear map \(H^l(M; \mathbb{R}) \times H^l(M; \mathbb{R}) \to \mathbb{R}\) by

\[
\sigma([\omega], [\eta]) = \int_M \omega \wedge \eta.
\]

By Poincaré duality, \(\sigma\) is actually a non-degenerate. If \(l\) is an even number, \(\sigma\) is symmetric obviously. Thus it doesn’t have zero eigenvalue. Therefore the sum of the number of positive eigenvalues \(b^+\) and the number of negative eigenvalues \(b^-\) is \(l\). The Hirzebruch signature is defined by

\[
\tau(M) \equiv b^+ - b^-.
\]

If \(l\) is odd, \(\tau(M)\) is defined to be zero. Now consider the operator which is

\[
\mathfrak{D} = d + d^+.
\]

We also define an operator \(\pi : \Omega^r(M)^C \to \Omega^{m-r}(M)^C\) by

\[
\pi = i^{r(r-1)+l\ast}.
\]

It is easy to check that \(\pi^2 = 1\) and that \(\pi\) anticommutes with \(\mathfrak{D}\). It has two eigenvalues which are 1 and -1. Now the space can be written in the form of a direct sum of the two eigenspaces:

\[
\Omega^*(M)^C = \Omega^+(M) \oplus \Omega^-(M).
\]

Now the signature complex is defined by the restriction of \(\mathfrak{D}\) on \(\Omega^+(M)\) which is

\[
\mathfrak{D}_+ : \Omega^+(M) \to \Omega^-(M).
\]

If we apply Atiyah-Singer index theorem to the signature complex, we can get the index of \(\mathfrak{D}_+\) immediately:

\[
\text{ind}\mathfrak{D}_+ = (-l)^l \int_M \text{ch}(\wedge^+ T^* M^C - \wedge^- T^* M^C) \frac{\text{Td}(TM^C)}{e(TM)} |_{\text{vol}}
\]

\[
= 2^l \int_M \prod_{i=1}^l \frac{x_i/2}{\tanh x_i/2} |_{\text{vol}}
\]

\[
= \int_M \prod_{i=1}^l \frac{x_i}{\tanh x_i} |_{\text{vol}}
\]

\[
= \int_M L(TM) |_{\text{vol}}.
\]
The last equality is the definition of $L$-polynomial. It is easy to see that if $m \equiv 2 \mod 4$, $\tau(M)$ is zero.

Now let us see how we can derive this formula for a $0 + 1$ nonlinear $\sigma$ model. Consider a theory which is

$$
L = \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j + \frac{1}{2} g_{ij}(x) \bar{\psi}^i \gamma^0 \frac{D}{dt} \psi^j + \frac{1}{12} R_{ijkl} \bar{\psi}^i \psi^k \bar{\psi}^j \psi^l.
$$

(3.2)

The Hamiltonian for this Lagrangian is

$$
H = \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j - \frac{1}{4} R_{ijkl} \bar{\psi}^i \psi^k \bar{\psi}^j \psi^l.
$$

(3.3)

The Lagrangian has a discrete symmetry:

$$
\psi^j \rightarrow \gamma_5 \psi^j.
$$

(3.4)

As pointed out in [5], $\text{Tr} Q_5 e^{-\beta H}$ depends on only zero modes and is independent of $\beta$. The supersymmetric transformations are

$$
\delta x^i = \bar{\epsilon}^i,
$$

$$
\delta \psi^j = \gamma^0 \bar{\epsilon}^j \psi^j - \Gamma^j_{ik} \bar{\epsilon}^i \psi^k.
$$

(3.4)

$\text{Tr} Q_5 e^{-\beta H}$ can be written in terms of path integral

$$
\text{Tr} Q_5 e^{-\beta H} = \int_{PBC} Dx D\psi_2 \int_{ABC} D\psi_1 e^{-S_E},
$$

(3.5)

where $\psi_1$ and $\psi_2$ belong to eigenspaces of eigenvalues 1 and -1 of $\gamma_5$. Notice we have to impose antiperiodic boundary condition on fermionic variables when we calculate tr. Therefore we have the above boundary condition because $Q_5$ is included in the calculation. Since $x^i$s and $\psi_2^i$s are periodic functions, we can expand them by Fourier series

$$
x^i(t) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{\infty} \xi^i_n e^{2\pi i n \beta t}, \psi_2^i = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{\infty} \psi_2^i_n e^{2\pi i n \beta t}.
$$

We expand the action $S$ around constant $(x, \psi_2)$ (fix $\psi_1$)

$$
L^{(2)} = \frac{1}{2} g_{ij}(x_0) \dot{x}^i \dot{x}^j + \frac{1}{2} (\psi_2^i - \psi_2^{i 20}) \frac{d}{dt} (\psi_2^i - \psi_2^{i 20}) g_{ij}(x) + \frac{1}{2} \psi_2^i \frac{d}{dt} g_{ij}(x_0) + \frac{1}{4} R_{ijkl} \psi_2^i \psi_2^j \psi_2^i \psi_2^j;
$$

(3.6)

the fluctuation operator for $\delta x = x - x_0$ is

$$
- \delta x \frac{d^2}{dt^2} + \frac{1}{2} R_{ijkl} \psi_2^i \psi_2^j \psi_2^i \psi_2^j \frac{d}{dt}.
$$

(3.7)

The fluctuation operator for $\delta \psi_2 = \psi_2 - \psi_2^{20}$ is

$$
\delta \psi_2 \frac{d}{dt},
$$

(3.8)

the fluctuation operator for $\psi_1$ is (we need to remind that the boundary condition for this operator is antiperiodic)

$$
\frac{d}{dt} + \frac{1}{2} R_{ijkl} \psi_2^i \psi_2^j.
$$

(3.9)
Now we can calculate the index of the operator $\mathbb{D}_+$. We have

$$
\text{Ind}\mathbb{D}_+ = \mathcal{N} \int \prod_{\mu=1}^{d} \frac{x_0}{\sqrt{2\pi}} d\psi_0 d\bar{\psi}_0 [\text{Det}_{PBC}'(-\delta_{ij} \frac{d^2}{dt^2} + \frac{1}{2} R_{ij} \frac{d}{dt})]^{-\frac{1}{2}} \times [\text{Det}_{PBC}'(\delta_{ij} \frac{d}{dt})]^\frac{1}{2} \times [\text{Det}_{APBC}'(\frac{d}{dt} + \frac{1}{2} R_{ij})]^\frac{1}{2} \times [\text{Det}_{APBC}'(\delta_{ij} \frac{d}{dt})]^\frac{1}{2} \times [\text{Det}_{APBC}'(\alpha_{ij} \frac{d}{dt})]^\frac{1}{2} \times [\text{Det}_{APBC}'(\beta_{ij} \frac{d}{dt})]^\frac{1}{2} \\
= \mathcal{N} \int \prod_{\mu=1}^{d} \frac{x_0}{\sqrt{2\pi}} d\psi_0 d\bar{\psi}_0.
$$

(3.10)

In order to calculate the right hand side of the equality above, we need to calculate $\mathcal{N}$ and the three determinants. First, let us calculate $\mathcal{N}_b$ which is the normalization factor of the second determinant in the integral. By the amplitude $|x,0; x,1\rangle = \frac{1}{\sqrt{2\pi}}$, we have a path integral

$$
\int \mathcal{N}_x e^{-\frac{1}{2} \int_0^1 \dot{x}^2 dt^2} = \mathcal{N}(\text{Det}_{\delta_{ij} + \beta \dot{t}})^{-\frac{1}{2}} \int \Pi_{\mu}^{2n} d\mu = (\sqrt{2\pi})^{-n} \int \prod_{\mu}^{2n} dx^\mu.
$$

(3.11)

Since the eigenvalues of $-\frac{d^2}{dt^2}$ are $\lambda_n = (\frac{2n\pi}{\beta})^2$, we have

$$
\text{Det}_{PBC}'(-\frac{d^2}{dt^2}) = \prod_{n \in \mathbb{Z}, n \neq 0} \left(\frac{2n\pi}{\beta}\right)^2.
$$

(3.12)

The spectral $\zeta$-function is

$$
\zeta_{-d^2/dt^2}(s) = \sum_{n \in \mathbb{Z}, n \neq 0} \left[\left(\frac{2n\pi}{\beta}\right)^2\right]^{-s} = 2\left(\frac{\beta}{2\pi}\right)^{2s} \zeta(2s).
$$

(3.13)

We have

$$
\zeta_{-d^2/dt^2}'(0) = 4 \log(\beta/2\pi) e^{2s \log(\beta/2\pi)} \zeta(2s) + 4e^{2s \log(\beta/2\pi)} \zeta'(2s)|_{s=0} = 4[\log(\beta/2\pi) \zeta(0) + \zeta'(0)] = -2 \log \beta.
$$

(3.14)

Hence the determinant is

$$
\text{Det}_{PBC}'(-\frac{d^2}{dt^2}) = e^{-\zeta_{-d^2/dt^2}'(0)} = \beta^2.
$$

(3.15)

If $\beta = 1$, $\text{Det}_{PBC}'(-d^2/dt^2) = 1$. Thus $\mathcal{N}_b = 1$. Let us evaluate $\mathcal{N}_x$. We know that as a fluctuation operator, $\text{Det}_{PBC}'(\delta_{ij}) > 0$. Therefore

$$
\text{Det}_{PBC}'(\delta_{ij} \frac{d}{dt}) = |\text{Det}_{PBC}'(\delta_{ij}) \frac{d^2}{dt^2}|^{\frac{1}{2}} = 1.
$$

(3.16)

By the theorem, we know that

$$
\text{Tr}_{\gamma^{2n+1}} = \int_{PBC} \mathcal{D}\psi e^{-\frac{1}{2} \int_0^1 \dot{\psi}^2 dt}
$$

$$
= \mathcal{N}_x \text{Det}_{PBC}'(\delta_{ij} \frac{d}{dt})^{\frac{1}{2}} \int d\psi_0^1 \cdots d\psi_0^{2n} = \mathcal{N} \int d\psi_0^1 \cdots d\psi_0^{2n}.
$$

(3.17)
We know that
\[(3.18) \quad \gamma_{2n+1} = i^{n-1}_0 \cdots \gamma_{0} = (2i)^n \psi_{02} \cdots \psi_{02},\]
and \(Tr\gamma_{2n+1}^2 = Tr I = 2^n\). Then we have the equation
\[(3.19) \quad 2^n = Tr\gamma_{2n+1}^2 = N_{\psi_2} \int d\psi_{02} \cdots d\psi_{2n} (2i)^n \psi_{02} \cdots \psi_{2n}\]
Thus \(N_{\psi_2} = i^n\).
Let us evaluate the third normalization factor \(N_{\psi_1}\).
We only need to evaluate \(Det_{APBC}(d/dt)\). Then immediately we can use the same method above. We add a harmonic oscillator \(\omega\) to the operator. As in [15], the partition function is
\[(3.20) \quad Tr(-1)^F e^{-\beta H} = 2 \cosh(\beta \omega/2) = e^{\beta \omega/2} Det_{APBC}'((1 - \epsilon \omega) d/dt + \omega).\]
Therefore, as \(\omega \to 0\),
\[(3.21) \quad Det_{APBC}'(d/dt) = \lim_{\omega \to 0} e^{-\beta \omega/2} \cosh(\beta \omega/2) = 1.\]
After evaluating the normalization factor, we need to evaluate the determinant in .. \(\mathcal{R}\) can be considered as a curvature two form which is an antisymmetric tensor, because of \(\psi_{02}\)'s are fermionic variables and anticommute. We can choose some local coordinates such that \(\mathcal{R}\) is diagonal of antisymmetric 2 by 2 matrices with vanishing diagonal entries. For the \(i\)th 2\times 2 matrix, the determinant is
\[(3.22) \quad I(\beta) = Det' \left( \begin{array}{cc} -d/dt & y_i \\
-y_i & -d/dt \end{array} \right) = Det'(d^2/dt^2 + y_i^2) = \prod_{n \neq 0} (y_i^2 - (2\pi n/\beta)^2) \]
\[= \prod_{n \leq 1} \left( \frac{2\pi n}{\beta} \prod_{n \leq 1} \left[ 1 - \left( \frac{y_i\beta}{2\pi n} \right)^2 \right] \right) = \left( \frac{\sin \beta y_i/2}{y_i/2} \right)^2. \]
When \(\beta = 1\), the right hand side of (3.22) is \((\sin y_i/2)^2\). Therefore
We can also use the above to evaluate \(Det_{APBC}(\delta_{ij}d/dt + \mathcal{R}_{ij})\). In fact, by the above,
\[(3.23) \quad I(2\beta) = \left( \frac{\sin \beta y_i}{y_i/2} \right)^2.\]
This is equivalent to substitute \(2n\) for \(n\) in the right hand side of above. We also notice that the eigenvalues for
\[Det_{APBC}' \left( \begin{array}{cc} -d/dt & y_i \\
-y_i & -d/dt \end{array} \right) = Det'(d^2/dt^2 + y_i^2) = \prod_{n \neq 0} (y_i^2 - (2\pi n/\beta)^2) \]
are actually \(n\pi \beta\) for odd \(n\).
Therefore we can have
\[
\text{Det}'_{APBC} \left( \begin{array}{cc}
-d/dt & y_i \\
y_i & -d/dt
\end{array} \right) = \text{Det}'(d^2/dt^2 + y_i^2) = I(2\beta)/I(\beta)
\]
\[
= (2 \cos \frac{\beta y_i}{2})^2.
\]
(3.24)

Now we are in a position to get the index of the operator. We fix $\beta = 1$ for the following calculations.

\[
\text{ind} D^+_+ = \int M \left( \prod_{\mu=1}^{2n} \frac{x^\mu_0}{\sqrt{2\pi}} \psi^\mu_0 \psi^\mu_1 \left( \prod_{i=1}^{n} \frac{y_i/2}{\sin \beta y_i/2} \right) \left( \prod_{i=1}^{n} 2 \cos \frac{\beta y_i}{2} \right) \right)
\]
(3.25)
\[
= \int M \left( \prod_{\mu=1}^{2n} \frac{x^\mu_0}{\sqrt{2\pi}} \psi^\mu_0 \psi^\mu_1 \left( \prod_{i=1}^{n} \frac{y_i}{\tan \beta y_i} \right) \right)
\]
(3.26)

We have to explain why the last two equalities hold. In fact, only $2n$ forms are picked up in the integrands. Thus if we multiply $i/2$ to each entries in the curvature matrices, since each entry is a 2-form, we then get the original $2n$ forms with an extra factor $i^n$. By the same reason, only when $n$ is even, the integrand is not zero. Therefore when $n$ is even, since $i^n = 1$ or $-1$, the last two equalities hold. Usually we denote the integrand by $L(M)$ which is a characteristic class of the manifold $M$. Then eventually we have proved the **Hirzebruch signature theorem**

Let’s prove another famous theorem which is Hirzebruch-Riemann-Roch theorem. Let us review what this theorem says. Consider product bundles $\Omega^{0,r} \otimes V$, where $V$ is a holomorphic vector bundle over $M$,

\[
... \rightarrow \Omega^{0,r-1}(M) \otimes V \rightarrow \Omega^{0,r}(M) \otimes V \rightarrow ...
\]

We can apply the Atiya-Singer index theorem to this complex, immediately we can get

\[
\text{ind} \partial V = \int Td(TM^+) \text{ch}(V).
\]
(3.27)

**3.2. Hirzebruch-Riemann-Roch theorem.** Now we prove it in two steps. First, under the assumption that $V$ is a trivial bundle, we prove the statement by the similar method as Hirzebruch signature theorem. Then we use a very intuitive argument to show the case that $V$ is not trivial.\(^\PageIndex{1}\)Pointed out the method to calculate the complex when $V$ is trivial. We give the full calculation here. Let us consider Dolbeault index on a complex manifold $M$. Here we have to impose a restriction on $M$ that is $M$ has a Kähler structure.\(^\PageIndex{18}\) showed that on a $\sigma$-model defined on $M$ admits two kind of supersymmetries. The requirement of the Kähler structure is the sufficient and necessary condition to get $N = 2$ supersymmetry.

We can decompose the exterior algebra by the direct sum of holomorphic $p, q$ forms by

\[
\Omega^*(M)^C = \bigoplus_{p,q=0}^{n} \Lambda^{p,q}(M),
\]
(3.28)
where $\Lambda^{p,q}$ denotes forms with $p$ holomorphic indices and $q$ antiholomorphic indices.

We can get the index of nontrivial $V$ by the following intuitive argument. First of all, since the index is an integer, the index remains the same when the spin field or the gauge field underlies a continuous deformation. In other word, mathematically, the index only depends on the fiber bundles of the spin field and the gauge field up to isomorphisms of fiber bundles. Therefore the index is equal to the integral of some index density which only depends on a local characteristic class that involves the two fiber bundles. We are able to find local coordinates such that the nontrivial charts of the two bundles don’t intersect. In the local charts in which spin bundle is nontrivial, the index density is identical with the case that there is no gauge field at all [5]; in the local charts in which the gauge field is nontrivial, the index density is just $ch(V)$ [3],[4]. In the local charts in which both fields are trivial, the index density is exactly 1. There is no conflict. Therefore we retrieve the Hirzebruch-Riemann-Roch theorem in a complex manifold with a Kähler structure.

4. CONCLUSION AND FURTHER QUESTIONS

The idea of calculating using supersymmetry is kind of simple although the real calculation is complicated. The most important thing in calculating the index is to clearly indentify the supercharge for the operator and find the corresponding physical theory. In the process of proceeding section, it seems hard to calculate the index of a Dirac operator coupled twisted by a gauge field from a supersymmetric theory. It is easy to deal with using Fujikawa’s method. The reason that the index of a Dirac operator in a spin field is easy to deal with in a supersymmetric theory is that supersymmetry actually exchanges the fermionic variables and the bosonic coordinates. Therefore supersymmetry founds a connection between the Dirac operator and the geometry of the manifold which is just the spin field. The difficulty lies that supersymmetry doesn’t provide the connection between a gauge field and the geometry of the manifold directly.

In addition, there is still a question. In the proof of the Hirzebruch-Riemann-Roch theorem, we assumed that the complex manifold has a Kähler structure, but the original theorem doesn’t require the existence of a Kähler structure. Since the Kähler condition is a necessary and sufficient of condition to yield supersymmetric $\sigma$-model for a manifold, a question arises which is how to prove the original theorem which is without the Kähler condition using supersymmetry.

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