Reduced second Zagreb index of product graphs

N. De
Department of Basic Sciences and Humanities (Mathematics),
Calcutta Institute of Engineering and Management, Kolkata, India
de.nilanjan@rediffmail.com

DOI 10.17586/2220-8054-2020-11-2-131-137

1. Introduction

Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For a graph $G$, let $d_G(v)$ denote the degree of a vertex $v$ in $G$, that is, the number of vertices adjacent with $v$. Throughout this article, we consider chemical graph [1, 2]. By chemical graph we mean a simple connected graph where vertices and edges are supposed to be atoms and chemical bonds between them respectively.

Topological indices, also called molecular structure descriptors are used in theoretical chemistry for design of chemical compounds with given physico-chemical properties and also to model chemical biological and pharmacological properties of the molecules. A topological index is a real number and it does not depend on the labelling of a graph and must be a structural invariant. The first topological index, Wiener index, was published in 1947 [3]. Due to the importance of topological indices in chemical research, lots of topological indices are developed in the chemical graph theory. Degree based topological indices are one of them that is applicable in quantitative structure property relationship and quantitative structure activity relationship [4, 5]. Among them Zagreb indices are most popular indices. First and second Zagreb indices are introduced by Gutman and Trinajstić [6], defined as follows:

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2,$$

$$M_2(G) = \sum_{u \in E(G)} d_G(u)d_G(v).$$

These graph invariants were proposed to measure the branching of carbon-atom skeleton [7]. For detail discussion on these indices, see [8–14]. Furtula et al. [15] proposed the reduced second Zagreb index of a graph $G$ to study the difference between $M_1$ and $M_2$, which is defined as follows:

$$RM_2(G) = \sum_{u \in E(G)} (d_G(u) - 1)(d_G(v) - 1).$$

The graphs having maximum and minimum reduced second Zagreb index in the class of cyclic graphs with cut edges are studied in [16]. Mahanta et al. [17] obtained Reduced the reduced second Zagreb index of four new graph operations based on tensor product. Several useful composite graphs can be obtained by operations of different graphs. The Cartesian product is one of that the operations and which is considered in the present work.

The Cartesian product $G_1 \times G_2$ of graphs $G_1$ and $G_2$ has the vertex set $V(G_1) \times V(G_2)$ and $(a, x)(b, y)$ is an edge of $G_1 \otimes G_2$ if $a = b$ and $xy \in E(H)$ or $ab \in E(G)$ and $x = y$. Using the Cartesian product of two graphs, the structure of several nanomaterials can be designed such as $C_4$-nanotube, $C_4$-nanotorus, multi-walled nanotorus etc. Carbon nanotube has diverse applications in cancer treatment, cardiac autonomic regulation, tissue regeneration etc. As topological index can predict different physico-chemical properties, it is worthy to compute that for different
2. Main results

Let \( G_i \) \((i = 1, 2, ..., k)\) be a connected graph with vertex set \( V(G_i) \) and \( E(G_i) \), so that, \( |V(G_i)| = n_i \) and \( |E(G_i)| = m_i \). In this section, we derive the reduced second Zagreb index of Cartesian product of \( k \)-number of connected graphs \( G_1, G_2, ..., G_k \). To do this first we prove the result for two connected graphs \( G_1 \) and \( G_2 \).

**Lemma 1.** [18] Let \( G_1 \) and \( G_2 \) be two connected graphs, then:

\[
\begin{align*}
(i) \quad |V(G_1 \otimes G_2)| &= |V(G_1)| \times |V(G_2)|, \\
(ii) \quad |E(G_1 \otimes G_2)| &= |E(G_1)||V(G_2)| + |E(G_2)||V(G_1)|, \\
(iii) \quad d_{G_1 \otimes G_2}(a, b) &= d_{G_1}(a) + d_{G_2}(b).
\end{align*}
\]

**Lemma 2.** [18] Let \( G_1, G_2, ..., G_k \) be \( k \)-number of graphs, then

\[
M_k = \sum_{i=1}^{k} \frac{M_i(G_i)}{n_i} + 4n \sum_{i \neq j, j=1}^{k} \frac{m_i m_j}{n_i n_j}.
\]

**Lemma 3.** [18] Let \( G_1, G_2, ..., G_k \) be \( k \)-number of graphs, then

\[
\begin{align*}
(i) \quad |V(\bigotimes_{i=1}^{k} G_i)| &= \prod_{i=1}^{k} n_i, \\
(ii) \quad |E(\bigotimes_{i=1}^{k} G_i)| &= \prod_{i=1}^{k} n_i \sum_{i=1}^{k} \frac{m_i}{n_i}.
\end{align*}
\]

**Theorem 1.** Let \( G_1 \) and \( G_2 \) be two connected graphs with \( n_1 \) and \( n_2 \) number of vertices and, \( m_1 \) and \( m_2 \) number of edges respectively, then:

\[
RM_2(G_1 \otimes G_2) = n_1 RM_2(G_2) + n_2 RM_2(G_1) + 3m_1 M_1(G_2) + 3m_2 M_1(G_1) - 8m_1 m_2.
\]

**Proof.** From definition of Cartesian product of two graphs, we have:

\[
RM_2(G_1 \otimes G_2) = \sum_{(a,b,c,d) \in E(G_1 \otimes G_2)} (d_{G_1}(a,b) - 1)(d_{G_2}(c,d) - 1)
\]

\[
= \sum_{u \in V(G_1) \quad bd \in E(G_2)} \sum_{v \in V(G_2) \quad ac \in E(G_1)} (d_{G_1}(u) + d_{G_2}(b) - 1)(d_{G_1}(u) + d_{G_2}(d) - 1)
\]

\[
+ \sum_{v \in V(G_2) \quad ac \in E(G_1)} \sum_{bd \in E(G_2) \quad u \in V(G_1)} (d_{G_1}(a) + d_{G_2}(v) - 1)(d_{G_1}(c) + d_{G_2}(v) - 1)
\]

\[
= C_1 + C_2, \quad (Say).
\]

Where,

\[
C_1 = \sum_{u \in V(G_1)} \sum_{bd \in E(G_2)} (d_{G_1}(u) + d_{G_2}(b) - 1)(d_{G_1}(u) + d_{G_2}(d) - 1)
\]

\[
= \sum_{u \in V(G_1)} \sum_{bd \in E(G_2)} (d_{G_2}(b) - 1)(d_{G_2}(d) - 1) + \sum_{bd \in E(G_2)} \sum_{u \in V(G_1)} d_{G_1}(u)^2
\]

\[
+ \sum_{bd \in E(G_2)} \sum_{u \in V(G_1)} d_{G_1}(u)(d_{G_2}(b) + d_{G_2}(d) - 2)
\]

\[
= n_1 RM_2(G_2) + m_2 M_1(G_1) + 2m_1 M_1(G_2) - 4m_1 m_2.
\]
Similarly,
\[
C_2 = \sum_{v \in V(G_2)} \sum_{a \in E(G_1)} (d_{G_1}(a) + d_{G_2}(v) - 1)(d_{G_1}(v) + d_{G_2}(v) - 1)
\]
\[= \sum_{v \in V(G_2)} \sum_{a \in E(G_1)} (d_{G_1}(a) - 1)(d_{G_1}(v) - 1) + \sum_{a \in E(G_1)} \sum_{v \in V(G_2)} d_{G_1}(v)^2
\]
\[+ \sum_{a \in E(G_1)} (d_{G_1}(a) + d_{G_1}(v) - 2) \sum_{v \in V(G_2)} d_{G_1}(v)
\]
\[= n_2 RM_2(G_1) + m_1 M_1(G_2) + 2m_2 M_1(G_1) - 4m_1 m_2.
\]
Therefore, combining the contributions of \(C_1\) and \(C_2\) we get the desired results.

In the following, we calculate reduced second Zagreb index of the Cartesian product of \(k\)-number of graphs \(G_i\) for \(i = 1, 2, ..., k\).

**Theorem 2.** Let \(G_1, G_2, ..., G_k\) be \(k\)-number of connected graphs, then
\[
RM_2(\bigotimes_{i=1}^{k} G_i) = n \sum_{i=1}^{k} \frac{RM_2(G_i)}{n_i} + 3 \sum_{i=1}^{k} \left( \frac{m_{i}}{n_i^2} - \frac{m_{i} m_{\bar{i}}}{n_i^2} \right) M_1(G_i)
\]
\[+ 4n \sum_{p,q,r=1, p \neq q \neq r} m_{p} m_{q} m_{r} \cdot \frac{1}{n_{p} n_{q} n_{r}} - 4n \sum_{i,j=1, i \neq j} \frac{m_{i} m_{j}}{n_{i} n_{j}}
\]
Where, \(n = |V(\bigotimes_{i=1}^{k} G_i)|\) and \(m = |E(\bigotimes_{i=1}^{k} G_i)|\).

**Proof.** Let us assume that, \(n' = \sum_{i=1}^{k-1} |V(\bigotimes_{i=1}^{k} G_i)|\) and \(m' = \sum_{i=1}^{k-1} |E(\bigotimes_{i=1}^{k} G_i)|\) so that, \(n' = \frac{n}{n_k}\) and \(m' = \frac{m m_k - m m_k}{n_k}\). Now by Theorem 1 and an inductive argument, we can have
\[
RM_2(\bigotimes_{i=1}^{k} G_i) = RM_2(\bigotimes_{i=1}^{k-1} G_i \otimes G_k)
\]
\[= n_k RM_2(\bigotimes_{i=1}^{k-1} G_i) + |V(\bigotimes_{i=1}^{k-1} G_i)| RM_2(G_k) + 3m_k M_1(\bigotimes_{i=1}^{k-1} G_i)
\]
\[+ 3|E(\bigotimes_{i=1}^{k-1} G_i)| M_1(G_k) - 8|E(\bigotimes_{i=1}^{k-1} G_i)||E(G_k)|
\]
Using Lemmas 1, 2, and 3 on the above result, we obtain:
\[
RM_2(\bigotimes_{i=1}^{k} G_i) = n_k [n'] \sum_{i=1}^{k-1} \frac{RM_2(G_i)}{n_i} + 3 \sum_{i=1}^{k-1} \left( \frac{m'}{n_i^2} - \frac{n' m_{\bar{i}}}{n_i^2} \right) M_1(G_i)
\]
\[+ 4n' \sum_{p,q,r=1, p \neq q \neq r} m_{p} m_{q} m_{r} \cdot \frac{1}{n_{p} n_{q} n_{r}} - 4n' \sum_{i,j=1, i \neq j} \frac{m_{i} m_{j}}{n_{i} n_{j}} + n' RM_2(G_k)
\]
\[+ 3m_k [n'] \sum_{i=1}^{k-1} \frac{M_1(G_i)}{n_i} + 4n' \sum_{i,j=1} \frac{m_{i} m_{j}}{n_i n_j} + 3n \sum_{i=1}^{k-1} \frac{m_i}{n_i}
\]
\[+ 8m_k n' \sum_{i=1}^{k-1} \frac{m_i}{n_i}
\]
After arranging the terms, we have:

\[
RM_2(\bigotimes_{i=1}^k G_i) = \left[ n_k n' \sum_{i=1}^{k-1} \frac{RM_2(G_i)}{n_i} + n'RM_2(G_k) \right] + \left[ 3n_k \sum_{i=1}^{k-1} \left( \frac{m'}{n_i} - \frac{n'm_i}{n_i^2} \right) M_{1}(G_i) \right] \\
+ 3m_k n' \sum_{i=1}^{k-1} \frac{M_1(G_i)}{n_i} + 3n' \sum_{i=1}^{k-1} \left( \frac{m}{n_i} M_1(G_k) \right) \\
+ [4n_k n' \sum_{p,q,r=1, p \neq q \neq r} m_p m_q m_r \frac{n_p n_q n_r}{n_k} + 12n' m_k \sum_{i,j=1}^{k-1} \frac{m_i m_j}{n_i n_j}] \\
- [4n \sum_{i,j=1}^{k-1} \frac{m_i m_j}{n_i n_j} + 8m_k n' \sum_{i=1}^{k-1} \frac{m_i}{n_i}]
\]

After some calculations, we get:

\[
RM_2(\bigotimes_{i=1}^k G_i) = \left[ n \sum_{i=1}^{k-1} \frac{RM_2(G_i)}{n_i} + n \frac{RM_2(G_k)}{n_k} \right] + \left[ 3 \sum_{i=1}^{k-1} \left( \frac{m'n_k + m_k n'}{n_i} \right) M_{1}(G_i) \right] \\
- \left( \frac{n'n_k m_i}{n_i^2} \right) M_{1}(G_i) + 3 \left( \frac{m'n_k - m_k n'}{n_i^2} \right) M_{1}(G_k) \\
+ [4n \sum_{p,q,r=1, p \neq q \neq r} m_p m_q m_r \frac{n_p n_q n_r}{n_k} + 12n \frac{m_k}{n_k} \sum_{i,j=1}^{k-1} \frac{m_i m_j}{n_i n_j}] \\
- [4n \sum_{i,j=1}^{k-1} \frac{m_i m_j}{n_i n_j} + 8n \frac{m_k}{n_k} \sum_{i=1}^{k-1} \frac{m_i}{n_i}]
\]

\[
= n \sum_{i=1}^{k} \frac{RM_2(G_i)}{n_i} + 3 \sum_{i=1}^{k} \left( \frac{m}{n_i} - \frac{n'm_i}{n_i^2} \right) M_{1}(G_i) \\
+ 4n \sum_{p,q,r=1, p \neq q \neq r} \frac{m_p m_q m_r}{n_p n_q n_r} - 4n \sum_{i,j=1}^{k} \frac{m_i m_j}{n_i n_j}
\]

which is the desired result. \qed

If \( G_1 = G_2 = \ldots = G_k = G \) then \( \bigotimes_{i=1}^k G_i = G^k \). Hence from the above theorem, the following corollary follows:

**Corollary 1.** If \( G \) be connected graph, then

\[
RM_2(G^k) = k|V(G)|^{k-3}|V(G)|^2RM_2(G) + 3(k-1)|V(G)||E(G)|M_{1}(G) \\
+ 4(k-1)(k-2)|E(G)|^3 - 4(k-1)|V(G)||E(G)|^2.
\]

### 3. Applications

Using various unary and binary graph operations on different elementary graphs, such as the path graph, cycle graph, complete graph etc, we can obtain several significant composite graphs having excellent usage in modern science and technology. The Cartesian product is one of those binary operations capable to construct different special structure. Using Cartesian product of two graphs, one can get ladder graph, \( C_4 \)-nanotube and nanotorus, rectangular grid, rook’s graph, hamming graph etc. The Hamming graphs are interesting in connection with error-correcting codes and association schemes. The rook’s graph represents all legal moves of rook on the chess board. So it is worth to investigate investigating topological indices for the above structures. In this section, the reduced second Zagreb index of aforesaid graphs is derived. For path, cycle, and complete graph of \( n \) vertices, the notations \( P_n \), \( C_n \), and \( K_n \) are used.

**Example 1.** The **Ladder graph** \( L_n \) (Fig. 1) is the Cartesian product of \( P_2 \) and \( P_{n+1} \), made by \( n \) sequences and \((2n + 2)\) vertices. So, using theorem 1, the reduced second Zagreb index of \( L_n \) is given by

\[
RM_2(L_n) = 12n - 10.
\]
Example 2. The Cartesian product of $P_n$ ($n \geq 2$) and $C_m$ ($m \geq 2$) is a $C_4$-nanotube $TUC_4(m, n)$, whose reduced second Zagreb index can be calculated using theorem 1 as follows:

$$RM_2(P_n \otimes C_m) = 18mn - 25m.$$ 

Example 3. The Cartesian product of $C_n$ ($n \geq 3$) and $C_m$ ($m \geq 3$) is a $C_4$-nanotorus $TC_4(m, n)$, whose reduced second Zagreb index is calculated from Theorem 1 as follows:

$$RM_2(C_n \otimes C_m) = 18mn.$$ 

Example 4. The rectangular grid (Fig.2) is the Cartesian product of the $P_n$ ($n \geq 2$) and $P_m$ ($m \geq 2$). So, using Theorem 1, its reduced second Zagreb index is given by

$$RM_2(P_n \otimes P_m) = 18mn - 25m - 25n + 28.$$ 

Example 5. The Cartesian product of $K_n$ and $K_m$ yields the rook’s graph (Fig. 3). So, using Theorem 1, its reduced second Zagreb index is given by

$$RM_2(K_m \otimes K_n) = \frac{mn}{2}[(m + n)^3 - 8m^2 - 8n^2 - 16mn + 21m + 21n - 14].$$
Example 6. The graph $G = \bigotimes_{i=1}^{N} K_{n_i}$ is known as a Hamming graph and is denoted by $H_{n_1,n_2,\ldots,n_N}$. So, applying previous theorem to compute reduced second Zagreb index of a Hamming graph as follows:

$$RM_2(H_{n_1,n_2,\ldots,n_N}) = \frac{1}{2} \prod_{i=1}^{N} n_i \sum_{i=1}^{N} (n_i - 1) \sum_{i=1}^{N} (n_i - 1) - 1)^2.$$  

If $n_1 = n_2 = \ldots = n_N = 2$, then the graph $G$ is known as a hypercube of dimension $N$ and denoted by $Q_n$ (Fig. 4). Thus from above, we directly get

$$RM_2(Q_n) = \frac{1}{2} N 2^N (N - 1)^2.$$  

![Fig. 4. Example of Hypercube](image)

Example 7. The reduced second Zagreb index of the torus $C_{n_1} \otimes C_{n_2} \otimes \ldots \otimes C_{n_k}$ is given by

$$RM_2(C_{n_1} \otimes C_{n_2} \otimes \ldots \otimes C_{n_k}) = k(2k - 1)^2 \prod_{i=1}^{k} n_i.$$  

Example 8. Let $T = T[p,q]$ be the molecular graph of a nanotorus (Fig. 5). Then $|V(T)| = pq$, $|E(T)| = \frac{9pq}{2}$, $M_1(T) = 9pq$. We consider a $q$-multi-walled nanotorus $G_n = P_n \otimes T$. It is easy to find that $RM_2(P_n) = n - 3$, $RM_2(T) = 6pq$, and $M_1(P_n) = 4n - 6$. Thus from theorem 1, we have the following result.

$$RM_2(G_n) = 5pq(8n - 9).$$  

We have plotted the result in Fig. 6

![Fig. 5. The graph of a nanotorus](image)
4. Conclusion

In this article, reduced second Zagreb index of Cartesian product graph is studied. Firstly $RM_2$ index is obtained for the product of two graphs and then the general case is considered. Applying that results, $RM_2$ index is investigated for some special structures. As future work, some other graph operations like composition, tensor product, corona product, strong product, splice, link etc. can be discussed in terms of the reduced second Zagreb index.

References

[1] Trinajstić N. Chemical Graph Theory. CRC Press, Boca Raton, 1993.
[2] Gutman I., Polansky O.E. Mathematical Concepts in Organic Chemistry. Springer, Berlin, 1986.
[3] Wiener H. Structural determination of paraffin boiling points. J. Am. Chem. Soc., 1947, 69, P. 7–20.
[4] Randić M. Quantitative Structure – Property Relationship: boiling points and planar benzenoids. New. J. Chem., 1996, 20(10), P. 1001–1009.
[5] Hawkins D.M., Basak S.C., Mills D. Assessing Model Fit by Cross Validation. J. Chem. Inf. Comput. Sci., 2003, 43, P. 579–586.
[6] Gutman I, Trinajstić N. Graph theory and molecular orbitals. Total π-electron energy of alternate hydrocarbons. Chem. Phys. Lett., 1972, 17, P. 535–538.
[7] Gutman I, Rušič B, Trinajstić N, Wilcox C.F. Graph theory and molecular orbitals. XIII. Acyclic polyenes. J. Chem. Phys., 1975, 62, P. 3399–3405.
[8] Das K.C., Xu K., Gutman I. On Zagreb and Harary indices. MATCH Commun. Math. Comput. Chem., 2013, 70, P. 301–314.
[9] Azari M., Iranmanesh A., Gutman I. Zagreb indices of bridge and chain graphs. MATCH Commun. Math. Comput. Chem., 2013, 70, P. 921–938.
[10] Hamzeh A., Reti T. An analogue of Zagreb index inequality obtained from graph ir-regularity measures. MATCH Commun. Math. Comput. Chem., 2014, 72, P. 669–683.
[11] Li S., Zhang M.J. Sharp upper bounds for the Zagreb indices of bipartite graphs with a given diameter. Appl. Math. Lett., 2011, 24, P. 131–137.
[12] Fonseca C.M.D., Stevanovic D. Further properties of the second Zagreb index. MATCH Commun. Math. Comput. Chem., 2014, 72, P. 655–668.
[13] Gutman I., Das K.C. The first Zagreb index 30 years after. MATCH Commun. Math. Comput. Chem., 2004, 50, P. 83–92.
[14] Mondal S., Bhosale A., De N., Pal A. Topological properties of some nanostructures. Nanosystems: Physics, Chemistry, Mathematics, 2020, 11(1), P. 14–24.
[15] Furtula B., Gutman I., Ediz S. On difference of Zagreb indices. Discrete Appl. Math., 2014, 178, P. 83–88.
[16] Horoldagva H., Buyantogtokh L., Dorjessem S. Difference of Zagreb indices and reduced second Zagreb index of cyclic graphs with cut edges. MATCH Commun. Math. Comput. Chem., 2017, 78, P. 337–349.
[17] Mahanta A., Buragohain J., Bharali A. Reduced first Zagreb index, Reduced second Zagreb index and hyper-Zagreb index of four new sums based on tensor product of graphs. Global journal of engineering science and researches, 2018, 5(11), P. 12–19.
[18] Khalifeh M.H., Azari H.Y, Ashrafi A.R. The first and second Zagreb indices of some graph operations. Discrete Appl. Math., 2009, 157, P. 804–811.
[19] Azari H.Y, Manochehrian B, Ashrafi A.R. The PI index of product graphs. Appl. Math. Lett., 2008, 21, P. 624–627.
[20] Klavzar S., Rajapakse A., Gutman I. The Szeged and the Wiener index of graphs. Appl. Math. Lett., 1996, 9, P. 45–49.
[21] Graovac A., Pisanski T. On the Wiener index of a graph. J. Math. Chem., 1991, 8, P. 53–62.
[22] De N., Nayeem S.M.A., Pal A. F-Index of some graph operations. 2016, Discrete Math. Algorithm Appl., 8, Article ID: 1650025.
[23] De N., Nayeem S.M.A., Pal A. The F-coindex of some graph operations. 2016, Springer Plus, 5, P. 221–233.
[24] De N., Nayeem S.M.A., Pal A. Reformulated First Zagreb Index of Some Graph Operations. Mathematics, 2015, 3(4), P. 945–960.
[25] De N., Pal A., Nayeem S.M.A. On some bounds and exact formulae for connective eccentric indices of graphs under some graph operations. Int. J. Comb., 2014, 2014, Article ID 579257.