Dynamics of fractals in Euclidean and measure spaces

To cite this article: Md. Shahidul Islam and Md. Jahurul Islam 2017 J. Phys.: Conf. Ser. 890 012058

View the article online for updates and enhancements.
Dynamics of fractals in Euclidean and measure spaces

Md. Shahidul Islam and Md. Jahurul Islam
Department of Mathematics, University of Dhaka, Dhaka -1000, Bangladesh
E-mail: mshahidul11@yahoo.com, jahurul93@gmail.com

Abstract. In this paper, we formulate iterated function system of the square fractal and three dimensional fractals such as the Menger sponge and the Sierpinski tetrahedron using affine transformation method and fixed points method of Devaney [1]. We show that these functions are asymptotically stable and also the Lebesgue measures of these fractals are zero.

1. Introduction
A Fractal, as defined by B. Mandelbrot, is a shape made of parts similar to the whole in some way [2]. Fractal is a geometric object that possesses the two properties: self-similar and non-integer dimensions. So a fractal is an object or quantity which displays self-similarity. The Cantor set is the prototypical fractal [1]. We studied the Cantor set and found generalized Cantor set and showed its fractal dimensions [3]. We studied these fractals in measure space and showed that these special types of sets are Borel set as well as Borel measurable and whose Lebesgue measure is zero [4]. We formulated iterated function system with probabilities of generalized Cantor sets and shown their invariant measures using Markov operator and Barnsley-Hutchison multifunction [5].

The main aim of the paper is to formulate iterated function system of higher dimensional fractals. To the best of my knowledge first time we formulate iterated function system of the square fractal and three dimensional fractals such as the Menger sponge, the Sierpinski tetrahedron. Also we study these fractals in measure space and show that the Lebesgue measures of these fractals are zero.

2. Preliminaries
Definition 2.1. Let \((X, \rho)\) be a metric space. A function \(f : X \to X\) is a contraction mapping, or contraction on \((X, \rho)\), with the property that there is some nonnegative real number \(0 \leq L < 1\) such that for all \(x, y \in X\),
\[
\rho(f(x), f(y)) \leq L \rho(x, y).
\]
The smallest such value of \(L\) is called the Lipschitz constant of \(f\).

Definition 2.2. The outer measure of any interval \(I\) on \(\mathbb{R}\) with endpoints \(a < b\) is \(b - a\) and is denoted as \(\lambda^*(I) = b - a\). A set \(E \subset \mathbb{R}\) is said to be outer measure (or measurable) if, for all \(A \subset \mathbb{R}\) one has \(\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c)\).
3. Iterated Function System and the Construction of Fractals

Any fractal has some infinitely repeating pattern. When creating such fractal, we would suspect that the easiest way is to repeat a certain series of steps which create that pattern. Iterated Function System (shortly IFS) is another way of generating fractals. It represents an extremely versatile method for conveniently generating a wide variety of useful fractal structures [6]. These iterated function systems are based on the application of a series of affine transformations. An affine transformation is a recursive transformation of the type

\[
\begin{pmatrix}
    x_{n+1} \\
    y_{n+1}
\end{pmatrix} =
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix}
\begin{pmatrix}
    x_n \\
    y_n
\end{pmatrix} +
\begin{pmatrix}
    e \\
    f
\end{pmatrix}
\]

where \(a, b, c, d, e\) and \(f\) are real numbers, which control rotation and scaling, while \(e\) and \(f\) control linear translation.

Now we consider \(w_1, w_2, \ldots, w_N\) as a set of affine linear transformations, and let \(A\) be the initial geometry. Then the collecting result from \(w_1(A), w_2(A), \ldots, w_N(A)\) can be represented by

\[
F(A) = \bigcup_{i=1}^{N} w_i(A)
\]

where \(F\) is the Barnsley-Hutchinson operator [7, 8].

**Alternative method to formulate iterated function system:** Let \(0 < \alpha < 1\). Let \(p_1, p_2, \ldots, p_N\) be points in the plane. Let \(A_i(p) = \alpha(p - p_i) + p_i\), where \(p = \begin{pmatrix} x \\ y \end{pmatrix}\) and for each \(i = 1, 2, \ldots, N\). The collection of functions \(\{A_1, A_2, \ldots, A_N\}\) is called an iterated function system [1].

4. Iterated Function System of Fractals

4.1. Iterated Function System of Two Dimensional Fractals

4.1.1. The Sierpinski gasket or equilateral triangle is obtained by the following iterated function system

\[
w_1(x, y) = \left(\frac{1}{2}, \frac{1}{2} \right), \quad w_2(x, y) = \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{2} \right), \quad w_3(x, y) = \left(\frac{1}{2} + \frac{1}{2}, \frac{1}{2} + \frac{1}{2} \right)
\]

On the literature, the analysis of the Sierpinski equilateral triangle is available for particular fractals and our interest is considered the problem for different types of triangular shapes, for example: isosceles triangle, isosceles right triangle and scalene triangle.

4.1.2. The Sierpinski isosceles triangle may be obtained by the following IFS

\[
w_1(x, y) = \left(\frac{1}{2}, \frac{1}{2} \right), \quad w_2(x, y) = \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{2} \right), \quad w_3(x, y) = \left(\frac{1}{2} + \frac{1}{2}, \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right)
\]

By (1) the attractor of IFS (2) is \(A_{n+1} = F(A_n), A_0 \in \{(0,0),(1,0),(1,1)\}, n = 0, 1, 2, \cdots\)

In general, \(A_n\) is the union of \(3^{n+1}\) vertices, each of size of isosceles triangle \(\frac{1}{3^{n+1}}\). We obtain a sequence \(A_0 \supset A_1 \supset A_2 \cdots\). Therefore, the Sierpinski isosceles triangle is \(A = \bigcap_{n=0}^{\infty} A_n\).
Similarly, we can formulate iterated function system of the Sierpinski isosceles right and scalene triangles.

4.1.3 The box fractal may be obtained by the following iterated function system

\[ w_1(x, y) = \left( \frac{1}{3} x, \frac{1}{3} y \right), \quad w_2(x, y) = \left( \frac{1}{3} x + \frac{2}{3}, \frac{1}{3} y \right), \quad w_3(x, y) = \left( \frac{1}{3} x + \frac{2}{3}, \frac{1}{3} y + \frac{2}{3} \right) \]

By (1) the attractor of IFS (3) is \( B_{n+1} = F(B_n), \ldots \). Thus the box fractal is \( B = \bigcap_{n=0}^{\infty} B_n \).

4.1.4 The square fractal (using Cantor 1/3-set) consists of four self-similar pieces, which is defined as follows:

\[ w_1(x, y) = \left( \frac{1}{3} x, \frac{1}{3} y \right), \quad w_2(x, y) = \left( \frac{1}{3} x + \frac{2}{3}, \frac{1}{3} y \right) \]

\[ w_3(x, y) = \left( \frac{1}{3} x, \frac{1}{3} y + \frac{2}{3} \right), \quad w_4(x, y) = \left( \frac{1}{3} x + \frac{2}{3}, \frac{1}{3} y + \frac{2}{3} \right) \]

By (1) the attractor of IFS (4) is \( S_{n+1} = F(S_n), \ldots \). Thus the square fractal is \( S = \bigcap_{n=0}^{\infty} S_n \).
Figure 3. The first three stages in the construction of the square fractal as an IFS.

4.2. Iterated Function System of Three Dimensional Fractals

4.2.1. The Menger sponge: It is interesting to see that the Mengersponge is presented by the following iterated function system

\[
w_1(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \quad w_2(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right),
\]

\[
w_3(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \quad w_4(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right),
\]

\[
w_5(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \quad w_6(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right),
\]

\[
w_7(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \quad w_8(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right),
\]

\[
w_9(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \quad w_{10}(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right),
\]

\[
w_{11}(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \quad w_{12}(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right),
\]

\[
w_{13}(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \quad w_{14}(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right),
\]

\[
w_{15}(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \quad w_{16}(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right),
\]

\[
w_{17}(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \quad w_{18}(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right),
\]

\[
w_{19}(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \quad w_{20}(x, y, z) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right).
\]

By (1) the attractor of IFS (5) is \( M_{n+1} = F(M_n) \), where \( M_0 \in \{(0,0,0), \ldots, (1,1,1)\} \).

In general, \( M_n \) is the union of \( 8 \cdot (20)^n \) vertices, each of size of the Mengersponge \( \frac{1}{3^n} \).

We obtain a sequence \( M_0 \supset M_1 \supset M_2 \cdots \). Thus the Menger sponge is \( M = \bigcap_{n=0}^{\infty} M_n \).
Figure 4. The first three stages in the construction of the Menger sponge as an IFS.

$M_0$ $M_1$ (1st Iteration) $M_2$ (2nd Iteration)

4.2.2. The Sierpinski tetrahedron may be obtained by the following iterated function system

$$w_1(x, y, z) = \left(\frac{x}{2}, \frac{y}{2}, \frac{z - \sqrt{3}}{2}\right),$$
$$w_2(x, y, z) = \left(\frac{x}{2}, \frac{y}{2}, \frac{z + 1}{2}\right),$$
$$w_3(x, y, z) = \left(\frac{x + 1}{2}, \frac{y}{2}, \frac{z}{2}\right),$$
$$w_4(x, y, z) = \left(\frac{x}{2}, \frac{y + 1}{2}, \frac{z}{2}\right).$$

By (1) the attractor of IFS (6) is $T_{n+1} = F(T_n)$, where $T_0 \in \{(0, 0, \sqrt{3}), (0, 2\sqrt{2/3}, \frac{1}{2}/\sqrt{3}), (\sqrt{2}, -\sqrt{2/3}, -\sqrt{1/3}), (\sqrt{2}, -\sqrt{2/3}, -\sqrt{1/3})\}$. In general, $T_n$ is the union of $4^{n+1}$ vertices, each of size of the tetrahedron $1/2^n$. We obtain a sequence $T_0 \supset T_1 \supset T_2 \ldots$. Hence the Sierpinski tetrahedron is $T = \bigcap_{n=0}^{\infty} T_n$.

Figure 5. The first four stages in the construction of the Sierpinski tetrahedron as an IFS.

$T_0$ $T_1$ (1st Iteration) $T_2$ (2nd Iteration) $T_3$ (3rd Iteration)

4.3. Dynamics of Fractals

Since Lipschitz constants of iterated function system of fractals $L_i < 1$ for $i \in I$, where $I$ is finite, then IFS of these fractals are asymptotically stable. Also we show that the fixed points of these functions are sink using the Jacobian matrix method [9].

5. Lebesgue Measure of Higher Dimensional Fractals

5.1. Lebesgue Measure of the Two Dimensional Fractals

Theorem 5.1.1. The Lebesgue measure of the Sierpinski equilateral triangle is zero.
Proof: We know that the Sierpinski equilateral triangle is $A = \bigcap_{n=0}^{\infty} A_n$. From the construction of Sierpinski equilateral triangle, we remove $3^{n-1}$ open triangles from each previous triangles and each having size $\frac{1}{2^n}$, where $n \geq 1$.

We remove a total area

$$\sum_{n=1}^{\infty} \frac{\sqrt{3}}{4} \left( \frac{1}{2^n} \right)^2 = \frac{\sqrt{3}}{4} \sum_{n=1}^{\infty} \left( \frac{3}{4} \right)^{n-1} = \frac{\sqrt{3}}{4} \sum_{n=0}^{\infty} \left( \frac{3}{4} \right)^n = \frac{\sqrt{3}}{4} \left( \frac{1}{1 - \frac{3}{4}} \right) = \frac{\sqrt{3}}{4}$$

Thus $\lambda(A \setminus A) = \frac{\sqrt{3}}{4}$. Since $\lambda(A_0) = \lambda(A) + \lambda(A_0 \setminus A)$, then $\lambda(A) = 0$.

Thus the Lebesgue measure of the Sierpinski equilateral triangle is zero. Similarly, we can prove the Lebesgue measures of the Sierpinski isosceles, isosceles right and scalene triangle are zero and also we can prove the following theorems:

**Theorem 5.1.2.** The Lebesgue measure of the box fractal is zero.

**Theorem 5.1.3.** The Lebesgue measure of the square fractal is zero.

5.2. *Lebesgue Measure of Three Dimensional Fractals*

**Theorem 5.2.1.** The Lebesgue measure of the Menger sponge is zero.

Proof: We know that the Menger sponge is $M = \bigcap_{n=0}^{\infty} M_n$. From the construction of the Menger sponge, we remove $7 \cdot (20)^{n-1}$ open cubes from each previous cube and each having edge length $\frac{1}{3^n}$, where $n \geq 1$. We remove a total volume

$$\sum_{n=1}^{\infty} 7 \cdot (20)^{n-1} \left( \frac{1}{3^n} \right)^3 = \frac{7}{27} \sum_{n=1}^{\infty} (20 / 27)^{n-1} = \frac{7}{27} \sum_{n=0}^{\infty} (20 / 27)^n = \frac{7}{27} \left( \frac{1}{1 - 20 / 27} \right) = 1$$

Thus $\lambda(M_0 \setminus M) = 1$. Since $\lambda(M_0) = \lambda(M) + \lambda(M_0 \setminus M)$, then $\lambda(M) = 0$.

Hence the Lebesgue measure of the Menger sponge is zero.

Similarly, we can prove the Lebesgue measure of the Sierpinski tetrahedron is zero.

6. Discussion and conclusion

We have formulated iterated function system of the square fractal and three dimensional fractal such as the Menger sponge, the Sierpinski tetrahedron. Also we have shown that these functions are asymptotically stable and the Lebesgue measures of these fractals are zero. These results may be extended to show the invariant measures for iterated function system of higher dimensional fractals.

References

[1] Devaney R. L. 1992 *A First Course in Chaotic Dynamical Systems*, 2nd edition, Boston University, Addison-Wesley, West Views Press.

[2] Addison P. S. 1997 *Fractals and Chaos: An Illustrated Course*, Institute of Physics, Bristol.

[3] Islam J. and Islam S. 2011 Generalized Cantor Set and its Fractal Dimension, Bangladesh J. Sci. Ind. Res. 46(4), 499-506.
[4] Islam J. and Islam S. 2015 Lebesgue Measure of Generalized Cantor Set, Annals of Pure and App. Math. 10(1), 75-86.

[5] Islam J. and Islam S. 2016 Invariant measures for Iterated Function System of Generalized Cantor Sets, German J. Ad. Math. Sci. 1(2), 41-47.

[6] Peitger H.O., Jungeus H. and Sourpe D. 1992 *Chaos and Fractals: New frontier of science*, Springer, New York.

[7] Barnsley M. F. 1993 *Fractals Everywhere*, Academic Press, Massachusetts.

[8] Hutchinson J. E. 1981 Fractals and self-similarity, Indiana Univ. Math. J. 30(5), 713-747.

[9] Alligood K. T., Sauer T. D., Yorke J. A., 1997 *Chaos, An Introduction to Dynamical Systems*, Springer, New York.