Considerations of a $k = +1$ Varluminopic Cosmology

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Abstract

Every relativistic particle has 4-speed equal to $c$, since $g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = c^2$. With the choice of $k = +1$ in the FRW metric, the cosmological scale factor $a(t)$ has the natural interpretation of the radius of the sphere $S^3 = \{x \in \mathbb{R}^4 : (x, x) = a^2\}$. Thus, a particle at rest in the cosmological frame has 4-speed equal to $\frac{da}{dt}$. This leads us to infer that $\ddot{a} = c$, which represents a simple kinematic constraint linking the speed of light to the cosmological scale factor. This drastically changes the $k = +1$ picture from a closed deaccelerating universe to an open accelerating universe, settles the horizon problem, and provides for a new cosmological model more appealing to our natural intuition. In this paper we shall consider ramifications of this model.

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Part I
Discussion and Overture

In this part of the paper, we will introduce our hypothesis, which will give us a simple kinematic constraint between the evolution of the scale factor and the speed of light. We will then consider various elements of cosmological theory that need modified if one is to consider varluminopic cosmological models. This will set the stage for our discussion of the dynamics in the upcoming parts of the paper.

1 Introduction

In subsection 1, we will review some empirical evidence which points to the $k = +1$ FRW Model. In subsection 2, we will introduce a kinematic constraint on this model which will change its resulting dynamics. In subsection 3, we will give an outline for the organization of this paper.

1.1 The Appeal of the $k = +1$ Cosmologies

1.1.1 Expansion

Cosmological observation shows us that our universe is expanding. This was first discovered by Edwin Hubble in 1929. As geometry codifies the interplay between gravitation and matter, this is a natural implication of general relativity. It can be mathematically described independent of your choice of universe, by including a time-varying scale factor in the space-time metric. However, in the $k = +1$ picture, expansion takes a very natural and elegant form. The universe takes the form of a three-dimensional expanding spherical surface in a four dimensional space, the radial component being time-like. In this picture, expansion is no more surprising than observing several dots on a balloon mutually moving farther apart as the balloon expands.
1.1.2 The Horizon Problem

Cosmologist have measured to temperature of the CMBR to be 2.728 K with anisotropies in temperature on the level of $10^{-5}$. The problem is that this is uniform in every direction in the sky. In particular, if we look at two points separated by 180°, we record the same background temperature. The radiation from each of these points has travelled approximately 98% of the horizon distance, from the last scattering surface, to reach us. Therefore, the two points where the radiation originated are separated by a distance of 196% of the horizon distance. If the universe is taken to be flat, we arrive at a fundamental paradox. How could two separate points in space-time, not causally connected, be in thermal equilibrium? However, if the universe is spherical, the paradox is resolved. All points in space are causally connected, as the entire universe originates from a single point in the past. Thus, by virtue of the geometry, our model solves the Horizon Problem without the need of inflation.

1.1.3 The Simultaneity Problem

Another cosmological problem is that the standard Big Bang model, when set in a $k = 0$ flat FRW universe, violates the principle of simultaneity. In this model, the Big Bang occurs everywhere simultaneously. That is, on a space-time diagram, there is a horizontal cutoff somewhere. Thus for every observer in motion relative to the cosmological frame (i.e. us), there are regions of space (on the surface $t = \text{now}$, where $t$ is coordinate time) where the big bang is currently going on, and there are regions of space where the universe has yet to be born. This problem is irrelevant in the spherical $k = +1$ FRW universe.

1.2 A New Kinematic Constraint

Despite these observations, the $k = +1$ FRW universe has been found unsatisfactory and inadequate by the cosmology community over the last several decades. We would like to introduce a fundamental constraint on the motion which radically changes the resulting dynamics of the $k = +1$ FRW picture. Taking the metric to have signature $(+1, -1, -1, -1)$, it is a basic result of relativity theory that

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = c^2 \tag{1}$$

For a particle at rest in the cosmological frame, and choosing a comoving coordinate system, the particle’s four-velocity must necessarily take the form

$$\left[ \frac{dx^\mu}{d\tau} \right] = \langle c, 0, 0, 0 \rangle \tag{2}$$

in order for the condition (1) to be satisfied.

Now consider the $k = +1$ FRW picture. In spherical coordinates, the particle’s position is $\langle a, \rho, \theta, \phi \rangle$. If the particle is at rest in the cosmological frame, then $\dot{\rho} = \dot{\theta} = \dot{\phi} = 0$, and the particle’s four-velocity becomes

$$\left[ \frac{dx^\mu}{d\tau} \right] = \langle \frac{da}{dt}, 0, 0, 0 \rangle = \langle \frac{da}{dt}, 0, 0, 0 \rangle \tag{3}$$

where $t$ is the proper age of the universe (i.e. cosmologically time).

Equating (2) with (3), we have the following fundamental cosmological hypothesis:

**Hypothesis 1** The universe is spherical and the speed of light varies with cosmological time $t$ subject to the constraint

$$\dot{a}(t) = c(t) \tag{4}$$

Having a speed of light which varies with cosmological time is no stranger than having a scale factor which varies with cosmological time.
1.3 Outline

In Part I we will devote our attention to necessary modifications of classical cosmology that should be taken into account when dealing with varluminopic theories. In §2 we will state the metric Ansatz connected with our hypothesis (1). In §3 we will make a brief comment on physical constants. In §4, we will show that an extra term appears in the geodesic equations when dealing with a varluminopic theory. This arises as the geodesic equations are geometric and not dynamical equations, and thus must be written with derivatives with respect to arclength $ds$. In converting to derivatives with respect to proper time, the chain rule now takes effect, thus modifying the standard geodesic equations.

In Part II, we will apply Einstein’s Field Equations to our metric Ansatz, and look at the resulting dynamics. Thus in §6, we form all of the tensor structures related to the metric, such as the Ricci Tensor, Einstein Tensor, Christoffel Symbols, and Ricci Scalar. In §6.2 we write the resulting field equations, which, unlike their isoluminopic FRW counterparts, do not yield an acceleration equation, rather pressure and density equations. An additional constraint is needed to specify the resulting dynamics. However, it uniquely establishes the equation of state in cosmological time, which was not previously the case. In §7 we find the resulting covariant divergence of the field equations, which contain additional terms due to the varying speed of light. One now has the freedom to constrain the total physical energy of the universe to be conserved, which was not previously possible. We will explore this scenario, the so-called isoergic case, in §8. Alternatively, we can impose an adiabatic condition, tantamount to constraining the covariant divergence of the stress-energy tensor to vanish. In this case, one is forced to pick up an additional dynamic variable, either $G$ or $\Lambda$ or both. Thus one would require an additional constraint to determine the resulting dynamics.

In Part III, we introduce a new set of Varluminopic Field Equations. We start from the hypothesis that the varluminopic and isoluminopic gravitational actions coincide. We then derive the resulting field equations, in parallel to their classical derivation, while including varluminopic effects. In §9 we show that the standard Lagrangian density can be recast in a form dependend only on the metric and its first derivative, as is done classically, even in the varluminopic case. In §10 we derive the resulting field equations. Finally, in §12 we impose the additional energy conservation constraint $\nabla_\mu T^\mu_\nu = 0$, and determine the resulting dynamical evolution of the universe. We note that the extra terms which appear in the Varluminopic Field Equations, as they appear in §10, save us from having to add additional dynamical variables and constraints on the motion, as is the case when one attempts to impose this constraint using Einstein’s Classical Field Equations.

We would additionally like to state that some work has been done in varluminopic cosmologies (commonly known as VSL or variable-speed-of-light theories). See Basset al [2] and the references contained therein, Barrow [1] and Magueijo [8], for instance. This paper will present a different approach on varluminopic theories.

2 Change of Variables

The FRW metric with $k = +1$ is equivalent to the form:

$$ds^2 = c^2 dt^2 - a(t)^2 \left( dp^2 + \sin^2(\rho) d\theta^2 + \sin^2(\rho) \sin^2(\theta) d\phi^2 \right)$$  \hspace{1cm} (5)

the spatial component of which represents the restriction of the standard Euclidean metric in $\mathbb{R}^4$ to the three-sphere $S^3_0$. Our paper diverges from the classical analysis of this problem by imposing the kinematic constraint

$$\frac{da}{dt} = c(t)$$  \hspace{1cm} (6)

We now introduce the change of variables $t \rightarrow a(t)$ defined by:

$$a(t) = \int_0^t c(s) \, ds$$  \hspace{1cm} (7)

Since $c(t) > 0$, $a(t)$ represents a monotone, increasing function of the cosmological (coordinate) time $t$. This transformation can therefore be inverted to express:

$$\tilde{c}(a) := c(t(a))$$  \hspace{1cm} (8)
Under this change of coordinates, our metric Ansatz becomes:

\[ ds^2 = \tilde{c}(a)^2 d\tau^2 = da^2 - a^2\{d\rho^2 + \sin^2(\rho)d\theta^2 + \sin^2(\rho)\sin^2(\theta)d\phi^2\} \] (9)

### 3 A Note on Fundamental Physical Constants

In Einstein’s derivation of the classical field equations, he supposes that the Einstein tensor is proportional to the stress-energy tensor of the field, i.e.

\[ G_{\mu\nu} = \sigma T_{\mu\nu} \] (10)

where \( \sigma \) works out to be

\[ \sigma = \frac{8\pi G}{c^4} \] (11)

where \( G \) is Newton’s constant, which appears in Newton’s Law of Gravitation:

\[ F = -\frac{GMm}{|r|^3} \] (12)

When considering varluminopic theories, we must discover whether it is \( \sigma \) or \( G \) which is to be held constant. Had Einstein come up with the field equations first, then Newton’s Law would have been written

\[ F = -\frac{\sigma c^4 Mm}{8\pi |r|^3} \] (13)

and it would have been \( \sigma \) which would have been considered the fundamental constant of gravitation.

In this paper, we will take \( G \) to be the fundamental constant. We would like to remark that this is not entirely obvious, and is an assumption. There is no distinction in the isoluminopic models, however it now makes a decided difference when we treat the speed of light as a dynamic variable.

### 4 Modified Geodesic Equations

If one considers the motion of a particle at rest in the cosmological frame (where \( ds^2 = da^2 \)), one would expect that the dynamics of the scale factor should be determined by solving the geodesic equations associated with the metric (10). One might naively mistake this evolution to be nontrivial, i.e. \( \ddot{a} = 0 \). This conclusion would be invalid. What is actually being observed is the obvious condition that \( \frac{d^2a}{ds^2} = 0 \).

In our theory; in fact, in any variable speed of light (VSL) theory, one can modify the geodesic equations to give dynamical equations of motion, as opposed to the standard geometric equations of motion. This distinction does not exist in classical general relativity, where the speed of light is a constant and can be moved freely through differential operators.

The geometric geodesic equations, which are valid even in the VSL case, are the standard geodesic equations of motion:

\[ \frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds} = 0 \] (14)

However, using the relation

\[ ds^2 = c(t)^2 d\tau^2 \] (15)

where \( t \) is the local cosmological time, one sees that:

\[ \frac{dx^\mu}{ds} = \frac{dx^\mu}{d\tau} \frac{1}{c(t)} \] (16)

\[ \frac{d^2x^\mu}{ds^2} = \frac{d^2x^\mu}{d\tau^2} \frac{1}{c(t)^2} \frac{dx^\mu}{d\tau} \frac{1}{c(t)^2} \frac{dc(t)}{dt} \frac{dt}{d\tau} \frac{dt}{d\tau} \] (17)
Substituting into the geometric geodesic equation (14), one obtains:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = \frac{dx^\mu}{d\tau} \frac{c'(t)}{c(t)} \frac{dt}{d\tau}$$  \hspace{1cm} (18)

Or, alternatively,

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = \frac{dx^\mu}{d\tau} \frac{d\tilde{c}(a)}{da} \frac{dt}{d\tau}$$  \hspace{1cm} (19)

We note that if a particle is at rest in the cosmological frame, then $dt = d\tau$. Substituting the Christoffel Symbols from (9) into the modified geodesic equations yield no information on the dynamics of the scale factor, i.e. they are trivially satisfied. To determine the dynamics of the scale factor, we now turn to Einstein’s Field Equations.

5 Redshifts and Distances

We derive the appropriate modifications for redshifts and luminosity distances, taking into account the changing speed of light. See Bergström [3] and Breton [4], among many others, for a review of the isoluminopic derivation.

5.1 Redshifts

Light waves travel along null-geodesics. Setting the $\theta$ and $\phi$ angular coordinate displacements to zero, we have, for photons travelling along constant $\rho$ world lines, the following:

$$\dot{a}^2 dt^2 - a^2 d\rho^2 = 0$$  \hspace{1cm} (20)

For a photon emitted at $t_e$ and absorbed at $t_0$, we thus have

$$\int_{t_e}^{t_0} \frac{\dot{a}(t)}{a(t)} \, dt = \int_0^\rho \, d\rho$$  \hspace{1cm} (21)

Suppose that the next wave peak is emitted and absorbed $\delta t_e$ and $\delta t_o$ later, respectively. Since they both travel through an angular displacement $\rho$, we have

$$\int_{t_e}^{t_e+\delta t_e} \frac{\dot{a}(t)}{a(t)} \, dt = \int_{t_o}^{t_o+\delta t_o} \frac{\dot{a}(t)}{a(t)} \, dt$$  \hspace{1cm} (22)

From which it becomes clear that

$$\int_{t_e}^{t_e+\delta t_e} \frac{\dot{a}(t)}{a(t)} \, dt = \int_{t_o}^{t_o+\delta t_o} \frac{\dot{a}(t)}{a(t)} \, dt$$  \hspace{1cm} (23)

Assuming that $\dot{a}(t)$ and $a(t)$ are approximately constant over the durations $\delta t_e$ and $\delta t_o$, we find that:

$$\frac{\delta t_o}{\delta t_e} = \frac{\dot{a}(t_e)\lambda_o}{\dot{a}(t_o)\lambda_e} = \frac{\dot{a}(t_e) a(t_o)}{\dot{a}(t_o) a(t_e)}$$  \hspace{1cm} (24)

where $\lambda$ is the wavelength. Therefore the cosmological redshift $z$ is given by

$$1 + z = \frac{\lambda_0}{\lambda_e} = \frac{a(t_o)}{a(t_e)}$$  \hspace{1cm} (25)

which is the regular formula.
5.2 The Luminosity Distance

The so-called luminosity distance $d_L$ is measured indirectly by measuring the arrival power flux $\mathcal{F}$ of light from distant objects with known intrinsic luminosities $\mathcal{L}$, by the relation

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi d_L^2}$$  \hspace{1cm} (26)

The luminosity distance is the distance we would be from the object if the universe was flat and expansionless. To measure the expansion of the universe, it is standard practice to derive a theoretical relationship between the luminosity distance and redshifts of incoming light from distant sources, and then compare this relationship to observation. As usual, we must tweak this theoretical relationship to find the appropriate one for our new cosmology.

Suppose a photon with energy $E_e$ is emitted from a distant source at time $t_e$ and with wavelength $\lambda_e$. Thus

$$E_e = \frac{h \dot{a}(t_e)}{\lambda_e}$$  \hspace{1cm} (27)

Its observed energy at arrival will be

$$E_o = \frac{h \dot{a}(t_o)}{\lambda_o} = \frac{\dot{a}(t_o) \lambda_e}{\dot{a}(t_e) \lambda_o} E_e = \frac{\dot{a}(t_o)}{\dot{a}(t_e)} \frac{1}{1+z} E_e$$  \hspace{1cm} (28)

The object’s intrinsic luminosity $\mathcal{L}$ is the power of a burst of photons emitted at time $t_e$:

$$\mathcal{L} = \frac{\delta E_e}{\delta t_e}$$  \hspace{1cm} (29)

Recalling (24), we see that the object’s power at arrival is given by

$$\mathcal{L}_o = \frac{\delta E_o}{\delta t_o} = \left[ \frac{\dot{a}(t_o)}{a(t_e)} \right]^2 \frac{1}{(1+z)^2} \mathcal{L}$$  \hspace{1cm} (30)

Suppose the constant angular distance to the source is $r$ (where we’ve replaced $\rho$ in (5) to avoid confusion with the energy density). Then the arrival flux is given by

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi a(t_o)^2 r^2 (1+z)^2} \left[ \frac{\dot{a}(t_o)}{\dot{a}(t_e)} \right]^2$$  \hspace{1cm} (31)

As light travels along a null geodesic from $t_e$ when it is emitted at the source to $t_o$ when it arrives at the detectors, we have

$$0 = \dot{a}(t) \, dt - a(t) \, dr$$  \hspace{1cm} (32)

Hence

$$r = \int_0^r \, d\tilde{r} = \int_{t_e}^{t_o} \frac{\dot{a}(t)}{a(t)} \, dt = \ln \left( \frac{a(t_o)}{a(t_e)} \right)$$  \hspace{1cm} (33)

This is the simple relation between the angular displacement of a distant object ($a(t_o)r$ being its current instantaneous distance) and the scale factor. Comparing (26) to (31), we have for the luminosity distance:

$$d_L = \frac{a(t_o)}{a(t_e)} \frac{\dot{a}(t_e)}{\dot{a}(t_o)} \ln \left( \frac{a(t_o)}{a(t_e)} \right)$$  \hspace{1cm} (34)

(26) combined with (34) provide a theoretical relation between the redshift and luminosity distance of detectable light. If one can find explicit solutions for $a(t)$ and $\dot{a}(t)$, one can then determine the theoretical relationship between $d_L$ and $z$ and compare with known data.

In practice, one also computes the distance modulus, defined by

$$\mu = 5 \log_{10} \left( \frac{d_L}{1 \text{ Mpc}} \right) + 25$$  \hspace{1cm} (35)

and plots this measure of distance vs. redshift.
Part II
The Classical Field Approach

In this part, we insert our metric Ansatz into Einstein’s Field Equations and determine the resulting relationships. For the standard FRW metric, the Field Equations produce an equation for the density (the Friedmann Equation) and an acceleration equation. Classically, one must in addition impose an equation of state to determine the dynamical outcome. One can gain insight as to how the universe should evolve with various equations of states, but the equation of state as a function of cosmological time has not been determined.

On the other hand, we will show that our metric Ansatz instead produces a density equation and a pressure equation, so that one can completely determine the equation of state as a function of the dynamic variables \(a(t)\) and \(c(t)\). In order to get the dynamics, one must pose an additional constraint on the system. At this point we will distinguish two possibilities. One may take a naive approach and constrain the total physical energy of the universe to be constant. In this case, our so-called “isoergic” model, one can determine the resulting dynamics of the system. This approach cannot be done in the isoluminopic models (as the covariant divergence of the Field Equations vanish), so it has not been seriously considered. However, its philosophical implication is that, for it to be correct, there can be no transfer of energy from the matter content to the geometry (i.e. gravitational field). The second, more standard approach, is to set the covariant divergence of the stress-energy tensor to zero. This is the so-called “adiabatic” condition. We will see that as a result of this condition, one must take either \(G\) or \(\Lambda\) as an additional dynamical variable, and thus the dynamical system remains underdetermined. We would like to point out that if instead of retaining Einstein’s Field Equations one retains the gravitational action, a new modified set of Vahluminopic Field Equations will result, as we will see in Part III. Using the modified Field Equations, the adiabatic condition will completely determine the resulting dynamics without the necessity of additional dynamical variables. The considerations of this part of the paper are nonetheless interesting: and as the dynamical solutions in both parts are similar, the work done in this part will provide insight into the dynamics of the latter.

6 Einstein’s Field Equations

6.1 The Metric Ansatz

In this section we state the Christoffel Symbols and nonzero components of the Einstein and Ricci Tensors for the metric ansatz stated in 9.

Christoffel Symbols

The nonzero Christoffel Symbols are:

\begin{align*}
\Gamma_{\rho \rho}^a &= a \\
\Gamma_{\theta \theta}^a &= a \sin^2 \rho \\
\Gamma_{\phi \phi}^a &= a \sin^2 \rho \sin^2 \theta \\
\Gamma_{\rho \theta}^a &= a^{-1} \\
\Gamma_{\rho \phi}^a &= - \sin \rho \cos \rho \\
\Gamma_{\phi \theta}^a &= - \sin \rho \sin^2 \theta \cos \rho \\
\Gamma_{\phi \phi}^a &= a^{-1} \\
\Gamma_{\rho \phi}^a &= \cot \rho \\
\Gamma_{\rho \phi}^a &= - \sin \theta \cos \theta \\
\Gamma_{\phi \phi}^a &= \cot \theta \\
\end{align*}
The Ricci Tensor

The nonzero components of the Ricci Tensor are:

\[ R_{\rho\rho} = 4 \]  
\[ R_{\theta\theta} = 4 \sin^2 \rho \]  
\[ R_{\phi\phi} = 4 \sin^2 \theta \sin^2 \rho \]  

The Ricci Scalar

The Ricci Scalar becomes:

\[ R = -\frac{12}{a^2} \]  

The Einstein Tensor

The nonzero components of the Einstein Tensor are:

\[ G_{\alpha\alpha} = \frac{6}{a^2} \]  
\[ G_{\rho\rho} = -2 \]  
\[ G_{\theta\theta} = -2 \sin^2 \rho \]  
\[ G_{\phi\phi} = -2 \sin^2 \rho \sin^2 \theta \]  

The Stress-Energy Tensor

As is standard, we will be taking the stress-energy tensor to be that of a perfect fluid

\[ T_{\mu\nu} = (\varepsilon + P)U_{\mu}U_{\nu} - P g_{\mu\nu} \]  

where \( \varepsilon \) is the energy density, \( P \) the pressure, and

\[ U_{\mu} = \frac{dx^\mu}{ds} = \langle 1, 0, 0, 0 \rangle \]  

the four-velocity of a particle in the cosmological reference frame. The stress-energy tensor can be expressed in matrix form as follows:

\[
[T_{\mu\nu}] = \begin{pmatrix}
\varepsilon & 0 & 0 & 0 \\
0 & -P_{\rho\rho} & 0 & 0 \\
0 & 0 & -P_{\theta\theta} & 0 \\
0 & 0 & 0 & -P_{\phi\phi}
\end{pmatrix}
\]  

6.2 Einstein’s Field Equations

Einstein’s Field Equations (with cosmological constant \( \Lambda \)) can be written in either of the following two forms:

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{a^4}T_{\mu\nu} + \Lambda g_{\mu\nu} \]  
\[ R_{\mu\nu} = \frac{8\pi G}{a^4} \left( T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu} \right) - \Lambda g_{\mu\nu} \]  

Einstein’s Field Equations in the form of (59) imply the conditions

\[ \frac{6}{a^2} = \frac{8\pi G}{a^4} \varepsilon + \Lambda \]  
\[ \frac{2}{a^2} = -\frac{8\pi G}{a^4} P + \Lambda \]
Alternatively, the Field Equations written as (60) imply the equivalent conditions

\[ 0 = \frac{4\pi G}{a^4} (\varepsilon + 3P) - \Lambda \]  
(63)

\[ \frac{4}{a^2} = \frac{4\pi G}{a^4} (\varepsilon - P) + \Lambda \]  
(64)

For independent equations of motion, we will choose the following:

\[ \varepsilon = \frac{\dot{a}^4}{8\pi G} \left( \frac{6}{a^2} - \Lambda \right) \]  
(65)

\[ P = \frac{\dot{a}^4}{8\pi G} \left( \Lambda - \frac{2}{a^2} \right) \]  
(66)

In §8 we will solve these equations explicitly for the case of a matter dominated universe, i.e. assuming \( E = \varepsilon V \) is constant.

We can define the following density parameters:

\[ \Omega_\varepsilon = \frac{8\pi G \varepsilon a^2}{3\dot{a}^4} \]  
(67)

\[ \Omega_\Lambda = \frac{\Lambda a^2}{3} \]  
(68)

\[ \Omega_k = -k \]  
(69)

where \( k = +1 \) in our model, as our metric Ansatz is equivalent to an \( k = +1 \) FRW metric with the additional kinematic constraint \( c(t)dt = da \). By convention, \( \Omega_\varepsilon \) is the density parameter of the total energy density: matter, radiation, and any additional vacuum energy besides the cosmological constant. With these definitions, (60) becomes:

\[ \Omega_{\text{tot}} = \Omega_\varepsilon + \Omega_\Lambda + \Omega_k = 1 \]  
(70)

which is just the regular Friedman’s Equation. We note that these definitions are commensurate with the standard ones if you simplify using the kinematic condition \( \dot{a} = c \).

We would also like to note that \( \Omega_\varepsilon = \Omega_R + \Omega_M + \Omega_V \) is taken to have contributions from radiation, matter, and nonlambdric vacuum energy; obtained by using \( \varepsilon_R, \varepsilon_M, \) and \( \varepsilon_V \) into (67), respectively.

### 6.3 The Equation of State

Since Einstein’s Field Equations determine the energy density \( \varepsilon \) and pressure \( P \) as a function of the scale factor, via (65) and (66), they therefore prescribe as well the equation of state. Introducing the state variable \( \chi = \Lambda a^2 \), we have:

\[ w(\chi) = \frac{P}{\varepsilon} = \frac{\chi - 2}{6 - \chi} \]  
(71)

In particular, we observe

\[ w(0) = \frac{1}{3} \]  
(72)

\[ w(2) = 0 \]  
(73)

\[ w(3) = \frac{1}{3} \]  
(74)

\[ \lim_{\chi \to 6^-} w(\chi) = +\infty \]  
(75)

\[ \lim_{\chi \to 6^+} w(\chi) = -\infty \]  
(76)

\[ \lim_{\chi \to \infty} w(\chi) = -1 \]  
(77)

In particular, we note the following important epochs.
6.3.1 Important Epochs

The Null-$\chi$ Epoch

When $\chi = 0$, we have the equation of state $P = -\varepsilon/3$.

The Dust Epoch

The universe can be treated as dust for $\chi = 2$. For $\chi < 2$, the pressure in the universe is negative. For $\chi > 2$, the pressure is positive. We are currently very close to this important epoch.

The Radiation Epoch

The universe can be treated as radiation for $\chi = 3$, when its equation of state is $P = \varepsilon/3$.

The Vanishing Epoch

As $\chi \to 6^-$, the energy density goes to zero. At $\chi = 6$, the universe is completely empty in net energy content. After this epoch passes, the universe (if it continues) shall have negative energy density for all subsequent time.

The $\chi$-Infinitum Epoch

As $\chi \to \infty$, the universe becomes dominated with a vacuum energy (unrelated to the cosmological constant term) with an ultimate equation of state $P = -\varepsilon$.

6.3.2 $\chi$ and the Cosmological Constant

As we are considering a class of cosmologies where the speed of light is allowed to vary with cosmological time, there is nothing which prohibits the cosmological constant $\Lambda$ from doing the same. For isolambdic cosmologies (with $\dot{\Lambda} = 0$), $\chi$ is an increasing function of cosmological time $t$, since $a(t)$ is. This is not necessarily true for their varlambdic counterparts, where evolution of $\chi$ is determined by $\dot{\chi} = 2\Lambda a \dot{a} + \Lambda a^2$. 

Figure 1: Equation of State vs. state variable $\chi = \Lambda a^2$
7 The Classical Thermodynamic Analogy

7.1 Covariant Divergence of the Field Equations

The Field Equations can be written in the form

\[ G^\mu_\nu = \frac{8\pi G}{a^4} T^\mu_\nu + \Lambda \delta^\mu_\nu \] (78)

where

\[ [T^\mu_\nu] = [g^{\mu\lambda}T_{\lambda\nu}] = \begin{pmatrix}
\varepsilon & 0 & 0 & 0 \\
0 & -P & 0 & 0 \\
0 & 0 & -P & 0 \\
0 & 0 & 0 & -P
\end{pmatrix} \] (79)

is the standard 1-1 stress-energy tensor.

For the moment, we will allow both \( G \) and \( \Lambda \) to vary with cosmological time, for the sake of generality. We only include the case \( \dot{G} \neq 0 \) momentarily, to leave room for future speculations. Recognizing

\[ \frac{\partial}{\partial a} = \frac{dt}{da} \frac{\partial}{\partial t} \] (80)

we obtain for the only nontrivially nonzero component of \( 0 = \nabla_\mu (8\pi GT^\mu_\nu/\dot{a}^4 + \Lambda \delta^\mu_\nu) \) the following:

\[ \frac{8\pi G}{\dot{a}^4} \left( \frac{\dot{\varepsilon}}{\dot{a}} + 3(\varepsilon + P) \right) + \frac{8\pi \varepsilon}{\dot{a}} \left( \frac{\dot{G}}{\dot{a}^3} - \frac{4G\dot{\varepsilon}}{\dot{a}^5} \right) + \frac{\dot{\Lambda}}{\dot{a}} = 0 \] (81)

This can be compactified into the following equation

\[ \frac{d}{dt} \left( \ln \left( \frac{\dot{a}^4}{\varepsilon a^3 G} \right) \right) = \frac{3w\dot{a}}{a} + \frac{\dot{a}^4 \dot{\Lambda}}{8\pi G \varepsilon} \] (82)

Thus

\[ \frac{\dot{a}^4}{\varepsilon a^3 G} \propto \exp \left( \int \frac{3w\dot{a}}{a} + \frac{\dot{a}^4 \dot{\Lambda}}{8\pi G \varepsilon} \, dt \right) \] (83)

We would like to point out that \( [S1] \) imposes no additional constraint on the evolutionary dynamics of the system. By \( \varepsilon \) and \( P \) satisfying \( [G] \) and \( [H] \), as solutions to the Field Equations, they automatically satisfy this condition. Whereas the standard FRW metric produces the Frieman Equation and the acceleration equation, and leaves open the equation of state; our modification, when submitted to the Field Equations, determines the equation of state and leaves open way for an additional constraint which one must impose on the system to determine the dynamics. For example, one now has room to impose the condition of isoergicity on the universe, or adiabaticity, for example, as we will see in the next section.

7.2 The First Law of Thermodynamics

In this section we shall proceed entirely by analogy. Viewing the universe as a closed and isolated system, we can compare various terms which arise in \( [S1] \) to terms which would arise doing a purely classical first law control volume analysis.

Allowing \( E = \varepsilon V \) be the total energy in the universe, where \( V = 2\pi^2 a^3 \) is the total volume of \( S_3^a \), we can apply the classical First Law of Thermodynamics and compare with our conservation condition \( [S1] \):

\[ \dot{E} = \dot{Q} - W = \dot{Q} - p\dot{V} \] (84)

For the given \( E = 2\pi^2 a^3 \varepsilon \) and \( V = 2\pi^2 a^3 \), we have

\[ 6\pi^2 a^2 \dot{\varepsilon} + 2\pi^2 a^3 \dot{\varepsilon} = \dot{Q} - 6p\pi^2 a^2 \dot{a} \] (85)
We can meanwhile rewrite (81) as follows:

\[ 6\pi^2 a^2 \dot{a} \varepsilon + 2\pi^2 a^3 \dot{\varepsilon} = \left( \frac{8\pi^2 a^3 \varepsilon \ddot{a}}{\dot{a}} - \frac{\Lambda \dot{a}^4 a^3 \pi}{4G} - 2\pi^2 \varepsilon a^3 \frac{\dot{G}}{G} \right) - 6\pi^2 a^2 \dot{a} \]  

(86)

Thus we can identify

\[ \dot{Q} = 8\pi^2 a^3 \varepsilon \ddot{a} - \frac{\Lambda \dot{a}^4 a^3 \pi}{4G} - 2\pi^2 \varepsilon a^3 \frac{\dot{G}}{G} \]  

Cosmological Heating  

(87)

as a Cosmological Heating term, in analogy to classical Thermodynamics.

In isoluminopic models, we have that \( \dot{Q} \equiv 0 \). And thus it is a direct result of the form of Einstein’s Field Equations that we must take \( \nabla_{\mu} T^{\mu}_{\nu} = 0 \) (which account for the remaining terms of (86)). In a varluminopic model, if one uses the classical Field Equations, we are left with the necessity of imposing an additional constraint, which should take the form of a conservation law. We now mention two philosophically different approaches.

7.2.1 The “Isoergic” Condition

In a so-called isoergic model, we would take \( \dot{E} = 0 \). The varying speed of light gives us room so that varluminopic effects (encapsulated in the \( \dot{Q} \) term) can do the expansion work for us, so that the total physical energy of the universe is conserved. In this choice, physical energy cannot be converted into the gravitational energy of the universe, as is normally done in the isoluminopic case (though we note that in the isoluminopic case, this choice does not exist, and one is forced to consider a transfer of physical energy to a gravitational cosmological energy). The condition that \( \dot{E} = 0 \), however, is enough to provide us with an acceleration equation, so that the dynamics of the evolving universe can be analyzed. We will do this in §98.

7.2.2 The “Adiabatic” Condition

On the other hand, we do not have to abandon adiabaticity, which would impose the following dynamical equation of motion

\[ \frac{8\pi^2 a^3 \varepsilon \ddot{a}}{\dot{a}} = \frac{\Lambda \dot{a}^4 a^3 \pi}{4G} + 2\pi^2 \varepsilon a^3 \frac{\dot{G}}{G} \]  

Adiabatic Condition  

(88)

This is tantamount to imposing the condition that \( \nabla_{\mu} T^{\mu}_{\nu} = 0 \). However, it is clear that if one is to obtain nontrivial dynamical solutions \( \ddot{a} \neq 0 \), one must include either \( G \) or \( \Lambda \) or both as dynamical variables. Thus the system remains underdetermined, and an additional constraint is needed to determine the dynamics. Such a constraint can be obtained by fixing the equation of state. We will explore this approach in §13.

It is interesting to note that this is not the case with the modified Field Equations we will derive in Part III. With the Varluminopic Field Equations one can impose the adiabatic condition \( \nabla_{\mu} T^{\mu}_{\nu} = 0 \) and completely determine the resulting dynamics, without the necessity of the introduction of additional dynamic variables.

8 Isoergic Models

As mentioned previously, in these new models, we now have the freedom to constrain the total energy \( E = \varepsilon V \) of the universe to be constant. In the isoergic model, the cosmological heating (87) is what does the expansion work, so that the total physical energy in the universe is conserved. We will take \( \dot{G} = \dot{\Lambda} = 0 \). Using these conditions in (82) leaves us with the following equations of motion

\[ \frac{a \ddot{a}}{\dot{a}^2} = \frac{3w}{4} \]  

(89)

Recalling (71) and defining \( c = \dot{a} \), we can rewrite this as

\[ \frac{dc}{c} = \frac{-16 - 3\Lambda a^2}{4 \cdot 6a - \Lambda a^3} \, da \]  

(90)
Integrating, we find
\[ \dot{a} = k(6a - \Lambda a^3)^{-1/4} \]  
(91)
Alternatively, we can proceed directly from the condition \( E = \text{const.} \):
\[ 8\pi G\varepsilon a^3 = \dot{a}^4 \left( 6a - \Lambda a^3 \right) = k^4 \]  
(92)
where \( k = \sqrt{4GE_{\text{tot}}/\pi} \). Solving for \( \dot{a} \) we arrive at:
\[ \frac{da}{dt} = \frac{k}{(6a - \Lambda a^3)^{1/4}} \]  
(93)
which agrees with (91), but does not require that \( G \) and \( \Lambda \) be constant. Assuming \( G \) and \( \Lambda \) are constants, we can integrate (93) to obtain
\[ kt = \frac{4}{5}\sqrt{6a^5} \, _2F_1 \left( \frac{-1}{4}, \frac{5}{8}, \frac{13}{8}; \frac{\Lambda a^2}{6} \right) \]  
(94)
where \( _2F_1 \) is the Gauss Hypergeometric Function (see [18]):
\[ _2F_1(\alpha, \beta; \gamma; \delta) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} \delta^n \]  
(95)
which has radius of convergence \( |\delta| < 1 \). Here, \((a)_n\) is the shifted factorial
\[ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \]  
(96)
and \( \Gamma(z) \) is the \( \Gamma \) function:
\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \]  
(97)
The solution (94) can be simplified as:
\[ kt = \frac{4}{5}\sqrt{6a^5} \sum_{n=0}^{\infty} \frac{\Gamma(n-1/4)}{(5+8n)n!} \left( \frac{\Lambda a^2}{6} \right)^n \]  
(98)
We plot the speed of light (93) in Fig. 2 left and the scale factor in Fig. 2 right.

Figure 2: \( a(t)/a(t_0) \) vs. \( t \) (left) and \( c(t)/c(t_0) \) vs. \( t \) (right)

The radius of convergence of (98) is \( \Lambda a^2 < 6 \). At the epoch \( \Lambda a^2 = 6 \), the total energy density of the universe vanishes. Beyond this point, the energy density would have to become negative for the universe.
to continue, a situation we view as unphysical. We can use (94) with the following mathematical fact (which holds for $c - b - a > 0$, as in our case):

$$2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - b - a)}{\Gamma(c - b)\Gamma(c - a)}$$

(99)

to see that this model predicts an end to the universe in the finite time

$$T_{\text{end}} = 4 \cdot 6^{7/8} \Gamma(13/8)\Gamma(5/4) \approx 20.4 \text{ Gyr}$$

(100)

where we used $a = 1.32 \times 10^2 m$ and $\Lambda a^2 = 2.1$ (so that $\Omega_\Lambda = .7$). These numbers imply the present age of the universe to be close to 12 Gyr, so that we are currently 8.4 Gyr from the end of the universe.

Moreover, using the redshift formula (25), the luminosity distance formula (34), and the definition (35), we can compare the theoretical curve produced by this model against data points from Supernova Ia data as recorded by Riess [13]. We find our theoretical curve matches the supernova quite well, as is shown in Fig. 3.

![Supernovae Ia data vs. theoretical predictions](image)

Figure 3: Supernovae Ia data vs. theoretical predictions

Without the Cosmological Constant $\Lambda$, this solution degenerates to

$$a(t) = \frac{1}{6^{1/5}} \left(\frac{5kt}{4}\right)^{4/5}$$

(101)

Inverting (100) gives the scale factor, $a(t)$, as a function of time.

**Part III**

**Varluminopic Field Theory**

In Isoluminopic Field Theory, Einstein’s Field Equations can be derived by varying the following action

$$S = \int \left\{ \frac{\alpha^4}{16\pi G} (\mathcal{R} + 2\Lambda) - 2\mathcal{R} \right\} dt d^3\theta$$

(102)

Gothicized variables denote tensor densities. For instance, $\mathcal{R} = R\sqrt{-g}$ is the tensor density of the Ricci scalar, and $\mathcal{U} = \sqrt{-g}$ is the tensor density of unity. $\mathcal{M}$ represents the Lagrangian density of the mass-energy which occupies the space-time, e.g., matter, radiation, etc. It is well known that one cannot
construct a scalar density form the metric and its first derivatives alone. Thus the Ricci scalar density represents the most general covariant candidate for a gravitational action. Even though it contains second derivatives of the metric, these have no effect when varying the action \([102]\), see Landau and Lifshitz \([7]\), Tolman \([10]\), or Pauli \([11]\), amongst others. Following the derivation in Landau and Lifshitz with appropriate modifications, we will show in \([9]\) that this is true even when taking the time-varying speed of light into account.

The action for any Varluminopic Field Theory should therefore reduce to \([102]\) in the isoluminopic limit. The most natural candidate for an action is therefore \([102]\) itself. In \([9]\) we will derive the resulting field equations from the action \([102]\), taking the varying speed of light into account. We will then apply these new field equations to the \(k = +1\) FRW model, including the kinematic constraint \(\dot{a} = c\), to determine the dynamical equations of motion of the resulting theory. We will see that imposition of the additional energy conservation constraint \(\nabla_\mu T^\mu_\nu = 0\) is enough to determine the resulting dynamics explicitly, as is the case with the classical Field Equations, where one was left with the necessity of having to also include \(G\) or \(\Lambda\) as dynamical variables.

In formulating the variational principle for gravity, we will find that the cosmological coordinates \((t, \rho, \theta, \phi)\) will be better suited for our needs. We will keep these coordinates in the back of our mind while we derive the modified field equations. There are two important features in doing this. Notice that all factors of \(c\) in \([102]\) are attached solely to the gravitational action. Thus, by letting \(x^0 = t\), as is in our prerogative to do so, we decouple the varluminopic effects from the matter action. In standard classical derivations of the Field Equations, the entire integrand of \([102]\) is typically divided by an extra factor of \(c\), in exchange for swapping the \(dt\) with a \(da\) (in the notation allowed by our insight \(da = cdt\)). Moreover, in this choice of variables we have \(\gamma_{00} = c^2\), so that variations of \(c\) can be easily related to variations of the metric, as will be shown in \([10]\).

9. The Action of the Gravitational Field

As is standard in relativity theory, one identifies the space-time metric with the gravitational field. It is well-known that there is no scalar density which depends only on the metric and its first derivatives. Thus the Ricci tensor is taken as the key ingredient of the Lagrangian density of the gravitational field.

Including factors of \(c\) which must be present for our later considerations, the action is similar to the nominal action

\[
S_n = \int c^4 \mathcal{R} dt d^3 \theta
\]

where

\[
\mathcal{R} = R \sqrt{-g}
\]

is the Ricci scalar density. Our aim is to show that this covariant Lagrangian density is equivalent to one which only involves the metric and its first derivatives (but depends on the coordinates). Thus we will show that

\[
\int c^4 \mathcal{R} dt d^3 \theta = \int \mathcal{E} dt d^3 \theta + \int \frac{\partial \mathbf{w}^i}{\partial x^i} dt d^3 \theta
\]

where \(\mathbf{w}^i\) is a vector density, which, by means of Gauss’ Theorem, can be converted to an integral over the boundary and therefore ignored, and \(\mathcal{E}\) is the pseudo-scalar density which depends only on the metric and its first derivative. We will follow closely the discussion in Landau and Lifshitz \([7]\) §93, with appropriate modification for the varluminopic effects.

The integrand may be expanded as

\[
c^4 \sqrt{-g} R = c^4 \sqrt{-g} g^{ik} R_{ik} = c^4 \sqrt{-g} \left\{ g^{ik} \frac{\partial \Gamma^l_{ik}}{\partial x^l} - g^{ik} \frac{\partial \Gamma^l_{il}}{\partial x^k} + g^{ik} \Gamma^l_{ikm} - g^{ik} \Gamma^m_{il} \Gamma^l_{km} \right\} \quad (103)
\]

Notice the first two terms may be written as

\[
c^4 \sqrt{-g} g^{ik} \frac{\partial \Gamma^l_{ik}}{\partial x^l} = \frac{\partial}{\partial x^l} \left( c^4 \sqrt{-g} g^{ik} \Gamma^l_{ik} \right) - \Gamma^l_{ik} \frac{\partial}{\partial x^l} \left( c^4 \sqrt{-g} g^{ik} \right) \quad (104)
\]

\[
c^4 \sqrt{-g} g^{ik} \frac{\partial \Gamma^l_{il}}{\partial x^k} = \frac{\partial}{\partial x^k} \left( c^4 \sqrt{-g} g^{ik} \Gamma^l_{il} \right) - \Gamma^l_{il} \frac{\partial}{\partial x^k} \left( c^4 \sqrt{-g} g^{ik} \right) \quad (105)
\]
The first term on the right hand side of either of these equations may be dropped as it may be converted into a boundary integral which vanishes, as we will take the variations to vanish on the boundary. Thus we have

\[
\mathcal{E} = c^4 \left\{ \Gamma^m_{im} \frac{\partial}{\partial x^i} \left( \sqrt{-g} g^{ik} \right) - \Gamma^l_{ik} \frac{\partial}{\partial x^i} \left( \sqrt{-g} g^{ik} \right) - c^4 \left( \Gamma^m_{il} \Gamma^l_{km} - \Gamma^l_{ik} \Gamma^m_{lm} \right) g^{ik} \sqrt{-g} \right\}
\]

\[+ \Gamma^m_{im} \sqrt{-g} g^{ik} \frac{\partial (c^4)}{\partial x^k} - \Gamma^l_{ik} \sqrt{-g} g^{ik} \frac{\partial (c^4)}{\partial x^l} \]  

\[= c^4 \sqrt{-g} \left( \Gamma^m_{il} \Gamma^l_{km} - \Gamma^l_{ik} \Gamma^m_{lm} \right) + \Gamma^m_{im} \sqrt{-g} g^{ik} \frac{\partial (c^4)}{\partial x^l} \]

\[= c^4 \sqrt{-g} \left( \Gamma^m_{il} \Gamma^l_{km} - \Gamma^l_{ik} \Gamma^m_{lm} \right) + \Gamma^m_{im} \sqrt{-g} g^{ik} \frac{\partial (c^4)}{\partial x^l} \]

This pseudo-scalar density only depends on the metric and its first derivative, and we have that

\[
\delta \int c^4 \mathcal{R} dtd^3 \theta = \delta \int \mathcal{E} dtd^3 \theta
\]

Thus we shall not hesitate in using \( \mathcal{R} \) as a Lagrangian density, even though it contains second derivatives of the metric, as they have no effect when one takes the variation.

### 10 The Varluminopic Field Equations

We wish to find the corresponding equations of motion by varying the action \[102\] with respect to the space-time metric. The only difference between our procedure and the classical approach is that \( c \) now varies with cosmological time. Hence we need to determine the effects of this on the resulting equations of motion. Noting that

\[
\delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} g_{ik} \delta g^{ik}
\]

we see that the variation of the first part of the integrand of \[102\] is

\[
\delta \left( c^4 \mathcal{R} \right) = \delta \left( c^4 g^{ik} R_{ik} \sqrt{-g} \right) = c^4 \left( \mathcal{R}_{ik} - \frac{1}{2} \mathcal{R} g_{ik} \right) \delta g^{ik} + c^4 g^{ik} \delta R_{ik} + 4 \mathcal{R} c^3 \delta c
\]

In order to deal with the \( c^4 g^{ik} \delta R_{ik} \) term, we will again closely follow the derivation in Landau and Lifshitz \[7\], with the appropriate modifications necessary for our varluminopic considerations. Choosing a locally geodesic frame, we have that

\[
c^4 g^{ik} \delta R_{ik} = c^4 g^{ik} \frac{\partial}{\partial x^l} \left( \delta \Gamma^l_{ik} \right) - c^4 g^{il} \frac{\partial}{\partial x^l} \left( \delta \Gamma^k_{ik} \right)
\]

Now consider the vector

\[
w^l = c^4 g^{ik} \delta \Gamma^l_{ik} - c^4 g^{il} \delta \Gamma^k_{ik}
\]

Taking the divergence with respect to \( x^l \), we find

\[
\frac{\partial w^l}{\partial x^l} = c^4 g^{ik} \frac{\partial}{\partial x^l} \left( \delta \Gamma^l_{ik} \right) - c^4 g^{il} \frac{\partial}{\partial x^l} \left( \delta \Gamma^k_{ik} \right) + (g^{ik} \delta \Gamma^l_{ik} - g^{il} \delta \Gamma^k_{ik}) \frac{\partial (c^4)}{\partial x^l}
\]

Returning to an arbitrary reference frame, we see that

\[
c^4 g^{ik} \delta R_{ik} = \frac{1}{\sqrt{-g}} \frac{\partial w^l}{\partial x^l} - (g^{ik} \delta \Gamma^l_{ik} - g^{il} \delta \Gamma^k_{ik}) \frac{\partial (c^4)}{\partial x^l}
\]

We will now show that the extra term which arises (second term on the right hand side) actually vanishes. Recalling that

\[
\delta g_{lp} = -g_{lk} g_{ps} \delta g^{ks}
\]
we have that
\[
\delta \Gamma_{ik}^k = \delta \left( g_{is} g^{ks} \Gamma_{ik}^l \right)
\]
(117)
\[
= g_{is} \Gamma_{ik}^l \delta g^{ks} + g^{ks} \Gamma_{ik}^l \delta g_{ls} + g_{is} g^{ks} \delta \Gamma_{ik}^l
\]
(118)
\[
= g_{is} \Gamma_{ik}^l \delta g^{ks} + g^{ls} \Gamma_{ik}^l \delta g_{ps} + g_{is} g^{ks} \delta \Gamma_{ik}^l
\]
(119)
\[
= g_{is} \Gamma_{ik}^l \delta g^{ks} - g^{ls} \Gamma_{lk}^r g_{ir} \delta g_{ps} + g_{is} g^{ks} \delta \Gamma_{ik}^l
\]
(120)
Thus
\[
g^{il} \delta \Gamma_{ik}^l = \Gamma_{sk}^l \delta g_{ks} - \Gamma_{ks}^l \delta g^{ks} + g^{ki} \delta \Gamma_{ik}^l
\]
(121)
and therefore
\[
g^{ik} \delta \Gamma_{ik}^l - g^{il} \delta \Gamma_{ik}^k = 0
\]
(122)
so that
\[
\int_{\Omega} c^4 g^{ik} \delta R_{ik} dtd^3 \theta = \int_{\partial \Omega} \partial w^l \frac{\partial}{\partial x^l} dtd^3 x = 0
\]
(123)
as we take all variations to vanish on the boundary.

We are therefore justified to write the variation of the full action (102)
\[
\delta \int \left\{ \frac{c^4}{16 \pi G} (R + 2 \Lambda) - \mathcal{M} \right\} dtd^3 \theta
\]
(124)
\[
= \int \left\{ \frac{c^4}{16 \pi G} \left( \mathcal{G}_{ik} - \Lambda g_{ik} - \frac{\Xi_{ik}}{2} \right) \delta g^{ik} + \frac{c^3}{4 \pi G} (R + 2 \Lambda) \frac{\delta \mathcal{G}}{\delta \delta_{ik}} \right\} dtd^3 \theta
\]
(125)
where
\[
\mathcal{G}_{ik} = \mathcal{R}_{ik} - \frac{1}{2} \mathcal{R} g_{ik}
\]
is the Einstein tensor density.

Writing the variation of (102) in the form (124), we are now able to see the full advantage of considering our variational principle using the coordinate \( x^0 = t \) as cosmological time as opposed to \( x^0 = a \), as is classically the choice. First, one no longer has a factor of \( c^{-1} \) appearing in the matter action, so that the varluminopic effects limit themselves to effects on the action of the gravitational field. Moreover, with the choice of \( t \) for our cosmological coordinates, we have the relation
\[
g^{00} = \frac{1}{c^2}
\]
(126)
so that
\[
\delta g^{00} = - \frac{2}{c^3} \delta c
\]
(127)
or
\[
\delta c = - \frac{c^3}{2} \delta g^{00}
\]
(128)
and thus the variation of (102) can be written, for our choice of cosmological coordinates, as:
\[
\delta \int \left\{ \frac{c^4}{16 \pi G} (R + 2 \Lambda) - \mathcal{M} \right\} dtd^3 \theta
\]
(129)
\[
= \int \left\{ \frac{c^4}{16 \pi G} \left( \mathcal{G}_{ik} - \Lambda g_{ik} - \frac{\Xi_{ik}}{2} \right) \delta g^{ik} + \frac{c^3}{8 \pi G} (R + 2 \Lambda) \delta_{ik} \delta_{00} \right\} dtd^3 \theta
\]
(130)
Thus, the modified Field Equations, in our choice of cosmological coordinates, become
\[
G_{\mu \nu} = \frac{8 \pi G}{c^4} T_{\mu \nu} + \Lambda g_{\mu \nu} + 2c^2 (R + 2 \Lambda) \delta_{\mu}^0 \delta_{\nu}^0
\]
(131)
Consider now a general coordinate system \( x^\mu \). To every point in space-time we identify a cosmological scale factor \( a(x^\mu) \), which represents a scalar function on the space-time. As the map \( a(t) \) is one-to-one, which identifies the cosmological age of the universe with the scale factor, we also have the scalar function
\[ G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda g_{\mu\nu} + 2c^2(R + 2\Lambda) \frac{\partial t}{\partial x^\mu} \frac{\partial t}{\partial x^\nu} \] (131)

Moreover, recalling \( da = c(t)dt \), we can alternatively express the extra term as a tensor product of the divergence of the scale factor \( a(x^\mu) \) with itself:

\[ G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda g_{\mu\nu} + 2\left(\frac{\partial a}{\partial x^\mu} \frac{\partial a}{\partial x^\nu}\right) \] (132)

Hence, even though we choose to work in the cosmological frame to simplify computations, the modified Field Equations can nonetheless be written in an entirely covariant fashion.

### 11 A Cosmological Action

Suppose we would like to add a cosmological action of the form

\[ S_c = \int \mathcal{C} dt d^3\theta \] (133)

to the action \( S \), where \( \mathcal{C} \) is of the form

\[ \mathcal{C} = C(a, c)\mathcal{M} \] (134)

We have

\[ \delta \mathcal{C} = \frac{\partial C}{\partial a} \mathcal{M} \delta a + \frac{\partial C}{\partial c} \mathcal{M} \delta c + C \delta \mathcal{M} \] (135)

However, noting that

\[ \delta c = \frac{dc}{dt} \delta t \] (136)

we can write \( \delta a \) in terms of a corresponding \( \delta c \). Using this relation with (110) and (128), we find

\[ \delta \mathcal{C} = \left\{ -\frac{c^3}{2} \left( \frac{\dot{a}}{\dot{c}} - \frac{\partial C}{\partial a} + \frac{\partial C}{\partial c} \right) \delta a \delta c - \frac{C g_{\mu\nu}}{2} \right\} \sqrt{-g} \delta g_{\mu\nu} \] (137)

Adding this to (129) and converting to an arbitrary coordinate system, we obtain the following field equations:

\[ G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} + \left( \Lambda + \frac{C}{2} \right) g_{\mu\nu} + \left( 2R + 4\Lambda + \frac{8\pi G}{c^4} \left( \frac{\dot{a}}{\dot{c}} - \frac{\partial C}{\partial a} + \frac{\partial C}{\partial c} \right) \right) \frac{\partial a}{\partial x^\mu} \frac{\partial a}{\partial x^\nu} \] (138)

However, for the rest of the discussion here we will take \( \mathcal{C} \equiv 0 \), so that our action is to coincide with the classical action for general relativity.

### 12 Application to our Varluminopic \( k = +1 \) Model

Using the Varluminopic Field Equations (132) with our metric Ansatz (9), and the stress-energy tensor for a perfect fluid (56), coupled with our hypothesis (4), we obtain the following Field Equations (compare with (61) and (62)):

\[ \frac{6}{a^2} = \frac{8\pi G}{\dot{a}^4} \varepsilon + \Lambda - \frac{24}{a^2} + 4\Lambda \] (139)

\[ \frac{2}{a^2} = -\frac{8\pi G}{\dot{a}^4} P + \Lambda \] (140)
These may alternatively be written in the form (compare with (65) and (66)):

\[ \varepsilon = \frac{5\dot{a}^4}{8\pi G} \left( \frac{6}{a^2} - \Lambda \right) \] (141)

\[ P = \frac{\dot{a}^4}{8\pi G} \left( \frac{2}{a^2} - \Lambda \right) \] (142)

These have the same qualitative features as those for the classical Field Equations approach as discussed in §6.3. In particular, notice that (141) implies a new Friedmann Equation:

\[ \Omega_\varepsilon + 5\Omega_\Lambda + 5\Omega_k = 1 \] (143)

Also, the equation of state is now given by

\[ w(\chi) = \frac{P}{\varepsilon} = \frac{\chi - 2}{5(6 - \chi)} \] (144)

where \( \chi = \Lambda a^2 \) (compare with (71)). The equation of state is plotted in Fig. 4. We also included the equation of state obtained by using the classical field equations (71) on the same plot for reference. The equation of state for the modified field equations, as given by (144), is the curve which has been vertically compressed by a factor of 5.

![Equation of State](image)

Figure 4: Equation of State vs. state variable \( \chi = \Lambda a^2 \) for classical and modified Field Equations

In particular, we note that the matter-dominated epoch occurs at \( \chi = \Lambda a^2 = 2 \) and that the radiation-dominated epoch occurs at \( \chi = \Lambda a^2 = 4.5 \). Recall that using the classical field equations, as in §6, these epochs occurred at \( \chi = 2 \) and \( \chi = 3 \), respectively.

Applying the additional constraint that \( \nabla_\mu T^\mu_\nu = 0 \), one can derive (either directly from the stress-energy tensor, or by incorporating this condition when taking the covariant divergence of the Varlu-minopic Field Equations (132)):

\[ \frac{a\ddot{a}}{a^2} = \frac{16 - 3\Lambda a^2}{5\,6 - \Lambda a^2} \] (145)

Setting \( c = \dot{a} \), we have

\[ \frac{1}{c} \frac{dc}{da} = \frac{16 - 3\Lambda a^2}{5\,6a - \Lambda a^3} \] (146)

And therefore, integrating, we obtain:

\[ \frac{da}{dt} = \frac{k}{(6a - \Lambda a^3)^{1/5}} \] (147)
where \( k \) is a constant of integration, different than the constant of integration which appears in (147). May now be integrated to find

\[
kt = \frac{5}{6} \sqrt{6a^6} \cdot 2F_1 \left( -\frac{1}{5}; \frac{3}{5}; \frac{8}{5}; \frac{\Lambda a^2}{6} \right) \tag{148}
\]

This can be simplified to

\[
kt = \frac{5 \sqrt{6a^6}}{2\Gamma(-1/5)} \sum_{n=0}^{\infty} \frac{\Gamma(n - 1/5)}{(3 + 5n)!} \left( \frac{\Lambda a^2}{6} \right)^n \tag{149}
\]

The scale factor (149) and the speed of light (147) are plotted against time in Fig. 5. We would like to point out that, despite appearances, we are slightly past the minimum value \( c_{\text{min}} \) in Fig. (147). We used \( a(t_0) = 1.32 \times 10^{28}\text{m} \) and \( \Lambda a^2 = 2.1 \) (so that \( \Omega_{\Lambda} = .7 \)).

\[
\mu_0 = 5\log\left( \frac{d_L}{1 \text{ Mpc}} \right) + 25
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{\( a(t)/a(t_0) \) vs. \( t \) (left) and \( c(t)/c(t_0) \) vs. \( t \) (right)}
\end{figure}

Moreover, using the redshift formula (147), the luminosity distance formula (54), and the definition (35), we can compare the theoretical curve produced by this model against data points from Supernova Ia data as recorded by Riess [13]. We find our theoretical curve matches the supernova quite well, as is shown in Fig. 6.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{Supernova Ia data vs. theoretical prediction}
\end{figure}
We would like to note that the scale factor evolution, as seen in Fig. 5, and the theoretical redshift-distance curve, Fig. 6, are nearly identical in appearance to those predicted by the isoergic model from §. See Fig. 2 and Fig. 3.

At the epoch \( \Lambda a^2 = 6 \), we see that the speed of light approaches infinity. This epoch also corresponds to the dissapearance of mass-energy, as is indicated by the density equation (141). We interpret this as an indication that the universe has “used up” the total amount of mass-energy in the universe by doing the expansion work. This is essentially the death of the universe. If one were to continue \( \Lambda a^2 \) beyond the value 6, one would obtain negative energy densities, which we view to be unphysical. Referring to (147), we can determine the cosmological time of the universe’s impending demise to be:

\[
T_{\text{end}} = \frac{5 \Gamma(8/5) \Gamma(6/5)}{6^{4/5} k \Lambda^{4/5} \Gamma(9/5)} \approx 21.0 \text{ Gyr}
\]  

This model predicts that the current age of the universe is approximately 12.3 Gyr, so that the universe would have about 8.7 Gyr of life left.

Alternatively, without a cosmological constant, the solution to (147) would reduce to

\[
a(t) = \frac{1}{6^{1/6}} \left( \frac{kt}{5} \right)^{5/6}
\]  

### 13 A Note About the Varlambdic Approach

By introducing a time-varying cosmological constant, one can effectively “control” the equations of state as given by the Classical Field Theory (CFT) and Varluminopic Field Theory (VFT) approaches. The radiation- and matter-dominated epochs occur at

\[
\begin{align*}
\chi^{\text{CFT}}_{\text{mat}} &= \chi^{\text{VFT}}_{\text{mat}} = 2 \\
\chi^{\text{CFT}}_{\text{rad}} &= 3 \\
\chi^{\text{VFT}}_{\text{rad}} &= 4.5
\end{align*}
\]

where \( \chi \) is the state variable \( \chi = \Lambda a^2 \), see (71) for the CFT case and (144) for the VFT case. Replacing \( \Lambda \) with \( \chi \) as a dynamical variable, we can choose a function \( \chi(t) \) which is initially constant with \( \chi_{\text{rad}} \) and whose value changes to \( \chi_{\text{mat}} \) during a rapid phase transition. This corresponds to a rapid phase transition from a radiation- to matter-dominated universe, with time varying cosmological constant.

The Field Equations can be rewritten as

\[
\begin{align*}
\varepsilon^{\text{CFT}} &= \frac{\dot{a}^4}{8\pi Ga^2} (6 - \chi) \\
\varepsilon^{\text{VFT}} &= \frac{5\dot{a}^4}{8\pi Ga^2} (6 - \chi) \\
P &= \frac{\dot{a}^4}{8\pi Ga^2} (\chi - 2)
\end{align*}
\]

see (65), (66) (CFT case) and (141), (142) (VFT case). For both cases, the conservation law \( \nabla_{\mu} T^{\mu}_{\nu} = 0 \) can be written as

\[
\frac{\dot{a}}{a} + 3(\varepsilon + P)a = 0
\]

Defining the acceleration parameter

\[
Q = \frac{a\ddot{a}}{a^2}
\]

we have that (108) produces

\[
\begin{align*}
Q^{\text{CFT}} &= \frac{1}{2} + \frac{a\dot{\chi} - 12\dot{a}}{4\dot{a}(6 - \chi)} \\
Q^{\text{VFT}} &= \frac{1}{2} + \frac{a\dot{\chi} - 12\dot{a}(7 - \chi)/5}{4\dot{a}(6 - \chi)}
\end{align*}
\]
Interestingly enough, recalling (152)-(154), we see that both approaches lead to the same steady-equation-of-state dynamics:

\[ Q_{\text{mat}} = -\frac{1}{4} \]
\[ Q_{\text{rad}} = -\frac{1}{2} \]  (162) (163)

Thus, for either the CFT or VFT approach, the scale factor becomes

\[ a(t)_{\text{mat}} \propto t^{4/5} \]  (164)
\[ a(t)_{\text{rad}} \propto t^{2/3} \]  (165)

for a matter-dominated and radiation-dominated universe, respectively.

Part IV
Conclusion

14 Conclusion

In this paper, we presented a new cosmology where the speed of light varies with cosmological time subject to a fundamental constraint (4). We explored the implications of this in Part II using the classical field equations and showed that it leads to a new and interesting dynamics of the scale factor. In Part III we took a different approach. Retaining the classical gravitational action (102), we showed that one obtains a new set of Field Equations (132). With the additional constraint \( \nabla_{\mu} T^{\mu}_{\nu} = 0 \), we solved these modified field equations explicitly (149) and found that they imply a possible end to the universe which would occur at the epoch \( \Delta a^2 = 6 \), or roughly 9 Gyr from now. The model predicts a universe which is currently around 12 Gyr old, which does not contradict known timelines. A key feature of the model is that it solves the Horizon Problem without the need of inflation.

We showed that, although the redshift formula (25) remains unaltered, the formula for the luminosity distance (34) inherits an extra factor in the vari-luminogenic case. Using the dynamical solutions as presented in §8 and §12, we plot the redshift-distance curve and showed that in both cases they match with the experimental measurements recorded in Riess [13], see Fig. 3 and Fig. 6. These results need to be examined more closely, but the preliminary figures included in this paper are a good sign.

Big Bang nucleosynthesis, on the other hand, could lead to problems. Our model changes the conditions at the time of nucleosynthesis quite drastically. When the universe was at the temperature \( T = 1 \text{ MeV} \), the speed of light, given by (147), was 229 times larger than its present day value. Thus it is not immediately obvious whether nucleosynthesis will still work in our model, and will considered in future works. Other astrophysical issues, such as the cosmic microwave background and the formation of structure, also need to be studied if this model is to remain viable.

Our model changes the history of the early universe. We no longer have the need for inflation. Phase transitions should work out to be different, e.g. electroweak and quark hadron. And further it changes the conditions during baryogenesis, which might also be different. All of these issues are uncertain in the standard model, and they may be better or worse here. This will be the focus of much further research.

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