Holomorphic blocks and the 5d AGT correspondence*

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Abstract
We review the holomorphic block factorisation of partition functions of supersymmetric theories on compact manifolds in various dimensions. We then show how to interpret 3d and 5d partition functions as correlation functions with underlying symmetry given by a deformation of the Virasoro algebra.

Keywords: holomorphic blocks, 5d AGT correspondence, Virasoro algebra

(Some figures may appear in colour only in the online journal)

1. Introduction

Over the last ten years, starting from the seminal work by Pestun [2], Witten’s localization has been extensively applied to supersymmetric theories defined on compact manifolds of various dimensions. This has led to the derivation of a large number of exact results such as the evaluation of partition functions and vevs of various BPS observables like Wilson-loops and surface operators.

Thanks to these results it has been possible to perform impressive large $N$ tests of various holographic dualities as summarised in the contributions contribution [3], contribution [4], contribution [5] of this review. It has also been possible to do precision checks of various non-perturbative dualities such as Seiberg-like dualities and 3d mirror symmetry. For example, as reviewed in contribution [6], 3d partition functions are protected under RG flow hence one can explicitly compute partition functions of pairs of UV Lagrangian supposed to flow to the same SCFT in the IR, and show that they are indeed equal.

Localization has also played a key role in the discovery of new surprising correspondence relating QFTs in different dimensions and with different types of symmetries. This is the case...
of the celebrated Alday–Gaiotto–Tachikawa (AGT) correspondence [7] relating $S^1$ partition functions of $\mathcal{N} = 2$ theories to Toda CFT correlators. This correspondence together with its variation involving the superconformal index or $S^1 \times S^1$ partition functions of $\mathcal{N} = 2$ theories is reviewed in contribution [8]. A similar correspondence relating 3d $\mathcal{N} = 2$ theories to complex Chern–Simons theories is reviewed in contribution [9].

The interest in studying SUSY theories on compact manifolds has led to the development of a comprehensive approach to the formulation of supersymmetric theories on curved space initiated by Festuccia and Seiberg [10] and reviewed in contribution [11]. In particular, it has been possible to derive general theorems to determine the amount of supersymmetry preserved by a given background and the dependence of partition functions on the data specifying the background.

It has also been observed that if the manifold $M$ on which the theory is formulated can be decomposed into simpler building blocks, as for example in the case of the solid tori decomposition of a three-sphere, then also the partition function $Z_M$ can be expressed in terms of the partition functions of the building blocks, the so-called holomorphic-blocks. In the first part of this review article we will illustrate several examples of this block decomposition in 3d, 4d and 5d.

Holomorphic blocks in various dimensions are interesting mathematical objects with intricate transformation properties under dualities which often involve non-trivial Stokes phenomena, for an extensive analysis of the properties of the 3d holomorphic blocks see [12]. 3d and 5d blocks are related to open and closed topological string amplitudes while 3d blocks appear also as Chern–Simons wave functions in the context of the 3d-3d correspondence discussed in contribution [9].

In this review article we focus instead on the interpretation of the holomorphic blocks in the context of AGT-like correspondences. Via the AGT correspondence partition functions of $\mathcal{N} = 2$ theories on $S^4$ can be mapped to Toda/CFT correlators and the holomorphic blocks, which in this case coincide with the two hemi-sphere partition functions, are mapped to the Toda chiral conformal blocks. Correlators involving degenerate operators are instead mapped to $\mathcal{N} = 2$ theories on $S^4$ with surface operators inserted on a codimension-two $S^2$ and the holomorphic blocks of the codimension-two defect theory correspond to degenerate chiral conformal blocks.

In the second part of this review we will argue that a similar correspondence can be established between $\mathcal{N} = 1$ theories on a large class of 5-manifolds and correlators with underlying symmetry given by a deformation of the Virasoro algebra. Also in this case codimension-two defect theories and 3d holomorphic blocks can be mapped to degenerate deformed Virasoro correlators.

The plan of the review is the following: we discuss the holomorphic block factorisation in 3d and 4d in section 2.1 and in 5d in section 2.2. In section 2.3 we consider the insertion of codimension-2 defect operators via Higgsing in 5d theories focusing on some simple cases. We then move to the discussion of the dual deformed Virasoro side. After introducing the deformed Virasoro algebra in section 3.1, we collect some of the evidence of the mapping of degenerate and non-degenerate deformed Virasoro chiral blocks to vortex and instanton partition functions in section 3.2. Finally in section 4 we discuss how to combine deformed Virasoro blocks to construct correlators and how these can be mapped to 3d and 5d partition functions.

2. Compact manifolds and holomorphic block factorisation

2.1. Factorisation and holomorphic blocks in 3d and 4d

In this section we discuss the holomorphic block factorisation in 4d theories defined on Hermitian manifolds of the form $M^{4d} = M^{4d} \times S^1$, where $M^{4d}$ is a possibly non-trivial $U(1)$ fibration over the 2-sphere, and their 3d reductions. More precisely we focus on the following
\( \mathcal{N} = 1 \) backgrounds: the \( S^1 \times S^1 \) and lens \( L_\ell \times S^1 \) indexes, the \( S^2 \times T^2 \) background and their 3d \( \mathcal{N} = 2 \) reductions: the squashed sphere \( S^3 \), the lens space \( L_\ell \), the \( S^2 \times S^1 \) index and the twisted \( S^2 \times S^1 \) index.

The 3-manifolds listed above can all be realised by gluing two solid tori \( D^2 \times S^1 \) with an element \( g \in SL(2, \mathbb{Z}) \), we call them \( M_g^{3d} \). Similarly all the 4-manifolds above can be constructed from the fusion of two solid tori \( D^2 \times T^2 \) with appropriate elements in \( g \in SL(3, \mathbb{Z}) \) and we call them \( M_g^{4d} \).

As reviewed in contribution [11] partition functions on these backgrounds are metric independent, they do however depend on other data specifying the background (for example in the 4d case they depend on the complex structure and holomorphic vector bundles associated to the flavour symmetries) so they are not properly topological objects, nevertheless, as we will review, there is evidence that the following chain of identities holds:

\[
Z_{M_g^{3d/4d}} = \sum_f \left\| \mathcal{Y}_{3d/4d} \right\|_g^2 = \sum_c \left\| B_c^{3d/4d} \right\|_g^2 = \sum_c \left\| \mathcal{Y}_{3d/4d} \right\|_g^2.
\]

(2.1)

The first equality states the factorisation into a ‘g-square’ of the integrand of the Coulomb branch partition function\(^1\) given by a classical and 1-loop contribution:

\[
Z_{cl}^{M_g^{3d/4d}} Z_{1-loop}^{M_g^{3d/4d}} = \left\| \mathcal{Y}_{3d/4d} \right\|_g^2.
\]

(2.2)

The functions \( \mathcal{Y}_{3d/4d} \) can be interpreted as integrands of the \( D^2 \times S^1 \) or \( D^2 \times T^2 \) partition functions. The data specializing the manifold \( M_g^{3d/4d} \) are all encoded in the gluing rule \( g \).

To explain this first equality we consider the case of the simplest 3d \( \mathcal{N} = 2 \) theory: the free chiral with (minus) half Chern–Simons unit, which we add to remove the parity anomaly. This theory is often referred to as the tetrahedron theory since in the context of the 3d-3d correspondence it computes the quantum volume of the ideal tetrahedron [13–15].

If we specialize to the squashed 3-sphere \( S^3_\theta = \{ (x, y) \in \mathbb{C}^2 \} \) \( b^2|x|^2 + b^{-2}|y|^2 = 1 \), as reviewed in contribution [6] the contribution of a charge plus chiral multiplet with canonical R-charge and real mass \( X \) for the background vector multiplet associated to the \( U(1) \) flavour symmetry, is given in terms of the double-sine function \( s_b \) defined in the appendix. When combined with the Gaussian contribution of the \( -1/2 \) CS unit, the partition function admits a factorised form:

\[
Z_{3d}^{S^3_\theta}(X) = e^{i\frac{\pi}{4} q_b (iQ/2 - X)} = \left( \frac{(qx^{-1}; q)_\infty}{(x^{-1}; q^{-1})_\infty} \right)^2 \left\| B^{3d}_\Delta(x; q) \right\|_{S^3}^2.
\]

(2.3)

where \( Q = b + b^{-1} \) and the holomorphic variables are defined as

\[
q = e^{2\pi i b}, \quad \tilde{q} = e^{2\pi i b}, \quad x = e^{-\pi i \ell}, \quad \tilde{x} = e^{2\pi i \ell}.
\]

(2.4)

The 3d holomorphic block \( B^{3d}_\Delta(x; q) = (qx^{-1}; q)_\infty \) is the partition function on the solid torus or Melvin cigar \( D^2 \times S^1 \) of the tetrahedron theory. Notice that when \( |q| < 1 \) we have \(|q| > 1 \) and

\[
(x; q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(q; q)_n} = \begin{cases} \prod_{n=0}^{\infty} \left(1 - q^n x\right) & \text{if } |q| < 1 \\ \prod_{n=0}^{\infty} \left(1 - q^{-n-1} x^{-1}\right) & \text{if } |q| > 1 \end{cases}
\]

(2.5)

\(^1\) The symbol \( \sum \) indicates that the Coulomb branch partition might include a sum over a discrete index besides the integration over the constant values of the fields parameterising the Coulomb branch.
Basically blocks in $x$, $q$, and $\tilde{x}, \tilde{q}$, share the same series expansion but they converge to different functions. This is a key feature of holomorphic blocks which has been extensively discussed in [12]. The two blocks are glued through the $S$-gluing acting on $\tau$, the modular parameter of the boundary $T^1$, and on the flavour variable $\chi$ as:

$$\tau \rightarrow \tilde{\tau} = -S(\tau) = \frac{1}{\tau}, \quad \chi \rightarrow \tilde{\chi} = \frac{\chi}{\tau}. \quad (2.6)$$

This gluing corresponds to the element $S \in SL(2, \mathbb{Z})$ (composed with orientation inversion) realising a three-sphere from a pair of solid tori.

There is a similar factorisation of the tetrahedron theory on the lens space $L_r$. This smooth 3-manifold is the free $\mathbb{Z}_r$ orbifold of the squashed 3-sphere with the identification

$$(x, y) \sim (e^{\frac{2\pi i}{r}} x, e^{-\frac{2\pi i}{r}} y). \quad (2.7)$$

In this case, as reviewed in contribution [6], the contribution of the chiral multiplet is expressed in terms of the modified double-sine function which takes into account the periodicity inherited from the $\mathbb{Z}_r$ quotient. The factorization properties of the modified double-sine are somewhat subtle and we refer the reader to [16] for details, in the end when combined with the appropriate half Chern–Simons unit one finds:

$$Z^L_{\Delta}(X, H) = \left| B^3_{\Delta}(x; q) \right|^2_r, \quad (2.8)$$

where we have turned on for the flavour symmetry a continuous real mass $X$ and a discrete holonomy $H \in \mathbb{Z}_r$, parameterising the topological sectors. The holomorphic variables are now defined as

$$x = e^{\frac{2\pi i}{r} H} e^{\frac{2\pi i}{r} H} = e^{2\pi i \chi} e^{\frac{2\pi i}{r} H}, \quad \tilde{x} = e^{\frac{2\pi i}{r} \chi} e^{-\frac{2\pi i}{r} H} = e^{2\pi i \chi} e^{-\frac{2\pi i}{r} H}, \quad q = e^{2\pi i \frac{r}{2}}, \quad \tilde{q} = e^{2\pi i \frac{r}{2}}. \quad (2.9)$$

The two blocks are glued through the $\tau$-pairing acting as

$$\tau \rightarrow \tilde{\tau} = -\tilde{r}(\tau) = \frac{\tau}{r \tau - 1}, \quad \chi \rightarrow \tilde{\chi} = \frac{\chi}{r \tau - 1}, \quad H \rightarrow \tilde{H} = r - H. \quad (2.10)$$

This gluing rule as expected coincides with the $\tilde{r} \in SL(2, \mathbb{Z})$ element (composed with the orientation inversion) realising the $L_r$ geometry from a pair of solid tori. The factorization on the more general $L(p, q)$ Lens spaces has been discussed in [17].

The next example we discuss is the $S^2 \times S^4$ background. Localization on this background is reviewed in contribution [6]. For the tetrahedron with a fugacity $\zeta$ which we take to be a phase, and the integer $m \in \mathbb{Z}$ for the background magnetic flux through $S^2$, we find:

$$Z^S_{\Delta} (\zeta, m) = \left| (q x^{-1}; q) \right|^2_{id}, \quad (\text{where the holomorphic variables})$$

$$x = \zeta q^m e^{2\pi i \chi}, \quad \tilde{x} = \zeta^{-1} q^m e^{2\pi i \chi}, \quad q = e^{2\pi i \chi}, \quad \tilde{q} = q^{-1} = e^{-2\pi i \chi}. \quad (2.11)$$

The two blocks are glued through the $id$-pairing (combined with orientation reverse) acting as

$$\tau \rightarrow \tilde{\tau} = -id(\tau) = -\tau, \quad \chi \rightarrow \tilde{\chi} = \chi. \quad (2.12)$$
This gluing rule as expected coincides with the id ∈ SL(2, ℤ) element (composed with orientation inversion). Finally as discussed in [16], there is a similar factorization also in the case of the twisted index background [18].

Generic interacting theories, with no parity anomaly, that is with integer effective Chern–Simons couplings, can be constructed by gauging products of tetrahedron theories and then adding integers units of Chern–Simons terms and the contribution of vector multiplets. This observation allow us to take the square root of the integrand of generic theories whenever there is no parity anomaly. Indeed we have just reviewed how to take tetrahedron theories as squares of tetrahedron blocks, thanks to these special functions identities we can easily factorize the matter and vector multiplet contributions. Chern–Simons terms at integer level, can instead be dealt with by using the properties of the theta function:

$$\theta(x; q) := (-q^{1/2}x; q)_{\infty}(-q^{1/2}x^{-1}; q)_{\infty}, \quad (2.13)$$

which for example satisfies

$$\left\| \theta((-q^{1/2}x^2; q)) \right\|_S^2 = C^{-2}e^{-i\pi \left( \frac{e^{2\pi i/3}}{x} + \frac{e^{2\pi i/3}}{x^{-1}} \right)}, \quad C = e^{-i\pi (b^2 + \tilde{b}^2)},$$

We can use this identity to express Chern–Simons terms on S^3 as squares of theta functions depending on the holomorphic variables. Chern–Simons terms on other 3-manifolds M^3_f are similarly factorised in terms of g-squares of theta functions.

To make a concrete example we consider the SQED partition function on S^3_f, with masses \( \tilde{m}_i \) for the \( N_f \) charge plus chirals, masses \( m_i \) for the \( N_f \) charge minus chirals and an FI parameter \( \xi \):

$$Z^{S^3_f}[\text{SQED}] = \int d\sigma e^{2\pi i \sigma \xi} \prod_{j,k=1}^{N_f} s_b(\sigma + m_j + iQ/2)s_b(-\sigma - \tilde{m}_k + iQ/2). \quad (2.14)$$

In this case the classical (FI term) and 1-loop term can be factorized as

$$\Upsilon^{3d}_{[\text{SQED}]} = \frac{\theta(x; q)}{\theta(x; q)\theta(x; q)} \prod_{j,k=1}^{N_f} \frac{(q x_j x_k^{-1}; q)_{\infty}}{(y_k x_k^{-1}; q)_{\infty}}, \quad (2.15)$$

with the following definition of holomorphic variables:

$$x = e^{2\pi i b}, \quad x_j = e^{2\pi i b_j}, \quad y_i = e^{2\pi i \tilde{m}_i}, \quad z = e^{2\pi i \xi}, \quad q = e^{2\pi i b^2},$$

$$\bar{x} = e^{2\pi i \sigma/b}, \quad \bar{x}_j = e^{2\pi i \sigma_j/b}, \quad \bar{y}_i = e^{2\pi i \tilde{m}_i/b}, \quad \bar{z} = e^{2\pi i \xi/b}, \quad \bar{q} = e^{2\pi i /b^2},$$

and

$$\prod_{j,k=1}^{N_f} x_j y_k^{-1} = r, \quad u = (-q^{1/2}r^{1/2}z^{-1}). \quad (2.16)$$

The discussion of the first equality in the chain of identities (2.1) for the 4-manifolds case is similar. Here the factorization of the integrand in terms of the \( D^2 \times T^2 \) integrand \( \Upsilon^{4d} \) again involves several non-trivial identities for the special functions appearing in the one-loop contributions but this time the necessary and sufficient condition for the factorization is the cancellation of the cubic anomaly. This was observed in [16] building on the discovery of the surprising relation between the modular properties of the superconformal index and the appearance of the anomaly polynomial [19].
The next equality in the chain of identities (2.1) is the block-factorization of the Coulomb branch partition function. 1-loop factors are meromorphic functions and it is possible to evaluate the integral by taking residues at their poles by choosing suitable convergent integration contours. The result takes the form of a sum over the supersymmetric vacua (critical points of the effective twisted superpotential) of the semiclassical $\langle 2, 2 \rangle$ theory on the $\mathbb{R}^2 \times S^1$ and $\mathbb{R}^2 \times T^2$ solid tori:

$$Z_{\text{C}}^{3d/4d} = \sum_c \left( Z_{\text{cl}}^{M^{3d/4d}} - Z_{\text{1-loop}}^{M^{3d/4d}} \right) \left\| Z_{\mathcal{V}}^{3d/4d,c} \right\|_g^2.$$  \hfill (2.18)

The contribution of each vacua consists of the product of classical and 1-loop terms evaluated at the $c$th vacuum and of the vortex $Z_{\mathcal{V}}^{3d/4d,c}$ partition function. This is the partition function of the theory placed on the cigars or $\mathbb{R}^2 \times S^1$ or $\mathbb{R}^2 \times T^2$ with the Omega background turned on $\mathbb{R}^2$ [20] and enumerates finite energy BPS vortex configuration. Typically vertex partition functions are expressed in terms of $q$-deformed or elliptic hypergeometric series.

We can also factorize the one-loop and classical contributions as discussed above and present the partition function as a sum of $g$-squares of 3d or 4d holomorphic blocks defined as:

$$B_c^{3d/4d} = \mathcal{Y}^{3d/4d} \left| \left. \frac{Z_{\mathcal{V}}^{3d/4d,c}}{Z_{\mathcal{V}}^{3d/4d,c}} \right| \right| = \sum_c e^{-2\pi i c m} \prod_{j,k=1}^{N} \frac{\sin(m_j - m_i + iQ/2)}{\sin(m_k - m_j - iQ/2)} \left\| Z_{\mathcal{V}}^{(i)} \right\|^2_S,$$  \hfill (2.19)

To make a concrete example we consider again the the SQED partition function on $S^3$. To evaluate the integral (2.14) we can close the contour in the upper half plane and take the contributions of poles located at $x = -m_i + im b + in / b$, see [21] for details. The result reads

$$Z_{\text{V}}^{3d/4d} = \sum_{n=0}^{N_i} \prod_{j,k=1}^{N_i} \frac{\sin(x_i - x_j)^{-1}}{\sin(x_i - x_j)} w^n = \sum_{n=0}^{N_i} \prod_{j,k=1}^{N_i} \frac{\sin(x_i - x_j)^{-1}}{\sin(x_i - x_j)} w^n \Phi_{N_i-1}(x_i^{-1} y_1, \ldots, x_i^{-1} y_{N_i} ; q, x_i^{-1} x_1, \ldots, x_i^{-1} x_{N_i} ; u).$$  \hfill (2.21)

where $\Phi_{N_i-1}$ is the basic hypergeometric function. The classical and 1-loop term can also be factorized to obtain the 3d block:

$$B^{3d} = \frac{\theta(x_i u ; q)}{\theta(x_i u ; q)} \prod_{j,k=1}^{N_i} \frac{\sin(x_i - x_j)^{-1}}{\sin(x_i - x_j)} \Phi_{N_i-1}(x_i^{-1} y_1, \ldots, x_i^{-1} y_{N_i} ; q, x_i^{-1} x_1, \ldots, x_i^{-1} x_{N_i} ; u).$$  \hfill (2.22)

Explicit examples of block factorization have been obtained for various theories and backgrounds including $S^3$, lens space $L_r$, the superconformal index $S^2 \times S^1$, the twisted index, [12, 16, 18, 21–25].

Similar residue computations yield the block factorization of $\mathcal{N} = 1$ theories on $S^3 \times S^1$, $L_r \times S^3$, $S^2 \times T^2$ [16, 26–28] and on the ellipsoid [29], and, as reviewed in contribution [30] in 2d $\mathcal{N} = (2, 2)$ theories on $S^3$ [31–33].

The block factorization of $M^{3d/4d}$ partition function can also be interpreted as the result of an alternative localization scheme know as Higgs branch localization. As reviewed in
contribution [30] the Higgs branch localization was originally introduced for the (2, 2) theories in [32, 34], and later applied to other backgrounds in [26, 27, 31, 35], see also the chapter contribution [30] in this review.

Another perspective on the factorization in the $S^3_b$ case was provided in [36], where it was shown that it is possible to deform the three-sphere geometry into two cigars $D^2 \times S^1$ connected by a long tube without changing the value of the partition function. This deformation has exactly the effect of projecting down the theory into the SUSY ground states which are the only states contributing to the overlap of the two blocks. It should be possible to extend this argument to other 3d and 4d backgrounds. In 2d a similar proof of the block-decomposition of the two-sphere was provided in [37]. The 2d holomorphic block in this case are the cigar partition functions appearing in the Cecotti–Vafa $tt^*$ set-up [38, 39] and their overlap computes the exact Kähler potential as reviewed in contribution [40].

To explain the last equality in (2.1) we begin by observing a key property of the 3d (or 4d) holomorphic blocks: they are annihilated by a set of difference equations which can be interpreted as Ward identities for Wilson loops (or surface operators) wrapping the circle $S^1$ (or the torus $T^2$) and acting at the tip of the cigar. There are in fact two commuting sets of difference operators annihilating respectively the holomorphic and the anti-holomorphic blocks. This set of difference operators can be systematically derived from the UV Lagrangian [13–15]. Building on this, an integral formalism was developed in [12], to compute the 3d holomorphic blocks by integrating the meromorphic one-form $\Upsilon^{3d}$ on a basis of middle-dimensional cycles in $(\mathbb{C}^*)^{|G|}$

$$B^{3d}_c = \oint_{\Gamma_c} \Upsilon^{3d}. \tag{2.23}$$

Each contour is associated to a critical point of the integrand, which in turn is related to a supersymmetric ground state and it is chosen so that the integral converges and solves the set of difference equations. The space of blocks can then be viewed as the vector space of solutions to the system of difference equations. Closely connected to this construction are the global transformations properties of the blocks in parameter space. It was shown that by fixing $q$ and varying $x$ the holomorphic blocks are subject to Stokes phenomena. We refer the reader to [12] for a detailed discussion of the block integrals and the interplay between mirror symmetry and Stokes phenomenon. See also [41] for a derivation of the block integrals from localization on $D^2 \times T^2$.

In the context of the 3d-3d correspondence reviewed in the contribution [11], 3d blocks are identified with complex Chern–Simons wave-functions. In the second part of this chapter we will instead see how 3d blocks can be been identified with $q$-deformed Virasoro correlators.

We should also mention that factorization and the definition of the blocks suffer an intrinsic ambiguity. By defining blocks as solutions to difference equations we have the possibility to multiply them by $q$-constant which satisfy $c(x; q) = c(x; q)$. By requiring that these constants don’t modify the compact space results we restrict to $q$-constants with unit $g$-square $\|c(x; q)\|^2_g = 1$. These are elliptic ratios of theta functions and have a trivial semiclassical limit $(q = e^{ih} \to 0)$. These functions represent our ambiguity.

A block integral formulation for 4d holomorphic blocks leading to the last equality in (2.1) has been proposed in [16]. The definition of the integration contours for 4d block integrals is quite subtle and a careful study of their properties is still missing. For example it would be interesting to study their behaviour under various 4d dualities. It should also be possible to re-derive the 4d block integrand prescription via localization on $D^2 \times T^2$. The relation of 4d block integrals to free field correlators in an elliptic deformation of the Virasoro algebra has been explored in [42].
2.2. Factorization and holomorphic blocks in 5d

As reviewed in contribution [43], localization can be applied to 5d $\mathcal{N} = 1$ theories formulated on a large class of 5d manifolds. The aim of this section is to show that partition functions on these manifolds can be obtained by gluing the so-called 5d holomorphic blocks $B^{5d}$, which are partition functions on $\mathbb{R}^4 \times S^1$. The gluing rule can be read out from the geometric data of the 5d manifolds.

2.2.1. Squashed $S^5$ partition functions and 5d holomorphic blocks. We begin our discussion with the squashed $S^5$, the simplest example of toric Sasaki–Einstein 5-manifold. It is convenient to think the $S^5$ as a $T^3$ fibration over the interior of a triangle, with the fiber degenerating to a torus on the faces and to a circle over the vertices.

As reviewed in contribution [43], the partition function of 5d $\mathcal{N} = 1$ theories on this background takes the following form:

$$Z_{S^5} = \int d\vec{\sigma} Z_{S^5}^{\text{cl}}(\vec{\sigma}) Z_{S^5}^{\text{loop}}(\vec{\sigma}, \vec{M}) Z_{S^5}^{\text{inst}}(\vec{\sigma}, \vec{M}).$$

The integral is over the zero-mode of the vector multiplets scalars $\vec{\sigma}$ taking value in the Cartan subalgebra of the gauge group, $\vec{M}$ indicate the hypermultiplet masses.

The non-perturbative $Z_{S^5}^{\text{inst}}(\vec{\sigma}, \vec{M})$ receives contributions from the three fixed points of the Hopf fibration over the $\mathbb{C}P^2$ base and takes the following factorized form [44, 45]:

$$Z_{S^5}^{\text{inst}}(\vec{\sigma}, \vec{M}) = \prod_{k=1}^{3} \left| Z_{S^5}^{\text{inst}}(\vec{\sigma}) \right|^3,$$

where $Z_{S^5}^{\text{inst}}$ coincides with the equivariant instanton partition function on $\mathbb{R}^4 \times S^1$ [46, 47] with Coulomb and mass parameters appropriately rescaled and with equivariant parameters $\epsilon_1 = e^{\frac{\omega_1}{e_3}}$ and $\epsilon_2 = e^{\frac{\omega_2}{e_3}}$:

$$Z_{S^5}^{\text{inst}} = \mathbb{Z}_{S^5}^{\mathbb{R}^4 \times S^1} \left( \frac{i\vec{\sigma}}{e_3}, \vec{m}; \epsilon_1, \epsilon_2 \right),$$

where $m = iM + E/2$ and $E = \omega_1 + \omega_2 + \omega_3$. The sub-index $k = 1, 2, 3$ in equation (2.25) refers to the following identification of the parameters $\epsilon_1, \epsilon_2, \epsilon_3$ with the squashing parameters $\omega_1, \omega_2, \omega_3$ in each sector:

| sector | $\epsilon_1$ | $\epsilon_2$ | $\epsilon_3$ |
|--------|-------------|-------------|-------------|
| 1      | $\omega_3$  | $\omega_2$  | $\omega_1$  |
| 2      | $\omega_1$  | $\omega_3$  | $\omega_2$  |
| 3      | $\omega_1$  | $\omega_2$  | $\omega_3$  |

(2.27)

The three sectors correspond to the loci where the Reeb vector forms closed orbits (in the round $S^5$ case they close everywhere). For more general toric SE manifolds, it is conjectured that the non-perturbative contributions are indeed localized at these isolated loci.

Actually the instanton partition function depends on the squashing parameters through the combination

$$q = e^{2\pi i\epsilon_1/e_1}, \quad t = e^{2\pi i\epsilon_2/e_2},$$

from which we see that the product $\left| \cdots \right|^3$ enjoys an $SL(3, \mathbb{Z})$ symmetry which acts as $S$-dualizing the couplings $q$ and $t$. 


The classical Yang-Mills action can also be expressed in the $SL(3, \mathbb{Z})$ factorized form as the instanton contribution [48]:

$$Z_{cl}^{s}(\sigma) = e^{-\frac{2\pi}{\omega_{12}\omega_{34}}} \text{Tr}(e^{i\sigma}) = e^{-\frac{2\pi}{\omega_{12}\omega_{34}}} \sum_{\alpha} |i\alpha(\sigma)|^2 = \left\| Z_{cl} \right\|^3,$$  

(2.29)

where

$$Z_{cl} = \prod_{\alpha} \Gamma_{q,t} \left( \frac{1}{c_{1}} \left( i\alpha(\sigma) + \frac{1}{g^{2}C_{3}(ad)} + \frac{\omega}{2} \right) \right),$$  

(2.30)

and we denoted by $\alpha$ the roots of the gauge group Lie algebra. To arrive at this expression we first need to write the Gaussian term as a combination of Bernoulli polynomial $B_{33}$ defined in appendix A.1 and then use the identity [49]

$$e^{-\frac{2\pi}{\omega} B_{33}(z)} = \prod_{k=1}^{3} \Gamma_{q,t} \left( \frac{z}{e_{3}} + \frac{z}{e_{3}} \right) = \left\| \Gamma_{q,t} \left( \frac{z}{e_{3}} + \frac{z}{e_{3}} \right) \right\|^3,$$  

(2.31)

where the elliptic gamma function $\Gamma_{q,t}$ is defined in the appendix. We can therefore write the partition function as:

$$Z_{cl}^{s} = \int d\sigma \ Z_{1-\text{loop}}^{s} (\sigma, \vec{M}) \left\| Z_{cl} \right\|^3,$$  

(2.32)

where $Z_{cl}$ and $Z_{\text{inst}}$ are given respectively in (2.30) and (2.26).

The 1-loop contribution to the partition function is expressed in terms of the triple sine function $S_{3}$ defined in the appendix:

$$Z_{1-\text{loop}}^{s} (\sigma, \vec{M}) = \prod_{\rho > 0} S_{3}(i\rho(\sigma)) S_{3}(E + i\rho(\sigma)) \prod_{\rho < 0} S_{2}(i\rho(\sigma) + iM + \frac{\omega}{2}),$$  

(2.33)

where $\rho$ is the weight of the representation $R$. As suggested in [44], using the relation (A.5), the 1-loop contribution to the partition function can be factorized as:

$$Z_{1-\text{loop}}^{s} (\sigma, \vec{M}) = \prod_{R} \prod_{\rho \in R} e^{-\frac{2\pi}{\omega} B_{33}(i\rho(\sigma)) - B_{33}(i\rho(\sigma) + m)} \prod_{k=1}^{3} \left( \frac{e^{\frac{2\pi}{\omega} \rho(\sigma)}}{e^{\frac{2\pi}{\omega} \rho(\sigma) + m}} \right)^{k} / 

\prod_{k=1}^{3} \left( \frac{e^{\frac{2\pi}{\omega} \rho(\sigma) + m}}{e^{\frac{2\pi}{\omega} \rho(\sigma) + m}} \right)^{k},$$  

(2.34)

where $(z, q, t) = \prod_{k=1}^{3} (1 - zq^{k}t^{k})$ denotes the double $(q, t)$-factorial and the sub-index $k$ indicates that $q, t$ defined in (2.28) are related to the squashing parameters according to the $k$th entry in table (2.27). Each factor of the $k$-product can in turn be identified with the 1-loop contribution to the $\mathbb{R}^{4} \times S^{1}$ theory:

$$Z_{1-\text{loop}} = Z_{1-\text{loop}}^{\mathbb{R}^{4} \times S^{1}} \left( \sigma, \vec{M} ; \vec{e}_{3} ; \vec{e}_{1} ; \vec{e}_{2} \right) = \prod_{R} \prod_{\rho \in R} \left( \frac{e^{\frac{2\pi}{\omega} \rho(\sigma)}}{e^{\frac{2\pi}{\omega} \rho(\sigma) + m}} \right)^{k} / \left( \frac{e^{\frac{2\pi}{\omega} \rho(\sigma) + m}}{e^{\frac{2\pi}{\omega} \rho(\sigma) + m}} \right)^{k},$$  

(2.35)

If we consider (pseudo) real representations, for each weight $\rho$ there is the opposite weight $-\rho$ and the sum of the Bernoulli is a quadratic polynomial. For example consider the case with $Nf$ fundamentals of mass $M_{f}$ and $Nf$ anti-fundamentals of mass $M_{f}$, with $f = 1, \ldots, Nf$, and $N_{\bar{a}}$.

\[2\text{To simplify this expression we defined } g^{2} = \frac{4\pi}{\omega} \text{ and used that } 2C_{3}(ad) \sum_{\rho} \rho(\sigma)^{2} = \sum_{\alpha} \alpha(\sigma)^{2}. \]
adjoints of mass \( M_a, \ a = 1, \ldots, N_a \). The total contribution from the Bernoulli’s in the 1-loop terms is (up to a \( \sigma \)-independent constant)
\[
\text{Ber} = e^{- \frac{1}{2 \pi} \sum_{a=1}^{N_a} \left( m_a \right)^2 \left[ \xi N_s - \frac{1}{4} \sum_{j} (m_j + \bar{m}_j) + C_2(\text{ad})(\xi(N_s - 1) - \sum_a m_a) \right]}, \tag{2.36}
\]
which, once combined with the classical terms, amounts to the shift of the coupling constant
\[
\frac{1}{g^2} \rightarrow \frac{1}{g_{\text{eff}}^2} = \frac{1}{g^2} + E \frac{1}{4} N_f - \frac{1}{4} \sum_f (m_f + \bar{m}_f) + C_2(\text{ad}) \left( \frac{E}{2} (N_a - 1) - \sum_a m_a \right). \tag{2.37}
\]
Combining all these observations one arrives at the completely factorized form \cite{48} (up to constant prefactors):
\[
Z^{S^5} = \int d\sigma \left| B^{5d} \right|_S^3, \tag{2.38}
\]
where \( B^{5d} \), the 5d holomorphic block, is defined as
\[
B^{5d} = Z_{1-\text{loop}} Z_{\text{cl}} Z_{\text{inst}}, \tag{2.39}
\]
with \( Z_{\text{cl}} \) defined as in equation (2.30) with \( g^2 \rightarrow g_{\text{eff}}^2 \). As in the 3d case there is an ambiguity in the definition of the 5d blocks, this is discussed in \cite{48}.

We will now see that the partition functions on a large class of 5-manifolds can be expressed in terms of the 5d blocks \( B^{5d} \).

### 2.2.2. Block-factorization of 5d toric Sasaki–Einstein partition functions

As reviewed in contribution \cite{43} localization can be performed on general simply connected toric Sasaki–Einstein (SE) manifolds \( \mathcal{M}^n \). These backgrounds preserve 2 supersymmetries.

As in the \( S^5 \) case it is convenient to think of these 5-manifolds as \( T^5 \) fibration over the interior of a polygon, with the fiber degenerating to a torus on the \( n \) faces and to circle over the \( n \) vertices.

The perturbative partition function on a SE manifold \( \mathcal{M}^n \) is again a Coulomb branch integral
\[
Z^{\mathcal{M}^n} = \int d\vec{\sigma} \ Z_{\text{cl}}^{\mathcal{M}^n}(\vec{\sigma}) \ Z_{1-\text{loop}}^{\mathcal{M}^n}(\vec{\sigma}, \vec{M}). \tag{2.40}
\]
The 1-loop contribution \( Z_{1-\text{loop}}^{\mathcal{M}^n}(\vec{\sigma}, \vec{M}) \) takes the same form as in the \( S^5 \) case (2.33) with the triple-sine functions replaced by the generalised triple-sine function defined as:
\[
S_3^{M^n}(X) \sim \prod_{\vec{m} \in C, (\mathcal{M}^n)^n = \mathbb{Z}^3} (\vec{m} \cdot \vec{R} + X)(\vec{m} \cdot \vec{R} + \vec{\xi} \cdot \vec{R} - X). \tag{2.41}
\]
In the above expression the product is over the integers in the moment map cone:
\[
C_{\mu}(\mathcal{M}^n) = \{ \vec{m} \in \mathbb{R}^3 | \vec{m} \cdot \vec{v}_i \geq 0, \ i = 1, \ldots, n \}, \tag{2.42}
\]
the three-component vector \( \vec{R} \) parameterizes the Reeb vector field, the vector \( \vec{\xi} \) satisfies \( \vec{\xi} \cdot \vec{v}_i = 1 \) for \( i = 1, \ldots, n \) and \( \vec{v}_i \) are the inward pointing normals of the \( n \) faces. The constant \( E \) in (2.33) is also replaced by \( E \rightarrow \vec{\xi} \cdot \vec{R} \).

In \cite{50, 51} it was derived a factorization formula for the generalised 3-ple sine functions:
\[
S_3^{M^n}(X) = \text{Ber} \prod_{k=1}^{n} \left( e^{\frac{x_{k\mu}}{\gamma_i}} e^{\frac{2\pi x_{k\mu}}{\gamma_i}} e^{\frac{2\pi x_{k\mu}}{\gamma_i}} \right)_k, \tag{2.43}
\]
where we denoted by $\text{Ber}$ the contribution of the exponential of the cubic Bernoulli polynomials and in each sector the equivariant parameters map to the toric data via the following table

| sector | $e_1$ | $e_2$ | $e_3$ |
|--------|-------|-------|-------|
| $k$    | $\det[\vec{R}, \vec{v}_{k+1}, \vec{n}]$ | $\det[\vec{v}_k, \vec{R}, \vec{n}]$ | $\det[\vec{v}_k, \vec{v}_{k+1}, \vec{R}]$ |

(2.44)

where $\vec{n}$ is chosen to satisfy the condition $\det[\vec{v}_k, \vec{v}_{k+1}, \vec{n}] = 1$. Proceeding as in the $S^5$ case, we apply the identity (2.43) to decompose the 1-loop part into $n$-copies of the 1-loop Nekrasov partition function $Z_{1\text{loop}}$.

The classical contribution is given by

$$Z_{\text{cl}}^{M^n} (\vec{\sigma}) = e^{\frac{2\pi i}{r} \rho \frac{g^2}{2} \text{Tr}(\vec{\sigma}^2)}$$

(2.45)

where $r$ is the overall scale of $\mathcal{M}^n$ and $\rho$ is the squashed volume normalized to $\text{vol}(S^3) = \pi^3$.

By repeatedly using the Gamma function identities (A.19) and (A.18) it is possible to show that

$$Z_{\text{cl}}^{M^n} (\vec{\sigma}) = \prod_{k=1}^n (Z_{\text{cl}} Z_{1\text{loop}})$$

(2.46)

with $Z_{\text{cl}}$ defined as in (2.30) and in each sector the equivariant parameters are related to the toric data according to the table (2.44).

By collecting the Bernoulli factors $\text{Ber}$ from the factorization of each generalized 3-ple sine one obtains a quadratic polynomial which produces the usual renormalization of the gauge coupling. This allows us to write

$$Z_{\text{cl}}^{M^n} (\vec{\sigma}) Z_{1\text{loop}}^{M^n} (\vec{\sigma}, \vec{M}) = \prod_{k=1}^n (Z_{\text{cl}} Z_{1\text{loop}})$$

(2.47)

where $Z_{\text{cl}}$ is defined as in equation (2.30) with $g^2 \rightarrow g_{\text{eff}}^2$.

In [50] it has been conjectured that the full non-perturbative partition function on $\mathcal{M}^n$ would receive contributions only from instantons solutions localized around closed Reeb orbits. Around each orbit the instanton contribution is given by a copy of the $\mathbb{R}^4 \times S^1$ instanton partition function $Z_{\text{inst}}$, leading to the following fully factorized proposal

$$Z_{\text{full}}^{M^n} = \int \prod_{k=1}^n (B_{5d}^{\text{eff}})$$

(2.48)

Proving this conjecture would require a careful study of the contact instanton equation which are quite difficult to analyse. At the moment we cannot rule out the possibility that other solutions will contribute to the full partition function.

In 3d and 4d, partition functions on the lens space $S^3/\mathbb{Z}_r$ and on the lens index $S^3/\mathbb{Z}_r \times S^1$ admit a block factorized form only when all the flat connections are summed over [16]. The contribution of a fixed flat connection to the partition function is not factorized. In the $M^n$ case instead, as we have just seen, the perturbative result in the trivial flat connection is already factorized. This fact could be a hint that the proposal (2.48) is indeed complete or perhaps just an accident. In conclusion further studies are necessary to shed light on this point.

2.2.3. $S^4 \times S^1$ partition functions and 5d holomorphic blocks. The next case we consider is 5d index or $S^4 \times S^1$ partition function. $\mathcal{N} = 1$ gauge theories on this background have been studied in [52–54]. The partition function takes the form of a Coulomb branch integral
\[ Z^{S^4 \times S^1} = \int d\bar{\sigma} \ Z^{S^4 \times S^1}_{-\text{loop}}(\bar{\sigma}, \tilde{M}) \ Z^{S^4 \times S^1}_{\text{inst}}(\bar{\sigma}, \tilde{M}), \]  

(2.49)

with instantons contributions from the fixed points at north and south poles of the \( S^4 \):

\[ Z^{S^4 \times S^1}_{\text{inst}} = \prod_{k=1}^{2} (Z_{\text{inst}})_{k} = \left\| Z_{\text{inst}} \right\|_{id}^2. \]  

(2.50)

Each pole contributes with a copy of the \( \mathbb{R}^4 \times S^1 \) partition function \( Z_{\text{inst}} \) (2.26) with \( m = iM + Q_0/2 \) and \( Q_0 = b_0 + 1/b_0 \) and the following identification

| sector | \( e_1 \) | \( e_2 \) | \( e_3 \) |
|--------|--------|--------|--------|
| 1      | \( b_0^{-1} \) | \( b_0 \) | \( 2\pi i/R \) |
| 2      | \( b_0^{-1} \) | \( b_0 \) | \( -2\pi i/R \) |

where \( R \) is the circumference of \( S^1 \) and \( b_0 \) the squashing parameter of \( S^4 \).

Due to the property (A.19) of the elliptic Gamma function, the classical term \( Z_{cl} \) defined in equation (2.30) ‘squares’ to one \( \left\| Z_{cl} \right\|_{id}^2 = 1 \), we can therefore write

\[ Z^{S^4 \times S^1} = \int d\sigma \ Z^{S^4 \times S^1}_{-\text{loop}}(\sigma, \tilde{M}) \ |Z_{cl} Z_{\text{inst}}|_{id}^2. \]  

(2.51)

The 1-loop contributions of vector and hyper multiplets are given by

\[ Z^{S^4 \times S^1}_{\text{vect}}_{-\text{loop}}(\sigma) = \prod_{\alpha > 0} \frac{\Upsilon R(i\alpha(\sigma)) \ Upsilon R(-i\alpha(\sigma))}{\prod_{\rho \in R} \ Upsilon R(i\rho(\sigma) + iM + \frac{b_0}{2})}. \]  

(2.52)

where the special function \( \Upsilon R \) is defined in appendix. Also in this case it is possible to bring the 1-loop term in a factorized form hence the 5d index can be factorized in terms of the same 5d blocks \( B_{5d} \) we found in the \( S^5 \) case [48]:

\[ Z^{S^4 \times S^1} = \int d\sigma \ \left\| B_{5d} \right\|_{id}^2. \]  

(2.53)

### 2.3. Codimension-two defects via Higgsing

Codimension-two half BPS defects in SUSY gauge theories, such as surface operators in 4d, are an important class of non-local operators which can be used to probe the dynamics of gauge theories. For a recent review on surface operators see [55] and contribution [56]. They can be defined by prescribing a singular behaviour of the fields on the codimension-two locus where defect operators live, this has the effect of breaking the gauge group \( G \) to a Levi subgroup \( L \). Another possibility to define defects in gauge theories is by a coupled system with extra degrees of freedom leaving on the defects. A related construction, the so called Higgsing procedure, introduced in [57] involves turning on a position dependent vev or vortex configuration in a UV theory \( T' \) and following the RG flow to an IR point described by a theory \( T \). This construction should be indeed equivalent to coupling the 4d gauge fields to a 2d sigma model with target space the vortex moduli space.

The Higgsing prescription gives rise to a large class of \( \mathcal{N} = 2 \) and \( \mathcal{N} = 1 \) theories with surface operators insertions. Some of these systems can be realised in Hanany–Witten brane
set ups with surface operators engineered by extra branes \cite{58} as show in figure 1 and admit a description in terms of a 2d GLSM coupled to the bulk theory \cite{59}. Although this is not the most general type of surface operator, we will restrict to this class in the following.

In recent years, there has been much progress in computing partition functions and indices of theories with the insertion of these operators. In 4d one can compute the superconformal index \((S^3 \times S^1\) partition function) of a theory \(T\) enriched by a surface operator via Higgsing by tuning the flavour fugacities of the theory \(T'\) to special values. This causes the integration contour to pinch a set of poles. The index is then evaluated by taking the residue at these poles and the result yields the index for the coupled 2d-4d system is decorated with the elliptic genus of the 2d GLSM \cite{60}.

It is also possible to compute the \(\mathbb{R}^4\) instanton partition function for 4d theory in presence of a surface defect. This takes the form of a double expansion (see for example \cite{20})

\[
\sum_{n=0}^{\infty} \sum_{m \in \Lambda} Q^m \cdot Z_{\text{inst}}^{n,m} + \text{vortex},
\]  

(2.54)
where \( n, Q \) are respectively the instanton number and the instanton counting parameter, 
\( m \in \Lambda \sim H_2(G/L; \mathbb{Z}) \) (where \( L \) is the Levi subgroup of the gauge group \( G \)) are the monopole numbers, and \( z \) is the vortex counting parameter. By decoupling the bulk theory (sending \( Q \to 0 \) and focusing on the \( n = 0 \) sector) one gets the purely 2d vortex counting partition function.

At the level of the Hanany–Witten brane setup, as shown in the second line of figure 2, decoupling the bulk theory amounts to sending to infinity all the NS5 branes far from the insertion point. The 4d theory is then just a collection of free 4d hypers coupled to the 2d theory on the stretched D2 brane. Also in this case the combined instanton-vortex counting partition function can be obtained via Higgsing, by tuning the mass parameters to special values [61].

Another option to compute the instanton-vortex partition function is geometric engineering, where \( N = 2 \) gauge theories are obtained via type II strings compactified on toric Calabi–Yau threefolds. The refined A-model topological string partition function on these targets [62, 63] coincides with the \( \mathbb{R}^4_{\epsilon_1, \epsilon_2} \times S^1 \) instanton partition function, while open topological strings amplitudes in presence of toric branes reproduce the instanton-vortex counting [20, 64–66].

In the context of the AGT correspondence [7], two types of surface defects in class \( S \) theories were discussed. The first type is realised by M2 branes with two directions extending in the 4d space-time and sitting at a point on the Gaiotto curve \( C_{g,n} \). The second type of surface operators is instead realised by M5 branes wrapping \( C_{g,n} \) and with two directions extending in the 4d space-time. The authors of [67] proposed to relate the first type class of surface operators to degenerate primary operators in the dual CFT. Indeed it was later shown that it is possible to match the instanton-vortex partition function for these surface operators to the conformal blocks with degenerate insertions as sketched in figure 2, [20, 64, 66].

The compact space version of the AGT + surface operators correspondence was proposed in [32, 68] where it was shown that degenerate correlators can be mapped to partition functions of class \( S \) theories on \( S^4 \), coupled to a 2d GLSM describing the surface defects defined on \( S^2 \).

As a simple example we consider the \( SU(2) \) theory with 4 flavours on \( S^4 \). When a combination of the masses is tuned to the ‘Higgsing condition’ \( m_1 + m_2 = -b_0 \), where \( b_0 \) is the squashing parameter of the ellipsoid, the integration contour pinches two poles. The sum of the residue of the partition function at these two points can be identified with the \( S^2 \) partition function of the (2,2) SQED with 2 flavours multiplied by the contributions of the free 4d hypers. Via the AGT dictionary, which relates the mass parameters to the four external momenta in the CFT correlator, the Higgsing condition is translated into the analytic continuation of one momentum to a degenerate value \( \alpha_2 \to -b_0/2 \). Summarising we have the following web of correspondence:

\[
\begin{array}{c}
Z^{S^4}[2 + SU(2) + 2] \xrightarrow{\text{Higgsing}} Z^{S^2}[U(1) + 2] \\
\text{AGT} \quad \updownarrow \quad \text{AGT} \quad \updownarrow \\
\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle \xrightarrow{\text{Degeneration}} \langle V_{\alpha_1} V_{-b_0/2} V_{\alpha_3} V_{\alpha_4} \rangle.
\end{array}
\]

3 Via the AGT dictionary the squashing parameter \( b_0 \) is identified with the parameter appearing in central charge \( c = 1 + 6(b_0 + b_0^{-1})^2 \) of the dual CFT.

4 The expressions obtained via Higgsing/degenerations have been normalised respectively by the contribution of 4d free hypers and by the 3-point function of the non-degenerate primaries.
We conclude our digression on surface operators in 4d theories by mentioning that for surface operators realised in terms of M5 branes the standard instanton moduli space is replaced by the ‘ramified instantons’ moduli space and the CFT duals, studied for example in [69, 70] and recently in [71], have affine sl(N) symmetries. More general surface operators and CFT duals have been studied in [72] and [73], for a review see [74].

Codimension-two half BPS defects can be introduced also in 5d \( \mathcal{N} = 1 \) theories. In [75] the Higgsing prescription has been applied to obtain the 5d index (\( S^4 \times S^1 \) partition function), decorated by the 3d index (\( S^3 \times S^1 \) partition function) of codimension-two defects. Here we will consider codimension-two defects defined via Higgsing on \( S^3 \) and \( S^1 \times S^1 \). We will restrict again to the simple case of free 5d hypers coupled to the codimension-two defect partition function. We consider the case of the the \( SU(2) \) theory with four fundamental hyper multiplets with masses \( M_f, f = 1, \ldots, 4 \) on \( S^5 \). As shown in [48], if we tune the masses to satisfy the condition:

\[
M_1 + M_2 = i(\omega_3 + E) \quad \text{or} \quad m_1 + m_2 = -\omega_3,
\]

where \( m_j = iM_j + E/2 \), the 1-loop factor develops poles which pinch the integration contour. The partition functions can then be defined via a meromorphic analytic continuation which prescribes to take the residues at the two poles trapped along the integration path located at

\[
a_1 = m_1 = -m_2 - \omega_3 = -a_2, \quad a_1 = m_1 + \omega_3 = -m_2 = -a_2.
\]

By analogy with the 4d case, we expect that by taking the residues at these poles, will reconstruct the \( S^5_0 \) partition function of codimension-two defects. Here

\[\begin{align*}
Z_{\text{inst}}^\mathcal{N} &
\rightarrow
Z_{\Phi_1}(A, B; C, e^{2\pi i \frac{m_1}{b}}; u) \\
\rightarrow
Z_{\Phi_1}^{(1)}(A, B; C, e^{2\pi i \frac{m_1}{b}}; u)
\end{align*}\]

The coefficients \( A, B, C \) of the basic hypergeometric function depend on the 5d mass parameters, if we identify them with those of the vortex partition function \( Z_{\Phi_1}^{(1)} \), we obtain the following dictionary between 3d and 5d masses:

\[
m_1^3 = -im_1, \quad m_2^3 = -im_2, \quad m_1^5 = im_3, \quad m_2^5 = im_4.
\]

while by matching the expansion parameters we find \( \frac{\xi}{\lambda} = 1/\tilde{g}^2 \). We also identify \( \omega_2 = \frac{1}{\tilde{g}^2} = b \).

In complete analogy, for the other pole, located at \( a_1 = m_1 + \omega_3 = -m_2 = -a_2 \) we find

\[
\rightarrow
Z_{\Phi_1}^{(2)}(A, B; C, e^{2\pi i \frac{m_1}{b}}; u)
\]

The last step is the identification of the residue at the \( i \)th pole of the \( S^5 \) classical and one loop contributions with \( S^5_0 \) classical and one loop contributions evaluated on the \( i \)th vacuum, in the end after normalizing by the contribution of the free 5d hypers, one obtains the promised result:

\[
Z^S_{[2 + SU(2) + 2]} \rightarrow \sum_{i=1,2} \left(Z_{\text{cl}}^S \xi_1^S \right) \left(Z_{\text{1-loop}}^S \right) \left(Z_{\Phi_1}^{(i)} \right) \left(Z_{\Phi_1}^{(i)} \right)
\]

\[
Z^S_{U(1) + 2}.
\]
Notice that there are two extra choices for the degeneration condition, which would have led to the same result:

\[ m_1 + m_2 = -\omega_{1,2}, \quad \text{with} \quad \omega_{1,2} = \omega_{2,1}^{-1} = b. \]  

(2.61)

The three possibilities correspond to the three maximal squashed 3-spheres inside the squashed 5-sphere.

In a similar manner, it is possible to show that the partition function of the SCQCD on \( S^4 \times S^1 \), when two of the masses satisfy the condition

\[ m_1 + m_2 = -b_0, \]  

(2.62)

reduces to the SQED partition function on \( S^2 \times S^1 \):

\[ Z_{S^4 \times S^1} [2 + SU(2) + 2] \rightarrow_{m_1 + m_2 = -b_0} \sum_{i=1,2} \left( Z_{S^4 \times S^1}^{S^2 \times S^1} \right) \left( \frac{Z_{S^2 \times S^1}^{(i)}}{Z_{S^2 \times S^1}} \right)^2 \]

= \[ Z_{S^2 \times S^1} [U(1) + 2], \]

(2.63)

with the 3d angular momentum fugacity \( q \) related to the 5d parameters by \( q = e^{R/b_0} \). Also in this case there is another possible degeneration condition \( m_1 + m_2 = -\frac{1}{b_0} \), which leads to the same result but with the identification \( q = e^{b_0} \). The two choices correspond to the two maximal \( S^2 \) inside the squashed \( S^4 \).

By paralleling the 4d case we will reinterpret the 5d-3d Higgsing (2.60), (2.63) as the analytic continuation of a 4-point \( q \)-deformed correlator to the \( q \)-correlator of three non-degenerate and one-degenerate primaries.

Before doing so in next section we will focus on the \( q \)-deformation of the chiral version of the AGT correspondence where we review how \( \mathbb{R}^4 \times S^1 \) instanton and \( \mathbb{R}^2 \times S^1 \) vortex counting can be mapped to chiral blocks of deformed Virasoro primaries.

### 3. Chiral 5d AGT

#### 3.1. Deformed Virasoro algebras and chiral blocks

Deformed Virasoro and \( \mathcal{W} \) algebras were introduced independently in the mid 90s by various groups \([76, 77],[78–80, 81, 82]\), see \([83]\) for a review. The Vir\(_{q,t} \) algebra is a deformation of the Virasoro algebra, it depends on two complex parameters \( q, t \) and is generated by an infinite set of generators \( T_n \) with \( n \in \mathbb{Z} \), satisfying the commutation relations

\[ [T_n, T_m] = -\sum_{l=1}^{+\infty} f_l (T_{m+l} - T_{m-l}) - \frac{(1-q)(1-t^{-1})(p^n - p^{-n})}{1-p} \delta_{m+n,0} \]

(3.1)

where \( p = \frac{q}{t} \) and the functions \( f_l \) are determined by the expansion

\[ f(z) = \sum_{l=0}^{+\infty} f_l z^l = \exp \left[ \sum_{l=1}^{+\infty} \frac{1}{n} (1-q^n)(1-t^{-n}) z^n \right]. \]

(3.2)

The algebra Vir\(_{q,t} \) is invariant under \((q, t) \rightarrow (q^{-1}, t^{-1})\) and \((q, t) \rightarrow (t, q)\).

As in the Virasoro algebra, the representations of Vir\(_{q,t} \) can be constructed using Verma modules \([76]\). The highest weight state \( |\lambda\rangle \) satisfies

\[ T_0 |\lambda\rangle = \lambda |\lambda\rangle, \quad T_n |\lambda\rangle = 0 \quad \text{for} \quad n > 0, \]

(3.3)
and the Verma module is constructed acting on the highest weight state $|\lambda\rangle$ with the operators $T_{-n}$ with $n > 0$. Singular states in the Verma module can be detected using the Kac determinant. In particular, there is a level two singular vector when $\lambda$ takes the following values

$$
\lambda_1 = p^{1/2} q^{1/2} + p^{-1/2} q^{-1/2},
\lambda_2 = p^{1/2} t^{-1/2} + p^{-1/2} t^{1/2}.
$$

(3.4)

The states $\lambda_1$ and $\lambda_2$ are mapped into each other by the exchange $(q, t) \to (t, q)$ and are left invariant by $(q, t) \to (q^{-1}, t^{-1})$.

The algebra $\mathcal{V}ir_{q,t}$ can be related to other known algebras by taking suitable limits on parameters $q, t$ [84]. In particular, the ordinary Virasoro algebra is recovered by setting $t = q^3$, $q = e^b$ and expanding the deformed Virasoro current $T(z) = \sum_{n \in \mathbb{Z}} T_n z^{-n}$:

$$
T(z) = 2 + T^{(2)} h^2 + T^{(4)} h^4 + \cdots
$$

(3.5)

In the second term of the expansion

$$
T^{(2)} = \beta \left( \frac{z^2 L(z) + (1 - \beta)^2}{4\beta} \right)
$$

(3.6)

we recognise the Virasoro current $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ where $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$ with central charge $c = 1 - 6(\sqrt{\beta} - \frac{1}{\sqrt{\beta}})^2$ and with the identification $\beta = -\frac{b_0}{2}$. From the expansion (3.5) we see that to control the deformed theory we need all the the higher spin currents $T^{(n)}(z)$, while in the undeformed case the current $L(z)$ constraints completely the conformal blocks. We also notice that since in the $h \to 0$ limit the $(q, t) \to (t, q)$ symmetry of $\mathcal{V}ir_{q,t}$ reduces to the $b_0 \leftrightarrow \frac{1}{b_0}$ symmetry, it is natural to identify the states $\lambda_1$, $\lambda_2$ (3.4) as the $q$-deformation of the level-two degenerate states $\alpha^{(1,2)} = -\frac{b_0}{2}$ and $\alpha^{(2,1)} = -\frac{1}{2b_0}$ of the undeformed Virasoro case.

Correlation functions in 2d CFTs can be computed by decomposing them into 3-point functions and conformal blocks, by the insertion of complete sets of Virasoro descendants. 3-point functions can be determined by the bootstrap approach by requiring the associativity of the OPE, which is equivalent to the modular invariance and single valuedness of the correlators [85]. Conformal blocks can in turn be computed as series expansions in powers of the cross ratios, with coefficients obtained by repeated applications of the commutations rules of the Virasoro generators with the primary operators such as:

$$
[L_m, V_\alpha(z)] = \varepsilon^{m+1} \frac{\partial}{\partial z} V_\alpha(z) + \Delta(\alpha)(m + 1) \varepsilon^m V_\alpha(z).
$$

(3.7)

There is also an alternative approach due to Dotsenko–Fateev [86] (see also [87]) which consists in deriving a Feigin–Fuchs integral representations for conformal blocks as $n$-point functions of Coulomb gas vertex operators.

In the deformed case, the analogue of equation (3.7) is not known and so far most of the results have been obtained via the free boson integral approach. In [76] the deformed Virasoro algebra and its free field realization was introduced to construct singular vectors which are eigenvectors of the Macdonald operator hence proportional to the Macdonald symmetric functions. In ordinary CFT there is an similar relation between singular vectors of the Virasoro algebra and the Jack symmetric functions which in turn appears in the description of excited states in the the Calogero–Sutherland model (CSM). Generalisation the CFT-CSM correspondence to the $q$-deformed case was indeed the motivation to study the deformed Virasoro algebra in [76].

The free boson integral formulation was also the central tool in the series of works [78–80] which led to an independent derivation of the deformed Virasoro algebra. In these works the authors focused on the algebra of chiral vertex operators.
\[ \Phi_{\Delta_1}^{o_1} (z_1) \Phi_{\Delta_2}^{o_2} (z_2) = \sum_{\nu_2} W_{\Delta_1 \Delta_2} \left[ \frac{\nu_2 \nu_1}{\nu_2 \nu_1} \right] \Phi_{\Delta_3}^{o_3} (z_3) \Phi_{\Delta_4}^{o_4} (z_4), \tag{3.8} \]

where the primary operators
\[ \Phi_{\lambda \nu}^{\Delta} (z) : \mathcal{L}_{\nu} \to \mathcal{L}_\lambda \otimes \mathbb{C} [z] z^{\Delta - \Delta}, \tag{3.9} \]
interpolate between irreducible representation of the algebra \( \mathcal{L}_\lambda \) specified by highest weight \( \Delta \).

In the Virasoro case the matrix \( W_{\Delta_1 \Delta_2} \) is a constant (independent on \( z_1 z_2^{-1} \)) solution of the Yang–Baxter equation. The idea of [78] was to consider a suitable deformation of the bosonization construction to realise chiral vertex operators which satisfy the commutation relations (3.8) with elliptic \( W_{\Delta_1 \Delta_2} \) matrices. Remarkably in this way they obtained the same deformed Virasoro algebra considered in [76]. This deformation of the Virasoro algebra was also identified as the dynamical symmetry of the Andrews–Baxter–Forester (ABF) model in [79].

Recently, the integral representation of the deformed Virasoro chiral blocks has been reconsidered in [88], before reviewing this work we record the integral representation of ordinary Virasoro conformal blocks in Liouville theory. The Liouville CFT is the theory of a scalar \( \phi \) with an exponential potential and with action
\[ A_{\text{Liouville}} = \int dz d\bar{z} \sqrt{g} \left( g^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi + Q_0 R \phi + e^{2b_0 \phi} \right), \tag{3.10} \]
where \( R \) is the Ricci scalar and \( Q_0 = b_0 + \frac{1}{b_0} \) the background charge. Correlators can be computed by treating the potential as a perturbation of the free boson theory. In particular, an \( M + 2 \)-point conformal blocks is obtained by inserting \( M + 2 \) vertex operators with momenta \( \alpha_a \) at positions \( z_a, a = 0, \ldots, M + 1 \), with \( z_{M+1} = \infty \), in the background of \( N \) screening charge integrals
\[ B_{M+2} = \left\langle \oint dw_1 \cdots \oint dw_N V_{\alpha_0} (0) \cdots V_{\alpha_M} (z_M) S (w_1) \cdots S (w_N) \right\rangle. \tag{3.11} \]

The vertex operators and screening charges are given in terms of the free boson \( \phi(z) \):
\[ V_\alpha (z) = : e^{-\frac{\alpha \phi(z)}{b}} : , \quad S(z) = : e^{2b_0 \phi(z)} :. \tag{3.12} \]

It is convenient to represent conformal blocks by means of comb diagrams as shown in figure 3. The momentum \( \alpha_{M+1} \) is determined by remaining \( M + 1 \) momenta and by the total number of screening charges \( N \)
\[ \sum_{a=0}^{M+1} \alpha_a + 2 \beta N = 2 - 2 \beta. \tag{3.13} \]

By expanding in modes the Liouville field:
\[ \phi(z) = \phi_0 + \frac{h_0 \log z}{b_0} + \frac{1}{b_0} \sum_{k \neq 0} h_k z^{-k}, \tag{3.14} \]
and using the modes commutation rule
\[ [h_n, h_m] = -\frac{b_0^2}{2} n \delta_{n+m,0}. \tag{3.15} \]
the conformal block reduces to the following Dotsenko–Fateev (DF) integral:
The integration contour is defined by splitting the $N$ screening integrals into $M$ groups with $N_a$ screening integrals each:

$$N = \sum_{a=1}^{N} N_a.$$  \hspace{1cm} (3.17)

The contour for the $a$th group encircles the segments $C_a$

$$C_a : \ [0, z_a], \quad a = 1, \ldots, M.$$  \hspace{1cm} (3.18)

Naively the integral (3.16) seems to miss some parameters. As shown in figure 3, the conformal block depends in fact on the internal momenta $a_1, \ldots, a_{M-1}$. The missing parameters are precisely the fractions of screening charges integrated along each contour [89–94], with the identifications

$$a_1 = \alpha_0 + \alpha_1 + 2\beta N_1, \quad a_k = a_{k-1} + \alpha_k + 2\beta N_k, \quad k = 2, \ldots, M-1.$$  \hspace{1cm} (3.19)

In the deformed Virasoro case one promotes the modes commutation rule to a $q$-deformed commutation

$$[h_n, r_m] = \frac{-b_0^2}{2} n \delta_{n+m,0} \quad \rightarrow \quad \{h_n, r_m\} = \frac{1}{1 + (t/q)^n} \left(1 - q^n\right) n \delta_{n+m,0},$$  \hspace{1cm} (3.20)

and defines bosonised vertex operators and screening charges as

$$\hat{V}_a(z) \equiv e^{\left(-\frac{2\phi_0}{n} + \frac{2b_0 \log z}{n} + \sum_{\alpha=1}^{a} \frac{1 + b_0 z^{-\alpha}}{1 + b_0 z^{-\beta}} \log z\right)} ;$$  \hspace{1cm} (3.21)

$$\hat{S}(z) \equiv e^{\left(2\phi_0 + 2b_0 \log z + \sum_{\alpha=1}^{a} \frac{1 + b_0 z^{-\alpha}}{1 + b_0 z^{-\beta}} \log z\right)} ;$$  \hspace{1cm} (3.22)

The $q$-deformed chiral block with $M + 2$ insertions then reads [76, 88]:

$$B_{M+2}^q = \left\langle \oint C_1 \cdots C_M \prod_{1 \leq i < j \leq N} (w_i - w_j) \prod_{a=0}^{M} \prod_{i=1}^{N} (w_i - z_a)^{\alpha_a} \right\rangle.$$  \hspace{1cm} (3.23)

By using the commutation rules (3.20) the block (3.23) reduces to the following $q$-deformed DF integral:

$$B_{M+2}^q = \frac{C}{\prod_{a=1}^{M} N_a} \oint_{C_1 \cdots C_M} \prod_{a=0}^{M} V_a(w, z_a).$$  \hspace{1cm} (3.24)
where
\[
\Delta^2_{q,f}(w) = \prod_{1 \leq i \neq j \leq N} \frac{(w_i/w_j; q)_{\infty}}{(w_i/w_j; q)^{\beta}_{\infty}}, \quad V_{\beta}(w, z_\alpha) = \prod_{j=1}^{N} \frac{(q^{\alpha} z_\alpha/w_j; q)_{\infty}}{(z_\alpha/w_j; q)_{\infty}}. \tag{3.25}
\]

The higher rank case, involving the \(q\)-\(\mathcal{W}\) algebra is studied in [95].

In the literature we find often another presentation of the DF integrals:
\[
B_{M+2} = \int_0^1 \int_0^{N_1-1} \cdots \int_0^{N_\beta-1} \prod_{1 \leq i \neq j \leq N} (1 - \frac{w_i}{w_j})^\beta \prod_{i=1}^{N_\beta} \bigg( 1 - \Lambda_i w_i \bigg)^2 \cdots \bigg( 1 - \Lambda_M w_i \bigg)^{\nu_i}, \tag{3.26}
\]

it is easy to find a dictionary relating the parameters \(v_i\) to the momenta \(\alpha_i\) and the cross ratios \(\Lambda_i\) to the \(z_i\) in equation (3.16), so the two presentations are equivalent. The DF integral (3.26) can be promoted to the \(q\)-deformed case by replacing
\[
\prod_{1 \leq i \neq j \leq N} (w_i - w_j)^\beta \rightarrow \prod_{1 \leq i \neq j \leq N} (w_i - q^k w_j), \quad (1 - x)^c \rightarrow \prod_{n=0}^{c-1} (1 - q^n x), \tag{3.27}
\]

and the integration measure by the Jackson measure \(dx \rightarrow dx^D\) defined as
\[
\int_0^1 f(x) dx = (1 - q) \sum_{k=0}^{\infty} q^k f(q^k). \tag{3.28}
\]

In the end one arrives at the \(q\)-deformed DF integral [96]:
\[
B^q_{M+2} = \int_0^1 \int_0^{N_1-1} \cdots \int_0^{N_\beta-1} \prod_{1 \leq i \neq j \leq N} (1 - q^k w_i)^{\nu_i} \prod_{k=0}^{\nu_i-1} (1 - q^k \Lambda_i w_i \cdots \prod_{k=0}^{\nu_M-1} (1 - q^k \Lambda_M w_i), \tag{3.29}
\]

which can be shown to be equivalent to (3.24) for integer values of \(\beta\) and of the momenta. Actually the expressions in (3.25) provide the analytic continuation to non-integer values of the products in (3.27).

### 3.2. \(\mathbf{\mathcal{W}}\)-\(\mathcal{R}\), \(\alpha\), \(\beta\) Chiral Blocks and Instanton-Vortex Counting

In this section we will discuss the map between \(\mathcal{W}_{\alpha,\beta}\) chiral blocks and \(\mathbb{R}^3 \times S^1\) instantons or \(\mathbb{R}^2 \times S^1\) vortex partition functions. The \(q, \tau\) parameter appearing in the deformed Virasoro algebra are identified with the equivariant parameters in the instanton partition function \(q = e^{\beta \tau_1}, \tau = e^{\tau_2}\) with \(R\) the radius of the \(S^1\).

The first evidence of this map is the very neat result of [97] (see also [98]) where the \(q\)-deformed version of the so-called Gaiotto–Whittaker states [99] was constructed. The deformed analogue of the Gaiotto–Whittaker states \(|G_n\rangle = \sum_{n=0}^{\infty} \Lambda^{2n} |G_n\rangle\) are states in the Verma module \(M_R\) satisfying:

\(^5\)The parameter \(\tau\) in this section is the inverse of the parameter \(\tau\) appearing in section 2.2.
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\[ T_n | G \rangle = \Lambda^2 G, \quad T_m | G \rangle = 0 \quad (m \geq 2), \quad | G \rangle = h \]

\[ Z_{\text{inst}} \left[ SU(2) \right] \]

\[ \langle G | G \rangle = Z_{\text{inst}} \left[ SU(2) \right], \]

where \( Z_{\text{inst}} \left[ SU(2) \right] \) is the instanton partition function of the 5d \( \mathcal{N} = 1 \) pure \( SU(2) \) theory and the parameter \( \Lambda^2 \) is identified with the instanton counting parameter. This result has been generalized to higher rank in [100].

Moving to multipoint correlators, we have to rely on the q-DF integral representation discussed in the previous section since, as we mentioned, the OPE approach in the q-deformed case is not well developed. This fact also implies that at the moment we cannot evaluate the correlators at generic values of external and internal momenta since they will have to satisfy conditions like (3.13), hence we can only test the correspondence with 5d instanton partition functions at special points in the moduli space.

The undeformed AGT correspondence associates to the sphere with \( M + 2 \) punctures the \( M + 2 \)-point correlator on the CFT side and the \( \mathcal{N} = 2, 2 + SU(2)^{M-1} + 2 \) linear quiver on the gauge theory side, one would then expect an analogous map between the \( M + 2 \)-point block in \( \text{Vir}_{q_d} \) and the 5d \( \mathcal{N} = 1, 2 + SU(2)^{M-1} + 2 \) quiver theory.

However, it is important to notice that in 5d the \( 2 + SU(2)^{M-1} + 2 \) theory is dual to the \( M + SU(M) + M \) theory (more generally the duality relates \( K + SU(K)^{M-1} + K \) to \( M + SU(M)^{M-1} + M \) theories) [101, 102]. A neat way to understand this duality is to consider the geometric engineering perspective. In fact both theories can be engineered by a IIA compactification on the same toric Calabi–Yau threefold, whose toric diagram for the \( M = 3 \) case is depicted in figure 4. The topological string partition function \( Z_{\text{top}} \) on this CY can be computed by means of the refined vertex formalism [63]. One has to glue trivalent vertices by summing over sets of representations associated to each internal leg. In general it is not possible to perform all the sums in a closed form and one typically gets a perturbative expansion in powers of the Kähler parameters of the legs with representations left to sum.
For example, in the case of figure 4 one can resum all the reps associated to the diagonal and vertical legs and obtain a perturbative expansion in the Kahler parameters of the horizontal legs with Young tableaux $Y_{1,1}, Y_{1,2}$ and $Y_{2,1}, Y_{2,2}$. The result one gets in this way can be identified with the 5d $2 + SU(2)^2 + 2$ instanton partition function with the Kahler parameters of the horizontal legs mapped to 5d gauge couplings of the two nodes.

Alternatively, one can resum first the diagonal and horizontal legs and obtain a perturbative expansion in the Kahler parameters associated to the vertical legs with Young tableaux $R_1, R_2, R_3$. This expansion can be mapped to the $3 + SU(3) + 3$ instanton partition function. The fact that the two ways of performing the sum are equivalent requires the non-trivial slicing invariance property of the refined topological vertex. The dictionary relating the topological string Kahler parameters to the Coulomb, gauge coupling and mass parameters in two dual gauge theories can be found in [103] (see also [104]).

As a result of the discussion above, the $M + 2$-point $q$-DF block is expected to be related to both these 5d theories. As we will see it turns out that there are two distinct evaluation methods which yield the two instanton expansions.

In [88] a procedure was devised to evaluate the $q$-DF $M + 2$-point block (3.24) by residues computation. The integrand in (3.24) is a meromorphic function, poles come from the zeros of the $q$-products $\prod_{k=1}^{\infty} (1 - q^k z)$ in the denominator located at $z = q^{-k}$. By taking into account carefully that certain poles are cancelled by the zeros of the $q$-products in the numerator, it can be shown that the poles enclosed by each integration contour $C_a$ where $N_a$ screening charges are integrated, are labelled by Young tableaux $R_a$ at most $N_a$ rows.

Basiclly one finds:

$$\frac{1}{N_{r_1}!} \oint_{C_{r_1}} dw_1^{N_{r_1}} \cdots \frac{1}{N_{r_M}!} \oint_{C_{r_M}} dw_{N_{r_M}} \to \sum_{\vec{R}} \cdots .$$

(3.32)

By taking the residues at these sets of poles in (3.24) the result very nicely organizes as the 5d Nekrasov partition function for the $M + SU(M) + M$ theory:

$$B_{M+2}^q = \sum_{\vec{R}} \Lambda^{r_{f\text{und}}(\vec{R})} \frac{z_{\text{fund}}(\vec{R})^{-q}}{z_{\text{fund}}(\vec{R})} = Z_{\text{inst}}^5 [M + SU(M) + M].$$

(3.33)

where $\vec{R} = (R_1, \cdots, R_M)$. For details on how the gauge theory parameters are mapped to the $q$-DF we refer the reader to [88, 105]. Notice that, since as discussed above the sets of representations are non-generic and the mass and Coulomb parameters via the dictionary are identified with momenta satisfying the conditions (3.19), the 5d theory is at a special point in the moduli space where the Higgs branch and the Coulomb branch meet at the origin. The non-restricted 5d theory was conjectured to emerge via geometric-transition in the large $N$ limit.

In [88, 95], it was also observed that the $M + 2$-point block (3.24) can be directly identified with the block integral $Z_{3d}$, discussed in section 2.1, of a 3d $U(N)$ theory with $2M$ flavours and one adjoint. Indeed the screening charge contributions $\Delta_{3d}(w)$ and the vertices $V_{3d}(w, z_a)$ in equation (3.25) can be respectively identified with the vector and hypermultiplets contribution to the 3d block integral. In the end one finds a triality relation:

$$B_{M+2}^q = Z_{3d}^5 [M + SU(M) + M].$$

(3.34)

The 3d theory is interpreted as the theory studied in [58, 106], on charge $N$ vortices in the 5d gauge theory.

The relation between free field correlators in $q$-Virasoro and 3d gauge theories partition functions has been recently discussed in [107]. The authors showed that it is possible to map
3d $\mathcal{N} = 2$ partition functions on the 3-manifolds $M_3^{\text{def}}$ to free field correlators of the $q$-Virasoro modular double.

We now turn to the second evaluation method of the $q$-DF integrals yielding the dual instanton expansion. We begin by recording the schematic form of the instanton expansion of the $2 + SU(2)^{M-1} + 2$ quiver theory:

$$Z_{\text{inst}}^{|SU(2)^{M-1} + 2|} = \sum \prod_{\vec{y}_a} |\vec{y}_1| \cdots |\vec{y}_M|$$

$$\sum_{\vec{z}_{\text{fund}}(\vec{y}_1)\vec{z}_{\text{fund}}(\vec{y}_2) \cdots \vec{z}_{\text{fund}}(\vec{y}_{M-1})} \vec{z}_{\text{vec}}(\vec{y}_1) \vec{z}_{\text{vec}}(\vec{y}_2) \cdots \vec{z}_{\text{vec}}(\vec{y}_{M-1}) \sim \sum_{\vec{y}_a} |\vec{y}_1| |\vec{y}_2| \cdots |\vec{y}_{M-1} | |\vec{y}_M|,$$

where $\vec{y}_a$ is a two component Young Tableau and $\Lambda_1 \cdots \Lambda_{M-1}$ the gauge couplings. In the 4d/undeformed case this structure suggested to look for an analogous decomposition of conformal blocks on a basis of states labelled by the Young tableaux $\vec{Y}_a$, this is the so-called Nekrasov decomposition of the conformal blocks:

$$\langle V_{\alpha_0}(0) V_{\alpha_1}(1) V_{\alpha_2}(-\infty) \cdots V_{\alpha_{M-1}}(\infty) \rangle \sim \sum_{\vec{y}_a} \langle V_{\alpha_0}(0) | V_{\alpha_1}(1) | \vec{\alpha}_1, \vec{y}_1 \rangle \langle \vec{\alpha}_1, \vec{y}_1 | V_{\alpha_2}(\Lambda_2) | \vec{\alpha}_2, \vec{y}_2 \rangle \cdots$$

$$\cdots \langle \vec{\alpha}_{M-1}, \vec{y}_{M-1} | V_{\alpha_M}(\Lambda_M) | V_{\alpha_{M+1}}(\infty) \rangle,$$

where the symbol $\sim$ is due to the omission of the so-called $U(1)$ factor that one needs to strip-off from the Nekrasov partition function in order to match with the CFT results. This factor plays a crucial role in \cite{108, 109} where this basis was identified as a special orthogonal basis of states for the algebra $\text{Vir} \otimes \mathcal{H}$, the tensor product of Virasoro and Heisenberg algebras. Besides, rendering much simpler the evaluation of the coefficients of the OPE, this basis has a clear interpretation as the class of fixed points in the equivariant cohomology of the instanton moduli space.

Unfortunately directly lifting this approach to the deformed case is problematic since, as we already mentioned, the OPE approach in the deformed case is not known, however one can try to find a similar decomposition of the $q$-DF integrals in terms of these states. This idea has been developed in a series of papers \cite{93, 94, 112, 113}. The authors initially focused on the 4-point block in the undeformed Virasoro case which can be schematically expressed as:

$$B_{4+2} = \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} d\mu(x) \ d\mu(y) \ \mathbb{P}^2(x,y),$$

where $x = (x_1, \cdots x_N)$ and $y = (y_1, \cdots y_N)$ indicate the two sets of screening charges variables integrated on the first and second contours respectively. We refer the reader to \cite{113} for the explicit definition of the factors $d\mu(x), d\mu(y)$, which can be identified with the so-called Selberg measure, and of the cross term $\mathbb{P}^2(x,y)$. The idea was to express the cross term by means of the completeness of a set of new orthogonal polynomials $\mathcal{K}_p(x)$ labelled by Young tableaux $\vec{Y} = Y_1, Y_2$. These polynomials are a generalization of Jack polynomials depending on two reps which, in the $\beta = 1$ case, reduce to the products of two Schur polynomials $\mathcal{K}_p(x) = \chi_{Y_1} \chi_{Y_2}$. Using the completeness of these polynomials

$$\mathbb{P}^2(x,y) = \sum_p \prod_{s} A_s^{\vec{y}_s} K_p(x) K_p(y),$$

$^6$The action of the $W_3$ algebra on the instanton moduli space is discussed \cite{110, 111}.
the 4-point function can be expressed as a double Selberg average:

$$B_{2+2} = \int_{C_1} d\mu(x) \int_{C_2} d\mu(y) \mathbb{P}^2(x, y) = \sum_{\vec{y}} \Lambda[\vec{y}] \int_{C_1} d\mu(x) K_{\vec{y}}(x) \int_{C_2} d\mu(y) K_{\vec{y}}^*(y). \tag{3.39}$$

The explicit evaluation of these Selberg averages remarkably yields

$$\int_{C_1} d\mu(x) K_{\vec{y}}(x) \int_{C_2} d\mu(y) K_{\vec{y}}^*(y) = \frac{z_{\text{fund}}(\vec{y}) z_{\text{antifund}}(\vec{y})}{z_{\text{vec}}(\vec{y})}, \tag{3.40}$$

hence one reconstructs the instanton expansion for the $2 + SU(2) + 2$ theory:

$$B_{2+2} = \sum_{\vec{y}} \Lambda[\vec{y}] \frac{z_{\text{fund}}(\vec{y}) z_{\text{antifund}}(\vec{y})}{z_{\text{vec}}(\vec{y})} = Z_{\text{inst}}[2 + SU(2) + 2]. \tag{3.41}$$

This result has been generalized to the $q$-deformed case for $q = t$ in [96] and then for generic $q, t$ in [114]. In this latter case one needs to introduce a new set of polynomials, combinations of Macdonalds polynomials to prove that

$$B_{2+2}^q = Z_{\text{inst}}^q[2 + SU(2) + 2]. \tag{3.42}$$

The generalization to multipoint blocks has been studied in [115]. We refer the reader to the original paper, here we record only the key steps leading to the final result. First one introduces an object called generalised bifundamental kernel $\tilde{N}_{\vec{y}_1, \vec{y}_2}$. For $q = t$ this kernel admits a factorized form in terms of skew Schur polynomials:

$$\tilde{N}_{\vec{y}_1, \vec{y}_2} = N_{\vec{y}_1, \vec{y}_2} N_{\vec{y}_2, \vec{y}_1}, \quad N_{A,B}[x] = \sum_C \chi_A^C[x] \chi_B^{\dagger}[y]. \tag{3.43}$$

The first result is that one can express the $M + 2$-point block in terms of $q$-Selberg averages of the generalised bifundamental kernels:

$$B_{M+2}^q = \sum_{\vec{y}_i} \prod_{a=1}^M \Lambda[\vec{y}_a] (\tilde{N}_{\vec{y}_a}), \tag{3.44}$$

with $\vec{y}_0 = \vec{y}_M = \emptyset$. One can then prove that:

$$\langle \tilde{N}_{\vec{y}_1, \vec{y}_2} \rangle \sim \frac{2^{2M} z_{\text{vec}}(\vec{y}_1) z_{\text{vec}}(\vec{y}_2)}{z_{\text{vec}}(\vec{y}_1)^{1/2} z_{\text{vec}}(\vec{y}_2)^{1/2}}, \tag{3.45}$$

which, by considering the form of the quiver instanton partition function in (3.35), leads to

$$B_{M+2}^q = Z_{\text{inst}}^q[2 + SU(2)^{M-1} + 2]. \tag{3.46}$$

This provides a proof of the ‘standard’ 5d lift of the AGT correspondence. However [115] managed to prove that there is a finer structure. They showed that the $q$-DF integral can in fact be directly decomposed in terms of resolved conifold topological string amplitudes $\mathcal{Z}_{\text{top}}[R_1, R_2]$, where $R_{1,2}$ and $Y_{1,2}$ are the representations, respectively carried by the external vertical and horizontal legs. In particular, the average of the bifundamental kernel decomposes as:
The sum over the representation $R$ is the result of the $q$ average. As we have already mentioned the integrands are typically meromorphic functions and integrals are evaluated by summing sets of poles labelled by Young tableaux (with finite number of rows). The Kähler parameter $Q_f$, associated to the internal vertical leg carrying the representation $R$, is related to Coulomb branch parameter. In conclusion, the $M+2$-point block can be directly mapped to the topological string partition function for the CY geometry depicted (for $M=3$) in figure 4:

$$B_{M+2} = \sum_{\vec{Y}_0} \prod_{a=1}^{M} A^{[\vec{R}_a]} R_a^{[\vec{Q}_a]} [Z_{\text{top}}] y_{a-1,1}, y_{a,1}^{R_T} [Z_{\text{top}}] y_{a-1,2}, y_{a,2}^{R_T},$$

(3.47)

with $\vec{Y}_0 = \vec{Y}_M = (0).$ This is the most fundamental decomposition of the $q$-deformed DF blocks.

Recently, it has been observed that DF and gauge theory matrix integrals can be considered as special cases of a more general class of matrix models, the so-called network matrix models which have a direct topological string interpretation. The symmetry of these matrix models is the Ding–Iohara–Miki algebra which has been conjectured to be the underlying symmetry of the AGT correspondence, see for example [116] and references therein.

We close this section by mentioning a further approach to the evaluation of (deformed) DF-integrals. In [92] it was shown that $q$-deformed blocks involving $3 - \text{generic}$ primaries operators plus any number of operators with specialized momentum (corresponding to level-2 degenerates) are given by multivariate basic hypergeometric series. For example, it is easy to see that if in the $q$-DF integral (3.24) we take arbitrary $\alpha_0, \alpha_1, \alpha_{M+1}$ and specialise $\alpha_i = -1$ for $i = 2, \ldots, M$ such that the corresponding vertices become

$$V_i(w, x_i) = \prod_{j=1}^{N} \frac{(q^{\alpha_j} x_i / w_j ; q)_{\infty}}{(x_i / w_j ; q)_{\infty}} \rightarrow \prod_{j=1}^{N} (1 - q^{-1} x_i / w_j),$$

(3.48)

the integral can be mapped to the Jackson integral studied in [117]. This integral can be exactly evaluated and yields (up to prefactors) a basic hypergeometric functions of $M$ variables:

$$B_{M+2}(x_1, \ldots, x_M) \sim 2\Phi_1(A, B; C; x_1, \ldots, x_M),$$

where the coefficients $A, B, C$ are functions of the 3 generic momenta. For example, in the case of a 4-point block with 3 generic and one degenerate insertion one finds:

$$B_{2+2}(z) = 2\Phi_1(A, B; C; z),$$

(3.49)

which is annihilated by the difference operator

$$D(A, B; C; q; z) B_{\alpha_0, \alpha_1, \alpha_2} (z) = 0,$$

(3.50)

The measure in [117] is different from the Macdonald measure in (3.24) but they actually give the same results up to prefactors independent on $x$, see for example the discussion in appendix C of [118].
\[ D(A, B; C; q; z) = h_2 \frac{\partial^2}{\partial q z^2} + h_1 \frac{\partial}{\partial q z} + h_0, \]  

(3.51)

where \( \frac{\partial}{\partial z} f(z) = \frac{f(qz) - f(z)}{q - 1} \) and the coefficients \( h_2, h_1, h_0 \) are defined by

\[
h_2 = z(C - ABqz),
\]

\[
h_1 = \frac{1 - C}{1 - q} + \frac{(1 - A)(1 - B) - (1 - ABq)z}{(1 - q)},
\]

\[
h_0 = -\frac{(1 - A)(1 - B)}{(1 - q)^2}.
\]

(3.52)

By removing the \( q \)-deformation this difference operator reduces to the familiar hypergeometric differential operator acting on level-two degenerate 4-point correlators.

As we discussed in section 2.3, we expect a map between \( \text{Vir}_{q,t} \) blocks with some of the momenta analytically continued to degenerate values and instanton-vortex partition functions associated to linear quivers with defects obtained via Higgsing. In particular the block \( B_{2+2} \) in (3.49), in analogy with the undeformed case, is expected to be mapped to the vortex partition function of the 3d \( \mathcal{N} = 2 \) QED with 2 flavours, describing the codimension-two defect theory obtained by Higgsing the 5d \( 2 + SU(2) + 2 \) theory (after normalizing by the contributions of 3-point functions and 5d free hypers). Indeed we see that \( B_{2+2} \) is given by a \( q \)-deformed hypergeometric series as the \( \mathcal{N} = 2 \) QED vortex partition function, so the two quantities can be mapped with a suitable dictionary.

4. 3d & 5d partition functions as \( q \)-deformed correlators

Having reviewed the identification of instanton/vortex partition functions with non-degenerate/degenerate chiral \( \text{Vir}_{q,t} \) blocks we now move to the construction of \( \text{Vir}_{q,t} \) correlators and discuss their map to 3d and 5d partition functions.

The first question one needs to address is how to combine \( \text{Vir}_{q,t} \) chiral blocks into a correlator. In the undeformed 2d CFT case, the underlying symmetry is the product of the holomorphic and anti-holomorphic copies of the Virasoro algebra and correlators are constructed by taking the modulus square of the holomorphic and anti-holomorphic conformal blocks. This ensures that the monodromies acquired by the chiral blocks under change of channel (or ordering of the OPE) cancel out in the physical correlators which are modular invariant and single-value objects. \( \text{Vir}_{q,t} \) chiral blocks don’t have monodromies since, they have lines of poles rather than brach-cuts. This is clear if we consider for example the degenerate chiral blocks discussed in the previous section, given by \( q \)-deformed hypergeometric series.

The authors of [48, 119] proposed to define deformed correlators by combining \( \text{Vir}_{q,t} \) chiral blocks with the \( SL(3, \mathbb{Z}) \) gluing rules described in section 2.2. In particular, the authors considered the \( S^4 \times S^1 \) and \( S^5 \) gluings and defined two families of correlators with, respectively, \( \text{Vir}_{q,t} \otimes \text{Vir}_{q,t} \) and \( \text{Vir}_{q,t} \otimes \text{Vir}_{q,t} \otimes \text{Vir}_{q,t} \) symmetry. In the first case the blocks are glued with the id-gluing (2.51), while in the second case with the S-gluing (2.27). Correspondingly these two families were called id- and S-correlators and the following 5d-lift of the AGT correspondence, relating \( M + 2 \)-point id- and S-correlators to \( S^4 \times S^1 \) and \( S^5 \) partition functions of the \( 2 + SU(2)^{M-1} \) + 2 linear quiver, was proposed:
\[ Z^5 [2 + SU(2)^{M-1} + 2] = \int d \sigma \ Z_{1-\text{loop}}^5(\sigma, M) \left\| Z_{\text{inst}} 5d \right\|_S^3 \]
\[ = \int d a \ C^5 \cdots C^5 \left\| \mathcal{C}_q \mathcal{B}_{M+2}^d \right\|_S^3 = (V_{\alpha\nu}(\infty)\nu_{\alpha\nu}(z_M) \cdots V_{\alpha}(z_1)\nu_{\alpha}(0))_S. \] (4.1)

\[ Z^5 \times 5^5 [2 + SU(2)^{M-1} + 2] = \int d \sigma \ Z_{1-\text{loop}}^5(\sigma, M) \left\| Z_{\text{inst}} 5d \right\|_{id}^2 \]
\[ = \int d a \ C^{id} \cdots C^{id} \left\| \mathcal{C}_q \mathcal{B}_{M+2}^d \right\|_{id}^3 = (V_{\alpha\nu}(\infty)\nu_{\alpha\nu}(z_M) \cdots V_{\alpha}(z_1)\nu_{\alpha}(0))_{id}. \] (4.2)

In detail the map goes as follows. The instanton and classical contributions are identified with the \(Vir_{q,d}\) chiral blocks:
\[ Z_{\text{inst}} 5d = \mathcal{C}_q \mathcal{B}_{M+2}. \] (4.3)

In the previous section we have discussed the map \( Z_{\text{inst}} 5d = \mathcal{B}_{M+2} \) for special values of the momenta and mentioned that the match should extend to generic momenta via geometric transition.

The classical terms are instead conjectured to map to the factors \( C_q \), which were interpreted as the deformed version of the conformal factors, to which they reduce in the \( q \to 1 \) limit. In the undeformed case these factors follow from the conformal Ward identities. In the \( q \)-deformed case at present it is not know how to derive the analogue of these identities, however an interesting discussion on the \( q \)-deformation of the SU(1, 1) Ward identities can be found in [120].

The key point then is to map the 1-loop factors to the 3-point functions contribution which we schematically indicated as \( C^5 \cdots C^5 \) or \( C^{id} \cdots C^{id} \). We will discuss this point in a moment, before doing so it is useful to note that the correspondence (4.1), (4.2), generate via analytic continuation/Higgsing, a series of secondary correspondence between \( id,S \)-correlators with degenerate insertions and \( S^i \times S^j, S^i \) partition functions decorated by the \( S^i \times S^j \), \( S^i \) partition functions describing the codimension-two defects. For example one expects the map of the \( id \) and \( S \)-correlators of 3 generic primaries and one level-two degenerate primary \( V_{\alpha\nu}(z) = V_{\text{deg}}(z) \) (normalized by the 3-point function of the non-degenerate primaries) to the partition functions of the \( N = 2 \) SQED with 2 flavours on \( S^2 \times S^1 \), \( S^3 \) (normalized by the contribution of the free 5d hypers):

\[ Z_{S^2}^{\text{SQED}} = \sum_i^2 G_{(i)}^{(i)} Z_V^{id} \right\|_S^2 = (V_{\alpha\nu}(\infty)\nu_{\alpha\nu}(1)\nu_{\text{deg}}(z)\nu_{\alpha}(0))_S. \] (4.4)

\[ Z_{S^2 \times S^1}^{\text{SQED}} = \sum_i^2 G_{(i)}^{(i)} Z_V^{id} \right\|_{id}^2 = (V_{\alpha\nu}(\infty)\nu_{\alpha\nu}(1)\nu_{\text{deg}}(z)\nu_{\alpha}(0))_{id}. \] (4.5)

The correspondence (4.4) and (4.5) indicate that the degenerate 4-point correlators will be \( SL(2, \mathbb{Z}) \) \( id \)- or \( S \)-squares of degenerate blocks hence, they will be annihilated by two hypergeometric difference operators (see also the discussion on the evaluation of the degenerate integral (3.49)):

\[ \langle V_{\alpha\nu}(\infty)\nu_{\alpha\nu}(1)\nu_{\text{deg}}(z)\nu_{\alpha}(0) \rangle_z \sim \mathcal{G}(z, \bar{z}), \] (4.6)

\[ D(A, B; C; q; z) G(z, \bar{z}) = 0, \quad D(\tilde{A}, \tilde{B}; \tilde{C}; \tilde{q}; \tilde{z}) G(z, \bar{z}) = 0. \] (4.7)
The parameters $A, B, C$ are functions of the momenta $\alpha_1, \alpha_3, \alpha_4$, un-tilded and tilded variables denote the parameters in the two chiral blocks and the subscript $*$ indicate either the $id$ or $S$ gluing.

This observation was used in [119] to derive 3-point functions by means of the bootstrap approach [85, 121]. Equations (4.7) imply that $G(z, \tilde{z})$ can be expressed as a bi-linear combination of solutions of the $q$-hypergeometric difference equation. Let $I^{(s)}_1, I^{(s)}_2$ be a basis of two linearly independent solutions in the neighbourhood of $z = 0$, then we can write:

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{\deg}(z) V_{\alpha_1}(0) \rangle_* \sim 2 \sum_{i=1}^{2} K^{(s)}_i \| I^{(s)}_i(z; q) \|^2_*, \quad (4.8)$$

where the coefficients $K^{(s)}_i$ are related to the 3-point function factor associated to the diagram on the left of figure 5.

Similarly in the $u$-channel the correlation function is a bilinear combination of solutions $I^{(u)}_1, I^{(u)}_2$ in the neighbourhood of $z = \infty$:

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(r) V_{\deg}(z) V_{\alpha_1}(0) \rangle_* \sim 2 \sum_{i=1}^{2} K^{(u)}_i \| I^{(u)}_i(z; q) \|^2_*, \quad (4.9)$$

with coefficient $K^{(u)}_i$ related to the 3-point functions factor associated to the diagram on the right of figure 5.

To bootstrap the 3-point functions we now impose crossing symmetry requiring the equality of $s$- and $u$-channel correlators:

$$K^{(s)}_1 \| I^{(s)}_1 \|^2_* + K^{(s)}_2 \| I^{(s)}_2 \|^2_* = K^{(u)}_1 \| I^{(u)}_1 \|^2_* + K^{(u)}_2 \| I^{(u)}_2 \|^2_*. \quad (4.10)$$

We then take the analytic continuation to the neighbourhood of $\infty$ of the solutions $I^{(s)}_i$ and express them as linear combination of $u$-channel solutions. At this point equation (4.10) yields a set of non-trivial equations for the coefficients $K^{(s)}_i$ and $K^{(u)}_i$ which determine the 3-point functions uniquely once the gluing rule is specified.

In the case of $id$-gluing there are two types of level-two degenerate primaries: $\alpha_2 = -\frac{k\pm 1}{2}$ (corresponding to the two $S^2 \times S^1$ defects inside $S^4 \times S^1$) and we can write two sets of equations for the 3-point function. The unique solution (up to $q$-constants) to these equations is given by:

$$C^{id}(\alpha_3, \alpha_2, \alpha_1) = \frac{1}{\prod_{i=1}^{3} \frac{\Gamma^{2}(2\alpha_i)}{\Gamma^{2}(2\alpha_i - 2\alpha_j)}}. \quad (4.11)$$
where $2\alpha_T = \alpha_1 + \alpha_2 + \alpha_3$ while the parameter $R$ is related to the deformation parameter of the algebra and to the $S^1$ radius of the $S^3 \times S^1$ geometry on the gauge theory side. The definition and useful properties of the $T^6(X)$ function are collected in the appendix.

Similarly, for $S$-correlators there are three types of degenerations $\alpha_2 = -\frac{1}{2}$ for $i = 1, 2, 3$ (corresponding to the three large $S^3$ inside $S^6$) and we can write three sets of equations with unique solution given by:

$$C^S(\alpha_3, \alpha_2, \alpha_1) = \frac{1}{S_3(\alpha_2 - E)} \prod_{i=1}^3 \frac{S_3(2\alpha_i)}{S_3(2\alpha_i - 2\alpha_{i-1})}.$$  \hspace{1cm} (4.12)

We can now complete the map between $q$-deformed correlators and partition functions. For example it is easy to check that the 3-point functions factor in the 4-point $S$-correlator can be mapped to the $S^3$-deformed correlators and partition functions. For $S$-correlators there are three types of degenerations $\alpha_2 = -\frac{1}{2}$ for $i = 1, 2, 3$ (corresponding to the three large $S^3$ inside $S^6$) and we can write three sets of equations with unique solution given by:

$$C^S(\alpha_3, \alpha_2, \alpha_1) = \frac{1}{S_3(\alpha_2 - E)} \prod_{i=1}^3 \frac{S_3(2\alpha_i)}{S_3(2\alpha_i - 2\alpha_{i-1})}.$$  \hspace{1cm} (4.12)

Similarly for $id$-correlators one can show that:

$$C_{id}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = Z_{1-\text{loop}}^{S \times S^1}(\sigma) \prod_{i=1}^3 Z_{1-\text{loop}}^{S \times S^1}(\sigma, m_i).$$  \hspace{1cm} (4.14)

with the following dictionary:

$$\alpha = i\sigma + \frac{Q_0}{2}, \quad \alpha_1 \pm \alpha_2 = im_{1,2} + Q_0, \quad \alpha_3 \pm \alpha_4 = im_{3,4} + Q_0.$$  \hspace{1cm} (4.15)

3-point functions factor in higher point $S$, $id$-correlators can be similarly mapped to 1-loop contributions in $S^3$ and $S^3 \times S^1$ linear quiver partition functions. In [48] it was also shown that the 3-point function contribution to the 1-punctured torus $S$, $id$-correlators can be mapped to the 1-loop factor of the $SU(2)$ theory with a massive adjoint hyper on $S^3$ and $S^3 \times S^1$.

It is possible to take a smooth limit which removes the $q$-deformation and reduces $id$-correlators to Liouville correlators. On the gauge theory side this limit corresponds to shrinking the $S^3$ radius and reducing to the $S^3$ partition function. Indeed for $R \to 0$, the 3-point function (4.11) smoothly reduces to the familiar DOZZ formula for the Liouville 3-point function [121–123], which via AGT is mapped to 1-loop factors on $S^3$. S-correlators instead don’t admit a smooth undeformed limit. In [48] reflection coefficients, constructed from $id$ and $S$ 3-point functions, were given a geometric interpretation as Harish-Chandra c-functions for certain quantum symmetric spaces. These c-functions were in turn related to the Jost functions describing scattering processes in two different limits of the XYZ spin chain.

The 3-point function (4.11) was earlier derived in [64], building on the results of [124], by using the topological string partition function on a particular toric CY threefold, the blow up of the $\mathbb{C}^2/[\mathbb{Z}_2 \times \mathbb{Z}_2]$ orbifold. The authors of [124] proposed that a five-dimensional version of the $T_N$ theory could be obtained in a IIB setup in terms of a junction of N D5-branes, N NS5-branes and N (1,1) 5-branes which realizes the blow up of the $\mathbb{C}^2/[\mathbb{Z}_N \times \mathbb{Z}_N]$ orbifold. The 4d $T_N$ theory is the non-Lagrangian theory of N M5 branes on the sphere with three full punctures and it is mapped via AGT to the Toda 3-point function. For this reason one expects
that the topological string partition function on the $T_N$ geometry maps to the $q$-deformed Toda 3-point function with 3 full primaries. The determination of the Toda 3-point function with 3 full primaries is a long standing open problem and the possibility of extracting the answer from the $T_N$ geometry has been explored in [125–127].

The work done on the correspondence between deformed $\text{Vir}_{q,t}$ correlators and partition functions has focused on establishing a direct map between the terms contributing to the partition functions and the terms contributing to the $\text{Vir}_{q,t}$ correlators. At a deeper level one would like to be able to identify how symmetries are mapped across the correspondence. In the AGT case the generalised S-duality of 4d $\mathcal{N} = 2$ gauge theories was beautifully identified with the Moore–Seiberg duality groupoid acting on 2d CFT correlators. This observation for example made it possible to borrow from the CFT the sophisticated machinery of Verlinde-loop operators and apply it to the exact computation of vevs of line operators on the gauge theory side [67, 128]. More recently, in [68] the map of symmetries has been understood also in the case where surface operators engineered by M2 branes are included. Remarkably in this case all the gauge theory dualities of the combined 4d-2d system describing the surface operator, can be identified with symmetries of the Toda correlators involving extra degenerate primaries.

It would be very interesting to establish an analogous complete map between 3d/5d gauge theory dualities and symmetries of $q$-correlators. An encouraging step in this direction was taken in [48] where it was pointed out that the crossing symmetry of the degenerate 4-point $\text{Vir}_{q,t}$ correlator, which was used to derive the 3-point functions, can be identified with the flop symmetry of the 3d SQED, which acts by swapping the sign of the Fayet–Iliopoulos and exchanging charge plus with charge minus chiral multiplets. Understanding the map of symmetries should open up the possibility to retrace in $q$-deformed case, the various applications of the gauge/CFT correspondence to the study of defects operators in gauge theories.

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Appendix

In this appendix we collect the definition and some properties of the special functions used in the main text.

A.1. Bernoulli polynomials

The Bernoulli polynomials $B_n(z|\vec{\omega})$ are defined by [129]

$$B_{11}(z|\vec{\omega}) = \frac{z}{\omega_1} - \frac{1}{2}$$

$$B_{22}(z|\vec{\omega}) = \frac{z^2}{\omega_1 \omega_2} - \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} z + \frac{\omega_1^2 + \omega_2^2 + 3 \omega_1 \omega_2}{\omega_1 \omega_2}$$

$$B_{33}(z|\vec{\omega}) = \frac{z^3}{\omega_1 \omega_2 \omega_3} - \frac{3}{2} \frac{(\omega_1 + \omega_2 + \omega_3)}{\omega_1 \omega_2 \omega_3} z^2 + \frac{\omega_1^2 + \omega_2^2 + \omega_3^2 + 3(\omega_1 \omega_2 + \omega_2 \omega_3 + \omega_3 \omega_1)}{2 \omega_1 \omega_2 \omega_3} z$$

$$- \frac{(\omega_1 + \omega_2 + \omega_3)(\omega_1 \omega_2 + \omega_2 \omega_3 + \omega_3 \omega_1)}{4 \omega_1 \omega_2 \omega_3}, \quad (A.1)$$
where Ω := (ω₁, ..., ωᵣ). We will use the shorthand \( B_r(z) := B_r(z|\bar{\omega}) \).

### A.2. Multiple Gamma and Sine functions

The Barnes \( r \)-Gamma function \( \Gamma_r(z|\bar{\omega}) \) can be defined as the \( \zeta \)-regularized infinite product [129]

\[
\Gamma_r(z|\bar{\omega}) \sim \prod_{\bar{\beta} \in \mathbb{Z}_0^r} \frac{1}{(z + \bar{\omega} \cdot \bar{n})}.
\]  

(A.2)

When there is no possibility of confusion, we will simply set \( \Gamma_r(z) := \Gamma_r(z|\bar{\omega}) \).

The \( r \)-Sine function is defined according to [129]

\[
S_r(z|\bar{\omega}) = \frac{\Gamma_r(E_r - z)^{(1)')} \Gamma_r(z)}{\Gamma_r(z|\bar{\omega})}
\]  

(A.3)

where we defined \( E_r := \sum_j \omega_j \). We will also denote \( S_r(z) := S_r(z|\bar{\omega}) \) when there is no confusion. Also, introducing the multiple \( q \)-shifted factorial

\[
(z;q_1, \ldots; q_r) := \prod_{k_1, \ldots, k_r \geq 0} \left(1 - z q_1^{k_1} \cdots q_r^{k_r}\right)
\]  

(A.4)

the \( r \)-sine function has the following product representation \( (r \geq 2) \) [129]

\[
S_r(z) = e^{(-1)^r z} \prod_{\beta \in \mathbb{Z}_0^r} \left( e^{\frac{iz}{2}}; e^{2\pi i \frac{z}{\omega_1}}; e^{2\pi i \frac{z}{\omega_2}}; \ldots; e^{2\pi i \frac{z}{\omega_r}} \right)
\]  

(A.5)

whenever \( \text{Im}(\omega_j/\omega_k) \neq 0 \) (for \( j \neq k \)). General useful identities are

\[
S_r(z) S_r(E_r - z)^{(1)'} = 1
\]  

(A.6)

\[
S_r(\lambda z|\bar{\omega}) = \lambda S_r(z|\bar{\omega}); \quad \lambda \in \mathbb{C}/\{0\}
\]  

(A.7)

\[
\frac{S_r(z + \omega_j)}{S_r(z)} = \frac{1}{\zeta_r(z_\omega_1, \ldots, \omega_{r-1}, \omega_{r+1}, \ldots, \omega_r)}.
\]  

(A.8)

For \( r = 2 \) the multi-sine function is related to the double sine function

\[
s_2(z) = S_2(Q/2 - iz|b, b^{-1}) \sim \prod_k \frac{n_1 \omega_1 + n_2 \omega_2 + Q/2 - iz}{n_1 \omega_1 + n_2 \omega_2 + Q/2 + iz}
\]  

(A.9)

where we take \( Q = \omega_1 + \omega_2 \) and \( b = \omega_1 = \omega_2^{-1} \).

### A.3. \( \Upsilon^R \) function

The \( \Upsilon^R \) function is defined as the regularized infinite product

\[
\Upsilon^R(z) \sim \prod_{n_1, n_2 \geq 0} \sinh \left[ \frac{R}{2} (z + n_1 b_0 + n_2 b_0^{-1}) \right] \sinh \left[ \frac{R}{2} (Q_0 - z + n_1 b_0 + n_2 b_0^{-1}) \right].
\]

Important defining properties are

\[
\Upsilon^R(z) = \Upsilon^R(Q_0 - z)
\]  

(A.10)
In the $R \to 0$ limit it reduces to the $\Upsilon(z)$ function appearing in Liouville field theory

$$\Upsilon(z) = \Gamma_2(z|b_0, b_0^{-1})^{-1}\Gamma_2(Q_0 - z|b_0, b_0^{-1})^{-1}$$  \hspace{1cm} (A.12)

where $Q_0 := b_0 + b_0^{-1}$.

### A.4. Jacobi Theta and elliptic Gamma functions

The Jacobi $\Theta$ function is defined by [130]

$$\Theta(z; \tau) = \left(e^{2\pi i z}; e^{2\pi i \tau}\right) \left(e^{2\pi i \tau} e^{-2\pi i z}; e^{2\pi i \tau}\right)$$  \hspace{1cm} (A.13)

and satisfies the functional relation

$$\frac{\Theta(\tau + z; \tau)}{\Theta(z; \tau)} = -e^{-2\pi i z}.$$  \hspace{1cm} (A.14)

Another relevant property is [129]

$$\Theta \left( \frac{\omega_1}{\omega_1}, \frac{\omega_2}{\omega_2} \right) \Theta \left( \frac{\omega_1}{\omega_2}, \frac{\omega_2}{\omega_1} \right) = e^{-i\pi B_{22}(z)}.$$  \hspace{1cm} (A.15)

The elliptic Gamma function $\Gamma_{q,t}$ is defined by [130]

$$\Gamma_{q,t}(z) = \frac{(qt e^{-2\pi i z}; q,t)}{(e^{2\pi i z}; q,t)}; \quad q = e^{2\pi i \sigma}; \quad t = e^{2\pi i}$$  \hspace{1cm} (A.16)

and satisfies the functional relations

$$\frac{\Gamma_{q,t}(\tau + z)}{\Gamma_{q,t}(z)} = \Theta(z; \sigma); \quad \frac{\Gamma_{q,t}(\sigma + z)}{\Gamma_{q,t}(z)} = \Theta(z; \tau).$$  \hspace{1cm} (A.17)

Other relevant properties are [49]

$$\Gamma_{q,t} \left( \frac{z}{e_3} \right) \Gamma_{q,t} \left( \frac{z}{e_3} \right) \Gamma_{q,t} \left( \frac{z}{e_3} \right) = e^{-i\pi B_{33}(z)}$$  \hspace{1cm} (A.18)

where $q, t$ are expressed via the $e_1, e_2, e_3$ parameters as described in (2.27), (2.28) and

$$\Gamma_{q,t} \left( \frac{z}{e_3} \right) \Gamma_{q,t} \left( \frac{e_1 + e_2 - z}{e_3} \right) = 1.$$  \hspace{1cm} (A.19)

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