ASSOCIATIVE ALGEBRA TWISTED BUNDLES OVER COMPACT TOPOLOGICAL SPACES

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Abstract. For the associative algebra $A(g)$ of an infinite-dimensional Lie algebra $g$, we introduce twisted fiber bundles over arbitrary compact topological spaces. Fibers of such bundles are given by elements of algebraic completion of the space of formal series in complex parameters, sections are provided by rational functions with prescribed analytic properties. Homotopical invariance as well as covariance in terms of trivial bundles of twisted $A(g)$-bundles is proven. Further applications of the paper’s results useful for studies of the cohomology of infinite-dimensional Lie algebras on smooth manifolds, $K$-theory, as well as for purposes of conformal field theory, deformation theory, and the theory of foliations are mentioned.

1. Introduction

It is natural to consider bundles of modules related to associative algebras. Playing important roles in clarification of the cohomology theory on smooth manifolds, they are also important for elliptic and Witten genera [21, 20], the highest weight representations of Heisenberg and affine Kac-Moody algebras, and provide important examples for the construction of related associative algebras. In this paper we introduce twisted fiber bundles of modules of associative algebras of infinite-dimensional Lie algebras twisted in the form of group of automorphisms torsors originating from local geometry. Our original motivation for this work is to understand continuous cohomology [3, 4, 7, 8, 9, 16, 19] of non-commutative structures over compact topological spaces. In particular [4], one hopes to relate cohomology of infinite-dimensional Lie algebras-valued series considered on complex manifolds to fiber bundles on auxiliary topological spaces [10]. We also plan to study applications of results of this paper for $K$-theories. Let $g$ be an infinite-dimensional Lie algebra [13]. Starting from algebraic completion $G_\mathbb{R}$ of the space of $g$-valued series in a few formal complex parameters, we introduce the category $\mathcal{C}_{A(g)}$ of associative algebra modules for the associative algebra $A(g)$ originating from $G_\mathbb{R}$ by means of factorization with respect to two natural multiplications [22]. Local parts of twisted bundles are constructed as principal bundles of products of Aut$(g)$ modules and spaces of all sets of local parameters of a $X$-covering. As in the untwisted case [4], this result is crucial in defining $A(g)$ $K$-groups and studying the cohomology their properties.

Key words and phrases. Associative algebras, fiber bundles, rational functions with prescribed properties.
2. Prescribed rational functions

In this section the space of prescribed rational functions is defined as rational functions with certain analytical and symmetric properties [12]. Such rational functions depend implicitly on an infinite number of non-commutative parameters.

2.1. Rational functions originating from matrix elements. Let us introduce the general notations used in this paper. We denote by boldface vectors of elements, e.g., $a_n = (a_1, \ldots, a_n)$, and the same for all types of objects used in the text. If $n$ is omitted then $a$ denotes any choice of $n \geq 0$. We also express as $(a_j)_n$ the $j$-th component of $a_n$. Let $I$ be set of positive integers, and $X_\alpha = \{X_\alpha, \alpha \in I\}$ be an open covering of a compact topological space $X$ which gives a local trivialization of the $A(g)$ fiber bundle. Let $g_*$ be an infinite-dimensional Lie algebra. Denote by $G$ a $g$-module. Denote by $G_{z_n}$ be the graded (with respect to a grading operator $K_G$) algebraic completion of the space of formal series individually in each of complex formal parameters $z_n$, and satisfying certain properties described below. We denote $x_n = (g_n, z_n)$ for $g_n$ of the $n$-th power $G_n = G^{\otimes n}$ of $g$-module $G$, and $G_{z_n}^*$ be the dual to $G_{z_n}$ with respect to non-degenerate bilinear pairing $(..)$. For fixed $\theta \in G_{z_n}^*$, and varying $x_n \in G_{z_n}$ we consider matrix elements $F(x_n)$ of the form

$$F(x_n) = (\theta, f(x_n)) \in \mathbb{C}((z)),$$

(2.1)

where $F(x_n)$ depends implicitly on $g_n \in G_n$. In this paper we consider meromorphic functions of several complex formal parameters defined on a compact topological space which are extendable to rational functions on larger domains on $X$. We denote such extensions by $R(f(x_n))$.

Definition 1. Denote by $F_n \mathbb{C}$ the configuration space of $n \geq 1$ ordered coordinates in $\mathbb{C}^n$, $F_n \mathbb{C} = \{z \in \mathbb{C}^n | z_i \neq z_j, i \neq j\}$.

In order to work with objects on $X$ for a set of $G_n$-elements $g_n$ we consider converging rational functions $f(x_n) \in G_{z_n}$ of $z_n \in F_n \mathbb{C}$.

Definition 2. For an arbitrary fixed $\theta \in G_{z_n}^*$, we call a map linear in $g_n$ and $z_n$,

$$F : x_n \mapsto R((\theta, f(x_n))),$$

(2.2)

a rational function in $z_n$ with the only possible poles at $z_i = z_j$, $i \neq j$. Abusing notations, we denote

$$F(x_n) = R((\theta, f(x_n))).$$

Definition 3. We define left action of the permutation group $S_n$ on $F(z_n)$ by

$$\sigma(F)(x_n) = F\left(g_n, z_{\sigma(i)}\right).$$

2.2. Conditions on $G_{z}$. For $G_{z}$ we assume [12] that is $G_{z} = \bigsqcup_{\alpha \in \mathbb{C}} G_{z, \lambda}$, where $G_{z, \lambda} = \{w \in G_{z}|K_0 w = \lambda w, \lambda = wt(w)\}$, such that $G_{z, \lambda} = 0$ when the real part of $\alpha$ is sufficiently negative. Moreover we require that $\dim G_{z, \lambda} < \infty$, i.e., it is finite, and for fixed $\lambda$, $G_{z, \lambda} = 0$, for all small enough integers $n$. In addition, assume that $G_{z}$ equipped with a map $\omega : G_{z} \rightarrow G((z, z^{-1}))$, $g \mapsto \omega_g(z) = \sum_{i \in \mathbb{C}} g_i z^i$. In addition to that, for $g \in g$ and $g \in G$, $\omega_g(z) w$ contains only finitely many negative power terms, that is, $\omega_g(z) w \in G((z))$. We denote by $G_{z}^* = \bigsqcup_{\lambda \in \mathbb{C}} G_{z, \lambda}^*$ the dual to $G_{z}$. Through


matrix elements \((2.1)\), locality and associativity conditions for \(g_1, g_2 \in \mathfrak{g}, w \in G, \theta \in G'\); for \(G_{z_n}^\perp\) are assumed, i.e., the series \((\theta, \omega_{g_1}(z_1) \omega_{g_2}(z_2) w), (\theta, \omega_{g_2}(z_2) \omega_{g_1}(z_1) w)\), \((\theta, \omega_{g_1}(z_1-z_2) \omega_{g_2}(z_2) w)\), are absolutely convergent in the regions \(|z_1| > |z_2| > 0, |z_2| > |z_1| > 0, |z_2| > |z_1 - z_2| > 0\), respectively, to a common rational function in \(z_1\) and \(z_2\) with the only possible poles at \(z_1 = 0 = z_2\) and \(z_1 = z_2\).

**Definition 4.** Let \(S \subset \text{Aut}(\mathfrak{g})\) be a subgroup of \(\text{Aut}(\mathfrak{g})\). We say that \(S\) acts on \(G_z\) as automorphisms if \(g \omega_h(z) g^{-1} = \omega_{gh}(z)\), on \(G_z\) for all \(g \in S, h \in \mathfrak{g}\).

**2.3. Conditions on rational functions.** Let \(z_n \in F_0\mathbb{C}\). Denote by \(T_G\) the translation operator \([12]\). We define now extra conditions on rational functions leading to the definition of restricted rational functions.

**Definition 5.** Denote by \((T_G)_i\), the operator acting on the \(i\)-th entry. We then define the action of partial derivatives on an element \(F(x_n)\)

\[
\frac{\partial}{\partial z_i} F(x_n) = F((T_G)_i, x_n),
\]

\[
\sum_{i \geq 1} \frac{\partial}{\partial z_i} F(x_n) = T_G F(x_n),
\]

(2.3)

and call it \(T_G\)-derivative property.

**Definition 6.** For \(z \in \mathbb{C},\) let

\[
e^{zT_G} F(x_n) = F(g_n, z_n + z).
\]

(2.4)

Let \(\text{Ins}_i(A)\) denote the operator of multiplication by \(A \in \mathbb{C}\) at the \(i\)-th position. Then we define

\[
F(g_n, \text{Ins}_i(z) z_n) = F(\text{Ins}_i(e^{zT_G}) x_n),
\]

(2.5)

are equal as power series expansions in \(z\), in particular, absolutely convergent on the open disk \(|z| < \min_{i \neq j} \{|z_i - z_j|\}\).

**Definition 7.** A rational function has \(K_G\)-property if for \(z \in \mathbb{C}^\times\) satisfies \((z z_n) \in F_n\mathbb{C}\),

\[
z^{K_G} F'(x_n) = F(z^{K_G} g_n, z z_n).
\]

(2.6)

**2.4. Rational functions with prescribed analytical behavior.** In this subsection we give the definition of rational functions with prescribed analytical behavior on a domain of \(X\). We denote by \(P_k : G \rightarrow G(k)\), \(k \in \mathbb{C}\), the projection of \(G\) on \(G(k)\). For each element \(g_i \in G\), and \(x_i = (g_i, z), z \in \mathbb{C}\) let us associate a formal series \(\omega_{g_i}(z) = \sum_{k \in \mathbb{C}} g_{ik} z^k, i \in \mathbb{Z}\). Following \([12]\), we formulate

**Definition 8.** We assume that there exist positive integers \(\beta(g_{i, j}, g_{r, j})\) depending only on \(g_{i, j}, g_{r, j} \in G\) for \(i, j = 1, \ldots, (l + k)n, k \geq 0, i \neq j, 1 \leq l', l'' \leq n\). Let \(L_n\) be a partition of \((l + k)n = \sum_{i \geq 1} l_i\), and \(k_i = l_1 + \cdots + l_{i-1}\). For \(\zeta_i \in \mathbb{C}\), define \(b_i = F(W_{g_{i+1}, l_1} (z_{ki} + 1 - \zeta_i)), i = 1, \ldots, n\). Then we call a rational function \(F\) satisfying properties \([2.3], [2.6]\), a rational function with prescribed analytical behavior, if under the following conditions on domains, \(|z_{ki+p} - \zeta_i| + |z_{kj+q} - \zeta_j| < |\zeta_i - \zeta_j|\), for \(i, j = 1, \ldots, n\), \(k = 0, \ldots, n\).
1, \ldots, k, i \neq j, and for \( p = 1, \ldots, l, i, q = 1, \ldots, l, j \), the function 
\[
\sum_{z_i, z_j \in \mathbb{C}} F(\mathbf{P}_r h_i; (\zeta)_l),
\]
is absolutely convergent to an analytical extension in \( z_{l+k} \), independently of complex 
parameters \( (\zeta)_l \), with the only possible poles on the diagonal of \( z_{l+k} \) of order less than 
or equal to \( \beta(\nu_i, i; \nu_{w,j}) \). In addition to that, for \( \mathbf{g}_{l+k} \in G \), the series 
\[
\sum_{q \in \mathbb{C}} F(\mathbf{W}(\mathbf{g}_{l+k}, z_{l+k}), (\zeta)_l),
\]
is absolutely convergent when \( z_i \neq z_j, i \neq j, |z_i| > |z_j| > 0 \), for 
\( i = 1, \ldots, k \) and \( s = k + 1, \ldots, l + k \) and the sum can be analytically extended 
to a rational function in \( z_{l+k} \) with the only possible poles at \( z_i = z_j \) of orders less than 
or equal to \( \beta(\nu_i, i; \nu_{w,j}) \).

For \( m \in \mathbb{N} \) and \( 1 \leq p \leq m-1 \), let \( J_{m,p} \) be the set of elements of \( S_m \) which preserve 
the order of the first \( p \) numbers and the order of the last \( m-p \) numbers, that is,
\[
J_{m,p} = \{ \sigma \in S_m \mid \sigma(1) < \cdots < \sigma(p), \; \sigma(p+1) < \cdots < \sigma(m) \}.
\]
Let \( J_{m,1}^{-1} = \{ \sigma \mid \sigma \in J_{m,p} \} \). In addition to that, for some rational functions require 
the property:
\[
\sum_{\sigma \in J_{m,1}^{-1}} (-1)^{|\sigma|} \sigma(F(\mathbf{g}_{\sigma(1)}), z_{n})) = 0. \tag{2.7}
\]
Then, we have

**Definition 9.** We define the space \( \Theta(n, k, G_{\mathbf{z}}, U) \) of matrix elements \( F(\mathbf{x}_n) \) of \( n \) for-
mal complex parameters as the space of restricted rational functions with prescribed 
analytical behavior on a \( F_n \mathbb{C} \)-domain \( U \subset X \), and satisfying \( T_G \) and \( K_G \)-properties 
(2.3)–(2.6), definition (8), and (2.7).

3. **Associative algebra** \( A(\mathfrak{g}) \) **of prescribed rational functions**

In this section we define a twisted bundle corresponding to the associative algebra 
\( A(\mathfrak{g}) \), and describe their properties.

3.1. **Associative algebra** \( A(\mathfrak{g}) \)** **out of** \( \mathfrak{g} \). In this subsection we recall \([22][5]\) a way 
how to derive an associative algebra \( A(\mathfrak{g}) \) out of an infinite-dimensional Lie algebra 
\( \mathfrak{g} \).

**Definition 10.** For any homogeneous vectors \( h, \tilde{h} \in G \), one defines the multiplications
\[
h \ast_e \tilde{h} = \text{Res}_z \left( (1 + z)^{\text{wt}(h)} \sum_{\kappa \in \mathbb{C}} h_{\nu} z^{-\kappa} \tilde{h} \right),
\]
for \( \kappa = 1, 2 \), and extend it bilinearly it to \( G \times G \).

Here, as usual, \( \text{Res}_z \) denotes the coefficient in front of \( z^{-1} \).

**Definition 11.** For \( h, \tilde{h} \in G \), define \( A(\mathfrak{g}) = G_{\mathbb{A}}/(\text{span}(h \ast_2 \tilde{h}))^{\theta} \).

For \( \theta = 0 \) we get back to \( G_{\mathbb{A}} \) with associativity property described in subsection 
(2) expressed via matrix elements, while for \( \theta = 1 \) we obtain an associative 
algebra associated to \( \mathfrak{g} \) with ordinary associativity. The following theorem is due 
to \([22][\S 2]\) (also see [5]).
Remark 1. For \( W = \bigoplus_{\lambda \in \mathbb{C}} W_\lambda \), \( L(W) = \bigoplus_{\lambda \in \mathbb{C}} L(W)_\lambda \) is naturally graded, and each homogeneous subspace \( L(W)_\lambda = L(W) \cap W_\lambda \) is finite dimensional.

It is easy to see the following

Lemma 1. For \( W, \tilde{W} \) be two \( A(\mathfrak{g}) \)-modules, and for \( \varphi : W \to \tilde{W} \) an \( A(\mathfrak{g}) \)-module homomorphism, one has \( \varphi(L(W)) \subset L(\tilde{W}) \). In particular, if \( \varphi \) is an isomorphism then \( \varphi(L(W)) = L(\tilde{W}) \).

3.2. Category \( \mathcal{O}_{A(\mathfrak{g})} \) of \( A(\mathfrak{g}) \)-modules. Let \( W_z \) be an \( A(\mathfrak{g}) \)-module and we denote the dual space of \( W \) with respect to the pairing \( (\cdot, \cdot) \) by \( W' \). The following lemma is obvious [6]:

Lemma 2. \( W' \) is an \( A(\mathfrak{g}) \)-module such that \( (a m', m) = (m', \nu(a) m) \), for \( a \in A(\mathfrak{g}) \), \( m' \in W' \), and \( m \in W \).

Definition 13. A pairing \( (\cdot, \cdot) \) defined on an \( A(\mathfrak{g}) \)-module \( W_z \) is called invariant if \( (a w_1, w_2) = (w_1, \nu(a) w_2) \) for \( w_i \in W_z \) and \( a \in A(\mathfrak{g}) \).

We also need to define the category \( \mathcal{O}_{A(\mathfrak{g})} \) of \( A(\mathfrak{g}) \)-modules.

Definition 14. An \( A(\mathfrak{g}) \)-module \( W \) is in \( \mathcal{O}_{A(\mathfrak{g})} \) if there exist \( \lambda, \mu \in \mathbb{C} \), such that \( W = \bigoplus_{\lambda, \mu \in \mathbb{C}} W_{\lambda, \mu} \), is a direct sum of finite dimensional \( A(\mathfrak{g}) \)-modules and \( \text{Hom}_{A(\mathfrak{g})}(W_{\lambda, \mu}) = 0 \), if \( \mu \neq \lambda \).

Theorem 2. Let \( W_0 \neq 0 \). Then the linear map \( o : W_z \to \text{End}(L(W_z)), g \to o(g)|L(W_z) \), induces a homomorphism from \( W_z \) to \( \text{End}(L(W_z)) \), and \( L(W_z) \) is a left \( A(\mathfrak{g}) \)-module. For all \( \lambda \in \mathbb{C} \), \( L(W_z)_\lambda \) is an finite-dimensional \( A(\mathfrak{g}) \)-module. \( \mathcal{O}_{A(\mathfrak{g})} \) is invariant with respect to definition of \( L \).

Note that for \( \lambda \neq \mu \), \( \text{Hom}_{A(\mathfrak{g})}(L(W_z)_\lambda, L(W_z)_\mu) = 0 \). Thus \( L(W_z) \) is an element of \( \mathcal{O}_{A(\mathfrak{g})} \).

4. Twisted \( A(\mathfrak{g}) \)-bundles

As it was shown in [6], it turns out that we can introduce corresponding bundles for a large class of associative algebras. In this section we define the main objects of this paper, associative algebra twisted \( A(\mathfrak{g}) \)-bundles.
4.1. Torsors and twists under groups of automorphisms. We now explain how to collect elements of the space $\Theta(n, k, W_{z, \lambda}, X_\alpha)$ of prescribed rational functions into sections of a twisted $A(\mathfrak{g})$-bundle on $X$. Let $\mathcal{H}$ be a subgroup of the group $\text{Aut}_x \mathcal{O}_X$ of independent formal parameters $z$ automorphisms on $X$. We recall here the notion of a torsor with respect to a group.

**Definition 15.** Let $\mathcal{H}$ be a group, and $\mathcal{X}$ a non-empty set. Then $\mathcal{X}$ is called a $\mathcal{H}$-torsor if it is equipped with a simply transitive right action of $\mathcal{H}$, i.e., given $\xi, \xi' \in \mathcal{X}$, there exists a unique $h \in \mathcal{H}$ such that $\xi \cdot h = \xi'$, where for $h, \tilde{h} \in \mathcal{H}$ the right action is given by $\xi \cdot (h \cdot \tilde{h}) = (\xi \cdot h) \cdot \tilde{h}$. The choice of any $\xi \in \mathcal{X}$ allows us to identify $\mathcal{X}$ with $\mathcal{H}$ by sending $\xi$ to $h$.

Using similar results for $W_z$ of [2], one shows that certain subspaces $W_z \subset G_z$ form $\mathcal{H}$-modules. Applying the definition of a group twist to the group $\mathcal{H}$ and its module $W_z$ we obtain

**Definition 16.** Given a $\mathcal{H}$-module $W_z$ and a $\mathcal{H}$-torsor $X_\alpha$, one defines the $X_\alpha$-twist of $W_z$ as the set

$$\mathcal{E}_{X_\alpha} = W_z \times_{\mathcal{H}} X_\alpha = W_z \times X_\alpha/\{(w, a \cdot \xi) \sim (aw, \xi)\},$$

for $\xi \in X_\alpha$, $a \in \mathcal{H}$, and $w \in W_z$.

Now we wish to attach to any $X_\alpha$ a twist $\mathcal{E}_{X_\alpha}$ of $W_z$. We have an isomorphism $i_{z, X_\alpha} : W_z \cong \mathcal{E}_{X_\alpha}$. The system of isomorphisms $i_{z, X_\alpha}$ should satisfy certain compatibility conditions. Namely, an automorphism $(i_{z, X_\alpha}^{-1} \circ i_{z, X_\alpha})$ of $W_z$ should define a representation on $W_z$ of the group $\mathcal{H}$. Then $\mathcal{E}_{X_\alpha}$ is canonically identified with the twist of $W_z$ by the $\mathcal{H}$-torsor $X_\alpha$. The elements of $\Theta(n, k, W_{z, \lambda}, X_\alpha)$ give rise to a collection of sections $F(x)$. The construction of local parts of a twisted $A(\mathfrak{g})$-bundle is grounded on the notion of a principal bundle for the group $\mathcal{H}$ naturally existing on $X$. Let $Aut_{X_\alpha}$ be the space of all sets of local parameters on $X_\alpha$. Next we have (cf. [2])

**Lemma 3.** The group $\mathcal{H}$ acts naturally on $Aut_{X_\alpha}$ which is a $\mathcal{H}$-torsor.

Thus, we can define the following twist.

**Definition 17.** We introduce the $\mathcal{H}$-twist of $W_z$ $\mathcal{E}_z = W_z \times_{\mathcal{H}} Aut_{X_\alpha}$. The original definition similar to [17] was given in [1 20].

4.2. Definition of the local part of prescribed rational functions bundle $\mathcal{E}(W_{z, \lambda})$. We now fix an infinite-dimensional Lie algebra $\mathfrak{g}$ satisfying requirements of subsection 2.2. Suppose that a $\mathcal{C}$-grading is generated by $K_0$ on $W_z$. Let $\mathfrak{g} \subset Aut(\mathfrak{g})$ be a $W_z$-grading preserving subgroup of $Aut(\mathfrak{g})$. Denote by $\mathcal{O}_{\mathfrak{g}, Aut(\mathfrak{g})}$ a subcategory of $\mathcal{O}_{A(\mathfrak{g})}$ consisting of $A(\mathfrak{g})$-modules $W_z$ such that $\mathfrak{g}$ acts on $W_z$ as automorphisms. By using the ideas of [2], we formulate here the definition of the local part $\mathcal{E}(W_{z, \lambda})$ of the fiber bundle associated to $\mathfrak{g}$ through matrix elements $F(x)$ with $x = (\mathfrak{g}, z)$, to the space $\Theta(n, k, W_{z, \lambda})$ for all $n, k \geq 0$, of prescribed rational functions on a finite part $\{X_{\alpha}, \alpha \in I_0\}$ of a covering $\{X_\alpha\}$ of $X$. For the fiber space provided by elements $f(x) \in W_z$, using the property of prescribed rational functions we form a principal
that purpose, replace \( W \) and transitively, i.e., the map \( a \rightarrow \zeta \cdot a \) is a homeomorphism. Thus, we have [2]

**Lemma 4.** The projection \( \text{Aut}_{X_a} \rightarrow X \) is a principal \( \mathcal{H} \)-bundle. The fiber of this bundle over \( X_a \) is the \( \mathcal{H} \)-torsor \( \text{Aut}_{X_a} \).

Then we obtain

**Definition 18.** Given a finite-dimensional \( \mathcal{H} \)-module \( W_{i_{n}, \lambda} \), let

\[
\mathcal{E}(W_{z, \lambda}) = \bigoplus_{n,k \geq 0} W_{i_{n}, \lambda} \times \mathcal{H} \text{ Aut}_{X_a},
\]

be the fiber bundle associated to \( W_{i_{n}, \lambda} \), \( \text{Aut}_{X_a} \), and with sections provided by elements of \( \Theta(n,k,W_{z, \lambda},X_{\alpha}) \), for \( n, k \geq 0 \).

On \( X \) we can choose \( \{X_{\alpha}\} \) such that the bundle \( \mathcal{E}(W_{z, \lambda}) \) over \( X_{\alpha} \) is \( X_{\alpha} \times F(x) \). The fiber bundle \( \mathcal{E}(W_{z, \lambda}) \) with fiber \( f(x_{\alpha}) \) is a map \( \mathcal{E}(W_{z, \lambda}): \mathbb{C}^{n} \rightarrow X \) where \( \mathbb{C}^{n} \) is the total space of \( \mathcal{E}(W_{z, \lambda}) \) and \( X \) is its base space. For every \( X_{\alpha} \) of \( X \), \( i_{z_{\alpha}}^{-1} \) is homeomorphic to \( X_{\alpha} \times \mathbb{C}^{n} \). Namely, we have for \( f(x_{\alpha}) \) : \( i_{z_{\alpha}}^{-1} \rightarrow X_{\alpha} \times \mathbb{C}^{n} \), that \( \mathcal{P} \circ f(x_{\alpha}) = i_{z_{\alpha}} \circ i_{z_{\alpha}}^{-1} (X_{\alpha}) \), where \( \mathcal{P} \) is the projection map on \( X_{\alpha} \).

4.3. **Definitions of a twisted \( A(\mathfrak{g}) \)-bundle.** In this subsection we formulate (generalizing examples of other cases considered in [3]) the definition of a twisted fiber bundle associated to \( A(\mathfrak{g}) \)-module \( W_{z} \in \mathcal{O}_{\mathfrak{g},A(\mathfrak{g})} \). We obtain

**Definition 19.** A twisted \( A(\mathfrak{g}) \)-bundle \( \mathcal{E} \) over \( X \) with fiber \( W_{z} \) and \( \Theta(n,k,W_{z},X), \) \( n, \) \( k \geq 0 \)-valued sections is a direct sum of vector bundles \( \mathcal{E} = \bigoplus_{\lambda \in \mathbb{C}} \mathcal{E}(W_{z, \lambda}) \), such that all transition functions are \( A(\mathfrak{g}) \)-module isomorphisms, and a family of continuous isomorphisms \( H_{\alpha} = \{H_{\alpha, \lambda}, \lambda \in \mathbb{C}\} \), of fiber bundles

\[
H_{\alpha, \lambda} : \mathcal{E}(W_{z, \lambda})|_{X_{\alpha}} \rightarrow W_{i_{n}, \lambda} \times _{\mathcal{H} \text{ Aut}_{X_a}},
\]

such that for transition functions \( g_{\alpha \beta, \lambda} = H_{\alpha, \lambda} \ast_{2} H_{\beta, \lambda}^{-1} \), for all \( \lambda \in \mathbb{C} \), then

\[
g_{\alpha \beta}(x) = (g_{\alpha \beta, \lambda}(\xi)) : W_{z} \rightarrow W_{z},
\]

is an \( A(\mathfrak{g}) \)-module isomorphism for any \( \xi \in (X_{\alpha} \cap X_{\beta}) \), where the transition functions \( g_{\alpha \beta}(x) \) are \( \mathfrak{g} \times \text{-} \text{valued.} \)

Note that definitions of direct sum of bundles, sub-bundles and quotient bundles appear accordingly. We are also able to define graded twisted \( A(\mathfrak{g}) \)-bundles. For that purpose, replace \( W_{z, \lambda} \rightarrow L(W_{z, \lambda}) \), \( \mathcal{E}(W_{z, \lambda}) \rightarrow \mathcal{E}^{gr}(W_{z, \lambda}) \), \( g_{\alpha \beta, \lambda} \) by \( \left(g_{\alpha \beta, \lambda}^{gr}\right)^{-1} \), and introduce \( H_{\alpha, \lambda}^{gr} = (H_{\alpha, \lambda}(\mathcal{E}^{gr}(W_{z, \lambda}))) \), for all \( \lambda \in \mathbb{C} \). Then using Theorem [2] we obtain

**Lemma 5.** The graded transition functions \( (g_{\alpha \beta, \lambda}^{gr})(\xi) \) provide an \( A(\mathfrak{g}) \)-module isomorphism \( \left(g_{\alpha \beta, \lambda}^{gr}\right)(x) : L(W_{z, \lambda}) \rightarrow L(W_{z, \lambda}) \), for any \( \lambda \in \mathbb{C} \) and \( x \in X_{\alpha} \cap X_{\beta} \). By Lemma [7] \( \mathcal{E}^{gr} \) is a twisted \( A(\mathfrak{g}) \)-bundle over \( X \).
Let $A(\mathfrak{g})$ and $\tilde{A}(\mathfrak{g})$ be two associative algebras with anti-involutions $\nu_A$ and $\nu_{\tilde{A}}$ respectively. Then, similar to [6], one has

**Lemma 6.** $A(\mathfrak{g}) \otimes \mathbb{C} \tilde{A}(\mathfrak{g})$ is an associative algebra with anti-involution $\nu_A \otimes \nu_{\tilde{A}}$.

### 5. Properties of twisted $A(\mathfrak{g})$-bundles

In this section we reveal properties of twisted $A(\mathfrak{g})$-bundles. In particular we show that twisted $A(\mathfrak{g})$-bundle behaves well under standard operations, and are invariant under homotopy transformations of $X$. Note that in the simplest case of $A(\mathfrak{g}) = \mathbb{C}$ the twisted $A(\mathfrak{g})$-bundle is a classical complex vector bundle over $X$. Let $\mathcal{E} \otimes \tilde{\mathcal{E}}$ be $A(\mathfrak{g})$ and $A(\tilde{\mathfrak{g}})$-bundles over $X$. Then we have

**Lemma 7.** Then $\mathcal{E} \otimes \tilde{\mathcal{E}}$ is an associative algebra with anti-involution $\nu_A \otimes \nu_{\tilde{A}}$.

**Proof.** For ordinary bundles this lemma was proven in [6]. Here we have to check (5.1) and multi-

### 5. Properties of twisted $A(\mathfrak{g})$-bundles

In this section we reveal properties of twisted $A(\mathfrak{g})$-bundles. In particular we show that twisted $A(\mathfrak{g})$-bundle behaves well under standard operations, and are invariant under homotopy transformations of $X$. Note that in the simplest case of $A(\mathfrak{g}) = \mathbb{C}$ the twisted $A(\mathfrak{g})$-bundle is a classical complex vector bundle over $X$. Let $\mathcal{E}, \tilde{\mathcal{E}}$ be $A(\mathfrak{g})$ and $A(\tilde{\mathfrak{g}})$-bundles over $X$. Then we have

**Lemma 7.** Then $\mathcal{E} \otimes \tilde{\mathcal{E}}$ is a $A(\mathfrak{g}) \otimes \mathbb{C} A(\tilde{\mathfrak{g}})$-bundle over $X$. In particular, if $A(\mathfrak{g}) = \mathbb{C}$ then $\mathcal{E} \otimes \tilde{\mathcal{E}}$ is again a $A(\mathfrak{g})$-bundle over $X$.

**Lemma 8.** The dual bundle $\mathcal{E}'$ is also a twisted $A(\mathfrak{g})$-bundle.

**Definition 20.** Let $\mathcal{E}, \tilde{\mathcal{E}}$ be two twisted $A(\mathfrak{g})$-bundles on $X$. A map $\eta : \mathcal{E} \to \tilde{\mathcal{E}}$, is called a twisted $A(\mathfrak{g})$-bundle morphism if there exist a family of continuous morphisms of fiber bundles

$\eta_\lambda : \mathcal{E}(W_{\alpha,\lambda}) \to \tilde{\mathcal{E}}(W_{\alpha,\lambda}),$

such that with $\eta = (\eta_\lambda)$, for all $\lambda \in \mathbb{C}$, and $\eta_\lambda : \mathcal{E} \to \tilde{\mathcal{E}}$, is an $A(\mathfrak{g})$-module morphism for any $\xi \in X$.

It follows from [6] that the following lemma is true for $A(\mathfrak{g})$.

**Lemma 9.** Let $\mathcal{E}$ be a twisted $A(\mathfrak{g})$-bundle on $X$. Then $\mathcal{E} \oplus \mathcal{E}'$ is a twisted $A(\mathfrak{g})$-bundle, with nondegenerate symmetric invariant bilinear pairing

$$(g_{\alpha\beta}^*(\xi)\theta, g_{\alpha\beta}(\xi)u) = (\theta, u), \quad (5.1)$$

which is an invariant of $\mathcal{E}$ (i.e., does not depend on $g_{\alpha\beta}$ for all $\alpha, \beta \in I$, $\xi \in X_\alpha \cap X_\beta$, $\xi \in G_{\mathfrak{g}}$, $\theta \in G_{\mathfrak{g}}$) induced from the natural bilinear from on $G_{\mathfrak{g}}$.

**Proof.** For ordinary bundles this lemma was proven in [6]. Here we have to check (5.1) in the twisted case. Namely, for particular $\lambda \in \mathbb{C}$, using the definition of $g_{\alpha\beta}$ and multiplication $*$, consider $g_{\alpha\beta}(\xi)\theta, g_{\alpha\beta}(\xi)u = (H_{\alpha,\lambda} * H_{\beta,\lambda}^{-1}) \theta, (H_{\alpha,\lambda} * H_{\beta,\lambda}^{-1}) u$.

One sees that it equals to $(\theta, u)$. Thus, the pairing $(g_{\alpha\beta}(\xi)\theta, g_{\alpha\beta}(\xi)u)$ does not depend on any $\alpha, \beta \in I$. □

It is useful to introduce the following

**Definition 21.** A twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$ is called trivial if there exists an $A(\mathfrak{g})$-bundle isomorphism $\varphi : \mathcal{E} \to W \times X$, here $W \times X$ is the natural $A(\mathfrak{g})$-bundle on $X$ with $W$ as fibers.

Finally, we provide proofs for generalizations of two propositions given in [6] for the case of twisted $A(\mathfrak{g})$-bundles.
**Proposition 1.** For any twisted $A(g)$-bundle $E$, there exists a twisted $A(g)$-bundle $\tilde{E}$ such that $E \oplus \tilde{E}$ is geometrically covariant for $E$, i.e., $E \oplus \tilde{E}$ is a trivial $A(g)$-bundle.

**Proof.** Due to the properties of non-degenerate bilinear pairing, it is naturally to use Proposition 1. Let $(\alpha, \beta) \in \mathcal{O}_{E, A(g)}$ over $X$, and $\chi_\alpha$ such that $\sum_{\alpha \in I_0} \chi_\alpha(\xi)\chi_\alpha(\xi) = Id_\mathbb{W}$ is the identity operator for $\xi \in \mathbb{X}$ on the covering $\{X_\alpha, \alpha \in I_0\}$. Define an $A(g)$-bundle injective and bilinear pairing preserving homomorphism $\psi : E \to W^\otimes k \times X$, $\psi(\epsilon) = ((\chi_\alpha(\xi) h_\alpha(\epsilon)), (\chi_\alpha(\xi) h_\alpha(\xi)))$, for $\alpha \in I_0$, where $\xi = \pi(\epsilon)$, and $k$ is the number of independent domains in the covering $\{X_\alpha\}$. The $\psi$ sends $E$ to the trivial $A(g)$-bundle $W^\otimes k \times X$. We are able to extend non-degenerate bilinear pairing $(.,.)$ to $W^\otimes k \times X$. By Lemma 2, the transition functions preserve the bilinear pairing on $W$, thus for any $\epsilon, \tilde{\epsilon} \in \mathcal{E}_X$ one finds

\[
(\psi(\epsilon), \psi(\tilde{\epsilon})) = \sum_{\alpha \in I_0} ((\chi_\alpha(\xi) h_\alpha(\epsilon)), (\chi_\alpha(\xi) h_\alpha(\tilde{\epsilon})))
= \sum_{\alpha \in I_0} \chi_\alpha^2(\xi) (g_\alpha(\xi) h_\beta(\epsilon), g_\alpha(\xi) h_\beta(\tilde{\epsilon}))
= (\epsilon, \tilde{\epsilon}).
\]

Thus the homomorphism $\psi$ preserves the bilinear pairing and the restriction of the bilinear pairing to $\psi(E)$ is nondegenerate. Let us take $\tilde{E} = \psi(E)^T$ with respect to the bilinear pairing. According to Lemma 8, $\tilde{E}$ is an $A(g)$-bundle on $X$, $(W^\otimes k \times X) = \tilde{E} \oplus \psi(E) \cong \tilde{E} \oplus \mathcal{E}$, and such $A(g) \oplus A(g)$-bundle is trivial is a geometrical covariant. 

A twisted $A(g)$ exhibits the following homotopy-stability property:

**Proposition 2.** The construction of a twisted $A(g)$-bundle $E$ is homotopy-invariant. I.e., let $\tilde{X}$ be a compact Hausdorff space, $\tau_t : \tilde{X} \to X$, for $0 \leq t \leq 1$, a homotopy and $E$ a twisted $A(g)$-bundle over $X$. Then $\tau_t^*(E) \cong \tau_t^*(E)$.

**Proof.** Denote by $\mathcal{I}$ the unit interval and let $\tau : \tilde{X} \times \mathcal{I} \to X$, be the homotopy, so that $\tau(\xi, t) = \tau_t(\xi)$, and let $\tau : \tilde{X} \times \mathcal{I} \to \tilde{X}$ denote the projection onto the first factor. For a collection of $\xi_i \in \tilde{X}$, $k \geq 1$, and an element $w_i \in W$, let us choose a finite open covering $(\tilde{X}_{\xi_i})_{i=1}^k$ of $X$ so that $\tau^*(\mathcal{E}) = w_i \times \xi_i$ is trivial over each $\tilde{X}_{\xi_i} \times \mathcal{I}$. For each $\xi \in \tilde{X}$ we can find open neighborhood $U_{\xi, k}$ in $\tilde{X}$, and a partition $\{t_i, 0 = t_0 < t_1 \cdots < t_k = 1\}$ of $[0, 1]$ such that the bundle is trivial over each $U_{\xi, i} \times [t_i, t_{i+1}]$. Set $U = U_{\xi, k} = \bigcap_{i=1}^k U_{\xi, i}$.

Then the twisted bundle $\tau^*(\mathcal{E})$ is trivial over $\tilde{X}_{\xi} \times \mathcal{I}$. Indeed, by choosing appropriate elements of $Aut_{\tau_t}$ we could find for $t_{i-1, i} = [t_{i-1}, t_i]$, isomorphisms of trivializations such that

\[
h_i : \tau^*(\mathcal{E})|_{\tilde{X}_{\xi} \times t_{i-1, i}} \to W \times \tilde{X}_{\xi} \times t_{i-1, i},
\]
for $1 \leq i \leq k$. For $u \in U$, we take the $A(g)$-bundle isomorphisms
\[ h_i(u, t_i) = (h_{i-1} \circ h_i^{-1})(u, t_i) \circ h_i(u, t_{i-1}) : \]
\[ \tau^*(E)|_{U_\bar{\xi} \times [t_i, t_{i+1}]} \to W \times U_\bar{\xi} \times [t_i, t_{i+1}], \]
then $h_i = h_i'$ on $U_\bar{\xi} \times \{t_i\}$, thus they define a trivialization on $\bar{X}_x \times [t_{i-1}, t_{i+1}]$, and thus $\tau^*(E)$ is trivial over $\bar{X}_x \times I$. Let $\chi_i$ be a partition of unity of $X$ with support of $\chi_i$ contained in $\bar{X}_x$. For $i \geq 0$, let $q_j = \sum_{i=1}^j \chi_i$. In particular, $q_0 = 0$ and $q_n = 1$. Consider the subspace of $\bar{X}_x \times I$ consisting of points of the pairing $p_\xi q_\bar{\xi}$, and let $\pi_i : E_i \to W_i$ be the restriction of the bundle $E$ over $W_i$. Since $E$ is trivial on $U_x \times I$, the projection homeomorphism $W_i \to W_{i-1}$ induces isomorphisms $\varepsilon_i : E_i \to E_{i-1}$, which is identity outside $\pi_i(U_{x,i})$, and which takes each fiber of $E_i$ isomorphically onto the corresponding fiber of $E_{i-1}$. The composition $\varepsilon = \prod_{i=1}^k \varepsilon_i$ is then an isomorphism from $E|_{U \times \{1\}}$ to $E|_{U \times \{0\}}$.

**Acknowledgments**

The author would like to thank H. V. Lê, D. Levin, A. Lytchak, and P. Somberg for related discussions.

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