WELL-POSEDNESS OF LINEAR FIRST ORDER PORT-HAMILTONIAN SYSTEMS ON MULTIDIMENSIONAL SPATIAL DOMAINS

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ABSTRACT. We consider a port-Hamiltonian system on a spatial domain $\Omega \subseteq \mathbb{R}^n$ that is bounded with Lipschitz boundary. We show that there is a boundary triple associated to this system. Hence, we can characterize all boundary conditions that provide unique solutions that are non-increasing in the Hamiltonian. As a by-product we develop the theory of quasi Gelfand triples. Adding “natural” boundary controls and boundary observations yields scattering/impedance passive boundary control systems. This framework can be applied to the wave equation, Maxwell equations and Mindlin plate model, and probably many more.

1. Introduction

The aim of this paper is to develop a port-Hamiltonian framework on multidimensional spatial domains that justifies existence and uniqueness of solutions. Those systems can be described by the following equations

$$\frac{\partial}{\partial t} x(t, \zeta) = \sum_{i=1}^{n} \frac{\partial}{\partial \zeta_i} P_i(\mathcal{H}(\zeta) x(t, \zeta)) + P_0(\mathcal{H}(\zeta) x(t, \zeta)),$$

$$x(0, \zeta) = x_0(\zeta),$$

where $P_i$ and $P_0$ are matrices, $\mathcal{H}$ is the Hamiltonian density, and $\Omega$ is a open subset of $\mathbb{R}^n$ with bounded Lipschitz boundary. We will restrict ourselves to the case, where the matrices $P_i$ have the block shape $[0, L_i]$ for $i \in \{1, \ldots, n\}$. We also introduce “natural” boundary controls and observations which makes the system a scattering passive (energy preserving) or impedance passive (energy preserving) boundary control system.

The port-Hamiltonian formulation has proven to be a powerful tool for the modeling and control of complex multiphysics systems. An introductory overview can be found in [vdSJ14]. For a one-dimensional spatial domain concerns about existence and uniqueness of solutions are covered in [JZ12].

Chapter 8 of the Ph.D. thesis [Vil07] also regards such port-Hamiltonian systems that have multidimensional spatial domains, but the results demand very strong assumptions, which are in case of the Maxwell equations and Mindlin plate model not satisfied. With the following approach we will overcome these limits.

The strategy is to find a boundary triple associated to the differential operator. The multidimensional integration by parts formula already suggests possible operators for a boundary triple, but unfortunately these operators cannot be extended to the entire domain of the differential operator. Hence, we need to adapt...
the codomain of these boundary operators, which will lead to the construction of suitable boundary spaces for this problem. These boundary spaces behave like a Gelfand triple with the original codomain as pivot space, but lack of a chain inclusion.

Up to the authors best knowledge there is no theory about this setting. So we will develop the notion of quasi Gelfand triples in section 4, which equips us with the tools to state the boundary condition in terms of the pivot space instead of the artificially constructed boundary spaces (Theorem 6.8).

The approach to the wave equation in [KZ15] perfectly fits the framework presented in this paper. In fact, many ideas from [KZ15] are generalized in this work. Also the Maxwell equations can be formulated as such a port-Hamiltonian system and the results in [WS13] can also be derived with the tools of this paper. Moreover, this theory can be applied on the model of Mindlin Plate in [BAPM18],[MMB05]. In section 7 we give examples of how this framework can be applied to these three PDEs.

2. Boundary Triple

In this section we state the most important properties of boundary triples for skew-symmetric operators for this work. More details can be found in [GG91, chapter 3.4] and [KZ15].

A linear relation \( T \) between two vector spaces \( X \) and \( Y \) is a linear subspace of \( X \times Y \). Clearly, every linear operator is also a linear relation. We will use the following notation

\[
\ker T := \{ x \in X : (x, 0) \in T \}, \quad \text{ran } T := \{ y \in Y : \exists x : (x, y) \in T \},
\]

\[
\text{mul } T := \{ y \in Y : (0, y) \in T \}, \quad \text{dom } T := \{ x \in X : \exists y : (x, y) \in T \}.
\]

Thus, \( T \) is single-valued, if \( \text{mul } T = \{ 0 \} \). For a linear relation \( T \) between two Hilbert spaces \( X \) and \( Y \) the adjoint relation is defined by

\[
T^* := \{ (u, v) \in Y \times X : \langle u, y \rangle_Y = \langle v, x \rangle_X \text{ for all } (x, y) \in T \}
\]

and the following holds true

\[
\ker T^* = (\text{ran } T)^\perp, \quad \text{mul } T^* = (\text{dom } T)^\perp \quad \text{and } \quad T^* = \begin{bmatrix} 0 & 1_Y \end{bmatrix} T^\perp,
\]

where \( \begin{bmatrix} 0 & 1_Y \end{bmatrix} \) is the adjoint of a skew-symmetric operator \( H \). A linear relation \( T \) on a Hilbert space \( H \) (between \( H \) and \( H \)) is dissipative, if \( \text{Re} \langle x, y \rangle_H \leq 0 \) for every \( (x, y) \in T \) and maximal dissipative, if additionally there is no proper dissipative extension of \( T \). More details can be found in [Cro98].

**Definition 2.1.** Let \( A_0 \) be a densely defined, skew-symmetric, and closed operator on a Hilbert space \( X \). By a boundary triple for \( A_0^\ast \) we mean a triple \((\mathcal{B}, B_1, B_2)\) consisting of a Hilbert space \( \mathcal{B} \), and two linear operators \( B_1, B_2 : \text{dom } A_0^\ast \to \mathcal{B} \) such that

(i) the mapping \( \begin{bmatrix} B_1 & B_2 \end{bmatrix} : \text{dom } A_0^\ast \to \mathcal{B} \times \mathcal{B}, x \mapsto \begin{bmatrix} B_1 x \\ B_2 x \end{bmatrix} \) is surjective, and

(ii) for \( x, y \in \text{dom } A_0^\ast \) there holds

\[
\langle A_0^\ast x, y \rangle_X + \langle x, A_0^\ast y \rangle_X = \langle B_1 x, B_2 y \rangle_{\mathcal{B}} + \langle B_2 x, B_1 y \rangle_{\mathcal{B}}. \tag{2.1}
\]

The operator \( A_0 \) can be restored from by restricting \(-A_0^\ast\) to \( \ker B_1 \cap \ker B_2 \) as the next lemma will show. However, if \( A_0^\ast \) wasn’t the adjoint of a skew-symmetric operator then this would not hold as Example A.1 demonstrates.

**Lemma 2.2.** Let \( A_0 \) be a densely defined, skew-symmetric, and closed operator on a Hilbert space \( X \) and \( (\mathcal{B}, B_1, B_2) \) be a boundary triple for \( A_0^\ast \). Then \( A_0 = -A_0^\ast |_{\ker B_1 \cap \ker B_2} \).
Proof. Let \( x \in \ker B_1 \cap \ker B_2 \) and \( y \in \dom A_0^* \). Then the right-hand-side of (2.1) is 0. Hence,
\[
\langle x, A_0^* y \rangle_X = \langle -A_0^* x, y \rangle_X \quad \text{for all} \quad y \in \dom A_0^*.
\]
This yields \( (x, -A_0^* x) \in A_0^* = A_0 \). Hence, \(-A_0^*|_{\ker B_1 \cap \ker B_2} \subseteq A_0\).

On the other hand if \( x \in \dom A_0 \), then \( A_0^* x = -A_0 x \) and consequently
\[
\langle A_0^* x, y \rangle_X + \langle x, A_0^* y \rangle_X = \langle -x, A_0^* y \rangle_X + \langle x, A_0^* y \rangle_X = 0.
\]
Therefore, using (2.1) yields \( \langle [B_2]_X [B_1]_X y \rangle_{\mathcal{B} \times \mathcal{B}} = 0 \) for all \( y \in \dom A_0^* \). Since \([B_1]_X\) is surjective on \( \mathcal{B} \times \mathcal{B} \), we have
\[
\begin{bmatrix}
[B_1]_X \\
[B_2]_X
\end{bmatrix}
\perp \mathcal{B} \times \mathcal{B},
\]
which yields \( x \in \ker B_1 \cap \ker B_2 \). \(\blacksquare\)

The following result is Theorem 2.2 from [KZ15].

**Proposition 2.3.** Let \( A_0 \) be a skew-symmetric operator and \((\mathcal{B}, B_1, B_2)\) be a boundary triple for \( A_0^* \). Consider the restriction \( A \) of \( A_0^* \) to a subspace \( \mathcal{D} \) containing \( \ker B_1 \cap \ker B_2 \). Define a subspace of \( \mathcal{B}^2 \) by \( \mathcal{C} := \left[\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\right] \mathcal{D} \). Then the following claims are true

(i) The domain of \( A \) can be written as
\[
\dom A = \mathcal{D} = \left\{ d \in \dom A_0^* : \left[\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} d \right] \in \mathcal{C} \right\}.
\]

(ii) The operator closure of \( A \) is \( A_0^* \) restricted to
\[
\mathcal{D}' := \left\{ d' \in \dom A_0^* : \left[\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} d' \right] \in \mathcal{C}' \right\},
\]
where \( \mathcal{C}' \) is the closure in \( \mathcal{B}^2 \). Therefore, \( A \) is closed if and only if \( \mathcal{C} \) is closed.

(iii) The adjoint \( A^* \) is the restriction of \(-A_0^* \) to \( \mathcal{D}' \), where
\[
\mathcal{D}' := \left\{ d' \in \dom A_0^* : \left[\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} d' \right] \in \left[\begin{bmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix} \mathcal{C}' \right] \right\}.
\]

(iv) The operator \( A \) is (maximal) dissipative if and only if \( \mathcal{C} \) is a (maximal) dissipative relation.

### 3. Differential Operators

Before we start analyzing port-Hamiltonian systems we will make some observation about the differential operators that will appear in the PDE. In this section we take care of all the technical details of these differential operators. Since it doesn’t really make a difference whether we use the scalar field \( \mathbb{R} \) or \( \mathbb{C} \) we will use \( K \in \{ \mathbb{R}, \mathbb{C} \} \) for the scalar field. The following assumption will be made for the rest of this work.

**Assumption 3.1.** Let \( m_1, m_2, n \in \mathbb{N} \), \( \Omega \subseteq \mathbb{R}^n \) be open with a bounded Lipschitz boundary, and \( L = (L_i)_{i=1}^n \) such that \( L_i \in \mathbb{K}^{m_1 \times m_2} \) for all \( i \in \{1, \ldots, n\} \). Corresponding to \( L \) we also have \( L^H := (L_i^H)_{i=1}^n \), where \( L_i^H \) denotes the complex conjugated transposed (Hermitian transposed) matrix.
We will write \( D(\Omega) \) for the set of all \( C^\infty(\Omega) \) functions with compact support in \( \Omega \). Its dual space, the space of distribution, will be denoted by \( D'(\Omega) \). Moreover, we will write \( D(\mathbb{R}^n)|_{\Omega} \) for \( \{ f|_{\Omega} : f \in D(\mathbb{R}^n) \} \). We will use \( \partial_i \) as a short notation for \( \frac{\partial}{\partial x_i} \).

Sometimes it can be confusing to pay attention to the antilinear structure of an inner product of a Hilbert space, when switching between the inner product and the dual pairing. Thus, for the sake of clarity we will always consider the antidual space instead of the dual space, which is the space of all continuous antilinear mappings from the topological vector space into its scalar field. Hence, both the inner product and the (anti)dual pairing is linear in one component and antilinear in the other. So also \( D'(\Omega) \) is actually the antidual space of \( D(\Omega) \).

**Definition 3.2.** Let \( L \) be as in Assumption 3.1. Then we define

\[
L_\phi := \sum_{i=1}^{n} \partial_i L_i \quad \text{and} \quad L_{\phi}^H := \sum_{i=1}^{n} \partial_i L_i^H
\]

as operators on \( D'(\Omega)^{m_2} \) and \( D'(\Omega)^{m_1} \), respectively. Furthermore, we define the spaces

\[
H(L_\phi, \Omega) := \{ f \in L^2(\Omega, \mathbb{K}^{m_2}) : L_\phi f \in L^2(\Omega, \mathbb{K}^{m_1}) \}
\]

and

\[
H(L_{\phi}^H, \Omega) := \{ f \in L^2(\Omega, \mathbb{K}^{m_1}) : L_{\phi}^H f \in L^2(\Omega, \mathbb{K}^{m_2}) \}.
\]

These spaces are endowed with the inner product

\[
\langle f, g \rangle_{H(L_\phi, \Omega)} := \langle f, g \rangle_{L^2(\Omega, \mathbb{K}^{m_2})} + \langle L_\phi f, L_\phi g \rangle_{L^2(\Omega, \mathbb{K}^{m_1})}
\]

and

\[
\langle f, g \rangle_{H(L_{\phi}^H, \Omega)} := \langle f, g \rangle_{L^2(\Omega, \mathbb{K}^{m_1})} + \langle L_{\phi}^H f, L_{\phi}^H g \rangle_{L^2(\Omega, \mathbb{K}^{m_2})}
\]

respectively. The space \( H_0(L_\phi, \Omega) \) is defined as \( \overline{D(\Omega)^{m_2} \cap H_0(L, \Omega)} \) and \( H_0(L_{\phi}^H, \Omega) \) analogously. We denote the trace operator by \( \gamma : H^1(\Omega) \to H^1(\partial \Omega) \) and the outward pointing normed normal vector on \( \partial \Omega \) by \( \nu \). We define

\[
L_{\nu} := \sum_{i=1}^{n} \nu_i L_i, \quad \text{and} \quad L_{\nu}^H := \sum_{i=1}^{n} \nu_i L_i^H.
\]

**Remark 3.3.** Clearly, \( H^1(\Omega, \mathbb{K}^{m_2}) \subseteq H(L_\phi, \Omega) \) and \( H^1(\Omega, \mathbb{K}^{m_1}) \subseteq H(L_{\phi}^H, \Omega) \). It is also easy to see that \( -L_{\phi}^H \) is the formal adjoint of \( L_\phi \).

For convenience we will write \( H^1(\Omega)^k \) instead of \( H^1(\Omega, \mathbb{K}^k) \) and \( L^2(\Omega)^k \) instead of \( L^2(\Omega, \mathbb{K}^k) \) for \( k \in \mathbb{N} \).

**Lemma 3.4.** The operator \( L_\phi \) with domain \( L_\phi = H(L_\phi, \Omega) \) is a closed operator from \( L^2(\Omega)^{m_2} \) to \( L^2(\Omega)^{m_1} \) and \( H(L_\phi, \Omega) \) endowed with the inner product \( \langle \cdot, \cdot \rangle_{H(L_\phi, \Omega)} \) is a Hilbert space.

**Proof.** Let \( \{ f_n, L_\phi f_n \} \) be a sequence in \( L_\phi \) that converges to a point \( (f, g) \in L^2(\Omega)^{m_2} \times L^2(\Omega)^{m_1} \). For an arbitrary \( \phi \in D(\Omega)^{m_1} \) we have

\[
\langle g, \phi \rangle_{D'(\Omega)^{m_1}, D(\Omega)^{m_1}} = \lim_{n \to \infty} \langle L_\phi f_n, \phi \rangle_{D'(\Omega)^{m_1}, D(\Omega)^{m_1}} = \langle f, -L_{\phi}^H \phi \rangle_{D'(\Omega)^{m_2}, D(\Omega)^{m_2}} = \langle f, L_{\phi}^H \phi \rangle_{D'(\Omega)^{m_2}, D(\Omega)^{m_2}} = \langle L_{\phi}^H f, \phi \rangle_{D'(\Omega)^{m_2}, D(\Omega)^{m_2}},
\]

which implies \( g = L_\phi f \). Since \( g \) is also in \( L^2(\Omega)^{m_1} \), we conclude that \( L_\phi \) is closed. \( \square \)
Example 3.5. Let us regard the following matrices

\[
L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad L_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Then we obtain the corresponding differential operators

\[
L_0 = \begin{bmatrix} \partial_1 & \partial_2 & \partial_3 \end{bmatrix} = \text{div} \quad \text{and} \quad L^H_0 = \begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix} = \text{grad}.
\]

The corresponding operator \(L_0\) that acts on \(L^2(\partial\Omega)\) can be written as an inner product

\[
L_0 f = \begin{bmatrix} \nu_1 & \nu_2 & \nu_3 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \nu \cdot f = \langle f, \nu \rangle_{L^2}.
\]

Clearly the previous example can be extended to any finite dimension.

Example 3.6. The following matrices will construct the rotation operator.

\[
L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad L_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

In this example we have \(L_1^H = -L_1\). Furthermore, the corresponding differential operator is

\[
L_0 = \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix} = \text{rot} = -L^H_0.
\]

The corresponding operator \(L_0\) that acts on \(L^2(\partial\Omega)\) can be written as a cross product

\[
L_0 f = \begin{bmatrix} 0 & -\nu_3 & \nu_2 \\ \nu_3 & 0 & -\nu_1 \\ -\nu_2 & \nu_1 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \nu \times f.
\]

Lemma 3.7. The adjoint of \(L_0\) with \(\text{dom} L_0 = H(L_0, \Omega)\) is \(L^*_0 = -L_0^H\) on \(\text{dom} L^*_0 \subseteq H(L^*_0, \Omega)\).

Proof. For an arbitrary \(g \in \text{dom} L_0^*\) and an arbitrary \(\phi \in \mathcal{D}(\Omega)\) we have (to shorten the notation we will write \(\langle \cdot , \cdot \rangle_{\mathcal{D}^*, \mathcal{D}}\) instead of \(\langle \cdot , \cdot \rangle_{\mathcal{D}^*(\Omega)^*, \mathcal{D}(\Omega)}\) and \(\langle \cdot , \cdot \rangle_{L^2}\) instead of \(\langle \cdot , \cdot \rangle_{L^2(\Omega)}\))

\[
\langle L_0^* g, \phi \rangle_{\mathcal{D}^*, \mathcal{D}} = \langle L_0^* g, \phi \rangle_{L^2} = \langle g, L_0 \phi \rangle_{L^2} = \langle g, L_0 \phi \rangle_{\mathcal{D}^*, \mathcal{D}} = \langle -L_0^H g, \phi \rangle_{\mathcal{D}^*, \mathcal{D}}.
\]

Therefore, \(L_0^* g = -L_0^H g\) and \(L_0^* g \in L^2(\Omega)\) implies \(L_0^H g \in L^2(\Omega)\). Consequently, \(\text{dom} L_0^* \subseteq H(L_0^*, \Omega)\).

Remark 3.8. If \(L\) contains only Hermitian matrices \((L_1^H = L_1)\), then \(L_0^*\) is skew-symmetric.

Remark 3.9. The mapping \(\iota : H(L_0, \mathbb{R}^n) \to H(L_0, \Omega), f \mapsto f |_{\Omega}\) is well-defined and continuous for any open set \(\Omega \subseteq \mathbb{R}^n\). In particular, \(L_0(f |_{\Omega}) = (L_0 f) |_{\Omega}\). Hence, we can always regard an \(f \in H(L_0, \mathbb{R}^n)\) as an element of \(H(L_0, \Omega)\), especially when \(\text{supp} f \subseteq \Omega\). Moreover, if \(f_n \to f\) in \(H(L_0, \mathbb{R}^n)\), then \(f_n \to f\) in \(H(L_0, \Omega)\).

Definition 3.10. A set \(O \subseteq \mathbb{R}^n\) is strongly star-shaped with respect to \(x_0\) if for every \(x \in \overline{O}\) the half-open line segment \(\{x_0 + \theta x : \theta \in [0, 1)\}\) is contained in \(O\). We call \(O\) strongly star-shaped, if there is a \(x_0\) such that \(O\) is strongly star-shaped with respect to \(x_0\).
Note that this is equivalent to
\[ \theta(O - x_0) + x_0 \subseteq O \] for all \( \theta \in [0, 1) \).

**Lemma 3.11.** Let \( f \in H(L_0, \mathbb{R}^n) \) and \( x_0 \in \mathbb{R}^n \). Furthermore, let \( f_\theta(x) := f(\theta(x - x_0) + x_0) \) for \( \theta \in (0, 1) \). Then \( f_\theta \in H(L_0, \mathbb{R}^n) \) and \( f_\theta \rightarrow f \) in \( H(L_0, \mathbb{R}^n) \) as \( \theta \rightarrow 1 \).

If there exists a strongly star-shaped set \( O \) such that \( \text{supp} \ f \subseteq \overline{O} \), then \( \text{supp} \ f_\theta \subseteq O \) for \( \theta \in (0, 1) \).

**Proof.** Let \( \alpha(x) := \frac{1}{\theta}(x - x_0) + x_0 \). Then we have \( f_\theta = f \circ \alpha \) and
\[
\langle L_\theta(f \circ \alpha), \phi \rangle_{D', D} = \langle f, -(L_\theta^H \phi) \circ \alpha^{-1} \theta^n \rangle_{D', D} = \left\langle f, -\sum_{i=1}^{n} L_\theta^H \partial_i \left( \frac{1}{\theta} \phi \circ \alpha^{-1} \theta^n \right) \right\rangle_{D', D} = \left\langle \frac{1}{\theta} (L_\theta f) \circ \alpha, \phi \right\rangle_{D', D}.
\]
Therefore, \( L_\theta f_\theta = \frac{1}{\theta} (L_\theta f)_\theta \) and \( f_\theta \in H(L_\theta, \mathbb{R}^n) \). We can also write \( f_\theta \) as \( T_{x_0} D_{\frac{1}{\theta}} - T_{x_0} f \) where \( T_y \) is the translation mapping \( f \mapsto f(. + y) \) and \( D_\theta \) is the dilation mapping \( f \mapsto f(\eta) \). Since \( T_y \) is bounded and \( D_\theta \) converges strongly to \( I \) as \( \eta \rightarrow 1 \), we conclude \( f_\theta \rightarrow f \) in \( L^2(\mathbb{R}^n)^{m_2} \) as \( \theta \rightarrow 1 \) and \( L_\theta f_\theta = \frac{1}{\theta} (L_\theta f)_\theta \rightarrow L_\theta f \) in \( L^2(\mathbb{R}^n)^{m_1} \) as \( \theta \rightarrow 1 \). Hence, \( f_\theta \rightarrow f \) in \( H(L_\theta, \mathbb{R}^n) \).

Let \( O \) be strongly star-shaped with respect to \( x_0 \) and \( \text{supp} \ f \subseteq \overline{O} \). Then for \( \theta \in (0, 1) \)
\[
\text{supp} \ f_\theta = \theta(\text{supp} \ f - x_0) + x_0 \subseteq \theta(O - x_0) + x_0 \subseteq O.
\]

**Remark 3.12.** If \( f \in H(L_\theta, \Omega) \) and \( \psi \in D(\mathbb{R}^n)|_\Omega \), then by the product rule for distributional derivatives also \( \psi f \in H(L_\theta, \Omega) \) and \( L_\theta(\psi f) = \psi L_\theta f + \sum_{i=1}^{n} (\partial_i \psi) L_i f \).

**Lemma 3.13.** For every \( f \in H(L_\theta, \mathbb{R}^n) \) exists a sequence \( (f_k)_{k \in \mathbb{N}} \) in \( H(L_\theta, \mathbb{R}^n) \) with \( \text{supp} \ f_k \) is compact that converges to \( f \) in \( H(L_\theta, \mathbb{R}^n) \).

**Proof.** Let \( \psi \in C^\infty(\mathbb{R}^n, \mathbb{R}) \) be such that
\[
\psi(\zeta) = \begin{cases} 1, & \text{if } ||\zeta|| \leq 1, \\ 0, & \text{if } ||\zeta|| \geq 2. \end{cases}
\]
Then \( f_k := \psi(\frac{1}{k}) f \in H(L_\theta, \mathbb{R}^n) \) and \( f_k \rightarrow f \) in \( L^2 \). By \( L_\theta f_k = \psi(\frac{1}{k}) L_\theta f + \frac{1}{k} \sum_{i=1}^{n} (\partial_i \psi) L_i f_k \), we conclude \( f_k \rightarrow f \) in \( H(L_\theta, \mathbb{R}^n) \).

The next lemma is similar to [DL90, Lemma 1, page 206] the main idea of the proof can be adopted.

**Lemma 3.14.** If \( f \in H(L_\theta, \Omega) \) is such that
\[
\langle L_\theta f, \phi \rangle_{L^2(\Omega)} + \langle f, L_\theta^H \phi \rangle_{L^2(\Omega)} = 0 \quad \text{for all} \quad \phi \in D(\mathbb{R}^n)^{m_1},
\]
then \( f \in H_0(\mathbb{R}^n) \).

**Proof.** Let \( f \in H(L_\theta, \Omega) \) such that it satisfies (3.1). Then we have to find a sequence \( (f_\alpha)_{\alpha \in \mathbb{N}} \) in \( D(\Omega)^{m_2} \) that converges to \( f \) with respect to \( \|\cdot\|_{H(L_\theta, \Omega)} \).

We define \( f_\alpha := L_{\alpha} f \) and \( \tilde{f}, \tilde{f}_\alpha \) as the extension of \( f \) and \( f_\alpha \) respectively on \( \mathbb{R}^n \) such that these functions are 0 outside of \( \Omega \). By
\[
\langle \tilde{f}_\alpha, \phi \rangle_{D'(\mathbb{R}^n), D(\mathbb{R}^n)} = \langle \tilde{f}, \phi \rangle_{L^2(\mathbb{R}^n)} = \langle \tilde{f}, (L_\theta^H \phi)_{L^2(\Omega)} \rangle_{D', D(\mathbb{R}^n)} = \langle \tilde{f}, -L_\theta^H \phi \rangle_{L^2(\Omega)} = \langle \tilde{f}, -L_\theta f \rangle_{D'(\mathbb{R}^n), D(\mathbb{R}^n)} \]
for \( \phi \in D(\mathbb{R}^n)^{m_1} \), we see that \( \tilde{f}_\alpha = L_{\alpha} \tilde{f} \) and \( \tilde{f} \in H(L_\theta, \mathbb{R}^n) \) with \( \text{supp} \ \tilde{f} \subseteq \overline{\Omega} \).

**Step 1. Assume that there is a bounded \( \Omega \subseteq \Omega \) with bounded Lipschitz boundary such that \( \text{supp} \ \tilde{f} \subseteq \overline{\Omega} \).** By [CDA02, Proposition 2.5.4, page 69] there is a finite
open covering \((O_i)_{i=1}^k\) of \(\overline{\Omega}\) such that \(O_i \cap \Omega'\) is strongly star-shaped. We employ a partition of unity and obtain \((\alpha_i)_{i=1}^k\), subordinate to this covering, that is
\[
\alpha_i \in \mathcal{D}(O_i), \quad \alpha_i(x) \in [0, 1] \quad \text{and} \quad \sum_{i=1}^k \alpha_i(x) = 1 \quad \text{for all} \quad x \in \Omega'.
\]
Hence, \(\tilde{f} = \sum_{i=1}^k \alpha_i \tilde{f}\) and we define \(f_\theta := \alpha_i \tilde{f}\). By construction \(f_\theta \in H(L_\theta, \mathbb{R}^n)\) and \(\text{supp} f_\theta \subseteq O_i \cap \Omega'\), where \(O_i \cap \Omega'\) is strongly star-shaped. Lemma 3.11 ensures that \(\text{supp} (f_\theta)_\theta \subseteq O_i \cap \Omega'\) for \(\theta \in (0, 1)\) and \(f_\theta \to f_\theta \in H(L_\theta, \mathbb{R}^n)\) for \(\theta \to 1\) - .

Let \(\rho_\varepsilon\) be a positive \(C^\infty\) mollifier with compact support. Then \(\rho_\varepsilon * g \to g\) in \(L^2(\mathbb{R}^n)\) for an arbitrary \(g \in L^2(\mathbb{R}^n)\). Since \(L_\theta(\rho_\varepsilon * h) = \rho_\varepsilon * L_\theta h\), we also have that \(\rho_\varepsilon * h \to h\) in \(H(L_\theta, \mathbb{R}^n)\) for \(h \in H(L_\theta, \mathbb{R}^n)\) and \(\rho_\varepsilon * h \in \mathcal{D}(\mathbb{R}^n)^{m_2}\).

For fixed \(\theta \in (0, 1)\) and \(\varepsilon\) sufficiently small, we can say \(\text{supp} \rho_\varepsilon * (f_\theta)_\theta \subseteq \Omega'\). This establishes the existence of a sequence \((\rho_\varepsilon * (f_\theta)_\theta)_{\theta \in \mathbb{N}}\) in \(\mathcal{D}(\Omega)^{m_2}\) converging to \(f_\theta\) in \(H(L_\theta, \mathbb{R}^n)\). Doing this for every \(i \in \{1, \ldots, k\}\) yields sequences \((f_{\theta,i})_{\theta \in \mathbb{N}}\) in \(\mathcal{D}(\Omega)^{m_2}\) converging to \(f_{i}\) in \(H(L_\theta, \mathbb{R}^n)\). Consequently, \(\sum_{i=1}^k (f_{\theta,i})_{\theta \in \mathbb{N}}\) is a sequence in \(\mathcal{D}(\Omega)^{m_2}\) that converges to \(\tilde{f}\) in \(H(L_\theta, \mathbb{R}^n)\) and by Remark 3.9 also in \(H(L_\theta, \Omega)\).

**Step 2. Without extra assumptions.** By the already shown each entry of the sequence \((f_{\theta,i})_{\theta \in \mathbb{N}}\) from Lemma 3.13 can be approximated by \(\mathcal{D}(\Omega)^{m_2}\) elements. A diagonalization argument yields the same for the limit \(\tilde{f}\). By Remark 3.9 this diagonal sequence also converges in \(H(L_\theta, \Omega)\). \(\square\)

**Theorem 3.15.** \(\mathcal{D}(\mathbb{R}^n)^{m_2}|_\Omega\) is dense in \(H(L_\theta, \Omega)\).

**Proof.** Suppose \(\mathcal{D}(\mathbb{R}^n)^{m_2}|_\Omega\) is not dense in \(H(L_\theta, \Omega)\). Then there exists a non zero \(f \in H(L_\theta, \Omega)\) such that
\[
\langle f, g \rangle_{H(L_\theta, \Omega)} = \langle f, g \rangle_{L^2} + \langle L_\theta f, L_\theta g \rangle_{L^2} = 0 \quad \text{for all} \quad g \in \mathcal{D}(\mathbb{R}^n)^{m_2}|_\Omega. \tag{3.2}
\]
In particular, for an arbitrary \(h \in \mathcal{D}(\Omega)^{m_2}\) we have
\[
\langle f, h \rangle_{\mathcal{D}'} = \langle f, h \rangle_{L^2} = -\langle L_\theta f, L_\theta h \rangle_{L^2} = \langle f, h \rangle_{\mathcal{D}'}, \quad \text{which implies that} \quad f = L^H_\theta f \in L^2(\Omega)^{m_2}.
\]
Hence we can rewrite (3.2) as
\[
\langle L^H_\theta f, g \rangle_{L^2(\Omega)} + \langle L_\theta f, L_\theta g \rangle_{L^2(\Omega)} = 0 \quad \text{for all} \quad g \in \mathcal{D}(\mathbb{R}^n)^{m_2}|_\Omega.
\]
By Lemma 3.14 (switching the roles of \(L_\theta\) and \(L^H_\theta\)), defining \(f_0 := L_\theta f\) yields \(f_0 \in H_0(L^H_\theta, \Omega)\). Since \(\mathcal{D}(\Omega)^{m_2}\) is dense in \(H_0(L^H_\theta, \Omega)\), there is a sequence \((f_n)_{n \in \mathbb{N}}\) in \(\mathcal{D}(\Omega)^{m_2}\) converging to \(f_0\) with respect to \(\|\cdot\|_{H_0(L^H_\theta, \Omega)}\). Note that \(f = L^H_\theta f_0\).

\[
\langle f_0, f_n \rangle_{H(L^H_\theta, \Omega)} = \langle f_0, f_n \rangle_{L^2} + \langle L^H_\theta f_0, L^H_\theta f_n \rangle_{L^2} = \langle L_\theta f_0, f_n \rangle_{L^2} + \langle f, L^H_\theta f_n \rangle_{L^2} = \langle L_\theta f_0, f_n \rangle_{\mathcal{D}'} - \langle L_\theta f_0, f_n \rangle_{\mathcal{D}'} = 0.
\]
Since \(\|f_0\|_{H_0(L^H_\theta, \Omega)}^2 \leq \limsup_{n \to \infty} \langle f_0, f_n \rangle_{H(L^H_\theta, \Omega)} = 0\), we have that \(f_0 = 0\), which implies \(f = L^H_\theta f_0 = 0\). Hence, \(\mathcal{D}(\mathbb{R}^n)^{m_2}|_\Omega\) is dense in \(H(L_\theta, \Omega)\). \(\square\)

**Lemma 3.16.** Let \(f \in H^1(\Omega)^{m_2}\) and \(g \in H^1(\Omega)^{m_2}\). Then we have
\[
\langle L_\theta f, g \rangle_{L^2(\Omega)^{m_2}} + \langle f, L^H_\theta g \rangle_{L^2(\Omega)^{m_2}} = \langle L_\theta f, \gamma_0 g \rangle_{L^2(\Omega)^{m_2}}
\]
\[
= \langle \gamma_0 f, L^H_\theta g \rangle_{L^2(\Omega)^{m_2}}. \tag{3.3}
\]
Lemma 4.3. \(H\) to the norm of \(H\) is isomorphic to the closure of another Hilbert space \(H\).

Remark 4.2. The completion is again a Hilbert space with the extension of \(\langle \cdot, \cdot \rangle\) on a dense \((w.r.t. \|\cdot\|)\) of \(H\).

Proof. By the definition of \(L_0\) and \(L^H_0\), and the linearity of the scalar product we can write the left-hand-side of (3.3) as
\[
\int_{\Omega} \sum_{i=1}^{n} (\partial_i L_i f, g) + (f, \partial_i L^H_i g) \, d\lambda = \int_{\Omega} \sum_{i=1}^{n} (\partial_i L_i f, g) + (L_i f, \partial_i g) \, d\lambda,
\]
where \(\lambda\) denotes the Lesbesgue measure. By the product rule for derivatives and the Gauss’s theorem (divergence theorem) this is equal to
\[
\int_{\partial \Omega} \sum_{i=1}^{n} \partial_i (L_i f, g) \, d\lambda = \int_{\partial \Omega} \sum_{i=1}^{n} \nu \gamma_0 (L_i f, g) \, d\mu = \int_{\partial \Omega} (L \gamma_0 f, \gamma_0 g) \, d\mu,
\]
where \(\nu\) denotes the outward pointing normed normal vector on \(\partial \Omega\) and \(\mu\) denotes the surface measure of \(\partial \Omega\).

Corollary 3.17. Let \(f \in H^1(\Omega)^m\) and \(g \in H^1(\Omega)^m\). Then we have
\[
|\langle L \gamma_0 f, \gamma_0 g \rangle_{L^2(\partial \Omega)^{m}}| \leq \|f\|_{H(L_0,\Omega)} \|g\|_{H(L_0,\Omega)}.
\]

Proof. Lemma 3.16, the triangular inequality and Cauchy Schwartz's inequality yield
\[
|\langle L \gamma_0 f, \gamma_0 g \rangle_{L^2(\partial \Omega)^{m}}| \leq |\langle L_0 f, g \rangle_{L^2(\Omega)^{m}}| + |\langle f, L^H_0 g \rangle_{L^2(\Omega)^{m}}|,
\]
\[
\leq \|L_0 f\| \|g\| + \|f\| \|L^H_0 g\|,
\]
\[
\leq \sqrt{\|L_0 f\|^2 + \|f\|^2} \sqrt{\|g\|^2 + \|L^H_0 g\|^2},
\]
\[
= \|f\| \|g\|_{H(L_0,\Omega)} \|g\|_{H(L_0,\Omega)}.
\]

4. Quasi Gelfand Triples

Normally when we talk about Gelfand triples we have a Hilbert space \(H_0\) and another Hilbert space \(H\) that can be continuously and densely embedded into \(H_0\).

We want to weaken this condition such that the norm of \(H\) isn’t necessarily related to the norm of \(H_0\).

We will have the following setting: Let \((X_0, \langle \cdot, \cdot \rangle_{X_0})\) be a Hilbert space and \(\langle \cdot, \cdot \rangle_{X_+}\) another inner product (not necessarily related to \(\langle \cdot, \cdot \rangle_{X_0}\)) which is defined on a dense \((w.r.t. \|\cdot\|_{X_+})\) subspace \(\tilde{D}_+\) of \(X_0\).

We denote the completion of \(\tilde{D}_+\) \(w.r.t. \|\cdot\|_{X_+} = \sqrt{\langle \cdot \rangle_{X_+}}\) by \(X_+\). This completion is again a Hilbert space with the extension of \(\langle \cdot, \cdot \rangle_{X_+}\), for which we use the same symbol. Now we have that \(\tilde{D}_+\) is dense in \(X_0\) \(w.r.t. \|\cdot\|_{X_0}\) and dense in \(X_+\) \(w.r.t. \|\cdot\|_{X_+}\).

Definition 4.1. Let \(X_0, X_+\) and \(\tilde{D}_+\) be as mentioned in the beginning of this section. Then we define
\[
\|g\|_{X_-} := \sup_{f \in \tilde{D}_+ \setminus \{0\}} \frac{|\langle g, f \rangle_{X_0}|}{\|f\|_{X_+}} \quad \text{and} \quad D_- := \left\{ g \in X_0 : \|g\|_{X_-} < +\infty \right\}.
\]

We denote the completion of \(D_-\) \(w.r.t. \|\cdot\|_{X_-}\) by \(X_-\).

Remark 4.2. By definition of \(D_-\) we can identify every \(g \in D_-\) with an element of \(X_+^0\) by the continuous extension of \(f \in \tilde{D}_+ \mapsto (g, f)_{X_0}\) to \(X_+\). The completion \(X_-\) is isomorphic to the closure of \(D_-\) in \(X_+^0\).

Lemma 4.3. \(D_-\) is complete with respect to \(\|g\|_{X_-}^2 := \|g\|_{X_0}^2 + \|g\|_{X_-}^2\).
Proof. Let \((g_n)_n\in\mathbb{N}\) be a Cauchy sequence in \(D_-\) with respect to \(\|\cdot\|_{X_+}\cap X_0\). Then \((g_n)_n\in\mathbb{N}\) is a convergent sequence in \(X_0\) (w.r.t. \(\|\cdot\|_{X_0}\)) and a Cauchy sequence in \(D_-\) (w.r.t. \(\|\cdot\|_{X_+}\)). We denote the limit in \(X_0\) by \(g_0\). We obtain
\[
\|g_n - g_m, f\|_{X_+} = \lim_{n,m\to\infty} \|g_n, f\|_{X_0} \leq \lim_{n\to\infty} \|g_n\|_{X_+} \leq C\|f\|_{X_+}
\]
and consequently \(g_0 \in D_-\).

Let \(\epsilon > 0\) be arbitrary. Since \((g_n)_n\in\mathbb{N}\) is a Cauchy sequence with respect to \(\|\cdot\|_{X_+}\), there is an \(n_0 \in \mathbb{N}\) such that for all \(f \in D_+\) with \(\|f\|_{X_+} = 1\)
\[
\|g_n - g_m, f\|_{X_+} \leq \frac{\epsilon}{2}, \quad \text{if} \quad n, m \geq n_0
\]
holds true. Furthermore, for all \(f \in D_+\) there exists an \(m_f \geq n_0\) such that
\[
\|g_0 - g_{m_f}, f\|_{X_+} \leq \frac{\epsilon}{2}
\]
This yields
\[
\|g_0 - g_n, f\|_{X_+} \leq \|g_0 - g_{m_f}, f\|_{X_+} + \|g_{m_f} - g_n, f\|_{X_+} \leq \epsilon.
\]
Since the right-hand-side is independent of \(f\), we obtain
\[
\|g_0 - g_n\|_{X_+} = \sup_{f \in D_+} \frac{\|g_0 - g_n, f\|_{X_+}}{\|f\|_{X_+}} \leq \epsilon.
\]
Hence, \(g_0\) is also the limit of \((g_n)_n\in\mathbb{N}\) with respect to \(\|\cdot\|_{X_+}\) and consequently the limit of \((g_n)_n\in\mathbb{N}\) with respect to \(\|\cdot\|_{X_+}\cap X_0}\). \(\square\)

Lemma 4.4. The embedding \(\iota_+: D_+ \subset X_+ \to X_0\) with \(f \mapsto f\) is a dense defined operator with \(\text{ran} \iota_+\) dense and \(\ker \iota_+ = \{0\}\).

Proof. By assumption on \(D_+\) the embedding \(\iota_+\) is dense defined and has a dense range. Clearly, \(\ker \iota_+ = \{0\}\). \(\square\)

Lemma 4.5. Let \(\iota_+^*\) denote the adjoint relation of the embedding mapping \(\iota_+\) in the previous lemma. Then \(\iota_+^*\) is single-valued (mul \(\iota_+^* = \{0\}\)) and \(\ker \iota_+^* = \{0\}\). Its domain coincides with \(D_-\) and \(\iota_+^*: D_- \to X_+\) is isometric w.r.t. \(\|\cdot\|_{X_+}\).

If \(\ker \overline{\iota_+^*} = \{0\}\), then \(\overline{\iota_+}\) is dense in \(X_+\).

Proof. The density of the domain of \(\iota_+\) yields \(\text{mul} \iota_+^* = (\text{dom} \iota_+)\perp = \{0\}\), and \(\overline{\text{ran} \iota_+^*}_0 = X_0\) yields \(\ker \iota_+^* = \{0\}\). The following equivalences show that \(\text{dom} \iota_+^* = D_-:\)
\[
g \in \text{dom} \iota_+^* \iff (g, \iota_+^* f)_{X_0} \text{ is continuous in } f \in D_+ \text{ w.r.t. } \|\cdot\|_{X_+}
\]
\[
\iff \sup_{f \in D_+} \frac{|(g, f)_{X_0}|}{\|f\|_{X_+}} < +\infty
\]
\[
\iff g \in D_-.
\]
For \(g \in D_-\) we have
\[
\|g\|_{X_+} = \sup_{f \in D_+} \frac{|(g, f)_{X_0}|}{\|f\|_{X_+}} = \sup_{f \in D_+} \frac{|\iota_+^* g, f\|_{X_+}|}{\|f\|_{X_+}} = \|\iota_+^* g\|_{X_+},
\]
which proves that \(\iota_+^*\) is isometric.

If \(\ker \overline{\iota_+^*} = \{0\}\), then the following equation yields the density of \(\text{ran} \iota_+^*\) in \(X_+\)
\[
\ker \overline{\iota_+^*} = \ker \iota_+^* = (\text{ran} \iota_+^*)\perp.
\]
\(\square\)

Proposition 4.6. The following assertions are equivalent.
(i) There is a topological vector space \((Z, \mathcal{T})\) that contains \(X_0\) and \(X_+\) such that \(\bar{D}_+ \subseteq X_+ \cap X_0\) in \(Z\), and the topology \(\mathcal{T}\) is coarser (weaker) than the topology of \(\|\cdot\|_{X_+}\) and coarser (weaker) than the topology of \(\|\cdot\|_{X_0}\).

(ii) If \(\bar{D}_+ \supseteq f_n \to 0\) \(\text{w.r.t.}\ \|\cdot\|_{X_+}\) and \(\lim_{n \in \mathbb{N}} f_n\) exists \(\text{w.r.t.}\ \|\cdot\|_{X_0}\) then this limit is also 0 and if \(\bar{D}_+ \not\supseteq f_n \to 0\) \(\text{w.r.t.}\ \|\cdot\|_{X_0}\) and \(\lim_{n \in \mathbb{N}} f_n\) exists \(\text{w.r.t.}\ \|\cdot\|_{X_+}\) then this limit is also 0.

(iii) \(\iota_+ : \bar{D}_+ \subseteq X_+ \to X_0, f \mapsto f\) is closable and its closure is injective.

(iv) \(D_-\) is dense in \(X_0\) and dense in \(X'_+\).

Proof. (i) \(\Rightarrow\) (ii): Let \((f_n)_{n \in \mathbb{N}}\) be a sequence in \(\bar{D}_+\) such that \(f_n \to 0\) \(\text{w.r.t.}\ \|\cdot\|_{X_+}\) and \(f_n \to f\ \text{w.r.t.}\ \|\cdot\|_{X_0}\). Since \(\mathcal{T}\) is coarser than both of the topologies induced by these norms, we also have

\[
\lim_{n \to \infty} f_n \xrightarrow{\mathcal{T}} f \quad \text{in } Z.
\]

Since \(\mathcal{T}\) is Hausdorff, we conclude \(f = 0\). Analogously, we can show the converse statement.

(ii) \(\Rightarrow\) (iii): If \((f_n, f_n)_{n \in \mathbb{N}}\) is a sequence in \(\iota_+\) that converges to \((0, f) \in X_+ \times X_0\), then \(f = 0\) by (ii). Hence, \(\text{mul}_{\mathcal{T}} = \{0\}\) and consequently \(\iota_+\) is closable. Analogously, we can show that \(\ker_{\mathcal{T}} = \{0\}\).

(iii) \(\Rightarrow\) (iv): We have \((\text{dom } \iota'_+)_{\mathcal{P}} = \text{mul}_{\mathcal{T}}\), where \(\mathcal{T}\) is the closure of \(\iota_+\). Since \(\iota_+\) is closable, we have \(\text{mul}_{\mathcal{T}} = \{0\}\), which yields \(\text{dom } \iota'_+ = \{0\}\). By Lemma 4.5 \(\text{dom } \iota'_+ \) coincides with \(D_-\). The second assertion of Lemma 4.5 yields that \(D_-\) is dense in \(X'_+\).

(iv) \(\Rightarrow\) (i): By Lemma 4.3 \(D_-\) is complete with respect to \(\|g\|_{X_+ \cap X_0} := \|g\|_{X_+} + \|g\|_{X_0}^2\). Since \(D_-\) dense in \(X_0\) and embedding into \(X_0\) is continuous, we can construct an ordinary Gelfand triple. Hence, \(Z\), the completion of \(X_0\) with respect to \(\|z\|_Z := \sup_{g \in D_- \setminus \{0\}} \frac{|\langle z, g \rangle_{X_0}|}{\|g\|_{X_+ \cap X_0}}\), is the dual space of \(D_-\) with respect to the pivot space \(X_0\). For \(f \in \bar{D}_+\) we have

\[
\|f\|_Z = \sup_{g \in D_- \setminus \{0\}} \frac{|\langle f, g \rangle_{X_0}|}{\|g\|_{X_+ \cap X_0}} \leq \sup_{g \in D_- \setminus \{0\}} \frac{\|f\|_{X_+} \|g\|_{X_0}}{\|g\|_{X_+ \cap X_0}} \leq \|f\|_{X_+}.
\]

Consequently, we can regard \(X_+\) as a subspace of \(Z\). By contraction the topology of \(\|\cdot\|_Z\) is coarser than the topology of \(\|\cdot\|_{X_0}\) and by the last inequality it is also coarser than the topology of \(\|\cdot\|_{X_+}\).

From now on we will assume that one and therefore all properties in Proposition 4.6 are satisfied. Therefore, \(X_+ \cap X_0\) is well-defined and complete with the norm \(\|\cdot\|_{X_+ \cap X_0} := \sqrt{\|\cdot\|_{X_+}^2 + \|\cdot\|_{X_0}^2}\).

**Lemma 4.7.** \(\bar{D}_+\) is dense in \(X_+ \cap X_0\) with respect to \(\|\cdot\|_{X_+ \cap X_0} := \|\cdot\|_{X_+} + \|\cdot\|_{X_0}^2\).

**Proof.** We define \(P_+ := X_+ \cap X_0 \) and we define \(P_-\) analogously to \(D_-\) in Definition 4.1. Clearly,

\[
\|g\|_{P_-} := \sup_{f \in P_- \setminus \{0\}} \frac{|\langle g, f \rangle_{X_0}|}{\|f\|_{X_+}} \geq \|g\|_{X_-}
\]

(4.1)

and consequently \(P_- \subseteq D_-\). Furthermore, we can define \(\iota_{P_-}\) analogously to \(\iota_+\). Then we have \(\text{dom } \iota_{P_-} = \text{dom } \iota_{P_+}\) and \(\iota_{P_-} \subseteq \iota_{P_+}\) and therefore \(\iota_{P_-} \subseteq \iota_{P_+}\). Let \(f \in P_+\).
Then there exists a sequence \((f_n)_{n \in \mathbb{N}}\) in \(\bar{D}_+\) that converges to \(f\) w.r.t. \(\|\cdot\|_{X_+}\). For \(g \in P_-\) we have
\[
\|g,f\|_{X_+} = \|\iota_{P_-,g,f}\|_{X_+} = \lim_{n \to \infty} \|\iota^*_n g, f_n\|_{X_+} \leq \lim_{n \to \infty} \|\iota^*_n g\|_{X_+} \|f_n\|_{X_+} = \|\iota g\|_{X_+} \|f\|_{X_+},
\]
which yields \(\|g\|_{P_-} \leq \|g\|_{X_-}\). Hence, \(\|\cdot\|_{P_-} = \|\cdot\|_{X_-}\), \(P_- = D_-\), \(\iota^*_P = \iota^*_+\) and \(\iota_P = \Pi^*_+\), which is equivalent to \(X_+ \cap X_0 = \overline{D_+} \cap X_0\).

We define \(D_+ := \overline{D_+} \cap X_0 = X_+ \cap X_0\) and we will denote the extension of \(\iota^*_+\) to \(D_+\), which is its closure, also by \(\iota^*_+\). The adjoint \(\iota^*_+\) is not affected by that.

**Theorem 4.8.** The mapping \(\iota^*_+\) can be uniquely extended to a bijective linear isometry \(\Psi : X_- \to X_+\). The space \(X_-\) is a Hilbert space with the inner product \(\langle g,f \rangle_{X_-} := \langle \Psi g, \Psi f \rangle_{X_+}\). Moreover, the induced norm of this inner product coincides with \(\|\cdot\|_{X_-}\).

**Proof.** By Lemma 4.5 \(\iota^*_+\) is a bounded linear mapping from \(D_-\) to \(X_+\) with \(\text{ran} \iota^*_+\) is dense. Since \(D_-\) is dense in \(X_-\) by construction, we can extend \(\iota^*_+\) by continuity to \(X_-\). We denote this extension by \(\Psi\). For an arbitrary \(g \in X_-\) there exists sequence \((g_n)_{n \in \mathbb{N}}\) in \(D_-\) that converges to \(g\). Hence,
\[
\|
\Psi g
\|_{X_+} = \lim_{n \to \infty} \|
\Psi g_n
\|_{X_+} = \lim_{n \to \infty} \|
\iota^*_n g_n
\|_{X_+} = \lim_{n \to \infty} \|
\Psi g
\|_{X_-} = \|
\Psi g
\|_{X_-}.
\]
This yields that \(\text{ran} \Psi\) is closed in \(X_+\). Since \(\text{ran} \Psi\) also contains the dense subspace \(\text{ran} \iota^*_+\), the mapping \(\Psi\) is surjective.

Since \(\Psi\) is bijective it is easy to see that \(X_-\) is a Hilbert space with the given inner product. By (4.2), we have
\[
\langle g,f \rangle_{X_-} = \langle \Psi g, \Psi f \rangle_{X_+} = \|
\Psi g
\|_{X_+}^2 = \|
\Psi g
\|_{X_-}^2.
\]

**Corollary 4.9.** The Hilbert space \(X_-\) can be identified with the (anti)dua space of \(X_+\) by
\[
\Lambda : \begin{cases}
X_- & \rightarrow X_+^*, \\
g & \mapsto \langle \Psi g, \cdot \rangle_{X_+},
\end{cases}
\]
where \(\Psi\) is the mapping from Theorem 4.8.

**Definition 4.10.** For \(f \in X_+\) and \(g \in X_-\) we define
\[
\langle g,f \rangle_{X_-X_+} := \langle \Lambda g, f \rangle_{X_+^*X_+} = \langle \Psi g, f \rangle_{X_+}.
\]
We call \((X_+, X_0, X_-)\) with this duality a quasi Gelfand triple. The space \(X_0\) will be refered as the pivot space and \(\Psi\) as the duality map in this setting.

Figure 1 illustrates the setting of a quasi Gelfand triple.

**Remark 4.11.** For \(f \in D_+\) and \(g \in D_-\) we have
\[
\langle g,f \rangle_{X_-X_+} = \langle \Psi g, f \rangle_{X_+} = \langle \iota^*_+ g, f \rangle_{X_+} = \langle g,f \rangle_{X_0}.
\]
Since these two sets are dense in \(X_+\) and \(X_-\) respectively, we have for \(f \in X_+\) and \(g \in X_-\)
\[
\langle g,f \rangle_{X_-X_+} = \lim_{(n,m) \in \mathbb{N}^2} \langle g_n, f_m \rangle_{X_0},
\]
where \((f_m)_{m \in \mathbb{N}}\) is a sequence in \(D_+\) that converges to \(f\) and \((g_n)_{n \in \mathbb{N}}\) is a sequence in \(D_-\) that converges to \(g\).
In contrast to “ordinary” Gelfand triple, the setting for quasi Gelfand triple is somehow “symmetric”, i.e. the roles of $X_+$ and $X_-$ are interchangeable. If we start with $D_-$, then

$$
\mathcal{L}_- : \begin{cases} 
D_- \subseteq X_- & \to \ X_0, \\
g \mapsto g, 
\end{cases}
$$

is closed by Lemma 4.3 and $(D_-)_- = D_+$. It is also easy to verify that the unitary operator from $X_+$ to $X_-$ resulting from the extension of $\iota^*$ is $\Psi^*$. In order to restore $\iota^*$ from $\Psi^*$ we only have to restrict $\Psi^*$ to $D_+$ or more exactly

$$
\iota^* = \Psi^* \iota_+^{-1}.
$$

(4.3)

**Proposition 4.12.** The space $D_+ \cap D_-$ is complete with respect to $\| \cdot \|_{X_+ \cap X_-} := \sqrt{\| \cdot \|_{X_+}^2 + \| \cdot \|_{X_-}^2}$.

**Proof.** For $f \in D_+ \cap D_-$ we have

$$
\|f\|_{X_0}^2 = \langle f, f \rangle_{X_0} = \|f\|_{X_+ \cap X_-} \leq \|f\|_{X_+} \|f\|_{X_-} \leq \|f\|_{X_+}^2 + \|f\|_{X_-}^2.
$$

Hence, every Cauchy sequence in $D_+ \cap D_-$ with respect to $\| \cdot \|_{X_+ \cap X_-}$ is also a Cauchy sequence with respect to $\| \cdot \|_{X_0}$, $\| \cdot \|_{X_+}$ and $\| \cdot \|_{X_-}$.

Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $D_+ \cap D_-$ with respect to $\| \cdot \|_{X_+ \cap X_-}$. Then the limit with respect to $\| \cdot \|_{X_0}$ and the limit with respect to $\| \cdot \|_{X_+}$ coincide by Lemma 4.3. Furthermore, by the closedness of $\iota_+$ the limit with respect to $\| \cdot \|_{X_0}$ and the limit with respect to $\| \cdot \|_{X_+}$ also coincide. Therefore, all these limits have to coincide and $(f_n)_{n \in \mathbb{N}}$ converges to that limit in $\| \cdot \|_{X_+ \cap X_-}$.

**Lemma 4.13.** The operator

$$
\begin{bmatrix} \iota_+ & \iota_- \end{bmatrix} : \begin{cases} 
D_+ \times D_- \subseteq X_+ \times X_- & \to \ X_0, \\
\begin{bmatrix} f \\ g \end{bmatrix} & \mapsto f + g,
\end{cases}
$$

is closed.

**Proof.** Let $((f_n, g_n))_{n \in \mathbb{N}}$ be a sequence in $\begin{bmatrix} \iota_+ & \iota_- \end{bmatrix}$ that converges to $((f, z)) \in X_+ \times X_- \times X_0$. Then we have

$$
\|z\|_{X_0}^2 = \lim_{n \to \infty} \|f_n + g_n\|_{X_0}^2 = \lim_{n \to \infty} (\|f_n\|_{X_0}^2 + \|g_n\|_{X_0}^2 + 2 \Re \langle f_n, g_n \rangle_{X_0}).
$$

Since $2 \Re \langle f_n, g_n \rangle_{X_0}$ converges to $2 \Re \langle f, g \rangle_{X_+ \cap X_-}$, we conclude that $\|f_n\|_{X_0}$ and $\|g_n\|_{X_0}$ are bounded. Hence, it exists a subsequence $(f_{n(k)})_{k \in \mathbb{N}}$ that converges weakly to an $f \in X_0$. Moreover, by Lemma A.2 there is a further subsequence such that $\frac{1}{j} \sum_{i=1}^{j} f_{n(k(i))}$ converges to $f$ strongly. The sequence $\left\{ \frac{1}{j} \sum_{i=1}^{j} f_{n(k(i))} \right\}_{j \in \mathbb{N}}$ has
Applying Theorem 4.15 to 
Proof.

Corollary 4.16. The set $D_+ \cap D_-$ is dense in $X_0$ with respect to $\| \cdot \|_{X_0}$.

Proof. By $\text{dom} \, \iota_+^* = D_+$ we have

$$X_0 = (\text{mul} \, [\iota_+ \iota_-])^\perp = \text{dom} \, [\iota_+ \iota_-] = \text{dom} \, \iota_+^* \cap \text{dom} \, \iota_-^* = D_+ \cap D_-.$$

The following theorem can be found in [Yos80, Theorem 2 p. 200], we just changed that the operator maps into a different space. Hence, we provide a proof.

Theorem 4.15. Let $T$ be a closed linear operator from the Hilbert spaces $X$ to the Hilbert space $Y$. Then $T^T$ and $TT^*$ are self-adjoint, and $(I_X + T^T)$ and $(I_Y + TT^*)$ are boundedly invertible.

Proof. Since $T^* = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} T^\perp$, we have $T \oplus \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} T^* = X \times Y$. Hence, for $(h, 0) \in X \times Y$ there are unique $x \in \text{dom} \, T$ and $y \in \text{dom} \, T^*$ such that

$$(h, 0) = (x, Tx) + (-T^* y, y).$$

(4.4)

Consequently, $h = x - T^* y$ and $y = -Tx$, which implies $x \in \text{dom} \, T^T$ and

$$h = x + T^T x.$$

Because of the uniqueness of the decomposition in (4.4), $x \in \text{dom} \, T^T$ is uniquely determined by $h \in X$. Therefore, $(I_X + T^T)^{-1}$ is a well-defined and everywhere defined operator.

For $h_1, h_2 \in X$, we define $x_1 := (I_X + T^T)^{-1} h_1$ and $x_2 := (I_X + T^T)^{-1} h_2$. Then $x_1, x_2 \in \text{dom} \, T^T$ and, by the closedness of $T$, $T^** = T$. Hence,

$$\langle (I_X + T^T) h_1, x_2 \rangle = \langle (I_X + T^T) x_1, x_2 \rangle = \langle x_1, x_2 \rangle + \langle T^T x_1, x_2 \rangle = \langle x_1, x_2 \rangle + \langle x_1, T^T x_2 \rangle = \langle x_1, (I_X + T^T) x_2 \rangle = \langle (I_X + T^T)^{-1} h_1, h_2 \rangle,$$

which yields that $(I_X + T^T)^{-1}$ is self-adjoint. Therefore $(I_X + T^T)$ and $T^T$ are also self-adjoint. Moreover, $(I_X + T^T)^{-1}$ is bounded as a closed and everywhere defined operator.

By $TT^* = (T^*)^* (T^*)$ the other statements follow by the already shown.

Applying this theorem to $S = \lambda T$ implies that $\mathbb{R}_-$ is contained in the resolvent set of $T^T$.

Corollary 4.16. The set $D_+ \cap D_-$ is dense in $X_+$ and $X_-$ with respect to their corresponding norms.

Proof. Applying Theorem 4.15 to $\iota_+$ yields $\iota_+^* \iota_+$ is self-adjoint. Hence, $\text{dom} \, \iota_+^* \iota_+$ is dense in $X_+$. By Lemma 4.5 $\text{dom} \, \iota_+^* \iota_- = D_+ \cap D_-$. An analogous argument for $\iota_- \iota_+^*$ yields $D_+ \cap D_- = D_+ \cap D_-$. 

Corollary 4.17. $D_+ + D_- = X_0$.

Proof. Applying Theorem 4.15 to $\iota_+$ yields $(I_{X_0} + \iota_+ \iota_+^*)$ is onto. Hence, for every $x \in X_0$ there exists a $g_x \in \text{dom} \, \iota_+ \iota_+^* \subseteq D_-$ such that

$$x = \underbrace{g_x}_{\in D_-} + \underbrace{\iota_+ \iota_+^* g_x}_{\in D_+}.$$ 

Since $g_x \in \text{dom} \, \iota_+ \iota_+^*$, we have $\iota_+^* g_x \in D_+$ and consequently $x \in D_+ + D_-$. 

$\blacksquare$
Proposition 4.18. Let $T$ be a bounded and boundedly invertible mapping on $\mathcal{X}_0$. Then $P_+ := TD_+$ equipped with $\|f\|_{\mathcal{Y}_+} := \|T^{-1}f\|_{\mathcal{X}_+}$ establishes a quasi Gelfand triple $(\mathcal{Y}_+, \mathcal{X}_0, \mathcal{Y}_-)$, where $\mathcal{Y}_+$ is the completion of $P_+$ and $\mathcal{Y}_-$ is the completion of $P_-$ defined as in Definition 4.1 where $D_+$ is replaced by $P_+$. Moreover, $P_- = (T^*)^{-1}D_-$, $\|g\|_{\mathcal{Y}_-} = \|T^*g\|_{\mathcal{X}_+}$, and $T$ and $(T^*)^{-1}$ can be continuously extended to linear bounded and boundedly invertible mappings from $\mathcal{X}_+$ and $\mathcal{X}_-$ to $\mathcal{Y}_+$ and $\mathcal{Y}_-$ respectively.

Proof. The mapping $T|_{D_+}$ is also bounded and boundedly invertible if we equipped its domain with $\|\cdot\|_{\mathcal{X}_+}$ and its codomain with $\|\cdot\|_{\mathcal{Y}_+}$. So the linear relation $\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}t_+ = \{(Tf, Tg) : (f, g) \in t_+\} \subseteq \mathcal{Y}_+ \times \mathcal{X}_0$ is closed. Since this linear relation coincides with the embedding $t_{P_+} : P_+ \subseteq \mathcal{Y}_+ \rightarrow \mathcal{X}_0$, $f \mapsto f$, Proposition 4.6 yields that all assumptions for a quasi Gelfand triple are satisfied. For $g \in \mathcal{X}_0$ we have

$$\|g\|_{\mathcal{Y}_-} = \sup_{h \in P_+} \frac{|\langle g, h \rangle|_{\mathcal{X}_0}}{\|h\|_{\mathcal{Y}_+}} = \sup_{f \in D_+} \frac{|\langle g, Tf \rangle|_{\mathcal{Y}_0}}{\|Tf\|_{\mathcal{Y}_+}} = \sup_{f \in D_+} \frac{|\langle T^*g, f \rangle|_{\mathcal{X}_0}}{\|f\|_{\mathcal{X}_+}} = \|T^*g\|_{\mathcal{X}_+}.$$  

Corollary 4.19. Let $S, T$ be a bounded and boundedly invertible mappings on $\mathcal{X}_0$. Then $\begin{bmatrix} ST|_{D_+} & S(T^*)^{-1}|_{D_-} \end{bmatrix}$ is a closed linear relation between $\mathcal{X}_+ \times \mathcal{X}_-$ and $\mathcal{X}_0$ with $\text{ran} \begin{bmatrix} ST|_{D_+} & S(T^*)^{-1}|_{D_-} \end{bmatrix} = \mathcal{X}_0$.

Proof. Let $P_+ = TD_+$. Then by Proposition 4.18 the corresponding $P_-$ can be obtained by $(T^*)^{-1}D_-$. The mapping

$$\Xi : \begin{bmatrix} \mathcal{X}_+ \times \mathcal{X}_- \times \mathcal{X}_0 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y}_+ \times \mathcal{Y}_- \times \mathcal{X}_0 \end{bmatrix}$$

$$\begin{bmatrix} f \\ g \\ z \end{bmatrix} \mapsto \begin{bmatrix} T & 0 & 0 \\ 0 & (T^*)^{-1} & 0 \\ 0 & 0 & S \end{bmatrix} \begin{bmatrix} f \\ g \\ z \end{bmatrix}$$

is linear bounded and boundedly invertible, where $\mathcal{Y}_+$ and $\mathcal{Y}_-$ are the spaces corresponding to $P_+$ and $P_-$ from Proposition 4.18. Since $(\mathcal{Y}_+, \mathcal{X}_0, \mathcal{Y}_-)$ is a quasi Gelfand triple, $[t_{P_+} \ t_{P_-}]$ is closed in $\mathcal{Y}_+ \times \mathcal{Y}_- \times \mathcal{X}_0$ and therefore also its pre-image under $\Phi$

$$\Xi^{-1}([t_{P_+} \ t_{P_-}]) = \begin{bmatrix} T^{-1} & 0 & 0 \\ 0 & T^* & 0 \\ 0 & 0 & S \end{bmatrix} \begin{bmatrix} t_{P_+} \\ t_{P_-} \end{bmatrix} = [ST_{t_+} \ S(T^*)^{-1}t_-]$$

is closed. Furthermore,

$$\text{ran} \begin{bmatrix} ST|_{D_+} & S(T^*)^{-1}|_{D_-} \end{bmatrix} = S\text{ran} [t_{P_+} \ t_{P_-}] = S\mathcal{X}_0 = \mathcal{X}_0.$$  

Lemma 4.20. Let $T$ be a bounded and boundedly invertible mapping on $\mathcal{X}_0$ and $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ be a quasi Gelfand triple such that $(\mathcal{X}_+, B_1, \Phi B_2)$ is a boundary triple for an operator $A$. Furthermore, let $\mathcal{Y}_+$ and $\mathcal{Y}_-$ be as defined in Proposition 4.18. Then $(\mathcal{Y}_+, \mathcal{X}_0, \mathcal{Y}_-)$ is also a quasi Gelfand triple such that $(\mathcal{Y}_+, TB_1, \Phi(T^*)^{-1}B_2)$ is a boundary triple for $A$, where $\Phi$ denotes the duality map of $(\mathcal{Y}_+, \mathcal{X}_0, \mathcal{Y}_-)$.

Proof. By Proposition 4.18 $(\mathcal{Y}_+, \mathcal{X}_0, \mathcal{Y}_-)$ is a quasi Gelfand triple. Note that $T$ and $(T^*)^{-1}$ can be extended to mappings from $\mathcal{X}_+$ to $\mathcal{Y}_+$ and $\mathcal{X}_-$ to $\mathcal{Y}_-$ respectively. For $x, y \in \text{dom} A$ we have

$$\langle B(x, \Phi B_2 y)_{\mathcal{X}_+} = \langle B_1 x, B_2 y \rangle_{\mathcal{X}_+, \mathcal{X}_-} = \langle TB_1 x, (T^*)^{-1} B_2 y \rangle_{\mathcal{Y}_+, \mathcal{Y}_-} = (TB_1 x, \Phi(T^*)^{-1} B_2 y)_{\mathcal{Y}_+}.$$  

$\square$
5. Boundary Spaces

In this section we will construct a suitable boundary space $V_L$ (Definition 5.5), where we will later formulate boundary conditions. This space will provide a quasi Gelfand triple with a subspace of $L^2(\partial \Omega)$ as pivot space.

**Definition 5.1.** We say $(\Gamma_j)_{j=1}^k$, where $\Gamma_j \subseteq \partial \Omega$, is a splitting with thin boundaries of $\partial \Omega$, if

(i) $\bigcup_{j=1}^k \Gamma_j = \partial \Omega$,

(ii) the sets $\Gamma_j$ are pairwise disjoint,

(iii) the sets $\Gamma_j$ are relatively open in $\partial \Omega$,

(iv) the boundaries of $\Gamma_j$ have zero measure w.r.t. the surface measure of $\partial \Omega$.

For $\Gamma \subseteq \partial \Omega$ we will denote by $P_\Gamma$ the orthogonal projection from $L^2(\partial \Omega)^{m_1}$ on $L^2_0(\Gamma) := \text{ran} P_\Gamma \subseteq L^2(\Gamma)^{m_1}$. Therefore, we can adapt (3.3) such that

$$\langle L_0 f, g \rangle_{L^2(\Omega)^{m_1}} + \langle f, L^H_0 g \rangle_{L^2(\Omega)^{m_2}} = \langle L_0 \gamma_0 f, P_{\partial \Omega} \gamma_0 g \rangle_{L^2(\partial \Omega)^{m_1}}.$$  

We define $\pi^\Gamma_L : H^1(\Omega)^{m_1} \to L^2_0(\Gamma) \subseteq L^2(\Gamma)^{m_1}$ by $\pi^\Gamma_L := P_\Gamma \gamma_0$ and $\pi_L := \pi^\partial L$. Since both $P_\Gamma$ and $\gamma_0$ are continuous, the mapping $\pi^\Gamma_L$ is also continuous. Therefore, ker $\pi^\Gamma_L$ is closed. Note that $P_\Gamma = 1_{\Gamma} P_{\partial \Omega}$ and consequently $\pi^\Gamma_L = 1_{\Gamma} \pi_L$, and $1_{\Gamma} L_0 = L_0 1_{\Gamma}$.

**Example 5.2.** Let $L$ be as in Example 3.5. Then $L_0 f = \langle f, \nu \rangle_{L^2}$ and $L_0$ is certainly surjective. Therefore, $L^2_0(\partial \Omega) = L^2(\partial \Omega)$, $\pi_L = \gamma_0$ and $\pi_L = 1_{\Gamma} \gamma_0$. Since $L^H_0 = \text{grad}$, we have that $H(L^H_0, \Omega) = H^1(\Omega)$.

**Lemma 5.3.** Let $\Gamma \subseteq \partial \Omega$ be relatively open. Then the subspace ker $\pi^\Gamma_L$ is closed in $H^1(\Omega)^{m_1}$ with respect to $\|\cdot\|_{H^1(\Omega)^{m_1}}$. This can also be formulated as

$$\Lambda \pi^\Gamma_L : H^1(\Omega)^{m_1} \cap H^1(\Omega)^{m_1} = \ker \pi^\Gamma_L.$$  

**Proof.** Let $(g_n)_{n \in \mathbb{N}}$ be a sequence in ker $\pi^\Gamma_L$, which converges to $g \in H^1(\Omega)^{m_1}$ with respect to $\|\cdot\|_{H^1(\Omega)^{m_1}}$. By Corollary 3.17 we have for an arbitrary $f \in H^1(\Omega)^{m_2} := \{ f \in H^1(\Omega)^{m_2} : 1_{\partial \Omega} \gamma_0 f = 0 \}$

$$\langle L_0 \gamma_0 f, \pi^\Gamma_L (g - g_n) \rangle_{L^2(\Gamma)} = \| f \|_{H(\partial \Omega, \Omega)} \| g - g_n \|_{H(L^H_0, \Omega)}.$$  

Since $\pi^\Gamma_L (g - g_n) = \pi^\Gamma_L g$ and the right-hand-side converges to 0, we can see that $\pi^\Gamma_L g \perp L_0 \gamma_0 H^1(\partial \Omega)^{m_2}$. By [TW09, Th. 13.6.10, Re. 13.6.12] $\gamma_0 H^1(\partial \Omega)^{m_2}$ is dense in $L^2(\Gamma)^{m_2}$, which implies $\pi^\Gamma_L g \perp \text{ran} P_\Gamma L_0$. By definition $\pi^\Gamma_L g$ is also in $\text{ran} 1_{\Gamma} L_0$, which leads to $\pi^\Gamma_L g = 0$. Hence, ker $\pi^\Gamma_L$ is closed in $H^1(\Omega)^{m_1}$ with respect to $\|\cdot\|_{H(L^H_0, \Omega)}$.

By the previous lemma we can endow $M_\Gamma := \text{ran} \pi^\Gamma_L$ with the norm

$$\| \phi \|_{M_\Gamma} := \inf \left\{ \| g \|_{H(L^H_0, \Omega)} : \pi^\Gamma_L g = \phi \right\},$$  

which makes it a pre-Hilbert space. The next lemma will clarify that.

**Lemma 5.4.** The space $(M_\Gamma, \| \cdot \|_{M_\Gamma})$ is a pre-Hilbert space. Furthermore, its completion denoted by $(\overline{M_\Gamma}, \| \cdot \|_{M_\Gamma})$ is isomorphic to $H(L^H_0, \Omega) / \ker \pi^\Gamma_L$. The mapping $\pi^\Gamma_L$ can be continuously extended to $H(L^H_0, \Omega)$. For the kernel of the extension $\overline{\pi^\Gamma_L}$ we have ker $\overline{\pi^\Gamma_L} = \ker \pi^\Gamma_L$. 

Proof. By Lemma 5.3 \( \ker \pi_L^1 \) is closed in \( H^1(\Omega)^m_1 \) with respect to \( \| \cdot \|_{H(L^m_0, \Omega)} \), which implies that \( \left( H^1(\Omega)^m_1/ \ker \pi_L^1, \| \cdot \|_{H(L^m_0, \Omega)/\ker \pi_L^1} \right) \) is a normed space. Since
\[
\| g \|_{H(L^m_0, \Omega)/\ker \pi_L^1} = \| \pi_L^1 g \|_{M_r},
\]
it is straightforward that \( (M_r, \| \cdot \|_{M_r}) \) is isomorphic to \( \left( H^1(\Omega)^m_1/ \ker \pi_L^1, \| \cdot \|_{H(L^m_0, \Omega)/\ker \pi_L^1} \right) \).

Clearly, \( (M_r, \| \cdot \|_{M_r}) \) has a completion \( (\overline{M_r}, \| \cdot \|_{\overline{M_r}}) \). By Definition of the norm \( \| \cdot \|_{M_r} \) we have for every \( g \in H^1(\Omega)^m_1 \)
\[
\| \pi_L^1 g \|_{\overline{M_r}} = \| \pi_L^1 g \|_{M_r} \leq \| g \|_{H(L^m_0, \Omega)}.
\]
Therefore, we can extend \( \pi_L^1 \) by continuity on \( H(L^m_0, \Omega) \). To avoid confusion in this proof we will use the symbol \( \overline{\pi_L^1} \) for this extension.

Let \( g \in H(L^m_0, \Omega) \) and \( (g_n)_{n \in \mathbb{N}} \) a sequence in \( H^1(\Omega)^m_1 \) which converges to \( g \). Then we have
\[
\| \overline{\pi_L^1} g \|_{\overline{M_r}} = \lim_{n \to \infty} \| \pi_L^1 g_n \|_{M_r} = \lim_{n \to \infty} \inf_{k \in \ker \pi_L^1} \| g_n + k \|_{H(L^m_0, \Omega)}.
\]
The triangular inequality yields
\[
\inf_{k \in \ker \pi_L^1} \| g + k \| - \| g_n - g \| \leq \inf_{k \in \ker \pi_L^1} \| g_n + k \| \leq \inf_{k \in \ker \pi_L^1} \| g + k \| + \| g_n - g \|.
\]
Hence, we have \( \| \overline{\pi_L^1} g \|_{\overline{M_r}} = \inf_{k \in \ker \pi_L^1} \| g + k \| = \inf_{k \in \ker \pi_L^1} \| g + k \| \) and consequently \( H(L^m_0, \Omega)/\ker \pi_L^1 \) is isomorphic to \( \overline{\pi_L^1} \). Since \( H(L^m_0, \Omega)/\ker \pi_L^1 \) is a Hilbert space, in particular complete, and \( M_r \subseteq \overline{\pi_L^1} \subseteq \overline{M_r} \), we have \( \overline{M_r} = \overline{\pi_L^1} \), which makes \( \overline{M_r} \) also a Hilbert space. We will use the symbol \( \pi_L^1 \) also for its continuous extension \( \overline{\pi_L^1} \).

**Definition 5.5.** Let \( \Gamma_0, \Gamma_1 \subseteq \partial \Omega \) be a splitting with thin boundaries. Then we define
\[
H_{\Gamma_0}(L^m_0, \Omega) := \ker \pi_L^1 \quad \text{and} \quad \mathcal{V}_{L,\Gamma_1} := \overline{\pi_L^1 L_{\Gamma_0}(L^m_0, \Omega)},
\]
where we endow \( H_{\Gamma_0}(L^m_0, \Omega) \) with \( \| \cdot \|_{H(L^m_0, \Omega)} \) and \( \mathcal{V}_{L,\Gamma_1} \) with \( \| \cdot \|_{\mathcal{V}_{L,\Gamma_1}} := \| \cdot \|_{\overline{M_r}} \). Instead of \( \mathcal{V}_{L,\partial \Omega} = \overline{\pi_L^1 L(\partial \Omega)} \) we just write \( \mathcal{V}_L \).

**Example 5.6.** Continuing Example 5.2 yields \( H_{\Gamma_0}(L^m_0, \Omega) = H^1(\Omega)^m_1 \) which already appeared in the proof of Lemma 5.3. Moreover, we have \( \pi_L = \gamma_0, \pi_L^1 = 1_{\Gamma_0 \setminus \gamma_0}, \mathcal{V}_L = H^{1/2}(\partial \Omega), \) and \( \mathcal{V}_{L,\Gamma_1} = \{ f \in H^{1/2}(\partial \Omega) : f|_{\Gamma_0} = 0 \} \).

**Remark 5.7.** \( H_{\Gamma_0}(L^m_0, \Omega) \) is again a Hilbert space and \( H^1(\Omega)^m_1 \cap H_{\Gamma_0}(L^m_0, \Omega) \) is dense in \( H_{\Gamma_0}(L^m_0, \Omega) \). The density follows from the assertion \( \ker \pi_L^1 |_{H^1(\Omega)^m_1} = \ker \pi_L^1 \) of the previous lemma. Moreover, \( \mathcal{V}_{L,\Gamma_1} \) is closed in \( \mathcal{V}_L \), since \( \pi_L^0 \circ \pi_L^{-1} \) is well-defined and continuous, and \( \mathcal{V}_{L,\Gamma_1} = \ker \pi_L^0 \circ \pi_L^{-1} \). Hence, \( \mathcal{V}_{L,\Gamma_1} \) is also a Hilbert space.

For \( g \in H^1(\Omega)^m_1 \cap H_{\Gamma_0}(L^m_0, \Omega) \), we have that \( \pi_L g = \pi_L^0 g \) as elements of \( L^2(\partial \Omega) \). So somehow it is possible to say that \( \mathcal{V}_{L,\Gamma_1} = \overline{\mathcal{V}_L} \), but the norms are different.

**Proposition 5.8.** The mapping \( 1_{\Gamma_0 \setminus \gamma_0} : H^1(\Omega)^m_2 \to L^2(\Gamma_1) \) can be extended to a linear continuous mapping
\[
1_{\Gamma_0 \setminus \gamma_0} : H(\partial \Omega, \Omega) \to \mathcal{V}_{L,\Gamma_1}.
\]
such that $\|1\Gamma_1 L_\nu f\|_{\mathcal{V}'_{L,\Gamma}} \leq \|f\|_{H(L_\nu, \Omega)}$.

**Proof.** Let $f \in H^1(\Omega)^{m_2}$. For $g \in H^1(\Omega)^{m_1} \cap H_{\Gamma_0}(L^H_0, \Omega)$ we have by Corollary 3.17

$$\|1\Gamma_1 L_\nu \gamma \phi f, \pi_L g\|_{L^2(\Gamma_1)} = \|L_\nu \gamma \phi f, \pi_L g\|_{L^2(\Gamma_1)} \leq \|f\|_{H(L_\nu, \Omega)} \|g\|_{H(L^H_0, \Omega)}.$$

By Remark 5.7 it is easy to see that the subspace $M := \text{ran } \pi_L \big|_{H^1(\Omega)^{m_1} \cap H_{\Gamma_0}(L^H_0, \Omega)} \subseteq L^2(\Gamma_1)^{m_1}$ of $\mathcal{V}_{L,\Gamma_1}$ is dense. For $\phi \in M$ there exists at least one $g \in H^1(\Omega)^{m_1} \cap H_{\Gamma_0}(L^H_0, \Omega)$ such that $\pi_L g = \phi$. Hence, we can rewrite the inequality by

$$\|1\Gamma_1 L_\nu \gamma \phi f, \phi\|_{L^2(\Gamma_1)} \leq \inf_{\pi_L g = \phi} \|f\|_{H(L_\nu, \Omega)} \inf_{\pi_L g = \phi} \|g\|_{H(L^H_0, \Omega)}.$$

We will extend the mapping $\phi \mapsto \|1\Gamma_1 L_\nu \gamma \phi f, \phi\|_{L^2(\Gamma_1)}$ by continuity on $\mathcal{V}_{L,\Gamma}$. We will denote this extension by $\Xi_f$. Therefore, we have

$$\|\Xi_f(\phi)\| \leq \|f\|_{H(L_\nu, \Omega)} \|\phi\|_{\mathcal{V}_{L,\Gamma_1}}.$$

This means that the mapping $f \mapsto \Xi_f$ is continuous, if we endow $H^1(\Omega)^{m_2}$ with $\|\cdot\|_{H(L_\nu, \Omega)}$. Once again, we will extend this mapping by continuity on $H(L_\nu, \Omega)$ and denote it by $1\Gamma_1 L_\nu.$

Instead of writing $1\partial \Omega L_\nu$ we will just write $L_\nu$.

**Remark 5.9.** In fact the extension of the $L^2(\Gamma_1)$ scalar product in the previous proof is nothing else but

$$(1\Gamma_1 L_\nu f)(\phi) = \langle L_\nu f, g \rangle \big|_{L^2(\Omega)^{m_1}} + \langle f, L^H_0 g \rangle \big|_{L^2(\Omega)^{m_2}},$$

where $g \in H_{\Gamma_0}(L^H_0, \Omega)$ is any element that satisfies $\pi_L g = \phi$.

**Remark 5.10.** Since $\mathcal{V}_{L,\Gamma_1}$ is a subspace of $\mathcal{V}_{L,\partial \Omega} = \mathcal{V}_L$ every element of $\mathcal{V}'_L$ can also be treated as an element of $\mathcal{V}'_{L,\Gamma_1}$. By definition of $1\Gamma_1 L_\nu$ and $L_\nu$ it is easy to see that $1\Gamma_1 L_\nu f = L_\nu f \big|_{\mathcal{V}_{L,\Gamma_1}}$ or equivalently $1\Gamma_1 L_\nu f$ and $L_\nu f$ coincide as elements of $\mathcal{V}_{L,\Gamma}$. The reason for even defining $1\Gamma_1 L_\nu$ is that the range of its restriction to $H^1(\Omega)^{m_2}$ is also contained in $L^2(\Gamma_1)$, which will be important for getting a quasi Gelfand triple.

**Corollary 5.11.** For $f \in H(L_\nu, \Omega)$ and $g \in H(L^H_0, \Omega)$ we have

$$\langle L_\nu f, g \rangle \big|_{L^2(\Omega)^{m_1}} + \langle f, L^H_0 g \rangle \big|_{L^2(\Omega)^{m_2}} = \langle L_\nu f, \pi_L g \rangle \big|_{\mathcal{V}'_{L,\Omega}} + \langle L^H_0 f, \pi_L g \rangle \big|_{\mathcal{V}'_{L,\Omega}},$$

and for $h \in H(L^H_0, \Omega)$ such that $L_\nu L^H_0 h \in L^2(\Omega)^{m_1}$ we have

$$\langle L_\nu L^H_0 g, h \rangle \big|_{L^2(\Omega)^{m_1}} + \langle L^H_0 h, L^H_0 g \rangle \big|_{L^2(\Omega)^{m_2}} = \langle L_\nu L^H_0 h, \pi_L g \rangle \big|_{\mathcal{V}'_{L,\Omega}}.$$

(5.2)

**Proof.** Since $H^1(\Omega)^{m_2}$ is dense in $H(L_\nu, \Omega)$ and $H^1(\Omega)^{m_1}$ is dense in $H(L^H_0, \Omega)$, the first equation follows from (5.1) by continuity. Switching the roles of $L_\nu$ and $L^H_0$ yields the second equation.

For the second assertion set $f = L^H_0 h$ in the first equation.

**Remark 5.12.** For $g \in H_0(L^H_0, \Omega)$ there is a sequence $(g_n)_{n \in \mathbb{N}}$ in $\mathcal{D}(\Omega)$ converging to $g$, which yields $\pi_L g = \lim_{n \to \infty} \pi_L g_n = 0$. Therefore, $H_0(L^H_0, \Omega) \subseteq \ker \pi_L = H_{\partial \Omega}(L^H_0, \Omega)$. On the other hand, if $g \in H_{\partial \Omega}(L^H_0, \Omega)$, then

$$\langle L_\nu f, g \rangle \big|_{L^2(\Omega)^{m_1}} + \langle f, L^H_0 g \rangle \big|_{L^2(\Omega)^{m_2}} = \langle L_\nu f, \pi_L g \rangle \big|_{\mathcal{V}'_{L,\Omega}} = 0.$$
Hence, by Lemma 3.14 \( g \in H_0(L_0^H, \Omega) \). Consequently, \( H_0(L_0^H, \Omega) = H_{01}(L_0^H, \Omega) \).

Clearly the same holds true for \( H(L_0, \Omega) \).

**Theorem 5.13.** The mapping \( \hat{L}_\nu : H(L_0, \Omega) \to \mathcal{V}_L' \) is linear, bounded and onto.

**Proof.** By Proposition 5.8 we already know that \( \hat{L}_\nu \) is linear and bounded, and maps \( H(L_0, \Omega) \) into \( \mathcal{V}_L' \).

Let \( \mu \in \mathcal{V}_L' \) be arbitrary. Since \( \pi_L \) is continuous from \( H(L_0^H, \Omega) \) to \( \mathcal{V}_L \), the mapping \( g \mapsto \langle \mu, \pi_L g \rangle_{\mathcal{V}_L' \cdot \mathcal{V}_L} \) is also continuous. Consequently, there exists an \( h \in H(L_0^H, \Omega) \) such that

\[
\langle h, g \rangle_{H(L_0^H, \Omega)} = \langle \mu, \pi_L g \rangle_{\mathcal{V}_L' \cdot \mathcal{V}_L} \quad \text{for all} \quad g \in H(L_0^H, \Omega).
\]

For a test function \( v \in \mathcal{D}(\Omega)^{m_2} \) we have

\[
0 = \langle \mu, \pi_L v \rangle_{\mathcal{V}_L' \cdot \mathcal{V}_L} = \langle h, v \rangle_{H(L_0^H, \Omega)} = \langle h, v \rangle_{L^2(\Omega)^{m_1}} + \langle L_0^H h, L_0^H v \rangle_{L^2(\Omega)^{m_2}} \\
= \langle (I - L_0 L_0^H) h, v \rangle_{\mathcal{D}'(\Omega)^{m_1}} + \langle L_0^H h, L_0^H v \rangle_{\mathcal{D}'(\Omega)^{m_1}} \\
= \langle (I - L_0 L_0^H) h, v \rangle_{\mathcal{D}'(\Omega)^{m_1}} + \langle L_0^H h, L_0^H v \rangle_{\mathcal{V}_L' \cdot \mathcal{V}_L}.
\]

This means \( L_0 L_0^H h = h \) in the sense of distributions. However, \( h \in H(L_0^H, \Omega) \) implies \( h \in L^2(\Omega) \), which in turn gives \( L_0 L_0^H h \in L^2(\Omega)^{m_1} \), and \( L_0^H h \in L^2(\Omega)^{m_2} \).

By (5.2) for \( g \in H(L_0^H, \Omega) \) we have

\[
\langle \mu, \pi_L g \rangle_{\mathcal{V}_L' \cdot \mathcal{V}_L} = \langle h, g \rangle_{H(L_0^H, \Omega)} = \langle h, g \rangle_{L^2(\Omega)^{m_1}} + \langle L_0^H h, L_0^H g \rangle_{L^2(\Omega)^{m_2}} \\
= \langle (I - L_0 L_0^H) h, g \rangle_{L^2(\Omega)^{m_1}} + \langle L_0^H h, \pi_L g \rangle_{\mathcal{V}_L' \cdot \mathcal{V}_L} \\
= \langle L_0^H h, \pi_L g \rangle_{\mathcal{V}_L' \cdot \mathcal{V}_L}.
\]

We define \( f := L_0^H h \in L^2(\Omega)^{m_2} \), which gives us \( L_0 f = L_0 L_0^H h = h \in L^2(\Omega) \) and consequently \( f \in H(L_0, \Omega) \). Hence, \( L_\nu f = \mu \) completes the proof.

By Remark 5.10 also \( \mathbb{1}_{\Gamma_1} L_\nu : H(L_0, \Omega) \to \mathcal{V}_L' \mathbb{1}_{\Gamma_1} \) is linear, bounded and onto.

**Proposition 5.14.** \( (\mathcal{V}_{L, \Gamma_1}, L^2_{\nu}(\Gamma_1), \mathcal{V}_{L, \Gamma_1}') \) is a quasi Gelfand triple.

**Proof.** Let \( \hat{D}_+ := \text{ran} \, \pi_L |_{H_{01}(\Omega)^{m_1}} \) and let \( D_- \) denote the corresponding set (Definition 4.1). Then \( \mathbb{1}_{\Gamma_1} L_\nu \gamma_0 \subseteq D_- \), which is dense in \( L^2_{\nu}(\Gamma_1) \) and by Proposition 5.8 and Theorem 5.13 also dense in \( \mathcal{V}_{L, \Gamma_1}' \). Hence, assertion (iv) of Proposition 4.6 is satisfied, which yields that the completion of \( \hat{D}_+ \) and \( D_- \) is a quasi Gelfand triple with pivot space \( L^2_{\nu}(\Gamma_1) \).

**Lemma 5.15.** \( \ker \pi_L = \ker \hat{L}^H_\nu \).

**Proof.** The following equivalences prove the statement

\[
g \in \ker \pi_L \Leftrightarrow \langle \pi_L g, \psi \rangle = 0 \quad \text{for all} \quad \psi \in \mathcal{V}_L' \Leftrightarrow \langle \pi_L g, L_\nu f \rangle = 0 \quad \text{for all} \quad f \in H(L_0, \Omega) \Leftrightarrow \langle \hat{L}^H_\nu g, \pi_L^\nu f \rangle = 0 \quad \text{for all} \quad f \in H(L_0, \Omega) \Leftrightarrow \langle \hat{L}^H_\nu g, \phi \rangle = 0 \quad \text{for all} \quad \phi \in \mathcal{V}_{L, \nu} \Leftrightarrow g \in \ker \hat{L}^H_\nu.
\]
6. Port Hamiltonian Systems

In this section we will introduce port-Hamiltonian systems on multidimensional spatial domains and formulate boundary conditions which justify existence and uniqueness of solutions. Moreover, we will parameterize all boundary conditions that provide solutions that are non-increasing in the Hamiltonian.

**Definition 6.1.** Let \( m \in \mathbb{N} \) and \( P = (P_i)_{i=1}^n \), where \( P_i \) is a Hermitian \( m \times m \) matrix. Moreover, let \( \mathcal{H} : \Omega \to \mathbb{K}^{m \times m} \) be such that \( \mathcal{H}(\zeta) = \mathcal{H}(\zeta) \) and \( \text{cl} \leq \mathcal{H}(\zeta) \leq \text{CT} \) for a.e. \( \zeta \in \Omega \) and some constants \( c, C \in \mathbb{R}_+ \) independent of \( \zeta \). Then we endow the space \( \mathcal{X}_H := L^2(\Omega)^m \) with the scalar product

\[
(f,g)_{\mathcal{X}_H} := \frac{1}{2} \int_{\Omega} (\mathcal{H}(\zeta), f(\zeta), g(\zeta))_{\mathbb{K}^m} \, d\lambda(\zeta).
\]

Furthermore, let \( \mathcal{P}_0 \in \mathbb{K}^{m \times m} \) be such that \( \mathcal{P}_0 \mathcal{H} = -\mathcal{P}_0 \). Then we call the differential equation

\[
\frac{\partial}{\partial t} x(t, \zeta) = \sum_{i=1}^n \frac{\partial}{\partial \nu_i} P_i (\mathcal{H}(\zeta) x(t, \zeta)) + P_0 (\mathcal{H}(\zeta) x(t, \zeta)), \quad t \in \mathbb{R}_+, \zeta \in \Omega, \quad x(0, \zeta) = x_0(\zeta), \quad \zeta \in \Omega
\]

a linear, first order port-Hamiltonian system, where \( x_0 \in L^2(\Omega)^m \) is the initial state. The associated Hamiltonian \( E : \mathcal{X}_H \to \mathbb{R}_+ \cup \{0\} \) is defined by

\[
E(x) := (x,x)_{\mathcal{X}_H} = \frac{1}{2} \int_{\Omega} \langle \mathcal{H}(\zeta) x(\zeta), x(\zeta) \rangle_{\mathbb{K}^m} \, d\lambda(\zeta),
\]

where \( \mathcal{H} \) is called the Hamiltonian density. We will refer to \( \mathcal{X}_H \) as the state space and to its elements as state variables or states.

In most applications the Hamiltonian describes the energy in the state space.

By the convention of regarding a function \( x : \mathbb{R}_+ \times \Omega \to \mathbb{K}^m \) as \( x : \mathbb{R}_+ \to L^2(\Omega; \mathbb{K}^m) \) by setting \( x(t) = x(t, \cdot) \), we can rewrite the PDE (6.1) as

\[
\dot{x} = \left( \sum_{i=1}^n \partial_\nu P_i + P_0 \right) \mathcal{H} x = (P_0 + P_0) \mathcal{H} x, \quad x(0) = x_0.
\]

We want to add the following assumptions.

**Assumption 6.2.** Let \( m, m_1, m_2 \in \mathbb{N} \) such that \( m = m_1 + m_2 \) and let \( L = (L_i)_{i=1}^n \) such that \( L_i \in \mathbb{K}^{m_1 \times m_2} \). Then we define \( P = (P_i)_{i=1}^n \) by

\[
P_i = \begin{bmatrix} 0 & L_i \\ L_i^* & 0 \end{bmatrix}.
\]

Clearly \( P \) contains only Hermitian matrices. Moreover, we have the identities

\[
P_0 = \begin{bmatrix} L_0^* & 0 \\ 0 & L_0 \end{bmatrix}, \quad P_0 = \begin{bmatrix} L_0 & 0 \\ 0 & L_0^* \end{bmatrix}, \quad \pi P = \begin{bmatrix} \pi L & 0 \\ 0 & \pi L^* \end{bmatrix}.
\]

Corresponding to those splittings we want to define \( (\mathcal{H} x)_1 \) and \( (\mathcal{H} x)_2 \), such that

\[
P_0 \mathcal{H} x = \begin{bmatrix} L_0 (\mathcal{H} x)_2 \\ \mathcal{L}_0 (\mathcal{H} x)_2 \end{bmatrix}, \quad [0 \ L_0] \mathcal{H} x = \mathcal{L}_0 (\mathcal{H} x)_2, \quad [\pi L \ 0] \mathcal{H} x = \pi L (\mathcal{H} x)_1.
\]

**Theorem 6.3.** The operator

\[
A_0 := -(P_0 + P_0) \mathcal{H}, \quad \text{dom } A_0 := \mathcal{H}^{-1}(\ker P_0)
\]

is closed, skew-symmetric, and densely defined on \( \mathcal{X}_H \). Its adjoint is

\[
A_0^* = (P_0 + P_0) \mathcal{H}, \quad \text{dom } A_0^* = \mathcal{H}^{-1}(H(P_0, \Omega)).
\]

Let \( B_1 = [\pi L \ 0] \mathcal{H}, \ B_2 = [0 \ L_0] \mathcal{H} \) and \( \Psi \) the duality map of \( (\mathcal{V}_L, L^2(\partial \Omega), V_L^*) \). Then \( (\mathcal{V}_L, B_1, \Psi B_2) \) is a boundary triple for \( A_0^* \).
Proof. Instead of considering $A_0^*$ as the adjoint of $A_0$, we just take it as a symbol. We will justify that it is in fact the adjoint of $A_0$ later in the proof.

By Lemma 3.4 $P_0$ is a closed operator on $H(P_0, \Omega)$. Since $\mathcal{H}$ is continuous, it is easy to see that $A_0^*$ is closed with domain $\mathcal{H}^{-1}(H(P_0, \Omega))$. Let $B^{**}$ denote the adjoint of $B$ with respect to $\langle \cdot, \cdot \rangle_H$ for any Hilbert space $H$. According to Remark 3.8 it is easy to see that the adjoint $((P_0 + P_0^\perp)\mathcal{H})^{\perp\times n}$ equals $((P_0^\perp)^{\perp\times n}) H = (P_0 + P_0^\perp)\mathcal{H}$ with domain $\mathcal{H}^{-1}(\text{dom } P_0^{\perp \times n}) \subseteq \mathcal{H}^{-1}(H(P_0, \Omega))$. Hence, $(A_0^*)^*$ is skew-symmetric on $\mathcal{A}_\mathcal{H}$. Since $A_0^*$ is closed, we have $(A_0^*)^{**} = A_0^*$.

Now we know that $A_0^*$ is the adjoint of a skew-symmetric operator. So we can talk about boundary triples for $A_0^*$. First we note that

$$\text{ran } [\Psi B_2] = \text{ran } \pi_L \times \text{ran } \Psi \bar{L}_0 = V_L \times V_L.$$

Since $\mathcal{H}$ is self-adjoint and $P_0$ is skew-adjoint, we have for $x \in \text{dom } A_0^*$

$$\langle A_0^* x, x \rangle_{\mathcal{X}_n} + \langle x, A_0^* x \rangle_{\mathcal{X}_n} = \langle P_0 \mathcal{H} x, \mathcal{H} x \rangle + \langle \mathcal{H} x, P_0 \mathcal{H} x \rangle = 2 \text{Re} \langle P_0 \mathcal{H} x, \mathcal{H} x \rangle.$$

The identity $P_0 = \begin{pmatrix} 0 & L_0 \\ L_0^* & 0 \end{pmatrix}$ and Corollary 5.11 yield

$$2 \text{Re} \langle P_0 \mathcal{H} x, \mathcal{H} x \rangle = 2 \text{Re} \begin{bmatrix} L_0(\mathcal{H} x)_2 \\ L_0^*(\mathcal{H} x)_1 \end{bmatrix} = 2 \text{Re} \langle \bar{L}_0(\mathcal{H} x)_2, \pi_L(\mathcal{H} x)_1 \rangle_{V_L \times V_L} = 2 \text{Re} \langle \Psi B_2 x, B_1 x \rangle_{V_L} = \langle B_1 x, \Psi B_2 x \rangle_{V_L} + \langle \Psi B_2 x, B_1 x \rangle_{V_L}.$$

The polarization identity implies that $(\mathcal{V}_L, B_1, \Psi B_2)$ is a boundary triple for $A_0^*$.

By Lemma 2.2 dom $A_0 = \ker B_1 \cap \ker B_2$, which is equal to

$$\ker B_1 \cap \ker B_2 = \mathcal{H}^{-1}((\ker [\pi_L \ 0]) \cap \ker [0 \ \bar{L}_0]) = \mathcal{H}^{-1}(\ker [\pi_L \times \ker \bar{L}_0]).$$

By Lemma 5.15 this is equal to $\mathcal{H}^{-1}(\ker P_0)$.

$\blacksquare$

Remark 6.4. We can replace $(\mathcal{V}_L, B_1, \Psi B_2)$ by $(\mathcal{V}_L', \Psi^* B_1, B_2)$ in the previous theorem.

Theorem 6.5. Let $A_0^*$ be the operator from the previous theorem and $\Phi$ the duality map associated to the quasi Gelfand triple $(\mathcal{V}_L, \Gamma_1, L_0^\perp(\Gamma_1), \mathcal{V}_L', \Gamma_1)$. Then we have $(\mathcal{V}_L, \Gamma_1, \pi_L \ 0) \mathcal{H}, \Phi [0 \ 1 \Gamma_1 \bar{L}_0 ] \mathcal{H})$ as a boundary triple for

$$A := A_0^* |_{\mathcal{H}^{-1}(\text{dom } L_0^\perp) \times \text{dom } \bar{L}_0}.$$ 

Proof. Since we already have a boundary triple for $A_0^*$, we can show that $A$ is the adjoint of a skew-symmetric operator by Proposition 2.3 (iii). Hence, we have to check, whether $[\pi_L \ 0] C^\perp \subseteq C$. For $B_1, B_2$ being the mappings from the previous theorem we have

$$C = \begin{pmatrix} B_1 & \Psi B_2 \\ 0 & 1 \end{pmatrix} \text{dom } A = \mathcal{V}_L, \Gamma_1 \times \mathcal{V}_L = \mathcal{V}_L' \times \mathcal{V}_L = C.$$

For $x, y \in \text{dom } A$ we have

$$\langle B_1 x, \Psi B_2 y \rangle_{\mathcal{V}_L} = \langle \pi_L(\mathcal{H} x)_1, \bar{L}_0(\mathcal{H} y)_2 \rangle_{\mathcal{V}_L \times \mathcal{V}_L} = \langle \pi_L(\mathcal{H} x)_1, 1 \Gamma_1 \bar{L}_0 \mathcal{H} y \rangle_{\mathcal{V}_L, \Gamma_1} = \langle [\pi_L \ 0] \mathcal{H} x, \Phi [0 \ 1 \Gamma_1 \bar{L}_0 ] \mathcal{H} y \rangle_{\mathcal{V}_L', \Gamma_1},$$

which yields item (ii) in Definition 2.1. By ran $\begin{pmatrix} \pi_L & 0 \\ 0 & \Phi \Gamma_1 \bar{L}_0 \end{pmatrix} [\mathcal{H}^{-1}(\text{dom } L_0^\perp) \times \text{dom } \bar{L}_0] = \mathcal{V}_L, \Gamma_1 \times \mathcal{V}_L', \Gamma_1$, the remaining item (i) is fulfilled.

$\blacksquare$

The next theorem is [KZ15, Theorem 2.5].
**Theorem 6.6.** Let $A_0$ be a skew-symmetric operator on a Hilbert space $X$ and $(B, B_1, B_2)$ be a boundary triple for $A_0^*$. Furthermore let $K$ be a Hilbert space, $W_B = [W_1, W_2]$, where $W_1, W_2 \in L_0(B, K)$, and $A := A_0|_{\text{dom} A}$, where $\text{dom} A = \ker W_B \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$. If $\text{ran} W_1 - W_2 \subseteq \text{ran} W_1 + W_2$ then the following assertions are equivalent.

(i) The operator $A$ generates a contractions semigroup on $X$.

(ii) The operator $A$ is dissipative.

(iii) The operator $W_1 + W_2$ is injective and the following operator inequality holds

$$W_1 W_2^* + W_2 W_1^* \geq 0.$$ 

We will reformulate this theorem to fit our situation.

**Corollary 6.7.** Let $K$ be some Hilbert space and $W = [W_1, W_2] : V_L, \Gamma_1 \times V_L, \Gamma_1 \to K$ a bounded linear mapping such that $\text{ran} W_1 - W_2 \subseteq \text{ran} W_1 + W_2$.

$$D := \left\{ x \in H^{-1}(H_{00}(L^H, \Omega) \times H(L_0, \Omega)) : W_1 \begin{bmatrix} \pi_L \\ 0 \end{bmatrix} H x + W_2 \Psi \begin{bmatrix} 0 \\ L_0 \end{bmatrix} H x = 0 \right\},$$

where $\Psi : V_{L, \Gamma_1} \to V_{L, \Gamma_1}$ is the duality mapping corresponding to the quasi Gelfand triple. Then the following assertions are equivalent.

(i) $(P_0 + P_0)H|_D$ generates a contractions semigroup.

(ii) $(P_0 + P_0)H|_D$ is dissipative.

(iii) The operator $W_1 + W_2$ is injective and the following operator inequality holds

$$W_1 W_2^* + W_2 W_1^* \geq 0.$$ 

Corollary 6.7 already gives a parameterization via $W$ for all boundary conditions that make $(P_0 + P_0)H$ a generator of a contractions semigroup. However, checking continuity for boundary operators which map into $V_L$ can be difficult. Hence, it would be appreciated to reduce the conditions on the boundary operators to conditions on better known spaces like $L^2(\partial \Omega)$.

So for the next theorem just imagine the quasi Gelfand triple to be $(V_L, L^2(\partial \Omega), V'_L)$ to get more satisfying conditions.

The following result is a generalization of [KZ15, Theorem 2.6] for quasi Gelfand triple and also fixes some minor issues.

**Theorem 6.8.** Let $(B_+, B_0, B_-)$ be a quasi Gelfand triple, $A_0$ be a skew-symmetric operator and $(B_+, B_1, B_2)$ be a boundary triple for $A_0^*$, where $\Psi$ is the duality map of the Gelfand triple. For $V_1, V_2 \in L_0(B_0, K)$ we define

$$D := \left\{ a \in \text{dom} A_0^* : B_1 a, B_2 a \in B_0 \text{ and } [V_1, V_2] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} a = 0 \right\}$$

and the operator $A := A_0^*|_D$. If

(i) $[V_1|_{B_0 \cap B_+}, V_2|_{B_0 \cap B_-}]$ is closed as an operator from $B_+ \times B_-$ to $B_0$,

(ii) $\ker [V_1, V_2]$ is dissipative as linear relation on $B_0$,

(iii) $V_1 V_2^* + V_2 V_1^* \geq 0$ as operator on $B_0$,

then $A$ is a generator of a contraction semigroup.

**Proof.** It is sufficient to show that $A$ is closed, and $A$ and $A^*$ are dissipative.
Step 1. Showing that $A$ is closed and dissipative.

$$a \in D \iff \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} a \in (B_0 \times B_0) \cap \ker [V_1 \ V_2]$$

$$\iff \begin{bmatrix} B_1 \\ \Psi B_2 \end{bmatrix} a \in \ker [V_1|_{B_0 \cap B_+} \ V_2 \Psi^*|_{(B_0 \cap B_-)}] =: C.$$ 

We can write

$$C = \left\{ \begin{bmatrix} q \\ p \end{bmatrix} \in B_+ \times B_+ : q \in B_0, \exists \bar{p} \in B_0 p = \Psi \bar{p}, V_1 q + V_2 \Psi^* p = 0 \right\}.$$

For $[\gamma] \in C$ we have

$$\text{Re}(q, p)_{B_+} = \text{Re}(q, \Psi \bar{p})_{B_+} = \text{Re}(q, \bar{p})_{B_+} = \text{Re}(q, \bar{p})_{B_0} \leq 0,$$

which implies the dissipativity of $A$ by Proposition 2.3. Assumption (i) implies that $C$ is closed, which implies the closedness of $A$ by Proposition 2.3.

Step 2. Showing that $A^*$ is dissipative. By Proposition 2.3 we can characterize the domain of $A^*$ by

$$d \in \text{dom } A^* \iff \begin{bmatrix} B_1 \\ \Psi B_2 \end{bmatrix} d \in \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} C^{-1} s^2_+$$

$$\iff \begin{bmatrix} \Psi B_2 \\ B_1 \end{bmatrix} d \in \text{ran} \begin{bmatrix} V_1|_{B_0 \cap B_+} \ V_2 \Psi^*|_{(B_0 \cap B_-)} \end{bmatrix} s^2_+.$$

Note that if $P$ is a bounded and everywhere defined operator, and $Q$ is a linear relation, then $(PQ)^* = Q^* P^*.$ Hence,

$$V_1|_{B_0 \cap B_+}^{s^2_+} = (V_1 \iota_+)^* = \iota_+^* V_1^* = \Psi V_1|_{V_1^{-1}(B_0 \cap B_-)}$$

and

$$V_2 \Psi^*|_{(B_0 \cap B_-)}^{s^2_+} = (V_2 \iota_- \Psi^*)^* = (\iota_- \Psi^*)^* V_2^*.$$

From $(\Psi \iota_+)^* = \iota_- \Psi^*$ and $\iota_-^* \iota_+^* = (\iota_- \Psi^*)^* = \Psi \iota_-$ follows $(\iota_- \Psi^*)^* = \Psi \iota_- = \iota_-^1.$ Consequently,

$$V_2 \Psi^*|_{(B_0 \cap B_-)}^{s^2_+} = \iota_+^* V_2^* = V_2^*|_{V_2^{-1}(B_0 \cap B_+)}.$$

Hence, for

$$\begin{bmatrix} x \\ y \end{bmatrix} \in \text{ran} \begin{bmatrix} V_1|_{B_0 \cap B_+}^{s^2_+} \\ V_2 \Psi^*|_{(B_0 \cap B_-)}^{s^2_+} \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} \Psi V_1^* \\ V_2^* \end{bmatrix} k : k \in V_1^{-1}(B_0 \cap B_-) \cap V_2^{-1}(B_0 \cap B_+) \right\}.$$

we have

$$\text{Re}(x, y)_{B_+} = \text{Re}(\Psi V_1^* k, V_2^* k)_{B_+} = \text{Re}(V_1^* k, V_2^* k)_{B_+} = \text{Re}(V_1^* k, V_2^* k)_{B_0} = \text{Re}(V_2 V_1^* k, k)_{B_0} \geq 0.$$

Therefore, $-A^*$ is accretive and $A^*$ is dissipative.

Remark 6.9. In the previous theorem, it is possible to replace the condition of $[V_2|_{B_0 \cap B_+} \ V_2|_{B_0 \cap B_-}]$ being closed by

$$\ker [V_2|_{B_0 \cap B_+} \ V_2|_{B_0 \cap B_-}] = \ker [V_2|_{B_0 \cap B_+} V_2|_{B_0 \cap B_-}].$$

Then instead of $A$ the operator closure $\overline{A}$ is a generator of contraction semigroup.
Example 6.10. Let $M \in \mathcal{L}_b(\mathcal{B}_0)$ be strictly positive. Then $V_1 := I$, $V_2 := M$ fulfill all conditions of the previous theorem.

(i) Setting $S = M \frac{\pi}{2}$ and $T = M \frac{3\pi}{2}$ in Corollary 4.19 yields $\left[ \left[ I_{\mathcal{B}_0 \cap \mathcal{B}_-} \right] [M|_{\mathcal{B}_0 \cap \mathcal{B}_-}] \right]$ being closed.

(ii) For $(x, y) \in \ker [V_1 \ V_2]$ we have $x = -M y$. Since $M$ is positive this yields
\[
\text{Re}(x, y)_{\mathcal{B}_0} = \text{Re}(-M y, y) \leq 0.
\]

(iii) $V_1 V_2^* + V_2 V_1^* = M^+ + M = 2 \text{Re} M \geq 0$.

Moreover, Corollary 4.19 also implies $\left[ t_{\mathcal{B}_0 \cap \mathcal{B}_-} \right] [M|_{\mathcal{B}_0 \cap \mathcal{B}_-}]$ being surjective. Actually, it would have been enough, if $M \in \mathcal{L}_b(\mathcal{B}_0)$ was boundedly invertible and accretive. Clearly, also $V_1 := M$, $V_2 := I$ fulfill all conditions.

7. Port-Hamiltonian Systems as Boundary Control Systems

We introduce the notion of boundary control systems, scattering passive and impedance passive in the manner of [MS07]. We will show that a port-Hamiltonian system can be described as such a system.

**Definition 7.1.** A colligation $\Xi := \left( \left[ \frac{G}{K} \right] ; \left[ \frac{U}{Y} \right] \right)$ consists of the three Hilbert spaces $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$, and the three linear maps $G$, $L$, and $K$, with the same domain $\mathcal{Z} \subseteq \mathcal{X}$ and with values in $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$, respectively.

**Definition 7.2.** A colligation $\Xi := \left( \left[ \frac{G}{K} \right] ; \left[ \frac{U}{Y} \right] \right)$ is an (internally well-posed) boundary control system, if

1. the operator $\left[ \frac{G}{K} \right]$ is closed from $\mathcal{X}$ to $\left[ \frac{U}{Y} \right]$,
2. the operator $G$ is surjective, and
3. the operator $A := L|_{\ker G}$ generates a contraction semigroup on $\mathcal{X}$.

We think of the operators in this definition as determining a system via
\[
\begin{align*}
&u(t) = G x(t), \\
&\dot{x}(t) = L x(t), \quad x(0) = x_0,
&y(t) = K x(t).
\end{align*}
\]
We call $\mathcal{U}$ the input space, $\mathcal{X}$ the state space, $\mathcal{Y}$ the output space and $\mathcal{Z}$ the solution space.

**Definition 7.3.** Let $\Xi := \left( \left[ \frac{G}{K} \right] ; \left[ \frac{U}{Y} \right] \right)$ be a colligation. If $\Xi$ is a boundary control system such that
\[
2 \text{Re}(L x, x)_{\mathcal{X}} + \| K x \|_{\mathcal{Y}}^2 \leq \| G x \|_{\mathcal{U}}^2,
\]
then it is scattering passive and it is scattering energy preserving if we have equality in (7.2).

We say $\Xi$ is impedance passive (energy preserving), if $\tilde{\Xi} := \left( \left[ \frac{L}{G+K} \right] ; \left[ \frac{U}{Y} \right] \right)$ is scattering passive (energy preserving).

Corresponding to a port-Hamiltonian system we want to introduce the following operators
\[
G_p := T \left[ \frac{\pi}{2} \ 0 \right] \mathcal{H}, \quad L_p := (P_0 + P_0) \mathcal{H} \quad \text{and} \quad K_p := (T^*)^{-1} \left[ 0 \ I_{\Gamma}, L_0 \right] \mathcal{H},
\]
where $T \in \mathcal{L}_b(L^2(\Gamma_1))$ is invertible. By Lemma 4.20 also $G_p$ and $K_p$ establish a boundary triple for $L_p$. For simplification $T$ can be imagined to be the identity mapping. We still have $\Gamma_0, \Gamma_1$ as a splitting with thin boundaries of $\partial \Omega$.  

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Corollary 7.4. The colligation \( \begin{bmatrix} G_p \\ L_p \\ K_p \end{bmatrix} \) with solution space 
\( Z = \mathcal{H}^{-1}(H_{Γ_0}(L^2_0, Ω) \times H(L_0, Ω)) \)
is a boundary control system.

Proof. Since \( L_p \) is closed with domain \( Z \), and \( G_p \) and \( K_p \) are continuous with the 
graph norm of \( L_p \), we have \( (G_p, L_p, K_p) \) is closed. By construction \( G_p \) is surjective. 
Since \( G_p \) is one operator of a boundary triple for \( L_p \), the restriction \( L_p|_{\ker G_p} \) is 
skew-adjoint and therefore a generator of a contraction semigroup. □

Proposition 7.5. Let \( R \in \mathcal{L}_b(TL^2_2(Γ_1)) \) be strictly positive. Then the colligation 
\( \Xi = \left( \begin{bmatrix} \left(2 Re(G_p + RK_p)\right)_{L_p} \\ X_{Γ_0} \end{bmatrix}, \begin{bmatrix} U \\ Y \end{bmatrix} \right) \) with \( U = Y = TL^2_2(Γ_1) \) endowed with \( \|f\|_U = \|f\|_Y = \|Re^{1/2}f\|_{L^2} \) and solution space 
\( Z = \{x \in \mathcal{H}^{-1}(H_{Γ_0}(L^2_0, Ω) \times H(L_0, Ω)) : G_p x, K_p x \in TL^2_2(Γ_1)\} \)
is a scattering energy preserving boundary control system.

Proof. Let \( (x_n, [G_p x_n, L_p x_n, K_p x_n]^T)_{n∈\mathbb{N}} \) be a sequence in \( [G_p, L_p, K_p]^T \) that converges to \( (x, [f y g]^T) \). Since \( L_p \) with domain \( H(Γ_0, Ω) \) is a closed operator 
and \( H_{Γ_0}(L^2_0, Ω) \times H(L_0, Ω) \) is closed in \( H(Γ_0, Ω) \), we conclude that \( x \in \mathcal{H}^{-1}(H_{Γ_0}(L^2_0, Ω) \times H(L_0, Ω)) \). Hence, \( G_p x_n \) converges in \( TL^2_2(Γ_1) \) to \( G_p x \). Since 
\( (TV_{L_2, Γ_1}, TL^2_2(Γ_1), (TV_{L_2, Γ_1})^T) \) is a quasi Gelfand triple \( G_p x = f \). Analogously, 
we conclude \( K_p x = g \). Therefore, \( x \in Z \) and \( [G_p, L_p, K_p]^T \) is closed, which implies 
that also \( \left(\frac{1}{\sqrt{2}}(G_p + RK_p)\right)_{L_p} \left(\frac{1}{\sqrt{2}}(G_p - RK_p)\right)^T \) is closed.

By Example 6.10 and Theorem 6.8 \( L_p|_{\ker G_p} \left(\frac{1}{\sqrt{2}}(G_p + RK_p)\right) \) generates a contraction 
semigroup.

Example 6.10 also gives the surjectivity of \( \frac{1}{\sqrt{2}}(G_p + RK_p) \).

Since \( (V_{L_2}, G_p, ΨK_p) \) is a boundary triple for \( L_p \), we have
\[ 2 Re\langle L_p x, x\rangle_{X_{Γ_0}} = 2 Re\langle G_p x, K_p x\rangle_{Y_{L_2}} = 2 Re\langle G_p x, K_p x\rangle_{L^2_2(Γ_1)} \]
\[ = \frac{1}{2} \left( \langle R^{-1}G_p x, G_p x\rangle_{L^2_2} + 2 Re\langle G_p x, K_p x\rangle_{L^2_2} + \langle RK_p x, K_p x\rangle_{L^2_2} \right) \]
\[ - \frac{1}{2} \left( \langle R^{-1}G_p x, G_p x\rangle_{L^2_2} - 2 Re\langle G_p x, K_p x\rangle_{L^2_2} + \langle RK_p x, K_p x\rangle_{L^2_2} \right) \]
\[ = \|\frac{1}{\sqrt{2}}(G_p + RK_p)x\|^2_U - \|\frac{1}{\sqrt{2}}(G_p - RK_p)x\|^2_Y, \]
which makes \( \Xi \) scattering energy preserving. □

Remark 7.6. Clearly, the previous theorem holds also true for the operator triple 
\( \left(\frac{1}{\sqrt{2}}(RK_p + G_p)\right)_{L_p} \left(\frac{1}{\sqrt{2}}(RK_p - G_p)\right)^T \) and for \( G_p \) and \( K_p \) being swapped. Moreover, replacing \( L_p \) by \( L_p + J \), where \( J \in \mathcal{L}_b(X_{Γ_0}) \) is dissipative, yields a scattering 
passive system.

So the port-Hamiltonian system with input \( u \) and output \( y \) described by the following equations is well-posed.

\[ \sqrt{2}u(t, ζ) = π_L(H(ζ)x(t, ζ))_2 + R\tilde{L}_u(H(ζ)x(t, ζ))_1, \quad t \in \mathbb{R}_+, ζ \in Γ_1, \]
\[ \frac{∂}{∂t}x(t, ζ) = \sum_{i=1}^n \frac{∂}{∂ζ}P_i(H(ζ)x(t, ζ)) + P_b(H(ζ)x(t, ζ)), \quad t \in \mathbb{R}_+, ζ \in Ω, \]
\[ \sqrt{2}y(t, ζ) = π_L(H(ζ)x(t, ζ))_2 - R\tilde{L}_y(H(ζ)x(t, ζ))_1, \quad t \in \mathbb{R}_+, ζ \in Γ_1, \]
\[ 0 = π_L(H(ζ)x(t, ζ))_2, \quad t \in \mathbb{R}_+, ζ \in Γ_0, \]
Proof. This is a direct consequence of Proposition 7.5 for \( R = 1 \). \( \square \)

Example 7.8 (Wave equation). Let \( \rho \in L^\infty(\Omega) \) be the mass density and \( T \in L^\infty(\Omega)^{n \times n} \) be the Young modulus, such that \( \frac{1}{\rho} \in L^\infty(\Omega) \), \( T(\zeta) = T(\zeta) \) and \( T(\zeta) \geq \delta I \) for a \( \delta > 0 \) and almost every \( \zeta \in \Omega \). Then the wave equation

\[
\frac{\partial^2}{\partial t^2} w(t, \zeta) = \frac{1}{\rho(\zeta)} \text{div} (T(\zeta) \text{grad} w(t, \zeta)),
\]

can be formulated as a port-Hamiltonian system by choosing the state variable

\[
x(t, \zeta) = \begin{bmatrix} \rho(\zeta) & 0 \\ \text{grad} w(t, \zeta) & T \end{bmatrix}.
\]

Then the PDE looks like

\[
\dot{x} = \begin{bmatrix} 0 & \frac{1}{\rho} \\ \text{grad} w(t, \zeta) & T \end{bmatrix} x.
\]

This is shown in section 3 of [KZ15]. This is exactly the port-Hamiltonian system we get from choosing \( L \) as in Example 3.5. From Example 5.2 and Example 5.6 we know that the boundary operators are \( \gamma_0 \) and the extension of \( \nu \cdot \gamma_0 \). So the system

\[
\sqrt{2} u(t, \zeta) = \nu \cdot (T(\zeta) \text{grad} w(t, \zeta)) + \frac{\partial}{\partial t} w(t, \zeta), \quad t \in \mathbb{R}_+, \zeta \in \Gamma_1,
\]

\[
\frac{\partial^2}{\partial t^2} w(t, \zeta) = \frac{1}{\rho(\zeta)} \text{div} (T(\zeta) \text{grad} w(t, \zeta)), \quad t \in \mathbb{R}_+, \zeta \in \Omega,
\]

\[
\sqrt{2} y(t, \zeta) = \nu \cdot (T(\zeta) \text{grad} w(t, \zeta)) - \frac{\partial}{\partial t} w(t, \zeta), \quad t \in \mathbb{R}_+, \zeta \in \Gamma_1,
\]

\[
0 = \frac{\partial}{\partial t} w(t, \zeta), \quad t \in \mathbb{R}_+, \zeta \in \Gamma_0,
\]

is well-posed.

Example 7.9 (Maxwell equations). Let \( L = (L_i)_{i=1}^3 \) be as in Example 3.6. In this example we have already showed \( L_0 = \text{rot} \) and \( L_c f = \nu \times f \). The corresponding differential operator for the port-Hamiltonian PDE is

\[
P_0 = \begin{bmatrix} 0 & L_0 \\ L_0^\top & 0 \end{bmatrix} = \begin{bmatrix} 0 & \text{rot} \\ -\text{rot} & 0 \end{bmatrix}.
\]

We write the state as \( x = [P_0 \mathcal{B}] \), where \( \mathcal{D}, \mathcal{B} \in \mathbb{K}^3 \). We also want to introduce the positive function \( \epsilon, \mu, g \) and \( r \) such that

\[
\epsilon, \frac{1}{\epsilon}, \mu, \frac{1}{\mu}, g \in L^\infty(\Omega) \quad \text{and} \quad r, \frac{1}{r} \in L^\infty(\Gamma_1).
\]

Furthermore, we define the Hamiltonian density by \( \mathcal{H}(\zeta) := \begin{bmatrix} \frac{\epsilon}{\mu} & 0 \\ 0 & \frac{\mu}{r} \end{bmatrix} \), where each block is a \( 3 \times 3 \) matrix. At last we define \( [P_0 \mathcal{B}] := \mathcal{H} [P_0 \mathcal{B}] \), so that we have the same notation as in [WS13].

The projection on \( \text{ran} \mathcal{L}_0 \) is given by \( g \mapsto (\nu \times g) \times \nu \), therefore \( \pi_L \) is the extension of \( g \mapsto (\nu \times \gamma_0 g) \times \nu \) to \( H(L_0, \Omega) \). The mapping \( \pi_\tau \) from [WS13] can be
It is easy to derive the corresponding PDE of the Mindlin plate. So the port-Hamiltonian PDE

\[ \sqrt{2}u(t, \zeta) = r(\zeta)\nu(\zeta) \times H(t, \zeta) + (\nu(\zeta) \times E(t, \zeta)) \times \nu(\zeta), \quad t \in \mathbb{R}^+, \zeta \in \Gamma, \]

\[ \epsilon(\zeta) \frac{\partial}{\partial t} E(t, \zeta) = \text{rot} H(t, \zeta) - g(\zeta) E(t, \zeta), \quad t \in \mathbb{R}^+, \zeta \in \Omega, \]

\[ \mu(\zeta) \frac{\partial}{\partial t} H(t, \zeta) = - \text{rot} E(t, \zeta), \quad t \in \mathbb{R}^+, \zeta \in \Omega, \]

\[ \sqrt{2} g(t, \zeta) = r(\zeta)\nu(\zeta) \times H(t, \zeta) - (\nu(\zeta) \times E(t, \zeta)) \times \nu(\zeta), \quad t \in \mathbb{R}^+, \zeta \in \Gamma, \]

\[ 0 = (\nu(\zeta) \times E(t, \zeta)) \times \nu(\zeta), \quad t \in \mathbb{R}^+, \zeta \in \Gamma_0, \]

and is scattering passive by Remark 7.6, where we set \( J = \begin{bmatrix} -g & 0 \\ 0 & 0 \end{bmatrix} \).

**Example 7.10** (Mindlin plate). Lets regard the differential operator \( P_3 \) and the skew-symmetric matrix \( b \) given by

\[
P_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \partial_1 & \partial_2 \\ 0 & 0 & \partial_1 & \partial_2 & 0 & 0 \\ 0 & \partial_1 & 0 & 0 & 0 & 0 \\ \partial_2 & 0 & 0 & 0 & 0 & 0 \\ \partial_1 & 0 & 0 & 0 & 0 & 0 \\ \partial_2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \frac{1}{\rho^2} := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_b := \begin{bmatrix} \frac{1}{\rho} & \rho^2 \omega_1 & \rho \omega_2 & \kappa_1 & \kappa_2 & \kappa_3 & \gamma_1 & \gamma_2 \end{bmatrix}^T, \]

where \( \rho, h \) are strictly positive function, \( D_b(\zeta) \) is a strictly positive \( 3 \times 3 \) matrix and \( D_s(\zeta) \) is strictly positive \( 2 \times 2 \) matrix, such that all conditions on \( H \) in Definition 6.1 are satisfied. We write the state variable \( x \) as

\[ x := \begin{bmatrix} \rho \omega v & \rho^2 \omega_1 & \rho \omega_2 & \kappa_1 & \kappa_2 & \kappa_3 & \gamma_1 & \gamma_2 \end{bmatrix}^T, \]

where we stick to the notation in [BAPM18] except that we changed the coordinates \( x, y \) and \( z \) to 1, 2 and 3. Furthermore, we have

\[ e := Hx = \begin{bmatrix} 0 & w_1 & w_2 & M_{1,1} & M_{1,2} & M_{1,3} & M_{2,1} & M_{2,2} & Q_1 & Q_2 \end{bmatrix}^T. \]

We don’t want to go into details about the physical meaning of these state variables. We just want to make it easier to express the results into the notation of [BAPM18]. So the port-Hamiltonian PDE

\[ \frac{\partial}{\partial t} x = (P_3 + P_0)x \quad \text{looks like} \quad \frac{\partial}{\partial t} x = (P_3 + P_0)e. \]
The corresponding boundary operator is

\[ L_\nu f = \begin{bmatrix} 0 & 0 & \nu_1 & \nu_2 \\ \nu_1 & 0 & \nu_2 & 0 \\ \nu_2 & \nu_1 & 0 & 0 \\ f_4 & f_3 & f_2 & f_1 \end{bmatrix} = \begin{bmatrix} \nu \cdot f_4 \\ \nu \cdot f_3 \\ \nu \cdot f_2 \\ \nu \cdot f_1 \end{bmatrix}. \]

Since \( \|\nu(\zeta)\| = 1 \), at least \( \nu_1(\zeta) \neq 0 \) or \( \nu_2(\zeta) \neq 0 \). This can be used to show that \( \text{ran } L_\nu = L^2(\partial\Omega)^3 \). Therefore, \( \pi_L \) is the extension of the boundary trace operator \( \gamma_0 \) to \( H(\mathcal{L}_0^H, \Omega) \).

Since there is no direct physical meaning to the boundary variables

\[ [0 \quad \bar{L}_\nu] e = \begin{bmatrix} \nu \cdot \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \\ M_{1,1} \\ M_{1,2} \\ M_{2,1} \\ M_{2,2} \end{bmatrix} \quad \text{and} \quad [\pi_L \quad 0] e = \begin{bmatrix} v \\ w \\ \nu \cdot \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \\ M_{1,1} \\ M_{1,2} \\ M_{2,1} \\ M_{2,2} \end{bmatrix} \]

we define \( \eta := \begin{bmatrix} -\nu_2 \\ \nu_1 \end{bmatrix} \) and apply the unitary transformation \( T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \nu_1 & \nu_2 & 0 \\ 0 & -\nu_2 & \nu_1 & 0 \end{bmatrix} \) to obtain

\[ \begin{bmatrix} Q_\nu \\ M_{\nu,\nu} \\ M_{\nu,\eta} \end{bmatrix} := T \begin{bmatrix} \nu \cdot \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \\ M_{1,1} \\ M_{1,2} \\ M_{2,1} \\ M_{2,2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v \\ w \nu \cdot \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \\ M_{1,1} \\ M_{1,2} \\ M_{2,1} \\ M_{2,2} \end{bmatrix} := (T^*)^{-1} \begin{bmatrix} v \\ w \nu \cdot \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \\ M_{1,1} \\ M_{1,2} \\ M_{2,1} \\ M_{2,2} \end{bmatrix} \].

Hence, by Corollary 7.7 the system

\[
\begin{align*}
u &= \begin{bmatrix} Q_\nu & M_{\nu,\nu} & M_{\nu,\eta} \end{bmatrix}^T, & \text{on } \mathbb{R}_+ \times \Gamma_1, \\
\frac{\partial}{\partial t} \alpha &= (P_0 + P_0)e, & \text{on } \mathbb{R}_+ \times \Omega, \\
y &= \begin{bmatrix} v & w_\nu & w_\eta \end{bmatrix}^T, & \text{on } \mathbb{R}_+ \times \Gamma_1, \\
o &= \begin{bmatrix} v & w_\nu & w_\eta \end{bmatrix}^T, & \text{on } \mathbb{R}_+ \times \Gamma_0,
\end{align*}
\]

for the Mindlin plate is impedance energy preserving.

**Appendix A.**

The next example shows that it is possible to have item (i) and item (ii) of a "boundary triple" for an operator \( A \) (Definition 2.1) without \( A \) being the adjoint of a skew-symmetric operator. Moreover, it shows that in this situation Lemma 2.2 does not hold. This demonstrates the importance of \( A \) being the adjoint of a skew-symmetric operator in the definition.
Example A.1. Let $A = \begin{bmatrix} \frac{1}{x} & -1 \\ 0 & \frac{1}{x} \end{bmatrix}$ be an operator on $L^2(0,1)^2$ with $\text{dom} \, A = H^1(0,1)^2$. By Remark 3.8 the operator $A$ is the adjoint of a skew-symmetric operator. Integration by parts yields

$$\langle Af, g \rangle + \langle f, Ag \rangle = \int_0^1 \left( \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} g_2 \\ f_2 \end{bmatrix} \right) \, \hat{A} \, \left( \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} \right) \, \text{d} \xi$$

for all $f, g \in \text{dom} \, A$. Then there exists a subsequence $\left( x_n \right)$ such that $\hat{A} |_{\ker F_1 \cap \ker F_2} = -A |_{H^2_0(0,1)^2} = A^*$.

The mapping $\left[ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} : \text{dom} \, A \to \mathbb{R}^4 \right]$ is surjective (this can be seen by choosing $f_1$ and $f_2$ to be linear interpolations). So $(\mathbb{R}^4, B_1, B_2)$ is a boundary triple for $A$.

We define $\hat{A}$ as the restriction of $A$ on $H^1_{(1)=0(0,1)} \times H^1_{(0)=1}(0,1)$, where

$$H^1_{(1)=0(0,1)} := \{ f \in H^1(0,1) : f(1) = 0 \}, \quad \text{and}$$

$$H^1_{(0)=1}(0,1) := \{ f \in H^1(0,1) : f(0) = f(1) \}.$$ 

Therefore, we can reformulate (A.1) for $f, g \in \text{dom} \, \hat{A}$

$$\langle \hat{A}f, g \rangle + \langle f, \hat{A}g \rangle = \langle B_1f, B_2g \rangle + \langle B_2f, B_1g \rangle.$$ 

Lemma A.2. Let $(x_n)_{n \in \mathbb{N}}$ be a weak convergent sequence in a Hilbert space $H$ with limit $x$. Then there exists a subsequence $(x_{n(k)})_{k \in \mathbb{N}}$ such that

$$\left\| \frac{1}{N} \sum_{k=1}^{N} x_{n(k)} - x \right\| \to 0.$$

Proof. We assume that $x = 0$. For the general result we just need to replace $x_n$ by $x_n - x$.

We define the subsequence inductively: $n(1) = 1$ and for $k > 1$ we choose $n(k)$ such that

$$|\langle x_{n(k)}, x_{n(j)} \rangle| \leq \frac{1}{k} \quad \text{for all} \quad j < k.$$
This is possible, because \((x_n)_{n \in \mathbb{N}}\) converges weakly to 0. Note that by the principle of uniform boundedness 
\[
\sup_{n \in \mathbb{N}} \|x_n\| \leq C.
\]

\[
\frac{1}{N} \sum_{k=1}^{N} x_n(k) \leq \frac{1}{N^2} \sum_{k=1}^{N} \sum_{j=1}^{N} x_n(k,x_n(j)) \leq \frac{1}{N^2} \sum_{k=1}^{N} \|x_n(k)\|^2 + \frac{1}{N^2} \sum_{j=k}^{N} \sum_{k=j+1}^{N} 2 \text{Re}\langle x_n(k),x_n(j)\rangle \leq \frac{1}{N} C^2 + 2 \sum_{j=1}^{N} \sum_{k=j+1}^{N} \frac{1}{k} \leq \frac{C^2}{N} + \frac{1}{N} \ln(N) \to 0.
\]

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