ANOMALIES AND ANALYTIC TORSION ON HYPERBOLIC MANIFOLDS

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Abstract. The global additive and multiplicative properties of the Laplacian on $j-$forms and related zeta functions are analyzed. The explicit form of zeta functions on a product of closed oriented hyperbolic manifolds $\Gamma \backslash \mathbb{H}^d$ and of the multiplicative anomaly are derived. We also calculate in an explicit form the analytic torsion associated with a connected sum of such manifolds.

1. Introduction

The additive and multiplicative properties of (pseudo-) differential operators as well as properties of their determinants have been studied actively during recent years in the mathematical and physical literature. The anomaly associated with product of regularized determinants of operators can be expressed by means of the non-commutative residue, the Wodzicki residue [1] (see also Refs. [2, 3]). The Wodzicki residue, which is the unique extension of the Dixmier trace to the wider class of (pseudo-) differential operators [4, 5], has been considered within the non-commutative geometrical approach to the standard model of the electroweak interactions [6, 7, 8] and the Yang-Mills action functional. Some recent papers along these lines can be found in Refs. [9, 10, 11, 12].

The product of two (or more) differential operators of Laplace type can arise in higher derivative field theories (for example, in higher derivative quantum gravity [3]). The zeta function associated to the product of Laplace type operators acting in irreducible rank 1 symmetric spaces and the explicit form of the multiplicative anomaly have been derived in [11].

Under such circumstances we should note that the conformal deformation of a metric and the corresponding conformal anomaly can also play an important role in quantum theories with higher derivatives. It is well known that evaluation of the conformal anomaly is actually

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possible only for even dimensional spaces and up to now its computation is extremely involved. The general structure of such an anomaly in curved $d$-dimensional spaces ($d$ even) has been studied in [14]. We briefly mention here analysis related to this phenomenon in constant curvature spaces. The conformal anomaly calculation for the $d-$ dimensional sphere can be found, for example, in Ref. [15]. The explicit computation of the anomaly (of the stress-energy tensor) in irreducible rank 1 symmetric spaces has been carried out in [16, 17, 18] using the zeta-function regularization and the Selberg trace formula.

Recently the topology of manifolds have been studied by means of quantum field theory methods. In this approach the partition function of quadratic functionals play an important role. It has been shown that the analytic or Ray-Singer torsion (a topological invariant) [19] occurs within quantum field theory as the partition function of a certain quadratic functional [20, 21]. Recall that Ray-Singer torsion $T_{an}(X)$ is defined for every closed Riemannian manifold $X$ and orthogonal representation $\chi$ of $\pi_1(X)$. The definition of the torsion involves the spectrum of the Laplacian on twisted $j-$ forms. It has been proved in [22, 23] that when $\chi$ is acyclic and orthogonal the value $T_{an}(X)$ coincides with the so-called Reidemeister torsion, which can be computed from twisted cochain complex of a finite complex by taking a suitable alternating product of determinants [24].

The purpose of the present paper is to investigate the spectral zeta functions associated with a product and Krönecker sum of Laplacians on $j-$ forms and to calculate in an explicit form the analytic torsion on closed oriented hyperbolic manifolds $\Gamma \setminus \mathbb{H}^d$ and on a connected sum of such manifolds.

2. The spectral zeta function and the trace formula

We shall be working with irreducible rank 1 symmetric spaces $X = G/K$ of non-compact type. Thus $G$ will be a connected non-compact simple split rank 1 Lie group with finite center and $K \subset G$ will be a maximal compact subgroup. Up to local isomorphism we choose $X = SO_1(d,1)/SO(d)$. Thus the isotropy group $K$ of the base point $(1,0,...,0)$ is $SO(d)$; $X$ can be identified with hyperbolic $d-$ space $\mathbb{H}^d$, $d = \dim X$. It is possible to view $\mathbb{H}^d$, for example, as one sheet of the hyperboloid of two sheets in $\mathbb{R}^{d+1}$ given by $q(x) = -x_0^2 + x_1^2 + ... + x_d^2 = -1$, $x_0 > 0$ with the metric induced by the quadratic form $q(x)$. Let $\Gamma \subset G$ be a discrete, co-compact, torsion free subgroup, and let $\chi(\gamma) = \text{trace}(\chi(\gamma))$ be the character of a finite-dimensional unitary representation $\chi$ of $\Gamma$ for $\gamma \in \Gamma$. Let $L^{(j)} \equiv \Delta^{(j)}_{\Gamma}$ be the Laplacian
on \( j \)-forms acting on the vector bundle \( V(X_T) \) over \( X_T = \Gamma \backslash G/K \) induced by \( \chi \). Note that the non-twisted \( j \)-forms on \( X_T \) are obtained by taking \( \chi = 1 \). One can define the heat kernel of the elliptic operator \( \mathcal{L}^{(j)} = \mathcal{L}^{(j)} + b^{(j)} \) by

\[
\text{Tr} \left( e^{-t \mathcal{L}^{(j)}} \right) = \frac{-1}{2\pi i} \text{Tr} \int_{C_0} e^{-zt} (z - \mathcal{L}^{(j)})^{-1} dz, \tag{2.1}
\]

where \( C_0 \) is an arc in the complex plane \( \mathbb{C} \); the \( b^{(j)} \) are endomorphisms of the vector bundle \( V(X_T) \). By standard results in operator theory there exist \( \varepsilon, \delta > 0 \) such that for \( 0 < t < \delta \) the heat kernel expansion holds

\[
\omega_T^{(j)}(t, b^{(j)}) = \sum_{\ell=0}^{\infty} n_\ell(\chi) e^{-(\lambda_\ell^{(j)} + b^{(j)})t} = \sum_{0 \leq \ell \leq \ell_0} a_\ell(L^{(j)}) t^{-\ell} + O(t^\varepsilon), \tag{2.2}
\]

where \( \{\lambda_\ell^{(j)}\}_{\ell=0}^{\infty} \) is the set of eigenvalues of operator \( L^{(j)} \) and \( n_\ell(\chi) \) denote the multiplicity of \( \lambda_\ell^{(j)} \). Eventually we would like also to take \( b^{(j)} = 0 \), but for now we consider only non-zero modes: \( b^{(j)} + \lambda_\ell^{(j)} > 0 \), \( \forall \ell : \lambda_0^{(j)} = 0, b^{(j)} > 0 \).

Let \( a_0, n_0 \) denote the Lie algebras of \( A, N \) in an Iwasawa decomposition \( G = KAN \). Since the rank of \( G \) is 1, \( \dim a_0 = 1 \) by definition, say \( a_0 = \mathbb{R}H_0 \) for a suitable basis vector \( H_0 \). One can normalize the choice of \( H_0 \) by \( \beta(H_0) = 1 \), where \( \beta : a_0 \to \mathbb{R} \) is the positive root which defines \( n_0 \); for more detail see Ref. [23]. Since \( \Gamma \) is torsion free, each \( \gamma \in \Gamma - \{1\} \) can be represented uniquely as some power of a primitive element \( \delta : \gamma = \delta^{j(\gamma)} \) where \( j(\gamma) \geq 1 \) is an integer and \( \delta \) cannot be written as \( \gamma_1^j \) for \( \gamma_1 \in \Gamma \), \( j > 1 \) an integer. Taking \( \gamma \in \Gamma \), \( \gamma \neq 1 \), one can find \( t_\gamma > 0 \) and \( m_\gamma \in M \defeq \{m_\gamma \in K | m_\gamma a = am_\gamma, \forall a \in A\} \) such that \( \gamma \) is \( G \)-conjugate to \( m_\gamma \exp(t_\gamma H_0) \), namely for some \( g \in G \), \( g\gamma g^{-1} = m_\gamma \exp(t_\gamma H_0) \). Besides let \( \chi_\sigma(m) = \text{trace}(\sigma(m)) \) be the character of \( \sigma \), for \( \sigma \) a finite-dimensional representation of \( M \).

**Theorem 2.1.** (Fried’s trace formula [24]) For \( 0 \leq j \leq d - 1 \),

\[
\text{Tr} \left( e^{-t \mathcal{L}^{(j)}} \right) = I^{(j)}(t, b^{(j)}) + I^{(j-1)}(t, b^{(j-1)}) + H^{(j)}(t, b^{(j)}) + H^{(j-1)}(t, b^{(j-1)}), \tag{2.3}
\]

where

\[
I^{(j)}(t, b^{(j)}) \defeq \frac{\chi(1) \text{Vol}(\Gamma \backslash G)}{4\pi} \int_{\mathbb{R}} \mu_{\sigma_j}(r) e^{-t[r^2 + b^{(j)} + (\rho_0 - j)^2]} dr, \tag{2.4}
\]

\[
H^{(j)}(t, b^{(j)}) \defeq \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in \Gamma \backslash \gamma_\Gamma(1)} \chi(\gamma) t_\gamma j(\gamma)^{-1} C(\gamma) \chi_{\sigma_j}(m_\gamma)
\]
\[ \rho_0 = \frac{d-1}{2}, \text{ and the function } C(\gamma), \quad \gamma \in \Gamma, \text{ defined on } \Gamma - \{1\} \text{ by} \]
\[ C(\gamma) \overset{\text{def}}{=} e^{-\rho_0 t_\gamma} \det_{t_0} (\text{Ad}(m_\gamma e^{t_\gamma H_0})^{-1} - 1)^{-1}. \quad (2.6) \]

For Ad denoting the adjoint representation of G on its complexified Lie algebra, one can compute \( t_\gamma \) as follows \cite{27}
\[ e^{t_\gamma} = \max\{ |c||c = \text{an eigenvalue of } \text{Ad}(\gamma) \}. \quad (2.7) \]

Here \( C_\Gamma \) is a complete set of representatives in \( \Gamma \) of its conjugacy classes; Haar measure on \( G \) is suitably normalized. In our case \( K \cong SO(d), M \cong SO(d-1) \). For \( j = 0 \) (i.e. for smooth functions or smooth vector bundle sections) the measure \( \mu_0(r) \) corresponds to the trivial representation of \( M \). For \( j \geq 1 \) there is a measure \( \mu_{\sigma}(r) \) corresponding to a general irreducible representation \( \sigma \) of \( M \). Let \( \sigma_j \) is the standard representation of \( M = SO(d-1) \) on \( \Lambda^j \mathbb{C}^{d-1} \). If \( d = 2n \) is even then \( \sigma_j \) (\( 0 \leq j \leq d-1 \)) is always irreducible; if \( d = 2n + 1 \) the every \( \sigma_j \) is irreducible except for \( j = (d-1)/2 = n \), in which case \( \sigma_n \) is the direct sum of two \((1/2)\)-spin representations \( \sigma^\pm : \sigma_n = \sigma^+ \oplus \sigma^- \). For \( j = n \) the representation \( \tau_n \) of \( K = SO(2n) \) on \( \Lambda^n \mathbb{C}^{2n} \) is not irreducible, \( \tau_n = \tau^+_n \oplus \tau^-_n \) is the direct sum of \((1/2)\)-spin representations. The Harish-Chandra Plancherel measures \( \mu_{\sigma_j}(r) \) are given by the following theorem.

**Theorem 2.2.** Let the group \( G = SO_1(2n, 1) \). Then
\[ \mu_{\sigma_j}(r) = \binom{2n-1}{j} \frac{\pi r}{2^{2n-4} \Gamma(n)^2} \prod_{i=2}^{j+1} \left[ r^2 + (n + \frac{3}{2} - i)^2 \right] \]
\[ \times \prod_{i=j+2}^{n} \left[ r^2 + (n + \frac{1}{2} - i)^2 \right] \tan(\pi r) \quad \text{for} \quad 0 \leq j \leq n-1, \quad (2.8) \]
\[ \mu_{\sigma_j}(r) = \binom{2n-1}{j} \frac{\pi r}{2^{2n-4} \Gamma(n)^2} \prod_{i=2}^{2n-j} \left[ r^2 + (n + \frac{3}{2} - i)^2 \right] \]
\[ \times \prod_{i=2n-j+1}^{n} \left[ r^2 + (n + \frac{1}{2} - i)^2 \right] \tan(\pi r) \quad \text{for} \quad n \leq j \leq 2n-1, \quad (2.9) \]
and \( \mu_{\sigma_j}(r) = \mu_{\sigma_{2n-j-1}}(r) \). 


For the group \( G = SO_1(2n+1, 1) \) one has

\[
\mu_{\sigma_j}(r) = \binom{2n}{j} \frac{\pi}{2^{4n-2\Gamma(n+\frac{1}{2})^2}} \prod_{i=1}^{j+1} \left[ r^2 + (n + 1 - i)^2 \right] 
\times \prod_{i=j+2}^{n} \left[ r^2 + (n - i)^2 \right] \quad \text{for} \quad 0 \leq j < n, \quad (2.10) 
\]

\[
\mu_{\sigma_j}(r) = \binom{2n}{j} \frac{\pi}{2^{4n-2\Gamma(n+\frac{1}{2})^2}} \prod_{i=1}^{2n-j+1} \left[ r^2 + (n + 1 - i)^2 \right] 
\times \prod_{i=2n-j+2}^{n} \left[ r^2 + (n - i)^2 \right] \quad \text{for} \quad n + 1 \leq j < 2n - 1. \quad (2.11) 
\]

We should note that the reason for the pair of terms \( \{I^{(j)}, I^{(j-1)}\}, \{H^{(j)}, H^{(j-1)}\} \) in the trace formula Eq. (2.3) is that \( \tau_j \) satisfies \( \tau_j|_M = \sigma_j \oplus \sigma_{j-1} \).

Finally using the result of Theorem 2.2, we have

\[
\mu_{\sigma_j}(r) = C^{(j)}(d)P(r, d) \times \begin{cases} 
\tanh(\pi r) & \text{for} \quad d = 2n \\
1 & \text{for} \quad d = 2n + 1 
\end{cases}, \quad (2.12) 
\]

\[
C^{(j)}(d) = \binom{d-1}{j} \frac{\pi}{2^{2d-4\Gamma(d/2)^2}}, \quad (2.13) 
\]

where the \( P(r, d) \) are even polynomials (with suitable coefficients \( a^{(j)}_{2\ell}(d) \)) of degree \( d - 1 \) for \( G \neq SO(2n+1, 1) \), and of degree \( d = 2n + 1 \) for \( G = SO_1(2n+1, 1) \) \([28, 29]\).

2.1. Case of the trivial representation. For \( j = 0 \) we take \( I^{(-1)} = H^{(-1)} = 0 \). Since \( \sigma_0 \) is the trivial representation \( \chi_{\sigma_0}(m_\gamma) = 1 \). In this case Fried’s formula Eq. (2.3) reduces exactly to the trace formula for \( j = 0 \) \([27, 29]\):

\[
\omega^{(0)}(t, b^{(0)}) = \frac{\chi(1)\text{vol}(\Gamma \backslash G)}{4\pi} \int_{\mathbb{R}} \mu_{\sigma_0}(r)e^{-(r^2+b^{(0)}+\rho_0^2)t}dr + H^{(0)}(t, b^{(0)}), \quad (2.14) 
\]

where \( \rho_0 \) is associated with the positive restricted (real) roots of \( G \) (with multiplicity) with respect to a nilpotent factor \( N \) of \( G \) in an Iwasawa decomposition \( G = KAN \). The function \( H^{(0)}(t, b^{(0)}) \) has the form
$$H(0)(t,b(0)) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in \mathcal{C}_\Gamma \setminus \{1\}} \chi(\gamma)t_\gamma j(\gamma)^{-1}C(\gamma)e^{-[b(0)t + \rho_0^2 t + t^2/(4t)]}.$$  

(2.15)

2.2. Case of zero modes. It can be shown that the Mellin transform of $H(0)(t,0)$ ($b(0) = 0$, i.e. the zero modes case)

$$\tilde{F}(0)(s) \overset{\text{def}}{=} \int_0^\infty H(0)(t,0)t^{s-1}dt,$$  

is a holomorphic function on the domain Re $s < 0$. Then using the result of Refs. [28, 25] one can obtain on Re $s < 0$,

$$\tilde{F}(0)(s) = \sum_{\gamma \in \mathcal{C}_\Gamma \setminus \{1\}} \chi(\gamma)t_\gamma j(\gamma)^{-1}C(\gamma)\int_0^\infty e^{-[\rho_0^2 t + t^2/(4t)]}t^{s-1}dt$$  

$$= \left(\frac{2\rho_0}{\sqrt{\pi}}\right)^{1-s} \sum_{\gamma \in \mathcal{C}_\Gamma \setminus \{1\}} \chi(\gamma)t_\gamma j(\gamma)^{-1}C(\gamma)t_\gamma^{-\frac{s}{2}}K_{\frac{1}{2}-s}(t_\gamma \rho_0),$$  

(2.17)

where $K_\nu(s)$ is the modified Bessel function, and finally

$$\tilde{F}(0)(s) = \frac{\sin(\pi s)}{\pi} \Gamma(s) \int_0^\infty \psi_{\Gamma}(t + 2\rho_0; \chi)(2\rho_0 t + t^2)^{-s}dt.$$  

(2.18)

Here $\psi_{\Gamma}(s; \chi) \equiv d(\log Z_{\Gamma}(s; \chi))/ds$, and $Z_{\Gamma}(s; \chi)$ is a meromorphic suitably normalized Selberg zeta function [31, 32, 33, 34, 35, 29, 37, 28].

3. The multiplicative anomaly

In this section the product of the operators on $j-$ forms $\otimes \mathcal{L}_p^{(j)}$, $\mathcal{L}_p^{(j)} = L^{(j)} + b^{(j)}_p$, $p = 1, 2$ will be considered. We are interested in multiplicative properties of determinants, the multiplicative anomaly [38, 2, 3].

The multiplicative anomaly $F(\mathcal{L}_1^{(j)}, \mathcal{L}_2^{(j)})$ reads

$$F(\mathcal{L}_1^{(j)}, \mathcal{L}_2^{(j)}) = \det_\zeta(\otimes \mathcal{L}_p^{(j)})\det_\zeta(\mathcal{L}_1^{(j)})\det_\zeta(\mathcal{L}_2^{(j)})^{-1},$$  

(3.1)

where we assume a zeta-regularization of determinants, i.e.

$$\det_\zeta(\mathcal{L}_p^{(j)}) \overset{\text{def}}{=} \exp \left(-\frac{\partial}{\partial s} \zeta(s|\mathcal{L}_p^{(j)})|_{s=0}\right).$$  

(3.2)

Generally speaking, if the anomaly related to elliptic operators is non-vanishing then the relation $\log \det(\otimes \mathcal{L}_p^{(j)}) = \text{Tr} \log(\otimes \mathcal{L}_p^{(j)})$ does not hold.
3.1. **The zeta function of the product of Laplacians.** The spectral zeta function associated with the product $\bigotimes_{p} \mathcal{L}_p^{(j)}$ has the form

$$\zeta(s|\bigotimes_{p} \mathcal{L}_p^{(j)}) = \sum_{\ell \geq 0} n_{\ell} \prod_{p} (\lambda_{\ell}^{(j)} + b_{p}^{(j)})^{-s}. \quad (3.3)$$

We shall always assume that $b_{1}^{(j)} \neq b_{2}^{(j)}$, say $b_{1}^{(j)} > b_{2}^{(j)}$. If $b_{1}^{(j)} = b_{2}^{(j)}$ then $\zeta(s|\bigotimes_{p} \mathcal{L}_p^{(j)}) = \zeta(2s|\mathcal{L}^{(j)})$ is a well-known function. For $b_{1}^{(j)}, b_{2}^{(j)} \in \mathbb{R}$, set $b_{+} \overset{\text{def}}{=} (b_{1}^{(j)} + b_{2}^{(j)})/2$, $b_{-} \overset{\text{def}}{=} (b_{1}^{(j)} - b_{2}^{(j)})/2$, thus $b_{1}^{(j)} = b_{+} + b_{-}$ and $b_{2}^{(j)} = b_{+} - b_{-}$.

**Theorem 3.1** [1]. The spectral zeta function can be written as follows:

$$\zeta(s|\bigotimes_{p} \mathcal{L}_p^{(j)}) = (2b_{-})^{\frac{s}{2}} \sqrt{\pi} \frac{\Gamma(s)}{\Gamma(s/2)} \int_{0}^{\infty} \omega_{\Gamma}^{(j)}(t, b_{+}) I_{s - 1/2}(b_{-}t) dt, \quad (3.4)$$

where the integral converges absolutely for $\text{Re} s > d/4$.

This formula is a main starting point to study the zeta function. It expresses $\zeta(s|\bigotimes_{p} \mathcal{L}_p^{(j)})$ in terms of the Bessel function $I_{s - 1/2}(b_{-}t)$ and $\omega_{\Gamma}^{(j)}(t, b_{+})$, where the trace formula applies to $\omega_{\Gamma}^{(j)}(t, b_{+})$. Let $B_{p}^{(j)} = (\rho_{0}(p) - j)^{2} + b_{p}^{(j)}$ and $A \overset{\text{def}}{=} \chi(1) \text{vol}(\Gamma \setminus G) C^{(j)}(d)/4$.

**Theorem 3.2.** For $\text{Re} s > d/4$ the explicit meromorphic continuation holds:

$$\zeta(s|\bigotimes_{p} \mathcal{L}_p^{(j)}) = A \sum_{\ell = 0}^{d-1} \left[ a_{2\ell}^{(j)}(d) \left( F_{\ell}^{(j)}(s) - E_{\ell}^{(j)}(s) \right) \right. + a_{2\ell}^{(j)}(d) \left. \left( F_{\ell}^{(j-1)}(s) - E_{\ell}^{(j-1)}(s) \right) \right] + I^{(j)}(s) + I^{(j-1)}(s), \quad (3.5)$$

where

$$E_{\ell}^{(j)}(s) \overset{\text{def}}{=} 4 \int_{0}^{\infty} \frac{dr_{1}^{2j+1}}{1 + e^{2\pi r_{1}}} \prod_{p} (r_{1}^{2} + B_{p}^{(j)})^{-s}, \quad (3.6)$$

which is an entire function of $s$,

$$F_{\ell}^{(j)}(s) \overset{\text{def}}{=} (B_{1}(j)B_{2}(j))^{-s} \ell! \left( \frac{2B_{1}(j)B_{2}(j)}{B_{1}(j)+B_{2}(j)} \right)^{\ell+1} \frac{(2s-1)(2s-2)\cdots(2s-(\ell+1))}{(2s-1)(2s-2)\cdots(2s-(\ell+1))}$$

$$\times F \left( \frac{\ell + 1}{2}, \frac{\ell + 2}{2}; s + \frac{1}{2}; \left( \frac{B_{1}(j) - B_{2}(j)}{B_{1}(j) + B_{2}(j)} \right)^{2} \right), \quad (3.7)$$
As a preliminary form of the zeta function

**Proposition 3.3.** A preliminary form of the zeta function \( \zeta(s) \otimes_p \mathcal{L}^{(j)}_p \) at \( s = 0 \) is

\[
\zeta(0) \otimes_p \mathcal{L}^{(j)}_p = A \sum_{\ell=0}^{q-1} \frac{(-1)^{\ell+1}}{2(\ell + 1)} \left[ \sum_p (a^{(j)}_{2\ell}(d) B_p(j))^{\ell+1} + a^{(j-1)}_{2\ell}(d) B_p(j-1)^{\ell+1} + (2 - 2^{-2\ell}) B_{2\ell+2} \left( a^{(j)}_{2\ell}(d) + a^{(j-1)}_{2\ell}(d) \right) \right],
\]

**Proposition 3.4.** The derivative of the zeta function at \( s = 0 \) has the form:

\[
\zeta'(0) \otimes_p \mathcal{L}^{(j)}_p = A \sum_{\ell=0}^{q-1} \left[ \sum_m \left( a^{(j)}_{2\ell}(d) \mathcal{E}^{(j)}_m + (a^{(j-1)}_{2\ell}(d) \mathcal{E}^{(j-1)}_m) \right) \right],
\]

where

\[
\mathcal{E}^{(j)}_1 = \ell! \left( B_1(j) + B_2(j) \right)^{\ell+1} \sum_{k=0}^{\ell} \frac{(-1)^{k+1}}{k!(\ell - k)!(j + 1 - k)!},
\]

\[
\mathcal{E}^{(j)}_2 = B_2(j)^{\ell+1} \left( \frac{B_1(j) - B_2(j)}{2B_1(j)} \right)^{\ell} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k + 1)!} \frac{(\ell + k + 1)!}{\ell!(\ell + 1)!} \times \sigma_n \left( \frac{B_1(j) - B_2(j)}{B_1(j)} \right)^k,
\]
\[ E^{(j)}_3 = \log(B_1(j)B_2(j)) \left( \frac{-1}{2(\ell + 1)} \right) (B_1(j)^{\ell+1} + B_2(j)^{\ell+1}) \]

\[ -4 \int_0^\infty r^{2\ell+1} \log \left( \frac{r^2+B_1(j)}{r^2+B_2(j)} \right) \, dr, \tag{3.16} \]

\[ E^{(j)}_4 \equiv \frac{d}{ds} \mathcal{T}^{(j)}(s)|_{s=0} = \int_0^\infty \left[ H^{(j)}(t, b_1^{(j)}) + H^{(j)}(t, b_2^{(j)}) \right] t^{-1} dt, \tag{3.17} \]

and \( \sigma_n \overset{\text{def}}{=} \sum_{k=1}^{n} k^{-1} \).

### 3.2. The residue formula and the multiplicative anomaly.

The value of \( F(\mathcal{L}_1, \mathcal{L}_2) \) can be expressed by means of the non-commutative Wodzicki residue [1]. Let \( O_p, p = 1, 2, \) be invertible elliptic (pseudo-) differential operators of real non-zero orders \( \alpha \) and \( \beta \) such that \( \alpha + \beta \neq 0 \). Even if the zeta functions for operators \( O_1, O_2 \) and \( O_1 \otimes O_2 \) are well defined and if their principal symbols satisfy the Agmon-Nirenberg condition (with appropriate spectra cuts) one has in general that \( F(O_1, O_2) \neq 1 \). For such invertible elliptic operators the formula for the anomaly of commuting operators holds:

\[ \mathcal{A}(O_1, O_2) = \mathcal{A}(O_2, O_1) = \log(F(O_1, O_2)) = \frac{\text{res} \left[ (\log(O_1^\beta \otimes O_2^{-\alpha}))^2 \right]}{2\alpha\beta(\alpha + \beta)}. \tag{3.18} \]

More general formulae have been derived in Refs. [2, 3]. Furthermore the anomaly can be iterated consistently. Indeed, using Eq. (3.18) we have

\[ \mathcal{A}(O_1, O_2, O_3) = \zeta'(0\|O_3) - \sum_j \zeta'(0\|O_j) - \mathcal{A}(O_1, O_2), \]

\[ \mathcal{A}(O_1, O_2, ..., O_n) = \zeta'(0\|O_n) - \sum_j \zeta'(0\|O_j) - \mathcal{A}(O_1, O_2) \]

\[- \mathcal{A}(O_1, O_2, O_3) ... - \mathcal{A}(O_1, O_2, ..., O_{n-1}). \tag{3.19} \]

In particular, for \( n = 2 \) and \( O_p \equiv \mathcal{L}^{(j)}_p \) the anomaly is given by the following theorem.

**Theorem 3.5.** The explicit formula for the multiplicative anomaly is

\[ \mathcal{A}(\mathcal{L}^{(j)}_1, \mathcal{L}^{(j)}_2) = A \sum_{\ell=0}^{n-1} \left[ \Omega^{(j)}_\ell + \Omega^{(j-1)}_\ell \right], \tag{3.20} \]
where
\[
\Omega_{\ell}^{(j)} = \frac{a_{2\ell}^{(j)}(d)(-1)^\ell}{2} \left[ \frac{\ell}{2}(B_1(j) - B_2(j))^2 B_2(j)^{\ell-1} + \frac{\ell(\ell - 1)}{4}(B_1(j) - B_2(j))^3 B_2(j)^{\ell-2} + \sum_{p=3}^{\ell} \frac{\ell!}{(p + 1)p!(\ell - p)!} \times \left( \frac{1}{p} + \frac{1}{p - 1} + \sum_{q=1}^{p-2} \frac{1}{p - q - 1} \right) (B_1(j) - B_2(j))^{p+1} B_2(j)^{\ell-p} \right].
\]
(3.21)

We note that for the four-dimensional space with \( G = SO(4,1) \), one derives from Theorem 3.5. the result
\[
A(L_1, L_2) = -A_G^{(j)} (b_1^{(j)} - b_1^{(j)})^2 - A_G^{(j-1)} (b_1^{(j-1)} - b_1^{(j-1)})^2,
\]
which also follows from Wodzicki’s formula (3.18), where we should set
\[
A_G^{(j)} = A_{21}^{(j)}(4)/4.
\]

4. The Conformal Anomaly and Associated Operator Products.

In this section we start with a conformal deformation of a metric and the conformal anomaly of the energy stress tensor. It is well known that (pseudo-) Riemannian metrics \( g_{\mu\nu}(x) \) and \( \tilde{g}_{\mu\nu}(x) \) on a manifold \( X \) are (pointwise) conformal if \( \tilde{g}_{\mu\nu}(x) = \exp(2f)g_{\mu\nu}(x) \), \( f \in C^\infty(\mathbb{R}) \). For constant conformal deformations the variation of the connected vacuum functional (the effective action) can be expressed in terms of the generalized zeta function related to an elliptic self-adjoint operator \( \mathcal{O} \) [13]:
\[
\delta W = -\zeta(0|\mathcal{O}) \log \mu^2 = \int_X <T_{\mu\nu}(x) > \delta g^{\mu\nu}(x) dx,
\]
(4.1)
where \( <T_{\mu\nu}(x) > \) means that all connected vacuum graphs of the stress-energy tensor \( T_{\mu\nu}(x) \) are to be included. Therefore Eq. (4.1) leads to
\[
<T_\mu^\mu(x) > = (\text{Vol}(X))^{-1}\zeta(0|\mathcal{O}).
\]
(4.2)

The formulae (3.5), (3.9), (3.10) and (3.11) give an explicit result for the conformal anomaly, namely
\[
<T_\mu^\mu(x) >_{(\sigma = \otimes L_p^{(j)})} = \frac{1}{(4\pi)^{d/2} \Gamma(d/2)} \sum_{\ell=0}^{d-1} \frac{(-1)^{\ell+1}}{2(\ell + 1)} \left\{ \sum_p \left[ a_{2\ell}^{(j)}(d)B_p(j)^{\ell+1} + a_{2\ell}^{(j-1)}(d)B_p(j - 1)^{\ell+1} \right] + (2 - 2^{-2\ell})B_{2\ell+2} \left( a_{2\ell}^{(j)}(d) + a_{2\ell}^{(j-1)}(d) \right) \right\},
\]
(4.3)
where $d$ is even. For $B_{1,2}(j) = B(j), B_{1,2}(j-1) = B(j-1)$ the anomaly (4.3) has the form

$$<T^\mu_\mu(x) >_{(\mathcal{L}^{(i)} \otimes \mathcal{L}^{(j)})} = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \sum_{\ell=0}^{d-1} \frac{(-1)^{\ell+1}}{2(\ell + 1)} \left\{ a_{2\ell}^{(j)}(d) B(j)^{\ell+1} + a_{2\ell}^{(j-1)}(d) B(j-1)^{\ell+1} \right\} + \left(2 - 2^{-2\ell}\right) B_{2\ell+2} \left( a_{2\ell}^{(j)}(d) + a_{2\ell}^{(j-1)}(d) \right) .$$

(4.4)

Note that for a minimally coupled scalar field of mass $m$, $B(0) = \rho_0^2 + m^2$. The simplest case is, for example, $G = SO_1(2,1) \simeq SL(2,\mathbb{R})$; besides $X = \mathbb{H}^2$ is a two-dimensional real hyperbolic space. Then we have $\rho_0^2 = 1/4$, $a_{20}^{(0)} = 1$, and finally

$$<T^\mu_\mu(x \in \Gamma \backslash \mathbb{H}^2) >_{(\mathcal{L}^{(0)} \otimes \mathcal{L}^{(0)})} = - \frac{1}{4\pi} \left( b + \frac{1}{3} \right) .$$

(4.5)

For real $d$-dimensional hyperbolic space the scalar curvature is $R(x) = -d(d-1)$. In the case of the conformally invariant scalar field we have $B(0) = \rho_0^2 + R(x)(d-2)/[4(d-1)]$. As a consequence, $B(0) = 1/4$ and

$$<T^\mu_\mu(x \in \Gamma \backslash \mathbb{H}^d) >_{(\mathcal{L}^{(0)} \otimes \mathcal{L}^{(0)})} = \frac{1}{(4\pi)^d\Gamma(d/2)} \sum_{\ell=0}^{d-1} \frac{(-1)^{\ell+1}}{\ell + 1} a_{2\ell}^{(0)}(d) \times \left\{ 2^{-2\ell-2} + (1 - 2^{-2\ell-1}) B_{2\ell+2} \right\} .$$

(4.6)

Thus in conformally invariant scalar theory the anomaly of the stress tensor coincides with one associated with operator product. This statement holds not only for hyperbolic spaces considered above but for all constant curvature manifolds as well [17].

5. Product of Einstein manifolds

In this section we consider the problem of the global existence of zeta functions on (pseudo-) Riemannian product manifolds, a product of two Einstein manifolds [39, 40]

$$(X, g, \mathcal{P}) = (X_1, g_1, \mathcal{P}_1) \otimes (X_2, g_2, \mathcal{P}_2),$$

(5.1)

where $g = g_1 \otimes g_2$ and the metric $g$ separates the variables, i.e.

$$ds^2 = g_{\alpha\beta}(x) dx^\alpha \otimes dx^\beta + g_{\mu\nu}(y) dy^\mu \otimes dy^\nu.$$  

(5.2)

The tangent bundle splits as $TX = TX_1 \oplus TX_2$ and $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$, where $\mathcal{P}_p$ ($p = 1, 2$) are the corresponding projections on $TX_p$.

$$\mathcal{P}^2 = Id, \quad g(\mathcal{P}_X, \mathcal{P}_Y) = g(X, Y), \quad X, Y \in \mathcal{G}(X),$$

(5.3)
and $G(X)$ being the Lie algebra of vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $X$. The trivial examples of an almost-product structure are given by the choices $\mathcal{P} = \pm Id$ (± identity).

We recall some facts about Einstein manifolds. An almost-product (pseudo-) Riemannian structure $(X, g, \mathcal{P})$ is integrable iff $\Delta \mathcal{P} = 0$ for the Levi-Civita connection $\Delta$ of $g$. The two integrable complementary subbundles, i.e. both foliations, are totally geodesic [39, 40]. Let $X$ be a pseudo-Kähler manifold. Such a manifold is an Einstein manifold iff $\mathcal{P}$ is trivial examples of an almost-product structure are given by the choices $g_1$ and $g_2$ are Einstein metrics for the same constant $\lambda$ [39, 40, 41], i.e.

$$\text{Ric}(g) = \lambda g.$$ (5.4)

Our consideration will be restricted to only locally decomposable manifolds. A wide class of (pseudo-) Riemannian manifolds includes non-locally decomposable manifolds as well, which are given by warped product space-times [12, 13, 14]. Note that many exact solutions of Einstein equations (associated with Schwarzschild, Robertson-Walker, Reissner-Nordström, de Sitter space-times) and $p$-brane solutions are, in fact, warped product space-times.

5.1. The explicit form of the zeta function. We study the zeta function

$$\zeta(s|\mathcal{L}^{(j)} \bigoplus \mathcal{L}^{(k)}) = \zeta_{\Gamma_1 \setminus \mathcal{X}} \otimes \zeta_{\Gamma_2 \setminus \mathcal{Y}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \omega_{\Gamma_1}^{(j)}(t) \omega_{\Gamma_2}^{(k)}(t) t^{s-1} dt, \quad \text{Re } s > d.$$ (5.5)

Let $B = B_1(j) + B_2(k)$, $A_p \overset{\text{def}}{=} \chi(1) \text{vol}(\Gamma_p \setminus G) C^{(j)}(d)/4$, $y_p(s; z) \overset{\text{def}}{=} s/2 + (-1)^{p-1} z$, $p = 1, 2$ ($s, z \in \mathbb{C}$). The explicit construction gives more, namely

**Theorem 5.1.** The function $\zeta(s|\mathcal{L}^{(j)} \bigoplus \mathcal{L}^{(k)})$ admits an explicit meromorphic continuation to $\mathbb{C}$ with at most a simple pole at $s = 1, 2, \ldots, d$. In particular on the domain $\text{Re } s < 1$,

$$\zeta(s|\mathcal{L}^{(j)} \bigoplus \mathcal{L}^{(k)}) = \frac{\pi^2}{2} A_1 A_2 \left[ C^{(j)}(d) + C^{(j-1)}(d) \right] \left[ C^{(k)}(d) + C^{(k-1)}(d) \right]$$

$$\times \sum_{m=0}^{n-1} \sum_{\ell=0}^{m} \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{\mu} \frac{a_{2m}^{(j)}(d) + a_{2m}^{(j-1)}(d)}{(m-\ell)! (\mu-\nu)!}$$

$$\times \int_0^\infty r^{2(\ell-\ell)} \text{sech}^2(\pi r) K_{\mu-\nu}(s - \ell - \nu - 1; r^2 + B, \pi) dr$$

$$\times \frac{1}{(s-1) \cdots (s - (\ell + 1))(s - (\ell + 2)) \cdots (s - (\ell + 1 + \nu + 1))}$$

$$+ C^{(j)}(d) V_1 \sin(\pi s) \sum_{m=0}^{n-1} \left[ a_{2m}^{(j)}(d) + a_{2m}^{(j-1)}(d) \right] \int_{\mathbb{R}} r^{2m+1} \tanh(\pi r)$$
\[ K - \Gamma = Re \rho_0 \Gamma, \quad (n - 1) \]

The simplest case is, for example, \( GH_1 \) we have

\[ \int_0^{\infty} \Psi_{1 \gamma} \left( \rho_0 - j + t + \sqrt{r^2 + B}; \chi_1 \right) \left( 2t \sqrt{r^2 + B} + t^2 \right)^{-s} dt \]

\[ + \Psi_2 \left( \rho_0 - k + t + \sqrt{r^2 + B}; \chi_2 \right) \left( 2t \sqrt{r^2 + B} + t^2 \right)^{-s} dt \]

\[ + C^{(j)}(d)V_2 \sin(\pi s) \sum_{m=0}^{n-1} \left[ a_{2m}^{(k)}(d) + a_{2m}^{(k-1)}(d) \right] \int_0^{\infty} r^{2m+1} \tanh(\pi r) \]

\[ \times \left[ \int_0^{\infty} \Psi_{1 \gamma} \left( \rho_0 - j + t + \sqrt{r^2 + B}; \chi_1 \right) \left( 2t \sqrt{r^2 + B} + t^2 \right)^{-s} dt \]

\[ + \Psi_2 \left( \rho_0 - k + t + \sqrt{r^2 + B}; \chi_2 \right) \left( 2t \sqrt{r^2 + B} + t^2 \right)^{-s} dt \] \[ dr \]

\[ + \frac{1}{2 \pi^3 i \Gamma(s)} \int_{Re z = \epsilon} dz \left[ \sin(\pi(z + s/2)) \left[ \sin(\pi(z + s/2)) \Gamma(z + s/2) \Gamma(z - s/2) \right] \right] \]

\[ \times \left[ \int_0^{\infty} \Psi_{1 \gamma} \left( \rho_0 - j + t + B_1^2; \chi_1 \right) \left( 2t B_1^2 + t^2 \right)^{-y_1(s,z)} \right. \]

\[ + \Psi_{1 \gamma} \left( \rho_0 - k + t + B_1^2; \chi_2 \right) \left( 2t B_1^2 + t^2 \right)^{-y_2(s,z)} \]

\[ \times \left[ \int_0^{\infty} \Psi_{1 \gamma} \left( \rho_0 - j + t + B_2^2; \chi_1 \right) \left( 2t B_2^2 + t^2 \right)^{-y_1(s,z)} \right] \]

\[ + \Psi_{1 \gamma} \left( \rho_0 - k + t + B_2^2; \chi_2 \right) \left( 2t B_2^2 + t^2 \right)^{-y_2(s,z)} \]

\[ dt \], \quad (5.6) \]

for any \(-1/2 \leq \epsilon \leq 1/2\). For \( a, \delta > 0, \nu \in \mathbb{N} \) the entire function \( K_\nu(s; \delta, a) \) of \( s \) is defined by

\[ K_\nu(s; \delta, a) \overset{\text{def}}{=} \int_{\mathbb{R}} \frac{r^{2\nu} \sech^2(ar) dr}{(\delta + r^2)^s}. \quad (5.7) \]

All of the integrals are entire functions of \( s \).

The simplest case is, for example, \( G = SO_1(2, 1) \simeq SL(2, \mathbb{R}) \); besides \( X = \mathbb{H}^2 \) is a two-dimensional real hyperbolic space. Then for \( j = k = 0, \Gamma_1 = \Gamma_2 = \Gamma \) we have \( a_{20}^{(0)}(2) = 1 \) and \( \mu_\sigma_0(r) = \pi r \tanh(\pi r) \) and for \( Re s < 1 \) we have

\[ \zeta(s | L^{(0)} \bigoplus L^{(0)}) = \frac{\pi A_1^2}{2(s-1)(s-2)} \int_0^{\infty} \sech^2(\pi r) K_0 \left( s - 2; r^2 + B, \pi \right) dr \]

\[ + \frac{2}{\pi} A_1 \sin(\pi s) \int \tanh(\pi r) dr \]

\[ \times \int_0^{\infty} \psi_T \left( \frac{1}{2} + t + \sqrt{r^2 + B}; \chi \right) \left( 2t \sqrt{r^2 + B} + t^2 \right)^{-s} dt \]

\[ + \frac{1}{2 \pi^3 i \Gamma(s)} \int_{Re z = \epsilon} \left[ \sin(\pi(z + s/2)) \left[ \sin(\pi(z + s/2) - z) \right] \Gamma(z + s/2) \Gamma(z - s/2) \right] dz \]
\[
\times \int_0^\infty \psi_T \left( \frac{1}{2} + t + B_0^{\frac{3}{2}}; \chi \right) \left( 2tB_0^{\frac{3}{2}} + t^2 \right)^{-y_1(s;z)} \, dt
\]
\[
\times \int_0^\infty \psi_T \left( \frac{1}{2} + t + B_0^{\frac{3}{2}}; \chi \right) \left( 2tB_0^{\frac{3}{2}} + t^2 \right)^{-y_2(s;z)} \, dt,
\] (5.8)

where \( B = \frac{1}{2} + 2b^{(0)} \) and \( B(0) = \frac{1}{4} + b^{(0)} \).

6. Quadratic Functional with Elliptic Resolvent and Analytic Torsion

Let \( \chi : \pi_1(X_\Gamma) \rightarrow O(V, \langle \cdot, \cdot \rangle_V) \) be a representation of \( \pi_1(X_\Gamma) \) on a real vector space \( V \). The mapping \( \chi \) determines (on the basis of a standard construction in differential geometry) a real flat vector bundle \( \xi \) over \( X_\Gamma \) and a flat connection map \( D \) on the space \( \Omega(X_\Gamma, \xi) \) of differential forms on \( X_\Gamma \) with values in \( \xi \). One can say that \( \chi \) determines the space of smooth sections in the vector bundle \( \Lambda(TX_\Gamma)^* \otimes \xi \).

Let \( D_j \) denote the restriction of \( D \) to the space \( \Omega^j(X_\Gamma, \xi) \) of \( j \)-forms and let

\[
H^j(D) = \ker(D_j) / \text{Im}(D_{j-1})^{-1}
\] (6.1)

be the corresponding cohomology spaces. There exists a canonical Hermitian structure on the bundle \( \chi \) which we denote by \( \langle \cdot, \cdot \rangle_V \). The above mentioned Hermitian structure determines for each \( x \in X_\Gamma \) a linear map \( \langle \cdot, \cdot \rangle_x : \xi_x \otimes \xi_x \rightarrow \mathbb{R} \), and the diagram for linear maps hold (see Ref. \[45\] for details)

\[
(\Lambda^p(T_xX_\Gamma)^* \otimes \xi_x) \otimes (\Lambda^q(T_xX_\Gamma)^* \otimes \xi_x) \xrightarrow{\bigwedge} \Lambda^{p+q}(TX_\Gamma)^* \otimes (\xi_x \otimes \xi_x)
\]

\[
\xrightarrow{\langle \cdot, \cdot \rangle_x} \Lambda^{p+q}(TX_\Gamma)^*,
\] (6.2)

where the image of \( \omega_x \otimes \tau_x \) under the first map has been denote by \( \langle \omega_x \wedge \tau_x \rangle_x \).

We define the quadratic functional \( S_D \) on \( \Omega^j(X_\Gamma, \xi) \) by

\[
S_D(\omega) = \int_{X_\Gamma} \langle \omega(x) \wedge (D_j \omega)(x) \rangle_x.
\] (6.3)

One can construct from the metric on \( X_\Gamma \) and Hermitian structure in \( \xi \) a Hermitian structure in \( \Lambda(T_xX_\Gamma)^* \otimes \xi \) and the inner products \( \langle \cdot, \cdot \rangle_j \) in the space \( \Omega^j(X_\Gamma, \xi) \). Thus

\[
S_D(\omega) = \langle \omega, T \omega \rangle_j, \quad T = *D_j,
\] (6.4)

where \( * \) is the Hodge-star map. Recall that the map \( T \) is formally self adjoint with the property \( T^2 = D^*_j D_j \). Let \( (J_p, D_p) \) be a complex, i.e. a sequence of vector space \( J_p \) and linear operators \( D_p \) acting from the
space \( J_p \) to the space \( J_{p+1} \), \((J_{-1} = J_{d+1})\) and satisfying \( D_{p+1}D_p = 0 \) for all \( p = 0, 1, \ldots, d \). Let us define the adjoint operators \( D_p^* : J_{p+1} \rightarrow J_p \) by \( \langle a, D_p b \rangle_{p+1} = \langle D_p^* a, b \rangle_p \). For the functional (6.3) there is a canonical topological elliptic resolvent \( R(S_D) \) (a chain of linear maps)

\[ 0 \rightarrow \Omega^0(X, \xi) \xrightarrow{D_0} \cdots \xrightarrow{D_{d-2}} \Omega^{d-1}(X, \xi) \xrightarrow{D_{d-1}} \ker(S_D) \xrightarrow{0} 0. \]  

(6.5)

From Eq. (6.5) it follows that for \( R(S_D) \) we have \( H^p(R(S_D)) = H^{d-p}(D) \) and \( \ker(S_D) \equiv \ker(T) = \ker(D_j) \).

Let us choose an inner product \( \langle \cdot, \cdot \rangle_{H^p} \) in each space \( H^p(R(S_D)) \). The partition function of \( S_D \) associated to points in the moduli space of flat gauge fields \( \omega(x) \) on \( X \) with the resolvent (6.5) can be written in the form (see for example Ref. 45)

\[ Z(\beta) \equiv Z(\beta; R(S_D), \langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle) = \mathcal{Z}(\beta, \xi, \eta) \tau(X, \chi, \langle \cdot, \cdot \rangle)^{1/2}, \]  

(6.6)

where \( \beta = i\lambda, \lambda \in \mathbb{R}, \mathcal{Z}(\beta, \xi, \eta) \) is known function of \( \beta \). The function \( \xi \) appearing in the partition function above can be expressed in terms of the dimensions of the cohomology spaces of \( D \),

\[ \xi \equiv \xi(0|T) = -\sum_{p=0}^{d} (-1)^p \dim H^p(R(S)) = (-1)^{d+1} \sum_{q=0}^{n} (-1)^q \dim H^q(D). \]  

(6.7)

The dependence of \( \eta = \eta(0|T_D) \) on the connection map \( D \) can be expressed with the help of formulae for the index of the twisted signature operator for a certain vector bundle over \( X \) \((0, 1) \)[10]. It can be shown that \( 15 \)

\[ \eta(s|B^{(t)}) = 2\eta(s|T_{D_D^{(0)}}), \]  

(6.8)

where the \( B^{(t)} \) are elliptic self adjoint maps on \( \Omega(X, \xi) \) defined on \( j \)-forms by

\[ B_j^{(t)} = (-i)^\lambda \left( *D^{(t)} + (-1)^{j+1} D^{(t)} * \right). \]  

(6.9)

In this formula \( \lambda(j) = (j+1)(j+2) + n+1 \) and for the Hodge star-map we have used that \( *\alpha \wedge \beta = \langle \alpha, \beta \rangle_{vol} \). From the Hodge theory

\[ \dim \ker B^{(t)} = \sum_{q=0}^{d} \dim H^q(D^{(t)}). \]  

(6.10)

The metric-dependence of \( \eta \) enters through \( L^r(TX) \) and \( \eta(0|T_{D_D^{(0)}}) \), where \( L^r(TX) \) is the \( r \)'th term in Hirzebruch’s \( L \)-polynomial (see
for detail Ref. [44] and $D^{(0)}$ is an arbitrary flat connection map on $\Omega(X_\Gamma, \xi)$. For $d = 3$ the only contribution of the $L$-polynomial is $L_0 = 1$ and the metric-dependance of $\eta$ is determined alone by $\eta(0|T^{(0)}_{D(\omega)})$.

The factor $\tau(X_\Gamma, \chi, \langle \cdot, \cdot \rangle_H)$ is independent of the choice of metric $g$ on $X$ [43]. In fact this quantity is associated with the analytic (Ray-Singer) torsion $T_{an}(X_\Gamma)$ of the representation $\chi$ of $\pi_1(X_\Gamma)$ constructed using the metric $g$. If $H^q(D) \neq 0$ and $H^q(D) = 0$ for $q = 1, \ldots, n$, $d = 2n + 1$ the dimension of $X$, then the product

$$\tau(X_\Gamma, \chi, \langle \cdot, \cdot \rangle_H) = T_{an}(X_\Gamma) \cdot \text{Vol}(X_\Gamma)^{-\dim H^0(D)},$$

(6.11)
is independent of the choice of metric $g$, i.e. the metric dependence of the Ray-Singer torsion $T_{an}(X_\Gamma)$ factors out as $\text{Vol}(X_\Gamma)^{-\dim H^0(D)}$.

6.1. Connected sum of 3-manifolds. Recall that an embedding of the cohomology $H(X_\Gamma; \xi)$ into $\Omega(X_\Gamma; \xi)$ as the space of harmonic forms induces a norm $\| \cdot \|_{RS}$ on the determinant line $\text{det} H(X_\Gamma; \xi)$. The Ray-Singer norm $\| \cdot \|_{RS}$ on $\text{det} H(X_\Gamma; \xi)$ is defined by [19]

$$\| \cdot \|_{RS} \overset{def}{=} | \cdot | \prod_{j=0}^{\dim X} \left[ \exp \left( -\frac{d}{ds} \zeta(s|L^{(j)}) |_{s=0} \right) \right]^{-1/2}$$

(6.12)

where the zeta function $\zeta(s|L^{(j)})$ of the Laplacian acting on the space of $j$–forms orthogonal to the harmonic forms has been used. For a closed connected orientable smooth manifold of odd dimension and for any Euler structure $\eta \in \text{Eul}(X)$ the Ray-Singer norm of its cohomological torsion $T_{an}(X_\Gamma, \eta) \in \text{det} H(X_\Gamma; \xi)$ is equal to the positive square root of the absolute value of the monodromy of $\xi$ along the characteristic class $c(\eta) \in H^1(X_\Gamma)$ [17]: $\| T_{an}(X_\Gamma) \|_{RS} = |\text{det}_c c(\eta)|^{1/2}$. In the special case where the flat bundle $\xi$ is acyclic ($H^q(X_\Gamma; \xi) = 0$ for all $q$) we have

$$[T_{an}(X_\Gamma, \eta)]^2 = |\text{det}_c c(\eta)| \prod_{j=0}^{\dim X} \left[ \exp \left( -\frac{d}{ds} \zeta(s|L^{(j)}) |_{s=0} \right) \right]^{(-1)^{j+1}}.$$  

(6.13)

If $\xi$ is unimodular then $|\text{det}_c c(\eta)| = 1$ and the torsion $T_{an}(X_\Gamma)$ does not depend on the choice of $\eta$. For odd-dimensional manifold the Ray-Singer norm is a topological invariant: it does not depend on the choice of metric on $X$ and $\xi$, used in the construction. But for even-dimensional $X$ this is not the case [43].

Suppose the flat bundle $\xi$ is acyclic. The analytic torsion $T_{an}(\Gamma \backslash \mathbb{H}^3)$ can be expressed in terms of the Selberg zeta functions $Z_j(s, \chi_j)$. Indeed the Ruelle zeta function in three dimension associated with the closed oriented hyperbolic manifold $X_\Gamma = \Gamma \backslash \mathbb{H}^3$ has the form
The function $R_\chi(s)$ extends meromorphically to the entire complex plane $\mathbb{C}$ \[ [49]. For the Ray-Singer torsion one gets \[ [50]$
abla_{\text{an}}(\Gamma \backslash \mathbb{H}^3) = R_\chi(0) = [Z_0(2, \chi_0)]^2 \exp \left( -\frac{\text{Vol}(\mathcal{F})}{3\pi} \right), \tag{6.15}$

where $\text{Vol}(\mathcal{F})$ is a volume of the fundamental domain $\mathcal{F}$ of $\Gamma \backslash \mathbb{H}^3$. In the presence of non-vanishing Betti numbers $b_j = b_j(X_\Gamma)$ we have \[ [50, 51]$
abla_{\text{an}}(\Gamma \backslash \mathbb{H}^3) = (b_1 - b_0)! [Z_0^{(b_0)}(2, \chi_0)]^2 \exp \left( -\frac{\text{Vol}(\mathcal{F})}{3\pi} \right). \tag{6.16}$

In Chern-Simons theory the partition function at level $k$ ($\lambda = 2\pi k$) depends on a framing (i.e. on a trivialization of the normal bundle to the link) of twice the tangent bundle as a $\text{Spin}(6)$ bundle, henceforth referred to as 2-framing. In particular for the $SU(2)$ theory in the large $k$– limit the partition function for a connected sum $\mathfrak{X} = X_{\Gamma,1} \# X_{\Gamma,2} \# \ldots \# X_{\Gamma,N}$ can be written as follows \[ [52]$
abla_{\text{sc}}(\mathfrak{X}) = \frac{\bigotimes_{\ell=1}^N \nabla_{\text{sc}}(X_{\Gamma,\ell})}{\nabla(S^3)^{N-1}}. \tag{6.17}$

Eq. (6.17) holds for any given 2-framings among $X_{\Gamma,p}$ and $X_{\Gamma,q}$, $p, q \leq N$, the induced 2-framing on $X_{\Gamma,p} \# X_{\Gamma,q}$, and a canonical 2-framing on $S^3$. Since the Ray-Singer torsion on $S^3$ is to be equal one, $\nabla(S^3) = \sqrt{2\pi k^{-3/2}}$, the partition function associated with the semiclassical approximation ($k \to \infty$) takes the form

$$
\nabla_{\text{sc}}(\mathfrak{X}) = \left( \frac{k^3}{2\pi^2} \right)^{\frac{N-1}{2}} \bigotimes_{\ell=1}^N \nabla_{\text{sc}}(X_{\Gamma,\ell}) = \sqrt{2\pi k^{-\frac{3}{2}}} \bigotimes_{\ell=1}^N \left| R_{\chi(\ell)}(0) \right|^{\frac{1}{4}}, \tag{6.18}
$$

while in the presence of non-vanishing Betti numbers $b_{j,\ell} = b_j(X_{\Gamma,\ell})$ one gets

$$
\nabla_{\text{sc}}(\mathfrak{X}) = \sqrt{2\pi k^{-\frac{3}{2}}} \bigotimes_{\ell=1}^N \left[ \frac{(b_{1,\ell} - b_{0,\ell})!(Z_0^{(b_{0,\ell})}(2, \chi_0))^2}{(b_{0,\ell})^2 Z_1^{(b_{1,\ell} - b_{0,\ell})}(1, \chi_1)} \right]^{\frac{1}{4}} 
\times \exp \left[ -\frac{1}{12\pi} \bigoplus_{\ell=1}^N \text{Vol}(\mathcal{F}_\ell) \right]. \tag{6.19}
$$
In the case of non-trivial characters $b_0(X_{\Gamma,\ell}) = 0$. If $b_1 = 0$ then Eq. (6.15) holds.

For the trivial character one has $b_0 = 1$ (for any closed manifold) and $b_1 = 0$ for an infinite number of $X_{\Gamma} = \Gamma \setminus \mathbb{H}^3$. The function $\mathcal{R}(s)$ has a zero at $s = 0$ of order 4 \[26\]. However, there is a class of compact sufficiently large hyperbolic manifolds which admit arbitrarily large values of $b_1(X_{\Gamma})$. Sufficiently large manifolds $X_{\Gamma}$ contain a surface $S$ whose fundamental group $\pi_1(S)$ is infinite and such that $\pi_1(S) \subset \pi_1(X_{\Gamma})$.

It seems that the most important problem in 3-topology is the classification problem. In general, hyperbolic manifolds have not been completely classified and therefore a systematic computation is not yet possible. However this is not the case for certain sufficiently large manifolds, the Haken manifolds \[53\]. There exists an algorithm for the enumeration of all Haken manifolds and there exists an algorithm for recognizing homeomorphy of the Haken manifolds \[54\]. Both algorithms depend on normal surface theory in the manifold, developed primarily by Haken. These manifolds give an essential contribution to the partition function (6.19).

7. Concluding remarks

We have obtained an explicit formula for the multiplicative anomaly (Theorem 3.5.). The anomaly is equal to zero for $d = 2$ and for the odd dimensional cases. We have preferred to limit ourselves here to discuss in detail various particular cases and emphasize the general picture. It seems to us that the explicit results for the anomaly (3.20), (3.22) are not only interesting as mathematical results but are of physical interest.

Besides we have considered product structures on closed real hyperbolic manifolds. In fact the explicit form of the zeta function on product spaces (Theorem 5.1.) is derived. As an example the zeta function associated with the Kronecker sum of Laplacians on twisted forms is calculated in two-dimensional case.

Finally the explicit formulae for analytic torsion $T_{an}(X_{\Gamma})$ (a topological invariant) on manifolds $X_{\Gamma} = \Gamma \setminus \mathbb{H}^d$ and on a connected sum of such manifolds are derived. We hope that proposed discussion of this invariants will be interesting in view of future applications to concrete problems in quantum field theory.

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