Axion cyclotron emissivity of magnetized white dwarfs and neutron stars

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The energy loss rate of a magnetized electron gas emitting axions $a$ due to the process $e^- \rightarrow e^- + a$ is derived for arbitrary magnetic field strength $B$. Requiring that for a strongly magnetized neutron star the axion luminosity is smaller than the neutrino luminosity we obtain the bound $g_{ae} \lesssim 10^{-10}$ for the axion electron coupling constant. This limit is considerably weaker than the bound derived earlier by Borisov and Grishina using the same method. Applying a similar argument to magnetic white dwarf stars results in the more stringent bound $g_{ae} \lesssim 9 \cdot 10^{-13} (T/10^7 K)^{5/4} (B/10^{16} G)^{-2}$, where $T$ is the internal temperature of the white dwarf.

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I. INTRODUCTION

The axion $a$ is a pseudoscalar boson introduced by Peccei and Quinn (PQ) to solve the strong CP-problem in a natural way \cite{PQ}. It results as a massless Goldstone boson from the spontaneous breaking of the PQ-symmetry by the VEV of a scalar field, $\langle \phi \rangle = f_{PQ}$. As noted first by Weinberg and Wilczek \cite{WW}, the chiral anomaly of QCD induces, at low temperatures $T \lesssim \Lambda_{QCD} \approx 200$ MeV, a mass $m_a \approx \Lambda_{QCD}^2 / f_{PQ}$ for the axion. The numerical value of the axion mass is given by

$$ m_a = \left( \frac{0.60 \times 10^7 \text{GeV}}{f_{PQ}} \right) \text{eV}. \quad (1) $$

Besides the coupling to two photons through the chiral anomaly, the axion interacts with fermions by derivative couplings, which are all inversely proportional to the axion mass. The interaction of axions with electrons is described by

$$ \mathcal{L}_{ae} = \frac{g_{ae}}{2m_e} \bar{\psi}_\gamma \gamma^5 \gamma \partial \mu \psi_a, \quad (2) $$

where the coupling constant

$$ g_{ae} = \frac{m_e}{f_{PQ}} c_e \quad (3) $$

depends on specific models through the effective PQ charge $c_e$. At tree level, the electron has $c_e = 1/3 \cos \beta$ in the DFSZ axion model, while it has $c_e = 0$ in the KSVZ model \cite{KSVZ}.

In invisible axion models, the axion mass $m_a$ is in principle arbitrary, but in fact severely constrained by astrophysical and cosmological considerations \cite{KSVZ, DFSZ}. The astrophysical bounds on $m_a$ arise because axion emission is an additional energy loss mechanism for stars, and therefore changes the stellar evolution. Since the axion couplings to matter are proportional to $m_a$, axion emission is suppressed for small values of $m_a$. Therefore astrophysics yields an upper bound for $m_a$. In ref. \cite{BS}, the bound $m_a \lesssim 9 \cdot 10^{-3}$ eV was derived for a DFSZ axion by studying the evolution of red giant stars. On the other hand, cosmology provides a lower bound for $m_a$. Axions would have been produced in the early Universe as coherent waves or by the decay of axion strings. Therefore they could constitute cold dark matter. Requiring that axions do not overclose the Universe yields the lower bound $10^{-5}$ eV for $m_a$.

It is the purpose of the present work to examine the emission of axions by electrons in the magnetic field $B$ of a strongly magnetized neutron star. The process considered, $e^- \rightarrow e^- + a$, is very similar to cyclotron emission, $e^- \rightarrow e^- + \gamma$, and, therefore, may be called axion cyclotron emission. This process was first examined by Borisov and Grishina \cite{BG} using a semiclassical expression for the transition probability of the elementary process $e^- \rightarrow e^- + a$, valid a priori only in the limit of low magnetic field strengths, $B \ll B_{cr} = m_e^2 / e \approx 4.41 \times 10^{13}$ Gauss, and of ultrarelativistic electrons, $E \gg m_e$. By comparing the axion cyclotron emission with neutrino cyclotron emission $e^- \rightarrow e^- + \nu + \bar{\nu}$, they concluded that the axion electron coupling constant is bounded from above by $g_{ae} \lesssim 5 \cdot 10^{-14}$. If this result, corresponding to $m_a \lesssim 6 \cdot 10^{-4}$ eV for $c_e = 1$, could be confirmed or improved, the window in the axion mass range would become very narrow, or even be closed. Consequently, it is imperative to recalculate the axion cyclotron emissivity and to extend the results obtained in ref. \cite{BS} to arbitrary magnetic field strengths and electron energies. We derive the probability of the elementary process $e^- \rightarrow e^- + a$ taking the magnetic field exactly into account, and then numerically evaluate an expression for the axion emissivity which is exact in the limit of a completely degenerate Fermi-Dirac gas. As the main result of this work we obtain the axion cyclotron luminosity of a degenerate star with arbitrary core magnetic field strength $B$. Although we confirm the analytical approximations of ref. \cite{BS} in the limit $B \ll B_{cr}$ and $E \gg m_e$, the bound derived by us for the axion electron coupling constant applying the energy-loss argument to neutron stars is three order of magnitudes weaker. The main reason for this is that all conventional neutrino emission mechanisms such as the modified URCA process, which we take into account, were neglected in ref. \cite{BS}. Finally, we consider the energy loss of white dwarf
stars due to axion cyclotron emissivity. Requiring that for a white dwarf star the axion luminosity is smaller than the photon surface luminosity we obtain the bound $g_{ae} \lesssim 9 \cdot 10^{-13} (T/10^7 \text{K})^{5/4} (B/10^{10} \text{G})^{-2}$.

II. ELEMENTARY PROCESS

We start by calculating the probability that an electron with polarization $\tau = \pm 1$ occupying an excited Landau level $N$ emits an axion $a$ with momentum $k = (\omega, \vec{k})$ and undergoes a transition to the Landau level $N'$. (Quantities describing the final electron are primed; compare also the similar calculations of $e^- \to e^- + \gamma$ in ref. [9] for more technical details.) The $S$-matrix element for this process,

$$S_{fi} = \frac{i g_{ae}}{\sqrt{2} \omega V} \int d^4 x \, \bar{\psi}'(x)\gamma^5 \psi(x) e^{i k x},$$  

(4)

follows directly from the Lagrangian (4) if we assume that there exists only one Goldstone boson.

We choose the homogeneous magnetic field in $z$ direction, $\vec{B} = B\hat{e}_z$, and as wave functions for the electron

$$\psi(x) = \left(\frac{eB}{\sqrt{L_y L_z}}\right)^{1/4} e^{-i(E_1p_y - p_z)} \left(\begin{array}{c} \Lambda_1 \phi_{N-1}(\xi) \\ \Lambda_2 \phi_N(\xi) \\ K_1 \phi_{N-1}(\xi) \\ K_2 \phi_N(\xi) \end{array}\right).$$

(5)

Here, $\phi_N(\xi = x + \frac{p_y}{\omega B})$ are the normalized eigenfunctions of the harmonic oscillator, and we have introduced the abbreviations

$$\Lambda_1 = H(1+\tau) + \Delta(1-\tau) = \Pi'_1$$

(6)

$$\Lambda_2 = \Delta(1+\tau) + H(1-\tau) = \Pi'_2$$

(7)

$$K_1 = Z(1+\tau) - \Gamma(1-\tau) = M'_1$$

(8)

$$K_2 = \Gamma(1+\tau) - Z(1-\tau) = M'_2$$

(9)

and

$$H = \sqrt{(E + E_0)(E_0 + m_e)}$$

$$\Delta = -ip_z \sqrt{E_0 - m_e}$$

$$Z = p_z \sqrt{E_0 + m_e}$$

$$\Gamma = i\sqrt{(E + E_0)(E_0 - m_e)}$$

$$E_0 = \sqrt{E^2 - p_z^2} = \sqrt{m_e^2 + 2NeB}.$$  

(10)

(11)

(12)

(13)

(14)

The time integration and two of the momentum integrations yield the usual conservation laws,

$$E = E' + \omega$$

$$p_y = p'_y + k_y$$

$$p_z = p'_z + k_z,$$

(15)

(16)

(17)

while the $x$ integration results, up to a phase factor, in Laguerre functions $I$,

$$I_{N' N}(\kappa) = \sqrt{\frac{N'!}{N!}} \kappa^{(N-N')/2} e^{-\kappa/2} L_{N' N}^{N-\kappa}(\kappa)$$

(18)

for $N > N'$. The argument of the $I$-functions is given by $\kappa = (k \sin \theta)^2/(2eB)$, $\theta$ is the angle between $\vec{B}$ and $\vec{k}$, and the Laguerre polynomials $L_{N' N}^{N-\kappa}(x)$ are defined as in ref. [10]. Using Eq. (15) and (17) we obtain the energy $\omega$ of the axion as a function of $\theta$,

$$\omega(\theta) = \left\{ E - p_z \cos \theta - \left[(E - p_z \cos \theta)^2 - 2(N - N')eB \sin^2 \theta\right]^{1/2} / \sin^2 \theta \right\},$$

(19)

where, as in the following, we have neglected the axion mass $m_a$.

The decay width of an electron in the Landau level $N$ into an Landau level $N'$,

$$d\Gamma_{\pm \to N'}(\vec{k}) = \lim_{T \to \infty} \frac{1}{T} \sum_{p_y',p_z',\tau'} |S_\pm|^2 \frac{V}{(2\pi)^3} d^3 k,$$

(20)

where denotes the (in general different) decay widths of Landau states with polarization $\tau = \pm$, becomes

$$\Gamma_{\pm \to N'} = \alpha_{ae} \sum_{\tau'} \int_0^\pi d\theta \frac{\omega \sin \theta |M|^2}{16 \pi^2 E E_0 E_0' (E' - p_z' \cos \theta)}$$

(21)

with

$$M = \left(\Pi'_1 K_1 - M'_1 \Lambda_1\right) I_{N'-1, N-1}(\kappa)$$

$$+ \left(\Pi'_2 K_2 - M'_2 \Lambda_2\right) I_{N', N}(\kappa)$$

(22)

and $\alpha_{ae} = g_{ae}^2/(4\pi)$.

1We will see below that $m_a \ll \omega \lesssim E_F$, where $E_F = O(m_e)$ is the Fermi energy of the degenerate electron gas.
III. AXION EMISSIVITY

The luminosity \( L_a \) emitted by \( N \) electrons occupying the volume \( V \) due to the process \( e^- \rightarrow e^- + a \) is given by

\[
L_a = \lim_{T \to \infty} \frac{1}{T} \sum_{\lambda, \lambda'} \sum_{\kappa} \omega |S_\pm|^2 S, \tag{23}
\]

where the summation index \( \lambda \) indicates the set of quantum numbers \( \lambda = \{N, \tau, p_y, p_z\} \),

\[
S = f(E)[1 - f(E')] \tag{24}
\]

and \( f(E) \) are Fermi-Dirac distributions functions

\[
f(E) = \left[ \exp(\beta(E - \mu)) + 1 \right]^{-1}.
\tag{25}
\]

For the electron density \( n_e = N/(L_x L_y L_z) \) and temperature \( T = \beta^{-1} \), the chemical potential \( \mu \) is fixed through

\[
N = \sum_{\lambda} f(E) = \frac{eB L_x L_y}{2\pi} \sum_{N=0}^\infty \sum_{\tau} \int_{-\infty}^\infty dp_z \frac{L_z}{2\pi} f(E).
\tag{26}
\]

Here the factor \( eB L_x L_y/(2\pi) \) takes into account the \( p_y \) degeneracy of the Landau states. For a nearly degenerate Fermi-Dirac gas, the \( p_z \) integral in Eq. (20) can be approximated (in a similar manner as in ref. [11]) by

\[
n_e = \frac{eB}{2\pi} \sum_{N=0}^{N_{\text{max}}} \sum_{\tau} \frac{1}{6} \left[ \frac{\sqrt{\mu^2 - E_0^2}}{\sqrt{\mu^2 - E_0^2}} T^2 + O(T^4) \right],
\tag{27}
\]

where \( N_{\text{max}} = \text{int} [(\mu^2 - m_e^2)/(2eB)] \).

The axion cyclotron emissivity \( \varepsilon_a \) of electrons, i.e. the rate of energy loss per volume due to the process \( e^- \rightarrow e^- + a \), follows from Eq. (28) as

\[
\varepsilon_a = \frac{eB}{(2\pi)^2} \sum_{N=1}^\infty \sum_{N' < N} \sum_{\tau = \pm} \int_{-\infty}^\infty dp_z \int_0^\pi d\theta \omega \frac{d\Gamma_{N \rightarrow N'}(k)}{d\theta} S.
\tag{28}
\]

The numerical evaluation of the terms in Eq. (28) becomes cumbersome already for moderate \( N \). Therefore we take advantage of the degeneracy of the electron gas inside a white dwarf or neutron star and employ an approximation commonly used in calculations of neutrino emission rates: With the identity

\[
S = \frac{1}{e^{\beta \omega} - 1} [f(E) - f(E')]
\tag{29}
\]

and \( f(E) \approx \theta(E - \mu), f(E') \approx \theta(E' - \mu) \), the integration of a slowly varying function \( g(E) \) of \( E \) over \( p_z \) becomes

\[
\int_{-\infty}^\infty dp_z g(E) S \approx \frac{2E_F}{p_z} g(E_F) \frac{\omega}{e^{\beta \omega} - 1},
\tag{30}
\]

and the Fermi energy \( E_F \) can be approximated by \( E_F \approx \mu \). Now the summation over the discrete Landau levels \( N \) breaks up at \( N_{\text{max}} = \text{int} [(E_F^2 - m_e^2)/(2eB)] \), while the momentum \( p_z \) of the initial electron is determined by \( p_z = \sqrt{E_F^2 - E_0^2} \). When \((E_F^2 - m_e^2)/(2eB) \) is an integer, the axion emissivity diverges because \( p_z \) is zero or because, more physically speaking, the density of initial states is infinite. This can be remedied by taking into account the finite life time of excited Landau states (cf. ref. [9]) but is unimportant for our purposes.

Although the two infinite sums over \( N \) and \( N' \) have been replaced by finite sums, for low magnetic field strengths or high densities it is still necessary to compute Laguerre polynomials with high index. This can be avoided in the semiclassical, ultrarelativistic case \((B/B_c \ll 1, E \gg m_e)\) by the use of a Bessel function approximation for the \( I \)-functions [8].

\[
I_{N',N}(\kappa) = \frac{1}{\pi \sqrt{3}} \left( 1 - \frac{\kappa}{\kappa_0} \right)^{1/2} K_{1/3}(z)
\tag{31}
\]

with

\[
\kappa_0 = \left( \sqrt{N} - \sqrt{N'} \right)^2
\tag{32}
\]

and

\[
z = \frac{2}{3} \left( \kappa_0^2 N N' \right)^{1/4} \left( 1 - \frac{\kappa}{\kappa_0} \right)^{3/2}.
\tag{33}
\]

This asymptotic expansion is only valid for \( \kappa \ll \kappa_0 \), i.e. \( \theta \approx \pi/2 \) is a necessary condition for its applicability. Therefore we evaluate \( d\Gamma \) for \( p_z = 0 \) and then carry out a Lorentz transformation to \( p_z = \sqrt{E_F^2 - E_0^2} \), viz.

\[
\cos \theta_0 = \frac{E \cos \theta - p_z}{E - p_z \cos \theta}
\tag{34}
\]

\[
d\Omega_0 = d\Omega \left( \frac{E_0}{E - p_z \cos \theta} \right)^2
\tag{35}
\]

\[
d\Gamma(k) = \frac{E_0}{E} d\Gamma_0(k_0)
\tag{36}
\]

Here quantities evaluated in the rest frame of the initial electron are denoted by the subscript 0. Inserting the approximation for \( S \) into Eq. (28) and using eqs. (24)–(36), we obtain as final result for the axion emissivity in the limit of a degenerate electron gas

\[
\varepsilon_a = \frac{eB}{2\pi^2} \sum_{N=1}^{N_{\text{max}}} \sum_{N' < N} \sum_{\tau = \pm} \int_0^\pi d\theta \sin \theta \frac{E_0}{p_z} \left( \frac{E_0}{E - p_z \cos \theta} \right)^2 \frac{d\Gamma_{N \rightarrow N'}(k)}{d\theta} \frac{\omega^2}{e^{\beta \omega} - 1}.
\tag{37}
\]
Finally we examine the classical limit $p_e^2 \ll m_e^2$ and $B \ll B_{\text{cr}}$. Inserting the approximations $E \approx m_e + N e B/m_e$, $E' \approx m_e + N' e B/m_e + p_e^2/(2m_e)$, and $\omega(\theta) \approx (N - N') e B/m_e$ in $|M|^2$, we see that $|M|^2$ is of order $O(B)$. Moreover, since $[\epsilon^2\omega - 1]^{-1} \approx T/\omega$, the axion emissivity is proportional to $B^4 T$ in this limit. If the condition $N^2 B/B_{\text{cr}} \ll 1$ is also valid, we can approximate the $I$-functions by

$$I_{N',N}(\kappa) = \kappa^{(N-N')/2} \frac{\sqrt{N!}}{\sqrt{N'!(N-N')}}. \quad (38)$$

IV. NUMERICAL RESULTS

A. Neutron stars

First, we will establish that Eq. (37) derived by us for general $B$ and $n_e$ has the correct semiclassical limit for $E \gg m_e$ and $B \ll B_{\text{cr}}$ as given in ref. [4],

$$\varepsilon_a^{1}\approx 8.67 \cdot 10^{-3} g_{ae} m_e^{5} \left( \frac{g_{F}}{m_{e}} \right)^{-2/3} \left( \frac{T}{m_{e}} \right)^{13/3} \left( \frac{B}{B_{\text{cr}}} \right)^{2/3}. \quad (39)$$

Since for a relativistic degenerate Fermi gas and $1/3 E_{B}^{2} \gg 2eB$ the usual relation $p_{F} = \sqrt{3\pi^{2}}n_{e}$ can be used [12], this approximation is applicable for $n_{e} \gg m_{e}^{3}/(3\pi^{2}) \approx 5.84 \times 10^{29} \text{cm}^{-3}$. In Fig. 4 we compare our numerically accurate results with the approximate results for $\varepsilon_{a} = \varepsilon_{a}/a_{1}eV$ as functions of $B/B_{\text{cr}}$ for $n_{e} = 10^{32} \text{cm}^{-3}$, and for temperatures $T = 0.04 m_{e}$ and $T = 0.1 m_{e}$, respectively. (We use the emissivity scaled by $a_{1}eV = a_{1eV}/(5.75 \times 10^{-22})$ to obtain a quantity independent of $m_{e}$.) One recognizes that the exact results (crosses and diamonds) reproduce the correct semiclassical $B$ and $T$ dependence (straight lines) already for $B/B_{\text{cr}} \approx 1$. To demonstrate the saw-tooth-like behavior caused by the root singularities in the number density of the Landau states, an amplification of a small excerpt of Fig. 1 is displayed in Fig. 2. Note that these singularities are cancelled out on average in the semiclassical approximation.

Fig. 3 shows the magnetic field dependence of the axion emissivity $\varepsilon_{a}$ for $n_{e} = 8.4 \times 10^{35} \text{cm}^{-3}$ and $T = 0.04 m_{e} \approx 2.4 \times 10^{9} \text{K}$. Assuming an ideal mixture of nucleons and electrons in $\beta$-equilibrium, this value of the electron density $n_{e}$ corresponds to the nuclear density $\rho_{0} = 2.8 \times 10^{14} \text{g/cm}^{3}$. It can be seen that the semiclassical approximation produces reasonable results even for $B/B_{\text{cr}} \approx 20$, and not only for $B/B_{\text{cr}} \ll 1$. A more suitable criterion for the applicability of Eq. (38) is $N_{\text{max}} \gg 1$, or, equivalently, for an ultrarelativistic degenerate electron gas, $E_{B}^{2} \gg 2eB$. Except for a factor of three, this criterion is the same as the condition derived in ref. [12] for the validity of the semiclassical treatment of a magnetized Fermi-Dirac gas. For relatively low $N_{\text{max}}$, the emissivity decreases exponentially until for $2eB > \mu^{2} - m_{e}^{2}$ all electrons populate the Landau ground state (for $T = 0$) and the emissivity becomes zero.

To obtain a bound for the coupling constant $g_{ae}$, we now compare the axion luminosity with other processes important for the cooling of neutron stars. A process specific to magnetized neutron stars is the cyclotron emission of neutrino pairs by electrons, $e^{-} \rightarrow e^{-} + \nu + \bar{\nu}$. Kaminker et al. [3] obtained

$$\varepsilon_{\nu,\text{cyc}} = (9 \times 10^{14} \text{erg cm}^{-3} \text{ s}^{-1}) \left( \frac{B}{10^{13} \text{G}} \right)^{2} \left( \frac{T}{10^{9} \text{K}} \right)^{5} \quad (40)$$

for the energy loss rate due to this process. The bound $g_{ae} \lesssim 5 \times 10^{-14}$ of ref. [6] was derived by requiring $\varepsilon_{a} \lesssim \varepsilon_{\nu,\text{cyc}}$ under conditions appropriate for the outer crust of a neutron star. Since only the total luminosity of stars is observable, a more suitable – but weaker – criterion is to demand that the axion luminosity $L_{a}$ is smaller than the total neutrino luminosity $L_{\nu}$. Other energy loss processes contributing to $L_{\nu}$ are neutrino bremsstrahlung in the Coulomb field of ions [14,16],

$$\varepsilon_{\text{ions}} = (2.1 \times 10^{30} \text{erg cm}^{-3} \text{ s}^{-1}) \frac{Z^{2}}{A} \frac{\rho}{\bar{\rho}_{0}} \left( \frac{T}{10^{9} \text{K}} \right)^{6} \quad (41)$$

and the modified URCA process,

$$\varepsilon_{\text{URCA}} = (2.7 \times 10^{21} \text{erg cm}^{-3} \text{ s}^{-1}) \left( \frac{m_{n}}{m_{p}} \right)^{3} \left( \frac{m_{n}}{m_{p}} \right)^{3} \times \left( \frac{\rho}{\bar{\rho}_{0}} \right)^{2/3} \left( \frac{T}{10^{9} \text{K}} \right)^{8} \quad (42)$$

The value $Z^{2}/A$ is in the range $1 - 10$ [17], and $m_{n}, m_{p}$ are effective masses of the Landau theory. In the following we use $Z^{2}/A = 5$ and $m_{n}/m_{p} = m_{p}/m_{n} = 1$. Recently, Pethick and Thorsson [18] recalculated the neutrino bremsstrahlung process. Instead of treating the interaction of the electrons with the ions immersed in a lattice in first-order perturbation theory, they used Bloch wave functions for the electrons and found that bremsstrahlung is exponentially suppressed at low temperatures, $T \lesssim m_{e}$.

As a model neutron star we choose a neutron star with radius $R = 10 \text{ km}$, constant density $\rho$ equal to nuclear density $\rho_{0}$ and temperature $T = 0.04 m_{e}$. Since the neutron star observations by ROSAT indicate that the standard cooling scenario without exotic matter describes the cooling curves [13], we assume the absence of a pion condensate. Young pulsars have magnetic dipole fields $10^{12} \text{G} \lesssim B_{\text{dipole}} \lesssim 3 \times 10^{13} \text{G}$. Since the magnetic field strength inside the neutron star is generally assumed to be stronger than $B_{\text{dipole}}$ (cf. e.g. [20]), we choose $B = O(1 - 10 B_{\text{cr}})$. 

4
First we consider the case where the core of the neutron star is not superfluid. Then the total neutrino luminosity is \( L_\nu = 4\pi/3 \, R^3 (\varepsilon_{\text{ion}} + \varepsilon_{\text{URCA}}) \approx 1.3 - 8.9 \cdot 10^{35}\text{erg/s} \).

Here the first number takes into account the exponentially suppression found in ref. \([18]\) by a factor \( \approx 0.02 \) for \( \varepsilon_{\text{ion}} \), while the second number uses \( \varepsilon_{\text{ion}} \) given by eq. \((41)\).

The luminosity due to cyclotron emission of \( \nu \) pairs \( \varepsilon \approx 10^{17}\text{erg cm}^{-3}\text{s}^{-1} \) from Fig. \([5]\) and obtain for the axion luminosity \( L_\alpha = 4 \cdot 10^{35} \bar{\alpha} \varepsilon_{\alpha} \text{erg/s} \).

About 5% of white dwarfs have surface magnetic fields in the range \( 10^8 \text{--} 10^{10}\text{G} \). Although the internal magnetic fields strengths are assumed to be in the range \( 10^8 \text{--} 10^{10}\text{G} \), they could approach even \( 10^{12}\text{G} \).

Next we comment on the case that in the core of the neutron star superfluid protons form a type II superconductor \([21]\). Then the initially homogenous magnetic field \( B \) becomes confined to quantized flux tubes (Abrikosov fluxoids).

Each fluxoid carries an elementary magnetic flux quantum \( \phi_0 = \pi/e \) and has the field profile

\[
B(r) \approx \frac{\phi_0}{2\pi \lambda^2} K_0 \left( \frac{r}{\lambda} \right),
\]

where \( \lambda \) is the penetration depth of the magnetic field and \( K_0 \) is a modified Bessel function. The above formula for \( B(r) \) is valid if the proton-proton correlation length is much smaller than the distance \( r \) from the fluxoid axis. The fluxoids are separated by the mean distance \( d_F = (2\phi_0/\sqrt{\lambda B})^{1/2} \).

We can approximate the mean axion emissivity \( \langle \varepsilon_a \rangle \) by the average over the lattice of fluxoids with a quasi-homogenous magnetic field given by eq. \((39)\),

\[
\langle \varepsilon_a \rangle \approx \frac{2}{d_F} \int_0^{d_F} \, dr \, r \varepsilon_a(r),
\]

if the coherent interaction length \( l_{\text{int}} \) is much smaller than the characteristic length scale \( \lambda \) on which the magnetic field varies. Typical values for \( \lambda \) are \( \lambda \approx 30\text{fm} \approx 0.1m_e \), while \( l_{\text{int}} \) is given by the uncertainty principle, \( l_{\text{int},x,y} \approx 1/\sqrt{\lambda B} \) and \( l_{\text{int},z} \approx 1/\rho_z \). Thus \( l_{\text{int}} \) has the same order of magnitude as \( \lambda \) and the quasi-uniform approximation is not well justified. Therefore the approximation \((44)\) should be seen only as a crude estimate. For initial magnetic field strengths \( B = B_{\text{cr}} = 100B_{\text{cr}} \), we obtain \( \langle \varepsilon_a \rangle \approx 3 \cdot 10^{15}\text{erg cm}^{-3}\text{s}^{-1} \) and \( \langle \varepsilon_a \rangle \approx 3 \cdot 10^{16}\text{erg cm}^{-3}\text{s}^{-1} \), respectively. This means that axion cyclotron emissivity is only weakly suppressed by the superfluidity of protons, while the neutrino emissivities \( \varepsilon_{\text{ion}}, \varepsilon_{\text{URCA}} \) are suppressed by a factor of \( \exp(-2\beta\Delta_n) \), where \( \Delta_n \) is the energy gap at \( T = 0 \).

The agreement with the assumed \( B^4T \) behavior is good. We want to give a simple argument in order to decide if eq. \((44)\) over- or underestimates the true emissivity because of the use of the low density \( n_e = 10^{26}\text{cm}^{-3} \). The ultrarelativistic approximation eq. \((39)\) yields, for the values \( B = 10^9\text{G}, T = 10^7\text{K} \) and \( n_e = 3 \cdot 10^{29}\text{cm}^{-3} \), a much higher emissivity than \( \varepsilon_{\text{fit}} \). Therefore the non-relativistic approximation eq. \((40)\) can be used as a lower bound for the true emissivity, if one assumes a smooth transition between the non-relativistic and the ultrarelativistic asymptotic behavior.

The distribution of hot strongly magnetized white dwarfs

B. White dwarfs

We now apply the same line of arguments to magnetic white dwarfs. The internal temperature \( T \) of a white dwarf with mass \( M \) is related to its surface photon luminosity \( L_\gamma \) by

\[
\frac{L_\gamma}{M} = 3.3 \cdot 10^{-3} \frac{\text{erg}}{\text{g s}} \left( \frac{T}{10^7\text{K}} \right)^{7/2}.
\]

About 5% of white dwarfs have surface magnetic fields in the range \( 10^8 \text{--} 10^{10}\text{G} \). Although the internal magnetic fields strengths are assumed to be in the range \( 10^8 \text{--} 10^{10}\text{G} \), they could approach even \( 10^{12}\text{G} \).

The mean density of a white dwarf depends on its total mass and chemical composition but nevertheless a typical mean density is \( 10^9\text{g/cm}^3 \) corresponding to \( n_e \approx 3 \cdot 10^{29}\text{cm}^{-3} \). Unfortunately, for this value of \( n_e \) none of the two approximations for the \( I \)-functions is applicable. Therefore, we compute \( \varepsilon_a \) for densities low enough such that the use of the approximation Eq. \((38)\) is possible. In Fig. \([3]\) \( \varepsilon_a \) is shown for \( n_e = 10^{26}\text{cm}^{-3} \) and the temperatures \( T = 10^6\text{K}, T = 10^7\text{K} \) and \( T = 10^8\text{K} \), respectively, together with the fit function

\[
\varepsilon_{\text{fit}} = 2.6 \frac{\text{erg}}{\text{cm}^3\text{s}} \left( \frac{B}{10^9\text{G}} \right)^4 \left( \frac{T}{10^7\text{K}} \right)^2.
\]

The agreement with the assumed \( B^4T \) behavior is good. We want to give a simple argument in order to decide if eq. \((40)\) over- or underestimates the true emissivity because of the use of the low density \( n_e = 10^{26}\text{cm}^{-3} \). The ultrarelativistic approximation eq. \((39)\) yields, for the values \( B = 10^9\text{G}, T = 10^7\text{K} \) and \( n_e = 3 \cdot 10^{29}\text{cm}^{-3} \), a much higher emissivity than \( \varepsilon_{\text{fit}} \). Therefore the non-relativistic approximation eq. \((40)\) can be used as a lower bound for the true emissivity, if one assumes a smooth transition between the non-relativistic and the ultrarelativistic asymptotic behavior.

We now require that magnetized white dwarfs do not emit more energy in axions than in photons, i.e. that \( L_\gamma/M \lesssim \varepsilon_a/\rho \), and obtain

\[
g_{ae} \lesssim 9 \cdot 10^{-13} \left( \frac{T}{10^7\text{K}} \right)^{5/4} \left( \frac{B}{10^{10}\text{G}} \right)^{-2}.
\]

Since this bound is quite sensitive to \( B \) and the knowledge of the internal magnetic field strengths of white dwarfs is poor, it is hard to derive a precise bound for \( g_{ae} \). However, since \( \varepsilon_a \propto T \), compared to \( L_\gamma \propto T^{7/2} \), axion emission becomes more and more important during the cooling history of magnetized white dwarfs. Therefore the fraction of magnetized white dwarfs among all white dwarfs should diminish drastically for low enough luminosities. In Table I, we compare the fraction of strongly magnetized white dwarfs with the fraction of all white dwarfs in three different temperature bins \([1,24]\).
adapted from Ref. [24] spans approximately three order of magnitudes in surface dipole field strength, $B_{\text{dipole}} = 3 \cdot 10^6 - 10^9$ G, and has a maximum at $B_{\text{dipole}} \approx 3 \cdot 10^7$ G. It consists of only 18 stars, so some caution in interpreting the data is appropriate. Nevertheless, the fraction of magnetized white dwarfs in the last temperature bin is considerably diminished compared to the total white dwarf population. One possible explanation for this could be the additional energy loss of magnetized white dwarfs due to axion cyclotron emission.

V. SUMMARY

We have derived the axion emissivity of a magnetized electron gas due to the process $e^- \rightarrow e^- + a$ for arbitrary magnetic field strengths $B$. Comparing the exact emissivity with the semiclassical approximation obtained earlier, we have shown that this approximation is not only valid for $B/B_{\text{cr}} \ll 1$, but also under the more general condition $E_2^2F \gg 2eB$. Applying axion cyclotron emission as an additional cooling mechanism to neutron stars and requiring that the axion luminosity is smaller than the neutrino luminosity we could constrain the axion electron coupling constant to a value as small as $g_{ae} < O(10^{-10})$. This bound is three orders of magnitude weaker than the bound derived in ref. [7] considering the same process. This discrepancy is not caused by differences in the calculation of the axion emissivity but by different assumptions about neutron star cooling. In the case of white dwarfs we could derive the more stringent limit eq. (47). We have noted that the lack of low-temperature magnetized white dwarfs could be interpreted as signature of an additional energy loss due to axion cyclotron emission.

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| $T_{\text{eff}}[10^3\text{K}]$ | Fraction of all WDs | Fraction of magnetized WDs |
|---------------------------|---------------------|---------------------------|
| 40-80                     | 1%                  | 0%                        |
| 20-40                     | 23%                 | 50%                       |
| 12-20                     | 76%                 | 50%                       |

**TABLE I.** Fraction of all white dwarfs (WDs) and of magnetized WDs in three temperature bins.
FIG. 1. Axion emissivity $\tilde{\varepsilon}_a$ due to $e^- \rightarrow e^- + a$ as a function of $B/B_{cr}$ for $n_e = 10^{32}\text{cm}^{-3}$ and for $T = 0.04m_e$ (bottom) and $T = 0.1m_e$ (top); lines: asymptotic formula, crosses (+) and diamonds (○): exact results.

FIG. 2. Amplification of a small part of Fig. 1 showing the saw-tooth-like behavior of the exact axion emissivity (for $T = 0.04m_e$).
FIG. 3. Axion emissivity $\tilde{\varepsilon}_a$ due to $e^- \rightarrow e^- + a$ as a function of $B/B_{cr}$ for $n_e = 8.4 \cdot 10^{35} \text{cm}^{-3}$ and $T = 0.04m_e$ (dashed line: asymptotic formula, diamonds: exact results).

FIG. 4. Axion emissivity $\tilde{\varepsilon}_a$ due to $e^- \rightarrow e^- + a$ as function of $B/B_{cr}$ for $n_e = 10^{26} \text{cm}^{-3}$ and $T = 10^8 \text{K}, T = 10^7 \text{K}, T = 10^6 \text{K}$ (lines: fit function Eq. (44), points: exact results).
Figure 1: Axion emissivity $\tilde{\epsilon}_a$ due to $e^- \rightarrow e^- + \alpha$ as a function of $B/B_{ce}$ for $n_e = 10^{32}\text{cm}^{-3}$ and for $T = 0.04m_e$ (bottom) and $T = 0.1m_e$ (top); lines: asymptotic formula, crosses (+) and diamonds (○): exact results.
Figure 2: Amplification of a small part of Fig. 1 showing the saw-tooth-like behavior of the exact axion emissivity (for $T = 0.04 m_e$).
Figure 3: Axion emissivity $\tilde{\varepsilon}_a$ due to $e^- \rightarrow e^- + a$ as a function of $B/B_{ct}$ for $n_e = 8.4 \cdot 10^{25} \text{cm}^{-3}$ and $T = 0.04 m_e$ (dashed line: asymptotic formula, diamonds: exact results).
Figure 4: Axion emissivity $\tilde{\varepsilon}_a$ due to $e^- \rightarrow e^- + a$ as function of $B/B_{cr}$ for $n_e = 10^{26}\text{ cm}^{-3}$ and $T = 10^8\text{ K}$, $T = 10^7\text{ K}$, $T = 10^6\text{ K}$ (lines: fit function Eq. (44), points: exact results).