Distributed inference for tail risks

Liujun Chen
International Institute of Finance, School of Management,
University of Science and Technology of China
Deyuan Li
School of Management, Fudan University
and
Chen Zhou
Econometric Institute, Erasmus University Rotterdam

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Abstract

For measuring tail risk with scarce extreme events, extreme value analysis is often invoked as the statistical tool to extrapolate to the tail of a distribution. The presence of large datasets benefits tail risk analysis by providing more observations for conducting extreme value analysis. However, large datasets can be stored distributedly preventing the possibility of directly analyzing them. In this paper, we introduce a comprehensive set of tools for examining the asymptotic behavior of tail empirical and quantile processes in the setting where data is distributed across multiple sources, for instance, when data are stored on multiple machines. Utilizing these tools, one can establish the oracle property for most distributed estimators in extreme value statistics in a straightforward way. The main theoretical challenge arises when the number of machines diverges to infinity. The number of machines resembles the role of dimensionality in high dimensional statistics. We provide various examples to demonstrate the practicality and value of our proposed toolkit.

Keywords: Oracle property, Tail empirical process, Tail quantile process, KMT inequality
1 Introduction

Financial risk management requires risk forecasting for rare but high-impact events, typically referred to as extreme events. Extreme value analysis, statistical methods for analyzing the tail of a distribution, is recognized as a useful tool for modeling and analyzing such extremes. In this paper, we consider tail risk analysis using a large dataset that is distributedly stored at various locations.

While the availability of large datasets in general benefits statistical analysis, such as extreme value analysis, it also presents at least three practical challenges to implementing conventional statistical procedures. Firstly, a combined dataset might not be available to one end user due to privacy concerns. For example, consider analyzing insurance claims from various insurance companies. Since insurance firms are contracted for protecting privacy of their customers, it is impossible to combine all claims from different insurers into one massive dataset. Secondly, the computation cost to analyze a massive dataset can be expensive when implementing statistical procedures involving an optimization algorithm, such as maximum likelihood or loss minimization. Thirdly, storage constraints can arise when dealing with massive datasets, for instance, when the size of a dataset exceeds a computer’s memory. Another example is to analyze online stream data, where data become available in a sequential manner (Gama et al., 2013).

One solution to overcome these challenges is to handle the massive datasets in batches, sometimes referred to as “distributedly stored”. Divide and Conquer (DC) algorithms are often invoked when data are distributedly stored in multiple machines. One first estimates a desired quantity or parameter based on part of the data stored on each machine and then combines the results obtained from all machines to calculate an aggregated estimate, often by a simple average. DC algorithms have at least three advantages. Firstly, DC algorithms help to preserve privacy. For example, insurance firms can share some statistical results
provided that that other companies cannot infer client level data from the shared results. Moreover, DC algorithms can significantly improve computational efficiency by utilizing parallel computing. Lastly, DC algorithms can overcome the challenge of storage constraint by analyzing the dataset in batches.

Theoretically a DC algorithm can be applied to a given statistical procedure only if it possesses the so-called oracle property: the aggregated estimator by averaging achieves the same statistical efficiency as the imaginary estimator using all observations. The latter is often referred to as the oracle estimator. The validity of applying a DC algorithm is not obvious for many statistical procedures, and often requires additional conditions, see Fan et al. (2019) for principal component analysis, Volgushev et al. (2019) for quantile regression and Li et al. (2013) for kernel density estimation, among others.

The validity of a DC algorithm to extreme value analysis may fail. For example, considering a distribution with a finite endpoint, a natural estimator for the endpoint is the sample maxima. If an oracle sample maxima cannot be obtained, one may consider collecting the maxima from data stored in each machine. Clearly, to obtain the oracle estimator, one needs to take the maximum of the machine-wise maxima instead of taking average. Therefore, in this specific example, the standard DC algorithm based on averaging may lead to an estimator that fails the oracle property. Since estimators in extreme value analysis are often based on high order statistics, the oracle property of the DC algorithm for such estimators are yet to be established. The goal of this paper is to fill in this gap.

In this paper, we provide general tools to establish the oracle property for distributed estimators in extreme value statistics. Chen et al. (2022) adapts a particular extreme value estimator, the Hill estimator, to the DC algorithm and proposes the distributed Hill estimator to estimate the extreme value index \( \gamma \) for the case \( \gamma > 0 \). They study the asymptotic behavior of the distributed Hill estimator and show sufficient, sometimes also necessary, conditions under which the distributed Hill estimator possesses the oracle property. The
proof therein relies on the specific construction of the Hill estimator, and cannot be gener-
alized to validate the oracle property of other estimators for the extreme value index, let
alone that of other estimators for practically relevant quantities such as high quantile, tail
probability and endpoint. By contrast, using the tools in this paper, the oracle property for
most estimators extreme value statistics based on the peak-over-threshold (POT) approach
can be established in a straightforward way.

In classical extreme value statistics, two key tools for establishing asymptotic theories
are the tail empirical process and the tail quantile process. Let \( l = l(N) \) be an intermediate
sequence such that as \( N \to \infty, l \to \infty, l/N \to 0 \). The tail empirical process is defined as
\[
Y_{N,l}(x) = \frac{N}{l} \bar{F}_N \left\{ a_0 \left( \frac{N}{l} \right) x + b_0 \left( \frac{N}{l} \right) \right\}, \quad x \in \mathbb{R},
\]
where \( \bar{F}_N := 1 - F_N \) and \( F_N \) denotes the empirical cumulative distribution function
\( F_N(x) := \frac{1}{N} \sum_{i=1}^{N} I(X_i \leq x) \). Here \( a_0 \) and \( b_0 \) are suitable versions of \( a \) and \( b \), respectively. \cite{Drees2006} shows a weighted approximation of the tail empirical process \( Y_{N,l}(x) \) (see Section 2 below) under some mild conditions. The approximation of the tail
empirical process is a useful tool in a wider context. For example, \cite{Drees2006} pro-
poses a test for the extreme value condition, \cite{deHaanFerreira2006, Example 5.1.5} estab-
ishes the asymptotic normality of the Hill estimator, both by using this result.

Analogous to the tail empirical process, \cite{Drees1998} shows a weighted approximation
of the tail quantile process (see Section 3 below). The tail quantile process is defined as
\[
Q_{N,l}(s) = \frac{X_{N-[s],N} - b_0 \left( \frac{N}{l} \right)}{a_0 \left( \frac{N}{l} \right)}, \quad s \in [0, 1],
\]
where \( X_{N,N} \geq \cdots \geq X_{1,N} \) are the order statistics of the sample \( \{X_1, \ldots, X_N\} \). Here and
thereafter, we use \([x]\) to denote the largest integer less than or equal to \( x \). Note that the
POT approach in extreme value statistics often uses high order statistics \( X_{N,N}, \ldots, X_{N-l,N} \).
Consequently, compared to the tail empirical process, the tail quantile process is more
straightforward for proving asymptotic theory for estimators in extreme value statistics
based on the POT approach. By writing such estimators as a functional of $Q_{N,i}(s)$ and using the weighted approximation of the tail quantile process, one can derive their asymptotic behavior. Examples of estimators for the extreme value index based on the POT approach are the probability weighted moment estimator (Hosking and Wallis, 1987), the maximum likelihood estimator (Drees et al., 2004) and the Pickands estimator (Pickands III, 1975). In addition, the asymptotic behavior for the estimators of high quantile, tail probability and endpoint can be derived from the approximation of the tail quantile process as well, see e.g. Chapter 4 in de Haan and Ferreira (2006).

In this paper, we establish weighted approximations of tail empirical processes and tail quantile processes for the distributed subsamples in a joint manner, with linking these approximations to that for the oracle sample. Such results lead to the oracle properties of many distributed estimators in extreme value statistics based on the POT approach, with some straightforward proofs. Mathematically, we show a stronger result than the classical oracle property: the difference between the distributed estimator and the oracle estimator diminishes faster than the speed of convergence of the oracle estimator. Note that Chen et al. (2022) only shows that the limiting distribution of the distributed Hill estimator coincides with that of the oracle estimator, whereas asymptotic behavior of the difference between the two estimators cannot be derived using the method therein. Daouia et al. (2021) achieves such result for the Hill estimator. However, the proof in Daouia et al. (2021) cannot be generalized to other estimators in extreme value statistics since it relies on the specific structure of the Hill estimator.

The main challenge for handling tail empirical process arises when the number of machines diverges to infinity. The number of machines resembles the role of dimensionality in high dimensional statistics. Observing that with equal subsample sizes across different machines, the tail empirical process for the oracle sample is the average of the tail empirical processes based on the distributed subsamples, it seems trivial that they can be
approximated by the same asymptotic limits. However, to aggregate the tail empirical processes in different machines, we need to make sure that the approximation errors in different machines are uniformly negligible. We achieve this mathematically difficult result by invoking Komlós-Major-Tusnády type inequalities (see e.g. Komlós et al. (1975)). Linking the weighted approximation of the tail empirical process based on the oracle sample to those of the tail empirical processes on each machine is an important intermediate step towards establishing similar links between the corresponding tail quantile processes.

By contrast, when handling tail quantile processes, we cannot follow similar steps as for tail empirical processes. The main difference is that the average of the tail quantile processes based on distributed subsamples in different machines is not equal to the tail quantile process based on the oracle sample. Linking the approximations of the tail quantile processes based on the distributed subsamples to that based on the oracle sample poses an additional layer of technical difficulty, which we will handle in Section 3.

The rest of this paper is organized as follows. Section 2 shows the weighted approximations of the tail empirical processes based on the distributed subsamples in a joint manner and links that to the weighted approximation of tail empirical process based on the oracle sample. Section 3 shows the analogous result for the weighted approximations of the tail quantile processes. We provide various examples in Section 4 to show how these tools can be used to prove the oracle property of extreme value estimators such as the estimators of extreme value index, high quantile, tail probability and endpoint. Section 5 extends the theoretical results to the case of heterogeneous subsample sizes. A real data application is given in Section 6. A concluding remark is made in Section 7. The technical proofs are deferred to the Supplementary Material, along with a simulation study showing the performance of the distributed estimators for the extreme value index and the high quantile.

Throughout the paper, $a(t) \sim b(t)$ means that $a(t)/b(t) \to 1$ as $t \to \infty$; $a(t) \asymp b(t)$
means that both \(|a(t)/b(t)|\) and \(|b(t)/a(t)|\) are \(O(1)\) as \(t \to \infty\).

2 Distributed Tail Empirical Process

Let \(X_1, \ldots, X_N\) be independently and identically distributed (i.i.d.) random variables with distribution function \(F\), which is in the maximum domain of attraction of an extreme value distribution \(G_\gamma\) with index \(\gamma \in \mathbb{R}\), i.e. there exist a positive function \(a\) and a real function \(b\) such that,

\[
\lim_{N \to \infty} F_N \{a(N)x + b(N)\} = G_\gamma(x) := \exp \left\{ -(1 + \gamma x)^{-1/\gamma} \right\},
\]

for all \(1 + \gamma x > 0\). We denote this assumption as \(F \in D(G_\gamma)\), where \(\gamma\) is the so called extreme value index. Extreme value statistics considers estimating the extreme value index \(\gamma\), the functions \(a\) and \(b\), as well as other practically relevant quantities such as high quantile of \(F\). For established results in extreme value statistics, we refer interested readers to monographs such as de Haan and Ferreira (2006) and Resnick (2007).

Write \(U = \{1/(1-F)\}^\leftarrow\), where \(\leftarrow\) denotes the left-continuous inverse function. Then the necessary and sufficient condition for \(F \in D(G_\gamma)\) with \(\gamma \in \mathbb{R}\) is

\[
\lim_{t \to \infty} \frac{U(tx) - U(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma},
\]

for all \(x > 0\). In this paper, we focus on the distributions which satisfy the second order refinement of condition (1) as follows (de Haan and Stadtmüller, 1996): there exists an eventually positive or negative function \(A\) with \(\lim_{t \to \infty} A(t) = 0\) and a real number \(\rho < 0\) such that for all \(x > 0\),

\[
\lim_{t \to \infty} \frac{U(tx) - U(t)}{a(t)} - \frac{x^\gamma - 1}{\gamma} = \frac{1}{\rho} \left( \frac{x^{\gamma + \rho} - 1}{\gamma + \rho} - \frac{x^\gamma - 1}{\gamma} \right).
\]

Under this condition, one can find suitable normalizing functions such that the convergence in (2) holds uniformly as follows, see Corollary 2.3.7 in de Haan and Ferreira (2006).
There exists functions \( a_0(t) \sim a(t), A_0(t) \sim A(t) \) and \( b_0(t) \) such that, for any \( \varepsilon, \delta > 0 \), there exists \( t_0 = t_0(\varepsilon, \delta) \) such that, for all \( t, tx \geq t_0 \),

\[
\left| \frac{U(tx) - b_0(t)}{a_0(t)} - \frac{x^{\gamma-1}}{\gamma} - \Psi(x) \right| \leq \varepsilon x^{\gamma+\rho} \max\left(x^\delta, x^{-\delta}\right),
\]

where

\[
\Psi(x) := \begin{cases} 
\frac{x^{\gamma+\rho}}{\gamma+\rho}, & \gamma + \rho \neq 0, \\
\log x, & \gamma + \rho = 0.
\end{cases}
\]

For the details of the expression of \( a_0, b_0 \) and \( A_0 \), see Corollary 2.3.7 in de Haan and Ferreira (2006).

Let \( l = l(N) \) be an intermediate sequence such that as \( N \to \infty, l \to \infty, l/N \to 0 \). The tail empirical process for the oracle sample is defined as

\[
Y_{N,l}(x) = \frac{N}{l} \bar{F}_N \left\{ a_0 \left( \frac{N}{l} \right) x + b_0 \left( \frac{N}{l} \right) \right\}, \quad x \in \mathbb{R},
\]

where \( \bar{F}_N := 1 - F_N \) and \( F_N \) denotes the empirical cumulative distribution function \( F_N(x) := N^{-1} \sum_{i=1}^{N} I(X_i \leq x) \).

Under the second order condition (2) and \( \sqrt{l}A(N/l) = O(1) \) as \( N \to \infty \), Drees et al. (2006) shows that, under proper Skorokhod construction, there exists a sequence of Brownian motions \( \{W^*_N\}_{N \geq 1} \), such that, for any \( v > 0 \),

\[
\sup_{x \in \mathbb{D}} \{z(x)\}^{v-1/2} \left| \sqrt{l} \{Y_{N,l}(x) - z(x)\} - W^*_N \{z(x)\} ight. \\
- \sqrt{l}A_0(N/l) \{z(x)\}^{1+\gamma} \Psi \{1/z(x)\} \right| = o_P(1), \tag{4}
\]

where \( z(x) \) and \( \mathbb{D} \) are defined in Theorem [1] below.

The result (4) is based on the oracle sample. We intend to provide an analogous result for the tail empirical processes based on the distributed subsamples in a joint manner. Assume that the \( N \) observations are distributedly stored in \( m \) machines with \( n \) observations in each machine and then \( N = nm \). We will extend our analysis to the case of heterogeneous subsample size in Section [5]. The tail empirical process based on the observations \( \{X_1^{(j)}, \ldots, X_n^{(j)}\} \) in machine \( j \) is defined as

\[
Y_{n,k}^{(j)}(x) = \frac{n}{k} \bar{F}_n^{(j)} \left\{ a_0 \left( \frac{n}{k} \right) x + b_0 \left( \frac{n}{k} \right) \right\}, \quad j = 1, \ldots, m,
\]
where $\bar{F}_n^{(j)} := 1 - F_n^{(j)}$ and $F_n^{(j)}$ denotes the empirical distribution function based on the observations in machine $j$. Here $k = k(N)$ is an intermediate sequence such that $k \to \infty$ and $k/n \to 0$, as $N \to \infty$.

We intend to relate the asymptotics of $Y_{N,l}(x)$ and $m^{-1} \sum_{j=1}^{m} Y_{n,k}^{(j)}(x)$ where $l = km$. Without causing any ambiguity, we use the simplified notation $Y_N(x)$ and $Y_n^{(j)}(x)$ for the tail empirical processes based on the oracle sample and the sample in machine $j$, respectively.

Throughout this paper, let $m, n, k$ be sequences of integers such that, $m = m(N) \to \infty$, $n = n(N) \to \infty$, $k = k(N) \to \infty$ and $k/n \to 0$ as $N \to \infty$. We assume the following conditions for the sequences $k$ and $m$:

(A1) $\sqrt{km} A(n/k) = O(1)$ as $N \to \infty$.

(A2) $\eta := \liminf_{N \to \infty} \log k / \log m - 1 > 0$.

(A3) $km(\log k)^2/n = O(1)$ as $N \to \infty$.

Remark 1. Note that $n/k = N/(nm)$. Condition (A1) is a typical condition assumed in extreme value analysis to guarantee finite asymptotic bias in the oracle estimator. Condition (A2) states that, the number of machines ($m$) should be smaller than the number of observations used in each machine ($k$). Similar conditions are assumed in the literature of distributed inference for other statistical procedures, see e.g. Corollary 3.4 in Volgushev et al. (2019) and Theorem 4 in Zhu et al. (2021). Condition (A3) is an additional technical condition, which requires that the number of observations ($n$) in each machine is at a sufficiently high level for given $k$ and $m$.

Remark 2. One example for $k$ and $m$ satisfying conditions (A1)-(A3) can be given as follows. Let $m \asymp n^a$ for some $0 \leq a < \frac{-1/\nu}{1-(1/\nu)\rho}$, where $\rho$ is the second parameter in (2), and $k \asymp n^b$ for some

$$a < b < \min \left(1-a, \frac{-2\rho-a}{-2\rho+1} \right),$$
then conditions (A1)-(A3) hold with \( \eta = b/a - 1 > 0 \).

The following theorem shows the weighted approximations of the tail empirical processes based on the distributed subsamples in a joint manner.

**Theorem 1.** Suppose that the distribution function \( F \) satisfies the second order condition (2) with \( \gamma \in \mathbb{R} \) and \( \rho < 0 \). Let \( m, n, k \) be sequences of integers satisfying conditions (A1)-(A3) and \( x_0 > -1/(\gamma \vee 0) \). Then under suitable Skorokhod construction, there exist \( m \) independent sequences of Brownian motions \( \{W_n^{(j)}\}_{n \geq 1}, j = 1, \ldots, m \), such that for any \( v \in ((2 + \eta)^{-1}, 2^{-1}) \), as \( N \to \infty \),

\[
\max_{1 \leq j \leq m} \sup_{x \in \mathbb{D}} \{z(x)\}^{v-1/2} \left| \sqrt{km} \left\{ Y_n^{(j)}(x) - z(x) \right\} - \sqrt{m} W_n^{(j)} \{z(x)\} \right| = o_p(1),
\]

where

\[
z(x) = (1 + \gamma x)^{-1/\gamma}, \quad \mathbb{D} = \left\{ x : x_0 \leq x < \frac{1}{(\gamma \vee 0)} \right\}.
\]

Moreover, as \( N \to \infty \),

\[
\sup_{x \in \mathbb{D}} \{z(x)\}^{v-1/2} \left| \sqrt{km} \left\{ Y_N(x) - z(x) \right\} - W_N \{z(x)\} \right| = o_p(1),
\]

where \( W_N = m^{-1/2} \sum_{j=1}^{m} W_n^{(j)} \) is a version of the Brownian motion \( W_N^* \) in (4).

For \( \gamma > 0 \), a similar but simpler result is given as follows.

**Theorem 2.** Suppose that the distribution function \( F \) satisfies the second order condition (2) with \( \gamma > 0 \) and \( \rho < 0 \). Let \( m, n, k \) be sequences of real numbers that satisfy conditions (A1)-(A3) and \( \tilde{x}_0 > 0 \). Then under suitable Skorokhod construction, there exist \( m \) independent sequences of Brownian motions \( \{W_n^{(j)}, n \geq 1\}, j = 1, \ldots, m \), such that for any \( v \in ((2 + \eta)^{-1}, 2^{-1}) \), as \( N \to \infty \),

\[
\max_{1 \leq j \leq m} \sup_{x \geq \tilde{x}_0} \left( \frac{x^{(1/2-v)/\gamma}}{\sqrt{km} \left\{ \frac{n}{k} \tilde{F}_n^{(j)}(xU(n/k)) - x^{-1/\gamma} \right\}} \right.
\]

\[
- \sqrt{m} W_n^{(j)}(x^{-1/\gamma}) - \sqrt{km} A_0 \left( \frac{n}{k} \right) x^{-1/\gamma} x^{\rho/\gamma \vee 1} \left| = o_p(1). \right.
\]
Moreover, as $N \to \infty$,
\[
\sup_{x \geq x_0} x^{(1/2-v)/\gamma} \left| \sqrt{k}m \left\{ \frac{n}{k} \bar{F}_N \left( xU(n/k) \right) - x^{-1/\gamma} \right\} \right.
- W_N(x^{-1/\gamma}) - \sqrt{k}m A_0 \left( \frac{n}{k} \right) x^{-1/\gamma} \frac{x^{\rho/\gamma} - 1}{\gamma \rho} \right| = o_P(1),
\]
where $W_N = m^{-1/2} \sum_{j=1}^m W_n^{(j)}$.

To prove these theorems, we need a fundamental inequality to bound the approximation error of the tail empirical process $Y_n^{(j)}(x)$ to the Gaussian process in machine $j$, which is of independent interest. Consider a positive sequence $t = t(N) \to 0$ as $N \to \infty$, satisfying
\[
(n/k)^{-1/2} \log k/t = O(1),
\]
\[
k^{1/2} A_0(n/k)/t = O(1),
\]
for some $\bar{\epsilon} > 0, \left\{ A_0(n/k) \right\}^{1/2-\bar{\epsilon}} / t = o(1).$

Proposition 1. Suppose that the distribution function $F$ satisfies the second order condition (2) with $\gamma \in \mathbb{R}$ and $\rho < 0$. Let $t$ be a sequence of real numbers satisfying conditions (5)- (7) and $x_0 > -1/(\gamma \vee 0)$. Then for sufficiently large $n$, under suitable Skorokhod construction, there exist $m$ independent sequences of Brownian motions $\left\{ W_n^{(j)}, n \geq 1 \right\}, j = 1, \ldots, m$ and a constant $C_1 = C_1(v) > 0$ such that, for any $v \in (0, 1/2),$
\[
P \left( \delta_n^{(j)} \geq t \right) \leq C_1 r^{-\frac{1}{1/2-v}},
\]
where
\[
\delta_n^{(j)} = \sup_{x \in \mathbb{D}} \left| \sqrt{k} \left\{ Y_n^{(j)}(x) - z(x) \right\} \right.
- W_n^{(j)} \left\{ z(x) \right\} - \sqrt{k} A_0(n/k) \left\{ z(x) \right\}^{1+\gamma} \Psi \left\{ 1/z(x) \right\},
\]
and $r = r(t, k)$ is defined by $k^{-v} r \log r = t$.

Proposition 1 guarantees that the approximation errors $\delta_n^{(j)}, j = 1, \ldots, m$ are uniformly negligible, which is a key step to prove Theorems 1 and 2.
3 Distributed Tail Quantile Processes

Drees (1998) provides a weighted approximation of the tail quantile process. Assume the second order condition (2) and \( \sqrt{l} A(N/l) = O(1) \) as \( N \to \infty \), with the same Brownian motions \( \{W^*_N\}_{N \geq 1} \) in (4), we have that, for any \( v > 0 \),

\[
\sup_{1/l \leq s \leq 1} s^{v+1/2+\gamma} \left| \sqrt{l} \left( Q_{N,l}(s) - \frac{s^{-\gamma} - 1}{\gamma} \right) - s^{-\gamma-1} W^*_N(s) - \sqrt{l} A_0 \left( \frac{N}{l} \right) \Psi(s^{-1}) \right| = o_P(1).
\]

Again, this result is based on the oracle sample. We intend to provide weighted approximations of the tail quantile processes based on the distributed subsamples in a joint manner.

The tail quantile process based on the observations in machine \( j \) is defined as

\[
Q_{n,k}^{(j)}(s) = \frac{X_{n-\lfloor ks \rfloor,n}^{(j)} - b_0 \left( \frac{n}{k} \right)}{a_0 \left( \frac{n}{k} \right)}, \quad j = 1, \ldots, m,
\]

where \( X_{n,n}^{(j)} \geq \cdots \geq X_{1,n}^{(j)} \) are the order statistics of the observations in machine \( j \).

We aim at linking the asymptotics of \( Q_{N,l}(s) \) and \( m^{-1} \sum_{j=1}^m Q_{n,k}^{(j)}(s) \) where \( l = km \). Again, without causing any ambiguity, we use the simplified notation \( Q_N(s) \) and \( Q_n^{(j)}(s) \) for the tail quantile process based on the oracle sample and the sample in machine \( j \), respectively. Since the average of the tail quantile processes based on distributed subsamples in \( m \) machines is not equal to the tail quantile process of the oracle sample, we cannot follow similar steps as in Section 2. Instead, we achieve our goal by “inverting” the result for the tail empirical processes. More specifically, we intend to replace \( x \) in Theorem 1 by \( Q_n^{(j)}(s) \) for \( s \in \left[ k^{-1+\delta}, 1 \right] \).

The following theorem shows that, with the same sequences of Brownian motions defined in Theorem 1 \( \left\{ W^{(j)}_n \right\}_{n \geq 1} \), \( j = 1, \ldots, m \), the approximation errors of the tail quantile processes are uniformly negligible for \( 1 \leq j \leq m \).

**Theorem 3.** Assume the same conditions as in Theorem 1. Then for any \( v \in ((2 +
\(\eta^{-1}, 1/2\) and \(\delta \in (0, 1)\), as \(N \to \infty\),

\[
\max_{1 \leq j \leq m} \sup_{k^{-1+\delta} \leq s \leq 1} s^{v+1/2+\gamma} \left| \frac{\sqrt{km}}{\gamma} \left( Q_n^{(j)}(s) - \frac{s^{-\gamma} - 1}{\gamma} \right) \right|
- \sqrt{m} s^{-\gamma-1} W_{n}^{(j)}(s) - \sqrt{km} A_0 \left( \frac{n}{k} \right) \Psi(s^{-1}) = o_P(1).
\]

Moreover, as \(N \to \infty\),

\[
\sup_{k^{-1+\delta} \leq s \leq 1} s^{v+1/2+\gamma} \left| \frac{\sqrt{km}}{\gamma} \left( Q_N(s) - \frac{s^{-\gamma} - 1}{\gamma} \right) \right|
- s^{-\gamma-1} W_N(s) - \sqrt{km} A_0 \left( \frac{n}{k} \right) \Psi(s^{-1}) = o_P(1).
\]

Here, \(\left\{ W_{n}^{(j)} \right\}_{n \geq 1}, j = 1, \ldots, m\) are the same Brownian motions constructed as in Theorem 1 and \(W_N = m^{-1/2} \sum_{j=1}^{m} W_{n}^{(j)}\). Consequently, \(m^{-1} \sum_{j=1}^{m} Q_n^{(j)}(s)\) has the same asymptotic expansion as that for \(Q_N(s)\), uniformly for \(s \in [k^{-1+\delta}, 1]\).

For \(\gamma > 0\), a similar but simpler result is given as follows.

**Theorem 4.** Assume the same conditions as in Theorem 2. Then for any \(v \in ((2 + \eta)^{-1}, 1/2)\) and \(\delta \in (0, 1)\), as \(N \to \infty\),

\[
\max_{1 \leq j \leq m} \sup_{k^{-1+\delta} \leq s \leq 1} s^{v+1/2+\gamma} \left| \frac{\sqrt{km}}{\gamma} \left( X_{n-[ks],n}^{(j)} U(n/k) - s^{-\gamma} \right) \right|
- \sqrt{m} \gamma s^{-\gamma-1} W_{n}^{(j)}(s) - \gamma \sqrt{km} A_0 \left( \frac{n}{k} \right) s^{-\gamma-1} \frac{s^{-\rho} - 1}{\rho} = o_P(1).
\]

Moreover, as \(N \to \infty\),

\[
\sup_{k^{-1+\delta} \leq s \leq 1} s^{v+1/2+\gamma} \left| \frac{\sqrt{km}}{\gamma} \left( X_{N-[kms],N} U(n/k) - s^{-\gamma} \right) \right|
- \gamma s^{-\gamma-1} W_N(s) - \gamma \sqrt{km} A_0 \left( \frac{n}{k} \right) s^{-\gamma-1} \frac{s^{-\rho} - 1}{\rho} = o_P(1).
\]

Here, \(\left\{ W_{n}^{(j)}, n \geq 1 \right\}, j = 1, \ldots, m\) are the same Brownian motions constructed as in Theorem 2 and \(W_N = m^{-1/2} \sum_{j=1}^{m} W_{n}^{(j)}\).

The following corollary, which is a direct consequence of Theorem 4 with applying the Cramér’s delta method, can be used for proving asymptotic theory of the distributed Hill estimator.
Corollary 1. Assume the same conditions as in Theorem 2. By the Cramér’s delta method, we can obtain that, as $N \to \infty$,
\[
\max_{1 \leq j \leq m} \sup_{k^{-1/4} \leq s \leq 1} s^{v+1/2} \left| \sqrt{km} \left( \frac{\log X_{n-[ks],n} - \log U \left( \frac{n}{k} \right)}{\gamma} + \log s \right) - \sqrt{m} \gamma s^{-1} W^{(j)}(s) - \gamma \sqrt{km} A_0 \left( \frac{n}{k} \right) \frac{1}{\rho} \left| s^{-\rho} - 1 \right| \right| = o_p(1).
\]

Theorem 3 provides a useful tool for establishing the oracle property of some extreme value estimators based on the POT approach. For example, using Theorem 3, one can immediately show that, the distributed Pickands estimator achieves the oracle property since the distributed Pickands estimator is a functional of the tail quantile processes $Q^{(j)}_n(s)$ at three points $s = 1/4, 1/2, 1/4$. We leave this to the readers. Instead, we focus on some other estimators, for which Theorem 3 alone may not be sufficient for proving their oracle property. We use the probability weighted moment (PWM) estimator as an example to explain the remaining issue.

The PWM estimator in machine $j$ is defined as
\[
\hat{\gamma}^{(j)}_{PWM} := \frac{P^{(j)}_n - 4Q^{(j)}_n}{P^{(j)}_n - 2Q^{(j)}_n},
\]
where
\[
P^{(j)}_n := \frac{1}{k} \sum_{i=1}^{k} X_{n-i+1,n}^{(j)} - X_{n-k,n}^{(j)}, \quad Q^{(j)}_n := \frac{1}{k} \sum_{i=1}^{k} i - \frac{1}{k} \left( X_{n-i+1,n}^{(j)} - X_{n-k,n}^{(j)} \right).
\]
The distributed PWM estimator is defined as the average of the $m$ estimates from each machine:
\[
\hat{\gamma}^D_{PWM} = \frac{1}{m} \sum_{j=1}^{m} \hat{\gamma}^{(j)}_{PWM}.
\]
To establish the asymptotic theory for $\hat{\gamma}^D_{PWM}$, we need to handle the asymptotic expansion of $P^{(j)}_n$ and $Q^{(j)}_n$ for $j = 1, \ldots, m$ in a joint manner. For $s \in [0, 1]$, define
\[
f^{(j)}_n(s) = Q^{(j)}_n(s) - \frac{s^{-\gamma} - 1}{\gamma} - \frac{1}{\sqrt{k}} s^{-\gamma-1} W^{(j)}(s) - A_0 \left( \frac{n}{k} \right) \Psi(s^{-1}).
\]
Then we can write $P_{n}^{(j)}$ as

$$
\frac{P_{n}^{(j)}}{a_0 \left( \frac{n}{k} \right)} = \int_0^1 \frac{X_{n-[ks],n} - X_{n-k,n}}{a_0 \left( \frac{n}{k} \right)} ds
$$

$$
= \int_0^1 \frac{s^{-\gamma} - 1}{\gamma} ds + \frac{1}{\sqrt{k}} \int_0^1 \left\{ s^{-\gamma} - W_n^{(j)}(s) - W_n^{(j)}(1) \right\} ds
$$

$$
+ \int_0^{k^{-1+\delta}} \left\{ f_n^{(j)}(s) - f_n^{(j)}(1) \right\} ds + \int_0^{k^{-1+\delta}} \left\{ f_n^{(j)}(s) - f_n^{(j)}(1) \right\} ds
$$

$$
= : I_1 + I_2 + I_3 + I_4 + I_5.
$$

The three terms $I_1, I_2$ and $I_3$ can be handled in a similar way as handling analogous terms in the oracle PWM estimator. The integral $I_4$ can be handled using Theorem 3. However, handling the last integral $I_5$ requires some additional tools to deal with the “corner” of the tail quantile processes. Similarly, for $Q_{n}^{(j)}$, we need to handle a different integral in the “corner”: $\int_0^{k^{-1+\delta}} s \left\{ f_n^{(j)}(s) - f_n^{(j)}(1) \right\} ds$. To complete the toolkit for our purpose, we provide a general result regarding the joint asymptotic behavior of weighted integrals of the tail quantile processes in the corner area $[0, k^{-1+\delta}]$.

**Proposition 2.** Assume the same conditions as in Theorem 1. Assume that a function $g$ defined on $(0,1)$ satisfies $0 < g(s) \leq Cs^\beta$ with $\beta > \gamma - \frac{\eta}{2(1+\eta)} + \frac{1}{1+\eta} \gamma I \{ \gamma > 0 \}$. Then, there exists a sufficiently small constant $\delta > 0$, such that, as $N \to \infty$,

$$
\sqrt{m} \max_{1 \leq j \leq m} \int_0^{k^{-1+\delta}} g(s) \left| \sqrt{k} \left( Q_n^{(j)}(s) - \frac{s^{-\gamma} - 1}{\gamma} \right) - s^{-\gamma} - W_n^{(j)}(s) - \sqrt{k}A_0 \left( \frac{n}{k} \right) \Psi(s^{-1}) \right| ds = o_P(1).
$$

The oracle property of most extreme value estimators, including the PWM estimator, in the extreme value statistics based on the POT approach can be established by applying Theorem 3 and Proposition 2 together. We demonstrate a few examples in Section 4.
4 Application

4.1 Distributed inference for the Hill estimator

In this subsection, we apply the approximations of the tail empirical processes based on the
distributed subsamples to establish the oracle property of the distributed Hill estimator.

The Hill estimator in machine $j$ is defined as

$$
\hat{\gamma}_H^{(j)} := \frac{1}{k} \sum_{i=1}^{k} \log X_{n-i+1,n}^{(j)} - \log X_{n-k,n}^{(j)}, \quad j = 1, \ldots, m.
$$

The distributed Hill estimator is defined as the average of the $m$ estimates from each ma-
cine: $\hat{\gamma}_D := \frac{1}{m} \sum_{j=1}^{m} \hat{\gamma}_H^{(j)}$. And the oracle Hill estimator using the top $l = km$ exceedance ratios is

$$
\hat{\gamma}_{\text{Oracle}} := \frac{1}{km} \sum_{i=1}^{km} \log X_{N-i+1,N} - \log X_{N-km,N}.
$$

Corollary 2. Suppose that the distribution function $F$ satisfies the second order condition
(2) with $\gamma > 0$ and $\rho < 0$. Let $m, n, k$ be sequences of real numbers that satisfy condi-
tions (A1)-(A3). Then, the distributed Hill estimator achieves the oracle property, i.e.

$$
\sqrt{km} (\hat{\gamma}_D - \hat{\gamma}_{\text{Oracle}}) = o_P(1), \text{ as } N \to \infty.
$$

Proof of Corollary 2. By applying the same techniques used in proving the asymptotic
normality of the oracle Hill estimator (cf. Example 5.1.5 in de Haan and Ferreira (2006)),
we have that, as $N \to \infty$,

$$
\hat{\gamma}_D - \gamma = \frac{1}{m} \sum_{j=1}^{m} \int_{X_{n-k,n}/U(n/k)}^{1} s^{-1/\gamma} \frac{ds}{s}
$$

$$
+ \frac{1}{m} \sum_{j=1}^{m} \int_{X_{n-k,n}/U(n/k)}^{1} \left[ \frac{n}{k} \left( 1 - F_n^{(j)} \left( sU \left( \frac{n}{k} \right) \right) \right) - s^{-1/\gamma} \right] \frac{ds}{s}
$$

$$
+ \frac{1}{m} \sum_{j=1}^{m} \int_{1}^{\infty} \left[ \frac{n}{k} \left( 1 - F_n^{(j)} \left( sU \left( \frac{n}{k} \right) \right) \right) - s^{-1/\gamma} \right] \frac{ds}{s}
$$

$$
= : I_1 + I_2 + I_3.
$$
For $I_1$, note that, as $N \to \infty$,
\[
I_1 = \frac{1}{m} \sum_{j=1}^{m} \left\{ \gamma \left( X_{n-k,n}^{(j)}/U(n/k) \right)^{-1/\gamma} - \gamma \right\}.
\]
By taking $s = 1$ in Theorem 4, we get that, as $N \to \infty$,
\[
\sqrt{m} \max_{1 \leq j \leq m} \left| \sqrt{k} \left( X_{n-k,n}^{(j)} \right)^{-1/\gamma} - \gamma W_n^{(j)}(1) \right| = o_P(1).
\] (9)
Thus, as $N \to \infty$,
\[
I_1 = -\gamma (km)^{-1/2} \frac{1}{\sqrt{m}} \sum_{j=1}^{m} W_n^{(j)}(1) + (km)^{-1/2} o_P(1).
\]
For $I_2$, the uniform convergence in (9) and Theorem 1 imply that as $N \to \infty$, $I_2 = (km)^{-1/2} o_P(1)$.

For $I_3$, since $F_N = m^{-1} \sum_{j=1}^{m} F_n^{(j)}$, we obtain that,
\[
I_3 = \int_{1}^{\infty} \left[ \frac{n}{k} \left\{ 1 - F_N \left( sU \left( \frac{n}{k} \right) \right) \right\} - s^{-1/\gamma} \right] \frac{ds}{s}.
\]
We can handle $\hat{\gamma}_H^{Oracle}$ in a similar way and get that,
\[
\hat{\gamma}_H^{Oracle} - \gamma = -\gamma \frac{1}{\sqrt{km}} W_N(1) + (km)^{-1/2} o_P(1) + I_3.
\]
The Corollary is proved by noting that $W_N = m^{-1/2} \sum_{j=1}^{m} W_n^{(j)}$.  \qed

**Remark 3.** *Chen et al. (2022)* only shows that the limiting distribution of the distributed Hill estimator coincides with that of the oracle Hill estimator, but does not investigate the difference between the two estimators.

### 4.2 Distributed inference for the PWM estimator

In this subsection, we take the distributed PWM estimator as an example to show how to apply Theorem 3 and Proposition 2 to establish its oracle property. The oracle PWM estimator is defined as
\[
\hat{\gamma}_{PWM}^{Oracle} := \frac{P_N - 4Q_N}{P_N - 2Q_N},
\]
where $P_N$ and $Q_N$ are counterparts of $P_n^{(j)}$ and $Q_n^{(j)}$ based on the oracle sample, respectively.
Corollary 3. Suppose that the distribution function $F$ satisfies the second order condition \[ \text{with } \gamma < 1/2 \text{ and } \rho < 0. \] Assume that conditions (A1)-(A3) hold with $\eta > \max \left\{ 0, \frac{2\gamma}{1/2 - \gamma} \right\}$. Then, the distributed PWM estimator achieves the oracle property, i.e.,
\[
\sqrt{km} (\hat{\gamma}_\text{PWM} - \hat{\gamma}_\text{Oracle}) = o_P(1) \text{ as } N \to \infty.
\]

Proof of Corollary 3. For a continuous function $f : [0,1] \to \mathbb{R}$, define an operator
\[
L(f) = (1 - \gamma)(2 - \gamma) \int_0^1 \{(1 - 4s) - \gamma(1 - 2s)\} \{f(s) - f(1)\} \, ds.
\]
It is obvious that $L$ is a linear operator.

Note that, for the oracle PWM estimator using top $km$ exceedances, we have that, as $N \to \infty$,
\[
\hat{\gamma}_\text{Oracle} - \gamma = \frac{1}{\sqrt{km}} L \left( s^{-\gamma-1}W_N(s) \right) + A_0 \left( \frac{n}{k} \right) L \left( \Psi(s^{-1}) \right) + \frac{1}{\sqrt{km}}o_P(1),
\]
see e.g. Section 3.6.1 in de Haan and Ferreira (2006). By using similar techniques, we obtain that, as $N \to \infty$,
\[
\frac{1}{m} \sum_{j=1}^m \hat{\gamma}_{j,\text{PWM}} - \gamma = \frac{1}{m} \sum_{j=1}^m \frac{1}{\sqrt{k}} L \left( s^{-\gamma-1}W_n^{(j)}(s) \right) + A_0 \left( \frac{n}{k} \right) L \left( \Psi(s^{-1}) \right)
\]
\[
+ o_P(1) \max_{1 \leq j \leq m} \int_0^1 |f_n^{(j)}(s)| \, ds + o_P(1) \max_{1 \leq j \leq m} \int_0^1 s |f_n^{(j)}(s)| \, ds.
\]
Recall that $L$ is a linear operator and $W_N = m^{-1/2} \sum_{j=1}^m W_n^{(j)}$, we get that
\[
\frac{1}{\sqrt{km}} L \left( s^{-\gamma-1}W_N(s) \right) = \frac{1}{m} \sum_{j=1}^m \frac{1}{\sqrt{k}} L \left( s^{-\gamma-1}W_n^{(j)}(s) \right).
\]
The Corollary is proved provided that, as $N \to \infty$, $I_1 := \max_{1 \leq j \leq m} \int_0^1 |f_n^{(j)}(s)| \, ds = (km)^{-1/2}o_P(1)$ and $I_2 := \max_{1 \leq j \leq m} \int_0^1 s |f_n^{(j)}(s)| \, ds = (km)^{-1/2}o_P(1)$.

For handling $I_1$, we divide $[0,1]$ into $[k^{-1+\delta},1]$ and $[0,k^{-1+\delta}]$. Thus,
\[
I_1 \leq \max_{1 \leq j \leq m} \int_0^{k^{-1+\delta}} |f_n^{(j)}(s)| \, ds + \max_{1 \leq j \leq m} \int_{k^{-1+\delta}}^1 |f_n^{(j)}(s)| \, ds
\]
\[
:= I_{1,1} + I_{1,2}.
\]
We first handle $I_{1,2}$. Note that, for $\eta > \frac{2\gamma}{1+2\gamma}$, we can always find a $v > \frac{1}{2+\eta}$ such that $v + \gamma < 1/2$. Then, by Theorem 3 as $N \to \infty$,

$$I_{1,2} = o_P(1)(km)^{-1/2} \int_{k^{-1}+\delta}^{1} s^{-v-1/2-\gamma} ds = (km)^{-1/2}o_P(1).$$

The term $I_{1,1}$ can be handled by Proposition 2 as follows. Choose $g(s) = 1$. Since $\gamma < 1/2$ and $\eta > \max\{0, \frac{2\gamma}{1+2\gamma}\}$, the conditions in Proposition 2 hold. The proposition yields that $I_{1,1} = (km)^{-1/2}o_P(1)$. Hence, we obtain $I_1 = (km)^{-1/2}o_P(1)$ as $N \to \infty$. The term $I_2$ can be handled in a similar way with choosing $g(s) = s$.

4.3 Distributed inference for the MLE

The MLE for the extreme value index and the scale parameter based on the sample on machine $j$ $(\hat{\gamma}^{(j)}_{\text{mle}}, \hat{\sigma}^{(j)}_{\text{mle}})$, is defined as the solution of the following equations:

$$\frac{1}{k} \sum_{i=1}^{k} \frac{1}{\gamma^2} \log \left( 1 + \frac{\gamma}{\sigma} \left( X_{n-i+1,n}^{(j)} - X_{n-k,n}^{(j)} \right) \right) - \left( \frac{1}{\gamma} + 1 \right) \frac{(1/\sigma) \left( X_{n-i+1,n}^{(j)} - X_{n-k,n}^{(j)} \right)}{1 + (\gamma/\sigma) \left( X_{n-i+1,n}^{(j)} - X_{n-k,n}^{(j)} \right)} = 0, \quad (10)$$

$$\sum_{i=1}^{k} \left( \frac{1}{\gamma} + 1 \right) \frac{(\gamma/\sigma) \left( X_{n-i+1,n}^{(j)} - X_{n-k,n}^{(j)} \right)}{1 + (\gamma/\sigma) \left( X_{n-i+1,n}^{(j)} - X_{n-k,n}^{(j)} \right)} = k.$$

The distributed MLE for the extreme value index and the scale parameter are defined as

$$\tilde{\gamma}^{D}_{\text{mle}} = \frac{1}{m} \sum_{j=1}^{m} \hat{\gamma}^{(j)}_{\text{mle}}, \quad \tilde{\sigma}^{D}_{\text{mle}} = \frac{1}{m} \sum_{j=1}^{m} \hat{\sigma}^{(j)}_{\text{mle}}.$$

The oracle MLE for the extreme value index and the scale parameter $(\hat{\gamma}^{\text{Oracle}}_{\text{mle}}, \hat{\sigma}^{\text{Oracle}}_{\text{mle}})$ are defined in a similar way by using the oracle sample.

**Corollary 4.** Suppose that the distribution function $F$ satisfies the second order condition $(2)$ with $\gamma > -1/2$ and $\rho < 0$. Assume that conditions $(A1),(A3)$ hold with $\eta > \max\left(0, \frac{2\gamma}{1+2\gamma}\right)$. Then, the distributed MLE for the extreme value index and the
scale parameter achieve the oracle property, i.e., as $N \to \infty$,
\[
\sqrt{km} \left( \hat{\gamma}_{mle} - \hat{\gamma}^{Oracle}_{mle} \right) = o_p(1).
\]
\[
\sqrt{km} \left( \hat{\sigma}_{mle} - \hat{\sigma}^{Oracle}_{mle} \right) = o_p(1).
\]

The proof is deferred to the Supplementary Material. Solving the likelihood equations (10) involves an optimization algorithm. The computation cost can be high when implementing an optimization algorithm for the oracle sample. We provide a simulation study to compare the computation cost of the oracle MLE and the distributed MLE in the Supplementary Material.

4.4 Distributed inference for the high quantile, endpoint and tail probability

In this subsection, we show how to establish the oracle property of the estimators for the high quantile, endpoint and tail probability. In order to estimate these quantities, we need to estimate the extreme value index $\gamma$, the scale parameter $a(n/k)$ and the location parameter $b(n/k)$, see e.g. de Haan and Ferreira (2006, Chapter 4). We focus on the PWM estimators for $\gamma$ and $a(n/k)$ as an example. Other estimators based on the POT approach can be treated in a similar way.

Based on the oracle sample, since $N/(km) = n/k$, one can estimate $a(n/k)$ and $b(n/k)$ as
\[
\hat{a}^{Oracle} \left( \frac{n}{k} \right) = \frac{2P_N Q_N}{P_N - 2Q_N}, \quad \hat{b}^{Oracle} \left( \frac{n}{k} \right) = X_{N-[km],N},
\]
see e.g. Hosking and Wallis (1987).

We apply the DC algorithm to estimate $a(n/k)$ and $b(n/k)$ based on distributed sub-samples. Define the distributed scale estimator as
\[
\hat{a}^{D} \left( \frac{n}{k} \right) := \frac{1}{m} \sum_{j=1}^{m} \tilde{a}^{(j)} \left( \frac{n}{k} \right) = \frac{1}{m} \sum_{j=1}^{m} \frac{2P_n^{(j)} Q_n^{(j)}}{P_n^{(j)} - 2Q_n^{(j)}},
\]
and the distributed location estimator as
\[
\hat{b}^D \left( \frac{n}{k} \right) = \frac{1}{m} \sum_{j=1}^{m} X_{n-k,n}^{(j)}.
\]

Following similar steps as in proving the oracle property of \( \hat{\gamma}_{PWM}^D \), we can show that, as \( N \to \infty \),
\[
\sqrt{km} \frac{\hat{a}^D \left( \frac{n}{k} \right) - \hat{a}^{\text{Oracle}} \left( \frac{n}{k} \right)}{a \left( \frac{n}{k} \right)} = o_P(1),
\]
\[
\sqrt{km} \frac{\hat{b}^D \left( \frac{n}{k} \right) - \hat{b}^{\text{Oracle}} \left( \frac{n}{k} \right)}{a \left( \frac{n}{k} \right)} = o_P(1).
\]

4.4.1 High quantile

Let \( x(p_N) := U(1/p_N) \), where \( p_N = o(k/n) \) as \( N \to \infty \), be the quantile we want to estimate. In finance management, the high quantile is often referred to as value at risk, which is the most prominent risk measure. The detailed procedures of the distributed estimator for high quantile \( x(p_N) \) are given as follows:

- On each machine \( j \), we calculate \( \hat{\gamma}^{(j)}_{PWM}, \hat{a}^{(j)} \left( \frac{n}{k} \right), X_{n-k,n}^{(j)} \) and transmit these values to the central machine.

- On the central machine, we take the average of the \( \hat{\gamma}^{(j)}_{PWM}, \hat{a}^{(j)} \left( \frac{n}{k} \right), X_{n-k,n}^{(j)} \) statistics collected from the \( m \) machines to obtain \( \hat{\gamma}^D_{PWM}, \hat{a}^D \left( \frac{n}{k} \right), \hat{b}^D \left( \frac{n}{k} \right) \).

- On the central machine, we estimate \( x(p_N) \) with \( p_N \to 0 \) by
\[
\hat{x}^D(p_N) = \hat{b}^D \left( \frac{n}{k} \right) + \hat{a}^D \left( \frac{n}{k} \right) \frac{\hat{\gamma}^D_{PWM} - 1}{\hat{\gamma}^D_{PWM}}.
\]

The oracle high quantile estimator \( \hat{x}^{Oracle}(p_N) \) is defined in an analogous way as \( \hat{x}^D(p_N) \), with replacing \( \hat{\gamma}^D_{PWM}, \hat{a}^D \left( \frac{n}{k} \right) \) and \( \hat{b}^D \left( \frac{n}{k} \right) \) by \( \hat{\gamma}^{Oracle}_{PWM}, \hat{a}^{Oracle} \left( \frac{n}{k} \right) \) and \( \hat{b}^{Oracle} \left( \frac{n}{k} \right) \) in (11), respectively. Following the lines of the proof for the asymptotics of the oracle high quantile estimator, we can obtain the asymptotic normality of \( \hat{x}^D(p_N) \). Moreover, since \( \hat{\gamma}^D_{PWM}, \hat{a}^D \left( \frac{n}{k} \right) \) and \( \hat{b}^D \left( \frac{n}{k} \right) \) possess the oracle property, \( \hat{x}^D(p_N) \) also achieves the oracle property.
due to applying the Cramér delta method. We present the result in the following corollary while omitting the proof.

**Corollary 5.** Assume the same conditions as in Corollary 3. Suppose that \( np_N = o(k) \) and \( \log(Np_N) = o(\sqrt{km}) \) as \( N \to \infty \). Then, as \( N \to \infty \),

\[
\sqrt{km} \frac{\hat{x}^D(p_N) - \hat{x}^{\text{Oracle}}(p_N)}{a\left(\frac{n}{k}\right) q_\gamma(d_N)} = o_P(1),
\]

where \( d_N = k/(np_N) \) and for \( t > 1 \), \( q_\gamma(t) := \int_1^t s^{\gamma - 1} \log s ds \).

### 4.4.2 Endpoint

Next, we consider the problem of estimating the endpoint of the distribution function \( F \).

Assume that \( F \in D(G_\gamma) \) for some \( \gamma < 0 \). In this case the endpoint \( x^* = \sup \{ x : F(x) < 1 \} \) is finite. The endpoint can be treated as a specific case of quantile by regarding \( p_N \) as 0. The distributed endpoint estimator can be defined as

\[
\hat{x}^{*,D} = \hat{b}^D \left( \frac{n}{k} \right) - \frac{\hat{a}^D \left( \frac{n}{k} \right)}{\hat{\gamma}^D_{PWM}}.
\]

The definition of the oracle endpoint estimator \( \hat{x}^{*,\text{Oracle}} \) is in an analogous way. Again, the distributed endpoint estimator achieves the oracle property as in the following corollary.

**Corollary 6.** Assume the same conditions as in Corollary 3 and \( \gamma < 0 \). Then, as \( N \to \infty \),

\[
\sqrt{km} \frac{\hat{x}^{*,D} - \hat{x}^{*,\text{Oracle}}}{a\left(\frac{n}{k}\right)} = o_P(1).
\]

### 4.4.3 Tail probability

Lastly, we consider the dual problem of estimating the high quantile: given a large value of \( x_N \), how to estimate \( p(x_N) = 1 - F(x_N) \) under the distributed inference setup. The detailed procedures for estimating the tail probability are similar to that for estimating the high quantile, except that on the central machine, we estimate the tail probability \( p(x_N) \)
by
\[ \hat{p}^D(x_N) = \frac{k}{n} \left\{ \max \left( 0, 1 + \gamma p_{PWM}^D x_N - \hat{b}^D \frac{\alpha}{\hat{D}(\alpha)} \right) \right\}^{-1/\gamma p_{PWM}} \].

The definition of the oracle tail probability estimator \( \hat{p}^{Oracle}(x_N) \) is in an analogous way.

Note that \( \hat{p}^{Oracle}(x_N) \) is valid only for \( \gamma > -1/2 \) (cf. Remark 4.4.3 in de Haan and Ferreira (2006)).

The oracle property of \( \hat{p}^D(x_N) \) is established in the following corollary.

**Corollary 7.** Assume the same conditions as in Corollary 3 and \( \gamma \in (-1/2, 1/2) \). Denote \( d_N = \frac{k}{n p(x_N)} \) and \( w_{\gamma}(t) = t^{-\gamma} \int_1^t s^{\gamma-1} \log s \, ds \) for \( t > 0 \). Suppose that \( d_N \to \infty \) and \( w_{\gamma}(d_N) = o(\sqrt{km}) \) as \( N \to \infty \), then
\[ \frac{\sqrt{km}}{w_{\gamma}(d_N)} \left( \hat{p}^D(x_N) - \hat{p}^{Oracle}(x_N) \right) \sim o_P(1). \]

## 5 Heterogeneous subsample sizes

In this section, we extend our results to the case of heterogeneous subsample sizes. We assume that the \( N \) observations are distributedly stored in \( m \) machines with \( n_j \) observations in machine \( j, j = 1, \ldots, m \), i.e. \( N = \sum_{j=1}^{m} n_j \). Moreover, we assume that all \( n_j, j = 1, \ldots, m \) diverge in the same order. Mathematically, that is,
\[ c_1 \leq \min_{1 \leq j \leq m} n_j m/N \leq \max_{1 \leq j \leq m} n_j m/N \leq c_2 \quad (12) \]
for some positive constants \( c_1 \) and \( c_2 \) and all \( N \geq 1 \).

The tail empirical process based on the observations in machine \( j \) is now defined as
\[ Y_{n_j, k_j}^{(j)}(x) = \frac{n_j}{k_j} \bar{F}_n^{(j)} \left\{ a_0 \left( \frac{n_j}{k_j} \right) x + b_0 \left( \frac{n_j}{k_j} \right) \right\}, \quad j = 1, \ldots, m, \]
where \( \bar{F}_n^{(j)} := 1 - F_{n_j}^{(j)} \) and \( F_{n_j}^{(j)} \) denotes the empirical distribution function based on the observations in machine \( j \).

We choose \( k_j, j = 1, \ldots, m \) such that the ratios \( k_j/n_j \) are homogenous across all the \( m \) machines, i.e.,
\[ k_1/n_1 = \cdots = k_m/n_m. \quad (13) \]
Denote \( K = \sum_{j=1}^{m} k_j \), clearly \( k_j/n_j = K/N, j = 1, \ldots, m \). Then, we have that,

\[
Y_{N,K}(x) = \sum_{j=1}^{m} \frac{n_j}{N} Y_{n_j,k_j}^{(j)}(x).
\]

In other words, the oracle tail empirical process is a weighted average of the tail empirical processes based on the distributed subsamples, where the weights equal to the fraction of the observations on each machine.

Following similar steps as in proving Theorem 1, we obtain the following result.

**Theorem 5.** Assume the same conditions as in Theorem 1 and conditions (12) and (13). Then under proper Skorokhod construction, there exist \( m \) independent sequences of Brownian motions \( \{W_{n_j}^{(j)}\}, j = 1, \ldots, m \), such that for any \( \nu \in ((2 + \eta)^{-1}, 1/2) \), as \( N \to \infty \),

\[
\max_{1 \leq j \leq m} \sup_{x \in \mathbb{D}} \left\{ z(x) \right\}^{\nu-1/2} \sqrt{k_j m} \left\{ Y_{n_j,k_j}^{(j)}(x) - z(x) \right\} \\
- \sqrt{k_j m} A_0(N/K) \left\{ z(x) \right\}^{1+\gamma} \Psi \left\{ 1/z(x) \right\} = o_P(1).
\]

Moreover, as \( N \to \infty \),

\[
\sup_{x \in \mathbb{D}} \left\{ z(x) \right\}^{\nu-1/2} \sqrt{K} \left\{ Y_{N,K}(x) - z(x) \right\} \\
- W_N \left\{ z(x) \right\} - \sqrt{K} A_0(N/K) \left\{ z(x) \right\}^{1+\gamma} \Psi \left\{ 1/z(x) \right\} = o_P(1),
\]

where \( W_N = \sum_{j=1}^{m} \frac{n_j}{N} W_{n_j}^{(j)} \) is also a Brownian motion.

Similar results hold for the tail quantile processes as in Theorem 3. Eventually, we can re-establish the oracle property of the distributed estimators as follows. Suppose the oracle estimator is based on \( K \) top order statistics in the oracle sample. On each machine, we use the top \( k_j = (n_j/N)K \) order statistics in the estimation. By taking a weighted average of the estimates from all machines using the weights \( n_j/N, j = 1, \ldots, m \), to obtain the distributed estimator, the oracle property holds under the same conditions as in the homogenous case with similar proofs.
6 Real Data Application

We use a dataset containing car insurance claims in five states of United States: Iowa ($n_1 = 2601$), Kansas ($n_2 = 798$), Missouri ($n_3 = 3150$), Nebraska ($n_4 = 1703$), and Oklahoma ($n_5 = 882$). The total sample size is $N = 9134$. We work under a hypothesis scenario: each state cannot share its own data to others, but they are willing to share their statistical results. Then one can apply a DC algorithm for conducting extreme value statistics. Our target is to estimate the common extreme value index of the total claim amount. We consider the MLE instead of the Hill estimator considered by Chen et al. (2022) since we do not assume heavy tail at the first place.

Let $K = \sum_{j=1}^{5} k_j$ be the total number of exceedances used by the five states. As suggested in Section 5, we choose $k_j$ as $k_j = \lceil K \frac{n_j}{N} \rceil$, and apply the MLE for each of the five states to obtain $\gamma_{mle}^{(j)}$, $j = 1, 2, \ldots, 5$. Then, we combine these five estimates to obtain the distributed MLE by

$$\hat{\gamma}_{mle}^D = \sum_{j=1}^{5} \frac{n_j}{N} \hat{\gamma}_{mle}^{(j)}.$$

The distributed MLE is plotted against different values of $K$ in Figure 1 along with its
95% confidence interval. We also plot the oracle MLE in this figure. The distributed MLE is close to the oracle MLE for almost all levels of $K$ and the oracle MLE always falls into the 95% confidence interval constructed based on the distributed MLE.

By choosing $K = 1000$, we obtain that the distributed MLE for the extreme value index is about 0.05. And we cannot reject the hypothesis that the extreme value index is 0 under the 5% significance level for this choice of $K$. This result shows that the insurance claims may not be heavy tailed. In turn, the distributed Hill estimator adopted in Chen et al. (2022) may not be suitable for this application.

7 Discussion

In this paper, we investigate the problem of distributed inference in extreme value analysis when the oracle sample $\{X_1, X_2, \ldots, X_N\}$ are i.i.d.. In fact, the assumption that all the data are drawn from the same distribution can be relaxed. In real applications, observations from different machines may follow different distributions, but nevertheless share some common properties such as the extreme value index.

We assume that all observations are independent, but only observations on the same machine follow the same distribution. Denote the common distribution function of the observations in machine $j$ as $F_{n,j}, j = 1, \ldots, m$. We assume that, there exists a continuous function $F$ which satisfies the second order condition (2) with $\gamma > 0$. In addition, assume that the series of constants $\{c_{n,j}\}_{1 \leq j \leq m}$ satisfies that $0 < \underline{c} \leq c_{n,j} \leq \bar{c} < \infty$ for all $1 \leq j \leq m$ and $n \in \mathbb{N}$, and $A_1(t)$ is a positive regularly varying function with index $\tilde{\rho} < 0$ such that as $t \to \infty$,

$$
\sup_{m \in \mathbb{N}} \max_{1 \leq j \leq m} \left| \frac{1 - F_{n,j}(t)}{1 - F(t)} - c_{n,j} \right| = O(A_1(t)).
$$

By restricting that $\sqrt{k m A_1(n/k)} \rightarrow 0$, Chen et al. (2022) gives a theoretical proof for the asymptotic theories of the distributed Hill estimator. Following similar steps, we can also
handle tail empirical processes and tail quantile processes. The details are omitted.

Supplementary Material

Appendix.pdf: The Supplementary Material contains all the technical proofs and simulation studies.

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