Finite boundary regularity for conformally compact Einstein manifolds of dimension 4

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Abstract

We prove that a 4−dimensional $C^{2,\sigma}$ conformally compact Einstein manifold with $C^{m,\alpha}$ boundary metric has a $C^{m,\alpha}$ compactification. We also study the regularity of the new structure and the new defining function. This is a supplementary proof of Anderson’s work and an improvement of Helliwell’s result in dimension 4.

1 Introduction

In 1985, Charles Fefferman and Robin Graham [8] introduced a new method to study the local conformal invariants of manifolds. Similar to $n$− sphere embedded into $n + 2$−dimensional Minkowski space, they tried to embed an arbitrary conformal $n$−manifold into an $n + 2$− dimensional Ricci-flat Lorentz manifold, which they called the ambient space. The ambient spaces were used to produce local scalar conformal invariants. An important part of the ambient space construction is the introduction of conformally compact Einstein metrics for a conformal manifold. The study of conformally compact Einstein metrics could tell us some relationship between the Riemannian structure in the interior and the conformal structure on the boundary. Much progress has been made since then. In recent years, the physics community has also become interested in conformally compact Einstein metrics because the introduction of AdS/CFT correspondence in the quantum theory of gravity in geometric physics by Maldacena [22].

Let $M$ be the interior of a compact $(n + 1)$-dimensional manifold $\overline{M}$ with non-empty boundary $\partial M$. We call a complete metric $g^+$ on $M$ is $C^{m,\alpha}(or W^{k,p})$ conformally compact if there exists a defining function $\rho$ on $\overline{M}$ such that the conformally equivalent metric

$$g = \rho^2 g^+$$

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can extend to a $C^{m,\alpha}(\text{or } W^{k,p})$ Riemannian metric on $\overline{M}$. The defining function is smooth on $\overline{M}$ and satisfies

$$\begin{cases}
\rho > 0 & \text{in } M \\
\rho = 0 & \text{on } \partial M \\
\partial \rho \neq 0 & \text{on } \partial M
\end{cases} \quad (1.1)$$

Here $C^{m,\alpha}$ and $W^{k,p}$ are usual Hölder space and the Sobolev space. We call the induced metric $h = g|_{\partial M}$ the boundary metric associated to the compactification $g$. It is easy to see that different defining function induces different boundary metric and every two of the boundary metrics are conformal equivalent. Then the conformal class $[h]$ is uniquely determined by $(M, g^+)$. We call $[h]$ the conformal infinity of $g^+$. If in addition, $g^+$ is Einstein, i.e.

$$Ric_{g^+} + ng^+ = 0,$$  
(1.2)

then we say $(M, g^+)$ is a conformally compact Einstein manifold.

There are some interesting problems concerning conformally compact Einstein metric. Such as the existence problem, see [2], [10], [12], [14], [17], [18] etc. The unique problem, see [1], [6]. The compactness problem, see [2], [5], [6].

In this paper, we deal with the boundary regularity problem. Given a conformally compact Einstein manifold $(M, g^+)$ and a compactification $g = \rho^2 g^+$, if the boundary metric $h$ is $C^{m,\alpha}$, is there a $C^{m,\alpha}$ compactification of $g^+$? This problem was first raised by Fefferman and Graham in 1985 in [8] and they observed that if $\dim M = n + 1$ is odd, the boundary regularity in general breaks down at the order $n$. If $\dim M = n + 1$ is even, the $C^{m,\alpha}$ compactification may exist.

In [4], Chruściel, Delay, Lee and Skinner used the harmonic diffeomorphism in infinity to construct a good structure near boundary where Einstein equation could be written as a degenerate elliptic PDE of second order. That is so-called ‘gauge-broken Einstein equation’. Then they use polyhomogeneity result of some specific degenerate equation to obtain a good result of the boundary regularity. They proved that if the boundary metrics are smooth, the $C^2$ conformally compact Einstein metrics have conformal compactifications that are smooth up to the boundary in the sense of $C^{1,\lambda}$ diffeomorphism in dimension 3 and all even dimensions, and polyhomogeneous smooth in odd dimensions greater than 3. This is certainly a very good result in the sense that they made good use of Einstein equation and gave us a suitable coordinate in infinity to study conformally compact Einstein metrics. I think their method is more geometrical. The condition of that the initial compactification is $C^2$ in all dimension should be sharp. However, their result only hold for smooth case. It is believed that their method could also be used to prove the finite regularity although we may loss half regularity in this situation.
In [1] and [2], M. T. Anderson considered the Bach tensor in dimension 4, and proved the finite regularity result. He only assume that the initial compacification $g$ is $W^{2,p}$ where $p > 4$. I am not sure whether the $W^{2,p}$ condition is good enough to prove his result. As a supplementary proof, we use Anderson’s method to prove his conclusion where we assume that the initial compactification $g$ is $C^{2,\sigma}$ for any $\sigma \in (0,1)$.

In [15], Helliwell solved the issue in all even dimensions by following Anderson’s method. He considered the Fefferman-Graham ambient obstruction tensor instead of Bach tensor in higher dimensions. It is conformally invariant and vanishes for Einstein metrics. Helliwell assumed the initial compactification $g$ is at least in $C_{n,\alpha}^{m,\alpha}$ for a $(n+1)-$ smooth manifold. It means the original compactification is $C^{3,\alpha}$ for a smooth manifold of dimension 4. Now we reduce the condition $C^{3,\alpha}$ to $C^{2,\sigma}$ to improve his result.

This is the main result:

**Theorem 1.1.** Let $(M, g^+)$ be a conformally compact Einstein manifold of dimension 4 with a $C^2$ compactification $g = \rho^2 g^+$. If the scalar curvature $S \in C^\sigma(M)$ and the mean curvature $H \in C^{1,\sigma}(\partial M)$ for some $\sigma > 0$, the boundary metric $h = g|_{\partial M} \in C_{m,\alpha}^{m,\alpha}(\partial M)$ with $m \geq 2, \alpha \in (0,1)$, then under a $C^{2,\lambda}$ coordinates change, $g^+$ has a $C^{m,\alpha}$ conformally compactification $\tilde{g} = \rho^2 g^+$ with the boundary $\tilde{g}|_{\partial M} = h$.

**Remark 1.2.** The new coordinates form $C^{m+1,\alpha}$ differential structure of $\overline{M}$. $\hat{\rho}$ is a $C^{m+1,\alpha}$ defining function.

If $g = \rho^2 g^+$ is $C^{2,\sigma}$, then the conditions of $S$ and $H$ in theorem 1.1 hold automatically. Hence the conclusion is also true.

If the boundary metric $h$ is smooth, then $g^+$ has a smoothly conformally compactification $\tilde{g}$ with the boundary $\tilde{g}|_{\partial M} = h$.

It is well known that (see [8]) if $(M, g^+)$ is a 4–dimensional conformally compact Einstein metric with boundary metric $h$ and $g = r^2 g^+$ is the geodesic compactification associated with $h$, then according to the Gauss lemma, $g_+ = r^{-2}(dr^2 + g_r)$.

$$g_r = h + g^{(2)} r^2 + g^{(3)} r^3 + \cdots$$

where $g^{(2)}$ is the Schouten tensor and is determined by $h$, $g^{(3)}$ is determined by $g^+$ and $h$ and hence it is a non-local term. The rest of power series is determined by $g^{(3)}$ and $h$. This property is also true for higher dimension. From this point of view, Helliwell’s condition of $C^{3,\alpha}$ initial compactification seems very nature. That we improve it to $C^{2,\sigma}$ is a big step as we don’t need any information of non-local term.
The outline of this paper is as follows. In section 2, we introduce some basic facts about conformally compact Einstein metrics. We show that the Yamabe compactification near infinity exists. The conditions in theorem 1.1 is unchanged under this compactification. We also consider the Bach equation in dimension 4 and it is an elliptic PDE of second order about Ricci tensor if the scalar curvature is constant. At last, we introduce the harmonic coordinates.

In section 3, we deduce some boundary conditions. Including the Dirichlet condition of metric and Ricci curvature, the Neumann condition of Ricci curvature and the oblique derivative condition of metric. We prove that these conditions are true even if the compactification $g$ is only $C^2$.

In section 4, we attempt to prove the main theorem. The first difficulty is $C^\alpha$ and $C^{1,\alpha}$ estimate of Ricci curvature. So we present the intermediate Schauder theory to solve the problem. Then we finish our proof with the classical Schauder theory. In the end, with the help of Bach equation, we prove the regularity of defining function in the new coordinates.

2 Preliminaries

Let $(M, g_+)$ be a $n+1$–dimensional conformally compact Einstein manifold and $g = \rho^2 g_+$ is a compactification. Then

$$K_{ab} = \frac{K_{+ab} + |\nabla \rho|^2}{\rho^2} - \frac{1}{\rho^2} [D^2 \rho(e_a, e_a) + D^2 \rho(e_b, e_b)],$$  \hspace{1cm} (2.1)

$$Ric = -(n-1) \frac{D^2 \rho}{\rho} + \frac{n(|\nabla \rho|^2 - 1)}{\rho^2} - \frac{\Delta \rho}{\rho} g,$$  \hspace{1cm} (2.2)

$$S = -2n \frac{\Delta \rho}{\rho} + n(n+1) \frac{|\nabla \rho|^2 - 1}{\rho^2}. \hspace{1cm} (2.3)$$

Here $K_{ab}, Ric, S$ are the sectional curvature, Ricci curvature and scalar curvature of $g$ and $D^2$ denote the Hessian.

If $g$ is a $C^2$ compactification, then from $2.3$, $|\nabla \rho| = 1$ on $\partial M$. Then by $2.1$, $K_{+ab}$ tends to $-1$ as $\rho \to 0$. Hence a $C^2$ conformally compact Einstein manifold is asymptotically hyperbolic. Let $D^2 \rho|_{\partial M} = A$ denote the second fundamental form of $\partial M$ in $(M, g)$. The equation $2.2$ further implies that $\partial M$ is umbilic.

2.1 Constant scalar curvature compactification

**Lemma 2.1.** Let $(M, g_+)$ be a conformally compact $n$-manifold with a $W^{2,p}$ conformal compactification $g = \rho^2 g_+$ where $p > n$. Suppose that $h = g|_{\partial M}$ is the
boundary metric. Then there exits a $W^{2,p}$ constant scalar curvature compactification $\hat{g} = \hat{\rho}^2 g_+$ with boundary metric $h$.

Proof. We only need to solve a Yamabe problem with Dirichlet data. Let $\hat{g} = \frac{u^4}{n-2} g$, then we consider the equation

$$\begin{cases}
    \Delta_g u - \frac{n-2}{4(n-1)}Su + \frac{n-2}{4(n-1)}\lambda u^{\frac{n+2}{n-2}} = 0 \\
    u > 0 \text{ in } M \\
    u \equiv 1 \text{ on } \partial M
\end{cases}$$

(2.4)

Let $\lambda = -1$, we can get a $W^{2,p}$ solution by standard PDE method. In fact, from [21] we know that the equation always has a $C^{2,\alpha}$ solution near the boundary if $g \in C^{2,\alpha}$. By the compactness result in [9], if a sequence of metrics $g_j \in C^{2,\alpha}$ converge to a metric $g$ in $W^{2,p}$ norm, then the Yamabe metric $\hat{g}_j \in [g_j]$ also converge to the Yamabe metric $\hat{g} \in [g]$ in $W^{2,p}$ norm. The corresponding factor $u \in W^{2,p}$, then $u |_{\partial M} = 1$. Hence the boundary metric $h$ is not changed.

If $g \in C^{2}$, $S_g \in C^\sigma$, for some $\sigma > 0$, we know that the equation [21] has a $C^{2,\sigma}$ solution $u$. Then $\hat{g} = u^2 g$ is still in $C^2$, and the new defining function $\hat{\rho} = \rho u \in C^{2,\sigma}$. Let $H$ and $\hat{H}$ denote the mean curvature on the boundary with respect to $g$ and $\hat{g}$, then by lemma [3.3]

$$\hat{H} = H - 3\frac{\partial u}{\partial x^0} \frac{\partial \rho}{\partial x^0} \in C^{1,\sigma}(\partial M)$$

In the following of this section, we don’t distinguish $g$ with $\hat{g}$. When we refer to the compactification $g$, we mean the scalar curvature of $g$ is $-1$ near the boundary and the defining function is $C^{2,\sigma}$.

2.2 The Bach equation

For a 4–dimensional manifold, the Bach tensor is a conformal invariant and vanishes for Einstein metric, see [3]. In local coordinates,

$$B_{ij} = P_{ij}^k - P_{ik}^j - P_{kl}^{kl}W_{kijl}$$

(2.5)

where $P_{ij} = \frac{1}{2} R_{ij} - \frac{S}{12} g_{ij}$ is the Schouten tensor.

Let $\{y^3\}_{i=1}^3$ be the smooth structure on $M$ and when restricted on $\partial M$, $\{y^i\}_{i=1}^3$ is smooth structure of $\partial M$. From above we can assume that $g \in C^\infty(M) \cap C^2(\overline{M})$, $S_g \equiv -1$. Then the fact that $g_+$ is Einstein and (2.5) imply that

$$\Delta Ric_{\alpha\beta} = \Gamma * \partial Ric + Q$$

(2.6)

in $y$-coordinates. Here $\Gamma$ is the Christoffel symbol of $g$, $\Gamma * \partial Ric$ denote the bilinear form of $\Gamma$ and $\partial Ric$ and $Q$ denote the quadratic term of curvature.

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2.3 The harmonic coordinates near boundary

In the rest of the paper, if there are no special instructions, any use of indices will follow the convention that Roman indices will range from 1 to \( n \), while Greek indices range from 0 to \( n \).

We call the coordinates \( \{ x^\beta \}_{\beta=0}^n \) harmonic coordinates with respect to \( g \) if

\[
\Delta_g x^\beta = 0
\]

for \( 0 \leq \beta \leq n \). We are now going to construct harmonic coordinates in a neighbourhood of \( \partial M \) if \( g \) is smooth.

In fact, if \( g \in C^1, \alpha \), \( \alpha \in (0, 1) \) for any point \( p \in \partial M \), there is a neighbourhood \( V \) and smooth structure \( \{ y^\beta \}_{\beta=0}^n \) where \( y^0|_{\partial M} = 0 \). Then by solving a local Dirichlet problem:

\[
\begin{cases}
  \Delta_g x^\beta = 0 & \text{in } V \\
  x^\beta|_{V \cap \partial M} = y^\beta|_{V \cap \partial M}
\end{cases}
\]

There is a \( C^{2,\alpha} \) solution by [13] and we have the Schauder estimate:

\[
\| x^\beta - y^\beta \|_{C^{2,\alpha}(V)} \leq C(\| \Delta(x^\beta - y^\beta) \|_{C^\alpha(V)} + \| x^\beta - y^\beta \|_{C^{2,\alpha}(\partial V)}) = C \| \Delta y \|_{C^\alpha(V)}
\]

We can assume that the \( y \)-coordinates is the normal coordinates at \( p \), then \( \Delta y(p) = 0 \). Hence if \( V \) is small enough, \( \| x^\beta - y^\beta \|_{C^{2,\alpha}(V)} \) tends to 0. \( \{ x^\beta \}_{\beta=0}^n, 0 \leq \beta \leq n \) is a coordinate around \( p \).

In particular, if \( g \in C^2 \), then the solution \( x \in C^{2,\alpha}(y) \) for any \( \alpha \in (0, 1) \). Hence

\[
g_{\alpha\beta} = g\left( \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right) \in C^{1,\alpha}(\overline{M})
\]

In harmonic coordinates \( \{ x^\beta \}_{\beta=0}^n \), the Ricci tensor could be written as:

\[
\Delta g_{ij} = -2R_{ij} + Q(g, \partial g)
\]

where \( Q(g, \partial g) \) is a polynomial of \( g \) and \( \partial g \). For more details, see [7].

3 The boundary conditions

In this section, we derive a boundary problem for \( g \) and Ricci curvature of a conformal compact Einstein manifold in the harmonic coordinates as defined in section 2. We do it locally, that is, for any \( p \in \partial M \), there is a neighborhood \( V \) contains \( p \) and local harmonic atlas \( \{ x^\beta \} \). Let \( D = V \cap \partial M \) be the boundary portion and let \( g \in C^2(V) \) be the Yamabe compactification. We will give the Dirichlet and Neumann boundary conditions of \( g \) and \( Ric(g) \) on \( D \). Here we state that the boundary conditions in this section hold for all dimension.
In fact, as it is showed in [15] and [16] that, if \( g \) is \( C^{3,\alpha} \) compact, we have following boundary conditions:

**Proposition 3.1.** Let \((M, g^+)\) be a \((n + 1)\)-dimensional conformally compact Einstein manifold with a \(C^{3,\alpha}\) Yamabe compactification \( g = \rho^2 g^+ \). \( g|_{\partial M} = h \) is the boundary metric. Suppose that \( \{x^\beta\}_{\beta=0}^n \) are any coordinates near the boundary such that \( x^0 \) is defining function and \( \{x^i\}_{i=0}^n \) are coordinates of \( \partial M \). We have:

\[
g_{ij} = h_{ij}. \tag{3.1}
\]

\[
R_{ij} = \frac{n-1}{n-2} (Rch)_{ij} + \left( \frac{1}{2n} S - \frac{1}{2(n-2)} S_h \right) h_{ij} + \frac{n-1}{2n^2} H^2 h_{ij}. \tag{3.2}
\]

\[
R_{0i} = - (g^{00})^{-\frac{1}{2}} \frac{n-1}{n} \frac{\partial H}{\partial x_i} - g_{ij} R_{ij}, \tag{3.3}
\]

\[
R_{00} = \left( g^{00} \right)^{-\frac{1}{2}} \left( g_{ij} g^{0j} R_{ij} + g^{00} \left( \frac{1}{2} (S - S_h) - \frac{n-1}{2n} H^2 \right) \right). \tag{3.4}
\]

\[
N(R_{0i}) = (g^{00})^{-\frac{1}{2}} \left( - g^{ij} \partial_j R_{0i} + g^{ij} \Gamma^r_{ij} R_{0r} \right) \tag{3.5}
\]

where \( N = \nabla x_0 |_{\partial M} = (g^{00})^{-\frac{1}{2}} g^{0\beta} \partial_\beta \) be the unit norm vector on \( \partial M \).

The formula (3.1) is trivial and (3.5) is deduced by the second Bianchi identity and the fact that the scalar curvature is constant near the boundary. Here we briefly recall the proof of the formula (3.2), (3.3) and (3.4). For a \( C^{3,\alpha} \) conformally compact Einstein metric, there is a unique \( C^{2,\alpha} \) geodesic compactification with the same boundary metric (lemma 5.1 in [20]). Then for such a \( C^{2} \) geodesic compactification, we have a good formula for Ricci curvature and scalar curvature on the boundary. At last, we use the Ricci formula under conformal change to get (3.2), (3.3) and (3.4).

In this section, we will show that the formula (3.2), (3.3) and (3.4) still hold for \( C^2 \) conformally compact Einstein metric.

In fact, if \( g \) is \( C^2 \) compact, then there exists a sequence of \( C^{3,\alpha}(\overline{M}) \) metrics \( g_k \) which converge to \( g \) in \( C^2 \) norm in smooth structure of \( \overline{M} \). However \( g_k \) are not conformal Einstein in general. In the following, we omit the index \( k \) and assume that \( g \) is a \( C^{3,\alpha} \) metric on \( \overline{M} \). By choosing a defining function \( \rho \) satisfying \( |\nabla \rho|_g = 1 \) on \( \partial M \), we make \( g^+ = \rho^{-2} g \). Then with Taylor theorem, there is a \( C^{2,\alpha} \) function \( b \) such that \(|\nabla \rho|^2 = 1 + b \rho \) near the boundary.

\[
Ric = - (n-1) \frac{D^2 \rho}{\rho} + \left( \frac{n(|\nabla \rho|^2 - 1)}{\rho^2} - \frac{\Delta \rho}{\rho} \right) g + \frac{F}{\rho}, \tag{3.6}
\]
\[ S = -2n \frac{\Delta \rho}{\rho} + n(n + 1) \frac{|\nabla \rho|^2 - 1}{\rho^2} + \frac{\text{tr} F}{\rho}, \]  
(3.7)

where \( F = \rho (\text{Ric}_g + \nu g) = \rho \text{Ric}_g + (n - 1)D^2 \rho - (nb - \Delta \rho)g \in C^{\alpha, \alpha}(\overline{M}). \)

Now we prove the following formulas:

\[ R_{0i} = -\left( g^{00} \right)^{-\frac{1}{2}} n - \frac{1}{n} \partial H - \frac{g^{ij}}{g^{00}} R_{ij} + Q(F, DF, h, Dg, H), \]

\[ R_{00} = \frac{1}{(g^{00})^2} \left( g^{0i} g^{0j} R_{ij} + g^{00} \left( \frac{1}{2} (S - S_h) - \frac{n-1}{2n} H^2 \right) \right) + Q(F, DF, h, Dg, H), \]

\[ R_{ij} = \frac{n}{n-2} \left( \text{Ric}_h \right)_{ij} + \left( \frac{1}{2n} S_h \right) h_{ij} + \frac{n-1}{2n} H h_{ij} + Q(F, DF, h, Dg, H). \]  
(3.8)

Here \( h = g|_\partial M, \) \( H \) is the mean curvature, \( Q \) is polynomial and \( Q(F, DF, h, Dg, H) = 0 \) if \( F = DF = 0 \) on \( \partial M. \) We will use three lemmas to prove (3.8).

Firstly, there is a unique \( C^{2, \alpha} \) geodesic compactification of \( g^+ \) with boundary metric \( h \) and denote it by \( \bar{g} = r^2 g^+ \). Let \( \bar{g} = u^2 g \) where \( u = \frac{r}{\rho} \) satisfying that \( \rho \equiv 1 \) on the boundary and \( u \in C^{2, \alpha}. \) Then \( \bar{F} = r (\text{Ric}_{g^+} + \nu g^+) = u \bar{F} \) is still \( C^{1, \alpha}(\overline{M}). \) We will calculate the boundary curvature of \( \bar{g} \) and notice that the second fundamental form of \( \bar{g} \) at \( \partial M \) is not 0, but determined by the tensor \( \bar{F}. \)

**Lemma 3.2.** Suppose that \( \bar{g} = r^2 g^+ \) is a \( C^2 \) conformally compactification of manifold \((M, g^+)\) with boundary metric \( h. \) Then on the boundary \( \partial M, \)

\[ \bar{S} = \frac{n}{n-1} (S_h) + Q(\bar{F}, D\bar{F}), \]  
(3.9)

\[ \bar{R}_{ij} = \frac{n-1}{n-2} (\text{Ric}_h)_{ij} - \frac{1}{2(n-1)(n-2)} S_h h_{ij} + Q(\bar{F}, D\bar{F}, h, D\bar{g}). \]  
(3.10)

Here \( \bar{S} \) and \( \bar{R}_{ij} \) are the scalar curvature and Ricci curvature of \( \bar{g}. \) \( Q \) is a polynomial satisfying \( Q(\bar{F}, D\bar{F}, h, D\bar{g}) = 0 \) if \( \bar{F} = D\bar{F} = 0. \)

**Proof.** Let us choose the coordinates \((r, y^1, \cdots, y^n)\), near \( \partial M \) such that \( \bar{g} = dr^2 + g_r, \) i.e.

\[ g_{ri} = g^{ri} = 0, \quad g_{rr} = g^{rr} = 1. \]

According to Gauss Codazzi equation,

\[
\bar{R}_{ij} = \bar{g}^{\alpha\beta} \bar{R}_{i\alpha j\beta} = \bar{g}^{kl}((R_h)_{ikl} + \bar{A}_{il} \bar{A}_{kj} - \bar{A}_{ij} \bar{A}_{kl}) + \bar{R}_{irr}j
\]

\[ = (R_h)_{ij} + \bar{g}^{kl} \bar{A}_{il} \bar{A}_{kj} + \bar{H} \bar{A}_{ij} + \bar{R}_{irr}j. \]  
(3.11)
Taking trace of $i$ and $j$,
\[ \bar{R}_{rr} = \frac{1}{2} (\bar{S} - S_h + \bar{H}^2 - \bar{g}^{ij} \bar{A}_i \bar{A}_j). \] (3.12)

Then
\[ \bar{R}_{iirj} = \bar{g}(\bar{\nabla}_i \bar{\nabla}_r, \partial_j) - \bar{g}(\bar{\nabla}_i \partial_r, \partial_j) - \bar{g}(\bar{\nabla}_{[i} \partial_{r]}, \partial_{j]}).
\] (3.13)

From (3.6) and (3.7), we have:
\[ \bar{R}_{ij} = -(n-1) \bar{A}_{ij} - \frac{\bar{\Delta}r}{r} \bar{g}_{ij} + \frac{\bar{F}_{ij}}{r}, \]
\[ \bar{R}_{ri} = \frac{\bar{F}_{ri}}{r}, \]
\[ \bar{R}_{rr} = -\frac{\Delta r}{r} + \frac{\bar{F}_{rr}}{r}, \]
\[ \bar{S} = -(n-1) \bar{F}_{rr} - \frac{\Delta r}{r} + \frac{\bar{F}_{rr} tr}{r}. \]

From (3.6) and (3.7), we have:
\[ \bar{A}_{ij} = \frac{1}{n-1}(\bar{F}_{rr} h_{ij} - \bar{F}_{ij}), \]
\[ \bar{H} = \bar{\Delta}r = \frac{\bar{F}_{rr}}{r} = \frac{1}{2n} tr \bar{F}. \]

Hence
\[ \bar{R}_{ij} = -(n-1) \bar{\Delta} \bar{A}_{ij} - \bar{\Delta} \bar{r} \bar{g}_{ij} - \bar{\Delta} \bar{r} \bar{g}_{ij} + \bar{\partial}_r \bar{F}_{ij}, \]
\[ \bar{R}_{ri} = \bar{\partial}_r \bar{F}_{ri}, \]
\[ \bar{R}_{rr} = -\bar{\partial}_r \Delta r + \bar{\partial}_r \bar{F}_{rr}, \]
\[ \bar{S} = -2n(\bar{\partial}_r \Delta r + \bar{\partial}_r tr \bar{F}). \]

Combining all the formulas above, we get that
\[ \bar{S} = \frac{n}{n-1} (S_h - \bar{H}^2 + |\bar{A}|^2 - \frac{1}{n} \bar{\partial}_r tr \bar{F}) \]
\[ = \frac{n}{n-1} (S_h - \bar{F}_{rr}^2 + \frac{1}{(n-1)^2} (n \bar{F}_{rr}^2 + \bar{F}_{rr} tr \bar{F} + |\bar{F}_{h}|^2) - \frac{1}{n} \bar{\partial}_r tr \bar{F}) \]
\[ = \frac{n}{n-1} (S_h - \bar{F}_{rr}^2 + \frac{1}{(n-1)^2} (n \bar{F}_{rr}^2 + \bar{F}_{rr} tr \bar{F} + |\bar{F}_{h}|^2) - \frac{1}{n} \bar{\partial}_r tr \bar{F}) \]
\[ \bar{R}_{ij} = \frac{n-1}{n-2} (\bar{F}_{ri})_{ij} + \bar{g}^{kl} \bar{A}_l \bar{A}_k + \bar{H} \bar{A}_{ij} - \bar{F}_{rr} \bar{\partial}_r \bar{g}_{ij} + \bar{\partial}_r \bar{F}_{ij} \]
\[ + \frac{1}{n-2} (\bar{A}_{ij}^2 - \frac{1}{2} (\bar{S} - S_h + \bar{H}^2 - |\bar{A}|^2) + \bar{\partial}_r \bar{F}_{rr}) h_{ij} \]
(3.18)
Noticing that $\bar{A}_{ij}$ is totally determined by $\bar{F}$ and $h$, hence (3.10) holds. \hfill \Box

**Lemma 3.3.** Let $g = \rho^2 g_+$ be $C^{3,\alpha}$ conformally compact metric of $(M, g_+)$ and $\bar{g} = r^2 g_+$ be $C^{2,\alpha}$ geodesic compactification with the same boundary metric $g|_{\partial M} = \bar{g}|_{\partial M} = h$. Let $r = u\rho, A = D^2\rho$, then $A|_{\partial M} = \bar{A} - u_r h$.

**Proof.** In the local coordinates $(r, y^1, y^2, \ldots, y^n)$ near $\partial M$, $\bar{A}_{ij} = -\bar{\Gamma}_{ij} r$. Then the relationship between the connection $\nabla$ of $g$ and $\bar{\nabla}$ of $\bar{g}$ is:

$$\Gamma_{ij} = \bar{\Gamma}_{ij} - \frac{1}{u}(\delta_i^j u_i + \delta_i^j u_j - g_{ij} u_r) = \frac{1}{u} u_r h_{ij}.$$ 

Lemma 3.3 tells us that $u_r = \frac{\bar{H} - H}{n}$. Using the fact that $u|_{\partial M} \equiv 1$,

$$\bar{\nabla} u = \frac{\bar{H} - H}{n} \bar{\nabla} r.$$ 

**Lemma 3.4.** Suppose that $g, \bar{g}$ are defined as lemma 3.3 then on the boundary $\partial M$,

$$R_{ri} = \partial_i \bar{F}(r_i) + \frac{n - 1}{n} \frac{\partial (\bar{H} - H)}{\partial x_i},$$

$$R_{rr} = \frac{1}{2}(S - S_h) - \frac{n - 1}{2n} H^2 + Q(F, DF, H),$$

$$R_{ij} = \bar{R}_{ij} + \left(\frac{1}{2n}(S - \bar{S})\right) h_{ij} + \frac{n - 1}{2n^2} H^2 h_{ij} + Q(\bar{F}, D\bar{F}, H).$$ (3.20)

Here $Q(\bar{F}, D\bar{F}, H) = 0$ if $F = DF = 0$.

**Proof.** Let $g = u^{-2} \bar{g}$, then

$$\text{Ric} = \bar{\text{Ric}} + (n - 1) \frac{D^2 u}{u} + \left(\frac{\bar{A} u}{u} + \frac{n|\nabla u|^2}{u^2}\right) \bar{g}.$$
We also know that
\[
\bar{\Delta} u = \text{div} \nabla u = \text{div} \left( \frac{\bar{H} - H}{n} \nabla r \right) = \frac{\bar{H} - H}{n} \bar{\Delta} r + \frac{\partial_r (\bar{H} - H)}{n},
\]
\[
\bar{D}^2 u(\partial_i, \partial_j) = \bar{g}(\nabla u, \nabla u) = \frac{\bar{H} - H}{n} A_{ij},
\]
\[
\bar{D}^2 u(\partial_i, \partial_r) = \bar{u}_{ir} = 1, \quad \bar{D}^2 u(\partial_r, \partial_r) = \partial_r (\bar{H} - H) n = \bar{\Delta} u - \frac{\bar{H} - H}{n} \bar{\Delta} r.
\]

Then on \( \partial M \), we conclude that
\[
R_{ri} = \bar{R}_{ri} + \frac{n - 1}{n} \partial_r (\bar{H} - H) \frac{\partial_i}{\partial x_i} = \partial_r \bar{R}_{ri} + \frac{n - 1}{n} \partial_i \frac{\partial_r (\bar{H} - H)}{\partial x_i},
\]
\[
R_{rr} = \bar{R}_{rr} + n \bar{\Delta} u - \frac{(n - 1)(\bar{H} - H)}{n} \bar{\Delta} r + \frac{(\bar{H} - H)^2}{n},
\]
\[
R_{ij} = \bar{R}_{ij} + (\bar{\Delta} u + \frac{(\bar{H} - H)^2}{n}) h_{ij} + (n - 1) \frac{\bar{H} - H}{n} \bar{A}_{ij}.
\] (3.21)

Taking trace of \( i \) and \( j \),
\[
S = \bar{S} + 2n \bar{\Delta} u + \frac{n + 1}{n} (\bar{H} - H)^2 - \frac{(n - 1)(\bar{H} - H)}{n} \bar{\Delta} r + \frac{n - 1}{n} (\bar{H} - H) \bar{H}.
\]

Then
\[
\bar{\Delta} u = \frac{1}{2n} (S - \bar{S} - \frac{n + 1}{n} (\bar{H} - H)^2 + \frac{(n - 1)(\bar{H} - H)}{n} \bar{\Delta} r - \frac{n - 1}{n} (\bar{H} - H) \bar{H}).
\] (3.22)

The result follows from (3.21) and (3.22). \( \square \)

In the end, lemma 3.2 and lemma 3.4 imply (3.8).

With the preparation above, let’s consider a \( C^2 \) conformally compact Einstein metric \( g = \rho^2 g^+ \) on \((M, \gamma)\). We can choose a sequence of \( C^{3,\alpha} \) metric \( g_k \) which converge to \( g \) in \( C^2(M) \) norm. Let \( \rho_k = \rho_k \rho_k^+ \), so \( \rho_k \in C^{3,\alpha}(\bar{M}) \) and \( |\nabla^k \rho_k|^g \equiv 1 \) on \( \partial M \). Let \( g^+_k = (\rho_k)^{-2} g_k \), then \( g_k \) is a \( C^{3,\alpha} \) conformally compactification of \((M, g_k^+)\) with defining function \( \rho_k \). Defining \( F_k = \rho_k (\text{Ric}_{g_k^+} + ng_k^+) \) as above, then the formula of \( \text{Ric}_{g_k} \) on \( \partial M \) is like (3.10) and \( F_k \) converge to 0 in \( C^1(\partial M) \) norm.

Finally, as the Ricci curvature of \( g_k \) converge to that of \( g \) continuously, we conclude that (3.7), (3.19) and (3.20) hold.
3.1 Other boundary conditions

We see that if the metric $g$ in lemma 3.3 is conformally Einstein, then $\bar{A} = 0$ on $\partial M$ and the boundary is umbilic. This conclusion is also true even if $g$ is $C^2$ compact and in this case the geodesic compactification is at least $C^1$. Then we have

$$A_{ij} = \frac{H}{n} h_{ij}.$$  

Taking the derivative of the equation above along $\partial M$,

$$\partial_k A_{ij} = \frac{\partial_k H}{n} h_{ij} + \frac{H}{n} \partial_k h_{ij}.$$  

Combining it with (3.3), we get that

$$\partial_k A_{ij} = - \frac{1}{n-1} (g^{00})^{\frac{1}{2}} (R_{0k} + \frac{g^{0j}}{g^{00}} R_{ij})$$  

(3.23)

Technically, this is not a boundary condition because both sides are of the second derivative of $g$. However, this plays an important role in proving the regularity and we will use the condition later.

If we choose the harmonic coordinates, we also have the following boundary condition:

$$g^{\eta \beta} \partial_\eta (g_{\alpha \beta} - \frac{1}{2} \partial_\alpha g_{\eta \beta}) = 0$$  

(3.24)

This is just the local expression of $\Delta_g x^\alpha = 0$.

4 Proof of the main theorem

We prove the main theorem in this section with the Bach equation in harmonic coordinates and some boundary conditions in last section. Firstly, let’s recall some intermediate Schauder theory of elliptic PDE in [11], [19], i.e. $C^\alpha$ and $C^{1,\alpha}$ estimate.

4.1 Intermediate Schauder estimate

Suppose $\Omega$ is a bounded convex domain in $\mathbb{R}^n$ and $a$ is a positive number satisfying $a = k + \beta$ ($k \in \mathbb{N}, \beta \in (0, 1]$) Define

$$|u|_a = \sum_{|\alpha| \leq k} |D^\alpha u|_0 + \sum_{|\alpha| = k} \sup_{x, y \in \Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\beta}.$$
Let \( H_a(\Omega) \) denote the Hölder space of functions with finite norm \( |u|_a \) on \( \Omega \), i.e. \( H_a(\Omega) = C^{k,\beta}(\Omega) \). Setting
\[
\Omega_\delta = \{ x \in \Omega | \text{dist}(x, \partial \Omega) > \delta \}
\]
Let \( b \) be a number satisfying \( a + b \geq 0 \) and define
\[
|u|^{(b)}_{a,\Omega} = \sup_{\delta > 0} \delta^{a+b}|u|_{a,\Omega_\delta}
\]
Let \( H^b_a(\Omega) \) denote the space of functions \( u \) in \( H^b_a(\Omega_\delta) \) \((\forall \delta > 0)\) such that \( |u|^{(b)}_{a,\Omega} \) is finite. Let \( H^{b-0}_a(\Omega) \) be the space of functions \( u \) in \( H^b_a(\Omega) \) such that if \( \delta \to 0 \), then \( \delta^{a+b}|u|_{a,\Omega_\delta} \to 0 \).

Basic properties: (the following constant \( C \) depends on \( a, b, \Omega \).)

1. \( H^{(a)}_{a}(\Omega) = H_a(\Omega) = C^{k,\beta}(\Omega) \). Noticing that if \( a \) is positive integer, \( H_a(\Omega) = C^{a-1,1}(\Omega) \);

2. If \( b \geq b' \), then \( |u|^{(b)}_{a,\Omega} \leq C|u|^{(b')}_{a,\Omega} \);

3. If \( 0 \leq a' \leq a, a' + b \geq 0 \) and \( b \) is not a non-positive integer, then \( |u|^{(b)}_{a',\Omega} \leq C|u|^{(b)}_{a,\Omega} \);

4. If \( 0 \leq c_j \leq a + b, a \geq 0, j = 1, 2 \), then
\[
|uv|^{(b)}_{a} \leq C(|u|^{(b-c_1)}_{a}|v|^{(c_1)}_{0} + |u|^{(c_2)}_{0}|v|^{(b-c_2)}_{a})
\]

Specially, if \( u \) and \( v \) are continuous functions (bounded), then \( |uv|^{(b)}_{a} \leq C(|u|^{(b)}_{a} + |v|^{(b)}_{a}) \).

With the preparation above, we could state the intermediate Schauder theory. Assuming that \( \Omega \) is a bounded \( C^\gamma \) domain where \( \gamma \geq 1 \) and \( a, b \) are not integer satisfying
\[
0 < b \leq a, \ a > 2, \ b \leq \gamma
\]
Let
\[
P = \sum_{|\alpha| \leq 2} p_\alpha(x) \partial^\alpha
\]
be the elliptic differential operator of second order on \( \overline{\Omega} \) where
\[
p_\alpha \in H^{(2-b)}_{a-2}(\Omega), \ if \ |\alpha| \leq 2
\]
\[ p_\alpha \in H_0(\Omega), \text{ if } |\alpha| = 2 \]
\[ p_\alpha \in H_{a-2}^{(2-|\alpha|-0)}(\Omega), \text{ if } b < |\alpha|. \]

Then we have:

**Lemma 4.1.** [Theorem 6.1 in [11]] Let \( P, a, b \) be defined as above. If \( p_0 \leq 0 \) and the principal part of \( P \) is positive, then the Dirichlet problem

\[ P u = f \text{ in } \Omega, \quad u = u_0 \text{ on } \partial \Omega \]

has a unique solution \( u \in H_a^{(-b)}(\Omega) \) for every \( f \in H_{a-2}^{(2-b)}(\Omega) \) and \( u_0 \in H_b(\partial \Omega) \), and we have

\[ u_a^{(-b)}(\Omega) \leq C(|u|_{b, \partial \Omega} + |f|_{a-2}^{(2-b)}(\Omega)) \]

We also have the regularity result:

**Lemma 4.2.** [Theorem 6.3 in [11]] Let \( \Omega, P, a, b \) satisfy the hypotheses in lemma 4.1, and let \( u \in C^0(\Omega) \cap C^2(\Omega), u|_{\partial \Omega} \in H_b(\partial \Omega), P u \in H_{a-2}^{(2-b)}(\Omega) \). Then it follows that \( u \in H_{a-1}^{(-b)}(\Omega) \).

For the boundary oblique derivative conditions, we have the following lemma:

**Lemma 4.3.** [Theorem 3 in [19]] Let \( a, b \) be non-integer and \( 1 < b \leq a, a > 2 \) and \( \Omega \subset \mathbb{R}^n \) be bounded domain with \( H_b \) boundary. Let

\[ P = \sum_{|\alpha| \leq 2} p_\alpha(x) D^\alpha \text{ in } \Omega, \quad M = \sum_{|\alpha| \leq 1} m_\alpha(x) D^\alpha \text{ on } \partial \Omega. \]

Here

\[ \sum_{|\alpha| = 2} p_\alpha \xi^\alpha \geq c |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \quad \sum_{|\alpha| = 1} m_\alpha v^\alpha > 0 \]

where \( c \) is a positive number. We also let

\[ p_\alpha \in H_{a-2}^{(2-b)}(\Omega), \text{ if } |\alpha| \leq 2; \quad m_\alpha \in H_{b-1}(\partial \Omega) \text{ if } |\alpha| \leq 1, \]
\[ p_\alpha \in H_{a-2}^{(0)} \text{ if } |\alpha| = 2 \text{ and } b < 2. \]

(a) If \( p_0 \leq 0, \quad m_0 < 0 \), then the oblique derivative problem

\[ P u = f \text{ in } \Omega, \quad M u = g \text{ on } \partial \Omega \quad (4.1) \]

has a unique solution \( u \in H_a^{(-b)}(\Omega) \) for every \( f \in H_{a-2}^{(2-b)}(\Omega) \) and \( g \in H_{b-1}(\partial \Omega) \). Moreover,

\[ u_a^{(-b)}(\Omega) \leq C(|g|_{b-1, \partial \Omega} + |f|_{a-2}^{(2-b)}(\Omega)). \]

(b) If \( u \in C^0(\overline{\Omega}) \cap C^2(\Omega) \) is a solution of (4.1) with \( f \in H_{a-2}^{(2-b)}(\Omega), g \in H_{b-1}(\partial \Omega) \) and the directional derivative \( \sum_{|\alpha| = 1} m_\alpha \) exists at each point of \( \partial \Omega \), then \( u \in H_a^{(-b)}(\Omega) \).
4.2 The $C^{1,\sigma}$ regularity of Ricci curvature

For a $C^2$ conformally compact Einstein metric $g = \rho^2 g_+ , \rho \in C^{2,\sigma}$, we know that $\text{Ric} \in C^0 (\overline{M})$ in the initial smooth $y$-coordinates. We observe that from (2.2)

$$ \rho \text{Ric} = -(n-1)D^2 \rho + \frac{n(|\nabla \rho|^2 - 1)}{\rho} - \Delta \rho \vert g = Q(\partial g, \partial^2 \rho) \in C^\sigma (\overline{M}, \{y\}). $$

Now we compute the metric and curvature in harmonic coordinates $\{x^\beta\}_{\beta=0}^3$. As $g$ is $C^2$, $x \in C^{2,\lambda}(y), \forall \lambda \in (0,1)$. Then in $x$-coordinates, we have that

$$ \text{Ric}(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}) = \frac{\partial y^\gamma}{\partial x^\alpha} \frac{\partial y^\tau}{\partial x^\beta} \text{Ric}(\frac{\partial}{\partial y^\gamma}, \frac{\partial}{\partial y^\tau}) \in C^0(\overline{M}, \{x\}) \quad (4.2) $$

$$ \rho \text{Ric}(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}) = \rho \frac{\partial y^\gamma}{\partial x^\alpha} \frac{\partial y^\tau}{\partial x^\beta} \text{Ric}(\frac{\partial}{\partial y^\gamma}, \frac{\partial}{\partial y^\tau}) \in C^\sigma(\overline{M}, \{x\}) \quad (4.3) $$

By lemma 4.4 below, we conclude that $\text{Ric} \in H^{(1-\sigma)}_\delta (\overline{M})$.

**Lemma 4.4.** Suppose that $f$ is a continuous function on $M$ and $\rho f \in C^\sigma (\overline{M})$, then $f \in H^{(1-\sigma)}_\delta (\overline{M})$.

**Proof.** As $|\nabla \rho| \equiv 1$ on $\partial M$, we can assume that $\frac{1}{2} \leq |\nabla \rho| \leq 2$ on $\partial M \times [0,\epsilon)$ for a small $\epsilon > 0$. Let

$$ \Omega_\delta = \{x \in M | \text{dist}(x, \partial M) > \delta\}, \quad M_\delta = \{x \in M | \rho(x) > \delta\}. $$

A direct calculation shows that

$$ \Omega_\delta \subset M_{\frac{\delta}{2}} \subset \Omega_{\frac{\delta}{4}} $$

So we don’t distinguish $\Omega_\delta$ and $M_\delta$ when studying the definition of $|u|^{\delta}_{0,\Omega}$. Since $\rho f \in C^\sigma (\overline{M})$, for any $x, y \in M_\delta$,

$$ |\rho(x)f(x) - \rho(y)f(y)| \leq C. $$

Then

$$ C \geq \frac{|\rho(x)f(x) - \rho(x)f(y) + \rho(x)f(y) - \rho(y)f(y)|}{d^\sigma(x,y)} \geq \rho(x) \frac{|f(x) - f(y)|}{d^\sigma(x,y)} - |f(y)| \frac{\rho(x) - \rho(y)}{d^\sigma(x,y)}, $$

which means

$$ \rho(x) \frac{|f(x) - f(y)|}{d^\sigma(x,y)} \leq C + |f|_{0,M_\delta} |\rho|_\delta. $$

By assumption, $f$ is continuous, so $f$ is bounded, then $|f|_{\sigma,M_\delta} < C'$ for any $\delta > 0$. This proves the lemma.

[15]
Lemma 4.5. In harmonic coordinates, \( g \in H_{2+\sigma}^{(-1-\sigma)}(\overline{M}) \).

Proof. In harmonic charts,
\[
\Delta g_{\alpha\beta} = -2R_{\alpha\beta} + Q(g, \partial g)
\]
Let \( a = 2 + \sigma, \ b = 1 + \sigma \), then according to lemma 4.2, \( g_{\alpha\beta} \in H_{2+\sigma}^{(-1-\sigma)}(\overline{M}) \). □

Now we have that \( g \in H_{2+\sigma}^{(-1-\sigma)}(\overline{M}) \), so the curvature \( Rm \in H_{\sigma}^{(1-\sigma)}(\overline{M}) \). By linear transformation of tensor in coordinate system (similar to (4.2)), \( Rm \) is still continuous in x-coordinates. Recall that \( Q \) in (2.6) is the quadratic term of curvature, then \( Q \in H_{\sigma}^{(1-\sigma)}(\overline{M}) \) from the basic property 4 in section 4.1.

As \( g \in C^2(\overline{M}, y) \), (3.2), (3.3) hold in y-coordinates on \( \partial M \). In the harmonic coordinates \( \{x^\beta\}_{\beta=0}^3 \),
\[
\frac{\partial x^\alpha}{\partial y^\beta}|_{\partial M} = \frac{\partial x^\alpha|_{\partial M}}{\partial y^\beta} = \delta^\alpha_\beta.
\]

Then on \( \partial M \),
\[
Ric\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^\alpha}\right) = \frac{\partial x^\gamma}{\partial y^0} \frac{\partial x^\tau}{\partial y^\beta} Ric\left(\frac{\partial}{\partial x^\gamma}, \frac{\partial}{\partial x^\tau}\right) = \delta^\gamma_\tau \Delta Ric\left(\frac{\partial}{\partial x^\gamma}, \frac{\partial}{\partial x^\tau}\right) = Ric\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^\alpha}\right),
\]
\[
Ric\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^\alpha}\right) = \frac{\partial y^\gamma}{\partial x^0} \frac{\partial y^\tau}{\partial x^\alpha} Ric\left(\frac{\partial}{\partial y^\gamma}, \frac{\partial}{\partial y^\tau}\right).
\]

For any \( p \in \partial M \), consider the \( C^{2,\lambda} \) harmonic chart \( (V, \{x^\theta\}_{\theta=0}^3) \) around \( p \). Let \( D = V \cap \partial M \) be the boundary portion. Then the Bach equation (3.3) could be written as
\[
\Delta Ric_{\alpha\beta} = \frac{\partial}{\partial y^\theta} f^\theta + Q.
\]
Here \( f^\theta = \Gamma * Ric \in H_{\sigma}^{(1-\sigma)}(\overline{M}) \), \( \theta = 0, 1, 2, 3 \). We will firstly deal with the \( R_{ij} \) term where \( 1 \leq i, j \leq 3 \). Consider the following equation:
\[
\begin{cases}
\Delta u^0 = f^0 \text{ in } V \\
\frac{\partial}{\partial y^\theta} u^0 = 0 \text{ on } D
\end{cases}
\]
and
\[
\begin{cases}
\Delta u^i = f^i \text{ in } V \\
u^i = 0 \text{ on } D
\end{cases}
\]
where \( i = 1, 2, 3 \). By lemma 4.1 and lemma 4.3, the 4 equations above have solutions in \( H_{2+\sigma, M}^{(-1-\sigma)} \). Let \( R_{\alpha\beta} = R_{\alpha\beta} - \partial_\theta u^\theta \), then
\[
\Delta R_{\alpha\beta} = Q(g, \partial g, \partial^2 g, \partial u, \partial^2 u) \in H_{\sigma, M}^{(1-\sigma)}
\]
From lemma 4.2 and the boundary conditions of $R_{ij}$, $\tilde{R}_{ij} \in H^{(-1-\sigma)}_{2+\sigma}(M)$, $\tilde{R}_{0\alpha} \in H^{(-1-\sigma)}_{2+\sigma}(M)$. So $R_{\alpha\beta} \in C^{\sigma}(M)$, then $\theta^{\alpha} \in C^{\sigma}$, then $u^{\alpha} \in C^{2,\sigma}$, finally, $R_{ij} \in C^{1,\sigma}$.

To study the regularity of $R_{0\alpha}$, we need to use the Neumann boundary condition (3.5).

$$N(R_{0\alpha}) = (g^{00})^{-\frac{1}{2}}(-g^{ij}\partial_{j}R_{i\alpha} + g^{\eta\beta}\Gamma^{r}_{i\beta}R_{\eta\tau}).$$

Let $\alpha = i$, from the regularity of $R_{ij}$ and lemma 4.3 we get that $R_{0i} \in C^{1,\sigma}$. Finally, let $\alpha = 0$, we have $R_{00} \in C^{1,\sigma}$.

Thus we have finished the first step of the proof, i.e. $Ric \in C^{1,\sigma}(M)$ in harmonic charts.

### 4.3 The $C^{m,\alpha}$ regularity of metric in harmonic charts

We have concluded that $g_{\alpha\beta} \in C^{1,\lambda}$ for any $\lambda \in (0, 1)$ in harmonic charts, then

$$\Delta g_{\alpha\beta} = -2R_{\alpha\beta} + Q(g, \partial g) \quad (4.7)$$

If $1 \leq i, j \leq 3$, we have the boundary conditions

$$g_{ij} = h_{ij},$$

So $g_{ij} \in C^{2,\lambda}$.

Let $A_{ij}$ be the second fundamental form,

$$A_{ij} = \frac{1}{2}(g^{00})^{\frac{1}{2}}g^{\alpha\beta}(\partial_{\beta}g_{ij} - \partial_{i}g_{\beta j} - \partial_{j}g_{\beta i}).$$

Since $Ric \in C^{1,\sigma}(M)$, according to (3.23), $A_{ij} \in C^{2,\sigma}(\partial M)$. Combining it with that $g_{ij} \in C^{2,\lambda}(M)$,

$$\partial_{j}g_{00} + \partial_{i}g_{j0} \in C^{2,\sigma}(\partial M) \quad (4.8)$$

Recall the boundary condition (3.24)

$$g^{\eta\beta}\partial_{\eta}(g_{\alpha\beta} - \frac{1}{2}\partial_{\alpha}g_{\eta\beta}) = 0.$$

Let $\alpha = 0$, and with (4.8) we conclude that

$$(g^{0\beta}\partial_{j} + \frac{1}{2}g^{00}\partial_{0})g_{00} \in C^{2,\sigma}(\partial M) \quad (4.9)$$

So $g_{00} \in C^{2,\lambda}(M)$. 

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Let $\alpha = i$ in (3.24), and with (4.8) we get that
\[
(g^{ij} \partial_j + \frac{1}{2} g^{00} \partial_0)g_{i0} \in C^{2,\sigma}(\partial M)
\]
So $g_{i0} \in C^{2,\lambda}(\overline{M})$. Now we have proved that $g$ is $C^{2,\lambda}$ in harmonic charts. Hence $\{x^0\}_{x^0=0}^3$ form a $C^{3,\lambda}$ differential structure of $\overline{M}$. Repeat the steps above, we could improve the regularity of metric $g$ gradually, and finally $g \in C^{m,\alpha}(\overline{M}, x)$. Hence $\{x^\theta\}_{x^\theta=0}^3$ form a $C^{m+1,\alpha}$ differential structure of $\overline{M}$.

### 4.4 Regularity of the defining function

We already show that $\rho \in C^{2,\sigma}(\overline{M})$ and $\rho$ is smooth in interior. Then the only thing is to study the boundary regularity of the defining function. For any $p \in \partial M$, take the harmonic chart $(V, x)$ of $p$ and let $D = V \cap \partial M$. We could also assume that $g_{0\alpha} = 1, g_{ij} = g_{02} = g_{03} = \cdots = g_{0n} = 0 (i \neq j), g_{01} = -\delta$ at $p$ where $\delta \in (0, 1)$ is sufficiently close to 1. according to (2.2) and (2.3)
\[
\text{Ric} - \frac{Sg}{n+1} = -(n-1) \frac{D^2 \rho}{\rho} + \frac{n-1}{n+1} \Delta \rho \rho g.
\]
Locally, when acting on $(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1})$,
\[
\Delta \rho - (n+1) \cdot g_{01}^{-1} \cdot D^2 \rho \left( \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1} \right) = \frac{n+1}{n-1} \cdot g_{01}^{-1} \cdot \rho \left( \text{Ric}_{01} - \frac{Sg_{01}}{n+1} \right)
\]
If $1 - \delta$ is small enough, then the left side of the formula above is a elliptic operator around $p$. Since $\rho|_D \equiv 0, \rho \in C^{m,\alpha}(x)$. In order to improve the $C^{m+1,\alpha}$ regularity of $\rho$, we need that $\rho(\text{Ric}_{01})$ in (4.11) is at least $C^{m-1,\alpha}$. Actually,
\[
\Delta (\rho \text{Ric}) = \rho \Delta (\text{Ric}) + \text{Ric} \Delta \rho + 2g(\nabla \rho, \nabla \text{Ric})
\]
The right side of this formula is $C^{m-3,\alpha}$ with the help of Bach equation. $\rho \text{Ric}|_{\partial M} \equiv 0$, so $\rho(\text{Ric}_{01}) \in C^{m-1,\alpha}$, and the defining function $\rho$ is $C^{m+1,\alpha}$.

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