A characterization of the family of secant lines to a hyperbolic quadric in $PG(3, q)$, $q$ odd

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Abstract

We give a combinatorial characterization of the family of lines of $PG(3, q)$ which meet a hyperbolic quadric in two points (the so called secant lines) using their intersection properties with the points and planes of $PG(3, q)$.

Keywords: Projective space, Hyperbolic quadric, Secant line, Combinatorial characterization

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1 Introduction

Throughout, $q$ is a prime power. Let $PG(3, q)$ denote the three-dimensional Desarguesian projective space defined over a finite field of order $q$. Characterizations of the family of external or secant lines to an ovoid/quadric in $PG(3, q)$ with respect to certain combinatorial properties have been given by several authors. A characterization of the family of secant lines to an ovoid in $PG(3, q)$ was obtained in [7] for $q$ odd and in [4] for $q > 2$ even, which was further improved in [6] for all $q > 2$. A characterization of the family of external lines to a hyperbolic quadric in $PG(3, q)$ was given in [5] for all $q$ (also see [9] for a different characterization in terms of a point-subset of the Klein quadric in $PG(5, q)$) and to an ovoid in $PG(3, q)$ was obtained in [6] for all $q > 2$. One can refer to [1, 2, 11, 12] for characterizations of external lines in $PG(3, q)$ with respect to quadric cone, oval cone and hyperoval cone. Here we give a characterization of the secant lines with respect to a hyperbolic quadric in $PG(3, q)$, $q$ odd.

Let $Q$ be a hyperbolic quadric in $PG(3, q)$, that is, a non-degenerate quadric of Witt index two. One can refer to [8] for the basic properties of the points, lines and planes of $PG(3, q)$ with respect to $Q$. Every line of $PG(3, q)$ meets $Q$ in 0, 1, 2 or $q + 1$ points. A line of $PG(3, q)$ is called secant with respect to $Q$ if it meets $Q$ in 2 points. The lines of $PG(3, q)$ that meet $Q$ in $q + 1$ points are called generators of $Q$. Each point of $Q$ lies on two generators. The quadric $Q$ consists of $(q + 1)^2$ points and $2(q + 1)$ generators.

The total number of secant lines of $PG(3, q)$ with respect to $Q$ is $q^2(q + 1)^2/2$. We recall the distribution of secant lines with respect to points and planes of $PG(3, q)$ which plays an important role in this paper. Each point of $Q$ lies on $q^2$ secant lines. Each point
of $\text{PG}(3, q) \setminus \mathcal{Q}$ lies on $q(q + 1)/2$ secant lines. Each plane of $\text{PG}(3, q)$ contains $q^2$ or $q(q + 1)/2$ secant lines. If a plane contains $q^2$ secant lines, then every pencil of lines in that plane contains 0 or $q$ secant lines. If a plane contains $q(q + 1)/2$ secant lines, then every pencil of lines in that plane contains $(q - 1)/2, (q + 1)/2$ or $q$ secant lines.

In this paper, we prove the following theorem when $q$ is odd.

**Theorem 1.1.** Let $\mathcal{S}$ be a family of lines of $\text{PG}(3, q)$, $q$ odd, for which the following properties are satisfied:

(P1) There are $q(q + 1)/2$ or $q^2$ lines of $\mathcal{S}$ through a given point of $\text{PG}(3, q)$. Further, there exists a point which is contained in $q(q + 1)/2$ lines of $\mathcal{S}$ and a point which is contained in $q^2$ lines of $\mathcal{S}$.

(P2) Every plane $\pi$ of $\text{PG}(3, q)$ contains $q(q + 1)/2$ or $q^2$ lines of $\mathcal{S}$. Further,

(P2a) if $\pi$ contains $q^2$ lines of $\mathcal{S}$, then every pencil of lines in $\pi$ contains 0 or $q$ lines of $\mathcal{S}$.

(P2b) if $\pi$ contains $q(q + 1)/2$ lines of $\mathcal{S}$, then every pencil of lines in $\pi$ contains $(q - 1)/2, (q + 1)/2$ or $q$ lines of $\mathcal{S}$.

Then either $\mathcal{S}$ is the set of all secant lines with respect to a hyperbolic quadric in $\text{PG}(3, q)$, or the set of points each of which is contained in $q^2$ lines of $\mathcal{S}$ form a line $l$ of $\text{PG}(3, q)$ and $\mathcal{S}$ is a hypothetical family of $q^3 + q^3 + 2q^2\over 2$ lines of $\text{PG}(3, q)$ not containing $l$.

## 2 Combinatorial results

Let $\mathcal{S}$ be a set of lines of $\text{PG}(3, q)$ for which the properties (P1), (P2), (P2a) and (P2b) stated in Theorem 1.1 hold. A plane of $\text{PG}(3, q)$ is said to be tangent or secant according as it contains $q^2$ or $q(q + 1)/2$ lines of $\mathcal{S}$. For a given plane $\pi$ of $\text{PG}(3, q)$, we denote by $\mathcal{S}_\pi$ the set of lines of $\mathcal{S}$ which are contained in $\pi$. By property (P2), $|\mathcal{S}_\pi| = q^2$ or $q(q + 1)/2$ according as $\pi$ is a tangent plane or not.

We first show that both tangent and secant planes exist. We call a point of $\text{PG}(3, q)$ black if it is contained in $q^2$ lines of $\mathcal{S}$.

**Lemma 2.1.** Let $l$ be a line of $\text{PG}(3, q)$. Then the number of tangent planes through $l$ is equal to the number of black points contained in $l$.

**Proof.** Let $t$ and $b$, respectively, denote the number of tangent planes through $l$ and the number of black points contained in $l$. We count in two different ways the total number of lines of $\mathcal{S} \setminus \{l\}$ meeting $l$. Any line of $\mathcal{S}$ meeting $l$ is contained in some plane through $l$. If $l \in \mathcal{S}$, then we get

$$t(q^2 - 1) + (q + 1 - t) \left( \frac{q(q + 1)}{2} - 1 \right) = b(q^2 - 1) + (q + 1 - b) \left( \frac{q(q + 1)}{2} - 1 \right).$$

If $l \notin \mathcal{S}$, then we get

$$tq^2 + (q + 1 - t) \frac{q(q + 1)}{2} = bq^2 + (q + 1 - b) \frac{q(q + 1)}{2}.$$ 

In both cases, it follows that $(t - b)\frac{q^2 - q}{2} = 0$ and hence $t = b$. \qed
Corollary 2.2. Both tangent and secant planes exist.

Proof. By property (P1), let \( x \) (respectively, \( y \)) be a point of \( PG(3, q) \) which is contained in \( q^2 \) (respectively, \( \frac{q(q+1)}{2} \)) lines of \( S \). Taking \( l \) to be the line through \( x \) and \( y \), the corollary follows from Lemma 2.1 using the facts that \( x \) is a black point but \( y \) is not a black point.

As a consequence of Lemma 2.1 we have the following.

Corollary 2.3. Every line of a tangent plane contains at least one black point.

Corollary 2.4. Every black point is contained in some tangent plane.

2.1 Tangent planes

Note that, by property (P2a), each point of a tangent plane \( \pi \) is contained in no line or \( q \) lines of \( S_\pi \).

Lemma 2.5. Let \( \pi \) be a tangent plane. Then there are \( q^2 + q \) points of \( \pi \), each of which is contained in \( q \) lines of \( S_\pi \). Equivalently, there is only one point of \( \pi \) which is contained in no line of \( S_\pi \).

Proof. Let \( A_\pi \) (respectively, \( B_\pi \)) be the set of points of \( \pi \) each of which is contained in no line (respectively, \( q \) lines) of \( S_\pi \). Then \( |A_\pi| + |B_\pi| = q^2 + q + 1 \). We show that \( |A_\pi| = 1 \) and \( |B_\pi| = q^2 + q \).

Observe that if \( l \) is a line of \( S_\pi \), then each of the \( q+1 \) points of \( l \) lies on \( q \) lines of \( S_\pi \) by property (P2a) and hence is contained in \( B_\pi \). Consider the following set of point-line pairs:

\[ X = \{(x, l) : x \in B_\pi, l \in S_\pi, x \in l\}. \]

Counting \( |X| \) in two ways, we get \( |B_\pi| \times q = |X| = q^2 \times (q + 1) \). This gives \( |B_\pi| = q^2 + q \) and hence \( |A_\pi| = 1 \).

For a tangent plane \( \pi \), there is a unique point of \( \pi \) which is contained in no line of \( S_\pi \) by Lemma 2.5. We denote this unique point by \( p_\pi \) and call it the pole of \( \pi \).

Corollary 2.6. Let \( \pi \) be a tangent plane. Then the \( q+1 \) lines of \( \pi \) not contained in \( S_\pi \) are precisely the lines of \( \pi \) through the pole \( p_\pi \).

2.2 Secant planes

By property (P2b), each point of a secant plane \( \pi \) is contained in \( \frac{q-1}{2}, \frac{q+1}{2} \) or \( q \) lines of \( S_\pi \). For a secant plane \( \pi \), we denote by \( \alpha(\pi), \beta(\pi) \) and \( \gamma(\pi) \) the set of points of \( \pi \) which are contained in \( \frac{q-1}{2}, \frac{q+1}{2} \) and \( q \) lines of \( S_\pi \), respectively. Similarly, for a line \( l \) of a secant plane \( \pi \), we denote by \( \alpha(l), \beta(l) \) and \( \gamma(l) \) the set of points of \( l \) which are contained in \( \frac{q-1}{2}, \frac{q+1}{2} \) and \( q \) lines of \( S_\pi \), respectively. We have \( \alpha(l) = l \cap \alpha(\pi), \beta(l) = l \cap \beta(\pi) \) and \( \gamma(l) = l \cap \gamma(\pi) \).

Lemma 2.7. Let \( \pi \) be a secant plane and \( l \) be a line of \( \pi \). Then the following hold:

(i) If \( l \in S_\pi \), then \( (|\alpha(l)|, |\beta(l)|, |\gamma(l)|) = (\frac{q-1}{2}, \frac{q+1}{2}, 2) \) or \((0, q, 1)\).
(ii) If \( l \notin S_\pi \), then \((|\alpha(l)|, |\beta(l)|, |\gamma(l)|) = (\frac{q+1}{2}, \frac{q-1}{2}, 0) \) or \((q, 0, 1)\).

Proof. We have \(|\alpha(l)| + |\beta(l)| + |\gamma(l)| = q + 1\), that is, \(|\gamma(l)| = q + 1 - |\alpha(l)| - |\beta(l)|\). We first assume that \( l \in S_\pi \). Counting the total number of lines of \( S_\pi \setminus \{l\} \) meeting \( l \), we get

\[
|\alpha(l)| \left(\frac{q-1}{2} - 1\right) + |\beta(l)| \left(\frac{q+1}{2} - 1\right) + |\gamma(l)|(q - 1) = |S_\pi| - 1 = \frac{q(q + 1)}{2} - 1.
\]

This gives

\[
|\alpha(l)| \left(\frac{q - 3}{2}\right) + |\beta(l)| \left(\frac{q - 1}{2}\right) + |\gamma(l)|(q - 1) = \frac{q^2 + q - 2}{2}.
\]

Putting the value of \(|\gamma(l)|\) in equation (1), we thus get

\[
|\alpha(l)| \left(\frac{q + 1}{2}\right) + |\beta(l)| \left(\frac{q - 1}{2}\right) = \frac{q(q - 1)}{2},
\]

that is,

\[
|\alpha(l)| \left(\frac{q + 1}{2}\right) = (q - |\beta(l)|) \left(\frac{q - 1}{2}\right).
\]

Since \( \frac{q+1}{2} \) and \( \frac{q-1}{2} \) are co-prime (being consecutive integers), it follows that \( \frac{q+1}{2} \) must divide \( q - |\beta(l)| \) and so

\[
|\beta(l)| \equiv q \mod \frac{q + 1}{2}.
\]

Since \( 0 \leq |\beta(l)| \leq q + 1 \), we have \(|\beta(l)| = \frac{q-1}{2} \) or \( q \). Consequently, \((|\alpha(l)|, |\beta(l)|, |\gamma(l)|) = (\frac{q-1}{2}, \frac{q+1}{2}, 2)\) or \((0, q, 1)\). This proves (i).

Now assume that \( l \notin S_\pi \). If \(|\beta(l)| = q + 1\), then the numbers lines of \( S \) which are contained in \( \pi \) would be \((q + 1) \left(\frac{q+1}{2}\right)\) which is greater than \(|S_\pi| = \frac{q(q+1)}{2}\), a contradiction. So \( 0 \leq |\beta(l)| \leq q \). Counting the total number of lines of \( S_\pi \setminus \{l\} \) meeting \( l \), we get

\[
|\alpha(l)| \left(\frac{q - 1}{2}\right) + |\beta(l)| \left(\frac{q + 1}{2}\right) + |\gamma(l)|q = |S_\pi| = \frac{q(q + 1)}{2}.
\]

Putting the value of \(|\gamma(l)|\) in equation (2), we get

\[
|\alpha(l)| \left(\frac{q + 1}{2}\right) + |\beta(l)| \left(\frac{q - 1}{2}\right) = \frac{q(q + 1)}{2},
\]

that is,

\[
|\beta(l)| \left(\frac{q - 1}{2}\right) = (q - |\alpha(l)|) \left(\frac{q + 1}{2}\right).
\]

If \(|\alpha(l)| = q\), then \(|\beta(l)| = 0\) and so \((|\alpha(l)|, |\beta(l)|, |\gamma(l)|) = (q, 0, 1)\). Suppose that \(|\alpha(l)| \neq q\). Since the integers \( \frac{q+1}{2} \) and \( \frac{q-1}{2} \) are co-prime, it follows that \( \frac{q+1}{2} \) divides \(|\beta(l)|\). Then the restriction on \(|\beta(l)|\) that \( 0 \leq |\beta(l)| \leq q \) implies \(|\beta(l)| = 0 \) or \( \frac{q+1}{2} \). Considering all the possibilities, we get \((|\alpha(l)|, |\beta(l)|, |\gamma(l)|) = (\frac{q+1}{2}, \frac{q+1}{2}, 0)\) or \((q, 0, 1)\). This proves (ii).
Recall that an arc in the projective plane $PG(2, q)$ is a nonempty set of points such that no three of them are contained in the same line. A line of $PG(2, q)$ is called external or secant with respect to a given arc according as it meets the arc in 0 or 2 points. If $q$ is odd, then the maximum size of an arc is $q + 1$. An oval is an arc of size $q + 1$.

**Corollary 2.8.** Let $\pi$ be a secant plane. Then the following hold.

(a) The set $\gamma(\pi)$ is an arc in $\pi$.

(b) The lines of $\pi$, which are secant with respect to the arc $\gamma(\pi)$, are contained in $S_\pi$.

**Proof.** (a) Note that the set $\gamma(\pi)$ is nonempty. This follows from the fact that $|\gamma(l)| \geq 1$ for any line $l \in S_\pi$ (Lemma 2.7(i)). If there is a line $l$ of $\pi$ containing at least three points of $\gamma(\pi)$, then $|\gamma(l)|$ would be at least 3, which is not possible by Lemma 2.7.

(b) If $l$ is a line of $\pi$ containing two points of $\gamma(\pi)$, then $|\gamma(l)| = 2$ and hence $l$ must be a line of $S_\pi$, which follows from Lemma 2.7. □

**Lemma 2.9.** Let $\pi$ be a secant plane. If $|\gamma(\pi)| = k$, then $|\alpha(\pi)| = \frac{k(k-1)}{2}$.

**Proof.** By Lemma 2.7(i), each line of $S_\pi$ contains either 1 or 2 points of $\gamma(\pi)$. Note that $|\gamma(l)| = 1$ for any line $l$ of $S_\pi$ which contains a unique point of $\gamma(\pi)$. For such a line $l$, we have $|\alpha(l)| = 0$ (follows from Lemma 2.7(i)) and hence $l$ does not contain any point of $\alpha(\pi)$. Similarly, $|\gamma(l)| = 2$ for any line $l$ of $S_\pi$ which contains two points of $\gamma(\pi)$ and in that case, $l$ contains $\frac{k-1}{2}$ points of $\alpha(\pi)$. Counting the cardinality of the set $Y = \{(x, l) : x \in \alpha(\pi), l \in S_\pi \text{ and } x \in l\}$, we get

$$|\alpha(\pi)| \times \frac{q-1}{2} = |Y| = 0 + \frac{k(k-1)}{2} \times \frac{q-1}{2}.$$ 

This gives $|\alpha(\pi)| = \frac{k(k-1)}{2}$. □

**Lemma 2.10.** For any secant plane $\pi$, the set $\gamma(\pi)$ is an oval in $\pi$ and so $|\gamma(\pi)| = q + 1$. Further, $S_\pi$ is precisely the set of lines of $\pi$ which are secant with respect to $\gamma(\pi)$.

**Proof.** Let $|\gamma(\pi)| = k \geq 1$. In order to prove that $\gamma(\pi)$ is an oval in $\pi$, it is enough to show that $k = q + 1$ by Corollary 2.8(a). We have $|\alpha(\pi)| = \frac{k(k-1)}{2}$ by Lemma 2.9.

Fix a point $x \in \gamma(\pi)$ and let $l_1, l_2, \ldots, l_{q+1}$ be the $q + 1$ lines of $\pi$ through $x$. Note that there are $q$ lines of $S_\pi$ through $x$. Since $\gamma(\pi)$ is a $k$-arc in $\pi$, there are $k - 1$ lines of $S_\pi$, say $l_1, l_2, \ldots, l_{k-1}$, through $x$ each of which contains two points of $\gamma(\pi)$. There are $q - (k - 1)$ lines of $S_\pi$, say $l_k, l_{k+1}, \ldots, l_q$, each of which contains a unique point (namely, $x$) of $\gamma(\pi)$. The line $l_{q+1}$ through $x$ is not a line of $S_\pi$ and contains one point (namely, $x$) of $\gamma(\pi)$. Since $l_i \in S_\pi$ for $1 \leq i \leq q$, we have $|\gamma(l_i)| = 2$ for $1 \leq i \leq k - 1$ and $|\gamma(l_i)| = 1$ for $k \leq i \leq q$. Then Lemma 2.7(i) implies that each of the lines $l_1, l_2, \ldots, l_{k-1}$ contains $\frac{q-1}{2}$ points of $\alpha(\pi)$ and each of the lines $l_k, l_{k+1}, \ldots, l_q$ contains no point $\alpha(\pi)$. Since $l_{q+1} \notin S_\pi$ and $|\gamma(l_{q+1})| = 1$, Lemma 2.7(ii) implies that the line $l_{q+1}$ contains $q$ points of $\alpha(\pi)$. Therefore, we get $|\alpha(\pi)| = (k-1) \times \frac{q-1}{2} + (q - (k-1)) \times 0 + q$. Thus, we have

$$(k - 1) \times \frac{q-1}{2} + q = \frac{k(k-1)}{2}.$$
On solving the above equation, we get $k = -1$ or $q + 1$. Since $k \geq 1$, we must have $k = q + 1$.

Since $\gamma(\pi)$ is an oval in $\pi$, the number of lines of $\pi$ which are secant to $\gamma(\pi)$ is equal to $\frac{q(q+1)}{2} = |S_\pi|$. Therefore, by Corollary 2.3(b), $S_\pi$ is precisely the set of secant lines to $\gamma(\pi)$. This completes the proof.

**Corollary 2.11.** Let $\pi$ be a secant plane. Then $|\alpha(\pi)| = \frac{q^2 + q}{2}$ and $|\beta(\pi)| = \frac{q^2 - q}{2}$.

*Proof.* We have $|\gamma(\pi)| = q + 1$ by Lemma 2.10 and so $|\alpha(\pi)| = \frac{q^2 + q}{2}$ by Lemma 2.9. Since $|\alpha(\pi)| + |\beta(\pi)| + |\gamma(\pi)| = q^2 + q + 1$, it follows that $|\beta(\pi)| = \frac{q^2 - q}{2}$.

## 3 Black points

Recall that every point of $PG(3, q)$ is contained in $q^2$ or $\frac{q(q+1)}{2}$ lines of $S$ by property (P1) and the black points are the ones which are contained in $q^2$ lines of $S$.

**Lemma 3.1.** If $\pi$ is a secant plane, then the set of black points in $\pi$ is contained in the oval $\gamma(\pi)$.

*Proof.* Let $x$ be a black point in $\pi$. Suppose that $x$ is not contained in $\gamma(\pi)$. Fix a line $l$ of $S_\pi$ through $x$ and consider the $q + 1$ planes of $PG(3, q)$ through $l$. There are $q^2$ lines of $S$ through $x$ and each of them is contained in some plane through $l$. Since $x \notin \gamma(\pi)$, the plane $\pi$ contains at most $\frac{q+1}{2}$ lines of $S$ through $x$. Each of the remaining $q$ planes through $l$ contains at most $q$ lines of $S$ through $x$. This implies that there are at most $\frac{q+1}{2} + q(q - 1)$ lines of $S$ through $x$. This is not possible, as $\frac{q+1}{2} + q(q - 1) < q^2$. So $x \in \gamma(\pi)$.

**Corollary 3.2.** Let $\pi$ be a secant plane and $x$ be a black point of $\pi$. Then there are exactly $q$ lines of $\pi$ through $x$ which are contained in $S$.

*Proof.* This follows from the fact that $x$ is contained in the oval $\gamma(\pi)$ by Lemma 3.1.

**Lemma 3.3.** The number of black points in a given secant plane is independent of that plane.

*Proof.* Let $\pi$ be a secant plane and $\lambda_\pi$ denote the number of black points in $\pi$. We count the total number of lines of $S$. The lines of $S$ are divided into two types:

(I) the $\frac{q(q+1)}{2}$ lines of $S$ which are contained in $\pi$,

(II) those lines of $S$ which meet $\pi$ in a singleton.

Let $\theta$ be the number of type (II) lines of $S$. In order to calculate $\theta$, we divide the points of $\pi$ into four groups:

(a) The $\lambda_\pi$ black points contained in $\pi$: These points are contained in $\gamma(\pi)$ by Lemma 3.1. Out of the $q^2$ lines of $S$ through such a point, $q$ of them are contained in $\pi$. 

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(b) The \(|\gamma(\pi)| - \lambda_{\pi}\) points of \(\gamma(\pi)\) which are not black: Out of the \(\frac{q(q+1)}{2}\) lines of \(S\) through such a point, \(q\) of them are contained in \(\pi\).

(c) The points of \(\alpha(\pi)\): Out of the \(\frac{q(q+1)}{2}\) lines of \(S\) through such a point, \(\frac{q-1}{2}\) of them are contained in \(\pi\).

(d) The points of \(\beta(\pi)\): Out of the \(\frac{q(q+1)}{2}\) lines of \(S\) through such a point, \(\frac{q+1}{2}\) of them are contained in \(\pi\).

Using the values of \(|\alpha(\pi)|, |\beta(\pi)|, |\gamma(\pi)|\) obtained in Lemma 2.10 and Corollary 2.11, we get

\[
\theta = \lambda_{\pi} (q^2 - q) + (q + 1 - \lambda_{\pi}) \left( \frac{q(q+1)}{2} - q \right) + |\alpha(\pi)| \left( \frac{q(q+1)}{2} - \frac{q-1}{2} \right)
+ |\beta(\pi)| \left( \frac{q(q+1)}{2} - \frac{q+1}{2} \right)
= \lambda_{\pi} \left( \frac{q^2 - q}{2} \right) + q^3(q+1) + q^4 + q^2 + q
\]

Then \(|S| = \theta + \frac{q(q+1)}{2} = \lambda_{\pi} \left( \frac{q^2 - q}{2} \right) + \frac{q^4 + q^3 + q^2 + q}{2}\). Since \(|S|\) is a fixed number, it follows that \(\lambda_{\pi}\) is independent of the secant plane \(\pi\).

By Lemma 3.3, we denote by \(\lambda\) the number of black points in a secant plane. From the proof of Lemma 3.3, we thus have the following equation involving \(\lambda\) and \(|S|\):

\[
\lambda \left( \frac{q^2 - q}{2} \right) + \frac{q^4 + q^3 + q^2 + q}{2} = |S|.
\]

As a consequence of Lemma 3.1, we have

**Corollary 3.4.** \(\lambda \leq q + 1\).

**Lemma 3.5.** The number of black points in a given tangent plane is independent of that plane.

**Proof.** Let \(\pi\) be a tangent plane with pole \(p_{\pi}\) and \(\mu_{\pi}\) be the number of black points in \(\pi\). We shall apply a similar argument as in the proof of Lemma 3.3 by calculating \(|S|\). The lines of \(S\) are divided into two types: (I) the \(q^2\) lines of \(S\) which are contained in \(\pi\), and (II) those lines of \(S\) which meet \(\pi\) in a singleton. Let \(\theta\) be the number of type (II) lines of \(S\). In order to calculate \(\theta\), we divide the points of \(\pi\) into two groups:

(a) The \(\mu_{\pi}\) black points contained in \(\pi\),

(b) The \(q^2 + q + 1 - \mu_{\pi}\) points of \(\pi\) which are not black.

If \(x\) is a point of \(\pi\) which is different from \(p_{\pi}\), then Lemma 2.15 implies that the number of lines of \(S\) through \(x\) which are not contained in \(\pi\) is \(q^2 - q\) or \(\frac{q(q+1)}{2} - q\) according as \(x\) is a black point or not. We consider two cases depending on \(p_{\pi}\) is a black point or not.
Case-1: $p_{\pi}$ is a black point. In this case, Lemma 2.5 implies that none of the $q^2$ lines of $S$ through $p_{\pi}$ is contained in $\pi$. Then

$$\theta = q^2 + (\mu_{\pi} - 1)(q^2 - q) + (q^2 + q + 1 - \mu_{\pi})\left(\frac{q(q+1)}{2} - q\right)$$

$$= \mu_{\pi}\left(\frac{q^2 - q}{2}\right) + \frac{q^4 + q}{2}.$$

Case-2: $p_{\pi}$ is not a black point. In this case, none of the $\frac{q(q+1)}{2}$ lines of $S$ through $p_{\pi}$ is contained in $\pi$ by Lemma 2.5. Then

$$\theta = \mu_{\pi}(q^2 - q) + \frac{q(q+1)}{2} + (q^2 + q - \mu_{\pi})\left(\frac{q(q+1)}{2} - q\right)$$

$$= \mu_{\pi}\left(\frac{q^2 - q}{2}\right) + \frac{q^4 + q}{2}.$$

In both cases, $|S| = \theta + q^2 = \mu_{\pi}\left(\frac{q^2 - q}{2}\right) + \frac{q^4 + 2q^2 + q}{2}$. Since $|S|$ is a fixed number, it follows that $\mu_{\pi}$ is independent of the tangent plane $\pi$. \hfill \Box

By Lemma 3.5 we denote by $\mu$ the number of black points in a tangent plane. From the proof of Lemma 3.5 we thus have the following equation involving $\mu$ and $|S|:

$$\mu\left(\frac{q^2 - q}{2}\right) + \frac{q^4 + 2q^2 + q}{2} = |S| \quad (4)$$

From equations (3) and (4), we have

$$\mu = \lambda + q. \quad (5)$$

**Lemma 3.6.** The following hold:

(i) Every line of $PG(3,q)$ contains 0, 1, 2 or $q+1$ black points.

(ii) If a line of $PG(3,q)$ contains exactly two black points, then it is a line of $S$.

**Proof.** Let $l$ be a line of $PG(3,q)$ and $b$ be the number of black points contained in $l$. Assume that $b > 2$. If there exists a secant plane $\pi$ through $l$, then Lemma 3.1 implies that the line $l$ contains $b \geq 3$ number of points of the oval $\gamma(\pi)$ in $\pi$, which is not possible. So all planes through $l$ are tangent planes. Then all the $q+1$ points of $l$ are black by Lemma 2.1. This proves (i).

If $b = 2$, then Lemma 2.1 implies that there exists a secant plane $\pi$ through $l$. By Lemma 3.1 $l$ is a secant line of $\pi$ with respect to the oval $\gamma(\pi)$. So $l$ is a line of $S_{\pi}$ by the second part of Lemma 2.10 and hence $l$ is a line of $S$. This proves (ii). \hfill \Box
4 Proof of Theorem 1.1

We shall continue with the notation used in the previous sections. We denote by \( \mathcal{H} \) the set of all black points of \( PG(3, q) \), and by \( \mathcal{H}_\pi \) the set of black points of \( PG(3, q) \) which are contained in a given plane \( \pi \).

**Lemma 4.1.** \(|\mathcal{H}| = \lambda(q + 1).\) In particular, \(|\mathcal{H}| \leq (q + 1)^2\).

**Proof.** Fix a secant plane \( \pi \). Let \( l \) be a line of \( \pi \) which is external to the oval \( \gamma(\pi) \). By Lemma 3.1, none of the points of \( l \) is black. Then, by Lemma 2.1, each plane through \( l \) is a secant plane. The number of black points contained in a secant plane is \( \lambda \). Counting all the black points contained in the \( q + 1 \) planes through \( l \), we get \(|\mathcal{H}| = \lambda(q + 1)\). Since \( \lambda \leq q + 1 \) by Corollary 3.4, we have \(|\mathcal{H}| \leq (q + 1)^2\). \( \square \)

The following result was proved by Bose and Burton in [3, Theorem 1]. We need it in the plane case.

**Proposition 4.2.** [3] Let \( B \) be a set of points of \( PG(n, q) \) such that every line of \( PG(n, q) \) meets \( B \). Then \( |B| \geq (q^n - 1)/(q - 1) \), and equality holds if and only if \( B \) is a hyperplane of \( PG(n, q) \).

**Lemma 4.3.** If \( \pi \) be a tangent plane, then \( \mathcal{H}_\pi \) contains a line.

**Proof.** By Corollary 2.3, every line of \( \pi \) meets \( \mathcal{H}_\pi \). By Proposition 4.2 (taking \( n = 2 \)), we have \(|\mathcal{H}_\pi| \geq q + 1 \), and equality holds if and only if \( \mathcal{H}_\pi \) itself is a line of \( \pi \).

Therefore, assume that \(|\mathcal{H}_\pi| > q + 1 \). Since \( q \) is odd, the maximum size of an arc in \( \pi \) is \( q + 1 \). So \( \mathcal{H}_\pi \) cannot be an arc and hence there exists a line \( l \) of \( \pi \) which contains at least three points of \( \mathcal{H}_\pi \). Then all points of \( l \) are black by Lemma 3.6(i) and so \( l \) is contained in \( \mathcal{H}_\pi \). \( \square \)

**Lemma 4.4.** Let \( \pi \) be a tangent plane. Then \( \mathcal{H}_\pi \) is either a line or union of two (intersecting) lines.

**Proof.** Since \( q \geq 3 \), using Lemmas 3.6(i) and 4.3 observe that there are only four possibilities for \( \mathcal{H}_\pi \):

1. \( \mathcal{H}_\pi \) is a line.
2. \( \mathcal{H}_\pi \) is the union of a line \( l \) and a point of \( \pi \) not contained in \( l \).
3. \( \mathcal{H}_\pi \) is the union of two (intersecting) lines.
4. \( \mathcal{H}_\pi \) is the whole plane \( \pi \).

We show that the possibilities (2) and (4) do not occur. If \( \mathcal{H}_\pi \) is the whole plane \( \pi \), then \( \mu = q^2 + q + 1 \) and so \( \lambda = q^2 + 1 \) by equation (3), which is not possible by Corollary 3.4.

Now suppose that \( \mathcal{H}_\pi \) is the union of a line \( l \) and a point \( x \) not on \( l \). If \( p_\pi \neq x \), then take \( t \) to be the line through \( p_\pi \) and \( x \) (note that \( p_\pi \) may or may not be on \( l \)). If \( p_\pi = x \), then take \( t \) to be any line through \( p_\pi = x \). Since \( \pi \) is a tangent plane, \( t \) is not a line of \( S_\pi \) by Corollary 2.6 and hence is not a line of \( S \). On the other hand, since \( t \) contains only two black points (namely, the point \( x \) and the intersection point of \( l \) and \( t \)), \( t \) is a line of \( S \) by Lemma 3.6(ii). This leads to a contradiction. \( \square \)
Lemma 4.5. Let $\pi$ be a tangent plane. If $H_{\pi}$ is a line of $PG(3, q)$, then the following hold:

(i) $H_{\pi}$ is not a line of $S$.

(ii) $H_{\pi} = H$.

(iii) $S$ is a set of $\frac{q^4 + q^3 + 2q^2}{2}$ lines of $PG(3, q)$ not containing the line $H$.

Proof. (i) Suppose that $H_{\pi}$ is a line of $S$. Then, by Corollary 2.6 the pole $p_{\pi}$ of $\pi$ must be a point of $\pi \setminus H_{\pi}$. Fix a line $m$ of $\pi$ through $p_{\pi}$. Note that $m \notin S$ again by Corollary 2.6. Let $x$ be the point of intersection of $m$ and $H_{\pi}$. Since $m$ contains only one black point (which is $x$), Lemma 2.1 implies that $m$ is contained in one tangent plane (namely, $\pi$) and $q$ secant planes. Since $x \neq p_{\pi}$, by Lemma 2.5 there are $q$ lines of $\pi$ through $x$ which are contained in $S$. In each of the $q$ secant planes through $m$, by Corollary 3.2 there are $q$ lines through $x$ which are contained in $S$. Since $m \notin S$, we get $q(q + 1) = q^2 + q$ lines of $S$ through $x$, which is not possible by property (P1).

(ii) Suppose that $x$ is a black point which is not contained in $H_{\pi}$. Let $\pi'$ be the plane generated by the line $H_{\pi}$ and the point $x$. We have $\pi \neq \pi'$ as $x$ is not a black point of $\pi$. Each of the planes through the line $H_{\pi}$ is a tangent plane by Lemma 2.1. In particular, $\pi'$ is a tangent plane. Note that $\pi$ contains $q + 1$ black points, whereas $\pi'$ contains at least $q + 2$ black points. This contradicts Lemma 4.5.

(iii) The line $H$ is not contained in $S$ by (i) and (ii). Since the tangent plane $\pi$ contains $q + 1$ black points, we have $\mu = q + 1$. Then equation (4) gives that $|S| = \frac{q^4 + q^3 + 2q^2}{2}$. □

In the rest of this section, we assume that $H_{\pi}$ is the union of two (intersecting) lines for every tangent plane $\pi$. So $\mu = 2q + 1$ and then equation (4) gives that $\lambda = q + 1$. From equation (4) and Lemma 4.1 we get

$$|S| = \frac{q^2(q + 1)^2}{2} \quad \text{and} \quad |H| = (q + 1)^2. \quad (6)$$

Lemma 4.6. Let $\pi$ be a tangent plane. If $H_{\pi}$ is the union of the lines $l$ and $l'$ of $\pi$, then the pole $p_{\pi}$ of $\pi$ is the intersection point of $l$ and $l'$.

Proof. Let $x$ be the intersection point of $l$ and $l'$. Suppose that $p_{\pi} \neq x$. Let $t$ be a line of $\pi$ through $p_{\pi}$ which does not contain $x$ (note that $p_{\pi}$ may or may not be contained in $l \cup l'$). Since $\pi$ is a tangent plane, $t$ is not a line of $S_{\pi}$ by Corollary 2.6 and hence is not a line of $S$. On the other hand, since $t$ contains two black points (namely, the two intersection points of $t$ with $l$ and $l'$), it is a line of $S$ by Lemma 3.6(ii). This leads to a contradiction. □

We call a line of $PG(3, q)$ black if it is contained in $H$.

Lemma 4.7. Every black point is contained in at most two black lines.
Proof. Let \( x \) be a black point. If possible, suppose that there are three distinct black lines \( l, l_1, l_2 \) each of which contains \( x \). Let \( \pi \) (respectively, \( \pi' \)) be the plane generated by \( l, l_1 \) (respectively, \( l, l_2 \)). Each plane through \( l \) is a tangent plane by Lemma 2.1. So \( \pi \) and \( \pi' \) are tangent planes. Since \( \mathcal{H}_x = l \cup l_1 \) and \( \mathcal{H}_{\pi'} = l \cup l_2 \), it follows that \( \pi \neq \pi' \). By Lemma 4.6, \( x \) is the pole of both \( \pi \) and \( \pi' \). So the lines through \( x \) which are contained in \( \pi \) or \( \pi' \) are not lines of \( \mathcal{S} \) by Corollary 2.6. Thus each line of \( \mathcal{S} \) through \( x \) is contained in some plane through \( l \) which is different from both \( \pi \) and \( \pi' \). It follows that the number of lines of \( \mathcal{S} \) through \( x \) is at most \( q(q-1) \), which contradicts to the fact that there are \( q^2 \) lines of \( \mathcal{S} \) through \( x \) (being a black point).

\[
\square
\]

Lemma 4.8. Every black point is contained in precisely two black lines.

Proof. Let \( x \) be a black point and \( l \) be a black line containing \( x \). The existence of such a line \( l \) follows from the facts that \( x \) is contained in a tangent plane (Corollary 2.3) and that the set of all black points in that tangent plane is a union of two black lines. By Lemma 2.1, let \( \pi_1, \pi_2, \ldots, \pi_{q+1} \) be the \( q+1 \) tangent planes through \( l \). For \( 1 \leq i \leq q+1 \), we have \( \mathcal{H}_{\pi_i} = l \cup l_i \) for some black line \( l_i \) of \( \pi_i \) different from \( l \). Let \( \{p_i\} = l \cap l_i \). Lemma 3.7 implies that \( p_i \neq p_j \) for \( 1 \leq i \neq j \leq q+1 \), and so \( l = \{p_1, p_2, \ldots, p_{q+1}\} \). Since \( x \in l \), we have \( x = p_j \) for some \( 1 \leq j \leq q+1 \). Thus, applying Lemma 3.7 again, it follows that \( x \) is contained in precisely two black lines, namely, \( l \) and \( l_j \).

We refer to [10] for the basics on finite generalized quadrangles. Let \( s \) and \( t \) be positive integers. A generalized quadrangle of order \((s, t)\) is a point-line geometry \( \mathcal{X} = (P, L) \) with point set \( P \) and line set \( L \) satisfying the following three axioms:

(Q1) Every line contains \( s+1 \) points and every point is contained in \( t+1 \) lines.

(Q2) Two distinct lines have at most one point in common (equivalently, two distinct points are contained in at most one line).

(Q3) For every point-line pair \((x, l) \in P \times L \) with \( x \notin l \), there exists a unique line \( m \in L \) containing \( x \) and intersecting \( l \).

Let \( \mathcal{X} = (P, L) \) be a generalized quadrangle of order \((s, t)\). Then, \(|P| = (s+1)(st+1)\) and \(|L| = (t+1)(st+1)\) [10, 1.2.1]. If \( P \) is a subset of the point set of some projective space \( PG(n, q) \), \( L \) is a set of lines of \( PG(n, q) \) and \( P \) is the union of all lines in \( L \), then \( \mathcal{X} = (P, L) \) is called a projective generalized quadrangle. The points and the lines contained in a hyperbolic quadric in \( PG(3, q) \) form a projective generalized quadrangle of order \((q, 1)\). Conversely, any projective generalized quadrangle of order \((q, 1)\) with ambient space \( PG(3, q) \) is a hyperbolic quadric in \( PG(3, q) \), this follows from [10, 4.4.8].

The following two lemmas complete the proof of Theorem 1.1.

Lemma 4.9. The points of \( \mathcal{H} \) together with the black lines form a hyperbolic quadric in \( PG(3, q) \).

Proof. We have \(|\mathcal{H}| = (q+1)^2 \) by (6). It is enough to show that the points of \( \mathcal{H} \) together with the black lines form a projective generalized quadrangle of order \((q, 1)\).
Each black line contains $q+1$ points of $\mathcal{H}$. By Lemma 4.8, each point of $\mathcal{H}$ is contained in exactly two black lines. Thus the axiom (Q1) is satisfied with $s = q$ and $t = 1$. Clearly, the axiom (Q2) is satisfied.

We verify the axiom (Q3). Let $l = \{x_1, x_2, \ldots, x_{q+1}\}$ be a black line and $x$ be a black point not contained in $l$. By Lemma 4.8, let $l_i$ be the second black line through $x_i$ (different from $l$) for $1 \leq i \leq q+1$. If $l_i$ and $l_j$ intersect for $i \neq j$, then the tangent plane $\pi$ generated by $l_i$ and $l_j$ contains $l$ as well. This implies that $\mathcal{H}$ contains the union of three distinct black lines (namely, $l, l_i, l_j$), which is not possible. Thus the black lines $l_1, l_2, \ldots, l_{q+1}$ are pairwise disjoint. These $q+1$ black lines contain $(q+1)^2$ black points and hence their union must be equal to $\mathcal{H}$. In particular, $x$ is a point of $l_j$ for unique $j \in \{1, 2, \ldots, q+1\}$. Then $l_j$ is the unique black line containing $x_j$ and intersecting $l$.

From the above two paragraphs, it follows that the points of $\mathcal{H}$ together with the black lines form a projective generalized quadrangle of order $(q, 1)$. This completes the proof.

\textbf{Lemma 4.10.} The lines of $\mathcal{S}$ are precisely the secant lines to the hyperbolic quadric $\mathcal{H}$.

\textbf{Proof.} By (6), we have $|\mathcal{S}| = \frac{q^2(q+1)^2}{2}$, which is equal to the number of secant lines to $\mathcal{H}$. It is enough to show that every secant line to $\mathcal{H}$ is a line of $\mathcal{S}$. This follows from Lemma 3.6(ii), as every secant line to $\mathcal{H}$ contains exactly two black points.

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