Homogeneous polynomial Lyapunov functions for the analysis of polytopic parameter-dependent descriptor systems

Ana Carolina dos Santos Paulino and Gabriela Iuliana Bara

ICube Laboratory, University of Strasbourg, Illkirch, France

ABSTRACT
In this paper, we present an extension of the homogeneous polynomial Lyapunov functions (HPLFs) approach previously developed for uncertain standard systems to the case of uncertain continuous-time descriptor ones. By employing a new power transformation of the state vector with respect to its dynamic and algebraic parts, new necessary and sufficient admissibility analysis conditions are developed based on such a class of Lyapunov functions. The presented results deal with time-varying parameters with both unbounded variation rates, based on parameter-independent HPLFs, and bounded variation rates lying inside a polytope by exploiting polytopic parameter-dependent HPLFs. Numerically, well tractable sufficient LMI conditions are also derived for both cases. A comparison with results from the literature is performed, showing the reduction of conservatism throughout the novel conditions presented in this article.

1. Introduction
Uncertain descriptor systems have been a topic of active research because they can represent altogether a comprehensive set of phenomena, such as time-varying dynamics, impulsive behaviour and algebraic constraints. It is commonly found in the literature that this mathematical representation serves to fields such as robotics, aeronautics, economics, electronics, among others, according to Mills and Goldenberg (1989), Lewis (1986) and Angelis (2001).

Lyapunov theory based on the notion of Lyapunov functions (LFs) is very appealing for the analysis and control of complex systems since LFs can be interpreted as an energy function and their generic formulation can be used along with many different classes of systems. Lyapunov functions have been massively employed in order to develop mathematical tools for the admissibility analysis and control synthesis of uncertain descriptor systems as they can provide non-conservative stability analysis criteria, according to Blanchini (1995) and Chesi et al. (2007).

The absence of conservatism is related to the concept of universality between a class of Lyapunov functions and a class of systems. As defined in Blanchini and Miani (1999), a given class $\mathcal{A}$ of LFs is said to be universal with respect to a class $\mathcal{B}$ of systems if the stability of system in $\mathcal{B}$ is equivalent to the existence of a LF in class $\mathcal{A}$. Notice that quadratic Lyapunov functions are universal for standard linear time-invariant (LTI) systems, yielding to necessary and sufficient stability conditions that can be easily translated into linear matrix inequalities (LMIs). However, the universality of quadratic LFs no longer holds for more complex classes of dynamical systems such as the uncertain systems class. Indeed, it has been shown in Zelentsovsky (1994), Blanchini (1995), Blanchini and Miani (1999) and Chesi et al. (2003a) that quadratic LFs provide conservative results when employed for the analysis of uncertain systems. This motivated the study of different classes of LFs, such as polyhedral (Blanchini & Miani, 1999), homogeneous polynomial (Chesi et al., 2003a, 2007; Zelentsovsky, 1994) and biquadratic (Barbosa et al., 2012, 2013).

Homogeneous polynomial Lyapunov functions (HPLFs) constitute a universal class for stability analysis of uncertain parameter-dependent systems (Chesi et al., 2003a, 2007; Zelentsovsky, 1994). Since HPLFs are constructed based on a power transformation of the state vector, it has been shown in Zelentsovsky (1994) and Chesi et al. (2003a) that the stability analysis of uncertain parameter-dependent systems based on HPLFs can be reformulated as a quadratic stability analysis problem of an associated extended system. Due to this fact, in Chesi et al. (2003a), robust LMI stability analysis conditions have been derived by exploiting the degree of freedom related to the complete square matricial representation (CSMR) of HPLFs. Moreover, it has been shown that HPLFs provide necessary and sufficient conditions for stability analysis once a suitable degree for the homogeneous polynomials is chosen. Notice that much of the work done with regards to the stability analysis of standard uncertain systems using HPLFs is yet to be transposed to the uncertain descriptor case. This is not straightforward since it raises the difficult question of how the power transformation should be applied with respect to the algebraic state apart from the dynamic state of the descriptor system. Indeed, this paper aims to provide a contribution not only in transferring techniques from the standard to the descriptor framework.
but also in supplying numerically exploitable, less restrictive, admissibility conditions to the latter.

Among the works related to the admissibility analysis of uncertain descriptor systems, the commonly employed Lyapunov functions are quadratic with respect to the states and either affine/polytopic parameter-dependent (Bara, 2011; Barbosa et al., 2012, 2013; dos Santos Paulino & Bara, 2017b) or quadratic parameter-dependent (Barbosa et al., 2012, 2013).

Each of these results use slack variables to decouple Lyapunov and state matrices. Among the derived LMI conditions that have affine/polytopic parameter dependence, Barbosa et al. (2012) and Barbosa et al. (2013) use parameter-independent slack variables, while dos Santos Paulino and Bara (2017b) employs polytopic slack variables and sum-of-squares polynomials. The LMI conditions obtained in the context of quadratic parameter dependence (Barbosa et al., 2012, 2013) use quadratic parameter-dependent slack variables. As far as we know, the only results available in the literature introducing HPLFs for the admissibility analysis of uncertain descriptor systems have been presented in our previous work (dos Santos Paulino & Bara, 2017a). In spite of using parameter-independent Lyapunov functions, the numerical examples show that the use of HPLFs may lead to less restrictive results.

In this paper, we present new results on the admissibility analysis of continuous-time uncertain descriptor systems by using parameter-independent and parameter-dependent HPLFs. First, by reviewing our previous work (dos Santos Paulino & Bara, 2017a), we present necessary and sufficient conditions for the existence of a parameter-independent HPLF of a given degree that evaluates the admissibility of a uncertain descriptor system with unbounded variation rate of the parameters. Necessary and sufficient numerically tractable LMI conditions are also derived. Then, the admissibility of uncertain descriptor systems is assessed through parameter-dependent HPLFs as an extension of the parameter-independent class. Necessary and sufficient conditions are presented and, subsequently, two sets of LMI conditions are derived. These novel conditions for parameter-dependent HPLFs are obtained by two different convexification techniques. Overall, the results presented in this paper extend the results proposed by Zelentsovsky (1994) and Chesi et al. (2003a) for uncertain standard systems to the case of uncertain descriptor ones. This is achieved by considering the descriptor system under its SVD normal form and by delivering an explicit formulation of the power transformation, clearly emphasising the dependence on the dynamic and algebraic states of the descriptor system, as well as of the associated extended descriptor system.

The paper is organised as follows: Section 2 presents the problem and previously developed mathematical tools; Section 3 brings the main contributions of this paper; Section 4 shows reduced conservatism of the proposed approach on numerical examples; and Section 5 summarises and concludes the manuscript.

Notations: The notation used in this manuscript is standard. The superscript $T$ over a vector or matrix stands for its transposition. The symbol $(*)$ in a matrix indicates a block induced by symmetry. For a vector $w$, $w_i$ stands for its $i$th component. The convex hull generated by the vectors $w^{(1)}, \ldots, w^{(N)}$ is given as $\text{Co}(w^{(1)}, \ldots, w^{(N)})$. The sum between a real matrix and its transposed version is indicated by $\text{He}(A) = A + A^T$ and a matrix composed by diagonal appendages of other matrices is given as $\text{Bdiag}(A, B) = [A \ 0 \ 0; 0 \ B \ 0]$. The Kronecker product between given matrices $A$ and $B$ is noted as $A \otimes B$. Scalars as subscripts indicate the dimensions of matrices e.g. $0_n$ stands for a zero square matrix in $\mathbb{R}^{n \times n}$ while $0_{n \times m}$ represents a zero rectangular matrix in $\mathbb{R}^{n \times m}$. The degree of the polynomial expression $S$ is indicated by $\text{deg}(S)$. For two given sets $A$ and $B$, $A \times B$ represents the cartesian product between them. The infinity norm of a real vector $d \in \mathbb{R}^n$ is given by $\|d\|_{\infty} = \max \{|d_1|, \ldots, |d_n|\}$. The orthogonal complement of a square matrix $E \in \mathbb{R}^{(n+n_0)\times(n+n_0)}$ of rank $n$ is the matrix $E_0 \in \mathbb{R}^{(n+n_0)\times n_0}$ such that $E^T E_0 = 0_{(n+n_0)\times n_0}$.

### 2. Problem statement and preliminaries

The current section presents the system of interest and recollects mathematical tools that are used in the main contribution of this paper.

#### 2.1 Polytopic parameter-dependent descriptor systems

Consider parameter-dependent continuous-time descriptor systems in the singular value decomposition (SVD) normal form

$$
\begin{aligned}
\dot{x}(t) &= A_{11}(w(t))x(t) + A_{12}(w(t))\xi(t) + B_1(w(t))u(t) \\
0_{n_2 \times 1} &= A_{21}(w(t))x(t) + A_{22}(w(t))\xi(t) + B_2(w(t))u(t)
\end{aligned}
$$

(1)

where $x(t) \in \mathbb{R}^{n}$ is the dynamic state, $\xi(t) \in \mathbb{R}^{n_0}$ is the algebraic state and $u(t) \in \mathbb{R}^{n_u}$ is the control input. The initial condition $(x(0), \xi(0)) = (x_0, \xi_0)$ is assumed to be consistent in the sense defined later in the paper.

The time-varying parameters $w(t) \in \mathbb{R}^m$ lie within the unit simplex and may have bounded rates of variation. Therefore,

$$
\begin{aligned}
w(t) \in \mathcal{W} = \left\{ w \in \mathbb{R}^m \ \Big| \ \sum_{i=1}^{m} w_i = 1, \ 0 \leq w_i \leq 1 \right\} \ \forall t \geq 0.
\end{aligned}
$$

The matrices $A_{ij}(w(t))$, for $i, j \in \{1, 2\}$, and $B_i(w(t))$, for $i \in \{1, 2\}$, are described within polytopes of matrices as

$$
\begin{aligned}
A_{ij}(w(t)) &= \sum_{k=1}^{m} w_k(t) A_{ij}^{(k)} \\
B_i(w(t)) &= \sum_{k=1}^{m} w_k(t) B_i^{(k)}
\end{aligned}
$$

(2)

where vertex matrices $A_{ij}^{(k)}$ and $B_i^{(k)}$ are known real-valued matrices of appropriate dimension. Notice that polytopic linear models have been widely used in the literature since they naturally arise as a description of uncertain systems or as an approximate and alternative description of nonlinear dynamical systems. For instance, this formalism has been employed for describing, to cite a few, rotational robotic manipulator systems (Angelis, 2001), power systems (Ramos et al., 2002) and aircraft dynamic systems (Dai et al., 2006) with multiple operating conditions.
For many physical systems, bounds on the variation rate of the parameters may be deduced from physical considerations (Dai et al., 2006). In the sequel, as already considered by Geromel and Colaneri (2006), Chesi et al. (2007) and Montagner et al. (2009) for continuous-time systems and by Oliveira and Peres (2009) for discrete-time ones, we assume that the parameters velocity space is specified as a polyhedral convex set, i.e. \( w(t) \in \mathcal{V} \forall t \geq 0 \) where

\[
\mathcal{V} = \left\{ v \in \mathbb{R}^m, v \in \text{Co}(d^{(1)}, \ldots, d^{(h)}) \right\},
\]

with \( d^{(j)} \), for \( j = 1, \ldots, h \), given vectors. This definition ensures that the constraint \( \sum_{i=1}^{m} \dot{w}_i(t) = 0 \) always holds considering that \( w(t) \) belongs to the unit simplex \( \mathcal{W} \). The interested reader can refer to Geromel and Colaneri (2006), Chesi et al. (2007), Oliveira and Peres (2008) and Oliveira and Peres (2009) for detailed procedures allowing to obtain the vectors \( d^{(j)} \) from known bounds on the parameters variation rate. The fact that \( \dot{w}(t) \in \mathcal{V} \) implies the following polytopic description for the parameters variation rate:

\[
\dot{w}(t) = \sum_{j=1}^{h} \sigma_j(t) d^{(j)}
\]

where \( \sigma(t) \in \left\{ \sigma \in \mathbb{R}^h | \sum_{i=1}^{h} \sigma_i = 1, 0 \leq \sigma_i \leq 1 \right\} \forall t \geq 0. \tag{4}
\]

The initial condition \((x_0, \xi_0)\) has to satisfy the algebraic constraint

\[
A_{21} w(0)x_0 + A_{22} w(0)\xi_0 + B_2 w(0)u(0) = 0.
\]

Given the polytopic structure described in (2) of system state matrices and since the initial value \( w(0) \) of the uncertain parameter vector \( w(t) \in \mathcal{W} \) is unknown, it follows, throughout the application of the multiconvexity principle, that this constraint is equivalent to a finite set of constraints obtained by evaluating the algebraic constraint on the vertices \( w^{(i)} \) of the unit simplex \( \mathcal{W} \). Therefore, the initial condition \((x_0, \xi_0)\) for system (1) is called consistent if the following conditions hold

\[
A_{21}^{(i)} x_0 + A_{22}^{(i)} \xi_0 + B_2^{(i)} u(0) = 0 \text{ for } i = 1, \ldots, m.
\]

We recall that a descriptor system in the general form

\[
E \dot{X}(t) = A(t) X(t), \tag{5}
\]

with \( X \in \mathbb{R}^{(n+n_\xi)} \), rank\(E) = n \leq n + n_\xi \) and \( E \) being a constant matrix, can be converted into the SVD normal form (1) through a constant, but not unique, coordinate transformation. As a matter of fact, the conversion can be done either by rank decomposition, according to Dai (1989), or singular value decomposition on the matrix \( E \), as presented by Bender and Laub (1987) and Ishihara and Terra (2002).

For systems in the form

\[
E \dot{X}(t) = A(w(t)) X(t) \tag{6}
\]

with constant derivative matrix \( E \), \( w(t) \in \mathcal{W} \) and \( w(t) \in \mathcal{V} \), an important definition follows.

**Definition 2.1:** The pair \((E, A(w(t)))\) is:

- **Regular** if \( \det(sE - A(w(t))) \neq 0 \forall w(t) \in \mathcal{W} \) and for some \( s \in \mathbb{C} \), indicating existence and unicity of solutions;
- **Impulse-free** if \( \text{deg}(\det(sE - A(w(t)))) = \text{rank} E \forall w(t) \in \mathcal{W} \), ensuring the absence of impulsive modes;
- **Robustly stable** if the finite modes are stable for all possible values and trajectories of parameters in the admissible parameter domain;
- **Robustly admissible** if it is regular, impulse-free and robustly stable for all possible values and trajectories of parameters in the admissible parameter domain.

**Remark 2.1:** The robust admissibility property of continuous-time uncertain descriptor system (6) can be expressed through its equivalent SVD normal form (1): regularity and impulse-free properties are equivalent to the invertibility of the matrix \( A_{22}(w(t)) \) for all possible values of the vector of parameters \( w(t) \); the stability of the finite modes is given by the stability of the equivalent standard form \( \dot{x}(t) = (A_{11}(w(t)) - A_{12}(w(t)) A_{22}^{-1}(w(t)) A_{21}(w(t))) x(t) \) for all possible values and trajectories of parameters in the admissible parameter domain.

**Remark 2.2:** The descriptor system in the general form (6), leading to form (1) through SVD decomposition, does not exhibit parameter dependence on the derivative matrix \( E \). In fact, in the case of **constant uncertain parameters**, a constant \( E \) matrix is not restrictive according to de Souza et al. (2008) and Barbosa et al. (2012) since an augmented descriptor system with constant derivative matrix can be obtained whenever the matrices \( E \) and \( A \) are affine parameter-dependent. Although this technique was proposed for discrete-time systems, it easily generalises to continuous-time ones. In the case of **time-varying parameters**, few approaches have been proposed for the robust admissibility analysis of descriptor systems with parameter dependence in the derivative matrix \( E \). For instance, the results of Barbosa et al. (2017), Barbosa et al. (2018) and Rodriguez et al. (2018) have been developed for discrete-time systems. The aim of our work is to generalise the HPLF framework developed for regular (standard) systems to the case of singular (descriptor) ones. Notice that the HPLF approach cannot easily be broaden to discrete-time parameter-dependent systems since the state power transformation, presented in the next section, generates a nonlinear parameter dependence on the extended system that is very difficult to deal with. Although (Fang, 2002) addressed continuous-time uncertain descriptor systems in addition to discrete-time ones, the proposed results, alike the aforementioned ones, use the quadratic Lyapunov...
functions framework that may be more conservative than the HPLF one as shown by our numerical examples. In our paper, the generalisation of the HPLF framework to the case of descriptor systems has been reached by proposing an explicit formulation of the state power transformation according to the dynamic and algebraic parts of the state vector of the original system. Handling general parameter-dependent derivative matrix $E$ in the HPLF framework is still a difficult problem and yet to be investigated.

In the following, we present the main concepts related to the HPLF framework.

### 2.2 Homogeneous polynomial Lyapunov functions

In the literature, HPLFs have been investigated for the robust stability analysis of uncertain standard systems (i.e. $E = I_n$) in Chesi et al. (2003a, 2003b, 2004, 2007). The main motivation in using this class of Lyapunov functions for the analysis of uncertain systems is that HPLFs can lead to non-conservative conditions once a suitable polynomial degree is chosen.

In the sequel, a compilation of important notions related to HPLFs is recalled.

**Definition 2.2 (homogeneous forms):** The vector $x^{\lfloor q\rfloor} \in \mathbb{R}^{|d_q|}$ denotes the base vector of homogeneous forms of degree $q$ in $x \in \mathbb{R}^n$. The base vector of homogeneous forms $x^{\lfloor q\rfloor}$ is a collection of all the integer powered monomials of degree $q$ that can be produced from the entries of the vector $x$. The components of $x^{\lfloor q\rfloor}$ are given as

\[
x^{\lfloor q\rfloor} = x_1^{i_1}x_2^{i_2} \cdots x_n^{i_n}, \quad i_1 + i_2 + \cdots + i_n = q,
\]

where $d_q$ is the dimension of $x^{\lfloor q\rfloor}$ given by $d_q = \binom{n+q-1}{n-1}$.

The mapping between $x$ and $x^{\lfloor q\rfloor}$ stated in Definition 2.2 is referred in the literature as the power transformation of degree $q$ of $x$ (see Barkin & Zelentsovsky, 1983; Brockett, 1973).

**Definition 2.3 (extended system):** The extended matrix $A_{\lfloor q\rfloor} \in \mathbb{R}^{|d_q| \times |d_q|}$ associated to the base vector of homogeneous forms of degree $q$ is given as

\[
\frac{d}{dt}x^{\lfloor q\rfloor} = A_{\lfloor q\rfloor}x^{\lfloor q\rfloor}.
\]

**Definition 2.4 (extended system):** Consider a parameter-dependent continuous-time standard state-space system

\[
\dot{x}(t) = \left( \sum_{i=1}^{m} w_i(t)A_i \right)x(t),
\]

and denote by $A_{\lfloor q\rfloor}$ the extended matrix of $A_i$, according to Definition 2.3. Then, the extended system associated to the base vector of homogeneous forms of degree $q$ is defined as:

\[
\frac{d}{dt}x^{\lfloor q\rfloor}(t) = A(w(t))_{\lfloor q\rfloor}x^{\lfloor q\rfloor}(t) = \left( \sum_{i=1}^{m} w_i(t)A_{i\lfloor q\rfloor} \right)x^{\lfloor q\rfloor}(t).
\]

This last definition specifies that in the case of continuous-time parameter-dependent systems, the associated extended system (8) preserves the linear parameter dependence of the original system (7). Note that this is not the case for discrete-time systems.

**Lemma 2.1** (Bara & dos Santos Paulino, 2016): There exist matrices $M_i$, $N_i$, for $i = 1, \ldots, d_q-1$, such that the extended matrix $A(w(t))_{\lfloor q\rfloor}$ is given by

\[
A(w(t))_{\lfloor q\rfloor} = \sum_{i=1}^{d_q-1} M_iA(w(t))N_i,
\]

where $d_q-1 = \dim(x^{\lfloor q-1\rfloor}) = \binom{(n+q-2)!}{(n-1)(q-1)!}$, and $M_i \in \mathbb{R}^{d_q \times n}$, $N_i \in \mathbb{R}^{n \times d_q}$ are constant matrices depending only on the order of monomials of degree $q$ stacked in the vector $x^{\lfloor q\rfloor}$.

**Definition 2.5 (homogeneous polynomial Lyapunov functions):** HPLFs of degree $2q$ are scalar functions given by weighted sums of the entries of the base vector of homogeneous forms of degree $2q$. They can be described through the square matricial representation (SMR), given by

\[
V_{2q}(x) = x^{\lfloor 2q\rfloor}^TPx^{\lfloor 2q\rfloor}. \tag{10}
\]

The matrix $P = P^T \in \mathbb{R}^{d_q \times d_q}$ is non-unique for a given HPLF. Then, consider the linear space

\[
\mathcal{P} = \left\{ P_0 = P_0^T \in \mathbb{R}^{d_q \times d_q} \mid x^{\lfloor 2q\rfloor}^TP_0x^{\lfloor 2q\rfloor} = 0 \ \forall \ x \in \mathbb{R}^n \right\} \tag{11}
\]

whose size is, according to Chesi et al. (2007):

\[
d_\mathcal{P} = \frac{1}{2} d_q(d_q + 1) - \frac{(n+2q-1)!}{(n-1)!(2q)!}.
\]

Therefore, the complete square matricial representation (CSMR) of the Lyapunov function (10) is

\[
V_{2q}(x) = x^{\lfloor 2q\rfloor}^T(P + \Gamma(\gamma))x^{\lfloor 2q\rfloor}. \tag{12}
\]

where $\gamma \in \mathbb{R}^{d_\mathcal{P}}$ and matrix $\Gamma(\gamma)$ is a linear parameterisation of $\mathcal{P}$ such as $\Gamma(\gamma) = \sum_{i=1}^{d_\mathcal{P}} \gamma_i P_{0,i}$, $P_0 \in \mathcal{P}$.

**Remark 2.3:** The power transformation of degree $q$ of $x$ stated in Definition 2.2 along with Definition 2.5 are particularly important since they lay the foundations of the HPLF approach. Indeed, the work of Zelentsovsky (1994) showed that a parameter-dependent system (7) is globally asymptotically stable if and only if the extended system (8) is also globally asymptotically stable. This is due to the fact that the time-derivative of the Lyapunov function $V_{2q}(x)$ given by (10) along the trajectories of the uncertain system (7) equals the time-derivative of $V_{2q}(x)$ along the trajectories of the extended system (8) meaning that $\frac{d}{dt}V_{2q}(x)|_{(7)} = \frac{d}{dt}V_{2q}(x)|_{(8)}$ as demonstrated by Zelentsovsky (1994) and Chesi et al. (2003a). This shows that the power transformation defined in Definition 2.2 allows to reformulate the problem of finding a HPLF of degree $2q$, as defined in Definition 2.5, for an uncertain system as an equivalent problem of finding a quadratic LF for the extended system.
3. Main results

Based on the concept of HPLFs used for standard uncertain systems and summarised in the previous section, the aim of this section is to extend the results proposed by Zelentsovsky (1994) and Chesi et al. (2003a) for uncertain standard systems to the case of uncertain descriptor ones. It is important to notice that the extension of the HPLF framework from standard systems to singular ones is not straightforward since it raises the difficult question of how the base vector of homogeneous forms should be constructed with regard to the dynamic and algebraic states of the descriptor system. In other words, unlike regular systems, the key question is how the power transformation should deal with the algebraic state apart from the dynamic state of the descriptor system. In the following, we address this question by considering the descriptor system under the SVD normal form and by presenting an explicit formulation of the power transformation, clearly emphasising the dependence on dynamic and algebraic states of the descriptor system, as well as of the associated extended descriptor system.

In the sequel, we propose new admissibility analysis conditions for continuous-time uncertain descriptor systems by means of HPLFs of degree 2q. Firstly, we tackle the case of parameters with arbitrary rate of variation by using parameter-independent HPLFs. Secondly, we address the case of parameters with bounded rates of variation by employing polytopic parameter-dependent HPLFs. For both cases, we introduce necessary and sufficient conditions for the robust admissibility analysis of uncertain descriptor systems, and then, we derive numerically tractable LMI conditions. The numerically tractable conditions pertaining to the parameter-independent case are necessary and sufficient whilst the two sets of numerically tractable conditions associated to the parameter-dependent case are only sufficient.

3.1 Admissibility analysis with parameter-independent HPLFs

This section presents parameter-independent HPLFs for the admissibility analysis of continuous-time uncertain descriptor systems subject to time-varying parameters with unbounded variation rates. This generalises the results of Zelentsovsky (1994) and Chesi et al. (2003a) to the descriptor systems case and has been introduced for the first time in our previous work (dos Santos Paulino & Bara, 2017a). The proposed method is reviewed in the following.

Based on Definition 2.1 and remark 2.1 in Section 2.1, we state the following definition.

Definition 3.1: The uncertain descriptor system under SVD normal form (1) is robustly admissible with a parameter-independent HPLF of degree 2q if the matrix \( A_{22}(w(t)) \) is invertible, for all possible values of the vector of parameters \( w(t) \in \mathcal{W} \), and its equivalent standard state-space system

\[
\dot{x}(t) = \Gamma(w(t))x(t), \quad \text{where}
\]

\[
\Gamma(w(t)) = A_{11}(w(t)) - A_{12}(w(t))A_{22}(w(t))^{-1}A_{21}(w(t))
\]  

\begin{equation}
(13)
\end{equation}
is robustly stable based on a parameter-independent HPLF of degree 2q, for all possible parameters trajectories of \( w(t) \in \mathcal{W} \).

The next lemma generalises the HPLF approach to the descriptor systems case and gives an explicit formulation of the base vector of homogeneous forms of degree q, according to the dynamic and algebraic states of the descriptor system, as well as of the extended descriptor system. This lemma, stated as a sufficient condition in our previous work (dos Santos Paulino & Bara, 2017a), is reviewed as a necessary and sufficient condition as follows.

Lemma 3.1: The uncertain descriptor system (1) is robustly admissible with a parameter-independent HPLF of degree 2q, for all possible trajectories \( w(t) \in \mathcal{W} \), if and only if the extended descriptor system

\[
\mathcal{E} \left[ \begin{array}{c}
\frac{d}{dt} x^{[q]}(t) \\
\frac{d}{dt} (x^{[q-1]}(t) \otimes \xi(t))
\end{array} \right] = \mathcal{A}_{[q]}(w(t)) \left[ \begin{array}{c}
x^{[q]}(t) \\
x^{[q-1]}(t) \otimes \xi(t)
\end{array} \right]
\]  

\begin{equation}
(14)
\end{equation}
is robustly admissible with a quadratic LF, for all possible trajectories \( w(t) \in \mathcal{W} \). The definitions of \( \mathcal{E} \) and \( \mathcal{A}_{[q]} \) are given in the following, the time dependence being omitted for compactness:

\[
\mathcal{E} = B \text{diag}[I_{d_x}, 0_{d_q-d_x}],
\]

\[
\mathcal{A}_{[q]}(w) = \begin{bmatrix}
A_{11}(w) & M_1A_{12}(w) & M_2A_{12}(w) & \ldots & M_{d_q-1}A_{12}(w) \\
A_{21}(w)N_1 & A_{22}(w) & A_{22}(w) & \ldots & A_{22}(w) \\
\vdots & & & \ddots & & \vdots \\
A_{21}(w)N_{d_q-1} & & & & A_{22}(w)
\end{bmatrix}
\]

Proof: Based on Definition 3.1, the descriptor system (1) is robustly admissible with a parameter-independent HPLF of degree 2q, for all possible trajectories \( w(t) \in \mathcal{W} \), if and only if \( A_{22}(w(t)) \) is invertible, for all possible values of \( w(t) \in \mathcal{W} \), and system (13) is robustly stable based on a parameter-independent HPLF of degree 2q, for all possible parameters trajectories of \( w(t) \in \mathcal{W} \).

The extended system obtained by applying a power transformation of degree q on the state vector \( x \) of system (13) is expressed, according to Definition 2.4, as:

\[
\frac{d}{dt} x^{[q]}(t) = \frac{\partial x^{[q]}(t)}{\partial x(t)} \Gamma(w(t))x = \Gamma_{[q]}(w(t))x^{[q]}(t).
\]  

\begin{equation}
(15)
\end{equation}

Note that the derivative of HPLF (10) of degree 2q along the trajectories of system (13) equals its derivative along the trajectories of system (15) and that HPLF (10) expresses a quadratic LF regarding system (15). It follows that the descriptor system (1) is robustly admissible with a parameter-independent HPLF of degree 2q, for all possible trajectories \( w(t) \in \mathcal{W} \), if and only if \( A_{22}(w(t)) \) is invertible, for all possible values of \( w(t) \in \mathcal{W} \), and the extended system (15) is quadratically stable for all possible parameters trajectories of \( w(t) \in \mathcal{W} \). Using Lemma 2.1,
there exist matrices $M_i, N_i$, for $i = 1, \ldots, d_q-1$, such that state matrix $\Gamma_{[q]}(w(t))$ in (15) can be rewritten as

$$
\Gamma_{[q]}(w(t)) = \sum_{i=1}^{d_q-1} M_i \Gamma(w(t)) N_i = \sum_{i=1}^{d_q-1} M_i A_{11}(w(t)) N_i - \sum_{i=1}^{d_q-1} M_i A_{12}(w(t)) A_{21}^{-1}(w(t)) A_{21}(w(t)) N_i. \quad (16)
$$

The expression $\sum_{i=1}^{d_q-1} M_i A_{11}(w(t)) N_i$ represents the extended matrix $A_{11}(w(t))$ through Lemma 2.1.

Introducing the notation $\mathcal{A}_{[q]}(w) = \begin{bmatrix} \mathcal{A}_{[q]11} & \mathcal{A}_{[q]12} \\ \mathcal{A}_{[q]21} & \mathcal{A}_{[q]22} \end{bmatrix}$ corresponding to the block-partitioning indicated in the statement of our theorem, descriptor system (14) is robustly admissible, for all possible trajectories $w(t) \in \mathcal{W}$, with a quadratic LF if and only if $\mathcal{A}_{[q]22}$ is invertible which is equivalent to the invertibility of $A_{22}(w(t))$, for all possible values of $w(t) \in \mathcal{W}$, and the following system

$$
\frac{d}{dt} x_{[q]}(t) = \left( \mathcal{A}_{[q]11} - \mathcal{A}_{[q]12} \mathcal{A}_{[q]22}^{-1} \mathcal{A}_{[q]21} \right) x_{[q]}(t)
$$

is quadratically stable. Note that this last dynamic equation is identical to the one of system (15) with description (16). This concludes the proof.

Based on the previous theorem, we are able to state that, when it comes to descriptor systems, the power transformation of degree $q$ acts in a different way on its algebraic state $\xi$ than on its dynamic state $x$. Indeed, the extended descriptor system is given by (14) and the associated state vector is $\begin{bmatrix} x_{[q]}^{(q-1)\xi} \end{bmatrix}$. This extended state vector is homogeneous of degree $q$ but does not involve all the possible monomials of degree $q$ of the vector $\begin{bmatrix} x \end{bmatrix}$. The upper part of this extended state vector, $x_{[q]}^{(q-1)\xi}$, represents the power transformation of degree $q$ of the dynamic state $x$ of system (1) while the lower part, $x_{[q]}^{(q-1)\xi} \otimes \xi$, represents the algebraic state of the extended descriptor system. This algebraic part of the extended descriptor system is homogeneous of degree $q$ and involves products between all algebraic state variables $\xi$ of the original system and all the monomials of degree $q-1$ of the dynamic state $x$.

**Remark 3.1:** The extended descriptor system (14) conserves the same parameter dependence as the original system (1). This is due to the fact that the block matrices composing the extended state matrix $\mathcal{A}_{[q]}(w)$ are linearly dependent on the matrices $A_{ij}(w), i = 1, 2, j = 1, 2$. Since matrices $A_{ij}(w)$ are, by definition, polytopic parameter-dependent, it follows that $\mathcal{A}_{[q]}(w)$ is also polytopic parameter-dependent. The vertices of the polytopic parameter-dependent matrix $\mathcal{A}_{[q]}(w)$ are denoted as $\mathcal{A}_{ij}[q], i = 1, \ldots, m$ in the following.

Based on Lemma 3.1, the quadratic Lyapunov function associated to the extended descriptor system (14) and therefore, the HPLF of degree $2q$ associated to the descriptor system (1) has the general form

$$
V_{2q} = \begin{bmatrix} x_{[q]}(t) \otimes \xi(t) \end{bmatrix}^T \sum_{j=1}^{d_p} \gamma_j \cdot Bdiag\{P_{0j}, 0_{d_{q-1}n_x} \} \begin{bmatrix} x_{[q]}(t) \otimes \xi(t) \end{bmatrix}, \quad (17)
$$

**Lemma 3.2:** The uncertain descriptor system (1) is robustly admissible with a parameter-independent HPLF of degree $2q$ for all possible trajectories of $w(t) \in \mathcal{W}$ if and only if there exists a lower-triangular matrix $P = \begin{bmatrix} P_{11} & 0_{d_q \times d_{q-1}n_x} \\ P_{21} & P_{22} \end{bmatrix}$, where $P_{11} \in \mathbb{R}^{d_q \times d_q}$, $P_{21} \in \mathbb{R}^{d_q \times d_{q-1}n_x}$, $P_{22} \in \mathbb{R}^{d_q \times d_{q-1}n_x}$, and $\Gamma(\gamma(w)) \in \mathcal{P}$, defined by (11), such that $P_{11} > 0$ and $\mathcal{A}_{[q]}(w(t))^T P + P \mathcal{A}_{[q]}(w(t)) + Bdiag\{\sum_{j=1}^{d_p} \gamma_j \cdot \mathcal{A}_{ij} \} < 0$ for all possible trajectories of $w(t) \in \mathcal{W}$.

**Proof:** The LMI conditions of this theorem derive from the strict LMI admissibility conditions proposed by Ishihara and Terra (2002). In fact, Ishihara and Terra (2002) demonstrated that a descriptor system $EX = AX$ is admissible if and only if there exist a positive-definite matrix $P$ and a matrix $Q$ such that $HEA^T + EQQ^T < 0$ where $E_0$ is the orthogonal complement matrix of $E$ i.e. such that $E_0^T E_0 = 0$. Note that the Lyapunov function is $V(X) = X^T E^T E X$. When the system is in the SVD form, $E = Bdiag\{t_{11}, \ldots, t_{kn} \}$ and then $E_0 = \begin{bmatrix} 0_{n \times n} & t_{11} \end{bmatrix}^T$. Using the notation $P = PE + E_0 Q$, it is straightforward to conclude that the above necessary and sufficient condition is equivalent to the existence of a block-partitioned matrix $P = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix}$ such that $P_{11} > 0$ and $A^T P + PA < 0$. The associated Lyapunov matrix is $V(X) = X^T E^T E X$.

This result can easily be generalised to the quadratic admissibility analysis of parameter-dependent systems. Therefore, the application of this generalisation to the polytopic parameter-dependent extended descriptor system (14) leads to the LMI admissibility conditions of our theorem with $\Gamma(\gamma) = 0_{d_q}$. Considering the block structure of the matrix $P$ and the definition of the derivative matrix of descriptor system (14), it follows that the Lyapunov function is $V_{2q} = x_{[q]}^T(t) P_{11} x_{[q]}(t)$. Therefore, the parameterisation $\Gamma(\gamma)$ of the linear space (11) associated to $P_{11}$ is exploited in the admissibility analysis inequality condition in order to introduce complementary degrees of freedom leading to less conservative results. Since the system state matrix is parameter-dependent, it follows that the complete parameterisation $\Gamma(\gamma(w))$ is also parameter-dependent i.e. a different parameterisation may be used for each value of the uncertain parameter in the parametric domain.

This last theorem leads to the following easily numerically tractable conditions.

**Theorem 3.3:** The polytopic parameter-dependent descriptor system (1) is robustly admissible with a parameter-independent HPLFs of degree $2q$ for all possible trajectories of $w(t) \in \mathcal{W}$ if and only if there exist a lower-triangular matrix $P = \begin{bmatrix} P_{11} & 0_{d_q \times d_{q-1}n_x} \\ P_{21} & P_{22} \end{bmatrix}$, where $P_{11} \in \mathbb{R}^{d_q \times d_q}$, $P_{21} \in \mathbb{R}^{d_q \times d_{q-1}n_x}$, $P_{22} \in \mathbb{R}^{d_q \times d_{q-1}n_x}$, and scalars $\gamma_j, i = 1, \ldots, N$, $j = 1, \ldots, d_p$, satisfying the following LMI conditions:

$$
\begin{cases}
P_{11} > 0 \\
\mathcal{A}_{[q]}^T P + P \mathcal{A}_{[q]} + \sum_{j=1}^{d_p} \gamma_j \cdot Bdiag\{P_{0j}, 0_{d_{q-1}n_x} \} < 0
\end{cases}
$$

for $i = 1, \ldots, m$. 


where matrices $P_{ij}$, for $j = 1, \ldots, d_P$, are a base of the linear space (11) (cf. Definition 2.5).

**Proof:** The LMI conditions of this theorem are derived from Lemma 3.2 by using the multiconvexity principle and the notations introduced in Remark 3.1, and by parameterising $\Gamma(\gamma(w))$ in Lemma 3.2 as $\Gamma(\gamma(w)) = \sum_{i=1}^{m} w_i \sum_{j=1}^{d_P} P_{ij} P_{ij}$.

**Remark 3.2:** Note that the triangular structure of the Lyapunov matrix $P$ employed by Lemma 3.2 and Theorem 3.3 is not restrictive. Indeed, as shown in the proof of Lemma 3.2, the matrix $P$ is obtained as $P_{ij} + \eta_{ij} P_{ij}$ where $P$ is the Lyapunov matrix in (17), $\eta_{ij}$ is the derivative of the extended descriptor system (14), $\eta_{ij}$ is the orthogonal complement matrix of $\eta$ and $\omega$ is an arbitrary matrix in $\mathbb{H}^{d_q \times n \times d_q \times n}$. Therefore, the triangular matrix $P$ includes the degree of freedom related to the use of orthogonal complement of the derivative matrix.

The representation (17) of the Lyapunov function is not unique and its complete square matricial representation (CSMR) is given by

$$ V_2 q = \begin{bmatrix} x^q[t] \backslash x^{q-1}[t] \otimes \xi(t) \end{bmatrix}^T \cdot \left[ \begin{array}{c} \phi^T P \cdot (w(t)) \cdot \phi \cdot B \text{diag}(\Gamma(\gamma), 0_{d_q \times n_q}) \cdot [x^{q-1}[t] \otimes \xi(t)] \end{array} \right] $$

(18)

where matrix $\Gamma(\gamma)$ belongs to the linear space $\mathcal{P}$ defined by (11) in Definition 2.5.

### 3.2 Admissibility analysis with affine parameter-dependent HPLFs

The previous section presented robust admissibility analysis results for polytopic parameter-dependent systems descriptor systems subject to arbitrarily fast parameters. However, for many parameter-dependent physical systems, there exists some information about the parameters’ velocities. In the following, we elaborate robust admissibility conditions that exploit the information available on the parameters variation rate, expressed as the polyhedral convex set $\mathcal{V}$ in (3), in order to provide less conservative results. We exploit two different convexification techniques to obtain numerically tractable conditions.

**Definition 3.2 (Polytopic parameter-dependent homogeneous polynomial Lyapunov functions – PPD-HPLFs):** A PPD-HPLF of degree $2q$ for the uncertain descriptor system (1) is given by:

$$ V_2 q = \begin{bmatrix} x^q[t] \backslash x^{q-1}[t] \otimes \xi(t) \end{bmatrix}^T \cdot \left[ \begin{array}{c} \phi^T P(w(t)) \cdot \phi \cdot B \text{diag}(\Gamma(\gamma), 0_{d_q \times n_q}) \cdot [x^{q-1}[t] \otimes \xi(t)] \end{array} \right] $$

(19)

where $P(w(t)) \in \mathbb{H}^{(d_q + d_q \times n_q) \times (d_q + d_q \times n_q)}$ and

$$ P(w(t)) = \sum_{i=1}^{m} w_i(t) P_i, \quad w(t) \in \mathcal{W} \quad \text{and} \quad \dot{w}(t) \in \mathcal{V}. $$

(20)

**Remark 3.3:** From the polytopic definition of $P(w(t))$ and since $\dot{w}(t) \in \mathcal{V}$ given in (3)–(4), it follows that:

$$ P(\dot{w}(t)) = \sum_{i=1}^{m} \dot{w}_i(t) P_i = \sum_{i=1}^{m} \sum_{j=1}^{d_P} \sigma_i(t) d_{ij} P_i = \sum_{i=1}^{h} \sigma_i(t) (d_{ij} P_i). $$

(21)

**Definition 3.3 (CSMR of PPD-HPLFs for descriptor systems):**

The CSMR associated to (19) is given by

$$ V_2 q = \begin{bmatrix} x^q[t] \backslash x^{q-1}[t] \otimes \xi(t) \end{bmatrix}^T \cdot \left[ \begin{array}{c} \phi^T P(w(t)) \cdot \phi \cdot B \text{diag}(\Gamma(\gamma), 0_{d_q \times n_q}) \cdot [x^{q-1}[t] \otimes \xi(t)] \end{array} \right]. $$

(22)

As stated in Definition 2.5, matrix $\Gamma(\gamma)$ is the span of the set $\mathcal{P}$ defined by (11).

**Remark 3.4:** In the work (Barbosa et al., 2013, Theorem 1), some necessary and sufficient conditions for the admissibility analysis of linear time-varying (LTV) descriptor systems have been presented. Note that these conditions are necessary and sufficient only for the existence of a quadratic Lyapunov function. In the present paper, we assess the admissibility of a parameter-dependent system by providing necessary and sufficient conditions for the existence of Lyapunov functions belonging to a larger class, namely the class of homogeneous polynomial Lyapunov functions. This class of Lyapunov functions is universal for uncertain systems, and its use might lead to less conservative robust admissibility results.

Our next lemma extends the results presented in Lemma 3.2 to PPD-HPLFs by exploiting the information available on the parameters variation rate. In this lemma, the degree of freedom related to the orthogonal complement of the derivative matrix $\eta$ appears explicitly in the formulation of the inequality conditions while, in Lemma 3.2, this degree of freedom is hidden in the $(2, 2)$-block of the Lyapunov matrix (see Remark 3.2).

**Lemma 3.4:** The uncertain descriptor system (1) subject to time-varying parameters with bounded rates of variation is robustly admissible with a PPD-HPLF of degree $2q$ if and only if one of the following equivalent sets of conditions is satisfied:

(i) There exist matrices $P(w(t)), Q(w(t)) \in \mathbb{H}^{(d_q + d_q \times n_q) \times (d_q + d_q \times n_q)}$ and $\Gamma(\gamma(w, \dot{w}))$ such that

$$ \begin{cases} P(w(t)) > 0 \\ He[\eta^T d_{ij} P(w(t)) \phi + \eta Q(w(t))] \\ +\phi^T d_{ij} P(w(t)) \phi + B \text{diag}(\Gamma(\gamma(w, \dot{w})), 0_{d_q \times n_q}) < 0 \end{cases} $$

for all $w(t) \in \mathcal{W}$ and $\dot{w}(t) \in \mathcal{V}$. 

(23)
There exist matrices $P(w(t))$, $F(w(t))$ and $G(w(t)) \in Σ((d_4+d_1−1)n_1) × (d_4+d_1−1)n_2$, $Q(w(t)) \in Σ((d_1−1)n_1) × (d_1−1)n_2$ and $Γ(γ(w, w))$ such that

$$
\begin{bmatrix}
P(w(t)) + P(w(t)) \epsilon^T + He[F(w(t)) \epsilon^T] + B\text{diag}[Γ(γ(w, w)), 0_{d_4−1n_1}] \\
\epsilon^T P(w(t)) + He[G(w(t))] - F^T(w(t))
\end{bmatrix} < 0,
$$

(24)

for all $w(t) \in W$ and $\dot{w}(t) \in V$.

Matrix $E_{\gamma} \in Σ((d_4+d_1−1)n_1) × (d_4+d_1−1)n_2$ is the orthogonal complement of $\epsilon$ while $A_{\gamma}[q](w(t))$ is defined in Lemma 3.1. The Lyapunov function associated to the above sets of conditions is given by (19).

**Proof:** (i): Following the lines of the proof of Lemma 3.1, the admissibility of a descriptor system by means of a HPLF of degree $2q$ and of a given parameter dependence is equivalent to the admissibility of its associated extended system by means of a LF that is quadratic in its extended state vector and that presents the same parameter dependence of the initial HPLF. This is due to the fact the LFs certifying the admissibility of both systems are identical. This means that the uncertain descriptor system (1) is admissible with a PPD-HPLF of degree $2q$ if and only if its extended system is admissible with a polytopic parameter-dependent LF that is quadratic on the states of the extended system.

Therefore, we look for a polytopic parameter-dependent LF candidate that is positive-definite and whose time derivative along the trajectories of the extended system (14) is negative-definite. Positive-definiteness of LF is verified in the first inequality of (i), while the second inequality of (i) results from the negativity condition of the LF time-derivative and the equality

$$
\begin{bmatrix}
\dot{\alpha}^{(1)}(t)

\dot{\alpha}^{(2)}(t)

\dot{\alpha}^{(1)}(t) \otimes \dot{\alpha}^{(2)}(t)
\end{bmatrix} = 0.
$$

Equivalences: (i) → (ii): Consider the second strict inequality of (23). Then, there exists a small scalar $\varepsilon$ such that

$$
\text{left side of inequality (23)} \leq -\frac{\varepsilon}{2} A_{\gamma}[q]^T(w(t)) A_{\gamma}[q](w(t)).
$$

This last inequality is equivalent, by Schur complement, to the second inequality in (24) when $F$ and $G$ are chosen as $F = \epsilon^T P(w(t)) + Q^T(w(t)) \epsilon^T$ and $G = \epsilon \cdot I_{d_4−1n_1}$.

(ii): Left- and right-multiplying the second inequality in (24) by $I_{d_4−1n_1} A_{\gamma}[q]^T(w(t))$ and its transpose, respectively, leads to the second inequality of (23).

**Remark 3.5:** Some complementary degrees of freedom associated to the conditions (i) and (ii) of Lemma 3.4 can be highlighted. The first of them is related to the CMSR described in Definition 2.5, caused by the power transformation of the state vectors. The second of them is related to the rank-deficiency of the matrix $E_{\gamma}$, and it can be exploited by the orthogonal complement of this matrix. The orthogonal complement has also been used in Barbosa et al. (2013, Theorem 1).

**Remark 3.6:** The negative-definiteness of the derivative of the Lyapunov function can accommodate the so-called slack variables, that appear through the use of the projection lemma and are referred to as $F(w(t))$ and $G(w(t))$ in (24). When the infinite amount of conditions to be observed (i.e. inequality (24) must be verified for every possible values of parameters and parameters rate of variation) is brought to a finite amount of conditions due to a certain choice on the structure of the unknown matrices, the slack variables might provide extra degrees of freedom, leading to less conservative results.

The admissibility conditions presented in Lemma 3.4 might lead to nonlinear matrix inequalities. In the following, two different structures of the involved slack variables are chosen in order to obtain new less restrictive LMI admissibility conditions for continuous-time uncertain descriptor systems subject to parameters with bounded rates of variation. The first structure consists of constant slack variables and is presented in the theorem below.

**Theorem 3.5:** The parameter-dependent uncertain descriptor system (1) subject to parameters with bounded rates of variation is robustly admissible based on a PPD-HPLF of degree $2q$ if there exist matrices $P_i > 0$, $Q_i = 1 \ldots m$ of the same dimensions as in Lemma 3.4, and a set of vectors $\gamma^{(i)} \in Σ^d P$, $i = 1 \ldots m$, $l = 1 \ldots h$, satisfying the LMI conditions below:

$$
\begin{bmatrix}
\epsilon^T P_i \epsilon^T + He[F_i \epsilon^T] + B\text{diag}[Γ(γ^{(i)}), 0_{d_4−1n_1}] \\
P_i \epsilon^T + \delta_0 Q_l + G\epsilon^T - F^T
\end{bmatrix} < 0,
$$

(25)

for $i = 1 \ldots m$ and $l = 1 \ldots h$. Matrices $A_{\gamma}[q]$ are the vertices of $A_{\gamma}[q](w)$, as referred in Remark 3.1.

**Proof:** Condition (25) is directly derived from (24) through the particular choice of constant slack variables $F(w(t)) = F$ and $G(w(t)) = G$, and by considering polytopic realisations of matrices $P(w(t))$, $F(w(t))$, and $Q(w(t))$:

$$
Q(w(t)) = \sum_{i=1}^{m} w_i(t) Q_i.
$$

We recall that the term $Γ(γ(w, w))$ in (24) spans a linear space, and that the dependence of $γ(w, w)$ with respect to the parameters and parameter’s velocities can be conveniently chosen in such a way that the term $Γ(γ(w, w))$ in (25) is convex with respect to these both independent sets: $Γ(γ(w, w)) = \sum_{i=1}^{m} \sum_{l=1}^{h} w_i(t) \Gamma(γ^{(i)})$.

**Remark 3.7:** Theorem 3.5 covers the results provided in Barbosa et al. (2013) that correspond to the particular case $q = 1$.

Our next theorem considers parameter-dependent slack variables.

**Theorem 3.6:** The uncertain descriptor system (1) subject to parameters with bounded rates of variation is robustly admissible
based on a PPD-HPLF of degree $2q$ if there exist matrices $P_i > 0$, $F_i$, $G_i$, and $Q_i$, $i = 1, \ldots, m$, and sets of vectors

$$\phi^{(i)} \in \Re^{d_p}, \quad \text{for } i = 1, \ldots, m, l = 1, \ldots, h$$

$$\Delta^{(ij)} \in \Re^{d_p}, \quad \text{for } i, j = 1, \ldots, m, i \neq j, l = 1, \ldots, h$$

$$\Omega^{(ijk)} \in \Re^{d_p}, \quad \text{for } i = 1, \ldots, m - 2, j = i + 1, \ldots, m - 1, k = j + 1, \ldots, m, l = 1, \ldots, h$$

such that the LMI conditions (27) are satisfied.

$$\begin{bmatrix}
   \varepsilon^T P(d^{(i)}) e^c + He[F_i e^{(i)}] + Bdiag\{\Gamma(\phi^{(i)}), 0_{d_q - 1, n_q}\} \\
   P_i e^c + \delta_0 Q_i + G_i e^{(i)} - F_i^T \\
   -He[G_i]
\end{bmatrix} < - \begin{bmatrix}
   I \\
   0
\end{bmatrix},$$

for $i = 1, \ldots, m, l = 1, \ldots, h$.

$$\begin{bmatrix}
   3 \varepsilon^T P(d^{(i)}) e^c + He[F_i e^{(i)}] + F_j e^{(i)} + F_i e^{(i)} + Bdiag\{\Gamma(\Delta^{(ij)}), 0_{d_q - 1, n_q}\} \\
   2(P_i e^c + \delta_0 Q_i - F_i^T) + (P_j e^c + \delta_0 Q_j - F_j^T) + G_i e^{(i)} + G_j e^{(i)} + G_k e^{(i)} \\
   -He[2G_i + G_j]
\end{bmatrix} < \frac{1}{(m - 1)^2} \begin{bmatrix}
   I \\
   0
\end{bmatrix},$$

for $i = 1, \ldots, m, j = 1, \ldots, m, j \neq i, l = 1, \ldots, h$.

$$\begin{bmatrix}
   6 \varepsilon^T P(d^{(i)}) e^c + He[F_i e^{(i)}] + F_j e^{(i)} + F_k e^{(i)} + Bdiag\{\Gamma(\Omega^{(ijk)}), 0_{d_3 - 1, n_3}\} \\
   2(P_i e^c + \delta_0 Q_i - F_i^T) + (P_j e^c + \delta_0 Q_j - F_j^T) + (P_k e^c + \delta_0 Q_k - F_k^T) + G_i e^{(i)} + G_j e^{(i)} + G_k e^{(i)} \\
   -2He[2G_i + G_j + G_k]
\end{bmatrix} < \frac{6}{(m - 1)^2} \begin{bmatrix}
   I \\
   0
\end{bmatrix},$$

for $i = 1, \ldots, m - 2, j = i + 1, \ldots, m - 1, k = j + 1, \ldots, m, l = 1, \ldots, h$.

**Proof:** Consider the polytopic definition of the system (1) and of matrices $P(w(t))$ in (20), $P(w(t))$ in (21), $Q(w(t))$ in (26), and $F(w(t))$, $G(w(t))$, respectively, as $F(w(t)) = \sum_{i=1}^{m} w_i(t)F_i$ and $G(w(t)) = \sum_{i=1}^{m} w_i(t)G_i$ for $w(t) \in \mathcal{V}$ and $w(t) \in \mathcal{V}$.

Using Lemma 3.4, system (1) is robustly admissible with a PPD-HPLF of degree $2q$ if and only if (24) holds. Once we replace the polytopic definitions of the matrices above in (24), we obtain a matrix inequality composed by terms with polynomial dependence on $w$ of degrees 0, 1 or 2, and of polynomial dependence on $\sigma$ of degrees 0 or 1. Inspired by the technique employed in Leite and Peres (2003) and Oliveira et al. (2008), we cast every term of the matrix inequality into a form with polynomial dependence of degree 3 in $w$ and of degree 1 in $\sigma$.

The matrix $\Gamma(\gamma)$ involved in the structure of the PPD-HPLF in (22) has no predefined dependence on the parameter vector $w$. A convenient parameterisation of $\gamma$ in order to provide one independent realisation of $P$ for every LMI condition to be generated from inequality (24) is given by:

$$\Gamma(\gamma(w, w)) = \sum_{i=1}^{h} \sum_{j=1}^{m} \sigma_i w_i^3 \Gamma(\phi^{(i)})$$

$$+ \sum_{i=1}^{h} \sum_{j=1}^{m} \sum_{j \neq i}^{m} \sigma_i w_i^2 w_j \Gamma(\Delta^{(ij)})$$

$$+ \sum_{i=1}^{h} \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{j \neq i}^{m} \sigma_i w_i w_j w_k \Gamma(\Omega^{(ijk)})$$.

With the recasting of each term and the defined structure of $\gamma$, the left side of (24) is given as in (28).

left-hand side of (24)

$$= \sum_{i=1}^{h} \sigma_i \sum_{j=1}^{m} w_j^3 \begin{bmatrix}
   \varepsilon^T P(d^{(i)}) e^c + He[F_i e^{(i)}] + Bdiag\{\Gamma(\phi^{(i)}), 0_{d_q - 1, n_q}\} \\
   P_i e^c + \delta_0 Q_i + G_i e^{(i)} - F_i^T \\
   -He[G_i]
\end{bmatrix}$$

$$+ \sum_{i=1}^{h} \sigma_i \sum_{j=1, j \neq i}^{m} \sum_{j=1, j \neq i}^{m} w_i w_j \begin{bmatrix}
   3 \varepsilon^T P(d^{(i)}) e^c + He[F_i e^{(i)}] + F_j e^{(i)} + F_i e^{(i)} + Bdiag\{\Gamma(\Delta^{(ij)}), 0_{d_q - 1, n_q}\} \\
   2(P_i e^c + \delta_0 Q_i - F_i^T) + (P_j e^c + \delta_0 Q_j - F_j^T) + G_i e^{(i)} + G_j e^{(i)} + G_k e^{(i)} \\
   -He[2G_i + G_j]
\end{bmatrix}$$

$$+ \sum_{i=1}^{h} \sigma_i \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^{m} w_i w_j w_k \begin{bmatrix}
   6 \varepsilon^T P(d^{(i)}) e^c + He[F_i e^{(i)}] + F_j e^{(i)} + F_k e^{(i)} + Bdiag\{\Gamma(\Omega^{(ijk)}), 0_{d_3 - 1, n_3}\} \\
   2(P_i e^c + \delta_0 Q_i - F_i^T) + (P_j e^c + \delta_0 Q_j - F_j^T) + (P_k e^c + \delta_0 Q_k - F_k^T) + G_i e^{(i)} + G_j e^{(i)} + G_k e^{(i)} \\
   -2He[2G_i + G_j + G_k]
\end{bmatrix}.$$
Whenever inequalities (27) hold, it follows that the left-hand side of (24), expressed as in (28), satisfies inequality (29). As detailed in Leite and Peres (2003a) and Leite and Peres (2003b), the scalar within the parentheses of the right side of inequality (29) can be rewritten as a polynomial sum of squares (SOS). This polynomial SOS is positive semi-definite, meaning that the right side of inequality (29) is negative semi-definite. Therefore, the left-hand side of (24) is negative definite i.e. (24) is satisfied, and the system (1) is admissible.

\[
\frac{\varepsilon^T P(w(t)) \varepsilon + \text{He}[F(w(t)) \mathcal{A}(q)(w(t))]}{+ \text{Bdiag}(\Gamma(\gamma, w), 0_{d_{k-1};n})} \geq 0\]

\[
\frac{P(w(t)) \varepsilon + \delta_0 Q(w(t))}{+ G(w(t)) \mathcal{A}(q)(w(t)) - F^T(w(t))} \geq -\text{He}[G(w(t))]
\]

\[
\left( - \sum_{l=1}^{h} a_l \left( \sum_{j=1}^{m} \frac{1}{w_j^2} - \sum_{j=1; j \neq i}^{m} \frac{1}{w_j^2} \right) - \frac{6}{(m-1)^2} \sum_{i=1}^{m-2} \sum_{j=i+1}^{m} \sum_{k=j+1}^{m} \frac{1}{w_i w_j w_k} \right) \left( I \quad 0 \right)
\]

(29)

\[\begin{bmatrix} 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 1 \\ -1 & -2 & -4 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \]

and

\[\begin{bmatrix} -2 \kappa & 1 & -\kappa & 0.5 \\ \kappa & -10 \kappa & 1 + 3 \kappa & 1 + \kappa \\ -1 + 3 \kappa & -2 - 4 \kappa & -4 - 2 \kappa & 0 \\ 1 - \kappa & 0 & 0 & 1 + 0.2 \kappa \end{bmatrix} \]

Since the parameter \( \alpha(t) \) variation rate is arbitrary, only parameter-independent LF can be used for admissibility analysis. Hence, the admissibility property has to be guaranteed for all possible, arbitrarily fast, trajectories of \( w(t) = \begin{bmatrix} w_1(t) & w_2(t) \end{bmatrix}^T \) in \( \mathcal{V} \).

The robust admissibility of this system is studied in the following using the result proposed in Theorem 3.3 for different degrees of HPLFs. For the sake of illustration, in the case of HPLFs of degree 4 (i.e. \( q = 2 \)), \( x^{[2]} \) is of dimension \( d_q = 6 \) and equals \( \begin{bmatrix} x_1^2 & x_2 x_1 & x_3 x_1 & x_2^2 & x_3 x_2 \end{bmatrix}^T \); the dimension of the linear space (11) is \( d_{P} = 6 \) and the parameterisation of the HLPF of degree 2 is

\[ P_0(\gamma) = \begin{bmatrix} 0 & 0 & 0 & -\gamma_1 & -\gamma_2 & -\gamma_3 \\ 0 & 2 \gamma_1 & \gamma_2 & 0 & -\gamma_4 & -\gamma_5 \\ 0 & \gamma_2 & 2 \gamma_3 & \gamma_4 & \gamma_5 & 0 \\ -\gamma_1 & 0 & \gamma_4 & 0 & 0 & -\gamma_6 \\ -\gamma_2 & -\gamma_4 & \gamma_5 & 0 & 2 \gamma_6 & 0 \\ -\gamma_3 & -\gamma_5 & 0 & -\gamma_6 & 0 & 0 \end{bmatrix} \]

Matrices \( M_i \) and \( N_i \), for \( i = 1, \ldots, d_{q-1} \) where \( d_{q-1} = 3 \), in Lemma 2.1, are given by

\[ M_1 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ N_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

When the parameter \( \alpha \) is uncertain and constant, by using the root-locus of the finite modes of the descriptor system, we obtain the largest value of the bound \( \kappa \) for which the system is admissible as 16.70083. When the parameter \( \alpha \) is time-varying with arbitrary variation rate, quadratic LFs (\( q = 1 \)) guarantee the system admissibility for all parameter trajectories when \( \kappa \leq 4.6667 \). Theorem 3.3 proves, based on HLPFs of degree 4

**Remark 3.8:** Remark 3.6 also covers results proposed in the literature. When using a quadratic Lyapunov function (i.e. \( q = 1 \)) and disregarding the degrees of freedom associated to the CSMR (i.e. \( \gamma = 0 \)) for the stability analysis of a standard continuous-time uncertain system (e.g. \( n_k = 0, E = I_n \)), our conditions cover the ones found in Leite and Peres (2003a).

The ensuing section brings numerical results that show reduction in the conservatism when using the proposed LMI conditions.

### 4. Numerical examples

In this section, numerical examples are proposed to evaluate the performance of the LMI conditions of Theorems 3.3, 3.5 and 3.6. First, we show that parameter-independent HPLFs can provide, through the LMI conditions of Theorem 3.3, less conservative admissibility analysis results for uncertain descriptor systems with arbitrary parameters variation rate. Then, we compare the admissibility analysis LMI conditions of Theorems 3.5 and 3.6 with results available in the literature for uncertain descriptor systems subject to different bounds on parameters’ velocities. In addition, we evaluate the influence of the degree of PPD-HPLFs on the performance of our new LMI results.

#### 4.1 Admissibility analysis using parameter-independent HPLFs

Consider the descriptor system defined by state matrices \( E = B \text{diag}(I_3, 0) \) and

\[ A(\alpha(t)) = \begin{bmatrix} -2 \alpha(t) & 1 & -\alpha(t) & 0.5 \\ \alpha(t) & -10 \alpha(t) & 1 + 3 \alpha(t) & 1 + \alpha(t) \\ -1 + \alpha(t) & -2 - 4 \alpha(t) & -4 + 2 \alpha(t) & 0 \\ 1 - \alpha(t) & 0 & 0 & 1 + 0.2 \alpha(t) \end{bmatrix} \]

where the parameter \( \alpha(t) \) is bounded, \( 0 \leq \alpha(t) \leq \kappa \), and has an arbitrary variation rate. The number of dynamic and algebraic states is \( n = 3 \) and, respectively, \( n_k = 1 \). This system can be rewritten in the polytopic parameter-dependent form (1) with \( m = 2 \) parameters, \( w_1(t) = \alpha(t)/\kappa \) and \( w_2(t) = 1 - \alpha(t)/\kappa \), and vertex state matrices

\[ A^{(1)} = \begin{bmatrix} 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 1 \\ -1 & -2 & -4 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \]

\[ A^{(2)} = \begin{bmatrix} -2 \kappa & 1 & -\kappa & 0.5 \\ \kappa & -10 \kappa & 1 + 3 \kappa & 1 + \kappa \\ -1 + 3 \kappa & -2 - 4 \kappa & -4 - 2 \kappa & 0 \\ 1 - \kappa & 0 & 0 & 1 + 0.2 \kappa \end{bmatrix} \]
(q = 2) and 6 (q = 3), that the admissibility property is preserved when \( \kappa \leq 16.3943 \) and, respectively, when \( \kappa \leq 16.7008 \). This shows a significant improvement, even for a slight augmentation of the HPLF degree, of the largest value of the bound \( \kappa \) for which the system is admissible. Therefore, this indicates that the HPLF approach proposed in this paper allows to obtain less conservative admissibility analysis results when a suitable polynomial degree 2q is chosen.

### 4.2 Admissibility analysis using parameter-dependent HPLFs

In order to provide a numerical comparison between our new results and results available in the literature, the following two systems are considered.

**System 1:** Consider \( E = B \text{diag} \{I_2, 0\} \),

\[
A(r(t)) = \begin{bmatrix}
-5 + 2r(t) & -2 + r(t) & 0.1r(t) \\
2 & -r(t) & 0 \\
0 & 10r(t) & 1 + 0.1r(t)
\end{bmatrix},
\]

where \( 0 \leq r(t) \leq \kappa \) and \( |\dot{r}(t)| \leq \beta \),

recast in the polytopic parameter-dependent form (1)–(2) with \( m = 2 \) parameters, \( w_1(t) = r(t)/\kappa \) and \( w_2(t) = 1 - r(t)/\kappa \), and vertex state matrices

\[
A^{(1)} = \begin{bmatrix}
-5 & -2 & 0 \\
2 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

\[
A^{(2)} = \begin{bmatrix}
-5 + 2\kappa & -2 + \kappa & 0.1\kappa \\
2 & -\kappa & 0 \\
0 & 10\kappa & 1 + 0.1\kappa
\end{bmatrix}.
\]

**System 2:** Consider \( E = B \text{diag} \{I_2, 0\} \),

\[
A(r(t)) = \begin{bmatrix}
0 & 1 & 0.2 \\
-6 - r(t) & -1 & 0 \\
0.1 & 0.2 & 1 + 0.1r(t)
\end{bmatrix},
\]

where \( -\kappa \leq r(t) \leq \kappa \) and \( |\dot{r}(t)| \leq \beta \),

reformulated in the polytopic parameter-dependent form (1)–(2) with \( m = 2 \) parameters, \( w_1(t) = r(t)/\kappa \) and \( w_2(t) = 1 - r(t)/\kappa \), and vertex state matrices

\[
A^{(1)} = \begin{bmatrix}
0 & 1 & 0.2 \\
-6 + \kappa & -1 & 0 \\
0.1 & 0.2 & 1 - 0.1\kappa
\end{bmatrix},
\]

\[
A^{(2)} = \begin{bmatrix}
0 & 1 & 0.2 \\
-6 - \kappa & -1 & 0 \\
0.1 & 0.2 & 1 + 0.1\kappa
\end{bmatrix}.
\]

For both systems, the number of dynamic and algebraic states is \( n = 2 \) and, respectively, \( n_2 = 1 \). From \( |\dot{r}(t)| \leq \beta \), it follows that \( |w_1(t)| \leq \beta/\kappa \) and \( |w_2(t)| \leq \beta/\kappa \). Then, the set of all admissible parameters trajectories is given by \( w(t) = [w_1(t) \ w_2(t)]^T \in \mathcal{W} \) along with \( \dot{w}(t) \in \mathcal{V} \) where the polyhedral convex set \( \mathcal{V} \) is defined by (3) with \( h = 2 \) and \( \{q^{(1)}, q^{(2)}\} = \left\{ \begin{bmatrix} -\beta/\kappa \\ \beta/\kappa \end{bmatrix} \right\} \). For the sake of illustration, since \( n = 2 \), if we consider HPLFs of degree 4 (q = 2) then \( d_q = 3 \) and \( d_{q-1} = 2 \). The power transformation of degree 2 of \( x \) is \( x^{[2]} = [x_1^2 \ x_1 x_2 \ x_2^2]^T \) and Lemma 2.1 holds with

\[
[M_1 \ | \ M_2] = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix},
\]

\[
[N_1 \ | \ N_2] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

For HPLFs of degree 6 (q = 3), we have \( d_q = 4 \), \( d_{q-1} = 3 \), \( x^{[3]} = [x_1^3 \ x_1 x_2^2 \ x_1 x_2 \ x_2^3]^T \) and the matrices \( M_i \) and \( N_i \) in Lemma 2.1 are given by

\[
[M_1^T \ | \ M_2^T] = \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0
\end{bmatrix},
\]

\[
[N_1 \ | \ N_2] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

We are interested in finding, for different values of \( \beta \), the largest value \( \kappa^{*}_q(\beta) \) of \( \kappa \) for which the uncertain parameter-dependent descriptor Systems 1 and 2 are robustly admissible in the uncertainty domain based on HPLFs of degree 2q.

#### 4.2.1 Evaluation of \( \kappa^{*}_q(\beta) \) in comparison with results from the literature

The values of \( \kappa^{*}_q(\beta) \) for Systems 1 and 2 have been assessed using our LMI conditions given in Theorems 3.3, 3.5 and 3.6, and the ones presented in the work of Bara (2011, Theorem 3.1) and Barbosa et al. (2013, Lemma 3). We recall that the technique provided in the paper of Barbosa et al. (2013, Lemma 3) is covered by Theorem 3.5 for the choice \( q = 1 \) (quadratic LF). The results of our evaluation are reported in Tables 1 and 2.

From Tables 1 and 2, we notice that LMI conditions of Theorems 3.5 and 3.6 lead to larger values of \( \kappa^{*}_q(\beta) \) than the techniques proposed in the literature. This is due to the fact that the result proposed in Barbosa et al. (2013) is covered by Theorem 3.5 while the condition proposed in Barbosa (2011) uses less degrees of freedom related to the use of slack variables than this latter reference. Note that Theorem 3.3 also leads to larger values of \( \kappa^{*}_q(\beta) \) for System 1 with \( \beta = 50 \) and for System 2 with \( \beta = 100 \).

Throughout Tables 1 and 2, we remark that, for a fixed degree of Lyapunov function, sometimes Theorem 3.5 gives better results than Theorem 3.6, other times the converse holds. This is due to the fact that Theorems 3.5 and 3.6, as the results
Theorem 3.6

Barbosa et al. (2013, Lemma 3) 5.0304 2.5897
Bara (2011, Theorem 3.1) 5.0139 2.5896

Table 2. $\kappa_{2q}^*(\beta)$ for System 2.

| $\kappa_{2q}^*(5)$ | $\kappa_{2q}^*(100)$ |
|-------------------|---------------------|
| Barbosa et al. (2013, Lemma 3) | 5.0304 25897 |

Theorem 3.3

$q = 1$ 2.4434
$q = 2$ 2.899
$q = 3$ 3.107
$q = 4$ 3.132
$q = 5$ 3.179
$q = 6$ 3.195
$q = 7$ 3.206

Theorem 3.5

$q = 1$ 2.4434
$q = 2$ 2.9097
$q = 3$ 3.1074
$q = 4$ 3.1449
$q = 5$ 3.1855
$q = 6$ 3.2022
$q = 7$ 3.213

Theorem 3.6

$q = 1$ 2.4471
$q = 2$ 2.9182
$q = 3$ 3.1101
$q = 4$ 3.1496
$q = 5$ 3.1853
$q = 6$ 3.2033
$q = 7$ 3.2064

Table 1. $\kappa_{2q}^*(50)$ for System 1.

| $\kappa_{2q}^*(50)$ |
|-----------------|
| Barbosa et al. (2013, Lemma 3) | 2.4434 |
| Barbosa et al. (2013, Lemma 3) | 2.4434 |

Figure 1. System 2: Maximum bound of $\beta$ and $\kappa_{2q}^*$ defining the uncertainty domain for which the admissibility is guaranteed based on Theorem 3.5 for PPD-HPLFs of degree 2q.

4.2.2 Conservativeness evaluation regarding the augmentation of the HPLFs degree

For System 2, the maximum bounds $\beta$ and $\kappa$ for which the admissibility is guaranteed based on Theorem 3.5 with PPD-HPLFs of different degrees, have been computed and plotted in Figure 1. We notice from this figure that the curves of different degrees of PPD-HPLFs overlap for this particular system. For instance, among the performed tests, the PPD-HPLF providing the biggest allowed velocity of parameters when $\kappa_{2q}^* = 4$ is the one of order 14, whilst for $\kappa_{2q}^* = 5$, the best choice of degree for a PPD-HPLF is 6.

As a matter of fact, the results reported in Figure 1 follow the same tendency displayed in the work of Chesi et al. (2007) for some standard uncertain systems ($E = I_n$). Indeed, in Chesi et al. (2007), it is highlighted that, for Lyapunov functions with parameter dependence of any degree higher than or equal to one, a mere augmentation on the degree of the state variables does not necessarily lead to less conservative results. That, once more, highlights one open problem concerning parameter-dependent homogeneous polynomial Lyapunov functions: there is no clear rule for choosing its best suitable degree for a given system and given parameters domain. It also shows that different PPD-HPLFs might provide different sets of necessary conditions, so that PPD-HPLFs of higher degrees do not necessarily span the same space of solutions of PPD-HPLFs of lower degrees. As a matter of fact, the authors of Chesi et al. (2007) state that, for standard uncertain systems, the joint augmentation on the degree of the homogeneous polynomial Lyapunov function with respect to the state and the parameter vectors might lead to less restrictive conditions. However, there is no guarantee that conservatism will decrease for an augmentation of the degree of the power transformation of the state vector alone. This fact has also been confirmed by our experiments in the case of descriptor uncertain systems.
5. Conclusion

The present paper presents new robust admissibility analysis conditions for continuous-time polytopic parameter-dependent uncertain descriptor systems. The case of parameters with unbounded variation rates is addressed by means of parameter-independent HPLFs while the case of parameters with bounded rates of variation is discussed through parameter-dependent HPLFs. For both cases, necessary and sufficient conditions for robust admissibility analysis are given whenever the degree of the HPLF is given. Then, using different convexification techniques, LMI admissibility analysis conditions are derived that cover previous findings in the literature. These conditions are necessary and sufficient for the parameter-independent case whilst only sufficient for the parameter-dependent case. To the best of the authors’ knowledge, the results presented in this paper are the first attempt extending the use of HPLFs developed for standard systems to the case of uncertain descriptor systems. The numerical examples show that the proposed approach leads to less conservative admissibility analysis results.

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

Ana Carolina dos Santos Paulino https://orcid.org/0000-0001-5936-9134

References

Angelis, G. (2001). System analysis, modelling and control with polytopic linear models [PhD thesis], Technische Universität Eindhoven.

Bara, G. I. (2011). Robust analysis and control of parameter-dependent uncertain descriptor systems. Systems & Control Letters, 60(5), 356–364. https://doi.org/10.1016/j.sysconle.2011.03.001

Bara, G. I., & dos Santos Paulino, A. C. (2016). Homogeneous polynomial Lyapunov functions for uncertain systems (Internal Report).

Barbosa, K. A., Coutinho, D., de Souza, C. E., & Rodrigues, C. (2017). Bounded real lemma for discrete linear time-varying descriptor systems. In Proceedings of the 11th asian control conference, Gold Coast, Australia (pp. 1835–1840).

Barbosa, K. A., de Souza, C. E., & Coutinho, D. (2012). Robust stability of discrete-time linear descriptor systems with time-varying uncertainties via parametric Lyapunov function. In Proceedings of the 51st IEEE conference on decision and control, Hawaii, USA (pp. 5134–5139).

Barbosa, K. A., de Souza, C. E., & Coutinho, D. (2013). Robust admissibility and $H_{\infty}$ performance of time-varying descriptor systems Robust admissibility and $H_{\infty}$ performance of time-varying descriptor systems. In 10th IEEE international conference on control and automation, Hangzhou, China (pp. 1138–1143).

Barbosa, K. A., de Souza, C. E., & Coutinho, D. (2018). Admissibility analysis of discrete linear time-varying descriptor systems. Automatica, 91(1), 136–143. https://doi.org/10.1016/j.automatica.2018.01.033

Barkin, A. I., & Zelentsovy, A. L. (1983). Method of power transformations for analysis of stability of nonlinear control systems. Systems & Control Letters, 3(5), 303–310. https://doi.org/10.1016/0167-6911(83)90030-0

Bender, D., & Laub, A. J. (1987). The linear-quadratic optimal regulator for descriptor systems. IEEE Transactions on Automatic Control, 32(8), 672–688. https://doi.org/10.1109/TAC.1987.1104694

Blanchini, F. (1995). Nonquadratic Lyapunov functions for robust control. Automatica, 31(3), 451–461. https://doi.org/10.1016/0005-1098(94)00133-4

Blanchini, F., & Miani, S. (1999). A new class of universal Lyapunov functions for the control of uncertain linear systems. IEEE Transactions on Automatic Control, 44(3), 641–647. https://doi.org/10.1109/9.751368

Brocke, R. W. (1973). Lie algebras and Lie groups in control theory. In D. Q. Mayne and R. W. Brockett (eds.) Geometric methods in system theory. NATO advanced study institutes series (Series C — mathematical and physical sciences), (Vol. 3, pp. 43–82). Reidel.

Chesi, G., Garulli, A., Tesi, A., & Vicino, A. (2003a). Homogeneous Lyapunov functions for systems with structured uncertainties. Automatica, 39(6), 1027–1035. https://doi.org/10.1016/S0005-1098(03)00039-6

Chesi, G., Garulli, A., Tesi, A., & Vicino, A. (2003b). Robust stability of polytopic systems via polynomially parameter-dependent Lyapunov functions. In 42nd IEEE conference on decision and control, Maui, Hawaii, USA (pp. 574–580).

Chesi, G., Garulli, A., Tesi, A., & Vicino, A. (2004). Parameter-dependent homogeneous Lyapunov functions for robust stability of linear time-varying systems. In 43rd IEEE conference on decision and control, Bahamas (Vol. 4, pp. 4095–4100).

Chesi, G., Garulli, A., Tesi, A., & Vicino, A. (2007). Robust stability of time-varying polytopic systems via parameter-dependent homogeneous Lyapunov functions. Automatica, 43(2), 309–316. https://doi.org/10.1016/j.automatica.2006.08.024

Dai, L. (1989). Singular control systems. Lecture Notes in Control and Information Sciences. Springer-Verlag.

Dai, S. L., Dimirovski, G. M., & Zhao, J. (2006). A Descriptor System Approach to Robust $H_{\infty}$ Control and Its Application to Flight Control A descriptor system approach to robust $H_{\infty}$ control and its application to flight control. In 2006 American control conference (pp. 1068–1073).

dos Santos Paulino, A. C., & Bara, G. I. (2017a). Homogeneous polynomial Lyapunov functions for the admissibility analysis of uncertain descriptor systems. In Proceedings of the 56th IEEE conference on decision and control, Melbourne, Australia (pp. 3187–3193).

dos Santos Paulino, A. C., & Bara, G. I. (2017b). New LMI conditions for admissibility analysis of time-varying descriptor systems. 20th IFAC World Congress, 50(1), 15477–15482. https://doi.org/10.1016/j.ifacol.2017.08.2113

Fang, C. H. (2002). Stability robustness analysis of uncertain descriptor systems – an LMI approach. In Proceedings of the 41st IEEE conference on decision and control, Las Vegas, Nevada, USA (pp. 1459–1460).

Geronel, J. C., & Colaneri, P. (2006). Robust stability of time-varying polytopic systems. Systems & Control Letters, 55(1), 81–85. https://doi.org/10.1016/j.sysconle.2004.11.016

Ishihara, J. Y., & Terra, M. H. (2002). On the Lyapunov theorem for singular systems. IEEE Transactions on Automatic Control, 47(11), 1926–1930. https://doi.org/10.1109/TAC.2002.804463

Leite, V. J. S., & Peres, P. L. D. (2003a). An improved LMI condition for robust $D$-stability of uncertain polytopic systems. In 2003 American Control Conference, Denver, Colorado, USA (pp. 833–838).

Leite, V. J. S., & Peres, P. L. D. (2003b). An improved LMI condition for robust $D$-stability of uncertain polytopic systems. IEEE Transactions on Automatic Control, 48(3), 500–504. https://doi.org/10.1109/TAC.2003.809167

Lewis, F. (1986). A survey of linear singular systems. Circuits, Systems, and Signal Processing, 5(1), 3–36. https://doi.org/10.1007/BF01600184

Mills, J., & Goldenberg, A. (1989). Force and position control of manipulation during constrained motion tasks. IEEE Transactions on Robotics and Automation, 5(1), 30–46. https://doi.org/10.1109/70.88015

Montagner, V. F., Oliveira, R. C., Peres, P. L., & Bliman, P. A. (2009). Stability analysis and gain-scheduled state feedback control for continuous-time systems with bounded parameter variations. International Journal of Control, 82(6), 1045–1059. https://doi.org/10.1080/00207170802403750

Oliveira, R. C. L. F., Bliman, P., & Peres, P. L. D. (2008). Robust LMIs with parameters in multi-simplex: Existence of solutions and applications. In Proceedings of the 47th IEEE Conference on Decision and Control, Cancun, Mexico (pp. 2226–2231).

Oliveira, R. C., & Peres, P. L. (2009). Time-varying discrete-time linear systems with bounded rates of variation: Stability analysis and control design. Automatica, 45(11), 2620–2626. https://doi.org/10.1016/j.automatica.2009.07.015
Oliveira, R. C. L. F., & Peres, P. L. D. (2008). Robust stability analysis and control design for time-varying discrete-time polytopic systems with bounded parameter variation. In Proceeding of the 2008 American control conference, Seattle, Washington, USA (pp. 3094–3099).

Ramos, R. A., Bretas, N. G., & Costa Alberto, L. F. (2002). Damping controller design for power systems with polytopic representation of operating conditions. In Proceedings of the IEEE power engineering society winter meeting (Vol. 2, pp. 1517–1521).

Rodriguez, C., Barbosa, K. A., & Coutinho, D. (2018). Robust \( H_\infty \) state-feedback design for discrete-time descriptor systems. Robust \( H_\infty \) state-feedback design for discrete-time descriptor systems. IFAC-PapersOnLine, 51(25), 78–83. https://doi.org/10.1016/j.ifacol.2018.11.085

Zelentsovsky, A. L. (1994). Nonquadratic Lyapunov functions for robust stability analysis of linear uncertain systems. IEEE Transactions on Automatic Control, 39(1), 135–138. https://doi.org/10.1109/9.273350