Delay regulated explosive synchronization in multiplex networks

Ajay Deep Kachhvah$^{1}$ and Sarika Jalan$^{1,2,3}$

$^1$ Complex Systems Lab, Indian Institute of Technology Indore—Simrol, Indore—453552, India
$^2$ Discipline of Biosciences and Biomedical Engineering, Indian Institute of Technology Indore—Simrol, Indore—453552, India
$^3$ Lobachevsky University, Gagarin Avenue 23, Nizhny Novgorod, 603950, Russia

E-mail: sarikajalan9@gmail.com

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Abstract

It is known that explosive synchronization (ES) in an isolated network of Kuramoto oscillators with inertia is significantly enhanced by the presence of time delay. Here we show that time delay in one layer of the multiplex network governs the transition to synchronization and ES in the other layers. We found that a single layer with time-delayed intra-layer coupling may experience a different type of transition to synchronization, e.g. ES or continuous, depending on the values of time delay. Importantly, the same type of transition is incorporated simultaneously in other layer(s) as well, irrespective of the intra-layer delay values. Hence, a suitable choice of time-delay in only one layer of a multiplex network can lead to a desired (either ES or continuous) transition simultaneously in the other layers, either directly or remotely connected to the delayed layer. These results offer a platform for a better understanding of the dynamics of those complex systems which are represented by the multilayered framework and contain time delays in the communication processes.

1. Introduction

In recent years, a novel synchronization process exhibiting hypersensitivity or explosiveness in the natural physical and biological systems, called explosive synchronization (ES), has drawn much attention in the scientific community [1–8]. The epileptic seizures in the brain [9], chronic pain in the Fibromyalgia brain [10], the cascading failure of power grids [11], and the jamming of the internet [12] have shown ES transitions with small initial perturbations. Furthermore, needless to say, the presence of time-delay is innate in a variety of communication systems and seems to have enhanced synchronization in such systems [13–18]. In recent studies [19, 20], the impact of time delay on ES transition in scale-free (SF) networks of Kuramoto oscillators and second-order Kuramoto oscillators, taking degree-frequency correlation into account, have been reported. For both the Kuramoto systems, it has been found that depending on the value of time-delay, the observed ES can either be enhanced or subsided.

Until recently, extensive studies on ES transition have been carried out in the context of single layer networks. However, several dynamical processes take place at the same time in different subsets of a same complex system and to understand the behavior of the system in a more precise manner an isolated network approach turns out to be inappropriate and hence, a multilayered structural approach is required. In the multilayered approach, the same set of the nodes, representing a complex system’s units, is laid out in several discrete layers, each one of them representing a different type of interaction process existing among the nodes [21–25]. For an instance, transport system of a country or state can be represented by a multilayer network in which cities or towns would be the nodes, and a distinct network of each bus, train and flight connectivity among the nodes (cities) would denote different layers of the multiplex network. Similarly, in an online social multilayer network, the users would be the nodes, and a network of each Facebook, Twitter and Instagram connectivity among the same users would represent the different layers of the multilayer network. In the multilayered networks, various dynamical processes evolving simultaneously in different layers having different structural and dynamical configurations may affect each other’s dynamics. A variety of synchronization transitions such as relay-synchronization, cluster-synchronization, and delay-induced synchronization have been investigated on
top of the multiplex frameworks [26–30]. Of late, the emergence of ES has been shown in the multilayer networks comprised of Kuramoto oscillators as well as second-order Kuramoto oscillators [31–33].

In this paper, we discuss the repercussions of inclusion of time delay on ES in the multilayer networks. We represent the nodes in multiplex networks by second-order Kuramoto oscillators [34–38] with natural frequencies positively correlated with their respective degrees, and study impact of the time delay in one layer on synchronization in other layers. It is perceived that inclusion of time delay in any one layer of the multiplex network induces either ES or non-ES transition concurrently in all the layers, irrespective of whether the layers are delayed or un-delayed, depending on the value of the inducted time-delay. It is further revealed that the occurrence of ES or non-ES transition is solely governed by the evolution of the average frequency of each multiplexed layer. Such occurrences of ES or non-ES transition were first revealed for the single layer time delayed networks [19, 20]. The important revelation coming out from our study is that by making an appropriate choice of time delay only in one layer, a desired type of transition can be achieved concurrently in all the multiplexed layers, either directly or remotely connected to the delayed layer (as shown for green and blue layers in figure 1). The impact of various structural and dynamical properties on the observed transition behavior has been discussed in details both numerically and analytically.

2. Delayed multiplex network model

In the current study, we investigate how the presence of time-delay between the each pair of nodes in a single layer affects transitions to synchronization in other layers of a multiplex network. For this purpose, we consider an undirected multiplex network comprised of $L$ layers of networks, each one having the same number of $N$ nodes. The dynamics of the nodes in all the layers are represented by the second-order Kuramoto oscillators. To bring in delay effect in the multiplex network, the nodes in one layer are subject to the time-delayed couplings while the nodes in remaining layers to un-delayed coupling (figure 1). We examine the impact of the time-delayed layer on the transition to intra-layer synchronization in the un-delayed layers.

The time-evolution of the nodes, denoted by the second-order Kuramoto oscillators, in the presence or absence of the time-delay in a layer are governed by

\[
m\ddot{\theta}_i^\alpha(t) + \dot{\theta}_i^\alpha(t) = \Omega_i^\alpha(0) + \lambda_\alpha \sum_{j=1}^{N} A_{ij}^\alpha \sin[\theta_j^\alpha(t) - \theta_i^\alpha(t)] + \sum_{\beta=1}^{L-1} \sigma_{\alpha\beta} \sin[\theta_j^\beta(t) - \theta_i^\beta(t)],
\]

\[
m\ddot{\theta}_i^\beta(t) + \dot{\theta}_i^\beta(t) = \Omega_i^\beta(0) + \lambda_\beta \sum_{j=1}^{N} A_{ij}^\beta \sin[\theta_j^\beta(t) - \theta_i^\beta(t)] + \sum_{\alpha=1}^{L-1} \sigma_{\beta\alpha} \sin[\theta_j^\alpha(t) - \theta_i^\alpha(t)],
\]

respectively, where $i = 1, \ldots, N$, $\Omega_i(0)$ is the natural frequency (the instantaneous phase) of $i$th oscillator, parameter $m$ is mass, $\lambda$ is coupling strength and $\tau$ is time-delay introduced between the phases of nodes in layer $\alpha$. Superscript $\alpha$ denotes the single delayed layer while $\beta \neq \alpha$ denotes all the other $L - 1$ un-delayed layers. $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$ denotes inter-layer coupling strength between layer $\alpha$ and $\beta$. The multiplex network is represented by $L$ layers encoded by a set of adjacency matrices, $\{A_1, \ldots, A_N, \ldots, A_L\}$, where element $A_{ij}^\beta = 1$ when the nodes $i$ and $j$ are connected and $A_{ij}^\beta = 0$, otherwise. Adjacency matrix (of size $NL \times NL$) of the multiplex network can be written in block matrix form as following

![Figure 1. Schematic diagram of a 3-layered multiplex network with intra-layer and inter-layer connections depicted by solid lines (cyan) and dashed lines (gray), respectively. It illustrates the effect of absence and presence of time delay in only one layer on the nature of synchronization transition in all the layers in the multiplex network. For the un-delayed case, all the layers manifest a regular second-order transition, however, as a time delay is employed in a single layer (layer 1), all the layers experience ES transition simultaneously for certain value of delay. $\lambda$ depicts over-all coupling strength, and $R_1$, $R_2$ and $R_3$ provide means of global synchronization as defined in section 2.](image)
\[ M = \begin{pmatrix} A_1 & \sigma_{12} I & \cdots & O \\ \sigma_{21} I & A_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_L \end{pmatrix}, \]

where \( I \) and \( O \) are identity and null matrices, respectively.

We quantify transition to synchronization in a layer by means of the global order parameter \( R \). We define the order parameters \( R_a \) and \( R_b \) at time \( t \) corresponding to the time-delayed and the un-delayed layers, respectively, as

\[ R_a(t)e^{\theta_a(t-t_\tau)} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_{aj}(t-t_\tau)}, \]

\[ R_b(t)e^{\theta_b(t)} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_{bj}(t)}, \]

where \( \theta(t) \) denotes average phase of the nodes. Hence, \( R \) is a measure of coherence of the collective dynamics of the nodes, i.e. the degree of synchronization of the network. \( R = 1 \) corresponds to a completely synchronized state, while \( R = 0 \) denotes an asynchronous state.

### 3. Simulation results

In this section, we present and discuss in details various simulations results exploring the effect of time delay employed in one layer on intra-layer synchronization in all the layers of the multiplex networks. We compute the order parameter as a function of the coupling strength to track the transition characteristics of the multiplexed layers. For the numerical study, we consider a duplex network comprised of two ER random networks [39] of average degrees \( \langle k \rangle = 12 \) and introduce a constant time delay between the phases of nodes in layer 1.

We compute the order parameters defined in (3) for both the delayed and un-delayed layers for different values of \( \tau \) as shown in figure 2. In the absence of time delay, i.e. \( \tau = 0 \), both the ER layers display second-order transitions and synchronize simultaneously. However, the presence of time delay significantly enhances the synchronization behavior of the multiplexed layers. For different values of the time delay, a different transition behavior, i.e. either strong or weak ES or second-order is witnessed for both the layers. Here, strong ES refers to an ES transition in which all the nodes abruptly get synchronized at once at the onset of transition. In the weak ES transition, not all the nodes get synchronized abruptly at the onset of transition, and few nodes still remain unsynchronized which later as coupling strength is increased beyond ES become part of the largest synchronized cluster.

In figure 2, for an instance, a rather weak ES is seen for \( \tau = 2.5 \) and 17, strong ES for \( \tau = 4 \) and 15, and very weak ES or second-order transition for \( \tau = 0.5 \), \( \tau = 7 \) and \( \tau = 11.5 \), respectively. Moreover, both the layers synchronize simultaneously exhibiting the same nature of the transition. The onset of transition to synchronization for different values of \( \tau \) is witnessed at different critical coupling strength. From these observations, one can infer that introduction of a time-delay not only enhances transition in the layer that includes the time delay, but also the same nature of transition is incorporated simultaneously in the un-delayed layer via the inter-layer coupling strength. Hence in the multiplex networks, a desired (ES or continuous) transition can be incorporated simultaneously in all the layers by making a suitable choice of the time-delay.

A dependency of average frequency \( \Omega \) on the time delay \( \tau \), i.e. \( \Omega = \Omega(\tau) \) is responsible for such interesting behavior. Therefore, to understand the underlying microscopic dynamics behind such time-delay dependent...
behavior observed in a multiplex network, we define effective frequency $\langle \Omega_i \rangle$ of each node and average frequency $\Omega$ of the network as

$$\langle \Omega_i \rangle = \frac{1}{T} \int_{t}^{t+T} \dot{\theta}_i(t) \, dt,$$

$$\Omega = \frac{1}{N} \sum_{i=1}^{N} \langle \Omega_i \rangle,$$

where $T$ is the total time of the averaging after eliminating the initial transients. We compute average frequencies $\Omega_1$ and $\Omega_2$ for the two ER layers as a function of time-delay $\tau$ (see figure 3). The initial (at $t = 0$) average frequencies of the first and the second layers are $\Omega_1^0 = 0.575$ and $\Omega_2^0 = 0.573$, respectively. In the presence of time-delay, the average frequencies of both the layers $\Omega_1$ and $\Omega_2$ simultaneously exhibit an oscillatory behavior around $\Omega_1^0$ and $\Omega_2^0$ (figure 3). Starting from 0, as the value of $\tau$ is increased, $\Omega_1$ and $\Omega_2$ exhibit a gradual decrease. Interestingly, at a certain value of $\tau$, $\Omega_1$ and $\Omega_2$ suddenly jump to higher values and again start decreasing along a new branch as $\tau$ is increased further.

From figure 3, it is apparent that the value of $\tau$ for which differences $|\Omega_1 - \Omega_1^0|$ and $|\Omega_2 - \Omega_2^0|$ are large, a strong ES is observed in both the layers (see figure 2 for $\tau = 0$ and $\tau = 15$). When both the differences are of intermediate values, a rather weak ES is observed (see figure 2 for $\tau = 2.5$ and $\tau = 17$). Moreover, if both the differences are low, a second-order transition is observed in both the layers (see figure 2 for $\tau = 0.5$ and $\tau = 11.5$). Hence, such oscillatory dependency of $\Omega_1$ and $\Omega_2$ on the time delay $\tau$ causes different types of transition in the multiplexed layers. We define $\Delta R$ as the jump height of an ES transition. In figure 3, the value of $\Delta R$ (square) as a function of $\tau$ varies according to the observed strong or weak ES transitions, and $\Delta R = 0$ for the second-order transitions.

To further understand the microscopic dynamics of ES with time delay, we analyze order parameter, corresponding effective frequencies and snapshot of stationary phases of the nodes as a function of $\lambda$ only for one (the delayed) layer in figure 4 as both the layers exhibit the same transition behavior as shown in figure 2. It is

![Figure 2. Synchronization transitions in multiplex networks comprised of two ER networks having $\langle k_i \rangle = \langle k_i \rangle = 12$, for different values of the time-delay introduced in layer 1.](image)

![Figure 3. Average frequencies $\Omega_1$ (circle) and $\Omega_2$ (sphere), and corresponding jump height of the ES transitions $\Delta R$ (square) for the two ER layers of the multiplex network as a function of time-delay $\tau$ for coupling strength $\lambda = 0.06$.](image)
apparent that for $\tau = 4$, ES transition takes place as effective frequency $\Omega_1^{(t)}$ and, in turn, phases $\theta_1^{(t)}$ get locked to their respective average values altogether at the critical coupling strength (see figure 4). For $\tau = 2.5$, most of the nodes are locked to their respective average frequencies and average phases at the critical coupling strength, however, rest of the nodes get locked later at a bit higher coupling strength. For $\tau = 0.5$ (see figure 4), the nodes in a gradual manner start merging to the large synchronized cluster, and $\langle \Omega_1^{(t)} \rangle$ and $\theta_1^{(t)}$ are eventually locked to their respective averages at the critical coupling strength, thereby leading to the continuous transitions.

Additionally, we demonstrate how the transition to synchronization in the presence of delay $\tau$ in an isolated network behave in comparison to that in a multiplex structure. Figure 5 depicts the oscillatory behavior of the average frequency $\Omega$ of an isolated ER network as a function of time-delay $\tau$ simulated for the synchronous state $\lambda = 0.06$.

Figure 4. Order parameter $R_1$ (top row panels), effective frequencies $\Omega_1^{(t)}$ (middle row panels) and snapshot of stationary phases $\theta_1^{(t)}$ (bottom layer panels) of nodes of the delayed layer of the multiplex network consisting of two ER networks. The effective frequencies and stationary phases of the nodes are plotted for each $\lambda$ in increasing order of values of their initial effective frequencies $\Omega_1^{(t)}$ which range between 0 and 1. The panels in left, middle and right columns displaying second-order, weak ES and ES transitions corresponding to time-delay $\tau = 0.5$, 2.5 and 4, respectively.

Figure 5. Average frequency $\Omega$ of an isolated ER network as a function of time-delay $\tau$ simulated for the synchronous state $\lambda = 0.06$.

3.1. Effect of the multiplexing strength $\sigma_{12}$ on the nature of transition:

Since we are studying the effect of time-delay on intra-layer synchronization within the purview of multiplexed architectures, it becomes essential to investigate the impact of multiplexing strength $\sigma_{12}$ on transition behavior.
of the multiplexed layers. For this purpose, we probe the average frequencies $\Omega_1$ and $\Omega_2$ of two multiplexed ER layers for different values of the inter-layer coupling strength $\sigma_{12}$ in figure 6. It is already seen in figure 3 that both the delayed ($\Omega_1$) and the non-delayed ($\Omega_2$) layers demonstrate exactly the same oscillatory behavior. Hence, display of the oscillatory behavior ($\Omega_1$ versus $\tau$) of one layer against inter-layer coupling strength $\sigma_{12}$ is sufficient to witness the changes (see figure 6). It is quite evident that $\Omega_1$ oscillatory branches shift along the $\tau$-axis as the multiplexing strength $\sigma_{12}$ is increased. Moreover, the oscillatory branches corresponding to different $\sigma_{12}$ remain exactly the same for their large segments along $\tau$-axis, hence any $\tau$ belonging to those segments for different $\sigma_{12}$ gives rise to the same type of transition. In addition, the oscillatory branches only differ in the vicinity of jumps associated with different $\sigma_{12}$, hence any $\tau$ in the proximity of the jumps may result in a different type of transition for each $\sigma_{12}$.

3.2. Effect of average degree of the multiplexed layers on nature of transition:

Next, we focus on the repercussions of the difference in average connectivities of the layers, on the nature of transition in the multiplex network. To investigate this, we fixed the average degree of the delayed ER layer to a value $\langle k_1 \rangle$ and chose an equal, a lower and a higher value for the average degree $\langle k_2 \rangle$ of the un-delayed ER layer (figure 7). For $\tau = 4$, ES and second-order transitions corresponding to $\langle k_2 \rangle = 6$ and 20, respectively, are observed at different critical coupling strengths as compared to that of $\langle k_2 \rangle = 12$. In the same fashion, weak ES transitions for $\tau = 11.5$ corresponding to $\langle k_2 \rangle = 6$ and 20 are observed at different critical coupling strengths as compared to the second-order transition observed for $\langle k_2 \rangle = 12$. Therefore, the variation in the average degree $\langle k_2 \rangle$ of the non-delayed layer shifts the critical coupling strength which, in turn, either enhances or subsides the transition accordingly. Different oscillatory dependencies of $\Omega_1$ and $\Omega_2$ on $\tau$, corresponding to $\langle k_2 \rangle = 6$ and 20 (see figure 8), account for this behavior. The different average degree $\langle k_2 \rangle$ causes a shift in the oscillatory branches along $\tau$-axis, which is quite evident for $\tau > 3$. This is the reason both the branches,
depending on the variation between their average degrees, may cause different types of transition for the same \( \tau \) value.

3.3. **Effect of number of multiplexed layers \( L \) on the transition:**

It is also important to verify whether the same set of observations hold for the multiplex networks having more than two layers. In order to investigate this, we consider a multiplex network consisting of three ER layers and compute the order parameter for each layer for different values of the time delay. From figure 9, it is quite apparent that for each value of the time delay \( \tau \), a different path to the synchronization, i.e. ES or continuous transition exists. It is to be noted that even though \( \mathcal{M} \) follows the connection matrix \( (2) \), which indicates that the delayed layer 1 and the un-delayed layer 3 are not directly interconnected, yet, by means of layer 2 that acts as a relay layer, the layer 3 exhibits exactly the same transition type as that of the layer 1. Hence, this is an instance of distant coordination similar to the remote (relay) synchronization in the multiplex networks [26]. On the basis of this finding we can say that for a multiplex network of a finite number of layers, the intra-layer delay introduced in one layer is sufficient to incorporate the same transition behavior in rest of the layers. However, as the number of the multiplexed layers \( L \) increases, the oscillatory dependence of \( \Omega - \tau \) for all the layers changes concurrently, i.e. a time-delay \( \tau \) induces different types of transition in the multiplexed network of different \( L \). For an instance, transitions corresponding to \( \tau = 2.5 \) and \( \tau = 4 \) are observed at different coupling strengths for the 2-layered (see figure 2) and the 3-layered (see figure 9) multiplex networks.

3.4. **Synchronization on a variety of multiplex networks:**

Next, we investigate the transition behavior in the presence of time delay on the multiplex networks constructed from different network topologies. To explore this, we consider multiplex networks comprised of two small-world (SW) [40], two SF [41], and ER and SF networks having time delay introduced in one layer (in ER layer in case of ER–SF multiplexed layers). We follow the same analysis as performed for the case of two ER layers. In the absence of time delay (\( \tau = 0 \)) (see figure 10), the two SW layers, the two SF layers, and the ER–SF layers exhibit...
ES, strong ES, and weak ES transitions, respectively, in contrary to the second-order transition observed in the case of two ER layers. However, depending on the value of time delay either a strong ES, weak ES or second-order transition is observed for both the SW–SW and ER–SF multiplexed layers (see figure 10) what has been observed in the case of ER–ER multiplexing. However, in the case of SF–SF multiplex network, no second-order transition is observed, only ES or weak ES transition exists for different values of $\tau$. This might be due to the strong structural heterogeneity of the SF layers which favors the ES transition in presence of degree-frequency correlation. To fathom the role of average frequency of each layer in determining the transition behavior of the selected pairs of multiplexed layers, we compute the average frequencies $\Omega_1$ and $\Omega_2$ for the layers as a function of the time delay as shown in figure 11. For these cases too, an oscillatory dependence of average frequencies $\Omega_1$ and $\Omega_2$ on the time delay $\tau$ is witnessed. In the case of two SF multiplexed layers, the observed oscillatory branches of average frequencies $\Omega_1$ and $\Omega_2$ are stretched further along the axis $\tau$ as compared to those belonging to the other pairs of the multiplexed layers.

Therefore, it is manifested that the time delay plays an important role in determining the nature of transition in a variety of multiplexed networks. We have shown that in multiplexed networks, time delay incorporates the same intra-layer transition properties in all the layers. In the presence of time delay, the difference in the average connectivity of the multiplexed layers also seems to either strengthen or subside ES transition in the layers.

4. Analytical treatment

In section 3, we have witnessed that the departure of average frequency $\Omega$ from its initial value $\Omega_0$ determines the nature of transition to synchronization. Additionally, it is numerically demonstrated that the average frequency $\Omega$ of a multiplexed layer depends not only on the time delay $\tau$ but also on the inter-layer coupling strength $\sigma_{12}$.
and the average degree $\langle k \rangle$ of all the layers. In this section we make an attempt to elucidate analytically the
dependence of average frequency $\Omega$ on dynamical as well as structural properties of the multiplex networks.
Therefore, to obtain average frequency, we rewrite (1) in the terms of the order parameters $R_\alpha(t)$ and $R_\beta(t)$ using
(3) and by defining degree of a node as

$$k_i = \sum_{j=1}^{N} A_{ij}$$

\[ m \dot{\theta}_i^\alpha(t) + \dot{\theta}_i^\alpha(t) = \Omega_i^\alpha(0) + N \lambda_{\alpha} R_\alpha k_i^\alpha R_\alpha \sin[\psi_i^\alpha(t - \tau) - \theta_i^\alpha(t)] + \sum_{j=1}^{L-1} \sigma_{i,j} \sin[\theta_j^\alpha(t) - \theta_i^\alpha(t)], \]

\[ m \dot{\theta}_i^\beta(t) + \dot{\theta}_i^\beta(t) = \Omega_i^\beta(0) + N \lambda_{\beta} R_\beta k_i^\beta \sin[\psi_i^\beta(t - \tau) - \theta_i^\beta(t)] + \sum_{j=1}^{L-1} \sigma_{i,j} \sin[\theta_j^\beta(t) - \theta_i^\beta(t)]. \]  

(6)

In the stationary state, $R_\alpha(t) \approx R_\alpha$ and $R_\beta(t) \approx R_\beta$. Next, we consider a separate rotating frame for each layer
with average phase $\psi_i(t) = \Omega_i t$. The new variables in the rotating frame are then defined as $\theta_i^\alpha = \theta_i^\alpha - \psi_i$. Sets of equations (6) get transformed in the following in the rotating frames

\[ m \ddot{\phi}_i^\alpha + \dot{\phi}_i^\alpha = [\Omega_i^\alpha(0) - \Omega_i] - N \lambda_{\alpha} R_\alpha k_i^\alpha \sin[\phi_i^\alpha + \Omega_i \tau] + \sum_{j=1}^{L-1} \sigma_{i,j} \sin[\phi_j^\alpha + \Omega_i \tau + (\Omega_j - \Omega_i) t], \]

\[ m \ddot{\phi}_i^\beta + \dot{\phi}_i^\beta = [\Omega_i^\beta(0) - \Omega_i] - N \lambda_{\beta} R_\beta k_i^\beta \sin[\phi_i^\beta + \Omega_i \tau] + \sum_{j=1}^{L-1} \sigma_{i,j} \sin[\phi_j^\beta + \Omega_i \tau + (\Omega_j - \Omega_i) t]. \]  

(7)

In synchronous state, it is safe to assume that all the nodes in each layer are locked to their respective mean field $\psi_i$, i.e. $\psi_i = \Omega_i t$. Therefore, summing over the set of $N$ equations for each layer, (7) yields the following relation for the average frequencies

\[ \Omega_i = \Omega_i^\alpha - N \lambda_{\alpha} R_\alpha \langle k_\alpha \rangle \sin[\Omega_i \tau] + \sum_{j=1}^{L-1} \sigma_{i,j} \sin[(\Omega_j - \Omega_i) t], \]

\[ \Omega_j = \Omega_j^\beta - \sum_{j=1}^{L-1} \sigma_{i,j} \sin[(\Omega_j - \Omega_i) t], \]  

(8)

where $\Omega_i^\alpha = \sum_{j=1}^{N} \Omega_i^\alpha(0)$ and $\Omega_j^\beta = \sum_{j=1}^{N} \Omega_j^\beta(0)$ are initial (t = 0) average frequencies, and average degree $\langle k_\alpha \rangle = \sum_{j=1}^{N} k_j^\alpha$. For the sake of convenience, we present our analysis to a two layered multiplex network, i.e. $L = 2$. For a duplex network, (8) can be expressed as

\[ \Omega_1 = \Omega_1^\alpha - N \lambda_1 R_1 \langle k_1 \rangle \sin[\Omega_1 \tau] + \sigma_{12} \sin[\Omega_2 - \Omega_1 \tau] t], \]

\[ \Omega_2 = \Omega_2^\beta - \sigma_{12} \sin[\Omega_2 - \Omega_1 \tau] t]. \]  

(9)

Further, we define $\Delta \Omega$, i.e. the difference between $\Omega_1$ and $\Omega_2$ as

\[ \Delta \Omega = \Omega_2 - \Omega_1 = \Delta \Omega_0 + \Omega_1^\alpha(\tau, \lambda_1) - 2 \sigma_{12} \sin[\Delta \Omega t], \]  

(10)

where constant $\Delta \Omega_0 = \Omega_2^\alpha - \Omega_1^\alpha = \langle k_2 \rangle / k_2^{\max} - \langle k_1 \rangle / k_1^{\max}$, and $\Omega_1(\tau, \lambda_1) = N \lambda_1 R_1 \langle k_1 \rangle \sin[\Omega_1 \tau]$ is also a constant as $\Omega_1(\tau)$ and $R_1$ would be constant in the stationary state for a certain $\tau$ and $\lambda_1 > \chi_1$. In the absence of time delay, i.e. $\tau = 0$, (10) reduces to

\[ \Delta \Omega = \Delta \Omega_0 - 2 \sigma_{12} \sin[\Delta \Omega t]. \]  

(11)

If $\Omega_1^\alpha$ and $\Omega_2^\alpha$ are equal, that, in turn, implies $\Delta \Omega_0 \approx 0$, consequently, (11) becomes

\[ \Delta \Omega = -2 \sigma_{12} \sin[\Delta \Omega t]. \]  

(12)

It is quite apparent from (9) or (10) that inter-dependent $\Omega_1$ and $\Omega_2$ depend on parameters $\tau, \lambda_1, \sigma_{12}, \langle k_1 \rangle$, and $\langle k_2 \rangle$, which is in further concurrence with the numerical results shown in section 3 that the multiplexing strength $\sigma_{12}$ and the average degree of the layers along with the time delay determine the characteristic of transition in the multilayered framework.

5. Discussions

In the present work, we have discussed the impact of time delay introduced in a single layer on the nature of phase transition in all the layers of multiplex networks. We demonstrate that the path of transition through which multiplexed layers achieve synchronous states, can be controlled by the means of the time delays. The multiplexed layers altogether choose a common transition-path, either a strong or weak ES or a second-order one, to reach to their respective synchronous states, which is solely determined by the value of introduced time-delay. Each value of the time delay deports the average frequencies of all the layers from their respective initial values leading to different but nearby values, and the extent of such departure governs the type of transition which is common to all the layers. If the extent of the departure is large, this leads to ES or first-order transition.
in all the layers. An intermediate departure gives rise to weak ES, whereas a small departure only results in a second-order transition. Our this finding holds true for a variety of multiplex networks constructed from different network topologies. Additionally, in the presence of time-delay, the difference in average connectivity of the multiplexed layers brings about simultaneous transitions in the remotely un-delayed layers too, demonstrating that 3-layered multiplex networks with no direct inter-layer connections between the remote layers, one delayed layer is capable of inducing the same type of transition in the remotely un-delayed layers too, demonstrating that multiplex networks favor remote (relay) synchronization. Such remote synchronization has found pertinency in the context of brain: the thalamus plays a role of relay in communication of information flow between the remote cortical regions via thalamo-cortical pathways [42]. We have also shown analytically that the average frequency of each layer depends on the inter-layer coupling strength and the average connectivity of all the layers along with the time-delay.

Lately, the evidence of ES is reported in the brains of patients suffering from fibromyalgia (FM), a condition described as an omnipresent, chronic pain [10]. The patients with FM have brain networks predisposed towards abrupt, global responses or abnormal hypersensitivity (ES) to minor changes. Hence, ES is worth investigating as it is detrimental to certain biological and physical circumstances, e.g. chronic pain (Fibromyalgia) and epileptic seizure in brains in which symptoms are abruptly turned on [9] and the power-grid failure in which power supplies are abruptly turned off [11]. Investigation of ES in real-world complex systems represented by multilayer framework will be promising as the multilayer architecture provides a more accurate representation of many real-world complex systems. Researchers have employed multiplex perspective to identify the most important functional region of the brain. A human functional brain network consists of peripheral and central or hub regions operating at different frequency bands, which can be modeled as a multiplex network in which each layer is identified by a single frequency band carrying unique topological information [43, 44]. Information processing from various functional or sensory regions of the brain is a cooperative process of neurons and time delay is naturally incorporated in the information processing to form a coherent and melded perception of the outer real world. Hence, time-delayed synchronization (ES and non-ES) of neurons, while employing multiplex network perspective in the representation of various functional sensory regions, in information processing is worth investigating. Hence, our study on ES on the multiplex framework can be prolific in not only providing better understanding but also regulating the dynamics of interwoven concurrent dynamical processes occurring in both biological and physical systems.

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