Black Holes in Two Dimensional Dilaton Gravity and Nonlinear Klein-Gordon Soliton

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Abstract

Two-dimensional dilaton gravity coupled to a Klein-Gordon matter field with a quartic interaction term is considered. The theory has a classical solution which exhibits black hole formation by a soliton. The geometry of black hole induced by a soliton is investigated.

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1. Introduction

Black hole solutions of the two-dimensional dilaton gravity coupled with massless scalar field have attracted much interest since the pioneering work by Callan, Giddings, Harvey and Strominger (CGHS).\(^1\) It was hoped that the model would unravel mysteries concerning the black hole evaporation and the resulting information loss. The theory, however, turned out to be intractable even in semi-classical approximations \(^2,3,4\) let alone in full quantum analysis.

Much of the analysis of two dimensional models about black hole formation was only concerned with free scalar fields.\(^1\) Recently black hole formation by an interacting scalar field was considered,\(^5,6\) where soliton solutions of Sine-Gordon theory provide energy-momentum stress tensor to cause the black hole. In two-dimensional spacetime there have been many types of partial differential equations besides Sine-Gordon theory which have soliton solutions. In particular Klein-Gordon theory with a quartic interaction offers another simple example of solitons.\(^7\) In this paper we consider black hole formation in two-dimensional dilaton gravity coupled to a Klein-Gordon matter field with a quartic interaction term.

In section 2 we introduce the model by giving the action, choose the convenient gauge, i.e., conformal gauge, and further reduce the equations by making use of subgauge freedom. In section 3 we solve the equations exactly using the known soliton solutions of quartic Klein-Gordon theory, and study the geometry with an emphasis on black hole formation. In the last section relativistic limit of our solution is compared with the shock wave solution of the CGHS model.\(^1\)
2. Action and Field Equations

The action of the model we are concerned is

\[ S = \frac{1}{2\pi} \int_M d^2x \sqrt{-g} \left[ e^{-2\phi} \left( R + 4(\nabla \phi)^2 + 4\lambda^2 \right) - \frac{1}{2} (\nabla f)^2 - \frac{\beta}{4} \left( f^2 - \frac{\mu^2}{\beta} \right)^2 e^{-2\phi} \right], \quad (1) \]

where \( g, \phi \) and \( f \) are metric, dilaton, and matter fields, respectively. \( R \) is the curvature scalar, \( \lambda^2 \) is a cosmological constant, \( \beta \) is a coupling constant of the matter field, and \( \pm \sqrt{\frac{\mu^2}{\beta}} \) is the minimum position of the quartic potential. Note that this is the CGHS action \(^1\) except the last quartic interaction term which is introduced in order to study formation of black holes by the solitons.

The classical theory described by (1) is most easily analyzed in the conformal gauge:

\[ ds^2 = -e^{2\rho} dx^+ dx^-, \quad (2) \]
\[ g_{++} = -\frac{1}{2} e^{2\rho}, \quad g^{+-} = -2e^{-2\rho}, \quad (3) \]
\[ R = 8e^{-2\rho} \partial_+ \partial_- \rho, \quad (4) \]

where \( x^\pm = t \pm x \). The action then reduces to

\[ S = \frac{1}{\pi} \int d^2x \left[ e^{-2\phi} \left( 2\partial_+ \partial_- \rho - 4\partial_+ \phi \partial_- \phi + \lambda^2 e^{2\rho} \right) + \frac{1}{2} \partial_+ f \partial_- f - \frac{\beta}{16} (f^2 - \frac{\mu^2}{\beta})^2 e^{2\rho - 2\phi} \right], \quad (5) \]

and the metric equations of motion are

\[ T_{++} = e^{-2\phi} (4\partial_+ \partial_- \rho - 4\partial_+ \phi \partial_- \phi + 2\partial_+^2 \phi + \frac{1}{2} (\partial_+ f)^2 - \frac{\beta}{16} (f^2 - \frac{\mu^2}{\beta})^2 e^{2\rho - 2\phi} = 0, \quad (6) \]
\[ T_{--} = e^{-2\phi} (4\partial_- \rho \partial_- \phi - 2\partial_-^2 \phi + \frac{1}{2} (\partial_- f)^2 = 0, \quad (7) \]
\[ T_{+-} = e^{-2\phi} (2\partial_+ \partial_- \phi - 4\partial_+ \phi \partial_- \phi - \lambda^2 e^{2\rho} + \frac{\beta}{16} (f^2 - \frac{\mu^2}{\beta})^2 e^{2\rho - 2\phi} = 0. \quad (8) \]

The dilaton and matter equations are

\[ (-4\partial_+ \partial_- \phi + 4\partial_+ \phi \partial_- \phi + 2\partial_+ \partial_- \rho) e^{-2\phi} + \left( \lambda^2 - \frac{\beta}{16} (f^2 - \frac{\mu^2}{\beta})^2 \right) e^{2\rho - 2\phi} = 0, \quad (9) \]
\[ \partial_+ \partial_- f + \frac{1}{4} (\beta f^3 - \mu^2 f) e^{2(\rho - \phi)} = 0. \] 

Adding the equations (8) and (9) we get \( \partial_+ \partial_- (\rho - \phi) = 0 \), and by making use of the subconformal gauge freedom we can let

\[ \rho = \phi. \]

Now the field equations are much simplified as

\[ \partial_+^2 (e^{-2\phi}) + \frac{1}{2} (\partial_+ f)^2 = 0, \] 

\[ \partial_-^2 (e^{-2\phi}) + \frac{1}{2} (\partial_- f)^2 = 0, \] 

\[ \partial_+ \partial_- (e^{-2\phi}) + \lambda^2 - \frac{\beta}{16} (f^2 - \frac{\mu^2}{\beta})^2 = 0, \] 

\[ \partial_+ \partial_- f + \frac{1}{4} (\beta f^3 - \mu^2 f) = 0. \]

In the rest of this paper we will consider solutions of the above equations in connection with solitons and their influence upon geometry. Notice that the last equation (15) is just a nonlinear Klein-Gordon equation whose solution is well known.

3. Blackhole Formation by a Kink

We begin with the soliton solution of the Klein-Gordon equation with a quartic interaction\(^7\):

\[ f(x, t) = \sqrt{\frac{\mu^2}{\beta}} \tanh \left[ \frac{\mu \gamma}{\sqrt{2}} \left((x - x_0) + v(t - t_0)\right) \right], \]

where

\[ \gamma = \frac{1}{\sqrt{1 - v^2}}, \]
and \((x_0, t_0)\) is the center of the soliton. This is a traveling wave of kink type with velocity \(-v\). It is helpful to rewrite the soliton solution in terms of \(x^\pm\) as

\[
f(x^+, x^-) = \sqrt{\frac{\mu^2}{\beta}} \tanh \left[ \frac{\mu}{2\sqrt{2}} (\alpha(x^+ - x_0^+) - \frac{1}{\alpha}(x^- - x_0^-)) \right]
\]

where

\[
\alpha = \sqrt{\frac{1 + v}{1 - v}}.
\]

In order to study black hole formation by a kink we solve the equations (12)-(14) with \(f\) given by (18). The equation (14) is explicitly given as

\[
\partial_+ \partial_- (e^{-2\phi}) = -\lambda^2 + \frac{\mu^4}{16\beta} \text{sech}^4 \left[ \frac{\mu}{2\sqrt{2}} (\alpha(x^+ - x_0^+) - \frac{1}{\alpha}(x^- - x_0^-)) \right],
\]

which can be integrated as

\[
e^{-2\phi} = a(x^+) + b(x^-) - \lambda^2 x^+ x^- - \frac{\mu^2}{3\beta} \left[ \log[\cosh(\Delta - \Delta_0)] + \frac{1}{4} \tanh^2(\Delta - \Delta_0) \right],
\]

where

\[
\Delta = \frac{\mu}{2\sqrt{2}} (\alpha x^+ - \frac{1}{\alpha} x^-).
\]

The two functions \(a(x^+)\) and \(b(x^-)\) are determined by the constraint equations (12) and (13). They are simply

\[
\frac{d^2a(x^+)}{dx^+} = 0, \quad \frac{d^2b(x^-)}{dx^-} = 0,
\]

which determines the dilaton field up to constants as

\[
e^{-2\phi} = C + ax^+ + bx^- - \lambda^2 x^+ x^- - \frac{\mu^2}{3\beta} \left[ \log[\cosh(\Delta - \Delta_0)] + \frac{1}{4} \tanh^2(\Delta - \Delta_0) \right],
\]

where \(a, b, C\) are constants. By choosing the origin of the coordinates suitably we can let \(a = b = 0\). Before we start analyzing the geometry we summarize the solution of kink type
as the followings:

\[ f(x^+, x^-) = \sqrt{\frac{\mu^2}{\beta}} \tanh[\Delta - \Delta_0], \]  

(25)

\[ e^{-2\rho} = e^{-2\phi} = C - \lambda^2 x^+ x^- - \frac{\mu^2}{3\beta} \left[ \log[\cosh(\Delta - \Delta_0)] + \frac{1}{4} \tanh^2(\Delta - \Delta_0) \right], \]  

(26)

\[ \Delta = \frac{\mu}{2\sqrt{2}} \left( \alpha x^+ - \frac{1}{\alpha} x^- \right) = \frac{\mu \gamma}{\sqrt{2}} (x + vt). \]  

(27)

In order to see the nature of the geometry induced by the soliton it is convenient to divide the spacetime into three regions: \( \Delta - \Delta_0 \ll -1 \), \( \Delta - \Delta_0 \simeq 0 \), and \( \Delta - \Delta_0 \gg 1 \). The first region \( \Delta - \Delta_0 \ll -1 \) is the area which has not yet been influenced by the soliton, and so we expect it to be a vacuum. We can indeed check that the equation (26) becomes in this region

\[ e^{-2\rho} \simeq -\lambda^2 \left( x^+ + \frac{\mu^3}{6\sqrt{2}\alpha\beta \lambda^2} \right) \left( x^- - \frac{\alpha \mu^3}{6\sqrt{2}\beta \lambda^2} \right) \]  

(28)

where the constant \( C \) is taken as

\[ C = \frac{\mu^2}{3\beta} (\Delta_0 + \frac{\mu^4}{24\beta \lambda^2} - \log 2 + \frac{1}{4}). \]  

(29)

This clearly exhibits that the metric is that of the linear dilaton vacuum when the soliton has not arrived to affect the spacetime. In the third region \( \Delta - \Delta_0 \gg 1 \), we expect the metric of a black hole, which can be checked by retaining only the first order term in (26) as

\[ e^{-2\rho} \simeq \frac{\mu^2}{3\beta} (2\Delta_0) - \lambda^2 \left( x^+ - \frac{\mu^3}{6\sqrt{2}\alpha\beta \lambda^2} \right) \left( x^- + \frac{\alpha \mu^3}{6\sqrt{2}\beta \lambda^2} \right). \]  

(30)

This represents the geometry of a black hole of mass \( \lambda \frac{\mu^2}{3\beta} (2\Delta_0) \) after shifting \( x^+ \) by \( \frac{\mu^3}{6\sqrt{2}\alpha\beta \lambda^2} \), and \( x^- \) by \( -\frac{\alpha \mu^3}{6\sqrt{2}\beta \lambda^2} \). The first region of the linear dilaton vacuum and the third one of a black hole are smoothly joined across the soliton wave. At near the center of the soliton \( \Delta - \Delta_0 \simeq 0 \) we see that

\[ e^{-2\rho} \simeq \frac{\mu^2}{3\beta} (\Delta_0 + \frac{\mu^4}{24\beta \lambda^2} - \log 2 + \frac{1}{4}) - \lambda^2 x^+ x^- \]  

(31)
which represents again the geometry of a black hole but with a different of mass $\frac{\mu^2}{24\beta^2} - \log 2 + \frac{1}{\lambda}$.

The position of the apparent horizon is given by $\partial_+ \phi = 0$ which is

$$x^- + \frac{\alpha \mu^3}{6\sqrt{2}\beta \lambda^2} \tanh(\Delta - \Delta_0)[1 + \frac{1}{2} \text{sech}^2(\Delta - \Delta_0)] = 0.$$  
(32)

At the center of the soliton wave $(\Delta - \Delta_0 = 0)$ it is simply $x^- = 0$, and at the third region $(\Delta - \Delta_0 \gg 1)$ it is $x^- \simeq -\frac{\alpha \mu^3}{6\sqrt{2}\beta \lambda^2}$ which coincides with the event horizon of the black hole.

4. Discussion

It is instructive to compare our results with those of the CGHS where a black hole is formed by a shock wave of massless scalar fields traveling in the $x^-$ direction with magnitude $A$ described by the stress tensor

$$\frac{1}{2}(\partial_+ f)^2 = A \delta(x^+ - x_0^+).$$  
(33)

The black hole geometry caused by this shock wave is described by the metric

$$e^{-2\rho} = -A(x^+ - x_0^+)\Theta(x^+ - x_0^+) - \lambda^2 x^+ x^-,$$  
(34)

where $\Theta$ is a step function. In our case we take the relativistic limit $(v \to 1)$ of the soliton.

From the soliton solution (25) we see that

$$\frac{1}{2}(\partial_+ f)^2 = \frac{1}{2} \left(\frac{\alpha \mu^2}{2\sqrt{2}\beta}\right)^2 \text{sech}^4 \left[\frac{\mu}{2\sqrt{2}}(\alpha(x^+ - x_0^+) - \frac{1}{\alpha}(x^- - x_0^-))\right]$$

$$\to \frac{\mu^3}{3\beta \sqrt{1 - v}} \delta(x^+ - x_0^+).$$  
(35)

Therefore the magnitude of the stress tensor of a ultrarelativistic soliton is

$$A_{\text{soliton}} = \frac{\mu^3}{3\beta \sqrt{1 - v}}.$$  
(36)
On the other hand the metric caused by the soliton becomes

\[ e^{-2\rho} \rightarrow \begin{cases} 
-\lambda^2 x^+(x^- - \frac{\mu^3}{6\beta \sqrt{1-v}}), & x^+ - x_0^+ < 0, \\
-\lambda^2 x^+(x^- + \frac{\mu^3}{6\beta^2 \sqrt{1-v}}) + \frac{\mu^3}{3\beta \sqrt{1-v}} x_0^+, & x^+ - x_0^+ > 0, \\
= -\lambda^2 x^+(x^- - \frac{\mu^3}{6\beta \sqrt{1-v}}) - \frac{\mu^3}{3\beta \sqrt{1-v}} (x^+ - x_0^+ \Theta(x^+ - x_0^+)), & (37)
\end{cases} \]

which again coincides with the metric of CGHS case with

\[ A = \frac{\mu^3}{3\beta \sqrt{1-v}}. \quad (39) \]

In this paper we limited our consideration to classical problems. It is natural to investigate quantum nature of the theory such as Hawking radiation, conformal anomaly, and semiclassical analysis which require considerable amount of work and not a straightforward extension of previous results on Hawking radiation because of simultaneous presence of black hole and soliton. We plan to deal with these topics in a separate article.

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