RIGID MULTIVIEW VARIETIES

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Abstract. The multiview variety from computer vision is generalized to images by \( n \) cameras of points linked by a distance constraint. The resulting five-dimensional variety lives in a product of \( 2n \) projective planes. We determine defining polynomial equations, and we explore generalizations of this variety to scenarios of interest in applications.

1. Introduction

The emerging field of Algebraic Vision is concerned with interactions between computer vision and algebraic geometry. A central role in this endeavor is played by projective varieties that arise in multiview geometry [5].

The set-up is as follows: A camera is a linear map from the three-dimensional projective space \( \mathbb{P}^3 \) to the projective plane \( \mathbb{P}^2 \), both over \( \mathbb{R} \). We represent \( n \) cameras by matrices \( A_1, A_2, \ldots, A_n \in \mathbb{R}^{3 \times 4} \) of rank 3. The kernel of \( A_j \) is the focal point \( f_j \in \mathbb{P}^3 \). Each image point \( u_j \in \mathbb{P}^2 \) of camera \( A_j \) has a line through \( f_j \) as its fiber in \( \mathbb{P}^3 \). This is the back-projected line.

We assume throughout that the focal points of the \( n \) cameras are in general position, i.e. all distinct, no three on a line, and no four on a plane. Let \( \beta_{jk} \) denote the line in \( \mathbb{P}^3 \) spanned by the focal points \( f_j \) and \( f_k \). This is the baseline of the camera pair \( A_j, A_k \). The image of the focal point \( f_j \) in the image plane \( \mathbb{P}^2 \) of the camera \( A_k \) is the epipole \( e_{k\leftarrow j} \). Note that the baseline \( \beta_{jk} \) is the back-projected line of \( e_{k\leftarrow j} \) with respect to \( A_j \) and also the back-projected line of \( e_{j\leftarrow k} \) with respect to \( A_k \). See Figure 1 for a sketch.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{two_view_geometry.png}
\caption{Two-view geometry}
\end{figure}

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Fix a point \( X \) in \( \mathbb{P}^3 \) which is not on the baseline \( \beta_{jk} \), and let \( u_j \) and \( u_k \) be the images of \( X \) under \( A_j \) and \( A_k \). Since \( X \) is not on the baseline, neither image point is the epipole for the other camera. The two back-projected lines of \( u_j \) and \( u_k \) meet in a unique point, which is \( X \). This process of reconstructing \( X \) from two images \( u_j \) and \( u_k \) is called triangulation [5, §9.1].

The triangulation procedure amounts to solving the linear equations

\[
B_{ij} \begin{bmatrix} X \\ -\lambda_j \\ -\lambda_k \end{bmatrix} = 0 \text{ where } B_{ij} = \begin{bmatrix} A_j & u_j & 0 \\ A_k & 0 & u_k \end{bmatrix} \in \mathbb{R}^{6 \times 6}.
\]

For general data we have \( \text{rank}(B_{ij}) = \text{rank}(B_{1ij}) = \cdots = \text{rank}(B_{6ij}) = 5 \), where \( B_{ij} \) is obtained from \( B_{ij} \) by deleting the \( i \)th row. Cramer’s Rule can be used to recover \( X \). Let \( \wedge_5 B_{ij} \in \mathbb{R}^6 \) be the column vector formed by the signed maximal minors of \( B_{ij} \). Write \( \wedge_5 B_{ij} \in \mathbb{R}^4 \) for the first four coordinates of \( \wedge_5 B_{ij} \). These are bilinear functions of \( u_j \) and \( u_k \). They yield

\[
X = \wedge_5 B_{1ij} = \wedge_5 B_{2ij} = \cdots = \wedge_5 B_{6ij}.
\]

We note that, in most practical applications, the data \( u_1, \ldots, u_n \) will be noisy, in which case triangulation requires techniques from optimization [1].

The multiview variety \( \mathcal{V}_A \) of the camera configuration \( A = (A_1, \ldots, A_n) \) was defined in [3] as the closure of the image of the rational map

\[
\phi_A : \mathbb{P}^3 \rightarrow \mathbb{P}^2 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^2,
X \mapsto (A_1X, A_2X, \ldots, A_nX).
\]

The points \( (u_1, u_2, \ldots, u_n) \in \mathcal{V}_A \) are the consistent views in \( n \) cameras. The prime ideal \( I_A \) of \( \mathcal{V}_A \) was determined in [3, Corollary 2.7]. It is generated by the \( \binom{n}{2} \) bilinear polynomials \( \det(B_{ij}) \) plus \( \binom{n}{3} \) further trilinear polynomials. See [8] for the natural generalization of this variety to higher dimensions.

The analysis in [3] was restricted to a single world point \( X \in \mathbb{P}^3 \). In this paper we study the case of two world points \( X, Y \in \mathbb{P}^3 \) that are linked by a distance constraint. Consider the hypersurface \( V(Q) \) in \( \mathbb{P}^3 \times \mathbb{P}^3 \) defined by

\[
Q = (X_0Y_3 - Y_0X_3)^2 + (X_1Y_3 - Y_1X_3)^2 + (X_2Y_3 - Y_2X_3)^2 - X_3^2Y_3^2.
\]

The affine variety \( \mathbb{V}_R(Q) \cap \{X_3 = Y_3 = 1\} \) in \( \mathbb{R}^3 \times \mathbb{R}^3 \) consists of pairs of points whose Euclidean distance is 1. The rigid multiview map is the rational map

\[
\psi_A : V(Q) \mapsto \mathbb{P}^3 \times \mathbb{P}^3 \mapsto (\mathbb{P}^2)^n \times (\mathbb{P}^2)^n,
(X, Y) \mapsto ((A_1X, \ldots, A_nX), (A_1Y, \ldots, A_nY)).
\]

The rigid multiview variety is the image of this map. This is a 5-dimensional subvariety of \( (\mathbb{P}^2)^{2n} \). Its multihomogeneous prime ideal \( J_A \) lives in the polynomial ring \( \mathbb{R}[u, v] = \mathbb{R}[u_{i0}, u_{i1}, u_{i2}, v_{i0}, v_{i1}, v_{i2} : i = 1, \ldots, n] \), where \( (u_{i0}:u_{i1}:u_{i2}) \) and \( (v_{i0}:v_{i1}:v_{i2}) \) are coordinates for the \( i \)th factor \( \mathbb{P}^2 \) on the left respectively right in \( (\mathbb{P}^2)^n \times (\mathbb{P}^2)^n \). Our aim is to determine the ideal \( J_A \). Knowing generators of \( J_A \) has the potential of being useful for designing optimization tools as in [1] for triangulation in the presence of distance constraints.

The choice of world and image coordinates for the camera configuration \( A = (A_1, \ldots, A_n) \) gives our problem the following group symmetries. Let \( N \) be an element of the Euclidean group of motions \( \text{SE}(3, \mathbb{R}) \), which is generated
by rotations and translations. We may multiply the camera configuration on the right by $N$ to obtain $AN = (A_1N, \ldots, A_nN)$. Then $J_A = J_{AN}$ since $V(Q)$ is invariant under $SE(3, \mathbb{R})$. For $M_1, \ldots, M_n \in GL(3, \mathbb{R})$, we may multiply $A$ on the left to obtain $A' = (M_1A, \ldots, M_nA)$. Then $J_{A'} = (M_1 \otimes \cdots \otimes M_n)J_A$.

This paper is organized as follows. In Section 2 we present the explicit computation of the rigid multiview ideal for $n = 2, 3, 4$. Our main result, to be stated and proved in Section 3, is a system of equations that cuts out the rigid multiview variety $V(A)$ for any $n$. Section 4 is devoted to generalizations. The general idea is to replace $V(Q)$ by arbitrary subvarieties of $R^n$ that represent polynomial constraints on $m \geq 2$ world points. We focus on scenarios that are of interest in applications to computer vision.

Our results in Propositions 1, 3, 4 and Corollary 2 are proved by computations with Macaulay2 [4]; for details see Appendix A. Following standard practice in computational algebraic geometry, we carry out the computation on many samples in a Zariski dense set of parameters, and then conclude that it holds generically.

2. TWO, THREE AND FOUR CAMERAS

In this section we offer a detailed case study of the rigid multiview variety when the number $n$ of cameras is small. We begin with the case $n = 2$. The prime ideal $J_A$ lives in the polynomial ring $\mathbb{R}[u, v]$ in 12 variables. This is the homogeneous coordinate ring of $(\mathbb{P}^2)^4$, so it is naturally $\mathbb{Z}^4$-graded. The variables $u_{10}, u_{11}, u_{12}$ have degree $(1, 0, 0, 0)$, the variables $u_{20}, u_{21}, u_{22}$ have degree $(0, 1, 0, 0)$, the variables $v_{10}, v_{11}, v_{12}$ have degree $(0, 0, 1, 0)$, and the variables $v_{20}, v_{21}, v_{22}$ have degree $(0, 0, 0, 1)$. Our ideal $J_A$ is $\mathbb{Z}^4$-homogeneous.

Throughout this section we shall assume that the camera configuration $A$ is generic in the sense of algebraic geometry. This means that $A$ lies in the complement of a certain (unknown) proper algebraic subvariety in the affine space of all $n$-tuples of $3 \times 4$-matrices. All our results in Section 2 were obtained by symbolic computations with sufficiently many random choices of $A$ (see Appendix A for details). Such choices of camera matrices are generic. They will be attained with with probability 1.

Proposition 1. For $n = 2$, the rigid multiview ideal $J_A$ is minimally generated by eleven $\mathbb{Z}^4$-homogeneous polynomials in twelve variables, one of degree $(1, 1, 0, 0)$, one of degree $(0, 0, 1, 1)$, and nine of degree $(2, 2, 2, 2)$.

We prove this result by sufficiently many random computations with Macaulay2. A slightly simplified version of the code is shown in Listing 1 in Appendix A.

Let us look at the result in more detail. The first two bilinear generators are the familiar $6 \times 6$-determinants

\[
\begin{vmatrix}
A_1 & u_1 & 0 \\
A_2 & 0 & u_2
\end{vmatrix}
\quad \text{and} \quad
\begin{vmatrix}
A_1 & v_1 & 0 \\
A_2 & 0 & v_2
\end{vmatrix}.
\]

These cut out two copies of the multiview threefold $V_A \subset (\mathbb{P}^2)^2$, in separate variables, for $X \mapsto u = (u_{10}, u_{20})$ and $Y \mapsto v = (v_{10}, v_{20})$. If we write the two bilinear forms in (6) as $u_1^\top Fu_2$ and $v_1^\top Fv_2$ then $F$ is a real $3 \times 3$-matrix of rank 2, known as the fundamental matrix [3, §9] of the camera pair $(A_1, A_2)$.
The rigid multiview variety $V(J_A)$ is a divisor in $V_A \times V_A \subset (\mathbb{P}^2)^2 \times (\mathbb{P}^2)^2$. The nine octics that cut out this divisor can be understood as follows. We write $B$ and $C$ for the $6 \times 6$-matrices in (6), and $B_i$ and $C_i$ for the matrices obtained by deleting their $i$th rows. The kernels of these $5 \times 6$-matrices are represented, via Cramer’s Rule, by $\wedge_5 B_i$ and $\wedge_5 C_i$. We write $\tilde{\lambda}_5 B_i$ and $\tilde{\lambda}_5 C_i$ for the vectors given by their first four entries. As in (2), these represent the two world points $X$ and $Y$ in $\mathbb{P}^3$. Their coordinates are bilinear forms in $(u_1, u_2)$ or $(v_1, v_2)$, where each coefficient is a $3 \times 3$-minor of $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$. For instance, writing $a_{ij}^k$ for the $(j, k)$ entry of $A_i$, the first coordinate of $\tilde{\lambda}_5 B_1$ is

\[
- (a_{12}^2 a_{23}^3 a_{34}^2 - a_{13}^2 a_{24}^3 a_{42}^1 + a_{13}^3 a_{24}^2 a_{42}^3 + a_{14}^2 a_{23}^3 a_{32}^1 - a_{14}^3 a_{23}^2 a_{32}^3) u_{11} u_{20} \\
+ (a_{12}^2 a_{33}^3 a_{42}^2 - a_{13}^2 a_{34}^3 a_{24}^1 + a_{13}^3 a_{34}^2 a_{24}^3 + a_{14}^2 a_{33}^3 a_{42}^3 - a_{14}^3 a_{33}^2 a_{42}^3) u_{11} u_{21} \\
- (a_{12}^2 a_{43}^3 a_{32}^1 - a_{13}^2 a_{42}^3 a_{34}^1 + a_{13}^3 a_{42}^2 a_{34}^3 + a_{14}^2 a_{43}^3 a_{32}^3 - a_{14}^3 a_{43}^2 a_{32}^3) u_{11} u_{22} \\
+ (a_{12}^2 a_{13}^3 a_{24}^2 - a_{13}^2 a_{14}^3 a_{23}^1 + a_{13}^3 a_{14}^2 a_{23}^3 + a_{14}^2 a_{13}^3 a_{24}^3 - a_{14}^3 a_{13}^2 a_{24}^3) u_{12} u_{21} \\
- (a_{13}^2 a_{22}^3 a_{34}^1 - a_{14}^2 a_{23}^3 a_{42}^1 + a_{14}^3 a_{23}^2 a_{42}^3 + a_{15}^2 a_{24}^3 a_{32}^1 - a_{15}^3 a_{24}^2 a_{32}^3) u_{12} u_{22} \\
+ (a_{14}^2 a_{22}^3 a_{33}^1 - a_{15}^2 a_{23}^3 a_{43}^1 + a_{15}^3 a_{23}^2 a_{43}^3 + a_{16}^2 a_{24}^3 a_{34}^1 - a_{16}^3 a_{24}^2 a_{34}^3) u_{14} u_{23}.
\]

Recall that the two world points in $\mathbb{P}^3$ are linked by a distance constraint (4), expressed as a biquadratic polynomial $Q$. We set $Q(X, Y) = T(X, Y, Z)$, where $T(\bullet, \bullet, \bullet, \bullet)$ is a quadrilinear form. We regard $T$ as a tensor of order 4. It lives in the subspace $\text{Sym}_2(\mathbb{R}^4)^{\otimes 3} \otimes \text{Sym}_2(\mathbb{R}^4)^{\otimes 2} \simeq \mathbb{R}^{100}$ of $\mathbb{R}^{100}$, or $\mathbb{R}^{256}$. Here $\text{Sym}_k(\cdot)$ denotes the space of symmetric tensors of order $k$.

We now substitute our Cramer’s Rule formulas for $X$ and $Y$ into the quadrilinear form $T$. For any choice of indices $1 \leq i \leq j \leq 6$ and $1 \leq k \leq l \leq 6$,

\begin{equation}
T(\tilde{\lambda}_5 B_i, \tilde{\lambda}_5 B_j, \tilde{\lambda}_5 C_k, \tilde{\lambda}_5 C_l)
\end{equation}

is a multihomogeneous polynomial in $(u_1, u_2, v_1, v_2)$ of degree $(2, 2, 2, 2)$. This polynomial lies in $J_A$ but not in the ideal $I_A(u) + I_A(v)$ of $V_A \times V_A$, so it can serve as one of the nine minimal generators described in Proposition 1.

The number of distinct polynomials appearing in (7) equals $\binom{9}{2} = 441$. A computation verifies that these polynomials span a real vector space of dimension 126. The image of that vector space modulo the degree $(2, 2, 2, 2)$ component of the ideal $I_A(u) + I_A(v)$ has dimension 9.

We record three more features of the rigid multiview with $n = 2$ cameras. The first is the multidegree (9) [8.5], or, equivalently, the cohomology class of $V(J_A)$ in $H^*(((\mathbb{P}^2)^4, \mathbb{Z}) = \mathbb{Z}[u_1, u_2, v_1, v_2]/(u_1^3, u_2^3, v_1^3, v_2^3)$. It equals

\[
2 u_1^2 v_1 + 2 u_1 u_2 v_1 + 2 u_2^2 v_1 + 2 u_1^2 v_2 + 2 u_1 u_2 v_2 + 2 u_2^2 v_2 \\
+ 2 u_1 v_1^2 + 2 u_1 v_2^2 + 2 u_2 v_1^2 + 2 u_2 v_2^2 + 2 u_1 v_1 v_2 + 2 u_2 v_1 v_2.
\]

This is found with the built-in command multidegree in Macaulay2.

The second is the table of the Betti numbers of the minimal free resolution of $J_A$ in the format of Macaulay2 [4]. In that format, the columns correspond to the syzygy modules, while rows denote the degrees. For $n = 2$ we obtain

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
\text{total:} & 1 & 11 & 25 & 22 & 8 & 1 \\
0: & 1 & . & . & . & . & . \\
1: & . & 2 & . & . & . & . \\
2: & . & 1 & . & . & . & . \\
7: & . & 9 & 24 & 22 & 8 & 1
\end{array}
\]
The column labeled 1 lists the minimal generators from Proposition 1. Since the codimension of $V(J_A)$ is 3, the table shows that $J_A$ is not Cohen-Macaulay. The unique 5th syzygy has degree $(3, 3, 3, 3, 3)$ in the $\mathbb{Z}^4$-grading.

The third point is an explicit choice for the nine generators of degree $(2, 2, 2, 2)$ in Proposition 1. Namely, we take $i = j \leq 3$ and $k = l \leq 3$ in (7).

Table 1 is also found by computation:

**Corollary 2.** The rigid multiview ideal $J_A$ for $n = 2$ is generated by $I_A(u) + I_A(v)$ together with the nine polynomials $Q(\hat{\Lambda}_5 B_i, \hat{\Lambda}_5 C_k)$ for $1 \leq i, k \leq 3$.

We next come to the case of three cameras:

**Proposition 3.** For $n = 3$, the rigid multiview ideal $J_A$ is minimally generated by 177 polynomials in 18 variables. Its Betti table is given in Table 1.

Proposition 3 is proved by computation. The 177 generators occur in eight symmetry classes of multidegrees. Their numbers in these classes are

\[(110000) : 1 \quad (220111) : 3 \quad (220220) : 9 \quad (211211) : 1 \]
\[(111000) : 1 \quad (211111) : 1 \quad (220211) : 3 \quad (1110111) : 1 \]

For instance, there are nine generators in degree $(2, 2, 0, 2, 2, 0)$, arising from Proposition 1 for the first two cameras. Using various pairs among the three cameras when forming the matrices $B_i, B_j, C_k$ and $C_l$ in (7), we can construct the generators of degree classes $(2, 2, 0, 2, 1, 1)$ and $(2, 1, 1, 2, 1, 1)$.

Table 1 shows the Betti table for $J_A$ in Macaulay2 format. The first two entries (6 and 2) in the 1-column refer to the eight minimal generators of $I_A(u) + I_A(v)$. These are six bilinear forms, representing the three fundamental matrices, and two trilinear forms, representing the trifocal tensor of the three cameras (cf. [2], [5, §15]). The entry 1 in row 5 of column 1 marks the unique sextic generator of $J_A$, which has $\mathbb{Z}^6$-degree $(1, 1, 1, 1, 1, 1)$.

For the case of four cameras we obtain the following result.

**Proposition 4.** For $n = 4$, the rigid multiview ideal $J_A$ is minimally generated by 1176 polynomials in 24 variables. All of them are induced from $n = 3$. Up to symmetry, the degrees of the generators in the $\mathbb{Z}^8$-grading are

\[(11000000) : 1 \quad (22001110) : 3 \quad (22002200) : 9 \quad (21102110) : 1 \]
\[(11100000) : 1 \quad (21101110) : 1 \quad (22002110) : 3 \quad (11101110) : 1 \]
We next give a brief explanation of how the rigid multiview ideals $J_A$ were computed with Macaulay2 [4]. For the purpose of efficiency, we introduce projective coordinates for the image points and affine coordinates for the world points. We work in the corresponding polynomial ring

$$\mathbb{Q}[u,v][X_0, X_1, X_2, Y_0, Y_1, Y_2].$$

The rigid multiview map $\psi_A$ is thus restricted to $\mathbb{R}^3 \times \mathbb{R}^3$. The prime ideal of its graph is generated by the following two classes of polynomials:

1. the $2 \times 2$ minors of the $3 \times 2$ matrices

$$[ A_i \cdot (X_0, X_1, X_2, 1)^\top \mid u_i ] , \ [ A_i \cdot (Y_0, Y_1, Y_2, 1)^\top \mid v_i ] ,$$

2. the dehomogenized distance constraint

$$Q((X_0, X_1, X_2, 1)^\top, (Y_0, Y_1, Y_2, 1)^\top).$$

From this ideal we eliminate the six world coordinates $\{X_0, X_1, X_2, Y_0, Y_1, Y_2\}$.

For a speed up, we exploit the group actions described in Section 1. We replace $A = (A_1, \ldots, A_n)$ and $Q = Q(X,Y)$ by $A' = (M_1 A_1 N, \ldots, M_n A_n N)$ and $Q' = Q(N^{-1}X, N^{-1}Y)$. Here $M_i \in \text{GL}_3(\mathbb{R})$ and $N \in \text{GL}_4(\mathbb{R})$ are chosen so that $A'$ is sparse. The modification to $Q$ is needed since we generally use $N \notin \text{SE}(3, \mathbb{R})$. The elimination above now computes the ideal $(M_1 \otimes \ldots \otimes M_n)J_A$, and it terminates much faster. For example, for $n = 4$, the computation took two minutes for sparse $A'$ and more than one hour for non-sparse $A$. For $n = 5$, Macaulay2 ran out of memory after 18 hours of CPU time for non-sparse $A$. The complete code used in this paper can be accessed via [http://www3.math.tu-berlin.de/combi/dmg/data/ rigidMulti/](http://www3.math.tu-berlin.de/combi/dmg/data/rigidMulti/).

One last question is whether the Gröbner basis property in [3, §2] extends to the rigid case. This does not seem to be the case in general. Only in Proposition 1 can we choose minimal generators that form a Gröbner basis.

Remark 5. Let $n = 2$. The reduced Gröbner basis of $J_A$ in the reverse lexicographic term order is a minimal generating set. For a generic choice of cameras the initial ideal equals

$$\text{in}(J_A) = \langle u_{10}v_{20}, v_{10}v_{20}, u_{10}^2u_{21}v_{10}^2v_{21}, u_{10}u_{21}^2v_{11}^2v_{20}, u_{10}^2u_{21}v_{11}v_{20}, u_{11}^2u_{20}v_{10}v_{21}, u_{11}u_{20}u_{21}v_{10}v_{21}, u_{11}u_{20}v_{11}^2v_{20}, u_{11}u_{20}u_{21}v_{11}v_{20}, u_{11}u_{20}u_{21}v_{11}v_{20}^2 \rangle.$$

For special cameras the exact form of the initial ideal may change. However, up to symmetry the degrees of the generators in the $\mathbb{Z}_+^4$-grading stay the same. In general, a universal Gröbner basis for the rigid multiview ideal $J_A$ consists of octics of degree $(2, 2, 2, 2)$ plus the two quadrics ([6]). This was verified using the Gfan [6] package in Macaulay2. Analogous statements do not hold for $n \geq 3$.

3. Equations for the Rigid Multiview Variety

The computations presented in Section 2 suggest the following conjecture.

Conjecture 6. The rigid multiview ideal $J_A$ is minimally generated by

$$\frac{1}{3}n^6 - \frac{1}{2}n^5 + \frac{1}{10}n^4 + \frac{1}{2}n^3 + \frac{1}{10}n^2 - \frac{1}{3}n \text{ polynomials.}$$

These polynomials come
Lemma 8. All points in $V_A$ found by triangulation. Algebraically, this means

$$X \subset \mathbb{P}^2 \times \mathbb{P}^2$$

by the 9(\(\binom{n}{2}\))^2 octic generators of degree class (220.220.). In other words, equations coming from any two pairs of cameras suffice set-theoretically.

At the moment we have a computational proof only up to $n = 5$. Table 2 offers a summary of the corresponding numbers of generators.

Conjecture 6 implies that $V(J_A)$ is set-theoretically defined by the equations coming from triples of cameras. It turns out that, for the set-theoretic description, pairs of cameras suffice. The following is our main result:

Theorem 7. Suppose that the $n$ focal points of $A$ are in general position in $\mathbb{P}^3$. The rigid multiview variety $V(J_A)$ is cut out as a subset of $V_A \times V_A$ by the 9(\(\binom{n}{2}\))^2 octic generators of degree class (220.220.). In other words, equations coming from any two pairs of cameras suffice set-theoretically.

With notation as in the introduction, the relevant octic polynomials are

$$T(\tilde{\lambda}_5 B_{i_1}^{j_1 k_1}, \tilde{\lambda}_5 B_{i_2}^{j_1 k_1}, \tilde{\lambda}_5 C_{i_3}^{j_2 k_2}, \tilde{\lambda}_5 C_{i_4}^{j_2 k_2}),$$

for all possible choices of indices. Let $H_A$ denote the ideal generated by these polynomials in $\mathbb{R}[u, v]$, the polynomial ring in 6n variables. As before, we write $I_A(u) + I_A(v)$ for the prime ideal that defines the 6-dimensional variety $V_A \times V_A$ in $(\mathbb{P}^2)^n \times (\mathbb{P}^2)^n$. It is generated by 2(\(\binom{n}{2}\)) bilinear forms and 2(\(\binom{n}{3}\)) trilinear forms, corresponding to fundamental matrices and trifocal tensors. In light of Hilbert’s Nullstellensatz, Theorem 7 states that the radical of $H_A + I_A(u) + I_A(v)$ is equal to $J_A$. To prove this, we need a lemma.

A point $u$ in the multiview variety $V_A \subset (\mathbb{P}^2)^n$ is triangulable if there exists a pair of indices $(j, k)$ such that the matrix $B^{j k}$ has rank 5. Equivalently, there exists a pair of cameras for which the unique world point $X$ can be found by triangulation. Algebraically, this means $X = \tilde{\lambda}_5 B_{i}^{j k}$ for some $i$.

Lemma 8. All points in $V_A$ are triangulable except for the pair of epipoles, $(e_{1-2}, e_{2-1})$, in the case where $n = 2$. Here, the rigid multiview variety $V(J_A)$ contains the threefolds $V_A(u) \times (e_{1-2}, e_{2-1})$ and $(e_{1-2}, e_{2-1}) \times V_A(v)$.

| $n$ | degree | 2  | 3  | 6  | 7  | 8  | total | timing (s) |
|-----|--------|----|----|----|----|----|-------|------------|
| 2   |        | 1  | 2  | 9  | 1  | <1  |       |            |
| 3   |        | 6  | 2  | 1  | 24 | 144 | 177   | 14         |
| 4   |        | 12 | 8  | 16 | 240| 900 | 1176  | 130        |
| 5   |        | 20 | 20 | 100| 1200|3600 |4940   |24064       |

Table 2. The known minimal generators of the rigid multiview ideals, listed by total degree, for up to five cameras. There are no minimal generators of degrees 4 or 5. Average timings (in seconds), using the speed up described above, are in the last column.
Proof. Let us first consider the case of $n = 2$ cameras. The first claim holds because the back-projected lines of the two camera images $u_1$ and $u_2$ always span a plane in $\mathbb{P}^3$ except when $u_1 = e_{1\leftarrow 2}$ and $u_2 = e_{2\leftarrow 1}$. In that case both back-projected lines agree with the common baseline $\beta_{12}$. Alternatively, we can check algebraically that the variety defined by the $5 \times 5$-minors of the matrix $B$ consists of the single point $(e_{1\leftarrow 2}, e_{2\leftarrow 1})$.

For the second claim, fix a generic point $X$ in $\mathbb{P}^3$ and consider the surface

$$(8) \quad X^Q = \{Y \in \mathbb{P}^3 : Q(X, Y) = 0\}.$$ 

Working over $\mathbb{C}$, the baseline $\beta_{12}$ is either tangent to $X^Q$, or it meets that quadric in exactly two points. Our assumption on the genericity of $X$ implies that no point in the intersection $\beta_{12} \cap X^Q$ is a focal point. This gives

$$(9) \quad (A_1 X, A_2 X, A_1 Y, A_2 Y) = (A_1 X, A_2 X, e_{1\leftarrow 2}, e_{2\leftarrow 1}).$$

The point $(A_1 X, A_2 X)$ lies in the multiview variety $V_A(u)$. Each generic point in $V_A(u)$ has this form for some $X$. Hence $\text{(9)}$ proves the desired inclusion $V_A(u) \times (e_{1\leftarrow 2}, e_{2\leftarrow 1}) \subset V(J_A)$. The other inclusion $\text{(e_{1\leftarrow 2}, e_{2\leftarrow 1}) \times V_A(v) \subset V(J_A)}$ follows by switching the roles of $u$ and $v$.

If there are more than two cameras then for each world point $X$, due to general position of the cameras, there is a pair of cameras such that $X$ avoids the pair’s baseline. This shows that each point is triangulable if $n \geq 3$. $\square$

Proof of Theorem $[7]$ It follows immediately from the definition of the ideals in question that the following inclusion of varieties holds in $(\mathbb{P}^2)^n \times (\mathbb{P}^2)^n$:

$$V(J_A) \subseteq V(I_A(u) + I_A(v) + H_A).$$

We prove the reverse inclusion. Let $(u, v)$ be a point in the right hand side.

Suppose that $u$ and $v$ are both triangulable. Then $v$ has a unique preimage $X$ in $\mathbb{P}^3$, determined by a single camera pair $\{A_{j_1}, A_{k_1}\}$. Likewise, $v$ has a unique preimage $Y$ in $\mathbb{P}^3$, also determined by a single camera pair $\{A_{j_2}, A_{k_2}\}$. There exist indices $i_1, i_2 \in \{1, 2, 3, 4, 5, 6\}$ such that

$$X = \tilde{\lambda}_5 B_{i_1}^{j_1 k_1} \quad \text{and} \quad Y = \tilde{\lambda}_5 C_{i_2}^{j_2 k_2}.$$ 

Suppose that $(u, v)$ is not in $V(J_A)$. Then $Q(X, Y) \neq 0$. This implies

$$Q(X, Y) = T(X, X, Y, Y) = T(\tilde{\lambda}_5 B_{i_1}^{j_1 k_1}, \tilde{\lambda}_5 B_{i_1}^{j_1 k_1}, \tilde{\lambda}_5 C_{i_2}^{j_2 k_2}, \tilde{\lambda}_5 C_{i_2}^{j_2 k_2}) \neq 0,$$

and hence $(u, v) \notin V(H_A)$. This is a contradiction to our choice of $(u, v)$.

It remains to consider the case where $u$ is not triangulable. By Lemma $\text{[8]}$ we have $n = 2$, as well as $v = (e_{1\leftarrow 2}, e_{2\leftarrow 1})$ and $(u, v) \in V(J_A)$. The case where $u$ is not triangulable is symmetric, and this proves the theorem. $\square$

The equations in Theorem $[7]$ are fairly robust, in the sense that they work as well for many special position scenarios. However, when the cameras $A_1, A_2, \ldots, A_n$ are generic then the number $9(n^2) + 2$ of octics that cut out the divisor $V(J_A)$ inside $V_A \times V_A$ can be reduced dramatically, namely to 16.

Corollary 9. As a subset of the 6-dimensional ambient space $V_A \times V_A$, the 5-dimensional rigid multiview variety $V(J_A)$ is cut out by 16 polynomials of degree class $(220, 220, \ldots)$. One choice of such polynomials is given by

$$Q(\tilde{\lambda}_5 B_{i_1}^{12}, \tilde{\lambda}_5 C_{k_2}^{12}), \quad Q(\tilde{\lambda}_5 B_{i_1}^{13}, \tilde{\lambda}_5 C_{k_2}^{13}),$$

$$Q(\tilde{\lambda}_5 B_{i_2}^{12}, \tilde{\lambda}_5 C_{k_2}^{12}), \quad Q(\tilde{\lambda}_5 B_{i_2}^{13}, \tilde{\lambda}_5 C_{k_2}^{13})$$

for all $1 \leq i, k \leq 2$. 
Proof. First we claim that for each triangulable point \( u \) at least one of the matrices \( B^{12} \) or \( B^{13} \) has rank 5, and the same for \( v \) with \( C^{12} \) or \( C^{13} \). We prove this by contradiction. By symmetry between \( u \) and \( v \), we can assume that \( \text{rk}(B^{12}) = \text{rk}(B^{13}) = 4 \). Then \( u_3 = e_{3+i} \), \( u_2 = e_{2+i} \), and \( u_1 = e_{1+i} = e_{1+i-3} \). However, this last equality of the two epipoles is a contradiction to the hypothesis that the focal points of the cameras \( A_1, A_2, A_3 \) are not collinear.

Next we claim that if \( B^{12} \) has rank 5 then at least one of the submatrices \( B_1^{12} \) or \( B_2^{12} \) has rank 5, and the same for \( B^{13} \), \( C^{12} \) and \( C^{13} \). Note that the bottom \( 4 \times 6 \) submatrix of \( B^{12} \) has rank 4, since the first four columns are linearly independent, by genericity of \( A_1 \) and \( A_2 \). The claim follows. \( \square \)

4. Other Constraints, More Points, and No Labels

In this section we discuss several extensions of our results. A first observation is that there was nothing special about the constraint \( Q \) in \((1)\). For instance, fix positive integers \( d \) and \( e \), and let \( Q(X,Y) \) be any irreducible polynomial that is bihomogeneous of degree \((d,e)\). Its variety \( V(Q) \) is a hypersurface of degree \((d,e)\) in \( \mathbb{P}^3 \times \mathbb{P}^3 \). The following analogue to Theorem \(7\) holds, if we define the map \( \psi_A \) as in \((5)\).

**Theorem 10.** The closure of the image of the map \( \psi_A \) is cut out in \( V_A \times V_A \) by \( 9(n\choose 2)^2 \) polynomials of degree class \((d,d,0,\ldots,e,e,0,\ldots)\). In other words, the equations coming from any two pairs of cameras suffice set-theoretically.

**Proof.** The tensor \( T \) that represents \( Q \) now lives in \( \text{Sym}_d(\mathbb{R}^4) \otimes \text{Sym}_e(\mathbb{R}^4) \). The polynomial \(7\) vanishes on the image of \( \psi_A \) and has degree \((d,d,e,e)\). The proof of Theorem \(7\) remains valid. The surface \( X^Q \) in \((8)\) is irreducible of degree \( e \) in \( \mathbb{P}^3 \). These polynomials cut out that image inside \( V_A \times V_A \). \( \square \)

**Remark 11.** In the generic case, we can replace \( 9(n\choose 2)^2 \) by 16, as in Corollary \(9\).

Another natural generalization is to consider \( m \) world points \( X_1,\ldots,X_m \) that are linked by one or several constraints in \( (\mathbb{P}^3)^m \). Taking images with \( n \) cameras, we obtain a variety \( V(J_A) \) which lives in \( (\mathbb{P}^2)^{mn} \). For instance, if \( m = 4 \) and \( X_1, X_2, X_3, X_4 \) are constrained to lie on a plane in \( \mathbb{P}^3 \), then \( Q = \det(X_1, X_2, X_3, X_4) \) and \( V(J_A) \) is a variety of dimension 11 in \( (\mathbb{P}^2)^{4n} \). Taking \( 6 \times 6 \)-matrices \( B, C, D, E \) as in \((1)\) for the four points, we then form
\[
(10) \quad \det(\tilde{\lambda}_S B_i, \tilde{\lambda}_S C_j, \tilde{\lambda}_S D_k, \tilde{\lambda}_S E_l) \quad \text{for all} \ 1 \leq i,j,k,l \leq 6.
\]

For \( n = 2 \) we verified with Macaulay2 that the prime ideal \( J_A \) is generated by 16 of these determinants, along with the four bilinear forms for \( V_A^{4} \).

**Proposition 12.** The variety \( V(J_A) \) is cut out in \( V_A^{4} \) by the \( 16(n\choose 2)^4 \) polynomials from \((10)\). In other words, the equations coming from any two pairs of cameras suffice set-theoretically.

**Proof.** Each polynomial \((10)\) is in \( J_A \). The proof of Theorem \(7\) remains valid. The planes \( (X_i, X_j, X_k)^4 \) intersect the baseline \( \beta_{12} \) in one point each. \( \square \)

To continue the theme of rigidity, we may impose distance constraints on pairs of points. Fixing a nonzero distance \( d_{ij} \) between points \( i \) and \( j \) gives
\[
Q_{ij} = (X_{i0}X_{j3} - X_{j0}X_{i3})^2 + (X_{i1}X_{j3} - X_{j1}X_{i3})^2 + (X_{i2}X_{j3} - X_{j2}X_{i3})^2 - d_{ij}^2 X_{i3}^2 X_{j3}^2.
\]
We are interested in the image of the variety $\mathcal{V} = V(Q_{ij} : 1 \leq i < j \leq m)$ under the multiview map $\psi_A$ that takes $(\mathbb{P}^3)^m$ to $(\mathbb{P}^2)^{mn}$. For instance, for $m = 3$, we consider the variety $\mathcal{V} = V(Q_{12}, Q_{13}, Q_{23})$ in $(\mathbb{P}^3)^3$, and we seek the equations for its image under the multiview map $\psi_A$ into $(\mathbb{P}^2)^{3n}$. Note that $\mathcal{V}$ has dimension 6, unless we are in the collinear case. Algebraically,

\begin{equation}
(d_{12}+d_{13}+d_{23})(d_{12}+d_{13}-d_{23})(d_{12}-d_{13}+d_{23})(-d_{12}+d_{13}+d_{23}) = 0.
\end{equation}

If this holds then $\dim(\mathcal{V}) = 5$. The same argument as in Theorem 7 yields:

**Corollary 13.** The rigid multiview variety $\psi_A(\mathcal{V})$ has dimension six, unless (11) holds, in which case the dimension is five. It has real points if and only if $d_{12}, d_{13}, d_{23}$ satisfy the triangle inequality. It is cut out in $V_A^3$ by $27(n^2)$ biquadratic equations, coming from the $9\binom{n}{2}$ equations for any two of the three points.

In many computer vision applications, the $m$ world points and their images in $\mathbb{P}^2$ will be unlabeled. To study such questions, we propose to work with the unlabeled rigid multiview variety. This is the image of the rigid multiview variety under the quotient map $((\mathbb{P}^2)^m)^n \to (\text{Sym}_m(\mathbb{P}^2))^n$.

Indeed, while labeled configurations in the plane are points in $(\mathbb{P}^2)^m$, unlabeled configurations are points in the Chow variety $\text{Sym}_m(\mathbb{P}^2)$. This is the variety of ternary forms that are products of $m$ linear forms (cf. [7, §8.6]). It is embedded in the space $\mathbb{P}^{(m+2)}-1$ of all ternary forms of degree $m$.

**Example 14.** Let $m = n = 2$. The Chow variety $\text{Sym}_2(\mathbb{P}^2)$ is the hypersurface in $\mathbb{P}^5$ defined by the determinant of a symmetric $3 \times 3$-matrix $(u_{ij})$. The quotient map $(\mathbb{P}^2)^2 \to \text{Sym}_2(\mathbb{P}^2) \subset \mathbb{P}^5$ is given by the formulas:

\begin{align*}
a_{00} &= 2u_{10}v_{10}, & a_{11} &= 2u_{11}v_{11}, & a_{22} &= 2u_{12}v_{12}, \\
a_{01} &= u_{11}v_{10} + u_{10}v_{11}, & a_{02} &= u_{12}v_{10} + u_{10}v_{12}, & a_{12} &= u_{12}v_{11} + u_{11}v_{12}.
\end{align*}

Similarly, for the two unlabeled images under the second camera we use

\begin{align*}
b_{00} &= 2u_{20}v_{20}, & b_{11} &= 2u_{21}v_{21}, & b_{22} &= 2u_{22}v_{22}, \\
b_{01} &= u_{21}v_{20} + u_{20}v_{21}, & b_{02} &= u_{22}v_{20} + u_{20}v_{22}, & b_{12} &= u_{22}v_{21} + u_{21}v_{22}.
\end{align*}

The unlabeled rigid multiview variety is the image of $V(J_A) \subset V_A \times V_A$ under the quotient map that takes two copies of $(\mathbb{P}^2)^2$ to two copies of $\text{Sym}_2(\mathbb{P}^2) \subset \mathbb{P}^5$. This quotient map is given by $(u_{1i}, v_1) \mapsto a_i$, $(u_{2j}, v_2) \mapsto b_j$.

We first compute the image of $V_A \times V_A$ in $\mathbb{P}^5 \times \mathbb{P}^5$, denoted $\text{Sym}_2(V_A)$. Its ideal has seven minimal generators, three of degree $(1,1)$, and one each in degrees $(3,0), (2,1), (1,2), (0,3)$. The generators in degrees $(3,0)$ and $(0,3)$ are $\det(a_{ij})$ and $\det(b_{ij})$. The five others depend on the cameras $A_1, A_2$.

Now, to get equations for the unlabeled rigid multiview variety, we intersect the ideal $J_A$ with the subring $R[a, b]$ of bisymmetric homogeneous polynomials in $R[u, v]$. This results in nine new generators which represent the distance constraint. One of them is a quartic of degree $(2,2)$ in $(a,b)$, and the other eight are quintics, four of degree $(2,3)$ and four of degree $(3,2)$.

Independently of the specific constraints considered in this paper, it is of interest to characterize the pictures of $m$ unlabeled points using $n$ cameras. This gives rise to the **unlabeled multiview variety** $\text{Sym}_m(V_A)$ in $(\mathbb{P}^{(m+2)}-1)^n$. It would be desirable to know the prime ideal of $\text{Sym}_m(V_A)$ for any $n$ and $m$. 

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Appendix A. Computations

We performed several random experiments in this paper. Our hardware was a cluster with Intel Xeon X2630v2 Hexa-Cores (2.8 GHz) and 64GB main memory per node. The software was Macaulay2, version 1.8.2.1. All computations were single-threaded.

The tests were repeated several times with random input. The exact running times vary, even with identical input; the Table lists the average values. It is not surprising that increasing $n$, the number of cameras, increases the running times considerably. Therefore we adapted the number of experiments according to $n$.

For all the statements in Section 2 regarding two cameras, (i.e. $n = 2$) we performed at least 1000 computations, with one exception. The statement regarding the universal Gröbner basis in Remark is based on 20 experiments. Regarding three and four cameras (i.e., $n \in \{3, 4\}$) we performed at least 100 computations each. For $n = 5$ we performed at least 20 computations each. Example was checked with 50 choices of random cameras.

In Listing we show Macaulay2 code which can be employed to establish Proposition. The complete code for all our results can be accessed via http://www3.math.tu-berlin.de/combi/dmg/data/rigidMulti/

Lines 1–4 define the rings in which the computations take place. Lines 6–7 produce random camera matrices. Here the code shown differs slightly from the code used. What we omitted is the extra code which checks the matrices before they are processed. To assert general position we check that none of the minors vanish as in §2. However, our experiments suggest that it suffices to check that the focal points of the cameras are in linear
general position. The multiview map $\phi_A$ from (3) is encoded in lines 11–14. Line 15 is the rigid constraint (4). The actual computation is the elimination in line 16. The rigid multiview ideal $J_A$ is defined in lines 17–18, and the final output are the multidegrees of $J_A$.

```
R1 = QQ[u_(1,0)..u_(1,2)] ** QQ[u_(2,0)..u_(2,2)];
R2 = QQ[X_0..X_2] ** QQ[Y_0..Y_2];
S = R1 ** R2;

n = 2;
for i from 1 to n do (A_i = random(ZZ^3,ZZ^4,Height=>20););

I = ideal();
for j from 1 to n do (A = random(ZZ^3,ZZ^4,Height=>20);
I = I + minors(2,A_j * (genericMatrix(S,X_0,3,1)||matrix{{1}}) |
            genericMatrix(S,u_(j,0),3,1));
I = I + minors(2,A_j * (genericMatrix(S,Y_0,3,1)||matrix{{1}}) |
            genericMatrix(S,v_(j,0),3,1));
I = I + ideal((X_0-Y_0)^2 + (X_1-Y_1)^2 + (X_2-Y_2)^2-1);
I = eliminate({X_0,X_1,X_2,Y_0,Y_1,Y_2},I);

F = map(R1,S);
J = F(I);
degrees(J)
```

Listing 1. Compute $J_A$ for two cameras

In these computations the world coordinates are dehomogenized by setting the last coordinate to 1, as explained at the end of Section 2. Notice that the code below line 7 does not need to be modified if we increase $n$.

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