Improved Bounds on the Randomized and Quantum Complexity of Initial-Value Problems

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Abstract

We study the problem, initiated in [8], of finding randomized and quantum complexity of initial-value problems. We showed in [8] that a speed-up in both settings over the worst-case deterministic complexity is possible. In the present paper we prove, by defining new algorithms, that further improvement in upper bounds on the randomized and quantum complexity can be achieved. In the Hölder class of right-hand side functions with $r$ continuous bounded partial derivatives, with $r$-th derivative being a Hölder function with exponent $\rho$, the $\varepsilon$-complexity is shown to be $O\left((1/\varepsilon)^{1/(r+\rho+1/3)}\right)$ in the randomized setting, and $O\left((1/\varepsilon)^{1/(r+\rho+1/2)}\right)$ on a quantum computer (up to logarithmic factors). This is an improvement for the general problem over the results from [8]. The gap still remaining between upper and lower bounds on the complexity is further discussed for a special problem. We consider scalar autonomous problems, with the aim of computing the solution at the end point of the interval of integration. For this problem, we fill up the gap by establishing (essentially) matching upper and lower complexity bounds. We show that the complexity in this case is $\Theta\left((1/\varepsilon)^{1/(r+\rho+1/2)}\right)$ in the randomized setting, and $\Theta\left((1/\varepsilon)^{1/(r+\rho+1)}\right)$ in the quantum setting (again up to logarithmic factors). Hence, this problem is essentially as hard as the integration problem.

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1 Introduction

Significant progress has been made in recent years in the field of quantum complexity of numerical problems. Integration ([11], followed by [3]) was the first problem to be so studied. Other problems were next analyzed, such as approximation [4, 5] and path integration [12]. The only paper that has studied the randomized and quantum complexity of initial-value problems for ordinary differential equations is [8]. This paper showed that we can achieve a nontrivial speed-up by going from the worst-case deterministic setting to the randomized or quantum settings. The idea in [8] was to use the optimal deterministic algorithm based on integral information [1], and replace integrals in a suitable way by optimal randomized or quantum approximations [10, 11]. We recall the results from [8] in Theorem 1.

In the present paper, we show that further improvement in upper bounds on the randomized and quantum complexity is possible. We first define a new deterministic integral algorithm for initial-value problems (Section 3). Although this algorithm is not optimal in the deterministic worst-case setting, it is better suited for randomization and implementation on a quantum computer than the algorithm used in [8]. Randomized and quantum algorithms are defined by a suitable application of optimal randomized and quantum algorithms for summation of real numbers [1, 9] (Section 4). The reduction of the total cost is achieved due to a better balance, compared to the algorithms from [8], between the deterministic and random components of the cost.

New upper bounds on the complexity are shown in Theorem 2 in Section 5. In the Hölder class of right-hand side functions with \( r \) continuous bounded partial derivatives, with \( r \)-th derivative being a Hölder function with exponent \( \rho \), the \( \varepsilon \)-complexity is shown to be (up to logarithmic factors) \( O \left( \frac{1}{\varepsilon} \right)^{1/(r+\rho+1/3)} \) in the randomized setting, and \( O \left( \frac{1}{\varepsilon} \right)^{1/(r+\rho+1/2)} \) on a quantum computer. Noticeable improvement in both settings is thus achieved, compared to the bounds from Theorem 1. The gap between upper and lower complexity bounds is reduced (but still not cancelled).

In order to further reduce the gap between the bounds, we turn to a special case of the general problem. We study in Section 6 the complexity of computing the solution of a scalar autonomous problem at one single point. In [8], we only showed (non-optimal) upper bounds on the randomized and quantum complexity of this problem. The question about lower bounds was left open. We provide essentially matching upper and lower complexity bounds in Theorem 3. Upper bounds are established by using a bisection argument, while lower bounds by reducing the problem to the summation of real numbers. Up to logarithmic factors, the complexity turns out to be \( \Theta \left( \frac{1}{\varepsilon} \right)^{1/(r+\rho+1/2)} \) in the randomized setting, and \( \Theta \left( \frac{1}{\varepsilon} \right)^{1/(r+\rho+1)} \) in the quantum setting. The gap between upper and lower bounds is thus essentially closed. Up to logarithmic factors, the problem considered turns out to be as difficult as the integration problem.

2 Preliminaries

We deal with the randomized and quantum solution of a system of ordinary differential equations...
with initial conditions
\[ z'(t) = f(z(t)), \quad t \in [a, b], \quad z(a) = \eta, \quad (1) \]
where \( f : \mathbb{R}^d \to \mathbb{R}^d \), the initial vector \( \eta \) is in \( \mathbb{R}^d \), and the solution \( z \) maps \( [a, b] \) into \( \mathbb{R}^d \). We assume that \( f(\eta) \neq 0 \).

This formulation covers nonautonomous systems \( z'(t) = f(t, z(t)) \) with \( f : \mathbb{R}^{d+1} \to \mathbb{R}^d \), which can be written in the form (1) by adding one scalar equation:
\[
\begin{bmatrix}
u'(t) \\
z'(t)
\end{bmatrix} = \begin{bmatrix}1 \\
f(u(t), z(t))\end{bmatrix}
\]
with an additional initial condition \( u(a) = a \). We assume that the right-hand side function \( f = [f^1, \ldots, f^d]^T \) belongs to the Hölder class \( F^{r, \rho} \). Given an integer \( r \geq 0 \), a number \( \rho \in (0, 1] \), positive numbers \( D_0, D_1, \ldots, D_r \) and \( H \), we set
\[
F^{r, \rho} = \{ f : \mathbb{R}^d \to \mathbb{R}^d \mid f \in C^r(\mathbb{R}^d), \quad |\partial^i f^j(y)| \leq D_i, \quad i = 0, 1, \ldots, r, \\
|\partial^i f^j(y) - \partial^i f^j(z)| \leq H \|y - z\|^\rho, \quad y, z \in \mathbb{R}^d, \quad j = 1, 2, \ldots, d \}, \quad (2)
\]
where \( \partial^i f^j \) represents all partial derivatives of order \( i \) of the \( j \)-th component of \( f \), and \( \| \cdot \| \) denotes the maximum norm in \( \mathbb{R}^d \). We assume that \( \rho = 1 \) for \( r = 0 \), which assures that \( f \) is a Lipschitz function.

We formulate the problem and shortly recall basic definitions concerning randomized and quantum settings. Our aim is to compute a bounded function \( F^{r, \rho} \) on \([a, b]\) that approximates the solution \( z \). Letting \( \{x_i\} \) be the uniform partition of \([a, b]\), so that \( x_i = a + ih \) with \( h = (b - a) / n \), we will construct \( l \) based on approximations \( a_i(f) \) to \( z(x_i) \) for \( i = 0, 1, \ldots, n \). We assume that available information about the right-hand side \( f \) is given by a subroutine that computes values of \( f \) or its partial derivatives. In the randomized setting, we allow for a random selection of points at which the values are computed. On a quantum computer, by subroutine calls we mean applications of a quantum query operator for \((\text{a component of}) f \), or evaluations of components of \( f \) or its partial derivatives on a classical computer. The transformation \( \phi \) that computes \( l \) based on available information is called an algorithm.

To be more specific, let \((\Omega, \Sigma, P)\) be a probability space. Let the mappings \( \omega \in \Omega \mapsto a^\omega_i(f) \) be random variables for each \( f \in F^{r, \rho} \). By an algorithm we mean a tuple
\[
\phi = \{(a^\omega_0(\cdot), a^\omega_1(\cdot), \ldots, a^\omega_n(\cdot))_{\omega \in \Omega}, \psi\}, \quad (3)
\]
where \( \psi \) is a mapping that produces a bounded function \( l^\omega \) based on \( a^\omega_0(\cdot), a^\omega_1(\cdot), \ldots, a^\omega_n(\cdot) \),
\[
l^\omega(t) = \psi(a^\omega_0(f), a^\omega_1(f), \ldots, a^\omega_n(f))(t), \quad (4)
\]
for \( t \in [a, b] \). The error of \( \phi \) at \( f \) is defined by
\[
e^\omega(\phi, f) = \sup_{t \in [a, b]} \|z(t) - l^\omega(t)\|. \quad (5)
\]
We assume that the mapping \( \omega \in \Omega \to e^\omega(\phi, f) \) is a random variable for each \( f \in F^{r, \rho} \).

In the randomized setting, the error of \( \phi \) in the class \( F^{r, \rho} \) is given by the maximal dispersion of \( e^\omega(\phi, f) \),
\[
e^{\text{rand}}(\phi, F^{r, \rho}) = \sup_{f \in F^{r, \rho}} (Ee^\omega(\phi, f)^2)^{1/2}, \quad (6)
\]
where $E$ is the expectation. (We could consider as well the maximal expected value of $e^\omega(\phi, f)$; this would only change the constants in our results.) The cost of an algorithm $\phi$ in the randomized setting is measured by a number of subroutine calls needed to compute an approximation. For a given $\varepsilon > 0$, by the $\varepsilon$-complexity of the problem, $\operatorname{comp}^\text{rand}(F^{r, \rho}, \varepsilon)$, we mean the minimal cost of an algorithm $\phi$ taken among all $\phi$ such that $e^{\text{rand}}(\phi, F^{r, \rho}) \leq \varepsilon$.

On a quantum computer, the output of an algorithm is also a random variable (taking a finite number of values). The randomness in the quantum setting results from quantum measurement operations $[3]$. The right-hand side function $f$ can be accessed through applications of a quantum query operator $Q_f$ on a quantum space (defined through values of components of $f$). Evaluations of components of $f$ or its partial derivatives on a classical computer are also allowed. For a detailed discussion of the quantum query operator, and of the effect of quantum measurement, the reader is referred to $[3]$. The error of an algorithm $\phi$ at $f$ in the quantum setting is again given by (5), and the error of $\phi$ in the class $F^{r, \rho}$ by

$$e^{\text{quant}}(\phi, F^{r, \rho}, \delta) = \sup_{f \in F^{r, \rho}} \inf \{ \alpha | \mathbb{P}\{e^\omega(\phi, f) > \alpha\} \leq \delta \},$$

for a given number $\delta$, where $0 < \delta < 1/2$. For $\varepsilon > 0$, $[6]$ implies that the bound $e^\omega(\phi, f) \leq \varepsilon$ holds with probability at least $1 - \delta$ for each $f$ iff $e^{\text{quant}}(\phi, F^{r, \rho}, \delta) \leq \varepsilon$. Hence, $1 - \delta$ is the (minimal) success probability in computing an $\varepsilon$-approximation.

The value of $\delta$ is usually set to $\delta = 1/4$. The success probability can then be increased to be at least $1 - \delta$ (for arbitrarily small $\delta$) by computing component by component the median of $c \log 1/\delta$ repetitions of the algorithm, where $c$ is a positive number independent of $\delta$, see $[4]$. The cost of an algorithm $\phi$ in the quantum setting is measured by the number of quantum queries, together with the number of classical evaluations of $f$ or its partial derivatives, needed to compute an approximation. For a given $\varepsilon > 0$, by the quantum $\varepsilon$-complexity of the problem, $\operatorname{comp}^{\text{quant}}(F^{r, \rho}, \varepsilon, \delta)$, we mean the minimal cost of a quantum algorithm $\phi$ taken among all $\phi$ such that $e^{\text{quant}}(\phi, F^{r, \rho}, \delta) \leq \varepsilon$.

We now recall upper and lower bounds on the randomized and quantum complexity for problem $[4]$ obtained in $[8]$. (We write below $\log$ for $\log_2$, although the base of the logarithm is not crucial.)

**Theorem 1** ($[8]$) For problem $[4]$, we have that

$$\operatorname{comp}^\text{rand}(F^{r, \rho}, \varepsilon) = O\left(\frac{1}{\varepsilon} \left(\frac{r + \rho + 3/2}{(r + \rho + 1/2)(r + \rho + 1)} \right) \log \frac{1}{\varepsilon}\right),$$

$$\operatorname{comp}^\text{quant}(F^{r, \rho}, \varepsilon, \delta) = O\left(\frac{1}{\varepsilon} \left(\frac{r + \rho + 2}{(r + \rho + 1)^2} \log \frac{1}{\varepsilon} + \log \frac{1}{\delta}\right)\right).$$

Moreover, for $d \geq 2$

$$\operatorname{comp}^\text{rand}(F^{r, \rho}, \varepsilon) = \Omega\left(\left(\frac{1}{\varepsilon}\right)^{\frac{r + \rho + 1/2}{(r + \rho + 1)^2}}\right),$$

$$\operatorname{comp}^\text{rand}(F^{r, \rho}, \varepsilon) = \Omega\left(\left(\frac{1}{\varepsilon}\right)^{\frac{1}{r + \rho + 1/2}}\right).$$
and, for $0 < \delta \leq 1/4$,

\[
\text{comp}^{\text{quant}}(F^{r,\rho}, \varepsilon, \delta) \geq \text{comp}^{\text{quant}}(F^{r,\rho}, \varepsilon, 1/4) = \Omega \left( \left( \frac{1}{\varepsilon} \right)^{\frac{1}{r+\rho}} \right). \tag{11}
\]

The constants in the $O$- and $\Omega$-notation only depend on the class $F^{r,\rho}$, and are independent of $\varepsilon$ and $\delta$.

In the deterministic worst-case setting, if only the values of $f$ or its partial derivatives can be accessed, the complexity of problem (I) is $\Theta(\varepsilon^{-1/(r+\rho)})$. Hence, Theorem 1 shows a speed-up in both randomized and quantum settings over the deterministic setting for all $r$ and $\rho$. Note also that there is a gap in the randomized and quantum settings between the upper and lower complexity bounds given in Theorem 1.

In this paper, we show that further improvement in upper bounds on the randomized and quantum complexities is possible (Theorem 2). We start in the next section by defining a new deterministic algorithm that will be used to design randomized and quantum algorithms in Section 4.

In the next sections we shall need results on randomized and quantum computation of the mean of real numbers, which we now recall. Suppose we wish to compute the value

\[
S = \frac{1}{s} \sum_{i=1}^{s} x_i, \tag{12}
\]

for $-1 \leq x_i \leq 1$. The $\varepsilon$-complexity of this problem in the randomized setting is defined as the minimal number of accesses to $x_1, \ldots, x_s$ that is sufficient to find a random approximation $A^\omega$ to $S$ with expected error at most $\varepsilon$, $E|A^\omega - S| \leq \varepsilon$. It is proportional to

\[
\min\{s, (1/\varepsilon)^2\} \tag{13}
\]

due to the result of Mathé, see for a discussion [6]. Note that $E|A^\omega - S| \leq \varepsilon$ implies that

\[
\mathbb{P}\{|A^\omega - S| > 4\varepsilon\} \leq 1/4. \tag{14}
\]

On a quantum computer we can do better than this. The probabilistic error criterion (14) is used in the quantum setting, and the cost of an algorithm is measured by a number of quantum queries (quantum accesses to $x_1, \ldots, x_s$). It is shown in [1] (upper bound) and [9] (lower bound) that the quantum complexity of computing the mean is proportional to

\[
\min\{s, 1/\varepsilon\}. \tag{15}
\]

### 3 Deterministic Algorithm

We define a deterministic integral algorithm for solving (I), which will be the subject to randomization and implementation on a quantum computer in the next section.

Let $m, n \geq 1$. Define $\{x_i\}$ to be $n+1$ equidistant partition points of $[a, b]$, so that $x_i = a + ih$ for
for sufficiently small $h$ and $\bar{h}$ independent of $l$. In the sequel, we shall need an error bound for $l^*_i$ by successive applications of Taylor’s method with step size $\bar{h}$. There exists a constant $\{l^*_i\}$ depending only on the parameters of the class $F^{r,\rho}$ (and independent of $i$, $y^*_i$ and $n$) such that

$$\sup_{t \in [x_i, x_{i+1}]} \|z^*_i(t) - l^*_i(t)\| \leq M \bar{h}^{r+\rho},$$

for sufficiently small $h$ (Lh $\leq \ln 2$, where $L$ is a Lipschitz constant for $f$).

The algorithm defined above is not optimal in the deterministic worst-case setting. It follows from this Lemma and the results from [8] that its worst-case error in $[a, b]$ in the class $F^{r,\rho}$ is $O(1/(n(nm)^{r+\rho}))$. This is achieved by using $\Theta(nm)$ evaluations. With the same number of evaluations it is however possible to get error $O(1/(nm)^{r+\rho+1})$, see [7].

In order to define randomized and quantum algorithms, we express (17) in an equivalent form. Defining

$$w^i_j(y) = \sum_{k=0}^{r} \frac{1}{k!} f^{(k)}(y^*_j)(y - y^*_j)^k$$

(20)
and
\[ g_{ij}(u) = \frac{1}{\bar{h}^{r+\rho}} \left( f(l^*_{ij}(z^j_i + u\bar{h})) - w^*_{ij}(l^*_{ij}(z^j_i + u\bar{h})) \right), \quad u \in [0, 1], \] (21)
we can write (17) as
\[ y^*_{i+1} = y^*_i + \sum_{j=0}^{m-1} \int_{z^j_i}^{z^j_{i+1}} w^*_{ij}(l^*_{ij}(t)) dt + \bar{h}^{r+\rho+1} \sum_{j=0}^{m-1} \int_0^1 g_{ij}(u) du. \] (22)

Arguments similar to those used in the proof of Lemma in [8] yield (after replacing the interval \([x_i, x_{i+1}]\) by \([z^j_i, z^j_{i+1}]\), \(h\) by \(\bar{h}\) and \(y^*_i, l^*_i, w^*_i\) by \(y^i_j, l^i_{ij}, w^i_{ij}\), respectively) that the functions \(g_{ij}\) are in \(C(r)\) ([0, 1]), and the derivatives of \(g_{ij}\) of order 0, 1, \ldots, \(r\) are bounded by constants depending only on the parameters of the class \(F_{r,\rho}\). Moreover,
\[ \|g^{(r)}_{ij}(u) - g^{(r)}_{ij}(\bar{u})\| \leq \bar{H}|u - \bar{u}|^\rho, \quad u, \bar{u} \in [0, 1], \]
where \(\bar{H}\) is a constant depending only on the parameters of \(F_{r,\rho}\).

4 Randomized and Quantum Algorithms

We shall denote approximations obtained in randomized and quantum algorithms by the same symbols as we did in the deterministic algorithm, omitting only the asterisk. In particular, the approximation to \(z(x_i)\) is denoted by \(y_i\). We start with \(y_0 = \eta\). For a given \(y_i\) we put \(y^0_i = y_i\), and denote by \(z_{ij}\) the solution of (14) (with the initial value \(y^0_j\) computed for \(y_i\)). We compute \(l_{ij}\) in a same way as \(l^*_ij\) (with \(y_i\) instead of \(y^*_i\)), and we set \(y^i_{i+1} = l_{ij}(z^j_{i+1})\). Approximations \(l\) in \([x_i, x_{i+1}]\) are defined to be equal to \(l_{ij}\) in each subinterval \([z^j_i, z^j_{i+1}]\), and the polynomial \(w_{ij}\) is constructed in the same way as \(w^*_ij\), with \(y^*_i\) replaced by \(y_i\).

The approximation at \(x_{i+1}\) is defined by
\[ y_{i+1} = y_i + \sum_{j=0}^{m-1} \int_{z^j_i}^{z^j_{i+1}} w_{ij}(l_{ij}(t)) dt + m\bar{h}^{r+\rho+1}A_i(f), \] (23)
where \(A_i(f)\) is a randomized or quantum approximation
\[ A_i(f) \approx \frac{1}{m} \sum_{j=0}^{m-1} \int_0^1 g_{ij}(u) du. \] (24)

The approximation \(l\) in \([a, b]\) is defined by \(l(t) = l_i(t)\) for \(t \in [x_i, x_{i+1}]\). For comparison, in [8] we had \(m = 1\) and \(A_i(f)\) was taken to be optimal randomized or quantum approximation to the integral \(\int_0^1 g_{00}(u) du\).
Here, we define $A_i(f)$ in a different way. Let $Q_i^N(f)$ be the mid-point rule approximation to $\int_0^1 g_{ij}(u) du$ based on $N$ points,

$$Q_i^N(f) = \frac{1}{N} \sum_{k=0}^{N-1} g_{ij}(u_k).$$

(25)

Consider the first-stage approximation (without computing it)

$$\frac{1}{m} \sum_{j=0}^{m-1} \int_0^1 g_{ij}(u) du \approx \frac{1}{mN} \sum_{j=0}^{m-1} \sum_{k=0}^{N-1} g_{ij}(u_k).$$

(26)

We define $A_i(f)$ to be the optimal randomized or quantum approximation (computed component by component) to the right-hand side mean of $mN$ vectors in (26).

Consider first the quantum setting. Let $\varepsilon_1 > 0$. For $i = 0, 1, \ldots, n-1$, let $A_i(f)$ be a random variable such that

$$\mathbb{P}\left\{ \left\| A_i(f) - \frac{1}{mN} \sum_{j=0}^{m-1} \sum_{k=0}^{N-1} g_{ij}(u_k) \right\| \leq \varepsilon_1 \right\} \geq \frac{3}{4}$$

(27)

for all $f \in F^{r,\rho}$. To compute $A_i(f)$ it suffices to use of order $\min\{mN, 1/\varepsilon_1\}$ quantum queries for computing each component of the mean, see [13]. (A number of repetitions dependent on $d$ is also needed to keep the success probability at least $3/4$ when passing from components to the vector norm. This changes the cost by a constant factor only.) To increase the success probability, we take the median (computed component by component) of $k$ results $A_i(f)$, where

$$k = \Theta \left( \log \frac{1}{1 - (1 - \delta)^{1/n}} \right) = O(\log n + \log 1/\delta)$$

(with absolute constants in the $\Theta$- and $O$-notation). We get a new approximation, denoted by the same symbol $A_i(f)$, such that

$$\mathbb{P}\left\{ \left\| A_i(f) - \frac{1}{mN} \sum_{j=0}^{m-1} \sum_{k=0}^{N-1} g_{ij}(u_k) \right\| \leq \varepsilon_1 \right\} \geq (1 - \delta)^{1/n}.$$ 

(28)

This yields that

$$\mathbb{P}\left\{ \left\| A_i(f) - \frac{1}{mN} \sum_{j=0}^{m-1} \sum_{k=0}^{N-1} g_{ij}(u_k) \right\| \leq \varepsilon_1 \right\} \geq 1 - \delta.$$ 

(29)

The cost of computing $A_i(f)$ is $O(n(\log n + \log 1/\delta) \min\{mN, 1/\varepsilon_1\})$ quantum queries.

In the randomized setting, we compute each component of the mean using the algorithm with expected error at most $\varepsilon_1/4$, and cost proportional to $\min\{mN, (1/\varepsilon_1)^2\}$, see [13]. Inequality (14) then holds with $\varepsilon := \varepsilon_1/4$. We next proceed as in the quantum case above to compute $A_i(f)$ such that (29) holds. For this, we need $O(n(\log n + \log 1/\delta) \min\{mN, (1/\varepsilon_1)^2\})$ function evaluations.
The deterministic part of the cost of algorithm (23) consists of computing coefficients of $l_{ij}$ and $w_{ij}$ for $j = 0, 1, \ldots, m-1$, for which we need $cm$ evaluations of partial derivatives of $f$ of order $0, 1, \ldots, r$, where $c$ only depends on $r$ and $d$. The computation of the integrals of $w_{ij}$ does not require new evaluations. Taking into account all indices $i$ and $j$, we need in total $cnm$ evaluations of $f$ or its partial derivatives.

5 Upper Bounds on the Randomized and Quantum Complexity

We now prove new upper bounds on the complexity of (1).

**Theorem 2** For problem (1), there exist constants $P_1$ and $P_2$ depending only on the parameters of the class $F^{r,\rho}$ such that for sufficiently small $\varepsilon$ and $\delta$,

\[
\text{comp}^{\text{rand}}(F^{r,\rho}, \varepsilon) \leq P_1 \left( \frac{1}{\varepsilon} \right)^{1/(r+\rho+1/3)} \log \frac{1}{\varepsilon},
\]

and

\[
\text{comp}^{\text{quant}}(F^{r,\rho}, \varepsilon, \delta) \leq P_2 \left( \frac{1}{\varepsilon} \right)^{1/(r+\rho+1/2)} \left( \log \frac{1}{\varepsilon} + \log \frac{1}{\delta} \right).
\]

**Proof** We analyze the error of the algorithm defined in the previous section. Let $e_i = z(x_i) - y_i$. Since

\[
z(x_{i+1}) = z(x_i) + \sum_{j=0}^{m-1} \int_{z_j^i}^{z_{j+1}^i} f(z(t)) \, dt,
\]

by subtracting (23) we get that

\[
e_{i+1} = e_i + \sum_{j=0}^{m-1} \int_{z_j^i}^{z_{j+1}^i} (f(z(t)) - f(l_{ij}(t))) \, dt + \sum_{j=0}^{m-1} \int_{z_j^i}^{z_{j+1}^i} (f(l_{ij}(t)) - w_{ij}(l_{ij}(t))) \, dt - m\tilde{h}^{r+\rho+1} A_i(f).
\]

Hence,

\[
\|e_{i+1}\| \leq \|e_i\| + \sum_{j=0}^{m-1} \int_{z_j^i}^{z_{j+1}^i} \|f(z(t)) - f(l_{ij}(t))\| \, dt + m\tilde{h}^{r+\rho+1} \left\| \frac{1}{m} \sum_{j=0}^{m-1} \int_0^{1} g_{ij}(u) \, du - A_i(f) \right\|
\]

for $i = 0, 1, \ldots, n-1$, where the function $g_{ij}$ is defined for $y_i$.

Let $\bar{z}_i$ be the solution of (19) with the initial condition $\bar{z}(x_i) = y_i$. Using the well known
dependence of the solution on initial conditions and the Lemma above, we get for \( t \in [z_j^i, z_{j+1}^i] \) that
\[
\|f(z(t)) - f(l_{ij}(t))\| \leq \|f(z(t)) - f(\bar{z}_i(t))\| + \|f(\bar{z}_i(t)) - f(l_{ij}(t))\|
\]
\[
\leq L\|z(t) - \bar{z}_i(t)\| + L\|\bar{z}_i(t) - l_{ij}(t)\|
\]
\[
\leq L\exp(Lh)\|e_i\| + LMh \bar{h}^{r+\rho}
\]
for \( Lh \leq \ln 2 \). Inequality \( (34) \) together with \( (29) \) yield now that the inequalities
\[
\|e_{i+1}\| \leq \|e_i\| (1 + hL \exp(hL)) + LMh^2 \bar{h}^{r+\rho}
\]
\[
+ h \bar{h}^{r+\rho} \left( \left\| \frac{1}{m} \sum_{j=0}^{m-1} \left( \int_0^1 g_{ij}(u) \, du - \frac{1}{N} \sum_{k=0}^{N-1} g_{ij}(u_k) \right) \right\| + \varepsilon_1 \right)
\]
(35)

hold for \( i = 0, 1, \ldots, n-1 \) with probability at least \( 1 - \delta \). We now take into account the error of the mid-point rule, and solve the resulting difference inequality with \( \varepsilon_0 = 0 \). With probability at least \( 1 - \delta \), we get that
\[
\|e_i\| \leq C(h + 1/N + \varepsilon_1) \bar{h}^{r+\rho}, \quad i = 0, 1, \ldots, n,
\]
(36)

for a constant \( C \) depending only on the parameters of the class \( F^{r,\rho} \). The total cost of computing \( y_0, y_1, \ldots, y_n \) is equal in its deterministic part to \( cnm \) evaluations of partial derivatives of \( f \). The non-deterministic part includes \( O(n(\log n + \log 1/\delta) \min\{mN, 1/\varepsilon_1\}) \) quantum queries in the quantum setting, and \( O(n(\log n + \log 1/\delta) \min\{mN, (1/\varepsilon_1)^2\}) \) evaluations of \( f \) in the randomized setting. It follows from \( (34) \) with \( N \geq n \) and \( \varepsilon_1 = 1/n \) that
\[
\|e_i\| \leq Ch \bar{h}^{r+\rho},
\]
(37)

for \( i = 0, 1, \ldots, n \), with probability at least \( 1 - \delta \) (and a different constant \( C \)). Passing to the approximation over \( [a, b] \), we get for \( t \in [x_i, x_{i+1}] \) the inequality
\[
\|z(t) - l(t)\| \leq \|z(t) - \bar{z}_i(t)\| + \|\bar{z}_i(t) - l_i(t)\| \leq \exp(hL)\|e_i\| + Mh \bar{h}^{r+\rho}.
\]

This yields that with probability at least \( 1 - \delta \), the error bound
\[
\sup_{t \in [a, b]} \|z(t) - l(t)\| \leq \tilde{C} \bar{h}^{r+\rho}
\]
(38)

holds, with the constant \( \tilde{C} \) depending only on the parameters of the class \( F^{r,\rho} \). Consider the quantum case. Neglecting for a while the logarithmic factors, we have that error \( O(1/(n(nm)^{r+\rho})) \) is achieved with cost \( O(nm + n^2) \). It is easy to see that the best choice in this case is \( m = n \). With a total number of \( k \) quantum queries and deterministic evaluations, we then achieve the error bound
\[
\sup_{t \in [a, b]} \|z(t) - l(t)\| \leq C_1 k^{-(r+\rho+1/2)},
\]
(39)

with probability at least \( 1 - \delta \). This holds for all \( f \in F^{r,\rho} \), and a constant \( C_1 \) depending only on the parameters of the class \( F^{r,\rho} \). Hence, to compute an \( \varepsilon \)-approximation \( l \) such that
\[
\sup_{t \in [a, b]} \|z(t) - l(t)\| \leq \varepsilon
\]
with probability at least \(1 - \delta\) for each \(f \in F^{r,\rho}\), the algorithm uses
\[
O \left( \left( \log 1/\varepsilon + \log 1/\delta \right) (1/\varepsilon)^{1/(r+\rho+1/2)} \right)
\]
quantum queries and deterministic evaluations (the logarithmic factors are again taken into account). This completes the proof of Theorem 2 in the quantum case.

In the randomized setting, we proceed in a similar way, with \(N \geq n^2\) and \(m = n^2\). With \(k\) calls of \(f\) or its partial derivatives (the logarithmic factors are for a while neglected), we get the error bound
\[
\sup_{t \in [a,b]} \|z(t) - l(t)\| \leq C_2 k^{-(r+\rho+1/3)}.
\] (40)
This holds with probability at least \(1 - \delta\) and a constant \(C_2\) depending, as above, only on \(F^{r,\rho}\). Denote the left-hand side random variable in (40) by \(X^w\), and the right-hand side by \(h(k)\). We note that
\[
E(X^w)^2 = \int_{X^w > h(k)} (X^w)^2 dP(\omega) + \int_{X^w \leq h(k)} (X^w)^2 dP(\omega) \leq K^2 \delta + h(k)^2
\]
for all \(f \in F^{r,\rho}\), where \(K\) is a positive constant, depending only on the parameters of the class \(F^{r,\rho}\), such that \(X^w \leq K\). To see that such a constant exists, note that the random variable \(A_i(f)\) in (27) can be assumed bounded by \(\|A_i(f)\| \leq 2M\), where \(M\) is a bound on \(\|g_{ij}\|\) (otherwise \(A_i(f) = 0\) would be a better approximation). Proceeding from (22) to (30) with \(\varepsilon_1 = 3M\), we see from (38) that \(X^w\) is bounded (in the deterministic sense) by \(C_2 k^{r+\rho}\). Hence, the constant \(K\) indeed exists.

Take now \(k\) to be the minimal number such that \(h(k) \leq \varepsilon/2\), so that \(k \asymp (1/\varepsilon)^{1/(r+\rho+1/3)}\), and set \(\delta = \min\{ 1/2, 3\varepsilon^2/(4K^2) \}\). Then \(E \left( \sup_{t \in [a,b]} \|z(t) - l(t)\| \right)^2 \leq \varepsilon^2\) for all \(f \in F^{r,\rho}\), which is achieved with cost \(O \left( \log(1/\varepsilon) \cdot (1/\varepsilon)^{1/(r+\rho+1/3)} \right)\). This proves Theorem 2 in the randomized setting.

The upper bounds obtained in Theorem 2 are better than those from Theorem 1 for all \(r\) and \(\rho\). For instance, for \(r = 0\) and \(\rho = 1\), if we neglect the logarithmic factors, Theorem 1 gives the bound \(O((1/\varepsilon)^{5/6})\) in the randomized setting, and \(O((1/\varepsilon)^{3/4})\) in the quantum setting. In Theorem 2 the respective bounds are \(O((1/\varepsilon)^{3/4})\) and \(O((1/\varepsilon)^{2/3})\). Nevertheless, we see from lower bounds in Theorem 1 that the gap still remains between the upper and lower bounds.

**Remark 1**

We comment on the proof of Theorem 2, and show a relation to Theorem 1. Looking at (22) we observe that, before starting randomized or quantum computations, we can separate the main part of \(\int_0^1 g_{ij}(u) \, du\) by replacing this integral with \(\int_0^1 s_{ij}(u) \, du + \int_0^1 (g_{ij}(u) - s_{ij}(u)) \, du\), where \(s_{ij}\) is an approximation to \(g_{ij}\). Using \(l\) evaluations of \(g_{ij}\) \((l \geq 1)\), we can define \(s_{ij}\) to have the error of order \(l^{-(r+\rho)}\), with the cost of one evaluation of \(s_{ij}\) independent of \(l\). We can next use randomized
or quantum algorithms to compute $\int_0^1 (g_{ij}(u) - s_{ij}(u)) \, du$. In this way, we get errors $\|e_i\|$ of order

$$\left( nm \right)^{-(r+\rho)} \left( n^{-1} + N^{-1} + \varepsilon_1 l^{-(r+\rho)} \right)$$

with cost (up to logarithmic factors)

$$nm + nml + n \min\{mN, (1/\varepsilon_1)^\kappa\}, \quad (41)$$

where $\kappa = 2$ in the randomized setting, and $\kappa = 1$ on a quantum computer. By selecting optimal parameters, we get that the minimal (upper bound on the) error achieved with cost $k$ is equal to $k^{-(r+\rho+1/3)}$ in the randomized setting, and $k^{-(r+\rho+1/2)}$ on a quantum computer. Hence, by admitting $l \geq 1$ and by allowing a selection of $s_{ij}$ we do not arrive at better bounds than those given in Theorem 2, in which the functions $s_{ij} = 0$ have simply been taken.

The upper bounds from Theorem 1 are a special case of (41), and can be obtained for sufficiently large $N$ by setting $m = 1$, $\varepsilon_1 = n^{-1/(2r+2\rho+1)}$ and $l = n^{2/(2r+2\rho+1)}$ in the randomized setting, and $m = 1$, $\varepsilon_1 = n^{-1/(r+\rho+1)}$ and $l = n^{1/(r+\rho+1)}$ on a quantum computer.

6 Scalar Autonomous Problems

In this section, we study the solution of a scalar autonomous problem. The aim is to compute the value of the solution at the end point of the interval of integration. We give essentially tight upper and lower bounds on the complexity of this problem. In our previous paper [8], no lower bounds for this problem were obtained. Upper bounds were discussed together with the general problem, which led to weaker results.

Note that the complexity of approximating the solution at only one single point may differ from that of approximating the solution over the whole interval of integration, which is the subject of the preceding part of this paper. In particular, upper bounds for the former problem need not be valid for the latter one.

Consider problem (I) with $d = 1$, and the right-hand side function $f$ belonging to the class

$$f \in \hat{F}^{r,\rho} = F^{r,\rho} \cap \{ f : |f(y)| \geq p, \ y \in \mathbb{R} \}, \quad (42)$$

for some $p > 0$. Our aim is to compute the value $z(b)$ with accuracy $\varepsilon$ by randomized or quantum algorithms. Since

$$t - a = \int_\eta^y \frac{1}{f(s)} \, ds,$$

we equivalently look for the solution $y^* = z(b)$ of the nonlinear equation $H(y) = 0$, where

$$H(y) = \int_\eta^y \frac{1}{f(s)} \, ds - (b - a). \quad (43)$$
(The idea of transforming a scalar autonomous problem into a nonlinear equation was exploited, for example, in \[2\] to derive a class of nonlinear Runge-Kutta methods.)

Note that
\[
\frac{1}{D_0}|y - \bar{y}| \leq |H(y) - H(\bar{y})| \leq \frac{1}{p}|y - \bar{y}|,
\]
for all \(y\) and \(\bar{y}\).

Given \(y\), the computation of \(H(y)\) reduces to the computation of the integral. Suppose that we have at our disposal a randomized or quantum algorithm for computing integrals, which computes a random approximation \(A(y)\) to \(H(y)\) such that
\[
|H(y) - A(y)| \leq \varepsilon_1
\]
for some (small) \(\varepsilon_1 > 0\), with probability at least \(3/4\), for any \(f\) and \(y\). We denote the cost of this algorithm (dependent on a current setting) by \(c(\varepsilon_1)\).

We now define algorithms for computing an approximation to \(y^* = z(b)\) with error at most \(\varepsilon\) with probability at least \(1 - \delta\), for all \(f \in \bar{P}^{\tau,\rho}\). We shall use the bisection method based on the values \(A(y)\). To get success probability at least \(1 - \delta\), we shall need inequality \([14]\) to hold with probability higher than \(3/4\). Let \(i^*\) be the minimal index \(i\) for which \(D_0(b-a)/(p2^{i+1}) \leq \varepsilon_1\), i.e.,
\[
i^* + 1 = \lceil \log (D_0(b-a)/(p\varepsilon_1)) \rceil.
\]
We need \([14]\) to hold with probability at least \(1 - \delta_1\), where \(\delta_1 = 1 - (1 - \delta)^{1/(i^*+1)}\). To increase the success probability in computing \(A(y)\) from \(3/4\) to \(1 - \delta_1\), we proceed in a standard way by computing the median of \(k\) repetitions of the algorithm, where
\[
k = O(\log 1/\delta_1) = O(\log (i^* + 1) + \log 1/\delta) = O(\log \log 1/\varepsilon_1 + \log 1/\delta).
\] (45)

Assume that \(f(y) > 0\) (the case \(f(y) < 0\) is analogous). We start the bisection method from the interval \([\alpha_0, \beta_0] = [\eta, \eta + D_0(b-a)]\) containing \(y^*\), and we set \(y_1 = (\alpha_0 + \beta_0)/2\). Given \([\alpha_i, \beta_i]\), we set \(y_{i+1} = (\alpha_i + \beta_i)/2\) and select the next interval \([\alpha_{i+1}, \beta_{i+1}]\) based on the sign of \(A(y_{i+1})\). We stop the iteration at first index \(i\), call it \(i^{\text{bis}}\), for which \(|A(y_i)| \leq 2\varepsilon_1\) (we shall discuss this termination criterion and the correctness of the selection of successive intervals in a while).

Note that for any \(j \leq i^*\), inequalities
\[
|H(y_{i+1}) - A(y_{i+1})| \leq \varepsilon_1 \quad i = 0, 1, \ldots, j
\]
(46)
hold (simultaneously) with probability at least \((1 - \delta_1)^{i^*+1} = 1 - \delta\).

Assume that \([16]\) is satisfied. We show that the number of bisection steps satisfies \(i^{\text{bis}} \leq i^* + 1\).

Suppose that the termination condition is not fulfilled by the \(i^*\)-th step, i.e., \(|A(y_i)| > 2\varepsilon_1\) for \(i = 1, 2, \ldots, i^*\). Then the selection of the interval \([\alpha_i, \beta_i]\) made on the basis of \(A(y_i)\), as well as the selection of all proceeding intervals, is correct. (In fact, it suffices for this that \(|A(y_i)| > \varepsilon_1\), since the signs of \(A(y_i)\) and \(H(y_i)\) are then the same by \([46]\).) Hence, we have
\[
|y_{i+1} - y^*| \leq |\alpha_i - \beta_i|/2 = |\alpha_0 - \beta_0|/2^{i^*+1},
\]
and
\[
|A(y_{i+1})| \leq |H(y_{i+1})| + |A(y_{i+1}) - H(y_{i+1})| \leq |y_{i+1} - y^*|/p + \varepsilon_1 \leq |\alpha_0 - \beta_0|/(p2^{i^*+1}) + \varepsilon_1 \leq 2\varepsilon_1.
\]
Since the termination condition is now satisfied, we have in this case that $i^{\text{bis}} = i^* + 1$. In any case, the desired bound on $i^{\text{bis}}$ holds, as claimed. In terms of $\varepsilon_1$, we have that

$$i^{\text{bis}} \leq \lceil \log (D_0(b-a)/(p\varepsilon_1)) \rceil \quad (47)$$

Take now $\varepsilon_1 = \varepsilon/(3D_0)$. Then, terminating after $i^{\text{bis}}$ steps, we arrive at the $\varepsilon$-approximation $y^{\text{bis}}$ to $y^*$, since

$$|y^{\text{bis}} - y^*| \leq D_0|H(y^{\text{bis}})| \leq D_0(|A(y^{\text{bis}})| + |H(y^{\text{bis}}) - A(y^{\text{bis}})|) \leq 3D_0\varepsilon_1 = \varepsilon \quad (48)$$

As in the case of (46), this holds with probability at least $1 - \delta$.

Summarizing, the described algorithm returns the approximation (random variable) $y^{\text{bis}}$ such that the bound $|y^{\text{bis}} - z(b)| \leq \varepsilon$ holds with probability at least $1 - \delta$, with total cost

$$c(\varepsilon_1) k i^{\text{bis}} = O(c(\varepsilon_1) (\log\log 1/\varepsilon + \log 1/\delta) \log 1/\varepsilon) \quad (49)$$

Using known results on integration, we now estimate $c(\varepsilon_1)$. Since $f \in \tilde{F}^{r,\rho}$, the function $1/f(s)$ is in the Hölder class $F^{r,\rho}$ (over the finite interval $s \in [\eta, \eta + D_0(b-a)]$) with certain parameters $\tilde{D}_0, \tilde{D}_1, \ldots, \tilde{D}_r, \tilde{H}$ depending on $D_0, D_1, \ldots, D_r, H$ and $p$.

Consider the quantum setting. There exists an algorithm for computing integrals of $1/f(s)$ with cost $c(\varepsilon_1) = O \left((1/\varepsilon)^{1/(r+\rho+1)}\right)$ quantum queries, see [3]. This leads to the following complexity bound for our problem

$$\text{comp}^{\text{quant}}(\tilde{F}^{r,\rho}, \varepsilon, \delta) = O \left((1/\varepsilon)^{1/(r+\rho+1)} (\log\log 1/\varepsilon + \log 1/\delta) \log 1/\varepsilon \right) \quad (50)$$

**Remark 2**

To establish the cost of an algorithm in the quantum setting, we have to count the number of applications of a quantum query operator $Q_f$ for $f$. Calculating the cost $c(\varepsilon_1)$ above we have taken into account the number of queries $Q_{1/f}$ for $1/f$. However, a query for $1/f$ for $f \in \tilde{F}^{r,\rho}$ can be simulated by a query for $f$ (and vice versa), see Lemma 4 in [3]. Hence, the upper bound in terms of both units remains the same.

Consider the randomized setting. There exists an algorithm approximating integrals with the mean square error (3) bounded by $\varepsilon_1/2$, and cost $c(\varepsilon_1) = O \left((1/\varepsilon_1)^{1/(r+\rho+1/2)}\right)$ evaluations of $f$. We use it to compute an approximation $A(y)$ to $H(y)$, for a given $y$. By the Markov inequality, error bound (14) holds for $A(y)$ with probability at least $3/4$. We now follow the steps between relations (44) and (49) above to get the approximation $y^{\text{bis}}$ to $z(b)$ such that

$$X^{\omega} := |y^{\text{bis}} - z(b)| \leq \varepsilon \quad (51)$$

with probability at least $1 - \delta$, for all $f$. By (49), the cost of computing $y^{\text{bis}}$ is

$$O \left((1/\varepsilon)^{1/(r+\rho+1/2)} (\log\log 1/\varepsilon + \log 1/\delta) \log 1/\varepsilon \right).$$

To estimate the mean square error of $y^{\text{bis}}$, we proceed in a similar way as we did in the final part of the proof of Theorem 2. We replace $\varepsilon$ in (51) by $\varepsilon/2$, which influences the cost only by
a constant factor, and we write

\[ E(X^\omega)^2 = \int_{X^\omega > \varepsilon/2} (X^\omega)^2 dP(\omega) + \int_{X^\omega \leq \varepsilon/2} (X^\omega)^2 dP(\omega) \leq K^2 \delta + \varepsilon^2/4 \]

for all \( f \in \hat{F}^{r,\rho} \). Here, \( K \) is a positive constant depending only on the parameters of the class \( \hat{F}^{r,\rho} \) such that \( X^\omega \leq K \). The choice \( \delta = \frac{3\varepsilon^2}{4K^2} \) gives the bound

\[ \sup_{f \in \hat{F}^{r,\rho}} (E(X^\omega)^2)^{1/2} \leq \varepsilon, \]

which is achieved with cost

\[ O \left( (1/\varepsilon)^{1/(r+\rho+1/2)} (\log \log 1/\varepsilon \log 1/\varepsilon) \right) = O \left( (1/\varepsilon)^{1/(r+\rho+1/2)} (\log 1/\varepsilon)^2 \right). \]

This yields an upper bound

\[ \text{comp}^\text{rand}(\hat{F}^{r,\rho}, \varepsilon) = O \left( (1/\varepsilon)^{1/(r+\rho+1/2)} (\log 1/\varepsilon)^2 \right) \quad (52) \]

on the complexity. Hence, up to logarithmic factors, we are able to solve our problem at cost of one single integration.

We now turn to lower bounds on the randomized and quantum complexity. Let \( \phi \) be any algorithm based on evaluations of \( f \) or its derivatives at possibly random points in the randomized setting, and on quantum queries for \( f \) and classical evaluations of \( f \) or its derivatives in the quantum setting. Assume that \( \phi \) computes an approximation to \( z(b) \) with error at most \( \varepsilon \), for any scalar problem \( [b] \) with \( f \in \hat{F}^{r,\rho} \). We estimate from below the number of evaluations (queries) used by \( \phi \), by reducing the problem to the summation of real numbers.

Without loss of generality, let \([a, b] = [0, 1] \). For \( n \geq 1 \), let \( \lambda_0, \lambda_1, \ldots, \lambda_{n-1} \) be numbers of at most unit absolute value, and define the function \( g(y) = 1/f(y) \) as follows.

Consider the uniform partition of \([\eta, \eta + 1/2] \) with points \( y_i = \eta + i/(2n) \) for \( i = 0, 1, \ldots, n \). We let \( h_i \in F^{r,\rho} \), where \( i = 0, 1, \ldots, n-1 \), be functions with the following properties:

\[ h_i \text{ has support } [y_i, y_{i+1}], \quad h_i(y) \geq 0, \]

\[ \max_{y_{i+1}} h_i(y) = h_i((y_i + y_{i+1})/2) = c_1(y_{i+1} - y_i)^{r+\rho}, \]

\[ \int_{y_i}^{y_{i+1}} h_i(y) dy = c_2(y_{i+1} - y_i)^{r+\rho+1}, \]

where \( c_1 \) and \( c_2 \) are known positive constants depending only on the parameters of the class \( F^{r,\rho} \) (and not on \( i \) and \( n \)). Such functions are often used in proving lower bounds and their construction is well known.

We define \( g(y) = 1 + \sum_{i=0}^{n-1} \lambda_i h_i(y) \). Then \( g \in \hat{F}^{r,\rho} \) for sufficiently large \( n \), and the same holds...
for $f$ (with different constants). Since $3/4 \leq f(y) \leq 3/2$ for sufficiently large $n$, we have that $\eta + 3/4 \leq z(1) \leq \eta + 3/2$, and we can write

$$1 = \int_\eta^{z(1)} 1/f(y) \, dy = \int_\eta^{\eta + 1/2} 1/f(y) \, dy + \int_{\eta + 1/2}^{z(1)} 1/f(y) \, dy = c_3 n^{-(r+\rho+1)} \sum_{i=0}^{n-1} \lambda_i + z(1) - \eta. \quad (53)$$

This yields that

$$\frac{1}{n} \sum_{i=0}^{n-1} \lambda_i = \frac{1 - z(1) + \eta}{c_3 n^{-(r+\rho)}} , \quad (54)$$

where $c_3$ is a known positive constant.

Consider first the quantum setting. Let $\zeta$ (a random variable) be an $\epsilon$-approximation to $z(1)$ computed by the algorithm $\phi$ for the right-hand side $f$ defined above. We have that $|z(1) - \zeta| \leq \epsilon$ with probability at least $3/4$ (we take $\delta \leq 1/4$). Hence,

$$\zeta_1 = \frac{1 - \zeta + \eta}{c_3 n^{-(r+\rho)}} \quad \text{is an approximation to} \quad S = \frac{1}{n} \sum_{i=0}^{n-1} \lambda_i$$

with error at most $\epsilon_2 = \epsilon n^{r+\rho}/c_3$ and probability at least $3/4$. The lower bound of Nayak and Wu, see (15), gives that the number of queries for $\Lambda = [\lambda_0, \lambda_1, \ldots, \lambda_{n-1}]$ must be at least $\Omega(\min\{n, 1/\epsilon_2\})$. This is also a lower bound on the number of queries for $1/f$ (and for $f$) needed in the algorithm $\phi$. We now take $n \asymp (1/\epsilon)^{1/(r+\rho+1)}$, and we conclude that

$$\text{comp}^{\text{quant}}(\hat{F}_{r,\rho}, \epsilon, 1/4) = \Omega \left( \left(1/\epsilon \right)^{1/(r+\rho+1)} \right).$$

In the randomized setting, let $\mathbb{E}|z(1) - \zeta|^2 \leq \epsilon$. Then $\mathbb{E}|S - \zeta_1|^2 \leq \epsilon n^{r+\rho}/c_3 = \epsilon_2$ and the same inequality holds for $\mathbb{E}|S - \zeta_1|$. Due to (13), the number of accesses to $\lambda_i$ must be at least $\Omega(\min\{n, (1/\epsilon_2)^2\})$. This is also a lower bound on the number of the number of evaluations of $f$ or its derivatives. We now take $n \asymp (1/\epsilon)^{1/(r+\rho+1/2)}$ to get

$$\text{comp}^{\text{rand}}(\hat{F}_{r,\rho}, \epsilon) = \Omega \left( \left(1/\epsilon \right)^{1/(r+\rho+1/2)} \right).$$

We have shown the following

**Theorem 3**  
Consider the scalar autonomous problem described in the beginning of this section, with a right-hand side $f$ in the class $\hat{F}_{r,\rho}$. There exist positive constants $P_i$ ($i = 3, 4, 5, 6$) depending only on the parameters of the class $\hat{F}_{r,\rho}$ such that, for sufficiently small $\epsilon$ and $\delta$, the following complexity bounds hold true.

In the randomized setting

$$\text{comp}^{\text{rand}}(\hat{F}_{r,\rho}, \epsilon) \leq P_i \left( \left( \frac{1}{\epsilon} \right)^{1/(r+\rho+1/2)} \left( \log \frac{1}{\epsilon} \right)^2 \right) \quad (55)$$
and

\[ \text{comp}^{\text{rand}}(\hat{F}^{r,\rho}, \varepsilon) \geq P_4 \left( \frac{1}{\varepsilon} \right)^{1/(r+\rho+1/2)}. \] (56)

In the quantum setting

\[ \text{comp}^{\text{quant}}(\hat{F}^{r,\rho}, \varepsilon, \delta) \leq P_5 \left( \left( \frac{1}{\varepsilon} \right)^{1/(r+\rho+1)} \left( \log \log \frac{1}{\varepsilon} + \log \frac{1}{\delta} \right) \log \frac{1}{\varepsilon} \right). \] (57)

and, for \( 0 < \delta \leq 1/4, \)

\[ \text{comp}^{\text{quant}}(\hat{F}^{r,\rho}, \varepsilon, \delta) \geq \text{comp}^{\text{quant}}(\hat{F}^{r,\rho}, \varepsilon, 1/4) \geq P_6 \left( \left( \frac{1}{\varepsilon} \right)^{1/(r+\rho+1)} \right). \] (58)

Note that in both randomized and quantum settings upper and lower bounds in Theorem 3 are matching, up to logarithmic factors. The question of finding matching upper and lower bounds for the general problem (1) still remains open.

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