Extrapolation for meson screening masses from imaginary to real chemical potential

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We first extend our formulation for the calculation of π- and σ-meson screening masses to the case of finite chemical potential μ. We then consider the imaginary-μ approach, which is an extrapolation method from imaginary chemical potential (μ = iμr) to real one (μ = μR). The feasibility of the method is discussed based on the entanglement Polyakov-loop extended Nambu–Jona-Lasinio (EPNJL) model in 2-flavor system. As an example, we investigate how reliable the imaginary-μ approach is for π- and σ-meson screening masses, comparing “screening masses at μR” in the method” with “those calculated directly at μR”. We finally propose the new extrapolation method and confirm its efficiency.

I. INTRODUCTION

T and μ dependence of hadron masses are closely related with those of the ground-state structure of hot QCD matter, where T is temperature and μ means quark-number chemical potential. In fact, medium modification of vector and η′ mesons has been measured in heavy-ion collisions [1, 2]. These results indicate the chiral and the effective U(1)A-symmetry restoration. It is, therefore, important to determine T and μ dependence of light hadron masses.

Lattice QCD (LQCD) is powerful tool to investigate the QCD matter at finite T and μ. In fact, many LQCD calculations have been done for low density (μ/T ≤ 1). The calculation in high density region is still challenging because of well-known “sign problem”. Several methods were proposed so far to circumvent the sign problem: the Taylor expansion method [3, 4], the reweighting method [5], the imaginary-μ method [6–9], the canonical approach [10], the complex Langevin method [11, 12], and the Lefschetz thimble theory [13, 14]. These have made great progress, but all the results are consistent only in μ/T ≤ 1 at the present stage. Among them, we pick up the the imaginary-μ method in the present paper. When one considers μ as complex variable, this method corresponds to the analytic continuation from the imaginary chemical potential (μ = iμr) to the real one (μ = μR).

In LQCD simulation for finite θ ≡ μA/T, the thermodynamic potential ΩQCD(θ) has the Roberge and Weiss (RW) periodicity: ΩQCD(θ) = ΩQCD(θ + 2π/3). The QCD phase diagram has the first-order phase transition (RW phase transition) at T ≥ TRW and θ = π/3, where TRW is RW transition temperature. The endpoint of RW phase transition is located at (θ, T) = (π/3, TRW). The order of the RW endpoint and the value of TRW have been investigated in 2-flavor LQCD simulations [6–8].

One can consider effective models as an complementary approach to the first-principle LQCD simulation. The Polyakov-loop extended Nambu–Jona-Lasinio (PNJL) model [18–38] qualitatively reproduces 2-flavor LQCD data in μR/T < 1, since the model can treat the chiral and the deconfinement transition simultaneously. In addition, the model is successful in accounting for 2-flavor LQCD data in 0 ≤ θ ≤ π/3 [34, 35], because it has the RW periodicity. The entanglement PNJL (EPNJL) model [39, 41] is improved version of PNJL model. The EPNJL model quantitatively reproduces 2-flavor LQCD data in 0 ≤ θ ≤ π/3 [39] and μR/T < 1 [40], since the model possesses the RW periodicity and the strong correlation between the chiral and the deconfinement transition.

Meson masses can be classified into “meson pole mass” and “meson screening mass”. In LQCD simulations at finite T, the derivation of meson screening mass is easier than that of meson pole mass, since the spatial lattice size is larger than the temporal one; see Appendix of Ref. [42] for the further explanation. Meanwhile, in NJL-type effective models, time-consuming calculations were needed for the meson screening mass compared with that of the meson pole mass. Recently, this problem was solved by our previous works [42, 43] for the case of μ = 0.

In this paper, for simplicity, we concentrate on the π-meson and σ-meson screening masses, Mπscr and Mσscr, in the framework of 2-flavor EPNJL model. We apply the method of Ref. [42, 43] for the case of finite μR and μL, and then investigate how reliable the imaginary-μ method is for Mπscr and Mσscr. For this purpose, we compare “the Mπscr extrapolated from iμR (extrapolating result)” with “the Mπscr calculated directly at μR (direct result)” for ξ = π, σ mesons.

In Sec. II, we explain a way of calculating the meson screening mass at finite μ. Numerical results are shown in Sec. III. Section IV is devoted to a summary.
II. FORMALISM

A. Model setting

The Lagrangian density of 2-flavor EPNJL model is defined by

\[
\mathcal{L} = \bar{\psi}(i\gamma_\nu D^\nu - m_0)\psi + GS(\Phi)[(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5 \tau \psi)^2] - U(\Phi[A], \bar{\Phi}[A], T) \tag{1}
\]

with u- and d-quark fields \( \psi = (u, d)^T \) and the isospin matrix \( \tau \). We assume isospin symmetry, i.e., u and d quarks have the same mass \( m_0 \). The gluon field \( A^\nu \) is introduced through the covariant derivative \( D^\nu = \partial^\nu + iA^\nu \) with \( A^\nu = \delta^\nu_0 g(A^0)_{\alpha\beta} / 2 - \delta^\nu_0 ig(A_4)_{\alpha\beta} / 2 \), where the matrices \( \lambda_\alpha \) are the Gell-Mann matrices in color space and \( g \) is the gauge coupling. Here, we consider only the time component \( A_4 \) of \( A_\mu \) and assume that the \( A_4 \) is a homogeneous and static background field.

In the EPNJL model, the Polyakov loop \( \Phi \) and its Hermitian conjugate \( \bar{\Phi} \) are defined by

\[
\Phi = \frac{1}{3} \text{tr}_c(L), \quad \bar{\Phi} = \frac{1}{3} \text{tr}_c(L^*) \tag{2}
\]

with \( L = \exp[i A_4 / T] = \exp[i \text{diag}(A_{11}^0, A_{22}^0, A_{33}^0) / T] \) for real classical variables \( A_{ij}^0 \) \((j = 1, 2, 3)\). The trace \( \text{tr}_c \) is taken in color space. The relation between \( A_{ij}^0 \) and \( \Phi \) or \( \bar{\Phi} \) is summarized in Appendix A. The coupling constant \( G_S \) of the four-quark interaction is assumed to depend on the Polyakov loop \( \Phi \) and \( \bar{\Phi} \):

\[
G_S(\Phi) = G_S(0) \times \left[ 1 - \alpha_1 \Phi \bar{\Phi} - \alpha_2 (\Phi^3 + \bar{\Phi}^3) \right]. \tag{3}
\]

We set the parameters \( \alpha_1, \alpha_2 \) to \( \alpha_1 = \alpha_2 = 0.2 \) to reproduce LQCD data on \( T \) dependence of chiral condensate [45] and Polyakov loop [46]; see Sec. IIIA for the further explanation.

The Polyakov loop \( \Phi \) and its Hermitian conjugate \( \bar{\Phi} \) are mainly governed by the Polyakov-loop potential \( U \) in Eq. (1). We use the logarithm-type Polyakov-loop potential \( U \) of Ref. [20]. The parameter set in \( U \) is determined from LQCD data on thermodynamical quantities in the pure gauge limit. The \( U \) has one dimensionful parameter \( T_0 \) and the value is often set to \( T_0 = 270 \text{ MeV} \) since the deconfinement transition occurs at \( T = 270 \text{ MeV} \) in the pure gauge limit. When one considers the dynamical quarks, the typical energy scale \( T_0 \) depends on the number of flavors \( (N_f) \). Hence we treat \( T_0 \) as an adjustable parameter and determine the value to reproduce the pseudocritical temperature \( T_0^\chi = 173 \pm 8 \text{ MeV} \) for chiral transition in 2-flavor LQCD simulations at zero chemical potential [45–47]. The parameter thus obtained is \( T_0 = 200 \text{ MeV} \).

Applying the mean field approximation to Eq. (1) leads to the linearized Lagrangian density

\[
\mathcal{L}^\text{MFA} = \bar{\psi}S^{-1}\psi - G_S(\Phi)\sigma^2 - U(\Phi[A], \bar{\Phi}[A], T), \tag{4}
\]

where the dressed quark propagator \( S \) is defined by

\[
S = \frac{1}{i\gamma_\nu \partial^\nu - i\gamma_\alpha A_4 - M} \tag{5}
\]

with the effective quark mass \( M = m_0 - 2G_S(\Phi)\sigma \) and the chiral condensate \( \sigma = \langle \bar{\psi}\psi \rangle \). One can make the path integral over the quark fields analytically, and the thermodynamic potential \( \Omega \) per unit volume is obtained by

\[
\Omega = U_M + U - 2N_f \int \frac{d^3p}{(2\pi)^3} [3E_p + \frac{1}{\beta}\ln \left[ 1 + 3(\Phi - \Phi e^{-\beta(E_p - \mu)})e^{-\beta(E_p - \mu)} - e^{-3\beta(E_p - \mu)} \right] + \frac{1}{\beta}\ln \left[ 1 + 3(\bar{\Phi} + \Phi e^{-\beta(E_p + \mu)})e^{-\beta(E_p + \mu)} - e^{-3\beta(E_p + \mu)} \right]] \tag{6}
\]

with \( E_p = \sqrt{p^2 + M^2} \) and \( U_M = G_S(\Phi)\sigma^2 \). The mean-field variables \( \sigma, \Phi, \bar{\Phi} \) are determined so as to minimize the potential \( \Omega \). For real \( \mu \), we take the approximation \( \Phi = \bar{\Phi} \) for simplicity. This approximation is pretty good for \( \mu_R / T \leq 1 \) and not so bad even for \( \mu_R / T \geq 1 \) [48].

In the \( \mu_R \) region, this thermodynamic potential \( \Omega \) has the RW periodicity [23, 33, 37]. The RW periodicity stems from the fact that \( \Omega \) is invariant under the extended \( Z_3 \) transformation [37] defined by

\[
\Phi \to e^{-i2\pi k/3} \Phi, \quad \bar{\Phi} \to e^{i2\pi k/3} \bar{\Phi}, \quad \theta \to \theta + \frac{2\pi k}{3} \tag{7}
\]

for integer \( k \).

The three-dimensional momentum \( p \) integral in Eq. (6) has ultraviolet divergence and needs to be regularized. In this paper, we use the Pauli–Villars (PV) regularization [48, 49]. When \( \Omega \) is divided into \( \Omega = U_M + U + \Omega_F(M) \), the function \( \Omega_F(M) \) is regularized in the PV scheme as

\[
\Omega_F^{\text{reg}}(M) = \sum_{\alpha=0}^{2} C_\alpha \Omega_F(M_\alpha), \tag{8}
\]

where \( M_\alpha = M \) and the \( M_\alpha (\alpha = 1, 2) \) mean masses of auxiliary particles. The parameters \( M_\alpha \) and \( C_\alpha \) are determined so as to satisfy the condition \( \sum_{\alpha=0}^{2} C_\alpha = \sum_{\alpha=0}^{2} C_\alpha M_\alpha^2 = 0 \) in order to remove the quartic, the quadratic and the logarithmic divergence in \( \Omega_F \). We then set \( (C_0, C_1, C_2) = (1, -2, 1) \) and \( (M_0^2, M_1^2, M_2^2) = (M^2, M^2 + A^2, M^2 + 2A^2) \). The parameter \( A \) should be finite even after the regularization [81], since the present model is non-renormalizable.

The EPNJL model has three parameters \( m_0, G_S(0), A \) in addition to \( T_0, \alpha_1, \alpha_2 \). We set \( m_0 = 6.3 \text{ MeV} \) and determine \( G_S(0), A \) to reproduce the experimental values of pion mass \( m_\pi = 138 \text{ MeV} \) and its decay constant \( f_\pi = 93.3 \text{ MeV} \) at vacuum. The EPNJL model parameters are summarized in Table I.

| Parameter | Value |
|-----------|-------|
| \( m_0 \) (MeV) | 6.3 |
| \( A \) (MeV) | 768 |
| \( G_S(0) \) (MeV) | 2.95 |
| \( \alpha_1 \) | 0.2 |
| \( \alpha_2 \) | 0.2 |
| \( T_0 \) (MeV) | 200 |
B. Meson screening mass at finite $T$ and $\mu$

Following the previous work \[43\], we first consider $\pi$ and $\sigma$ mesons at $T = \mu = 0$. The current operator is expressed by

$$J_\xi(x) = \bar{\psi}(x) \Gamma_\xi \psi(x) - \langle \bar{\psi}(x) \Gamma_\xi \psi(x) \rangle$$  \quad (9)

with $x = (t, \mathbf{x})$ for meson species $\xi = \pi, \sigma$, where $\Gamma_\pi = 1$ for $\pi$ meson and $\Gamma_\sigma = i\gamma_5 \tau_3$ for $\sigma$ meson. The mesonic correlation function in coordinate space is defined by

$$\zeta_{\xi\xi}(t, \mathbf{x}) = \langle 0 | T \left( J_\xi(t, \mathbf{x}) J_\xi^\dagger(0) \right) | 0 \rangle.$$  \quad (10)

Here, the symbol $T$ stands for the time-ordered product. The Fourier transform $\chi_{\xi\xi}(q_0, \mathbf{q}^2)$ of $\zeta_{\xi\xi}(t, \mathbf{x})$ is obtained by

$$\chi_{\xi\xi}(q_0, \mathbf{q}^2) = i \int d^4x e^{i\mathbf{q}\cdot\mathbf{x}} \zeta_{\xi\xi}(t, \mathbf{x})$$  \quad (11)

for an external momentum $q = (q_0, \mathbf{q})$ and $\tilde{q} = \pm |\mathbf{q}|$. When we take the random-phase approximation, we can get $\chi_{\xi\xi}$ as

$$\chi_{\xi\xi} = \frac{\Pi_{\xi}}{1 - 2G_S(\Phi)\Pi_{\xi}}$$  \quad (12)

for $\xi = \pi, \sigma$. The one-loop polarization function $\Pi_{\xi}$ is explicitly calculated by

$$\Pi_\sigma = (-2i) \int \frac{d^4p}{(2\pi)^4} \text{tr}_{c.d} (iS(p+q)iS(p)) = 4i[I_1 + I_2 - (q^2 - 4M^2) I_3]$$  \quad (13)

for $\sigma$ meson and

$$\Pi_\pi = (-2i) \int \frac{d^4p}{(2\pi)^4} \text{tr}_{c.d} ((i\gamma_5)iS(p+q)(i\gamma_5)iS(p)) = 4i[I_1 + I_2 - q^2 I_3]$$  \quad (14)

for $\pi$ meson, where the trace $\text{tr}_{c.d}$ is taken in color and Dirac spaces. Three functions in Eqs. (13) and (14) are defined by

$$I_1 = \int \frac{d^4p}{(2\pi)^4} \text{tr}_{c} \left[ \frac{1}{p^2 - M^2} \right],$$  \quad (15)

$$I_2 = \int \frac{d^4p}{(2\pi)^4} \text{tr}_{c} \left[ \frac{1}{(p+q)^2 - M^2} \right],$$  \quad (16)

$$I_3 = \int \frac{d^4p}{(2\pi)^4} \text{tr}_{c} \left[ \frac{1}{(p^2 - M^2)((p+q)^2 - M^2)} \right].$$  \quad (17)

These functions are regularized with the same procedure as shown in Eq. (8).

In the two cases of (a) finite $T$ and $\mu = \mu_\text{h}$ and (b) finite $T$ and $\mu = i\mu_\text{h}$, one can get the final equations by taking the following replacement

$$p_0 \rightarrow i\omega_n + iA_4 + \mu = i(2n+1)\pi T + iA_4 + \mu,$$

$$\int \frac{d^4p}{(2\pi)^4} \rightarrow iT \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3}.$$  \quad (18)

Consequently, we obtain the regularized function $I_3^{\text{reg}}$ as an

$$M_{\xi}^{\text{cr}} = - \lim_{r=|x|\rightarrow\infty} \left( \frac{d}{dr} \ln \zeta_{\xi\xi}(0, x) \right),$$  \quad (19)

where the correlation function $\zeta_{\xi\xi}(0, x)$ in coordinate space is obtained by the Fourier transformation of the correlation function $\chi_{\xi\xi}(0, \mathbf{q}^2)$ in momentum space as

$$\zeta_{\xi\xi}(0, x) = \frac{1}{4\pi^2 ir} \int_{-\infty}^{\infty} d\tilde{q} \hat{\chi}_{\xi\xi}(0, \tilde{q}^2)e^{i\tilde{q}r};$$  \quad (20)

see Fig. 1 to understand the meaning of $\tilde{q}$ integral.

![Fig. 1: Singularities of $\chi_{\xi\xi}(0, \mathbf{q}^2)$ in the complex-$\tilde{q}$ plane. Cuts are denoted by the wavy lines and poles are denoted by points. The threshold masses correspond to the endpoints of cuts. The original contour $C$ in Eq. (20) is deformed into $C_1$, $C_2$ (cut contributions) and $C_3$ (pole contribution). For the definition of threshold masses, see Eq. (23).](image)

NJL-type effective models have two problems in the calculation of Eq. (20). The first problem stems from the regularization. The three-dimensional-momentum cutoff regularization commonly used explicitly breaks Lorentz invariance, and induces unphysical oscillations in $\zeta_{\xi\xi}(0, x)$ \[48\]. This problem can be solved by taking the PV regularization \[49\]. We then use the PV regularization in this paper. The second problem is the fact that direct numerical calculations of $\tilde{q}$ integral is quite difficult because the integrand is highly oscillating at large $r$ where $M_{\xi}^{\text{cr}}$ is defined. In order to overcome this problem, one can rewrite the $\tilde{q}$ integral to the complex $\tilde{q}$ axis by using the Cauchy’s integral theorem. However, it is shown in Ref. \[48\] that the complex function $\chi_{\xi\xi}(0, \tilde{q}^2)$ has logarithmic cuts in the vicinity of the real $\tilde{q}$ axis. The evaluation of the cuts still demands time-consuming numerical calculations. Our previous works \[43, 44\] showed that the emergence of these logarithmic cuts is avoidable by making the $p$ integration analytically before taking the Matsubara $(\ii)$ summation in Eqs. (12)–(18).
infinite series of analytic functions:

\[
T^\text{res}(0, \tilde{q}^2) = iT \sum_{n=0}^{\infty} \sum_{j=1}^{N_x} \sum_{\alpha=0}^{2} C_{\alpha} \\
\times \int \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{p^2 + M^2} \frac{1}{(p + \tilde{q})^2 + M^2} \right] \\
= \frac{T}{8\pi \tilde{q}} \sum_{j,n,\alpha} C_{\alpha} \log \left( \frac{2M + i\tilde{q}}{2M - i\tilde{q}} \right)
\]

with a complex valued thermal mass

\[
\mathcal{M}(M_\alpha, \omega_n, A^{ij}_n, \mu) = \sqrt{M^2_\alpha + (\omega_n + A^{ij}_n - i\mu)^2},
\]

where we take the principle value for logarithm in Eq. (21) and the square root in Eq. (22). Each term in last line of Eq. (21) has four cuts starting at \( \tilde{q} = \pm 2iM(M_\alpha, \omega_n, A^{ij}_n, \mu) \) and \( \tilde{q} = \pm 2iM(M_\alpha, \omega_n, -A^{ij}_n, -\mu) \), as shown in Fig. 1. For later convenience, we define the threshold mass \( M_{\text{th}} \) and the decay width \( \Gamma_{\text{th}} \) by the \( \mathcal{M} \) located at the lowest branch point in the upper-half plane: Namely

\[
2M_{\text{lowest}} = M_{\text{th}} - i\frac{\Gamma_{\text{th}}}{2},
\]

where \( M_{\text{th}} (\Gamma_{\text{th}}) \) is the real (imaginary) part of \( 2M_{\text{lowest}} \). Meson screening mass \( M^{\text{scr}}_\xi \) is a pole of \( \chi_{\xi\xi} \) and is calculated by

\[
[1 - 2G_S(\Phi)\Pi(0, \tilde{q}^2)]|_{\tilde{q}=iM^{\text{scr}}_\xi} = 0,
\]

when the pole is located below the lowest branch point. This condition leads to

\[
M^{\text{scr}}_\xi \leq M_{\text{th}}.
\]

### III. NUMERICAL RESULTS

#### A. Deconfinement and chiral transition lines in \( \theta-T \) plane

Figure 2 shows \( T \) dependence of \( \sigma \) and \(|\Phi|\) for the case of \( \theta = 0 \). The EPNJL-model results with the parameter set of Table I well simulate LQCD data [46, 47]. This means that the present EPNJL model is reliable at least for \( \theta = 0 \).

Figure 3 shows the deconfinement and chiral transition lines in the imaginary-\( \mu \) region, where the transition temperatures are determined from peak positions of chiral and Polyakov-loop susceptibilities. \( \theta \) dependence of the transition lines are well fitted in \( 0 \leq \theta \leq \pi/3 \) by using

\[
\frac{T^\chi(\theta)}{T^\chi} = 1 + c_1^\chi \theta^2 + c_2^\chi \theta^4,
\]

where the superscript “\( X = d \)” means the deconfinement transition and “\( X = \chi \)” corresponds to the chiral transition. The results of the fitting are summarized in Table [I].

#### B. \( \theta \) dependence of \( \pi \) and \( \sigma \) meson screening masses

First we have confirmed that \( \pi \)- and \( \sigma \)-meson screening masses have the RW periodicity and charge symmetry:

\[
M^{\text{scr}}_\xi (\theta) = M^{\text{scr}}_\xi \left( \theta + \frac{2\pi k}{3} \right), \quad M^{\text{scr}}_\xi (\theta) = M^{\text{scr}}_\xi (-\theta)
\]

for \( \xi = \pi, \sigma \), where \( k \) is an arbitrary integer. This result stems from the fact that Eqs. (15)–(18) and the threshold mass \( M_{\text{th}} \) are invariant under the extended \( Z_3 \) transformation defined by Eq. (7).
In the next subsection, we will extrapolate the meson screening masses from $\mu = i\mu_I$ to $\mu = \mu_R$. For this purpose, we first fit our model results with the polynomial function,

$$M_{\xi}^{\text{scr}}(T, i\mu_I) = \sum_{n=0}^{n_{\text{max}}} a_{\xi}^{(n)}(T) \theta^{2n}, \quad (28)$$

in $0 \leq \theta \leq \pi/3$. We take $n_{\text{max}} = 1, 2, 3, 4$ in order to confirm convergence of the expansion. $\theta$ dependence of $M_{\pi}^{\text{scr}}$ and $M_{\sigma}^{\text{scr}}$ is well fitted with $n_{\text{max}} = 4$. In this procedure, $\theta$ is varied in the range $0 \leq \theta \leq \pi/3$, although $T$ is fixed.

We consider the following two cases:

(A) $T = 250$ MeV in Fig. 3. The system is in both the deconfinement and the chiral-symmetry restored phase for any $\theta$, since $T \geq T_c^{\chi}(\pi/3)$.

(B) $T = 180$ MeV in Fig. 3. This case satisfies $T_c^{\chi}(0) \leq T \leq T_{\text{HW}}$. The system is in the deconfinement phase for $0 \leq \theta \leq 0.697$ but in the confinement phase in $0.697 \leq \theta \leq \pi/3$. The system is in the chiral-symmetry restored phase for $0 \leq \theta \leq 0.403$ but in the chiral-symmetry broken phase for $0.403 \leq \theta \leq \pi/3$.

Figure 4 explains $\theta$ dependence of $\pi$-meson screening masses for two cases (A) and (B). The $M_{\pi}^{\text{scr}}$ monotonically decrease as $\theta$ increases for two cases (A) and (B).

Figure 5 shows $\theta$ dependence of $\sigma$-meson screening masses for two cases (A) and (B). The $M_{\sigma}^{\text{scr}}$ have non-monotonic $\theta$ dependence for case (B). As for case (A), the $\pi$- and $\sigma$-meson screening masses agree with each other due to the chiral symmetry restoration.

### C. Extrapolation from $\mu_I$ to $\mu_R$ region

We compare the extrapolating result with the direct one for finite $\mu_R$ in order to confirm applicability of the analytic continuation. One can easily make the analytic continuation by replacing $\theta$ with $-i\mu_I/T$:

$$M_{\xi}^{\text{scr}}(T, \mu_R) = \sum_{n=0}^{n_{\text{max}}} (-1)^n a_{\xi}^{(n)}(T) \left( \frac{\mu_R}{T} \right)^{2n}. \quad (29)$$

Figure 6 explains $\mu_R$ dependence of $\pi$-meson screening masses for two cases (A) and (B). In $\mu_R/T \leq 0.4$, the $M_{\pi}^{\text{scr}}$ converge to the direct results as $n_{\text{max}}$ increases for both the two cases.

Figure 7 shows $\mu_R/T$ dependence of $M_{\pi}^{\text{scr}}$ for case (B), i.e., $T = 180$ MeV. We skip case (A) since chiral symmetry is restored in case (A), and $\theta$ dependence of $M_{\pi}^{\text{scr}}$ is almost same as that of $M_{\pi}^{\text{scr}}$. The extrapolating results tends to the direct ones for $0 \leq \mu_R/T \leq 0.4$, and the deviation in $0.4 \leq \mu_R/T$ can not be improved by taking the higher order terms.

The origin of the deviation can be understood when one considers the relation between $\sigma$-meson screening mass and chiral susceptibility. Equation (19) indicates that the inverse of $M_{\sigma}^{\text{scr}}$ corresponds to the correlation length in the fluctuation of $\langle \bar{\psi}(x)\psi(x) \rangle$; see Ref. [50] for the further explanation, and note that screening mass is referred to be the frequency of “sound mode” there. Hence $M_{\sigma}^{\text{scr}}$ is related to the chiral susceptibility $\chi_\sigma$ as

$$M_{\sigma}^{\text{scr}} \propto \chi_\sigma^{-1/2}. \quad (30)$$

Particularly for the chiral limit, $\mu_R$ and $\mu_I$ dependence of $M_{\sigma}^{\text{scr}}$ is non-analytic on the chiral phase transition line in $\mu_R$—
Fig. 6: Comparison between the extrapolating and the direct results on $\mu_R/T$ dependence of $M_\sigma^{scr}$. We draw direct-result lines only when the inequality $M_\sigma^{scr} < M_{th}$ in Eq. (25) is satisfied.

$T$ and $\mu_\Pi$-$T$ plane, since $\chi_\sigma$ is non-analytic on the chiral phase transition line. As for finite quark mass, a remnant of the non-analyticity makes the accuracy of the analytic continuation less accurate.

D. Phase-transition-line extrapolation

We propose the new extrapolation method by modifying a trajectory of $(T, \theta)$ in fitting. In standard extrapolation, $\theta$ is varied with fixed $T$. In new method, we also vary $T$ so that the trajectory runs along the phase transition line. We then assume $\theta$ dependence of $T$ as

$$T = T_{PTL}^X(\theta) = R \times T_\sigma^X(\theta)$$

(31)

Fig. 7: Comparison between the extrapolating and the direct results on $\mu_R/T$ dependence of $M_\sigma^{scr}$ in case (B), i.e., $T = 180$ MeV.

with any constant $R$ that is introduced to cover the $\theta$-$T$ plane; see Fig. 8 for the understanding. The symbol $X$ means the chiral transition ($X = \chi$) or deconfinement transition ($X = d$). In this paper, we refer to the modified extrapolation as "phase-transition-line (PTL) extrapolation".

From now on, we consider the chiral transition ($X = \chi$). We fit $\theta$ dependence of $\sigma$-meson screening masses with a polynomial series:

$$\frac{M_\sigma^{scr}(\theta)}{T_{PTL}^X(\theta)} = \sum_{n=0}^{n_{max}} b_{\sigma}^{(n)}(R) \theta^{2n}.$$  (32)

In Eq. (31), the extrapolation line does not pass through the chiral transition line, we can use all range of $\theta$ for fitting, i.e., $0 \leq \theta \leq \pi/3$.

Fig. 8: A schematic figure of PTL extrapolation and standard-extrapolation. The arrows stand for the standard extrapolation and the PTL extrapolation. Transition line for chiral symmetry restoration is denoted by the dotted line.
We then extrapolate \( M^{\text{scrt}}_\sigma(\theta) \) and \( T^\chi_{\text{PTL}}(\theta) \) from finite \( \mu_1 \) region to \( \mu_R \) Region. Figure 9 shows the comparison between the direct result and the extrapolating ones for \( M^{\text{scrt}}_\sigma \), where we set \( T^\chi_{\text{PTL}}(0) = 180 \) MeV. The extrapolating results rapidly converge to direct-calculated one in \( \mu_R/T^\chi_{\text{PTL}} \leq 0.8 \). The PTL extrapolation yields better agreement than the standard extrapolation.

We also check the reliability of extrapolation by estimating the radius of convergence in Eq. (32) based on the d’Alembert ratio test. The coefficients \( b_{\sigma}^{(n)} \) in Eq. (32) are summarized in Table. III. The radius of convergence \( r_\sigma \) is calculated by

\[
r_\sigma \equiv \sqrt{b_{\sigma}^{(n_{\text{max}}-1)}}/b_{\sigma}^{(n_{\text{max}})} \approx 0.84, \text{ whose value is consistent with the upper bound of the agreement region.}
\]

| Table III | Coefficients and convergence radii for \( M_{\sigma}^{\text{scrt}} \) and \( M_{\pi}^{\text{scrt}} \) with \( n_{\text{max}} = 4 \). |
|------------|-----------------------------------------------------------------------------------------------------------------|
|            | \( b_{\sigma}^{(0)} \) | \( b_{\sigma}^{(1)} \) | \( b_{\sigma}^{(2)} \) | \( b_{\sigma}^{(3)} \) | \( b_{\sigma}^{(4)} \) | \( r_\sigma \) |
| \( \sigma \) meson | 1.677 | -0.312 | -0.010 | -0.012 | 0.017 | 0.84 |
| \( \pi \) meson | 1.254 | -0.327 | 0.168 | -0.090 | 0.0184 | 2.21 |

Parallel discussion is possible for \( M^{\text{scrt}}_\pi \), as shown in Fig. 10. We can obtain good agreement between direct results and extrapolating ones for \( \mu_R/T^\chi_{\text{PTL}} \leq \pi/3 \).

IV. SUMMARY

We first showed a method of calculating screening masses for finite \( \mu_R \) and \( \mu_1 \) in the framework of the 2-flavor EPNJL model.

Next, we investigated how reliable the imaginary-\( \mu \) approach is for \( M^{\text{scrt}}_\pi \) and \( M^{\text{scrt}}_\sigma \) by comparing “the results extrapolated from imaginary \( \mu \)” with “those calculated directly in real \( \mu \)”. In the standard extrapolation, the agreement between the direct and the extrapolating results is seen in \( \mu_R/T^\chi_{\text{PTL}} \leq 0.4 \) for \( M^{\text{scrt}}_\pi \) and \( M^{\text{scrt}}_\sigma \) for \( T = 180 \) and 250 MeV. Especially for \( \sigma \) meson, the disagreement in \( 0.4 \leq \mu_R/T \) can not be improved by taking higher order terms.

We can understand the difficulty of extrapolation when one remembers that \( M^{\text{scrt}} \) is nothing but the inverse of correlation length in fluctuation of local chiral condensate. The \( M^{\text{scrt}}_\sigma \) is thus related with the chiral susceptibility \( \chi_\sigma \) as \( M^{\text{scrt}}_\sigma \propto \chi_\sigma^{-1/2} \). When one set quark mass to zero, \( \chi_\sigma \) becomes non-analytic on the chiral transition line \( T = T^\chi_\sigma(\theta) \), and so does \( M^{\text{scrt}}_\sigma \). Even for finite quark mass, a remnant of this non-analyticity makes the accuracy of extrapolation less accurate, since quark mass is much smaller to temperature and negligible around the chiral phase transition. This indicates that the simple extrapolation is not useful for \( M^{\text{scrt}}_\sigma(T, \mu_R) \).

In order to circumvent this problem, we propose the PTL extrapolation. In the method, the agreement between the direct and the extrapolating results is seen in \( \mu_R/T^\chi_{\text{PTL}} \leq 0.8 \) for \( M^{\text{scrt}}_\pi \) and in \( \mu_R/T^\chi_{\text{PTL}} \leq \pi/3 \) for \( M^{\text{scrt}}_\sigma \) with \( T^\chi_{\text{PTL}}(0) = 180 \) MeV. The extrapolating results tend to the direct results as higher order terms are taken into account. The PTL extrapolation thus makes better extrapolating results than the standard one.

The difficulty of the simple extrapolation may be in common with other scalar, vector and pseudovector mesons composed of \( u \) and \( d \) quarks, since these meson masses are sensitive to the chiral transition. The application of PTL extrapolation to such mesons is thus interesting as a future perspective.

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Appendix: The relation between $A_4$ and $\Phi, \bar{\Phi}$

The diagonal components $A_{4}^{11}, A_{4}^{22}, A_{4}^{33}$ of the gluon field are related with the Polyakov loop $\Phi$ and its conjugate $\bar{\Phi}$ as

$$\Phi = \frac{1}{3}(\phi_1 + \phi_2 + \phi_3),$$

$$\bar{\Phi} = \frac{1}{3}(\phi_1^{*} + \phi_2^{*} + \phi_3^{*}) = \frac{1}{3}(\phi_1 \phi_2 + \phi_2 \phi_3 + \phi_3 \phi_1)$$

(A.1)

(A.2)

with $\phi_j \equiv \exp(iA_{4}^{jj}/T)$ ($j = 1, 2, 3$). Furthermore, the traceless condition for $A_4$ leads to

$$\phi_1 \phi_2 \phi_3 = 1.$$  

(A.3)

One can confirm that $\phi_1, \phi_2, \phi_3$ are solutions of following equation:

$$\phi^3 - 3\Phi \phi^2 + 3\bar{\Phi} \phi - 1 = 0.$$  

(A.4)

by considering Vieta’s formulas. Once we get $\Phi$ and $\bar{\Phi}$, we can obtain $\phi_1, \phi_2, \phi_3$ by solving above equation analytically and get the gluon field as $A_{4}^{jj} = -iT \log \phi_j$. The relation between $A_{4}^{jj}$ and $\phi_j$ has an ambiguity coming from the replacement $A_4 \to A_4 + 2\pi n T$ for integer $n$, but this ambiguity does not change any physical observables and we simply assume $n = 0$. If we take the approximation $\Phi \simeq \bar{\Phi}$, we can simply obtain the gluon fields as

$$A_{4}^{11} = -A_{4}^{22} = T \cos^{-1} \left(\frac{3\Phi - 1}{2}\right), A_{4}^{33} = 0.$$  

(A.5)
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