Modeling Nonlinear Evolution of Baryon Acoustic Oscillations: Convergence Regime of N-body Simulations and Analytic Models

Takahiro NISHIMICHI,1 Akibito SHIRATA,1,2 Atsushi TARUYA,1,3,5 Kazuhiro YAHATA,1
Shun SAITO,1 Yasushi SUTO,1,3 Ryuichi TAKAHASHI,4 Naoki YOSHIDA,4,5 Takahiko MATSUBARA,4
Naoshi SUGIYAMA,4,5 Issha KAYO,4,5 Yipeng JING,6 and Kohji YOSHIKAWA;
1Department of Physics, School of Science, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033
2Department of Physics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551
3Research Center for the Early Universe, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033
4Department of Physics and Astrophysics, Nagoya University, Chikusa-ku, Nagoya 464-8602
5Institute of Physics and Mathematics of the Universe, The University of Tokyo, 5-1-5 Kashiwa-no-ha, Kashiwa, Chiba 277-8582
6Shanghai Astronomical Observatory, 80 Nandan Road, Shanghai 200030, China
7Center for Computational Sciences, University of Tsukuba, Tsukuba, Ibaraki 305-8577
nishimichi@utap.phys.s.u-tokyo.ac.jp

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Abstract

We used a series of cosmological N-body simulations and various analytic models to study the evolution of the matter power spectrum in real space in a Λ cold dark matter universe. We compared the results of N-body simulations against three analytical model predictions; standard perturbation theory, renormalized perturbation theory, and a closure approximation. We included the effects from a finite simulation box size under comparison. We determined the values of the maximum wavenumbers, \( k^{\text{lim}_{\Lambda}} \) and \( k^{\text{lim}_{\Lambda}} \), below which the analytic models and the simulation results agree with accuracy to within 1 and 3 percent. We then provided a simple empirical function that describes the convergence regime determined by comparisons between our simulations and the analytical models. We found that if we use the Fourier modes within the convergence regime alone, the characteristic scale of baryon acoustic oscillations can be determined with an accuracy of 1% from future surveys with a volume of a few \( h^{-3} \) Gpc\(^3\) at \( z \sim 1 \) or \( z \sim 3 \) in the absence of any systematic distortion of the power spectrum.

Key words: cosmology: large-scale structure of universe — methods: theoretical

1. Introduction

The nature of dark energy remains one of the most fundamental questions in physics and cosmology. It is accessible only through astronomical observations, and a number of large galaxy redshift surveys and weak-lensing observations are expected to yield tight constraints on the dark energy equation of state parameter, \( w_{\text{DE}} \equiv p_{\text{DE}}/\rho_{\text{DE}} \). The use of baryon acoustic oscillations (hereafter BAOs) is a promising tool for determining \( w_{\text{DE}} \). The characteristic scale of BAOs can be used as a robust standard ruler, which helps us to reconstruct the expansion history of the universe.

Substantial improvements in theories as well as observations are required to constrain \( w_{\text{DE}} \) with accuracy to within a few percent, which is the goal of next-generation BAO surveys. Nishimichi et al. (2007) showed that measurements of the angular diameter distance and of the Hubble parameter must be very accurate, with errors of less than 1%, in order to constrain \( w_{\text{DE}} \) at a 5% level at \( z \sim 1 \) in the special case where the other cosmological parameters are fixed. Clearly, theoretical models are required to predict the matter power spectrum with even higher accuracy.

Recently, significant progress has been made in developing analytic models of the matter power spectrum by extending conventional perturbation theory (e.g., Crocce & Scoccimarro 2006a, 2006b, 2008; Matarrese & Pietroni 2007, 2008; Valageas 2007; McDonald 2007; Taruya & Hiramatsu 2008; Matsubara 2008). Cosmological N-body simulations are often used as a reference to calibrate these models. It is not clear, however, whether or not N-body simulations provide sufficiently accurate results at the required percent level, even at weakly nonlinear scales where perturbation theory is generally supposed to be reasonably accurate.

Indeed a variety of systematic effects needs to be considered in interpreting the results of N-body simulations. It is well known that discreteness of a density field in N-body simulations causes a spurious behavior of the power spectrum at length scales comparable to or smaller than the mean inter-particle separation (Melott et al. 1997; Splinter et al. 1998; Hamana et al. 2002; Baertschiger et al. 2002; Marcos et al. 2006; Joyce & Marcos 2007a, 2007b; Joyce et al. 2008; Romeo et al. 2008). Adopting a finite-size simulation box and imposing a periodic boundary condition also systematically bias the growth of density fluctuations at almost all the length scales (Seto 1999). Recently, Takahashi et al. (2008) examined the effect of a finite number of Fourier modes in a single N-body realization due to a finite box size. They found that the evolution of the power spectrum measured from each realization deviates from the prediction of perturbation theory by more than a few percent, even at very large scales (e.g., \( k \lesssim 0.1 \) h Mpc\(^{-1}\)), and that this deviation results from having only a finite number of modes, which can be explained...
by including second-order perturbation contributions. This implies that a subpercent level accuracy both in theories and in simulations is very demanding, and we need to explore this in detail.

In the present paper, we consider a specialized case of the matter power spectrum in real space. We critically compared the evolution of the matter density fluctuations in the weakly nonlinear regime in cosmological simulations to that of theoretical predictions. We measured the matter power spectra using the simulation outputs, and compared them with analytic models. In doing so we corrected the finite-mode effect using the simulation outputs, and compared them with analytic models. We determined the ranges of wavenumbers where both theories and $N$-body simulations agree within a given accuracy.

The rest of the present paper is organized as follows. Section 2 briefly outlines the analytic models that we examined for the nonlinear matter power spectrum. Section 3 describes the details of our simulations. Our methods for measuring the matter power spectrum and for correcting the finite volume effect are shown in section 4. We show our results of $N$-body simulations and determine the convergence regime in wavenumber in section 5. We also discuss the phase information of BAOs and present the parameter forecasting from the modes in this convergence regime alone in section 6. Finally, section 7 summarizes the present paper. The convergence tests of our $N$-body simulations are discussed in the Appendix.

### 2. Analytic Nonlinear Models

Various analytic models have been proposed to account for nonlinear evolution of the matter power spectra in real space. In this paper, we compare $N$-body simulations with three different analytic models, while specifically focusing on the leading-order contributions. The models include standard perturbation theory (SPT: e.g., Bernardeau et al. 2002), renormalized perturbation theory (RPT: Crocce & Scoccimarro 2006a, 2006b, 2008), and a closure approximation (CLA: Taruya & Hiramatsu 2008).

SPT is a straightforward expansion of the fluid equations around their linear solution, assuming that the fluctuation amplitudes are small. Schematically, the expansion is written as

$$P(k, z) = P^1(k, z) + P^{1\text{-loop}}(k, z) + \cdots,$$

where $P^1(k, z)$ is the linear power spectrum, which grows as \(\propto D^1(z)\) with a linear growth rate, $D^1(z)$. The second term, $P^{1\text{-loop}}(k, z)$, is a one-loop correction to the power spectrum, which represents the contributions coming from the second- and third-order solutions. The one-loop correction is roughly proportional to $P^1(k, z)$ with $\Delta^2 = k^3 P^1(k, z)/(2\pi^2)$, and this term subsequently exceeds the linear term at lower redshift and smaller scales. The explicit expressions for the one-loop correction, $P^{1\text{-loop}}(k, z)$, may be found in the literature (e.g., Makino et al. 1992; Jain & Bertscherger 1994; Scoccimarro & Frieman 1996; Jeong & Komatsu 2006; Nishimichi et al. 2007).

Both RPT and CLA are constructed from the renormalized expression of a perturbation series obtained from SPT, by introducing nonperturbative statistical quantities, such as the propagator and the vertex function. The resultant renormalized expressions for the power spectrum are given as an infinite series of irreducible diagrams composed of a propagator, a vertex function, and the power spectra themselves in a fully nonperturbative way. Then, employing the Born approximation, RPT of Croce and Scoccimarro (2008) perturbatively evaluates the renormalized diagrams under the tree-level approximation of vertex functions. In contrast, CLA proposed by Taruya and Hiramatsu (2008) first truncates the renormalized expansion at the one-loop order under the tree-level approximation of vertex functions, and obtains a closed set of equations for the power spectrum and the propagator. In practice, they further applied the Born approximation to the truncated diagram. As a result, the leading-order calculations of the power spectra in CLA are reduced to the same analytic expression as in the case of RPT, leading to

$$P(k, z) = G^2(k, z) P^1(k, z) + P^{1\text{-loop}}_{\text{MC}}(k, z) + \cdots. \quad (2)$$

Here, the function $G(k, z)$ is the propagator, and the term $P^{1\text{-loop}}_{\text{MC}}(k, z)$ represents the corrections generated by mode-mode coupling at smaller scales, constructed from the one-loop diagram. In the above expression, the only difference between CLA and RPT is the asymptotic behavior of the propagator. Explicit expressions for the propagator together with the mode-coupling term are summarized in Croce and Scoccimarro (2006b, 2008) and Taruya and Hiramatsu (2008).

We note here that the most important difference between SPT and RPT/CLA is the convergence properties dictated by the propagator. In the same manner as $P^{1\text{-loop}}(k, z)$ of SPT, the term $P^{1\text{-loop}}_{\text{MC}}(k, z)$ is roughly proportional to $P^1(k, z)$ at high $k$. Therefore, to the one-loop level, RPT and CLA are expected to be the most accurate at low $k$ (below the propagator damping scale), but the predictions at low $k$ are expected to be very accurate for RPT/CLA compared to SPT.

### 3. Simulations

In this section we describe our fiducial simulations. We also ran some simulations with different settings (initial
Fig. 1. Flow chart to illustrate our methodology to correct the effect of finite box size.

conditions, integrators, and box sizes), which are discussed in
the Appendix.

We begin with an explanation of our initial conditions for
the N-body simulations. We computed the linear power spec-
trum at an initial redshift of $z_{ini} = 31$ using CAMB (Lewis
et al. 2000). For all of the simulations discussed in the
present paper, we adopted the standard $\Lambda$ cold dark matter
($\Lambda$CDM) model with the cosmological parameters from the
Wilkinson Microwave Anisotropy Probe 3 (WMAP3) results
(Spergel et al. 2007):

$$\Omega_m = 0.234, \Omega_\Lambda = 0.766, \Omega_b = 0.0414,$$
$$h = 0.734, \sigma_8 = 0.76, \text{ and } n_s = 0.961,$$

which are the current matter, cosmological constant, baryon densities in
units of the critical density, the Hubble constant in units of
$100 \text{ km s}^{-1} \text{Mpc}^{-1}$, the density fluctuation amplitude smoothed
with a top-hat filter of radius $8h^{-1}\text{Mpc}$, and the scalar spec-
tral index, respectively (see table 1). We then generated
a linear overdensity field in Fourier space while assuming
Gaussianity; the amplitude follows the Rayleigh distribution,
and the phase is uniformly distributed. We employed 512
dark matter particles within a cube of $1000h^{-1}\text{Mpc}$ in each
side. We displaced these particles from regular grid preinitial
positions using second-order Lagrangian perturbation theory
(2LPT; e.g., Crocce et al. 2006).

We next describe the time integration scheme. We
adopted a freely available parallel cosmological N-body solver
Gadget2 (Springel 2005). The number of meshes used in
the particle-mesh computation was $1024^3$. We adopted a softening
length of $0.1h^{-1}\text{Mpc}$ for the tree forces. We selected three
output redshifts: $z = 3, 1, \text{ and } 0$, where we measured the power
spectrum (for details see section 4).

In the Appendix, we discuss the convergence of our simulations against different initial conditions, box sizes, and codes.
We found that the current setup provides a convergent result
within 1% of the amplitude of the power spectrum at our scales
of interest ($k \lesssim 0.4h\text{Mpc}^{-1}$). We used this setup as our fiducial
model, and ran 4 different realizations for this model.

4. Power Spectrum Analysis

4.1. Matter Density Field and Power Spectrum in N-Body Simulations

Here we briefly describe the notation for various density
fields and power spectra, which we use throughout the present
paper (see figure 1 and table 2).

We denote the Gaussian-sampled density field from
$P_L(k,z_{ini})$ by $\delta_{n,k,n}(z_{ini})$, where the subscript $n$ stands for the
$n$-th realization. We then denote by $\delta_{n,\text{body}}(z_{ini})$ the density
field realized by particles using a 2LPT displacement. Note
that $\delta_{L,n}(z_{ini})$ is different from $\delta_{\text{body},n}(z_{ini})$; the former is
strictly Gaussian, whereas the latter slightly deviates from
Gaussian because of the nonlinearity and discreteness in the
process of displacement. The measured density contrast
from the simulation output at redshift $z$ is also expressed as
$\delta_{k,n}(z)$. In measuring $\delta_{k,n}(z)$ from the simulation
output, we first assigned particles onto a $1024^3$ mesh using
a cloud-in-cell interpolation (Hockney & Eastwood 1981).
We then used a fast Fourier transform (FFT) to calculate density
contrasts in Fourier space, and divided each Fourier mode by
the Fourier transform of the window function (Angulo et al.
2008). We made sure that the details of the interpolation
scheme and the number of mesh points would not significantly
affect the results on the scales of our interest ($k \lesssim 0.4h\text{Mpc}^{-1}$).
See also Jing (2005) for a discussion of the effects of aliasing.

We squared $\delta_{k,n}(z)$ and took an average over realiza-
tions and modes:
The matter power spectrum measured from simulations deviates from the prediction for the ideal ensemble average, which can be obtained only in the limit of an infinite number of realizations or an infinite box size. This deviation is actually important for interpreting the results of N-body simulations, as shown by Takahashi et al. (2008); the matter power spectrum measured from their N-body simulations does not agree with the predictions of linear theory nor SPT, even at very large scales (e.g., $k \lesssim 0.1 h\text{Mpc}^{-1}$). They examined the effect of a finite box size (hence a finite number of modes), and showed that the finite-mode effect is actually responsible for the anomalous growth rate. Here, we briefly summarize their formulation of the correction.

We follow the standard notation used in cosmological perturbation theory [see Bernardeau et al. (2002) for a review]. Let us expand the density perturbations in $k$-space for the $n$-th $N$-body realization as

$$\delta_{k,n}^L(z) = \delta_{k,n}^L(z) + \delta_{k,n}^{(2)}(z) + \ldots.$$  

Here, the second-order term is expressed as a sum of contributions from mode couplings between two modes:

$$\delta_{k,n}^{(2)}(z) = \sum_p F^{(2)}(p, k - p; z)\delta_{p,n}^L(z)\delta_{k-p,n}^L(z),$$

where the symmetrized second-order kernel $F^{(2)}$ is expressed as

$$F^{(2)}(x, y; z) = \frac{1}{2} \left(1 + \epsilon(z)\right) + \frac{1}{2} \frac{x \cdot y}{x y} \left(\sum_{p=1}^{4} \frac{1}{x^2 y^2} - 1\right),$$

$$\epsilon(z) \approx \frac{3}{7}\Omega^{-1/14}_m(z).$$

In practice, the time dependence of the kernel function is very weak, and thus we simply set $\epsilon = 3/7$ in what follows. The power spectrum up to the third order in $\delta_{k,n}^L(z)$ averaged over modes in the $i$-th wavenumber bin and over realizations is

$$\hat{P}_{\text{PT}}(k_i, z) \equiv \left(\left|\hat{P}_{\text{PT}}^L(k, z)\right|^2\right)_{i},$$

$$\left|\hat{P}_{\text{PT}}^L(k, z)\right|^2 = 2\Re\left[\delta_{k,n}^L(z)\delta_{k+n,n}^{(2)}(z)\right],$$

where $\Re[...]$ stands for the real part of a complex number. Though the second term should vanish for an ensemble average over infinite modes, it does not vanish exactly if the number of Fourier modes is finite. The first term grows as $\propto D_0^2(z)$, while the second term grows as $\propto D_1^2(z)$, and thus the second term becomes important at late time (i.e., at low redshifts).

The method for correcting the deviation from the ideal ensemble average is to multiply $\hat{P}_{\text{N-body}}(k_i, z)$ by the ratio of
5. Results

5.1. Comparison between N-Body Simulations and Analytic Models

As mentioned above, the accuracy of N-body simulations, themselves, is not perfect, and we do not regard them as being perfect calibrators of theoretical models. Instead, we compare the power spectrum of our simulations with those from theoretical predictions while aiming at determining the reliable range of wavenumbers in which both simulations and theoretical models agree.

We first compare the averaged power spectrum over the four realizations without any corrections to the theoretical predictions. The left panel of figure 2 plots the fractional differences from the linear power spectrum, \( P_{\text{L}}(k,z) \), computed from no wiggle formula of Eisenstein and Hu (1998). The error bars indicate the standard errors of the estimated mean value [equation (5)]. To clarify the differences between the N-body results and theoretical predictions, we also plot the residuals from RPT in the right panel of figure 2.

Since large error bars at \( k \lesssim 0.1 \, h \, \text{Mpc}^{-1} \) are expected to come mostly from the finite-volume effect, we next correct this effect. In figure 3, we plot the power spectra, based on the procedure in subsection 4.2, but we truncate the expansion of equation (11) at the first term (left, the fractional difference from the no-wiggle formula; right, residuals from RPT prediction). The error bars become significantly smaller compared with those in figure 2, because the finiteness effect is reduced by our methodology. Nevertheless, the results of N-body simulation still exhibit an error of a few percent, even after the correction.

Next, we include the second term of equation (11) which comes from the mode couplings among finite modes for the correction. Figure 4 shows the results for the fractional difference between the simulations and the no-wiggle formula (left), and the residuals from RPT predictions (right). Now the size of the error bars is further reduced to the subpercent level. All of the theoretical predictions plotted in figure 4 and N-body simulations agree with each other well within...
Fig. 3. Same as figure 2, but we correct the effect of the finite volume. We truncate the expansion of equation (11) at the first term. The error bars show equation (13).

Fig. 4. Same as figure 2, but we correct the effect of finite volume, including the second term of equation (11). We also show the 1% limit wavenumbers, $k_{\text{lim}}^{1\%}$, for LIN, SPT, and RPT/CLA by vertical arrows (from left to right).

A limitation of the error bars at large scales up to some wavenumbers (we determine the range of convergence in the next subsection). Among the four theoretical predictions, linear theory deviates at the smallest wavenumber. The range of the agreement in SPT is wider than in linear theory, because SPT includes the leading-order contribution of nonlinear growth. RPT and CLA seem to agree well with $N$-body simulations compared with SPT, although all of the three nonlinear models include their own leading-order nonlinear corrections. This difference in the agreement ranges corresponds to their different convergence properties; RPT and CLA possess the property of converging at the scales where the nonlinearity is very weak.

5.2. Convergence Regime in Wavenumber

We are now able to quantitatively estimate the convergence regime of wavenumbers where the theories and $N$-body simulations agree. We define two characteristic wavenumbers, $k_{\text{lim}}^{1\%}$ and $k_{\text{lim}}^{3\%}$, such that the results of $N$-body simulations and theoretical predictions agree within a limitation of 1% at $k < k_{\text{lim}}^{1\%}$ and within a limitation of 3% at $k < k_{\text{lim}}^{3\%}$. We confirmed that if we add 1% (3%) Gaussian errors on the power spectra binned...
by $\Delta k = 0.005 \, h \, \text{Mpc}^{-1}$ that leads to a $\sim 2\% \ (\sim 6\%)$ error on the dark-energy equation of state parameter, $w_{\text{DE}}$, at $z = 1$ using the phase information of BAOs, which is roughly the goal of upcoming (ongoing) surveys.

Before determining the wavenumbers, we briefly summarize three convergence criteria previously introduced in the literature. The first one is

$$\Delta^2(k, z) \equiv \frac{k^3 P_{\text{SPT}}(k, z)}{2\pi^2} < 0.4, \quad (14)$$

introduced by Jeong and Komatsu (2006) as a 1% convergence regime of SPT. The second one is introduced by Sefusatti and Komatsu (2007):

$$\sigma^2(R_{\text{min}}, z) \equiv \int d^3q \hat{W}^2(qR_{\text{min}}) P^L(q, z) = 0.25, \quad (15)$$

$$k_{\text{max}} = \frac{\pi}{2R_{\text{min}}}, \quad (16)$$

where $\hat{W}(qR_{\text{min}})$ denotes the Fourier transform of a top-hat window function with radius $R_{\text{min}}$. The third one is proposed by Matsubara (2008):

$$k^2 \sigma_z^2 \equiv \frac{k^2}{6\pi^2} \int_0^\infty P^L(q, z) dq < 0.25, \quad (17)$$

where $\sigma_z^2$ is the one-dimensional linear velocity dispersion. This gives a scale where the nonlinearity becomes important. Note that Crocce and Scoccimarro (2006b) presented similar criteria.

We show these criteria by dotted lines against the redshift in figure 5. The dotted curve labeled with “JK06” represents the $k_{1\%}^{\text{lim}}$ determined from equation (14). And the $k_{\text{max}}$ of equation (16) is described by “SK07” and the nonlinear scale of equation (17) by “M08”. Among these three criteria, the first two have a similar redshift dependence. This is because they are based on a local value of the power spectrum or its Fourier transform convolved with a window function. The other criterion (Matsubara 2008) refers to the integrated value of the power spectrum over scale, and thus this shows a rather mild redshift dependence of the maximum wavenumber.

The values of $k_{1\%}^{\text{lim}}$ and $k_{3\%}^{\text{lim}}$ can be directly read off from figure 4, which are shown in table 3 and also in figure 5 ($k_{1\%}^{\text{lim}}$ by filled symbols and $k_{3\%}^{\text{lim}}$ by open symbols: squares, RPT/CLA; triangles, SPT; circles, LIN). We empirically find that a modified version of equation (17),

$$\frac{k^2}{6\pi^2} \int_0^k P^L(q, z) dq < C, \quad (18)$$

well reproduces the results in a unified fashion. The constant $C$ in the right-hand side depends on the choice of the theoretical model and the threshold value (1% or 3% in this paper). We find that $C_{1\%}^{\text{RPT/CLA}} = 0.35$, $C_{1\%}^{\text{SPT}} = 0.18$, and $C_{1\%}^{\text{LIN}} = 0.06$ well reproduce the 1% agreement limit of RPT/CLA, SPT and linear theory from figure 4, respectively. We plot equation (18) with these constant values as solid lines in figure 5, and also as vertical arrows in figure 4. Similarly, we find that the corresponding 3% limits are given by $C_{3\%}^{\text{RPT/CLA}} = 0.5$, $C_{3\%}^{\text{SPT}} = 0.3$, and $C_{3\%}^{\text{LIN}} = 0.13$ for the three theoretical predictions (dashed lines in figure 5).

The convergence regimes of our criteria [equation (18)] are narrower than those previously proposed. Note, however, that criteria (18) with the above constant values reasonably describe the valid range of analytic models, even at higher redshifts ($z = 7$ and 15). One of the possible sources of the difference between our criterion and the previous ones is that they use simulations with smaller box sizes to achieve better resolution.

![Fig. 5. Upper limits of reliable wavenumbers, $k_{1\%}^{\text{lim}}$ and $k_{3\%}^{\text{lim}}$, described in the text. Symbols show the values in figure 4 gauged with eye. Circles, linear theory; triangles, SPT; squares, RPT/CLA. Filled symbols correspond to $k_{1\%}^{\text{lim}}$, while open ones represent $k_{3\%}^{\text{lim}}$. The three solid lines plot equation (18): $k_{1\%}^{\text{lim}}$ for linear theory ($C = 0.06$), SPT ($C = 0.18$), and RPT/CLA ($C = 0.35$) from left to right, and the dashed lines correspond to $k_{3\%}^{\text{lim}}$ ($C = 0.13$, 0.3, and 0.5, respectively). We also show the nonlinear wavenumbers proposed by Jeong and Komatsu (2006), Sefusatti and Komatsu (2007), and Matsubara (2008) as dotted lines with JK06, SK07, and M08. The two shaded regions mean the redshift range planned by the WFMOS survey (with minimum wavenumbers, $2\pi / V^{1/3}$, where $V$ is the survey volume).](https://academic.oup.com/pasj/article-abstract/61/2/321/1591554)
which enhances a systematic error due to the finiteness of the box size. More details are discussed in the Appendix.

6. Implications for the Phases of BAOs

6.1. Extraction of the Phases of BAOs from the Nonlinear Power Spectra

So far we have focused on the amplitudes of the matter power spectra, but the phase information of BAOs is also important. This is imprinted mainly in the leading term, which is the product of the linear power spectrum and the propagator in RPT and CLA [see equation (2)]. Since the other contributions, the mode-coupling terms, are fairly smooth functions of the wavenumber, the phase information of BAOs is almost erased by convolution of the smoothed kernel and the propagator. Therefore, almost all of the BAO information comes from the zeroth-order term in RPT/CLA. We thus expect that the phases of BAOs predicted by N-body simulations and theories agree with each other for a wider range of wavenumbers, compared to the amplitudes, since the amplitudes are sensitive to higher-order corrections.

We tested the convergence of the predicted phases of BAOs as follows. We first constructed a smooth reference power spectrum using a spline technique developed by Percival et al. (2007) and Nishimichi et al. (2007). We adopted a basis spline (B-spline) fitting function as the reference spectrum, \( P^{\text{B-spline}}(k) \), with break points at \( k_{\text{break}} = 0.001 \, \text{h Mpc}^{-1} \) and \( k_{\text{break}} = (0.05 j - 0.025) \, \text{h Mpc}^{-1} \), where \( j \) is a positive integer. The data points to be fitted were set at \( k_i \) of equation (4) with a bin width of \( \Delta k = 0.005 \, \text{h Mpc}^{-1} \) for both the N-body simulations results and the theoretical predictions. We used the N-body power spectrum, \( \hat{P}_{\text{N-body}}(k, z) \), in which the effect of finite volume was corrected by using perturbation theory. We assigned equal weights to all data points. A smooth reference spectrum was computed for the average of the simulations and theoretical models. We then divided the model and N-body power spectra by their respective smooth reference power spectra, \( P^{\text{B-spline}}(k) \), which are shown in figure 6. The symbols and lines have the same meanings as in figure 4. Among the three nonlinear models, RPT and CLA lie almost on top of each other, and they are indistinguishable. These two models agree with the SPT model, which is shown in figure 6. The symbols, lines, and vertical arrows are analogous to those in figure 4.

6.2. Recovery of the BAO Scale from WFMOS Survey

By our criteria for trustable ranges of simulations and analytic models, we are able to discuss future cosmological constraints using BAOs. In a number of simulation papers, they attempted to present parameter forecasting of the dark energy equation of state parameter, \( w_{\text{DE}} \equiv p_{\text{DE}}/\rho_{\text{DE}} \), using BAOs as a standard ruler (i.e., Meiksin et al. 1999; Springel et al. 2005; Angulo et al. 2005, 2008; Eisenstein et al. 2007a, 2007b; Seo & Eisenstein 2003, 2005; Seo et al. 2008; Smith et al. 2007, 2008; White 2005; Huff et al. 2007; Jeong & Komatsu 2006, 2009). We follow a similar procedure, taking into account the reliable range of wavenumbers in the prediction of the BAO scale.

We considered the Wide-Field Multi-Object Spectrograph (WFMOS; Bassett et al. 2005) as a specific example in the present analysis. We constructed a template power ratio, \( P/P^{\text{B-spline}}(k) \), using RPT and linear theory, as described above. We created mock power spectra adding Gaussian random errors (Feldman et al. 1994),

\[
\Delta \left( \frac{P(k)}{P^{\text{B-spline}}(k)} \right) \approx \frac{\Delta P(k)}{P(k)} = \sqrt{\frac{2}{N_{\text{modes}}}} \left[ 1 + \frac{1}{n_g P(k)} \right].
\]

(19)

We neglected errors arising from the construction process of \( P^{\text{B-spline}}(k) \). In the above expression, \( N_{\text{modes}} \) is the number of modes in the wavenumber bin \( k \sim k + \Delta k \),

\[
N_{\text{modes}} = \frac{k^2 \Delta k V}{2\pi^2},
\]

(20)

where \( V \) denotes the survey volume, and \( n_g \) in equation (19) is the mean number density of galaxies. We set \( \Delta k \) to be very small \((0.001 \, \text{h Mpc}^{-1})\), so as not to affect the results and to use the nonlinear matter power spectrum of Smith et al. (2003) to calculate the shot noise contribution, \( [n_g P(k)]^{-1} \), just for the sake of simplicity.

We fit these mock data to the corresponding theoretical template, \( P/P^{\text{B-spline}}(ak) \), with a single free parameter, \( \alpha \).
due to these effects. The resulting error of $\sigma$ was 0.7%(1.0%) at $z = 1(3)$ when we adopted $k_{\text{max}}^{1/3}$ of RPT/CLA as $k_{\text{max}}$. These values were slightly improved and became 0.6%(1.0%) if we increased $k_{\text{max}}$ to 0.4$h^{-1}$Mpc$^{-1}$. We can put a strong constraint on $\sigma$, even if we do not include the modes at $k > k_{\text{lim}}^{1/3}$ of RPT in the analysis. In reality, however, we need to take account of galaxy clustering in redshift space in addition to what we did in the current work, which will be presented in a future paper.

7. Summary

We carried out a systematic comparison of power spectra calculated from $N$-body simulations and various theoretical models. We corrected the effect of a finite volume of $N$-body simulations using a perturbation-theory-based methodology developed in Takahashi et al. (2008).

We found that our simulations agree with all of the theoretical models used in the present paper at large length scales ($k \lesssim 0.1h^{-1}$Mpc$^{-1}$). At smaller length scales, where nonlinearity is mild, our simulations, RPT (Crocce & Scoccimarro 2006a, 2006b, 2008), and CLA (Taruya & Hiramatsu 2008) agreed with accuracy of 1% and 3% up to $k_{\text{lim}}^{1/3}$ and $k_{\text{lim}}^{1/3}$, as presented in figure 5. These convergence regimes depend on the redshift, and can be explained by a simple empirical formula of equation (18). We also showed that the phase information of BAOs extracted from power spectra using B-spline fitting is more robust than the amplitudes: predicted phases of nonlinear theories and $N$-body simulations agree well in a wider range of wavenumbers than the amplitudes of the power spectrum.

The currently achieved accuracies of both theoretical predictions and simulations are sufficient to interpret the data from future surveys, like WFMS; one can put a tight constraint on the BAO scale ($\sigma_g \lesssim 1\%$) using modes within our convergence regimes. There is a potential to extract more than simply the BAO scale from the power spectrum, such as the sum of the neutrino masses, if one is able to exploit smaller scale information (e.g., Saito et al. 2008). To this end, it is important to consider higher order corrections (i.e., 2-loop terms) on the theory end and higher resolution simulations. It is also important to investigate the accuracy of the velocity field in $N$-body simulations to model accurately in redshift space. These are future investigations, and a study in redshift space is now in progress.

We also checked the convergence of the matter power spectrum in $N$-body simulations while changing the initial conditions, $N$-body solvers, and box sizes. Among these we showed that $N$-body simulations with small box sizes ($\lesssim 500h^{-1}$Mpc) suffer from systematic enhancements of the matter power spectra at BAO scales (e.g., Springel et al. 2005; Seo & Eisenstein 2005; Jeong & Komatsu 2006).

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Appendix. Convergence Check of Different \(N\)-body Codes and Dependence on Simulation Parameters

A.1.1. Initial Condition Generator

We tested two methods for generating cosmological initial conditions. One was the Zel’dovich approximation (Zeldovich 1970; hereafter ZA), commonly used for cosmological simulations; the other was second-order Lagrangian perturbation theory (2LPT: e.g., Crocce et al. 2006). Starting from the same linear density field, \(\delta_L\), we generated two initial conditions at \(z_{ini} = 31\) from \(N\)-body runs using ZA and 2LPT. We then compared the power spectra evolved with Gadget2.

Figure 8 shows the ratio of the two matter power spectra from simulations at various output redshifts. The overall trend that \(P^{ZA}(k)\) has smaller amplitudes at all scales than \(P^{2LPT}(k)\) is consistent with the result of Crocce, Pueblas, and Scoccimarro (2006). This is because the higher order terms in ZA do not necessarily correspond to the growing solutions. The plot indicates that differences in the initial conditions affect the power spectrum at the few-percent level. We confirmed that this difference between 2LPT and ZA reduces to subpercent when we increase the starting redshift to \(z_{ini} = 127\). However, we need a very large number of mesh points in the calculation of the Particle-Mesh (PM) force to avoid large errors introduced by the tree force that arise when the density field is nearly uniform. This is very time consuming. Thus, we needed the 2LPT initial condition for the purpose of present analysis if we started the simulations at \(z_{ini} = 31\). We also confirmed that our 2LPT result was not affected when we changed the starting redshift to \(z_{ini} = 63\) or \(z_{ini} = 127\).

A.1.2. Comparison Using Different \(N\)-body Solvers

Next, we compare three different \(N\)-body solvers; a Tree-Particle-Mesh (Tree-PM) solver Gadget2 (Springel 2005), another Tree-PM code TPM of Bode and Ostriker (2003), and a Particle-Particle-Particle-Mesh (P³M) code developed in Jing and Suto (1998, 2002; hereafter JS02). We used identical initial conditions as that of one of our fiducial runs: \(512^3\) particles in a box of \(1000\, h^{-1}\,\text{Mpc}\). We stopped the simulations at \(z = 3\). In addition to the Tree-PM and P³M calculations, we also ran simulations using only the PM part of the three codes. We measured the power spectra for the simulation output at \(z = 3\), and corrected the effect of a finite volume (see section 4), so that the comparison became clearer.

Figure 9 shows the difference relative to the RPT prediction, \(\left[\hat{P}^{N\text{-body corrected}}(k) - P^{RPT}(k)\right]/P_{\text{nw}}(k)\) at \(z = 3\). Different line types correspond to TPM (dotted), JS02 (dashed), and Gadget2 (solid). The vertical arrows show the limit wavenumber \(k_{lim}\) for RPT/CLA. The error bars are given by equation (13) for Gadget2. Upper: Results of Tree-PM and PP-PM calculations. Lower: Results of PM only calculations.
the results of the three codes when we used only PM forces. The degree of dispersion among the three is larger than that in the upper panel, and is ~3% at $k^\text{lim}_{\text{sim}}$ of RPT/CLA. The short-range force is helpful in increasing the accuracy at this level at this scale. We conclude that our main result was not affected so much by the choice of a particular $N$-body solver when we turned on the short-range force.

### A.1.3. Effects of Finite Box Size and Resolution

The finite box size introduces an additional variance in the power spectrum, as shown in Takahashi et al. (2008) and this work. It is possible that this finiteness affects not only the variance, but also the mean value. The resolution of simulations may also affect the result. We thus investigated the box size/resolution dependence of the power spectra from $N$-body simulations.

In order to increase the spatial resolution, we ran simulations varying the box size [1000 (4 realizations; fiducial), 500 (7 realizations), and 250 $h^{-1}$Mpc (9 realizations)], but keeping the number of particles to be the same as our fiducial runs (512$^3$) to check the convergence. For this test, we used outputs at $z = 3$. The results are shown in figure 10. The symbols correspond to $L_{\text{box}} = 1000 h^{-1}$Mpc (circle), 500 $h^{-1}$Mpc (triangle), and 250 $h^{-1}$Mpc (square). We adopted the bin width $\Delta k = 0.01 h$Mpc$^{-1}$ in this section for clarity. We also calculated the RPT predictions with the effect of a finite box size introducing a cutoff scale ($k_{\text{box}} = 2\pi/k^\text{lim}_{\text{box}}$) in the integration, which are shown in the left panel. The solid line is the normal RPT result (without cutoff), and the other lines take account of the effect of a finite box size (short-dashed, 1000 $h^{-1}$Mpc; long-dashed, 500 $h^{-1}$Mpc; dot-dashed, 250 $h^{-1}$Mpc). Note that we show only the results without any cutoff in the text. In the right panel, we plot a comparison of the SPT prediction (no cutoff) to show the dependence on the box size of the valid ranges of SPT.

In the left panel, one may notice the difference of theoretical lines at large wavenumbers. This reflects the finiteness of the box; short-dashed line ($L_{\text{box}} = 1000 h^{-1}$Mpc case) deviates from solid line (infinite volume case) only at $k > 0.3 h$Mpc$^{-1}$ and is very small (~0.5% at $k = 0.4 h$Mpc$^{-1}$), whereas long-dashed and dash-dotted ones ($L_{\text{box}} = 500$ and 250 $h^{-1}$Mpc cases) diverge at smaller wavenumbers and to a greater degree (~1.5% and ~3% at $k = 0.4 h$Mpc$^{-1}$, respectively). Although the results of $N$-body simulations with smaller box sizes have larger variances due to smaller total volumes than the fiducial runs, the box size dependence appears at small scales. The runs with smaller box size tend to have larger amplitudes, which is consistent with RPT predictions with cutoff scales. We thus conclude that the results of simulations also suffer from a systematic effect due to finite box size, and adopted $L_{\text{box}} = 1000 h^{-1}$Mpc for our fiducial model, where the effect is <0.5%.

Next, the convergence regime of SPT seems to be wider if one believes the results of the smaller box size runs in the right panel. We consider that these wider valid ranges just coincide due to systematic effects in the simulations with smaller box sizes, and should not be trusted. One should keep this in mind when one uses the results of simulations with a small box size. Even the Millennium simulation (Springel et al. 2005) has only 500 $h^{-1}$Mpc on each side, although it has many more particles (2160$^3$) than ours, and thus the spatial resolution is better. This box size introduces $\lesssim 1\%$ systematics at $k \gtrsim 0.35 h$Mpc$^{-1}$ according to our figure 10. Part of the disagreement in the valid range of SPT between our results and that of Jeong and Komatsu (2006) may be due to this effect, because they use $L_{\text{box}} = 512, 256, 128, and 64 h^{-1}$Mpc.

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**Fig. 10.** Compared results of $N$-body simulations for different box sizes at $z = 3$. Circle, 1000 $h^{-1}$Mpc; triangle, 500 $h^{-1}$Mpc; square, 250 $h^{-1}$Mpc. Left: Comparison with the RPT prediction (dotted line). We also plot predictions of RPT taking account of the finiteness of the volume by introducing a cutoff scale: 1000 $h^{-1}$Mpc (dashed), 500 $h^{-1}$Mpc (long-dashed), and 250 $h^{-1}$Mpc (dot-dashed). Right: Comparison with the SPT prediction, in which we do not take account of the finiteness effect.
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