ELLIPITC EQUATIONS WITH VMO A, B \( \in \mathbb{L}^d \), AND 
C \( \in \mathbb{L}^{d/2} \)

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Abstract. We consider elliptic equations with operators 
\[ L = a^{ij}D_{ij} + b^iD_i - c \] 
with \( a \) being almost in VMO, \( b \in \mathbb{L}_d \) and \( c \in \mathbb{L}_q, \ c \geq 0, \ d > q \geq d/2 \). We prove the solvability of 
\[ Lu = f \in \mathbb{L}_p \] 
in bounded \( C^{1,1} \)-domains, \( 1 < p \leq q \), and of \( \lambda u - Lu = f \) in the whole space for 
any \( \lambda > 0 \). Weak uniqueness of the martingale problem associated with 
such operators is also obtained.

1. Introduction

Let \( \mathbb{R}^d \) be a \( d \)-dimensional Euclidean space of points \( x = (x^1, ..., x^d) \) with 
\( d \geq 2 \). We are dealing with a uniformly elliptic operator

\[ Lu(x) = a^{ij}(x)D_{ij}u(x)+b^i(x)D_iu(x)-c(x)u(x), \quad D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_iD_j, \]

acting on functions given on \( \mathbb{R}^d \). Throughout the article the numbers \( p, q \in (1, \infty) \) are fixed and assumed to satisfy either

\[ \frac{d}{2} < q < d, \quad 1 < p \leq q, \quad (1.1) \]

or

\[ q = \frac{d}{2}, \quad 1 < p < \frac{d}{2} \quad (\text{and} \quad d \geq 3). \quad (1.2) \]

We assume that \( b \in \mathbb{L}_d(\mathbb{R}^d) \) and \( c \in \mathbb{L}_q(\mathbb{R}^d), \ c \geq 0 \). Note that the case that \( q = d/2 \) is not excluded. We also assume that \( a \) is bounded and almost in VMO and prove the unique solvability results for the equation

\[ Lu = f \in \mathbb{L}_p(G) \] 
in regular domains \( G \) in the class \( W^{2,0}_p(G) \) and for the equation 
\[ \lambda u - Lu = f \] 
in \( \mathbb{R}^d \) for any \( \lambda > 0 \) in the class \( W^{2,0}_p(\mathbb{R}^d) \). We apply these results to prove that the corresponding solutions of Itô’s stochastic equation possess the weak uniqueness property.

To the best of the author’s knowledge these results are new even if \( a^{ij} = \delta^{ij} \), however, much work was done in this case.

G. Stampacchia in [13] (1965) was probably the first author who presented the \( W^{2,0}_2 \)-solvability theory of divergence form equations with \( b \in \mathbb{L}_d(\mathbb{R}^d) \) and \( c \in \mathbb{L}_{d/2}(\mathbb{R}^{d/2}) \), with some additional restrictions on \( c \) but without assuming the smallness of the norms of \( b \) and \( c \) as well as without assuming that the

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domain $G$ in which the equation is solved is small. The restriction on $c$ can be summarized as follows (see Theorem 3.4 in [13]): $c = c' + \lambda$, where the parameter $\lambda$ should not belong to a countable set, which is known to be lying below some $\bar{\lambda} > 0$. There is a plethora of other important results in [13], but we will discuss only the one mentioned above which is most related to our own results in case $a^{ij} = \delta^{ij}$. The free term [13] is taken in the divergence form $f = D_{i}g_{i}$, where $g_{i} \in L_{2}$. To match this with our $f \in L^{p}$ under condition (1.2) we have to have $p > 2d/(d+2)$ (and $d \geq 3$). Then in this level-ground situation we have a solution $u \in W^{2}_{p}$ and [13] guarantees only $u \in W^{1}_{2}$. At the same time $W^{2}_{p} \subset W^{1}_{r} \subset W^{1}_{2}$ for an $r > 2$. By the way, in the statement of Theorem 3.4 of [13] the condition that $d \geq 3$ is not included, but it is actually tacitly imposed (see pages 200-201 there where embedding theorems are applied).

The estimates leading to Theorem 3.4 of [13] are also found in O.A. Ladyzhenskaya and N.N. Ural’tseva book [10] (1973), see pages 189-191 there, where the condition $d \geq 3$ is explicitly imposed.

N.S. Trudinger in [14] (1973) in the setting of generally degenerate divergence form operators, among many other things, removed the condition on $c$ in [13] and replaced it with just $c \geq 0$.

In a recent article [3] Byungsoo Kang and Hyunseok Kim present a deep investigation of divergence type equations with $b \in L_{d}$ (no $c$) and solutions in $W^{1}_{p}$. In case $a^{ij} = \delta^{ij}$ we obtain solutions in $W^{2}_{p}$.

It is also worth mentioning the recent article [1] by Apushkinskaya, Nazarov, Palagachev, and Softova, in which nondivergence form equations with VMO $a$, $b \in L_{d}$, and $c \in L_{d/2}$ are considered and a priori estimates are obtained but on the right in these estimates the zeroth order norm of the unknown function is present.

Saisai Yang and Tusheng Zhang in [15] (2018) use probabilistic approach to prove, among other things, that there exists a unique $C^{1+\alpha}$-solution to the Dirichlet boundary value problem in case $a^{ij} = \delta^{ij}$ and $b$ and $c$ are signed measures of Kato class $K_{d,1-\alpha}$, $\alpha \in (0,1)$. Although the pointwise regularity of solutions in [15] is stronger than ours $u \in C^{2-d/q}$ (see Corollary 2.14), it is worth mentioning that a general $f \in L_{r}$ is in $K_{d,1-\alpha}$ only if $(1-\alpha)r > d$. Therefore, generally $b \in L_{d}$ is not in any $K_{d,1-\alpha}$ and $c \in L_{d/2}$ is way out of $K_{d,1-\alpha}$. Therefore, the results of [15] are not applicable in our case.

The above discussion seems to support that even in the case of $a^{ij} = \delta^{ij}$ our PDE results were unknown. We prove them when $a \in BMO$. Regarding numerous issues for equations with BMO main coefficients and bounded lower order coefficients we refer the reader to [4] and the references therein.

In what concerns the weak uniqueness of solutions of the corresponding stochastic equation with drift in Lebesgue spaces, much work has been done mostly in the time nonhomogeneous case $b = b(t,x)$, mostly when $a^{ij} = \delta^{ij}$. A good source of recent results and bibliography is the paper by L. Beck, F. Flandoli, M. Gubinelli, and M. Maurelli [2] (2019). In this paper the authors
prove (see there Theorem 5.4) an existence and uniqueness (stronger than the pathwise uniqueness) theorem applicable to our situation ($a^{ij} = \delta^{ij}$, $b \in L^d_1$) but only for solutions with initial starting point having density and only in a class of solutions possessing certain properties. We prove weak uniqueness but for any solution starting from any fixed point and our $a \in BMO$. From the probabilistic point of view this is also a new result complementing the information in [5].

2. Equations in bounded domains

Fix numbers $\delta \in (0, 1)$ and $\|b\|, \|c\| \in [0, \infty)$.

Assumption 2.1. The coefficients of $L$ are measurable, the matrices $a(x) = (a^{ij}(x))$ are symmetric and satisfy

$$\delta^{-1} |\lambda|^2 \geq a^{ij}(x)\lambda^i\lambda^j \geq \delta|\lambda|^2$$

(2.1)

for all $\lambda, x \in \mathbb{R}^d$. Also $c \geq 0$.

$$\|b\|_{L_1(\mathbb{R}^d)} \leq \|b\|, \quad \|c\|_{L_1(\mathbb{R}^d)} \leq \|c\|.$$

To state one more assumption we set $B_r(x)$ to be the open ball in $\mathbb{R}^d$ of radius $r$ centered at $x$, $B_r = B_r(0)$. Denote

$$\text{osc } (a, B_\rho(x)) = |B_\rho|^{-2} \int_{y,z \in B_\rho(x)} |a(s,y) - a(s,z)| dy dz,$$

$$a^\#_r = \sup_{x \in \mathbb{R}^d} \sup_{\rho < r} \text{osc } (a, B_\rho(x)).$$

Set

$$L_0 u = a^{ij} u_{x^i} x^j.$$

Fix a bounded domain $G \subset \mathbb{R}^d$ of class $C^{1,1}$. Here is a particular case of Theorem 8 of [4].

**Lemma 2.2.** Under Assumption 2.1 for any $s \in (1, \infty)$ there exists $\theta_0 = \theta_0(d, \delta, s)$ such that, if there is $r_0 > 0$ for which $a^\#_{r_0} \leq \theta_0$, then there exist $\lambda_0 \geq 1, N_0$, depending only on $d, \delta, s, r_0$, and $G$, such that, for any $u \in W^2_s(G)$ and $\lambda \geq \lambda_0$,

$$\|D^2 u\|_{L_s(G)} + \sqrt{\lambda} \|Du\|_{L_s(G)} + \lambda \|u\|_{L_s(G)} \leq N_0 \|L_0 u - \lambda u\|_{L_s(G)}.$$  (2.2)

Furthermore, for any $f \in L_s(G)$ there exists a unique $u \in W^2_s(G)$ such that $L_0 u - \lambda u = f$.

We fix $r_0 > 0$ and impose the following.

**Assumption 2.3** ($p, r_0$). We have $a^\#_{r_0} \leq \theta_0(d, \delta, p)$, where $\theta_0$ is taken from Lemma 2.2.

Below by $\lambda_0$ we mean the one from Lemma 2.2 for $s = p$. 
Theorem 2.4. Under Assumptions 2.1 and 2.3 \((p, r_0)\) introduce \(N^* = N^*(p, q, d, G)\) as the best constant such that
\[
\|Du\|_{L_p((d-p)(G))} + \|u\|_{L_q((q-p)(G))} \leq N^*(\|D^2u\|_{L_p(G)} + \|u\|_{L_p(G)})
\]
for any \(u \in W^2_p(G)\). Assume that
\[
2N_0N^*(\|b\|_{L_d(G)} + \|c\|_{L_q(G)}) \leq 1. \tag{2.3}
\]
Then for any \(u \in W^2_p(G)\) and \(\lambda \geq \lambda_0\),
\[
\|D^2u\|_{L_p(G)} + \sqrt{\lambda}\|Du\|_{L_p(G)} + \lambda\|u\|_{L_p(G)} \leq 2N_0\|Lu - \lambda u\|_{L_p(G)}. \tag{2.4}
\]
Furthermore, for any \(f \in L_p(G)\) there exists a unique \(u \in W^2_p(G)\) such that \(Lu - \lambda u = f\).

Proof. To prove (2.4) observe that
\[
N_0\|Lu - \lambda u\|_{L_p(G)} \leq N_0\|Lu - \lambda u\|_{L_p(G)} + N_0(\|b\| \|Du\|_{L_p(G)} + |c|u\|_{L_p(G)}),
\]
where the last term by Hölder’s inequality, embedding theorems, and (2.3) is less than
\[
N_0\|b\|_{L_d(G)}\|Du\|_{L_p((d-p)(G))} + N_0\|c\|_{L_q(G)}\|u\|_{L_p((q-p)(G))}
\leq N_0(\|b\|_{L_d(G)} + |c|\|L_q(G)\|)N^*(\|D^2u\|_{L_p(G)} + \|u\|_{L_p(G)})
\leq (1/2)(\|D^2u\|_{L_p(G)} + \|u\|_{L_p(G)}).
\]
This shows that (2.4) follows from (2.2). The existence assertion of the theorem follows as usual by the method of continuity. The theorem is proved.

Remark 2.6. The above estimates show that the operator \(L\) is bounded as an operator from \(W^2_p(G)\) to \(L_p(G)\) as long as \(b \in L_d(G)\) and \(c \in L_q(G)\).

Next for our fixed \(b\) and \(c\) there exist \(b_0, c_0 \geq 0\) such that
\[
2N_0N^*(\|bI_{|b|\geq b_0}\|_{L_d(G)} + |cI_{|c|\geq c_0}\|_{L_q(G)}) \leq 1. \tag{2.5}
\]

Theorem 2.6. Under Assumptions 2.1 and 2.3 \((p, r_0)\) there exist \(\lambda_1 \geq 1, N\), depending only on \(d, \delta, p, r_0, b_0, c_0\), and \(G\), such that, for any \(u \in W^2_p(G)\) and \(\lambda \geq \lambda_1\),
\[
\|D^2u\|_{L_p(G)} + \sqrt{\lambda}\|Du\|_{L_p(G)} + \lambda\|u\|_{L_p(G)} \leq N\|Lu - \lambda u\|_{L_p(G)}. \tag{2.6}
\]
Furthermore, for any \(f \in L_p(G)\) there exists a unique \(u \in W^2_p(G)\) such that \(Lu - \lambda u = f\).

Proof. As usual, it suffices to prove the a priori estimate (2.6). By Theorem 2.4 its left hand side is dominated by
\[
2N_0\|Lu - \lambda u\|_{L_p(G)} + 2N_0\|b^I_{|b|\leq b_0} + cI_{|c|\leq c_0}\|_{L_p(G)},
\]
where the last term, by interpolation inequalities is less than
\[
N(\|Du\|_{L_p(G)} + \|u\|_{L_p(G)}) \leq (1/2)(\|D^2u\|_{L_p(G)} + N_1\|u\|_{L_p(G)}).
\]
This yields (2.6) for $\lambda \geq 2N_1$ and proves the theorem.

We denote the solution from Theorem 2.6 by $R_{\lambda+c}f$.

**Remark 2.7.** By taking here $\lambda = \lambda_1$ in (2.6) we see that for the same kind of $N$ as in (2.6) and any $u \in W^2_1(G)$

$$\|u\|_{W^2_1(G)} \leq N\left(\|Lu\|_{L^p(G)} + \|u\|_{L^p(G)}\right).$$

The next result, the proof of which is left to the reader, is a standard consequence of Theorem 2.6

**Theorem 2.8.** Let $a^n, b^n, c^n$, $n = 1, 2, \ldots$, be a sequence of symmetric $d \times d$-matrix valued, $\mathbb{R}^d$-valued, and $[0, \infty)$-valued, respectively, measurable functions, satisfying Assumptions 2.1 and 2.3 $(p, r_0)$ (with the same $\delta$, $\|b\|$, $\|c\|$, and $\theta_0$ as above). Let $f, f^n \in L^p(G)$ and suppose that $a^n \to a$ on $\mathbb{R}^d$ (a.e.) and

$$\|b - b^n\|_{L^d(G)} + \|c^n - c\|_{L^q(G)} + \|f^n - f\|_{L^p(G)} \to 0$$

as $n \to \infty$. Let $\lambda \geq \lambda_1$, where $\lambda_1$ is taken from Theorem 2.6, and introduce $u^n$ as unique $W^2_1(G)$-solutions of $\lambda u^n - L^n u^n = f$, where the operator $L^n$ is constructed from $a^n, b^n, c^n$. Then

$$\lim_{n \to \infty} \|u^n - R_{\lambda+c}f\|_{W^2_1(G)} = 0.$$

By using mollifiers and properties of solutions of equations with smooth coefficients we easily arrive at the following.

**Corollary 2.9.** Under Assumptions 2.1 and 2.3 $(p, r_0)$ for $\lambda \geq \lambda_1$, where $\lambda_1$ is taken from Theorem 2.6, and any $f \in L^p(G)$ we have $|R_{\lambda+c}f| \leq R_{\lambda+c}|f| \leq R_{\lambda+c}|f|$ (a.e.).

Next we turn to some properties of equations with $b \in L_d$ and $c \in L_q$. The main goal of these further results, important in their own rights, is to prepare the necessary tools to be able to treat the equations in the whole space for any $\lambda > 0$ and in domains when $\lambda = 0$.

**Lemma 2.10.** Under Assumption 2.1 let $0 < R \leq R_0 < \infty$, $\varepsilon \in (0, 1]$,

$$d \geq t \geq p, \quad \frac{d}{p} < 1 + \frac{d}{t},$$

$u \in W^2_1(G), \quad \zeta \in C_0^\infty(\mathbb{R}^d)$ and let $\zeta$ have support in a ball $B$ of radius $R$ with center in $G$ and satisfy $1 \leq \zeta \leq 1$, $|D\zeta| \leq K_0 R^{-1}$, $|D^2\zeta| \leq K_0^2 R^{-2}$, where $K_0 \geq 1$ is a constant. Introduce

$$L'u := Lu + cu, \quad Mu := u L'\zeta + 2a^{ij} D_i \zeta D_j u \quad (= L(\zeta u) - \zeta L u).$$

Then there exists a constant $N$, depending only on $R_0$, $d, \delta, p, \|b\|$, and $G$ (but not on $b_0, c_0, \theta_0$, or $\theta_0$), such that

$$\|Mu\|_{L^t(G)} \leq \varepsilon R^{-2\tau_2 \gamma} D^2 u\|_{L^p(G \cap B)}$$

$$+ N \left(\varepsilon^{-\alpha} K_0^2 + \varepsilon^{-\beta} K_0^2\right) R^{-2 - 2\tau_2} \|u\|_{L^p(G \cap B)},$$

(2.9)
where
\[ \alpha = \tau_1/(1 - \tau_1), \quad \beta = \tau_2/(1 - \tau_2), \quad \gamma = (1 - \tau_2)^{-1} \]
and \( \tau_1, \tau_2 \) are specified in the proof.

Proof. Make the change of coordinates \( y = x/R \) and, accordingly, set \( u(x) = v(y) \). Under this change \( B \) will be transformed into a ball of radius one, the domain \( G \) will also change, but, what is important (due to \( R \leq R_0 \)), the embedding theorems we need in the transformed domain \( B \cap G \) will hold with constants comparable to the ones in the original \( B \cap G \). Also observe that, if \( Mu(x) = f(x) \), then
\[
v(y)\tilde{L}\tilde{\zeta}(y) + 2\tilde{a}^{ij}(y)D_i\tilde{\zeta}D_jv(y) = R^2f(Ry),
\]
where \( \tilde{L} = \tilde{a}^{ij}(Ry)D_i + Rb_i(Ry)D_i \), \( \tilde{a}^{ij}(y) = a^{ij}(Ry) \), \( \tilde{\zeta}(y) = \zeta(Ry) \). It is easy to check that the \( L_d \)-norm of the new \( b \) remains the same. It follows that we may concentrate on \( R = 1 \).

In that case use Hölder’s inequality and embedding theorems (see, in particular, Corollary 1.4.7/2 in [11]). Observe that,
\[
I := \|u|D\zeta|\|_{L_t(G)} \leq K_0\|b\|_{L_d(G)}\|u\|_{L_{td/(d-t)}(G \cap B)},
\]
and since
\[
\frac{d}{p} - 2 < \frac{d(d - t)}{td},
\]
we have
\[
I \leq \varepsilon\|D^2u\|_{L_p(G \cap B)} + N\varepsilon^{-\tau_1/(1 - \tau_1)}K_0^2\|u\|_{L_p(G \cap B)},
\]
where
\[
\tau_1 = \frac{1}{2}\left(1 + \frac{d}{p} - \frac{d}{t}\right).
\]
Also
\[
\frac{d}{p} - 2 < \frac{d}{t} - 1,
\]
so that
\[
\|D^2u|D\zeta|\|_{L_t(G)} \leq K_0\|D^2u\|_{L_t(G \cap B)}
\]
\[
\leq \varepsilon\|D^2u\|_{L_p(G \cap B)} + N\varepsilon^{-\tau_1/(1 - \tau_1)}K_0^2\|u\|_{L_p(G \cap B)}.
\]

Finally,
\[
\|u|D^2\zeta|\|_{L_t(G)} \leq K_0\|u\|_{L_t(G \cap B)}
\]
\[
\leq \varepsilon\|D^2u\|_{L_p(G \cap B)} + N\varepsilon^{-\tau_2/(1 - \tau_2)}K_0^{2/(1 - \tau_2)}\|u\|_{L_p(G \cap B)},
\]
where
\[
\tau_2 = (1/2)\left(\frac{d}{p} - \frac{d}{t}\right).
\]
Upon combining these estimates and observing that \( K_0 \geq 1 \) and \( \varepsilon \leq 1 \), we come to (2.9) with \( R = 1 \). The lemma is proved.

The following theorem allows us, in particular, to obtain interior estimates.
Theorem 2.11. Under Assumptions 2.1 and 2.3 \((p, r_0)\) let \(0 < R \leq \text{diam}(G)\), \(z \in G\). Denote

\[ G_r = G \cap B_r(z). \]

Suppose that

\[ \zeta u \in W^2_p(G_3R) \quad \forall \zeta \in C_0^\infty(B_{3R}(z)), \quad Lu \in L_p(G_3R). \]  \tag{2.10}

Then there exists a constant \(N\), depending only on \(d, \delta, p, r_0, b_0, c_0, G, \) and \(\| b \|\), such that

\[ \| u \|_{W^2_p(G_R)} \leq N \| Lu \|_{L_p(G_2R)} + NR^{-2} \| u \|_{L_p(G_2R)} \]. \tag{2.11}

Proof. We may and will assume that \(z = 0\). In that case set

\[ R_m = R \sum_{j=0}^m 2^{-j}, \quad D_m = G_{R_m}, \quad m = 0, 1, 2, \ldots \]

We need some functions \(\zeta_m \in C_0^\infty(\mathbb{R}^d)\) such that \(\zeta_m(x) = 1\) in \(B_{R_m}\), \(\zeta_m(x) = 0\) outside \(B_{R_{m+1}}\) and

\[ |D\zeta_m| \leq NR^{-1}2^m, \quad |D^2\zeta_m| \leq NR^{-2}2^{2m}, \]

where \(\rho = 1/2\) and \(N = N(d)\). To construct them, take an infinitely differentiable function \(h(t), t \in (-\infty, \infty)\), such that \(h(t) = 1\) for \(t \leq 0\), \(h(t) = 0\) for \(t \geq 1\) and \(0 \leq h \leq 1\). After this define

\[ \zeta_m(x) = h(2^{m+1}R^{-1}(|x| - R_m)). \]

Now we put \(u\zeta_m\) in (2.7) to get

\[ \| u \|_{W^2_p(D_m)} \leq \| u\zeta_m \|_{W^2_p(G)} \leq N(\| L(u\zeta_m) \|_{L_p(G)} + \| u\zeta_m \|_{L_p(G)}) \]

\[ \leq N\| Lu \|_{L_p(G_2R)} + \| M_m u \|_{L_p(G)} + N\| u \|_{L_p(G_2R)}), \]

where

\[ M_m u := uL^\prime \zeta_m + 2a^{rs} D_r \zeta_m D_s u. \]

By Lemma 2.10 with \(t = p\) when \(\tau_2 = 0\) (and \(K_0 \sim 2^m\))

\[ \| M_m u \|_{L_p(G)} \leq (1/8)\| u\zeta_{m+1} \|_{W^2_p(G)} + NR^{-2}2^{2m} \| u \|_{L_p(G_2R)}. \]

Then

\[ \| u\zeta_m \|_{W^2_p(G)} \leq N\| Lu \|_{L_p(G_2R)} + (1/8)\| u\zeta_{m+1} \|_{W^2_p(G)} \]

\[ + NR^{-2}2^{2m} \| u \|_{L_p(G_2R)}, \]

\[ (1/8)^m\| u\zeta_m \|_{W^2_p(G)} \leq N(1/8)^m\| Lu \|_{L_p(G_2R)} \]

\[ + (1/8)^{m+1}\| u\zeta_{m+1} \|_{W^2_p(G)} + (N + c_1)R^{-2}2^{-m} \| u \|_{L_p(G_2R)}). \]

By summing up over \(m = 0, 1, \ldots\) and cancelling like terms we obtain

\[ \| u\zeta_m \|_{W^2_p(G)} \leq \| Lu \|_{L_p(G_2R)} + NR^{-2} \| u \|_{L_p(G_2R)}. \]

This proves (2.11) and the theorem.

In the following theorem we show that in our estimates on the right one can have the \(L_p\) norm of \(u\) with lower \(p\) (see (2.14)).
**Theorem 2.12.** (i) Let $q > d/2$, $q > p$, $0 < R \leq \text{diam} G$, $z \in \bar{G}$. Denote

$$G_r = G \cap B_r(z).$$

(ii) Introduce

$$\gamma = 1 + \frac{2q - d}{d} \cdot \frac{p}{q}. \quad (2.12)$$

Observe that $\gamma > 1$ and introduce $p(n) = p_0^n$, $n = 0, \ldots, m - 1$, where $m - 1$ is the largest $n$ such that $p(n) \leq q$. Then set $p(m) = q$ and suppose that Assumption 2.1 is satisfied and Assumptions 2.3, $p(n), r_0$) are satisfied with the above $p(n)$’s, $n = 0, \ldots, m$.

Then

$$\zeta u \in \dot{W}^2_p(G_{2R}) \quad \forall \zeta \in C_0^\infty(B_{2R}(z)), \quad Lu \in L_q(G_{2R})$$

$$\implies \zeta u \in \dot{W}^2_p(G_{2R}) \quad \forall \zeta \in C_0^\infty(B_{2R}(z)). \quad (2.13)$$

Furthermore, there exists a constant $N$, depending only on $R, d, \delta, p, q, r_0, b_0, c_0, c_1, \|b\|$, and $G$, such that, if the condition of the implication (2.13) holds, then

$$\|u\|_{\dot{W}^2_p(G_{2R})} \leq N(\|Lu\|_{L_q(G_{2R})} + \|u\|_{L_p(G_{2R})}). \quad (2.14)$$

Proof. Take $\lambda$ so large (see Theorem 2.6) that $\lambda - L$ is invertible as an operator acting from $\dot{W}^2_p(G)$ onto $L_p(G)$ for $n = 0, \ldots, m$. Also take a $u$ such that the condition of the implication (2.13) holds, take a $\zeta \in C_0^\infty(B_{2R}(z))$, notice that $\zeta u \in \dot{W}^2_p(G)$ and denote

$$f = Lu, \quad g = (L - \lambda)(\zeta u) = \zeta f + 2a^{ij} u_{x_i} \zeta_{x_j} + u(L' - \lambda) \zeta.$$

Observe that for $1 \leq n < m$

$$\frac{d}{p(n - 1)} - \frac{d}{p(n)} = \frac{d}{p \gamma^n} (\gamma - 1) \leq \frac{d}{p} (\gamma - 1) \leq \frac{2q - d}{q} = 2 - \frac{d}{q} < 1 \quad (2.15)$$

and $p(n - 1) \leq p(n) \leq q < d$. Therefore condition (2.8) is satisfied with $t = p(n)$ and $p(n - 1)$ in place of $p$. If $n = m$ and $p(m - 1) = q$, then the left-hand side of (2.15) vanishes for $n = m$, and if $p(m - 1) < q$, then $q \leq p(m - 1)$ and

$$\frac{d}{p(m - 1)} - \frac{d}{p(m)} = \frac{d}{p(m - 1)} - \frac{d}{q} \leq \frac{d}{p(m - 1)} - \frac{d}{p(m - 1)} \gamma < 1.$$

Hence, Lemma 2.10 is applicable with $t = p(n)$ and $p(n)$ in place of $p$ for any $n = 1, \ldots, m$.

It follows that $g \in L_p(G)$ if $\zeta u \in \dot{W}^2_p(G)$, $n = 1, \ldots, m$. By the choice of $\lambda$ the equation

$$(L - \lambda)w = g$$
has a solution in $\tilde{W}^2_{p(1)}(G) \subset \tilde{L}^2_p(G)$ which in addition is unique in $\tilde{W}^2_p(G)$. Hence for $n = 1$

$$w = \zeta u \in \tilde{W}^2_{p(n)}(G), \quad \forall \zeta \in C^\infty_0(B_{2R}).$$

(2.16)

If $p(1) < q$, then by repeating this argument with $p(1)$ in place of $p$, we get (2.16) for $n = 2$. In this way we get this inclusion for all $n$ and this proves (2.13).

To prove (2.14), we accompany the above argument with estimates. By the choice of $\lambda$ and Lemma 2.10, for $n \geq 1$ and any $\zeta, \eta \in C^\infty_0(B_{2R}(z))$ such that $\eta = 1$ on the support of $\zeta$, we have

\[
\|\zeta u\|_{W^2_{p(n)}(G)} \leq N\|\zeta f + 2a^{ij}u,_{x_i} \zeta x^j \|_{L_p(B_{2R})} + u(L' - \lambda)\zeta\|_{L_p(B_{2R})}.
\]

By iterating the inequality between the extreme terms, we obviously get that for any $\zeta \in C^\infty_0(B_{2R}(z))$ there is an $\eta \in C^\infty_0(B_{7R/4}(z))$ such that

\[
\|\zeta u\|_{W^2_{p(n)}(G)} \leq N\|\zeta f\|_{L_p(G_{2R})} + \|\eta u\|_{W^2_{p(n)}(G)}.
\]

Finally, recall that by Theorem 2.11

\[
\|\eta u\|_{W^2_{p(n)}(G)} \leq N\|\zeta u\|_{W^2_{p(n)}(G)} \leq N\|\zeta f\|_{L_p(G_{7R/4})} + \|\zeta f\|_{L_p(G_{7R/4})}.
\]

This yields (2.14) with $7R/2$ in place of $2R$. However, obviously Theorem 2.11 is also true with any number $> 1$ in place of 2. Then on the right in the above inequality one can take $2R (> 7R/4)$ in place of $7R/2$ and get (2.14) in its original form. The theorem is proved.

Remark 2.13. The author does not know that, if $q = d/2$ and $p < q$, the implication (2.13) is true or not. In that case a simple modification of the above argument shows that $\zeta u \in \tilde{W}^2_r(G_{2R})$ with any $r < d/2$. The reason behind this restriction lies in Theorem 2.6 in which the case $d/2 = q = p$ is not allowed.

Corollary 2.14. Under the assumptions of Theorem 2.12 if $u \in W^2_{p,loc}(G)$ satisfies $Lu = 0$ in $G$, then $u \in W^2_{q,loc}(G)$. In particular, $u \in C^{2-d/q}_{loc}(G)$.

Below we use the constant $d_0 = d_0(d, \delta) \in (d/2, d)$ introduced in [9]. From Corollary 6.3 of [9] and Corollary 2.14 we obtain the following.

Corollary 2.15 (Harnack inequality). Under the assumptions of Theorem 2.12, let $q \geq d_0$, $R \in (0, \infty]$ and let $u \in W^2_p(B_{2R})$ be a nonnegative function satisfying $Lu = 0$ (a.e.) in $B_{2R}$ with $c \equiv 0$. Then for any $x, y \in B_R$ we have $u(x) \leq Nu(y)$, where $N = N(d, \delta, \|b\|)$.

It would be interesting to know if this result can be obtained by using purely PDE methods as in [12].

The following theorem will be used when $c \equiv 0$, so that we can take $q$ as close to $d$ as we like.
Theorem 2.16. Under Assumption 2.1 suppose that either (a) $q > d/2$ and the condition (ii) of Theorem 2.12 is satisfied or (b) $p > d/2$. Set $\lambda = \lambda_1(d, \delta, p, r_0, b_0, c_0, G)$ where the latter is introduced in Theorem 2.6. Then there exists an integer $m_0$, depending only on $p$ and $d$, and a constant $N$, depending only on $d, \delta, p, r_0, b_0, c_0, \|b\|, G$, such that for any $f \in L_p(G)$ we have
\[
\sup_{x \in G} |R_{\lambda+c}^{m_0} f(x)| \leq N \|f\|_{L_p(G)}.
\] (2.17)

Proof. If $p > d/2$, we are done due to Theorem 2.6 and embedding theorems. In case $p \leq d/2$ we also have $p < q$ and we use $p(n) = \gamma^n p$, $n = 0, 1, \ldots, m - 1$, $p(m) = q$, and set
\[
u_0 = f, \quad u_n = R_{\lambda+c}^n f, \quad n \geq 1.
\]
Observe that, for $n \geq 0$, we have
\[
\Delta u_{n+1} - Lu_{n+1} = u_n,
\]
so that $u_{n+1} \in W^2_p(G)$ and
\[
\|u_{n+1}\|_{W^2_p(G)} \leq N \|u_n\|_{L_p(G)} \leq N \|u_n\|_{L_{p(n)}(G)}.
\]
By Theorem 2.12
\[
\|u_{n+1}\|_{W^2_{p(n)}(G)} \leq N \|u_n\|_{L_{p(n)}(G)} + N \|u_{n+1}\|_{L_p(G)}.
\]
Hence
\[
\|u_{n+1}\|_{L_{p(n)}(G)} \leq N \|u_n\|_{L_{p(n)}(G)},
\] (2.18)
and by embedding theorems
\[
\|u_{n+1}\|_{L_{p(n)}(G)} \leq N \|u_n\|_{L_{p(n)}(G)}.
\]
Iterating this yields that for $n \geq 0$
\[
\|u_n\|_{L_{p(n)}(G)} \leq N \|u_0\|_{L_{p(0)}(G)} = N \|f\|_{L_p(G)},
\] (2.19)
where the constants $N$ depend on the data as in the statement of the theorem and they also depend on $n$. Now we fix an $n = n(p, d)$ ($\leq m$) so that $p(n) > d/2$ and from (2.18) and (2.19) and embedding theorems conclude that
\[
\sup_{x \in G} |u_{n+1}(x)| \leq N \|u_{n+1}\|_{W^2_{p(n)}(G)} \leq N \|f\|_{L_p(G)},
\]
which shows that (2.17) holds with $m_0 = n + 1$. The theorem is proved.

3. Two auxiliary results using probability theory

Here we assume that the coefficients of $L$ satisfy only Assumption 2.1 and are infinitely differentiable.

Lemma 3.1. Let $\lambda \geq \nu > 0$ and let $u \in W^2_\nu(G)$ satisfy $\lambda u - Lu \leq 1$ in $G$. Then $\lambda u \leq \mu$, where $\mu < 1$ is a constant depending only on $\nu, d, \delta, \|b\|$, and the diameter of $G$. 
Proof. In light of the maximum principle we may assume that $c \equiv 0$. Fix $x \in G$ and define the random process $x_t$, $t \geq 0$, as a solution of the Itô equation
\begin{equation}
x_t = x + \int_0^t \sqrt{2a(x_s)} \, dw_s + \int_0^t b(x_s) \, ds
\end{equation}
on some probability space with some $d$-dimensional Wiener process $w_t$. By Itô’s formula
\begin{equation*}
u u(x) = E \int_0^\tau f(x_t) e^{-\lambda t} \, dt,
\end{equation*}
where $\tau$ is the first exit time of $x_t$ from $G$ and $f = \lambda u - Lu$. It follows that \( \lambda u(x) \leq E \int_0^\tau \lambda e^{-\lambda t} \, dt = 1 - E e^{-\lambda \tau} \leq 1 - e^{-\nu \tau} \).

By Corollary 2.7 of [7] there exist constants $\kappa = \kappa(d, \delta, \|b\|, \text{diam}(G)) > 0$ and $N = N(d, \delta, \|b\|)$ such that, for any $T > 0$
\begin{equation}
P(\tau > T) \leq Ne^{-\kappa T}.
\end{equation}
Hence,
\begin{equation*}
e^{-\nu \tau} \geq e^{-\nu T} P(\tau \leq T) \geq e^{-\nu T} (1 - N e^{-\kappa T})
\end{equation*}
and $\lambda u(x) \leq 1 - e^{-\nu T} (1 - N e^{-\kappa T}) =: \mu$, where $\mu < 1$ for an appropriate choice of $T$. The lemma is proved.

Lemma 3.2. Let $\lambda \geq \nu > 0$ and let $u \in W^2_d(B_R)$ satisfy $\lambda u - Lu \leq 0$ in $B_R$. Then
\begin{equation}
u u(0) \leq 2e^{-\kappa \sqrt{\nu R}} \max_{\partial B_R} u_+,
\end{equation}
where $\kappa = \kappa(d, \delta, \|b\|) > 0$.

This lemma is a direct corollary of Theorem 2.10 of [7] because in light of Itô’s formula
\begin{equation*}
u u(0) = E \left( e^{-\lambda \tau} - \phi \right) u(x_\tau) + \int_0^\tau e^{-\lambda t - \phi_t} (\lambda - L) u(x_t) \, dt \leq E e^{-\lambda \tau} \max_{\partial B_R} u_+,
\end{equation*}
where
\begin{equation*}
\phi_t = \int_0^t c(x_s) \, ds,
\end{equation*}
$\tau$ is the first exit time of $x_t$ from $B_R$ and $x_t$ is the solution of (3.1) with $x = 0$.

4. Solvability of $\lambda u - Lu = f$ in $G$ for $\lambda \geq 0$

Here we follow the line of arguments from Section 11.3 of [6]. We take $p, q$ as in Section 1. We also take any $q' > d/2, q' > p$, produce $\gamma$ by using (2.12) with $q'$ in place of $q$, then introduce $m$ and $p(n)$ as in the statement of Theorem 2.12 with $q'$ in place of $q$. We also take $\lambda \geq 0$.

Assumption 4.1. Assumption 2.1 is satisfied and either (a) $q = p$ (which is only possible if $p > d/2$) and Assumption 2.3 $(p, r_0)$ is satisfied or (b) $q > p$ and Assumptions 2.3 $(p(n), r_0)$ are satisfied with the above $p(0), ..., p(m)$. 
This assumption is supposed to be satisfied throughout the section.

**Theorem 4.2.** There exists a constant $N$ depending only on $d, \delta, p, q, r_0, b_0, c_0, \|b\|,$ and $G$, such that for any $u \in \dot{W}^2_p(G)$

$$\|u\|_{\dot{W}^2_p(G)} \leq N\|\lambda - L\|_{L^p(G)}. \quad (4.1)$$

Furthermore, for any $f \in L^p(G)$ there exists a unique $u \in \dot{W}^2_p(G)$ such that

$$\lambda u - Lu = f \text{ in } G.$$

**Proof.** In light of the method of continuity it suffices to prove the first assertion, while proving which we may assume that the coefficients of $L$ are infinitely differentiable. If $\lambda \geq \bar{\lambda}$, with $\bar{\lambda}$ taken from Theorem 2.16, the result is known from Theorem 2.6. Therefore we will only concentrate on $0 \leq \lambda < \bar{\lambda}$. Define

$$f = \lambda u - Lu$$

so that

$$\bar{\lambda}u - Lu = (\bar{\lambda} - \lambda)u + f, \quad u = (\bar{\lambda} - \lambda)R_{\bar{\lambda} + c}u + R_{\bar{\lambda} + c}f,$$

and by induction on $n$

$$u = [(\bar{\lambda} - \lambda)R_{\bar{\lambda} + c}]^n u + \sum_{i=0}^{n-1}[(\bar{\lambda} - \lambda)R_{\bar{\lambda} + c}]^i R_{\bar{\lambda} + c}f,$$

where $n$ is any integer $\geq 1$. We thus have the beginning of the von Neumann series.

Introduce the constants $N_1$ and $M_n$ so that

$$\|R_\lambda g\|_{L^p(G)} \leq N_1 \|g\|_{L^p(G)} \quad \forall g \in L^p(G), \quad M_n = \sum_{i=0}^{n-1} \bar{\lambda}^{i+1} N_1^{i+1}.$$

Finally, let $|G|$ be the volume of $G$ and take $m_0$ from Theorem 2.16 when $c \equiv 0$ and $q$ is replaced by $q' = p(m)$. For $n > m_0$, in light of Corollary 2.9

$$\|u\|_{L^p(G)} \leq |G|^{1/p} \bar{\lambda}^n \sup_{x \in G} R_{\lambda}^{n-m_0} \|u\|_{L^p(G)} + M_n \|f\|_{L^p(G)}.$$

By Lemma 3.1 the above supremum is dominated by

$$\bar{\lambda}^{m_0-n} \mu^{n-m_0} \sup_{x \in G} R_{\lambda}^{m_0} |u|(x),$$

where $\mu < 1$, which by Theorem 2.16 is less than

$$N_2 \bar{\lambda}^{m_0-n} \mu^{n-m_0} \|u\|_{L^p(G)}.$$

Hence,

$$\|u\|_{L^p(G)} \leq N_2 |G|^{1/p} \bar{\lambda}^m \mu^{n-m_0} \|u\|_{L^p(G)} + M_n \|f\|_{L^p(G)}.$$

We fix $n$ so that $N_2 |G|^{1/p} \bar{\lambda}^m \mu^{n-m_0} \leq 1/2$ and then arrive at

$$\|u\|_{L^p(G)} \leq 2M_n \|f\|_{L^p(G)}.$$
Now to get (4.1) it only remains to refer to Remark 2.7. The theorem is proved.

**Corollary 4.3** (Maximum principle). Let \( u \in \dot{W}^2_p(G) \). Then
\[
\|u_+\|_{L_p(G)} \leq N\|\lambda u - Lu\|_{L_p(G)}
\]
where \( N \) depends only on \( \delta, d, p, q, r_0, b_0, c_0, \|b\|, \) and the diameter of \( G \). In particular, if \( u \in \dot{W}^2_p(G) \) and \( Lu - \lambda u \geq 0 \) in \( G \), then \( u \leq 0 \) in \( G \).

This corollary is derived from Theorem 4.2 in the same way as Theorem 11.3.3 of [6] is derived from Theorem 11.3.2.

5. EQUATIONS IN THE WHOLE SPACE WITH \( \lambda \) LARGE

We take \( p, q \) as in Section 1. The following is a slight restatement of part of Theorem 6.4.1 of [6].

**Lemma 5.1.** Let \( s \in (1, \infty) \). If Assumptions 2.1 and 2.3 \((s, r_0)\) are satisfied, then there exist \( \lambda_0 = \lambda_0(d, \delta, s) \) and \( N_0 = N_0(d, \delta, s, r_0) \) such that, for any \( u \in W^2_s(\mathbb{R}^d) \) and \( \lambda \geq \lambda_0 \),
\[
\|D^2u\|_{L_s(\mathbb{R}^d)} + \sqrt{\lambda}\|Du\|_{L_s(\mathbb{R}^d)} + \lambda\|u\|_{L_s(\mathbb{R}^d)} \leq N_0\|L_0u - \lambda u\|_{L_s(\mathbb{R}^d)}.
\]
Furthermore, for any \( f \in L_s(\mathbb{R}^d) \) there exists a unique \( u \in W^2_s(\mathbb{R}^d) \) such that \( L_0u - \lambda u = f \).

In this section we suppose that Assumptions 2.1 and 2.3 \((p, r_0)\) are satisfied. Below by \( \lambda_0 \) we mean the one from Lemma 5.1 for \( s = p \).

**Theorem 5.2.** Introduce \( N^* = N^*(p, d) \) as the best constant such that
\[
\|Du\|_{L_{pd/(d-p)}(\mathbb{R}^d)} \leq N^*\|D^2u\|_{L_p(\mathbb{R}^d)}.
\]
for any \( u \in W^2_p(\mathbb{R}^d) \). Assume that
\[
2N_0N^*(\|b\|_{L_d(\mathbb{R}^d)} + \|c\|_{L_q(\mathbb{R}^d)}) \leq 1.
\]
Then for any \( u \in W^2_p(\mathbb{R}^d) \) and \( \lambda \geq \lambda_0 \),
\[
\|D^2u\|_{L_p(\mathbb{R}^d)} + \sqrt{\lambda}\|Du\|_{L_p(\mathbb{R}^d)} + \lambda\|u\|_{L_p(\mathbb{R}^d)} \leq 2N_0\|Lu - \lambda u\|_{L_p(\mathbb{R}^d)}.
\]
Furthermore, for any \( f \in L_p(\mathbb{R}^d) \) there exists a unique \( u \in W^2_p(\mathbb{R}^d) \) such that \( Lu - \lambda u = f \).

The proof of this theorem is achieved by repeating that of Theorem 2.4.

**Remark 5.3.** Similarly to Remark 2.5 we observe that the operator \( L \) is bounded as an operator from \( W^2_p(\mathbb{R}^d) \) to \( L_p(\mathbb{R}^d) \) as long as \( b \in L_d(\mathbb{R}^d) \) and \( c \in L_q(\mathbb{R}^d) \).

Next, for our fixed \( b \) and \( c \) there exists a \( b_0, c_0 \geq 0 \) such that
\[
2N_0N^*(\|bI|b| \geq b_0\|_{L_d(\mathbb{R}^d)} + \|cI|c| \geq c_0\|_{L_q(\mathbb{R}^d)}) \leq 1.
\]
Obviously we may take the same \( b_0, c_0 \) in (2.5) and (5.4).
Theorem 5.4. There exist \( \lambda_1 \geq 1, N, \) depending only on \( d, \delta, p, r_0, b_0, \) and \( c_0, \) such that, for any \( u \in W^2_p(\mathbb{R}^d) \) and \( \lambda \geq \lambda_1, \)

\[
\|D^2u\|_{L^p(\mathbb{R}^d)} + \sqrt{\lambda}\|Du\|_{L^p(\mathbb{R}^d)} + \lambda\|u\|_{L^p(\mathbb{R}^d)} \leq N\|Lu - \lambda u\|_{L^p(\mathbb{R}^d)}. \tag{5.5}
\]

Furthermore, for any \( f \in L^p(\mathbb{R}^d) \) there exists a unique \( u \in W^2_p(\mathbb{R}^d) \) such that \( Lu - \lambda u = f. \)

One proves this theorem in the same way as Theorem 2.6.

We denote the solution from Theorem 5.4 by \( R_{\lambda+c}f. \)

Remark 5.5. By taking \( \lambda = \lambda_1 \) in (5.5) we see that for the same kind of \( N \) as in (5.5) and any \( u \in W^2_p(\mathbb{R}^d) \)

\[
\|u\|_{W^2_p(\mathbb{R}^d)} \leq N\left(\|Lu\|_{L^p(\mathbb{R}^d)} + \|u\|_{L^p(\mathbb{R}^d)}\right). \tag{5.6}
\]

6. Equations in the whole space with \( \lambda \) small

We take \( p, q \) as in Section 1. We also take any \( q' > d/2, q' > p, \) produce \( \gamma \) by using (2.12) with \( q' \) in place of \( q, \) then introduce \( m \) and \( p(n) \) as in the statement of Theorem 2.12 with \( q' \) in place of \( q. \) We also take \( \lambda > 0. \)

Assumption 6.1. Assumption 2.1 is satisfied and either (a) \( q = p \) and Assumption 2.3 \((p, r_0)\) is satisfied or (b) \( q > p \) and Assumptions 2.3 \((p(n), r_0)\) are satisfied with the above \( p(0), ..., p(m). \)

Recall that \( B_R = \{x : |x| < R\}. \) Take the constant \( \kappa \) from Lemma 3.2 and define \( R \geq 4 \) so that

\[
2e^{-\kappa\sqrt{N}(R-2)} \leq 1/2.
\]

Lemma 6.2. Under Assumption 6.1 let \( u \) and \( f \) be bounded infinitely differentiable functions. Assume that \( f = 0 \) outside \( B_1 \) and \( \lambda u - Lu = f \) in \( \mathbb{R}^d. \) Also assume that the coefficients of \( L \) are infinitely differentiable.

Then there exists a constant \( N, \) depending only on \( \lambda, R, d, \delta, p, q, r_0, b_0, c_0, \) and \( \|b\|, \) such that

\[
\|u/v\|_{L^p(\mathbb{R}^d)} \leq N\|f\|_{L^p(\mathbb{R}^d)},
\]

where \( v(x) = e^{-\kappa\sqrt{N}|x|}. \)

Proof. We follow the proof of Lemma 11.6.1 of [6]. Relying on classical results, define \( h \in W^2_q(B_R) \) as a unique solution of

\[
\lambda h - Lh = 0 \quad \text{in} \quad B_R \quad \text{such that} \quad w := h - u \in W^2_q(B_R).
\]

By regularity results \( h \) is infinitely differentiable in \( \bar{B}_R \) and \( h = u \) on \( \partial B_R. \) Hence \( w \) is infinitely differentiable in \( B_R, \) vanishes on \( \partial B_R, \) and satisfies

\[
\lambda w - Lw = f.
\]

Notice that \( \lambda u - Lu = 0 \) outside \( B_1 \) and by the maximum principle

\[
|u(x)| \leq \max_{|x| = 2} |u| \quad \text{for} \quad |x| \geq 2.
\]
Taking this into account, taking \( x \) as the new origin, and using Lemma 3.2, we obtain
\[
|u(x)| \leq 2e^{-\kappa\sqrt{\lambda}|x|-2} \max_{|x|=2} |u| \quad \text{for } |x| \geq 2.
\] (6.1)

Also observe that by the maximum principle
\[
|h| \leq \max_{|x|=R} |u|
\]
in \( B_R \).

Now we claim that to prove the lemma, it suffices to prove that
\[
|w(x)| \leq N \|f\|_{L^p(B_R)} \quad \text{for } |x| = 2.
\] (6.2)

Indeed, if (6.2) holds, then
\[
\max_{|x|=2} |u| \leq \max_{|x|=2} |h| + \max_{|x|=2} |w| \leq \max_{|x|=R} |u| + N \|f\|_{L^p(\mathbb{R}^d)}
\]
\[
+ N \|w\|_{L^p(B_R)} \leq 2e^{-\kappa\sqrt{\lambda}(R-2)} \max_{|x|=2} |u| + N \|f\|_{L^p(\mathbb{R}^d)},
\]
which for our choice of \( R = R(\kappa, \lambda) \) yields
\[
\max_{|x|=2} |u| \leq N \|f\|_{L^p(\mathbb{R}^d)} + N \|w\|_{L^p(B_R)}.
\]

By Theorem 4.2 we have
\[
\|w\|_{L^p(B_R)} \leq N \|f\|_{L^p(B_R)}.
\]

Coming back to (6.1) we get that
\[
\|u/v\|_{L^p(B_2)} \leq N \|f\|_{L^p(\mathbb{R}^d)}.
\]

The remaining part of the norm is also bounded by \( N \|f\|_{L^p(\mathbb{R}^d)} \) since \( |u| \leq |h| + |w| \) and
\[
\max_{B_R} |h| \leq \max_{|x|=R} |u| \leq \max_{|x|=2} |u| \leq N \|f\|_{L^p(\mathbb{R}^d)}.
\]

Thus, indeed we need only prove (6.2).

By the maximum principle \(|w| \leq \psi|\), where \( \psi \) is a \( W^2_0(B_R) \)-solution of \( L\psi = -|f| \). So it suffices to estimate \( \psi \) on \(|x| = 2\). Take a point \( x_0 \) with \(|x_0| = 2\) and observe that by embedding theorems we have
\[
|\psi(x_0)| \leq N \|\psi\|_{W^2_0(B_{1/2}(x_0))}.
\]

Next, we use the local regularity result from Theorem 2.12. Then we find
\[
\|\psi\|_{W^2_0(B_{1/3}(x_0))} \leq N \|L\psi\|_{L^q(B_1(x_0))} + N \|\psi\|_{L^p(B_1(x_0))}.
\]

Here the first term on the right is zero since \( f = 0 \) outside of \( B_1 \) and the second term is less than \( N \|f\|_{L^p(B_R)} \) by Theorem 4.2. The lemma is proved.

The above proof of Lemma 6.2 is slightly different from the proof of Lemma 11.6.1 of [6] and is drift-specific because we needed to use Lemma 3.2, whose counterpart in [6] was obtained by using simple barriers. Contrary to
that the following theorem is derived from Lemma 6.2 by literally repeating the derivation of Theorem 11.6.2 of [6] from Lemma 11.6.1 of [6].

**Theorem 6.3.** Under Assumption 6.1 for any \( f \in L^p(\mathbb{R}^d) \) there exists a unique \( u \in W^2_p(\mathbb{R}^d) \) such that \( \lambda u - Lu = f \). Moreover, there exists a constant \( N \), depending only \( \lambda, R, d, \delta, p, q, r_0, b_0, c_0, \) and \( \| b \| \), such that

\[
\| u \|_{W^2_p(\mathbb{R}^d)} \leq N \| f \|_{L^p(\mathbb{R}^d)}.
\]

One more result concerning elliptic equations which will be proved in the next section is the following stability theorem in which Assumption 6.1 is not imposed.

**Theorem 6.4.** Let \( q = p \geq d_0 \), where \( d_0 = d_0(d, \delta) \in (d/2, d) \) is taken from [9], and suppose that Assumptions 2.1 and 2.3 \((p, r_0)\) are satisfied. Let \( a^n, b^n, c^n, n = 1, 2, \ldots, \) be sequences of smooth bounded functions with values in the set of symmetric \( d \times d \) matrices having all eigenvalues in \([\delta, \delta^{-1}]\), in \( \mathbb{R}^d \), and in \([0, \infty)\), respectively, such that \( a^n \to a \) on \( \mathbb{R}^d \) (a.e.) and

\[
\| b - b^n \|_{L^d(\mathbb{R}^d)} + \| c^n - c \|_{L^q(\mathbb{R}^d)} \to 0
\]
as \( n \to \infty \). Take \( \lambda > 0 \), \( f \in L^p_q(\mathbb{R}^d) \), and introduce \( u^n \) as unique smooth bounded solutions of \( \lambda u^n - L^n u^n = f \), where the operators \( L^n \) are constructed from \( a^n, b^n, c^n \). Then at each point of \( \mathbb{R}^d \) we have \( u^n \to u \) as \( n \to \infty \), where \( u \in W^2_p(\mathbb{R}^d) \) is a unique solution of \( \lambda u - Lu = f \).

The author does not know if this theorem holds for \( q \in (d/2, d_0) \). In this range we have unique solvability in \( W^2_p(\mathbb{R}^d) \) with \( 1 < p \leq q \), but could it happen that there is no stability?

### 7. Weak uniqueness of solutions of stochastic equations

Here we let \( q = p \geq d_0 \), where \( d_0 = d_0(d, \delta) \in (d/2, d) \) is taken from [9], and suppose that Assumptions 2.1 and 2.3 \((p, r_0)\) are satisfied. Take \( x \in \mathbb{R}^d \). Recall that according to Theorem 1.1 of [8] there exists a probability space \((\Omega, \mathcal{F}, P)\), a filtration of \( \sigma\)-fields \( \mathcal{F}_t \subset \mathcal{F}, t \geq 0 \), a process \( w_t, t \geq 0 \), which is a \( d \)-dimensional Wiener process relative to \( \{\mathcal{F}_t\} \), and an \( \mathcal{F}_t \)-adapted process \( x_t \) such that (a.s.) for all \( t \geq 0 \)

\[
x_t = x + \int_0^t \sqrt{2a(x_s)} \, dw_s + \int_0^t b(x_s) \, ds. \tag{7.1}
\]

Take \( f \in L^p_q(\mathbb{R}^d) \) and \( \lambda > 0 \). By Theorem 6.3 there is a unique \( u \in W^2_q(\mathbb{R}^d) \) such that \( \lambda u - Lu = f \). By Theorem 1.3 of [8] Itô’s formula is applicable so that for all \( t \geq 0 \) and \( L_s = a^{ij}(x_s) D_{ij} + b^i(x_s) D_i \)

\[
u(x_t) = u(x) + \int_0^t L_s u(x_s) \, ds + \int_0^t D_i u(x_s) \sigma_s^{ik} \, dw_s^k. \tag{7.2}
\]
and the last term is a square integrable martingale. By Theorem 1.5 of [8]
\[
E \int_0^\infty e^{-\lambda t} c(x_t) \, dt < \infty.
\]
Therefore, Itô’s formula is applicable to
\[
u(x_t) \exp(-\lambda t - \int_0^t c(x_s) \, ds).
\] (7.3)

**Remark 7.1.** Applying Itô’s formula to (7.3) yields that
\[
u(x) = E \int_0^\infty f(x_t) \exp(-\lambda t - \int_0^t c(x_s) \, ds) \, dt
\]
These facts and the standard argument based on considering resolvent operators (see, for instance, the arguments in [8] after Theorem 1.1 there) immediately proves the following weak uniqueness theorem.

**Theorem 7.2.** All solutions of (7.1) on all possible probability spaces have the same distribution on \(C([0, \infty), \mathbb{R}^d)\).

**Proof of Theorem 6.4.** Let \(x^n_t\) be solutions of
\[
x^n_t = x + \int_0^t \sqrt{2a^n(x^n_s)} \, dw_s + \int_0^t b^n(x^n_s) \, ds
\] (7.4)
on the same probability space as \(x_t\) or on different ones. By Theorem 1.1 of [8] the set of distributions of \(x^k\) on \(C([0, \infty), \mathbb{R}^d)\) is tight and any weakly converging subsequence of distributions converges weakly to the distribution of one of solutions of (7.1), which is the only one in light of Theorem 7.2. Hence, the whole sequence of distributions of of \(x^n\) weakly converges to the distribution of \(x\). In particular, for any \(n_0\) and smooth bounded \(g\)
\[
\lim_{n \to \infty} E \int_0^\infty g(x^n_t) \exp(-\lambda t - \int_0^t c^{n_0}(x^n_s) \, ds) \, dt = E \int_0^\infty g(x_t) \exp(-\lambda t - \int_0^t c^{n_0}(x_s) \, ds) \, dt.
\] (7.5)
At this point it is appropriate to mention that by Itô’s formula
\[
u^n(x) = E \int_0^\infty f(x^n_t) \exp(-\lambda t - \int_0^t c^n(x^n_s) \, ds) \, dt.
\]
Next, since \(\|b^n\|_{L_d(\mathbb{R}^d)} \leq \|b\|_{L_d(\mathbb{R}^d)} + 1\) for sufficiently large \(n\), by Theorem 1.5 of [8] for any \(\lambda > 0\), \(r \geq d_0\), and \(g(x)\) given on \(\mathbb{R}^d\) we have
\[
E \int_0^\infty e^{-\lambda t} |g(x^n_t)| \, dt \leq N \lambda^{d/(2r) - 1} \|g\|_{L_r(\mathbb{R}^d)},
\] (7.6)
where \(N\) is independent of \(f\) and \(n\). Below all constants like this one are called \(N\). The same estimate holds for \(x_t\) in place of \(x^n_t\). Hence also taking
into account (7.5) we get that, for $\varepsilon > 0$ and smooth bounded $g$ such that $\|f - g\|_{L_q(\mathbb{R}^d)} \leq \varepsilon$,

$$\lim_{n \to \infty} u^n(x) \geq -N\varepsilon + \lim_{n \to \infty} E \int_0^\infty g(x^n_t) \exp(-\lambda t - \int_0^t c^n(x^n_s) \, ds) \, dt$$

$$\geq -N\varepsilon + E \int_0^\infty g(x_t) \exp(-\lambda t - \int_0^t c^{n_0}(x_s) \, ds) \, dt$$

$$- \sup |g| E \int_0^\infty e^{-\lambda t} \left( \int_0^t |c^n(x^n_s) - c^{n_0}(x^n_s)| \, ds \right) dt.$$

Integrating by parts we see that the last expectation equals

$$\lambda^{-1} E \int_0^\infty e^{-\lambda t} |c^n(x^n_t) - c^{n_0}(x^n_t)| \, dt \leq N\|c^n - c^{n_0}\|_{L_q(\mathbb{R}^d)}.$$

It follows that

$$\lim_{n \to \infty} u^n(x) \geq -N\varepsilon - N \sup |g|\|c - c^{n_0}\|_{L_q(\mathbb{R}^d)}$$

$$+ E \int_0^\infty g(x_t) \exp(-\lambda t - \int_0^t c^{n_0}(x_s) \, ds) \, dt.$$

By using similar estimates for $u(x)$ we conclude that

$$\lim_{n \to \infty} u^n(x) \geq -N\varepsilon - N \sup |g|\|c - c^{n_0}\|_{L_q(\mathbb{R}^d)} + u(x).$$

By letting $\varepsilon \downarrow 0$ and $n_0 \to \infty$ we arrive at

$$\lim_{n \to \infty} u^n(x) \geq u(x).$$

This result is also true if we replace $f$ with $-f$ and this, certainly, proves the theorem.

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