From generalized directed animals to the asymmetric simple exclusion process

N Haug\textsuperscript{1,2}, S Nechaev\textsuperscript{2,3,4} and M Tamm\textsuperscript{4,5}

\textsuperscript{1} School of Mathematical Sciences, Queen Mary University of London, London, E1 4NS, UK
\textsuperscript{2} LPTMS, Université de Paris Sud 11, 91405 Orsay Cedex, France
\textsuperscript{3} P.N. Lebedev Physical Institute of the Russian Academy of Sciences, 119991, Moscow, Russia
\textsuperscript{4} Physics Department, Lomonosov Moscow State University, 119991, Moscow, Russia
\textsuperscript{5} Department of Applied Mathematics, International Research University Higher School of Economics, 101000, Moscow, Russia
E-mail: n.a.haug@qmul.ac.uk

Received 25 June 2014
Accepted for publication 22 August 2014
Published 9 October 2014

Online at stacks.iop.org/JSTAT/2014/P10013
doi:10.1088/1742-5468/2014/10/P10013

Abstract. Using the generalized normally ordered form of words in a locally-free group of \( n \) generators, we show that in the limit \( n \to \infty \), the partition function of weighted directed lattice animals on a semi-infinite strip coincides with the partition function of stationary configurations of the asymmetric simple exclusion process (ASEP) with arbitrary entry/escape rates through open boundaries. We relate the features of the ASEP in the different regimes of the phase diagram to the geometric features of the associated generalized directed animals by showing the results of numerical simulations. In particular, we show how the presence of shocks at the first order transition line translates into the directed animal picture. Using the evolution equation for generalized, weighted Lukasiewicz paths, we also provide a straightforward calculation of the known ASEP generating function.

Keywords: solvable lattice models, exact results, stochastic particle dynamics (theory), wetting (theory)

ArXiv ePrint: 1210.8060
1. Introduction

The concept of heaps of pieces was introduced by G Viennot in 1986 [1] (see also [2] for a review). Informally, a heap of pieces is a collection of elements which are piled together. If two elements intersect in their horizontal projections, then the resulting heap depends on the order in which the two are placed. In this case, the element which is placed second is said to be above the element placed first. On the other hand, the resulting heap does not depend on the order in which two elements are placed if their horizontal projections do not intersect. A special case of heaps are heaps of dimers. The dimers can be drawn as unit squares (boxes) which are not allowed to touch each other with their vertical edges. This means that if we place a box in the $k$th column and one in the $(k \pm 1)$th column afterwards, then the resulting heap is different to the one obtained from placing the boxes in the inverse order. Figure 1(c) shows an example of a heap of dimers.

Heaps of dimers are particularly interesting due to their relation to the model of directed animals (DA). The term ‘lattice animals’ is used as a collective name for several related models describing the growth of aggregates, for example, molecular layers on substrates. 2D directed animals are structures of occupied and unoccupied nodes on a lattice strip of width $n$ and infinite height. In this paper we only consider triangular lattices. The occupied sites on the lowest row are called roots (or source points) and the DA has to satisfy the condition that each occupied site can be reached from at least one root along a directed path containing only occupied sites via diagonal or vertical edges (for the triangular lattice). Figure 1(a) shows an example of a directed animal on such a triangular lattice and figures 1(b) and (c) illustrate the bijection between directed animals and heaps of dimers, which works as follows. Given a directed animal, we draw boxes around the occupied sites. In this way, the DA shown in figure 1(a) is redrawn as shown in figure 1(b). If a box is not supported from below, then we shift it downwards so...
that it is now supported. In this way, we obtain figure 1(c). It can be easily seen that this mapping is in fact invertible. Namely, for a given heap like in figure 1(c), we shift each box upwards which sits on top of a box in the same column and with it all the boxes which are above it. Then we redraw the boxes as black circles, place white circles everywhere else and connect everything by a triangular lattice to obtain our DA. The arrows in figure 1(c) illustrate the so-called ‘Mikado’ enumeration of the nodes which will be explained below. Since we are only considering heaps of dimers in this paper, we will refer to them as heaps from now on. Also, due to the described bijection, we use the term directed animals and heaps synonymously.

The typical problem for $N$-site DA concerns the computation of the number $\Omega(N,n|\{C\})$ of all distinct DA configurations in the bounding box of $n$ columns for a given configuration of roots (base) $\{C\}$ (for example, the base of the DA in figure 1 is $\{C\} = \{3, 5\}$). This function was computed exactly for the first time by Hakim and Nadal [3] by using algebraic methods dealing with the transfer matrix diagonalization for some spin system.

In this work, we review a different algebraic approach to directed animals which consists of representing each DA on a lattice strip of width $n$ by an ordered word, spelled by the generators of a locally free semi-group of $n$ generators, and we will show that there is a deep connection between this group-theoretical approach and the asymmetric simple exclusion process (ASEP) on an open line.

The ASEP is a stochastic process on a chain of $N$ sites which can be either occupied by a particle or can be empty. A particle hops to its right with rate 1 if the right neighbouring site is empty. For a detailed introduction and review of the important results of this process, we refer the reader to [4]. The ASEP can be considered both with periodic and open boundary conditions. In this paper, we consider open boundaries, where particles enter the chain from the left with a rate $\alpha$ and exit the chain on the right with a rate $\beta$ (see figure 2). For this case, the probability distribution of the stationary state has been derived in [5] by a matrix ansatz. Combinatorial interpretations of the steady state weights of the configurations have already been given in terms of pairs of paths (for $\alpha = \beta = 1$) [6], in terms of weighted permutation tableaux [7], and in terms of weighted binary trees [8].

In the following sections we will give a new combinatorial interpretation of the stationary weights of the ASEP on an open line in terms of directed animals. More precisely, we will demonstrate that the partition function of the ASEP steady state on
an $N$-site segment with entrance and exit rates equal to one coincides with the partition function of $(N + 1)$-site directed animals on a triangular semi-infinite lattice strip with the topmost particle (the ‘roof’) being located at the left boundary. This correspondence can be extended towards the arbitrary entrance and exit rates by defining an appropriate weighting of the position of the leftmost root and a ‘sticky’ left boundary. It is then possible to relate the features of the steady state distribution of the ASEP in the different regimes of the phase diagram to the geometric features of the associated generalized directed animals.

2. Algebraic approach to directed animals

The algebraic approach to directed animals consists of assigning to each DA-configuration an equivalence class of words in some semi-group with special local commutation relations corresponding to local particle configurations, as shown in figure 3. To be specific, define the locally free semi-group, $F_n^+$ with $n$ generators $g_1, \ldots, g_n$, determined by the relations

$$g_k g_m = g_m g_k, \quad |k - m| \geq 2.$$  

Each pair of neighbouring generators, $(g_k, g_{k+1})$ produces a free sub-semigroup of $F_n^+$. The statistical properties of locally free groups and semi-groups were investigated in detail in [9], where it was shown that the partition function of an $N$-site heap in a bounding box of size $n$ coincides with the partition function of an $N$-step Markov chain on $F_n^+$, or, equivalently, by the total number of equivalence classes of $N$-letter words in $F_n^+$. Namely, to any configuration of DA one can bijectively associate an equivalence class of words in $F_n^+$. Now each equivalence class contains exactly one word which is in a normal form, which means that in this word, the generators with smaller indices are pushed as far left as possible in accordance with the commutation relations (1). Consequently, the word

$$W = g_{s_1} g_{s_2} \cdots g_{s_N},$$

doi:10.1088/1742-5468/2014/10/P10013
is in an ordered form if and only if the indices $s_1, \ldots, s_N$ satisfy the following conditions, graphically represented in figure 4.

(a) If $s_i = 1$ then $s_{i+1} \in \{1, 2, \ldots, n\}$;

(b) If $s_i = x$ ($2 \leq x \leq n-1$) then $s_{i+1} \in \{x-1, x, x+1, \ldots, n\}$;

(c) If $s_i = n$ then $s_{i+1} \in \{n-1, n\}$.

Thus, any $N$-site heap in a bounding box of $n$ columns can be uniquely represented by an $N$-letter ordered word, ‘spelled’ by the generators of $F_n^+$. For example, the normally ordered word

$$W = g_3 g_2 g_1 g_1 g_2 g_5 g_4 g_5 g_3 g_6 g_6$$

(3) uniquely represents the 12-site directed animal shown in figure 1(a).

For a given heap, the corresponding normally ordered word can be obtained by an algorithm which sets a constructive geometrical way of normal ordering. We call this enumeration procedure the ‘Mikado ordering’ since it resembles the famous Mikado game, the goal of which consists of the sequential removal of the boxes from a random pile, one-by-one, without disturbing the other elements. To proceed, define in a heap a set of top sites, each of which can be removed from the heap without disturbing the rest of the pile. We call these elements the ‘roof’, $T$, of the heap. Remove the rightmost element of $T$. In the updated roof, $T'$, remove again the rightmost element to get $T''$, and so on, until the heap is empty. The sequence of one-by-one removed elements is ordered normally and uniquely enumerates the heap (i.e. the directed lattice animal). This fact is established in lemma 3 of [9]. For the heap shown in figure 1(c), the Mikado ordering is depicted by the sequence of arrows and coincides with (3) (note that the topmost element in the 4th column does not belong to the roof as it cannot be removed without disturbing the topmost element in the 3rd column, which is above it).
3. Matrix ansatz for generalized DA and ASEP

We now introduce the partition function \( \Omega_{i,j}(N+1, n) \), which enumerates all the \((N+1)\)-particle heaps in the bounding box of \( n \) columns whose Mikado ordering has its first element in the \( i \)-th and its last element in the \( j \)-th column. This function can be expressed in terms of a local \((n \times n)\) transfer matrix, \( M \), with transitions described by the rules (a)–(c) (see also figure 4), namely

\[
\Omega_{i,j}(N+1, n) = \langle v_i | M^{N} | v_j \rangle ,
\]

where \( \langle v_k \rangle = (0, \ldots, 0, 1, 0 \ldots 0) \) with a one in the \( k \)-th position, and, as usual, \( |v_k \rangle = |v_k \rangle^T \).

For reasons which will become clear in the following, we are mostly interested in the values of \( \Omega_{i,j}(N+1, 1) \). It is instructive to introduce the generating function

\[
Z_{N+1}(n, \alpha) = \sum_{i=1}^{n} \Omega_{i,1}(N+1, n) \alpha^{1-i} = \langle v_{in} | M^{N} | v_{1} \rangle,
\]

where \( \langle v_{in} \rangle = (1, \alpha^{-1}, \alpha^{-2}, \alpha^{-3} \ldots) \). Now the transfer matrix \( M \) allows a natural decomposition in ‘forward’ \( (D) \) and ‘backward’ \( (E) \) parts, associated with arbitrarily large jumps to the right and one-step jumps to the left (see figure 4). Namely, we can write \( M = D + E \), where

\[
D = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
0 & 1 & 1 & \ldots & 1 \\
0 & 0 & 1 & \ldots & 1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 1
\end{pmatrix}; \quad E = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix} .
\]

There is a striking similarity between this result and the celebrated exact solution of the asymmetric simple exclusion process (ASEP) on a line \([5]\). Let us briefly recall this well-known result. The steady state of this process can, according to \([5]\), be calculated via the following procedure known as the ‘matrix ansatz’. Introduce two formal operators \( \tilde{D} \) and \( \tilde{E} \) which satisfy

\[
\tilde{D} + \tilde{E} = \tilde{D}\tilde{E}
\]

and two vectors \( \langle \tilde{v}_{in} \rangle \) and \( |\tilde{v}_{out} \rangle \), such that

\[
\langle \tilde{v}_{in} | \tilde{E} = \alpha^{-1} \langle \tilde{v}_{in} ; \tilde{D} | \tilde{v}_{out} \rangle = \beta^{-1} |\tilde{v}_{out} \rangle .
\]

Now one can show that the probability of observing any given ASEP configuration in the steady state is proportional to a matrix element of the type \( \langle \tilde{v}_{in} | E \tilde{D} E \tilde{D} \tilde{E} \tilde{D} \tilde{E} | \tilde{v}_{out} \rangle \), where for the dots one should insert a sequence of the operators \( \tilde{D} \) and \( \tilde{E} \), with \( \tilde{D} \) and \( \tilde{E} \) corresponding to occupied and empty sites, respectively. For example, the probability of the configuration shown in figure 2 is proportional to \( \langle \tilde{v}_{in} | E \tilde{D} \tilde{D} \tilde{E} \tilde{D} \tilde{E} \tilde{D} \tilde{E} \tilde{D} | \tilde{v}_{out} \rangle \). The sum \( Z_N(\alpha, \beta) \)
of matrix elements over all possible configurations, which plays a role very similar to the partition function of the steady-state ASEP, can be written as

\[ \tilde{Z}_N(\alpha, \beta) = \langle \tilde{v}_{\text{in}} | (\tilde{E} + \tilde{D})^N | \tilde{v}_{\text{out}} \rangle = \langle \tilde{v}_{\text{in}} | (\tilde{M})^N | \tilde{v}_{\text{out}} \rangle. \]  

(9)

For arbitrary \( \alpha \) and \( \beta \), the algebra defined by (7) and (8) has no finite-dimensional representations. However, there exist many infinite-dimensional representations, among which the most interesting for us is constructed as follows. Set \( \langle \tilde{v}_{\text{in}} | = (1, \alpha^{-1}, \alpha^{-2}, \alpha^{-3}, \ldots) \), \( \langle \tilde{v}_{\text{out}} | = (1, 0, 0, \ldots) \) and choose the matrices \( \tilde{D} \) and \( \tilde{E} \) as

\[ \tilde{D} = \begin{pmatrix} 1 \beta & 1 \beta & 1 \beta & \cdots \\ 0 & 1 & 1 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}; \quad \tilde{E} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \]  

(10)

It is easy to check that conditions (7) and (8) are satisfied. Furthermore, the similarity between (6) and (10) is strikingly clear. Indeed, we immediately get

\[ \tilde{Z}_N(\alpha, \beta = 1) = \lim_{n \to \infty} Z_{N+1}(n, \alpha). \]  

(11)

This explicitly demonstrates that in the limit \( n \to \infty \), the generating function of \( N+1 \)-particle heaps with a topmost particle in the first column and activity \( \alpha \) associated to the position of the first particle (which is the leftmost particle in the lowest row), coincides with the ‘partition function’ of the stationary ASEP chain of \( N \) sites in the case \( \beta = 1 \).

The correspondence between DA and the ASEP for arbitrary values of \( \alpha \) and \( \beta \) is established as follows. Each ASEP configuration corresponds to a set of pyramids [1], i.e. heaps with a roof consisting of a single element in column 1. Now we can associate the sequence of ‘backward’ and ‘forward’ jumps with the ASEP configuration so that backwards jumps correspond to a hole in the ASEP sequence, and forward jumps to a particle—see figure 4 (here ‘no jump’ is considered as a ‘forward jump’ with length zero). Since there can be forward jumps of different lengths, there are, generally speaking, many different DAs corresponding to a single ASEP configuration. This fact is depicted in figure 5, where we show two heaps of dimers corresponding to the same ASEP configuration.

Now the weight of a given ASEP configuration in the steady state on a line is proportional to the sum of the weights of all corresponding heaps, where the weight of a given heap equals \( \alpha^{1-x} \beta^{1-y} \) with \( x \) being the coordinate of the column in which the leftmost root is located, and \( y \) being the number of elements of the heap in column 1. Using the Mikado enumeration, one can rephrase this statement as follows: (a) the first letter in the normally ordered word associated to a specific DA has a weight \( \alpha^{1-x} \), (b) each generator \( g_i \) (except for the last letter) carries the weight \( \beta^{-1} \), while all the other generators have a weight 1, (c) the last letter in the normally ordered word is always \( g_1 \), (d) to get a weight of an ASEP configuration one must sum over all corresponding DAs in which any pair \( g_i g_k \) with \( k \geq i \) \((i, k) \in \{1, \ldots, n\}^2\) corresponds to a particle, while a pair \( g_i g_{i-1} \) \((i \in \{2, \ldots, n\}\) corresponds to a hole. In table 1, we summarize the correspondence between the stationary ASEP and the DA. We note several important facts about this correspondence.

doi:10.1088/1742-5468/2014/10/P10013
From generalized directed animals to the asymmetric simple exclusion process

Two generalized directed animals corresponding to the same ASEP configuration (below) and the Markov chains representing the associated ordered words. The horizontal coordinate stands for the index of a letter in the word. All jumps starting from position \( x = 1 \) carry the weight \( \beta^{-1} \) and a first letter \( g_x \) contributes the weight \( \alpha^{-(x-1)} \). The last letter is always \( g_1 \).

Table 1. Correspondence between the asymmetric simple exclusion process and directed animals on a strip of width \( n \) in the limit \( n \to \infty \).

| ASEP | Directed animals |
|------|------------------|
| \( N \), the system size \( (i.e. \) the segment length) | \( N + 1 \), number of letters in a word \( (i.e. \) number of sites in an animal) |
| \( \alpha \), entrance probability at a position 1 of a segment | Weight \( \alpha^{1-x} \) of a 1st letter in the column \( x \) |
| \( \beta \), exit probability at a position \( N \) of a segment | Weight, \( \beta^{-1} \), of a letter \( g_1 \) in a normally ordered word\(^a\) |
| Particle | Any pair of letters \( g_i g_j \) with \( j \geq i \) in a normally ordered word |
| Hole | Any pair of letters \( g_i g_{i-1} \) in a normally ordered word |

\(^a\) Except for the last letter which has a weight 1.

Firstly, there is no straightforward analogue of time in the heap picture, and there is no time evolution imposed on the DAs. From the mapping revealed above one immediately gets the steady state probabilities of the ASEP configurations, but not the underlying time evolution.

Secondly, formally, the mapping is exact only for \( n \to \infty \) (i.e. for directed animals on a quarter-plane with no boundary on the right. However, the condition that the last particle should be put in column 1 dictates that all \( N \)-step trajectories stay in columns \( x \leq N \), so \( n \geq N \) is enough to make the mapping exact.
Thirdly, the original ASEP problem has a well-known particle-hole symmetry, i.e. if one replaces the particles with holes and vice versa, and reverses the direction of the flow and interchanges $\alpha \leftrightarrow \beta$, one returns to the original problem. This symmetry is evident in the formal algebraic matrix ansatz, but broken down by the representation (10), which makes it a bit artificial in the original ASEP model. The interpretation in terms of Mikado-ordered DAs gives, however, a natural, intuitive meaning to the representation (10). The connection between the DA and the ASEP problems also shows that there is actually a hidden symmetry in the DA model, namely a symmetry between the position of the leftmost root and the number of visits of the column $x = 1$. In other words, the partition function of $N$-particle DAs with a single roof particle in the first column, $s$ particles with weight $\beta^{-1}$ in the first column and the leftmost root in column $k$ with weight $\alpha^{1-k}$, coincides with the partition function of $N$-particle DAs with a single roof particle in the first column, $k$ particles in the first column with weight $\alpha$ in total and the leftmost root in column $s$ with weight $\beta^{1-s}$.

Now we can understand more clearly the statistics of standard, non-weighted directed animals ($\alpha = \beta = 1$). The point $(\alpha, \beta) = (1, 1)$ lies deeply in the maximum-current phase of the ASEP ([4], see also the next section). This means that the corresponding ASEP steady state is dominated by configurations where the particle density is equal to 1/2. In terms of heaps, this means that for a large enough $N$, there are typically equal numbers of left and right jumps in the Mikado ordering (compare this with the numeric results shown in the next section). This might seem counter-intuitive, as rightward jumps can have an arbitrarily large length while leftward jumps always have a length 1. Note, though, that ‘rightward’ jumps can also have a length 0, so the average length of a rightward jump can turn out to be 1.

Small changes in $\alpha$ or $\beta$ do not move the ASEP out of the maximum-current phase, and thus the concentration of forward and backward jumps also remains equal for $\alpha$ and $\beta$ slightly differing from 1. One has to go as far as $\alpha = 1/2$ or $\beta = 1/2$ to see a significant change in the behaviour of the heaps. We suggest the reader should compare this result to the adsorption-desorption transition of a random walk (polymer chain) on a half-line with a potential well at $x = 0$, where the change in typical trajectory also occurs for the potential well depth $\beta^{-1} = 2$ [10].

4. Simulation of generalized heaps

For different values of $\alpha$ and $\beta$, we have numerically generated corresponding generalized heaps of $N = 150$ particles. The simulation was carried out by generating random $N = 150$-step generalized Lukasiewicz paths with a fixed endpoint at $x = 1$ [11] and weighting of the steps and initial position according to figure 5. The algorithm we used is described in [12]. Figure 6 shows the resulting pictures.

In the high density phase $\beta < \alpha < \frac{1}{2}$, the typical heaps are roughly vertical piles in the first column with only a few boxes sticking out into the second column. The number of boxes which are supported from below right, corresponding to a hole in the associated ASEP configuration, is very small, the particle density of the corresponding ASEP configuration is close to 1 in this case—see figure 6(a).
In the low density phase $\alpha < \beta < \frac{1}{2}$, the typical heap roughly follows a diagonal line, going from the bottom right to the top left. This corresponds to a very low particle density of the corresponding ASEP configuration—see figure 6(b).

For both $\alpha$ and $\beta$ greater than $\frac{1}{2}$, one obtains less regular pictures with, on average, as many boxes which are supported from the bottom left or sit on top of another box as boxes supported from the bottom right. This means that the corresponding ASEP configuration has a particle density close to $\frac{1}{2}$. The maximal current case $\alpha = \beta = 1$ is depicted in figure 6(c).

At the first-order transition line $\alpha = \beta < \frac{1}{2}$, one observes heaps which roughly consist of a diagonal line below, followed by a straight vertical pile in the first column. This means that the corresponding ASEP configuration is divided into a region with very low density on the left and a region with very high density on the right. However, the size of the two
From generalized directed animals to the asymmetric simple exclusion process

5. Stationary ASEP as polymer wetting

Let us sketch the derivation of the stationary ASEP partition function (9). Although the answer is well known since the pioneering works [5] and has been derived with different nuances in subsequent works (see, for example, [13,14]), we would like to emphasize the deep analogy of the ASEP generating function with the generating function of the wetting problem on a 1D adsorbing substrate [10,15]. In a general setting, wetting implies the interface pinning by an impenetrable solid. The problem of interface statistics in the presence of a hard wall was addressed in many publications (see, for example, [16,17] and references therein). The most interesting question concerns the nature of the wetting or pinning-depinning transition of the interface controlled by the parameters of its interaction with the substrate. To the best of our knowledge, the similarity of the analytic structures of the generating functions with asymmetric exclusion and wetting has only been briefly explored in the review [13]. The connection between the ASEP and the pinned interface statistics allows us, as we saw in the previous section, to get a simple and transparent view of the nature of the shocks. Conversely, this connection raises open questions about whether the fluctuations of the interface density in the vicinity of the pinning-depinning transition could exhibit the KPZ scaling seen near the ASEP shock profiles as pointed out in [18].

Define $Z_N(x, \alpha, \beta)$, the partition function of the $N$-step trajectories on the semi-infinite discrete line ($n \to \infty$) with allowed steps and weighting as shown in figure 4, and with the final position in $x$. For brevity we write it as $Z_N(x) \equiv Z_N(x)$. This function can be expressed in terms of a matrix product as

$$Z_N(x) = \langle v_{in} | \tilde{M}^{N-1} | v_x \rangle.$$  \hspace{1cm} (12)

The quantity of our interest is $Z_N(x = 1)$. From (12), we obtain the following recursion relation, valid for any $N \geq 0$ (compare to [5]):

$$\begin{cases} Z_{N+1}(x) = \beta^{-1}Z_N(1) + \sum_{y=2}^{x+1} Z_N(y) & x = 1, 2, \ldots, \\ Z_{N=0}(x) = \alpha^{x-1} & x = 1, 2, \ldots, \\ Z_N(x) = 0 & x = 0. \end{cases}$$  \hspace{1cm} (13)

Introduce the generating function

$$W(s, x) = \sum_{N=0}^{\infty} Z_N(x) s^N; \ Z_N(x) = \frac{1}{2\pi i} \oint_{C} \frac{W(s, x)}{s^{N+1}} ds$$

with a suitably chosen closed contour $C$ around the origin. In what follows we denote $W(s, x) \equiv W(x)$ for brevity.
Defining now $Q(x) = s^{x/2}W(x)$ and using the Kronecker $\delta$-symbol, where $\delta_{x,1} = 1$ for $x = 1$, and 0 otherwise, we can rewrite (13) as a single equation in a symmetrized form, which has a straightforward interpretation in terms of the wetting generating function [15] on a semi-infinite line $x \geq 0$. We get

$$
\left\{ Q(x) - \sqrt{s}(Q(x - 1) + Q(x + 1)) - s^{x/2}\alpha^{1-x}(1 - \alpha) \right\} (1 - \delta_{x,1})
+ \left\{ Q(x) \frac{\beta - s}{\beta} - \sqrt{s}Q(x + 1) - s^{x/2} \right\} \delta_{x,1} = 0.
$$

(14)

Applying the Fourier transform

$$
Q(q) = \sum_{x=0}^{\infty} Q(x) \sin qx; \quad Q(x) = \frac{2}{\pi} \int_{0}^{\pi} Q(q) \sin q x \, dq
$$

to equation (14), we obtain

$$(1 - 2\sqrt{s} \cos q)Q(q) - \frac{s}{\beta} \sin q Q(1) - (\alpha - \alpha^2) \sum_{x=2}^{\infty} \left( \frac{\sqrt{s}}{\alpha} \right)^x \sin qx - \sqrt{s} \sin q = 0.
$$

(15)

The solution for $Q(q)$ reads

$$
Q(q) = \frac{\frac{s}{\beta} \sin q Q(1) + f(q)}{1 - 2\sqrt{s} \cos q}.
$$

(16)

where we have defined $f(q)$ as

$$
f(q) = (\alpha - \alpha^2) \sum_{x=2}^{\infty} \left( \frac{\sqrt{s}}{\alpha} \right)^x \sin qx + \sqrt{s} \sin q.
$$

$$
= \frac{\alpha^2(1 - 2\sqrt{s} \cos q) + s\alpha}{\alpha^2 - 2\sqrt{s}\alpha \cos q + s} \sqrt{s} \sin q,
$$

(17)

Remembering that $W(1) = s^{-1/2}Q(1)$, inserting the expression (17) for $f(q)$ into (16) and applying the inverse Fourier transform, we end up with

$$
W(s, 1) = \frac{2}{\pi \sqrt{s}} \int_{0}^{\pi} \frac{f(q) \sin q}{1 - 2\sqrt{s} \cos q} \, dq
\frac{1 - 2s}{\pi \beta} \int_{0}^{\pi} \frac{\sin^2 q}{1 - 2\sqrt{s} \cos q} \, dq
$$

$$
= \frac{4\alpha \beta}{(2\alpha - 1 + \sqrt{1 - 4s})(2\beta - 1 + \sqrt{1 - 4s})}
$$

(18)

as an explicit expression for the generating function of the stationary ASEP partition function, $Z_N(x = 1, \alpha, \beta)$. Note that the roots in the denominator are positive for $s < \frac{1}{4}$. For $\alpha$ and $\beta < \frac{1}{2}$, the generating function (18) has two pole singularities at

$$
s_1 = \alpha(1 - \alpha), \quad s_2 = \beta(1 - \beta),
$$

(19)

which are both smaller or equal to $\frac{1}{2}$. For $\alpha$ and $\beta > \frac{1}{2}$, these poles leave the real axis and the branching point $s_3 = \frac{1}{4}$ becomes the dominant singularity. Depending on which singularity is dominant, one recovers the known phase diagram of the ASEP, with the three borders $\alpha = \beta < \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$ [4].

doi:10.1088/1742-5468/2014/10/P10013
As one can see, both generating functions, of the ASEP and of the wetting problem, have similar analytic structures; they diverge at branching points, which signals the existence of a phase transition. However, the behaviour of the function $W(s, 1)$ is far more rich: it has two possible singularities $s_1$ and $s_2$ controlled by two independent parameters, $\alpha$ and $\beta$. Thus, in the thermodynamic limit $N \to \infty$, the ASEP ‘free energy’, $f(\alpha, \beta)$ depends strongly on the parameters $\alpha$ and $\beta$ and is determined by the singularity which is closest to zero:

$$f(\alpha, \beta) = -\ln \min\{s_1(\alpha), s_2(\beta), s_3\}. \quad (20)$$

6. Summary

In this article we have established the connection between generalized directed animals on a semi-infinite strip with an adsorbing boundary and special initial particle distribution with the stationary state configurations of the asymmetric simple exclusion process. Given the relation between directed animals and the ASEP, we analysed how the features of one model translate into the features of the other one. We simulated generalized directed animals (heaps respectively) in the different regimes of the ASEP phase diagram and discussed the shape of the typical pictures obtained. In particular, we were able to observe shock configurations at the first order transition line between the low and the high density phase of the ASEP. We also noted a hidden symmetry of the directed animals model by making use of the known particle-hole symmetry in the ASEP.

The random walk picture of directed animals which resulted from the normal order representation of directed animal configurations (associated with the locally free group), allowed us to regard the stationary ASEP as a sort of wetting model on a 1D adsorbing substrate. Using the evolution equation for this random walk, we provided a simple derivation of the ASEP generating function on a 1D line.

Acknowledgments

The authors are grateful to A Vershik for numerous discussion of the problem. NH and MT would like to thank the LPTMS for its warm hospitality. This work was partially supported by the grants ANR-2011-BS04-013-01 WALKMAT, FP7-PEOPLE-2010-IRSES 269139 DCP-PhysBio, as well as by a MIT-France Seed fund and the Higher School of Economics program for Basic Research.

References

[1] Viennot G X 1993 Heaps of pieces I: Basic definitions and combinatorial lemmas (Ann. NY Acad. Sci. 576 542–70)
[2] Krattenthaler C The theory of heaps and the Cartier–Foata monoid www.mat.univie.ac.at/~kratt
[3] Hakim V and Nadal J P 1983 Exact results for 2D directed lattice animals on a strip of finite width J. Phys. A: Math. Gen. 16 L213–8
[4] Derrida B 1983 An exactly soluble non-equilibrium system: the asymmetric simple exclusion process Phys. Rep. 301 65–83

doi:10.1088/1742-5468/2014/10/P10013
From generalized directed animals to the asymmetric simple exclusion process

[5] Derrida B, Evans M R, Hakim V and Pasquier V 1993 Exact solution of a 1D asymmetric exclusion model using a matrix formulation J. Phys. A: Math. Gen. 26 1493–517

[6] Shapiro L W and Zeilberger D 1982 A Markov chain occurring in enzyme kinetics J. Math. Biol. 15 351–7

[7] Corteel S and Williams L K 2007 Permutation tableaux and the asymmetric exclusion process Adv. Appl. Math. 39 293–310 (http://igm.univ-mlv.fr/ fpsac/FPSAC07/SITE07/PDF-Proceedings/Talks/4.pdf)

[8] Viennot X G 2009 Catalan tableaux and the asymmetric simple exclusion process http://arxiv.org/pdf/0905.3081.pdf

[9] Vershik A M, Nechaev S K and Bikbov R 2000 Statistical properties of locally free groups with applications to braid groups and growth of random heaps Commun. Math. Phys. 212 469–501

[10] Naidenov A and Nechaev S K 2001 Adsorption of a random heteropolymer at a potential well revisited: location of transition point and design of sequences J. Phys. A: Math. Gen. 34 5625–34

[11] Lehner F 2003 Cumulants, lattice paths, and orthogonal polynomials Discrete Math. 1–3 177–91

[12] Frank-Kamenetskiǐ M D and Vologoskiǐ A V 1981 Topological aspects of the physics of polymers: the theory and Its biophysical applications Usp. Fiz. Nauk 134 641–73

[13] Blythe R A and Evans M R 2007 Nonequilibrium steady states of matrix-product form: a solver’s guide J. Phys. A: Math. Theor. 40 R333

[14] Depken M and Stinchcombe R B 2004 Exact joint density-current probability function for the asymmetric exclusion process Phys. Rev. Lett. 93 040602

[15] Gangardt D M and Nechaev S K 2008 Wetting transition on a 1D disorder J. Stat. Phys. 130 483–502

[16] Abraham D B 1980 Solvable model with a roughening transition for a planar ising ferromagnet Phys. Rev. Lett. 44 1165

[17] Abraham D B 1986 Phase Transitions and Critical Phenomena vol 10 ed C Domb and J L Lebowitz (London: Academic) pp 1–74

[18] Janowsky S and Lebowitz J 1992 Finite-size effects and shock fluctuations in the asymmetric simple-exclusion process Phys. Rev. A 45 618

doi:10.1088/1742-5468/2014/10/P10013