Critical points in a relativistic bosonic gas induced by the quantum structure of spacetime

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It is well known that phase transitions arise if the interaction among particles embodies an attractive as well as a repulsive contribution. In this work it will be shown that the breakdown of Lorentz symmetry, characterized through a deformation in the relation dispersion, plus the bosonic statistics predict the emergence of critical points. In other words, in some quantum gravity models the structure of spacetime implies the emergence of critical points even when no interaction among the particle has been considered.

I. INTRODUCTION

The possibility that Lorentz symmetry is just an approximation to quantum space-time can be interpreted, in some cases, as a deformation in the relation dispersion \[ E^2 = p^2 \left[ 1 - \alpha \left( E \ell_p \right)^n \right] + m^2. \] (1)

Here \( \alpha \) is a coefficient, whose precise value depends upon the considered quantum gravity model, and \( n \), the lowest power in Planck’s length leading to a non-vanishing contribution, is also model dependent.

In ordinary units we have

\[ E^2 = p^2 c^2 \left[ 1 - \alpha \left( E \sqrt{G/(c^5 \hbar)} \right)^n \right] + \left( mc^2 \right)^2. \] (2)

The most difficult aspect in the search of experimental hints relevant for the quantum-gravity problem is the smallness of the involved effects. However, a modified dispersion relation emerges as an excellent way for searching phenomenological effects in this kind of theories. In the experimental quest for this kind of effects, interferometry has played a fundamental role, through the energy dependence of the speed of light \[ \frac{E}{c}. \]

The perspective concerning the development of this type of theories in quantum gravity appears intimately attached to the experimental confirmation of some of their predictions.

The idea in the present work is to introduce a deformed dispersion relation as a fundamental fact for the statistical mechanics of massive relativistic bosons. Afterwards, we analyze the effects of this assumption upon the thermodynamics and critical phenomena of the corresponding gas.

In a physical system comprising a large number of particles either we consider interaction among the particles or neglect it \[ \Box. \] In the first case the existence of phase transitions and critical phenomena is practically null, with the exception of the Bose-Einstein condensation, which appears without the presence of interaction among the components of the system. The phenomenon of Bose-Einstein condensation is caused exclusively by quantum effects.

In the subsequent development of this work we will analyze a relativistic boson gas without including interactions among the corresponding particles. It will be proved that the bosonic statistics plus the breakdown of Lorentz symmetry entail the emergence of critical points. Indeed, we may consider the quantal properties of the system as an attractive pseudo-interaction, whereas the breakdown of Lorentz symmetry may be understood as a repulsive pseudo-interaction. These two ingredients, this is the main issue in our work, imply the appearance of critical points.

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II. 2. DEFORMED RELATION DISPERSION AND BOSONIC STATISTICS

As mentioned above, several quantum-gravity models suggest a deformation in the dispersion relation, the one can be characterized phenomenologically, through corrections hinging upon Planck’s length, $\ell_p$ ($E_p = c^5 \hbar/G$ denotes Planck’s energy, where $G$ is the Newtonian gravitational constant, $c$ denotes the speed of light, and $\hbar$ is the Planck’s constant divided by $2\pi$), and in ordinary units

$$E^2 = p^2 c^2 \left[ 1 - \alpha \left( \frac{E}{E_p} \right)^n \right] + \left( mc^2 \right)^2. \quad (3)$$

Let us consider massive bosons and then the relation between energy and momentum becomes now:

$$p = \frac{1}{c} \sqrt{\frac{E^2 - m^2 c^4}{1 - \alpha \left( \frac{E}{E_p} \right)^n}}. \quad (4)$$

The number of microstates is given by [8]

$$\Sigma = \frac{s}{(2\pi \hbar)^3} \int \int \int \dd r \dd p. \quad (5)$$

In this last expression, $s$ is a weight factor arising from the internal structure of the particles, i.e., spin. If our gas is inside a container of volume $V$.

$$\Sigma = \frac{4\pi s V}{(2\pi \hbar)^3} \int \int p^2 \dd p. \quad (6)$$

And then we obtain the density of states per energy unit

$$\Omega(E) = \frac{4\pi s V}{(2\pi \hbar)^3} \left( \frac{E^2 - m^2 c^4}{1 - \alpha \left( \frac{E}{E_p} \right)^n} \right)^{\frac{1}{n}} \left\{ \frac{E + \alpha(n - 1)E^2 - nm^2 c^4 \left( E^{n-1}/E_p^n \right)}{[1 - \alpha \left( E/E_p \right)^n]^2} \right\}. \quad (7)$$

If in (7), we set $\alpha = 0$, $s = 2$ and $m = 0$, then we recover the density of states for photons [7, 8]

$$\Omega(E) = \frac{8\pi V}{(2\pi \hbar)^3} E^2. \quad (8)$$

Let us analyze the pressure of our gas. In the grand canonical ensemble for an ideal Bose-Einstein gas [7], we have

$$\frac{PV}{\kappa T} = - \sum_E \ln(1 - \lambda e^{-\beta E}). \quad (9)$$

Where $\lambda \equiv exp(\mu/\kappa T)$ is the so-called fugacity and $\kappa$ is Boltzmann’s constant [7].

With these assumptions we may obtain that the pressure is given by [6]

$$P = \frac{4\pi s}{(2\pi \hbar)^3} \left\{ \frac{1}{3} \int_0^\infty \frac{E^4}{\sqrt{E^2 + m^2 c^4}} \frac{dE}{\lambda^{-1} \exp[\sqrt{E^2 + m^2 c^4/\kappa T}] - 1} \right\}$$

$$+ \alpha(n + 3/2) \left( \frac{\kappa T}{T_p} \right)^4 \sum_{l=0}^n \frac{n!}{l!(n-l)!} \times \left( \frac{mc^2}{\kappa T} \right)^l \Gamma(n + 3 - l) g_{n+4-l} (\lambda \exp(-mc^2/\kappa T)). \quad (10)$$
Where \( T_p = E_p/\kappa \), denotes Planck’s temperature, \( g_\nu(x) \) are the Bose-Einstein functions \([7, 8]\), and \( \Gamma(x) \) are the Gamma functions.

It is important to mention that if \( \alpha > 0 \), then the pressure grows, with respect to the case in which Lorentz symmetry is present. This last remark allows us to interpret the breakdown of Lorentz symmetry for massive bosons as a repulsive interaction, if \( \alpha > 0 \). Indeed, the presence of a repulsive interaction (among the particles of a gas) entails the increase of the pressure, compared against the corresponding value for an ideal gas. It is in this sense that we say that the loss of symmetry appears, at the bulk level, as the emergence of a repulsive interaction (if \( \alpha > 0 \)). The quantum statistics can be contemplated as a pseudo–attractive interaction, in the sense that a series expansion in the parameter \( N \Lambda^3/V \) entails a pressure lower than that related to an ideal gas \([7]\). We have then the main ingredients associated to systems showing phase transitions.

The Bose-Einstein functions are defined by \([7, 8]\)

\[
g_\nu(\lambda) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} dx}{\lambda^{\nu} e^x - 1} = \lambda + \frac{\lambda^2}{2^\nu} + \frac{\lambda^3}{3^\nu} + ... \tag{11}\]

On the other hand \([7]\)

\[
g_{3/2}(\lambda) = \frac{\Lambda^3 N}{V} = \frac{\lambda^3}{v}. \tag{12}\]

As a first and roughly approximation:

\[
\frac{\lambda^2}{2^{3/2}} + \lambda - \frac{\Lambda^3 N}{V} = 0. \tag{13}\]

Then we have the fugacity in terms of the volume

\[
\lambda = -2^{1/2} + \sqrt{2 + 2^{3/2} \frac{\Lambda^3 N}{V}}. \tag{14}\]

Where \( \Lambda = h/(2\pi m \kappa T)^{1/2} \) is the mean thermal wavelength \([7]\). We take the case in which \( n = 2 \) then the pressure is given by

\[
P = \frac{4\pi s}{(2\pi \hbar)^3} \left\{ \lambda e^{-mc^2/\kappa T} \left[ \frac{1}{3} f(T) + \alpha \left( g(T) + h(T) \lambda e^{-mc^2/\kappa T} + j(T) \lambda^2 e^{-2mc^2/\kappa T} \right) \right] \right\}. \tag{15}\]

Where \( f(T), g(T), h(T) \) and \( j(T) \), are functions of the temperature, given by

\[
f(T) = 6(\kappa T)^4 + 6(m^2)(\kappa T)^3 + 3(m^2)^2(\kappa T)^2 + (m^2)^3(\kappa T). \tag{16}\]

\[
g(T) = \frac{7}{2} (\kappa T)^4 \left\{ \frac{T}{T_p} \right\}^2 \left\{ \Gamma(5) + 2\Gamma(4) \left( \frac{mc^2}{\kappa T} \right) + \Gamma(3) \left( \frac{mc^2}{\kappa T} \right)^2 \right\}. \tag{17}\]

\[
h(T) = \frac{7}{2} (\kappa T)^4 \left\{ \frac{T}{T_p} \right\}^2 \left\{ \frac{\Gamma(5)}{2^6} + \frac{\Gamma(4)}{2^4} \left( \frac{mc^2}{\kappa T} \right) + \frac{\Gamma(3)}{2^4} \left( \frac{mc^2}{\kappa T} \right)^2 \right\}. \tag{18}\]

\[
j(T) = \frac{7}{2} (\kappa T)^4 \left\{ \frac{T}{T_p} \right\}^2 \left\{ \frac{\Gamma(5)}{3^6} + \frac{2\Gamma(4)}{3^5} \left( \frac{mc^2}{\kappa T} \right) + \frac{\Gamma(3)}{3^4} \left( \frac{mc^2}{\kappa T} \right)^2 \right\}. \tag{19}\]

If we set \( \alpha = 0, m = 0, \) and \( \lambda = 1 \) we recover the expression for the pressure of a gas of photons \([7]\)

\[
P \sim (\kappa T)^4. \tag{20}\]
Let us expand the fugacity (14) in terms of \( \Lambda^3 N/V \) then, pressure is given by

\[
P = \frac{4\pi s}{(2\pi \hbar)^3} \left[ F(T) \left( \frac{\Lambda^3 N}{V} \right) + G(T) \left( \frac{\Lambda^3 N}{V} \right)^2 + H(T) \left( \frac{\Lambda^3 N}{V} \right)^3 + J(T) \left( \frac{\Lambda^3 N}{V} \right)^4 + \ldots \right]
\]  

(21)

Where:

\[
F(T) = \frac{1}{3} f(T)e^{-mc^2/\kappa T} + \alpha e^{-mc^2/\kappa T} g(T) + \ldots,
\]

(22)

\[
G(T) = -\frac{2^{3/2}}{3} f(T)e^{-mc^2/\kappa T} + \alpha \left( h(T)e^{-2mc^2/\kappa T} - 2^{3/2} g(T)e^{-mc^2/\kappa T} \right) + \ldots,
\]

(23)

\[
H(T) = \frac{2^{-2}}{3} f(T)e^{-mc^2/\kappa T}
+ \alpha \left( j(T)e^{-3mc^2/\kappa T} - 2^{5/2} h(T)e^{-2mc^2/\kappa T} + 2^{-2} g(T)e^{-mc^2/\kappa T} \right) + \ldots,
\]

(24)

\[
J(T) = \alpha \left( (-2^{3/2} + 2^{-1}) h(T)e^{-2mc^2/\kappa T} - (2^{5/2} + 2^{3/2}) j(T)e^{-3mc^2/\kappa T} \right) + \ldots
\]

(25)

At the critical points the first and the second derivatives for the pressure respect to the volume must vanish

\[
\left( \frac{\partial P}{\partial V} \right)_T = 0 \quad \text{and} \quad \left( \frac{\partial^2 P}{\partial V^2} \right)_T = 0.
\]

(26)

Using this fact we can obtain an expression for the critical volume in terms of the temperature and our additional parameter \( \alpha \)

\[
\frac{V_c}{N} = \nu_e = \frac{1}{2} \Lambda^3 \left\{ -3 \frac{H(T)}{G(T)} + \sqrt{9 \left( \frac{H(T)}{G(T)} \right)^2 - 24 \left( \frac{J(T)}{G(T)} \right) \Theta(T)} \right\}.
\]

(27)

Let us analyze the expression (27), fixing \( \alpha = 0 \). If we introduce this condition, the critical volume goes to zero, and the critical pressure goes to infinite. In other words, we recover the ideal behavior \( \text{(10)} \). This fact allows us to interpret the parameter \( \alpha \) as the intensity of a repulsive interaction among the particles of our bosonic gas. Let us remark that this interpretation is related to the quantum structure of spacetime, and not properly with a real interaction among the particles of the system.

Let us analyze the critical volume with the experimental values, \( T = 50 \times 10^{-9} \) K and the mass of Rb\(_{87} \) \( \text{(11)} \), then (27) becomes

\[
\nu_e = \frac{V_c}{N} \approx \alpha 10^{-57} m^3.
\]

(28)

Due to the limitations in the calculation of the critical temperature, in terms of the parameter \( \alpha \), let us impose a limit on the experimental parameters. We now analyze the behavior of the critical volume in the region where the quantum effects are predominant. In other words, we analyze the behavior of the critical volume when \( T \to 0 \). The critical volume, when \( T \to 0 \), is given by

\[
\nu_e = \frac{V_c}{N} = \frac{1}{2} \Lambda^3 \left\{ 3\eta - \sqrt{9\eta^2 - 24\alpha|\Theta(T)|} \right\}.
\]

(29)

Where

\[
\Theta(T) = \frac{J(T)}{G(T)} = \frac{(-2^{3/2} + 2^{-1}) h(T)e^{-2mc^2/\kappa T} - 2^{5/2} j(T)e^{-3mc^2/\kappa T}}{-2^{3/2} f(T)} > 0.
\]

(30)

If we set in (29) \( \alpha = 0 \) we recover the ideal behavior.
III. CONCLUSIONS

Accepting the breakdown of Lorentz symmetry as a fundamental fact for the statistics of a massive boson gas, we observe that the pressure of a massive boson gas grows respect to the usual case, if we assume that $\alpha > 0$. This fact allows us interpret the breakdown of Lorentz symmetry as a repulsive pseudo-interaction. It has been proved that under these conditions phase transitions shall emerge. Unfortunately, the critical parameters are very complicated functions of the temperature, and therefore we do not obtain an analytic expression for the critical temperature and the critical pressure in terms of the parameter $\alpha$. In other words, some quantum gravity models predict the emerge of critical points, due to the breakdown of Lorentz symmetry in the form of a modified dispersion relation, even for systems in which no interaction among the particles exist.

Clearly, a realistic system does include interaction among its particles, therefore the present model shall be improved introducing this fact. Additionally, according to the laws of thermodynamics, the critical exponents, related to phase transitions, are not independent from one another\cite{7, 9, 10}. Therefore we may wonder, in this context, if the corresponding exponents arising from this scheme satisfy this condition, and what kind of information, concerning quantum gravity, could be elicited from these exponents.

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