FIBER FUNCTORS ON TEMPERLEY-LIEB CATEGORIES

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Abstract. Fiber functors on Temperley-Lieb categories are determined with the help of classification results on non-degenerate bilinear forms. The case of unitary fiber functors is also investigated.

1. Introduction

Tensor categories are often realized as categories of finite-dimensional comodules of Hopf algebras, which can be seen in other ways as giving a faithful exact monoidal functor from an abelian tensor category into that of finite-dimensional vector spaces. The converse implication holds if one imposes rigidity on the tensor category in question; given a faithful exact monoidal functor $F$ from a rigid abelian tensor category $T$ into the tensor category of finite-dimensional vector spaces, there exists a Hopf algebra $A$ such that $T$ is monoidally equivalent to the tensor category of finite-dimensional $A$-comodules with the functor identified with the one forgetting $A$-coactions ([11, 12]).

An exact monoidal functor from an abelian tensor category into the tensor category $\mathcal{V}ec$ of finite-dimensional vector spaces is usually called a fiber functor.

Tensor categories we shall work with are the so-called Temperley-Lieb categories $\mathcal{K}_d$ ($d \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$): $\mathcal{K}_d$ is a rigid tensor category associated with planar tangles, whose objects are parametrized natural numbers $n = 0, 1, 2, \ldots$ and $\text{End}(n) = \text{Hom}(n, n)$ is in an obvious way identified with the $n$-th Temperley-Lieb algebra (generated by unit and elements $h_1, \ldots, h_{n-1}$ satisfying $h_j^2 = dh_j$, $h_i h_j = h_j h_i$ ($|i-j| \geq 2$) and $h_i h_j h_i = h_i$ ($|i-j| = 1$)). The Temperley-Lieb category $\mathcal{K}_d$ admits a compatible $\text{C}^*$-structure if and only if $|d| \geq 2$ with $d \in \mathbb{R}$ or $d = \pm 2 \cos(\pi/l)$ with $l = 3, 4, \ldots$. In that case, the compatible $\text{C}^*$-structure is unique up to $\text{C}^*$-monoidal equivalences (see, for example, [19] and references therein).

Although the Temperley-Lieb category $\mathcal{K}_d$ is never abelian, the idempotent-completion of $\mathcal{K}_d$ after adding finite-direct sums (first add direct sums and then add subobjects) turns out to be semisimple and therefore abelian unless $d = q + q^{-1}$ with $q^2$ a non-trivial root of unity. Moreover, any monoidal functor $F : \mathcal{K}_d \to \mathcal{V}ec$ is uniquely extended to this abelianized Temperley-Lieb category, which is automatically exact by semisimplicity.

With these backgrounds in mind, we shall refer to a monoidal functor $\mathcal{K}_d \to \mathcal{V}ec$ simply as a fiber functor in what follows. When $\mathcal{K}_d$ is a $\text{C}^*$-tensor category, we can naturally talk about the unitarity of fiber functors; a monoidal functor
F : K\textsubscript{d} → Hilb (Hilb denotes the C*-tensor category of finite-dimensional Hilbert spaces) is called a **unitary fiber functor** if F preserves the *-operation. For a unitary fiber functor, the reconstructed Hopf algebra is naturally regarded as defining a compact quantum group \( G \) with its representation category (the Tannaka dual of \( G \)) monoidally equivalent to the C*-tensor category \( K\textsubscript{d} \).

Our main concern here is how fiber functors are determined on Temperley-Lieb categories and we shall present a classification result with the help of similar works on non-degenerate bilinear forms (see [13] and references therein). By Saavedra Rivano-Ulbrich's theorem on Tannaka-Krein duality, this produces mutually non-isomorphic Hopf algebras with the isomorphic representation categories as far as \( d \) is generic in the sense that \( K\textsubscript{d} \) is semisimple. Since the relevant Hopf algebras turn out to be the ones introduced by M. Dubois-Violette and G. Launer ([4]), we particularly obtain an isomorphism classification of Hopf algebras in that class.

When \( |d| \geq 2 \) with \( d \in \mathbb{R} \) and fiber functors are restricted to be unitary, we also give their complete descriptions, resulting in a classification of compact quantum groups whose Tannaka duals are isomorphic to \( K\textsubscript{d} \) as monoidal categories. In this case, the relevant Hopf algebras are identified with the free orthogonal quantum groups investigated by T. Banica and S.Z. Wang ([1], [3], [14]), and the present analysis enables us to get access to some of their results from the viewpoint of tensor categories.

### 2. Fiber Functors

Let \( F : K\textsubscript{d} → \mathcal{V}ec \) be a fiber functor. By definition, it consists of a linear functor \( F \) together with a natural family of isomorphisms \( \mu_{m,n} : F(X^m) \otimes F(X^n) → F(X^{m+n}) \) which makes the following diagram commutative

\[
\begin{array}{ccc}
F(X^l) \otimes F(X^m) \otimes F(X^n) & \xrightarrow{1 \otimes \mu_{m,n}} & F(X^l) \otimes F(X^{m+n}) \\
\downarrow \mu_{l,m} \otimes 1 & & \downarrow \mu_{l,m+n} \\
F(X^{l+m}) \otimes F(X^n) & \xrightarrow{\mu_{l+m,n}} & F(X^{l+m+n})
\end{array}
\]

Then isomorphisms \( F(X^n) → F(X)^{\otimes n} \) obtained as repetitions of \( \mu \)'s are identical and define a single isomorphism (the coherence theorem), which satisfies the commutativity of the diagram

\[
\begin{array}{ccc}
F(X^m) \otimes F(X^n) & \xrightarrow{\mu_{m,n}} & F(X^{m+n}) \\
\downarrow & & \downarrow \\
F(X)^{\otimes m} \otimes F(X)^{\otimes n} & \xrightarrow{1^{\otimes m+n}} & F(X)^{\otimes (m+n)}
\end{array}
\]

In other words, the functor \( F \) is monoidally equivalent to strict one. Then we see that the functor \( F \) in its strict version is specified by the choice of a bilinear form \( F(\epsilon) : F(X) \otimes F(X) → \mathbb{C} = F(I) \), which is non-degenerate by rigidity (cf. [19] Lemma 6.1], tensor categories being assumed strict without qualifications there).

Conversely, given a non-degenerate bilinear form \( \mathcal{E} : V \otimes V → \mathbb{C} \) on a finite-dimensional vector space \( V \), we can determine the copairing \( \mathcal{D} : \mathbb{C} → V \otimes V \) by the rigidity identity and, if the dimension relation \( \mathcal{E} \mathcal{D} = d \) is satisfied furthermore, we can recover the strict monoidal functor \( F : K\textsubscript{d} → \mathcal{V}ec \) so that \( F(X) = V \) and \( F(\epsilon) = \mathcal{E} \).
Two fibre functors $F$, $G : K_d \to Vec$ are said to be (monoidally) equivalent if there is a natural equivalence \{\( t_n : F(X^n) \to G(X^n) \)\} fulfilling the commutativity of the diagram

\[
\begin{array}{ccc}
F(X^m) \otimes F(X^n) & \xrightarrow{\mu_F} & F(X^{m+n}) \\
\downarrow t_m \otimes t_n & & \downarrow t_{m+n}.
\end{array}
\]

By using the natural isomorphisms $F(X^n) \to F(X)^{\otimes n}$ and $G(X^n) \to G(X)^{\otimes n}$ explained above, the natural equivalence \{\( t_n \)\} is uniquely determined by the initial isomorphism $t_1 : F(X) \to G(X)$ which makes the following diagram commutative.

\[
\begin{array}{ccc}
F(X) \otimes F(X) & \xrightarrow{t_1 \otimes t_1} & G(X) \otimes G(X) \\
\downarrow F(\epsilon) & & \downarrow G(\epsilon) \\
\mathbb{C} & & \mathbb{C}
\end{array}
\]

where $F(I) = \mathbb{C} = G(I)$.

Conversely, starting with a linear isomorphism $t_1 : F(X) \to G(X)$ making the above diagram commutative, the choice $t_n = (t_1)^{\otimes n}$ gives a monoidal equivalence between $F$ and $G$.

Thus, if we take the canonical column vector spaces $\mathbb{C}^N$ with the canonical basis \{\( e_i \)\} as representatives of finite-dimensional vector spaces and describe a non-degenerate bilinear form $\mathcal{E}$ on $\mathbb{C}^N$ by the associated invertible matrix $E \in GL_N(\mathbb{C})$

\[
\mathcal{E}(e_i \otimes e_j) = E_{i,j},
\]

then the accompanied copairing $D = D(1) \in \mathbb{C}^N \otimes \mathbb{C}^N$ is given by

\[
D = \sum_{i,j} D_{i,j} e_i \otimes e_j,
\]

with the matrix $D = \{D_{i,j}\}$ characterized as the inverse of $E$. The dimension condition $d = \mathcal{E}D$ then takes the form

\[
d = \text{tr}(tEE^{-1}).
\]

Let $M_N(d)$ be the set of such matrices in $GL_N(\mathbb{C})$. Since $tTET$ belongs to $M_N(d)$ whenever $E \in M_N(d)$ and $T \in GL_N(\mathbb{C})$, fiber functors are completely classified up to monoidal equivalences by the orbit space $M_N(d)/GL_N(\mathbb{C})$, where the right action of $GL_N(\mathbb{C})$ on the set $M_N(d)$ is defined by

\[
E \mapsto tTET, \quad T \in GL_N(\mathbb{C}), \quad E \in M_N(d).
\]

This kind of classification problem on transposed similarity has been investigated in various contexts and we know a complete structural analysis ([13, 11, 5]). To describe the relevant results, we introduce some notations.

Let $\Theta : GL_N(\mathbb{C}) \to GL_N(\mathbb{C})$ be the map defined by $\Theta(E) = (E^{-1})(tE)$ for $E \in GL_N(\mathbb{C})$. Then the relation $\Theta(tTET) = T^{-1}\Theta(E)T$ reveals that $\Theta$ is equivariant with respect to the transposed similarity action on the domain and the ordinary similarity action on the range. The similarity class of an matrix $M$ is in turn completely described by the multiplicity sequence $\mu_M(z) = (\mu_M^{(1)}(z), \mu_M^{(2)}(z), \ldots)$ with $z \in \mathbb{C}$, where $\mu_M^{(k)}(z)$ denotes the multiplicity of $z$-Jordan block of size $k$ in $M$. 
Thus $\mu_M(z) = 0$ unless $z$ is an eigenvalue of $M$ and the sequence $\mu_M(z)$ vanishes after the $N$-th component.

**Theorem 2.1** ([13], cf. also [10, 5]).

(i) Two invertible matrices $E$ and $E'$ are equivalent relative to the transposed similarity if and only if $\Theta(E)$ and $\Theta(E')$ are conjugate in $GL_N(\mathbb{C})$, i.e., we can find $T \in GL_N(\mathbb{C})$ such that $\Theta(E') = T^{-1}\Theta(E)T$.

(ii) An invertible matrix $M \in GL_N(\mathbb{C})$ belongs to the image of $\Theta$ if and only if (i) $\mu_M(z) = \mu_M(z^{-1})$ for $z \in \mathbb{C}^\times$, (ii) the multiplicity $\mu_M^{(k)}(1)$ is even for even $k$ and (iii) the multiplicity $\mu_M^{(k)}(-1)$ is even for odd $k$.

For a multiplicity sequence $\mu = (\mu^{(1)}, \mu^{(2)}, \ldots)$, define its total multiplicity $|\mu|$ by

$$|\mu| = \sum_{k \geq 1} k \mu^{(k)}.$$ 

Note that the function $|\mu_M(z)|$ is in one-to-one correspondence with the characteristic polynomial

$$\det(\lambda I - M) = \prod_{z}(\lambda - z)^{|\mu_M(z)|}.$$ 

In particular, the trace $\text{tr}(\Theta(E))$ of $\Theta(E)$ is an invariant of the transposed similarity of $E$.

If we combine this result with our previous arguments, the following is obtained.

**Theorem 2.2.** There is a one-to-one correspondence between isomorphism classes of fiber functors on the Temperley-Lieb category $Kd$ and multiplicity functions $\mu(z) = (\mu^{(1)}(z), \mu^{(2)}(z), \ldots)$ for $z \in \mathbb{C}^\times$ satisfying the conditions (i)-(iii) in the above theorem and

$$d = \sum_{z \in \mathbb{C}^\times} |\mu(z)| \cdot z.$$ 

If the multiplicity function $\mu(z)$ meets the condition that $\mu^{(k)}(\pm 1)$ is even except for $\mu^{(1)}(1)$, then we can describe a matrix $E$ in the following simple way: let $\{q_i\}$ be a half of the spectral set $\sigma = \{ z \in \mathbb{C}^\times; \mu(z) \neq 0 \}$ satisfying $\sigma \setminus \{ \pm 1 \} = \{ q_i, q_i^{-1} \}$. Choose an invertible (square) matrix $Q$ so that

$$m_Q(z) = \begin{cases} 
\mu(q_i) & \text{if } z = q_i \text{ for some } i, \\
\mu(-1)/2 & \text{if } z = -1, \\
0 & \text{otherwise}
\end{cases}$$

and set

$$E = \begin{pmatrix}
0 & I & 0 \\
Q^{-1} & 0 & 0 \\
0 & 0 & I
\end{pmatrix}.$$ 

Then it is immediate to see that the multiplicity function of

$$\Theta(E) = \begin{pmatrix}
Q & 0 & 0 \\
0 & tQ^{-1} & 0 \\
0 & 0 & I
\end{pmatrix}$$

is given by the initial function $\mu(z)$. 
Example 2.3. The generic orbits in $GL_2(\mathbb{C})$ are represented by

$$E_q = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} \quad \text{with} \quad \Theta(E_q) = -\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

for $q \in \mathbb{C}^\times$.

The stabilizer at $E_q$ is given by

$$\left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} ; t \in \mathbb{C}^\times \right\} \cong \mathbb{C}^\times$$

for $q \neq \pm 1$,

$$\left\{ \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} , \begin{pmatrix} 0 & t \\ t^{-1} & 0 \end{pmatrix} ; s, t \in \mathbb{C}^\times \right\} \cong \mathbb{C}^\times \times \mathbb{Z}_2$$

for $q = -1$ and $SL_2(\mathbb{C})$ for $q = 1$.

There remains one more orbit, which is represented by

$$E = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{with} \quad \Theta(E) = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} ,$$

where the stabilizer is given by

$$\left\{ \begin{pmatrix} \pm 1 & z \\ 0 & \pm 1 \end{pmatrix} ; z \in \mathbb{C} \right\} \cong \mathbb{C} \times \mathbb{Z}_2 .$$

Thus there are unique fiber functors when $d \neq -2$ while we have two non-isomorphic ones for the quantum dimension $d = -2$.

By utilizing these explicit descriptions, we can write down the associated Hopf algebra $A$ as well. If we rename the generator $\{a_{ij}\}_{1 \leq i,j \leq 2}$ of the associated Hopf algebra $A$ by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and apply the description in the appendix, then the quantum group $SL_q(2, \mathbb{C})$ is recovered (see [6] for more information on $SL_q(2, \mathbb{C})$): $A$ is the unital algebra $A_q$ generated by $a, b, c$ and $d$ with the relations

$$ab = q^{-1}ba, \quad ac = q^{-1}ca, \quad bd = q^{-1}db, \quad cd = q^{-1}dc,$$

$$bc = cb, \quad ad - q^{-1}bc = da - qbc = 1$$

and the coproduct is given by

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

In view of these commutation relations, we notice that the (polynomial) function algebra on the matrix group $SL(2, \mathbb{C})$ comes out from the orbit specified by $q = 1$.

Next consider the singular orbit, which is represented by matrices

$$\begin{pmatrix} \tau & 1 \\ -1 & 0 \end{pmatrix} , \quad 0 \neq \tau \in \mathbb{C} .$$

Although they belong to the same orbit, this redundancy will be helpful in identifying the associated Hopf algebra $A$. Keeping the notation for generators of $A$,
the Hopf algebra $A$ turns out to be the quantum group dealt with in [17]; $A$ is the unital algebra generated by $a$, $b$, $c$ and $d$ with the relations
\[
\begin{align*}
\tau a^2 - ba + ab &= \tau 1, \\
\tau d^2 - bd + db &= \tau 1, \\
\tau ac - bc + ad &= 1, \\
\tau dc - bc + da &= 1, \\
\tau ca - da + cb &= -1, \\
\tau cd - ac + ca &= -1, \\
\tau c^2 - dc + cd &= 0, \\
\tau c^2 - ac + ca &= 0
\end{align*}
\]
with the coproduct given by the same formula as above. In particular, we see that all these Hopf algebras are isomorphic irrelevant to the choice of $0 \neq \tau \in \mathbb{C}$.

3. UNITARY FIBER FUNCTORS

Let $\pm d \geq 2$ and furnish the Temperley-Lieb category $\mathcal{K}_d$ with a canonical $C^*$-structure so that it is a $C^*$-tensor category (see [19, Proposition 6.2]).

In general, given a $C^*$-tensor category $\mathcal{T}$, a monoidal functor $F$ from $\mathcal{T}$ into the $C^*$-tensor category $\mathbf{Hilb}$ of finite-dimensional Hilbert spaces is called a unitary fiber functor if the multiplicativity isomorphisms $m_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ are unitary and the linear maps $F : \text{Hom}(X,Y) \to \text{Hom}(F(X), F(Y))$ on hom-spaces are $*$-preserving.

Two unitary fiber functors $F, G : \mathcal{T} \to \mathbf{Hilb}$ are said to be unitarily equivalent if there is a monoidal natural equivalence \{ $\varphi_X : F(X) \to G(X)$ \} given by unitary maps.

For the $C^*$-tensor category $\mathcal{K}_d$, giving a unitary fiber functor is equivalent to specifying a finite-dimensional Hilbert space $V$ and linear maps $F(\epsilon) : V \otimes V \to \mathbb{C}$ and $F(\delta) : \mathbb{C} \to V \otimes V$ satisfying the rigidity relations and the unitarity condition $F(\epsilon)^* = (d/|d|)F(\delta)$.

Recall here some categorical operations on Hilbert spaces. Given a Hilbert space $V$, denote by $V^*$ the conjugate Hilbert space of $V$. By the ‘self-duality’ of Hilbert spaces, $V^*$ is identified with the dual Hilbert space of $V$. As a notation, we use $\overline{\cdot}$ to stand for the associated vector in $V^*$ so that $\overline{v}$ defines a linear functional by $\langle \overline{v}, w \rangle = (v|w)$ for $v, w \in V$ (the inner product $(v|w)$ being linear in $w$ by our convention). Given a (bounded) linear map $f : V \to W$ between Hilbert spaces, we denote by $\overline{f} : V^* \to W^*$ the linear map defined by $\overline{f}(\overline{v}) = \overline{f(v)}$, which is referred to as the conjugation of $f$. The operation of conjugation commutes with that of taking hermitian conjugate. The transposed operation is then defined by $^t f = (\overline{f})^* = \overline{(\overline{f})}$, which is a linear map $W \to V$. Thus, the three operations $^t f$, $f^*$ and $\overline{f}$ are mutually commutative. Moreover, for invertible linear maps, these operations preserve inverses.

Now, given a non-degenerate bilinear form $E : V \otimes V \to \mathbb{C}$, define a linear map $\Phi : V \to V^*$ by
$E(v \otimes v') = (\overline{v} | \Phi v')$.

Let $\mathcal{D} : \mathbb{C} \to V \otimes V$ be the associated coparing. If we identify $\mathcal{D}$ with a vector $\mathcal{D}(1)$ in $V \otimes V$, then an expression
$$\mathcal{D} = \sum_j v_j \otimes \delta_j$$
with \( \{ v_j \} \) an orthonormal basis in \( V \) satisfies the rigidity relation if and only if 
\[
\delta_j = t^* \Phi^{-1} v_j,
\]
i.e.,
\[
D = \sum_j v_j \otimes t^* \Phi^{-1} v_j.
\]
As a benefit of this expression, we have
\[
d = \mathcal{E} \circ D = \sum_j (v_j^* | \Phi^* \Phi^{-1} v_j) = \text{tr}(\Phi^* \Phi^{-1}).
\]

The unitarity condition \( \mathcal{E}^* = (d/|d|) D \) is then equivalent to
\[
(\mathcal{E}^* \Phi w) = \mathcal{E}(v \otimes w) = (\mathcal{E}^* | v \otimes w) = \frac{d}{|d|} (D | v \otimes w) = \frac{d}{|d|} \sum_j (v_j \otimes t^* \Phi^{-1} v_j | v \otimes w)
\]
\[
= \frac{d}{|d|} \sum_j (v_j^* | \Phi t^* v_j) | \Phi^* \Phi^{-1} v_j^* | v \otimes w) = \frac{d}{|d|} (v_j^* | \Phi t^* v_j) | (\Phi^* \Phi^{-1})^* w),
\]
i.e., the invertible map \( \Phi : V \to \overline{V} \) should satisfy the relation
\[
\Phi^{-1} = \frac{d}{|d|} \overline{\Phi},
\]
which is equivalent to \( \Phi^{-1} = (d/|d|) \overline{\Phi^*} \). Note here that this implies the relation
\[
\text{tr}(\Phi \Phi^*) = \frac{|d|}{d} \text{tr}(\Phi^* \Phi^{-1}) = |d|.
\]
Conversely, starting with an invertible linear map \( \Phi \) satisfying \( \Phi^{-1} = (d/|d|) \overline{\Phi} \) and \( \text{tr}(\Phi^* \Phi) = |d| \), we can recover a unitary fiber functor.

To rephrase the monoidal equivalence of fibre functors, consider an isomorphism of vector spaces \( T : V \to W \). Given a rigidity pair \( F(\epsilon) : V \otimes V \to \mathbb{C} \) and \( F(\delta) : \mathbb{C} \to V \otimes V \), the composite maps
\[
F(\epsilon)(T^{-1} \otimes T^{-1}) : W \otimes W \to \mathbb{C}, \quad (T \otimes T)F(\delta) : \mathbb{C} \to W \otimes W
\]
satisfy the rigidity relation. Hence, given another rigidity pair \( G(\epsilon) : W \otimes W \to \mathbb{C} \) and \( G(\delta) : \mathbb{C} \to W \otimes W \), \( G(\epsilon) = F(\epsilon)(T^{-1} \otimes T^{-1}) \) if and only if \( G(\delta) = (T \otimes T)F(\delta) \).

In terms of the associated linear maps \( \Phi : V \to \overline{V} \) and \( \Psi : W \to \overline{W} \), the condition is further equivalent to requiring \( \Phi = t^* T \Psi T \).

Since
\[
\Psi^{-1} = T \Phi^{-1} t^* T, \quad \overline{\Phi} = (T^*)^{-1} \overline{\Phi} \overline{T}^{-1},
\]
the unitarity of \( \Psi \) follows from that of \( \Phi \) if \( T \) is a unitary.

To conclude the discussions so far, we introduce the following notation: Given a finite-dimensional Hilbert space \( V \) and \( d \in \mathbb{R}^\times = \mathbb{R} \setminus \{0\} \), set
\[
\Gamma_d(V) = \left\{ \Phi : V \to \overline{V} ; \Phi^{-1} = \frac{d}{|d|} \overline{\Phi} \right\}.
\]
Lemma 3.1. Let $\Phi \in \Gamma_d(V)$ and $\Psi \in \Gamma_d(W)$ with $V$ and $W$ finite-dimensional Hilbert spaces. Then these give rise to unitarily equivalent unitary fiber functors if and only if we can find a unitary map $T : V \to W$ satisfying $\Phi = {}^t T \Psi T$.

Conversely, given $\Phi \in \Gamma_d(V)$ and a unitary map $T : V \to W$, the linear map $\Psi = {}^t T^{-1} \Phi T^{-1} : W \to W$ belongs to the set $\Gamma_d(W)$.

Thus, for each dimensionality of $V$ with $\dim V = n$, unitary fiber functors are classified up to unitary equivalence by the orbit space

$$
\Gamma_d(V)/U(V),
$$

where the unitary group $U(V)$ of $V$ acts on the set $\Gamma_d(V)$ by $\Phi T = {}^t T \Phi T$ (a right action).

We shall now rephrase the above orbit space by spectral data of $\Phi$. Let $\Phi \in \Gamma_d(V)$ and $\Phi = U|\Phi|$ be its polar decomposition with $U : V \to V$ unitary and $|\Phi| : V \to V$ positive. By taking inverses and substituting $\Phi^{-1}$ with $\Phi^{-1}$, we have

$$
U^{-1}U|\Phi|^{-1}U^{-1} = \pm \overline{|\Phi|}
$$

and then, by the uniqueness of polar decompositions,

$$
U^{-1} = \pm \overline{U}, \quad U|\Phi|^{-1}U^{-1} = \overline{|\Phi|}.
$$

To utilize these identities, we introduce an antiunitary operator $C : V \to V$ by

$$
Cv = \overline{Uv} = \overline{\Phi} v, \quad v \in V,
$$

which satisfies

$$
C^2 v = \overline{U \overline{U} v} = \frac{d}{|d|} v,
$$

i.e., $C^2 = \pm 1_V$. The equality $U|\Phi|^{-1}U^{-1} = \overline{|\Phi|}$ is then equivalent to the commutation relation

$$
|\Phi| C = C |\Phi|^{-1}
$$

because of

$$
|\Phi| C v = |\Phi| \overline{U v} = \overline{|\Phi| U v} = \overline{U |\Phi|^{-1} v} = C |\Phi|^{-1} v.
$$

Thus, the spectral structure of the positive operator $|\Phi|$ bears the symmetry of taking inverses on values under the operation of $C$; if $|\Phi| v = hv$ with $h > 0$, then

$$
|\Phi| C v = C |\Phi|^{-1} v = h^{-1} C v.
$$

In other words, $CV_h = V_h^{-1}$ if we denote the spectral subspace by $V_h = \{ v \in V ; |\Phi| v = hv \}$. In particular, the antiunitary $C$ leaves $V_1$ invariant and here a dichotomy is in order according to $C^2 = \pm 1_V$.

For $C^2 = 1_V$, i.e., $d > 0$, $C$ gives a real structure on $V$ and on the invariant subspace $V_1$ by restriction. Thus we can find an orthonormal basis $\{ w_k \}$ of $V_1$ which is real in the sense that $Cw_k = w_k$.

For the case $C^2 = -1_V$ ($d < 0$), the anti-unitarity of $C$ is used to see

$$
(v|Cv) = (C^2 v|Cv) = -(v|Cv),
$$

i.e., $v \perp C v$ for any $v \in V$. Hence we can find an orthonormal system $\{ w_k \}$ in $V_1$ such that $\{ w_k, Cw_k \}$ constitutes an orthonormal basis of $V_1$. Note that this occurs only when $V_1$ is even-dimensional.

In both cases, the multiplicity information of the spectrum of $|\Phi|$ completely determines the operator $\Phi$ up to unitary equivalence:
Theorem 3.2. The orbit space $\Gamma_d(V)/U(V)$ is completely parametrized by the eigenvalue list $\{h_j\}$ of $|\Phi|$ (including the multiplicity but irrelevant to the order), satisfying $\{h_j^{-1}\} = \{h_j\}$,  
\[ \sum_j h_j^2 = |d| \]
and $(d/|d|)^m = 1$, where $m$ denotes the multiplicity of an eigenvalue 1 ($m = \dim \ker(|\Phi| - 1_V)$).

Corollary 3.3. The set $\Gamma_d(V)$ is empty unless $(d/|d|)^{\dim V} = 1$.

Example 3.4. For a two-dimensional Hilbert space $V$ with an orthonormal basis $\{v_1, v_2\}$, the orbit space $\Gamma_d(V)/U(V)$ consists of one-point for each $\pm d \geq 2$. More concretely, with the expression $d = \pm(h^2 + h^{-2})$ ($h \geq 1$), the orbit is represented by a linear map $\Phi : V \to \mathcal{V}$, where  
\[ \Phi(v_1) = h\overline{v_2}, \quad \Phi(v_2) = \pm h^{-1}\overline{v_1}. \]

In other words, a unitary fiber functor $F : K_d \to \text{Hilb}$ such that $\dim F(X) = 2$ is unique up to unitary equivalence and realized by the choice  
\[ F(\epsilon)(v_1 \otimes v_2) = \pm h^{-1}, \quad F(\epsilon)(v_2 \otimes v_1) = h, \]
\[ F(\epsilon)(v_1 \otimes v_1) = F(\epsilon)(v_2 \otimes v_2) = 0. \]

With the notation in Example 2.3, the Hopf algebra is identified with $A_q$ for the choice $q = \mp h^{-2}$. The associated $*$-structure is then computed by the procedure in the appendix; let $\xi = (\xi_1, \xi_2)$ be the frame given by the orthonormal basis $\{v_1, v_2\}$ and $\eta = (\eta_1, \eta_2)$ be the dual frame relative to $F(\epsilon)$. Then we have  
\[ (\xi_1, \xi_2) = (\eta_1, \eta_2)T \quad \text{with} \quad T = \begin{pmatrix} 0 & h \\ \pm h^{-1} & 0 \end{pmatrix} \]
and then  
\[ a^\eta = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = Ta^\xi T^{-1} = \begin{pmatrix} 0 & h \\ \pm h^{-1} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & h \\ \pm h^{-1} & 0 \end{pmatrix}^{-1}, \]
i.e.,  
\[ a^* = d, \quad b^* = -q^{-1}c, \]
which coincides with the choice in [15].

Appendix A.

We shall here review some points in Tannaka-Krein duality on Hopf algebras following the basic references [11, 12] (cf. [2, §5.1] also for a concise description and [7] for a further generalization) and present the reconstruction part in an informal way, which will be helpful in understanding the structure of relevant Hopf algebras without going into deep formalities of category theory.
A.1. Saavedra Rivano-Ulbich’s Theorem. Let \( \mathcal{C} \) be a \( \mathbb{C} \)-linear category and consider a faithful linear functor \( F : \mathcal{C} \to \text{Vec} \), where \( \text{Vec} \) is the tensor category of finite-dimensional \( \mathbb{C} \)-vector spaces. By imbedding \( \text{Vec} \) into the tensor category \( \text{Vec} \) of (not necessarily finite-dimensional) \( \mathbb{C} \)-vector spaces, we regard \( F \) as a functor from \( \mathcal{C} \) into \( \text{Vec} \).

If we assume that the functor \( F \) is exact, then we can define a \( \mathbb{C} \)-coalgebra \( A \) as a solution of the universality problem for natural transformations \( F \to F \otimes B \), where \( F \otimes B \) for an object \( B \) in \( \text{Vec} \) is the functor specified by \( (F \otimes B)(X) = F(X) \otimes B \). We shall now present a naive (and less formal) approach to the results described above. Start with an essentially (F)-monoidal, then \( A \) is a Hopf algebra with the product \( m : A \otimes A \to A \) specified by the commutativity of the diagram

\[
\begin{array}{ccc}
F(X) & \xrightarrow{\rho_X} & F(X) \otimes A \\
\downarrow_{\rho_X} & & \downarrow_{1_{F(X)} \otimes \phi} \\
F(X) \otimes A & \xrightarrow{\rho_X \otimes 1_A} & F(X) \otimes A \otimes A \\
\end{array}
\]

\[
\begin{array}{ccc}
F(X) & \xrightarrow{\rho_X} & F(X) \otimes A \\
\downarrow_{\rho_X} & & \downarrow_{1_{F(X)} \otimes \epsilon} \\
F(X) \otimes A & \xrightarrow{\rho_X \otimes 1_A} & F(X) \otimes A \otimes A \\
\end{array}
\]

and the unit \( 1_A \in A \) given by the relation \( \rho_I(1) = 1 \otimes 1_A \) (recall that \( F(I) = \mathbb{C} \) and \( \rho_I : \mathbb{C} \to \mathbb{C} \otimes A \)),

The Ulbrich’s theorem states that the constructed bialgebra \( A \) is a Hopf algebra if and only if the tensor category \( \mathcal{C} \) is rigid.

A.2. From Tensor Categories to Hopf Algebras. We shall now present a naive (and less formal) approach to the results described above. Start with an essentially small linear category \( \mathcal{C} \) and a faithful linear functor \( F : \mathcal{C} \to \text{Vec} \) again. Given an object \( X \) of \( \mathcal{C} \), we denote a linear basis for the vector space \( F(X) \) by the corresponding greek letter such as \( \xi \), which is referred to as a frame. We also use the notation \( |\xi| \) to stand for the object \( X \). Let \( \xi = (\xi_1, \ldots, \xi_m) \) and \( \eta = (\eta_1, \ldots, \eta_n) \) be frames with \( X = |\xi| \) and \( Y = |\eta| \). For a linear map \( T : F(X) \to F(Y) \), the matrix
(t_{ji}) is denoted by \([\eta|T|\xi]\), where

\[ T(\xi_i) = \sum_{j=1}^{n} t_{ji} \eta_j, \quad \text{i.e.,} \quad T\xi = \eta [\eta|T|\xi]. \tag{1} \]

Given a frame \(\xi = (\xi_1, \ldots, \xi_m)\), we introduce symbols \(a_{ij}^\xi\) and let \(A\) be the vector space spanned by all these symbols with the relations (called the covariance condition)

\[ \sum_{l=1}^{n} t_{li} a_{kl}^\eta = \sum_{j=1}^{m} t_{kj} a_{ji}^\xi, \quad 1 \leq i \leq m, \quad 1 \leq k \leq n \tag{2} \]

for various \(X, Y\) and \(T \in \text{Hom}(X, Y)\). Note here that a basis-change \(\sum_{j} t_{ji} \xi_j' = \xi_i\) induces the relation

\[ \sum_{j} a_{ij}^\xi t_{jk} = \sum_{j} t_{ij} a_{jk} \tag{3} \]

since \([\xi'|1_X|\xi] = (t_{ij}).

More precisely, we consider a free vector space generated by symbols \(a_{ij}^\xi\) and \(A\) is defined to be the quotient space by the linear subspace spanned by (we have restricted ourselves to essentially small categories to avoid set-theoretical difficulties)

\[ \sum_{l=1}^{n} t_{li} a_{kl}^\eta - \sum_{j=1}^{m} t_{kj} a_{ji}^\xi. \tag{4} \]

The symbol \(a_{ij}^\xi\) then is used to denote the quotient element of \(a_{ij}^\xi\).

If we denote by \(a^\xi\) the matrix arrangement of elements \(a_{ij}^\xi\) in \(A\) \((a^\xi \in M_m(A))\), then the covariance condition takes the form of matrix equations

\[ [\eta|T|\xi] a^\xi = a^\eta [\eta|T|\xi], \quad T : X \rightarrow Y. \tag{5} \]

A natural transformation \(\rho : F \rightarrow F \otimes A\) is now defined so that

\[ \rho_X (\xi_i) = \sum_{j=1}^{m} \xi_j \otimes a_{ji}^\xi, \quad 1 \leq i \leq m, \quad X = |\xi|, \quad \xi = (\xi_1, \ldots, \xi_m). \tag{6} \]

The well-definedness of \(\rho\) is exactly the covariance condition. The natural transformation \(\rho\) is a solution to the universality problem. In fact, given another natural transformation \(\sigma : F \rightarrow F \otimes B\), the linear map \(\varphi : A \rightarrow B\) is well-defined by \(\varphi(a_{ji}^\xi) = b_{ji}\) with

\[ \sigma_X (\xi_i) = \sum_{j=1}^{m} \xi_j \otimes b_{ji}. \tag{7} \]

The coproduct \(\Delta : A \rightarrow A \otimes A\) is well-defined by

\[ \Delta(a_{ij}^\xi) = \sum_{k=1}^{m} a_{ik}^\xi \otimes a_{kj}^\xi. \tag{8} \]
because it preserves the covariance condition:
\[
\sum_{l=1}^{n} t_{li} \sum_{s=1}^{n} a_{ks}^l \otimes a_{si}^l - \sum_{j=1}^{m} t_{kj} \sum_{r=1}^{m} a_{jr}^i \otimes a_{ri}^i = \sum_{s=1}^{n} a_{ks}^i \otimes \sum_{r=1}^{m} t_{sr} a_{ri}^i - \sum_{r=1}^{m} \sum_{s=1}^{n} t_{sr} a_{ks}^i \otimes a_{ri}^i = 0.
\]

The coproduct \( \Delta \) is obviously coassociative by its form and the counit \( \epsilon : A \to C \) for \( \Delta \) is also well-defined by \( \epsilon(a_{ij}^k) = \delta_{ij} \) simply because of
\[
\sum_{l=1}^{n} t_{li} \delta_{kl} - \sum_{j=1}^{m} t_{kj} \delta_{ji} = 0.
\]

The coproduct is chosen so that each \( \rho_A : F(X) \to F(X) \otimes A \) gives a corepresentation of \( A \). The resulting right comodule is denoted by \( F^{A} \). In this way, we have obtained a coalgebra \( A \) and the functor \( F \) is lifted to a faithful linear functor of \( C \) into the category \( M^{A} \) of finite-dimensional right \( A \)-comodules. (Note that the \( A \)-colinearity is exactly the covariance condition.)

Non-trivial is the fact that the functor \( C \to M^{A} \) gives an equivalence of categories if and only if \( C \) is abelian and \( F \) is exact ([11] Theorem 2.3.5). For a semisimple \( C \), however, the equivalence theorem is an easy consequence of irreducible decompositions of objects as it will be discussed below.

Now assume that \( C \) is a tensor category with unit object \( I \) and \( F \) is monoidal. We may assume \( F \) is strictly monoidal without loss of generality. Then \( A \) is an algebra with the multiplication map \( \mu : A \otimes A \to A \) defined by
\[
a_{ij}^k a_{kl} = a_{i,k;j,i}.
\]

where \( \xi \otimes \eta = \{ \xi_i \otimes \eta_k \} \) is a frame of \( X \otimes Y \) obtained from \( \xi \) and \( \eta \) by taking tensor products. This is again well-defined because it preserves the covariance condition; the relation
\[
\sum_{l} a_{kl}^\eta t_{li} - \sum_{j} t_{kj} a_{ji}^\xi = \sum_{l} a_{k,l,s}^\eta t_{li} - \sum_{j} t_{kj} a_{j,s}^\xi = 0.
\]
is, for example, associated to the morphism \( T \otimes 1_{Z} : X \otimes Z \to Y \otimes Z \) in \( C \).

Clearly \( \mu \) is associative and the coproduct \( \Delta \) is compatible with \( \mu \);
\[
\Delta(a_{i,k;j,i}) = \sum_{r,s} a_{i,k;r,s}^\xi \otimes a_{r,s;j,i}^\eta = \sum_{r,s} a_{i,s}^\xi a_{ks,r}^\eta \otimes a_{r,j}^\eta = \Delta(a_{i}^\xi) \Delta(a_{k}^\eta).
\]
The unit \( 1_{A} \) for \( \mu \) is given by \( \rho_{I} : 1 \in \mathbb{C} = A \) (\( F(I) = \mathbb{C} \)).

At this point, we have established a bialgebra structure on \( A \) so that \( C \to M^{A} \) is a monoidal imbedding.

Now assume the rigidity on \( C \) and we shall show that \( A \) is a Hopf algebra. Let \( \xi \) be a frame for \( X \) and choose a dual object \( X^{*} \) together with a non-degenerate morphism \( \epsilon_X : X^{*} \otimes X \to I \) (\( \epsilon_X \) is said to be non-degenerate if we can find a morphism \( \delta_X : I \to X \otimes X^{*} \) fulfilling \( (1_X \otimes \epsilon_X)(\delta_X \otimes 1_X) = 1_X \) and \( (\epsilon_X \otimes 1_X^{*})(1_X^{*} \otimes \delta_X) = 1_X^{*} \)). We then define a frame \( \xi^{*} \) for \( X^{*} \) by the relation
\[
F(\epsilon_X)(\xi_i^* \otimes \xi_j) = \delta_{ij},
\]
i.e., $\xi^*$ is the dual basis of $\xi$ with respect to the non-degenerate bilinear form $F(\epsilon_X) : F(X^*) \otimes F(X) \to \mathbb{C}$. By the covariance condition, we see that the matricial element $\delta^s \in M_m(A)$ does not depend on the choice of either $X^*$ or $\epsilon_X$. The linear isomorphism $S : A \to A$ is then well-defined by $S(a^s_{ij}) = a^s_{ji}$. The property of antipode

$$
\sum_r S(a^s_{ij}) a^s_{tr} = \epsilon(a^s_{ij}) 1_A = \sum_r a^s_{ir} S(a^s_{rj})
$$

follows from the commutativity of diagram

$$
\begin{array}{ccc}
F(X^*) \otimes F(X) & \xrightarrow{\rho_{X^* \otimes X}} & F(X^*) \otimes F(X) \otimes A \\
F(\epsilon_X) \downarrow & & \downarrow F(\epsilon_X) \otimes 1_A \\
\mathbb{C} & \xrightarrow{\rho_I} & A \\
F(\delta_X) \downarrow & & \downarrow F(\delta_X) \otimes 1_A \\
F(X) \otimes F(X^*) & \xrightarrow{\rho_{X \otimes X^*}} & F(X) \otimes F(X^*) \otimes A \\
\end{array}
$$

(15)

Notice here that the linear map $F(\delta_X) : \mathbb{C} \to F(X) \otimes F(X^*)$ takes the form

$$
F(\delta_X) : 1 \mapsto \sum_{j=1}^m \xi_j \otimes \xi_j^*
$$

by the uniqueness of the coparing map in $\mathcal{V}ec$.

As a conclusion of discussions so far, we have

**Proposition A.1.** Let $\mathcal{C}$ be an essentially small rigid tensor category and $F : \mathcal{C} \to \mathcal{V}ec$ be a faithful monoidal functor from $\mathcal{C}$ to the tensor category $\mathcal{V}ec$ of finite-dimensional vector spaces. Then we can find a Hopf algebra $A$ and a faithful monoidal functor from $\mathcal{C}$ to the tensor category $\mathcal{M}^A$ of finite-dimensional right $A$-comodules so that $F$ is the forgetful functor of this.

We shall now assume that the linear category $\mathcal{C}$ is semisimple in the sense that any object is isomorphic to a direct sum of simple objects, where an object $X$ in $\mathcal{C}$ is said to be simple if $\text{End}(X) = \text{Hom}(X, X) = \mathbb{C}1_X$. Let $S$ be the set of isomorphism classes of simple objects of $\mathcal{C}$ and choose a representative family $\{X_s\}_{s \in S}$ of simple objects. Furthermore, select a frame $\xi$ for each $X_s$ and denote the associated element of $A$ by $a^{(s)}_{ij}$.

Then by decomposing objects into direct sums of $X_s$’s, we see that the family $\{a^{(s)}_{ij}\}_{s, i, j}$ constitutes a linear basis of $A$. Observing the coproduct formula

$$
\Delta(a^{(s)}_{ij}) = \sum_k a^{(s)}_{ik} \otimes a^{(s)}_{kj},
$$

(17)

this implies that the coalgebra $A$ is semisimple with all simple comodules supplied by $F(X_s)^A$ ($s \in S$). Thus we have arrived at

**Corollary A.2.** Assume that $\mathcal{C}$ is semisimple furthermore. Then $\mathcal{C}$ is monoidally equivalent to the tensor category $\mathcal{M}^A$.

Assume that the (strict) tensor category $\mathcal{C}$ is generated by an object $X$; objects of $\mathcal{C}$ are of the form $X \otimes^n = X \otimes \cdots \otimes X$ (the $n$-th tensor power of $X$) for $n = 0, 1, 2, \ldots$. 
Here $X^\otimes 0$ is the unit object $I$ by definition. Choose a frame $\xi$ of $X$ once for all and write $a_{ij} = a_{\xi ij}$.

Let $F : \mathcal{C} \to \text{Vec}$ be a faithful strictly monoidal functor and $A$ be the associated bialgebra. Then $A$ is generated by the elements $a_{ij}$ as a unital algebra with the relations given by the commutativity of diagrams

\[
\begin{align*}
F(X^m) & \xrightarrow{\rho_m} F(X^m) \otimes A \\
F(f) & \downarrow \quad F(f) \otimes 1_A \\
F(X^n) & \xrightarrow{\rho_n} F(X^n) \otimes A
\end{align*}
\]

for various morphisms $f : X^m \to X^n$, with the coproduct $\Delta : A \to A \otimes A$ specified by

\[
\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}
\]

as already discussed.

Now we restrict ourselves to the Temperley-Lieb category $\mathcal{K}_d$. In this case, $\text{Hom}(X^m, X^n)$ is generated by basic arcs as tensor cateogories, which implies that the algebra $A$ is generated by $a_{ij}$ with the relations given by the commutativity of two diagrams

\[
\begin{align*}
F(X) \otimes F(X) & \xrightarrow{\rho_X \otimes 1} F(X) \otimes F(X) \otimes A \\
F(\epsilon_X) & \downarrow \quad F(\epsilon_X) \otimes 1_A \\
\mathbb{C} & \xrightarrow{=} \mathbb{C} \otimes A \\
F(\delta_X) & \downarrow \quad F(\delta_X) \otimes 1_A \\
F(X) \otimes F(X) & \xrightarrow{\rho_X \otimes 1} F(X) \otimes F(X) \otimes A
\end{align*}
\]

If we write

\[
E_{ij} = F(\epsilon_X)(\xi_i \otimes \xi_j), \quad F(\delta_X)(1) = \sum_{ij} D_{ij} \xi_i \otimes \xi_j,
\]

then $D = (D_{ij})$ is the inverse matrix of $E = (E_{ij})$ and the above commutativity takes the form

\[
\sum_{k,l} E_{k,l} a_{kl} a_{lj} = E_{ij} 1_A, \quad \sum_{i,j} D_{ij} a_{ki} a_{lj} = D_{kl} 1_A
\]

of quadratic relations.

A.3. From Unitary Fiber Functors to Compact Quantum Groups. From here on, $\mathcal{C}$ is assumed to be a rigid $C^*$-tensor category and $F : \mathcal{C} \to \text{Hilb}$ be a unitary fiber functor (see [16, 18, 9]). The associated Hopf algebra $A$ is then a Hopf $*$-algebra. In fact, let $\xi$ be a frame consisting of orthonormal vectors and define a conjugate-linear map $A \ni a \mapsto a^* \in A$ so that $(a_{ij}^\xi)^* = S(a_{\xi ji}) = a_{ij}^\xi$. This is well-defined because it preserves the covariance condition as seen from

\[
\sum_{l=1}^n \overline{t_{li}} a_{kl}^\xi - \sum_{j=1}^m \overline{t_{kj}} (a_{ij}^\xi)^* = S \left( \sum_{l=1}^n \overline{t_{li}} a_{lk}^\xi - \sum_{j=1}^m \overline{t_{kj}} a_{ij}^\xi \right),
\]
the relation such as \( \xi \)

The dual frame Remark.

If we use the change-of-basis \( \eta \)

(28)

\[
\Delta((\xi_j^\ast \eta_k^\ast)) = \delta_{k,l}, \quad \text{i.e.,} \quad \sum_k (\xi_j^\ast \eta_k^\ast)(\xi_j^\ast \eta_k^\ast) = \delta_{i,j}.
\]

which is used in (27) to obtain

(29)

\[
\left( a_{ij}^\xi \right)^\ast = \sum_{i,k} (\xi_j^\ast \eta_k^\ast)(\xi_j^\ast \eta_k^\ast) a_{ik}^\eta.
\]

If we use the change-of-basis \( \eta_i^\ast = \sum_l (\xi_l^\ast \eta_l^\ast) \xi_l \) in the last expression, it takes the form

(30)

\[
\sum_{i,k} (\xi_j^\ast \eta_k^\ast)(\xi_j^\ast \eta_k^\ast) a_{ii}^\xi = \sum_i \delta_{i,j} a_{ii}^\xi = a_{ij}^\xi,
\]

showing \((a_{ij}^\xi)^\ast = (a_{ij}^\xi)^\ast = a_{ij}^\xi\).

Remark. The dual frame \( \xi^\ast \) is not necessarily orthonormal and we cannot expect the relation such as \( \xi^\ast \ast = \xi \).

It is now immediate to check that the involution \( * \) makes \( A \) a Hopf \( * \)-algebra. In fact, given orthonormal frames \( \xi \) and \( \eta \), their tensor product frame \( \xi \otimes \eta \) is orthonormal and we have

(31) \[
(a_{i,j}^\xi a_{l,k}^\eta)^\ast = (a_{i,k,l,j}^\xi \otimes \eta)^\ast = S(a_{i,k,l,j}^\xi \otimes \eta) = S(a_{l,k}^\eta)S(a_{i,j}^\xi) = (a_{l,k}^\eta)^\ast (a_{i,j}^\xi)^\ast
\]

whereas the compatibility with the coproduct is seen from

(32) \[
\Delta((a_{ij}^\xi)^\ast) = \Delta(a_{ij}^\xi) = \sum_k a_{ik}^\xi \otimes a_{kj}^\xi = \sum_k (a_{ik}^\xi)^\ast \otimes (a_{kj}^\xi)^\ast = (\Delta(a_{ij}^\xi))^\ast.
\]

Finally observe that each \( A \)-comodule \( \rho_X : F(X) \to F(X) \otimes A \) is unitary in the sense (see [15], cf. also [8]) that

(33) \[
\langle \rho_X(v)|\rho_X(w) \rangle = (v|w)1_A \quad \text{for} \ v, w \in F(X),
\]
where $\langle \; \mid \; \rangle$ on the left side is the $A$-valued inner product defined by

$$\langle v \otimes a \mid w \otimes b \rangle = (v\mid w)a^*b, \quad a, b \in A.$$  

This follows from

$$\langle \rho_X(\xi_i) \mid \rho_X(\xi_j) \rangle = \left\langle \sum_k \xi_k \otimes a^\xi_{ki} \left| \sum_l \xi_l \otimes a^\xi_{lj} \right. \right\rangle$$

$$= \sum_{k,l} (\xi_k \mid \xi_l)(a^\xi_{ki})^*a^\xi_{lj}$$

$$= \sum_k (a^\xi_{ki})^*a^\xi_{kj} = \sum_k S(a^\xi_{ki})a^\xi_{kj} = \delta_{ij}1_A.$$

**Remark.** In the definition of compatible $*$-operations, we do not use the positivity of inner products and can equally well work with $*$-tensor categories and (finite-dimensional) indefinite inner product spaces, which in fact arises from the Temperley-Lieb category $K_d$ for $-2 < d < 2$ (the associated Hopf $*$-algebra is then $SU_q(1,1)$ with $|q| = 1$, another real form of $SL_q(2,\mathbb{C})$).

Thus we have derived the algebraic part in Woronowicz’ Tannaka-Krein duality (\cite{10}).

**Proposition A.3.** Let $C$ be a rigid abelian $C^*$-tensor category with simple unit object and $F: C \to \mathcal{H}ilb$ be a unitary fiber functor. Then we can find a $*$-Hopf algebra so that $C$ is $C^*$-monoidally equivalent to the $C^*$-tensor category of finite-dimensional unitary $A$-comodules with $F$ identified with the associated forgetful functor.

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