EXISTENCE AND MULTIPLICITY OF PERIODIC SOLUTIONS TO AN INDEFINITE SINGULAR EQUATION WITH TWO SINGULARITIES. THE DEGENERATE CASE

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ABSTRACT. We analyze the existence of $T$–periodic solutions to the second-order indefinite singular equation

$$u'' = \beta \frac{h(t)}{\cos^2 u}$$

which depends on a positive parameter $\beta > 0$. Here, $h$ is a sign-changing function with $h = 0$ and where the nonlinear term of the equation has two singularities. For the first time, the degenerate case is studied, displaying an unexpected feature which contrasts with the results known in the literature for indefinite singular equations.

1. Introduction and main results. In recent decades many authors have studied the existence and multiplicity of solutions to the Neumann, and to the periodic boundary value problem for second order differential equations of the form

$$u'' = \beta h(t)g(u),$$

where $h \in L(R/T \mathbb{Z})$, $\beta > 0$ is a parameter, and $g : (A, B) \to (0, +\infty)$ is a continuous function with $-\infty \leq A < B \leq +\infty$. See, e.g., [1, 3, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 19, 29, 30]. An elementary observation is that any solution of (1) satisfying the above-mentioned boundary conditions verifies

$$\int_0^T h(t)g(u)dt = 0,$$

and, consequently the function $h$ (called the weight function) has to change its sign. As a result, the equations of type (1) are so-called indefinite equations. The

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terminology indefinite was introduced by Hess and Kato in [23] under a rather different context.

To the best knowledge, the history of these equations starts with Waltman [31], who considered oscillatory solutions for the non-singular equation \( u'' + h(t)u^{2n+1} = 0 \), with \( n \geq 1 \). A major breakthrough came with the work of Butler [12] who proposed an open problem related with the number of these solutions (see [28] for a brief historical survey about this problem).

When singular nonlinearities are taken into account (that is \(-\infty < A \) or \( B < +\infty \)), both the periodic and the Neumann problem become slightly more tricky and still remain insufficiently investigated. In what follows we will describe some important contributions in this direction. To start the discussion, the first works concerning this case dealt when the nonlinearity \( g : (0, +\infty) \to (0, \infty) \) is assumed to be a decreasing homeomorphism. As a model equation, we have in mind the classical generalized Emden-Fowler equation

\[
u'' = \frac{h(t)}{u^\lambda},
\]

where \( \lambda > 0 \), that corresponds to the choice \( g(x) := x^{-\lambda} \) (sometimes we will refer to (2) instead of (1) to underline certain aspects of the singularity and to simplify the explanation). First Bravo and Torres in [11] opened the door to handle with problems of this form. Later in [29], Ureña established the existence of a \( T \)-periodic solution when \( \lambda \geq 1 \) and the weight function has only simple zeroes or it is piecewise constant with a finite number of pieces. Afterwards, this property was extended in [21] by Hakl and Zamora, proving the existence of such solution under a subtle relation between the multiplicity of every zero of the weight function and the order of the singularity of the nonlinear term. Let us denote this relation by \([R]\) which was already observed in [8] dealing with a different boundary value problem, provided also that \( \lambda \geq 1 \). It was proven later in [30] that somehow the relation \([R]\) was essential. Finally, to see results dealing with \( \lambda \in (0, 1) \) we refer to [17, 18].

In a recent series of papers [1, 20, 22], the authors have dealt with indefinite singular equations with two singularities, i.e., they considered the equation (1) with \( A, B \in \mathbb{R} \). The appearance of another singularity substantially changes the attitude to the problem, mainly due to the lack of monotony of the nonlinear term. This aspect carries serious difficulties to apply directly the arguments used above and, consequently, new strategies have to be proposed. Up to our knowledge, the first work devoted to the investigation of these equations appears in [20]. In that paper, the equation

\[
u'' = \beta \frac{h(t)}{\cos^2 u}
\]

was considered under very restrictive conditions on \( h \) (assuming that \( h \) is piecewise constant function with two pieces of different sign), establishing the following result: There exists a critical value \( \beta_0 > 0 \) such that for every \( \beta > \beta_0 \), there is no \( T \)-periodic solution, for \( \beta = \beta_0 \), there exists at least one \( T \)-periodic solution, and for \( \beta \in (0, \beta_0) \) there exist at least two \( T \)-periodic solutions. The question here is: is it true the result if we do not require any further special condition on \( h \)?

A first step toward the answer to the above question comes in the paper [22]. For the case when \( h \neq 0 \), the results proved in [21] were applied in order to provide
a partial answer and to better understand how this problem works. Of course, the above-mentioned procedure necessarily involves the condition [R], which together with [F] and assuming that \( \bar{h} \neq 0 \), allows to prove the following assertion: There exist \( 0 < \beta_* \leq \beta^* < +\infty \) such that for every \( \beta > \beta^* \) there is no \( T \)-periodic solution, for \( \beta = \beta_* \) there exists at least one \( T \)-periodic solution, and for \( \beta \in (0, \beta_*) \) there exist at least two \( T \)-periodic solutions. Here, the condition [F] is referred with the fact that the number of sign changes of the weight function \( h \) is finite.

The case when \( \bar{h} = 0 \) cannot be resolved as in the previous issue, assuming the results established in [21]. The reason is that the relation \( \bar{h} < 0 \) is not only essential in the proof of the mentioned result, but even necessary (see [Remark 1.2][21]). In addition, the approximation of \( h \) by some functions \( h_n \) with \( \bar{h}_n < 0 \) and \( \bar{h}_n \to 0 \) does not allow to apply the ideas of [21]. Indeed, each \( T \)-periodic solution \( u \) of (2) has to change its sign, but the \( T \)-periodic solutions \( u_n \) (corresponding to the weights \( h_n \)) are always non-positive. For this reason this is considered a degenerate case which was proposed as an open problem in [Section 5][22].

The objective of this paper is to investigate this degenerate case, i.e., the case when \( \bar{h} = 0 \). At first glance one may expect to extend this case to the results proved in [22] under the same conditions, i.e., assuming that [R]-[F] hold. However, this special case keeps some mystery as we will see below.

Now we are going to describe our setting in more detail. By \( \bar{h} \) we understand the mean function of \( h \), i.e., \( \bar{h} := \frac{1}{T} \int_0^T h(s)ds \). From (3) it follows that the nonlinear term of the equation has singularities at the points \( x = -\pi/2 \) and \( x = \pi/2 \). We are interested in solutions that avoid these singularities. More precisely, we are looking for \( T \)-periodic functions \( u : \mathbb{R} \to (-\pi/2, \pi/2) \) which are continuous together with their first derivative on \([0, T]\) and satisfy the equality (3) for almost every \( t \in [0, T] \).

In addition, and only to simplify the presentation, we will assume that \( h \) is bounded, i.e., \( h \in L^\infty(\mathbb{R}/T\mathbb{Z}) \). The reader who is acquainted with the classical Carathéodory-formulation theory of lower and upper functions for the periodic should be able to adapt the results.

In the present paper, the first result can be stated as follows.

**Theorem 1.** Let us assume the hypothesis:

\[
\bar{h} = 0, \quad h \neq 0. \quad (H)
\]

Then, there exists \( \beta_* > 0 \) such that the equation (3) has at least one \( T \)-periodic solution for every \( 0 < \beta \leq \beta_* \). Moreover, if there exist \( [a, b] \subseteq [0, T] \) and \( \delta \in \{-1, 1\} \) such that

\[
\int_a^b h(s)ds \neq 0, \quad \delta h(t) \geq 0 \text{ for a.e. } t \in [a, b],
\]

then there exists \( \beta^* \geq \beta_* \) such that (3) has no \( T \)-periodic solution for every \( \beta > \beta^* \).

In particular, we display an unexpected feature: for the first time, the degeneracy of the zeroes of the weight function \( h \) has no influence on the existence of a \( T \)-periodic solution. In addition, we can prove the existence of solutions in situations when the weight function changes sign an arbitrary number of times. In this manner, Theorem 1 shows that the conditions [R]-[F] can be dropped (see Example 1 in Section 5).
Remark 1. It is sufficient to integrate both sides of (3) to observe that
$h \not\equiv 0$ is a necessary condition for the existence of a $T$-periodic solution. Hence, our condition (H) in Theorem 1 is necessary and sufficient to prove the result. Up to our knowledge, for the first time, under a necessary and sufficient condition is guaranteed the existence of a periodic solution to a equation with indefinite singularity.

Remark 2. Condition (4) is necessary to guarantee that there exists a subinterval of $[0, T]$ where the function $h$ does not change sign and it is also not trivial. Of course, if our weight function is continuous, this would be true in any case because $h \not\equiv 0$; but otherwise we need to impose this condition to study the non-existence of periodic solutions.

In our second result we obtain various $T$-periodic solutions in return for strengthening the hypotheses on the weight function $h$.

Theorem 2. Let us assume (H), and that there exist pairwise disjoint intervals $(a_k, b_k) \subseteq [0, T]$ and numbers $\delta_k \in \{-1, 1\}$ ($k = 1, \ldots, n$) such that
\[
\bigcup_{k=1}^{n} [a_k, b_k] = [0, T],
\]
\[
\delta_k h(t) \geq 0 \text{ for a.e. } t \in [a_k, b_k] \quad (k = 1, \ldots, n).
\]

Moreover, let $c_k \in (a_k, b_k)$ ($k = 1, \ldots, n$) be such that
\[
\lim_{x \to -\pi/2} \int_{0}^{x} \frac{|h(s + t_0)|}{\cos^2 s} \, ds = +\infty \text{ for a.e. } t_0 \in [a_k, c_k] \quad (k = 1, \ldots, n),
\]
\[
\lim_{x \to -\pi/2} \int_{0}^{x} \frac{|h(t_0 - s)|}{\cos^2 s} \, ds = +\infty \text{ for a.e. } t_0 \in [c_k, b_k] \quad (k = 1, \ldots, n).
\]

Then, there exists $\beta_* > 0$ such that the equation (3) has at least two $T$-periodic solutions for every $0 < \beta < \beta_*$ and at least one $T$-periodic solution for $\beta = \beta_*$. Moreover, there exists $\beta^* \geq \beta_*$ such that the equation (3) has no $T$-periodic solution for every $\beta > \beta^*$.

The conditions of Theorem 2 are the ones expected according to the literature available on the subject, see e.g., [1, 21, 22, 29, 30]. Hence, at least for this case, the conditions $\[R]\cdot[F]$ seem to be connected more with the multiplicity of the periodic problem than with its mere solvability.

Before ending this introduction, it is worth mentioning here that equations which can be put in the form of (3) appear in a model describing a Kepler problem on the sphere. By Kepler problem on the sphere we understand the study of the dynamic of a charged particle moving on the sphere submitted to the influence of an electric field created by a charge of a time-depending magnitude fixed in the north pole, see e.g., [2] or [20].

The paper is organized as follows. In Section 2, our second-order equation is rewritten as a suitable equation with a more convenient structure. This transformation allows us to formulate a fixed point problem for an operator equation whose solutions are connected with the ones of (3) satisfying certain boundary conditions. Section 3 takes advantage of such solutions to find lower and upper functions in
reverse order associated with the periodic problem to the equation (3). The theoretical part of this paper concludes with Section 4, where by applying topological techniques we prove the main results. In the last section we discuss the applicability of our results, describing the main finding of our manuscript.

We collect here, for the reader’s convenience, the main notation used throughout the paper. Given \( h \in L(R/T \mathbb{Z}) \) (in our case \( h = 0 \)), we let

\[
\sigma(t) := \int_0^t h(s)ds, \quad \overline{\sigma} = \frac{1}{T} \int_0^T \sigma(s)ds
\]

\[
\tilde{\sigma}(t) := \sigma(t) - \overline{\sigma}, \quad \text{for} \quad t \in [0, T].
\]

Now, we fix some constants that will be later used. From now on we fix \( \kappa > 1 \) and \( \alpha > 0 \) such that \( \alpha > \sqrt{-1 + \sqrt{\kappa}} \).

Moreover, we will introduce the function

\[
\varepsilon_\ast : (0, +\infty) \to \mathbb{R}, \quad \varepsilon_\ast(\varepsilon) := \varepsilon + \varepsilon T (\varepsilon + \kappa \|\tilde{\sigma}\|_\infty).
\]

In what follows we will denote by \( C := C([0, T]; \mathbb{R}) \) the Banach space of the continuous functions \( u : [0, T] \to \mathbb{R} \) endowed with the standard norm \( \|u\|_\infty := \max_{t \in [0, T]} |u(t)| \). Also, as usual, we write \( m_u := \min_{t \in [0, T]} u(t) \) and \( M_u := \max_{t \in [0, T]} u(t) \).

Let \( \Omega \subseteq \mathbb{R}^3 \times C \) be a set and \( \beta_0 > 0 \), we denote by

\[
\Omega^+ := \{ (\beta, x, y, f) \in \Omega : \beta \geq 0 \},
\]

\[
\Omega_{\beta_0} := \{ \eta \in \mathbb{R}^2 \times C : (\beta_0, \eta) \in \Omega \},
\]

respectively, the right-hand side part of \( \Omega \) and its vertical section along \( \beta_0 \).

Let us go now into the abstract part of the paper. Our theoretical arguments require the use of the following notation in order to avoid handle too long mathematical expressions. Let \( \xi := (\beta, x, y) \in \mathbb{R}^3 \) be a real vector, and \( f \in C \), we set

\[
A_\xi[f](t) := x + |\beta| \int_0^t f(s)ds,
\]

\[
B_\xi[f](t) := (\alpha + x)y + \int_0^t \left| \beta \right| \left( \frac{1 + 2\alpha A_\xi[f](s) + 3A_\xi[f]^2(s)}{1 + A_\xi[f]^2(s)} \right) f^2(s) + h(s)(\alpha + A_\xi[f](s))(1 + A_\xi[f]^2(s))^2 ds.
\]

The rest of the notation, will be introduced later in the corresponding sections.
Remark 3. Before passing to the mathematical details of the paper, we observe that there is no loss of generality in adding the condition
\[ \int_0^T \int_0^s h(r)drds \neq 0 \] (9)
in our hypothesis (H). Indeed, for every \( \gamma \in [0, T] \), the transformation \( v(t) = u(t+\gamma) \) leads to the equation
\[ v'' = \frac{h_{\gamma}(t)}{\cos^2 v}, \]
where \( h_{\gamma}(t) := h(t + \gamma) \). In other words, looking for a \( T \)-periodic solution on \([0, T]\) is the same that looking for it on the interval \([\gamma, \gamma + T]\). However, when \( h \) is not trivial one can easily find a \( \gamma \in [0, T] \) such that (9) holds (with \( h \equiv h_{\gamma} \)). Consequently, we will assume
\[ T = 0, \quad h \neq 0, \quad \int_0^T \int_0^s h(r)drds \neq 0 \] (H)
in the rest of the paper.

2. From a singular equation to an equivalent non-singular equation.
Consider the family of equations
\[ u'' = |\beta| \frac{h(t)}{\cos^2 u} \] (10)
In this section, we show how to handle equation (10) in order to bring the research for its \( T \)-periodic solutions.
Using the change of variable
\[ \rho(t) := \tan u(t) \text{ for } t \in [0, T], \] (11)
we obtain the equation
\[ \rho''(t) = \frac{2 \rho(t)}{1 + \rho^2(t)} \rho^2(t) + |\beta|h(t)(1 + \rho^2(t))^2. \] (12)
Notice that (12) takes the advantage of being a regular equation (it has no singularity). Moreover, multiplying by \( \rho(t) + \alpha \) we pass to the equation
\[ \frac{d}{dt}(\rho'(\rho(t) + \alpha)) = \left( \frac{1 + 2\alpha \rho(t) + 3\rho^2(t)}{1 + \rho^2(t)} \right) \rho^2(t) + |\beta|h(t)(\rho(t) + \alpha)(1 + \rho^2(t))^2. \] (13)
Although our aim is to address the periodic problem associated with (3), we consider firstly a weaker problem, which consists in finding solutions of (13) (or (12)) verifying the condition \( \rho(0) = \rho(T) \). To deal with this issue, we will formulate our problem in the space \( R^3 \times C \), being sufficient to find points of the form \((\beta, x, y, f)\) satisfying certain properties.
Proposition 1. Assume that there exists $\xi = (\beta, x, y) \in \mathbb{R}^3$ and $f \in C$ such that $A_\xi[f](t) > -\alpha$ for $t \in [0, T]$, and satisfying the following properties:

$$\int_0^T f(s) ds = 0,$$

$$ (A_\xi[f](t) + \alpha)f(t) = B_\xi[f](t) \quad \text{for } t \in [0, T].$$

Then, the function $\rho(t) := A_\xi[f](t)$ is a solution of the equation (13) which satisfies $\rho(0) = \rho(T)$.

Proof. From (15) we have

$$\rho'(t) = |\beta| f(t) = \frac{|\beta| B_\xi[f](t)}{\alpha + A_\xi[f](t)}.$$ 

Thus, a direct computation shows that $\rho$ is a solution of the equation (13). Moreover, (14) implies that $\rho(0) = \rho(T)$. This concludes the proof. \qed

Solutions appearing in Proposition 1 are connected with solutions of (10) satisfying the same boundary conditions. They will play an important role in Section 3, where more properties will be analyzed. Now, our objective is to obtain several solutions of this type. With this proposal, we will use continuation arguments of the degree. Of course, first of all we need to introduce an adequate functional framework.

2.1. Continuation of solutions with nonzero degree. We are going to rewrite the problem of finding solutions to (14)-(15) as a fixed point problem for an operator equation. That is described briefly in what follows.

Let $X := \mathbb{R}^2 \times C$ be the Banach space endowed with the standard norm $\|(x, y, f)\| := |x| + |y| + \|f\|_\infty$. Problem (14)-(15) now becomes

$$(x, y, f) = F[\beta, x, y, f],$$

where

$$F[\beta, x, y, f] := \left( x - \int_0^T f(s) ds, y - \int_0^T \int_0^t h(s) ds dt, B_\xi[f](t) \right).$$

Solutions of the problem (16) make sense on the set

$$\Lambda := \{(\beta, x, y, f) \in \mathbb{R} \times X : \quad A_\xi[f](t) > -\alpha, \text{ for } t \in [0, T]\},$$

which is an unbounded open set on $\mathbb{R} \times X$. Thus, the operator $F : \Lambda \to X$ is continuous and maps open bounded subsets of $\Lambda$ whose closure is contained on $\Lambda$ into relative compact sets of $X$ (i.e., $F : \Lambda \to X$ is completely continuous). So that, for every $\beta \in \mathbb{R}$, the Leray-Schauder degree to the operator $I - F_\beta$ can be defined on subsets of $\Lambda$ with the above-mentioned property.

The key point in the following is that the family of operators introduced above contains an operator with finite-dimensional range. Thus, the standard degree arguments allow to compute the degree. This opens the door to the use of continuation techniques.
Lemma 1. Let $\mathcal{B}_i := B_{\|\cdot\|}(p_i, \varepsilon) \subset \mathbb{R} \times X$ $(i = 1, 2)$ be open balls with $\varepsilon > 0$ sufficiently small such that $\mathcal{B}_i \subset \Lambda$ $(i = 1, 2)$ and $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$. Here,

$$p_i = \left(0, (-1)^{i+1}\sqrt{-1 + \sqrt{\kappa}}, \kappa\sigma, \kappa\tilde{\sigma}\right).$$  \hspace{1cm} (17)

Then, there exists a connected set $\mathcal{C}_i \subseteq \mathcal{B}_i$ $(i = 1, 2)$ of solutions to (14)-(15) verifying the conditions:

(i) $\mathcal{B}_i \cap \mathcal{C}_i \cap \{(0) \times X\} \neq \emptyset$,

(ii) $\text{dist}(\mathcal{C}_i, \partial \mathcal{B}_i^+) = 0$.

Proof. Observe that $\mathcal{B}_i$ is an open and bounded subset of $\mathbb{R} \times X$ whose closure is contained in $\Lambda$. We first check that $F[0, x, y, f] \neq (x, y, f)$ for every $(x, y, f) \in (\partial \mathcal{B}_i)_0$ $(i = 1, 2)$. Indeed, if $(x, y, f) \in (\partial \mathcal{B}_i)_0$ then

$$\left| x + (-1)^i\sqrt{-1 + \sqrt{\kappa}} + |y - \kappa\sigma| + \|f - \kappa\tilde{\sigma}\|_{\infty} \right| = \varepsilon \quad (i = 1, 2).$$ \hspace{1cm} (18)

On the other hand, if $F(0, x, y, f) = (x, y, f)$, one easily checks that it is contained in $\mathbb{R}^2 \times \text{span}\{1, \sigma\}$, which is a finite-dimensional subspace of $X$. Hence,

$$f(t) = \delta + \gamma\sigma(t), \text{ for } t \in [0, T],$$

for $\delta$ and $\gamma$ adequate constants such that (16) holds. In particular,

$$\int_0^T (\delta + \gamma\sigma(s))ds = 0,$$

$$y = -\kappa\sigma,$$

$$\delta = y,$$

$$(1 + x^2)^2 = \gamma.$$ \hspace{1cm} (19)

Solving this system we obtain two solutions, one of them in $\mathcal{B}_1$ and the other one in $\mathcal{B}_2$, i.e., we get explicitly

$$x_i = (-1)^{i+1}\sqrt{-1 + \sqrt{\kappa}}, \quad y_i = -\kappa\sigma = \delta_i, \quad \gamma_i = \kappa.$$ \hspace{1cm} (20)

Hence, the fact that $F[0, \cdot]$ does not have fixed points on $(\partial \mathcal{B}_i)_0$ is a direct consequence of (20) and the equality (18). Thus, the Leray-Schauder degree of $(I - F)[0, \cdot]$ is well defined. Moreover, since $F[\cdot, \mathcal{B}_i]$ is included in a 4-dimensional subspace of $X$, by [Theorem 8.7][14] we obtain that

$$\text{d}_{LS}(I - F[0, \cdot], \mathcal{B}_i, 0) = dB(I_{\mathbb{R}^4} - F[0, \cdot]_{\mathbb{R}^4}, \mathcal{B}_i \cap \mathbb{R}^4, 0),$$ \hspace{1cm} (21)

where $\mathbb{R}^2 \times \text{span}\{1, \sigma\}$ is identified by $\mathbb{R}^4$,

$$(I_{\mathbb{R}^4} - F[0, \cdot]_{\mathbb{R}^4})(x, y, \delta, \gamma) = \left( \int_0^T (\delta + \gamma\sigma(s))ds, y + \kappa\sigma, \delta - y, \gamma - (1 + x^2)^2 \right).$$
Since $\mathcal{B}_i \cap \mathbb{R}^4$ contains only a solution of (19), by the properties of the degree we have that

$$\text{sgn } J_i = d_B \left( I_{\mathbb{R}^4} - F[0, \cdot]_{\mathbb{R}^4}, \mathcal{B}_i \cap \mathbb{R}^4, 0 \right),$$

(22)

where $J_i$ denotes the determinant of the Jacobian matrix associated to the function $I_{\mathbb{R}^4} - F[0, \cdot]_{\mathbb{R}^4}$ at the points $(x_i, y_i, \delta_i, \gamma_i) \ (i = 1, 2)$ defined by (20); i.e.,

$$J_i = \begin{vmatrix} 0 & 0 & T & T\sigma \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -4(1 + x^2)x & 0 & 0 & 1 \end{vmatrix} = 4T\sigma(1 + x^2)x \neq 0$$

because $\sigma \neq 0$. So that, by (21) and (22) we get that $d_{LS}(I - F[0, \cdot], \mathcal{B}_i, 0) \neq 0$ for $i = 1, 2$.

Under these conditions the classical Leray-Schauder continuation theorem (see, e.g. [25, 26, 27]) provides the existence of connected set $\mathcal{C}_i \ (i = 1, 2)$ of solutions to (14)-(15) verifying (i)-(ii) (see Figure 1).

Figure 1. Open balls $\mathcal{B}_i$ and connected sets $\mathcal{C}_i$ for $i = 1, 2$

3. T-periodic lower and upper functions in reverse order. In this section we will see how to handle with the solutions of the set $\mathcal{C}_i \ (i = 1, 2)$ in order to solve the periodic problem associated with the equation (10). It is in that context where we shall investigate the connection between the solutions of (10) and the ones of (12) (or (13)).

First of all, we fix $\varepsilon > 0$ small enough such that

$$\varepsilon_* := \varepsilon_*(\varepsilon) < \sqrt{-1 + \sqrt{\kappa}}. $$

(23)

Consider then the points $p_i \ (i = 1, 2)$ defined in Lemma 1 and the balls $\mathcal{B}_i = B_{\mathbb{R} \times X}(p_i, \varepsilon)$ for $i = 1, 2$, with the norm of $\mathbb{R} \times X$. We continue now by identifying the space $\mathbb{R} \times X$ with $C$ through the relation

$$(\beta, x, y, f) \equiv \rho(\beta, x, y, f)(t) := x + |\beta| \int_0^t f(s)ds.$$
Thus, we take advantage of the fact that if \((\beta, x, y, f) \in \mathbb{R} \times X\) is a solution of (14)-(15), performing the above identification, the function \(\rho(\beta, x, y, f)\) is a solution of (12) satisfying \(\rho(\beta, x, y, f)(0) = \rho(\beta, x, y, f)(T)\) (see Proposition 1).

Furthermore, the assertion below allows to control the sign of the functions lying on \(\mathcal{B}_i\) \((i = 1, 2)\).

**Lemma 2.** For every \((\beta, x, y, f) \in \mathcal{B}_i\) \((i = 1, 2)\) the following relation holds.

\[
\left| \rho_i(t) + (-1)^i \sqrt{-1 + \sqrt{\kappa}} \right| < \varepsilon_*. 
\]

In particular,

\[
\text{sgn } \rho_i(t) = \text{sgn } (-1)^{i+1} \text{ for } t \in [0, T] \quad (i = 1, 2).
\]

**Proof.** We will assume that \(i = 1\) holds; in the case when \(i = 2\) one can argue analogously. Indeed, we let \((\beta, x, y, f) \in \mathcal{B}_1\), then

\[
|\beta| + \left| x - \sqrt{-1 + \sqrt{\kappa}} \right| + |\kappa \sigma + y| + \|f - \kappa \tilde{\sigma}\|_{\infty} < \varepsilon.
\]

The latter inequality implies

\[
\left| \rho_1(t) - \sqrt{-1 + \sqrt{\kappa}} \right| = \left| x + \beta \int_0^t f(s)ds - \sqrt{-1 + \sqrt{\kappa}} \right|
\]

\[
\leq \varepsilon + \varepsilon T(\|f - \kappa \tilde{\sigma}\|_{\infty} + \kappa \|\tilde{\sigma}\|_{\infty})
\]

\[
< \varepsilon_*.
\]

Finally, by (23),

\[
\rho_1(t) > \sqrt{-1 + \sqrt{\kappa}} - \varepsilon_* > 0\text{ for } t \in [0, T].
\]

This concludes the proof. \(\square\)

After doing the corresponding identification, solutions of (14)-(15) may no give rise to periodic solutions to the equation (10) because the derivatives at times \(t = 0\) and \(t = T\) may be different, they are indeed so. However, the curves contained in the set \(\mathcal{B}_i\) \((i = 1, 2)\) are, respectively, \(T\)-periodic lower and upper functions.

The terminology \(T\)-periodic lower and upper functions is referred to the classical concept of lower and upper functions to the periodic problem associated with the equation (10). The result below is devoted to check this fact.

**Lemma 3.** Under the previous setting the following assertions hold.

(i) \(\mathcal{B}_1 \cap \mathcal{C}_1 \setminus \{p_1\}\) is a set of solutions of (12) which are strict \(T\)-periodic lower functions (i.e., \(\rho'(0) > \rho'(T)\));

(ii) \(\mathcal{B}_2 \cap \mathcal{C}_2 \setminus \{p_2\}\) is a set of solutions of (12) which are strict \(T\)-periodic upper functions (i.e., \(\rho'(0) < \rho'(T)\)).

**Proof.** We will assume that (i) holds; in the case when (ii) holds one can argue analogously.
Let, as before, \( (\beta, x, y, f) \in B_1 \cap C_1 \backslash \{p_1\} \). By identifying this point with \( \rho(\beta, x, y, f) \) and by applying Lemma 2, we obtain that \( \rho \) is a positive solution of (12) such that \( \rho(0) = \rho(T) \). Thus, to prove the result we need to check that \( \rho'(0) > \rho'(T) \). Indeed, we assume, on the contrary, that \( \rho'(0) \leq \rho'(T) \). Then, by (11), the function
\[
\rho(t) := \arctan(\rho(t)) \quad \text{for} \quad t \in [0, T]
\]
is a positive solution of (10) such that \( \rho(0) = \rho(T) \). Hence, since \( m_u > 0 \) and \( \rho'(0) \leq \rho'(T) \) we deduce that \( \rho \) is constant. But this is a contradiction because \( h \not\equiv 0 \).

We close this section with a final comment about the \( T \)-periodic lower and upper functions provided in Lemma 3. In the proof of this result we have used implicitly the following relation: if \( \rho \) is a solution of (12) such that \( \rho(0) = \rho(T) \), then the function \( u : [0, T] \to \mathbb{R} \) defined in (24) is a solution of (10) such that \( u(0) = u(T) \). Moreover,
\[
\rho'(0) \geq \rho'(T) \quad \text{(resp.} \quad \rho'(0) \leq \rho'(T)\text{)}
\]
if and only if
\[
u'(0) \geq u'(T) \quad \text{(resp.} \quad u'(0) \leq u'(T)\).
\]

In short, roughly speaking, and performing the above-mentioned identifications, the elements of \( B_i \cap C_i \backslash \{p_i\} \) \((i = 1, 2)\) are connected with the \( T \)-periodic lower and upper functions in reverse order to the equation (10). This is the key ingredient of this section, or even more, this is the key point in this paper.

4. Main results. This section is devoted to the proof of the main results. The second part of Theorems 1 and 2 is specially simple and it is based on the following assertion.

**Lemma 4.** Assume that there exist \([a, b] \subseteq [0, T]\) and \( \delta \in \{-1, 1\} \) such that (4) holds. Then there exists \( \beta^* > 0 \) such that, for every \( \beta \geq \beta^* \), there is no \( T \)-periodic solutions to (3).

**Proof.** Without loss of generality we can assume that the function \( h \) is non-negative and that
\[
\int_a^b h(s)ds > 0.
\]
Let there exists a \( T \)-periodic solution of (3). Then
\[
u''(t) \geq \beta h(t)\quad \text{for \ a. e. \ } t \in [a, b]
\]
and thus, for \( t \in [a, b] \) we have
\[
u(t) \leq \frac{u(b)(t - a) + u(a)(b - t)}{b - a} \times \left[ (b - t) \int_a^t (s - a)h(s)ds + (t - a) \int_t^b (b - s)h(s)ds \right].
\]
Consequently, from the latter inequality, for \( t = (a + b)/2 \) we obtain
\[
\frac{\beta}{2} \left[ \int_a^{a+b} (s-a)h(s)ds + \int_{a+b}^{b} (b-s)h(s)ds \right] \leq \frac{u(b) + u(a) - u\left(\frac{a+b}{2}\right)}{2} < \frac{\pi}{2}
\]

Therefore, the assertion holds. \( \square \)

**Proof of Theorem 1.** As we saw in the previous section, and following with its notation, we can choose \( \varepsilon > 0 \) such that (23) holds. Then, we define the points \( p_i \) \( (i = 1, 2) \) by (17) and we consider the set \( B_i \cap C_i \setminus \{p_i\} \) \( (i = 1, 2) \) provided by Lemma 3. We proved that, for every \( (\beta, x^{(i)}_\beta, y^{(i)}_\beta, f^{(i)}_\beta) \in B_i \cap C_i \setminus \{p_i\} \) \( (i = 1, 2) \), the corresponding identification
\[
u^{(i)}_\beta(t) = \arctan\left(x^{(i)}_\beta + \beta \int_0^t f^{(i)}_\beta(s)ds\right)
\]
for \( t \in [0, T] \)
is, respectively, a \( T \)-periodic lower and upper function to the equation (3) if either \( i = 1 \) or \( i = 2 \). Moreover,
\[
u^{(2)}_\beta(t) < 0 < \nu^{(1)}_\beta(t) \text{ for } t \in [0, T],
\]

\[
\|u^{(i)}_\beta\|_\infty \leq \arctan\left(\varepsilon + \sqrt{-1 + \sqrt{\kappa}}\right) \text{ for } i = 1, 2.
\]
The last inequality follows from Lemma 2. To simplify the notation, we set \( u_0 := \arctan\left(\varepsilon + \sqrt{-1 + \sqrt{\kappa}}\right) \). We now define \( \beta_* > 0 \) such that, for every \( 0 < \beta \leq \beta_* \), one has
\[
\frac{2\beta \|h\|_{\infty}}{\cos^2 u_0} < \left(\frac{T}{T}\right)^2,
\]
\[
(\{\beta\} \times X) \cap B_i \cap C_i \neq \emptyset \text{ for } i = 1, 2.
\]
In view of the nonlinearity \( (t, x) \mapsto \beta h(t) \cos^{-2} x \), (25) and (26), we find a \( T \)-periodic solution by applying the theory of lower and upper functions in reverse order (see e.g., [13]), i.e., for every \( 0 < \beta \leq \beta_* \), there exists \( u_\beta \) a \( T \)-periodic solution of (3) such that
\[
\|u_\beta\|_{\infty} < \arctan\left(\varepsilon + \sqrt{-1 + \sqrt{\kappa}}\right).
\]
The existence of \( \beta^* \geq \beta_* \) follows directly from Lemma 4. The proof now is complete. \( \square \)

**4.2. Proof of Theorem 2.** To prove Theorem 2 we need to establish first some preliminary results. We point out that these are not completely new. Some of them can be found and formulated in a more general context in [22], but for the reader’s convenience we shall provide below the details of their proofs.
A priori bounds. Our starting point consists in proving the existence of a priori bounds for the set of $T$–periodic solutions of the equation (3) in terms of a fixed $\beta_0 > 0$. To deal with this issue we need to introduce a new hypothesis, roughly speaking: $h$ is a function with a finite number of changing signs on the interval $[0,T]$ and the degeneracy of its zeroes is small in comparison with the order of the singularity of the equation (3) (see conditions [R]–[F] in our introduction). Firstly, the following assertion holds.

**Lemma 5.** Let $\beta_0 > 0$ and let $[a,b] \subseteq [0,T]$ be such that

$$h(t) \geq 0 \text{ for a. e. } t \in [a,b].$$

Let, moreover, be $c \in (a,b)$ such that

$$\lim_{x \to -\frac{c}{2}} \int_{t_0}^{t_0 + \frac{b-c}{2}} \int_{t_0}^{t_0 + \frac{a-c}{2}} \frac{h(\xi)}{\cos^2 \left( x + \frac{\pi(\xi-t_0)}{b-c} \right)} d\xi ds > \frac{\pi}{2\beta_0} \text{ for every } t_0 \in [a,c],$$

$$\lim_{x \to -\frac{c}{2}} \int_{t_0}^{t_0 + \frac{b-c}{2}} \int_{t_0}^{t_0 + \frac{a-c}{2}} \frac{h(\xi)}{\cos^2 \left( x + \frac{\pi(\xi-t_0)}{c-a} \right)} d\xi ds > \frac{\pi}{2\beta_0} \text{ for every } t_0 \in [c,b].$$

Then, there exists $m(a,b;\beta_0) \in (-\pi/2,0)$ such that, for every $\beta \geq \beta_0$, an arbitrary solution $u$ to the equation (3) admits

$$m(a,b;\beta_0) \leq u(t) \text{ for } t \in [a,b].$$

**Proof.** Assume on the contrary that for every $n \in \mathbb{N}$ there exists $\beta_n > 0$, a solution $u_n$ to (3) with $\beta = \beta_n$, and $t_n \in [a,b]$ such that $u_n(t_n) = \min_{t \in [a,b]} u_n(t) < 0$ and $u_n(t_n) \to -\pi/2$. Obviously, $u_n'(t_n) = 0$ if $t_n \in (a,b)$, $u_n'(t_n) \geq 0$ if $t_n = a$, and $u_n'(t_n) \leq 0$ if $t_n = b$. Moreover, without loss of generality (passing to a subsequence if necessary) we have either

a) $t_n \in [a,c]$ for every $n \in \mathbb{N}$, or

b) $t_n \in [c,b]$ for every $n \in \mathbb{N}$.

We will assume that a) holds; the case when b) holds can be considered analogously. Notice that $u_n'(t_n) \geq 0$ for every $n \in \mathbb{N}$. Moreover, without loss of generality we can assume that there exists $t_0 \in [a,c]$ such that $t_n \to t_0$. On the other hand, since $u_n''(t) \geq 0$ for $t \in [a,b]$, we have

$$u_n(t) \leq u_n(t_0) + \frac{u_n(b) - u_n(t_0)}{b-t_n}(t-t_n)$$

$$\leq u_n(t_0) + \frac{\pi}{b-c}(t-t_n)$$

$$\leq 0, \text{ for all } t \in [t_n,s_n].$$

Here, $s_n := t_n - \frac{b-c}{\pi}(u_n(t_n))$. Thus, using this in (3) we obtain

$$u_n''(t) \geq \frac{\beta h(t)}{\cos^2 u_n(t)} \geq \frac{\beta_0 h(t)}{\cos^2 \left( u_n(t_0) + \frac{\pi}{b-c}(t-t_n) \right)} \text{ for } t \in [t_n,s_n].$$

(30)
Remark 4. Conditions (28), (33); respectively (29), (34); are fulfilled, i.e., if

\[
\frac{\pi}{2\beta_0} < \int_{t_0}^{t_0 - \frac{h - n}{\pi}} \int_{t_0}^{s} \frac{h(\xi)\,d\xi\,ds}{\cos^2\left(x_0 + \frac{\pi}{b-c}(\xi - t_0)\right)}. \tag{31}
\]

Furthermore, since \(u_n(t) < 0\) for every \(t \in [t_n, s_n]\) and by integrating twice in (30), for \(n \geq n_0\) we have

\[
\frac{\pi}{2\beta_0} > \int_{t_n}^{t_n - \frac{h_n - n}{\pi}} u_n(t_n) \int_{t_n}^{s} \frac{h(\xi)\,d\xi\,ds}{\cos^2\left(u_n(t_n) + \frac{\pi}{b-c}(\xi - t_n)\right)}. \tag{32}
\]

Now, passing to the limit as \(n\) tends to \(+\infty\), in the last inequality it follows that

\[
\frac{\pi}{2\beta_0} \geq \int_{t_0}^{t_0 - \frac{h - n}{\pi}} \int_{t_0}^{s} \frac{h(\xi)\,d\xi\,ds}{\cos^2\left(x_0 + \frac{\pi}{b-c}(\xi - t_0)\right)}. \tag{33}
\]

However, (32) contradicts (31). \(\square\)

Analogously one can prove the following assertion:

**Lemma 6.** Let \(\beta_0 > 0\) and let \([a, b] \subseteq [0, T]\) be such that

\[h(t) \leq 0\] for a. e. \(t \in [a, b]\).

Let, moreover, \(c \in (a, b)\) be such that

\[
\lim_{x \to a} \int_{a}^{t_0 + \frac{h - x}{\pi}} \int_{a}^{s} \frac{|h(\xi)|}{\cos^2\left(x - \frac{\pi(\xi - t_0)}{b-c}\right)}\,d\xi\,ds > \frac{\pi}{2\beta_0} \text{ for every } t_0 \in [a, c], \tag{33}
\]

\[
\lim_{x \to b} \int_{t_0}^{t_0 + \frac{h - x}{\pi}} \int_{t_0}^{s} \frac{|h(\xi)|}{\cos^2\left(x - \frac{\pi(\xi - t_0)}{b-c}\right)}\,d\xi\,ds > \frac{\pi}{2\beta_0} \text{ for every } t_0 \in [c, b]. \tag{34}
\]

Then, there exists \(M(a, b; \beta_0) \in (0, \pi/2)\) such that, for every \(\beta \geq \beta_0\), an arbitrary solution \(u\) to the equation (3) admits

\[M(a, b; \beta_0) \geq u(t)\] for \(t \in [a, b]\).

**Remark 4.** Conditions (28), (33); respectively (29), (34); are fulfilled, e.g., if

\[
\lim_{t \to t_0} \int_{t}^{t_0 + \frac{h - x}{\pi}} \frac{|h(s)|}{\cos^2\left(-\frac{s}{2} + \frac{\pi(s - t_0)}{b-c}\right)}\,ds = +\infty \text{ for every } t_0 \in [a, c],
\]

respectively

\[
\lim_{t \to t_0} \int_{t_0}^{t} \frac{|h(s)|}{\cos^2\left(\frac{s}{2} + \frac{\pi(t_0 - s)}{c-a}\right)}\,ds = +\infty \text{ for every } t_0 \in [c, b].
\]
In particular, since the order of both singularities of the non-linear term is two, the conditions (7)-(8) (with $c_k := c, a_k := a$ and $b_k := b$) imply the conclusions of Lemmas 5 and 6.

Combining Lemma 5 and Lemma 6 one can easily prove the following assertion.

**Lemma 7.** Let $\beta_0 > 0$, and let $(a_k, b_k) \subseteq [0, T]$ be pairwise disjoint intervals and numbers $\delta_k \in \{-1, 1\}$ $(k = 1, \ldots, n)$ such that (5)-(6) hold. Let, moreover, $c_k \in (a_k, b_k)$ $(k = 1, \ldots, n)$ be such that verify (7)-(8). Then, there exist $m(\beta_0) \in (-\pi/2, 0)$ and $M(\beta_0) \in (0, \pi/2)$ such that, for every $\beta \geq \beta_0$, an arbitrary $T-$periodic solution $u_\beta$ to the equation (3) admits the estimate

$$m(\beta_0) \leq u_\beta(t) \leq M(\beta_0) \text{ for } t \in [0, T].$$

We end this part with a remarkable comment about the integral conditions appearing in Theorem 2. In general, under the hypothesis (H) of Theorem 1, there are no a priori bounds for the set of $T-$periodic solutions of the equation (3) with $\beta = \beta_0$. However, these bounds did not play any role in the proof of Theorem 1, where a $T-$periodic solution of (3) for small $\beta > 0$ was guaranteed. Hence, the additional conditions of Theorem 2 are connected only with the presence of various $T-$periodic solutions. It is worth mentioning here that this phenomenon is quite unexpected according to recent works dealing with indefinite singular equations (see e.g., [8, 21, 22, 29, 30]).

**4.2.2. The homotopy.** We devote this section to investigate the topological degree of an operator depending on $\beta > 0$ whose fixed points are periodic solutions to the equation (3). The idea is to compute the degree of the operator for certain $\beta > 0$ and by using the results obtained in the above section to prove that this is preserved.

For this proposal we need to describe briefly a classical framework. First of all, we consider the subspace

$$C_T = \{u \in C : u(0) = u(T), u'(0) = u'(T)\}.$$            

We denote by $P, Q : C \rightarrow C$ the continuous projectors

$$P[u](t) = u(0), \quad Q[u](t) = \varpi.$$            

Furthermore, let $K : C \rightarrow C$ be a continuous linear operator given by

$$K[u](t) = \int_0^t u(s)ds \text{ for } t \in [0, T].$$

All $T-$periodic solutions to (3) make sense on the open set

$$\Delta = \{u \in C_T : -\frac{\pi}{2} < u(t) < \frac{\pi}{2} \text{ for } t \in [0, T]\}.$$            

Thus, for each $\beta > 0$, the operator $S_\beta : \Delta \rightarrow C_T$ is defined by

$$S_\beta = P + Q N_\beta + K(I - Q)K(I - Q)N_\beta,$$            

(35)
where $N_\beta$ is the Nemitsky operator associated with (3), i.e., $N_\beta[u] := \beta h(t) \cos^{-2} u$. Obviously, the operator $S_\beta$ is completely continuous on $\Delta$. Now the periodic boundary value problem for (3) becomes equivalent to the fixed-point problem for an operator equation

$$u = S_\beta[u], \quad u \in \Delta. \quad (36)$$

We can assert even more than in Lemma 4: the assertion below is an elemental consequence of Lemma 4 and the generalized invariance property to the degree in combination with the a priori bounds obtained in Lemma 7.

**Lemma 8.** Let all the assumptions of Theorem 2 be fulfilled and let $\beta_0 > 0$ be. There exists $U_{\beta_0} \subseteq \Delta$ with the following property:

$$d_{LS}(I - S_{\beta_0}, U, 0) = 0$$

for each open set $U \subseteq C_T$ such that $U_{\beta_0} \subseteq U \subseteq U \subseteq \Delta$.

### 4.2.3. The proof of Theorem 2.

First of all, set

$$A = \{ \tilde{\beta} > 0 :$$

there exist at least two $T$-periodic solutions to (3) for every $0 < \beta \leq \tilde{\beta} \}.$$

**Claim.** $A \neq \emptyset$. Theorem 1 implies that there exists $\beta_0 > 0$ such that, for every $0 < \beta \leq \beta_0$, there is a $T$-periodic solution $u_\beta$ to the equation (3) verifying (27).

Let us define

$$\tilde{\beta} := \min \left\{ \beta_0, 2 \left( \frac{\pi}{T} \right)^2 \frac{\cos^3 u_0}{\|h\|_\infty} \right\}.$$

Here, $u_0$ was defined in the proof of Theorem 1 to simplify the notation. In principle, $\tilde{\beta} \neq \beta_0$, but by the construction of $\beta_0$ one has equality (this observation is not needed for the proof). We now consider the homotopy (36), where the operator $S_\beta$ is defined by (35). Obviously, the operator $S_\beta$ is differentiable at $u_\beta$ and we claim that, for every $0 < \beta < \tilde{\beta},$

$$I - dS_\beta[u_\beta] : C_T \to C_T$$

is an isomorphism.

Indeed, this is equivalent to prove that the equation

$$v'' = \beta q_\beta(t)v \quad (37)$$

has a nontrivial $T$-periodic solution, where

$$q_\beta(t) := \frac{2h(t) \sin u_\beta(t)}{\cos^3 u_\beta(t)} \quad \text{for a. e. } t \in [0, T].$$

First of all we observe that

$$\|q_\beta\|_\infty \leq \frac{2\|h\|_\infty}{\cos^3 u_0} \quad (38)$$
which does not depend on $\beta$. Here, the inequality (27) has been exploited. On the other hand, a direct computation shows:

$$
\int_0^T q_\beta(s)ds = 2 \int_0^T \frac{h(s)}{\cos^2 u_\beta} \tan u_\beta ds
= 2 \int_0^T u_\beta'' \tan u_\beta ds
= -2 \int_0^T \frac{u_\beta'^2}{\cos^2 u_\beta} ds.
$$

Since $u_\beta$ is a $T$-periodic solution to the equation (3), it cannot be constant, and therefore $\bar{q}_\beta < 0$. By using now [Theorem 1.1][24], on account of (38) we have the unique solvability of the boundary periodic problem associated with the equation (37). At this moment, for the computation of the degree of $I - S_\beta$ we apply the linearization principle, i.e., we have that, for every $0 < \beta \leq \bar{\beta}$, there exists $V_\beta$ a neighborhood of $u_\beta$ such that

$$
d_{LS}(I - S_\beta, V_\beta, 0) = d_{LS}(I - dS_\beta[u_\beta], B(0, \delta), 0) \neq 0.
$$

where $\delta > 0$. Finally, we fix $0 < \beta_* \leq \bar{\beta}$ and, by applying Lemma 8, we find $U_\beta$, such that $V_\beta \subseteq U_\beta \subseteq \overline{U_\beta} \subseteq \Delta$ verifying

$$
d_{LS}(I - S_\beta, U_\beta, 0) = 0.
$$

Now, by the excision property of the degree we get $d_{LS}(I - S_\beta, U_\beta, \Delta_\beta, 0) \neq 0$, whence one deduces the existence of at least two $T$-periodic solutions for $\beta = \beta_*$. This concludes the proof of the claim.

We now define

$$
\beta_* := \sup A
$$

and $\beta^* \geq \beta_*$ given by Lemma 4. An elementary analysis of these numbers proves Theorem 2 (notice that the estimate of the solutions obtained in Lemma 7 together with the Arzel`a-Ascoli Theorem imply the existence of a solution for $\beta = \beta_*$).

**Conclusions and final remarks.** Our results can be of substantial interest in two areas: physics and mathematics. Indeed, they are relevant in the development of the recent theory devoted to investigation of $T$-periodic solutions to indefinite singular equations. On other hand, from physical point of view, our results allow to better understand the existence and multiplicity of periodic motions of a charged particle in an oscillating magnetic field on the sphere, under many reasonable cases which are still unexplored (for instance when $h(t) := \sin t$).

We turn now our attention to (H) in order to understand more about the applicability of our results. As we mentioned in the introduction, the classical relation between the multiplicity of every zero of the weight function and the order of the singularity of the nonlinear term loses its importance in the solvability of the periodic problem associated with the equation (3) when $\bar{h} = 0$. The example below reflects this situation (see also Figure 2).
Example 1. Consider

\[ u'' = \frac{\sin^m t}{\cos^2 u}, \]

where \( m \geq 1 \) is an odd number. Then, there exist \( 0 < \beta_* \leq \beta^* < +\infty \) such that for every \( \beta > \beta^* \) there is no \( 2\pi \)-periodic solution, and for \( \beta \leq \beta_* \) there exists at least one \( 2\pi \)-periodic solution.

![Figure 2. A numerical approximation to a closed orbit of the equation (3) for \( \beta = 0.01 \), with \( h(t) = \sin^{99} t \).](image1)

![Figure 3. A numerical approximation to a closed orbit of the equation (3) for \( \beta = 0.01 \), with \( h(t) = \sin^{99} t \).](image2)

In addition, (H) covers the classical situations which have already been investigated in the literature. For instance, we can generalize in somehow the result proved in [20]. The following assertion follows from our Theorem 2.

**Corollary 1.** Let \( \overline{h} = 0 \), and let us assume that \( h \) is a piecewise constant function with both positive and negative values. Then, there exists \( \beta_* > 0 \) such that (3) has at least two \( T \)-periodic solutions for every \( 0 < \beta < \beta_* \) and at least one \( T \)-periodic solution for \( \beta = \beta_* \). Moreover, there exists \( \beta^* \geq \beta_* \) such that the equation (3) has no \( T \)-periodic solution for \( \beta > \beta^* \).

See [Corollary 6][22] for the case when \( \overline{h} \neq 0 \).

Before closing this paper, let us underline that attitude of the periodic problem associated with (3) when \( \overline{h} = 0 \) substantially differs from earlier cases studied in the literature. Indeed, only when \( \overline{h} = 0 \), any \( T \)-periodic solution \( u_\beta \) of (3) goes to zero as \( \beta \to 0 \) (otherwise, either \( u_\beta \to -\pi/2 \) if \( \overline{h} < 0 \) or \( u_\beta \to \pi/2 \) if \( \overline{h} > 0 \)). Hence, it is not possible to construct a universal lower and upper functions for all \( \beta > 0 \) small in the direction of Lemma (3).

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PERIODIC SOLUTIONS TO AN INDEFINITE SINGULAR EQUATION

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