Model-free Reinforcement Learning for Content Caching at the Wireless Edge via Restless Bandits

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Abstract—An explosive growth in the number of on-demand content requests has imposed significant pressure on current wireless network infrastructure. To enhance the perceived user experience and support latency-sensitive applications, edge computing has emerged as a promising computing paradigm. The performance of a wireless edge depends on contents that are cached. In this paper, we consider the problem of content caching at the wireless edge with unreliable channels to minimize average content request latency. We formulate this problem as a restless bandit problem, which is provably hard to solve. We begin by investigating a discounted counterpart, and prove that it admits an optimal policy of the threshold-type. We then show that the result also holds for the average latency problem. Using these structural results, we establish the indexability of the problem, and employ Whittle index policy to minimize average latency. Since system parameters such as content request rate are often unknown, we further develop a model-free reinforcement learning algorithm dubbed Q-Whittle learning that relies on our index policy. We also derive a bound on its finite-time convergence rate. Simulation results using real traces demonstrate that our proposed algorithms yield excellent empirical performance.

Index Terms—Wireless Edge, Content Caching, Whittle Index Policy, Restless Bandits, Reinforcement Learning, Finite Time Analysis

I. INTRODUCTION

T HE dramatic growth of wireless traffic due to an enormous increase in the number of mobile devices is posing many challenges to the current mobile network infrastructures. In addition to this increase in the volume of traffic, many emerging applications such as Augmented/Virtual Reality, autonomous vehicles and video streaming, are latency-sensitive. In view of this, the traditional approach of offloading the tasks to remote data centers is becoming less attractive. Furthermore, since these emerging applications typically require unprecedented computational power, it is not possible to run them on mobile devices, which are typically resource-constrained.

To support such stringent timelines and computational requirements, mobile edge computing architectures have been proposed as a means to improve the quality of experience (QoE). These move servers from the cloud to edges, often wirelessly that are closer to end users. Such edge servers are often empowered with a small wireless base station, e.g., the storage-assisted future mobile Internet architecture and cache-assisted 5G systems [1]. By using such edge servers, content providers are able to ensure that contents or services are provided with a high QoE (with minimal latency). The success of edge servers relies upon “content caching”, in which popular content such as movies, videos, software are placed at the cache associated with the wireless edge. If the content requested by end users is available at the wireless edge, then it is promptly delivered to them. Otherwise, the request is forwarded to the remote server, and this gives rise to an increase in the latency. While the remote server often resides in a well-provisioned data center, resources are typically limited at the edge, so that the amount of content that can be cached at the wireless edge is often limited. These issues are further exacerbated in the case of wireless edges, where the requested content is delivered over unreliable channels.

In this work, we are interested in minimizing the average latency incurred while delivering contents to end users connected to a wireless edge via unreliable channels. We design dynamic policies that decide which content should be cached at the wireless edge so as to minimize the average latency of end users, while simultaneously satisfying resource constraints associated with the wireless edge?

We pose this problem as a Markov decision process (MDP) in Section III. This MDP turns out to be a restless multi-armed bandit (RMAB) problem [3]. Even though in theory RMAB can be solved by using relative value iteration [2], [4], this approach suffers from the curse of dimensionality, and fails to provide an insight into the solution. Thus, it is desirable to derive low-complexity solutions and provide guarantees on their performance. A celebrated policy for RMAB is the Whittle index policy [5]. We propose to use the Whittle index policy for solving the problem of optimal caching.

Following the approach taken by Whittle [5], we begin by relaxing the hard constraints of the original MDP, which requires that the number of cached contents at each time is exactly equal to the cache size. These are relaxed to a constraint which requires that the number of cached contents is equal to the cache size on average. We then consider the Lagrangian of this relaxed problem, which yields us a set of decoupled average-reward MDPs, which we call the per-content MDP. Instead of optimizing the average cost (latency) of this per-content MDP, we firstly consider a discounted per-content RMAB which can be solved by using relative value iteration [2]. This structural result is then shown to also hold for the average latency problem. We use this structural result to show that our problem is indexable [3]. We derive Whittle indices for each content. Whittle index policy then prioritizes contents in decreasing order of their Whittle indices, and caches the maximum possible content that is allowed with the available cache size. Whittle index policy is computationally tractable
since its complexity increases linearly with the number of contents. Moreover it is known to be asymptotically optimal \cite{5, 6} as the number of contents and the cache size are scaled up, while keeping their ratio as a constant. Our contribution in Section \[IV\] is non-trivial since establishing indexability of RMAB problems is typically intractable \cite{7}, and Whittle indices of many practical problems remain unknown except for a few special cases.

Note that Whittle index policy needs to know controlled transition probabilities of the underlying MDP, which in our case amounts to knowing the statistics of the content request process associated with end users, and also the reliability of wireless channels. However, these parameters are often unknown, and time-varying. Hence in Section \[V\] we design an efficient reinforcement learning (RL) algorithm to make optimal content caching decisions dynamically when these parameters are unknown. We do not directly apply off-the-shelf RL methods such as UCRL2 \cite{8} and Thompson Sampling \cite{9} since the size of state-space grows exponentially with the number of contents, and hence the computational complexity and learning regret would also grow exponentially. Thus, the resulting algorithms would be too slow to be of any practical use. To overcome these challenges, we derive a model-free RL algorithm called Q-Whittle learning. By coupling conventional Q-functions with Whittle indices \cite{10}, \cite{11}, our Q-Whittle learning leverages the threshold-structure of the optimal policy to learn only Q-values of state-action pairs following the current threshold policy. This novel update rule significantly improves the sample efficiency of Q-learning with conventional \(\varepsilon\)-greedy policy. Finally, our Q-Whittle learning can be viewed through the lens of a two-timescale stochastic approximation (2TSA) \cite{12}, \cite{13}. We prove a bound on its finite-time convergence rate in Section \[VI\] To the best of our knowledge, our work is perhaps the first to consider a RL approach towards a Whittle index policy derived from a MDP in the context of content caching at the wireless edge with unreliable channels, and the first to provide a finite-time analysis of a Whittle index based Q-learning algorithm.

Finally, we provide extensive numerical results using both synthetic and real traces to support our theoretical findings in Section \[VII\], which demonstrate that our proposed algorithms produce significant performance gain over state of the arts.

II. RELATED WORK

We overview two areas closely related to our work: content caching and restless bandits, and provide a brief discussion of our design methodology in the context of prior work.

Content Caching. The content caching problem has been studied in numerous domains with different objectives such as minimizing expected delay \cite{14} or operational costs \cite{15}. The joint content caching and request routing has also been investigated, e.g., \cite{16}, \cite{17}. Most prior works formulated the problem as a constrained/stochastic optimization problem, etc. None of those works provided a formulation using the RMAB framework and developed an index based caching policy. Furthermore, all above works assumed full knowledge of request processes and hence did not incorporate a learning component. A recent line of works considered content caching from an online learning perspective, e.g., \cite{18}--\cite{19}, and used the performance metric of learning regret or competitive ratio. Works such as \cite{20}, \cite{21} used deep RL methods. However, deep RL methods lack of theoretical performance guarantees. Our model, objective and formulation significantly depart from those considered in aforementioned works, where we pose the content caching problem at the wireless edge as a MDP and develop a simple index policy with performance guarantee that can be easily learned through a model-free RL framework.

Restless Bandits. The RMAB is a general framework for sequential decision making problems, e.g., \cite{22}, \cite{23}. However, RMAB is notoriously intractable \cite{25}. One celebrated heuristic is the Whittle index policy \cite{3}. However, Whittle index is well-defined only when the indexability condition is satisfied, which is in general hard to verify. Further, even when an arm is indexable, finding its Whittle index can still be intractable \cite{7}. A few successes e.g. \cite{23}, \cite{24} are all under specific assumptions and hard to be generalized. Additionally, the application of above Whittle index requires full system knowledge, which is often not the case in practice. Thus it is important to examine RMAB from a learning perspective, e.g., \cite{26}--\cite{32}. However, these methods do not exploit the special structure available in the problem and contend directly with an extremely high dimensional state-action space yielding the algorithms to be too slow to be of any practical use. Recently, RL based algorithms have been developed \cite{10}, \cite{11}, \cite{33} to explore the problem structure through index policies. However, \cite{10}, \cite{11} lack finite-time performance analysis and multi-timescale SA algorithms usually suffer from slow convergence, and \cite{33} depends on a simulator for a finite-horizon setting which cannot be directly applied here since it is difficult to build a perfect simulator in complex wireless edge environments. In contrast, we provide a finite-time analysis of our Q-Whittle learning algorithm. The closest work is \cite{34}, which characterized the convergence rate for a general non-linear 2TSA. We generalize the result to the proof of convergence rate of our Q-Whittle learning algorithm.

Our Design Philosophy. We make contributions to both areas in this paper. First, we formulate the content caching problem at the wireless edge with unreliable channels so as to minimize the average content request latency as an average-reward MDP, which turns out to be a RMAB. Second, we consider this RMAB from an online perspective given that the knowledge of content request process and unreliable wireless channel is often unknown and time-varying. A key differentiator between our approach and existing ones stems from two perspectives: (i) we focus on designing index policy for content caching at the wireless edge, which operate on a much smaller dimensional subspace by exploiting the inherent structure of our problem; and (ii) our index-based approach naturally lends itself to a lightweight model-free RL based framework that can fully exploit the structure of our index policy so as to reduce the high computational complexity.

III. SYSTEM MODEL AND PROBLEM FORMULATION

In this section, we present the system model and formulate the average latency minimization problem.
outstanding requests for content succeeds with probability due to noise or interference. We assume that the transmission user through a wireless channel, which is often by a user is available with the cache, it is delivered to the end users directly through a wireless channel that is unreliable. Otherwise, the request is sent to the remote server at the cost of a longer latency. The goal of the wireless edge system is to decide which content to cache, subject to cache capacity constraint, so that the average content request latency experienced by end users is minimal.

Content Request and Delivery Model. We assume that requests for content \( m \in \mathcal{M} \) arrive at the wireless edge from end users according to a Poisson process with arrival rate \( \lambda_m \). The time taken to deliver content \( m \) to end users is a random variable that is exponentially distributed with mean \( 1/\mu_m \).

Unreliable Channel. In the case the content which is requested by a user is available with the cache, it is delivered to the user through a wireless channel, which is often unreliable due to noise or interference. We assume that the transmission succeeds with probability \( q \in (0,1] \).

Queue Model. To each content \( m \), we associate a “request queue” at the wireless edge, which stores the number of outstanding requests for content \( m \) at time \( t \). This is denoted by \( S_{m,t} \). This assumption is justified since the wireless edge is closer to end users and receives content requests at a much faster timescale as compared with the content downloads or updates in the wireless edge from the remote server. The content requested from an end user might not be served immediately, so that there will be latency associated with the user getting content. This motivates us to consider a queuing model that captures the latency experienced by end users.

Decision Epochs. The decision epochs are the moments when the states of request queues change. At each decision epoch/time \( t \), the wireless edge determines for each content whether or not it should be cached, and then delivers cached contents to desired end users.

1. Our model can be generalized to the case of contents with variable sizes by dividing contents into unit-sized chunks.
2. Poisson arrivals have been widely used in the literature, e.g., [16], [17] and references therein. However, our model holds for general stationary process and our RL based algorithm and analysis in Section V holds for any request process.

Fig. 1. System model for content caching at the wireless edge with unreliable channels.

A. System Model

Consider a wireless edge system connected to a remote server (e.g., data center) through backhaul links as shown in Figure 1. The wireless edge is equipped with a cache of size \( B \) units in which it stores contents that are provided to end users. We denote the set of distinct contents as \( \mathcal{M} = \{1, \ldots, M\} \) with \( |\mathcal{M}| = M \). Without loss of generality (W.l.o.g.), we assume that all contents are of unit size\(^1\) End users make requests for different contents to the wireless edge. If the requested content is available at the wireless edge, then it is delivered to end users directly through a wireless channel that is unreliable. Otherwise, the request is sent to the remote server at the cost of a longer latency. The goal of the wireless edge is to decide which content to cache, subject to cache capacity constraint, so that the average content request latency experienced by end users is minimal.

B. System Dynamics

We now formulate the problem of average latency minimization for the above model as a MDP.

States. We denote the state of the wireless edge at time \( t \) as \( S_t := (S_{1,t}, \ldots, S_{M,t}) \in \mathbb{N}^M \), where \( S_{m,t} \) is the number of outstanding requests for content \( m \in \mathcal{M} \) from end users at time \( t \) as described above.

Actions. The action for content \( m \) at time \( t \) is denoted as \( A_{m,t} \), where \( A_{m,t} = 1 \) means that content \( m \) is cached in the wireless edge at time \( t \), and \( A_{m,t} = 0 \) otherwise. Denote \( \mathbf{A}_t := (A_{1,t}, \ldots, A_{M,t}) \). Taking the cache capacity constraint at the wireless edge into account, we have that \( A_t \) must satisfy the following constraints,

\[
\sum_{m=1}^{M} A_{m,t} \leq B, \quad \forall t. \tag{1}
\]

A content caching policy \( \pi \) maps the state of the wireless edge \( S_t \) to the caching decision action \( \mathbf{A}_t \), i.e., \( \mathbf{A}_t = \pi(S_t) \).

Transition Kernel. The state of the \( m \)-th request queue can change from \( S_m \) to either \( S_m + 1 \), or \( S_m - 1 \) at the beginning of each decision epoch. More specifically,

\[
S = \begin{cases} 
S + e_m, & \text{with transition rate } b_m(S_m, A_m), \\
S - e_m, & \text{with transition rate } d_m(S_m, A_m),
\end{cases}
\tag{2}
\]

where \( e_m \) is a \( M \)-dimension vector with the \( m \)-th entry being 1 and all others being 0, \( b_m(S_m, A_m) = \lambda_m \) and \( d_m(S_m, A_m) = \mu_m(S_m) q \) with \( \mu_m(0) = 0 \). Note that we allow for state-dependent content delivery rates, which enables us to model realistic settings [23], [24]. In particular, we consider the classic \( M/M/k \) queue, i.e., \( d_m(S_m, A_m) = \mu_m S_m A_m q \).

C. Problem Formulation

We denote by \( C_{m,t}(S_{m,t}, A_{m,t}) \) the instantaneous cost incurred by content \( m \) at time \( t \). Note that this depends upon its state \( S_{m,t} \) and also the action \( A_{m,t} \) that is applied to it. It follows from Little’s Law [36] that minimizing the average latency is equivalent to minimizing the average total number of outstanding requests in the system. Hence, with this choice of instantaneous cost, the average cost represents the average latency for end users. We denote the immediate total cost at time \( t \) as

\[
C_t(S_t, \mathbf{A}_t) = \sum_{m=1}^{M} C_{m,t}(S_{m,t}, A_{m,t}) = \sum_{m=1}^{M} S_{m,t}. \tag{3}
\]

Our objective is to derive a policy \( \pi \) that makes decisions regarding which content should be cached at the capacity-constrained wireless edge to minimize the average latency. This problem can be formulated as the following MDP:

\[
\min_{\pi \in \Pi} C_\pi := \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_\pi \left[ \int_{0}^{T} S_{m,t} dt \right] \tag{4}
\]

s.t. \( \sum_{m=1}^{M} A_{m,t} \leq B, \quad \forall t, \)
where the subscript denotes the fact that the expectation is taken with respect to the measure induced by policy $\pi$, and $\Pi$ is the set of all feasible content caching policies.

For simplicity, we convert the continuous-time MDP problem \((4)\) into an equivalent discrete-time MDP problem by using the method of uniformization \([2]\). Thus, a time slot corresponds to either a new arrival (of request), or a departure (content is delivered to user). Let $\mathcal{N} = \{1, \ldots, N\}$ denote the set of time slots such that the state of the wireless edge does not change during each time slot. Denote the system state at time slot $n \in \mathcal{N}$ as $S_n$. W.l.o.g., we scale time, and that the transition probabilities for the $m$-th request queue in the MDP are defined as

$$\begin{align*}
\Pr(S_m, A_{m,n}, S_{m+1}) &= \frac{\lambda_m}{\sum_{m=1}^{M} \left( \lambda_m + \mu_m S_m A_{m,n} q \right)} ,
\Pr(S_m, A_{m,n}, S_m - 1) &= \frac{\mu_m S_m A_{m,n} q}{\sum_{m=1}^{M} \left( \lambda_m + \mu_m S_m A_{m,n} q \right)},
\end{align*}$$

\(5\)

such that $\sum_{m=1}^{M} \left[ \Pr(S_m, A_{m,n}, S_{m+1}) + \Pr(S_m, A_{m,n}, S_m - 1) \right] = 1, \forall n, m$. The equivalent discrete-time MDP, obtained after uniformization, is as follows:

$$\begin{align*}
\min_{\pi \in \Pi} \limsup_{N \to \infty} \frac{1}{N} \mathbb{E}_{\pi} \left[ \sum_{n=1}^{N} S_{m,n} \right] \\
s.t. \sum_{m=1}^{M} A_{m,n} \leq B, \forall n.
\end{align*}$$

\(6\)

Henceforth, we refer to \((6)\) as the “original MDP”. Since it is an infinite-horizon average cost per stage problem, in principle it can be solved by using the relative value iteration \([2, 4]\).

**Lemma 1.** Given the transition probabilities in \((5)\), there exist a $\beta^*$ and a function $V(\cdot)$ that satisfy the following dynamic programming (DP) equation \((2)\):

$$V(S) + \beta^* = \min_{A \leq B} \left\{ \sum_{m=1}^{M} \left[ S_m + \lambda_m V(S + e_m) + \mu_m S_m A_{m,n} q V(S - e_m) \right] \right\}.$$  

\(7\)

One can always obtain an optimal policy $\pi^*$ which fulfills $\beta^* = \min_{\pi} C_\pi$ with methods such as the relative value iteration. However, this approach suffers from the curse of dimensionality, i.e., the computational complexity grows exponentially in the size of state space as a function of the distinct content number $M$, rendering such a solution impractical. Furthermore, this approach fails to provide insight into the structure of the solution. To this end, many efforts have been focused on developing computationally appealing solutions.

### D. Lagrangian Relaxation

In this subsection, we discuss the Lagrangian relaxation of the original MDP \((6)\) and the corresponding per-content problems. The Lagrangian multipliers together with these per-content problems form the building block of our Whittle index policy, that will be formally introduced in Section IV.

Following Whittle’s approach \([3]\), we first consider the following “relaxed problem,” which relaxes the “hard” constraint in \((6)\) to an average constraint:

$$\begin{align*}
\min_{\pi} \limsup_{N \to \infty} \frac{1}{N} \mathbb{E}_{\pi} \left[ \sum_{n=1}^{N} S_{m,n} \right] \\
s.t. \limsup_{N \to \infty} \frac{1}{N} \mathbb{E}_{\pi} \left[ \sum_{n=1}^{N} A_{m,n} \right] \leq B.
\end{align*}$$

\(8\)

Next, we consider the following Lagrangian relaxation \([37]\). The Lagrangian can be written as,

$$L(\pi, W) = \limsup_{N \to \infty} \frac{1}{N} \mathbb{E}_{\pi} \left[ \sum_{n=1}^{N} \left\{ \sum_{m=1}^{M} S_{m,n} - W \left( B - \sum_{m=1}^{M} A_{m,n} \right) \right\} \right].$$

\(9\)

Next, we consider the following Lagrangian relaxation \([37]\). The Lagrangian can be written as,

$$L(\pi, W) = \limsup_{N \to \infty} \frac{1}{N} \mathbb{E}_{\pi} \left[ \sum_{n=1}^{N} \left\{ \sum_{m=1}^{M} S_{m,n} - W \left( B - \sum_{m=1}^{M} A_{m,n} \right) \right\} \right].$$

\(9\)

Given the Lagrangian multiplier $W$, the relaxed problem decouples into $M$ “per-content MDPs,” where the MDP for the $m$-th content is given as follows,

$$\begin{align*}
\min_{\pi_m} \limsup_{N \to \infty} \frac{1}{N} \mathbb{E}_{\pi_m} \left[ \sum_{n=1}^{N} \bar{C}(S_{m,n}, A_{m,n}) \right],
\end{align*}$$

\(11\)

where $\bar{C}(S_{m,n}, A_{m,n}) = S_{m,n} - W(1 - A_{m,n})$, and $\pi_m$ is the policy for content $m$. With this decomposition, in order to evaluate the dual function at $W$, it suffices to iteratively solve all $M$ independent per-content MDPs \((11)\) \([2, 4]\). The relaxed problem \((8)\) can be solved by solving each of these $M$ per-content MDPs, and then combining their solutions, i.e., to each content $m$ we apply the solution corresponding to its individual MDP. Note that this solution does not always provide a content caching policy that is feasible for the original problem \((6)\), since the original problem requires that the cache capacity constraint \((1)\) must be met at all times, rather than just in the average sense as in the constraint \((8)\). Whittle index policy combines these solutions in such a way that the resulting allocation is also feasible for the original problem \((6)\), i.e., it satisfies hard constraints.

### IV. Whittle Index Policy

We now design the Whittle index policy for content caching at the wireless edge with unreliable channels, as illustrated in Figure 2. To the best of our knowledge, Whittle index policy has not been used in order to solve this problem in the literature. Specifically, the content caching problem \((6)\) can be posed as a RMAB problem in which each content $m \in M$ corresponds to an arm $m$. At each time slot $n$, the queue length $S_{m,n}$ of the corresponding request queue is the state of arm $m$, and $A_{m,n}$ is the action taken for the content $m$. We let $A_{m,n} = 1$ denote caching for content $m$ at time $n$, while $A_{m,n} = 0$ denote not caching. It is well known that the Whittle index policy is a computationally tractable solution to the RMAB \([3]\), which has a computational complexity that scales linearly with $M$. 


### Section IV-B

| Proposition 1 |
|-------------------------------|

**Optimal policy to (12) is of threshold-type**

Propositions 2 & 3

**Threshold policy to (11) and MDP (11) is indexable**

Propositions 4 & 5

**Whittle index policy for the original problem (6)**

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**A. Indexability and Whittle Index**

Our proposed Whittle index policy is based on the solution to the relaxed problem \([3]\). In order to derive this policy, we need to establish that our MDP is *indexable*. Roughly speaking, this property requires to show that as the Lagrangian multiplier \(W\) increases, the collection of states in which the optimal action is passive (i.e., not to cache) increases. This property was first introduced by Whittle \([3]\) and we define it formally here for completeness.

**Definition 1. (Indexability)** We denote \(D_m(W)\) as the set of states \(S\) for which the optimal action for the \(m\)-th MDP \([11]\) is to choose a passive action, i.e., \(A_m = 0\). Then the \(m\)-th MDP is said to be indexable if \(D_m(W)\) increases with \(W\), i.e., if \(W > W'\), then \(D_m(W) \supset D_m(W')\). The original MDP \([4]\) is indexable if all \(M\) per-content MDPs are indexable.

Following the indexability property, the Whittle index in a particular state \(S\) is defined as follows.

**Definition 2. (Whittle Index)** The Whittle index in state \(S\) for the indexable \(m\)-th MDP \([11]\) is the smallest value of the Lagrangian multiplier \(W\) such that the optimal policy for content \(m\) at state \(S\) is indifferent towards actions \(A_m = 0\) and \(A_m = 1\). We denote such a Whittle index as \(W_m(S) := \inf_{W \geq 0} \{S \in D_m(W)\}\).

### Section IV-C

**The per-content MDP \([11]\) is indexable**

Our proof of indexability relies on the “threshold” property of the optimal policy for each per-content MDP \([11]\), i.e., content \(\forall m \in M\) is cached in the wireless edge only when the number of outstanding requests for content \(m\) is above a certain threshold. To this end, we focus on the per-content MDP \([11]\) for a particular content \(m\), and drop the subscript \(m\) in the rest of this subsection for ease of exposition.

**1) Optimal Threshold Policy:** To avoid convergence issues in the presence of bounded value function \([2]\), we start with the problem of minimizing the expected total discounted latency of content requests over the wireless edge, i.e., the discounted latency problem. Then we will extend our results to the average latency problem \([11]\). The discounted latency problem in the equivalent discrete-time MDP is given as

\[
\min_\pi \mathbb{E} \left[ \lim_{N \to \infty} \sum_{n=1}^{N} \alpha^{n-1} C(S_n, A_n) | S_1 = s \right],
\]

where \(0 < \alpha < 1\) is a discount factor. It is known that there exists an optimal deterministic stationary policy for the discounted latency problem \([2]\). Hence we only need to consider the class of deterministic stationary policies. We apply the value iteration method to find the optimal policy.

We assume that value functions of the initial state \(s \in \mathbb{N}\) are bounded real-value functions, i.e., the system is stable. Let \(U\) denote the Banach space of bounded real-value functions on \(\mathbb{N}\) with supremum norm. Define operator \(T : U \to U\) as

\[
(T u)(s) = \min_{a \in \{0, 1\}} \bar{C}(s, a) + \alpha \mathbb{E}[u(s')],
\]

where \(u(\cdot) \in U\) and the expectation is taken over all possible next state \(s'\) when action \(a\) is taken at state \(s\). Let \(J^*(s)\) denote the optimal expected total discounted cost of initial state \(s\). Then we have that \(J^*(s) = T J^*(s)\), i.e., \(J^*(s)\) is a solution of the Bellman equation satisfying

\[
J^*(s) = \min_{a \in \{0, 1\}} \bar{C}(s, a) + \alpha \mathbb{E}[J^*(s')].
\]

As described in \([2]\), the next state \(s'\) can be either \(s-1\) or \(s+1\). Define \(P_s = \lambda s + 1\) for ease of expression. Hence we can further write \([14]\) as

\[
J^*(s) = \min_{a \in \{0, 1\}} \bar{C}(s, a) + \alpha \left( P_s J^*(s+1) + (1-P_s) J^*(s-1) \right).
\]

The corresponding state-action value function satisfies

\[
Q^*(s, a) = \bar{C}(s, a) + \alpha \left( P_s J^*(s+1) + (1-P_s) J^*(s-1) \right).
\]

Therefore, we have \(J^*(s) = \min_{a \in \{0, 1\}} Q^*(s, a)\).

As shown in \([3]\), for an extreme large value of the Lagrangian multiplier \(W\), it is optimal to keep the arm passive (never cache the content), i.e., \(a = 0\), for each state \(s\). Hence, we make the following assumption.

**Assumption 1.** The Lagrangian multiplier \(W\) is a finite positive real number such that there exists at least one state \(s\) satisfying \(Q^*(s, 1) \leq Q^*(s, 0)\).

We now show that the optimal policy for \([12]\) is of the threshold-type under a fixed \(W\).

**Proposition 1.** Consider the discounted latency MDP \([12]\) with a fixed \(W \geq 0\). There exists an optimal policy for \([12]\) of threshold-type with the threshold depending upon \(W\).

Proof is provided in Appendix \(A\).

**Remark 1.** Existing works \([28, 29]\) among others has also shown that a threshold policy is optimal to an MDP-based problem formulation to show the Whittle indexability. The key is to show that if the optimal action at state \(a\) is active, i.e., \(Q^*(s, 1) \leq Q^*(s, 0)\), the optimal action at state \(s+1\) is also active, i.e., \(Q^*(s+1, 1) \leq Q^*(s+1, 0)\). This depends on showing the convexity \([28]\) or monotonicity \([29]\) of discounted value function \(J^*(s)\), \(\forall s\) by leveraging dynamic value iteration. These properties hold under a fixed and state-independent kernel \([28, 29]\), however, it is hard to show or
might not hold at all when transition probabilities are state-dependent as in \cite{15}. To this end, existing results cannot be directly applied here, and the analyses in this paper and those in \cite{28}, \cite{29} are significantly different.

The following proposition extends our results in Proposition 1 for the discounted latency problem in \cite{12} to the original average latency in \cite{11}.

**Proposition 5.** The stationary distribution of the threshold policy is required to compute the Whittle indices. We now compute this stationary distribution under our model.

**Proof.** According to \cite{40}, the optimal expected total discounted latency \( J_{\pi^*_\alpha} \) under optimal policy \( \pi^*_\alpha \) with discount factor \( \alpha \), and the optimal average latency \( J_{\pi^*} \) under optimal policy \( \pi^* \) satisfy \( \lim_{\alpha \to 1}(1-\alpha)J_{\pi^*_\alpha}(s) = J_{\pi^*}(s) \), \( \forall s \). Since our action set is finite, there exists an optimal stationary policy for the average latency problem such that \( \pi^*_\alpha \to \pi^* \) \cite{40}, which implies the optimal policy for \cite{11} is of the threshold-type. \( \Box \)

2) **Indexability of the per-content MDP \cite{11}:** We are now ready to show that the per-content MDP \cite{11} is indexable.

**Proposition 3.** The per-content MDP \cite{11} is indexable.

**Proof.** The proof is provided in Appendix B.

**Proposition 4.** Define the stationary distribution of state \( s \) under threshold policy \( \pi = R \) as \( \phi^*_R(s) \). The Whittle index for the per-content MDP \cite{11} is given as follows,

\[
W(R) = \frac{\sum_{s \in S} \phi^*_R(s) \left( \sum_{s' \in S} \phi^*_R(s') - \phi^*_R(s) \right)}{\sum_{s \in S} \phi^*_R(s) - \sum_{s' = 0}^{R-1} \phi^*_R(s')},
\]

when the right hand side of \eqref{eq:16} is non-decreasing in \( R \).

**Proof.** The proof is based on the definition of Whittle index and can be found in \cite{23}, \cite{24}.

**Remark 2.** Since the cost function and stationary probabilities are known, \eqref{eq:16} can be numerically computed. From \eqref{eq:16}, it is clear that the index of content \( m \) does not depend on the number of requests to other contents \( m' \), \( m' \neq m \). Therefore, it provides a systematic way to derive simple policies that are easy to implement.

From \eqref{eq:16}, it is clear that the stationary distribution of the threshold policy is required to compute the Whittle indices. We now compute this stationary distribution under our model.

**Proposition 5.** The stationary distribution of the threshold policy \( \pi = R \) satisfies

\[
\begin{align*}
\phi^*_R(R') & = 0, \quad \phi^*_R(R) = \frac{\mu q(R + 1)}{\lambda + \mu q(R + 1)} \cdot \phi^*_R(R + 1), \\
\phi^*_R(R + 1) & = 1 / \left( 1 + \frac{\mu q(R + 1)}{\lambda + \mu q(R + 1)} \right) \\
& + \sum_{l=2}^{\infty} \prod_{j=2}^{l-1} \frac{\lambda}{\lambda + \mu q(R + j - 1)} \cdot \frac{\lambda + \mu q(R + j)}{\mu q(R + j)}, \\
\phi^*_R(R + l) & = \phi^*_R(R + 1) \prod_{j=2}^{l} \frac{\lambda}{\lambda + \mu q(R + j - 1)}
\end{align*}
\]

\[
\lambda + \mu q(R + j) \quad , \quad l = 2, 3, \cdots , (17)
\]

where \( R' \) is a dummy state representing state 0 to \( R - 1 \).

**Proof.** Provided in Appendix C.

**C. Whittle Index Policy**

We now describe how the solutions to the relaxed problem \cite{9} are used to obtain a policy for the original problem \cite{6}. It is clear that the optimal solutions to \cite{9} are not always feasible for \cite{6}, since in the latter at most \( B \) contents can be cached at the wireless edge. To this end, Whittle \cite{3} proposed a heuristic, referred to as Whittle index policy, which assigns an index to the \( m \)-th MDP \cite{11} for all \( m \in M \), which depends on its current state and current time. The Whittle index policy then activates the \( B \) arms with the highest Whittle indices \( W_m(S_m) \) given that the \( m \)-th arm is in state \( S_m \) at the current time. Although Whittle index policy is not optimal to the original problem \cite{6} in general, it has been shown that such a policy is asymptotically optimal \cite{4}, \cite{6} as the number of contents and cache size are scaled up, while keeping their ratio as a constant.

**V. Q-WHITTLE LEARNING**

The exact knowledge of the transition kernel associated with the MDP described in Section III-B is needed to compute the Whittle index policy developed in Section IV. However, such knowledge is often unavailable and varying over time at wireless edges. We now adopt a learning perspective on top of the Whittle index policy. Specifically, we design a novel model-free reinforcement learning augmented algorithm entitled Q-Whittle learning which leverages the threshold structure of the optimal policy developed in Section IV while learning Q-functions for different state-action pairs.

**A. Preliminaries**

We first review some preliminaries for Q-learning augmented Whittle index policy, which was first proposed in \cite{11}, and further generalized in \cite{10}. The per-content MDP in \cite{11} can be formulated as a DP \cite{2}, \cite{3}, i.e.,

\[
V(s) + \beta^* = \min_{a \in \{0, 1\}} \left\{ a \left( s + \sum_{s'} p(s'|s, 1)V(s') \right) \right. \\
+ \left. (1-a) \left( s - W + \sum_{s'} p(s'|s, 0)V(s') \right) \right\}, \quad (18)
\]

where \( \beta^* \) is the minimal long-term average cost of this MDP with parameter \( W \in \mathbb{R} \), and \( V(s) \) is the optimal state value up to an additive constant, which depends on the parameter \( W \). The Q-function can then be defined as \cite{3}

\[
Q(s, a) + \beta^* = s - (1-a)W(s) + \sum_{s'} p(s'|s, a)V(s'), \quad (19)
\]

such that \( V(s) = \min_{a \in \{0, 1\}} Q(s, a) \).

The Whittle index associated with state \( s \) \cite{3} is defined as the value \( W(s) \) such that actions 0 and 1 are equally favorable
in state $s$ with a “subsidy” $W(s)$, i.e., $Q(s, 0) = Q(s, 1)$. Combining with (19), the closed-form for $W(s)$ satisfies (20).

$$-W(s) + \sum_{s'} p(s'| s, 0) V(s') = \sum_{s'} p(s'| s, 1) V(s').$$

However, the unknown transition probabilities $p(s'| s, \cdot)$ hinder us to directly evaluate the Whittle index according to (20). Next, we overcome this limitation by proposing a Q-learning based algorithm to jointly learn the Q-function and Whittle index by leveraging the inherent structure in our problem. Since the Q-function $Q(s, a), \forall s, a$ and Whittle index $W(s)$ are coupled, it requires a two-timescale iteration wherein the faster timescale performs Q-function update with a fixed $W(s)$, which is updated in a slower timescale.

### B. Q-Whittle Learning

Since the parameter $\tilde{\beta}^*$ introduced by the long-term average MDP is unknown, a widely-adopted approach is to learn the discounted MDP (with the same states, actions, cost function, and transition) instead, for some discount factor $\alpha$ according to the Blackwell optimality theorem [41]. This indicates that there exists an optimal solution of the $\alpha$-discounted cumulative cost of the discounted MDP for all $\alpha$, and when $\alpha$ is close to 1, this solution is also long-run average optimal. Such a technique has been also applied to the study of average-reward MDP [42] and references therein. Thus we focus on the discounted Q-learning below.

As the optimal policy for the per-content MDP [11] is of the threshold-type, i.e., provided a threshold $R$, the arm is made passive for $s < R$, and active $s > R$, our key insight is that this appealing property can significantly reduce the exploration overhead for the update of Q-functions. Specifically, the Q-learning under such a threshold policy $\pi = R$ only needs to update Q-functions $Q(s, 0), \forall s < R$ with all other state-action values unchanged since the optimal action for $s < R$ is deterministic, i.e., $a = 0$. Similarly, only $Q(s, 1)$ is updated for $s > R$. When the arm is in state $R$, it randomly chooses actions $a = 0$ or $a = 1$. For simplicity, we assume that these two actions are equally chosen in state $R$. This key observation dramatically reduces the complexity of Q-learning on top of Whittle index compared to existing methods, e.g., [10], [11].

More specifically, with this key observation, our Q-functions are updated as follows:

**Case 1:** When $I_{(s > R)} = 1$, we have

$$Q_{n+1}(S_n, 1) \leftarrow (1 - \gamma_n)Q_n(S_n, 1) + \gamma_n S_n + \frac{\alpha I_{(s > R)} Q_n(S_n + 1, 1)}{\text{Term}_1} + \frac{\alpha I_{(s < R)} Q_n(S_n, 1, 0)}{\text{Term}_2} + \frac{\alpha I_{(s = R)} \min_a Q_n(S_n + 1, a)}{\text{Term}_3},$$

**Case 2:** When $I_{(s < R)} = 1$, we have

$$Q_{n+1}(S_n, 0) \leftarrow (1 - \gamma_n)Q_n(S_n, 0) + \gamma_n (S_n - W) + \frac{\alpha I_{(s > R)} Q_n(S_n + 1, 1)}{\text{Term}_1} + \frac{\alpha I_{(s < R)} Q_n(S_n, 1, 0)}{\text{Term}_2} + \frac{\alpha I_{(s = R)} \min_a Q_n(S_n + 1, a)}{\text{Term}_3},$$

**Case 3:** When $I_{(s = R)} = 1$, $Q_{n+1}(S_n, A_n)$ is updated according to either (21) or (22) with equal probability.

We denote by $Q_n$ the Q value at the beginning of time step $n$ under the threshold policy $\pi = R$. With the above Q-function updates, we have

$$Q_{n+1}(s, a) = \begin{cases} (21) & \text{if } (s, a) = (S_n, A_n); \\ (22) & \text{otherwise.} \end{cases}$$

Given the above Q-function updates, the parameter $W$ under the threshold policy $\pi = R$ at time step $n$ is updated as

$$W_{n+1}(R) = W_n(R) + \eta_n \left( Q_n^R(R, 0) - Q_n^R(R, 1) \right),$$

where $\sum_n \eta_n = \infty$ and $\sum_n \eta_n^2 < \infty$.

We summarize the Q-Whittle learning in Algorithm 1.

Since the wireless edge can cache at most $B$ contents, an easy implementation is to find the possible activation set $C := \{ m \in M | S_m(n) \geq R \}$ for threshold $R$ at epoch $n$ and activate the $\min(B, |C|)$ arms with highest Whittle indices $W_{m,n}(S_m(n))$. To leverage the exploration gain, with probability $\epsilon$, the algorithm randomly select $\min(B, |C|)$ arms from $C$ (lines 6-9). The Q-function and parameter $W$ updates can be simply repeated for all $M$ contents (lines 11-12).

**Remark 3.** Some definitions (e.g., $W(s)$) in this paper are similar to those in [10], [11], which also studied Whittle index policy based Q-learning through a two-timescale update. However, our Q-Whittle learning algorithm significantly differs from those in [10], [11]. First, [10], [11] adopted the conventional $\epsilon$-greedy rule for Q-value updates. In contrast, we leverage the property of optimal threshold-type policy into Q-value updates as in (21) and (22). Such a threshold-type Q-value update dramatically reduces the computational complexity (e.g., by reducing the exploration space by at least half) since each state only has a fixed action to explore. Second, the threshold policy further enables us to update
and (24) in the form of a 2TSA.

In this section, we provide a finite-time analysis of our Q-Whittle learning updates in (23) and (24) into a standard 2TSA. For ease of exposition, we present the synchronous update where every entry of Q-function is updated at each time step [44], [45]. Our goal is to rewrite the updates (23) and (24) as

\[ Q_{n+1} = Q_n + \gamma_n [h(Q_n, W_n) + \xi_{n+1}], \]
\[ W_{n+1} = W_n + \eta_n [g(Q_n, W_n) + \psi_{n+1}], \]

where \( Q_n \) represents Q-function update at time step \( n \); \{\xi_n\} is an appropriate martingale difference sequence conditioned on \( \sigma \)-field \( \mathcal{F}_n = \{Q_0, W_0, \xi_0, \ldots, Q_n, W_n, \xi_n\} \) generated by iterations and trajectory up to time step \( n \); \{\psi_n\} is a suitable error sequence; \( h \) and \( g \) are appropriate Lipschitz functions defined below that satisfy the conditions needed for our ODE analysis, and the step sizes satisfy Assumption 4 below.

Specifically, using the operator in [13], we rewrite the Q-learning update in (23) for \( \forall(s, a) \in \mathcal{S} \times \mathcal{A} \) as

\[ Q_{n+1}(s, a) = Q_n(s, a) + \gamma_n \left[ TQ_n(s, a) - Q_n(s, a) + \xi_{n+1}(s, a) \right], \]

where

\[ TQ_n(s, a) = s - (1-a)W_n + \alpha \sum_{a'} \pi(s'|s, a) Q_n(s', a'), \]
\[ \xi_{n+1}(s, a) = s - (1-a)W_n + \alpha \min_a Q_n(s, a) - TQ_n(s, a). \]

Hence, we have

\[ h(Q_n, W_n) := [TQ_n] - Q_n, \]

which is Lipschitz in both \( Q \) and \( W \). Similarly, we have

\[ g(Q_n, W_n) := Q_n(R, 0) - Q_n(R, 1), \]

which is Lipschitz in \( Q \). W.l.o.g., we assume \( \psi_n = 0, \forall n \) since the update of \( W \) in (24) is deterministic. With identifications of these two functions, the asymptotic convergence of our 2TSA can be established by using the ODE method following the solution of suitably defined differential equations [10], [12], [34], [45], [46]. For ease of exposition, we temporally assume fixed step size here, then the 2TSA is reduced to the following differential equations:

\[ \dot{Q}(t) = h(Q(t), W(t)), \]
\[ \dot{W}(t) = \frac{\eta}{\gamma} g(Q(t), W(t)), \]

where the ratio \( \eta/\gamma \) represents the difference in timescale between these two updates. Our focus here is on characterizing the finite-time convergence rate of \( (Q_n, W_n) \) to the global asymptotically optimal equilibrium point \( (Q^R, W(R)) \) of (29) for each \( R \). Using an idea of [34], the key part of our analysis is based on the choice of two step sizes and a Lyapunov function. We first define the following two error terms as

\[ \hat{Q}_n = Q_n - f(W_n), \]
\[ \hat{W}_n = W_n - W(R), \]

which characterizes the coupling between \( Q_n \) and \( W_n \). If \( \hat{Q}_n \) and \( \hat{W}_n \) go to zero simultaneously, \( (Q_n, W_n) \to (Q^R, W(R)) \) is established. Thus, to prove the convergence of \( (Q_n, W_n) \) of our 2TSA to its true value \( (Q^R, W(R)) \), we instead study the
convergence of \((Q_n, W_n)\) by providing the finite-time analysis for the mean squared error generated by \((25)-(26)\). To couple the two updates, we define the following Lyapunov function

\[
M(Q_n, W_n) := \frac{\eta_n}{\gamma_n} \left\| \hat{Q}_n \right\|^2 + \left\| W_n \right\|^2 = \frac{\eta_n}{\gamma_n} \left\| Q_n - f(W_n) \right\|^2 + \left\| W_n - W(R) \right\|^2.
\]

We make the following assumptions in our analysis.

**Assumption 2.** Provided \(W\), there exists an operator \(f\) such that \(Q = f(W)\) is the unique solution to \(h(f(W), W) = 0\), where \(h\) and \(f\) are Lipschitz continuous with positive constants \(L_h\) and \(L_f\) such that

\[
\|f(W) - f(W')\| \leq L_f \|W - W'\|, \quad \|h(Q, W) - h(Q', W')\| \leq L_h (\|Q - Q'\| + \|W - W'\|). \tag{32}
\]

The operator \(g\) is Lipschitz continuous with constant \(L_g\), i.e.,

\[
\|g(Q, W) - g(Q', W')\| \leq L_g (\|Q - Q'\| + \|W - W'\|).
\]

**Assumption 3.** Random variables \(\xi_n\) are independent of each other and across time, with zero mean and common variances \(\xi\).

**Assumption 4.** The step sizes \(\gamma_n\) and \(\eta_n\) satisfy \(\sum_{n=0}^{\infty} \gamma_n = \infty\), \(\sum_{n=0}^{\infty} \gamma_n^2 + \eta_n^2 < \infty\), \(\eta_n/\gamma_n\) is non-increasing as \(n\) and \(\lim_{n \to \infty} \eta_n/\gamma_n = 0\).

These assumptions are standard in SA literature \([10], [12], [34], [45], [56]\). Assumption 2 guarantees the existence of solutions to \((29)\). Assumption 3 holds since \(\xi_n(s, a) = \min_{s'} Q_n(S_{n+1}, a) - \alpha \sum_{a'} p(s'|s, a) \min_{a'} Q_n(s', a')\), thus \(\mathbb{E}[\xi_n|F_{n-1}] = 0\). Assumption 4 is needed in the proof.

### B. Finite-Time Analysis of Q-Whittle Learning

**Theorem 1.** Suppose that Assumptions 4 hold. Furthermore, we assume that \(\gamma_n^2 \leq \min\left(\frac{1-\mu^2}{L_h^2 + 2L_h^2\gamma_n(\gamma_n^2 + \eta_n^2)}, \frac{(n+1)^{-5/3}}{2L_h^2 + 2L_h^2\gamma_n^2 + L_h^2 L_h + 3L_h^2}\right)\).

Then, we have for \(\forall n \geq 0\)

\[
\mathbb{E}[M(Q_{n+1}, W_{n+1})|F_n] \leq \frac{1}{(n+1)^2} \mathbb{E}[M(Q_0, W_0)] + \frac{1}{(n+1)^2/3} \left( D_1 \left( \left\| \hat{Q}_0 \right\|^2 + \left\| W_0 \right\|^2 \right) + (D_2 + 1)A \right), \tag{33}
\]

where \(D_1 := \prod_{n=0}^{\infty} (1 + x_n)\) and \(D_2 := \left( \gamma_n + \sum_{i=1}^{n} (1 + \gamma_i)^{\gamma_n-1} \right)\), with \(x_n = \left( 2L_h^2 \gamma_n^2 + 2L_h^2 \gamma_n^2 (L_h + 1)^2 + 2\eta_n^2 L_h^2 (L_h + 1)^2 + L_h L_h (L_h + 1) + 3L_h \gamma_n \right)\).

Proof is provided in Appendix 1\footnote{We evaluate the impact of \(\mu\) and relegate the results to Appendix E}.

**Remark 4.** Our finite-time analysis of Q-Whittle learning consists of two steps. First, we rewrite Q-Whittle updates into a 2TSA in \((25)-(26)\). The key is to identify two critical terms \(h\) and \(g\). Second, we prove a bound on finite-time convergence rate of Q-Whittle learning by leveraging and generalizing the machinery of non-linear two-timescale stochastic approximation \([34]\). The key is to the choice of two step sizes (as characterized in Theorem 1 and a Lyapunov function given in \((37)\)). Though the main steps of our proofs are motivated by \([34]\), we need to characterize the specific requirements for our settings as aforementioned. Need to mention that we do not need the assumption that \(h\) and \(g\) are strongly monotone as in \([34]\), and hence requires a derivation of the main results.

### VII. Numerical Results

In this section, we numerically evaluate the performance of our Whittle index policy and Q-Whittle learning algorithm using both synthetic and real traces.

#### A. Baselines

**Content Caching Algorithms.** We compare our Whittle index policy to state-of-the-art methods when system parameters are known: (a) Greedy policy that stores contents with the largest \(B\) request queues; (b) Continuous-Greedy with Power Series approximation (CG-PS) \([47]\), an optimization based algorithm using 500 samples with a gradient estimator based on power series expansion; (c) Projected Gradient Ascent (PGA-10) \([16]\), an optimization based method with a measure period 10; and (d) Least-Recently-Used (LRU). For CG-PS and PGA-10, we adopt the default settings in \([47]\) and \([16]\) and omit their descriptions. We refer interested readers to \([47]\) and \([16]\) for detail due to space constraints.

**Q-learning based Algorithms.** We compare our Q-Whittle learning to existing Q-learning algorithms (see Remark 3) when system parameters are unknown: (a) Q-learning Whittle Index Controller (Fu) \([17]\); (b) Q learning for Whittle index (AB) \([10]\); (c) Whittle Index Q-learning (WIQL) \([43]\); and (d) our Whittle policy, i.e., assume full knowledge of underlying transition probabilities. The discount factor is \(\alpha = 0.8\), learning rates are initialized to \(\gamma(0) = 0.1\) and \(\eta(0) = 0.01\), and are decayed by half every 1,000 time steps. The exploration and exploitation parameter parameter is set as \(\epsilon = 0.05\).

#### B. Evaluation Using Synthetic Traces

We simulate a system with the number of distinct contents \(\Gamma\) ranging from 200 to 10,000 with a step size of 200. For each case, content requests are drawn from a Zipf distribution with Zipf parameters \(\theta\) of 0.6, 0.9 and 1.2. As we consider a state-dependent delivery rate in our model \((AB)\), we set the “unit rate” \(\mu = 20\) with the true delivery rate of \(\mu SA_{\Gamma}\), and the total number of requests varies across each \(\Gamma\). The successful delivery probability over unreliable channels is \(q = 0.9\) and the cache size is \(B = 10\).

The accumulated costs of content caching algorithms and Q-learning based algorithms are presented in Figures 3 and 4 where we use the Monte Carlo simulation with 10,000 independent trails. It is evident from Figure 3 that our Whittle index policy significantly outperforms existing algorithms for content caching to minimize average latency. As expected,
LRU and Greedy are myopic, CG-PS and PGA-10 improve the performance as shown in their papers which require to solve stochastic optimization problems. In contrast, our Whittle index policy is well-known to computationally efficient and scalable, which is highly desirable for content caching at the wireless edge since the number of distinct contents could be very large. From Figure 4 it is also clear that our Q-Whittle learning consistently outperforms its counterparts. In particular, WIQL outperforms Fu and AB, which is the same trend observed in [43]. Moreover, our learning algorithm Q-Whittle performs close to the Whittle index policy.

We further demonstrate the convergence of our Q-Whittle learning algorithm in Figure 5. Due to the decoupled nature of our framework (see Section V), we randomly draw one content from the trace with Zipf parameter 0.9. For ease of exposition, we only show results of Whittle indices of three states for this particular content. We observe that the Whittle indices obtained by Q-Whittle learning gradually converge to the true Whittle index, which is obtained under the assumption that system parameters are known. Similar observations hold for other contents in other traces.

Finally, we validate the asymptotic optimality of our Whittle index policy by comparing with the optimal policy derived from (7). Since the state space of relative value iteration grows exponentially with the number of contents, we cannot directly evaluate a very large system. Hence, we consider a simple configuration with 20 contents. Even in this simple case, the dimension of state-action space for relative value iteration reaches to $20^4 \times 20$. It is clear from Figure 6 (Left) that Whittle index policy achieves optimal performance as the number of distinct contents grows large. Finally, Figure 6 (Right) presents the switching curves for serving any one of two contents under these two policies. The curves divide the entire space into two distinct regions, in which content 1 and content 2 are served, respectively. It is clear that our Whittle index policy coincides with the optimal policy almost across all state space, and hence captures the qualitative structure of the optimal policy.

C. Evaluation Using Real Traces

We further evaluate our algorithms using two real traces: (i) Iqiyi [43], which contains mobile video behaviors; and (ii) YouTube [49], which contains trace data about user requests for specific YouTube content collected from a campus network. For the Iqiyi (resp. YouTube) trace, there are more than 67 (resp. 0.6) million requests for more than 1.4 million (resp. 0.3) unique contents over a period of 335 (resp. 336) hours. We evaluate the accumulated cost over rough 14 days for each trace with a cache size of $B = 4,000$ (resp. 2,000) for Iqiyi (resp. YouTube). We choose these values based on the observation of average number of active contents in the trace. The accumulated costs of content caching algorithms and Q-learning based algorithms are shown in Figures 7 and 8 respectively. Again, we observe that our Whittle index policy and Q-Whittle learning significantly outperform their counterparts with smaller costs. Finally, we note that Q-Whittle learning can quickly learn the system dynamics and perform close to the Whittle index policy, which matches well with our theoretical results.

VIII. Conclusion

In this paper, we studied the content caching problem at the wireless edge with unreliable channels. Our goal is to derive an optimal policy for making content caching decisions so as to minimize the average content request latency from end users. We posed the problem in the form of a Markov decision process, and showed that the optimal policy has a simple threshold-structure and presented a closed form of Whittle indices for each content. We then developed a novel reinforcement learning algorithm entitled Q-Whittle learning that can fully exploit the structure of the optimal policy when the system parameters are unknown. We mathematically

\footnote{A content is said to be active at time $t$ if $t$ lies between the first and the last requests for the content.}
characterized the performance of Q-Whittle learning and also numerically demonstrated its empirical performance.

**Appendix A**

**Proof of Proposition 1**

*Proof.* According to Assumption 1, we denote the lowest state with no preference among active and passive actions as $R$, i.e., $Q^a(R, 1) = Q^a(R, 0)$. This implies the following two facts. First, for state $s < R$, the optimal action is 0, i.e.,

$$J^a(R - 1) = R - 1 - W + \alpha J^a(R).$$

Second, equal preference of both actions at state $R$ implies

$$R - W + \alpha J^a(R + 1) = R + \alpha P_R J^a(R + 1) + \alpha(1 - P_R) J^a(R - 1),$$

which is equivalent to the following with respect to $W$

$$W = \alpha(1 - P_R)(J^a(R + 1) - J^a(R - 1)). \quad (34)$$

With these two facts, we establish the connection between value functions of states $R - 1$ and $R + 1$, i.e.,

$$J^a(R - 1) = R - 1 - W + \alpha(R - W + \alpha J^a(R + 1)). \quad (35)$$

Substituting (35) into (34), we have

$$J^a(R + 1) = \frac{W}{\alpha(1 - P_R)} + \frac{R - 1 - W + \alpha(R - W)}{1 - \alpha^2}. \quad (36)$$

As a result, $J^a(R + 1)$ can be updated as

$$\begin{cases} R + 1 - W + \alpha J^a(R + 2), & \text{if } a = 0 \\ R + 1 + \alpha P_{R+1} J^a(R + 2) + \alpha(1 - P_{R+1}) J^a(R), & \text{if } a = 1. \end{cases} \quad (37)$$

In the following, we show that it is optimal to choose action 1 at state $R + 1$. We first show that $a = 0$ is not optimal by contradiction, and then verify that $a = 1$ is optimal. Assume that the optimal action at state $R + 1$ is $a = 0$. Then, we have

$$W \geq \alpha(1 - P_{R+1})(J^a(R + 2) - J^a(R))$$

$$= \alpha(1 - P_{R+1})\left(\frac{J^a(R + 1) - (R + 1 - W)}{\alpha} - (R - W + \alpha J^a(R + 1))\right) = \frac{1 - P_{R+1}}{\alpha(1 - P_R)} W, \quad (38)$$

where the inequality is due to the fact that optimal action is 0 at state $R + 1$ and the last equality directly comes by plugging the closed-form expression of $J^a(R + 1)$. Since $\frac{1 - P_{R+1}}{\alpha(1 - P_R)} > 1$, the inequality does not hold and it occurs an contradiction. This means that action 0 is not optimal for state $R + 1$. We further verify that $a = 1$ is optimal. When optimal action at state $R + 1$ is $a = 1$, we have

$$W \leq \alpha(1 - P_{R+1})\left(\frac{J^a(R + 1) - (R + 1 - W)}{\alpha} - (R - W + \alpha J^a(R + 1))\right) \leq \alpha(1 - P_{R+1})\left(\frac{J^a(R + 1) - (R + 1) - \alpha(1 - P_{R+1}) J^a(R)}{\alpha P_{R+1}}\right)$$

$$= \alpha(1 - P_{R+1})\left(J^a(R + 2) - J^a(R)\right), \quad (39)$$

where (a) directly follows the contradiction implied by (36), (b) holds as $\frac{J^a(R + 1) - (R + 1) - \alpha(1 - P_{R+1}) J^a(R)}{\alpha P_{R+1}} \geq \frac{J^a(R + 1) - (R + 1 - W)}{\alpha}$. Thus the optimal action for $R + 1$ is 1.

Following the same idea, the above results can be easily generalized to the larger state $s \geq R + 1$, and hence we omit the detail here. To this end, it is clear that the optimal policy of the discounted MDP (12) is of the threshold-type.

**Appendix B**

**Proof of Proposition 3**

*Proof.* Since the optimal policy for (11) is of the threshold-type, for a given subsidy $W$, the optimal average cost under a threshold $R$ satisfies

$$f(W) := \min_R \left\{ h^R(W) := \sum_{s=0}^{\infty} s \phi_R(s) - W \sum_{s=0}^{R} \phi_R(s) \right\}, \quad (38)$$

$\phi_R(s)$ represents the stationary probability of state $s$ under threshold policy $\pi = R$. It is easy to show that $h^R(W)$ is concave non-increasing in $W$ since it is a lower envelope of linear non-increasing functions in $W$, i.e., $h^R(W) > h^R(W')$ if $W < W'$. Thus we can choose a larger threshold $R$ when $W$ increases to further decrease the total cost according to (38), i.e., $D(W) \leq D(W')$ when $W < W'$.
APPENDIX C
PROOF OF PROPOSITION [5]
Proof. It is clear that \( \forall R' < R \) is transient because the state keeps increasing. Therefore, \( \phi_R(R') = 0, \forall R' < R \). Note that for threshold state \( R \), the stationary probability satisfies
\[
\phi_R(R) = \frac{\mu q(R+1)}{\lambda + \mu q(R+1)} \cdot \phi_R(R+1).
\]
Based on birth-and-death process, the stationary probabilities for states \( R + l, \forall l = 2, 3, \cdots \) of content \( n \) satisfies
\[
\phi_R(R+l) = \frac{\lambda}{\lambda + \mu q(R+l)} \cdot \phi_R(R+l+1).
\]
Therefore, we have the following relation
\[
\phi_R(R + l) = \phi_R(R + l) \cdot \prod_{j=2}^{l} \frac{\lambda}{\lambda + \mu q(R + j - 1)} \cdot \phi_R(R + j).
\]
Since \( \phi_R(R) + \phi_R(R + 1) + \cdots = 1 \), we have
\[
\phi_R(R + 1) = \frac{1}{1 + \frac{\mu q(R+1)}{\lambda + \mu q(R+1)}} + \sum_{l=2}^{\infty} \prod_{j=2}^{l} \frac{\lambda}{\lambda + \mu q(R + j - 1)} \cdot \phi_R(R + j).
\]
\[\square\]

APPENDIX D
PROOF OF THEOREM [1]
To prove Theorem [1] we need the following three key lemmas about the error terms defined in (30). First, we study the property of \( \tilde{Q}_n \).

Lemma 2. Let the sequence \( \{Q_n, W_n\} \) be generated by (25)-(26). Then, under Assumptions 23, we have for all \( n \geq 0 \),
\[
\mathbb{E} \left[ \left\| \tilde{Q}_{n+1} \right\|_F^2 | F_n \right] \leq \gamma_n^2 \lambda + (1 + L^2 h^2)_n \left\| Q_n \right\|_F^2
\]
\[+ 2L^2 \tilde{Q}_n h^2 \left\| \tilde{Q}_n \right\|_F^2 + 2L^2 \tilde{Q}_n h \left( L_f + 1 \right) \eta_n \left\| W_n \right\|_F^2
\]
\[+ L_f \eta_n (1 + L_h \gamma_n) \left( L_g (L_f + 3) \left\| \tilde{Q}_n \right\|_F^2 + L_g (L_f + 1) \right) \left\| W_n \right\|_F^2 \].
\]
Proof. According to definition in (30), we have
\[
\tilde{Q}_{n+1} = Q_{n+1} - f(W_{n+1}) = \tilde{Q}_n + \gamma_n h(Q_n, W_n) + \gamma_n \xi_n + f(W_n) - f(W_{n+1}),
\]
which leads to
\[
\left\| \tilde{Q}_{n+1} \right\|_F^2 = \left\| \tilde{Q}_n + \gamma_n h(Q_n, W_n) + \gamma_n \xi_n + f(W_n) - f(W_{n+1}) \right\|_F^2
\]
\[= \left\| \tilde{Q}_n + \gamma_n h(Q_n, W_n) \right\|_F^2 + \left\| \gamma_n \xi_n + f(W_n) - f(W_{n+1}) \right\|_F^2
\]
\[+ 2 \left( \tilde{Q}_n + \gamma_n h(Q_n, W_n) \right)^T (f(W_n) - f(W_{n+1}))
\]
\[+ 2 \gamma_n \left( \tilde{Q}_n + \gamma_n h(Q_n, W_n) \right)^T \xi_n \].
\]
The second equality due to the fact that \( \left\| x + y \right\|_F^2 = \left\| x \right\|_F^2 + \left\| y \right\|_F^2 + 2 \left\| x \right\|_F \left\| y \right\|_F \). We next analyze the conditional expectation of each term in \( \left\| \tilde{Q}_{n+1} \right\|_F^2 \) on \( F_n \). We first focus on Term 1.
\[
\mathbb{E} \left[ \text{Term}_1 | F_n \right] = \frac{\left\| \tilde{Q}_n \right\|_F^2 + 2 \gamma_n \tilde{Q}_n h(Q_n, W_n) + \gamma_n h(Q_n, W_n) \right\|_F^2
\]
\[= \frac{(a1) \left\| \tilde{Q}_n \right\|_F^2 + 2 \gamma_n \tilde{Q}_n h(Q_n, W_n) - h(f(W_n), W_n) \right\|_F^2
\]
\[+ \gamma_n h(Q_n, W_n) - h(f(W_n), W_n) \right\|_F^2
\]
\[\leq \frac{(a2) \left\| \tilde{Q}_n \right\|_F^2 + L^2 \gamma_n \left\| \tilde{Q}_n \right\|_F^2 \right. \]
\[ + \frac{Q_f}{2} \left\| W_n \right\|^2, \]

where (c1) is due to the Lipschitz continuity of \( f \) and (c2) holds because \( \sqrt{x^2} \leq \|x\|^2 + \|y\|^2 \). Since \( \mathbb{E} \left\| F_n \right\| = 0 \), combining all terms leads to the final expression in (39). \( \square \)

**Lemma 3.** Let \( \{Q_n, W_n\} \) be generated by (25)-(26). Then under Assumptions \( 2 \) and \( 3 \) for any \( k \geq 0 \), we have

\[
\mathbb{E} \left[ \left\| \tilde{W}_{n+1} \right\|^2 | F_n \right] \leq \left\| \tilde{W}_n \right\|^2 + 2\eta_n \tilde{W}_n^T g(Q_n, W_n) + \eta_n^2 \left\| g(Q_n, W_n) \right\|^2
\]

\[
\leq \left\| \tilde{W}_n \right\|^2 + 2\eta_n \tilde{W}_n^T g(Q_n, W_n) + \eta_n^2 \left\| g(Q_n, W_n) \right\|^2
\]

(41)

**Proof.** According to (30), we have \( \tilde{W}_{n+1} = W_{n+1} - W(R) = W_n + \eta_n g(Q_n, W_n) \), which leads to

\[
\mathbb{E} \left[ \left\| \tilde{W}_{n+1} \right\|^2 | F_n \right] \leq \left\| \tilde{W}_n \right\|^2 + 2\eta_n \tilde{W}_n^T g(Q_n, W_n) + \eta_n^2 \left\| g(Q_n, W_n) \right\|^2
\]

(42)

where (d1) is due to \( 2\eta_n \tilde{W}_n^T g(Q_n, W_n) \leq 0 \) and (d2) is due to (b3)-(b5). \( \square \)

**Lemma 4.** Under Assumptions \( 2 \) and \( 3 \), there exist constants \( C_1 \) and \( C_2 \) such that the sequence \( \{Q_n, W_n\} \) satisfies

\[
\mathbb{E} \left[ \left\| \tilde{Q}_n \right\|^2 + \left\| \tilde{W}_n \right\|^2 | F_n \right] \leq D_1 \left( \left\| Q_0 \right\|^2 + \left\| W_0 \right\|^2 \right) + D_2 \Lambda,
\]

(43)

where \( D_1 := \prod_{i=0}^{n} (1 + \eta_i) \) and \( D_2 := \left( \gamma_n^2 + 2 \sum_{i=1}^{n} (1 + \eta_i) \gamma_{n-i} \right) \), with \( \eta_n = (2L_\gamma \gamma_n + 2L_f^2 L_\gamma \gamma_n^2 (L_f + 1)^2 + 2\eta_n L_\gamma \gamma_n \left\| Q \right\| \left\| W \right\| ) \).

**Proof.** Providing Lemma \( 2 \) and Lemma \( 3 \) we have

\[
\mathbb{E} \left[ \left\| \tilde{Q}_n \right\|^2 + \left\| \tilde{W}_n \right\|^2 | F_n \right] \leq \gamma_n^2 \Lambda + (1 + L_\gamma \gamma_n^2) \left\| \tilde{Q}_n \right\|^2 + \left( 2L_f^2 L_\gamma \gamma_n^2 + 2L_f \left\| \tilde{Q}_n \right\| + 2\eta_n^2 \left\| W \right\| \right)^2
\]

\[
+ L_f \eta_n L_\gamma \gamma_n \left( L_\gamma (L_f + 3) + \left\| \tilde{Q}_n \right\| + L_f \left\| \tilde{W}_n \right\| \right)^2
\]

\[
\leq \left\| \tilde{Q}_n \right\|^2 + \left\| \tilde{W}_n \right\|^2 + \gamma_n^2 \Lambda + \left( 2L_f^2 L_\gamma \gamma_n^2 + 2L_f \left\| \tilde{Q}_n \right\| + 2\eta_n^2 \left\| W \right\| \right)^2
\]

\[
+ L_f \eta_n L_\gamma \gamma_n \left( L_\gamma (L_f + 1)^2 + 2\eta_n \left\| W \right\| \right)
\]

\[
\leq \left\| \tilde{Q}_n \right\|^2 + \left\| \tilde{W}_n \right\|^2 + \gamma_n^2 \Lambda + \left( 2L_f^2 L_\gamma \gamma_n^2 + 2L_f \left\| \tilde{Q}_n \right\| + 2\eta_n \left\| W \right\| \right)^2
\]

\[
+ L_f \eta_n L_\gamma \gamma_n \left( L_\gamma (L_f + 1)^2 + 2\eta_n \left\| W \right\| \right)
\]

\[
\leq \left\| \tilde{Q}_n \right\|^2 + \left\| \tilde{W}_n \right\|^2 + \gamma_n^2 \Lambda
\]

\[
\leq \left\| \tilde{Q}_n \right\|^2 + \left\| \tilde{W}_n \right\|^2 + \gamma_n^2 \Lambda
\]

\[
\leq \left( 1 + L_\gamma \gamma_n^2 \right) \left( \left\| Q_0 \right\|^2 + \left\| W_0 \right\|^2 \right) + D_2 \Lambda,
\]

(44)

Multiplying both sides of (44) with \( (n + 1)^2 \), we have

\[
(n + 1)^2 \mathbb{E} \left[ \left\| M(Q_n+1, W_n+1) \right\| F_n \right] \leq (n + 1)^2 \left( 1 + L_\gamma \gamma_n^2 \right) \mathbb{E} \left[ \left\| M(Q_n, W_n) \right\| F_n \right] + (n + 1) \gamma_n^2 \Lambda
\]

\[
+ x_{n+1}(n+1)^2 \mathbb{E} \left[ \left\| \tilde{Q}_n \right\|^2 + \left\| \tilde{W}_n \right\| \right] \mathbb{E} \left[ \left\| F_n \right\| \right]
\]

\[
\leq n^2 \mathbb{E} \left[ \left\| M(Q_n, W_n) \right\| F_n \right] + (n + 1)^{1/3} \left( D_1 \left( \left\| Q_0 \right\|^2 + \left\| W_0 \right\|^2 \right) + D_2 \Lambda + \Lambda \right),
\]

(45)

where the last inequality holds due to \( \gamma_n^2 \leq \min \left( \frac{1}{\lambda^3}, \frac{1}{2L_\gamma^2 + 2L_f(L_f + 1)^2 + L_f \eta_n(L_\gamma + 3L_f)} \right) \).

Summing (45) from time step 0 to time step \( n \), we have

\[
(n + 1)^2 \mathbb{E} \left[ \left\| M(Q_n+1, W_n+1) \right\| F_n \right] \leq \mathbb{E} \left[ \left\| M(Q_0, W_0) \right\| F_n \right] + (n + 1)^{1/3} \left( D_1 \left( \left\| Q_0 \right\|^2 + \left\| W_0 \right\|^2 \right) + (D_2 + 1) \Lambda \right).
\]

Finally, dividing both sides by \( (n + 1)^2 \) yields the results in Theorem 1.

**Appendix E**

**Additional Numerical Results**

We provide additional numerical results complementary to Section VII. In particular, we consider the same settings as in Section VII-B and investigate the impact of “unit rate” \( \mu \), as shown in Figure 9. Again, it is clear that our Whittle
index policy consistently outperforms existing content caching algorithms across all settings. Furthermore, we observe that a larger delivery rate lowers the accumulated cost. This is intuitive since each content will experience a lower waiting time in the request queue when the delivery rate $\mu$ is larger.



Fig. 9. Accumulated cost of content caching algorithms vs departure rates using synthetic traces.



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