Hadwiger’s Conjecture for ℓ-Link Graphs

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Abstract: In this article, we define and study a new family of graphs that generalizes the notions of line graphs and path graphs. Let G be a graph with no loops but possibly with parallel edges. An ℓ-link of G is a walk of G of length ℓ ⩾ 0 in which consecutive edges are different. The ℓ-link graph Lℓ(G) of G is the graph with vertices the ℓ-links of G, such that two vertices are joined by μ ⩾ 0 edges in Lℓ(G) if they correspond to two subsequences of each of μ (ℓ + 1)-links of G. By revealing a recursive structure, we bound from above the chromatic number of ℓ-link graphs. As a corollary, for a given graph G and large enough ℓ, Lℓ(G) is 3-colorable. By investigating the shunting of ℓ-links in G, we show that the Hadwiger number of a nonempty Lℓ(G) is greater or equal to that of G. Hadwiger’s conjecture states that the Hadwiger number of a graph is at least the chromatic number of that graph. The conjecture has been proved by Reed and

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Seymour (Eur J Combin 25(6) (2004), 873–876) for line graphs, and hence 1-link graphs. We prove the conjecture for a wide class of $\ell$-link graphs.

Keywords: $\ell$-link graph; path graph; chromatic number; graph minor; Hadwiger’s conjecture

1. INTRODUCTION AND MAIN RESULTS

We introduce a new family of graphs, called $\ell$-link graphs, which generalizes the notions of line graphs and path graphs. Such a graph is constructed from a certain kind of walk of length $\ell \geq 0$ in a given graph $G$. To ensure that the constructed graph is undirected, $G$ is undirected. To avoid loops, $G$ is loopless, and the consecutive edges in each walk are different. Such a walk is called an $\ell$-link. For example, a 0-link is a vertex, a 1-link is an edge, and a 2-link consists of two distinct edges with an end vertex in common. An $\ell$-path is an $\ell$-link without repeated vertices. We use $L_\ell(G)$ and $P_\ell(G)$ to denote the sets of $\ell$-links and $\ell$-paths of $G$, respectively. There have been a number of families of graphs constructed from $\ell$-links. For example, the line graph $L(G)$, introduced by Whitney [23], is the simple graph with vertex set $E(G)$, in which two vertices are adjacent if their corresponding edges are incident to a common vertex. More generally, the $\ell$-path graph $P_\ell(G)$ is the simple graph with vertex set $P_\ell(G)$, where two vertices are adjacent if the union of their corresponding $\ell$-paths forms a path or a cycle of length $\ell + 1$. Note that an $\ell$-path contains $\ell$ distinct edges and $\ell + 1$ distinct vertices. So $P_\ell(G)$ is the $P_{\ell+1}$-graph of $G$ introduced by Broersma and Hoede [4]. Inspired by these graphs, we define the $\ell$-link graph $L_\ell(G)$ of $G$ to be the graph with vertex set $L_\ell(G)$, in which two vertices are joined by $\mu \geq 0$ edges in $L_\ell(G)$ if they correspond to two subsequences of each of $\mu$ ($\ell + 1$)-links of $G$. More strict definitions can be found in Section 2, together with some other related graphs.

This article studies the structure, coloring, and minors of $\ell$-link graphs including a proof of Hadwiger’s conjecture for a wide class of $\ell$-link graphs. By default $\ell \geq 0$ is an integer. And all graphs are finite, undirected, and loopless. Parallel edges are admitted unless we specify the graph to be simple.

1.1. Graph Coloring

Let $t \geq 0$ be an integer. A $t$-coloring of $G$ is a map $\lambda : V(G) \rightarrow [t] := \{1, 2, \ldots, t\}$ such that $\lambda(u) \neq \lambda(v)$ whenever $u, v \in V(G)$ are adjacent in $G$. A graph with a $t$-coloring is $t$-colorable. The chromatic number $\chi(G)$ is the minimum $t$ such that $G$ is $t$-colorable. Similarly, a $t$-edge-coloring of $G$ is a map $\lambda : E(G) \rightarrow [t]$ such that $\lambda(e) \neq \lambda(f)$ whenever $e, f \in E(G)$ are incident to a common vertex in $G$. The edge-chromatic number $\chi'(G)$ of $G$ is the minimum $t$ such that $G$ admits a $t$-edge-coloring. Let $\chi_\ell(G) := \chi(L_\ell(G))$, and $\Delta(G)$ be the maximum degree of $G$. Brooks’ theorem [5] states that, the chromatic number of a connected graph $G$ equals $\Delta(G) + 1$ if $G$ is an odd cycle or a complete graph with at least one vertex, and is at most $\Delta(G)$ otherwise. Shannon [18] proved that $\chi_1(G) = \chi'(G) \leq \frac{\Delta}{2} \Delta(G)$. We prove a recursive structure for $\ell$-link graphs, which leads to the following upper bounds for $\chi_\ell(G)$.

**Theorem 1.1.** Let $G$ be a graph, $\chi := \chi(G)$, $\chi' := \chi'(G)$, and $\Delta := \Delta(G)$.

- (1) If $\ell \geq 0$ is even, then $\chi_\ell(G) \leq \min\{\chi, \lfloor (\frac{\ell}{2})^{\ell/2} (\chi - 3) \rfloor + 3\}$. 

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(2) If $\ell \geq 1$ is odd, then $\chi_\ell(G) \leq \min\{\chi', \lfloor \frac{1}{2} (\chi' - 3) \rfloor + 3\}.$

(3) If $\ell \not= 1$, then $\chi_\ell(G) \leq \Delta + 1.$

(4) If $\ell \geq 2$, then $\chi_\ell(G) \leq \chi_{\ell-2}(G).$

Theorem 1.1 implies that $\mathbb{L}_\ell(G)$ is 3-colorable for large enough $\ell.$

**Corollary 1.2.** For each graph $G$, $\mathbb{L}_\ell(G)$ is 3-colorable in the following cases:

1. $\ell \geq 0$ is even, and either $\chi(G) \leq 3$ or $\ell > 2 \log_{1.5}(\chi(G) - 3).$
2. $\ell \geq 1$ is odd, and either $\chi'(G) \leq 3$ or $\ell > 2 \log_{1.5}(\chi'(G) - 3) + 1.$

As explained in Section 2, this corollary is related to and implies a result by Kawai and Shibata [15].

### 1.2. Graph Minors

A connected graph with two or more vertices is *biconnected* if it cannot be disconnected by removing a vertex. By contracting an edge we mean identifying its end vertices and deleting possible resulting loops. A graph $H$ is a *minor* of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges. An $H$-minor is a minor of $G$ that is isomorphic to $H.$ The *Hadwiger number* $\eta(G)$ of $G$ is the maximum integer $t$ such that $G$ contains a $K_t$-minor. Denote by $\delta(G)$ the minimum degree of $G.$ The *degeneracy* $d(G)$ of $G$ is the maximum $\delta(H)$ over the subgraphs $H$ of $G.$ We prove the following.

**Theorem 1.3.** Let $\ell \geq 1,$ and $G$ be a graph such that $\mathbb{L}_\ell(G)$ contains at least one edge. Then $\eta(\mathbb{L}_\ell(G)) \geq \max\{\eta(G), d(G)\}.$

By definition $\mathbb{L}(G)$ is the underlying simple graph of $\mathbb{L}_1(G).$ And $\mathbb{L}_\ell(G) = \mathbb{P}_\ell(G)$ if $girth(G) > \lfloor \ell, 2 \rfloor.$ Thus Theorem 1.3 can be applied to path graphs.

**Corollary 1.4.** Let $\ell \geq 1,$ and $G$ be a graph of girth at least $\ell + 1$ such that $\mathbb{P}_\ell(G)$ contains at least one edge. Then $\eta(\mathbb{P}_\ell(G)) \geq \max\{\eta(G), d(G)\}.$

As a far-reaching generalization of the four-color theorem, in 1943, Hugo Hadwiger [10] conjectured the following.

**Hadwiger’s conjecture:** $\eta(G) \geq \chi(G)$ for every graph $G.$

Hadwiger’s conjecture was proved by Robertson, Seymour, and Thomas [17] for $\chi(G) \leq 6.$ The conjecture for line graphs, or equivalently for 1-link graphs, was proved by Reed and Seymour [16]. We prove the following.

**Theorem 1.5.** Hadwiger’s conjecture is true for $\mathbb{L}_\ell(G)$ in the following cases:

1. $\ell \geq 1$ and $G$ is biconnected.
2. $\ell \geq 2$ is an even integer.
3. $d(G) \geq 3$ and $\ell > 2 \log_{1.5}(\Delta(G) - 2)/2 + 3.$
4. $\Delta(G) \geq 3$ and $\ell > 2 \log_{1.5}(\Delta(G) - 2) - 3.83.$
5. $\Delta(G) \leq 5.$

The corresponding results for path graphs are listed below.

**Corollary 1.6.** Let $G$ be a graph of girth at least $\ell + 1.$ Then Hadwiger’s conjecture holds for $\mathbb{P}_\ell(G)$ in the cases of Theorem 1.5 (1)–(5).
2. DEFINITIONS AND TERMINOLOGY

We now give some formal definitions. A graph $G$ is null if $V(G) = \emptyset$, and non-null otherwise. A non-null graph $G$ is empty if $E(G) = \emptyset$, and nonempty otherwise. A unit is a vertex or an edge. The subgraph of $G$ induced by $V \subseteq V(G)$ is the maximal subgraph of $G$ with vertex set $V$. And in this case, the subgraph is called an induced subgraph of $G$. We may not distinguish between $V$ and its induced subgraph. For $\emptyset \neq E \subseteq E(G)$, the subgraph of $G$ induced by $E \cup V$ is the minimal subgraph of $G$ with edge set $E$, and vertex set including $V$. The diameter $\text{diam}(G)$ of $G$ is $+\infty$ if $G$ is disconnected, and the maximum distance between two vertices of $G$ otherwise.

Let $G$ be a graph, and $H$ be a subgraph of $G$. Let $Q$ be a partition of $V(H)$ such that every $V \in Q$ induces a connected subgraph of $H$. Let $M$ be the graph obtained from $H$ by contracting each $V \in Q$ into a vertex. Then $M$ is a minor of $G$. And $V$ is called a branch set of $M$.

For more accurate analysis, we need to define $\ell$-arcs. An $\ell$-arc (or $*\text{-arc}$ if we ignore the length) of $G$ is an alternating sequence $\overrightarrow{L} := (v_0, e_1, \ldots, e_\ell, v_\ell)$ of units of $G$ such that the end vertices of $e_i \in E(G)$ are $v_{i-1}$ and $v_i$ for $i \in [\ell]$, and that $e_i \neq e_{i+1}$ for $i \in [\ell - 1]$. The direction of $\overrightarrow{L}$ is its vertex sequence $(v_0, v_1, \ldots, v_\ell)$. In algebraic graph theory, $\ell$-arcs in simple graphs have been widely studied [3, 19, 20, 22]. Note that $\overrightarrow{L}$ and its reverse $\overrightarrow{-L} := (v_\ell, e_\ell, \ldots, e_1, v_0)$ are different unless $\ell = 0$. The $\ell$-link (or $*\text{-link}$ if the length is ignored) $L := [v_0, e_1, \ldots, e_\ell, v_\ell]$ is obtained by taking $\overrightarrow{L}$ and $\overrightarrow{-L}$ as a single object. For $0 \leq i \leq j \leq \ell$, the $(j - i)$-arc $\overrightarrow{L}(i, j) := (v_i, e_{i+1}, \ldots, e_j, v_j)$ and the $(j - i)$-link $\overrightarrow{L}[i, j] := [v_i, e_{i+1}, \ldots, e_j, v_j]$ are called segments of $\overrightarrow{L}$ and $L$, respectively. We may write $\overrightarrow{L}(j, i) := -\overrightarrow{L}(i, j)$, and $\overrightarrow{L}[j, i] := \overrightarrow{L}[i, j]$. These segments are called middle segments if $i + j = \ell$. $L$ is called an $\ell$-cycle if $\ell \geq 2$, $v_0 = v_\ell$, and $\overrightarrow{L}[0, \ell - 1]$ is an $(\ell - 1)$-path.

Denote by $\mathcal{L}(G)$ and $\mathcal{C}_\ell(G)$ the sets of $\ell$-arcs and $\ell$-cycles of $G$, respectively. Usually, $\overrightarrow{L}(v_0, v_\ell)$ is called an arc for short. In particular, $v_0, v_\ell, e_1, e_\ell, \overrightarrow{e_1}$, and $\overrightarrow{e_\ell}$ are called the tail vertex, head vertex, tail edge, head edge, tail arc, and head arc of $\overrightarrow{L}$, respectively.

Godsil and Royle [9] defined the $\ell$-arc graph $\mathcal{A}_\ell(G)$ to be the digraph with vertex set $\mathcal{L}(G)$, such that there is an arc, labeled by $\overrightarrow{Q}$, from $\overrightarrow{Q}(0, \ell)$ to $\overrightarrow{Q}(1, \ell + 1)$ in $\mathcal{A}_\ell(G)$ for every $\overrightarrow{Q} \in \mathcal{L}_{\ell+1}(G)$. The $t$-dipole graph $D_t$ is the graph consists of two vertices and $t \geq 1$ edges between them. (See Figure 1 for $D_3$, and Figure 1 b the 1-arc graph of $D_3$.)
The $\ell$th iterated line digraph $A_\ell^{\ell}(G)$ is $A_1(G)$ if $\ell = 1$, and $A_1(A_{\ell-1}(G))$ if $\ell \geq 2$ (see [2]). Examples of undirected graphs constructed from $\ell$-arcs can be found in [12, 13].

**Shunting** of $\ell$-arcs was introduced by Tutte [21]. We extend this notion to $\ell$-links. For $\ell, s \geq 0$, and $\vec{Q} \subseteq L_{\ell+s}(G)$, let $L_i := \vec{Q}[i, \ell + i]$ for $i \in [s] \cup \{0\}$, and $Q_i := \vec{Q}[i - 1, \ell + i]$ for $i \in [s]$. Let $Q^{\ell} := \{L_0, Q_1, L_1, \ldots, L_{s-1}, Q_s, L_s\}$. We say $L_0$ can be **shunted** to $L_s$ through $\vec{Q}$ or $\vec{Q}' := \{L_0, L_1, \ldots, L_s\}$ is the set of **images** during this shunting. For $L, R \in L_\ell(G)$, we say $L$ can be shunted to $R$ if there are $\ell$-links $L = L_0, L_1, \ldots, L_s = R$ such that $L_{i-1}$ can be shunted to $L_i$ through some $\ell$-arc $\vec{Q}_i$ for $i \in [s]$. In Figure 2, $[u_0, f_0, v_0, e_0, v_1]$ can be shunted to $[v_1, e_0, v_0, e_1, v_1]$ through $(u_0, f_0, v_0, e_0, v_1, f_1, u_1)$ and $(u_1, f_1, v_1, e_0, v_0, e_1, v_1)$.

For $L, R \in L_\ell(G)$ and $Q \subseteq L_{\ell+1}(G)$, denote by $Q(L, R)$ the set of $Q \subseteq Q$ such that $L$ can be shunted to $R$ through $Q$. We show in Section 3 that $|Q(L, R)|$ is 0 or 1 if $G$ is simple, and can be up to 2 if $\ell \geq 1$ and $G$ contains parallel edges. A more formal definition of $\ell$-link graphs is given below.

**Definition 2.1.** Let $L \subseteq L_\ell(G)$, and $Q \subseteq L_{\ell+1}(G)$. The partial $\ell$-link graph $L(L, \ell, Q)$ of $G$, with respect to $L$ and $Q$, is the graph with vertex set $L$, such that $L, R \in L$ are joined by exactly $|Q(L, R)|$ edges. In particular, $L(L, \ell, Q) = L(L, \ell, Q)$ is the $\ell$-link graph of $G$.

**Remark.** We assign exclusively to each edge of $L(L, \ell, Q)$ between $L, R \in L_\ell(G)$ a $Q \subseteq L_{\ell+1}(G)$ such that $L$ can be shunted to $R$ through $Q$, and refer to this edge simply as $Q$. In this sense, $Q^{\ell} := [L, Q, R]$ is a $1$-link of $L(L, \ell, Q)$.

For example, the $1$-link graph of $D_3$ can be seen in Figure 1 c. A 2-link graph is given in Figure 2 b, and a 2-path graph is depicted in Figure 2 d.

Reed and Seymour [16] pointed out that proving Hadwiger’s conjecture for line graphs of multigraphs is more difficult than for that of simple graphs. This motivates us to work on the $\ell$-link graphs of multigraphs. Diestel [7, page 28] explained that, in some situations, it is more natural to develop graph theory for multigraphs. We allow parallel edges in $\ell$-link graphs in order to investigate the structure of $L(L, \ell, Q)$ by studying the shunting of
\(\ell\)-links in \(G\) regardless of whether \(G\) is simple. The observation below follows from the definitions.

**Observation 2.2.** \(\mathbb{L}_0(G) = G\), \(\mathbb{P}_1(G) = \mathbb{L}_1(G)\), and \(\mathbb{P}_\ell(G)\) is the underlying simple graph of \(\mathbb{L}_\ell(G)\) for \(\ell \in \{0, 1\}\). For \(\ell \geq 2\), \(\mathbb{P}_\ell(G) = \mathbb{L}_\ell(G)\), \(\mathbb{P}_{\ell+1}(G) \cup \mathcal{E}_{\ell}(G)\) is an induced subgraph of \(\mathbb{L}_\ell(G)\). If \(G\) is simple, then \(\mathbb{P}_\ell(G) = \mathbb{L}_\ell(G)\) for \(\ell \in \{0, 1, 2\}\). Further, \(\mathbb{P}_\ell(G) = \mathbb{L}_\ell(G)\) if \(\text{girth}(G) > \max\{\ell, 2\}\).

Let \(\tilde{Q} \in \mathcal{L}_{\ell+1}(G)\), and \([L_0, Q_1, L_1, \ldots, L_{\ell-1}, Q_{\ell}, L_{\ell}] := Q^{[\ell]}\). From the remark above, for \(i \in [s]\), \(Q_i\) is an edge of \(H := \mathbb{L}_\ell(G)\) between \(L_{i-1}, L_i \in V(H)\). So \(Q^{[\ell]}\) is an \(s\)-link of \(H\). In Figure 2 b, \([u_0, f_0, v_0, e_0, v_1, e_1, v_0, e_0, v_1]\) is a 2-path of \(H\).

We say \(H\) is homomorphic to \(G\), written \(H \rightarrow G\), if there is an injection \(\alpha : V(H) \cup E(H) \rightarrow V(G) \cup E(G)\) such that for \(w \in V(H), f \in E(H)\) and \([u, e, v] \in \mathcal{L}_1(H)\), their images \(w^\alpha \in V(G), f^\alpha \in E(G)\) and \([u^\alpha, e^\alpha, v^\alpha] \in \mathcal{L}_1(G)\). In this case, \(\alpha\) is called a homomorphism from \(H\) to \(G\). The definition here is a generalisation of the one for simple graphs by Godsil and Royle [9, page 6]. A bijective homomorphism is an isomorphism. By Hell and Nešetřil [11], \(\chi(H) \leq \chi(G)\) if \(H \rightarrow G\). For instance, \(\tilde{L} \mapsto L\) for \(\tilde{L} \in \mathcal{L}_\ell(G)\) and \(L \in \mathcal{L}_\ell(G)\) can be seen as a homomorphism from \(\mathbb{A}_\ell(G)\) to \(\mathbb{L}_\ell(G)\). By Bang-Jensen and Gutin [1], \(\mathbb{A}_\ell(G)\) is isomorphic to \(\mathbb{A}^\ell(G)\). So \(\chi(\mathbb{A}^\ell(G)) = \chi(\mathbb{A}_\ell(G)) \leq \chi(\mathbb{L}_\ell(G)) = \chi_\ell(G)\). We emphasize that \(\chi(\mathbb{A}^\ell(G))\) might be much less than \(\chi_\ell(G)\). For example, as depicted in Figure 1, when \(t \geq 3\), \(\chi(\mathbb{A}^\ell(D_t)) = 2 < t = \chi_\ell(D_t)\). Kawai and Shibata proved that \(\mathbb{A}^\ell(G)\) is 3-colorable for large enough \(\ell\). By the analysis above, Corollary 1.2 implies this result.

A graph homomorphism from \(H\) is usually represented by a vertex partition \(\mathcal{V}\) and an edge partition \(\mathcal{E}\) of \(H\) such that (a) each part of \(\mathcal{V}\) is an independent set of \(H\), and (b) each part of \(\mathcal{E}\) is incident to exactly two parts of \(\mathcal{V}\). In this situation, for different \(U, V \in \mathcal{V}\), define \(\mu(U, V)\) to be the number of parts of \(\mathcal{E}\) incident to both \(U\) and \(V\). The quotient graph \(H_{(\mathcal{V}, \mathcal{E})}\) of \(H\) is defined to be the graph with vertex set \(\mathcal{V}\), and for every pair of different \(U, V \in \mathcal{V}\), there are exactly \(\mu(U, V)\) edges between them. To avoid ambiguity, for \(V \in \mathcal{V}\) and \(E \in \mathcal{E}\), we use \(V_V\) and \(E_E\) to denote the corresponding vertex and edge of \(H_{(\mathcal{V}, \mathcal{E})}\), which defines a graph homomorphism from \(H\) to \(H_{(\mathcal{V}, \mathcal{E})}\). Sometimes, we only need the underlying simple graph \(H_V\) of \(H_{(\mathcal{V}, \mathcal{E})}\).

For \(\ell \geq 2\), there is a natural partition in an \(\ell\)-link graph. For each \(R \in \mathcal{L}_{\ell-2}(G)\), let \(\mathcal{L}_{\ell}(G, R)\), or \(\mathcal{L}_0(R)\) for short, be the set of \(\ell\)-links of \(G\) with middle segment \(R\). Clearly, \(\mathcal{V}_0(G) := \{\mathcal{L}_0(R) \neq \emptyset | R \in \mathcal{L}_{\ell-2}(G)\}\) is a vertex partition of \(\mathcal{L}_{\ell}(G)\). And \(\mathcal{E}_0(G) := \{\mathcal{L}_{\ell+1}(R) \neq \emptyset | R \in \mathcal{L}_{\ell-1}(G)\}\) is an edge partition of \(\mathcal{L}_{\ell}(G)\). Consider the 2-link graph \(H\) in Figure 2 b. The vertex and edge partitions of \(H\) are indicated by the dotted rectangles and ellipses, respectively. The corresponding quotient graph is given in Figure 2 c.

Special partitions are required to describe the structure of \(\ell\)-link graphs. Let \(H\) be a graph admitting partitions \(\mathcal{V}\) of \(V(H)\) and \(\mathcal{E}\) of \(E(H)\) that satisfy (a) and (b) above. \((\mathcal{V}, \mathcal{E})\) is called an *almost standard partition of \(H* if further:

1. Each part of \(\mathcal{E}\) induces a complete bipartite subgraph of \(H\),
2. Each vertex of \(H\) is incident to at most two parts of \(\mathcal{E}\),
3. For each \(V \in \mathcal{V}\), and different \(E, F \in \mathcal{E}\), \(V\) contains at most one vertex incident to both \(E\) and \(F\).

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If \( \ell \geq 2 \) is an even integer, and \( G \) is a simple graph, then \( \mathbb{L}_\ell(G) \) is isomorphic to the \((2, \ell/2)\)-double star graph of \( G \) introduced by Jia [12]. While this article focuses on the combinatorial properties including connectedness, coloring, and minors of \( \mathbb{L}_\ell(G) \), a series of companion papers have been composed to contribute to the recognition and determination problems and algorithms. For example, a joint work by Ellingham and Jia [8] shows that, for a given graph \( H \), there is at most one pair \((G, \ell)\), where \( \ell \geq 2 \), and \( G \) is a simple graph of minimum degree at least 3, such that \( \mathbb{L}_\ell(G) \) is isomorphic to \( H \). Moreover, such a pair can be determined from \( H \) in linear time.

3. GENERAL STRUCTURE OF \( \ell \)-LINK GRAPHS

We begin by determining some basic properties of \( \ell \)-link graphs, including their multiplicity and connectedness. The work in this section forms the basis for our main results on coloring and minors of \( \ell \)-link graphs.

Let us first fix some concepts by two observations.

Observation 3.1. The number of edges of \( \mathbb{L}_\ell(G) \) is equal to the number of vertices of \( \mathbb{L}_{\ell+1}(G) \). In particular, if \( G \) is \( r \)-regular for some \( r \geq 2 \), then this number is \( |E(G)|(r-1)^\ell \). If further \( \ell \geq 1 \), then \( \mathbb{L}_\ell(G) \) is \( 2(r-1) \)-regular.

**Proof.** Let \( G \) be \( r \)-regular, \( n := |V(G)| \) and \( m := |E(G)| \). We prove that \( |\mathcal{L}_{\ell+1}(G)| = m(r-1)^\ell \) by induction on \( \ell \). It is trivial for \( \ell = 0 \). For \( \ell = 1 \), \( |\mathcal{L}_2(v)| = \binom{n}{2} \), and hence \( |\mathcal{L}_2(G)| = \binom{n}{2} n = m(r-1) \). Inductively assume \( |\mathcal{L}_{\ell+1}(G)| = m(r-1)^{\ell-2} \) for some \( \ell \geq 2 \). For each \( R \in \mathcal{L}_{\ell+1}(G) \), we have \( |\mathcal{L}_{\ell+1}(R)| = (r-1)^2 \) since \( r \geq 2 \). Thus \( |\mathcal{L}_{\ell+1}(G)| = |\mathcal{L}_{\ell-1}(G)|(r-1)^3 = m(r-1)^{\ell} \) as desired. The other assertions follow from the definitions.

Observation 3.2. Let \( n, m \geq 2 \). If \( \ell \geq 1 \) is odd, then \( \mathbb{L}_\ell(K_{n,m}) \) is \((n+m-2)\)-regular with order \( nm[(n-1)(m-1)]^{\ell-1} \). If \( \ell \geq 2 \) is even, then \( \mathbb{L}_\ell(K_{n,m}) \) has average degree \( \frac{2nm(n+m-2)(n-1)(m-1)}{n+m-2} \), and order \( \frac{1}{2} nm(n+m-2)(n-1)(m-1)^{\ell-1} \).

**Proof.** Let \( \ell \geq 1 \) be odd, and \( L \) be an \( \ell \)-link of \( K_{n,m} \) with middle edge incident to a vertex \( u \) of degree \( n \) in \( K_{n,m} \). It is not difficult to see that \( L \) can be shunted in one step to \( n-1 \) \( \ell \)-links whose middle edge is incident to \( u \). By symmetry, each vertex of \( \mathbb{L}_\ell(K_{n,m}) \) is incident to \( (n-1)+(m-1) = n+m-2 \) edges. Now we prove \( |\mathcal{L}_\ell(K_{n,m})| = nm[(n-1)(m-1)]^{\ell-1} \) by induction on \( \ell \). Clearly, \( |\mathcal{L}_1(K_{n,m})| = |E(K_{n,m})| = nm \). Inductively assume \( |\mathcal{L}_{\ell+2}(K_{n,m})| = nm[(n-1)(m-1)]^{\ell-1} \) for some \( \ell \geq 3 \). For each \( R \in \mathcal{L}_{\ell+2}(K_{n,m}) \), we have \( |\mathcal{L}_3(R)| = (n-1)(m-1) \). So \( |\mathcal{L}_\ell(K_{n,m})| = |\mathcal{L}_{\ell-2}(K_{n,m})|(n-1)(m-1) = nm[(n-1)(m-1)]^{\ell-1} \) as desired. The even \( \ell \) case is similar.

3.1. Loops and Multiplicity

Our next observation is a prerequisite for the study of the chromatic number since it indicates that \( \ell \)-link graphs are loopless.

Observation 3.3. For each \((\ell+1)\)-arc \( \bar{Q} \), we have \( \bar{Q}[0, \ell] \neq \bar{Q}[1, \ell+1] \).

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Let \( G \) be a graph, and \( \tilde{Q} := (v_0, e_1, \ldots, e_{\ell+1}, v_{\ell+1}) \in \mathcal{L}_{\ell+1}(G) \). Since \( G \) is loopless, \( v_0 \neq v_1 \) and hence \( \tilde{Q}(0, 0) \neq \tilde{Q}(1, 1) \). So the statement holds for \( \ell = 0 \). Moreover, \( \tilde{Q}(0, \ell) \neq \tilde{Q}(1, \ell + 1) \). Now let \( \ell \geq 1 \). Suppose for a contradiction that \( \tilde{Q}(0, \ell) = -\tilde{Q}(1, \ell + 1) \). Then \( v_i = v_{i+1} \) and \( e_{i+1} = e_{\ell+1-i} \) for \( i \in \{0, 1, \ldots, \ell\} \).

If \( \ell = 2s \) for some integer \( s \geq 1 \), then \( v_s = v_{s+1} \), contradicting that \( G \) is loopless. If \( \ell = 2s + 1 \) for some integer \( s \geq 0 \), then \( e_{s+1} = e_{s+2} \), contradicting the definition of a \(*\)-arc.

The following statement indicates that, for each \( \ell \geq 1 \), \( \mathbb{L}_\ell(G) \) is simple if \( G \) is simple, and has multiplicity exactly \( 2 \) otherwise.

**Observation 3.4.** Let \( G \) be a graph, \( \ell \geq 1 \), and \( L_0, L_1 \in \mathcal{L}_\ell(G) \). Then \( L_0 \) can be shunted to \( L_1 \) through two \((\ell + 1)\)-links of \( G \) if and only if \( G \) contains a 2-cycle \( O := [v_0, e_0, v_1, e_1, v_0] \), such that one of the following cases holds:

1. \( \ell \geq 1 \) is odd, and \( L_i = [v_i, e_i, v_{i-1}, e_{i-1}, \ldots, v_{i-j}, e_{i-j}] \in \mathcal{L}_\ell(O) \) for \( i \in \{0, 1\} \). In this case, \( [v_i, e_i, v_{i-1}, e_{i-1}, \ldots, v_{i-j}, e_{i-j}] \in \mathcal{L}_{\ell+1}(O) \), for \( i \in \{0, 1\} \), are the only two \((\ell + 1)\)-links available for the shunting.
2. \( \ell \geq 2 \) is even, and \( L_i = [v_i, e_i, v_{i-1}, e_{i-1}, \ldots, v_{i-j}, e_{i-j}] \in \mathcal{L}_\ell(O) \) for \( i \in \{0, 1\} \). In this case, \( [v_i, e_i, v_{i-1}, e_{i-1}, \ldots, v_{i-j}, e_{i-j}] \in \mathcal{L}_{\ell+1}(O) \), for \( i \in \{0, 1\} \), are the only two \((\ell + 1)\)-links available for the shunting.

**Proof.** \((\Leftarrow)\) is trivial. For \((\Rightarrow)\), since \( L_0 \) can be shunted to \( L_1 \), there exists \( \tilde{L} := (v_0, e_0, v_1, e_1, \ldots, e_{\ell}, v_{\ell+1}) \in \mathcal{L}_{\ell+1}(G) \) such that \( L_i = \tilde{L}[i, \ell + i] \) for \( i \in \{0, 1\} \). Let \( \tilde{R} \in \mathcal{L}_{\ell+1}(G) \setminus \{\tilde{L}\} \) such that \( L_i = \tilde{R}[i, \ell + i] \). Then \( \tilde{L}(i, \ell + i) \) equals \( \tilde{R}(i, \ell + i) \) or \( \tilde{R}(\ell + i, i) \). Suppose for a contradiction that \( \tilde{L}(0, \ell) = \tilde{R}(0, \ell) \). Then \( \tilde{L}(1, \ell) = \tilde{R}(1, \ell) \). Since \( \tilde{\ell} \neq \tilde{R} \), we have \( \tilde{L}(1, \ell + 1) \neq \tilde{R}(1, \ell + 1) \). Thus \( \tilde{L}(1, \ell + 1) = \tilde{R}(\ell + 1, 1) \), and hence \( \tilde{L}(2, \ell + 1) = \tilde{R}(\ell, 1) = \tilde{L}(\ell, 1) \), contradicting Observation 3.3. So \( \tilde{L}(0, \ell) = \tilde{R}(0, \ell) \).

Similarly, \( \tilde{L}(1, \ell + 1) = \tilde{R}(\ell + 1, 1) \). Consequently, \( \tilde{L}(0, \ell - 1) = \tilde{R}(\ell, 1) = \tilde{L}(2, \ell + 1) \); that is, \( v_j = v_0 \) and \( e_j = e_0 \) if \( j \in [0, \ell) \) is even, while \( v_j = v_1 \) and \( e_j = e_1 \) if \( j \in [0, \ell + 1) \) is odd.

### 3.2. Connectedness

This subsection characterizes when \( \mathbb{L}_\ell(G) \) is connected. Let \( L := [v_0, e_1, \ldots, e_\ell, v_\ell] \) be an \( \ell \)-link of \( G \), and \( m := \lceil \frac{\ell}{2} \rceil \). The middle unit \( c_L \) of \( L \) is defined to be \( v_m \) if \( \ell \) is even, and \( e_m \) if \( \ell \) is odd. Denote by \( G(\ell) \) the subgraph of \( G \) induced by the middle units of \( \ell \)-links of \( G \).

The lemma below is important in dealing with the connectedness of \( \ell \)-link graphs. Before stating it, we define a conjunction operation, which is an extension of an operation by Biggs [3, Chapter 17]. Let \( \tilde{L} := (v_0, e_1, v_1, \ldots, e_\ell, v_\ell) \in \mathcal{L}_\ell(G) \) and \( \tilde{R} := (u_0, f_1, u_1, \ldots, f_s, u_s) \in \mathcal{L}_s(G) \) such that \( v_\ell = u_0 \) and \( e_\ell \neq f_1 \). The conjunction of \( \tilde{L} \) and \( \tilde{R} \) is \( (\tilde{L} \tilde{R}) := (v_0, e_1, \ldots, e_\ell, v_\ell = u_0, f_1, \ldots, f_s, u_s) \in \mathcal{L}_{\ell+s}(G) \) or \( [\tilde{L} \tilde{R}] := [v_0, e_1, \ldots, e_\ell, v_\ell = u_0, f_1, \ldots, f_s, u_s] \in \mathcal{L}_{\ell+s}(G) \).

**Lemma 3.5.** Let \( \ell, s \geq 0 \), and \( G \) be a connected graph. Then \( G(\ell) \) is connected. And each \( s \)-link of \( G(\ell) \) is a middle segment of a \((2\lceil \frac{\ell}{2} \rceil + s)\)-link of \( G \). Moreover, for \( \ell \)-links \( L \) and \( R \) of \( G \), there is an \( \ell \)-link \( L' \) with middle unit \( c_{L'} \), and an \( \ell \)-link \( R' \) with middle unit \( c_R \), such that \( L' \) can be shunted to \( R' \).

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Proof. For \( \ell \in \{0, 1\} \), since \( G \) is connected, \( G(\ell) = G \) and the lemma holds. Let \( \ell := 2m \geq 2 \) be even. Then for \( u, v \in V(G(\ell)) \) if and only if they are middle vertices of some \( \vec{L}, \vec{R} \in \bar{L}_\ell(G) \), respectively. Since \( G \) is connected, there exists some \( \vec{P} := (u = v_0, e_1, \ldots, e_s, v_s = v) \in \bar{L}_\ell(G) \). By Observation 3.3, \( \bar{L}[m - 1, m] \neq \bar{L}[m, m + 1] \). For such an \( s \)-arc \( \vec{P} \), without loss of generality, \( e_1 \neq \bar{L}[m - 1, m] \), and similarly, \( e_s \neq \bar{R}[m, m + 1] \). Then \( \vec{P} \) is a middle segment of \( \vec{Q} := (\bar{L}(0, m).\vec{P}.\bar{R}(m, 2m)) \in \bar{L}_{\ell+s}(G) \). So \( L' := \bar{Q}[0, \ell] \) can be shunted to \( R' := \bar{Q}[s, \ell + s] \) through \( \bar{Q} \). Moreover, for each \( i \in \{0, \ldots, s\} \), \( v_i \) is the middle vertex of \( \bar{Q}[i, \ell + i] \in \bar{L}_\ell(G) \). Hence \( \bar{P} \) is an \( s \)-arc of \( G(\ell) \) from \( u \) to \( v \). So \( G(\ell) \) is connected. The odd \( \ell \) case is similar.

Sufficient conditions for \( \mathbb{A}_\ell(G) \) to be strongly connected can be found in [9, page 76]. The following corollary of Lemma 3.5 reveals a strong relationship between the shunting of \( \ell \)-links and the connectedness of \( \ell \)-link graphs.

Corollary 3.6. For a connected graph \( G \), \( \mathbb{L}_\ell(G) \) is connected if and only if every pair of \( \ell \)-links of \( G \) with the same middle unit can be shunted to each other.

Proof. On the one hand, if \( \mathbb{L}_\ell(G) \) is connected, then every pair of \( \ell \)-links of \( G \) can be shunted to each other. On the other hand, let \( L \) and \( R \) be two \( \ell \)-links of \( G \). Since \( G \) is connected, by Lemma 3.5, there are \( \ell \)-links \( L' \) and \( R' \) with \( c_L = c_L' \) and \( c_R = c_R' \) such that \( L' \) can be shunted to \( R' \). Hence if \( L \) can be shunted to \( L' \) and \( R \) can be shunted to \( R' \), then \( L \) can be shunted to \( R \). So if every pair of \( \ell \)-links of \( G \) with the same middle unit can be shunted to each other, then \( \mathbb{L}_\ell(G) \) is connected.

We now present our main result of this section, which plays a key role in dealing with the graph minors of \( \ell \)-link graphs in Section 5.

Lemma 3.7. Let \( G \) be a graph, and \( X \) be a connected subgraph of \( G(\ell) \). Then for every pair of \( \ell \)-links \( L \) and \( R \) of \( X \), \( L \) can be shunted to \( R \) under the restriction that in each step, the middle unit of the image of \( L \) belongs to \( X \).

Proof. First we consider the case that \( c_L \) is in \( R \). Then there is a common segment \( Q \) of \( L \) and \( R \) of maximum length containing \( c_L \). Without loss of generality, assign directions to \( L \) and \( R \) such that \( \bar{L} = (\bar{L}_0.\bar{Q}1,\bar{L}_1) \) and \( \bar{R} = (\bar{R}_0.\bar{Q}1,\bar{R}_0) \), where \( \bar{L}_i \in \bar{L}_\ell(X) \) and \( \bar{R}_i \in \bar{L}_\ell(X) \) for \( i \in \{0, 1\} \) such that \( s_1 \geq s_0 \). Then \( \ell \geq \ell_0 + \ell_1 = s_0 + s_1 \geq s_1 \). Let \( x \) be the head vertex and \( e \) be the head edge of \( \bar{L} \). Since \( c_L \) is in \( Q \), \( \ell_0 \leq \ell/2 \). Since \( X \) is a subgraph of \( G(\ell) \), by Lemma 3.5, there exists \( \bar{L}_2 \in \bar{L}_{\ell_0}(G) \) with tail vertex \( x \) and tail edge different from \( e \). Let \( y \) be the tail vertex and \( f \) be the tail edge of \( \bar{R} \). Then there exists \( \bar{R}_2 \in \bar{L}_{\ell_0}(G) \) with head vertex \( y \) and head edge different from \( f \). We can shunt \( L \) to \( R \) first through \( (\bar{L}1.\bar{L}_2) \in \bar{L}_{\ell_0+\ell_1}(G) \), then \( (\bar{R}_2.\bar{R}_1.\bar{Q}1.\bar{Q}_1.\bar{L}_1.\bar{L}_2) \in \bar{L}_{\ell_0+\ell_0+\ell_1}(G) \), and finally \( (\bar{R}_2.\bar{R}) \in \bar{L}_{\ell_0+\ell_1}(G) \). Since \( \ell_0 \leq \ell/2 \) and \( s_0 \leq s_1 \leq \ell/2 \), the middle unit of each image is inside \( L \) or \( R \).

Second, we consider the case that \( c_L \) is not in \( R \). Then there exists a segment \( Q \) of \( L \) of maximum length that contains \( c_L \), and is edge-disjoint with \( R \). Since \( X \) is connected, there exists a shortest \( \ell \)-arc \( \bar{P} \) from a vertex \( v \) of \( R \) to a vertex \( u \) of \( L \). Then \( \bar{P} \) is edge-disjoint with \( Q \) because of its minimality. Without loss of generality, assign directions to \( L \) and \( R \) such that \( u \) separates \( \bar{L} \) into \( (\bar{L}_0.\bar{L}_1) \) with \( c_L \) on \( \bar{L}_1 \), and \( v \) separates \( \bar{R} \) into \( (\bar{R}_1.\bar{R}_0) \), where \( \bar{L}_i \) is of length \( \ell_i \) while \( R_i \) is of length \( s_i \) for \( i \in \{0, 1\} \), such that \( s_1 \geq s_0 \). Then \( \ell_0, s_0 \leq \ell/2 \). Let \( x \) be the head vertex and \( e \) be the head edge of \( \bar{L} \). Since \( \ell_0 \leq \ell/2 \) and \( X \) is a subgraph of \( G(\ell) \), by Lemma 3.5, there exists an \( \ell_0 \)-arc \( \bar{L}_2 \) of \( G \) with tail vertex \( x \)

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and tail edge different from \( e \). Let \( y \) be the tail vertex and \( f \) be the tail edge of \( \tilde{R} \). Then there exists an \( s_0 \)-arc \( \tilde{R}_2 \) of \( G \) with head vertex \( y \) and head edge different from \( f \). Now we can shunt \( L \) to \( R \) through \((L,\tilde{L}_2), (\tilde{R}_2,\tilde{R}_1,\tilde{P},\tilde{L}_1,\tilde{L}_2)\) and \((\tilde{R}_2,\tilde{R})\) consecutively. One can check that in this process the middle unit of each image belongs to \( L, P, \) or \( R \). ■

From Lemma 3.7, the set of \( \ell \)-links of a connected \( G(\ell) \) serves as a “hub” in the shunting of \( \ell \)-links of \( G \). More explicitly, for \( L, R \in \mathcal{L}_\ell(G) \), if we can shunt \( L \) to \( L' \in \mathcal{L}_\ell(G(\ell)) \), and \( R \) to \( R' \in \mathcal{L}_\ell(G(\ell)) \), then \( L \) can be shunted to \( R \) since \( L' \) can be shunted to \( R' \). Thus we have the following corollary that provides a more efficient way to test the connectedness of \( \ell \)-link graphs.

**Corollary 3.8.** Let \( G \) be a graph such that \( G(\ell) \) contains at least one \( \ell \)-link. Then \( \mathbb{L}_\ell(G) \) is connected if and only if \( G(\ell) \) is connected, and each \( \ell \)-link of \( G \) can be shunted to an \( \ell \)-link of \( G(\ell) \).

### 4. CHROMATIC NUMBER OF \( \ell \)-LINK GRAPHS

In this section, we reveal a recursive structure of an \( \ell \)-link graph \( H \), which leads to an upper bound for the chromatic number of \( H \). To achieve this, we need to show that when \( \ell \geq 2 \), \( H \) admits an almost standard partition defined in Section 2.

**Lemma 4.1.** Let \( G \) be a graph and \( \ell \geq 2 \) be an integer. Then \((V, E) := (V(\ell)(G), E(\ell)(G))\) is an almost standard partition of \( H := \mathbb{L}_\ell(G) \). Further, \( H(V, E) \) is isomorphic to an induced subgraph of \( \mathbb{L}_{\ell-2}(G) \).

**Proof.** First we verify that \((V, E)\) satisfies conditions (a)–(e) in the definition of an almost standard partition in Section 2.

(a) We prove that, for each \( R \in \mathcal{L}_{\ell-2}(G) \), \( V := \mathcal{L}_\ell(R) \in V \) is an independent set of \( H \). Suppose not. Then there are \( \tilde{L}, \tilde{L}' \in \tilde{\mathcal{L}}_\ell(G) \) such that \( L, L' \in V \), and \( L \) can be shunted to \( L' \) in one step. Then \( R = \tilde{L}[1, \ell - 1] \) can be shunted to \( R = \tilde{L}'[1, \ell - 1] \) in one step, contradicting Observation 3.3.

(b) Here we show that each \( E \in E \) is incident to exactly two parts of \( V \). By definition there exists \( P \in \mathcal{L}_{\ell-1}(G) \) with \( \mathcal{L}_{\ell+1}(P) = E \). Let \( \{L, R\} := P^{(\ell-2)} \). Then \( \mathcal{L}_\ell(L) \) and \( \mathcal{L}_\ell(R) \) are the only two parts of \( V \) incident to \( E \).

(c) We explain that each \( E \in E \) is the edge set of a complete bipartite subgraph of \( H \). By definition there exists \( \tilde{P} \in \tilde{\mathcal{L}}_{\ell-1}(G) \) with \( \tilde{\mathcal{L}}_{\ell+1}(P) = E \). Let \( A := \{[\tilde{e}, \tilde{P}] \in \tilde{\mathcal{L}}_\ell(G)\} \) and \( B := \{[\tilde{e}, \tilde{P}] \in \tilde{\mathcal{L}}_\ell(G)\} \). One can check that \( E \) induces a complete bipartite subgraph of \( H \) with bipartition \( A \cup B \).

(d) We prove that each \( v \in V(H) \) is incident to at most two parts of \( E \). By definition there exists \( Q \in \mathcal{L}_{\ell}(G) \) with \( Q = v \). Then the set of edge parts of \( E \) incident to \( v \) is \( \mathcal{L}_{\ell+1}(L) = \emptyset \) or \( L \in \mathcal{L}(\ell-1)[L] \) with cardinality at most 2.

(e) Let \( v \) be a vertex of \( V \in V \) incident to different \( E, F \in E \). We explain that \( v \) is uniquely determined by \( V, E, \) and \( F \).

By the analysis above, \((V, E)\) is an almost standard partition of \( H \).

By definition there exists \( \tilde{P} \in \tilde{\mathcal{L}}_{\ell-2}(G) \) such that \( V = \mathcal{L}_\ell(P) \). There also exists \( Q := [\tilde{e}, \tilde{P}, \tilde{e}] \in \tilde{\mathcal{L}}_\ell(P) \) such that \( v = Q \). Besides, there are \( L, R \in \mathcal{L}_{\ell-1}(G) \) such that \( E = \mathcal{L}_{\ell+1}(L) \) and \( F = \mathcal{L}_{\ell+1}(R) \). Then \( \{L, R\} = Q^{(\ell-1)} \) since \( L \neq R \). Note that \( Q \) is uniquely determined by \( V, E, \) and \( F \).
determined by \( \mathcal{Q}(k-1) \) and \( c_Q = c_P \). Thus it is uniquely determined by \( E = \mathcal{L}_{l+1}(L), F = \mathcal{L}_l(R), \) and \( V = \mathcal{L}_l(P) \).

Now we show that \( H(\mathcal{V}, \mathcal{E}) \) is isomorphic to an induced subgraph of \( \mathbb{I}_{l-2}(G) \). Let \( X \) be the subgraph of \( \mathbb{I}_{l-2}(G) \) of vertices \( L \in \mathbb{I}_{l-2}(G) \) such that \( \mathcal{L}_l(L) \neq \emptyset \), and edges \( Q \in \mathbb{L}_{l-1}(G) \) such that \( \mathcal{L}_{l+1}(Q) \neq \emptyset \). One can check that \( X \) is an induced subgraph of \( \mathbb{I}_{l-2}(G) \). An isomorphism from \( H(\mathcal{V}, \mathcal{E}) \) to \( X \) can be defined as the injection sending \( \mathcal{L}_l(L) \neq \emptyset \) to \( L \), and \( \mathcal{L}_{l+1}(Q) \neq \emptyset \) to \( Q \).

Below we give an interesting algorithm for coloring a class of graphs.

**Lemma 4.2.** Let \( H \) be a graph with a \( t \)-coloring such that each vertex of \( H \) is adjacent to at most \( r \) differently colored vertices. Then \( \chi(H) \leq \left\lfloor \frac{t}{r+1} \right\rfloor + 1 \).

**Proof.** The result is trivial for \( t = 0 \) since, in this case, \( \chi(H) = 0 \). If \( r + 1 \geq t + 1 \), then \( \left\lfloor \frac{t}{r+1} \right\rfloor = \left\lfloor \frac{t}{r+1} \right\rfloor = t - 1 \), and the lemma holds since \( t \geq \chi(H) \).

Now assume \( t \geq r + 2 \). Let \( U_1, U_2, \ldots, U_t \) be the color classes of the given coloring. For \( i \in [t] \), denote by \( i \) the color assigned to vertices in \( U_i \). Run the following algorithm: For \( j = 1, \ldots, t \), and for each \( u \in U_{i-j} \), let \( s \in [t] \) be the minimum integer that is not the color of a neighbor of \( u \) in \( H \); if \( s < t - j + 1 \), then recolor \( u \) by \( s \).

In the algorithm above, denote by \( C_i \) the set of colors used by the vertices in \( U_i \) for \( i \in \{1, \ldots, t\} \). Let \( k := \left\lfloor \frac{t}{r+1} \right\rfloor \). Then \( t - 1 \geq k(r + 1) \geq k \geq 1 \). We claim that after \( j \in [0, k] \) steps, \( C_{i+1} \subseteq [r+1] \) for \( i \in [j] \), and \( C_i = \{i\} \) for \( i \in [t - j] \). This is trivial for \( j = 0 \).

Inductively assume it holds for some \( j \in [0, k-1] \). In the \( (j+1) \)-th step, we change the color of each \( u \in U_{i-j} \) from \( t - j \) to the minimum \( s \in [t] \) that is not used by the neighborhood of \( u \). It is enough to show that \( s \leq (j + 1)r + 1 \).

First suppose that all neighbors of \( u \) are in \( \bigcup_{i \in [t-j-1]} U_i \). By the analysis above, \( t - j - 1 \geq r - k \geq kr + 1 \geq r + 1 \). So at least one part of \( S := \{U_i | i \in [t-j-1]\} \) contains no neighbor of \( u \). From the induction hypothesis, \( C_i = \{i\} \) for \( i \in [t-j-1] \).

Hence at least one color in \([r+1]\) is not used by the neighborhood of \( u \); that is, \( s \leq r + 1 \leq (j+1)r + 1 \).

Now suppose that \( u \) has at least one neighbor in \( \bigcup_{i \in [t-j+1]} U_i \). By the induction hypothesis, \( \bigcup_{i \in [t-j+1]} C_i \subseteq [r+1] \). At the same time, \( u \) has neighbors in at most \( r - 1 \) parts of \( S \). So the colors possessed by the neighborhood of \( u \) are contained in \([jr + 1 + r - 1] = (j+1)r \). Thus \( s \leq (j+1)r + 1 \). This proves our claim.

The claim above indicates that, after the \( k \)-th step, \( C_{i+1} \subseteq [r+1] \) for \( i \in [k] \), and \( C_i = \{i\} \) for \( i \in [t-k] \). Hence we have a \((r-k)\)-coloring of \( H \) since \( t - k \geq kr + 1 \).

Therefore, \( \chi(H) \leq t - k = \left\lfloor \frac{r}{r+1} \right\rfloor = \left\lfloor \frac{r}{r+1} \right\rfloor + 1 \).

**Lemma 4.1** indicates that \( \mathbb{I}_{l+2}(G) \) is homomorphic to \( \mathbb{I}_{l-2}(G) \) for \( \ell \geq 2 \). So by [6, Proposition 1.1], \( \chi_l(G) \leq \chi_{l-2}(G) \). By Lemma 4.1, every vertex of \( \mathbb{I}_{l+2}(G) \) has neighbors in at most two parts of \( \mathcal{V}_l(G) \), which enables us to improve the upper bound on \( \chi_l(G) \).

**Lemma 4.3.** Let \( G \) be a graph, and \( \ell \geq 2 \). Then \( \chi_l(G) \leq \left\lfloor \frac{3}{2} \chi_{l-2}(G) \right\rfloor + 1 \).

**Proof.** By Lemma 4.1, \((\mathcal{V}, \mathcal{E}) := (\mathcal{V}_l(G), \mathcal{E}_l(G))\) is an almost standard partition of \( H := \mathbb{I}_{l+2}(G) \). So each vertex of \( H \) has neighbors in at most two parts of \( \mathcal{V} \). Further, \( H_V \) is a subgraph of \( \mathbb{I}_{l-2}(G) \). So \( \chi_l(G) \leq \chi := \chi(H_V) \leq \chi_{l-2}(G) \).

We now construct a \( \chi \)-coloring of \( H \) such that each vertex of \( H \) is adjacent to at most two differently colored vertices. By definition \( H_V \) admits a \( \chi \)-coloring with color classes \( K_1, \ldots, K_\chi \). For \( i \in [\chi] \), assign the color \( i \) to each vertex of \( H \) in \( U_i := \bigcup_{V \in K_i} V \). One
can check that this is a desired coloring. In Lemma 4.3, letting \( t = \chi \) and \( r = 2 \) yields that \( \chi_\ell(G) \leq \lfloor \frac{1}{3} \chi \rfloor + 1 \). Recall that \( \chi \leq \chi_{\ell-2}(G) \). Thus the lemma follows.

As shown below, Lemma 4.3 can be applied recursively to produce an upper bound for \( \chi_\ell(G) \) in terms of \( \chi(G) \) or \( \chi'(G) \).

**Proof of Theorem 1.1.** When \( \ell \in \{0, 1\} \), it is trivial for (1)(2) and (4). By [7, Proposition 5.2.2], \( \chi_0 = \chi \leq \Delta + 1 \). So (3) holds. Now let \( \ell \geq 2 \). By Lemma 4.1, \( H := \mathbb{L}_{\ell}(G) \) admits an almost standard partition \( (V, E) := (V_\ell, E_\ell) \), such that \( H(V, E) \) is an induced subgraph of \( \mathbb{L}_{\ell-2}(G) \). By definition each part of \( V \) is an independent set of \( H \). So \( H \to \mathbb{L}_{\ell-2}(G) \), and \( \chi_\ell \leq \chi_{\ell-2} \). This proves (4). Moreover, each vertex of \( H \) has neighbors in at most two parts of \( V \). By Lemma 4.3, \( \chi_\ell := \chi_\ell(G) \leq \frac{2\chi_{\ell-2} + 1}{3} \). Continue the analysis, we have \( \chi_\ell \leq \chi_{\ell-2} \), and \( \chi_\ell - 3 \leq (\frac{2}{3})^i (\chi_{\ell-2i} - 3) \) for \( 1 \leq i \leq \lceil \ell/2 \rceil \). Therefore, if \( \ell \) is even, then \( \chi_\ell \leq \chi_0 = \chi \leq \Delta + 1 \), and \( \chi_\ell - 3 \leq (\frac{2}{3})^{\lfloor \ell/2 \rfloor} (\chi - 3) \).

Thus (1) holds. Now let \( \ell \geq 3 \) be odd. Then \( \chi_\ell \leq \chi_1 = \chi' \), and \( \chi_\ell - 3 \leq (\frac{2}{3})^{\ell/2} (\chi' - 3) \).

This verifies (2). As a consequence, \( \chi_\ell \leq \chi_3 \leq \frac{2}{3} (\chi' - 3) + 3 = \frac{2}{3} \chi' + 1 \). By Shannon [18], \( \chi' \leq \frac{2}{3} \Delta \). So \( \chi_\ell \leq \Delta + 1 \), and hence (3) holds.

The following corollary of Theorem 1.1 implies that Hadwiger’s conjecture is true for \( \mathbb{L}_{\ell}(G) \) if \( G \) is regular and \( \ell \geq 4 \).

**Corollary 4.4.** Let \( G \) be a graph with \( \Delta := \Delta(G) \geq 3 \). Then \( \chi_\ell(G) \leq 3 \) for all \( \ell > 2 \log_{1.5}(\Delta - 2) + 3 \). Further, Hadwiger’s conjecture holds for \( \mathbb{L}_{\ell}(G) \) if \( \ell > 2 \log_{1.5}(\Delta - 2) - 3.83 \), or \( d := d(G) \geq 3 \) and \( \ell > 2 \log_{1.5} \frac{\Delta - 2}{d - 2} + 3 \).

**Proof.** By Theorem 1.1, for each \( t \geq 3 \), \( \chi_t := \chi_\ell(G) \leq t \) if \( (\frac{2}{3})^{\ell/2} (\Delta - 2) < t - 2 \) and \( (\frac{2}{3})^{\ell/2} \frac{(t - 2)}{3} < t - 2 \). Solving these inequalities gives \( \ell > 2 \log_{1.5}(\Delta - 2) - 2 \log_{1.5}(t - 2) + 3 \). Thus \( \chi_\ell \leq 3 \) if \( \ell > 2 \log_{1.5}(\Delta - 2) + 3 \). So the first statement holds.

By Robertson et al. [17] and Theorem 1.3, Hadwiger’s conjecture holds for \( \mathbb{L}_{\ell}(G) \) if \( \ell \geq 1 \) and \( \chi_\ell \leq \max\{6, d\} \). Letting \( t = 6 \) gives that \( \ell > 2 \log_{1.5}(\Delta - 2) - 4 \log_{1.5} 2 + 3 \). Letting \( t = d \geq 3 \) gives that \( \ell > 2 \log_{1.5} \frac{\Delta - 2}{d - 2} + 3 \). So the corollary holds since \( 4 \log_{1.5} 2 - 3 > 3.83 \).

**Proof of Theorem 1.5(3)(4)(5).** (3) and (4) follow from Corollary 4.4. Now consider (5). By Reed and Seymour [16], Hadwiger’s conjecture holds for \( \mathbb{L}_1(G) \). If \( \ell \geq 2 \) and \( \Delta \leq 5 \), by Theorem 1.1(3), \( \chi_\ell(G) \leq 6 \). In this case, Hadwiger’s conjecture holds for \( \mathbb{L}_{\ell}(G) \) by Robertson et al. [17].

**5. COMPLETE MINORS OF \( \ell \)-LINK GRAPHS**

It has been proved in the last section that Hadwiger’s conjecture is true for \( \mathbb{L}_{\ell}(G) \) if \( \ell \) is large enough. In this section, we further investigate the minors, especially the complete minors, of \( \ell \)-link graphs. To see the intuition of our method, let \( v \) be a vertex of degree \( t \) in a graph \( G \). Then \( \mathbb{L}_1(G) \) contains a \( K_t \)-subgraph whose vertices correspond to the edges of \( G \) incident to \( v \). For \( \ell \geq 2 \), roughly speaking, we extend \( v \) to a subgraph \( X \) of diameter less than \( \ell \), and extend each edge incident to \( v \) to an \( \ell \)-link of \( G \) starting from a vertex of \( X \). By studying the shunting of these \( \ell \)-links, we find a \( K_t \)-minor in \( \mathbb{L}_{\ell}(G) \).

Let \( \{u, e, v\} \) be a 1-link of \( G \). Since \( G \) is undirected, \( e \) has no direction. But we can choose a direction, say \( u \) to \( v \), for \( e \) to get an arc \( e' := (u, e, v) \) of \( G \). For subgraphs \( X, Y \)
of $G$, let $E(X, Y)$ be the set of edges of $G$ between $V(X)$ and $V(Y)$, and $\bar{E}(X, Y)$ be the set of arcs of $G$ from $V(X)$ to $V(Y)$. Figure 3 illustrates the proofs of Lemmas 5.1 and 5.2.

**Lemma 5.1.** Let $\ell \geq 1$ be an integer, $G$ be a graph, and $X$ be a subgraph of $G$ with $\text{diam}(X) < \ell$ such that $Y := G - V(X)$ is connected. If $t := |E(X, Y)| \geq 2$, then $L_t(G)$ contains a $K_t$-minor.

**Proof.** Let $\vec{e}_1, \ldots, \vec{e}_i$ be distinct arcs in $\bar{E}(Y, X)$. Say $\vec{e}_i = (y_i, e_i, x_i)$ for $i \in [t]$. Since $\text{diam}(X) < \ell$, there is a dipath $P_{ij}$ of $X$ from $x_i$ to $x_j$ of length $\ell_{ij} \leq \ell - 1$ such that $P_{ij} = P_{ji}$. Since $Y$ is connected, it contains a dipath $Q_{ij}$ from $y_i$ to $y_j$. Since $t \geq 2$, $O_i := [\vec{P}_{ij} - \vec{e}_i, \vec{Q}_{ij}, \vec{e}_i]$ is a cycle of $G$, where $\vec{e}_i := (i \text{ mod } t) + 1$. Thus $H := L_t(G)$ contains a cycle $L_{ij}(O_i)$, and hence a $K_t$-minor. Now let $t \geq 3$, and $\vec{L}_i \in \mathcal{L}_t(O_i)$ with head arc $\vec{e}_i$. Then $[\vec{L}_i, \vec{P}_{ij}]^{[i]} \in \mathcal{L}_{\ell_t}(H)$. And the union of the units of $[\vec{L}_i, \vec{P}_{ij}]^{[i]}$ over $j \in [t]$ is a connected subgraph $X_i$ of $H$. In the remainder of the proof, for distinct $i, j \in [t]$, we show that $X_i$ and $X_j$ are disjoint. Further, we construct a path in $H$ between $X_i$ and $X_j$ that is internally disjoint with its counterparts, and has no inner vertex in any of $V(X_1), \ldots, V(X_t)$. Then by contracting each $X_i$ into a vertex, and each path into an edge, we obtain a $K_t$-minor of $H$.

First of all, assume for a contradiction that there are different $i, j \in [t]$ such that $X_i$ and $X_j$ share a common vertex that corresponds to an $\ell$-link $R$ of $G$. Then by definition, there exists some $p \in [t]$ such that $R$ can be obtained by shunting $L_i$ along $\bar{L}_i, \bar{P}_{ip}$ by some $s_i \leq \ell_{ip}$ steps. So $R = [\bar{L}_i(s_i), \ell] \bar{P}_{ip}(0, s_i)]$. Similarly, there are $q \in [t]$ and $s_j \leq \ell_{ij}$ such that $R = \bar{L}_j(s_j, \ell) \bar{P}_{jq}(0, s_j)]$. Recall that $E(X) \cap E(X, Y) = E(Y) \cap E(X, Y) = \emptyset$. So $e_i = \bar{L}_i[\ell - 1, \ell]$ and $e_j = \bar{L}_j[\ell - 1, \ell]$ belong to both $L_i$ and $L_j$. By the definition of $O_i$, this happens if and only if $i = j'$ and $j = i'$, which is impossible since $t \geq 3$. 

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Second, for distinct \( i, j \in [t] \), we define a path of \( H \) between \( X_i \) and \( X_j \). Clearly, \( L_q \) can be shunted to \( L_j \) through \( \tilde{R}_{ij} := \tilde{L}_q(\ell, \ell - L_j) \) in \( G \). In this shunting, \( L'_j := [\tilde{L}_j(\ell, \ell - L_j)] \) is the first image corresponding to a vertex of \( X_j \), while \( L'_j := [\tilde{R}_{ij}(\ell, \ell - L_j)] \) is the first image corresponding to a vertex of \( X_j \). Further, \( L'_i \) can be shunted to \( L'_j \) through \( \tilde{R}_{ij} := \tilde{L}_j(\ell, \ell - L_j) \) in \( \mathcal{L}_{2\ell-L_j}(G) \), which is a subsequence of \( R_{ij} \). Then \( R_{ij}^{(\ell)} \) is an \((\ell - \ell_i)\)-path of \( H \) between \( X_i \) and \( X_j \). We show that for each \( p \in [t] \), \( X_p \) contains no inner vertex of \( R_{ij}^{(\ell)} \). When \( \ell - \ell_i = 1 \), \( R_{ij}^{(\ell)} \) contains no inner vertex. Now assume \( \ell - \ell_i \geq 2 \). Each inner vertex of \( R_{ij}^{(\ell)} \) corresponds to some \( Q_{ij} \) \( \in \mathcal{L}_{2\ell-L_j}(G) \), where \( \ell_i + 1 \leq s_i \leq \ell - 1 \). Assume for a contradiction that for some \( p \in [t] \), \( X_p \) contains a vertex corresponding to \( Q_{ij} \). By definition there exists \( q \in [t] \) such that \( Q_{ij} = [\tilde{L}_p(s_p, \ell), \tilde{R}_{pq}(0, s_p)] \), where \( 0 \leq s_p \leq \ell - p \). Without loss of generality, 

\[
(\tilde{L}_i(s_i, \ell), \tilde{P}_{ij}(\ell, \ell + s_i - s_i)) = (\tilde{L}_p(s_p, \ell), \tilde{R}_{pq}(0, s_p)).
\]

Since \( e_p \) and \( e_q \) are not in \( \tilde{Q}_{pq} \), hence \( \tilde{c}_p \) belongs to \( -\tilde{L}_p \) and \( \tilde{c}_q \) belongs to \( -\tilde{L}_j \). By the definition of \( \tilde{L}_i \), this happens only when \( j = p' \) \( \text{ and } p = j' \), contradicting \( t \geq 3 \).

We now show that \( R_{ij}^{(\ell)} \) and \( R_{pq}^{(\ell)} \) are internally disjoint, where \( i \neq j \), \( p \neq q \) and \( \{i, j\} \neq \{p, q\} \). Suppose not. Then by the analysis above, there are \( s_i \) and \( s_p \) with \( \ell_i + 1 \leq s_i \leq \ell - 1 \) and \( \ell_p + 1 \leq s_p \leq \ell - 1 \) such that \( Q_{ij} = Q_{pq} \). Without loss of generality, 

\[
(\tilde{L}_i(s_i, \ell), \tilde{P}_{ij}(\ell, \ell + s_i - s_i)) = (\tilde{L}_p(s_p, \ell), \tilde{R}_{pq}(0, s_p)).
\]

If \( s_i = s_p \), then \( \tilde{c}_i = \tilde{c}_p \) and \( \tilde{c}_q = \tilde{c}_p \) since \( E(X) \cap E(Y) = \emptyset \); that is, \( i = p \) and \( j = q \), contradicting \( \{i, j\} \neq \{p, q\} \). Otherwise, with no loss of generality, \( s_i > s_p \). Then \( \tilde{e}_q \) and \( \tilde{e}_i \) belong to \( \tilde{L}_i \) and \( \tilde{L}_p \), respectively; that is, \( i = p \) and \( j = q \), again contradicting \( \{i, j\} \neq \{p, q\} \).

In summary, \( X_1, \ldots, X_t \) are vertex-disjoint connected subgraphs, which are pairwise connected by internally disjoint \( \ast \)-links \( R_{ij}^{(\ell)} \) of \( H \), such that no inner vertex of \( R_{ij}^{(\ell)} \) is in \( V(X_1) \cup \cdots \cup V(X_t) \). So by contracting each \( X_i \) to a vertex, and \( R_{ij}^{(\ell)} \) to an edge, we obtain a \( K_{t+1} \)-minor of \( H \).

\textbf{Lemma 5.2.} Let \( \ell \geq 1 \), \( G \) be a graph, and \( X \) be a subgraph of \( G \) with \( \text{diam}(X) < \ell \) such that \( Y := G - V(X) \) is connected and contains a cycle. Let \( t := |E(X, Y)| \). Then \( \mathbb{L}_\ell(G) \) contains a \( K_{t+1} \)-minor.

\textbf{Proof.} Let \( O \) be a cycle of \( Y \). Then \( H := \mathbb{L}_\ell(O) \) contains a cycle \( \mathbb{L}_\ell(O) \) and hence a \( K_2 \)-minor. Now assume \( t \geq 2 \). Let \( \tilde{c}_1, \ldots, \tilde{c}_t \) be distinct arcs in \( \tilde{E}(Y, X) \). Say \( \tilde{c}_i = (y_i, e_i, x_i) \) for \( i \in [t] \). Since \( Y \) is connected, there is a dipath \( \tilde{P}_i \) of \( Y \) of minimum length \( s_i \geq 0 \) from some vertex \( y_i \) of \( O \) to \( y_j \). Let \( \tilde{Q}_i \) be an \( \ast \)-arc of \( O \) with head vertex \( y_i \). Then \( \tilde{L}_i := (\tilde{Q}_i, \tilde{P}_i, \tilde{c}_i)(s_i + 1, \ell + s_i + 1) \in \mathcal{L}_{2\ell}(G) \). Since \( \text{diam}(X) \leq \ell - 1 \), there is a dipath \( \tilde{P}_{ij} \) of \( X \) of length \( \ell - 1 \) from \( x_i \) to \( x_j \) such that \( P_{ij} = P_{ij} \).

Clearly, \( [\tilde{L}_i, \tilde{P}_{ij}] \) \( \ell_i \)-link of \( H \). And the union of the units of \( [\tilde{L}_i, \tilde{P}_{ij}] \) over \( j \in [t] \) induces a connected subgraph \( X_i \) of \( H \). For different \( i, j \in [t] \), let \( R_{ij} := [\tilde{L}_i(\ell, \ell - L_j), \tilde{P}_{ij}(\ell, \ell - L_j)] \in \mathcal{L}_{2\ell-L_j}(G) \). Then \( R_{ij}^{(\ell)} \) is an \((\ell - \ell_i, \ell - \ell_j)\)-path of \( H \) between \( X_i \) and \( X_j \). As in the proof of Lemma 5.1, it is easy to check that \( X_1, \ldots, X_t \) are vertex-disjoint connected subgraphs of \( H \), which are pairwise connected by internally disjoint paths \( R_{ij}^{(\ell)} \). Further, no inner vertex of \( R_{ij}^{(\ell)} \) is in \( V(X_1) \cup \cdots \cup V(X_t) \). So a \( K_{t+1} \)-minor of \( H \) is obtained accordingly.

Finally, let \( Z \) be the connected subgraph of \( H \) induced by the units of \( \mathbb{L}_\ell(O) \) and \( [\tilde{Q}_i, \tilde{P}_j] \) over \( i \in [t] \). Then \( Z \) is vertex-disjoint with \( X_i \) and with the paths \( R_{ij}^{(\ell)} \). Moreover, \( Z \) sends an edge \( (\tilde{Q}_i, \tilde{P}_j, \tilde{c}_i)(s_i + 1, \ell - s_i + 1) \) to each \( X_i \). Thus \( H \) contains a \( K_{t+1} \)-minor. \hfill \Box
In the following, we use the “hub” (described after Lemma 3.7) to construct certain minors in \( \ell \)-link graphs.

**Corollary 5.3.** Let \( \ell \geq 0 \), \( G \) be a graph, \( M \) be a minor of \( G(\ell) \) such that each branch set contains an \( \ell \)-link. Then \( \mathbb{L}_\ell(G) \) contains an \( M \)-minor.

**Proof.** Let \( X_1, \ldots, X_t \) be the branch sets of an \( M \)-minor of \( G(\ell) \) such that \( X_i \) contains an \( \ell \)-link for each \( i \in [t] \). For any connected subgraph \( Y \) of \( G(\ell) \) contains at least one \( \ell \)-link, let \( \mathbb{L}_\ell(G, Y) \) be the subgraph of \( H := \mathbb{L}_\ell(G) \) induced by the \( \ell \)-links of \( G \) of which the middle units are in \( Y \). Let \( H(Y) \) be the union of the components of \( \mathbb{L}_\ell(G, Y) \), which contains at least one vertex corresponding to an \( \ell \)-link of \( Y \). By Lemma 3.7, \( H(Y) \) is connected.

By definition each edge of \( M \) corresponds to an edge \( e \) of \( G(\ell) \) between two different branch sets, say \( X_i \) and \( X_j \). Let \( Y \) be the graph consisting of \( X_i, X_j \), and \( e \). Then \( H(X_i) \) and \( H(X_j) \) are vertex-disjoint since \( X_i \) and \( X_j \) are vertex-disjoint. By the analysis above, \( H(X_i) \) and \( H(X_j) \) are connected subgraphs of the connected graph \( H(Y) \). Thus there is a path \( Q \) of \( H(Y) \) joining \( H(X_i) \) and \( H(X_j) \) only at end vertices. Further, if \( \ell \) is even, then \( Q \) is an edge; otherwise, \( Q \) is a 2-path whose middle vertex corresponds to an \( \ell \)-link \( L \) of \( Y \) such that \( c_L = e \). This implies that \( Q \) is internally disjoint with its counterparts and has no inner vertex in any branch set. Then, by contracting each \( H(X_i) \) to a vertex, and \( Q \) to an edge, we obtain an \( M \)-minor of \( H \).

Now we are ready to give a lower bound for the Hadwiger number of \( \mathbb{L}_\ell(G) \).

**Proof of Theorem 1.3.** Since \( H := \mathbb{L}_\ell(G) \) contains an edge, \( t := \eta(H) \geq 2 \). We first show that \( t \geq d := d(G) \). By definition there exists a subgraph \( X \) of \( G \) with \( \delta(X) = d \). We may assume that \( d \geq 3 \) and \( \ell \geq 2 \). Then \( X \) contains an \( (\ell - 1) \)-arc \( \tilde{P} := (u, e, \ldots, f, v) \).

Since the degree of \( u \) in \( X \) is at least \( d \), there are \( d - 1 \) distinct arcs \( \tilde{e}_1, \ldots, \tilde{e}_{d-1} \) of \( X \) with head vertex \( u \) such that \( e_i \neq e \) for \( i \in [d - 1] \). Similarly, there are \( d - 1 \) distinct arcs \( \tilde{f}_1, \ldots, \tilde{f}_{d-1} \) of \( X \) with tail vertex \( v \) such that \( f_j \neq f \) for \( j \in [d - 1] \). Then the \( \ell \)-link \( L \) can be shunted to the \( \ell \)-link \( R_j := [\tilde{P}, \tilde{f}_j] \) through the \( (\ell + 1) \)-link \( Q_{ij} := [\tilde{e}_i, \tilde{P}, \tilde{f}_j] \).

So \( H \) contains a \( K_{d-1,d-1} \)-subgraph with bipartition \( \{L_j | j \in [d - 1]\} \cup \{R_j | j \in [d - 1]\} \) and edge set \( \{Q_{ij} | i, j \in [d - 1]\} \). By Zelinka [25], \( K_{d-1,d-1} \) contains a \( K_d \)-minor. Thus \( t \geq d \) as desired.

We now show that \( t \geq \eta := \eta(G) \). If \( \eta = 3 \), then \( G \) contains a cycle \( O \) of length at least \( 3 \), and \( H \) contains a \( K_3 \)-minor contracted from \( \mathbb{L}_\ell(O) \). Now assume that \( G \) is connected with \( \eta \geq 4 \). Repeatedly delete vertices of degree 1 in \( G \) until \( \delta(G) \geq 2 \). Then \( G = G(\ell) \). Clearly, this process does not reduce the Hadwiger number of \( G \). So \( G \) contains branch sets of a \( K_\eta \)-minor covering \( V(G) \) (see [24]). If every branch set contains an \( \ell \)-link, then the statement follows from Corollary 5.3. Otherwise, there exists some branch set \( X \) with \( diam(X) < \ell \). Since \( \eta \geq 4 \), \( Y := G - V(X) \) is connected and contains a cycle. Thus by Lemma 5.2, \( H \) contains a \( K_\eta \)-minor since \( |E(X, Y)| \geq \eta - 1 \).

Here we prove Hadwiger’s conjecture for \( \mathbb{L}_\ell(G) \) for each \( \ell \geq 2 \).

**Proof of Theorem 1.5(2).** Let \( d := d(G) \), \( \ell \geq 2 \) be an even integer, and \( H := \mathbb{L}_\ell(G) \).

By [7, Proposition 5.2.2], \( \chi := \chi(G) \leq d + 1 \). So by Theorem 1.1, \( \chi(H) \leq \min\{d + 1, \frac{d}{2} + \frac{1}{2}\} \). If \( d \leq 4 \), then \( \chi(H) \leq 5 \). By Robertson et al. [17], Hadwiger’s conjecture holds for \( H \) in this case. Otherwise, \( d \geq 5 \). By Theorem 1.3, \( \eta(H) \geq d \geq \frac{d}{2} + \frac{5}{2} \geq \chi(H) \) and the statement follows.
We end this article by proving Hadwiger’s conjecture for \( \ell \)-link graphs of biconnected graphs for \( \ell \geq 1 \).

**Proof of Theorem 1.5(1).** By Reed and Seymour [16], Hadwiger’s conjecture holds for \( H := L_1(G) \) for \( \ell = 1 \). By Theorem 1.5(2), the conjecture is true if \( \ell \geq 2 \) is even. So we only need to consider the situation that \( \ell \geq 3 \) is odd. If \( G \) is a cycle, then \( H \) is a cycle and the conjecture holds [10]. Now let \( v \) be a vertex of \( G \) with degree \( \Delta := \Delta(G) \geq 3 \).

By Theorem 1.1, \( \chi(H) \leq \Delta + 1 \). Since \( G \) is biconnected, \( Y := G - v \) is connected. By Lemma 5.2, if \( Y \) contains a cycle, then \( \eta(H) \geq \Delta + 1 \geq \chi(H) \). Now assume that \( Y \) is a tree, which implies that \( G \) is \( K_4 \)-minor free. By Lemma 5.1, \( \eta(H) \geq \Delta + 1 \). By Theorem 1.1, \( \chi(H) \leq \chi' := \chi'(G) \). So it is enough to show that \( \chi' = \Delta \).

Let \( U := \{ u \in V(Y) | \deg_Y(u) \leq 1 \} \). Then \( |U| \geq \Delta(Y) \). Let \( \hat{G} \) be the underlying simple graph of \( G \), \( t := \deg_{\hat{G}}(v) \geq 1 \) and \( \hat{\Delta} := \Delta(\hat{G}) \geq t \). Since \( G \) is biconnected, \( U \subseteq N_G(v) \). So \( t \geq |U| \geq \Delta(Y) \). Let \( u \in U \). When \( |U| = 1 \), \( t = \deg_{\hat{G}}(u) = 1 \). When \( |U| \geq 2 \), \( \deg_{\hat{G}}(u) = 2 \leq |U| \leq t \). Thus \( t = \hat{\Delta} \). Juvan et al. [14] proved that the edge-chromatic number of a \( K_4 \)-minor free simple graph equals the maximum degree of this graph. So \( \hat{\chi}' := \chi'(\hat{G}) = \hat{\Delta} \) since \( \hat{G} \) is simple and \( K_4 \)-minor free. Note that all parallel edges of \( G \) are incident to \( v \). So \( \chi' = \hat{\chi}' + \deg_G(v) - t = \hat{\Delta} + \Delta - \hat{\Delta} = \Delta \) as desired. ■

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