Some new inequalities involving the Hardy operator

Ludmila Nikolova1 | Lars-Erik Persson2,3 | Natasha Samko2

1Department of Mathematics and Informatics, Sofia University, Sofia, Bulgaria
2UiT, The Arctic University of Norway, Narvik, Norway
3University of Karlstad, Karlstad, Sweden

Correspondence
Lars-Erik Persson, UiT, The Arctic University of Norway, Narvik, Norway; University of Karlstad, Karlstad, Sweden.
Email: Lars.E.Persson@uit.no; larserik6pers@gmail.com

Funding information
Sofia University SRF, Grant/Award Number: 80-10-13/2018

Abstract
In this paper we derive some new inequalities involving the Hardy operator, using some estimates of the Jensen functional, continuous form generalization of the Bellman inequality and a Banach space variant of it. Some results are generalized to the case of Banach lattices on \((0, b], 0 < b \leq \infty\).

KEYWORDS
Banach lattice, Bellman’s inequality, Hardy operator, Hardy-type inequalities, Jensen functional, Jensen’s inequality, the Jensen gap

MSC (2010)
Primary: 26D10; Secondary: 26D20

1 | INTRODUCTION

In 1928 G. H. Hardy (see [6]) himself proved the following generalization of his famous inequality from 1925 (see [5]) namely

\[
\int_0^\infty \left( \frac{1}{x} \int_0^x f(y) dy \right)^p x^\alpha dx \leq \left( \frac{p}{p-1-\alpha} \right)^p \int_0^\infty f^p(x) x^\alpha dx,
\]

whenever \( p \geq 1 \) and \( \alpha < p - 1 \). The constant \( \left( \frac{p}{p-1-\alpha} \right)^p \) is sharp.

Here, and in the sequel in this paper, all functions are supposed to be nonnegative and measurable (and if some negative power of the function appears, then we assume that the function is strictly positive a.e.).

The first Hardy inequality was for the case \( \alpha = 0 \). The continued history of Hardy-type inequalities up to 2007 can be found in [8] and some complements in [9]. It was recently pointed out in [16] (see also [9]) that (1.1) is not a genuine generalization of this first inequality, because both are equivalent to the “fundamental” Hardy inequality

\[
\int_0^\infty \left( \frac{1}{x} \int_0^x g(y) dy \right)^p \frac{dx}{x} \leq \int_0^\infty g^p(x) \frac{dx}{x}.
\]

This equivalence follows by just doing the substitutions \( f(x) = g \left( x^{\frac{p-1}{p}} \right)^{-\frac{1}{p}} x^{-\frac{\alpha-1}{p}} \) and \( f(x) = g \left( x^{\frac{p-1}{p}} \right)^{-\frac{1}{p}} x^{-\frac{\alpha+1}{p}} \), respectively.

The classical Hardy operator \( H \) is defined by

\[
H f(x) := \frac{1}{x} \int_0^x f(t) dt.
\]
We are dealing with nonnegative functions. Hence, in order to prove (1.1) we need only to prove (1.2) and this, in its turn, is just to use Jensen’s inequality to see that

\[(H(f(x))^p \leq H((f(x))^p)\]  

(1.3)

and reverse the order of integration. The same technique works also for finite intervals and we have that for \(0 < d \leq \infty\) the inequality

\[\int_0^d \left( \frac{1}{x} \int_0^x g(y) \, dy \right) p \, dx / x \leq 1 \cdot \int_0^d g^p(x) \left( 1 - \frac{x}{d} \right) dx / x, \]

(1.4)

\(p \geq 1\) or \(p < 0\), holds and with the same substitutions, is equivalent to the inequality

\[\int_0^{d_0} \left( \frac{1}{x} \int_0^x f(y) \, dy \right)^p x^a \, dx \leq \left( \frac{p}{p - 1 - a} \right)^p \int_0^{d_0} f^p(x)x^a \left( 1 - \left( \frac{x}{d_0} \right)^{p-1-a} \right) dx, \]

(1.5)

where \(0 < d_0 \leq \infty\) (formally \(d_0 = d^{\frac{p}{p-1-a}}\)) and \(p \geq 1, \alpha < p - 1\) or \(p < 0, \alpha > p - 1\). The constants are sharp since the constant 1 in (1.4) is sharp. See [9], p. 340.

In connection to (1.4) it is also natural to study the so-called Jensen functional

\[J_p(f(x)) = \int_0^d g^p(x) \left( 1 - \frac{x}{d} \right) dx / x - \int_0^d \left( \frac{1}{x} \int_0^x g(y) \, dy \right)^p dx / x \]

(1.6)

as a measure of the so-called “Jensen gap”.

In Section 2 we shall discuss further the Jensen functional in more general situations and prove some new estimates of the Hardy type operator and a corresponding new Hardy-type inequality with this basic idea (see Theorem 2.8).

In Section 3 we prove some new reverse Hardy-type inequalities on the cone of non-increasing functions in the setting of a general Banach lattice \(E = E(0, b), 0 < b \leq \infty\). As applications we get some well-known such reverse Hardy-type inequalities but now including also the case with finite intervals.

Finally, in Section 4 we study the problem, to find conditions which guarantee that the left–hand side of (1.1) is finite even in cases when \(\alpha\) do not satisfy the restriction \(p \geq 1, \alpha < p - 1\). Here we use some fairly new continuous generalizations of Bellman’s inequality (see [14] and [15]) in a crucial way.

2 \ TWO-SIDED ESTIMATES OF THE JENSEN FUNCTIONAL AND HARDY OPERATORS

A simple form of the Jensen’s inequality reads:

**Theorem 2.1.** Let \(I\) be an interval on \(R\), let \(w(x)\) be a weight function on \(I\) such that \(\int_I w(x) \, dx = 1\). If \(f\) is a measurable function on \(I\) and \(\phi\) is a convex function on an open interval, which contains the image of \(f\), then

\[\phi \left( \int_I f(x)w(x) \, dx \right) \leq \int_I \phi(f(x))w(x) \, dx.\]

It is also well known that in special cases of convex functions we can also find good upper estimates of the so-called “Jensen gap”

\[J_\phi(f) = \int_I \phi(f(x))w(x) \, dx - \phi \left( \int_I f(x)w(x) \, dx \right),\]

see e.g. [1], the new book [13] and the references there. Note also that, by the Jensen inequality mentioned above, \(J_\phi(f) \geq 0\) and, since this inequality holds in the reversed direction for concave functions, that \(J_\phi(f) \leq 0\) in this case. For later purposes in this paper we need the following result of this type.
Lemma 2.2. Let \( \phi \) be a differentiable convex function on an open interval, which contains the image of \( f \). Moreover, let \( I \) be an interval on \( \mathbb{R} \) and let the weight \( w(x) \) be such that \( \int_I w(x) \, dx = 1 \). Then the inequalities

\[
0 \leq \int_I \phi(f(x))w(x) \, dx - \phi\left(\int_I f(x)w(x) \, dx\right)
\]

\[
\leq \int_I \phi'(f(x))f(x)w(x) \, dx - \int_I \phi'(f(x))w(x) \, dx - \int_I f(x)w(x) \, dx
\]

hold.

If instead \( \phi \) is concave, then these inequalities hold in the reversed directions.

A discrete version of this lemma is proved in [3] and a related result can be found in [4] for isotonic normalized functionals. We present the following proof.

Proof. For a differentiable convex function \( \phi \), we have that

\[
\phi(a) - \phi(b) \geq \phi'(b)(a - b) \tag{2.1}
\]

and (2.1) hold in reversed direction when \( \phi \) is concave. This follows e.g. from a result of O. Stolz, see e.g. Theorem 1.4.2 in [13]. Now, for \( \phi \) convex, putting \( a = \int_I f(t)w(t) \, dt \) and \( b = f(x) \) we get

\[
w(x)\phi\left(\int_I w(t)f(t) \, dt\right) - w(x)\phi(f(x)) \geq w(x)\phi'(f(x))\left(\int_I w(t)f(t) \, dt - f(x)\right).
\]

After integrating over \( I \) we get the second inequality, the first one is just Jensen’s inequality.

The proof in the case of \( \phi \) concave is analogous. \( \square \)

Next we formulate the following consequence of the Jensen and reverse Jensen inequalities.

Lemma 2.3. Let \( H = H(f(x)), \) \( 0 < x < \infty, \) be the Hardy operator.

(a) If \( p \geq 1 \) or \( p < 0 \), then

\[
(H(f(x)))^p \leq H\left(f^p(x)\right). \tag{2.2}
\]

(b) If \( 0 < p \leq 1 \), then (2.2) holds in the reversed direction.

By using this lemma we can conclude that when \( p \geq 1 \) or \( p < 0 \) the Jensen gap

\[
H\left(f^p(x)\right) - (H(f(x)))^p
\]

is nonnegative. In the case \( 0 < p < 1 \) the Jensen gap is non-positive.

Next we state the following two-sided estimate:

Theorem 2.4. Let \( H = H(f(x)), \) \( 0 < x < \infty, \) be the Hardy operator. Then we have the following two-sided estimates:

(a) If \( p \geq 1 \) or \( p < 0 \), then

\[
0 \leq H\left(f^p(x)\right) - (H(f(x)))^p \leq pH\left(f^p(x)\right) - pH\left(f^{p-1}(x)\right).H((f(x))). \tag{2.3}
\]

(b) If \( 0 < p \leq 1 \), then both inequalities in (2.3) hold in the reversed direction.

Proof. (a) The first inequality is just Jensen’s inequality (see Lemma 2.3). We use Lemma 2.2 with \( I = [0, x] \) and \( w(t) = \frac{1}{x}, 0 \leq t \leq x, \) and obtain

\[
0 \leq \frac{1}{x} \int_0^x \phi(f(t)) \, dt - \phi\left(\frac{1}{x} \int_0^x f(t) \, dt\right) \leq \frac{1}{x} \int_0^x \phi'(f(t))f(t) \, dt - \frac{1}{x} \int_0^x \phi'(f(t)) \, dt - \frac{1}{x} \int_0^x f(t) \, dt.
\]
Apply these inequalities with the convex function \( \phi(t) = t^p \), \( p \geq 1 \) or \( p < 0 \) to find that

\[
0 \leq H\left(f^p(x)\right) - \left(H(f(x))\right)^p \leq pH\left(f^p(x)\right) - pH\left(f^{p-1}(x)\right) \cdot H(\left(f(x)\right)).
\]

(b) The proof is completely similar since both used inequalities hold in reversed direction in this case.

\[\square\]

Remark 2.5. Of course the right–hand side in (2.3) must be nonnegative when \( p \geq 1 \) or \( p < 0 \). In fact, this is a special case of the Chebyshev inequality

\[
\int_0^x u(t) \, dt \cdot \int_0^x f(t) g(t) \, dt \geq \int_0^x f(t) u(t) \, dt \int_0^x g(t) u(t) \, dt,
\]

yielding for similarly ordered functions \( f \) and \( g \). It holds in reversed direction if \( f \) and \( g \) are oppositely ordered. By applying this inequality with \( u(t) = \frac{1}{x}, 0 \leq t \leq x \), for \( g(t) = f^{p-1}(t) \) (hence \( f \) and \( g \) are similarly ordered when \( p > 1 \) and oppositely ordered when \( p < 1 \)). We get

\[
PH\left(f^p(x)\right) \geq PH\left(f(x)\right) \cdot H\left(f^{p-1}(x)\right)
\]

in the case \( p \geq 1 \) or \( p < 0 \) and the reverse inequality holds in the case \( 0 < p \leq 1 \).

Corollary 2.6. Let \( H = H(f(x)), 0 < x < \infty \), be the Hardy operator. If \( p \geq 2 \), then the following refinement of (2.2) holds:

\[
(H(f(x)))^p \leq \frac{1}{p-1} \left[ pPH\left(f^{p-1}(x)\right) H\left(f(x)\right) - (H(f(x)))^p \right] \leq H\left(f^p(x)\right).
\]

\[\text{Corollary 2.6.} \] Let \( H = H(f(x)), 0 < x < \infty \), be the Hardy operator. If \( p \geq 2 \), then the following refinement of (2.2) holds:

\[
(H(f(x)))^p \leq \frac{1}{p-1} \left[ pPH\left(f^{p-1}(x)\right) H\left(f(x)\right) - (H(f(x)))^p \right] \leq H\left(f^p(x)\right).
\]

Proof. We get the second inequality in (2.4) by just rearranging the terms in the second inequality in (2.3) and dividing by \( p - 1 \).

Next we apply the inequality (2.2) for \( p - 1 \geq 1 \) and get

\[
(H(f(x)))^{p-1} \leq H\left(f^{p-1}(x)\right),
\]

multiply both sides of this inequality with \( pH(f(x)) \) and subtract \( (H(f(x)))^p \) from both sides. So we get the first inequality. \(\square\)

By using Corollary 2.6 we get the following refinement of the fundamental Hardy inequality (1.4) for \( p \geq 2 \):

Lemma 2.7. Let \( 0 < d \leq \infty \) and let \( g \) be a measurable function on \( (0, d] \). If \( p \geq 2 \), then

\[
\int_0^d \left( \frac{1}{x} \int_0^x g(s) \, ds \right)^p \frac{dx}{x} \leq \frac{1}{p-1} \left[ \int_0^d \left( \int_0^x g^{p-1}(s) \, ds \right) \, dx \right] - \int_0^d \left( \frac{1}{x} \int_0^x g(s) \, ds \right)^p \frac{dx}{x} \cdot \int_0^d g^p(s) \left( 1 - \frac{s}{d} \right) \, ds.
\]

\[\text{Lemma 2.7.} \] Let \( 0 < d \leq \infty \) and let \( g \) be a measurable function on \( (0, d] \). If \( p \geq 2 \), then

\[
\int_0^d \left( \frac{1}{x} \int_0^x g(s) \, ds \right)^p \frac{dx}{x} \leq \frac{1}{p-1} \left[ \int_0^d \left( \int_0^x g^{p-1}(s) \, ds \right) \, dx \right] - \int_0^d \left( \frac{1}{x} \int_0^x g(s) \, ds \right)^p \frac{dx}{x} \cdot \int_0^d g^p(s) \left( 1 - \frac{s}{d} \right) \, ds.
\]

Proof. Multiply (2.4) by \( \frac{dx}{x} \), integrate over the interval \( [0, d] \) and reverse the order of integration in the last integral

\[
\int_0^d \int_0^x g^p(s) \, ds \, \frac{dx}{x} = \int_0^d g^p(s) \left( \int_s^d \frac{1}{x^2} \, dx \right) \, ds = \int_0^d g^p(s) \left( 1 - \frac{s}{d} \right) \, ds.
\]

By using this lemma and the same substitution as before we can derive the following refinement of the Hardy inequality (1.5) when \( p \geq 2 \).

Theorem 2.8. Let \( 0 < d_0 \leq \infty \) and let \( f \) be a measurable function on \( (0, d_0] \). If \( p \geq 2 \), then

\[
\int_0^{d_0} \left( \frac{1}{x} \int_0^x f(y) \, dy \right)^p \frac{x^a \, dx}{x} \leq I \leq \left( \frac{p}{p-1-a} \right)^p \int_0^{d_0} f^p(x) x^a \left( 1 - \left( \frac{x}{d_0} \right)^{\frac{p-1-a}{p}} \right) \, dx.
\]

\[\text{Theorem 2.8.} \] Let \( 0 < d_0 \leq \infty \) and let \( f \) be a measurable function on \( (0, d_0] \). If \( p \geq 2 \), then

\[
\int_0^{d_0} \left( \frac{1}{x} \int_0^x f(y) \, dy \right)^p \frac{x^a \, dx}{x} \leq I \leq \left( \frac{p}{p-1-a} \right)^p \int_0^{d_0} f^p(x) x^a \left( 1 - \left( \frac{x}{d_0} \right)^{\frac{p-1-a}{p}} \right) \, dx.
\]
where $\alpha < p - 1$ and

\[
I = \frac{p}{p-1} \left( \frac{p}{p-1-\alpha} \right)^{p-2} \int_0^{d_0} \left( \frac{1}{x^{\frac{3p-2a-2}{p}}} \int_0^x f^{p-1}(y)y^\beta \ dy \cdot \int_0^x f(y) \ dy \right) dx
\]

\[-\frac{1}{p-1} \int_0^{d_0} \frac{1}{x^{p-a}} \left( \int_0^x f(y) \ dy \right)^p dx
\]

with $\beta = (1+a)(p-2)/p$.

**Proof.** We use Lemma 2.7 with $g(s) = f \left( \frac{s^p}{s^{p-1-a}} \right)^{\frac{p}{p-a+1}}$ and $d_0 = \frac{p}{p-a+1}$ and multiply all the terms in the obtained inequality by $q^{p+1}$, where $q = \frac{p}{p-1-a}$. The substitution $x = s^{p-1-a}$ and simple calculations show that

\[
\left( \frac{p}{p-\alpha-1} \right)^{p+1} \int_0^d g^p(s) \left( 1 - \left( \frac{s}{d} \right)^p \right) \ ds = \left( \frac{p}{p-\alpha-1} \right)^p \int_0^{d_0} f^p(x)x^\alpha \left( 1 - \left( \frac{x}{d_0} \right)^{p-1-a} \right) dx
\]

and

\[
I_1 = \left( \frac{p}{p-\alpha-1} \right)^{p+1} \int_0^d \left( \frac{1}{x} \int_0^x g(s) \ ds \right)^p dx = \int_0^{d_0} \left( \frac{1}{x} \int_0^x f(y) \ dy \right)^p x^\alpha dx.
\]

We want to replace $g$ by $f \left( g(s) = f \left( \frac{s^p}{s^{p-1-a}} \right)^{\frac{p}{p-a+1}} \right)$ also in the “middle term” $I := \frac{1}{p-1}(I_2 - I_1)$, where

\[
I_2 = \left( \frac{p}{p-\alpha-1} \right)^{p+1} \int_0^d p \left( \frac{1}{x^2} \int_0^x g^{p-1}(s) \ ds \right) \int_0^x g(s) \ ds \ dx x
\]

In the integral

\[
\int_0^x g^{p-1}(s) \ ds = \int_0^x f^{p-1}(s)^{q(p-1)} \ ds
\]

we make the substitution $s^q = y$ and get

\[
\frac{1}{q} \int_0^{x^q} f^{p-1}(y)y^{\frac{(q-1)(p-1)}{q}+\frac{1}{q}} \ dy = \frac{1}{q} \int_0^{x^q} f^{p-1}(y)y^\beta \ dy,
\]

where

\[
\beta = \frac{(q-1)(p-2)}{q} = \frac{(1+a)(p-2)}{p}.
\]

In particular,

\[
\int_0^x g(s) \ ds = \frac{1}{q} \int_0^{x^q} f(y) \ dy
\]

and then

\[
I_3 := \int_0^d p \left( \frac{1}{x^2} \int_0^x g^{p-1}(s) \ ds \int_0^x g(s) \ ds \right) \ dx = \frac{p}{q^2} \int_0^d \left( \frac{1}{x^3} \int_0^{x^q} f^{p-1}(y)^\beta \ dy \cdot \int_0^{x^q} f(y) \ dy \right) dx.
\]

Next we make the substitution $x^q = t$ and get

\[
I_3 = \frac{p}{q^3} \int_0^{d_0} \left( \frac{t^{\frac{1}{q}}}{t^{\frac{3}{q}}} \right) \int_0^t f^{p-1}(y)^\beta \ dy \cdot \int_0^t f(y) \ dy \ dt.
\]
Then the middle term in Theorem 2.8 will be equal to

\[ I = \frac{p}{p-1} q^{p-2} \int_0^{d_0} \left( \frac{1}{x^{\frac{1}{p}+1}} \int_0^x f^{p-1}(y)y^\beta \, dy \cdot \int_0^x f(y) \, dy \right) \, dx - \frac{1}{p-1} \int_0^{d_0} \left( \frac{1}{x} \int_0^x f(y) \, dy \right)^p x^\alpha \, dx \]

\[ = \frac{p}{p-1} q^{p-2} \int_0^{d_0} \left( \frac{1}{x^{\frac{1}{p}+1}} \int_0^x f^{p-1}(y)y^\beta \, dy \cdot \int_0^x f(y) \, dy \right) \, dx - \frac{1}{p-1} \int_0^{d_0} \frac{1}{x^{\frac{1}{p}+1}} \left( \int_0^x f(y) \, dy \right)^p \, dx \]

with \( q = \frac{p}{p-1-\alpha} \) and \( \beta = (1+\alpha)(p-2)/p \). The proof is complete.

Remark 2.9. By using other estimates of Jensen functionals than that in Lemma 2.2 we obtain other refinements of the second estimate in Corollary 2.6. For example, if we use the fairly new estimate (see [1]).

\[ \int f \Phi(f) \, d\mu - \Phi \left( \int f \, d\mu \right) \geq \int f^2 \, d\mu - \left( \int f \, d\mu \right)^2 \left( \frac{\Phi(x) - \Phi(0)}{x} \right)^\prime \bigg|_{x=f} \, f \, d\mu, \] (2.7)

where \( \Phi = x\phi(x) + \Phi(0) \) with \( \phi \) convex increasing on \( [0,d) \), \( 0 < d \leq \infty \), we can derive in a similar way the following estimate for the Hardy operator:

\[ H(f^{p+1}(x)) \geq p(H(f(x)))^{p-1}H(f^2(x)) - (p-1)(H(f(x)))^{p+1}. \]

3 | ON REVERSED HARDY INEQUALITIES

As M. Milman writes in [12] the subtle point is that Hardy’s operator \( H = H(f) \) is not invertible on \( L_p(0,\infty) \)-spaces, and therefore it is not possible to find a reverse Hardy inequality of the form

\[ \|f\|_{L_p(0,\infty)} \leq c(p)\|H(f)\|_{L_p(0,\infty)} \]

for any finite \( c(p) > 0 \) and holding for all positive functions.

On the other hand, if \( f \) is positive and non-increasing, we trivially have that \( f(t) \leq H(f(t)) \), and that e.g. the inequality (1.1) holds for positive non-increasing functions with \( c(p) = 1 \). It turns out, however, that this inequality can be considerably sharpened and the obtained inequality is sharp. For example Gehring’s lemma reads:

Lemma 3.1. If \( p > 1 \) and \( f(x) \) is a positive non-increasing function on \( (0,\infty) \), then

\[ \left( \int_0^\infty (H f(x))^p \, dx \right)^{\frac{1}{p}} \geq \left( \frac{p}{p-1} \right)^{\frac{1}{p}} \left( \int_0^\infty (f(x))^p \, dx \right)^{\frac{1}{p}}. \] (3.1)

Results about Gehring’s lemma in Orlicz spaces and the connection to interpolation and approximation spaces can be found e.g. in [10,11], etc.

Here we first consider the possibility to prove a reverse type estimate for the Hardy operator for a general r.i. space.

Remember that the Banach lattice \( E \) is \( q \)-concave if there exists a positive constant \( M \) such that, for every finite set \( x_1, x_2, \ldots, x_n \) of elements in \( E \), we have

\[ \left( \sum_{i=1}^n \|x_i\|^q_E \right)^{\frac{1}{q}} \leq M \left\| \left( \sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}} \right\|_E. \] (3.2)

The smallest constant \( M \) satisfying (3.2) is called the constant of \( q \)-concavity. As it was mentioned in [15], from a result of A. Shep we have that if \( E \) is \( 1 \)-concave with constant of concavity equal to \( M \), then

\[ M \left\| \int_0^b g(x,t) \, dt \right\|_E \geq \int_0^b \|g(x,t)\|_E \, dt. \] (3.3)
Theorem 3.2. Let \( E = E(0, b), 0 < b \leq \infty \), be a Banach lattice having the Fatou property and which is 1-concave with constant of 1-concavity equal to \( M \). If \( f(x) \) is a positive and non-increasing function on \( (0, \infty) \) and \( p \geq 1 \), then

\[
\| (H f(x))^p \|_E \geq M^{-1} p \int_0^b t^{p-1} (f(t))^p \frac{1}{x^p} x_{[t,b]}(x) \, dt,
\]

where \( x_{[t,b]} \) denotes the characteristic function of the interval \([t, b]\).

**Proof.** Since \( f(x) \) is non-increasing we have \( \int_0^x f(t) \, dt \geq xf(x) \). Therefore

\[
\frac{d}{dx}(xH f(x))^p = \frac{d}{dx} \left( \int_0^x f(t) \, dt \right)^p = pf(x) \left( \int_0^x f(t) \, dt \right)^{p-1} \geq px^{p-1}(f(x))^p.
\]

By integrating we get that

\[
(H f(x))^p \geq \frac{1}{xp} \int_0^x pt^{p-1}(f(t))^p \, dt.
\]

Thus

\[
\| (H f(x))^p \|_E \geq \frac{p}{xp} \int_0^x t^{p-1}(f(t))^p \, dt \geq M^{-1} p \int_0^b t^{p-1} x_{[t,b]}(x) \frac{1}{x^p} \, dt = M^{-1} p \int_0^b t^{p-1} x_{[t,b]}(x) \, dt.
\]

Therefore, according to the Schep estimate (3.3) we have that

\[
\| (H f(x))^p \|_E \geq M^{-1} p \int_0^b t^{p-1} x_{[t,b]}(x) \, dt \geq M^{-1} p \int_0^b \frac{t^{p-1} x_{[t,b]}(x)}{x^p} \, dt.
\]

The proof is complete. \( \square \)

**Corollary 3.3.** If \( p > 1 \) and \( f(x) \) is a positive and non-increasing function on \( (0, b), 0 < b \leq \infty \), then

\[
\left( \int_0^b (H f(x))^p \, dx \right)^{\frac{1}{p}} \geq \left( \frac{p}{p-1} \right)^{\frac{1}{p}} \left( \int_0^b (f(x))^p \left( 1 - \left( \frac{x}{b} \right)^{p-1} \right) \, dx \right)^{\frac{1}{p}}. \tag{3.4}
\]

**Proof.** Apply Theorem 3.2 with \( E = L_1(0, b) \) and we find that

\[
\left\| \frac{1}{x^p} x_{[t,b]}(x) \right\|_E = \int_t^b \frac{1}{x^p} \, dx = \frac{1}{p-1} \left( t^{1-p} - b^{1-p} \right) = \frac{t^{1-p}}{p-1} \left( 1 - \left( \frac{t}{b} \right)^{p-1} \right)
\]

and the proof follows. \( \square \)

**Remark 3.4.** For \( b = \infty \) we have just Gehring’s inequality (3.1).

Next we recall the following generalization of Lemma 3.1:

**Lemma 3.5.** Let \( p > 1 \), let \( a < p - 1 \) and let \( f(x) \) be a positive and non-increasing function. Then, for \( b, 0 < b \leq \infty \),

\[
\left( \int_0^\infty (H f(x))^p x^a \, dx \right)^{\frac{1}{p}} \geq \left( \frac{p}{p-1-\alpha} \right)^{\frac{1}{p}} \left( \int_0^\infty (f(x))^p x^a \, dx \right)^{\frac{1}{p}}. \tag{3.5}
\]

This is a special case of Theorem 3.11 in [2] (the case \( \beta = 0 \)). The constant is sharp so also the constant in Gehring’s lemma is sharp.

Moreover, we observe that by following the proof of Theorem 3.2 we find that the following generalization fitting to Lemma 3.5 holds:

**Theorem 3.6.** Let \( E = E(0, b), 0 < b \leq \infty \), be a Banach lattice having the Fatou property and which is 1-concave with constant of 1-concavity equal to \( M \). If \( f(x) \) is a positive and non-increasing function, then for \( p \geq 1, a < p - 1 \)

\[
\| (H f(x))^p x^a \|_E \geq M^{-1} p \int_0^b t^{p-1} (f(t))^p \frac{1}{x^p-a} x_{[t,b]}(x) \, dt.
\]
Remark 3.7. By making the similar calculations as in the proof of Corollary 3.3 we get the following weighted version of the inequality (3.4):

\[
\left(\int_0^b (Hf(x))^p x^a \, dx\right)^\frac{1}{p} \geq \left(\frac{p}{p-1} \right)^\frac{1}{p} \left(\int_0^b \left(f(x) - \left(\frac{x}{b}\right)^{p-1}a\right) \, dx\right)^\frac{1}{p}
\]

for \(p \geq 1, \alpha < p - 1\), yielding for all positive and non-increasing functions. Especially, for \(b = \infty\) we get (3.5).

4 | INEQUALITIES, BASED ON NEW FORMS OF BELLMAN’S INEQUALITY

A recently presented continuous form of Bellman’s inequality reads (see [14]):

**Theorem 4.1.** Let \(u(x)\) and \(v(y)\) be weight functions on the measure spaces \((X, \mu)\) and \((Y, \nu)\), respectively, let \(g(x, y)\) be a positive measurable function on \(X \times Y\) and assume that

\[
f_0^p(x) > \int_Y g^p(x, y) v(y) \, d\nu(y),
\]

for all \(x \in X\).

Then, for \(p \geq 1\),

\[
\int_X \left[ f_0^p(x) - \int_Y g^p(x, y) v(y) \, d\nu(y) \right]^\frac{1}{p} u(x) \, d\mu(x) \leq \int_X f_0(x) u(x) \, d\mu(x) - \int_Y \left[ \int_X g(x, y) u(x) \, d\mu(x) \right]^\frac{1}{p} v(y) \, d\nu(y).
\]

Using this theorem we can derive the first main result in this section:

**Theorem 4.2.** Let \(p \geq 1\) and \(\alpha \in \mathbb{R}\). Then the inequality

\[
\int_0^\infty (Hf(y))^p y^a \, dy \leq \left[ \int_0^1 f_0^p(x) \, dx \right]^\frac{1}{p} - \left[ \int_0^1 f_0^p(x) - \int_0^\infty f^p(xy) y^a \, dy \right]^\frac{1}{p} \int_0^\infty f^p(x) s^a \, ds
\]

holds for any positive function \(f(x)\) whenever the function \(f_0(x)\) satisfies

\[
f_0^p(x) > \int_0^\infty f_0^p(xy) y^a \, dy \quad \text{or, which is the same,} \quad f_0^p(x) > \frac{1}{x^{\alpha+1}} \int_0^\infty f^p(s) s^a \, ds
\]

for all \(x \in (0, 1)\).

**Proof.** Consider the case \(X = (0, 1)\), \(Y = (0, \infty)\), \(u(x) \equiv 1\) and \(v(y) \equiv 1\) in Theorem 4.1. Put \(g(x, y) = f(xy)^{\frac{p}{p-1}}\). The change of variables \(xy = s\) gives

\[
\int_0^\infty g^p(x, y) \, dy = \int_0^\infty f^p(xy) y^a \, dy = \frac{1}{x^{\alpha+1}} \int_0^\infty f^p(s) s^a \, ds.
\]

On the other hand the change of variables \(t = xy\) implies that

\[
\int_0^\infty (Hf(y))^p y^a \, dy = \int_0^\infty \left( \frac{1}{y} \int_0^y f(t) \, dt \right)^p y^a \, dy
\]

\[
= \int_0^\infty \left( \frac{1}{y} \int_0^1 f(xy)^{\frac{p}{p-1}} y \, dx \right)^p y^a \, dy = \int_0^\infty \left( \int_0^1 g(x, y) \, dx \right)^p \, dy.
\]

Hence, according to Theorem 4.1 the proof is complete. \(\square\)
Remark 4.3. We note that (1.1) holds only under the restriction \( \alpha < p - 1 \) i.e. we can estimate the quantity
\[
\int_0^{d_0} \left( \frac{1}{x} \int_0^x f(y) \ dy \right) \frac{1}{x^\alpha} \ dx
\]
only under this restriction. However, according to Theorem 4.2 this quantity can be estimated also without this restriction if we instead put the following restriction on \( f \):

There exists a function \( f_0(x) \) such that
\[
f_0^p(x) > \frac{1}{x^{\alpha + 1}} \int_0^x f^p(y)y^\alpha \ dy.
\]

The following more general continuous version of the Bellman inequality in Banach lattices was proved in [15].

Theorem 4.4. Let \( X \) and \( Y \) be measure spaces, let \( g(x, y) \) be a positive measurable function on \( X \times Y \) and assume that \( p \geq 1 \) and that \( f_0(x) \) is a function on \( X \) such that
\[
f_0^p(x) > \| g^p(x, \cdot) \|_{E, y},
\]
where \( E \) is a Banach function space on \( Y \) for all \( x \in X \). Assume that \( E \) has Fatou property. Then the inequality

\[
\left( \int_X \left[ f_0^p(x) - \| g^p(x, \cdot) \|_{E, y} \right] \frac{1}{x} \ dx \right)^p \leq \left[ \int_0^1 f_0(x) \ dx \right]^p - \left\| \left[ \int_X g(x, \cdot) \ dx \right]^p \right\|_E
\]

holds provided that all integrals exist.

By using this theorem we can get a Banach lattice variant of Theorem 4.2.

We recall that (see [7]) the dilatation operator \( \sigma_r f(t) = f(\tau^{-1}t) \), \( \tau > 0 \), in the space \( S(0, \infty) \) is bounded. Moreover, if \( E \) is a symmetric Banach function space, then \( \| \sigma_r \|_E \leq \max\{1, \tau\} \). A Banach function space on the half line \( (0, \infty) \) is said to be symmetric if

1. it has Banach lattice property;
2. If \( y \in E \) and the function \( x(t) \) is equi-measurable with the function \( |y(t)| \), i.e.

\[
\text{mes}\{t : x(t) > \tau\} = \text{mes}\{t : |y(t)| > \tau\},
\]

then \( x \in E \) and \( \|x\|_E = \|y\|_E \).

The conditions (1) and (2) are equivalent to one condition:

3. If \( y \in E \) and for the non-increasing rearrangement invariant functions we have that \( x^+(t) \leq y^+(t) \) for all \( t \in (0, \infty) \), then \( x \in E \) and \( \|x\|_E \leq \|y\|_E \).

Here \( x^+ \) stands for the non-increasing rearrangement of \( x \), i.e., the non-increasing, right-continuous function on \( (0, \infty) \), equimeasurable with \( x \). It can be defined by the formula

\[
x^+(t) = \inf\left\{ \tau : n_x(\tau) < t \right\}.
\]

Note that \( \|x\|_E = \|y\|_E \) whenever \( x^+(t) = y^+(t) \).

With this in mind we can formulate the last main result in this section.

Theorem 4.5. Let \( p \geq 1 \) and let \( f_0(x) \) be a function on \( (0, 1) \) such that \( f_0^p(x) > \frac{1}{x^{\alpha + 1}} \left\| f^p(y)y^\alpha \right\|_E \), where \( E = E(0, \infty) \) is a symmetric Banach function space on \( Y = (0, \infty) \) for all \( x \in (0, 1) \). Assume that \( E \) has the Fatou property. Then the inequality

\[
\| (Hf)^p(y)y^\alpha \|_E \leq \left[ \int_0^1 f_0(x) \ dx \right]^p - \left( \int_0^1 \left[ f_0^p(x) - \| f^p(x)y^\alpha \|_E \right] \frac{1}{x} \ dx \right)^p
\]

holds.
Proof. Apply Theorem 4.4 for the case $X = (0, 1)$, $Y = (0, \infty)$, $u(x) = 1$, $v(y) = 1$ and $E = E(0, \infty)$. Put $g(x, y) = f(xy)^{\alpha}$. Then
\[
\|g^p(x, y)\|_E = \|f^p(xy)^{\alpha}\|_E = \frac{1}{x^\alpha}\|f^p(xy(x)^\alpha\|_E = \frac{1}{x^\alpha}\|\sigma_{x^{-1}}(f^p(y)^{\alpha})\|_E \leq \frac{1}{x^\alpha}\|\sigma_{x^{-1}}\| \cdot \|f^p(y)^{\alpha}\|_E
\]

On the other hand the change of variables $t = xy$ implies that
\[
\|(Hf)^p(y)^{\alpha}\|_E = \left\| \left( \frac{1}{y} \int_0^y f(t) \, dt \right)^{\frac{1}{p}} ight\|_E = \left\| \left( \frac{1}{y} \int_0^1 f(xy)^{\frac{2}{p}} y \, dx \right)^{\frac{1}{p}} \right\|_E
\]
\[
= \left\| \left( \int_0^1 g(x, y) \, dx \right)^{\frac{1}{p}} \right\|_E = \left\| \left[ \int_X g(x, \cdot) \, dx \right]^{\frac{1}{p}} \right\|_E.
\]

Hence, by Theorem 4.4 the proof is complete. \qed

ACKNOWLEDGMENTS

The research of Ludmila Nikolova was partially supported by the Sofia University SRF under contract 80-10-13/2018.

REFERENCES

[1] S. Abramovich and L.-E. Persson, Some new estimates of the “Jensen gap”, J. Inequal. Appl. 2016 (2016), no. 39, available at https://doi.org/10.1186/s13660-016-0985-4.
[2] J. Bergh, V. Burenkov, and L.-E. Persson, Best constants in reversed Hardy’s inequalities for quasimonotone functions, Acta. Sci. Math. (Szeged) 59 (1994), 221–239.
[3] S. S. Dragomir and N. M. Jonescu, Some converse of Jensen’s inequality and applications, Anal. Numér. Théor. Approx. 23 (1994), 71–78.
[4] S. S. Dragomir, On a reverse of Jessen’s inequality for isotonic linear functionals, JIPAM J. Inequal. Pure Appl. Math. 2 (2001), no. 3, Article 36.
[5] G. H. Hardy, Notes of some points in the integral calculus, LX: An inequality between integrals, Messenger of Math. 54 (1925), 150–156.
[6] G. H. Hardy, Notes of some points in the integral calculus, LXIV. Further inequalities between integrals, Messenger of Math. 57 (1928), 12–16.
[7] S. G. Krein, Yu. I. Petunin, and E. M. Semenov, Interpolation of linear operators, Transl. Math. Monogr., vol. 54, Amer. Math. Soc., Providence, R.I., 1982.
[8] A. Kufner, L. Maligranda, and L.-E. Persson, The Hardy inequality. About its history and some related results, Vydavatelski Servis Publishing House, Pilsen, 2007.
[9] A. Kufner, L.-E. Persson, and N. Samko, Weighted inequalities of Hardy type, 2nd ed., World Scientific Publishing Co, New Jersey, 2017.
[10] J. Martin and M. Milman, Reverse Hölder inequalities and approximation spaces, J. Approx. Theory. 109 (2001), 82–109.
[11] M. Mastylo and M. Milman, A new approach to Gehring’s lemma, Indiana Univ. Math. J. 49 (2000), no. 2, 655–679.
[12] M. Milman, An note on reversed Hardy inequalities and Gehring’s lemma, Comm. Pure Appl. Math. 50 (1997), no. 4, 311–315.
[13] C. Niculescu and L.-E. Persson, Convex functions and their applications. A contemporary approach, 2nd ed., CMS Books in Mathematics, Springer, 2018.
[14] L. Nikolova, L.-E. Persson, and S. Varošanec, Continuous forms of classical inequalities, Mediterr. J. Math. 13 (2016), no. 5, 3483–3497.
[15] L. Nikolova, L.-E. Persson, and S. Varošanec, A new look at classical inequalities involving Banach lattice norms, J. Inequal. Appl. 2017 (2017), no. 1, 302.
[16] L.-E. Persson and N. Samko, What should have happened if Hardy had discovered this? J. Inequal. Appl. 2012 (2012), no. 29, available at https://doi.org/10.1186/1029-242X-2012-29.

How to cite this article: Nikolova L, Persson L-E, Samko N. Some new inequalities involving the Hardy operator. Mathematische Nachrichten. 2020;293:376–385. https://doi.org/10.1002/mana.201900080