Super-Whittaker vector at $c = 3/2$

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Abstract

The degenerate Whittaker vector of the superconformal algebra can be represented in terms of Jack superpolynomials. However, in this representation the norm of the Whittaker vector involves a scalar product with respect to which the Jack superpolynomials are not orthogonal. In this note, we point out that this defect can be cured at $c = 3/2$ by means of a trick specific to the supersymmetric case. At $c = 3/2$, we thus end up with a closed-form expression for the norm of the degenerate super-Whittaker vector. Granting the super-version of the AGT conjecture, this closed-form expression should be equal to the $\mathbb{Z}_2$-symmetric SU(2) pure-gauge instanton partition function—the corresponding equality taking the form of a rather nontrivial combinatorial identity.

Keywords: Whittaker vector, superconformal algebra, Jack superpolynomials, AGT conjecture

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1. Introduction

In the context of the Alday–Gaiotto–Tachikawa (AGT) conjecture [1] applied to asymptotically free theories and in the absence of matter, Gaiotto [14] has considered vectors not annihilated by $L_1$—dubbed degenerate Whittaker vectors. An elegant representation for these vectors has been obtained as a sum of Jack polynomials [22] (see also [3]). However, the immediate usefulness of the resulting expression is not clear since the norm of the Whittaker vector—the degenerate conformal block [19]—is then expressed in terms of a scalar product with respect to which the Jack polynomials $P^{(\alpha)}_\lambda$ are not orthogonal, namely

$$\langle P^{(\alpha)}_\lambda | P^{(\alpha')}_{\omega} \rangle_{2\alpha'},$$

whose expression is not known.
These results have been generalized to the superconformal case in [12]. There, it is shown that the degenerate super-Whittaker vector is represented as a sum of Jack superpolynomials (sJacks) \( P^{(\alpha)}_{\Lambda}/\Lambda_1 \) (where \( \Lambda_1 \) is now a superpartition instead of a partition) and that its norm is formulated in terms of the scalar product
\[
\langle P^{(\alpha)}_{\Lambda}/\Lambda_1 | P^{(\alpha)}_{\Omega}/\Omega_1 \rangle_{\alpha},
\]
whose explicit form is again unknown. However, notice that the value \(-2\alpha\) in the Virasoro case gets replaced by \(-\alpha\) in the superconformal case. This numerical simplification allows us to derive a closed-form expression for the norm of super-Whittaker vector at \( \alpha = 1 \), that is, at the value \( c = 3/2 \) of the central charge. The corresponding superconformal field theory (SCFT) is the super-analogue of the \( c = 1 \) CFT which has a rich structure in itself [13]. In order to place this result in context, we first briefly review, in section 2, certain relevant results of [12].

2. Superpolynomial representation of the degenerate Neveu–Schwarz (NS) Whittaker vector

2.1. The degenerate Neveu–Schwarz (NS) Whittaker vector

In the Neveu–Schwarz (NS) sector, the superconformal degenerate Whittaker vector in the Verma module with highest-weight state \(|h\rangle^{NS}_0\), is defined order by order from the following recursion relations [4]:
\[
G_{1/2} |h\rangle^{NS}_k = |h\rangle^{NS}_{k-1}, \quad G_{1/2} |h\rangle^{NS}_k = 0, \quad \forall k \in \mathbb{N}_2,
\]
where \(|h\rangle^{NS}_k\) denotes a descendant of \(|h\rangle^{NS}_0\). Through the free-field representation of the superconformal algebra and the symmetric polynomial representation of the free modes (see [2, 8, 20] in the Virasoro case), we can represent the NS Whittaker vector in terms of superpolynomials. In the NS-sector, the free-field representation of the superconformal algebra
\[
L_n = -\gamma (n + 1)a_n + \frac{1}{2} \sum_{m \in \mathbb{Z}} : a_m a_{n-m} : + \frac{1}{4} \sum_{k \in \mathbb{Z} + \frac{1}{2}} (n - 2k) : b_k b_{n-k} :
\]
\[
G_k = -2\gamma \left( k + \frac{1}{2} \right) b_k + \sum_{m \in \mathbb{Z}} a_m b_{k-m},
\]
where
\[
[a_n, a_m] = n \delta_{n+m,0} \quad \text{and} \quad \{b_k, b_l\} = k \delta_{k+l,0}.
\]
The indices \( n, m \) are integer, while \( k \) and \( l \) are half-integer. The central charge is \( c = 3/2 - 12\gamma^2 \), which we parametrize as
\[
c = \frac{15}{2} - 3 \left( \alpha + \frac{1}{\alpha} \right) \quad \Rightarrow \quad \gamma = \frac{1}{2} \left( \sqrt{\alpha} - \frac{1}{\sqrt{\alpha}} \right).
\]
The highest-weight states in the Fock space \( \mathcal{F} \) are characterized by a complex number \( \eta \) and satisfy
\[
a_0 |\eta\rangle = \eta |\eta\rangle, \quad \text{and} \quad a_n |\eta\rangle = b_k |\eta\rangle = 0, \quad \forall n, k > 0.
\]
The full Fock space \( \mathcal{F} \) is the linear span over \( \mathbb{C} \) of all monomials
\[
b_{-k_1} \cdots b_{-k_l} a_{-m_1} \cdots a_{-m_r} |\eta\rangle, \quad k_i, l_i > 0.
\]
Finally, the conformal dimension of \(|\eta\rangle\) is \( h = \frac{1}{2} \eta^2 - \gamma \eta \).
2.2. Fock states and symmetric superpolynomials

The correspondence between Fock states and symmetric functions is made via the generalization to superspace of the power-sum symmetric functions. The \( n \)th power-sum in infinitely many variables, denoted \( p_n \), and its fermionic partner \( \tilde{p}_n \) are defined as \[ p_n = \sum_i x_i^n \quad (n > 0) \quad \text{and} \quad \tilde{p}_n = \sum_i \theta_i x_i^n, \quad (n \geq 0), \tag{9} \]

where \( \theta_1, \theta_2, \ldots \) are anti-commuting variables. Both \( p_n \) and \( \tilde{p}_n \) are symmetric superpolynomials, that is, polynomials in the variables \( x_1, x_2, \ldots \) and \( \theta_1, \theta_2, \ldots \) that remain unchanged under any simultaneous permutation of the form \((x_i, \theta_i) \leftrightarrow (x_j, \theta_j)\). Any element of the space \( \mathcal{F} \) of all symmetric superpolynomials in infinitely many variables with coefficients in \( \mathbb{C} \) can be uniquely written as a polynomial in \( p_1, p_2, \ldots \) and \( \tilde{p}_0, \tilde{p}_1, \ldots \) (see for instance [9]).

The announced correspondence between the free-field modes and the differential operators acting on \( \mathcal{F} \) reads:

\[
\begin{align*}
a_{-n} &\longleftrightarrow \frac{(-1)^{n-1}}{\sqrt{\alpha}} p_n & a_n &\longleftrightarrow n(-1)^{n-1} \frac{\partial}{\partial p_n} \\
b_{-k} &\longleftrightarrow \frac{(-1)^{k-1/2}}{\sqrt{\alpha}} \tilde{p}_{k-1/2} & b_k &\longleftrightarrow (-1)^{k-1/2} \frac{\partial}{\partial \tilde{p}_{k-1/2}},
\end{align*}
\tag{10}
\]

where \( k, n > 0 \) and \( \alpha \) is a non-zero free parameter. This implies the correspondence

\[
b_{-k_1} \cdots b_{-k_m} a_{-n_1} \cdots a_{-n_p} \left| \eta \right> \longleftrightarrow \xi \tilde{p}_{k_1 - \frac{1}{2}} \cdots \tilde{p}_{k_m - \frac{1}{2}} p_{n_1} \cdots p_{n_p},
\tag{11}
\]

with \( k_i > k_{i+1} \geq \frac{1}{2} \) and \( n_i \geq n_{i+1} \geq 1 \), and \( \xi \) is a constant read off (10). We then relabel the indices as

\[
\begin{align*}
k_i - \frac{1}{2} &= \Lambda_i & \text{for} \quad 1 \leq i \leq m, \quad (\Rightarrow \Lambda_i > \Lambda_{i+1} \geq 0 \quad \text{for} \quad 1 \leq i \leq m - 1), \\
n_i &= \Lambda_{i+m} & \text{for} \quad 1 \leq i \leq p = \ell - m, \quad (\Rightarrow \Lambda_i \geq \Lambda_{i+1} \geq 1 \quad \text{for} \quad m + 1 \leq i \leq \ell - 1).
\end{align*}
\tag{12}
\]

Thus, any state in \( \mathcal{F} \) is in correspondence with a polynomial in \( \mathcal{R} \) indexed by two partitions: \( \Lambda' = (\Lambda_1, \ldots, \Lambda_m) \), whose elements are strictly decreasing with \( \Lambda_m \geq 0 \), and \( \Lambda^s = (\Lambda_{m+1}, \ldots, \Lambda_{\ell}) \), which is a standard partition with \( \ell \) non-zero elements. Together, the partitions \( \Lambda' \) and \( \Lambda^s \) form the superpartition \( \Lambda = (\Lambda_1, \ldots, \Lambda_{\ell}; \Lambda_{m+1}, \ldots, \Lambda_{\ell}) \) (see [9] and references therein). The non-negative integer \( m \) is called the fermionic degree of the superpartition \( \Lambda \) while its bosonic degree is given by \( |\Lambda| = \sum_{i=1}^{\ell} \Lambda_i \). From \( \Lambda \), we construct two partitions: \( \Lambda^* \), which is obtained by removing the semi-colon and reordering the parts in non-increasing order, and \( \Lambda^{\#} \), which is obtained similarly but from \((\Lambda_1 + 1, \ldots, \Lambda_m + 1; \Lambda_{m+1}, \ldots, \Lambda_{\ell})\). Clearly, \( \Lambda \) is uniquely specified by the pair \((\Lambda^*, \Lambda^{\#})\). The diagram of \( \Lambda \) is that of \( \Lambda^* \) but with circles added to the end of the \( m \) rows for which \( \Lambda^*_i - \Lambda^*_{i+1} = 1 \). The conjugate of \( \Lambda \), denoted \( \Lambda' \), is the superpartition associated with the transposed diagram of \( \Lambda \) obtained by reflecting along the main diagonal. For instance, for \( \Lambda = (3, 1, 0; 2, 1) \), we have

\[
\begin{align*}
\Lambda : & \quad \begin{array}{c}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array} & \quad \begin{array}{c}
\Lambda^s : & \quad \begin{array}{c}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array} & \quad \Lambda^* : & \quad \begin{array}{c}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array} & \quad \Lambda' : & \quad \begin{array}{c}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array}
\end{array}
\end{align*}
\tag{13}
\]
2.3. sJack representation of the degenerate NS Whittaker vector

Bases of the space $\mathcal{B}$ can be naturally indexed by superpartitions [9–11]. For instance, the power-sum basis is defined as

$$p_\Lambda = \tilde{p}_\Lambda, \cdots, \tilde{p}_{\Lambda_n} p_{\Lambda_{n+1}} \cdots p_{\Lambda_1}. \quad (14)$$

Thus, any symmetric superpolynomial can be expanded as a sum over the $p_\Lambda$’s, and in particular, the Jack superpolynomials (sJack) $P^{(\alpha)}_\Lambda$.

The degenerate NS super-Whittaker vector at level $k$ can thus be represented as a sum over sJacks as

$$|\bar{h}^{\text{NS}}_k \rangle \leftrightarrow W_k = \sum_{\Lambda, \text{level} (\Lambda) = k} w_\Lambda P_\Lambda, \quad (15)$$

where the level of the superpartition $\Lambda$ is defined as

$$\text{level} (\Lambda) = \frac{1}{2} (|\Lambda^*| + |\Lambda^0|) = \frac{|\Lambda^*| + m}{2}. \quad (16)$$

For instance, $(3, 1, 0; 2, 1)$ has level $17/2$.

The free-field representation (4) and the correspondence (10) immediately imply that the generators $G_i$ and $L_n$ can be represented as differential operators acting on the space $\mathcal{B}$ of symmetric superpolynomials. Let us denote these differential representations by $\mathcal{G}_i$ and $\mathcal{L}_n$ respectively. For instance, we have

$$\mathcal{G}_1 \times W_k = W_{k-1}, \quad \mathcal{G}_1 W_k = 0, \quad \forall k > 0. \quad (19)$$

The expression for the coefficients in (15) has been reported in [12]:

$$w_\Lambda (\alpha, \tilde{\eta}) = \frac{(-1)^{|\Lambda^0|} \alpha^{|\Lambda^*|}}{\tilde{\eta} h^*_{\Lambda}} \left[ \prod_{(i,j) \in \Lambda^0 \setminus \{1,1\}} \frac{1}{\tilde{\eta} + i - \alpha j} \right] A(\Lambda), \quad (20)$$

where (using $\lambda = \Lambda^*$)

$$A(\Lambda) = \prod_{(i,j) \in \Lambda^*_n} \frac{[2 \tilde{\eta} + 1 + i + \lambda'_j - \alpha (1 + j + \lambda_i)]}{2 \tilde{\eta} + 1 + 2i - \alpha (1 + 2j)}$$

$$\times \prod_{(i,j) \in \mathcal{F} \Lambda} \frac{1}{2 \tilde{\eta} + 1 + i + \lambda'_j - \alpha (1 + j + \lambda_i)}. \quad (21)$$

In the previous equation, $\Lambda^*_{n*}$ stands for the cells of $\Lambda^*$ that are not removable corners (a removable corner of a partition is a cell that lies both at the end of its row and at the end of its column), and $\mathcal{F} \Lambda$ denotes the set of boxes $s = (i, j)$ in the diagram of $\Lambda$ that belong at the same time in a fermionic row and in a fermionic column (a row/column is said to be fermionic if it terminates with a circle). For example, in the diagram of $\Lambda = (3, 1, 0; 2, 1)$ displayed in (13), the boxes in $\mathcal{F} \Lambda$ are $(1, 1)$, $(1, 2)$ and $(3, 1)$. Finally, $h^*_{\Lambda}$ is defined below in (46).

Albeit conjectural, the closed-form expression (20)–(21) of the coefficients $w_\Lambda$ has been checked extensively via its norm (cf (32) below).
2.4. Norm of the degenerate NS Whittaker vector

The motivation for studying the degenerate Whittaker vector is rooted in the observation that the square of its norm, \((|\Lambda\rangle, |\Omega\rangle)_{c,h}\), is equal to the degenerate limit of the four-point conformal block at level \(k\) \([4, 14, 19]\). Here \(|\Lambda\rangle \equiv |h\rangle_{NS}^{\Lambda} \equiv |h\rangle^{\Lambda}_{NS}\) denotes the level-\(k\) degenerate Whittaker vector, and \((\ , \ )_{c,h}\) stands for the usual Hermitian Shapovalov form on the NS highest-weight module. The latter form is non-degenerate and characterized, up to a constant, by the following invariance property:

\[
(\ G_{r}|\Lambda\rangle, |\Omega\rangle)_{c,h} = (|\Lambda\rangle, G_{-r}|\Omega\rangle)_{c,h}
\]

for all half-integers \(r\) and for all basic states \(|\Lambda\rangle, |\Omega\rangle\) of the NS highest-weight module, where we made use of the shorthand notation

\[
|\Lambda\rangle = G_{-\Lambda_{1}}^{-\frac{1}{2}} \cdots G_{-\Lambda_{n}}^{-\frac{1}{2}} L_{-\Lambda_{n+1}} \cdots L_{-\Lambda_{k}} |h\rangle_{NS}^{\Lambda}.
\]

Our interest is to rephrase the norm squared \((|k\rangle, |k\rangle)_{c,h}\) in the language of symmetric superpolynomials. Given equations \((4)\) and \((10)\), we already know how to represent any element of the highest-weight module over the NS sector as an element of the space \(\mathcal{R}\) of symmetric superpolynomials:

\[
G_{-r_{1}} \cdots G_{-r_{n}} L_{-n_{1}} \cdots L_{-n_{k}} |h\rangle_{NS} \longleftrightarrow G_{-r_{1}} \cdots G_{-r_{n}} L_{-n_{1}} \cdots L_{-n_{k}} (1) = \sum_{\Lambda\geq1} u_{\Lambda} P^{(\alpha)}_{\Lambda},
\]

where the sum runs over all superpartitions whose level is equal to \(\sum r_{i} + \sum j_{i} \) and where the \(u_{\Lambda}\)'s denote complex coefficients depending upon \(\alpha\) and \(\eta\).

The space \(\mathcal{R}\) is naturally equipped with the scalar product \(\langle \ | \ \rangle_{\Omega}\) defined as

\[
\langle p_{\Lambda} \ | \ q_{\Omega} \rangle_{\Omega} = (-1)^{\ell(\Lambda)} \alpha^{\Lambda} \delta_{\Lambda,\Omega}, \quad \text{where} \quad \alpha^{\Lambda} = \prod_{i\geq1} p^{(s_{i}(i))} n^{(i)}_{\Lambda^{'}}(i)!,
\]

with \(n_{\Lambda^{'}}(i)\) the number of parts in \(\Lambda^{'}\) equal to \(i\). It turns out that the sJacks are orthogonal with respect to the scalar product \((25)\):

\[
\langle P^{(\alpha)}_{\Lambda} \ | \ P^{(\beta)}_{\Omega} \rangle_{\Omega} = 0 \quad \text{when} \quad \Lambda \neq \Omega.
\]

In order to make this scalar product compatible with the invariance property \((22)\), we make a slight change of parametrization,

\[
\eta = \rho + \gamma \quad \Longrightarrow \quad h = \frac{1}{2}(\rho + \gamma)(\rho - \gamma),
\]

and set \(\varphi(\rho) = -\rho\) and \(\varphi(\gamma) = \gamma\), where \(\varphi\) denotes complex conjugation. We then define

\[
\langle f|g\rangle_{\rho} = \langle \varphi(f)|\varphi(g)\rangle_{\rho},
\]

which is a Hermitian form on \(\mathcal{R}\), where \(\beta\) is a function of \(\alpha\) which is determined by enforcing the invariance \((22)\). A close examination reveals that \(\beta = -\alpha\) \([12]\). Therefore, if \(|\Lambda\rangle \longleftrightarrow f\) and \(|\Omega\rangle \longleftrightarrow g\), then

\[
(|\Lambda\rangle, |\Omega\rangle)_{c,h} = \langle f|g\rangle_{\rho}^{w_{\alpha}}.
\]

By writing

\[
W_{k}(\alpha, \bar{\rho}) = \sum_{\Lambda, \text{level}(\Lambda)=k} w_{\Lambda}(\alpha, \bar{\rho}) P^{(\alpha)}_{\Lambda},
\]

where we now use \(w_{\Lambda}(\alpha, \bar{\rho})\) instead of \(w_{\Lambda}(\alpha, \bar{\eta})\) as in \((20)\) to emphasize that \(w_{\Lambda}\) only depends on \(\alpha\) and \(\bar{\rho}\) (given that \(\bar{\eta} = \bar{\rho} + \bar{\gamma} = \bar{\rho} + (\alpha - 1)/2\)), we have

\[
(|k\rangle, |k\rangle)_{c,h} = \langle W_{k}(\alpha, \bar{\rho})|W_{k}(\alpha, \bar{\rho})\rangle_{\eta}^{w_{\alpha}} = \langle W_{k}(\alpha, -\bar{\rho})|W_{k}(\alpha, \bar{\rho})\rangle_{\eta}^{w_{\alpha}},
\]

where \(\eta = \rho + \gamma\) and \(\gamma\) denotes complex conjugation.
so that\(^4\),
\[
(|k\rangle, |k\rangle)_{c,b} = \sum_{\text{level}(\Lambda) = k} w_\Lambda(\alpha, -\tilde{p}) w_\Omega(\alpha, \tilde{p}) \langle \hat{P}_\Lambda^{(\alpha)} | \hat{P}_\Omega^{(\alpha)} \rangle_a. \tag{32}
\]

This completes our review of \([12]\). Let us point out that the equality (32) had only been tested up to level 5/2 in \([12]\). We have now extended these tests up to level 13/2.

We further point out that the norm on the left-hand side is equal to the coefficient, in \(|k\rangle\), of the term containing solely the operators \(G_{-\frac{1}{2}}\) and \(L_{-1}\). More explicitly, if
\[
|k\rangle = \sum_{\text{level}(\Lambda) = k} c_\Lambda |\Lambda\rangle, \tag{33}
\]
where we used the notation (23), then
\[
(|k\rangle, |k\rangle)_{c,b} = c_{\Lambda^0} \quad \text{where} \quad |\Lambda^0\rangle = G_{-\frac{1}{2}}^{2(k-|k\rangle)} L_{-1}^{-|\hat{h}|} |\hat{h}\rangle_{\text{NS}}. \tag{34}
\]

This is an easy consequence of the fact that the only non-vanishing actions of positive super-Virasoro modes on \(|k\rangle\) are \(G_{\frac{1}{2}}\) and \(L_{1}\). (This is analogous to the situation in the Virasoro case \([4, 19]\)).

Given that \(\langle \hat{P}_\Lambda^{(\alpha)} | \hat{P}_\Omega^{(\alpha)} \rangle_a\) is not known, the usefulness of expression (32) is questionable. However, we will see that at \(\alpha = 1\) it can be turned into a closed-form expression.

### 3. Norm of the NS Whittaker vector at \(c = 3/2\)

#### 3.1. Duality transformations

By replacing \(\alpha\) by \(-\alpha\) in the scalar product (25), we get
\[
\langle \hat{P}_\Lambda | \hat{P}_\Omega \rangle_{-\alpha} = (-1)^{C_\alpha + m + \ell(\Lambda^\vee)} \omega^{\ell(\Lambda)} z_\Lambda^\vee \delta_{\Lambda\Omega} \tag{35}
\]
where we used the relation \(\ell(\Lambda) = m + \ell(\Lambda^\vee)\). Introduce the operator \(\tilde{\omega}\) whose action on the elementary power-sums is defined as \([10]\)
\[
\tilde{\omega}(p_r) = (-1)^{r-1} p_r \quad \text{and} \quad \tilde{\omega}(\tilde{p}_r) = (-1)^r \tilde{p}_r, \tag{36}
\]
so that on the full power-sum, we get
\[
\tilde{\omega}(p_\Lambda) = (-1)^{|\Lambda| + \ell(\Lambda^\vee)} p_\Lambda. \tag{37}
\]

We can thus relate the scalar product at \(-\alpha\) to the one evaluated at \(\alpha\) at the price of acting with \(\tilde{\omega}\) on one of the terms:
\[
\langle \hat{P}_\Lambda | \hat{P}_\Omega \rangle_{-\alpha} = (-1)^{|\Lambda|} \langle \tilde{\omega}(p_\Lambda) | \hat{P}_\Omega \rangle_a. \tag{38}
\]

Since the sJacks can be expanded linearly in terms of the power-sums, this readily implies
\[
\langle \hat{P}_\Lambda^{(\alpha)} | \hat{P}_\Omega^{(\alpha)} \rangle_{-\alpha} = (-1)^{|\Lambda|} \langle \tilde{\omega}(P_\Lambda^{(\alpha)}) | \hat{P}_\Omega^{(\alpha)} \rangle_a. \tag{39}
\]

This is still not a convenient expression since there is no known explicit expansion of \(\tilde{\omega}P_\Lambda^{(\alpha)}\) in terms of sJacks. We thus consider a further simplification. Notice that
\[
\langle \tilde{\omega}(p_\Lambda) | \hat{P}_\Omega \rangle_a = \langle \tilde{\omega}_b p_\Lambda | \hat{P}_\Omega \rangle_a = 1. \tag{40}
\]

\(\text{\footnotesize{\textsuperscript{4}} Our expressions for the left-hand side differ slightly from those of \([4]\), listed there for } \frac{1}{2} \leq k \leq \frac{5}{2} \text{ (and which we label BF): }
\]
\[
(|k\rangle, |k\rangle)_{c,b}^{\text{BF}} = (|k\rangle, |k\rangle)_{c,b} \times 4^{-|\hat{h}|}.
\]
where
\[ \hat{\omega}_\alpha(p_r) = (-1)^{r-1} \alpha p_r \quad \text{and} \quad \hat{\omega}_\alpha(\tilde{p}_r) = (-1)^r \alpha \tilde{p}_r. \] (41)

This again implies that
\[ \| \hat{\omega}_\alpha P_A^{(a)} | P_{a}^{(a)} \|_a = \| \hat{\omega}_\alpha P_A^{(a)} | P_{a}^{(a)} \|_{a=1}. \] (42)

Explicitly, the action of \( \hat{\omega}_\alpha \) on \( P_A^{(a)} \) reads [10]:
\[ \hat{\omega}_\alpha P_A^{(a)} = (-1)^{\binom{c}{2}} j_\lambda(\alpha) P_A^{(1/a)}, \] (43)

where \( j_\lambda(\alpha) \) is essentially the norm of \( P_A^{(a)} \):
\[ \| P_A^{(a)} | P_A^{(a)} \|_a = (-1)^{\binom{c}{2}} j_\lambda(\alpha). \] (44)

This normalization factor takes the form [11, 16]
\[ j_\lambda(\alpha) = \alpha^m h_\lambda^{+1}, \] (45)

where \( h_\lambda^{+1} \) are defined as follows:
\[ h_\lambda^{+1} = \prod_{s \in B \Lambda} h_\lambda^+(s), \quad h_\lambda^+(s) = l_{(s)}(s) + \alpha(a_{(s)}(s) + 1), \]
\[ h_\lambda^{+1} = \prod_{s \in B \Lambda} h_\lambda^+(s), \quad h_\lambda^+(s) = l_{(s)}(s) + 1 + \alpha a_{(s)}(s). \] (46)

In the previous expressions, \( B \Lambda \) corresponds to the boxes \( s = (i, j) \) in the diagram of \( \Lambda \) that do not belong to \( F \Lambda \) [11]. Given the box \( s = (i, j) \) (ith row and jth column) of a partition \( \lambda \), the quantities \( a_s(s) \) and \( l_s(s) \) are defined as [17]
\[ a_s(s) = \lambda' - j \quad \text{and} \quad l_s(s) = \lambda' - i, \] (47)

where \( \lambda' \) stands for the conjugate of \( \lambda \), obtained by interchanging rows and columns.

Summing up, we have obtained the sequence of equalities:
\[ \| P_A^{(a)} | P_A^{(a)} \|_a = (-1)^{\binom{c}{2}} \| \hat{\omega}_\alpha P_A^{(a)} | P_{a}^{(a)} \|_a \]
\[ = (-1)^{\binom{c}{2}} \| \hat{\omega}_\alpha P_A^{(a)} | P_{a}^{(a)} \|_{a=1} \]
\[ = (-1)^{\binom{c}{2} + \binom{c}{2}} j_\lambda(\alpha) \| P_A^{(1/a)} | P_{a}^{(a)} \|_{a=1}. \] (48)

In other words, by introducing the operation \( \hat{\omega} \), the scalar product evaluated at \(-\alpha\) is transformed into the one evaluated at \(\alpha\), and by trading \( \hat{\omega} \) for \( \hat{\omega}_\alpha \), this scalar product is then changed into the one evaluated at \(\alpha = 1\). Therefore, if we had an expression for \( P_A^{(a)} \) (which would obviously provide one for \( P_A^{(1/a)} \) in terms of the Schur analogues \( P_{a}^{(1)} \) or even in terms of the power-sums \( p_{a} \), we could obtain the norm of the degenerate super-Whittaker vector in closed-form for any value of \( \alpha \). This is an interesting combinatorial problem which deserves further study. However, our immediate purpose is to point out a dramatic simplification that occurs when \( \alpha = 1 \), that is, at \( c = 3/2 \).

3.2. The case \( \alpha = 1 \)

At \( \alpha = 1 \), the orthogonality condition together with the normalization (44) yield
\[ \| P_A^{(1)} | P_{a}^{(1)} \|_{a=1} = (-1)^{\binom{c}{2}} j_\lambda(1) \delta_{\Lambda, \Omega}. \] (49)
so that the last equality in (48) reduces to
\[
\left\langle p^{(1)}_\lambda | p^{(1)}_\omega \right\rangle_{-1} = (-1)^{|\lambda|} j_\lambda(1) j_\omega(1) \delta_{\lambda, \omega} = (-1)^{|\lambda|} \delta_{\lambda, \omega},
\]  
(50)
since it is easily checked that the product \( j_\lambda(1) j_\omega(1) \) reduces to \( 1^5 \).

When \( \alpha = 1 \), we have \( \gamma = 0 \) and \( \bar{\eta} = \bar{\rho} = \rho \). Hence (32) can be written in this case as
\[
\langle k, k \rangle_{r=3/2, h} = \sum_{\lambda, \omega, k} w_\lambda(1, -\rho) w_\omega(1, \rho) (-1)^{|\lambda|} \delta_{\lambda, \omega}.
\]  
(51)
In the previous expression, the product \( w_\lambda(1, -\rho) w_\lambda(1, \rho) \) can be written compactly in the form
\[
w_\lambda(1, -\rho) w_\lambda(1, \rho) = \frac{(-1)^{|\lambda|+1}(\gamma)}{4^{|\lambda|+1/2} \prod_{(i,j) \in \lambda^\omega} (\rho - i - j + \lambda_j)^2} \prod_{(i,j) \in \mathcal{F}_\lambda} (2 \rho - i - j + \lambda_j)^2
\]  
(52)
where in the last term \( \lambda = \Lambda^* \), and where \( \epsilon_{ij}^r = 1 \) if \( (i, j) \in \Lambda^*_a \) and 0 otherwise.

### 3.3. A digressing remark

The exact expression for the super-Whittaker vector depends crucially upon the proper choice of the precise proportionality coefficient relating the free-field modes \( a_{-n} \) to \( p_n \) and \( b_{-r} \) to \( \bar{p}_{r-1/2} \). For instance, in the Virasoro case, the required relationship is (with \( n > 0 \))
\[
a_{-n} \leftrightarrow (-1)^{n-1} \frac{\sqrt{\alpha}}{2} p_n \quad \text{so that} \quad a_n \leftrightarrow n(-1)^{n-1} \frac{\sqrt{\alpha}}{\partial p_n}
\]  
(53)
which differs from the corresponding relationship in the supersymmetric version (10) in that \( \sqrt{\alpha} \) is replaced by \( \sqrt{2\alpha} \). To be clear, if we let
\[
a_{-n} \leftrightarrow (-1)^{n-1} \frac{\sqrt{\alpha}}{\eta} p_n \quad \text{and} \quad a_n \leftrightarrow (-1)^{n-1} \frac{n\sqrt{\alpha}}{\partial p_n}
\]  
(54)
we obtain an expansion in terms of Jack polynomials but in general the expansion coefficients are very complicated and do not factorize. This factorization is observed only for \( \bar{\kappa} = \sqrt{2} \), which is equivalent to the choice of parametrization made in (53) [18].

In the expression for the norm, the value of \( \beta \) at which the scalar product of the Jack polynomials is evaluated (that is, the value of \( \beta \) in \( \left\langle p^{(1)}_\lambda | p^{(1)}_\mu \right\rangle \)) depends crucially upon the

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5 Let us stress that a slack reduces to a Jack in absence of fermionic variables, that is, when \( \Lambda^a = \emptyset \). Therefore, the chain of equalities (48), as well as equation (50), still hold in the non-supersymmetric case. However, this is not relevant because in the non-supersymmetric case at \( \alpha = 1 \) (\( c = 1 \)), we need an expression for \( \left\langle p^{(1)}_\lambda | p^{(1)}_\mu \right\rangle \) and not for \( \left\langle p^{(1)}_\lambda | p^{(1)}_\mu \right\rangle_{-1} \) (cf the introduction). The usefulness of the trick summarized in equation (48) is thus specific to the supersymmetric case.

6 If we replace \( \kappa \) in (54) by a parameter \( \kappa_0 \) depending on \( n \), then we observe the factorization not only for \( \kappa_0 = \sqrt{2} \), but also for \( \kappa_0 = (-1)^{n-1} \sqrt{2}/\alpha \). The correspondence associated with the latter value of \( \kappa_0 \) is \( a_{-n} \leftrightarrow \sqrt{\alpha/2} p_n \) and \( a_n \leftrightarrow n(-1)^{n-1} \sqrt{\alpha/2} p_n \), which is that used in [2, 22]. However, we easily get this second correspondence from the first one by acting with \( \bar{a}_0 \) on the symmetric-function-side of (53). Moreover, \( \bar{a}_0 (c_\lambda) = (-1)^{\lambda} c_{\lambda^*} \), where \( c_{\lambda^*} \) is obtained from \( c_\lambda \) by changing \( (a, \rho, \gamma) \) into \( (a^*, -\rho, -\gamma) \). Thus, if \( W_k(a, \rho, \gamma) \) denotes the Whittaker vector at level \( k \) computed using \( \kappa_0 = \sqrt{2} \), then that computed using \( \kappa_0 = (-1)^{n-1} \sqrt{2}/\alpha \) is equal to \( (-1)^n W_k(a^{-1}, -\rho, -\gamma) \).
coefficient relating \( a_{\alpha,\beta} \) to \( p_{\alpha} \). With (54) one finds that \( \beta = -\kappa^2 \alpha \). The argument goes as follows: using the relations (54), we get
\[
L_1(\rho) = \kappa \sqrt{\alpha} (\rho - \gamma) \partial_1 - (n + 1)p_{\alpha} \partial_{n+1},
\]
\[
L_{-1}(\rho) = \frac{1}{\kappa \sqrt{\alpha}} (\rho + \gamma) p_1 - \sum_{n=0}^{\infty} n p_{n+1} \partial_n.
\] (55)
Now enforce
\[
\langle \langle L_1(-\rho) f | g \rangle \rangle \beta = \langle \langle f | L_{-1}(\rho) g \rangle \rangle \beta,
\] (56)
using \( f = p_1^\ell \) and \( g = p_1^{\ell+1} \). This yields \( \beta/\alpha = -\kappa^2 \) as claimed. Therefore, it is possible to get \( \beta = -\alpha \) also in the Virasoro case, but we then lose the possibility of obtaining an explicit expression of the Jack expansion coefficients. Clearly, it would be interesting to find a good argument that would fix \textit{a priori} the relationship between the free-field modes and the power sums that yields nice factorized coefficients in the (s)Jack basis.

Finally, this discussion shows that it is a noteworthy property of the supersymmetric case that the corresponding value of \( \kappa \) for which the coefficients of the degenerate Whittaker vector factorize is 1 instead of 2, allowing the derivation of an explicit expression for its norm at \( \alpha = 1 \).

4. Relation to the instanton formula

The proper way of modifying the original AGT conjecture in order to link the four-dimensional supersymmetric \( SU(2) \) gauge theory to superconformal blocks turns out to restrict the pure-gauge instanton partition function to the \( \mathbb{Z}_2 \)-symmetric sector [4] (see also [5–7, 15, 21]). The resulting expression for the instanton partition function reads:
\[
Z(\alpha; q) = \sum_{k \in \mathbb{N}/2} q^k \sum_{\vec{Y}} \frac{1}{Z_{\text{vec}}^{\text{sym}}(\vec{a}, \vec{Y})} \left( \epsilon = 0, \frac{1}{2} \right).
\] (57)
It is expressed in terms of the data of a pair of Young diagrams \( \vec{Y} = (Y_1, Y_2) \), each drawn on a chessboard with top-left box white, with total number of boxes \( |Y_1| + |Y_2| = N_+ + N_- \), where \( N_+ \) (resp. \( N_- \)) is the number of white (resp. black) boxes. Terms with integer (resp. half-integer) powers of \( q \) have \( N_+ - N_- = 0 \) (resp. \( N_+ - N_- = 1 \)), in which case we take \( \epsilon = 0 \) (resp. \( \epsilon = \frac{1}{2} \)). The quantity \( Z_{\text{vec}}^{\text{sym}}(\vec{a}, \vec{Y}) \) is given by
\[
Z_{\text{vec}}^{\text{sym}}(\vec{a}, \vec{Y}) = \prod_{\alpha, \beta = 1}^{2} \prod_{s \subset Y_\alpha(\beta)} E(\mathbf{a}_\alpha - \mathbf{a}_\beta, Y_\alpha, Y_\beta|s)(Q - E(\mathbf{a}_\alpha - \mathbf{a}_\beta, Y_\alpha, Y_\beta|s))
\]
where \( Q = b + b^{-1}, \mathbf{a}_1 = -\mathbf{a}_2 = \mathbf{a} \) and
\[
E(\mathbf{a}_\alpha - \mathbf{a}_\beta, Y_\alpha, Y_\beta|s) = (\mathbf{a}_\gamma - \mathbf{a}_\beta + b(l_{\gamma_\alpha}(s) + 1) - b^{-1}a_{\gamma_\beta}(s))
\]
with \( l_{\gamma}(s) \) et \( a_{\gamma}(s) \) being respectively the leg and the arm of the box \( s \in Y \). The set \( Y_\alpha(\beta) \) is defined as follows
\[
Y_\alpha(\beta) = \{ s \mid l_{\gamma_\alpha}(s) \neq a_{\gamma_\beta}(s) \mod 2 \}.
\]
The SCFT version of the AGT conjecture is thus [4] (the factor \( 2^{-2k} \) is absent there):
\[
Z(\alpha; q) = \sum_{k \in \mathbb{N}/2} q^{k} 2^{-2k} (|k|, |k|)_{-\kappa, \kappa}.
\] (58)
or, equivalently, at a fixed order,
\[
\sum_{\vec{Y}} \frac{1}{Z^{\text{sym}}_{c} (\vec{a}, \vec{Y})} = 2^{-2k} (|k\rangle, |k\rangle)_{c,b}.
\] (59)

We have verified this relation up to 12 instantons.

Let us specialize this expression for \(c = \frac{3}{2}\), so that \(b = i\) (\(Q = 0\)) and \(\rho = i\vec{a}\). In this case, we end up with the identity
\[
\sum_{\vec{Y}} \frac{1}{Z^{\text{sym}}_{c} (\vec{a}, \vec{Y})}_{|b=1} = \frac{1}{2^{2k}} \sum_{\text{level}(\Lambda) = k} \sum_{\Lambda} w_{\Lambda} (1, -i\vec{a}) w_{\Lambda'} (1, i\vec{a}) (-1)^{|\Lambda|^\omega}. \] (60)

The rhs provides an alternative closed-form expression for the \(\mathbb{Z}_2\)-symmetric instanton partition function (for \(b = i\)) in terms of a sum over super-diagram’s data. We have not been able to prove this identity but given its highly nontrivial character, we expect its presentation to be of interest.

5. Conclusion

The verification of the expression (32) of the degenerate super-Whittaker vector in terms of \(s\)Jack polynomials conjectured in [12] has been considerably extended, from level 5/2 to 13/2. This remains rather formal since there is no known expression for \(\langle P_{\Lambda}^{(1)} (\vec{a}) | P_{\Omega} (\vec{a}) \rangle_{-1}\).

However, we have pointed out that for \(\alpha = 1\), which corresponds to \(c = \frac{3}{2}\) in SCFT, there is a sequence of manipulations done directly on the power-sums (that are in correspondence with the free-field modes) which allows us to evaluate explicitly \(\langle P_{\Lambda}^{(1)} (\vec{a}) | P_{\Omega} (\vec{a}) \rangle_{-1}\). This results in an expression for the degenerate super-Whittaker vector in terms of a sum over superpartitions of a function expressed solely in terms of the superdiagram’s data. Then, granting the super-version of the AGT conjecture proposed in [4], we have lifted this to an identity between the \(\mathbb{Z}_2\)-symmetric part of the instanton partition function and the previously mentioned sum over super-diagram’s data, namely equation (60). Note that the former is actually a double sum over chessboard-type Young diagrams with constraints, while the latter involves a single summation. Again, this relation has been tested to relatively large order.

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