The supersymmetric technique for random-matrix ensembles with zero eigenvalues

D. A. Ivanov

Institut für Theoretische Physik,
ETH-Hönggerberg, CH-8093 Zürich, Switzerland

March 29, 2001

Abstract

The supersymmetric technique is applied to computing the average spectral density near zero energy in the large-\(N\) limit of the random-matrix ensembles with zero eigenvalues: \(B\), \(D\)III-odd, and the chiral ensembles (classes \(A\)III, \(BD\)I, and \(C\)II). The supersymmetric calculations reproduce the existing results obtained by other methods. The effect of zero eigenvalues may be interpreted as reducing the symmetry of the zero-energy supersymmetric action by breaking a certain abelian symmetry.

There exists a remarkable correspondence between large families of random-matrix ensembles and symmetric superspaces. It has been shown by Zirnbauer that in the large-\(N\) limit (\(N\) is the matrix dimension) correlation functions in random-matrix ensembles may be represented as integrals over appropriate Riemannian symmetric superspaces (with dimensions independent of \(N\)) [1]. This relation to symmetric superspaces is based on Efetov’s supersymmetric technique introducing auxiliary anticommuting (Grassmann) variables in order to directly average correlation functions over the statistical ensemble [2].

At the same time, the random-matrix ensembles are known to be in one-to-one correspondence with symmetric spaces (Cartan symmetry classes) [3, 4]. The classification of Zirnbauer thus establishes a correspondence between large families of symmetric spaces and Riemannian symmetric superspaces [1]. The random-matrix ensembles with zero eigenvalues were not included in the original classification, and later it became apparent that the zero eigenvalues in random-matrix ensembles are related to the reducibility of the corresponding symmetric superspaces [5, 6].

In this paper, I study this relation by explicitly calculating the average density of states in all random-matrix ensembles with zero eigenvalues. There are five such ensembles (Table 1): class \(B\) [so(\(N\)) matrices at odd \(N\)], class \(D\)III-odd [so(2\(N\))/u(\(N\)) matrices at odd \(N\)], and the three chiral ensembles: unitary \(A\)III, orthogonal \(BD\)I, and symplectic \(C\)II. In a physical context, the ensembles \(B\) and \(D\)III-odd appear in vortices in superconductors with odd pairing [5, 7], the chiral classes — in QCD [8, 9]. The zero levels in these ensembles occur as a

| Cartan class | Symmetric space (compact type) | \(\beta\) | \(\alpha\) | Number of zero eigenvalues \(m\) |
|--------------|-------------------------------|--------|--------|-----------------|
| \(B-D\)      | SO(\(N\))                     | 2      | 2\(m\) | \(m=0\), even \(N\) |
| \(D\)III     | SO(2\(N\))/U(\(N\))          | 4      | 1+4\(m\) | \(m=1\), odd \(N\) |
| \(A\)III     | SU(\(p+q\))/S(U(\(p\))×U(\(q\))) | 2      | 1+2\(m\) | \(m=|p-q|\) |
| \(BD\)I      | SO(\(p+q\))/SO(\(p\))×SO(\(q\)) | 1      | \(m\)   | \(m=0\), even \(N\) |
| \(C\)II      | Sp(\(p+q\))/Sp(\(p\))×Sp(\(q\)) | 4      | 3+4\(m\) | \(m=1\), odd \(N\) |
consequence of the symmetry inverting energy \((E \to -E)\) combined with the odd dimension (for classes \(B\) and \(DIII\)-odd) or with the dimensional imbalance between the two chiral sectors (for the chiral classes). Table 1 also lists the values of the parameters \(\alpha\) and \(\beta\) appearing in the joint probability distribution for energy levels \(\omega_i\):

\[
dP(\omega_1, \ldots, \omega_n) \propto \prod_{i<j} |\omega_i^2 - \omega_j^2|^{\beta} \prod_i \omega_i^\alpha d\omega_i
\]

(\(\beta\) determines the strength of repulsion between levels, \(\alpha\) — the strength of repulsion from zero).

Previously, a supersymmetric calculation of the microscopic spectral density for the chiral unitary case was done in Ref. [6], and the case of class \(B\) was studied in Ref. [5] (in the context of class \(BD\), which is the average of classes \(B\) and \(D\)). I include these cases for completeness in the corresponding sections.

From the supersymmetric calculations for the five random-matrix ensembles, we find that zero levels in random-matrix ensembles manifest themselves in reducing the symmetry of the supersymmetric action at zero energy. In the absence of zero levels, this action (a function of the supermatrix \(Q\) in Efetov’s technique [2]) is invariant with respect to the full supergroup preserving the linear constraints on \(Q\) (latter being determined by the symmetries of random matrices). For ensembles with zero levels, the zero-energy action is invariant with respect to only a normal subgroup of this supergroup, but breaks the remaining abelian symmetry. In the large-\(N\) limit, the integral over \(Q\) is dominated by the saddle-point manifold. This manifold is a Riemannian symmetric superspace [1], and for ensembles admitting zero levels it is not irreducible: it may be split into orbits of the normal subgroup of the full symmetry (super) group. The quotient by this normal subgroup is an abelian (conventional, not super) group (\(Z_2\) for classes \(B–D\) and \(DIII\), and \(GL(1)\) for the chiral classes). If the random-matrix ensemble contains zero levels, the action is not invariant with respect to this residual abelian group, but transforms according to one of its one-dimensional representations.

The paper is organized as follows. In the next section, I review the results for the average spectral density in the random-matrix ensembles with zero levels. Next, I describe the details of the supersymmetric calculations for each of the five random-matrix ensembles. The calculation for the ensemble \(B–D\) is presented in somewhat more detail, and in the subsequent sections the repeating steps of the derivations are described only briefly. In the last section I discuss common features of these calculations specific for ensembles with zero levels.

## 1 Spectral density in random-matrix ensembles with zero levels

In this section I review the results for the average spectral density in the vicinity of the zero eigenvalue. All these results are known and were previously derived by other methods. The reader may use this section as a quick reference.

In what follows we consider zero-curvature random-matrix ensembles and treat them as quantum-mechanical Hamiltonians. Accordingly we use quantum-mechanical terminology such as “energy levels”, “inter-level spacing”, etc.

In an ensemble of random matrices of size \(N\), with a fixed dispersion of matrix elements, the inter-level spacing in the middle of the spectrum scales as \(N^{-1/2}\) for large \(N\). If we measure the energy \(E\) in the units of this inter-level spacing \(\Delta\), the correlation functions in the vicinity of zero energy (middle of the spectrum) have a finite and universal limit as \(N \to \infty\). In this paper I am interested in the average density of states \(\rho(x)\) as a function of dimensionless energy \(x = E/\Delta\). This function gives the average number of energy levels in any interval \([a; b]\):

\[
\langle n \rangle_{[a; b]} = \int_a^b \rho(x) dx
\]

(In the case of symplectic ensembles \(DIII\) and \(CII\), all energy levels are doubly degenerate, and for counting purposes every degenerate pair of states will be counted as a single level). The function \(\rho(x)\) is symmetric \(\rho(x) = \rho(-x)\) and has the normalization \(\lim_{x \to \infty} \rho(x) = 1\). For an ensemble with \(m\) zero levels, \(\rho(x)\) has a \(\delta\)-functional contribution at \(x = 0\): \(\rho(x) = m\delta(x) + \tilde{\rho}(x)\), where \(\tilde{\rho}(x)\) is continuous at \(x = 0\).

The results for the average density of states \(\rho(x)\) in the ensembles studied in this paper are the following (defining \(y = 2\pi |x|\), \(m\) is the number of zero levels in the chiral ensembles).

---
Class $D$:

\[ 1 + \frac{\sin y}{y} \] (3)

Class $B$:

\[ 1 - \frac{\sin y}{y} + \delta(x) \] (4)

Class DIII-even:

\[ \frac{\pi}{2} y \left[ J'_1(y) J_0(y) + J'_1(y) \right] + \frac{\pi}{2} J_1(y) \] (5)

Class DIII-odd:

\[ \frac{\pi}{2} y \left[ J'_1(y) J_0(y) + J'_1(y) \right] - \frac{\pi}{2} J_1(y) + \delta(x) \] (6)

Class AIII (chiral unitary):

\[ \frac{\pi}{4} y \left[ J^2_{m}(\frac{y}{2}) - J_{m-1}(\frac{y}{2}) J_{m+1}(\frac{y}{2}) \right] + m\delta(x) \] (7)

Class BDI (chiral orthogonal):

\[ \frac{\pi}{2} \left( \frac{y}{2} \left[ J^2_{m}(\frac{y}{2}) - J_{m-1}(\frac{y}{2}) J_{m+1}(\frac{y}{2}) \right] + J_m(\frac{y}{2}) R_m(\frac{y}{2}) \right) + m\delta(x) \] (8)

Class CII (chiral symplectic):

\[ \frac{\pi}{2} \left( y \left[ J^2_{2m}(y) - J_{2m-1}(y) J_{2m+1}(y) \right] - J_{2m}(y) \tilde{R}_{2m}(y) \right) + m\delta(x) \] (9)

where the functions $R_n$ and $\tilde{R}_n$ are defined as:

\[ \tilde{R}_n(z) = 1 - R_n(z) = \int_0^z J_n(z')dz'. \] (10)

These results were previously derived by other methods. The results (3) and (4) are presented in the book of Mehta [10]. They are also straightforward to obtain from mapping of level statistics onto free fermions. A supersymmetric approach to classes $B$ and $D$ was developed in Ref. [5]. The result (5) was found by Nagao and Slevin [11] and by Altland and Zirnbauer [12] (contrary to their claim, their result is identical to the result of Nagao and Slevin after some algebraic manipulations with Bessel functions). The result (7) was obtained in the works of Verbaarschot and Zahed [13], Nagao and Slevin [14], and Forrester [15]. Also, a supersymmetric calculation of (7) at $m = 0$ was reported in [16], and then at arbitrary $m$ in [6]. To make this paper self-contained, I repeat the derivation of Ref. [6] in the corresponding section. The particular case of the formula (8) at $m = 1$ can be found in [11]. The case of arbitrary $m$ was treated in [17] and [18]. The latter work also contains the answer for the ensemble CII. The results of [17] and [18] are presented in the form of complicated integrals. The simple formulas (8) and (9) were later reported in Refs. [19, 20, 21].

The average spectral densities (3)–(9) are plotted in Fig. 1.

2 Remarks about notation, supergroups and superspaces

In this section I explain some notational conventions used in the subsequent sections. The calculations involve supermatrices acting in a superspace which has the structure $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, depending on the symmetry class. One of the $\mathbb{C}^2$ factors refers to the Fermi–Bose (FB) sectors and defines the supersymmetric grading. The one or two remaining $\mathbb{C}^2$ factors are either produced by additionally doubling the dimension to take into account the symmetries of the random-matrix ensemble (in classes $B$–$D$, DIII, and BDI) or are originally present in the matrix structure in the random-matrix ensemble (in class DIII and in the three chiral classes). These $\mathbb{C}^2$ will be labeled to as “particle-hole” (PH) or “1-2” sectors, without stressing the physical meaning of this terminology. In supermatrices, the FB sectors will be graphically divided by solid lines [see, for example,
Figure 1: The average spectral density $\rho(x)$ for ensembles $B-D$, DIII, AIII, BDI, and CII.
eq. (12)], with the BB sector in the upper left corner, and the FF sector in the lower right corner. When the matrices also act in the PH or 1-2 spaces, in order to avoid confusion, these spaces will be explicitly mentioned as a subscript, from the outermost division to the innermost subdivision [see, for example, eqs. (58)–(60)]. The empty spaces in matrices denote zeroes.

The supergroups $GL(n|m)$ and $OSp(n|2m)$ appearing in our supersymmetric constructions are defined as follows. The complex supergroup $GL(n|m)$ consists of all invertible supermatrices of dimension $n + m$. The complex supergroup $OSp(n|2m)$ is the subgroup of $GL(n|2m)$ obeying the relation

$$g^{-1} = g^{T} \gamma^{-1},$$

where

$$\gamma = \begin{pmatrix} \frac{1}{\sqrt{N}} & 0 & \ldots & 0 \\ 0 & \ddots & \ldots & \vdots \\ \ldots & \ldots & \ddots & 0 \\ -1_m & \ldots & 0 & -1_m \end{pmatrix}.$$  

(12)

(11)

Its support is the direct product of $O(n)$ and $Sp(m)$. More details about these supergroups may be found in Refs. [22, 1]. The reader may also refer to Ref. [2] for conventions regarding manipulations with supermatrices. The Lie superalgebras of $OSp(n|2m)$ and $GL(n|m)$ are denoted as $osp(n|2m)$ and $gl(n|m)$.

To distinguish between fermionic and bosonic sectors [which is important when performing integration, either compact or non-compact, see below], we shall reserve the notation $OSp(n|2m)$ for the supergroup with $O(n)$ in the bosonic and $Sp(m)$ in the fermionic sector. The same supergroup with $O(n)$ in the fermionic and $Sp(m)$ in the bosonic sector we denote as $SpO(n|2m)$. Also, we use the notation $SpSO(n|2m)$ for the connected component of $SpO(n|2m)$ [with the unit superdeterminant].

The notation $Sp(m)$ in this paper refers to the symplectic group of $2m \times 2m$ matrices. This notation agrees with Refs. [1, 3, 7], but differs from Ref. [22] where the same group is denoted $Sp(2m)$.

## 3 Classes $B$ and $D$

In this section we use the supersymmetric technique to compute the density of states for the $so(N)$ random matrices (class $D$ for even $N$, class $B$ for odd $N$). This is the simplest of the five examples considered in this paper, and we describe it in more detail to demonstrate the technique of the calculation. The calculation follows the prescription described in detail by Zirnbauer [1].

The random-matrix ensembles $B$–$D$ is unitary (with $\beta = 2$). The supermatrix $Q$ used in the calculation of the average density of states has dimension $2 + 2$ (2 bosonic and 2 fermionic dimensions), is parameterized by $4 + 4$ independent variables and is an element of $osp(2|2)$ Lie superalgebra. The saddle-point manifold has dimension $2 + 2$, and thus the density of states in the large-$N$ limit is computed as an integral over two commuting and two Grassmann variables.

The random-matrix ensemble $B$–$D$ consists of purely imaginary antisymmetric matrices $H$

$$H_{ab} = H_{ba}^{*} = -H_{ba}, \quad a, b = 1, \ldots, N. \quad (13)$$

The matrix elements have independent Gaussian distributions:

$$dP(H) \propto \prod_{a \geq b} \exp \left( -\frac{|H_{ab}|^2}{2v^2} \right) dH_{ab},$$

(14)

so that the averages of any number of matrix elements are given by the Wick rule together with the pair average

$$\langle H_{ab} H_{a'b'} \rangle = v^2 (\delta_{a'b'} \delta_{ba'} - \delta_{aaa'} \delta_{b'b'}).$$

(15)

From the calculation below we shall see that the energy unit defined as

$$\Delta = \frac{\pi v}{\sqrt{N}}$$

(16)
plays the role of the average level spacing near zero energy.

The average density of states can be found by differentiating the generating function

$$Z(\omega_B, \omega_F) = \int dP(H) \frac{\det(H - \omega_F \Delta)}{\det(H - \omega_B \Delta)},$$

(17)

where the integration is performed over the ensemble of random matrices $H$; $\omega_B$ and $\omega_F$ are auxiliary variables (complex numbers). We included the energy scale $\Delta$ in the definition (17) to make $\omega_B$ and $\omega_F$ dimensionless.

The ensembles considered in this paper have an $E \rightarrow -E$ symmetry, which leads to the symmetry of the generating function:

$$Z(\omega_B, \omega_F) = Z(-\omega_B, -\omega_F) = (-1)^m Z(-\omega_B, \omega_F),$$

(18)

where $m$ is the number of zero levels. In the supersymmetric calculation below we neglect the overall sign of $Z(\omega_B, \omega_F)$, but restore it at the end of the calculation from the condition $Z(\omega, \omega) = 1$ and from positiveness of the density of states.

The two determinants in (17) may be written as Gaussian integrals over bosonic and fermionic variables (auxiliary fields). Introducing the $(N + N)$-component supervector $\psi_a = (\psi_{Ba}, \psi_{Fa})$, $a = 1, \ldots, N$, and performing the integration over $dP(H)$, we arrive at the partition function for interacting superfields [the common energy scale $\nu$ drops out already at this step, thanks to our including $\Delta$ in (17)]:

$$Z(\omega_B, \omega_F) = \int D(\psi^\dagger, \psi) \exp \left( -\frac{i\pi \omega_B}{\sqrt{N}} \psi^\mu \psi^\dagger \mu - \frac{1}{2} \left[ (\psi^\dagger \mu \psi^\dagger \nu)(\psi^\dagger \nu \psi^\dagger \mu) - (\psi^\dagger \mu \psi^\dagger \nu)(\psi^\dagger \nu \psi^\dagger \mu) \right] \right),$$

(19)

where $\mu, \nu$ are fermion-boson indices. In the integral, the Grassmann components in $\psi$ and $\psi^\dagger$ are treated as independent variables (total $2N$ Grassmann variables). The integral over bosonic components of $\psi$ and $\psi^\dagger$ is taken over the $2N$-dimensional real submanifold $(\psi^\dagger)^{Ba} = (\psi^{Ba})^*$.

To decouple the interaction with the $Q$-matrix, it is necessary to double the dimension of vector $\psi$. Combine the old superfields $\psi_{\mu a}$ into the new ones:

$$\Psi_a = \begin{pmatrix} \psi_B^a \\ \psi_F^a \end{pmatrix}, \quad \overline{\Psi}_a = \begin{pmatrix} \psi_B^a \\ -\psi_B^a \\ \psi_F^a \\ \psi_F^a \end{pmatrix},$$

(20)

so that

$$\overline{\Psi} = (\gamma \Psi)^T,$$

(21)

where

$$\gamma = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$  

(22)

In terms of the supervectors $\Psi$ and $\overline{\Psi}$, the partition function may be rewritten as

$$Z(\omega_B, \omega_F) = \int D\Psi \exp \left( -\frac{1}{2} \text{Str} \left[ \frac{i\pi}{\sqrt{N}} \Psi_a \overline{\Psi}_a \hat{\omega} + \frac{1}{2} (\Psi_a \overline{\Psi}_a)^2 \right] \right),$$

(23)

where

$$\hat{\omega} = \begin{pmatrix} \omega_B \\ \omega_F \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(24)

(With this definition of $\hat{\omega}$ we in fact change the sign of $\omega_F$, which may result in the change of sign of $Z(\omega_B, \omega_F)$, according to (18). We shall not control the overall sign of $Z(\omega_B, \omega_F)$, but restore the correct sign at the end of the calculation.)

6
The matrix $(\Psi_a \overline{\Psi}_a)$ has the explicit form

$$\Psi_a \overline{\Psi}_a = \begin{pmatrix} Q_B & -X & \sigma & \rho \\ -\sigma & -Q_B & \rho & \bar{\sigma} \\ \bar{\rho} & -\rho & 0 & \bar{Q}_F \\ \bar{Q}_F & 0 & -Q_F & 0 \end{pmatrix}. \tag{25}$$

This is also the form of the supermatrix $Q$ used to decouple the interaction via Hubbard-Stratonovich transformation:

$$Z(\omega_B, \omega_F) = \int DQ \int D\Psi \exp \left( -\frac{1}{2} S \text{Tr} \left[ \frac{N}{2} Q^2 + i \sqrt{N} (Q - \frac{\pi}{N} \hat{\omega}) \Psi_a \overline{\Psi}_a \right] \right). \tag{26}$$

The integration is performed in the space of matrices $Q$ of the form (25) which is equivalent to the linear constraint

$$\gamma Q \gamma^{-1} = -Q^T. \tag{27}$$

More precisely, the integral is taken along the real subspace in the complex space (27) where $Q_B$ is real, $Q_F$ is purely imaginary, and $\overline{X} = X^*$ (for convergence of the integral).

After integrating in $\Psi$, we arrive at

$$Z(\omega_B, \omega_F) = \int DQ \left[ \text{SDet}(Q - \frac{\pi}{N} \hat{\omega}) \right]^{-N/2} \exp \left( -\frac{N}{2} S \text{Tr} \frac{Q^2}{2} \right). \tag{28}$$

Since we are interested in small energy scales (of the order of several level spacings from zero), we can expand the action to terms linear in $\omega$ and obtain

$$Z(\omega_B, \omega_F) = \int DQ \exp \left( -\frac{N}{2} S \text{Tr} \left[ \frac{Q^2}{2} + \ln Q \right] + S \text{Tr} \frac{\pi}{2} \hat{\omega} Q^{-1} \right). \tag{29}$$

At large $N$, the integral is determined by the saddle points of the action

$$S_0(Q) = \text{Tr} \left( \frac{Q^2}{2} + \ln Q \right). \tag{30}$$

By varying the action, the equation of the saddle-point manifold is

$$Q^2 = -1. \tag{31}$$

By deforming the integration contour onto the saddle-point manifold, the integral reduces to

$$Z(\omega_B, \omega_F) = \int_{\Gamma} DQ \exp \left( -\frac{N}{2} S_0(Q) - S \text{Tr} \frac{\pi}{2} \hat{\omega} Q \right) \tag{32}$$

(the transversal directions do not contribute to the integral because of the supersymmetry).

The contour of integration $\Gamma$ on the saddle-point manifold should be determined from the condition that the original contour of integration can be deformed onto it without making integrals divergent (see also discussion of this procedure in [1]). For the convergence of the integral (19) over the bosonic components of $\psi$ and $\psi^\dagger$, the energy $\omega_B$ must have an infinitesimal imaginary part $\text{Im} \omega_B < 0$. Then, for the convergence of the integral (32), the matrix $Q$ must satisfy $\text{Im} Q_B > 0$ at infinity on the contour $\Gamma$ if $Q_B$ is the bosonic diagonal element, as shown in (25)]. Besides, the contour $\Gamma$ must be compact in the fermionic and non-compact in the bosonic sector (see, e.g., [1, 2, 23]). It is shown in [1] that $\Gamma$ is the Riemannian symmetric superspace $\text{SpO}(2|2)/\text{GL}(1|1)$ (class $\text{Cl}D\text{III}$).

The key observation, important for taking into account the parity of $N$, is that the saddle-point manifold $\Gamma$ consists of two connected components, which are the images of the two components of the symmetry group
SpO(2|2) acting on $\Gamma$. This symmetry group acts on $Q$ by conjugation: $Q \mapsto UQU^{-1}$. Explicitly, the two connected pieces of $\Gamma$ may be obtained by rotating by the connected component of the symmetry group SpSO(2|2) the two representative matrices

$$Q_1 = \begin{pmatrix} i & -i \\ -i & i \end{pmatrix}, \quad Q_2 = \begin{pmatrix} i & -i \\ -i & i \end{pmatrix}. \quad (33)$$

The action (30) is invariant with respect to SpSO(2|2), but acquires an additional shift by $2\pi i$ between the two connected components of the saddle-point manifold. It is this property of the supersymmetric action that allows to distinguish between odd and even $N$ in the large-$N$ limit: in the even-$N$ case (class $D$, no zero levels), the contributions from the two pieces of the saddle-point manifold come with equal signs, and in the odd-$N$ case (class $B$, one zero level) — with opposite signs:

$$Z(\omega_B, \omega_F) = Z_1(\omega_B, \omega_F) + (-1)^N Z_2(\omega_B, \omega_F). \quad (34)$$

The average spectral density $\rho(\omega)$ may be found as

$$\rho(\omega) = -\frac{1}{\pi} \imath \text{Im} \frac{\partial Z(\omega_B, \omega_F)}{\partial \omega_B} \bigg|_{\omega_B=\omega_F=\omega-i0} = \frac{1}{\pi} \imath \text{Im} \frac{\partial Z(\omega_B, \omega_F)}{\partial \omega_F} \bigg|_{\omega_B=\omega_F=\omega-i0}. \quad (35)$$

To take the integral over $\Gamma$, we need to parameterize the integration contour: in this section, the parameterization involves the two commuting parameters $x, \phi$ and the two anticommuting $\xi$ and $\bar{\xi}$ (we never use complex conjugation of anticommuting variables in this paper, and so $\xi$ and $\bar{\xi}$ should be treated as independent variables). The expressions for $\rho(\omega)$ in coordinates takes the form:

$$\rho(\omega) = \text{Im} \int DQ \, Q_F \exp \left[ -\pi \omega (Q_B - Q_F) \right], \quad (36)$$

where $Q_B(x, \phi, \xi, \bar{\xi})$ and $Q_F(x, \phi, \xi, \bar{\xi})$ are diagonal matrix elements of $Q$ in a particular parameterization, and the measure of integration $DQ$ is

$$DQ = \frac{1}{2\pi} J(x, \phi, \xi, \bar{\xi}) dx \, d\phi \, d\xi \, d\bar{\xi}. \quad (37)$$

The Jacobian $J(x, \phi, \xi, \bar{\xi})$ may be found from expressing the invariant metric $\text{STr}(dQ)^2$ in coordinates

$$\text{STr}(dQ)^2 = g^{ij} dx_i dx_j \quad (38)$$

and taking its superdeterminant

$$J(\{x_i\}) = (\text{SDet} \, g^{ij})^{1/2}. \quad (39)$$

In parameterizing the saddle-point manifold we use the usual trick of splitting the rotation of the supermatrix $Q_i$ into the two rotations by even and odd generators of the supergroup [2]. Namely, we parameterize

$$Q = U_\xi Q_2 U_\xi^{-1}, \quad (40)$$

where $Q_2$ is obtained from $Q_1$ or $Q_2$ by even rotations parameterized by $x$ and $\phi$ (and without mixing between boson-bosonic and fermion-fermionic blocks), and

$$U_\xi = \exp(A), \quad (41)$$

where $A$ is an odd infinitesimal rotation linear in $\xi$ and $\bar{\xi}$.

Supersymmetric calculations of this sort often lead to singularities in superintegrals which need to be resolved by properly taking into account boundary terms (see e.g. [2, 1]). In this paper I avoid such singularities by an appropriate choice of parameterization of the odd rotation $U_\xi$. 

8
We shall also employ the symmetry relating the two components of the saddle-point manifold. Namely, conjugation by the matrix

\[
T = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 1 \\
1 & 0
\end{pmatrix}
\] (42)

transforms \(Q_1\) into \(Q_2\) (\(Q_2 = T^{-1}Q_1T\)) and the two components of the saddle-point manifold into each other.

Thus we first parameterize the component generated by \(Q_1\), and then the parameterization of the other component may be obtained by applying the operator \(T\).

The even rotations of \(Q_1\) may be parameterized as

\[
Q_z = i \begin{pmatrix}
\cosh x & -e^{i\phi} \sinh x \\
e^{-i\phi} \sinh x & -\cosh x \\
-1 & 0 \\
0 & 1
\end{pmatrix}
\] (43)

(with \(x \in [0; +\infty), \phi \in [0; 2\pi]\)).

The symmetry (27) of the matrix \(Q\) imposes a similar symmetry constraint on the matrix \(A\) in (41). The latter constraint admits four independent parameters in the boson-fermion and fermion-boson sectors of \(A\). However, when acting on \(Q_z\), only two of them are independent. At this stage we have a freedom of choosing two of the four infinitesimal rotations for our parameterization. The final result does not depend on our choice (provided the Jacobian (39) is non-degenerate), but a good choice of parameterization may considerably simplify the calculation.

We choose

\[
A = \begin{pmatrix}
\xi & 0 \\
0 & 0 \\
0 & \xi
\end{pmatrix}
\] (44)

which leads to

\[
U_\xi = \begin{pmatrix}
1 - \frac{1}{2} \xi \xi & 0 & 0 \\
0 & 1 - \frac{1}{2} \xi \xi & 0 \\
0 & 0 & 1 + \frac{1}{2} \xi \xi
\end{pmatrix}
\] (45)

The Jacobian calculation may be simplified using the simple algebraic identity [2]

\[
ds^2 = \frac{1}{2} \text{STr}(dQ)^2 = \frac{1}{2} \text{STr}(dQ_z)^2 + \frac{1}{2} \text{STr}[Q_z, \delta U_\xi]^2 + \text{STr}(\delta U_\xi [Q_z, dQ_z]),
\] (46)

where \(\delta U_\xi = U_\xi^{-1}dU_\xi\).

After some calculation, we find for the parameterization chosen

\[
ds^2 = dx^2 + \sinh^2 x \, d\phi^2 - \left[4(\cosh x + 1) + 2 \sinh^2 x \, \xi \xi \right] d\xi \, d\xi + 2i \sinh^2 x(\xi \, d\xi + \xi \, d\xi) d\phi
\] (47)

and

\[
J(x, \phi, \xi, \bar{\xi}) = \frac{1}{2} \tanh \frac{x}{2}.
\] (48)

Also, by a direct calculation,

\[
Q_{B1} = i[\cosh x - (\cosh x + 1)\xi], \quad Q_{F1} = i[1 + (\cosh x + 1)\xi].
\] (49)
Using the operator \( T \) to relate the two connected components of the saddle-point manifold, we find for the second component

\[
Q_{B2} = Q_{B1}, \\
Q_{F2} = -Q_{F1},
\]

and the Jacobian obviously remains the same (48).

As a consistency check, one may verify that

\[
Z_{\pm}(\omega, \omega) = \int \frac{1}{2\pi} J(x, \phi, \xi, \bar{\xi}) dx \, d\phi \, d\xi \, d\bar{\xi} \, \exp[-\pi \omega (Q_{B1} - Q_{F1})] \pm \exp[-\pi \omega (Q_{B1} + Q_{F1})] = 1
\]

(51)

(up to a sign).

Now the calculation of the integral (36) is easily done:

\[
\rho_1(\omega) = \Re \int_0^\infty dx \int_0^{2\pi} d\phi \int d\xi \, d\bar{\xi} \frac{1}{4\pi} \tanh \frac{x}{2} \left[ 1 + (\cosh x + 1)\xi \bar{\xi} \right] \times \exp \left( -i\pi \omega \left[ (\cosh x - 1) - 2(\cosh x + 1)\bar{\xi} \right] \right) \frac{1}{2} \delta(x) + 1,
\]

\[
\rho_2(\omega) = \Re \int_0^\infty dx \int_0^{2\pi} d\phi \int d\xi \, d\bar{\xi} \frac{1}{4\pi} \tanh \frac{x}{2} \left[ 1 + (\cosh x + 1)\xi \bar{\xi} \right] \times \exp \left( -i\pi \omega \left[ \cosh x + 1 \right] \right) \frac{1}{2} \delta(x) - \frac{\sin(2\pi \omega)}{2\pi \omega}.
\]

[all the calculations are performed up to an overall sign]. The \( \delta \)-function terms are obtained from imaginary \( 1/i\omega \) terms by shifting \( \omega \) to the lower half-plane \( \omega \to \omega - i0 \).

Combining these results with proper signs, we arrive at the final expressions (3) and (4). The asymptotic value \( \rho(\omega \to \infty) = 1 \) proves that \( \Delta \) given by (16) is indeed the average level spacing.

Note that \( \rho_1(\omega) \) appeared in Ref. [5] as the spectral density in class \( BD \) (which is the average of \( B \) and \( D \)).

### 4 Classes DIII-even and DIII-odd

For classes DIII-even and DIII-odd, the calculation is similar to that of the previous section. The saddle-point manifold again consists of two connected components, and taking their contributions with different signs distinguishes between odd and even matrix dimension.

The ensembles DIII-even and DIII-odd are symplectic (have \( \beta = 4 \)). In the calculation of the average spectral density in these ensembles, the matrix \( Q \) has dimension 4+4 and belongs to a (8+8)-dimensional linear space. The saddle-point manifold is (4+4)-dimensional.

The ensembles DIII are defined as consisting of \( 2N \times 2N \) matrices

\[
H = i \begin{pmatrix} H_1 & H_2 \\ H_2 & -H_1 \end{pmatrix},
\]

(54)

where \( H_1 \) and \( H_2 \) are real \( N \times N \) antisymmetric matrices (\( H_1^T = -H_1, H_2^T = -H_2 \)). Depending on whether \( N \) is even or odd, this defines the ensemble DIII-even or DIII-odd, respectively. The matrix elements of \( H_1 \) and \( H_2 \) are assumed to be distributed independently with a Gaussian distribution, and produce the following pair correlation function for the matrix elements of \( H \):

\[
(H_{ai,bj} H_{ai',b'j'}) = i^2 \left( \delta_{ii'} \delta_{jj'} - (1)^{i+j} \delta_{ii'} \delta_{jj'} \right) \left( \delta_{ab} \delta_{ba'} - \delta_{aa'} \delta_{bb'} \right),
\]

(55)

where the indices \( i, j \) take values 1 or 2 and distinguish between the two \( N \)-dimensional sectors in the \( 2N \)-dimensional linear space, and \( \delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) in this “1-2” space.
We express energy in the units
\[ \Delta = \frac{\sqrt{2\pi v}}{\sqrt{N}} \] (56)
(as the result of the calculation, this is the average level spacing).

The space of \( \Psi \)-vectors needs to be doubled. Instead of a single vector \( \Psi_a \) we introduce a pair of vectors \( \Psi_{1a} \) and \( \Psi_{2a} \) (here \( a \) takes values 1, \ldots, \( N \)):

\[
\Psi_{1a} = \begin{pmatrix} \psi_{1B} \\ \psi_{2B} \\ \psi_{1F}^\dagger \\ \psi_{2F} \end{pmatrix}_a, \quad \Psi_{2a} = \begin{pmatrix} \psi_{1B}^\dagger \\ -\psi_{1B} \\ \psi_{2B}^\dagger \\ -\psi_{2B} \end{pmatrix}_a, \quad \overline{\Psi}_{1a} = \begin{pmatrix} \psi_{1B} \\ -\psi_{1B} \\ \psi_{1F}^\dagger \\ -\psi_{1F} \end{pmatrix}_a, \quad \overline{\Psi}_{2a} = \begin{pmatrix} \psi_{2B} \\ -\psi_{2B} \\ \psi_{2F}^\dagger \\ -\psi_{2F} \end{pmatrix}_a. \quad (57)
\]

The two sets of vectors \( \Psi, \overline{\Psi} \) are necessary to reproduce the four terms in the interaction induced by (55).

The corresponding supermatrix \( Q \) has the form

\[
Q = \begin{pmatrix} Q_B & X_B & -Y_B & 0 \\ X_B & -Q_B & 0 & Y_B \\ -Y_B & 0 & Q_B & X_B \\ 0 & -Y_B & X_B & Q_B \end{pmatrix}
\begin{pmatrix} \sigma_1 & -\sigma_2 & \rho_1 & -\rho_2 \\ -\sigma_2 & -\sigma_1 & -\rho_2 & \rho_1 \\ \rho_1 & -\rho_2 & \sigma_1 & -\sigma_2 \\ -\rho_2 & -\rho_1 & -\sigma_2 & \sigma_1 \end{pmatrix}
\begin{pmatrix} \bar{\sigma}_1 & -\bar{\sigma}_2 & \bar{\rho}_1 & -\bar{\rho}_2 \\ -\bar{\sigma}_2 & -\bar{\sigma}_1 & -\bar{\rho}_2 & \bar{\rho}_1 \\ \bar{\rho}_1 & -\bar{\rho}_2 & \bar{\sigma}_1 & -\bar{\sigma}_2 \\ -\bar{\rho}_2 & -\bar{\rho}_1 & -\bar{\sigma}_2 & \bar{\sigma}_1 \end{pmatrix}
\begin{pmatrix} \bar{Q}_F & -\bar{X}_F & 0 & -\bar{Y}_F \\ -\bar{X}_F & \bar{Q}_F & -\bar{Y}_F & 0 \\ 0 & -\bar{Y}_F & \bar{Q}_F & -\bar{X}_F \\ -\bar{Y}_F & 0 & -\bar{X}_F & \bar{Q}_F \end{pmatrix}_{FB,PH,12} \quad (58)
\]

Equivalently, \( Q \) may be described as obeying the two linear constraints:

\[ \gamma_{12} Q \gamma_{12}^{-1} = -Q, \quad \gamma_{12} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{12}, \quad (59) \]

and

\[ \gamma_{PH} Q \gamma_{PH}^{-1} = Q^T, \quad \gamma_{PH} = \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}_{FB,PH} \quad \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{12}, \quad (60) \]

where \( FB \) and \( PH \) indices specifies that the operator acts in the Fermi-Bose and “particle-hole” spaces (the doubling of dimension by combining \( \psi \) and \( \psi^\dagger \) in a single vector \( \Psi \)), and “1-2” denotes the space corresponding to the two \( N \)-dimensional sectors in the original Hamiltonian (54).

Similarly to the previous section, we find for the generating function \( Z(\omega_B, \omega_F) \)

\[
Z(\omega_B, \omega_F) = \int D\Psi \exp \left( -\frac{1}{4} \text{Str} \left[ \frac{i\pi \sqrt{2}}{\sqrt{N}} \overline{\psi_{1a}} \overline{\psi_{1a}} \omega + \frac{1}{2} (\psi_{1a} \overline{\psi_{1a}})^2 \right] \right) =
\]

\[
= \int DQ \int D\Psi \exp \left( -\frac{1}{2} \text{Str} \left[ \frac{N}{2} Q^2 + i\sqrt{\frac{N}{2}} (Q - \frac{\pi}{N} \omega) \psi_{1a} \overline{\psi_{1a}} \right] \right) =
\]

\[
= \int DQ \left[ \text{SDet} (Q - \frac{\pi}{N} \omega) \right]^{-N/2} \exp \left( \frac{N}{2} \text{Str} \frac{Q^2}{2} \right), \quad (61)
\]

11
where
\[
\hat{\omega} = \begin{pmatrix} \omega_B \\ \omega_F \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in_{12}.
\] (62)

At small energies $\omega_B, \omega_F$ this leads to the formulas (29)–(32), albeit with the new definitions of $Q$ and $\hat{\omega}$.

The saddle-point manifold consists of the two connected pieces represented by

\[
Q_1 = \begin{pmatrix} i & 0 \\ 0 & -i \\ 0 & i \\ i & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} -i & 0 \\ 0 & i \\ 0 & -i \\ -i & 0 \end{pmatrix}.
\] (63)

Similarly to the procedure described in the previous section, first the supermatrices $Q_1, Q_2$ are rotated by even symmetry-group generators. These rotations do not mix bosonic and fermionic components, i.e. the matrix $Q_z$ contains only boson-boson and fermion-fermion blocks:

\[
Q_z = \begin{pmatrix} Q_z^{(BB)} \\ Q_z^{(FF)} \end{pmatrix}.
\] (64)

We shall use the following parameterization of these blocks:

\[
Q_z^{(BB)} = \begin{pmatrix} i \cosh \theta_B & n_1 \sinh \theta_B & (n_2 - in_3) \sinh \theta_B & 0 \\ n_1 \sinh \theta_B & -i \cosh \theta_B & 0 & -(n_2 - in_3) \sinh \theta_B \\ (n_2 + in_3) \sinh \theta_B & 0 & n_1 \sinh \theta_B & i \cosh \theta_B \\ 0 & -(n_2 + in_3) \sinh \theta_B & n_1 \sinh \theta_B & i \cosh \theta_B \end{pmatrix}_{PH,12}.
\] (65)

where $(n_1, n_2, n_3)$ is a vector of a real two-dimensional unit sphere ($n_1^2 + n_2^2 + n_3^2 = 1$). The boson-boson block is the same for the two sectors of the saddle-point manifold.

The fermion-fermion blocks for the two components of the saddle-point manifold are:

\[
Q_z^{(FF)} = \begin{pmatrix} 0 & 0 & 0 & -ie^{i\theta_F} \\ 0 & 0 & -ie^{-i\theta_F} & 0 \\ 0 & -ie^{-i\theta_F} & 0 & 0 \\ -ie^{-i\theta_F} & 0 & 0 & 0 \end{pmatrix}, \quad Q_z^{(FF)} = \begin{pmatrix} -i \cos \theta_F & -i \sin \theta_F & 0 & 0 \\ -i \sin \theta_F & i \cos \theta_F & 0 & 0 \\ 0 & 0 & -i \sin \theta_F & -i \cos \theta_F \\ 0 & 0 & i \cos \theta_F & -i \sin \theta_F \end{pmatrix}_{PH,12}.
\] (66)

The parameter range is $\theta_B \in [0; +\infty), \theta_F \in [0; 2\pi]$, and the vector $\mathbf{n}$ runs over the two-dimensional unit sphere $S^2$.

Like in the previous section, we first do the calculation in the first component of the saddle-point manifold, and then obtain the answers for the second component by using the symmetry operator $T$ mapping one component (generated by $Q_1$) onto the other (generated by $Q_2$). One possible choice of such a matrix $T$ is

\[
T = \begin{pmatrix} 1 & 1 & 1 \\ 1/2 & -i/2 & -i/2 & 1/2 \\ -i/2 & 1/2 & 1/2 & i/2 \\ -i/2 & 1/2 & -i/2 & i/2 \\ 1/2 & i/2 & i/2 & 1/2 \end{pmatrix}_{FB,PH,12}.
\] (67)
Returning to the parameterization for the first component of the saddle-point manifold, the matrix $A$ involved in the odd rotation $(40,41)$ is chosen as follows:

$$A = \begin{pmatrix}
\xi & \nu & 0 & 0 \\
-\nu & \xi & 0 & 0 \\
\bar{\nu} & \bar{\xi} & 0 & 0 \\
-\bar{\xi} & \bar{\nu} & 0 & 0
\end{pmatrix}.$$

(68)

This matrix satisfies the flatness condition $[A, dA] = 0$, and this leads to $\delta U_\xi = U_\xi^{-1} dU_\xi = dA$. Using the algebraic identity (46), together with (38), (39), and

$$DQ = \frac{1}{(2\pi)^2} J(\theta_B, \theta_F, n, \xi, \nu, \bar{\nu}) d\theta_B d\theta_F d^2n d\xi d\nu d\bar{\nu},$$

one finds after some calculation the explicit form for the invariant measure in the coordinates chosen:

$$DQ = \frac{1}{16\pi^2} e^{-2i\theta_F} \sinh^2 \theta_B d\theta_B d\theta_F d^2n d\xi d\nu d\bar{\nu}.$$

(here $d^2n$ is the integration over the solid angle on the unit sphere).

The explicit expressions for the diagonal entries of the $Q$-matrix $Q_B$ and $Q_F$ are found to be

$$Q_{B1} = i[\cosh \theta_B + e^{i\theta_F}(\bar{\xi}\xi - \bar{\nu}\nu)],$$
$$Q_{F1} = -ie^{i\theta_F}(\xi\bar{\xi} + \bar{\nu}\nu).$$

(71)

Using the operator $T$ defined in (67), for the second component of the saddle-point manifold we find

$$Q_{B2} = Q_{B1},$$
$$Q_{F2} = i[\cos \theta_F + 2\xi\bar{\xi}\nu\nu e^{i\theta_F} + \cosh \theta_B(\bar{\nu}\nu - \xi\bar{\xi}) + in_1 \sinh \theta_B(\bar{\nu}\xi - \bar{\xi}\nu) - (n_2 + in_3)\nu\xi \sin \theta_B + (n_2 - in_3)\nu\bar{\xi} \sin \theta_B].$$

(72)

After some calculation, one verifies the normalization:

$$Z_1(\omega, \pm \omega) = \int dQ \exp[-2\pi\omega(Q_{B1} \mp Q_{F1})] = 0,$$
$$Z_2(\omega, \pm \omega) = \int dQ \exp[-2\pi\omega(Q_{B2} \mp Q_{F2})] = 1.$$

(73)

The density of states is found in terms of Bessel functions:

$$\rho_1(\omega) = \text{Im} \int dQ Q_{F1} \exp[-2\pi\omega(Q_{B1} - Q_{F1})] = \frac{1}{2}\delta(\omega) - \frac{\pi}{2} J_1(2\pi\omega)$$

(74)

in the first sector, and

$$\rho_2(\omega) = \text{Im} \int dQ Q_{B2} \exp[-2\pi\omega(Q_{B2} - Q_{F2})] = \frac{1}{2}\delta(\omega) + \pi^2 \omega \left[ J_1'(2\pi\omega)J_0(2\pi\omega) + J_1^2(2\pi\omega) \right].$$

(75)

where in (74) and (75) we assumed $\omega > 0$ and are careless about the overall sign of the answers.

Taking these contributions with proper signs, we obtain the answers (5) and (6). Remarkably, in (6), the contributions of $\rho_1(\omega)$ and of $\rho_2(\omega)$ cancel each other to the third order at small $\omega$, producing the correct behaviour of the total density of states $\rho(\omega) \propto \omega^5$.  

13
5 Class AIII (chiral unitary)

The supersymmetric calculations for the three chiral random-matrix ensembles differ from those for classes $B-D$ and $DIII$ in that the broken symmetry of the saddle-point manifold is not the discrete $\mathbb{Z}_2$, but the continuous $\text{GL}(1)$. The representations of $\text{GL}(1)$ are enumerated by the integer winding number whose absolute value equals the number of zero levels in the random-matrix ensemble.

The chiral unitary ensemble considered in this section has $\beta = 2$ (unitary bulk statistics). In the calculation of the average density of states, the supermatrix $Q$ has the block form (in the “1-2” space)

$$Q = \begin{pmatrix} 0 & Q_1 \\ Q_2 & 0 \end{pmatrix},$$

(76)

where $Q_1$ and $Q_2$ are $(1+1)$-dimensional supermatrices without linear constraints. Thus the linear space of matrices $Q$ is 4+4-dimensional, and the saddle-point manifold has dimension 2+2.

The (Gaussian) chiral unitary ensemble (class AIII in Cartan notation) consists of the matrices of the form

$$H = \begin{pmatrix} 0 & \tilde{H} \\ \tilde{H}^\dagger & 0 \end{pmatrix},$$

(77)

where $\tilde{H}$ is a rectangular matrix $p \times q$ with complex matrix elements. The matrix elements of $\tilde{H}$ have independent Gaussian distributions:

$$dP(H) \propto \prod_{a,b} \exp \left( -\frac{|\tilde{H}_{ab}|^2}{\nu^2} \right) d\text{Re}\tilde{H}_{ab} d\text{Im}\tilde{H}_{ab}.$$  

(78)

The spectrum of such a matrix $H$ consist of $N = \min(p,q)$ pairs of eigenvalues $\pm E_i$ and of $m = |p - q|$ zero eigenvalues. In this paper we are interested in the average density of states near zero in the limit of large $Q$, where $\beta = 2$.

At small energies, expanding the supersymmetric action to terms linear in $\omega_B, \omega_F$, we find

$$Z(\omega_B, \omega_F) = \int D\psi \exp \left( -\text{STr} \left[ i\frac{\pi}{2\sqrt{N}} (\psi_{1a} \overline{\psi}_{1a} + \psi_{2a} \overline{\psi}_{2a}) \hat{\omega} + (\psi_{1a} \overline{\psi}_{1a}) (\psi_{2b} \overline{\psi}_{2b}) \right] \right) =$$

$$= \int D(Q_1, Q_2) \int D\psi \exp \left( -\text{STr} \left[ NQ_1Q_2 + i\sqrt{N} \left( Q_1 - \frac{\pi}{2N} \hat{\omega} \right) \psi_{2b} \overline{\psi}_{2b} + i\sqrt{N} \left( Q_2 - \frac{\pi}{2N} \hat{\omega} \right) \psi_{1a} \overline{\psi}_{1a} \right] \right) =$$

$$= \int D(Q_1, Q_2) \left[ \text{SDet}(Q_1 - \frac{\pi}{2N} \hat{\omega}) \right]^{-q} \left[ \text{SDet}(Q_2 - \frac{\pi}{2N} \hat{\omega}) \right]^{-p} \exp (-N \text{STr} Q_1Q_2),$$

(82)

where the original integration contour in $Q$ is at $Q_2 = Q_1^\dagger$, and

$$\hat{\omega} = \left( \frac{\omega_B}{\omega_F} \right).$$

(83)

At small energies, expanding the supersymmetric action to terms linear in $\omega_B, \omega_F$, we find

$$Z(\omega_B, \omega_F) = \int DQ \ [\text{SDet} Q_1]^m \exp \left[ -NS_0(Q_1, Q_2) + \frac{\pi}{2} \text{STr} \hat{\omega}(Q_1^{-1} + Q_2^{-1}) \right]$$

(84)
with
\[ S_0(Q_1, Q_2) = \text{STr} \left( Q_1 Q_2 + \ln Q_1 + \ln Q_2 \right). \] (85)

If \( Q_1 \) and \( Q_2 \) are combined into a single supermatrix \( Q \) according to (76), this action coincides with the standard form (30). The saddle-point manifold \( \Gamma \) is given by the condition
\[ Q_1 Q_2 = -1 \] (86)

[which is equivalent to \( Q^2 = -1 \)], and the generating function may be written as the integral over the saddle-point manifold
\[ Z_m(\omega_B, \omega_F) = \int_\Gamma DQ \left[ \text{SDet} Q_1 \right]^m \exp \left[ -\frac{\pi}{2} \text{STr} \hat{\omega}(Q_1 + Q_2) \right]. \] (87)

A parameterization and the calculation of the integral was previously performed in Ref. [6], and I outline their calculation here for completeness. The matrices \( Q_1 \) and \( Q_2 \) are parameterized as
\[ Q_1 = -Q_2^{-1} = \begin{pmatrix} ie^x & 0 \\ 0 & ie^{i\phi} \end{pmatrix} \exp \left( \frac{0}{\xi} \right). \] (88)

The invariant measure deduced from the metric \( ds^2 = \text{STr}(Q_1 Q_2) \) leads to the trivial Jacobian \( J = 1 \) and to the integration measure
\[ DQ = \frac{1}{2\pi} dx d\phi d\xi d\bar{\xi}. \] (89)

From the parameterization (88), the diagonal elements of the matrices \( Q_1 \) and \( Q_2 \) are easily computed. The generating function (87) involves the average of the diagonal elements of \( Q_1 \) and \( Q_2 \):
\[ Q_{BB} = \frac{1}{2} \left[ (Q_1)_{BB} + (Q_2)_{BB} \right] = i \cosh x \left[ 1 - \frac{1}{2} \xi \right], \]
\[ Q_{FF} = \frac{1}{2} \left[ (Q_1)_{FF} + (Q_2)_{FF} \right] = i \cos \phi \left[ 1 + \frac{1}{2} \xi \right]. \] (90)

Also, the \( m \)-dependent prefactor in (87) is a plane wave generated by
\[ \text{SDet} Q_1 = e^{x-i\phi}. \] (91)

The normalization can be verified by computing the integral
\[ Z_m(\omega, \omega) = \int \frac{1}{2\pi} dx d\phi d\xi d\bar{\xi} e^{m(x-i\phi)} \exp \left[ -i\pi \omega \left( \cosh x \left[ 1 - \frac{1}{2} \xi \right] + \cos \phi \left[ 1 + \frac{1}{2} \xi \right] \right) \right] = \pm 1. \] (92)

The average spectral density is calculated from (35) as the integral (again, up to an overall sign):
\[ \rho_m(\omega) = \text{Re} \int \frac{1}{2\pi} dx d\phi d\xi d\bar{\xi} e^{m(x-i\phi)} \cosh x \left[ 1 - \frac{1}{2} \xi \right] \exp \left[ -i\pi \omega \left( \cosh x \left[ 1 - \frac{1}{2} \xi \right] + \cos \phi \left[ 1 + \frac{1}{2} \xi \right] \right) \right] \]
which after some algebra produces the result (7).

6 Class BDI (chiral orthogonal)

Similarly to the three Wigner–Dyson random-matrix ensembles, the chiral ensembles form the three classes: unitary, orthogonal, and symplectic, depending on the structure of the matrix \( \hat{H} \) in the block form (77). In the chiral orthogonal ensemble (class BDI in Cartan notation), the matrix \( \hat{H} \) is real. In this ensemble, the bulk repulsion of the levels corresponds to the orthogonal regime: \( \beta = 1 \). The supermatrix \( Q \) involved in the calculation of the average spectral density has the block form (76), but now the matrices \( Q_1 \) and \( Q_2 \) have
dimensions 2+2 each, with one linear constraint, so the dimension of the linear space of all the matrices $Q$ is 8+8. The saddle-point equation selects the saddle-point manifold of dimension 4+4.

The entries of the matrix $\tilde{H}$ in (77) are assumed to be real, with independent Gaussian distributions:

$$dP(H) \propto \prod_{a,b} \exp\left(-\frac{|\tilde{H}_{ab}|^2}{2v^2}\right) d\tilde{H}_{ab}.$$  \hspace{1cm} (94)

As in the chiral unitary ensemble, the spectrum consists of $N$ pairs of opposite energies, and of $m = |p - q|$ zero-energy levels. The average level spacing is given by the same expression (79) [note however a difference between the definitions of $v$ in (78) and in (94), depending on whether $\tilde{H}_{ab}$ are complex or real].

To account for the matrix elements $\tilde{H}_{ab}$ being real, we need to double the dimensions of the superfields $\Psi_i$ and $\overline{\Psi}_i$. The $p$-component superfields $\Psi_1$ and $\overline{\Psi}_1$ are defined as

$$\Psi_1a = \begin{pmatrix} \psi_{1B}^a \\ \psi_{1F}^a \\ \psi_{2F}^a \end{pmatrix}, \quad \overline{\Psi}_1a = \begin{pmatrix} \psi_{1B}^{\dagger} \\ \psi_{1F}^{\dagger} \\ -\psi_{1F}^{\dagger} \end{pmatrix}^T, \quad a = 1, \ldots, p,$$

and similarly, the $q$-component superfields $\Psi_2$ and $\overline{\Psi}_2$:

$$\Psi_2b = \begin{pmatrix} \psi_{2B}^b \\ \psi_{2F}^b \\ \psi_{2F}^b \end{pmatrix}, \quad \overline{\Psi}_2b = \begin{pmatrix} \psi_{2B}^{\dagger} \\ \psi_{2F}^{\dagger} \\ -\psi_{2F}^{\dagger} \end{pmatrix}^T, \quad b = 1, \ldots, q.$$

Repeating the steps of the derivation (82), we arrive at the following expression for the generating function [in place of (84)]:

$$Z_m(\omega_B, \omega_F) = \int DQ \left[ S\text{Det} Q_1 \right]^{m/2} \exp \left[ -\frac{N}{2} S_0(Q_1, Q_2) + \frac{\pi}{4} S\text{Tr} \hat{\omega}(Q_1^{-1} + Q_2^{-1}) \right],$$  \hspace{1cm} (97)

where $\hat{\omega}$ and $S_0(Q_1, Q_2)$ are given by the old expressions (83) and (85). However, the matrices $Q_1$ and $Q_2$ are now two times bigger. Each of them has the explicit form:

$$Q_i = \begin{pmatrix} Q_{Bi} & -X_i & \sigma_i & -\rho_i \\ X_i & Q_{Bi} & \bar{\sigma}_i & -\bar{\rho}_i \\ \sigma_i & \rho_i & Q_{Fi} & 0 \\ \bar{\rho}_i & \bar{\sigma}_i & 0 & Q_{Fi} \end{pmatrix}.$$  \hspace{1cm} (98)

Equivalently, this form of the matrices $Q_i$ may be described by the linear constraints

$$\gamma Q_i \gamma^{-1} = Q_i^T,$$

where

$$\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}. $$  \hspace{1cm} (100)

The saddle-point manifold $\Gamma$ is determined by the condition (86), and the generating function is expressed as

$$Z_m(\omega_B, \omega_F) = \int_{\Gamma} DQ \left[ S\text{Det} Q_1 \right]^{m/2} \exp \left[ -\frac{\pi}{4} S\text{Tr} \hat{\omega}(Q_1 + Q_2) \right].$$  \hspace{1cm} (101)
In the present section we choose a slightly different form of parameterization than in the previous one. Namely, parameterize
\[ Q_1 = U_1 Q_{z1} U_2^{-1}, \quad Q_2 = U_2 Q_{z2} U_1^{-1}, \]
where the matrices \( Q_{z1} \) and \( Q_{z2} \) contain only boson-boson and fermion-fermion blocks (contain only even rotations), and the matrices \( U_1 \) and \( U_2 \) contain only odd rotations. The explicit form of this parameterization is as follows:

\[
Q_{z1} = \begin{pmatrix}
  e^x \cosh \theta & e^{x+iy} \sinh \theta \\
  e^{x-iy} \sinh \theta & e^x \cosh \theta
\end{pmatrix}, \quad Q_{z2} = \begin{pmatrix}
  e^{-x} \cosh \theta & -e^{-x+iy} \sinh \theta \\
  -e^{-x-iy} \sinh \theta & e^{-x} \cosh \theta
\end{pmatrix}
\]

\[ U_1 = \exp(A_1), \quad U_2 = \exp(A_2), \]

\[ A_1 = \begin{pmatrix}
  -\xi & \lambda \\
  -\bar{\lambda} & \bar{\xi}
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
  \bar{\xi} & \lambda \\
  \bar{\lambda} & \bar{\xi}
\end{pmatrix}.
\]

After some calculation, the Jacobian is found to be
\[ J = \sinh \theta e^{2(x-i\phi)}, \]

and therefore the measure of integration is
\[ DQ = \frac{1}{(2\pi)^2} \sinh \theta e^{2(x-i\phi)} \, dx \, dy \, d\theta \, d\bar{\xi} \, d\xi \, d\bar{\lambda} \, d\lambda. \]

Finally, from the explicit calculation of the diagonal elements of \( Q_1 \) and \( Q_2 \):

\[
Q_{BB} = \frac{1}{2} \left[ (Q_1)_{BB} + (Q_2)_{BB} \right] = i \left[ \cosh x \cosh \theta - \frac{1}{2} e^{i\phi} (\xi - \bar{\lambda}) \right],
\]

\[
Q_{FF} = \frac{1}{2} \left[ (Q_1)_{FF} + (Q_2)_{FF} \right] = i \left[ \cos \phi + \frac{1}{2} e^{-x} \left( \cosh \theta |\xi - \bar{\lambda}| - \sinh \theta [e^{iy} \bar{\xi} + e^{-iy} \lambda] \right) \right].
\]

Now, after verifying the normalization
\[ Z_m(\omega, \pm \omega) = \int DQ e^{m(x-i\phi)} \exp[-\pi\omega(Q_{BB} \pm Q_{FF})] = 1, \]

we compute the average density of states as the following integral:
\[ \rho_m(\omega) = \text{Re} \int DQ e^{m(x-i\phi)} Q_{BB} \exp[-\pi\omega(Q_{BB} + Q_{FF})], \]

[where \( DQ, Q_{BB}, \) and \( Q_{FF} \) are defined in (107) and (108)]. After some algebra and manipulations with Bessel functions, this produces the result (8).

7 Class CII (chiral symplectic)

The last symmetry class considered in this paper is CII in Cartan notation: the chiral symplectic one. It consists of the matrices \( H \) of the block form (77), where the matrix \( \bar{H} \) has an internal \( 2 \times 2 \) structure:

\[ \bar{H} = \begin{pmatrix}
  a & b \\
  -b^* & a^*
\end{pmatrix}, \]
where $a$ and $b$ are rectangular $p \times q$ matrices [the matrix $\tilde{H}$ thus has the dimensions $2p \times 2q$, and the Hamiltonian (77) has the dimension $2(p + q)$]. The spectrum of such a Hamiltonian consists of the $N = \min(p, q)$ pairs of doubly degenerate levels at opposite energies $\pm E_i$, and of $2m = 2|p - q|$ zero-energy levels. Since each level has degeneracy two (or a multiple of two), we divide the density of states by two for the purpose of level counting, and count each degenerate level as a single one.

As in the two previous sections, the supermatrix $Q$ is of the form (76). Similarly to the chiral orthogonal case, the matrices $Q_1$ and $Q_2$ in this calculation have dimension $2+2$, with one linear constraint. The dimension of the linear space of matrices $Q$ is thus $8+8$, and the dimension of the saddle-point manifold is $4+4$.

The entries of the matrix (111) are assumed to be normalized as

$$dP(A) \propto \prod_{a,b} \exp \left( -\frac{|g_{ab}|^2}{v^2} \right) d\text{Re}a_b \, d\text{Im}a_b \, \prod_{a,b} \exp \left( -\frac{|b_{ab}|^2}{v^2} \right) d\text{Re}b_{ab} \, d\text{Im}b_{ab}. \quad (112)$$

With this normalization, the average level spacing is

$$\Delta = \frac{\pi v}{\sqrt{2N}}. \quad (113)$$

Similarly to the DIII ensemble discussed before, we need to additionally double the set of the superfields to account for the symplectic matrix structure. Namely, in addition to the “1-2” sector, distinguishing between the size-$p$ and size-$q$ blocks in (77), we introduce the “particle-hole” (PH) sector referring to the two sectors inside the matrix $A$ (111). Thus we arrive to the four pairs of superfields $\Psi_{P1}$, $\Psi_{P2}$, $\Psi_{H1}$, $\Psi_{H2}$, $\Psi_{H1}$, $\Psi_{H2}$:

$$\Psi_{P1a} = \begin{pmatrix} \psi_{P1B}^1 \\ \psi_{P1F}^1 \\ \psi_{P1B}^2 \\ \psi_{P1F}^2 \end{pmatrix}, \quad \Psi_{P1a} = \begin{pmatrix} \psi_{P1B}^1 \\ \psi_{P1F}^1 \\ -\psi_{P1B}^2 \\ -\psi_{P1F}^2 \end{pmatrix}, \quad \Psi_{H1a} = \begin{pmatrix} \psi_{H1B}^1 \\ -\psi_{H1B}^2 \\ \psi_{H1F}^1 \\ -\psi_{H1F}^2 \end{pmatrix}, \quad \Psi_{H1a} = \begin{pmatrix} \psi_{H1B}^1 \\ -\psi_{H1B}^2 \\ -\psi_{H1F}^1 \\ -\psi_{H1F}^2 \end{pmatrix}, \quad a=1,\ldots,p, \quad (114)$$

and similarly for the sector-2 fields $\Psi_{P2}$, $\Psi_{P2}$, $\Psi_{H2}$, $\Psi_{H2}$.

This results in the following form of matrices $Q_i$:

$$Q_i = \begin{pmatrix} Q_{Bi} & 0 & -\sigma_i & \bar{\rho}_i \\ 0 & Q_{Bi} & \rho_i & \bar{\sigma}_i \\ -\bar{\rho}_i & \bar{\sigma}_i & X_i & Q_{Fi} \\ \rho_i & \sigma_i & X_i & Q_{Fi} \end{pmatrix}_{FB,PH}, \quad (115)$$

which differs from the chiral orthogonal ensemble by interchanging the bosonic and fermionic sectors (this duality was already described in Ref. [1]).

Performing the standard steps of the derivation [similarly to (82)], we arrive at the following answer in the saddle-point approximation:

$$Z_m(\omega_B, \omega_F) = \int DQ \, [\text{SDet} \, Q_1]^m \exp \left[ -\frac{\pi}{2} \text{STr} \, \hat{\omega} (Q_1 + Q_2) \right]. \quad (116)$$

[Note the differences from the orthogonal result (101)!]

Finally, the calculation may be performed using the parameterization obtained from that of the previous section by interchanging bosonic and fermionic sectors. Note that under this interchange, the compact variables become non-compact and vice versa. Explicitly, eqs. (103), (105)–(107) are replaced by

$$Q_{z1} = i \begin{pmatrix} e^x & 0 & e^{i \phi} \cos \theta & e^{i(\phi + y)} \sin \theta \\ 0 & e^x & -e^{i(\phi - y)} \sin \theta & e^{i \phi} \cos \theta \\ e^{i \phi} \cos \theta & e^{i(\phi + y)} \sin \theta & e^{-x} & 0 \\ -e^{i(\phi - y)} \sin \theta & e^{i \phi} \cos \theta & 0 & e^{-x} \end{pmatrix}, \quad Q_{z2} = i \begin{pmatrix} e^{-x} & 0 & e^{-i \phi} \cos \theta & e^{-i(\phi + y)} \sin \theta \\ 0 & e^{-x} & e^{-i(\phi - y)} \sin \theta & e^{-i \phi} \cos \theta \\ e^{-i \phi} \cos \theta & e^{-i(\phi + y)} \sin \theta & e^x & 0 \\ e^{-i(\phi - y)} \sin \theta & e^{-i \phi} \cos \theta & 0 & e^x \end{pmatrix}. \quad (117)$$
\[
A_1 = \begin{pmatrix}
\xi & \lambda \\
-\bar{\lambda} & -\bar{\xi}
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
\xi & \lambda \\
\bar{\lambda} & \bar{\xi}
\end{pmatrix},
\]

\[
DQ = \frac{1}{(2\pi)^2} \sin \theta e^{2(x-i\phi)} dx \, dy \, d\theta \, d\phi \, d\xi \, d\bar{\lambda} \, d\lambda.
\]

Eq. (108) is replaced by
\[
Q_{BB} = \frac{1}{2} [(Q_1)_{BB} + (Q_2)_{BB}] = i \left[ \cosh x + \frac{1}{2} e^{i\phi} (\cos \theta [\xi \bar{\xi} + \lambda \bar{\lambda}] + \sin \theta [e^{iy}\bar{\lambda} x + e^{-iy}\bar{\xi}]) \right]
\]
\[
Q_{FF} = \frac{1}{2} [(Q_1)_{FF} + (Q_2)_{FF}] = i \left[ \cos \phi \cos \theta - \frac{1}{2} e^{-x} (\bar{\xi} \xi + \bar{\lambda} \lambda) \right].
\]

Taking the integral
\[
\rho_m(\omega) = \Re \int DQ e^{2m(x-i\phi)} Q_{BB} \exp[-2\pi \omega (Q_{BB} + Q_{FF})],
\]
we obtain the final result (9). [Note that in (121) we divided the density of states by two to prevent double counting of the doubly degenerate states.]

8 Zero levels and reduced supersymmetry of the action

After presenting the calculations of the average density of states in the five random-matrix ensembles with zero levels, in this section we discuss the specifics of the supersymmetric method due to the zero levels. I do not present here a consistent mathematical analysis of this problem, leaving it for future study. Instead I only summarize the common features of the above calculations specific for the ensembles with zero levels.

The standard supersymmetric procedure to calculate spectral correlation functions in any random-matrix ensemble starts with introducing bosonic and fermionic fields $\psi_B$ and $\psi_F$ [1, 2]. Integrating over the Gaussian disorder produces a four-term interaction. This interaction is then decoupled via Hubbard–Stratonovich transformation by a supermatrix $Q$ whose dimension is independent on the matrix size $N$ in the original random-matrix ensemble. Integrating over the superfields $(\psi_B, \psi_F)$, one arrives at an effective action for the supermatrix $Q$. The supermatrix $Q$ obeys certain linear symmetry relations and thus belongs to a linear superspace depending on the symmetries of the original random-matrix ensemble. In the superspace $L$, there acts a supergroup $G$ inherited from the supersymmetry mixing the bosonic and fermionic fields $\psi_B$ and $\psi_F$.

Namely, an element $g \in G$ is a supermatrix acting on $Q$ by conjugation:
\[
g : Q \mapsto gQg^{-1}.
\]

The group $G$ then depends on the symmetries of the matrix $Q$ and hence on the symmetry of the random-matrix ensemble.

The effective action $S(Q)$ may, at small energies $\omega$, be expanded to the linear in $\omega$ term:
\[
S(Q) = S_{\omega=0}(Q) + \omega \text{STr} \Lambda Q^{-1},
\]
where $\Lambda \in L$ is a particular supermatrix with $\Lambda^2 = 1$.

In ensembles without zero eigenvalues, the zero-energy action $S_{\omega=0}(Q)$ is invariant with respect to the supergroup $G$. The zero-energy action $S_{\omega=0}(N)$ scales linearly with $N$, and in the large-$N$ limit the integral over $Q$ is determined by the saddle points of $S_{\omega=0}(Q)$. The saddle-point equation, with the appropriate normalization, reads
\[
Q^2 = -1.
\]

This equation is solved by $Q = i\Lambda$, as well as by any matrix obtained from $i\Lambda$ by $G$-rotations (122). The matrix $\Lambda$ is invariant under a subgroup $H \subset G$, and the saddle-point manifold $\Gamma$ is the quotient $G/H$. 
the logarithm of the pre-exponent in (87), (101), (116) is of the form $S$ zero eigenvalues in the random-matrix ensemble. The action $-\interchanging fermionic and bosonic sectors, in the supergroup $OSp(2\times2)$Z_2$.

Where

1. **Class**: $B-D$
2. **L**: $osp(2|2)$
3. **G**: $SpO(2|2)$
4. **H**: $GL(1|1)$
5. **$\Gamma = G/H$**: $SpO(2|2)/GL(1|1)$
6. **$G_0$**: $SpSO(2|2)$
7. **$G/G_0$**: $Z_2$

DIII

1. **L**: $osp(4|4)/\oplus osp(2|2)\oplus osp(2|2)$
2. **G**: $SpO(2|2)\times SpO(2|2)$
3. **H**: $SpO(2|2)$
4. **$\Gamma = G/H$**: $SpO(2|2)/SpO(2|2)$
5. **$G_0$**: $SpSO(2|2)$
6. **$G/G_0$**: $Z_2$.

AII

1. **L**: $gl(1|1)\oplus gl(1|1)$
2. **G**: $GL(1|1)\times GL(1|1)$
3. **H**: $GL(1|1)$
4. **$\Gamma = G/H$**: $GL(1|1)$
5. **$G_0$**: $SL(2|2)$
6. **$G/G_0$**: $SL(2|2)$

BDI

1. **L**: $[g[2|2]/\oplus osp(2|2)\oplus [g[2|2]/\oplus osp(2|2)]$
2. **G**: $GL(2|2)$
3. **H**: $OSp(2|2)$
4. **$\Gamma = G/H$**: $GL(2|2)/OSp(2|2)$
5. **$G_0$**: $SL(2|2)$
6. **$G/G_0$**: $SL(2|2)$

CII

1. **L**: $[g[2|2]/\oplus osp(2|2)\oplus [g[2|2]/\oplus osp(2|2)]$
2. **G**: $GL(2|2)$
3. **H**: $SpO(2|2)$
4. **$\Gamma = G/H$**: $GL(2|2)/SpO(2|2)$
5. **$G_0$**: $SL(2|2)$
6. **$G/G_0$**: $SL(2|2)$

* $H$ is diagonal in $G$: $H = \{g, g\}$ in classes DIII and AIII
** $S[H \times H]$ denotes here the subgroup $\{(g_1, g_2)|SDet g_1 = SDet g_2\}$
*** $SL(2|2) = \{g \in GL(2|2)|SDet g = 1\}$; $SL(2|2) = \{g \in GL(2|2)|SDet g = \pm 1\}$

More precisely, $\Gamma$ is a Riemannian (real) supermanifold in $G/H$, which makes it a Riemannian symmetric superspace as defined in Ref. [1]. This real submanifold should be determined geometrically from deforming the real integration subspace in $L$ while keeping the integral convergent. A good geometric understanding of this contour deformation still needs to be developed, but the rule of thumb for choosing the real integration manifold in $\Gamma$ is to take the bosonic sector non-compact and the fermionic one compact (this choice also provides a metric of a definite sign on $\Gamma$).

For the random-matrix ensembles with zero eigenvalues, the invariance properties of the effective action $S_{\omega=0}(Q)$ are modified. In this case, $S_{\omega=0}(Q)$ is invariant with respect not to the whole supergroup $G$, but only with respect to its normal subgroup $G_0$. The subgroup $G_0$ must contain $H$, and the factorgroup $G/G_0$ is an abelian group (an ordinary group, not a supergroup). The exponent $\exp[S_{\omega=0}(Q)]$ transforms according to one of its representations. The degree of this representation equals the number of zero eigenvalues in the random-matrix ensemble.

In the ensembles $B-D$ and DIII, the group $G/G_0$ is discrete $Z_2$, and the two representations of $Z_2$ correspond to the ensembles with and without zero eigenvalues (odd $N$ and even $N$, respectively). Specifically, the action $S_{\omega=0}(Q)$ has the form

$$S_{\omega=0}(Q) = NS_0(Q),$$

where $S_0(Q)$ gets incremented by $i\pi$ under the action of the generator of $G/G_0 = Z_2$. Hence, $\exp[S_{\omega=0}(Q)]$ transforms according to the even/odd representation of $Z_2$ for even/odd $N$.

In the chiral random-matrix ensembles AIII, BDI, and CII, the group $G/G_0$ is the continuous $GL(1)$, with its representations labeled by the integer “winding number” $m$. The absolute value of $m$ equals the number of zero eigenvalues in the random-matrix ensemble. The action $S_{\omega=0}(Q)$ in the chiral ensembles [which includes the logarithm of the pre-exponent in (87), (101), (116)] is of the form

$$S_{\omega=0}(Q) = NS_0(Q) + mS_1(Q),$$

where $S_0(Q)$ is invariant under $G$, and $S_1(Q)$ produces phase shifts under $G/G_0 = GL(1)$. The exponent $\exp[S_{\omega=0}(Q)]$ then transforms as the representation of $GL(1)$ of degree $m$.

The summary of the “building blocks” of the supersymmetric calculations in this paper (the calculation of the average spectral density) is presented in Table 2. This table is compiled using the results of Ref. [1] and the calculations in the previous sections. The definitions of the supergroups involved in this table may be found in Section 2. To keep track of the bosonic (non-compact) and fermionic (compact) sectors of the supergroups, we use the notation $OSp$ for the orthosymplectic supergroup with the orthogonal part in the bosonic, and the symplectic part in the fermionic sectors. In the opposite case of orthogonal fermionic and symplectic bosonic sector, we denote the same supergroup $SpO$. It is important for reducing the action symmetry in ensembles $B-D$ and DIII, that $SpO(2n|2n)$ has two disconnected components with superdeterminants 1 and $-1$ [the former of them denoted as $SpSO(2n|2n)$]. Incidentally, in the ensembles $C$ and $CI$, dual to $B-D$ and DIII by interchanging fermionic and bosonic sectors, in the supergroup $OSp(2n|2n)$, with the non-compact orthogonal sector, the second component (with $SDet = -1$) plays no role and should be disregarded as it always corresponds

| Class | L | G | H | $\Gamma = G/H$ | $G_0$ | $G/G_0$ |
|-------|---|---|---|----------------|------|---------|
| $B-D$ | $osp(2|2)$ | $SpO(2|2)$ | $GL(1|1)$ | $SpO(2|2)/GL(1|1)$ | $SpSO(2|2)$ | $Z_2$ |
| DIII | $osp(4|4)/\oplus osp(2|2)\oplus osp(2|2)$ | $SpO(2|2)\times SpO(2|2)$ | $SpO(2|2)$ | $SpO(2|2)/SpO(2|2)$ | $SpSO(2|2)$ | $Z_2$ |
| AIII | $gl(1|1)\oplus gl(1|1)$ | $GL(1|1)\times GL(1|1)$ | $GL(1|1)$ | $GL(1|1)$ | $GL(1|1)$ | $GL(1)$ |
| BDI | $[g(2|2)/\oplus osp(2|2)\oplus [g(2|2)/\oplus osp(2|2)]$ | $GL(2|2)$ | $OSp(2|2)$ | $GL(2|2)/OSp(2|2)$ | $SL(2|2)$ | $GL(1)$ |
| CII | $[g(2|2)/\oplus osp(2|2)\oplus [g(2|2)/\oplus osp(2|2)]$ | $GL(2|2)$ | $SpO(2|2)$ | $GL(2|2)/SpO(2|2)$ | $SL(2|2)$ | $GL(1)$ |
to a divergent integral. The supergroup $\text{GL}(n|n)$ is not simple either. Firstly, it has a one-dimensional center consisting of scalar matrices. Secondly, it has a normal subgroup $\text{SL}(n|n)$ consisting of matrices with unit superdeterminant. This latter reduction of the supergroup $\text{GL}(n|n)$ is crucial for the symmetry classification in the case of the chiral ensembles (the last three lines in Table 2).

Finally, it is worth mentioning that the conclusion about the reduced supersymmetry of the zero-energy effective action $S_{\omega=0}(Q)$ may be extended to higher-order correlation functions involving averaging several Green's functions [the average spectral density requires averaging only one Green's function]. In Ref. [1] a general procedure of calculating correlation functions of arbitrary order (with the number of zero levels $m = 0$) was described, and the saddle-point manifold (a Riemannian symmetric superspace) $\Gamma$ was found to be always reducible for ensembles admitting zero levels. Thus the extension of the calculation of the present paper to higher-order correlations is straightforward (however, explicit parameterization and integral evaluation immediately becomes much more complicated).

I thank N. Nekrasov, P. Ostrovsky, and M. Skvortsov for teaching me different aspects of supersymmetry, and Swiss National Foundation for financial support.

References

[1] M. R. Zirnbauer, “Riemannian symmetric superspaces and their origin in random-matrix theory”, J. Math. Phys. 37 (1996), 4986 [math-ph/9808012].

[2] K. Efetov, “Supersymmetry in disorder and chaos”, Cambridge University Press, Cambridge 1997.

[3] S. Helgason, “Differential geometry, Lie groups, and symmetric spaces”, Academic Press, New York 1978.

[4] M. Caselle, “A new classification scheme for random matrix theories”, cond-mat/9610017.

[5] M. Bocquet, D. Serban, and M. R. Zirnbauer, “Disordered 2d quasiparticles in class D: Dirac fermions with random mass, and dirty superconductors”, Nucl. Phys. B 578 (2000), 628 [cond-mat/9910480].

[6] P. H. Damgaard, J. C. Osborn, D. Toublan, and J. J. M. Verbaarschot, “The microscopic spectral density of the QCD Dirac operator”, Nucl. Phys. B 547 (1999), 305.

[7] D. A. Ivanov, “Random-matrix ensembles in p-wave vortices”, to appear in “Vortices in unconventional superconductors and superfluids — microscopic structure and dynamics”, Springer, 2001 [cond-mat/0103089].

[8] J. Verbaarschot, “Spectrum of the QCD Dirac operator and chiral random matrix theory”, Phys. Rev. Lett. 72 (1994), 2531.

[9] J. J. M. Verbaarschot and T. Wettig, “Random matrix theory and chiral symmetry in QCD”, hep-ph/0003017.

[10] M. L. Mehta, “Random Matrices”, Academic Press, Boston 1991.

[11] T. Nagao and K. Slevin, “Laguerre ensembles of random matrices: Nonuniversal correlation functions”, J. Math. Phys. 34 (1993), 2317.

[12] A. Altland and M. R. Zirnbauer, “Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures”, Phys. Rev. B 55 (1997), 1142 [cond-mat/9602137].

[13] J. J. M. Verbaarschot and I. Zahed, “Spectral density of the QCD Dirac operator near zero virtuality”, Phys. Rev. Lett. 70 (1993), 3852.

[14] T. Nagao and K. Slevin, “Nonuniversal correlations for random matrix ensembles”, J. Math. Phys. 34 (1993), 2075.

[15] P. J. Forrester, “The spectrum edge of random matrix ensembles”, Nucl. Phys. B 402 (1993), 709.
[16] A. V. Andreev, B. D. Simons, and N. Taniguchi, “Supersymmetry applied to the spectrum edge of random matrix ensembles”, Nucl. Phys. B 432 (1994), 487.

[17] J. Verbaarschot, “The spectrum of the Dirac operator near zero virtuality for $N_c = 2$ and chiral random matrix theory”, Nucl. Phys. B 426 (1994), 559.

[18] T. Nagao and P. J. Forrester, “Asymptotic correlations at the spectrum edge of random matrices”, Nucl. Phys. B 435 (1995), 401.

[19] T. Wettig, “Random-matrix theory in quantum chromodynamics and on the lattice”, Habilitationsschrift, Ruprecht–Karls–University Heidelberg, April 1998.

[20] B. Klein and J. J. M. Verbaarschot, “Spectral universality of real random matrix ensembles”, Nucl. Phys. B 588 (2000), 483.

[21] J.-Z. Ma, T. Guhr, and T. Wettig, “Statistical properties at the spectrum edge of the QCD Dirac operator”, Eur. Phys. J. A2 (1998), 87, erratum ibid. A2 (1998), 425 [hep-lat/9712026].

[22] B. DeWitt, “Supermanifolds”, Cambridge University Press, Cambridge, 1992 (second edition).

[23] J. J. M. Verbaarschot, H. A. Weidenmüller, and M. R. Zirnbauer, “Grassmann integration in stochastic quantum physics: the case of compound-nucleus scattering”, Phys. Rep. 6 (1985), 367.