Stress concentration analysis in technical structures with boundary integral equations

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Abstract. The direct boundary integral equation (DBIE) method in the multi-domain formulation is used for three-dimensional stress concentration problems solution. The computational results of several new stress concentration problems are presented and analyzed. The following problems have been studied: flat plates with oblique, conical and round-edges holes under in-plane tensile loading for different angles of holes inclination. The stress concentration problem for an aviation engine turbine disk with cooling hole has also been considered.

1. Introduction
Holes of different shapes and orientations are widely used in the modern technologies. Through-thickness holes with axes inclined to surface (oblique holes) are used in cylindrical and spherical shells, turbine blades and disks [1, 2]. In the neighborhood of holes the stress state is triaxial in nature and the problem should be considered as the three-dimensional one. General analytical solutions for stress distributions around holes exist only for a few two-dimensional problems. For essentially three-dimensional stresses concentration problems (for example, oblique holes) analytical solutions do not exist and only experimental analysis and numerical modelling can be used. Stress concentration near one oblique hole or arrays of such holes have been analyzed by photoelasticity method [3, 4, 5]. The problems with one oblique hole in flat plates and in thick-walled cylinders were considered by finite and boundary element methods [6, 7, 8].

The purpose of this paper is the expansion of 3D computational modelling to the stress concentration analysis of some technical structures with one and two oblique holes, with conical holes for different angles of holes inclination.

2. Boundary integral equations formulation
Within the multi-domain formulation of the DBIE, the direct boundary integral equations for elasticity problems are used for each homogeneous subregion of the structure. The supplementary boundary conditions at interfacial boundaries of subregions are introduced and used to eliminate additional variables on joint boundaries of subregions. For elasticity problems without volume forces the DBIE for any homogeneous subregion of the 3D structure is given by [9]:

\[ c_{ij} (p) u_i (p) = \int_{\Gamma} \left[ G_{ij} (p, q) t_i (q) - T_{ij} (p, q) u_i (q) \right] d\Gamma (q) , \quad i, j = 1, 2, 3 \] (1)
here \( c_{ij}(p) \) depends on local boundary geometry of the surface \( \Gamma \) (for a smooth boundary \( c_{ij}(p) = 0.5\delta_{ij} \)). \( G_{ij}(p,q) \) and \( T_{ij}(p,q) \) are Kelvin’s fundamental solutions for displacements and tractions, respectively, \( u_i(q) \) and \( t_i(q) \) are displacements and traction on the subregion surface; the location of source and field points belonging to the subregion surface \( \Gamma \) are defined by the coordinates of points \( q \) and \( p \).

The displacement continuity and the traction equilibrium supplementary relations are used at the interfacial boundaries of subregions with conditions of ideal contact in the following form

\[
    u_i^k(p) = u_i^n(p), \quad t_i^k(p) = -t_i^n(p), \quad i = 1, 2
\]

where \( k \) and \( n \) are numbers of join subregions, \( u_i(p) \) and \( t_i(p) \) are displacements and traction components at the boundary point \( p \).

For numerical solution of the BIE (1) in three-dimensional case surfaces of all subregions are subdivided into eight-node quadrilateral quadratic isoparametric elements. We use the collocation scheme to discretize BIE and collocation points coincided with the nodal points of boundary elements. The kernels in equations (1) are singular: the kernel \( T_{ij}(p,q) \) is strongly singular. In the used collocation scheme these singularity are transferred to discrete equations. In these equations the integrals with a weak singularity, associated with the kernel \( G_{ij}(p,q) \), are calculated using the transformation to the polar coordinate system [10]. The sum of the integrals with a strong singularity corresponding to the kernels \( T_{ij}(p,q) \) and free term \( c_{ij}(p) \) in equations (1) are calculated using the rigid body motion approach, [9]. Integrating over all subregions boundaries we obtain the final discrete BIE, corresponding to equations (1). The numerical algorithm of equations system (1)-(2) solving has been implemented into the author’s computer code, [10, 11, 12]. Under computer implementation this algorithm has demonstrated high accuracy and low time consuming versus the published results.

3. Results of numerical modelling

In this part of the paper we present some new results of the stress concentration analysis for plates with one and two oblique holes, plates with rounded edge of holes and plates with conical holes. The problem of stress concentration for the cooling oblique hole in the rim of the turbine disk is also considered. All computations were performed for Poisson’s ratio \( \nu = 0.3 \) under uniaxial tension loading \( \sigma_0 \).

The plate with oblique holes is the model of a turbine blade wall with cooling holes. At first, we consider the computation results for in-plane uniaxial tension of the plate with one oblique hole. Note, that cuts set of this plate, parallel to its surface are ellipses with semi-axes \( a = R \) and \( b = R\sin\alpha \), where \( R \) is the radius of hole cylindrical surface. The plane \( XOZ \) is the plane of symmetry for uniaxial plate tension in the direction \( OX \) and \( OY \). The boundary element model with accounting of the problem symmetry is presented in Fig. 1, axis \( OY \) is the normal to the plane \( XOZ \). The computational analysis has been shown that under uniaxial tension in the direction \( OY \) (orthogonal to the hole axis) for hole obliquity angles more than \( 75^\circ \) the position of the normal stress maximum \( (\sigma_y) \) occurs at the plate middle section (see Fig. 2). If the hole axis turns in the plane normal to the direction of tensile loading, then the position of the normal stress maximum is shifted to the hole acute-angled edge (for angles less then \( 75^\circ \)). For tension in the orthogonal direction \( OX \) the largest stress \( \sigma_y \) is compressive and occurs also at the hole acute-angled edge.

For engineering applications it will be convenient to present the results of oblique holes stress analysis as a simple relation for normal stress \( \sigma_y \) versus a plate thickness. The maximal normal stress at the edge of an elliptical hole with semi-axes \( a \) and \( b \) in \( 2D \) infinite plate is

\[
    \sigma_y = \sigma_0(1 + 2a/b)
\]
Figure 1. Plate with the oblique hole, boundary element model, number of nodes - 194, $\alpha = 30^\circ$, $W/T = 10$, $T/R = 2$.

Figure 2. Normal stress distribution over thickness of the plate with the oblique hole (along line $AB$, Fig. 1), $\eta = (Z/T) + 0.5$, uniaxial tension.

where $b/a = \sin\alpha$ for cuts of 3D plates with oblique holes, $\alpha$ is the angle of hole obliquity.

The second term in (3) corresponds to amplification of the stress $\sigma_y$ (above the nominal stress) at the hole edge in 2D problem. Let’s assume that in 3D problems the stress $\sigma_y$ amplification (above the 2D stress concentration) at the hole acute-angled edge is inversely proportional to $\sin\alpha$. Hence, the stress at the hole acute-angle edge (point $A$, Fig. 1) defines as

$$\sigma_y(-T/2) = \sigma_0 \left( 1 + \frac{2}{\sin^2\alpha} \right)$$

The normal stress near the hole blunt angle (point $B$, Fig. 1) is decreased in comparison by the second term in (3) and it is assumed that this stress decreasing is proportional to $\sin^3\alpha$. Consequently, stress at the hole blunt edge is

$$\sigma_y(+T/2) = \sigma_0 \left( 1 + 2\sin^2\alpha \right)$$

Note, that value (3) can be attributed as the stress at the middle section of the plate, $z = 0$. We can obtain the parabolic approximation of the normal stress at the oblique hole edge by using introduced above three values of the normal stress along the plate thickness (3), (4) and (5)

$$\sigma_y(\xi) = \sigma_0 \left( C_1\xi^2 + C_2\xi + C_3 \right) f(W/T, T/R)$$

where $\xi = Z/T$, $|\xi| \leq 0.5$, $C_{1,2,3}$ depend on the angle obliquity $\alpha$ and $f(W/T, T/R)$ is the function for accounting of plates sizes, which can be obtained experimentally or numerically.

Dependences of the normal stress at the hole acute-angle versus of hole axis obliquity,
which were obtained numerically by DBIE, were calculated on the base of relation (4) (with \( \xi = -0.5, f(W/T, T/R) = 1 \) this relation identically to (6)) and the results (3) obtained for 2D problem are shown in Fig. 3. Deviation between the numerical results and the approximation by relation (6) for obliquity angles \( 35^\circ \leq \alpha \leq 90^\circ \) is less than 3—5\% that is convenient for the most technical applications.

The rounding of the hole edge is used to decrease the stress concentration at the hole acute-angle. Computations by DBIEM were performed for plates with rounded edges of oblique holes (see Fig. 4), where \( \varrho \) is the radius of rounding and values of \( \varrho/T = 0.25; 0.125 \) were used. The boundary conditions for this task are similar to the previous problem (see Fig. 1). The graphs of computation results are presented in Fig. 5. For rounded hole edges the normal stress at the surface of the plate becomes less than for an acute angle of hole. The position of the stress maximum shifts along the hole axis away of the hole edge. For \( \alpha = 90^\circ \) and \( \alpha = 60^\circ \) the stresses decreasing are not noticeable, but the maximum stress position is shifted from the hole edge in the distance about rounded radius. If \( \alpha = 45^\circ \) and \( \alpha = 30^\circ \) the stress decreasing are more noticeable, it is up to 20\% for \( \alpha = 30^\circ \).

![Figure 4](image)

**Figure 4.** Plate with the hole rounded edge, \( W/T = 10, T/R = 2, \alpha = 45^\circ, \varrho/T = 0.25 \), tension is transversally to the hole axis.

![Figure 5](image)

**Figure 5.** Normal stress distribution over thickness of the plate (along line \( AB \), Fig. 4), rounded edge of hole, \( \varrho/T = 0.25, \eta = Z/T \).

Effects analogous to those observed at the edges of oblique holes are also possible near the edges of conical holes. Numerical analysis of the stress state under uniaxial tension was performed for flat plates with holes in the form of a truncated cone. The radius of the small circle of the truncated cone was assigned to be equal to the size of small semi-axis of the elliptical section of oblique hole. The numerical DBIEM model of a quarter of the plate for one of the calculation variants is shown in Fig. 6. The tensile stresses distribution along the plate thickness as a function of the slope of the hole generatrix are shown in Fig. 7. In the acute edge zone of the conical hole, as in the case of oblique holes, a considerable stress concentration is observed. The maximum of stress magnitude is somewhat lower for conical holes than on the acute edge of the “normal” oblique hole.

Some products (for example, cooled blades of gas turbine engines) contain systems of closely located oblique holes. The stress state of a plate with two oblique holes was calculated to evaluate the interference of such holes. The plane \( XOZ \) is the symmetry plane in this problem, uniaxial tension is applied transversely to this plane, similarly as for a plate with one hole. A discrete model of a plate half with two oblique holes is shown in Fig. 8. The dimensions of the plate, the elastic constants and the boundary conditions are the same as those adopted for
Figure 6. Plate with a conical hole, boundary element model, number of nodes - 200, the cone generatrix slope is \( \alpha = 45^\circ \), uniaxial tension.

Figure 7. Distribution of normal stress over thickness of the plate at the hole edge, the cone generatrix slope variation, \( \eta = Z/T \).

Figure 8. Plate with two oblique hole, boundary element model, number of nodes - 308, \( \alpha = 30^\circ \), uniaxial tension.

Figure 9. Distribution of normal stress over thickness of plates with the oblique holes (along lines \( AB \), Fig. 1 and Fig. 8), \( \eta = (Z/T) + 0.5 \).

Oblique holes in the rim of turbine disks serve to supply of cooling air to blades. The angle between the axis of the cooling hole and the radial direction usually does not exceed \( \beta = 20^\circ \) (see Fig. 10), which corresponds to \( \alpha \geq 70^\circ \) (see Fig. 1). The effect of the hole axis inclination for such angles is relatively small (see Fig. 2), and the predominant effect is intersection of the stress concentrators (holes and wells), in the hole transition zone at the groove for attaching.
of the blade tail. A discrete model of this problem with the hole zone and the part of the groove for attaching the blade is shown in Fig. 11. On the rear invisible part of the model (Fig. 11) uniform tensile loading simulating the action of circumferential stresses, was applied. On the lower invisible part of the model surface and in the plane passing through the axis of the hole, normal displacements and tangential tractions were assumed to be zero. The sides and top faces of the model, as well as the hole surface, were assumed as free of loads. The computation by DBIE gives the coefficient of the normal stress concentration at the hole edge in intersection between the hole and well as $K = 8.02$. This value is rather high and the hole and well intersection region must be modified to increase the structure durability.

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