QUANTUM GALILEI GROUP
AS SYMMETRY OF MAGNONS.

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Abstract. Inhomogeneous quantum groups are shown to be an effective algebraic tool in the study of integrable systems and to provide solutions equivalent to the Bethe ansatz. The method is illustrated on the 1D Heisenberg ferromagnet whose symmetry is shown to be the quantum Galilei group $\Gamma_q(1)$ here introduced. Both the single magnon and the $s = 1/2$ bound states of $n$–magnons are completely described by the algebra.

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In very recent times it has been observed that physical systems with a fundamental length scale have a symmetry of a inhomogeneous quantum group [1]. Fruitful results have been obtained in nuclear physics, where the quantum parameter is related to the time scale of strong interactions [2], as well as in solid state physics, where a fundamental length arises naturally [3]. In this letter we show that, actually, the quantum group symmetry yields an algebraic scheme consistent with the Bethe ansatz [4,5] for solving the dynamics of quantum integrable models.

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We shall illustrate the method on the concrete example of the Heisenberg ferromagnet. By looking at the single magnon excitations, the symmetry of the model is found to be a $q$-deformation of the Galilei group in one dimension, where the deformation parameter is the lattice spacing. The coalgebra structure will then define naturally the many magnon systems, both producing the global variables and describing bound states for $s = 1/2$ as kinematical properties of the structure.

The Hamiltonian of the Heisenberg ferromagnet with periodic conditions $\vec{S}_{N+1} = \vec{S}_1$ is given by [6]:

$$H = -J \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1} + g\mu_B H \sum_{i=1}^{N} S_i^z, \quad (1)$$

where $\vec{S}_i$ are spin $s$ operators, $H$ is an external homogeneous magnetic field and the exchange integral $J$ is positive. The energy of the ground state, reads $E_0 = -JNs^2 - g\mu_B HNs$ and the states with one spin deviate are $\psi = \sum_i f_i S_i^+ |0\rangle$ with the usual definition of $S_i^\pm$. The eigenvalue equation $H \psi = E \psi$ is equivalent to the algebraic system

$$-sJ(f_{i-1} + f_{i+1} - 2f_i) + g\mu_B H f_i = (E - E_0) f_i. \quad (2)$$

The diagonalization of equation (2) leads to the following dispersion relation for plane waves with wave vector $k$:

$$E - E_0 = 2sJ(1 - \cos k) + g\mu_B H. \quad (3)$$

Let us now show that a quantum group symmetry can give an algebraic description of the system. Adopting the same point of view of ref. [3], we see that the solutions of system (2) are obtained by evaluating at integer multiples of the lattice spacing $a$ the solutions of the differential equation

$$\left(2sJ(1 - \cos(-ia\partial_x)) + g\mu_B H\right) f(x) = (E - E_0) f(x). \quad (4)$$
In the limit \( a \to 0 \), (4) is just the Schrödinger equation in a constant potential, with an effective mass \((2sJ a^2)^{-1}\) and with the symmetry of the 1-dim Galilei group.

We introduce a new inhomogeneous quantum algebra, that we call \( \Gamma_q(1) \), which is a deformation of the Galilei algebra and is generated by the four elements \( B, M, P, T \). The commutation relations are:

\[
[B, P] = iM , \quad [B, T] = (i/a) \sin(aP) , \quad [P, T] = 0 ,
\]

the generator \( M \) being central.

The coproducts read

\[
\Delta B = e^{-iaP} \otimes B + B \otimes e^{iaP} , \quad \Delta M = e^{-iaP} \otimes M + M \otimes e^{iaP} ,
\]

\[
\Delta P = 1 \otimes P + P \otimes 1 , \quad \Delta T = 1 \otimes T + T \otimes 1 ,
\]

and the antipodes

\[
\gamma(B) = -B - aM , \quad \gamma(M) = -M , \quad \gamma(P) = -P , \quad \gamma(T) = -T .
\]

The Casimir of \( \Gamma_q(1) \) is

\[
C = MT - (1/a^2) (1 - \cos(aP)) .
\]

A differential realization of this quantum algebra is given by

\[
B = mx , \quad M = m , \quad P = -i\partial_x , \quad T = (ma^2)^{-1}(1 - \cos(-ia\partial_x)) + c/m ,
\]

(5)

where \( c \) is the constant value of the Casimir \( C \): for \( m = (2sJ a^2)^{-1} \) and \( c = mg\mu_B H \) we see that the expression of \( T \) coincides with the left hand side of (4).
The algebra is invariant under $P \mapsto P + (2\pi/a) n$, so that we can choose for $P$ any domain of size $2\pi/a$. Like in the Galilei algebra the position operator is defined as $X = (1/M) B$, with time derivative

$$\dot{X} = i[T, X] = 2sJa \sin(aP),$$

in agreement with the canonical equation. The properties concerning the single magnon states are then obtained using the symmetry under the quantum algebra $\Gamma_q(1)$.

We are now going to study a system of two magnons in a Heisenberg ferromagnet described by the state $\psi = \sum_{i>j} f_{ij} S_i^+ S_j^+ |0\rangle$, with $f_{ij} = f_{ji}$. The eigenvalue equation for the Hamiltonian leads to the following system for $f_{ij}$:

$$(E - E_0 - 2g\mu_B H - 4sJ) f_{ij} + s \sum_n (J_{nj} f_{in} + J_{in} f_{nj})$$

$$= \frac{1}{2} J_{ij} (f_{ii} + f_{jj} - f_{ij} - f_{ji}),$$

where the bonds $J_{ij}$ are equal to $J$ when the label $(ij)$ are nearest neighbor pairs and vanish otherwise. For $s = 1/2$ the amplitudes $f_{ii}$ are unphysical: however such amplitudes appear in both sides of (6) for $i = j + 1$ and cancel. The left hand side of (6) constitutes the equation for the free part of the system, while the terms of the right hand side are responsible for the interaction. The quantum group $\Gamma_q(1)$ is a symmetry of the free system: nevertheless it is able to describe many magnon states, provided that the interaction can be treated as a “boundary condition” ensuring that the homogeneous free equation is satisfied at every pair of sites. This is indeed the Bethe ansatz, which imposes the separate vanishing of the two sides of equation (6).

The coproduct describes the composition of elementary systems. When the symmetry is given by a Lie algebra, since the generators are primitive, the global
symmetry of a composite system is obtained by summing the generators of the algebra of the elementary constituents. In the quantum group context we can have non primitive generators, but the coproducts can be used for the same purpose.

From the coproduct of $T$, $\Delta T = 1 \otimes T + T \otimes 1$, we find for the energy $T_{12}$ of a two magnon system

$$T_{12} = T_1 + T_2$$

$$= (M_1 a^2)^{-1}(1 - \cos(aP_1)) + (M_2 a^2)^{-1}(1 - \cos(aP_2)) + C_1/M_1 + C_2/M_2 .$$

Considering magnons with the same value of $s$, the eigenvalue of $M_1 = M_2 = M$ is equal to $(2sJa^2)^{-1}$ and $C_2/M_2 = C_1/M_1 = g\mu_B H$. Using the differential realization $P_1 = -i\partial_{x_1}$ and $P_2 = -i\partial_{x_2}$ we get

$$T_{12} f(x_1, x_2) = 4sJ f(x_1, x_2) + 2g\mu_B H f(x_1, x_2) - sJ \left( f(x_1, x_2 + a) + f(x_1, x_2 - a) + f(x_1 + a, x_2) + f(x_1 - a, x_2) \right)$$

and the eigenvalue equation for $T_{12}$ coincides with the vanishing of the left hand side of equation (6). Plane waves solve the equation (7) with energy

$$E - E_0 = 2sJ \left( 2 - \cos(ap_1) - \cos(ap_2) \right) + 2g\mu_B H .$$

This is the energy of the continuum, the eigenfunction of the two magnons states being obtained by imposing the periodic boundary conditions on the plane waves.

An important point is that the interaction coefficients $J_{ij}$ depend on the relative coordinates, so that the total momentum is always a constant of the motion. This is rephrased in our algebraic approach by the fact that $P$ is a primitive generator, i.e. $\Delta P = 1 \otimes P + P \otimes 1$; the quantum group implies then

$$P_{12} = P_1 + P_2 ,$$
and the total momentum has the correct symmetry and invariance in the composite
system.

For \( s = 1/2 \) a solution for the bound states based on the Bethe ansatz is
known [4]. We show here that our treatment based on \( \Gamma_q(1) \) reproduces this result
in a very natural way. For higher spin neither the Bethe ansatz nor our quantum
group approach apply, since the model does not reduce to a free equation plus
“boundary conditions”.

The generator \( M \) is a central element in the algebra, but the coalgebra shows
that it combines in a nontrivial way:

\[
M_{12} = M_1 e^{iaP_2} + M_2 e^{-iaP_1} .
\]

We then rewrite \( T_{12} \) in terms of global and “relative energy”

\[
T_{12} = (a^2 M_{12})^{-1} (1 - \cos(aP_{12})) + C_1/M_1 + C_2/M_2 - \frac{(M_{12} - M_1 - M_2)^2}{2a^2 M_{12} M_1 M_2} ,
\]

where the “relative energy” is invariant with respect to the global variables and in
the continuum limit \( a \to 0 \) gives the Galilei relative energy \( \frac{M_1 M_2}{2(M_1 + M_2)} (P_1/M_1 - P_2/M_2)^2 \). The only new feature with respect to the Lie case is that the effective
mass of the composite system is a complex variable. For \( s = 1/2 \) magnons we have
\( M_1 = M_2 = M = (J a^2)^{-1} \).

In (8) the relative energy is vanishing on the states for which \( M_{12} = 2M \),
namely

\[
M_{12} f(x_1, x_2) = M \left( f(x_1, x_2 + a) + f(x_1 - a, x_2) \right) = 2M f(x_1, x_2) .
\]

Searching a solution of the form

\[
f(x_1, x_2) = e^{ip(x_1+x_2)/2} F(x_1 - x_2) , \quad p = p_1 + p_2 , \quad x_1 > x_2 ,
\]
i.e. solving the problem at fixed $P_{12}$ and using the invariance of $p$ under translations of $2\pi n/a$, equation (9) for $F(x_1 - x_2)$ takes the form $F(x_1 - x_2) = |\cos(ap/2)| F(x_1 - x_2 - a)$ and is solved by

$$F(x_1 - x_2) = \text{const} \cdot |\cos(ap/2)|^{(x_1-x_2)/a}.$$  

This solution is consistent with periodic conditions only for asymptotically large values of $N$; the evaluation of the energy on $f(x_1, x_2)$ results in

$$T_{12} f(x_1, x_2) = \left((J/2)(1 - \cos(ap)) + 2g\mu_B H\right) f(x_1, x_2).$$

This is the energy and the eigenfunction found by Bethe [4]. Indeed the Bethe equation for bound states is nothing else than the constraint on the coproduct, $M_{12} = 2M$:

$$\cot(aP_1/2) - \cot(aP_2/2) = -2i \quad \Leftrightarrow \quad e^{-iaP_1} + e^{iaP_2} = 2 \quad (10)$$

It seems therefore clear that these bound states are not a consequence of an actual interaction: their existence is due to the lattice and they merge in the continuum for $a \to 0$.

Let us now consider the position operator of the composite system; from the coproduct of $B$ we can define

$$X_{12} = \frac{B_{12}}{M_{12}} = \frac{x_2 e^{-iaP_1} + x_1 e^{iaP_2}}{e^{-iaP_1} + e^{iaP_2}} = \frac{x_1 + x_2}{2} + i \frac{x_1 - x_2}{2} \tan\left(\frac{aP_{12}}{2}\right),$$

which satisfies $[X_{12}, P_{12}] = i$. For the bound state we find

$$\dot{X}_{12} = i[T_{12}, X_{12}] = i[(J/2)(1 - \cos(aP_{12})) + 2g\mu_B H, X_{12}]$$

$$= (J/2) a \sin(aP_{12}), \quad (11)$$
showing the same structure as for the one magnon states with twice the value of the effective mass. For continuous spectra of $P_1$ and $P_2$, i.e. $N \to \infty$, the eigenvalue equation for $X_{12}$ can be defined in the momentum space:

$$X_{12} f(p, k) = \left( i\partial_p - \frac{1}{2} \tan(ap/2) \partial_k \right) f(p, k) = \lambda f(p, k) , \quad (12)$$

where $p = p_1 + p_2$ and $k = (p_1 - p_2)/2$. The characteristic curves of equation (12) are

$$k + (i/a) \ln |\cos(ap/2)| = \text{const} .$$

Thus, for hermitian $X_{12}$ and $P_{12}$, $k$ is imaginary and compatible with a constant $\Delta M$, and thus with (10).

Let us now consider the generalization to the $n$–magnon case. Here also the quantum group yields the relations giving the total energy and the energy of the bound states. An induction procedure gives

$$T_{12..n} = \sum_{k=1}^{n} T_k = (a^2 M_{12..n})^{-1} \left( 1 - \cos(aP_{12..n}) \right) + \sum_{k=1}^{n} \left( C_k/M_k \right) - \frac{1}{2a^2} \sum_{k=2}^{n} \left( \frac{M_{12..k} - M_{12..(k-1)} - M_k}{M_{12..k}M_{12..(k-1)}M_k} \right) \quad (13)$$

where $P_{12..n} = \sum_{k=1}^{n} P_k$ and $M_{12..n}$ is defined by iterating the coproduct and using the coassociativity:

$$M_{l..k} = M_{l..(h-1)} e^{i\alpha(P_h+...+P_k)} + M_{h..k} e^{-i\alpha(P_l+...+P_{h-1})} , \quad l < h \leq k .$$

In analogy with the two magnon case, the bound states are obtained by imposing the vanishing of the relative energy in (13):

$$E^{R}_{12..n} = \frac{J}{n} \left( 1 - \cos(ap) \right) + n g \mu B H , \quad p = \sum_{k=1}^{n} p_k .$$
The conditions we get, namely \( M_{1\ldots k} = kM \) for \( k = 2, \ldots, n \) are equivalent to the Bethe conditions \( M_{(k-1)k} = 2M \).

We conclude by giving some remarks.

The quantum group \( \Gamma_q(1) \) is applicable to any elementary physical system described by a discretized Schrödinger equation; moreover it can be used in the same way as the Galilei group in the continuum to account for the asymptotic states in the presence of well behaved potential.

A Galilei continuous symmetry is present in the Bose gas model where the Bethe ansatz consists in boundary conditions in the first derivatives \([5,7]\); we notice that our approach contains the description of this case in the limit \( a \to 0 \) once we modify the coproduct constraint in \( M_{12} = 2M + aCM \), where \( C \) is the delta potential strength.

It is then interesting to realize that it is just the appropriate quantum group symmetry that indicates the Bethe ansatz and then the integrability of the system.

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**References.**

[1] E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, J. Math. Phys. 31, 2548 (1990); J. Math. Phys. 32, 1155 (1991); J. Math. Phys. 32, 1159 (1991); E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, “Contractions of quantum groups”, Proceedings of the first semester on quantum groups, Eds. L.D. Faddeev and P.P. Kulish, Leningrad October 1990, Springer-Verlag, in press.
[2] E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, “Quantum Groups of Motion and Rotational Spectra of Heavy Nuclei.”, Phys. Lett. B (1992), in press.

[3] F. Bonechi, E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, “Inhomogeneous Quantum Groups as Symmetry of Phonons.”, University of Florence Preprint, DFF 152/12/91.

[4] H. Bethe, Z. Phys. 71, 205 (1931).

[5] V.E. Korepin, A.G. Izergin and N.M. Bogoliubov, “Quantum Inverse Scattering Method and Correlation Function. Algebraic Bethe Ansatz”, (Cambridge, to appear in 1992).

[6] D.C. Mattis, “The Theory of Magnetism, I”, (Springer–Verlag, Berlin Heidelberg 1981).

[7] E.H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963).