Local exact controllability to constant trajectories for Navier-Stokes-Korteweg model

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ABSTRACT
In this article, we study the local exact controllability to a constant trajectory for a compressible Navier-Stokes-Korteweg system on the torus in dimension \( d \in \{1, 2, 3\} \) when the control acts on an open subset. To be more precise, we obtain the local exact controllability to the constant state \((\rho_\ast, 0)\) for arbitrary small positive times and without any geometric condition on the control region. In order to do so, we analyze the control properties of the linearized equation, and present a detailed study of the observability of the adjoint equations. In particular, we shall exhibit the parabolic (possibly also dispersive) structure of these adjoint equations. Based on that, we will be able to recover observability of the adjoint system through Carleman estimates.

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1 Introduction

In this work, we are interested in the control properties of the Navier-Stokes-Korteweg system. This system describes a compressible and viscous fluid on a region of \( \mathbb{R}^d \) with \( d \in \{1, 2, 3\} \), of density \( \rho = \rho(t, x) \) and velocity field \( u = u(t, x) \) and reads

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \mathcal{A}(\rho)u + \nabla (P(\rho)) &= \text{div}(\mathcal{K}(\rho)),
\end{aligned}
\]  

\( (1.1) \)

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where \( P(\rho) \) is the pressure function assumed to depend only on the density \( \rho \),
\[
A(\rho)u := \text{div} (2\mu(\rho)D_S(u)) + \nabla (\nu(\rho) \text{div} (u))
\]
is the viscosity part, \( D_S(u) := \frac{1}{2}(\nabla u + \nabla u^T) \) is the symmetric gradient and the capillarity tensor \( \mathcal{K}(\rho) \) is given by
\[
\mathcal{K}(\rho) := \rho \text{div}(\kappa(\rho) \nabla \rho)I_{\mathbb{R}^d} + \frac{1}{2} (\kappa(\rho) - \rho\kappa'(\rho)) |\nabla \rho|^2 I_{\mathbb{R}^d} - \rho \kappa(\rho) \nabla \rho \otimes \nabla \rho,
\]
where \( I_{\mathbb{R}^d} \) denotes the \( d \times d \) identity matrix, see [2]. The coefficients \( \nu = \nu(\rho) \) and \( \mu = \mu(\rho) \) designate the bulk and shear viscosity, respectively and \( \kappa = \kappa(\rho) \) the capillarity function. Note that all these coefficients are assumed to be functions of the density.

The Navier-Stokes-Korteweg system describes a two-phase compressible and viscous fluid in the case of diffuse interface, in which the change of phase corresponds to a fast but regular transition zone for the density and the velocity. We refer to [9] for the modeling of phase transition and to [12] for the full derivation of the Navier-Stokes-Korteweg model. Note that this model includes as a special case the quantum Navier-Stokes equation see for instance [4] for its derivation from the Wigner equation.

Based on physical considerations, it is natural to assume that
\[
\nu \geq 0, \quad 2\mu + \nu \geq 0, \quad \kappa \geq 0. \tag{1.2}
\]
Note that if the viscosity parameters satisfy \( \mu > 0 \) and \( \mu + \nu > 0 \), and the capillarity coefficient \( \kappa \) vanishes, then System (1.1) coincides with the compressible Navier-Stokes system. Below, we will be focusing on the case \( \mu > 0, 2\mu + \nu > 0 \) and \( \kappa > 0 \), at least locally, which corresponds to the Navier-Stokes-Korteweg model.

**Main results.** Before going further, let us remark that system (1.1) possesses some specific stationary states given by constant states \((\rho_*, u_*)\) with \( \rho_* > 0 \) and \( u_* \in \mathbb{R}^d \). Our goal is to analyse the local exact controllability property of (1.1) around these constant states \((\rho_*, u_*)\). For simplicity, we will reduce our analysis only to the case \( u_* = 0 \) (the case \( u_* \in \mathbb{R}^d \) can be handled similarly).

Let us describe the geometrical setting. We work in the \( d \)-dimensional torus \( \mathbb{T}_L := (\mathbb{R}/L\mathbb{Z})^d \) identified with \([0, L]^d \) with periodic boundary conditions, where \( L > 0 \), and the controls will be assumed to act on some non-empty open subset \( \omega \) of \( \mathbb{T}_L \).

Our main result is the following one:

**Theorem 1.1.** Let \( d \in \{1, 2, 3\} \), \( L > 0 \), and \( \omega \) be a non-empty open subset of \( \mathbb{T}_L \).

Let \( \rho_* > 0 \), and let us assume that

(H1) \( \kappa(\rho_*), \mu(\rho_*) \) and \( 2\mu(\rho_*) + \nu(\rho_*) \) are positive;

(H2) there exists \( \eta \in (0, \rho_*) \) such that \( \mu \) and \( \nu \) belong to \( C^2([-\eta + \rho_*, \eta + \rho_*]) \) and \( P \) and \( \kappa \) belong to \( C^3([-\eta + \rho_*, \eta + \rho_*]) \).

Then there exists \( \delta > 0 \) such that, for all \( (\rho_0, u_0) \in H^2(\mathbb{T}_L) \times H^1(\mathbb{T}_L) \) satisfying
\[
\| (\rho_0 - \rho_*, u_0) \|_{H^2(\mathbb{T}_L) \times H^1(\mathbb{T}_L)} \leq \delta,
\]
there exist a control \((v_\rho, v_u) \in L^2(0, T; H^2(\mathbb{T}_L)) \times L^2(0, T; H^1(\mathbb{T}_L))\) and a corresponding controlled trajectory \((\rho, u)\) solving
\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= v_\rho 1_\omega & \text{in} & \quad (0, T) \times \mathbb{T}_L, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - A(\rho)u + \nabla (P(\rho)) &= \text{div}(\mathcal{K}(\rho)) + v_u 1_\omega & \text{in} & \quad (0, T) \times \mathbb{T}_L, \\
|\rho, u|_{t=0} &= (\rho_0, u_0) & \text{in} & \quad \mathbb{T}_L,
\end{aligned}
\tag{1.3}
\]
and satisfying
\[(\rho, u)|_{t=T} = (\rho_*, 0) \quad \text{in} \quad T_L.
\]
Besides, the controlled trajectory \((\rho, u)\) enjoys the following regularity
\[
\rho \in C([0, T]; H^2(T_L)) \cap L^2(0, T; H^3(T_L)) \cap H^1(0, T; H^1(T_L)),
\]
\[
u \in C([0, T]; H^1(T_L)) \cap L^2(0, T; H^2(T_L)) \cap H^1(0, T; L^2(T_L)),
\]
and the following positivity condition
\[
\inf_{(t,x) \in [0,T] \times T_L} \rho(t,x) > 0.
\]

Remark 1.2. Hypothesis \((H1)\) concerning the sign of \(\kappa, \mu\) and \(\nu\) at the point \(\rho_*\) guarantees the parabolic-type structure of the linearized system, see afterwards. Let us point out that, here, we work with strong solutions on bounded intervals, so that, we do not need any assumption on the monotonicity of the pressure \(P\).

Hypothesis \((H2)\) concerning the regularity of \(\mu, \nu, \kappa\) and \(P\) is mainly technical and is needed to handle the non-linear terms in the proof of Theorem 1.1.

Since the torus is a rather academic example, let us start by pointing out that this result leads to an exact controllability result to constant trajectories \((\rho_*, 0)\) in bounded domains \(\Omega\) when the controls act on the whole boundary \(\partial \Omega\). To be more precise, we have the following immediate corollary:

Corollary 1.3. Let \(d \in \{1, 2, 3\}\), \(\Omega\) be a smooth bounded domain of \(\mathbb{R}^d\). Let \(\rho_* > 0\), and let us assume conditions \((H1)\) and \((H2)\) of Theorem 1.1.

Then there exists \(\delta > 0\) such that, for all \((\rho_0, u_0) \in H^2(\Omega) \times H^1(\Omega)\) satisfying
\[
\|(\rho_0 - \rho_*, u_0)\|_{H^2(\Omega) \times H^1(\Omega)} \leq \delta,
\]
there exists a controlled trajectory \((\rho, u)\) solving
\[
\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0 & \text{in} \quad (0, T) \times \Omega, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - A(\rho)u + \nabla(P(\rho)) = \text{div}(\mathcal{K}(\rho)) & \text{in} \quad (0, T) \times \Omega, \\
(\rho, u)|_{t=0} = (\rho_0, u_0) & \text{in} \quad \Omega,
\end{cases}
\]
and satisfying
\[(\rho, u)|_{t=T} = (\rho_*, 0) \quad \text{in} \quad \Omega.
\]
Besides, the controlled trajectory \((\rho, u)\) enjoys the following regularity
\[
\rho \in C([0, T]; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)) \cap H^1(0, T; H^1(\Omega)),
\]
\[
u \in C([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)),
\]
and the following positivity condition
\[
\inf_{(t, x) \in [0,T] \times \Omega} \rho(t, x) > 0.
\]

Let us note that the controls do not appear explicitly in the equation (1.5). In fact, they are hidden in the boundary conditions, which do not appear in (1.5).

We will not give the complete details of the proof of Corollary 1.3, and we only sketch it hereafter. Since \(\Omega\) is bounded, it can be embedded into some torus \(T_L\), where \(T_L\) is identified to \([0,L]^d\) with periodic condition. We then consider the control problem on the torus \(T_L\) with controls appearing as source terms supported in \(T_L \setminus \overline{\Omega}\), starting from an initial datum obtained as an extension of \((\rho_0, u_0)\)
gives the existence of a controlled trajectory solving (1.1) if $\omega$ is and in any positive time. This is in sharp contrast with the control results for the linearized compressible Navier-Stokes equation, corresponding to $\kappa = 0$. Indeed, for those equations, the linearized models have some initial data which cannot be controlled in short time, as a result of the transport-parabolic structure of the linearized compressible Navier-Stokes equations, see in particular the works [8] and [23]. However, one can obtain local exact controllability result around trajectories $(\rho_*, u_*)$ provided suitable geometric conditions related to the flow of the target velocity field are satisfied, see for instance [14], [15] and [16].

Let us now explain the reason for this striking difference between the control properties for the compressible Navier-Stokes equations and the Navier-Stokes-Korteweg system. The crucial point is to notice that the Navier-Stokes-Korteweg system actually behaves like a parabolic system. In order to see this property, let us consider the linearized version of (1.1) around $(\rho_*, 0)$, which is

$$\begin{cases}
\partial_t a + \text{div}(u) = 0 \\
\partial_t u - \rho_*^{-1}(\mu(\rho_*) \Delta u - (\mu(\rho_*) + \nu(\rho_*) \nabla \text{div}(u)) + P'(\rho_*) \nabla a - \rho_* \kappa(\rho_*) \nabla \Delta a = 0
\end{cases} \text{ in } (0, T) \times \mathbb{T}_L,$$

where $a := \rho/\rho_* - 1$. To simplify the argument we omit the term $P'(\rho_*) \nabla a$ in the second equation, since it is of lower order compared to $\rho_* \kappa(\rho_*) \nabla \Delta a$. By taking the Laplacian of the first equation and the divergence of the second equation, setting $b = -\Delta a$ and $q := \text{div}(u)$, we obtain the following subsystem of two scalar equations

$$\begin{cases}
\partial_t b - \Delta q = 0 \\
\partial_t q - \rho_*^{-1}((2\mu(\rho_*) + \nu(\rho_*)) \Delta q + \rho_* \kappa(\rho_*) \Delta b = 0
\end{cases} \text{ in } (0, T) \times \mathbb{T}_L. \quad (1.6)
$$

Accordingly, it is essential to analyze the structure of the matrix

$$A := \begin{pmatrix} 0 & -1 \\ \rho_* \kappa(\rho_*) & -\rho_*^{-1}(2\mu(\rho_*) + \nu(\rho_*)) \end{pmatrix}. \quad (1.7)$$

Explicit computations show that:

1. If $\rho_*^{-2}(2\mu(\rho_*) + \nu(\rho_*))^2 > 4\rho_* \kappa(\rho_*)$, then $A$ is diagonalizable and has two real eigenvalues:

$$-\rho_*^{-1}(2\mu(\rho_*) + \nu(\rho_*)) \pm \sqrt{\rho_*^{-2}(2\mu(\rho_*) + \nu(\rho_*))^2 - 4\rho_* \kappa(\rho_*)}.$$

2. If $\rho_*^{-2}(2\mu(\rho_*) + \nu(\rho_*))^2 < 4\rho_* \kappa(\rho_*)$, then $A$ is diagonalizable and has two complex eigenvalues:

$$-\rho_*^{-1}(2\mu(\rho_*) + \nu(\rho_*)) \pm i\sqrt{4\rho_* \kappa(\rho_*) - \rho_*^{-2}(2\mu(\rho_*) + \nu(\rho_*))^2}.$$

3. If $\rho_*^{-2}(2\mu(\rho_*) + \nu(\rho_*))^2 = 4\rho_* \kappa(\rho_*)$, then $A$ is similar to an upper triangular matrix having the double eigenvalue

$$-\rho_*^{-1}(2\mu(\rho_*) + \nu(\rho_*)).$$

Assumption (H1) implies that, in any of the three above cases, the real parts of the eigenvalues are negative. This therefore implies that the system (1.6) has a parabolic/dispersive structure.

Note that this parabolic/dispersive structure has already been pointed out in the literature. Indeed, the analytic smoothing effect in space variable for this equation for both the velocity field and the density has been shown in [7] (see also [27] and [26]) and is precisely based on the derivation of
dissipative estimate on the Fourier modes of the solutions to the linearized system, underlying the parabolic structure of the system. In views of the terminology in [22], we say that the system is \textit{purely parabolic} in the cases 1 and 3. In the case 2, we have dispersion in addition to the dissipation (note that this case contains the quantum Navier-Stokes system, see for instance [3]) and the system should thus be considered as \textit{parabolic/dispersive}.

Our analysis will be based on a similar discussion, but on the adjoint of the linearized equations of (1.5) which involves the transpose matrix $^tA$. In fact, our analysis will be split into two cases:

- $^tA$ is diagonalizable which correspond to Items 1 and 2 above;
- $^tA$ is nondiagonalizable which correspond to Item 3 above.

We end this section by mentioning some related results and open problems.

\textbf{Cauchy theory.} Although we will not use any result on the Cauchy theory of the Navier-Stokes-Korteweg system, let us point out that the existence of strong solutions, respectively weak, has been obtained in [19, 20], respectively [21]. Also note that, in the case of the whole space by using Fourier analysis methods related to the parabolic structure mentioned above, the well-posedness of the Cauchy problem in critical Besov spaces for global and local solutions and in dimension $d \geq 2$ is established in [11].

\textbf{Open problems.} In Theorem 1.1, we consider internal control which appear both in the continuity equations and the momentum equations. In order to get a more physically relevant interpretation of the internal controllability as forces, it would be interesting to investigate the controllability of the Navier-Stokes-Korteweg system when the control acts only in the momentum equations. In such case, one could rely for instance on the algebraic solvability method as in [9, 10], respectively [11]. Note that such results have also been obtained in different contexts: for 1-d compressible Navier-Stokes equation (see for instance [16]), for coupled hyperbolic-parabolic system (see for instance [17]) and for Kuramoto-Sivashinsky system (see for instance [5]).

Another interesting question concerns the controllability of the Navier-Stokes-Korteweg system (1.1) on bounded open domain of $\mathbb{R}^d$ with controls localized on an nonempty open subset of the boundary. The additional difficulty compared to the ones in Theorem 1.1 is that the algebraic manipulations used to make explicitly appear the parabolic structure of the system create intricate boundary conditions on the systems, which are difficult to handle. We point out that such intricate terms also appear, with a different coupling, in the context of non-homogeneous incompressible Navier-Stokes system, in which they can be handled through appropriate weighted energy estimates, see [1].

\textbf{Outline of the article.} In Section 2, we start by recalling and developing the controllability results obtained for the heat equations, which will be used as a building block in all our proofs. Then, in Section 3, we present the strategy to prove Theorem 1.1 based on the analysis of the controllability of the linearized version of (1.3): by duality, we only have to show an observability estimate on the adjoint system, which is obtained by identifying a closed subsystem of the adjoint on which the parabolic structure appears in a somewhat decoupled manner. In Sections 4 and 5 we prove the controllability of the adjoint of the resulting system depending on the diagonalizability of $^tA$ (Section 4) or not (Section 5). In Section 6 we use the results of Sections 4 and 5 to show the controllability of the linearized model in suitably regular Sobolev spaces. In Section 7, we show the local exact controllability of (1.3) using a fixed point argument. In the Appendix we give the proof of the Carleman estimates used in this article for the complex coefficient heat equation (which coincides with the one in [1] when the coefficients are real).

\textbf{Notation.} We set, for any $(\ell, \sigma, p) \in \mathbb{Z} \times \mathbb{R} \times ([1, +\infty] \cup \{\infty\})$

$$H^\ell(H^\sigma) := H^\ell(0, T; H^\sigma(\mathbb{T}_L)) \quad \text{and} \quad L^p(H^\sigma) := L^p(0, T; H^\sigma(\mathbb{T}_L));$$

and in the same spirit

$$\| \cdot \|_{H^\ell(H^\sigma)} := \| \cdot \|_{H^\ell(0, T; H^\sigma(\mathbb{T}_L))}, \quad \| \cdot \|_{L^p(H^\sigma)} := \| \cdot \|_{L^p(0, T; H^\sigma(\mathbb{T}_L))} \quad \text{and} \quad \| \cdot \|_{H^\sigma} := \| \cdot \|_{H^\sigma(\mathbb{T}_L)}.$$
Throughout this article, we will also use the notation $f \lesssim g$ to express that there exists a positive constant $C$, such that $f \leq Cg$. In some proofs, it will be important to underline the fact that the positive constant does not depend on some parameter. When this occurs, this will be said within the proof.

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2 Controllability of the heat equation

In this section we recall and develop the control results for the complex-valued heat equation. Let $L$ be a positive real number and $\omega$ a non empty open subset of $T_L$ such that $\overline{\omega} \subset T_L$. Let $\zeta$ be a complex number, such that $\Re(\zeta) > 0$.

In order to add a margin on the control zone $\omega$, we introduce a non-negative smooth cut-off function $\chi_0$ such that there exist two proper open subsets $\omega_0$ and $\omega_1$ of $T_L$ such that

$$\omega_0 \subset \text{supp}(\chi_0) \subset \omega_1 \subset \omega \quad \text{and} \quad \chi_0 = 1 \quad \text{on} \quad \omega_0.$$ (2.2)

We consider the following controllability problem: Given $r_0$ and $f$, find a control function $v_r$ such that the solution $r$ of

$$\begin{align*}
\partial_t r - \zeta \Delta r &= f + v_r \chi_0 \quad \text{in} \quad (0, T) \times T_L, \\
\text{and} \quad r|_{t=0} &= r_0 \quad \text{in} \quad T_L,
\end{align*}$$ (2.3)

satisfies

$$r|_{t=T} = 0 \quad \text{in} \quad T_L.$$ (2.4)

We introduce Carleman estimates derived from Carleman estimates for real coefficients heat equation established in [1]. In our context we need this Carleman estimate also for complex coefficients heat equation which we establish in the appendix.

2.1 Construction of the weight function

Let $\psi$ be a function $\psi$ in $C^2(T_L, \mathbb{R})$ such that, for every $x \in T_L$

$$\psi(x) \in [6, 7],$$ (2.5)

and

$$\inf_{T_L \setminus \overline{\omega}} \{ |\nabla \psi| \} > 0.$$ (2.6)

Such a function exists according to [28, Theorem 9.4.3, p.299].

We choose $T_0 > 0$ and $\frac{1}{T} \geq T_1 > 0$ small enough, so that

$$T_0 + 2T_1 < T.$$ (2.7)

For any $m \geq 2$, we introduce a weight function $\theta_m \in C^2([0, T))$ such that

$$\theta_m(t) = \begin{cases} 
1 + \left(1 - \frac{t}{T_0}\right)^m & \text{for all } t \in [0, T_0], \\
1 & \text{for all } t \in [T_0, T - 2T_1], \\
\theta_m \text{ is increasing} & \text{in } [T - 2T_1, T - T_1], \\
\frac{1}{T - t} & \text{for all } t \in [T - T_1, T].
\end{cases}$$ (2.7)
Then we consider the following weight function, given for $s \geq 1$ and $\lambda \geq 1$, and for any $(t, x) \in [0, T) \times \mathbb{T}_L$ by

$$\varphi_{s, \lambda}(t, x) := \theta_m(t)(\lambda e^{12\lambda} - e^{\lambda \psi(x)}), \quad \text{where} \quad m = s\lambda^2 e^{2\lambda}, \quad (2.8)$$

which is always larger than 2.

In the following, for simplifying notations, we will always denote $\varphi_{s, \lambda}$ and $\theta_m$ simply by $\varphi$ and $\theta$.

Note that $\theta$ is bounded by below by a positive constant, more precisely

$$\theta \geq 1 \quad \text{in} \quad [0, T). \quad (2.9)$$

We point out that, due to the definition of $\psi$ and to Condition (2.5), and using that $\lambda \geq 1$, we have the following bounds for any $(t, x)$ in $[0, T) \times \mathbb{T}_L$:

$$\frac{3}{4} \Phi(t) \leq \varphi(t, x) \leq \Phi(t), \quad (2.10)$$

where\(^1\)

$$\Phi(t) := \theta(t)\lambda e^{12\lambda}. \quad (2.11)$$

### 2.2 Controllability results for the complex coefficients heat equation

In this section we give some tools related to the controllability of the heat equation. We use classical method to study the controllability properties of (2.3), which is based on the observability of the adjoint system, obtained here with the following Carleman estimates for the heat equation which we introduce in the following lemma.

**Lemma 2.1.** Let $T > 0$ and $\zeta$ a complex number satisfying (2.1). There exist three positive constants $C$, $s_0 \geq 1$ and $\lambda_0 \geq 1$, large enough, such that for any smooth function $w$ on $[0, T] \times \mathbb{T}_L$ and for all $s \geq s_0$, we have

$$s^2 \| \theta_{s} \psi we^{-s\varphi} \|_{L^2(L^2)} + s^2 \| \theta_{s} \nabla we^{-s\varphi} \|_{L^2(L^2)} + s \| w(0)e^{-s\varphi(0)} \|_{L^2}$$

$$\leq C \left( \| (-\partial_t - \zeta \Delta) we^{-s\varphi} \|_{L^2(L^2)} + s^2 \| \theta_{s} \chi_0 we^{-s\varphi} \|_{L^2(L^2)} \right).$$

We give the proof of this lemma in Appendix A. Note that, when $\zeta$ is a positive real number, this is established in [1]. When $\zeta$ is a complex number satisfying (2.1), the Carleman estimates in Theorem 2.1 have not been done in the literature with the weight function $\varphi$ defined in (2.8). However, similar Carleman estimates have been obtained when the weight function is the one in [18] (see also [28] Subsections 9.4, 9.5 and 9.6 for more comments on the control of parabolic equations), which is singular at $t = 0$ and at $t = T$, see in particular the works [17] or [25].

Still, the proof of Theorem 2.1 in the case of non-real parameter $\zeta$ is not completely contained in [1] nor in [17, 25], and some terms need to be handled carefully, which is why we give the complete proof of Lemma 2.1 in Appendix A.

Note that we chose to take the weight function $\varphi$ in (2.8), which is not singular at time $t = 0$, since by duality, as we will see afterwards, it allows to solve directly the control problem (2.3)–(2.4) without relying on the well-posedness of the equations. Regarding the heat equation (2.3), this is of course not a big issue, but this is more subtle when dealing with systems.

The proof of the following results in this subsection are left to the reader since they are straightforward adaptations of the corresponding results from [14] and [24]. As in [14], these estimates lead to the following controllability result which is an adaptation of Theorem 3.3 of [14] to the case of complex coefficients heat equation.

\(^1\)In [14], the authors use $\frac{14}{15}\Phi \leq \varphi$ in $[0, T]\times\mathbb{T}_L$, while in this article, we only use $\frac{4}{7}\Phi \leq \varphi$ in $[0, T]\times\mathbb{T}_L$, which is of course weaker.
Theorem 2.2. Let $T > 0$. Assume that $\zeta$ satisfies (2.1). There exist constants $C > 0$ and $s_0 \geq 1$ such that for all $s \geq s_0$, for all $f \in L^2(0, T; L^2(\mathbb{T}_L))$ satisfying
\[ \|\theta^{-\frac{3}{2}} f e^{s\varphi}\|_{L^2(L^2)} < +\infty \] (2.12)
and $r_0 \in L^2(\mathbb{T}_L)$, there exists a solution $(r, v_r)$ of the control problem (2.3)-(2.4) which furthermore satisfies the following estimate:
\[ s^\frac{3}{2} \|r e^{s\varphi}\|_{L^2(L^2)} + \|\theta^{-\frac{3}{2}} \chi_0 v_r e^{s\varphi}\|_{L^2(L^2)} + s^\frac{1}{2} \|\theta^{-1} \nabla r e^{s\varphi}\|_{L^2(L^2)} \leq C \left( \|\theta^{-\frac{3}{2}} f e^{s\varphi}\|_{L^2(L^2)} + s^\frac{1}{2} \|r_0 e^{s\varphi(0)}\|_{L^2} \right). \] (2.13)
Moreover, the solution $(r, v_r)$ can be obtained through a linear operator in $(r_0, f)$.

We need also to know what can be done when the source term $f$ is more regular and lies in $L^2(0, T; H^1(\mathbb{T}_L))$ or in $L^2(0, T; H^2(\mathbb{T}_L))$ (c.f. [1] Proposition 3.4). By adapting the proof of Proposition 3.4 of [14], we deduce the following lemma.

Lemma 2.3. Let $T > 0$. Consider the solution $(r, v_r)$ construct in Theorem 2.2. Then, with the above notation, for some constant $C > 0$ independent of $s$, we have the following properties:

1. ($H^2$ regularity of the control $v_r \in L^2(0, T; H^2(\mathbb{T}_L))$ and
\[ \|\chi_0 v_r e^{3s\Phi/4}\|_{L^2(H^2)} \leq C \left( \|\theta^{-\frac{3}{2}} f e^{s\varphi}\|_{L^2(L^2)} + \|r_0 e^{s\varphi(0)}\|_{L^2} \right). \]

2. ($H^3$ regularity estimate for the state) if $r_0 \in H^2(\mathbb{T}_L)$, $f e^{3s\Phi/4} \in L^2(0, T; H^1(\mathbb{T}_L))$ and $\theta^{-\frac{3}{2}} f e^{s\varphi} \in L^2(0, T; L^2(\mathbb{T}_L))$, then $r \in L^2(0, T; H^3(\mathbb{T}_L))$ and
\[ \|r e^{3s\Phi/4}\|_{L^2(H^3)} \leq C \left( \|f e^{3s\Phi/4}\|_{L^2(H^1)} + \|\theta^{-\frac{3}{2}} f e^{s\varphi}\|_{L^2(L^2)} + \|r_0 e^{s\varphi(0)}\|_{H^2} \right). \]

For later use, we also present the following results, which contain a shift in the weight $\theta$ compared to Theorem 2.2 and Lemma 2.3. The following results are modified versions of Theorem 3.3 of [24] which also can be adapted to the case of complex coefficients heat equation.

Theorem 2.4. Let $T > 0$. Assume that $\zeta$ satisfies (2.1). There exist a positive constant $C$ and a real number $s_0 \geq 1$ such that for all $s \geq s_0$ and for all $f$ satisfying
\[ \|\theta^{-1} f e^{s\varphi}\|_{L^2(0, T; L^2(\mathbb{T}_L))} < +\infty \] (2.14)
and $r_0 \in H^1(\mathbb{T}_L)$, the solution $(r, v_r)$ of the control problem (2.3)-(2.4) satisfies
\[ s^\frac{3}{2} \|\theta^\frac{1}{2} r e^{s\varphi}\|_{L^2(L^2)} + \|\theta^{-1} \chi_0 v_r e^{s\varphi}\|_{L^2(L^2)} + s^\frac{1}{2} \|\theta^{-\frac{1}{2}} \nabla r e^{s\varphi}\|_{L^2(L^2)} + s^{-\frac{1}{2}} \|\theta^{-\frac{3}{2}} r^2 e^{s\varphi}\|_{L^2(L^2)} \leq C \left( \|\theta^{-1} f e^{s\varphi}\|_{L^2(L^2)} + s^\frac{1}{2} \|r_0 e^{s\varphi(0)}\|_{L^2} + s^{-\frac{1}{2}} \|\nabla r_0 e^{s\varphi(0)}\|_{L^2} \right). \] (2.15)
Moreover, the solution $(r, v_r)$ can be obtained through a linear operator in $(r_0, f)$.

Lemma 2.5. Let $T > 0$. Consider the solution $(r, v_r)$ constructed in Theorem 2.4. Then, with the above notations, for some constant $C > 0$ independent of $s$, we have the following properties:

1. ($H^2$ regularity of the control $v_r \in L^2(0, T; H^2(\mathbb{T}_L))$ and
\[ \|\chi_0 v_r e^{3s\Phi/4}\|_{L^2(H^2)} \leq C \left( \|\theta^{-1} f e^{s\varphi}\|_{L^2(L^2)} + \|r_0 e^{s\varphi(0)}\|_{L^2} \right). \]
2. \((H^3\) regularity estimate for the state) if \(r_0 \in H^2(T_L), \; fe^{3\sqrt{\Phi}/4} \in L^2(0, T; H^1(T_L))\) and \(\theta^{-1} fe^{\varepsilon_2} \in L^2(0, T; H^2(T_L))\) and
\[
\|re^{3\sqrt{\Phi}/4}\|_{L^2(H^2)} \leq C \left( \|fe^{3\sqrt{\Phi}/4}\|_{L^2(H^1)} + \|\theta^{-1} fe^{\varepsilon_2}\|_{L^2(L^2)} + \|r_0 e^{\varepsilon(0)}\|_{H^2} \right).
\]

3 Strategy

In order to perform a perturbative argument by fixed-point, we first recast the system into a more user-friendly shape. Let us set
\[
a := \frac{\rho}{\rho_*} - 1.
\]

(3.1)

We are then led to study the following system
\[
\begin{aligned}
\partial_t a + \text{div}(u) &= f_a(a, u) + v_a 1_\omega \\
\partial_t u - \mu_* \Delta u - (\mu_* + \nu_*) \nabla \text{div}(u) + p_* \nabla a - \kappa_* \nabla \Delta a &= f_u(a, u) + v_u 1_\omega
\end{aligned}
\]
in \((0, T) \times T_L, \) (3.2)

where
\[
\kappa_* := \rho_\kappa(\rho_*), \quad \mu_* := \rho_*^{-1}\mu(\rho_*), \quad \nu_* := \rho_*^{-1}\nu(\rho_*), \quad p_* := P'(\rho_*)
\]

and
\[
\begin{aligned}
\begin{cases}
f_a(a, u) := -(a + 1) u \cdot \nabla a, \\
f_u(a, u) := \frac{1}{\mu} \left[ \text{div}(2\mu(a) D_S(u)) + \nabla(\mu(a) \text{div} u) \right], \\
\partial_u a, \\
\partial_u a, \\
P'(\rho) \nabla a, \\
(a + 1) \nabla \left( \frac{a}{\mu} \nabla a + \nabla \kappa(a) \cdot \nabla a \right)
\end{cases}
\end{aligned}
\]

and
\[
\begin{aligned}
\kappa(a) &:= \rho_\kappa(\rho_* a + \rho_*) - \kappa(\rho_*), \\
\mu(a) &:= \rho_*^{-1}(\mu(\rho_* a + \rho_*) - \mu(\rho_*)), \\
\nu(a) &:= \rho_*^{-1}(\nu(\rho_* a + \rho_*) - \nu(\rho_*)), \\
P'(\rho) &:= P'(\rho_* a + \rho_*) - P'(\rho_*)
\end{aligned}
\]

This form of System (1.3) is more convenient since the linearized equations around \((\rho_*, 0)\) explicitly appear in the left-hand side of (3.2).

Also note that we can recover strong solutions of (1.1) from strong solutions of (3.2) by using (3.1) under the form
\[
\rho = \rho_* a + \rho_*.
\]

This leads to studying the distributed null controllability problem associated to (3.2) on \(T_L\). Then we consider the following control problem: Given \((a_0, u_0)\) small enough, find control functions \((v_a, v_u)\) on \((0, T) \times T_L\) such that the solution \((a, u)\) of
\[
\begin{aligned}
\partial_t a + \text{div}(u) &= f_a(a, u) + v_a 1_\omega \\
\partial_t u - \mu_* \Delta u - (\mu_* + \nu_*) \nabla \text{div}(u) + p_* \nabla a - \kappa_* \nabla \Delta a &= f_u(a, u) + v_u 1_\omega
\end{aligned}
\]
in \((0, T) \times T_L, \) (3.4)

satisfies
\[
(a, u)|_{t=T} = (0, 0) \quad \text{in} \ T_L.
\]

Then, Theorem 1.3 is equivalent to the following theorem.
**Theorem 3.1.** Let $d \in \{1, 2, 3\}$ and $T > 0$. There exists $\delta > 0$ such that, for all $(a_0, u_0) \in H^2(\mathbb{T}_L) \times H^1(\mathbb{T}_L)$ satisfying
\[
\|(a_0, u_0)\|_{H^2 \times H^1} \leq \delta,
\] (3.6)
there exist control functions $(v_a, v_u)$ in $L^2(0, T; H^2(\mathbb{T}_L)) \times L^2(0, T; H^1(\mathbb{T}_L))$ and a corresponding controlled trajectory $(a, u)$ solving (3.4) with initial data $(a_0, u_0)$ and satisfying the control requirement (3.5). Besides, the controlled trajectory $(a, u)$ enjoys the following regularity
\[
a \in C([0, T]; H^2(\mathbb{T}_L)) \cap L^2(0, T; H^3(\mathbb{T}_L)) \cap H^1(0, T; H^1(\mathbb{T}_L)),
\]
\[
u \in C([0, T]; H^1(\mathbb{T}_L)) \cap L^2(0, T; H^2(\mathbb{T}_L)) \cap H^1(0, T; L^2(\mathbb{T}_L)).
\]
and the following bound from below
\[
\inf_{(t, x) \in [0, T] \times \mathbb{T}_L} a(t, x) > -1.
\]

To take into account the support of the control functions $(v_a, v_u)\mathbf{1}_\omega$ and facilitate regularity issues, we replace $\mathbf{1}_\omega$ by a smooth cut-off function $\chi \in C_c^\infty(\omega, [0, 1])$ satisfying
\[
\chi = 1 \text{ on } \omega_1,
\]
recall (2.2) for the definition of $\omega_1$ compared to $\omega$. In other words, we consider the following control problem: Given $(a_0, u_0)$ small in $H^2(\mathbb{T}_L) \times H^1(\mathbb{T}_L)$, find control functions $(v_a, v_u)$ in $L^2(H^1) \times L^2(L^2)$ such that the solution $(a, u) \in L^2(H^3) \times L^2(H^2)$ of
\[
\begin{aligned}
\begin{cases}
\partial_t a + \text{div}(u) = f_a(a, u) + v_a \chi & \text{in } (0, T) \times \mathbb{T}_L, \\
\partial_t u - \mu_\ast \triangle u - (\mu_\ast + \nu_\ast) \text{div}(u) + p_\ast \nabla a - \kappa_\ast \nabla \triangle a = f_u(a, u) + v_u \chi & \text{in } (0, T) \times \mathbb{T}_L,
\end{cases}
\end{aligned}
\] (3.7)
with initial data
\[
(a, u)|_{t=0} = (a_0, u_0) \quad \text{in } \mathbb{T}_L,
\] (3.8)
satisfies (3.5).

Of course, this will be done by the analysis of the control properties of the linearized system
\[
\begin{aligned}
\begin{cases}
\partial_t a + \text{div}(u) = f_a + v_a \chi & \text{in } (0, T) \times \mathbb{T}_L, \\
\partial_t u - \mu_\ast \triangle u - (\mu_\ast + \nu_\ast) \text{div}(u) + p_\ast \nabla a - \kappa_\ast \nabla \triangle a = f_u + v_u \chi & \text{in } (0, T) \times \mathbb{T}_L,
\end{cases}
\end{aligned}
\] (3.9)
where $f_a$ and $f_u$ are given\(^2\). Namely, we establish the following theorem whose proof is given in Section 6.

**Theorem 3.2.** Let $T > 0$. There exist a real number $s_0 \geq 1$ and a positive constant $C$, such that for any $(a_0, u_0) \in H^2(\mathbb{T}_L) \times H^1(\mathbb{T}_L)$ and $f_a, f_u$ such that $f_a e^{\frac{4s_0}{3}} \in L^2(0, T; H^1(\mathbb{T}_L))$ and $f_u e^4 \in L^2(0, T; L^2(\mathbb{T}_L))$, there exist two control functions $v_a$ and $v_u$ and a corresponding controlled trajectory $(a, u)$ solving (3.9) with initial data $(a_0, u_0)$, satisfying the controllability requirements (3.5)\(^2\).

\(^2\)Note that index $a$ and $u$ aims to indicate in which equations $f_a$ and $f_u$ appear in the lines of the system according to the terms $\partial_t a$ and $\partial_t u$ respectively. In particular, $f_a$ and $f_u$ do not depend on $a$ and $u$ except if it explicitly appear, as in $f_a(a, u)$ or $f_u(a, u)$. In this article, we use similar notations for source terms in (3.9), (3.11), (3.12), (3.15), (3.16), (3.19) and (3.20).
and depending linearly on the data \((a_0, u_0, f_a, f_u)\). Besides, we have the following estimate

\[
\| (\partial_t a, \partial_x u) e^{2\rho_0 \Phi} \|_{L^2(H^1) \times L^2(L^2)} + \| (a, u) e^{2\rho_0 \Phi} \|_{L^2(H^1) \cap L^\infty(H^2) \times L^2(H^2) \cap L^\infty(H^1)} \\
+ \| \chi(v, v_u) e^{2\rho_0 \Phi} \|_{L^2(H^1) \times L^2(L^2)} \leq C \left( \| (f_a, f_u) e^{4\rho_0 \Phi} \|_{L^2(H^1) \times L^2(L^2)} + \| (a_0, u_0) \|_{H^n \times H^1} \right).
\]

(3.10)

This allows us to define a linear operator \(G\) defined on the space

\[
\left\{ (a_0, u_0, f_a, f_u) \in H^2 \times H^1 \times L^2(H^1) \times L^2(L^2) \mid f_a e^{4\rho_0 \Phi} \in L^2(H^1) \text{ and } f_u e^{4\rho_0 \Phi} \in L^2(L^2) \right\},
\]

by

\[
G(a_0, u_0, f_a, f_u) = (a, u),
\]

where \((a, u)\) is the controlled trajectory, with initial condition \((a_0, u_0)\) and forces \((f_a, f_u)\), satisfying the control requirement (3.5) and Estimate (3.10).

Since System (3.9) is linear, its controllability in \(L^2(H^3) \times L^2(H^2)\) is equivalent to the following observability estimate

\[
\| \sigma e^{-\rho_0 \Phi} \|_{L^2(H^{-1})} + \| \sigma(0) e^{-\rho_0 \Phi(0)} \|_{H^{-2}} + \| ze^{-\frac{4\rho_0 \Phi}{3}} \|_{L^2(L^2)} + \| z(0) e^{-\frac{4\rho_0 \Phi(0)}{3}} \|_{H^{-1}} \\
\leq \| (g_\sigma, g_z) e^{-\frac{3\rho_0 \Phi}{2}} \|_{L^2(H^{-3}) \times L^2(H^{-2})} + \| \chi(\sigma, z) e^{-\frac{3\rho_0 \Phi}{4}} \|_{L^2(H^{-1}) \times L^2(L^2)},
\]

where \((\sigma, z)\) is a solution of the following adjoint system

\[
\begin{align*}
-\partial_t \sigma - p_* \text{div}(z) + \kappa_* \Delta \text{div}(z) &= g_\sigma & \text{in } (0, T) \times T_L, \\
-\partial_t z - \nabla \sigma - \mu_* \Delta z - (\mu_* + \nu_*) \nabla \text{div}(z) &= g_z & \text{in } (0, T) \times T_L,
\end{align*}
\]

(3.11)

with \((g_\sigma, g_z) \in L^2(H^{-1}) \times L^2(L^2)\).

The main idea to get this observability estimate for (3.11) is based on the fact that, with

\[
q := \text{div}(z),
\]

\((\sigma, q)\) satisfies the closed subsystem

\[
\begin{align*}
-\partial_t \sigma - p_* q + \kappa_* \Delta q &= g_\sigma & \text{in } (0, T) \times T_L, \\
-\partial_t q - \Delta \sigma - (2\mu_* + \nu_*) \Delta q &= g_q & \text{in } (0, T) \times T_L,
\end{align*}
\]

(3.12)

where \(g_q := \text{div}(g_z)\).

We will thus base our analysis on the following observability estimate

\[
\| (\sigma, q) e^{-\rho_0 \Phi} \|_{L^2(H^{-1}) \times L^2(L^{-2})} + \| (\sigma(0), q(0)) e^{-\rho_0 \Phi(0)} \|_{H^{-2} \times H^{-2}} \\
\leq \| (g_\sigma, g_q) e^{\frac{3\rho_0 \Phi}{4}} \|_{L^2(H^{-3}) \times L^2(H^{-2})} + \| \chi(\sigma, q) e^{-\frac{3\rho_0 \Phi}{4}} \|_{L^2(H^{-1}) \times L^2(H^{-1})},
\]

for the solutions \((\sigma, q)\) of (3.12), where \(\chi_0\) is the cut-off function in (2.2).

In order to do that and to underline the parabolic behavior of (3.12), we rely on the analysis of the matrix

\[
\begin{pmatrix}
0 \\
-1 - (2\mu_* + \nu_*)
\end{pmatrix}
\]

(3.13)

which coincides with the matrix \(^tA\), where \(A\) is given in (1.7).

As said in the introduction, our analysis will then be divided into two cases: when \(^tA\) is diagonalizable (equivalently \(A\)), and when \(^tA\) (equivalently \(A\)) is not diagonalizable.

When \(A\) is diagonalizable, System (3.12) is a parabolic or parabolic/dispersive system in which
the coupling is done through lower order terms. We can then use directly Lemma 2.1 to obtain $L^2$ observability results for System (3.12). However, there still remains an additional difficulty to obtain observability results for System (3.12) in negative index Sobolev spaces. This is done by duality by obtaining controllability results for the adjoint of System (3.12) in spaces of higher regularity, based on Theorem 2.2 and Lemma 2.3.

When $A$ is not diagonalizable, System (3.12) is a parabolic system in which the coupling is done through the leading order. This does not prevent us to follow the same strategy, but one needs to perform a slight shift of a power of the function $\theta$ in the control results. This is why we introduced Theorem 2.4 and Lemma 2.5.

### 3.1 Diagonalizable case

We assume in this subsection that (3.13) is diagonalizable. Then, setting

$$\zeta_+ := \frac{(2\mu_* + \nu_*) - D}{2}, \quad \zeta_- := \frac{(2\mu_* + \nu_*) + D}{2}, \quad \text{where} \quad 3 \ D := \sqrt{(2\mu_* + \nu_*)^2 - 4\kappa_*},$$

(note that $\Re(\zeta_+) > 0$ according to (H1)), the matrix $A$ is equivalent to the diagonal matrix $\text{diag}(\zeta_+, \zeta_-)$. This can be done through the $2 \times 2$ invertible matrix $Q$ given by

$$Q := \begin{pmatrix} \frac{\zeta_+}{D} & \frac{\kappa_*}{D} \\ -\frac{\zeta_-}{D} & \frac{\kappa_*}{D} \end{pmatrix}.$$  \hspace{1cm} (3.14)

In particular, $(\sigma, q)$ solves System (3.12) if and only if the new unknowns

$$\begin{pmatrix} y^+ \\ y^- \end{pmatrix} := Q \begin{pmatrix} \sigma \\ q \end{pmatrix},$$

satisfy

$$\begin{cases} -\partial_t y^+ - \zeta_+ \Delta y^+ = g_{y^+} + \alpha_1 y^+ + \alpha_2 y^- \quad &\text{in} \ (0, T) \times \mathbb{T}_L, \\ -\partial_t y^- - \zeta_- \Delta y^- = g_{y^-} + \alpha_3 y^+ + \alpha_4 y^- \quad &\text{in} \ (0, T) \times \mathbb{T}_L, \end{cases}$$  \hspace{1cm} (3.15)

with

$$\alpha_1 := \frac{\zeta_+ - p_*}{(\zeta_+ + \zeta_-)\kappa_*}, \quad \alpha_2 := \frac{\zeta_- p_*}{(\zeta_+ + \zeta_-)\kappa_*}, \quad \alpha_3 := \frac{\zeta_+ + p_*}{\zeta_+ + \zeta_-}, \quad \alpha_4 := -\frac{\zeta_- + p_*}{\zeta_+ + \zeta_-},$$

and

$$\begin{pmatrix} g_{y^+} \\ g_{y^-} \end{pmatrix} := Q \begin{pmatrix} g_{\sigma} \\ g_{q} \end{pmatrix}.$$

The $L^2$-observability for this system is well-known and follows directly from the Carleman estimates for parabolic system (see [18] and [28] Subsections 9.4, 9.5 and 9.6 for more comments). A part of the difficulty here is to obtain this observability estimate in negative index Sobolev spaces. To this aim, we will adapt the strategy from [14], and prove the following controllability result in Sobolev spaces of strong regularity:

*Given $(r_0^+, r_0^-) \in H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L)$, find two control $(v_{r^+}, v_{r^-})$ in $L^2(H^1) \times L^2(H^1)$ such that the solution $(r^+, r^-)$ of

$$\begin{cases} \partial_t r^+ - \zeta_+ \Delta r = f_{r^+} + \overline{\alpha}_1 r^+ + \overline{\alpha}_3 r^- + \chi_0 v_{r^+} &\text{in} \ (0, T) \times \mathbb{T}_L, \\ \partial_t r^- - \zeta_- \Delta r^- = f_{r^-} + \overline{\alpha}_2 r^+ + \overline{\alpha}_4 r^- + \chi_0 v_{r^-} &\text{in} \ (0, T) \times \mathbb{T}_L, \end{cases}$$  \hspace{1cm} (3.16)

satisfies

$$(r^+, r^-)_{t=0} = (r_0^+, r_0^-) \quad \text{and} \quad (r^+, r^-)_{t=T} = (0, 0) \quad \text{in} \ \mathbb{T}_L,$$  \hspace{1cm} (3.17)
and belongs to $L^2(H^3) \times L^2(H^3)$. We treat this problem in Section 4. The proof is based on the estimates of Theorem 2.2 and Lemma 2.3.

### 3.2 Non-diagonalizable case

In this subsection we assume that (3.13) is non-diagonalizable. In this case we set

$$\zeta = \frac{2\mu_\ast + \nu_\ast}{2} > 0,$$

and, to make $\mathcal{A}$ triangular, we consider the $2 \times 2$ invertible matrix $R$ given by

$$R := \begin{pmatrix} 1 & 0 \\ \frac{\zeta}{2} & 1 \end{pmatrix}. \quad (3.18)$$

Through explicit computations, we then check that $(\sigma, q)$ solves System (3.12) if and only if the new unknowns

$$\begin{pmatrix} y^+ \\ y^- \end{pmatrix} := R \begin{pmatrix} \sigma \\ q \end{pmatrix}$$

satisfies

$$\begin{cases} -\partial_t y^+ - \zeta \triangle y^+ = g_y^+ + \beta_1 y^+ + \beta_2 y^- - \kappa_\ast \triangle y^- & \text{in } (0, T) \times \mathbb{T}_L, \\ -\partial_t y^- - \zeta \triangle y^- = g_y^- + \beta_3 y^+ + \beta_4 y^- & \text{in } (0, T) \times \mathbb{T}_L, \end{cases} \quad (3.19)$$

with

$$\beta_1 := -\frac{p_\ast}{\zeta}, \quad \beta_2 := p_\ast, \quad \beta_3 := -\frac{p_\ast}{\zeta}, \quad \beta_4 := \frac{p_\ast}{\zeta}.$$

Since System (3.19) is linear, its observability is equivalent to the controllability statement for this adjoint system written in the dual variables $(r^+, r^-)$, where the adjoint is taken with respect to the variables $(y^+, y^-)$. And again, to get an observability result in Sobolev spaces of weak regularity on $(y^+, y^-)$, we will consider the controllability of the adjoint in Sobolev spaces of high regularity as follows.

Given $(r^+_0, r^-_0)$ in $H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L)$, find two control $(v_{r^+}, v_{r^-})$ in $L^2(H^1) \times L^2(H^1)$ such that the solution $(r^+, r^-)$ of the

$$\begin{cases} \partial_t r^+ - \zeta \triangle r^+ = f_{r^+} + \beta_1 r^+ + \beta_3 r^- + \chi_0 v_{r^+} & \text{in } (0, T) \times \mathbb{T}_L, \\ \partial_t r^- - \zeta \triangle r^- = f_{r^-} + \beta_2 r^+ + \beta_4 r^- - \kappa_\ast \triangle r^+ + \chi_0 v_{r^-} & \text{in } (0, T) \times \mathbb{T}_L, \end{cases} \quad (3.20)$$

satisfies

$$(r^+, r^-)_{|t=0} = (r^+_0, r^-_0) \quad \text{and} \quad (r^+, r^-)_{|t=T} = (0, 0) \quad \text{in } \mathbb{T}_L,$$

and belongs to $L^2(H^3) \times L^2(H^3)$.

We treat this problem in Section 5. The proof is based on Theorem 2.4 and Lemma 2.5.

### 4 Controllability of (3.16): The diagonalizable case

This section is devoted to the controllability of (3.16). We recall that this system corresponds to the case in which (3.13) is diagonalizable. We aim to establish the following theorem.

**Lemma 4.1.** Let $T > 0$. Let $(r^+_0, r^-_0) \in H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L)$. There exist a positive constant $C$ and a
real number \( s_0 \geq 1 \) such that for all \( s \geq s_0 \), for all \( f_{r+} \) and \( f_{r-} \) in \( L^2(L^2) \) such that

\[
\|\theta^{-\frac{3}{2}}(f_{r+}, f_{r-})e^{s\varphi}\|_{L^2(L^2)} < +\infty \quad (4.1)
\]

and

\[
(f_{r+} + f_{r-})e^\Phi \in L^2(0, T; H^1(\mathbb{T}_L)),
\]

there exists a controlled trajectory \((r^+, r^-)\) solving (3.16) and satisfying the following estimate

\[
\| (r^+, r^-)e^{3s\Phi / 4} \|_{L^2(H^3)} + \| \chi_0(v_{r+}, v_{r-})e^{3s\Phi / 4} \|_{L^2(H^1)} \\
\leq C \left( \| (f_{r+}, f_{r-})e^{\Phi} \|_{L^2(H^1)} + \| (r^+_0, r^-_0)e^{\Phi(0)} \|_{H^2} \right),
\]

\[
(4.3)
\]

**Proof.** We will prove the controllability of (3.16) by a fixed-point argument based on the control results obtained in Theorem 2.2 and Lemma 2.3. The proof follows the strategy of [14].

**Existence of the solution to the control problem.** We construct the controlled trajectory using a Banach fixed-point argument. We introduce the linear space

\[
C_s := \{ r \in L^2(0, T; L^2(\mathbb{T}_L)) \mid re^{s\varphi} \in L^2(0, T; L^2(\mathbb{T}_L)) \}.
\]

For \( \hat{r}^+ \) and \( \hat{r}^- \) in \( C_s \), we introduce

\[
\begin{align*}
\hat{f}_{r+} &:= \hat{f}_{r+}(\hat{r}^+, \hat{r}^-) = f_{r+} + \alpha_1 \hat{r}^+ + \alpha_3 \hat{r}^-,
\hat{f}_{r-} &:= \hat{f}_{r-}(\hat{r}^+, \hat{r}^-) = f_{r-} + \alpha_2 \hat{r}^+ + \alpha_4 \hat{r}^-.
\end{align*}
\]

As \( f_{r+} \) and \( f_{r-} \) satisfy (4.1), for every \((\hat{r}^+, \hat{r}^-)\) in \( C_s \times C_s \), \( \hat{f}_{r+} \) and \( \hat{f}_{r-} \) satisfy Assumption (2.12) of Theorem 2.2. In fact, Theorem 2.2 provides two linear maps \((r^+_0, f_{r+}) \mapsto (r^+, v_{r+})\) and \((r^-_0, f_{r-}) \mapsto (r^-, v_{r-})\). Therefore, one can define a map \( \Lambda_s \) on \( C_s \times C_s \) which to a data \((\hat{r}^+, \hat{r}^-)\) in \( C_s \times C_s \) associates \((r^+, r^-)\) where \( r^+ \) and \( r^- \) are respectively the solutions of the following controlled systems

\[
\begin{align*}
\partial_t r^+ - \triangle r^+ &= \hat{f}_{r+} + v_{r+} \chi_0 \text{ in } (0, T) \times \mathbb{T}_L, \\
r^+_{l=0} &= r^+_0, \quad r^+_{l=T} = 0 \text{ in } \mathbb{T}_L,
\end{align*}
\]

and

\[
\begin{align*}
\partial_t r^- - \triangle r^- &= \hat{f}_{r-} + v_{r-} \chi_0 \text{ in } (0, T) \times \mathbb{T}_L, \\
r^-_{l=0} &= r^-_0, \quad r^-_{l=T} = 0 \text{ in } \mathbb{T}_L,
\end{align*}
\]

given by Theorem 2.2. As in [1], using estimates of Theorem 2.2, we show that for any \((r^+_1, r^+_1)\) and \((r^-_2, r^-_2)\) in \( C_s \times C_s \), we have

\[
\|(\Lambda_s(\hat{r}^+_1, \hat{r}^-_2) - \Lambda_s(\hat{r}^+_1, \hat{r}^-_2))e^{s\varphi}\|_{L^2(L^2)} \leq C_1 s^{-\frac{3}{2}} \|(\hat{r}^+_1, \hat{r}^-_2) - (\hat{r}^+_1, \hat{r}^-_2)\|_{L^2(L^2)} e^{s\varphi},
\]

\[
(4.4)
\]

for large enough \( s \geq 1 \), where \( C_1 \) is a constant which does not depend on \( s \). By equipping the space \( C_s \times C_s \) with the following norm given for any \((r^+, r^-)\) in \( C_s \times C_s \) by

\[
\|(r^+, r^-)\|_{C_s \times C_s} := \|(r^+, r^-)e^{s\varphi}\|_{L^2(L^2)},
\]

then \( C_s \times C_s \) is a Banach space. Then, for \( s \geq 1 \) large enough, it follows from (4.4) that the map \( \Lambda_s \) is a strict contraction from \( C_s \times C_s \) into itself. Applying Banach’s fixed-point theorem, we deduce that \( \Lambda_s \) has a unique fixed-point \((r^+, r^-)\) in \( C_s \times C_s \). By construction, this fixed-point \((r^+, r^-)\) solves the
controllability problem (3.16)-(3.17). Furthermore, applying Theorem 2.2 to \( r^+ \) and \( r^- \), it follows that
\[
\begin{align*}
\frac{3}{2} \| (r^+, r^-) e^{s\varphi} \|_{L^2(L^2)} + \frac{1}{2} \| \theta^{-1} \nabla (r^+, r^-) e^{s\varphi} \|_{L^2(L^2)} + \| \theta^{-\frac{3}{2}} \chi_0 (v_{r^+}, v_{r^-}) e^{s\varphi} \|_{L^2(L^2)} \\
\leq C_2 \left( \left\| \theta^{-\frac{3}{2}} (f_{r^+}, f_{r^-}) e^{s\varphi} \right\|_{L^2(L^2)} + \left\| \theta^{-\frac{3}{2}} (r^+, r^-) e^{s\varphi} \right\|_{L^2(L^2)} + \frac{3}{2} \left\| (r^+_0, r^-_0) e^{s\varphi(0)} \right\|_{L^2(L^2)} \right),
\end{align*}
\]
where \( C_2 \) is a positive constant, which does not depend on \( s \). Since \( \theta^{-\frac{3}{2}} \leq 1 \), by taking \( s \geq 1 \) large enough to absorb the second term of the right-hand side of the above estimate, we finally obtain
\[
\begin{align*}
\frac{3}{2} \| (r^+, r^-) e^{s\varphi} \|_{L^2(L^2)} + \frac{1}{2} \| \theta^{-1} \nabla (r^+, r^-) e^{s\varphi} \|_{L^2(L^2)} + \| \theta^{-\frac{3}{2}} \chi_0 (v_{r^+}, v_{r^-}) e^{s\varphi} \|_{L^2(L^2)} \\
\leq C_2 \left( \left\| \theta^{-\frac{3}{2}} (f_{r^+}, f_{r^-}) e^{s\varphi} \right\|_{L^2(L^2)} + s_{\Delta} \left\| (r^+_0, r^-_0) e^{s\varphi(0)} \right\|_{L^2(L^2)} \right). \tag{4.5}
\end{align*}
\]

**5 Controllability of (3.20): The non-diagonalizable case**

In this section we are interested in the controllability of System (3.20).

**Lemma 5.1.** Let \( T > 0 \). Let \((r^+_0, r^-_0) \in H^2(T_L) \times H^2(T_L)\). There exist a positive constant \( C \) and a real number \( s_0 \geq 1 \) such that for all \( s \geq s_0 \), for all \( f_{r^+} \) and \( f_{r^-} \) in \( L^2(L^2) \) such that
\[
\left\| \left( \theta^{-\frac{3}{2}} f_{r^+}, \theta^{-1} f_{r^-} \right) e^{s\varphi} \right\|_{L^2(L^2)} < +\infty
\]
and
\[
\left( f_{r^+}, f_{r^-} \right) e^{s\varphi} \in L^2(0, T; H^1(T_L)),
\]
there exists a controlled trajectory \((r^+, r^-)\) solving (3.20) and satisfying the following estimate
\[
\begin{align*}
\left\| (r^+, r^-) e^{3s\Phi/4} \right\|_{L^2(H^2)} + \left\| \chi_0 (v_{r^+}, v_{r^-}) e^{3s\Phi/4} \right\|_{L^2(H^2)} \\
\leq C \left( \left\| (f_{r^+}, f_{r^-}) e^{s\varphi} \right\|_{L^2(H^1)} + \left\| (r^+_0, r^-_0) e^{s\varphi(0)} \right\|_{H^2} \right). \tag{5.1}
\end{align*}
\]

**Proof.** In this proof the constant implied by the symbol \( \lesssim \) is independent from the parameter \( s \). Let us introduce the following two functional spaces
\[
\begin{align*}
C^{-}_s := \{ r \in L^2(0, T; L^2(T_L)) \mid r e^{s\varphi} \in L^2(0, T; L^2(T_L)) \}
\end{align*}
\]
and
\[
\begin{align*}
C^{+}_s := \{ r \in L^2(0, T; H^2(T_L)) \mid \theta^\frac{2}{3} r e^{s\varphi}, \theta^{-\frac{2}{3}} \triangle r e^{s\varphi} \in L^2(0, T; L^2(T_L)) \}
\end{align*}
\]
which we equip respectively with the norms
\[
\left\| r \right\|_{C^{-}_s} := s_{\Delta} \left\| r e^{s\varphi} \right\|_{L^2(L^2)}
\]
and
\[
\left\| r \right\|_{C^{+}_s} := s_{\Delta} \left\| \theta^\frac{2}{3} r e^{s\varphi} \right\|_{L^2(L^2)} + s_{\Delta} \left\| \theta^{-\frac{2}{3}} \triangle r e^{s\varphi} \right\|_{L^2(L^2)}.
\]
Endowed with these norms, \( C^{-}_s \) and \( C^{+}_s \) are Hilbert (hence Banach) spaces. For \( \tilde{r}^+ \) in \( C^{+}_s \) and \( \tilde{r}^- \) in \( C^{-}_s \), we introduce
\[
\begin{align*}
\begin{cases}
\dot{f}_{r^+} := f_{r^+} (\tilde{r}^+, \tilde{r}^-) = f_{r^+} + \beta_1 \tilde{r}^+ + \beta_2 \tilde{r}^-,
\dot{f}_{r^-} := f_{r^-} (\tilde{r}^+, \tilde{r}^-) = f_{r^-} + \beta_2 \tilde{r}^+ + \beta_3 \tilde{r}^- - \kappa \triangle \tilde{r}^+.
\end{cases}
\end{align*}
\]
Using Theorem 2.4 for the equation on \( r^- \) and Theorem 2.2 for the equation on \( r^+ \), one can define a map \( \Lambda_\theta \) on \( C^+_s \times C^-_s \) which to a data \((\tilde{r}^+_1, \tilde{r}^-_1)\) in \( C^+_s \times C^-_s \) associates \((r^+, r^-)\) where \( r^+ \) and \( r^- \) are respectively solutions of the controlled problem
\[
\begin{align*}
\left\{ \begin{array}{ll}
\partial_t r^+ - \zeta \Delta r^+ &= \tilde{f}_r^+ + v_r^+ \chi_0 & \text{in} \ (0, T) \times \mathbb{T}_L, \\
\partial_t r^- - \zeta \Delta r^- &= \tilde{f}_r^- + v_r^- \chi_0 & \text{in} \ (0, T) \times \mathbb{T}_L, \\
(r^+, r^-)_{|t=0} &= (\tilde{r}^+_1, \tilde{r}^-_1), & (r^+, r^-)_{|t=T} = (0, 0) \quad \text{in} \ \mathbb{T}_L \\
\end{array} \right.
\end{align*}
\]
given by Theorem 2.4 and Theorem 2.2. Let \((\tilde{r}^+_1, \tilde{r}^-_1)\) and \((\tilde{r}^+_2, \tilde{r}^-_2)\) in \( C^+_s \times C^-_s \). We set \((R^+, R^-) := \Lambda_\theta(\tilde{r}^+_1, \tilde{r}^-_1) - \Lambda_\theta(\tilde{r}^+_2, \tilde{r}^-_2), \) \( \tilde{f}_{R^+} := f_r^+(\tilde{r}^+_1, \tilde{r}^-_1) - f_r^+(\tilde{r}^+_2, \tilde{r}^-_2) \) and \( \tilde{f}_{R^-} := f_r^-(\tilde{r}^+_1, \tilde{r}^-_1) - f_r^-(\tilde{r}^+_2, \tilde{r}^-_2) \) so that \((R^+, R^-)\) is a solution of the following control problem
\[
\begin{align*}
\left\{ \begin{array}{ll}
\partial_t R^+ - \zeta \Delta R^+ &= \tilde{f}_{R^+} + v_{R^+} \chi_0 & \text{in} \ (0, T) \times \mathbb{T}_L, \\
\partial_t R^- - \zeta \Delta R^- &= \tilde{f}_{R^-} + v_{R^-} \chi_0 & \text{in} \ (0, T) \times \mathbb{T}_L, \\
(R^+, R^-)_{|t=0} &= (0, 0), & (R^+, R^-)_{|t=T} = (0, 0) \quad \text{in} \ \mathbb{T}_L \\
\end{array} \right.
\end{align*}
\]
From Theorem 2.4 and Theorem 2.2, we deduce that
\[
\begin{align*}
\frac{s^2}{2} \| \theta^{-\frac{5}{2}} R^+ e^{s\varphi} \|_{L^2(L^2)} + s^{\frac{1}{2}} \| \theta^{-\frac{5}{2}} \Delta R^+ e^{s\varphi} \|_{L^2(L^2)} &\lesssim \| \theta^{-\frac{1}{2}} R^+ e^{s\varphi} \|_{L^2(L^2)} + \| \theta^{-\frac{1}{2}} R^- e^{s\varphi} \|_{L^2(L^2)} \\
(5.2)
\end{align*}
\]
and
\[
\begin{align*}
\frac{s^2}{2} \| R^- e^{s\varphi} \|_{L^2(L^2)} &\lesssim \| \theta^{-\frac{1}{2}} \Delta R^- e^{s\varphi} \|_{L^2(L^2)} + \| \theta^{-\frac{1}{2}} R^- e^{s\varphi} \|_{L^2(L^2)} + \| \theta^{-\frac{1}{2}} \Delta R^+ e^{s\varphi} \|_{L^2(L^2)}.
(5.3)
\end{align*}
\]
Multiplying (5.2) by \( s \) and combining the resulting estimate with (5.3), and then, using that \( \theta \geq 1 \) and \( s \geq 1 \), we get
\[
\begin{align*}
\frac{s^2}{2} \| \theta^{-\frac{5}{2}} R^+ e^{s\varphi} \|_{L^2(L^2)} + s^{\frac{1}{2}} \| \theta^{-\frac{5}{2}} \Delta R^+ e^{s\varphi} \|_{L^2(L^2)} &\lesssim s \| \theta^{-\frac{1}{2}} R^+ e^{s\varphi} \|_{L^2(L^2)} + s \| R^- e^{s\varphi} \|_{L^2(L^2)} + \| \theta^{-\frac{1}{2}} \Delta R^+ e^{s\varphi} \|_{L^2(L^2)} \\
&\lesssim s^{-\frac{1}{2}} \left( \frac{s^2}{2} \| R^+ e^{s\varphi} \|_{L^2(L^2)} \right) + \frac{1}{2} \left( \frac{s^2}{2} \| R^- e^{s\varphi} \|_{L^2(L^2)} \right) + s^{-\frac{1}{2}} \left( \frac{s^2}{2} \| \theta^{-\frac{1}{2}} \Delta R^+ e^{s\varphi} \|_{L^2(L^2)} \right) \\
&\lesssim s^{-\frac{1}{2}} \left( \frac{s^2}{2} \| R^+ e^{s\varphi} \|_{L^2(L^2)} + s^2 \| R^- e^{s\varphi} \|_{L^2(L^2)} + s^2 \| \theta^{-\frac{1}{2}} \Delta R^+ e^{s\varphi} \|_{L^2(L^2)} \right).
(5.4)
\end{align*}
\]
Then, (5.4) can be rewritten as follows
\[
\| \Lambda_\theta(\tilde{r}^+_1, \tilde{r}^-_1) - \Lambda_\theta(\tilde{r}^+_2, \tilde{r}^-_2) \|_{C^+_s \times C^-_s} \leq C s^{-\frac{1}{2}} \| (\tilde{r}^+_1, \tilde{r}^-_1) - (\tilde{r}^+_2, \tilde{r}^-_2) \|_{C^+_s \times C^-_s},
\]
where \( C \) is a positive constant that does not depend on \( s \geq s_0 \). From the Banach fixed-point theorem, we deduce that for \( s \) large enough, \( \Lambda_\theta \) admits a unique fixed-point in \( C^+_s \times C^-_s \). Let \((r^+, r^-)\) in \( C^+_s \times C^-_s \) be the fixed-point of \( \Lambda_\theta \) and let \((v_{r^+}, v_{r^-})\) be the associated control.

**Regularity estimates.** Recall that \((f_{r^+}, f_{r^-})e^{s\varphi} \in L^2(H^1)\). We have \( r^-e^{3s\varphi/4} \in L^2(L^2) \) and \( r^+e^{3s\varphi/4} \in L^2(L^2) \), hence \( f_{r^+}(r^+, r^-)e^{3s\varphi/4} \in L^2(L^2) \). Lemma 2.3 thus implies that \( r^+e^{3s\varphi/4} \in L^2(H^2) \). It follows that \( f_{r^-}(r^+, r^-)e^{3s\varphi/4} \in L^2(L^2) \). Then, from Lemma 2.5, we get that \( r^-e^{3s\varphi/4} \in L^2(H^2) \). Accordingly, \((f_{r^+}(r^+, r^-)e^{3s\varphi/4} \in L^2(H^1) \) and using again Lemma 2.3, we get \( r^+e^{3s\varphi/4} \in L^2(H^3) \). We thus deduce that \( f_{r^-}(r^+, r^-)e^{3s\varphi/4} \in L^2(H^1) \), and finally \( r^-e^{3s\varphi/4} \in L^2(H^3) \) from Lemma 2.5. Each of the application of Lemma 2.3 and Lemma 2.5 comes with estimates, which directly yield (5.1).

6 Proof of Theorem 3.2

**Proof of Theorem 3.2.** Step 1: Observability of (3.12). Let us consider \( s_0 \geq 1 \) large enough so that Lemma 4.1 and 5.1 hold. We will recover the observability of (3.12) from the controllability of (3.16)
in the diagonalizable case obtained in Lemma 4.1 and of (3.20) in the non-diagonalizable case obtained in Lemma 5.1.
Let us first focus on the diagonalizable case. Let \((y^+, y^-)\) be a solution of (3.15). By definition of the dual norm, we have

\[
\|(y^+, y^-)e^{-s_0\Phi}\|_{L^2(H^{-1})} + \|(y^+(0), y^-(0))e^{-s_0\Phi(0)}\|_{H^{-2}} = \sup_{\|(r_0^+, r_0^-)e^{s_0\Phi(0)}\|_{H^2} \leq 1} \{ \Re((f_{r^+}, f_{r^-}, (y^+, y^-))_{L^2(1, L^2(1))} + \Re((r_0^+, r_0^-), (y^+(0), y^-(0)))_{H^2, H^{-2}}) \}.
\]

(6.1)

Now, for \((r_0^+, r_0^-) \in H^2(T_L)\) and \((f_{r^+}, f_{r^-}) \in L^2(H^1)\), such that \((f_{r^+}, f_{r^-})e^{s_0\Phi} \in L^2(H^1)\), we can associate the controlled trajectory \((r^+, r^-)\) of (3.16) with corresponding controls \((v^+, v^-)\) given by Lemma 4.1, and we obtain (recall that \((y^+, y^-)\) be a solution of (3.15) with source term \((g^+, g^-)\))

\[
\Re((f_{r^+}, f_{r^-}, (y^+, y^-))_{L^2(1), L^2(1)}) + \Re((r_0^+, r_0^-), (y^+(0), y^-(0)))_{H^2, H^{-2}} = \Re((g_{y^+}, g_{y^-}), (r^+, r^-))_{L^2(3), L^2(1)}) + \Re((y^+, y^-), \chi_0(r^+, r^-))_{L^2(1), L^2(2)\rangle}.
\]

Consequently, using (4.3) and (6.1), we get

\[
\|(y^+, y^-)e^{-s_0\Phi}\|_{L^2(H^{-1})} + \|(y^+(0), y^-(0))e^{-s_0\Phi(0)}\|_{H^{-2}} \lesssim \|(g_{y^+}, g_{y^-})e^{-\frac{3s_0\Phi}{4}}\|_{L^2(H^{-3})} + \|\chi_0(y^+, y^-)e^{-\frac{3s_0\Phi}{4}}\|_{L^2(H^{-1})}.
\]

(6.2)

In order to obtain observability for (3.12) from (6.2), we simply remind that solutions \((y^+, y^-)\) of (3.15) correspond to solutions \((\sigma, q)\) of (3.12) through the transform

\[
\begin{pmatrix} \sigma \\ q \end{pmatrix} := Q^{-1} \begin{pmatrix} y^+ \\ y^- \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} g_{\sigma} \\ \text{div}(g_z) \end{pmatrix} := Q^{-1} \begin{pmatrix} g_{y^+} \\ g_{y^-} \end{pmatrix},
\]

where the matrix \(Q\) is the one in (3.14). When \(A\) is not diagonalizable, the same strategy applies line to line, based on the duality between the control result in Lemma 5.1 for the system (3.20) and the observability of (3.19), and the correspondence between the solutions \((y^+, y^-)\) of system (3.19) and the solutions \((\sigma, q)\) of system (3.12) through the matrix \(R\) in (3.18) by

\[
\begin{pmatrix} \sigma \\ q \end{pmatrix} := R^{-1} \begin{pmatrix} y^+ \\ y^- \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} g_{\sigma} \\ \text{div}(g_z) \end{pmatrix} := R^{-1} \begin{pmatrix} g_{y^+} \\ g_{y^-} \end{pmatrix}.
\]

In both cases, we have obtained that solutions \((\sigma, q)\) of (3.12) satisfies,

\[
\|(\sigma, q)e^{-s_0\Phi}\|_{L^2(H^{-1}) \times L^2(H^{-1})} + \|(\sigma(0), q(0))e^{-s_0\Phi(0)}\|_{H^{-2} \times H^{-2}} \lesssim \|(g_{\sigma}, g_z)e^{-\frac{3s_0\Phi}{4}}\|_{L^2(H^{-3}) \times L^2(H^{-2})} + \|\chi_0(\sigma, q)e^{-\frac{3s_0\Phi}{4}}\|_{L^2(H^{-1}) \times L^2(H^{-1})},
\]

(6.3)

that is the observability estimate for (3.12).

**Step 2: Observability of (3.11).** Let us rewrite the equation on \(z\) in (3.11) as

\[-\partial_t z - \mu_+ \Delta z = g_z + \nabla \sigma + (\mu_+ + \nu) \nabla q \quad \text{in} \quad (0, T) \times T_L.
\]

To recover the estimate on \(z\), we use the duality with the following controllability problem for the heat equation

\[
\begin{cases}
\partial_t y - \mu_+ \Delta y = \tilde{f}_y + v_y \chi_0 & \text{in} \quad (0, T) \times T_L, \\
y|_{t=0} = y_0 \quad \text{and} \quad y|_{t=T} = 0 & \text{in} \quad T_L. 
\end{cases}
\]

(6.4)
Replacing $s$ by $4s_0/3$ in Lemma 2.3, Items 1 and 2, we define a map $\Xi$ which to two functions $y_0$ in $H^1(T_L)$ and $\tilde{f}_y$ such that $\tilde{f}_y e^{4s_0 \Phi}$ belongs in $L^2(L^2)$, associates the solution $(y, \chi_0 v_y)$ of (6.4) satisfying

$$
\| y e^{s_0 \Phi} \|_{L^2(H^2)} + \| \chi_0 v_y e^{s_0 \Phi} \|_{L^2(L^2)} \lesssim \| \tilde{f}_y e^{4s_0 \Phi} \|_{L^2(L^2)} + \| y_0 e^{4s_0 \Phi(0)} \|_{H^1}.
$$

By duality and according to the above estimate, we get

$$
\| z e^{-\frac{4s_0 \Phi}{3}} \|_{L^2(L^2)} + \| z(0) e^{-\frac{4s_0 \Phi(0)}{3}} \|_{H^{-1}}
\lesssim \sup \{ \mathcal{R}(\tilde{f}_y, y_0), (z, z(0)) \} \|_{L^2(L^2) \times H^2, L^2(L^2) \times H^{-1}}
\lesssim \sup \{ \mathcal{R}(\Xi(f_y, y_0), (g_z + \nabla \sigma + (\nu_\ast + \mu_\ast) \nabla q, z)) \} \|_{L^2(H^2) \times L^2(L^2), L^2(H^{-2}) \times L^2(L^2)}
\lesssim \| g_z e^{-\frac{4s_0 \Phi}{3}} \|_{L^2(H^{-2})} + \| (\sigma, q) e^{-s_0 \Phi} \|_{L^2(H^{-1})} + \| \chi_0 z e^{-s_0 \Phi(0)} \|_{L^2(L^2)}
$$

Then, according to (6.3) it follows that

$$
\| z e^{-\frac{4s_0 \Phi}{3}} \|_{L^2(L^2)} + \| z(0) e^{-\frac{4s_0 \Phi(0)}{3}} \|_{H^{-1}}
\lesssim \| (g_\sigma, g_z) e^{-\frac{4s_0 \Phi}{3}} \|_{L^2(H^{-3}) \times L^2(H^{-2})} + \| \chi_0 (\sigma, q) e^{-\frac{4s_0 \Phi}{3}} \|_{L^2(H^{-1}) \times L^2(H^{-1})} + \| \chi_0 z e^{-s_0 \Phi} \|_{L^2(L^2)}
$$

As $\chi = 1$ on $\text{supp}(\chi_0)$, we have $\chi_0 \chi = \chi$ and $\chi_0 \text{div}(z) = \chi_0 \text{div}(\chi z)$. Therefore, using that the multiplication by $\chi_0$ maps $H^{-1}$ to itself, we get

$$
\| \chi_0 q e^{-\frac{3s_0 \Phi}{2}} \|_{L^2(H^{-1})} \lesssim \| \chi z e^{-\frac{3s_0 \Phi}{2}} \|_{L^2(L^2)}
$$

and combining the above estimate with (6.3), we obtain the following observability estimate for solutions $(\sigma, z)$ of (3.11)

$$
\| \sigma e^{-s_0 \Phi} \|_{L^2(H^{-1})} + \| (\sigma(0) e^{-s_0 \Phi(0)}) \|_{H^{-2}} + \| z e^{-\frac{4s_0 \Phi}{3}} \|_{L^2(L^2)} + \| z(0) e^{-\frac{4s_0 \Phi(0)}{3}} \|_{H^{-1}}
\lesssim \| (g_\sigma, g_z) e^{-\frac{4s_0 \Phi}{3}} \|_{L^2(H^{-3}) \times L^2(H^{-2})} + \| (\sigma, z) e^{-\frac{3s_0 \Phi}{2}} \|_{L^2(H^{-1}) \times L^2(L^2)}.
$$

**Step 3: Conclusion.** Since solutions $(\sigma, z)$ of (3.11) satisfy the above observability estimate, we again argue by duality to deduce that System (3.9) is controllable and that the following estimate holds

$$
\| (a, u) e^{4s_0 \Phi} \|_{L^2(H^1) \times L^2(H^1)} + \| \chi(v_u, u) e^{4s_0 \Phi} \|_{L^2(H^1) \times L^2(L^2)}
\lesssim \| (f_a e^{s_0 \Phi}, f_a e^{4s_0 \Phi}) \|_{L^2(H^1) \times L^2(L^2)} + \| (a_0 e^{s_0 \Phi(0)}, u_0 e^{4s_0 \Phi(0)}) \|_{H^2 \times H^1}
\lesssim \| (f_a, f_u) e^{-\frac{4s_0 \Phi(0)}{3}} \|_{L^2(H^1) \times L^2(L^2)} + \| (a_0, u_0) e^{-\frac{s_0 \Phi(0)}{3}} \|_{H^2 \times H^1}.
$$

Then, it remains to establish the regularity estimate (3.10). We perform the regularity estimate on the equation satisfied by $(a, u) e^{2s_0 \Phi}$, that induces a small loss in the parameter $s_0$, which is reflected in the fact that we estimate $(a, u) e^{\frac{2s_0 \Phi}{3}}$ instead of $(a, u) e^{-\frac{s_0 \Phi}{3}}$ to apply the above estimate, and (3.10) follows.

Finally, it can be easily checked that the above control process, based on duality arguments, provides a linear operator $\mathcal{G}$, which, to any initial conditions $(a_0, u_0) \in H^2 \times H^1$ and source terms $(f_a, f_u)$ such
that \((f_a, f_u) e^{4\eta \Phi} \in L^2(H^1) \times L^2(L^2)\), provides a controlled trajectory \((a, u)\) in \(L^2(H^3) \times L^2(H^2)\) for (3.9), as claimed in Theorem 3.2.

\[\Box\]

7 Proof of Theorem 3.1

Proof of Theorem 3.1. In order to prove the controllability of System (3.7), we will perform a fixed-point argument.  

We start by fixing the parameter \(s_0\) so that Theorem 3.2 holds.  

We then define the space \(X \times Y\), where \(X := L^2(H^3) \cap L^\infty(H^2) \cap H^1(H^1)\) and \(Y := L^2(H^2) \cap L^\infty(H^1) \cap H^1(L^2)\), and the following closed subset

\[B_R := \{(\bar{a}, \bar{u}) \in X \times Y \mid \|(\bar{a}, \bar{u}) e^{2\eta \Phi}\|_{X \times Y} + \|\partial_t(\bar{a}, \bar{u}) e^{2\eta \Phi}\|_{L^2(H^1) \times L^2(L^2)} \leq R\},\]

where \(R\) is a positive real number which will be chosen later. Let \((a_0, u_0) \in H^3(T_L) \times H^2(T_L)\). Our goal is to find a fixed-point of the map given as follows

\[\mathcal{F}(\bar{a}, \bar{u}) := \mathcal{G}(a_0, u_0, f_a(\bar{a}, \bar{u}), f_u(\bar{a}, \bar{u})), \tag{7.1}\]

for \((\bar{a}, \bar{u})\) in \(B_R\) and, \(f_a(a, \bar{u})\) and \(f_u(a, \bar{u})\) are given (recall (3.3)) by

\[
\begin{align*}
\begin{cases}
f_a(\bar{a}, \bar{u}) := -\bar{u} \cdot \nabla \bar{a}, \\
f_u(\bar{a}, \bar{u}) := f^1_u(\bar{a}, \bar{u}) + f^2_u(\bar{a}, \bar{u}) + f^3_u(\bar{a}, \bar{u}) + f^4_u(\bar{a}, \bar{a}) + f^5_u(\bar{a}, \bar{a}).
\end{cases}
\end{align*}
\]

with

\[
\begin{align*}
\begin{cases}
f^1_u(\bar{a}, \bar{u}) := -(\bar{a} + 1)\bar{u} \cdot \nabla \bar{u}, \\
f^2_u(\bar{a}, \bar{u}) := \text{div} \left(2\mu(\bar{a})D_S(\bar{u})\right) + \nabla(\mu(\bar{a}) \text{ div} \bar{u}) , \\
f^3_u(\bar{a}, \bar{u}) := \partial_t \bar{u} \bar{a}, \\
f^4_u(\bar{a}, \bar{a}) := \mathcal{P'}(\bar{a}) \nabla \bar{a}, \\
f^5_u(\bar{a}, \bar{a}) := -\bar{a} + 1 \nabla(\mu(\bar{a}) \Delta \bar{a} + \nabla \mathcal{E}(\bar{a}) \cdot \nabla \bar{a}).
\end{cases}
\end{align*}
\]

For this purpose, we prove that:

1. for \(R > 0\) small enough, \(\mathcal{F}\) is well-defined on \(B_R\);  
2. for \(R > 0\) and \(\delta > 0\) small enough, we have \(\mathcal{F}(B_R) \subset B_R\);  
3. for \(R > 0\) and \(\delta > 0\) small enough, \(\mathcal{F}\) is a strict contraction from \(B_R\) to \(B_R\).

We will next conclude by the application of Banach Picard fixed-point theorem. In all that follows, the constant implied by the symbol \(\lesssim\) is assumed to be independent from \(R\), the parameter \(\delta\) in (3.6).

Step 1: \(\mathcal{F}\) is well-defined on \(B_R\) for all \(R \in (0, R_*)\), with \(R_*\) small enough. We begin by showing the following lemma.

Lemma 7.1. Let \(s_0 \geq 1\) as in Theorem 3.2. There exist positive real numbers \(R_*\) and \(C > 0\) such that for any \(R \in (0, R_*)\), such that for all \((\bar{a}, \bar{u}) \in B_R\) the quantities \(f_a(\bar{a}, \bar{u})\) and \(f_u(\bar{a}, \bar{u})\) are well-defined and

\[\|(f_a(\bar{a}, \bar{u}), f_u(\bar{a}, \bar{u})) e^{4\eta \Phi})\|_{L^2(H^1) \times L^2(L^2)} \lesssim CR^2. \tag{7.2}\]

Estimate of the nonlinear terms are based on the following classical lemma, whose proof is left to the reader as it is an adaptation of [6, Lemma 4.10.2 p.134].

Lemma 7.2. Let \(\ell > \frac{d}{2}\) be an integer and \(\eta\) a positive real number. Let \(F\) be a function in \(C^\ell([-\eta, \eta])\) such that \(F(0) = 0\). Then there exists a constant \(C\) such that for any \(u\) and \(v\) in \(H^\ell(T_L)\) satisfy such
that \( \|u\|_{L^\infty} + \|v\|_{L^\infty} + \|u\|_{H^t} \leq \eta \), we have
\[
\|F(u)\|_{H^t} \leq C\|u\|_{H^t} \quad \text{and} \quad \|F(u) - F(v)\|_{H^t} \leq C\|u - v\|_{H^t}.
\]

**Proof of Lemma 7.1.** Let us choose \( R_* > 0 \) such that if we denote by \( K \) the constant of the Sobolev embedding \( H^2(T_L) \hookrightarrow L^\infty(T_L) \), then
\[
KR_* < \frac{\eta}{\rho_*},
\]
where \( \eta \) is given by (H2). In this case, if \((\tilde{a}, \tilde{u})\) belongs to \( B_{R_*} \), then we have
\[
\|\tilde{a}\|_{L^\infty([0,T] \times T_L)} \leq K\|\tilde{a}\|_{L^\infty(H^2)} < \frac{\eta}{\rho_*} .
\]

Accordingly, since \( \mu \) and \( \nu \), and \( \kappa \) and \( P \) are respectively \( C^2 \) and \( C^3 \) in a neighborhood of \( \rho_* \), we can apply Lemma 7.1 for any elements \( \tilde{a} \) such that \((\tilde{a}, \tilde{u})\) belongs to \( B_{R_*} \), for some \( \tilde{u} \) in \( Y \). Let \( R \in [0, R_*] \) and \((\tilde{a}, \tilde{u})\) in \( B_R \).

We will repeatedly use that the weight function \( \Phi \) depends only on the time variable and for \( d \in \{1, 2, 3\} \) that the product is continuous from \( H^1(T_L) \times H^2(T_L) \) to \( H^1(T_L) \) and from \( H^2(T_L) \times H^2(T_L) \) to \( H^2(T_L) \). We will also repeatedly use that \( 4/3 = 2/3 + 2/3 \).

**Estimate on \( f_u \).** We have
\[
\left\| f_u(\tilde{a}, \tilde{u}) e^{\frac{4\nu_0}{3}} \right\|_{L^2(H^1)} \lesssim \|\tilde{u}e^{\frac{2\nu_0}{3}}\|_{L^\infty(H^1)} \|\tilde{u}e^{\frac{2\nu_0}{3}}\|_{L^2(H^2)} \lesssim \|\tilde{u}e^{\frac{2\nu_0}{3}}\|_Y \|\tilde{a}e^{\frac{2\nu_0}{3}}\|_X.
\]

**Estimate on \( f_u^1 \).** We have, according to (7.4)
\[
\left\| f_u^1(\tilde{a}, \tilde{u}) e^{\frac{4\nu_0}{3}} \right\|_{L^2(L^2)} \lesssim \|(\tilde{a} + 1)\|_{L^\infty(L^\infty)} \|\tilde{u}e^{\frac{2\nu_0}{3}}\|_{L^2(L^\infty)} \|\tilde{u}e^{\frac{2\nu_0}{3}}\|_{L^\infty(L^\infty)} \lesssim \|\tilde{u}e^{\frac{2\nu_0}{3}}\|_Y^2.
\]

**Estimate on \( f_u^2 \).** We have
\[
\left\| f_u^2(\tilde{a}, \tilde{u}) e^{\frac{4\nu_0}{3}} \right\|_{L^2(L^2)} \lesssim \left\| \left( \text{div}(\nu(\tilde{a}) D_S(\tilde{u})) + \nabla(\nu(\tilde{a}) \text{div}(\tilde{u})) \right) e^{\frac{4\nu_0}{3}} \right\|_{L^2(L^2)},
\]

Moreover, we have
\[
\| \text{div}(\nu(\tilde{a}) D_S(\tilde{u})) e^{\frac{4\nu_0}{3}} \|_{L^2(L^2)} \lesssim \|\nu(\tilde{a}) e^{\frac{2\nu_0}{3}}\|_{L^2(L^2)} \|\tilde{u}e^{\frac{2\nu_0}{3}}\|_{L^2(L^2)} \|\tilde{u}e^{\frac{2\nu_0}{3}}\|_{L^2(L^2)}.
\]

and
\[
\| \nabla(\nu(\tilde{a}) \text{div}(\tilde{u})) e^{\frac{4\nu_0}{3}} \|_{L^2(L^2)} \lesssim \|\nu(\tilde{a}) e^{\frac{2\nu_0}{3}}\|_{L^\infty(L^\infty)} \|\tilde{u}e^{\frac{2\nu_0}{3}}\|_{L^2(L^2)}.
\]

Furthermore, according to (7.4) and Hypothesis (H2), we deduce from Lemma 7.2 that
\[
\|\nu(\tilde{a}) e^{\frac{2\nu_0}{3}}\|_{L^\infty(L^\infty)} \lesssim \|\tilde{u}e^{\frac{2\nu_0}{3}}\|_{L^\infty(L^\infty)} \quad \text{and} \quad \|\nu(\tilde{a}) e^{\frac{2\nu_0}{3}}\|_{L^\infty(L^\infty)} \lesssim \|\tilde{u}e^{\frac{2\nu_0}{3}}\|_{L^\infty(L^\infty)}.
\]

Then we deduce that
\[
\left\| f_u^2(\tilde{a}, \tilde{u}) e^{\frac{4\nu_0}{3}} \right\|_{L^2(L^2)} \lesssim \|(\tilde{a}, \tilde{u}) e^{\frac{2\nu_0}{3}}\|_X^2 \|Y\|.
\]

**Estimate on \( f_u^3 \).** We first have
\[
\left\| f_u^3(\tilde{a}, \tilde{u}) e^{\frac{4\nu_0}{3}} \right\|_{L^2(L^2)} \lesssim \|\tilde{a} e^{\frac{2\nu_0}{3}}\|_{L^\infty(L^\infty)} \|\tilde{u} e^{\frac{2\nu_0}{3}}\|_{L^2(L^2)}.
\]

Then, we have
\[
\left\| f_u^3(\tilde{a}, \tilde{u}) e^{\frac{4\nu_0}{3}} \right\|_{L^2(L^2)} \lesssim R^2.
\]

**Estimate on \( f_u^4 \).** We have...
Thus, according to Lemma 7.2 applying to \( P' \), we conclude that

\[
\| f_u^4(\tilde{a}, \tilde{a}) e^{4\Phi/3} \|_{L^2(L^2)} \lesssim \| P'_u(\tilde{a}) e^{2\Phi/3} \|_{L^\infty(H^2)} \| \tilde{a} e^{2\Phi/3} \|_{L^2(H^1)}.
\]

Estimate on \( f_{u}^5 \). According to (7.4), we have

\[
\| f_{u}^5(\tilde{a}, \tilde{a}) e^{4\Phi/3} \|_{L^2(L^2)} \lesssim \| \tilde{a} + 1 \|_{L^\infty(L^\infty)} \| \tilde{a} \|_{L^2(H^1)} + \| \tilde{a} + 1 \|_{L^\infty(L^\infty)} \| \tilde{a} \|_{L^2(H^1)} \lesssim \| \tilde{a} \|_{L^\infty(H^2)} \| \tilde{a} e^{2\Phi/3} \|_{L^2(L^2)} + \| \tilde{a} \|_{L^2(H^1)} \| \tilde{a} e^{2\Phi/3} \|_{L^2(L^2)}
\]

In view of Hypothesis (H2) and Lemma 7.2, it follows that

\[
\| \tilde{a} \|_{L^\infty(H^2)} \lesssim \| \tilde{a} e^{2\Phi/3} \|_{L^\infty(H^2)}.
\]

Then, we obtain that

\[
\| f_{u}^5(\tilde{a}, \tilde{a}) e^{4\Phi/3} \|_{L^2(L^2)} \lesssim \| (\tilde{a}, \tilde{u}) e^{2\Phi/3} \|_{X \times Y}.
\]

Combining all the above estimates, we conclude that for all \((\tilde{a}, \tilde{u}) \in B_R\)

\[
\| f_u(\tilde{a}, \tilde{u}) e^{4\Phi/3} \|_{L^2(H^1)} + \| f_u(\tilde{a}, \tilde{u}) e^{4\Phi/3} \|_{L^2(L^2)} \leq CR^2,
\]

where \( C \) is a positive constant independent of \( R \), which concludes the proof of Lemma 7.1.

Let \( R \in (0, R_*] \). From Lemma 7.1, we deduce that if \((\tilde{a}, \tilde{u}) \in B_R\), then \( f_u(\tilde{a}, \tilde{u}) e^{4\Phi/3} \in L^2(H^1) \) and \( f_u(\tilde{a}, \tilde{u}) e^{4\Phi/3} \in L^2(L^2) \). Since \((a_0, u_0) \in H^2(T_L) \times H^1(T_L)\), this shows, by using Estimate (3.10) of Theorem 3.2, that for all \((\tilde{a}, \tilde{u}) \in B_R\) the definition of \( F \) by (7.1) is meaningful. Furthermore, according to (3.10) and (7.2), it follows that for all \((\tilde{a}, \tilde{u}) \in B_R\) we have

\[
\| \partial_t F(\tilde{a}, \tilde{u}) e^{2\Phi/3} \|_{L^2(H^1) \times L^2(L^2)} + \| F(\tilde{a}, \tilde{u}) e^{2\Phi/3} \|_{X \times Y} \leq C \left( R^2 + \| e^{s_0 \Phi(0)}(a_0, u_0) \|_{H^2 \times H^1} \right).
\]

Step 2: For \( R > 0 \) and \( \delta > 0 \) small enough, \( F(B_R) \subset B_R \). From the estimate (7.5) and the smallness assumption (3.6), we obtain

\[
\| \partial_t F(\tilde{a}, \tilde{u}) e^{2\Phi/3} \|_{L^2(H^1) \times L^2(L^2)} + \| F(\tilde{a}, \tilde{u}) e^{2\Phi/3} \|_{X \times Y} \leq \tilde{C}(R^2 + \delta),
\]

where \( \tilde{C} := C e^{s_0 \Phi(0)} \). Then, setting \( R_0 := \min \left( \frac{1}{C}, R_* \right) \), for all \( R \in (0, R_0) \), there exists a positive real number \( \delta_R \), given by

\[
\delta_R := \frac{R}{C} - R^2,
\]

such that, if

\[
\|(a_0, u_0)\|_{H^2 \times H^1} \leq \delta_R,
\]

for all \((\tilde{a}, \tilde{u}) \in B_R\), we have the bound

\[
\| \partial_t F(\tilde{a}, \tilde{u}) e^{2\Phi/3} \|_{L^2(H^1) \times L^2(L^2)} + \| F(\tilde{a}, \tilde{u}) \|_{X \times Y} \leq R.
\]

This shows that for all \( R \in (0, R_0) \), if \( \delta \leq \delta_R \), then \( F(B_R) \subset B_R \). From now on, for any \( R \in (0, R_0) \),
the smallness of the initial data parameter $\delta$ in (3.6) will be automatically $\delta_R$.

**Step 3:** $F$ is a strict contraction from $(B_R, d)$ into itself. On $B_R$, we consider the distance $d$ given by

$$d(V, W) := \|(V - W)e^{\frac{2\gamma \Phi}{3}}\|_{X \times Y} + \|\hat{\partial}(V - W)e^{\frac{2\gamma \Phi}{3}}\|_{L^2(H^1) \times L^2(L^2)} \quad (V, W \in B_R).$$

This step is based on the following lemma.

**Lemma 7.3.** Let $s_0 \geq 1$ as in Theorem 3.2. There exist $C > 0$ such that for any $R \in (0, R_0)$, for all $(\bar{a}_1, \bar{u}_1)$ and $(\bar{a}_2, \bar{u}_2)$ in $B_R$ we have

$$d(F(\bar{a}_1, \bar{u}_1), F(\bar{a}_2, \bar{u}_2)) \leq CRd((\bar{a}_1, \bar{u}_1), (\bar{a}_2, \bar{u}_2)).$$

The proof is based on Lemma 7.2 and follows the same lines as the one of Lemma 7.1.

**Proof of Lemma 7.3.** Let $(\bar{a}_1, \bar{u}_1)$ and $(\bar{a}_2, \bar{u}_2)$ be two elements of $B_R$. As in the proof of Lemma 7.1, we use systematically that $4/3 = 2/3 + 2/3$ and the continuity of the product from $H^1(\mathbb{T}_L) \times H^2(\mathbb{T}_L)$ to $H^1(\mathbb{T}_L)$ and from $H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L)$ to $H^2(\mathbb{T}_L)$.

**Estimate on $f_a$.** We can write

$$f_a(\bar{a}_1, \bar{u}_1) - f_a(\bar{a}_2, \bar{u}_2) = f_a(\bar{a}_1, \bar{u}_1 - \bar{u}_2) + f_a(\bar{a}_1 - \bar{a}_2, \bar{u}_2).$$

Then, as in the proof of Lemma 7.1, we have

$$\|f_a(\bar{a}_1, \bar{u}_1) - f_a(\bar{a}_2, \bar{u}_2)e^{\frac{4\gamma \Phi}{3}}\|_{L^2(H^1)} \lesssim Rd((\bar{a}_1, \bar{u}_1), (\bar{a}_2, \bar{u}_2)).$$

**Estimate on $f_1^1$.** Let us first note that

$$f_1^1(\bar{a}_1, \bar{u}_1) - f_1^1(\bar{a}_2, \bar{u}_2) = - (\bar{a}_1 - \bar{a}_2)\bar{u}_1 \cdot \nabla \bar{u}_1 - \bar{a}_2(\bar{u}_1 - \bar{u}_2) \cdot \nabla \bar{u}_1 - \bar{a}_2\bar{u}_2 \cdot \nabla (\bar{u}_1 - \bar{u}_2).$$

Then, we have

$$\|f_1^1(\bar{u}_1, \bar{u}_1) - f_1^1(\bar{u}_2, \bar{u}_2)e^{\frac{4\gamma \Phi}{3}}\|_{L^2(L^2)} \lesssim \|\bar{u}_1\|_{L^\infty(L^\infty)} \|\bar{u}_1 - \bar{a}_2||_{L^2(L^\infty)}^2 + \|\bar{u}_2\|_{L^\infty(L^\infty)}\|\bar{u}_1 - \bar{a}_2||_{L^2(L^\infty)}^2 \lesssim Rd((\bar{a}_1, \bar{u}_1), (\bar{a}_2, \bar{u}_2)).$$

**Estimate on $f_2^1$.** We first remark that

$$f_2^1(\bar{a}_1, \bar{u}_1) - f_2^1(\bar{a}_2, \bar{u}_2) = 2 \text{div}((\mu(\bar{a}_1) - \mu(\bar{a}_2))D_S\bar{u}_1) + \text{div}((\nu(\bar{a}_1) - \nu(\bar{a}_2)) \text{div}(\bar{u}_1))$$

$$+ 2 \text{div}(\mu(\bar{a}_2)D_S(\bar{u}_1 - \bar{u}_2)) + \text{div}(\nu(\bar{a}_2) \text{div}(\bar{u}_1 - \bar{u}_2)).$$

We will apply Lemma 7.2 with $F = \mu$ and $F = \nu$ respectively according to Hypothesis (H2) and (7.4). Moreover, we have

$$\|2 \text{div}((\mu(\bar{a}_1) - \mu(\bar{a}_2))D_S\bar{u}_1) + \text{div}((\nu(\bar{a}_1) - \nu(\bar{a}_2)) \text{div}(\bar{u}_1))e^{\frac{4\gamma \Phi}{3}}\|_{L^2(L^2)} \lesssim Rd((\bar{u}_1, \bar{u}_1), (\bar{a}_2, \bar{u}_2)).$$
We also have
\[
\| \frac{1}{2} \div (\mu(\bar{u}_2) D_S(\bar{u}_1 - \bar{u}_2)) + \nabla (\mu(\bar{u}_2) \div (\bar{u}_1 - \bar{u}_2)) \| e^{\frac{4q_0\phi}{3}} \| L^2(L^2) \lesssim Rd((\bar{a}_1, \bar{u}_1), (\bar{a}_2, \bar{u}_2)).
\]

We deduce that
\[
\| (f^3_u(\bar{a}_1, \bar{u}_1) - f^3_u(\bar{a}_2, \bar{u}_2)) e^{\frac{4q_0\phi}{3}} \| L^2(L^2) \lesssim Rd((\bar{a}_1, \bar{u}_1), (\bar{a}_2, \bar{u}_2)).
\]

**Estimate on \( f^3_u \).** By using the bilinearity of \( f^3_u \), as for \( f_a \), we deduce similarly as for the estimate of \( f^3_u \) in the proof of Lemma 7.1 that
\[
\| (f^3_u(\bar{a}_1, \bar{u}_1) - f^3_u(\bar{a}_2, \bar{u}_2)) e^{\frac{4q_0\phi}{3}} \| L^2(L^2) \lesssim Rd((\bar{a}_1, \bar{u}_1), (\bar{a}_2, \bar{u}_2)).
\]

**Estimate on \( f^4_u \).** We have
\[
f^4_u(\bar{a}_1, \bar{u}_1) - f^4_u(\bar{a}_2, \bar{u}_2) = (P'(\bar{a}_1) - P'(\bar{a}_2)) \nabla \bar{a}_1 + P'(\bar{a}_2) \nabla (\bar{a}_1 - \bar{a}_2).
\]

Then, by applying Lemma 7.2, we get
\[
\| (P'(\bar{a}_1) - P'(\bar{a}_2)) \nabla \bar{a}_1 e^{\frac{4q_0\phi}{3}} \| L^2(L^2) \lesssim \| (P'(\bar{a}_1) - P'(\bar{a}_2)) e^{\frac{2q_0\phi}{3}} \| L^\infty(L^\infty) \| \bar{a}_1 e^{\frac{2q_0\phi}{3}} \| L^2(H^1) \lesssim Rd((\bar{a}_1, \bar{u}_1), (\bar{a}_2, \bar{u}_2))
\]
and
\[
\| P'(\bar{a}_2) \nabla (\bar{a}_1 - \bar{a}_2) e^{\frac{4q_0\phi}{3}} \| L^2(L^2) \lesssim \| P'(\bar{a}_2) e^{\frac{2q_0\phi}{3}} \| L^\infty(L^\infty) \| (\bar{a}_1 - \bar{a}_2) e^{\frac{2q_0\phi}{3}} \| L^2(H^1)
\]
\[
\lesssim Rd((\bar{a}_1, \bar{u}_1), (\bar{a}_2, \bar{u}_2)).
\]

We deduce that
\[
\| (f^4_u(\bar{a}_1, \bar{u}_1) - f^4_u(\bar{a}_2, \bar{u}_2)) e^{\frac{4q_0\phi}{3}} \| L^2(L^2) \lesssim Rd((\bar{a}_1, \bar{u}_1), (\bar{a}_2, \bar{u}_2)).
\]

**Estimate on \( f^5_u \).** We have
\[
f^5_u(\bar{a}_1, \bar{a}_1) - f^5_u(\bar{a}_2, \bar{a}_2) = (\bar{a}_1 + 1) \nabla (\kappa(\bar{a}_1) - \kappa(\bar{a}_2)) \triangle \bar{a}_1 + \nabla (\kappa(\bar{a}_1) - \kappa(\bar{a}_2)) \cdot \nabla \bar{a}_1
\]
\[
+ (\bar{a}_1 + 1) \nabla (\kappa(\bar{a}_2) \triangle (\bar{a}_1 - \bar{a}_2) + \nabla \kappa(\bar{a}_2) \cdot \nabla (\bar{a}_1 - \bar{a}_2))
\]
\[
+ (\bar{a}_1 - \bar{a}_2) \nabla (\kappa(\bar{a}_2) \triangle \bar{a}_2 + \nabla \kappa(\bar{a}_2) \cdot \nabla \bar{a}_2).
\]

To estimate the first term of the right-hand side, we apply Lemma 7.2 to \( F = \kappa \) according to (7.4). We get
\[
\| (\bar{a}_1 + 1) \nabla (\kappa(\bar{a}_1) - \kappa(\bar{a}_2)) \triangle \bar{a}_1 + \nabla (\kappa(\bar{a}_1) - \kappa(\bar{a}_2)) \cdot \nabla \bar{a}_1 e^{\frac{4q_0\phi}{3}} \| L^2(L^2)
\]
\[
\lesssim \| \bar{a}_1 + 1 \| L^\infty(L^\infty) \| \kappa(\bar{a}_1) - \kappa(\bar{a}_2) \| L^\infty(L^\infty) \| \triangle \bar{a}_1 e^{\frac{2q_0\phi}{3}} \| L^2(H^1)
\]
\[
+ \| \bar{a}_1 + 1 \| L^\infty(L^\infty) \| \nabla (\kappa(\bar{a}_1) - \kappa(\bar{a}_2)) \| L^2(H^2) \| \nabla \bar{a}_1 e^{\frac{2q_0\phi}{3}} \| L^2(H^1)
\]
\[
\lesssim \| \kappa(\bar{a}_1) - \kappa(\bar{a}_2) \| L^\infty(L^\infty) \| \bar{a}_1 e^{\frac{2q_0\phi}{3}} \| L^2(H^1)
\]
\[
\lesssim Rd((\bar{a}_1, \bar{u}_1), (\bar{a}_2, \bar{u}_2)).
\]
Similarly, we also have
\[ \| (\tilde{a}_1 + 1) \nabla \left( \kappa(\tilde{a}_2) \triangle \tilde{a}_2 + \nabla \kappa(\tilde{a}_2) \cdot \nabla (\tilde{a}_1 - \tilde{a}_2) \right) e^{\frac{4\rho_{\phi}}{3}} \|_{L^2(L^2)} \lesssim R d((\tilde{a}_1, \tilde{u}_1), (\tilde{a}_2, \tilde{u}_2)) \]
and
\[ \| (\tilde{a}_1 - \tilde{a}_2) \nabla \left( \kappa(\tilde{a}_2) \triangle \tilde{a}_2 + \nabla \kappa(\tilde{a}_2) \cdot \nabla \tilde{a}_2 \right) e^{\frac{4\rho_{\phi}}{3}} \|_{L^2(L^2)} \lesssim R d((\tilde{a}_1, \tilde{u}_1), (\tilde{a}_2, \tilde{u}_2)). \]

Then we obtain that
\[ \| (f^5_u(\tilde{a}_1, \tilde{a}_1) - f^5_u(\tilde{a}_2, \tilde{a}_2)) e^{\frac{4\rho_{\phi}}{3}} \|_{L^2(L^2)} \lesssim (R + R^2) d((\tilde{a}_1, \tilde{u}_1), (\tilde{a}_2, \tilde{u}_2)). \]

This concludes the proof of Lemma 7.3.

According to Lemma 7.3, we choose
\[ R := \frac{1}{2} \min \left( \frac{1}{C}, R_0 \right) \quad \text{and} \quad \delta := \delta_R, \]
so that if \((a_0, u_0)\) satisfies
\[ \| (a_0, u_0) \|_{H^2 \times H^1} \leq \delta, \]
the map \(F\) maps \(B_R\) into itself by Step 2 and is contractive on \(B_R\) (with the topology induced by the distance \(d\) for which \(B_R\) is complete) by Lemma 7.3.

**Step 4: Conclusion.** We conclude from the Banach fixed-point theorem that \(F\) admit a fixed-point \((a, u)\) in \(B_R\). Moreover, it follows from Theorem 3.2 and Lemma 7.1 that \((a, u)\) have the wanted regularity. Finally, according to (7.3) and (7.4), it follows from Hypothesis \((H2)\) that \(\rho(t, x) > \rho_* - \eta > 0\) for any \((t, x) \in (0, T) \times \mathbb{T}_L\).

A Proof of the Carleman estimate

In this appendix we are interested in establishing the following Carleman estimate

**Lemma A.1.** Let \(T > 0\). Let us consider a complex number \(\zeta\) such that \(\Re(\zeta) > 0\). There exist three positive constants \(C, s_0 \geq 1\) and \(\lambda_0 \geq 1\), large enough, such that for all smooth function \(w\) on \([0, T] \times \mathbb{T}_L\) and for all \(s \geq s_0\) and \(\lambda \geq \lambda_0\), we have

\[ s^2 \lambda^2 \| \nabla^2 w e^{-s\phi} \|_{L^2(L^2)} + s^2 \lambda \| \nabla w e^{-s\phi} \|_{L^2(L^2)} + s \lambda^2 e^{s\lambda} \| w(0) e^{-s\phi(0)} \|_{L^2} \lesssim C \left( \| (\partial_t - \zeta \Delta) w e^{-s\phi} \|_{L^2(L^2)} + s^2 \lambda^2 \| \nabla w e^{-s\phi} \|_{L^2(L^2)} \right), \]

where we have set
\[ \xi(t, x) := \theta(t) e^{\lambda^2(x)}. \]

**Proof.** Let us set \(\zeta = \alpha + i \beta\), where \(\alpha\) and \(\beta\) are real numbers with \(\alpha > 0\). Let \(w\) be a smooth complex valued function on \([0, T] \times \mathbb{T}_L\) and set
\[ f := -\partial_t w - \zeta \Delta w. \]

We shall deal with the function
\[ w := e^{-s\phi} w. \]

According to the definition of \(\theta\), \(w\) satisfies
\[ w(T, x) = 0 \quad \text{and} \quad \nabla w(T, x) = 0, \quad x \in \mathbb{T}_L. \]
Let us define the conjugate of $-\zeta \partial_t - \triangle$ by

$$P_{\phi} := e^{-s\phi}(-\partial_t - \zeta \triangle)e^{s\phi}.$$ 

Then

$$P_{\phi}w = -\partial_tw - s\partial_t\varphi w - \zeta \triangle w - 2s\zeta \nabla \varphi \cdot \nabla w - s^2\zeta |\nabla \varphi|^2 w - s\zeta \triangle \varphi w$$

and

$$e^{-s\phi}f = P_{\phi}w.$$ 

Inspired by the strategy to prove Carleman estimate (see [25]), we now define quantities $P_1w$ and $P_2w$ from the symmetric and antisymmetric part of $P_{\phi}$ by setting

$$P_1w := \frac{1}{2}(P_{\phi} + P_{\phi}^*)w = -\alpha(\triangle w + s^2|\nabla \varphi|^2 w) - i\beta(2s\nabla \varphi \cdot \nabla w + s \triangle \varphi w) - s\partial_t\varphi w$$

and

$$P_2w := \frac{1}{2}(P_{\phi} - P_{\phi}^*)w = -\partial_tw - i\beta(\triangle w + s^2|\nabla \varphi|^2 w) - \alpha(2s\nabla \varphi \cdot \nabla w + s \triangle \varphi w),$$

so that

$$P_{\phi}w = P_1w + P_2w.$$ 

Since $P_1w + P_2w = e^{-s\phi}f$, we get

$$\iint_{[0,T] \times T_L} |P_1w|^2 + \iint_{[0,T] \times T_L} |P_2w|^2 + 2\Re\left(\iint_{[0,T] \times T_L} P_1w \overline{P_2w}\right) \leq \iint_{[0,T] \times T_L} e^{-2s\phi}|f|^2. \quad (A.1)$$

The main part of the proof $L^2$-Carleman estimate consists to estimate from below the real part of the scalar product of $P_1w$ with $P_2w$. We begin by setting

$$\Re\left(\iint_{[0,T] \times T_L} P_1w \overline{P_2w}\right) = \sum_{1 \leq k,l \leq 3} \Re(I_{k,l}), \quad (A.2)$$

where $I_{k,l}$ is the scalar product of the $k$-th term of $P_1w$ with the $l$-th term of $P_2w$. Note that by $L$-periodicity, all the boundary terms generated from integration by parts with respect to the space variable vanish.

**Step 1: Computation of the scalar product.** Let us begin by remarking that

$$\Re(I_{2,3}) = \Re(I_{1,2}) = 0.$$
Computation of $\Re(I_{1,3}) + \Re(I_{2,2})$. We get

$$
\Re(I_{1,3}) + \Re(I_{2,2}) = (\alpha^2 + \beta^2) \Re\left( \int_{[0,T] \times T_L} (\Delta w + s^2 |\nabla \varphi|^2 w)(2s \nabla \varphi \cdot \nabla w + s \Delta \varphi w) \right)
$$

$$
= 2|\zeta|^2 s \Re\left( \int_{[0,T] \times T_L} \Delta w \nabla \varphi \cdot \nabla w \right) - |\zeta|^2 s \Re\left( \int_{[0,T] \times T_L} \nabla \Delta \varphi \cdot \nabla w \right)
$$

$$
- |\zeta|^2 s \int_{[0,T] \times T_L} \Delta \varphi |\nabla w|^2
$$

$$
- |\zeta|^2 s \int_{[0,T] \times T_L} \text{div}(|\nabla \varphi|^2 \nabla \varphi)|w|^2 + |\zeta|^2 s \int_{[0,T] \times T_L} |\nabla \varphi|^2 \Delta \varphi |w|^2
$$

$$
= 2|\zeta|^2 s \Re\left( \int_{[0,T] \times T_L} \Delta w \nabla \varphi \cdot \nabla w \right) + \frac{|\zeta|^2 s}{2} \int_{[0,T] \times T_L} \Delta^2 \varphi |w|^2
$$

$$
- |\zeta|^2 s \int_{[0,T] \times T_L} \Delta \varphi |\nabla w|^2
$$

$$
- |\zeta|^2 s \int_{[0,T] \times T_L} \text{div}(|\nabla \varphi|^2 \nabla \varphi)|w|^2 + |\zeta|^2 s \int_{[0,T] \times T_L} |\nabla \varphi|^2 \Delta \varphi |w|^2
$$

$$
= -2|\zeta|^2 s \Re\left( \int_{[0,T] \times T_L} D^2 \varphi(\nabla w, \nabla w) \right) + \frac{|\zeta|^2 s}{2} \int_{[0,T] \times T_L} \Delta^2 \varphi |w|^2
$$

$$
- |\zeta|^2 s \int_{[0,T] \times T_L} \text{div}(|\nabla \varphi|^2 \nabla \varphi)|w|^2 + |\zeta|^2 s \int_{[0,T] \times T_L} |\nabla \varphi|^2 \Delta \varphi |w|^2. \quad (A.3)
$$

Computation of $\Re(I_{1,1})$. By integrating by parts and using that $w(T) = \nabla w(T) = 0$, we deduce

$$
\Re(I_{1,1}) = \alpha \Re\left( \int_{[0,T] \times T_L} (\Delta w + s^2 |\nabla \varphi|^2 w) \partial_t \nabla w \right)
$$

$$
= \alpha \Re\left( \int_{[0,T] \times T_L} \Delta w \partial_t \nabla w \right) - \frac{\alpha s^2}{2} \int_{T_L} |\nabla \varphi(0)|^2 |w(0)|^2 - \frac{\alpha s^2}{2} \int_{[0,T] \times T_L} \partial_t |\nabla \varphi|^2 |w|^2
$$

$$
= \frac{\alpha}{2} \int_{T_L} |\nabla w(0)|^2 - \frac{\alpha s^2}{2} \int_{T_L} |\nabla w(0)|^2 |w(0)|^2 - \frac{\alpha s^2}{2} \int_{[0,T] \times T_L} \partial_t |\nabla \varphi|^2 |w|^2. \quad (A.4)
$$

Computation of $\Re(I_{2,1})$. Keeping in mind that $w(T) = \nabla w(T) = 0$ and using that for any complex
number $z$ we have $\Im(z) + \Im(\overline{z}) = 0$, by integrating by parts, we have

$$
\Re(I_{2,1}) = \Re \left( i\beta \int_{[0,T] \times T_L} (2s \nabla \varphi \cdot \nabla w + s \Delta \varphi w) \partial_t \overline{w} \right) \\
= 2s \Re \left( i\beta \int_{[0,T] \times T_L} \nabla \varphi \cdot \nabla w \partial_t \overline{w} \right) - s \Re \left( i\beta \int_{[0,T] \times T_L} \Delta \varphi w \partial_t \overline{w} \right) \\
= s \Re \left( i\beta \int_{[0,T] \times T_L} \nabla \varphi \cdot \nabla w \partial_t \overline{w} \right) - s \Re \left( i\beta \int_{[0,T] \times T_L} \nabla \varphi \cdot w \partial_t \nabla \overline{w} \right) \\
= -s \Re \left( i\beta \int_{T_L} \nabla \varphi(0) \cdot \nabla w(0) \overline{w}(0) \right) - s \Re \left( i\beta \int_{[0,T] \times T_L} \partial_t \nabla \varphi \cdot \nabla \overline{w} \right) \\
= -s \Re \left( i\beta \int_{T_L} \nabla \varphi(0) \cdot \nabla w(0) \overline{w}(0) \right) - s \Re \left( i\beta \int_{[0,T] \times T_L} \partial_t \nabla \varphi \cdot \nabla \overline{w} \right). \quad (A.5)
$$

Computation of $\Re(I_{3,1})$. Since $w(T) = 0$, we obtain

$$
\Re(I_{3,1}) = s \Re \left( \int_{[0,T] \times T_L} \partial_t \varphi w \partial_t \overline{w} \right) = -s \int_{T_L} \partial_t \varphi(0) |w(0)|^2 - s \int_{[0,T] \times T_L} \partial_t^2 \varphi |w|^2. \quad (A.6)
$$

Computation of $\Re(I_{3,2})$. We have

$$
\Re(I_{3,2}) = -\Re \left( i\beta \int_{[0,T] \times T_L} s \partial_t \varphi w (\Delta \overline{w} + s^2 |\nabla \varphi|^2 \overline{w}) \right) \\
= -s \Re \left( i\beta \int_{[0,T] \times T_L} \partial_t \varphi \cdot \Delta \overline{w} \right) \\
= s \Re \left( i\beta \int_{[0,T] \times T_L} \partial_t \nabla \varphi \cdot \nabla \overline{w} \right). \quad (A.7)
$$

Computation of $\Re(I_{3,3})$. We have

$$
\Re(I_{3,3}) = \alpha s^2 \Re \left( \int_{[0,T] \times T_L} (2 \nabla \varphi \cdot \nabla \overline{w} + \Delta \varphi \overline{w}) \partial_t \varphi w \right) \\
= -s^2 \alpha \int_{[0,T] \times T_L} \text{div}(\nabla \varphi \partial_t \varphi) |w|^2 + s^2 \alpha \int_{[0,T] \times T_L} \nabla \varphi \partial_t |w|^2 \\
= -s^2 \int_{[0,T] \times T_L} \nabla \varphi \cdot \partial_t \nabla \varphi |w|^2 \\
= -\frac{\alpha s^2}{2} \int_{[0,T] \times T_L} \partial_t |\nabla \varphi|^2 |w|^2. \quad (A.8)
$$

From (A.5) and (A.7), we deduce that

$$
\Re(I_{3,2}) + \Re(I_{2,1}) = -s \Re \left( i\beta \int_{T_L} \nabla \varphi(0) \cdot \nabla w(0) \overline{w}(0) \right) + 2s \Re \left( i\beta \int_{[0,T] \times T_L} \partial_t \nabla \varphi \cdot \nabla \overline{w} \right). \quad (A.9)
$$
By combining (A.3), (A.4), (A.6), (A.8) and (A.9), it follows that

\[
\Re(\langle P_1 w, P_2 w \rangle)
= \frac{\alpha}{2} \int_{T_L} |\nabla w(0)|^2 - \frac{s}{2} \int_{T_L} \partial_t \varphi(0) |w(0)|^2 - \frac{s^2 \alpha}{2} \int_{T_L} |\nabla \varphi(0)|^2 |w(0)|^2 - s \Re \left( i \beta \int_{T_L} \nabla \varphi(0) \cdot \nabla w(0) \bar{w}(0) \right) \tag{A.10}
\]

\[
+ \frac{s |\zeta|^2}{2} \int_{[0,T] \times T_L} \triangle^2 \varphi |w|^2 + (|\nabla \varphi|^2 \triangle \varphi - \div(\nabla \varphi \nabla \varphi)) |w|^2 - s^2 \alpha \int_{[0,T] \times T_L} \partial_t |\nabla \varphi|^2 |w|^2 - \frac{s}{2} \int_{[0,T] \times T_L} \partial_t^2 \varphi |w|^2
\]

\[
+ s \left( -2 |\zeta|^2 \Re \left( \int_{[0,T] \times T_L} D^2 \varphi(\nabla w, \nabla \bar{w}) \right) + 2 \Re \left( i \beta \int_{[0,T] \times T_L} \partial_t \nabla \varphi \cdot \nabla \bar{w} \right) \right) \tag{A.13}
\]

**Step 2: Lower bound of the scalar product.** We now give a lower bound for the scalar product \(\Re(P_1 w, P_2 w)\). Along the rest of the proof, we will take the parameters \(s\) and \(\lambda\) large enough in order to absorb lower order terms with respect to the power of these parameters. In the following, to simplify notations, we will denote by \(C^*\) a generic large positive constant which do not depends on \(s\) and \(\lambda\) and by \(C_*\) a generic small positive constant independent of \(s\) and \(\lambda\). The constants may change from line to line.

**Lower bound of (A.10).** We have

\[-\partial_t \varphi(0) = \frac{s \lambda^2 e^{2\lambda(\lambda e^{12\lambda} - e^{\lambda \psi})}}{T_0}.
\]

Since \(\psi \leq 7\), we deduce that

\[-\partial_t \varphi(0) \geq C_* s \lambda^3 e^{14\lambda}.
\]

Thus we obtain that

\[-\frac{s}{2} \int_{T_L} \partial_t \varphi(0) |w(0)|^2 \geq C_* s^2 \lambda^2 e^{14\lambda} \int_{T_L} |w(0)|^2.
\]

Besides, since \(\nabla \varphi(0) = -2\lambda \nabla \psi e^{\lambda \psi}\) and \(\psi \leq 7\), we deduce that

\[-\frac{\alpha s^2}{2} \int_{T_L} |\nabla \varphi(0)|^2 |w(0)|^2 \geq -C^* s^2 \lambda^2 e^{14\lambda} \int_{T_L} |w(0)|^2 \geq -C^* s^2 \lambda^2 e^{14\lambda} \int_{T_L} |w(0)|^2 \geq -C^* s^2 \lambda^2 e^{14\lambda} \int_{T_L} |w(0)|^2
\]

and

\[-s \Re \left( i \beta \int_{T_L} \nabla \varphi(0) \cdot \nabla w(0) \bar{w}(0) \right) \geq -C^* s^2 \lambda^2 e^{14\lambda} \int_{T_L} |w(0)|^2 - \frac{C^*}{\lambda^2} \int_{T_L} |\nabla w(0)|^2.
\tag{A.14}
\]

According to \(\alpha > 0\), we conclude that

\[
\frac{\alpha}{2} \int_{T_L} |\nabla w(0)|^2 - \frac{s}{2} \int_{T_L} \partial_t \varphi(0) |w(0)|^2 - \frac{s^2 \alpha}{2} \int_{T_L} |\nabla \varphi(0)|^2 |w(0)|^2
\]

\[
\geq C_* \int_{T_L} |\nabla w(0)|^2 + C_* s^2 \lambda^3 e^{14\lambda} \int_{T_L} |w(0)|^2 - C^* s^2 \lambda^2 e^{14\lambda} \int_{T_L} |w(0)|^2 - \frac{C^*}{\lambda^2} \int_{T_L} |\nabla w(0)|^2
\geq C_* \int_{T_L} |\nabla w(0)|^2 + C_* s^2 \lambda^3 e^{14\lambda} \int_{T_L} |w(0)|^2.
\tag{A.15}
\]
Lower bound of (A.11). We first have
\[ \frac{s|\zeta|^2}{2} \int_{[0,T] \times T_L} \Delta^2 \varphi |w|^2 \geq -C^* s \lambda^4 \int_{[0,T] \times T_L} \xi^3 |w|^2. \] (A.16)

Besides, we have
\[ -|\zeta|^2 s^2 \int_{[0,T] \times T_L} \text{div}(|\nabla \varphi|^2 \nabla \varphi) |w|^2 + |\zeta|^2 s^3 \int_{[0,T] \times T_L} |\nabla \varphi|^2 \Delta \varphi |w|^2 \]
\[ = -|\zeta|^2 s^3 \int_{[0,T] \times T_L} \nabla |\nabla \varphi|^2 \cdot \nabla \varphi |w|^2 \]
\[ \geq -C^* s^3 \lambda^3 \int_{[0,T] \times T_L} \xi^2 |w|^2 + |\zeta|^2 s^3 \lambda^4 \int_{[0,T] \times T_L} |\nabla \psi|^4 \xi^3 |w|^2. \]

Moreover, since \( \inf \{ |\nabla \psi| \} > 0 \) on \( T_L \setminus \overline{w} \), and \( |\zeta| > 0 \), we deduce that
\[ |\zeta|^2 s^3 \lambda^4 \int_{[0,T] \times T_L} |\nabla \psi|^4 \xi^3 |w|^2 \geq C^* s^3 \lambda^4 \int_{[0,T] \times T_L} \xi^3 |w|^2 - C^* s^3 \lambda^4 \int_{[0,T] \times \omega} \xi^3 |w|^2. \]

Thus, we have
\[ \frac{s|\zeta|^2}{2} \int_{[0,T] \times T_L} \Delta^2 \varphi |w|^2 - |\zeta|^2 s^3 \int_{[0,T] \times T_L} \text{div}(|\nabla \varphi|^2 \nabla \varphi) |w|^2 + |\zeta|^2 s^3 \int_{[0,T] \times T_L} |\nabla \varphi|^2 \Delta \varphi |w|^2 \]
\[ \geq C^* s^3 \lambda^4 \int_{[0,T] \times T_L} \xi^3 |w|^2 - C^* s^3 \lambda^4 \int_{[0,T] \times \omega} \xi^3 |w|^2. \] (A.17)

Lower bound of (A.12). According to the definition of \( \varphi \), we have
\[ \partial^2_t \varphi = \frac{\partial^2 \theta}{\theta} \varphi. \]

Furthermore, in views of the definition of \( \theta \), on \([0,T_0] \times T_L\), we have
\[ 0 \leq \partial^2_t \theta \leq C^* s^2 \lambda^4 e^{4\lambda}. \]

Thus, since \( \psi \geq 6 \) and \( \theta \geq 1 \), we obtain
\[ -\partial^2_t \varphi \geq -C^* s^2 \lambda^5 e^{16\lambda} \geq -C^* s^2 \lambda^3 \xi^3, \]
on \([0,T_0] \times T_L\). On the other hand, on \([T_0, T] \times T_L\), we have
\[ -\partial^2_t \varphi \geq -C^* \lambda \xi^2. \]

We deduce that
\[ -\frac{s}{2} \int_{[0,T] \times T_L} \partial^2_t \varphi |w|^2 \geq -C^* s^3 \lambda^3 \int_{[0,T] \times T_L} \xi^3 |w|^2. \]

Moreover, it follows from the definition of \( \theta \) that
\[ \frac{\partial \theta}{\theta} \leq 0 \text{ on } [0, T - 2T_1] \text{ and } \frac{\partial \theta}{\theta} \leq C^* \xi \text{ on } [T - 2T_1, T). \]
Then, we deduce that
\[-\frac{\alpha s^2}{2} \iint_{[0,T] \times T_L} \partial_t \rho |\nabla \varphi|^2 |w|^2 \geq -C^* s^2 \lambda^2 \iint_{[0,T] \times T_L} \xi^3 |w|^3.\]

Then, we conclude that
\[-\frac{s}{2} \iint_{[0,T] \times T_L} \partial_t \rho |\nabla \varphi|^2 |w|^2 \geq -\frac{\alpha s^2}{2} \iint_{[0,T] \times T_L} \partial_t \rho |\nabla \varphi|^2 |w|^2 \geq -C^* s^2 \lambda^3 \iint_{[0,T] \times T_L} \xi^3 |w|^2. \tag{A.18}

**Lower bound of (A.13).** From the definition of \( \varphi \), we deduce that
\[-2|\zeta|^2 s \Re \left( \iint_{[0,T] \times T_L} D^2 \varphi(\nabla w, \nabla w) \right) = 2|\zeta|^2 s \lambda \Re \left( \iint_{[0,T] \times T_L} \xi D^2 \psi(\nabla w, \nabla w) \right) \]
\[\geq 2|\zeta|^2 s \lambda \Re \left( \iint_{[0,T] \times T_L} \xi \nabla \psi \cdot \nabla w \right)^2 \]
\[\geq -C^* s \lambda \iint_{[0,T] \times T_L} \xi |\nabla w|^2.\]

Furthermore, we have
\[-s \Re \left( i \beta \iint_{[0,T] \times T_L} \partial_t \nabla \psi \cdot \nabla w \right) = s \lambda \Re \left( \frac{\partial_t \theta}{\theta} \xi \nabla \psi \cdot \nabla w \right).

On \([0, T-2T_1] \), we have
\[s \lambda \xi \frac{|\partial \theta|}{\theta} \leq \frac{s^2 \lambda^3 \xi e^{2\lambda}}{T_0},\]
thus
\[s \lambda \Re \left( i \beta \iint_{[0,T-2T_1] \times T_L} \frac{\partial_t \theta}{\theta} \xi \nabla \psi \cdot \nabla w \right) \geq -C^* s^3 \lambda^4 e^{2\lambda} \iint_{[0,T] \times T_L} \xi^2 |w|^2 - C^* s \lambda^2 e^{2\lambda} \iint_{[0,T] \times T_L} |\nabla w|^2 \]
\[\geq -C^* s^3 \lambda^3 \iint_{[0,T] \times T_L} \xi^3 |w|^2 - C^* s \lambda \iint_{[0,T] \times T_L} \xi |\nabla w|^2.

Besides, on \([T-2T_1, T)\), we have
\[s \lambda \xi \frac{|\partial \theta|}{\theta} \leq C^* s \lambda^2 \xi^2,\]
and then
\[s \lambda \Re \left( i \beta \iint_{[0,T] \times T_L} \frac{|\partial \theta|}{\theta} \xi \nabla \psi \cdot \nabla w \right) \geq -C^* s \lambda \iint_{[0,T] \times T_L} \xi^3 |w|^2 - C^* s \lambda \iint_{[0,T] \times T_L} \xi |\nabla w|^2.

We deduce that
\[-s \Re \left( i \beta \iint_{[0,T] \times T_L} \partial_t \nabla \psi \cdot \nabla w \right) \geq -C^* s^3 \lambda^3 \iint_{[0,T] \times T_L} \xi^3 |w|^2 - C^* s \lambda \iint_{[0,T] \times T_L} \xi |\nabla w|^2.\]
If we denote by $L_2$ the expression of Line (A.13), we deduce that
\[
L_2 \geq -C^* s^3 \lambda^3 \iint_{[0,T] \times \mathbb{T}_L} \xi^3 |w|^2 - C^* s \lambda \iint_{[0,T] \times \mathbb{T}_L} \xi |\nabla w|^2.
\] (A.19)

We deduce from (A.15), (A.17), (A.19) and (A.18), that
\[
\Re \langle P_1 w, P_2 w \rangle \geq C_\star \int_{\mathbb{T}_L} |\nabla w(0)|^2 + C_\star s^2 \lambda \iint_{[0,T] \times \omega} |w(0)|^2
+ C_\star s^3 \lambda^4 \iint_{[0,T] \times \mathbb{T}_L} \xi^3 |w|^2 - C^* s^3 \lambda^4 \iint_{[0,T] \times \mathbb{T}_L} \xi^3 |w|^2 - C^* s \lambda \iint_{[0,T] \times \mathbb{T}_L} \xi |\nabla w|^2.
\]

**Step 3: Observation on $[0,T] \times \omega$.** Using the previous estimate and (A.1), we obtain that
\[
\iint_{[0,T] \times \mathbb{T}_L} e^{-2s\varphi} |f|^2 + C^* s \lambda \iint_{[0,T] \times \mathbb{T}_L} \xi |\nabla w|^2 + C^* s^3 \lambda^4 \iint_{[0,T] \times \omega} \xi^3 |w|^2
\geq C_\star s^3 \lambda^4 \iint_{[0,T] \times \mathbb{T}_L} \xi^3 |w|^2 + C_\star s^2 \lambda \iint_{[0,T] \times \mathbb{T}_L} |w(0)|^2 + C_\star \int_{\mathbb{T}_L} |\nabla w(0)|^2
+ C_\star \int_{[0,T] \times \mathbb{T}_L} |P_1 w|^2 + C_\star \int_{[0,T] \times \mathbb{T}_L} |P_2 w|^2.
\] (A.20)

Moreover, we have the following lemma.

**Lemma A.2.** For any $\lambda \geq 1$ and $s \geq 1$ large enough, we have
\[
s\lambda^2 \int_{[0,T] \times \mathbb{T}_L} \xi |\nabla w|^2 \leq C^* s^3 \lambda^4 \int_{[0,T] \times \mathbb{T}_L} \xi^3 |w|^2 + C^* \int_{[0,T] \times \mathbb{T}_L} |P_1 w|^2.
\] (A.21)

**Proof.** We have
\[
s\lambda^2 \int_{[0,T] \times \mathbb{T}_L} \xi |\nabla w|^2 = -s \lambda^2 \Re \left( \int_{[0,T] \times \mathbb{T}_L} \nabla \xi \cdot \nabla w \bar{w} \right) - s \lambda^2 \Re \left( \int_{[0,T] \times \mathbb{T}_L} \xi \Delta w \bar{w} \right)
= \frac{s \lambda^2}{2} \int_{[0,T] \times \mathbb{T}_L} \Delta \xi |w|^2 - s \lambda^2 \Re \left( \int_{[0,T] \times \mathbb{T}_L} \xi \Delta w \bar{w} \right).
\]

Then, from
\[
- \Delta w = \frac{1}{\alpha} \left( P_1 w + \alpha s^2 |\nabla \varphi|^2 w + i\beta (2s \nabla \varphi \cdot \nabla w + s \lambda \Delta \varphi w) + s \partial_t \varphi w \right) \quad \text{on} \quad [0,T) \times \mathbb{T}_L,
\]
it follows that

\[
\begin{align*}
  s\lambda^2 \int_{[0,T] \times T_L} \xi |w|^2 &= \frac{s\lambda^2}{2} \int_{[0,T] \times T_L} \Delta \xi |w|^2 + \frac{s\lambda^2}{\alpha} \Re \left( \int_{[0,T] \times T_L} P_1 w \xi \overline{w} \right) \\
  &\quad + s^3 \lambda^2 \int_{[0,T] \times T_L} \xi |\nabla \varphi|^2 |w|^2 \\
  &\quad + \frac{s\lambda^2}{\alpha} \Re \left( ib \int_{[0,T] \times T_L} \left( 2s \nabla \varphi \cdot \nabla w + s\lambda \Delta \varphi w \right) \xi \overline{w} \right) \\
  &\quad + \frac{s^2 \lambda^2}{\alpha} \int_{[0,T] \times T_L} \xi \partial_t \varphi |w|^2 \\
  &= \frac{s\lambda^2}{2} \int_{[0,T] \times T_L} \Delta \xi |w|^2 + \frac{s\lambda^2}{\alpha} \Re \left( \int_{[0,T] \times T_L} P_1 w \xi \overline{w} \right) \\
  &\quad + s^3 \lambda^2 \int_{[0,T] \times T_L} \xi |\nabla \varphi|^2 |w|^2 \\
  &\quad + \frac{2s^2 \lambda^2}{\alpha} \Re \left( ib \int_{[0,T] \times T_L} \nabla \varphi \cdot \nabla w \xi \overline{w} \right) \\
  &\quad + \frac{s^2 \lambda^2}{\alpha} \int_{[0,T] \times T_L} \xi \partial_t \varphi |w|^2.
\end{align*}
\]

Furthermore, since \( \partial_t \varphi \leq 0 \) on \([0, T - 2T_1] \times T_L\), we deduce that

\[
\begin{align*}
  s^2 \lambda^2 \int_{[0,T] \times T_L} \xi \partial_t \varphi |w|^2 &\leq s^2 \lambda^2 \int_{[T - 2T_1, T] \times T_L} \xi \partial_t \varphi |w|^2. \quad (A.22)
\end{align*}
\]

Moreover, by using the definition of \( \theta \) and (2.10), we deduce that

\[
|\partial_t \varphi| \leq \left| \frac{\partial_t \theta}{\theta} \right| \varphi \leq C^* \theta \varphi \leq C^* \theta^2 \lambda e^{12\lambda} \quad \text{on} \quad [T - 2T_1, T) \times T_L.
\]

Since \( \psi \geq 6 \), we obtain that

\[
\partial_t \varphi \leq C^* \lambda \xi^2 \quad \text{on} \quad [T - 2T_1, T) \times T_L.
\]

Hence, from (A.22), it follows that

\[
\begin{align*}
  s^2 \lambda^2 \int_{[0,T] \times T_L} \xi \partial_t \varphi |w|^2 &\leq C^* s^2 \lambda^3 \int_{[0,T] \times T_L} \xi^3 |w|^2. \quad (A.23)
\end{align*}
\]

On the other hand, by using the Young estimate, we get

\[
\begin{align*}
  s\lambda^2 \Re \left( \int_{[0,T] \times T_L} P_1 w \xi \overline{w} \right) &\leq C^* \int_{[0,T] \times T_L} |P_1 w|^2 + C^* s^2 \lambda^4 \int_{[0,T] \times T_L} \xi^2 |w|^2. \quad (A.24)
\end{align*}
\]
Furthermore, it follows from the Young estimate that
\[
\frac{2s^2\lambda^2}{\alpha^2} \Re \left( i\beta \int_{[0,T] \times T_L} \nabla \varphi \cdot \nabla w \xi w \right) \\
\leq s^2\lambda^2|\beta| \left( \frac{1}{s(|\beta| + 1)} \int_{[0,T] \times T_L} \xi|\nabla w|^2 + \frac{4(|\beta| + 1)s}{\alpha^2} \int_{[0,T] \times T_L} |\nabla \varphi|^2 |\xi w|^2 \right) \\
\leq \frac{|\beta|}{|\beta| + 1}s\lambda^2 \int_{[0,T] \times T_L} \xi|\nabla w|^2 + C^* s^3\lambda^4 \int_{[0,T] \times T_L} \xi^3|w|^2.
\]  
(A.25)

Then, by combining (A.23), (A.24) and (A.25), we get
\[
s\lambda^2 \int_{[0,T] \times T_L} \xi|\nabla w|^2 \leq C^* \int_{[0,T] \times T_L} |P_1w|^2 + C^* s^3\lambda^4 \int_{[0,T] \times T_L} \xi^3|w|^2 \\
+ \frac{|\beta|}{|\beta| + 1}s\lambda^2 \int_{[0,T] \times T_L} \xi|\nabla w|^2 \\
+ \frac{s\lambda^2}{2} \int_{[0,T] \times T_L} \Delta \xi|w|^2 + s^3\lambda^2 \int_{[0,T] \times T_L} \xi|\nabla \varphi|^2 |\xi w|^2 \\
\leq C^* \int_{[0,T] \times T_L} |P_1w|^2 + C^* s^3\lambda^4 \int_{[0,T] \times T_L} \xi^3|w|^2 \\
+ C^* s^4 \int_{[0,T] \times T_L} \xi^3|w|^2,
\]
that is
\[
\left( 1 - \frac{|\beta|}{|\beta| + 1} \right) s\lambda^2 \int_{[0,T] \times T_L} \xi|\nabla w|^2 \leq C^* \int_{[0,T] \times T_L} |P_1w|^2 + C^* s^3\lambda^4 \int_{[0,T] \times T_L} \xi^3|w|^2 \\
+ C^* s^3\lambda^4 \int_{[0,T] \times T_L} \xi^3|w|^2.
\]

We apply Lemma A.2. Then, by using (A.20) to estimate the last term of the right-hand side of (A.21) and by absorbing the first term of the right-hand side, we deduce that
\[
C_2 s\lambda^2 \int_{[0,T] \times T_L} \xi|\nabla w|^2 \leq s^2 \int_{[0,T] \times T_L} \xi|\nabla w|^2 \\
\leq C^* \int_{[0,T] \times T_L} |P_1w|^2 + C^* s^3\lambda^4 \int_{[0,T] \times T_L} \xi^3|w|^2 \\
\leq C^* \int_{[0,T] \times T_L} e^{-2s\tau}|f|^2 + C^* s^3\lambda^4 \int_{[0,T] \times \omega} \xi^3|w|^2 \\
+ C^* s\lambda \int_{[0,T] \times T_L} \xi|\nabla w|^2.
\]  
(A.26)

By combining estimates (A.20) and (A.26), we obtain
\[
C^* s^3\lambda^4 \int_{[0,T] \times T_L} \xi^3|w|^2 + C_2 s\lambda^2 \int_{[0,T] \times T_L} \xi|\nabla w|^2 + C^* s^3\lambda^4 e^{14\lambda} \int_{T_L} |w(0)|^2 + C_2 \int_{T_L} |\nabla w(0)|^2 \\
\leq C^* \int_{[0,T] \times T_L} e^{-2s\tau}|f|^2 + C^* s^3\lambda^4 \int_{[0,T] \times \omega} \xi^3|w|^2.
\]  
(A.27)
Step 4: Observation on $[0, T] \times \text{supp}(\chi_0)$. Let us remark that the observation is done on $\omega \subset \{\chi_0 = 1\} \subset \text{supp}(\chi_0)$. Thus, we have

$$\int_{[0, T] \times \omega} \xi^3 |w|^2 \leq \int_{[0, T] \times T_L} \chi_0 \xi^3 |w|^2.$$ 

Then, we deduce from (A.27) that

$$s^3 \lambda^4 \int_{[0, T] \times T_L} \xi^3 |w|^2 + s \lambda^2 \int_{[0, T] \times T_L} \xi |\nabla w|^2 + s^2 \lambda e^{14 \lambda} \int_{T_L} |w(0)|^2 + \int_{T_L} |\nabla w(0)|^2 \leq C^* \left( \int_{[0, T] \times T_L} e^{-2s \varphi} |f|^2 + s^3 \lambda^4 \int_{[0, T] \times T_L} \chi_0 \xi^3 |w|^2 \right). \quad (A.28)$$

Step 5: Conclusion. It is enough to recover the estimate on $z$ from (A.28). Since $w = \text{we}^{s \varphi}$, we get

$$|w|^2 e^{-2s \varphi} = |w|^2$$

and

$$|\nabla w|^2 e^{-2s \varphi} \leq 2|\nabla w|^2 + 2s^2 |\nabla \varphi|^2 |w|^2 \leq 2|\nabla w|^2 + 2C^* s^2 \lambda^2 \xi^2 |w|^2.$$ 

Combining the above estimate and (A.28), we conclude the proof of Lemma A.1. \qed

References

[1] M. Badra, S. Ervedoza, and S. Guerrero. Local controllability to trajectories for non-homogeneous incompressible Navier-Stokes equations. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 33(2):529–574, 2016.

[2] S. Benzoni-Gavage, R. Danchin, S. Descombes, and D. Jamet. Structure of Korteweg models and stability of diffuse interfaces. *Interfaces and Free Boundaries*, 7:371–414, 2005.

[3] D. Bresch, M. Gisclon, and I. Lacroix-Violet. On Navier-Stokes-Korteweg and Euler-Korteweg systems: application to quantum fluids models. *Arch. Ration. Mech. Anal.*, 233(3), 2019.

[4] S. Brull and F. Méhats. Derivation of viscous correction terms for the isothermal quantum Euler model. *ZAMM Journal of applied mathematics and mechanics: Zeitschrift für angewandte Mathematik und Mechanik*, 90:219–230, 03 2010.

[5] N. Carreño and E. Cerpa. Local controllability of the stabilized Kuramoto-Sivashinsky system by a single control acting on the heat equation. *J. Math. Pures Appl. (9)*, 106(4):670–694, 2016.

[6] T. Cazenave. *Semilinear Schrodinger Equations*. Courant lecture notes in mathematics. American Mathematical Society, 2003.

[7] F. Charve, R. Danchin, and J. Xu. Gevrey analyticity and decay for the compressible Navier-Stokes system with capillarity. *Indiana University Mathematics Journal*, 2018.

[8] S. Chowdhury, M. Ramaswamy, and J. P. Raymond. Controllability and stabilizability of the linearized compressible Navier-Stokes system in one dimension. *SIAM J. Control. Optim.*, 50:2959–2987, 2012.

[9] F. Coquel, D. Diehl, C. Merkle, and C. Rohde. Sharp and diffuse interface methods for phase transition problems in liquid-vapour flows. *IRMA Lect. Math. Theor. Phys.*, 7, 09 2009.
[10] J.-M. Coron and P. Lissy. Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components. *Invent. Math.*, 198(3):833–880, 2014.

[11] R. Danchin and B. Desjardins. Existence of solutions for compressible fluid models of Korteweg type. *Annales de l’Institut Henri Poincaré C, Analyse non linéaire*, 18(1):97–133, 2001.

[12] J. E. Dunn and J. Serrin. On the thermomechanics of interstitial working. *Arch. Rational Mech. Anal.*, 88(2):95–133, 1985.

[13] M. Duprez and P. Lissy. Indirect controllability of some linear parabolic systems of m equations with m-1 controls involving coupling terms of zero or first order. *Journal de Mathématiques Pures et Appliquées*, 106(5):905–934, 2016.

[14] S. Ervedoza, O. Glass, and S. Guerrero. Local exact controllability for the two- and three-dimensional compressible Navier–Stokes equations. *Communications in Partial Differential Equations*, 41:1660 – 1691, 2015.

[15] S. Ervedoza, O. Glass, S. Guerrero, and J.-P. Puel. Local exact controllability for the one-dimensional compressible Navier-Stokes equation. *Arch. Ration. Mech. Anal.*, 206(1):189–238, 2012.

[16] S. Ervedoza and M. Savel. Local boundary controllability to trajectories for the 1D compressible Navier-Stokes equations. *ESAIM Control Optim. Calc. Var.*, 24(1):211–235, 2018.

[17] X. Fu. Null controllability for the parabolic equation with a complex principal part. *Journal of Functional Analysis*, 257:1333–1354, 09 2009.

[18] A. V. Fursikov and O. Y. Imanuvilov. Controllability of evolution equations. *Seoul National University*, 1996.

[19] B. Haspot. Global strong solution for the Korteweg system with quantum pressure in dimension $n \geq 2$. *Mathematische Annalen*, 367(1):667–700, 2017.

[20] H. Hattori and D. Li. Global solutions of a high dimensional system for Korteweg materials. *Journal of Mathematical Analysis and Applications*, 198:84–97, 1996.

[21] A. Jüngel. Global weak solutions to compressible Navier-Stokes equations for quantum fluids. *SIAM J. Math. Anal.*, 42:1025–1045, 2010.

[22] S. Kawashima, Y. Shibata, and J. Xu. Dissipative structure for symmetric hyperbolic-parabolic systems with Korteweg-type dispersion. *Comm. Partial Differential Equations*, 47(2):378–400, 2022.

[23] D. Maity. Some controllability results for linearized compressible Navier-Stokes system. *ESAIM: Control, Optimisation and Calculus of Variations*, 21(4):1002–1028, 2015.

[24] N. Molina. Local exact boundary controllability for the compressible Navier-Stokes equations. *SIAM J. Control. Optim.*, 57:2152–2184, 2019.

[25] L. Rosier and B.-Y. Zhang. Null Controllability of the Complex Ginzburg-Landau Equation. *Annales de l’I.H.P. Analyse non linéaire*, 26(2):649–673, 2009.

[26] Z. Song and J. Xu. Global existence and analyticity of $L^p$ solutions to the compressible fluid model of Korteweg type. *J. Differential Equations*, 370:101–139, 2023.

[27] A. Tendani Soler. Analytic regularity for Navier-Stokes-Korteweg model on pseudo-measure spaces. *Dynamics of Partial Differential Equations*, 20(1):1–21, 2023.
[28] M. Tucsnak and G. Weiss. *Observation and Control for Operator Semigroups*. Birkhäuser Advanced Texts Basler Lehrbücher. Birkhäuser Basel, 2009.