MSGUT Reborn?

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ABSTRACT

We present examples of fits of fermion mass data using the $10 - 120 - \overline{126}$ FM Higgs system in Susy SO(10) GUTs that follow the scenario[1, 2] in which the $120$ -plet collaborates with the $10$-plet to fit the charged fermion masses while small $\overline{126}$-plet couplings enhance the Type I seesaw neutrino masses to viable values and make the fit to light fermions accurate. Restricting ourselves to the CP conserving case we use a linear perturbative technique to obtain accurate charged fermion fits ($\chi^2_{\text{tot}} < .2$) to all the charged fermion masses and angles. The resulting fits imply Type I neutrino masses that are generically $10^2 - 10^3$ times larger than those obtained in the $10 - \overline{126}$ scenario precisely because of the small $\overline{126}$ coupling. Thus the difficulty of obtaining sufficiently large neutrino masses in the context of these Next to Minimal Susy GUTs is essentially removed. The remaining free parameters allow one to obtain the correct ratio of neutrino mass squared splitting and a large($\theta_{23}^{PMNS}$) and small($\theta_{13}^{PMNS}$) mixing angle. The $\theta_{12}^{PMNS}$ is however small and this indicates that -as in the Type I $10 - \overline{126}$ case - a fully realistic fit to the lepton mixing data also requires CP violation.

1 Introduction

The discovery of neutrino oscillations in 1998, followed by the measurement of increasingly accurate values of the neutrino mass squared differences and mixing an-
gles has driven an intense wave of research into supersymmetric seesaw mechanisms. Left-right symmetric models and particularly all aspects of the structure of renormalizable Susy SO(10) GUTs. In particular the minimal Supersymmetric Grand Unified Theory (MSGUT) \[3, 4, 5\] namely a Supersymmetric SO(10) GUT based on a \(210 - 10 - \overline{126} - 126\) Higgs system has received a great deal of attention over the last seven years. Detailed calculations of every aspect of its spectra\[6, 7, 8, 9\], couplings\[6, 9\] and their evolution\[6, 9\] have been published by several groups. Besides the question of gauge unification the central problem of the second phase of the Grand Unification program namely that of accounting for the observed fermion mass spectrum and mixing structure, for both charged fermions and neutrinos, using the mass relations dictated by the simplest possibly realistic \(10 - \overline{126}\) Higgs structure \[10\], has inspired a great deal of detailed analysis of the fitting problem\[12, 13, 14, 15\].

This so called Babu-Mohapatra(BM) program had, until last year, met with uniform success in accounting for the observed charged fermion spectrum, neutrino mass splitting and mixing using the generic structure of the fermion mass matrices dictated by the choice of Higgs \(10 - \overline{126}\) and the Type Ib and Type II \[20\] Seesaw mechanisms naturally present in SO(10) models. Successful fits of all the fermion mass data available including the neutrino mixing data \[22\] using Type I , Type II and mixed seesaw mechanisms were obtained\[11, 12, 13, 14, 15\]. It was, however, just assumed that the overall scale and the relative magnitudes of the Type I and Type II seesaw masses required for the fits would be realizable in appropriate Susy GUTs. The only such available theory, where in fact the calculations of spectra\[6, 9, 7, 8\] and couplings\[6, 9, 16\] necessary to perform precisely such a test had almost contemporaneously been completed, was the MSGUT itself. The very first survey\[16\] revealed that there were serious difficulties in obtaining Type II over Type I dominance as well as in obtaining large enough Type I neutrino masses. Using a convenient parametrization of the MSGUT spectra and couplings in terms of the single “fast” parameter \((x)\) which controlled MSGUT symmetry breaking \[5, 7, 9, 17\] a complete proof was then given\[11\] of the failure of the Seesaw mechanism in the context of the MSGUT. The nature of the obstruction uncovered by us also led us to suggest a very natural scenario that might deal with the problem in the MSGUT extended by adding the \(120\) FM Higgs representation(Next to Minimal Susy GUT or NMSGUT). The \(120\) had been optimistically excluded in the Babu-Mohapatra program to foster the predictivity of the MSGUT in the neutrino sector. In this alternative scenario\[1\] the \(120\) -plet collaborates with the \(10\) -plet to fit the dominant charged fermion masses.

The small \(126\) -plet couplings give appreciable contributions only to light charged fermion masses and enhance the Type I seesaw masses to viable values since the Type one seesaw masses are inversely proportional to these masses. To begin analysis of the fitting problem in this new and qualitatively different fitting scenario we analyzed\[2\] the 2-3 generation case as a toy model of the dominant core of the complete hierarchical fermion mass system. We found that ansatz consistency requires
\( m_b - m_s = m_r - m_\mu \) at the GUT scale \( M_X \) and predicts near maximal (PMNS) mixing in the leptonic sector for central values of charged fermion parameters and for wide ranges of the other relevant parameters (right-handed neutrino masses and relative strength of contributions of the two doublet pairs from the \( 120 \)-plet to the effective MSSM Higgs pair).

In the current contribution we report on progress in extending the analysis of [2] to a 3 generation CP conserving but otherwise realistic NMSGUT model. We first briefly review the failure of the seesaw mechanism in the MSGUT and the results of our analysis of the 2-3 generation toy model. This is followed by a description of the fitting equations in the CP conserving 3 generation case and our procedure for obtaining accurate solutions of the 3 generation equations by an effectively linear perturbation expansion (in the Wolfenstein parameter \( \epsilon \sim \sin \theta_{12} \sim \sqrt{\sin \theta_{23} \sim .2} \)) around the dominant 2-3 generation mass matrix structures. We then present examples of solutions that are realistic in all respects except that of the omitted CP violation and the lack of two large mixing angles. We argue that since even in the \( 10 - 126 \) Type I seesaw a solution with two large angles and the observed small ratio of mass splittings[22] could be found only when CP violation was also included[11, 15] our results should be construed positively. The study of the CP violating case is now in progress[24]. Thus the main obstruction of too small Type I masses is already overcome even in the CP conserving model but the fit for the CP violating case remains to be done.

2 Seesaw Failure in the MSGUT

The Type I and Type II seesaw Majorana masses of the light neutrinos in the MSGUT are[1] :

\[
M^I_\nu = (1.70 \times 10^{-3} eV) F_I \hat{n} \sin \beta \\
M^{II}_\nu = (1.70 \times 10^{-3} eV) F_{II} \hat{f} \sin \beta \\
\hat{n} = (\hat{h} - 3\hat{f}) \hat{f}^{-1}(\hat{h} - 3\hat{f})
\]

where \( v = 174 \text{ GeV} \) and \( \hat{h}, \hat{f} \) are the Yukawa coupling matrices of \( 10, 126 \) to the \( 16 \) plets containing fermion families and \( \beta \) is the MSSM Higgs doublet mixing angle. The functions \( F_I, F_{II} \) are completely specified by in the MSGUT and explicit forms may be found in [3, 16, 11, 17].

In typical BM-Type II fits[13, 14, 15] the maximal value of \( \hat{f} \) eigenvalues is \( \sim 10^{-2} \) while the corresponding values for \( \hat{h} \) are about \( 10^2 \) times larger. As a result \( \hat{n}_{\text{max}} \sim 10^2 \). This implies that \( R = F_I/F_{II} \leq 10^{-3} \) in order that the pure BM-Type II not be overwhelmed by the BM-Type I values it implies. Such \( R \) values were shown[16, 1] to be un-achievable anywhere in parameter space of the MSGUT (while preserving
baryon stability, perturbativity etc). Furthermore in the BM-Type I fit of [11, 15] the requirement of large neutrino mixing yields, typically, $\hat{n}_{\text{max}} \sim 5 \hat{f}_{\text{max}} \sim .5$ for the maximal eigenvalues of $\hat{n}$. Thus values of $F_1 \sim 100$ are required in order to reach realistic values of seesaw masses for the heaviest neutrino. We demonstrated[16, 1] that such values are not achievable anywhere over the complex $x$ plane except where they also violate some aspect of successful unification. Recently another group has verified our results[18].

3 The new $10 - 120 - \underline{126}$ scenario

To resolve the difficulty with the overall neutrino mass scale we proposed[1][2] that the $\underline{126}$ couplings be reduced far below the level where they are important for 2-3 generation masses while introducing a $\underline{120}$ plet to do the work of charged fermion mass fitting previously accomplished by $\underline{126}$ couplings almost comparable to those of the $\underline{10}$-plet.

The Dirac masses in such GUTs are then generically given by[6][9][1][2]

$$\hat{m}^u = v(\hat{h} + \hat{f} + \hat{g})$$
$$\hat{m}_\nu = v(\hat{h} - 3\hat{f} + (r_5 - 3)\hat{g}) \equiv v(\hat{h} - 3\hat{f} + r'_5\hat{g})$$
$$\hat{m}^d = v(r_1\hat{h} + r_2\hat{f} + r_6\hat{g})$$
$$\hat{m}^l = v(r_1\hat{h} - 3r_2\hat{f} + r_7\hat{g})$$

See [6][9][1][2] for the form of the coefficients $r_i$.

The right handed neutrino mass is $M_\nu = \hat{f}\hat{\sigma}$ and the Type I seesaw formula is

$$M_\nu = vr_4\hat{n} ; \; \hat{n} = (\hat{h} - 3\hat{f} - r'_5\hat{g})\hat{f}^{-1}(\hat{h} - 3\hat{f} + r'_5\hat{g})$$

where $\hat{\sigma} = \frac{i \sqrt{3}}{\alpha_{3/2} \sin \beta}$, $\hat{\sigma}$ is the GUT scale vev of the $\underline{126}$ while $\alpha_i$, $\bar{\alpha}_i$ refer to fractions of the MSSM doublets contributed by various doublets present in the GUT Higgs representations [5][7][9][16][1].

The essence of our proposal is that the $10, 120$ multiplet Yukawas dominate the contributions of the $\underline{126}$ coupling in the charged fermion masses i.e $(\hat{h}, \hat{g})_{\text{max}} >> \hat{f}$. We shall work with the assumption that the fermion Yukawa couplings and coefficients $r_i$ are both real. The latter assumption is almost certainly not valid[6][9][7]. However if a fit could be found with all complexity introduced via these coefficients and the Yukawa couplings were kept real then the resulting NMSGUT would contain only 12 Yukawa coupling parameters i.e 3 less than in the MSGUT with complex Yukawas. Thus even with the new couplings due to the $\underline{120}$ the NMSGUT with only spontaneous CP violation would still be a strong contender as far as minimality of parameters is considered as decisive.

The mass terms above must be matched to the renormalized mass matrices of the MSSM evaluated at the GUT scale. While doing so one must allow[16] for the
possibility that the fields of the GUT are only unitarily related to those of the MSSM at \( M_X \). This introduces several unitary matrices into the fitting problem which, besides conventional ambiguities, specify just how the MSSM lies within the MSGUT once conventions are fixed. Although unphysical in the MSSM these matrices are of vital relevance for calculation of the exotic signatures (e.g B-L violation) of the GUT in question\[16\]. Thus when matching the charged fermion dirac mass matrices we get

\[
\hat{m}_u = V_{u}^{T}D_uQ \\
\hat{m}_d = V_{d}^{T}D_dR \\
\hat{m}_l = V_{l}^{T}D_lL
\]  

(3)

Where \( D_{u,d,l} \) are the charged fermion masses at \( M_X \) and \( V_u, Q, V_d, R = C^\dagger Q, L, V_l \) are arbitrary unitary matrices (\( C \) is the CKM matrix). These matrices must be fixed by convention – where allowed by conventional ambiguities – or determined by the fitting procedure in terms of the low energy data, or left as parameters to be determined by future experiments sensitive to degrees of freedom and couplings (e.g baryon violating couplings) that the low energy data is not. We shall make the SO(10) basis choice \( R = 1 \) in what follows.

To put our equations in a form transparent enough to clearly separate the contributions of the \( 10, 120 \) (and -eventually- \( 126 \)) plets\[24\] we write the matrices \( V_{u,d,l} \) associated with the anti-fermion fields as new unitary matrices \( \Phi_{u,d,l} \) times the fermion field matrices \( Q, R, L \).

\[
V_d = \Phi_d R \quad ; \quad V_u = \Phi_u Q \quad ; \quad V_l = \Phi_l L
\]  

(4)

We then separate symmetric and antisymmetric parts for each charged fermion equation

\[
Z = \Phi^T D + D \Phi \quad ; \quad A = \Phi^T D - D \Phi
\]  

(5)

Solving the mass formulae for the symmetric yukawa couplings one finds :

\[
\hat{h} = \frac{\hat{r}_1}{2v} R^T (3Z_d + DZ_lD^T) R \\
\hat{f} = \frac{\hat{r}_2}{2v} R^T (Z_d - DZ_lD^T) R
\]  

(6)

while the antisymmetric couplings are

\[
\hat{g} = \frac{1}{2v r_7} R^T A_l R
\]  

(7)

and where \( \hat{r}_i = 1/(4r_i) \).

Then the remaining equations are the “sum rule”

\[
(3\hat{r}_1 + \hat{r}_2)Z_d + (\hat{r}_1 - \hat{r}_2)DZ_lD^T - C^T Z_u C = 0
\]  

(8)
3.1 23 generation toy model

We first briefly recapitulate the analysis of the 2 generation CP conserving toy model given in [2]. As explained, in this case the contribution of $\hat{f}$ to the charged fermion masses is taken to be negligible. Then using (say) just the d-type quark equations to solve for $\hat{h}, \hat{g}$, one obtains for the rest:

$$
C^*Z_dC^\dagger = r_1Z_u \ ; \ Z_d = \mathcal{D}Z_l\mathcal{D}^T \ ; \ \mathcal{D} = R^*L^T \\
C^*A_dC^\dagger = r_6A_u \ ; \ r_7A_d = r_6\mathcal{D}A_l\mathcal{D}^T
$$

In this 2 generation case the antisymmetric equations (9) serve only to fix the parameters $r_6, r_7$ and thus play no further role. On the other hand since $C, \mathcal{D}$ are orthogonal matrices it follows that $r_1 = \text{Tr}(Z_d)/\text{Tr}(Z_u)$ so that we can write these two equations in the (dimensionless) form

$$
\hat{S}_1 = \frac{CZ_dC^\dagger - Z_u}{\text{Tr}Z_d - \text{Tr}Z_u} = 0 \\
\hat{S}_2 = \frac{Z_d - \mathcal{D}Z_l\mathcal{D}^T}{\text{Tr}Z_d} = 0
$$

In [2] we gave an analytic solution of the toy model charged fermion mass fitting relations. We parametrized the matrices $\Phi_{u,d,l}, C, \mathcal{D}$ as

$$
\Phi_{u,d,l} = \begin{pmatrix} 
\cos \chi_{u,d,l} & \sin \chi_{u,d,l} \\
-\sin \chi_{u,d,l} & \cos \chi_{u,d,l}
\end{pmatrix} \\
C, \mathcal{D} = \begin{pmatrix} 
\cos \chi_{c,D} & \sin \chi_{c,D} \\
-\sin \chi_{c,D} & \cos \chi_{c,D}
\end{pmatrix}
$$

Then we found that the equations $\hat{S}_1 = 0$ yield ($\alpha = \frac{(d_1-d_2)(u_3+u_2)}{(d_4+d_2)(d_3-u_2)}$)

$$
\chi_u = \chi_d - 2\chi_c \\
\tan \chi_d = \frac{\csc 2\chi_c}{\alpha} - \cot 2\chi_c
$$

Since $\alpha = 1 + (\epsilon^2)$, it is clear that the leading ($\sim \epsilon^{-2}$) contributions cancel leaving behind an $O(1)$ result. Similarly two of the three equations $\hat{S}_2 = 0$ yield

$$
\chi_D = (\chi_l - \chi_d)/2 \ ; \ \chi_l = \pm \tilde{\chi}_l \\
\tilde{\chi}_l \equiv \cos^{-1} \frac{T_d \cos \chi_d}{T_l}
$$

6
To leading order $\chi_l = \pm \chi_d$, this defines two branches of the solution which we call the (+) and (-) branches following the sign between $\chi_l$ and $\bar{\chi}_l$. The third $\hat{S}_2 = 0$ equation yields the all important consistency condition

$$d_3 - d_2 = \Delta d = \Delta l = l_3 - l_2$$

(14)

i.e

$$m_b(M_X) - m_\tau(M_X) = m_s(M_X) - m_\mu(M_X)$$

(15)

Using this solution we determined the 23 sector PMNS mixing angle as a function of the parameters $r'_5, \rho = \hat{f}_3/\hat{f}_2$ ($\hat{f}_i$ are the eigenvalues of $\hat{f}$) and then showed that the 23 sector PMNS mixing was near maximal over much of the parameter space of the theory. Thus the toy model provides a cartoon of the core of the 23 generation dominated fermion hierarchy that is satisfactory and coherent with our scenario in all respects.

### 3.2 3 generation fitting formulae

We now wish to generalize our 2 generation toy model to the 3 generation case, while keeping the CP conserving approximation for simplicity, to enquire how closely one may approach the actual fermion data. The resulting equations are however too complicated to solve analytically. Thus one must resort to some variety of numerical approximation technique or perturbation theory. Our approach is to consider the 23 sector as the dominant “core” of the fermion mass hierarchy and thus to expand around this core by rewriting all the charged fermion mass parameters and mixing angles relative to the dominant normative magnitudes of the third generation by introducing a small parameter $\epsilon \sim \sqrt{\chi_c} \sim \sin \theta_{12}^c \sim \sqrt{\theta_{13}^c} \sim \sqrt{d_2/d_3}$ etc. The explicit analytic solution of the toy model given above coincides, order by order in perturbation theory, including the $b - \tau = s - \mu$ constraint, with the solution found by expanding all angles $\chi$ in powers of $\epsilon$ and solving the equations $\hat{S}_1 = 0 = \hat{S}_2$ order by order in $\epsilon$. The $b - \tau = s - \mu$ unification constraint arises at order $\epsilon^2$. In the three generation case the analytic solution is not available but the complete regularity observed in the perturbation theory in $\epsilon$ and the extreme smallness of the first generation perturbations makes an expansion in $\epsilon$ well motivated. Thus we shall cast the fitting equations in a form such that they reduce to the toy model equations up to order $\epsilon^2$ since $\hat{f}, d_1, u_1, l_1$ appear only at order $\epsilon^3$ or even higher.

We continue with our CP conserving restriction to maintain tractability and parametrize each of the orthogonal matrices in the standard Kobayashi-Masakawa form for real unitary, i.e orthogonal, matrices:

$$O = O_{23}(\chi) \cdot O_{13}(\phi) \cdot O_{12}(\theta)$$

(16)
where $\chi, \phi, \theta$ are generic symbols for rotation angles in the 23, 13, 12 sectors. We shall always work in the SO(10) basis fixed by the condition $R = 1$.

Now the antisymmetric equations are no longer empty but read

\begin{align*}
\hat{A}_{u,d,l} &\equiv \frac{A_{u,d,l}}{\sqrt{\bar{A}_{u,d,l}}} \\
\hat{A}_1 &= C \hat{A}_d C^T - \hat{A}_u = 0 \\
\hat{A}_2^\pm &= \hat{A}_d \mp D \hat{A}_l D^T = 0
\end{align*}

(17)

where

\[ \bar{A}^2 = A_{12}^2 + A_{13}^2 + A_{23}^2 \quad (18) \]

and the sign in the last of equations (17) is chosen corresponding to the branch of the 2 generation model that one is expanding about i.e $\hat{A}_2^\pm = 0$ for $\chi_l = \pm \bar{\chi}_l$. This is because we solved for the coefficients $r_6^2, r_7^2$ in terms of square roots of ratios of $TrA_f^2$ ($f = u, d, l$). So we should retain an option for the sign of the square root: which must be exercised as noted, for consistency, when considering the equations appropriate for expansion around the two branches of the toy model solution discussed above. The necessity of our choice can be easily understood by considering the two generation example where it can be easily checked that $\hat{A}_2^\pm$ cannot vanish on the (-) branch.

Defining

\begin{align*}
\hat{X} &= \frac{3Z_d + DZ_l D^T}{3TrZ_d + TrZ_l} \\
\hat{S}_X &= \hat{X} - C^T \frac{Z_u}{TrZ_u} C
\end{align*}

(19)

we have finally

\[ \hat{S}_3 = \hat{S}_X + \hat{r}_2 \frac{TrZ_d}{TrZ_u} (\hat{S}_2 - \frac{(TrZ_d - TrZ_l)}{TrZ_d} \hat{X}) \quad (20) \]

Since the up sector fermion masses are about $10^2$ times larger than the down or charged lepton sector it is convenient to rescale the values of the up sector masses by $10^2$: $TrZ_u = 10^2 \bar{Tr}Z_u$ and $\hat{r}_2 = 10^2 \hat{r}_2$. The definitions are chosen so that when $\hat{S}_2 = 0$ then $\hat{S}_3 = 0$ reduces to $\hat{S}_1 = 0$. Then, in view of the encouraging results of our 2 generation model for the 23 dominated fermion hierarchy, it is natural to look for solutions to the fitting problem not \textit{ab initio}, i.e not for a solution of hopelessly
non-linear equations but by an expansion (in powers of \( \epsilon \sim \sqrt{\theta_{23}} \sim \theta_{12} \sim .2 \)) around the 2 generation results. Since the 2 generation case gave \( \hat{S}_2 \sim O(\epsilon^3) \) it follows that:

\[
\hat{f} = \frac{1}{2v} \hat{r}_2 Tr Z_d \hat{S}_2
\]  

The \( O(10^{-2}) \) suppression provided by the ratio \( d_3/v \) in the above equation implies that \( (\hat{f}) \sim .01\epsilon^{3+\delta} \) when \( \hat{r}_2 \sim \epsilon^{\delta} \). This will then ensure the enhancement of Type I neutrino masses that is the rationale for this fitting scenario.

We chose the SO(10) basis where \( R = 1 \) and expand all the orthogonal matrix angles \( \theta_{u,d,l}, \phi_{u,d,l,D}, \chi_{u,d,l,D} \) as well as the free parameter \( \hat{r}_2 \) in powers of \( \epsilon \). The expansion is started by assuming either

\[
\chi^{(0)}_d = \chi^{(0)}_u = \chi^{(0)}_l = \chi^{(0)}_D
\]  

(22)
corresponding to the \( \chi_l = +\bar{\chi}_l \) solution, or

\[
\chi^{(0)}_d = \chi^{(0)}_u = -\chi^{(0)}_l = -\chi^{(0)}_D
\]  

(23)
corresponding to the \( \chi_l = -\bar{\chi}_l \) solution. Furthermore, since the off-diagonality of the sum rule eqn. [5] is driven by that of the CKM matrix which has \( \phi_c \sim \epsilon^3 \) we make the weak and well justified assumption (further justified post hoc by actually finding solutions of this type) that the angles \( \phi_{u,d,l,D} \) are \( O(\epsilon) \) or smaller and thus their expansions begin at order \( \epsilon \).

For order \( \epsilon^{0,1,2} \) the expansion simply reproduces the expansion of the toy model already described above in Section 3 together with some additional constraints on the new angles. For example for the (+) type solutions, at order \( \epsilon^0 \), one gets \( \theta_u^0 = \theta_d^0 = \theta_l^0 - \theta_d^0 \). At every order of perturbation we always solve for parameters in which the equations at that order are linear and of as high order as possible. This obviates the difficulty of multiple solutions of nonlinear equations and their extreme sensitivity to the grossly mismatched coefficients implied by the fermion mass hierarchy which stretches over 5 order of magnitude. At every order beyond \( \epsilon^7 \) (where the smallest of the mass parameters i.e \( \tilde{u}_1 \sim \epsilon^7 \) enters the \( \hat{S}_3 = 0 \) equation) we evaluate the \( \chi^2 \) parameters for the charged fermion fit by setting all undetermined parameters to zero. The expansion is pursued till as high as 20th order (!) and the solutions with smallest \( \chi^2 \) retained. Then using the truncated values at the chosen order and the values of the determined expansion coefficients we reconstruct the fermion mass data via the reconstructed Yukawa couplings and mass relation coefficients. We defer a report of the details of these solutions to [24] but two points are crucial: the value of \( \chi_d^0 \) and the constraint \( d_3 = l_3 - l_2 + d_2 \) are the same as in the toy model. Thus in line with our discussion of that model we shall fit the central values of the charged fermion masses and mixing angles except for the value of \( d_3 \) which we shall take to be exactly \( d_3 = m_\tau - m_\mu + m_s \).
Till order $\epsilon^2$ the equations are $\hat{S}_1 = \hat{S}_2 = 0$, thereafter since $\tilde{r}_2 \sim \epsilon^{3+\delta}$ if $\hat{r}_2 \sim \epsilon^{3+\delta}$. It follows that the complete equation $\hat{S}_3 = 0$ reduces to $\hat{S}_X = 0$ till one reaches order $\epsilon^{6+\delta}$. Thereafter the full equation $S_3 = 0$ must be used with the expansion of $\tilde{r}_2$ beginning at order $\epsilon^{3+\delta}$. Since the up quark mass enters the equations only at order $\epsilon^7$ there is no question of finding accurate solutions before that order. Furthermore many of the angle expansion coefficients are left undetermined at any given order after all the equations available at that order have been solved. These undetermined coefficients are simply set to zero before evaluating the results numerically. Note that this cannot affect the Orthogonality of the matrices in which the relevant angles occur and that this luxury is a consequence of our parametrization of the sum rule. Such a procedure may look mathematically questionable particularly since the convergence appears to be very weak and typically the values of the masses and mixing angles oscillate in an apparently haphazard manner. The reason for this may actually be the absence of complexity or the poorness of $\epsilon$ as an analytic expansion parameter. Nevertheless the proof of the pudding lies in the eating of it! Our aim -as that of any competing numerical fitting procedure - is to find a “close” fit i.e to obtain values of the coefficient matrices $\hat{f}, \hat{g}, \hat{h}$ and coefficient quantities $r_1, r_2, r_5, r_6, r_7$ such that when we use the generic form of the SO(10) GUT mass formulae we generate charged and neutral fermion masses compatible with the low energy fermion data extrapolated over an MSSM desert to the GUT scale. As in the fitting done using the downhill simplex method [19, 18] a simple criterion to pick out an acceptable solution is to define $\chi^2$ functions and ask that they be appropriately small:

$$
\chi^2_m = \sum_i \left( \frac{m_i - \bar{m}_i}{\delta m_i} \right)^2
$$

$$
\chi^2_{CKM} = \sum_a \left( \frac{\sin \theta_a - \sin \bar{\theta}_a}{\delta \sin \theta_a} \right)^2
$$

$$
\chi^2_{tot} = \chi^2_m + \chi^2_{CKM}
$$

For the central quark masses and angles we used the Das-Parida central values the fermion data (for $\tan \beta(M_S) = 55$ and at $M_X = 2 \times 10^{16} GeV$ ) (except for $d_3$ as explained above):

$$
\bar{m}_u = .000724 ; \quad \bar{m}_c = .2105 ; \quad \bar{m}_t = 95.148
$$

$$
\bar{m}_d = .001497 ; \quad \bar{m}_s = .0298 ; \quad \bar{m}_b = \bar{m}_\tau + \bar{m}_s - \bar{m}_\mu = 1.5835
$$

$$
\bar{m}_c = .000356 ; \quad \bar{m}_\mu = .0753 ; \quad \bar{m}_\tau = 1.629
$$

$$
\sin \theta_c = .2248 ; \quad \sin \chi_c = .03278 ; \quad \sin \phi_c = .00216
$$

For the $1 - \sigma$ error estimates we used (all masses in GeV):

$$
\bar{m}_u = .000544 ; \quad \bar{m}_c = .1895 ; \quad \bar{m}_t = 91.784
$$

$$
\bar{m}_d = .001231 ; \quad \bar{m}_s = .0260 ; \quad \bar{m}_b = \bar{m}_\tau + \bar{m}_s - \bar{m}_\mu = 1.5625
$$

$$
\bar{m}_c = .000208 ; \quad \bar{m}_\mu = .0711 ; \quad \bar{m}_\tau = 1.609
$$

$$
\sin \theta_c = .2190 ; \quad \sin \chi_c = .03098 ; \quad \sin \phi_c = .00206
$$

For the $1 - \sigma$ error estimates we used (all masses in GeV):
\[ \begin{align*}
\delta m_t &= 40.0 ; \\
\delta m_b &= 0.34 ; \\
\delta m_{\tau} &= 0.038 ; \\
\delta m_c &= 0.018 ; \\
\delta m_s &= 0.0042 ; \\
\delta m_d &= 0.00013 ; \\
\delta \sin \theta_c &= 0.0016 ; \\
\delta \sin \phi_c &= 0.0005 ; \\
\delta \sin \chi_c &= 0.0013
\end{align*} \] (26)

For each parameter at an average of (say) half a $1 - \sigma$ deviation away from the central value one might expect a contribution to $\chi^2$ of, roughly .25 so that $\chi_m^2 \sim 2.25, \chi_{CKM}^2 \sim 0.75$. Determining fits much more accurate than these values is academic as regards the actual values. It is informative only inasmuch as the ability to find finely matched fits to an *a priori* un-sacrosanct set of central values for only approximately known parameters implies only that *any* such set of data is likely to be achievable via a fit of the type indicated. In view of the fact that we pushed $m_b$ up from its central value by about $\delta m_b/2$ to begin with one might even expect larger deviations from the “correct” values (the quotes reflect the fact that the internal correlations of central values and errors in the extrapolated data at $M_X$ are at present very poorly understood). Moreover since the correct scenario for 3 generations must also explain CP violation, it may be that the slow and oscillatory convergence we observe is due to an obstruction in the approach to the true solution due to a projection onto a real subspace of the actual parameter space. Surprisingly our weakly convergent perturbative method frequently gives solutions with $\chi_{tot}^2$ as small as .2 while solutions with $\chi^2 \sim 2$ are hardly difficult to find. In the next section we discuss examples of the solutions found and of how much closer they have brought us to realizing our scenario of a fully successful, NMSGUT compatible, fermion data fit.

### 3.3 Neutrino Masses and Mixing

As shown and discussed in [16, 1, 2] the the Type I seesaw mass formula may be written

\[
\hat{M}_\nu^I \simeq \frac{v^2}{2\sigma}(\hat{h} + r_5'\hat{g} - 3\hat{f})^T \hat{f}^{-1}(\hat{h} + r_5'\hat{g} - 3\hat{f})
\]

\[
= (1.70 \times 10^{-3} eV)R^T \hat{n} RF \sin \beta
\]

\[
= L^T \mathcal{P} D_\nu \mathcal{P}^T L
\] (27)

Where $\mathcal{P}$ is the Lepton mixing (PMNS) matrix[21] in the basis with diagonal leptonic charged current and $D_\nu$ the light neutrino masses extrapolated to $M_X$.

The PMNS matrix $\mathcal{P}$ and the ratio of solar to atmospheric mass squared splittings can then be identified as

\[
\mathcal{P} = D^\dagger N ; \quad \frac{m_{sol}^2}{m_{atm}^2} = \left| \frac{n_{\mu}^2 - n_{\tau}^2}{n_{\mu}^2 - n_{\tau}^2} \right|
\] (28)
where $\mathcal{N}$ diagonalizes $\hat{n} : \hat{n} = \mathcal{N} \hat{n}_{\text{diag}} \mathcal{N}^T$. Since it is very reasonable to assume that the coefficient function $F_I$ will generically have the same magnitudes in the NMSGUT as it did in MSGUT i.e $\leq 1$, it follows that we can verify whether the enhancement of $\hat{n}$ makes the NMSGUT generically viable simply by comparing it with the typical values $\hat{n}_{\text{max}} \sim .5$ which failed (for generic $F_I$ values) in the MSGUT case by about two orders of magnitude and for special ones by about one order of magnitude.

4 Examples of 3 generation fits

Our purpose in this contribution is no more than to outline the likely features of 3 generation fits that follow the scenario suggested [1, 2] by us as a route to overcoming the generic debility of neutrino masses [16, 1] in the Babu-Mohapatra [10] program. Therefore we here only give as examples a few quasi-realistic fits, of the type sought, and a few others for comparison, obtained using the method outlined in the previous section. As described there we simply took the central values of [25] and, after implementing the constraint of $b = \tau + s - \mu$ unification [2] required by the consistency of the fitting equations, aimed for the central values of the other parameters. Thereafter we fit to the neutrino data as closely as possible. This is what we report here. No attempt was made by us as yet to search the parameter space for “optimal” fits. A detailed numerical survey will be reported in [24]. Of course improvement of the procedure regarding the connection to the low energy data is also possible, but, in our opinion this will be called for only after the gross features of the viable fermion data fit are established and CP violation is included. In our view this fit should not be and is not so unstable and delicately poised as to change its qualitative features due to shifts (due to better RG flow control) of a few per cent in the central values that one is trying to fit.

Our procedure was first to fit the ratio of neutrino mass squared splittings to the current [22] central value of the solar to atmospheric mass squared splitting ratio $\Delta m^2_{\odot}/\Delta m^2_{\text{atm}} \sim .032$ by adjusting and thus fixing the value of $r'_5$. This gave a number of candidate $r'_5$ values for a given charged fermion data fit. Each value $r'_5$ corresponds to a certain neutrino mass hierarchy and determines a certain value for the eigenvalues of $\hat{n}$ : the crucial matrix whose maximum eigenvalue and coefficient in the Type I seesaw mass were generically found [16, 1] to be too small in the MSGUT. Using the values of the angles of the orthogonal matrix $D$ found as part of the solution procedure it is easy to generalize the formulae of Section 3 and obtain the three PMNS mixing angles.
4.1 (+) Type fit

In the case of the (+) type solutions we find that the solutions obtained assuming \( r_2 \sim \epsilon^{0.1,2} \) all yield final \( \hat{r}_2 \) values that differ between the solutions with lowest \( \chi^2 \) by no more than about 15%. The \( \chi^2 \) values are also very similar (\( \chi^2_{\text{tot}} \sim .14 \) or about .01 per variable ! : an excellent fit by normal standards). Since the effect of \( \hat{r}_2 \) on the other quantities in the fit is extremely weak, the resultant values of the Yukawa couplings and fermion masses and angles are almost identical. This strengthens our notion of a unique stable realistic solution/map, of the experimentally found hierarchical type, tying together the Yukawa coupling matrices and coefficients to the fermion data. Nevertheless a detailed survey is required to determine the actual ranges of mixing angles and neutrino mass-squared splitting obtainable and even perhaps alternative branches of fits that may actually arise. The values given below have been truncated to 4 decimal places but the actual calculation was done retaining 10 digit accuracy.

In the representative I+ case where \( \hat{r}_2 \) is assumed to be \( \sim \epsilon \) to begin with we find

\[
\hat{r}_1 = 15.27 \quad ; \quad \hat{r}_2 = 0.255 \quad ; \quad r_6 = 0.0187 \quad ; \quad r_7 = 0.023305
\]

\[
\hat{h} = \begin{pmatrix}
-0.0002988 & 0.0000394 & -0.0222691 \\
0.0000394 & 0.009291 & -0.134823 \\
-0.0222691 & -0.134823 & 0.481333
\end{pmatrix}
\]

\[
\hat{f} = \begin{pmatrix}
0.0000132 & 0.0000277 & -0.0001 \\
0.0000277 & -0.0000146 & 0.000057 \\
-0.0001 & 0.000057 & 0.000040
\end{pmatrix}
\]

\[
\hat{g} = \begin{pmatrix}
0 & 0.001644 & -0.02478 \\
-0.001644 & 0 & -0.119710 \\
0.02478 & 0.119710 & 0
\end{pmatrix}
\]

The eigenvalues of \( \hat{h}, \hat{f}, \hat{g} \) are

\[
\hat{h} : .518 \quad ; \quad .0277 \quad ; \quad 1.92 \times 10^{-5}
\]

\[
\hat{f} : 1.33 \times 10^{-4} \quad ; \quad 1.11 \times 10^{-4} \quad ; \quad 0.171 \times 10^{-4}
\]

\[
\hat{g} : \pm 0.122 \quad ; \quad 0
\]

Note how the premises of our scenario[1, 2] are indeed respected.

Then the reconstructed values of the charged fermion masses are(in GeV)

\[
M_U = \{95.148, 0.2105, 0.00077\}
\]
\[ M_D = \{1.5835, 0.02981, 0.0015\} \]
\[ M_l = \{1.629, 0.0753, 0.000356\} \]  

Note that we always fit the charged lepton masses exactly. The reconstructed CKM matrix is

\[ \hat{C} = \begin{pmatrix} 0.974449 & -0.224599 & 0.002155 \\ 0.224548 & 0.973911 & -0.032784 \\ 0.00526452 & 0.324303 & 0.99946 \end{pmatrix} \]  

Which yields the CKM angle magnitudes

\[ \theta_{12} = 0.2265 \quad ; \quad \theta_{13} = 0.002155 \quad ; \quad \theta_{23} = 0.03279 \]  

Then one finds for this solution

\[ \chi^2_m = 0.126 \quad ; \quad \chi^2_{CKM} = 0.016 \quad ; \quad \chi^2_{tot} = 0.142 \]  

To determine the neutrino mixing data the free parameter \( r'_5 \) must be fixed. If it is known one can calculate \( \hat{n} \) and then since \( D \) is known from the solution the mixing angles are easily determined. We fix \( r'_5 \) by enquiring for which values of \( r'_5 \) the six possible ratios

\[ R_{ijk} = \left| \frac{\hat{n}^2_i - \hat{n}^2_j}{\hat{n}^2_i - \hat{n}^2_k} \right| \]  

can attain the current\textsuperscript{[22]} central value \( R = m^2_{\text{sol}}/m^2_{\text{atm}} = 0.32 \). It is then evident that then the indices j,i,k correspond to the e, \( \mu \), \( \tau \) neutrinos respectively and so we labelled such a solution as an \( ijik \) type solution. The software we used allots the index 1 to the largest eigenvalue, 2 the next largest and 3 the smallest, this accounts for the solution labels encountered below.

For the I+ case in question one finds the following set of solutions:

\[ I + 2321 : \quad (a) \ r'_5 = -0.33285 \quad ; \quad (b) \ r'_5 = 0.48265 \]
\[ I + 3231 : \quad (a) \ r'_5 = -0.33140 \quad ; \quad (b) \ r'_5 = 0.481314 \]
\[ I + 2123 : \quad (a) \ r'_5 = -0.14427 \quad ; \quad (b) \ r'_5 = -0.13932 \]
\[ \quad (c) \ r'_5 = 0.301785 \quad ; \quad (d) \ r'_5 = 0.30650 \]
\[ I + 1213 : \quad (a) \ r'_5 = -0.14435 \quad ; \quad (b) \ r'_5 = -0.139237 \]
\[ \quad (c) \ r'_5 = 0.306578 \quad ; \quad (d) \ r'_5 = 0.30171 \]  

It is clear that the cases I+2321 and I+3231 and cases I+2123 and I+1213 are almost identical. Note also how close I+2123(a) and I+2123(b) and I+2123(c) and I2123(d) are. In fact these pairs give almost identical leptonic mixing parameters
and \( \hat{n} \) eigenvalues. Case I+2321 corresponds to the hierarchy \( m_\nu > m_\mu > m_\text{e} \) while case I+3231 corresponds to \( m_\mu > m_\text{e} > m_\nu \). Similarly case I+2123 corresponds to \( m_\text{e} > m_\mu > m_\nu \) while case I+3123 has highly hierarchical \( \hat{n} \) eigenvalues while in cases I+2123, I+1213 which have an inverted hierarchy there are two large almost degenerate eigenvalues and one very small one. Because of the close similarities we give only the values for cases I+2321a, I+2321b and I+2123a, I+2123c in Table 2. Only in case I+2312 is the 23 sector mixing angle large and the 13 mixing angle reasonably small. However the value of the 12 sector mixing is very low. Clearly then two large and one small PMNS mixing angles are hard to achieve. On the other hand the large (about 200 times larger than the Type I fits in the \( 10 - 126 \) scenario) value of the largest \( \hat{n} \) eigenvalue together with the satisfactory value of the mass squared splitting ratio means that the problem with too small neutrino masses is unlikely to appear even for generic values of the GUT scale breaking, leave alone regions where the coefficient function \( F_I \) is itself large.

Even though they are “captured” at different orders in perturbation theory the solutions obtained when \( \hat{r}_2 \) is assumed to be \( \sim 1 \) or \( \sim \epsilon^2 \) to begin with are very close to the one displayed above which was obtained by assuming that \( \hat{r}_2 \sim \epsilon \). In fact the final values of \( \hat{r}_2 \) in these two cases are \( 0.00185, 0.00225 \). Some of the details can be compared in Table I.

Focussing on the case I+2312(a) we find that using the value \( r'_5 = -0.333 \) and the values of the \( D \) angles found for the I+ solution:

\[
\phi_D = 0.0525 \quad ; \quad \theta_D = 0.43481 \quad ; \quad \chi_D = 0.02568
\]

then gives, on using eqn. (28) the following values for the eigenvalues of \( \hat{n} \)

\[
\hat{n}_1 = 113.8 \quad ; \quad n_2 = 20.045 \quad ; \quad n_3 = 1.97 \times 10^{-5}
\]

Which clearly shows the required 100-300 fold enhancement relative to the BM fitting scenario in the overall scale of the Type I seesaw masses via an enhancement of \( \hat{n} \) that motivates our scenario. However the PMNS angles are found to be:

\[
\sin^2 \theta_{12}^{PMNS} = .073 \quad ; \quad \sin^2 \theta_{23}^{PMNS} = .77 \quad ; \quad \theta_{13} = .176
\]

Although one might hope to improve the value of the 23 and 13 angles towards more realistic values it is clear that raising the 12 sector mixing from such a small value will be difficult if the fit is indeed stable as we propose it is. A final conclusion must await a detailed survey. However recall that even in the \( 10 - 126 \) scenario no Type I solution was found till CP violation was included in the charged fermion sector. Results of the lepton sector parameters for some for some of the other distinct cases are given for comparison in Table 2.
Table 1: Charged fermion fitting parameters and Yukawa eigenvalues for 6 sample fits. N is the order of perturbation theory at which the fit was “captured” while the exponent δ specifies the initial assumption \( \hat{r}_2 \sim \epsilon^\delta \).

To the accuracy displayed the value of \( \hat{r}_1 = 15.27 \) for every solution.

### 4.2 (-) Type fits

For the branch defined by imposing \( \chi_0^d = \chi_0^d = -\chi_0^l = -\chi_0^D \) on the perturbative iteration, the fits we obtain are, at \( \chi^2 > 1.6 \), somewhat poorer than the ones found for the (+) case. Moreover even such fits first occur at quite high order in the perturbation expansion and the differences between the final \( \hat{r}_2 \) values are somewhat greater. The acceptable cases with final values \( \hat{r}_2 \sim .01 \) are shown in Table 1. as cases I-,II- along with another fit III- where an initial value \( \hat{r}_2 \sim \epsilon^{-1} \sim \) leads to a distinct solution with \( \hat{r}_2 \sim .05 \). This type of solution has \( \hat{n} \) eigenvalues \{4.3, .75, .012\} and shows clearly that \( \hat{f} < .01 \epsilon^3 \) is necessary to raise the value of \( \hat{n} \) enough to overcome the difficulty uncovered for the MSGUT [16, 1]. The rest of the procedure is identical to the (+) case and the relevant results are collected in Tables 1 and 2.

For the readers convenience we give the values of the Yukawa couplings, coefficients \( r_i \) for one of the (-) cases. Namely I-2312a.

\[
\hat{r}_1 = 15.27 \; ; \; \hat{r}_2 = 1.21 \; ; \; r_6 = .01886 \; ; \; r_7 = -.02340
\]

\[
\hat{h} = \begin{pmatrix}
-0.000149706 & -0.00054677 & -0.014659 \\
-0.00054677 & 0.00927299 & -0.134398 \\
-0.014659 & -0.134398 & 0.48307
\end{pmatrix}
\]
Table 2: Values of neutrino mass and mixing parameters determined assuming central value for the ratio of solar to atmospheric mass splitting.

\[
\hat{f} = \begin{pmatrix}
0.0000521467 & 0.000139 & -0.00044 \\
0.000139 & -0.000039 & 0.000182 \\
-0.00044 & 0.000182 & 0.000133
\end{pmatrix}
\]

\[
\hat{g} = \begin{pmatrix}
0 & 0.00116 & -0.01753 \\
-0.00116 & 0 & -0.1191 \\
0.01753 & 0.1191 & 0
\end{pmatrix}
\]

\[
\phi_D = -0.04057 ; \quad \theta_D = -0.134924 ; \quad \chi_D = -0.57541
\]

\[
M_U = \{95.148, 0.2105, 0.00077\}
\]

\[
M_D = \{1.5835, 0.0298017, 0.0014962\}
\]

\[
M_l = \{1.629, 0.0753, 0.000356\}
\]

5 Discussion, Conclusions and Outlook

In this paper we have reported progress in accomplishing our program for a completely realistic fit of all known charged fermion and neutrino mass data.
using the mass relations and RG evolution common to any SO(10) Susy GUT with a \(10 - 120 - \overline{126}\) FM Higgs system fermion. Specifically, we have shown that in the quasi realistic 3 generation but CP preserving(real) case we are able to obtain accurate charged fermion fits, neutrino mass parameters and a PMNS mixing pattern that can be large at least in the 23 sector and small in the 13 sector. The remaining deficiencies, namely the absence of simultaneous large mixing in the 12 sector and the fit of the MSSM CKM CP phase in the first quadrant can presumably be remedied in the complex 3 generation case, in close analogy with the BM program where a successful Type I fit could be found only when CP violation was introduced. For the case where the couplings are all real and CP violation arises from the phases present in the coefficients \(r\); the CP violation can be traced to the GUT scale symmetry breaking and fine tuning[5, 9, 7] that defines the light Higgs pair of the MSSM. In this case the EW doublet([1,2,\pm1]) mass matrix formulae for the NMSGUT that we have already furnished[1] will permit the evaluation of the feasibility of the generic fit for realizability in the parameter space of the NMSGUT. The report on the investigation of these questions will be given in [26].

We carried a somewhat novel expansion in a single parameter \(\epsilon\), in terms of which the very strong (\(\sim 10^5\)) hierarchy visible in the MSSM fermion data can be compactly formulated, to high orders of the perturbation expansion. The explicit and systematic incorporation of the hierarchical structure into our fitting procedure makes it well suited to verification of the viability of the hierarchical fitting scenario we proposed[1, 2]. On the other hand the convergence in the parameter \(\epsilon\) in the three generation case(in sharp contrast to the 23 generation toy model[2]) is only oscillatory[24]. The 2 generation Toy model is necessarily CP conserving in the charged fermion sector whereas the true solution in the 3 generation case is not. Thus the failure to converge may reflect an obstruction to fitting the data to the absolute minimum when the channels to reach it on the complex plane have been blocked by our reality assumption. Our method is complementary to alternative numerical procedures such as the “downhill simplex” method[19, 18] which it can verify as well as provide initial search points for. Inasmuch as it is only the actual hierarchical mass data that must be fitted the true solution must necessarily be “close” to the 23 generation toy model caricature. Thus the “missed solutions anxiety” associated with these methods, may find some palliation when the fitting problem is considered in the light of the insight provided by these two complementary methods jointly. It is only after the successful completion of this program that we may, finally, be in possession of a well defined Susy GUT compatible with all low energy data as well as information on the embedding of the MSSM in the MSGUT: which emerges as the most valuable corollary product of the fitting procedure[16]. It is only then that we will be able to enter the third phase of the GUT program in which the exotic process (\(\Delta B \neq 0\) etc ) predictions will finally begin to be formulated in a sufficiently tight manner as to make the comparison with data a falsifiability test rather than a
hopeful check on a lottery bet.

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7 Note Added

On 18 July 2006, as this paper was being readied for submission to the archive, we received [27] in which a successful fit to the complete fermion data including CP violation, for \( \tan \beta \sim 10 \), with spontaneous CP violation, and with an additional \( Z_2 \) symmetry imposed to further reduce the number of degrees of freedom, is reported. The results are consonant the scenario proposed in [12] and the results of this paper.

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