An Efficient Generalized Shift-Rule for the Prefer-Max De Bruijn Sequence

Gal Amram\textsuperscript{a} and Amir Rubin\textsuperscript{b}

\textsuperscript{a}Department of Electrical Engineering, Technion-Israel Institute of Technology, Haifa, Israel
\textsuperscript{b}Department of Computer Science, Ben-Gurion University of the Negev, Beer-sheva, Israel

Abstract

One of the fundamental ways to construct De Bruijn sequences is by using a shift-rule. A shift-rule receives a word as an argument and computes the digit that appears after it in the sequence. An optimal shift-rule for an \((n, k)-\)De Bruijn sequence runs in time \(O(n)\). We propose an extended notion we name a generalized-shift-rule, which receives a word, \(w\), and an integer, \(c\), and outputs the \(c\) digits that comes after \(w\). An optimal generalized-shift-rule for an \((n, k)-\)De Bruijn sequence runs in time \(O(n + c)\). We show that, unlike in the case of a shift-rule, a time optimal generalized-shift-rule allows to construct the entire sequence efficiently. We provide a time optimal generalized-shift-rule for the well-known prefer-max and prefer-min De Bruijn sequences.

1 Introduction

De Bruijn sequences were rediscovered many times over the years, starting from 1894 by Flye-Sainte Marie \cite{10}, and finally by De Bruijn himself in 1946 \cite{5}. For two positive non-zero integers, \(k\) and \(n\), an \((n, k)-\)De Bruijn \((n, k)-\text{DB}\), for abbreviation) sequence is a cyclic sequence over the alphabet \(\{0, \ldots, k-1\}\) in which every word of length \(n\) over \(\{0, \ldots, k-1\}\) appears exactly once as a subword. It is cyclic in the sense that some words are generated by concatenating the suffix of length \(m < n\) of the sequence, with the prefix of length \(n - m\).

A construction for a family of \((n, k)-\text{DB}\) sequences is an algorithm that receives the two arguments, \(n\) and \(k\) (occasionally, \(k\) is fixed and only \(n\) is given as argument), and outputs an \((n, k)-\text{DB}\) sequence. Obviously, a trivial time lower bound for a construction is \(\Omega(k^n)\), as this is the exact length of an \((n, k)-\text{DB}\) sequence. Many constructions for variety families of De Bruijn sequences are known, (for example, \cite{12, 6, 9, 14, 13, 16, 21, 24, 23}) and some of them are also time optimal.

A specifically famous family of \((n, k)-\text{DB}\) sequences is the prefer-max family \cite{11, 21}, which is constructed by the well-known “granddaddy” greedy algorithm \cite{21} (see also \cite{17} Section 7.2.1.1). The algorithm constructs the sequence digit by digit, where at each step, the maximal value is added to the initial segment constructed so far (assuming the alphabet is linearly ordered), so that the new suffix of length \(n\) does not appear elsewhere. A symmetric approach produces the prefer-min DB sequence. Besides this highly inefficient algorithm, many other constructions for the prefer-max and prefer-min sequences have been proposed in the literature. A classical result by Fredricksen and Kessler \cite{12}, and Fredricksen and Maiorana \cite{13} shows that the prefer-max sequence is in fact a concatenation of certain (Lyndon) words, a result we use in this work. This block construction was later proved to be time optimal in \cite{23}. Another efficient block concatenation construction was suggested in \cite{22}.

A common and important way of generating DB sequences is by using a shift-rule (also named a shift-register). A shift-rule for an \((n, k)-\text{DB}\) sequence receives a word of length \(n\), \(w\), as an input, and outputs the digit that follows \(w\) at the sequence. Here, \(n\) and \(k\) are parameters of the algorithm. Obviously, a shift-rule
must run in \(\Omega(n)-\text{time}\) since it must read every digit in its input to produce the correct output. Shift-rules are important since, unlike block constructions, they can be applied on words that appear at the middle of the sequence.

Several efficient shift-rules for DB sequences are known for \(k = 2\) (see [24] for a comprehensive list). However, only recently efficient shift-rules were discovered for non-binary sequences. Sawada et al. [24] introduced a new family of DB sequences and provided a linear time shift-rule for these sequences. Amram et al. [2] introduced an efficient shift-rule for the famous prefer-max and prefer-min DB sequences.

We note that, generally, a construction for a DB sequence provides an exponential-time shift-rule, since, on many inputs, it is required to construct almost the whole sequence to find the desired digit the shift-rule should output. On the other hand, a shift-rule for a DB sequence provides a construction in \(O(nk^n)\) time, by finding the next digits one by one, which is not an optimal approach.

We see that none of these two methods, a general construction and a shift-rule, dominates the other, and we propose here a third way, which generalizes both methods, that we name a Generalized-Shift-Rule (GSR for abbreviation). A GSR for an \((n,k)\)-DB sequence is an algorithm that receives two arguments: a word, \(w\), of length \(n\), and an integer, \(c > 0\). The GSR outputs the \(c\) letters that follow \(w\) at the sequence. Since the algorithm must read its input and must write \(c\) letters, \(\Omega(n+c)\) is a trivial time lower bound for a GSR. An optimal GSR provides an optimal shift-rule when used only with \(c = 1\). In addition, an optimal GSR provides an optimal construction by invoking it with \(c = k^n - n\), or by invoking it \(\frac{c}{n}\) times with \(c = n\) (for example).

Although a GSR is defined here for the first time, researchers have noted the advantages behind this notion, and mentioned that their shift-rule poses the properties we seek for in this paper. In [25] Sawada et al. described a shift-rule with \(O(1)\)-amortized time per bit. As this seems to contradict the trivial time lower bound mentioned earlier, this statement requires a clarification. The shift-rule proposed in [25] has the interesting property that after using it once, it can be invoked \(c\) more times and, by carefully retaining data from one invocation to another, it can produce the next \(c\) digits in \(O(c)\)-amortized time. Hence, in fact, Sawada et al. noted and mentioned that their shift-rule also forms a time optimal GSR. A similar remark can be found in [24].

In this paper, we present an optimal GSR for the well-known prefer-max and prefer-min DB sequences. Our construction relies on the classical result of [13]. Namely, that the prefer-max \((n,k)\)-DB sequence and the prefer-min \((n,k)\)-DB sequence can be constructed by concatenating certain Lyndon words. Our GSR construction takes advantage of this result in the following manner.

The prefer-min sequence is a concatenation of Lyndon words: \(L_1 \cdots L_i\). In Section 4 we note that a GSR can be constructed by solving a similar problem, named filling-the-gap. The problem is to find a word, \(x\), that completes \(w\) into a suffix of \(L_1 \cdots L_i\) for some \(L_i\). If this suffix is of length smaller than \(c\), we use another algorithm, presented in Section 3, which finds the Lyndon words that follow \(L_i\) in the block-construction of the prefer-min DB sequence. In Section 6 we present a filling-the-gap algorithm which provides, as explained, a GSR for the prefer-min and prefer-max sequences. In addition, in Section 2 we define notations used throughout the paper, and a discussion is given in Section 6.

## 2 Preliminaries

For an integer \(k\), consider the alphabet \(\{0,1,\ldots,k-1\}\), ordered naturally by \(0 < 1 < \cdots k-1\). Hence, the set of all words over \(\{0,1,\ldots,k-1\}\), \(\{0,\ldots,k-1\}^*\), is totally ordered by the lexicographic order, which we simply denote by \(<\). We say that a word, \(w\), is an \(m\)-word if \(|w| = m\). Furthermore, the \(m\)-prefix of \(w\) is the prefix of \(w\) of length \(m\), and the \(m\)-suffix of \(w\) is the suffix of \(w\) of length \(m\). This notions are defined, of course, only when \(|w| \geq m\).

A word, \(v\), is a rotation of a word, \(w\), if \(w = xy\) and \(v = yx\). In addition, \(v\) is a non trivial rotation of \(w\), if \(x \neq w\) and \(y \neq w\). Note that a word \(w\) can be equal to some of its non-trivial rotations. This happens when \(w = x^t\) for some non-empty word, \(x\), and an integer \(t > 1\). In this case, \(w\) is said to be periodic. A Lyndon word [20] is a non-empty word that is strictly smaller than all its non-trivial rotations. Hence, in particular, a Lyndon word is aperiodic.
The prefer-max \((n,k)\)-DB sequence is a cyclic sequence constructed by a greedy algorithm that starts with \(0^n\), and repeatedly adds the largest possible digit in \(\{0,\ldots,k-1\}\) so that no \(n\)-word appears twice as a subword of this sequence, until the sequence length is \(k^n\), and then rotates the obtained sequence to the left \(n\) times. As an example, for \(n = 2\) and \(k = 3\), the greedy procedure produces the sequence: 002212011 and the prefer-max \((2,3)\)-DB sequence is: 221200110. Analogously, the prefer-min \((n,k)\)-DB sequence is produced by a greedy process which starts with \((k-1)^n\), and repeatedly concatenates the smallest possible digit, so that no repetition occurs, and afterwards rotates the resulting sequence to the left \(n\) times.

We note that the prefer-min \((n,k)\)-DB sequence and the prefer-max \((n,k)\)-DB sequence can be derived one from the other, by replacing each digit, \(m\), with \(k-1-m\). Therefore, a GSR for one of these sequences can be easily transformed into a GSR for the other one as well. We present here a GSR algorithm for the prefer-min \((n,k)\)-DB sequence.

From this point on, we refer to \(n\) and \(k\) as fixed, unknown, parameters, larger than 1 (to avoid trivialities). We measure time complexity of all algorithms given here in terms of the parameter \(n\), assuming that arithmetic operations can be computed in constant time, regardless of how large the numbers they are applied on.

Let \(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots\) be the (finite) sequence of all Lyndon words over \(\{0,\ldots,k-1\}\) of length at most \(n\), ordered lexicographically. Let \(N(n,k)\) be the number of all Lyndon words over \(\{0,\ldots,k-1\}\) whose length divides \(n\). Let \(L_1, L_2, \ldots, L_{N(n,k)}\) be an enumeration of all Lyndon words whose length divides \(n\), ordered lexicographically.

For a Lyndon word \(L_i\), let \(r_i = \frac{n}{|L_i|}\). Since \(|L_i|\) divides \(n\), \(r_i\) is a positive integer. Note that every \(L_i\) is equal to some \(\mathcal{L}_j\) where \(j \geq i\). The main result of [13] (with a straightforward adaptation) is:

**Theorem 1.** The prefer-min \((n,k)\)-DB sequence is: \(L_1 L_1 \cdots L_{N(n,k)}\).

As an example, for \(n = k = 3\) we concatenate in an increasing order all Lyndon words of length one or three. We get the following sequence, decomposed into Lyndon words:

\[
\text{prefer-min } (3,3)\text{-DB} = 0|001|002|011|012|022|1|112|122|2
\]

As said, our strategy in constructing a GSR for the prefer-min sequence is to fill the gap between the input, \(w\), to a word \(L_i\) in the sequence, and then concatenate Lyndon words until we find the required \(c\) digits that follow \(w\). To this end, we refer to the sequence \(L_1, \ldots, L_{N(n,k)}\) as cyclic, meaning that, for \(t \in \mathbb{Z}\) and \(i \in \{1,\ldots,N(n,k)\}\), we set \(L_{t,N(n,k)+i} = L_i\).

### 3 An Efficient \(\text{Lnext}(L)\) Algorithm

As a first step in constructing a GSR algorithm, we analyze a relatively simple case. By Theorem 1 for every \(i \leq N(n,k)\), \(L_1 \cdots L_i\) is a prefix of the prefer-min sequence. We consider the case where we are given an \(n\)-word, \(w\), that happens to be a suffix of \(L_1 \cdots L_i\). To find the next \(c\) digits, we need to find words: \(L_{i+1} \cdots L_{i+c}\) so that the length of this sequence is at least \(c\).

For dealing with this restricted case, we design an algorithm that computes, efficiently, the function: \(\text{Lnext}(L_i) = L_{i+1}\). Moreover, for technical reasons that will arise later, we also want to apply the algorithm over Lyndon words whose length does not necessarily divides \(n\). Therefore, for a Lyndon word, \(L_j \neq L_{N(n,k)}\), we define \(\text{Lnext}(L_j)\) to be the Lexicographically smallest \(L_i\) such that \(L_j < L_i\), and \(\text{Lnext}(L_{N(n,k)}) = L_1\) (that is, \(\text{Lnext}(k-1) = 0\)). In this section, we present an \(\text{Lnext}\) algorithm with \(O(n)\) time complexity.

**Proposition 2.** \(\forall L_i \text{ Algorithm 3 computes } \text{Lnext}(L_i) \text{ in } O(n) \text{ time.}\)

In [8], Duval describes an algorithm to build the next Lyndon word from a given one.

In [8], Duval proved:

**Theorem 3.** On every input \(L_i \neq k-1\), Algorithm 4 returns \(L_{i+1}\) in \(O(n)\) time.
Algorithm 1 Duval

Input: A Lyndon word, $L_i \neq k - 1$
Output: $L_{i+1}$

1: $x \leftarrow (L_i)^t u$, where $u$ is a proper prefix of $L_i$ and $|(L_i)^t u| = n$
2: remove largest suffix of $x$ of the form $(k - 1)^{l}$
3: increase the last digit of $x$ by one
4: return $x$

Algorithm 2 Naive-Lnext

Input: a Lyndon word, $L_i$
Output: $L_{\text{next}}(L_i)$

1: if $L_i = k - 1$ then
2: return 0
3: end if
4: $x \leftarrow \text{Duval}(L_i)$
5: if $|x| | n$ then
6: return $x$
7: end if
8: if $|x| < \frac{n}{2}$ then
9: $x \leftarrow \text{Duval}(x)$
10: end if
11: length $\leftarrow |x|$
12: while length | n do
13: $x \leftarrow \text{Duval}(x)$, length $\leftarrow |x|$
14: end while
15: return $x$.

Note that the length of the output of the algorithm may not divide $n$. However, we use it to construct a naive algorithm which achieves that goal, with a runtime complexity of $O(n^2)$. We first describe this naive version, and then improve it to run in linear time.

Note that $\forall L_i$, Algorithm 2 outputs $L_{\text{next}}(L_i)$. At each iteration of the loop in lines 12-14, the algorithm invokes Duval’s algorithm, until it finds a Lyndon word whose length divides $n$. This establishes a worst case runtime of $\Theta(n^2)$.

The reader may note that the if instructions in lines 5-7 and lines 8-10 can be omitted. However, we aim to construct a liner-time $L_{\text{next}}$ algorithm, and we do that by modifying the while loop. Then, it will be important that the loop acts on words whose length is larger than $\frac{n}{2}$. Thus, lines 5-7 and 8-10 are added to simplify the comparison between this naive $L_{\text{next}}$ version and our linear time $L_{\text{next}}$ version.

To improve the runtime of this algorithm, we identify cases in which the outcome of several loop iterations can be computed directly. These are the cases in which calling Duval’s algorithm again and again results in concatenating the same sequence several times. For illustration, consider the word $L_i = 00(k - 1)^l$, where $\frac{n}{2} \leq l + 2$. The Lyndon words that follow $L_i$ are:

$$00(k - 1)^l 01, 00(k - 1)^l 0101, 00(k - 1)^l 010101, \ldots, 00(k - 1)^l (01)^{\lceil \frac{n-l-2}{2} \rceil}.$$

Instead of applying Duval $t = \lceil \frac{n-l-2}{2} \rceil$ times, we can save time by computing $t$, and go to $00(k - 1)^l (01)^t$ without traversing all words in that list. This allows us to compute $L_{\text{next}}(L_i)$ in linear time, as we do in Algorithm 3.

In order to prove Algorithm 3 correctness, we show that both algorithms, Algorithm 2 and Algorithm 3, have the same output for any legal input. We start with two observation, derived from Duval’s algorithm.
Corollary 4. \( \forall L_i \neq k - 1, \text{ if } |L_i| \leq \frac{n}{2}, \text{ then } |L_{i+1}| > \frac{n}{2}. \)

Corollary 5. \( \forall L_i \neq k - 1, \text{ if } |L_i| < n, \text{ then } |L_i| < |L_{i+1}|. \)

From these two observations we can deduce the following conclusion, discussing the similarity between the two algorithms when entering the while loop:

Corollary 6. The following invariant holds for both Algorithm 2 and Algorithm 3 whenever the while loop starts: \( |x| \nmid n \) and \( |x| > \frac{n}{2}. \)

The next lemma shows that every execution of the while loop in Algorithm 3 corresponds to several executions of the while loop in Algorithm 2.

Lemma 7. Let \( x \) be a Lyndon word so that \( |x| > \frac{n}{2}. \) Let \( m, u, u', t \) be as in lines 9-12 of Algorithm 3. Then, for each \( j \leq t: \)

1. \( xu^{j} \) is a Lyndon word.
2. If \( j < t, \) then \( \text{Duval}(xu^{j}) = xu^{j+1}. \)

Proof. The proof is by induction on \( j. \) If \( j = 0, \) item 1 holds as \( x \) is a Lyndon word, and if \( j > 0 \) item 1 holds by applying the induction hypothesis on item 2 with \( j - 1. \) It remains to prove that item 2 holds thus we assume that \( j < t. \)

Write \( u' = v(\sigma + 1) \) for a word \( v \) and \( \sigma + 1 < k. \) Hence, the \( m \)-prefix of \( x, u, \) is \( u = v\sigma(k-1)^l \) for some \( l \geq 0. \) Namely, \( x = v\sigma(k-1)^l u' \) so that \( m = n - |x| = |u| = |v\sigma(k-1)^l| \) (note that since \( |x| > \frac{n}{2}, \) the \( m \)-prefix of \( x \) is defined). Thus, \( xu^j = v\sigma(k-1)^l v'(v(\sigma+1)^l). \) Let \( \hat{m} = n - |xu^j| \leq m = |u| = |v\sigma(k-1)^l|. \) Note that since \( j < t, \) \( \hat{m} \geq |u'| = |v\sigma| \) (this holds because \( t \) is the largest integer so that \( |xu^2| \leq n). \)

To summary, \( \hat{m} \leq m = |v\sigma(k-1)^l| \) and \( \hat{m} \geq |v\sigma|. \) Hence, the \( \hat{m} \)-prefix of \( xu^j = v\sigma(k-1)^l v'u'^j \) is \( v\sigma(k-1)^l \) for some \( l' \leq l. \) Therefore, \( \text{Duval}(xu^j) \) is obtained by concatenating \( v\sigma(k-1)^l \) to \( xu^j, \) removing the suffix: \( (k-1)^l \) and increasing \( \sigma \) by one. Namely, \( \text{Duval}(xu^j) = xu^j v(\sigma + 1) = xu^j+1, \) as required. \( \square \)
Lemma 8 states that each execution of the while loop of Algorithm 3 corresponds to \( t \) executions of the while loop of Algorithm 2. Therefore, we conclude:

**Corollary 8.** \( \forall L \), the output of Algorithm 3 is equal to the output of Algorithm 2.

It is left to prove that our runtime is linear. For this purpose, consider an execution of Algorithm 3 on input \( L \), and assume that \( L \neq k - 1 \) (otherwise, the algorithm terminates after \( O(1) \) steps). Let us denote by \( m_i, u_i, u'_i, t_i \) and \( x_i \) the values assigned to variables \( m, u, u', t \), respectively, at the \( i \)-th iteration of the while loop. In addition, let \( x \) be the value of variable \( x \) before entering the while loop for the first time, and let \( r \) denote the number of loop iterations at the execution.

**Lemma 9.** For \( 0 < i \leq r \), \( m_i = n - |x_{i-1}| \leq \frac{n}{2} \).

**Proof.** By induction on \( i \). The base case \( i = 1 \) is trivial, as \( |x_0| > \frac{n}{2} \). For the induction step, take \( m_{i+1} = n - |x_i| \) and note that \( x_i = x_{i-1}u'^{t_i} \), where \( |u_i'| \leq |u_i| = m_i \), and \( t_i \) is the largest integer so that \( |u'^{t_i}| \leq m_i \). Therefore, \( |u'^{t_i}| \geq \frac{m_i}{2} \), and hence, by the induction hypothesis we get: \( |m_{i+1}| = n - |x_i| = n - |x_{i-1}| - |u'^{t_i}| = m_i - |u'^{t_i}| \leq m_i - \frac{m_i}{2} = \frac{m_i}{2} \leq \frac{n}{2} \).

Relaying on this lemma, we can now analyze the runtime of our algorithm, and prove Proposition 2.

**Proof.** By Corollary 8, the algorithm computes \( L_{\text{next}}(L) \) correctly. We shall prove that the algorithm runs in \( O(n) \) time. If \( L \neq k - 1 \), the execution terminates in constant number of steps and we are done. Otherwise, by Lemma 8 we have:

1. \( r \leq \log(n) \).
2. For every \( 0 < i \leq r \), \( m_i \leq \frac{n}{2} \).

In each iteration of the while loop, finding \( u \) and \( u' \) are the most time-consuming steps, each costs \( m_i \). Therefore, the global runtime is \( O(n + m_1 + m_2 + \cdots + m_r) \leq O(n + \frac{n}{2} + \frac{n}{2} + \cdots + \frac{n}{2}) = O(n^2) \).

## 4 A GSR Algorithm Based on a Reduction to FTG

The fact that \( L_{\text{next}} \) can be computed efficiently is useful for designing an efficient GSR algorithm. Given an \( n \)-word, \( w \), assume that \( wx \) is a suffix of \( L_1 \cdots L_i \). In this case, several invocations of our \( L_{\text{next}} \) algorithm produce the \( c \)-word that follows \( w \) at the prefer-min sequence. For taking this approach, first, it is required to find a Lyndon word \( L_i \) and a word \( x \) so that \( wx \) is a suffix of \( L_1 \cdots L_i \). This implies that a GSR algorithm for the prefer-min sequence can be derived from a solution to another problem we propose in this section, which we name: Filling-The-Gap (FTG for abbreviation).

**Definition 10.** For an \( n \)-word, \( w \), \( FTG(w) = (L_i, x) \) if

1. \( wx \) is a suffix of \( L_1 \cdots L_i \).
2. If \( wy \) is a suffix of \( L_1 \cdots L_j \), then \( |x| \leq |y| \).

We leave for the reader to verify that \( FTG(w) \) is well defined, meaning that for every \( n \)-word, \( w \), only a single pair, \((L_i, x)\), satisfies the conditions of Definition 10. We remark that it is possible that \( FTG(w) = (L_i, x) \) where \( i > N(n, k) \). This occurs in the case where \( w \) is a concatenation of a suffix of the prefer-min sequence with a prefix of it. For example, if \( w = (k - 1)10^{k-1} \), then \( FTG(w) = (L_{N(k)+2}, 01) \) since \( w01 \) is a suffix of \( L_1 \cdots L_{N(k)+1}L_{N(k)+2} = L_1 \cdots L_{N(k)}L_1L_2 \).

Note that \( FTG(w) \) can be trivially computed by concatenating Lyndon words and searching for \( w \). However, this naive solution is highly inefficient as \( w \) may appear anywhere in the prefer-min sequence. Hence, for constructing an efficient GSR in the way described above, we need an efficient \( FTG \)-algorithm.
There is also another issue concerning the suggested approach, which requires attention. If \( FTG(w) = (L_i, x) \), for computing the \( c \)-word that comes after \( w \) we need to invoke Algorithm 3 several times. It is required to explain why the number of Lyndon words we concatenate is proportional to the suffix we seek for. More precisely, we need to show that the total number of invocations of Algorithm 3 consumes \( O(n + c) \) time. This is settled by the next lemma, which claims that there are no two consecutive words, \( L_i, L_{i+1} \), both of length smaller than \( n \):

**Lemma 11.** For every \( 1 \leq i < N(n, k) \), if \( |L_i| < n \), then \( |L_{i+1}| = n \).

**Proof.** Write \( L_i = L_j \) and \( L_{i+1} = L_{j+m} \), for \( m \geq 1 \). Since \( |L_j| < n \) and \( |L_j| \times n, |L_j| \leq \frac{n}{k} \). By Corollary 3, \( |L_{j+1}| > \frac{n}{2} \). Hence, if \( L_{j+1} = L_{i+1} \), then \( |L_{i+1}| = n \), since \( n \) is the only number in range \( \{1, \ldots, n\} \), which is larger than \( \frac{n}{2} \) and divides \( n \). Otherwise, the length of each word among \( L_{j+1}, \ldots, L_{j+m-1} \) is, in particular, smaller than \( n \). Thus, by Corollary 5, \( \frac{n}{2} < |L_{j+1}| < \cdots < |L_{j+m-1}| < |L_{j+m}| \), and since \( |L_{j+m}| \) divides \( n \), by the same argument as before, we get that \( n = |L_{j+m}| = |L_{i+1}| \).

We can now present, in Algorithm 4, a GSR algorithm based on a reduction to the \( FTG \) problem.

**Algorithm 4** generalized_shift_rule

**Input:** \( (w, c), |w| = n \)

**Output:** a word \( w' \) of length \( c \) that appears after \( w \) at prefer-min

1. \( (L, x) \leftarrow FTG(w) \)
2. **while** \( |x| < c \) **do**
3. \( L \leftarrow \text{Length}(L) \)
4. \( x \leftarrow xL \)
5. **end while**
6. return the \( c \)-prefix of \( x \)

Consider the **while** loop in Algorithm 4 and use Lemma 11 to conclude that after \( O(1) \) loop iterations, which consumes \( O(n) \) time, \( |x| \) increases by at least \( n \) digits. It follows that the loop halts in \( O(n + c) \) steps and hence, we get the following:

**Proposition 12.** Assume that \( FTG(w) \) can be computed in \( O(n) \) time. Then, Algorithm 4 forms a GSR for the prefer-min \((n, k)\)-DB sequence with \( O(n + c) \) time complexity.

5 **An \( FTG \) Algorithm**

In this section we construct an efficient \( FTG \)-algorithm. This is done in two steps. First, we define the notion of a **cover** of an \( n \)-word \( w \), and show how a cover for \( w \) can be transformed into \( FTG(w) \) efficiently. Then, we show how to find a cover for an \( n \)-word, \( w \), in linear time.

5.1 **Finding \( FTG(w) \) By Means of a Cover**

The \( FTG \) problem, applied on an \( n \)-word, \( w \), is to extend \( w \) into a suffix of \( L_1 \cdots L_i \). For solving this problem, we introduce a similar notion.

**Definition 13.** For an \( n \)-word, \( w \neq (k - 1)^p0^{n-p} \), cover \((w) = (L_i, x) \) if

1. \( wx \) is a suffix of \( L_1 \cdots L_i-1L_i^v \).
2. \( wy \) is a suffix of \( L_1 \cdots L_{j-1}L_j^v \), then \( |x| \leq |y| \).

In addition, we say that \( w \) is covered by \( L_i \), if cover \((w) = (L_i, x) \) for some word \( x \).
Also here, we leave for the reader to verify that \( \text{cover}(w) \) is well-defined. We focus on \( n \)-words different from \((k-1)p^n-p\) from technical reasons, as it allows us to provide a simpler presentation of our results. Otherwise, many parts in our analysis should be rephrased, and some proofs should be rewritten, to include more details. However, it is simple to show that the \( \text{FTG} \) algorithm we provide at the end of this section, works for every \( n \)-word.

The two notions, \( \text{cover}(w) \) and \( \text{FTG}(w) \) are similar, but different. The difference between these notions can be bridged by observing that if \((L_i,x)\) is a cover for \( w \), then \( w \) is a subword of \( L_1 \cdots L_i \). To clear this issue, we deal with the relationships between two consecutive Lyndon words in the following Lemma.

**Lemma 14.** For every \( 1 \leq i < N(n,k) \), if \( |L_i| < n \), then \( L_{i+1} = L_i^{r-1}z \), for some \( |L_i| \)-word \( z \).

**Proof.** Let \( L_i = L_j \) and \( L_{i+1} = L_{j+m} \). By an examination of Algorithm \( 1 \) and since \( L_i \) does not start with \( k-1 \) (the only such word is \( L_N \), and \( 1 \leq i < N(n,k) \)), we conclude that \( L_{j+1} = L_i^{r-1}u_1 \), where \( u_1 \) is constructed by removing the suffix of \( L_i \) that includes only occurrences of \( k-1 \), and increasing the last digit of the obtained word by one. A simple inductive argument shows that for \( t = 1, \ldots, m \), \( L_{j+t} = L_i^{r-1}u_1 \cdots u_t \) for some non-empty words \( u_1, \ldots, u_t \). By Lemma \( 14 \) \( |L_{i+1}| = n \) thus \( L_{i+1} = L_{j+1} = L_i^{r-1}u_1 \cdots u_m \), and \( z = u_1 \cdots u_m \) is an \( |L_i| \)-word.

Note that it follows that for each \( 1 \leq i < N(n,k) \), \( L_1 \cdots L_i L_i^{r-1}z = L_1 \cdots L_{i+1} \). Now we turn to deal with the relationships between \( \text{cover}(w) \) and \( \text{FTG}(w) \).

**Lemma 15.** Assume that \( \text{cover}(w) = (L_i,x) \) for an \( n \)-word, \( w \neq (k-1)p^n-p \).

1. If \( |x| \geq L_i^{r-1} \), \( \text{FTG}(w) = (L_i,y) \) where \( y \) is the \( (|x| - |L_i^{r-1}|) \)-prefix of \( x \).

2. Otherwise, \( \text{FTG}(w) = (L_{i+1},xz) \), where \( z \) is the \( |L_i| \)-suffix of \( L_{i+1} \).

**Proof.** We start by proving the first item. As \( \text{cover}(w) = (L_i,x) \), \( wx \) is a suffix of \( L_1 \cdots L_{i-1} L_i^{r} \). Since \( |x| \geq L_i^{r-1} \), \( wxy \) is a suffix of \( L_1 \cdots L_i \). Moreover, the minimality of \( x \) guarantees that \( w \) is not a subword of \( L_1 \cdots L_{i-1} \) thus \( \text{FTG}(w) = (L_i,y) \) as required.

We turn to prove the second item, in which \( wx \) is a suffix of \( L_1 \cdots L_{i-1} L_i^{r} \) and \( |x| < L_i^{r-1} \). Then, it follows that \( |L_i| < n \) since otherwise we get the false equation: \( |x| < |\varepsilon| \). Moreover, as \( w \neq (k-1)p^n-p \), \( 1 \leq i < N(n,k) \). Hence, Lemma \( 14 \) can be invoked and we get that \( L_{i+1} = L_i^{r-1}z \) where \( z \) is the \( |L_i| \)-suffix of \( L_{i+1} \). As a result, \( wxyz \) is a suffix of \( L_1 \cdots L_{i-1} L_i^{r-1}z = L_1 \cdots L_i \). Furthermore, since \( wx \) is a suffix of \( L_1 \cdots L_{i-1} L_i^{r} \) where \( |x| < |L_i^{r-1}| \), \( w \) is not a suffix of \( L_1 \cdots L_i \). Therefore, \( \text{FTG}(w) = (L_{i+1},xz) \) as required.

Using the above, Algorithm \( 5 \) transforms \( \text{cover}(w) \) into \( \text{FTG}(w) \) in linear time.

**Algorithm 5** \( \text{cover} \rightarrow \text{FTG} \)

**Input:** a pair \((L,x) = \text{cover}(w)\)

**Output:** \( \text{FTG}(w) \)

1: \( r \leftarrow \frac{n}{|L|} \)
2: if \( |x| \geq |L^{r-1}| \) then
3: \( y \leftarrow (|x| - |L^{r-1}|) \)-prefix of \( x \)
4: return \((L,y)\)
5: end if
6: \( z \leftarrow |L| \)-suffix of \( \text{Lnext}(L) \)
7: return \((\text{Lnext}(L),xz)\)

We conclude this section with the next corollary. Its first item follows by Lemma \( 14 \) and its second item follows immediately from the code.
Corollary 16. Let \( w \neq (k-1)^n \) be an \( n \)-word.

1. If \( w \neq (k-1)^{p0^n-p} \) and \( \text{cover}(w) = (L, x) \), then Algorithm \( \text{cover} \) returns \( \text{FTG}(w) \) on the input \((L, x)\).
2. If \( w = (k-1)^{p0^n-p} \), then Algorithm \( \text{cover} \) returns \( \text{FTG}(w) \) on the input \((0, 0^p)\).

In the next subsection we show how to compute \( \text{cover}(w) \) efficiently.

5.2 Computing \( \text{cover}(w) \)

In this section we show how to compute \( \text{cover}(w) \), efficiently. Assume that an \( n \)-word, \( w \neq (k-1)^{p0^n-p} \), is covered by \( L_i \). Then, \( w \) is a subword of \( L_{i-1}L_i^r \). As described below in Algorithm \( \text{cover} \) in some cases, we compute \( L_{i-1} \) and use it to find \( \text{cover}(w) \), and in other cases we compute directly the suffix of \( L_{i-1}L_i^r \) that follows \( w \). The way this goal is achieved, relies on the analysis we provide here, which we divide to two parts. First, we show how to construct \( L_i^{r_i+1} \) from \( L_i^r \), by concatenating certain words to \( L_i^{r_i-1} \). Then, we present a structural characterization of \( w \) which will use us to compute \( \text{cover}(w) \).

5.2.1 Modifying \( L_i^{r_i} \) into \( L_i^{r_i+1} \)

Assume that an \( n \)-word, \( w \neq (k-1)^{p0^n-p} \) is covered by \( L_i \). Hence, \( w \) is a subword of \( L_1 \ldots L_{i-1}L_i^r \), but not of \( L_1 \ldots L_{i-2}L_i^{r_i-1} \). Clearly, Lemma \( \text{cover} \) implies that \( L_1 \ldots L_{i-2}L_i^{r_i-1} \) is a prefix of \( L_1 \ldots L_{i-1}L_i^r \), but what is the difference between these two sequences? The first aim of our analysis is to show how to construct \( L_i^{r_i+1} \) from \( L_i^r \), by concatenating a suffix to \( L_i^r \).

Definition 17. Let \( L_i \neq k-1 \). We define a sequence of words: \( v_{i,1}, v_{i,2}, \ldots, v_{i,m_i} \), and a sequence of indices: \( k_1, \ldots, k_{m_i+1} \) by induction, where \( k_i \) indicates the amount of characters left to calculate in step \( i \): Write \( k_1 = |L_i| \) and assume that \( v_{i,1} \ldots v_{i,j-1} \) were defined, together with \( k_1, \ldots, k_j \).

- If \( k_j = 0 \), \( j-1 = m_i \) and we are done.
- Otherwise, \( v_{i,j} \) is obtained as follows: take the prefix of \( L_i \) of size \( k_j \), remove its suffix that includes only occurrences of \( k-1 \), and increase the last digit by one. In addition, let \( k_{j+1} = k_j - |v_{i,j}| = |L_i| - |v_{i,1}| - |v_{i,2}| - \cdots - |v_{i,j}| \).

We show now how the words \( v_{i,1}, \ldots, v_{i,m_i} \) form as buliding blocks for constructing \( L_i^{r_i+1} \) from \( L_i^r \). We divide the analysis into three Lemmas, to deal with the different cases.

Lemma 18. Take \( L_i \neq k-1 \), and consider the words \( v_{i,1}, \ldots, v_{i,m_i} \), as defined in Definition \( \text{cover} \) If \( r_i > 1 \), then:

1. \( \text{Duval}(L_i) = L_i^{r_i-1}v_{i,1} \).
2. For each \( 1 \leq j < m_i \), \( \text{Duval}(L_i^{r_i-1}v_{i,1} \cdots v_{i,j}) = L_i^{r_i-1}v_{i,1} \cdots v_{i,j}v_{i,j+1} \).

Proof. The first item follows immediately from the definition of \( v_{i,1} \). For proving the second item, note that \( k_{j+1} = |L_i| - |v_{i,1} \cdots v_{i,j}| \). But since \( |L_i^r| = n \), \( k_{j+1} = n - |L_i^{r_i-1}v_{i,1} \cdots v_{i,j}| \). Let \( \sigma(k-1)^l \) be the \( k_{j+1} \)-prefix of \( L_i \), where \( \sigma < k-1 \) and \( l \geq 0 \). Hence, \( \text{Duval}(L_i^{r_i-1}v_{i,1} \cdots v_{i,j}) = L_i^{r_i-1}v_{i,1} \cdots v_{i,j}\sigma \text{Duval}(\sigma+1) \).

Also, by Definition \( \text{cover} \), \( v_{i,j+1} = \sigma \text{Duval}(\sigma+1) \), which completes the proof.

Lemma 19. Take \( L_i \neq k-1 \), and consider the words \( v_{i,1}, \ldots, v_{i,m_i} \), as defined in Definition \( \text{cover} \) If \( r_i = 1 \) and \( v_{i,1} \mid n \), then:

1. \( \text{Duval}(L_i) = v_{i,1} = L_i+1 \).
2. \( v_{i,1} = v_{i,2} = \cdots = v_{i,m_i} \) and \( m_i = r_i+1 \).
Proof. For the first item, note that by Definition 17 and by the fact that \( r_i = 1 \) (namely, \( |L_i| = n \)), \( \text{Duval}(L_i) = v_{i,1} \). Moreover, \( v_{i,1} = L_{i+1} \), as \( v_{i,1} \mid n \). We turn to prove the second item. Write \( L_i = vr(k-1)^j \), where \( \sigma < k - 1 \). Hence, since \( k_1 = n, v_{i,1} = v(\sigma + 1) \), we show now by induction that \( v_{i,j} = v_{i,j} \) for every \( j = 1, \ldots, m_i \). The induction basis trivially holds, as \( v_{i,1} = v_{i,1} \). Assume, now, that \( j > 1 \) and \( v_{i,1} = v_{i,2} = \cdots = v_{i,j-1} \). Since \( j \leq m_i, k_j > 0 \). But \( k_j = |L_i| - |v_{i,1}| - \cdots - |v_{i,j-1}| = n - (j-1)|v_{i,1}| > 0 \). Since \( |v_{i,1}| \mid n \), it follows that \( k_j = n - (j-1)|v_{i,1}| \leq |v_{i,1}| \). Therefore, the \( |k_j| \)-prefix of \( L_i \) is \( v\sigma(k-1)^j \)' for some \( 0 \leq k < l \). It follows that \( v_{i,j} = v(\sigma + 1) = v_{i,1} \), as required. Moreover, \( k_{m_i+1} = 0 = n - |v_{i,1}|v_{i,2}\cdots|v_{i,m_i}| \). Thus, \( n = |v_{i,1}| \cdots |v_{i,m_i}| = |(v_{i,1})^{m_i}| = |(L_{i+1})^{m_i}| \) which proves that \( m_i = r_{i+1} \).

Lemma 20. Take \( L_i \neq k - 1 \), and consider the words \( v_{i,1}, \ldots, v_{i,m_i} \), as defined in Definition 17. If \( r_i = 1 \) and \( v_{i,1} \mid n \), then:

1. \( \text{Duval}(L_i) = v_{i,1} \).

2. Let \( j_0 \) be the maximal integer such that \(|(v_{i,1})^{j_0}| < n \) (equality cannot hold since \( |v_{i,1}| \mid n \)). Then, \( v_{i,1} = \cdots = v_{i,j_0} \) and \( \text{Duval}(v_{i,1}) = v_{i,1}v_{i,2} \cdots v_{i,j_0}v_{i,j_0+1} \).

3. For \( j_0 \) as defined above, if \( j_0 < j < m_i \), then \( \text{Duval}(v_{i,1} \cdots v_{i,j}) = v_{i,1} \cdots v_{i,j+1} \).

Proof. As in the two former lemmas, the first item trivially holds by Definition 17. For the second item, let \( j_0 \) be maximal such that \(|(v_{i,1})^{j_0}| < n \). Then, \( \text{Duval}(v_{i,1}) = (v_{i,1})^{j_0}v' \). By using the same argument as in the proof of item 2 of Lemma 19 it can be shown that \( v_{i,1} = \cdots = v_{i,j_0} \). We prove that \( v' = v_{i,j_0+1} \). Since \( r_1 = 1, |L_i| = n \) thus \( k_{j_0+1} = n - |v_{i,1}| \cdots v_{i,j_0} = n - |(v_{i,1})^{j_0}| \). By the maximality of \( j_0 \), \( k_{j_0+1} < |v_{i,1}| \). Let \( x(\sigma - 1)^j \) be the \( k_{j_0+1} \)-prefix of \( v_{i,1} \), where \( \sigma < k - 1 \). Hence, removing the suffix of \( (k-1)^j \), and increasing the last digit by one results in \( x(\sigma + 1) = v' \). Now, since \( k_{j_0+1} < |v_{i,1}| \), \( x(\sigma - 1)^j \) is also the \( k_{j_0+1} \)-prefix of \( L_i \). Therefore, \( v_{i,j_0+1} = x(\sigma + 1) = v' \) as required. We leave for the reader to verify that the same argument proves item 3 as well.

Now we can show how to construct \( L_{i+1}^{r_{i+1}} \) from \( L_i \).

Lemma 21. Take \( L_i \neq k - 1 \), and consider the words \( v_{i,1}, \ldots, v_{i,m_i} \), as defined in Definition 17. Then, \( L_i^{r_{i+1}}v_{i,1} \cdots v_{i,m_i} = L_{i+1}^{r_{i+1}} \).

Proof. The proof is divided into three parts, in accordance with Lemmas 18, 19 and 20. First, assume that \( r_i > 1 \). By Lemma 18 the following is a sequence of consecutive Lyndon words: \( L_i, L_i^{r_i-1}, \ldots, L_i^{r_i-1}v_{i,1}, \ldots, v_{i,m_i} \). Note that \( \frac{\theta}{\sigma} < |L_i^{r_i-1}v_{i,1}| < |L_i^{r_i-1}v_{i,2}| < \cdots < |L_i^{r_i-1}v_{i,1}v_{i,2} \cdots v_{i,m_i}| = n \). Hence, since \( |L_{i+1}| \mid n, L_i^{r_i-1}v_{i,1}v_{i,2} \cdots v_{i,m_i} = L_{i+1} \). In addition, by Corollary 3 \( r_{i+1} = 1 \), and the claim follows.

Now, consider the case where \( r_i = 1 \) and \( |v_{i,1}| \mid n \). Since \( |v_{i,1}| \mid n \), by the first part of Lemma 19 \( v_{i,1} = L_{i+1} \). Moreover, by the second part of Lemma 19 \( v_{i,1} = \cdots = v_{i,m_i} \) and \( m_i = r_{i+1} \). Therefore, \( L_i^{r_i-1}v_{i,1} \cdots v_{i,m_i} = v_{i,1} \cdots v_{i,m_i} = v_{i,1}^{m_i} = L_{i+1}^{r_{i+1}} \), as required.

It is left to deal with case where \( r_i = 1 \) and \( |v_{i,1}| \mid n \). By Lemma 20 the following is a sequence of consecutive Lyndon words:

\[
L_i, v_{i,1}v_{i,1}v_{i,2} \cdots v_{i,j_0+1}, v_{i,1}v_{i,2} \cdots v_{i,j_0}v_{i,j_0+2}, \cdots, v_{i,1}v_{i,2} \cdots v_{i,m_i},
\]

where \( j_0 \) is the maximal integer so that \(|(v_{i,1})^{j_0}| < n \). As in the former case, \( \frac{\theta}{\sigma} < |v_{i,1} \cdots v_{i,j_0+1}| < \cdots < |v_{i,1} \cdots v_{i,m_i}| = n \). Since \( |L_{i+1}| \mid n \), we conclude that \( L_{i+1} = v_{i,1} \cdots v_{i,m_i} \). Moreover, we get that \( |L_{i+1}| = n \) thus \( r_{i+1} = 1 \) and the lemma follows.

By the former lemma and by Lemmas 18, 19 and 20 we also conclude:

Corollary 22. Take \( L_i \neq k - 1 \), and consider the words \( v_{i,1}, \ldots, v_{i,m_i} \), as defined in Definition 17.

1. If \( \mathcal{L} = L_i^{r_{i+1}}v_{i,1} \cdots v_{i,j} \) is a Lyndon word where \( j \leq m_i \), then \( L_i < \mathcal{L} \leq L_{i+1} \).

2. \( L_i^{r_i}v_{i,1} \cdots v_{i,m_i} = L_iL_{i+1}^{r_{i+1}} \).
5.2.2 Analyzing the Structure of $w$

We are ready to present our analysis concerning the structure of $w$, for extracting information that will use us to compute $\text{cover}(w)$. First, we identify a distinguished simple case, and define:

**Definition 23.** An $n$-word, $w$, is said to be an expanded-Lyndon-word, if $w = L_i^r$ for some $i \leq N(n,k)$.

If $w = L_i^r$ is an expanded-Lyndon-word, then $\text{cover}(w) = (L_i, z)$. The reader may observe that procedures findroot and isLyndon, both described in subsection 5.2.3 can be used to decide if $w$ is an expanded-Lyndon-word efficiently, and to extract $L_i$ in linear time in those cases.

But what shall we do in the general case? Namely, if $w$ is a subword of $L_{i-1}L_i^r$, but not a suffix of this sequence? As a first step for answering this question, we invoke Corollary 22 which establishes relationships between $w$ and the words defined in Definition 17 as the next lemma elaborates.

**Lemma 24.** Assume that an $n$-word, $w \neq (k-1)^00^{n-p}$ is not an expanded-Lyndon-word, and $w$ is covered by $L_{i+1}$. Then, $w = yL_i^{r_i-1}v_{i,1}v_{i,2} \cdots v_{i,j}z$, where $y$ is a proper suffix of $L_i$, and $z$ is a proper prefix of $v_{i,j+1}$.

**Proof.** $w$ is a subword of $L_1 \cdots L_i L_i^{r_i+1}$. By Corollary 22 $L_1 \cdots L_i L_i^{r_i+1} = L_1 \cdots L_i L_i^r v_{i,1} \cdots v_{i,m_i}$. Since $w$ is not covered by $L_i$, $w$ is a subword, but not a prefix, of $L_i^r v_{i,1} \cdots v_{i,m_i}$. Furthermore, since $w$ is not an expanded-Lyndon-word, $w$ is not a suffix of $L_i^r v_{i,1} \cdots v_{i,m_i}$, which proves that $w$ is of the required form. \[\square\]

From the proof of Lemma 24 we also conclude:

**Corollary 25.** Assume that an $n$-word, $w \neq (k-1)^00^{n-p}$ is not an expanded-Lyndon-word, and $w$ is covered by $L_{i+1}$. Write $w = yL_i^{r_i-1}v_{i,1}v_{i,2} \cdots v_{i,j}z$ as in Lemma 24. Then, $\text{cover}(w) = (L_{i+1}, x)$ where $x$ is the $|y|$-suffix of $L_i L_i^{r_i+1}$.

This corollary suggests a direction for computing $\text{cover}(w)$. Namely, finding $L_{i+1}$ and finding the $|y|$-suffix of $L_i L_i^{r_i+1}$. For extracting this data, first, we check if the subword of $w$: $v_{i,1} \cdots v_{i,j}$ is not empty (i.e. if $j > 0$). In case that $j = 0$, $w$ is just a rotation of $L_i^r$, and we use this fact to find $L_{i+1}$ and the $|y|$-suffix of $L_i L_i^{r_i+1}$. In case that $j > 0$, we use Lemma 24 and Corollary 22 to find a Lyndon word, $L$, such that $L_i < L \leq L_{i+1}$. Then, $L_{i+1}$ can be found by applying Algorithm 3 on $L$. It is also required to compute $|y|$ in this case. To summary, we set three goals for our analysis:

1. Deciding if $j = 0$.
2. If $j > 0$, finding a Lyndon word, $L$, such that $L_i < L \leq L_{i+1}$.
3. If $j > 0$, computing $|y|$.

We start with the first goal. The next lemma provides a criterion equivalent to $j = 0$.

**Lemma 26.** Assume that an $n$-word, $w \neq (k-1)^00^{n-p}$ is not an expanded-Lyndon-word, and $w$ is covered by $L_{i+1}$. Write $w = yL_i^{r_i-1}v_{i,1}v_{i,2} \cdots v_{i,j}z$ as in Lemma 24. Then, $j > 0$ if and only if $y = (k-1)^{|y|}$.

**Proof.** Write $v_{i,1} = v(\sigma + 1)$. Thus, $L_i = v\sigma(k-1)^l$ where $l = |L_i| - |v_{i,1}|$. Assume that $j > 0$ and observe that, since $|w| = n = |L_i^r|, |y| + |v_{i,1}| \leq |L_i|$. Therefore, $|y| \leq l$. Hence, since $y$ is a suffix of $L_i$, $y = (k-1)^{|y|}$.

For the other direction assume that $j = 0$. Hence, $w = yL_i^{r_i-1}z$ where $z$ is a proper prefix of $v_{i,1}$. Write $v_{i,1} = v(\sigma + 1) = zu(\sigma + 1)$, and note that $z$ is also a prefix of $L_i$. As $|w| = n = |yL_i^r z|$, we get that $L_i = zy$ and hence, $y = u\sigma(k-1)^l$. Since $\sigma < \sigma + 1 \leq k - 1$, the claim follows. \[\square\]

At first glance, it seems that this lemma is not useful at all, since we aim to compute $|y|$, but we have to know $|y|$ to determine if $y = (k-1)^{|y|}$. In fact, Lemma 26 actually serves as an intermediate property, which is equivalent to another property that concerns the structure of $w$, and can be computed efficiently.

**Definition 27.** An $n$-word, $w$, is said to be almost-Lyndon, if $w = (k-1)^l u$, and $u(k-1)^l$ is an Expanded-Lyndon-word.
**Lemma 28.** Assume that an n-word, \( w \neq (k - 1)^0 \) is not an expanded-Lyndon-word, and \( w \) is covered by \( L_{i+1} \). Write \( w = yL_i^{r_i-1}v_{i,1}v_{i,2} \cdots v_{i,j}z \) as in Lemma 24. Then, \( w \) is almost-Lyndon if and only if \( j > 0 \).

**Proof.** Before we prove the lemma, we mention a simple claim whose straightforward proof can be found in [2] (Lemma 6).

Claim. Let \( u\sigma(k-1)^l \) be an expanded-Lyndon word. Then, \( u(k-1)^{l+1} \) is an expanded-Lyndon-word.

We turn now to prove the lemma. First, assume that \( j > 0 \) and hence, by Lemma 26, \( y = (k - 1)^{|y|} \). By Lemma 24, \( L_i^{r_i-1}v_{i,1} \cdots v_{i,m_i} = L_{i+1}^{r_{i+1}} \). Since \( z \) is a prefix of \( v_{i,j+1} \), \( L_i^{r_i} = L_i^{r_i-1}v_{i,1} \cdots v_{i,j}z \), where \( |x| = |y| \). As \( L_i^{r_i} \) is an expanded-Lyndon word, by applying the mentioned claim \(|y| \) times, we get that \( L_i^{r_i-1}v_{i,1} \cdots v_{i,j}z \neq (k - 1)^{|y|} \) is also an expanded-Lyndon word. Therefore, \( w = yL_i^{r_i-1}v_{i,1} \cdots v_{i,j}z = (k - 1)^{|y|}L_i^{r_i-1}v_{i,1} \cdots v_{i,j}z \) is almost-Lyndon.

For proving the other direction of the equivalence, assume that \( j = 0 \) and we shall prove that \( w \) is not almost-Lyndon. Since \( j = 0 \), \( w = yL_i^{r_i-1}z \) where \( L_i = yz \). Moreover, since \( w \) is not covered by \( L_i \), \( z \neq \varepsilon \). By Lemma 26, \( y = (k - 1)^{|y|} \) thus \( y = (k - 1)^{|y|}y' \) where \( y' \neq \varepsilon \) does not start with \( k - 1 \). Therefore, \( L_i^{r_i} = L_i^{r_i-1}z(k-1)^{|y'|} \), and we claim that \( L_i^{r_i} < yL_i^{r_i-1}z(k-1)^{|y|} \). Indeed, this inequality holds since \( y' \) is a proper suffix of \( L_i \) and \( L_i \) is a Lyndon word, which guarantees that \( y' \) is strictly larger than the prefix of \( L_i \) of length \(|y'| \). Clearly, the CFL-factorization of \( w \) cannot strictly be smaller than one of its non-trivial rotations thus this inequality proves that \( yL_i^{r_i-1}z(k-1)^{|y|} \) is not an expanded-Lyndon word. Therefore, \( w = (k - 1)^{|y|}yL_i^{r_i-1}z \) is not almost-Lyndon.

So far, we identified a structural property of \( w \) which testifies if \( j = 0 \) or not. We turn now to achieve our second and third goals, which are finding a Lyndon word \( L \) such that \( L_i < L \leq L_{i+1} \), and computing \(|y| \), when \( j > 0 \). For this purposes, we use the classical result by Chen, Fox and Lyndon [3]. The authors of [3] (see also [7]) proved that every non-empty word, \( w \), can be uniquely factorized into Lyndon words: \( w = x_1 \cdots x_l \) so that \( x_1 \geq x_2 \geq \cdots \geq x_l \). We name this decomposition: the CFL-factorization of \( w \). In the next lemma we show the connection between the CFL-factorizing of an n-word, \( w \), and the structure of \( w \) as characterized in Lemma 24.

**Lemma 29.** Assume that an n-word, \( w \neq (k - 1)^0 \) is not an expanded-Lyndon-word, and \( w \) is covered by \( L_{i+1} \). Write \( w = yL_i^{r_i-1}v_{i,1}v_{i,2} \cdots v_{i,j}z \) as in Lemma 24 where \( j > 0 \). Let \( y = y_1 \cdots y_l \) be the CFL-factorization of \( y \) and let \( z = z_1 \cdots z_m \) be the CFL-factorization of \( z \).

1. If \( r_i = 1 \) and \( v_{i,1} = \cdots = v_{i,j} \), then the CFL-factorization of \( w \) is \( w = y_1 \cdots y_l |v_{i,1}| \cdots |v_{i,j}|z_1| \cdots |z_m \).

2. Otherwise, the CFL-factorization of \( w \) is \( w = y_1 \cdots y_l |L_i^{r_i-1}v_{i,1} \cdots v_{i,j}|z_1| \cdots |z_m \).

**Proof.** The two statements are proved by similar arguments thus we prove only the second claim. We leave for the reader to observe that \( L_i^{r_i-1}v_{i,1} \cdots v_{i,j} \) is indeed a Lyndon word. If \( r_i > 1 \), this implied by Lemma 13 and if \( r_i = 1 \) it follows from Lemma 14 since in this case \( -(v_{i,1} = \cdots = v_{i,j}) \).

Therefore, \( y_1 \cdots y_l |L_i^{r_i-1}v_{i,1} \cdots v_{i,j}|z_1| \cdots |z_m \) is a factorization of \( w \) into Lyndon words. Since \( y_1 \cdots y_l \) and \( z_1| \cdots |z_m \) are the CFL-factorization of \( y \) and \( z \), respectively, it is left to prove:

(a) \( y_l \geq L_i^{r_i-1}v_{i,1} \cdots v_{i,j} \).

(b) \( L_i^{r_i-1}v_{i,1} \cdots v_{i,j} \geq z_1 \).

Since \( j > 0 \), \( y = (k - 1)^{|y|} \) thus \( y_l = k - 1 \), which proves (a). Now, (b) holds since \( z_1 \) is prefix of \( L_i \). Indeed, if \( r_i > 1 \), \( L_i^{r_i-1}v_{i,1} \cdots v_{i,j} \geq z_1 \) since \( z_1 \) is a prefix of \( L_i^{r_i-1}v_{i,1} \cdots v_{i,j} \). In addition, if \( r_i = 1 \), since \( z_1 \) is a prefix of \( L_i \), by the definition of \( v_{i,1}, v_{i,1} > L_i > z_1 \) thus \( v_{i,1} \cdots v_{i,j} > z_1 \) as required.
By this lemma, and by Lemmas 20 and 22 we conclude the following consequence.

**Corollary 30.** Assume that an n-word, \( w \neq (k-1)p0^{n-p} \) is not an expanded-Lyndon-word, and \( w \) is covered by \( L_{i+1} \). Write \( w = yL_{i+1}r_{i+1}^{-1}v_{i+1,1}v_{i+1,2} \cdots v_{i+1,z} \) as in Lemma 24 and assume \( j > 0 \). Let \( x_1 \cdots |x_t \) be the CFL-factorization of \( w \), and let \( x_1 \) be the first word in this factorization, different from \( k-1 \). Then,

1. \( L_i \leq x_t \leq L_{i+1} \).
2. \( |y| = t-1 \).

### 5.2.3 Linear-time FTG algorithm

We are finally ready to present a linear-time FTG algorithm. In addition to the algorithms described earlier, we use the following procedures, all of which can be computed in linear time:

- **find_root** \((v)\). The root of a word, \( v \neq \varepsilon \), is its shortest non-empty prefix, \( x \), such that \( v = x^t \) for some integer \( t \). The word \( x \) can be found by searching \( v \) within \( v'v \), where \( v' \) is the \(|v| - 1\)-suffix of \( v \). This can be done, for example, by invoking the KMP-algorithm 15, which runs in linear-time. See 16 for a presentation of this technique. Extensions and a detailed discussion can be found in [19], chapter 8.

- **find_suffix** \((u, v)\). This procedure receives two words, \( u, v \), where \( u \) is subword of \( v \). The procedure returns a word, \( x \), so that \( ux \) is a suffix of \( v \). This procedure can be implemented by modifying the KMP-algorithm. The procedure runs in \( O(|u| + |v|) \) time. We apply this procedure only on inputs of size at most \( 3n \) thus we refer to this procedure as a linear-time procedure.

- **find_min_rot** \((w)\). Returns the lexicographically minimal rotation of \( w \). It can be implemented by invoking Booth’s algorithm 3, or Shilaoch’s Algorithm 20, both runs in linear time.

- **is_Lyndon** \((L)\). Tests if \( L \) is a Lyndon word. \( L \) is a Lyndon word if and only if it is equal to its minimal rotation (tested by invoking **find_root**) and it is equal to its minimal rotation (tested by invoking the former procedure).

- **CFL** \((w)\). Returns the CFL-factorization of \( w \). See 7 for details and runtime analysis.

- **is_almost_Lyndon** \((w)\). Given an \( n \)-word, \( w \), this procedure checks if it is almost-Lyndon. If \( w = (k-1)^t u \), where \( u \) does not start with \( k-1 \), we test if \( w' = u(k-1)^l \) is an Expanded-Lyndon-word. This occurs if and only if \( is_Lyndon\( (find_root(u')) \) holds.

**Proposition 31.** Algorithm 7 computes \( FTG\( (w) \) \) in \( O(n)\)-time.

**Proof.** The algorithm clearly terminates after \( O(n) \) steps, and we prove its correctness. First, use Corollary 16 to observe that lines 1-7 handle correctly the case where \( w \) is an expanded Lyndon word (use the second item of Corollary 16 for the case where \( w = 0^n \)). Furthermore, if \( w = (k-1)p0^{n-p} \), where \( 0 < p < n \), then \( w \) is almost-Lyndon, and the algorithm terminates in line 25 and returns **cover_to_FTG**(0,0\( p \)). Again, item 2 of Corollary 16 shows that a correct value is returned by the algorithm in those cases.

It remains to deal with the general case, where \( w \neq (k-1)p0^{n-p} \) and \( w \) is not an expanded-Lyndon-word. Take such \( w \), which is covered by \( L_i \). Write \( w = yL_{i+1}r_{i+1}^{-1}v_{i-1,1}v_{i-1,2} \cdots v_{i-1,z} \) as in Lemma 24 and assume, first, that \( j = 0 \). Hence, by Lemma 28 the test in line 8 returns a negative response, and the computation proceeds to line 11. Now, since \( j = 0 \), \( w \) is a rotation of \( L_{i+1}r_{i+1}^{-1} \). As a result, in line 11, the word \( L_{i+1}r_{i+1}^{-1} \) is assigned to variable \( u \), in line 12, \( L \) is assigned with \( L_{i-1} \), and in line 13, \( L' \) is assigned with \( L_i \). Since \( w \) is a subword of \( L_{i-1}L_{i+1}r_{i+1}^{-1} \), line 15 assigns to \( x \) the value that satisfies: **cover** \((w) = (L_i, x) \). Therefore, by Corollary 30, the invocation of the **cover_to_FTG** procedure in line 16 returns **FTG** \((w) \).

Now, consider the case where \( j > 0 \). By Lemma 28 the test in line 8 returns true, and the computation traverses to line 17. By Corollary 31 in line 19 a Lyndon word is assigned to variable \( L \) such that \( L_{i-1} < L \leq L_i \). Thus, when the computation reaches line 23, \( L \) stores the value \( L_i \). Moreover, by the same corollary, in line 24, \( x \) is assigned with the \(|y|\)-prefix of \( L_{i+1}r_{i+1}^{-1} \). Therefore, by Corollary 28 **cover** \((w) = (L_i, x) \) and hence, by Corollary 16 the method invocation in line 25 returns **FTG** \((w) \). \( \square \)
Algorithm 6 filling_the_gap

Input: a word \( w \) of length \( n \)
Output: \( \text{FTG}(w) \)

1. if \( w = (k - 1)^n \) then // 1-7: edge cases, \( w = L_i' \)
2. return \((k - 1, \varepsilon)\)
3. end if
4. \( L \leftarrow \text{find_root}(w) \)
5. if \( \text{isLyndon}(L) \) then
6. return \( \text{cover_to_FTG}(L, \varepsilon) \)
7. end if
8. if \( \text{isalmostLyndon}(w) \) then // 8-10: test if \( j > 0 \)
9. goto 17
10. end if
11. \( u \leftarrow \text{find_min_rot}(w) \) // 11-16: if \( j = 0 \)
12. \( L \leftarrow \text{find_root}(u) \)
13. \( L' \leftarrow \text{next_n_Lyndon}(L) \)
14. \( r \leftarrow n/|L'|, \)
15. \( x \leftarrow \text{find_suffix}(w, LL' r) \)
16. return \( \text{cover_to_FTG}(L', x) \)
17. \( x_1 | \cdots | x_t \leftarrow \text{CFL}(w) \) // 17-25: if \( j > 0 \)
18. \( t \leftarrow \text{first integer such that } x_t \neq k - 1 \)
19. \( L \leftarrow x_t \)
20. if \( |L| \nmid n \) then
21. \( L \leftarrow \text{Lnext}(L) \)
22. end if
23. \( r \leftarrow n/|L| \)
24. \( x \leftarrow (t - 1)\)-suffix of \( L^r \)
25. return \( \text{cover_to_FTG}(L, x) \)

By Propositions 12 and 31 we conclude:

Theorem 32. Algorithm 4 forms a generalized-shift-rule for the prefer-min DB sequence that runs in \( O(n + c) \)-time.

6 Conclusions

We proposed the notion of a generalized-shift-rule for a De Bruijn sequence which, unlike a shift-rule, allows to construct the entire De Bruijn sequence efficiently. We noted that a generalized-shift-rule for an \((n, k)\)-DB sequence runs in time \( \Omega(n + c) \), and presented an \( O(n + c) \)-time generalized-shift-rule for the well-known prefer-min De Bruijn sequence. By imposing a trivial reduction, as explained at the preliminaries section, our results provide a generalized-shift-rule for the prefer-max De Bruijn sequence as well.

References

[1] Abbas M Alhakim. A simple combinatorial algorithm for de bruijn sequences. American Mathematical Monthly, 117(8):728–732, 2010.
[2] Gal Amram, Yair Ashlagi, Amir Rubin, Yotam Svoray, Moshe Schwartz, and Gera Weiss. An efficient shift rule for the prefer-max de Bruijn sequence. arXiv preprint arXiv:1706.01106, 2017.

[3] Kellogg S Booth. Lexicographically least circular substrings. Information Processing Letters, 10(4-5):240–242, 1980.

[4] Kuo-Tsai Chen, Ralph H Fox, and Roger C Lyndon. Free differential calculus, iv. the quotient groups of the lower central series. Annals of Mathematics, pages 81–95, 1958.

[5] N.G. de Bruijn. A combinatorial problem. Nederlandse Akademie v. Wetenschappen, 43:758–764, 1946.

[6] Patrick Baxter Dragon, Oscar I Hernandez, and Aaron Williams. The grandmama de Bruijn sequence for binary strings. In Latin American Symposium on Theoretical Informatics, pages 347–361. Springer, 2016.

[7] Jean Pierre Duval. Factorizing words over an ordered alphabet. Journal of Algorithms, 4(4):363–381, 1983.

[8] Jean-Pierre Duval. Génération d’une section des classes de conjugaison et arbre des mots de Lyndon de longueur bornée. Theoretical computer science, 60(3):255–283, 1988.

[9] Tuvi Etzion. Self-dual sequences. Journal of Combinatorial Theory, Series A, 44(2):288–298, 1987.

[10] T. Marie-Flaye-Sainte. Solution of problem 58. Intermediare des Mathematiciens, pages 107–110, 1894.

[11] L. R. Ford. A cyclic arrangement of n-tuples. Technical Report P-1071, Rand Corporation, Santa Monica, California, 1957.

[12] Harold Fredricksen and Irving Kessler. Lexicographic compositions and de Bruijn sequences. Journal of Combinatorial Theory, Series A, 22(1):17–30, 1977.

[13] Harold Fredricksen and James Maiorana. Necklaces of beads in k colors and k-ary de Bruijn sequences. Discrete Mathematics, 23(3):207–210, 1978.

[14] Daniel Gabric and Joe Sawada. A de Bruijn sequence construction by concatenating cycles of the complemented cycling register. In International Conference on Combinatorics on Words, pages 49–58. Springer, 2017.

[15] Pawel Gawrychowski, Dalia Krieger, Narad Rampersad, and Jeffrey Shallit. Finding the growth rate of a regular of context-free language in polynomial time. In Developments in language theory, pages 339–358. Springer, 2008.

[16] Yuejiang Huang. A new algorithm for the generation of binary de Bruijn sequences. Journal of Algorithms, 11(1):44–51, 1990.

[17] Donald E Knuth. The Art of Computer Programming, Volume 4A: Combinatorial Algorithms, Part 1. Pearson Education India, 2011.

[18] Donald E Knuth, James H Morris, Jr, and Vaughan R Pratt. Fast pattern matching in strings. SIAM journal on computing, 6(2):323–350, 1977.

[19] M Lothaire. Applied combinatorics on words, volume 105. Cambridge University Press, 2005.

[20] Roger C Lyndon. On burnsides problem. Transactions of the American Mathematical Society, 77(2):202–215, 1954.

[21] Monroe H Martin. A problem in arrangements. Bulletin of the American Mathematical Society, 40(12):859–864, 1934.
[22] Anthony Ralston. A new memoryless algorithm for de Bruijn sequences. *Journal of Algorithms*, 2(1):50–62, 1981.

[23] Frank Ruskey, Carla Savage, and Terry Min Yih Wang. Generating necklaces. *Journal of Algorithms*, 13(3):414–430, 1992.

[24] Joe Sawada, Aaron Williams, and Dennis Wong. A surprisingly simple de Bruijn sequence construction. *Discrete Mathematics*, 339(1):127–131, 2016.

[25] Joe Sawada, Aaron Williams, and Dennis Wong. A simple shift rule for k-ary de Bruijn sequences. *Discrete Mathematics*, 340(3):524–531, 2017.

[26] Yossi Shiloach. Fast canonization of circular strings. *Journal of Algorithms*, 2(2):107–121, 1981.