Finitely presented groups and the Whitehead nightmare

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Abstract

In this paper we show that any finitely presented group is “easily-representable”, in the sense that it admits an easy (inverse)-representation. More precisely, to any finitely presented group $\Gamma$ one can associate a singular compact 3-manifold $M(\Gamma)$ whose fundamental group is $\Gamma$, and an easy-representation of such a group presentation is a non-degenerate simplicial map $f$ from some locally-finite 2-dimensional GSC (geometrically simply connected) simplicial complex $X$ to the universal covering space $\tilde{M}(\Gamma)$ such that the map $f$ is zippable (meaning that the smallest equivalence relation on $X$, compatible with $f$, which kills all the singularities of $f$, i.e. the non-immersive points, also kills all the double points) and such that the sets of double points of $f$ and $f(X)$ are both closed in $X$ and $\tilde{M}(\Gamma)$ respectively.

Our statement is based on the result by the second author (V.P.) proving the qsf (Brick’s quasi-simple filtration) for all finitely presented group, of which it is a direct consequence. In other terms, we actually prove that qsf groups are “easy”.

Keywords: Geometric simple connectivity (GSC), quasi-simple filtration (qsf), (inverse)-representations, finitely presented groups, universal covering spaces, singularities.

MSC Subject: 57 M 05, 57 M 10, 57 N 35.

1 Introduction

The present paper is a direct continuation of the 3-parts work [37, 38, 39] of the second author (V.P.). An overview of this work can be found on-line in [36].

What that previous work had done was to show that any finitely presented group $\Gamma$, and no other groups will ever be considered here, has the QSF property of Brick, Mihalik and Stallings [4, 48] (roughly speaking, a space is QSF if any compact subspace of it can be “approximated” by an (abstract) simply connected compact). We will only give for it right here the following equivalent definition, but this equivalence is actually a theorem of the first author (D.O.) and of L. Funar [10]: the finitely presented group $\Gamma$ is QSF if and only if there exists a smooth compact manifold $M$ such that $\pi_1 M = \Gamma$ and $\tilde{M}$ is geometrically simply connected (GSC), i.e. has a...
handlebody decomposition where all the 1-handles cancel with (a subset of) the 2-handles. But one should keep in mind that this more mundane and transparent GSC definition is not group-presentation invariant, while the original definition of Brick and Mihalik, which we will recall in the next section, is. This is actually one of the important virtues of the concept QSF.

The present work presents a first application of the fact that any group $\Gamma$ is QSF, and we will state this result a bit later (see Theorem 1 below).

For us, a presentation of a finitely presented group $\Gamma$ will be any finite simplicial complex $K$ such that $\pi_1 K = \Gamma$. Now, in this present paper, as well as in [22, 30, 37, 38, 39], we deal with finitely presented groups $\Gamma$ which are not submitted to any additional condition, but we will be very choosy when it comes to presentations. These will be always compact 3-manifolds with singularities, locally as $(\text{figure Y}) \times \mathbb{R}^2$; this is enough for catching all the groups we may want. Let us denote $M(\Gamma)$ such a compact singular 3-manifold with $\pi_1 M(\Gamma) = \Gamma$.

Our basic tool for dealing with $\Gamma$’s will be REPRESENTATIONS, which we will define more formally in the next section. It will suffice to say, for right now, that contrary to the more usual group representations which, for $\Gamma$, take the general form “$\Gamma \rightarrow \text{something}$”, our (inverse-)representations, which we will always write in capital letters, take the dual form “$X(= \text{some space with special features}) \rightarrow \tilde{M}(\Gamma)$”, and here the universal covering space $\tilde{M}(\Gamma)$ is the same thing as the group $\Gamma$, up to quasi-isometry [14].

This triple $X \rightarrow \tilde{M}(\Gamma)$ is endowed with the following features: $X$ is a not necessarily locally-finite simplicial complex which is geometrically simply connected (GSC), $f$ is a non-degenerate simplicial map, meaning that $f(d\text{-simplex}) = d\text{-simplex}$, and, furthermore, this map is also zippable, by which we intend that the “smallest” equivalence relation on $X$ which is compatible with $f$ and which is also such that the quotient space immerses into $\tilde{M}(\Gamma)$, via the obviously induced map, is the trivial equivalence relation induced by $f$ itself, namely: $x \sim y \iff f(x) = f(y)$. In other words, what zippability means is that the “cheapest” way to kill all of $\text{Sing}(f) \equiv \{\text{the points } x \in X \text{ where } f \text{ is not locally an embedding, i.e. the non-immersive points of } X\}$, is to kill all the double points $M_2(f)$; this will actually happen via folding maps.

Here are some additional explanations concerning this long definition, which was given now rather informally, and which will be restated more formally in the next sections.

To begin with, the GSC, which is a notion stemming from the differential topology, makes perfectly well sense for arbitrary cell-complexes, locally-finite or not (this will be precisely explained in Section 2). There is also a related notion, to be used too in this paper, the weak geometric simple connectivity (WGSC). A locally finite simplicial complex $X$ is said to be WGSC if it admits an exhaustion by simply connected compact (i.e. finite) sub-complexes. This notion has been introduced by L. Funar [8, 9], and also studied by him and the first author (D.O.), in the present context of geometric group theory (see e.g. [10, 20, 21]).

Via classical arguments à la Smale [47], when we are in high dimensions, and in the DIFF context, GSC and WGSC are equivalent notions (see [9]), and, in a form which is useful for us, this argument will be reviewed, later, in this paper. But it should be stressed that, otherwise, GSC and WGSC are far from being equivalent concepts. The WGSC is a sort of an asymptotic version of the simple connectivity, actually for open 3-manifolds $V^3$ there is an implication

$$V^3 \text{ is WGSC } \implies \pi_1^\infty V^3 = 0 \text{ (i.e. } V^3 \text{ is simply connected at infinity)},$$

which is certainly false in higher dimensions. On the other hand, GSC is closely related to the
issue of the collapsibility of the 2-skeleton. And certainly for 2-complexes, as everybody knows, even in the compact case, between mere simple-connectivity and collapsibility, there is a deep chasm.

Enough having been said about gsc, let us go back now to the non-degeneracy of \( f \). This means, among other things, that the dimension of the representation space \( X \), source of \( f \), is restricted to \( \dim X \leq 3 \). The only serious cases are actually \( \dim X = 2 \) and \( \dim X = 3 \), each interesting in its own right.

We will speak about 2d-representations and 3d-representations, and the capital letters should remind the reader that we are not talking about the mundane group representations, where the dimension of the representation means quite a different thing. Retain, also, that our representations \( X \xrightarrow{f} \tilde{M}(\Gamma) \) are sort of a resolutions of \( \Gamma \simeq \tilde{M}(\Gamma) \) into the gsc space \( X \).

With all these things, here is what is probably the most striking result of this paper, and, in the next sections, after the representations are more formally defined, the statement below melts into the more comprehensive statements of the Theorems 4 and 5.

**Theorem 1.** For any finitely presented group \( \Gamma \) there is a 2d-representation \( X^2 \xrightarrow{f} \tilde{M}(\Gamma) \), such that (the gsc simplicial complex) \( X^2 \) is locally finite and so that, moreover, we have:

(i) both \( fX^2 \subset \tilde{M}(\Gamma) \) and the double points set \( M_2(f) \subset X^2 \) are closed subsets;

(ii) if one relaxes the condition that \( X^2 \) be gsc to wgsc, then one can, in addition to the things above, get an \( X^2 \) with a free \( \Gamma \)-action \( \Gamma \times X \to X \) such that \( f \) is equivariant, i.e. \( f(\gamma x) = \gamma x \) for all \( \gamma \in \Gamma \), \( x \in X \).

**Definition 1.** A representation satisfying (i) above will be called easy and, in the context of (ii), we will talk about wgsc-representations.

Now, at least informally, we propose the following (impertinent) definition:

**Definition 2.** A finitely presented group \( \Gamma \) is called easy (or easily-representable) if it admits a 2-dimensional representation \( X^2 \xrightarrow{f} \tilde{M}^3(\Gamma) \) which is easy, in the sense just defined (namely with closed \( fX \subset \tilde{M}(\Gamma) \) and \( M_2(f) \subset X^2 \)).

With this definition we may rephrase Theorem 1 as follows:

**Theorem 2** (Reformulation of Theorem 1). Any finitely presented group \( \Gamma \) is easy. Or, equivalently, all finitely presented groups admit easy-representations.

**Remark 1.1.**

1. As it will become clear, the proof of Theorem 1 relies very heavily on the previous 3-parts work of the second author (V.P.), \([37, 38, 39]\), where it is showed that all finitely presented \( \Gamma \)'s are QSF. So, a fast way to sum up the present paper is to say that, for finitely presented groups, we actually have the implication

\[
\Gamma \in \text{QSF} \implies \Gamma \text{ is easy}.
\]

The converse implication is already a theorem proved by us two (D.O. and V.P.), see \([22]\). Both implications are not very hard to prove, certainly much easier with respect to the proof that all \( \Gamma \)'s are QSF.
2. One should not be misled by the adjective “easy”, as used above. Both our Theorem 1 as well as the one stating that all finitely presented group are QSF \[36\], on which it relies, are valid for all such \(\Gamma\)’s but, contrary to the common belief that no non-trivial result can be true for all groups, they really are non-trivial. Here is an explicit measure of their non-triviality. Consider the case of those \(\Gamma\)’s which are \(\Gamma = \pi_1(M^3)\), with \(M^3\) a closed 3-manifold and, in terms of M. Gromov’s theory of random groups \[14\], these should be quite rare events indeed. Now, for these particular groups it was actually previously known already that they are QSF (and hence easy), since this is a corollary of the Thurston Geometrization Conjecture, proved by G. Perelman (see \[26\], \[27\], \[28\], and \[1\], \[2\], \[16\], \[18\]). But even for this special case, there are no other roads of access, to the best of our knowledge, then either to deduce it from the Ricci flow, or from the more general theorem \(\forall \Gamma \in \text{QSF}\).

1.1 Historical remarks

There is also a long story behind Theorem 1 above, which we will review now. About twenty-five years or so ago, independently of each other, Andrew Casson and the second author (V.P.) have devised a technique for proving results of the following general type. Take \(\Gamma = \pi_1M^3\), where \(M^3\) is a closed 3-manifold, and assume that it satisfies some “nice geometric conditions” (and we will soon explain what that is supposed to mean); then \(\widetilde{M}^3\) is simply connected at infinity, i.e. \(\pi_1^\infty\widetilde{M}^3 = \pi_1^\infty\Gamma = 0\). Alternatively, this conclusion may also be phrased as follows: if in addition to those nice conditions we also assume that \(M^3\) is irreducible, then \(\widetilde{M}^3 = \mathbb{R}^3\).

For these old papers, see for instance \[12\], \[30\], \[31\], \[33\], \[40\], \[41\], and, for a somehow different approach to these same issues, see also \[3\]. A whole little industry was developed in those old days around these various papers, but, of course, they are all superseded, by now, by Perelman’s work \[26\], \[27\], \[28\] (see also \[16\]). However, it is worthy to note that the list of “nice geometric conditions” for \(\Gamma\) includes \textit{hyperbolicity} (in the sense of Gromov \[13\]), or more generally \textit{almost-convexity} (in the sense of Cannon), the \textit{automatic property}, or, more generally, \textit{combability} (in the sense of Thurston et al. \[6\]), Casson’s condition \(\hat{C}_\alpha \[12\], a.s.o. [in particular, this class of groups is quite big].

Although these papers mentioned above are today superseded, it may still be appropriate to tell here, with hindsight, from our present vantage point and expressed in a more recent jargon, what was actually done there. This may be summarized as follows.

(a) Let \(\Gamma\) be any finitely presented group which satisfies some nice geometric condition, like above. Then \(\Gamma\) is easy in the sense of Definition 2 above. This was actually the main result of these old papers, the only one which may still deserve to be remembered today. Of course, things were not phrased this way, the author (V.P.) being at that time too concerned with the universal covers of 3-manifolds, to be able to see the world beyond.

(b) Hence \(\Gamma\) is also QSF (but of course, the concept did not exist then). It should be noted that neither (a) nor (b) have anything to do with 3-manifolds.

(c) Now, finally, if one also assumes that \(\Gamma = \pi_1M^3\), then, by a result of Brick and Mihalik \[4\], \(\pi_1^\infty\widetilde{M}^3 = 0\) (in the old papers one actually used a variant of Dehn’s lemma \[30\]).
At that time, this issue of $\pi_1^\infty \tilde{M}^3 = 0$ was thought to be the punch line of the papers, whereas, today, this is certainly superseded by Perelman’s work.

In [31, 33], on the way from $\pi_1 M^3 \in \{\text{Gromov hyperbolic and/or s.o.}\}$ to $\pi_1^\infty \tilde{M}^3 = 0$, an intermediate link was the following concept, called “Dehn-exhaustibility”:

**Definition 3.** A locally compact simplicial complex $X$ is called **Dehn-exhaustible** (DE) if for any compact simplicial complex $k \subset X$ there is a commutative diagram

\[
\begin{array}{ccc}
 k & \xrightarrow{j} & K \\
 \downarrow i & & \downarrow f \\
 X & \xrightarrow{f} & \\
\end{array}
\]

where $K$ is a simply-connected and compact complex, $i$ is the canonical injection, $j$ is another injection, and $f$ is an immersion satisfying the “Dehn condition”: $M_2(f) \cap j(k) = \emptyset$ (where $M_2(f) \subset K$ denotes the set of double points of $f$).

Actually, in [30, 31, 33], only the special case when $X$ is an open smooth 3-manifold with $f$ a smooth immersion was explicitly considered, since then the Dehn-exhaustibility from diagram (1) can be plugged then into a variant of the classical Dehn Lemma which was useful for that old approach to $\pi_1^\infty \tilde{M}^3 = 0$ (see point (c) above).

Of course, the notion of Dehn-exhaustibility can be phrased, also, in a completely **DIFF** setup. Here is then a typical result:

**Theorem 3** (V. Poénaru). Let $V^n$ be a smooth open manifold, such that there is a $p \geq 1$ for which $V^n \times B^p$ is GSC. Then $V^n$ is Dehn-exhaustible, in the **DIFF** category.

For $n = 3$ this is actually explicitly proved in [30], but the argument can be adapted for any $n$. Actually, in the third paper from the second author’s QSF trilogy [37, 38, 39] something like Theorem 3 is even done when $\partial V^n \neq \emptyset$.

Anyway, a few years after these old papers were written, Brick and Mihalik [4] abstracted the notion QSF (meaning **quasi-simple filtration**) from the earlier work of Casson [12] and the second author (V.P.) [30, 31, 33], as follows:

**Definition 4.** We are now in the simplicial category and the locally compact simplicial complex $X$ is **QSF** iff for any compact subcomplex $k \subset X$ there is a simply-connected compact (abstract) complex $K$ endowed with an inclusion $k \hookrightarrow K$ and with a simplicial map $K \xrightarrow{f} X$ satisfying the Dehn condition $M_2(f) \cap j(k) = \emptyset$, and entering in a commutative diagram like (1) above (but now $f$ is no longer an immersion, it is just a simplicial map).

In other words, a locally finite simplicial complex $X$ is QSF if it satisfies something like the Dehn-exhaustibility (see Definition 3 above), with the condition on $f$ relaxed from immersion to $f$ being a mere simplicial map (see here [4, 10, 49]). Most of the good virtues of the Dehn-exhaustibility (DE) are preserved for the more general QSF, like for instance the implication

$$V^3 \text{ open, irreducible and QSF} \implies V^3 = \mathbb{R}^3.$$
But what one gains is very valuable, since, unlike $\text{DE}$, the $\text{QSF}$ turns out to be a group theoretical, presentation-independent notion: if $K_1, K_2$ are two presentations (i.e. two presentation complexes) for the same (finitely presented) group $\Gamma$, then $\widetilde{K}_1 \in \text{QSF} \iff \widetilde{K}_2 \in \text{QSF}$ \footnote{\[.\]}

If that happens, then we will say that the group $\Gamma$ itself is $\text{QSF}$. [Note that only finitely presented groups are considered in this paper and the $K_1, K_2$ are always compacts.]

\textbf{Remark 1.2.} We do not know whether the notion $\Gamma \in \text{QSF}$ is expressible in more algebraic terms, let us say via homological algebra or $K$-theory.

For locally finite $X$’s, there are trivial implications $\text{DE} \implies \text{QSF}$ and $\text{GSC} \implies \text{QSF}$. In the group theoretical context, L. Funar and the first author (D.O.) \cite{10} have proved weak converses to these implications, where “weak” means things of the following type. If $\Gamma$ has a presentation $K_1$ such that $\widetilde{K}_1 \in \text{QSF}$, then it also has one presentation $K_2$ such that $\widetilde{K}_2 \in \text{GSC}$ (and, of course, conversely, but this is a triviality). But, unlike $\text{QSF}$, neither $\text{GSC}$ nor $\text{DE}$ nor $\text{WGSF}$ are presentation independent (see \cite{10, 20}).

Coming back now to the REPRESENTATIONS, with which this paper deals, they have their own little story too. They were already present at the level of the very initial step in the approach of the second author (V.P.) to the Poincaré Conjecture. In the papers \cite{32, 43} homotopy 3-spheres $\Sigma^3$ and open 3-manifolds were REPRESENTED, rather than $\widetilde{M}(\Gamma)$’s, like here. It is in \cite{32}, under a different name, that REPRESENTATIONS first appeared (see also \cite{34} and \cite{21}). [For a complete overview of the second author’s (V.P.) approach to the Poincaré Conjecture see the ArXiv paper \cite{35}.]

\textbf{Remark 1.3.} It should also be stressed at this point that the definition of REPRESENTATIONS $X \xrightarrow{f} Y$ is such that the object $Y$ which is REPRESENTED, automatically comes with $\pi_1 Y = 0$.

In the old papers \cite{30, 31, 33, 34} it was the $\widetilde{M}^3$’s which were REPRESENTED. Then in \cite{42}, REPRESENTATIONS of the classical Whitehead manifold $\text{Wh}^3$ \cite{51} were investigated and what was found there, was that for the simplest and most natural 2$^d$-REPRESENTATIONS of $\text{Wh}^3$, $X^2 \xrightarrow{f} \text{Wh}^3$, not only $M_\text{Wh}(f) \subset X^2$ is not a closed subset, but its accumulation pattern is chaotic. Very explicitly, the pattern in question is guided by a specific class of Julia sets generated by the infinite iteration of real quadratic polynomials; the same feed-back loop occurs in both cases.

We end here this historical digression and go back to the notion of REPRESENTATIONS of a group $\Gamma$, $X^2 \xrightarrow{f} \widetilde{M}(\Gamma)$, which, as such, has apriori nothing group-theoretical about it, except that it allows the possibility of a free action $\Gamma \times X \rightarrow X$, with an equivariant $f$, i.e. $f(gx) = gf(x)$; the point (ii) in Theorem \footnote{\[.\]} brings this option to life. Incidentally, REPRESENTATIONS of $\Gamma$ which are both locally finite and equivariant are essential for the second author’s (V.P.) proof that any $\Gamma \in \text{QSF}$ \cite{37, 38, 39}.

In the next section we will state more formally, and with more details, what the paper actually proves. Then Theorem \footnote{\[.\]} will appear as a piece of some bigger, more comprehensive statement. This will deal with $3^d$-REPRESENTATIONS too, and then the “Whitehead nightmare” appearing in the title of this paper will be explained too.

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2 Definitions and statements of the results

We will give now, with full details, the definition of the representations for finitely presented groups $\Gamma$, which were only very informally presented in the last section.

To begin with, like in [36, 37], we consider presentations for $\Gamma$ which are singular compact 3-manifolds with non-empty boundary $M(\Gamma)$. The structure of such an $M(\Gamma)$ is very simple (see e.g. [22]). Start with a compact 3-dimensional handlebody of some appropriate genus $g$, call it $H$; this embodies the generators of the group $\Gamma$. Then 2-handles are attached to $H$, embodying the relations. Explicitly, the attaching zones are given by an immersion

$$\sum_{j=1}^{k} (S^1_j \times [0,1]) \rightarrow \partial H,$$

which injects on each individual $S^1_j \times I$, the double points coming from (singular) little squares $S \in \partial H$, where $\phi(S^1_l \times I)$ and $\phi(S^1_m \times I)$, for $m \neq l$, go through each other. These immortal singularities $S$ are the points where $M(\Gamma)$ fails to be a 3-manifold.

Now we are ready to give the precise and formal definition of representations for finitely presented groups, leaving more details and comments just after the definition.

Definition 5. A representation of a finitely presented group $\Gamma$ is a (simplicial) map

$$X \xrightarrow{f} \tilde{M}(\Gamma),$$

which satisfies the following list of conditions:

(3−1) the space $X$ is an (at most) countable simplicial complex which is not necessarily assumed to be locally finite; but the complex $X$ is assumed to be gsc (i.e. geometrically simply connected);

(3−2) the simplicial map $f$ is non-degenerate, which also means that $\dim X \leq 3$. Hence, once the meaningless case $\dim X = 1$ is discarded, we are left with the two meaningful cases $\dim X = 2$ and $\dim X = 3$, namely with 2-dimensional and 3d- representations;

(3−3) the equality $\Psi(f) = \Phi(f)$ holds (see the explanation here below), and in this case we say that $f$ is zippable;

(3−4) the map $f$ is “essentially surjective”, which means the following: if $\dim X = 3$, then $\text{Im} f = \tilde{M}(\Gamma)$, and if $\dim X = 2$, then $\tilde{M}(\Gamma) = \text{Im} f + \{\text{cells of dimension 2 and 3}\}$.

Here, some remarks and details are needed. First of all, notice that the gsc concept, which stems from the differential topology, makes sense for arbitrary cell-complexes. More precisely:
Definition 6. We will say that a cell complex $X$ is gsc if it admits a cell decomposition (or handle-decomposition), with $T =$ infinite tree and $H^\lambda = \lambda$-cells (or handles of index $\lambda$), of the following form

$$X = T + \sum_{i=1}^{\infty} H^1_i + \sum_{j=1}^{\infty} H^2_j + \{\text{more cells (or handles) } H^\lambda \text{ with } \lambda \geq 2\},$$

where $\sum_{j=1}^{\infty} H^2_j$ is a selected set of 2-cells, in bijection with the set of 1-cells $\sum_{i=1}^{\infty} H^1_i$, and then the geometric intersection matrix is of the following form, which we call easy id+nilpotent

$$\partial H^2_j \cdot \partial H^1_i = \delta_{ji} + a_{ji}, \text{ where } a_{ji} > 0 \iff j > i.$$

When one is in a non-compact situation, like now, then this is the correct way of having the 1-handles in canceling position with the 2-handles. Also, if in the little equation above we replace the “$a_{ji} > 0 \iff j > i$” with the dual equation “$a_{ji} > 0 \iff j < i$”, which is called the difficult id+nilpotent, then this no longer implies gsc.

For instance, the classical Whitehead manifold $Wh^3$ has a handlebody decomposition of the difficult id+nil type, and it certainly is NOT gsc! [Proof. Assume $Wh^3$ is gsc, then $Wh^3 \times B^n$ is gsc too. Theorem 3 implies then that $Wh^3$ is DE. But, in dimension 3, we have the implication $DE \implies \pi^\infty_1 = 0$, and it is well-known that $\pi^\infty_1 Wh^3 \neq 0$.]

Secondly, concerning the point (3-3) above, consider a non-degenerate simplicial map $g : A \to B$, like, for instance, our map $f$ from (3); for any such a map we define the set of mortal singularities, $Sing(g) \subset A$, as being the set of those points $x \in A$, at which $g$ fails to be immersive. There are two interesting equivalence relations on $A$, in this context. To begin with, we have the trivial one

$$\Phi(g) \subset A \times A, \text{ where } (x, y) \in \Phi(g) \iff gx = gy.$$  

Then (and see here [29, 37] for more details) there is the following more subtle equivalence relation $\Psi(g) \subset \Phi(g)$, which is defined as follows (and it can be proved that this definition makes sense, see [29]): $\Psi(g) \subset A \times A$ is the “smallest” equivalence relation compatible with $g$, which kills all the mortal singularities, i.e. which is such that in the following diagram the map $g_1$ is an immersion (i.e. $Sing(g_1) = \emptyset$)

$$
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\downarrow{\pi} & \searrow{g_1} \\
A/\Psi(g). & & \\
\end{array}
$$

It can be shown that there is a UNIQUELY well-defined equivalence relation $\Psi(g)$ (constructed via folding maps) with the properties listed above, and that it has the additional property that the following induced map is SURJECTIVE

$$\pi_1(A) \xrightarrow{(g_1)_*} \pi_1(A/\Psi(g)).$$

Details concerning the equivalence relations $\Psi$ and $\Phi$ can be found in [29, 37].

So, we have finally completed all the explanations concerning our definition of REPRESENTATIONS for finitely presented groups $\Gamma$. 
Remark 2.1. Notice that, it can be actually shown that, for any such a $\Gamma$, representations as above always exist \[37\]; but, the simplest representations which one stumbles upon fail, generally speaking, to be locally finite. As already said earlier, many other objects can be represented, not only groups $\Gamma \left( \simeq \tilde{M}(\Gamma) \right)$, provided they are simply connected. The definition is always the same, but what is special when we represent groups, which comes automatically with the canonical action $\Gamma \times \tilde{M}(\Gamma) \rightarrow \tilde{M}(\Gamma)$, is that there is then the possibility that the representation $X \xrightarrow{f} \tilde{M}(\Gamma)$ may be equivariant, meaning that there may be a second free action $\Gamma \times X \rightarrow X$, coming with $f(\gamma x) = \gamma f(x)$ for all $\gamma \in \Gamma$, $x \in X$. It is a non-trivial fact \[37 \ 38 \ 39\] that such equivariant representations exist for all $\Gamma$’s, and this is one of the key ingredients in the proof that any $\Gamma$ is qsf. One of the first steps in the proof that all $\Gamma$’s are qsf, is actually to show that for any $\Gamma$ one can construct an equivariant representation, the representation space $X$ of which is locally finite. This is certainly not a trivial step.

In this paper only representations of $\Gamma$ with locally finite $X$ will be considered. But unless it is specially mentioned, equivariance will not be required now.

Without any additional assumption on the couple $(X, f)$ from (3) above, there is a metric structure, well-defined up to quasi-isometry, which permeates this whole story. Chose any Riemannian metric on $M(\Gamma)$, and what we mean by this is the following. On each individual 3-dimensional handle $H^\lambda_i$ of $M(\Gamma)$, a Riemannian metric is given and, whenever two handles are incident, it is required that the induced metrics on the intersection should coincide. Then, using the non trivial free group action $\Gamma \times \tilde{M}(\Gamma) \rightarrow \tilde{M}(\Gamma)$, the arbitrarily chosen Riemannian metric on $M(\Gamma)$ lifts to an equivariant metric on $\tilde{M}(\Gamma)$. Finally, one lifts this metric on $X$, via the non-degenerate map $X \xrightarrow{f} \tilde{M}(\Gamma)$. Thus, $X$ becomes a metric space and, up to quasi-isometry, this metric on $X$ is canonical, i.e. independent of the original choice of Riemannian metric on $M(\Gamma)$.

Let us fix now a compact fundamental domain $\delta \subset \tilde{M}(\Gamma)$, such that

$$\tilde{M}(\Gamma) = \bigcup_{\gamma \in \Gamma} \gamma \delta.$$ 

In a similar vein, we consider “large fundamental domains”, and a locally finite decomposition of $X$ into such domains

$$X = \bigcup_{j \in J} \Delta_j,$$

where $J$ is some countable set of indices. Since there is no group action on $X$ (in the general case, at least), what we will ask now from the compact pieces $\Delta_j$ above, apart from the obvious condition that their interiors should be disjoined, is the existence of two positive constants $C_2 > C_1 > 0$ such that we should have

$$(3 - 5) \quad C_1 \leq \|\Delta_j\| \leq C_2, \; \forall \; j \in J.$$ 

Here $\|\Delta_j\|$ is the diameter of $\Delta_j$. Our large fundamental domains $\Delta_j$ could be, for instance, maximal dimensional cells of the cell-decomposition of the representation space $X$ occurring in (3), satisfying the metric condition $(3 - 5)$, when $j \rightarrow \infty$. The next Theorem \[4\] stated below, has two parts corresponding to the dimension of $X$, in a 3-dimensional representation this
is $X = X^3$, while in a 2-dimensional representation it is $X = X^2$. In both cases we have also immortal singularities $\text{Sing}(\hat{M}(\Gamma)) \subset \hat{M}(\Gamma)$ and mortal singularities $\text{Sing}(f) \subset X$.

At least in the 2d case, we will want to be a bit more specific about the singularity issues, and so, when it comes to the 2-dimensional part of the Theorems 4 and 5 stated below, the following condition will be imposed too

(3 - 6) the set of mortal singularities $\text{Sing}(f) \subset X^2$ is discrete and,

at each $x \in \text{Sing}(f)$, there is the following local model. There is an open neighborhood $P = P_1 \cup P_2$ of $x$ in $X^2$ and an embedding $\mathbb{R}^3 \rightarrow \hat{M}(\Gamma)$ (which, a priori, might happily go through $\text{Sing} \hat{M}(\Gamma)$), through which $P \xrightarrow{f} \hat{M}(\Gamma)$ factorizes. At the source $X^2$, the $P_1, P_2$ are two planes $\mathbb{R}^2$ glued along a half-line $[0, \infty)$ with $x = 0$, $x$ being here our mortal singularity.

In the diagram below

\begin{center}
\begin{tikzpicture}
  \node (M) {$\hat{M}(\Gamma)$};
  \node (P) [above left of=M] {$P$};
  \node (R) [above of=M, xshift=2cm] {$\mathbb{R}^3$};
  \draw[->] (P) to node [above] {$f$} (M);
  \draw[->] (P) to node [left] {$j$} (R);
\end{tikzpicture}
\end{center}

each $j|P_1, j|P_2$ injects, the two being transverse. So there is a double line in $M_2(f)$ starting at the mortal singularity $x$. This is a local model already used by the second author (V.P.) in his work on the Poincaré Conjecture [32, 35]. According to a suggestion of Barry Mazur, these singularities were called “undrawable” in [32].

For our (3) we will also assume that

(3 - 7) $f(\text{Sing}(f)) \cap \text{Sing}(\hat{M}(\Gamma)) = \emptyset$.

But at the later stages in the zipping of $f$, this condition may be violated. Then, besides the $\text{Sing}(f) \subset X^2$, there is also a set of immortal singularities $\text{Sing}(X^2) \subset X^2 - \text{Sing}(f)$, which is also discrete. This comes with the inclusion $f(\text{Sing}(X^2)) \subset \text{Sing}(\hat{M}(\Gamma))$. At the points $x \in \text{Sing}(X^2)$, there are no local factorizations

\begin{center}
\begin{tikzpicture}
  \node (M) {$\hat{M}(\Gamma)$};
  \node (V) [below left of=M] {$\mathbb{R}^3$};
  \node (X) [below of=M] {$X^2 \supset U$};
  \draw[->] (V) to node [left] {$f$} (X);
  \draw[->] (X) to node [below] {$f$} (M);
\end{tikzpicture}
\end{center}

and it is their absence which makes the $x \in \text{Sing}(X^2)$ be an immortal singularity, never to be killed by the zipping. But in purely topological terms, and forgetting about $f$, at one immortal singularity $x \in X^2$, the $X^2$ looks exactly alike as at a mortal singularity. This ends our digression on $\text{Sing}(f)$.

We are now ready to state precisely the two main results of the present paper.

**Theorem 4.**

1. (3d-part) For any (finitely presented group) $\Gamma$ there exists a locally finite 3-dimensional representation $X^3 \xrightarrow{f} \hat{M}(\Gamma)$ and also a function $\mathbb{Z}^+ \xrightarrow{\mu} \mathbb{Z}^+$, such that the following condition is satisfied for any $\gamma \in \Gamma$:

(4) $\#\{\Delta_i, \text{s.t. } f(\Delta_i) \cap \gamma \delta \neq \emptyset\} < \mu(\|\gamma\|)$,
where $\|\gamma\|$ is the word-length of $\gamma \in \Gamma$.

In particular, any given domain $\gamma \delta \subset \bar{M}(\Gamma)$ downstairs, can only be hit finitely many times by the image of a large domain $\Delta \subset X^3$ from upstairs.

2. (2$^d$-part) For any $\Gamma$ there exists a 2$^d$-representation $X^2 \xrightarrow{f} \bar{M}(\Gamma)$ such that

\begin{equation}
\text{both } \text{Im}(f) = fX^2 \subset \bar{M}(\Gamma) \text{ and also } M_2(f) \subset X^2 \text{ are closed subsets.}
\end{equation}

For a generic 3$^d$-representation $X^3 \xrightarrow{f} \bar{M}(\Gamma)$, one normally finds the following situation, at the opposite pole with respect to our (4) above, and which, in papers like [34], the second author (V.P.) has called the Whitehead nightmare

\begin{equation}
\# \{ \Delta_i, \text{ s.t. } f(\Delta_i) \cap \gamma \delta \neq \emptyset \} = \infty, \forall \gamma \in \Gamma.
\end{equation}

Our present Whitehead nightmare under discussion, should remind the reader of the basic structure of the classical Whitehead manifold $Wh^3$ [51] (whence the name of our nightmare), of the Casson Handle [15], or of the gropes of M. Freedman and F. Quinn [7].

So, the first part of our Theorem 4 means that any finitely presented group can avoid the Whitehead nightmare, and this is what the title of the present paper refers to.

The 2$^d$ counterpart of the Whitehead nightmare (4$^*$) is the following condition

\begin{equation}
M_2(f) \subset X^2 \text{ is not closed.}
\end{equation}

This is the generic situation for 2$^d$-representations and one has to start by living with it and look at the accumulation pattern of $M_2(f)$ inside $X^2$, all this being done in [37, 38, 39], before one can, eventually, avoid both (4$^*$) and (5$^*$).

For the next theorem we will need the notion of WGSC, already mentioned in the introduction, but which we restate now more formally. The WGSC (weak geometric simple connectivity) is a weakening of the GSC, introduced by L. Funar in [8, 9] (see also [10, 20]).

**Definition 7.** A locally compact space $X$ (let us say a locally finite $X$ in the context of (3)) is said to be weakly geometrically simply connected (WGSC), if it has an exhaustion by compact (finite), simply connected subcomplexes $K_1 \subset K_2 \subset \ldots \subset X$.

**Remark 2.2.** Notice that for an open 3-manifold $V^3$, $\pi_1^\infty V^3 = 0$ and $V^3 \in \text{WGSC}$ are equivalent, but this little fact is immaterial for us now. Apart from the fact that our group-presentations are 3-dimensional, the present paper has not much to do with 3-manifolds.

When in the context of (3−1) we replace GSC by WGSC, without any other change, then by definition the (3) becomes a WGSC-representation.

**Theorem 5.**

1. (3$^d$-part) For any $\Gamma$ there exists a locally finite 3$^d$ WGSC-representation $X^3 \xrightarrow{f} \bar{M}(\Gamma)$ satisfying the following conditions:

\begin{equation}
\text{there is a free action } \Gamma \times X^3 \longrightarrow X^3, \text{ and } f \text{ is equivariant;}
\end{equation}

\begin{equation}
\text{there is a constant } C = C(\delta) > 0 \text{ s.t. } \forall \gamma \in \Gamma \text{ one has: } \# \{ \Delta_i \text{ s.t. } f\Delta_i \cap \gamma \delta \} < C.
\end{equation}
2. (2d-part) For any $\Gamma$ there exists a locally finite 2d WGSC-representation $X^2 \rightarrow \tilde{M}(\Gamma)$ which is both equivariant, like in (6), and which also satisfies (5).

Remark 2.3.

1. The representation spaces $X$ occurring in the two theorems above are, of course, distinct spaces, although not quite totally unrelated, as we shall see.

2. Condition (7) of Theorem 5 can also be replaced by the following variant: there exist equivariant triangulations for $\tilde{M}(\Gamma)$ and for $X^3$, and also a constant $C'$ such that, for any simplex $\sigma \subset \tilde{M}(\Gamma)$, we should have

   (7 − bis) $\# \{\text{simplices } S \subset X^3, \text{ s.t. } fS \cap \sigma \neq \emptyset\} < C'$.

3. We believe that one cannot both avoid the Whitehead nightmare and retain equivariance, without paying the price of going from gsc to wgsc. But we have no proof for this conjecture.

3 A prentice on $\pi_1^\infty \tilde{M}^3 = 0$

Immediately next to the Poincaré Conjecture, the statement that for all closed $M^3$'s one has $\pi_1^\infty \tilde{M}^3 = 0$, or equivalently that for an irreducible $M^3$ one has $\tilde{M}^3 = \mathbb{R}^3$, has been a highly desired result in low-dimensional topology. Of course, today, once G. Perelman [26, 27, 28] has proved the full Thurston Geometrization Conjecture [50] (for detailed proofs see [1, 2, 16, 18]), this $\pi_1^\infty \tilde{M}^3 = 0$ is just a corollary of that work. But the $\pi_1^\infty \tilde{M}^3 = 0$ also follows easily from the second author’s (V.P.) result that all $\Gamma$’s are qsf [36].

In the introduction to this paper, we have shown how Casson’s and the second author’s (V.P.) efforts to prove $\pi_1^\infty \tilde{M}^3 = 0$, some twenty or so years ago, fit into the more contemporary framework. As a very last glance to those, by now, old issues, and maybe also as a last good-bye to 3-manifolds, we will give now a simple geometric argument, stemming essentially from [37], proving the following implication:

Corollary 1.

\{ Point 1. in Theorem 7 \} $\implies \pi_1^\infty \tilde{M}^3 = 0$.

We believe this little prentice might be useful for some readers.

Proof. To begin with, as a general comment, when $\Gamma = \pi_1 M^3$, then our whole present theory, as well as the 3-parts work [37, 38, 39], functions automatically with $M^3(\Gamma) = M^3$, $\tilde{M}^3(\Gamma) = M^3$.

So, accordingly to Theorem 3 above, we have now a locally finite representation

(8 − 1) $X^3 \rightarrow \tilde{M}^3 = \bigcup_{\gamma \in \pi_1 M^3} \gamma \delta$, such that

(**) for any $\gamma \delta \in \tilde{M}^3$, there are only finitely many large fundamental domains $\Delta \subset X^3$ such that $f \Delta \cap \gamma \delta \neq \emptyset$. 

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Since $X^3$ is $\text{gsc}$ we also have $X^3 \in \text{wgsc}$, and hence there is an exhaustion by finite, simply connected subcomplexes $Z^3_0 \subset Z^3_1 \subset \ldots \subset X^3$.

We will use the notation $\Phi_n \equiv \Phi(f|Z^3_n) \supset \Psi_n \equiv \Psi(f|Z^3_n)$. Using the compact exhaustion above, for any $k \subset \tilde{M}^3$ compact, we find a $Z^3_m$ such that $fZ^3_m \supset k$. Next, using now (**) we may find a higher $Z^3_n$, with $n > m$, such that $f^{-1}fZ^3_m \subset Z^3_n$.

Our $(8-1)$ being a representation, we also have $\Psi(f) = \Phi(f)$. Via a little compactness argument (see [30]), one deduces from this the existence of a function $\mathbb{Z}_+ \ni n \mapsto N(n) \in \mathbb{Z}_+$ with $N(n) >> n$, having the property that

$$\Psi_N|Z^3_n = \Phi_n.$$  

Remark 3.1. It would be nice if one could connect the asymptotic behavior of the function $N$ above, to the more mundane asymptotic properties of $\Gamma = \pi_1 M^3$.

By now we have a sequence of inclusions

$$k \subset Z^3_m|\Phi_m \subset Z^3_n|\Phi_n \subset Z^3_N|\Psi_N,$$

from which one extracts the following commutative diagram

\[
\begin{array}{ccc}
Z^3_N|\Psi_N & \xrightarrow{g} & \tilde{M}^3 \\
\downarrow j & & \downarrow i \\
Z^3_m|\Phi_m & \xrightarrow{i} & \tilde{M}^3 \\
\end{array}
\]

where $i$ is the canonical inclusion, $j$ some other inclusion, $\pi_1(Z^3_N|\Psi_N) = 0$ and $g$ is an immersion. Moreover, because of $f^{-1}fZ^3_m \subset Z^3_n$, we also have the following Dehn-type property: $Z^3_N|\Psi_N \supset j(k) \cap M_2(g) = \emptyset$.

What we have actually achieved, so far, was to show that $\tilde{M}^3$ is Dehn-exhaustible. If one plugs in here the Dehn-type lemma from [30], one can deduce that $\pi_1^\infty M^3 = 0$. This ends the proof of Corollary 4.

This also ends our prentice on the simple connectivity at infinity of $\tilde{M}^3$.

4 Preliminaries lemmas

We give now the beginning of the proofs of the Theorems 4 and 5 above. Some technicalities will be postponed until the next section. Our arguments will rely heavily on the second author’s (V.P.) previously proved result already mentioned, that all finitely presented groups are $\text{qsf}$ (see [36, 37, 38, 39]). Once we know that $\Gamma \in \text{qsf}$, this also means that $\tilde{M}(\Gamma) \in \text{qsf}$ (because $M(\Gamma)$ is a compact presentation of $\Gamma$). Since our 3-dimensional $\tilde{M}(\Gamma)$ is singular, we prefer to replace it by a smooth, albeit higher dimensional, object.

Let $\mathcal{R}$ be a resolution of the singularities of $M(\Gamma)$ (and see here [32], or better, our recent joined work [22], where all this issue is explained in a context which is very much akin to the present one). Given a choice of $\mathcal{R}$, we get a smooth 4-manifold $\Theta^4(M(\Gamma), \mathcal{R})$, and, as soon as one takes the product with $B^m$, $m \geq 1$, and one goes to $\Theta^4(M(\Gamma), \mathcal{R}) \times B^m$, then the $\mathcal{R}$-dependence is washed away, and everything becomes then canonical.
In particular, there is now a free action of $\Gamma$ on $\Theta^4(\tilde{M}(\Gamma), R) \times B^m$, for $m \geq 1$, and one has that
\[
(\Theta^4(\tilde{M}(\Gamma), R) \times B^m)/\Gamma = \Theta^4(M(\Gamma), R) \times B^m.
\]
We take now $n = m + 4 \geq 5$, and then we get
\[
(9) \quad M^n = \Theta^4(\tilde{M}(\Gamma), R) \times B^{n-4} = (\Theta^4(M(\Gamma), R) \times B^{n-4})^\sim.
\]
This $M^n$ is a smooth non-compact manifold, of very large boundary. Also, because $\Gamma \in \text{qsf}$, we also have $M^n \in \text{qsf}$.

**Lemma 1.** If $N >> n$, then $W^p \equiv M^n \times B^N$, for $p = n + N$, is $\text{wgsc}$.

**Proof.** This is actually a result from the first author’s (D.O.) PhD Thesis (see [10]), but for completeness, we prove it here again, in the present set-up.

We start with an exhaustion by compact submanifolds of codimension zero, now without any $\pi_1 = 0$ assumption,
\[
k_0 \subset k_1 \subset \cdots \subset M^n = \bigcup_i k_i.
\]
Thus, because $\partial M^n \neq \emptyset$, generally speaking, we find that $\partial k_i \cap \partial M^n \neq \emptyset$. Since, by (9), $M^n$ is $\text{qsf}$, there is a commutative diagram
\[
(10) \quad k_0 \subset i \partial k_0 \subset j_0 k_0 \subset K_0 \subset f_0 M^n
\]
where $i$ is the canonical inclusion, $j_0$ is an inclusion, $K_0$ and $f_0$ are simplicial, $K_0$ is compact with $\pi_1 K_0 = 0$ and where $j_0(k_0) \cap M_2(f_0) = \emptyset$.

Without any loss of generality, we may assume that the injection $j_0$ extends to
\[
k_0 \subset k_0 \cup (\partial k_0 \times [0, 1]) \subset K_0,
\]
where $k_0 \cup (\partial k_0 \times [0, 1]) \subset K_0$ is an open subset, and where the Dehn-condition extends to $k_0 \cup (\partial k_0 \times [0, 1])$.

We claim now that, for high enough $N$ (actually $N > n + 2$ suffices), we can construct a commutative diagram with injective $\Phi$ as follows
\[
(11) \quad K_0 \setminus j_0(k_0) \xrightarrow{f_0(K_0 \setminus j_0 k_0)} M^n \setminus k_0 = (M^n \setminus k_0) \times (\ast_N)
\]
\[
\downarrow \Phi \quad \downarrow \Phi
\]
\[
(\ast_N) \subset (M^n \setminus k_0) \times B^N,
\]
(where $\ast_N$ denotes the center of $B^N$) such that inside $(M^n \setminus k_0) \times B^N$, the diagram (11) above commutes modulo a deformation not budging the $\partial k_0 \times [0, 1] \subset K_0 \setminus j_0 k_0$, where the $f_0$ in formula (11) is already injective. There is here, of course, a standard argument, but some care is required at the boundary. Here is how it goes.
Consider $\partial(M^n - k_0)$, which is the union of a contribution from $\partial M^n - k_0$ and another contribution from $\partial k_0 - \partial M^n$. Have in mind here that $(k_0, \partial k_0) \subset (M^n, \partial M^n)$. Next, we also consider

$$\partial[(M^n - k_0) \times B^N] = \partial(M^n - k_0) \times B^N + \{\text{other additional terms}\}.$$ 

With this, we go to the composite map

$$(11 - 1) \quad [(K_0 - j_0 k_0) \cap \partial(M^n - k_0)] \xrightarrow{f_0} \partial(M^n - k_0) \times (*_N) \subset \partial(M^n - k_0) \times B^N.$$ 

In the simplicial context of (10), it may be happily assumed that $K_0$ is an $n$-manifold, like $M^n$. With this, as soon as $2n + 1 \leq N + n - 1$, we may demolish the double points of (11-1) by a standard argument. So, for (11), at the level of the boundary, we have already an embedding and, rel this, we may continue in the interior; from now on, things are really standard stuff.

The offshoot of the story above, is the existence of a compact bounded submanifold

$$X_0^{n+N} = N^{n+N} (K_0 - j_0 k_0) \bigcup_{\partial k_0 \times B^N} k_0 \times B^N \subset M^n \times B^N,$$

which is such that $\pi_1(X_0) = 0$.

We can find a $k_{n_1}$ in our original compact exhaustion which is such that $X_0^{n+N} \subset k_{n_1} \times B^N$, and then, like in (10), go to

$$k_{n_1} \xrightarrow{j_{n_1}} K_1, \quad \text{with } \pi_1 K_1 = 0.$$

From here on, just like before, we can construct a simply-connected manifold $X_1^{n+N}$ having the feature that $k_{n_1} \times B^N \subset X_1^{n+N} \subset M^n \times B^N$, a.s.o. This proves Lemma 1. □

**Lemma 2.** The manifold $W^p = M^n \times B^N$ is actually GSC.

**Proof.** We will proceed via the standard Smale-type arguments, but with a certain amount of additional care in view of our present non-compact context with non-empty large boundary.

What Lemma 1 tells us is that there is an exhaustion by compact, simply-connected, codimension zero submanifolds, each embedded in the interior of the next

$$(12) \quad K_1 \subset K_2 \subset \cdots \subset W^p = \bigcup_{i=1}^\infty K_i.$$ 

In what follows we will use the notation $\delta K_i \equiv \partial K_i - \partial W$. The $\delta K_i$ are disjoined and, for the sake of a simpler exposition, we will assume the following

$$(13) \quad \text{for each } i, \text{ both } \delta K_i \text{ and } (K_{i+1} - K_i) \text{ are connected.}$$

Nota Bene: We could certainly do without (13), at the price of some complications, essentially notational. But then, we could also do something else: start by assuming that the group $\Gamma$ has
exactly one end, which would make (13) essentially automatic. Then one notices that proving our theorems for one-ended groups suffices, this implies easily the general case.

From now on, we will constantly play the following game, back and forth:

\[
\begin{array}{c|c}
\text{Context I} & \text{Context II} \\
\hline
\end{array}
\]

\[
\{\text{smooth triangulations of } W^p\} \equiv \{\text{smooth handlebody decompositions of } W^p\}
\]

Here the arrow I \(\Rightarrow\) II is the bijection \{\lambda-dimensional simplexes\} \(\mapsto\) \{handles of index \(\lambda\)\}, which is gsc preserving. While the arrow II \(\Rightarrow\) I is supposed to be any “nice” gsc-preserving subdivision. This could be for instance the following succession of steps, which is also gsc-preserving.

Read II as a cell-decomposition, then apply to it a Siebenmann bisection (as defined in [16]), change this into a triangulation via a stellar subdivision, then, if necessary, subdivide barycentrically so as to reduce the size of the basic units, or any other appropriate combination of these kind of steps. The elementary steps just mentioned are not only gsc-preserving, but they are also wgsc-preserving. We will call them admissible subdivisions.

In the context of (14), when we are in the situation I, the \(\delta K_i\)’s are subcomplexes of codimension one, while in the case II, they are non-singular level hypersurfaces.

Whether we are in the context I or II of (14), the gsc property is always expressed by the following scheme

\[
W = T + (\lambda = 1) + (\lambda = 2)_1 + (\lambda \geq 2)_2
\]

where \(T\) is an infinite tree and where the two sets of \(\lambda\)-cells, or \(\lambda\)-handles, \((\lambda = 1)\) and \((\lambda = 2)_1\), are in bijection, and have a geometric intersection matrix of the form \(\text{id} + \text{nil}\), of the easy type (as in Definition 6).

When we are in the smooth context, then the \(T\) in formula (15) should be read \(N^p(T) \subset W^p\), for a properly embedded tree \(T \subset W^p\).

Here is how the \(T\) in (15) fares in the context (14)

\[
\text{I } \Rightarrow \text{ II } \Rightarrow \text{ I}, \quad \text{i.e.}
\]

\(T \Rightarrow \text{thickening } N^p(T)\) of the same \(T \Rightarrow \text{extension of } T\) to a (possibly much) denser new \(T\).

We start now with a smooth equivariant triangulation for \(W^p\) (in the context I), out of which a smooth equivariant handlebody decomposition \(H(0)\) is to be gotten. The corresponding triangulation from which we started might be called \(H(0)\) too. Of course, the compact exhaustion (12) is not equivariant and neither is (15); this would imply the existence of a free action \(\Gamma \times T \rightarrow T\), forcing our \(\Gamma\) to be a free group.

But our \(H(0)\) is only wgsc and not (yet) gsc, so that (15) is not yet with us (anyway). What we will ask from \(H(0)\) right now is the following (and we are here in the context I)

\[
W = T \text{ (tree) } + (\lambda \geq 1) \text{ (meaning cells of dimensions } \lambda \geq 1), \quad \text{such that}
\]

for each \(i\), each of the following graphs in (16–1) below is a maximal tree inside the respective 1-skeleton of the compact complexes \(K_1, \delta K_i\) or \((K_{i+1} - (K_i - \delta K_i))\)

\[
T \cap K_1 \subset K_1, \ T \cap \delta K_i \subset \delta K_i, \ T \cap (K_{i+1} - (K_i - \delta K_i)) \subset K_{i+1} - (K_i - \delta K_i).
\]
Furthermore, we also ask that the compact object $\overline{T}\cap (K_{i+1} - K_i)$ be a tree of foot $T\cap (K_{i+1} - K_i)\cap (T\cap \delta K_i) = \{\text{a single point}\}$. This ends (16).

Here is how we can achieve (16). Start with the trivial remark that if $X$ is any graph and $T \subset X$ a tree, then $T$ can always be extended to a maximal tree. With this, we perform the following infinite cascade of successive steps.

Choose a maximal tree $T \cap \delta K_1$, continue this with a maximal tree $T \cap K_1$, then choose a maximal tree $T \cap \delta K_2$ and join it to $T \cap \delta K_1$ by a simple path $\lambda \subset K_2 - K_1$, then continue $(T\cap \delta K_1) \cup \lambda \cup (T\cap \delta K_2)$ to a maximal tree $T\cap (K_2 - K_1)$, a.s.o. It should also be understood that the $H(0)$, in terms of which both (12) and (16) are written, is nice, in the sense that any handle $H_i^\lambda \subset K_i - (K_{i-1} - \delta K_{i-1})$ has its attaching zone glued to

$$T \cap (K_i - (K_{i-1} - \delta K_{i-1})) + \sum \{\text{the handles } H_i^{\mu<\lambda} \text{ in } K_i - (K_{i-1} - \delta K_{i-1})\}.$$

Once we have all these things, we will proceed via the following infinite sequence of steps.

**Step I.** We think now in the context II, whenever the contrary is not explicitly stated. With this, we have

$$H(0)|K_1 = K_1 \cap T + \sum H_1^1 + \sum H_1^2 + K_1^{\text{residual}}(\lambda \geq 3),$$

where $H_1^\lambda$ are the $\lambda$-handles of $K_1$, with their $\partial H_1^\lambda$ attached to $K_1 \cap T + H_1^{\mu<\lambda}$. We introduce the notation

$$L_1 \equiv K_1 \cap T + \sum H_1^1 + \sum H_1^2,$$

and one should notice that, because $\pi_1 K_1 = 0$, we also have

$$\pi_1 L_1 = 0,$$

and, moreover, since $\dim L_1 \gg \{\text{the indices } \lambda \leq 2 \text{ of the handles involved in } L_1\}$, we also have $\pi_1 \partial L_1 = 0$.

At this point we throw in additional handles, namely pairs of 2-handles and 3-handles in canceling position $h_1^2 + h_1^3$, which we call Smale pairs, coming with an abstract bijection

$$(*) \quad \{\text{Smale pairs } (h_1^2 + h_1^3)\} \longleftrightarrow \{\text{1-handles } H_1^1\}.$$  

The Smale pairs are attached to $\partial (K_1 \cap T) - \sum (H_1^1 + H_1^2) \subset \partial L_1$, far from $\partial W$, i.e. deep inside the interior of $K_1$, just between $K_1 \cap T$ and the $K_1^{\text{residual}}$. Since $\pi_1 \partial L_1 = 0$, one may slide the system $h_1^2$ along $\partial L_1$, until one has the following condition satisfied:

We start by drawing on $\partial L_1$, for each individual $H_1^1$, some simple closed loop $l(H_1^1) \subset \partial (K_1 \cap T + H_1^1)$ such that $l(H_1^1)$ meets $\delta H_1^1 \equiv \{\text{the lateral surface of the 1-handle } H_1^1\}$ exactly once; if $h_1^2 = h_1^2(H_1^1)$ corresponds to our $H_1^1$ via the isomorphism $(*)$ above, then at the end of the sliding move we should find

$$\{\text{attaching curve of the slided 2-handle } h_1^2(H_1^1)\} = l(H_1^1),$$

with the following global result for the geometric intersection matrix

$$\partial h_i^2(\text{slided}) \cdot \delta H_j^1 = \delta_{ij},$$

17
i.e. the handles $h_2$(slided) are now in canceling position with the $H^1$'s. But notice that our handles sliding is brushing through $\partial W^p \cap K_1$ too and this forces the manner in which we will proceed next.

In the context of (18), it should be understood that the various $(H^1)$'s are disjoined and that the sliding isotopy avoids the foot of $T \cap (K_2 - K_1)$, where $T \cap (K_2 - K_1)$ rests on $T \cap \delta K_1 \subset T \cap K_1$. Also, our sliding of $h_2^2$ drags along $h_3^1$, producing, apriori, contacts

\[(18-1) \quad \partial H^3_2 \cap \delta h^3$(slided), $\lambda = 2$ or $3$.\]

These non-generic contacts can easily be disposed of without disturbing the story below, and leaving us with a nice $H(0)_1$. With the sliding above we get a smooth transformation

\[(19) \quad L_1 \xrightarrow{D_1} L_1 + (h_1^2 + h_3^2)(slided).\]

where $D_1$ is a diffeomorphism from $L_1$ to the RHS of (19).

If one simply replaces the $L_1 \subset H(0)|K_1$ by the RHS of (19), then one gets a new $K_1$, with a handlebody decomposition $H(0)_1$, and which is diffeomorphic to the initial $K_1$,

\[(19-1) \quad K_1 \xrightarrow{D_3} \text{new } K_1.\]

We have then

\[K_2 - (K_1 - \delta K_1) \supset \delta K_1 \xrightarrow{D_1^{-1}} \text{new } K_1,\]

allowing us to define the smooth manifold $W^p_1$, which is diffeomorphic to our original $W^p$,

\[(K_2 - (K_1 - \delta K_1)) \cup_{D_1(\delta K_1)} \text{new } K_1 \subset W^p_1 \equiv (W - (K_1 - \delta K_1)) \cup_{D_1(\delta K_1)} \text{new } K_1,\]

and the disposal of the non-generic (18-1) mentioned above, does not affect the topologies.

We find, canonically, the embedding $T \cap \delta K_1 \subset D_1(\delta K_1) \subset \text{new } K_1$ but, when we put together the handlebody decomposition $H(0)|_1$ new $K_1$ with $H(0)|_1(W^p - (K_1 - \delta K_1))$, we get a handlebody decomposition for $W^p_1$ which is not necessarily nice, because of things of type (18-1). Nevertheless, we can render it nice without changing neither $H(0)|_1$ new $K_1$, nor the relevant topologies (of $W^p_1$ for instance) and the new handlebody decomposition for $W^p_1$, which is now nice, is denoted again $H(0)_1$.

Notice that, at the end of our Step I we have also created the manifold

\[(20) \quad \Lambda_1 \equiv T \cap K_1 + \sum H^1_1 + \sum h_1^2$(slided),\]

which comes with $\pi_1 \Lambda_1 = \pi_1 \partial \Lambda_1 = 0$.

Step II. We work now with the $W^p_1$ from Step I above and (12) is then modified into

\[\text{new } K_1 \subset (K_2 - (K_1 - \delta K_1)) \cup \text{new } K_1 \subset (K_3 - (K_1 - \delta K_1)) \cup \text{new } K_1 \subset \ldots \subset W^p_1.\]

Then one starts from

\[H(0)_1|(K_2 - (K_1 - \delta K_1)) = T \cap (K_2 - (K_1 - \delta K_1)) + \sum H^1_2 + \sum H^2_2 + K_{2}^{\text{residual}}(\lambda \geq 3).\]
Here we have $H^3_2 \subset (K_2 - (K_1 - \delta K_1))$. We also introduce

\begin{equation}
L_2 \equiv \left( T \cap (K_2 - (K_1 - \delta K_1)) + \sum H^1_2 + \sum H^2_2 \right) \cup_{T \cap \delta K_1} \Lambda_1,
\end{equation}

where we may certainly find contacts $\partial H^2_2 \cap \delta H^1_1$. One can notice that $\pi_1 L_2 = \pi_1 \partial L_2 = 0$. [Let us say that, when we consider the group $\pi_1 K_2 = \pi_1 \left( (K_2 - (K_1 - \delta K_1)) \cup \text{new } K_1 \right) = 0$, then a good bona fide presentation (in our sense) of it is provided by (21)].

We attach now Smale pairs $(h^2_2 + h^2_2)$ along

\[ \partial \left( T \cap (K_2 - (K_1 - \delta K_1)) \right) - \left( \sum H^1_2 + \sum H^2_2 \right) \subset \partial L_2. \]

Then, we make use of $\pi_1 \partial L_2 = 0$ in order to slide the $h^2_2$'s into new positions

\[ \partial h^2_2(\text{slided}) \subset \partial \left[ \left( T \cap (K_2 - (K_1 - \delta K_1)) \right) + \sum H^1_2 \right] \]

in canceling position with respect to the $h^2_2$'s . . .

**Step** $i + 1$. Inductively, by now, the initial piece of (12) up to level $i$ included, has already been Smale-treated and so we replace this initial (12) by another similar simply-connected compact exhaustion, which starts directly at level $i$, namely

\begin{equation}
\text{(22) new } K_i \subset (K_{i+1} - (K_i - \delta K_i)) \cup \text{new } K_i \subset (K_{i+2} - (K_i - \delta K_i)) \cup \text{new } K_i \subset \ldots \subset W_i^p \equiv \left( W^p - (K_i - \text{new } K_i) \right) \cup_{D_i(\delta K_i)} \text{new } K_i,
\end{equation}

coming with a diffeomorphism $K_i \xrightarrow{D_i} \text{new } K_i$, which also induces a bigger diffeomorphism

\[ W_i^p \xrightarrow{D_i} W_i^p. \]

The (22) also comes with a nice handlebody decomposition for $W_i^p$, denoted $H(0)_i$. Inside the new $K_i$, the 1-handles $H^{1}_{j \leq i} \subset \text{new } K_i$ are all in cancelling position with the corresponding $h^2_j$ (slided), each coming with $\partial h^2_j(\text{slided}) \subset \partial \left[ T \cap (K_j - (K_j - \delta K_j)) + H^1_j \right]$.

By analogy with (21) we have now the

\[ L_i \equiv \left( T \cap (K_i - (K_{i-1} - \delta K_{i-1})) + \sum H^1_i + \sum H^2_i \right) \cup_{T \cap \delta K_{i-1}} \Lambda_i, \]

and the isotopic sliding move

\[ h^2_{j \leq i} \xrightarrow{\text{Smale-treatment}} h^2_{j \leq i}(\text{slided}) \]

takes place inside $\partial L_{j \leq i}$, which, inductively, is simply-connected.

Next, we find the co-dimension zero submanifold

\begin{equation}
\Lambda_i \equiv T \cap (K_i - (K_{i-1} - \delta K_{i-1})) + H^1_i + h^2_i(\text{slided}) \subset \text{new } K_i,
\end{equation}

which comes with $\pi_1 \Lambda_i = 0$, because of the Smale-treatment. Like above this implies that $\pi_1 \partial \Lambda_i = 0$ too.
The $H_i^{2^+}$'s are not included in (23) and, partially, they rest on $T \cap \delta K_{i-1} + \sum H_{i-1}^1 | \delta K_{i-1}$. Similarly, the $H_i^{2^+}$ may rest partially on $H_i^2$ and not on the useful $\Lambda_i$ (which they may nevertheless touch). Because of this, in the context of (22) we introduce the following object, indistinguishable $\pi_1$-wise from $K_{i+1}$

$\text{(24)} \quad \text{mock } K_{i+1} = \left\{ (K_{i+1} - (K_1 - \delta K_i)) \cup \text{ new } K_i \right\}$,

with all the handles $H_i^{\lambda \geq 3} \subset K_{i+1} - (K_1 - \delta K_i)$ deleted (or at least with their cores deleted).

We define now

$\text{(25)} \quad L_{i+1} = \left( T \cap (K_{i+1} - (K_i - \delta K_i)) + \sum H_{i+1}^1 + \sum H_{i+1}^2 \right) \cup \Lambda_i,$

where we certainly also have contacts $\partial H_i^{2^+} \cap \delta H_i^1$. We have a decomposition

$\text{(26)} \quad \text{mock } K_{i+1} = (\text{new } K_i) \cup L_{i+1},$

which comes with $(\text{new } K_i) \cap L_{i+1} = \Lambda_i$.

In the context of (26), the $H_i^{1+1}$ is attached to $T \cap (K_{i+1} - (K_i - \delta K_i))$, which itself rests with its foot on $\Lambda_i$, while $H_i^{2^+}$ is attached to $T \cap (K_{i+1} - (K_i - \delta K_i)) + \sum H_{i+1}^1 + \Lambda_i$. We have

$$\pi_1(\text{mock } K_{i+1}) = \pi_1(\text{new } K_i) = \pi_1 \Lambda_i = 0$$

and so, by Van Kampen, $\pi_1 L_{i+1} = 0$, and hence $\pi_1 \partial L_{i+1} = 0$ too.

At this point we throw in new Smale pairs $(h_i^{2^+} + h_i^{3^+})$, in abstract bijection with the set of 1-handles $H_i^{1+1}$. They are added along

$$\partial\left( T \cap (K_{i+1} - (K_i - \delta K_i)) - \sum (H_{i+1}^1 + H_{i+1}^2) \right) \subset \partial L_{i+1},$$

deep inside the interior of $(K_{i+1} - (K_i - \delta K_i))$. Making use of $\pi_1 \partial L_{i+1} = 0$, we can slide each individual $h_i^{2^+}$ into a new position, such that

$$\partial h_i^{2^+}(\text{slided}) \subset \partial \left[ \left( T \cap (K_{i+1} - (K_i - \delta K_i)) \right) + H_i^{1+1} \right],$$

in canceling position with its canonically attached $H_i^{1+1}$.

This process sweeps through parts of $\partial W_i^p$ and drags $h_i^{3^+}$ along. So, we have now a transformation

$\text{(27)} \quad L_{i+1} \Rightarrow L_{i+1} + (h_i^{2^+} + h_i^{3^+})(\text{slided}),$

(where $D_{i+1}$ is a diffeomorphism of the LHS to the RHS) and, if we replace $L_{i+1} \subset K_{i+1} - K_i + \text{new } K_i$ by the RHS of (27), we get a new $K_{i+1}$, with a handlebody decomposition $H(0)_{i+1}$ which comes with a diffeomorphism

$\text{(27 - 1)} \quad K_{i+1} \sim D_{i+1} \rightarrow \text{new } K_{i+1}.$
With \((K_{i+1}, D_{i+1})\) we can start a new sequence like (22), this time ready for the next level \(i + 2\). This process continues now indefinitely.

Forgetting now temporarily about the \(H^{\lambda \geq 3}\)'s, notice that our process also builds up an infinite sequence of non-compact manifolds

\[(28) \quad T + H^1 + H^2 + (h_i^2 + h_i^3)(slided) \subset \ldots \subset T + H^1 + H^2 + \sum_{j=1}^{i} (h_j^2 + h_j^3)(slided) \subset \ldots \subset T + H^1 + H^2 + (h^2 + h^3)(slided),\]

where the last object is a manifold where all the 1-handles cancel with the 2-handles.

Very importantly, in the context of (28), the various finite packages \((h_n^2 + h_n^3)(slided)\) do not accumulate at finite distance, which we express symbolically by

\[
\lim_{n=\infty} (h_n^2 + h_n^3)(slided) = \infty.
\]

With this comes then a diffeomorphism

\[(28-1) \quad T + H^1 + H^2 (= \text{the original 2-skeleton of } W^p) \longrightarrow T + H^1 + H^2 + (h^2 + h^3)(slided).\]

We can replace now the \(T + \sum H^1 + \sum H^2 \subset W^p\), by the diffeomorphic model which is provided by the RHS of the formula (28-1); thereby we get a new diffeomorphic model for \(W^p\) itself, which we call \(W^p_\infty\).

This comes naturally endowed with a handlebody decomposition which is not necessarily nice because of the non-generic contacts \(\partial H^3 \cap \partial h^3(slid)\). However, this can be easily changed into a nice gsc handlebody decomposition \(H(1)\) for \(W^p_\infty\). This ends the proof of Lemma 2.

**Remark 4.1.** Notice that it is not claimed that \(H(1)\) is equivariant for the obvious action of \(\Gamma\) on \(W^p_\infty \simeq W^p\), contrary to the \(H(0)\) with which we started the proof of our lemma, and which is only wgsc.

As a last general comment concerning these matters, both Lemma 1 and Lemma 2 have a large overlap with previous work by L. Funar and the first author (D.O.) [10], but we have preferred to redo here things from scratch, in a manner which is adapted for our purposes.

5 The proofs of Theorems 4 and 5

In this section we will give the proofs of our two main theorems, but we will actually start by proving the part 1. of Theorem 4 and only afterwards we will deal with Theorem 5.

**Lemma 3.** Given \(\Gamma\), we can choose our presentation \(M(\Gamma)\) so that for any desingularization \(\mathcal{R}\), the smooth 4-manifold \(Y^4 \equiv \Theta^4(M(\Gamma), \mathcal{R})\) should be parallelizable. It will follow then, via the \(h\)-principle for immersions and/or submersions (which in this particular case boils down to the standard Smale-Hirsh theory), that there is a smooth submersion

\[(29) \quad Y^4 \equiv \Theta^4(M(\Gamma), \mathcal{R}) \xrightarrow{\phi_1} \mathbb{R}^4.\]
Proof. Along each singular square $S \subset \text{Sing}M(\Gamma)$, the $M(\Gamma)$ has three smooth branches (see (2))

$$U_1 \subset H (= \text{the 3d handlebody}), \ U_2 \subset D^2_{j_1} \times [0, 1], \ U_3 \subset D^2_{j_2} \times [0, 1],$$

coming with $S = S^1_{j_1} \times [0, 1] \cap S^1_{j_2} \times [0, 1] \subset \partial H$, where $\partial D_j = S^1_j$.

Each of the $U_1 \cup U_2$ and $U_1 \cup U_3$ is a smooth 3-manifold and this induces for $M(\Gamma)$ a structure of smooth 3-dimensional train-track manifold. For each $x \in S$ there is a canonical identification $T_x(U_1 \cup U_2) = T_x(U_1 \cup U_3)$ defining the $T_xM(\Gamma)$ for $x \in S$. For the smooth points of $M(\Gamma)$ this tangent space is obvious.

Claim (30). For each $\Gamma$, we can chose the $M(\Gamma)$ so that there is a smooth submersion of our train-track into the Euclidean 3-space

$$M(\Gamma) \xrightarrow{\psi_0} \mathbb{R}^3.$$ 

Here is the proof of the claim. Start with an arbitrary chosen presentation for our $\Gamma$

$$M(\Gamma)_0 = H \cup \sum_{j=1}^{k} D^2_j \times [0, 1]$$

where each $D^2_j \times [0, 1]$ is glued to $H$ via the $\phi|S^1_j \times [0, 1]$ in (2), with, of course, $S^1_j = \partial D^2_j$.

Next, take any embedding $H \subset \mathbb{R}^3$, the standard one if one wants, but it does not matter. If $\phi(S^1_j \times [0, 1]) \subset \mathbb{R}^3$ extends now to a submersion, we are ok, in the sense that our $H \subset \mathbb{R}^3$ extends to a submersion of $H \cup D^2_j \times [0, 1]$. If not, we can change the embedding $S^1_j \times [0, 1] \subset H$ by letting it spiral around $H$ so that now we get a regular homotopy class

$$\phi|S^1_j \times [0, 1] \longrightarrow \mathbb{R}^3$$

which does extend to an immersion $D^2_j \times [0, 1] \longrightarrow \mathbb{R}^3$.

This process can be performed in such a way that the homotopy class of $\sum_{j=1}^{k} S^1_j \longrightarrow H$ should stay unchanged. Of course, more singularities $S$ get created, the $S^1_j \times [0, 1]$’s are only immersed and not embedded, but all this is ok. The Claim (30) is then proved.

It provides us with a smooth field of frames

$$F^3(x) \in \{\text{Frames of } T_xM(\Gamma)\} \simeq SO(3),$$

for each $x \in M(\Gamma)$. We consider now the composite maps

$$M(\Gamma) \xrightarrow{\phi_0} \mathbb{R}^3 = \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4 = \mathbb{R}^3 \times (-\infty < t < +\infty),$$

starting from which, any desingularization $\mathcal{R}$ of $M(\Gamma)$

$$\{U_2, U_3\} \xrightarrow{\mathcal{R}_s} \{s, n\},$$

produces a smooth train-track immersion

$$M(\Gamma) \xrightarrow{\Phi_0} \mathbb{R}^4,$$
simply by pushing the $s$-branch in (33) towards $t = +1$ and the $n$-branch towards $t = -1$. With this (as explained in [11, 22, 32]), we have

\begin{equation}
\Theta^4(M(\Gamma), \mathcal{R}) = \{ \text{the 4$^\text{th}$ smooth regular neighborhood of } M(\Gamma), \text{ induced by } \Phi_0 \}. \tag{35}
\end{equation}

At each $x \in M(\Gamma)$, the $(\Phi_0, F^3(x))$ (with $F^3$ like at (31)) is a 3-frame of the tangent space $T_{\Phi_0(x)}\mathbb{R}^4$. By adding appropriately a fourth orthogonal vector, we can complete this 3-frame into an oriented 4-frame. This is then a trivialization of the tangent space

\[ T(\Theta^4(M(\Gamma), \mathcal{R}))|M(\Gamma), \]

which then easily induces a parallelization for $\Theta^4(M(\Gamma), \mathcal{R})$. This proves Lemma 3.

When the $\phi_1$ of (29) is extended to a larger version of $Y^4$,

\[ Y^4_1 \equiv Y^4 \cup \partial Y^4 \times [0, 1) \supset Y^4 \]

we get a locally finite affine structure on the extension $Y^4_1$ of $Y_4$, i.e. a riemannian (not necessarily complete) metric with sectional curvature $K = 0$. There exists also a second structure on $Y^4$, namely a foliated structure, to be described next. Both the affine and the foliated structures are compatibles with the natural $\text{Diff}^r$ structure of $Y^4 = \Theta^4(M(\Gamma))$.

Let $L_3 \equiv \partial \Theta^4(M(\Gamma), \mathcal{R})$ and let consider the following natural retraction coming from (35):

\begin{equation}
L^3 \subset \Theta^4(M(\Gamma), \mathcal{R}) \xrightarrow{r} M(\Gamma). \tag{37}
\end{equation}

**Lemma 4.**

1. The map $r|L^3$ is simplicially non-degenerate, and, outside of some very simple fold-type singularities, it is an immersion into the train-track $M(\Gamma)$.

2. There is an isomorphism

\begin{equation}
(Y^4, L^3) \equiv \left( M(\Gamma) \cup_{L^3 \times \{0\}} L^3 \times [0, 1), L^3 \times \{1\} \right), \tag{38}
\end{equation}

where the map $r|L^3 \times \{0\}$ is used for glueing together $M(\Gamma)$ and $L^3 \times [0, 1]$.

This lemma tells us that $Y^4$ admits a codimension one foliation $\mathcal{F}$, given by

\[ Y^4 = \bigcup_{t \in [0, 1]} L^3_t, \]

where, for $t > 0$, we have $L^3_t = L^3$ and where $L^3_0 \equiv M(\Gamma)$ is a unique singular leaf.

Returning now to the affine structure which $\Phi_1$ (29) induces on $Y^4_1 \supset Y^4$, we endow $\mathbb{R}^4$ with a very fine affine triangulation, which we afterwards pull back on $Y^4_1$, so that $L^3 = \partial Y^3$ becomes a polyhedral hypersurface. Next, with an appropriate $N_1 \in \mathbb{Z}_+$, in the context of the Lemmas 1 and 2 we have that

\[ M^n \times B^N = (\tilde{Y}^4 \times B^{N_1}) = (Y^4 \times B^{N_1})^\sim \]
and \( B_{N_1} = [0, 1]^{N_1} \) has its own canonical affine structure, putting now affine structures on \( Y^4 \times B_{N_1} \) and on \( M^n \times B^N \) (here \( M^n \) is like in (9)).

Remember that the proof of Lemma 2 starts with a smooth equivariant wgsc cell decomposition \( H(0) \) (the context (I) of (14)). Without any loss of generality, there is an affine triangulation \( \Theta \) of \( Y^4 \times B_{N_1} \) such that \( H(0) = \tilde{\Theta} \equiv \{ \text{the lift of } \Theta \text{ from } Y^4 \times B_{N_1} \text{ to } (Y^4 \times B_{N_1})^\sim \} \).

[Remember that we are proving now Theorem 5, where there is no question of gsc, only of wgsc.]

Our strategy will be now to work downstairs, at the level of \( Y^4 \times B_{N_1} \) and use only admissible subdivisions for our cell-decompositions. When we will lift these things, afterwards, at the level \( M^n \times B^N = (Y^4 \times B_{N_1})^\sim \), equivariance will be automatic, the admissible condition, which is local, is verified upstairs too, and there it will preserve the wgsc property which \( H(0) = \Theta \) initially had. [Note that “admissible” would preserve gsc too, but that is not the question now].

We will be interested now in \( \epsilon \)-skeleta of \( \Theta \), for \( \epsilon = 3 \) or 4, denoted by \( Z^\epsilon \equiv (Y^4 \times B_{N_1})^{(\epsilon)} = \Theta^{(\epsilon)}. \) These come with maps

\[
Z^4 \xrightarrow{F=\pi|X^4} Y^4, \quad Z^3 \xrightarrow{f=r\circ F} M(\Gamma).
\]

**Lemma 5.** After a small perturbation of the \( 0 \)-skeleton \( \Theta^{(0)} \), followed by a global isotopic perturbation of \( \Theta \), which leaves it affine, we can make so that the maps

\[
Z^4 \xrightarrow{F} Y^4, \quad Z^3 \supset F^{-1}\partial Y^4 \xrightarrow{F|F^{-1}\partial Y^4} \partial Y^4
\]

are non-degenerate simplicial surjections, the restrictions of which, on each simplex, are affine.

**Proof.** The proof is left to the reader; one can see here also the argument analogous to this one in [30].

So, in the context of (40) we have now affine triangulations \( \Theta(Z^4) \), \( \Theta(Y^4) \), connected by a simplicial non-degenerate map \( F \).

We introduce now a second class of triangulations, compatible with the same differential structure as the \( \Theta(Y^4) \), but related now to the foliation \( F \) too. These triangulations are denoted \( \Theta_F(Y^4) \), and will be subjected to the following conditions

\begin{align*}
(40-1) \ M(\Gamma) \ &\text{is a subcomplex of } \Theta_F(Y^4); \\
(40-2) \ &\text{there is a distinguished, quite dense, set of leaves, all subcomplexes of } \Theta_F(Y^4), \\
&\quad L_0^3 = M(\Gamma), \ L_1^3, L_2^3, \ldots, L_q^3 = L^3 \times \{1\} = \partial Y^4, \\
&\quad \text{such that every } 4 \text{-simplex } \sigma^4 \text{ of } \Theta_F(Y^4) \text{ rests on two consecutive distinguished leaves } L_i^3, L_{i+1}^3; \\
(40-3) \ &\text{the } 3 \text{-simplices of } \Theta_F(Y^4) \text{ are all essentially parallel to } F, \text{ always transversal to the fibers of } r \ (37), \text{ and such that } r|\sigma^3 \text{ injects.}
\end{align*}
It is assumed that the triangulation $\Theta_F(Y^4)|M(\Gamma)$ is sufficiently fine so that $r\sigma^3$ is a subcomplex. [Our “triangulations” may happily be Siebenmann cell-decompositions].

In the context of $\Theta_F$ we will have $\mathcal{F}$-admissible subdivisions

$$
\Theta_F(Y^4) \xrightarrow{\text{admissible}} \Theta^1_F(Y^4)
$$

which are both admissible and respect the conditions (40–1) to (40–3), with a possibly denser, bigger subset of distinguished leaves.

**Lemma 6.** Once both $\Theta(Y^4)$ and $\Theta_F(Y^4)$ are given, there exists then admissible, respectively $\mathcal{F}$-admissible, subdivisions for each of them, yielding isomorphic cell-decompositions, like in the diagram below (where all the vertical arrows are subdivisions)

$$
\Theta_F(Y^4) \xleftarrow{\text{cell-decompositions}} Y^4 \xrightarrow{\text{cell-dec.}} \Theta(Y^4) \xrightarrow{\mathcal{F}} \Theta(Z^4)
$$

$$
M(\Gamma) \xleftarrow{r} \Theta^1_F(Y^4) \xleftarrow{\text{isomorphism } I} \Theta^1(Y^4) \xrightarrow{F^1} \Theta^1(Z^4)
$$

where both $F$ and $F^1$ in the diagram are simplicial and non-degenerates.

**Proof.** Both $\Theta(Y^4)$ and $\Theta_F(Y^4)$ are compatible with the same $D\text{IFF}$ structure on $Y^4$, and, via the smooth Hauptvermutung, they have isomorphic subdivisions. From there on one uses Siebenmann’s cellulations and his very transparent version of the old Alexander lemma. 

*End of the proof of Theorem 5, point 1.* By taking the universal cover of the lower long composite arrow in (42), we get the following map

$$
X^3 \equiv \Theta^1(\tilde{Z}^4)^{(3)} \xrightarrow{f} \tilde{M}(\Gamma),
$$

where $f \equiv (r \circ I \circ F^1)^\sim$, which has the following features:

(43–1) since both $F^1$ and $r$ are non-degenerate, so is $f$;

(43–2) we have started from $H(0) = \tilde{\Theta}$ which was WGSC and, from there on, all the subdivisions were admissible: this implies that $X^3$ is also WGSC;

(43–3) the map $f$ is surjective and, moreover, it admits the section $\tilde{M}(\Gamma) \subset X^3$ (see (41-1)), which is such that $f|\tilde{M}(\Gamma) = id$.

From this point on, there is a standard argument showing that $\Psi(f) = \Phi(f)$ (and see here the proof of Lemma 2.8 in [30] too). In a nutshell, this argument is the following. Assume $\Psi(f) \subseteq \Phi(f)$, then the induced map

$$
\Theta^1(\tilde{X}^4)^{(3)}/\Psi(f) \longrightarrow \tilde{M}(\Gamma)
$$

would have singularities, which is a contradiction.

So, by now, we have already shown that (43) is an equivariant, WGSC 3$^d$-representation of $\tilde{M}(\Gamma)(\sim \Gamma)$. It remains to check (7) or, equivalently, (7–bis).
The fibers of $\Theta^4(M(\Gamma), R) \times B^{N_1} \xrightarrow{\text{rop}} M(\Gamma)$ are compact, hence so are also those of $\Theta^4(Z^4) \xrightarrow{\text{rop}F^1} M(\Gamma)$ and of

\begin{equation}
\Theta^4(Z^4)^{(3)} \xrightarrow{\text{rop}F^1} M(\Gamma).
\end{equation}

This means that, in the context of (44), for any 3-simplex $\sigma^3$ of $M(\Gamma)$, the inverse image consists of a finite number of 3-simplexes, this number being clearly uniformly bounded.

By equivariance, the same is true for $X^3 \xrightarrow{f} \widetilde{M}(\Gamma)$, and the point 1. in Theorem 5 is by now proved.

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$\Box$
\end{flushright}

5.1 Proof of part 2. of Theorem 5

We want to move now from the 3-dimensional representation (43) to a 2-dimensional representation

\begin{equation}
X^2 \xrightarrow{f} \widetilde{M}(\Gamma),
\end{equation}

which should be WGSC, equivariant, and also satisfying (5).

The general idea is that for the passage (43) $\implies$ (45), there is a similar step in [38], the second paper in the trilogy $\forall \Gamma \in \text{qsf}$, and the techniques used there can be adapted here too. We will only give here the main lines of the argument.

Since we want to have equivariance, we will work downstairs at level $M(\Gamma)$, taking universal coverings in the end. From the lower line in (42), we pick now the map

\begin{equation}
Y^3 \equiv \Theta^4(Z^4)^{(3)} \xrightarrow{\text{rop}F^1} M(\Gamma), \ (Y^3 = X^3/\Gamma),
\end{equation}

choosing to read $Y^3 \equiv \Theta^4(Z^4)^{(3)}$ like a singular handlebody decomposition (see here [37, 38, 43]). For each 3-handle of our $Y^3$ of (46), there are three mutually orthogonal, not everywhere well-defined foliations

\begin{equation}
\mathcal{F}_0(\text{blue}), \mathcal{F}_1(\text{red}), \mathcal{F}_2(\text{black}).
\end{equation}

Each 3-handle is endowed with the three foliations, but, $\mathcal{F}_\lambda(\text{color})$ is natural for the handles of index $\lambda$. There, it is essentially a product foliation of copies of the lateral surface of the handle in question, namely $\partial(\text{cocore}) \times \text{core}$. The reader is invited to look at the figures in [43]. The paper [43] was written, of course, in the non-singular context of $\widetilde{M}^3$ rather then of $\widetilde{M}(\Gamma)$, but, for these individual handles, the story is the same.

Now, at the initial time of the trilogy [37, 38, 39], the QS\(F\) theorem had not yet been proved, and the Whitehead nightmare had to be faced frontally, and there was no question of proving (7). The author’s (V.P.) concern there was to control and tame the then unavoidable accumulation of double points at finite distance. This forced him to work with non-compact bicollared handles of dimension three.
But our present map (46) is devoid of any infinitistic pathologies, and so we can afford now to work with usual compact handles $H^\lambda$ (of index $\lambda$ and dimension 3). For each of these handles $H^\lambda$ we consider now a very dense 2-skeleton, which uses only finitely many leaves of the foliations (47).

Putting these things together, we get a simplicial non-degenerate map

\[(48) \quad Y^2 \xrightarrow{f} M(\Gamma)\]

about which the following items may be assumed without any loss of generality:

1. we have subsets $\text{Sing}(f) \subset Y^2$, $\text{Sing}Y^2 \subset Y^2 - \text{Sing}(f)$ just like in (3–6), and, outside $\text{Sing}(f)$, the $M_2(f) \subset Y^2$ are transversal intersection points;

2. the image $fY^2 \subset M(\Gamma)$ is very dense, i.e. the complement $M(\Gamma) - fY^2$ consists of a disjoint union of three copies of $\mathbb{R}^3_+ = \mathbb{R}^2$.

Next, we take the universal cover of (48),

\[(49) \quad X^2 \equiv \tilde{Y}^2 \xrightarrow{\tilde{f}} \tilde{M}(\Gamma).\]

Here is what we can say about (49).

Our (49) is automatically equivariant and, since $X^3$ (43) was wgsc, so is our present $X^2$ too, since it is the 2-skeleton of it. The fact that in the context of (43) we had $\Psi(f) = \Phi(f)$, together with the fact that $X^2$ is very dense, make that in the context of (49) we also have $\Psi(f) = \Phi(\tilde{f})$, so that (49) is an equivariant wgsc $2^d$-representation, for which local finiteness should be obvious.

Locally, (49) is exactly like (48), where $Y^2$ is a finite complex. The (6) follows from here automatically. Theorem 5 is by now completely proved.

5.2 Proof of 1. in Theorem 4

At the end of Lemma 2 we had gotten a new model for the smooth manifold

$$W^p = \Theta^4(\tilde{M}(\Gamma), \mathcal{R}) \times B^{p-4}, \ (p - 4 = N_1),$$

we called it $W^p_{\infty}$, endowed with a handlebody (and/or cell-decomposition) $H(1) \in \text{gsc}$ and with a diffeomorphism

\[(50) \quad W^p_{\infty} \xrightarrow{\Phi \equiv (D_{\infty})^{-1}} W^p = \tilde{Y}^4 \times B^{N_1}.\]

**Lemma 7.**

1. One may assume $\Theta^1(Z^4)$ (see (42)) dense enough so that $\Phi(\{(W^p_{\infty})^4\}) \subset \Theta^1(Z^4)$.

2. When we go to $X^3_{\infty} \equiv \Phi((W^p_{\infty})^3)$ this is contained inside the $X^3$ (46) and with the $f$ (46) we get a gsc $3^d$-representation

\[(51) \quad X^3_{\infty} \xrightarrow{f} \tilde{M}(\Gamma).\]
3. Because the \( f(46) \) satisfies condition (7) (and/or condition (7-bis)), so does \( f(41) \), as long as we are outside of the contribution \( \sum_i^\infty (h_i^2 + h_i^3) \) (slided) \( \subset X_3^\infty \). On the other hand, inside \( X_3^\infty \), we have

\[
\lim_{n \to \infty} (h_n^2 + h_n^3) \text{(slided)} = \infty
\]

so that the condition (4) is now satisfied too.

Proof. The function \( Z_+ \xrightarrow{\mu} Z_+ \) coming with the condition (4) for our (51) depends on how far backwards the individual terms \( (h_n^2 + h_n^3) \) (slided) have to hit. This makes that, in terms of the sequence of compacta (12), the function \( \mu \) is determined by the infinite sequence of numbers

\[
\text{diam}(K_2 - K_1), \, \text{diam}(K_3 - K_2), \ldots, \, \text{diam}(K_{n+1} - K_n), \ldots .
\]

But, the important point is that the gsc representation (51) avoids the Whitehead night-mare.

Like in the context of Lemma 6 we have to use the smooth Hauptvermutung in order to get our \( f \) (51).

This ends the proof of part 1. of Theorem 4. From here on, the implication 1. \( \implies \) 2. in the context of Theorem 4 uses the same kind of arguments as in the context of Theorem 5. Of course, equivariance is now no longer with us, but, locally, (51) is finitistic and that suffices. Theorem 4 is by now completely proved.

6 Closing remarks

Once our results proven, one could ask: what’s next? Here we give some few general comments about this problem. The first natural question coming in mind after the statement that all finitely presented groups are easy, is the following: where are then the “difficult objects”, in the realm of groups?

We propose a very general and conjectural idea concerning this issue. We think that there may be a new big category, larger than that of finitely presented groups, in the interior of which finitely presented groups should be something like rational numbers among real numbers. The objects of this category could possibly be related to the aperiodic tilings of Roger Penrose or to the quasi-crystals from physics. And here, A. Connes’ non-commutative geometry should come in help for a proper definition.

Eventually, this could give a concrete good explanation of the (apparent) paradoxical fact that, in the category of finitely presented groups, there exists a property (the qsf) which is both universal and (highly) non-trivial.

But all this is, for the moment, just a philosophical vague thought.

References

[1] L. Bessières, G. Besson, M. Boileau, S. Maillot and J. Porti, Geometrisation of 3-manifolds. EMS Tracts in Mathematics, volume 13. European Mathematical Society, Zurich, 2010.
2. G. Besson, *Preuve de la Conjecture de Poincaré en déformant la métrique par la courbure de Ricci (d’après G. Perelman).* Séminaire Bourbaki, Vol. 2004/2005, exposés 938–951. Paris, SMF. Astérisque 307, 309–347, Exp. No. 947 (2006).

3. M. Bestvina and G. Mess, *The boundary of negatively curved groups,* J. Am. Math. Soc. 4 (1991), No. 3, 469–481.

4. S.G. Brick and M.L. Mihalik, *The QSF property for groups and spaces,* Math. Zeit., 220 (1995), 207–217.

5. M.W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidian Spaces,* Ann. Math., 117 (1983), 293–324.

6. D. Epstein, J.W. Cannon, D. Holt, S. Levy, M. Paterson, W.P. Thurston, *Word processing in groups.* Boston, MA etc.: Jones and Bartlett Publishers, 1992.

7. M.H. Freedman and F. Quinn, *Topology of 4-manifolds.* Princeton Mathematical Series, vol. 39, Princeton University Press, Princeton, NJ, 1990.

8. L. Funar, *Simple homotopy type and open 3-manifolds.* Rev. Roumaine Math. Pures Appl., 46 (2001), no. 5, 617–637.

9. L. Funar and S. Gadgil, *On the geometric simple connectivity of open manifolds,* I.M.R.N., 24 (2004), 1193–1248.

10. L. Funar and D.E. Otera, *On the wgsc and QSF tameness conditions for finitely presented groups,* Groups, Geometry and Dynamics, 4 (2010), 549–596.

11. D. Gabai, *Valentin Poénaru's program for the Poincaré conjecture,* in Yau, S.-T. (ed.), “Geometry, topology and physics for Raoul Bott”. Cambridge, MA: International Press. Conf. Proc. Lect. Notes Geom. Topol. 4, 139–166 (1995).

12. S.M. Gersten and J.R. Stallings, *Casson's idea about 3-manifolds whose universal cover is \( \mathbb{R}^3 \),* Int. Jour. Alg. Comput., 1 (1991), 395–406.

13. M. Gromov, *Hyperbolic groups,* Essays in Group Theory (S. Gersten Ed.), MSRI publications, n. 8, Springer-Verlag (1987).

14. M. Gromov, *Asymptotic invariants of infinite groups,* Geometric group theory, Vol. 2 (Sussex, 1991), 1–295, London Math. Soc. Lecture Note Ser. 182, Cambridge Univ. Press, Cambridge, 1993.

15. L. Guillou and A. Marin (eds.), *A la recherche de la topologie perdue.* Progress in Mathematics, Vol. 62, Birkhauser, 1986.

16. B. Kleiner and J. Lott, *Notes on Perelman’s papers,* Geometry & Topology, 12 (2008), 2587–2855.

17. M.L. Mihalik and S.T. Tschantz, *Tame combings of groups,* Trans. AMS, 349 (1997), 4251–4264.
[18] J. Morgan, G. Tian, Ricci flow and the Poincaré conjecture. Clay Mathematics Monographs 3. Providence, RI: American Mathematical Society (AMS); Cambridge, MA: Clay Mathematics Institute. xlii + 521 pp., 2007.

[19] D.E. Otera, *On the proper homotopy invariance of the Tucker property*, Acta Math. Sinica, Eng. Ser. (3) **23** (2007), 571–576.

[20] D.E. Otera, *A topological property for groups*, Contemporary geometry and topology and related topics, 227–236. Proceedings of the 8th International Workshop on Differential Geometry and its Applications, Cluj-Napoca, Romania, August 19–25, 2005. Cluj University Press, 2008.

[21] D.E. Otera, *Topological tameness conditions of groups: Results and developments*, Preprint, 2014.

[22] D.E. Otera and V. Poénaru, “Easy” *Representations and the QSF property for groups*, Bull. Belgian Math. Soc. - Simon Stevin, **19** (2012), n.3, 385–398.

[23] D.E. Otera and V. Poénaru, *Tame combings and easy groups*, preprint 2014.

[24] D.E. Otera, V. Poénaru and C. Tanasi, *On Geometric Simple Connectivity*, Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie. Nouvelle Série, Tome 53 (101) (2010), n.2, 157–176.

[25] D.E. Otera and F. Russo, *On the wgsc property in some classes of groups*, Medit. Jour. Math., **6** (2009), 501–508.

[26] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, ArXiv: math/0211.159 [math.DG] (2002).

[27] G. Perelman, *Ricci flow with surgery on three-manifolds*, ArXiv: math/0303.109 [math.DG] (2003).

[28] G. Perelman, *Finite extinction time for the solutions of the Ricci flow on certain three-manifolds*, ArXiv: math/0307.245 [math.DG] (2003).

[29] V. Poénaru, *On the equivalence relation forced by the singularities of a non-degenerate simplicial map*, Duke Math. J., **63** (1991), 431–447.

[30] V. Poénaru, *Killing handles of index one stably and π₁∞*, Duke Math. J., **63** (1991), 431–447.

[31] V. Poénaru, *Almost convex groups, Lipschitz combing, and π₁∞ for universal covering spaces of closed 3-manifolds*, Jour. Diff. Geom., **35** (1992), 103–130.

[32] V. Poénaru, *The collapsible pseudo-spine representation theorem*, Topology, **31** (1992), n.3, 625–636.
[33] V. Poénaru, *Geometry “à la Gromov” for the fundamental group of a closed 3-manifold M³ and the simple connectivity at infinity of M³*, Topology, 33 (1994), n.1, 181–196.

[34] V. Poénaru, *π₁∞ and simple homotopy type in dimension 3*, Contemporary Math. AMS, 239 (1999), 1–28.

[35] V. Poénaru, *On the 3-Dimensional Poincaré Conjecture and the 4-Dimensional Smooth Schoenflies Problem*, Preprint (2006), arXiv: math/0612554 [math.GT].

[36] V. Poénaru, *Discrete symmetry with compact fundamental domain, and geometric simple connectivity - A provisional Outline of work in Progress*, Preprint (2007), arXiv: 0711.3579 [math.GT].

[37] V. Poénaru, *Equivariant, locally finite inverse representations with uniformly bounded zipping length, for arbitrary finitely presented groups*, Geom. Ded., 167 (2013), 91–121.

[38] V. Poénaru, *Geometric simple connectivity and finitely presented groups*, Preprint (2014), arXiv: 1404.4283 [math.GT].

[39] V. Poénaru, *All finitely presented groups are qsf*, Preprint (2014), arXiv: 1409.7325 [math.GT].

[40] V. Poénaru and C. Tanasi, *Hausdorff combing of groups and π₁∞ for universal covering spaces of closed 3-manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 20 (1993), n.3, 387–414.

[41] V. Poénaru and C. Tanasi, *k-weakly almost convex groups and π₁∞  M³*, Geom. Ded., 48 (1993), n.1, 57–81.

[42] V. Poénaru and C. Tanasi, *Representation of the Whitehead manifold Wh³ and Julia sets*, Ann. Fac. Sci. Toulouse, Sér. 6, 4 (1995), n.3, 665–694.

[43] V. Poénaru and C. Tanasi, *Equivariant, Almost-Arborescent representations of open simply-connected 3-manifolds; A finiteness result*, Memoirs of the AMS, 800, 88 pp., 2004.

[44] V. Poénaru and C. Tanasi, *A group-theoretical finiteness theorem*, Geom. Ded., 137 (2008), n.1, 1–25.

[45] L.C. Siebenmann, *On detecting Euclidean space homotopically among topological manifolds*, Invent. Math., 6 (1968), 263–268.

[46] L.C. Siebenmann, *Les bisections expliquent le théorème de Reidemeister-Singer, un retour aux sources*, Prepublications mathématiques d’Orsay 80T16, 1980.

[47] S. Smale, *On the structure of manifolds*, Amer. J. Math., 84 (1962), 387–399.
[48] J.R. Stallings, *The piecewise linear structure of the Euclidean space*, Proc. of the Cambridge Math. Phil. Soc., **58** (1962), 481–488.

[49] J.R. Stallings, *Brick’s quasi-simple filtrations for groups and 3-manifolds*, Geometric group theory, Vol. 1 (Sussex, 1991), 188–203, London Math. Soc. Lecture Note Ser., **181**, Cambridge Univ. Press, Cambridge, 1993.

[50] W.P. Thurston, Three-dimensional geometry and topology. Vol. 1. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997.

[51] J.H.C. Whitehead, *A certain open manifold whose group is unity*, Quart. Jour. of Math., **6** (1935), 268–279.