THE LEAST H-EIGENVALUE OF ADJACENCY TENSOR OF HYPERGRAPHS WITH CUT_vertices

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Abstract. Let $G$ be a connected hypergraph with even uniformity, which contains cut vertices. Then $G$ is the coalescence of two nontrivial connected sub-hypergraphs (called branches) at a cut vertex. Let $\mathcal{A}(G)$ be the adjacency tensor of $G$. The least H-eigenvalue of $\mathcal{A}(G)$ refers to the least real eigenvalue of $\mathcal{A}(G)$ associated with a real eigenvector. In this paper we obtain a perturbation result on the least H-eigenvalue of $\mathcal{A}(G)$ when a branch of $G$ attached at one vertex is relocated to another vertex, and characterize the unique hypergraph whose least H-eigenvalue attains the minimum among all hypergraphs in a certain class of hypergraphs which contain a fixed connected hypergraph.

1. Introduction

Recently, the spectral hypergraph theory developed rapidly, the adjacency tensors $\mathcal{A}(G)$ [4] of uniform hypergraphs $G$ were introduced to investigating the structure of hypergraphs, just like adjacency matrices to simple graphs. As $\mathcal{A}(G)$ is nonnegative, by using Perron-Frobenius theorem [2, 11, 22, 23, 24], many results about its spectral radius are presented [3, 4, 5, 8, 16, 18].

For the least H-eigenvalue of $\mathcal{A}(G)$ of a $k$-uniform connected hypergraph $G$, Khan and Fan [15] discussed the limit points of the least H-eigenvalue of $\mathcal{A}(G)$ when $G$ is non-odd-bipartite. Let $\rho(G)$ be the spectral radius of $\mathcal{A}(G)$. Shao et al. [21] proved that the $-\rho(G)$ is an H-eigenvalue of $\mathcal{A}(G)$ if and only if $k$ is even and $G$ is odd-bipartite. Some other equivalent conditions are summarized in [9]. Note that $-\rho(G)$ is an eigenvalue of $\mathcal{A}(G)$ if and only if $k$ is even and $G$ is odd-colorable [9]. So, there exist odd-colorable but non-odd-bipartite hypergraphs [7, 19], for which $-\rho(G)$ is an N-eigenvalue. An odd-colorable hypergraph has a symmetric spectrum. One can refer [6] for more results on the spectral symmetry of nonnegative tensors and hypergraphs.

To our knowledge, the least H-eigenvalue of $\mathcal{A}(G)$ receives little attention except the above work. If $k$ is even, then the least H-eigenvalue of $\mathcal{A}(G)$ is a solution of minimum problem over a real unit sphere; see Eq. (2.3). So, throughout of this paper, when discussing the least H-eigenvalue of $\mathcal{A}(G)$, we always assume that $G$ is connected with even uniformity $k$. For convenience, the least H-eigenvalue of $\mathcal{A}(G)$ is simply called the least eigenvalue of $G$ and the corresponding H-eigenvectors are called the first eigenvectors of $G$.

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In this paper we give a perturbation result on the least H-eigenvalue of $\mathcal{A}(G)$ when a branch of $G$ attached at one vertex is relocated to another vertex, and characterize the unique hypergraph whose least H-eigenvalue attains the minimum among all hypergraphs in a certain class of hypergraphs which contain a fixed connected hypergraph. The perturbation result in this paper is a generalization of that on the least eigenvalue of the adjacency matrix of a simple graph in [10].

2. Preliminaries

2.1. Tensors and eigenvalues. A real tensor (also called hypermatrix) $\mathcal{A} = (a_{i_1i_2,...,i_k})$ of order $k$ and dimension $n$ refers to a multi-dimensional array with entries $a_{i_1i_2,...,i_k} \in \mathbb{R}$ for all $i_j \in [n] := \{1,2,\ldots,n\}$ and $j \in [k]$. Clearly, if $k = 2$, then $\mathcal{A}$ is a square matrix of dimension $n$. The tensor $\mathcal{A}$ is called symmetric if its entries are invariant under any permutation of their indices.

Given a vector $x \in \mathbb{C}^n$, $\mathcal{A}x^k \in \mathbb{C}$ and $\mathcal{A}x^{k-1} \in \mathbb{C}^n$, which are defined as follows:

$$\mathcal{A}x^k = \sum_{i_1,i_2,...,i_k \in [n]} a_{i_1i_2,...,i_k}x_{i_1}x_{i_2}\cdots x_{i_k},$$

$$\langle \mathcal{A}x^{k-1} \rangle_i = \sum_{i_2,...,i_k \in [n]} a_{i_1i_2,...,i_k}x_{i_2}\cdots x_{i_k}, i \in [n].$$

Let $I = (i_1i_2...i_k)$ be the identity tensor of order $k$ and dimension $n$, that is, $i_1i_2...i_k = 1$ if $i_1 = i_2 = \cdots = i_k \in [n]$ and $i_1i_2...i_k = 0$ otherwise.

Definition 2.1 ([17, 20]). Let $\mathcal{A}$ be a real tensor of order $k$ dimension $n$. For some $\lambda \in \mathbb{C}$, if the polynomial system $(\lambda I - \mathcal{A})x = 0$, or equivalently $\mathcal{A}x^{k-1} = \lambda x^{k-1}$, has a solution $x \in \mathbb{C}^n \setminus \{0\}$, then $\lambda$ is called an eigenvalue of $\mathcal{A}$ and $x$ is an eigenvector of $\mathcal{A}$ associated with $\lambda$, where $x^{[k-1]} := (x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1})$.

In the above definition, $(\lambda, x)$ is called an eigenpair of $\mathcal{A}$. If $x$ is a real eigenvector of $\mathcal{A}$, surely the corresponding eigenvalue $\lambda$ is real. In this case, $\lambda$ is called an H-eigenvalue of $\mathcal{A}$. A real symmetric tensor $\mathcal{A}$ of even order $k$ is called positive semidefinite (or positive definite) if for any $x \in \mathbb{R}^n \setminus \{0\}$, $\mathcal{A}x^k \geq 0$ (or $\mathcal{A}x^k > 0$). Denote by $\lambda_{\text{min}}(\mathcal{A})$ the least H-eigenvalue of $\mathcal{A}$.

Lemma 2.2 ([20, Theorem 5]). Let $\mathcal{A}$ be a real symmetric tensor of order $k$ and dimension $n$, where $k$ is even. Then the following results hold.

1. $\mathcal{A}$ always has H-eigenvalues, and $\mathcal{A}$ is positive definite (or positive semidefinite) if and only if its least H-eigenvalue is positive (or nonnegative).

2. $\lambda_{\text{min}}(\mathcal{A}) = \min \{\mathcal{A}x^k : x \in \mathbb{R}^n, ||x||_k = 1\}$, where $||x||_k = (\sum_{i=1}^n |x_i|^k)^{1/k}$. Furthermore, $x$ is an optimal solution of the above optimization if and only if it is an eigenvector of $\mathcal{A}$ associated with $\lambda_{\text{min}}(\mathcal{A})$.

2.2. Uniform hypergraphs and eigenvalues. A hypergraph $G = (V(G), E(G))$ is a pair consisting of a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and an edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, where $e_j \subseteq V(G)$ for each $j \in [m]$. If $|e_j| = k$ for all $j \in [m]$, then $G$ is called a $k$-uniform hypergraph. The degree $d_G(v)$ or simply $d(v)$ of a vertex $v \in V(G)$ is defined as $d(v) = |\{e_j : v \in e_j\}|$. A walk in a $G$ is a sequence of alternate vertices and edges: $v_0e_1v_1e_2\cdots e_tv_t$, where $v_i, v_{i+1} \in e_i$ for $i = 0, 1, \ldots, t - 1$. A walk is called a path if all the vertices and edges appeared on the walk are distinct. A hypergraph $G$ is called connected if every two vertices of $G$ are connected by a walk or path.
If a hypergraph is both connected and acyclic, it is called a hypertree. The $k$-th power of a simple graph $H$, denoted by $H^k$, is obtained from $H$ by replacing each edge (a 2-set) with a $k$-set by adding $(k - 2)$ additional vertices \[13\]. The $k$-th power of a tree is called power hypertree, which is surely a $k$-uniform hypertree. In particular, the $k$-th power of a star (as a simple graph) with $m$ edges is called a hyperstar, denote by $S^k_m$. In a $k$-th power hypertree $T$, an edge is called a pendent edge of $T$ if it contains $k - 1$ vertices of degree one, which are called the pendent vertices of $T$.

**Lemma 2.3** (\[1\]). If $G$ is a connected $k$-uniform hypergraph with $n$ vertices and $m$ edges, then $G$ is a hypertree if and only if $m = \frac{n}{2} - 1$.

**Definition 2.4** (\[12\]). Let $k$ be even. A $k$-uniform hypergraph $G = (V, E)$ is called odd-bipartite, if there exists a bipartition $\{V_1, V_2\}$ of $V$ such that each edge of $G$ intersects $V_1$ (or $V_2$) in an odd number of vertices (such bipartition is called an odd-bipartition of $G$); otherwise, $G$ is called non-odd-bipartite.

The odd-bipartite hypergraphs are considered as generalizations of the ordinary bipartite graphs. The examples of non-odd-bipartite hypergraphs can be found in \[14\] \[19\]. Note that the odd-bipartite hypergraphs are also called odd-transversal bipartite graphs. The examples of non-odd-bipartite hypergraphs can been found \[13\].

Let $G$ be a $k$-uniform hypergraph on $n$ vertices $v_1, v_2, \ldots, v_n$. The adjacency tensor of $G$ \[4\] is defined as $A(G) = (a_{i_1i_2\ldots i_k})$, an order $k$ dimensional $n$ tensor, where

$$
a_{i_1i_2\ldots i_k} = \begin{cases} 
\frac{1}{(k-1)!}, & \text{if } \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in E(G); \\
0, & \text{otherwise.}
\end{cases}
$$

Observe that the adjacency tensor of a hypergraph is symmetric.

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$. Then $x$ can be considered as a function defined on the vertices of $G$, that is, each vertex $v_i$ is mapped to $x_i := x_{v_i}$. If $x$ is an eigenvector of $A(G)$, then it defines on $G$ naturally, i.e., $x_v$ is the entry of $x$ corresponding to $v$. If $G_0$ is a sub-hypergraph of $G$, denote by $x|_{G_0}$ the restriction of $x$ on the vertices of $G_0$, or a subvector of $x$ indexed by the vertices of $G_0$.

Denote by $E_G(v)$, or simply $E(v)$, the set of edges of $G$ containing $v$. For a subset $U$ of $V(G)$, denote $x^U := \Pi_{v \in U} x_v$. Then we have

$$
A(G)x^k = \sum_{e \in E(G)} kx^e,
$$

and for each $v \in V(G)$,

$$
(A(G)x^{k-1})_v = \sum_{e \in E(v)} x^{e\setminus\{v\}}.
$$

So the eigenvector equation $A(G)x^{k-1} = \lambda x^{[k-1]}$ is equivalent to that for each $v \in V(G)$,

$$
\lambda x^{k-1} = \sum_{e \in E(v)} x^{e\setminus\{v\}}.
$$

\[2.1\]
From Lemma 2.2, if $k$ is even, then $\lambda_{\text{min}}(G) := \lambda_{\text{min}}(A(G))$ can be expressed as

$$\lambda_{\text{min}}(G) = \min_{x \in \mathbb{R}^n, \|x\|_k = 1} \sum_{e \in E(G)} kx^e.$$  

Note than if $k$ is odd, the Eq. (2.3) does not hold. The reason is as follows. If $G$ contains at least one edge, then by Perron-Frobenius theorem, the spectral radius $\rho(A(G))$ is positive associated with a unit nonnegative eigenvector $x$. Now

$$-\rho(A(G)) \leq \lambda_{\text{min}}(G) \leq A(G)(-x)^k = -A(G)x^k = -\rho(A(G)),$$

so $\lambda_{\text{min}}(G) = -\rho(A(G))$, which implies that $k$ is even and $G$ is odd-bipartite [21,9], a contradiction.

**Lemma 2.5.** Let $G$ be a $k$-uniform hypergraph, and $(\lambda, x)$ be an eigenpair of $A(G)$. If $E(u) = E(v)$ and $\lambda \neq 0$, then $x_u = x_v$.

**Proof.** Consider the eigenvector equation of $x$ at $u$ and $v$ respectively,

$$\lambda x_u = \sum_{e \in E(u)} x^e, \quad \lambda x_v = \sum_{e \in E(v)} x^e.$$

As $E[u] = E[v]$, $\sum_{e \in E(u)} x^e = \sum_{e \in E(v)} x^e$. The result follows. \qed

**Lemma 2.6.** Let $G$ be a $k$-uniform hypergraph with at least one edge, where $k$ is even. Let $G_0$ be a induced sub-hypergraph of $G$. Then $\lambda_{\text{min}}(G) \leq \lambda_{\text{min}}(G_0)$. In particular, $\lambda_{\text{min}}(G) \leq -1$.

**Proof.** Let $x$ be a unit first eigenvector of $G_0$. Define $\tilde{x}$ on $G$ such that $\tilde{x}(v) = x(v)$ if $v \in V(G_0)$ and $\tilde{x}(v) = 0$ otherwise. Then by Lemma 2.2

$$\lambda_{\text{min}}(G_0) = A(G_0)x^k = A(G)\tilde{x}^k \geq \min_{x \in \mathbb{R}^n, \|x\|_k = 1} A(G)x^k = \lambda_{\text{min}}(G).$$

If letting $G_0$ be an edge, then $\lambda_{\text{min}}(G_0) = -1$. The result follows. \qed

### 3. Properties of the first eigenvectors

We will give some properties of the first eigenvectors of a connected $k$-uniform $G$. We stress that $k$ is even in this and the following section.

Let $G_1, G_2$ be two vertex-disjoint hypergraphs, and let $v_1 \in V(G_1), v_2 \in V(G_2)$. The **coalescence** of $G_1, G_2$ with respect to $v_1, v_2$, denoted by $G_1(v_1) \circ G_2(v_2)$, is obtained from $G_1, G_2$ by identifying $v_1$ with $v_2$ and forming a new vertex $u$. The graph $G_1(v_1) \circ G_2(v_2)$ is also written as $G_1(u) \circ G_2(u)$. If a connected hypergraph $G$ can be expressed in the form $G = G_1(u) \circ G_2(u)$, where $G_1, G_2$ are both nontrivial and connected, then $u$ is called a **cut vertex** of $G$, and $G_1$ is called a **branch** of $G$ with **root** $u$. Clearly $G_2$ is also a branch of $G$ with root $u$ in the above definition.

**Lemma 3.1.** Let $G = G_0(u) \circ H(u)$ be a connected $k$-uniform hypergraph. Let $x$ be a first eigenvector of $G$. Then the following results hold.

1. If $H$ is odd-bipartite, then $x^e \leq 0$ for each $e \in E(H)$, and there exists a first eigenvector of $G$ such that it is nonnegative on one part and nonpositive on the other part for any odd-bipartition of $H$.
2. If $x_u = 0$, then $\sum_{e \in E_G(u)} x^e(u) = \sum_{e \in E_H(u)} x^e(u) = 0$. If further $H$ is odd-bipartite, then $x^e(u) = 0$ for each $e \in E_H(u)$. 

Proof. Let \( \{U, W\} \) be an odd-bipartition of \( H \), where \( u \in U \). Without loss of generality, we assume that \( \|x\|_k = 1 \) and \( x_u \geq 0 \). Let \( \tilde{x} \) be such that

\[
\tilde{x}_v = \begin{cases} x_v, & \text{if } v \in V(G_0) \backslash \{u\}; \\ |x_v|, & \text{if } v \in U; \\ -|x_v|, & \text{if } v \in W. 
\end{cases}
\]

Note that \( \|\tilde{x}\|_k = \|x\|_k = 1 \), and for each \( e \in E(H) \), \( \tilde{x}^e \leq x^e \) with equality if and only if \( x^e \leq 0 \).

We prove the assertion (1) by a contradiction. Suppose that there exists an edge \( e \in E(H) \) such that \( x^e > 0 \). Then \( \tilde{x}^e < x^e \). By Eq. (2.3), we have

\[
\lambda_{\min}(G) \leq \mathcal{A}(G)\tilde{x}^k < \mathcal{A}(G)x^k = \lambda_{\min}(G),
\]
a contradiction. So \( x^e \leq 0 \) for each \( e \in E(H) \), and \( \tilde{x} \) is also a first eigenvector as \( \mathcal{A}(G)\tilde{x}^k = \mathcal{A}(G)x^k \). The assertion (1) follows.

For the assertion (2), let \( \tilde{x} \) be such that

\[
\tilde{x}_v = \begin{cases} x_v, & \text{if } v \in V(G_0) \backslash \{u\}; \\ -x_v, & \text{if } v \in V(H). 
\end{cases}
\]

Then \( \tilde{x} \) is also a first eigenvector of \( G \) as \( \mathcal{A}(G)\tilde{x}^k = \mathcal{A}(G)x^k \). Note that \( x_u = 0 \) and consider the eigenvector equation Eq. (2.3) of \( x \) and \( \tilde{x} \) at \( u \), respectively.

\[
\lambda_{\min}(G)x_{u}^{k-1} = 0 = \sum_{e \in E_{G_0}(u)} x^{\backslash\{u\}} + \sum_{e \in E_H(u)} x^{\backslash\{u\}},
\]

\[
\lambda_{\min}(G)\tilde{x}^{k-1}_u = 0 = \sum_{e \in E_{G_0}(u)} \tilde{x}^{\backslash\{u\}} + \sum_{e \in E_H(u)} \tilde{x}^{\backslash\{u\}}
\]

\[
= \sum_{e \in E_{G_0}(u)} x^{\backslash\{u\}} - \sum_{e \in E_H(u)} x^{\backslash\{u\}}.
\]

Thus \( \sum_{e \in E_{G_0}(u)} x^{\backslash\{u\}} = \sum_{e \in E_H(u)} x^{\backslash\{u\}} = 0 \).

If further \( H \) is odd-bipartite, applying the above result to \( \tilde{x} \) (also a first eigenvector of \( G \)), we have \( \sum_{e \in E_H(u)} \tilde{x}^{\backslash\{u\}} = 0 \). As \( \tilde{x}^{\backslash\{u\}} \leq 0 \) for each edge \( e \in E_H(u) \), \( \tilde{x}^{\backslash\{u\}} = 0 \) for each \( e \in E_H(u) \). The assertion (2) follows by the definition of \( \tilde{x} \). \( \square \)

Lemma 3.2. Let \( G = G_0(u) \odot H(u) \) be a connected \( k \)-uniform hypergraph. Then

\[
\lambda_{\min}(G_0) \geq \lambda_{\min}(G),
\]

with equality if and only if for any first eigenvector \( y \) of \( G_0 \), \( y_u = 0 \) and \( \bar{y} \) is a first eigenvector of \( G \), where \( \bar{y} \) is defined by

\[
\bar{y}_v = \begin{cases} y_v, & \text{if } v \in V(G_0); \\ 0, & \text{otherwise}. 
\end{cases}
\]

Proof. Suppose that \( y \) is a first eigenvector of \( G_0 \), \( \|y\|_k = 1 \). Let \( e_0 \) be an edge of \( H \) containing \( u \). Define \( \bar{y} \) by

\[
\bar{y}_v = \begin{cases} y_v, & \text{if } v \in V(G_0); \\ -y_v, & \text{if } v \in e_0 \backslash \{u\}; \\ 0, & \text{otherwise}. 
\end{cases}
\]
Then \(\|\bar{y}\|_k^k = 1 + (k - 1)\bar{y}_k^k\), and
\[
A(G)\bar{y}^k = \sum_{e \in E(G)} ky^e + \sum_{e \in E(H)} ky^e
\]
\[
= A(G_0)\bar{y}^k - ky^e
\]
\[
= \lambda_{\text{min}}(G_0) - ky^e.
\]
By Eq. (2.2), we have
\[
\lambda_{\text{min}}(G) \leq \frac{A(G)\bar{y}^k}{\|\bar{y}\|_k^k} = \frac{\lambda_{\text{min}}(G_0) - ky^e}{1 + (k - 1)\bar{y}_k^k} \leq \lambda_{\text{min}}(G_0),
\]
where the first equality holds if and only if \(\bar{y}\) is also a first eigenvector of \(G\), and the second equality holds if and only if \(y_u = 0\). The result now follows. \(\Box\)

**Corollary 3.3.** Let \(G = G_0 \diamond H(u)\) be a connected \(k\)-uniform hypergraph.
(1) If \(y\) is a first eigenvector of \(G_0\) with \(y_u \neq 0\), then
\[
\lambda_{\text{min}}(G_0) > \lambda_{\text{min}}(G).
\]
(2) If \(x\) is a first eigenvector of \(G\) such that \(x_u = 0\) and \(x|_{G_0} \neq 0\), then
\[
\lambda_{\text{min}}(G_0) = \lambda_{\text{min}}(G).
\]

**Proof.** By Lemma 3.2 we can get the assertion (1) immediately. Let \(x\) be a first eigenvector of \(G\) as in (2). By Lemma 3.1 (2), \(\sum_{e \in E(G_0)} x^e(u) = 0\). Considering the eigenvector equation \(\bar{y}^k\) of \(x\) at each vertex of \(V(G_0)\), we have
\[
A(G_0)(x|_{G_0})^{k-1} = \lambda_{\text{min}}(G_0)x^{k-1}.
\]
So \(x|_{G_0}\), the restriction of \(x\) on \(G_0\), is an eigenvector of \(A(G_0)\) associated with the eigenvalue \(\lambda_{\text{min}}(G_0)\). The result follows by Lemma 3.2 \(\Box\)

**Lemma 3.4.** Let \(G = G_0 \diamond H(u)\) be a connected \(k\)-uniform hypergraph. If \(x\) is a first eigenvector of \(G\), then
\[
\alpha_H(x) := \sum_{e \in E_H(u)} x^e \leq 0.
\]
Furthermore, if \(\alpha_H(x) = 0\) and \(x|_{G_0} \neq 0\), then \(x_u = 0\) and \(\lambda_{\text{min}}(G_0) = \lambda_{\text{min}}(G)\); or equivalently if \(x_u \neq 0\), then \(\alpha_H(x) < 0\).

**Proof.** Let \(\lambda := \lambda_{\text{min}}(G)\). By Eq. (2.2), for each \(v \in V(G_0) \{u\}\),
\[
((A(G) - \lambda I)x^{k-1})_v = ((A(G_0) - \lambda I)(x|_{G_0})^{k-1})_v = 0.
\]
For the vertex \(u\),
\[
\lambda x_u^{k-1} = (A(G)x^{k-1})_u = \sum_{e \in E_G(u)} x^e(u) + \sum_{e \in E_H(u)} x^e(u)
\]
\[
= (A(G_0)(x|_{G_0})^{k-1})_u + \sum_{e \in E_H(u)} x^e(u).
\]
So,
\[
((A(G_0) - \lambda I)(x|_{G_0})^{k-1})_u = -\sum_{e \in E_H(u)} x^e(u).
\]
By Lemma 3.2 and Lemma 2.2(1), \( A(G_0) - \lambda I \) is positive semidefinite. Then \((A(G_0) - \lambda I)y^k \geq 0\) for any real and nonzero \(y\). So, by Eq. (3.2) and Eq. (3.3), we have
\[
0 \leq (A(G_0) - \lambda I)(x|_{G_0})^k = (x|_{G_0})^\top ((A(G_0) - \lambda I)(x|_{G_0})^{k-1}
\sum_{e \in E_H(u)} x_{e \setminus \{u\}} = -x_u \sum_{e \in E_H(u)} x_{e \setminus \{u\}} = -\alpha_H(x).
\]
So we have \(\alpha_H(x) \leq 0\).

Suppose that \(\alpha_H(x) = 0\) and \(x|_{G_0} \neq 0\). If \(x_u = 0\), by Corollary 3.3(2), \(\lambda_{\text{min}}(G_0) = \lambda_{\text{min}}(G)\). If \(x_u \neq 0\), then \(\sum_{e \in E_H(u)} x_{e \setminus \{u\}} = 0\). By Eq. (3.2) and Eq. (3.3), \((\lambda_{\text{min}}(G), x|_{G_0})\) is an eigenpair of \(A(G_0)\), implying that \(\lambda_{\text{min}}(G) = \lambda_{\text{min}}(G_0)\) by Lemma 3.2. However, \((x|_{G_0})_u = x_u \neq 0\), a contradiction to Corollary 3.3(1).

4. Perturbation of the least eigenvalues

We will give a perturbation result on the least eigenvalues under relocating a branch. Let \(G_0, H\) be two vertex-disjoint hypergraphs, where \(v_1, v_2\) are two distinct vertices of \(G_0\), and \(u\) is a vertex of \(H\) (called the root of \(H\)). Let \(G = G_0(v_2) \circ H(u)\) and \(\tilde{G} = G_0(v_1) \circ H(u)\). We say that \(\tilde{G}\) is obtained from \(G\) by relocating \(H\) rooted at \(u\) from \(v_2\) to \(v_1\); see Fig. 4.1.

**Figure 4.1.** Relocating \(H\) from \(v_2\) to \(v_1\)

**Lemma 4.1.** Let \(G = G_0(v_2) \circ H(u)\) and \(\tilde{G} = G_0(v_1) \circ H(u)\) be connected \(k\)-uniform hypergraphs. If \(x\) is a first eigenvector of \(G\) such that \(|x_{v_1}| \geq |x_{v_2}|\), then
\[
(4.1) \quad \lambda_{\text{min}}(\tilde{G}) \leq \lambda_{\text{min}}(G),
\]
with equality if and only if \(x_{v_1} = x_{v_2} = 0\), and \(\tilde{x}\) defined in (4.3) is a first eigenvector of \(\tilde{G}\).

**Proof.** Let \(x\) be a first eigenvector of \(G\) such that \(\|x\|_k = 1\) and \(x_{v_1} \geq 0\). We divide the discussion into three cases. Denote \(\lambda := \lambda_{\text{min}}(G)\).

**Case 1:** \(x_{v_2} > 0\). Write \(x_{v_1} = \delta x_{v_2}\), where \(\delta \geq 1\). Define \(\tilde{x}\) on \(\tilde{G}\) by
\[
(4.2) \quad \tilde{x}_v = \begin{cases} 
  x_v, & \text{if } v \in V(G_0); \\
  \delta x_v, & \text{if } v \in V(H) \setminus \{u\}. 
\end{cases}
\]
Then \( \| \tilde{x} \|^k_k = 1 + (\delta^k - 1) \sum_{v \in V(H) \setminus \{u\}} x^k_v \), and

\[
\mathcal{A}(\tilde{G})\tilde{x}^k = \sum_{e \in E(\tilde{G})} k\tilde{x}^e
\]

\[
= \mathcal{A}(G)x^k + (\delta^k - 1) \sum_{e \in E(H)} kx^e
\]

\[
= \lambda + (\delta^k - 1) \sum_{e \in E(H)} kx^e.
\]

By the eigenvector equation of \( x \) at each vertex \( v \in V(H) \setminus \{u\} \),

\[
(4.3) \quad \sum_{e \in E(H)} H^v \{u\} x^e = \lambda x^k_v.
\]

By the eigenvector equation of \( x \) at \( u \),

\[
(4.4) \quad \sum_{e \in E(G)} G^u x^e = \alpha H^u x^k + \gamma G_0^u x^k = \lambda x^k_u,
\]

where \( \gamma G_0^u(x) := \sum_{e \in E(G^u)} x^e \). By Eq. (4.3) and Eq. (4.4), we have

\[
\gamma G_0^u(x) + \sum_{e \in E(H)} kx^e = \lambda \sum_{v \in V(H)} x^k_v.
\]

So

\[
\sum_{e \in E(H)} kx^e = \lambda \sum_{v \in V(H)} x^k_v - \gamma G_0^u(x)
\]

\[
= \lambda \sum_{v \in V(H)} x^k_v - (\lambda x^k_u - \alpha H(x))
\]

\[
= \lambda \sum_{v \in V(H) \setminus \{u\}} x^k_v + \alpha H(x).
\]

Thus

\[
\mathcal{A}(\tilde{G})\tilde{x}^k = \lambda + (\delta^k - 1) \left( \lambda \sum_{v \in V(H) \setminus \{u\}} x^k_v + \alpha H(x) \right)
\]

\[
= \lambda \left( 1 + (\delta^k - 1) \sum_{v \in V(H) \setminus \{u\}} x^k_v \right) + (\delta^k - 1) \alpha H(x)
\]

\[
= \lambda \| \tilde{x} \|^k_k + (\delta^k - 1) \alpha H(x).
\]

As \( x_{v_2} \neq 0 \), \( \alpha_H(x) < 0 \) by Lemma 3.3,

\[
\lambda_{\text{min}}(\tilde{G}) \leq \frac{\mathcal{A}(\tilde{G})\tilde{x}^k}{\| \tilde{x} \|^k_k} = \lambda + \frac{(\delta^k - 1) \alpha H(x)}{\| \tilde{x} \|^k_k} \leq \lambda = \lambda_{\text{min}}(G),
\]

where the first equality holds if and only if \( \tilde{x} \) is a first eigenvector of \( \tilde{G} \), and the second equality holds if and only if \( \delta = 1 \), i.e. \( x_{v_1} = x_{v_2} \). So, the equality holds if and only if \( x_{v_1} = x_{v_2} \) and \( \tilde{x} \) is a first eigenvector of \( \tilde{G} \). By the eigenvector equations of \( x \) and \( \tilde{x} \) at \( v_2 \) respectively, we will get \( \alpha_H(x) = 0 \), a contradiction. So, in this case, \( \lambda_{\text{min}}(\tilde{G}) < \lambda_{\text{min}}(G) \).
Case 2: $x_{v_2} = 0$. First assume $x_{v_1} = 0$. Define $\tilde{x}$ on $\tilde{G}$ by

$$\tilde{x}_v = \begin{cases} x_v, & \text{if } v \in V(G_0); \\ x_v, & \text{if } v \in V(H) \setminus \{u\}. \end{cases}$$

Then $\|\tilde{x}\|^k_k = 1$, and

$$\lambda_{\min}(\tilde{G}) \leq A(\tilde{G})\tilde{x}^k = A(G)x^k = \lambda_{\min}(G),$$

with equality if and only if $\tilde{x}$ is a first eigenvector of $\tilde{G}$.

Now assume that $x_{v_1} > 0$. By Corollary 4.3(2) and its proof, $\lambda_{\min}(G) = \lambda_{\min}(G_0)$ as $x_u = 0$ and $x|G_0 \neq 0$; furthermore, $x|G_0$ is a first eigenvector of $G_0$. By Corollary 4.3(1), $\lambda_{\min}(G_0) > \lambda_{\min}(\tilde{G})$ as $(x|G_0)v_1 \neq 0$, thinking of $v_1$ a coalescence vertex between $G_0$ and $H$ in $\tilde{G}$. So $\lambda_{\min}(\tilde{G}) < \lambda_{\min}(G)$.

Case 3: $x_{v_2} < 0$. Write $x_{v_1} = -\delta x_{v_2}$, where $\delta \geq 1$. Define $\tilde{x}$ on $\tilde{G}$ by

$$\tilde{x}_v = \begin{cases} x_v, & \text{if } v \in V(G_0); \\ -\delta x_v, & \text{if } v \in V(H) \setminus \{u\}. \end{cases}$$

By a similar discussion to Case 1 by replacing $\delta$ by $-\delta$, we also have $\lambda_{\min}(\tilde{G}) < \lambda_{\min}(G)$. \hfill $\square$

Denoted by $\mathcal{T}_m(G_0)$ the class of hypergraphs with each obtained from a fixed connected hypergraph $G_0$ by attaching some hypertrees at some vertices of $G_0$ respectively (i.e. identifying a vertex of a hypertree with some vertex of $G_0$ each time) such that the number of its edges equals $\varepsilon(G_0) + m$. A hypergraph is called a minimizing hypergraph in a certain class of hypergraphs if its least eigenvalue attains the minimum among all hypergraphs in the class.

We will characterize the minimizing hypergraph(s) in $\mathcal{T}_m(G_0)$. Denote by $G = G_0(u) \circ S^k_m(u)$ the coalescence of $G_0$ and $S^k_m$ by identifying one vertex of $G_0$ and the central vertex of $S^k_m$ and forming a new vertex $u$.

**Theorem 4.2.** Let $G_0$ be a connected $k$-uniform hypergraph. If $G$ is a minimizing hypergraph in $\mathcal{T}_m(G_0)$, then $G = G_0(u) \circ S^k_m(u)$ for a unique vertex $u$ of $G_0$.

**Proof.** Suppose that $G$ is a minimizing hypergraph in $\mathcal{T}_m(G_0)$, and $G$ has no the structure as desired in the theorem. We will get a contradiction by the following three cases.

**Case 1:** $G$ contains hypertrees attached at two or more vertices of $G_0$. Let $T_1, T_2$ be two hypertrees attached at $v_1, v_2$ of $G_0$ respectively. Let $x$ be a first eigenvector of $G$. Assume $|x_{v_1}| \geq |x_{v_2}|$. Relocating $T_2$ rooted at $v_2$ and attaching to $v_1$, we will get a hypergraph $\tilde{G} \in \mathcal{T}_m(G_0)$ such that $\lambda_{\min}(\tilde{G}) \leq \lambda_{\min}(G)$ by Lemma 4.1. Repeating the above operation, we finally arrive at a hypergraph $G^{(1)}$ with only one hypertree $T^{(1)}$ attached at one vertex $u_0$ of $G_0$ such that $\lambda_{\min}(G^{(1)}) \leq \lambda_{\min}(G)$.

**Case 2:** $T^{(1)}$ contains edges with three or more vertices of degree greater than one, i.e. $T^{(1)}$ is not a power hypertree. Let $e$ be one of such edges containing $u, v, w$ with $d(u), d(v), d(w)$ all greater than one. Let $x$ be a first eigenvector of $G^{(1)}$, and assume that $|x_u| \geq |x_w|$. Relocating the hypertree rooted at $w$ and attaching to $u$, we will get a hypergraph $\tilde{G} \in \mathcal{T}_m(G_0)$ such that $\lambda_{\min}(\tilde{G}) \leq \lambda_{\min}(G^{(1)})$ by Lemma 4.1. Repeating the above operation on the edge $e$ until $e$ contains exactly 2 vertices of degree greater than one, and on each other edges like $e$, we finally arrive at a
hypergraph $G^{(2)}$ such that the unique hypertree $T^{(2)}$ attached at $u_0$ is a power hypertree, and $\lambda_{\min}(G^{(2)}) \leq \lambda_{\min}(G^{(1)})$.

Case 3: $T^{(2)}$ contains pendant edges not attached at $u_0$. Let $x$ be a first eigenvector of $G^{(2)}$. We assert that $|x_{u_0}| = \max_{v \in V(G_0)} |x_v|$. Otherwise, there exists a vertex $v_0$ of $G_0$ such that $|x_{v_0}| > |x_{u_0}|$. Relocating $T^{(2)}$ rooted at $u_0$ and attaching to $v_0$, we will get a hypergraph $\hat{G} \in \mathcal{T}_m(G_0)$ such that $\lambda_{\min}(\hat{G}) < \lambda_{\min}(G^{(2)})$ by Lemma 4.1. Then $\lambda_{\min}(\hat{G}) < \lambda_{\min}(G)$, a contradiction to $G$ being minimizing.

We also assert that $|x_{u_0}| = \max_{v \in V(G^{(2)})} |x_v|$. Otherwise, let $v_0 \in V(T^{(2)}) \setminus \{u_0\}$ such that $|x_{w_0}| = \max_{v \in V(G^{(2)})} |x_v| > |x_{u_0}|$. Then $w_0$ must have degree greater than one; otherwise letting $e$ be the only edge containing $w_0$, and letting $w_0$ a vertex in $e$ with degree greater than one, relocating the branch attached at $w_0$ and attaching to $w_0$, we will get a hypergraph with smaller least eigenvalue by Lemma 4.1. A contradiction. Now relocating $G_0$ rooted at $u_0$ and attaching to $w_0$, we also get a hypergraph with smaller least eigenvalue by Lemma 4.1. By a similar discussion, $u_0$ is the unique vertex satisfying $|x_{u_0}| = \max_{v \in V(G^{(2)})} |x_v|$. Finally let $e$ be a pendant edge of $T^{(2)}$ attached at $v \neq u_0$. Relocating $e$ from $v$ to $u_0$, we get a hypergraph with smaller least eigenvalue by Lemma 4.1 a contradiction. The result now follows.

If taking $G_0$ to be a single edge in Theorem 4.2, then we get the following corollary immediately.

**Corollary 4.3.** Among all $k$-uniform hypertree with $m$ edges, the minimizing hypergraph is unique, namely the hyperstar $S^k_m$.

It is easy to verify that a hypertree $T$ is odd-bipartite (by induction on the number of edges). So the spectrum of $A(T)$ is symmetric with respect to the origin $2\pi i$, and hence $\lambda_{\min}(T) = -\rho(T)$. Therefore, Corollary 4.3 implies a result of 10 when $k$ is even, which says that $S^k_m$ is the unique hypergraph with maximum spectral radius among all $k$-uniform hypertrees with $m$ edges.

Next we consider the case that $G_0$ is non-odd-bipartite. Let $C_{2n+1}$ be an odd cycle of length $2n+1$ (as a simple graph). Let $k \geq 4$ be a positive even integer. Then $C^{k, \frac{2}{k}}_{2n+1}$ is a non-odd-bipartite $k$-uniform hypergraph [14]. Let $K^k_n$ be a complete $k$-uniform hypergraph on $n \geq k + 1$ vertices. It is easy to verify that $K^k_n$ is non-odd-bipartite. If taking $G_0$ be $C^{k, \frac{2}{k}}_{2n+1}$ or $K^k_n$ in Theorem 4.2 we get the following results immediately.

**Corollary 4.4.** The minimizing hypergraph in $\mathcal{T}_m(C^{k, \frac{2}{k}}_{2n+1})$ is the unique hypergraph $C^{k, \frac{2}{k}}_{2n+1}(u) \circ S^k_m(u)$ for an arbitrary vertex $u$ of $C^{k, \frac{2}{k}}_{2n+1}$, up to isomorphism.

**Corollary 4.5.** The minimizing hypergraph in $\mathcal{T}_m(K^k_n)$ is the unique hypergraph $K^k_n(u) \circ S^k_m(u)$ for an arbitrary vertex $u$ of $K^k_n$, up to isomorphism.

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