INTRODUCTION

This paper is a continuation of [EK3]. In [EK3], we introduced the Hopf algebra $F(R)_z$ associated to a quantum R-matrix $R(z)$ with a spectral parameter defined on a 1-dimensional connected algebraic group $\Sigma$, and a set of points $z = (z_1, ..., z_n) \in \Sigma^n$. This algebra is generated by entries of a matrix power series $T_i(u), i = 1, ..., n$, subject to Faddeev-Reshetikhin-Takhtajan type commutation relations, and is a quantization of the group $GL_N[[t]]$.

In this paper we consider the quotient $F_0(R)_z$ of $F(R)_z$ by the relations $q\det_R(T_i) = 1$, where $q\det_R$ is the quantum determinant associated to $R$ (for rational, trigonometric, or elliptic R-matrices). This is also a Hopf algebra, which is a quantization of the group $SL_N[[t]]$.

This paper was inspired by [FR]. The main goal of this paper is to study the representation theory of the algebra $F_0(R)_z$ and of its quantum double, and show how the consideration of coinvariants of this double (quantum conformal blocks) naturally leads to the quantum Knizhnik-Zamolodchikov equations of Frenkel and Reshetikhin [FR]. Our construction for the rational R-matrix is a quantum analogue of the standard derivation of the Knizhnik-Zamolodchikov equations in the Wess-Zumino-Witten model of conformal field theory [TUY], and for the elliptic R-matrix is a quantum analogue of the construction of [KT].

Our result is a generalization of the construction of Enriques and Felder [EF], which appeared while this paper was in preparation. Enriques and Felder gave a derivation of the quantum KZ equations from coinvariants in the case of the rational R-matrix and $N=2$.

The results of this paper for the rational R-matrix (the Yangian case) can be directly generalized to the case of any simple Lie algebra $\mathfrak{g}$ (what we do here corresponds to $\mathfrak{g} = \mathfrak{sl}_N$). We did not include this generalization here since for a general $\mathfrak{g}$ it is more difficult to write explicit formulas.

We note that this paper does not use the results from [EK1,EK2] on the existence of quantization.

Finally, we would like to explain the relationship between the present paper and the papers [FR,KS], which are devoted to the same subject. The papers [FR,KS]
generalize to the quantum case the construction of Tsuchiya-Kanie ([TK]), which
represents conformal blocks as intertwiners between a highest weight representation
and the tensor product of a highest weight and a finite dimensional representation
of an affine Lie algebra. This allows to obtain the quantum KZ equations, but does
not allow to define the quantum analogue of fusion of highest weight modules for
affine Lie algebras. On the contrary, the approach of the present paper generalizes
the coinvariant approach [TUY], and allows to define a quantum analogue of fusion
(i.e. of vertex operator calculus).

However, the results of our paper hold only for formal quantum parameter $h$,
while the results of [FR,KS] are valid for numerical $h$. It is an interesting question
what is the analogue of our results for numerical $h$.

In the next paper we will discuss the notion of a quantum vertex operator algebra,
which naturally arises from the setting of “Quantization of Lie bialgebras III and
IV”, and describe the quantum analogue of the affine vertex operator algebra.

ACKNOWLEDGEMENTS

The authors were supported by NSF grant DMS-9700477.

1. The algebra $F_0(R)$

1.1. R-matrices with crossing symmetry.

We recall the setting of [EK3].

Let $k$ be an algebraically closed field of characteristic 0.
Recall [EK3] that $R_{rat}(u)$, $R_{tr}(u)$, $R_{ell}(u)$ denote the rational, the trigonometric,
and the elliptic R-matrix respectively. For example, $R_{rat}(u) = 1 - \frac{h(\sigma - 1/N)}{N(u - h/N^2)}$, where
$\sigma$ is the permutation of components.

Let $R(u) \in \text{End}(k^N \otimes k^N)((u))[[h]]$ be $R_{rat}(u/N)$, $R_{tr}(u/N)$ or
$R_{ell}(u/N)$.

For any connected 1-dimensional algebraic group $\Sigma$ over $k$ (i.e. $\mathbb{G}_a$, $\mathbb{G}_m$, or
an elliptic curve), we let $u$ be a canonical formal parameter on $\Sigma$ near the origin (it is
defined uniquely up to scaling). Given $R$, let $(\Sigma, u)$ be such that $R \in \text{End}(k^N \otimes
k^N) \otimes k(\Sigma)[[h]]$ and has a trivial stabilizer in $\Sigma$.

As we mentioned in [EK3], the function $R(u)$ satisfies the quantum Yang-Baxter
equation

\begin{equation}
R_{12}^{13}(u_1 - u_2)R_{13}^{23}(u_1 - u_3)R_{23}^{12}(u_2 - u_3) = R_{23}^{21}(u_2 - u_3)R_{13}^{12}(u_1 - u_3)R_{12}^{11}(u_1 - u_2),
\end{equation}

and the condition

\begin{equation}
\lim_{s \to 0} R(su, sh) = R_{rat}(u/N) = 1 - \frac{h(\sigma - 1/N)}{u - h/N^2}.
\end{equation}

Besides, it is known that the function $R(u)$ has the property of crossing symmetry
(see [Ha] for the elliptic case):

\begin{equation}
(((R(u)^{-1})_{t_1}^{-1})_{t_1} = (((R(u)^{-1})_{t_2}^{-1})_{t_2} = g(u)R(u + Nh).
\end{equation}

where $t_i$ denotes transposition in the $i$-th component, and $g \in 1 + h\mathbb{C}((u))[h]$ is a
scalar function. We will explain the meaning of this condition in section 1.4.

1.2. The Hopf algebra $F(R)$.

Recall from [EK3] the definition and properties of the algebra $F(R)$. 
Definition. The algebra $F(R)$ is the $h$-adic completion of the algebra over $k[[h]]$ whose generators are the entries of the coefficients of formal series $T(u)^{\pm 1} \in \text{End}(k^N) \otimes F(R)[[u]]$, $T(u) = T_0 + T_1 u + ...$, and the defining relations are

\begin{equation}
T(u)T(u)^{-1} = T(u)^{-1}T(u) = 1,
\end{equation}

As we saw in [EK3], this algebra is a (topological) Hopf algebra, with the coproduct, counit, and antipode defined by

\begin{equation}
\Delta(T(u)) = T^{12}(u)T^{13}(u), \quad \varepsilon(T(u)) = 1, \quad S(T(u)) = T^{-1}(u).
\end{equation}

Let $r$ be defined by $R = 1 - hr + O(h^2)$, and $GL_N(r)$ be the corresponding (infinite-dimensional) Poisson group defined in Chapter 2 of [EK3]. By Proposition 3.9(b) of [EK3], equations (1.1) and (1.2) imply that the Hopf algebra $F(R)$ is a quantization of $GL_N(r)$.

1.3. The quantum determinant for $F(R)$.

In [EK3], we introduced the quantum determinant for the dual Yangian. Let us generalize this construction to the trigonometric and elliptic cases.

Recall that for a coalgebra $A$, a right comodule over $A$ is a vector space $V$ with a linear map $\pi^* : V \to A \otimes V$ such that $(1 \otimes \pi^*) \circ \pi^* = (\Delta^{op} \otimes 1) \circ \pi^*$.

Let $E(u)$ be the basic comodule for $F(R)$ over $k[[u]]$. Namely, let $E(u)$ be the space $k^N[[u]]$ with the coaction $\pi^*_{E(u)}(v) = T^{21}(u)(1 \otimes v)$. (it is easy to check that this formula indeed defines a right comodule).

Consider the tensor product of comodules $E(u - \frac{h(N-1)}{2}) \otimes E(u - \frac{h(N-3)}{2}) \otimes \cdots \otimes E(u - h(N-1)\frac{1}{2})$. It follows from the representation theory of $F(R)$ that this comodule has a unique 1-dimensional subcomodule $QDet$ (it is enough to use the functor from comodules to modules defined in Section 2.3, and then apply the results of [Ch] for the elliptic case). Denote the generator of $QDet$ by $v_0$. Then we have $\pi_{QDet}(v_0) = q(u) \otimes v_0$, where $q(u) \in F(R)[[u]]$. The element $q(u)$ is central and grouplike, and it is called the quantum determinant. We will denote it by $qdet_R(T(u))$. For the case of the rational $R$-matrix, the quantum determinant was discussed in Section 3.3 of [EK3].

Let us write down an explicit expression for the quantum determinant. Let $e_i$ be the standard basis of $k^N$. Let $v_0 = \sum C_{i_1 \ldots i_N} e_{i_1} \otimes \cdots \otimes e_{i_N}$, and normalize $v_0$ in such a way that $C_{1 \ldots N} = 1$. Let $T = \sum E_{ij} \otimes T_{ij}$. Then

\begin{equation}
qdet_R(T(u)) = \sum_{i_1 \ldots i_N} C_{i_1 \ldots i_N} T_{i_1 i_1}(u - h(N-1)/2) \cdots T_{N i_N}(u + h(N-1)/2).
\end{equation}

In the rational case, $C_{i_1 \ldots i_N}$ is nonzero only if $i_1, \ldots, i_N$ are distinct, and equals the sign of the permutation $[1, \ldots, N] \to [i_1, \ldots, i_N]$. In this case, the formula for quantum determinant is formula (3.11) in [EK3] (with $u$ replaced by $u/N$).

This motivates the following definition. Let $A$ be any algebra and $X(u) \in \text{End}(k^N) \otimes A((u))[[h]]$. In this case, define

\begin{equation}
qdet_R(X(u)) = \sum_{i_1 \ldots i_N} C_{i_1 \ldots i_N} X_{i_1 i_1}(u - h(N-1)/2) \cdots X_{N i_N}(u + h(N-1)/2).
\end{equation}

We will need this definition below.
1.4. The normalized $R$-matrix.

For any $f \in k(\Sigma)[[h]]$, $f = \sum f_n h^n$, and $z \in \Sigma(k[[h]])$, we say that $f$ is regular at $z$ if $f_i$ are regular at $z_0$, where $z_0$ is the reduction of $z$ mod $h$.

For any $a \in \Sigma(k[[h]])$ where $R$ and $R^{-1}$ are regular, let $\tilde{E}(a)$ be the shifted basic module over $F(R)$: as a vector space $\tilde{E}(a) = k^N$, and $\pi_{\tilde{E}(a)}(T(u)) = R(u - a)$.

**Remark.** The crossing symmetry equations (1.3) have the following interpretation: the double dual of the basic module $E(0)$ over $F(R)$ is isomorphic to the tensor product of the module $E(-Nh)$ and a 1-dimensional module.

For any function $f \in k(\Sigma)[[h]]$ which is regular at $a$ together with its inverse, we can twist $\tilde{E}$ by $f$ and define a new module $\tilde{E}_f(a)$ by $\pi_{\tilde{E}_f(a)}(T(u)) = f(u-a)R(u-a)$.

**Proposition 1.1.** There exists a unique function $f_0 = 1 + O(h)$ such that $\pi_{\tilde{E}_f_0(a)}(qdetR(T(u))) = 1$.

**Proof.** Let $\pi_{\tilde{E}(a)}(qdetR(T(u))) = \rho(u-a)$, where $\rho \in k(\Sigma)[[h]]$. Then $f_0$ is defined by the condition $f_0(u - h(N - 1)/2 - a)\ldots f_0(u + h(N - 1)/2 - a) = \rho(u - a)$. This equation has a unique solution in $k(\Sigma)[[h]]$, which obviously does not depend on $a$. \□

We will call $f_0$ the normalizing function for $R$, and denote $f_0R$ by $\overline{R}$. We will call $\overline{R}$ the normalized $R$-matrix.

**Remark.** The function $f_0$ does not have a simple explicit expression even in the rational case.

**Proposition 1.2.** The normalized $R$-matrix satisfies the crossing symmetry equations and the unitarity condition:

\begin{align*}
((\overline{R}(u)^{-1})^{t_1})^{-1} &= \overline{R}(u + Nh) \\
((\overline{R}(u)^{-1})^{t_2})^{-1} &= \overline{R}(u + Nh) \\
\overline{R}(u)\overline{R}^{21}(-u) &= 1
\end{align*}

(1.6)

**Proof.** It follows from (1.3) that $((\overline{R}(u)^{-1})^{t_2})^{-1} = \phi(u + Nh)\overline{R}(u + Nh)$ for a suitable scalar function $\phi(u)$. This implies that $\tilde{E}_{f_0}(a)** = \tilde{E}_{f_0\phi}(a-Nh)$. However, for any representation $U$ of $F(R)$ the quantum determinant acts in $U$ and $U^{**}$ in the same way, since it is a grouplike element. Thus, since the quantum determinant acts trivially on $\tilde{E}_{f_0}(a)$, we get $\phi = 1$. This proves the second identity of (1.6). The first identity of (1.6) follows immediately, and the third one follows from the first two and the fact that $R(u)R^{21}(-u)$ is a scalar. \□

1.5. The $\partial$-copseudotriangular structure on the algebra $F(R)$.

In [EK3], we defined the notion of a copseudotriangular structure on a Hopf algebra $A$.

Recall from [EK3] the construction of a $\partial$-copseudotriangular structure on $F(R)$, i.e. a bilinear form $B : F(R) \otimes F(R) \rightarrow k(\Sigma)[[h]]$, satisfying equations (1.17)-(1.20) of [EK3].

Let $\partial$ be the derivation of $F(R)$ given by $\partial T(u) = T'(u)$. Let $f \in k(\Sigma)[[h]]$, and $R^f(u) = f(u)R(u)$. Then, as we showed in [EK3], the form $B^f$ defined by the
formula

\[ B^f(T^{1,p+q+1}(u_1) \ldots T^{p,p+q+1}(u_p), T^{p+1,p+q+1}(v_1) \ldots T^{p+q,p+q+1}(v_q))(y) = \]

(1.7)

\[
\prod_{i=1}^{p} \prod_{j=q}^{1} (R^f)^{i,j}(u_i - v_j + y)
\]

is a \( \partial \)-copseudotriangular structure on \( F(R) \).

We will be especially interested in the case when \( f = f_0 \), i.e. \( R^f = \bar{R} \). In this case, we denote \( B^f \) by \( B^0 \).

**Proposition 1.3.** For any \( X \in F(R) \), one has

(1.8) \[ B^0(q\text{det}_R(T(u)), X)(y) = B^0(X, q\text{det}_R(T(v)))(y) = \varepsilon(X). \]

**Proof.** Since the quantum determinant is grouplike, it is enough to show that

(1.9) \[ B^0(q\text{det}_R(T(u)), T(v))(y) = B^0(T(u), q\text{det}_R(T(v)))(y) = 1. \]

By (1.6), \( \bar{R} \) satisfies the unitarity condition \( \bar{R}(u)\bar{R}^{21}(-u) = 1 \), which implies that \( B_{12}^0(u)B_{12}^0(-u)(\Delta(X) \otimes \Delta(Y)) = \varepsilon(X)\varepsilon(Y) \). Thus, it suffices to prove only the first equality of (1.9). This equality follows by taking the quantum determinant in the identity \( B^0(T(u), T(v))(y) = \bar{R}(u - v + y) \). \( \square \)

**Corollary 1.4.** The ideal \( I \subset F(R) \) generated by the relation \( q\text{det}_R(T(u)) = 1 \) belongs to the kernel of \( B^0 \) on right and left.

1.5. **The Hopf algebra \( F_0(R) \).**

Define the Hopf algebra \( F_0(R) \) to be the quotient of \( F(R) \) by the relation \( q\text{det}_R(T(u)) = 1 \); that is, \( F_0(R) = F(R)/I \). This is a flat deformation of the algebra \( k[SL_N[[t]]] \).

By Corollary 1.4, the form \( B^0 \) descends to \( F_0(R) \) and defines a copseudotriangular structure on \( F_0(R) \). Let us denote this structure by \( B \). It is defined by the formula

(1.10) \[ B(T^{1,p+q+1}(u_1) \ldots T^{p,p+q+1}(u_p), T^{p+1,p+q+1}(v_1) \ldots T^{p+q,p+q+1}(v_q))(y) = \]

\[
\prod_{i=1}^{p} \prod_{j=q}^{1} R^{i,j}(u_i - v_j + y)
\]

**Proposition 1.5.** The form \( B \) is nondegenerate (i.e. has trivial kernel) on both sides.

**Proof.** Let \( R_s(u, h) = R(su, sh) \), and \( B_s \) be the form on \( F_0(R_s) \) constructed as above. If \( B \) is degenerate, then \( B_s \) is degenerate for all nonzero values of \( s \), therefore for \( s = 0 \). But for \( s = 0 \) we have \( R_s(u) = R_{rat}(u/N) \), so \( F_0(R) \) is the dual Yangian with opposite product, and by Drinfeld’s uniqueness theorem ([Dr1]; see also Proposition 3.2 in [EK3]) the form \( B_0 \) is the form on the dual Yangian with opposite product corresponding to the unique pseudotriangular structure \( \mathbb{R}(u) \in Y(sl_N) \otimes Y(sl_N) \).
It remains to show that the form $B_0$ is nondegenerate, i.e. that it is left-
nondegenerate and right-nondegenerate (the left and the right kernels are zero).
First of all, the right-nondegeneracy follows from left-nondegeneracy and the identity
$B(Y, X)(-u) = B(S(X), Y)(u)$, which is a consequence of the identities
$\mathbb{R}(u)\mathbb{R}^{21}(-u) = 1$ and $(S \otimes 1)(\mathbb{R})\mathbb{R} = 1$.

The left-nondegeneracy can be shown, for example, as follows: as we pointed
out in [EK3], $B_0$ is the quantization of the standard $\partial$-copseudotriangular structure $\beta$
on the Lie bialgebra $t^{-1}sl_N[t^{-1}]$, which is nondegenerate, so $B_0$ is left-nondegenerate
by Proposition 1.22 in [EK3].

Here is another proof of left-nondegeneracy of $B_0$, which does not use Proposition
1.22 of [EK3]. Recall that the Yangian $Y(sl_N)$ is generated by
$t^*(u) = t_{-1}^* u^{-1} + t_{-2}^* u^{-2} + ...$,
with defining relations (1.4),(1.5), where instead of $T(u)$ one has substituted
$T^*(u) = 1 + ht^*(u) = 1 + T_{-1}^* u^{-1} + T_{-2}^* u^{-2} + ...$, and an additional relation

$\text{qdet}_{R_0}(T^*(u)) = 1$.

The bilinear form $B_0$ defines a Hopf algebra homomorphism $\Theta : F_0(R_0) \rightarrow Y(sl_N) \otimes_k k((u))$, by $\Theta(X)(Y) = B_0(X, Y)$. (the tensor product is completed). It is easy to
check that this homomorphism is defined by the formula $\Theta(T(u)) = T^*(u + v)$
(where $T^*(v + u) := \sum T^*(m)(v)u^m/m!$).

From this formula, it is clear that $\Theta$ is injective. Indeed, let $G = SL_N$, and $\tilde{G}$
be the group of $G$-valued regular functions on $\mathbb{P}^1 \setminus 0$ which equal to 1 at infinity. Quasiclassically, $\Theta$ defines the group homomorphism $\theta : \tilde{G}(k) \rightarrow G[[u]](k((v)))$, which assigns to every element $g(s) \in \tilde{G}$ its Taylor expansion at a generic point $v$.
This homomorphism has Zariski dense image (any power series can be approximated
by polynomials), which implies that the corresponding map of function algebras is
injective.

Since $\Theta$ is injective, $B_0$ is left-nondegenerate.

The proposition is proved. $\square$

1.6. Square of the antipode.

One has the following well known proposition.

Proposition 1.6. In $F_0(R)$, one has

$S^2(T(u)) = T(u + Nh)$.

Proof. First of all, observe that the form $B$ is invariant under the antipode $S$:
$B(S(x), S(y)) = B(x, y)$. This fact follows from the fact that $B$ satisfies the hexagon
axioms.

Thus,

$B(S(T^{-1}(u)^{0,n+1}), T(v_n)^{n,n+1}...T(v_1)^{1,n+1})(z) =$

$B(T^{-1}(u)^{0,n+1}, S^{-1}(T(v_1)^{1,n+1})...S^{-1}(T(v_n)^{n,n+1}))(z)$. 

(1.13)
The last expression can be easily computed from the properties of $B$, using that $T(u)T^{-1}(u) = 1$. The answer is

$$B(S(T^{-1}(u)^{0,n+1})T(v_n)^{n,n+1}...T(v_1)^{1,n+1})(z) =$$

$$\theta^2(\overline{R}^{01}(u - v_1 + z)...\overline{R}^{bn}(u - v_n + z)) =$$

$$\theta^2(\overline{R}^{01}(u - v_1 + z))...\theta^2(\overline{R}^{bn}(u - v_n + z)) =$$

(1.14)

$$\overline{R}^{01}(u + Nh - v_1 + z)...\overline{R}^{bn}(u + Nh - v_n + z),$$

where $\theta(X) = (X^{-1})^{\epsilon_1}$. Here we have used the crossing symmetry of $\overline{R}$.

Analogously, we have

$$B(T(u + Nh)^{0,n+1}, T(v_n)^{n,n+1}...T(v_1)^{1,n+1})(z) =$$

$$\overline{R}^{01}(u + Nh - v_1 + z)...\overline{R}^{bn}(u + Nh - v_n + z).$$

(1.15)

By nondegeneracy of $B$, we have (1.12). □

2. The algebra $F_0(R)_z$ and its finite-dimensional representations.

2.1. The definition of $F_0(R)_z$.

Now let us define the factored Hopf algebra $F_0(R)_z$ with $n$ factors $F_0(R)$, corresponding to a collection of points $z_1,...,z_n \in \Sigma(k[[h]])$ (cf. [EK3]).

Let $z = (z_1,...,z_n)$, $z_i \in \Sigma(k[[h]])$ be such points that $R(u)^{\pm 1}$ is regular at $z_i - z_j$ for $i \neq j$. Denote the space of such $z$ by $\Sigma_n$.

Definition. The algebra $F_0(R)_z$ is the $h$-adic completion of the algebra over $k[[h]]$ whose generators are the entries of the coefficients of formal series $T_i(u)^{\pm 1} \in \text{End}(k^N) \otimes F_0(R)_z[[u]]$, $T_i(u) = T_0^i + T_1^i u + ...$, $i = 1,...,n$ and the defining relations are

$$T_i(u)T_i(u)^{-1} = T_i(u)^{-1}T_i(u) = 1,$$

$$R^{12}(u - v + z_i - z_j)T_i^{13}(u)T_j^{23}(v) = T_j^{23}(v)T_i^{13}(u)R^{12}(u - v + z_i - z_j),$$

(2.1)

$$\text{qdet}_{R}(T_i) = 1.$$

This algebra is a Hopf algebra, with the coproduct, counit, and antipode defined by

(2.2)  \[ \Delta(T_i(u)) = T_i^{12}(u)T_i^{13}(u), \varepsilon(T_i(u)) = 1, \quad S(T_i(u)) = T_i^{-1}(u) \]

(cf. [EK3]).

It is obvious that $F_0(R)_z = F_0(R)_{z+a}$, where $a = (a,...,a)$, $a \in \Sigma(k[[h]])$. In particular, if $n = 1$ then $F_0(R)_z$ does not depend on $z$, and is isomorphic to $F_0(R)$.

Remark 1. We showed in [EK3] that $F_0(R)_z$ can be identified with the tensor product $F_0(R)_{z_1} \otimes ... \otimes F_0(R)_{z_n}$ (i.e. with $F_0(R)^{\otimes n}$) as a coalgebra, using the map $b_1 \otimes ... \otimes b_n \rightarrow b_1...b_n$. This factorization will be useful below.

Remark 2. Let $\overline{R} = 1 - h\overline{r} + O(h^2)$. Then $\overline{r} \in \mathfrak{sl}_N \otimes \mathfrak{sl}_N$. Let $SL_N(r)_z$ be the Poisson group defined in Section 2.8 of [EK3]. Then $F_0(R)_z$ is a quantization of $SL_N(r)_z$. 7
Analogously to $F_0(R)$, the algebra $F_0(R)_L$ has a $\partial$-cospseudotriangular structure $B_L : F_0(R)_L \otimes F_0(R)_L \to k(\Sigma)[[h]]$. This structure is defined by the formula

\[ B_L(T^{1,p+q+1}(u_1) \ldots T^{p,p+q+1}(u_p), T^{p+1,p+q+1}(v_1) \ldots T^{p+q,p+q+1}(v_q))(y) = \prod_{i=1}^{p} \prod_{j=q}^{1} \frac{R^i,j}{R^i,j}(u_i - v_j + z_{m_i} - z_{l_j} + y) \]

(2.3)

In the remainder of this Chapter, we discuss modules and comodules over $F_0(R)$ and $F_0(R)_L$. In this paper, all modules will be left modules and all comodules will be right comodules, so we will subsequently omit the words “left” and “right”.

### 2.2. Finite-dimensional representations of $F_0(R)_L$.

By a representation of $F_0(R)_L$, we mean a $k$-vector space $W$ together with a $k[[h]]$-linear homomorphism $F_0(R)_L \to \text{End}W[[h]]$.

Let us describe a special class of finite dimensional representations of $F_0(R)_L$, which we call rational representations.

**Definition.** A finite dimensional representation $U$ of $F_0(R)_L$ is said to be rational if there exists a matrix function $L(u) \in \text{End}k^N \otimes \text{End}U \otimes k(\Sigma)[[h]]$, regular and invertible at $z_1, \ldots, z_n$, such that the representation is defined by the assignment $T_i(u) \rightarrow L(u + z_i)$.

**Remark.** It is obvious that the direct sum and tensor product of rational representations is a rational representation. Further, a subrepresentation and a dual representation of a rational representation is rational.

It is clear that given a rational representation $U$ of $F_0(R)_L$, the corresponding function $L(u)$ is uniquely determined. We will denote this function by $L_U(u)$.

For any rational representation $U$ of $F_0(R)_L$, and any point $a \in \Sigma(k[[h]])$, define the shifted representation $U(a)$ by the equality $L_U(a)(u) = L_U(u - a)$. The shifted representation is defined only if $L_U$ is regular and invertible at $z_i - a$.

**Examples**

1. The trivial representation: $U = k$, $L_U = 1$.
2. The basic representation $\bar{E}$: $U = k^N$, $L_U(u) = \bar{R}(u)$.
3. The shifted basic representation $\bar{E}(a)$: $U = k^N$, $L_U(u) = \bar{R}(u - a)$ ($a \in \Sigma(k[[h]])$).

Since rational representations form a tensor category, we can construct rational representations of the form $\bar{E}(a_1) \otimes \ldots \otimes \bar{E}(a_n)$.

### 2.3. Comodules over $F_0(R)$.

In this section we are interested in finite-dimensional $F_0(R)$-comodules $V$.

**Examples of $F_0(R)$-comodules:**

1. The trivial comodule: $V = k$, $\pi^*(1) = 1$.
2. The basic comodule $E$: $V = k^N$, $\pi^*(v) = T^{21}(0)(1 \otimes v)$.
3. The shifted basic comodule $E(a)$ ($a \in h(k[[h]])$: $V = k^N$,

\[ \pi^*(v) = T^{21}(a)(1 \otimes v). \]

(2.4)

Comodules over $F_0(R)$ form a tensor category. From now on, it will be convenient to us to define the tensor product in the way which is opposite to the usual one:

\[ \pi_{V \otimes W} = (m^{op} \otimes 1 \otimes 1) \circ \sigma_{23} \circ (\pi_V^* \otimes \pi_W^*). \]
In particular, we have comodules $E(a_1) \otimes \ldots \otimes E(a_n)$.

Now we will define a functor $\phi_u$ from the category of $F_0(R)$-comodules to the category of $F_0(R)$-modules, parametrized by points $u \in \Sigma(k[[h]])$ such that $R^{\pm 1}(y)$ is regular at $y = u$.

Let $V$ be a $F_0(R)$-comodule. Define the $F_0(R)$-module $\phi_u(V)$ to be $V$ as a vector space, with the following action of $F_0(R)$:

$$Xv = B_{12}(X \otimes \pi^*(v))(u).$$

The fact that this is indeed an action follows from the identity $B(XY, Z) = B(X \otimes Y, \Delta(Z))$, which is a basic property of $B$.

The functor $\phi_u$ is a tensor functor. This follows from the property that $B(X, YZ) = B(\Delta(X), Z \otimes Y)$.

Using the functor $\phi_u$, for any $z_1, \ldots, z_n$ such that $R(y)$ is regular and invertible at $z_1, \ldots, z_n$ we can construct a functor $\phi_{z_1, \ldots, z_n}$ from the category of finite dimensional comodules over $F_0(R)$ to the category of rational representations of $F_0(R)_z$. This functor is defined by the condition that $\phi_{z_1, \ldots, z_n}(U)|_{F_0(R)_{z_i}} = \phi_{z_i}(U)$.

Example. $\phi_{z_1, \ldots, z_n}(E) = \tilde{E}$.

In view of this example, the representation $\tilde{E}$ will further be denoted simply by $E$.

2.4. The $R$-matrix for comodules.

For any two finite dimensional $F_0(R)$-comodules $V, W$, define the $R$-matrix

$$R_{VW}(u) : V \otimes W \to V \otimes W((u))[[h]]$$

by the formula

$$R_{VW}(u)(v \otimes w) = B_{13}(u)(\pi^*(v) \otimes \pi^*(w)).$$

The function $R_{VW}(u)$ is defined on $\Sigma$ outside of singularities of $R(u)$. Note that $R_{EE}(u) = R(u)$.

It follows from the properties of $B$ that this $R$-matrix has the following properties:

(i) the hexagon relations:

$$R_{V_1 \otimes V_2, W}(u) = R_{V_1, W}(u)R_{V_2, W}(u), R_{V, W_1 \otimes W_2}(u) = R_{V, W_1}(u)R_{V, W_2}(u);$$

(ii) the unitarity condition: $\sigma R_{VW}(u)\sigma R_{WV}(-u) = 1$, where $\sigma$ is the permutation.

(iii) the operator $\beta_{VW}(a) := \sigma R_{VW}^{-1}(a) : V \otimes W \to W \otimes V$ is an intertwiner $\phi_u(V(a)) \otimes \phi_u(W) \to \phi_u(W) \otimes \phi_u(V(a))$ for $a \in \Sigma(k[[h]])$ (whenever it is defined).

3. Dimodules over $F_0(R)_z$

In Kac-Moody theory, one is interested in the representation theory of the (centrally extended) double of the Lie bialgebra of functions on a punctured curve, rather than in representation theory of this algebra itself. Similarly, here we are interested in representation theory of the quantum double of $F_0(R)_z$. Unfortunately, in general we do not have a convenient explicit description of this double. Therefore, we will use the notion of a dimodule over a Hopf algebra (a module and comodule simultaneously, with a consistency condition), which is equivalent to the notion of a module over the quantum double of this Hopf algebra. This notion was defined by us in [EK2], but we repeat the main definitions for the reader’s convenience.
3.1. Dimodules.

Definition. Let $A$ be a Hopf algebra over $k$. A vector space $X$ is said to be equipped with the structure of an $A$-dimodule if it is endowed with two linear maps $\pi : A \otimes X \to X$, $\pi^* : X \to A \otimes X$, such that $\pi$ is an action of $A$ on $X$ as an algebra, $\pi^*$ is a coaction of $A$ on $X$ as a coalgebra, and they agree according to the formula (cf [Dr1], p. 816)

$$\pi^* \circ \pi = (m_3 \otimes \pi) \circ \sigma_{13} \sigma_{24} \circ (S^{-1} \otimes 1^\otimes 4) \circ (\Delta_3 \otimes \pi^*),$$

where $m_3 := m \circ (m \otimes 1)$, and $\Delta_3 := (\Delta \otimes 1) \circ \Delta$.

Similarly one defines the notion of a dimodule in the case when $A, X$ are topologically free $k[[h]]$-modules.

An $A$-dimodule is called trivial if it is trivial both as a module and a comodule.

There is an obvious notion of tensor product of modules and comodules over $A$. Namely, for any two modules (comodules) $V, W$

$$\pi_{V \otimes W} = (\pi_V \otimes \pi_W) \circ \sigma_{23} \circ (\Delta \otimes 1 \otimes 1); \pi^*_{V \otimes W} = (m^{op} \otimes 1 \otimes 1) \circ \sigma_{23} \circ (\pi^*_V \otimes \pi^*_W).$$

The tensor product of dimodules is just the tensor product of the underlying modules and comodules. It follows from [Dr1], p. 816, and can be checked by a direct computation, that in this way one indeed obtains a new dimodule.

Thus, dimodules over $A$ form a tensor category.

According to the results of Drinfeld [Dr1], the category of dimodules over $A$ has a natural structure of a braided tensor category. The braiding is defined by the formula

$$\beta = \sigma \circ R, R = (\pi \otimes 1) \circ \sigma_{12} \circ (1 \otimes \pi^*).$$

Drinfeld proved that (3.3) satisfies the hexagon relations.

The meaning of the notion of a dimodule is clarified by the following construction ([Dr1]). Let $A$ be a finite dimensional Hopf algebra over $k$, and $M$ a dimodule over $A$. Then $M$ has a natural structure of a module over the quantum double $D(A) = A \otimes A^{op}$. Namely, $A$ already acts in $M$, and the action of $A^{op}$ is defined by $\pi(a^* \otimes v) = (a^* \otimes 1)(\pi^*(v))$, $a^* \in A^*$, $v \in M$. It can be checked that these actions are compatible. Thus, we have defined a functor $F$ from the category of $A$-dimodules to the category of $D(A)$-modules.

**Proposition 3.1.** The functor $F$ with trivial tensor structure is an equivalence from the braided tensor category of $A$-dimodules to the braided tensor category of $D(A)$-modules.

**Proof.** Follows from [Dr1]. □

Now let $B \subset A$ be a Hopf subalgebra, and $I \subset A$ be a two-sided Hopf ideal.

**Definition.** A vector space ($k[[h]]$-module) $X$ is said to be equipped with the structure of an $(B, A, I)$-dimodule if it is endowed with two linear maps $\pi : B \otimes X \to X$, $\pi^* : X \to A/I \otimes X$, such that $\pi$ is an action of $B$ on $X$ as an algebra, $\pi^*$ is a coaction of $A/I$ on $X$ as a coalgebra, and they agree according to formula (3.1).

If $I = 0$, we will refer to $(B, A, I)$-dimodules as $(B, A)$-dimodules.
Analogously to Proposition 3.1, one can show that if $A$ is finite-dimensional, then the tensor category of $(B, A, I)$-dimodules is equivalent to the tensor category of modules over the Hopf subalgebra $B \otimes (A/I)^{op} \subset D(A)$.

Now let us define the induction functor $\text{Ind}^{(A,A,I)}_{(B,A,I)}$ from the category of $(B, A, I)$-dimodules to the category of $(A, A, I)$-dimodules. Let $V$ be an $(B, A, I)$-dimodule, with a coaction $p^* : V \to A/I \otimes V$.

**Proposition 3.2.** There exists a unique $(A, A, I)$-dimodule $\text{Ind}^{(A,A,I)}_{(B,A,I)}V$ equal to $A \otimes_B V$ as a vector space, such that

\begin{equation}
\pi(a \otimes b \otimes v) = ab \otimes v, \quad \pi^*(1 \otimes v) = \sigma_{12} \circ (1 \otimes p^*(v)), \quad a, b \in A, \quad v \in V,
\end{equation}

where $\sigma_{12}$ denotes the permutation.

**Proof.** The uniqueness is obvious. The existence follows from the fact that $\text{Ind}^{(A,A,I)}_{(B,A,I)}V$ is isomorphic to $\text{Ind}^{A}_{B}V$ as an $A$-module. The coaction of $A/I$ on $V$ can be computed using (3.4) and (3.1). $\square$

### 3.2. Finite-dimensional dimodules.

For $i = 1, \ldots, n$, set $z_i := (z_1, \ldots, \hat{z}_i, \ldots, z_n) \in \Sigma_{n-1}$. Then $F_0(R)_{z_i}$ is the subalgebra in $F_0(R)_z$ generated by the entries of $T_j(u), j \neq i$. Let $V$ be a finite dimensional comodule over $F_0(R)$. Consider the functor $\phi_{z_1 - z_i, \ldots, z_n - z_i}$ (where $z_i - z_i$ is omitted). Applying this functor to $V$, we will get a rational representation of $F_0(R)_{z_i}$. Denote this rational representation by $V(z_i).

The space $V(z_i)$ is equipped with a structure of an $F_0(R)_{z_i}$-comodule, induced by the embedding $F_0(R) = F_0(R)_{z_i} \to F_0(R)_z$. Thus, $V(z_i)$ is an $F_0(R)_{z_i}$-module and $F_0(R)_{z_i}$-comodule, and these structures are connected by the formula $av = B_{z_i 12}(0)(a, \pi^*(v)), a \in F_0(R)_{z_i}$.

**Remark.** Although $B$ has a pole at 0, the function $B_z(a, b)(u)$ is regular at $u = 0$ (for generic $z_i$) if $a \in F_0(R)_{z_i}, b \in F_0(R)_{z_j}, i \neq j$.

**Proposition 3.3.** The space $V(z_i)$, equipped with the coaction of $F_0(R)_{z_i}$ and an action of $F_0(R)_{z_i}$, defined above, is an $(F_0(R)_{z_i}, F_0(R)_{z_i})$-dimodule.

**Proof.** The proof amounts to checking identity (3.1). Clearly, it is enough to check it on elements $a \otimes v$, where $v \in V(z_i)$, and $a$ is among the generators of the algebra $F_0(R)_{z_i}$. In other words, it is enough to check (3.1) on the formal series $T_j(u) \otimes v$ for all $j \neq i, v \in V$.

In the following computation, indices above a tensor will denote the numbers of components of the tensor product where this tensor appears.

Let the comodule structure of $V(z_i)$ be given by

\begin{equation}
\pi^*(v) = T_V^{21}(1 \otimes v),
\end{equation}

where $T_V \in \text{End}V \otimes F_0(R)$.

From (3.5), we get

\begin{equation}
\pi^*(\pi(T_j^{01}(u)v^2)) = T_V^{21}T_j^{02}(u)v^2.
\end{equation}
Computation of the right hand side of (3.1) on $v$ yields

$$RHS = (m_3 \otimes \pi)(S^{-1}(T_j^{03}(u))T_j^{04}(u)T_j^{01}(u)T_V^{52})v^5 = $$(3.7)

$$(m_3 \otimes \pi)(S^{-1}(T_j^{03}(u))T_j^{04}(u)T_V^{52}T_j^{01}(u)v^5).$$

Using the substitution $T_j^{01}(u) = S^{-1}(T_j^{01}(u)^{-1})$ and $T_V = S^{-1}(T_V^{-1})$, we can simplify (3.7) as follows:

$$RHS = (1 \otimes S^{-1} \otimes 1)(T_j^{01}(u)T_j^{02}(u)(T_V^{21})^{-1}(T_j^{01})^{-1}(u)v^2).$$

Thus, Proposition 3.3 is a consequence of the following Lemma.

**Lemma.** We have the following version of the Yang-Baxter equation:

$$T_j^{01}(u)T_j^{02}(u)T_V^{12} = T_V^{12}T_j^{02}(u)T_j^{01}(u).$$

in $\text{End}^k N \otimes \text{End} V \otimes F_0(R)_z$.

Indeed, transposing components 1 and 2 in (3.9) and multiplying both sides by $(T_V^{21})^{-1}$ from the right, we get that the RHS of (3.6) and (3.8) are the same.

**Proof of the Lemma.** We have

$$T_j^{01}(u)T_j^{02}(u)T_V^{12} = B_{z,43}(0)(T_j^{04}(u)T_j^{02}(u)T_V^{13}T_V^{12}) =$$

$$m_{32}B_{z,54}(0)(T_V^{14}T_V^{12}T_j^{05}(u)T_j^{03}(u)) = m_{35}B_{z,24}(0)(T_V^{14}T_V^{15}T_j^{02}(u)T_j^{03}(u)).$$

On the other hand

$$T_V^{12}T_j^{02}(u)T_j^{01}(u) = B_{z,43}(0)(T_V^{12}T_j^{02}(u)T_V^{04}(u)T_V^{13}) =$$

$$m_{42}B_{z,35}(0)(T_V^{14}T_V^{15}T_j^{02}(u)T_j^{03}(u)).$$

Expressions (3.10) and (3.11) are equal by property (1.18) in [EK3] of the copseudotriangular structure $B_z$. The Lemma and Proposition 3.3 are proved. □

We will call the dimodules $V(z_i)$ the evaluation dimodules.

**Example.** Let $V = E$ be the basic comodule over $F_0(R)$, defined by the formula $T_V = T$. In this case, $V(z_i) = E(z_i)$ is the shifted basic representation of $F_0(R)_z$.

### 3.3. Central extensions.

In this section we will define useful central extensions $\hat{F}_0(R), \hat{F}_0(R)_z$ of the Hopf algebras $F_0(R), F_0(R)_z$, which will also be Hopf algebras.

Define $\hat{F}_0(R) := F_0(R) \otimes_k k[c]$, where $c$ is central and the tensor product is $h$-adically complete. Equip $\hat{F}_0(R)$ with the coproduct such that $c$ is a primitive element, and

$$\hat{\Delta}(a) = e^{\frac{h}{c} \otimes \partial - \partial \otimes c}) \Delta(a), a \in F_0(R),$$

(3.12)

where $\partial$ is the derivation defined above. Define the counit on $\hat{F}_0(R)$ to be the extension of the counit on $F_0(R)$. It is clear that these coproduct and counit satisfy the axioms of a bialgebra.
It is easy to show that the antipode of $F_0(R)$ extends to the antipode on $\hat{F}_0(R)$ by setting $S(c) = -c.$ Thus, $\hat{F}_0(R)$ is a Hopf algebra.

The Hopf algebra $\hat{F}_0(R)$ is defined by the generators $T(u), c$ with defining relations \((1.4)\) and $[T(u), c] = 0$. By \((3.12)\), the coproduct in this Hopf algebra is given by

\[
\Delta(T(u)) = T^{12}(u - hc^3/2)T^{13}(u + hc^2/2).
\]

Now introduce the multi-point analog of $\hat{F}_0(R)$ – the algebra $\hat{F}_0(R)_z$. To do this, consider the $\partial$-copseudotriangular structure $\hat{B}$ on $\hat{F}_0(R)$ obtained by pulling back the $\partial$-copseudotriangular structure from $F_0(R)$ under the natural morphism of Hopf algebras $\hat{F}_0(R) \to F_0(R)$ ($c \to 0$). Define $\hat{F}_0(R)_z$ to be the factored Hopf algebra $A_z$ for $A = \hat{F}_0(R)$ and $\partial$-copseudotriangular structure $\hat{B}$ (see \cite{EK3}).

It is checked directly from the definition, analogously to Proposition 3.25 of \cite{EK3}, that $\hat{F}_0(R)_z$ is the $h$-adically complete algebra generated by entries of $T_i(u)^{\pm 1}, i = 1, \ldots, n,$ and central elements $c_1, \ldots, c_n,$ with the relations

\[
R^{12}(u - v + z_i - z_j - \frac{h}{2}(c_i^3 - c_j^3))T^{13}_i(u)T^{23}_j(v) =
\]

\[
T^{23}_j(v)T^{13}_i(u)R^{12}(u - v + z_i - z_j + \frac{h}{2}(c_i^3 - c_j^3)),
\]

\[
T_i(u)T^{-1}_i(u) = T_i^{-1}(u)T_i(u) = 1.
\]

It is easy to see that there exists a unique Hopf algebra structure on $\hat{F}_0(R)_z$ such that $c_i$ are primitive elements, and

\[
\Delta(T_i(u)) = T^{12}_i(u - hc_i^3/2)T^{13}_i(u + hc_i^2/2), \varepsilon(T_i(u)) = 1, S(T_i(u)) = T_i(u)^{-1}.
\]

Now fix $K \in k[[h]]$. Let $\langle c \rangle$ be the ideal in $\hat{F}_0(R)$ generated by $c$. Let $V$ be a finite dimensional $F_0(R)$-comodule. Then we can extend $V$ to a $(k[c], \hat{F}_0(R), \langle c \rangle)$-dimodule by setting

\[
c|_V = K\text{Id}.
\]

We denote the obtained dimodule by $V_K$, and call it an evaluation dimodule with central charge $K$.

It is easy to generalize this construction to the multi-point situation. Let $C$ be the ideal in $\hat{F}_0(R)_z$ generated by $c_1, \ldots, c_n$. Let $V(z_i)$ be as above. Then we can extend $V(z_i)$ to a $(\hat{F}_0(R)_z \otimes k[c_i], \hat{F}_0(R)_z, C)$-dimodule by setting

\[
c_i|_{V(z_i)} = K\text{Id}, c_j|_{V(z_i)} = 0, j \neq i.
\]

We denote the obtained dimodule by $V_K(z_i)$, and call it an evaluation dimodule with central charge $K$.
3.4. Local dimodules.

Let $V$ be a finite dimensional $F_0(R)$-comodule, and $V_K$ be the corresponding evaluation dimodule with central charge $K$. Define an infinite-dimensional $(\hat{F}_0(R), \hat{F}_0(R), \langle \cdot, \cdot \rangle)$-dimodule

\[
\hat{V}_K := \text{Ind}^\langle \hat{F}_0(R), \hat{F}_0(R), \langle \cdot, \cdot \rangle \rangle_{\langle k[c], F_0(R), \langle \cdot, \cdot \rangle \rangle}(V_K).
\]

**Remark.** The dimodule $\hat{V}_K$ is analogous to highest weight (Weyl) modules over affine Kac-Moody algebras.

This construction has a multi-point analogue. Let $V_K(z_i)$ be the evaluation dimodule with central charge $K$ defined above. Define an infinite-dimensional $(\hat{F}_0(R)_z, \hat{F}_0(R)_z, C)$-dimodule

\[
\hat{V}_K(z_i) := \text{Ind}^\langle \hat{F}_0(R)_z, \hat{F}_0(R)_z, C \rangle_{\langle \hat{F}_0(R)_z \otimes k[c], \hat{F}_0(R)_z, C \rangle}(V_K(z_i)).
\]

We will call $\hat{V}_K(z_i)$ a local dimodule at the point $z_i$.

Clearly, the space $\hat{V}_K(z_i)$ is naturally isomorphic to $\hat{V}_K$, via the identification $\hat{F}_0(R)_{z_i} \to \hat{F}_0(R)$. Therefore, $\hat{V}_K(z_i)$ is naturally an $F_0(R)$-comodule.

Let the comodule structure on $\hat{V}_K$ be defined by the formula $\pi^*(v) = T_{\hat{V}_K}^{21}(1 \otimes v)$, where $T_{\hat{V}_K} \in \text{Hom}(\hat{V}_K, \hat{V}_K \otimes F_0(R))$.

The action of $\hat{F}_0(R)_{z_i}$ on $\hat{V}_K(z_i)$ has a convenient description in terms of the element $T_{\hat{V}_K}$.

**Proposition 3.4.** The action of $F_0(R)_{z_i}$ on $\hat{V}_K(z_i)$ is given by the formula

\[
T_j(u) \to B_{34}(z_j - z_i)(T_{\hat{V}_K}^{13}(u), T_{\hat{V}_K}^{24}), j \neq i.
\]

**Proof.** On the top component $V(z_i) \subset \hat{V}_K(z_i)$, formula (3.19) follows from the definition of the evaluation dimodule $V(z_i)$. To go from the case of the top component to the general case, it is enough to observe that both sides of (3.19) have the same commutation relations with $T_i(u)$.  

Proposition 3.4 implies that $\hat{V}_K(z_i)$ is a locally rational representation of $F_0(R)_{z_i}$, i.e. an (h-adically completed) inductive limit of finite dimensional rational representations.

3.5. Global dimodules and quantum conformal blocks.

Now let $V^1, \ldots, V^n$ be finite dimensional $F_0(R)$-comodules. Define the $(\hat{F}_0(R)_z, \hat{F}_0(R)_z, C)$-dimodule

\[
M_K(V^1, \ldots, V^n, z) := \hat{V}_K^1(z_1) \otimes \ldots \otimes \hat{V}_K^n(z_n).
\]

We will call $M_K(V^1, \ldots, V^n)$ a global dimodule. Sometimes we will shortly denote it by $M_K(z)$.

Let $\hat{F}_0(R)_z = \hat{F}_0(R)_z/(c_i = c_j, i \neq j)$. It is clear that $\hat{F}_0(R)_z$ is a Hopf algebra. We will denote by $c$ the image of $c_i$ in this quotient. An important property of $\hat{F}_0(R)_z$, which $\hat{F}_0(R)_z$ does not have, is that $F_0(R)_z$ is a subalgebra of $\hat{F}_0(R)_z$. 


4.1. The Hopf algebra $F_T$ equipped with a weak topology. Let us define a Laurent series

$$B_K(V^1, ..., V^n, z) = B_K(z) := \text{Hom}_{F_0(R)_z}(M_K(z), k[[h]]).$$

The elements of $B_K(z)$ are quantum analogues of conformal blocks in the Wess-Zumino-Witten model. Therefore we will call them “quantum conformal blocks”.

We have a natural evaluation map $\xi : B_K(z) \to (V^1 \otimes ... \otimes V^n)^*$ defined by

$$\xi(f)(v_1 \otimes ... \otimes v_n) = f(v_1 \otimes ... \otimes v_n), \quad v_i \in V^i(z_i) \subset \hat{V}_K(z_i).$$

**Proposition 3.5.** The map $\xi$ is a linear isomorphism.

**Proof.** The statement follows from Frobenius reciprocity. \(\square\)

Thus we have constructed a vector bundle $B_K(z)$ over the space $\Sigma_n$ and a trivialization $\xi$ of this bundle.

4. The dual Hopf algebra and the double of $F_0(R)$

4.1. The Hopf algebra $F_0(R)^{\ast \text{op}}$.

Consider the dual Hopf algebra $F_0(R)^{\ast \text{op}}$. It is a topological Hopf algebra, equipped with a weak topology. Let us define a Laurent series $T^*(v) \in \text{Mat}_N(k) \otimes F_0(R)^{\ast \text{op}}((v))$ (where $F_0(R)^{\ast \text{op}}((v)) := F_0(R)^{\ast \text{op}} \otimes_k k((v))$ is the completed tensor product), which topologically generates $F_0(R)^{\ast \text{op}}$.

Let $\Theta : F_0(R) \to F_0(R)^{\ast \text{op}}((v))$ be defined by $\Theta(X)(Y) = B(X, Y)$. This map is a Hopf algebra homomorphism. By properties of $B$, $\Theta$ is injective, and its image (tenored with $k((v))$) is dense.

Define

$$T^*(v) = \Theta(T(0)) \in F_0(R)^{\ast \text{op}}((v)).$$

Since the image of $\Theta$ is dense, we get that the entries of $T^*(v)$ topologically generate $F_0(R)^{\ast \text{op}}$.

Since $\Theta$ is a Hopf algebra map, the series $T^*(v)$ satisfies equations (1.4),(1.5) (with $T^*$ instead of $T$) and the equation $\text{qdet}_R(T^*(v)) = 1$.

**Remark.** The series $T^*(v)$ is always infinite in the negative direction. However, it is convergent in the weak topology, i.e. $T_{-m} \to 0$ when $m \to +\infty$.

For Yangians $(R(u) = R_{\text{rat}}(u/N))$, the series $T^*(u)$ has the form $1 + \sum T^*_{-j} u^{-j}$, but in general it is infinite in both directions. However, the algebras $F_0(R)^{\ast \text{op}}$ should have “the same size” for all $R$. This means that we should expect that nonnegative coefficients of $T^*(v)$ are not independent generators, but in fact express via negative coefficients. Indeed, we have

**Proposition 4.1.** The entries of the matrices $T^*_{-j}$ for $j \geq 1$ topologically generate $F_0(R)^{\ast \text{op}}$ (i.e. $T^*(v)$ can be uniquely reconstructed from its “principal part”).

**Proof.** We argue as in the proof of Proposition 1.5. Consider the R-matrix $R_s(u, h) = R(su, sh)$. Suppose that negative $T^*_i$ generate a smaller algebra than $F_0(R)^{\ast \text{op}}$. Then it is so for $F_0(R_s)$ for all $s \neq 0$. But then the same statement holds for
Now consider the dual Hopf algebra $\hat{F}_0(R)^{\text{op}}$ to the Hopf algebra $\hat{F}_0(R)$. Let $\partial \in h^{-1}\hat{F}_0(R)^{\text{op}}$ be the primitive element which acts according to the rule $\partial(c) = -1/h, \partial(T(u)) = 0$. Then we have in $\hat{F}_0(R)^{\text{op}}$:

$$[\partial, T_*(u)] = \frac{dT_*(u)}{du}.$$  

(4.2)

This explains the notation $\partial$ for this element.

4.2. The double of $F_0(R)$ and the universal R-matrix.

Let $\mathcal{R}$ denote the universal R-matrix of the double $D(F_0(R)) = F_0(R) \otimes F_0(R)^{\text{op}}$, and $\hat{\mathcal{R}}$ the universal R-matrix of $D(\hat{F}_0(R)) = \hat{F}_0(R) \otimes \hat{F}_0(R)^{\text{op}}$ (the tensor products are completed). Since have a natural decomposition $\hat{F}_0(R) = F_0(R) \otimes k[c]$ as algebras, we can consider both $\mathcal{R}, \hat{\mathcal{R}}$ as elements of $\hat{F}_0(R) \otimes \hat{F}_0(R)^{\text{op}}$.

Proposition 4.2. One has

$$\hat{\mathcal{R}} = e^{-h(c \otimes \partial)/2} \mathcal{R} e^{-h(c \otimes \partial)/2}.$$  

(4.3)

Proof. By Drinfeld’s theorem [Dr1] of uniqueness of the R-matrix for the double, it is enough to check that (4.3) satisfies the axioms of a quasitriangular structure. The hexagon axioms are immediate, and the fact that $\hat{\mathcal{R}}$ transforms the coproduct to the opposite coproduct is checked by a straightforward computation. □

5. Quantum Knizhnik-Zamolodchikov connection

5.1. The quantum Sugawara construction.

Consider the $(\hat{F}_0(R), \hat{F}_0(R), \hat{c})$-dimodule $\hat{\mathcal{V}}$ defined by (3.19). This dimodule is automatically equipped with a coaction of $F_0(R)$, which is obtained from the coaction of $\hat{F}_0(R)$ by setting $c = 0$. Moreover, the algebra $F_0(R)$ embeds naturally in $\hat{F}_0(R)$ as a subalgebra, which means that it acts in $\hat{\mathcal{V}}$. Let us denote these actions and coactions by $\hat{\pi}$ and $\hat{\pi}^*$ (warning: in general they do not combine into a dimodule structure).

Define the operator $Q : \hat{\mathcal{V}} \to \hat{\mathcal{V}}$, given by

$$Qx = \hat{\pi}((\tau_{kh/2}S^{-1} \otimes 1)(\hat{\pi}^*(x))),$$  

where $\tau_a(T(u)) := T(u - a)$. We call the operator $Q$ the Sugawara operator.

Consider the evaluation comodule $V$ and let $T_V \in \text{End}(V) \otimes F_0(R)$ be such that $\pi^*(v) = T_V^{21}(1 \otimes v)$. For example, if $V = E(a), a \in h[k[h]],$ then $T_V = T(a)$.

Let $v \in V \subset \hat{\mathcal{V}}(0)$. Then from the definition of $Q$ we get

$$Qv = m_{12}(\tau_{kh/2}S^{-1}(T_V^{21}))(1 \otimes v),$$  

(5.2)

where $m_{12}$ denotes the multiplication of components.

From now on we set $\kappa = K + N$.

Let $\alpha$ be the automorphism of $\hat{F}_0(R)$ defined by $\alpha(T(u)) = T(u - \kappa h), \alpha(c) = c$. Denote by $\hat{\mathcal{V}}_K$ the dimodule $\hat{\mathcal{V}}$ twisted by $\alpha$, i.e. it is the same as a vector space, and $T(u)|_{\hat{\mathcal{V}}_K} = \alpha(T(u))|_{\hat{\mathcal{V}}_K}$.
Proposition 5.1. The linear operator $Q$ is an isomorphism of dimodules: $\hat{V}_K \to \hat{V}_K^\alpha$. In particular,

\[
QT(u)Q^{-1} = T(u - \kappa h),
\]
\[
QT^*(u)Q^{-1} = T^*(u - \kappa h),
\]
on $\hat{V}_K$.

Proof. Following Drinfeld, consider the quantum Casimir operator on $\hat{V}_K(0): U = m_{12}((S^{-1} \otimes 1)(\mathcal{R}))$. By Drinfeld’s theorem [Dr2], it satisfies the following equation:

\[
U^{-1}XU = S^2(X),
\]
\[
X \in F_0(R).
\]

On the other hand, using Proposition 4.2 and the definitions of $\bar{\pi}, \bar{\pi}^*$, we can express $U$ in terms of $Q$ as follows:

\[
U = m_{12}((S^{-1} \otimes 1)(e^{-h(c\otimes\partial)/2}\mathcal{R}e^{-h(c\otimes\partial)/2})) = m_{12}(e^{h(c\otimes\partial)/2}(S^{-1} \otimes 1)(\mathcal{R})e^{h(c\otimes\partial)/2}) = Qe^{Kh\partial}.
\]

Combining (5.4) and (5.5), we get

\[
Q^{-1}XQ = e^{-Kh\partial}U^{-1}XUe^{Kh\partial} = e^{Kh\partial}S^2(X)e^{-Kh\partial}.
\]

Using Proposition 1.6, we get the desired result. □

5.2. The braiding.

Let $V^1, ..., V^n$ be finite-dimensional $F_0(R)$-comodules. Let $i, j \in \{1, ..., n\}, i \neq j$. Recall the notations and definitions of Chapter 3.

Proposition 5.2. There exists a unique isomorphism of dimodules $\beta_{ij} : \hat{V}_K^i(z_i) \otimes \hat{V}_K^j(z_j) \to \hat{V}_K^j(z_j) \otimes \hat{V}_K^i(z_i)$, such that $\beta_{ij}|_{V^i \otimes V^j} = \sigma R_{V^i \otimes V^j}^{-1}(z_i - z_j)$.

Proof. It follows from our definitions that the operator $\sigma R_{V^i \otimes V^j}(z_i - z_j)^{-1} : V^i(z_i) \otimes V^j(z_j) \to V^j(z_j) \otimes V^i(z_i)$ is an isomorphism of evaluation dimodules. As the dimodules $\hat{V}_K^i(z_i) \otimes \hat{V}_K^j(z_j), \hat{V}_K^j(z_j) \otimes \hat{V}_K^i(z_i)$ are induced from $V^i(z_i) \otimes V^j(z_j), V^j(z_j) \otimes V^i(z_i)$, this operator extends uniquely to an isomorphism of dimodules. □

It is clear from Section 2.4 that $\beta_{ij}$ satisfy the braid relations and the unitarity condition $\beta_{ij}\beta_{ji} = 1$. We will call them the braiding maps.

Now we compute $\beta_{ij}$ explicitly. Recall that the dimodules $\hat{V}_K^i(z_i), \hat{V}_K^j(z_j)$ have the structure of $F_0(R)$-comodules.

Let $\mathbb{R}(u) \in F_0(R)^{op} \otimes F_0(R)^{op}((u))$ be the element defined by the condition $B(X, Y)(u) = \langle Y \otimes X, \mathbb{R}^{-1}(u) \rangle$.

Proposition 5.3. $\beta_{ij}(v \otimes w) = \sigma \mathbb{R}(z_i - z_j)(v \otimes w)$.

Proof. On the top component $V^i(z_i) \otimes V^j(z_j)$, the operators $\beta_{ij}$ and $\sigma \mathbb{R}(z_i - z_j)$ are the same by the definition of $\beta_{ij}$. Thanks to Proposition 5.2, it is now enough to show that the map $v \otimes w \to \sigma \mathbb{R}(z_i - z_j)(v \otimes w)$ commutes with $T_i^{ij}(u)$ and $T_j^{ij}(u)$. 

17
According to (3.13), on $\hat{V}_K^i(z_i) \otimes \hat{V}_K^j(z_j)$ we have
\begin{equation}
T_j^{0,ij}(v) = T_j^{0,i}(v - Kh/2)T_j^{0,j}(v).
\end{equation}
Since $\hat{V}_K^i(z_i)$ is a locally rational representation of $F(R)_{z_i}$ obtained by applying of the functor $\phi_{z_i}$ to some $F_0(R)$-comodule, we obtain from (5.6)
\begin{equation}
T_j^{0,ij}(v) = T_j^{*0,i}(v - z_i + z_j - Kh/2)T_j^{0,j}(v).
\end{equation}
Thus, we need to check the relation
\begin{equation}
\mathbb{R}^{ij}(u)T^{0;i}(v - u - Kh/2)T^{0;i}(v) = T^{0;j}(v)T^{0;i}(v - u + Kh/2)\mathbb{R}^{ij}(u).
\end{equation}
in $\text{Mat}_N(k) \otimes D(F_0(R))_K \otimes D(F_0(R))_K$, where $D(F_0(R))_K := D(F_0(R))/(c = K)$.
(the relation that $\sigma \mathbb{R}(z_i - z_j)$ commutes with $T_j^{ij}$ is (5.8) for $u = z_i - z_j$).
Recall the quasirationality property of $\tilde{R}$:
\begin{equation}
\tilde{R}^{12}T^{01}(v - Kh/2)T^{02}(v + Kh/2) = T^{02}(v - Kh/2)T^{01}(v + Kh/2)\tilde{R}^{12}.
\end{equation}
Using Proposition 4.2, we see that in terms of $\mathcal{R}$, this property can be written as
\begin{equation}
\mathcal{R}^{12}T^{01}(v - Kh/2)T^{02}(v) = T^{02}(v)T^{01}(v + Kh/2)\mathcal{R}^{12}.
\end{equation}
Let us apply the map $\Theta(-u)$ defined in Chapter 4 to the component 1 of (5.10).
Using (4.1) and the identity $(\Theta(-u) \otimes 1)(\mathcal{R}) = \mathbb{R}(u)$, we get exactly (5.8).
The fact that $\sigma \mathbb{R}(z_i - z_j)$ commutes with $T_j^{ij}$ is checked similarly. The Proposition is proved.

5.3 The quantum Knizhnik-Zamolodchikov connection.

Let $\{e_i\}$ be the standard basis in $k^n$, i.e. $(e_i)_j = \delta_{ij}$. Let $\mathcal{E}$ be a vector bundle over the space $\Sigma_n$, and $a \in k[[h]]$. By an $a$-connection on $E$ we mean any collection of maps $A_i(z) : E_z \to E_{z - ahe_i}$, defined for every $z \in \Sigma_n$, where $E_z$ denotes the fiber of $\mathcal{E}$ over $z$. An $a$-connection $\{A_i\}$ is called flat if $A_j(z - ahe_i)A_i(z) = A_i(z - ahe_j)A_j(z)$.

In the previous sections we defined an infinite-dimensional vector bundle $M_K(z) = \hat{V}_K^1(z_1) \otimes \ldots \otimes \hat{V}_K^n(z_n)$ over the configuration space $\Sigma_n$. This bundle has a natural trivialization, since $\hat{V}_K^i(z_i) = V^i \otimes F_0(R)$ as vector spaces. For any $z, z' \in \Sigma_n$, denote by $I_{z,z'} : M_K(z) \to M_K(z')$ the identification map defined by this trivialization.

Now we define a flat $(-\kappa)$-connection in the vector bundle $M_K(z)$. We do it by prescribing, for any $i \in \{1, \ldots, n\}$, the holonomy operator $A_i(z) : M_K(z) \to M_K(z + \kappa he_i)$.

Set
\begin{equation}
A_i(z) = I_{z,z + \kappa he_i}Q_i,
\end{equation}
where $Q_i$ is the operator $Q$ described in section 5.1 which acts in the $i$-th component of the tensor product. It is obvious that $\{A_i\}$ define a flat $(-\kappa)$-connection. This $(-\kappa)$-connection is called the quantum Knizhnik-Zamolodchikov (KZ) connection.
Proposition 5.4. We have the following identities:

\[ A_i(z)T_i^{0,1...n}(u) = T_i^{0,1...n}(u - \kappa h)A_i(z), \]
\[ A_i(z)T_j(u) = T_j(u)A_i(z), \quad j \neq i. \]  

(5.12)

Proof. We know that \( \hat{V}_K^i(z_i) \) is a locally rational representation of \( F_0(R)z_i \). Let \( L(u) \) be the corresponding matrix function. Let \( \theta_{ij} = 1 \) if \( i > j \), \(-1\) if \( i < j \), and 0 if \( i = j \). Recall that in \( \hat{V}_K^1(z_1) \otimes ... \otimes \hat{V}_K^n(z_n) \) one has

\[ \Delta_p(T_i^{01}(u)) = T_i^{01}(u + \frac{K\theta_{1i}}{2})...T_i^{0p}(u + \frac{K\theta_{pi}}{2}). \]

(5.13)

Therefore, by the definition of \( \hat{V}_K^i(z_i) \), we have in \( \hat{V}_K^1(z_1) \otimes ... \otimes \hat{V}_K^i(z_i) \otimes \hat{V}_K^n(z_n) \):

\[ T_i^{0,1...n}(u) = \]
\[ L^{01}(u + z_i - z_1 - \frac{K\theta_{1i}}{2})...L^{0,i-1}(u + z_i - z_i-1 - \frac{K\theta_{pi}}{2})T_i^{0i}(u) \times \]
\[ L^{0,i+1}(u + z_i - z_{i+1} + \frac{K\theta_{j1}}{2})...L^{0n}(u + z_i - z_n + \frac{K\theta_{pn}}{2}). \]

(5.14)

Using the first relation of (5.3), we obtain

\[ A_i(z)T_i^{0,1...n}(u) = \]
\[ L^{01}(u + z_i - z_1 - \frac{K\theta_{1i}}{2})...L^{0,i-1}(u + z_i - z_i-1 - \frac{K\theta_{pi}}{2})T_i^{01}(u - \kappa h) \times \]
\[ L^{0,i+1}(u + z_i - z_{i+1} + \frac{K\theta_{j1}}{2})...L^{0n}(u + z_i - z_n + \frac{K\theta_{pn}}{2})A_i(z) = \]
\[ T_i^{0,1...n}(u - \kappa h)A_i(z). \]

(5.15)

This proves the first relation in (5.12).

Similarly, using the second relation in (5.3), one deduces the second relation in (5.12). □

Proposition 5.5. Let \( f \in B_K(z + \kappa e_i) \). Then the functional \( A_i^*(z)f \) on \( M_K(z) \) defined by

\[ Q_i^*f(x_1 \otimes ... \otimes x_i \otimes ... \otimes x_n) := f(x_1 \otimes ... \otimes Q_ix_i \otimes ... \otimes x_n), \quad x_j \in \hat{V}_K^j(z_j). \]

(5.16)

belongs to \( B_K(z) \).

Proof. The proposition follows immediately from Proposition 5.2 and the definition of \( B_K(z) \). □

Proposition 5.5 shows that the \((-\kappa)\)-connection \( \{ A_i \} \) gives rise to a well defined flat \( \kappa \)-connection \( \{ \hat{A}_i \} \) on the vector bundle \( B_K(z) \), defined by the formula \( \hat{A}_i(z) = A_i^*(z - \kappa e_i) \). This \( \kappa \)-connection is called the quantum KZ connection on quantum conformal blocks.
Proposition 5.6. The connection \( \{ A_i \} \) commutes with the braiding maps \( \beta_{i,i+1} \). That is, \( \beta_{i,i+1} A_i(z) = A_{i+1}(\sigma_{i,i+1} z) \beta_{i,i+1} \), where \( \sigma_{i,i+1} \) is the transposition \((i,i+1)\).

Proof. Using the second relation of (5.3), we obtain the identity \( Q^{-1}_i \mathbb{R}^{ij}(u) Q_i = \mathbb{R}^{ij}(u - \kappa) \). Together with Proposition 5.3, this implies the proposition. □

5.4. Quantum KZ equations.

Since the bundle \( B_K(z) \) is equipped with a trivialization \( B_K(z) \to (V^1 \otimes \ldots \otimes V^n)^* \), the \( \kappa \)-connection \( \{ A_i \} \) in \( B_K(z) \) defines a \( \kappa \)-connection \( \{ \nabla_i(z) \} \) in the trivial bundle over \( \Sigma_n \) with fiber \( V^1 \otimes \ldots \otimes V^n \) (\( \nabla_i(z) \) acts from the fiber over \( z \) to the fiber over \( z - \kappa h e_i \)). This connection is defined by the formula \( \nabla_i = (A_i^*)^{-1} \). In this section we compute the \( \kappa \)-connection \( \{ \nabla_i \} \) explicitly.

Theorem 5.7.

\[
\nabla_i(z) = R^{i-1,i}_{V_{i-1}V_i}(z_{i-1} - z_i + \kappa h) \ldots R^{1i}_{V_1V_i}(z_1 - z_i + \kappa h) \times \]

\[
R^{ni}_{V_nV_i}(z_n - z_i) \ldots R^{i+1i}_{V_{i+1}V_i}(z_{i+1} - z_i).
\]

(5.17)

Proof. We have to compute the expression \( f(v_1 \otimes \ldots \otimes Q^{-1}_i v_i \otimes \ldots \otimes v_n) \) for an invariant functional \( f \in B_K(z - \kappa h e_i) \).

Denote by \( \mathcal{R}_i \) the element \( \mathcal{R} \) for the algebra \( F_0(R) \). Set \( \mathcal{R}_i(a) := e^{-a \partial^1} \mathcal{R}_i e^{a \partial^1} \). Set \( \mathcal{R}_i(Kh/2) = \sum_j a_j \otimes b_j \). Then we have \( Q_i = \sum_j S^{-1}(a_j)b_j \).

Using the fact that \( \hat{\mathcal{R}} \) satisfies the hexagon relations, we deduce

\[
(\Delta_n \otimes 1)(\mathcal{R}_i(Kh/2)) = (\prod_{j=1}^{i-1} \mathcal{R}_i^{j,n+1}(K)) \mathcal{R}_i^{i,n+1}(K) (\prod_{j=i+1}^n \mathcal{R}_i^{j,n+1}(K)).
\]

(5.18)

in \( \hat{V}_K(z_1) \otimes \ldots \otimes \hat{V}_K(z_n) \otimes F_0(R)^{\text{op}} \).

Let \( \mathcal{R}_{st} = (S^{-1} \otimes 1)(\mathcal{R}_i) = (1 \otimes S)(\mathcal{R}_i) \). Then from (5.18) we get

\[
(\Delta_n \otimes 1)(\mathcal{R}_{st}(Kh/2)) = (\prod_{j=n}^{i+1} \mathcal{R}_{st}^{j,n+1}(K)) \mathcal{R}_{st}^{i,n+1}(K) (\prod_{j=i-1}^1 \mathcal{R}_{st}^{j,n+1}(K)).
\]

(5.19)

We do the rest of the computation for \( i = n \). Using the invariance of \( f \) and the identity (5.19), we get

\[
f(Q_n \mathcal{R}_{sn}^{-1,n}(K) \ldots \mathcal{R}_{sn}^{1,n}(K)(v_1 \otimes \ldots \otimes v_n)) = f(v_1 \otimes \ldots \otimes v_n),
\]

where \( v_i \in V^i \). From (5.20), we obtain

\[
f(v_1 \otimes \ldots \otimes Q^{-1}_n v_n) = f(\mathcal{R}_{sn}^{-1,n}(K) \ldots \mathcal{R}_{sn}^{1,n}(K)v_1 \otimes \ldots \otimes v_n).
\]

(5.21)

Observe that since \( (S \otimes 1)(\mathcal{R}) = \mathcal{R}^{-1} \), we have \( \mathcal{R}_{st} = (S^{-2} \otimes 1)(\mathcal{R}_i^{-1}) = \mathcal{R}_i(hN)^{-1} \).

Making this substitution, we bring (5.20) to the form

\[
f(v_1 \otimes \ldots \otimes Q^{-1}_n v_n) = f(\mathcal{R}_{sn}^{-1,n}(K) \ldots \mathcal{R}_{sn}^{1,n}(K)^{-1}v_1 \otimes \ldots \otimes v_n).
\]

(5.22)
From the definition of $V^j(z_j)$ and the definition of $R_{VW}$ it follows that

$$R_n^{ij} |_{V^1(z_1) \otimes \ldots \otimes V^n(z_n)} = R^{jn}_{V^1V^n}(z_j - z_n)^{-1}.$$  

This shows that

(5.23) \[ \nabla_n(z) = R^{n-1}_{V^{n-1}V^n}(z_{n-1} - z_n + \kappa h) \ldots R^{1n}_{V^1V^n}(z_1 - z_n + \kappa h), \]

which coincides with (5.17).

To go from the case $i = n$ to the general case, it is enough to use the invariance of the quantum KZ connection with respect to the braiding. Indeed, by Proposition 5.6,

(5.24) \[ \nabla_i(z) = \beta^{-1}_{ii+1} \ldots \beta^{-1}_{n-1n} \nabla_{n'}(z') \beta_{n-1n} \ldots \beta_{ii+1}, \]

where $z' := (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n, z_i)$. Using Proposition 5.3, and (5.23), we get

\begin{align*}
\nabla_i(z) &= R^{n-1}_{V^{i+1}V^n}(z_i - z_{i+1} - \kappa h) \sigma_{i+1} \ldots R^{1n}_{V^1V^n}(z_i - z_n + \kappa h) \sigma_{n-1n} \\
&= R^{n-1}_{V^iV^n}(z_i - z_n + \kappa h) \ldots R^{i+1}_{V^iV^{i+1}}(z_{i+1} - z_i + \kappa h) R^{i-1}_{V^iV^{i+1}}(z_{i-1} - z_i + \kappa h) \ldots R^{n}_{V^nV^n}(z_n - z_i + \kappa h) \\
&= \sigma_{n-1n} R^{n-1}_{V^iV^n}(z_i - z_n)^{-1} \ldots \sigma_{i+1} R^{i+1}_{V^iV^{i+1}}(z_i - z_{i+1})^{-1}.
\end{align*}

(5.25)

After cancelations we get (5.17). The theorem is proved. □

Let $F(z)$ be a section of the trivial bundle over $\Sigma_n$ with fiber $V^1 \otimes \ldots \otimes V^n$, i.e. a function on $\Sigma_n$ with values in $V^1 \otimes \ldots \otimes V^n$. We say that $F$ is flat with respect to the $\kappa$-connection $\nabla = (\nabla_i)$ if for any $i = 1, \ldots, n$ \( \nabla_i(z)F(z) = F(z - \kappa h e_i) \).

The condition for $F$ to be flat is equivalent to the following system of difference equations:

\begin{align*}
F(z - \kappa h e_i) &= R^{i-1}_{V^iV^{i+1}}(z_{i-1} - z_i + \kappa h) \ldots R^{i+1}_{V^iV^{i+1}}(z_i - z_{i+1} + \kappa h) \\
&= R^{ni}_{V^nV^i}(z_n - z_i) \ldots R^{i+1i}_{V^{i+1}V^i}(z_{i+1} - z_i) F(z).
\end{align*}

(5.26)

This system is called the quantum KZ equations. More precisely, it can be identified with the quantum KZ equations of [FR] by reversing the $R$-matrix.

References

[Ch] I.V. Cherednik, *On Irreducible representations of elliptic quantum R-algebras*, Soviet Math. Dokl.; 34 (1987), no. 3.

[Dr1] V.G. Drinfeld, *Quantum groups*, Proc. Int. Congr. Math. (Berkeley, 1986) 1, 798-820.

[Dr2] V.G. Drinfeld, *On almost cocommutative Hopf algebras*, Len. Math. J. 1 (1990), 321-342.

[EK1] P. Etingof and D. Kazhdan, *Quantization of Lie bialgebras, I*, q-alg 9506005, Selecta Math. 2 (1996), no. 1, 1-41.

[EK2] P. Etingof and D. Kazhdan, *Quantization of Lie bialgebras, II, (revised version)*, q-alg 9701038 (1996).

[EK3] P. Etingof and D. Kazhdan, *Quantization of Lie bialgebras, III, (revised version)*, 9701038 (1996).

[EF] Enriques, B., and Felder, G., *Coinvariants of Yangian doubles and quantum Knizhnik-Zamolodchikov equations*, q-alg 9707012 (1997).

[FR] Frenkel, I.B., and Reshetikhin, N.Yu., *Quantum affine algebras and holonomic difference equations*, Comm. Math. Phys. 146 (1992), 1-60.
[Ha] Hasegawa, K., *Crossing symmetry in elliptic solutions of the Yang-Baxter equation and a new L-operator for Belavin's solution*, J.Phys. A: Math.Gen. **26** (1993), 3211-3228.

[KS] Kazhdan, D., and Soibelman, Y., *Representations of quantum affine algebras*, Selecta Math. **1** (1995), no. 3.

[KT] Kuroki, G., and Takebe, T., *Twisted Wess-Zumino-Witten models on elliptic curves*, q-alg/9612033 (1996).

[TK] Tsuchiya, A., Kanie, Y., *Vertex operators in conformal field theory on $P^1$ and monodromy representations of braid group*, Adv. Stud. Pure Math., vol. 16, 1988, pp. 297–372.

[TUY] Tsuchiya, A., Ueno, K. and Yamada, Y., *Conformal field theory on universal family of stable curves with gauge symmetries*, Adv. Stud. Pure Math., vol. 19, 1992, pp. 459–566.