Infinitely many sign-changing solutions for a class of elliptic problem with exponential critical growth.

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Abstract
In this work we prove the existence of infinitely many nonradial solutions that change signal to the problem $-\Delta u = f(u)$ in $B$ with $u = 0$ on $\partial B$, where $B$ is the unit ball in $\mathbb{R}^2$ and $f$ is a continuous and odd function with exponential critical growth.

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1 Introduction
Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and $f : \mathbb{R} \to \mathbb{R}$ be a $C^1$ function with $f(-t) = -f(t)$. Consider the following problem

$$\begin{cases}
-\Delta u = f(u), & \text{in } \Omega, \\
Bu = 0, & \text{on } \partial \Omega,
\end{cases} \quad (P)$$

when $N \geq 4$, $Bu = u$ and $f(t) = |t|^{\frac{4}{N-2}} + \lambda t$, Brézis-Niremberg proved that $(P)$ admits a non-trivial positive solution, provided $0 < f'(0) < \lambda_1(\Omega), \quad *denilsonsp@dme.ufcg.edu.br
where $\lambda_1(\Omega)$ is the first eigenvalue of $(-\Delta, H^1_0(\Omega))$. In [6], Cerami-Solimini-Struwe proved that if $N \geq 6$, problem $(P)$ admits a solution with changes sign. Using this, they also proved that when $n \geq 7$ and $\Omega$ is a ball, $(P)$ admits infinitely many radial solution which change sign.

Comte and Knaap [7] obtained infinitely many non-radial solutions that change sign for $(P)$ on a ball with Neumann boundary condition $\mathcal{B}u = \partial u / \partial \nu$, for every $\lambda \in \mathbb{R}$. They obtained such solutions by cutting the unit ball into angular sectors. This approach was used by Cao-Han [5], where the authors dealt with the scalar problem $(P)$ involving lower-order perturbation and by de Morais Filho et al. [10] to obtain multiplicity results for a class of critical elliptic systems related to the Brézis-Nirenberg problem with the Neumann boundary condition on a ball.

When $N = 2$, the notion of “critical growth” is not given by the Sobolev imbedding, but by the Trudinger-Moser inequality (see [13] and [11]), which claims that for any $u \in H^1_0(\Omega)$,

$$\int_{\Omega} e^{\alpha u^2} dx < +\infty, \text{ for every } \alpha > 0. \quad (1.1)$$

Moreover, there exists a positive constant $C = C(\alpha, |\Omega|)$ such that

$$\sup_{||u||_{H^1_0(\Omega)} \leq 1} \int_{\Omega} e^{\alpha u^2} dx \leq C, \quad \forall \alpha \leq 4\pi. \quad (1.2)$$

Motivated by inequality in (1.2), we say that the nonlinearity $f$ has exponential critical growth if $f$ behaves like $e^{\alpha_0 s^2}$, as $|s| \to \infty$, for some $\alpha_0 > 0$. More precisely,

$$\lim_{|s| \to \infty} \frac{|f(s)|}{e^{\alpha s^2}} = 0, \quad \forall \alpha > \alpha_0 \text{ and } \lim_{|s| \to \infty} \frac{|f(s)|}{e^{\alpha s^2}} = +\infty \quad \forall \alpha < \alpha_0.$$ 

In this case, Adimurthi [11] proved that $(P)$ admits a positive solution, provided that $\lim_{t \to \infty} tf(t)e^{\alpha t^2} = +\infty$ (See also Figueiredo-Miyagaki-Ruf [8] for a more weaker condition). In [3], Adimurthi-Yadava proved that $(P)$ has a solution that changes sign, and when $\Omega$ is a ball in $\mathbb{R}^2$, $(P)$ has infinitely many radial solutions that change sign. Inspired in [7], this paper is concerned with the existence of infinitely many non-radial sign changing solutions for $(P)$ when $f$ has exponential critical growth and $\Omega$ is a ball in $\mathbb{R}^2$. Our main result complements the studies made in [7] and [10], because we consider the case where $f$ has critical exponential growth in $\mathbb{R}^2$. It is important to notice that in both studies mentioned above was considered the Neumann
boundary condition in order that the Pohozaev identity (see [12]) ensures that
the problem \((P)\) with the Dirichlet boundary condition, has no solutions for \(\lambda < 0\) and \(N \geq 3\). Since the Pohozaev identity is not available in dimension two, in our case we can use the Dirichlet boundary condition.

Here we suppose the following assumptions

\((f_1)\) There is \(C > 0\) such that
\[|f(s)| \leq Ce^{4\pi|s|^2}\] for all \(s \in \mathbb{R}\);

\((f_2)\) \(\lim_{s \to 0} \frac{f(s)}{s} = 0\);

\((H_1)\) There are \(s_0 > 0\) and \(M > 0\) such that
\[0 < F(s) := \int_0^s f(t)dt \leq M|f(s)|, \text{ for all } \|s\| \geq s_0.\]

\((H_2)\) \(0 < F(s) \leq \frac{1}{2} f(s)s, \text{ for all } s \in \mathbb{R} \setminus \{0\}.\)

\((H_3)\) \(\lim_{s \to \infty} sf(s)e^{-4\pi s^2} = +\infty\)

Our main result is the following:

**Theorem 1.1** Let \(f\) be an odd and continuous function satisfying \((f_1) - (f_2)\) and \((H_1) - (H_3)\). Then, problem \((P)\) has infinitely many sign-changing solutions.

## 2 Notation and auxiliary results

For each \(m \in \mathbb{N}\), we define
\[A_m = \left\{ x = (x_1, x_2) \in B : \cos\left(\frac{\pi}{2m}\right)|x_1| < \sin\left(\frac{\pi}{2m}\right)x_2 \right\} .\]

So \(A_1\) is a half-ball, \(A_2\) an angular sector of angle \(\pi/2\), and \(A_3\) an angular sector of angle \(\pi/4\) and so on (see figure 1).

Using the above notation, we consider the following auxiliary Dirichlet problem
\[\begin{cases}
-\Delta u = f(u), \text{ in } A_m, \\
u = 0, \text{ on } \partial A_m,
\end{cases}\]

\((P)_m\)
We will use the Mountain Pass Theorem to obtain a positive solution of $(P)_m$. Using this solution together with an anti-symmetric principle, we construct a sign-changing solution of problem $(P)$.

According to Figueiredo, Miyagaki and Ruf [8], to obtain a positive solution of $(P)_m$ it is sufficient to assume that the limit in $(H)_3$ verifies

$$
(H_3)' \lim_{s \to +\infty} s f(s) e^{-4\pi s^2} \geq \beta > \frac{1}{2\pi d_m^2},
$$

where $d_m$ is the radius of the largest open ball contained in $A_m$. The hypothesis $(H_3)$ was initially considered in Adimurthi [1]. This hypothesis will be fundamental to ensure not only the existence but also the multiplicity of sign-changing solutions. As we will see bellow, assuming $(H_3)$ in place of $(H_3)'$, we have the existence of positive solution of $(P)_m$, for every $m \in \mathbb{N}$. This is the content of the next result.

**Theorem 2.1** Under the assumptions $(f_1) - (f_2)$ and $(H_1) - (H_3)$, problem $(P)_m$ has a positive solution, for every $m \in \mathbb{N}$. 

3 Proof of Theorem 2.1

Since we are interested in positive solutions to the problem \((P)_m\), we assume that
\[ f(s) = 0, \quad \forall s \leq 0. \]

Associated with problem \((P)_m\), we have the functional \(I : H^1_0(A_m) \to \mathbb{R}\) given by
\[ I(u) = \frac{1}{2} \int_{A_m} |\nabla u|^2 - \int_{A_m} F(u). \]

In our case, \(\partial A_m\) is not of class \(C^1\). However, the functional \(I\) is well defined. In fact, for each \(u \in H^1_0(A_m)\), let us consider \(u^* \in H^1_0(B)\) the zero extension of \(u\) in \(B\) defined by
\[ u^*(x) = \begin{cases} u(x), & \text{if } x \in A_m \\ 0, & \text{if } x \in B \setminus A_m. \end{cases} \]

Clearly
\[ \|u\|_{A_m} = \|u^*\|_B. \]

Then, from \((f_1)\) and the Trudinger-Moser inequality (1.1)
\[ \left| \int_{A_m} F(u) \right| = \left| \int_{B} F(u^*) \right| \leq \int_{B} |F(u^*)| \leq C \int_{B} e^{4\pi|u^*|^2} < \infty. \]

Moreover, using a standard argument we can prove that the functional \(I\) is of class \(C^1\) with
\[ I'(u)v = \int_{A_m} \nabla u \nabla v - \int_{A_m} f(u)v, \quad \forall u, v \in H^1_0(A_m). \]

Therefore, critical points of \(I\) are precisely the weak solutions of \((P_m)\).

The next lemma ensures that the functional \(I\) has the mountain pass geometry.

Lemma 3.1 (a) There exist \(r, \rho > 0\) such that
\[ I(u) \geq \rho > 0, \quad \text{for all } \|u\|_{A_m} = r. \]

(b) There is \(e \in H^1_0(A_m)\) such that
\[ \|e\|_{A_m} > r \quad \text{and} \quad I(e) < 0. \]
Proof. Using the definition of $I$ and the growth of $f$, we obtain
\[
I(u) \geq \frac{1}{2} \int_{A_m} |\nabla u|^2 - \frac{\epsilon}{2} \int_{A_m} |u|^2 - C \int_{A_m} |u|^q e^{\beta |u|^2},
\]
or equivalently,
\[
I(u) \geq \frac{1}{2} \int_{B} |\nabla u^*|^2 - \frac{\epsilon}{2} \int_{B} |u^*|^2 - C \int_{B} |u^*|^q e^{\beta |u^*|^2}.
\]
By the Poincar inequality,
\[
I(u) \geq \frac{1}{2} \int_{B} |\nabla u^*|^2 - \frac{\epsilon}{2 \lambda_1} \int_{B} |\nabla u^*|^2 - C \int_{B} |u^*|^q e^{\beta |u^*|^2},
\]
where $\lambda_1$ is the first eigenvalue of $(-\Delta, H_0^1(B))$. Fixing $\epsilon > 0$ sufficiently small, we have $C_1 := \frac{1}{2} - \frac{\epsilon}{2 \lambda_1} > 0$, from where it follows that
\[
I(u) \geq C_1 \int_{B} |\nabla u^*|^2 - C \int_{B} |u^*|^q e^{\beta |u^*|^2}.
\]
Notice that, from Trudinger-Moser inequality (1.2)
\[
e^{\beta |u^*|^2} \in L^2(B)
\]
and by continuous embedding
\[
|u^*|^q \in L^2(B).
\]
Since $H_0^1(B) \hookrightarrow L^{2q}(B)$ for all $q \geq 1$, by Hölder inequality
\[
\int_{B} |u^*|^q e^{\beta |u^*|^2} \leq \left( \int_{B} |u^*|^{2q} \right)^{1/2} \left( e^{2\beta |u^*|^2} \right)^{1/2}
\]
\[
\leq |u^*|^{q}_{\frac{2q}{2q},B} \left( \int_{B} e^{2\beta |u^*|^2} \right)^{1/2}
\]
\[
\leq C \|u^*\|_{B}^q \left( \int_{B} e^{2\beta |u^*|^2} \right)^{1/2}.
\]
We claim that for $r > 0$ small enough, we have
\[
\sup_{\|u^*\|_{B}=r} \int_{B} e^{2\beta |u^*|^2} < \infty.
\]
In fact, note that
\[ \int_B e^{2\beta\|u^*\|^2} = \int_B e^{2\beta\|u^*\|^2} \left( \frac{|u^*|}{\|u^*\|_B} \right)^2. \]
Choosing \( 0 < r \approx 0 \) such that \( \alpha := 2\beta r^2 < 4\pi \) and using the Trudinger-Moser inequality (1.2),
\[ \sup_{\|u^*\|_B=r} \int_B e^{2\beta\|u^*\|^2} \leq \sup_{\|v\|_B \leq 1} \int_B e^{\alpha|v|^2} < \infty. \]
Thus,
\[ I(u) \geq C_1\|u^*\|^2_B - C_2\|u^*\|_B^q \]
Fixing \( q > 2 \), we derive
\[ I(u) \geq C_1 r^2 - C_2 r^q := \rho > 0, \]
for \( r = \|u\|_{A_m} = \|u^*\|_B \) small enough, which shows that the item (a) holds.

**Claim 1.** For each \( \epsilon > 0 \), there exists \( \overline{s}_\epsilon > 0 \) such that
\[ F(s) \leq \epsilon f(s)s, \quad \text{for all} \quad x \in A_m, \quad |s| \geq \overline{s}_\epsilon. \]
In fact, from hypothesis \( (H_1) \)
\[ \frac{|F(s)|}{sf(s)} \leq M \frac{|s|}{|s|}, \quad \forall |s| \geq s_0. \]
For \( p > 2 \), the claim 1 with \( \epsilon = 1/p > 0 \), guarantees the existence of \( \overline{s}_\epsilon > 0 \) such that
\[ pF(s) \leq f(s)s, \quad \forall s \geq \overline{s}_\epsilon, \]
which implies the existence of constant \( C_1, C_2 > 0 \) verifying
\[ F(s) \geq C_1|s|^p - C_2, \quad \forall s \geq 0. \]
Thus, fixing \( \varphi \in C_0^\infty(A_m) \) with \( \varphi \geq 0 \) and \( \varphi \not= 0 \). For \( t \geq 0 \), we have
\[ \int_{A_m} F(t\varphi) \geq \int_{A_m} (C_1|t\varphi|^p - C_2) \]
\[ \geq C_1|t|^p \int_{A_m} |\varphi| - C_2|A_m|, \]
from where it follows that
\[ \int_{A_m} F(t\varphi) \geq C_3|t|^p - C_4. \] (3.1)
From (3.1), if \( t \geq 0 \),
\[
I(t\varphi) \leq \frac{t^2}{2}\|\varphi\|_{A_m}^2 - C_3|t|^p + C_4.
\]
Since \( p > 2 \),
\[
I(t\varphi) \to -\infty, \quad \text{as} \quad t \to +\infty.
\]
Fixing \( t_0 \approx +\infty \) and let \( e = t_0\varphi \), we get
\[
\|e\|_{A_m} \geq \rho \quad \text{and} \quad I(e) < 0.
\]

The next lemma is crucial to prove that the energy functional \( I \) satisfies the Palais-Smale condition and its proof can be found in [8].

**Lemma 3.2** Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain and \( (u_n) \) be a sequence of functions in \( L^1(\Omega) \) such that \( u_n \) converging to \( u \in L^1(\Omega) \) in \( L^1(\Omega) \). Assume that \( f(u_n(x)) \) and \( f(u(x)) \) are also \( L^1 \) functions. If
\[
\int_{\Omega} |f(u_n)u_n| \leq C, \quad \text{para todo} \quad n \in \mathbb{N},
\]
then \( f(u_n) \) converges in \( L^1(\Omega) \) to \( f(u) \).

**Lemma 3.3** The functional \( I \) satisfies \((PS)_d\) condition, for all \( d \in (0, 1/2) \).

**Proof.** Let \( d < 1/2 \) and \( (u_n) \) be a \((PS)_d\) sequence for the functional \( I \), i.e.,
\[
I(u_n) \to d \quad \text{and} \quad I'(u_n) \to 0, \quad \text{as} \quad n \to +\infty.
\]
For each \( n \in \mathbb{N} \), let us define \( \epsilon_n = \sup \{|I'(u_n)v|\} \), then
\[
|I'(u_n)v| \leq \epsilon_n \|v\|_m,
\]
for all \( v \in H^1_0(A_m) \), where \( \epsilon_n = o_n(1) \). Thus
\[
\frac{1}{2} \int_{A_m} |\nabla u_n|^2 - \int_{A_m} F(u_n) = d + o_n(1), \quad \forall n \in \mathbb{N} \quad (3.2)
\]
and
\[
\left| \int_{A_m} \nabla u_n \nabla v - \int_{A_m} f(u_n)v \right| \leq \|v\|_m \epsilon_n, \quad \forall n \in \mathbb{N}, \quad v \in H^1_0(A_m). \quad (3.3)
\]
From (3.2) and Claim 1, for any $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that
\[
\frac{1}{2}\|u_n\|_m^2 - \frac{1}{2} \int_{A_m} |\nabla u_n|^2 \leq \epsilon + d + \int_{A_m} F(u_n) \leq C\epsilon + \epsilon \int_{A_m} f(u_n)u_n,
\]
for all $n \geq n_0$. Using (3.3) with $v = u_n$, we get
\[
\left(\frac{1}{2} - \epsilon\right)\|u_n\|_m^2 \leq C\epsilon + \epsilon\|u_n\|_m, \quad \forall n \geq n_0.
\]
Thus, the sequence $(u_n)$ is bounded. Since $H^1_0(A_m)$ be a reflexive Banach space, there exits $u \in H^1_0(A_m)$ such that, for some subsequence,
\[
u_n \rightharpoonup u \quad \text{in} \quad H^1_0(A_m).
\]
Furthermore, from compact embedding,
\[
u_n \to u \quad \text{in} \quad L^q(A_m), \quad q \geq 1
\]
and
\[
\nu_n(x) \to u(x) \quad \text{a.e. in} \quad A_m.
\]
On the other hand, using (3.3) with $v = u_n$, we get
\[
-\epsilon\|u_n\|_m \leq \int_{A_m} |\nabla u_n|^2 - \int_{A_m} f(u_n)u_n,
\]
which implies
\[
\int_{A_m} f(u_n)u_n \leq \|u_n\|_m^2 - \epsilon\|u_n\|_m \leq C, \quad \forall n \in \mathbb{N}.
\]
From Lemma 3.2 $f(u_n) \to f(u)$ in $L^1(A_m)$. Then, there is $h \in L^1(A_m)$ such that
\[
|f(u_n(x))| \leq h(x), \quad \text{a.e. in} \quad A_m,
\]
and from $(H_1)$,
\[
|F(u_n)| \leq Mh(x), \quad \text{a.e. in} \quad A_m.
\]
Furthermore,
\[
F(u_n(x)) \to F(u(x)) \quad \text{a.e. in} \quad A_m.
\]
Consequently, by the Lebesgue’s dominated convergence,
\[
\int_{A_m} F(u_n) - \int_{A_m} F(u) = o_n(1).
\]
Thus, from (3.2),
\[
\frac{1}{2} \| u_n \|_m^2 - \int_{A_m} F(u) - d = o_n(1),
\]
which implies,
\[
\lim_{n \to \infty} \| u_n \|_m^2 = 2 \left( d + \int_{A_m} F(u) \right).
\]  \hspace{1cm} (3.4)

Using again (3.3) with \( v = u_n \), we obtain
\[
\left| \| u_n \|_m^2 - \int_{A_m} f(u_n) u_n \right| \leq o_n(1),
\]
from where we derive
\[
\left| \int_{A_m} f(u_n) u_n - 2 \left( d + \int_{A_m} F(u) \right) \right| \leq \left| \| u_n \|_m^2 - \int_{A_m} f(u_n) u_n \right|
+ \left| \| u_n \|_m^2 - 2 \left( d + \int_{A_m} F(u) \right) \right|.
\]

Then,
\[
\lim_{n \to \infty} \int_{A_m} f(u_n) u_n = 2 \left( d + \int_{A_m} F(u) \right).
\]

Furthermore, from \((H_2)\),
\[
2 \int_{A_m} F(u) \leq 2 \lim_{n \to \infty} \int_{A_m} F(u_n)
\leq \lim_{n \to \infty} \int_{A_m} f(u_n) u_n = 2d + 2 \int_{A_m} F(u),
\]
which implies that \( d \geq 0 \).

**Claim 2.** For any \( v \in H_0^1(A_m) \),
\[
\int_{A_m} \nabla u \nabla v = \int_{A_m} f(u) v.
\]

In fact, let us fix \( v \in H_0^1(A_m) \) and notice that
\[
\left| \int_{A_m} \nabla u \nabla v - \int_{A_m} f(u) v \right| \leq \left| \int_{A_m} \nabla u_n \nabla v - \int_{A_m} \nabla u \nabla v \right| + \left| \int_{A_m} f(u_n) v - \int_{A_m} f(u) v \right|
+ \left| \int_{A_m} \nabla u_n \nabla v - \int_{A_m} f(u_n) v \right|.
\]
Using Lemma 3.2, the weak convergence $u_n \rightharpoonup u$ in $H_0^1(A_m)$ and the estimate in (3.3), we derive
\[
|\int_{A_m} \nabla u \nabla v - \int_{A_m} f(u)v| \leq o_n(1) + \|v\|o_n(1),
\]
and the proof of Claim 2 is complete.
Notice that from (H2) and Claim 2,
\[
J(u) \geq \frac{1}{2} \int_{A_m} |\nabla u|^2 - \frac{1}{2} \int_{A_m} f(u)u = 0.
\]
Now, we split the proof into three cases:

**Case 1.** The level $d = 0$. By the lower semicontinuity of the norm,\[
\|u\|_m \leq \liminf_{n \to \infty} \|u_n\|_m,
\]
then\[
\frac{1}{2} \|u\|^2_m \leq \frac{1}{2} \|u_n\|^2_m.
\]
Using (3.4),
\[
0 \leq I(u) \leq \frac{1}{2} \liminf \|u_n\|^2 - \int_{A_m} F(u),
\]
which implies\[
0 \leq I(u) \leq \int_{A_m} F(u) - \int_{A_m} F(u) = 0,
\]
from where $I(u) = 0$, or equivalently,
\[
\|u\|^2_m = 2 \int_{A_m} F(u).
\]
Using again (3.4), we derive
\[
\|u_n\|^2_m - \|u\|^2_m = o_n(1),
\]
since $H_0^1(A_m)$ be a Hilbert espace,
\[
u_n \rightharpoonup u \text{ in } H_0^1(A_m).
\]
Therefore, $I$ verifies the Palais-Smale at the level $d = 0$.

**Case 2.** The level $d \neq 0$ and the weak limit $u \equiv 0$. 


We will show that this can not occur for a Palais-Smale sequence. **Claim 3.** There are \( q > 1 \) and a constant \( C > 0 \) such that
\[
\int_{A_m} |f(u_n)|^q < C, \quad \forall n \in \mathbb{N}.
\]
In fact, from (3.4), for each \( \epsilon > 0 \)
\[
\|u_n\|^2_m \leq 2d + \epsilon, \quad \forall n \geq n_0,
\]
for some \( n_0 \in \mathbb{N} \). Furthermore, from \( (f_1) \),
\[
\int_{A_m} |f(u_n)|^q \leq C \int_{A_m} e^{4\pi q u_n^2} = C \int_B e^{4\pi q u_n^2 (\frac{u_n}{\|u_n\|})^2}.
\]
By the Trudinger-Moser inequality (1.2), the last integral in the equality above is bounded if \( 4\pi q \|u_n^*\|^2 < 4\pi \) and this occur if we take \( q > 1 \) sufficiently close to 1 and \( \epsilon \) small enough, because \( d < 1/2 \), which proves the claim.

Then, using (3.3) with \( v = u_n \), we obtain
\[
\left| \int_{A_m} |\nabla u_n|^2 - \int_{A_m} f(u_n)u_n \right| \leq \epsilon_n \|u_n\|_m \leq \epsilon_n C, \quad \forall n \in \mathbb{N}.
\]
Thus,
\[
\|u_n\|_m \leq o_n(1) + \int_{A_m} f(u_n)u_n, \quad \forall n \in \mathbb{N}. \tag{3.5}
\]
Furthermore, from Hölder inequality, we can estimate the integral above as follows
\[
\int_{A_m} f(u_n)u_n \leq \left( \int_{A_m} |f(u_n)|^q \right)^{1/q} \left( \int_{A_m} |u_n|^q \right)^{1/q'}, \quad \forall n \in \mathbb{N},
\]
and since \( u_n \to 0 \) in \( L^{q'}(A_m) \),
\[
\int_{A_m} f(u_n)u_n = o_n(1).
\]
Then, from (3.5),
\[
\|u_n\|_m^2 \to 0, \quad \text{as } n \to \infty, \tag{3.6}
\]
which contradicts (3.4), because
\[
\|u_n\|_m^2 \to 2d \neq 0, \quad \text{as } n \to \infty,
\]
proving that \( d \neq 0 \) and \( u = 0 \) does not occur.

**Case 3.** The level \( d \neq 0 \) and the weak limit \( u \neq 0 \). Since

\[
I(u) = \frac{1}{2} \|u\|^2_m - \int_{A_m} F(u) \leq \liminf_n \left( \frac{1}{2} \|u_n\|^2_m - \int_{A_m} F(u_n) \right) = d.
\]

we have \( I(u) \leq d \).

**Claim 4.** \( I(u) = d \).

In fact, suppose by contradiction that \( I(u) < d \), from definition of \( I \),

\[
\|u\|^2_m < 2 \left( d + \int_{A_m} F(u) \right).
\]  \hspace{1cm} (3.7)

On the other hand, if we consider the functions

\[
v_n = \frac{u_n^*}{\|u_n^*\|}, \quad n \in \mathbb{N}
\]

and

\[
v = u^* \left[ 2 \left( d + \int_B F(u^*) \right) \right]^{-1/2},
\]

we have \( \|v_n\|_B = 1 \) e \( \|v\|_B < 1 \). Furthermore, since

\[
\int_B \nabla v_n \nabla \varphi = \|u_n\|^{-1} \int_{A_m} \nabla u_n \nabla \varphi \to \left[ 2 \left( d + \int_B F(u^*) \right) \right]^{-1/2} \int_B \nabla u \nabla \varphi = \int_B \nabla v \nabla \varphi,
\]

for every \( \varphi \in C_0^\infty(B) \), i.e.,

\[
\int_B \nabla v_n \nabla \varphi - \int_B \nabla v \nabla \varphi = o_n(1),
\]

we have

\[
v_n \rightharpoonup v \quad \text{in} \quad H_0^1(B).
\]

**Claim 3.4** There are \( q > 1 \) and \( n_0 \in \mathbb{N} \) such that

\[
\int_{A_m} |f(u_n)|^q < C, \quad \forall n \geq n_0.
\]

To prove this, we need the following result due to P.L. Lions [9].
Proposition 3.5 \ Let \((u_n)\) be a sequence in \(H^1_0(\Omega)\) such that \(|\nabla u_n|_2 = 1\) for all \(n \in \mathbb{N}\). Furthermore, suppose that \(u_n \rightharpoonup u\) in \(H^1_0(\Omega)\) with \(|\nabla u|_2 < 1\). If \(u \neq 0\), then for each \(1 < p < \frac{1}{1 - |\nabla u|_2^2}\), we have
\[
\sup_{n \in \mathbb{N}} \int_{\Omega} e^{4\pi pu_n^2} < \infty.
\]

From hypothesis \((f_1)\),
\[
\int_{A_m} |f(u_n)|^q \leq C \int_{A_m} e^{4\pi q u_n^2} = C \int_{B} e^{4\pi q \|u_n^*\|^2 v_n^2}.
\]

The last integral in the above expression is bounded. In fact, by Proposition 3.5 it is sufficient to prove that there are \(q, p > 1\) and \(n_0 \in \mathbb{N}\) such that
\[
q \|u_n^*\|^2 \leq p < \frac{1}{1 - \|v\|^2_B}, \quad \forall n \geq n_0.
\]

To prove that (3.9) occur, notice that \(I(u) \geq 0\) and \(d < 1/2\), which implies that
\[
2 < \frac{1}{d - I(u)},
\]
from where it follows that
\[
2 \left( d + \int_{B} F(u^*) \right) < \frac{d + \int_{B} F(u^*)}{d - I(u)} = \frac{1}{1 - \|v\|^2_B}.
\]

Thus, for \(q > 1\) sufficiently close to 1,
\[
2q \left( d + \int_{B} F(u) \right) < \frac{1}{1 - \|v\|^2_B}.
\]

From (3.4), there are \(p > 1\) and \(n_0 \in \mathbb{N}\) such that
\[
q \|u_n^*\|^2 \leq p < \frac{1}{1 - \|v\|^2_B}, \quad \forall n \geq n_0.
\]

Thus, (3.9) occur. Therefore, Claim 3.4 holds.
Now, we will show that $u_n \to u$ in $H_0^1(A_m)$. First, notice that from Hölder inequality and (3.4),
\[
\int_{A_m} f(u_n)(u_n - u) \leq \int_{A_m} (|f(u_n)|)^{1/q} \left( \int_{A_m} |u_n - u|^{q'} \right)^{1/q'} \leq C|u_n - u|_{q',A_m},
\]
where $1/q + 1/q' = 1$. Since $u_n \to u$ in $L^{q'}(A_m)$,
\[
\int_{A_m} f(u_n)(u_n - u) = o_n(1).
\] (3.10)
Using (3.3) with $v = u_n - u$ and (3.10), we obtain $\langle u_n - u, u_n \rangle = o_n(1)$, and since $u_n \rightharpoonup u$ in $H_0^1(A_m)$,
\[
\|u_n - u\|_m^2 = \langle u_n - u, u_n \rangle - \langle u_n - u, u \rangle = o_n(1).
\]
Then, $\|u_n\|_m^2 \to \|u\|_m^2$ and this together with (3.4) contradicts (3.7). Which proves that $I(u) = d$, i.e.,
\[
\|u\|_m^2 = 2 \left( d + \int_{A_m} F(u) \right).
\]
Furthermore, from (3.4), $\|u_n\|_m \to \|u\|_m$ as $n \to \infty$. Therefore
\[
u_n \to u \quad \text{in} \quad H_0^1(A_m).
\]
From Lemma 3.1 and the Mountain pass Theorem without compactness conditions (see [14]), there is a $(PS)_{c_m}$ sequence $(u_n) \subset H_0^1(A_m)$ such that
\[
I(u_n) \to c_m \quad \text{and} \quad I'(u_n) \to 0,
\]
where
\[
c_m = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))
\]
and
\[
\Gamma = \{ \gamma \in C([0,1], H_0^1(A_m)) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0 \}.
\]

To conclude the proof of existence of positive solution for $(P)_m$, it remains to show that $c_m \in (-\infty, 1/2)$. For this, we introduce the following Moser’s functions (see [11]):
\[
\overline{w}_n(x) = \begin{cases} 
\frac{(ln(n))^{1/2}}{\sqrt{2\pi}}, & 0 \leq |x| \leq 1/n \\
\frac{ln|\frac{1}{|x|}|}{(ln(n))^{1/2}}, & 1/n \leq |x| \leq 1 \\
0, & |x| \geq 1
\end{cases}
\]
Let $d_m > 0$ and $x_m \in A_m$ such that $B_{d_m}(x_m) \subset A_m$ and define

$$w_n(x) = w_n \left( \frac{x - x_m}{d_m} \right),$$

we have $w_n \in H_0^1(A_m)$, $\|w_n\|_{A_m} = 1$ and supp $w_n \subset B_{d_m}(x_m)$.

We claim that there exists $n \in \mathbb{N}$ such that

$$\max_{t \geq 0} I(tw_n) < \frac{1}{2}. \tag{3.11}$$

In fact, suppose by contradiction that this is not the case. Then, there exist $t_n > 0$ such that

$$\max_{t \geq 0} I(tw_n) = I(t_n w_n) = \frac{1}{2}. \tag{3.11}$$

It follows from (3.11) and (H1) that

$$t_n^2 \geq 1. \tag{3.12}$$

Furthermore, $\frac{d}{dt} I(tw_n) \big|_{t=t_n} = 0$, i.e.,

$$t_n^2 = \int_{A_m} f(t_n w_n) t_n w_n, \tag{3.13}$$

which implies that

$$t_n^2 \geq \int_{B_{d/n}(x_m)} f(t_n w_n) t_n w_n. \tag{3.14}$$

In what follows, we fix a positive constant $\beta_m$ verifying

$$\beta_m > \frac{1}{2\pi d_m^2}. \tag{3.15}$$

From (H3), there exists $s_m = s_m(\beta_m) > 0$ such that

$$f(s) s \geq \beta_m e^{4\pi s^2}, \forall s \geq s_m. \tag{3.16}$$

Using (3.16) in (3.14) and the definition of $w_n$ in $B_{d/n}(0)$, we obtain

$$t_n^2 \geq \beta_m \pi \frac{d_m^2}{n^2} e^{2s_n^2/n}. \tag{3.17}$$
for \( n \) large enough, or equivalently,
\[
t^2_n \geq \beta_m \pi d_m^2 e^{2h_n(t^2_n - 1)},
\]
(3.18)

it implies that the sequence \((t_n)\) is bounded. Moreover, from (3.18) and (3.12),
\[
t^2_n \to 1, \quad \text{as} \quad n \to \infty.
\]

Now, let us define
\[
C_n = \{ x \in B_{d_m}(x_m) : t_n w_n(x) \geq s_m \}
\]
and
\[
D_n = B_{d_m}(x_m) \setminus C_n.
\]

With the above notations and using (3.13),
\[
t^2_n \geq \int_{B_{d_m/n}(x_m)} f(t_n w_n) t_n w_n = \int_{C_n} f(t_n w_n) t_n w_n + \int_{D_n} f(t_n w_n) t_n w_n
\]
and by (3.16),
\[
t^2_n \geq \int_{D_n} f(t_n w_n) t_n w_n + \beta_m \int_{C_n} e^{4\pi t^2_n w^2_n}
\]
or equivalently,
\[
t^2_n \geq \int_{D_n} f(t_n w_n) t_n w_n + \beta_m \int_{B_{d_m}(x_m)} e^{4\pi t^2_n w^2_n} - \beta_m \int_{D_n} e^{4\pi t^2_n w^2_n}.
\]
(3.19)

Notice that
\[
w_n(x) \to 0 \quad \text{a.e. in} \quad B_{d_m}(x_m),
\]
\[
\chi_{D_n}(x) \to 1 \quad \text{a.e. in} \quad B_{d_m}(x_m)
\]
and
\[
e^{4\pi t^2_n w^2_n} \chi_{D_n} \leq e^{4\pi t^2_n s^2_m} \in L^1(B_{d_m}(x_m)).
\]

Then, by Lebesgue’s dominated convergence
\[
\lim_n \int_{D_n} e^{4\pi t^2_n w^2_n} = \lim_n \int_{B_{d_m}(x_m)} e^{4\pi t^2_n w^2_n} \chi_{D_n} = \int_{B_{d_m}(x_m)} 1 = \pi d_m^2.
\]
(3.20)

Furthermore,
\[
f(t_n w_n) t_n w_n \chi_{D_n} \leq C t_n w_n e^{4\pi t^2_n w^2_n} \leq C s_m e^{4\pi s^2_m} \in L^1(B_{d_m}(x_m))
\]
\[ f(t_n w_n(x)) t_n w_n(x) \chi_{D_n}(x) \to 0 \text{ a.e. in } B_{d_m}(x_m). \]

Thus, using again Lebesgue’s dominated convergence,

\[ \lim_n \int_{B_d} f(t_n w_n) t_n w_n = 0 \]  \hspace{1cm} \text{(3.21)}

Passing to the limit \( n \to \infty \) in (3.19) and using (3.20) and (3.21),

\[ 1 \geq \beta_m \lim_n \int_{B_{d_m}(x_m)} e^{4\pi t_n^2 w_n^2} - \beta_m \pi d_m^2. \]

Since \( t_n^2 \geq 1 \), we get

\[ 1 \geq \beta_m \lim_n \left[ \int_{B_{d_m}(x_m)} e^{4\pi w_n^2} \right] - \beta_m \pi d_m^2. \]  \hspace{1cm} \text{(3.22)}

On the other hand, since

\[ \int_{B_{d_m}(x_m)} e^{4\pi w_n^2} = d_m^2 \int_{B_1(0)} e^{4\pi \pi^2 n} = d_m^2 \left\{ \frac{\pi}{n^2} e^{4\pi \frac{1}{n} \log(n)} + 2\pi \int_1^{1/n} e^{4\pi \frac{1}{2\pi} \frac{\log(1/r)^2}{\ln(n)} r} \right\}, \]

making a changing of variables \( s = \log(1/r)/\log(n) \),

\[ \int_{B_{d_m}(x_m)} e^{4\pi w_n^2} = \pi d_m^2 + 2\pi d_m^2 \log(n) \int_0^1 e^{2s^2 \ln(n) - 2s \ln(n)} \]

and since

\[ \lim_{n \to \infty} \left[ 2\ln(n) \int_0^1 e^{2\ln(n)(s^2-s)} ds \right] = 2, \]

we have

\[ \lim_{n \to \infty} \int_{B_{d_m}(x_m)} e^{4\pi w_n^2} = \pi d_m^2 + 2\pi d_m^2 = 3\pi d_m^2. \]

Using the last limit in (3.22), we get

\[ 1 \geq 3\beta_m \pi d_m^2 - \beta \pi d_m^2 = 2\beta \pi d_m^2, \]

from where we derive

\[ \beta_m \leq \frac{1}{2\pi d_m^2}, \]

which contradicts the choice of \( \beta_m \) in (3.15). Then,

\[ \max_{t \geq 0} I(t w_n) < \frac{1}{2}, \]

proving that \( c_m < 1/2 \), for any \( m \in \mathbb{N} \) fixed arbitrarily.
4 Proof of Theorem 4.1

To prove Theorem 4.1, we will use the following proposition.

**Proposition 4.1** Let $A$ be an angular sector contained on the positive semiplane of $\mathbb{R}^2$ such that one of its boundary lies in $x_2$ axis, and denote such boundary of $A$ by $B_0 = \{x = (x_1, x_2) \in A : x_2 = 0\}$. Consider $A'$ the reflection $A$ with respect to $x_2$ axis. Suppose that $u$ is a solution of the following problem:

$$
\begin{align*}
(P) \quad \begin{cases}
-\Delta u = f(u), & \text{in } A, \\
u = 0, & \text{on } B_0,
\end{cases}
\end{align*}
$$

where $f$ is a real, continuous and odd function. Then, the function $\tilde{u}$ such that $\tilde{u} = u$ in $A$ and $\tilde{u}$ is antisymmetric with respect to $x_2$ axis,

$$
\tilde{u}(x_1, x_2) = \begin{cases}
u(x_1, x_2), & \text{in } A \\
-u(x_1, -x_2), & \text{in } A' \\
0, & \text{on } B_0
\end{cases}
$$

satisfies

$$-\Delta \tilde{u} = f(\tilde{u}) \quad \text{in } A \cup A'.
$$

**Proof.** Since $u$ be a solution of $(P)$, we have

$$\int_A \nabla u \nabla \varphi = \int_A f(u) \varphi, \quad \forall \varphi \in C_c^\infty(A).$$

we want to prove that

$$\int_{A \cup A'} \nabla \tilde{u} \nabla \phi = \int_{A \cup A'} f(\tilde{u}) \phi, \quad \forall \phi \in C_c^\infty(A \cup A').$$

For any $\phi \in C_c^\infty(A \cup A')$,

$$\int_{A \cup A'} f(\tilde{u}) \phi = \int_A f(u(x_1, x_2))\phi(x_1, x_2) + \int_{A'} f(-u(x_1, -x_2))\phi(x_1, x_2).$$

Since $f$ be an odd function,

$$\int_{A \cup A'} f(\tilde{u}) \phi = \int_A f(u(x_1, x_2))\phi(x_1, x_2) + \int_{A'} f(-u(x_1, -x_2))\phi(x_1, x_2)
\begin{align*}
&= \int_A f(u(x_1, x_2))\phi(x_1, x_2) - \int_{A'} f(u(x_1, -x_2))\phi(x_1, x_2) \\
&= \int_A f(u(x_1, x_2))\phi(x_1, x_2) - \int_A f(u(x_1, x_2))\phi(x_1, -x_2).
\end{align*}$$
Thus
\[
\int_{A \cup A'} f(\bar{u})\phi = \int_A f(u)\psi, \tag{4.1}
\]
where \( \psi(x_1, x_2) = \phi(x_1, x_2) - \phi(x_1, -x_2) \). On the other hand,
\[
\int_{A \cup A'} \nabla \bar{u} \nabla \phi = \int_A \nabla u(x_1, x_2) \nabla \phi(x_1, x_2) - \int_{A'} \nabla u(x_1, -x_2) \nabla \phi(x_1, x_2)
\]
\[
= \int_A \nabla u(x_1, x_2) \nabla \phi(x_1, x_2) - \int_A \nabla u(x_1, x_2) \nabla (\phi(x_1, -x_2))
\]
\[
= \int_A \nabla u(x_1, x_2) \nabla (\phi(x_1, x_2) - \phi(x_1, -x_2)).
\]

Then,
\[
\int_{A \cup A'} \nabla \bar{u} \nabla \phi = \int_A \nabla u \nabla \psi. \tag{4.2}
\]

The function \( \psi \) does not in general belong to \( C_0^\infty(A) \). Therefore, \( \psi \) can not be used as a function test (in the definition of weak solution on \( H^1(A) \)).

On the other hand, if we consider the sequence of functions \( (\eta_k) \) in \( C^\infty(\mathbb{R}) \), defined by
\[
\eta_k(t) = \eta(kt), \ t \in \mathbb{R}, \ k \in \mathbb{N},
\]
where \( \eta \in C^\infty(\mathbb{R}) \) is a function such that
\[
\eta(t) = \begin{cases} 
0, & \text{if } t < 1/2, \\
1, & \text{if } t > 1.
\end{cases}
\]

Then
\[
\varphi_k(x_1, x_2) := \eta_k(x_2) \psi(x_1, x_2) \in C_0^\infty(A),
\]
which implies that
\[
\int_A \nabla u \nabla \varphi_k = \int_A f(u)\varphi_k, \ k \in \mathbb{N}. \tag{4.3}
\]

From (4.1), (4.2) and (4.3), we can conclude the proof, in view of the following limits
\[
(I) \quad \int_A \nabla u \nabla \varphi_k \to \int_A \nabla u \nabla \psi
\]
\[
(II) \quad \int_A f(u)\varphi_k \to \int_A f(u)\psi,
\]
as $k \to \infty$ occur. To see that (I) occur, notice that
\[ \int_A \nabla u \nabla \varphi_k = \int_A \eta_k \nabla u \nabla \psi + \int_A \frac{\partial u}{\partial x_2} k \eta'(kx_2) \psi. \]

Clearly,
\[ \int_A \eta_k \nabla u \nabla \psi \to \int_A \nabla u \nabla \psi, \text{ as } k \to \infty. \]

Assim, resta mostrar que
\[ \int_A \frac{\partial u}{\partial x_2} k \eta'(kx_2) \psi \to 0 \text{ as } k \to \infty. \quad (4.4) \]

In fact,
\[ \left| \int_A \frac{\partial u}{\partial x_2} k \eta'(kx_2) \psi \right| \leq kMC \int_{0<x_2<1/k} \left| \frac{\partial u}{\partial x_2} \right| x_2 \leq MC \int_{0<x_2<1/k} \left| \frac{\partial u}{\partial x_2} \right|, \]

where $C = \sup_{t \in [0,1]} |\eta'(t)|$ and $M > 0$ is such that
\[ |\psi(x_1, x_2)| \leq M|x_2|, \forall (x_1, x_2) \in A \cup A', \]

and since
\[ \int_{0<x_2<1/k} \left| \frac{\partial u}{\partial x_2} \right| \to 0, \text{ as } k \to \infty, \]

the limit in (4.4) occur. The item (II) is an immediately consequence of the Lebesgue’s dominated convergence.

Now, for each $m \in \mathbb{N}$, we apply the Proposition 4.1 to the solution $u$ of problem $(P)_m$. Let $A'_m$ be the reflection of $A_m$ in one of its sides. On $A_m \cup A'_m$, we can define the function $\tilde{u}$ such that $\tilde{u} = u$ on $A_m$, and $\tilde{u}$ is antisymmetric with respect to the side of reflection. Now, let $A''_m$ be the reflection of $A_m \cup A'_m$ in one of its sides and $\tilde{\tilde{u}}$ the function defined on $A_m \cup A'_m \cup A''_m$ such that $\tilde{\tilde{u}} = \tilde{u}$ on $A_m \cup A'_m$ and $\tilde{\tilde{u}}$ is antisymmetric with respect to the side of reflection. Repetindo este procedimento, após um número finito de reflexões. Repeating this procedure, after finite steps, we finally obtain a function defined on the whole unit ball $B$, denoted by $u_m$. Clearly, $u_m$ satisfies the Dirichlet condition on the boundary $\partial B$. That is, $u_m$ is a sign-changing solution of problem $(P)$. Since for every $m \in \mathbb{N}$, problem $(P)_m$ admits a positive solution, hence there exist infinitely many different sign-changing solutions, and the proof of Theorem 1.1 is complete.
Remark. In figure 2 we represent the signal of three solutions, corresponding to the cases \( m = 1 \), \( m = 2 \) and \( m = 3 \), respectively. The blue color represents the regions where the solutions are negative and the red color, the regions where the solutions are positive.

![Figure 2: Signal of solutions](image)

We show in Figure 3 the profile of solution for the case \( m = 2 \).

![Figure 3: Case m = 2](image)

**Remark 4.2** It is possible to make a version of Theorem 1.1 with Neumann boundary condition using the same arguments that we used here, but we have to work with another version of Trudinger-Moser inequality in \( H^1(\Omega) \) due to Adimurthi-Yadava [2], which says that if \( \Omega \) is a bounded domain with smooth boundary, then for any \( u \in H^1(\Omega) \),

\[
\int_{\Omega} e^{\alpha u^2} < +\infty, \quad \text{for all } \alpha > 0. \tag{4.5}
\]

Furthermore, there exists a positive constant \( C = C(\alpha, |\Omega|) \) such that

\[
\sup_{\|u\|_{H^1(\Omega)} \leq 1} \int_{\Omega} e^{\alpha u^2} \leq C, \quad \forall \alpha \leq 2\pi. \tag{4.6}
\]
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