Topological Field Theory and Quantum Holonomy
Representations of Motion Groups

Richard J. Szabo

Department of Physics and Astronomy, University of British Columbia
6224 Agricultural Road, Vancouver, B.C. V6T 1Z1, Canada

and

The Niels Bohr Institute
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

szabo@nbi.dk

Abstract

Canonical quantization of abelian $BF$-type topological field theory coupled to extended sources on generic $d$-dimensional manifolds and with curved line bundles is studied. Sheaf cohomology is used to construct the appropriate topological extension of the action and the topological flux quantization conditions, in terms of the Čech cohomology of the underlying spatial manifold, as required for topological invariance of the quantum field theory. The wavefunctions are found in the Hamiltonian formalism and are shown to carry multi-dimensional projective representations of various topological groups of the space. Expressions for generalized linking numbers in any dimension are thereby derived. In particular, new global aspects of motion group presentations are obtained in any dimension. Applications to quantum exchange statistics of objects in various dimensionalities are also discussed.
1. Introduction

Topological gauge theories involving higher-rank antisymmetric tensor fields have been of much interest over the years. The simplest example of such a model is $BF$ theory [1]–[3] which provides a quantum field theoretical framework for understanding various important topological invariants. In the abelian case, the partition function computes the Ray-Singer analytic torsion of the underlying spacetime manifold [1, 3, 4] while the correlation functions of higher-rank holonomy operators (the appropriate generalizations of Wilson loop operators) compute linking and intersection numbers of manifolds of various dimensionalities [3, 5]. These models have also been of interest in a wide variety of physical applications in which non-local holonomy effects arising from processes involving adiabatic transports of extended objects, such as strings, play a significant role [6]. Recent interest in non-abelian $BF$ theory has been sparked by the realization that it serves as a dual model for Yang-Mills theory and quantum chromodynamics in four dimensions [7] and thereby provides a computational tool in which the non-perturbative sector of the relevant quantum field theory can be analysed quantitatively. $BF$ theory has also played a significant role in various models of low dimensional quantum gravity [8]. An introduction to the applications of $BF$ theory in four dimensional quantum gravity may be found in [8], while a general concise introduction to topological quantum field theory is given in [10].

In this paper we will study a general class of abelian topological $BF$ theories and use it to describe some topological invariants of manifolds of generic dimension. The basic mathematical motivation comes from a well-known classification problem in geometric topology. In the classification theory of three-dimensional manifolds $M_3$, an important invariant is the topological class of a mapping of the circle $S^1$ into $M_3$ such that no two points of the loop intersect on $M_3$. The study of all such embeddings is known as knot theory. Knots in dimensions other than three are always trivial, but in three dimensions there is a vast collection of such topological classes which live in the fundamental homotopy group $\pi_1(M_3)$ of the manifold. An algebraic entity related to knot theory is the braid group of a two-dimensional space [11, 12]. A braid of $N$ strands can be viewed as a collection of overlapping lines on the plane. It is a fundamental theorem of geometric topology that any collection of knots or links can be constructed by joining the ends of a certain braid. The braid group is therefore a useful tool for the classification of three-manifolds. These structures can all be generalized to higher dimensional embeddings into higher dimensional manifolds to form objects known as motion groups [13, 14] which are the appropriate generalizations of braid groups. The motion group is a useful tool in the topological classification of manifolds whose properties have only been touched upon briefly in the mathematics literature.

Polynomial invariants of knots in three dimensions and of two-knots in four dimensions are derived in [13, 14] from observables in various versions of abelian and non-abelian $BF$
field theories. In the following we will obtain a special class of multi-dimensional abelian unitary representations of generic motion groups in manifolds of generic dimension. This analysis sheds some light on the global structures involved in the algebraic and geometric definitions of motion groups in general. Just as the braid group and its representations via Chern-Simons gauge theory \[17\]–\[19\] arise from the statistical exchange holonomies between quantum mechanical point particles in two spatial dimensions, the quantum holonomies induced in the wavefunctions for BF gauge fields coupled minimally to higher dimensional worldvolumes lead to the appropriate motion group representations and are relevant to the description of exchange statistics between extended objects in higher spatial dimensions. The possibility of exotic statistics between strings in three dimensions was considered in several different contexts in \[6\] and analysed in some more generality in \[20\]. Field theoretical models using four dimensional BF theory were described in \[21\]–\[23\] and related to features of the motion group. The class of quantum field theories analysed in this paper is relevant to more general models of D-branes and M-branes which have played a fundamental role in the present understanding of the dynamics of superstring theory and M-theory. The results obtained in the following describe various geometrical aspects of antisymmetric tensor fields with non-trivial topological charges and present an essential description of the quantum field theory for these extended objects.

We will be particularly interested in global aspects of these topological field theories and their associated motion group representations. The basic modifications which arise when the line bundle of the gauge theory is topologically non-trivial can be understood easiest in terms of the global gauge group of the BF theory. For a BF field theory constructed from a \(p\)-form field and a corresponding dual \(d-p-1\)-form field which are sections of a trivial vector bundle over a \(d\)-dimensional manifold \(\mathcal{M}_d\), the harmonic zero mode contributions of the fields to the corresponding path integral produce a Grassmann parity graded direct sum of deRham cohomology groups \[3\],

\[
Z(\mathcal{M}_d) = \bigoplus_{n=0}^{2p-d} (-1)^n H_D^{p-n}(\mathcal{M}_d) \oplus \left(1 + (-1)^{d+1}\right) \bigoplus_{m=2p-d+1}^{p} (-1)^m H_D^{p-m}(\mathcal{M}_d) \quad (1.1)
\]

The appropriate modification for curved vector bundles of the treatment of the harmonic zero modes is described in general in \[24\] and applied to abelian BF theory in \[25\]. It is found there that the zero mode space \((1.1)\) is modified by the Čech cohomology of the manifold \(\mathcal{M}_d\). In the following we will describe this modification within the framework of canonical quantization. We shall use sheaf cohomology in a rather straightforward way \[13, 26\] that is much simpler than the covariant approach of \[24\] which requires the use of Čech hyper-cohomology complexes and higher rank bundles whose properties are not very well-understood except in some lower dimensional cases \[27, 28\]. As we will show, the present sheaf cohomological approach is particularly well-suited for the canonical quantization of the source-coupled BF theory, but is extremely cumbersome for a path integral treatment along the lines of \[24, 25\]. Conversely, the former approach is not very useful in a canonical framework because the various degrees of freedom in the
decompositions of the fields are not very well understood. Having obtained the appropriate 
modification of the $BF$ action which incorporates global effects, we will then be able 
to deduce the corresponding constraints that the generators of the motion group on a 
homologically non-trivial manifold must satisfy.

The outline of this paper is as follows. In section 2 we describe, using sheaf cohomol-
ogy, the modifications of the $BF$ field theory action that must be made when the relevant 
gauge fields are sections of a non-trivial line bundle. We show how this modification sim-
plifies within a canonical framework and also derive a topological quantization constraint 
on the fluxes of the gauge fields in terms of the Čech cohomology of the manifold which 
ensures that the quantum field theory is indeed topologically invariant. In section 3 we 
introduce the parametrizations of the fields that will be used and study the canonical 
structure of the field theory. We illustrate here how the appropriate modification to (1.1) 
arises in the canonical formalism through strong gauge constraints which relate the global 
fluxes of the gauge fields to the external charges of the theory. In section 4 we solve 
explicitly for the wavefunctions of the quantum field theory in the functional Schrödinger 
representation and show how the sheaf cohomological constraints ensure that they are 
indeed topological invariants. One important ingredient in this construction is the derivation 
of a generalized solid angle function which computes the adiabatic linking numbers of 
embedded surfaces of generic dimension. In section 5 we introduce the motion group 
and describe some of its formal mathematical properties, and then proceed to describe 
the holonomy representations of it carried by the wavefunctions of the canonical quantum 
field theory. Section 6 then concludes with some examples, applications and directions for 
further generalizations. We show how the standard braid group representations induced 
from Chern-Simons gauge theory on a Riemann surface [18, 19] are recovered within the 
present formalism but with some global modifications. These global modifications are 
also shown to occur for the quantum exchange holonomies of equal rank extended objects 
in higher dimensions. The main feature is that the $BF$ formalism avoids the cumber-
some self-linking numbers that arise in Chern-Simons theory [13] which are regularization 
dependent. Physically this is understood as an inducing of an intrinsic spin to the $p - 1$-
branes by the doubling of field theoretical degrees of freedom in the present case, which 
than cancels the statistical phases which appear because the spin-statistics theorem holds 
in these special situations. This is in contrast to the case of strings in three spatial di-
mensions, whereby the spin-statistics theorem need not hold in certain cases [20]. The 
generalizations of the standard Gauss linking integral are also derived.

2. Global Aspects of Topological $BF$ Field Theory

In this section we shall describe some general aspects of $BF$-type topological field 
theories. Starting with a quick review of the well-known situation when the theory is
defined using flat vector bundles, we then describe the required modifications of the field theory when the associated bundle lives in a non-trivial topological class.

2.1. **BF Theory on Trivial Line Bundles**

Consider the field theory of a real-valued \( p \)-form field \( B \in \Omega^p(\mathcal{M}_{d+1}) \) and a real-valued \( d-p \)-form field \( A \in \Omega^{d-p}(\mathcal{M}_{d+1}) \) defined on the spacetime manifold \( \mathcal{M}_{d+1} \) of dimension \( d+1 \) with metric of Minkowski signature. We assume in this subsection that these differential forms take values in some flat vector bundle over \( \mathcal{M}_{d+1} \). The BF-action is given by the space-time integral of a \( d+1 \)-form

\[
S = \int_{\mathcal{M}_{d+1}} \frac{k}{2\pi} B \wedge dA \tag{2.1}
\]

where \( k \in \mathbb{R} \) is a coupling constant. This action is invariant under the \( U(1) \) gauge transform

\[
A \rightarrow A + \chi \tag{2.2}
\]

where \( \chi \) is a closed \( d-p \)-form, \( d\chi = 0 \), and it transforms by a surface term under

\[
B \rightarrow B + \xi \tag{2.3}
\]

where \( \xi \) is a closed \( p \)-form, \( d\xi = 0 \). In the present field theory without sources any closed forms are allowed in \eqref{2.2} and \eqref{2.3}. However, when this topological field theory is coupled to sources we also require gauge invariance of the holonomy operators

\[
W[\Sigma_p] = \exp i \oint_{\Sigma_p} B, \quad W[\Sigma_{d-p}] = \exp i \oint_{\Sigma_{d-p}} A \tag{2.4}
\]

This restricts the class of closed forms allowed in \eqref{2.2} and \eqref{2.3} to those of integer-valued cohomology, so that

\[
\oint_{\Sigma_p} \xi = 2\pi n_p, \quad \oint_{\Sigma_{d-p}} \chi = 2\pi n_{d-p} \tag{2.5}
\]

for some integers \( n_p \) and \( n_{d-p} \) and for any compact, closed orientable submanifolds \( \Sigma_p \) and \( \Sigma_{d-p} \) of \( \mathcal{M}_{d+1} \). In the following, we shall assume this restricted gauge symmetry.

The partition function is given by the path integral

\[
\langle 1 \rangle = \int_{\mathcal{M}} D\mu(A, B) \exp i \int_{\mathcal{M}_{d+1}} \frac{k}{2\pi} B \wedge dA \tag{2.6}
\]

which is normalized by the volume of the gauge group. The functional measure \( D\mu(A, B) \) on the moduli space \( \mathcal{M} \) of gauge orbits is obtained by the standard gauge-fixing procedure to give\(^1\)

\[
D\mu(A, B) = DA \ DB \ \Delta_{FP}[A] \Delta_{FP}[B] \ \delta(F[A]) \delta(G[B]) \tag{2.7}
\]

\(^1\)This is the conventional parametrization for the partition function. In \([4]\) an alternative parametrization in terms of Hodge decompositions of the fields (see section 3.1) is implemented which avoids using the BRST gauge fixing procedure.
where $\Delta_{\text{FP}}$ denotes the usual Faddeev-Popov determinant, and $\mathcal{F}$ and $\mathcal{G}$ are gauge-fixing functions. This path integral is related to the Ray-Singer analytic torsion which is a topological invariant of $\mathcal{M}_{d+1}$ given by properties of the spectrum of the differential operators $d$ and $\ast d$ and the Laplacian ($\ast$ denotes the Hodge duality operator of $\mathcal{M}_{d+1}$). Here it is given explicitly by the ratio of determinants [3, 5, 10]

$$\langle 1 \rangle = \prod_{k=0}^{p} \det_{\perp}^{\mu_k} \Box_{p-k} \prod_{l=0}^{d-p} \det_{\perp}^{\mu_l} \Box_{d-p-l} \quad , \quad \mu_k \equiv (-1)^{k+1} \frac{2k+1}{4}$$

(2.8)

where $\Box_k$ is the Laplacian acting on $\Omega^k(\mathcal{M}_{d+1})$ and $\det_{\perp}$ denotes the regularized determinant with zero modes arising from gauge invariance excluded.

Gauge and topologically invariant operators are given by $p$-cycle holonomies of $B$ and $d-p$-cycle holonomies of $A$. The expectation value of the operators $\langle \Sigma, A \rangle$ is given by the path integral with sources,

$$\langle W[\Sigma_p], W[\Sigma_{d-p}] \rangle_{\langle 1 \rangle} = \int_{\mathcal{M}} D\mu(A, B) \exp \left( i \int_{\mathcal{M}_{d+1}} \frac{k}{2\pi} B \wedge dA + i \int_{\Sigma_{d-p}} \delta(B, A) + \int_{\Sigma_p} B \wedge dA \right)$$

(2.9)

This functional integral is independent of the metric of $\mathcal{M}_{d+1}$ and is formally a topological invariant. It is related to the topological linking number of disjoint, closed contractible hypersurfaces $\Sigma_p$ and $\Sigma_{d-p}$, which can be seen by explicitly performing the integral to obtain [3, 4, 10]

$$\langle W[\Sigma_p], W[\Sigma_{d-p}] \rangle_{\langle 1 \rangle} = \exp - \frac{2\pi i}{k} L(\Sigma_p, \Sigma_{d-p})$$

(2.10)

where

$$L(\Sigma_p, \Sigma_{d-p}) = \int_{S(\Sigma_p)} \Delta_{\Sigma_{d-p}} = (-1)^{(p-1)(d-p)} \int_{S(\Sigma_{d-p})} \Delta_{\Sigma_p}$$

(2.11)

is the standard expression for the signed linking number of two cycles. Here $S(\Sigma_p)$ is a hypersurface spanned by the $p$-cycle $\Sigma_p$, and $\Delta_{\Sigma_p}$ is the (singular) deRham current $d - p + 1$-form [29] which is the delta-function supported Poincaré dual to the embedding $X_p : \Sigma_p \to \mathcal{M}_{d+1}$. It is closed, $d\Delta_{\Sigma_p} = 0$, and locally it can be expressed as

$$\Delta_{\Sigma_p}(x) = \oint_{\Sigma_p} \delta^{(p,d-p+1)}(X_p(\sigma), x) \ast \oint_{\Sigma_p} d\sigma(X_p) \delta^{(d+1)}(X_p(\sigma), x)$$

(2.12)

where

$$d\sigma^{\mu_1 \ldots \mu_p}(x) = \epsilon^{\alpha_1 \ldots \alpha_p} \prod_{k=1}^{p} \frac{\partial X_{\mu_k}^p(\sigma)}{\partial \sigma_{\alpha_k}} \, d\sigma^k$$

(2.13)

is the induced volume element of $\Sigma_p$ in $\mathcal{M}_{d+1}$ and $\delta^{(p,d-p+1)}(x, y)$ is the Dirac delta-function in the exterior algebra $\Omega^p(\mathcal{M}_{d+1}(x)) \otimes \Omega^{d-p+1}(\mathcal{M}_{d+1}(y))$, i.e.

$$\int_{\mathcal{M}_{d+1}(y)} \delta^{(p,d-p+1)}(x, y) \wedge \alpha(y) = \alpha(x) \quad \forall \alpha(x) \in \Omega^p(\mathcal{M}_{d+1}(x))$$

(2.14)
A more complete picture of this system is obtained by canonical quantization. For this, we choose the spacetime to be the product manifold \( \mathbb{R}^1 \times \mathcal{M}_d \), where \( \mathbb{R}^1 \) parametrizes the time \( t \) and \( \mathcal{M}_d \) is a compact, path-connected, orientable \( d \)-dimensional manifold without boundary.\(^2\) The field \( B \) may then be decomposed according to

\[
B = B^0 \wedge dt + \tilde{B} \tag{2.15}
\]

where \( B^0 \) is the \( p - 1 \)-form on \( \mathcal{M}_d \) with local components \( B^0_{i_1 \cdots i_{p-1}} = B_{i_1 \cdots i_{p-1} 0} \) and \( \tilde{B} \) is the restriction of \( B \) to \( \mathcal{M}_d \) (and similarly for the other fields of the theory). The action is now written as

\[
S(\Sigma_p, \Sigma_{d-p}) = \int_{-\infty}^{\infty} dt \int_{\mathcal{M}_d} \left( \frac{k}{2\pi} B \wedge dA + Q_p B \wedge \Delta_{\Sigma_p} + Q_{d-p} A \wedge \Delta_{\Sigma_{d-p}} \right) \tag{2.16}
\]

where \( Q_p, Q_{d-p} \in \mathbb{R} \) are worldvolume charges, \( \Sigma_p \) and \( \Sigma_{d-p} \) are disjoint hypersurfaces in \( \mathcal{M}_{d+1} \), and we use the local worldvolume reparametrization invariance to fix the gauge in which the temporal embedding coordinate parametrizes the hypersurface \( \Sigma_p \), i.e. \( X^0(\sigma^1, \ldots, \sigma^p) = \sigma^1 \). The temporal components of the fields are Lagrange multipliers which enforce the local gauge constraints

\[
\frac{k}{2\pi} d\tilde{A} + Q_p \tilde{\Delta}_{\Sigma_p} \approx 0 \quad , \quad (-1)^{p(d-p)} \frac{k}{2\pi} d\tilde{B} + Q_{d-p} \tilde{\Delta}_{\Sigma_{d-p}} \approx 0 \tag{2.17}
\]

The remaining action is of first order in time derivatives and is therefore already expressed in phase space with the spatial components of \( A \) and \( B \) being the canonically conjugate variables. The canonical quantum commutator is

\[
[\tilde{A}_{i_1 \cdots i_{d-p}}(x), \tilde{B}_{j_1 \cdots j_p}(y)] = \frac{2\pi i}{k} \epsilon_{0i_1 \cdots i_{d-p}j_1 \cdots j_p} \delta^{(d)}(x, y) \tag{2.18}
\]

Note that the factors of \( \text{det} \ g \), where \( g \) is the metric of \( \mathcal{M}_d \), which would make the delta-function on the right-hand side of (2.18) generally covariant cancel similar factors coming from the totally antisymmetric tensor \( \epsilon \). The canonical commutator is therefore independent of the metric of \( \mathcal{M}_d \). The Hamiltonian in the temporal gauge \( A^0 = B^0 = 0 \) is

\[
H = -\int_{\mathcal{M}_d} \left( Q_{d-p} \tilde{A} \wedge \Delta_{\Sigma_{d-p}}^0 + Q_p \tilde{B} \wedge \Delta_{\Sigma_p}^0 \right) \tag{2.19}
\]

However, there is a technical problem with the way that we have set up the canonical formalism for this topological field theory. The difficulty lies in the fact that the local gauge constraints (2.17), which must be imposed as physical state conditions in the quantum field theory, imply that the forms \( \tilde{A} \) and \( \tilde{B} \) have non-vanishing flux around cycles of the manifold \( \mathcal{M}_d \), i.e. generically we have \( \int_{\Sigma_{d-p+1}} d\tilde{A} \neq 0 \) and \( \int_{\Sigma_{d+1}} d\tilde{B} \neq 0 \). This implies

\(^2\)In the following Greek indices \( \mu = 0, 1, \ldots, d \) will label spacetime directions in \( \mathcal{M}_{d+1} \) while Latin indices \( i = 1, \ldots, d \) label spatial directions in \( \mathcal{M}_d \). Furthermore, explicit metric factors required to make quantities diffeomorphism invariant will be typically omitted.
that $A$ and $B$ cannot be considered as globally defined differential forms and must be
defined locally on patches covering the manifold. In turn, the definition of the topologi-
cal field theory must be appropriately modified. This will be the subject of the next
subsection. Here we note only one final aspect of this model in the absence of sources
($Q_p = Q_{d-p} = 0$). In that case the constraints (2.17) imply that $A$ and $B$ restricted to
$\mathcal{M}_d$ are closed forms. Their worldvolume integrals are therefore topological invariants
and (2.18) implies that they obey the operator algebra

$$\left[ \oint_{\tilde{\Sigma}_{d-p}} A, \oint_{\tilde{\Sigma}_p} \tilde{B} \right] = \frac{2\pi i}{k} \nu[\tilde{\Sigma}_{d-p}, \tilde{\Sigma}_p]$$  \hspace{1cm} (2.20)

where

$$\nu[\tilde{\Sigma}_{d-p}, \tilde{\Sigma}_p] = \sum_{x \in \tilde{\Sigma}_p \cap \tilde{\Sigma}_{d-p}} \text{sgn}(x)$$  \hspace{1cm} (2.21)

is the signed intersection number of the embedded hypersurfaces $\tilde{\Sigma}_{d-p}$ and $\tilde{\Sigma}_p$ on $\mathcal{M}_d$
taken over all intersections $x$ with orientation $\text{sgn}(x) = \pm 1$ (Generically, in $d$ dimensions,
a $p$-surface and a $d-p$-surface intersect at discrete points). This number is a topological
invariant, so that if either $\tilde{\Sigma}_{d-p}$ or $\tilde{\Sigma}_p$ is a contractible hypersurface, then $\nu[\tilde{\Sigma}_{d-p}, \tilde{\Sigma}_p]$
vanesishes and they intersect an even number of times with cancelling orientations. Thus,
in (2.20), the commutator is non-trivial only for those worldvolumes which are non-trivial
elements of the $p$-th and $d-p$-th homology groups $H_p(\mathcal{M}_d)$ and $H_{d-p}(\mathcal{M}_d)$. This property
will be the crucial aspect of the topological group representations that we shall find.

2.2. BF Theory on Non-trivial Line Bundles

We shall now consider the case where the fields $A$ and $B$ are sections of some non-
trivial bundle over the spacetime manifold $\mathcal{M}_{d+1}$. Since these sections have rank which
is in general larger than 1, they are actually sections of a higher-rank bundle [28] over
$\mathcal{M}_{d+1}$, i.e. a fiber bundle whose fibers are groupoids, rather than some Lie group. Such
generalized fiber bundles are known as gerbes. As pointed out in [25], in this case the
zero mode contribution to the path integral (2.6) is modified and the resulting topological
invariant represents not just the Ray-Singer torsion (2.8), but also the Čech cohomology
of the underlying manifold. Thus the canonical quantization of the $BF$ field theory in
this case will yield not only interesting quantum field theoretical representations of the
derRham complex of the spatial manifold $\mathcal{M}_d$, but also of the more general Čech complex
which encodes the possibility of passage from local to global data on $\mathcal{M}_d$ and which
classifies the topological line bundles over $\mathcal{M}_d$. Gerbes have also been used recently for
some general analyses in quantum field theory [30] and in the context of massive D-brane
configurations in Type II superstring theory [31].

We assume that the spatial manifold $\mathcal{M}_d$ admits a finite open Leray cover $\mathcal{U} = \{U_a\}$. For each ordered collection $(U_{a_0}, U_{a_1}, \ldots, U_{a_q})$ of open sets of $\mathcal{U}$ with non-empty intersec-
along with a formal orientation defined by $U_{a_{r(0)}a_{r(1)}\cdots a_{r(q)}} = \text{sgn}(\pi) U_{a_0a_1\cdots a_q}$ for $\pi \in S_{q+1}$. The abelian group of all formal linear combinations with integer coefficients of objects of the form (2.22) is the $q$-chain group $C_q(\mathcal{U})$ of the cover $\mathcal{U}$. Using a $\mathbb{Z}$-linear boundary operator $\partial$ defined on $q$-chains by

$$\partial U_{a_0a_1\cdots a_q} = \sum_{k=0}^{q} (-1)^k U_{a_0a_1\cdots a_{k-1}a_{k+1}\cdots a_q}$$

one may define the $q$-th Čech homology group $H_q(\mathcal{M}_d)$ which is independent of the choice of Leray cover (If $\mathcal{U}$ is not a Leray cover, then the homology must be defined by taking an inductive limit over all possible open coverings of the space).

The cover $\mathcal{U}$ naturally defines a simplicial decomposition of the manifold $\mathcal{M}_d$. Namely, to each elementary $q$-chain (2.22) we can naturally associate a $q$-simplex $\triangle_q$. In dimension $d$, we may label the simplices with a number of indices which is dual to their dimension, by the inductive definition that a $q$-simplex $\triangle_q^{(a_1\cdots a_{d-q})}$ is obtained as the intersection of $d - q + 1$ simplices of dimension $q + 1$,

$$\triangle_q^{(a_1\cdots a_{d-q})} = \triangle_{q+1}^{(a_1\cdots a_{d-q-1})} \cap \triangle_{q+1}^{(a_{d-q-1}a_{d-q}a_1\cdots a_{d-q-3})} \cap \triangle_{q+1}^{(a_{d-q-3}a_{d-q-2}a_{d-q-1}a_{d-q-1}a_{d-q}a_1\cdots a_{d-q-5})} \cap \cdots \cap \triangle_{q+1}^{(a_{2}\cdots a_{d-q})}$$

along with the appropriate orientation induced by the supports of the cover $\mathcal{U}$. With this convention the boundary operator acts on a $q$-simplex as

$$\partial \triangledown_q^{(a_1\cdots a_{d-q})} = \sum_a \sum_{k=1}^{d-q+1} (-1)^{k+1} \triangle_{q-1}^{(a_1\cdots a_k\cdots a_{d-q})}$$

where the first sum in (2.23) runs through all $q - 1$-simplices with the appropriate index labellings. In this way the Čech homology of the manifold $\mathcal{M}_d$ coincides with its simplicial homology (This is also true if $\mathcal{M}_d$ doesn’t admit a Leray cover).

On each open set $U_a$ of the covering $\mathcal{U}$ there is a $(d - p)$-form gauge field $A^{(a)}$ and a $p$-form gauge field $B^{(a)}$, which are elements of the 0-cochain group $C^0(\mathcal{U}, \Omega^{d-p})$ with coefficients in the sheaf $\Omega^{d-p}$ of real-valued differential $d - p$-forms on $\mathcal{M}_d$ and of the 0-cochain group $C^0(\mathcal{U}, \Omega^p)$ with coefficients in the sheaf $\Omega^p$ of $p$-forms, respectively. There are gauge transformations $\Lambda_1^{(ab)}$ and $\Xi_1^{(ab)}$ defined on the non-empty intersection $U_{ab}$ of any two open sets by

$$A^{(a)} - A^{(b)} = d\Lambda_1^{(ab)} \quad , \quad B^{(a)} - B^{(b)} = d\Xi_1^{(ab)}$$

with $\Lambda_1^{(ab)} = -\Lambda_1^{(ba)}$ and $\Xi_1^{(ab)} = -\Xi_1^{(ba)}$. These local forms are elements of the 1-cochain groups $C^1(\mathcal{U}, \Omega^{d-p-1})$ and $C^1(\mathcal{U}, \Omega^{p-1})$, respectively. Since each $\Omega^p$ is a fine sheaf, the corresponding Čech cohomology is trivial, and so there are secondary gauge transformations
\( \Lambda_2^{(abc)} \in C^2(\mathcal{U}, \Omega^{d-p-2}) \) and \( \Xi_2^{(abc)} \in C^2(\mathcal{U}, \Omega^{p-2}) \) defined on non-empty triple intersections \( \mathcal{U}_{abc} \) by

\[
\Lambda_1^{(ab)} + \Lambda_1^{(bc)} + \Lambda_1^{(ca)} = d \Lambda_2^{(abc)} \quad , \quad \Xi_1^{(ab)} + \Xi_1^{(bc)} + \Xi_1^{(ca)} = d \Xi_2^{(abc)} \quad (2.27)
\]

This procedure can be iterated inductively to higher degree cochains over higher degree chains. Namely, for each \( 1 \leq q < d - p \) there are cochains \( \Lambda_q^{(a_0a_1 \cdots a_q)} \in C^q(\mathcal{U}, \Omega^{d-p-q}) \) and for each \( 1 \leq q < p \) we have \( \Xi_q^{(a_0a_1 \cdots a_q)} \in C^q(\mathcal{U}, \Omega^{p-q}) \) which, on each non-trivial elementary \( q + 1 \)-chain \( \mathcal{U}_{a_0a_1 \cdots a_{q+1}} \), satisfy the overlap relations

\[
\Lambda_q^{(a_0a_1 \cdots a_q)} + \Lambda_q^{(a_0a_q+1a_0a_1 \cdots a_{q-2})} + \Lambda_q^{(a_q-2a_q-1a_q+1a_0a_1 \cdots a_{q-4})} \\
+ \ldots + \Lambda_q^{(a_{q+1})} = d \Lambda_{q+1}^{(a_0a_1 \cdots a_{q+1})} \quad (2.28)
\]

\[
\Xi_q^{(a_0a_1 \cdots a_q)} + \Xi_q^{(a_0a_q+1a_0a_1 \cdots a_{q-2})} + \Xi_q^{(a_q-2a_q-1a_q+1a_0a_1 \cdots a_{q-4})} \\
+ \ldots + \Xi_q^{(a_{q+1})} = d \Xi_{q+1}^{(a_0a_1 \cdots a_{q+1})} \quad (2.29)
\]

Finally, on each \( p \)-chain \( \mathcal{U}_{a_0a_1 \cdots a_p} \) and each \( d - p \)-chain \( \mathcal{U}_{a_0a_1 \cdots a_{d-p}} \) we have

\[
\Lambda_{d-p}^{(a_0a_1 \cdots a_{d-p})} + \Lambda_{d-p}^{(a_{d-p}a_{d-p+1}a_0a_1 \cdots a_{d-p-2})} + \Lambda_{d-p}^{(a_{d-p-2}a_{d-p-1}a_0a_1 \cdots a_{d-p-4})} \\
+ \ldots + \Lambda_{d-p}^{(a_{d-p})} = \lambda_{a_0a_1 \cdots a_{d-p+1}} \quad (2.30)
\]

\[
\Xi_{d-p}^{(a_0a_1 \cdots a_p)} + \Xi_{d-p}^{(a_0a_{p+1}a_0a_1 \cdots a_{p-2})} + \Xi_{d-p}^{(a_{p-2}a_{p-1}a_0a_1 \cdots a_{p-4})} \\
+ \ldots + \Xi_{d-p}^{(a_{d-p})} = \xi_{a_0a_1 \cdots a_{p+1}} \quad (2.31)
\]

The locally constant functions \( \lambda_{a_0a_1 \cdots a_{d-p+1}} \) and \( \xi_{a_0a_1 \cdots a_{p+1}} \) are, respectively, \( d - p + 1 \)-cocycles and \( p+1 \)-cocycles of the Čech cohomology groups \( H^{d-p+1}_C(M_d, \mathbb{R}) \) and \( H^{p+1}_C(M_d, \mathbb{R}) \) of the manifold \( M_d \) with coefficients in the constant sheaf \( \mathbb{R} \) (Again this is independent of the choice of Leray cover). These Čech cohomology groups are naturally isomorphic to the corresponding deRham cohomology groups, \( H^{d}_C(M_d, \mathbb{R}) \cong H^{d}_D(M_d) \) [26].

The \( \lambda \)'s and \( \xi \)'s satisfy the usual cocycle relations

\[
\lambda_{a_0a_1 \cdots a_{d-p+1}} + \lambda_{a_{d-p}a_{d-p+1}a_0a_1 \cdots a_{d-p-1}} + \ldots + \lambda_{a_1 \cdots a_{d-p+2}} = 0 \quad (2.32)
\]

\[
\xi_{a_0a_1 \cdots a_{p+1}} + \xi_{a_{p+1}a_{p+2}a_0a_1 \cdots a_{p-1}} + \ldots + \xi_{a_1 \cdots a_{p+2}} = 0 \quad (2.33)
\]

on \( \mathcal{U}_{a_0a_1 \cdots a_{d-p+2}} \) and \( \mathcal{U}_{a_0a_1 \cdots a_{p+2}} \), respectively, and they are completely antisymmetric in their indices. Given a \( d - p + 1 \)-cycle \( \Sigma_{d-p+1} \) and a \( p + 1 \)-cycle \( \Sigma_{p+1} \) of \( M_d \), the corresponding fluxes of the gauge fields \( A \) and \( B \) can be represented in terms of the Čech cohomology classes using the overlap relations (2.26)–(2.31) and repeated application of Stokes' theorem to write

\[
F_0(\bar{\Sigma}_{d-p+1}) \equiv \oint_{\bar{\Sigma}_{d-p+1}} dA = \sum_{\Delta_0} (a_0a_1 \cdots a_{d-p+1}) (\bar{\Sigma}_{d-p+1}) \lambda_{a_0a_1 \cdots a_{d-p+1}} \quad (2.34)
\]

\[
H_0(\bar{\Sigma}_{p+1}) \equiv \oint_{\bar{\Sigma}_{p+1}} dB = \sum_{\Delta_0} (a_0a_1 \cdots a_{p+1}) (\bar{\Sigma}_{p+1}) \xi_{a_0a_1 \cdots a_{p+1}} \quad (2.35)
\]
where the sums are taken over all 0-simplices \( \Delta_0^{(a_0 a_1 \ldots a_{d-p+1})}(\tilde{\Sigma}_{d-p+1}) \) and \( \Delta_0^{(a_0 a_1 \ldots a_{p+1})}(\tilde{\Sigma}_{p+1}) \) with respect to the induced simplicial decompositions of \( \tilde{\Sigma}_{d-p+1} \) and \( \tilde{\Sigma}_{p+1} \), respectively, from the restrictions of the cover \( \mathcal{U} \) to these submanifolds of \( \mathcal{M}_d \) (via a refinement of \( \mathcal{U} \) if necessary). Equations (2.34) and (2.35) demonstrate explicitly the relationship between \( B \) and deRham cohomologies.

Now we come to the appropriate extension of the action (2.16) for the present situation. For this, we shall compactify the time direction on a circle, so that \( \mathcal{M}_{d+1} = S^1 \times \mathcal{M}_d \), and extend the Leray cover \( \mathcal{U} \) with its corresponding simplicial decomposition of \( \mathcal{M}_d \) trivially through the time direction. This means that we shall consider only periodic motions on \( \mathcal{M}_d \). Since the gauge fields \( A \) and \( B \) are only locally defined on \( \mathcal{M}_{d+1} \), the \( BF \) action must be modified so that it is independent of the simplicial decomposition (or covering) used to define the fields. Consider the term

\[
\int_{\mathcal{M}_{d+1}} B \wedge dA = \sum_{\Delta^{(a)}_{d+1}} \int_{\Delta^{(a)}_{d+1}} B^{(a)} \wedge dA \tag{2.36}
\]

where the sum runs through all \( d+1 \)-simplices of \( \mathcal{M}_{d+1} \). If we deform the simplex \( \Delta^{(a)}_{d+1} \), then the corresponding change of integrand in (2.36) is \( d(\Xi^{(ab)}_1 \wedge dA) \), so that we must add the term \(- \sum_{a,b} f_{\Delta^{(ab)}_1} \Xi^{(ab)}_1 \wedge dA \) in order to cancel this variation. In turn, we must cancel the dependence of this additional term on deformations of the simplices \( \Delta^{(ab)}_d \), and so on. Now we must repeat this procedure for the deRham currents appearing in (2.16). Since \( \Delta^{(a)}_{\Sigma_p} \) is a closed form representing the Poincaré cohomology class of the cycle \( \Sigma_p \), by Poincaré’s lemma we have

\[
\Delta^{(a)}_{\Sigma_p} = d\delta^{(a)}_{\Sigma_p} \tag{2.37}
\]

on \( U_a \in \mathcal{U} \), with \( \delta^{(a)}_{\Sigma_p} \in C^0(\mathcal{U}, \Omega^{d-p}) \). Proceeding as before, we then obtain a set of cochains \( \mathcal{X}_q^{(a_0 a_1 \ldots a_q)} \in C^q(\mathcal{U}, \Omega^{d-p-q}) \) for \( 1 \leq q < d - p \) defined by

\[
\delta^{(a)}_{\Sigma_p} - \delta^{(b)}_{\Sigma_p} = d\mathcal{X}_1^{(ab)} \tag{2.38}
\]

\[
\mathcal{X}_q^{(a_0 a_1 \ldots a_q)} + \mathcal{X}_q^{(a_q a_{q+1} a_0 a_1 \ldots a_{q-2})} + \mathcal{X}_q^{(a_{q-2} a_{q-1} a_q a_{q+1} a_0 a_1 \ldots a_{q-4})} + \ldots + \mathcal{X}_q^{(a_1 \ldots a_{q+1})} = d\mathcal{X}_{q+1}^{(a_0 a_1 \ldots a_q+1)} \tag{2.39}
\]

on \( U_{ab} \) and \( U_{a_0 a_1 \ldots a_{q+1}} \), respectively.

Using the method described above, we arrive at the consistent topological extension of the source coupled \( BF \) action (2.16),

\[
S(\Sigma_p, \Sigma_{d-p}) = \sum_{a} \int_{\Delta^{(a)}_{d+1}} \left[ B^{(a)} \wedge \left( \frac{k}{2\pi} dA + Q_p \Delta^{(a)}_{\Sigma_p} \right) + Q_{d-p} A^{(a)} \wedge \Delta^{(a)}_{\Sigma_{d-p}} \right] + \sum_{q=1}^{p} (-1)^q \sum_{a_0, a_1, \ldots, a_{q+1}} \int_{\Delta^{(a_0 a_1 \ldots a_q+1)}_{d-q+1}} \Xi^{(a_0 a_1 \ldots a_q)} \wedge \left( \frac{k}{4\pi} dA + Q_p \Delta^{(a_0 a_1 \ldots a_q)}_{\Sigma_p} \right)
\]

10
where the sums all run over the simplicial decomposition of the spacetime manifold $\mathcal{M}$.

It is straightforward to verify that the complicated expression (2.40) ensures that the total integrals. In this way, we arrive at a topologically invariant action for the field theory. However, it can be simplified by noticing that the gauge field may be rearranged permutation, i.e. for $\pi \in S_d$ we have

$$X^{(\pi_1 \cdots \pi_p)} Y^{(\pi_{p+1} \cdots \pi_d)} = \text{sgn}(\pi) X^{(1 \cdots p)} Y^{(p+1 \cdots d)}$$

(2.41)

It is straightforward to verify that the complicated expression (2.40) ensures that the total action is independent of the simplicial decomposition of $\mathcal{M}_{d+1}$ used to evaluate each of the integrals. In this way, we arrive at a topologically invariant action for the BF field theory with gauge fields that are sections of a non-trivial $U(1)$ bundle over the manifold.

The action (2.40) as it stands is difficult to deal with, especially for the quantum field theory. However, it can be simplified by noticing that the gauge field $A$ may be decomposed in terms of an arbitrary globally-defined differential form $A \in \Omega^{d-p}(\mathcal{M}_{d+1})$ and a singular form $\tilde{A}$ which is an explicit representative of the topological bundle of $A$ (and similarly for $B$). This latter degree of freedom may be constructed as follows. Let $N_p$ be the rank of the singular homology groups $H_p(\mathcal{M}_d)$ and $H_{d-p}(\mathcal{M}_d)$, and let $\tilde{\Sigma}_p^{(k)}$ and $\tilde{\Sigma}_{d-p}^{(k)}$ be sets of corresponding generators. The associated intersection matrix of $\mathcal{M}_d$ is

$$f^{(p)kl} = \int_{\tilde{\Sigma}_{d-p}^{(l)}} \int_{\tilde{\Sigma}_p^{(k)}} \delta^{(d-p,p)}(x,y)$$

(2.42)
Then the gauge field $A$ may be written as

$$A = \sum_{k=1}^{N_{d-1}} \tilde{A}_k + A$$

(2.43)

where the singular forms $\tilde{A}_k$ are defined by

$$d \tilde{A}_k = (-1)^{p(d-p)} \sum_{l=1}^{N_{d-1}} I_{lk}^{(p-1)} F_0(\tilde{\Sigma}_{d-p+1}^{(l)}) \tilde{\Delta}_{\Sigma_{d-p+1}}^{(k)}$$

(2.44)

and they ensure that $A$ has the correct periods (2.34) around cycles of $\mathcal{M}_d$. Here $I_{lk}^{(p)}$ is the matrix inverse of the intersection matrix (2.42), and $\tilde{\Delta}$ denotes the deRham current which is Poincaré dual to a cycle in the spatial manifold $\mathcal{M}_d$ (for ease of notation the tildes are suppressed on Dirac delta-functions over $\mathcal{M}_d$, as in (2.42)). This means that we have chosen to localize the flux of the gauge fields over submanifolds of $\mathcal{M}_d$, rather than the full spacetime manifold $\mathcal{M}_{d+1}$. This choice is a necessary requirement in the canonical formalism, and is possible due to the topological triviality of the time direction of $\mathcal{M}_{d+1}$. Similarly, the gauge field $B$ can be written as

$$B = \sum_{k=1}^{N_{d+1}} \mathbb{B}_k + B$$

(2.45)

where $B \in \Omega^p(\mathcal{M}_{d+1})$ is a globally-defined differential form and $\mathbb{B}_k$ are singular forms defined by

$$d \mathbb{B}_k = \sum_{l=1}^{N_{d+1}} I_{lk}^{(p+1)} H_0(\tilde{\Sigma}_{d-p+1}^{(l)}) \Delta_{\Sigma_{d-p-1}}^{(k)}$$

(2.46)

which ensure that $B$ has the correct periods (2.35). Finally, the deRham currents are written as

$$\Delta_{\Sigma_p} = \sum_{k,l=1}^{N_{d-1}} I_{lk}^{(p-1)} \nu[\tilde{\Sigma}_{d-p+1}^{(l)}, \partial \Sigma_p(0)] \Delta_{\Sigma_{d-p-1}}^{(k)} + d \delta_{\Sigma_p}$$

(2.47)

with $\delta_{\Sigma_p} \in \Omega^{d-p}(\mathcal{M}_{d+1})$. Here $\Sigma_p(t)$ represents the embedded hypersurface $X_p(\Sigma_p) \subset \mathcal{M}_{d+1}$ projected onto $\mathcal{M}_d$ with boundary the $p-1$-brane $X_p(t, \sigma^2, \ldots, \sigma^p)$ at time $t$, and we have localized the period integrals of the deRham currents onto a fixed patch of $\mathcal{M}_d$ at $t = 0$ (as with the fluxes in (2.44) and (2.46)).

Substituting the decompositions (2.43)–(2.47) into the action (2.16) and integrating by parts we have

$$S(\Sigma_p, \Sigma_{d-p}) = \sum_{m=1}^{N_{d-1}} \sum_{l=1}^{N_{d+1}} \int dt \int_{\mathcal{M}_d} \left( \frac{k}{2\pi} \mathbb{B}_l \wedge d\tilde{A}_m ight.$$

$$+ Q_p \sum_{n=1}^{N_{d-1}} I_{nm}^{(p-1)} \nu[\tilde{\Sigma}_{d-p+1}^{(n)}, \partial \Sigma_p(0)] \mathbb{B}_l \wedge \tilde{\Delta}_{\Sigma_{d-p-1}}^{(m)}$$

$$+ Q_{d-p} \sum_{n=1}^{N_{d+1}} I_{nl}^{(p+1)} \nu[\tilde{\Sigma}_{d-p+1}^{(n)}, \partial \Sigma_{d-p}(0)] \mathbb{A}_m \wedge \tilde{\Delta}_{\Sigma_{d-p-1}}^{(l)} \right)$$

12
\[ + \int dt \int_{\mathcal{M}_d} \left[ \sum_{m=1}^{N_p-1} \left( \frac{k}{2\pi} B + (-1)^{p(d-p)} Q_{d-p} \delta_{\Sigma_{d-p}} \right) \wedge d A_m \right. \\
+ \left. (-1)^p \sum_{m=1}^{N_p+1} \left( \frac{k}{2\pi} A + Q_p \delta_{\Sigma_p} \right) \wedge d \mathcal{B}_m \right. \\
+ \left. \frac{k}{2\pi} B \wedge d A + Q_{d-p} A \wedge \Delta_{\Sigma_{d-p}} + Q_p B \wedge \Delta_{\Sigma_p} \right] \] (2.48)

The first set of integrals in (2.48) are defined using the topological extension (2.40). Remembering that the simplicial decomposition of \( \mathcal{M}_d \) is extended trivially through the periodic time direction of \( \mathcal{M}_{d+1} \), using the decompositions (2.43)--(2.47) and some calculation we find that the only non-vanishing contributions are

\[ S_{\text{sing}} = (-1)^d \sum_{k=1}^{N_p-1} \sum_{l=1}^{N_p+1} \left[ Q_{d-p} \sum_{m=1}^{N_p+1} I_{ml}^{(p+1)} \nu[\bar{\Sigma}_{d-p}^{(l)}, \partial \Sigma_{d-p}(0)] \right. \\
\times \left. \sum_{a_0, a_1, \ldots, a_{d-p+1}} \frac{1}{\Delta_p(a_0 a_1 \cdots a_{d-p+1})} \int_{\bar{\Sigma}_{d-p}^{(l)}} \delta_{\Sigma_{d-p}^{(l)}}(a_{d-p+1}) \right. \\
+ \left. Q_p \sum_{m=1}^{N_p+1} I_{mk}^{(p-1)} \nu[\bar{\Sigma}_{d-p+1}^{(m)}, \partial \Sigma_{p}(0)] \right. \\
\times \left. \sum_{a_0, a_1, \ldots, a_{p+1}} \frac{1}{\Delta_{d-p}^{(a_0 a_1 \cdots a_{p+1})}} \int_{\bar{\Sigma}_{d-p}^{(l)}} \delta_{\Sigma_{d-p}^{(l)}}(a_{p+1}) \right) \] (2.49)

The contribution (2.49) is not a topological invariant because it now changes under deformations of the simplices of \( \mathcal{M}_d \). Using the period relations (2.34) and (2.33), we see that the two sets of integrals in (2.49) are defined modulo terms of the form \( Q_{d-p} F_0 \) and \( Q_p H_0 \) (for periodic motions). In the quantum field theory, such terms would then appear as phases \( e^{iS_{\text{sing}}} \) and thus the ambiguity can be removed by a flux relation among the gauge fields. The required consistency condition ensuring topological invariance of the quantum field theory is thus

\[ 1 = \prod_{l=1}^{N_p-1} \prod_{k=1}^{N_p+1} \exp i Q_{d-p} F_0(\bar{\Sigma}_{d-p}^{(l)}) \sum_{m=1}^{N_p+1} I_{ml}^{(p+1)} \nu[\bar{\Sigma}_{d-p}^{(l)}, \partial \Sigma_{d-p}(0)] \]
\[ \times \prod_{l=1}^{N_p-1} \prod_{k=1}^{N_p+1} \exp i Q_p H_0(\bar{\Sigma}_{d-p}^{(l)}) \sum_{m=1}^{N_p+1} I_{mk}^{(p-1)} \nu[\bar{\Sigma}_{d-p+1}^{(m)}, \partial \Sigma_{p}(0)] \] (2.50)

In the following sections we will see that this condition does indeed lead to a sensible Hilbert space representation. We shall see later on that it implies some noteworthy features of the motion group on topologically non-trivial spaces.

Next, let us consider the integrals

\[ \int_{\mathcal{M}_d} dt \int \delta_{\Sigma_p} \wedge d \mathcal{B}_k = \sum_{l=1}^{N_p+1} I_{lk}^{(p+1)} H_0(\bar{\Sigma}_{p+1}^{(l)}) \int_{\bar{\Sigma}_{d-p}^{(l)}} \delta_{\Sigma_p}^{(l)} \wedge dt \] (2.51)
The integration in (2.51) can be set to 0 via a judicious choice of decomposition of the fixed deRham currents. This will be done in the next section. The essential feature is that the deRham currents are decomposed in (2.47) so that the only non-vanishing period integrals come from cycles which lie entirely in the spatial manifold $M_d$. A similar calculation shows that the fourth and fifth lines of (2.48) can be taken to be 0 in the temporal gauge. The final result is the action

$$S(\Sigma_p, \Sigma_{d-p}) = S_{\text{sing}} + \oint dt \int_{M_d} \left( \frac{k}{2\pi} B \wedge dA + Q_{d-p} A \wedge \Delta \Sigma_{d-p} + Q_p B \wedge \Delta \Sigma_p \right)$$ \hspace{1cm} (2.52)

This form of the action, along with the constraint (2.50), will be used to construct the canonical quantum field theory in the following sections.

### 3. Canonical Quantization

Having introduced the required modifications of the topological field theory that are required over non-trivial bundles, we shall now proceed to study the structure of the phase space of this system which will be used in the next section to construct the wavefunctions of the canonical quantum field theory.

#### 3.1. Hodge Decompositions

To deal with the quantum field theory associated with the action (2.52), it will be convenient to exploit the fact that $A$ and $B$ are globally defined differential forms on $M_d$ and therefore admit Hodge decompositions [29]. In this way we may write the field degrees of freedom in terms of their local exact and co-exact components, and their global components which take into account the topological degrees of freedom. For this, we consider the intersection matrix $I^{(p)kl}$ of $M_d$ whose matrix inverse can be defined by the bilinear form

$$I_{kl}^{(p)} = \int_{M_d} \alpha_l^{(p)} \wedge \beta_k^{(p)}$$ \hspace{1cm} (3.1)

where $\{\alpha_l^{(p)}\}_{l=1}^{N_p}$ and $\{\beta_k^{(p)}\}_{k=1}^{N_p}$ are bases of generators of $H_{D}^{d-p}(M_d)$ and $H_{D}^{p}(M_d)$, respectively, which are orthonormal in the inner product

$$\int_{M_d} \alpha_l^{(p)} \wedge * \alpha_k^{(p)} = \int_{M_d} \beta_l^{(p)} \wedge * \beta_k^{(p)} = \delta_{lk}$$ \hspace{1cm} (3.2)

Here $*$ is the Hodge duality operator on $M_d$ with respect to the restriction of the spacetime metric of $M_{d+1}$. With this definition, the harmonic forms $\alpha_l^{(p)}$ and $\beta_k^{(p)}$ are the (non-singular) Poincaré duals of the corresponding homology generators $\tilde{\Sigma}_d^{(l)}$ and $\tilde{\Sigma}_p^{(k)}$ of the free parts of the singular homology groups $H_{d-p}(M_d)$ and $H_p(M_d)$, respectively.
torsion components of the homology will play no role in what follows). The natural bilinear pairing on deRham cohomology between any closed $d - p$-form $\alpha$ and any closed $p$-form $\beta$ may then be written as

$$
\int_{\mathcal{M}_d} \alpha \wedge \beta = \sum_{k,l=1}^{N_p} \left( \oint_{\Sigma_{d-p}^{(k)}} I^{(p)}_{lk} \right) \left( \oint_{\Sigma_{d-p}^{(l)}} \beta \right)
$$

(3.3)

We shall denote by $\nabla_p^2 = *d*d$ the Laplacian operator acting on co-closed $p$-forms in $\Omega^p(\mathcal{M}_d)$.

The field $\tilde{A}$ may then be expressed in terms of its Hodge decomposition over $\mathcal{M}_d$ as

$$
\tilde{A} = d\theta + *dP_K + \sum_{l=1}^{N_p} a^l(t) \alpha_{l}^{(p)}
$$

(3.4)

where $\theta \in \Omega^{d-p-1}(\mathcal{M}_d)$ and $P_K \in \Omega^{p-1}(\mathcal{M}_d)$ with

$$
\nabla_{d-p-1}^2 \theta = *d* \tilde{A} \quad , \quad \nabla_{p-1}^2 P_K = *d \tilde{A}
$$

(3.5)

$$
a^l(t) = \sum_{k=1}^{N_p} I^{(p)kl} \int_{\mathcal{M}_d} \tilde{A} \wedge \beta_{k}^{(p)} = \oint_{\Sigma_{d-p}^{(l)}} \tilde{A}
$$

Since, by assumption, $\tilde{A}$ is a globally defined differential form on $\mathcal{M}_d$, the form $*d \tilde{A}$ contains no zero modes (harmonic forms) of the Laplacian operator $\nabla_{p-1}^2$. Moreover, the local and global parts of the gauge transformations (2.2) may be expressed in terms of the above degrees of freedom as

$$
\theta \rightarrow \theta + \chi' \quad , \quad a^l \rightarrow a^l + 2\pi n_{d-p}^l
$$

(3.6)

where $d\chi'$ is the local exact part of the closed $d - p$-form $\chi$ and $n_{d-p}^l$ labels the winding numbers of the gauge field $\tilde{A}$ around the Poincaré dual homology basis element $\Sigma_{d-p}^{(l)}$. Using the time-independent gauge transformations (3.6), we can remove the Laplacian zero modes of the form $*d* \tilde{A}$. Similarly, the Hodge decomposition of the field $\tilde{B}$ over $\mathcal{M}_d$ is

$$
\tilde{B} = *dP_\theta + dK + \sum_{l=1}^{N_p} b^l(t) \beta_{l}^{(p)}
$$

(3.7)

where $P_\theta \in \Omega^{d-p-1}(\mathcal{M}_d)$ and $K \in \Omega^{p-1}(\mathcal{M}_d)$ with

$$
\nabla_{d-p-1}^2 P_\theta = *d\tilde{B} \quad , \quad \nabla_{p-1}^2 K = *d \tilde{B}
$$

(3.8)

$$
b^l(t) = \sum_{k=1}^{N_p} I^{(p)kl} \int_{\mathcal{M}_d} \tilde{B} \wedge \alpha_{k}^{(p)} = \oint_{\Sigma_{d-p}^{(l)}} \tilde{B}
$$

and the gauge transformations (2.3) may be written as

$$
K \rightarrow K + \xi' \quad , \quad b^l \rightarrow b^l + 2\pi n_p^l
$$

(3.9)
where \( d\xi' \) is the local exact part of the closed \( p \)-form \( \xi \). It follows that the harmonic modes of the differential forms \(*d\tilde{B}\) and \(*d*\tilde{B}\) may be set to 0.

It will prove convenient to use a holomorphic polarization for the harmonic degrees of freedom of the gauge fields. For this, we consider the \( 2N_p \)-dimensional symplectic vector space

\[
\mathcal{P} = H_D^p(\mathcal{M}_d) \oplus H_D^{d-p}(\mathcal{M}_d)
\]

which, according to the gauge constraints (2.17), is the reduced classical phase space of the source-free \( BF \) field theory and is spanned by the topological degrees of freedom \( a^l \) and \( b^k \) of the gauge fields. On this finite dimensional vector space we may introduce a complex structure which is parametrized by an \( N_p \times N_p \) symmetric complex-valued matrix \( \tau \) such that \(-\tau \) lives in the Siegel upper half-plane. Its imaginary part defines a metric

\[
G^{lk} = -2 \sum_{m,n=1}^{N_p} I^{(p)ml}(\text{Im} \tau_{mn}) I^{(p)nk}
\]

on the topological phase space \( \mathcal{P} \). Note that the topological invariance property of the \( BF \) field theory implies that all observables will be independent of the phase space complex structure. The desired holomorphic polarization is then defined by the complex variables

\[
\gamma^l = a^l + \sum_{k,m=1}^{N_p} I^{(p)ml} \tau_{mk} b^k, \quad \bar{\gamma}^l = a^l + \sum_{k,m=1}^{N_p} I^{(p)ml} \bar{\tau}_{mk} b^k
\]

in terms of which the large gauge transformations take the form

\[
\gamma^l \rightarrow \gamma^l + 2\pi \left( n^l_{d-p} + \sum_{k,m=1}^{N_p} I^{(p)ml} \tau_{mk} n^k_p \right), \quad \bar{\gamma}^l \rightarrow \bar{\gamma}^l + 2\pi \left( n^l_{d-p} + \sum_{k,m=1}^{N_p} I^{(p)ml} \bar{\tau}_{mk} n^k_p \right)
\]

Now we come to the Hodge decompositions for the non-singular parts of the deRham currents (2.47). We have

\[
* d\delta\Sigma_p = \bar{\delta}\Sigma_p^0 \wedge dt + \tilde{\delta}\Sigma_p
\]

where the \( p \)-form \( \tilde{\delta}\Sigma_p \) may be decomposed as

\[
\tilde{\delta}\Sigma_p = dw_p + *d\rho_p + \sum_{k,l=1}^{N_p} \Sigma^{(p)}_l(t) I^{(p)kl} \star \alpha_k^{(p)}
\]

with \( \omega_p \in \Omega^{p-1}(\mathcal{M}_d) \) and \( \rho_p \in \Omega^{d-p-1}(\mathcal{M}_d) \). From the source continuity equation \( d^2\delta\Sigma_p = 0 \) we have

\[
\frac{\partial}{\partial t} \bar{\delta}\Sigma_p^0 = - *d \star \tilde{\delta}\Sigma_p = - \nabla^2_{p-1} \omega_p
\]

and from the definitions we find

\[
* d\tilde{\delta}\Sigma_p = \nabla^2_{d-p-1} \rho_p
\]

\[
\Sigma^{(p)}_l(t) = \sum_{k=1}^{N_p} I^{(p)}_{lk} \int_{\mathcal{M}_d} \tilde{\delta}\Sigma_p \wedge \alpha_k^{(p)} = \frac{d}{dt} \int_{\Sigma_p(t)} \beta^{(p)}_l
\]
In arriving at the second equality in (3.13) we have used Poincaré-Hodge duality and the local form (2.12) of the deRham current. Again the globally defined differential form \(*d\tilde{\Sigma}_p\) contains no zero modes of the Laplacian operator \(\nabla^2_{d-p-1}\), and the continuity equation (3.16) implies the current “gauge symmetry”

\[ \rho_p \to \rho_p + \tilde{\Lambda} \quad (3.19) \]

with \(\tilde{\Lambda}\) an arbitrary \(d-p-1\)-form, which allows one to remove the harmonic components of \(*d\tilde{\Sigma}_p\).

Finally, we shall write the Hodge decompositions (3.15) of the deRham currents in terms of a generalized eigenfunction expansion on \(\mathcal{M}_d\). For this, we introduce, for each \(p\), a basis of co-exact \(p\)-forms \(\psi^{(p)}_{\lambda_p}\) which constitute the complete system of eigenstates of the Laplacian operator \(\nabla^2_p\) with eigenvalues \(\lambda_p^2 \geq 0\):

\[ *d* \psi^{(p)}_{\lambda_p} = 0 , \quad \nabla^2_p \psi^{(p)}_{\lambda_p} = *d* \psi^{(p)}_{\lambda_p} = \lambda_p^2 \psi^{(p)}_{\lambda_p} \quad (3.20) \]

and which are orthonormal:

\[ \int_{\mathcal{M}_d} \psi^{(p)}_{\lambda_p} \wedge *\psi^{(p)}_{\lambda_p'} = \delta_{p,\lambda_p} \delta_{\lambda_p,\lambda_p'} \quad (3.21) \]

Because of Hodge duality, we may identify \(\psi^{(p)}_{\lambda_p} = *d\psi^{(d-p-1)}_{\lambda_{d-p-1}}\) and so it suffices to consider only \([\frac{d}{2}]\) of these \(p\)-form eigenfunctions. Note that the zero modes of \(\nabla^2_p\) are just the harmonic \(p\)-forms, \(\psi^{(p)}_0 = \{\beta^{(p)}_k\}_{k=1}^{N_p}\).

These eigenstates are particularly useful for expanding the Dirac delta-functions which act on the exterior algebras of \(\mathcal{M}_d\) in terms of completeness relations. For example, we can readily write down the following distribution-valued Hodge decompositions over the appropriate exterior algebras:

\[ \delta^{(d)}(x,y) = \sum_{\lambda_0} \psi^{(0)}_{\lambda_0}(x) \psi^{(0)}_{\lambda_0}(y) \]

\[ \delta^{(d-p,p)}(x,y) = - \sum_{\lambda_{d-p-1} \neq 0} \frac{1}{\lambda_{d-p-1}^2} d\psi^{(d-p-1)}_{\lambda_{d-p-1}}(x) \otimes *d\psi^{(d-p-1)}_{\lambda_{d-p-1}}(y) \]

\[ + \sum_{\lambda_{d-p} \neq 0} \psi^{(d-p)}_{\lambda_{d-p}}(x) \otimes *\psi^{(d-p)}_{\lambda_{d-p}}(y) + \sum_{k,l=1}^{N_p} \alpha^{(p)}_k(x) \otimes \beta^{(p)}_l(y) \]

\[ (3.22) \]

Then, by equating the exact and co-exact parts of \(\delta^{(d-p,p)}(x,y)\) in (3.22) with those of \(*d\delta_{\Sigma_p}\) in (3.13), we arrive the following generalized eigenfunction expansions for the components of the deRham currents:

\[ \tilde{\delta}_{\Sigma_p}^{(0)} = \sum_{\lambda_{p-1}} \psi^{(p-1)}_{\lambda_{p-1}} \int_{\partial \Sigma_p(t)} \psi^{(p-1)}_{\lambda_{p-1}} \]

\[ \rho_p = - \sum_{\lambda_{d-p-1} \neq 0} \frac{1}{\lambda_{d-p-1}^2} \psi^{(d-p-1)}_{\lambda_{d-p-1}} \frac{d}{dt} \int_{\Sigma_p(t)} *d\psi^{(d-p-1)}_{\lambda_{d-p-1}} \]

17
\[
\omega_p = - \sum_{\lambda_{p-1} \neq 0} \frac{1}{\lambda_{p-1}^2} \psi_{\lambda_{p-1}}^{(p-1)} \frac{d}{dt} \oint_{\partial \Sigma_{p}(t)} \psi_{\lambda_{p-1}}^{(p-1)} \tag{3.23}
\]

where we have used (3.16), (3.18), (2.12), and Hodge duality. Note that with the decomposition of \(\delta^0_{\Sigma_p}\) in (3.23), the integral (2.51) vanishes because it yields the intersection number of the projected hypersurface \(\Sigma_p(t)\) with the time direction of \(\mathcal{M}_{d+1}\), which is 0.

### 3.2. Canonical Structure

We shall now proceed to describe the canonical quantization of the field theory. First we examine the local gauge constraints (2.17). Upon substitution of the decompositions (2.43), (2.44) and (2.47), we may integrate the multi-valued part of (2.17) over a suitable cycle of \(\mathcal{M}_d\) and obtain the relation

\[
F_0(\tilde{\Sigma}^{(l)}_{d-p+1}) = - \left( -1 \right)^{(d-1)(p-1)} \frac{k}{2\pi} \nabla^2_{p-1} P_K + Q_p \delta_{\Sigma_p} \approx 0 \tag{3.24}
\]

Note that this relation is a strong equality since neither the flux \(F_0\) nor the charge \(Q_p\) will be a dynamical degree of freedom. The remaining non-singular part of (2.17) can be expressed in terms of the decompositions (3.5) and (3.14) to give the weak equality

\[
\left( -1 \right)^d \frac{k}{2\pi} \nabla^2_{d-p} P_\theta + Q_{d-p} \delta_{\Sigma_{d-p}} \approx 0 \tag{3.25}
\]

Similarly, from the local gauge constraints associated with the \(\tilde{B}\) field in (2.17) we obtain, using (2.45) and (2.46), the strong equality

\[
H_0(\tilde{\Sigma}^{(l)}_{p+1}) = - \left( -1 \right)^{p(d-p)} \frac{2\pi}{k} Q_{d-p} \nu[\tilde{\Sigma}^{(l)}_{d-p+1}, \partial \Sigma_{d-p}(0)] \tag{3.26}
\]

and from (3.8) the weak equality

\[
- \left( -1 \right)^d \frac{k}{2\pi} \nabla^2_{d-p} P_\theta + Q_{d-p} \delta_{\Sigma_{d-p}} \approx 0 \tag{3.27}
\]

Using the strong equalities (3.24) and (3.26), we can now rewrite the consistency condition (2.50) in the form

\[
1 = \prod_{r,l=1}^{N_{p-1}} \prod_{m,n=1}^{N_{p+1}} \exp \left( - \frac{2\pi i}{k} Q_p Q_{d-p} \nu[\tilde{\Sigma}^{(m)}_{p+1}, \partial \Sigma_{d-p}(0)] \nu[\tilde{\Sigma}^{(l)}_{d-p+1}, \partial \Sigma_{p}(0)] \left( I_{mn}^{(p+1)} + I_{rl}^{(p-1)} \right) \right) \tag{3.28}
\]

The constraint (3.28) is a topological quantization condition determined by the intersection numbers of the homology cycles of \(\mathcal{M}_d\). It is the appropriate generalization to the present case of the usual Dirac charge quantization condition. The global gauge constraints of the \(BF\) field theory will be described in the next section.

Having described the gauge constraints of the model, we may now write down the canonical quantum commutators. For this, we need to examine the source-free \(BF\) action in (2.52). Using the Hodge decompositions of the previous subsection we may write
down explicitly the remaining BF action without the gauge constraints (i.e. in the temporal gauge $A^0 = B^0 = 0$). After some algebra and integrations by parts over $\mathcal{M}_d$ it is straightforward to arrive at

$$S = \oint dt \frac{k}{2\pi} \left[ \int_{\mathcal{M}_d} \left( *\theta \wedge \nabla^2_{d-p-1} P_{\theta} + \nabla^2_{p-1} K \wedge *\hat{P}_K \right) + (-1)^{p(d-p)} \frac{i}{2} \sum_{k,l=1}^{N_p} G_{kl} \left( \gamma^k \gamma^l - \gamma^k \gamma^l \right) \right]$$

(3.29)

where $G_{kl}$ is the matrix inverse of the topological phase space metric (3.11). From the action (3.29) we can immediately read off the non-vanishing canonical Poisson brackets

$$\left\{ \theta(x) \otimes P_{\theta}(y) \right\} = -\frac{2\pi}{k} \frac{1}{(\nabla^2_{d-p-1})^\perp} \Pi^{(d-p-1)} \delta^{(d)}(x,y)$$

$$\left\{ K(x) \otimes P_{K}(y) \right\} = -\frac{2\pi}{k} \frac{1}{(\nabla^2_{p-1})^\perp} \Pi^{(p-1)} \delta^{(d)}(x,y)$$

$$\left\{ \gamma^k, \gamma^l \right\} = -\frac{2\pi i}{k} (-1)^{p(d-p)} G^{kl}$$

(3.30)

where $\Pi^{(d)} : \Omega^p(\mathcal{M}_d) \rightarrow \Omega^p(\mathcal{M}_d)$ is the symmetric, transverse orthogonal projection onto the subalgebra of co-closed $p$-forms of $\Omega^p(\mathcal{M}_d)$, and $(\nabla^2_p)^\perp$ denotes the Laplacian operator $\nabla^2_p$ with its zero modes arising from gauge invariance removed.

In the quantum field theory, Poisson brackets are mapped onto quantum commutators according to the correspondence principle. In the functional Schrödinger picture, we may treat the fields $\theta$, $K$ and $\gamma^k$ as “coordinates” on the (infinite dimensional) configuration space $[17]$. Then the canonical commutation relations corresponding to (3.30) are represented by writing the canonical momentum differential forms as the derivative operators

$$P_{\theta}(x) = \frac{2\pi i}{k} \frac{1}{(\nabla^2_{d-p-1})^\perp} \Pi^{(d-p-1)} \delta \frac{\delta}{\delta \theta(x)}$$

$$P_{K}(x) = \frac{2\pi i}{k} \frac{1}{(\nabla^2_{p-1})^\perp} \Pi^{(p-1)} \delta \frac{\delta}{\delta K(x)}$$

$$\gamma^l = \frac{2\pi}{k} (-1)^{p(d-p)} G^{dk} \frac{\partial}{\partial \gamma^k}$$

(3.31)

As we shall see, the projection operators in (3.31) have the effect of ensuring the invariance of the physical state wavefunctions under the time-independent secondary gauge symmetries

$$\theta \rightarrow \theta + d\chi'' , \quad K \rightarrow K + d\zeta''$$

(3.32)

These symmetries are a consequence of the feature that the topological gauge theory has first-stage, off-shell reducible gauge symmetries [3, 10], and they can be regarded, through the minimal couplings of the gauge fields to the deRham currents, as being dual to the current symmetries (3.19).
Finally, we can express the Hamiltonian \((2.19)\) as well in terms of the various Hodge decompositions of the previous subsection. Using the fact that the singular parts of the deRham currents make no contributions, after some algebra and integrations by parts we arrive at the classical Hamiltonian

\[
H = -\int_{\mathcal{M}_d} \left( (-1)^d Q_{d-p} \theta \wedge * \nabla_{d-p}^2 \omega_{d-p} - (-1)^{(d-1)(p-1)} Q_p K \wedge * \nabla_{p-1}^2 \omega_p \right)
+ \left( (-1)^d Q_{d-p} \ast \rho_{d-p} \wedge \nabla_{p-1}^2 P_K - (-1)^{(d-1)(p-1)} Q_p \ast \rho_p \wedge \nabla_{d-p-1}^2 P_{\theta} \right)
+ (-1)^{(d-1)p} \sum_{m=1}^{N_p} \left( Q_p \Sigma_m^{(p)} - Q_{d-p} \sum_{n,l=1}^{N_p} \tau_{mn} I^{(p)nl} \Sigma_{l}^{(d-p)} \right) \sum_{r,k=1}^{N_p} I^{(p)mr} G_{rk} \gamma^k
- (-1)^{(d-1)p} \sum_{m=1}^{N_p} \left( Q_p \Sigma_m^{(p)} - Q_{d-p} \sum_{n,l=1}^{N_p} \tau_{mn} I^{(p)nl} \Sigma_{l}^{(d-p)} \right) \sum_{r,k=1}^{N_p} I^{(p)mr} G_{rk} \gamma^k
\]

Using the continuity equations \((3.16)\) and the Schrödinger representations \((3.31)\), we arrive at the corresponding quantum Hamiltonian operator:

\[
H = -\int_{\mathcal{M}_d} \left( (-1)^d Q_{d-p} \theta \wedge * \frac{\partial}{\partial t} \delta_{\Sigma_{d-p}}^0 - (-1)^{(d-1)(p-1)} Q_p K \wedge * \frac{\partial}{\partial t} \delta_{\Sigma_p}^0 \right)
+ \frac{2\pi i}{k} Q_{d-p} \ast \rho_{d-p} \wedge \Pi^{(p-1)} \frac{\delta}{\delta K} + \frac{2\pi i}{k} Q_p \ast \rho_p \wedge \Pi^{(d-p-1)} \frac{\delta}{\delta \theta}
+ (-1)^{(d-1)p} \sum_{m=1}^{N_p} \left( Q_p \Sigma_m^{(p)} - Q_{d-p} \sum_{n,l=1}^{N_p} \tau_{mn} I^{(p)nl} \Sigma_{l}^{(d-p)} \right) \sum_{r,k=1}^{N_p} I^{(p)mr} G_{rk} \gamma^k
- \frac{2\pi i}{k} \sum_{m=1}^{N_p} \left( Q_p \Sigma_m^{(p)} - Q_{d-p} \sum_{n,l=1}^{N_p} \tau_{mn} I^{(p)nl} \Sigma_{l}^{(d-p)} \right) \sum_{k=1}^{N_p} I^{(p)mk} \frac{\partial}{\partial \gamma^k}
\]

\[(3.33)\]

\[(3.34)\]

### 3.3. Global Phase Space

We close this section by comparing the present canonical approach to the path integral formalism used in [25] for the quantization of abelian BF theory defined over non-trivial line bundles. In the former approach, the zero mode vector space \((3.10)\) comes from the single-valued parts of the gauge fields and would yield the same overall volume factor in the path integral measure as that for the quantum field theory defined on a trivial vector bundle. The sum over all line bundles localizes onto the topological class determined by the external charges \(Q_p\) and \(Q_{d-p}\) through the flux relationships \((3.24)\) and \((3.26)\). The non-triviality of the bundle is then encoded in the strong gauge constraints \((3.28)\) which will be imposed on the wavefunctions in the next section. The topological sum may then be taken over all quantized charges obeying \((3.28)\). In the latter approach, the sum over non-trivial line bundles is explicitly carried out in the source-free path integral and is
shown to modify the (ungraded) space of harmonic zero modes to

\[ \tilde{\mathcal{P}} = \bigoplus_{n=p,d-p} \left( H^0_C(\mathcal{M}_d, \mathbb{Z}) / H^{n-1}_C(\mathcal{M}_d, \mathbb{R}/\mathbb{Z}) \right) \oplus \left( H^{d-n}_C(\mathcal{M}_d, \mathbb{R}/\mathbb{Z}) \otimes H^{d-n}_D(\mathcal{M}_d) \right) \]

Here the Čech cohomology group \( H^{d-p}_C(\mathcal{M}_d, \mathbb{R}/\mathbb{Z}) \) classifies the higher-rank bundles of degree \( d - p \) which admit flat \( d - p \)-form connections (equivalently constant transition functions), while the quotient cohomology in (3.35) classifies the higher-rank bundles of degree \( d - p \) modulo those with constant transition functions. Thus, although the functional integration reproduces the usual Ray-Singer invariant, the overall volume factor is now modified to take into account of the non-trivial Čech cohomology. In the following we shall see how this cohomology is represented explicitly by the wavefunctions of the canonical quantum field theory.

4. Construction of the Physical States

In this section we shall use the canonical formalism for the quantum field theory developed in the previous section to explicitly solve for the physical state wavefunctions of \( BF \) theory. From the form of the Hamiltonian operator (3.34) we see that they may be separated into two pieces \( \Psi_L \) and \( \Psi_T \) representing the local and global topological parts in the corresponding decomposition of the \( BF \) field theory:

\[ \Psi_{\text{phys}}[\theta, K, \gamma; t] = \Psi_L[\theta, K; t] \Psi_T(\gamma; t) \]

We shall see that each component in (4.1) plays an important role in the topological group representations that we will find in the next section.

4.1. Local Gauge Symmetries

The local components of the full wavefunction must satisfy the weak equalities (3.25) and (3.27) which are imposed as physical state conditions in the quantum field theory and which truncate the full Hilbert space onto the physical, gauge invariant subspace. Using the functional Schrödinger representations in (3.31), we thereby arrive at the quantum equations which express the local gauge constraints of the theory:

\[
\begin{align*}
\left[ (-1)^{(d-1)(p-1)} i \Pi^{(p-1)} \frac{\delta}{\delta K} + Q_p \delta_\Sigma_p \right] \Psi_L[\theta, K; t] &= 0 \\
\left[ (-1)^d i \Pi^{(d-p-1)} \frac{\delta}{\delta \theta} + Q_{d-p} \delta_{\Sigma_{d-p}} \right] \Psi_L[\theta, K; t] &= 0
\end{align*}
\]

They are solved by wavefunctionals of the form

\[ \Psi_L[\theta, K; t] = \exp \left[ i \int_{\mathcal{M}_d} \left( (-1)^{(d-1)(p-1)} Q_p \ K \wedge \delta_\Sigma_p^0 - (-1)^d Q_{d-p} \ \theta \wedge \delta_{\Sigma_{d-p}}^0 \right) \right] \tilde{\Psi}_L(t) \]
which yield a projective representation of the local gauge symmetries in terms of a non-trivial, local \(U(1) \times U(1)\) one-cocycle:

\[
\Psi_{\text{phys}}[\theta + \chi', K + \xi', \gamma; t] = \exp \left[ i \int_{\mathcal{M}_d} (-1)^{(d-1)(p-1)} Q_p \xi' \wedge \delta^0_{\Sigma_p} - (-1)^d Q_{d-p} \chi' \wedge \delta^0_{\Sigma_{d-p}} \right] \Psi_{\text{phys}}[\theta, K, \gamma; t]
\]

(4.4)

Note that the wavefunction (4.3) can be written in a more explicit form using the decompositions (2.47) of the deRham currents.

4.2. Schrödinger Wave Equation

To determine the remaining part \(\tilde{\Psi}_L(t)\) of the wavefunction (4.3), we solve the corresponding Schrödinger wave equation

\[
i \frac{\partial}{\partial t} \Psi_{\text{phys}}[\theta, K, \gamma; t] = H \Psi_{\text{phys}}[\theta, K, \gamma; t]
\]

(4.5)

for the local degrees of freedom of the gauge fields. From (3.34) we readily arrive at

\[
\tilde{\Psi}_L(t) = \exp -\frac{2\pi i}{k} Q_p Q_{d-p} (-1)^{(d-1)} \int_0^t dt' \int_{\mathcal{M}_d} \left( -\star \rho_{d-p} \wedge \delta^0_{\Sigma_p} + \star \rho_p \wedge \delta^0_{\Sigma_{d-p}} \right)
\]

(4.6)

Substituting in the eigenfunction expansions (3.23) gives

\[
\tilde{\Psi}_L(t) = \exp -\frac{2\pi i}{k} Q_p Q_{d-p} (-1)^{(d-1)}
\]

\[
\times \int_0^t dt' \left[ -\sum_{\lambda_{d-p} \neq 0} \left( \frac{d}{dt'} \int_{\Sigma_p(t')} \psi^{(d-p)}_{\lambda_{d-p}} \right) \left( \int_{\Sigma_p(t')} \star \psi^{(d-p)}_{\lambda_{d-p}} \right) 
\]

\[
+ \sum_{\lambda_{d-p-1} \neq 0} \frac{1}{\lambda^2_{d-p-1}} \left( \frac{d}{dt'} \int_{\Sigma_p(t')} \star \psi^{(d-p-1)}_{\lambda_{d-p-1}} \right) \left( \int_{\Sigma_{d-p}(t')} \psi^{(d-p-1)}_{\lambda_{d-p-1}} \right) \right] \}
\]

\[
= \exp -\frac{2\pi i}{k} Q_p Q_{d-p} (-1)^{(d-1)}
\]

\[
\times \int_0^t dt' \left[ \frac{d}{dt'} \sum_{\lambda_{d-p-1} \neq 0} \frac{1}{\lambda^2_{d-p-1}} \left( \int_{\Sigma_p(t')} \star \psi^{(d-p-1)}_{\lambda_{d-p-1}} \right) \left( \int_{\Sigma_{d-p}(t')} \psi^{(d-p-1)}_{\lambda_{d-p-1}} \right) 
\]

\[- \sum_{l,m=1}^{N_p} \int_{\Sigma_p(t')} \beta^{(p)}_l \left( \int_{\Sigma_{d-p}(t')} \alpha^{(p)}_m \right) 
\]

\[- \sum_{\lambda_{d-p-1} \neq 0} \frac{1}{\lambda^2_{d-p-1}} \left( \int_{\Sigma_p(t')} \star \psi^{(d-p-1)}_{\lambda_{d-p-1}} \right) \left( \int_{\Sigma_{d-p}(t')} \psi^{(d-p-1)}_{\lambda_{d-p-1}} \right) \right] 
\]

(4.7)
where \( t_{\lambda d-p} : \Omega^q(\mathcal{M}_d) \to \Omega^{d-1}(\mathcal{M}_d) \) is the nilpotent interior multiplication with respect to the vector field \( \frac{\partial}{\partial t} X_{d-p}(t, \sigma^2, \ldots, \sigma^{d-p}) \). From (4.7) we arrive finally at

\[
\tilde{\Psi}_L(t) = \exp\left\{ -\frac{2\pi i}{k} Q_p Q_{d-p} (-1)^{p(d-1)} \times \frac{1}{\Omega_{d-1}} \oint dt' \left( \frac{d\Phi_p(t')}{dt'} + \sum_{l,m=1}^{N_p} \int dt'' \int_0^t dt'' \int_0^t \int_0^t \delta^{(p)}(t'') \left( \frac{\partial \Phi_p(t'')}{\partial \Sigma_p(t'')} \right) \right) \right\}
\]

(4.8)

where we have introduced the function

\[
\Phi_p(t) = \Omega_{d-1} \int_0^t dt' \left[ \oint \left( \tilde{\Psi}_L(t', \sigma^2, \ldots, \sigma^{d-p}) \right) \right]
\]

(4.9)

and

\[
\Omega_{d-1} = \text{vol}(\mathbb{S}^{d-1}) = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}
\]

(4.10)

is the \( d-1 \)-dimensional solid angle. Note that the above derivation can be carried out in the same way to obtain a final expression which is explicitly symmetric in \( \Sigma_{d-p}(t) \) and \( \Sigma_p(t) \), and which thereby exhibits the manifest (Hodge) duality symmetry between the two hypersurfaces. However, in order to keep the formulas from getting overly lengthy, we do not exhibit this symmetrization explicitly.

Let us examine the dependence of the function (4.9) on the topological classes of the projected hypersurfaces. For this, we fix the \( p-1 \)-brane embedding function \( X_p(t, \sigma^2, \ldots, \sigma^p) \) and choose another hypersurface \( \Sigma_p(t) \) in the same topological class as \( \Sigma_p(t) \), i.e. \( \Sigma_p(t) - \Sigma_p(t) = \partial S_p(t) \) for some \( p+1 \)-volume \( S_p(t) \). It is then straightforward to compute the change in the second term in (4.3):

\[
\delta \Phi_p(t) = \Omega_{d-1} \sum_{\lambda_{d-p} \neq 0} \frac{1}{\lambda_{d-p}^2} \left[ \frac{1}{\partial \Sigma_{d-p}(t)} \right] \left( \oint \left( \tilde{\Psi}_L(t', \sigma^2, \ldots, \sigma^{d-p}) \right) \right]
\]

(4.11)
where we have used Stokes’ theorem, and \( s_p(t) \) is a local coordinate system on \( S_p(t) \). This shows that if we continuously deform \( \Sigma_p(t) \) in \( M_d \), then, modulo the harmonic forms in (4.11), the second term in (4.9) does not change unless the deformation crosses the \( d - p - 1 \)-brane at \( X_{d-p}(t, \sigma^2, \ldots, \sigma^{d-p}) \). In that case the change is then \( \Omega_{d-1} \), which is cancelled by the delta-function term in (4.9). As for the harmonic part of (4.11), we symmetrize the expression (4.8) in \( \Sigma_p(t) \) to each time slice of \( M \) to obtain a manifestly Hodge duality symmetric function. Then the harmonic term in (4.11) becomes

\[
-\Omega_{d-1} \left[ \sum_{l,m=1}^{N_p+1} \left( \int_{\partial \Sigma_{d-p}(t)} \alpha_m^{(p+1)} I^{(p+1)lm} \right) \left( \int_{S_p(t)} \beta_l^{(p+1)} \right) \\
+ \sum_{l,m=1}^{N_{d-1}} \left( \int_{\partial \Sigma_{d-p}(t)} \beta_m^{(p-1)} I^{(p-1)ml} \right) \left( \int_{S_{d-p}(t)} \alpha_m^{(d-p)} \right) \right] \quad (4.12)
\]

Using the bilinear identity (3.3), we see that the change (4.12) will contribute a phase factor to (4.8) which is simply unity due to the topological phase constraint (3.28) (applied to each time slice of \( M_{d+1} \)). The condition (3.28) thereby represents a fundamental global constraint that must be satisfied by the external charges \( Q_p, Q_{d-p} \) for a consistent (topologically invariant) solution of the \( BF \) quantum field theory. We shall see later on that this imposes a corresponding global constraint that must be met by all consistent well-defined representations of the motion group on topologically non-trivial manifolds \( M_d \).

The function \( \Phi_p(t) \) therefore depends only on the topological classes of the trajectories \( \Sigma_p(t) \) and \( \Sigma_{d-p}(t) \) in \( M_d \), and not on their particular representatives, provided that they do not intersect. The above argument also shows that if \( \Sigma_{d-p}(t) \) is kept constant in time \( t \) while \( \Sigma_p(t) \) sweeps out a closed hypersurface in a given time span, then the only contribution to \( \Phi_p(t) \) is from the second term in (4.9) which gives \( \Omega_{d-1} \). Furthermore, if \( \Sigma_p(t) \) is fixed and \( \Sigma_{d-p}(t) \) sweeps out a closed hypervolume in a given time span, then the second term in (4.9) is invariant while the first term counts exactly the number of times the hypersurface \( \Sigma_{d-p}(t) \) links \( \Sigma_p(t) \), giving a contribution of \( \Omega_{d-1} \) each time. Thus, \( \Phi_p(t) \) gives the \( d \)-dimensional relative solid angle between \( \Sigma_p(t) \) and \( \Sigma_{d-p}(t) \) in adiabatic linking processes in \( M_d \). It is the generalized, adiabatic linking function that will yield the appropriate holonomy phase changes in the wavefunctions for the motion group representations that we will obtain in the following. Furthermore, for infinitesimal paths it is readily seen that (4.9) reduces to the usual solid angle function on \( M_d = \mathbb{R}^d \).

We will describe the transformation properties of the wavefunctions under homologically non-trivial motions of the hypersurfaces \( \Sigma_p(t) \) and \( \Sigma_{d-p}(t) \) later on.

We now come to the remaining, topological part of the Schrödinger equation (4.5) for the global harmonic degrees of freedom of the gauge fields. From (3.34) it follows that
this equation may be solved in the form

\[
\Psi_T(\gamma; t) = \prod_{m,n,r=1}^{N_p} \exp \left[ (-1)^{p(d-1)} \int_0^t dt' \left( Q_p \Sigma_m^{(p)}(t') - Q_{d-p} \sum_{s,l=1}^{N_p} \tau_{ms} I^{(p)st} \Sigma_l^{(d-p)}(t') \right) \right] \\
\times I^{(p)mr} G_{rn} \gamma^n \\
\times -2\pi \left( -1 \right)^{p(d-1)} \int_0^t dt' \left( Q_p \Sigma_m^{(p)}(t') - Q_{d-p} \sum_{u,l=1}^{N_p} \tau_{mu} I^{(p)nl} \Sigma_l^{(d-p)}(t') \right) \\
\times \sum_{q=1}^{N_p} I^{(p)mq} G_{qn} I^{(p)rn} \int_0^t dt'' \left( Q_p \Sigma_n^{(p)}(t'') - Q_{d-p} \sum_{s,v=1}^{N_p} \tau_{rv} I^{(p)us} \Sigma_s^{(d-p)}(t'') \right) \\
\times \Psi_0(\gamma; t)
\]  

(4.13)

where the function \(\Psi_0(\gamma; t)\) is a solution of the partial differential equation

\[
\frac{\partial \Psi_0(\gamma; t)}{\partial t} = -2\pi \left( -1 \right)^{p(d-1)} \sum_{m=1}^{N_p} \left( Q_q \Sigma_m^{(p)}(t) - Q_{d-p} \sum_{n,l=1}^{N_p} \tau_{mn} I^{(p)nl} \Sigma_l^{(d-p)}(t) \right) \\
\times \sum_{r=1}^{N_p} I^{(p)mr} \frac{\partial \Psi_0(\gamma; t)}{\partial \gamma^r}
\]  

(4.14)

which is solved by any function of the form

\[
\Psi_0(\gamma^l; t) = \Psi_0 \left( \gamma^l - \frac{2\pi}{k} \left( -1 \right)^{p(d-1)} \sum_m I^{(p)ml} \int_0^t dt' \left[ Q_p \Sigma_m^{(p)}(t') - Q_{d-p} \sum_{n,r} \tau_{mn} I^{(p)nr} \Sigma_r^{(d-p)}(t') \right] \right)
\]  

(4.15)

The function \(\Psi_0\) may be fixed by requiring that the wavefunctions respect the large gauge transformations of the fields which are not connected to the identity in the topological phase space (3.14). This will be done in the next subsection.

4.3. Global Gauge Symmetries

For a consistent quantum theory, we must demand that, when there are no sources present \((Q_p = Q_{d-p} = 0)\), the wavefunctions \(\Psi_0\) coincide with the cohomological states that represent the invariance of the quantum field theory under large gauge transformations. In this case, the local gauge constraints (4.2) imply that the full physical state wavefunctions depend only on the global harmonic degrees of freedom \(\gamma^l\). Furthermore, the Hamiltonian then vanishes (since the pure source-free \(BF\) field theory is topological) so that the states are also time-independent. This means that in the absence of any sources the wavefunctions carry information only about the topology of the manifold \(\mathcal{M}_d\).

To construct these states, we consider the classical translation operators which generate the appropriate shifts (3.13) of the holomorphic gauge degrees of freedom:

\[
C(n_p, n_{d-p}) = \prod_{l=1}^{N_p} \exp \left[ 2\pi \left( n_p^l + \sum_{m,r=1}^{N_p} I^{(p)ml} \tau_{mr} n_{d-p}^r \right) \frac{\partial}{\partial \gamma^l} \right]
\]  

25
$$+ 2\pi \left( n'_{p} + \sum_{m,r=1}^{N_p} I^{(p)ml}_r \tau_{mr} n'_{d-p} \right) \frac{\partial}{\partial \gamma^l} \right]$$ (4.16)$$

Using the Schrödinger representation (3.31) we can then write down the corresponding quantum operators which implement the global gauge symmetries:

$$U(n_p, n_{d-p}) = \prod_{l=1}^{N_p} \exp \left[ 2\pi \left( n'_{p} + \sum_{m,r=1}^{N_p} I^{(p)ml}_r \tau_{mr} n'_{d-p} \right) \frac{\partial}{\partial \gamma^l} \right] - (-1)^{p(d-p)} k \left( n'_{p} + \sum_{m,r=1}^{N_p} I^{(p)ml}_r \tau_{mr} n'_{d-p} \right) \prod_{q=1}^{N_p} G_{lq} \gamma^q \right] (4.17)$$

For the remainder of this paper we will assume that the coefficient $k$ of the pure $BF$ action is of the form

$$k = \mathcal{I}^{(p)} \frac{k_1}{k_2} \quad (4.18)$$

where $\mathcal{I}^{(p)} > 0$ is the integer-valued determinant of the intersection matrix $I^{(p)lm}$ and $k_1, k_2$ are positive integers with gcd($\mathcal{I}^{(p)} k_1, k_2) = 1$.

In contrast to the classical operators (4.16), the operators $U(n_p, n_{d-p})$ do not commute with each other. Using the Baker-Campbell-Hausdorff formula it is straightforward to compute that these operators generate the global $U(1) \times U(1)$ two-cocycle algebra:

$$U(n_p, n_{d-p}) U(m_p, m_{d-p}) = \prod_{l,r=1}^{N_p} \exp(2\pi i(-1)^{p(d-p)} I^{(p)} I_{lr} r_{n_{d-p}} - n_{p} m_{d-p} ) \right] U(m_p, m_{d-p}) \right] U(n_p, n_{d-p}) \right]$$ (4.19)

A similar calculation shows that their action on the wavefunctions is given by

$$U(n_p, n_{d-p}) \Psi_0(\gamma^l) = \prod_{l,q=1}^{N_p} \exp(-1)^{p(d-p)} \left\{ - k \left( n'_{p} + \sum_{m,r=1}^{N_p} I^{(p)ml}_r \tau_{mr} n'_{d-p} \right) G_{lq} \gamma^q \right. \right.$$ 

$$\left. - \pi k \left( n'_{p} + \sum_{m,r=1}^{N_p} I^{(p)ml}_r \tau_{mr} n'_{d-p} \right) G_{lq} \left( n'_{p} + \sum_{s,u=1}^{N_p} I^{(p)su}_r \tau_{su} n'_{d-p} \right) \right\} \right.$$ 

$$\times \Psi_0 \left( \gamma^l + 2\pi \left( n'_{p} + \sum_{m,r} I^{(p)ml}_r \tau_{mr} n'_{d-p} \right) \right$$ (4.20)

On the other hand, the cocycle algebra (4.19) implies that the operators $U(k_2 n_p, k_2 n_{d-p})$ commute with all of the other gauge transformation generators, so that they lie in the center of the global $U(1) \times U(1)$ gauge group and their action on the Hilbert space is represented simply as multiplication by some phases $e^{i\phi(n_p, n_{d-p})}$. This then implies the transformation law:

$$\Psi_0 \left( \gamma^l + 2\pi k_2 \left( n'_{p} + \sum_{m,r} I^{(p)ml}_r \tau_{mr} n'_{d-p} \right) \right)$$ 

$$= \exp \left[ i\phi(n_p, n_{d-p}) + \sum_{l=1}^{N_p} (-1)^{p(d-p)} k_1 \left( n'_{p} + \sum_{m,r=1}^{N_p} I^{(p)ml}_r \tau_{mr} n'_{d-p} \right) \right.$$ 

$$\left. + \pi k_1 k_2 \left( n'_{p} + \sum_{m,r=1}^{N_p} I^{(p)ml}_r \tau_{mr} n'_{d-p} \right) \right] \Psi_0(\gamma^l)$$ (4.21)
These algebraic constraints are uniquely solved by the \((\mathcal{I}^{(p)}k_1k_2)^{N_p}\) independent holomorphic wavefunctions

\[
\Psi_0^{(q)}(a,b)(\gamma) = \left( \prod_{l,r=1}^{N_p} e^{\frac{1}{2\pi i} \gamma^r G_{lr} \gamma^l} \right) \times \Theta \left( \frac{\mathcal{I}^{(p)}k_1k_2}{b} \right) \left( \frac{\mathcal{I}^{(p)}k_1}{2\pi} (-1)^{p(d-p)} \sum_{m=1}^{N_p} I_{ml}^{(p)} \gamma^m \right) - k_1 k_2 \mathcal{I}^{(p)} \tau \right) \quad (4.22)
\]

where \(q^l = 1, 2, \ldots, \mathcal{I}^{(p)}k_1k_2 (l = 1, \ldots, N_p)\), and we have introduced the standard (multi-dimensional) Jacobi theta-functions:

\[
\Theta \left( \frac{a}{b} \right)(z|\tau) = \sum_{n \in \Gamma} \prod_{\ell=1}^{\text{rank}(\Gamma)} \exp \left[ -i\pi \sum_{k=1}^{\text{rank}(\Gamma)} \tau_{lk} (n^k + a^k) + 2\pi(n^l + a^l)(z_l + b_l) \right]
\]

where \(a^l, b_l \in [0, 1]\) and \(\Gamma\) is some lattice. The functions \((4.22)\) are orthogonal in the canonical coherent state measure on the reduced topological phase space \(\mathcal{P}/\Gamma \oplus \Gamma^*\) which leads to the inner product:

\[
\left( \Psi_0^{(q)} | \Psi_0^{(q')} \right) = \int_{\mathcal{P}/\Gamma \oplus \Gamma^*} d\tau^m d\gamma^m d\tau^n d\gamma^n \prod_{m=1}^{N_p} \prod_{k,l=1}^{N_p} e^{-\frac{1}{2\pi i} \gamma^k G_{kl} \gamma^l} \left( \det G \right)^{-1/2} \delta^{qq'} (\gamma)
\]

where we have implicitly divided out by the volume of the global gauge group used to define the complex \(N_p\)-torus \(\mathcal{P}/\Gamma \oplus \Gamma^*\) (as a consequence of the large gauge invariances). The states \((4.22)\) thereby provide a complete, orthonormal basis of the full physical Hilbert space, and they are well-defined functions on \(\mathcal{P}/\Gamma \oplus \Gamma^*\). Furthermore, under a large gauge transformation these wavefunctions transform as

\[
U(n_p, n_{d-p}) \Psi_0^{(q)}(a,b)(\gamma) = \sum_{q'} U(n_p, n_{d-p})_{qq'} \Psi_0^{(q')}(a,b)(\gamma) \quad (4.25)
\]

where the unitary matrices

\[
U(n_p, n_{d-p})_{qq'} = \prod_{l,m=1}^{N_p} \exp \left\{ \frac{2\pi i}{k_2} (-1)^{p(d-p)} I_{ml}^{(p)} \right\} \times \left( a^l n_p^m + \sum_{r=1}^{N_p} I_{lr} b_r n_{d-p}^l + q^l n_p^m - \frac{\mathcal{I}^{(p)}k_1}{2} n_p^m n_{d-p}^l \right) \times \delta_{q'^{-k_1 \mathcal{I}^{(p)} n_{d-p}^l}, q^l} \quad (4.26)
\]
generate a \((k_2)^{N_p}\)-dimensional projective representation of the group \(\Gamma \oplus \Gamma^*\) of large gauge transformations. Here the projective phases are non-trivial global \(U(1) \times U(1)\) one-cocycles which are cyclic with period \(k_2\). The topological part of the full wavefunction thereby carries a non-trivial multi-dimensional projective representation of the discrete gauge group representing the windings of the \(BF\) gauge fields around the appropriate non-trivial homology cycles of \(M_d\). This symmetry partitions the Hilbert space into superselection sectors labelled by the integer (Čech) cohomology classes of the spatial manifold.

The wavefunctions (4.22) possess some noteworthy modular transformation properties. The automorphism group of the reduced topological phase space \(\mathcal{P}/\Gamma \oplus \Gamma^*\) with its complex structure \(\tau\) and associated metric \(G\) is \(Sp(2N_p, \mathbb{Z})\). It acts on the geometrical parameters as

\[
\gamma' = (-C\tau + D)^{-1\top} \gamma \\
\tau' = -(-A\tau + B)(-C\tau + D)^{-1} \\
G' = (-C\tau + D)^{-1\top} G (-C\tau + D)^{-1}
\]

(4.27)

where \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2N_p, \mathbb{Z})\). The corresponding transformation of the Jacobi theta-functions (4.23) is given by

\[
\Theta\left(\begin{array}{c} a' \\ b' \end{array}\right) (\gamma' | -\tau') = e^{-i\pi\phi} \sqrt{\det(-C\tau + D)} e^{i\pi\gamma (C\tau + D)^{-1} \gamma} \Theta\left(\begin{array}{c} a \\ b \end{array}\right) (\gamma | -\tau)
\]

(4.28)

where \(\phi\) is an irrelevant phase and

\[
a' = Da - Cb - \frac{1}{2} (CD\top)_{\text{diag}}, \quad b' = -Ba + Ab - \frac{1}{2} (AB\top)_{\text{diag}}
\]

(4.29)

It follows that a modular invariant set of wavefunctions exists only when the quantity \(I^{(p)} k_1 k_2\) is an even integer, in which case we set \(a_l = b_l = 0\) (and also \(\phi = 0\)). Otherwise, we may take \(a_l, b_l \in \{0, \frac{1}{2}\}\), which corresponds to a choice of spin structure on the complex \(N_p\)-torus which is the reduced topological phase space. The totality of wavefunctions labelled by the \(a_l, b_l\) then increases by \(4^{N_p}\), and they now transform non-trivially under modular transformations representing their transformation properties under a change in choice of spin structure. These latter choices of \(a_l, b_l\) are the only ones for which the reflection symmetry \(\gamma \rightarrow -\gamma\) closes on the set of wavefunctions (4.22). In this way, the physical states turn out to be effectively independent of the phase space complex structure, as required by the topological nature of the quantum field theory.

5. Representations of Motion Groups

The various components of the full physical wavefunction (4.1) can now be combined together using the results of the previous section and section 3.1. After some algebra, we
arrive finally at
\[
\Psi_{\text{phys}}^{(q)} \left( \frac{a}{b} \right) |[\theta, K, \gamma_l; t] = \exp \left[ i \oint_{\partial \Sigma_{d-p}(t)} K - i \sum_{m,l=1}^{N_p-1} I_{lm}^{(p-1)} \nu_{\Sigma_{d-p+1}, \partial \Sigma_{d-p}(0)} | \oint_{\Sigma_{d-p+1}(m)} K \right]
\]
\[
+ (-1)^{p(d-1)} i \oint_{\partial \Sigma_{d-p}(t)} \theta - (-1)^{p(d-1)} i \sum_{l,m=1}^{N_p+1} I_{lm}^{(p+1)} \nu_{\Sigma_{d-p+1}, \partial \Sigma_{d-p}(0)} | \oint_{\Sigma_{d-p+1}(m)} \theta \right]
\]
\[
\times \exp \left\{ -\frac{2\pi i}{k} Q_p Q_{d-p} (-1)^{p(d-p)} \left[ \Phi_p(t) - \Phi_p(0) \right] \right\}
\]
\[
- (-1)^{p(d-p)} \frac{2\pi i}{k} Q_p Q_{d-p} \int_0^t dt' \sum_{l,m=1}^{N_p} \Sigma_{m}^{(p)}(t') I_{lm} \int_0^{t'} dt'' \Sigma_{l}^{(d-p)}(t'')
\]
\[
- (-1)^{p(d-p)} \frac{\pi i}{k} Q_{d-p}^2 \sum_{m,n,l,q=1}^{N_p} \int_0^t dt' \Sigma_{m}^{(d-p)}(t') I_{nm}^{(p)} I_{nl}^{(p)} \frac{k}{4\pi} \sum_{l,r=1}^{N_p} \gamma^l G_{lr} \gamma^r
\]
\[
- (-1)^{p(d-p)} \frac{\pi i}{k} Q_{d-p}^2 \sum_{m,n,l,q=1}^{N_p} \int_0^t dt' \Sigma_{m}^{(d-p)}(t') I_{nm}^{(p)} I_{nl}^{(p)} \int_0^t dt'' \Sigma_{q}^{(d-p)}(t'')
\]
\[
\times \Theta \left( \frac{a+q}{b} \right) \left( \frac{T^{(p)} k_1 k_2}{2\pi} (-1)^{p(d-p)} \sum_{m=1}^{N_p} I_{m}^{(p)} \gamma^m \right)
\]
\[
- (-1)^{p(d-p)} k_2 \int_0^t dt' \left[ Q_p \Sigma_{l}^{(p)}(t') - Q_{d-p} \sum_{n,q=1}^{N_p} \tau_{ln} I_{n}^{(p)} I_{q}^{(d-p)}(t') \right] - k_1 k_2 T^{(p)} \right)
\]
(5.1)

where \(q^l = 1, \ldots, T^{(p)} k_1 k_2, l = 1, \ldots, N_p\), the \(BF\) coefficient \(k\) is given by (4.18), and the harmonic parts of the source degrees of freedom \(\Sigma_{l}^{(p)}(t)\) are determined by the period integrals (3.18) of the corresponding harmonic forms over the trajectories \(\Sigma_{l}^{(p)}(t)\). The external charge parameters in (5.1) are in addition constrained by the topological quantization condition (3.28). In this section we will study various aspects of the transformation properties of the physical states (5.1) in connection with the representation theory of the associated motion group. To set the stage for this, we begin by describing some general aspects of motion groups.

5.1. The Dahm Motion Group

The Dahm motion group \([13, 14]\) of a compact subspace \(\Sigma \subset \mathcal{M}_d\) is the group of essentially different ways of continuously propagating \(\Sigma\) in \(\mathcal{M}_d\) so that at the end of the motion, \(\Sigma\) returns to its original configuration in \(\mathcal{M}_d\). This topological structure generalizes the Artin braid group \([11]\), whereby a braid is viewed as a continuous one-
parameter family of trajectories of \( N \) distinct points in the plane, where at each time \( t_0 \), the configuration is given by the intersection of the braid at height \( z = t_0 \). Thus, the motion group has its origins in the Artin braid group, and therefore the present topological field theory yields the appropriate generalization of fractional statistics of (extended) objects to any dimension. These applications will be discussed in the next section.

Let us start by recalling the definition of the braid group \( B_N(\mathcal{M}_2) \) of a connected Riemann surface \( \mathcal{M}_2 \) of genus \( g \). It can be constructed as the fundamental group of the quantum mechanical configuration space \( Q_N(\mathcal{M}_2) \) for the motion of \( N \) identical particles on \( \mathcal{M}_2 \) with a hard-core repulsive interaction between them:

\[
Q_N(\mathcal{M}_2) = (\mathcal{M}_2^N - \Delta_N)/S_N
\]  

(5.2)

where \( \Delta_N = \{(x, x, \ldots, x) | x \in \mathcal{M}_2\} \) is the diagonal subspace of \( \mathcal{M}_2^N \). Let \( \vec{z} \in \mathcal{M}_2^N \) be some configuration of \( N \) points in \( \mathcal{M}_2 \). A motion of \( \vec{z} \) in \( \mathcal{M}_2 \) is a loop \( \vec{z}(t) \in Q_N(\mathcal{M}_2) \), \( t \in [0, 1] \), based at \( \vec{z} \). The group of motions of \( \vec{z} \) in \( \mathcal{M}_2 \) is then defined as the fundamental homotopy group \( \pi_1(Q_N(\mathcal{M}_2); \vec{z}) \) of loops in \( Q_N(\mathcal{M}_2) \) based at \( \vec{z} \). Since the configuration space (5.2) is a connected manifold (\( \pi_0(Q_N(\mathcal{M}_2)) = 0 \)), one may prove that

\[
\pi_1(Q_N(\mathcal{M}_2); \vec{z}) = B_N(\mathcal{M}_2)
\]  

(5.3)

The generators of \( B_N(\mathcal{M}_2) \) are given by the operators \( \sigma_n \), \( n = 1, \ldots, N-1 \), which braid the trajectories of particles \( n \) and \( n + 1 \), along with the usual presentation of the Artin braid group of the plane \( \mathcal{M}_2 = \mathbb{R}^2 \) \([11]\). In addition, there are \( 2g \) generators associated with carrying a particle trajectory around each homology generator of \( \mathcal{M}_2 \) \([12]\).

It is precisely this homotopy definition of the braid group that generalizes and gives the general notion of a motion group, whereby we replace the Riemann surface \( \mathcal{M}_2 \) with an arbitrary \( d \)-manifold \( \mathcal{M}_d \) and the collection \( \vec{z} \) of \( N \) points in \( \mathcal{M}_2 \) by any compact subspace \( \Sigma \subset \mathcal{M}_d \). We then define a motion \( f \) of \( \Sigma \) in \( \mathcal{M}_d \) to be a path \( f_t, t \in [0, 1] \), of homeomorphisms of \( \mathcal{M}_d \) with compact support such that \( f_0 = \mathbb{1}_{\mathcal{M}_d} \) and \( f_1(\Sigma) = \Sigma \). A stationary motion of \( \Sigma \) in \( \mathcal{M}_d \) is a motion \( f \) for which \( f_t(\Sigma) = \Sigma \ \forall t \in [0, 1] \). The product \( f \cdot g \) of two motions is the path

\[
(f \cdot g)_t = \begin{cases} 
g_{2t}, & 0 \leq t \leq \frac{1}{2} 
g_{2(t-\frac{1}{2})} \circ g_1, & \frac{1}{2} \leq t \leq 1
\end{cases}
\]  

(5.4)

while the inverse \( f^{-1} \) of a motion \( f \) is the path \( f_{1-t} \circ f_t^{-1} \). We say that two motions \( f, g \) of \( \Sigma \) in \( \mathcal{M}_d \) are equivalent, \( f \equiv g \), if \( f \cdot g \) is homotopic to a stationary motion. In particular, stationary motions are equivalent to the trivial motion \( f_t = i_\Sigma \ \forall t \in [0, 1] \), where \( i_\Sigma : \Sigma \hookrightarrow \mathcal{M}_d \) is the canonical inclusion. It may then be shown that the corresponding set of equivalence classes of motions of \( \Sigma \) in \( \mathcal{M}_d \), with the multiplication induced by that in (5.4), forms a group \( \mathbb{M}_{\Sigma}(\mathcal{M}_d) \) which is called the Dahm motion group of \( \Sigma \) in \( \mathcal{M}_d \) \([13, 14]\).
To understand what this group represents in terms of configuration space homotopy, let \( e(\mathcal{M}_d, \Sigma) \) denote the space of embeddings of \( \Sigma \) in \( \mathcal{M}_d \) and \( h(\mathcal{M}_d) \) the space of homeomorphisms of \( \mathcal{M}_d \) of compact support, both equipped with the compact-open topology. Let \( h(\mathcal{M}_d, \Sigma) \subset h(\mathcal{M}_d) \) be the subspace of homeomorphisms which leave \( \Sigma \) fixed. Note that both \( h(\mathcal{M}_d) \) and \( h(\mathcal{M}_d, \Sigma) \) are topological groups, so that one may define the fundamental relative homotopy group \( \pi_1(h(\mathcal{M}_d), h(\mathcal{M}_d, \Sigma); \mathbb{I}_{\mathcal{M}_d}) \) as the set of homotopy classes of paths in \( h(\mathcal{M}_d) \) which begin at \( \mathbb{I}_{\mathcal{M}_d} \in h(\mathcal{M}_d, \Sigma) \) and end in \( h(\mathcal{M}_d, \Sigma) \), and with multiplication induced by that of \( h(\mathcal{M}_d) \). Then by definition it follows that

\[
\mathbb{M}_\Sigma(\mathcal{M}_d) = \pi_1(h(\mathcal{M}_d), h(\mathcal{M}_d, \Sigma); \mathbb{I}_{\mathcal{M}_d}) \tag{5.5}
\]

Note that the analog of the quantum configuration space \((5.2)\) in the situation at hand is the quotient space \( e(\mathcal{M}_d, \Sigma) / \sim \), where \( f \sim f' \) if \( f(\Sigma) = f'(\Sigma) \), but the topology of this space is unmanageable. However, it is straightforward to show \(^{14}\) that if \( f, g \) are motions of \( \Sigma \) in \( \mathcal{M}_d \), then \( f \equiv g \) if and only if \( f \) is homotopic to a motion \( f' \) of \( \Sigma \) in \( \mathcal{M}_d \) with \( f'_t(\Sigma) = g_t(\Sigma) \quad \forall t \in [0,1] \). With this property, it is evident that the above definition of a motion coincides with the notion of a loop in a quantum configuration space.

Let us now consider some basic examples of motion groups. First, we note that any \( k \)-isotopy of \( N \) distinct points in a manifold \( \mathcal{M}_d \) extends to a \( k \)-isotopy of all of \( \mathcal{M}_d \). From this fact one may prove that, if \( P_N = \{x_1, \ldots, x_N\} \subset \mathcal{M}_d \) is a collection of \( N \) distinct points of \( \mathcal{M}_d \), then the restriction map

\[
(h(\mathcal{M}_d), h(\mathcal{M}_d, P_N), \mathbb{I}_{\mathcal{M}_d}) \xrightarrow{\rho} (e(\mathcal{M}_d, P_N), e(P_N, P_N), \mathbb{I}_{P_N}) \tag{5.6}
\]

induces isomorphisms

\[
\rho^* : \pi_n(h(\mathcal{M}_d), h(\mathcal{M}_d, P_N); \mathbb{I}_{\mathcal{M}_d}) \xrightarrow{\sim} \pi_n(e(\mathcal{M}_d, P_N), e(P_N, P_N); \mathbb{I}_{P_N}) \tag{5.7}
\]

for all \( n \geq 0 \) \(^{14}\). It follows that the group of motions of a point \( x \in \mathcal{M}_d \) coincides with the fundamental group

\[
\mathbb{M}_x(\mathcal{M}_d) = \pi_1(\mathcal{M}_d, x) \tag{5.8}
\]

and, if \( \mathcal{M}_d \) is connected, the motion group of \( P_N \) is the braid group

\[
\mathbb{M}_{P_N}(\mathcal{M}_d) = B_N(\mathcal{M}_d) \tag{5.9}
\]

Notice that, if \( \dim \mathcal{M}_d > 2 \), then \(^{13}\)

\[
\mathbb{M}_{P_N}(\mathcal{M}_d) \cong \bigoplus_{n=1}^{N} \pi_1(\mathcal{M}_d; x_n) \tag{5.10}
\]

from which it follows that the braiding phenomenon disappears in manifolds of dimension larger than 2. A more interesting example is provided by the Dahm group of motions of \( N \) unlinked and unknotted circles in \( \mathbb{R}^3 \). The generators are the motions which flip a circle, exchange two circles, and move one circle through another \(^{14}\). We shall return to
this example in the next section. For a description of the Dahm group of a certain class of non-trivial links in $S^3$, see [33].

The present $BF$ field theory approach that we are ultimately interested in actually focuses on two disjoint, compact submanifolds $\Sigma = \partial \Sigma_p(0)$ and $\Sigma' = \partial \Sigma_{d-p}(0)$ which lead to motions $\Sigma_p(t)$ and $\Sigma_{d-p}(t)$ of dual dimension in $\mathcal{M}_d$. For this, we need a slight generalization of the Dahm motion group above [14]. Let $h(\mathcal{M}_d, \Sigma, \Sigma')$ be the subspace of $h(\mathcal{M}_d)$ of homeomorphisms which leave fixed both of the submanifolds $\Sigma$ and $\Sigma'$. Then the motion group of the pair $(\Sigma, \Sigma')$ in $\mathcal{M}_d$ is defined to be the relative fundamental homotopy group:

$$M_{\Sigma, \Sigma'}(\mathcal{M}_d) = \pi_1(h(\mathcal{M}_d), h(\mathcal{M}_d, \Sigma, \Sigma'); \mathbb{I}_{\mathcal{M}_d})$$ (5.11)

Note that this definition differs from that of (5.3) with $\Sigma$ replaced by $\Sigma \sqcup \Sigma'$, since this latter group uses motions of $\Sigma$ and $\Sigma'$ which return to their original value only modulo permutation of the two subspaces $\Sigma, \Sigma'$. The former group, on the other hand, is the one that is required when the statistics of the branes corresponding to $\Sigma, \Sigma'$ are non-identical, as is the case in the canonical formulation of $BF$ field theory which utilizes brane source couplings to two independent gauge fields. In fact, the relation between the motion group (5.11) and the Dahm groups (5.3) is given by the following theorem [14]. Let $\varepsilon : M_{\Sigma, \Sigma'}(\mathcal{M}_d - \Sigma) \to M_{\Sigma, \Sigma'}(\mathcal{M}_d)$ be the group homomorphism induced by the map

$$(h(\mathcal{M}_d - \Sigma), h(\mathcal{M}_d - \Sigma, \Sigma'), \mathbb{I}_{\mathcal{M}_d - \Sigma}) \longrightarrow (h(\mathcal{M}_d), h(\mathcal{M}_d, \Sigma, \Sigma'), \mathbb{I}_{\mathcal{M}_d})$$ (5.12)

which sends each homeomorphism $f \in h(\mathcal{M}_d - \Sigma)$ to its extension $\varepsilon(f) \in h(\mathcal{M}_d)$ with $\varepsilon(f)|_{\Sigma} = \mathbb{I}_{\Sigma}$. Let $\omega : M_{\Sigma, \Sigma'}(\mathcal{M}_d) \to M_{\Sigma}(\mathcal{M}_d)$ be the group homomorphism induced by the map

$$(h(\mathcal{M}_d), h(\mathcal{M}_d, \Sigma, \Sigma'), \mathbb{I}_{\mathcal{M}_d}) \longrightarrow (h(\mathcal{M}_d), h(\mathcal{M}_d, \Sigma), \mathbb{I}_{\mathcal{M}_d})$$ (5.13)

which sends each $f \in h(\mathcal{M}_d, \Sigma, \Sigma')$ to $f \in h(\mathcal{M}_d, \Sigma)$. Then the sequence of groups

$$M_{\Sigma, \Sigma'}(\mathcal{M}_d - \Sigma) \xrightarrow{\varepsilon} M_{\Sigma, \Sigma'}(\mathcal{M}_d) \xrightarrow{\omega} M_{\Sigma}(\mathcal{M}_d)$$ (5.14)

is exact.

We close this subsection with a final useful computational property of the Dahm motion groups. Namely, there is a map, called the Dahm homomorphism [13, 14], which is a homomorphism

$$\mathcal{D} : M_{\Sigma}(\mathcal{M}_d) \longrightarrow \text{Aut}(\pi_1(\mathcal{M}_d - \Sigma))$$ (5.15)

from the group of motions of $\Sigma$ in $\mathcal{M}_d$ to the automorphism group of $\pi_1(\mathcal{M}_d - \Sigma)$ induced at the end of a given motion. This yields another presentation of the motion group which may be thought of as the fundamental homotopy group of an appropriate quantum configuration space. For some other computational aspects of the motion groups, see [14]. In the following we will derive a class of representations of the motion group $M_{\Sigma, \Sigma'}(\mathcal{M}_d)$ which
illustrates some new general aspects of these topological groups, and thereby extends the generally unprobed theory of motion groups. This will present a highly non-trivial application and thereby demonstrate the usefulness of the present topological field theory approach.

### 5.2. Abelian Holonomy Representations

Let us now examine the various transformation properties of the wavefunctions (5.1) and describe the ensuing representations of the motion group. Consider an adiabatic motion of the hypersurface $\Sigma_p(t)$ about $\Sigma_{d-p}(t)$. First we consider the homologically trivial motions. If one of the hypersurfaces $\Sigma_p(t)$ or $\Sigma_{d-p}(t)$ traces out a contractible volume as it moves, then the topological current integrals in the full wavefunction (5.1) vanish (c.f. eq. (3.18)). These currents therefore contribute nothing to the physical states under these types of motions. The solid angle function $\Phi_p(t) - \Phi_p(0)$, on the other hand, has delta-function singularities and thereby contributes whenever the hypersurfaces link each other. The change in $\Phi_p$ whenever such a linking occurs is $\Omega_{d-1}$, but this function is nevertheless independent of the choice of representative of the topological classes of the source trajectories, as shown in section 4.2. Thus under such an adiabatical linking, the wavefunctions (5.1) acquire the phase

$$\left(\sigma_p\right)^2 = e^{-\frac{2\pi i}{k_1} (-1)^{p(d-p)} Q_p Q_{d-p}}$$  (5.16)

Now let us examine the case of a homologically non-trivial motion. Consider the source motion whereby $\Sigma_p(t)$ is fixed in time and $\Sigma_{d-p}(t)$ winds $w_{l}^{d-p}$ times, in a time span $t_0$, around the $l$-th homology $d-p$-cycle of $\mathcal{M}_d$, and then afterwards $\Sigma_{d-p}(t)$ is fixed and $\Sigma_p(t)$ winds $w_{p}^{l}$ times, up to some time $t > t_0$, around the $l$-th homology $p$-th cycle of $\mathcal{M}_d$. According to (3.18), this motion can be summarized by the following equations:

$$\int_{t_0}^{t} dt' \Sigma_l^{(d-p)}(t') = w_l^{d-p}$$
$$\int_{t_0}^{t} dt' \Sigma_l^{(p)}(t') = 0$$
$$\int_{t_0}^{t} dt' \Sigma_l^{(d-p)}(t') = 0$$
$$\int_{t_0}^{t} dt' \Sigma_l^{(p)}(t') = w_p^{l}$$  (5.17)

The holonomies arising from possible linkings of these motions are taken into account by the solid angle function and constitute the phase operators (5.16) for the motion group. The remaining part of the periodic motion is readily found to change the wavefunctions (5.1) according to

$$\Psi_{\text{phys}}^{(q)}[\theta, K, \gamma; t] \rightarrow \sum_{q'} [M(w^p, w^{d-p})]_{qq'} \Psi_{\text{phys}}^{(q')}[\theta, K, \gamma; 0]$$  (5.18)

where the unitary matrices

$$[M(w^p, w^{d-p})]_{qq'} = \prod_{l=1}^{N_p} \exp \left[ -\frac{2\pi i}{k_1 I(p)} (-1)^{p(d-p)} Q_p Q_{d-p} \sum_{m=1}^{N_p} w_m^{d-p} \right]$$

where the unitary matrices

$$[M(w^p, w^{d-p})]_{qq'} = \prod_{l=1}^{N_p} \exp \left[ -\frac{2\pi i}{k_1 I(p)} (-1)^{p(d-p)} Q_p Q_{d-p} \sum_{m=1}^{N_p} w_m^{d-p} \right]$$
generate a \((k_1)^{N_p}\)-dimensional projective representation of the group \(\Gamma \oplus \Gamma^*\) of large gauge transformations. Their products also determine a global \(U(1) \times U(1)\) two-cocycle algebra:

\[
M(w^p, w^{d-p})M(v^p, v^{d-p}) = \prod_{l,m=1}^{N_p} e^{-\frac{2\pi i}{k} (-1)^{p(d-p)} Q_p Q_{d-p} (w^p I_{m} - v^p I_{m}) I_{l}^{p l m} m \times M(v^p, v^{d-p}) M(w^p, w^{d-p})
\]

which may be viewed as dual to the algebra (4.19) of the winding translation generators, in that the integers \(k_1\) and \(k_2\) are interchanged and the intersection matrix \(I_{lm}^{p}\) is replaced by its inverse through the combination \(Q_p Q_{d-p} I_{l m}^{p}\) (The appearence of the charges here owes to the fact that the algebra (5.20) comes from the windings of the sources, whereas the algebra (4.19) comes about from the windings of the gauge fields themselves).

To describe the appropriate motion group representation, we define \(w^{p(l)}_k = \delta_k^l\) and introduce the unitary operators

\[
\eta^{(l)}_p = M(w^{p(l)}, 0) , \quad \mu^{(m)}_p = M(0, w^{d-p(m)})
\]

for each \(l, m = 1, \ldots, N_p\). Then, together with the phase operators \(\sigma_p \mathbb{I}_{(k_1)^{N_p}}\), these operators generate the following \((k_1)^{N_p}\) dimensional representation of the pertinent motion group:

\[
\left[ \eta^{(l)}_p, \eta^{(m)}_p \right] = \left[ \mu^{(l)}_p, \mu^{(m)}_p \right] = 0
\]

\[
\left[ \sigma_p, \eta^{(l)}_p \right] = \left[ \sigma_p, \mu^{(l)}_p \right] = 0
\]

\[
\eta^{(l)}_p \mu^{(m)}_p = (\sigma_p)^{2l(m)_{lm}} \mu^{(m)}_p \eta^{(l)}_p
\]

and, according to (3.28), the topology of the manifold \(\mathcal{M}_d\) imposes the following global constraint on these generators:

\[
1 = \prod_{r,l=1}^{N_p-1} \prod_{m,n=1}^{N_p+1} (\sigma_p)^{2l[\Sigma_p^{(m)}, \partial \Sigma_{d-p}(0)] \nu[\Sigma_p^{(l)}, \partial \Sigma_p(0)] (I_{mn}^{p+1} + I_{rl}^{p-1})
\]

The collection of \(2N_p + 1\) unitary operators \(\{\sigma_p, \eta^{(l)}_p, \mu^{(m)}_p\}\) with the relations (5.22)–(5.25) constitute a subset of the full set of generators of a \((k_1)^{N_p}\) dimensional representation of the motion group \(\mathbb{M}_{\partial \Sigma_p(0), \partial \Sigma_{d-p}(0)}(\mathcal{M}_d)\). Presumably there are more generators and relations for this group (see the next section), but due to the abelian nature of the present formalism such operators are represented trivially on the physical Hilbert space of the BF field theory. Notice that the constraint (5.25) is represented in terms of the intersection matrices \(I^{(p-1)}\) and \(I^{(p+1)} = -I^{(d-p-1)}\) (by Poincaré-Hodge duality) which arise from the
initial $p-1$-brane and $d-p-1$-brane configurations $\partial \Sigma_p(0)$ and $\partial \Sigma_{d-p}(0)$ used to define the appropriate motion group (and which come about from the relevant Čech cohomology groups).

The global constraint \ref{global-constraint} comes about from the fact that there is always a trajectory $\Sigma_p(t)$ which encircles $\Sigma_{d-p}(t)$ and traces the homology generators of $\mathcal{M}_d$ in such a way that it forms a trivial motion, i.e. one that is equivalent to a stationary motion of $\partial \Sigma_p(0)$ in $\mathcal{M}_d$. As a simple example of this constraint, consider the motion in $\mathcal{M}_d$ whereby, initially at time $t = 0$, the $p-1$-brane intersects only with the $l_0$-th homology $d-p+1$-cycle of $\mathcal{M}_d$ exactly once, and likewise for the $d-p-1$-brane with the $m_0$-th homology $p+1$-cycle. Then the global restriction \ref{global-constraint} simplifies to the form

$$
(\sigma_p)^2 \sum_{n} i^{(p+1)}_{m_0 n} + 2 \sum_{r} i^{(p-1)}_{r l_0} = 1
$$

This relation is a fundamental constraint that must be met the external charges $Q_p, Q_{d-p}$ of the quantum field theory in order to yield a well-defined motion group representation. Generally, the relations \ref{linking-relations}–\ref{global-constraint} between the linking operator $\sigma_p$ and the generators of homologically non-trivial motions $\eta^{(l)}_p, \mu^{(m)}_p$ reflect the non-trivial relationships that exist between motions around the various cycles of $\mathcal{M}_d$. The relation \ref{global-constraint} is very natural, since it tells us that the operations of moving $\Sigma_p$ and $\Sigma_{d-p}$ around the pertinent cycles commute only when these cycles do not intersect. Otherwise, they differ by a holonomy factor that depends precisely on the intersection number of the cycles and represents the number of linking operations required to unravel the motion to a stationary one. Together with the exact sequence \ref{exact-sequence}, these relationships may determine at least a large portion of the full motion group for a wide class of submanifold embeddings in terms of their individual motions in $\mathcal{M}_d$. These relationships thereby reflect a highly non-trivial application of the present topological field theory to the theory of motion groups. In the next section we shall describe briefly how the present model may be modified so as to potentially produce the full set of generators and motions of $\mathcal{M}_d|\Sigma_p(0), \partial \Sigma_{d-p}(0)(\mathcal{M}_d)$.

6. Applications

In this final section we will briefly describe some examples and applications of the formalism above. We will also mention possible generalizations which could probe deeper into the structure of motion groups on topologically non-trivial manifolds.

6.1. The Braid Group

For our first example, we illustrate how the well-known holonomy representations of the braid group \cite{17}–\cite{19} appear within our more general formalism. We set $d = 2$, $p = 1$ and $N_1 = 2g$, where $g$ is the genus of a compact Riemann surface $\mathcal{M}_2$. For any $g > 0$,
we may view $M_2$ as the connected sum $(T^2)^{#g}$ of two-tori and hence as an embedded submanifold of $\mathbb{R}^3$. The canonical basis of $H_1(M_2, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ is defined by the $2g$ generators $a^l$ and $b^m$, $l, m = 1, \ldots, g$, where $a^l$ corresponds to the class of the outer cycle of $T^2$ in the $l$-th component of the connected sum $(T^2)^{#g}$ while $b^m$ corresponds to the class of the inner cycle of $T^2$ in the $m$-th component. The intersection indices of these one-cycles are given by

$$\nu[a^l, a^k] = \nu[b_l, b_k] = 0$$

$$\nu[a^l, b_k] = -\nu[b_k, a^l] = \delta_k^l$$

and the corresponding intersection matrix is

$$I^{(1)kl} = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

with $k, l = 1, \ldots, 2g$. Furthermore, we have $I^{(0)} = -I^{(2)} = \pm 1$, where the sign is chosen with respect to a given orientation of the surface $M_2$ when viewed as an embedded submanifold of $\mathbb{R}^3$.

In the present case, we denote the generators $\eta^{(l)}_l$ (and $\mu^{(l)}_l$) by $\alpha^l$ for $l = 1, \ldots, g$ and by $\beta_{m=2g+1-l}$ for $l = g + 1, \ldots, 2g$. The operator which represents the braiding of the trajectories of two particles of charges $Q_1$ and $Q_2$ is

$$\sigma = e^{\frac{\pi i}{k} Q_1 Q_2}$$

Then, according to the relations (5.22)–(5.24), these operators have the following presentation which represents various equivalent braids:

$$[\sigma, \alpha^l] = [\sigma, \beta_l] = 0$$

$$[\alpha^l, \alpha^m] = [\beta_l, \beta_m] = 0$$

$$[\alpha^l, \beta_m] = 0 \quad \text{for} \; l \neq m$$

$$\alpha^l \beta_l = \sigma^2 \beta_l \alpha^l$$

for each $l, m = 1, \ldots, g$. However, the global constraint (5.23) for this particular braid group representation is an identity. The topology of the Riemann surface $M_2$ in the present case does not affect the linking operator $\sigma$, because the BF field theory representation in effect generates an abelian holonomy representation of the unpermuted braid group of $M_2$, i.e. that associated with the quantum mechanical configuration space of a system of non-identical particles. For a system of $N$ identical particles, the corresponding braid group representation would have to satisfy the additional global constraint $\sigma^{2(N+g-1)} = 1$ for a closed manifold [12, 19]. Normally, such a constraint would come geometrically in part from a framing of the corresponding three-dimensional manifold which is required to regulate the self-linkings of the particle trajectories [34]. Here the linking numbers induced by the BF field theory contain no such ambiguous self-linking terms. In effect,
the present holonomy factors induce a representation of the two colour braid group \([35]\) generated by exchanging ribbon-like configurations. The ribbons can themselves twist, leading to intrinsic spin phases which cancel exactly with the statistical exchange phases as a result of the spin-statistics theorem (This may be checked explicitly by computing the action of the energy-momentum tensor on the physical states \((5.1)\)). Note that the argument of the solid angle function \((4.9)\) defines an eigenfunction expansion of the prime form of the Riemann surface \(M_2\) which in turn produces the appropriate generalization of the usual multi-valued angle function of the plane \([17][19]\).

6.2. Quantum Exchange Statistics of Extended Objects

The generic properties of the braid group representations described above can be generalized to any spatial manifold \(\mathcal{M}_d\) of even dimension \(d = 2p\). This in turn provides a field theoretical model which generalizes the phenomenon of fractional statistics of quantum mechanical point particles in two dimensions to that of non-identical \(p−1\)-branes in \(2p\)-dimensions (strings in four spatial dimensions, membranes in six spatial dimensions, etc.). Note that to describe a system of \(N > 2\) non-interacting objects, one considers the worldvolumes to be disjoint unions \(\Sigma_p = \bigsqcup_{n=1}^{N} \Sigma_p^{(n)}\) and writes the corresponding deRham current of \(\Sigma_p\) as a sum over those of the \(\Sigma_p^{(n)}\). In the present case, we again have that \(I^{(p-1)} = -I^{(p+1)}\) (by Poincaré-Hodge duality) and so the global constraint \((5.25)\) again simplifies considerably. In particular, if the initial configurations at time \(t = 0\) of both \(p−1\)-branes intersect the homology \(p+1\)-cycles of \(M_{2p}\) in the same way, then this global constraint is once again an identity and there are no further constraints on the linking operators \(\sigma_p\) of the holonomical motion group representation. This feature again owes to the fact that we obtain a representation of the unpermutated, two-colour motion group. As in the case of particles, we attribute this global cancellation as being due to an induced spin of the extended motions which cancels the holonomy factors. In particular, we may deduce from this cancellation that the standard spin-statistics theorem holds for such configurations of extended objects (in contrast to the generic case \([20]\)).

In the special case where the spatial manifold is flat infinite Euclidean space \(\mathcal{M}_d = \mathbb{R}^d\), the constructions of the previous sections can be made somewhat more explicit (essentially because there are no harmonic zero modes in this case). In particular, it is known \([14]\) that in this case the motion groups count the connected components of the space of orientation-preserving homeomorphisms of \(\mathbb{R}^d\) which preserve \(\Sigma\),

\[
\mathcal{M}_\Sigma(\mathbb{R}^d) \cong \pi_0(h^+(\mathbb{R}^d, \Sigma)) \tag{6.5}
\]

Furthermore, the Euclidean Green’s function of the scalar Laplacian for \(p > 1\) is given by

\[
\left( x \mid \nabla_0^{-2} \mid y \right) = -\frac{1}{\Omega_{2p-1} \cdot |x - y|^{2(p-1)}} \tag{6.6}
\]

Then, using \((2.12), (3.17)\) and \((4.6)\), we arrive at an explicit expression for the (symmetrized) \(2p−1\)-dimensional solid angle formed between two \(p−1\)-brane configurations
at time $t$:

$$
\Phi_p(t) = \frac{2(p-1)}{\Omega_{2p-1}} \int_0^t \int d^{p-1}\sigma \int d^{p-1}\sigma' \epsilon_{\sigma_0...\sigma_p} \\
\times \left[ \prod_{k=1}^{p-1} \frac{\partial X_p^{i_k}(t', \sigma)}{\partial \sigma^k} \frac{\partial X_p^{i_{p+1}}(t', \sigma')}{\partial \sigma^l} \prod_{l=p+2}^{2p} \frac{\partial X_p^{i_l}(t', \sigma')}{\partial \sigma^l} \\
- \prod_{k=1}^{p-1} \frac{\partial X_p^{i_k}(t', \sigma)}{\partial \sigma^k} \frac{\partial X_p^{i_{p+1}}(t', \sigma)}{\partial \sigma^l} \prod_{l=p+2}^{2p} \frac{\partial X_p^{i_l}(t', \sigma)}{\partial \sigma^l} \right] \left( X_p(t', \sigma) - X_p'(t', \sigma') \right)^{i_p} \\
\right)^{2p} (6.7)
$$

The present topological field theory formalism therefore produces very explicit higher-dimensional generalizations of the standard multi-valued angle functions which have been extensively studied and utilized in the physics of planar systems [17]–[19]. Eq. (6.7) generalizes the standard expression for the adiabatic limit of the Gauss linking number in three-dimensions for two curves ($p = 1$) [34]. In this way the function (6.7) may in fact be thought of as giving the appropriate higher-rank generalization of the electromagnetic Faraday law, but in a way that avoids the cumbersome self-linking number terms that arise in the standard two dimensional formulations [17]–[19], [34].

6.3. Quantum Exchange Statistics in Odd Dimensions

Fractional statistics of identical extended objects in odd dimensional spaces may be attained by modifying the topological class of one of the deRham currents appearing in the source-coupled BF action. An important example is the case of the motion of $N$ non-interacting identical strings in flat Euclidean three-space $\mathbb{R}^3$. The relevant motion group $\mathbb{M}_{\Sigma(N)}(\mathbb{R}^3)$ in this case is constructed from the submanifold $\Sigma^{(N)} = \bigcup_{n=1}^N C_n$ which is a collection of $N$ unknoted and unlinked circles $C_n = S^1$ in $\mathbb{R}^3$. Then

$$
\pi_1(\mathbb{R}^3 - \Sigma^{(N)}) \cong \langle \ell_1, \ldots, \ell_N \rangle
$$

is the free group on $N$ generators $\ell_n, n = 1, \ldots, N$. The generating automorphisms for the Dahm subgroup $\mathcal{D}(\mathbb{M}_{\Sigma(N)}(\mathbb{R}^3))$ of $\text{Aut}(\langle \ell_1, \ldots, \ell_N \rangle)$ are then $\tau_n, \sigma_n$ and $\rho_{nm}$ [14], where

$$
\tau_n(\ell_k) = \begin{cases} 
(\ell_n)^{-1} & , k = n \\
\ell_k & , k \neq n
\end{cases}
$$

$$
\sigma_n(\ell_k) = \begin{cases} 
\ell_n & , k = n+1 \\
\ell_k & , k \neq n, n+1
\end{cases}
$$

$$
\rho_{nm}(\ell_k) = \begin{cases} 
\ell_m \ell_n (\ell_m)^{-1} & , k = n \\
\ell_k & , k \neq n
\end{cases}
$$

(6.9)
These automorphisms correspond, respectively, to a rotation through angle $\pi$ of the $n$-th circle $C_n$ about its diameter, to interchanging the $n$-th and $n+1$-th circles, and to transporting the $n$-th circle through the $m$-th circle. Some of the relations of this motion group can also be thereby deduced to be [21]:

\[
(\tau_n)^2 = \mathbb{1}, \quad \tau_n \rho_{n,n+1} (\tau_n)^{-1} = \rho_{n,n+1}, \\
(\sigma_n)^2 = \mathbb{1}, \quad \sigma_n \sigma_m = \sigma_m \sigma_n \quad \text{for} \quad |m-n| \geq 2, \\
\sigma_n \sigma_{n+1} \sigma_n = \sigma_{n+1} \sigma_n \sigma_{n+1}, \\
(\tau_n \sigma_n)^4 = (\tau_n \rho_{n,n+1})^2 = \mathbb{1}
\] (6.10)

We shall now construct the appropriate topological BF field theory to describe this group [21]. We consider the usual BF action (2.16) for $d=3, p=2$ with the standard coupling of the two-form field $B$ to the total de Rham current of the string worldsheets $\Sigma_2 = \bigsqcup_{n=1}^N \Sigma_2^{(n)}$ and with

\[
\Delta \Sigma_2 = \sum_{n=1}^N \phi_n \Delta \Sigma_2^{(n)}
\] (6.11)

where $\phi_n$ is the flux of string $n$ (Here we set the overall charge parameters $Q_1, Q_2$ equal to 1). The crucial modification is in the coupling $A \wedge * J$ of the one-form field $A$, where the new current $J$ is represented by the vector field [36]

\[
J^\mu(x) = \sum_{n=1}^N \int d^2 \sigma \, \delta^{(4)}(x - X_n(\sigma)) \frac{\partial X_\mu^\alpha(\sigma)}{\partial \sigma^\alpha} J_\alpha^\nu(\sigma)
\] (6.12)

which is defined in terms of the conserved worldsheet current

\[
J_\alpha^\nu(\sigma) = \epsilon^{\alpha\beta} \frac{\partial \varphi_\nu(\sigma)}{\partial \sigma^\beta}
\] (6.13)

Here $X_n : \Sigma_2^{(n)} \to \mathbb{R}^3$ is the worldsheet embedding of string $n$ and $\varphi_n(\sigma)$ is some continuous function on $\Sigma_2^{(n)}$. We assume that $d\varphi_n \in \Omega^1(\Sigma_2^{(n)})$ is a globally defined differential one-form on the string worldsheet, but that the function $\varphi_n$ itself is multi-valued. Performing a canonical split of the coordinates of the surface $\Sigma_2^{(n)}$ with $\sigma^2 \in [0,1]$ parametrizing the loop of the closed string, we see that the current (6.13) induces a non-zero $U(1)$ charge on the worldsheet of string $n$:

\[
q_n = \int_0^1 d\sigma^2 \, J_\alpha^{\nu}(\sigma^1, \sigma^2) = \varphi_n(\sigma^1, 1) - \varphi_n(\sigma^1, 0)
\] (6.14)

This charge is a constant of the motion because both currents (6.12) and (6.13) define closed differential forms on $\mathbb{R}^3$ and $\Sigma_2^{(n)}$, respectively. The current (6.12) can thereby be thought of as a smeared particle current which serves as a smoothed-out induced de Rham current and which induces an electric charge on the string worldsheets.
The worldsheet scalar fields $\varphi_n(\sigma)$ can be regarded as dynamical degrees of freedom in the field theory, in which case they are associated with the reparametrization invariances of the string surfaces. Fixing these functions to some prescribed form, with $d\varphi_n$ in the cohomology classes appropriate to the charges $q_n$, ruins the invariance of the strings under diffeomorphisms of $S^1$. Nevertheless, we will set $\varphi_n(\sigma^1, \sigma^2) = q_n \sigma^2$ and calculate the resulting holonomies that arise in the modified wavefunctions of the quantum field theory. Because the current (6.12) is a closed one-form, the relevant phase factor that appears in the wavefunctions can be calculated in the same way as before, by using the Hodge decompositions for (6.12) analogous to those of the usual (singular) deRham currents. Following the steps which led to (6.7) using the Euclidean Green’s function for the three-dimensional scalar Laplacian operator, this straightforward calculation produces the holonomy function

$$
\Phi_{2,2}(t) = \sum_{n,m=1}^{N} \frac{q_n \phi_m}{4\pi} \int_{0}^{t} dt' \int_{0}^{1} d\sigma \int_{0}^{1} d\sigma' \epsilon_{0ijk} \left[ \frac{\partial X^i_n(t', \sigma)}{\partial t'} \frac{\partial X^k_m(t', \sigma')}{\partial \sigma'} \right] \frac{(X_n(t', \sigma) - X_m(t', \sigma'))^j}{|X_n(t', \sigma) - X_m(t', \sigma')|^3}
$$

(6.15)

In this abelian holonomy representation, the images of the flip operators $\tau_n$ and the exchange generators $\sigma_n$ are all trivial on the Hilbert space (see (6.9)). This owes to the property that the strings have no abelian linking in three-dimensions, and also that the function (6.13) yields no representation of the “self-interactions” of a given string configuration. The slide operators $\rho_{nm}$, on the other hand, produce non-trivial quantum phases in the wavefunctions under the adiabatic transport of string $n$ through the loop of string $m$. The resulting one-dimensional, unitary holonomy representation is therefore given by

$$
\tau_n = \sigma_n = 1 \quad \rho_{nm} = e^{\frac{\pi \xi}{k} q_n \phi_m}
$$

(6.16)

The adiabatic holonomy in (6.16) arises from the fact that electric charge and flux can link in three-dimensions, so that the sliding operation has the same effect as adiabatically transporting a charge $q_n$ around a flux $\phi_m$ [22]. Whether or not this model, with a particular fixed configuration for the worldsheet scalar fields $\varphi_n(\sigma)$, leads to a sensible Hilbert space representation is a point which deserves further investigation. In any case, this simple example shows the possibilities that exist for constructing representations of arbitrary motion groups for any dimensionalities of the manifolds involved. It would also be interesting to analyse global aspects of these sorts of couplings to $BF$ gauge fields, along the lines developed in earlier sections of this paper. This construction would then yield the extra (multi-dimensional) generators and relations of the motion group which arise due to homological effects, and also compute the curved space version of the holonomy function.
A holonomy function for strings in generic three-dimensional spatial manifolds has been derived in [23] based on a marginal deformation of the canonical $BF$ field theory.

The representations obtained in this paper are all abelian (although multi-dimensional when the space contains non-contractible cycles) and as such lead to very simple representations of the generators and relations of the motion group. For this reason they do not completely probe the algebraic structures of the motion group, although they do provide geometrical and field theoretical origins for various aspects of it. More interesting representations may be attainable using non-abelian $BF$ theories, whereby the increase in colour symmetry is expected to give rise to richer invariants of the embedded submanifolds and of the spatial manifolds themselves. The holonomy operators in four-dimensional non-abelian $BF$-theory have been studied recently in [16] where it was shown that surface observables yield possibly new invariants of immersed surfaces in four-manifolds. A non-abelian version of the $BF$ model described in this subsection is analysed briefly in [21]. It would also be interesting to incorporate “interactions” of various extended objects into the framework of this paper. This would lead to a quantum field theoretical description of, for example, the motion groups associated with non-trivial knots and links immersed in $\mathbb{R}^3$ such as those studied in [33]. Furthermore, the incorporation of deformations of the standard $BF$ action, such as those studied in [23], would produce canonical versions of the topological invariants obtained in [15].

Acknowledgements

The author would like to thank G. Semenoff for hospitality at the University of British Columbia, where this work was completed. Part of this work was carried out during the PIFms/APCTP/CRM Workshop “Parties, Fields and Strings ’99” which was held at the University of British Columbia during the summer of 1999. The author thanks the organizers and participants for having provided a stimulating environment in which to work. This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.
References

[1] A.S. Schwarz, Lett. Math. Phys. 2 (1978) 247; Commun. Math. Phys. 67 (1979) 1.
[2] G.T. Horowitz, Commun. Math. Phys. 125 (1989) 417.
[3] M. Blau and G. Thompson, Ann. Phys. 205 (1991) 130.
[4] J. Gegenberg and G. Kunstatter, Ann. Phys. 231 (1994) 270.
[5] G.T. Horowitz and M. Srednicki, Commun. Math. Phys. 130 (1990) 83.
[6] M.J. Bowick, S.B. Giddings, J.A. Harvey, G.T. Horowitz and A. Strominger, Phys. Rev. Lett. 61 (1988) 2823; X. Fustero, R. Gambini and A. Trias, Phys. Rev. Lett. 62 (1989) 1964; J.A. Harvey and J.T. Liu, Phys. Lett. B240 (1990) 369; B. Harms and Y. Leblanc, Phys. Rev. D45 (1992) 2880; M.I. Polikarpov, U.J. Wiese and M.A. Zubkov, Phys. Lett. B309 (1993) 133; M. Sato and S. Yahikozawa, Nucl. Phys. B436 (1995) 100; E.T. Akhmedov, M.N. Chernodub, M.I. Polikarpov and M.A. Zubkov, Phys. Rev. D53 (1996) 2087; H. Fort and R. Gambini, Phys. Lett. B372 (1996) 226; Phys. Rev. D54 (1996) 1778; E.T. Akhmedov, M.N. Chernodub and M.I. Polikarpov, JETP Lett. 67 (1998) 389.
[7] K.-I. Izawa, Progr. Theor. Phys. 90 (1993) 911; A.S. Cattaneo, P. Cotta-Ramusino, A. Gamba and M. Martellini, Phys. Lett. B355 (1995) 245; H. Reinhardt, in: Quark Confinement and the Hadron Spectrum II, eds. N. Brambilla and G.M. Prosperi (World Scientific, 1997); F. Fucito, M. Martellini and M. Zeni, Nucl. Phys. B496 (1997) 259; A.S. Cattaneo, P. Cotta-Ramusino, F. Fucito, M. Martellini, M. Rinaldi, A. Tanzini and M. Zeni, Commun. Math. Phys. 197 (1998) 571; K.-I. Kondo, Phys. Rev. D58 (1998) 105019.
[8] A.H. Chamseddine and D. Wyler, Nucl. Phys. B340 (1990) 595; D. Birmingham, R. Gibbs and S. Mokhtari, Phys. Lett. B263 (1991) 176; C.G. Callan, S.B. Giddings, J.A. Harvey and A. Strominger, Phys. Rev. D45 (1992) 1005; H. Verlinde, in: String Theory and Quantum Gravity '91, eds. J.A. Harvey, R. Iengo, K.S. Narain, S. Randjbar-Daemi and H. Verlinde (World Scientific, 1992), p. 178; G. Grignani and G. Nardelli, Phys. Rev. D45 (1992) 2719; Nucl. Phys. B412 (1994) 320; D. Cangemi and R. Jackiw, Ann. Phys. 225 (1993) 229; Phys. Lett. B229 (1993) 24; M. Abe and N. Nakanishi, Progr. Theor. Phys. 89 (1993) 501; I. Oda and S. Yahikozawa, Class. Quant. Grav. 11 (1994) 2653; J.P. Lupi, A. Restuccia and J. Stephany, Phys. Rev. D54 (1996) 3861; V. Husain and S. Major, Nucl. Phys. B500 (1997) 381; J.-H. Lee and O.K. Pashaev, J. Math. Phys. 39 (1998) 102; L. Friedel and K. Krasnov, Class. Quant. Grav. 16 (1999) 351.
[9] J.C. Baez, An Introduction to Spin Foam Models of Quantum Gravity and BF Theory, gr-qc/9905087.
[10] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Rep. 209 (1991) 129.
[11] E. Fadell and L. Neuwirth, Math. Scand. 10 (1962) 111; R.H. Fox and L. Neuwirth, Math. Scand. 10 (1962) 119; J.S. Birman, Comm. Pure. Appl. Math. 22 (1968) 41.
[12] Y. Ladegaillerie, Bull. Soc. Math. 100 (1976) 255.

[13] D.M. Dahm, *A Generalization of Braid Theory*, Ph.D. Thesis, Princeton University (1961), unpublished.

[14] D.L. Goldsmith, Michigan Math. J. 28 (1981) 3.

[15] P. Cotta-Ramusino and M. Martellini, in: *Knots and Quantum Gravity*, ed. J.C. Baez (Oxford University Press, 1994); A.S. Cattaneo, P. Cotta-Ramusino and M. Martellini, Nucl. Phys. B436 (1995) 355; A.S. Cattaneo, P. Cotta-Ramusino, J. Fröhlich and M. Martellini, J. Math. Phys. 36 (1995) 6137; A.S. Cattaneo, J. Math. Phys. 37 (1996) 3664; Commun. Math. Phys. 189 (1997) 795.

[16] A.S. Cattaneo, P. Cotta-Ramusino and M. Rinaldi, Commun. Math. Phys. 204 (1999) 493.

[17] G.V. Dunne, R. Jackiw and C.A. Trugenberger, Ann. Phys. 194 (1989) 197; I.I. Kogan, Comm. Nucl. Part. Phys. 19 (1990) 305.

[18] M. Bos and V.P. Nair, Phys. Lett. B223 (1989) 61.

[19] M. Bergeron and G.W. Semenoff, Ann. Phys. 245 (1996) 1.

[20] A.P. Balachandran, W.D. McGlinn, L. O’Raifeartaigh, S. Sen, R.D. Sorkin and A.M. Srivastava, Mod. Phys. Lett. A7 (1992) 1427; C. Aneziris, Mod. Phys. Lett. A7 (1992) 3789; A.P. Balachandran and P. Teotonio-Sobrinho, Int. J. Mod. Phys. A9 (1994) 1569.

[21] C. Aneziris, A.P. Balachandran, L.H. Kauffman and A.M. Srivastava, Int. J. Mod. Phys. A6 (1991) 2519.

[22] M. Bergeron, G.W. Semenoff and R.J. Szabo, Nucl. Phys. B437 (1995) 695.

[23] R.J. Szabo, Nucl. Phys. B531 (1998) 525.

[24] M.I. Caicedo, I. Martín and A. Restuccia, *On the Geometry of Antisymmetric Fields*, hep-th/9711122.

[25] M.I. Caicedo and A. Restuccia, Class. Quant. Grav. 15 (1998) 3749.

[26] O. Alvarez, Commun. Math. Phys. 100 (1985) 279; A.P. Polychronakos, Nucl. Phys. B281 (1987) 241.

[27] P. Freund and R. Nepomechie, Nucl. Phys. B199 (1982) 482; R. Nepomechie, Phys. Rev. D31 (1985) 1921.

[28] J.-L. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization* (Birkhäuser, 1992).

[29] R. Bott and L.W. Tu, *Differential Forms in Algebraic Topology*, (Springer-Verlag, New York, 1986).

[30] A. Carey, J. Mickelsson and M. Murray, Commun. Math. Phys. 183 (1997) 707; hep-th/9711133.

[31] J. Kalkkinen, J. High Energy Phys. 9907 (1999) 002.

[32] D. Mumford, *Tata Lectures on Theta* (Birkhäuser, Basel, 1983).
[33] D.L. Goldsmith, Bull. Amer. Math. Soc. 80 (1974) 62; Math. Scand. 50 (1982) 167.

[34] A.M. Polyakov, Mod. Phys. Lett. A3 (1988) 325;
    C.-H. Tze, Int. J. Mod. Phys. A3 (1988) 1959;
    C.-H. Tze and S. Nam, Ann. Phys. 193 (1989) 419;
    E. Witten, Commun. Math. Phys. 121 (1989) 351.

[35] D. Rolfsen, Knots and Links (Publish or Perish, 1976);
    D. Eliezer and G.W. Semenoff, Phys. Lett. B266 (1991) 375.

[36] A.P. Balachandran, A. Stern and B.-S. Skagerstam, Phys. Rev. D20 (1979) 439.