COUNTING TWISTED HIGGS BUNDLES

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Abstract. We count invariants of the moduli spaces of twisted Higgs bundles on a smooth projective curve.

1. Introduction

Let $X$ be a smooth projective curve of genus $g$ defined over a finite field $\mathbb{F}_q$. Let $L$ be a line bundle of degree $\ell$ over $X$ and let $\mathcal{M}_L(r,d)$ be the moduli space of semistable $L$-twisted Higgs bundles over $X$. It parametrizes pairs $(E, \phi)$, where $E$ is a vector bundle of rank $r$ and degree $d$ over $X$ and $\phi : E \rightarrow E \otimes L$ is a homomorphism. A formula for the computation of the number of points of $\mathcal{M}_L(r,d)$ for coprime $r,d$ was conjectured in [18] and is proved in this note.

The above conjecture was obtained as a solution of a recursive formula, called an ADHM recursion, conjectured by Chuang, Diaconescu, and Pan [3]. The ADHM recursion was itself based on a conjectural wall-crossing formula for the refined Donaldson-Thomas invariants on a noncompact 3CY variety $Y = L \oplus (\omega_X \otimes L^{-1})$, where $\omega_X$ is the canonical bundle of $X$, as well as a conjectural formula for the asymptotic ADHM invariants. The latter invariants can be interpreted as Pandharipande-Thomas invariants of $Y$ [21]. The formula counting them was derived in [3] by string theoretic methods, hence remains conjectural from the mathematical point of view.

On the other hand, the formula for $\mathcal{M}_L(r,d)$ conjectured in [18] can be considered as a generalization of the conjecture by Hausel and Rodriguez-Villegas [11] in the case of usual Higgs bundles, where the twisting line bundle $L$ is equal to $\omega_X$. A breakthrough for the counting of usual Higgs bundles was made by Schiffmann [23] who proved an explicit, albeit rather complicated formula for these invariants, quite different from the conjecture of [11]. An equivalence between these formulas was proved recently by purely combinatorial methods in a brilliant series of papers by Mellit [14, 16, 15].

Results on the invariants of moduli spaces of Higgs bundles for small rank and degree were obtained in [10, 11, 12, 9, 7, 22, 3]. The conjecture of Hausel and Rodriguez-Villegas was proved for the $y$-genus in [6]. An alternative general formula for twisted Higgs bundles on $\mathbb{P}^1$ – in terms of quiver representations – was obtained in [19]. Other interesting results related to counting of Higgs bundles can be found in [1, 2, 4, 5].

In this paper we will apply Mellit’s methods in order to prove a formula for general $L$-twisted Higgs bundles. This task will be rather straightforward as Schiffmann’s computation was generalized earlier for twisted Higgs bundles in [20]. More precisely, let $\mathcal{M}_L(r,d)$ be the moduli stack of semistable $L$-twisted Higgs bundles over $X$. Given a finite type algebraic stack $\mathcal{X}$ over $\mathbb{F}_q$, define its volume (see §2.4 for more details on volumes)

$$[\mathcal{X}] = (\#\mathcal{X}(\mathbb{F}_{q^n}))_{n \geq 1}, \quad \#\mathcal{X}(\mathbb{F}_{q^n}) = \sum_{x \in \mathcal{X}(\mathbb{F}_{q^n})/\sim} \frac{1}{\# \text{Aut}(x)}.$$  (1)
Define (integral) Donaldson-Thomas invariants $\Omega_{r,d}$ using the plethystic logarithm (see §2.2)

$$\sum_{d/r=\tau} \Omega_{r,d} T^r z^d = (q-1) \log \left( \sum_{d/r=\tau} (-q^{\frac{1}{r}})^{-\ell r^2} [\mathcal{M}_L(r,d)] T^r z^d \right), \quad \tau \in \mathbb{Q}. \quad (2)$$

Note that if $r, d$ are coprime, then every $E \in \mathcal{M}_L(r,d)$ is stable and $\text{End}(E) = \mathbb{F}_q$ (see Remark 3.1). Therefore

$$\frac{[\mathcal{M}_L(r,d)]}{q-1} = [\mathcal{M}_L(r,d)] = (-q^{\frac{1}{r}})^{-\ell r^2} \Omega_{r,d} q^{(1)}$$

hence we can recover $[\mathcal{M}_L(r,d)]$ from $\Omega_{r,d}$. Consider the zeta function of the curve $X$

$$Z_X(t) = \exp \left( \sum_{n \geq 1} \frac{\#X(\mathbb{F}_q^n)}{n} t^n \right) = \frac{\prod_{i=1}^{g} (1 - \alpha_i t)(1 - \alpha_i^{-1} qt)}{(1-t)(1/qt)},$$

where $\alpha_i$ are the Weil numbers of $X$ (see §2.4). The following result was conjectured in [18] (cf. §4.3). We formulate it in the case $\deg L > 2g - 2$ (see Remark 4.5 for the case $L = \omega_X$).

**Theorem 1.1** (cf. Theorem 4.4). Assume that $p = \ell - (2g - 2) > 0$. Given a partition $\lambda$ and a box $s \in \lambda$, let $a(s)$ and $l(s)$ denote its arm and leg lengths respectively (see §2). Define

$$\hat{\Omega}_p(T, z) = \sum_{\lambda} T^{\ell \lambda} \prod_{s \in \lambda} (-q^{a(s)} z^{l(s)})^p \prod_{i=1}^{q} \frac{(q^{a(s)} - \alpha_i^{-1} z^{l(s)+1})(q^{a(s)} z^{l(s)+1} - \alpha_i z^{l(s)})}{(q^{a(s)} - z^{l(s)+1})(q^{a(s)+1} - z^{l(s)})}, \quad (4)$$

$$\sum_{r \geq 1} \Omega_r(z) T^r = (q-1)(1-z) \log \hat{\Omega}_p(T, z). \quad (5)$$

Then $\Omega_r(z) \in \mathbb{Z}[q, z, \alpha_1^{\pm 1}, \ldots, \alpha_g^{\pm 1}]$ and $\Omega_{r,d} = q^{pr/2} \Omega_r(1)$ for all $d \in \mathbb{Z}$. In particular, if $r, d$ are coprime, then

$$[\mathcal{M}_L(r,d)] = (-1)^p q^{(g-1)r^2 + p} \Omega_r(1). \quad (6)$$

2. Preliminaries

2.1. Partitions. A partition is a sequence of integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ such that $\lambda_n = 0$ for $n \gg 0$. We define its length $l(\lambda) = \# \{ i \mid \lambda_i \neq 0 \}$ and its weight $|\lambda| = \sum \lambda_i$. Define its Young diagram (also denoted by $\lambda$)

$$d(\lambda) = \{ (i, j) \in \mathbb{Z}^2 \mid i \geq 1, 1 \leq j \leq \lambda_i \}. \quad (7)$$

An element $s = (i, j) \in \lambda$ is called a box of the Young diagram located at the $i$-th row and $j$-th column. Define the conjugate partition $\lambda'$ with $\lambda'_j$ equal the number of boxes in the $j$-th column of $\lambda$. Given a box $s = (i, j) \in \lambda$, define its arm and leg lengths respectively

$$a(s) = \lambda_i - j, \quad l(s) = \lambda'_j - i. \quad (8)$$

Define the hook length $h(s) = a(s) + l(s) + 1$.

**Figure 1.** Young diagram for $\lambda = (4, 4, 2)$. Here $\lambda' = (3, 3, 2, 2)$, $s = (2, 1)$, $a(s) = 3$, $l(s) = 1$, $h(s) = 5$. 

\[ \begin{array}{ccc} 
\, & & \, \\
\, & s & \, \\
\, & & \, \\
\end{array} \]
Define
\[ n(\lambda) = \sum_{s \in \lambda} l(s) = \sum_{i \geq 1} \binom{\lambda_i}{2} = \sum_{i \geq 1} (i-1)\lambda_i, \] (9)

\[ \langle \lambda, \lambda \rangle = \sum_{i \geq 1} (\lambda'_i)^2 = 2n(\lambda) + |\lambda|. \] (10)

Define
\[ N_{\lambda}(u, q, t) = \prod_{s \in \lambda} (q^{a(s)} - ut^{l(s)+1})(q^{a(s)+1} - ut^{l(s)}). \] (11)

One can show that
\[ N_{\lambda}(u, q, t) = N_{\lambda'}(u, t, q). \] (12)

2.2. \( \lambda \)-rings and symmetric functions. For simplicity we will introduce only \( \lambda \)-rings without \( \mathbb{Z} \)-torsion. To make things even simpler we can assume that our rings are algebras over \( \mathbb{Q} \). The reason is that in this case the axioms of a \( \lambda \)-ring can be formulated in terms of Adams operations.

Define the graded ring of symmetric polynomials
\[ \Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}, \]
where \( \deg x_i = 1 \). Define the ring of symmetric functions \( \Lambda = \lim \Lambda_n \), where the limit is taken in the category of graded rings. For any commutative ring \( R \), define \( \Lambda_R = \Lambda \otimes \mathbb{Z} R \). As in [13], define generators of \( \Lambda \) (complete symmetric and elementary symmetric functions)
\[ h_n = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}, \quad e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}, \]
and generators of \( \Lambda_{\mathbb{Q}} \) (power sums)
\[ p_n = \sum_i x_i^n. \]

The elements \( h_n, e_n, p_n \) have degree \( n \). We also define \( h_0 = e_0 = p_0 = 1 \) for convenience. For any partition \( \lambda \) of length \( \leq n \), define monomial symmetric polynomials \( m_\lambda = \sum x^\alpha \in \Lambda_n \), where the sum runs over all distinct permutations \( \alpha = (\alpha_1, \ldots, \alpha_n) \) of \( (\lambda_1, \ldots, \lambda_n) \). They induce monomial symmetric functions \( m_\lambda \in \Lambda \) which form a \( \mathbb{Z} \)-basis of \( \Lambda \).

A \( \lambda \)-ring \( R \) is a commutative ring equipped with a pairing, called plethysm,
\[ \Lambda \times R \to R, \quad (f, a) \mapsto f \circ a = f[a], \]
such that with \( \psi_n = p_n[-] : R \to R \), called Adams operations, we have
\begin{enumerate}
  \item The map \( \Lambda \to R, \ f \mapsto f[a], \) is a ring homomorphism, for all \( a \in R \).
  \item \( \psi_1 : R \to R \) is an identity map.
  \item The map \( \psi_n : R \to R \) is a ring homomorphism, for all \( n \geq 1 \).
  \item \( \psi_m \psi_n = \psi_{mn} \), for all \( m, n \geq 1 \).
\end{enumerate}

Remark 2.1.
\begin{enumerate}
  \item The first axiom implies that it is enough to specify just Adams operations \( \psi_n \) or \( \sigma \)-operations \( \sigma_n = h_n[-] \) or \( \lambda \)-operations \( \lambda_n = e_n[-] \). It also implies that \( 1[a] = 1 \), for all \( a \in R \).
  \item Usually we equip algebras of the form \( \mathbb{Q}[x_1, \ldots, x_k], \mathbb{Q}(x_1, \ldots, x_k), \mathbb{Q}[x_1, \ldots, x_k] \) with a \( \lambda \)-ring structure by the formula
    \[ p_n[f(x_1, \ldots, x_k)] = f(x_1^n, \ldots, x_k^n). \]
\end{enumerate}
The ring $\Lambda$ can be itself equipped with a $\lambda$-ring structure using the same formula

$$p_m[f] = f(x_1^m, x_2^m, \ldots), \quad f \in \Lambda.$$ 

In particular $p_m[p_n] = p_{mn}$.

(4) If $R$ is a $\lambda$-ring, then $f \circ (g \circ a) = (f \circ g) \circ a$ for all $f, g \in \Lambda$ and $a \in R$.

The ring $\Lambda$ can be considered as a free $\lambda$-ring with one generator in the following sense. Consider the category $\text{Ring}_\lambda$ of $\lambda$-rings (with morphisms that respect plethystic operations). The forgetful functor $F: \text{Ring}_\lambda \to \text{Set}$ has a left adjoint

$$\text{Sym}: \text{Set} \to \text{Ring}_\lambda.$$ 

Given a finite set $\{X_1, \ldots, X_n\}$, we denote $\text{Sym} \{X_1, \ldots, X_n\}$ by $\text{Sym}[X_1, \ldots, X_n]$. Then, for a one-point set $\{X\}$, there is an isomorphism of $\lambda$-rings

$$\text{Sym}[X] \cong \Lambda, \quad X \mapsto p_1.$$ 

We will usually identify $\Lambda$ and $\text{Sym}[X]$ using this isomorphism.

Define a filtered $\lambda$-ring $R$ to be a $\lambda$-ring equipped with a filtration $R = F^0 \supset F^1 \supset \ldots$ such that $F^i F^j \subset F^{i+j}$ and $\psi_n(F^i) \subset F^{ni}$. It is called complete if the natural homomorphism $R \to \lim_{\leftarrow} R/F^i$ is an isomorphism. For example, the ring $\Lambda$ is graded, where $\deg h_n = n$. Hence we have a decomposition $\Lambda = \bigoplus_{k \geq 0} \Lambda^k$ into graded components. We equip $\Lambda$ with the filtration $F^k \Lambda = \bigoplus_{i \geq k} \Lambda^i$ and define the completion

$$\hat{\Lambda} = \lim_{\leftarrow} \Lambda/F^k \Lambda \simeq \mathbb{Z}[h_1, h_2, \ldots].$$ 

(13)

This ring can be considered as a free complete $\lambda$-ring with one generator. One can see that if $R$ is a complete $\lambda$-ring then the plethystic pairing extends to

$$\hat{\Lambda} \times F^1 R \to R.$$ 

In particular, the element

$$\text{Exp}[X] = \sum_{n \geq 0} h_n[X] = \exp \left( \sum_{n \geq 1} \frac{p_n[X]}{n} \right) = \prod_{i \geq 1} \frac{1}{1 - x_i} \in \hat{\Lambda},$$ 

(14)

called a plethystic exponential, induces a map $\text{Exp}: F^1 R \to 1 + F^1 R$ which satisfies

$$\text{Exp}[a + b] = \text{Exp}[a] \text{Exp}[b].$$ 

(15)

This map has an inverse, called a plethystic logarithm,

$$\text{Log}: 1 + F^1 R \to F^1 R, \quad \text{Log}[1 + a] = \sum_{n \geq 1} \frac{\mu(n)}{n} p_n[\text{log}(1 + a)].$$ 

(16)

2.3. Modified Macdonald polynomials. For an introduction to modified Macdonald polynomials see [8] or [15]. Let $P_n$ denote the set of partitions $\lambda$ with $|\lambda| = n$. Define the natural partial order on $P_n$ by

$$\lambda \preceq \mu \iff \sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i \quad \forall k \geq 1.$$ 

One can show that $\lambda \preceq \mu \iff \mu' \preceq \lambda'$ [13, 1.1.11]. Let $\Lambda^\lambda \subset \Lambda$ be the subspace spanned by monomial symmetric functions $m_\mu \in \Lambda$ with $\mu \preceq \lambda$. 


Let \( F = \mathbb{Q}(q,t) \) and \( \Lambda_F = \Lambda \otimes_{\mathbb{Z}} F \). For any symmetric function \( f \in \Lambda_F \), we will sometimes denote \( f[X] \) by \( f[X; q,t] \) to indicate dependence on \( q,t \). Let \( P_{\lambda}[X; q,t] \in \Lambda_F \) be Macdonald polynomials \([13, \S 6]\). Define modified Macdonald polynomials \( \widetilde{H}_\lambda[X; q,t] \in \Lambda_F \) \([8, I.8-I.11]\)

\[
\widetilde{H}_\lambda[X; q,t] = H_\lambda \left[ X; q, t^{-1} \right] \cdot t^n(\lambda), \quad H_\lambda[X] = P_\lambda \left[ \frac{X}{1-t} \right] \cdot \prod_{s \in \lambda} (1-q^s t^{l+1}).
\] (17)

Alternatively, one can uniquely determine \( \widetilde{H}_\lambda[X; q,t] \in \Lambda_F \) by the properties

(1) \( \widetilde{H}_\lambda[(1-t)X] \in \Lambda_F^\wedge\lambda \).

(2) **Cauchy identity:**

\[
\sum_{\lambda} \frac{\widetilde{H}_\lambda[X] \widetilde{H}_\lambda[Y]}{\prod_{s \in \lambda} (q^a - t^{l+1})(q^{a'} - t^l)} = \text{Exp} \left[ \frac{XY}{(q-1)(1-t)} \right].
\]

We have by \([8, \text{Cor. 2.1}]\) (see also \([13, 6.6.17]\))

\[
\widetilde{H}_\lambda[1-u; q,t] = \prod_{s \in \lambda} (1-q^{u}(s) t^{u(s)}),
\] (18)

where \( a'(s) = j - 1, t'(s) = i - 1 \) for \( s = (i,j) \in \lambda \). This implies \( \widetilde{H}_\lambda[1; q,t] = 1 \). The symmetric function \( \widetilde{H}_\lambda \) has degree \( |\lambda| \), hence, applying it to \( z \in F[z] \), we obtain

\[
\widetilde{H}_\lambda[z; q,t] = z^{|\lambda|}.
\] (19)

Finally, we have by \([8, \text{Cor. 2.2}]\)

\[
\widetilde{H}_\lambda[X; q,t] = \widetilde{H}_\lambda[X; t,q].
\] (20)

### 2.4 Volume ring.

Following \([17]\), we will introduce in this section a \( \lambda \)-ring which is an analogue of the Grothendieck ring of algebraic varieties or the ring of motives. We define it to be the ring \( \mathcal{V} = \prod_{n \geq 1} \mathbb{Q} \) with Adams operations

\[
\psi_m(a) = (a_{mn})_{n \geq 1}, \quad a = (a_n)_{n \geq 1} \in \mathcal{V},
\] (21)

and call it the volume ring or the ring of counting sequences \([17]\).

Given an algebraic variety \( X \) over a finite field \( \mathbb{F}_q \), define its volume

\[
[X] = (\#X(\mathbb{F}_{q^n}))_{n \geq 1} \in \mathcal{V}.
\] (22)

More generally, given a finite type algebraic stack \( \mathcal{X} \) over \( \mathbb{F}_q \), we define its volume

\[
[\mathcal{X}] = (\#\mathcal{X}(\mathbb{F}_{q^n}))_{n \geq 1} \in \mathcal{V},
\] (23)

where we define, for a finite groupoid \( \mathcal{G} = \mathcal{X}(\mathbb{F}_{q^n}) \),

\[
\# \mathcal{G} = \sum_{x \in \mathcal{G}/\sim} \frac{1}{\# \text{Aut}(x)}.
\] (24)

Next, let us fix a projective curve \( X \) over the field \( \mathbb{F}_q \) and consider its zeta function

\[
Z_X(t) = \exp \left( \sum_{n \geq 1} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n \right) = \prod_{i=1}^g (1-\alpha_i t)(1-\alpha_i^{-1} q t)/(1-t)(1-qt),
\] (25)

\[
\#X(\mathbb{F}_{q^n}) = 1 + q^n - \sum_{i=1}^g \alpha_i^n - q^n \sum_{i=1}^g \alpha_i^{-n} \quad \forall n \geq 1.
\] (26)
Consider the algebra
\[ R_g = \mathbb{Q}[q^{\pm 1}, \alpha_1^{\pm 1}, \ldots, \alpha_g^{\pm 1}, (q^n - 1)^{-1} : n \geq 1], \tag{27} \]
equipped with the usual \( \lambda \)-ring structure
\[ \psi_n(f) = f(q^n, \alpha_1^n, \ldots, \alpha_g^n) \quad \forall f \in R_g. \]
Consider an algebra homomorphism
\[ \sigma: R_g \to \mathbb{C}, \quad q \mapsto q, \ \alpha_i \mapsto \alpha_i, \]
and a \( \lambda \)-ring homomorphism
\[ \bar{\sigma}: R_g \to \mathcal{V}_\mathbb{C} = \prod_{n \geq 1} \mathbb{C}, \quad f \mapsto (\sigma(\psi_n(f)))_{n \geq 1}. \tag{28} \]
It restricts to an (injective) \( \lambda \)-ring homomorphism
\[ \bar{\sigma}: R_g^{S_g \times S_2} \to \mathcal{V}, \tag{29} \]
where \( S_g \) permutes variables \( \alpha_i \) and the \( i \)-th copy of \( S_2 \) permutes \( \alpha_i \) and \( q \alpha_i^{-1} \). Given elements \( a \in \mathcal{V} \) and \( f \in R_g \), we will write \( a = f \) if \( a = \bar{\sigma}(f) \). All equalities in this paper should be understood in this sense. For example \( [K^1] = (q^n)_{n \geq 1} = q \) and (26) implies
\[ [X] = 1 + q - \sum_{i=1}^g \alpha_i - q \sum_{i=1}^g \alpha_i^{-1}. \]
In what follows we will write \( q \) and \( \alpha_i \) instead of \( q \) and \( \alpha_i \) respectively, hoping it will not lead to confusion.

3. Positive Higgs bundles

In this section we will review the formula from [20] counting positive Higgs bundles. Then we will simplify it using an approach from [16]. Let \( X \) be a smooth projective curve of genus \( g \) over a field \( k \) and let \( L \) be a line bundle of degree \( \ell \) over \( X \). Given a coherent sheaf \( E \in \text{Coh} \ X \), we define its slope \( \mu(E) = \deg E / \text{rk} E \) and we call \( E \) semistable if \( \mu(F) \leq \mu(E) \) for all \( F \subset E \).

**Remark 3.1.** We call \( E \) stable if \( \mu(F) < \mu(E) \) for all proper \( F \subset E \). In this case \( K = \text{End}(E) \) is a finite-dimensional division algebra over \( k \) by Schur’s lemma. In particular, \( K = k \) if \( k \) is algebraically closed. If \( \text{rk} E, \deg E \) are coprime and \( E \) is semistable, then \( E \) is automatically stable. If, moreover, \( k \) is a finite field, then \( K = k \). Indeed, \( K \) is a finite (Galois) field extension of \( k \) by Wedderburn’s little theorem. We can decompose \( E_K = E \otimes_k K \) over \( X_K = X \times_{\text{Spec} k} \text{Spec} K \) as a direct sum \( \bigoplus_{\sigma \in \text{Gal}(K/k)} F^\sigma \), where \( F^\sigma \) have the same rank and degree [17]. But this would imply that \( \text{rk} E, \deg E \) are not coprime if \( [K:k] > 1 \).

Every coherent sheaf \( E \in \text{Coh} \ X \) has a unique filtration, called a Harder-Narasimhan filtration,
\[ 0 = E_0 \subset E_1 \subset \ldots \subset E_n = E \]
such that \( E_i/E_{i-1} \) are semistable and \( \mu(E_i/E_0) > \cdots > \mu(E_n/E_{n-1}) \). We will say that \( E \) is positive if \( \mu(E_n/E_{n-1}) \geq 0 \). Equivalently, for any semistable sheaf \( F \) with \( \mu(F) < 0 \), we have \( \text{Hom}(E, F) = 0 \).

Recall that an \( L \)-twisted Higgs sheaf is a pair \((E, \phi)\), where \( E \) is a coherent sheaf over \( X \) and \( \phi: E \to E \otimes L \) is a homomorphism. We will say that \((E, \phi)\) is positive if \( E \) is positive.
Let $\text{Higgs}_L(X)$ be the category of $L$-twisted Higgs sheaves and $\text{Higgs}_+^L(X)$ be the category of positive $L$-twisted Higgs sheaves. We will say that $(E, \phi) \in \text{Higgs}_L^+(X)$ is semistable if 

$$
\mu(F) \leq \mu(E) \quad \text{for every} \quad (F, \phi') \subset (E, \phi).
$$

Let $\mathcal{M}_L^+(r, d)$ denote the stack of semistable Higgs bundles and $\mathcal{M}_L^0(r, d)$ denote the stack of positive Higgs bundles (not necessarily semistable) having rank $r$ and degree $d$. Assuming that $k$ is a finite field $\mathbb{F}_q$, we define (exponential) DT invariants

$$
\hat{\Omega}_{r,d} = (-q^{\frac{1}{2}})^{-\ell r^2 |\mathcal{M}_L^+(r, d)|}
$$

and define (integral) DT invariants by the formula

$$
\sum_{d/r} \Omega_{r,d} T^r z^d = (q - 1) \log \left( \sum_{d/r} \hat{\Omega}_{r,d} T^r z^d \right), \quad \tau \in \mathbb{Q};
$$

On the other hand, consider the series

$$
\hat{\Omega}^+(T, z) = \sum_{r,d} (-q^{\frac{1}{2}})^{-\ell r^2 |\mathcal{M}_L^+(r, d)|} T^r z^d
$$

and define positive (integral) DT invariants by the formula

$$
\sum_{r,d} \Omega^+_{r,d} T^r z^d = (q - 1) \log \hat{\Omega}^+(T, z).
$$

The following result was proved in [20]:

**Theorem 3.2.** For every $r \geq 1$, we have

1. $\hat{\Omega}_{r,d+r} = \hat{\Omega}_{r,d}$.
2. $\Omega_{r,d+r} = \Omega_{r,d}$.
3. $\Omega_{r,d} = \Omega^+_{r,d}$ for $d \gg 0$.

The last result implies that it is enough to find the positive DT invariants $\Omega^+_{r,d}$ in order to determine the usual DT invariants $\Omega_{r,d}$. The following explicit formula for the series $\hat{\Omega}^+(T, z)$ was proved in [20] (although the power of $z$ was missing there).

**Theorem 3.3.** Assuming that $p = \ell - (2g - 2) > 0$, we have

$$
\hat{\Omega}^+(T, z) = \sum_{\lambda} (-q^{\frac{1}{2}})^{\ell(\lambda, \lambda)} z^{m(\lambda)} J_\lambda(z) H_\lambda(z) T^{|\lambda|},
$$

where the sum runs over all partitions $\lambda$ and $J_\lambda(z), H_\lambda(z)$ are certain complicated expressions defined in [20].

The following simplification of the above expression was obtained in [16, Prop. 3.1].

**Proposition 3.4.** For every partition $\lambda$ of length $n$ define

$$
f(z_1, \ldots, z_n; q, \alpha) = \prod_{i=1}^{n} \prod_{k=1}^{g} \frac{1 - \alpha_k^{-1}}{1 - \alpha_k^{-1} z_i} \prod_{i>j} \left( \frac{1}{1 - z_i/z_j} \prod_{k=1}^{g} \frac{1 - \alpha_k^{-1} z_i/z_j}{1 - q \alpha_k^{-1} z_i/z_j} \right) \prod_{i \geq 2} (1 - z_i), \quad \sigma \in S_n
$$

$$
f_\lambda = f(z_1, \ldots, z_n; q, \alpha), \quad z_i = q^{i-n} z^{\lambda_i}, \quad i = 1, \ldots, n.
$$
where $\bar{\alpha} = (\alpha_1, \ldots, \alpha_g)$. Then (see (11) for the definition of $N_\lambda$)
\begin{equation}
q^{(s-1)(\lambda, \lambda)} J_\lambda(z) H_\lambda(z) = \frac{\prod_{i=1}^{g} N_\lambda(\alpha_i^{-1}, z, q)}{N_\lambda(1, z, q)} f_\lambda.
\end{equation}

The last two results imply

**Corollary 3.5.** Assuming that $p = \ell - (2g - 2) > 0$, we have
\begin{equation}
\hat{\Omega}^+(q^{-p/2}T, z) = \sum_{\lambda} \left((1)^{\lambda} q^{n(\lambda)} z^{n(\lambda)}\right)^p \frac{\prod_{i=1}^{g} N_\lambda(\alpha_i^{-1}, z, q)}{N_\lambda(1, z, q)} f_\lambda \cdot (q^{p/2}T)^{|\lambda|}.
\end{equation}

**Proof.** Using the fact that $\langle \lambda, \lambda \rangle = 2n(\lambda) + |\lambda|$ (see (10)), we obtain

\begin{equation}
\hat{\Omega}^+(T, z) = \sum_{\lambda} \left((1)^{\lambda} q^{n(\lambda)} z^{n(\lambda)}\right)^p \frac{\prod_{i=1}^{g} N_\lambda(\alpha_i^{-1}, z, q)}{N_\lambda(1, z, q)} f_\lambda \cdot (q^{p/2}T)^{|\lambda|}.
\end{equation}

Now we sum over conjugate partitions and apply (12). \qed

**Lemma 3.6.** We have
\[
f \in \mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}; q^{\pm 1}][\alpha_1^{-1}, \ldots, \alpha_g^{-1}].
\]

**Proof.** The factors $(1 - z_i/z_j)$ disappear from the denominator of $f$ when we sum over $S_n$, so looking at the remaining factors we see that
\[
f(z_1, \ldots, z_n) \cdot \prod_{k=1}^{g} \left( \prod_{i=1}^{n} (1 - \alpha_k^{-1} z_i) \prod_{i \neq j} (1 - q\alpha_k^{-1} z_i/z_j) \right)
\]
is a Laurent polynomial. The result follows on observing that every factor in the brackets is invertible in $\mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}; q^{\pm 1}][\alpha_1^{-1}, \ldots, \alpha_g^{-1}]$. \qed

**Proposition 3.7** (see [16, §4.2]). We have
\[
f(1, z_1, \ldots, z_n) = f(qz_1, \ldots, qz_n).
\]

4. MAIN RESULT

4.1. Admissibility. Let $R$ be a $\lambda$-ring flat over $\mathbb{Q}(q)[t^{\pm 1}]$ and let $R^* = R \otimes_{\mathbb{Q}(q)[t^{\pm 1}]} \mathbb{Q}(q, t)$. We will say that $F \in R^*$ is admissible if $(1 - t) \log F$ is contained in $R$ (usually $R$ will be clear from the context). In view of Proposition 3.7, we introduce the following concept.

**Definition 4.1.** Let $q \in R$ be an invertible element. For every $n \geq 0$, consider rings $\bar{\Lambda}_n = R[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]^{S_n}$ and ring homomorphisms
\[
\pi_n: \bar{\Lambda}_{n+1} \to \bar{\Lambda}_n, \quad (\pi_n f)(z_1, \ldots, z_n) = f(1, q^{-1}z_1, \ldots, q^{-1}z_n).
\]

Define a $q$-twisted symmetric function $f = (f_n)_{n \geq 0}$ to be an element of $\bar{\Lambda} = \varprojlim \bar{\Lambda}_n$.

Given a $q$-twisted symmetric function $f$, define for any partition $\lambda$ (cf. (35))
\begin{equation}
f_\lambda = f_n(z_1, \ldots, z_n), \quad z_i = q^{-n} t^\lambda_i, \quad n \geq l(\lambda).
\end{equation}

Note that this expression is independent of the choice of $n \geq l(\lambda)$.

**Remark 4.2.** The following result is a reformulation of [16, Lemma 5.1]. Here we exchange the roles of $q, t$ and use conjugate partitions. We also add an invertible factor $(q - 1)$. 
Theorem 4.3. Let \( f(u) = \sum_{i \geq 0} f^{(i)} u^i \in \mathbb{A}[u] \) be a power series with \( f^{(0)} = 1 \) and let
\[
\hat{\Omega}[X; u] = \sum_{\lambda} c_{\lambda} \tilde{H}_\lambda[X; q, t] f_{\lambda'}(u), \quad \Omega[X; u] = (q - 1)(1 - t) \log \hat{\Omega}[X; u],
\]
where \( c_{\lambda} \in \mathbb{R}^* \) and \( c_\phi = 1 \). If \( \hat{\Omega}[X; 0] \) is admissible, then \( \Omega[X; u] - \Omega[X; 0] \) has coefficients in \((t - 1)R\). In particular, \( \Omega[X; u] \) is independent of \( u \) at \( t = 1 \).

4.2. Proof of the main theorem. Now we are ready to prove Theorem 1.1 from the introduction. For this section we will use the variable \( t \) in place of \( z \) as it is customary in the theory of orthogonal symmetric polynomials.

Theorem 4.4 (cf. Theorem 1.1). Assume that \( p = \ell - (2g - 2) > 0 \). Define (see (11) for the definition of \( N_\lambda \))
\[
\hat{\Omega}^p(T, q, t) = \sum_{\lambda} \left( (-1)^{|\lambda|} q^{n(\lambda)} t^{n(\lambda)} \right)^p \frac{\prod_{i=1}^g N_\lambda(\alpha_i^{-1}, q, t)}{N_\lambda(1, q, t)} T^{|\lambda|}, \tag{39}
\]
\[
\Omega^p(T, q, t) = \sum_{r \geq 1} \Omega_r^+(q, t) T^r = (q - 1)(1 - t) \log \hat{\Omega}^p(T, q, t). \tag{40}
\]
Then \( \Omega_r^+(q, t) \in \mathbb{Z}[q, t, \alpha_1^\pm, \ldots, \alpha_g^\pm] \) and
\[
\Omega_{r,d} = q^{pr/2} \Omega_r^+(q, 1) \quad \forall d \in \mathbb{Z}.
\]

Proof. According to Theorem 3.2 it is enough to show that \( \Omega_{r,d}^+ = q^{pr/2} \Omega_r^+(q, 1) \) for \( d \gg 0 \), where \( \Omega_{r,d}^+ \) are determined by (33) and Corollary 3.5:
\[
\hat{\Omega}^+(q^{-p/2} T, q, t) = \sum_{\lambda} \left( (-1)^{|\lambda|} q^{n(\lambda)} t^{n(\lambda)} \right)^p \frac{\prod_{i=1}^g N_\lambda(\alpha_i^{-1}, q, t)}{N_\lambda(1, q, t)} f_{\lambda'} T^{|\lambda|}, \tag{41}
\]
\[
\Omega^+(T, q, t) = \sum_r \Omega_r^+(q, t) T^r = \sum_{r, d} \Omega_{r,d}^+ T^r t^d = (q - 1)(1 - t) \log \hat{\Omega}^+(T, q, t). \tag{42}
\]

We will compare the series \( \hat{\Omega}^+(q^{-p/2} T, q, t) \) to the series \( \hat{\Omega}^p(T, q, t) \) using Theorem 4.3 with the ring of Laurent series
\[
R = \mathbb{Q}(q)[t^{\pm 1}](\alpha_1^{-1}, \ldots, \alpha_g^{-1})
\]
and the series \( \hat{f}(u) = \sum_{i \geq 0} \hat{f}^{(i)} u^i \) which is a deformation of \( f \) (34) defined by
\[
\hat{f}^{(i)} = (\hat{f}_n^{(i)})_{n \geq 0}, \quad \hat{f}_n(z_1, \ldots, z_n; u) = \sum_{i \geq 0} \hat{f}_n^{(i)} u^i = f(z_1, \ldots, z_n; q, u^{-1} a),
\]
where every \( \alpha_i \) is substituted by \( u^{-1} \alpha_i \). It follows from Lemma 3.6 that
\[
\hat{f}_n \in \mathbb{Q}[q^{\pm 1}, \alpha_1^{\pm 1}, \ldots, \alpha_g^{\pm 1}][z_1^{\pm 1}, \ldots, z_n^{\pm 1}] S_n[u],
\]
and hence by Proposition 3.7 the coefficients \( \hat{f}^{(i)} \) are \( q \)-twisted symmetric functions over \( R \). It follows from [16, Theorem 5.2] that \( \hat{f}_n|_{u=0} = 1 \), hence \( \hat{f}(0) = 1 \).

As before, define
\[
\tilde{f}_{\lambda}(u) = \tilde{f}_n(z_1, \ldots, z_n; u), \quad z_i = q^{-n} t^{\lambda_i}, \quad n \geq l(\lambda),
\]
and consider the series of symmetric functions
\[
\hat{\Omega}[X; u, q, t] = \sum_{\lambda} \left( (-1)^{|\lambda|} q^{n(\lambda)} t^{n(\lambda)} \right)^p \frac{\prod_{i=1}^g N_\lambda(\alpha_i^{-1}, q, t)}{N_\lambda(1, q, t)} \tilde{H}_\lambda[X; q, t] \tilde{f}_{\lambda'}(u), \tag{44}
\]
\[
\Omega[X; u, q, t] = (q - 1)(1 - t) \log \hat{\Omega}[X; u, q, t]. \tag{45}
\]
Then (39) and (41) translate to

\[
\hat{\Omega}[T; 0, q, t] = \hat{\Omega}^\circ(T, q, t), \quad \Omega[T; 0, q, t] = \Omega^\circ(T, q, t), \\
\hat{\Omega}[T; 1, q, t] = \hat{\Omega}^+(q^{-p/2}T, q, t), \quad \Omega[T; 1, q, t] = (1 - t)\Omega^+(q^{-p/2}T, q, t).
\]

In order to apply Theorem 4.3 we need to show that

\[
\hat{\Omega}[X; 0, q, t] = \sum_\lambda \left((-1)^{|\lambda|}q^{n(X)} t^{n(\lambda)}\right) \frac{\prod_{i=1}^p N_\lambda(\alpha_i^{-1}, q, t)}{N_\lambda(1, q, t)} \tilde{H}_\lambda[X; q, t]
\]
is admissible. The series

\[
\sum_\lambda \frac{\prod_{i=1}^p N_\lambda(\alpha_i^{-1}, q, t)}{N_\lambda(1, q, t)} \tilde{H}_\lambda[X; q, t]
\]
is admissible according to [14]. The operator \(\nabla\) defined by

\[
\tilde{H}_\lambda \mapsto (-1)^{|\lambda|}q^{n(X)} t^{n(\lambda)} \tilde{H}_\lambda
\]
preserves admissibility by [14, Cor. 6.3]. Therefore the series \(\hat{\Omega}[X; 0, q, t]\) is also admissible (one actually obtains from [14] that the coefficients of \(\Omega[X; 0, q, t]\) are in \(\mathbb{Z}[q, t, \alpha_1^{\pm 1}, \ldots, \alpha_g^{\pm 1}]\), hence the same is true for \(\Omega^\circ(T, q, t)\)).

We conclude from Theorem 4.3 that

\[
\Omega[T; u, q, t] - \Omega[T; 0, q, t] \in (1 - t)R[T, u]. \quad (46)
\]

By Lemma 3.6 we can consider \(\hat{\Omega}[T; u, q, t]\) (44) as a series with polynomial coefficients in \(u\)

\[
\hat{\Omega}[T; u, q, t] \in \mathbb{Q}(q, t)[u][\langle \alpha_1^{-1}, \ldots, \alpha_g^{-1} \rangle][T].
\]
The same then applies to \(\Omega[T; u, q, t]\) and we can set \(u = 1\) in (46). We obtain

\[
(1 - t)\Omega^+(q^{-p/2}T, q, t) - \Omega^\circ(T, q, t) \in (1 - t)R[T].
\]
This implies that \((1 - t)q^{-pr/2}\Omega^+_r(q, t) - \Omega^\circ_r(q, t) = (1 - t)g\) for some \(g \in R\). Therefore

\[
q^{-pr/2} \sum_{d \geq 0} \Omega^+_r d^d = \frac{\Omega^\circ_r(q, t)}{1 - t} + g.
\]
Comparing the coefficients of the monomials in \(\alpha_1, \ldots, \alpha_g\) and using the fact that \(\Omega^+_r(q, 1) = \Omega^+_r(q, 1)\) for \(d \gg 0\), we conclude that \(q^{-pr/2}\Omega^+_r \rightarrow \Omega^+_r(q, 1)\) for \(d \gg 0\). \(\Box\)

**Remark 4.5.** Let us also formulate the result in the case \(L = \omega_X\) (the canonical bundle) for completeness [16]. In this case we have \(\ell = 2g - 2\) and \(p = \ell - (2g - 2) = 0\). Define as before

\[
\hat{\Omega}^\circ(T, q, t) = \sum_\lambda \prod_{i=1}^p N_\lambda(\alpha_i^{-1}, q, t) T^{\lambda} \quad (47)
\]

\[
\Omega^\circ(T, q, t) = \sum_{r \geq 1} \Omega^\circ_r(q, t) T^r = (q - 1)(1 - t) \log \hat{\Omega}^\circ(T, q, t). \quad (48)
\]

Using results of [20] and the same proof as before, we obtain the formula for integral Donaldson-Thomas invariants \(\Omega_{r,d} = q\Omega^\circ_r(q, 1)\) (note the additional factor \(q\)). These invariants are related to the invariants \(A_{r,d}\) counting absolutely indecomposable vector bundles of rank \(r\) and degree \(d\) over \(X\): \(\Omega_{r,d} = qA_{r,d}\) [20]. This implies \(A_{r,d} = \Omega^\circ_r(q, 1)\), as was proved by Mellit [16].
4.3. Alternative formulation. The following result was conjectured in [18, Conj. 3].

**Theorem 4.6.** Assume that \( p = \ell - (2g - 2) > 0 \). Consider the series

\[
\mathcal{H}(T, q, t) = \sum_{\lambda} T^{[\lambda]} \prod_{s \in \lambda} (-t^{-a(s)}q^{a(s)})^{p_g(1-g)(2l+1)} Z_X(t^{h(s)}q^{a(s)}),
\]

\[
\mathcal{H}^p(T, q, t) = \sum_{r \geq 1} \mathcal{H}_r^p(q, t) T^r = (1 - t)(1 - qt) \log \mathcal{H}(T, q, t).
\]

Then \( \mathcal{H}_r^p(q, t) \in \mathbb{Z}[q, t, \alpha_1, \ldots, \alpha_g] \) and \( \Omega_{r,d} = q^{pr/2} \mathcal{H}_r^p(q, t) \).

**Proof.** Using the substitution \( t \mapsto t^{-1} \), we obtain

\[
\mathcal{H}(T, q, t^{-1}) = \sum_{\lambda} T^{[\lambda]} \prod_{s \in \lambda} (-t^{-a}q^a)^p l^{(g-1)(2l+1)} Z_X(t^{-h}q^a)
\]

\[
= \sum_{\lambda} T^{[\lambda]} \prod_{s \in \lambda} (-t^{-a}q^a)^p \prod_{i=1}^{g} \frac{(t^{l+1} - \alpha_i^{-1} pq^a)(t^l - \alpha_i^{-1} t^{-a-1}q^a)}{(t^{l+1} - t^{-a}q^a)(t^l - t^{-a}q^a)},
\]

while

\[
t \mathcal{H}^p(T, q, t^{-1}) = (1 - t)(t^{-1}q - 1) \log \mathcal{H}(T, q, t^{-1}).
\]

Using the substitution \( q \mapsto qt \), we obtain

\[
\mathcal{H}(T, qt, t^{-1}) = \sum_{\lambda} T^{[\lambda]} \prod_{s \in \lambda} (-t^aq^a)^p \prod_{i=1}^{g} \frac{(t^{l+1} - \alpha_i^{-1} q^a)(t^l - \alpha_i^{-1} t^{-1}q^a)}{(t^{l+1} - q^a)(t^l - q^a)}
\]

\[
= \sum_{\lambda} T^{[\lambda]} \left( (-1)^{|\lambda|} q^{n(\lambda)} t^{n(\lambda)} \right)^p \prod_{i=1}^{g} N_{\lambda}(\alpha_i^{-1}, q, t) \frac{N_{\lambda}(1, q, t)}{N_{\lambda}(1, q, t)}.
\]

Now the result follows from Theorem 4.4. \( \square \)

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