Free particles from Brauer algebras in complex matrix models

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QMUL

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Based on arXiv:0911.4408 with Yusuke Kimura and Sanjaye Ramgoolam
Motivations

1. Half-BPS sector of $\mathcal{N} = 4$ super Yang-Mills: holomorphic, $U(N)$ singlet sector of a free $N \times N$ complex matrix model.

2. Description in terms of $N$ free fermions - eigenvalues

Corley, Jevicki, Ramgoolam
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   Balasubramanian, de Boer, Jejjala, Simon
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5. Free particle descriptions in other sectors? Non-holomorphic sectors?
Overview

1. Review of free particles in matrix models and AdS/CFT
2. Introduction to Brauer algebra basis
3. Emergence of free particles in complex matrix models
4. Counting and stringy exclusion principle
5. Open Questions
Free particles in unitary matrix quantum mechanics

Consider the free Unitary matrix quantum mechanics with Hamiltonian

$$H = \text{tr} \left( U \frac{\partial}{\partial U} \right)^2$$

- $U(N)$ symmetry $U \rightarrow gUg^\dagger$, \hspace{0.5cm} $g \in U(N)$.
- An orthonormal basis of $U(N)$ invariant wavefunctions is given by $U(N)$ characters.
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- An orthonormal basis of **U(N)** invariant wavefunctions is given by **U(N)** characters.

**U(N)** representations built from tensor products of the fundamental are specified by a Young diagram \( R \) with \( c_1(R) \leq N \) and their characters are the Schur polynomials:

\[ \chi_R(U) = \sum_{\sigma \in S_n} \chi_R(\sigma) U_{i_{\sigma_1}}^{i_1} \cdots U_{i_{\sigma_n}}^{i_n}. \]

The same is true for tensor copies of the antifundamental with \( U \leftrightarrow U^\dagger \).
More general representations are specified by a composite Young diagram \((R, \bar{S})\), where

- \(R\) controls the fundamental indices
- \(S\) controls the antifundamental indices

For \(U(N)\) (and everywhere in this talk) a composite Young diagram has \(N\) rows so we require \(c_1(R) + c_1(S) \leq N\):
Use symmetry \( U \rightarrow gUg^\dagger \) to diagonalise \( U \):

\[
U = gDg^\dagger, \quad D = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_N}), \quad g \in U(N).
\]

This introduces jacobian \( \Delta(u) = \prod_{i<j}(e^{i\theta_i} - e^{i\theta_j}) \).
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This introduces jacobian $\Delta(u) = \prod_{i<j}(e^{i\theta_i} - e^{i\theta_j})$.

The Hamiltonian becomes

$$H = -\sum_i \left[ \frac{1}{\tilde{\Delta}} \frac{d^2}{d\theta_i^2} \tilde{\Delta} \right] - \frac{1}{12} N(N^2 - 1) + \text{off-diag},$$

where

$$\tilde{\Delta} = \prod_{i<j} \sin \frac{\theta_i - \theta_j}{2} = \frac{\Delta(u)}{(\det U)^{\frac{N-1}{2}}}.$$

Douglas '93
Free particles in unitary matrix quantum mechanics

Absorb $\tilde{\Delta}$ into wavefunctions and Hamiltonian:

$$\psi_f = \tilde{\Delta} \psi, \quad H_f = \tilde{\Delta} H \frac{1}{\tilde{\Delta}} = \sum_i \frac{\partial}{\partial \theta_i^2} - \frac{1}{12} N(N^2 - 1)$$

Wavefunctions $\psi_f$ antisymmetric under exchange of any pair $\theta_i \leftrightarrow \theta_j$. 

Singlet eigenfunctions are Slater determinants - $N$-fermion wavefunctions, $\psi_p = \det_{j,k} e^{i \theta_j p_k}$ which are related to Schur polynomials via $\Psi_p = \Delta(u) \chi_R(U)$ where if $r_j$ are the rows of the Young diagram $R$, $p_j = r_j + (n_F + 1 - j)$, $n_F = \frac{N-1}{2}$. This sector is thus equivalent to $N$ free fermions on a circle.

Douglas '93
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$$p_j = r_j + (n_F + 1 - j), \quad n_F = \frac{N - 1}{2}.$$
Fermions on a circle

- States of fermion on a circle: quantised momentum $p \in \mathbb{Z}$
- Energy $E = p^2$
- $N$ fermions: Fermi sea with two Fermi levels.
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- Energy $E = p^2$
- $N$ fermions: Fermi sea with two Fermi levels.

- Excitations labelled by composite Young diagram $(R, \bar{S})$:
  length of row $j$ is excitation energy of fermion $j$
- Natural interpretation of $c_1(R) + c_1(S) \leq N$. 
Free particles in hermitian matrix models

Consider the Gaussian hermitian matrix quantum mechanics with Lagrangian

$$\mathcal{L} = \text{tr} \left( \frac{1}{2} \dot{\Phi}^2 - \frac{1}{2} \Phi^2 \right)$$

which is invariant under the global $U(N)$ action

$$\Phi \rightarrow g \Phi g^\dagger, \quad g \in U(N).$$
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$$
\Phi \rightarrow g \Phi g^\dagger, \quad g \in U(N).
$$

Decompose $\Phi$ into diagonal and off-diagonal d.o.f.:

$$
\Phi = U \Lambda U^\dagger, \quad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N), \quad U \in U(N).
$$

The jacobian is $\Delta = \prod_{i<j}(\lambda_i - \lambda_j)$ and the Hamiltonian becomes

$$
H_\Lambda = \frac{1}{2} \sum_i \left( -\frac{1}{\Delta} \frac{\partial^2}{\partial \lambda_i^2} \Delta + \lambda_i^2 \right) + \text{off-diag},
$$
Absorb $\Delta$ into wavefunctions and Hamiltonian:

$$
\psi^f(\lambda) = \Delta \psi(\lambda)
$$

$$
H^f = \Delta H \frac{1}{\Delta} = \frac{1}{2} \sum_i \left( -\frac{d^2}{d\lambda_i^2} + \lambda_i^2 \right)
$$
Free particles in hermitian matrix models

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Singlet eigenfunctions are Slater determinants - $N$ fermion wavefunctions,

$$
\psi^f_{\vec{E}} = \det_{i,j} \lambda_i \mathcal{E}_j e^{-\frac{1}{2} \text{tr} \Phi^2}
$$

which are related to Schur polynomials as in the UMM via

$$
\psi^f_{\vec{E}} = \Delta \mathcal{O}_R(\Phi) e^{-\frac{1}{2} \text{tr} \Phi^2}, \quad \mathcal{E}_i = r_i + (N - i)
$$

where $r_i$ are the rows of the Young diagram $R$. 
Fermions in 1D SHO

- States of SHO: \((n + \frac{1}{2})\hbar\)
- Ground state of \(N\) fermion system: Fermi sea
Fermions in 1D SHO

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Excitations labelled by single Young diagram: length of row $j$ is excitation energy of fermion $j$

- Natural interpretation of $c_1(R) \leq N$
\( \mathcal{N} = 4 \) SYM

Gauge group \( U(N) \)

't Hooft coupling

\[ \lambda = g_{YM}^2 N \]

\[ \sqrt{\lambda} \longleftrightarrow \frac{L^2}{\alpha'} \]

\[ \frac{\lambda}{N} \longleftrightarrow g_s \]

- **Strong form of conjecture:** equivalence for all \( \lambda, N \).
- **This talk:** \( \lambda = 0, \ N \) finite.
Field Content of $\mathcal{N} = 4$ SYM

- Gauge field, 4 Weyl fermions
- 3 Complex scalars $X, Y, Z$
- All fields in adjoint of $U(N)$.

Restrict attention to one complex scalar - say $Z$.

Holomorphic polynomials in $Z$ are $\frac{1}{2}$-BPS operators: they preserve half of the supersymmetries.

- Relevant part of the Lagrangian is:

$$\mathcal{L}_Z = \text{tr} \left( D_\mu Z^\dagger D^\mu Z \right)$$
Consider $\mathcal{N} = 4$ SYM on $S^3 \times \mathbb{R}$:

- $\frac{1}{2}$-BPS states correspond to s-wave modes.
- Fields $Z(t), A_0(t)$.
- $A_0$ non-dynamical - imposes Gauss’s Law (gauge invariance).
Spherical Harmonics and Dimensional Reduction

Consider $\mathcal{N} = 4$ SYM on $S^3 \times \mathbb{R}$:

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Dimensionally reduced Lagrangian: extra term from conformal coupling to curvature of $S_3$ - absorbing constants, this becomes

$$\mathcal{L} = \text{tr} \left( \dot{Z} \dot{Z}^\dagger - ZZ^\dagger \right)$$

$\rightarrow U(N)$ singlet sector of complex matrix quantum mechanics in a simple harmonic oscillator potential.

Hashimoto '00, Corley, Jevicki, Ramgoolam '01
Schur polynomials (yet again)

The Schur polynomials generalised to a complex matrix, $\mathcal{O}_R(\Phi)$ are polynomials of degree $n$ labelled by a representation $R$ of $S_n$, where the first column of $R$ has length at most $N$:

$$\mathcal{O}_R(\Phi) = \sum_{\sigma \in S_n} \chi_R(\sigma) \Phi_{i_{\sigma_1}}^1 \cdots \Phi_{i_{\sigma_n}}^n$$

In this basis the two-point function is diagonal:

$$\left\langle \mathcal{O}_R(\Phi)^\dagger \mathcal{O}_S(\Phi) \right\rangle = f_R \delta_{RS}$$

Corley, Jevicki, Ramgoolam '01
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Corley, Jevicki, Ramgoolam '01

At $n = 2$ the Schur polynomials are

$$\mathcal{O}_{[2]}(\Phi) = \frac{1}{2} (\text{tr } \Phi \text{ tr } \Phi + \text{tr } \Phi^2)$$

$$\mathcal{O}_{[1^2]}(\Phi) = \frac{1}{2} (\text{tr } \Phi \text{ tr } \Phi - \text{tr } \Phi^2)$$

where $[2] = \begin{array}{c} \end{array}$ (symmetric), $[1^2] = \begin{array}{c} \end{array}$ (antisymmetric).
$U(N)$ not sufficient to diagonalise $Z$; use Schur Triangularisation:

$$Z = U T U^\dagger, \quad U \in U(N), \quad T \text{ upper triangular.}$$
Schur Triangularisation

$U(N)$ not sufficient to diagonalise $Z$; use Schur Triangularisation:

$$Z = UTU^\dagger, \quad U \in U(N), \quad T \text{ upper triangular}.$$

- $z_i$: diagonal entries - eigenvalues of $Z$
- $t_{jk}$: off-diagonal entries for $j < k$.

Since

$$\text{tr} \ Z^p = \text{tr} \ T^p = \sum_i z_i^p$$

the holomorphic GIOs are symmetric polynomials in the $z_i$, related to Schur polynomials as for the Hermitian matrix model.
Change of variables

Since the holomorphic GIOs are symmetric polynomials in the $z_i$, change variables

$$Z_{ij} \rightarrow \{z_i, t_{jk}, U\}$$
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Since the holomorphic GIOs are symmetric polynomials in the $z_i$, change variables

$$Z_{ij} \rightarrow \{z_i, t_{jk}, U\}$$

- Jacobian $\Delta = \prod_{j<k} (z_j - z_k)$
- Absorb $\Delta$ into wavefunctions - interpret the $z_i$ as fermions.

Fermions $z_i$ are complex - target space is a plane. Holomorphic dynamics is effectively one-dimensional. Thus the holomorphic, $U(N)$ singlet sector of the matrix SHO quantum mechanics is equivalent to a theory of $N$ fermions in a 1D SHO potential.
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Fermion phase space and LLM

- SHO Fermions on $\mathbb{R}$ have a 2D phase space plane
- Quantize: each fermion occupies area $\hbar$
- System occupies area $N\hbar$ of phase space

So phase space configurations are colourings of the plane into black/white regions.
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$\frac{1}{2}$-BPS solutions to IIB supergravity with $SO(4) \times SO(4)$ isometry: (LLM)

- Coordinates $t, y, x_1, x_2, S^3, \tilde{S}^3$
- Geometries determined by function $u(x_1, x_2)$
- Smoothness condition: $u(x_1, x_2) = 0$ or $1$
- $x_1 - x_2$ plane identified with fermion phase space above.

Lin, Lunin, Maldacena ’04
Beyond the Holomorphic Sector

- So far: $U(N)$ singlet, holomorphic sector of Complex Matrix Model
- Natural extension: relax holomorphic constraint
- GIIOs now functions of $Z, Z^\dagger$, equivalently $z_i, t_{jk}$ - this takes us beyond eigenvalues.
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New features:
1. This sector is non-BPS - no non-renormalisation theorems.
2. However at zero coupling, $Z, Z^\dagger$ sector remains a consistent truncation of $\mathcal{N} = 4$ SYM
3. Is there a string dual of this sector at zero coupling?
The symmetric group $S_n$ in diagrams

Symmetric group elements may be represented by diagrams:

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3
\end{array}
\]

= \ (123)

Products are obtained by stacking diagrams: e.g.

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3
\end{array}
\]

= \ \begin{array}{c}
1 \\
1 \\
2 \\
3
\end{array}
\begin{array}{c}
1 \\
2 \\
3
\end{array}

represents the product \((12)(123) = (23)\).
The (walled) Brauer algebra $B_N(m, n)$ contains the group algebra of $S_m \times S_n$ along with ‘contraction’ elements, which cross a wall:

$$
\begin{array}{ccc}
1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow \\
1 & 2 & 3 \\
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{ccc}
\bar{1} & \bar{2} \\
\uparrow & \uparrow \\
1 & 2 \\
\end{array}
= C_{3\bar{1}}
$$
The (walled) Brauer algebra $B_N(m, n)$ contains the group algebra of $S_m \times S_n$ along with ‘contraction’ elements, which cross a wall:

\[
\begin{array}{cccc}
1 & 2 & 3 & 1 \\
1 & 2 & 3 & 1 \\
\end{array}
\]

along with the rule that in a product, a closed loop is replaced by multiplication by the parameter $N$:

\[
\begin{array}{cccc}
1 & 2 & 3 & 1 2 \\
1 & 2 & 3 & 1 2 \\
\end{array}
\] = $N$

which represents the product $C_{3\bar{1}}[(12)C_{3\bar{1}}] = N(12)C_{3\bar{1}}$. 

David Turton (QMUL)
Free particles from Brauer algebras
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Application to GIOs

The index structure of $\text{tr} \, ZZ^\dagger$ can be represented diagrammatically using $Z$ and $Z^\dagger$, using a symmetric group element and a trace:

$$Z \, Z^\dagger = Z^i_j \, Z^\dagger_j^i = \text{tr} \, ZZ^\dagger$$

Alternatively, $\text{tr} \, ZZ^\dagger$ can be represented with $Z$ and $Z^*$, using a Brauer algebra contraction and a trace:

$$Z \, Z^* = Z^i_j \, Z^*_{j^i} = Z^i_j \, Z^\dagger_j^i = \text{tr} \, ZZ^\dagger$$
More generally, any GIO may be written using $Z, Z^*$ and a Brauer algebra element $b$ as

$$Z^m Z^{*n} b = \text{tr} \left( b Z^m Z^{*n} \right)$$
Brauer basis of operators

The Brauer algebra can be used to build an orthogonal basis, as follows:

- The representations of the Brauer algebra are labelled by \( \gamma = (k, \gamma_+, \gamma_-) \) where
  - \( k \) is an integer in the range \( 0 \leq k \leq \min(m, n) \)
  - \( (\gamma_+, \gamma_-) \) have \( m - k \) and \( n - k \) boxes respectively and form a composite Young diagram; \( c_1(\gamma_+) + c_1(\gamma_-) \leq N \)
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  - $k$ is an integer in the range $0 \leq k \leq \min(m, n)$
  - $(\gamma_+, \gamma_-)$ have $m - k$ and $n - k$ boxes respectively and form a form a composite Young diagram; $c_1(\gamma_+) + c_1(\gamma_-) \leq N$
- A representation $\gamma$ can be decomposed into irreps $A = (\alpha, \beta)$ of the $\mathbb{C}[S_m \times S_n]$ sub-algebra, where $\alpha \vdash m, \beta \vdash n$. 
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  - \( k \) is an integer in the range \( 0 \leq k \leq \min(m, n) \)
  - \((\gamma_+, \gamma_-)\) have \( m - k \) and \( n - k \) boxes respectively and form a form a composite Young diagram; \( c_1(\gamma_+) + c_1(\gamma_-) \leq N \)

- A representation \( \gamma \) can be decomposed into irreps \( A = (\alpha, \beta) \) of the \( \mathbb{C}[S_m \times S_n] \) sub-algebra, where \( \alpha \vdash m, \beta \vdash n \).

- Let the irrep \( A \) appear with multiplicity \( M^\gamma_A \), let \( i \) run over this multiplicity and let \( |\gamma; A, m_A; i\rangle \) be the state in the representation \( \gamma \) which transforms in the \( i \) th copy of the state \( m_A \) of the irrep \( A \) of the sub-algebra.
Brauer basis of operators

The Brauer basis, formed of particular linear combinations of such traces, is a generalisation of the Schur Polynomials to $Z, Z^\dagger$ operators,

$$O^\gamma_{\alpha\beta;ij}(Z, Z^\dagger) = \text{tr} \left( Q^\gamma_{\alpha\beta;ij} Z^m Z^{*n} \right)$$

where

$$Q^\gamma_{\alpha\beta;ij} = |\gamma; A, m_A; i\rangle \langle \gamma; A, m_A; j| .$$

This basis diagonalises the two-point function.

Kimura, Ramgoolam '07
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The Brauer basis, formed of particular linear combinations of such traces, is a generalisation of the Schur Polynomials to $Z, Z^\dagger$ operators,

$$O_{\alpha\beta;ij}^\gamma(Z, Z^\dagger) = \text{tr} \left(Q_{\alpha\beta;ij}^\gamma Z^m Z^{*n}\right)$$

where

$$Q_{\alpha\beta;ij}^\gamma = |\gamma; A, m_A; i\rangle\langle\gamma; A, m_A; j| .$$

This basis diagonalises the two-point function.

Kimura, Ramgoolam ’07

For example, when $m = 1, \ n = 1$, suppressing non-essential labels:

$$O_{[1,[\bar{1}]}^{k=0} = \text{tr} Z \text{tr} Z^\dagger - \frac{1}{N} \text{tr}(ZZ^\dagger)$$

$$O_{[1,[\bar{1}]}^{k=1} = \frac{1}{N} \text{tr}(ZZ^\dagger) .$$

- Note that the coefficients depend on $N$. 
The $k = 0$ sector

The $k = 0$ operators are special:

- They do not require point-splitting regularisation.
- In the $k = 0$ sector $\gamma = (0, \alpha, \beta)$ so operators are labelled simply by $\alpha$ and $\beta$.
- To connect with the notation of the unitary matrix model, we write $\alpha = R$ and $\beta = S$. 
The $k = 0$ sector

If $S = \emptyset$, then the $k = 0$ operator is a holomorphic Schur polynomial:

$$O_{R, \emptyset}^{k=0}(Z, Z^\dagger) = \chi_R(Z).$$

If $R = \emptyset$, then the $k = 0$ operator is an anti-holomorphic Schur polynomial:

$$O_{\emptyset, S}^{k=0}(Z, Z^\dagger) = \chi_S(Z^\dagger).$$
The $k = 0$ sector

If $S = \emptyset$, then the $k = 0$ operator is a holomorphic Schur polynomial:

$$\mathcal{O}_{R,\emptyset}^{k=0}(Z, Z^\dagger) = \chi_R(Z).$$

If $R = \emptyset$, then the $k = 0$ operator is an anti-holomorphic Schur polynomial:

$$\mathcal{O}_{\emptyset,\overline{S}}^{k=0}(Z, Z^\dagger) = \chi_S(Z^\dagger).$$

If both $\alpha$ and $\beta$ are nontrivial, the leading order term in the expansion of $\mathcal{O}^{k=0}$ begins with the product of the holomorphic and antiholomorphic Schur polynomials:

$$\mathcal{O}_{R,\overline{S}}^{k=0}(Z, Z^\dagger) = \chi_R(Z)\chi_S(Z^\dagger) + \cdots ,$$

where the dots denote terms with at least one $ZZ^\dagger$ inside a trace.
The $k = 0$ sector

The $k = 0$ operators are the generalisations of the characters of the composite representations of unitary matrix model to a complex matrix - if we replace $Z$ by a unitary matrix, we obtain:

$$O_{RS}^{k=0}(U, U^\dagger) = d_R d_S \chi_{RS}(U) .$$

This gives an isomorphism between the $k = 0$ sector and the states of the Unitary matrix model.

- Motivation to look for free fermions on a circle in $k = 0$ sector.
Free particles from Brauer Algebra

Strategy:
- Seek free particle physics in the Brauer basis at particular values of \( k \)

Results:
- Free particle descriptions in two sectors: \( k = 0 \) and \( k = m = n \)
- \( k = 0 \) sector: Explicit expressions at \( N = 2 \) for momenta of free fermions on a circle in terms of combinations of \( z_i, t_{jk} \), implicit generalisation to arbitrary \( N \)
- \( k = m = n \) sector: map to free fermions in harmonic oscillator of hermitian matrix model for arbitrary \( N \).

see also Masuku & Rodrigues, 0911.2846
Let us examine more closely the Schur Decomposition,

\[ Z = U T U^\dagger \]

where \( t_{ii} = z_i \) and \( t_{jk} = 0 \) for \( j < k \).

Residual symmetries:

- \( S_N \) permutes eigenvalues \( z_i \) (\& transforms \( t_{jk} \))
- \( U(1)^{N-1} \) acts on phases of the \( t_{jk} \).

The parameter space of inequivalent adjoint \( U(N) \) orbits, \( \mathcal{M}_N \) can be obtained by fixing an ordering of \( z_i \) and setting \( t_{j,j+1} \in \mathbb{R} \).
Schur Triangularisation revisited

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Residual symmetries:
- \( S_N \) permutes eigenvalues \( z_i \) (& transforms \( t_{jk} \))
- \( U(1)^{N-1} \) acts on phases of the \( t_{jk} \).

The parameter space of inequivalent adjoint \( U(N) \) orbits, \( M_N \) can be obtained by fixing an ordering of \( z_i \) and setting \( t_{j,j+1} \in \mathbb{R} \).

At \( N = 2 \) setting \( t_0 \in \mathbb{R} \) we have

\[ T = \begin{pmatrix} z_1 & t_0 \\ 0 & z_2 \end{pmatrix} \].
Differential Gauss’s law

Recall the relevant part of the $\mathcal{N} = 4$ SYM Lagrangian

$$ \mathcal{L}_Z = \text{tr} \left( D_\mu Z^\dagger D^\mu Z \right). $$

A convenient gauge choice is to set $A_0 = 0$. The e.o.m. for $A_0$ leads to Gauss’s Law:

$$ Z^\dagger \dot{Z} + Z \dot{Z}^\dagger - \dot{Z} Z^\dagger - \dot{Z}^\dagger Z = 0. $$

Upon canonical quantization this leads to the differential form of Gauss’s Law,

$$ G = G_1 + G_2 + G_3 + G_4 = 0 $$

where $G_i$ are defined as:

$$ (G_1)_j^i = Z^\dagger_k \left( \frac{\partial}{\partial Z^\dagger} \right)_j^k $$

$$ (G_2)_j^i = Z_k^i \left( \frac{\partial}{\partial Z} \right)_j^k $$

$$ (G_3)_j^i = -Z^\dagger_k^j \left( \frac{\partial}{\partial Z^\dagger} \right)_i^k $$

$$ (G_4)_j^i = -Z_k^j \left( \frac{\partial}{\partial Z} \right)_i^k $$
Casimir Operators

Given generators \( \{ e_i \} \) of a Lie algebra \( \mathcal{G} \), with

\[
[e_i, e_j] = c_{ij}^k e_k , \quad (*)
\]

the algebra formed from linear combinations of products of the \( \{ e_i \} \), subject to (\( * \)), is called the \textit{universal enveloping algebra} \( \mathcal{G}^U \) of \( \mathcal{G} \).

- Elements in the centre of \( \mathcal{G}^U \) are called \textit{Casimir operators}.
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- Elements in the centre of \( \mathcal{G}^U \) are called \textit{Casimir operators}.

Given a representation of \( \mathcal{G} \), \( e_i \rightarrow \rho(e_i) = E_i \), then given a Casimir operator \( c \),

\[
C = \rho(c)
\]

is called a \textit{Casimir operator of the representation} \( \rho \).

- By Schur’s Lemma, Casimir operators of irreducible representations take constant values.
Casimir Operators

The Gauss Law operators $G_i$ each form a representation of $U(N)$ on GIOs. There are thus associated Casimir operators: recalling the definitions

$$(G_2)^i_j = Z^i_k \left( \frac{\partial}{\partial Z} \right)^k_j, \quad (G_3)^i_j = -Z^\dagger_k j \left( \frac{\partial}{\partial Z^\dagger} \right)^i_k$$

and defining $G_L = G_2 + G_3$, one may define

$$H_1 = \text{tr } G_2 \quad H_2 = \text{tr } G_2^2$$
$$\bar{H}_1 = \text{tr } G_3 \quad \bar{H}_2 = \text{tr } G_3^2 \quad H_L = \text{tr } G_L^2.$$

These can be thought of as Hamiltonians acting on GIOs.
Casimirs and Young diagrams

Recall that a composite Young diagram $R$ with arbitrary integer row lengths $r_i$ labels momenta $p_i$ of $N$ free fermions on a circle given in terms of the Fermi energy $n_F = \frac{N-1}{2}$ by

$$p_i = r_i + (n_F + 1 - i).$$

Given a Young diagram $R$, the linear and quadratic Casimirs of the $U(N)$ representation $R$ are expressible in terms of $r_i$ or $p_i$:

$$C_1(R) = \sum_i r_i = \sum_i p_i = n$$

$$C_2(R) = nN + \sum_i r_i(r_i - 2i + 1) = \sum_{i=1}^{N} p_i^2 - \frac{N}{12}(N^2 - 1)$$
Casimir Operators

Acting on a Brauer basis operator $O_{\alpha\beta;ij}^\gamma(Z, Z^\dagger)$,

- $H_1, H_2$ measure $C_1(\alpha), C_2(\alpha)$
- $\bar{H}_1, \bar{H}_2$ measure $C_1(\beta), C_2(\beta)$
- $H_1 - \bar{H}_1$ measures $C_1(\gamma)$, $H_L$ measures $C_2(\gamma)$.

Generalized Casimir operators such as $\text{tr}(G^2_1 G_3)$ are sensitive to the labels $i, j$.

At $N = 2$ the labels $i, j$ are trivial and it suffices to consider linear and quadratic Casimirs.

Kimura, Ramgoolam '08

At $N = 2$ the labels $i, j$ are trivial and it suffices to consider linear and quadratic Casimirs.
Free fermions in the $k = 0$ sector

At $N = 2$, a $k = 0$ operator is determined by the composite Young diagram $\gamma$ which has two integer rows $r_1^\gamma$, $r_2^\gamma$. We shift the row lengths to obtain fermion momenta:

$$p_1 = r_1 + \frac{1}{2}, \quad p_2 = r_2 - \frac{1}{2}.$$ 

The linear and quadratic Casimirs at $N = 2$ become

$$C_1 = p_1 + p_2$$
$$C_2 = p_1^2 + p_2^2 - \frac{1}{2}$$

which may be inverted to

$$p_1 = \frac{C_1}{2} + \sqrt{\frac{C_2}{2} - \frac{C_1^2}{4} + \frac{1}{4}}$$
$$p_2 = C_1 - p_1.$$
Free fermions in the $k = 0$ sector

We have identified differential operators which measure Casimirs, in particular:

- $H_1 - \bar{H}_1$ measures $C_1(\gamma)$, $H_L$ measures $C_2(\gamma)$.
Free fermions in the \( k = 0 \) sector

We have identified differential operators which measure Casimirs, in particular:

- \( H_1 - \bar{H}_1 \) measures \( C_1(\gamma) \), \( H_L \) measures \( C_2(\gamma) \).

We may thus write the fermion momenta as differential operators:

\[
\hat{p}_1 = \frac{H_1 - \bar{H}_1}{2} + \sqrt{\frac{H_L}{2} - \frac{(H_1 - \bar{H}_1)^2}{4}} + \frac{1}{4}
\]
\[
\hat{p}_2 = H_1 - \bar{H}_1 - \hat{p}_1
\]

In terms of the matrix entries, \( H_1, \bar{H}_1, H_L \) are combinations of the eigenvalues \( z_i \) and off-diagonal entries \( t_{jk} \)...
Hamiltonians in terms of matrix elements

Introducing for convenience $z_c = z_1 + z_2$, $z = z_1 - z_2$ and

\[
L_1 = z_1 \frac{\partial}{\partial z_1} \quad \bar{L}_1 = \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \\
L_2 = z_2 \frac{\partial}{\partial z_2} \quad \bar{L}_2 = \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \quad L_t = \frac{t_0}{2} \frac{\partial}{\partial t_0},
\]
Hamiltonians in terms of matrix elements

Introducing for convenience $z_c = z_1 + z_2$, $z = z_1 - z_2$ and

$L_1 = z_1 \frac{\partial}{\partial z_1}$ \hspace{1cm} $\bar{L}_1 = \bar{z}_1 \frac{\partial}{\partial \bar{z}_1}$

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Then $H_1$, $\bar{H}_1$, $H_L$ in terms of the entries of $Z$ at $N = 2$ are:

$H_1 = L_1 + L_2 + L_t$, \hspace{1cm} $\bar{H}_1 = \bar{L}_1 + \bar{L}_2 + \bar{L}_t$

$H_L = (L_1 - \bar{L}_1)^2 + (L_2 - \bar{L}_2)^2 + \frac{z_c}{z} (L_1 - L_2) + \frac{\bar{z}_c}{\bar{z}} (\bar{L}_1 - \bar{L}_2)$

$- \frac{2}{|z|^2} \left\{ t_0^2 (L_1 - L_2)(\bar{L}_1 - \bar{L}_2) + \frac{1}{t_0} (z_1 \bar{z}_1 - z_2 \bar{z}_2)^2 L_t^2$

$- (z_1 \bar{z}_1 - z_2 \bar{z}_2) \left[ (L_1 - L_2) + (\bar{L}_1 - \bar{L}_2) \right] L_t - (z_1 \bar{z}_1 + z_2 \bar{z}_2) L_t \right\}$
Free fermions in the $k = 0$ sector

The construction carried out explicitly at $N = 2$ generalises to general $N$ in a slightly weaker form.

At general $N$, we have $N$ independent Casimirs leading to a degree $N$ polynomial for the $p_i$.

- To obtain closed form expressions for the $p_i$ in terms of the $C_i$ would require one to solve arbitrary order polynomials, however for any specific values of the $C_i$ one may solve for $p_i$.
- This gives an implicit map to free fermion momenta for any $N$. 
Free fermions in the $k = m = n$ sector

The label $k$ is related to the number of contractions in an operator.

- $k = m = n$: all terms in an operator involve the maximum number of contractions
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- These operators are multi-traces of the matrix $Y = Z^\dagger Z$
- $Y$ is hermitian so we find a map to the $N$ fermions of the hermitian matrix model...
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In this sector $\gamma = (k = m, \gamma_+ = \emptyset, \gamma_- = \emptyset)$ and $\alpha = \beta$ so operators in this sector labelled by $\alpha$ alone.
Free fermions in the $k = m = n$ sector

The operators may be written as

$$\mathcal{O}_{\alpha}^{k=m}(Z, Z^\dagger) = \frac{d_\alpha}{\text{Dim}_\alpha} \text{tr}_k(p_\alpha Y \otimes_k)$$

where

- $d_\alpha$ is the dimension of the $S_k$ representation $\alpha$
- $\text{Dim}_\alpha$ is the dimension of the $U(N)$ representation $\alpha$.
- $p_\alpha$ is the projector onto the $S_k$ representation $\alpha$.

Thus operators in this sector are Schur polynomials constructed from $Y$. 

Free fermions in the $k = m = n$ sector

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- $p_\alpha$ is the projector onto the $S_k$ representation $\alpha$.

Thus operators in this sector are Schur polynomials constructed from $Y$.

As discussed earlier, Schur polynomials in a hermitian matrix correspond to the states of $N$ free fermions in a harmonic oscillator potential.

- The harmonic oscillator fermions observed here are a second emergence of free particles, distinct from those of the $k = 0$ sector.
Counting

It is important to check whether the Brauer basis matches the counting of operators at finite $N$ described by other bases for two-matrix models.

Bhattacharyya, Collins, de Mello Koch & collaborators; Brown, Heslop, Ramgoolam

To count Brauer basis operators $O_{\alpha\beta;ij}(Z, Z^\dagger)$, we must calculate the multiplicity $M_A^\gamma$ of the restriction $\gamma \rightarrow A = (\alpha, \beta)$ of $S_m \times S_n$,

$$M_A^\gamma = \sum_{\delta \vdash k} \sum_{\delta} g(\gamma_+, \delta; \alpha) g(\gamma_-, \delta, \beta)$$

since the indices $i, j$ run from 1 to $M_A^\gamma$. The number of operators in the Brauer basis is thus

$$Q_b^N(m, n) = \sum_{\gamma, A} (M_A^\gamma)^2.$$  

For $m + n \leq N$, this formula counts multi-traces correctly.
What if $N$ isn’t big enough?

- For $m + n > N$, further constraints must be added to $c_1(\gamma_+) + c_1(\gamma_-) \leq N$
- At $N = 2$, we have a conjecture, as follows.
What if $N$ isn’t big enough?

- For $m + n > N$, further constraints must be added to $c_1(\gamma_+) + c_1(\gamma_-) \leq N$
- At $N = 2$, we have a conjecture, as follows.

Firstly, replace the reduction multiplicities by

$$M_{\alpha,\beta}^{\gamma; N=2} = \begin{cases} 1 & \text{if } M_{\alpha,\beta}^{\gamma} > 0 \\ 0 & \text{otherwise} \end{cases}$$

and secondly, constrain $\alpha, \beta$ as follows:

1. $c_1(\alpha) + c_1(\beta) \leq N + k$
2. $[c_1(\alpha) + c_1(\beta)] + [c_2(\alpha) + c_2(\beta)] \leq 2N + k$

and in general for each $p = 1, 2, \ldots, \min(m, n)$, constrain

$$\sum_{r=1}^{p} (c_r(\alpha) + c_r(\beta)) \leq pN + k.$$
We can also express this as a constraint on $k$:

$$k \geq \min(r_2, \bar{r}_2) + \min(\min(r_1, \bar{r}_1), \max(r_2, \bar{r}_2))$$

- We have numerically checked our conjecture up to $(m, n) = (15, 15)$.
- Is there a physical meaning to these extra constraints?
We found free particle signatures in two sectors: 
\[ k = 0 \text{ and } k = m = n \]

- **\( k = 0 \) sector**: Explicit expressions at \( N = 2 \) for momenta of free fermions on a circle in terms of combinations of \( z_i, t_{jk} \), implicit generalisation to arbitrary \( N \).
- **\( k = m = n \) sector**: Map to free fermions in harmonic oscillator of hermitian matrix model for arbitrary \( N \).
- **Brauer basis counts correctly for** \( N \geq m + n \); interesting subtleties for \( m + n > N \).
Is it possible to realise these emergent fermions more explicitly?
- We have their momenta - can we find the dual coordinates?
- Can we express the wavefunctions as Slater determinants?
Open questions

- Is it possible to realise these emergent fermions more explicitly?
  - We have their momenta - can we find the dual coordinates?
  - Can we express the wavefunctions as Slater determinants?

- The label $k$ seems to interpolate between degrees of freedom described by
  - Free fermions on a circle for $k = 0$
  - Free fermions on a line for $k = m = n$.

Can these be interpreted as ‘radial’ and ‘angular’ degrees of freedom?
Open questions

- There is a family of bubbling $\frac{1}{4}$-BPS and $\frac{1}{8}$-BPS asymptotically $AdS_5 \times S^5$ geometries
  
  Chen, Cremonini, Donos et al

- Recent progress on identifying $\frac{1}{4}$-BPS operators at non-zero $\lambda$
  
  Brown; Kimura

- Can we find a description for $\frac{1}{4}$-BPS operators in terms of fermions?
Open questions

At zero coupling, this model is a consistent truncation of $\mathcal{N} = 4$ SYM.

- Does it have a string dual at zero coupling?
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- Does it have a string dual at zero coupling?

Some speculations on this conjectured string dual:

- $z_i$: positions of $N$ branes in two space dimensions.
- $t_{ij}$: strings connecting brane $i$ to $j$

Here the triangular constraint ($t_{ij} = 0$ for $i > j$) will make the dual qualitatively different from the standard system of strings and branes.

c.f. Witten ’95
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- Does any of this physics survive at $\lambda \neq 0$, $\lambda \to \infty$ in SYM?
- Can it be compared to supergravity? Near-extremal AdS black holes?
Thanks!