Hiding pebbles when the output alphabet is unary

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Abstract
Pebble transducers are nested two-way transducers which can drop marks (named “pebbles”) on their input word. Blind transducers have been introduced by Nguyên et al. as a subclass of pebble transducers, which can nest two-way transducers but cannot drop pebbles on their input.

In this paper, we study the classes of functions computed by pebble and blind transducers, when the output alphabet is unary. Our main result shows how to decide if a function computed by a pebble transducer can be computed by a blind transducer. We also provide characterizations of these classes in terms of Cauchy and Hadamard products, in the spirit of rational series. Furthermore, pumping-like characterizations of the functions computed by blind transducers are given.

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1 Introduction

Transducers are finite-state machines obtained by adding outputs to finite automata. In this paper, we assume that these machines are always deterministic and have finite inputs, hence they compute functions from finite words to finite words. In particular, a deterministic two-way transducer consists of a two-way automaton which can produce outputs. This model computes the class of regular functions, which is often considered as one of the functional counterparts of regular languages. It has been largely studied for its numerous regular-like properties: closure under composition [4], equivalence with logical transductions [9] or regular transducer expressions [5], decidable equivalence problem [11], etc.

Pebble transducers and blind transducers. Two-way transducers can only describe functions whose output size is at most linear in the input size. A possible solution to overcome this limitation is to consider nested two-way transducers. In particular, the nested model of pebble transducers has been studied for a long time (see e.g. [10, 5]).

A k-pebble transducer is built by induction on k ≥ 1. For k = 1, a 1-pebble transducer is just a two-way transducer. For k ≥ 2, a k-pebble transducer is a two-way transducer that, when on any position i of its input word, can launch a (k−1)-pebble transducer. This submachine works on the original input where position i is marked by a “pebble”. The original two-way transducer then outputs the concatenation of all the outputs returned by the submachines that it has launched along its computation. The intuitive behavior of a 3-pebble transducer is depicted in Figure 2. It can be seen as program with 3 nested loops. The class of word-to-word functions computed by k-pebble transducers for some k ≥ 1 is known as polyregular functions. It has been quite intensively studied over the past few years due to its regular-like properties such as closure under composition [8], equivalence with logical interpretations [3] or other transducer models [2], etc.
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A subclass of pebble transducers named blind transducers was recently introduced in \[13\]. For \( k = 1 \), a 1-blind transducer is just a two-way transducer. For \( k \geq 2 \), a \( k \)-blind transducer is a two-way transducer that can launch a \((k-1)\)-blind transducer like a \(k\)-pebble transducer. However, there is no pebble marking the input of the submachine (i.e. it cannot see the position \( i \) from which it was called). The behavior of a 3-blind transducer is depicted in Figure 2. It can be seen as a program with 3 nested loops which cannot see the upper loop indexes. We call polyblind functions the class of functions computed by blind transducers. It is closed under composition and deeply related to lambda-calculus \[12\].

We study here polyregular and polyblind functions whose output alphabet is unary. Up to identifying a word with its length, we thus consider functions from finite words to \( \mathbb{N} \). With this restriction, we show that one can decide if a polyregular function is polyblind, and connect these classes of functions to rational series.

Relationship with rational series. Rational series over the semiring \((\mathbb{N}, +, \times)\). are a well-studied class of functions from finite words to \( \mathbb{N} \). They can be defined as the closure of (unary output) regular functions under sum \( + \), Cauchy product \( \odot \) (product for formal power series) and Kleene star \( * \) (iteration of Cauchy products). It is also well known that rational series are closed under Hadamard product \( \odot \) (component-wise product) \[1\].

The first result of this paper states that polyregular functions are exactly the subclass of rational series “without star”, that is the closure of regular functions under \( + \) and \( \odot \) \( (\odot \) can also be used but it is not necessary). This theorem is obtained by combining several former works. Our second result establishes that polyblind functions are exactly the closure of regular functions under \( + \) and \( \odot \). It is shown in a self-contained way.

The aforementioned classes are depicted in Figure 3. All the inclusions are strict and this paper provides a few separating examples (some of them were already known in \[6\]).
Class membership problems. We finally show how to decide whether a polyregular function with unary output is polyblind. It is by far the most involved and technical result of this paper. Furthermore, the construction of a blind transducer is effective, hence this result can be viewed as program optimization. Indeed, given a program with nested loops, our algorithm is able to build an equivalent program using “blind” loops if it exists.

In general, decision problems for transductions are quite difficult to solve, since contrary to regular languages, there are no known “canonical” objects (such as a minimal automaton) to represent (poly)regular functions. It is thus complex to decide an intrinsic property of a function, since it can be described in several seemingly unrelated manners. Nevertheless, the membership problem from rational series to polyregular (resp. regular) functions was shown to be decidable in [7, 6]. It is in fact equivalent to checking if the output of the rational series is bounded by a polynomial (resp. a linear function) in its input’s length.

However, both polyregular and polyblind functions can have polynomial growth rates. To discriminate between them, we thus introduce the new notion of repetitiveness (which is a pumping-like property for functions) and show that it exactly captures the polyregular functions that are polyblind. The proof is a rather complex induction on the depth \( k \geq 1 \) of the \( k \)-pebble transducer representing the function. We show at the same time that repetitiveness is decidable and that a blind transducer can effectively be built whenever this property holds. Partial results were obtained in [6] to decide “blindness” of the functions computed by 2-pebble transducers. Some of our tools are inspired by this paper, such as the use of bimachines and factorization forests. Nevertheless, our general result requires new proof techniques (e.g. the induction techniques which insulate the term of “highest growth rate” in the function) and concepts (e.g. repetitiveness).

Outline. We first describe in Section 2 the notions of pebble and blind transducers. In the case of unary outputs, we recall the equivalent models of pebble, marble and blind bimachines introduced in [6]. These bimachines are easier to handle in the proofs, since they manipulate a monoid morphism instead of having two-way moves. In Section 3, we recall the definitions of sum +, Cauchy product \( \otimes \), Hadamard product \( \circ \) and Kleene star \( * \) for rational series. We then show how to describe polyregular and polyblind functions with these operations. Finally, we claim in Section 4 that the membership problem from polyregular to polyblind functions is decidable. The proof of this technical result is sketched in sections 5 and 6. Due to space constraints we focus on the most significant lemmas.
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2 Preliminaries

\( \mathbb{N} \) is the set of nonnegative integers. If \( i \leq j \), the set \([i:j]\) is \( \{i, i+1, \ldots, j\} \subseteq \mathbb{N} \) (empty if \( j < i \)). The capital letter \( A \) denotes an alphabet, i.e. a finite set of letters. The empty word is denoted by \( \varepsilon \). If \( w \in A^* \), let \(|w| \in \mathbb{N} \) be its length, and for \( 1 \leq i \leq |w| \) let \( w[i] \) be its \( i \)-th letter. If \( I = \{i_1 < \cdots < i_r\} \subseteq \{1:|w|\} \), let \( w[I] := w[i_1] \cdots w[i_r] \). If \( a \in A \), let \(|w|_a\) be the number of letters \( a \) occurring in \( w \). We assume that the reader is familiar with the basics of automata theory, in particular one-way and two-way automata, and monoid morphisms.

Two-way transducers. A deterministic two-way transducer is a deterministic two-way automaton (with input in \( A^* \)) enhanced with the ability to produce outputs (from \( B^* \)) when performing a transition. The output of the transducer is defined as the concatenation of these productions along the unique accepting run on the input word (if it exists): it thus describes a (partial) function \( A^* \rightarrow B^* \). Its behavior is depicted in Figure 4. A formal definition can be found e.g. in [1]. These machines compute the class of regular functions.

\[ \begin{array}{c}
\text{Input word} \\
\hline
\text{Run of the machine}
\end{array} \]

Figure 4 Behavior of a two-way transducer

Example 2.1. The function \( a_1 \cdots a_n \mapsto a_1 \cdots a_n \# a_n \cdots a_1 \) can be computed by a two-way transducer which reads its input word from left to right and then from right to left.

From now on, the output alphabet \( B \) of our machines will always be a singleton. By identifying \( B^* \) and \( \mathbb{N} \), we assume that the functions computed have type \( A^* \rightarrow \mathbb{N} \).

Example 2.2. Given \( a \in A \), the function \( nb_a : A^* \rightarrow \mathbb{N}, w \mapsto |w|_a \) is regular.

Blind and pebble transducers. Blind and pebble transducers extend two-way transducers by allowing to “nest” such machines. A 1-blind (resp. 1-pebble) transducer is just a two-way transducer. For \( k \geq 2 \), a \( k \)-blind (resp. \( k \)-pebble) transducer is a two-way transducer which, when performing a transition from a position \( 1 \leq i \leq |w| \) of its input \( w \in A^* \), can launch a \((k-1)\)-blind (resp. \((k-1)\)-pebble) transducer with input \( w \) (resp. \( w[1:i-1]w[i+1:|w|] \)) i.e. \( w \) where position \( i \) is marked). The two-way transducer then uses the output of this submachine as if it was the output produced along its transition. The intuitive behaviors are depicted for \( k = 3 \) in figures 1 and 2. Formal definitions can be found e.g. in [13, 6].

Example 2.3. Let \( a_1, \ldots, a_k \in A \), then \( nb_{a_1 \cdots a_k} : w \mapsto |w|_{a_1} \times \cdots \times |w|_{a_k} \) can be computed by a \( k \)-blind transducer. The main transducer processes its input from left to right, and it calls inductively a \((k-1)\)-blind transducer for \( nb_{a_1 \cdots a_{k-1}} \) each time it sees an \( a_k \).

Example 2.4. The function \( 2 \)-powers : \( a^m b \cdots a^m b \mapsto \sum_{i=1}^{m} n_i^2 \) can be computed by a 2-pebble transducer. Its main transducer ranges over all the \( a \) of the input, and calls a 1-pebble (= two-way) transducer for each \( a \), which produces \( n_i \) if the \( a \) is in the \( i \)-th block (it uses the pebble to detect which block is concerned). Similarly, the function \( k \)-powers : \( a^m b \cdots a^m b \mapsto \sum_{i=1}^{m} n_i^k \) for \( k \geq 1 \) can be computed by a \( k \)-pebble transducer.

Definition 2.5. We define the class of polyregular functions (resp. polyblind functions) as the class of functions computed by a \( k \)-pebble (resp. \( k \)-blind) transducer for some \( k \geq 1 \).

It is not hard to see that polyblind functions are a subclass of polyregular functions. Indeed, a blind transducer is just a pebble transducer “without pebbles”.
Bimachines. In this paper, we shall describe formally the regular, polyregular and polyblind functions with another computation model. A bimachine is a transducer which makes a single left-to-right pass on its input, but it can use a morphism into a finite monoid to check regular properties of the prefix (resp. suffix) ending (resp. starting) in the current position. This notion of bimachines enable us to easily use algebraic techniques in the proofs, and in particular factorization forests over finite monoids.

Definition 2.6. A bimachine $M := (A, M, \mu, \lambda)$ is:
- an input alphabet $A$; a finite monoid $M$ and a monoid morphism $\mu : A^* \to M$;
- an output function $\lambda : M \times A \times M \to N$.

$M$ computes $f : A^* \to N$ defined by $f(w) := \sum_{i=1}^{\left\lfloor \frac{1}{w[i+1]:|w|} \right\rfloor} \lambda(\mu(w[1:i-1]), w[i], \mu(w[i+1]:|w|))$ for $w \in A^*$.

\[ \text{Input word} \quad \begin{array}{c|c|c}
|w[1:i-1]| & |w[i]| & |w[i+1]:|w|| \\
\hline 
\mu(w[1:i-1]) & \mu(w[i+1]:|w|) \\
\end{array} \]

\[ \text{Figure 5} \quad \text{Behavior of a bimachine when producing } \lambda(\mu(w[1:i-1]), w[i], \mu(w[i+1]:|w|)) \]

When dealing with bimachines, we consider without loss of generality total functions such that $f(\varepsilon) = 0$ (the domains of two-way transducers are regular languages [14], and the particular image of $\varepsilon$ does not matter). In this context, it is well known that regular functions of type $A^* \to N$ are exactly those computed by bimachines (in other words, it means that regular functions and rational functions are the same). Now we recall how [6] has generalized this result to $k$-blind and $k$-pebble transducers. Intuitively, a bimachine with external functions is a bimachine enriched with the possibility to launch a submachine for each letter of the input (it outputs the sum of all the outputs returned by these submachines).

Definition 2.7 ([6]). A bimachine with external pebble (resp. external blind, resp. external marble) functions $M = (A, M, \mu, \lambda)$ consists of:
- an input alphabet $A$;
- a finite monoid $M$ and a morphism $\mu : A^* \to M$;
- a finite set $\mathcal{H}$ of external functions $\mathcal{H} : (A \uplus \Delta)^* \to N$ (resp. $A^* \to N$, resp. $A^* \to N$);
- an output function $\lambda : M \times A \times M \to \mathcal{H}$.

Given $1 \leq i \leq |w|$ a position of $w \in A^*$, let $\mathcal{H}_i := \lambda(\mu(w[1:i-1]), w[i], \mu(w[i+1]:|w|)) \in \mathcal{H}$. A bimachine $M$ with external pebble functions computes a function $f : A^* \to N$ defined by $f(w) := \sum_{1 \leq i \leq |w|} \mathcal{H}_i(w[1:i-1], w[i], \mu(w[i+1]:|w|))$ (that is “$w$ with a pebble on position $i$”), and uses the result $\mathcal{H}_i(w[1:i-1], w[i], \mu(w[i+1]:|w|))$ of this function in its own output.

For a bimachine with external blind (resp. marble) functions, the output is defined by $f(w) := \sum_{1 \leq i \leq |w|} \mathcal{H}_i(w)$ (resp. $f(w) := \sum_{1 \leq i \leq |w|} \mathcal{H}_i(w[1:i])$). In this case $M$ calls $\mathcal{H}_i$ with argument $w$ (resp. the prefix of $w$ ending in position $i$).

Definition 2.8 ([6]). Given $k \geq 1$, a $k$-pebble (resp. $k$-blind, resp. $k$-marble) bimachine is:
- for $k = 1$, a bimachine (without external functions, see Definition 2.6);
- for $k \geq 2$, it a bimachine with external pebble (resp. external blind, resp. external marble) functions (see Definition 2.7) which are computed by $(k-1)$-pebble (resp. $(k-1)$-blind, resp. $(k-1)$-marble) bimachines. These $(k-1)$-bimachines are implicitly fixed and given by the external functions of the main bimachine.
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Remark 2.9. For pebble bimachines, a natural question is whether the inner bimachines can ask which pebble was dropped by which ancestor, or whether they can only see that "there is a pebble". Both models are in fact equivalent, since the number of pebbles is bounded.

Now we recall in Theorem 2.10 that pebble and blind bimachines are respectively equivalent to the aforementioned pebble and blind transducers. More interestingly, the marble bimachines (which call "prefixes") also correspond to pebble transducers. In our proofs, we shall preferentially use the marble model to represent polyregular functions.

Theorem 2.10. For all $k \geq 1$, $k$-pebble transducers, $k$-pebble bimachines and $k$-marble bimachines compute the same class of functions. Furthermore, $k$-blind transducers and $k$-blind bimachines compute the same class of functions.

3 From pebbles to rational series

The class of rational series (which are total functions $A^+ \to \mathbb{N}$) is the closure of regular functions under sum, Cauchy product and Kleene star. It is well known that it can be described by weighted automata, furthermore this class is closed under Hadamard product (see e.g. [1, Theorem 5.5]). Let us recall the definition of these operations for $f, g : A^+ \to \mathbb{N}$:

- the sum $f + g : w \mapsto f(w) + g(w)$;
- the Cauchy product $f \otimes g : w \mapsto \sum_{i=1}^{|w|} f([w[i:i+1]])g([w[i+1:|w|]])$;
- the Hadamard product $f \otimes g : w \mapsto f(w)g(w)$;
- if and only if $f(\varepsilon) = 0$, the Kleene star $f^* \coloneqq \sum_{n \geq 0} f^n$ where $f^0 : \varepsilon \mapsto 1, u \neq \varepsilon \mapsto 0$ is neutral for Cauchy product and $f^{n+1} \coloneqq f \otimes f^n$.

Example 3.1. In Example 2.3 we have $\text{nb}_{a_1, \ldots, a_k} = \text{nb}_{a_1} \otimes \cdots \otimes \text{nb}_{a_k}$.

Example 3.2. Let $f, g : \{a, b\}^+ \to \mathbb{N}$ defined by $f(wa) = 2$ for $w \in A^+$ and $f(w) = 0$ otherwise; $g(a^nbw) = n$ for $w \in A^+$ and $g(w) = 0$ otherwise. It is easy to see that $f \otimes g(a^nb \cdot a^m) = \sum_{i=1}^m n_i(n_i-1)$. Hence 2-powers $= \text{nb}_a + f \otimes g$ (see Example 2.4).

We are now ready to state the first main results of this paper. The first one shows that polyregular functions correspond to the subclass of rational series where the use of star is disallowed (and it also corresponds to rational series whose “growth” is polynomial).

Theorem 3.3. Let $f : A^+ \to \mathbb{N}$, the following conditions are (effectively) equivalent:

1. $f$ is polyregular;
2. $f$ is a rational series with polynomial growth, i.e. $f(w) = O(|w|^k)$ for some $k \geq 1$;
3. $f$ belongs to the closure of regular functions under sum and Cauchy product;
4. $f$ belongs to the closure of regular functions under sum, Cauchy and Hadamard products.

Proof sketch. [1] $\iff$ [2] follows from [7, 6]. For $[2] \implies [3]$ it is shown in [1, Exercise 1.2 of Chapter 9] that the rational series of polynomial growth can be obtained by sum and Cauchy products from the characteristic series of rational languages (which are regular functions). Having $[3] \implies [4]$ is obvious. Finally for $[4] \implies [2]$ it suffices to note that regular functions have polynomial growth, and this property is preserved by $+$, $\otimes$ and $\circ$.

Note that Hadamard products do not increase the expressive power of this class. However, removing Cauchy products gives polyblind functions, as shown in Theorem 3.4.

Theorem 3.4. Let $f : A^+ \to \mathbb{N}$, the following conditions are equivalent:

1. $f$ is polyblind;
2. $f$ belongs to the closure of regular functions under sum and Hadamard product.
Proof sketch. We first show $[1] \Rightarrow [2]$ by induction on $k \geq 1$ when $f$ is computed by a $k$-blind bimachine. For $k = 1$, it is obvious. Let us describe the induction step from $k \geq 1$ to $k+1$. Let $f$ be computed by a bimachine $M = (A, M, \mu, \delta, \lambda)$, with external blind functions computed by $k$-blind bimachines. Given $h \in {\mathcal {H}}$, let $f'_{h}$ be the function which maps $w \in A^{*}$ to the cardinal $\{1 \leq i \leq |w| : \lambda (\mu (w[1:i-1]), w[i], \mu (w[i+1:|w|])) = h\}$. Then $f'_{h}$ is a regular function and $f = \sum_{h \in {\mathcal {H}}} f'_{h} \circ h$. The result follows by induction hypothesis.

For $[2] \Rightarrow [1]$, we show that functions computed by blind bimachines are closed under sum and Hadamard product. For the sum, we use the blind transducer model to simulate successively the computation of the two terms of a sum. For the Hadamard product, we show by induction on $k \geq 1$ that if $f$ is computed by a $k$-blind bimachine and $g$ is polyblind, then $f \circ g$ is polyblind. If $k = 1$, we transform the bimachine for $f$ in a blind bimachine computing $f \circ g$ by replacing each output $n \in \mathbb {N}$ by a call to a machine computing $n \times g$. If $k \geq 2$, using the notations of the previous paragraph we have $f = \sum_{h \in {\mathcal {H}}} f'_{h} \circ h$, thus $f \circ g = \sum_{h \in {\mathcal {H}}} f'_{h} \circ h \circ g$. By induction hypothesis, each $h \circ g$ is polyblind. Hence we compute $f \circ g$ by replacing in $M$ each external function $h$ by the function $h \circ g$.

We conclude this section by recalling that polyregular (resp. polyblind) functions with unary output enjoy a “pebble minimization” property, which allows to reduce the number of nested layers depending on the growth rate of the output.

Definition 3.5. We say that a function $f : A^{*} \rightarrow \mathbb {N}$ has growth at most $k$ if $f(w) = O(|w|^k)$.

Theorem 3.6 ([7] & [13]). A a polyregular (resp. polyblind) function $f$ can be computed by a $k$-pebble (resp. $k$-blind) transducer if and only it has growth at most $k$. Furthermore this property can be decided and the construction is effective.

Remark 3.7. The result also holds for blind transducers with non-unary outputs [13]. However, it turns out to be false for pebble transducers with non-unary outputs (unpublished work of the author of [2]).

In the next section, we complete the decidability picture by solving the membership problem from polyregular to polyblind functions.

4 Membership problem from polyregular to polyblind

In this section, we state the most technical and interesting result of this paper, which consists in deciding if a polyregular function is polyblind. We also give in Theorem 4.6 a semantical characterization of polyregular functions which are polyblind, using the notion of repetitiveness. Intuitively, a $k$-repetitive function is a function which, when given two places in a word where the same factor is repeated, cannot distinguish between the first and the second place. Hence its output only depends on the total number of repetitions of the factor.

Definition 4.1 (Repetitive function). Let $k \geq 1$. We say that $f : A^{*} \rightarrow \mathbb {N}$ is $k$-repetitive if there exists $\eta \geq 1$, such that the following holds for all $\alpha, \alpha_{0}, u_{1}, \alpha_{1}, \ldots , u_{k}, \alpha_{k}, \beta \in A^{*}$ and $\omega \geq 1$ multiple of $\eta$. Let $W : \mathbb {N}^{k} \rightarrow A^{*}$ defined by:

$$W : X_{1}, \ldots , X_{k} \mapsto \left( \alpha_{0} \prod_{i=1}^{k} u_{i}^{\omega \alpha_{i}} \right).$$

and let $w := W(1, \ldots , 1)$. Then there exists a function $F : \mathbb {N}^{k} \rightarrow \mathbb {N}$ such that for all $X := X_{1}, \ldots , X_{k} \geq 3$ and $Y := Y_{1}, \ldots , Y_{k} \geq 3$, we have:

$$f(\alpha \omega^{2\omega -1} W(X) w^{\omega -1} W(Y) w^{\omega \beta}) = F(X_{1} + Y_{1}, \ldots , X_{k} + Y_{k})$$
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Remark 4.2. If \( f \) is \( k \)-repetitive, \( f \) is also \((k-1)\)-repetitive, by considering an empty \( u_k \).
Let us now give a few examples in order so see when this criterion holds, or not.

Example 4.3. The function \( nb_{a_1,\ldots,a_k} \) (see Example 2.3) is \( k \)-repetitive for all \( k \geq 1 \).

Example 4.4. If \( f : A^* \rightarrow \mathbb{N} \) where \( A = \{a\} \) is a singleton, then \( f \) is \( k \)-repetitive for all \( k \geq 1 \). Indeed, \( \alpha \omega_{2k-1}^\omega X w_{\omega-1}^\omega W(Y) w_\omega \beta \in A^* \) is itself a function of \( X_1 + Y_1, \ldots, X_k + Y_k \), hence so is its image \( f(\alpha \omega_{2k-1}^\omega X w_{\omega-1}^\omega W(Y) w_\omega) \).

Example 4.5. For all \( k \geq 2 \), the function \( k \)-powers : \( a^{n_1} b \cdots a^{n_m} b \mapsto \sum_{i=1}^m n_i^k \) is not \( 1 \)-repetitive. Let us choose any \( \omega \geq 1 \) and fix \( \alpha = \beta = \varepsilon \), \( u_1 := a \) and \( a_0 = \alpha_1 := b \), then:

\[
-k\text{-powers}(W(X_1,Y_1)) = k\text{-powers} ((ba^\omega b)^{2\omega-1}ba^\omega X_1;b(ba^\omega b)^{\omega-1} ba^\omega Y_1 b(ba^\omega b)^\omega + \omega^k X_1 + \omega^k Y_1^k as a function of } \]

which is not a function of \( X_1 + Y_1 \) for \( k \geq 2 \).

We are now ready to state our main result of this paper.

Theorem 4.6. Let \( k \geq 1 \) and \( f : A^* \rightarrow \mathbb{N} \) be computed by a \( k \)-pebble transducer. Then \( f \) is polyblind if and only if it is \( k \)-repetitive. Furthermore, this property can be decided and one can effectively build a blind transducer for \( f \) when it exists.

Remark 4.7. By Theorem 3.6, we can even build a \( k \)-blind transducer.

Theorem 4.6 provides a tool to show that some polyregular functions are not polyblind:

Example 4.8. By Example 4.5, the polyregular function \( k \)-powers is not \( k \)-repetitive for \( k \geq 2 \). Therefore it is not polyblind. Also note that it is computable by a \( k \)-pebble transducer, but not by an \( \ell \)-pebble transducer for \( \ell < k \) (Theorem 3.6).

An immediate consequence of Theorem 4.6 is obtained below for functions which have both a unary output alphabet and a unary input alphabet. This result was already obtained by the authors of [13] using other techniques, in an unpublished work.

Corollary 4.9. Polyblind and polyregular functions over a unary input alphabet coincide.

Proof. Polyregular functions with unary input are \( k \)-repetitive by Example 4.4.

5 Repetitive functions and permutable bimachines

The rest of this paper is devoted to the proof of Theorem 4.6. Since \( k \)-pebble transducers and \( k \)-marble bimachines compute the same functions (Theorem 2.10), it follows directly from Theorem 5.1 below (the notions used are defined in the next sections).

From now on, we assume that a \( k \)-marble bimachine uses the same morphism \( \mu : A^* \rightarrow M \) in its main bimachine, and inductively in all its submachines computing the external functions.

We do not lose any generality here, since it is always possible to get this situation by taking the product of all the morphisms used. We also assume that \( \mu \) is surjective (we do no lose any generality, since it is possible to replace \( M \) by the image \( \mu(M) \)).

Theorem 5.1. Let \( k \geq 2 \) and \( f : A^* \rightarrow \mathbb{N} \) be computed by \( \mathcal{M} = (A,M,\mu,\delta,\lambda) \) a \( k \)-marble bimachine. The following conditions are equivalent:

1. \( f \) is polyblind;
2. \( f \) is \( k \)-repetitive;
3. \( \mathcal{M} \) is \( 2^{[\mathcal{M}]} \)-permutable (see Definition 5.4) and the function \( f'' \) (built from Proposition 6.1) is polyblind.

Furthermore this property is decidable and the construction is effective.

Let us now give the skeleton of the proof of Theorem 5.1 which is rather long and involved. The propositions on which it relies are shown in the two following sections.

**Proof of Theorem 5.1.** [1] \( \Rightarrow \) [2] follows from Proposition 5.2. For [2] \( \Rightarrow \) [3] if \( f \) is \( k \)-repetitive then \( \mathcal{M} \) is \( 2^{[\mathcal{M}]} \)-permutable by Proposition 5.12. Then Proposition 6.1 gives \( f = f' + f'' \) where \( f' \) is polyblind and \( f'' \) is computable by a \((k-1)\)-marble bimachine. By Proposition 5.2 \( f' \) is \((k-1)\)-repetitive. Since \( f \) is also \((k-1)\)-repetitive, then \( f'' = f - f' \) is also \((k-1)\)-repetitive by Claim 5.3. Thus by induction hypothesis \( f'' \) is polyblind. For [3] \( \Rightarrow \) [4] since in Proposition 6.1 we have \( f = f' + f'' \) where \( f' \) and \( f'' \) are polyblind, then \( f \) is (effectively) polyblind. The decidability is also obtained by induction: one has to check that \( \mathcal{M} \) is \( 2^{[\mathcal{M}]} \)-permutable (which is decidable) and inductively that \( f'' \) is polyblind. \( \blacksquare \)

Now we present the tools used in this proof. We first show that polyblind functions are repetitive. Then we introduce the notion of *permutability*, and show that repetitive functions are computed by polypermutable bimachines.

### 5.1 Polyblind functions are repetitive

Intuitively, a blind transducer cannot distinguish between two “similar” iterations of a given factor in a word, since it cannot drop a pebble for doing so. We get the following using technical but conceptually easy pumping arguments.

▸ **Proposition 5.2.** A polyblind function is \( k \)-repetitive for all \( k \geq 1 \).

**Proof idea.** We first show that a regular function (computed by a bimachine without external functions) is \( k \)-repetitive for all \( k \geq 1 \). In this case, \( \eta \) is chosen as the idempotent index of the monoid used by the bimachine. Then, it is easy to conclude by noting that \( k \)-repetitiveness is preserved under sums and Hadamard products.

“The induction which proves Theorem 5.1 also requires the following result. Its proof is obvious since \( k \)-repetitiveness is clearly preserved under subtractions.”

▸ **Claim 5.3.** If \( f, g \) are \( k \)-repetitive and \( f - g \geq 0 \), then it is \( k \)-repetitive.

### 5.2 Repetitive functions are computed by permutable machines

In this subsection, we show that \( k \)-marble bimachines which compute \( k \)-repetitive functions have a specific property named *permutability* (which turns out to be decidable).

**Productions.** We first introduce the notion of production. In the rest of this paper, the notation \( \{\cdots\} \) represents a multiset (i.e., a set with multiplicities). If \( S \) is a set and \( m \) is a multiset, we write \( m \subseteq S \) to say that each element of \( m \) belongs to \( S \) (however, there can be multiplicities in \( m \) but not in \( S \)). For instance \( \{1,1,2,3,3\} \subseteq \mathbb{N} \).

▸ **Definition 5.4 (Production).** Let \( \mathcal{M} = (A, M, \mu, \delta, \lambda) \) be a \( k \)-marble bimachine, \( w \in A^* \). We define the production of \( \mathcal{M} \) on \( \{i_1, \ldots, i_k\} \subseteq [1:]w \) as follows if \( i_1 \leq \cdots \leq i_k \):

- if \( k = 1 \), \( \text{prod}_\mathcal{M}^w (\{i_1\}) := \lambda(\mu(w[i_1+1]) - 1), w[i_1], \mu(w[i_1+1]; w[\cdot])) \in \mathbb{N} \);
- if \( k \geq 2 \), let \( h := \lambda(\mu(w[i_1+1]) - 1), w[i_1], \mu(w[i_1+1]; w[\cdot])) \in \delta \) and \( \mathcal{M}_h \) be the \((k-1)\)-marble bimachine computing \( h \). Then \( \text{prod}_\mathcal{M}^w (\{i_1, \ldots, i_k\}) := \text{prod}_{\mathcal{M}_h}^{w[i_1]} (\{i_1, \ldots, i_{k-1}\}) \).
The value $\prod_{M}^{w}(\{i_{1}, \cdots, i_{k}\})$ with $i_{1} \leq \cdots \leq i_{k}$ thus corresponds to the value output when doing a call on position $i_{1}$, then on $i_{k-1}$, etc. Now let $\{I_{1}, \cdots, I_{k}\}$ be a multiset of sets of positions of $w$ (i.e., for all $1 \leq i \leq k$ we have $I_{i} \subseteq \{1:|w|\}$). We define the production of $M$ on $\{I_{1}, \cdots, I_{k}\}$ as the combination of all possible productions on positions among these sets:

$$\prod_{M}^{w}(\{I_{1}, \cdots, I_{k}\}) := \sum_{\{i_{1}, \cdots, i_{k}\} \subseteq \{1:|w|\} \text{ with } i_{j} \in I_{j} \text{ for } 1 \leq j \leq k} \prod_{M}^{w}(\{i_{1}, \cdots, i_{k}\}).$$



**Remark 5.5.** Note that we no longer have $i_{1} \leq \cdots \leq i_{k}$.

By rewriting the sum which defines the function $f$ from $M$, we get the following.

**Lemma 5.6.** Let $M$ be a $k$-marble bimachine computing a function $f : A^{*} \rightarrow \mathbb{N}$. Let $w \in A^{*}$ and $J_{1}, \ldots, J_{n}$ be a partition of $\{1:|w|\}$:

$$f(w) = \sum_{\{i_{1}, \ldots, i_{k}\} \subseteq (J_{1}, \ldots, J_{n})} \prod_{M}^{w}(\{I_{1}, \ldots, I_{k}\}).$$

Following the definition of bimachines, the production $\prod_{M}^{w}(\{i_{1}, \cdots, i_{k}\})$ should only depend on $w[i_{1}], \ldots, w[i_{k}]$ and of the image under $\mu$ of the factors between these positions. Now we formalize this intuition in a more general setting.

**Definition 5.7 (Multicontext).** Given $x \geq 0$, an $x$-multicontext consists of two sequences of words $v_{0}, \cdots, v_{x} \in A^{*}$ and $u_{1}, \cdots, u_{x} \in A^{*}$. It is denoted $v_{0}[u_{1}]v_{1} \cdots [u_{x}]v_{x}$.

Let $w := v_{0}u_{1} \cdots u_{k}v_{k} \in A^{*}$. For $1 \leq i \leq k$, let $I_{i} \subseteq \{1:|w|\}$ be the set of positions corresponding to $u_{i}$. We define the production on the multicontext $v_{0}[u_{1}]v_{1} \cdots [u_{k}]v_{k}$ by:

$$\prod_{M}(v_{0}[u_{1}]v_{1} \cdots [u_{k}]v_{k}) := \prod_{M}^{w}(\{I_{1}, \ldots, I_{k}\}).$$

As stated in Proposition-Definition 5.8, this quantity only depends on the $u_{i}$ and the images of the $v_{i}$ under the morphism $\mu$ of $M$, which leads to a new notion of productions.

**Proposition-Definition 5.8.** Let $M = (A, M, \mu, A, \lambda)$ be a $k$-marble bimachine. Let $v_{0}u_{1} \cdots u_{k}v_{k} \in A^{*}$ and $v_{0}'u_{1}' \cdots v_{k}v_{k}' \in A^{*}$ be such that $\mu(u_{i}) = \mu(v_{i})$ for all $0 \leq i \leq k$. Then:

$$\prod_{M}(v_{0}[u_{1}]v_{1} \cdots [u_{k}]v_{k}) = \prod_{M}(v_{0}'[u_{1}]v_{1}' \cdots [u_{k}]v_{k}').$$

Let $m_{i} := \mu(u_{i}) = \mu(v_{i})$, we define $\prod_{M}(m_{0}[u_{1}]m_{1} \cdots [u_{k}]m_{k})$ as the previous value.

**Remark 5.9.** In the following, we shall directly manipulate multicontexts of the form $m_{0}[u_{1}]m_{1} \cdots [u_{k}]m_{k}$ with $m_{i} \in M$ and $u_{i} \in A^{*}$. Note that an $x$-multicontext and a $y$-multicontext can be concatenated to obtain an $(x+y)$-multicontext which corresponds to an $x$-multicontext in which the words $u_{i}$ have length at most $K > 0$ and have idempotent images under $\mu$. Recall that $e \in M$ is idempotent if and only if $ee = e$.

**Definition 5.10 (Iterator).** Let $x, K \geq 0$ and $\mu : A^{*} \rightarrow M$. An $(x, K)$-iterator for $\mu$ is an $x$-multicontext of the form $m_{0}(\prod_{i=1}^{x} c_{i}[u_{i}]e_{i}m_{i})$ such that $m_{0}, \ldots, m_{x} \in M$ and $u_{1}, \ldots, u_{x} \in A^{*}$ are such that for all $1 \leq i \leq x$, $|u_{i}| \leq K$ and $e_{i} = \mu(u_{i})$ is in idempotent.

**Permutable $k$-marble bimachines.** We are now ready to state the definition of permutability for marble bimachines. Intuitively, this property means that, under some idempotency conditions, $\prod_{M}(m_{0}[u_{1}]m_{1} \cdots [u_{k}]m_{k})$ only depends on the 1-multicontexts of each $[u_{i}]$, which are $m_{0}(u_{i}) \cdots m_{i}[u_{i}]m_{i+1} \mu(u_{i+1}) \cdots m_{k} \in M$ for $1 \leq i \leq k$. In particular, it does not depend on the relative position of the $u_{i}$ nor on the $m_{i}$ which separate them. Hence, it will be possible to simulate a permutable $k$-marble bimachine by a $k$-blind bimachine which can only see the 1-multicontext of one position at each time.
**Definition 5.11.** Let $\mathcal{M}$ be a $k$-marble bimachine using a surjective morphism $\mu : A^* \to M$. Let $K \geq 0$, we say that $\mathcal{M}$ is $K$-permutable if the following holds whenever $\ell + x + r = k$:

- for all $(\ell, K)$-iterator $L$ and $(r, K)$-iterator $R$;
- for all $(x, K)$-iterator $m_0 (\prod_{i=1}^{x} e_i[u_i]e_i m_1)$ such that $e := m_0 (\prod_{i=1}^{x} e_i m_i)$ idempotent;
- for all $1 \leq j \leq x$, define the left and right contexts:

\[
\text{left}_j := e \left( \prod_{i=1}^{j} m_{i-1} e_i \right) \quad \text{and} \quad \text{right}_j := \left( \prod_{i=j}^{x} e_i m_i \right) e.
\]

Then if $\sigma$ is a permutation of $[1:x]$, we have:

\[
\prod_{\mathcal{M}} \left( L \ e m_0 \left( \prod_{i=1}^{x} e_i[u_i] e_i m_1 \right) \ e \ R \right) = \prod_{\mathcal{M}} \left( L \ \prod_{i=1}^{x} \ \text{left}_{\sigma(i)} \ [u_{\sigma(i)}] \ \text{right}_{\sigma(i)} \ \ R \right).
\]

An visual description of permutability is depicted in Figure 6.

\[\text{(a) Initial multicontext and definition of the left}_j / \text{right}_j \text{ for } 1 \leq j \leq 3\]

\[\text{(b) Multicontext which separates the factors using the left}_j / \text{right}_j \text{ and } \sigma\]

**Figure 6** Productions which must be equal in Definition 5.11 with $x = 3$ and $\sigma = (3, 1, 2)$

The next result follows from a technical proof based on iteration techniques (it can be understood as a kind of pumping lemma on iterators).

**Proposition 5.12.** Let $\mathcal{M}$ be a $k$-marble bimachine computing a $k$-repetitive function. Then $\mathcal{M}$ is $K$-permutable for all $K \geq 0$.

Let us finally note that being $K$-permutable (for a fixed $K$) is a decidable property. Indeed, it suffices to range over all $\ell + x + r = k$ and $(\ell, K)$-, $(x, K)$- and $(r, K)$-iterators (there are finitely many of them, since they correspond to bounded sequences which alternate between monoid elements and words of bounded lengths), and compute their productions.

### 6 From permutable bimachines to polyblind functions

The purpose of this section is to show Proposition 6.1, which allows us to perform the induction step in the proof of Theorem 5.1 by going from $k$ to $k-1$ marbles.

**Proposition 6.1.** Let $\mathcal{M} = (A, M, \mu, \delta, \lambda)$ be a $2^{|M|}$-permutable $k$-marble bimachine computing a function $f : A^* \to \mathbb{N}$. One can build a polyblind function $f' : A^* \to \mathbb{N}$ and a function $f'' A^* \to \mathbb{N}$ computed by a $(k-1)$-marble bimachine such that $f = f' + f''$.  

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Proof overview. It follows from Equation 2 and Lemma 6.14 that:

\[
\begin{align*}
f &= (\text{sum-dep}_M + \text{sum-ind}_M) \circ \text{forest}_\mu \\
&= \text{sum-ind}'_M \circ \text{forest}_\mu + (\text{sum-dep}_M + \text{sum-ind}'_M) \circ \text{forest}_\mu \\
&= f''
\end{align*}
\]

By Lemma 6.14 one has \(f'' \geq 0\). Furthermore, \(\text{sum-ind}'_M\) is polyblind, hence \(f'\) is polyblind since the class of polyblind functions is closed under pre-composition by regular functions (even when the outputs are not unary, see [13]). Similarly, it follows from lemmas 6.12 and 6.14 that \(\text{sum-dep}_M\) and \(\text{sum-ind}'_M\) are polyregular with growth at most \(k-1\) (see Definition 3.5), hence so is their sum and its pre-composition by a regular function [2]. Thus \(f''\) is polyregular and has growth at most \(k-1\). By theorems 2.10 and 3.6 it can be computed by a \((k-1)\)-marble bimachine. \(\square\)

Now we give the statements and the proofs of lemmas 6.12 and 6.14. An essential tool is the notion of factorization forest, which is recalled below.

### 6.1 Factorization forests

If \(\mu : A^* \to M\) is a morphism into a finite monoid and \(w \in A^*\), a factorization forest (called \(\mu\)-forest in the following) of \(w\) is an unranked tree structure defined as follows.

**Definition 6.2 (Factorization forest [15]).** Given a morphism \(\mu : A^* \to M\) and \(w \in A^*\), we say that \(F\) is a \(\mu\)-forest of \(w\) if:

- either \(F = a\) and \(w = a \in A\);
- or \(F = \langle F_1 \rangle \cdots \langle F_n \rangle\), \(w = w_1 \cdots w_n\) and for all \(1 \leq i \leq n\), \(F_i\) is a \(\mu\)-forest of \(w_i \in A^+\).

Furthermore, if \(n \geq 3\) then \(\mu(w_1) = \cdots = \mu(w_n)\) is an idempotent of \(M\).

**Remark 6.3.** If \(\langle F_1 \rangle \cdots \langle F_n \rangle\) is a \(\mu\)-forest, then so is \(\langle F_i \rangle \langle F_{i+1} \rangle \cdots \langle F_j \rangle\) for \(1 \leq i \leq j \leq n\). The empty word \(\varepsilon\) is the unique \(\mu\)-forest of the empty word \(\varepsilon\).

We shall use the standard tree vocabulary of “height” (a leaf is a tree of height 1), “child”, “sibling”, “descendant” and “ancestor” (both defined in a non-strict way: a node is itself one of its ancestors/descendants), etc. We denote by \(\text{Nodes}(F)\) the set of nodes of \(F\). In order to simplify the statements, we identify a node \(t \in \text{Nodes}(F)\) with the subtree rooted in this node. Thus \(\text{Nodes}(F)\) can also be seen as the set of subtrees of \(F\), and \(F \in \text{Nodes}(F)\). We say that a node is idempotent if is has at least 3 children (see Definition 6.2).

Given \(\mu : A^* \to M\), we denote by \(\text{Forests}_\mu(w)\) the set of \(\mu\)-forests of \(w \in A^*\). If \(K \geq 0\), let \(\text{Forests}^K_\mu(w)\) be the \(\mu\)-forests of \(w\) of height at most \(K\). Note that \(\text{Forests}^K_\mu(w)\) is a set of tree structures which can also be seen as a language over \(\hat{A} := A \cup \{\langle,\rangle\}\). Indeed, a forest of \(w\) can also be seen as “the word \(w\) with brackets” in Definition 6.2.

```
| a | a | c |
|---|---|---|
|   |   |   |
```

**Figure 7** The \(\mu\)-forest \(\langle aa \rangle \langle c(a(cbcb)) \rangle \langle bb \rangle \) on \(aaccacbcbcbcb\).
Example 6.4. Let \( A = \{a, b, c\} \), \( M = \{(1, -1, 0), \times\} \) with \( \mu(a) \equiv -1 \) and \( \mu(b) \equiv \mu(c) \equiv 0 \). Then \( \mathcal{F} := \langle aa|c|a|c(bb)b\rangle b|c(bbc) \) \( \in \text{Forests}_\mu^3(aac|cbbbc) \) (we dropped the brackets around single letters for readability) is depicted in Figure 7. Double lines are used to denote idempotent nodes (i.e. with more than 3 children).

A celebrated result states that for any word, a forest of bounded height always exist and it can be computed by a bimachine (with non-unary output alphabet), or a two-way transducer. The following theorem can also be found in [2, Lemma 6.5].

Theorem 6.5 ([15]). Given a morphism \( \mu : A^* \to M \), we have \( \text{Forests}_\mu^3(w) \neq \emptyset \) for all \( w \in A^* \). Furthermore, one can build a two-way transducer (with a non-unary output alphabet) which computes a total function \( \text{forest}_\mu : A^* \to (\bar{A})^* \), \( w \mapsto \mathcal{F} \in \text{Forests}_\mu^3(w) \).

6.2 Iterable nodes and productions

We define the iterable nodes \( \text{Iters}(\mathcal{F}) \subseteq \text{Nodes}(\mathcal{F}) \) as the set of nodes which have both a left and right sibling. Such nodes are thus exactly the middle children of idempotent nodes.

Definition 6.6. Let \( \mathcal{F} \in \text{Forests}_\mu(w) \), we define inductively the iterable nodes of \( \mathcal{F} \):

\[
\begin{align*}
\text{if } \mathcal{F} = a & \in A \text{ is a leaf, } \text{Iters}(\mathcal{F}) := \emptyset; \\
\text{otherwise if } \mathcal{F} = \langle \mathcal{F}_1 \rangle \cdots \langle \mathcal{F}_n \rangle, \text{ then:} \\
\text{Iters}(\mathcal{F}) := \{\mathcal{F}_i : 2 \leq i \leq n-1\} \cup \bigcup_{1 \leq i \leq n} \text{Iters}(\mathcal{F}_i).
\end{align*}
\]

Now we define a notion of skeleton which selects the right-most and left-most children.

Definition 6.7. Let \( \mathcal{F} \in \text{Forests}_\mu(w) \) and \( t \in \text{Nodes}(\mathcal{F}) \), we define the skeleton of \( t \) by:

\[
\begin{align*}
\text{if } t & = a \in A \text{ is a leaf, then } \text{Skel}(t) := \{t\}; \\
\text{otherwise if } t = \langle \mathcal{F}_1 \rangle \cdots \langle \mathcal{F}_n \rangle, \text{ then } \text{Skel}(t) := \{t\} \cup \text{Skel}(\mathcal{F}_1) \cup \text{Skel}(\mathcal{F}_n).
\end{align*}
\]

Intuitively, \( \text{Skel}(t) \subseteq \text{Nodes}(\mathcal{F}) \) contains all the descendants of \( t \) except those which are descendant of a middle child. We then define the frontier of \( t \), denoted \( \text{Fr}_\mathcal{F}(t) \subseteq \{1, \ldots, |w|\} \) as the set of positions of \( w \) which belong to \( \text{Skel}(t) \) (when seen as leaves of \( \mathcal{F} \)).

Example 6.8. In Figure 7 the top-most blue node \( t \) is iterable. Furthermore \( \text{Skel}(t) \) is the set of blue nodes, \( \text{Fr}_\mathcal{F}(t) = \{4, 5, 9\} \) and \( w[\text{Fr}_\mathcal{F}(t)] = aeb \).

Using the frontiers, we can naturally lift the notion of productions of a \( k \)-marble bimachine from multisets of positions to multisets of nodes in a forest.

Definition 6.9. Let \( \mathcal{M} = (A, M, \mu, \delta, \lambda) \) be a \( k \)-marble bimachine, \( w \in A^* \), \( \mathcal{F} \in \text{Forests}_\mu(w) \) and \( t_1, \ldots, t_k \in \text{Nodes}(\mathcal{F}) \). We let \( \text{prod}_\mathcal{F}^w(\{t_1, \ldots, t_k\}) := \text{prod}_\mathcal{M}^w(\{\text{Fr}_\mathcal{F}(t_1), \ldots, \text{Fr}_\mathcal{F}(t_k)\}) \).

Using Lemma 5.6, we can recover the function \( f \) from the productions over the nodes. Lemma 6.10 roughly states over all possible tuples of calling positions of \( \mathcal{M} \).

Lemma 6.10. Let \( f : A^* \to \mathbb{N} \) be computed by a \( k \)-marble bimachine \( \mathcal{M} = (A, M, \mu, \delta, \lambda) \). If \( w \in A^* \) and \( \mathcal{F} \in \text{Forests}_\mu(w) \), we have:

\[
f(w) = \sum_{\{t_1, \ldots, t_k\} \subseteq \text{Iters}(\mathcal{F}) \cup \{\mathcal{F}\}} \text{prod}_\mathcal{F}^w(\{t_1, \ldots, t_k\}).
\]

Proof. It follows from 6.8 that for all \( w \in A^* \) and \( \mathcal{F} \in \text{Forests}_\mu(w) \), the set of frontiers \( \{\text{Fr}_\mathcal{F}(t) : t \in \text{Iters}(\mathcal{F}) \cup \{\mathcal{F}\}\} \) is a partition of \( [1:w] \). We then apply Lemma 5.6. \( \blacksquare \)
6.3 Dependent multisets of nodes

The multisets \( \{t_1, \ldots, t_k\} \subseteq \text{Iters}(\mathcal{F}) \cup \{\mathcal{F}\} \) will be put into two categories. The independent multisets are those whose nodes are distinct and “far enough” in \( \mathcal{F} \). The remaining ones are said dependent; their number is bounded by a polynomial of degree \( k-1 \).

\[ \text{Definition 6.11 (Independent multiset). Let } \mu : A^* \to M, \ w \in A^* \text{ and } \mathcal{F} \in \text{Forests}_\mu(w), \text{ we say that a multiset } \{t_1, \ldots, t_k\} \subseteq \text{Iters}(\mathcal{F}) \text{ is independent if for all } 1 \leq i \neq j \leq k:\]

\[ \begin{align*}
&= t_i \text{ is not an ancestor of } t_j; \\
&= t_i \text{ is not the immediate left sibling of an ancestor of } t_j; \\
&= t_i \text{ is not the immediate right sibling of an ancestor of } t_j.
\end{align*} \]

Note that if \( \mathcal{T} \) is independent, then \( \mathcal{F} \notin \mathcal{T} \) since it is not an iterable node. We denote by \( \text{Ind}^k(\mathcal{F}) \) the set of independent multisets. Conversely, let \( \text{Dep}^k(\mathcal{F}) \) be the set of multisets \( \{t_1, \ldots, t_k\} \subseteq \text{Iters}(\mathcal{F}) \cup \{\mathcal{F}\} \) which are dependent (i.e. not independent). By Lemma 6.10, if \( \mathcal{M} = (A, M, \mu, \delta, \lambda) \) computes \( f : A^* \to \mathbb{N} \) and \( \mathcal{F} \in \text{Forests}_\mu(w) \), then:

\[ f(w) = \sum_{\mathcal{T} \in \text{Ind}^k(\mathcal{F})} \prod_{M}(\mathcal{T}) + \sum_{\mathcal{T} \in \text{Dep}^k(\mathcal{F})} \prod_{M}(\mathcal{T}) \quad (2) \]

The idea is now to compute these two sums separately. We begin with the second one.

\[ \text{Lemma 6.12. Given a } k\text{-marble binachine } \mathcal{M} = (A, M, \mu, \delta, \lambda), \text{ the following function is (effectively) polyregular and has growth at most } k-1:\]

\[ \text{sum-dep}_\mathcal{M} : (\hat{A})^* \to \mathbb{N}, \mathcal{F} \mapsto \begin{cases} 
\sum_{\mathcal{T} \in \text{Ind}^k(\mathcal{F})} \prod_{M}(\mathcal{T}) & \text{if } \mathcal{F} \in \text{Forests}^{\mathcal{M}}(w) \text{ for some } w \in A^*; \\
0 & \text{otherwise.} \end{cases} \]

\[ \text{Remark 6.13. For this result, we do not need to assume that } \mathcal{M} \text{ is permutable.} \]

6.4 Independent multisets of nodes

In order to complete the description of \( f \) from Equation 2, it remains to treat the productions over independent multisets of nodes. When \( \{t_1, \ldots, t_k\} \in \text{Ind}^k(\mathcal{F}) \), all the \( t_i \) must be distinct, hence we shall denote it by a set \( \{t_1, \ldots, t_k\} \). We define the counterpart of \( \text{sum-dep}_\mathcal{M} \):

\[ \text{sum-ind}_\mathcal{M} : (\hat{A})^* \to \mathbb{N}, \mathcal{F} \mapsto \begin{cases} 
\sum_{\mathcal{T} \in \text{Ind}^k(\mathcal{F})} \prod_{M}(\mathcal{T}) & \text{if } \mathcal{F} \in \text{Forests}^{\mathcal{M}}(w) \text{ for some } w \in A^*; \\
0 & \text{otherwise.} \end{cases} \]

\[ \text{Lemma 6.14. Given a } k\text{-marble binachine } \mathcal{M} \text{ which is } 2^{\mathcal{M}}\text{-permutable, one can build a polyblind function } \text{sum-ind}'_\mathcal{M} : (\hat{A})^* \to \mathbb{N} \text{ and a polyregular function } \text{sum-ind}''_\mathcal{M} : (\hat{A})^* \to \mathbb{N} \text{ with growth at most } k-1, \text{ such that } \text{sum-ind}_\mathcal{M} = \text{sum-ind}'_\mathcal{M} + \text{sum-ind}''_\mathcal{M}. \]

If \( \text{sum-ind}_\mathcal{M} \) was a polynomial, then \( \text{sum-ind}'_\mathcal{M} \) should roughly be its term of highest degree and \( \text{sum-ind}''_\mathcal{M} \) corresponds to a corrective term.

The rest of this section is devoted to the proof of Lemma 6.14. In order to simplify the notations, we extend a morphism \( \mu \) to its \( \mu \)-forests by \( \mu(\mathcal{F}) := \mu(w) \) when \( \mathcal{F} \in \text{Forests}_\mu(w) \). Given \( t \in \text{Nodes}(\mathcal{F}) \), we denote by \( \text{depth}^\mathcal{F}(t) \) its depth in the tree structure \( \mathcal{F} \) (the root has depth 1, and it is defined inductively as usual). Now we introduce the notion of linearization of \( t \in \text{Nodes}(\mathcal{F}) \), which is used to describe \( w[\text{Fr}_\mathcal{F}(t)] \) as a 1-multicontext.
Definition 6.15 (Linearization). Let $\mu : A^* \to M$, $w \in A^*$ and $F \in \text{Forests}_{\mu}(w)$. The linearization of $t \in \text{Nodes}(F)$ is a 1-multicontext $m[1]m'$ defined by induction:

- if $t = F$ then $\text{lin}^F(t) := [w[\text{Fr}_t(F)]]$;
- otherwise $F = (F_1) \cdot \ldots \cdot (F_n)$, and $t \in \text{Nodes}(F_i)$ for some $1 \leq i \leq n$. We define:
  $$\text{lin}^F(t) := \mu(F_1) \cdot \ldots \cdot \mu(F_{i-1}) \cdot \text{lin}^F(t) \cdot \mu(F_{i+1}) \cdot \ldots \cdot \mu(F_n).$$

We finally introduce the notion of architecture. Intuitively, it is a simple tree which describes the positions of a set of nodes $\Xi \in \text{Ind}^k(F)$ in its forest $F$. We build it inductively on the example depicted in Figure 8. At the root, we see that there is no node of $\Xi$ in the left subtree, hence we replace it by its image under $\mu$. The right subtree is an idempotent node whose leftmost and rightmost subtrees have no node in $\Xi$. We thus replace this idempotent node by a leaf containing the multisets of the linearizations and depths of the $t \in \Xi$. Since our machine $M$ is permutable, this simple information will be enough to recover $\text{prod}_{\text{Fr}_t}(\Xi)$.

![Diagram](a) In blue, a set $\Xi$ of 3 independent nodes in the forest from Figure 7

$$\mu(aa) = \{(\mu(\text{ca}), \text{bb})[\mu(\text{cc})[\mu(\text{bb})[\mu(\text{c})[\mu(\text{c})]]]] \}$$

(b) The corresponding architecture

Definition 6.16 (Architecture). Let $w \in A^*$, $F \in \text{Forests}_{\mu}(w)$ and $\Xi \in \text{Ind}^k(F)$. We define the architecture of $\Xi$ in $F$ by induction as follows:

- if $F = e$, then $k = 0$. We define $\text{arc}^F(\Xi) := \varepsilon$;
- if $F = a$, then $k = 0$. We define $\text{arc}^F(\Xi) := \mu(a)$;
- otherwise $F = (F_1) \cdot \ldots \cdot (F_n)$ with $n \geq 1$:
  - if $k = 0$, we set $\text{arc}^F(\Xi) := (\mu(\text{Fr}_t(F)))$;
  - else if $\Xi_1 := \Xi \cap \text{Nodes}(F_1) \neq \emptyset$, then $\Xi_1 \in \text{Ind}^{\text{Fr}_t(F)}_1(F_1)$ (since $F_1 \notin \Xi$ by iterability) and $\Xi = \Xi_1 \otimes \Xi_2 \in \text{Ind}^{\text{Fr}_t(F)}_1(F_2) \otimes \ldots \otimes (F_n)$ (since $F_2 \notin \Xi$ by independence). We set:
    $$\text{arc}^F(\Xi) := (\text{arc}^F(\Xi_1)) \cdot \text{arc}^F(\Xi_2) \otimes \ldots \otimes (\text{arc}^F(\Xi_n)).$$
  - else if $\Xi_n := \Xi \cap \text{Nodes}(F_n) \neq \emptyset$, we define symmetrically:
    $$\text{arc}^F(\Xi) := (\text{arc}^F(\Xi_1)) \cdot \text{arc}^F(\Xi_2) \otimes \ldots \otimes (\text{arc}^F(\Xi_{n-1})) \cdot (\text{arc}^F(\Xi_n)).$$
  - else $\Xi_1 = \Xi_2 = \emptyset$ but $k > 0$, thus $n \geq 3$ and $\mu(\mathcal{F})$ is idempotent. We define:
    $$\text{arc}^F(\Xi) := \{(\text{lin}^F(t), \text{depth}^F(t)) : t \in \Xi \}.$$

Given a morphism, the set of architectures over bounded-height forests is finite.

Claim 6.17. The set $\text{Arcs}^{[3|M|]}_\mu := \{\text{arc}^F(\Xi) : \Xi \in \text{Ind}^k(F), F \in \text{Forests}_{\mu}^{[3|M|]}(w), w \in A^*\}$ is finite, given $k \geq 0$ and a morphism $\mu : A^* \to M$.

Proof. The architectures from $\text{Arcs}^{[3|M|]}_\mu$ are tree structures of height at most $3|M|$. Furthermore they have a branching bounded by $k+3$ and their leaves belong to a finite set (they are either idempotents $e \in M$, or multisets of at most $k$ elements of the form $(m[1]m', d)$ with $m, m' \in M$, $|u| \leq 2|M|$ and $1 \leq d \leq 3|M|).$
Hiding pebbles when the output alphabet is unary

Using the permutability of the $k$-marble bimachine, we show that the production over a set of independent nodes only depends on its architecture. This result enables us to define the notion of production over an architecture.

**Proposition-Definition 6.18.** Let $\mathcal{M} = (A, \lambda, \mu, H, \lambda)$ be a 2$|\mathcal{M}|$-permutable $k$-marble bimachine. Let $w, w' \in A^*$, $F \in \text{Forests}_{\lambda}^{|\mathcal{M}|}(w)$ and $F' \in \text{Forests}_{\lambda}^{|\mathcal{M}|}(w')$, $\mathcal{I} \in \text{Ind}^k(F)$ and $\mathcal{I}' \in \text{Ind}^k(F')$ such that $\lambda := \text{arc}^F(\mathcal{I}) = \text{arc}^{F'}(\mathcal{I}')$. Then \(\prod_{\mathcal{M}}^F(\mathcal{I}) = \prod_{\mathcal{M}}^{F'}(\mathcal{I}')\).

We define \(\prod_{\mathcal{M}}^F(A)\) as the above value.

By using the previous statements, we get for all $w \in A^*$ and $F \in \text{Forests}_{\lambda}^{|\mathcal{M}|}(w)$:

\[
\sum_{\mathcal{I} \in \text{Ind}^k(F)} \prod_{\mathcal{M}}^F(\mathcal{I}) = \sum_{\lambda \in \text{Arcs}_{\lambda}^{|\mathcal{M}|}} \sum_{\mathcal{I} \in \text{Ind}^k(F)} \prod_{\mathcal{M}}^F(\mathcal{I}) = \sum_{\lambda \in \text{Arcs}_{\lambda}^{|\mathcal{M}|}} \prod_{\mathcal{M}}(\lambda) \times \text{count}_{\lambda}(F)
\]

where \(\text{count}_{\lambda}(F) := |\{\mathcal{I} \in \text{Ind}^k(F) : \text{arc}^F(\mathcal{I}) = \lambda\}|\). It describes the number of multisets of independent nodes which have a given architecture. Now we show how to compute this function as a sum of a polyblind function and a polyregular function with lower growth.

**Lemma 6.19.** Let $\mu : A^* \rightarrow M$. Given an architecture $\lambda \in \text{Arcs}_{\lambda}^{|\mathcal{M}|}$, one can build:
- a polyblind function $\text{count}_{\lambda}' : (\hat{A})^* \rightarrow \mathbb{N}$;
- a polyregular function $\text{count}_{\lambda}'' : (\hat{A})^* \rightarrow \mathbb{N}$ with growth at most $k-1$;

such that $\text{count}_{\lambda}(F) = \text{count}_{\lambda}'(F) + \text{count}_{\lambda}''(F)$ for all $F \in \text{Forests}_{\lambda}^{|\mathcal{M}|}(w)$ and $w \in A^*$.

To conclude the proof of Lemma 6.14, we define the function (which is polyblind):

\[
\sum_{\mathcal{I} \in \text{Ind}^k(F)} \prod_{\mathcal{M}}(\lambda) \times \text{count}_{\lambda}'(F)
\]

We define similarly the following function which is polyregular and has growth at most $k-1$:

\[
\sum_{\mathcal{I} \in \text{Ind}^k(F)} \prod_{\mathcal{M}}(\lambda) \times \text{count}_{\lambda}''(F)
\]

**7 Outlook**

This paper provides a technical solution to a seemingly difficult membership problem. This result can be interpreted both in terms of nested transducers (i.e. programs with visible or blind recursive calls) and in terms of rational series. We conjecture that the new techniques introduced here (especially the induction techniques), and the concepts of productions on words and forests, give an interesting toolbox to tackle other decision problem for transducers such as equivalence or membership issues. It could also be interesting to characterize polyblind functions as the series computed by specific weighted automata over $(\mathbb{N}, +, \times)$. 

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**A Proof of Proposition 5.2**

Since polyblind functions correspond to the closure of regular functions under sums and Hadamard products, the result is an immediate consequence of lemmas [A.1] and [A.2] below.

- **Lemma A.1.** A regular function is $k$-repetitive for all $k \geq 1$.

- **Lemma A.2.** Let $f, g : A^* \to \mathbb{N}$ be $k$-repetitive for some $k \geq 1$. Then $f + g$ and $f \odot g$ are $k$-repetitive.

The proof of Claim 5.3 (for $f - g$) is exactly the same as that of Lemma A.2.

**A.1 Proof of Lemma A.1**

Let $f : A^* \to \mathbb{N}$ be computed by a bimachine $\mathcal{M} = (A, M, \mu, \lambda)$. In this proof, we shall use the notion of *productions* of a $k$-marble bimachine introduced in Subsection 5.2 and the properties of this productions shown in Appendix B. Note that we only use them for the notion of polyblind functions, the result is an immediate consequence of lemmas A.1 and A.2 below.

Let $\mu$ be the idempotence index of $M$, that is the smallest $\mu > 0$ such that $\mu^3$ is idempotent for all $m \in M$. Let $k \geq 1$, $\alpha, \alpha_0, u_1, \alpha_1, \ldots, u_k, \alpha_k, \beta \in A^*$ and $\omega \geq 1$ be a multiple of $\mu$. Let $W : \mathbb{N}^k \to A^*$ defined by:

$$W : X_1, \ldots, X_k \mapsto \left( \alpha_0 \prod_{i=1}^k u_i^{\omega X_i \alpha_i} \right).$$

and let $w := W(1, \ldots, 1)$. By definition of $\mu$, we have for all $1 \leq i \leq k$ that $e_i := \mu(u_i^\omega)$ is an idempotent. Hence $p := \mu(w) = \mu(\alpha_0) \prod_{i=1}^k e_i \mu(\alpha_i) = \mu \left( \alpha_0 \prod_{i=1}^k u_i^{\omega X_i \alpha_i} \right)$ is independent of $X_1, X_2, \ldots, X_k \geq 1$ and $e := p^\omega$ is idempotent.

Let $\overline{X} := X_1, \ldots, X_k \geq 3$ and $\overline{Y} := Y_1, \ldots, Y_k \geq 3$, then we can decompose by Lemma B.8

$$f(\alpha w^{2\omega-1} W(\overline{X}) w^{\omega-1} W(\overline{Y}) w^\beta)$$

$$= \prod_{\mathcal{M}} \left( \prod_{i=1}^k u_i^{2\omega_1} \right) pp^\omega \mu(\beta) + \prod_{\mathcal{M}} \left( \alpha_0 \prod_{i=1}^k u_i^{\omega X_i \alpha_i} \right) p^{\omega-1} pp^\omega \mu(\beta)$$

$$+ \prod_{\mathcal{M}} \left( \alpha_0 \prod_{i=1}^k u_i^{\omega X_i \alpha_i} \right) p^{\omega-1} pp^\omega \mu(\beta)$$

$$+ \prod_{\mathcal{M}} \left( \alpha_0 \prod_{i=1}^k u_i^{\omega X_i \alpha_i} \right) p^{\omega-1} pp^\omega \mu(\beta)$$

$$= \prod_{\mathcal{M}} \left( \prod_{i=1}^k u_i^{2\omega_1} \right) e \mu(\beta) + \prod_{\mathcal{M}} \left( \alpha_0 \prod_{i=1}^k u_i^{\omega X_i \alpha_i} \right) e \mu(\beta)$$

$$+ \prod_{\mathcal{M}} \left( \alpha_0 \prod_{i=1}^k u_i^{\omega X_i \alpha_i} \right) e \mu(\beta)$$

The three first terms are constants, we only need to focus on the two last ones. For this, we decompose their productions using the fact that the $e_i = \mu(u_i^\omega)$ are idempotents.

- **Sublemma A.3.** For $m, n \in M$, there exists a polynomial $L \in \mathbb{Z}[X_1, \ldots, X_k]$ of degree 1 such that for all $\overline{X} = X_1, \ldots, X_k \geq 3$ we have:

$$\prod_{\mathcal{M}} \left( \prod_{i=1}^k u_i^{\omega X_i \alpha_i} \right) m = L(\overline{X}).$$
Hence we have
\[ \text{elements } s_1, \ldots, s_n \text{ in which } r_i \geq 0 \text{ is the multiplicity of } s_i \text{ (thus the cardinal is } r_1 + \cdots + r_n). \]

Proof. By cutting the productions with Lemma B.8 we get:

\[
\prod_{M} \left( m \prod_{i=1}^{k} u_i^{X_i} \alpha_i \right) n = \sum_{j=0}^{k} \prod_{M} \left( m \left( \prod_{i=0}^{j-1} \mu(\alpha_i) e_i(\alpha_i) \right) \left( \prod_{i=j}^{k} e_i(\alpha_i) \right) n \right) + \sum_{j=1}^{k} \prod_{M} \left( m \left( \mu(\alpha_0) \prod_{i=1}^{j-1} e_i(\alpha_i) \right) \left( (u_j^\omega)^{X_j} \right) \mu(\alpha_j) \left( \prod_{i=j+1}^{k} e_i(\alpha_i) \right) n \right).
\]

Since \( e_j = \mu(u_j^\omega) \), by item 2 of Lemma B.9 we see that each term of the second line is a polynomial of degree at most 1 in \( X_j \). Furthermore the sum in the first line is a constant.

Let \( L \) be the polynomial given by applying Sublemma A.3 to any of the two last terms of its equation. It follows that for all \( \bar{X}, \bar{Y} \geq 3 \) we have for some \( C \in \mathbb{N} \):

\[
f(\alpha u^{2\omega-1} W(\bar{X}) w^{\omega-1} W(\bar{Y}) w^\omega \beta) = L(\bar{X}) + L(\bar{Y}) + C.
\]

Since \( L \) is a polynomial of degree 1, we finally obtain the function \( F \) of Lemma A.1 (which is in fact a polynomial) by grouping the terms in \( X_i \) and \( Y_i \) together.

A.2 Proof of Lemma A.2

Let \( f \) and \( g \) be two \( k \)-repetitive functions for some \( k \geq 1 \). We show that \( f \circ g \geq 0 \) is \( k \)-repetitive (the other cases are very similar). Let \( \eta_f \) and \( \eta_g \) be the \( \eta \) of each functions given in Definition 4.1, we set \( \eta_f \circ g = \eta_f \times \eta_g \).

Let \( \alpha, \alpha_0, u_1, \alpha_1, \ldots, u_k, \alpha_k, \beta \in A^* \) and \( \omega \geq 1 \) multiple of \( \eta_f \circ g \). Note that \( \omega \) is also a multiple of \( \eta_f \) and \( \eta_g \). Let \( w := \alpha_0 \prod_{1 \leq i \leq k} u_i^\omega \alpha_i \) and define \( W : \mathbb{N}^{2k} \rightarrow A^* \):

\[
W : X_1, Y_1, \ldots, X_k, Y_k \mapsto \alpha_0 \prod_{i=1}^{k} u_i^{\omega X_i} \alpha_i w^{\omega-1} \prod_{i=1}^{k} u_i^{\omega Y_i} \alpha_i w^\omega \beta
\]

Then there exists \( F_f, F_g \) such that for all \( X_1, Y_1, \ldots, X_k, Y_k \geq 3 \):

\[
\begin{align*}
& f(W(X_1, Y_1, \ldots, X_k, Y_k)) = F_f(X_1 + Y_1, \ldots, X_k + Y_k) \\
& g(W(X_1, Y_1, \ldots, X_k, Y_k)) = F_g(X_1 + Y_1, \ldots, X_k + Y_k).
\end{align*}
\]

Hence we have

\[
(f \circ g)(W(X_1, Y_1, \ldots, X_k, Y_k)) = F_f(X_1 + Y_1, \ldots, X_k + Y_k) \times F_g(X_1 + Y_1, \ldots, X_k + Y_k)
= (F_f \cup F_g)(X_1 + Y_1, \ldots, X_k + Y_k)
\]

which concludes the proof with \( F_f \circ g := F_f \circ F_g \).

B Properties of productions

In the forthcoming appendices, we denote by \( \{s_1^{r_1}, \ldots, s_n^{r_n}\} \) a multiset with \( n \) distinct elements \( s_1, \ldots, s_n \), in which \( r_i \geq 0 \) is the multiplicity of \( s_i \) (thus the cardinal is \( r_1 + \cdots + r_n \)).
B.1 Proof of Lemma 5.6

We want to show that if \( J_1, \ldots, J_n \) is a partition of \([1:w]\), then:

\[
f(w) = \sum_{\{I_1, \ldots, I_k\} \subseteq (J_1, \ldots, J_n)} \prod_{M_i}^w (\{I_1, \ldots, I_k\}) .
\]

► Sublemma B.1. Let \( M \) be a \( k \)-marble bimachine which computes a function \( f : A^* \to \mathbb{N} \). Then for all \( w \in A^* \) we have:

\[
f(w) = \sum_{\{i_1, \ldots, i_k\} \subseteq [1:w]} \prod_{M_i}^w (\{i_1, \ldots, i_k\}) = \prod_{M_i}^w (\{[1:w] \mid \{k\}\}) .
\]

Proof. The second equality is obtained by definition of \( \prod_{M_i}^w (\{[1:w] \mid \{k\}\}) \). We show the first one by induction on \( k \geq 1 \). The base case is trivial, let us assume that \( k \geq 2 \). Let \( M = (A, M, \mu, \mathcal{F}, \lambda) \) be the \( k \)-marble bimachine which computes \( f \). For \( 1 \leq i_k \leq |w| \), let \( h_{i_k} = \lambda(\mu([w|i_k-1]), [w|i_k], \mu([w|i_k+1:w])) \in \mathcal{F} \) and \( M_{i_k} \) be the \((k-1)\)-marble bimachine which computes it. Then we have:

\[
f(w) = \sum_{i_k=1}^{[w]} h_{i_k} ([w] \mid 1:i_k) \quad \text{by definition of a bimachine with external marble functions}
\]

\[
= \sum_{i_k=1}^{[w]} \sum_{i_1 \leq \ldots \leq i_k} \prod_{M_{i_k}}^w (\{i_1, \ldots, i_k\}) \quad \text{by induction hypothesis}
\]

\[
= \sum_{i_1 \leq \ldots \leq i_k \leq [1:w]} \prod_{M_{i_k}}^w (\{i_1, \ldots, i_k\})
\]

\[
= \prod_{M_i}^w (\{i_1, \ldots, i_k\}) \quad \text{by definition of} \prod_{M_i}^w .
\]

We now show how to decompose a production when one set is split.

► Definition B.2. We say that \((r_1, \ldots, r_n) \in \mathbb{N}^n\) is a \( k \)-sum if \( n \geq 0 \) and \( r_1 + \cdots + r_n = k \). We denote by \( \Sigma_k \) the set of \( k \)-sums.

► Sublemma B.3. Let \( M \) be a \( k \)-marble bimachine, \( w \in A^* \), \( I_1, \ldots, I_n \) be disjoint subsets of \([1:w]\) and \((r_1, \ldots, r_n) \in \Sigma_k \). Assume that \( I_1 = J \cup J' \), then:

\[
\prod_{M_i}^w (\{I_1 \triangleright r_1, \ldots, I_n \triangleright r_n\}) = \sum_{(j,j') \in \Sigma_{r_1}} \prod_{M_i}^w (\{J_1 \triangleright j, J' \triangleright j', I_2 \triangleright r_2, \ldots, I_n \triangleright r_n\}) .
\]

Proof. By definition of productions and since the sets are disjoint, we have:

\[
\prod_{M_i}^w (\{I_1 \triangleright r_1, \ldots, I_n \triangleright r_n\})
\]

\[
= \sum_{\{i_1^1, \ldots, i_n^1\} \subseteq I_1} \cdots \sum_{\{i_1^n, \ldots, i_n^n\} \subseteq I_n} \prod_{M_i}^w (\{i_1^1, \ldots, i_n^n\})
\]

\[
= \sum_{0 \leq j \leq r_1} \sum_{\{i_1^1, \ldots, i_n^1\} \subseteq J} \cdots \sum_{\{i_1^n, \ldots, i_n^n\} \subseteq J} \prod_{M_i}^w (\{i_1^1, \ldots, i_n^n\})
\]

The last line is obtained since \( J \) and \( J' \) are disjoint. It is easy to see that it corresponds to the expression we are looking for.
This result is generalized by induction to any partition of $I_1$ in Sublemma B.4.

**Sublemma B.4.** Let $\mathcal{M}$ be a $k$-marble bimachine, $w \in A^*$, $I_1, \ldots, I_n$ be disjoint subsets of $[1:]$ and $(r_1, \ldots, r_n) \in \Sigma_k$. Assume that $I_1 = J_1 \cup \cdots \cup J_p$, then:

$$\prod^w_{\mathcal{M}} (\langle I_1 \upharpoonright r_1, \ldots, I_n \upharpoonright r_n \rangle) = \sum_{(j_1, \ldots, j_p) \in \Sigma_r} \prod^w_{\mathcal{M}} (\langle J_1 \upharpoonright j_1, \ldots, J_p \upharpoonright j_p, I_2 \upharpoonright r_2, \ldots, I_n \upharpoonright r_n \rangle).$$

**Proof.** The result is shown by induction on $p$ with Sublemma B.3. ▶

**Remark B.5.** This result is in fact stronger than what we need for Lemma 5.6, but it shall be re-used in the following sections.

We conclude this subsection by showing Lemma 5.6 as follows:

$$f(w) = \sum_{(r_1, \ldots, r_n) \in \Sigma_k} \prod^w_{\mathcal{M}} (\langle J_1 \upharpoonright r_1, \ldots, J_n \upharpoonright r_n \rangle)$$

by Sublemma B.1 and Sublemma B.3.

$$= \sum_{\vec{J} \subseteq \langle I_1, \ldots, I_n \rangle} \prod^w_{\mathcal{M}} (\langle I_1, \ldots, I_n \rangle).$$

**B.2 Proof of Proposition-Definition 5.8**

We shall in fact show Proposition-Definition B.7, which is a generalization of Proposition-Definition 5.8. It shall be useful in the rest of these appendices.

**Sublemma B.6.** Let $\mathcal{M} = (A, M, \mu, \delta, \lambda)$ be a $k$-marble bimachine. Let $a_1, \ldots, a_n \in A$, $(r_1, \ldots, r_n) \in \Sigma_k$, $w := v_0 a_1 \cdots a_n v_n \in A^*$ and $w' := v_0' a_1 \cdots a_n v_n' \in A^*$ such that $\mu(v_j) = \mu(v_j')$ for all $0 \leq j \leq n$. For $1 \leq j \leq n$ let $i_j \in [1:]$ (resp. $i_j' \in [1:]$) be the position of $a_j$ in $w$ (resp. $w'$). Then

$$\prod^w_{\mathcal{M}} (\langle i_1 \upharpoonright r_1, \ldots, i_n \upharpoonright r_n \rangle) = \prod^{w'}_{\mathcal{M}} (\langle i_1' \upharpoonright r_1, \ldots, i_n' \upharpoonright r_n \rangle).$$

**Proof.** The proof is done by induction on $k \geq 1$. For $k = 1$ the base case is obvious, let us assume that $k = 2$. Without loss of generality assume that $r_1 = 1$. Let $b := \lambda(w[1:i_1-1], w[i_n], \mu(w[i_n+1:w')))$. Let $M_b$ be the submachine computing $b$, we get:

$$\prod^w_{\mathcal{M}} (\langle i_1 \upharpoonright r_1, \ldots, i_n \upharpoonright r_n \rangle) = \prod^{w[1:i_1]}_{\mathcal{M}_b} (\langle i_1 \upharpoonright r_1, \ldots, i_n \upharpoonright (r_n - 1) \rangle)$$

by definition

$$= \prod^{w[1:i_1]}_{\mathcal{M}_b} (\langle i_1' \upharpoonright r_1, \ldots, i_n' \upharpoonright (r_n - 1) \rangle)$$

by induction hypothesis

$$= \prod^{w'}_{\mathcal{M}} (\langle i_1' \upharpoonright r_1, \ldots, i_n' \upharpoonright r_n \rangle)$$

by definition. ▶

Let us now give an extension of Equation 1 to state our generalization of Proposition-Definition 5.8. Let $(r_1, \ldots, r_n) \in \Sigma_k$, $w := v_0 u_1 \cdots u_n v_n \in A^*$, and for $1 \leq i \leq n$, let $I_i \subseteq [1:]$ be the set of positions corresponding to $u_i$. We define:

$$\prod_{\mathcal{M}} (v_0 \llbracket u_1 \rrbracket, v_1 \cdots \llbracket u_n \rrbracket, v_n) := \prod^w_{\mathcal{M}} (\langle I_1 \upharpoonright r_1, \ldots, I_n \upharpoonright r_n \rangle).$$

Note that Equation 1 is the case when $r_i = 1$ for all $1 \leq i \leq n$ (thus $n = k$). On the other hand, if $r_i = 0$, then the element $\llbracket u_i \rrbracket_{r_i}$ can be equivalently replaced by $\mu(u_i)$ (it means that in fact $u_i$ is not used when making the production).
**Proposition-Definition B.7.** Let \( M = (A, M, \mu, \delta, \lambda) \) be a \( k \)-marble bimachine. Let \((r_1, \ldots, r_n) \in \Sigma_k\), \(v_0u_1 \cdots u_n v' \in A^*\) and \( v_0' u_1' \cdots u_n' v'' \in A^*\) such that \( \mu(v_i) = \mu(v'_i) \) for all \( 0 \leq i \leq n \). Then we have \( \prod_M (v_0[u_1]_{r_1}v_1 \cdots [u_n]_{r_n} v_n) = \prod_M (v'_0[u_1]_{r_1}v'_1 \cdots [u_n]_{r_n} v''_n) \). Let \( m_i := \mu(v_i) = \mu(v'_i) \), we define \( \prod_M (m_0[u_1]_{r_1}m_1 \cdots [u_n]_{r_n} m_n) \) as the previous value.

**Proof.** Let \( w := v_0u_1 \cdots u_n v_n \) and \( w' := v_0'u_1' \cdots u_n' v''_n \). Let \( I_1, \ldots, I_n \subseteq [1:|w|] \) (resp. \( I'_1, \ldots, I'_n \subseteq [1:|w'|] \)) be the sets of positions of \( u_1, \ldots, u_n \) in \( w \) (resp. \( w' \)). For \( 1 \leq i \leq n \), let \( \sigma_i : n_i \rightarrow n'_i \) be the unique monotone (for \(<\)) bijection between these sets. Then if \( \{i_1, \ldots, i_n\}_r \subseteq I_1, \ldots, \{i'_1, \ldots, i'_n\}_{r'} \subseteq I'_n \), it follows from Sublemma B.6 that:

\[
\prod_M^w \left( \prod_{\{i_1, \ldots, i_n\}_r} w \prod_{\{i'_1, \ldots, i'_n\}_{r'}} w' \right) = \prod_M^w \left( \prod_{\{\sigma_1(i_1), \ldots, \sigma_1(i_n)\}_\mu} \prod_{\{\sigma_1(i'_1), \ldots, \sigma_1(i'_n)\}_{\nu}} \right).
\]

Finally we get \( \prod_M^w (\prod I_1^{r_1}, \ldots, I_n^{r_n}) = \prod_M^w (\prod I'_1^{r_1}, \ldots, I'_n^{r_n}) \) by summing all the terms of the above form. ◀

**B.3 Further results**

In this subsection, we establish further properties of productions which are to be used later. We first give an analogous of Sublemma B.4 in order to “split” the productions on the factors.

**Lemma B.8.** Let \( M = (A, M, \mu, \delta, \lambda) \) be a \( k \)-marble bimachine. Let \( v_1 \cdots v_n u_1 \cdots v'_n \in A^* \) and \((\delta_1, \ldots, \delta_n, r, \delta'_1, \ldots, \delta'_n) \in \Sigma_k\). Let \( L := [v_1]\delta_1 \cdots [v_n]\delta_n \) and \( R := [v'_1]\delta'_1 \cdots [v'_n]\delta'_n \), then we have:

\[
\prod_M (L [u_1]_{r_1} \cdots [u_X]_{r_X} R) = \sum_{(r_1, \ldots, r_X) \in \Sigma_r} \prod_M (L [u_1]_{r_1} \cdots [u_X]_{r_X} R).
\]

**Proof.** We consider those productions as productions of sets of positions within the word \( v_1 \cdots v_n u_1 \cdots u_X v'_1 \cdots v'_{n'} \). The result follows from Sublemma B.4 by choosing \( J_1, \ldots, J_X \) as the positions corresponding to \( u_1, \ldots, u_X \), and using Proposition-Definition B.7. ◀

We now want to describe in a more precise way what happens in Lemma B.8 when \( u_1 = \cdots = u_X = u \) and \( \mu(u) \) is idempotent. This is the purpose of Lemma B.9 below. Given \( r \geq 0 \) and \((r_1, \ldots, r_X) \in \Sigma_r\) we define \( \text{shape}(r_1, \ldots, r_X) \) as the tuple obtained from \((r_1, \ldots, r_X) \) by replacing the maximal blocks of the form 0, 0, 0, 0, 0, 1, 2 = (0, 0, 1, 2) \( \in \Sigma_4 \). Let \( \text{Shapes}_r \) be the set of all shapes of elements of \( \Sigma_r \) (note that \( \text{Shapes}_r \) is a finite subset of \( \Sigma_r \)).

**Lemma B.9.** Let \( M = (A, M, \mu, \delta, \lambda) \) be a \( k \)-marble bimachine.

Let \((\delta_1, \ldots, \delta_n, r, \delta'_1, \ldots, \delta'_n) \in \Sigma_k\) and \( v_1, \ldots, v_n, u, v'_1, \ldots, v'_{n'} \in A^* \). Assume that \( e := \mu(u) \) is idempotent. Let \( L := [v_1]\delta_1 \cdots [v_n]\delta_n \) and \( R := [v'_1]\delta'_1 \cdots [v'_{n'}]\delta'_{n'} \). \( \text{If } X \geq 2r+1, \text{ then:} \)

1. \( \prod_M (L [u_X]_{r_X} R) = \sum_{s=(x_1, \ldots, x_{r}) \in \text{Shapes}_r} P_s(X) \times \prod_M (L [u_1]_{x_1} \cdots [u_X]_{x_X} R) \)

where \( P_s \) is a polynomial in \( X \) of degree at most \( r \) which is independent of \( L, R \) and \( u \);

2. \( \prod_M (L [u_X]_{r_X} R) \) is a polynomial in \( X \) of degree at most \( r \);

3. the coefficient in \( X \) of the polynomial \( \prod_M (L [u_X]_{r_X} R) \) is:

   a. 0 if \( r = 0 \);

   b. \( \prod_M (L e[u] e R) \) if \( r = 1 \).
Proof. We have $\prod_M(\mathcal{L}[u]^X, \mathcal{R}) = \sum_{(r_1, \ldots, r_X) \in \Sigma_r} \prod_M(\mathcal{L}[u]_{r_1} \cdots [u]_{r_X}, \mathcal{R})$ by Lemma B.8.

We shall recombine several terms of this sum thanks to the claim below.

\begin{claim}[B.10] Let $(r_1, \ldots, r_X) \in \Sigma_r$ and $(s_1, \ldots, s_Y) \in \text{shape}(r_1, \ldots, r_X)$, then:
$$\prod_M(\mathcal{L}[u]_{r_1} \cdots [u]_{r_X}, \mathcal{R}) = \prod_M(\mathcal{L}[u]_{s_1} \cdots [u]_{s_Y}, \mathcal{R})$$
\end{claim}

\begin{proof}
Transforming several consecutive $[u]_0$ in a single one corresponds to transforming the concatenation of several idempotent $e = \mu(u)$ in a single one.
\end{proof}

Given $s \in \text{Shapes}_r$, let $P_s(X)$ be the number of tuples $(r_1, \ldots, r_X) \in \Sigma_r$ such that $\text{shape}(r_1, \ldots, r_X) = s$. This quantity does not depend on $\mathcal{L}, \mathcal{R}$ nor $u$. Furthermore, it follows from Claim [B.10] that:
$$\prod_M(\mathcal{L}[v]^X, \mathcal{R}) = \sum_{s=(s_1, \ldots, s_Y) \in \text{Shapes}_s} P_s(X) \times \prod_M(\mathcal{L}[v]_{s_1} \cdots [v]_{s_Y}, \mathcal{R})$$

We still need to show that the $P_s$ are polynomials of degree at most $r$, this is Claim B.11.

\begin{claim}[B.11] For $X \geq 2r+1$, $P_s$ is a polynomial in $X$ of degree at most $r$.
\end{claim}

\begin{proof}
If $s := (s_1, \ldots, s_Y) \in \text{Shapes}_r$, one has $Y \leq 2r+1$ since there are no two consecutive 0. Let $0 \leq z \leq r+1$ be the number of zeros in $s$, then $P_z(X)$ is the number of tuples $(q_1, \ldots, q_z) \in \Sigma_{X-Y}$ (it describes how many times we count each 0). We show by induction on $z \geq 0$ that it is a polynomial in $X$ of degree at most $z-1$ whenever $X-Y \geq 0$.

It follows directly from Claim B.11 and Equation 3 that $\prod_M(\mathcal{L}[u]^X, \mathcal{R})$ is a polynomial in $X$ for $X \geq 2r+1$. It remains to deal specifically with the cases $r = 0$ (which is obvious, we only have a constant) and $r = 1$. Note that $\text{Shapes}_1 = \{(0,1), (1,0), (0,1,0)\}$. By definition of $P_s$ we get $P_{(0,1)}(X) = P_{(1,0)}(X) = 1$ and $P_{(0,1,0)}(X) = X - 2$ for $X \geq 3$. Hence the coefficient in $X$ is $\prod_M(\mathcal{L}[u]_0 [u]_1 [u]_0, \mathcal{R})$ which can be rewritten $\prod_M(\mathcal{L} e[u] e, \mathcal{R})$.
\end{proof}

\section{Proof of Proposition 5.12}

In order to show Proposition 5.12, we shall use Lemma C.1 below. Intuitively, it says that under the same conditions as those of Definition 5.11 if we extract one factor $u_j$ while preserving left $j$ and right $j$, then the production will be the same (see Figure 0).

\begin{lemma}[C.1]
Let $K \geq 0$. Let $f$ be a $k$-repetitive function computed by a $k$-marble bimachine $M = (A, M, \mu, G, \lambda)$. The following holds whenever $\ell + x + r = k$:

- let $\mathcal{L}$ (resp. $\mathcal{R}$) be a $(\ell, K)$-iterator (resp. $(r, K)$-iterator);
- let $m_0 \left( \prod_{i=1}^x e_i [u_i] e_i m_i \right)$ be an $(x, K)$-iterator with $e := m_0 \left( \prod_{i=1}^x e_i m_i \right)$ idempotent;
- choose $1 \leq j \leq x$, and define:
$$\text{left} : = e \left( \prod_{i=1}^j \prod_{i=j}^x e_i m_i \right)$$
and
$$\text{right} : = \left( \prod_{i=1}^j e_i m_i \right) e$$

then we have:
$$\prod_M(\mathcal{L} e m_0 \left( \prod_{i=1}^x e_i [u_i] e_i m_i \right) e, \mathcal{R}) = \prod_M(\mathcal{L} e m_0 \left( \prod_{i=1}^{j-1} e_i [u_i] e_i m_i \right) e j m_j \left( \prod_{i=j+1}^x e_i [u_i] e_i m_i \right) e (\text{left} [u_j] \text{right} j), \mathcal{R})$$
\end{lemma}

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Proposition 5.12 is obtained by induction on $x$ from Lemma C.1. The proof is depicted in Figure 9 for $x = 3$ and $\sigma = (3, 1, 2)$. Since $u_2$ has to be the last element after substitution, we first apply Lemma C.1 with $j = 2$ to send it “on the right”, then we do the same with $u_1$.

(a) Initial production with $x = 3$

(b) Production after applying Lemma C.1 once with $x = 3$

(c) Production after applying Lemma C.1 again with $x = 2$

Proof of Lemma C.1. The rest of this section is devoted to the proof of Lemma C.1. The idea is to build a word in which the two productions compared in Lemma C.1 occur. Then, by iterating some factors in this word, we use the $x$-repetitiveness (since $x \leq k$) of $f$ to show that these productions must be equal. Let $\omega$ be given by Definition 4.1, assume that $\omega \geq 3$. Assume that:

$$L = p_0 \left( \prod_{i=1}^{\ell} f_i[u_i]f_i p_i \right) \quad \text{and} \quad R = p'_0 \left( \prod_{i=1}^{r} f'_i[u'_i]f'_i p'_i \right)$$

For every $m \in M$ in these notations, we denote $\overline{m} \in A^*$ a word such that $\mu(\overline{m}) = m$ (it exists since $\mu$ is supposed surjective). We then define:

- for $\overline{L} := L_1, \ldots, L_\ell \geq 0$, $U(\overline{L}) := \overline{p_0} \left( \prod_{i=1}^{\ell} v_i^{L_i} \right)$;
- for $\overline{X} := X_1, \ldots, X_r \geq 0$, $W(\overline{X}) := \overline{m_0} \left( \prod_{i=1}^{x} u_i^{X_i} \right)$;
- for $\overline{R} := R_1, \ldots, R_r \geq 0$, $V(\overline{R}) := \overline{p'_0} \left( \prod_{i=1}^{r} v'_i^{R_i} \right)$.

Let $w := W(1, \ldots, 1)$, note that for $\overline{X} \geq 1$, $\mu(W(\overline{X})) = \mu(w) = e$. We define:

$$P : \mathbb{N}^{k+1} \to \mathbb{N}$$

$$\overline{L}, \overline{X}, X'_j, \overline{R} \mapsto f(U(\overline{L})w^{2\omega - 1}W(\overline{X})w^{\omega - 1}W(3, \ldots, 3, X'_j, 3, \ldots, 3)w^\omega V(\overline{R}))$$

where $X'_j$ is in position $j$ of $W(3, \ldots, 3, X'_j, 3, \ldots, 3)$. 

\[ \text{Figure 9} \quad \text{Proof idea for Proposition 5.12 with } x = 3 \text{ and } \sigma = (3, 1, 2) \]
Let $T := L_1 \cdots L_t X_1 \cdots X_x R_1 \cdots R_r / X_j$. By applying Sublemma C.4 stated in Section C.1 (it does not use the fact that $f$ is $k$-repetitive) and the definitions of our words, we get:

\[ \text{Claim C.2.} \quad \text{For } T, \overline{X}, X'_j, \overline{R} \geq 2k + 1, \text{ } P(T, X, X'_j, \overline{R}) \text{ is a polynomial and:} \]

- the coefficient in $TX_j$ of $P$ is:

\[ c := \prod_M \left( \mathcal{L} \text{ em}_0 \left( \prod_{i=1}^x e_i \left[ u_i \right] e_i m_i \right) e_j \right); \]

- the coefficient in $TX'_j$ of $P$ is:

\[ c' := \prod_M \left( \mathcal{L} \text{ em}_0 \left( \prod_{i=1}^{j-1} e_i \left[ u_i \right] e_i m_i \right) e_j m_j \left( \prod_{i=j+1}^x e_i \left[ u_i \right] e_i m_i \right) e \left( \text{left}_j \left[ u_j \right] \text{right}_j \right) \right). \]

Since $f$ is $k$-repetitive, we also get the following from our construction.

\[ \text{Claim C.3.} \quad \text{There exists a function } F : \mathbb{N}^k \rightarrow \mathbb{N} \text{ such that for } T, \overline{X}, X'_j, \overline{R} \geq 2k + 1: \]

\[ P(T, X, X'_j, \overline{R}) = F(T, X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_x, \overline{R}, X_j + X'_j). \]

**Proof.** The function $f$ is $x$-repetitive since it is $k$-repetitive and $x \leq k$. Let $T$ and $\overline{R}$ be fixed, using the definition of $P$ and Definition 4.1 (with $\alpha = U(T)$ and $\beta = V(\overline{R})$) we have that $P$ only depends on $X_1 + 3, \ldots, X_{j-1} + 3, X_j + X'_j, X_{j+1} + 3, \ldots, X_x + 3$. \hfill \triangle

From Claim C.3, we deduce that for $T, \overline{X}, X'_j, \overline{R} \geq 2k + 1$:

\[ P(T, X_1, \ldots, X_{j-1}, X_j, X_{j+1}, \ldots, X_x, X'_j, \overline{R}) = P(T, X_1, \ldots, X_{j-1}, 2k+1, X_{j+1}, \ldots, X_x, X'_j + X_j - (2k+1), \overline{R}). \tag{4} \]

By developing the second expression in Equation 4, it is easy to see that the coefficients in $TX_j$ and in $TX'_j$ of $P$ are equal. Hence $c = c'$ in Claim C.2.

### C.1 Statement and proof of Sublemma C.4

Intuitively, the result below says that if we iterate $k+1$ idempotent factors in a word, then the output will be a multivariate polynomial in the number of iterations. Furthermore, specific terms of the polynomial allow to recover the productions over these factors.

\[ \text{Sublemma C.4.} \quad \text{Let } M = (A, M, \mu, \delta, \lambda) \text{ be a } k \text{-marble bimachine.} \]

- Let $\alpha_0 u_1 \alpha_1 \cdots u_{k+1} \alpha_{k+1} \in A^*$. Let $m_i := \mu(\alpha_i), e_i := \mu(u_i)$ and assume that the $e_i$ are idempotent. Then $P : X_1, \ldots, X_{k+1} \mapsto f \left( \alpha_0 u_1 \cdots u_{k+1} \alpha_{k+1} \alpha_{k+1} \right)$ is a polynomial for $X_1, \ldots, X_{k+1} \geq 2k + 1$. For $1 \leq j \leq k + 1$, the coefficient of $P$ in $X_1 \cdots X_{k+1} / X_j$ is:

\[ \prod_M \left( m_0 \prod_{i=1}^j e_i \left[ u_i \right] e_i m_i \right) e_j m_j \left( \prod_{i=j+1}^{k+1} e_i \left[ u_i \right] e_i m_i \right). \]

We shall prove a stronger result by induction.
Induction hypothesis. Given \(1 \leq \ell \leq k + 1\), let \(\beta(X_1, \ldots, X_\ell) := \alpha_0 u_{X_1}^{\ell_1} \cdots u_{X_\ell}^{\ell_\ell} e_{\ell_\ell}\). The induction hypothesis on \(\ell\) states that for all \(v_1, \ldots, v_n \in A^*\), for all \((\delta_1, \ldots, \delta_n, r) \in \Sigma_k\), if \(\mathcal{L} = [v_1]_{\delta_1} \cdots [v_n]_{\delta_n}\), then \(X_1, \ldots, X_\ell \mapsto \prod M ([\beta(X_1, \ldots, X_\ell)]_{r}, \mathcal{L})\) is a polynomial for \(X_1, \ldots, X_\ell \geq 2k + 1\). Furthermore:

1. if \(\ell \geq 1\), the coefficient in \(X_1 \cdots X_\ell\) is:
   a. \(0\) if \(r < \ell\);
   b. \(\prod M \left( m_0 \left( \prod_{i=1}^{\ell} e_i[u_i]e_{m_i} \right) \right)\) if \(r = \ell\);

2. if \(\ell \geq 2\) and \(1 \leq j \leq \ell\), the coefficient in \(X_1 \cdots X_\ell/X_j\) is:
   a. \(0\) if \(r < \ell - 1\);
   b. \(\prod M \left( m_0 \prod_{i=1}^{j-1} e_i[u_i]e_{m_i} e_j m_j \left( \prod_{i=\ell+1}^{\ell} e_i[u_i]e_{m_i} \right) \right)\) if \(r = \ell - 1\).

Proof by induction. Let \(\ell \geq 1\) and assume that the result holds for \(\ell - 1\) (define \(\beta() := \alpha_0\), then the result is true by emptiness for \(\ell = 0\)). Then:

\[
\prod M ([\beta(X_1, \ldots, X_\ell)]_{r}, \mathcal{L}) = \prod M ([\beta(X_1, \ldots, X_{\ell-1})u_{X_\ell}^{\ell_\ell} e_{\ell_\ell}]_{r}, \mathcal{L})
\]

\[
= \sum_{(r_1, r_2, r_3) \in \Sigma_r} \prod M \left( [\beta(X_1, \ldots, X_{\ell-1})]_{r_1} [u_{X_\ell}^{\ell_\ell}]_{r_2} [e_{\ell_\ell}]_{r_3} \mathcal{L} \right)
\]

(5)

We now want to cut the factor \(u_{X_\ell}^{\ell_\ell}\) in several pieces, for this we use item 2 of Lemma B.9. It follows from Equation 5 that for all \(X_\ell \geq 2k + 1\):

\[
\prod M ([\beta(X_1, \ldots, X_\ell)]_{r}, \mathcal{L}) = \sum_{(r_1, r_2, r_3) \in \Sigma_r} \sum_{s = (s_1, \ldots, s_\ell) \in \text{Shapes}_{s_\ell}} (P_s(X_\ell)
\]

\[
\times \prod M \left( [\beta(X_1, \ldots, X_{\ell-1})]_{r_1} [u_{X_j}^{s_j}]_{r_2} \cdots [u_{X_\ell}^{s_\ell}]_{r_3} \mathcal{L} \right)
\]

(6)

Each last term is by induction hypothesis a polynomial in \(X_1, \ldots, X_{\ell-1}\) if they are \(\geq 2k+1\). Since \(P_s\) only depends on \(s\) by Lemma B.9 we conclude that \(\prod M ([\beta(X_1, \ldots, X_\ell)]_{r}, \mathcal{L})\) is a polynomial in \(X_1, \ldots, X_\ell\). Now let us treat the specific cases:

1. consider the coefficient in \(X_1 \cdots X_\ell\) in \(\prod M ([\beta(X_1, \ldots, X_\ell)]_{r}, \mathcal{L})\):
   a. if \(r < \ell\), then by induction the only term in the first sum of Equation 6 which contains a factor in \(X_1 \cdots X_{\ell-1}\) with a possibly non-zero coefficient is for \(r_1 = r\) and \(r_2 = r_3 = 0\), but then we have no \(X_\ell\). Hence the coefficient in \(X_1 \cdots X_\ell\) is \(0\);
   b. if \(r = \ell\), then by induction hypothesis three terms can provide a factor in \(X_1 \cdots X_\ell\):
      = \(r_1 = \ell, r_2 = 0, r_3 = 0\) or \(r_1 = \ell - 1, r_2 = 0, r_3 = 1\), but then we also have no \(X_\ell\);
      = \(r_1 = \ell - 1, r_2 = 1, r_3 = 0\). Then by using item 3 of Lemma B.9 and induction hypothesis, the coefficient in \(X_1 \cdots X_\ell\) has the good shape;

2. now let \(\ell \geq 2\) and \(1 \leq j \leq \ell\), and consider the coefficient in \(X_1 \cdots X_\ell/X_j\). We assume that of \(j < \ell\) and \(\ell \geq 3\) (the other cases can be treated using similar techniques):
   a. if \(r < \ell - 1\), then by induction the only term in Equation 6 which contains a factor in \(X_1 \cdots X_{\ell-1}/X_j\) with a possibly non-zero coefficient is for \(r_1 = r - 1\) and \(r_2 = r_3 = 0\), but then we have no \(X_j\). Hence the coefficient in \(X_1 \cdots X_{\ell-1}/X_j\) is \(0\);
   b. if \(r = \ell - 1\), then by induction hypothesis three terms can provide a factor in \(X_1 \cdots X_\ell\):
      = \(r_1 = \ell - 1, r_2 = 0, r_3 = 0\) or \(r_1 = \ell - 2, r_2 = 0, r_3 = 1\), but then we also no \(X_j\);
      = \(r_1 = \ell - 2, r_2 = 1, r_3 = 0\). Then by using item 3 of Lemma B.9 and induction hypothesis, the coefficient in \(X_1 \cdots X_\ell\) has the good shape.
D  Proof of Lemma 6.12

Given a \( k \)-marble bimachine \( M = (A, M, \mu, \delta, \lambda) \), we want to show that the following function is polyregular and has growth at most \( k-1 \):

\[
\text{sum-dep}_{M} : (\tilde{A})^{*} \to \mathbb{N}, F \mapsto \begin{cases} \prod_{\Sigma \in \text{Dep}^{k}(F)} \mu(\Sigma) & \text{if } F \in \text{Forests}_{3}^{M}(w) \text{ for some } w \in A^{*}; \\ 0 & \text{otherwise.} \end{cases}
\]

We show the two parts of this statement separately. We first recall a result from [6].

Lemma D.1 ([6]). If \( \mu : A^{*} \to M, w \in A^{*} \) and \( F \in \text{Forests}_{\mu}(w) \), then:
\( \{\text{Fr}_{F}(t) : t \in \text{Iters}(F) \cup \{F\}\} \) is a partition of \([1:w]\] .

D.1  Proof that \( \text{sum-dep}_{M}(F) = O(|F|^{k-1}) \)

We first show that each term of the sum defining \( \text{sum-dep}_{M} \) is bounded independently from \( w, F \) and \( \Sigma \). This is the following sublemma.

Sublemma D.2. If \( M = (A, M, \mu, \delta, \lambda) \) is a \( k \)-marble bimachine, then exists \( K \geq 0 \) such that for all \( w \in A^{*}, F \in \text{Forests}_{\mu}(w) \), for all \( \Sigma \subseteq \text{Iters}(F) \cup \{F\} \), we have \( \prod_{\Sigma} (\mu(\Sigma)) \leq K \).

Proof. Let us assume that \( \Sigma = \{t_1 \downarrow r_1, \ldots, t_n \downarrow r_n\} \). Then the frontiers of the \( t_i \) are disjoint by Lemma D.1 and by definition we have:

\[
\prod_{\Sigma} = \sum_{\{i_1, \ldots, i_n\} \subseteq \text{Fr}_{F}(t_1)} \cdots \sum_{\{i_1, \ldots, i_n\} \subseteq \text{Fr}_{F}(t_n)} \prod_{\Sigma} (\{i_1, \ldots, i_n\}) \tag{7}
\]

Note that for \( t \in \text{Nodes}(F) \), we have \( |\text{Fr}_{F}(t)| \leq 2|F|^{3} \). Indeed, the frontier are the leaves of the skeleton which is a binary tree whose height is bounded by \( 3|M| \) since \( F \in \text{Forests}_{\mu}(w) \). Furthermore, it is easy to see that if \( \{i_1, \ldots, i_k\} \subseteq [1:w] \), one has \( \prod_{\Sigma} (\{i_1, \ldots, i_k\}) \leq B \) for some constant \( B \) depending only on \( M \). Hence the sums in Equation (7) are sums of bounded terms over a bounded range: they are bounded.

To conclude the proof, it remains to show that the number of multisets in \( \text{Dep}^{k}(F) \) grows in \( O(|F|^{k-1}) \). For this we note that:

\[
|\text{Dep}^{k}(F)| \leq \left| \{(t_1, \ldots, t_k) : \{t_1, \ldots, t_k\} \in \text{Dep}^{k}(F)\} \right|
\leq \sum_{1 \leq i \neq j \leq k} \left| \{(t_1, \ldots, t_k) \in (\text{Nodes}(F))^k : \{t_i, t_j\} \text{ is not independent}\} \right|
\leq 12|M|k^2 |\text{Nodes}(F)|^{k-1} = O(|F|^{k-1})
\]

Indeed, given \( t_i \in \text{Iters}(F) \cup \{F\} \), there are at most \( 6|M| \) nodes \( t_j \) whose depth is smaller than \( t_i \) and such that \( \{t_i, t_j\} \) is not independent.

D.2  Proof that \( \text{sum-dep}_{M} \) is polyregular

We first note that the set of \( \{F \in \text{Forests}_{\mu}(w) : w \in A^{*}\} \) is regular. Since regular properties can be checked by pebble transducers (e.g. by launching a one-way automaton to test whether the property holds before beginning the computation), we can without loss of generality assume that our inputs belongs to this set.
Let \( w \in A^* \) and \( F \in \text{Forests}_\mu(w) \). If \( 1 \leq i \leq |w| \), by Lemma D.1 let \( \text{Fr}_F^{-1}(i) \) be the unique \( t \in \text{Ilers}(F) \cup \{F\} \) such that \( i \in \text{Fr}_F(t) \). We then describe by Algorithm 1 the behavior of a \( k \)-pebble transducer which computes \( \text{sum-dep}_{M}^w \). We show its correctness as follows:

\[
\text{sum-dep}_{M}^w(F) = \sum_{\|t_1, \ldots, t_k\| \in \text{Dep}^k(F)} \prod_{i_j \in \text{Fr}_F(t_j)} \text{prod}^w\chi(\|i_1, \ldots, i_k\|)
\]

where \( \chi \in \mathbb{K} \) denotes the characteristic function.

Algorithm 1: Computing \( \text{sum-dep}_{M}^w \) with a \((k+1)\)-pebble transducer

\[
\begin{algorithm}
\text{Function } \text{sum-dep}_{M}^w(F) \\
\quad w \leftarrow \text{word factored by } F; \\
\quad \text{for } 1 \leq i_1 \leq \cdots \leq i_k \leq |w| \text{ do} \\
\quad \quad \text{let } t_1, \ldots, t_k \leftarrow \text{Fr}_F^{-1}(i_1), \ldots, \text{Fr}_F^{-1}(i_k) \\
\quad \quad \text{if } \|t_1, \ldots, t_k\| \in \text{Dep}^k(F) \text{ then} \\
\quad \quad \quad \text{Output } \text{prod}_{M}^w(\|i_1, \ldots, i_k\|) \\
\quad \text{end}
\end{algorithm}
\]

Then we justify that it can be implemented by a \( k \)-pebble transducer. The loop over \( i_1 \leq \cdots \leq i_k \) is implemented using \( k \) pebbles ranging over the leaves of \( F \). Whether \( \|t_1, \ldots, t_k\| \in \text{Dep}^k(F) \) is a regular property which is checked in a standard way with a lookahead. Finally, the \( \text{prod}_{M}^w(\|i_1, \ldots, i_k\|) \) are bounded independently from \( w \) (see Subsection D.1) and checking if \( \text{prod}_{M}^w(\|i_1, \ldots, i_k\|) = C \) is also a regular property.

### E Linearizations

We presented in Definition E.1 the linearization of a node, which is a 1-multicontext. We now generalize it to any set of independent nodes in Definition E.1. It is easy to see that coincides on singletons with the former definition.

**Definition E.1** (Linearization). Let \( \mu : A^* \rightarrow M \), \( w \in A^* \) and \( F \in \text{Forests}_\mu(w) \). If \( \mathcal{I} \in \bigcup_{k \geq 0} \text{Ind}^k(F) \) or \( \mathcal{I} = \{F\} \), we define its linearization by induction:

- if \( |\mathcal{I}| = 0 \), then \( \text{lin}^F(\mathcal{I}) := \mu(F) \);
- if \( \mathcal{I} = \{F\} \) then \( \text{lin}^F(\mathcal{I}) := \|w[\text{Fr}_F(F)]\| \);
- otherwise \( F = (F_1) \cdots (F_n) \) and \( F \not\subseteq \mathcal{I} \), then for \( 1 \leq j \leq n \) we let \( \mathcal{I}_j := \mathcal{I} \cap \text{Nodes}(F_j) \). Then we define \( \text{lin}^F(\mathcal{I}) := \text{lin}^{F_1}(\mathcal{I}_1) \cdots \text{lin}^{F_n}(\mathcal{I}_n) \).

In the following subsections, we give some properties of linearizations which shall be useful for the proofs of appendices F and G.
E.1 Productions and linearizations

We show here that the production on a set of independent nodes is the same as the production on its linearization. This result will be extremely useful in order to transform the permutability of $\mathcal{M}$ in a property about the productions on its independent sets of nodes.

\begin{lemma}
Let $\mathcal{M} = (A, M, \mu, \delta, \lambda)$ a $k$-marble bimachine. Let $w \in A^*$, $\mathcal{F} \in \text{Forests}_w(u)$ and $\mathcal{T} \in \text{Ind}^b(\mathcal{F})$. Then $\prod^w_M(\mathcal{T}) = \prod^w_M(\text{lin}^F(\mathcal{T}))$.
\end{lemma}

The rest of this subsection is devoted to the proof of Lemma E.2.

Assume that $\mathcal{T} = \{t_1, \ldots, t_n\}$ and for all $1 \leq i \leq k$ let $I_i := \text{Fr}_{\mathcal{T}}(t_i)$. Since their is no ancestor-relationship in $\{t_1, \ldots, t_k\}$, their frontiers do not overlap and one can assume that $\min(I_1) \leq \max(I_1) < \min(I_2) \leq \max(I_2) < \cdots < \min(I_k) \leq \max(I_k)$ (and they exist since the frontiers are always non-empty). For $0 \leq i \leq k$, let $v_i := w[\max(I_i) + 1 : \min(I_{i+1}) - 1]$ (with the convention that $\max(I_0) = 0$ and $\min(I_{k+1}) = |w| + 1$). It is easy to check by construction of the linearization that $\text{lin}^F(\mathcal{T}) = \mu(v_0)\mu[I_1]\mu(v_1)\cdots\mu[I_k]\mu(v_k)$.

Therefore by Proposition-Definition 5.8 it follows:

$$\prod^w_M(\text{lin}^F(\{t_1, \ldots, t_k\})) = \prod^w_M(v_0[w[I_1]]v_1\cdots[w[I_k]]v_k) = \prod^w_M(\{I'_1, \ldots, I'_k\})$$

where $w' := v_0[w[I_1]v_1\cdots[w[I_k]]v_k \in A^*$ and $I'_1, \ldots, I'_k \subseteq [1:|w'|]$ denote the sets of positions of $w[I_1], \ldots, w[I_k]$ in $w'$. On the other hand one has by definition of productions:

$$\prod^w_M(\{I_1, \ldots, I_k\}) = \prod^w_M(\{I'_1, \ldots, I'_k\}).$$

We now show that all the terms occurring in the sums defining $\prod^w_M(\{I'_1, \ldots, I'_k\})$ and $\prod^w_M(\{I'_1, \ldots, I'_k\})$ are equal. For $1 \leq i \leq k$, let $\sigma_i : I_i \rightarrow I'_i$ be the unique monotone bijection. To conclude the proof, we show that for all $i_1, \ldots, i_k \in I_1, \ldots, I_k$, we have:

$$\prod^w_M(\{i_1, \ldots, i_k\}) = \prod^w_M(\{\sigma_1(i_1), \ldots, \sigma_k(i_k)\})$$

For this it is sufficient to show (see Sublemma B.6) that:

$$\mu(w[1:i-1])\mu(w[i_1])\cdots\mu(w[i_k])\mu(w[i+k+1:|w|]) = \mu(w'[1:|\sigma_1(i_1)-1|])\mu(w'[\sigma_1(i_1)])\cdots\mu(w'[|\sigma_k(i_k)|])\mu(w'[|\sigma_k(i_k)| + 1 : |w'|])$$

Equation (8) directly follows from Claim E.3.

\begin{claim}
Let $u \in A^*$, $\mathcal{F} \in \text{Fact}_w(u)$, $I := \text{Fr}_{\mathcal{F}}(\mathcal{F})$, $u' := u[I]$ and $\sigma : I \rightarrow [1:|u'|]$ be the unique monotone bijection. Then for all $i \in I$ we have:

$$\mu(u[1:i-1])[u[i]]\mu(u[i+1:|u|]) = \mu(u'[1:|\sigma_1(i)-1|])\mu(u'[\sigma_1(i)])\mu(u'[\sigma_1(i)+1 : |u'|]).$$

\end{claim}

\begin{remark}
This result implies in particular that $\mu(u) = \mu(u') = \mu(u[\text{Fr}_{\mathcal{F}}(\mathcal{F})])$.
\end{remark}

\begin{proof}
We show the result by induction on the factorization as follows:

- If $\mathcal{F} = a \in A$, one has $u = u' = a$ and it is obvious;
- If $\mathcal{F} = (\mathcal{F}_1) \cdots (\mathcal{F}_n)$ and $u = u_1 \cdots u_n$ is the according factorization. Then we have $u' = u'_1 u'_2$, where $u'_j := u_j[\text{Fr}_{\mathcal{F}_j}(\mathcal{F}_j)]$ for $j \in \{1, n\}$. We suppose that $n \geq 3$ (the other case is easier) hence $\mu(u_1) \cdots \mu(u_n)$ is an idempotent. Assume by symmetry that
Hiding pebbles when the output alphabet is unary

\(i \in \text{Fr}_x(F),\) then \(1 \leq \sigma(i) \leq |u_i|\) and \(\sigma\) restricted to \([1,|u_1|]\) is the monotone bijection between \(\text{Fr}_x(F)\) and \([1,|u_1|]\). We have:

\[
\mu(u[1:i-1])u[i]u[i+1:] = \mu(u[1:i-1])u[i]\mu(u[i+1:]u_1)u_2 \cdots \mu(u_n)
\]

\[
= \mu(u[1:i-1])u[i]\mu(u[i+1:]u_1)\mu(u_n)
\]

by idempotence of \(\mu(u_n)\).

\[
= \mu(u_1[1:|\sigma(i)|-1]|\sigma(i)|)]u[i+1:]u_1)\mu(u_n)
\]

by induction hypothesis

\[
= \mu(u'[|\sigma(i)|-1]|\sigma(i)|)]\mu(u'[\sigma(i)+1:]u'][\mu(u'])
\]

which concludes the proof.

\[\blacktriangledown\]

### E.2 Linearizations are iterators

Here we show that the linearization of an independent set of nodes is an iterator. Furthermore, the linearizations of its elements can (up to re-ordering them) be recovered from it.

**Lemma E.5.** Let \(\mu : A^* \to M\) and \(K \geq 0\). Let \(w \in A^*, F \in \text{Forests}_\mu^K(w)\) and \(X = \{x_1, \ldots, x_k\} \in \text{Ind}^k(F)\), then:

1. \(\text{lin}^F(X)\) is a \((k,2^K)\)-iterator of the form \(m_0\left(\prod_{i=1}^k e_i[u]e_i \right);\)
2. \(\mu(F) = m_0\left(\prod_{i=1}^k e_i [u] e_i \right);\)
3. up to a permutation of \([1:k]\) we have for all \(1 \leq j \leq k:\)

\[
\text{lin}^F(t_j) = m_0\left(\prod_{i=1}^{j-1} e_i m_i\right) e_j[u_j] e_j \left(\prod_{i=j+1}^k m_i e_i \right) m_k.
\]

**Proof.** The proof is done by induction. For \(k = 0\), then \(\text{lin}^F(\varnothing) = \mu(F)\) and the result is obvious. Otherwise we have \(F = \langle F_1, \ldots, F_n \rangle\). For all \(1 \leq j \leq n\) let \(X_j := X \cap \text{Nodes}(F_j)\), so that we have \(\text{lin}^F(X) = \text{lin}^F(X_1) \cdots \text{lin}^F(X_n)\).

For the \(j\) such that \(X_j \neq \{F_j\}\), then \(\text{lin}^F_j(X_j)\) is \((|X_j|,2^K)\)-iterator by induction hypothesis (since in that case we have \(X_j \subset \text{Ind}^{[1,j]}(F_j)\)). Now if \(X_j = \{F_j\}\), we must have \(1 < j < n\) (since this node is iterable) and \(X_j = 0\) (by definition of independent multisets). Hence we have: \(\text{lin}^{F_{j-1}}(X_{j-1})\text{lin}^{F_j}(X_j)\text{lin}^{F_{j+1}}(X_{j+1}) = e^w[\text{Fr}_x(F_j)]e = \text{lin}^F(X_j)\) where \(e = \mu(F)\) is idempotent since \(n \geq 3\) and \(e = \mu(w[\text{Fr}_x(F_j)])\) by Remark E.4. Furthermore, by definition of the frontier we have \(|\text{Fr}_x(F_j)| \leq 2^K\).

\[\blacktriangledown\]

### F Proof of Proposition-Definition 6.18

Let \(M = (A,M,\mu,\delta,\lambda)\) be a \(2^{3|M|}\)-permutable \(k\)-marble bimachine. We show that the production on a set of independent nodes only depends on its architecture.

**Definition F.1 (Rank).** Let \(w \in A^*, F \in \text{Forests}_\mu^{3|M|}(w), X \in \text{Ind}^d(F).\) We say that the architecture \(\text{arc}^F(X)\) has rank \(x\).

**Remark F.2.** An architecture has only one rank (it is sum of the sizes of its multisets).

We show by induction on the structure of \(A\) of rank 0 \(\leq x \leq k\) the following result:

1. if \(w, w' \in A^*\) with \(F \in \text{Forests}_\mu^{3|M|}(w), F' \in \text{Forests}_\mu^{3|M|}(w')\);
2. if \(X \in \text{Ind}^d(F)\) and \(X' \in \text{Ind}^d(F')\) such that \(A = \text{arc}^F(X) = \text{arc}^{F'}(X')\);
3. and if \((r, t) \in \Sigma_{k-2}\) and \(L\) (resp. \(R\)) is a \((t, 3^{3|M|})\)-iterator (resp. \((r, 3^{3|M|})\)-iterator); then we have \(\text{prod}_M(L \text{lin}^F(T) R) = \text{prod}_M(L \text{lin}^{F'}(T') R)\).
F.1 Cases for $x = 0$

Three cases are possible. If $A = \langle m \rangle$ (resp. $A = a$, resp. $A = \epsilon$), then $F = \langle F_1 \rangle \cdots \langle F_n \rangle$ and $F' = \langle F'_1 \rangle \cdots \langle F'_{n'} \rangle$ with $n, n' \geq 1$ and $\mu(F) = \mu(F') = m$ (resp. $F = F' = a$, resp. $F = F' = \epsilon$). Hence we have $\text{lin}_F(\emptyset) = \text{lin}_F'(\emptyset) = m$ (resp. $= \mu(a)$, resp. $= \mu(\epsilon)$).

F.2 Case $x \geq 1$ and $A = \langle A_1 \rangle \langle A_2 \rangle \cdots \langle A_p \rangle$ with $p \geq 1$

Let us first assume that $A_1$ is an architecture of rank $x_1 \geq 1$ (otherwise, we must have that $A_p$ has rank $x_p \geq 1$, and the reasoning is similar). Let $y := x-x_1 \geq 0$ be the rank of $B := \langle A_2 \rangle \cdots \langle A_p \rangle$. The only way to have $\text{arc}_F(F) = A$ for $F \in \text{Ind}^k(F)$ is that $F = \langle F_1 \rangle \langle F_2 \rangle \cdots \langle F_n \rangle$ with $n \geq 1$. Let $G := \langle F_2 \rangle \cdots \langle F_n \rangle$ and $\mathcal{I}_1 := \mathcal{I} \cap \text{Nodes}(F_1)$. Then we must have $\text{arc}_F'(\mathcal{I}_1) = A_1$ and $\text{arc}_G(\mathcal{I} \setminus \mathcal{I}_1) = B$ (indeed we have $\mathcal{I} \setminus \mathcal{I}_1 \in \text{Ind}^k(G)$). Both are iterators by Lemma E.5. It follows from the definition of linearizations that $\text{lin}_F(\mathcal{I}) = \text{lin}_{F_1}(\mathcal{I}_1) \text{lin}_G(\mathcal{I} \setminus \mathcal{I}_1)$. Similar results hold for $F' = \langle F'_1 \rangle G'$ and we have:

$$\prod_{\mathcal{M}} \left( L \text{ lin}_F(t) R \right) = \prod_{\mathcal{M}} \left( L \text{ lin}_F(\mathcal{I}_1) \text{ lin}_G(\mathcal{I} \setminus \mathcal{I}_1) R \right) = \prod_{\mathcal{M}} \left( L \text{ lin}_F(\mathcal{I}_1) \text{ lin}_G(\mathcal{I} \setminus \mathcal{I}_1) R \right) \text{ by induction hypothesis}$$

F.3 Case $x \geq 1$ and $A = \langle T \rangle$

In this case $A = \langle T \rangle$ where $T$ is a non-empty multiset of elements of the form $(m[u]m', d)$. Note that $x = |T|$ and let $e$ be the idempotent such that $e = m\mu(u)m'$ for all $(m[u]m', d) \in T$ (by construction and item 2 of Lemma E.5 it is the same for all the elements occurring in $T$).

It follows from the definitions that we must have $F = \langle F_1 \rangle \cdots \langle F_n \rangle$ and $F' = \langle F'_1 \rangle \cdots \langle F'_{n'} \rangle$, with $n, n' \geq 3$ and $\mu(F) = \mu(F') = e$. For all $1 \leq i \leq n$ (resp. $1 \leq i \leq n'$), let $\mathcal{I}_i := \mathcal{I} \cap F_i$ (resp. $\mathcal{I}'_i := \mathcal{I} \cap F'_i$). By construction of architectures we must have $\mathcal{I}_i = \mathcal{I}_n = \mathcal{I}'_i = \mathcal{I}'_{n'} = \emptyset$, thus $\mathcal{I} := \bigcup_{i=0}^{n-1} \mathcal{I}_i$ and $\mathcal{I}' := \bigcup_{i=0}^{n'-1} \mathcal{I}'_i$.

\begin{itemize}
  \item \textbf{Claim F.3.} $\text{lin}_F(\mathcal{I})$ is a $(x, 2^{|M|})$-iterm of the form $em_0(\prod_{i=1}^{x} e_i[u_i] e_i m_i) e$ such that $m_0(\prod_{i=1}^{x} e_i m_i) = e$.
  \item \textbf{Proof.} Since $\mathcal{I} \in \text{Ind}^k(F)$, by applying Lemma E.5 we get a $(x, 2^{|M|})$-iterm, therefore $\text{lin}_F(\mathcal{I}) = m_0(\prod_{i=1}^{x} e_i[u_i] e_i m_i) e$ with $m_0(\prod_{i=1}^{x} e_i m_i) = \mu(F) = e$.
  
  But since $\text{lin}_F(\mathcal{I}_1) = \mu(F_1) = \text{lin}_F(\mathcal{I}_n) = \mu(F_n) = e$, then by definition of linearizations we must have $m_0 e_1 = em_0' e_1$ and $e_x m_x = em_x e_2$. In particular $m_0 e_1 = em_0 e_1$ and $e_x m_x = em_x e_2$ since $e$ is idempotent. Thus we get $m_0(\prod_{i=1}^{x} e_i[u_i] e_i m_i) = em_0(\prod_{i=1}^{x} e_i[u_i] e_i m_i) e$ which concludes the proof.
  \end{itemize}

We get with Claim F.3 a similar result for $F'$ (note that it uses the same $e$).

\begin{itemize}
  \item \textbf{Claim F.4.} $\text{lin}_F'(\mathcal{I})$ is a $(k, 2^{|M|})$-iterm of the form $em_0'(\prod_{i=1}^{x} e_i'[u_i'] e_i' m_i') e$ such that $m_0'(\prod_{i=1}^{x} e_i m_i') = e$.
  \end{itemize}
We show that the productions on \( \text{lin}^F(\Sigma) \) and \( \text{lin}^{F'}(\Sigma') \) are equal. For \( 1 \leq j \leq x \) define:

\[
\begin{align*}
\text{left}_j & := e \left( \prod_{i=1}^{j} m_{i-1} e_i \right) \quad \text{and} \quad \text{right}_j := \left( \prod_{i=j}^{x} e_i m_i \right) e \\
\text{left}'_j & := e \left( \prod_{i=1}^{j} m_{i-1} e'_i \right) \quad \text{and} \quad \text{right}'_j := \left( \prod_{i=j}^{x} e'_i m'_i \right) e
\end{align*}
\]

\( \square \text{Claim F.5.} \) There exists a permutation \( \sigma \) of \([1:x]\) such that for all \( 1 \leq i \leq x \), \( u'_i = u_{\sigma(i)} \), \( \text{left}'_i = \text{left}_{\sigma(i)} \) and \( \text{right}'_i = \text{right}_{\sigma(i)} \).

Proof. By definition of \( \Sigma \) we have \( \{\text{lin}^F(\{t\}) : t \in \Sigma \} = \{\text{lin}^{F'}(\{t'\}) : t' \in \Sigma' \} \). Hence by applying item 3 of Lemma F.5 twice, we get the suitable bijection \( \sigma \).

Let \((\ell, r) \in \Sigma_{k-1} \), and \( \mathcal{L} \) (resp. \( \mathcal{R} \)) be a \((\ell, 2^{3^{|M|}})\)-iterator (resp. \((r, 2^{3^{|M|}})\)-iterator), then:

\[
\begin{align*}
\text{prod}_M \left( \mathcal{L} \text{ lin}^F(\Sigma) \mathcal{R} \right) &= \text{prod}_M \left( \mathcal{L} \left( e m_0 \left( \prod_{i=1}^{x} e_i [u_i] e_i m_i \right) e \right) \mathcal{R} \right) \quad \text{by Claim F.3} \\
&= \text{prod}_M \left( \mathcal{L} \left( \prod_{i=1}^{x} \text{left}_{\sigma(i)} [u_{\sigma(i)}] \text{right}_{\sigma(i)} \right) \mathcal{R} \right) \quad \text{since } |u_i| \leq 2^{3^{|M|}} \text{ and } M \text{ is } 2^{3^{|M|}} \text{-permutable} \\
&= \text{prod}_M \left( \mathcal{L} \left( \prod_{i=1}^{x} \text{left}'_i [u'_i] \text{right}'_i \right) \mathcal{R} \right) \quad \text{by Claim F.5} \\
&= \text{prod}_M \left( \mathcal{L} \left( e m'_0 \left( \prod_{i=1}^{x} e'_i [u'_i] e'_i m'_i \right) e \right) \mathcal{R} \right) \quad \text{since } |u'_i| \leq 2^{3^{|M|}} \text{ and } M \text{ is } 2^{3^{|M|}} \text{-permutable} \\
&= \text{prod}_M \left( \mathcal{L} \text{ lin}^{F'}(\Sigma') \mathcal{R} \right) \quad \text{by Claim F.4}
\end{align*}
\]

\textbf{G \quad Proof of Lemma 6.19}

For \( A \in \text{Arctime}_{\mu}^{3^{|M|}} \) of rank \( k \geq 0 \), we had defined \( \text{count}_A(\mathcal{F}) := \{|\Sigma \in \text{Ind}^k(\mathcal{F}) : \text{arc}^F(\Sigma) = A\}| \).

We show by induction on the inductive structure of \( A \) that one can build two functions \( \text{count}'_A, \text{count}''_A : (\hat{A})^* \to \mathbb{N} \) such that \( \text{count}_A = \text{count}'_A + \text{count}''_A \) and:

\begin{itemize}
\item \( \text{count}'_A \) is polyblind and has growth at most \( k \);
\item \( \text{count}''_A \) is polyregular and has growth at most \( k-1 \).
\end{itemize}

The most interesting case is that of \( A = (\Sigma) \) treated in Subsection \( \text{G.3} \).

Once more, it is enough to describe our functions on the set of inputs \( \mathcal{F} \in (\hat{A})^* \) such that \( \mathcal{F} \in \text{Forest}_{\mu}^{3^{|M|}}(w) \) for some \( w \in A^* \). Indeed, this domain is regular.

\textbf{G.1 Cases for } \( k = 0 \)

Three cases occur, and we treat them in a similar way. If \( A = (m) \) (resp. \( A = a \), resp. \( A = \varepsilon \)), then \( \text{count}_A(\mathcal{F}) = 1 \) if \( \mathcal{F} = (\mathcal{F}_1) \cdots (\mathcal{F}_n) \) with \( n \geq 1 \) and \( m(\mathcal{F}) = m \) (resp. \( \mathcal{F} = a \), resp. \( \mathcal{F} = \varepsilon \)) and 0 otherwise. In this cases the function \( \text{count}_A \) is the indicator function of a regular language, thus it is polyblind with growth at most 0. We define \( \text{count}'_A := \text{count}_A \) and \( \text{count}''_A := 0 \).
G.2 Case $k \geq 1$ and $A = \langle A_1 \rangle \langle A_2 \rangle \cdots \langle A_p \rangle$ with $p \geq 1$

We assume that $A_1$ is an architecture of rank $k_1 \geq 1$ (otherwise, we must have that $A_p$ has rank $k_p \geq 1$, and the reasoning is similar). Let $b = k - k_1 \geq 0$ be the rank of $B := \langle A_2 \rangle \cdots \langle A_p \rangle$.

\[ \text{Claim G.1. We have for } w \in A^* \text{ and } F \in \text{Forests}_{\beta, M}^B(w): \]
\[ \text{count}_A(F) = \begin{cases} 0 & \text{if } F \text{ is not of the form } \langle F_1 \rangle \langle F_2 \rangle \cdots \langle F_n \rangle \text{ with } n \geq 1 \\ \text{count}_{A_1}(F_1) \times \text{count}_B((F_2) \cdots \langle F_n \rangle) & \text{otherwise.} \end{cases} \]

Proof. The only way to get $\text{arc}^A(F) = \langle A_1 \rangle B$ for $\Sigma \in \text{Ind}^k(F)$ is $F = \langle F_1 \rangle \langle F_2 \rangle \cdots \langle F_n \rangle$ with $n \geq 1$ and $\Sigma \cap \text{Nodes}(F_1) \neq \emptyset$. If $F$ is of this form, let $G := \langle F_2 \rangle \cdots \langle F_n \rangle$, we have:
\[ |\{ \Sigma \in \text{Ind}^k(F) : \text{arc}^A(F) = A \} | = |\{ (\Sigma_1, \Sigma_2) : \Sigma_1 \in \text{Ind}^{k_1}(F_1), \text{arc}^{F_1}(\Sigma_1) = A_1 \text{ and } \Sigma_2 \in \text{Ind}^{k-(G)}, \text{arc}^{G}(\Sigma_2) = B \}|. \]

Indeed, the function: $\Sigma \mapsto (\Sigma \cap \text{Nodes}(F_1)), (\Sigma \cap \text{Nodes}(G))$ is a bijection between these two sets. First note that we have $\Sigma \cap \text{Nodes}(F_1) \in \text{Ind}^{k_1}(F_1)$ (since $F_1 \notin \Sigma$). Furthermore $F_2 \notin \Sigma$ since otherwise it would be the sibling of an ancestor of a node of $\Sigma$, thus $\Sigma \cap \text{Nodes}(G) \in \text{Ind}^{k-(G)}$. The inverse of the bijection is clearly $((\Sigma_1, \Sigma_2)) \mapsto \Sigma_1 \cup \Sigma_2$.

By applying Claim G.1 and induction hypothesis, we get:
\[ \text{count}_A(F) = (\text{count}_{A_1}(F_1) + \text{count}_B'(G)) (\text{count}_A^2(G) + \text{count}_B'(G)) = \text{count}_{A_1}(F_1) \text{count}_B^2(G) + \text{count}_{A_1}(F_1) \text{count}_B'(G) + \text{count}_{A_1}(F_1) \text{count}_B'(G). \]

We first note that the functions $f_1 : \langle F_1 \rangle G \mapsto \text{count}_{A_1}(F_1)$ and $f_2 : \langle F_1 \rangle G \mapsto \text{count}_B'(G)$ (defined only on inputs of the form $\langle F_1 \rangle G \in \text{Forests}_{\beta, M}^B(w)$ for some $w \in A^*$) are polyblind. Indeed, since the height of the forest is bounded, a bimachine can detect the which matches the first $\langle$, and simulate the computation of $\text{count}_{A_1}$ (resp. $\text{count}_B$) on $F_1$ (resp. $G$). Hence $\text{count}_{A_1} = f_1 \otimes f_2$ is polyblind. Furthermore $f_1$ (resp. $f_2$) has growth at most $k_1$ (resp. $b$), thus $\text{count}_{A_1}$ has growth at most $k_1 + b = k$.

Similarly, $\text{count}_{A} = \text{count}_{A_1} - \text{count}_{A_1}$ is a polyregular function (it also detects $F_1$ and $G$). Furthermore each of its terms has growth at most $k-1$ by induction hypothesis.

G.3 Case $k \geq 1$ and $A = \langle \Sigma \rangle$

In this case $A = \langle \Sigma \rangle$ where $\Sigma$ is a non-empty multiset of elements of the form $(m[u]m', d)$. Note that $x = \langle \Sigma \rangle$ and let $e$ be the idempotent such that $e = m\mu(u)m'$ for all $(m[u]m', d) \in \Sigma$ (by construction and item 2 of Lemma 1.5 it is the same for all elements occurring in $\Sigma$)

\[ \text{Definition G.2. Let } w \in A^*, F \in \text{Forests}_{\beta, M}^B(w) \text{ and } \Sigma \in \text{Nodes}(F), \text{ we define the multiset } \text{Types}^F(\Sigma) := \{ \langle \text{lin}^F(t), \text{depth}^F(t) \rangle : t \in \Sigma \}. \text{ It can have multiplicities even if } \Sigma \text{ is a set.} \]

By definition of architectures and count, we have:
\[ \text{count}_{\langle \Sigma \rangle}(F) = \begin{cases} 0 & \text{if } F \text{ is not of the form } \langle F_1 \rangle \cdots \langle F_n \rangle \\ \text{otherwise.} \end{cases} \]
Hiding pebbles when the output alphabet is unary

From now, we assume that $\mathcal{F} = \langle \mathcal{F}_1 \cdots \mathcal{F}_n \rangle$ where $\mu(\mathcal{F}) = e$ (this is a regular property which can be checked). Sublemma G.3 directly concludes the proof if it is applied inductively.

**Sublemma G.3.** Let $\tau = (m'[\mu]n', d)$. Assume that $T := T_1 \otimes \{ \tau \}$ where $r > 0$, $\tau \notin T_1$ and for all $(m'[\mu]n', d') \in T_1$, one has $d \leq d'$. Then one can build:
- a polyblind function $g' : (\hat{A})^* \to \mathbb{N}$ with growth at most $r$;
- a polyregular function $g'' : (\hat{A})^* \to \mathbb{N}$ with growth at most $k-1$.

such that $\text{count}_{(\tau)}(\mathcal{F}) = g'(\mathcal{F}) \times \text{count}_{(\tau_1)}(\mathcal{F}) + g''(\mathcal{F})$.

The rest of this subsection is devoted to the proof of Sublemma G.3. We suppose that the $\mathcal{F}_1$ and $\mathcal{F}_n$ have no iterable nodes, thus $\text{count}_{(\tau)}(\mathcal{F}) = |\{ \tau \in \text{Ind}^k(\mathcal{F}) : \text{Types}^\mathcal{F}(\tau) = T \}|$ (this assumption is just used to simplify the notations). We first show how to decompose $\text{count}_{(\tau)}$ as a sum indexed by the independent sets of type $T_1$. Let $k_1 := k-r \geq 0$ (this way $|T_1| = k_1$) and $A(\mathcal{F}) := \{ t \in \text{Iters}(\mathcal{F}) : \text{Types}^\mathcal{F}(\{t\}) = \tau \}$.

**Claim G.4.** $\text{count}_{(\tau)}(\mathcal{F}) = \sum_{\mathcal{T}_1 \in \text{Ind}^{k_1}(\mathcal{F})} \left| \left\{ \mathcal{T}_2 \subseteq A(\mathcal{F}) : \mathcal{T}_1 \cup \mathcal{T}_2 \in \text{Ind}^k(\mathcal{F}) \right\} \right|$.

Proof. Since $\tau \notin T_1$, the function $\mathcal{T} \mapsto (\mathcal{T} \cap \{ t : \text{Types}^\mathcal{F}(\{t\}) \in T_1 \}, \mathcal{T} \cap \{ t : \text{Types}^\mathcal{F}(\{t\}) = \tau \})$ is a bijection between the set of sets $\{ \mathcal{T} \in \text{Ind}^k(\mathcal{F}) : \text{Types}^\mathcal{F}(\mathcal{T}) = T \}$ and the set of couples of sets $\{(\mathcal{T}_1, \mathcal{T}_2) : \text{Types}^\mathcal{F}(\mathcal{T}_1) = T_1, \mathcal{T}_2 \subseteq A(\mathcal{F}) \text{ and } \mathcal{T}_1 \cup \mathcal{T}_2 \in \text{Ind}^k(\mathcal{F}) \}$.

We shall give two constructions for $g'$ and $g''$, depending on whether $|A(\mathcal{F})| < 3k_1 + 2r$ or not. Since this condition is a regular property of $\mathcal{F}$, it can be checked before the computation and does not matter. To simplify the notations, we shall describe transducers which range over the nodes of $\mathcal{F}$. This can be implemented in practice by ranging over the opening $\langle$ corresponding to the nodes. Using this convention, the ordering $<$ on the positions of $\mathcal{F}$ (seen as a word) induces a total ordering $<$ on $\text{Nodes}(\mathcal{F})$. Furthermore, the transducer can access (using a lookahead) the type of a node, check if two nodes marked by its pebbles are independent or not, or if one is $<$ than another. Finally, note that ranging over ordered tuples of nodes $t_1 < \ldots < t_k$ exactly corresponds to ranging over sets of $k$ nodes.

**First case: if $|A(\mathcal{F})| < 3k_1 + 2r$.** We set $g'(\mathcal{F}) := 0$ and $g''(\mathcal{F}) := \text{count}_{(\tau)}(\mathcal{F})$. We show in Algorithm 2 how to implement $g''$ with $k$ pebbles which range over the sets of independent nodes. Furthermore, $g''$ has growth at most $k_1 \leq k-1$ by Claim G.4, since for a given set $\mathcal{T}_1$, there is only a bounded number of sets $\mathcal{T}_2 \subseteq A(\mathcal{F})$ such that $\mathcal{T}_1 \cup \mathcal{T}_2 \in \text{Ind}^k(\mathcal{F})$.

**Algorithm 2** First case: computing $\text{count}_{(\tau)}$ with a pebble transducer

```
Function count_{(\tau)}(\mathcal{F})
    for $t_1 < \ldots < t_k \in \text{Iters}(\mathcal{F})$ do
        if $\{t_1, \ldots, t_k\} \in \text{Ind}^k(\mathcal{F})$ and $\text{Types}^\mathcal{F}(\{t_1, \ldots, t_k\}) = T$ then
            Output 1
    end
end
```
**Second case:** if $|A(F)| \geq 3k_1 + 2r$. Given $\mathfrak{T}_1 \in \text{Ind}^{k_1}(F)$ such that $\text{Types}^F(\mathfrak{T}_1) = \mathfrak{T}_1$, we define $A_{\mathfrak{T}_1}(F) := \{t \in A(F) : \{t\} \cup \mathfrak{T}_1 \in \text{Ind}^{k_1+1}(F)\} \subseteq A(F)$.

\[\triangleright \text{Claim G.5.} \quad \text{If } \text{Types}^F(\mathfrak{T}_1) = \mathfrak{T}_1, \text{ we have } |A(F) \setminus A_{\mathfrak{T}_1}(F)| \leq 3k_1\]

**Proof.** The nodes of $A$ have a fixed depth $d$, which is $\leq$ the depths of the nodes of $\mathfrak{T}_1$. Hence $A(F) \setminus A_{\mathfrak{T}_1}(F)$ contains the nodes of $A(F)$ which are either an ancestor or the sibling of an ancestor of a node from $\mathfrak{T}_1$, and there are at most $3|\mathfrak{T}_1| = 3k_1$ such nodes. $\triangleright$

Since $\prec$ is a total ordering on $\text{Nodes}(F)$, let $B_{\mathfrak{T}_1}(F)$ denote the $3k_1 - |A(F) \setminus A_{\mathfrak{T}_1}(F)| \geq 0$ first elements of $A_{\mathfrak{T}_1}(F)$ (with respect to $\prec$) and $C_{\mathfrak{T}_1}(F) := A_{\mathfrak{T}_1}(F) \setminus B_{\mathfrak{T}_1}(F)$. It follows immediately that $|C_{\mathfrak{T}_1}(F)| = |A(F)| - 3k_1 \geq 2r$. We say that two nodes $t \prec t' \in C_{\mathfrak{T}_1}(F)$ are **close** if there is no $t'' \in C_{\mathfrak{T}_1}(F)$ such that $t \prec t'' \prec t'$. Since $\mathfrak{T}_1 \in \text{Ind}^{k_1}(F)$, we have:

\[
\begin{align*}
\mathfrak{T}_2 \subseteq A(F) : \mathfrak{T}_1 \cup \mathfrak{T}_2 &\in \text{Ind}^k(F) \\
\mathfrak{T}_2 \subseteq A_{\mathfrak{T}_1}(F) : \mathfrak{T}_1 \cup \mathfrak{T}_2 &\in \text{Ind}^k(F) \\
\mathfrak{T}_2 \subseteq C_{\mathfrak{T}_1}(F) : \mathfrak{T}_1 \cup \mathfrak{T}_2 &\in \text{Ind}^k(F) \text{ and no } t, t' \in \mathfrak{T}_2 \text{ are close} \\
\mathfrak{T}_2 \subseteq C_{\mathfrak{T}_1}(F) : \mathfrak{T}_1 \cup \mathfrak{T}_2 &\in \text{Ind}^k(F) \text{ and } \exists t, t' \in \mathfrak{T}_2 \text{ which are close} \\
\mathfrak{T}_2 \subseteq C_{\mathfrak{T}_1}(F) : \mathfrak{T}_1 \cup \mathfrak{T}_2 &\in \text{Ind}^k(F) \text{ and } \mathfrak{T}_2 \cap B_{\mathfrak{T}_1}(F) \neq \emptyset.
\end{align*}
\]

Let us consider the function $P_r : \mathbb{N} \rightarrow \mathbb{N}$ which maps $X \geq 0$ to the cardinal of the set $W$ of words $w \in \{0, 1\}^X$ such that $|w|_1 = r$ and there are no two consecutive 1 in $w$.

\[\triangleright \text{Claim G.6.} \quad \text{For all } \mathfrak{T}_1 \in \text{Ind}^{k_1}(F) \text{ such that } \text{Types}^F(\mathfrak{T}_1) = \mathfrak{T}_1, \text{ we have } P_r(|A(F)|-3k_1) = |\{\mathfrak{T}_2 \subseteq C_{\mathfrak{T}_1}(F) : \mathfrak{T}_1 \cup \mathfrak{T}_2 \in \text{Ind}^k(F) \text{ and no } t, t' \in \mathfrak{T}_2 \text{ are close}\}|.\]

**Proof.** We first see that $\{\mathfrak{T}_2 \subseteq C_{\mathfrak{T}_1}(F) : \mathfrak{T}_1 \cup \mathfrak{T}_2 \in \text{Ind}^k(F) \text{ and no } t, t' \in \mathfrak{T}_2 \text{ are close}\}$ $= \{\mathfrak{T}_2 \subseteq C_{\mathfrak{T}_1}(F) : \text{no } t, t' \in \mathfrak{T}_2 \text{ are close}\}$. Finally, we note that $|C_{\mathfrak{T}_1}(F)| = |A(F)| - 3k_1$ and that $P_r$ exactly counts subsets of $r$ nodes such that no two of them are close. $\triangleright$

Since this cardinal does not depend on $\mathfrak{T}_1$, we define $g'(F) := P_r(|A(F)|-3k_1)$.

\[\triangleright \text{Claim G.7.} \quad \text{The function } g' \text{ is polyblind and has growth at most } r.\]

**Proof.** The function which maps some $w \in W$ to itself in which each 10 factor (excepted the last one) is replaced by 1, is a bijection between $W$ and $\{w \in \{0, 1\}^X : |w|_r = 1\}$. Hence $P_r(X) = \left(\frac{X-r+1}{r}\right) = \frac{\prod_{i=0}^{r-1}(|A(F)|-3k_1-r-i+1)}{\prod_{i=0}^{r-1}(|A(F)|-r-i+1)}$. It is clear that $r \times g'$ is a polyblind function, since it is the Hadamard product of $r$ regular functions. Then, dividing by $r!$ consists in a post-composition by a regular function (with both unary input and output alphabets), which preserves polyblindness [13]. $\triangleright$

If we denote by $c_{\mathfrak{T}_1}(F)$ the cardinal of the two last terms of Equations [3] we get:

\[
\text{count}(\mathfrak{T}_1)(F) = g'(F) \times \text{count}(\mathfrak{T}_1)(F) + \sum_{\substack{\mathfrak{T}_1 \in \text{Ind}^{k_1}(F) \\
\text{Types}^F(\mathfrak{T}_1) = \mathfrak{T}_1 \\
g'(F)}} c_{\mathfrak{T}_1}(F).
\]

The function $g''$ is polyregular (it can be computed by ranging over all possible sets $\mathfrak{T}_1$ as in Algorithm [2] and then sets $\mathfrak{T}_2$). Furthermore, it has growth at most $k-1$ since $c_{\mathfrak{T}_1}(F)$ has growth at most $r-1$ (and the bound is independent from $\mathfrak{T}_1$). Indeed, it has either two elements which are close in $A_{\mathfrak{T}_1}(F)$, or one element which is among the $3k_1$ first ones: in both cases there is one less degree of freedom (like in Appendix [1]).