A NEW APPROACH BASED ON INVENTORY CONTROL USING INTERVAL DIFFERENTIAL EQUATION WITH APPLICATION TO MANUFACTURING SYSTEM

Md Sadikur Rahman, Subhajit Das, Amalesh Kumar Manna, Ali Akbar Shaikh and Asoke Kumar Bhunia

Department of Mathematics
The University of Burdwan
Burdwan-713104, India

Ali Ahmadian
Institute of IR 4.0
The National University of Malaysia
43600 Bangi, Malaysia
Department of Mathematics
Near East University
Nicosia, TRNC, Mersin 10, Turkey

Soheil Salahshour*
Faculty of Engineering
Bahcesehir Universiti
Istanbul, Turkey

Abstract. Now-a-days, uncertainty conditions play an important role in modelling of real-world problems. In this regard, the aim of this study is two folded. Firstly, the concept of system of interval differential equations and its solution procedure in the parametric approach have been proposed. To serve this purpose, using parametric representation of interval and its arithmetic, system of linear interval differential equations is converted to the system of differential equations in parametric form. Then, a mixing problem with three liquids is considered and the mixing process is governed by system of interval differential equations. Thereafter, the mixing liquid is used in the production process of a manufacturing firm. Secondly, using this concept, a production inventory model for single item has been developed by employing mixture of liquids and the proposed production system is formulated mathematically by using system of interval differential equations. The corresponding interval valued average profit of the proposed model has been obtained in parametric form and it is maximized by centre-radius optimization technique. Then to validate the proposed model, two numerical examples have been solved using MATHEMATICA software. In addition, we have shown the concavity of the objective function graphically using the code of 3D plot in MATHEMATICA. Finally, the post optimality analyses are carried out with respect to different system parameters.

2020 Mathematics Subject Classification. 65G40.
Key words and phrases. Parametric approach, interval number, system of interval linear differential equation, mixing problem, production inventory.

*Corresponding author: Soheil Salahshour (soheil.salahshour@eng.bau.edu.tr).
1. **Introduction.** System of differential equations is an important branch of mathematics. Based on this branch, the valuable research works in science and technology are growing up to enrich the human society. Also, the system of non-linear differential equations gives a lot of contributions in the research of Medical sciences/Dynamical System. From the theory of system of non-linear differential equations, it is known to us that, the non-linear system cannot be solved analytically. However, a non-linear system can be transformed into linear one by giving a small perturbation. The nature of solutions of the non-linear system and corresponding linearized system are same. In other words, the solution spaces of non-linear system and the corresponding linearized system are topologically equivalent. Again, a non-homogeneous system can be converted into a homogeneous system by giving a suitable translation. Thus, the solution procedure of a linear system is very important. The standard form of a system of linear homogeneous equation can be represented as follows:

\[
\begin{align*}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\quad \quad \quad \vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n 
\end{align*}
\]  

\text{(A)}

where \(x_i\) (\(i = 1, 2, \ldots, n\)) are dependent variables of \(t\) such that \(\dot{x}_i = \frac{dx_i}{dt}\) and \(a_{ij} \in \mathbb{R}\). 

The system (A) can be solved simply by converting the equations of (A) into the higher order linear differential equations. Also, the system (A) can be solved by using eigenvector technique. In this connection, one may refer to some of the excellent books on the system of differential equations with different solution techniques written by Gronwall (1919)[21], Chicone (2006)[8], Thongmoon and Pusjuso (2010)[44], etc.

In reality, there are so many situations (like: critical situations due to pandemic, flood, tornado, storm etc.) where most of the things can’t be determined accurately or predicted with certainty. In these situations, different parameters of various dynamical problems, viz. economic model, food chain model, epidemic model are not precise. To represent those dynamical problems under these uncertain situations, all the imprecise parameters are expressed in the form of either fuzzy numbers or random variables with proper probability distribution functions or interval numbers. So, the system of differential equations related to each dynamical system becomes either system of fuzzy differential equations or system of stochastic differential equations or system of interval differential equations. The concepts of system of fuzzy differential equations were first introduced by Buckley and Feuring (2000)[7]. Later, it is developed by several researchers in which the works of Bede et al. (2007)[4], da Costa et al. (2018)[12], Ahmady (2019)[2], Pandit and Singh (2020)[35] are considerably mentionable. On the other hand, the concepts of system of stochastic differential equations were first introduced by Bergstrom (1966)[5]. In this area, a lot of works were accomplished. Among those, the works of Arnold (1974)[3], Ikeda and Watanabe (1977)[23], Mao and Yuan (2006)[30] and Liu et al. (2020)[28] are worth mentioning.
However, the representation of imprecise parameters of system of differential equation in the form of fuzzy set or random variable makes the system more complicated. Because in these representations, membership functions or distribution functions are involved which make the complications. Thus, to study the imprecise system of differential equations, a simple representation of impreciseness is required and this requirement may be fulfilled by the interval number. If the imprecise parameters of a system of imprecise differential equations are represented by interval numbers, then that system is called system of interval differential equations. The first order system of linear differential equations with interval coefficients is the particular type of system of interval differential equations. The generalized form of a system of interval linear differential equations is given below:

\[
\frac{d}{dt}[\bar{x}_1(t), \bar{x}_2(t), ..., \bar{x}_n(t)] = [\bar{a}_{11}, \bar{a}_{12}, ..., \bar{a}_{1n}]\bar{x}_1(t) + [\bar{a}_{21}, \bar{a}_{22}, ..., \bar{a}_{2n}]\bar{x}_2(t) + ... + [\bar{a}_{n1}, \bar{a}_{n2}, ..., \bar{a}_{nn}]\bar{x}_n(t)
\]

\[
\frac{d}{dt}[\bar{x}_1(t), \bar{x}_2(t), ..., \bar{x}_n(t)] = [\bar{a}_{11}, \bar{a}_{12}, ..., \bar{a}_{1n}]\bar{x}_1(t) + [\bar{a}_{21}, \bar{a}_{22}, ..., \bar{a}_{2n}]\bar{x}_2(t) + ... + [\bar{a}_{n1}, \bar{a}_{n2}, ..., \bar{a}_{nn}]\bar{x}_n(t)
\]

where \([\bar{x}_i, \bar{x}_i] (i = 1, 2, ..., n)\) are interval-valued dependent variables of \(t\) and \(\bar{a}_{ij}\), \(\bar{a}_{ij} \in \mathbb{R}\) such that \(\bar{a}_{ij} \leq \bar{a}_{ij}, \forall i = 1, 2, ..., n\).

The concept of first order interval differential equation (IDE) and its solution methodology were already introduced by several researchers (Stefanini and Bede, (2009)[43], Ramezanadeh (2015)[39] and Costa et al. (2018)[12], Salahshour et al. (2018, 2019)[41, 42], Gasilov and Amrahov (2020)[19], etc.). Among the solution methodology of first order IDE, parametric approach proposed by Ramezanadeh et al. (2015)[39] is simpler and more significant. However, the solution methodology of system IDE were reported by very few researchers among those the work of Gasilov and Amrahov (2018)[18] on solving of non-homogeneous system of IDEs is of great interest.

The system of linear differential equations has many applications in the area of dynamical systems, mathematical biology and many other fields. In the area of mathematical biology, Engelborghs et al. (2001)[17] studied numerical bifurcation of system of delay differential equations. Materi and Wishart (2007)[31] developed the computational method for solving system of differential equations in drug discovery. Recently, Kumar et al. (2018)[26] developed a new method for solving fractional differential equations arising in mathematical biology. Pal et al. (2018)[34] introduced a new approach for studying stability analyses of imprecise prey-predator model using interval simultaneous differential equations. Ghanbari et al. (2020)[20] studied three species pre-predator model using system of fractional differential equations. Mahata et al. (2021)[29] analysed the stability of a food chain model with harvesting under fuzzy uncertainty. Apart from the mathematical biology, the differential equations are also used in different real life optimization problems such as inventory control (Rahman et al. (2020a)[36], Rout et al. (2020)[40], Liao et al. (2020)[27], Das et al. (2021)[14] etc.). However, till now no one used the concept of system of IDEs in inventory control. To point out the research gap and contribution of this work, a brief survey of literature has been made and it is shown in Table 1.
Table 1. Some previous works on applications of differential equations

| Reported Works          | Simultaneous /Single differential equations | Nature of equations (Crisp/Fuzzy/Stochastic/Interval) | Area of applications |
|-------------------------|--------------------------------------------|----------------------------------------------------|----------------------|
| Cui and Friedman (2003)[10]| Simultaneous                      | Crisp (ordinary)                          | Mathematical biology |
| Das et al. (2008)[13]     | Single                                 | Fuzzy                                | Inventory            |
| Guchhait et al. (2013)[22]| Single                                 | Fuzzy                               | Production inventory |
| Jalari et al. (2016)[24]  | Simultaneous                          | Fuzzy                              | Mathematical biology |
| da Costa Campos (2019)[11] | Simultaneous                          | Crisp (ordinary)                      | Dynamical System     |
| Tsoularis (2019)[45]      | Single                                 | Stochastic                              | Inventory            |
| Kanekiyo and Agata (2019)[25]| Single                              | Stochastic                               | Bio Science          |
| Overstall et al. (2020)[33]| Single                                 | Crisp (ordinary)                        | Production inventory |
| De et al. (2020)[16]      | Single                                 | Fuzzy                               | Dynamical system     |
| Agocs et al. (2020)[37]   | Single                                 | Interval                               | Inventory            |
| Rahman et al. (2020b)[37] | Single                                 | Interval                             | Production inventory |
| Das et al. (2020)[15]     | Simultaneous                          | Interval                              | Production inventory |
| This work                | Simultaneous                          | Interval                              | Production inventory |

1.1. Novelty of this work. The system of differential equations plays a significant role to study the dynamical behaviour of a physical or biological system. However, due to the occurrence of randomness or impreciseness of different parameters involved in such systems, the study of the dynamical behaviour of these system under uncertainty using system of crisp differential equations is not possible. To study these systems, we must have to know the concept of imprecise system of differential equations and its solution methodology. Though the solution methodology of system of IDEs already derived by Gasilov and Amrahov (2020)[19], this work also introduces the solution methodology of system of IDEs in an alternative way. Here, the system of IDEs can be converted into the system of parametric differential equations by using parametric representation of intervals. Then, using the existing solution methodology, the solution procedure of system of IDEs is discussed. This is a newness of this work. Additionally, for the first time, these concept of system of IDEs is applied to formulate an inventory problem of mixture of liquids with interval parameters. These all together prove the originality of this work.

2. Generalized system of interval linear equations. A system of linear differential equations with both the interval valued dependent variables and coefficients is defined as follows:

\[
\frac{d}{dt}[x_1(t), x_2(t)] = [a_{11}(t), a_{12}(t)]x_1(t) + [a_{21}(t), a_{22}(t)]x_2(t) + ... + [a_{n1}(t), a_{n2}(t)]x_n(t)
\]

\[
\frac{d}{dt}[x_1(t), x_2(t)] = [a_{11}(t), a_{12}(t)]x_1(t) + [a_{21}(t), a_{22}(t)]x_2(t) + ... + [a_{n1}(t), a_{n2}(t)]x_n(t)
\]

\[
\frac{d}{dt}[x_1(t), x_2(t)] = [a_{11}(t), a_{12}(t)]x_1(t) + [a_{21}(t), a_{22}(t)]x_2(t) + ... + [a_{n1}(t), a_{n2}(t)]x_n(t)
\]

where \([x_i, x_j]\) \((i = 1, 2, ..., n)\) are interval-valued dependent variables of \(t\) and \(a_{ij}\), \(a_{ij} \in R\) such that \(a_{ij} \leq \bar{a}_{ij}\), \(\forall i = 1, 2, ..., n\).

Now, the corresponding parametric form of (1) is of the form

\[
\hat{x}_1(t, \eta) = a_{11}(\eta)x_1(t, \eta) + a_{12}(\eta)x_2(t, \eta) + ... + a_{n1}(\eta)x_n(t, \eta)
\]

\[
\hat{x}_2(t, \eta) = a_{21}(\eta)x_1(t, \eta) + a_{22}(\eta)x_2(t, \eta) + ... + a_{n2}(\eta)x_n(t, \eta)
\]

\[
\vdots
\]

\[
\hat{x}_n(t, \eta) = a_{n1}(\eta)x_1(t, \eta) + a_{n2}(\eta)x_2(t, \eta) + ... + a_{nn}(\eta)x_n(t, \eta)
\]

(2)
where \( \dot{x}_i(t, \eta) = \dot{x}_i(t, \eta) + \eta(\dot{x}_i(t, \eta) - \dot{x}_i(t, \eta)) \); \( a_{ij}(\eta) = a_{ij}(\eta) + \eta(\pi_{ij}(\eta) - a_{ij}(\eta)) \); 
\( \dot{x}_i(t, \eta) = \frac{dx_i(t, \eta)}{dt} \) for \( i, j = 1, 2, \ldots, n \).

**Theorem 1.** \([\pi_1^1(t), \pi_1^2(t)], [\pi_2^1(t), \pi_2^2(t)], \ldots, [\pi_n^1(t), \pi_n^2(t)]\) be the solutions of the system of interval linear differential equations (1) if and only if \( \dot{x}_i(t, \eta) = \dot{x}_i(t, \eta) + \eta(\pi_i(t, \eta) - \dot{x}_i(t, \eta)) \) for \( \eta \in [0, 1] \), be the solutions of the system of differential equations (2).

**Proof.** Let \([\pi_1^1(t), \pi_1^2(t)], [\pi_2^1(t), \pi_2^2(t)], \ldots, [\pi_n^1(t), \pi_n^2(t)]\) be the solutions of the system of interval linear differential equations (1). Therefore,

\[
\frac{d}{dt}[\pi_i^1(t), \pi_i^2(t)] = [a_{i1}, \pi_{i1}][\pi_i^1(t), \pi_i^2(t)] + [a_{i2}, \pi_{i2}][\pi_i^1(t), \pi_i^2(t)] + \ldots + [a_{in}, \pi_{in}][\pi_i^1(t), \pi_i^2(t)]
\]

Using Definition A.2 (1) (cf. Appendix), we get

\[
\begin{align*}
\{ \dot{x}_1^1(t, \eta) : \eta \in [0, 1] \} & = \{ a_{11}(\eta)x_1^1(t, \eta) + a_{12}(\eta)x_2^1(t, \eta) + \ldots + a_{1n}(\eta)x_n^1(t, \eta) : \eta \in [0, 1] \} \\
\{ \dot{x}_1^2(t, \eta) : \eta \in [0, 1] \} & = \{ a_{21}(\eta)x_1^2(t, \eta) + a_{22}(\eta)x_2^2(t, \eta) + \ldots + a_{2n}(\eta)x_n^2(t, \eta) : \eta \in [0, 1] \} \\
\{ \dot{x}_n^1(t, \eta) : \eta \in [0, 1] \} & = \{ a_{n1}(\eta)x_1^1(t, \eta) + a_{n2}(\eta)x_2^1(t, \eta) + \ldots + a_{nn}(\eta)x_n^1(t, \eta) : \eta \in [0, 1] \}
\end{align*}
\]

where \( \dot{x}_i^1(t, \eta) = \dot{x}_i^2(t, \eta) + \eta(\pi_i^1(t, \eta) - \dot{x}_i^2(t, \eta)) \); \( a_{ij}(\eta) = a_{ij}(\eta) + \eta(\pi_{ij}(\eta) - a_{ij}(\eta)) \) for \( i, j = 1, 2, \ldots, n \) and \( \eta \in [0, 1] \).

Again, using Definition A.2 (6) (cf. Appendix), we get,

\[
\begin{align*}
\{ \dot{x}_1^1(t, \eta) & = a_{11}(\eta)x_1^1(t, \eta) + a_{12}(\eta)x_2^1(t, \eta) + \ldots + a_{1n}(\eta)x_n^1(t, \eta) \\
\dot{x}_2^1(t, \eta) & = a_{21}(\eta)x_1^1(t, \eta) + a_{22}(\eta)x_2^1(t, \eta) + \ldots + a_{2n}(\eta)x_n^1(t, \eta) \\
& \vdots \\
\dot{x}_n^1(t, \eta) & = a_{n1}(\eta)x_1^1(t, \eta) + a_{n2}(\eta)x_2^1(t, \eta) + \ldots + a_{nn}(\eta)x_n^1(t, \eta)
\end{align*}
\]

where \( \eta \in [0, 1] \).

**Lemma 1.** The systems (1) and (2) are equivalent.

**Proof.** Proof follows from Theorem 1.

### 2.1. Solution procedure of system of linear interval differential equations.

To discuss the procedure of solution of the system of linear interval differential equations, for the sake of simplicity, we take two-dimensional linear system of interval differential equations. The standard form of the two-dimensional interval system of linear differential equations is given below:

\[
\begin{align*}
\frac{d}{dt}[\pi(t), \pi(t)] & = [a_1, \pi_1][\pi(t), \pi(t)] + [b_1, \beta_1][\pi(t), \pi(t)] \\
\frac{d}{dt}[y(t), y(t)] & = [a_2, \pi_2][y(t), y(t)] + [b_2, \beta_2][y(t), y(t)]
\end{align*}
\]
We define a system of linear differential equations in parametric form as follows:

\[
\begin{align*}
\dot{x}(t, \eta) &= a_1(\eta)x(t, \eta) + b_1(\eta)y(t, \eta) \\
\dot{y}(t, \eta) &= a_2(\eta)x(t, \eta) + b_2(\eta)y(t, \eta)
\end{align*}
\]  

(4)

where \(x(t, \eta) = \frac{d}{dt}x(t) + \eta\{\mathcal{F}(t) - x(t)\}; \ y(t, \eta) = \frac{d}{dt}y(t) + \eta\{\mathcal{G}(t) - y(t)\}; \ a_1(\eta) = a_1(\eta) + \eta(\mathcal{F}_1(\eta) - a_1(\eta)); \ b_1(\eta) = b_1(\eta) + \eta(\mathcal{G}_1(\eta) - b_1(\eta)); \ \eta \in [0, 1].

Case-I. Let \(I_1 \subset [0, 1]\) such that, the discriminant

\[
\{a_1(\eta) + b_2(\eta)\}^2 - 4\{a_1(\eta)b_2(\eta) - a_2(\eta)b_1(\eta)\} \geq 0 \quad \text{for} \ \eta \in I_1
\]

\[
\Rightarrow \{a_1(\eta) - b_2(\eta)\}^2 + 4a_2(\eta)b_1(\eta) \geq 0 \quad \text{for} \ \eta \in I_1
\]

Then the general solution of (5) is given by

\[
x(t, \eta) = C_1 \exp\left(\frac{a_1(\eta) + b_2(\eta) + \sqrt{\{a_1(\eta)b_2(\eta) - a_2(\eta)b_1(\eta)\}^2 + 4a_2(\eta)b_1(\eta)} t}{2}\right)
+ C_2 \exp\left(\frac{a_1(\eta) + b_2(\eta) - \sqrt{\{a_1(\eta)b_2(\eta) - a_2(\eta)b_1(\eta)\}^2 + 4a_2(\eta)b_1(\eta)} t}{2}\right), \ \eta \in I_1
\]

where \(C_1\) and \(C_2\) are arbitrary constants.

Case-II. Let \(I_2 \subset [0, 1]\) such that, the discriminant

\[
\{a_1(\eta) + b_2(\eta)\}^2 - 4\{a_1(\eta)b_2(\eta) - a_2(\eta)b_1(\eta)\} < 0 \quad \text{for} \ \eta \in I_1
\]

\[
\Rightarrow \{a_1(\eta) - b_2(\eta)\}^2 + 4a_2(\eta)b_1(\eta) < 0 \quad \text{for} \ \eta \in I_1
\]

Then the general solution of (5) is given by

\[
x(t, \eta) = \exp\left(\frac{a_1(\eta) + b_2(\eta)}{2}\right)\left\{C_1 \cos\left(\frac{\sqrt{4a_2(\eta)b_1(\eta)} - \{a_1(\eta)b_2(\eta) - a_2(\eta)b_1(\eta)\}^2}{2} t\right)
+ C_2 \sin\left(\frac{\sqrt{4a_2(\eta)b_1(\eta)} - \{a_1(\eta)b_2(\eta) - a_2(\eta)b_1(\eta)\}^2}{2} t\right)\right\}, \ \eta \in I_1
\]

where \(C_1\) and \(C_2\) are arbitrary constants.

Clearly \(I_1 \cap I_2\) is an empty set.

Hence the required solution \(x(t, \eta)\) of the system of linear differential equation (4) is
Similarly, the solution for \( y(t, \eta) \) can be determined from the equation given below,

\[
\dot{x}(t, \eta) = a_1(\eta)x(t, \eta) + b_4(\eta)y(t, \eta), \text{if} \eta(t, \eta) \neq 0 \dot{y}(t, \eta) = a_2(\eta)x(t, \eta) + b_2(\eta)y(t, \eta), \text{if} \eta(t, \eta) = 0
\]

Let the solution for \( y(t, \eta) \) of the system (4) be

\[
\{y(t, \eta) : \eta \in [0, 1]\} = \{y(t, \eta) : \eta \in I_3\} \cup \{y(t, \eta) : \eta \in I_4\}, I_3, I_4 \in [0, 1]
\]

Clearly, \( I_3 \cap I_4 \) is an empty set.

Now, our claim is that \( \{x(t, \eta) : \eta \in [0, 1]\} \) is a closed and connected interval of \( R \).

Again, \( x(t, \eta) = x(t) + \eta(\pi(t) - x(t)) \) is continuous if \( t \) is kept fixed. Now since \([0, 1]\) is connected and compact intervals of \( R \), so \( \{x(t, \eta) : \eta \in [0, 1]\} \) is also a connected and a compact interval of \( R \) for \( k = 1, 2 \) and \( x(t, \eta) \) has a maximum as well as a minimum value over \([0, 1]\). Therefore, \([x(t), \pi(t)]\) can be obtained as

\[
[x(t), \pi(t)] = [\min\{x(t, \eta) : \eta \in [0, 1]\}, \max\{x(t, \eta) : \eta \in [0, 1]\}]
\]

Hence the required solution of the system of linear differential equations (3) is given by

\[
[x(t), \pi(t)] = [\min\{x(t, \eta) : \eta \in [0, 1]\}, \max\{x(t, \eta) : \eta \in [0, 1]\}]
\]

\[
[y(t), \overline{y}(t)] = [\min\{y(t, \eta) : \eta \in [0, 1]\}, \max\{y(t, \eta) : \eta \in [0, 1]\}]
\]

**Example 1.** Consider a system

\[
\begin{cases}
\frac{d}{dt}[x(t), \pi(t)] = [0, 4][y(t), \overline{y}(t)] \\
\frac{d}{dt}[y(t), \overline{y}(t)] = [-4, 0][x(t), \pi(t)]
\end{cases}
\]

This system of equations is equivalent to the following system of equations (using Lemma 1)

\[
\dot{x}(t, \eta) = 4\eta y(t, \eta), \text{for} \eta \in 0 \\
\dot{y}(t, \eta) = (4\eta - 4)x(t, \eta), \text{for} \eta \in 0
\]

Now the system (6) can be expressed as

\[
\dot{x}(t, \eta) = 4\eta(4\eta - 4)x(t, \eta), \text{for} \eta \in 0
\]

(7)

Clearly, the equation (7) is a second order ordinary differential equation with constant coefficients (function of \( \eta \)).

Now since \( \eta \in [0, 1] \) so \( (4\eta - 4) \leq 0 \). Now, solving (7), we get

\[
x(t, \eta) = C_1\cos(\sqrt{4\eta(4\eta - 4)}t) + C_2\sin(\sqrt{4\eta(4\eta - 4)}t), \text{if} \eta \in [0, 1]
\]

Now from (6), we get
\[ y(t, \eta) = \sqrt{4\eta(4\eta - 4)} \left[ -C_1 \sin \left( \sqrt{4\eta(4\eta - 4)} t \right) + C_2 \cos \left( \sqrt{4\eta(4\eta - 4)} t \right) \right], \text{if } \eta \in [0, 1] \]

3. Application of system of linear interval differential equations. In this section, a production inventory model of mixed liquid product is formulated as an application of the theory of system of linear interval differential equations. The model is formulated under an uncertain situation with several realistic assumptions.

The fundamental assumptions and notation for this model are described below:

3.1. Notation.

| Notations | Description |
|-----------|-------------|
| \[x(t), \bar{x}(t)\] | Interval valued volume of liquid in container-I at time \(t\) |
| \[y(t), \bar{y}(t)\] | Interval valued volume of mixed liquid in container-II at time \(t\) |
| \[\bar{q}(t), \bar{q}(t)\] | Interval valued inventory level at time \(t\) |
| \(A\) | Capacity of container-I |
| \(B\) | Capacity of container-II |
| \[\alpha, \alpha\] | Interval valued incoming rate of the liquid with concentration \(\eta\) in container-I |
| \[\beta, \beta\] | Interval valued outgoing rate of the liquid from container-I to container-II |
| \[\gamma, \gamma\] | Interval valued incoming rate of the liquid from container-II to container-I |
| \[\delta, \delta\] | Interval valued outgoing rate of the mixture liquid from container-II |
| \(\xi\) | Concentration of incoming liquid of container-I |
| \(k\) | Concentration of the liquid in container-II |
| \(p\) | Selling price per unit |
| \[a, \alpha, b, \beta\] | Location and shape parameters in demand |
| \[D(p), \bar{D}(p)\] | Interval valued demand rate of customer |
| \[P(t), \bar{P}(t)\] | Interval valued production rate |
| \[\theta, \theta\] | Interval valued wastage rate during production |
| \[c_p, \bar{c}_p\] | Interval valued processing cost |
| \[c_o, \bar{c}_o\] | Interval valued set up cost per cycle |
| \[h, \bar{h}\] | Interval valued holding cost per unit per unit time |
| \(T\) | Cycle length |
| \(t_1\) | Production time |

3.2. Assumptions.

(i) A mixture is produced of a liquid of different concentrations by using two containers (container-I and container-II). Container-I is filled with liquid with zero concentration and container-II is filled with the liquid of concentration \(k\). After mixing of different concentrations of liquid, the liquid is used in the production process as a raw material.

(ii) In the production process, the rate is taken as interval value function of time which is proportional to the volume of mixed liquid. Mathematical form of production rate is

\[ [P(t), \bar{P}(t)] = \frac{[\delta, \delta]}{B} [y(t), \bar{y}(t)] \]

(iii) Generally, selling price of a product imposes a negative impact on the demand of the product, i.e., the customers’ demand of the product decreases as selling price of the product increases. From this idea, the demand function can be considered as \(D(p) = a - bp\) which decreases with the increase of selling price. In physical sense, ‘\(a\)’ is the fixed demand without price effect and ‘\(b\)’ is the selling price effective parameter of demand. However, in the proposed work, uncertainty in customers’ demand is considered. Due to the uncertainty, the demand parameter ‘\(a\)’ will not be fixed, it must be flexible and this flexibility is presented by an interval \([a, \bar{a}]\). Also, under uncertain customers’ demand,
how much demand is affected by selling price that will not be fixed and its uncertainty is presented by the interval \([b, \bar{b}]\). Therefore, in this proposed model, mathematical form of demand rate is \([D(p), \overline{D}(p)] = [a, \overline{a}] - [b, \bar{b}]p = [a - \bar{b}p, \overline{a} - \bar{b}p]\) (according to definition A.3) with \(a - \bar{b}p > 0\). To run the production process with both lower and upper inventory levels positive, the parameters of production rate and demand rate are chosen in such a way that \(P(t) \geq D(p)\), and automatically \(\overline{P}(t) \geq \overline{D}(p)\), \(0 < t < t_1\).

(iv) Deterioration rate is taken as interval-valued.
(v) All the inventory costs are taken as interval-valued.
(vi) The planning horizon is infinite and lead time is constant.
(vii) Shortages are not allowed.

3.3. Problem description. At first, a mixing problem is considered and then it is embedded in the production inventory problem. In mixing problem, two types of liquid with different densities are contained in container-I and container-II. It is also assumed that the container-I is filled with liquid having density zero whereas container-II is filled with liquid of density \(k\). Later a mixture of liquid (liquid of container-I and container-II) having density \(\xi\) is passed through the container-I at the rate \([\alpha, \overline{\alpha}]\) and then from container-I to container-II with the rate \([\beta, \overline{\beta}]\). Again, the mixed liquid is returned back from container-II to container-I with the rate \([\gamma, \overline{\gamma}]\) and this process is continued up to the time to get the desired mixing liquid. Finally, the desired mixing liquid exits from container-II at the rate \([\delta, \overline{\delta}]\). Then in the production problem, the desired mixture is taken as raw material and single product is produced at the production rate \([P(t), \overline{P}(t)] = [\delta, \overline{\delta}]\). The pictorial representation of the problem is shown in Figure 1.

![Figure 1. Representation of mixing procedure in production process](image-url)
3.4. Mathematical formulation. The mixing liquid is produced at a rate \([P(t), \bar{P}(t)]\) and it fulfills the customers’ demand \(([D(p), \bar{D}(p))]\). Then the excess amount is stocked at the rate \((|P(t), \bar{P}(t)| - |D(p), \bar{D}(p)|)\) per unit time. At time \(t = t_1\), the inventory level reaches the maximum level \([Q, \bar{Q}]\). After that, the inventory level decreases due to demand of the customers and at time \(t = T\), it reaches to zero level. Therefore, the variation of inventory level at any time \(t\) is represented by the interval governing differential equation (8a, b) and the mixing process can be represented mathematically by the system of interval differential equation (1).

The mathematical representation of mixing possess is given by the following system of interval differential equations:

\[
\frac{d}{dt}[x(t), \bar{x}(t)] = \xi[\alpha, \bar{\alpha}] - \frac{[\beta, \bar{\beta}]}{A}[x(t), \bar{x}(t)] + \frac{[\gamma, \bar{\gamma}]}{B}[y(t), \bar{y}(t)]
\]

\[
\frac{d}{dt}[y(t), \bar{y}(t)] = \frac{[\beta, \bar{\beta}]}{A}[x(t), \bar{x}(t)] - \frac{[\gamma, \bar{\gamma}] + [\delta, \bar{\delta}]}{B}[y(t), \bar{y}(t)]
\]

subject to \([x(0), \bar{x}(0)] = [0, 0]\) and \([y(0), \bar{y}(0)] = kB\).

According to the principle of flow, inward flow=outward flow i.e., \([\beta, \bar{\beta}] = [\alpha, \bar{\alpha}] + [\gamma, \bar{\gamma}]\) (For container-I) \([\beta, \bar{\beta}] = [\gamma, \bar{\gamma}] + [\delta, \bar{\delta}]\) (For container-II) where \(\beta > \gamma > \delta\) and \(0 < k < 1\).

The mathematical form of the variation of inventory level at any time, \(t\) is given by the following system of differential equations

\[
\frac{d}{dt}[q(t), \bar{q}(t)] + [\bar{\beta}, \bar{\beta}][q(t), \bar{q}(t)] = [P(t), \bar{P}(t)] - [D(p), \bar{D}(p)], \text{ for } 0 \leq t \leq t_1
\]

\[
\frac{d}{dt}[q(t), \bar{q}(t)] + [\bar{\beta}, \bar{\beta}][q(t), \bar{q}(t)] = -[D(p), \bar{D}(p)], \text{ for } t_1 < t \leq T
\]

with the conditions \([q(0), \bar{q}(0)] = [0, 0], [q(t_1), \bar{q}(t_1)] = [Q, \bar{Q}]\) and \([q(T), \bar{q}(T)] = [0, 0]\).

System (9 a, b) is equivalent to its parametric form

\[
\frac{d}{dt}x(t, \eta) = \xi[\alpha, \bar{\alpha}] - \frac{\beta(\eta)}{A}x(t, \eta) + \frac{\gamma(eta)}{B}y(t, \eta)
\]

\[
\frac{d}{dt}y(t, \eta) = \frac{\beta(\eta)}{A}x(t, \eta) - \frac{\gamma(\eta) + \delta(\eta)}{B}y(t, \eta)
\]

subject to \(x(0, \eta) = 0\) and \(y(0, \eta) = kB\).

The parametric form of the system (9a, b) is given by

\[
\frac{d}{dt}q(t, \eta) + \theta(\eta)q(t, \eta) = P(t, \eta) - D(p, \eta), \text{ for } 0 \leq t \leq t_1
\]

\[
\frac{d}{dt}q(t, \eta) + \theta(\eta)q(t, \eta) = -D(p, \eta), \text{ for } t_1 < t \leq T
\]

with the conditions \(q(0, \eta) = 0, q(t_1, \eta) = Q(\eta)\) and \(q(T, \eta) = 0\).

Now, using the solution procedure of system of interval differential equations, system (10a) can be rewritten as

\[
\frac{d^2x(t, \eta)}{dt^2} = -\frac{\beta(\eta)}{A}\frac{dx(t, \eta)}{dt} + \frac{\gamma(eta)}{B}\frac{dy(t, \eta)}{dt} = -\frac{\beta(\eta)}{A}\frac{dx(t, \eta)}{dt} + \frac{\gamma(\eta) + \delta(\eta)}{B}y(t, \eta), \text{ by using (10b)}
\]

\[
= -\left(\frac{\beta(\eta)}{A} + \frac{\gamma(\eta) + \delta(\eta)}{B}\right)\frac{dx(t, \eta)}{dt} = -\left(\frac{\beta(\eta)\delta(\eta)}{AB}\right)\frac{dx(t, \eta)}{dt} + \frac{\beta(\eta)\delta(\eta)}{AB}x(t, \eta) + \xi[\alpha, \bar{\alpha}](\gamma(\eta) + \delta(\eta)), \text{ by using (10a)}
\]
Therefore, \[ \frac{d^2x(t, \eta)}{dt^2} + \left\{ \frac{\beta(\eta) + \gamma(\eta) + \delta(\eta)}{A} \right\} \frac{dx(t, \eta)}{dt} + \frac{\beta(\eta)\delta(\eta)}{AB} x(t, \eta) = \xi_{\eta}(\eta + \delta(\eta)) \] (12)

Let \( k_1(\eta) = \left\{ \frac{\beta(\eta)}{A} + \frac{\gamma(\eta) + \delta(\eta)}{B} \right\} \), \( k_2(\eta) = \frac{\beta(\eta)\delta(\eta)}{AB} \) and \( k_3(\eta) = \xi_{\eta}(\eta + \delta(\eta)) \).

So, the equation (12) can be rewritten of the form
\[ \frac{d^2x(t, \eta)}{dt^2} + k_1(\eta) \frac{dx(t, \eta)}{dt} + k_2(\eta)x(t, \eta) = k_3(\eta) \] (13)

The auxiliary equation of (13) is of the form
\[ m^2 + k_1(\eta)m + k_2(\eta) = 0 \]

The discriminant of this equation is
\[ k_1^2(\eta) - 4k_2(\eta) = \frac{\beta(\eta)}{A^2} + \left\{ \frac{\gamma(\eta) + \delta(\eta)}{B} \right\}^2 - \frac{2\beta(\eta)\gamma(\eta) - \delta(\eta)}{AB} \] (14)

which is strictly greater than zero for all \( \eta \in [0, 1] \).

Therefore, the solution of the equation (13) is of the following form:
\[ x(t, \eta) = \exp \left( -\frac{k_1(\eta)}{2} t \right) \left\{ c_1 \exp \left( k_4(\eta)t \right) + c_2 \exp \left( -k_4(\eta)t \right) \right\} + \frac{k_3(\eta)}{k_2(\eta)} \] (15)

and \( y(t, \eta) = \frac{B}{\gamma(\eta)} \left\{ \frac{\beta(\eta)}{A} x(t, \eta) - \xi_{\eta}(\eta) \right\} \)
\[ = \frac{B}{\gamma(\eta)} \left\{ c_1 \left\{ k_4(\eta) - \frac{k_1(\eta)}{2} \right\} + \frac{\beta(\eta)\gamma(\eta)}{A} \left\{ k_4(\eta) - \frac{k_1(\eta)}{2} \right\} t \right\} + \frac{c_2 \left\{ -k_4(\eta) - \frac{k_1(\eta)}{2} \right\} + \frac{\beta(\eta)\gamma(\eta)}{A} \left\{ -k_4(\eta) - \frac{k_1(\eta)}{2} \right\} t}{\eta_{\eta}(\eta) + Bk_3(\eta)\beta(\eta)\gamma(\eta)} \] (16)

where \( k_4(\eta) = \sqrt{k_1^2(\eta) - 4k_2(\eta)} \), \( c_1 \) and \( c_2 \) are determined using the boundary conditions.

Using the condition \( x(0, \eta) = 0 \) and \( y(0, \eta) = kB(\eta) \), we get
\[ c_1 = -\frac{\gamma}{k_2} + \frac{1}{4k_2} \left\{ \gamma^2 + \xi_{\eta} + \frac{\gamma}{k_2} \left\{ \frac{\beta(\eta)}{A} - \frac{\eta_{\eta}(\eta)}{B} \right\} \right\} \tag{17} \]
\[ c_2 = -\frac{\gamma}{k_2} + \frac{1}{4k_2} \left\{ \gamma^2 + \xi_{\eta} + \frac{\gamma}{k_2} \left\{ \frac{\beta(\eta)}{A} - \frac{\eta_{\eta}(\eta)}{B} \right\} \right\} \tag{18} \]

The solution of (11a) and (11b) are of the form
\[ q(t, \eta) = \frac{D(p, \eta)}{\gamma(\eta)} \left\{ \exp \left( -\theta(\eta)(t - t_1) \right) \right\} \tag{19} \]
\[ + \frac{c_1 \left\{ k_4(\eta) - \frac{k_1(\eta)}{2} \right\} + \frac{\beta(\eta)\gamma(\eta)}{A} \left\{ k_4(\eta) - \frac{k_1(\eta)}{2} \right\} t}{k_4(\eta) - \frac{k_1(\eta)}{2} + \theta(\eta)} \exp \left( \left( k_4(\eta) - \frac{k_1(\eta)}{2} \right) t \right) - \exp(\theta(\eta)t) \tag{20} \]
From the continuity of \(q(t, \eta)\) at \(t = t_1\), we have

\[
T = t_1 + \frac{1}{2\theta} \log \left[ \frac{\theta}{D(p)} \left\{ \frac{k_3 \beta \delta}{Ak_2 \theta^2} - \frac{D(p)}{\theta} - \frac{\xi \alpha \delta}{\theta^2} \right\} \left\{ 1 - \exp(-\theta t) \right\} \right] + \frac{c_1}{\theta} \left[ \left( k_4 - \frac{k_1}{2} \right) \frac{\theta^2}{\beta^2} + \frac{\beta \delta}{\theta^2} \right] \left[ \exp \left\{ - \left( k_4 - \frac{k_1}{2} \right) t_1 \right\} - \exp(-\theta t) \right] + 1
\]

\[
+ \frac{c_2}{\theta} \left[ \left( k_4 + \frac{k_1}{2} \right) \frac{\theta^2}{\beta^2} - \frac{\beta \delta}{\theta^2} \right] \left[ \exp \left\{ - \left( k_4 + \frac{k_1}{2} \right) t_1 \right\} - \exp(-\theta t) \right] + 1 \quad (21)
\]

3.5. Various components of the model in parametric form.

Sales revenue. \(SR(\eta) = sp\eta \int_0^T D(p, \eta) \, dt = sp\eta D(p, \eta) T\)

Ordering cost. \(C_o(\eta) = C_o + \eta(\bar{C}_o - C_o)\)

Production cost. \(PC(\eta) = c_p(\eta) \int_0^{t_1} P(t, \eta) \, dt\)
Average profit in parametric form. The average profit of the system in parametric form is given by

\[ Z(t_1, p, \eta) = \frac{1}{T} \left[ SR(\eta) - PC(\eta) - HC(\eta) - C_o(\eta) \right], \ \eta \in [0, 1] \]

The, centre and radius of average profit are respectively given by

\[ Z_c(t_1, p) = \frac{Z(t_1, p, 1) + Z(t_1, p, 0)}{2} \text{ and } Z_r(t_1, p) = \frac{Z(t_1, p, 1) - Z(t_1, p, 0)}{2} \]

Therefore, the corresponding optimization problem of the proposed problem can be expressed as follows:

Maximize \( Z(t_1, p, \eta), \ \eta \in [0, 1] \)
subject to \( t_1 > 0, \ p > 0 \) \hfill (22)

Bounds of the average profit. The bounds of the average profit can be obtained as

\[
\begin{align*}
Z(t_1, p) &= \min \left\{ Z(t_1, p, \eta), \ \eta \in [0, 1] \right\} \\
\bar{Z}(t_1, p) &= \max \left\{ Z(t_1, p, \eta), \ \eta \in [0, 1] \right\}
\end{align*}
\]

(23)

3.6. Solution methodology. To develop the solution methodology of the interval optimization problem in parametric form (22), the concepts of interval order relations, definition of maximizer of interval-valued function and characterization theorem for existence of maximizer of interval-valued are required. So, first of all, we have to introduce these concepts.

3.6.1. Interval order relation. According to the Bhunia and Samanta (2014), an order relation between two intervals in parametric form can be defined as follows:

**Definition 1.** Let \( A_1 = [a, \bar{a}] \) and \( B_1 = [b, \bar{b}] \) be two intervals with parametric forms whose parametric representations are \( A_1 = \{ a(\eta) = a + \eta(\bar{a} - a) : \eta \in [0, 1] \} \) and \( B_1 = \{ b(\eta) = b + \eta(\bar{b} - b) : \eta \in [0, 1] \} \). Then the order relation between \( A_1 \) and \( B_1 \) is denoted by \( \geq_{\text{max}} \) and is defined by \( A_1 \geq_{\text{max}} B_1 \iff \begin{cases} A_c > B_c, & \text{if } A_c \neq B_c \\ A_c \leq B_c, & \text{if } A_c = B_c \end{cases} \)

where \( A_c = \frac{a(0) + a(1)}{2}, \ A_r = \frac{a(1) - a(0)}{2}, \ B_c = \frac{b(0) + b(1)}{2} \) and \( B_r = \frac{b(1) - b(0)}{2} \).

**Definition 2.** Let \( F : D \subseteq \mathbb{R}^n \rightarrow I_c \) be an interval-valued function given in the parametric form, \( F(t) = [\underline{F}(t), \overline{F}(t)] = \{ F(t, \eta) : \eta \in [0, 1] \} \), where \( F(t, \eta) = \underline{F}(t) + \eta(\overline{F}(t) - \underline{F}(t)) \). Then a point \( t = t^* \in D \) is said to be the maximizer of \( F \) if \( \underline{F}(t^*) \geq_{\text{max}} \overline{F}(t), \forall t \in B(t^*, s) \cap D \). Where \( B(t^*, s) \) is the open ball centre at \( t = t^* \) with radius \( s(> 0) \) and \( I_c \) is the set of all compact intervals i.e., \( I_c = \{ [a, \bar{a}] : a \leq \bar{a} \} \).

**Theorem 2.** A point \( t = t^* \in D \subseteq \mathbb{R}^n \) is the maximizer of \( F(t) = \{ F(t, \eta) : \eta \in [0, 1] \} \) if and only if \( \begin{cases} t = t^* \text{is a maximizer of } F_c(t), \text{ when } F_c(t) \neq \text{constant} \\ t = t^* \text{is a minimizer of } F_r(t), \text{ when } F_r(t) = \text{constant} \end{cases} \)

where \( F_c(t) = \frac{F(t, 0) + F(t, 1)}{2}, \ F_r(t) = \frac{F(t, 1) - F(t, 0)}{2} \).
Proof. A point \( t = t^* \in D \subseteq R^n \) is the maximizer of \( F(t) \) if and only if \( F(t^*) \geq \max F(t) \)

\[
\iff \left\{ \begin{array}{l}
F(t^*) > F(t) \text{ if } F_c(t^*) \neq F_c(t), \forall t \in D \text{ and } t \neq t^* \\
F(t^*) \leq F(t) \text{ if } F_c(t^*) = F_c(t), \forall t \in D 
\end{array} \right.
\]

\[
\iff \left\{ \begin{array}{l}
F(t^*) > F(t) \text{ if } F_c(t) \neq \text{ constant} \\
F(t^*) \leq F(t) \text{ if } F_c(t) = \text{ constant} 
\end{array} \right.
\]

\[
\iff \left\{ \begin{array}{l}
t = t^* \text{ is a maximizer of } F_c(t), \text{ when } F_c(t) \neq \text{ constant} \\
t = t^* \text{ is a minimizer of } F_c(t), \text{ when } F_c(t) = \text{ constant} 
\end{array} \right.
\]

This completes the proof.

3.6.2. Solution algorithm. The maximization problem (22) is solved by using the following steps:

**Step-1.** Input the values of known inventory parameters.

**Step-2.** Calculate \( Z_c(t_1, p), Z_r(t_1, p) \).

**Step-3.** Check \( Z_c(t_1, p) = \text{constant or not} \).

**Step-4.** If \( Z_c(t_1, p) \neq \text{constant} \), go to Step-5. Otherwise go to Step-6.

**Step-5.** Maximize \( Z_c(t_1, p) \) using 'NMaximize' comment in MATHEMATICA, and obtained the maximizer \( t_1 = t_1^*, p = p^* \).

**Step-6.** Minimize \( Z_r(t_1, p) \) using 'NMinimize' comment in MATHEMATICA, and obtained the minimizer \( t_1 = t_1^*, p = p^* \).

**Step-7.** Find the values of \( T = Z(t_1^*, p^*, \eta) \) for different values of \( \eta \in [0, 1] \).

3.7. Numerical examples. To justify the validity of the proposed production inventory model, two numerical examples are taken as follows:

**Example 2 (interval environment).** The values of known inventory parameters in interval environment are taken as follows: \( A = 105 \text{ Lt.}, B = 152 \text{ Lt.}, [\alpha, \beta] = [10, 12] \text{Lt./time}, [\gamma, \eta] = [75, 77] \text{Lt./time}, [\delta, \theta] = [10, 12] \text{Lt./time}, [\alpha, \beta] = [220, 230], [\alpha, \beta] = [1.2, 1.3], [\alpha, \beta] = [0.10, 0.12], [\varepsilon_p, \tau_p] = [20, 22], [C_c, C_o] = [500, 505], [\lambda, \lambda] = [0.6, 0.8].

**Example 3 (Crisp environment).** The values of inventory parameters in crisp environment are taken in the form of interval as follows: \( A = 105 \text{ Lt.}, B = 152 \text{Lt.}, [\alpha, \beta] = [11, 11] \text{Lt./time}, [\gamma, \eta] = [76, 76] \text{Lt./time}, [\delta, \theta] = [11, 11] \text{Lt./time}, [\alpha, \beta] = [225, 225], [\alpha, \beta] = [1.25, 1.25], [\alpha, \beta] = [0.11, 0.11], [\varepsilon_p, \tau_p] = [21, 21], [C_c, C_o] = [503, 503], [\lambda, \lambda] = [0.7, 0.7].

Examples 2 and 3 are solved numerically by using interval order relation and MATHEMATICA software and the obtained optimal results are presented in Tables 2-4. The pictorial representation of production rate is presented in Figure 2. Also, the pictorial representations of center of average profit, bounds of the average profit and different parametric values of average profit are displayed in Figures 3-6.
Table 2. Optimal results of Example 2

| Variable                               | Optimal result          |
|----------------------------------------|-------------------------|
| Production time ($t_1$)                | 1.743 year              |
| Selling price ($p$)                    | $102.03/Lit.            |
| Cycle length ($T$)                     | 1.865 year              |
| Centre of the average profit ($Z_c$)   | $7282.09/year           |
| Interval valued average profit ([$Z$, $Z$]) | [$7168.11, $7397.71]/year |

Table 3. Optimal average profit for different values of ‘$\eta$’ of Example 2

| $\eta$ | Average profit ($Z(\eta)$) |
|--------|-----------------------------|
| 0.0    | $7397.71                    |
| 0.2    | $7351.26                    |
| 0.4    | $7305.08                    |
| 0.5    | $7282.09                    |
| 0.6    | $7259.17                    |
| 0.8    | $7213.52                    |
| 1.0    | $7168.11                    |

Table 4. Optimal results of Example 3

| Variable                               | Optimal result          |
|----------------------------------------|-------------------------|
| Production time ($t_1$)                | 1.746 year              |
| Selling price ($p$)                    | $102.042/Lit.           |
| Cycle length ($T$)                     | 1.868 year              |
| Centre of the average profit ($Z_c$)   | $7281.62/year           |
| Interval valued average profit ([$Z$, $Z$]) | [$7281.62, $7281.62]/year |

Figure 2. Pictorial representation of Production rate for different values of ‘$\eta$’ for Example 2
Figure 3. Pictorial representation of centre of interval-valued average profit for Example 2

Figure 4. Pictorial representation of average profit for different values of ‘η’ for Example 2
Discussions. From the computational results of numerical examples, the following findings are observed.

(i) Comparing Tables 2 and 4, it is observed that the average profit for Example 2 is lying between the bounds of the average profit for Example 3.
(ii) On the other hand, from Table 3, it is clear that the parametric value of average profit decreases with the increase of values of the parameter ‘η’.

(iii) From the Figures 3-5, it is observed that the centre of average profit, bounds of average profit and different parametric values of average profit are concave. Thus, these graphical representations give a geometrical proof of the optimality of all the results of the Examples 2 and 3.

3.8. Sensitivity analyses. To inspect the effects of different inventory parameters on the centre of average profit and other decision variables related to the proposed model, sensitivity analyses are performed considering Example2. Since all the inventory parameters for the Example2 are interval valued, both bounds of an inventory parameter are changed from -20% to +20% keeping other parameters as same and the computational results are presented graphically in Figures 7-12.
Figure 9. Effect of $[H, \bar{H}]$ on optimal policy

Figure 10. Effect of $[C_o, \bar{C}_o]$ on optimal policy

Figure 11. Effect of $[v_p, \bar{v}_p]$ on optimal policy
From Figures 7-12, it is observed that

(i) the centre of average profit \( Z_c \) is highly sensitive with the reverse effect with the changes of both the bounds of \([a, \bar{a}]\), \([b, \bar{b}]\) and \([c_p, \bar{c}_p]\), whereas it is slightly sensitive with respect to \([\theta, \bar{\theta}]\), \([h, \bar{h}]\) and \([C_o, \bar{C}_o]\).

(ii) The selling price \( p \) is moderately sensitive with the changes of both the bounds of \([b, \bar{b}]\). Whereas, it is less sensitive with respect to \([\theta, \bar{\theta}]\), \([c_p, \bar{c}_p]\), \([h, \bar{h}]\) and \([C_o, \bar{C}_o]\).

(iii) The production time \( t_1 \) and the cycle length \( T \) are less sensitive with the changes of both the bounds of \([b, \bar{b}]\), \([\theta, \bar{\theta}]\), \([c_p, \bar{c}_p]\) and \([h, \bar{h}]\). Whereas these are moderately sensitive with respect to \([a, \bar{a}]\) and \([C_o, \bar{C}_o]\).

4. Conclusions. In this article, for the first time, the concept of system of linear interval differential equations and its solution procedure in parametric form are introduced. Also, this concept is applied in the area of inventory control in an impressive way. In the derivations of solution methodology, all the theories are presented in the parametric form. Then to visualize the solution methodology, some simple examples are considered and solved. After that, a production inventory model for mixed product of liquid (cold drinks, liquid foods, different type of liquid medicine, etc) with interval parameters are formulated mathematically as an application of system of linear interval differential equations and by using interval order relation and MATHEMATICA software, the corresponding interval-valued average profit is computed. To justify the proposed inventory model, two numerical examples are solved and presented the average profits for different values of parameters lying between 0 and 1. From the concavity representations of centre of the average profit, bounds of the average profit and different parametric values of average profit, it can be said that all the obtained results are optimal. Finally, from the sensitivity analyses, it can be also concluded that a decision maker should give more concentration about the demand parameters and production cost.

For future research, the concept of this work may be extended in Type-2 fuzzy (Moreno et al., 2020)[32], Type-2 interval (Rahman et al., 2020c)[38] environments. Also, the concept of this work can be applied in the bioeconomic model, supply chain model etc. Finally, the model of this work can be extended by including
Appendix. Basic concepts of interval and interval-valued function.

Let $I_c$ denote the space of all compact intervals of $R$, i.e., $I_c = \{[a, \overline{a}] : a, \overline{a} \in \mathbb{R} \land \overline{a} \geq a \}$

Definition A.1. The parametric representation of an interval $A_1 = [a, \overline{a}] \in I_c$ is defined as follows:

1. $A_1 = \{(a(\eta) = a + \eta(\overline{a} - a) : \eta \in [0, 1]\}$ in increasing form.
2. $A_1 = \{(a(\eta) = \overline{a} + \eta(a - \overline{a}) : \eta \in [0, 1]\}$ in decreasing form.

Definition A.2. Let $A_1 = \{(a(\eta) = a + \eta(\overline{a} - a) : \eta \in [0, 1]\}$ and $B_1 = \{(b(\eta) = b + \eta(\overline{b} - b) : \eta \in [0, 1]\}$ be the parametric representations of two intervals $A_1 = [a, \overline{a}]$ and $B_1 = [b, \overline{b}]$ respectively. Then the arithmetic operations are defined as follows:

1. $A_1 + B_1 = \{(a(\eta) + b(\eta) : \eta \in [0, 1]\}$
2. $A_1B_1 = \{(a(\eta)b(\eta) : \eta \in [0, 1]\}$
3. $\frac{A_1}{B_1} = \left\{\frac{a(\eta)}{b(1-\eta)} : \eta \in [0, 1]\right\}$, provided $b(1-\eta) \neq 0$
4. $cA_1 = \{ca(\eta) : \eta \in [0, 1]\} \land c \in R$
5. $A_1 - pB_1 = \{(a(\eta) - b(\eta) : \eta \in [0, 1]\}$, where $-p$ denotes the parametric difference
6. $A_1 = B_1 \iff a(\eta) = b(\eta), \eta \in [0, 1]$

The definition of interval arithmetic can also be represented in upper and lower bounds of the form given below:

Definition A.3. Let two intervals are given by $A_1 = [a, \overline{a}]$ and $B_1 = [b, \overline{b}]$ whose parametric representations are $\{a(\eta) : \eta \in [0, 1]\}$ and $\{b(\eta) : \eta \in [0, 1]\}$. Then different arithmetic operations on $I_c$ are defined as follows:

1. $A_1 + B_1 = [\min_{\eta \in [0, 1]}\{a(\eta) + b(\eta)\}, \max_{\eta \in [0, 1]}\{a(\eta) + b(\eta)\}]$
2. $A_1 - B_1 = [\min_{\eta \in [0, 1]}\{a(\eta) - b(\eta)\}, \max_{\eta \in [0, 1]}\{a(\eta) - b(\eta)\}]$
3. $A_1B_1 = [\min_{\eta \in [0, 1]}\{a(\eta)b(\eta)\}, \max_{\eta \in [0, 1]}\{a(\eta)b(\eta)\}]$
4. $\frac{A_1}{B_1} = [\min_{\eta \in [0, 1]}\{\frac{a(\eta)}{b(1-\eta)}\}, \max_{\eta \in [0, 1]}\{\frac{a(\eta)}{b(1-\eta)}\}]$, provided $b(1-\eta) \neq 0$
5. $cA_1 = [\min_{\eta \in [0, 1]}\{ca(\eta)\}, \max_{\eta \in [0, 1]}\{ca(\eta)\}] \land c \in R$
6. $A_1 - pB_1 = [\min_{\eta \in [0, 1]}\{a(\eta) - b(\eta)\}, \max_{\eta \in [0, 1]}\{a(\eta) - b(\eta)\}]$
where $-p$ denotes the parametric difference

Definition A.4. Let $F : D \subseteq R \to I_c$ be an interval-valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$. Then the parametric representation of the function $F(t)$ is given by

$$\{F(t, \eta) = \underline{F}(t) + \eta(\overline{F}(t) - \underline{F}(t)) : \eta \in [0, 1]\} \text{ in the increasing form}$$

$$\{F(t, \eta) = \overline{F}(t) - \eta(\overline{F}(t) - \underline{F}(t)) : \eta \in [0, 1]\} \text{ in the decreasing form}$$
REFERENCES

[1] F. J. Agocs, W. J. Handley, A. N. Lasenby and M. P. Hobson, Efficient method for solving highly oscillatory ordinary differential equations with applications to physical systems, Physical Review Research, 2 (2020), 013030.

[2] N. Ahmady, A numerical method for solving fuzzy differential equations with fractional order, International Journal of Industrial Mathematics, 11 (2019), 71–77.

[3] L. Arnold, Stochastic Differential Equations, New York, 1974.

[4] B. Bede, I. J. Rudas and A. L. Bencsik, First order linear fuzzy differential equations under generalized differentiability, Information Sciences, 177 (2007), 1648–1662.

[5] A. R. Bergstrom, Non recursive models as discrete approximations to systems of stochastic differential equations, Econometrica: Journal of the Econometric Society, (1966), 173–182.

[6] A. K. Bhunia and S. S. Samanta, A study of interval metric and its application in multi-objective optimization with interval objectives, Computers & Industrial Engineering, 74 (2014), 169–178.

[7] J. J. Buckley and T. Feuring, Fuzzy differential equations, Fuzzy Sets and Systems, 110 (2000), 43–54.

[8] C. Chicone, Ordinary Differential Equations with Applications, 34, Springer Science & Business Media, 2006.

[9] D. P. Covei and T. A. Pirvu, An elliptic partial differential equation and its application, Applied Mathematics Letters, 101 (2020), 106059.

[10] S. Cui and A. Friedman, A free boundary problem for a singular system of differential equations: An application to a model of tumour growth, Transactions of the American Mathematical Society, 355 (2003), 3537–3590.

[11] L. M. B. da Costa Campos, Non-linear differential equations and dynamical systems, CRC Press, 2019.

[12] T. M. da Costa, Y. Chalco-Cano, W. A. Lodwick and G. N. Silva, A new approach to linear interval differential equations as a first step toward solving fuzzy differential, Fuzzy Sets and Systems, 347 (2018), 129–141.

[13] B. Das, N. K. Mahapatra and M. Maiti, Initial-valued first order fuzzy differential equation in Bi-level inventory model with fuzzy demand, Mathematical Modelling and Analysis, 13 (2008), 493–512.

[14] S. Das, M. A. A Khan, E. E. Mahmoud, A. H. Abdel-Aty, K. M. Abualnaja and A. A. Shaikh, A production inventory model with partial trade credit policy and reliability, Alexandria Engineering Journal, 60 (2021), 1325–1338.

[15] S. Das, A. K. Manna, E. E. Mahmoud, A. H. Abdel-Aty and A. A. Shaikh, Product replacement policy in a production inventory model with replacement period-, stock- and price-dependent demand, Journal of Mathematics, (2020).

[16] M. De, B. Das and M. Maiti, EPL models with fuzzy imperfect production system including carbon emission: A fuzzy differential equation approach, Soft Computing, 24 (2020), 1293–1313.

[17] K. Engelborghs, V. Lemaire, J. Belair and D. Roose, Numerical bifurcation analysis of delay differential equations arising from physiological modeling, Journal of Mathematical Biology, 42 (2001), 361–385.

[18] N. A. Gasilov and S. E. Amrahov, Solving a nonhomogeneous linear system of interval differential equations, Soft Computing, 22 (2018), 3817–3828.

[19] N. A. Gasilov and S. E. Amrahov, On differential equations with interval coefficients, Mathematical Methods in the Applied Sciences, 43 (2020), 1825–1837.

[20] B. Ghanbari, H. Günerhan and H. M. Srivastava, An application of the Atangana-Baleanu fractional derivative in mathematical biology: A three-species predator-prey model, Chaos, Solitons & Fractals, 138 (2020), 109910.

[21] T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Annals of Mathematics, (1919), 292–296.

[22] P. Guchhait, M. K. Maiti and M. Maiti, A production inventory model with fuzzy production and demand using fuzzy differential equation: An interval compared genetic algorithm approach, Engineering Applications of Artificial Intelligence, 26 (2013), 766–778.

[23] N. Ikeda and S. Watanabe, A comparison theorem for solutions of stochastic differential equations and its applications, Osaka Journal of Mathematics, 14 (1977), 619–633.
[24] R. Jafari, W. Yu and X. Li, Fuzzy differential equations for nonlinear system modelling with Bernstein neural networks, *IEEE Access*, 4 (2016), 9428–9436.

[25] H. T. Kaneko and S. Agata, Optimal control in an inventory management problem considering replenishment lead time based upon a non-diffusive stochastic differential equation, *Journal of Advanced Mechanical Design, Systems, and Manufacturing*, 13 (2019), JAMDSM0008-JAMDSM0008.

[26] D. Kumar, A. R. Seadawy and A. K. Joardar, Modified Kudryashov method via new exact solutions for some conformable fractional differential equations arising in mathematical biology, *Chinese Journal of Physics*, 56 (2018), 75–85.

[27] H. Liao and L. Li, Environmental sustainability EOQ model for closed-loop supply chain under market uncertainty: A case study of printer remanufacturing, *Computers & Industrial Engineering*, (2020), 106525.

[28] W. Liu, M. Rockner, X. Sun and Y. Xie, Averaging principle for slow-fast stochastic differential equations with time dependent locally Lipschitz coefficients, *Journal of Differential Equations*, 268 (2020), 2910–2948.

[29] A. Mahata, S. P. Mondal, B. Roy and S. Alam, Study of two species prey-predator model in imprecise environment with MSY policy under different harvesting scenario, *Environment, Development and Sustainability*, (2021), 1–25.

[30] X. Mao and C. Yuan, Stochastic differential equations with Markovian switching, Imperial college press, 2006.

[31] W. Materi and D. S. Wishart, Computational systems biology in drug discovery and development: Methods and applications, *Drug Discovery Today*, 12 (2007), 295–303.

[32] J. E. Moreno, M. A. Sanchez, O. Mendoza, A. Rodriguez-Diaz, O. Castillo, P. Melin and J. R. Castro, Design of an interval type-2 fuzzy model with justifiable uncertainty, *Information Sciences*, 513 (2020), 206–221.

[33] A. M. Overstall, D. C. Woods and B. M. Parker, Bayesian optimal design for ordinary differential equation models with application in biological science, *Journal of the American Statistical Association*, (2020), 1–16.

[34] D. Pal, G. S. Mahapatra and G. P. Samanta, New approach for stability and bifurcation analysis on predator-prey harvesting model for interval biological parameters with time delays, *Computational and Applied Mathematics*, 37 (2018), 3145–3171.

[35] P. Pandit and P. Singh, Fully Fuzzy Semi-linear Dynamical System Solved by Fuzzy Laplace Transform Under Modified Hukuhara Derivative, In Soft Computing for Problem Solving, Springer, Singapore, 2020, 155–179.

[36] M. S. Rahman, A. K. Manna, A. A. Shaikh and A. K. Bhunia, An application of interval differential equation on a production inventory model with interval-valued demand via center-radius optimization technique and particle swarm optimization, *International Journal of Intelligent Systems*, 35 (2020), 1280–1326.

[37] M. S. Rahman, A. Duary, A. A. Shaikh and A. K. Bhunia, An application of parametric approach for interval differential equation in inventory model for deteriorating items with selling-price-dependent demand, *Neural Computing and Applications*, (2020), 1–17.

[38] M. S. Rahman, A. A. Shaikh and A. K. Bhunia, On type-2 interval with interval mathematics and order relations: Its applications in inventory control, *International Journal of Systems Science: Operations & Logistics*, (2020), 1–13.

[39] M. Ramezanzadeh, M. Heidari, O. Fard and A. Borzabadi, On the interval differential equation: Novel solution methodology, *Advances in Difference Equations*, (2015).

[40] C. Rout, D. Chakraborty and A. Goswami, An EPQ model for deteriorating items with imperfect production, two types of inspection errors and rework under complete backordering, *International Game Theory Review*, 22 (2020), 2040011.

[41] S. Salahshour, A. Ahmadian, S. Abbasbandy and D. Baleanu, M-fractional derivative under interval uncertainty: Theory, properties and applications, *Chaos, Solitons and Fractals*, (2018), 121–125.

[42] S. Salahshour, A. Ahmadian, M. Salimi, M. Ferarra and D. Baleanu, Asymptotic solutions of fractional interval differential equations with nonsingular kernel derivative, *Chaos: An Interdisciplinary Journal of Nonlinear Science*, AIP, 29 (2019), 083110.

[43] L. Stefanini and B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, *Nonlinear Analysis: Theory, Methods & Applications*, 71 (2009), 1311–1328.
[44] M. Thongmoon and S. Pusjuso, The numerical solutions of differential transform method and the Laplace transform method for a system of differential equations, *Nonlinear Analysis: Hybrid Systems*, 4 (2010), 425–431.

[45] A. Tsoularis, A stochastic differential equation inventory model, *International Journal of Applied and Computational Mathematics*, 5 (2019), 8.

Received October 2020; 1st revision April 2021; final revision June 2021; early access November 2021.

E-mail address: mdsadikur.95@gmail.com
E-mail address: mathsubhajitdas@gmail.com
E-mail address: akmanna1987@gmail.com
E-mail address: aliashaikh@math.buruniv.ac.in
E-mail address: akbhunia@math.buruniv.ac.in
E-mail address: ali.ahmadian@ukm.edu.my
E-mail address: soheil.salahshour@eng.bau.edu.tr