A solution to the problem posed by Byland and Scialom

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Recently, Byland and Scialom studied the evolution of the Bianchi I, the Bianchi III and the Kantowski–Sachs universe on the basis of dynamical systems methods (Phys. Rev. D57, 6065 (1998), gr-qc/9802043). In particular, they have pointed out a problem to determine the stability properties of one of the degenerate critical points of the corresponding dynamical system. Here we give a solution, showing that this point is unstable both to the past and to the future. We also discuss the asymptotic behavior of the trajectories in the vicinity of another critical point.

In one of their recent works, Byland and Scialom studied the evolution of the Bianchi I, the Bianchi III and the Kantowski–Sachs universe in a model with a real scalar field and a convex positive potential [1]. A considerable part of the investigation was devoted to the analysis of the asymptotic behavior and the stability properties of the solutions of the Einstein–Klein–Gordon equations

\[ \ddot{\theta} = -\frac{1}{3} \dot{\theta}^2 - 2\sigma^2 + V(\varphi) - \psi^2, \]
\[ \dot{\dot{\sigma}} = -\frac{1}{3\sqrt{3}} \sigma^2 - \theta \sigma + \frac{1}{\sqrt{3}} \left( \sigma^2 + V(\varphi) + \frac{1}{2} \psi^2 \right), \]
\[ \dot{\varphi} = \psi, \]
\[ \dot{\psi} = -\theta \psi - \frac{dV}{d\varphi}, \]

where \( \theta \) is the function of the expansion rate, \( \sigma_{\mu\nu} \) is the shear tensor of the hypersurface of constant time, \( \sigma = \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu} \), \( \varphi \) is the scalar field, and \( V \) is a convex positive potential; an overdot stands for derivatives with respect to \( t \) (see Eqs. (8)–(11) in [1]).

The analysis was based on determining the stability properties of the critical points of the dynamical system

\[ S' = -\frac{1}{3\sqrt{3}} - \frac{2}{3} S + \frac{1}{2\sqrt{3}} (2S^2 + 2U^2 + P^2) + FS, \]
\[ U' = \left( 1 - \frac{\lambda}{2} P + F \right) U, \]
\[ P' = -\frac{2}{3} P + \lambda U^2 + FP, \]

where \( S = \sigma/\theta, U = \sqrt{V}/\theta, P = \psi/\theta, F = 2S^2 - U^2 + P^2, \lambda \) is given by \( V = V_0 e^{-\lambda \varphi} \), and a prime stands for derivatives with respect to \( \tau \) defined by \( d\tau = \theta dt \) (see Eqs. (24)–(26) in [1]). Here we omit an equation for \( \theta \), since Eqs. (1) do not contain this function.

It was found in [1] that the dynamical system (1) has the following critical points:

\[ P_1: \quad S = \frac{1}{2\sqrt{3}}, \quad U = 0, \quad P = 0 \]
\[ P_2: \quad S = 0, \quad U = \sqrt{\frac{6 - \lambda^2}{18}}, \quad P = \frac{\lambda}{3}, \quad \lambda \leq \sqrt{6} \]

\[ P_3: \quad S = \frac{1}{2\sqrt{3}} \left( 1 + \frac{\lambda^2}{\lambda} \right), \quad U = \frac{\sqrt{2 + \lambda^2}}{\sqrt{2} \left( 1 + \lambda^2 \right)}, \quad P = \frac{\lambda}{1 + \lambda^2} \]

Thus, it is stable for \( \lambda < \sqrt{2} \) and unstable for \( \sqrt{2} < \lambda < \sqrt{6} \). It was also found that \( P_2 \) is also unstable for \( \lambda = \sqrt{2} \).

The problem is to determine the stability properties of \( P_2 \) for the case \( \lambda = \sqrt{6} \), in which \( P_2 \) is degenerate with the eigenvalues \( \varepsilon_1 = 1 \) (twice) and \( \varepsilon_2 = 4/3 \). Here we shall demonstrate that the point \( P_2 \) is unstable (both to the future and to the past).

Notice that for the case \( \lambda = \sqrt{6} \), \( P_2 \) belongs to the ellipsis \( \Sigma \). Thus, one of the zero eigenvalues corresponds to the fact that \( \Sigma \) is a one-dimensional critical set. In order to prove that the critical point \( P_2 \) is unstable both to the past and to the future, it is sufficient to show that there is a projection of this point, which is unstable. To do this, we rewrite the dynamical system (1) as

\[ S' = 2\alpha \left( \frac{1}{2\sqrt{3}} + S \right) \bar{P} + \frac{1}{2\sqrt{3}} (2S^2 + 2U^2 + \bar{P}^2) + FS, \]
\[ U' = \left( \frac{\alpha}{2} \bar{P} + \bar{F} \right) U, \]
\[ P' = \frac{4}{3} \bar{P} + \alpha (3U^2 + 2\bar{P}^2) + (\alpha + \bar{P}) \bar{F}, \]

where \( \bar{P} = P - \alpha, \alpha = \sqrt{2}/3, \bar{F} = 2S^2 - U^2 + \bar{P}^2 \) is the function of the expansion rate. The point \( P_2 \) now corresponds to the origin, \( (S, U, P) = (0, 0, 0) \).

Consider the projection of (2) in the plane \( S = 0 \). The equations for \( U \) and \( \bar{P} \) may now be written as

\[ \frac{U'}{2} = \left( \frac{\alpha}{2} \bar{P} - U^2 + \bar{P}^2 \right) U, \quad \bar{P}' = \frac{4}{3} \bar{P} + (2\alpha - \bar{P}) U^2 + (3\alpha + \bar{P}) \bar{P}^2. \]

Denote the right hand sides of Eqs. (3a) and (3b) as \( \bar{U}(U, \bar{P}) \) and \( \bar{P}(U, \bar{P}) \) respectively. The idea is to find a solution \( \bar{P} = f(U) \) of the equation \( \bar{P}(U, \bar{P}) = 0 \) in a neighborhood of \( U = 0 \), to substitute it into \( \bar{U}(U, \bar{P}) \):
that starting around the $m$-dimensional stable manifold $W^s$, the trajectories tend to \( \omega \) (see, e.g., [2]). It is easy to see that the corresponding solution in our case assumes the form

\[
\tilde{P} = f(U) = - \sqrt{\frac{3}{2}} U^2 + O(U^4).
\]

Therefore,

\[
U(U, f(U)) = - \frac{3}{2} U^3 + O(U^7).
\]

Thus, $m = 3$ is odd, and $a_m$ is negative. It follows immediately from the standard results for the two-dimensional dynamical systems that the point \((U, \tilde{P}) = (0, 0)\) is a topological saddle. Hence, it is unstable both to the future and to the past.

Let us mention that the same conclusion may be obtained basing on the results of the center manifold theory (see, e.g., [3]). We remark that $P_2$ is not the only degenerate point of $\Sigma$. Recall that the eigenvalues of $\Sigma$ are

\[
\varepsilon_1 = 0, \quad \varepsilon_2 = 1 - \frac{\lambda}{2} P, \quad \varepsilon_3 = \frac{2}{3}(2 + \sqrt{3} S).
\]

Thus, for any point of $\Sigma$, except for the points \((P, S) = (0, \pm 1/\sqrt{3})\), there exists $\lambda = 2/P, \lambda \in [-\sqrt{6}, \sqrt{6}]$ such that $\varepsilon_2$ turns to zero, thus making this point degenerate. The stability properties of these degenerate points may be studied in a way similar to the above.

Finally, let us make some comments on the behavior of the trajectories of (4) in the vicinity of the point $P_1$. It was found in [4] that this point has the eigenvalues $\varepsilon_{S, P} = -1/2$ and $\varepsilon_U = 1/2$. It was also claimed that starting around $P_1$, the critical point can never be reached by any solutions of the dynamical system. This is not quite correct. It follows from the standard results of the dynamical systems theory (see, e.g., [4], Theorem 6.1) that in a neighborhood of $P_1$ there exists a two-dimensional stable manifold $W^s$ and a one-dimensional unstable manifold $W^u$. The trajectories lying on these manifolds tend to $P_1$ as $\tau \to \infty$ and $\tau \to -\infty$ respectively. One can easily find the asymptotic behavior of these solutions.

Namely, let us introduce $\tilde{S} = S + 1/(2\sqrt{3})$. Then the dynamical system (4) reads as

\[
\tilde{S}' = -\frac{1}{2} \tilde{S} - \frac{1}{2\sqrt{3}} (4S^2 - 3U^2) + \tilde{F} \tilde{S},
\]

\[
U' = \frac{1}{2} \left(1 - \lambda P - \frac{4}{\sqrt{3}} \tilde{S} + 2 \tilde{F}\right) U,
\]

\[
P' = -\frac{1}{2} P - \frac{2}{\sqrt{3}} S \tilde{P} + \lambda U^2 + \tilde{F} P,
\]

where $\tilde{F} = 2S^2 - U^2 + P^2$. The point $P_1$ now corresponds to the origin, $(S, U, P) = (0, 0, 0)$. The eigenvectors of this point are $\zeta_S = (1, 0, 0), \zeta_P = (0, 0, 1), \zeta_U = (0, 1, 0)$. Now one can see that the trajectories on $W^s$ take the form

\[
U \equiv 0, \quad S \approx -\frac{1}{2\sqrt{3}} + C_S e^{-\tau/2}, \quad P \approx C_P e^{-\tau/2},
\]

as $\tau \to \infty$, where $C_S$ and $C_P$ are arbitrary constants, $C_S^2 + C_P^2 \neq 0$. (One can also obtain these solutions in a parametric and partially in an explicit form (for $S \equiv 0$).) Notice that $S$ here is just a linear function of $P$. We also mention that one of the trajectories on $W^s$, namely,

\[
U \equiv 0, \quad S = -\frac{1}{2\sqrt{3}} + \frac{1}{\sqrt{2}\sqrt{3}} P \quad \text{for} \quad P \in \left[0, \frac{\sqrt{6}}{2}\right],
\]

joins $P_1$ to $P_2$ if $\lambda = \sqrt{6}$.

In order to obtain the asymptotic behavior of the trajectories on $W^u$, we notice that for these trajectories $\tilde{S} = o(U)$ and $\tilde{P} = o(U)$ as $U \to 0$. This allows us to consider only the leading terms in (4):

\[
\tilde{S}' = -\frac{1}{2} \tilde{S} + \frac{3}{2} U^2, \quad U' = \frac{1}{2} U, \quad P' = -\frac{1}{2} P + \lambda U^2.
\]

It follows immediately that the outgoing trajectories take the form

\[
U \approx C_U e^{\tau/2}, \quad S \approx -\frac{1}{2\sqrt{3}} + \frac{1}{\sqrt{3}} U^2, \quad P \approx \frac{2}{3} \lambda U^2
\]

as $\tau \to -\infty$, where $C_U$ is a nonzero constant.

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[1] S. Byland and D. Scialom, Phys. Rev. D57, 6065 (1998)
[2] L. Perko, Differential Equations and Dynamical Systems (Springer-Verlag, New York, 1991).
[3] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos (Springer-Verlag, New York, 1990).
[4] P. Hartman, Ordinary Differential Equations (John Wiley & Sons, New York, 1964).