CONVERGENCE TO STABLE LAWS FOR A CLASS OF MULTIDIMENSIONAL STOCHASTIC RECURSIONS

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Abstract. We consider a Markov chain \( \{X_n\}_{n=0}^{\infty} \) on \( \mathbb{R}^d \) defined by the stochastic recursion
\[
X_n = M_n X_{n-1} + Q_n, \tag{1.1}
\]
where \( (Q_n, M_n) \) are i.i.d. random variables taking values in the affine group \( H = \mathbb{R}^d \rtimes \text{GL}(\mathbb{R}^d) \). Assume that \( M_n \) takes values in the similarity group of \( \mathbb{R}^d \), and the Markov chain has a unique stationary measure \( \nu \), which has unbounded support. We denote by \( |M_n| \) the expansion coefficient of \( M_n \) and we assume \( E|M|^\alpha = 1 \) for some positive \( \alpha \). We show that the partial sums \( S_n = \sum_{k=0}^{n} X_k \), properly normalized, converge to a normal law (\( \alpha \geq 2 \)) or to an infinitely divisible law, which is stable in a natural sense (\( \alpha < 2 \)). These laws are fully nondegenerate, if \( \nu \) is not supported on an affine hyperplane. Under a natural hypothesis, we prove also a local limit theorem for the sums \( S_n \).

1. Introduction and main results

We consider the vector space \( V = \mathbb{R}^d \) endowed with the scalar product \( \langle x, y \rangle = \sum_{i=1}^{d} x_i y_i \) and the norm \( |x| = \sum_{i=1}^{d} |x_i|^2 \). Let \( H = V \rtimes \text{GL}(V) \) be the affine group of \( V \) i.e. \( H \) is a semi-direct product of the linear group \( \text{GL}(V) \) and the group of translations of \( V \). The action of \( h = (b, g) \), \( b \in V \), \( g \in \text{GL}(V) \) on \( x \in V \) is
\[
h x = gx + b.
\]
We denote by \( u^* \) the adjoint operator of \( u \in \text{End}V \).

Given a probability measure \( \mu \) on \( H \) and \( x \in V \), we consider the recurrence relation with random coefficients
\[
\begin{align*}
X_0^x &= x, \\
X_n^x &= M_n X_{n-1}^x + Q_n,
\end{align*}
\]
where the random pairs \( (Q_n, M_n) \in H \) are independent and distributed according to \( \mu \). We assume that a unique stationary law \( \nu \) for this recursion exists and has unbounded support. We denote by \( \overline{\mu} \) the projection of \( \mu \) on \( \text{GL}(V) \) and by \( G_{\overline{\mu}} \) the closed subgroup generated by \( \text{supp}\overline{\mu} \).

We are interested in the limiting behavior of the sum \( S_n^x = \sum_{k=0}^{n} X_k^x \) of the non independent random variables \( X_k^x \) (\( 0 \leq k \leq n \)). Such a problem was considered in [K], and convergence to stable laws for sums like \( S_n^x \), but with i.i.d. increments, was stated there. Under some conditions, the homogeneity at infinity of stationary laws was proved and was an essential aspect of the limit
affine subspace. The transition operator of the chain 
\( \text{supp}\, \nu \)
\( \text{Hypothesis} \) \( \theta \)
\( v \)
and \( \eta \)
\( S \) of the limit law for
\( v \)
case here and the limit law has a tail, which depends linearly of the tails of the stationary laws
\( \text{the chain (see for example} [GLJ], \text{for the case of continuous} \]
\( \text{fraction expansion]). This is not the}
\( \text{same limiting behavior as if the increments were i.i.d. with law equal to the stationary measure of} \)
\( \text{for functionals of Markov chains, which are considered in th}
\( \text{e literature, the Birkhoff sums have the}
\( \alpha \)
\( \text{observe that} \Lambda \text{is homogeneous of degree} \alpha \text{with respect to} G^\alpha \). \text{In contrast to the general case of}
\( \text{recursion (1.1), we observe that here, modulo a compact subgroup,} G^\alpha \text{is isomorphic to} \mathbb{R} \text{or} \mathbb{Z}. \text{If} \alpha \leq 2, \text{this fact will be reflected in the form of the limit laws. If} G^\alpha \text{contains} \mathbb{R}_+^* \text{or if} \alpha > 2, \text{then} \nu \text{belongs to the domain of attraction of a stable law. More generally, if} \alpha < 2 \text{the concept of semistability and normalization along a subsequence of integers is relevant (see} [L]). \text{We show}
\( \text{that the limiting law of the properly normalized sum} S^n_x \text{exists, is infinitely divisible and stable in a natural sense. If} \text{supp}\mu \text{has no invariant affine subspace, this law is fully nondegenerate. If} \alpha \leq 2, \text{the tail} \Lambda \text{enters as an essential component in the description of the limit law. If} \alpha \geq 2, \text{this law is normal and if} \alpha > 2 \text{its covariance form is a simple modification of the covariance form of} \nu. \text{In particular, if} \mu \text{varies continuously and satisfies very general moment conditions, one passes from Gaussian asymptotics} (\alpha > 2) \text{to non Gaussian ones} (\alpha < 2). \text{This is analogous to a phase transition, as in statistical physics (see} [S, DLNP]).
\( \text{If} \alpha < 2, \text{in particular in the non normal case, the description of the parameters of the limit law}
\( \text{for} S^n_x \) \text{involves another family of Markov chains and stationary measures. For any fixed nonzero} \)
\( v \) \text{in} \( V \), \text{the Markov chain on} \( V \) \text{defined by the recursion}
\begin{align*}
W_0 &= 0, \\
W_n &= M^n_x(W_{n-1} + v),
\end{align*}
\( (1.2) \)
\( \text{has also a finite stationary measure} \eta_v. \text{It turns out that the tails of family} \eta_v \text{enter in the expression}
\( \text{of the limit law for} S^n_x. \text{We observe, that in most cases of convergence to non normal stable laws}
\( \text{for functionals of Markov chains, which are considered in the literature, the Birkhoff sums have the}
\( \text{same limiting behavior as if the increments were i.i.d. with law equal to the stationary measure of}
\( \text{the chain (see for example} [GLJ], \text{for the case of continuous fraction expansion). This is not the}
\( \text{case here and the limit law has a tail, which depends linearly of the tails of the stationary laws} \nu \text{and} \eta_v. \text{In order to state our main results, we need some notations. For} v \in V \text{we write} \chi_v(x) = e^\langle v, x \rangle, \text{v}(x) = \langle v, x \rangle \text{and the characteristic function of a probability measure} \theta \text{on} V \text{will be written}
\hat{\theta}(x) = \int_V \chi_x(y)\theta(dy).
\text{We will say that} \mu \text{or recursion (1.1) satisfies hypothesis} \text{H if}
\begin{itemize}
\item \text{No point of} V \text{is invariant under the action of} \text{supp}\mu.
\item \text{There exists} \alpha > 0 \text{with} E|M|^{\alpha} = 1.
\item \text{There exists} \alpha > 0 \text{with} E|M|^{\alpha} |M| < \infty \text{and} E|Q|^{\alpha} < \infty.
\end{itemize}
\text{Hypothesis} \text{H implies} E|\log |M|| < 0, \text{hence (see} [Bra]) \text{the Markov chain defined by (1.1) has a}
\text{unique stationary measure} \nu \text{and the support of} \nu \text{is unbounded. The affine subspace generated by}
\text{supp}\nu \text{is} \text{supp}\mu - \text{invariant and, if useful, we can assume that there is no proper} \text{supp}\mu - \text{invariant}
\text{affine subspace. The transition operator of the chain} \{X_n\}_n \text{will be denoted by} P, \text{hence}
\begin{align*}
P\phi(x) = E[\phi(X_0)] &= \int_H \phi(gx + b)\mu(dh).
\end{align*}
Also the series \( Q_0 + \sum_{k=1}^{\infty} M_0 \ldots M_{k-1} Q_k \) converges \( \mathbb{P} \)-a.e. to a \( V \)-valued random variable \( R \) and \( \nu \) is the law of \( R \). Similar properties are valid for the Markov chain (1.2) associated to the transition operator \( T_v \) \( (v \in V) \) given by

\[
T_v \phi(x) = \int_G \phi(g^*(x + v)) \mu(dg)
\]

and we denote by \( \eta_v \) its unique stationary measure, i.e. the law of \( (\sum_{k=1}^{\infty} M_0^* \ldots M_{k-1}^*) v \).

The group \( G \) is the direct product of \( \mathbb{R}^+ \) and the orthogonal group \( K = O(V) \). We denote by \( R_{\mathbb{R}} \) the projection of \( G_{\mathbb{R}} \) on \( \mathbb{R}^+ \). The center of \( G_{\mathbb{R}} \) will be denoted by \( Z_{\mathbb{R}} \). Let \( K_{\mathbb{R}} = G_{\mathbb{R}} \cap K \). Since \( \mathbb{P}([M_n] = 1) < 1 \), and \( R_{\mathbb{R}} \) is closed we have

\[
R_{\mathbb{R}} = \mathbb{R}^+ \quad \text{or} \quad R_{\mathbb{R}} = \{ p^n : n \in \mathbb{Z} \} \quad \text{for} \quad p > 1.
\]

There exists a closed subgroup \( A_{\mathbb{R}} \subset G_{\mathbb{R}} \) such that the projection \( g \mapsto |g| \) defines an isomorphism of \( A_{\mathbb{R}} \) onto \( R_{\mathbb{R}} \), and \( G_{\mathbb{R}} = A_{\mathbb{R}} \ltimes K_{\mathbb{R}} \) is the semidirect product of \( A_{\mathbb{R}} \) and \( K_{\mathbb{R}} \). Furthermore, \( A_{\mathbb{R}} \) can be chosen to contain a central subgroup of \( G_{\mathbb{R}} \) as a finite index subgroup. In particular the center \( Z_{\mathbb{R}} \) of \( G_{\mathbb{R}} \) is the product of \( Z_{\mathbb{R}} \cap K \) by a subgroup isomorphic to \( \mathbb{R} \) or \( \mathbb{Z} \). Below, the elements of \( Z_{\mathbb{R}} \) (if \( 0 < \alpha \leq 2 \)) or \( \mathbb{R}^+ \) (if \( \alpha > 2 \)) will be used to normalize the sums \( S_n^\alpha \). If \( G_{\mathbb{R}} \supset \mathbb{R}^+ \), or if \( \alpha > 2 \), the normalization is as usual, by positive numbers. See Appendix A for some further discussion on the structure of \( G_{\mathbb{R}} \) and \( Z_{\mathbb{R}} \).

We denote by \( \Sigma_1 \) the fundamental domain of \( A_{\mathbb{R}} \) on \( V \setminus \{ 0 \} \) given by: \( \Sigma_1 = \{ x \in V ; 1 \leq |x| < p \} \) if \( R_{\mathbb{R}} = \langle p \rangle \), \( \Sigma_1 = S_1 \), the unit sphere of \( V \), if \( R_{\mathbb{R}} = \mathbb{R}^+ \). Then we write \( x = a(x) \bar{a} \) with \( a(x) \in A_{\mathbb{R}} \) and \( \bar{a} \in \Sigma_1 \). Then \( r(x) = |a(x)| \leq |x| \) takes values in \( R_{\mathbb{R}} \), and \( r(x) = |x| \) if \( R_{\mathbb{R}} = \mathbb{R}^+ \).

It is shown in [BDGHU] that under hypothesis \( \mathbf{H} \), the following \( G_{\mathbb{R}} \)-homogeneous Radon measure \( \Lambda \) is well defined by the following weak convergence on \( V \setminus \{ 0 \} \)

\[
\Lambda = \lim_{|g| \to 0, g \in G_{\mathbb{R}}} |g|^{-\alpha} g \nu.
\]

Then \( \Lambda \) is called the tail measure (or tail) of \( \nu \) and the support of \( \Lambda \) is studied under natural conditions. Here we need the fact that \( \Lambda \) is nonzero and this is a consequence of hypothesis \( \mathbf{H} \) only.

In the case \( d = 1 \) and \( R_{\mathbb{R}} = \mathbb{R}^+ \) the measure \( \Lambda \) is defined by

\[
\Lambda(dx) = C_+ 1_{(0, \infty)}(x) \frac{dx}{x^{\alpha+1}} + C_- 1_{(-\infty, 0)}(x) \frac{dx}{|x|^{\alpha+1}}.
\]

In general \( \Lambda \) has a product form. Let \( l \) be the Haar measure on \( A_{\mathbb{R}} \) i.e. either \( l(da) = \frac{d|a|}{|a|} \) if \( R_{\mathbb{R}} = \mathbb{R}^+ \) or \( l \) is the counting measure multiplied by \( \log p \), if \( R_{\mathbb{R}} = \langle p \rangle \). Define \( l^\alpha(da) = |a|^{-\alpha} l(da) \), then there exists a finite measure \( \sigma \) on the fundamental domain \( \Sigma_1 \) such that \( \Lambda \) can be written as the product of \( l^\alpha \) and \( \sigma \):

\[
\int_{V \setminus \{ 0 \}} f(x) \Lambda(dx) = \int_{A_{\mathbb{R}}} \int_{\Sigma_1} f(aw) l^\alpha(da) \sigma(dw).
\]

Also, if \( \alpha > 1 \), we will denote by \( m = \int_v x \nu(dx) \) the mean of \( \nu \), by \( q(x,y) = \int_V \langle x, \zeta - m \rangle \langle y, \zeta - m \rangle \nu(d\zeta) \), the covariance form of \( \nu \). We write also \( z = \mathbb{E}[M_n] \) for the averaged operator of \( M_n \).

It will be shown that \( \eta_v \) has also a tail \( \Delta_v \) given by

\[
\Delta_v = \lim_{|g| \to 0, g \in G_{\mathbb{R}}} |g|^{-\alpha} g^* \eta_v.
\]
We denote by \( \tilde{\Lambda} \) the stability relation can be reduced, using translations, to a semigroup \( \theta \). Assume that the probability measure \( \hat{\theta} \) say that a probability measure \( \theta \) belongs to a one parameter convolution semigroup \( \theta^t \) \((t \geq 0)\) and for every \( u \in U \), there exists \( \beta(u) \in V \) such that
\[
\hat{\theta}(\lambda) = \hat{\theta}(\lambda) \theta(u) \theta(u) = \theta(u) \theta(u) = \theta(u) \theta(u)
\]
This equation implies that \( \hat{\theta}(\lambda) \) do not vanish and if \( \phi(\lambda) = \log \hat{\theta}(\lambda) \), then for any \( u \in U \), \( \phi(u^\ast \lambda) = \phi(u) \phi(\lambda) + i\beta(u, \lambda) \). Conversely these conditions imply the \((U, \tilde{\theta})\) stability of \( \theta \), and in particular \( \theta \) belongs to a well defined one parameter convolution semigroup.

Given a closed subgroup \( U \) of \( GL(V) \) and a continuous homomorphism \( \pi \) of \( U \) in \( \mathbb{R}_+^\ast \), we will say that a probability measure \( \hat{\theta} \) on \( V \) is \((U, \pi)\) stable if \( \hat{\theta} \) belongs to a one parameter convolution semigroup \( \theta^t \) \((t \geq 0)\) and for every \( u \in U \), there exists \( \beta(u) \in V \) such that
\[
u(\theta) = \theta^{\pi}(u) \ast \beta(u).
\]

Main Theorem 1.5. Assume that the probability measure \( \mu \) on \( H \) satisfies hypothesis \( H \). Then for any \( x \in V \).

1. If \( \alpha > 2 \), \( \frac{1}{\sqrt{\pi}}(S^x_n - nm) \) converges in law to the normal law with the Fourier transform
\[
\Phi_{2+}(v) = \exp \left( -q(v, v)/2 - q(v, (I - z^\ast)^{-1} z^\ast v) \right).
\]
2. If \( \alpha \in (0, 2) \), assume \( c_n \in Z_{\pi} \) is related to \( n \in \mathbb{N} \) by \( |c_n|^{-\alpha} \) and define \( d_n = 0, n = n_\xi(c_n) = nc_n u, \) resp. if \( \alpha < 1, \) \( n_\xi(c) = \int_V \frac{c}{1 + |c|^\alpha} v(dx) \) for \( c \in Z_{\pi} \).

Then \( c_nS_n^\ast - d_n \) converges in law towards the \((Z_{\pi}, \pi)\)-stable law with the Fourier transform
\[
\Phi_\alpha(v) = \exp C_\alpha(v),
\]
where
\[
C_\alpha(v) = \alpha m_\alpha(\tilde{\Lambda})^2, \quad \text{if } \alpha \neq 1,
\]
\[
C_\alpha(v) = m_\alpha(\tilde{\Lambda})^2 + i\gamma(v), \quad \text{if } \alpha = 1,
\]
where \( \gamma(v) \in \mathbb{R}, \) \( R_{\pi} = \mathbb{R}_+^\ast \). If \( R_{\pi} = \{p\} \), the same formulas are valid, where \( \alpha \Delta_\alpha(\tilde{\Lambda})^2 \) is replaced by \( \frac{1}{\log p} \Delta_\alpha(\tilde{\Lambda})^2 \). Furthermore if \( \alpha = 1, \) then for any constant \( I(v) > 0: \)
\[
|\xi(c)| \leq I(v)|c| \log |c|, \quad \text{if } |c| < 1/2,
\]
\[
|\xi(c)| \leq I(v)|c|, \quad \text{if } |c| \geq 1/2.
\]
3. If \( \alpha = 2 \), assume \( c_n \in Z_{\pi} \) satisfy \( \lim_{n \to \infty} |c_n|^{-\alpha} n \log n = 1 \), then \( c_n(S_n^\ast - nm) \) converges in law to the normal law with the Fourier transform \( \Phi_2(v) = \exp(C_2(v)), \) with
\[
C_2(v) = \frac{1}{4} \int_{\Sigma_1} \left( \langle v, w \rangle^2 + 2\langle v, w \rangle \eta_\epsilon(w^\ast) \right) \sigma(dw) = 2\Delta_\epsilon(\tilde{\Lambda})^2.
\]
if \( R_\pi = \mathbb{R}^*_+ \). If \( R_\pi = (r) \), the same formula is valid with \( \frac{1-p^{-2}}{\log p} \) instead of 2. In both cases
\[ C_2(c^*v) = |c|^2 C_2(v) \] if \( c \in Z_\pi \)

If no affine subspace of \( V \) is suppy invariant, then the limit laws are fully nondegenerate i.e., their supports are not contained in a proper subspace of \( V \).

**Remarks**

a) If \( G_\pi \supset \mathbb{R}^*_+ \) or if \( \alpha > 2 \), \( \Phi_\alpha(v) \) is the characteristic function of a multidimensional stable law in the sense of [L] (p. 213-224).

b) In case \( d = 1 \) and \( G_\pi = \mathbb{R}^*_+ \), the analogue of Theorem 1.5 has been proved in [GL1]. For another proof of assertion (1) in Theorem 1.5 in a more general context and under a moment condition of order 4, see [HH2].

c) If \( R_\pi = (p), \alpha < 2 \) the sequence \( c_n \) given in Theorem 1.5 is lacunary, hence also the sequence of integers defined there. However the limit law is infinitely divisible; in general the tail of \( \nu \) has a nontrivial periodic multiplicative part, hence

\[ \text{stable law (see [F], p. 577)}, \] then the limit law is only semistable in the sense of [L]. If \( \alpha = 2 \) and \( R_\pi = (p) \), the sequence \( c_n \) is also lacunary but the limit law is normal.

d) If \( \alpha = 2 \), since \( C_2(c^*v) = |c|^2 C_2(v) \) if \( c \in Z_\pi \) and \( C_2(v) \) is a quadratic form, the corresponding normal law is invariant under the subgroup of \( K \), which is the projection of \( Z_\pi \) on \( K \).

e) As in [GL1] the proofs follow the Fourier analytic approach of [GH] (see also [BDP, HH1]). However, here the dominant eigenvalue of the Fourier operator is not analytic and even not differentiable if \( \alpha < 2 \). Thus, an important point is to get explicit asymptotic fractional expansions. This is based on the homogeneity at infinity of stationary measures, studied in [BDGHU] and a remarkable intertwining relation. Moreover, instead of the analytic perturbation theory used in [GH], we need to use here the operator perturbation theorem of [KL].

**Main Theorem 1.6** (Local Limit Theorem). Assume that \( R_\pi = \mathbb{R}^*_+ \), hypothesis H is satisfied, no affine subspace of \( V \) is suppy-invariant and \( \alpha \notin \{1,2\} \). Then for every \( v \in V \) and domain \( I \subset \mathbb{R}^d \) with negligible boundary
\[
\lim_{n \to \infty} n^\chi \mathbb{P}[S_n \in I] = p_\alpha(0) \lambda(I),
\]
where
- \( \chi = \frac{4}{\alpha} \frac{d}{2} \) if \( \alpha < 2, > 2 \), resp.
- \( d_n = 0, = \alpha \) if \( \alpha < 1, > 1 \), resp.
- \( p_\alpha \) is the density of the corresponding limit law in Theorem 1.5;
- \( \lambda(I) \) denotes the Lebesgue measure of \( I \).

**Remark.** This theorem can be interpreted as a local limit theorem for a random walk defined by \( \mu \) on a homogeneous space \( \tilde{V} \) of a larger group \( \tilde{H} \) (see Section 8). Then we see that the exponent \( \chi \) of the corresponding local limit asymptotics is determined by the geometry of \( (\tilde{H}, \tilde{V}) \) if \( \alpha > 2 \), while it depends strongly of \( \mu \) if \( \alpha < 2 \). Such a situation, in case of Lie groups, was considered in [V].

In order to get an idea of what happens in general case we consider also the more general situation of generalized similarities. We will say that \( g \in \text{GL}(V) \) is a componentwise similarity if \( V \) is an orthogonal direct sum \( V = \oplus_{j=1}^l V_j \) and \( g \) acts on \( V_j \) through a similarity \( g_j \), i.e. for any \( x_j \in V_j \), \( gx_j \in V_j \) and \( |gx_j| = |g| |x_j| \). We write \( x = \sum_{j=1}^l x_j \), \( g = (g_1, \ldots, g_l) \). Here we fix positive numbers \( 1 = \lambda_1 < \lambda_2 \cdots < \lambda_l \) and an orthogonal direct sum \( V = \oplus_{j=1}^l V_{\lambda_j} \). We consider a ’homogeneous norm’ \( \tau \), i.e. \( \tau(x) = \sum_{j=1}^l |x_j|^{\lambda_j} \) and we observe that if \( a > 0 \) and \( \gamma_a \in \text{GL}(V) \) is given by \( \gamma_a(x_j) = a^{\lambda_j} x_j \), then \( \gamma_a \) is a componentwise similarity, which satisfies \( \tau(\gamma_a x) = a \tau(x) \). We denote \( D = \{ \gamma_a ; a \in \mathbb{R}^*_+ \} \), \( |g| = \sup_{\tau(x)=1} \tau(gx) \), \( G = \{ g \in \text{GL}(V) : \tau(gx) = |g| \tau(x), \forall x \in V \} \). Then any
$g \in G$ is a componentwise similarity, with $V_{\lambda_1} = V_j$. If $g \in G$ we call $g$ a $\tau$-similarity. If $l = 1$, we are back in the situation of similarities. Here we will use the same notations; their meaning will be clear from the context. We also denote $K = \{ g \in G; |g| = 1 \}$. Then, if $K_j = K \cap \text{GL}(V_j)$, $G_j = G \cap \text{GL}(V_j)$, we have: $G_j = \mathbb{R}_+^* \times K_j$, $K = \prod_{j=1}^l K_j$, $G = D \times K$, where $K_j$ is identified with a subgroup of $G$. For $\gamma \geq 1$, we define subspaces of $V$: $V_{\gamma_1} = \oplus_{\lambda_1 < \gamma} V_{\lambda_1}$, $V_{\gamma_1} = \oplus_{\lambda_1 > \gamma} V_{\lambda}$. Moreover for $\gamma_1 < \gamma_2$ we define $V_{\gamma_1, \gamma_2} = V_{\gamma_1} \cap V_{\gamma_2}$. For $x \in V$, $x_{\gamma_1, +}$, $x_{\gamma_1, -}$, $x_{\gamma_1, \gamma_2}$ will denote the projections of $x$ onto the corresponding subspaces.

Here we will assume that $M_n \in G$, hence $G_{\mathbb{P}} \subset G$. See Appendix for more information on the structure of $G_{\mathbb{P}}$ and in particular for the fact that $G_{\mathbb{P}}$ has a finite index subgroup, which is the product of $G_{\mathbb{P}} \cap K$ by a subgroup isomorphic to $\mathbb{R}$ or $\mathbb{Z}$. Also the center $Z_{\mathbb{P}}$ of $G_{\mathbb{P}}$ has the same form. Here $R_{\mathbb{P}}$ is defined as the projection of $G_{\mathbb{P}}$ on $D$, modulo $K$. Moreover, the action of $G_{\mathbb{P}}$ on $V$ is reducible and non isotropic. This property is reflected in the mixture of Gaussian and non Gaussian asymptotics in the theorem below.

If $\alpha > 1$, we define the mean of $\nu$ as above, i.e. $m_{\alpha, -} = \int_{V_{\alpha, -}} x \nu(dx)$. Also if $\alpha > 2$ we define the averaged operator of $M_n$ by $z = \mathbb{E}[M_n|V_{\alpha, -}]$ and the covariance form $q$ on $V_{\alpha, -}$ by $q(x, y) = \int_{V_{\alpha, -}} (x, \zeta - m)\langle y, \zeta - m \rangle \nu(d\zeta)$. For a description of $\Lambda$ in this situation see [BDG], Appendix.

**Main Theorem 1.7.** Assume that the probability measure $\mu$ on $H$ satisfies hypothesis H. Let $\{c_n\}$ be a sequence of elements of $Z_{\mathbb{P}}$ such that $|c_n|^{-\alpha} = n$ and put $d_n = 0$, $n \xi_1 (c_n) = n \xi_2 (c_n)$, resp. if $\alpha < 1$, $\alpha \in [1, 2)$, $\alpha > 2$, resp., where $\xi_1 (c) = cm_{\alpha, -} + \int_V \frac{c}{1 + |x|} \nu(dx)$ and $\xi_2 (c) = cm_{\frac{\alpha}{2}, \alpha} + \int_V \frac{c}{1 + |x|} \nu(dx)$.

1. If $\alpha \in (0, 2)$, then $c_n S_{\alpha}^* - d_n$ converges in law to the $(Z_{\mathbb{P}}, \mathbb{P})$ stable law with Fourier transform

$$
\Phi_\alpha(v) = \exp \left[ \int_V (\chi_v(x) - 1) \hat{\nu}(x) \Lambda(dx) \right], \quad \text{if } \alpha < 1,
$$

$$
\Phi_1(v) = \exp \left[ \int_V (\chi_v(x) - 1) \hat{\nu}(x) - \frac{i \langle v, x \rangle}{1 + |x|^2} \Lambda(dx) \right], \quad \text{if } \alpha = 1,
$$

$$
\Phi_\alpha(v) = \exp \left[ \int_V (\chi_v(x) - 1) \hat{\nu}(x) - i \langle v, x_{\alpha, -} \rangle - \frac{i \langle v, x \rangle}{1 + |x|^2} \Lambda(dx) \right], \quad \text{if } 1 < \alpha < 2.
$$

2. If $\alpha > 2$ and $V_{\alpha} = \{0\}$, then $\frac{1}{n} (S_{\alpha}^* - n \hat{\mu})_{\alpha, -} + (c_n S_{\alpha}^* - d_n)_{\alpha, +}$ converges in law to the direct product of a normal law on $V_{\alpha, -}$ and a $(Z_{\mathbb{P}}, \mathbb{P})$ stable law on $V_{\alpha, +}$ with Fourier transforms

$$
\Phi_{\alpha, +}(v) = \exp \left( - \frac{q(v, v)}{2} - q(v, (I - z^*)^{-1} z^* v) \right)
$$

and

$$
\Phi_{\alpha, +} = \exp \left[ \int_{V_{\alpha, +}} (\chi_v(x) - 1) \hat{\nu}(x) - i \langle v, x_{\alpha, -} \rangle - \frac{i \langle v, x \rangle}{1 + |x|^2} \Lambda(dx) \right].
$$

Moreover in all cases, if no affine subspace of $V$ is supported invariant, then the limiting laws are fully nondegenerate i.e., their supports are not contained in proper subspaces of $V$.

**Remarks**

a) If $\alpha \in (1, 2)$ and $V_{\alpha} = \{0\}$, the formulas for $\Phi_\alpha(v)$ simplify. In this case, they extend the formulas of the stable or semistable laws (see [L], p. 213-224).

b) If $\alpha > 2$, then use of different normalizations depending of the components allows to get fully nondegenerate laws. Furthermore, the result allows to predict the value of the exponent $\chi$ in the
local limit asymptotics, as in Theorem 1.6: \( \chi = \frac{1}{\alpha} \text{dim} V_{\alpha,-} + \frac{1}{\alpha} \text{dim} V_{\alpha,+} \). The product form of the limit law is reminiscent of the results of [GLJ] and [BaP].

(c) Here, modulo \( Z_{\bar{\tau}} \cap K \), the normalization operators \( a_n = (\frac{1}{\sqrt{n}}, c_n) \) are suitable powers of a single matrix, which is a component-wise similarity. For a general approach to normalizations by linear operators and limit laws of iterated convolutions see [JM]. It turns out that if \( \tilde{R}_{\bar{\tau}} = D \), then the limit law in Theorem 1.7 is \('\text{operator stable}'\) as defined in [JM], but its parameters are different from those of the limit law corresponding to \( \nu^{*n} \). In the non-normal case considered here detailed information (see Appendix) on \( G_{\bar{\tau}} \), \( Z_{\bar{\tau}} \) is needed for the construction of the normalization operators.

2. Stochastic recursions and some properties of their stationary measures

In sections 2 - 4, we assume that \( V \) is equipped with a homogeneous norm \( \tau \) and we study recursion (1.1), if \( M_n \) is a \( \tau \)-similarity.

Here we will describe some further properties of stationary measures \( \nu \) and \( \eta_\nu \) of recursions (1.1) and (1.2), respectively, that will be used in the remaining part of the paper. If \( M_n \in G \), recursion (1.1) is studied in [BDGHU] and proofs of all its properties listed below can be found there. For general information on recursion (1.1) see [Bra].

We define \( \kappa(s) = E[M|^s] \). Under hypothesis H, the function \( \kappa \) is well defined for \( s \in [0, \alpha] \) and it is strictly convex, hence \( \kappa(s) < 1 \) for \( s < \alpha \). It is known that the sequence \( \{X^x_n\}_{n=0}^\infty \) converges in distribution to a random variable \( R \) with law \( \nu \), and finite \( \theta \)-moments for \( \theta < \alpha \):

\[
\nu(\tau^\theta) = E[\tau(R)^\theta] \leq \sup_n E[\tau(X_n)^\theta] < \infty.
\]

Furthermore the tail of the stationary measure \( \nu \) is well understood i.e. there exists a \( G_{\bar{\tau}} \)-homogeneous Radon measure \( \Lambda \) on \( V \setminus \{0\} \) such that

\[
\lim_\{g|\to 0, g \in G_{\bar{\tau}}\} |g|^{-\alpha} \nu(g(f)) = \lim_\{g|\to 0, g \in G_{\bar{\tau}}\} |g|^{-\alpha} \int_V f(gx) \nu(dx) = \Lambda(f)
\]

and the convergence is valid for every function \( f \) such that the set of discontinuities of \( f \) has \( \Lambda \) measure 0 and for some \( \varepsilon > 0 \)

\[
\sup_{x \neq 0} \left( \tau(x)^{-\alpha} |\log \tau(x)|^{1+\varepsilon} |f(x)| \right) < \infty.
\]

\( G_{\bar{\tau}} \)-homogeneity of \( \Lambda \) means that for every \( g \in G_{\bar{\tau}} \)

\[
\Lambda(f \circ g) = |g|^\alpha \Lambda(f).
\]

In particular \( \Lambda \) is \( K_{\bar{\tau}} \)-invariant.

**Lemma 2.5.** Assume \( \mu \) satisfies hypothesis H. Then \( \nu \) has no atom. If furthermore there is no proper supp\( \mu \)-invariant affine subspace, then \( \nu \) gives zero measure to every affine subspace.

**Proof.** The first assertion is a special case of Proposition 2.4 in [BDGHU]. We give a simple proof, for the sake of completeness. Let \( X \) be the set of atoms of \( \nu \). Since \( \sum_{x \in X} \nu(x) \leq 1 \), \( \nu(x) \) reaches its maximum value \( a \) and \( X_0 = \{y \in X; \nu(y) = a\} \) is finite. On the other hand \( \mu \)-stationarity of \( \nu \) implies that \( X_0 \) is supp\( \mu \)-invariant. Hence the barycenter of \( X_0 \) is supp\( \mu \)-invariant. This contradicts the first condition in hypothesis H. It follows \( X_0 = \emptyset \), hence \( \nu \) has no atom.

For the second assertion we can repeat the first part of the above argument. Thus we consider the set \( W \) of affine subspaces \( L \) of minimal dimension such that \( \nu(L) > 0 \). From the definition of \( W \): \( \nu(L \cap L') = 0 \) if \( L, L' \in W \) and \( L \neq L' \). Hence \( \sum_{L \in W} \nu(L) \leq 1 \), and there exists \( N \in W \) with \( a' = \nu(N) = \sup_{L \in W} \nu(L) \). Let \( W_0 = \{N \in W; \nu(N) = a'\} \). Then as above \( W_0 \) is finite and supp\( \mu \)-invariant, hence \( H_{\mu} \)-invariant, where \( H_{\mu} \) is the closed subgroup of \( H \) generated by supp\( \mu \). It follows
that \( H'_\mu = \{ h \in H : hN = N, \forall N \in W_0 \} \) is a finite index subgroup of \( H_\mu \). Let \( h = (b, g) \) be an element of \( H'_\mu \) with \( |g| < 1 \), and \( h^+ \) its unique fixed point. Then for every \( v \in V \), \( \lim_{n \to \infty} h^nv = h^+ \). In particular, let \( v_i \) be a point of \( N_i \in W_0 \) (1 \( \leq i \leq p \)). Then \( \lim_{n \to \infty} h^n v_i = h^+ \in V_i \) (1 \( \leq i \leq p \)). It follows \( h^+ \in \bigcap_{N_i \in W_0} N_i \), in particular \( \bigcap_{N_i \in W_0} N_i \neq \emptyset \) is a supp\( \mu \)-invariant affine subspace. This contradicts the hypothesis, hence \( W = \emptyset \), i.e. \( \nu(L) = 0 \) for every affine subspace \( L \) of \( V \). 

We complete the result of [BDGHU] concerning nondegeneracy of the tail measure and following methods described in [Gr2, Go, B] we prove it under hypothesis \( H \), without any further assumptions.

**Proposition 2.6.** The tail measure \( \Lambda \) is nonzero. In particular, if \( \mu \) satisfies hypothesis \( H \), there exists \( k > 0 \) with \( \mathbb{P}[|R| > t] \geq kt^{-\alpha} \) for \( t \) large enough.

**Proof.** Define the backward process \( \vec{R}_n \):

\[
\begin{align*}
\vec{R}_0 &= 0, \\
\vec{R}_n &= \pi_V((Q_1, M_1) \cdots (Q_n, M_n)) = Q_1 + \Pi_1 Q_2 + \cdots + \Pi_{n-1} Q_n,
\end{align*}
\]

where \( \Pi_k = M_1 \cdots M_k \). Recall that \( R_n \) converges pointwise to \( R \), and \( R = \vec{R}_n + \Pi_n \vec{R}^n \), where \( \vec{R}^n = \sum_{k=n+1}^{\infty} (M_{n+1} \cdots M_{k-1}) Q_k \), hence for any \( n \), \( \vec{R}^n \) and \( R \) have the same distribution.

Fix two positive numbers \( \eta \) and \( \delta \) and a point \( u \in \text{supp}\mu \). For any ball \( U \) of center \( u \) and radius \( \delta \), \( \varepsilon = \mathbb{P}[|R| \in U] \) is positive. We have, using independence of \( \vec{R}^n \) and \( (\vec{R}_i, \Pi_i) \) for \( i < n \),

\[
\mathbb{P}[\inf_{x \in U} |\vec{R}_n + \Pi_n x| > t \text{ for some } n] = \sum_n \mathbb{P}[\max_{i < n} \inf_{x \in U} |\vec{R}_i + \Pi_i x| \leq t \text{ and } \inf_{x \in U} |\vec{R}_n + \Pi_n x| > t] \leq \frac{1}{\varepsilon} \sum_n \mathbb{P}[\max_{i < n} \inf_{x \in U} |\vec{R}_i + \Pi_i x| \leq t \text{ and } \inf_{x \in U} |\vec{R}_n + \Pi_n x| > t] \mathbb{P}[\vec{R}^n \in U]
\]

\[
\leq \frac{1}{\varepsilon} \sum_n \mathbb{P}[\max_{i < n} \inf_{x \in U} |\vec{R}_i + \Pi_i x| \leq t \text{ and } \inf_{x \in U} |\vec{R}_n + \Pi_n x| > t \text{ and } |R| > t] \leq \frac{1}{\varepsilon} \mathbb{P}[|R| > t].
\]

Since \( \mathbb{P}[Mu + Q = u] < 1 \), there exist a positive number \( \eta \) such that

\[
\theta = \mathbb{P}[|Q + (M - I)u| > 2\eta] > 0.
\]

Moreover there is a large number \( N \) such that

\[
\mathbb{P}[|M| \geq N] \leq \frac{\theta}{2}.
\]

Choose \( \delta = \frac{\eta}{N + 1} \) and define

\[
U_n = \vec{R}_n + \Pi_n u - (\vec{R}_{n-1} + \Pi_{n-1} u) = \Pi_{n-1}(Q_n + (M_n - I)u).
\]

Then

\[
\mathbb{P}[|R| > t] \geq \varepsilon \mathbb{P}[\inf_{x \in U} |\vec{R}_n + \Pi_n x| > t \text{ for some } n] \geq \varepsilon \mathbb{P}[|\vec{R}_n + \Pi_n u| - |\Pi_n|\delta > t \text{ for some } n] \geq \varepsilon \mathbb{P}[|U_n| - (|\Pi_n| + |\Pi_{n-1}|)\delta > t \text{ for some } n] = \varepsilon \mathbb{P}[|\Pi_{n-1}|(|Q_n + (M_n - I)u| - (|M_n| + 1)\delta > 2t \text{ for some } n]
\]
We define for \( n \geq 0 \), \( Y_n = |Q_n + (M_n - I)u| - (|M_n| + 1)\delta \) and we observe that, using independence

\[
\mathbb{P}[|\tilde{\Pi}_{n-1}| Y_n > 2t \text{ for some } n] = \mathbb{P}[|\tilde{\Pi}_{n-1}| Y_0 > 2t \text{ for some } n] \geq \mathbb{P}[|\tilde{\Pi}_{n-1}| > 2t/\eta \text{ for some } n] \mathbb{P}[Y_0 > \eta] \geq \mathbb{P}[\max_{n \geq 1} |\tilde{\Pi}_{n-1}| > 2t/\eta] \mathbb{P}[Y_0 > \eta].
\]

On the other hand

\[
\mathbb{P}[Y_0 > \eta] \geq \mathbb{P}[|Q + Mu - u| > 2\eta \text{ and } |M| < N] \geq \mathbb{P}[|Q + Mu - u| > 2\eta] - \mathbb{P}[|M| \geq N] \geq \frac{\theta}{2}.
\]

Since \( \mathbb{E}[\log |M_1|] < 0 \), \( \mathbb{E}[|M_1|^\alpha] = 1 \), we can use Cramer estimate of ruin (see [F], p. 411) for \( \mathbb{P}[\max_{n \geq 1} |\tilde{\Pi}_{n-1}| > 2t/\eta] \). This gives the existence of \( C > 0 \) (depending of \( \mu_r \) only) such that

\[
\mathbb{P}[\max_{n \geq 1} |\tilde{\Pi}_{n-1}| > 2t/\eta] \geq C \eta^\alpha t^{-\alpha}. \quad \text{Finally,}
\]

\[
\mathbb{P}[|R| > t] \geq \frac{\theta C \eta^\alpha}{2} t^{-\alpha}.
\]

Hence we can take \( k = \frac{\theta C \eta^\alpha}{2} \). By definition of \( \Lambda \), \( \Lambda \neq 0 \).

**Corollary 2.7.** The function \( x \mapsto |x|^\alpha \) is not \( \nu \)-integrable.

**Proof.** The relation \( \int_V |x|^\alpha \nu(dx) = \infty \) follows from \( \mathbb{P}[|R| > t] \geq kt^{-\alpha} \).

**Corollary 2.8.** Assume furthermore that there is no supp\( \mu \)-invariant affine subspace. Then, for every affine subspace \( W \) of \( V \), \( \Lambda(W) = 0 \).

**Proof.** We use the formula for \( \Lambda \) obtained in [BDGHU], Theorem 1.6:

\[
\Lambda = \frac{1}{m_\alpha} \int_V g(\nu - \overline{\nu} \ast \nu) \lambda^\alpha(dg),
\]

where \( \lambda^\alpha \) is a Radon measure on \( \mathbb{G} \), equivalent to the Haar measure of \( \mathbb{G} \) and \( m_\alpha \) was defined in hypothesis \( H \). Let \( W \) be an affine subspace of \( V \) and \( X \subset W \) a compact subset with \( \Lambda(X) > 0 \). Then

\[
\int_V (\nu - \overline{\nu} \ast \nu)(g^{-1}X) \lambda^\alpha(dg) > 0.
\]

Hence, for some \( g \in \mathbb{G} \):

\[
(\nu - \overline{\nu} \ast \nu)(g^{-1}X) > 0
\]

and

\[
\nu(g^{-1}X) = \overline{\nu} \ast \nu(g^{-1}X) + (\nu - \overline{\nu} \ast \nu)(g^{-1}X) > 0.
\]

In particular \( \nu(g^{-1}W) > 0 \), which contradicts Lemma 2.5. The conclusion follows. \( \square \)

The properties of \( \eta_v \) that will be useful are contained in the following Lemma

**Lemma 2.9.** Assume that \( \mu \) satisfies hypothesis \( H \). Then the sequence

\[
(2.10) \quad Z_n = \sum_{k=1}^{n} M_{k-1} \ldots M_0
\]

converges \( \mathbb{P} \)-a.e. to \( Z = \sum_{k=1}^{\infty} M_{k-1} \ldots M_0 \). For any \( v \in V \), the law of \( Z^* v \) is the unique stationary measure \( \eta_v \) of the Markov chain on \( V \) defined by (1.2).
If \( v \neq 0 \) then for any \( x \in V \), \( \mathbb{P}[M^*(x + v) = x] < 1 \). In particular the recursion (1.2) satisfies hypothesis \( \mathcal{H} \), the measure \( \eta_v \) has no atoms and has all moments smaller than \( \alpha \), i.e. \( \eta_v(\tau^\alpha) < \infty \) for \( \theta < \alpha \) and \( \eta_v(\tau^\alpha) = \infty \).

Moreover for every \( c \in Z(\mathbb{P}) \), the centralizer of \( G_\mathbb{P} \) in \( G \), \( \eta_{c^*v}(f) = \eta_v(f \circ c^*) \) for \( f \in C_b(V) \).

Proof. If suffices to show the convergence of \( \langle Z_n(\omega)x, y \rangle \) for any \( x, y \in V \). But \( \langle Z_n(\omega)x, y \rangle = \langle x, Z_n^*(\omega)y \rangle \) and since \( \mathbb{E}[\log |M^*|] < 0 \)

\[
Z^*(\omega) = \lim_{n \to \infty} \left( \sum_{k=1}^{\infty} M^*_k(\omega) \cdots M^*_1(\omega) \right)
\]

exists \( \mathbb{P} \)-a.e. and also the existence and uniqueness of \( \eta_v \) is clear (see [BDGHU] for some further explanations).

If \( x \in V \) satisfies \( M^*(x + v) = x \) \( \mathbb{P} \)-a.e. then for any \( g \in \text{supp} \mathbb{P}, v = (g^*)^{-1}x - x \). Therefore putting into the last equation two arbitrary elements belonging to the support of \( \mathbb{P} \), say \( g \) and \( g' \), we obtain \( (g^*)^{-1}x = (g'^*)^{-1}x \). If \( x \neq 0 \) this implies \( |g| = |g'| \), which contradicts hypothesis \( \mathcal{H} \), since \( |g| (g \in \text{supp} \mathbb{P}) \) takes at least two different values. Thus, hypothesis \( \mathcal{H} \) is valid and by Lemma 2.5, \( \eta_v \) no atoms and \( \eta_v(\tau^\alpha) < \infty \) if \( \theta < \alpha \). Also \( \eta_v(\tau^\alpha) = \infty \), by Corollary 2.7.

For the last assertion notice that if \( f = 1_U \) for some \( U \subset V \) and \( c \in Z(\mathbb{P}) \), then \( \eta_{c^*v}(1_U) = \mathbb{P}[Z^*c^*v \in U] = \mathbb{P}[Z^*v \in (c^*)^{-1} \cdot U] = \eta_v(1_U \circ c^*) \).

\( \square \)

3. Fourier operators and their spectral properties

3.1. Analysis of the Fourier operators. On continuous functions on \( V \) we introduce as in [LP1] the seminorm

\[
[f]_{\varepsilon,\lambda} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\tau(x - y)^\varepsilon (1 + \tau(x))^\lambda (1 + \tau(y))^\lambda}
\]

and the two norms

\[
[f]_\theta = \sup_x \frac{|f(x)|}{(1 + \tau(x))^\theta}
\]

\[
\|f\|_{\theta,\varepsilon,\lambda} = \|f\|_\theta + [f]_{\varepsilon,\lambda}
\]

Notice, that if \( \lambda + \varepsilon \leq \theta \) (that will be always assumed), then \( [f]_{\varepsilon,\lambda} < \infty \) implies \( |f|_\theta < \infty \).

Define Banach spaces

\[
\mathbb{C}_\theta = \left\{ f : |f|_\theta < \infty \right\},
\]

\[
\mathbb{B}_{\theta,\varepsilon,\lambda} = \left\{ f : \|f\|_{\theta,\varepsilon,\lambda} < \infty \right\}.
\]

On \( \mathbb{C}_\theta \) and \( \mathbb{B}_{\theta,\varepsilon,\lambda} \) we consider the transition operator

\[
Pf(x) = \mathbb{E}\left[f(Mx + Q)\right] = \int_H f(gx + b)\mu(db),
\]

where \( (Q, M) \) is a random variable distributed according to the measure \( \mu \). We consider also the Fourier operator \( P_v \) defined by

\[
P_vf(x) = \mathbb{E}\left[e^{i(v,Mx+Q)}f(Mx + Q)\right] = P(\chi_vf)(x),
\]

where \( v \in V \) and \( \chi_v(x) = e^{i(v,x)} \). Notice \( P_0 = P \). We will prove later (Lemma 3.7 and Proposition 3.9) that the operators \( P_v \) are bounded on \( \mathbb{B}_{\theta,\varepsilon,\lambda} \) for appropriately chosen parameters \( \theta, \varepsilon, \lambda \). It
follows from the inequality in Proposition 3.9 below and the Theorem of Ionescu Tulcea and Marinescu [ITM] that all the operators $P_v$ have at most finitely many eigenvalues of modulus 1, they have finite multiplicity, and the rest of the spectrum is contained in a ball centered at the origin of radius less than 1. Moreover, for $|v|$ small, the perturbation theorem of Keller and Liverani [KL] provides uniform control of these spectrums. Namely the spectrum and spectral properties of $P$ can be approximated in an appropriate way by the corresponding features of operators $P_v$. All the details will be given below. For an operator $A$ we denote by $\sigma(A)$ its spectrum and by $r(A)$ its spectral radius. After a few lemmas we will apply [KL] to our situation.

For random variables $\{X_n^y\}$ defined in (1.1) we consider partial sums $S_n^x = \sum_{k=1}^n X_k^y$.

The following simple lemma is the basis of the use of spectral methods in limit theorems for functionals of Markov chains.

**Lemma 3.5.** We have

$$P_n^f(x) = E\left[\chi_v(S_n^x)f(X_n^x)\right]$$

**Proof.** If $n = 1$, then the formula above coincides with definition (3.4). Assume the result holds for $n$. If $(Q,M)$ is independent of $S_n^x$ we write

$$P_n^{v+1}f(x) = E\left[\chi_v(Mx + Q)P_n^f(Mx + Q)\right]$$

$$= E\left[\chi_v(Mx + Q)\chi_v(S_{n+1}^xMx + Q)f(X_{n+1}^x)\right] = E\left[\chi_v(S_{n+1}^x)f(X_{n+1}^x)\right],$$

that completes the proof. \qed

We will need the following inequality, valid for any $\beta \in [0, 1]$:

$$|e^{i(x,y)} - 1| \leq 2\tau(x)\beta\tau(y)\beta, \quad \text{for every } 0 < \beta \leq 1.$$ (3.6)

**Lemma 3.7.** For every $v \in V$, $n \in \mathbb{N}$ and $\theta < \alpha$. We have

$$|P_n^f|_\theta \leq D|f|_\theta$$

with $D = 3^\theta[2 + \sup_n E|\tau(X_n)\theta]| < \infty$.

**Proof.** Notice first that

$$X_n^x = X_n^y + \Pi_n(x - y),$$

where $\Pi_n = M_nM_{n-1}\ldots M_1$. Therefore by Lemma 3.5, for every $x \in V$ we have

$$\frac{|P_n^f(x)|}{(1 + \tau(x))^\theta} \leq E\left[\frac{|f(X_n^x)|}{(1 + \tau(x))^\theta} \cdot \frac{(1 + \tau(X_n^x))^\theta}{(1 + \tau(x))^\theta}\right]$$

$$\leq |f|_\theta E\left[\frac{(1 + \tau(X_n) + |\Pi_n|\tau(x))^\theta}{(1 + \tau(x))^\theta}\right] \leq 3^\theta|f|_\theta(1 + E\tau(X_n)\theta + \kappa^n(\theta)).$$

Since $\theta < \alpha$, in view of (2.1) the factor of $|f|_\theta$ above is bounded by $D = 3^\theta[2 + \sup_n E|\tau(X_n)\theta]| < \infty$ and the lemma follows. \qed

**Proposition 3.9.** Assume $2\lambda + \varepsilon < \alpha$, $\varepsilon < 1$ and $\theta < 2\lambda$. Then there exist constants $C_1, C_2$ and $\rho < 1$ independent of $v$ such that for every $n \in \mathbb{N}$, $f \in B_{\theta,\varepsilon,\lambda}$, $v \in V$

$$|P_n^f|_{\varepsilon,\lambda} \leq C_1\rho^n|f|_{\varepsilon,\lambda} + C_2\tau(v)^\varepsilon|f|_\theta.$$
Proof. We have
\[ P^n f(x) - P^n f(y) = \mathbb{E} \left[ \chi_v(S^n)(f(X^n_x) - f(X^n_y)) \right] + \mathbb{E} \left[ (\chi_v(S^n) - \chi_v(S^n)) f(X^n_x) \right] \]
Let us denote these two functions above by \( \Delta_1 \) and \( \Delta_2 \), respectively, and estimate first \( \Delta_1 \)
\[
\frac{|\Delta_1(x, y)|}{\tau(x - y)^{\beta} (1 + \tau(x))^{\lambda} (1 + \tau(y))^\lambda} \leq [f]_{\varepsilon, \lambda} \cdot \mathbb{E} \left[ \frac{\tau(x^n - y^n)^{\beta} (1 + \tau(x^n)^{\beta})^{\lambda} (1 + \tau(y)^{\lambda})^{\lambda}}{\tau(x - y)^{\beta} (1 + \tau(x))^{\lambda} (1 + \tau(y))^\lambda} \right]
\]
\[
\leq [f]_{\varepsilon, \lambda} \mathbb{E} \left[ |\Pi_n|^{\beta} (1 + \tau(x))^{\lambda} (1 + \tau(y))^\lambda \right] \leq 3^{2\lambda} [f]_{\varepsilon, \lambda} \mathbb{E} \left[ |\Pi_n|^{\beta} \right]
\]
Expanding the expression in brackets we obtain a sum of 6 factors of the form \( \mathbb{E}[\Pi_n^{\beta} \tau(X_n)^\gamma] \) for \( \beta + \gamma \leq \varepsilon + 2\lambda < \alpha \). Applying the Hölder inequality with parameters \( p = \frac{\beta + \gamma}{\beta} \), \( q = \frac{\varepsilon + \gamma}{\varepsilon} \), in view of (2.1), we have
\[
\mathbb{E}[\Pi_n^{\beta} \tau(X_n)^\gamma] \leq \kappa^{\frac{\beta}{\beta}}(\beta + \gamma) \left( \mathbb{E}[\tau(X_n)^{\beta + \gamma}] \right)^{\frac{\gamma}{\beta}} = C_{\beta, \gamma} \rho_{\beta, \gamma}^{\beta},
\]
for \( \rho_{\beta, \gamma} = \kappa(\beta + \gamma)^\frac{\beta}{\gamma} \), strictly smaller than 1. Therefore if \( C_1 = 3^{2\lambda + 2} \sup_{\beta, \gamma} C_{\beta, \gamma} \) and \( \rho = \sup_{\beta, \gamma} \rho_{\beta, \gamma} < 1 \), then
\[
(3.10) \quad \frac{|\Delta_1(x, y)|}{\tau(x - y)^{\beta} (1 + \tau(x))^{\lambda} (1 + \tau(y))^\lambda} \leq C_1 \rho^n [f]_{\varepsilon, \lambda}.
\]
Now we are going to estimate \( \Delta_2 \). Define the random variable \( B_n = 1 + |\Pi_1| + \cdots + |\Pi_n| \), then for \( \delta < \min\{1, \alpha\} \) we obtain
\[
\mathbb{E}[B_n^\delta] = \mathbb{E}[1 + M_n |B_{n-1}|^\delta] \leq 1 + \kappa(\delta) \mathbb{E}[B_{n-1}^\delta] \leq \sum_{j=0}^n \kappa_j(\delta).
\]
Therefore
\[
\sup_n \mathbb{E}[B_n^\delta] = \frac{1}{1 - \kappa(\delta)} < \infty.
\]
Assume \( \tau(y) \geq \tau(x) \). Applying (3.6), we write
\[
|\chi_v(S^n_x) - \chi_v(S^n_y)| \leq 2\tau(v)^{\varepsilon} \tau(Z^n)^{\varepsilon} (x - y)^{\varepsilon} \leq 2\tau(v)^{\varepsilon} B_n^\varepsilon \tau(x - y)^{\varepsilon},
\]
therefore
\[
\frac{|\Delta_2(x, y)|}{\tau(x - y)^{\beta} (1 + \tau(x))^{\lambda} (1 + \tau(y))^\lambda} \leq 2\tau(v)^{\varepsilon} \left[ f \right]_{\phi} \mathbb{E} \left[ \frac{B_n^\varepsilon \tau(x - y)^{\varepsilon} (1 + \tau(Z^n)^{\varepsilon})^{\lambda} (1 + \tau(y))^\lambda}{\tau(x - y)^{\beta} (1 + \tau(x))^{\lambda} (1 + \tau(y))^\lambda} \right]
\]
\[
\leq 2\tau(v)^{\varepsilon} \left[ f \right]_{\phi} \mathbb{E} \left[ \frac{B_n^\varepsilon (1 + \tau(X_n) + |\Pi_n|^{\beta})^{\lambda} (1 + \tau(y))^\lambda}{(1 + \tau(x))^{\lambda} (1 + \tau(y))^\lambda} \right] \leq 2 \cdot 3^\varepsilon \tau(v)^{\varepsilon} \left[ f \right]_{\phi} \mathbb{E} \left[ B_n^\varepsilon \tau(X_n)^{\beta} + B_n^\varepsilon (1 + |\Pi_n|^\beta) \right]
\]
Applying as before the Hölder inequality we prove that the expression in brackets above is bounded and
\[
(3.11) \quad \frac{|\Delta_2(x, y)|}{\tau(x - y)^{\beta} (1 + \tau(x))^{\lambda} (1 + \tau(y))^\lambda} \leq C_2 \tau(v)^{\varepsilon} \left[ f \right]_{\phi},
\]
with \( C_2 = 2 \cdot 3^\varepsilon \sup_{n} \mathbb{E}[B_n^\varepsilon \tau(X_n)^{\beta} + B_n^\varepsilon (1 + |\Pi_n|^\beta)] \). Finally combining (3.10) and (3.11) we obtain the Lemma.
Lemma 3.12. If $\lambda + 2\varepsilon < \theta < \alpha$ and $\delta \leq \varepsilon$, then there exists a constant $C$, such that for every $\gamma$ satisfying $\lambda + 2\varepsilon \leq \gamma \leq \theta$ and $v, w \in V$:

$$|(P_v - P_w)f| \leq C\tau(v - w)^{\delta}\|f\|_{\theta, \varepsilon, \lambda}.$$ 

Proof. Using (3.6) we have

$$\frac{|(P_v - P_w)f(x)|}{(1 + \tau(x))^{\gamma}} \leq \frac{|(P_v - P_w)(f - f(0))(x)|}{(1 + \tau(x))^{\gamma}} + \frac{|(P_v - P_w)(f(0))(x)|}{(1 + \tau(x))^{\gamma}}$$

$$\leq \mathbb{E}\left[|1 - e^{i(v - w, Mx + Q)}| \cdot \frac{|f(Mx + Q) - f(0)|}{(1 + \tau(x))^{\gamma}} + |f(0)| \cdot \mathbb{E}\left[\frac{|1 - e^{i(v - w, Mx + Q)}|}{(1 + \tau(x))^{\gamma}}\right]\right]$$

$$\leq 2[f]_{\varepsilon, \lambda}\tau(v - w)^{\delta} \cdot \mathbb{E}\left[\tau(Mx + Q)^{\delta} \cdot (\tau(Mx + Q)^{\delta}(1 + \tau(Mx + Q)^{\delta}))^{1 + \tau(x)}\right]$$

$$\leq 2\tau(v - w)^{\delta}|f(0)| \cdot \mathbb{E}\left[\tau(Mx + Q)^{\delta} \cdot (1 + \tau(x))^{\gamma}\right] \leq C\tau(v - w)^{\delta}\|f\|_{\theta, \varepsilon, \lambda},$$

where $C = 2\sup_{x} \mathbb{E}\left[(\tau(Mx + Q)^{\lambda} + 1)/(1 + \tau(x))^{\lambda}\right]$ for $\lambda' = \lambda + \varepsilon + \delta < \theta$. \qed

Lemma 3.13. The unique eigenvalue of modulus 1 for $P$ acting on $\mathbb{C}_\theta$ is 1. The corresponding eigenspace is $\mathbb{C}1$ and the projection on $\mathbb{C}1$ along the hyperplane $\text{Ker}\nu = \{f \in \mathbb{C}_\theta : \nu(f) = 0\}$ is given by the map $f \mapsto \nu(f)$. 

Proof. Of course constant functions are eigenfunctions of $P$ with eigenvalue 1 and $P$ acts on $\mathbb{C}_\theta$ in view of Lemma 3.7. In fact there are no other elements of $\mathbb{C}_\theta$ satisfying $P\nu = \nu$. Indeed, let $f$ be such a function, then for every $x \in V$, $\lim_{n \to \infty} P^n f(x) = f(x)$. On the other hand, $\lim_{n \to \infty} P^n f(x) = \lim_{n \to \infty} \mathbb{E} f(X^n_x) = \nu(f)$ (recall that the law of $\{X^n_x\}$ tends in distribution to $\nu$). Hence $f(x) = \nu(f)$ for every $x \in V$, and $f$ must be a constant. Furthermore, we observe that $\mathbb{C}_\theta = \text{Ker}\nu \oplus \mathbb{C}1$ and $f = (f - \nu(f))1 + \nu(f)$. The assertion for the projection of $f$ follows.

To prove that there are no other eigenvalues of modulus 1 we proceed similarly. Assume that for some $z$ of modulus 1 and a nonzero function $f \in \mathbb{C}_\theta$ we have $P\nu = z\nu$. Then $\lim_{n \to \infty} P^n f(x) = \nu(f)$, but $P^n f(x) = z^n f(x)$ and if $\eta$ would be different than 1, the sequence $z^n f(x)$, for every $x$ such that $f(x) \neq 0$, couldn’t converge to a constant. This implies $z = 1$ and finishes the proof. \qed

Lemma 3.14. Assume that no affine subspace of $V$ is supp$\mu$-invariant. Then for every $v \neq 0$, the equation $P_v f = zf$, $|z| = 1$, $f \in \mathbb{B}_{\theta, \varepsilon, \lambda}$ implies $f = 0$. In particular the spectral radius of $P_v$ is strictly smaller than 1.

Proof. We proceed as in [GL1]. Assume that

$$P_v f = zf$$

for some nonzero $f \in \mathbb{B}_{\theta, \varepsilon, \lambda}$ and $|z| = 1$. Then the function $f$ is bounded. Indeed for every $n$

$$|f(x)| = |z^n f(x)| \leq P^n(|f|)(x)$$

hence

$$|f(x)| \leq \lim_{n \to \infty} P^n(|f|)(x) = \nu(|f|).$$

Next observe $\langle \nu, \nu(|f|) - |f| \rangle = 0$, therefore, since $f$ is continuous, on the support of $\nu$ the function $|f|$ is equal to its maximum and without any loss of generality we may assume that this maximum is 1. A convexity argument, Lemma 3.5 and (3.15) imply that for every $n$ and $x \in \text{supp}\nu$,

$$z^n f(x) = e^{i(v, S^n_x)} f(X^n_x) \quad \text{P a.s.}$$
Hence for any \( x, y \in \text{supp} \nu \)
\[
(3.16) \quad \frac{f(x)e^{i(v,Z_n(y-x))}}{f(y)} = \frac{f(X_n^x)}{f(X_n^y)},
\]
where \( Z_n \) was defined in (2.10). Observe that taking \( p = \frac{2\lambda+\varepsilon}{\varepsilon} \) and \( q = \frac{2\lambda+\varepsilon}{2\lambda} \), by the Hölder inequality we have
\[
\limsup_{n \to \infty} E \left| \frac{f(X_n^x)}{f(X_n^y)} - 1 \right| \leq [f]_{\varepsilon,\lambda} \limsup_{n \to \infty} E \left[ \tau(X_n^x - X_n^y)^{\varepsilon}(1 + \tau(X_n^x))^{\lambda}(1 + \tau(X_n^y))^{\lambda} \right]
\]
\[
= [f]_{\varepsilon,\lambda} \limsup_{n \to \infty} E \left[ \tau(M_n \cdots M_1(x-y))^{\varepsilon}(1 + \tau(X_n^x))^{\lambda}(1 + \tau(X_n^y))^{\lambda} \right]
\]
\[
\leq [f]_{\varepsilon,\lambda} \tau(x-y)^{\varepsilon} \limsup_{n \to \infty} \left( E[M_n \ldots M_1(2\lambda+\varepsilon)]^{\frac{1}{\lambda}} \cdot \limsup_{n \to \infty} \left( E \left[ (1 + \tau(X_n^x))^{\lambda+\frac{\varepsilon}{2}}(1 + \tau(X_n^y))^{\lambda+\frac{\varepsilon}{2}} \right] \right)^{\frac{1}{\lambda+\frac{\varepsilon}{2}}} \right).
\]
In view of (2.1) the last term is finite and since \( 2\lambda + \varepsilon < \alpha \), \( \lim_{n \to \infty} \alpha^{\frac{1}{n}}(2\lambda + \varepsilon) = 0 \), hence
\[
\lim_{n \to \infty} E \left| \frac{f(X_n^x)}{f(X_n^y)} - 1 \right| = 0.
\]
Therefore for \( \mathbb{P} \) a.e. trajectory \( \omega \) there exists a sequence \( \{n_k\} \) such that
\[
\lim_{n_k \to \infty} \frac{f(X_n^{x_k})}{f(X_n^{y_k})} = 1.
\]
Notice that, in view of Lemma 2.9 \( \lim_{n \to \infty} Z_n(\omega) = Z(\omega) \) exists \( \mathbb{P} \)-a.e. Hence passing with \( n \) to infinity in (3.16) we obtain \( \frac{\langle f, \chi_{\nu} \rangle}{\langle f, \chi_{\nu} \rangle} = e^{-i(Z(\nu)(x-y))} \) for \( \omega \in \Omega_0 \) with \( \mathbb{P}(\Omega_0) = 1 \). Then for every \( \omega \in \Omega_0 \), \( e^{-i(Z(\nu)(x-y))} \langle f, \chi_{\nu} \rangle = 1 \). We are going to prove that this leads to a contradiction whenever \( v \neq 0 \). We choose points \( x_j, y_j \in \text{supp} \nu, j = 1, \ldots, d \) with \( v_j = x_j - y_j \) spanning \( V \) as a vector space. Such points exist because the support of \( \nu \) as a set invariant under the action of \( \text{supp} \mu \) is not contained in some proper affine subspace of \( V \). Let \( \eta_v \) be the law of \( W(\omega) \) in \( \mathbb{P}(\omega) \). Then for every \( j \) the support of \( \eta_v \) is contained in the union of affine hyperplanes \( \bigcup_{n \in \mathbb{Z}} \{ H_j + n s_j v_j \} \), where \( H_j \) is some hyperplane orthogonal to \( v_j \) and \( s_j \) are appropriately chosen constants. Taking intersection of all such sets defined for every \( j \) we conclude that \( \text{supp} \eta_v \) is contained in some discrete set of points, hence \( \text{supp} \eta_v \) is discrete. This contradicts Lemma 2.5.

For the last assertion we observe that in view of Theorem of Ionescu Tulcea and Marinescu [ITM], if \( z \) belongs to the spectrum of \( P_v \) and \( |z| = 1 \) then \( z \) is an eigenvalue of \( P_v \).

3.2. A perturbation theorem. For \( c \in Z(\mathbb{R}) \cup \{0\} \) we denote \( P_{c,v} = P_{v,c} \) and we write \( c \to 0 \) for \( |c| \to 0 \). We observe that \( Z(\mathbb{R}) \), the centralizer of \( G^*_\mathbb{R} \) in \( G \), contains \( \mathbb{R}^*_+ \), \( Z(\mathbb{R}) \) and
\[
P_{c,v} f(x) = \int_H \chi_v(c(gx + b)) f(gx + b) \mu(dh).
\]
In view of Lemmas 3.7, 3.12 and Proposition 3.9 we may use the perturbation theorem of Keller and Liverani [KL] for the family \( P_{c,v} \) (the hypothesis concerning the essential radius is fulfilled by a result of Hennion [H], Corollary 1). Their result is stated for the case when the parameter is real, but what they really use is the Hölder continuity in Lemma 3.12, which is valid also in our more general settings. Then we obtain the following.

Proposition 3.17. Assume \( \varepsilon < 1 \), \( \lambda + 2\varepsilon < \theta < 2\lambda \), \( 2\lambda + \varepsilon < \alpha \), \( v \in V \) is fixed then there exist \( t_0 > 0 \), \( \delta > 0 \), \( \rho < 1 - \delta \) such that for every \( c \in Z(\mathbb{R}) \cup \{0\} \) with \( |c| \leq t_0 \):
The spectrum of \( P_{c,v} \) acting on \( \mathbb{B}_{\theta,\varepsilon,\lambda} \) is contained in \( \mathcal{D} = \{ z : |z| \leq \rho \} \cup \{ z : |z - 1| < \delta \} \).

The set \( \sigma(P_{c,v}) \cap \{ z : |z - 1| < \delta \} \) consists of exactly one eigenvalue \( k(c,v) \), the corresponding eigenspace is one dimensional and moreover \( \lim_{x \to 0} k(c,v) = 1 \).

If \( \pi_{c,v} \) is the projection of \( P_{c,v} \) onto the mentioned above eigenspace, then there exists an operator \( Q_{c,v} \) such that \( \|Q_{c,v}\| \leq \rho \), \( \pi_{c,v}Q_{c,v} = Q_{c,v}\pi_{c,v} = 0 \) and for every \( n \)

\[
P_{c,v}^n f = k(c,v)^n \pi_{c,v}(f) + Q_{c,v}^n(f), \quad f \in \mathbb{B}_{\theta,\varepsilon,\lambda}.
\]

For any \( z \) belonging to the complement of \( \mathcal{D} \):

\[
\|(z - P_{c,v})^{-1}f\|_{\theta,\varepsilon,\lambda} \leq D\|f\|_{\theta,\varepsilon,\lambda},
\]

for some constant \( D \) independent of \( c \).

Define for small \( |c| \) the function \( g_{c,v} = \pi_{c,v}(1) \). Then for every function \( f \) belonging to \( \mathbb{B}_{\theta,\varepsilon,\lambda} \), we define \( \nu_{c,v}(f) \in \mathcal{C}_0 \) by \( \pi_{c,v}(f) = \nu_{c,v}(f)g_{c,v} \).

**Proposition 3.18.** Assume additionally that \( \lambda + 3\varepsilon < \theta \), \( 2\lambda + 3\varepsilon < \alpha \). The identity embedding of \( \mathbb{B}_{\theta,\varepsilon,\lambda} \) into \( \mathbb{B}_{\theta,\varepsilon,\lambda+\varepsilon} \) is continuous and the decomposition \( P_{c,v} = k(c,v)\pi_{c,v} + Q_{c,v} \) coincides on both spaces. Moreover there exist constants \( D \) and \( t_1 \) such that for \( |c| \leq t_1 \), \( c \in \mathbb{Z}([\tau]) \cup \{0\} \) we have, if \( \tau(v) \leq 1 \)

i) \( \|(P_{c,v} - P)f\|_{\theta,\varepsilon,\lambda+\varepsilon} \leq D\|c\|\|f\|_{\theta,\varepsilon,\lambda} \)

ii) \( \|(k(c,v)\pi_{c,v} - \pi_0)f\|_{\theta,\varepsilon,\lambda+\varepsilon} \leq D\|c\|\|f\|_{\theta,\varepsilon,\lambda} \)

iii) \( \|(k(c,v) - \pi_0)f\|_{\theta,\varepsilon,\lambda+\varepsilon} \leq D\|c\|\|f\|_{\theta,\varepsilon,\lambda} \)

iv) \( \|Q_{c,v} - Q)f\|_{\theta,\varepsilon,\lambda+\varepsilon} \leq D\|c\|\|f\|_{\theta,\varepsilon,\lambda} \)

v) \( |g_{c,v} - 1|_{\theta,\varepsilon,\lambda} \leq D|c| \)

vi) \( |k(c,v) - 1| \leq D|c| \)

vii) \( \nu_{c,v} \) is bounded on \( \mathbb{B}_{\theta,\varepsilon,\lambda} \) and \( |\nu_{c,v}(f) - \nu(f)| \leq D|c|\|f\|_{\theta,\varepsilon,\lambda} \).

**Proof.** The triple \( (\theta, \varepsilon, \lambda + \varepsilon) \) satisfies assumptions of Proposition 3.17, and of course

\[
\| \cdot \|_{\theta,\varepsilon,\lambda+\varepsilon} \leq \| \cdot \|_{\theta,\varepsilon,\lambda},
\]

therefore considering the family of operators \( \{ P_{c,v} \} \) on both Banach spaces \( \mathbb{B}_{\theta,\varepsilon,\lambda+\varepsilon} \) and \( \mathbb{B}_{\theta,\varepsilon,\lambda} \) we obtain the same decomposition of \( P_{c,v} \).

To prove i), in view of Lemma 3.12, it is enough to estimate

\[
\left[ (P_{c,v} - P)f \right]_{\varepsilon,\lambda+\varepsilon} = \sup_{x \neq y} \left| (P_{c,v} - P)f(x) - (P_{c,v} - P)f(y) \right|
\]

\[
\leq \sup_{x \neq y} \frac{\mathbb{E}[|\chi_v(cX^x_t) - 1||f(X^x_t) - f(X^y_t)|]}{\tau(x - y)^\varepsilon(1 + \tau(x))^{\lambda+\varepsilon}(1 + \tau(y))^{\lambda+\varepsilon}} + \sup_{x \neq y} \frac{\mathbb{E}[|\chi_v(cX^y_t) - \chi_v(cX^x_t)||f(X^y_t)|]}{\tau(x - y)^\varepsilon(1 + \tau(x))^{\lambda+\varepsilon}(1 + \tau(y))^{\lambda+\varepsilon}}
\]

\[
= \Delta_1 + \Delta_2.
\]

Next we have, using (3.6)

\[
\Delta_1 \leq 2|c|\|f\|_{\theta,\varepsilon,\lambda} \sup_{x \neq y} \mathbb{E} \left[ \frac{\tau(x^x)^\varepsilon \tau(x^x_t - X^y_t)^\varepsilon(1 + \tau(x^x))^{\lambda}(1 + \tau(y))^{\lambda}}{\tau(x - y)^\varepsilon(1 + \tau(x))^{\lambda+\varepsilon}(1 + \tau(y))^{\lambda+\varepsilon}} \right]
\]

\[
\leq 2|c|\|f\|_{\theta,\varepsilon,\lambda} \mathbb{E} \left[ (|M_1| + \tau(Q_1))^{\varepsilon} |M_1|^\varepsilon (1 + |M_1| + \tau(Q_1))^{2\lambda} \right]
\]
Reasoning as in Proposition 3.9, one can prove that the expected value above is finite. Similarly we estimate $\Delta_2(x, y)$:

$$\Delta_2 \leq \sup_{x \neq y} \mathbb{E} \left[ \frac{|\chi_v(cX^x_t) - \chi_v(cX^y_t)| \cdot (|f(X^x_t) - f(0)| + |f(0)|)}{\tau(x - y)^c(1 + \tau(x))(1 + \tau(y))^{\lambda + \varepsilon}} \right]$$

$$\leq |c|^c \|f\|_{\alpha, c, \lambda} \sup_{x \neq y} \mathbb{E} \left[ \frac{\tau(X^x_t - X^y_t)^c (\tau(X^x_t)^c(1 + \tau(X^x_t))^{\lambda + 1} + 1)}{\tau(x - y)^c(1 + \tau(x))(1 + \tau(y))^{\lambda + \varepsilon}} \right]$$

$$\leq D|c|^c \|f\|_{\alpha, c, \lambda} \mathbb{E} \left[ |M_1|^c (1 + |M_1| + \tau(Q_1))^{\lambda + \varepsilon} \right].$$

Again arguments from Proposition 3.9 prove that the foregoing value is finite. Similarly we prove $|(P_{c,v} - P)(f)|_\theta \leq 2|c|^c |f|_\theta$ that gives i).

In order to prove ii) and iii) we will use the fact that both $\pi_{c,v}$ and $Q_{c,v}$ can be expressed in terms of the resolvent of $P_{c,v}$. We follow arguments in [LP2]

$$k(c, v)\pi_{c,v} = \frac{1}{2\pi i} \int_{|z| = \delta'} z(z - P_{c,v})^{-1} dz,$$

$$\pi_{c,v} = \frac{1}{2\pi i} \int_{|z| = \delta'} (z - P_{c,v})^{-1} dz,$$

$$Q_{c,v} = \frac{1}{2\pi i} \int_{|z| = \rho'} z(z - P_{c,v})^{-1} dz,$$

for appropriately chosen constants $\delta'$ and $\rho'$. Then combining the formulas above with

$$(z - P)^{-1} - (z - P_{c,v})^{-1} = (z - P)^{-1}(P - P_{c,v})(z - P_{c,v})^{-1},$$

the point i) and estimates of the norm of resolvent (Proposition 3.17) we conclude ii), iii) and iv). v) is an immediate consequence of ii). To prove vi) we write

$$(k(c, v) - 1)\pi_0 = k(c, v)\pi_{c,v} - \pi_0 - k(c, v)(\pi_{c,v} - \pi_0),$$

apply $(k(c, v) - 1)\pi_0$ to 1 and we use ii) and iii). Similarly, writing

$$(\nu_{c,v}(f) - \nu_0(f))1 = \pi_{c,v}(f) - \pi_0(f) - \nu_{c,v}(f)(g_{c,v} - 1)$$

and applying iii) and v) we obtain vii), that finishes the proof.

\[ \square \]

**Lemma 3.19.** For fixed $v$ the function $c \mapsto r(P_{c,v})$ defined on $Z(\mathbb{F}) \cup \{0\}$ is continuous at 0. Moreover for any $f \in \mathbb{B}_{\theta, c, \lambda}$, $c_0 \in Z(\mathbb{F})$ with $r_f(P_{c,v}) = \limsup_n \|P^n_{c,v}f\|^{\frac{1}{n}}$, we have $\limsup_{c \rightarrow c_0} r_f(P_{c,v}) = r_f(P_{c_0,v}).$

**Proof.** From Proposition 3.17, $r(P_{c,v}) = |k(c, v)|$ for $c$ small, hence the continuity of $r(P_{c,v})$ follows from Proposition 3.18. Using again Proposition 3.18, for any fixed $f$ and $n, \|P^n_{c,v}f\|^{\frac{1}{n}}$ depends continuously on $c$. Hence $\limsup_{n \rightarrow \infty} \|P^n_{c,v}f\|^{\frac{1}{n}} = r_f(P_{c,v})$ is upper semicontinuous in $c$. \[ \square \]

### 3.3. Eigenfunctions of $P_{c,v}$

Proposition 3.18 says that the dominant eigenvalues $k(c, v)$ of $P_{c,v}$ tend to 1 with rate at least $|c|^c$. However to prove our Main Theorem we will need more precise information concerning the asymptotic expansion of $k(c, v)$, that will be described in Theorem 5.1. For this purpose, following ideas of [GL1] we will express the eigenfunction corresponding to $k(c, v)$ in a more explicit way.
For \( c \in Z(\mathfrak{p}) \cup \{0\} \), let us define a family of operators on the Banach space \( \mathbb{B}_{\theta,\varepsilon,\lambda} \):

\[
T_{c,v} f(x) = \mathbb{E} \left[ e^{iQ(c^*(x+v))} f(M^*(x+v)) \right] = \int_H \chi_b(c^*(x+v))f(g^*(x+v))\mu(dx).
\]

Then \( T_v = T_{0,v} \) and we have \( T_v \eta_v = \eta_v \), where \( \eta_v \) is the stationary measure of the Markov chain \( \{W_n\} \) defined in (1.2) (see also Lemma 2.9).

It turns out that the family \( \{T_{c,v}\} \) satisfies assumptions of the perturbation theorem of [KL] and also the analogue of Lemma 3.13 is valid in these settings, i.e. the set of peripherical eigenvalues consists of one element 1 and \( \eta_v \) is the projection onto the corresponding one dimensional eigenspace.

We omit the argument, because this can be proved exactly in the same way as for the family \( \{P_{c,v}\} \). Therefore for small values of \(|c|\), the spectrum of \( T_{c,v} \) intersected with some neighbourhood of 1, consists of exactly one point \( k'(c,v) \), which is the dominant eigenvalue of \( T_{c,v} \). Let us denote the corresponding projection by \( \pi'_{c,v} \) and as before define, for every \( f \in \mathbb{B}_{\theta,\varepsilon,\lambda} \), \( \eta_{c,v}(f) \) to be the unique number such that \( \pi'_{c,v}(f) = \eta_{c,v}(f)\pi'_{c,v}(1) \), hence \( \eta_{c,v}(1) = 1 \) and \( T_{c,v} \eta_{c,v} = k'(c,v)\eta_{c,v} \).

Reasoning as in Proposition 3.18 one can prove

**Proposition 3.20.** There exist constants \( t_2 \), \( C' \) and \( D' \) such that for \(|c| \leq t_2 \), \( c \in Z(\mathfrak{p}) \), and every \( f \in \mathbb{B}_{\theta,\varepsilon,\lambda} \) we have

\[
|\eta_{c,v}(f) - \eta_v(f)| \leq C'|c|^\varepsilon \|f\|_{\theta,\varepsilon,\lambda}.
\]

In particular

\[
\|\eta_{c,v}\|_{\theta,\varepsilon,\lambda} \leq 1 + C'|c|^\varepsilon \leq D',
\]

if \(|c| \leq t_2 \).

One easily verifies that for any \( x \) the function \( \chi_x \) is an element of \( \mathbb{B}_{\theta,\varepsilon,\lambda} \) and moreover

\[
(3.21) \quad \|\chi_x\|_{\theta,\varepsilon,\lambda} \leq 1 + 2\tau(x)\varepsilon.
\]

It follows that for any \( \eta \in \mathbb{B}_{\theta,\varepsilon,\lambda} \) the Fourier transform \( \hat{\eta}(x) = \eta(\chi_x) \) is well-defined and

\[
|\hat{\eta}(x)| \leq 1 + 2\tau(x)\varepsilon.
\]

The following intertwining relation between \( P_{c,v} \) and \( T_{c,v} \) plays an essential role in the calculation of the expansion of \( k(c,v) \) \((c \in Z(\mathfrak{p}))\).

**Lemma 3.22.** For any \( c \in Z(\mathfrak{p}) \cup \{0\} \), \( v \in V \), \( \eta \in \mathbb{B}_{\theta,\varepsilon,\lambda} \)

\[
P_{c,v}(\hat{\eta} \circ c) = (T_{c,v}^{*})(\hat{\eta}) \circ c.
\]

**Proof.** We observe that, since \( c \in Z(\mathfrak{p}) \cup \{0\} \)

\[
T_{c,v}(\chi_{ce})(y) = \int_H \chi_{cb}(y+v)\chi_{ge}(y+v)\mu(dy) = \int_H \chi_{c(gx+b)}(y+v)\mu(dy)
\]

On the other hand

\[
P_{c,v}(\hat{\eta} \circ c)(x) = \int_V \int_V \chi_{c}(gx+b)\chi_{c}(gx+b)\eta(dy)\mu(dx)
\]

\[
= \int_H \int_V \chi_{y+v}(c(gx+b))\eta(dy)\mu(dx)
\]

\[
= \eta(T_{c,v}(\chi_{ce}))(x) = (T_{c,v}^{*})(\hat{\eta})(ce).
\]

The lemma follows.
Lemma 3.23. Suppose \( \varepsilon < 1/2 \). There exists \( t_3 \) such that for \( |c| \leq t_3 \), \( c \in \mathbb{Z}(\mathbb{P}) \cup \{0\} \) the function

\[ \psi_{c,v} = \hat{\eta}_{c,v} \circ c \]

is a nonzero element of \( \mathbb{B}_{\theta,c,\lambda} \) and is an eigenfunction of \( P_{c,v} \) corresponding to the eigenvalue \( k(c,v) \), i.e.

\[ P_{c,v}(\psi_{c,v}) = k(c,v)\psi_{c,v}. \]

Moreover \( k'(c,v) = k(c,v) \) and

\[ (k(c,v) - 1)\nu(\psi_{c,v}) = \nu(\psi_{c,v}(\chi_{c,v} - 1)). \]

Proof. First we shall prove that \( \psi_{c,v} \) is an element of \( \mathbb{B}_{\theta,c,\lambda} \). In view of Proposition 3.20 and (3.21) we have

\[ |\psi_{c,v}| = \sup_x \left( \frac{\eta_{c,v}(\chi_{cx})}{(1 + \tau(x))}\right) \leq ||\eta_{c,v}|| \cdot \sup_x \frac{||\chi_{cx}||_{\theta,c,\lambda}}{(1 + \tau(x))} \leq (1 + C|t_2|^\varepsilon)(1 + \lambda|t_2|^\varepsilon). \]

To estimate \( |\psi_{c,v}|_{\varepsilon,\lambda} \) we define the function

\[ g_{c,x,x'}(y) = \frac{\chi_{cx}(y) - \chi_{cx'}(y)}{\tau(x - x')^\varepsilon(1 + \tau(x))^\lambda(1 + \tau(x'))^\lambda}. \]

Then we have

\[ \sup_{x \neq x'} |g_{c,x,x'}|_{\varepsilon,\lambda} \leq 2 \sup_y \frac{|c| \varepsilon \tau(x - x')^\varepsilon \tau(y)^\varepsilon}{\tau(x - x')^\varepsilon(1 + \tau(x))^\lambda(1 + \tau(x'))^\lambda(1 + \tau(y))^\lambda} \leq 2|c|\varepsilon. \]

Next we have, since \( 2\varepsilon \leq \{1, \lambda\} \)

\[
\begin{align*}
\sup_{x \neq x'} |g_{c,x,x'}|_{\varepsilon,\lambda} &= \sup_{x \neq x', y \neq y'} \frac{|\chi_{cx}(y) - \chi_{cx'}(y) - \chi_{cx}(y') + \chi_{cx'}(y')|}{\tau(x - x')^\varepsilon \tau(y - y')^\varepsilon(1 + \tau(y))^\lambda(1 + \tau(y'))^\lambda(1 + \tau(x))^\lambda(1 + \tau(x'))^\lambda} \\
&\leq \sup_{\tau(x - x') \leq \tau(y - y')} \frac{|\chi_{cx}(y - y') - 1 + |\chi_{cx'}(y' - y') - 1|}{\tau(y - y')^\varepsilon(1 + \tau(y))^\lambda(1 + \tau(y'))^\lambda(1 + \tau(x))^\lambda(1 + \tau(x'))^\lambda} \\
&\leq \frac{2(|c|^{2\varepsilon} \tau(x)^{2\varepsilon} + \tau(x')^{2\varepsilon})}{2|c|^{2\varepsilon} \tau(y)^{2\varepsilon}(1 + \tau(y))^\lambda(1 + \tau(y'))^\lambda(1 + \tau(x))^\lambda(1 + \tau(x'))^\lambda} \\
&\leq 2|c|^{2\varepsilon},
\end{align*}
\]

that proves \( \sup_{x \neq x'} \|g_{c,x,x'}\|_{\theta,c,\lambda} \leq 4|c|^{2\varepsilon} \). Finally

\[ [\psi_{c,v}]_{\varepsilon,\lambda} = \sup_{x \neq x'} (\eta_{c,v}, g_{c,x,x'}) \leq ||\eta_{c,v}||_{\theta,c,\lambda} \sup_{x \neq x'} ||g_{c,x,x'}||_{\theta,c,\lambda} \leq 4|c|\varepsilon(1 + C|c|\varepsilon) < \infty, \]

and we obtain \( \psi_{c,v} \in \mathbb{B}_{\theta,c,\lambda} \). Next notice, using Lemma 3.22

\[ P_{c,v}(\psi_{c,v})(x) = (T_{c,v}^e \eta_{c,v})(\chi_{cx}) = k'(c,v) \hat{\eta}_{c,v}(cx) = k'(c,v)\psi_{c,v}(x), \]

but for \( |c| \) small enough there exists only one eigenvalue of \( P_{c,v} \) close to 1, hence \( k'(c,v) = k(c,v) \). The last relation is a direct consequence of \( P_{c,v}\psi_{c,v} = k(c,v)\psi_{c,v} \) and the form of \( P_{c,v} \). \( \square \)

4. SOME TECHNICAL LEMMAS

4.1. Some further properties of the stationary measure \( \nu \). In the next section very often will appear expressions of the form \( \int_{\mathbb{P}} f(c, x)\nu(dx), c \in \mathbb{Z}(\mathbb{P}) \cup \{0\} \) and we will be interested in their behavior for small values of \( |c| \). We denote \( Z_1(\mathbb{P}) = \{ c \in \mathbb{Z}(\mathbb{P}) \cup \{0\} ; |c| \leq 1 \} \). We will need the following:
Lemma 4.1. Let $f$ be any continuous function on $Z_1(\mathcal{P}) \times V$ satisfying

$$
|f(c, x)| \leq D_{\delta, \beta}|c|^\delta \tau(x)^\beta, \quad \text{for } \tau(cx) \geq 1,
$$

$$
|f(c, x)| \leq D_{\delta, \gamma}|c|^\delta \tau(x)^\gamma, \quad \text{for } \tau(cx) \leq 1,
$$

where $\beta < \alpha$, $\gamma + \delta > \alpha$ and $\delta > 0$. Then

$$
\lim_{c \to 0, c \in Z(\mathcal{P})} \frac{1}{|c|^\alpha} \int_V f(c, x)\nu(dx) = 0.
$$

Proof. Notice that taking the function $f(x) = 1_{\{\tau(x) \geq 1\}}$, by (2.2), there exists $C > 0$ such that

$$
\nu(\tau(x) > t) \leq Dt^{-\alpha},
$$

for any $t > 0$. We divide the integral into three parts and study each of them independently. For appropriately small values of $|c|$ we have

$$
\frac{1}{|c|^\alpha} \int_{\tau(x) \leq \frac{1}{|c|}} f(c, x)\nu(dx) \leq D_{\delta, \gamma}|c|^\gamma \int_{\tau(x) \leq \frac{1}{|c|}} \tau(x)^\gamma \nu(dx)
$$

$$
\frac{1}{|c|^\alpha} \int_{1 < \tau(x) \leq \frac{1}{|c|}} f(c, x)\nu(dx) \leq D_{\delta, \gamma}|c|^\gamma \int_{1 < \tau(x) \leq \frac{1}{|c|}} \tau(x)^\gamma \nu(dx)
$$

$$
\leq D_{\delta, \gamma}|c|^\gamma \sum_{n=0}^{\lceil -\log_2 |c| \rceil} \frac{1}{2^{n+1}|c|^\gamma} \nu\left(\frac{1}{2^n+1} |c| < \tau(x)\right)
$$

$$
\leq D_{\delta, \gamma}|c|^\gamma \sum_{n=0}^{\lceil -\log_2 |c| \rceil} \frac{1}{2^n |c|^\gamma} \left(2^n |c|^\gamma\right) \leq D_{\delta, \gamma}|c|^\gamma \sum_{n=0}^{\lceil -\log_2 |c| \rceil} 2^{n(\alpha - \gamma)}
$$

If $\alpha \leq \gamma$ then the foregoing sum can be estimated by some constant or $|\log |c||$ and the expression converges to zero. On the other hand if $\alpha > \gamma$, the sum is smaller than $D_{\delta, \gamma}|c|^\gamma$ and multiplied by $|c|^\delta$ converges to zero. Finally

$$
\frac{1}{|c|^\alpha} \int_{\tau(x) > \frac{1}{|c|}} f(c, x)\nu(dx) \leq D_{\delta, \beta}|c|^\beta \sum_{n=0}^{\infty} \int_{2^n |c| < \tau(x) \leq 2^{n+1} |c|} \tau(x)^\beta \nu(dx)
$$

$$
\leq D_{\delta, \beta}|c|^\beta \sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{|c|}\right)^\beta \nu\left(\frac{2^n}{|c|} < \tau(x)\right) \leq D_{\delta, \beta}|c|^\beta \sum_{n=0}^{\infty} 2^n |c|^\alpha \frac{2^n |c|^\alpha}{2^{n\alpha}} \leq D_{\delta, \beta}|c|^\delta
$$

Of course the same proof gives the second part of the Lemma.

4.2. Some properties of the eigenfunction $\psi_{c, v}$. Up to now, we have not taken any care about precise values of parameters $\theta, \varepsilon, \lambda$. However, we will need some further hypotheses, and from now on, we will assume additionally that

$$
\begin{align*}
\text{if } 1 < \alpha < 2, \text{ then } &1 + \lambda + \varepsilon > \alpha, \\
\text{if } \alpha = 2, \text{ then } &\lambda + 2\varepsilon > 1, \\
\text{if } \alpha > 2, \text{ then } &\lambda = 1.
\end{align*}
$$

It can be easily proved that there exist $\theta, \varepsilon, \lambda$ satisfying all the assumptions of Propositions 3.17, 3.18 and the condition above.
Lemma 4.5. There exists $D''$ such that
\[
|\psi_{c,v}(x) - \tilde{\eta}_c(cx)| \leq D''|c|^{2\tau(x)} \varepsilon, \quad \text{for } \tau(cx) > 1,
\]
\[
|\psi_{c,v}(x) - \tilde{\eta}_c(cx)| \leq D''|c|^{\tau(cx)} \varepsilon, \quad \text{for } \tau(cx) \leq 1,
\]
for $\eta = \min\{1, \lambda + \varepsilon\}$.

Proof. Let us first estimate the norm $\|\chi_{cx} - 1\|_{\theta, \varepsilon, \lambda}$. Let $0 < \beta < 1$. We have, since $\eta < 1$
\[
|\chi_{cx} - 1| \leq 2\tau(cx)^{\eta},
\]
\[
|\chi_{cx} - 1| \leq \sup_{y,z} \frac{2\tau(cx)^{\eta}(y-z)^{\eta}}{\tau(y-z)^{\lambda}(1 + \tau(z))^{\lambda}} \leq 2\tau(cx)^{\eta}
\]
which proves
\[
(4.6) \quad \|\chi_{cx} - 1\|_{\theta, \varepsilon, \lambda} \leq 4\tau(cx)^{\eta}.
\]
Therefore, by Proposition 3.20, and the expression of $\psi_{c,v}$ given by Lemma 3.23
\[
|\psi_{c,v}(x) - \tilde{\eta}_c(cx)| \leq \|\eta_{c,v} - \eta_v, \chi_{cx} - 1\| \leq C'|c|^{\tau(cx)} \|\chi_{cx} - 1\|_{\theta, \varepsilon, \lambda} \leq 4C'|c|^{\tau(cx)^{\eta}}.
\]
For $\tau(cx) > 1$ we need better estimates. By Proposition 3.20 and (3.21) we have
\[
|\psi_{c,v}(x) - \tilde{\eta}_c(cx)| = |\eta_{c,v} - \eta_v, \chi_{cx} - 1\| \leq C'|c|^{\tau(cx)^{\eta}} \|\chi_{cx}\|_{\theta, \varepsilon, \lambda} \leq C'|c|^{\tau(cx)^{\eta}} \leq 3C'|c|^{\tau(cx)^{\eta}},
\]
if $\tau(cx) \geq 1$. The result follows with $D'' = 4C'$. \qed

Corollary 4.7. If $\alpha \leq 2$, then
\[
\lim_{c \to 0, c \in Z(\mathbb{P})} \frac{1}{|c|^\alpha} \int_V (\chi_{c,v} - 1)(\psi_{c,v}(x) - \tilde{\eta}_c(cx))\nu(dx) = 0.
\]

Proof. We will apply Lemma 4.1 for $f(c,x) = (\chi_{c,v} - 1)(\psi_{c,v}(x) - \tilde{\eta}_c(cx))$. Let’s check that its hypotheses are satisfied. For $\tau(cx) \geq 1$, by Lemma 4.5, we have
\[
|f(c,x)| \leq D|c|^{2\tau(x)} \varepsilon.
\]
Next for $\tau(cx) \leq 1$ and $\eta$ as above
\[
|f(c,x)| \leq D|c|^{\tau(cx)^{1+\eta}}
\]
and in view of (4.4), $1 + \eta + \varepsilon > \alpha$, so the hypotheses of Lemma 4.1 are fulfilled. \qed

4.3. Eigenvalue $k(c,v)$ of $P_{c,v}$. From Lemma 3.23 we obtain, if $c \in Z(\mathbb{P}) \cup \{0\}$
\[
(4.8) \quad (k(c,v) - 1)\nu(\psi_{c,v}) = \nu(\psi_{c,v}(\chi_{c,v} - 1)),
\]
for $\psi_{c,v} = \tilde{\eta}_{c,v} \circ c$ The formula will be crucial in sections 5 and 7 to describe asymptotic behavior of the function $c \mapsto k(c,v)$ near zero. One can easily prove that $\nu(\psi_{c,v})$ goes to 1, hence to understand behavior of $k(c,v)$ near 1 one has to describe the integral above for small $|c|$. For some technical reasons we will need also speed of convergence of $\nu(\psi_{c,v})$ to 1.

Lemma 4.9. Assume $v$ is fixed. Then there exists $D''' > 0$ and $t_3 > 0$ such that for $|c| < t_3$, $c \in Z(\mathbb{P}) \cup \{0\}$, we have
\[
|1 - \langle \nu, \psi_{c,v} \rangle| \leq D'''|c|^{\min\{1, \lambda + \varepsilon\}}.
\]

Proof. We use the formula $\psi_{c,v}(x) = \eta_{c,v}(\chi_{cx})$ for $|c| \leq t_3$. Then by Lemma 3.23 and formulae (4.6) and (2.1) we have, with $C' = \sup_{|c| \leq t_3} \|\eta_{c,v}\|_{\theta, \varepsilon, \lambda}$
\[
|1 - \nu(\psi_{c,v})| \leq \int_V |\eta_{c,v} - 1 - \chi_{cx}|\nu(dx) \leq D' \int_V |1 - \chi_{cx}|\nu(dx)
\]
\[
\leq D'|c|^{\min\{1, \lambda + \varepsilon\}} \int_V \tau(x)^{\min\{1, \lambda + \varepsilon\}}\nu(dx) \leq D'''|c|^{\min\{1, \lambda + \varepsilon\}},
\]
with $D'' = D' \int_V \tau(x)^{\min\{1, \lambda + \epsilon\}} \nu(dx)$.

5. Asymptotic expansions of eigenvalues $k(c, v)$ in the Euclidean case

The purpose of this section is to give asymptotic expansions of the eigenvalues $k(c, v)$ when $|c|$ goes to 0. First to present main ideas of the proof we will consider the Euclidean case, then $|x|^2 = \sum x_i^2$. If $\alpha \leq 2$, we take $c \in \mathbb{Z}_\mathbb{R}$. If $\alpha > 2$ we take $c = t \in \mathbb{R}^*_+$. The main result of this section is the following

**Theorem 5.1.**

1. If $0 < \alpha < 1$ then
   \[
   \lim_{c \to 0, c \in \mathbb{Z}_\mathbb{R}} \frac{k(c, v) - 1}{|c|^{\alpha}} = C_\alpha(v)
   \]
   where
   \[
   C_\alpha(v) = \int_V (\chi_v(x) - 1) \tilde{\eta}_v(x) \Lambda(dx)
   \]
   and $C_\alpha(c^* v) = |c|^{\alpha} C_\alpha(v)$ for any $c \in \mathbb{Z}_\mathbb{R}$.

2. If $\alpha = 1$ then
   \[
   \lim_{c \to 0, c \in \mathbb{Z}_\mathbb{R}} \frac{k(c, v) - 1 - i\langle v, \xi(c) \rangle}{|c|} = C_1(v),
   \]
   for
   \[
   C_1(v) = \int_V (\chi_v(x) - 1) \tilde{\eta}_v(x) - \frac{i\langle v, x \rangle}{1 + |x|^2} \Lambda(dx),
   \]
   and $\xi(c) = \int_V \frac{z}{1 + |z| - |c|^2} \nu(dz)$. Moreover $|\xi(c)| \leq I(v)|c|^{\log |c|}$ for $|c| < 1/2$ and $|\xi(c)| \leq I(v)|c|$ for $|c| > 1/2$. Furthermore $C_1(c^* v) = |c| C_1(v) + i\langle v, \beta(c) \rangle$ with $\beta(c) = \int_V \left( \frac{x}{1 + |x|^2} \right) \Lambda(dx)$.

3. If $1 < \alpha < 2$
   \[
   \lim_{c \to 0, c \in \mathbb{Z}_\mathbb{R}} \frac{k(c, v) - 1 - i\langle v, cm \rangle}{|c|^{\alpha}} = C_\alpha(v),
   \]
   where
   \[
   C_\alpha(v) = \int_V \left( (\chi_v(x) - 1) \tilde{\eta}_v(x) - i\langle v, x \rangle \right) \Lambda(dx)
   \]
   and $C_\alpha(c^* v) = |c|^{\alpha} C_\alpha(v)$ for any $c \in \mathbb{Z}_\mathbb{R}$.

4. If $\alpha = 2$
   \[
   \lim_{c \to 0, c \in \mathbb{Z}_\mathbb{R}} \frac{k(c, v) - 1 - i\langle v, cm \rangle}{|c|^2 \log |c|} = 2C_2(v),
   \]
   where
   \[
   C_2(v) = -\frac{1}{4} \int_{\mathcal{E}_1} \left( \langle v, w \rangle^2 + 2\langle v, w \rangle \eta_v(w^*) \right) \sigma(dw)
   \]
   and $C_2(c^* v) = |c|^2 C_2(v)$ for every $c \in \mathbb{Z}_\mathbb{R}$.

5. If $\alpha > 2$ then
   \[
   \lim_{t \to 0^+} \frac{k(t, v) - 1 - i\langle v, tm \rangle}{t^2} = C_{2+}(v),
   \]
   \[
   C_{2+}(v) = -\frac{1}{2} \tau(v, v) - \frac{1}{2} \langle v, m \rangle^2 - q(v, (I - z^*)^{-1} z^* v).
   \]

To prove the Theorem we shall consider each case separately.
5.1. Case: \( \alpha < 1 \). Let us write

\[
\frac{1}{|c|^{\alpha}} \int_V (\chi_v(cx) - 1) \psi_{c,v}(x) \nu(dx)
\]

\[
= \frac{1}{|c|^{\alpha}} \int_V (\chi_v(cx) - 1) \cdot (\psi_{c,v}(x) - \widehat{\eta}_v(cx)) \nu(dx) + \frac{1}{|c|^{\alpha}} \int_V (\chi_v(cx) - 1) \widehat{\eta}_v(cx) \nu(dx)
\]

and notice that by Corollary 4.7 the first term of the sum above tends to zero. To describe the second one, observe that the function \( f_v \) and notice that by Corollary 4.7 the first term of the sum above tends to zero. To describe the second one, observe that the function \( f_v = (\chi_v - 1)\widehat{\eta}_v \) satisfies (2.3). In fact the characteristic function \( \widehat{\eta}_v \) is bounded by 1, hence also \( f_v \) is bounded, and for \( |x| < 1 \), we have \( |f_v(x)| \leq 2|x| \).

Therefore by (2.2) the expression above tends to the constant \( C_{\alpha}(v) \).

Finally by (4.8) and Lemma 4.9

\[
\lim_{c \to 0} \frac{k(c,v) - 1}{|c|^{\alpha}} = \lim_{c \to 0} \frac{1}{\nu(\psi_{c,v})|c|^{\alpha}} \cdot \int_V (\chi_v(cx) - 1) \psi_{c,v}(x) \nu(dx)
\]

\[
= \lim_{c \to 0} \nu(\psi_{c,v})|c|^{\alpha} \cdot \int_V (\chi_v(cx) - 1) \widehat{\eta}_v(cx) \nu(dx) = C_{\alpha}(v)
\]

as \( c \) goes to 0. The last assertion is an immediate consequence of the homogeneity of \( \eta_v \) given by Lemma 2.9 and of the homogeneity of \( \Lambda \) mentioned in Section 2.

5.2. Case: \( \alpha = 1 \).

Lemma 5.3.

\[
\lim_{c \to 0} \frac{1}{|c|} \left( \langle \nu, (\chi_{c,v} - 1) \psi_{c,v} \rangle - i \langle \nu, \xi(c) \rangle \right) = C_1(v).
\]

Proof. We have

\[
\langle \nu, (\chi_{c,v} - 1) \psi_{c,v} \rangle = \int_V (\chi_v(cx) - 1) \psi_{c,v}(x) \nu(dx)
\]

\[
= \int_V (\chi_v(cx) - 1) \cdot (\psi_{c,v}(x) - \widehat{\eta}_v(cv)) \nu(dx)
\]

\[
+ \int_V (\chi_v(cx) - 1)(\widehat{\eta}_v(cv) - 1) \nu(dx)
\]

\[
+ \int_V (\chi_v(cx) - 1 - i\langle \nu, cx \rangle \frac{1}{1 + |cx|^2}) \nu(dx) + i\langle \nu, \xi(c) \rangle
\]

\[
= \sum_{j=0}^2 W_j(c) + i\langle \nu, \xi(c) \rangle
\]

By Corollary 4.7, \( \frac{W_0(c)}{|c|} \) converges to 0, as \( c \) goes to 0. Next observe that the function \( f_1 = (\chi_v - 1)(\widehat{\eta}_v - 1) \) satisfies (2.3). Indeed \( f_1 \) is bounded and for \( |x| \leq 1 \), from (4.6)

\[
|f_1(x)| \leq |\chi_v(x) - 1||\widehat{\eta}_v(x) - 1| \leq 2|v||x||\eta_v(\chi_v - 1)| \leq 2|v||x||\chi_x - 1| \leq 8|v||x|^{1+\lambda+\varepsilon}
\]

Similarly one can prove that \( f_2(x) = \chi_v(x) - 1 - i\langle \nu, cx \rangle \frac{1}{1 + |cx|^2} \) fulfills (2.3). Thus, by 2.2

\[
\lim_{c \to 0} \left( \frac{W_1(c)}{|c|} + \frac{W_2(c)}{|c|} \right) = \int_V (f_1(x) + f_2(x)) \Lambda(dx) = C_1(v),
\]

which finishes the proof. \( \square \)
Proof of Theorem 5.1, part (2).

By (4.8) and Lemmas 4.9, 5.3, 5.5 we have

\[ |\xi(c)| \leq \begin{cases} I(v)|c|, & \text{for } |c| \geq \frac{1}{2}, \\ I(v)|c||\log |c||, & \text{for } |c| < \frac{1}{2}. \end{cases} \]

Proof. For $|c| \geq 1$ the Lemma is obvious. For $|c| < 1$ we write

\[ |\xi(c)| \leq \int_{|v|>1} \frac{|cx|}{1 + |c|^2|x|^2} \nu(dx) = \left( \int_{|x| \leq 1} + \int_{1 < |x| \leq \frac{1}{|c|}} + \int_{|c| < |x|} \right) \left( \frac{|cx|}{1 + |c|^2|x|^2} \right) \nu(dx). \]

The first integral is dominated by $|c|$. In view of Corollary 4.7 it is enough to consider $\int_{|x|>1} \frac{|x|}{1 + |x|^2} \Lambda(dx)$, as $s$ goes to $0$. Finally applying (4.3) we estimate the second integral by

\[
\sum_{k=0}^{\log |c|} \int_{|x| \leq 2^{k+1}} \frac{|cx|}{1 + |c|^2|x|^2} \nu(dx) \leq |c| \sum_{k=0}^{\log |c|} 2^{k+1} \nu[2^{k} < |x|] \\
\leq C |c| \sum_{k=0}^{\log |c|} 2^{k+1} \cdot 2^{-k} \leq C |c| \log |c|.
\]

\[ \square \]

Proof of Theorem 5.1, part (2). By (4.8) and Lemmas 4.9, 5.3, 5.5 we have

\[
\lim_{c \to 0} \frac{k(c,v) - 1 - i\langle v, \xi(c) \rangle}{|c|} = \lim_{c \to 0} \left[ \frac{\nu((\chi_v - 1)\psi_{c,v}) - i\langle v, \xi(c) \rangle}{\nu(\psi_{c,v})|c|} + \frac{i(1 - \nu(\psi_{c,v}))\langle v, \xi(c) \rangle}{\nu(\psi_{c,v})|c|} \right] = C_1(v)
\]

\[ \square \]

5.3. Case: $1 < \alpha < 2$.

Lemma 5.6.

\[ \lim_{c \to 0, c \in \mathbb{R}} \frac{1}{|c|^{\alpha}} \left( \nu, (\chi_v - 1)\psi_{c,v} - i\langle v, cm \rangle \right) = C_\alpha(v). \]

Proof. In view of Corollary 4.7 it is enough to consider

\[
\int_V (\chi_v(cx) - 1)\tilde{\psi}_v(cx)\nu(dx) = \int_V (\chi_v(cx) - 1) \cdot (\tilde{\psi}_v(cx) - 1)\nu(dx) + \int_V (\chi_v(cx) - 1 - i\langle v, cx \rangle)\nu(dx) + i\langle v, cm \rangle.
\]

Reasoning as in previous cases we prove that the functions $f_1 = (\chi_v - 1) \cdot (\tilde{\psi}_v - 1)$ and $f_2 = \chi_v - 1 - iv^*$ satisfy (2.3), therefore (2.2) implies the Lemma.

\[ \square \]

Proof of Theorem 5.1, part (3). By (4.8), Lemma 4.9 and (4.4)

\[
\lim_{c \to 0} \frac{k(c,v) - 1 - i\langle v, cm \rangle}{|c|^{\alpha}} = \lim_{c \to 0} \left[ \frac{\nu(\psi_{c,v}) (k(c,v) - 1 - i\langle v, cm \rangle)}{\nu(\psi_{c,v})|c|^{\alpha}} + \frac{i\langle v, cm \rangle (1 - \nu(\psi_{c,v}))}{\nu(\psi_{c,v})|c|^{\alpha}} \right] = C_\alpha(v)
\]

\[ \square \]
5.4. Case: \( \alpha = 2 \).

Lemma 5.7. Suppose we are given two functions on \( V, f \) and \( h \) such that \( h(x) = \langle x, v_1 \rangle \langle x, v_2 \rangle \) for some \( v_1, v_2 \in V \), \( \lim_{x \to 0} \frac{f(x)}{|x|^{1+\eta}} = C_0 \) and \( |f(x)| \leq C|x|^{1+\eta} \) for some positive constants \( C_0, C \) and \( \eta < 1 \). Then

\[
\lim_{g \to 0, g \not\in G \atop |g| \to 0} \frac{1}{|g|^2 \log |g|} \int_V f(gx) \nu(dx) = C_0 \int_{\Sigma_1} h(w) \sigma(dw),
\]

where \( \sigma \) is the measure on the fundamental domain \( \Sigma_1 \) defined in (1.4).

Moreover the function \( \tilde{\Lambda}(v) = \int_{\Sigma_1} \langle v, w \rangle^2 \sigma(dw) \) is \( G^{\Phi} \)-homogeneous, i.e. \( \tilde{\Lambda}(g^*v) = |g|^2 \tilde{\Lambda}(v) \) for every \( g \in G^{\Phi} \).

Proof. Fix \( \beta \in R^{\Phi} \) such that \( \beta > 1 \) and denote by \( U \) the annulus \( U = \{ x \in V : 1 < |x| \leq \beta \} \). Next we fix arbitrary small number \( \delta > 0 \). Then there exists \( \varepsilon \) such that

\[
(5.8) \quad \left| \frac{f(x)}{h(x)} - C_0 \right| < \delta, \quad \text{for } |x| < \varepsilon.
\]

Without any lose of generality we may assume \( |v_1| = |v_2| = 1 \). Given \( y_1, y_2 \in V \) we define a function on \( V, h_{y_1, y_2}(x) = \langle x, y_1 \rangle \langle x, y_2 \rangle \). We are going to prove that there exists large \( A \in R^{\Phi} \) such that

\[
(5.9) \quad \left| \frac{1}{|g|^2} \int_V 1_U(gx) h_{y_1, y_2}(gx) \nu(dx) - \int_U h_{y_1, y_2}(x) \Lambda(dx) \right| \leq \delta,
\]

for any \( g \in G^{\Phi} \) such that \( |g| < \frac{\varepsilon}{2} \) and all \( y_1, y_2 \) belonging to \( S_1 \), the unit sphere in \( V \).

Of course the last assertion, by (2.2), is clear for fixed vectors \( y_1 \) and \( y_2 \). However, we will justify that also uniform estimates are valid.

Fix \( y_1, y_2 \in S_1 \). Then in view of (2.2) there exists \( M \in R^{\Phi} \) such that

\[
\left| \frac{1}{|g|^2} \int_V 1_U(gx) h_{y_1, y_2}(gx) \nu(dx) - \int_U h_{y_1, y_2}(x) \Lambda(dx) \right| \leq \frac{\delta}{2},
\]

and

\[
\left| \frac{1}{|g|^2} \int_V 1_U(gx) |gx|^2 \nu(dx) - \int_U |x|^2 \Lambda(dx) \right| \leq 1,
\]

for \( |g| \leq \frac{\varepsilon}{4} \). Choose \( \delta' < \frac{\delta}{4(2\beta^2 \Lambda(U) + 1)} \). Define \( B_{y_1, y_2}(\delta') \) to be the ball in \( V \times V \) centered at \( (y_1, y_2) \) of radius \( \delta' \) and take \( (y'_1, y'_2) \in B_{y_1, y_2}(\delta') \). Notice

\[
| h_{y'_1, y'_2}(x) - h_{y_1, y_2}(x) | \leq | \langle y_1 - y'_1, x \rangle \langle y_2, x \rangle | + | \langle y'_1, x \rangle \langle y_2 - y'_2, x \rangle | \leq 2\delta' |x|^2,
\]

therefore

\[
\left| \frac{1}{|g|^2} \int_V 1_U(gx) h_{y'_1, y'_2}(gx) \nu(dx) - \int_U h_{y'_1, y'_2}(x) \Lambda(dx) \right|
\leq \left| \frac{1}{|g|^2} \int_V 1_U(gx) \left( h_{y'_1, y'_2}(gx) - h_{y_1, y_2}(gx) \right) \nu(dx) \right| + \left| \int_U \left( h_{y'_1, y'_2}(x) - h_{y_1, y_2}(x) \right) \Lambda(dx) \right| + \frac{\delta}{2}
\leq \frac{2\delta'}{|g|^2} \int_V 1_U(gx) |gx|^2 \nu(dx) + 2\delta' \int_U |x|^2 \Lambda(dx) + \frac{\delta}{2}
\leq 2\delta'(2\beta^2 \Lambda(U) + 1) + \frac{\delta}{2} < \delta.
\]

So, we may find finitely many pairs \( \{(y_{i,1}, y_{i,2})\}_{1 \leq i \leq N} \) and positive numbers \( M_i \in R^{\Phi} \) such that the balls \( B_{y_{i,1}, y_{i,2}}(\delta') \) cover \( S_1 \). Then choosing \( A_1 = \max_{1 \leq i \leq N} M_i \) we deduce that the first line of (5.9)
is satisfied for $|g| < \frac{1}{\lambda^1}$. Next we repeat our argument for $|h_{y_1, y_2}|$ instead of $h_{y_1, y_2}$, we find $A_2$ and finally choosing $A = \max\{A_1, A_2\}$ we obtain (5.9).

For $|g| < \frac{1}{\lambda^1}$ (that will be assumed from now), we divide the integral of $f$ into three parts:

\begin{equation}
\int_V f(gx)\nu(dx) = \int_{|x| \leq A} f(gx)\nu(dx) + \int_{A < |x| < |x|} f(gx)\nu(dx) + \int_{|x| > |x|} f(gx)\nu(dx).
\end{equation}

Notice first that

\[ \left| \int_{|x| \leq A} f(gx)\nu(dx) \right| \leq (|C_0| + \delta) \int_{|x| \leq A} |h(gx)|\nu(dx) \leq (|C_0| + \delta)|g|^2 A^2 \]

and by (2.2)

\[ \lim_{g \to 0, g \in G \frac{1}{|g|^2 \log |g|}} \int_{|x| \leq A} f(gx)\nu(dx) = \int_{|x| \geq \epsilon} f(x)\Lambda(dx). \]

Hence

\[ \lim_{g \to 0, g \in G \frac{1}{|g|^2 \log |g|}} \left( \int_{|x| \leq A} f(gx)\nu(dx) + \int_{|x| \geq \epsilon} f(gx)\nu(dx) \right) = 0. \]

Therefore we have to handle with the middle term in (5.10). We will prove

\begin{equation}
\lim_{g \to 0, g \in G \frac{1}{|g|^2 \log |g|}} \int_{A < |x| < |x|} f(gx)\nu(dx) = \frac{C_0}{\log \beta} \int_U h(x)\Lambda(dx).
\end{equation}

Applying (5.8), we write

\begin{equation}
\left| \frac{1}{|g|^2 \log |g|} \int_{A < |x| < |x|} f(gx)\nu(dx) - \frac{C_0}{\log \beta} \int_U h(x)\Lambda(dx) \right| 
\leq |C_0| \cdot \frac{1}{|g|^2 \log |g|} \int_{A < |x| < |x|} h(gx)\nu(dx) - \frac{1}{\log \beta} \int_U h(x)\Lambda(dx) 
+ \frac{\delta}{|g|^2 \log |g|} \int_{A < |x| < |x|} |h(gx)|\nu(dx).
\end{equation}

We estimate the first expression. For this purpose we define $K = \left| \log \frac{A}{|g|} / \log \beta \right| - 1$. For $r \in R_\pi$ we will denote by $g(r)$ any element of $G_\pi$ such that $|g(r)| = r$. To simplify our notation we define elements of $G_\pi$: $g_n = g(A \beta^n)$ and annulus $U_n = \{x : A \beta^n < |x| \leq A \beta^{n+1}\}$. Notice that $|g_n| > A$,
Combining the formula above and (5.11) we prove the first part of the Lemma.

Hence passing with $\beta$ we obtain

$$\left| \frac{1}{|g|^2 \log |g|} \int_{A < |x| < \frac{1}{g}} h(gx) \nu(dx) - \frac{1}{\log \beta} \int_U h(x) \Lambda(dx) \right|$$

$$\leq \left| \frac{1}{|g|^2 \log |g|} \sum_{n=0}^{K} \int_{U_n} h(gx) \nu(dx) - \frac{1}{\log \beta} \int_U h(x) \Lambda(dx) \right| + \frac{1}{|g|^2 \log |g|} \int_{U_{K+1}} |h(gx)| \nu(dx)$$

$$= \left| |g|| \sum_{n=0}^{K} |g_n|^2 \int_V 1_U(g_n^{-1}x) h(g_{nk}^{-1}) \frac{h(g_{nk}^{-1})}{|g_{nk}|} (g_{nk}^{-1}x) \nu(dx) - \frac{1}{\log \beta} \int_U h(x) \Lambda(dx) \right|$$

$$\leq \left| |g|| \sum_{n=0}^{K} |g_n|^2 \int_U h(gg_n x) \Lambda(dx) - \frac{1}{\log \beta} \int_U h(x) \Lambda(dx) \right| + \delta(K+1)$$

$$= \left( \frac{K+1}{|g|} - \frac{1}{\log \beta} \right) \cdot \left( \int_U h(x) \Lambda(dx) + \delta \right).$$

The second term in (5.12) can be estimated exactly using exactly the same arguments. Thus, we obtain

$$\left| \frac{1}{|g|^2 \log |g|} \int_{A < |x| < \frac{1}{g}} h(gx) \nu(dx) \right| \leq \frac{\delta(K+2)}{|g|}. \left( \int_U |h(x)| \Lambda(dx) + \delta \right).$$

Therefore passing to the limit in (5.12) we obtain

$$\limsup_{g \to 0, g \in \mathcal{C}_\pi} \left| \frac{1}{|g|^2 \log |g|} \int_{A < |x| < \frac{1}{g}} f(gx) \nu(dx) - \frac{C_0}{\log \beta} \int_U h(x) \Lambda(dx) \right|$$

$$\leq \frac{\delta}{\log \beta} \left( 1 + \delta + \int_U |h(x)| \Lambda(dx) \right),$$

but $\delta$ can be arbitrary small, hence we obtain (5.11).

Finally to conclude we choose $\beta = p$ if $R_\pi = \{p\}$. Otherwise, if $R_\pi = \mathbb{R}_+^*$, we compute the limit as $\beta$ tends to 1. For this purpose, given $a \in A_\pi$ and $w \in V$ we will write $aw = |a|\theta(w)$, where $|\theta(w)| = |w|$. Then $\theta(w)$ tends to $w$, if $|a|$ tends to 1. By (1.4) we write

$$\frac{1}{\log \beta} \int_{1 < |x| \leq \beta} h(x) \Lambda(dx) = \frac{1}{\log \beta} \int_{1 < |a| \leq \beta} \int_{\Sigma_1} h(aw) \sigma(dw) \frac{da}{|a|^3}$$

$$= \frac{1}{\log \beta} \int_{1 < |a| \leq \beta} \int_{\Sigma_1} (h(\theta(w)) - h(w)) \sigma(dw) \frac{da}{|a|} + \int_{\Sigma_1} h(w) \sigma(dw).$$

Hence passing with $\beta$ to the limit we obtain

$$\lim_{\beta \to 1} \frac{1}{\log \beta} \int_{1 < |x| \leq \beta} h(x) \Lambda(dx) = \int_{\Sigma_1} h(w) \sigma(dw).$$

Combining the formula above and (5.11) we prove the first part of the Lemma.
To prove the last assertion, assume $v = v_1 = v_2$ and notice that the limit (5.11) does not depend on $\beta$, hence if for $\beta \in \mathbb{R}$ we define

$$H_\beta(v) = \frac{1}{\log \beta} \int_{1 < x \leq |\beta|} \langle x, v \rangle^2 \Lambda(dx),$$

then $H_\beta$ in fact does not depend on $\beta$ and moreover $H_\beta(v) = \tilde{\Lambda}(v)$. Therefore it is enough to prove that

$$\lim_{\beta \to -\infty, \beta \in \mathbb{R}} H_\beta(g^*v) = |g|^2 \lim_{\beta \to -\infty, \beta \in \mathbb{R}} H_\beta(v), \quad \text{for } g \in G_\mathbb{R},$$

because then

$$\tilde{\Lambda}(g^*v) = \lim_{\beta \to -\infty, \beta \in \mathbb{R}} H_\beta(g^*v) = |g|^2 \lim_{\beta \to -\infty, \beta \in \mathbb{R}} H_\beta(v) = |g|^2 \tilde{\Lambda}(v), \quad \text{for } g \in G_\mathbb{R}.$$ 

Assume $|g| > 1$. We apply $G_\mathbb{R}$ homogeneity of $\Lambda$ and write

$$H_\beta(g^*v) = \frac{1}{\log \beta} \int_{1 < |x| \leq |\beta|} \langle v, gx \rangle^2 \Lambda(dx)$$

$$= \frac{|g|^2}{\log \beta} \int_{|v| < |g| \leq |\beta|} \langle v, x \rangle^2 \Lambda(dx)$$

$$= \frac{|g|^2 \log (|\beta|)}{\log \beta} H_{|v|,|\beta|}(v) - \frac{|g|^2 \log |g|}{\log \beta} H_{|g|,|v|}(v).$$

Passing with $\beta$ to infinity we obtain (5.13) and finish the proof of the lemma.

**Lemma 5.14.** We have

$$\lim_{c \to 0, c \in \mathbb{Z}, |c|^2 \log |c|} \frac{1}{|c|^2 \log |c|} \left( \langle \nu, (\chi_{c+v} - 1)\psi_{c,v} \rangle - i \langle v, cm \rangle \right) = 2C_2(v).$$

**Proof.** We begin as in previous cases and write

$$\langle \nu, (\chi_{c+v} - 1)\psi_{c,v} \rangle = \int_V (\chi_v(cx) - 1) \cdot \left( \psi_{c,v}(x) - \tilde{\eta}_v(cx) \right) \nu(dx)$$

$$+ \int_V (\chi_v(cx) - 1) \cdot \eta_v(\chi_{cx} - 1 - i(cx)^*) \nu(dx)$$

$$+ \int_V (\chi_v(cx) - 1) \cdot \eta_v(i(cx)^*) \nu(dx) + \int_V (\chi_v(cx) - 1 - i\langle v, cx \rangle) \nu(dx)$$

$$+ i \langle v, cm \rangle.$$

The first term, in view of Corollary 4.7, divided by $|c|^2$ goes to zero. The second one divided by $|c|^2$, by (2.2) has a finite limit. Hence both divided by $|c|^2 \log |c|$ tend to 0. To handle with the third and the fourth expression we will use Lemma 5.7. Notice

$$\lim_{x \to 0} \frac{\chi_v(x) - 1 - i\langle v, x \rangle}{\langle x, v \rangle^2} = -1,$$

$$\lim_{x \to 0} \frac{\chi_v(x) - 1 - i\langle v, x \rangle}{\langle x, v \rangle^2} = -\frac{1}{2},$$

where $m_v = \int_V \eta_v(dy)$ is the mean of $\eta_v$. Hence all the assumptions of Lemma 5.7 are satisfied, thus

$$\lim_{c \to 0, c \in \mathbb{Z}, |c|^2 \log |c|} \frac{1}{|c|^2 \log |c|} \left( \int_V (\chi_v(cx) - 1) \cdot \eta_v(i(cx)^*) \nu(dx) + \int_V (\chi_v(cx) - 1 - i\langle v, cx \rangle) \nu(dx) \right) = 2C_2(v)$$

and the Lemma follows. \qed
Proof of Theorem 5.1, part (4). First we will improve Lemma 4.9 and we will show that if $\alpha = 2$ then

\[(5.15) \quad |1 - \langle \nu, \psi_{c,v} \rangle| \leq C|c|.
\]

Indeed, applying Lemma 4.5 and (4.4), we have

\[
\begin{align*}
|1 - \langle \nu, \psi_{c,v} \rangle| & \leq \int_V |\psi_{c,v}(x) - \tilde{\eta}_c(cx)| \nu(dx) + \int_V \int_V |\chi_y(cx) - 1| \eta_y(dy) \nu(dx) \\
& \leq C|c|^{\lambda + 2\varepsilon} \int_V |x|^{\lambda + \varepsilon} \nu(dx) + C|c| \int_V |x| \nu(dx) \int_V |y| \eta_y(dy) \\
& \leq C|c|,
\end{align*}
\]

which proves (5.15). Finally, applying Lemma 5.14 and (5.15), we write

\[
\lim_{c \to 0, c \in \mathbb{Z}_+} \frac{k(c,v) - 1 - i(v,cm)}{|c|^2 \log |c|} = \lim_{c \to 0, c \in \mathbb{Z}_+} \frac{\nu(\psi_{c,v})(k(c,v) - 1) - i(v,cm)}{\nu(\psi_{c,v})|c|^2 \log |c|} = C_2(v).
\]

\[
\square
\]

5.5. Case: $\alpha > 2$. Here we replace $\mathbb{Z}_+$ by $\mathbb{R}_+$, hence $c = t \in \mathbb{R}_+$. We use expression of $\psi_{t,v}$ given by (4.8)

**Lemma 5.16.**

\[
\lim_{t \to 0} \frac{1}{t^2} \left( \langle \nu, (\chi_{tv} - 1) \psi_{t,v} \rangle - it \langle v, m \rangle \right) = C^1_{2+}(v),
\]

where

\[
C^1_{2+}(v) = -\frac{1}{2} \int_V \langle v, x \rangle^2 \nu(dx) - \int_V \langle v, x \rangle \eta_v(x) \nu(dx).
\]

**Proof.** We write

\[
\begin{align*}
\int_V (\chi_{tv}(x) - 1) \psi_{t,v}(x) \nu(dx) &= \int_V (\chi_{tv}(x) - 1) \nu(dx) + \int_V (\chi_{tv}(x) - 1) (\psi_{t,v}(x) - 1) \nu(dx) \\
&= W_1(t) + W_2(t).
\end{align*}
\]

Notice that for any $\delta < 1$ there exists $C$ such that

\[
|e^{is} - 1 - is + \frac{s^2}{2}| \leq C|s|^{2+\delta}
\]

for every $s \in \mathbb{R}$. Therefore choosing $\delta < \min\{1, \alpha - 2\}$ and applying (2.1) with $2 < \theta < \alpha$ we have

\[
\lim_{t \to 0} \frac{1}{t^2} \left( W_1(t) - it \langle v, m \rangle \right) = -\frac{1}{2} \int_V \langle v, x \rangle^2 \nu(dx) + \lim_{t \to 0} \frac{1}{t^2} \int_V \left( e^{it\langle v, x \rangle} - 1 - it \langle v, x \rangle + \frac{t^2}{2} \langle v, x \rangle^2 \right) \nu(dx)
\]

\[
= -\frac{1}{2} \int_V \langle v, x \rangle^2 \nu(dx).
\]

To handle $W_2$ we will prove first that

\[(5.17) \quad \|\chi_x - 1 - ix\|_{\theta, 1} \leq C|x|^{1+\delta},
\]
for some $\delta > 0$. Indeed, recall that in view of (4.4) we may assume $\lambda = 1$ and $1 < \theta < 2$. We have $|\chi_x - 1 - i x^*|^\alpha \leq C|x|^{1+\delta}$ and

\[
[\chi_x - 1 - i \langle x, \cdot \rangle]_{\epsilon,1} \leq \sup_{y \neq y'} \frac{\left|\chi_x(y) - 1 - i \langle x, y \rangle - \chi_x(y') - 1 - i \langle x, y' \rangle\right|}{|y - y'|^{\epsilon} (1 + |y|)(1 + |y'|)}
\]

\[
\leq \sup_{y \neq y'} \min \left\{ \frac{|\chi_x(y) - 1 - i \langle x, y \rangle|}{|y - y'|^{\epsilon} (1 + |y|)(1 + |y'|)}, \frac{|\chi_x(y') - 1 - i \langle x, y' \rangle|}{|y - y'|^{\epsilon} (1 + |y|)(1 + |y'|)}, \frac{|\chi_x(y) - \chi_x(y')| + |\langle x, y - y' \rangle|}{|y - y'|^{\epsilon} (1 + |y|)(1 + |y'|)} \right\}
\]

\[
\leq C \sup_{y \neq y'} \left[ \frac{1}{(1 + |y|)(1 + |y'|)} \min \left\{ \frac{|x|^{1+\epsilon}(|y|^{1+\epsilon} + |y'|^{1+\epsilon})}{|y - y'|^{\epsilon}}, \frac{|x||y - y'|^{1-\epsilon}}{1} \right\} \right]
\]

\[
\leq C \sup_{y \neq y'} \left[ \frac{1}{(1 + |y|)(1 + |y'|)} \left( \frac{|x|^{1+\epsilon}(|y|^{1+\epsilon} + |y'|^{1+\epsilon})}{|y - y'|^{\epsilon}} \right)^{1-\epsilon} \cdot \left( \frac{|x||y - y'|^{1-\epsilon}}{1} \right)^{\epsilon} \right]
\]

\[
\leq C|x|^{1+\epsilon-\epsilon^2},
\]

which proves (5.17).

Now applying Proposition 3.20 and (5.17) we have

\[
\lim_{t \to 0} \psi_{t,v}(x) - t = \lim_{t \to 0} \frac{\langle \eta_{t,v}, \chi_{x,v} - 1 - itx^* \rangle}{t} + i \lim_{t \to 0} \langle \eta_{t,v} - \eta_v, x^* \rangle + i \langle \eta_v, x^* \rangle
\]

\[
= i \eta_v(x^*).
\]

Therefore we have

\[
\lim_{t \to 0} \frac{\psi_{t,v}(x) - 1}{t} = \lim_{t \to 0} \frac{\langle \eta_{t,v}, \chi_{x,v} - 1 - itx^* \rangle}{t} = \lim_{t \to 0} \psi_{t,v}(x) (x^*) \nu(dx) = - \int_V \langle v, x \rangle \eta_v(x^*) \nu(dx)
\]

hence the Lemma. \qed

**Proof of Theorem 5.1, part (5).** First we will prove

\[
\lim_{t \to 0^+} \frac{\nu(\psi_{t,v}) - 1 - it \int_V \eta_v(x^*) \nu(dx)}{t} = 0.
\]

Applying inequality (5.17) and Proposition 3.20 we have

\[
\nu(\psi_{t,v}) - 1 - it \int_V \eta_v(x^*) \nu(dx) = \int_V \left( \eta_{t,v}(\chi_{x,v} - 1 - itx^*) + i \langle \eta_{t,v} - \eta_v, x^* \rangle \right) \nu(dx)
\]

\[
\leq C \int_V \| \chi_{t,v} - 1 - itx^* \|_{\theta, t, \lambda} \nu(dx) + t^{1+\epsilon} \int_V \| x^* \|_{\theta, t, \lambda} \nu(dx) \leq Ct^{1+\delta}
\]

Therefore by (4.8) and Lemma (5.16)

\[
\lim_{t \to 0^+} \frac{k(t,v) - 1 - it \langle v, m \rangle}{t^2} = \lim_{t \to 0^+} \left[ \frac{\nu(\psi_{t,v}) (k(t,v) - 1) - it \langle v, m \rangle}{\nu(\psi_{t,v}) t^2} - \frac{i(\nu(\psi_{t,v}) - 1)t \langle v, m \rangle}{\nu(\psi_{t,v}) t^2} \right]
\]

\[
= C_{2+}^1(v) - \lim_{t \to 0^+} \frac{it \langle v, m \rangle (\nu(\psi_{t,v}) - 1 - it \int_V \eta_v(x^*) \nu(dx))}{\nu(\psi_{t,v}) t^2} + \lim_{t \to 0^+} \frac{\langle v, m \rangle \cdot \int_V \eta_v(x^*) \nu(dx)}{\nu(\psi_{t,v})}
\]

\[
= -\frac{1}{2} \int_V \langle v, x \rangle^2 \nu(dx) - \int_V \langle x - m, v \rangle \eta_v(x^*) \nu(dx).
\]
Finally, since $\kappa(1) < 1$ the matrix $I - z^* = \mathbb{E}[I - M^*]$ is invertible, therefore

$$
\int_V \langle x - m, v \rangle \eta_c(x^*) \nu(dx) = \int_V \langle x - m, v \rangle \mathbb{E} \left[ \sum_{k=1}^{\infty} M_0^* \cdots M_{k-1}^* v \right] \nu(dx)
$$

$$
= \int_V \langle x - m, v \rangle \int_{\sum_{k=1}^{\infty} (z^*)^k v \nu(dx)
$$

$$
= \int_V \langle x - m, v \rangle \left( x - m, (I - z^*)^{-1} z^* v \right) \nu(dx)
$$

$$
= q(v, (I - z^*)^{-1} z^* v)
$$

Also

$$\int_V \langle x, x \rangle^2 \nu(dx) = q(v, v) + (v, m)^2,$$

which proves Theorem 5.1.

5.6. Calculations of $C_\alpha(v)$ in terms of tails ($0 < \alpha \leq 2$). Observe first that the function $\tilde{\Lambda}$ defined in Introduction is $G^*_\mathbb{T}$ homogeneous, i.e. $\tilde{\Lambda}(g^\alpha y) = |g|^\alpha \tilde{\Lambda}(y)$ if $g \in G^*_\mathbb{T}$ Indeed for $\alpha < 2$ this follows from (2.4) and for $\alpha = 2$ this was proved in Lemma 5.7. As in [BDGHU] we define the polar coordinates $(a(x), \overline{r})$ of $x \in V \setminus \{0\}$, using the decomposition $G^*_\mathbb{T} = A^*_\mathbb{T} \ltimes K^*_\mathbb{T}$. We denote by $\Sigma_1$ the natural fundamental domain of $A^*_\mathbb{T}$ on $V \setminus \{0\}$, i.e.

- $\Sigma_1 = \{ x : 1 \leq |x| < p \}$ if $R^*_\mathbb{T} = \langle p \rangle$,
- $\Sigma_1 = S_1$ the unit sphere, if $R^*_\mathbb{T} = \mathbb{R}^*_+$.

Then we write $x = a(x) \overline{r}$ with $a(x) \in A^*_\mathbb{T}$ and $\overline{r} \in \Sigma_1$. Then $r(x) = |a(x)|$ takes values in $R^*_\mathbb{T}$, and if $R^*_\mathbb{T} = \mathbb{R}^*$, $r(x) = |x|$.

We will write $\tilde{\Lambda}_s(y) = r^{s-\alpha}(y) \tilde{\Lambda}(y)$, so that $\tilde{\Lambda}_s(y) = r^s(y) \tilde{\Lambda}(\overline{r})$ is well defined by its restriction to $\Sigma_1$, and is $G^*_\mathbb{T}$-homogeneous of degree $s$. Also we denote $\Lambda^s(y) = \tilde{\Lambda}(\overline{r}) 1_{[1, \infty)}(r(y))$. We recall that the tail measure $\Delta_\alpha$ of $\eta_c$ exists i.e.

$$\Delta_\alpha = \lim_{|g| \to 0, g \in G^*_\mathbb{T}} |g|^{-\alpha} \langle g^\alpha \eta_c \rangle.
$$

Also, in view of Proposition 2.6 and Lemma 2.9, $\Delta_\alpha \neq 0$.

Proposition 5.19. Assume $R^*_\mathbb{T} = \mathbb{R}^*_+$, then

- if $\alpha \in (0, 1) \cup (1, 2]$ then $C_\alpha(v) = \alpha m_\alpha \Delta_\alpha(\Lambda^1)$;
- if $\alpha = 1$, then $C_1(v) = m_1 \Delta_\alpha(\Lambda^1) + \gamma(v)$, with $\gamma(v) \in \mathbb{R}$.

If $R^*_\mathbb{T} = \langle p \rangle$, the same formulas are valid, where $\alpha \Delta_\alpha(\Lambda^1)$ is replaced by $\frac{1 - p^{-\alpha}}{m_\alpha p^\alpha} \Delta_\alpha(\Lambda^1)$.

Proof. If $\alpha \neq 1$, we have, by definition of $\tilde{\Lambda}$:

$$C_\alpha(v) = \int_{\Sigma_1} (\tilde{\Lambda}(y + v) - \tilde{\Lambda}(y)) \eta_c(dy).$$

Indeed if $\alpha \in (0, 1) \cup (1, 2)$, this follows immediately from the formulas given in Theorem 5.1 and for $\alpha = 2$ we write

$$C_2(v) = -\frac{1}{4} \int \int_{\Sigma_1} \langle w, v \rangle^2 + 2\langle w, v \rangle \eta_c(w^*) \sigma(dw)$$

$$= -\frac{1}{4} \int \int_{\Sigma_1} \langle w, v + y \rangle^2 - \langle w, y \rangle^2 \sigma(dw) \eta_c(dy)$$

$$= \int \langle \tilde{\Lambda}(y + v) - \tilde{\Lambda}(y) \rangle \eta_c(dy).$$
If $\alpha = 1$, then
\[
C_1(v) = \int_V (\overline{A}(y + v) - \overline{A}(y)) \eta_v(dy) + i\gamma(v)
\]
with
\[
\gamma(v) = \int_V \int_V \left( -\frac{\langle v, x \rangle}{1 + |x|^2} - \frac{\langle y, x \rangle}{1 + |y|^2 |x|^2} + \frac{\langle v + y, x \rangle}{1 + |v + y|^2 |x|^2} \right) \Lambda(dx) \eta_v(dy).
\]
We write if $s < \chi$
\[
C_{\alpha, s} = \int_V (\overline{A}_s(y + v) - \overline{A}_s(y)) \eta_v(dy)
\]
and we observe: $\lim_{s \to \alpha^{-}} - C_{\alpha, s} = C_\alpha(v)$. On the other hand $Z^* v$ satisfies $Z^* v = M^*_1(Z^*_1 v + v)$, where $Z^*_1$ is another copy of $Z^*$, independent of $M^*_v$. Since $\overline{A}_s$ is $G^*_v$-homogeneous and $s < \alpha$,
\[
\mathbb{E}[\overline{A}_s(Z^* v)] = \kappa(s) \mathbb{E}[\overline{A}_s(Z^* v + v)],
\]
hence
\[
C_{\alpha, s} = \mathbb{E}[\overline{A}_s(Z^* v + v)] - \mathbb{E}[\overline{A}_s(Z^* v)]
\]
\[
= (1 - \kappa(s)) \mathbb{E}[\overline{A}_s(Z^* v + v)].
\]
Since $\lim_{s \to \alpha^{-}} \frac{1 - \kappa(s)}{s} = m_\alpha$, we need to evaluate $\lim_{s \to \alpha^{-}} (\alpha - s) \mathbb{E}[\overline{A}_s(Z^* v + v)]$. For the sake of brevity, we work with $\lim_{s \to \alpha^{-}} (\alpha - s) \mathbb{E}[\overline{A}_s(Z^* v)]$ and we show that this quantity depends only of the tail of $\eta_v$. This will give the required result, since the tails of $\eta_v$ and $\delta_v * \eta_v$ are the same.

Assume first $R_{\overline{\eta}} = \mathbb{R}^+$ and write $F_v(t) = \int_{|\zeta| \geq 1} \overline{\Lambda}(\zeta) \eta_v(d\zeta)$. Then $\lim_{t \to \infty} |F_v(t)| \leq \sup_{\zeta \in \Sigma_1} |\overline{\Lambda}(\zeta)| < \infty$, hence $F_v$ is a bounded function. Also, if $g \in G_{\overline{\eta}}$, $|g| = t$:
\[
t^\alpha F_v(t) = \int_{|\zeta| \geq 1} \overline{\Lambda}(\zeta) |g|^\alpha (g^{-1} \eta_v)(d\zeta).
\]
Hence, using the convergence of $|g|^\alpha (g^{-1} \eta_v)$ to $\Delta_v$ if $|g| \to \infty$,
\[
(5.20) \quad t^\alpha F_v(t) = \Delta_v(\overline{\Lambda}) + o(t), \quad \text{as } t \to \infty.
\]
By definition of $F_v$:
\[
\mathbb{E}[\overline{A}_s(Z^* v)] = \int_V |y|^s \overline{\Lambda}(y) \eta_v(dy)
\]
\[
= \int_V \left( \int_{0 < t < |y|} s^{\alpha - 1} dt \right) \overline{\Lambda}(y) \eta_v(dy)
\]
\[
= \int_0^\infty s F_v(t) t^{\alpha - 1} dt.
\]
Let $r$ be any positive increasing function on $(0, \alpha)$ satisfying
\[
(5.21) \quad \lim_{s \to \alpha^{-}} r(s) = +\infty, \quad \lim_{s \to \alpha^{-}} (\alpha - s) r^\alpha(s) = 0, \quad \lim_{s \to \alpha^{-}} r^{s - \alpha}(s) = 1.
\]
One can take for example $r(s) = (\alpha - s)^{-\frac{1}{\alpha}}$. Then to compute the required limit we decompose the integral of $F_v$ above according to the function $r(s)$ and apply (5.20), which gives the asymptotics of $F_v(t)$:
\[
\lim_{s \to \alpha^{-}} (\alpha - s) \int_V |y|^s \overline{\Lambda}(y) \eta_v(dy) = \lim_{s \to \alpha^{-}} (\alpha - s) \int_0^{r(s)} s F_v(t) t^{\alpha - 1} dt
\]
\[
+ \lim_{s \to \alpha^{-}} (\alpha - s) \int_{r(s)}^\infty s \Delta_v(\overline{\Lambda}) t^{-\alpha + s - 1} dt + \lim_{s \to \alpha^{-}} (\alpha - s) \int_{r(s)}^\infty o(t) t^{-\alpha + s - 1} dt.
\]
Notice that the first and third limit are 0. Indeed, by (5.21),

\[
\lim_{s \to \alpha-} \left| (\alpha - s) \int_0^r sF_v(t)t^{s-1}dt \right| \leq \lim_{s \to \alpha-} (\alpha - s)r^s(s) \sup_{t \geq 0} |F_v(t)| = 0.
\]

To compute the third limit take for any \( \varepsilon > 0 \), as observed in Section 1, stability follows from the last assertions in Theorem 5.1 (1, 2 and \( \alpha < s \)).

\[
\lim_{s \to \alpha-} \left| (\alpha - s) \int_0^\infty o(t)t^{-\alpha+s^{-1}}dt \right| \leq \varepsilon \lim_{s \to \alpha-} r^{s-\alpha}(s) = \varepsilon.
\]

Since \( \varepsilon \) was arbitrary we obtain that the limit above is in fact 0. As a result, using (5.21),

\[
C_\alpha(v) = \lim_{s \to \alpha-} (1 - \kappa(s))E[\bar{\Lambda}_s(Z^*v)]
\]

\[
= m_\alpha \cdot \lim_{s \to \alpha-} (\alpha - s) \int_{r(s)}^\infty s\Delta_v(\bar{\Lambda}^1)t^{-\alpha+s^{-1}}dt
\]

\[
= \alpha m_\alpha \Delta_v(\bar{\Lambda}^1).
\]

If \( R_\pi = \langle p \rangle \), the calculation runs parallel, using the formula

\[
E[\bar{\Lambda}_s(Z^*v)] = \int_V \bar{\Lambda}_s(\zeta)\eta_v(d\zeta)
\]

we decompose \( \{ \zeta \in V; |\zeta| > 1 \} \) into shells of the form \( \{ \zeta \in V; p^k \leq \zeta \leq p^{k+1} \} \) and use geometric series instead of the integrals above. Using Theorem 1.4 of [BDGHU] which gives a formula for \( \Delta_v \), we get

\[
C_\alpha(v) = m_\alpha \frac{1 - p^{-\alpha}}{\log p} \Delta_v(\bar{\Lambda}^1).
\]

### 6. Proof of Main Theorem 1.5

To prove the Theorem, in view of the continuity theorem, it is enough to justify that the corresponding characteristic functions converge pointwise to a function, which is continuous at zero. If \( \alpha < 2 \), as observed in Section 1, stability follows from the last assertions in Theorem 5.1 (1, 2 and 3).

#### 6.1. Case \( \alpha < 1 \)

Let \( \phi_n^\alpha \) be the characteristic function of the random variable \( c_nS_n^\alpha \). Then by Lemma 3.5 and Proposition 3.17 we have

\[
\phi_n^\alpha(v) = \mathbb{E} \left[ \chi_v(c_nS_n^\alpha) \right] = (P_{c_n,v}(1))(x) = k^n(c_n, v)\phi_{\pi(c_n,v)(1)}(x) + (Q_{c_n,v}(1))(x).
\]

The second factor tends to 0 as \( n \) goes to infinity, because \( \|Q_{c_n,v}\| < 1 \). Moreover, by Proposition 3.18, \( (\pi(c_n,v)(1))(x) \) converges to 1. Therefore it is enough to compute

\[
\lim_{n \to \infty} k^n(c_n, v) = \lim_{n \to \infty} \left( 1 + k(c_n, v) - 1 \right)^{1 - \frac{1}{n(k(c_n, v) - 1)} - n(k(c_n, v) - 1)} = \lim_{n \to \infty} e^{n \cdot (k(c_n, v) - 1)}
\]

Notice that by (5.2)

\[
\lim_{n \to \infty} n \cdot (k(c_n, v) - 1) = \lim_{n \to \infty} \frac{k(c_n, v) - 1}{|c_n|^\alpha} = C_\alpha(v).
\]

This proves pointwise convergence of \( \phi_n^\alpha \) to \( \Phi_\alpha \). Continuity of \( \Phi_\alpha \) at zero follows easily from the Lebesgue dominated theorem.
6.2. **Case** $\alpha = 1$. Let $\phi_n^1$ be the characteristic function of $c_n S_n^x - n\xi(c_n)$. Then arguing as above we prove that

$$\lim_{n \to \infty} \phi_n^1(v) = \lim_{n \to \infty} \mathbb{E} \left[ \chi_v(c_n S_n^x - n\xi(c_n)) \right] = \lim_{n \to \infty} \left[ \chi_v(-n\xi(c_n))(P_{c_n \cdot v}^n(1))(x) \right] = \lim_{n \to \infty} \left[ \chi_v(-\xi(c_n))k(c_n, v) \right]^n = e^{\lim_{n \to \infty} \left[ n(\chi_v(-\xi(c_n))k(c_n, v) - 1) \right]}$$

Let us compute the limit in the exponent

$$\lim_{n \to \infty} \left[ n(\chi_v(-\xi(c_n))k(c_n, v) - 1) \right] = \lim_{n \to \infty} \left[ \chi_v(-\xi(c_n)) \cdot \frac{k(c_n, v) - 1 - i(v, \xi(c_n))}{|c_n|^2} + n\chi_v(-\xi(c_n))(1 + i(v, \xi(c_n))) - n \right]
= C_1(v) + \lim_{n \to \infty} \left[ n \cdot \left( 1 - i(v, \xi(c_n)) + O\left( |v, \xi(c_n)|^2 \right) \right) \left( 1 + i(v, \xi(c_n)) \right) \right] - n
= C_1(v)$$

To prove continuity of $\Phi_1$ at zero, it is enough to observe

$$g_v(x) = \left( e^{i(v, x)} - 1 \right) \cdot \hat{\eta}_v(x) - \frac{i(v, x)}{1 + |x|^2} = \left( e^{i(v, x)} - 1 \right) \left( \hat{\eta}_v(x) - 1 \right) + e^{i(v, x)} - 1 - \frac{i(v, x)}{1 + |x|^2} \leq C|x|^{1 + \delta},$$

for $|x| < 1$ and some constants $C$ and $\delta > 0$, independent of $v$, and next one can apply the Lebesgue dominated theorem.

6.3. **Case** $1 < \alpha < 2$. Denote by $\phi_n^\alpha$ the characteristic function of $c_n(S_n^x - nm)$. We reason as in previous cases and obtain

$$\lim_{n \to \infty} \phi_n^\alpha(v) = \lim_{n \to \infty} \mathbb{E} \left[ \chi_v(c_n(S_n^x - nm)) \right] = \lim_{n \to \infty} \left[ \chi_v(-c_n m)k(c_n, v) \right]^n = e^{\lim_{n \to \infty} \left[ n(\chi_v(-c_n m)k(c_n, v) - 1) \right]},$$

and we have

$$\lim_{n \to \infty} \left[ n(\chi_v(-c_n m)k(c_n, v) - 1) \right] = \lim_{n \to \infty} \left[ \chi_v(-c_n m) \cdot \frac{k(c_n, v) - 1 - i(v, c_n m)}{|c_n|^2} + n\chi_v(-c_n m)(1 + i(v, c_n m)) - n \right]
= C_\alpha(v) + \lim_{n \to \infty} \left[ n \left( 1 - i(v, c_n m) + O(n^{-\frac{1}{2}}) \right) \left( 1 + i(v, c_n m) \right) \right] - n
= C_\alpha(v).$$

To prove that $\Phi_\alpha$ is continuous at zero and stable, we proceed as before.

6.4. **Case** $\alpha = 2$. Let $\phi_n^2$ be the characteristic function of $c_n(S_n^x - nm)$. Arguing as in previous case we show

$$\log \Phi_2(v) = \log \lim_{n \to \infty} \phi_n^2(v) = \lim_{n \to \infty} \left[ n(\chi_v(-c_n m)k(c_n, v) - 1) \right]
= \lim_{n \to \infty} \left[ n|c_n|^2 \log |c_n| \right] \cdot \left( \frac{k(c_n, v) - 1 - ic_n(v, m)}{|c_n|^2 \log |c_n|} \right) + \lim_{n \to \infty} \left[ n\chi_v(-c_n m)(1 + ic_n(v, m)) - n \right]
= C_2(v).$$
6.5. Case $\alpha > 2$. We argue as in the previous case. Let $\phi_n^{2+}$ the characteristic function of $\frac{1}{\sqrt{n}}(S_n^+-nm)$. Then
\[
\lim_{n \to \infty} \phi_n^{2+}(v) = e^{\lim_{n \to \infty} [n(\chi_v(-m/\sqrt{n})k(1/\sqrt{n},v) - 1)]}
\]
An elementary calculation, using the asymptotics of $k(1/\sqrt{n},v)$ given in Theorem 5.1, 5) proves
\[
\log \Phi_{2+}(v) = \lim_{n \to \infty} \left[ n(\chi_v(-m/\sqrt{n})k(1/\sqrt{n}, v) - 1) \right] = C_{2+}(v) + \frac{1}{2}(v,m)^2.
\]

6.6. Nondegeneracy of the limit law for $0 < \alpha \leq 2$. In order to prove that the limit law is fully nondegenerate (i.e. its support is not contained in some lower dimensional subspace of $V$) it is enough to justify that the function $F_\alpha(v) = \Re \log \Phi_\alpha(v)$, defined on $V$, does not vanishes outside zero. We use the expression of $C_\alpha(v)$ given in Proposition 5.19.

**Proposition 6.1.** For every $v \in V \setminus \{0\}$, $F_\alpha(v) = \Re C_\alpha(v)$ is negative.

**Proof.** If $R_{\Pi}=\mathbb{R}_+^*$, the expression of $C_\alpha(v)$ in Proposition 5.19 gives $F_\alpha(v) = \Delta_v(\Re \Lambda^1)$. The definition of $\Lambda$ gives
\[
\Re \Lambda(y) = \int_{V \setminus \{0\}} (\cos(x,y) - 1) \Lambda(dx).
\]
Using Corollary 2.8 and Lemma 2.9, we know that $\text{supp}\Lambda$ is not contained in a hyperplane. Since $(\cos(x,y) - 1) \leq 0$, we get that for any $y \neq 0$, $\Re \Lambda(y) < 0$. In particular, for any $y$ with $|y| \geq 1$: $\Re \Lambda^1(y) < 0$. Since $\Delta_v$ is $G_{\Pi}$-homogeneous, nonzero and $G_{\Pi}$ is not compact, we have $\text{supp}\Delta_v \cap \{y \in V; |y| \geq 1\} \neq \emptyset$. It follows $\Delta_v(\Re \Lambda^1) < 0$ if $v \neq 0$.

If $R_{\Pi} = \langle p \rangle$, a simple modification of the argument above give the same result. For $\alpha = 2$ we reason analogously. \hfill \Box

6.7. Nondegeneracy of the limit law for $\alpha > 2$. Notice first that if $G_{\Pi} \subset \mathbb{R}_+^*$ then nondegeneracy of the limiting random variable follows immediately from the formula of its characteristic function. Namely we may write

\[
- \log \Phi_{2+}(v) = \frac{1}{2} \int_V \langle x-m,v \rangle^2 \nu(dx) + \int_V \langle x-m,v \rangle \eta_v(x^*) \nu(dx)
\]

\[
= \frac{1}{2} \int_V \langle x-m,v \rangle^2 \nu(dx) + \int_V \langle x-m,v \rangle \mathbb{E} \left[ \sum_{1}^{\infty} |M_1| \cdots |M_k| v \right] \nu(dx)
\]

\[
= \left( \frac{1}{2} + \sum_{n=1}^{\infty} \kappa^n(1) \right) \int_V \langle x-m,v \rangle^2 \nu(dx)
\]

\[
= \frac{1 + \kappa(1)}{2(1 - \kappa(1))} q(v,v),
\]

with $\frac{1 + \kappa(1)}{2(1 - \kappa(1))} > 0$. If the value above were zero, the support of $\nu$ would be contained in some hyperplane of $V$ orthogonal to $v$, but this contradicts to hypothesis H.

In general we cannot use the foregoing argument hence we apply ideas of [GH] (see also [HH1]). Define $\sigma_\alpha^2 = -\log \Phi_{2+}(v)$, i.e. $\sigma_\alpha^2$ is equal to the quadratic form $q(v,v)/2 + q((I-a^*)^{-1}a^* v, v)$. Since $e^{-\sigma_\alpha^2}$ is the characteristic function of a probability measure, $\sigma_\alpha^2 \geq 0$. Given a function $f$ on $V$ and $y \in V$ we define $f_y(x) = f(x-y)$.
**Lemma 6.2.** We have
\[ \sigma_v^2 = \frac{1}{2} \nu((v^*_v)^2) + \nu(v^* \zeta), \]
where \( \zeta \in \mathcal{B}_{\theta, \varepsilon, \lambda} \) is uniquely defined by the equations:
\[ (6.3) \quad \nu(\zeta) = 0, \quad (I - P)(\zeta) = P(v^*_m). \]

**Proof.** Let \( h_t \) be the eigenfunction of \( P_{t,v} \):
\[ (6.4) \quad P_{t,v}(h_t) = k(t, v) h_t \]
such that \( \nu(h_t) = 1 \). The function \( t \mapsto h_t = \frac{\pi_{t,v}(s)}{\nu(\pi_{t,v}(s))} \) is differentiable in \( \mathcal{B}_{\theta, \varepsilon, \lambda} \) for appropriately chosen \( \theta \) and \( \lambda \) (see below). First we prove that \( P_{t,v} \) is differentiable. Let
\[ M_{t,v} f(x) = i \int \chi_{t,v}(gx + b) \langle v, gx + b \rangle f(gx + b) \mu(dh). \]
Then
\[ (6.5) \quad \frac{\left| P_{t+\Delta t,v} f - P_{t,v} f \right|}{\Delta t} - M_{t,v} f \to 0 \quad \text{when} \quad \Delta t \to 0 \]
for \( f \in \mathcal{B}_{\theta', \varepsilon', \lambda'} \) with sufficiently small \( \theta', \varepsilon' \). In particular (6.5) applies to \( h_t \). Using the resolvent we write
\[ \pi_{t,v} = \frac{1}{2\pi i} \int_{|z| = \delta} (z - P_{t,v})^{-1} \, dz \]
and we differentiate \( \pi_{t,v}(e) \). We need to take triples \( (\theta', \varepsilon, \lambda') \) and \( (\theta, \varepsilon, \lambda) \) in the way that not only (6.5) is satisfied but also all the assumptions of section 3 to assure that the resolvent is bounded both on both \( \mathcal{B}_{\theta', \varepsilon, \lambda'} \) and \( \mathcal{B}_{\theta, \varepsilon, \lambda} \). Taking \( \varepsilon \) sufficiently small, \( \lambda' = 5 \varepsilon, \theta' = 9 \varepsilon, \lambda = 1 + 10 \varepsilon, \theta = 1 + 14 \varepsilon \) will do. Clearly, \( h_t \in \mathcal{B}_{\theta', \varepsilon, \lambda'} \). Finally,
\[ h_t - 1 \quad \nu(\pi_{t,v}(e)) \left[ \frac{\pi_{t,v}(e) - 1}{t} + \nu \left( \frac{1 - \pi_{t,v}(e)}{t} \right) \right] \]
and so \( \lim_{t \to \infty} \frac{h_t - 1}{t} = \zeta \) exists.

We apply \( \nu \) to both sides of (6.4) and we obtain
\[ (6.6) \quad \nu(\chi_{t,v} h_t) = k(t, v). \]
Next differentiating the equation \( \nu(h_t) = 1 \) with respect to \( t \) at 0 we obtain \( \nu(\zeta) = 0 \). Computing the second order term of asymptotic expansion of \( k(t, v) \), in view of Theorem 5.1 we have
\[
-\sigma_v^2 - \frac{1}{2}(v, m)^2 = \lim_{t \to 0} \frac{k(t, v) - 1 - it(v, m)}{t^2} = \lim_{t \to 0} \frac{\nu(\chi_{t,v} h_t) - 1 - it(v, m)}{t^2} = \lim_{t \to 0} \nu \left( \frac{(1 + itv^* - t^2(v^*)^2)h_t - 1 - itv^*}{t^2} \right) = -\frac{1}{2} \nu((v^*)^2) + \nu(v^* \cdot h_t - 1) = -\frac{1}{2} \nu((v^*)^2) - \nu(v^* \zeta),
\]
that gives the required formula for \( \sigma_v^2 \). To prove that the function \( \zeta \) satisfies the Poisson equation (6.3) we differentiate (6.4) at zero, i.e. applying Theorem 5.1 we write
\[
\lim_{t \to 0} \frac{k(t, v) h_t - 1}{t} = \lim_{t \to 0} \left[ \frac{k(t, v) - 1}{t} \cdot h_t + \frac{h_t - 1}{t} \right] = i((v, m) + \zeta).
\]
On the other side we obtain
\[
\lim_{t \to 0} \frac{P_{t,v}(h_t) - 1}{t} = \lim_{t \to 0} \left[ \frac{(P_{t,v} - P)(h_t)}{t} + P\left(\frac{h_t - 1}{t}\right) \right] = i(P(\nu^*) + P(\zeta)).
\]
Comparing both equations we prove (6.3).

Finally, in order to prove that $\zeta$ is uniquely determined by these two conditions, assume that some $\zeta_1$ satisfies $\nu(\zeta_1) = 0$ and $P(v^*_m) = (I - P)(\zeta_1)$, then $(I - P)(\zeta - \zeta_1) = 0$, that implies $\zeta = \zeta_1 + C$. Since $\nu(\zeta) = \nu(\zeta_1)$, we get $\zeta = \zeta_1$.

**Lemma 6.7.** Let $u_0$ be the unique solution of $(I - P)u_0 = v^*_m$, $\nu(u_0) = 0$. Then $2\sigma_v^2 = E_\nu \left[ u_0(X_1) - Pu_0(X_0) \right]^2$. In particular, if $\sigma_v^2 = 0$ then $r(P_{t,v}) = 1$

*Proof.* Since $I - P$ is invertible on the space $\{ g : \langle \nu, g \rangle = 0 \}$, the system of equations $(\nu, f) = 0$ and $(I - P)f = g$, $f \in \mathbb{B}_{\theta, r, \lambda}$ has a unique solution for $g$ such that $\langle \nu, g \rangle = 0$. Therefore, equation $(I - P)f = v^*_m$ has unique solution satisfying $\nu(f) = 0$, and we denote this solution by $u_0$. Then $\nu(u_0^2) < \infty$. Indeed, the function $u_0$ belongs to $\mathbb{B}_{\theta, r, \lambda}$, therefore $|u_0(x)|^2 \leq C(1 + |x|)^{2+2r}$ and by (2.1) $u_0^2$ is integrable with respect to $\nu$.

Notice that $\zeta = Pu_0$. Indeed, it is enough to prove that $Pu_0$ satisfies (6.3). For this purpose we write
\[
(I - P)(Pu_0) = (I - P)(u_0 - v^*_m) = (I - P)u_0 - v^*_m + P(v^*_m) = P(v^*_m),
\]
and
\[
\nu P(u_0) = \nu (u_0) = 0.
\]

Next we write
\[
2\sigma_v^2 = \nu((v^*_m)^2) + 2\nu(v^*_m \zeta) = \nu((v^*_m)^2) + 2v^*_mPu_0 = \langle \nu, (u_0 - Pu_0)(u_0 + Pu_0) \rangle
\]
\[
= \int_H \int_V \left( u_0^2(h \cdot x) - (Pu_0)^2(x) \right) \nu(dx) \mu(dh) = \int_H \int_V \left( u_0(h \cdot x) - (Pu_0)(x) \right)^2 \nu(dx) \mu(dh)
\]
\[
= E_\nu \left[ u_0(X_1) - Pu_0(X_0) \right]^2 = E_\nu \left[ v^*_m(X_1) + Pu_0(X_1) - Pu_0(X_0) \right]^2
\]
If $\sigma_v^2 = 0$, then
\[
v^*_m(X_1) = Pu_0(X_0) - Pu_0(X_1), \quad P_\nu \text{ a.s.}
\]
and
\[
e^{it\langle v, x \rangle}e^{itPu_0(x)} = e^{it\langle v, m \rangle}e^{itPu_0(x)},
\]
hence, taking the expected value, we have
\[
P_{t,v}(e^{itPu_0}) = \int_H e^{it\langle v, h \cdot x \rangle}e^{itPu_0(h \cdot x)} \mu(dh) = e^{it\langle v, m \rangle}e^{itPu_0(x)},
\]
that proves $r(P_{t,v}) = 1$.

Nondegeneracy of the limit follows immediately from Lemmas 3.14 and 6.7.

7. Proof of Theorem 1.7

In order to prove Theorem 1.7 we proceed as in the Euclidean case. However, now we have to handle with general dilations of $V$, that requires some additional arguments. We omit these parts of the proof that are similar in both cases. The crucial step is to describe asymptotic expansion of $k(c, v)$ as $c$ goes to 0.
7.1. Asymptotic expansion of $k(c,v)$ for $0 < \alpha < 2$.

**Proposition 7.1.**  
(1) If $0 < \alpha < 1$ then

$$
\lim_{|c| \to 0, c \in \mathbb{Z}_+} \frac{k(c,v) - 1}{|c|^\alpha} = C_\alpha(v)
$$

where

$$
C_\alpha(v) = \int_V \left( (\chi_v(x) - 1) \cdot \tilde{\eta}_v(x) - i \langle v, x_{\alpha,-} \rangle - \frac{i \langle v, x_{\alpha} \rangle}{1 + |x_{\alpha}|^2} \right) \Lambda(dx)
$$

In particular $C_\alpha(c^*v) = |c|^\alpha C_\alpha(v)$, if $c \in \mathbb{Z}_+^*$.  

(2) Assume $1 \leq \alpha < 2$. Let $\xi_1(c) = cm_{\alpha,-} + \int_V \frac{c^2}{1 + |x_{\alpha}|^2} \nu(dx)$. Then

$$
\lim_{|c| \to 0, c \in \mathbb{Z}_+} \frac{k(c,v) - 1 - i \langle v, \xi_2(c) \rangle}{|c|^\alpha} = C_\alpha(v),
$$

where

$$
C_\alpha(v) = \int_V \left( (\chi_v(x) - 1) \cdot \tilde{\eta}_v(x) - i \langle v, x_{\alpha,-} \rangle - \frac{i \langle v, x_{\alpha} \rangle}{1 + |x_{\alpha}|^2} \right) \Lambda(dx)
$$

and $\lim_{c \to 0} |c|^{-1} |\xi(c)| = m_1$ for $m_1 = \int_V x_{\alpha} \nu(dx)$. In particular, if $c \in \mathbb{Z}_+^*$, $C_\alpha(c^*v) = |c|^\alpha C_\alpha(v) + i \langle v, \beta(c) \rangle$, with $\beta(c) = \int_V \left( \frac{\tilde{\eta}_v}{1 + |x_{\alpha}|^2} - \frac{\tilde{\eta}_v}{1 + |x_{\alpha}|^2} \right) \Lambda(dx)$.

**Proof.** For $\alpha < 1$ the proof is exactly the same as in section 5. Assume $1 \leq \alpha < 2$. First we will prove that

$$
(7.2) \lim_{|c| \to 0} \frac{1}{|c|^\alpha} \left( \langle \nu, (\chi_{c^*v} - 1) \psi_{c,v} \rangle - i \langle v, \xi_1(c) \rangle \right) = C_\alpha(v).
$$

For this purpose we decompose $V = V_{\alpha,-} \oplus V_\alpha \oplus V_{\alpha,+}$ and write $x = x_{\alpha,-} + x_{\alpha} + x_{\alpha,+}$. Then

$$
\int_V (\chi_v(cx) - 1) \psi_{c,v}(x) \nu(dx) = \int_V (\chi_v(cx) - 1) (\psi_{c,v}(x) - \tilde{\eta}_v(cx)) \nu(dx)
$$

$$
+ \int_V (\chi_v(cx) - 1) (\tilde{\eta}_v(cx) - 1) \nu(dx) + \int_V (\chi_v(cx) - 1 - i \langle v, cx_{\alpha,-} \rangle - \frac{i \langle v, cx_{\alpha} \rangle}{1 + |cx_{\alpha}|^2} ) \nu(dx)
$$

To handle the first and the second integrals we use the same arguments as in the proof of Theorem (5.1), i.e. the first one converges to 0 as $c$ goes to 0, and the second one tends to

$$
\int_V (\chi_v(x) - 1) (\tilde{\eta}_v(x) - 1) \nu(dx)
$$

For the third one we are going to prove that

$$
f(x) = \chi_v(x) - 1 - i \langle v, x_{\alpha,-} \rangle - \frac{i \langle v, x_{\alpha} \rangle}{1 + |x_{\alpha}|^2}
$$

satisfies (2.3). Let $D_+ = \min\{\lambda_j : \lambda_j > \alpha\}$, $D_- = \max\{\lambda_j : \lambda_j < \alpha\}$, $x_{\alpha} = C \tau(x) x_{\alpha}$, $x_{\alpha} = C \tau(x) x_{\alpha}$, $x_{\alpha} = C \tau(x) x_{\alpha}$, $x_{\alpha} = C \tau(x) x_{\alpha}$, $x_{\alpha} = C \tau(x) x_{\alpha}$. Then $\tau(x) \geq 1$ we have $|f(x)| \leq C(1 + \sum_{\lambda_j < \alpha} |x_{\lambda_j}|) \leq C \tau(x) D_-$. Then for $\tau(x) \leq 1$ we have $\sum_{\lambda_j > \alpha} |x_{\lambda_j}| \leq C \tau(x) D_+$, $|x_{\alpha}| \leq C \tau(x)^\alpha$ and we obtain

$$
|f(x)| \leq |e^{i \langle v, x \rangle} - 1 - i \langle v, x \rangle| + |\langle v, x_{\alpha,+} \rangle| + |\langle v, x_{\alpha} \rangle| \cdot \left| 1 - \frac{1}{1 + |x_{\alpha}|^2} \right|
$$

$$
\leq C \tau(x)^2 + C \tau(x)^{D_+} + C \tau(x)^\alpha \cdot |x_{\alpha}|^2 \leq C \tau(x)^{\min(2,D_+)}.
$$
Hence (2.2) implies (7.2). One can easily prove that for \(|c| < 1/2\) we have \(|\int_V |c|^\alpha \nu(dx)| < |c|^\alpha \log |c|\) (compare proof of Lemma 5.5) and \(\lim_{|c| \to 0} \frac{\xi_1(c)}{|c|} = m_1\). Finally by (4.4) and Lemma 4.9 we have

\[
\lim_{|c| \to 0} \frac{k(c,v) - 1 - i\langle v, \xi_1(c) \rangle}{|c|^{\alpha}} = \lim_{|c| \to 0} \left( \frac{\nu(\psi_{c,v})(k(c,v) - 1) - i\langle v, \xi_1(c) \rangle}{\nu(\psi_{c,v})|c|^{\alpha}} + \frac{i\langle v, \xi_1(a) \rangle(1 - \nu(\psi_{c,v}))}{\nu(\psi_{c,v})|c|^{\alpha}} \right) = C_\alpha(v)
\]

\[\square\]

7.2. Asymptotic expansion of \(k(c,v)\) for \(\alpha > 2\). In order to get fully nondegenerate laws, we have to normalize \(S^x_n\) in inhomogeneous way. Let

\[
V_- = V_\llangle \llcorner = \bigoplus \lambda_j < V_\lambda_j, \\
V_+ = V_\llcorner = \bigoplus \lambda_j > V_\lambda_j.
\]

We assume that \(V_\llangle = \{0\}\) and so \(V = V_- \oplus V_+\). For \(x \in V\) we write \(x = x_- + x_+\), where \(x_- \in V_-\), \(x_+ \in V_+\). Let \(c_n \in \mathbb{Z}_n^n\) be such that

\[
|c_n| = \sup_{\tau(x) \leq 1} \tau(c_n x) = \frac{1}{n^\alpha}.
\]

The right normalization in the case \(\alpha > 2\) is

\[
\frac{1}{\sqrt{n}} (S^x_n - nm)_- + (c_n S^x_n - d_n)_+.
\]

We need to modify accordingly the operators \(P_{c,v}\) and \(T_{c,v}\) and so we consider the following linear transformations:

\[
b_n(x) = \frac{1}{\sqrt{n}} x_-,
\]

\[
c_n(x) = c_n(x_+),
\]

and

\[
a_n(x) = b_n(x) + c_n(x).
\]

Notice that the operators \(P_{b_n,v}\) and \(T_{b_n,v}\) are defined both on \(B_{\theta,\epsilon,\lambda}(V)\) and \(B_{\theta,\epsilon,\lambda}(V_-)\). If \(f(x) = f(x_-)\) then \(f \in B_{\theta,\epsilon,\lambda}(V_-)\) and \(f \in B_{\theta,\epsilon,\lambda}(V)\) if and only if \(f \in B_{\theta,\epsilon,\lambda}(V_-)\) and

\[
P_{b_n,v}f(x) = P_{b_n,v} f(x_-),
\]

\[
T_{b_n,v}f(x) = T_{b_n,v} f(x_-).
\]

The same holds for \(P_{c_n,v}\), \(T_{c_n,v}\) and functions \(f\) depending only on \(x_+\). Therefore, it is convenient to refer to operators \(P_{b_n,v}, P_{c_n,v}, T_{b_n,v}, T_{c_n,v}\) while they act on \(B_{\theta,\epsilon,\lambda}(V)\) and to \(P_{b_n,v_-}, T_{b_n,v_-}, P_{c_n,v_+}, T_{c_n,v_+}\) when they are considered on \(B_{\theta,\epsilon,\lambda}(V_-)\) or \(B_{\theta,\epsilon,\lambda}(V_+)\). Clearly, the peripheral eigenvalues \(k(b_n, v_-), k(c_n, v_+)\) of \(P_{b_n,v_-}\) and \(P_{c_n,v_+}\) are equal to the peripheral eigenvalues \(k(b_n, v), k(c_n, v)\) of \(P_{b_n,v}\) and \(P_{c_n,v}\), respectively. They are closely related to \(k(a_n, v)\). Indeed, we are going to prove that

\[
\lim_{n \to \infty} n(k(a_n, v) - k(b_n, v) - k(c_n, v) + 1) = 0.
\]

Moreover we describe the asymptotic behavior of \(k(b_n, v_-)\) and \(k(c_n, v_+)\) restricting our attention to \(B_{\theta,\epsilon,\lambda}(V_-)\) and \(B_{\theta,\epsilon,\lambda}(V_+)\).

The content of Section 3 is needed for both operators \(P_{b_n,v_-}\) and \(P_{c_n,v_+}\). The inhomogeneous dilations in \(P_{b_n,v_-}\) imply a slight modification of Propositions 3.18 and 3.20. They hold with \(|b_n|^\alpha = \left(\frac{1}{\sqrt{n}}\right)^\alpha\) instead of \(|c|^\alpha\), where \(\lambda_{b_0} = \max\{\lambda_j : \lambda_j < \frac{\theta}{2}\}\).
From now on we assume not only that: \( \varepsilon < 1, \theta \leq 2\lambda, \lambda + 3\varepsilon < \theta, 2\lambda + 3\varepsilon < \alpha \), but also \( \lambda > \lambda_{k_0}, \frac{\theta}{2} < \lambda + 2\varepsilon < \lambda_{k_0+1} \) (notice that \( \lambda_{k_0+1} = \min \{ \lambda_j : \lambda_j > \frac{\theta}{2} \} \)).

To study \( P_{c_n, \nu} \) we need a further decomposition of \( V_+ \), i.e.
\[
V_+ = V_{\alpha} \oplus V_0 \oplus V_{\alpha,+}
\]
and for \( x_+ \in V_+ \) we write \( x_+ = u + u_+ + u_{\alpha} + u_+ \). Let \( \nu_-, \nu_+ \) be projections of the Poisson kernel \( \nu \) on \( V_- \) and \( V_+ \) and let \( \Lambda_+ \) be the tail measure (1.3) for \( \nu_+ \).

**Proposition 7.3.** If \( \alpha > 2 \), then
\[
(7.4) \quad \lim_{n \to \infty} n(k(b_n, v) - 1 - i(b_n(v), m_0)) = C_-(v),
\]
where
\[
C_-(v) = -\frac{1}{2}q(v_-, v_-) - \frac{1}{2}(v_-, m_0)^2 - q(v_-, (I - z_+)^{-1}z_+ v_-),
\]
\[
m_0 = m_{\nu,-} = \int_{V_-} x\nu_-(dx) \text{ and } z_- = E[M_{\nu}].
\]

Moreover
\[
(7.5) \quad \lim_{n \to \infty} n(k(c_n, v) - 1 - i(v, \xi_2(c_n))) = C_+(v),
\]
where
\[
C_+(v) = \int_{V_+} \left( (\chi_+(u) - 1) \cdot \eta_+(u) - i(v, u_-) - \frac{i(v, u_{\alpha})}{1 + |u_\alpha|^2} \right) \Lambda_+(du),
\]
\[
\xi_2(c_n) = \int_{V_+} \left( c_n u_- + \frac{c_n u_{\alpha}}{1 + |c_n u_{\alpha}|^2} \right) \nu_+(du)
\]
and if \( V_{\alpha} \neq \{0\} \), then
\[
\lim_{n \to \infty} \frac{\xi_2(c_n)}{|c_n|^{\lambda_{k_0+1}}} = \int_{V_+} u_{\lambda_{k_0+1}} \nu_+(du).
\]

**Proof.** To prove (7.4) we consider \( P_{b_n, \nu_-} \) and we proceed as in the proof of Theorem 5.1 (5). The crucial estimate is
\[
(7.6) \quad |x_-| \leq (1 + \tau(x_-))^{\lambda_{k_0}},
\]
where \( |\cdot| \) is the Euclidean norm on \( V_- \), which implies
\[
(7.7) \quad \int_{V_-} |x_-|^2 \nu(dx) < \infty
\]
and so the integrals \( \int_{V_-} \langle v_-, x_- \rangle^2 \nu(dx), \int_{V_-} |\langle v_-, x_- \rangle| \eta_{\nu_-}(x_-) \nu(dx), \int_{V_-} |\eta_{\nu_-}(x_-)| \nu(dx) \) are finite. Moreover in view of (7.6) we have
\[
\|\chi_{x_-} - 1 - ix_- \|_{\theta, \varepsilon, \lambda} \leq C|x_-|^{1+\varepsilon-\varepsilon^2},
\]
which is also needed.

To prove (7.5) we proceed as in Proposition 7.1, that is we prove
\[
(7.8) \quad \lim_{n \to \infty} \frac{1}{|c_n|^3} \int_{V_+} (\chi_{c_n}(c_n u) - 1)(\psi_{c_n}(u) - \eta_{c_n}(c_n u)) \nu_2(du) = 0,
\]
\[
(7.9) \quad \lim_{n \to \infty} \frac{1}{|c_n|^3} \int_{V_+} (\chi_{c_n}(c_n u) - 1)(\eta_{c_n}(c_n u) - 1) \nu_2(du) = \int_{V_+} (\chi_{c_n}(u) - 1)(\eta_{c_n}(u) - 1) \Lambda_2(du),
\]
\[
(7.10) \quad \lim_{n \to \infty} \frac{1}{|c_n|^\alpha} \int_{V_n} \left( \chi_{v_+}(c_n u) - 1 - i\langle v_+, c_n u \rangle - \frac{i\langle v, c_n u \rangle}{1 + |c_n u|^2} \right) \nu_2(du)
\]

\[
= \int_{V_+} \left( \chi_{v_+}(u) - 1 - i\langle v_+, u \rangle - \frac{i\langle v, u \rangle}{1 + |u|^2} \right) \Lambda_2(du).
\]

For (7.8) we need
\[
|\psi_{\xi_n,v_+}(u)| \leq C|c_n|^\tau \tau(c_n u)^{\lambda + \varepsilon} \quad \text{if } \tau(c_n u) \leq 1
\]
\[
|\psi_{\xi_n,v_+}(u)| \leq |c_n|^\tau \tau(c_n u)^\varepsilon, \quad \text{if } \tau(c_n u) \geq 1
\]

(7.12) was proved in Lemma (4.5) and (7.11) follows from (7.14) below. Moreover, the assumption $\lambda + 2\varepsilon > \frac{\alpha}{2}$ guarantees $\lambda_{k_0} + 2\varepsilon + \lambda > \alpha$, that is used in the calculations.

The function in (7.9) satisfies 2.3, because
\[
|\chi_{v_+}(u) - 1| \leq \max(1, |u|^2) \leq \max(1, \tau(u_+)^{2\lambda_{k_0} + 1})
\]
and $\lambda < 2\lambda_{k_0} + 1$. Therefore, (7.9) follows. Finally, $\chi_{v_+}(u) - 1 - i\langle v_+, u \rangle - \frac{i\langle v, u \rangle}{1 + |u|^2}$ is estimated as in the proof of Proposition 7.1

In order to compare $k(a_n, v)$ with $k(b_n, v)$ and $k(c_n, v)$ we need two technical lemmas.

**Lemma 7.13.** For every $s \leq \lambda_{k_0} + 1$

\[
(7.14) \quad |\chi_g(a_n x) - 1| \leq C \left( \frac{1}{\sqrt{n}} |x_-| + \frac{1}{n^{\tau}} \tau(x_+)^\tau(y_+)^s \right).
\]

Moreover
\[
(7.15) \quad \|\chi_{a_n x} - 1\|_{\theta, \varepsilon, \lambda} \leq C \left( \frac{1}{\sqrt{n}} |x_-| + \frac{1}{n^{\tau}} \tau(x_+)^{\lambda + \varepsilon} \right)
\]

**Proof.** We use the following inequality
\[
(7.16) \quad |e^{i\sum_{j=1}^m \alpha_j} - e^{i\sum_{j=1}^m \beta_j}| \leq \sum_{j=1}^m |e^{i\alpha_j} - e^{i\beta_j}|,
\]
which holds for real $\alpha_j, \beta_j, 1 \leq j \leq m$.

In view of (7.16) we have
\[
|\chi_g(a_n x) - 1| \leq |e^{i(y_\beta_n x)} - 1| + |e^{i(y_{\xi_n} x)} - 1|
\]
\[
\leq |y_-||b_n x_-| + \sum_{j > k_0} \left| e^{i(y_{\lambda_j} c_n x_{\lambda_j})} - 1 \right|
\]
\[
\leq \frac{1}{\sqrt{n}} |y_-| |x_-| + \sum_{j > k_0} \left( |y_{\lambda_j}||c_n x_{\lambda_j}| \right)^{\frac{\varepsilon}{\tau}}
\]
\[
\leq \frac{1}{\sqrt{n}} |y_-| |x_-| + \frac{1}{n^{\tau}} \tau(y_+)^\tau(x_+)^s,
\]
and (7.14) follows.

For (7.15) we have
\[
|y_-| \leq (1 + \tau(y_-))^{\lambda_{k_0}} \leq (1 + \tau(y_-))^\theta
\]
and
\[
\tau(y_+)^{\lambda + \varepsilon} \leq (1 + \tau(y_+))^\theta,
\]
hence we obtain the required estimate for $|\chi_{a_n x} - 1|_{\theta}$. Applying (7.16) again we have
Proposition 7.21. We have

\[ \lim_{n \to \infty} n(k(a_n, v) - k(b_n, v) - k(c_n, v) + 1) = 0 \]
Proof. By (4.8) we have
\[
k(a_n, v) - k(b_n, v) - k(c_n, v) + 1
= \frac{1}{\nu(\psi_{a_n,v})}\nu\left(\psi_{a_n,v}(\chi_{a_n^*v} - 1)\right) - \frac{1}{\nu(\psi_{b_n,v})}\nu\left(\psi_{b_n,v}(\chi_{b_n^*v} - 1)\right) - \frac{1}{\nu(\psi_{c_n,v})}\nu\left(\psi_{c_n,v}(\chi_{c_n^*v} - 1)\right)
= I_1 + I_2 + I_3,
\]
where
\[
I_1 = \frac{1}{\nu(\psi_{a_n,v})}\nu\left(\psi_{a_n,v}(\chi_{b_n^*v} - 1)(\chi_{c_n^*v} - 1)\right),
I_2 = \frac{1}{\nu(\psi_{a_n,v})}\nu\left(\psi_{a_n,v}(\chi_{b_n^*v} - 1)\right) - \frac{1}{\nu(\psi_{b_n,v})}\nu\left(\psi_{b_n,v}(\chi_{b_n^*v} - 1)\right),
I_3 = \frac{1}{\nu(\psi_{a_n,v})}\nu\left(\psi_{a_n,v}(\chi_{c_n^*v} - 1)\right) - \frac{1}{\nu(\psi_{c_n,v})}\nu\left(\psi_{c_n,v}(\chi_{c_n^*v} - 1)\right).
\]
Applying Lemma 7.14 with \(s = \lambda + 2\varepsilon\) we have
\[
|I_1| \leq \int_V \frac{1}{\sqrt{n}} |x_-| \frac{1}{n} \tau(x_+)^s \nu(dx)
\leq \frac{1}{n^{\frac{1}{2} + \frac{3}{2}}} \int_V (1 + \tau(x)^{\lambda_{k_0} + \lambda + 2\varepsilon}) \nu(dx)
= o\left(\frac{1}{n}\right),
\]
because \(\frac{\gamma}{\alpha} > \frac{1}{2}\) and \(\lambda_{k_0} + \lambda + 2\varepsilon < 2\lambda + 2\varepsilon < \alpha\).

For \(I_2\) we have
\[
I_2 = \frac{1}{\nu(\psi_{a_n,v})}\nu\left((\psi_{a_n,v} - \psi_{b_n,v})(\chi_{b_n^*v} - 1)\right) + \frac{\nu(\psi_{b_n,v} - \psi_{a_n,v})}{\nu(\psi_{a_n,v})\nu(\psi_{b_n,v})}\nu\left(\psi_{b_n,v}(\chi_{b_n^*v} - 1)\right).
\]
Therefore by Lemmas 7.14 and 7.18
\[
|I_2| \leq \frac{1}{n^{\frac{1}{2} + \frac{3}{2}}} \left(\int_V (1 + \tau(x)^{\lambda + 2\varepsilon}) |x_-| \nu(dx) + \int_V (1 + \tau(x)^{\lambda + 2\varepsilon}) \nu(dx) \cdot \int_V |x_-| \nu(dx)\right)
= o\left(\frac{1}{n}\right),
\]
because \(\lambda + 2\varepsilon > \frac{\gamma}{\alpha}\) and \(|x_-| \leq (1 + \tau(x_-))^{\lambda_{k_0}}\). For \(I_3\) we have
\[
I_3 = \frac{1}{\nu(\psi_{a_n,v})}\nu\left((\psi_{a_n,v} - \psi_{c_n,v})(\chi_{c_n^*v} - 1)\right) + \frac{\nu(\psi_{c_n,v} - \psi_{a_n,v})}{\nu(\psi_{a_n,v})\nu(\psi_{c_n,v})}\nu\left(\psi_{c_n,v}(\chi_{c_n^*v} - 1)\right)
\]
and so by Lemmas 7.14 and 7.18, with \(s = \lambda + 2\varepsilon\)
\[
|I_3| \leq \frac{1}{n^{\frac{1}{2} + \frac{3}{2}}} \left(\int_V (1 + \tau(x)^{\lambda + \varepsilon}) \tau(x)^{\lambda + 2\varepsilon} \nu(dx) + \int_V (1 + \tau(x)^{\lambda + \varepsilon}) \nu(dx) \int_V \tau(x)^{\lambda + 2\varepsilon} \nu(dx)\right)
= o\left(\frac{1}{n}\right),
\]
because \(2\lambda + 3\varepsilon < \alpha\) and both integrals are finite.\(\square\)

Proof of Theorem 1.7. For \(\alpha < 2\) the Theorem is an immediate consequence of Proposition 7.1. To prove existence of limit of appropriately normed sums \(S_n\) we proceed exactly as in paragraphs 6.1 - 6.5. Also continuity at 0 of the characteristic function and stability require only a repetition of the previous arguments, that will be omitted. Finally we have to justify nondegeneracy of the limiting random variable. For this purpose take \(v \in V_\gamma\) for some nonempty subspace \(V_\gamma\) of \(V\). Notice that
\[(c_n S_n^\pi - d_n, v) = (\pi^\gamma(c_n S_n^\pi - d_n), v),\] where \(\pi^\gamma\) denotes projection onto \(V_\gamma\). Then \(\pi^\gamma(S_n^\pi)\) are partial sums of \(X_n^\gamma\) defined by the recursion \(X_n^\gamma = M_n^\gamma X_{n-1}^\gamma + \pi^\gamma(Q_n),\) where \(M_n^\gamma\) is the restriction of the action of \(M_n\) to \(V_\gamma\) (it is well defined, because \(V_\gamma\) is invariant under the action of \(G_\gamma\)). The law \(\mu_\gamma\) of \((\pi^\gamma(Q_n), M_n^\gamma)\) is the projection of \(\mu\) on \(\pi^\gamma(G_\gamma)\) under the natural homomorphism. Since \(\mu\) doesn’t admit invariant affine subspaces, there is \(\gamma\) such that there is no affine subspace invariant under \(\pi^\gamma(G_\gamma)\). Then we have to study \(\pi^\gamma(S_n)\) on \(V_\gamma\) i.e. we reduce the problem in fact to the Euclidean settings and last part of Theorem 1.5 implies that the limit \((\pi^\gamma(c_n S_n^\pi - d_n), v)\) is nonzero.

If \(\alpha > 2\) we proceed as previously, however for the reader convenience and to underline the role of Proposition 7.21 we will present part of the proof in more details.

Let \(\phi_n\) be the characteristic function of
\[b_n(S_n^\pi - nm)\mathbb{1}_{\mathbb{R}_+} - (c_n S_n^\pi - d_n)\mathbb{1}_{\mathbb{R}_+} = a_n S_n^\pi - nb_n m_0 - d_n.\]

Then
\[
\lim_{n \to \infty} \phi_n(v) = \lim_{n \to \infty} E\left[\chi_v(a_n S_n^\pi - nb_n m_0 - n\xi_2(c_n))\right] \\
= \lim_{n \to \infty} \left[\chi_v(-b_n m_0 - \xi_2(c_n)) k(a_n, v)\right]^n \\
= e^{\lim_{n \to \infty} \left[n(\chi_v(-b_n m_0 - \xi_2(c_n)) k(a_n, v) - 1]\right]}. 
\]

Applying Propositions 7.3 and 7.21 we obtain
\[
\log \left(\lim_{n \to \infty} \phi_n(v)\right) = \lim_{n \to \infty} \left[n \chi_v(-b_n m_0 - \xi_2(c_n)) \left(k(a_n, v) - 1 - i(v, b_n m_0) - i(v, \xi_2(c_n))\right) + n \chi_v(-b_n m_0 - \xi_2(c_n)) \left(1 + i(v, b_n m_0) + i(v, \xi_2(c_n))\right) - n\right] \\
= \lim_{n \to \infty} n \left(k(b_n, v) - 1 - i(b_n v, m_0)\right) + \lim_{n \to \infty} n \left(k(c_n, v) - 1 - i(v, \xi_2(c_n))\right) \\
+ \lim_{n \to \infty} \left[n \chi_v(-b_n m_0 - \xi_2(c_n)) \left(1 + i(v, b_n m_0) + i(v, \xi_2(c_n))\right) - n\right] \\
= C_-(v) + C_+(v) + \frac{1}{2}(v, m_0)^2.
\]

To prove nondegeneracy of the limit we use exactly the same argument as above. \(\square\)

8. Local Limit Theorem

Proof of Theorem 1.6. In this section we will study the Euclidean case and assume \(\alpha \notin \{1, 2\}\). Take \(x = 0\). In view of Theorem 10.7 \([Br]\) it is enough to prove that
\[
\lim_{n \to \infty} n^x E[h(S_n - d_n)] = p_\alpha(0) \int_e h(v)dv,
\]
for every function \(h \in L^1\) such that the Fourier transform of \(h\) is compactly supported. By Propositions (3.17), (3.18) and Lemma 3.19, using the Fourier inversion formula
\[
E[h(S_n^\pi - d_n)] = \frac{1}{(2\pi)^d} \int_V \mathbb{E}[e^{i(v, S_n^\pi - d_n)}] \hat{h}(v)dv = \frac{1}{(2\pi)^d} \int_V e^{-i(v, d_n)}\mathbb{P}_n^{(1)}(0)\hat{h}(v)dv.
\]
Take \(N = [-\delta, \delta]^d\) and denote by \(J\) the support of \(\hat{h}\). By Lemma 3.14, \(r(P_v) < 1\), if \(v \neq 0\) hence using Lemma 3.19 with \(f = 1\) there exists \(\beta > 0\) such that for \(v \in J \setminus N\): \(r(P_v) < 1 - \beta\). Therefore
\[
\lim_{n \to \infty} n^x \left|\int_{J \setminus N} e^{-i(v, d_n)}(\mathbb{P}_n^{(1)}(0))\hat{h}(v)dv\right| \leq \lim_{n \to \infty} Cn^x(1 - \beta)^n = 0.
\]
Hence we have reduced the problem to computing the limit
\[
\lim_{n \to \infty} n^\chi \int_{\mathbb{R}^d} e^{-i\langle v, d_n \rangle} \left( P_n^\chi(1)(0) \hat{h}(v) \right) dv = \lim_{n \to \infty} \frac{n^\chi}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle v, d_n \rangle} \left[ k_n(v)\pi_n(1)(0) + Q_n^\chi(1)(0) \right] \hat{h}(v) dv
\]
Next by Proposition 3.17 there exists \( \gamma > 0 \) such that \( \|Q_v\| \leq 1 - \gamma \) for \( v \in N \) hence
\[
\lim_{n \to \infty} n^\chi \int_{\mathbb{R}^d} e^{-i\langle v, d_n \rangle} (Q_n^\chi(1)(0) \hat{h}(v)) dv \leq \lim_{n \to \infty} Cn^\chi(1 - \gamma)^n = 0.
\]
To handle with the remaining term we take a similarity \( c_n \) such that \( |c_n| = n^{-\frac{\chi}{2}} \) change variables \( v \mapsto c_nv \) and obtain
\[
(8.1) \quad \lim_{n \to \infty} n^\chi \int_{\mathbb{R}^d} e^{-i\langle v, d_n \rangle} k_n(v)\pi_n(1)(0) \hat{h}(v) dv
\]
\[
= \lim_{n \to \infty} \int_{|v| < \delta n^\chi} \left( e^{-i\langle c_nv, m \rangle} k(c_nv) \right)^n \pi_{c_nv}(1)(0) \hat{h}(c_nv) dv,
\]
where \( m = 0 \) if \( \alpha < 1 \). Now we are going to use the Lebesgue theorem. For this purpose we need that for every \( v \in V \) there are \( \delta > 0 \) and \( D > 0 \) such that
\[
(8.2) \quad |e^{i\langle v, m \rangle} k_v(v)| \leq e^{-D|v|^\chi}.
\]
Indeed assume first \( \alpha > 2 \). Then by Theorem 5.1, for small values of \( |v| \) we have
\[
e^{-i\langle v, m \rangle} k(v) = \left( 1 - i\langle v, m \rangle - \frac{1}{2} \left( v, m \right)^2 + o(|v|) \right) \cdot \left( 1 + i\langle v, m \rangle + (C_{2+}(v) + o(|v|)) \right)
\]
\[
= 1 + C_{2+}(v) - \frac{1}{2} (v, u)^2 + o(|v|)
\]
Moreover nondegeneracy of the limit in Theorem 1.5 implies \( C_{2+}(v) - \frac{1}{2} (v, m)^2 < 0 \), that gives (8.2) in this case.
If \( \alpha < 2 \), then by Theorem 5.1
\[
k(v) = 1 + i\langle v, m \rangle + |v|^\alpha \left( C_{\alpha}(v) + o(|v|) \right),
\]
with \( \Re C_{\alpha}(v) < 0 \). Therefore
\[
\left| e^{-i\langle v, m \rangle} k_v(v) \right|^2 = \left| \left( 1 - i\langle v, m \rangle + o(|v|) \right) \left( 1 + i\langle v, m \rangle + |v|^\alpha \left( C_{\alpha}(v) + o(|v|) \right) \right) \right|^2
\]
\[
= \left| 1 + |v|^\alpha \left( C_{\alpha}(v) + o(|v|) \right) \right|^2 = 1 + |v|^\alpha \left( 2\Re C_{\alpha}(v) + o(|v|) \right) + O(|v|^{2\alpha})
\]
\[
\leq e^{-D|v|^\alpha},
\]
that proves (8.2). Therefore we may use the Lebesgue dominated theorem and pass in (8.1) to the limit under the integral. Then reasoning as in the proof of Theorem 1.5 we obtain that the limit above is equal to
\[
\hat{h}(0) \cdot \int_{V} \Phi_{\alpha}(v) dv = (2\pi)^d p_{\alpha}(0) \int_{V} h(v) dv, \quad \text{if } \alpha \in (0, 1) \cup (1, 2)
\]
and
\[
\hat{h}(0) \cdot \int_{V} \Phi_{2+}(v) dv = (2\pi)^d p_{\alpha}(0) \int_{V} h(v) dv, \quad \text{if } \alpha > 2,
\]
that proves the Theorem. \( \square \)
Theorem 1.6 can be interpreted as a local limit theorem for a random walk on a homogeneous space. This interpretation brings up some aspects already encountered for the case of groups in [V], namely that the degree of the monomial part of the asymptotics depends of the measure, hence is not determined by the geometry.

One is led to consider the Markov chain on $\tilde{V} = V \times \mathbb{R}^d$ defined by $x_n = X^x_n$, $y_n = y + S^r_n = y_{n-1} + x_{n-1}$. We denote by $\tilde{P}$ its transition kernel. Then $\tilde{P}$ is a fibered Markov kernel over $P$ (see [GH]), with typical fiber $\mathbb{R}^d$. Clearly the ‘vertical translations’ $(x, y) \mapsto (x, y + r) = (x, y) \circ r$ with $r \in \mathbb{R}^d$ commutes with $\tilde{P}$. We denote also by $\circ$ the convolution operation between measures on $\tilde{V}$ and on $\mathbb{R}^d$, and by $\lambda$ a Lebesgue measure on $\mathbb{R}^d$. Since $\tilde{V} = V \times \mathbb{R}^d$, we can identify the measure $\nu$ on $V$ with a measure $\tilde{\nu}$ on $\tilde{V}$. Then we observe that $\tilde{\nu} \circ \lambda$ is a $\tilde{P}$ stationary measure. Furthermore $h = (g, b) \in H$ acts on $V \times \mathbb{R}^d$ by $h(x, y) = (gx + b, y + x)$. This is an affine action of $H$ which is part of a natural action of a larger group $\tilde{H}$ on $\tilde{V}$ considered as a homogeneous space as follows.

Let $T$ be the real Lie group of $2d \times 2d$ matrices of the form $\xi = \begin{pmatrix} g & 0 \\ u & I \end{pmatrix}$, where $g \in G$, $u \in \text{End}(V)$. Then $T$ acts on $\mathbb{R}^{2d}$ and we consider the corresponding semidirect product $\tilde{H} = T \ltimes \mathbb{R}^{2d}$. Then $\tilde{V} = \mathbb{R}^{2d}$ is a homogeneous space of $\tilde{H}$, i.e. $\tilde{V} = \tilde{H}/T$. The action of $h = (\xi, \eta) (\xi \in T, \eta = (b, c) \in \mathbb{R}^{2d})$ on $\tilde{v} = (x, y) \in \tilde{V}$ is given by $x' = gx + b$, $y' = y + ux + c$. Hence this action commutes with the ‘vertical translations’ on $\tilde{V}$. We recover the $H$-action on $V$ as a factor of the $\tilde{H}$-action by the vertical translations.

In particular we denote by $\tilde{\mu}$ the push forward of $\mu$ by the map $(g, b) \mapsto (\xi, \eta)$ with $\xi = \begin{pmatrix} g & 0 \\ I & I \end{pmatrix}$, $\eta = \begin{pmatrix} b \\ 0 \end{pmatrix}$. Then we can write $\tilde{P}(\tilde{v}, \cdot) = \tilde{\mu} \circ \delta_\tilde{v}$, hence $\tilde{P}^n(\tilde{v}, \cdot) = \tilde{\mu}^n \circ \delta_\tilde{v}$. We know that $X^x_n$ converges in law to $\nu$. Furthermore the theorem tells us that if $\mu_n$ denotes the law of $y_n = y + S^r_n$, then the sequence of measures $n^x(\mu_n \circ \delta_{-d_n})$ converges weakly to $\mu_\alpha(0) \lambda$. Then, following the analysis of [GH] for local limit asymptotics in the context of fibered Markov kernels we get

**Corollary 8.3.** With the above notations, for any $\tilde{v} \in \tilde{V}$ we have the weak convergence:

$$\lim_{n \to \infty} n^x(\tilde{\mu}^n \circ \delta_{-d_n}) \circ \delta_{-d_n} = \mu_\alpha(0) \tilde{\nu} \circ \lambda.$$
Proof. We observe that the projection map $\pi$ of $G$ on $D$ has compact kernel hence $\pi$ is proper. It follows that $R_1 = \pi(G_1)$ is closed, hence $\pi(G_1)$ is either $\{1\}$ or $\{p\}$ ($c > 1$). Since $G_1$ is non compact $\pi(G_1) = \{1\}$ is excluded, hence the first assertion. On the other hand $K_1$ is normal in $G_1$. Every element $X$ of $G$ can be written as $X = (\lambda, \theta)$ with $\lambda \in \mathbb{R}$ and $\theta \in K$, an antisymmetric matrix.

The quadratic form $q$ on $G$ defined by $q(X) = \lambda^2 - \text{Tr} \theta^2$ is positive definite and $\text{Ad}G$-invariant.

If $R_1 = D$, the result is proved as follows. Indeed, the Lie algebra $G_1$ is $\text{Ad}G_1$-invariant and $G_1$ contains an element $Y_1$ with $\pi(Y_1) \neq 0$. It follows $G_1 = \langle \exp Y_1 \rangle \ltimes K_1$, $G_1 = \mathbb{R} Y_1 \ltimes K_1$. Since $K_1$ and $G_1$ are $\text{Ad}G_1$-invariant, the same is true for the orthogonal line $K_1^\perp \subset G_1$. On the other hand, we have for any $t \in \mathbb{R}$, $g \in G_1$: $ge^{tY_1}g^{-1}e^{-tY_1} \in K_1$, hence $\text{Ad}(Y_1) \subset Y_1 + K_1$. Then the affine hyperplane $Y_1 + K_1$ of $G_1$ is $\text{Ad}G_1$-invariant, hence the point of intersection $Y$ of $K_1^\perp$ and $Y_1 + K_1$ satisfies $\text{Ad}G_1(Y) = Y$. It follows $G_1 = A_1 \times K_1$, $Z_1 = A_1 \times (Z_1 \cap K)$. 

If $R_1 = \langle p \rangle$, we consider $y \in G_1$ with $\pi(y) = p$, and the Zariski closure $L$ of the subgroup $\langle y \rangle$. Then $L$ is a closed abelian Lie group with a finite number of connected components. Let $L^0$ be the connected component of $e$ in $L$. Let $r \in \mathbb{N}$ with $y^r = \exp Y_0 \in L^0$ with $Y_0 \in \tilde{G}$, hence $\exp \mathbb{R} Y_0 \subset L^0$, $\text{Ad}y(Y_0) = Y_0$. Since for any $g \in G_1$, $n \in \mathbb{Z}$, $gg^ny^{-n} \in K_1$ and $K_1$ is algebraic, we have also, $gg^{-1}z^{-1} \in K_1$ for any $z \in L$. In particular for any $t \in \mathbb{R}$, $ge^{tY_0}g^{-1}e^{-tY_0} \in K_1$. It follows $\text{Ad}g(Y_0) - Y_0 \in K_1$, hence the affine hyperplane $Y_0 + K_1$ of $\mathbb{R} Y_0 + K_1$ is $\text{Ad}G_1$-invariant. Since $K_1$ and $\mathbb{R} Y_0 + K_1$ and are $\text{Ad}G_1$-invariant, we can repeat the argument used if $R_1 = D$: the point $Y_1 = Y_0 + U$, $U \subset K_1$ of intersection of $K_1^\perp$ and $Y_0 + K_1$ satisfies $\text{Ad}G_1(Y_1) = Y_1$. In particular $[Y_1, K_1] = [Y_1, U] = \{0\}$, $\exp Y_1 = \exp Y_0 \exp U$. Then, using $y^r = \exp Y_0 \in G_1$, $\exp U \in K_1$, we get $\exp Y_1 \in G_1$, hence $\exp Z_1$ is a central subgroup of $G_1$. Since $G_1 = \langle y \rangle \times K_1$ and $y^r = \exp Y_0$, we conclude that $A_1 \times K_1$ has finite index in $G_1$. Furthermore $\text{Ad}Y_1 = Y_1$ and $\text{Ad}y(Y_0) = Y_0$ imply $\text{Ad}U = U$, hence $(y \exp \frac{U}{r})^r = y^r \exp U = \exp Y_1$. Since $|y \exp \frac{U}{r}| = p$ and $g_1 = y \exp \frac{U}{r} \in G_1$ we conclude $G_1 = \langle \exp (g_1) \rangle \times K_1$, with $g_1^r = \exp Y_1$. Using the above we can write $Z_1 = (z) \times (Z_1 \cap K_1)$. Also $A_1 \times (Z_1 \cap K_1)$ is a subgroup of finite index in $Z_1$. This follows from the fact that $\pi$ defines an isomorphism of $(z)$ onto a cyclic subgroup of $D$, which contains $\pi(A_1)$ as a finite index subgroup. Then, for some $n \in N$, $z^n = u_1 a_1$ with $a_1 \in A_1$, $u \in (Z_1 \cap K)^0$. We can write $u^{-1} = v^n$ with $v \in (Z_1 \cap K)^0$, hence $(zv)^n = z^n u^{-1} = a_1$. Then with $z_1 = zv$ we have $Z_1 = \langle z_1 \rangle \times (Z_1 \cap K_1)$, $\langle z_1 \rangle \cap A_1 \supset \langle a_1 \rangle$, hence $\langle z_1 \rangle / \langle z_1 \rangle \cap A_1$ is finite.

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