On the stratorotational instability in the quasi-hydrostatic semi-geostrophic limit

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ABSTRACT

The linear normal-mode stratorotational instability (SRI) is analytically reexamined in the inviscid limit where the length scales of horizontal disturbances are large compared to their vertical and radial counterparts. Boundary conditions different than channel walls are also considered. This quasi-hydrostatic, semi-geostrophic (QHSG) approximation allows one to examine the effect of a vertically varying Brunt-Vaisala frequency, $N^2$. It is found that the normal-mode instability persists when $N^2$ increases quadratically with respect to the disc vertical coordinate. However we also find that the SRI seems to exist in this inviscid QHSG extreme only for channel wall conditions: when one or both of the reflecting walls are removed there is no instability in the asymptotic limit explored here. It is also found that only exponential-type SRI modes (as defined by Dubrulle et al. 2005) exist under these conditions. These equations also admit non-normal mode behaviour. Fixed Lagrangian pressure conditions on both radial boundaries predicts there to be no normal mode behaviour in the QHSG limit. The mathematical relationship between the results obtained here and that of the classic Eady (1949) problem for baroclinic instability is drawn. We conjecture as to the mathematical/physical nature of the SRI.

The general linear problem, analyzed without approximation in the context of the Boussinesq equations, admits a potential vorticity-like quantity that is advectively conserved by the shear. Its existence means that a continuous spectrum is a generic feature of this system. It also implies that in places where the Brunt-Vaisala frequency becomes dominant the linearized flow may two-dimensionalize by advectively conserving its vertical vorticity.

Key words: accretion, accretion discs – hydrodynamics – instabilities – linear theory

The question whether or not hydrodynamic activity can emerge in protoplanetary discs is experiencing a Renaissance. Given the absence of an inflection point in the basic Keplerian flow of discs, it was natural for investigations, beginning in the early 90’s, to consider other physical effects as a possible source of supercritical linear instabilities. The MRI (Balbus, 2003) an instability (non-conservative) involving the joint interplay of rotation and magnetic effects, has proven itself to be a viable linear mechanism which could lead to globally sustained activity in discs.

However, fresh analysis of simplified models of protoplanetary discs, like the shearing sheet approximation (Goldreich & Lynden-Bell, 1965) utilized in many hydrodynamic and magnetohydrodynamic investigations of circumstellar Keplerian discs, have shown that a number of alternative routes to long-term activity, both linear and nonlinear, can occur for purely hydrodynamic disturbances. Bracco et al (1999) demonstrated that anticyclonic vortices can live for long periods of time before ultimately decaying away in viscous two-dimensional, global disc models of incompressible flow executing Keplerian motion. This work has since led to many other investigations which consider an array of dynamical processes, sometimes even linearly stable, that could operate in a purely hydrodynamic manner and these include (but not limited to): large amplitude defects of vorticity in the flow (Li et al. 2000) leading to inflection point "secondary" - instabilities, steady forcing of 2D linearized global viscous disc disturbances which have shown to lead to strong transiently growing structures and patterns (Ioumou & Kakouris, 2002). Transient growth in 2D disturbances, an effect primarily driven by the Orr-mechanism of vortex-tilting (Orr 1907) and which is fundamentally a non-normal mode effect (Schmid & Henningson, 2001, Chagelishvilli et al. 2003, Tevzadze et al., 2003, Yecko 2004) can lead to sustained nonlinear activity in 2D, including robust and coherent anticyclones (Umurhan & Regev 2004).

Recently, Barranco and Marcus (2005), considering an analogous set of equations describing the shearing-sheet limit of a circumstellar disc, showed that nonlinear disturbances in 3D, where

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the disc vertical direction and effect of gravity are included, lead to long-lived vortex structures with vorticity pointing parallel to the direction of gravity: something akin to steady anticyclones persisting away from the disc midplane. This is a striking feature because it implies that in these regimes the flow behaves nearly two-dimensionally. This goes against implications of the shearung sheet simulations of Balbus, Hawley & Stone (1996) and Hawley, Balbus & Winters (1999) which have suggested that there is slim possibility of sustained dynamics in purely hydrodynamic systems subject to strong rotation and shear.

The effect of buoyancy on disturbances has only recently been reconsidered. According to the Solberg-Hoiland criterion it seemed that a sufficient condition for a naturally occurring non-magnetic instability in a disk is for there to be an adverse entropy gradient in the direction normal to the disc plane. However the work of Yavneh et al. (2002) has recently turned this wisdom on its head. They demonstrated that a linear stability analysis of the classic Tyler Couette problem (that is, flow between two concentric rotating cylinders) in which the fluid is stratified (in the direction parallel to the cylinder axes) leads to normal-mode instability even if the stratification is stable to buoyant oscillations.

Dubrulle et al., 2005 (D05) showed that 3D linearized normal-mode disturbances of stratified local sections of a protoplanetary disc can manifest this instability. The effect, termed the Stratortational Instability (SRI), has been proposed as a mechanism, free of non-conservative effects like magnetic field interactions, which could actively operate in the disc, possibly lead to turbulence (D05), and has been suggested to be relevant in connection to the steady vortices observed in the simulations of Barranco & Marcus (2005) (Shalybkov & Rudiger, 2005).

The SRI effect, as revealed by Yavneh et al (2000), Shalybkov & Rudiger (2005) and in D05, was considered for flows in which the radial boundaries of the system are rigid: in other words, under boundary conditions in which there is no normal flow across the bounding channel walls. In order to affect a tractable analysis the investigations thus far have assumed that the disc vertical component of gravity, $g_{\phi}$, and the vertical gradient of the basic state temperature, $\partial_{z}T_b$ (or, equivalently, the basic state entropy gradient, $\partial_{z}S_b$) are constant with respect to the disc vertical coordinate. Although this may be a good approximation for atmospheric flows like the Earth’s (Pedlosky, 1987), this is not an accurate representation of conditions near the midplane of a circumstellar disc.

Furthermore, the question as to what are reasonable radial boundary conditions for shearung-sheet sections of Keplerian discs remains very open. Shearung-sheet sections of Keplerian discs, centered about the disc midplane, have both a $g$ and a $\partial_{z}T_b$ ($\partial_{z}S_b$), whose product is proportional to the square of the Brunt-Vaisaila frequency, $N^2$, which varies quadratically with respect to the height from the midplane, $z$. It is natural then to ask (i) whether or not the SRI is a dynamical effect driven primarily by the reflecting property of the system’s radial boundaries and, (ii) whether or not it persists under conditions in which the Brunt-Vaisaila frequency varies with respect to the vertical coordinate - something one would expect in a protoplanetary disc.

A purpose of this study is to revisit, primarily via asymptotic means, the properties of the SRI instability as discussed by D05. The main motivation of this inquiry is to see whether or not the no-flow radial boundary conditions employed by their study and/or whether or not the assumption of the constancy of $N^2$ significantly alters the SRI effect uncovered by previous investigations.

We would like to make it clear here that the inclusion of stratification in quantities like the vertical component of gravity and the temperature/entropy gradients mathematically results in a normal mode problem who is primarily non-separable in the radial and vertical coordinates. It is for this reason a general analysis of this physical scenario is quite difficult for both analytical and numerical reasons.

We approach these questions by reconsidering the SRI within the context of two model equations describing a local section of a circumstellar disc. We will experiment with the effects of differing boundary conditions.

In Section 1, we introduce the first of the equations, the Large-Shearing Box (LSB), which are the basic equations appropriate to shearung-sheet sections of accretion discs centered about their midplanes. These are then used to motivate a second, simpler, model equation set: the incompressible Boussinesq equations (BE) in inviscid three dimensional rotating plane Couette flow (or rpfCf, see Yecko, 2004). This simpler set is mathematically equivalent to the set studied, in the inviscid limit, in D05. Boundary conditions are considered which share a common property in the BE, namely those that cause the total disturbance energy, called $E$, to change solely due to the energy exchanged between disturbances and the shear through the Reynolds stress term and not due to work done on the system from outside. This is achieved by enforcing periodicity in the azimuthal direction: either periodicity in the disc vertical direction (when constant $N^2$ is assumed) or zero normal (vertical) velocity fluctuations at some fiducial vertical disc boundary (when $N^2$ varies with disc height). There a number of radial boundary conditions which achieves these objective and we investigate these:

(a) no-normal flow at the radial boundaries, sometimes referred to in this manuscript as “channel-wall conditions”, (b) no Lagrangian variation of the pressure on the moving radial boundaries, (c) or some mixed combination of these two.

In Section 2 normal mode solutions of the BE are considered. First it is shown that there exists an advectively conserved quantity of the flow. The conservation of this quantity in certain limits implies that the linearized flow is nearly two-dimensional - in that the vertical vorticity is the dominant quantity that is preserved by the shear advected flow. We then proceed towards a normal mode analysis by initially assuming the constancy of the Brunt-Vaisaila frequency, $N^2$. We asymptotically analyze disturbances whose radial ($x$) and vertical length scales are dwarfed by comparison to the azimuthal scales ($y$) - a circumstance which is referred to as the quasi-hydrostatic semi-geostrophic approximation (QHSG). This limit predisposes the resulting equations into admitting simple analytical solutions. This asymptotic limit shares many of the qualities of the quasi-WKB analysis considered in D05. Implementing the variety of boundary conditions enumerated above shows that, in this limit, only the no normal (radial) flow boundary conditions admit unstable normal modes. We also find that there is always a continuous spectrum in this problem, irrespective of the radial boundary conditions, and we briefly discuss some of its features. The section is rounded out by relaxing the constancy of $N^2$ and considering the case where it varies quadratically with respect to the disc height in this same QHSG limit. There appears to be a potential vorticity-like quantity which is advectively conserved in the linear limit. Moreover, it turns out that it is possible to construct separable solutions (in $x$ and $z$) for the normal mode prob-
lem. The main conclusion from this is that the normal mode stability behaviour is unaffected by the stratification of $N^2$. Unlike in D05, where the SRI appears for both "exponential"-modes and "oscillatory"-modes, the results of this section show that the SRI only occurs for "exponential"-modes.

In Section 3 the same QHSG approximation is applied directly to the LSB. These yield, similarly, a conserved potential vorticity like quantity but which now includes the effects of weak compressibility and a finite soundspeed.\footnote{Note that incompressibility has the mathematical effect of an infinite soundspeed.} The equations for the normal-modes are also separable here and it is found, again, that the stability behaviour appears to be unaffected by the inclusion of stratification and weak compressibility effects. This has been analyzed only for the case of constant soundspeed (i.e. constant disc background temperature profile). In Section 4 we discuss these results and conjecture as to the disappearance of the SRI in this inviscid QHSG limit.

The main results of this paper can be summarized here: (a) the SRI is also present in the LSB model equations, (b) we find that in the inviscid-QHSG approximation only the doubly reflecting boundary conditions, i.e. no-normal flow at the two radial boundaries, seem to admit unstable normal modes, (c) the inclusion of stratification realistic for a disc (again, in the QHSG approximation) does not alter the SRI as obtained in previous investigations, and (d) there is an advectively conserved quantity in the BE which has the character of a potential vorticity; its existence implies both that this type of flow always has a continuous spectrum associated with it and that there exists certain conditions, including large Brunt-Vaisaila frequencies, in which the underlying flow behaves two dimensionally and conjecture as to the disappearance of the SRI in this inviscid QHSG limit.

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about the disc’s symmetry plane (i.e. $z = 0$). Again, for details of this procedure see Umurhan & Regev (2004).

The steady state density and pressure functions are given by $\rho_b, \rho_0$. These relate to each other according to the aforementioned local hydrostatic equilibrium relationship,

$$\partial_z p_b = -\rho_0 g(z),$$

with $g(z) = \Omega_0^2 z$. The detailed solution for the steady states are then determined once something has been said relating the steady quantities. For simple theoretical investigations this comes in the form of a barotropic equation of state, namely, the statement that $p_b = p_b(\rho_b)$. Note that in this paper, especially in Section 4, we explicitly assume that these steady quantities depend only on $z$ and not on $x$. The dynamic pressures and densities, $p$ and $\rho$, represent deviations about their corresponding steady state quantities.

We want to gain some insight as to the effects that gravity and gradients of state quantities have on the local dynamics. We take an incremental approach towards this goal by considering a more simplified version of these equations. In more concrete terms, the linear equation (5) with an evolution equation for the temperature fluctuation, $\theta$, gives rise to a pair of acoustic modes, (ii) a pair of inertial-gravity modes and (iii) an entropy mode (Tevzadze et al., 2003). Although the acoustic modes are interesting, we chose to consider the dynamics of a system in which the acoustic modes are effectively filtered out.

To do this we invoke the Boussinesq approximation which, in this sense, we replace the continuity equation (1) with the statement

$$\nabla \cdot \nabla \theta = \frac{\partial p}{\partial T} \theta,$$

In other words, $\alpha_p$ is the coefficient of thermal expansion at constant pressure. This is the typical formulation of the Boussinesq approximation (Spiegel and Veronis, 1960). We furthermore posit that in this limit the basic state density profile is a constant and in order to distinguish this from a spatially varying density profile we denote the former with $\bar{\rho}_0$. The resulting model set of equations are similar to those assumed in the studies by Yavneh et al. (2001), Dubrulle, et al. (2005) (these being viscous studies) and Rudiger et al. (2005) (a cylindrical Taylor-Couette analysis). We have then,

$$\nabla \cdot u' = 0,$$  

$$(\partial_t - q\Omega_0 x \partial_y) u' + \nabla \theta' = -2\Omega_0 v' = -\frac{\partial p}{\bar{\rho}_0},$$  

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These are otherwise known as the Boussinesq equations (BE) in plane-Couette shear. The term $T_b$ is the way in which the basic state temperature profile varies with height in this model formulation of the disc system. Such a term would vary according to $\partial_z T_b = T_{zz} z$, in other words, the gradient of the basic state temperature has a linear dependence with respect to the disc height in which $T_{zz}$ is the parameter that controls the slope of this variation. For situations in which $T_{zz}$ is negative, the atmosphere is thought of as being classically buoyantly unstable which could lead to Rayleigh-Benard convection (see for instance, Cabot, 1996).

The BE equations here are mathematically equivalent to the inviscid limit of the equations studied in D05. The only difference here is in interpretation. Whereas we follow a temperature perturbation, $\theta$, and steady temperature profile, $T_b$, they follow an entropy perturbation, denoted by $h$, and steady entropy distribution $H$. The two disturbance quantities are related to each other via

$$h = -\frac{\gamma \alpha_p C_V}{\bar{\rho}_b} \theta.$$  

(Also see Appendix B, $\gamma$ is the ratio of specific heats and $C_V$ is the specific heat at constant volume. The equations (7-11) are the basis of the discussion in Section 3. The steady configuration of these equations which will be perturbed is

$$u' = v' = w' = \theta = 0, \quad p = \bar{p} = \text{constant}. \quad (12)$$

The set (13), including their corresponding analogous boundary conditions will be considered in Section 4.

### 1.2 Boundary conditions

There is no obvious choice of boundary conditions for this reduced set of inviscid flow equations. We consider all disturbances to be periodic in the $y$ direction. Because these are equations meant to model what happens near the midplane of a circumstellar disc, we can consider periodic conditions in the vertical direction only under special circumstances (i.e. the constant Brunt-Väisälä frequency approximation of the BE in Section 3, see below). In a more general sense we will distinguish, instead, between either varicose or sinuous modes. By varicose modes we mean to say disturbances which have even symmetry with respect to the $z = 0$ plane in all disturbances except the vertical velocities, which have odd symmetry with respect to the $z = 0$ plane. Sinuous modes have the reverse symmetry of the varicose modes. In situations in which a boundary condition needs to be specified on vertical boundaries of the atmosphere, we assume that there is no normal flow.

The more troublesome of the boundary conditions has to do with what to say about the flow variables in the radial direction. This is because an injudicious choice of a boundary condition might incite disturbances of the fluid into instability by drawing energy across the boundaries. It is our interest here to minimize this potential as much as possible. Of the myriad of possible choices, we see the following three sets of boundary conditions as ones that achieve this objective (and motivated further in Appendix A): (a) that the flow be confined between channel walls lying at $x = 0, 1$, which means in practice that the radial velocities are set to zero there, i.e., no normal-flow conditions; (b) the flow has zero Lagrangian pressure fluctuations (defined below) on both of the moving radial boundaries and; (c) a mixture of these two conditions, for example, by requiring there to be no normal-flow on the inner boundary while there is no Lagrangian pressure fluctuation at the outer boundary.

The vanishing of the Lagrangian pressure fluctuation on the undulating radial bounding surface $S_r$ in linear theory translates to requiring

$$p' + \xi_\delta \partial_r \bar{p} = 0,$$

where $p'$ is the pressure fluctuation about the steady state. The position of any particular radial surface, initially at rest at coordinate $x$, is denoted by $\xi_\delta (y, z)$, and evolves according to its Lagrangian equation of motion (Drazin & Reid, 1984)
There exists a conserved quantity in these equations. Operating on  
\[ u'(x, y, z, t) = \frac{\partial u}{\partial t} = (\partial_t - q\Omega_0 x \partial_y)\xi_x + v' \partial_y \xi_x + w' \partial_z \xi_x. \tag{13} \]
Because the steady state pressure configurations of both the BE  
(\bar{p}) and the LSB (p_b) are constant with respect to x, the condition  
simplifies to requiring  
\[ p' = 0, \tag{14} \]
at \( x = 0, 1. \)

## 2. LINEAR DYNAMICS OF THE BOUSSINESQ EQUATIONS

In this inviscid limit there emerges a natural timescale defined as the  
*Brunt-Vaisälä frequency*, \( N \). This time scale is defined through the  
product of the vertical temperature gradient and vertical gravity via  
\[ N^2 \equiv \frac{g}{\rho_0} \alpha_T \partial_z T, \tag{15} \]
which is, in general, a function of the vertical coordinate \( z \). Throughout  
this study \( N \) is taken to be real (buoyantly stable). Linearization of (\ref{eq:11})  
reduces to,  
\[ \begin{align*}  
(\partial_t - q\Omega_0 x \partial_y) u - 2\Omega_0 v &= -\frac{\mu}{\rho_0} \partial_z P, \tag{16} 
(\partial_t - q\Omega_0 x \partial_y) v + \Omega_0 (2 - q) u &= -\frac{\mu}{\rho_0} \partial_y P, \tag{17} 
(\partial_t - q\Omega_0 x \partial_y) w &= -\frac{\mu}{\rho_0} \partial_z P + \Theta, \tag{18} 
\partial_t \xi_u + \partial_y \xi_v + \partial_z \xi_w &= 0, \tag{19} 
\end{align*} \]
where the temperature variable has been slightly redefined as  
\[ \Theta = \frac{g \alpha_T}{\rho_0} \theta, \]
\( \rho_0 \) is set to 1 from here on out. It is now to be understood  
that unprimed velocity expressions (i.e. \( u, v, w \)) represent linearized  
disturbances.

### 2.1 A conserved quantity for linearized flow

There exists a conserved quantity in these equations. Operating on  
(\ref{eq:10}) by \( \partial_y \) followed by operating (\ref{eq:11}) by \( \partial_x \) and subtracting  
the result reveals  
\[ (\partial_t - q\Omega_0 x \partial_y)(\partial_x v - \partial_y u) = \Omega_0 (2 - q) \partial_z w, \tag{21} \]
where the incompressibility condition has been used. The term on  
the LHS of this expression is the *vertical vorticity*, i.e. \( \zeta = \partial_x v - \partial_y u \). With a similar tack one can multiply (\ref{eq:10})  
by \( \Omega_0 (2 - q)/N^2 \) and then operate on the result with \( \partial_z \) to get  
\[ (\partial_t - q\Omega_0 x \partial_y)\left( \frac{\partial}{\partial z} \frac{\Omega_0 (2 - q)}{N^2} \Theta \right) = -\Omega_0 (2 - q) \partial_z w. \tag{22} \]
Adding the results together yields a general conserved quantity of  
linearized flow of this type:  
\[ (\partial_t - q\Omega_0 x \partial_y) \Xi = 0, \tag{23} \]
where  
\[ \Xi = \zeta + \frac{\partial}{\partial z} \frac{\Omega_0 (2 - q)}{N^2} \Theta. \tag{24} \]

The quantity \( \Xi \) can be thought of as a generalized potential vorticity for  
this type of flow whose analogous quantity is discussed in  
Tevfadze et al. (2003). The conservation of \( \Xi \) immediately implies  
that there always exists a continuous spectrum (see Schmid &  
Henningson, 2001, Tevzadze et al., 2003) for this type of physical  
system. Note also that the system of linearized equations (i.e.  
\( \Xi = 0 \)) is third order in time. One may suppose that there are three  
independent normal modes for any given set of quantum numbers  
of the system (see below), however, given that there exists a  
conserved quantity, together with its associated continuous spectrum,  
means that there are at most only two normal modes for any given  
quantum number set.

Inspection of \( \Xi \) shows that disturbances behave in a quasi two-  
dimensional fashion in some limits. One of these is when \( q = 2, \)  
that is at the critical Rayleigh condition (Drazin & Reid, 1984):  
it follows that the vertical vorticity is conserved by the flow. The  
second of these is to notice that if the temperature fluctuation re-  
mains an order one quantity as \( N^2 \) gets large then the flow again  
shows quasi two-dimensionality with the vertical vorticity being  
conserved. We reflect upon the consequences of this conserved  
quantity some more in the Discussion.

### 2.2 Constant \( N^2 \)

Since part of the purpose of this work is to further develop some  
amount of intuition about the dynamics of such disc environments  
primarily through analytical means, it will be more tractable for us  
to first treat the vertical gravity component \( g(z) \) to be  
\[ g(z) = g_0 \text{sgn}(z). \tag{25} \]
The non-dimensional constant \( g_0 \) is technically arbitrary. In a similar  
vein we approximate the steady state temperature gradient by saying  
\[ \partial_z T_k = T_{\text{s}} \text{sgn}(z), \tag{26} \]
in which \( T_{\text{s}} \) is another non-dimensional parameter. The consequence  
of this is that \( N^2 \) is a constant for \( z \neq 0 \), and is zero at \( z = 0 \). At this stage, these assumptions are qualitatively no  
different than what has been done in Yavneh et al. (2001), D05 and  
Shalybkov & Rudiger (2005), although we take a more realistic  
interpretation of a constant \( N^2 \) (and see below). From here on out  
we set \( \rho_b = 1 \). Additionally, we restrict analysis of the dynamics  
\( z > 0 \) and keeping in mind that modes are considered to have  
either sinuous or varicose spatial character in the vertical.

We write general solutions into the form  
\[ \begin{pmatrix} u \\ v \\ P \\ \Theta \end{pmatrix} = \begin{pmatrix} u_{\alpha \beta} \\ v_{\alpha \beta} \\ P_{\alpha \beta} \\ \Theta_{\alpha \beta} \end{pmatrix} \begin{pmatrix} \cos \beta z \\ \sin \beta z \\ e^{i\omega t + i\alpha y} + \text{c.c.} \\ -\sin \beta z \\ \cos \beta z \\ e^{i\omega t + i\alpha y} + \text{c.c.} \end{pmatrix}, \tag{27} \]
while for the other variables  
\[ \begin{pmatrix} w \\ \Theta \end{pmatrix} = \begin{pmatrix} w_{\alpha \beta} \\ \Theta_{\alpha \beta} \end{pmatrix} \begin{pmatrix} -\sin \beta z \\ \cos \beta z \\ e^{i\omega t + i\alpha y} + \text{c.c.} \end{pmatrix}, \tag{28} \]
The terms above in the curly brackets represent varicose disturbances while the terms below are the sinuous disturbances. In this sense the “quantum numbers” of the system are given by \( \alpha, \beta \) (varicose  
or sinuous) and a radial overtone number (if there are more than one) subject to solution of the normal mode boundary value problem below.\(^2\). Note that because this is a single Fourier expansion we restrict our considerations to \( 0 < \alpha < \infty \) together with  

\(^2\) The use of quantum numbers should be considered only in terms of conventional nomenclature. There is no real quantization in the horizontal and vertical directions per se since we allow these quantities to take on any value from the continuum of real numbers.
0 < \beta < \infty. Insertion of Eq. (28) into the governing linear equations gives,
\[ i(\omega - q\Omega_0\alpha)x)u_{\alpha\beta} - 2\Omega_0v_{\alpha\beta} = -\partial_x P_{\alpha\beta}, \tag{29} \]
\[ i(\omega - q\Omega_0\alpha)v_{\alpha\beta} + \Omega_0q(2 - q)u_{\alpha\beta} = -i\alpha P_{\alpha\beta}, \tag{30} \]
\[ i(\omega - q\Omega_0\alpha)w_{\alpha\beta} = -\beta P_{\alpha\beta} - \Theta_{\alpha\beta}, \tag{31} \]
\[ i(\omega - q\Omega_0\alpha)x\alpha_{\alpha\beta} = -N^2u_{\alpha\beta}, \tag{32} \]
\[ \partial_x u_{\alpha\beta} + i\alpha v_{\alpha\beta} - \beta w_{\alpha\beta} = 0, \tag{33} \]

It turns out that it is much more tractable to consider the linearized normal-mode behavior in terms of equations describing the pressure fluctuation and radial velocities. This is entirely analogous to what was done in D05 and the following equations should be compared to the ones quoted in D05 as Equations (21-22).

\[
\left(\frac{\beta^2\sigma^2}{N^2 - \sigma^2} - \alpha^2\right)P_{\alpha\beta} = -\sigma\partial_x u_{\alpha\beta} + \Omega_0q(2 - q)\alpha u_{\alpha\beta},
\]
where, for the sake of compact notation we use the expression \( \sigma \equiv \omega - q\Omega_0\alpha x \). The epicyclic frequency \( \omega^e \) is equivalent to the expression \( 2(2 - q)\Omega_0^2 \).

We are mainly interested in analytically expressible solutions to the above set of equations. To achieve this in an asymptotically rigorous manner the following scalings seem natural: when the horizontal wavenumber is small, it follows that the frequency scales similarly. Using \( \varepsilon \) to measure this smallness it follows,
\[
\alpha = \varepsilon\alpha_1, \quad \omega = \varepsilon\omega_1 + \cdots.
\]

To lowest order it also follows that \( \sigma = \varepsilon\sigma_1 + \cdots \). The pressure and velocities are consequently expanded by
\[
P_{\alpha\beta} = P_0 + \varepsilon P_1 + \cdots, \]
\[
u_{\alpha\beta} = \varepsilon u_1 + \varepsilon^2 u_2 + \cdots.
\]

Implementing these expansions into the governing equations yields at lowest order in \( \varepsilon \) a single equation for the pressure perturbation,
\[
(\omega_1 - q\Omega_0\alpha_1 x) \left( \partial_x^2 - F_c^2\beta^2 \right) P_0 = 0. \tag{35}
\]

Normal-mode type solutions to Eq. (35) are,
\[
P_0 = A \cosh k_F x + B \sinh k_F x, \tag{36}
\]
where the Froude-wavenumber, \( k_F \), is defined as \( k_F^2 \equiv \omega^2 \beta^2/N^2 = \beta^2 F_c^2 \). The epicyclic-Froude number is denoted by \( F_c \). This mathematical structure of Eq. (35) is identical to the operator describing the evolution of plane-Couette disturbances in a channel (e.g. Case, 1960).

We compare the analytical solutions generated here with numerical solutions generated for the boundary value problem defined by the un-approximated full linearized equation set Eq. (35). A second order correct (in the \( x \) direction derivatives) Newton-Raphson-Kantorovich (NRK) relaxation scheme on a grid of approximately 1000 to 2000 points is used for the verification. Relative convergence was checked by doubling the size of the domain. Eigenvalues are determined with errors that were no more than \( O(5 \times 10^{-6}) \). As such the eigenvalues generated asymptotically are considered to be valid in all cases where normal modes exist.

Because this is a relaxation method reasonably good initial guesses are required, both in the eigenfunction and the eigenvalue, in order to accurately obtain an answer. When the initial guesses were far off from the actual solution the scheme submitted solutions belonging to the continuous spectrum. This occurs for all sets of boundary condition but is the only solution possible for the case of fixed pressure conditions (see below). Therefore we discuss the general features of the continuous spectrum in Section 2.2.2 Nonetheless, we find that it helps to avoid the continuous spectrum if the initial eigenvalue guess is set so that \( \text{Im}(\omega) \neq 0 \). The possibility of the solution jumping onto a random continuous mode solution is satisfactorily bypassed in this way (also see below).

In what follows we consider the discussion for each of the three boundary conditions. We finally note that the resulting mathematical structure resulting from this asymptotic limit is closely similar to the WKB analysis done in D05. Whereas in this study the small parameter is the horizontal wavenumber, in D05 the small parameter is the shear term \( q\Omega_0 \) (denoted by \( S \) in their equation 3). In this sense, their asymptotic form is valid for small values of the shear while ours is valid for small values of \( \alpha \).

### 2.2.1 No-normal-flow conditions

The wall conditions, i.e. that \( u_{\alpha\beta} = 0 \) at \( x = 0, 1 \), becomes in terms of the pressure the requirement
\[
\omega^2 u_1 = -i(\sigma_1 \partial_x P_0 + 2\Omega_0\alpha_1 P_0) = 0, \quad at \ x = 0, 1, \tag{37}
\]

at lowest order. Implementing these conditions and a little algebra gives a dispersion relation for \( \omega_1 \)
\[
\omega_1 = \alpha_1 q\Omega_0 \left( \frac{1}{2} \pm \frac{1}{2k_F q \Delta_F} \right), \tag{38}
\]

As the dispersion relation clearly indicates, if \( \Delta_F < 0 \) then there appears a pair of complex modes, one which grows and one which decays. When \( \Delta_F > 0 \) there are two propagating modes oscillating with no overall growth in amplitude. The character of the stability is dictated only by the two parameters \( q \) and \( k_F \). The limit where \( N^2 \rightarrow 0 \) (i.e. \( k_F \rightarrow \infty \)) reveals that \( \omega_1 \rightarrow 0, q\alpha_1\Omega_0 \).

The striking feature of this general solution is that there exists a band of vertical wavenumbers for which a stable/unstable solution exists. In Figure 2 we plot this dispersion curve for the general features of the continuous spectrum in Section 2.2.2. We finally note that the resulting mathematical structure resulting from this asymptotic limit is closely similar to the WKB analysis done in D05. Whereas in this study the small parameter is the horizontal wavenumber, in D05 the small parameter is the shear term \( q\Omega_0 \) (denoted by \( S \) in their equation 3). In this sense, their asymptotic form is valid for small values of the shear while ours is valid for small values of \( \alpha \).

The boundaries of the unstable band for general values of \( q \) may be inferred from the expression for the growth rate: this means determining the function \( q_S(k_F) \) that satisfies \( \Delta_F(q, k_F) = 0 \) as defined in Eq. (38). The two functions are defined as \( q_S = 4\tanh(k_F/2)/k_F \) and \( q_+ = 4\coth(k_F/2)/k_F \). We see that the band structure for the instability range persists until \( q = 2 \), which happens to also correspond to the Rayleigh instability line. Beyond \( q = 2 \) the instability range is bounded from below by zero vertical wavenumber but it is still bounded from above by a finite \( \beta \). As the shear becomes weak, the band of unstable modes gets correspondingly thinner. Note that this instability disappears in the non-shearing limit, that is when \( q \rightarrow 0 \). Figure 4 graphically summarizes these results.
2.2.2 Fixed pressure conditions

The boundary conditions on the lowest order pressure conditions becomes
\[ P_0 = 0, \]

at both boundaries \( x = 0, 1 \). Given the form of the underlying equations it turns out that there is no normal mode solution possible. The numerical procedure admits solutions, however, these are always modes associated with the continuous spectrum of the linearized system. They are not true normal modes in the usual sense because they exhibit a discontinuity in some quantity; here being in the horizontal velocity, \( v \), and manifesting explicitly as a step in the quantity \( \partial_x P \). Discontinuities of this sort, referred to sometimes as singular eigenfunctions, are typical features of modes associated with a continuous spectrum (Case, 1960, Balmforth & Morrison, 1999). In all cases, the location of the discontinuity is at some value of \( x = x_c \), which is the location of the critical layer; in other words, the place where the quantity \( \omega_1 - q\Omega_0\alpha_1 x \) is zero. This means (and this is verified numerically) for given values of \( \alpha, q, \Omega_0 \) there will be a continuum frequencies, \( \omega_c(\alpha, q, \Omega_0) \), existing between 0 and \( q\Omega_0\alpha_1 \).

Figure 3 displays examples of this continuum mode together with an analytic representation of the continuum mode developed in Appendix B.

2.2.3 Mixed conditions

We define terms by identifying mixed-A boundary conditions with zero radial velocities at \( x = 0 \) and zero pressure perturbation at \( x = 1 \), while mixed-B boundary conditions indicate zero pressure fluctuations at \( x = 0 \) with zero radial velocity perturbations at \( x = 1 \). Both boundary conditions yield single normal mode solutions Consequently for mixed-A conditions the frequency response is
\[ \omega_1 = \Omega_0\alpha_1 - \frac{2\Omega_0\alpha_1}{k_f} \tanh k_{p}, \]

while for mixed-B conditions we have
\[ \omega_1 = \frac{2\Omega_0\alpha_1}{k_{p}} \tanh k_{p}. \]

In Figure 3 we plot sample eigenfunctions for all boundary conditions we considered in these sections, as well as comparisons between the analytic and numerical solutions obtained.

It is important to mention that when discrete normal modes have frequencies which sit in the continuous sea, i.e. when \( 0 < \omega < q\Omega_0\alpha_1 \) it becomes challenging for the numerical method to not mistake it with a mode belonging to the continuous spectra. To circumvent this possible ambiguity (and to properly numerically verify this limit) we follow modes in \( \beta \) starting initially with values of \( \alpha, q, \Omega_0 \) and, using these as a starting point, one may incrementally move into the realm where normal modes exist within the continuous sea. This is depicted in Figure 4. In all boundary condition cases investigated, we successfully trace the discrete mode spectrum.

2.3 The QHSG approximation: vertically varying \( N^2 \)

The asymptotic results of the last section hints toward a tractable approach in evaluating the generality of the SRI under a variety of conditions. The limit where the horizontal wavenumbers (i.e. \( \alpha \)) are small suggests that there exists well-posed reduction of the governing equations of motion. With the azimuthal scales of disturbances scaling as \( O(1/\alpha) \) it followed that the temporal disturbances scale as \( O(\alpha) \) and, as such, it implied that the radial and vertical velocities similarly scale as \( O(\alpha) \) while the pressure, the temperature fluctuation and the azimuthal velocities scale as \( O(1) \). We therefore propose that when the horizontal scales are large compared to the corresponding vertical and radial ones the following scalings hold, in general, with respect to quantities and operators of the system:
\[ \partial_t, \partial_y, u, w \sim O(\alpha), \quad \partial_x, \partial_z, p, v, \theta \sim O(1). \]
The above set is similar to the quasi-geostrophic, quasi-hydrostatic approximation used in the study of atmospheric flows (e.g. Pedlosky, 1987, Salmon, 2002). Whereas in the terrestrial analog full quasi-geostrophy involves retaining the inertial terms in the radial momentum equation, here they are absent (cf. 44), and it is for this reason we consider the above set of equations to be a sort of semi-geostrophic limit. The semi-geostrophic nature of this set shares some similarities with the so-called elongated-vortices equations derived in Barranco et al. (2000). The set presented here differs from that work in that the elongated-vortex equations do not make the hydrostatic approximation as is a natural and necessary consequence here.

The power in this reduced set of equations, aside from exactly reproducing the asymptotic limit explored in the previous section, is that it allows one to investigate the effects that a position-dependent function of gravity and background state temperature gradient, i.e. $g(z)$ and $\partial_z T_b$, have on the SRI. In this sense, unlike the approximation utilized in Section 2.2, we relax the condition that $g$ and $\partial_z T_b$ are constants and let them be general functions of $z$. It therefore means that the Brunt-Vaisala frequency is now $z$-dependent.

When specific forms are considered here we assume that these quantities are simply proportional to $z$, that is to say,

$$g(z) = \Omega^2 z; \quad \partial_z T_b = \tilde{T}_{zz} z.$$  

The constant $\tilde{T}_{zz}$ sets the severity of the background temperature gradient (see Section 1.1).

We linearize the set (43-47) about the quiet state $u = v = w = \theta = 0$, $p = \text{constant}$. Disturbances are denoted with primes. A little algebra shows that these equations may be simplified into a single one for the pressure perturbation:

$$\left(\partial_t - \Omega q x \partial_y\right) \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 0,$$

where the $z$-dependent Brunt-Vaisala frequency is given as

$$N^2(z) = \frac{1}{\rho_b} \alpha_p g \partial_z T_b = \frac{1}{\rho_b} \Omega^2 \tilde{T}_{zz} \alpha_p z^2.$$

This all also means that we can consider a $z$-dependent Froude number according to

$$F_e^2 = \frac{\omega^2}{N^2(z)} = F_0^2 \frac{1}{z^2},$$

because all variables and quantities have been non-dimensionalized, the Froude-number scale $F_0$ should be considered in parallel to the Froude-number $F_e$ treated in Section 2.2.
The expression inside the square brackets of (49) looks analogous to the potential-vorticity of atmospheric flows. When \( N^2(z) \) is a constant then the expression within the square brackets exactly recovers the asymptotically valid governing equation for the pressure perturbation in (55).

Separable solutions are assumed of the form

\[
p' = \Pi(z) P_0(x) e^{i\omega t + i\alpha y} + \text{c.c.}
\]

and applied to the governing equation (49). This results in two ODE’s, one for the vertical structure function and one for the radial structure function,

\[
\frac{\partial^2 F}{\partial z^2} \frac{\partial \Pi}{\partial z} = -k_p^2 \Pi, \\
\frac{\partial^2 P_0}{\partial x^2} = k_p^2 P_0
\]

The separation constant for this procedure is \( k_p \). We notice immediately that the equation for the radial structure function is mathematically the equivalent to (55). Since the boundary conditions and associated relationships are identical in this asymptotic limit, it follows then the same stability properties that was determined in Section 2.2 carry over here to this particular example of a z-dependent function of \( N^2 \) if the allowed values of \( k_p \) are real.

In remaining consistent with the terminology introduced in D05, modes for which \( k_p \) are real are referred hereafter as exponential, or as e-modes, since this describes the quality of the radial structure function that results from solving (53). By contrast, modes in which the \( k_p \) are imaginary are referred to as oscillating or as o-modes, (again see D05) and these in principle will have different stability properties than those determined for the e-modes.

Thus the task that remains is to determine the allowed values of the separation constant. The analysis below indicates that the only types of disturbances permitted for \( N^2(z) \sim z^2 \) are e-modes on a finite domain.

### 2.3.1 Exponential modes

Though these may be artificial, for the sake of simplicity and comparison we consider only the vanishing of the normal velocities at the boundaries \( z = \pm 1 \) (e.g. Barranco & Marcus, 2005). When assuming the specific forms for \( g \) and \( \partial_sT_b \) as in (43) we find two possible solution forms to (52)

\[
\Pi(z) = \begin{cases} 
\Pi_{(\text{varicose})} \\
\Pi_{(\text{sinuous})}
\end{cases} = \begin{cases} 
z^2 J_\frac{1}{2} \left( \frac{1}{2} k_p z^2 \right) \\
z^2 J_\frac{3}{2} \left( \frac{1}{2} k_p z^2 \right)
\end{cases},
\]

where the symbol \( J \) denotes the Bessel function of the first kind. The expression \( k_p / F_e \) can be considered to be parallel to the vertical wavenumber \( \beta \) introduced and treated in Section 2.2. A Taylor Series expansion of these solutions near \( z = 0 \) verifies the even (odd) symmetry of the varicose (sinuous) solutions. Given these inherent symmetries we are left with the task of setting to zero \( w' \) at \( z = 1 \) which, given the relationships between (49) and (52), is equivalent to setting \( \partial_z p' = 0 \) there. Given the general properties of Bessel Functions (Abramowitz & Stegun, 1972) the set of \( k_p \) values that satisfy the boundary conditions are always real. In Figure 5 we display the functions \( \Pi(z) \) for the first three values of the separation constant \( k_p \). It should be noted that the asymptotic expansion of the solution forms presented above are characterized by an amplitude function which grows as \( \sqrt{z} \) for \( |z| \to \infty \).

![Vertical Eigenfunctions: First 3 modes](image)

Figure 5. The vertical eigenfunctions \( \Pi(z) \) for e-modes are depicted for the first three modes (varicose and sinuous) that satisfy the boundary condition that the vertical velocity is zero at \( z = \pm 1 \). Due to the inherent symmetries in the modes, the behaviour of the function is depicted for \( z \geq 0 \).

### 2.3.2 Oscillating modes

Consideration of o-modes starts by redefining the separation constant \( k_p \) according to

\[
k_p = i\kappa_p,
\]

where \( \kappa_p \) is real. Therefore, as in the previous section there are two independent solutions to (52) with \( k_p \) so defined given by

\[
\sim z^\frac{3}{2} I\left( \frac{1}{2} \frac{\kappa_p}{F_e} z^2 \right), \quad z^\frac{5}{2} I\left( \frac{1}{2} \frac{\kappa_p}{F_e} z^2 \right),
\]

respectively corresponding to sinuous and varicose disturbances. However, Modified Bessel Functions as these do not have zeros for real values of the ratio \( \kappa_p / F_e \). It means that it is not possible to satisfy the same sort of boundary conditions considered in the previous section (e.g. \( w = 0 \) at \( z = \pm 1 \)) which, in turn, means that o-modes are not allowed solutions on a finite vertical domain with no-vertical flow boundary conditions.

However, a consideration of the problem of (52) on an infinite domain (i.e. \( z \to \pm \infty \)) where, instead, we require boundedness and asymptotic decay of all quantities as \( |z| \to \infty \), suggests that the solution for \( \Pi(z) \) could be the following,

\[
\Pi(z) = z^\frac{3}{2} K_{\frac{3}{2}} \left( \frac{1}{2} \frac{\kappa_p}{F_e} z^2 \right),
\]

where \( K_{\frac{3}{2}}(z) \) is the Modified Bessel Function of the Second Kind (Abramowitz & Stegun, 1972). Indeed an asymptotic expansion of the leading order behavior of \( K_{3/2}(z^2) \) in the large argument limit shows that it behaves like a Gaussian, i.e. \( \exp \left( -\frac{1}{2} \frac{\kappa_p}{F_e} z^2 \right) \), for large \( z^2 \). On its surface this behaviour seems to satisfy the requirements on the functions as \( z \to \pm \infty \). Although the above conclusion is correct for \( z \to \infty \), to extend this conclusion as \( z \to -\infty \), based on the above representation would be wrong. In fact, given the functional form (55) it becomes a matter of subtlety as to how one must cross the point \( z = 0 \). In Appendix C it is shown that the behaviour of (55) for \( z < 0 \) is
where we have introduced the basic state entropy $c^2\equiv \frac{\partial \omega^2}{\partial z}\left(\frac{\partial p}{\partial z}\right)^2$.

Inspection reveals exponential divergence as $z \to -\infty$ since $\mathcal{I}_c(x)$ behaves exponentially as $x \to \infty$. It appears that this analysis indicates that there are no bounded solutions possible for $\Pi(z)$ and, consequently, it implies there are no $\alpha$-modes permitted when $N^2(z) \sim z^2$ in the context of the BE model.

3 LARGE SHEARING BOX EQUATIONS: THE QHSG APPROXIMATION AND LINEARIZED DYNAMICS

Given the clues revealed by using the QHSG for the BE, we consider the same scalings expressed in (42) and apply them to the full LSB equations (1-5). The major departure here, of course, is that disturbances are now not incompressible. One scaling relation is to say that the density and pressure variables are of comparable scale, i.e. $\mathcal{O}(\rho) = \mathcal{O}(p) \sim 1$. Therefore the QHSG reduction of the nonlinear LSB becomes

\[
(\partial_t - q\Omega_0 x \partial_y)\partial_y + \nabla \cdot (\rho \partial_t + \rho u) = 0,
\]

\[
0 = 2\Omega_0 v - \frac{\partial p}{\partial x} + \mathcal{O}(\alpha^2),
\]

\[
(\partial_t - q\Omega_0 x \partial_y)\partial_y + \nabla \cdot (\rho \partial_t + \rho u + (2 - q)\Omega_0 u) = -\frac{\partial p}{\rho_0} + \rho
\]

\[
0 = -\frac{\partial p}{\rho_0} - \rho_0 q\Omega_0 u + \mathcal{O}(\alpha^2),
\]

\[
(\partial_t - q\Omega_0 x \partial_y)\Sigma + \nabla \cdot (\Sigma \partial_x + \Sigma) = 0,
\]

where we have introduced the basic state entropy $\Sigma_0$ and its dynamically varying counterpart $\Sigma$ which are defined by

\[
\Sigma_0 = \ln \frac{\rho_0}{\rho_0}, \quad \Sigma = \ln \frac{1 + \frac{p}{\rho_0}}{1 + \frac{\rho}{\rho_0}},
\]

where $\gamma$ is the the usual thermodynamic ratio of specific heats. Linearizing (57-61) and sorting through the algebra (see Appendix D) leaves us with a single master equation for the pressure perturbation

\[
(\partial_t - q\Omega_0 x \partial_y)\left[\frac{\omega^2}{g}\partial_z \rho_0 + \frac{\omega^2}{N^2_c G_0^2} \left(\frac{\rho}{\rho_0} \partial_z p + \partial_z \rho_0\right) + \partial_z^2 p\right] = 0.
\]

The generalized Brunt-Vaisala frequency is defined by

\[
N^2_c \equiv \frac{g}{\gamma} \partial_t \ln \frac{\rho_0}{\rho_0},
\]

while the (non-dimensional) adiabatic soundspeed $c^2$ is defined by

\[
c^2 \equiv \frac{\gamma p_0}{\rho_0}.
\]

(53) is the LSB equivalent, in this QHSG limit, of a potential vorticity for a local section of a circumstellar disc. Comparing this equation for the potential-vorticity with the analogous one for the BE in (49) reveals some differences between them being, namely,

\[
\frac{\omega^2}{g} \partial_z \rho_0, \quad \frac{\partial \omega^2}{\partial z} \frac{g}{N^2_c G_0^2} \partial_z p.
\]

The first of these is associated with the time rate of change of the density fluctuation in the continuity equation. This is explicitly absent in the BE due to the assumption of incompressibility. The second of these is associated with the generalized entropy fluctuation and is inversely proportional to the soundspeed. This term is absent in the Boussinesq Equations because the assumption of incompressibility is equivalent to the interpretation that the soundspeed is infinite.

Having $N^2_c < 0$ is equivalent to the Schwarzschild condition for buoyant instability (Tassoul, 2000). As before, we assume that the atmosphere is stable to buoyant oscillations ($N^2_c > 0$). However we must also say something about the soundspeed $c$: for the sake of this discussion we will assume that it is a constant with respect to $z$, that is, we assume the atmosphere is isothermal. We proceed toward determining normal mode solutions of the expression inside the square brackets of (53).

\[
\frac{\omega^2}{g} \partial_z \rho_0 + \frac{\partial \omega^2}{\partial z} \left(\frac{g}{c^2} \partial_z \rho_0 + \partial_z \rho_0\right) + \partial_z^2 \rho_0 = 0
\]

Assuming separable solutions of the form (54) we find, once again, the following two problems to solve:

\[
\frac{\omega^2}{g} \partial_z \Pi + \frac{\partial \omega^2}{\partial z} \left(\frac{g}{c^2} \Pi + \partial_z \Pi\right) = -k^2 \Pi,
\]

\[
\partial_z^2 \Pi_0 = k^2 \Pi_0
\]

The separation constant $k_x$ is the same as before.

Because the equation for $\Pi_0$ is the same as in the BE model, cf. (53), it immediately follows that the same stability properties that was determined for the BE apply here too if the set of $k_x$ values are all real (i.e. e-modes). For this study we restrict our attention to finite vertical domains (see below). It means, then, that the task that remains is to determine the eigenvalues of $k_x$ by seeking solutions of (57) subject to the boundary condition that the vertical velocity vanishes at $z = \pm 1$. In the LSB, this condition amount to setting

\[
\frac{1}{N^2_c G_0^2} \left(\frac{g}{c^2} \partial_z \Pi + \partial_z \Pi\right) = 0,
\]

at $z = \pm 1$. (see Appendix D). Also, as before, we consider solutions to $\Pi$ that are either sinusuous or varicose.

We emphasize that we are restricting our attention here to solutions on a finite $z$ domain. This is because attention needed to treat such problems on an infinite domain is challenging and it is, thus, outside the scope of this current work (see Discussion).

Aside from very special values of the parameters, there are no simple or analytically tractable solutions to the ordinary differential equation posed by (57). Therefore we numerically solve for this equation and $k_x$ using a fourth-order variant of the NRK scheme discussed in the previous section. We use a grid of 300 points which lets us determine solutions up to machine accuracy (i.e. an error of less than $10^{-11}$). The solutions were all normalized by setting $\Pi = 1$ at $z = 1$. We verify the robustness of the numerical scheme by using it to solve the simpler equation (49) and comparing the numerically generated results against the exact solutions (54).

There are two parameters that govern the solutions. The first of these is the scale measure, $\bar{F}_S$, of the height-dependent Froude-number

\[
\bar{F}_S = \frac{\omega^2}{N^2_c G_0^2} = \bar{F}_S^2 \left(\frac{\pi}{2}\right)^2.
\]

Given that this atmosphere is isothermal this Froude-number scale is measured by the parameter

\[
\bar{F}_S^2 = \frac{\omega^2 \gamma}{\Omega_0^2 (\gamma - 1)}.
\]

For a medium dominated by molecular hydrogen $\gamma \approx 7/5$. It means that in a Keplerian flow $\bar{F}_S^2 \approx 7/2$. The second parameter is

\[3 General solutions of this equation are linear combinations of hypergeometric functions which require numerical evaluation anyway.
the relative measure of the vertical scale height of the atmosphere defined by $H$ and given to be

$$H^2 \equiv \frac{c^2}{\Omega_0^2} = \frac{\gamma R_\mu \hat{T}}{\Omega_0^2},$$

in which $R_\mu$ is the non-dimensionalized gas constant for the given composition, $\hat{T}$ is the non-dimensionalized temperature of the atmosphere. This quantity is essentially the same as the classic $\epsilon$-parameter governing thin-disc theory (Shakura and Sunyaev, 1973, Lynden-Bell and Pringle, 1974) When $H$ is small, the atmosphere is very shallow and, consequently, cold.

The solutions that we scan all show that the $k_F$ values are always real indicating these are $e$-modes (cf. Section 2.3). We were unable to find purely imaginary solutions for $k_F$ on this finite domain. Thus it implies that the stability properties determined for $e$-modes (i.e. Section 2) carry over to here too and that this system does not support $o$-modes on this finite-domain.

4 SUMMARY AND DISCUSSION

4.1 The QHSG and the persistence of the SRI with height dependence in $N^2$

We have achieved an extension of the SRI to models which takes into consideration the vertical structure of the physical environment. One of the departures taken here from previous work is to explicitly include the effects of a vertically varying Brunt-Vaisala frequency. The results of the previous sections shows that the SRI, under channel-wall boundary conditions, persists unaltered irrespective of the model equations considered (i.e. either the BE or the LSB) or type of mode (i.e. $e$-mode or $o$-mode) so long as one is in the inviscid-QHSG asymptotic limit. This is not to say that if one relaxes the restriction of long horizontal length scale disturbances (i.e. small $\alpha$) then this instability will not continue - we merely mean to say that its existence under those conditions remains open. It does seem likely, however, given the pattern of the presence of the SRI in D05, that it will do so also in the $O(\alpha) \sim 1$ case too.

The advantage of the inviscid-QHSG approximation employed here is that an analytical analysis of normal-modes is possible through a separation of variables. In general cases where both $g$ and the other state variables like $\rho_0$ and $p_0$ are functions of the vertical coordinate $z$, the resulting linear equations and mode structure are generally non-separable. This makes for assessing the normal-mode behaviour of disturbances to be challenging (at best) although not impossible, as has been shown here. In this sense it appears that this is a reasonable peek into situations where complicating background structure in both the radial direction (here the shear) and vertical direction (here gravity and other state variables) can be taken into account together.

It is also worth noting that in the inviscid QHSG limit of both the BE and LSB the radial structure of the eigenmodes (be they either exponential or oscillatory) are unaffected by the vertical stratification: in other words, the radial eigenfunctions are always the same. As a result, the stability criterion turns out to be insensitive to vertical stratification in the state variables of the system.

The QHSG limiting process is achieved by looking at disturbances with horizontal length scales that are large compared to the other dimensions which, therefore, implies that the associated time scale of disturbances can scale with proportional smallness. From the framework of the the LSB equations, the implication is that one recovers the part of the dynamics of inertia-gravity waves modified, to some extent, by the effect of weak compressibility (dilatation) and a finite sound speed. The inclusion of these effects is not to say that some facet of acoustic disturbances are recovered in this limit: this is precisely because the time scales associated with acoustics are much shorter than the time scales explored here. In this sense this asymptotic limit naturally filters out direct acoustic effects and preserves the dynamics of disturbances that would usually be associated with Rossby waves in geophysical flows (Pedlosky, 1987). This is also not meant to imply that acoustic effects are unimportant (an issue that is far from settled), it merely means that this limit is effective at isolating the dynamics of these waves.

4.2 On the nature of $e$-modes and $o$-modes with height dependent $N^2$

D05 showed that there are two types of SRI modes which are characterized by the quality of the mode’s radial structure function: either exponential or oscillatory. Because both the vertical background temperature gradient and component of gravity are constant D05 showed that both $e$-modes and $o$-modes are supported for such a model atmosphere.

For example, $e$-modes, which have vertical structure functions which are sinusoidal with respect to the vertical coordinate, are permitted in D05 even if the atmosphere extends mathematically to infinity. Because $N^2$ is constant with respect to $z$, the corresponding vertical structure functions have no envelope growth or decay with respect to the vertical coordinate. As such, such solutions satisfy reasonable expectations of boundedness as the atmosphere extends indefinitely.

By contrast the analysis of the QHSG limit of the BE equations, in which $N^2$ depends on $z$ quadratically, shows that the re-
sulting vertical structure functions for $e$-modes to have envelope structures which grow ($\sim \sqrt{|z|}$). This fact makes it impossible to meaningfully impose boundary conditions or boundedness conditions on solutions as $|z| \to \infty$. The analysis performed here seems to indicate that $e$-modes are restricted to BE systems involving finite vertical domains with height dependent $N^2$.

$O$-modes in model atmospheres with quadratic dependence of $N^2$ with respect to $z$ are not allowed. In atmospheres with constant $N^2$ $o$-modes are admitted on account of the exponential decay of the vertical structure function. For $N^2(z) \sim z^2$ (cf. 2.3.2) it appears there is no way to construct bounded solutions in the directions $z \to \pm \infty$ simultaneously. It was also demonstrated that $o$-modes are ruled out on finite domains.

The conclusions regarding the existence of $o$-modes is mainly based on the analysis of the BE with $N^2(z) \sim z^2$ in the QHSG limit. We showed also in Section 3 that similar conclusions seem to hold for SRI $e$-modes and $o$-modes in the isothermal-QHSG limit of the LSB model set when considered only on a finite domain. It is not entirely clear how the existence properties of SRI modes are affected when: (i) the domain mathematically extends to infinity, (ii) the soundspeed varies with height. These questions are reasonable points of departure for further investigation.

4.3 The absence of the SRI for non-reflecting boundaries

The troubling aspect of this investigation is that when one considers boundary conditions other than no-flow conditions on the radial boundaries, the instability appears to vanish in the asymptotic limit considered. When the Lagrangian pressure is zero on both radial boundaries the analysis predicts that there are no normal-modes with the time scales assumed and, furthermore, there are only modes associated with the continuous spectrum. It is probably safe to conclude that for this type of boundary condition, that there are no normal-modes whose frequencies have magnitudes on the order of or greater than the small scaling parameter ($\alpha$, the horizontal wavenumber) of the limit explored here. In situations where there is a mixture of no-flow conditions on one boundary and no Lagrangian perturbations on the other, only normal-mode is admitted by the system which propagates with no growth or decay.

The circumstance encountered here shares a number of similarities with the Eady problem of baroclinic instability in geophysical shear flows (Eady 1949, Criminale & Drazin, 1990). Drazin and Reid (1994) show that the Eady problem is essentially equivalent to the stability of inviscid plane Couette flow (pCf) subject to boundary conditions where the pressure perturbations are fixed on the channel walls, instead of the usual condition in which the normal velocities are set to zero. As Case (1960) showed, inviscid pCf flows with no-normal flow boundary conditions admit only continuous spectrum modes and no discrete modes. By contrast, Criminale & Drazin (1990) showed that the Eady problem has, in addition to the continuous spectrum, a number of discrete modes present (which are possibly unstable under suitable conditions of the disturbances) when disturbances in inviscid pCf flow have fixed pressures at the boundaries (Criminale et al. 2003).

For the SRI investigated here, an entirely analogous situation occurs: the linear operator governing the system is mathematically equivalent to the one characterizing inviscid pCf, but, the variables and stability characteristics are interchanged. Whereas in inviscid pCf the operator operates on the radial velocity (wall-normal) in the SRI case here it operates on the pressure perturbation. Thus the inviscid pCf problem admits normal modes (no normal modes) for fixed pressure (no normal flow) boundary conditions while the SRI problem admits normal modes (no normal modes) for no normal flow (fixed pressure) conditions. Of course, both scenarios reveal the presence of a continuous spectrum irrespective of the boundary conditions employed. It seems as though the inviscid pCf and SRI problems have properties and stability characteristics that are interchanged.

4.4 Questions and a conjecture

From a more physically motivated standpoint, we have experimented with these set of boundary conditions because they allow one to exert some comparative control between conditions. It is shown in Appendix A that disturbances in the BE, subject to these boundary conditions, have a total disturbance energy, $E$, which evolves according to the exchange of energy that takes place between the (Keplerian) shear and disturbance modes via the Orr-Mechanism and measured by the Reynolds Stress term, i.e. the RHS of $(\ref{eq:A4})$. Conditions other than these would cause there to be some net work (positive or negative) to be performed on the layer during the ensuing course of the disturbances (Schmid & Hemmingson, 2001).

It is a puzzle, then, that in this QHSG limit there is an instability in the case of no-normal flow conditions and none otherwise. Although this is merely a conjecture, is it possible that the SRI occurs because of the double reflecting boundary conditions? The instability shares many of the same properties of the acoustic instability uncovered by Papaloizou & Pringle (1984,1985), otherwise known as the PP instability (Li et al. 2000). It is an instability of an acoustic disturbance in a domain like this with reflecting inner and outer walls in which there exists a critical layer, sometimes referred to as a corotation radius (Li et al., 2000). The waves grow in a resonant fashion because the reflecting walls, either one or both, allows for repeated passages of the wave across the critical layer which allows it to draw energy from the shear (Drury, 1985). It was found that the PP instability vanishes when the amplifying agent, usually the second reflecting boundary, is removed (Narayan, Goldreich & Goodman, 1987).

Like the PP instability, the SRI as determined in this work are waves existing in a domain containing a critical point along with reflecting boundaries. When one of the boundaries no longer reflects, there is no instability. Although these are neither compressible modes nor two-dimensional is it possible that the SRI arises in an analogous way due to the pathology that afflicts the PP instability? This is an open question which should be clarified in future work.

A clue towards this end might be found in the observation that there exists a second energy integral, as developed in Appendix D, involving a total energy expression $F$ which says something interesting. The domain integral of the quantity $F$ is conserved under no-normal radial flow conditions whereas it is not for the others. Perhaps the instability is related to this constraint placed on the dynamics of the system?

4.5 Flow two-dimensionalization and another conjecture

We demonstrated in Section 2.1 that there exists a conserved quantity ($\Sigma$) of the general linearized system of the BE that is advected by the local basic shear. This quantity, which looks like a potential vorticity, is conserved independent of the QHSG asymptotic limit explored. Its analog is implied to exist in the for the LSB model.
We also noted that in the limit where the buoyancy oscillations become very strong the term associated with it in $\Xi$ may become less important. In this circumstance it implies that the flow will take on a nearly two-dimensional character.

In particular if the quantity $N^2$ becomes large then, according to (19) one possible scaling between quantities in an initial value problem is to have $\Theta$ remain an order 1 quantity while the vertical velocity, $w$, be $O\left(N^{-2}\right)$. If all other quantities remain corresponding order 1, that is to say if $u, v, P, \partial_x, b, \rho, \partial_t \sim O\left(1\right)$, then to lowest order it would imply that the disturbances are dynamically in hydrostatic equilibrium and it would imply that the flow is nearly two-dimensional conserving its vertical component of vorticity (cf. (23-24)).

Barranco and Marcus (2005) demonstrated, in their shearing sheet simulations of a stratified fluid, the appearance of coherent vortical structures with vorticity vector pointing in the vertical direction. When they manifested themselves, the anticyclonic vortices appear near the vertical boundaries of the system, in other words, in that part of the atmosphere where the vertical component of gravity is greatest in magnitude. They also demonstrated the robustness and persistence of these anticyclones by artificially removing the vortex structure (after having developed) and replacing the flow field with noise. They show that the noisy spectrum quickly redeveloped into coherent anticyclones(s) much as it is known to do so in two dimensional shear flows (e.g. Umurhan & Regev, 2004).

This fact is consistent with the implications suggested by the advected conservation of the linear quantity $\Xi$. Of course, only a non-linear reformulation and reexamination of $\Xi$ can offer a more solid basis to any connection that may exist here.

Is it possible that it is a generic feature of stratified flow with a Couette shear profile and a vertically dependent Brunt-Vaisala frequency (e.g. appropriate for a local representation of a circumstellar disk as here) to behave two-dimensionally in substantial parts of the atmosphere significantly away from the disk midplane, i.e. those regions dominated by a large Brunt-Vaisala frequency?

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APPENDIX A: AN INTEGRAL STATEMENT FOR THE BOUSSINESQ EQUATIONS

It is instructive to consider integrals of the system since they can help guide one into deciding which boundary conditions to be used. We begin with by noticing that for the situations considered in here, the functional forms relating $g$ and $\partial_t T_b$ are always constant multiplicative factors of each other (see Sections 2.1 and 2.2). Therefore we take the ratio of these two quantities to be always a constant, that is,

$$\partial_t T_b / g = \text{constant}$$

over the full spatial domain under consideration. With this assumption in hand one may (i) take the scalar product of (8-10) and $\rho_0 u'$, (ii) multiply (11) by $\theta g \alpha_p / \partial_t T_b$ and (iii) adding the results of (i) and (ii) together and making use of the incompressibility condition (7) to reveal

$$(\partial_t - \partial_0 \alpha_p \partial_p) \mathcal{E} + \mathbf{v} \cdot \nabla (\mathcal{E} + p) = 0,$$  \hspace{1cm} (A1)

where

$$\mathcal{E} = \frac{\partial_0 u'^2}{2} + \frac{g \alpha_p}{\partial_t T_b} \frac{\partial^2}{2}$$

Using condition (7) once more, we may integrate (A1) over the full spatial domain to find,
\[ \frac{dE}{dt} = -\int_S (\mathcal{E} + p) u' \cdot \hat{n} dS - \int_V qu' v' dV \quad (A2) \]

with
\[
E \equiv \int_V \left( \frac{\bar{\rho}_0 u'^2}{2} + g \alpha \frac{\theta^2}{\bar{T}_0} \right) dV,
\]
in which \( V \) and \( S \) is the volume and surface-boundary of the domain in which \( \hat{n} \) is the unit normal of the surface. The above result is true in general for both linear and nonlinear perturbations. We interpret the quantities in \( \mathcal{E} \) in the following way: the term \( \frac{\bar{\rho}_0 u'^2}{2} \) represents the kinetic energy in the disturbances while the term \( g \alpha \frac{\theta^2}{\bar{T}_0} \) represents the energy in thermal processes. By definition \( \mathcal{E} \) is zero in steady state, while the steady pressure is constant, denoted by \( \bar{p} \).

The global integral \( E \), which we refer to as the total energy in disturbances, can change due to the influx of \( \mathcal{E} \) across the boundaries, through work done upon the system externally as represented by the boundary flux term \( \int_S pu' \cdot \hat{n} dS \), and finally due to the interaction with the background shear via the Reynolds stress term \( -\int_V qu' v' dV \) (for a discussion of this see Schmid & Henningson, 2002).

The general evaluation of (A2) may proceed once boundary conditions are specified. As we have stated earlier, we will consider disturbances to be periodic in the \( y \) and \( z \) directions (sinuous or varicose for the latter). The radial boundary conditions and the motivation for their choices deserve some additional reflection. We remind the reader that one of the goals of this study is to assess whether or not the SRI is an intrinsic instability of the fluid and not some artifact of boundary conditions. One reasonable control is to require that there is neither work done on the system from outside nor there be any flux of energy across the bounding walls. This requirement requires that either the normal velocities are zero on either of the two walls or that (for linear disturbances only) the Lagrangian pressure perturbations are zero at the two bounding surfaces. A mixture of these can also affect the same outcome. That is to say, for example, one could require that the normal velocity at one bounding surface is zero while the Lagrangian pressure perturbation is zero at the other surface. Imposing these conditions therefore implies that \( E \) can change only due to the interactions of the perturbations directly with the shear, in other words, all such disturbances behave according to
\[
\frac{dE}{dt} = -q \int_V u' v' dV. \quad (A3)
\]

In this sense, then, these solutions share some common property that allows for some comparison.

**APPENDIX B: DEVELOPMENT OF THE CONTINUOUS SPECTRUM MODE**

The discussion here largely follows the tack taken by Case (1960). Beginning with (35) we can say that
\[
(\partial^2_t - F_0^2 \beta^2) P_0 = 0, \quad (B1)
\]
is true for \( x \neq x_c \) where
\[
x_c = \frac{\omega_1}{q \bar{T}_0 \alpha_1}.
\]

Taking \( P_0^{\pm} \) to denote the solution of (B1) to the left and right (respectively) of \( x_c \), and, assuming zero pressure conditions at \( x = 0, 1 \) we have that

\[ P_0^- = A^- \sinh [k_p x], \quad P_0^+ = A^+ \sinh [k_p (1 - x)] \quad (B2) \]

Then, to enforce continuity of the pressure at \( x = x_c \) we see that
\[
A^- = A^+ \frac{\sinh [k_p (1 - x_c)]}{\sinh [k_p x_c]}.
\]

Once some normalization is specified, that is, a value of \( A^+ \) is assumed, the solution is complete. For numerically generated solutions, we set \( P(x = 0.99) = 0.01 \). In summary, then, we have the continuous mode pressure eigenfunction, \( P_0^{\pm} \), is
\[
P_0^{\pm}(x) = A^+ \begin{cases} \frac{\sinh [k_p (1 - x_c)]}{\sinh [k_p x_c]} \sinh [k_p x] & 0 < x < x_c, \\ \sinh [k_p (1 - x)] & x_c < x < 1. \end{cases} \quad (B3)
\]
in which \( A^+ = 0.01/\sinh [0.99 \times k_p] \). Note that it means that any value of \( \omega_1 \), which satisfies the requirement \( 0 < x_c < 1 \) is an allowed solution. This is the nature of the continuous spectrum (Schmid & Henningson, 2001). Also, the mode associated with the continuous spectrum is not technicall defined at \( x = x_c \).

**APPENDIX C: SUBTLETY IN THE \( K_c \) OSCILLATING MODE SOLUTION**

A closer analysis of (35) reveals that it is everywhere analytic and, therefore, an entire function along the real \( z \) axis, despite the presence of a branch point at \( z = 0 \) for the \( K_c \) Bessel function. It
means, therefore, that in order to express the nature of this function as you one passes through $z = 0$ one must be very careful about the relative phases that are incurred by crossing the $z = 0$ point. To be more specific, let us define

$$\zeta \equiv \left( \frac{F_p}{F_c} \right)^{1/2} z$$

and rewrite the solution (85) according to its composition of Modified Bessel Functions of the first kind,

$$\Pi(z) \sim \zeta^3 \left[ I_0 \left( \frac{1}{2} \zeta^2 \right) - I_1 \left( \frac{1}{2} \zeta^2 \right) \right],$$

which, according to the series representation of $I_m$ would be

$$\zeta^4 \left[ 3(Y(3/4, \zeta)) - \zeta^2 Y(-3/4, \zeta) \right] \left( \frac{d}{d z} \right) \zeta$$

where

$$\gamma(\nu, \zeta) = \left( \frac{1}{\nu} \right)^\nu \sum_{k=0}^{\infty} \frac{(\nu k)^k}{k!} \Gamma(\nu + k + 1).$$

(Abramowitz & Stegun, 1972). The function $\gamma$ is symmetric with respect to the reflections $\zeta \rightarrow -\zeta$. However, the pre-factors appearing in the above expressions imply that one must be very careful in interpreting the function as $\zeta$ crosses over zero. This is best illustrated by considering the behaviour of (85) for $\zeta < 0$. Expired initially without restriction of $\zeta$ is,

$$\zeta^4 \gamma(3/4, \zeta) - \zeta^2 \gamma(-3/4, \zeta)$$

Then if we restrict attention to $\zeta < 0$ by defining the variable $s = -\zeta$ and restricting attention to $s > 0$ we see that the above becomes

$$= -s^3 \gamma(3/4, s) - s \gamma(-3/4, s)$$

$$= s^2 \left[ \frac{1}{2} s^2 Y(3/4, s) - s^2 Y(-3/4, s) \right]$$

$$= s^2 \left[ -s^2 Y(3/4, s) - s^2 Y(-3/4, s) \right]$$

$$= s^2 \left[ -2s^2 Y(3/4, s) + K(2s^2) \right]$$

rewriting the above we see that the solution $\Pi(z)$ for $z < 0$ becomes as it is expressed in the text (85).

APPENDIX D: QHSG LINEARIZATION OF THE LSB

We linearize (77-81). It now means that $\rho'$ and $p'$ are the linearized density and pressure fluctuations. The resulting equations become

$$\frac{\partial}{\partial t} - qQm_u + \partial_x m_v + \partial_z m_w = 0,$$  

$$\frac{\partial}{\partial t} - qQm_v + \partial_y m_u + \partial_z m_w = 0,$$  

$$\frac{\partial}{\partial t} - qQm_w + \partial_y m_v + \partial_z m_u = 0,$$  

and the perturbed entropy

$$\Sigma' = \frac{\gamma}{\rho_b} \left( \frac{p'}{c^2} - \rho' \right).$$

Operating on (D3) with $\partial_x$ and then making use of (D1) reveals

$$(\partial_t - qQ \partial_x) \left( \partial_x m_v - (2 - q)Q \partial_x \rho' \right) = (2 - q)Q \partial_x \rho'$$

This is followed by multiplying (D5) by $Q \partial_x S_b$ and then operating on the result with $\partial_z$. This gives

$$(\partial_t - qQ \partial_x) \frac{\partial}{\partial z} Q \partial_x (2 - q) \frac{\rho_b}{\partial_z S_b} = -(2 - q)Q \partial_x \partial_z m_w.$$

Adding these two equations gives

$$(\partial_t - qQ \partial_x) \left[ \partial_z m_v - (2 - q)Q \partial_z \rho' + \frac{\partial}{\partial z} Q \partial_x (2 - q) \frac{\rho_b}{\partial_z S_b} \right] = 0.$$

Given the definition of $\Sigma'$, the hydrostatic relationship (D4), and the radial geostrophic balance (D2) recovers (85).

APPENDIX E: A SECOND INTEGRAL STATEMENT OF THE BOUSSINESQ EQUATIONS

Following the steps in Section X one may generate a second energy integral. The dynamical equations (7-11) describe the evolution of the stratorotational instability, i.e. the velocity fluctuations over and above the steady state Keplerian velocity, and the temperature fluctuations. We denote the total velocity of disturbances in the frame of the shear box as $U$ and given to be

$$U \equiv -qQ \partial_x \hat{y} + u' \equiv \{ u', v', w' \} = -qQ \partial_x \hat{y}$$

As such the governing equations of motion (8-11) are more concisely written in vector form as

$$\frac{\partial}{\partial t} U + U \cdot \nabla U = -\frac{1}{\rho_b} \nabla p + \frac{2 \Omega_0}{\rho_b} \hat{z} \times (U + qQ \partial_x \hat{y}) + \frac{1}{\rho_b} g \alpha \hat{y} \theta \hat{z}$$

$$\frac{\partial}{\partial t} \theta + U \cdot \nabla \theta = -w \partial_z T_b$$

$$\nabla \cdot U = 0.$$  

As we have posited $\partial_t, T$, and $q$ multiplicative factors of each other over the full spatial domain under consideration. With this assumption in hand one may (i) multiply (E2) by $\rho_b \nu$, (ii) multiply (E3) by $\theta g \alpha \partial \rho / \partial T_b$ and (iii) adding the results of (i) and (ii) together to yield

$$\partial_t \mathcal{F} + U \cdot \nabla (\mathcal{F} + p) = 0,$$

where

$$\mathcal{F} \equiv \frac{\rho_b U^2}{2} + \frac{g \alpha \theta^2}{2 \partial T_b} - qQ \partial_x \hat{y}^2.$$

With use of the incompressibility condition (E4) we may integrate (E5) over the full spatial domain to find,

$$\frac{d \Phi}{dt} = -\int_S (\mathcal{F} + p) U \cdot \hat{n} dS,$$

with

$$\Phi \equiv \int_V \mathcal{F} dV = \int_V \left( \frac{\rho_b U^2}{2} + \frac{g \alpha \theta^2}{2 \partial T_b} - qQ \partial_x \hat{y}^2 \right) dV,$$

in which $V$ and $S$ are as they were defined before. We interpret the quantities in $\mathcal{F}$ in the following way: the term $\rho_b U^2 / 2$ represents the kinetic energy, the term $-qQ \partial_x \hat{y}^2$ is like a potential energy and $g \alpha \theta^2 / 2 \partial T_b$ represents the energy in thermal processes. The global integral $\Phi$ can change due to the influx of $\mathcal{F}$ across the dynamically undulating boundaries as well as through the work done upon the system externally as represented by the boundary flux term $\int_S p U \cdot \hat{n} dS$.
The point of this exercise is to note that only for no-normal flow boundary conditions does $\Phi$ remain fixed for disturbances. The other conditions, like fixing the Lagrangian pressure, can cause $\Phi$ to vary over the course of its evolution. This is because although $\bar{\rho}$ may be constant in steady state, the total quantity $F$ is not constant in steady state where a simple inspection of its definition clearly reveals. By fixing only the Lagrangian pressure fluctuation, the otherwise moving boundary can allow $F$ to seep in and out of the domain.

Perhaps, then, the reason for the existence of the instability under no-normal flow boundary conditions arises because of this preserved property of the disturbances. The reflection property of the boundaries perhaps traps energy in a way that causes growth to be encouraged. In this case, the energy of the disturbances must come from the energy contained in the background shear state and because there is an overall trapping of the energy, a runaway extraction processes takes place - ironically, leaving the total energy budget, $\Phi$, fixed over the course of the evolution.