We only require generalized chiral symmetry and $\gamma_5$-hermiticity, which leads to a large class of Dirac operators describing massless fermions on the lattice, and use this framework to give an overview of developments in this field. Spectral representations turn out to be a powerful tool for obtaining detailed properties of the operators and a general construction of them. A basic unitary operator is seen to play a central role in this context. We discuss a number of special cases of the operators and elaborate on various aspects of index relations. We also show that our weaker conditions lead still properly to Weyl fermions and to chiral gauge theories.

1. Introduction

In investigations of chiral fermions on the lattice the Dirac operator $D$ is largely required to satisfy the Ginsparg-Wilson (GW) relation

$$\{\gamma_5, D\} = \rho^{-1} D \gamma_5 D,$$

with a real constant $\rho$, and to be $\gamma_5$-hermitian,

$$D^\dagger = \gamma_5 D \gamma_5.$$

In this GW case Lüscher has pointed out that the classical action has a generalized chiral symmetry which corresponds to the condition

$$\gamma_5 D + D \hat{\gamma}_5 = 0 \quad \text{with} \quad \hat{\gamma}_5 = \gamma_5 (1 - \rho^{-1} D)$$

for the Dirac operator $D$.

We note that conditions (1.1) and (1.2) imply that the operator $1 - \rho^{-1} D$ is unitary and $\gamma_5$-hermitian. Therefore requiring $D$ to have the form

$$D = \rho (1 - V) \quad \text{with} \quad V^\dagger = V^{-1} = \gamma_5 V \gamma_5$$

is equivalent to imposing (1.1) and (1.2). Further this indicates that (1.3) actually has the general form

$$\gamma_5 D + D \gamma_5 V = 0 \quad \text{with} \quad V^\dagger = V^{-1} = \gamma_5 V \gamma_5.$$

We have recently found that, instead of imposing (1.1) and (1.2) as is usually done, only requiring (1.3) and (1.2) leads to a large class of Dirac operators.
describing massless fermions on the lattice, which includes GW fermions, the ones proposed by Fujikawa and the extension of the latter as special cases. Nevertheless it has turned out that our weaker conditions still lead properly to Weyl fermions and to chiral gauge theories.

On the basis of our conditions we have performed a detailed analysis of the properties of the Dirac operators as well as worked out a general construction of them. This has become possible because for the general class $D$ gets a function of $V$, so that the spectral representation of $V$ can be used to deal with $D = F(V)$. In addition to the conditions (1.5) and (1.2), we have required that $F$ must allow for a nonvanishing index of the Dirac operator.

In the present paper we use this framework to give an overview of developments in the field of massless fermions on the lattice. We start with general properties of the Dirac operators and index relations. We then turn to the general construction of such operators and after this to the discussion of special cases. Next various issues related to the index are addressed. Finally we consider chiral gauge theories.

In Sec. 2 we first give the basic relations which immediately follow from our general conditions. We then introduce spectral representations and obtain related properties. After this we derive various relations for the index of $D$ in a general way. It turns out that solely the operator $V$ enters the index for the general class of $D$. The result generalizes ones obtained in the overlap formalism and in the GW case before. Similarly the sum rule, first observed by Chiu in the GW case, is found to hold in general and to rely only on $V$.

In Sec. 3 we describe the general construction of Dirac operators of the class. We use spectral functions $f$, which on the one hand side describe the location of the spectrum of $D$, and on the other provide a powerful tool for the construction. The general form of $f$ is found to involve two functions, $w$ and $h$, which must have specific properties and determine the form of $D$. Particular choices of $h$ turn out to require related forms of $V$. We give a realization of such $V$ which generalizes the $V$ implicit in the overlap Dirac operator of Neuberger.

In Sec. 4 we discuss the special cases which can be found literature. We start with the GW relation, also adding some remarks about properties of its general form. We then show that the extended Fujikawa type belongs to the general class and bring $D$ of the respective original proposal into the form $D = F(V)$.

In Sec. 5 we study further special cases. First, to see effects of nontrivial (non-constant) $w$ we use trivial $h(x) = x$. In a special case of this $D$ is given by an expansion in powers of $V$. In examples of nontrivial $w$ together with nontrivial $h(x)$ it becomes still explicit how conditions on $w$ are related to behaviors at the eigenvalues $±1$ of $V$. We then present a method for getting a subclass of nontrivial functions $h(x)$ and work out a particular example of this.

In Sec. 6 we consider the relation between the index of the Dirac operator $D$ and the spectral flows of the hermitian Wilson-Dirac operator. This relation, which is important in the overlap formalism, turns out to extend to the generalized Wilson-Dirac operator considered here. We also derive the differential equation.
which describes these spectral flows in an exact way.

In Sec. 7 we address questions which arise for the index relations in the continuum limit. After an overview of the relevant issues, we make sure that the sum rule for the index still holds in the limit. For this purpose we firstly give the proper definition of the trace expressions involved and secondly show that a continuous part of the spectrum does not contribute. We then elaborate on the structural difference to the Atiyah-Singer framework which is reflected by the sum rule.

In Sect. 8 we turn to the alternative form of \( V \) realized in work by Chiu and found not to describe topological properties on the finite lattice correctly. This form is of the Cayley-transform type for which we have shown that correct properties generally only arise in the limit. After some remarks about explicit \( V \) we present the analysis of Cayley-type \( V \). We then also make details of the spaces involved precise.

In Sec. 9 we show that the general class of operators considered leads still properly to Weyl fermions and to chiral gauge theories. In this context it turns out that quantities like the chiral degrees of freedom or the gauge anomaly only involve \( V \) for the whole class. We also add some remarks about proper accounting for zero modes of \( D \).

2. Properties of Dirac Operators

2.1. Basic Relations

As pointed out in the introduction, we impose the conditions
\[
\gamma_5 D + D \gamma_5 V = 0 \quad \text{with} \quad V^\dagger = V^{-1} = \gamma_5 V \gamma_5, \tag{2.1}
\]
\[
D^\dagger = \gamma_5 D \gamma_5 \tag{2.2}
\]
on \( D \). We first note that using (2.2) one gets from (2.1)
\[
D + D^\dagger V = 0, \quad D^\dagger + D V^\dagger = 0, \tag{2.3}
\]
by which it follows that \( V \) and \( D \) commute and that \( D \) is normal,
\[
[V, D] = 0, \quad DD^\dagger = D^\dagger D. \tag{2.4}
\]
To account for this we require \( D \) to be a function of \( V \),
\[
D = F(V), \tag{2.5}
\]
i.e. to depend on \( V \) and possibly on constants, however, not on any other operator.

2.2. Spectral Representations

On the finite lattice \( V \) has the spectral representation \( V = \sum_k v_k P_k \) with eigenvalues satisfying \(|v_k| = 1\) and orthogonal projections \( P_k = P_k^\dagger \). Therefore the operator functions \( F(V) \) can be represented by \( F(V) = \sum_k f(v_k) P_k \) with spectral functions \( f(v) \). The use of this will be an important tool in the following.
Because of the $\gamma_5$-hermiticity of $V$ its part related to real eigenvalues commutes with $\gamma_5$ and its complex eigenvalues come in pairs. Taking $V^\dagger = \gamma_5 V \gamma_5$ into account we obtain the more detailed representation

\[ V = P_1(+) + P_1(-) - P_2(+) - P_2(-) + \sum_{k \ (0 < \varphi_k < \pi)} (e^{i\varphi_k} P_k^{(I)} + e^{-i\varphi_k} P_k^{(II)}) , \]  

in which the projections satisfy

\[ \gamma_5 P_j^{(\pm)} = P_j^{(\pm)} \gamma_5 = \pm P_j^{(\pm)} , \quad \gamma_5 P_k^{(I)} = P_k^{(II)} \gamma_5 . \]  

The spectral representation of $D = F(V)$ then becomes

\[ D = f(1)(P_1(+) + P_1(-)) + f(-1)(P_2(+) + P_2(-)) \]
\[ + \sum_{k \ (0 < \varphi_k < \pi)} \left( f(e^{i\varphi_k}) P_k^{(I)} + f(e^{-i\varphi_k}) P_k^{(II)} \right) , \]

in which $D$ is characterized by the function $f(e^{i\varphi})$. Clearly this function enters only at the values $e^{i\varphi} = e^{i\varphi_k}$. However, since we want to define $D$ in general (in particular, for any gauge field configuration), we have to specify $f(v)$ for all $v = e^{i\varphi}$.

Inserting the general form (2.8) into (2.2) and (2.3) we obtain the conditions

\[ f^*(v) = f(v^*) , \quad (2.9) \]
\[ f(v) + f^*(v) v = 0 , \quad (2.10) \]

respectively, for the functions $f(v)$. These conditions imply

\[ f(1) = 0 , \quad f(-1) \text{ real} . \]

The operator form of (2.3) is

\[ F^\dagger(V) = F(V^\dagger) . \]

In addition to giving $D$ by (2.8), the functions $f(v)$ obviously describe the location of its spectrum. For continuous $f(e^{i\varphi})$ the eigenvalues of $D$ reside on a closed curve in the complex plane which is symmetric to the real axis and meets this axis at zero and at the value $f(-1)$ (which is required to be nonzero below).

2.3. Index Relations

Denoting the dimensions of the right-handed and of the left-handed eigenspace for eigenvalue $\pm 1$ of $V$ by $N_+(\pm 1)$ and $N_-(\pm 1)$, respectively, we have from (2.7)

\[ \text{Tr}(\gamma_5 P_1^{(\pm)}) = \pm N_\pm(1) , \quad \text{Tr}(\gamma_5 P_2^{(\pm)}) = \pm N_\pm(-1) , \]
\[ \text{Tr}(\gamma_5 P_k^{(I)}) = \text{Tr}(\gamma_5 P_k^{(II)}) = 0 . \]  

With this because of $f(1) = 0$, using the resolvent $(D - \zeta I)^{-1}$, we obtain for the index of $D$

\[ \lim_{\zeta \to 0} \text{Tr} \left( \gamma_5 \frac{-\zeta}{D - \zeta I} \right) = \begin{cases} N_+(1) - N_-(1) & \text{for } f(-1) \neq 0 \\ N_+(1) - N_-(1) + N_+(\pm 1) - N_-(\pm 1) & \text{for } f(-1) = 0 \end{cases} . \]  

(2.14)
and also find
\[
\lim_{\zeta \to 0} \text{Tr} \left( \gamma_5 \frac{D}{D - \zeta} \right) = \begin{cases} 
N_+(-1) - N_-(-1) & \text{for } f(-1) \neq 0 \\
0 & \text{for } f(-1) = 0 .
\end{cases}
\] (2.15)

Adding up (2.14) and (2.15) the sum on the l.h.s. gets \( \text{Tr}(\gamma_5 \mathbb{1}) = 0 \) so that in any case
\[
N_+(1) - N_-(1) + N_+(-1) - N_-(-1) = 0 .
\] (2.16)

Because of (2.14) and (2.16), to admit a nonvanishing index we have to impose the condition
\[
f(-1) \neq 0 .
\] (2.17)

After having (2.17), according to (2.16) to allow for a nonvanishing index one has also to require that in addition to 1 the eigenvalue \(-1\) of \(V\) occurs. The sum rule (2.16) corresponds to the one found in Ref. 9 for the special case of Dirac operators which satisfy the GW relation (1.1).

Using (2.6) with (2.13) we find
\[
\text{Tr}(\gamma_5 V) = N_+(1) - N_-(1) - N_+(-1) + N_-(-1) ,
\] (2.18)
so that with (2.16) we generally get for the index of the Dirac operators \(D\)
\[
N_+(1) - N_-(1) = \frac{1}{2} \text{Tr}(\gamma_5 V) .
\] (2.19)

Thus it turns out that solely the operator \(V\) enters for the whole class. This generalizes the results obtained in the overlap formalism and in the GW case before.

In this context it is to be noted that the spectral representation of \(\gamma_5 V\) on the basis of (2.6) becomes
\[
\gamma_5 V = P_1^{(+)} - P_1^{(-)} - P_2^{(+)} + P_2^{(-)} + \sum_k (\tilde{P}_k^{[+]} - \tilde{P}_k^{[-]}),
\] (2.20)

where the projections \(\tilde{P}_k^{[\pm]}\) are expressed in terms of \(P_k^{(I)}\) and \(P_k^{(II)}\) by
\[
\tilde{P}_k^{[\pm]} = \frac{1}{2} (P_k^{(I)} + P_k^{(II)} \pm e^{i\varphi_k} \gamma_5 P_k^{(II)} \pm e^{-i\varphi_k} \gamma_5 P_k^{(I)}) , \quad 0 < \varphi_k < \pi .
\] (2.21)

With (2.13) one gets
\[
\text{Tr} \tilde{P}_k^{[+]} = \text{Tr} \tilde{P}_k^{[-]} = \text{Tr} P_k^{(I)} = \text{Tr} P_k^{(II)}
\] (2.22)
for the respective dimensions.
3. Construction of Dirac Operators

3.1. Construction with Spectral Functions

To develop a general construction of the Dirac operators of the class we use the functions $f$ of the spectral representation (2.8) as a tool. We start noting that condition (2.10), $f(e^{i\varphi}) + f^*(e^{i\varphi})e^{i\varphi} = 0$, can be written as

$$(i\nu e^{-i\varphi/2} f(e^{i\varphi}))^* = i\nu e^{-i\varphi/2} f(e^{i\varphi}).$$

(3.1)

This shows that $f$ is of form

$$f(e^{i\varphi}) = -i\nu e^{i\varphi/2} g(\varphi), \quad g(\varphi) \text{ real}.$$  

(3.2)

Then from condition (2.9), $f^*(e^{i\varphi}) = f(e^{-i\varphi})$, the requirement

$$g(-\varphi) = -g(\varphi)$$

(3.3)

follows. Further, with $(2\pi)$-periodicity in $\varphi$ of $f(e^{i\varphi})$, the function $g(\varphi)$ has to satisfy

$$g(\varphi + 2\pi) = -g(\varphi).$$

(3.4)

We now see that forms of $D = F(V)$ can be obtained by determining functions $g(\varphi)$ which are real, odd, and satisfy (3.4).

In view of the indicated requirements the basic building blocks for the construction of $g$ are the functions $\sin \nu \varphi/2$ and $\cos \mu \varphi/2$ with integer $\nu$ and $\mu$. Using them one arrives at the form

$$g = \sum_{\nu} s_{\nu} w_{\nu}(t_1, t_2, \ldots),$$

(3.5)

where $w_{\nu}$ are real functions and

$$s_{\nu} = \sin(2\nu + 1)\varphi/2, \quad t_\mu = \cos \mu \varphi, \quad \nu, \mu \text{ integer}.$$  

(3.6)

Because of the identity

$$s_\nu = s_0 \left(1 + 2 \sum_{\mu=1}^{\nu} t_\mu\right)$$

(3.7)

(3.5) can be simplified to

$$g = s_0 w(t_1, t_2, \ldots)$$

(3.8)

with a real function $w$. Further, since the $t_\nu$ are given by polynomials of $t_1$,

$$t_{2\mu} = d_2 t_1^{2\mu} + d_2 - 2d_2 t_1^{2\mu - 2} + \ldots + d_0,$$

$$t_{2\mu+1} = d_{2\mu+1} t_1^{2\mu+1} + d_{2\mu-1} t_1^{2\mu-1} + \ldots + d_1 t_1,$$

(3.9)

(3.8) can be cast into the still simpler form

$$g = s_0 w(t_1).$$

(3.10)
We next note that given a function \( g \) with the required properties, then \( h(g) \) is again a function with such properties provided that \( h \) is odd and real,

\[
h(-x) = -h(x), \quad h^*(x) = h(x) \quad \text{for real } x.
\]  

(3.11)

With this the form (3.10) generalizes to

\[
g = h(s_0 w(t_1)).
\]

(3.12)

To satisfy condition (2.17), \( f(-1) \neq 0 \), we need \( g(\pi) \neq 0 \) or \( h(w(-1)) \neq 0 \). Therefore we have to impose

\[
w(-1) \neq 0,
\]

(3.13)

which is sufficient if \( h(x) \) gets only zero for \( x = 0 \). To guarantee this we in addition require strict monotony,

\[
h(x_2) > h(x_1) \quad \text{for } x_2 > x_1.
\]

(3.14)

Then also the inverse function \( \eta(y) \) with

\[
\eta(h(x)) = x
\]

(3.15)

is uniquely defined and strictly monotonous, which we will need below.

With (3.2), (3.12) and (3.6) we now have the general form

\[
f(e^{i\varphi}) = -ie^{i\varphi/2} h(s_0 w(t_1)) = -ie^{i\varphi/2} h\left(\sin\frac{\varphi}{2} w(\cos \varphi)\right)
\]

(3.16)

doing the spectral function \( f \).

3.2. Nontrivial Choices of \( h(x) \)

Looking for functions \( h \) one has to note that because of the identity

\[
s_0^{2k+1} = s_0 \sum_{\nu=0}^{k} \left(\frac{2k+1}{\nu}\right)(-1)^\nu \left(1 + 2 \sum_{\mu=1}^{k-\nu} t_\nu \right), \quad k = 0, 1, 2, \ldots,
\]

(3.17)

(3.12) reduces to the form (3.10) if \( h(x) \) is a polynomial or allows an expansion in powers of \( x \). This limits the possibilities for nontrivial choices of \( h(x) \), i.e. of ones by which (3.12) gives something beyond (3.10).

The nontrivial choices of \( h \) actually are equivalence classes, i.e. equivalent ones are not to be counted as different. For example, \( h(x) = x^{1/(2k+1)} \) and \( h(x) = x^{1/(2k+1)} r(x) \) with \( r(-x) = r(x) \) are equivalent, because with \( r(x) = r(|x|) \) and \( |s_0| = \sqrt{\frac{1}{2}(1-t_1)} \) they give the same form in (3.12). Also forming in addition odd powers because of (3.17) gives nothing new.

To get a criterion for triviality of \( h \) we note that, given \( g = h(s_0 w(t_1)) \), in the trivial case \( \hat{w}(t_1) \) should exist so that this could also be expressed as \( g = s_0 \hat{w}(t_1) \). For \( \varphi \neq 0 \) one gets simply \( \hat{w}(t_1) = h(s_0 w(t_1))/s_0 \), while at \( \varphi = 0 \) conditions on \( h \) and \( w \) arise from

\[
\lim_{s_0 \to 0} |h(s_0 w(t_1))/s_0| < \infty,
\]

(3.18)

which has to hold in the trivial case.
3.3. Operator Form of Construction

The operator $D$ for the general construction is obtained inserting (3.16) into (2.8), which gives

$$D = -i \frac{V}{2} H \left( \frac{1}{2i} (V^\dagger - V) W \left( \frac{1}{2}(V + V^\dagger) \right) \right),$$

(3.19)

where $V^\frac{1}{2}$ is defined with $+e^{i\varphi/2}$ in its spectral representation corresponding to $e^{i\varphi}$ in that of $V$. According to the reality of $w$ and to (3.13) $W$ has to satisfy

$$W^\dagger(X) = W(X^\dagger), \quad W(-P) \neq 0 \text{ for } P = P^\dagger = P^2 > 0.$$

(3.20)

Because of (3.11) and (3.14) for the function $H$ one needs

$$H(-X) = -H(X), \quad H^\dagger(X) = H(X)$$

$$H(X) > H(X_1) \text{ for } X > X_1 \quad \text{with } X^\dagger = X, X_1^\dagger = X_1.$$

(3.21)

Then corresponding to (3.15) also the inverse function $E$ of $H$

$$E(H(X)) = X$$

(3.22)

exists and is odd, hermitian and strictly monotonous, too, i.e. satisfies analogous relations as for $H$ are given in (3.21). Equation (3.19) demonstrates the drastic increase of possible forms of Dirac operators which occurs as compared to the GW case (1.1).

3.4. Realization of $V$

To specify $V$ explicitly we introduce the normalization-type definition

$$V = -D^{(n)}_W \left( \frac{1}{2i} \sum \mu \gamma_\mu (\nabla_\mu - \nabla^\dagger_\mu) \right) + E \left( \frac{r}{2} \sum \mu \nabla^\dagger_\mu \nabla_\mu \right) + E(m1).$$

(3.23)

with

$$D^{(n)}_W = iE \left( \frac{1}{2i} \sum \mu \gamma_\mu (\nabla_\mu - \nabla^\dagger_\mu) \right) + E \left( \frac{r}{2} \sum \mu \nabla^\dagger_\mu \nabla_\mu \right) + E(m1).$$

(3.24)

Obviously for $E(X) = X$ (3.24) reduces to the overlap form of Neuberger [44].

We note that the strict use of hermitian functions of hermitian operators in (3.23) with (3.24) makes it generally applicable. Then monotony is also defined in terms of operators. In particular, it will allow below to evaluate the limit in a general way. Each term of (3.24) then may be described by the spectral representation of its argument. To avoid the square root of noncommuting operators in (3.23), one can use the alternative representation

$$V = -D^{(n)}_W \left( \frac{1}{\pi} \int_{-\infty}^{\infty} ds \frac{1}{(D^{(n)}_W)^2 + s^2} \right).$$

(3.25)
To check the continuum limit we note that in the free case and infinite volume with the Fourier representation $V_{\kappa'\kappa} = \tilde{V}(\kappa') \delta(\kappa' - \kappa)$ one gets with $\kappa_\mu = a p_\mu$ at the corners of the Brillouin zone $\tilde{V} = -1$ and at zero

$$\tilde{V} \rightarrow 1 - \frac{i}{|\eta(m)|} \tilde{E} \left( a \sum_\mu \gamma_\mu p_\mu \right) \quad \text{for} \quad a \rightarrow 0.$$ (3.26)

Then imposing the requirement

$$\tilde{W}(-1) \neq 0,$$ (3.27)

because of the monotony of $E(X)$, doublers are suppressed for $-2r < m < 0$ as usual. Condition (3.27) corresponds to (2.17) since at the corners of the Brillouin zone the eigenvalue $-1$ of $V$ occurs. On the lattice working with dimensionless quantities, the limit to be considered is $\tilde{D}/a \rightarrow \tilde{D}_{\text{cont}}$. Because of $H(E(X)) = X$, putting

$$\tilde{W}(1) = 2|\eta(m)|$$ (3.28)

the correct result with the usual normalization of the propagator is obtained in this quite general way.

4. Special Cases from Literature

4.1. GW Relation with Constant $\rho$

The Dirac operators satisfying the GW relation (1.1), which have already been discussed in Sec. [1], represent the simplest special case of the general class. They correspond to the trivial choices $h(x) = x$ and $w = \text{const} = 2\rho$. This gives the spectral function

$$f(e^{i\varphi}) = -2i\rho e^{i\varphi/2} \sin \frac{\varphi}{2},$$ (4.1)

which in the complex plane describes the well-known circle through zero around $\rho$, and which inserted into (2.8) gives the special case (3.4) of (3.19). According to (3.28) one has to put $\rho = |m|$.

4.2. General GW Relation with $[R, D] \neq 0$

The Dirac operators satisfying the general GW relation

$$\{\gamma_5, D\} = 2D\gamma_5 RD \quad \text{with} \quad [R, D] \neq 0,$$ (4.2)

where $R^\dagger = R$ and $[\gamma_\mu, R] = 0$, do not belong to the class. In fact, with $\gamma_5$-hermiticity of $D$ and $[\gamma_5, R] = 0$ from this relation one gets $[D, D^\dagger] = 2D^\dagger[R, D]D^\dagger$. Thus for $[R, D] \neq 0$ the Dirac operator $D$ is not normal, which is in contrast to what is required in (2.4).

It is to be noted that for such operators the analysis of the index gets rather subtle. For them one obtains the relation

$$\text{Tr}(\gamma_5(P_j + RQ_j)) + \text{Tr}(\gamma_5 RD) = 0$$ (4.3)
in which the projections $P_j$ need not to be orthogonal and where eigen-nilpotents $Q_j$ can occur if the dimensions of the respective algebraic and geometric eigenspaces differ. In a lengthy proof it has been shown that for the eigenvalue $\lambda_k = 0$ of $D$ these dimensions are equal so that $Q_k = 0$ and $P_k = P_k^\dagger$. Thus for zero eigenvalue the unwanted term with $Q_k$ disappears and $P_k$ gets orthogonal as needed. However, the precise effect of $R$ in the term $\text{Tr}(\gamma_5 RD)$ remains still to be investigated.

Dirac operators for which (4.2) holds occur in fixed-point QCD. There it has been proposed to to apply the rescaling

$$D_1 = \rho^{-1}(2R)^\frac{1}{2}D(2R)^\frac{1}{2}$$  \hspace{1cm} (4.4)

to them, by which one can switch to the operator $D_1$ which satisfies (1.1). Thus tracing the operators $D$ with (4.2) back to the normal operators $D_1$, one can work with $D_1$ belonging to the general class.

### 4.3. Extended Fujikawa Type

The Dirac operators of a recent extension of the Fujikawa proposal satisfy

$$\{\gamma_5, D\} = \rho^{-1}D\gamma_5 D \Phi((2\rho)^{-2}(\gamma_5 D)^2),$$  \hspace{1cm} (4.5)

where the operator function $\Phi$ is subject to

$$\Phi(X)^\dagger = \Phi(X) \quad \text{for} \quad X^\dagger = X.$$  \hspace{1cm} (4.6)

Using the identity $[\gamma_5, D^\dagger D] = [\{\gamma_5, \gamma_5 D\}, \gamma_5 D]$ with (4.5) since $D^\dagger D = (\gamma_5 D)^2$ we get

$$[\gamma_5, D^\dagger D] = 0.$$  \hspace{1cm} (4.7)

With this we have $D^\dagger D = \gamma_5 D^\dagger D \gamma_5$ and find

$$[D^\dagger, D] = 0.$$  \hspace{1cm} (4.8)

With (4.8) one sees that the Dirac operators in (4.5) can also be considered as satisfying the general GW relation with $2R = \rho^{-1}\Phi((2\rho)^{-2}D^\dagger D)$, where, however, in contrast to (4.2), one has $[R, D] = 0$.

Using the $\gamma_5$-hermiticity of $D$ we can write (4.3) in the form

$$D + D^\dagger = \rho^{-1}D^\dagger D \Phi((2\rho)^{-2}D^\dagger D)$$  \hspace{1cm} (4.9)

and comparing with the basic relation (2.3), $D + D^\dagger V = 0$, it follows that

$$V = 1 - \rho^{-1}D \Phi((2\rho)^{-2}D^\dagger D).$$  \hspace{1cm} (4.10)

Because of (4.7) and $\Phi^\dagger((2\rho)^{-2}D^\dagger D) = \Phi((2\rho)^{-2}D^\dagger D)$ this is seen to be $\gamma_5$-hermitian and using (4.8) and (4.5) it can be checked to be unitary. It thus turns out that the operators satisfying (4.7) are also a special case of the general class.

With (4.10) and (2.3) one gets the equation

$$\rho^{-1}D \Phi(- (2\rho)^{-2}V^{-1}D^2) + V = 1$$  \hspace{1cm} (4.11)

for $D$ and $V$, the solution of which gives $D = F(V)$.
4.4. Proposal of Fujikawa

The original proposal of Fujikawa is given by the choice

$$\Phi(X) = X^k, \quad k = 1, 2, \ldots$$  

(4.12)

of the function $\Phi$ in (4.11). With this (4.11) becomes

$$2(-V)^{-k}((2\rho)^{-1}D)^{2k+1} + V = 1,$$  

(4.13)

which can be solved for $D$ giving

$$D = 2\rho\left(\frac{1}{2}(1 - V)(-V)^k\right)^{1/(2k+1)},$$  

(4.14)

where of the $(2k+1)$-th roots the one which satisfies (2.12) is to be chosen. Thus with (4.14) we have indeed the form $D = F(V)$ (while in Ref. only the form $D = 2\rho\gamma_5\gamma_5(1 - V)^{1/(2k+1)}$ involving also $\gamma_5$ occurs).

The functions $f$ of the spectral representation (2.8) for $D$, in the present case are obtained from (4.14) as

$$f(e^{i\phi}) = -2i\rho\frac{e^{i\phi/2}}{\sin\left(\frac{\phi}{2}\right)}^{1/(2k+1)}$$  

(4.15)

where of the $(2k+1)$-th roots the real one is to be taken. They give the curves describing the location of the eigenvalues of $D$, which for $k > 0$ arise as deformations of the circle for $k = 0$. All of them meet the real axis at zero and at the value $f(-1) = 2\rho$.

Comparing with (3.16) it is seen that in the present case one has the choice

$$h(x) = x^{1/(2k+1)}, \quad k = 1, 2, \ldots$$  

(4.16)

where the real one of the $(2k+1)$-th roots is to be taken. The function $w$ is obviously constant here, $w = (2\rho)^{2k+1}$, and according to (3.28) one has to put $w = 2|m|^{2k+1}$.

In Ref. in the definition of the generalized Wilson-Dirac operator the factors $i$ we have in (3.24) are not included. For $h(x) = x^{1/(2k+1)}$ both formulations are possible. However, in the general case only our formulation with the strict use of hermitian functions of hermitian operators appears appropriate.

5. Further Special Cases

5.1. General $w$ with $h(x) = x$

For $h(x) = x$ (3.19) specializes to

$$D = \frac{1}{2}(1 - V) W\left(\frac{1}{2}(V + V^\dagger)\right).$$  

(5.1)

Since $E(X) = X$ for all operators of this subclass the overlap form of $V$ is suitable.

It is instructive to consider the eigenvalues of $V$ in the free case, which here can be calculated explicitly as those of $V$,

$$e^{i\phi} = -(\tau \pm i\sqrt{s^2})/\sqrt{\tau^2 + s^2},$$  

(5.2)
where
\[ s^2 = \sum_{\mu} \sin \kappa_{\mu}^2, \quad \tau = m + r \sum_{\mu} (1 - \cos \kappa_{\mu}), \quad -2r < m < 0. \quad (5.3) \]

The real eigenvalues in (5.2) obviously occur for \( \kappa_{\mu} = 0, \pi \) and one gets +1 if all \( \kappa_{\mu} = 0 \) and −1 at each corner of the Brillouin zone. Noting that
\[ \frac{1}{2} (\tilde{V}(\kappa) + \tilde{V}^\dagger(\kappa)) = -\tau / \sqrt{\tau^2 + s^2} = \cos \varphi \quad (5.4) \]
it becomes explicit that the conditions (3.28) and (3.27) on \( \tilde{W}(1) \) and \( \tilde{W}(-1) \) are related to the behavior of \( V \) at its eigenvalues +1 and −1, respectively.

5.2. Expansion in Powers of \( V \)

In the case where \( w(t_1) \) allows for an expansion in a series or a polynomial we obtain for (3.10)
\[ g = s_0 w(t_1) = s_0 \left( b_0/2 + \sum_{\mu \geq 1} b_\mu t_\mu \right) = 2 \sum_{\nu \geq 0} c_\nu s_\nu, \quad (5.5) \]
which follows using (3.7) and (3.9) and relating the coefficients by \( b_\mu = 2 \sum_{\nu \geq \mu} c_\nu \).

With (5.5) we then have in terms of operators
\[ D = \sum_{\nu \geq 0} c_\nu (V^{-\nu} - V^{\nu+1}). \quad (5.6) \]
Condition (2.17) now gets the form
\[ f(-1) = 2 \sum_{\nu \geq 0} (-1)^\nu c_\nu \neq 0 \quad (5.7) \]
and (3.28) corresponds to
\[ \sum_{\nu \geq 0} (2\nu + 1)c_\nu = |m|. \quad (5.8) \]
Obviously the special case (1.4) of (5.6) arises by putting \( c_\nu = \rho \delta_{\nu 0} \).

For the functions \( f \) in the spectral representation of the Dirac operators (5.6) we obtain the form
\[ f(e^{i\varphi}) = -2ie^{i\varphi/2} \sum_{\nu \geq 0} c_\nu \sin(2\nu + 1)\varphi/2. \quad (5.9) \]
The eigenvalues of \( D \) thus reside on a closed curve in the complex plane which is given by a linear combination of rosette functions \(-ie^{i\varphi/2} \sin(2\nu + 1)\varphi/2 \) and which meets the real axis at zero and at the value (5.7).

In case of an infinite number of terms of the expansion (5.6), convergence properties can be studied considering the Fourier series \( \sum_{\nu \geq 0} c_\nu \sin(2\nu + 1)\varphi/2 \) in (5.11). Uniform convergence then is guaranteed by the condition
\[ \sum_{\nu \geq 0} |c_\nu| < \infty. \quad (5.10) \]
5.3. Nontrivial $w$ with $h(x) = x^{1/(2k+1)}$

As examples of nontrivial $w$ together with nontrivial $h$ we use $h(x) = x^{1/(2k+1)}$ with $k = 0, 1, 2, \ldots$ to have simple explicit forms of the latter. Inserting (3.2) into (2.8) here gives

$$D = \left( \frac{1}{2} (1 - V)(-V)^k W\left( \frac{1}{2}(V + V') \right) \right)^{1/(2k+1)},$$

(5.11)

where of the $(2k+1)$-th roots the one which satisfies (2.12) is to be chosen and where $W$ is subject to (3.20). Obviously (5.11) generalizes (4.14) replacing the constant $2\rho$ there by the function $W^{1/(2k+1)}$. The requirement (3.28) now becomes

$$\tilde{W}(1) = 2|m|^{2k+1}.$$

The eigenvalues of $V$ in the free case can again be determined,

$$e^{i\varphi} = -\left( \tau_k \pm i\sqrt{(s^2)^{2k+1}} / \sqrt{\tau_k^2 + (s^2)^{2k+1}} \right),$$

(5.12)

with

$$s^2 = \sum \sin \kappa^2_\mu, \quad \tau_k = m^{2k+1} + \left( r \sum \mu (1 - \cos \kappa_\mu) \right)^{2k+1}, \quad -2r < m < 0,$$

(5.13)

and are $+1$ if all $\kappa_\mu = 0$ and $-1$ at each corner of the Brillouin zone. From

$$\frac{1}{2}(\tilde{V}(\kappa) + \tilde{V}^\dagger(\kappa)) = -\tau_k / \sqrt{\tau_k^2 + (s^2)^{2k+1}} = \cos \varphi$$

(5.14)

the relation of the conditions (3.28) and (3.27) on $\tilde{W}(1)$ and $\tilde{W}(-1)$ to the behavior of $V$ at its eigenvalues $+1$ and $-1$, respectively, gets again explicit.

5.4. Polynomial Form of $\eta(y)$

A possibility to get further concrete examples of nontrivial $h$ is to start from $\eta$ being a polynomial,

$$\eta(y) = \sum_{\nu=0}^N B_\nu y^{2\nu+1},$$

(5.15)

with real coefficients $B_\nu$. Because of $\eta(h(x)) = x$ to obtain $h(x)$ we have to solve the algebraic equation

$$\sum_{\nu=0}^N B_\nu h^{2\nu+1} - x = 0.$$
5.5. \( h(x) \) from Cubic Equation

In the case where only the coefficients \( B_0 \) and \( B_1 \) in (5.16) are nonvanishing we have the cubic equation

\[
h^3 + 3ph + 2q = 0, \quad p = \frac{B_0}{3B_1}, \quad q = -\frac{x}{2B_1}.
\]

(5.18)

Requiring \( B_0B_1 > 0 \) it follows that \( p^3 + q^2 > 0 \), which implies that one gets the real solution

\[
h = \sqrt[3]{-q + \sqrt{q^2 + p^3}} + \sqrt[3]{-q - \sqrt{q^2 + p^3}}.
\]

(5.19)

We thus have in more detail

\[
h(x) = \sqrt[3]{\frac{x}{2B_1}} + \sqrt[3]{\left(\frac{x}{2B_1}\right)^2 + \left(\frac{B_0}{3B_1}\right)^3} + \sqrt[3]{\frac{x}{2B_1} - \sqrt[3]{\left(\frac{x}{2B_1}\right)^2 + \left(\frac{B_0}{3B_1}\right)^3}},
\]

(5.20)

with the required properties for \( h(x) \) of being odd and strictly monotonous. The related operator expression (3.19) for \( D \) then becomes

\[
D = \sqrt[3]{\frac{1}{4B_1}} V(V-1)W + \sqrt[3]{\left(\frac{1}{4B_1}\right) V(V-1)W^2 + \left(\frac{B_0}{3B_1}\right)^3}

+ \sqrt[3]{\frac{1}{4B_1}} V(V-1)W - \sqrt[3]{\left(\frac{1}{4B_1}\right) V(V-1)W^2 + \left(\frac{B_0}{3B_1}\right)^3}
\]

(5.21)

with \( W = W\left(\frac{1}{4}(V + V^\dagger)\right) \). Condition (3.28) here requires \( \tilde{W}(1) = 2|B_0m + B_1m^3| \).

6. Index and Spectral Flows

6.1. Relation to Flows

The spectral flows of the hermitian Wilson-Dirac operator \( \mathcal{H} \) are of fundamental importance in the overlap formalism. They provide a further description of the index of \( D \). To make contact to the formulations here we note that \( \mathcal{H} = \gamma_5 D_w \) where \( D_w \) is \( D^{(n)}_w \) of (3.24) specialized to \( E(X) = X \). Then according to (3.23) one has

\[-\gamma_5 V = \mathcal{H}(\sqrt{\mathcal{H}^2})^{-1} = \epsilon(\mathcal{H}). \]

(6.1)

With (6.1) relation (2.19) for the index of \( D \) becomes

\[N_+(1) - N_-(1) = \frac{1}{2} \text{Tr}(\gamma_5 V) = -\text{Tr}\epsilon(\mathcal{H}). \]

(6.2)

This shows that the index of \( D \) is also given by the difference of the numbers of positive and negative eigenvalues of \( \mathcal{H} \), which is the view introduced in the overlap formalism. It has led there to investigations of spectral flows, i.e. of the eigenvalues as a functions of the mass parameter.

We note that this view extends to the case of our general functions \( E(X) \), because considering the individual terms in (3.24) one can still confirm \( \gamma_5 \)-hermiticity.
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of \( D^{(n)} \). For the function with \( \gamma_\mu \) there for this purpose one has to use the spectral representation of its argument. Then with the generalized hermitian Wilson-Dirac operator \( H^{(n)} = \gamma_5 D^{(n)}_W \) instead of \( \frac{1}{2} \) one obtains

\[
N_+(1) - N_-(1) = \frac{1}{2} \text{Tr}(\gamma_5 V) = -\text{Tr} \epsilon(H^{(n)}).
\]

### 6.2. Differential Equation

For the description of the spectral flows of \( H \) an exact differential equation has been derived and a complete overview of its solutions has been given. To get this differential equation one notes that for \( H \) one has the eigenequation

\[
H \phi_l = \alpha_l \phi_l. \tag{6.4}
\]

Multiplying \( (6.4) \) by \( \phi_l^\dagger \gamma_5 \) one gets \( \phi_l^\dagger \gamma_5 H \phi_l = \alpha_l \phi_l^\dagger \gamma_5 \phi_l \) and summing this and its hermitian conjugate one has \( \phi_l^\dagger \{ \gamma_5, H \} \phi_l = 2 \alpha_l \phi_l^\dagger \gamma_5 \phi_l \). From this by inserting the explicit form of \( H \) one obtains

\[
\alpha_l \phi_l^\dagger \gamma_5 \phi_l = m + g_l(m), \quad g_l(m) = \frac{r}{2} \sum_\mu ||\nabla_\mu \phi_l||^2. \tag{6.5}
\]

For \( g_l(m) \) using \( ||\nabla_\mu \phi_l|| \leq ||(\nabla_\mu - 1)\phi_l|| + ||\phi_l|| = 2 \) one gets \( 0 \leq g_l(m) \leq 8r \).

Further, abbreviating \( (d \alpha_l)/(d m) \) by \( \dot{\alpha}_l \), we obtain

\[
\frac{d\phi_l^\dagger H \phi_l}{d m} = \phi_l^\dagger H \phi_l + \phi_l^\dagger H \phi_l + \phi_l^\dagger H \phi_l = \phi_l^\dagger \gamma_5 \phi_l + \alpha_l \frac{d\phi_l^\dagger \phi_l}{d m} \tag{6.6}
\]

which means that we have

\[
\dot{\alpha}_l = \phi_l^\dagger \gamma_5 \phi_l. \tag{6.7}
\]

Combining \( (6.5) \) and \( (6.7) \) we get the differential equation

\[
\dot{\alpha}_l(m) \alpha_l(m) = m + g_l(m) \tag{6.8}
\]

for the eigenvalue flows of the hermitean Wilson-Dirac operator \( H \).

### 7. Index and Limit

#### 7.1. Introductory Remarks

So far we have considered only relations on the finite lattice, except for checking the continuum limit of the propagator for free fermions in Sec. 3.4. In this context we have derived condition \( (3.27) \) needed for the suppression of doublers and condition \( (3.28) \) giving the usual normalization of the propagator in the limit. It has turned out that \( (3.27) \) is related to condition \( (2.17) \) which is necessary to allow for a non-vanishing index. The connection of these conditions to behaviors at the eigenvalues \(-1\) and \(+1\) of \( V \), respectively, has become explicit from \( (6.4) \) and \( (5.14) \).

We now address some further issues of the limit related to the index of \( D \). For the sum rule \( (2.16) \) of the index we have to make sure that it still holds in the
continuum limit, in which the unitary space in which $D$ acts gets a Hilbert space (see Sec. 8.3 for precise details of the occurring spaces). Therefore, we firstly have to define the trace $\text{Tr}(\gamma^5 1_l)$, which we have used to derive the sum rule, in infinite space, too. Secondly we have to show that a continuous part of the spectrum of $V$ does not contribute. This will be done in Secs. 7.2 and 7.3, respectively. Having thus the validity of the sum rule established also in the limit, the fundamental structural difference to the Atiyah-Singer framework it reflects is pointed out in Sec. 7.4.

One further has to check for fermions in a background gauge field that the r.h.s. side of (2.19) gives the correct limit of the topological charge,

$$\frac{1}{2} \text{Tr}(\gamma^5 V) \rightarrow -\frac{1}{32\pi^2} \int d^4 x \sum_{\mu\nu\lambda\tau} \epsilon_{\mu\nu\lambda\tau} \text{tr} \left( F_{\mu\nu}(x) F_{\lambda\tau}(x) \right),$$

and that correspondingly for the chiral anomaly, with tr denoting the trace in Dirac and gauge field space only,

$$\frac{1}{2} \text{tr}(\gamma^5 V_{nn}) \frac{1}{a^4} \rightarrow -\frac{1}{32\pi^2} \sum_{\mu\nu\lambda\tau} \epsilon_{\mu\nu\lambda\tau} \text{tr} \left( F_{\mu\nu}(x) F_{\lambda\tau}(x) \right)$$

holds. In case of the subclass with $H(X) = X$ for this one can rely on the fact that for the overlap $V$ it is safely known that one gets the correct results (see Ref. 17 for a proof and a discussion of literature). This is so for all $W$ because only $V$ enters relation (2.19). For the case $H(X) = X^{2k+1}$ a check has been presented in Ref. 18.

### 7.2. Trace Expressions

The obvious condition for defining the indicated $\text{Tr}$-expression is to require it to be the limit of the respective finite-lattice results. We thus consider the sequence of results of $\text{Tr}(\gamma^5 1_l)$ for larger and larger lattices. Since all members of this sequence are zero, the limit is zero, too, and one gets

$$\text{Tr}(\gamma^5 1_l) = 0.$$  \hspace{1cm} (7.3)

In practice it is desirable to have also formal descriptions leading to (7.3). An immediate possibility is to use the factorization of the identity into parts related to Dirac space, gauge group space and volume $N^4$, and to define $\text{Tr}(\gamma^5 1) := \lim_{N \to \infty} \text{Tr}(\gamma^5 \otimes 1_U \otimes 1_{N^4})$. Another possibility is to introduce a regularization suitable after taking the infinite-volume limit. For this purpose one can, for example, use decompositions of type $1 = \sum_{j} \hat{P}_j$ with any orthogonal projections $\hat{P}_j$ of finite dimension and trivial in Dirac space (i.e. of form $\hat{P}_j = \gamma^5 \otimes \hat{p}_j$), and define $\text{Tr}(\gamma^5 1) := \sum_{j} \text{Tr}(\gamma^5 \hat{P}_j)$. The point here obviously is that the summation occurs after taking the traces.

Introducing $\hat{P}_j^\pm = \frac{1}{2}(1 \pm \gamma^5)\hat{P}_j$, the latter definition can also be written as $\text{Tr}(\gamma^5 1) := \sum_{j} \text{Tr}(\hat{P}_j^{(+)} - \hat{P}_j^{(-)})$. Thus a possible alternative is obtained using functions $c_j(t)$ of a real parameter $t$ with $c_j(0) = 1$ and $\sum_{j} \text{Tr}(c_j(t)\hat{P}_j^{\pm}) < \infty$. Now with each of the sums being separately well-defined, because of $\text{Tr}(\hat{P}_j^{(+)} - \hat{P}_j^{(-)})$
one gets \( \sum_j \text{Tr}(c_j(t)(\hat{P}^{(+)}_j - \hat{P}^{(-)}_j)) = 0 \). Since the result is independent of \( t \), letting \( t \to 0 \) at the end is no problem. Therefore this provides a further definition of \( \text{Tr}(\gamma_5 \mathbb{1}) \). Its principle is that of the heat-kernel regularization, to which we come back in Sec. 7.4.

For the projections \( P_R = \frac{1}{2}(1 + \gamma_5) \mathbb{1} \) and \( P_L = \frac{1}{2}(1 - \gamma_5) \mathbb{1} \), which project on the total right-handed and left-handed space, respectively, \( P_R - P_L = \gamma_5 \mathbb{1} \) holds. Therefore, with the definition of \( \text{Tr}(\gamma_5 \mathbb{1}) \) in infinite space in place, which gives (7.3), we obtain

\[
\text{Tr}(P_R - P_L) = 0 \quad (7.4)
\]

for the difference of the respective dimensions.

### 7.3. Continuous Spectrum

In Hilbert space the spectra of operators can also get continuous parts. To show that such parts do not contribute to the sum rule (2.16), we note that the derivation of this rule is based on expressions of form \( \text{Tr}(\gamma_5 \Phi(V)) \). To evaluate this we can use the general spectral representation of \( V \), which is given by the operator Stieltjes integral

\[
V = \int_{-\pi}^{\pi} e^{i\varphi} \, dE_{\varphi}, \quad (7.5)
\]

with which the expressions of interest are represented by

\[
\text{Tr}(\gamma_5 \Phi(V)) = \int_{-\pi}^{\pi} \phi(e^{i\varphi}) \, d(\text{Tr}(\gamma_5 E_{\varphi})). \quad (7.6)
\]

Its discrete part is given in terms of (2.6). Here it remains to consider the continuous part, for which we obtain

\[
V_{\text{con}} = \int_{-\pi}^{\pi} e^{i\varphi} \, dE_{\text{con}, \varphi} = \int_{0}^{\pi} e^{i\varphi} \, dE_{\text{con}, \varphi} - \int_{0}^{\pi} e^{-i\varphi} \, dE_{\text{con}, -\varphi}, \quad (7.7)
\]

where the subdivision into two integrals is possible because the projector function \( E_{\text{con}, \varphi} \) is purely continuous. With \( V^\dagger = \gamma_5 V \gamma_5 \) and \( E_{\text{con}, \varphi}^\dagger = E_{\text{con}, \varphi} \), using (7.7) one gets \( \gamma_5 E_{\text{con}, \varphi} = -E_{\text{con}, -\varphi} \gamma_5 \). Further, because \( E_{\text{con}, \varphi} E_{\text{con}, \varphi'} = E_{\text{con}, \varphi} E_{\text{con}, \varphi'} \), it follows that

\[
\text{Tr}(\gamma_5 E_{\text{con}, \varphi}) = 0. \quad (7.8)
\]

With this we find that the continuous part of (7.6) indeed vanishes,

\[
\text{Tr}(\gamma_5 \Phi(V_{\text{con}})) = \int_{-\pi}^{\pi} \phi(e^{i\varphi}) \, d(\text{Tr}(\gamma_5 E_{\text{con}, \varphi})) = 0. \quad (7.9)
\]
7.4. Differences to the Atiyah-Singer Case

The sum rule for the index of the Dirac operator, which turns out to be important in lattice theory, is related to a basic structural difference to the framework of the Atiyah-Singer Dirac operator [2]. Because this appears to be not sufficiently realized, we here point out some details.

The definition in the Atiyah-Singer case is based on (Weyl) operators $D_{AS}^{(+)}$ and $D_{AS}^{(-)}$ which map from the total right-handed space $\mathcal{E}^+$ to the total left-handed space $\mathcal{E}^-$ and back, respectively. It is given in the combined space $\mathcal{E}^+ \oplus \mathcal{E}^-$ by $D_{AS} = \hat{D}_{AS}^{(+)} + \hat{D}_{AS}^{(-)}$, where $\hat{D}_{AS}^{(\pm)} = D_{AS}^{(\pm)}$ on $\mathcal{E}^{(\pm)}$ and $\hat{D}_{AS}^{(\pm)} = 0$ on $\mathcal{E}^{(\mp)}$. Because of $D_{AS}^{(+)\dagger} = D_{AS}^{(-)}$ the Dirac operator $D_{AS}$ is self-adjoint and since it acts on a compact manifold its spectrum is discrete. Thus it is represented by $D_{AS} = \sum_j \lambda_{AS,j} P_{AS,j}$ which implies $D_{AS}^2 = \sum_j \lambda_{AS,j}^2 P_{AS,j}$.

On the other hand, one gets $D_{AS}^2 = \hat{D}_{AS}^{(+)} \hat{D}_{AS}^{(-)} + \hat{D}_{AS}^{(-)} \hat{D}_{AS}^{(+)}$, which can be evaluated noting that the operators $D_{AS}^{(+)} D_{AS}^{(-)}$ and $D_{AS}^{(+)\dagger} D_{AS}^{(-\dagger)}$ map within $\mathcal{E}^+$ and $\mathcal{E}^-$, respectively, and are selfadjoint and nonnegative. With the eigenequation $D^{(-)} D^{(+)} \Phi_j = \lambda_j \Phi_j$ in $\mathcal{E}^+$ one gets the eigenequation $D^{(+)} D^{(-)} (D^{(+)\dagger} \Phi_j) = \lambda_j (D^{(+)\dagger} \Phi_j)$ in $\mathcal{E}^-$. Further, from $\langle D^{(+)} \Phi_{jr} | D^{(+)\dagger} \Phi_{jr} \rangle = \langle \Phi_{jr} | D^{(-)} D^{(+)\dagger} \Phi_{jr} \rangle = \lambda_j (D^{(+)\dagger} \Phi_j)$ one sees that, except for $\lambda_j = 0$, for each of the common eigenvalues $\lambda_j$ the eigenspaces must have the same dimension. Therefore, except for $\lambda_j = 0$, the operators $D_{AS}^{(-)} D_{AS}^{(+)}$ and $D_{AS}^{(+)\dagger} D_{AS}^{(-\dagger)}$ have the same spectra. Denoting the projections on their eigenspaces by $P_{AS,j}^{(+)}$ and $P_{AS,j}^{(-)}$, respectively, and comparing the above expressions for $D_{AS}^2$ it then follows that $D_{AS} = \sum_j \lambda_{AS,j} (P_{AS,j}^{(+)} + P_{AS,j}^{(-)})$ and that one always has

$$\hat{N}^+_j(\lambda_{AS,j}) = \hat{N}^-_j(\lambda_{AS,j}) \text{ for } \lambda_{AS,j} \neq 0 \quad (7.10)$$

for the dimensions $\hat{N}^+_j(\lambda_{AS,j}) = \text{Tr} P_{AS,j}^{(+)}$ of the eigenspaces.

For $\lambda_j = \lambda_{AS,j} = 0$ one obtains $\text{ker} D_{AS}^{(+)} = \text{ker} D_{AS}^{(-)} = D_{AS}^{(+)\dagger}$ and $\text{ker} D_{AS}^{(+)\dagger} = \text{ker} D_{AS}^{(-\dagger)}$ (as is obvious from left to right and follows from right to left from $\langle \Phi | D_{AS}^{(-\dagger)} D_{AS}^{(+)\dagger} \Phi \rangle = \langle D_{AS}^{(+)\dagger} \Phi | D_{AS}^{(+)\dagger} \Phi \rangle$). Thus the eigenspaces with eigenvalue zero have the dimensions $\hat{N}_j^\pm(0) = \dim \text{ker} D_{AS}^{(+)\dagger}$ and the index becomes

$$\hat{N}^+_j(0) - \hat{N}^-_j(0) = \dim \text{ker} D_{AS}^{(+)} - \dim \text{ker} D_{AS}^{(+)\dagger} \quad (7.11)$$

It is now seen that, while according to (7.10) pairs of subspaces with the same dimension and opposite chirality occur for $\lambda_{AS,j} \neq 0$, with (7.11) in the topologically nontrivial case the chiral subspaces for $\lambda_{AS,j} = 0$ contribute differently to the dimensions of the total right-handed and left-handed space. This is obviously different from the situation in lattice theory, where (7.3) (on which the sum rule is based) reflects symmetry between right-handed and left-handed space.

To compare this in more detail with the lattice case, we consider the projections $P_+ = \sum_j P_{AS,j}^{(+)}$ and $P_- = \sum_j P_{AS,j}^{(-)}$ which here project on the total right-handed and left-handed space, respectively. Analogous regularizations as in in Sec. 7.2 can be used to evaluate the trace of the difference. The simple prescription of taking the
trace before summing gives \( \text{Tr}(P_R - P_L) := \sum_j \text{Tr}(P^{(j)}_{A^+} - P^{(j)}_{A^-}) = \hat{N}_+(0) - \hat{N}_-(0) \).

Using a function \( c \), which satisfies \( \sum_j \text{Tr}(c(D^2t)P^{(j)}_{A^+} - P^{(j)}_{A^-}) < \infty \) for \( t > 0 \) and \( c(0) = 1 \), with \( \text{Tr}(P_R - P_L) := \lim_{t \to 0} \text{Tr}(c(D^2t)(P_R - P_L)) = \hat{N}_+(0) - \hat{N}_-(0) \) independently of \( t \), i.e. the same result. For \( c(x) = \exp(-x) \) this is the heat-kernel regularization frequently used in works on the Atiyah-Singer case. Thus we here have

\[
\text{Tr}(P_R - P_L) = \hat{N}_+(0) - \hat{N}_-(0) .
\] (7.12)

The comparison of (7.12) and (7.4) reveals the precise difference of the space structures. One should note that (7.12) introduces a dependence on the particular gauge field configuration, while (7.4) does not.

8. Cayley-Transform Type V

8.1. Explicit Forms

Looking for explicit forms of the unitary operator \( V \) one notes that there are three standard constructions. They are up to constant phase factors given by

(i) \( X(\sqrt{X^\dagger X})^{-1} \), normalizing an operator \( X \), not requiring particular mathematical properties of \( X \) (apart from \( X^\dagger X \neq 0 \)),

(ii) the Cayley transform \( (Y - i\mathbb{1})(Y + i\mathbb{1})^{-1} \) of a hermitian operator \( Y \),

(iii) \( \exp(iZ) \) with a hermitian generator \( Z \).

In addition to unitarity one gets \( \gamma_5 \)-hermiticity of the constructed operators \( V \) by requiring \( X, iY, iZ \) to be \( \gamma_5 \)-hermitian.

Construction (i) has been used by Neuberger in the overlap Dirac operator\(^\text{10}\) and here in the general realization of \( V \) by (3.23) with (3.24).

With respect to (ii) we observe that it is actually the basis of a Dirac operator which has been introduced by Chiu\(^\text{13}\) in the GW case (1.1). In fact, the form \( D = 2\rho r D_e(\mathbb{1} + rD_e)^{-1} \) in Ref. 13, with an appropriate antihermitian operator \( D_e \) and a suitable positive constant \( r \), amounts to putting

\[
V = -(Y - i\mathbb{1})(Y + i\mathbb{1})^{-1} \quad \text{with} \quad Y = i\rho r D_e .
\] (8.1)

This operator has turned out, due to the sum rule, not to admit a nonvanishing index\(^\text{14}\). We have shown\(^\text{14}\) that the respective phenomenon generally occurs for Cayley-type operators on the finite lattice, while in the continuum limit this defect is no longer there.

Construction (iii) has not been applied in the present context. For this one could think of using an operator \( Z \) of the form of \( Y \) mentioned above and tune \( r \) to cover the spectrum appropriately. However, since different tunings would be necessary for different gauge field configurations, in practice (iii) appears not useful.
8.2. Analysis of Cayley-Type $V$

In our investigation of Construction (ii) above we consider the general choice

$$V = -(Y - i\mathbb{1})(Y + i\mathbb{1})^{-1} = 2(\mathbb{1} + Y^2)^{-1} - i 2Y(\mathbb{1} + Y^2)^{-1}. \quad (8.2)$$

It is immediately obvious from (8.2) that for the eigenvalue $y = 0$ of $Y$ one gets the eigenvalue $v = 1$ of $V$.

On the finite lattice the operators act in a unitary space of finite dimension. Therefore, requiring $Y$ to be a well-defined hermitian operator, its spectrum consists of a finite number of real eigenvalues (which are discrete and finite). Introducing $s = \max(|y_{\min}|, |y_{\max}|)$, where $y_{\min}$ and $y_{\max}$ denote the smallest and the largest eigenvalue of $Y$, respectively, according to (8.2) we have

$$\Re v \geq \frac{2}{1 + s^2} - 1 \quad \text{for all } v, \quad |\Im v| \geq \frac{2s}{1 + s^2} \quad \text{for } \Re v < 0. \quad (8.3)$$

We see from this that the eigenvalue $v = -1$ of $V$ cannot be reached. Thus on the finite lattice Construction (ii) does generally not meet the basic requirement needed to allow for a nonvanishing index.

The obvious obstacle which prevents from reaching the eigenvalue $-1$ of $V$ is that on the finite lattice $Y$ is bounded. A related problem is that there the inverse Cayley transform,

$$Y = -i(V - \mathbb{1})(V + \mathbb{1})^{-1}, \quad (8.4)$$

is not valid for all unitary operators but only for the subset for which the spectrum does not extend to $-1$.

The crucial observation now is that the indicated restrictions no longer hold in Hilbert space, where one gets a well-defined connection between general unitary operators $V$ and selfadjoint operators $Y$ which can also be unbounded. To recall how this comes about we start from the general spectral representation of unitary operators,

$$V = \int_{-\pi}^{\pi} e^{i\varphi} dE_{\varphi}, \quad (8.5)$$

where the projection function $E_{\varphi}$ accounts for discrete as well as for continuous contributions. Naive insertion of (8.5) into (8.4) does not generally make sense because $-i(e^{i\varphi} - 1)(e^{i\varphi} + 1)^{-1} = \tan\frac{\varphi}{2}$ is not bounded, diverging for $\varphi = \pm\pi$ where the value $-1$ of the spectrum of $V$ is reached. However, $Y$ is well-defined on Hilbert space vectors $f$ by

$$Yf = \lim_{\varphi \to \pi} \int_{-\varphi}^{\varphi} \tan\frac{\varphi}{2} d(E_{\varphi}f) \quad (8.6)$$

in the sense of strong convergence. This is seen noting that with $f = (\mathbb{1} + V)g$ one gets $\tan^2\frac{\varphi}{2} d\|E_{\varphi}f\|^2 = 4\sin^2\frac{\varphi}{2} d\|E_{\varphi}g\|$ over which the integral from $-\pi$ to $\pi$ is obviously finite.
Thus with unbounded operators $Y$ in (8.4) we indeed get unitary operators $V$ in (8.3) with a spectrum extending to $-1$, as is necessary in order that the sum rule (2.16) can admit a nonvanishing index. It is seen that for this a Hilbert space is necessary, which not only has infinite dimension but also includes its limit elements. Obviously the latter here is of crucial importance.

We thus find that on the finite lattice Cayley-type type operators generally do not allow for a nonvanishing index, while in the continuum limit they do. The respective problems therefore turn out to be not restricted to the particular realization, in which it they have been observed by Chiu, but to occur generally for Cayley-type constructions.

Similar remarks apply to the chiral anomaly, which on the lattice is given by $\frac{1}{2} \text{tr}(\gamma_5 V_{nn})$. Not reaching the eigenvalue $-1$ of $V$ on the finite lattice means that one remains with $P_2^{(\pm)} \equiv 0$ in (2.20), which is clearly felt by the anomaly. On the other hand, calculating the continuum form of the anomaly, which implies that the limit has been performed, the correct result is to be expected.

### 8.3. Spaces for Fermions

For completeness we now point out in more detail how precisely the Hilbert space considered above arises.

The fermion operators in the dimensionless formulation on the finite lattice act in a unitary space, in a basis of which a vector is related to a lattice site $n$, a Dirac index $\beta$, and a gauge field index $\alpha$. Taking the infinite-volume limit this unitary space gets of infinite dimension and, in order to be able to perform limits in it, it must be completed to a Hilbert space. This space is the Hilbert space of sequences $l_2$. The unitarily equivalent Hilbert space $L_2(\pi; \kappa)$ of functions $f(\kappa)$ with $-\pi \leq \kappa_\mu \leq \pi$ (Dirac and gauge-group indices being suppressed) is obtained from $l_2$ by a Fourier transformation. Introducing the lattice spacing $a$ and variables $p = \kappa/a$, the space $L_2(\pi; \kappa)$ becomes $L_2(\pi/a; p)$. By the limit $a \to 0$ one then gets the operators in $L_2(\infty; p)$ from the ones in $L_2(\pi/a; p)$. The space $L_2(\infty; x)$, unitarily equivalent to $L_2(\infty; p)$, is again obtained by a Fourier transformation. Instead of proceeding in the more instructive way sketched, one can also realize the direct way from $l_2$ to $L_2(\infty; x)$. The equivalent spaces $L_2(\infty; x)$ and $L_2(\infty; p)$ are the ones one has in the continuum limit.

In detail the definition of the operators of interest by the indicated limit needs some care. Firstly, since the limit element cannot be given explicitly, we resort to the definition by all matrix elements, i.e. by weak operator convergence. Secondly, because two spaces are involved, the usual weak limit is to be slightly generalized. To show that this can be properly done, we introduce $f_a(p) = f(p)\Pi_p \Theta(\pi/a - p_\mu)$ with $f(p) \in L_2(\infty; p)$ and the operator $\hat{O}_a$, requiring $\hat{O}_a(p', p)$ of $L_2(\infty; p)$ to be equal to $O_a(p', p)$ of $L_2(\pi/a; p)$ for $-\pi/a \leq p_\mu \leq \pi/a$. Then $\langle f_a|O_a g_a \rangle$ in $L_2(\infty; p)$ equals $\langle f_a|O_a g_a \rangle$ in $L_2(\pi/a; p)$ for all finite $a$ and $\langle f_a|\hat{O}_a g_a \rangle \to \langle f|\hat{O} g \rangle$ for $a \to 0$ defines the operator $\hat{O}$ in $L_2(\infty; p)$. 


It is now seen that the practical procedure of first calculating the desired matrix elements of the operators or of the functions of operators of interest on the infinite lattice and then performing the $a \to 0$ limit can be precisely formulated and justified in Hilbert space. Furthermore, one notes that the mapping from $O_a$ to $\hat{O}$ is not invertible so that the spectra of $O_a$ and $\hat{O}$ can be substantially different. Thus the operators $Y$ in Sec. 8.4 can get unbounded in the limit as required.

9. Chiral Gauge Theories

9.1. Basic Relations for Weyl Operators

The chiral projection operators implicit in the overlap formalism of Narayanan and Neuberger\cite{6} and used in the formulation of Lüscher\cite{2} are of form

$$ P_\pm = P_\mp = \frac{1}{2}(1 \pm \gamma_5)\mathbb{I}, \quad \tilde{P}_\pm = \tilde{P}_\mp = \frac{1}{2}(1 \pm \gamma_5 V)\mathbb{I}, \quad (9.1) $$

with $V^\dagger = V^{-1} = \gamma_5 V \gamma_5$. Obviously only $\gamma_5$ and $V$ are involved in (9.1) so that we can start with it only requiring (2.1) and (2.2) for $D$. From condition (2.1) we get the identity $D = \frac{1}{2}(D - \gamma_5 D \gamma_5 V)$ and inserting $\gamma_5 = P_+ - P_-$ and $\gamma_5 V = \tilde{P}_+ - \tilde{P}_-$ into it we obtain

$$ D = P_+ D \tilde{P}_- + P_- D \tilde{P}_+. \quad (9.2) $$

With this we have the relations

$$ P_\pm D \tilde{P}_+ = D \tilde{P}_+ = P_\pm D, \quad (9.3) $$

which generalize the expressions for the Weyl operators in terms of the Dirac operator familiar in continuum theory.

With respect to possible forms of (9.3) one should be aware of the fact that the relations

$$ P_\pm \gamma_5 = \pm P_\pm, \quad \gamma_5 V \tilde{P}_+ = \mp \tilde{P}_+, \quad (9.4) $$

allow to absorb parts of $D$. In the special case of the Dirac operator (1.4), with (9.4) one gets $P_+ \rho(1 - V) \tilde{P}_- = 2 \rho P_+ \tilde{P}_-$, which relates the different forms of the chiral determinant in Ref. 3 and in Ref. 1. Considering the general class of operators $D$ here, we have to observe that (9.3) is the generally valid form and that modifications by (9.4) depend on the particular choice of $D$.

9.2. Degrees of Freedom

For the numbers of the degrees of freedom $\text{Tr} P_+$ and $\text{Tr} \tilde{P}_-$ of the Weyl fermions in $P_+ D \tilde{P}_-$ one gets from (1.1)

$$ \text{Tr} P_+ - \text{Tr} \tilde{P}_- = \frac{1}{2} \text{Tr}(\gamma_5 V), \quad (9.5) $$

which agrees with the result (2.19) for the index of the Dirac operators $D$ of the general class.
These degrees of freedom are exhibited in more detailed form representing the projections by
\[ P_+ = \sum_j u_j u_j^\dagger, \quad \tilde{P}_- = \sum_k \tilde{u}_k \tilde{u}_k^\dagger, \quad u_i^\dagger u_j = \delta_{ij}, \quad \tilde{u}_k^\dagger \tilde{u}_l = \delta_{kl}. \quad (9.6) \]

Clearly the choice of the bases here is not unique, however, different ones of them must represent the same projection, respectively, and thus are related by unitary transformations,
\[ v_j = \sum_l u_l S_{lj}, \quad \tilde{v}_k = \tilde{u}_l \tilde{S}_{lk}. \quad (9.7) \]

According to (9.6) the basis vectors are normalized vectors satisfying the eigenequations
\[ P_+ u_j = u_j, \quad \tilde{P}_- \tilde{u}_k = \tilde{u}_k, \quad (9.8) \]

or equivalently
\[ \gamma_5 u_j = u_j, \quad \gamma_5 V \tilde{u}_k = -\tilde{u}_k. \quad (9.9) \]

This shows that, apart from \( \gamma_5 \), only the unitary operator \( V \) is involved in the determination of \( \tilde{u}_k \) for the general class of operators considered here.

9.3. Correlation Functions

Associating Grassmann variables \( \bar{\chi}_j \) and \( \chi_k \) to the degrees of freedom, the fermion field variables get
\[ \bar{\psi} = \sum_j \bar{\chi}_j u_j^\dagger, \quad \psi = \sum_k \tilde{u}_k \chi_k. \quad (9.10) \]

The fermion action then is given by
\[ S_f = \bar{\psi} D \psi = \sum_{j,k} \bar{\chi}_j M_{jk} \chi_k, \quad (9.11) \]

where one has for the matrix \( M \)
\[ M_{jk} = u_j^\dagger D \tilde{u}_k. \quad (9.12) \]

Considering fermionic correlation functions \( \langle \psi_{\sigma(j)} \bar{\psi}_{\sigma(r_1)} \cdots \psi_{\sigma(j)} \bar{\psi}_{\sigma(r_f)} \rangle_t \) with equal numbers of fields \( \psi \) and \( \bar{\psi} \) (\( \sigma \) standing for the combination \( (n, \alpha, \beta) \)) with \( n, \beta \) and \( \alpha \) being related to position space, Dirac space and gauge-group space, respectively) one obtains
\[ \langle \psi_{\sigma(1)} \bar{\psi}_{\sigma(r_1)} \cdots \psi_{\sigma(j)} \bar{\psi}_{\sigma(r_f)} \rangle_t = \int \prod_l (d \bar{\chi}_l d \chi_l) \exp(-S_f) \psi_{\sigma(1)} \bar{\psi}_{\sigma(r_1)} \cdots \psi_{\sigma(j)} \bar{\psi}_{\sigma(r_f)} = \sum_{s_1, s_2, \ldots, s_f} \epsilon_{s_1 s_2 \ldots s_f} (\tilde{P}_- D^{-1} P_+)^{\sigma(s_1)}_{\sigma(r_1)} \cdots (\tilde{P}_- D^{-1} P_+)^{\sigma(s_f)}_{\sigma(r_f)} \det M \quad (9.13) \]
for gauge-field configurations with \( \text{Tr} \tilde{P}_- = \text{Tr} P_+ \), while one gets

\[
\langle \psi_{\sigma(r_1)} \bar{\psi}_{\sigma(r_1)} \cdots \psi_{\sigma(r_f)} \bar{\psi}_{\sigma(r_f)} \rangle_t = 0 \tag{9.14}
\]

for ones with \( \text{Tr} \tilde{P}_- \neq \text{Tr} P_+ \). Then including the gauge-field integrations the full correlation functions become

\[
\frac{\int [dU] \exp(-S_g) \langle \psi_{\sigma(r_1)} \bar{\psi}_{\sigma(r_1)} \cdots \psi_{\sigma(r_f)} \bar{\psi}_{\sigma(r_f)} \rangle_t}{\int [dU] \exp(-S_g)(1)_t} \tag{9.15}
\]

We note that the operator product \( \tilde{P}_- D^{-1} P_+ \) in (9.13) in the presence of zero modes of \( D \) is not yet defined. Replacing \( D^{-1} \) by \( (D - \zeta)^{-1} \) there and letting the parameter \( \zeta \) go to zero after the evaluation we get a well-defined finite result. This follows since with (2.7) and (2.6) we obtain \( P_1 = P_1^{(+)} + P_1^{(-)} \) on the space of zero modes of \( D \).

### 9.4. Remarks on the Chiral Determinant

To study gauge-invariance properties of the chiral determinant \( \det M \) in (9.13), in Ref. 2 the variation \( \delta \ln \det M = \text{Tr}(M^{-1} \delta M) \) has been considered. For this expression using (9.6) and (9.3)

\[
\text{Tr}(M^{-1} \delta M) = \text{Tr}(\tilde{P}_- D^{-1} \delta D) + \sum_k \tilde{u}_k^\dagger \delta \tilde{u}_k \tag{9.16}
\]

is obtained, which with \( \delta D = [\mathcal{G}, D] \), where \( \mathcal{G} \) is the generator of the gauge transformation, becomes

\[
\text{Tr}(M^{-1} \delta M) = \frac{1}{2} \text{Tr}(\gamma_5 \mathcal{G} V) + \sum_k \tilde{u}_k^\dagger \gamma_5 \delta \tilde{u}_k. \tag{9.17}
\]

We note, however, that this only holds in the absence of zero modes of \( D \), because only then

\[
(M^{-1})_{kl} = \tilde{u}_k^\dagger D^{-1} u_l \tag{9.18}
\]

is properly defined. To get the appropriate definition in the presence of zero modes, too, we replace \( D^{-1} \) in (9.18) by \( (D - \zeta)^{-1} \). Then in the evaluation using

\[
D (D - \zeta)^{-1} \rightarrow \, \mathbb{1} - P_1, \quad -\zeta (D - \zeta)^{-1} \rightarrow P_1 \quad \text{for} \quad \zeta \rightarrow 0 \tag{9.19}
\]

we obtain the further terms

\[
- \text{Tr}(\gamma_5 \mathcal{G} P_1) + \sum_k \tilde{u}_k^\dagger \gamma_5 P_1 \delta \tilde{u}_k \tag{9.20}
\]

which are to be added on the r.h.s. of (9.17). The projector \( P_1 = P_1^{(+)} + P_1^{(-)} \) on the space of zero modes of \( D \) according to (2.6) and (2.8) may be expressed in terms of \( V \) by

\[
P_1 = \lim_{\zeta \rightarrow 0} -\zeta (V - \mathbb{1} - \zeta \mathbb{1})^{-1}. \tag{9.21}
\]
It is seen now that, apart from $\gamma_5$ and $G$, also only $V$ enters $\text{Tr}(M^{-1}\delta M)$ for the general class of operators considered.

With $G_{\nu n} = i \delta_{\nu n} \sum_{\ell} \omega_n^{\ell} T^{\ell}$, where $T^{\ell}$ are hermitian generators and the $\omega_n^{\ell}$ real, the first term on the r.h.s. of (9.17) can be written in more detail as

$$\frac{1}{2} \text{Tr}(\gamma_5 GV) = i \sum_{\nu,\ell} \omega_n^{\ell} \frac{1}{2} \text{tr}(\gamma_5 T^{\ell} V_{\nu n}).$$

The trace expression $\text{tr}(\gamma_5 T^{\ell} V_{\nu n})$ here differs from that in (7.22) only by the insertion of the factor $T^{\ell}$. Since the inclusion of such factor in the derivation of (7.22) for the chiral anomaly is straightforward, it becomes obvious that in the continuum limit one gets for the gauge anomaly

$$\frac{1}{2} \text{tr}(\gamma_5 T^{\ell} V_{\nu n}) \frac{1}{a} \rightarrow -\frac{1}{32\pi^2} \sum_{\mu\nu\lambda\tau} \epsilon_{\mu\nu\lambda\tau} \text{tr}(T^{\ell} F_{\mu\nu}(x) F_{\lambda\tau}(x)),$$

which is thus seen to derive generally from $V$ (and not from $D$).

**Acknowledgement**

I wish to thank Michael Müller-Preussker and his group for their kind hospitality.

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