A GENERALIZATION OF NEWTON-MACLAURIN’S INEQUALITIES

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Abstract. In this paper, we prove Newton-Maclaurin type inequalities for functions obtained by linear combination of two neighboring primary symmetry functions, which is a generalization of the classical Newton-Maclaurin inequality.

1. Introduction

The $k$-th elementary symmetric function of the variables $x_1, x_2, \ldots, x_n$ is defined by

$$\sigma_k(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}, \quad 1 \leq k \leq n,$$

where $x = (x_1, x_2, \cdots, x_n)$. For example, when $n = 3$,

$$\sigma_1(x) = x_1 + x_2 + x_3, \quad \sigma_2(x) = x_1x_2 + x_1x_3 + x_2x_3, \quad \sigma_3(x) = x_1x_2x_3.$$

It will be convenient to define $\sigma_0(x) = 1$, and define $\sigma_k(x) = 0$ if $k < 0$ or $k > n$.

Furthermore, define a $k$-th elementary symmetric mean as

$$E_k(x) = \frac{\sigma_k(x)}{C^k_n}, \quad k = 0, 1, \cdots, n,$$

where $C^k_n = \frac{n!}{k!(n-k)!}$.

Theorem 1.1. (Newton [19] and Maclaurin [17]) Let $x = (x_1, \cdots, x_n)$ be an $n$-tuple non-negative real numbers. Then

(1.1) \quad $E_k^2(x) \geq E_{k-1}(x)E_{k+1}(x), \quad k = 1, 2 \cdots, n-1,$

(1.2) \quad $E_1(x) \geq E_2^{1/2}(x) \geq \cdots \geq E_n^{1/n}(x)$,

and the inequality is strict unless all entries of $x$ coincide.

The inequalities (1.1) in Theorem 1.1 is a consequence of a rule stated by Newton [19] which gives a lower bound on the number of nonreal roots of a real polynomial. Since Newton did not give a proof of his rule, the proof of Theorem 1.1 is due to MacLaurin [17]. Actually, the Newton’s inequalities (1.1) work for $n$-tuples of real, not necessarily positive elements. For an inductive proof in the case where $x_1, x_2, \cdots, x_n$ are nonnegative, see [9] § 2.22. For a proof by differential calculus in the case where $x_1, x_2, \cdots, x_n$ are real, see [9]

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Several reformulations/generalisations of Newton’s inequalities have been given over the years, see for instance [18, 21, 20, 2]. The Newton-Maclaurin inequalities play important roles in deriving theoretical results for fully nonlinear partial differential equations and geometric analysis. There are many important results that need to use the Newton-Maclaurin inequalities, such as [3, 4, 5, 6, 7, 11] etc. This is because the following $k$-Hessian equations and curvature equations

$$\sigma_k(\lambda(u_{ij})) = f(x, u, \nabla u),$$
$$\sigma_k(\kappa(X)) = f(X, \nu)$$

are central studies in the field of fully nonlinear partial differential equations and geometric analysis. Their left-hand side $k$-Hessian operator $\sigma_k$ is the primary symmetric function about the eigenvalues $\lambda = (\lambda_1, \cdots, \lambda_n)$ of the Hessian matrix $(u_{ij})$ or the principal curvature $\kappa = (\kappa_1, \cdots, \kappa_n)$ of the surface. In recent years, the fully non-linear equations derived from linear combinations of primary symmetric functions have received increasing attention. For example, the following special Lagrangian equation

$$\text{Im} \det(\delta_{ij} + i u_{ij}) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k \sigma_{2k+1}(\lambda(u_{ij})) = 0,$$

which derived by Harvey and Lawson in their study of the minimal submanifold problem [10]. The fully non-linear partial differential equations

$$P_m(u_{ij}) = \sum_{k=0}^{m-1} (l^+_k)^{m-k}(x) \sigma_k(u_{ij}) = g^{m-1}(x)$$

studied by Krylov [12] and Dong [1]. In [13], Li-Ren-Wang studied the concavity of operators $\sum_{s=0}^{k} \alpha_s \sigma_s$ and $\sigma_k + \alpha \sigma_{k-1}$, and discussed the curvature estimates for the convex surface of the corresponding equations

$$\sum_{s=0}^{k} \alpha_s \sigma_s(\kappa(X)) = f(X, \nu(X)), \quad X \in M.$$

Guan and Zhang [8] investigated the curvature estimates of the following curvature equation

$$\sigma_k(W_u(x)) + \alpha(x) \sigma_{k-1}(W_u(x)) = \sum_{l=0}^{k-2} \alpha_l(x) \sigma_l(W_u(x)), \quad x \in S^n.$$

Recently, Liu and Ren [16] discussed the Pogorelov-type $C^2$ estimates for $(k-1)$-convex and $k$-convex solutions of the following Sum Hessian equations

$$\sigma_k(\lambda(u_{ij})) + \alpha \sigma_{k-1}(\lambda(u_{ij})) = f(x, u, \nabla u),$$

and established a rigidity theorem when the right-hand side of the equation is constant.

In the further study of the above problem, the Newton-Maclaurin type inequalities for the operators derived from the left-hand side of the equations are always needed. A
natural question is that whether the Newton-Maclaurin type inequalities for the operators of linear combinations of these primary symmetric functions still hold?

Now the main result of this paper is stated as follows.

Theorem 1.2. Let \( n \geq 3, 1 \leq k \leq n - 2 \). For any real number \( \alpha \in \mathbb{R} \) and any \( n \)-tuple real numbers \( x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \), we have

\[
\alpha E_k(x) + E_{k+1}(x) \geq \alpha E_{k-1}(x) + E_k(x) \geq \alpha E_{k+1}(x) + E_k(x).
\]

The inequality is strict unless \( x_1 = \cdots = x_n \), or

\[
\frac{\alpha E_k(x) + E_{k+1}(x)}{\alpha E_{k-1}(x) + E_k(x)} = -\alpha.
\]

A special case in which the above equality holds is that there are \( n - 1 \) elements of \( x_1, x_2, \cdots, x_n \) taking the value \(-\alpha\).

Furthermore, if \( \alpha \geq 0 \) and

\[
\alpha E_m(x) + E_{m+1}(x) \geq 0, \quad \text{for all } m = 0, 1, \cdots, k,
\]

then

\[
\alpha + E_1(x) \geq \alpha E_1(x) + E_2(x) \geq \cdots \geq \alpha E_k(x) + E_{k+1}(x) \geq \alpha E_{k+1}(x) + E_k(x).
\]

Theorem 1.2 is the Newton-Maclaurin type inequality for the function obtained by linear combination of two neighboring terms \( E_k \) and \( E_{k+1} \). Obviously, it is the classical Newton-Maclaurin inequality when \( \alpha = 0 \). Furthermore, one may ask if the Newton-Maclaurin type inequality still holds for the functions which obtained by linear combination of general primary symmetric functions. That is, for \( \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n \), \( x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \), whether the following inequalities hold?

\[
\left( \sum_{k=1}^{n} \alpha_k E_k(x) \right)^2 \geq \left( \sum_{k=1}^{n} \alpha_k E_{k+1}(x) \right) \left( \sum_{k=1}^{n} \alpha_k E_{k-1}(x) \right).
\]

The answer is no in general. For example, let \( n = 3 \), \( \alpha = (1, 0, 1) \), \( x = (4, 4, \frac{1}{4}, \frac{1}{4}) \), we have

\[
E_1(x) = \frac{1}{4} (4 + 4 + \frac{1}{4} + \frac{1}{4}) = \frac{17}{8},
\]

\[
E_2(x) = \frac{1}{6} (4 \times 4 + 4 \times 4 \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4}) = \frac{107}{32},
\]

\[
E_3(x) = \frac{1}{4} (2 \times 4 \times 4 \times \frac{1}{4} + 2 \times 4 \times \frac{1}{4} \times \frac{1}{4}) = \frac{17}{8},
\]

\[
E_4(x) = 4 \times 4 \times \frac{1}{4} \times \frac{1}{4} = 1,
\]

therefore,

\[
[E_1(x) + E_3(x)]^2 - [1 + E_2(x)]E_2(x) + E_4(x)] = -\frac{825}{1024} < 0.
\]

Maybe inequalities (1.5) hold when \( \alpha = (\alpha_1, \cdots, \alpha_n) \) satisfy some structural conditions.
A straightforward conclusion of Newton’s inequality (1.1) is
\[
\sigma_k^2(x) - \sigma_{k-1}(x)\sigma_{k+1}(x) \geq \theta \sigma_k^2(x),
\]
where \(0 < \theta < 1\) is a constant depending on \(n\) and \(k\). The inequality (1.6) is also a common sense which is widely used in the study of \(k\)-Hessian equations and curvature equations, see [4, 5, 6, 7, 11, 13, 14, 15], etc. For the Sum Hessian operators \(\sigma_k + \alpha \sigma_{k-1}\), if \(\alpha > 0\) and \(x = (x_1, \cdots, x_n) \in \Gamma_k\), Liu and Ren [16] proved
\[
[\sigma_k(x) + \alpha \sigma_{k-1}(x)]^2 \geq [\sigma_{k-1}(x) + \alpha \sigma_{k-2}(x)][\sigma_{k+1}(x) + \alpha \sigma_k(x)]
\]
by Newton’s inequality (1.6). Here \(\Gamma_k\) is the Garding’s cone
\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n \mid \sigma_m(\lambda) > 0, \quad m = 1, \cdots, k \}.
\]

The requirement \(x \in \Gamma_k\) of above inequalities restrict their applications in the study of Sum Hessian equations. In this paper, we remove the restrictions from \(\alpha\) and \(x\).

**Theorem 1.3.** Let \(n \geq 3, 0 \leq k \leq n - 1\). For any real number \(\alpha \in \mathbb{R}\) and any \(n\)-tuple real numbers \(x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n\), we have
\[
(1 - \theta)[\alpha \sigma_k(x) + \sigma_{k+1}(x)]^2 - [\alpha \sigma_{k-1}(x) + \sigma_k(x)][\alpha \sigma_{k+1}(x) + \sigma_{k+2}(x)] \geq 0,
\]
where \(0 < \theta < 1\) is a constant depending only on \(n\) and \(k\).

**Remark 1.4.** When \(k = 0\) or \(k = n - 1\), (1.3) does not hold, one can take \(x_1 = x_2 = \cdots = x_n\) for verification. It is a difference between (1.7) and (1.3).

2. The proof of Theorem 1.2

First, we consider the following inequality in Euclidean space \(\mathbb{R}^3\).

**Lemma 2.1.** For any real number \(\alpha \in \mathbb{R}\) and \(z = (z_1, z_2, z_3) \in \mathbb{R}^3\), we have
\[
[\alpha E_1(z) + E_2(z)]^2 \geq [\alpha E_0(z) + E_1(z)][\alpha E_2(z) + E_3(z)].
\]
The inequality is strict unless \(z_1 = z_2 = z_3\) or
\[
\frac{\alpha E_1(z) + E_2(z)}{\alpha E_0(z) + E_1(z)} = \frac{\alpha E_2(z) + E_3(z)}{\alpha E_1(z) + E_2(z)} = -\alpha.
\]

**Proof.** A straightforward calculation shows
\[
18[\alpha E_1(z) + E_2(z)]^2 - 18[\alpha E_0(z) + E_1(z)][\alpha E_2(z) + E_3(z)]
\]
\[
= [(z_1 + \alpha)(z_2 + \alpha) - (z_1 + \alpha)(z_3 + \alpha)]^2 + [(z_1 + \alpha)(z_2 + \alpha) - (z_2 + \alpha)(z_3 + \alpha)]^2
\]
\[
+ [(z_1 + \alpha)(z_3 + \alpha) - (z_2 + \alpha)(z_3 + \alpha)]^2 \geq 0.
\]
Obviously, the inequality is strict unless \(z_1 = z_2 = z_3\) or any two elements of \(z_1, z_2, z_3\) valued \(-\alpha\). Without loss of generality, we assume \(z_1 = z_2 = -\alpha\). Then we obtain
\[
\alpha E_0(z) + E_1(z) = \frac{z_3 + \alpha}{3}, \quad \alpha E_1(z) + E_2(z) = \frac{-\alpha(z_3 + \alpha)}{3}, \quad \alpha E_2(z) + E_3(z) = \frac{\alpha^2(z_3 + \alpha)}{3}.
\]
Thus,
\[
\frac{\alpha E_1(z) + E_2(z)}{\alpha E_0(z) + E_1(z)} = \frac{\alpha E_2(z) + E_3(z)}{\alpha E_1(z) + E_2(z)} = -\alpha.
\]
The following Lemma is a useful tool to prove Newton’s inequalities [11] from Sylvester [22, 20].

Lemma 2.2. If
\[ F(x, y) = c_0x^m + c_1x^{m-1}y + \cdots + c_my^m \]
is a homogeneous function of the n-th degree in \(x\) and \(y\) which has all its roots \(x/y\) real, then the same is true for all non-identical 0 equations \( \frac{\partial^{i+j} F}{\partial x^i \partial y^j} = 0 \), obtained from it by partial differentiation with respect to \(x\) and \(y\). Further, if \(E\) is one of these equations, and it has a multiple root \(\alpha\), then \(\alpha\) is also a root, of multiplicity one higher, of the equation from which \(E\) is derived by differentiation.

Let us now present the proof of Theorem 1.2.

Proof. We assume \(P(t)\) is a polynomial of \(n\)-degree, with real roots \(x_1, x_2, \ldots, x_n\). Then \(P(t)\) is represented as
\[ P(t) = \prod_{i=1}^{n} (t - x_i) = E_0(x)t^n - C_1^1E_1(x)t^{n-1} + C_2^2E_2(x)t^{n-2} - \cdots + (-1)^nE_n(x), \]
and we shall apply Lemma 2.2 to the associated homogeneous polynomial
\[ F(t, s) = E_0(x)t^n - C_1^1E_1(x)t^{n-1}s + C_2^2E_2(x)t^{n-2}s^2 - \cdots + (-1)^nE_n(x)s^n. \]

Considering the case of the derivatives \(\frac{\partial^{n-3} F}{\partial t^{n-2-k}\partial s^{k-1}}\) (for \(k = 1, \ldots, n - 2\)), we arrive to the fact that all the cubic polynomials
\[ E_{k-1}(x)t^3 - 3E_k(x)t^2s + 3E_{k+1}(x)ts^2 - E_{k+2}(x)s^3 \]
for \(k = 1, \ldots, n - 2\) also have real roots.

If \(E_{k-1}(x) = E_{k+2}(x) = 0\), it is easy to get
\[
[\alpha E_k(x) + E_{k+1}(x)]^2 - [\alpha E_{k-1}(x) + E_k(x)][\alpha E_{k+1}(x) + E_{k+2}(x)]
= \frac{1}{2}[\alpha E_k(x) + E_{k+1}(x)]^2 + \frac{\alpha^2}{2}E_{k+2}(x) + \frac{1}{2}E_{k+1}(x) \geq 0.
\]

So we divide it in two cases to deal with (1.3): \(E_{k-1}(x) \neq 0\) or \(E_{k+2}(x) \neq 0\).

Case A: When \(E_{k-1}(x) \neq 0\), by (2.2), the polynomial
\[ t^3 - \frac{3E_k(x)}{E_{k-1}(x)}t^2 + \frac{3E_{k+1}(x)}{E_{k-1}(x)}t - \frac{E_{k+2}(x)}{E_{k-1}(x)} \]
has three real roots. We denote the roots by \(z_1, z_2, z_3\) and denote \(z = (z_1, z_2, z_3)\), then
\[ E_1(z) = \frac{E_k(x)}{E_{k-1}(x)}, \quad E_2(z) = \frac{E_{k+1}(x)}{E_{k-1}(x)}, \quad E_3(z) = \frac{E_{k+2}(x)}{E_{k-1}(x)}. \]
Using Lemma 2.1 we obtain (1.3).
3.1. Special cases.

By canceling the common factor.

Thus, Combining (1.3) and the inequalities above, we get

\[ x \]

A special case in which the above equality holds is that there are \( n - 1 \) elements of \( x_1, x_2, \ldots, x_n \) valued \(-\alpha\).

Now, we prove (1.4). When \( k = 0 \), we get from (1.1),

\[
[\alpha + E_1(x)]^2 - [\alpha E_1(x) + E_2(x)] = \alpha^2 + \alpha E_1(x) + E_1^2(x) - E_2(x) \\
\geq \alpha (\alpha + E_1(x)) \geq 0.
\]

Therefore,

\[
[\alpha + E_1(x)] \geq [\alpha E_1(x) + E_2(x)]^{1/2}.
\]

For \( 1 \leq k \leq n - 2 \), we assume

\[
[\alpha E_k(x) + E_{k+1}(x)]^{1/(k-1)} \geq [\alpha E_{k-1}(x) + E_k(x)]^{1/k}.
\]

Combining (1.3) and the inequalities above, we get

\[
[\alpha E_{k-1}(x) + E_k(x)]^2 \geq [\alpha E_k(x) + E_{k+1}(x)] [\alpha E_k(x) + E_{k+1}(x)] \\
\geq [\alpha E_{k-1}(x) + E_k(x)]^{(k-1)/k} [\alpha E_k(x) + E_{k+1}(x)].
\]

Thus,

\[
[\alpha E_{k-1}(x) + E_k(x)]^{1/k} \geq [\alpha E_k(x) + E_{k+1}(x)]^{1/(k+1)}
\]

by canceling the common factor.

\[ \square \]

3. The proof of Theorem 1.3

We divide it in two cases to prove Theorem 1.3

(i) Special cases: \( k = 0, k = n - 1 \) or \( n = 3, k = 1 \);

(ii) General cases: \( n \geq 4, 1 \leq k \leq n - 2 \).

3.1. Special cases.

- When \( k = 0 \), choose \( \theta = 1/2 \) in (1.7). Using the identity \( \sigma_1^2(x) = \sum_{i=1}^{n} x_i^2 + 2\sigma_2 \), we have

\[
\frac{1}{2} [\alpha + \sigma_1(x)]^2 - [\alpha \sigma_1(x) + \sigma_2(x)] = \frac{\alpha^2}{2} + \frac{1}{2} \sigma_1^2(x) - \sigma_2(x) \geq 0.
\]

- When \( k = n - 1 \), choose \( \theta = 1/2 \). By using the identity

\[
\sigma_{n-1}^2(x) = \sum_{i=1}^{n} \frac{\sigma_i^2(x)}{x_i^2} + 2\sigma_{n-2}(x)\sigma_n(x),
\]

has three real roots. Then the proof of (1.3) is similar to Case A.

By Lemma 2.1, the inequalities of (1.3) are strict unless \( x_1 = x_2 = \cdots = x_n \) or

\[
\frac{\alpha E_k(x) + E_{k+1}(x)}{\alpha E_{k-1}(x) + E_k(x)} = \frac{\alpha E_{k+1}(x) + E_k(x)}{\alpha E_k(x) + E_{k+1}(x)} = -\alpha.
\]

Case B: When \( k = n - 1 \), choose \( \theta = 1/2 \) in (1.7). Using the identity

\[
\sigma_{n-1}^2(x) = \sum_{i=1}^{n} \frac{\sigma_i^2(x)}{x_i^2} + 2\sigma_{n-2}(x)\sigma_n(x),
\]

for \( 1 \leq k \leq n - 2 \), we have

\[
\sigma_{k}^2(x) = \sum_{i=1}^{n} \frac{\sigma_i^2(x)}{x_i^2} + 2\sigma_{k-1}(x)\sigma_{k+1}(x) - \sigma_{k}^2(x).
\]

...
we get
\[ \frac{1}{2} [\alpha \sigma_{n-1}(x) + \sigma_n(x)]^2 - [\alpha \sigma_{n-2}(x) + \sigma_{n-1}(x)] \alpha \sigma_n(x) \]
\[ = \frac{\alpha^2}{2} \sigma_{n-1}^2(x) + \frac{1}{2} \sigma_n^2(x) - \frac{\alpha^2}{2} \sigma_{n-2}(x) \sigma_n(x) \geq 0. \]

- When \( n = 3, k = 1 \), choose \( \theta = \frac{1}{2} \). Using the identities
\[
\sigma_1(x)^2 = x_1^2 + x_2^2 + x_3^2 + 2\sigma_2(x), \quad \sigma_2(x) = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + 2\sigma_1(x)\sigma_3(x),
\]
we obtain
\[
\frac{1}{2} [\alpha \sigma_1(x) + \sigma_2(x)]^2 - [\alpha + \sigma_1(x)][\alpha \sigma_2(x) + \sigma_3(x)]
\]
\[ = \frac{\alpha^2}{2} \sigma_1^2(x) + \frac{1}{2} \sigma_2^2(x) - \frac{\alpha^2}{2} \sigma_2(x) - \alpha \sigma_3(x) - \sigma_1(x) \sigma_3(x) \]
\[ \geq \frac{\alpha^2}{2} x_1^2 + \frac{1}{2} x_2^2 x_3 - \alpha x_1 x_2 x_3 \geq 0. \]

3.2. General cases. For \( n \geq 4, 1 \leq k \leq n - 2 \), we denote
\[
(3.1) \quad a = C_n^{k-1}, \quad b = C_n^k, \quad c = C_n^{k+1}, \quad d = C_n^{k+2}.
\]
Then the inequalities (1.7) can be rewritten as
\[
(1 - \theta) [abE_k(x) + cE_{k+1}(x)]^2 - [\alpha aE_{k-1}(x) + bE_k(x)][\alpha cE_{k+1}(x) + dE_{k+2}(x)] \geq 0.
\]
From Lemma 2.2 and by a similar proof to (1.3), we shall complete the proof Theorem 1.3 via establishing the following inequalities.
\[
(1 - \theta) [abE_1(z) + cE_2(z)]^2 - [\alpha a + bE_1](\alpha c E_2 + dE_3(z)] \geq 0.
\]
Here \( z = (z_1, z_2, z_3) \in \mathbb{R}^3 \). Furthermore, it is equivalent to prove
\[
(3.2) \quad L(z) := (ab \sigma_1 + c \sigma_2)^2 - (3aa + b \sigma_1)(ac \sigma_2 + 3d \sigma_3) \geq \theta (ab \sigma_1 + c \sigma_2)^2,
\]
where
\[
\sigma_1 = z_1 + z_2 + z_3, \quad \sigma_2 = z_1 z_2 + z_1 z_3 + z_2 z_3, \quad \sigma_3 = z_1 z_2 z_3.
\]
Now we prove (3.2). Let us expand the left-hand and right-hand sides of (3.2) separately. By using identities
\[
\sigma_1^2 = z_1^2 + z_2^2 + z_3^2 + 2\sigma_2, \quad \sigma_2^2 = z_1^2 z_2^2 + z_1^2 z_3^2 + z_2^2 z_3^2 + 2\sigma_1 \sigma_3,
\]
we have
\[ L(z) = \alpha^2 b^2 \sigma_1^2 + abc \sigma_1 \sigma_2 + c^2 \sigma_2^2 - 3\alpha ac \sigma_2 - 9\alpha ad \sigma_3 - 3bd \sigma_1 \sigma_3 \]
\[ = \alpha^2 b^2 \sum_{i=1}^{3} z_i^2 + \alpha^2 (2b^2 - 3ac) \sigma_2 + abc \sigma_1 \sigma_2 \]
\[ + c^2 \sum_{p<q} z_p^2 z_q^2 + (2c^2 - 3bd) \sigma_1 \sigma_3 - 9\alpha ad \sigma_3, \]
(3.3)
\[ (ab \sigma_1 + c \sigma_2)^2 = \alpha^2 b^2 \sum_{i=1}^{3} z_i^2 + 2\alpha^2 b^2 \sigma_2 + 2abc \sigma_1 \sigma_2 + c^2 \sum_{p<q} z_p^2 z_q^2 + 2c^2 \sigma_1 \sigma_3. \]
(3.4)

For any \( z = (z_1, z_2, z_3) \in \mathbb{R}^3 \), we define a nonnegative function
\[ W(z, t) = (z_1 - z_2)^2 (\alpha + tz_3)^2 + (z_1 - z_3)^2 (\alpha + tz_2)^2 + (z_2 - z_3)^2 (\alpha + tz_1)^2. \]

Expanding \( W(z, t) \) we get
\[ W(z, t) = 2\alpha^2 \sum_{i=1}^{3} z_i^2 - 2\alpha^2 \sigma_2 + 2\alpha t \sigma_1 \sigma_2 + 2t^2 \sum_{p<q} z_p^2 z_q^2 - 2t^2 \sigma_1 \sigma_3 - 18\alpha t \sigma_3. \]
(3.5)

We want to absorb the terms \( \sigma_1 \sigma_2, \sigma_1 \sigma_3 \) and \( \sigma_2 \) in \( L(z) \) by using \( (ab \sigma_1 + c \sigma_2)^2 \) and \( W(z, t) \). So let
\[ L(z) = \theta_1 (ab \sigma_1 + c \sigma_2)^2 + \theta_2 W(z, t) + V(z), \]
where \( \theta_1 \) and \( \theta_2 \) are constants which will be determined later, \( V(z) \) is the remaining part. By (3.3)-(3.6), comparing the coefficients of \( \sigma_1 \sigma_2, \sigma_1 \sigma_3 \) and \( \sigma_2 \), we have
\[ \begin{cases} 
bc = 2bc \theta_1 + 2t \theta_2, & (\sigma_1 \sigma_2) \\
2c^2 - 3bd = 2c^2 \theta_1 - 2t^2 \theta_2, & (\sigma_1 \sigma_3) \\
(2b^2 - 3ac) = 2b^2 \theta_1 - 2t \theta_2. & (\sigma_2) 
\end{cases} \]
We solve the equations above to find
\[ \theta_1 = \frac{3(2ac^3 + 2b^3 d - b^2 c^2 - 3abcd)}{6ac^3 + 6b^3 d - 4b^2 c^2}, \quad \theta_2 = \frac{c^2 (3ac - b^2)}{6ac^3 + 6b^3 d - 4b^2 c^2}, \quad t = \frac{b(3bd - c^2)}{c(3ac - b^2)}. \]
(3.7)

Using (3.6) again, we obtain
\[ V(z) = A_1 \alpha^2 \sum_{i=1}^{3} z_i^2 + A_2 \sum_{p<q} z_p^2 z_q^2 + A_3 \alpha \sigma_3, \]
where
\[ A_1 = b^2 - b^2 \theta_1 - 2t \theta_2, \quad A_2 = c^2 - c^2 \theta_1 - 2t^2 \theta_2, \quad A_3 = 18t \theta_2 - 9ad. \]
We claim that for \( n \geq 4, 1 \leq k \leq n - 2, \)
\[ \theta_1 > 0, \quad \theta_2 > 0, \quad V(z) \geq 0. \]
(3.8)
Notice that \( W(z, t) \) is nonnegative, so by (3.6) we obtain (3.2). That is, Theorem 1.3 is proved. Next, we will prove the claim.

**Lemma 3.1.** For \( n \geq 4 \), \( 1 \leq k \leq n - 2 \), we have
\( i \) \( 3bd - c^2 > 0 \), \( 3ac - b^2 > 0 \);
\( ii \) \( 2ac^3 + 2b^3d - b^2c^2 - 3abcd > 0 \).

**Proof.** (i) By (3.1) we have
\[
\begin{align*}
a & = \frac{C_{n-1}^k}{C_n^k} = \frac{k}{n-k+1}, \\
b & = \frac{C_n^k}{C_n^{k+1}} = \frac{k+1}{n-k}, \\
c & = \frac{C_{n-1}^{k+1}}{C_n^{k+2}} = \frac{k+2}{n-k-1}. \\
d & = \frac{C_{n-2}^{k+1}}{C_n^{k+3}} = \frac{k+3}{n-k-2}.
\end{align*}
\]
By (3.9), one has
\[
\begin{align*}
3bd - c^2 &= c^2 \left( \frac{3bd}{c^2} - 1 \right) = c^2 \left( \frac{3(k+1)(n-k-1)}{(n-k)(k+2)} - 1 \right) \\
&= c^2 \cdot \frac{-2k^2 + 2(n-2)k + (n-3)}{(n-k)(k+2)}.
\end{align*}
\]
Since the following quadratic function
\[
f_1(k) := -2k^2 + 2(n-2)k + (n-3)
\]
with respect to variable \( k \) is increasing on the interval \( [1, \frac{n-2}{2}] \), decreasing on the interval \( [\frac{n-2}{2}, n-2] \), and
\[
f_1(1) = 3(n-3) > 0, \quad f_1(n-2) = n-3 > 0,
\]
f_1(k) > 0 on \( [1, n-2] \). Combining (3.10), we obtain \( 3bd - c^2 > 0 \).

Similarly,
\[
3ac - b^2 = b^2 \cdot \frac{-2k^2 + 2nk - n-1}{(n-k+1)(k+1)},
\]
and the function \(-2k^2 + 2nk - n-1\) respect to variable \( k \) is positive on \( [1, n-2] \), so we get \( 3ac - b^2 > 0 \).

(ii) By (3.9), we have
\[
2ac^3 + 2b^3d - b^2c^2 - 3abcd = b^2c^2 \left( \frac{2ac}{b^2} + \frac{2bd}{c^2} - 1 - \frac{3ad}{bc} \right)
\]
\[
= b^2c^2 \cdot \frac{2(n+1)(-k^2 + kn - k - 1)}{(k+1)(k+2)(n-k+1)(n-k)}.
\]
It is easy to get \(-k^2 + (n-1)k - 1 \geq 0\) for \( 1 \leq k \leq n-2 \).
\[
\square
\]

**Lemma 3.2.** For \( n \geq 4 \), \( 1 \leq k \leq n - 2 \), we have
\( i \) \( A_1 > 0 \);
\( ii \) \( A_2 > 0 \);
\( iii \) \( A_1A_2 - A_3^2/36 > 0 \).
Proof. (i) Substituting (3.7) into $A_1$, we have

$$A_1 = b^2(1 - \theta_1) - 2\theta_2 = \frac{9ab^3cd - b^4c^2}{6ac^3 + 6b^3d - 4b^2c^2} - \frac{2c^2(3ac - b^2)^2}{6ac^3 + 6b^3d - 4b^2c^2}$$

$$= \frac{3c(4ab^2c^2 + 3ab^3d - 6a^2c^3 - b^4c)}{6ac^3 + 6b^3d - 4b^2c^2} \cdot (4 + \frac{3bd}{c^2} - \frac{6ac}{b^2} - \frac{b^2}{ac}).$$

(3.11)

By (3.9),

$$\frac{4(n + 1)}{k(k + 1)(k + 2)(n - k + 1)(n - k)} = 2(n - 5)(n - 5)k + 3n - 2 = 0.$$ 

Let us show the cubic function

$$f_2(k) := -k^3 + (n - 5)k^2 + (3n - 2)k - n - 1 > 0.$$ 

Differentiating $f_2(k)$, we solve the quadratic equation

$$f'_2(k) = -3k^2 + 2(n - 5)k + 3n - 2 = 0.$$ 

The quadratic formula gives

$$k_1 = \frac{2(n - 5) - \sqrt{4n^2 - 4n + 76}}{6}, \quad k_2 = \frac{2(n - 5) + \sqrt{4n^2 - 4n + 76}}{6}.$$ 

It is easy to get

$$k_1 < \frac{2(n - 5) - (2n - 1)}{6} < 0,$$

$$\frac{4n - 11}{6} < k_2 < \frac{2(n - 5) + 3n}{6} = \frac{5(n - 2)}{6},$$

and $f'_2(k) \geq 0$ on $[1, k_2]$, $f'_2(k) \leq 0$ on $[k_2, n - 2]$. So function $f_2(k)$ is increasing on $[1, k_2]$ and decreasing on $[k_2, n - 2]$. On the other hand, $f_2(1) = f_2(n - 2) = 3(n - 3) > 0$, this implied $f_2(k) > 0$ on $[1, n - 2]$. Combining (3.11) and (i) of Lemma 3.1, we obtain $A_1 > 0$.

(ii) Substituting (3.7) into $A_2$, we have

$$A_2 = c^2(1 - \theta_1) - 2t^2\theta_2 = \frac{9abc^3d - b^4c^4}{6ac^3 + 6b^3d - 4b^2c^2} - \frac{2b^2(3bd - c^2)^2}{6ac^3 + 6b^3d - 4b^2c^2}$$

$$= \frac{3b(4b^2c^2d + 3ac^3d - 6b^3d^2 - bc^4)}{6ac^3 + 6b^3d - 4b^2c^2} \cdot (4 + \frac{3ac}{b^2} - \frac{6bc}{bd} - \frac{c^2}{bd}).$$

(3.12)

By using (3.9), we get

$$\frac{4ac}{bd} - \frac{c^2}{bd} = 2(n + 1)\left[\frac{k^3 - (2n + 2)k^2 + (n^2 + 3n - 5)k - n^2 + 2n - 3}{(k + 1)(k + 2)(n - k + 1)(n - k)(n - k - 1)}\right].$$

Denote

$$f_3(k) := k^3 - (2n + 2)k^2 + (n^2 + 3n - 5)k - n^2 + 2n - 3.$$
Differentiating $f_3(k)$ and solving the quadratic equation

$$f'_3(k) = 3k^2 - 2(2n + 2)k + n^2 + 3n - 5 = 0,$$

we get

$$k_1 = \frac{2(2n + 2) - \sqrt{4n^2 - 4n + 76}}{6}, \quad k_2 = \frac{2(2n + 2) + \sqrt{4n^2 - 4n + 76}}{6}.$$ 

Obviously,

$$\frac{n + 4}{6} = \frac{2(2n + 2) - 3n}{6} < k_1 < \frac{2(2n + 2) - (2n - 1)}{6} = \frac{2n + 5}{6},$$

$$k_2 > \frac{2(2n + 2) + 2n - 1}{6} > n,$$

and $f'_3(k) \geq 0$ on $[1, k_1]$, $f'_3(k) \leq 0$ on $[k_1, n - 2]$. So function $f_3(k)$ is increasing on $[1, k_1]$ and decreasing on $[k_1, n - 2]$. Since $f_3(1) = f_3(n - 2) = 3(n - 3) > 0$, we have $f_3(k) > 0$ on $[1, n - 2]$. Combining (3.12) and (i) of Lemma 3.1, we obtain $A_2 > 0$.

(iii) Substituting (3.7) into $A_3$, we have

$$A_3 = 18t\theta_2 - 9ad = \frac{9bc(3ac - b^2)(3bd - c^2)}{3ac^3 + 3b^3d - 2b_2c^2} - 9ad$$

$$= 9\left(\frac{b^3c^3 - 3abc^4 - 3b^4cd + 11ab^2c^2d - 3a^2c^3d - 3ab^3d^2}{3ac^3 + 3b^3d - 2b_2c^2}\right).$$

By using the expressions $A_1$ and $A_2$ in (3.11) and (3.12) respectively, we have

$$A_1A_2 - A^2_3/36$$

$$= \frac{9(12ab^3c^3d + 20a^2b^2c^2d^2 - ab^2c^5 - b^5c^2d - 12a^2bc^4d - 12ab^4cd^2 - 3a^3c^3d^2 - 3a^2b^3d^3)}{4(3ac^3 + 3b^3d - 2b^2c^2)}$$

$$= \frac{9a^2b^2c^2d^2}{4(3ac^3 + 3b^3d - 2b^2c^2)} \cdot \left(\frac{12bc}{ad} + 20 - \frac{c^3}{ad^2} - \frac{b^3}{a^2d} - \frac{12c^2}{bd} - \frac{12b^2}{ac} - \frac{3ac}{b^2} - \frac{3bd}{c^2}\right).$$

From (3.9), we get

$$\left(\frac{12bc}{ad} + 20 - \frac{c^3}{ad^2} - \frac{b^3}{a^2d} - \frac{12c^2}{bd} - \frac{12b^2}{ac} - \frac{3ac}{b^2} - \frac{3bd}{c^2}\right)$$

$$= \frac{4(n + 1)^2[\text{-(n+5)}k^2 + (n^2 + 4n - 5)k - n^2 + 1]}{k^2(k + 1)(k + 2)(n - k + 1)(n - k)(n - k - 1)^2}. $$

Denote

$$f_4(k) := -(n + 5)k^2 + (n^2 + 4n - 5)k - n^2 + 1.$$ 

It is easy to get that function $f_4(k)$ is increasing on $[1, n - 1/2]$ and decreasing on $[n - 1/2, n - 2]$, and

$$f_4(1) = f_4(n - 2) = 3(n - 3) > 0,$$

so we have $f_4(k) > 0$ for $1 \leq k \leq n - 2$. Combining formulas above we obtain

$$A_1A_2 - A^2_3/36 > 0.$$
- **Proof of the Claim (3.3).**

Lemma 3.1 implies $\theta_1 > 0, \theta_2 > 0$. By Lemma 3.2,

\[
A_1 \alpha^2 z^2_1 + A_2 z^2_2 z_3^2 \geq 2\sqrt{A_1 A_2} |\alpha \sigma_3| > \frac{1}{3} |A_3 \alpha \sigma_3|,
\]

\[
A_1 \alpha^2 z^2_2 + A_2 z^2_1 z_3^2 \geq 2\sqrt{A_1 A_2} |\alpha \sigma_3| > \frac{1}{3} |A_3 \alpha \sigma_3|,
\]

\[
A_1 \alpha^2 z^2_3 + A_2 z^2_1 z_2^2 \geq 2\sqrt{A_1 A_2} |\alpha \sigma_3| > \frac{1}{3} |A_3 \alpha \sigma_3|,
\]

Combing the inequalities above, we get

\[
V(z) = A_1 \alpha^2 \sum_{i=1}^{3} z_i^2 + A_2 \sum_{p<q}^{3} z_p^2 z_q^2 + A_3 \alpha \sigma_3 \geq 0.
\]

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