Matrix model version of AGT conjecture and generalized Selberg integrals

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Abstract

Operator product expansion (OPE) of two operators in two-dimensional conformal field theory includes a sum over Virasoro descendants of other operator with universal coefficients, dictated exclusively by properties of the Virasoro algebra and independent of choice of the particular conformal model. In the free field model, these coefficients arise only with a special “conservation” relation imposed on the three dimensions of the operators involved in OPE. We demonstrate that the coefficients for the three unconstrained dimensions arise in the free field formalism when additional Dotsenko-Fateev integrals are inserted between the positions of the two original operators in the product. If such coefficients are combined to form an n-point conformal block on Riemann sphere, one reproduces the earlier conjectured β-ensemble representation of conformal blocks, thus proving this (matrix model) version of the celebrated AGT relation. The statement can also be regarded as a relation between the 3j-symbols of the Virasoro algebra and the slightly generalized Selberg integrals Iγ, associated with arbitrary Young diagrams. The conformal blocks are multilinear combinations of such integrals and the remaining part of the original AGT conjecture relates them to the Nekrasov functions which have exactly the same structure.

1 Introduction

The AGT conjecture [1] unifies and identifies a number of different domains in modern theory, what makes it a very interesting and promising subject, attracting a lot of attention [2]-[45]. In its original form the AGT conjecture relates the conformal blocks in 2d conformal field theory (CFT) [46, 47] and the Nekrasov functions [48], obtained by expansion of the LNS multiple contour integrals [49]. In this form it is now proved only in three cases: in the limit of large central charge c [19], when the conformal blocks and the Nekrasov functions reduce to (generic) hypergeometric series; in the case of special value of one of the external dimensions [14, 6], when they are also hypergeometric series (however, different from the first case); and in the case of a 1-point toric function [33], when one can use the powerful Zamolodchikov recurrent relation [50, 22]. Following earlier considerations in [14, 11, 29, 30, 31, 32] in [39] a simpler version of the AGT relation was suggested, identifying conformal blocks with the Dotsenko-Fateev β-ensemble integral [51, 52, 58, 11], which can be considered as a new avatar of the old proposal in the free field approach to CFT [51, 53] and, at the same time, as a concrete application of the more recent theory of Dijkgraaf-Vafa (DV) phases of matrix models [54]. Ref.[39] contains absolutely explicit formulas for generic conformal blocks, made from the free field correlators with screening integral insertions, analytically continued in the number of screenings. The only problem is that these formulas are very tedious to derive and their meaning from the point of view of representation theory of the Virasoro algebra, an underlying algebra for the standard construction of the conformal blocks in [46, 47], remains no less obscure than in the original AGT relation of [1].

In this paper we provide a similar, but conceptually different derivation of the same formulas of [32, 39], which involves nothing but the Virasoro representation theory and by now the elementary Selberg integrals [55]. Calculations remain tedious but now they are conceptually clear and straightforward. We give only basic examples, but a full constructive proof can definitely be worked out in this way, and in this sense one may say that the simplified version [39] of the AGT conjecture [1] is now practically established. In fact, the conceptual proof is readily available and is given in s.11.1 in the conclusion. It can be further promoted to a straightforward proof of the original AGT conjecture: after the recent progress in [45] there remain just a few combinatorial details to fix. In this sense the program to prove the AGT conjecture through the technique of the

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Dotsenko-Fateev (matrix model like) integrals, which was formulated in [14] and [11], is nearly completed. Still it would be interesting to work out some other proof, establishing an explicit relation between the Dotsenko-Fateev and LNS integrals, perhaps, making use of the duality between Gaussian and Kontsevich models: this is, however, only mentioned in s.11.2 in the Conclusion and remains beyond the scope of the present paper.

In this paper we consider the triple functions, the coefficients in the operator product expansion involving a sum over Virasoro descendants, in the free field formalism. We begin with the case of the single free field, i.e. with the pure Virasoro chiral algebra (in the AGT terminology this corresponds to the $U(2)^{\otimes(n-3)}$ case), the extension to $k$ fields, the $W_{k+1}$ chiral algebra and the $U(k+1)$ quiver is straightforward. In terms of the Virasoro primary, elements of the Verma modules look like $:L_{-y_1}V_{\Delta_1}(0):$, and the operator product expansion in the corresponding algebra is

$$
: L_{-y_1}V_{\Delta_1}(0) : : L_{-y_2}V_{\Delta_2}(q) : = \sum_{n} q^{\Delta_1+\Delta_2-|Y_1|-|Y_2|} S^{\Delta_1}_{\Delta_2} \sum_{Y} q^{V_{\Delta_1,Y_1}V_{\Delta_2,Y_2}} : L_{-Y}V_{\Delta}(0) : ,
$$

(1)

Here $V_{\Delta}$ denotes the Virasoro primary with dimension $\Delta$: $L_{n}V_{\Delta} = 0$ for $n > 0$, $L_0V_{\Delta} = \Delta V_{\Delta}$, and $L_{-Y} = \ldots L_{-n_2}L_{-n_1}$, denotes the ”negative” (raising) Virasoro operator, labeled by the Young diagram $Y = \{n_1 \geq n_2 \geq \ldots \}$. The coefficients $S^{\Delta_1}_{\Delta_2}$ depend on choice of the conformal model. In particular, their values, directly provided by the free field formalism below, are usually referred to as the Liouville model structure constants [56], we do not consider other choices in this paper. In contrast to $S^{\Delta_1}_{\Delta_2}$, the coefficients $C^{\Delta_1,Y_1}_{\Delta_2,Y_2}$ are universal, depend only on the properties of the Virasoro algebra, and these are the quantities we are going to investigate. Moreover, we further restrict our consideration to the case of $Y_2 = 0$, this is enough to reproduce the spherical 4-point conformal blocks, studied in [1, 35, 39].

The coefficients $C$ can be straightforwardly found by standard CFT methods, see [5] for a detailed review. Coming back to free fields, they provide an alternative derivation, somewhat simpler and more transparent, see [5] and s.2. The only problem is that in the free field model there is a ”conservation law”: the primaries are represented as $V_{\Delta} = e^{\alpha \phi}$ with

$$
\Delta = \alpha(Q - \alpha), \quad c = 1 - 6Q^2,
$$

(2)

and in the sum at the r.h.s. of (1)

$$
\alpha = \alpha_1 + \alpha_2
$$

(3)

Thus, only a restricted set of the triple functions $C$, namely, $C^{\alpha_1+\alpha_2,Y}_{\alpha_1,Y_1;\alpha_2,Y_2}$ can be defined in this model (from now on we label these functions with $\alpha$- rather than $\Delta$-parameters). It is a long-standing problem in CFT, how the free field formalism can be used to obtain arbitrary $C^{\alpha,Y}_{\alpha_1,Y_1;\alpha_2,Y_2}$ with $\alpha \neq \alpha_1 + \alpha_2$. The results of [32, 39] imply that the operator product

$$
: L_{-y_1} e^{\alpha_1 \phi(0)} : : L_{-y_2} e^{\alpha_2 \phi(q)} : \left( \int_0^q : e^{\phi(z)} : dz \right)^{N} = \tilde{C}^{\alpha_1+\alpha_2+bN,Y}_{\alpha_1,Y_1;\alpha_2,Y_2} : L_{-Y} e^{(\alpha_1+\alpha_2+bN)\phi(0)} : ,
$$

(4)

1In terms of parametrization from the Appendix of [6], normalization conventions for free fields in the present paper are as follows: $k_1 = 1/2$, $k_2 = 2$, $p = 1/\lambda = 1$, where

$$
\phi(z)\phi(0) = k_2 \log z, \\
T = \frac{1}{2k_2} (\phi)^2 + k_1 Q \partial^2 \phi, \\
c = 1 - 12k_1^2 k_2 Q^2, \\
V_0 = e^{p\alpha\phi}, \quad V_0(z)V_0(0) = z^{p^2 k_2 \alpha^2}, \\
\Delta(V_0) = \frac{p^2 k_2}{2} \alpha(\alpha - Q), \quad \sum \alpha_i = 2k_1 Q/p
$$

Then

$$
: L_{-1} V_0 : = \partial^2 V_0 =: (p^2 \alpha^2 (\partial \phi)^2 + p \alpha \partial^2 \phi) V_0 : ,
$$

$$
: L_{-2} V_0 : =: \left( \frac{1}{2k_2} (\partial \phi)^2 + (k_1 Q + p \alpha) \partial^2 \phi \right) V_0 : 
$$


where $b$ is the Dotsenko-Fateev screening charge, i.e. $Q = b - 1/b$, has exactly the same expansion coefficients as OPE,

$$C_{α_1,α_2} = C_{α_1,α_2},$$  \tag{5}$$

and we demonstrate below that this is indeed the case. Eq. (5) is the main claim of the present paper, supported by a number of examples. In other words, the r.h.s. of eq.(4) is identically the same as (1) provided the structure constants $C$ and $C$ are related by a change of variables (2) and additionally

$$α = α_1 + α_2 + bN$$  \tag{6}$$

Thus, (4) resolves the above mentioned problem in the sense of analytical continuation: the coefficients $C_{α_1,Y_1; α_2,Y_2}$ are rational functions of $α$ and they are fully defined by their values at discrete points $α = α_1 + α_2 + bN$.

If the original two fields are primaries, $Y_1 = Y_2 = ∅$, then eq.(4) is derived in three steps.

A) First, one uses the basic free field relation,

$$e^{α_1φ(0)} : e^{α_2φ(q)} : \prod_{i=1}^{N} e^{bφ(z_i)} : = \left\{ q^{2α_1α_2} \prod_{i<j}^{N} (z_i - z_j)^{2b^2} \prod_{i=1}^{N} z_i^{2bα_1} (q - z_i)^{2bα_2} \right\} : e^{α_1φ(0)+α_2φ(q)+b\sum_i φ(z_i)} :$$  \tag{7}$$

and then expands the exponential in powers of $q$ and $z_i$:

$$e^{α_1φ(0)+α_2φ(q)+b\sum_i φ(z_i)} : = \sum_{Y,Y'} q^{[Y]-[Y']} H_{Y,Y'} z^{Y'} : L_{-Y} e^{(α_1+α_2+Nb)φ(0)} :$$  \tag{8}$$

Here $z^{Y'} = \prod z_i^{n_i}$ for a Young diagram $Y' = \{n_1 ≥ n_2 ≥ ... \}$ and the sum goes over all pairs of Young diagrams with $|Y| ≥ |Y'|$. At this step, one evaluates the $z$-independent coefficients $H_{Y,Y'}$, as functions of $α_1,α_2$ and $N$.

B) Next, one takes the integrals over $z_i$,

$$I_{Y'} = \prod_{i=1}^{N} \int_0^q dz_i \left\{ z^{Y'} \prod_{i<j}^{N} (z_i - z_j)^{2b^2} \prod_{i=1}^{N} z_i^{2bα_1} (q - z_i)^{2bα_2} \right\}$$  \tag{9}$$

For the single-line Young diagrams $Y' = \{1^n\}$ these are the well known Selberg integrals, which generalize the Euler $B$-function and are equal (after the standard analytical continuation from integer powers in the integrand) to the ratio of $Gamma$-factors. For generic $Y'$, the integrals generalize the Selberg integrals producing extra non-factorizable polynomial factors, which can be explicitly evaluated. Being polynomial, they do not complicate the analytical continuation.

C) Combining the results of steps A and B, one gets the structure

$$\tilde{C}_{α_1,α_2} = \sum_{|Y'| ≤ |Y|} H_{Y,Y'} I_{Y'} \bigg|_{q=1}$$  \tag{10}$$

in the form of a finite sums over Young diagrams.

D) The last step is to compare the $C$ with the known expressions for the conformal theory structure constants $C$ (the $3j$-symbols of the Virasoro algebra), transformed with the help of (2).

For non-trivial $Y_1$ and $Y_2$ the calculation goes the same way, with additional powers of $z_i^{-1}$ and $(z_i - q)^{-1}$ emerging in the integrand.

In this letter we provide in full detail a sample calculation of this kind for the two simplest cases of $\{Y_1, Y_2, Y\} = \{[0],[0],[1]\}, \{[0],[0],[2]\}$ and $\{[0],[0],[11]\}$. It is enough to demonstrate the principle and can be straightforwardly computerized to provide more examples. There is small doubt that all such examples would confirm the relation, which at the moment looks like a non-trivial statement, identifying the $3j$-symbols of the Virasoro algebra with linear combinations of the generalized Selberg integrals $I_Y$. 

3
2 The free field formulas for $C_{\alpha_1+\alpha_2}^{\alpha_1\alpha_2}$

Evaluation of the operator product coefficients in the free field model is considered in detail in [5]. The simplest example is:

$$e^{\alpha_1\phi(0)} : e^{\alpha_2\phi(q)} : = q^{2\alpha_1\alpha_2} : e^{\alpha_1\phi(0)+\alpha_2\phi(q)} : = q^{2\alpha_1\alpha_2} \left( 1 + q\alpha_2\partial\phi(0) + \frac{q^2}{2} \left( \alpha_2\partial^2\phi(0) + \alpha_2^2(\partial\phi(0))^2 \right) + \ldots \right) e^{(\alpha_1+\alpha_2)\phi(0)} :$$ (11)

Now, using

$$e^{\alpha\phi(0)} : e^{\alpha\phi(q)} : = : \alpha\partial\phi e^{\alpha\phi} :$$

from

$$e^{\alpha_1\phi(0)} : e^{\alpha_2\phi(q)} : = : \sum_{Y} q^{\Omega} C_{\alpha_1+\alpha_2, L-Y}^{\alpha_1+\alpha_2} : e^{(\alpha_1+\alpha_2)\phi(0)} :$$ (12)

one obtains

$$C_{\alpha_1+\alpha_2, L-1}^{\alpha_1+\alpha_2} = \frac{\alpha_2}{\alpha_1 + \alpha_2}$$

$$C_{\alpha_1+\alpha_2, L-2}^{\alpha_1+\alpha_2} = \frac{4\alpha_1^2 + \alpha_2(2\alpha_2Q + 4\alpha_1\alpha_2 - 1)}{2 \left( 4(\alpha_1 + \alpha_2)^2 + 2Q(\alpha_1 + \alpha_2) - 1 \right) (\alpha_1 + \alpha_2)}$$ (13)

3 The CFT formulas for $C_{\alpha_1\alpha_2}^{\alpha}$

According to [47, eq.(5.16)], for three generic dimensions one has, instead of (13),

$$C_{\alpha_1\alpha_2, L-1}^{\alpha} = \frac{\Delta_2 + \Delta - \Delta_1}{2\Delta}$$ (14)

Similarly,

$$C_{\alpha_1\alpha_2, L-2}^{\alpha} = \frac{8\Delta^3 + c\Delta^2 + 16\Delta^2\Delta_2 + 2\Delta\Delta_2c - 16\Delta^2\Delta_1 - 2\Delta\Delta_1c - 4\Delta^2 + c\Delta + 8\Delta^2\Delta_1}{4\Delta(16\Delta^2 - 10\Delta + c + 2c\Delta)} +$$

$$+ \frac{\Delta_2^2c - 16\Delta_1\Delta_2c - 2\Delta_1\Delta_2c - 16\Delta_1\Delta_2c + 8\Delta_2\Delta + \Delta_2^2c + 4\Delta\Delta_1 - \Delta_1c}{4\Delta(16\Delta^2 - 10\Delta + c + 2c\Delta)}$$ (15)

$$C_{\alpha_1\alpha_2, L-2}^{\alpha} = \frac{\Delta^2 + 2\Delta\Delta_2 + 2\Delta\Delta_1 - \Delta - 3\Delta_2^2 + 6\Delta_1\Delta_2 + \Delta_2 - 3\Delta_2^2 + \Delta_1}{16\Delta^2 - 10\Delta + c + 2c\Delta}$$ (16)

and so on.

Only in the case of the conservation law condition,

$$\alpha = \alpha_1 + \alpha_2$$ (17)

these expressions are reproduced by the free-field formula (13).
4 Operator product with screening insertions

In order to relax the $U(1)$ conservation law (17) we insert the Dotsenko-Fateev screening charges into the l.h.s. of (11). Then, in the integrand one has

\[ e^{\alpha_1 \phi(0)} : e^{\alpha_2 \phi(q)} : \prod_{i=1}^{N} e^{b \phi(z_i)} : = \sum_{i<j}^{N} (z_i - z_j)^{2 \beta} \prod_{i=1}^{N} \phi(z_i) : q^{2 \alpha_1 \alpha_2} \prod_{i<j}^{N} (z_i - z_j)^{2 \beta} \prod_{i=1}^{N} \phi(z_i) : e^{\alpha_1 \phi(0) + \alpha_2 \phi(q) + b \sum_i \phi(z_i)} : = \]

\[ = q^{2 \alpha_1 \alpha_2} \prod_{i<j}^{N} (z_i - z_j)^{2 \beta} \prod_{i=1}^{N} \phi(z_i) : 1 + \left( \alpha_2 q + b \sum_{i=1}^{N} z_i \right) \frac{\partial \phi(0)}{2} + \left( \alpha_2 q^2 + b \sum_{i=1}^{N} z_i^2 \right) \frac{\partial^2 \phi(0)}{2} + \left( \alpha_2 q^3 + b \sum_{i=1}^{N} z_i^3 \right) \frac{\partial^3 \phi(0)}{6} + \ldots \] (18)

\[ + \frac{4(\alpha_1 + \alpha_2 + b N + 2 Q) \left( \alpha_2 q + b \sum_{i=1}^{N} z_i \right)^2 - \left( \alpha_2 q^2 + b \sum_{i=1}^{N} z_i^2 \right)}{2(\alpha_1 + \alpha_2 + b N)(4(\alpha_1 + \alpha_2 + b N)^2 + 2 Q (\alpha_1 + \alpha_2 + b N) - 1)} \left( \alpha_1 + \alpha_2 + b N \right) L_{-1} + \]

\[ + 2 \frac{\alpha_2 q^2 + b \sum_{i=1}^{N} z_i^2}{4(\alpha_1 + \alpha_2 + b N)^2 + 2 Q (\alpha_1 + \alpha_2 + b N) - 1} \left( \alpha_1 + \alpha_2 + b N \right) L_{-2} + \]

\[ + \frac{A L_{-3} + 2(\alpha_1 + \alpha_2 + b N) B L_{-3} + 2(\alpha_1 + \alpha_2 + b N) C L_{-3}}{6(\alpha_1 + \alpha_2 + b N)(4(\alpha_1 + \alpha_2 + b N)^2 + 2 Q (\alpha_1 + \alpha_2 + b N) - 1)((\alpha_1 + \alpha_2 + b N)^2 + Q (\alpha_1 + \alpha_2 + b N) - 1)} + \ldots \] (20)

where

\[ A = \left( \alpha_2 q^3 + b \sum_{i=1}^{N} z_i^3 \right) - (\alpha_1 + \alpha_2 + b N + Q) \left( \alpha_2 q + b \sum_{i=1}^{N} z_i \right) \left( \alpha_2 q^2 + b \sum_{i=1}^{N} z_i^2 \right) + \]

\[ + \left( 4(\alpha_1 + \alpha_2 + b N)^2 + 6 Q (\alpha_1 + \alpha_2 + b N) + 2 Q^2 (\alpha_1 + \alpha_2 + b N) - 2 \right) \left( \alpha_2 q + b \sum_{i=1}^{N} z_i \right)^3 \]

\[ B = -4(\alpha_1 + \alpha_2 + b N)^2 \left( \alpha_2 q^3 + b \sum_{i=1}^{N} z_i^3 \right) + \]

\[ + 4(\alpha_1 + \alpha_2 + b N)^2 ((\alpha_1 + \alpha_2 + b N) + Q) \left( \alpha_2 q + b \sum_{i=1}^{N} z_i \right) \left( \alpha_2 q^2 + b \sum_{i=1}^{N} z_i^2 \right) + \]

\[ + 4 \left( 1 - 3(\alpha_1 + \alpha_2 + b N)^2 - 3 Q (\alpha_1 + \alpha_2 + b N) \right) \left( \alpha_2 q + b \sum_{i=1}^{N} z_i \right)^3 \] (21)
\[
C = 2(\alpha_1 + \alpha_2 + bN)^2 + 2Q(\alpha_1 + \alpha_2 + bN) - 1)\left((\alpha_1 + \alpha_2 + bN)^2 \left(\alpha_2 q^3 + b \sum_{i=1}^{N} z_i^3\right) + \right.
\]

\[
- (\alpha_1 + \alpha_2 + bN) \left(\alpha_2 q + b \sum_{i=1}^{N} z_i\right) \left(\alpha_2 q^2 + b \sum_{i=1}^{N} z_i^2\right) + 2 \left(\alpha_2 q + b \sum_{i=1}^{N} z_i\right)^3 \right)
\]

Underlined by one, two and three lines are the contributions at levels one, two and three respectively.

5 Structure constants \( C_{\alpha_1\alpha_2}^{\alpha_1\alpha_2+bN} \) from Selberg integrals. Level one

Now, as suggested in \([32, 39]\), we take integrals over \( z_i \) along an open contour which connects positions of the two original operators. In order to perform integration at level one, i.e. to integrate the first line in OPE, (18), one needs the integrals which are given by the now standard formulas from ref.[55] (see also the Appendix in the present paper):

\[
q^{2\alpha_1\alpha_2}\prod_{i=1}^{N} \int_0^1 dz_i z_i^{2\alpha_1}(q - z_i)^{2b\alpha_2} \prod_{i<j}^{N}(z_i - z_j)^{2b^2} = \]

\[
= q^{N+b^2N(N+1)} q^{2(\alpha_1+Nb)(\alpha_2+Nb)} \prod_{i=1}^{N-1} \Gamma\left(1 + 2b\alpha_1 + j b^2\right) \Gamma\left(1 + 2b\alpha_2 + j b^2\right) \Gamma\left(1 + (j+1)b^2\right) \Gamma\left(1 + b^2\right) (22)
\]

and

\[
q^{2\alpha_1\alpha_2}\prod_{i=1}^{N} \int_0^1 dz_i z_i^{2\alpha_1}(q - z_i)^{2b\alpha_2} \prod_{i<j}^{N}(z_i - z_j)^{2b^2} \left(q\alpha_2 + b \sum_{i=1}^{N} z_i\right) = \]

\[
= q^{(N+1)(1+b^2N)} q^{2(\alpha_1+Nb)(\alpha_2+Nb)} \prod_{i=1}^{N-1} \Gamma\left(1 + 2b\alpha_1 + j b^2\right) \Gamma\left(1 + 2b\alpha_2 + j b^2\right) \Gamma\left(1 + (j+1)b^2\right) \Gamma\left(1 + b^2\right) \cdot \left(\alpha_2 + Nb\frac{1+2b\alpha_1 + (N-1)b^2}{2 + 2b\alpha_1 + 2b\alpha_2 + 2(N-1)b^2}\right) (23)
\]

One can now extract the structure constants from the integrals of (18). We do it first for \( N = 1 \) where formulas are just a little simpler, and then for arbitrary \( N \).

- For \( N = 1 \) these integrals are just the Euler B-functions:

\[
< 1 > = \int_0^1 dz z^{2b\alpha_1}(1 - z)^{2b\alpha_2} = \frac{\Gamma(1 + 2b\alpha_1)\Gamma(1 + 2b\alpha_2)}{\Gamma(2 + 2b\alpha_1 + 2b\alpha_2)},
\]

\[
< \alpha_2 + bz > = \int_0^1 dz z^{2b\alpha_1}(1 - z)^{2b\alpha_2}(\alpha_2 + bz) = \frac{\Gamma(1 + 2b\alpha_1)\Gamma(1 + 2b\alpha_2)}{\Gamma(2 + 2b\alpha_1 + 2b\alpha_2)} \left(\alpha_2 + b\frac{1 + 2b\alpha_1}{2 + 2b\alpha_1 + 2b\alpha_2}\right) (24)
\]

According to (4) and (18), the first of these formulas defines

\[
\tilde{C}_{\alpha_1\alpha_2}^{\alpha_1\alpha_2+b} = \frac{\Gamma(1 + 2b\alpha_1)\Gamma(1 + 2b\alpha_2)}{\Gamma(2 + 2b\alpha_1 + 2b\alpha_2)}, (25)
\]

while the second one is proportional to the product \( \tilde{S}_{\alpha_1\alpha_2}^{\alpha_1\alpha_2+b} \tilde{C}_{\alpha_1\alpha_2}^{\alpha_1\alpha_2+b,L-1} \). Therefore, \( \tilde{C}_{\alpha_1\alpha_2}^{\alpha_1\alpha_2+b,L-1} \) is given by the ratio of the two integrals (up to an additional factor):

\[
\tilde{C}_{\alpha_1\alpha_2}^{\alpha_1\alpha_2+b,L-1} = \frac{< \alpha_2 + bz >}{< 1 >} \frac{1}{\alpha_1 + \alpha_2 + b} (24) \equiv \left(\alpha_2 + b\frac{1 + 2b\alpha_1}{2 + 2b\alpha_1 + 2b\alpha_2}\right) \frac{1}{\alpha_1 + \alpha_2 + b} (26)
\]
At the same time, from (14) in this case one has, taking into account that $Q = b - 1/b$:

$$C_{\alpha_1, \alpha_2}^{\alpha_1 + \alpha_2, b, L-1} = \frac{\Delta_2 + \Delta - \Delta_1}{2\Delta} \overset{(2)}{=} \frac{\alpha_2(\alpha_2 - Q) + (\alpha_1 + \alpha_2 + b)(\alpha_1 + \alpha_2 + b - Q) - \alpha_1(\alpha_1 - Q)}{2(\alpha_1 + \alpha_2 + b)(\alpha_1 + \alpha_2 + b - Q)} =$$

$$= \frac{2\alpha_2(\alpha_1 + \alpha_2) + 2b\alpha_1 + 2\alpha_2/b + 1}{2(\alpha_1 + \alpha_2 + b)(\alpha_1 + \alpha_2 + 1/b)} \overset{(24)}{=} \frac{1}{\alpha_1 + \alpha_2 + b} \frac{1 + 2b\alpha_1}{2 + 2b\alpha_1 + 2\alpha_2/b + 1} \overset{(26)}{=} \tilde{C}_{\alpha_1, \alpha_2}^{\alpha_1 + \alpha_2, b, L-1} \overset{(27)}{=}$$

- Similarly, for arbitrary $N$:

$$C_{\alpha_1, \alpha_2}^{\alpha_1 + \alpha_2, bN, L-1} = \frac{\Delta_2 + \Delta - \Delta_1}{2\Delta} = \frac{\alpha_2(\alpha_2 - Q) + (\alpha_1 + \alpha_2 + bN)(\alpha_1 + \alpha_2 + bN - Q) - \alpha_1(\alpha_1 - Q)}{2(\alpha_1 + \alpha_2 + bN)(\alpha_1 + \alpha_2 + bN - Q)} =$$

$$= \left(\frac{\alpha_2 + bN}{\alpha_1 + \alpha_2 + bN}\right) \frac{1}{2 + 2b\alpha_1 + 2\alpha_2/b + 1} \overset{(28)}{=} \tilde{C}_{\alpha_1, \alpha_2}^{\alpha_1 + \alpha_2, bN, L-1}$$

and

$$\tilde{S}_{\alpha_1, \alpha_2}^{\alpha_1 + \alpha_2, bN} \overset{(29)}{=} \prod_{j=0}^{N-1} \frac{\Gamma\left(1 + 2b\alpha_1 + jb_2\right)\Gamma\left(1 + 2b\alpha_2 + jb_2\right)}{\Gamma\left(2 + 2b\alpha_1 + 2b\alpha_2 + (N - 1 + j)b^2\right)} \prod_{j=1}^{N} \frac{\Gamma\left(1 + jb_2\right)}{\Gamma\left(1 + b_2^2\right)}$$

Formulas for the structure constants $\tilde{C}$ are rational, therefore, they can be straightforwardly analytically continued in $N$ to arbitrary values of $\alpha = \alpha_1 + \alpha_2 + bN$. The analytical continuation of the above expression for $S$ is somewhat more ambiguous (and, anyway, there is nothing to compare them with, since the coefficients $S$ generally do not factorize into holomorphic and anti-holomorphic parts).

### 6 Level two

At level two, one needs to integrate (20). The ordinary Selberg integrals (70) and (71) are not sufficient for this purpose, one needs also the generalized one (73) from the Appendix. Then, the two integrals that one needs in (20) turn out to be

$$q^{2\alpha_1 \alpha_2} \prod_{i=1}^{N} \int_{0}^{q} dz_i \frac{z_i^{2b\alpha_1}(q - z_i)^{2b\alpha_2}}{\prod_{i<j}^{N}(z_i - z_j)^{2b^2}} \left(\alpha_2 q + b \sum_{i=1}^{N} z_i\right) =$$

$$= q^{(N+1)(1+b^2)N+1} \cdot q^{2(\alpha_1+Nb)(\alpha_2+Nb)} \prod_{i=1}^{N} \int_{0}^{1} dz_i z_i^{2b\alpha_1}(1 - z_i)^{2b\alpha_2} \prod_{i<j}^{N}(z_i - z_j)^{2b^2} \left(\alpha_2 + b \sum_{i=1}^{N} z_i\right)^{b(70)} \overset{(73)}{=}$$

$$= q^{(N+1)(1+b^2)N+1} \cdot q^{2(\alpha_1+Nb)(\alpha_2+Nb)} \prod_{j=0}^{N-1} \frac{\Gamma\left(1 + 2b\alpha_1 + jb_2\right)\Gamma\left(1 + 2b\alpha_2 + jb_2\right)\Gamma\left(1 + (j + 1)b^2\right)}{\Gamma\left(2 + 2b\alpha_1 + 2b\alpha_2 + (N - 1 + j)b^2\right)\Gamma\left(1 + b^2\right)} \left[\alpha_2 + Nb \times \overset{(30)}{=}ight.$$

$$\left.\frac{(4\alpha_1^2b^2 + 4\alpha_1\alpha_2b^2 + 6\alpha_1b^3N - 8\alpha_1b^3 + 4b^3\alpha_2 - 4b^3\alpha_2 + 8\alpha_1b + 4\alpha_2^2 + 4 + 3b^4N^2 - 7b^4N + 4b^4 + 7b^2N - 9b^2)}{2(2\alpha_1b + 2\alpha_2b + 2b^2N - 3b^2 + 2)(2\alpha_1b + 2\alpha_2b + 2b^2N - 3b^2 + 2)(\alpha_1b + \alpha_2b + b^2N - b^2 + 1)} \right]$$

and

$$q^{2\alpha_1 \alpha_2} \prod_{i=1}^{N} \int_{0}^{q} dz_i z_i^{2b\alpha_1}(q - z_i)^{2b\alpha_2} \prod_{i<j}^{N}(z_i - z_j)^{-2b^2} \left(\alpha_2 q + b \sum_{i=1}^{N} z_i\right)^{2} =$$

$$= q^{(N+1)(1+b^2)N+1} \cdot q^{2(\alpha_1+Nb)(\alpha_2+Nb)} \prod_{i=1}^{N} \int_{0}^{1} dz_i z_i^{2b\alpha_1}(1 - z_i)^{2b\alpha_2} \prod_{i<j}^{N}(z_i - z_j)^{2b^2} \left(\alpha_2 + b \sum_{i=1}^{N} z_i\right)^{2} \overset{(70)}{=}$$

$$= q^{(N+1)(1+b^2)N+1} \cdot q^{2(\alpha_1+Nb)(\alpha_2+Nb)} \prod_{j=0}^{N-1} \frac{\Gamma\left(1 + 2b\alpha_1 + jb_2\right)\Gamma\left(1 + 2b\alpha_2 + jb_2\right)\Gamma\left(1 + (j + 1)b^2\right)}{\Gamma\left(2 + 2b\alpha_1 + 2b\alpha_2 + (N - 1 + j)b^2\right)\Gamma\left(1 + b^2\right)} \overset{(31)}{=}$$
\[\left(\alpha_1^2 + 2N\alpha_2b + \frac{1 + 2b\alpha_1 + (N - 1)b^2}{2 + 2b\alpha_1 + 2bN + 2(N - 1)b^2} + N(N - 1)b^2 - \frac{(2\alpha_1b + b^2N - b^2 + 1)(2\alpha_1b + b^2N - b^2 + 1) - N\alpha_1b}{2(\alpha_1b + b^2N - b^2 + 1)} \times \frac{(4\alpha_1^2b^2 + 4\alpha_1b^2 + 6\alpha_1^2b^3 - 8\alpha_1b^3 + 4b^3\alpha_2N - 4b^3\alpha_2 + 8\alpha_1b + 4b^3\alpha_2 + 4 + 3b^2N^2 - 7b^2N + 4b^2 + 7b^2N - 9b^2)(2\alpha_1b + b^2N - b^2 + 1)}{2(2\alpha_1b + 2b^2N - 3b^2 + 2)(2\alpha_1b + 2b^2N - 3b^2 + 2)}\right)\]

\section{Higher levels}

The detailed explicit formulas are quite lengthy already at level two. Writing them down for higher levels is simply impossible: they take several pages. However, in every particular case eq.\((4)\) can be easily validated by simple computer calculations, provided one knows the following set of matrices.

- The action of Virasoro generators on the free field primaries,
  \[L^{-\alpha}\phi^\alpha = \sum_{|Y'|=|Y|} \mathcal{L}_{Y'|Y}^\alpha : J^{Y'} e^{\alpha \phi} :\]
  where \(J^Y = \partial^{n_1} \phi \partial^{n_2} \phi \ldots\), produces the matrix \(\mathcal{L}_{Y'|Y}\) (see s.2).

- The expansion
  \[e^{\alpha_1 \phi(0) + \alpha_2 \phi(q) + b \sum_i \phi(z_i)} = \sum_Y \mathcal{E}_Y(q, \bar{z}) J^Y(0) e^{(\alpha_1 + \alpha_2 + bN) \phi(0)}\]
  gives the vector \(\mathcal{E}_Y\) actually expressed through the Schur polynomials. Up to level 3 this vector, and also
  \[\mathcal{E}^Y = \sum_{|Y'|=|Y|} \mathcal{L}_{Y'|Y} \mathcal{E}_{Y'}\]
  with \(\mathcal{L}_{Y'|Y}\) being the inverse of \(\mathcal{L}_{Y'|Y}\), are explicitly given in eqs.\((18)-(21)\).

- If \(Y_1\) and \(Y_2\) in \((4)\) are non-trivial, then one actually needs a more sophisticated triple-vertex \(\mathcal{E}_{Y_1 Y_2}^Y\), describing the expansion
  \[J_{Y_1}^{\alpha_1 \phi(0)} : J_{Y_2}^{\alpha_2 \phi(q)} : : e^{b \sum_i \phi(z_i)} := \sum_Y \mathcal{E}_{Y_1 Y_2}^Y(q, \bar{z}) : J^Y(0) e^{(\alpha_1 + \alpha_2 + bN) \phi(0)} :\]
  In this case, one has to consider also the quantity
  \[\mathcal{E}_{Y_1 Y_2}^Y = \sum_{Y_1, Y_2 : Y'} \mathcal{L}_{Y_1 Y_1}^{Y_1} \mathcal{L}_{Y_2 Y_2}^{Y_2} \mathcal{L}_{Y'}^{Y'} \mathcal{L}^{Y Y'}\]

- These \(\mathcal{E}_Y\) are actually functions of \(\{z_i\}\), i.e. have the form
  \[\mathcal{E}_Y = \sum_{Y'} \hat{\mathcal{E}}_{YY'} z^{Y'}\]
  with one extra index \(Y'\). This time the sizes of Young diagrams can be different, only \(|Y'| \leq |Y|\).

- Integration over \(z\) converts \(\hat{\mathcal{E}}\) into
  \[< \mathcal{E}_Y > = \sum_{Y'} \hat{\mathcal{E}}_{YY'} I^{Y'}\]
  where \(I^Y = < z^Y >\) are generalized Selberg integrals, described in the Appendix below. In fact, as emphasized in [45], they are expressed through the simpler quantities, the averages of Jack polynomials with the help of one more matrix,
  \[I^Y = \sum_{|Y'|=|Y|} \mathcal{P}_{Y'Y}^Y z^{Y'} < P_{Y'} >\]
  inverse to the matrix of expansion of the Jack polynomials into monomials,
  \[P_Y = \sum_{|Y'|=|Y|} \mathcal{P}_{Y'Y} z^{Y'}\]
Putting all the things together, one obtains for the Dotsenko-Fateev representation of the conformal triple function:

\[
\tilde{C}_{Y_1 Y_2} = \mathcal{L}_{Y_1 Y_1'} (\alpha_1) \mathcal{L}_{Y_2 Y_2'} (\alpha_2) \mathcal{L}^{Y' Y} (\alpha_1 + \alpha_2 + bN) < \mathcal{E}^{Y_1 Y_2'},
\]

\[
< \mathcal{E}^{Y_1 Y_2'} = \mathcal{E}^{Y_1 Y_2'} < z^Y >, \quad < z^Y > = P^Y Y'' < Y''^N \]  

Summation over repeated indices is implied.

This should be compared with the usual CFT expression

\[
C^Y_{Y_1 Y_2} = \sum_{Y'} \tilde{\Gamma}_{Y_1 Y_2 Y'} Q^{Y' Y}
\]

Details of this calculation are described in [5].

An explicit check of the relation

\[
\tilde{C}_{Y_1 Y_2} = C^Y_{Y_1 Y_2}
\]  

at levels 1 and 2 and for \( Y_1 = Y_2 = 0 \) is described above in ss.5-6. Since all the matrices are explicitly presented there also for the case of level 3, it is a trivial computer exercise to make a check also at this level, and, of course, it also confirms relation (43).

To check it at other levels for \( Y_1 = Y_2 = 0 \) one needs to know four matrices, \( \mathcal{L}, \hat{\mathcal{E}}, P, Q \), and two vectors, \( \tilde{\Gamma}_Y \) and \( \hat{\Gamma}_Y \). When \( Y_1 \) and \( Y_2 \) are non-trivial, \( \hat{\mathcal{E}} \) and \( \hat{\Gamma} \) acquire an additional pair of indices, \( Y_1, Y_2 \). These entries belong to different sciences: \( \mathcal{L} \) and \( \mathcal{E} \) to the free field calculus, \( Q \) and \( \hat{\Gamma} \) to CFT, \( P \) to the theory of orthogonal polynomials, \( \tilde{\mathcal{E}} \) to the theory of Selberg integrals\(^2\). Eq.(43), the weak (matrix model) form of the AGT conjecture establishes a concrete relation between the seemingly unrelated quantities from these different subjects.

8 Virasoro intertwiners

Expansion rule (1) implies that the structure constants \( C \) are the components of the Virasoro intertwining operator between Verma modules \( \Delta_1, \Delta_2 \) and \( \Delta \). Comultiplication in the Virasoro algebra is somewhat non-trivial [57]:

\[
\begin{array}{c}
\Delta(L_{-1}) = L_{-1} \otimes I + I \otimes L_{-1}, \\
\Delta(L_{-0}) = L_0 \otimes I + I \otimes L_0 + q \otimes L_0 + q^2 \otimes L_0, \\
\Delta(L_1) = L_0 \otimes I + I \otimes L_0 + q \otimes L_0 + q^2 \otimes L_0 + \Delta^{2} \otimes L_{-1}, \\
\end{array}
\]

As the simplest example, this means that \( L_0 \) acts on the operator product expansion of two primaries as

\[
L_0 \left( V_{\Delta_1}(0)V_{\Delta_2}(q) \right) = \left( \Delta_1 + \Delta_2 + q \frac{\partial}{\partial q} \right) \left( V_{\Delta_1}(0)V_{\Delta_2}(q) \right)
\]

This is, of course, in a perfect agreement with (1):

\[
V_{\Delta_1}(0)V_{\Delta_2}(q) = q^{\Delta_1+\Delta_2 - \Delta_0} \sum_{Y} q^{Y \mid C_{\Delta_1 \Delta_2} L_{-N} V_{\Delta}(0)}^Y
\]

On one hand, \( (\Delta_1 + \Delta_2 + q \frac{\partial}{\partial q}) q^{\Delta_1-\Delta_2+\mid Y \rangle} = (\Delta + \mid Y \rangle q^{\Delta_1-\Delta_2+\mid Y \rangle}, \) on the other hand, \( L_0 L_{-N} V_{\Delta} = (\Delta + \mid Y \rangle L_{-N} V_{\Delta} \). It is instructive to see how this works also for representation (4). Then, at the l.h.s., one has a product of many operators, \( V_{\Delta_1}(0)V_{\Delta_2}(q) \left( \int_{0}^{q} V_1(z)dz \right)^N \), and the multiple comultiplication (it is associative) now acts as

\[
L_0 \otimes I \otimes I^{\otimes N} + I \otimes (L_0 + qL_{-1}) \otimes I^{\otimes N} + I \otimes I \otimes \left( (L_0 + z_1 L_{-1}) \otimes I^{\otimes (N-1)} + \ldots + I^{\otimes (N-1)} \otimes (L_0 + z_N L_{-1}) \right)
\]  

\(^2\)Note that the Selberg integrals (see the Appendix) produce in the denominators the products automatically presenting the decomposition of the Kac determinants in terms of \( \alpha \)-variables, [51, 47].
Since \( V_1 = e^{b_0} : \) has unit dimension, \((L_0 + zL_{-1})V_1(z) = \frac{d}{dz}(zV_1(z))\). Our definition of Selberg integrals is the analytical continuation from the points where all \(2\alpha_i b\) and \(b^2\) are positive integers, therefore, all \(z\)-derivatives are always integrated to zero as if one uses the closed contours, so that one actually gets

\[
L_0 \left\{ V_{\Delta_1}(0)V_{\Delta_2}(q) \left( \int_0^q V_1(z)dz \right)^N \right\} = \left( \Delta_1 + \Delta_2 + q \frac{\partial}{\partial q} \right) V_{\Delta_1}(0)V_{\Delta_2}(q) \left( \int_0^q V_1(z)dz \right)^N \bigg|_{q=q} \tag{48}
\]

Note that the \(q'\)-derivative does not act on the upper limit of the integrals. The multiple integral at the r.h.s.

\[
\text{of this formula depends on } q \text{ through the factors } q^{2\alpha_1 \alpha_2} \text{ and } \prod_{i=1}^N (q - z_i)^{2\alpha_i b} \text{ in the integrand. The action of the logarithmic } q\text{-derivative on the first factor gives the factor } 2\alpha_1 \alpha_2, \text{ while the action on the others gives } -2\alpha_2 b \sum_{i=1}^N \frac{q}{q - z_i}. \text{ When } N = 1 \text{ this simply means that the Selberg (Euler) integral has its argument } c = 2\alpha_2 b \text{ shifted by } -1:
\]

\[
2\alpha_2 b \int_0^q z^{2\alpha_1 b}(1 - z)^{2\alpha_2 b - 1}dz = (2\alpha_1 + 2\alpha_2 b + 1) \int_0^q z^{2\alpha_1 b}(1 - z)^{2\alpha_2 b}dz \tag{49}
\]

For \(N > 1\) this is a similar, but a little more complicated exercise (see eq.(61) below for a similar evaluation of \(\left\langle \sum_i \frac{1}{a_i} \right\rangle\), which gives

\[
2\alpha_2 b \left( \sum_{i=1}^N \frac{1}{q - z_i} \right) = N \left( 2\alpha_1 b + 2\alpha_2 b + (N - 1)b^2 + 1 \right) \tag{50}
\]

Substituting all this together with (4) and \(\Delta_i = \alpha_i(\alpha_i - b + 1/b)\) into the r.h.s. of (48), one obtains for the coefficient in front of \(V_{\Delta}\)

\[
\alpha_1^2 + \alpha_2^2 - (\alpha_1 + \alpha_2)(b - 1/b) + 2\alpha_1 \alpha_2 + N \left( 2\alpha_1 b + 2\alpha_2 b + (N - 1)b^2 + 1 \right) = (\alpha_1 + \alpha_2 + bN)(\alpha_1 + \alpha_2 + bN - b + 1/b) = \Delta, \tag{51}
\]

as needed. In a similar way, one can act with any other \(L_k\) on (4) and check in detail that it is indeed consistent with the comultiplication rule (44). This, of course, follows from the general argument of [51], since the screening insertion is an integral of the dimension one operator, and, once again, our definition of Selberg integrals actually allows one to consider the integration contour as closed. As we demonstrated in this section, an explicit check confirms this general claim.

9 From OPE to conformal blocks

Eq.(4) is very well suited for constructing arbitrary conformal blocks. If we denote the operator product expansion at the l.h.s. of Eq.(4) through \(V_1(0) *_N V_2(q)\), then conformal block is the value of a linear form on an ordered product, for example,

\[
\left\langle \left( \left( V_1(x_1) *_{N_{12}} V_2(x_2) \right) *_{N_{123}} V_3(x_3) \right) *_{N_{1234}} V_4(x_4) \right\rangle \tag{52}
\]

for Fig.1 or

\[
\left\langle \left( \left( V_1(x_1) *_{N_{12}} V_2(x_2) \right) *_{N_{1234}} V_3(x_3) \right) *_{N_{1234}} V_4(x_4) \right\rangle \tag{53}
\]

for Fig.2, analytically continued in all the \(N\)-variables, which are in this way converted into arbitrary intermediate dimensions.

The linear form here is defined by the usual rule

\[
\left\langle L_{-Y} e^{\alpha \phi(x)} \right\rangle \sim \delta_{Y,0} \delta_{\alpha,Q} \tag{54}
\]

Note that the product \(*_N\) is defined in (4) asymmetrically: the result is an operator at point \(0\), i.e. at the position of the first entry of the product. This makes \(*_N\) non-associative:

\[
\left( V_1(x_1) *_{N} V_2(x_2) \right) *_{M} V_3(x_3) \equiv V_1(x_1)V_2(x_2)V_3(x_3) \left( \int_{x_1}^{x_2} : e^{b \phi} : \right)^N \left( \int_{x_1}^{x_3} : e^{b \phi} : \right)^M \tag{55}
\]
This nice exponential formula first appeared in [45]. The "matrix-model" or Dotsenko-Fateev representation, a weak form of the AGT relation, is readily available and provided by A star-like conformal block, for which the AGT relation (the corresponding set of Nekrasov function) is yet unknown.

Figure 2: A comb-like conformal block from [1, 35], for which the AGT relation is known in the case of $0 = x_1 \ll x_2 \ll x_3 \ll x_4 \ll \ldots$ Shown are the $\alpha$-parameters, the dimensions are equal to $\Delta = \alpha_2 - 1/b$. The intermediate dimensions are parameterized by the $N$-variables, after analytical continuation in $N$ they take arbitrary (continuum) values.

while

$$V_1(x_1) *_N \left( V_2(x_2) *_M V_3(x_3) \right) \equiv V_1(x_1) V_2(x_2) V_3(x_3) \left( \int_{x_1}^{x_2} : e^{b\phi} : \right)^N \left( \int_{x_2}^{x_3} : e^{b\phi} : \right)^M$$

and the difference is in the integration segments in the last items. Thus, the brackets are essential in the above expressions for the conformal blocks. In practice, CFT calculations are determined by the ordering of $x$-arguments: $0 = x_1 \ll x_2 \ll x_3 \ll x_4 \ll \ldots$ in (52) and $0 = x_1 \ll x_2 \ll x_3 \ll x_5$, $x_4 - x_3 \ll x_3$ in (53) and, hence, these two cases correspond to different regions of the values of variables in the conformal block.

10 Towards a proof of the AGT conjecture

As explained in [32, 39] and further developed in [45], representations like (52) can be directly used to prove the original AGT conjecture. For example, for the 4-point conformal block (52) implies that

$$B = q^{\alpha_1 \alpha_2} (1 - q)^{\alpha_2 \alpha_3} \prod_{k=1}^{N_{12}} \int_0^1 y_k^a (1 - y_k)^c (y_k - q)^\gamma dy_k \prod_{i=1}^{N_{12}} \int_0^1 z_i^a (1 - z_i)^c (z_i - q)^\gamma dz_i \prod_{i<k} z_{ij}^{2\beta} \prod_{i,k} y_k^{2\beta} \prod (z_i - y_k)^{2\beta}$$

where $q = \frac{x_1 x_3}{x_2 x_4}$, $x_1 = 0$, $x_2 = q$, $x_3 = 1$, $N_1 = N_{12}$, $N_2 = N_{(12)3}$ and $a = 2b\alpha_1$, $\beta = 2\gamma$, $c = 2b\alpha_3$, $\gamma = 2b\alpha_2$. In order to make use of the Selberg integrals from the Appendix, which are all along the segment $[0,1]$, we rescale $y_k \to q y_k$ and expand in powers of $q$:

$$B = q^{\deg(B)} \prod_{i=1}^{N_{12}} \int_0^1 z_i^{a+\gamma+2\beta N_1} (1 - z_i)^c dz_i \prod_{i<j} z_{ij}^{2\beta} \prod_{k=l}^{N_{12}} \int_0^1 y_k^a (1 - y_k)^c dy_k \prod_{k<l} y_k^{2\beta} \cdot \exp \left\{ -2 \sum_{m=1}^{\infty} \frac{q^m}{m} \left( \alpha_2 + b \sum_{i=1}^{N_{12}} \frac{1}{z_i^1} \right) \left( \alpha_3 + b \sum_{k=1}^{N_{12}} \frac{1}{y_k^m} \right) \right\}$$

This nice exponential formula first appeared in [45].
It can be now expanded in the Schur/Jack polynomials so that the result is a bilinear combination of the generalized Selberg integrals over the $z$ and $y$ variables. The integrals are labeled by Young diagrams (see the Appendix below), thus, one naturally obtains a bilinear expansion in Young diagrams. In the formulation of [45], the AGT conjecture is now reduced to the claim that there are two different expansions of this type: in triple vertices and in the Nekrasov functions. Denoting independent averaging over the $z$ and $y$ variables by the double angle brackets, we have [45]:

\[
\left\langle \exp \left\{ -2 \sum_{m=1}^{\infty} \frac{q^m}{m} \left( \alpha_2 + b \sum_{i=1}^{N_{123}} \frac{1}{z_i^m} \right) \left( \alpha_3 + b \sum_{k=1}^{N_{12}} y_k^m \right) \right\} \right\rangle \rightarrow \sum_{Y_1,Y_2} \bar{\Gamma}(Y_1)Q^{-1}(Y_1,Y_2)\Gamma(Y_2) \quad (59)
\]

\[
\rightarrow (1-q)^{2\alpha_2\alpha_3} \sum_{Y_1,Y_2} Z_{Nek}(Y_1,Y_2)
\]

- Despite this is already done in [32, 39, 45], for the sake of completeness we explicitly illustrate the situation at the first level of the $q$-expansion.

At this level, one needs just two explicit expressions for the Selberg integrals from the Appendix:

\[
<\sum y> = \frac{N_{12}I[1]}{I[0]} = \frac{N_{12} - a + (N_{12} - 1)\beta + 1}{a + \gamma + (2N_{12} - 2)\beta + 2}
\quad (60)
\]

and

\[
<\sum z> = \frac{N_{123}I[-1]}{I[0]} = \frac{N_{123}I_{a' - 1}[1^{N_{123} - 1}]}{I_{a'[0]}} = \frac{N_{123} a' + c + (N_{123} - 1)\beta + 1}{a'}
\quad (61)
\]

Here $a' = a + \gamma + 2N_1\beta$ and

\[
- \left( \alpha_2\alpha_3 + 2\beta \sum_i \frac{1}{z_i} \sum_k y_k + \gamma \sum_i \frac{1}{z_i} + c \sum_k y_k \right) = \frac{(\Delta + \Delta_2 - \Delta_1)(\Delta + \Delta_3 - \Delta_4)}{2\Delta}
\quad (62)
\]

with [39]

\[
\Delta = a'(a + \gamma + (2N_1 - 2)\beta + 2)/2\beta = (a + \gamma + 2N_1\beta)(a + \gamma + 2N_1\beta + 2 - 2\beta)/2\beta,
\]

\[
\Delta_1 = (a)(a + 2 - 2\beta)/2\beta,
\]

\[
\Delta_2 = (\gamma)(\gamma + 2 - 2\beta)/2\beta,
\]

\[
\Delta_3 = (c)(c + 2 - 2\beta)/2\beta,
\]

\[
\Delta_4 = (a + c + \gamma + 2\beta(N_1 + N_2))(a + c + \gamma + 2\beta(N_1 + N_2) + 2 - 2\beta)/2\beta
\quad (63)
\]

thus reproducing the expression for the $q$-linear contribution to the conformal block [46, 47, 4].

- In general the $q^m$-term of the $B$-expansion contains bilinear combinations of the integrals $I_{N_1}[Y]$ and $I_{N_2}[-Y']$ with $|Y'|,|Y'| \leq m$. Generalization to the multi-point conformal blocks with the multi-linear expansion in Young diagrams is also straightforward.

11 Conclusion

In this paper we justified the claim that the coefficients of the operator product expansions in arbitrary conformal theory are fully controlled by the free field model, provided one allows insertions of the Dotsenko-Fateev screening operators between the points, where the original operators are located, and analytical continuation in the number of these insertions. The well-known complexity of the operator expansion coefficients appears related to that of the generalized Selberg integrals, which are defined for arbitrary Young diagrams, but contain non-trivial non-factorizable polynomial factors when the diagrams are different from $[1^n]$. Since the single line diagrams $[1^n]$ are associated with the hypergeometric series [14], one may say that the non-triviality of the Selberg integrals for other diagrams is responsible for the deviation of the conformal blocks from the hypergeometric functions and, thus, it is what requires the generic Nekrasov functions to appear in description of the conformal blocks.
11.1 The proof of the matrix-model version of AGT conjecture

Despite the present paper does not contain a full constructive proof, hopefully, it provides a conceptually clear explanation of the weak form of the AGT conjecture [32, 39], identifying the conformal block with the analytically continued matrix model partition function in the DV phase [11, 30, 29, 31]. Moreover, for the 3-point functions this identification can be implicitly (not constructively) proved with the following chain of arguments:

- The structure constants $C_{\alpha,Y;\alpha_2,Y_2}$ in (1), i.e. components of the intertwining operator, are unambiguously defined by representation theory of the Virasoro algebra.
- Free field + DF – induced structure constants $\tilde{C}_{\alpha,Y;\alpha_2,Y_2}$ in (4) are also components of the Virasoro intertwining operator, but for a triple of concrete and explicitly realized Verma modules. Instead they are defined only for discrete values of $a = \alpha_1 + \alpha_2 + bN$.
- The both $C$ and $\tilde{C}$ are rational functions of their arguments $a$ and $\alpha$. The rational analytical continuation in $a$ of the function $\tilde{C}$ is unique, therefore, such an analytical continuation coincides with $C$:

$$C_{\alpha,Y;\alpha_2,Y_2} = \tilde{C}_{\alpha,Y;\alpha_2,Y_2}$$

Note that with this technique one obtains ”matrix-model” representations for arbitrary conformal blocks, not only for (52), but also for (53). At the same time, the literal AGT relations are currently applicable only for the case of (52), their generalization (an extension of the set of the Nekrasov functions) to arbitrary conformal blocks remains to be found.

11.2 Towards a proof of the remaining part of the AGT conjecture

After the matrix model version of the AGT conjecture is proved, the original AGT conjecture is reduced to an exercise, outlined above in section 10. There are still some combinatorial identities to be proved in this direction, but this seems rather straightforward. F complete proof of the AGT conjecture on this track is now clearly within reach.

Much more interesting would be to prove the AGT conjecture differently, without any direct use of the Nekrasov functions. Given the result of the present paper, it turns into a puzzling observation that the two is now clearly within reach.

Direction, but this seems rather straightforward.

where $\sum_{\alpha_1,2,3}$ and $\alpha_4 \equiv \alpha_2 + \alpha_3 + bN_{12}3$ [4]. Thus, the positions of poles in the LNS integral (dictated by $\alpha_1$ and, hence, by $N_{12}$) become a number of integrations in the DF case, while the number of integrations in the LNS integral depends on the degree of expansion into $q$, i.e. on the level of expansion of the conformal block into descendant contributions.

Thus, the AGT acquires form of a duality relation, where the number of integrations on one side is a parameter in the integrand at the other side and vice versa. This type of duality may seem mysterious, but it is well-known in the theory of matrix models [58]. The simplest example is provided by conversion of the Gaussian model into the Kontsevich type model [59]:

$$\int_{N \times N} dM e^{-\frac{1}{2} \text{tr} M^2 + \sum_k t_k \text{tr} M^k} \sim \int_{n \times n} dX (\text{det} X)^N e^{-\frac{1}{2} \text{tr} X^2 - \text{itr} \Lambda X}$$

$$\int_{N \times N} dM e^{-\frac{1}{2} \text{tr} M^2 + \sum_k t_k \text{tr} M^k} \sim \int_{n \times n} dX (\text{det} X)^N e^{-\frac{1}{2} \text{tr} X^2}$$

where $\text{tr}$, $\text{det}$ and $\text{det}$ denote the traces and determinants of the $N \times N$ and $n \times n$ matrices, and $t_k = \frac{1}{k!} \text{tr} \Lambda^{-k}$. Both models are of the eigenvalue type and clearly the number $N$ of integrations over the eigenvalues at the l.h.s. appears just as a parameter at the r.h.s., where the number of integrations is a fully independent parameter $n$. This identity, (66) can also be rewritten as

$$\int_{N \times N} dM \text{Det} \left( I \otimes I - M \otimes \Lambda^{-1} \right) e^{-\frac{1}{2} \text{tr} M^2} \sim \int_{n \times n} dX \text{ det} \left( I - i \frac{X}{\Lambda} \right)^N e^{-\frac{1}{2} \text{tr} X^2}$$

$$\int_{N \times N} dM \text{Det} \left( I \otimes I - M \otimes \Lambda^{-1} \right) e^{-\frac{1}{2} \text{tr} M^2} \sim \int_{n \times n} dX \text{ det} \left( I - i \frac{X}{\Lambda} \right)^N e^{-\frac{1}{2} \text{tr} X^2}$$
which, if expanded in powers of $\Lambda^{-1}$, becomes an identity for the Gaussian correlators:

$$1 - \frac{1}{2} \text{Tr} \frac{1}{\Lambda^2} << \text{Tr} M^2 >>_N + \frac{1}{2} \left( \text{tr} \frac{1}{\Lambda} \right)^2 << (\text{Tr} M)^2 >>_N + \ldots =$$

$$= 1 + \frac{N}{2} \left( \text{tr} X \frac{1}{\Lambda} X \frac{1}{\Lambda} \right)_n - \frac{N^2}{2} \left( \left( \text{tr} X \frac{1}{\Lambda} \right)^2 \right)_n + \ldots$$

(68)

Since $<< M_{ij} M_{kl} >> = \delta_{jk} \delta_{il}$ and $< X_{ab} X_{cd} > = \delta_{bc} \delta_{ad}$, the two sides of the equality coincide, but, as usual for dualities, the second term at the l.h.s. is equal to the third term at the r.h.s. and vice versa.

Eq. (66) can be used (at $\beta = 1$) to further transform the multiple integrals (57) so that $N$ becomes a parameter in the integrand. Indeed, such integrals arise from the l.h.s. of (66) for $t_k = \sum_a \frac{\mu_k}{q_a} q_a$ which for the integer values of $\mu_a$ correspond to $\Lambda$ matrices of the block form $\Lambda = \sum_a q_a I_{\mu_a}$ at the r.h.s. of (66). The number of integrations at the r.h.s. is then equal to $n = \sum_a \mu_a$. It is natural to conclude that the AGT relation between (57) and (65) is a further generalization of the duality relation (66) continued to the $\beta$-ensembles (to $\beta \neq 1$) and to non-integer values of $n$. Details of this analysis will be presented elsewhere.

## 11.3 Extension from Virasoro to $W$

Another mystery associated with our result in this paper concerns extension to the case of several free fields. In conformal theory, this corresponds to switching from the Virasoro to $W$ chiral algebras. The problem is that representation theory of the $W$ algebras is not sufficient to specify unambiguously arbitrary conformal blocks. Additional constraints should therefore be imposed by brute force. It is, however, unclear what are the parallel restrictions in the free field formalism and its Dotsenko-Fateev extension described in the present paper, which seems easily generalizable to an arbitrary number of free fields. This subject also remains open for future investigation.

## Acknowledgements

Al.Mor. is indebted for the hospitality and support to Uppsala University, where part of this work was done.

Our work is partly supported by Russian Federal Nuclear Energy Agency, Federal Agency for Science and Innovations of Russian Federation under contract 02.740.11.5194, by RFBR grants 10-01-00536 (A.Mir. and Al.Mor.) and 10-01-00836 (An.Mor.), by joint grants 09-02-90493-Ukr, 09-02-93105-CNRLS, 09-01-92440-CE, 09-02-91005-ANF, 10-02-92109-Yaf-a.
Appendix. Selberg integrals and their generalizations

The Selberg integrals

\[ I_Y = \prod_{i=1}^{N} \Gamma(\beta j + 1) \prod_{j=0}^{N-1} \frac{\Gamma(a + \beta j + 1)\Gamma(c + \beta j + 1)}{\Gamma(a + c + (N - j)\beta + 2)} \]  

(69)

with \( z^Y = z_1^{n_1}z_2^{n_2} \ldots \) for \( Y = \{n_1 \geq n_2 \geq \ldots\} \) are direct generalizations of the Euler Beta-function, also represented as products of elementary Gamma-function factors. The Selberg integrals are naturally labeled by Young diagrams \( Y \), and well-known are only integrals for the single line diagrams \([1^n]\). For more complicated diagrams, the integrals contain additional polynomial factors, which are not further factorized into linear expressions. However, they are needed for comparison with DF 3-point functions in this paper.

- From [55] one knows that

\[ I[0] = \prod_{j=1}^{N} \Gamma(\beta j + 1) \prod_{j=0}^{N-1} \frac{\Gamma(a + \beta j + 1)\Gamma(c + \beta j + 1)}{\Gamma(a + c + (N - j)\beta + 2)} \]  

(70)

\[ I[1] = \frac{a + (N - 1)\beta + 1}{a + c + (2N - 2)\beta + 2} I[0] \]  

(71)

and, in general,

\[ I[1^n] = I[0] \prod_{j=1}^{n} \frac{a + (N - j)\beta + 1}{a + c + (2N - j - 1)\beta + 2} \]  

(72)

- If the Young diagram \( Y \) contains \( k > 1 \) lines, then [55] is not sufficient, and actually \( I[Y] \) acquires additional factors, which are polynomials of degree \( 2k - 2 \). In particular,

\[ I[2] = \frac{a^2 + ac + (3N - 4)\alpha + 2(N - 1)\beta c + 4a + 2c + 4 + (N - 1)(3N - 4)\alpha^2 + (7N - 9)\beta}{(a + c + (2N - 3)\beta + 2)(a + c + (2N - 2)\beta + 3)} I[1] \]  

(73)

\[ I[21] = \frac{(a + (N - 2)\beta + 1)(a + (N - 1)\beta + 1)}{(a + c + (2N - 4)\beta + 2)(a + c + (2N - 3)\beta + 2)(a + c + (2N - 2)\beta + 2)(a + c + (2N - 2)\beta + 3)} \]  

\[ \cdot \left( a^2 + ac + (3N - 5)\alpha + (2N - 3)\beta c + 3(N - 1)(N - 2)\beta^2 + 4a + 2c + (7N - 12)\beta + 4 \right) \]  

(74)

and in general

\[ I[21^n] = I[0] \left( \frac{a^2 + ac + (3N - 4)(N - 1)\beta^2 + (3N - 4)\alpha + 2(N - 1)\beta c + 4a + 2c + (7N - 9)\beta + 4}{a + c + (2N - 2)\beta + 3} - n\beta \right) \]  

\[ \cdot \prod_{j=1}^{n+1} \frac{a + (N - j)\beta + 1}{a + c + (2N - j - 1)\beta + 2} \]  

(75)

Moving further,

\[ I[3] = \frac{(a + (N - 1)\beta + 1) I[0] P[3]}{(a + c + (2N - 4)\beta + 2)(a + c + (2N - 3)\beta + 2)(a + c + (2N - 2)\beta + 2)(a + c + (2N - 2)\beta + 3)(a + c + (2N - 2)\beta + 4)} \]  

(76)

where

\[ P[3] = (a + 3)(a + 2)(a + c - \beta + 2)(a + c - 2\beta + 2) + \]
Indeed, according to [60], the Selberg integrals of Jack polynomials are factorized:

\[ I[11] = \frac{1 + N\beta}{1 + \beta} \frac{(a + (N - 1)\beta + 1)(a + (N - 1)\beta + 2)}{(a + c + (2N - 2)\beta + 2)(a + c + (2N - 2)\beta + 3)} I[0] \]  

(77)

Note that not only a decomposition into linear factors is obtained in this way, also the factor \((a + c + (2N - 3)\beta + 2)\), which was present in the denominators of both \(I[2]\) and \(I[11]\), is canceled in this combination.

As noted in [45] the relevant linear combinations are actually the Jack polynomials \(P^{(1/\beta)}[Y]\):

\[ P^{(1/\beta)}[1^n](z) = m_{(1^n)}(z) = \sum_{1 \leq i_1 < i_2 < \ldots < i_n} \prod_{k=1}^n z_{i_k}, \]

\[ P^{(1/\beta)}[2] = m_2(z) + \frac{2\beta}{1 + \beta} m_{(1^2)}(z) = \sum_i z_i^2 + \frac{2\beta}{1 + \beta} \sum_{1 < i < j} z_i z_j, \]

\[ P^{(1/\beta)}[3] = m_3(z) + \frac{3\beta}{1 + 2\beta} m_{(2,1)}(z) + \frac{6\beta^2}{(1 + \beta)(1 + 2\beta)} m_{(1^3)}(z), \]

\[ P^{(1/\beta)}[2,1] = m_{(2,1)}(z) + \frac{6\beta}{1 + 2\beta} m_{(1^3)}(z), \]

\[ \ldots (78) \]

Indeed, according to [60], the Selberg integrals of Jack polynomials are factorized:

\[ < P^{(1/\beta)}[Y] > = c[Y] I[0] \prod_{i \geq 1} \prod_{j=0}^{m_i - 1} \frac{a + (N - i)\beta + 1 + j}{a + c + (2N - 1 - i)\beta + 2 + j} \]  

(79)

and the \(\beta\)- and \(N\)-dependent coefficient is

\[ c[Y] = \prod_{i \geq 1} \prod_{j=0}^{n_i - 1} \frac{(N + 1 - i)\beta + j}{(n_i - j + (n_j - i + 1)\beta)} \]  

(80)

where \(\tilde{n}\) parameterizes the transposed Young diagram \(\tilde{Y} = \{\tilde{n}_1 \geq \tilde{n}_2 \geq \ldots\}\). In particular,

\[ c[2] = \frac{N\beta(N\beta + 1)}{(n_1 - 1 + \tilde{n}_2\beta)(n_1 - 2 + \tilde{n}_2\beta)} = \frac{N\beta + 1}{\beta + 1}, \]  

(81)

in accordance with (77).

The Selberg integrals \(I[Y]\) satisfy a set of sum rules. Since \(\prod_i (1 - z_i) = 1 - \sum_{i=1}^N z_i + \sum_{i < j} z_i z_j - \ldots\) one has

\[ I_{c+1}[0] = I_c[0] - N I_c[1] + \frac{N(N - 1)}{2} I_c[11] - \ldots = \sum_{n=0}^N \frac{(-1)^n N!}{n!(N - n)!} I_c[1^n] \]  

(82)

what is indeed true for (72). This sum rule involves only the single row Young diagrams.

Similarly, from the expansion \(\prod_i (1 - z_i)^2 = 1 - 2 \sum_{i=1}^N z_i + \sum_{i < j} z_i^2 + 2 \sum_{i < j} z_i z_j - \ldots\) one gets

\[ I_{c+2}[0] = I_c[0] - 2 N I_c[1] + N I_c[2] + N(N - 1) I_c[11] - \ldots \]  

(83)

which includes only the double row Young diagrams (of which the single row diagram is a particular case with \(k_2 = 0\)).

Similarly, expanding \(\prod_i (1 - z_i)^m\), one can deduce the expansion of \(I_{c+m}[0]\) into a sum of the \(m'\)-row Young diagrams with \(m' \leq m\). Moreover, such sum rules can also be written for \(I_{c+m}[Y]\) with arbitrary \(Y\).
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