Operator error estimates for homogenization of the nonstationary Schrödinger-type equations: sharpness of the results

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ABSTRACT
In $L^2(\mathbb{R}^d; \mathbb{C}^n)$, we consider a self-adjoint matrix strongly elliptic second-order differential operator $A_\varepsilon$ with periodic coefficients depending on $x/\varepsilon$. We find approximations of the exponential $e^{-i\tau A_\varepsilon}$, $\tau \in \mathbb{R}$, for small $\varepsilon$ in the $(H^s \to L^2)$-operator norm with suitable $s$. The sharpness of the error estimates with respect to $\tau$ is discussed. The results are applied to study the behavior of the solution $u_\varepsilon$ of the Cauchy problem for the Schrödinger-type equation $i\partial_\tau u_\varepsilon = A_\varepsilon u_\varepsilon + F$.

1. Introduction
The paper concerns homogenization for periodic differential operators (DOs). A broad literature is devoted to homogenization problems; first of all, we mention the books [1–3]. For homogenization problems in $\mathbb{R}^d$, one of the methods is the spectral approach based on the Floquet–Bloch theory; see, e.g. [2, Chapter 4], [3, Chapter 2], [4–6].

1.1. The class of operators
Let $\Gamma$ be a lattice in $\mathbb{R}^d$, and let $\Omega$ be the elementary cell of the lattice $\Gamma$. For $\Gamma$-periodic functions in $\mathbb{R}^d$, we denote $\varphi^\varepsilon(x) := \varphi(\varepsilon^{-1}x)$, $\varepsilon > 0$. In $L^2(\mathbb{R}^d; \mathbb{C}^n)$, we consider self-adjoint elliptic matrix DOs of the following form:

$$A_\varepsilon = (f^\varepsilon(x))^*b(D)^*g^\varepsilon(x)b(D)f^\varepsilon(x).$$ (1)

Here $b(D)$ is a homogeneous first-order matrix DO with constant coefficients. We assume that the symbol $b(\xi)$ is an $(m \times n)$-matrix of rank $n$ ($m \geq n$). Next, $g(x)$ is a $\Gamma$-periodic bounded and uniformly positive definite $(m \times m)$-matrix-valued function and $f(x)$ is a $\Gamma$-periodic bounded together with its inverse $(n \times n)$-matrix-valued function.

It is convenient to start with a simpler class of operators

$$\widehat{A}_\varepsilon = b(D)^*g^\varepsilon(x)b(D)$$ (2)

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corresponding to the case where \( f = I_n \). Many operators of mathematical physics can be represented in the form (1) or (2); see, e.g. [7, Chapter 4]. The simplest example is the acoustics operator \( \hat{A}_\varepsilon = D^* g^\varepsilon(x) D = -\text{div} g^\varepsilon(x) \nabla \).

### 1.2. Survey

In 2001, M. Birman and T. Suslina (see [8]) suggested an operator-theoretic (spectral) approach to homogenization problems in \( \mathbb{R}^d \), based on the scaling transformation, the Floquet–Bloch theory and the analytic perturbation theory. With the help of this method, the so-called operator error estimates were obtained. In the case of elliptic and parabolic problems, this approach was developed in detail, see [7, 9–17].

A different approach to operator error estimates for elliptic and parabolic problems (the “shift method”) was suggested by V. Zhikov and S. Pastukhova, see [18–21].

The operator error estimates for nonstationary Schrödinger-type and hyperbolic equations have been studied to a lesser extent. The papers [22–26] were devoted to such problems; see also [27] and [28], where a wider class of operators with the lower order terms was considered. In operator terms, the behavior of the operator-valued functions \( e^{-i\tau \hat{A}_\varepsilon}, \cos(\tau \hat{A}_\varepsilon^{1/2}), \) and \( \hat{A}_\varepsilon^{-1/2} \sin(\tau \hat{A}_\varepsilon^{1/2}) \) (where \( \tau \in \mathbb{R} \)) for small \( \varepsilon \) was studied. Let us dwell on the results for the nonstationary Schrödinger-type equations. In [22], the following estimate was obtained:

\[
\|e^{-i\tau \hat{A}_\varepsilon} - e^{-i\tau \hat{A}_0}\|_{H^3(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C(1 + |\tau|) \varepsilon. \tag{3}
\]

Here \( \hat{A}_0 = b(D)^* g^0 b(D) \) is the effective operator with the constant effective matrix \( g^0 \). Next, in [26] it was shown that, in the general case, this estimate is sharp with respect to the type of the operator norm. On the other hand, under some additional assumptions (formulated in the spectral terms near the lower edge of the spectrum) this result was improved:

\[
\|e^{-i\tau \hat{A}_\varepsilon} - e^{-i\tau \hat{A}_0}\|_{H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C(1 + |\tau|^{1/2}) \varepsilon. \tag{4}
\]

### 1.3. Main results of the paper

The present paper is devoted to error estimates for the operator exponential; a special attention is paid to the dependence of the estimates on time. We show that, in the general case, the factor \( (1 + |\tau|) \) in (3) cannot be replaced by \( (1 + |\tau|^\alpha) \) with \( \alpha < 1 \). On the other hand, we prove that estimate (4) (which holds under some additional assumptions) can be improved:

\[
\|e^{-i\tau \hat{A}_\varepsilon} - e^{-i\tau \hat{A}_0}\|_{H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C(1 + |\tau|^{1/2}) \varepsilon.
\]

This result allows us to obtain qualified estimates for large time \( \tau = O(\varepsilon^{-\alpha}) \) with \( \alpha < 2 \). Analogous results are obtained also for the more general operator (1). It turns out that it is convenient to study the operator \( f^\varepsilon e^{-i\tau \hat{A}_\varepsilon} (f^\varepsilon)^{-1} \) (the operator exponential sandwiched between rapidly oscillating factors).

The results given in the operator terms are applied to study the behavior of the solution \( u_\varepsilon(x, \tau), x \in \mathbb{R}^d, \tau \in \mathbb{R}, \) of the problem

\[
\frac{\partial u_\varepsilon(x, \tau)}{\partial \tau} = (\hat{A}_\varepsilon u_\varepsilon)(x, \tau) + F(x, \tau), \quad u_\varepsilon(x, 0) = \phi(x). \tag{5}
\]

A more general problem with the operator \( A_\varepsilon \) is also studied.
1.4. Method

The results are obtained with the help of the operator-theoretic approach. The scaling transformation reduces investigation of the difference of exponentials under the norm sign in (3) to studying the difference \( e^{-i\tau z^2}A - e^{-i\tau z^2}A_n \), where \( A = b(D)^*g(x)b(D) \). Next, with the help of the unitary Gelfand transformation, the operator \( \hat{A} \) expands into the direct integral of the operators \( \hat{A}(k) \) depending on the quasimomentum \( k \) and acting in the space \( L_2(\Omega; \mathbb{C}^n) \). According to [9], we distinguish the one-dimensional parameter \( t = |k| \) and consider the family \( \hat{A}(k) \) as a quadratic operator pencil with respect to the parameter \( t \). Here, a good deal of constructions can be done in the framework of an abstract operator-theoretic setting. In the abstract scheme, the operator family \( A(t) \) acting in some Hilbert space \( \mathcal{H} \) and admitting a factorization of the form \( A(t) = X(t)^*X(t) \), where \( X(t) = X_0 + tX_1 \), is considered.

1.5. The plan of the paper

The paper consists of three chapters. Chapter 1 (Sections 1.6–3) contains necessary abstract operator-theoretic material. In Chapter 2 (Sections 4–6), periodic DOs acting in \( L_2(\mathbb{R}^d; \mathbb{C}^n) \) are studied. Chapter 3 (Sections 7–8.3) is devoted to homogenization problems for nonstationary Schrödinger-type equations. In Section 7, the main results of the paper in operator terms are obtained. Next, in Section 8 these results are applied to homogenization of the Cauchy problem (5) and a more general problem with the operator \( \mathcal{A}_\varepsilon \). Section 8.3 is devoted to applications of the general results to particular equations.

1.6. Notation

Let \( \mathcal{H} \) and \( \mathcal{H}_s \) be complex separable Hilbert spaces. The symbols \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) and \( \| \cdot \|_{\mathcal{H}} \) denote the inner product and the norm in \( \mathcal{H} \). The symbol \( \| \cdot \|_{\mathcal{H}\to\mathcal{H}_s} \) stands for the norm of a bounded linear operator from \( \mathcal{H} \) to \( \mathcal{H}_s \). Sometimes we omit the indices if this does not lead to confusion. By \( I = I_{\mathcal{H}} \), we denote the identity operator in \( \mathcal{H} \). If \( A: \mathcal{H}\to\mathcal{H}_s \) is a linear operator, then \( \text{Dom} A \) and \( \text{Ker} A \) stand for its domain and kernel, respectively. If \( \mathcal{M} \) is a subspace in \( \mathcal{H} \), then \( \mathcal{M}^\perp := \mathcal{H} \ominus \mathcal{M} \). If \( P \) is the orthogonal projection of \( \mathcal{H} \) onto \( \mathcal{M} \), then \( P^\perp \) is the orthogonal projection of \( \mathcal{H} \) onto \( \mathcal{M}^\perp \).

The symbols \( \langle \cdot, \cdot \rangle \) and \( | \cdot | \) stand for the standard inner product and the norm in \( \mathbb{C}^n \); \( I_n \) is the unit \((n \times n)\)-matrix. Next, \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), \( iD_j = \frac{\partial}{\partial x_j}, j = 1, \ldots, d \), \( D = -i\nabla = (D_1, \ldots, D_d) \).

The \( L_p \)-classes of \( \mathbb{C}^n \)-valued functions in a domain \( \mathcal{O} \subset \mathbb{R}^d \) are denoted by \( L_p(\mathcal{O}; \mathbb{C}^n) \), \( 1 \leq p \leq \infty \). The Sobolev classes of \( \mathbb{C}^n \)-valued functions in a domain \( \mathcal{O} \subset \mathbb{R}^d \) of order \( s \) and integrability index \( p \) are denoted by \( W^s_p(\mathcal{O}; \mathbb{C}^n) \). For \( p = 2 \), we use the notation \( H^s(\mathcal{O}; \mathbb{C}^n) \), \( s \in \mathbb{R} \). If \( n = 1 \), we write simply \( L_p(\mathcal{O}), W^s_p(\mathcal{O}), H^s(\mathcal{O}) \), etc., but sometimes we use such abbreviated notation also for spaces of vector-valued or matrix-valued functions. Various constants in estimates are denoted by \( C, c, \mathcal{C}, \mathcal{C} \) (probably, with indices and marks).

Chapter 1. Abstract operator-theoretic scheme

2. Quadratic operator pencils

2.1. The operators \( X(t) \) and \( A(t) \)

Let \( \mathcal{H} \) and \( \mathcal{H}_s \) be complex separable Hilbert spaces. Suppose that \( X_0: \mathcal{H} \to \mathcal{H}_s \) is a densely defined and closed operator, and \( X_1: \mathcal{H} \to \mathcal{H}_s \) is a bounded operator. On the domain \( \text{Dom} X_0 \), we introduce the operator \( X(t) := X_0 + tX_1, t \in \mathbb{R} \). Consider the family of self-adjoint (and nonnegative) operators \( A(t) := X(t)^*X(t) \) in \( \mathcal{H} \). The operator \( A(t) \) is generated by the closed quadratic form \( \|X(t)u\|_{\mathcal{H}_s}^2, u \in \text{Dom} X_0 \). Denote \( \mathcal{M} := \text{Ker} A(0) = \text{Ker} X_0, \mathcal{M}_s := \text{Ker} X_0^* \). We impose the following condition.
Condition 2.1: The point $\lambda_0 = 0$ is an isolated point in the spectrum of $A(0)$, and $0 < n := \dim \mathcal{M} < \infty$, $n \leq n_* := \dim \mathcal{M}_* \leq \infty$.

Denote by $d^0$ the distance from the point $\lambda_0 = 0$ to the rest of the spectrum of $A(0)$. Let $P$ and $P_*$ be the orthogonal projections of $\mathcal{M}$ onto $\mathcal{M}$ and of $\mathcal{M}_*$ onto $\mathcal{M}_*$, respectively. Denote by $F(t; [a, b])$ the spectral projection of $A(t)$ for the interval $[a, b]$, and put $\mathfrak{G}(t; [a, b]) := F(t; [a, b])\mathcal{M}_*$. We fix a number $\delta > 0$ such that $8\delta < d^0$. We write $F(t)$ in place of $F(t; [0, \delta])$ and $\mathfrak{G}(t)$ in place of $\mathfrak{G}(t; [0, \delta])$. Next, we choose a number $t^0 > 0$ such that

$$
t^0 \leq \delta^{1/2}\|X_1\|^{-1}.
$$

(6)

According to [9, Chapter 1, Proposition 1.2], $F(t; [0, \delta]) = F(t; [0, 3\delta])$ and $\text{rank}F(t; [0, \delta]) = n$ for $|t| \leq t^0$.

2.2. The operators $Z, R, S, Z_2, and R_2$

Now we introduce some operators appearing in the analytic perturbation theory considerations; see [9, Chapter 1, §1] and [10, §1].

Let $\omega \in \mathfrak{M}$, and let $\psi = \psi(\omega) \in \text{Dom} X_0 \cap \mathfrak{M}^\perp$ be a (weak) solution of the equation $X_0^* (X_0 \psi + X_1 \omega) = 0$. We define a bounded operator $Z: \tilde{\mathcal{M}} \to \mathfrak{M}$ by the relation $Zu = \psi(Pu)$, $u \in \tilde{\mathcal{M}}$. Next, we define the operator $R := X_0Z + X_1: \mathfrak{M} \to \mathfrak{M}_*$. Another representation for $R$ is given by $R = P_*X_1|_{\mathfrak{M}}$. According to [9, Chapter 1, §1], the operator $S := R^* R: \mathfrak{M} \to \mathfrak{M}$ is called the spectral germ of the operator family $A(t)$ at $t = 0$. The germ can be represented as $S = PX_1^* P_*X_1|_{\mathfrak{M}}$. The spectral germ is said to be non-degenerate if $\text{Ker}S = \{0\}$.

We need to introduce the operators $Z_2$ and $R_2$ defined in [13, §1]. Let $\omega \in \mathfrak{M}$, and let $\phi = \phi(\omega) \in \text{Dom} X_0 \cap \mathfrak{M}^\perp$ be a (weak) solution of the equation

$$
X_0^* (X_0 \phi + X_1 Z \omega) = -P^1 X_1^* R \omega.
$$

The right-hand side of this equation belongs to $\mathfrak{M}^\perp = \text{Ran} X_0^*$, so the solvability condition is fulfilled. We define an operator $Z_2: \tilde{\mathcal{M}} \to \mathfrak{M}$ by the relation $Z_2u = \phi(Pu)$, $u \in \tilde{\mathcal{M}}$. Finally, let $R_2 := X_0Z_2 + X_1Z: \mathfrak{M} \to \mathfrak{M}_*$.

2.3. The analytic branches of eigenvalues and eigenvectors of $A(t)$

According to the general analytic perturbation theory (see [29]), for $|t| \leq t^0$ there exist real-analytic functions $\lambda_l(t)$ (the branches of eigenvalues) and real-analytic $\mathfrak{M}$-valued functions $\varphi_l(t)$ (the branches of eigenvectors) such that $A(t) \varphi_l(t) = \lambda_l(t) \varphi_l(t)$, $l = 1, \ldots, n$, and the set $\varphi_l(t)$, $l = 1, \ldots, n$, forms an orthonormal basis in $\mathfrak{M}$. Moreover, for $|t| \leq t_*$, where $0 < t_* \leq t^0$ is sufficiently small, we have the following convergent power series expansions:

$$
\lambda_l(t) = \gamma_l t^2 + \mu_l t^3 + \nu_l t^4 + \ldots, \quad \gamma_l \geq 0, \quad \mu_l, \nu_l \in \mathbb{R}, \quad l = 1, \ldots, n, 
$$

(7)

$$
\varphi_l(t) = \omega_0 + t\psi_l^{(1)} + t^2\psi_l^{(2)} + \ldots, \quad l = 1, \ldots, n.
$$

(8)

We agree to use the numeration such that $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$. The elements $\omega_0 = \varphi(0)$, $l = 1, \ldots, n$, form an orthonormal basis in $\mathfrak{M}$. In [9, Chapter 1, §1] and [10, §1] it was checked that $\tilde{\omega}_l := \psi_l^{(1)} - Z_0\omega_l \in \mathfrak{M}$,

$$
S\omega_l = \gamma_l \omega_l, \quad l = 1, \ldots, n, 
$$

(9)

$$
(\tilde{\omega}_j, \omega_k) + (\omega_j, \tilde{\omega}_k) = 0, \quad j, k = 1, \ldots, n.
$$

(10)

Thus the numbers $\gamma_l$ and the elements $\omega_l$ defined by (7) and (8) are eigenvalues and eigenvectors of the germ $S$. We have $P = \sum_{l=1}^{n} \varphi(\cdot, \omega_l)\omega_l$ and $SP = \sum_{l=1}^{n} \gamma_l(\cdot, \omega_l)\omega_l$. 

2.4. Threshold approximations

The following statements were obtained in [9, Chapter 1, Theorems 4.1 and 4.3] and [10, Theorem 4.1]. In what follows, we agree to denote by $\beta_j$ various absolute constants (which can be controlled explicitly) assuming that $\beta_j \geq 1$.

Theorem 2.2 ([9]): Under the assumptions of Section 2.1, for $|t| \leq t^0$ we have

$$
\|F(t) - P\| \leq C_1|t|, \quad C_1 = \beta_1 \delta^{-1/2}\|X_1\|, \quad \|A(t)F(t) - t^2SP\| \leq C_2|t|^3, \quad C_2 = \beta_2 \delta^{-1/2}\|X_1\|^3.
$$

(11)

Theorem 2.3 ([10]): Under the assumptions of Section 2.1, for $|t| \leq t^0$ we have

$$
A(t)F(t) = t^2SP + t^3K + \Xi(t), \quad \|\Xi(t)\| \leq C_3t^4, \quad C_3 = \beta_3 \delta^{-1/2}\|X_1\|^4.
$$

The operator $K$ is represented as $K = K_0 + N = K_0 + N_0 + N_*$, where $K_0$ takes $\mathcal{M}$ to $\mathcal{M}^\perp$ and $\mathcal{M}^\perp$ to $\mathcal{M}$, while $N = N_0 + N_*$ takes $\mathcal{M}$ to itself and $\mathcal{M}^\perp$ to $\{0\}$. In terms of the power series coefficients, the operators $K_0$, $N_0$, $N_*$ are given by $K_0 = \sum_{l=1}^n \gamma_l((\cdot, \omega_1)\omega_l + (\cdot, \omega_l)\omega_1)$,

$N_0 = \sum_{l=1}^n \mu_l((\cdot, \omega_l)\omega_l, \quad N_* = \sum_{l=1}^n \gamma_l((\cdot, \tilde{\omega}_l)\omega_l + (\cdot, \omega_l)\tilde{\omega}_l).

(12)

In the invariant terms, we have $K_0 = ZSP + SP^*Z$ and $N = Z^*X_r^*RP + (RP)^*X_1Z$.

Remark 1: 1°. If $Z = 0$, then $K_0 = 0, N = 0$, and $K = 0$.

2°. In the basis $\{\omega_l\}_{l=1}^n$ the operators $N, N_0, N_*$ (restricted to the subspace $\mathcal{M}$) are represented by matrices of size $n \times n$. The operator $N_0$ is diagonal: $(N_0\omega_j, \omega_k) = \mu_j \delta_{jk}, j, k = 1, \ldots, n$. The matrix entries of $N_*$ are given by $(N_*\omega_j, \omega_k) = \gamma_k(\omega_j, \tilde{\omega}_k) + \gamma_j((\cdot, \omega_j)\tilde{\omega}_k)(\cdot, \omega_k) = (\gamma_j - \gamma_k)(\omega_j, \omega_k), j, k = 1, \ldots, n$. So, the diagonal elements of $N_*$ are equal to zero. Moreover, $(N_*\omega_j, \omega_k) = 0$ if $\gamma_j = \gamma_k$.

3°. If $n = 1$, then $N_* = 0$ and $N = N_0$.

2.5. The nondegeneracy condition

Below we impose the following additional condition.

Condition 2.4: There exists a constant $c_* > 0$ such that $A(t) \geq c_* t^2I$ for $|t| \leq t^0$.

From Condition 2.4, it follows that $\lambda_j(t) \geq c_* t^2, l = 1, \ldots, n$, for $|t| \leq t^0$. By (7), this implies $\gamma_l \geq c_* > 0, l = 1, \ldots, n$, i.e. the spectral germ is nondegenerate:

$$
S \geq c_* I_{\mathcal{M}}.
$$

(13)

2.6. The clusters of eigenvalues of $A(t)$

The content of this section is borrowed from [26, Section 2] and concerns the case where $n \geq 2$.

Suppose that Condition 2.4 is satisfied. Now, it is convenient to change the notation tracing the multiplicities of the eigenvalues of the operator $S$. Let $p$ be the number of different eigenvalues of the germ $S$. We enumerate these eigenvalues in the increasing order and denote them by $\gamma_j^*, j = 1, \ldots, p$. Let $k_1, \ldots, k_p$ be their multiplicities (obviously, $k_1 + \ldots + k_p = n$). Denote $\mathcal{M}_j := \text{Ker}(S - \gamma_j^* I_{\mathcal{M}}), j = 1, \ldots, p$. Then $\mathcal{M} = \sum_{j=1}^p \mathcal{M}_j$. Let $P_j$ be the orthogonal projection of $\mathcal{M}_j$ onto $\mathcal{M}_j$. Then $P = \sum_{j=1}^p P_j$, and $P_jP_l = 0$ for $j \neq l$. 

Remark 2: By Remark 1, we have \( P_jN_0P_j = 0 \) and \( P_lN_0P_j = 0 \) for \( l \neq j \). Hence, the operators \( N_0 \) and \( N_* \) admit the invariant representations:

\[
N_0 = \sum_{j=1}^{p} P_jN_0P_j, \quad N_* = \sum_{1 \leq i,j \leq p; i \neq j} P_lN_0P_j.
\] (14)

We divide the first \( n \) eigenvalues of the operator \( A(t) \) in \( p \) clusters for \( |t| \leq t^0 \); the \( j \)th cluster consists of the eigenvalues \( \lambda_i(t), l = i, \ldots, i+k_j-1 \), where \( i = i(j) = k_1 + \cdots + k_{j-1} + 1 \).

For each pair of indices \((j,l)\), \( 1 \leq j \leq l \leq p, j \neq l \), denote \( c_{jl}^0 := \min\{c_i, n^{-1} |\lambda_j - \lambda_i|\} \). Clearly, there exists a number \( i_0 = i_0(j,l) \), where \( j \leq i_0 \leq l - 1 \) if \( j < l \) and \( l \leq i_0 \leq j - 1 \) if \( l < j \) such that \( \gamma_{i_0,j+1}^\circ - \gamma_{i_0,j}^\circ \geq c_{jl}^0 \). It means that on the interval between \( \gamma_{j}^\circ \) and \( \gamma_{i}^\circ \) there is a gap in the spectrum of \( S \) of length at least \( c_{jl}^0 \). If such \( i_0 \) is not unique, we agree to take the minimal possible \( i_0 \) (for definiteness).

Next, we choose a number \( t_{0j}^0 \leq t^0 \) such that \( t_{0j}^0 \leq (4C_2)^{-1}c_{jl}^0 = (4\beta)^{-1}t_{0}^2/\mu_{ij} \) and \( \Delta_j^1(l) := [\gamma_{i_0,j}^\circ - c_{jl}^0/4, \gamma_{i_0,j}^\circ + c_{jl}^0/4] \) and \( \Delta_j^2(l) := [\gamma_{i_0,j+1}^\circ - c_{jl}^0/4, \gamma_{i_0,j}^\circ + c_{jl}^0/4] \). The distance between the segments \( \Delta_j^1(l) \) and \( \Delta_j^2(l) \) is at least \( c_{jl}^0/2 \). As was shown in [26, Section 2], for \( |t| \leq t_{0j}^0 \) the operator \( A(t) \) has exactly \( k_1 + \cdots + k_{i_0} \) eigenvalues (counted with multiplicities) in the segment \( t^2\Delta_j^1(l) \) and exactly \( k_{i_0+1} + \cdots + k_p \) eigenvalues in the segment \( t^2\Delta_j^2(l) \).

2.7. The coefficients \( \nu_l, l = 1, \ldots, n \)

We need to establish a relationship between the coefficients \( \nu_l, l = 1, \ldots, n \), and some eigenvalue problem.

In [13, (1.34), (1.37)], it was checked that \( \psi_l^{(2)} - Z\omega_l - Z_2\omega_l =: \omega_l^{(2)} \in H, l = 1, \ldots, n, \)

\[
(\omega_l^{(2)}, \omega_k) + (Z\omega_l, Z\omega_k) + (\omega_l, \omega_k) + (\omega_l, \omega_k^{(2)}) = 0, \quad l, k = 1, \ldots, n.
\] (15)

Next, by [13, (2.47), the formula below (2.46)], we have

\[
(N_1\omega_l, \omega_k) - \mu_l(\omega_l, \omega_k) - \mu_k(\omega_k, \omega_l) - \nu_l(\omega_k^{(2)}, \omega_k) - \nu_k(\omega_l, \omega_k^{(2)}) - (S\omega_l, \omega_k) = \nu_l\delta_{lk},
\]

\[
l, k = 1, \ldots, n.
\] (16)

where \( N_1 := N^0_1 - Z^*ZSP - SPZ^*Z, N^0_1 := Z^*_2X^*_1RP + (RP)^*X^*_1Z_2 + R^*_1R^*_2P \).

Let \( \gamma_{i}^q \) be the \( q \)th eigenvalue of problem (9) of multiplicity \( k_i \) (i.e. \( \gamma_{i}^q = \gamma_l = \cdots = \gamma_{i+k_i-1} \) for \( i = i(q) = k_1 + \cdots + k_{q-1} + 1 \)). Consider the eigenvalue problem (see Remark 1)

\[
P_qN\omega_l = \mu_l\omega_l, \quad l = i, \ldots, i + k_q - 1.
\] (17)

Assume that \( \mu_{i,l} = l, i = \ldots, i + k_q - 1 \), are enumerated in the increasing order. Let \( p'(q) \) be the number of different eigenvalues of problem (17) and denote by \( k_{q,1}, \ldots, k_{p'(q),q} \) their multiplicities (of course, \( k_{1,q} + \cdots + k_{p'(q),q} = k_q \)). We also change the notation and denote by \( \mu_{i,q}^q, j = 1, \ldots, p(q), \) the different eigenvalues of problem (17), enumerating them in the increasing order. Denote \( \omega_l := \text{Ker}(P_qN|\omega_l - \mu_{i,q}^q|\omega_l), j = 1, \ldots, p'(q) \). Then \( \omega_l = \sum_{j=1}^{p'(q)} \omega_{l,j} \) and \( \omega_{l,j} \). Let \( P_{l,j} \) be the orthogonal projection of \( \omega_l \) onto \( \omega_{l,j} \). Then \( P_q = \sum_{j=1}^{p'(q)} P_{l,j} \) and \( P_{l,j}P_{l,j'} = 0 \) for \( j \neq j' \).

Let \( \mu_{i',q}^q \) be the \( q \)th eigenvalue of problem (17) of multiplicity \( k_{i',q}^q \), i.e. \( \mu_{i',q}^q = \mu_{i'+k_{i',q}^q-1} = \mu_{i'+k_{i',q}^q-1} = \mu_{i'+k_{i',q}^q-1} = \mu_{i'+k_{i',q}^q-1} \). Using (10) and (15) and taking into
account that \( y_l = y_k = \gamma_q^l \), \( \mu_l = \mu_k = \mu_q^l \), \( l, k = i', \ldots, i' + k^-q - 1 \), from (16) we deduce

\[
\begin{align*}
(N_1 \omega_l, \omega_k) + y_l(Z\omega_l, Z\omega_k) + y_l(\tilde{\omega}_l, \tilde{\omega}_k) - (S\tilde{\omega}_l, \tilde{\omega}_k) = v_l\delta_{lk}, \\
l, k = i', \ldots, i' + k^-q - 1.
\end{align*}
\]  
(18)

Next, by virtue of Remark 1, we have

\[
y_l(\tilde{\omega}_l, \tilde{\omega}_k) - (S\tilde{\omega}_l, \tilde{\omega}_k)
= \sum_{i' = 1}^{n} (y_l - y_l)(\tilde{\omega}_l, \omega_l)(\omega_l, \tilde{\omega}_k)
= \sum_{i' \in \{1, \ldots, n\}} \sum_{l' \neq i', i' + k^-q - 1} (N\omega_l, \omega_l)(\omega_l, N\omega_k) y_q^l - y_q^l
= \sum_{j \in \{1, \ldots, p\}} (P_j N\omega_l, N\omega_k) y_q^l - y_q^l := n_{0}(q', q)[\omega_l, \omega_k],
\]

\( l, k = i', \ldots, i' + k^-q - 1 \).

Relations (18) can be treated as the eigenvalue problem for the operator \( N(q', q') \):

\[
N(q', q') \omega_l = v_l \omega_l, \quad l = i', \ldots, i' + k^-q - 1,
\]  
(19)

where \( N(q', q') := P_q' \omega_l (N_1 - \frac{1}{2} Z^* ZSP - \frac{1}{2} SPZ^* Z) |_{\Omega q', q'} + N_0(q', q') \) and \( N_0(q', q') \) is the operator acting in \( \Omega q', q' \) and generated by the form \( n_{0}(q', q') [\cdot, \cdot] \).

**Remark 3**: Let \( N_0 = 0 \). By (12), this condition is equivalent to the relations \( \mu_l = 0 \) for all \( l = 1, \ldots, n \). In this case, we have \( \Omega_{1,q} = \Omega_q, q = 1, \ldots, p \). Then we shall write \( N(q) \) instead of \( N(q, q) \). Suppose, in addition, that \( N(q) \neq 0 \) for some \( q \in \{1, \ldots, p\} \). By (19), this assumption means that \( v_l \neq 0 \) for some \( l \in \{1, \ldots, n\} \).

### 3. Approximation of the operator \( e^{-it\tau e^{-2A(t)}p} \)

#### 3.1. Approximation of the operator \( e^{-it\tau e^{-2A(t)}p} \)

Let \( \varepsilon > 0 \). We study the behavior of the operator \( e^{-it\tau e^{-2A(t)}p} \) for small \( \varepsilon \). We shall multiply this operator by the “smoothing factor” \( e^s(t^2 + \varepsilon^2)^{-s/2}P \), where \( s > 0 \). (The term is explained by the fact that in applications to DOs this factor turns into the smoothing operator.) Our goal is to find an approximation of the smoothed operator exponential with an error of order \( O(\varepsilon) \) for minimal possible \( s \).

We rely on the following statements proved in [22, Theorem 2.1] and [26, Corollaries 3.3, 3.5].

**Theorem 3.1 ([22]):** For \( \tau \in \mathbb{R} \) and \( |t| \leq t^0 \) we have

\[
\|e^{-it\tau A(t)}P - e^{-it\tau^2 SP}P\| \leq 2C_1 |t| + C_2 |\tau||t|^3.
\]  
(20)

**Theorem 3.2 ([26]):** Suppose that \( N = 0 \). Then for \( \tau \in \mathbb{R} \) and \( |t| \leq t^0 \) we have

\[
\|e^{-it\tau A(t)}P - e^{-it\tau^2 SP}P\| \leq 2C_1 |t| + C_4 |\tau||t|^4,
\]  
(21)

where \( C_4 = \beta_4 \delta^{-1} \|X_1 \|^4 \).
Theorem 3.3 ([26]): Suppose that \( N_0 = 0 \). Then for \( \tau \in \mathbb{R} \) and \( |t| \leq t^0 \) we have
\[
\|e^{-irA(t)}P - e^{-ir^2 SP}p\| \leq C_5|t| + C_6|\tau|t^4.
\]
Here \( t^0 \) is subject to the restriction
\[
t^0 \leq (4\beta_2)^{-1}\frac{1}{\delta}\|X_1\|^{-3}c^\circ,
\]
\( c^\circ := \min_{(j,l) \in Z} c^\circ_{jl}, \quad Z := \{(j,l): 1 \leq j, l \leq p, j \neq l, P_jNP_l \neq 0\}. \)
The constants \( C_5, C_6 \) are given by
\[
C_5 = \beta_5\delta^{-1/2}(\|X_1\| + n^2\|X_1\|^3(c^\circ)^{-1}), \quad C_6 = \beta_6\delta^{-1}(\|X_1\|^4 + n^2\|X_1\|^8(c^\circ)^{-2}).
\]

Now, we apply the formulated results and we start with Theorem 3.1. Let \( |t| \leq t^0 \). By (20) (with \( \tau \) replaced by \( \epsilon^{-2}\tau \)),
\[
\|e^{-ir\epsilon^{-2}A(t)}p - e^{-ir\epsilon^{-2}r^2 SP}p\|\epsilon^3(t^2 + \epsilon^2)^{-3/2} \leq (2C_1|t| + C_2\epsilon^2|\tau||t|^3)\epsilon^3(t^2 + \epsilon^2)^{-3/2} \leq (C_1 + C_2|\tau|)\epsilon.
\]
We arrive at the following result which has been proved before in [22, Theorem 2.6].

Theorem 3.4 ([22]): For \( \tau \in \mathbb{R} \) and \( |t| \leq t^0 \) we have
\[
\|e^{-ir\epsilon^{-2}A(t)}p - e^{-ir\epsilon^{-2}r^2 SP}p\|\epsilon^3(t^2 + \epsilon^2)^{-3/2} \leq (C_1 + C_2|\tau|)\epsilon.
\]
The constants \( C_1, C_2 \) are majorated by polynomials of the variables \( \delta^{-1/2}, \|X_1\| \).

Theorem 3.2 allows us to improve the result of Theorem 3.4 in the case where \( N = 0 \).

Theorem 3.5: Suppose that \( N = 0 \). Then for \( \tau \in \mathbb{R} \) and \( |t| \leq t^0 \) we have
\[
\|e^{-ir\epsilon^{-2}A(t)}p - e^{-ir\epsilon^{-2}r^2 SP}p\|\epsilon^3(t^2 + \epsilon^2)^{-3/2} \leq (C_1 + C_4|\tau|^{1/2})\epsilon.
\]
The constants \( C_1, C_4 \) are majorated by polynomials of the variables \( \delta^{-1/2}, \|X_1\| \).

Proof: Note that for \( |t| \geq \epsilon^{1/2}/|\tau|^{1/4} \) we have \( \frac{\epsilon^2}{t^2 + \epsilon^2} \leq \frac{\epsilon|\tau|^{1/2}}{1 + \epsilon|\tau|^{1/2}} \leq \epsilon|\tau|^{1/2} \), whence the left-hand side of (24) does not exceed \( 2|\tau|^{1/2}\epsilon \).

Using (21) with \( \tau \) replaced by \( \epsilon^{-2}\tau \), for \( |t| < \epsilon^{1/2}/|\tau|^{1/4} \) we obtain
\[
\|e^{-ir\epsilon^{-2}A(t)}p - e^{-ir\epsilon^{-2}r^2 SP}p\|\epsilon^3(t^2 + \epsilon^2)^{-3} \leq (2C_1|t| + C_4\epsilon^2|\tau|t^3)\epsilon^2(t^2 + \epsilon^2)^{-1} \leq C_1\epsilon + C_4|\tau||t|^2 \leq C_1\epsilon + C_4|\tau|^{1/2}\epsilon.
\]
The required estimate (24) follows with the constant \( C_4 = \max\{2, C_4\} \).

Similarly, using Theorem 3.3, one can deduce the following result.

Theorem 3.6: Suppose that \( N_0 = 0 \). Then for \( \tau \in \mathbb{R} \) and \( |t| \leq t^{00} \) we have
\[
\|e^{-ir\epsilon^{-2}A(t)}p - e^{-ir\epsilon^{-2}r^2 SP}p\|\epsilon^2(t^2 + \epsilon^2)^{-3} \leq (C_5 + C_6|\tau|^{1/2})\epsilon.
\]
Here \( t^{00} \) is subject to (22), the constants \( C_5, C_6 \) are majorated by polynomials of the variables \( \delta^{-1/2}, \|X_1\|, n, (c^\circ)^{-1} \).

Remark 4: Theorems 3.5 and 3.6 improve the results of Theorems 4.2 and 4.3 from [26] with respect to dependence of the estimates on \( \tau \).
3.2. Sharpness of the results with respect to the smoothing factor

Now, we show that the obtained results are sharp with respect to the smoothing factor. The following theorem proved in [26, Theorem 4.4] confirms the sharpness of Theorem 3.4.

**Theorem 3.7 ([26]):** Suppose that $N_0 \neq 0$. Let $\tau \neq 0$ and $0 \leq s < 3$. Then there does not exist a constant $C(\tau) > 0$ such that the estimate

$$
\|e^{-ir\varepsilon^2 A(t)} P - e^{-ir\varepsilon^2 t^2 SP} e^s (t^2 + \varepsilon^2)^{-s/2} \| \leq C(\tau) \varepsilon
$$

holds for all sufficiently small $|t|$ and $\varepsilon$.

Next, we confirm the sharpness of Theorems 3.5 and 3.6.

**Theorem 3.8:** Suppose that $N_0 = 0$ and $N^{(q)} \neq 0$ for some $q \in \{1, \ldots, p\}$. Let $\tau \neq 0$ and $0 \leq s < 2$. Then there does not exist a constant $C(\tau) > 0$ such that estimate (25) holds for all sufficiently small $|t|$ and $\varepsilon$.

**Proof:** We start with preliminary remarks. Since $F(t)^\perp P = (P - F(t))P$, from (11) it follows that

$$
\|e^{-ir\varepsilon^2 A(t)} F(t)^\perp P \| e(t^2 + \varepsilon^2)^{-1/2} \leq C_1 |t| \varepsilon(t^2 + \varepsilon^2)^{-1/2} \leq C_1 \varepsilon, \quad |t| \leq t^0.
$$

Next, for $|t| \leq t^0$ we have

$$
e^{-ir\varepsilon^2 A(t)} F(t) = \sum_{l=1}^n e^{-ir\varepsilon^2 \lambda_l(t)} (\cdot, \varphi_l(t)) \varphi_l(t).
$$

From the convergence of the power series expansions (8) it follows that

$$
\|\varphi_l(t) - \omega_l\| \leq c_1 |t|, \quad |t| \leq t^*_l, \quad l = 1, \ldots, n.
$$

It suffices to assume that $1 \leq s < 2$. Let us fix $0 \neq \tau \in \mathbb{R}$. We prove by contradiction. Suppose that for some $1 \leq s < 2$ there exists a constant $C(\tau) > 0$ such that (25) is valid for all sufficiently small $|t|$ and $\varepsilon$. By (26)–(28), this assumption is equivalent to the existence of a positive constant $\tilde{C}(\tau)$ such that the estimate

$$
\left\| \sum_{l=1}^n \left( e^{-ir\varepsilon^2 \lambda_l(t)} - e^{-ir\varepsilon^2 t^2} \gamma_l \right) (\cdot, \omega_l) \omega_l \right\| e^s (t^2 + \varepsilon^2)^{-s/2} \leq \tilde{C}(\tau) \varepsilon
$$

is valid for all sufficiently small $|t|$ and $\varepsilon$.

By Remark 3, the conditions $N_0 = 0$ and $N^{(q)} \neq 0$ for some $q \in \{1, \ldots, p\}$ mean that in the expansions (7) $\mu_l = 0$ for all $l = 1, \ldots, n$ and $v_j \neq 0$ at least for one $j$. Then $\lambda_j(t) = \gamma_j t^2 + v_j t^4 + O(|t|^5)$. 
Assume that $t_\ast$ is sufficiently small so that
\begin{equation}
\frac{1}{2} |\omega_j| t^4 \leq |\lambda_j(t) - \gamma_j t^2| \leq \frac{3}{2} |\omega_j| t^4, \quad |t| \leq t_\ast.
\end{equation}

Applying the operator under the norm sign in (29) to $\omega_j$. Then
\begin{equation}
\left| e^{-it^2 \lambda_j(t)} - e^{-it^2 \gamma_j} \right| \epsilon^{s} (t^2 + \epsilon^2)^{-s/2} \leq \tilde{C} (\epsilon)
\end{equation}
for all sufficiently small $|t|$ and $\epsilon$. The left-hand side of (31) can be written as
\begin{equation}
2 \left| \sin \left( \frac{1}{2} \tau \epsilon^{-2} (\lambda_j(t) - \gamma_j t^2) \right) \right| \epsilon^{s} (t^2 + \epsilon^2)^{-s/2}.
\end{equation}

Now, assuming that $\epsilon$ is sufficiently small so that $\epsilon \leq \pi^{-1/2} |\omega_j t^2 |^{1/2} t_\ast^2$, we put $t = t(\epsilon) = \pi^{1/4} |\omega_j t|^{-1/4} e^{1/2} = c \epsilon^{1/2}$. Then $t(\epsilon) \leq t_\ast$ and, by (30),
\begin{equation}
2 \left| \sin \left( \frac{1}{2} \tau \epsilon^{-2} (\lambda_j(t(\epsilon)) - \gamma_j t(\epsilon))^2 \right) \right| \geq \sqrt{2},
\end{equation}
whence (31) implies $\sqrt{2} c^2 (c^2 + \epsilon^2)^{-s/2} \leq \tilde{C} (\epsilon)$. It follows that the function $\epsilon^{s/2 - 1} (c^2 + \epsilon)^{-s/2}$ is uniformly bounded for small $\epsilon$. But this is not true if $s < 2$. This contradiction completes the proof.

\begin{equation}
\boxed{3.3. \textbf{Sharpness of the results with respect to time}}
\end{equation}

Now, we prove the following statement confirming the sharpness of Theorem 3.4 with respect to dependence of the estimate on time.

\begin{theorem}
Suppose that $N_0 \neq 0$. Let $s \geq 3$. Then there does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and estimate (25) holds for all $\tau \in \mathbb{R}$ and all sufficiently small $|t|$ and $\epsilon > 0$.
\end{theorem}

\begin{proof}
We prove by contradiction. Suppose that there exists a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and (25) is valid for all sufficiently small $|t|$ and $\epsilon$. By (26)–(28), this assumption is equivalent to the existence of a positive function $\tilde{C}(\tau)$ such that $\lim_{\tau \to \infty} \tilde{C}(\tau)/|\tau| = 0$ and estimate (29) holds for all sufficiently small $|t|$ and $\epsilon$.

The condition $N_0 \neq 0$ means that $\mu_j \neq 0$ at least for one $j$. Then $\lambda_j(t) = \gamma_j t^2 + \mu_j t^3 + O(t^4)$. Assume that $t_\ast$ is sufficiently small so that
\begin{equation}
\frac{1}{2} |\mu_j| t^3 \leq |\lambda_j(t) - \gamma_j t^2| \leq \frac{3}{2} |\mu_j| t^3, \quad |t| \leq t_\ast.
\end{equation}

Applying the operator under the norm sign in (29) to $\omega_j$, we obtain
\begin{equation}
2 \left| \sin \left( \frac{1}{2} \tau \epsilon^{-2} (\lambda_j(t) - \gamma_j t^2) \right) \right| \epsilon^{s} (t^2 + \epsilon^2)^{-s/2} \leq \tilde{C} (\epsilon)
\end{equation}
for all sufficiently small $|t|$ and $\epsilon$.
Let \( \tau \neq 0 \), and let \( \varepsilon \leq \varepsilon_0 |\tau|^{1/2} \), where \( \varepsilon_0 = (2\pi)^{-1/2} |\mu_j|^{1/2} t_\nu^3/2 \). We put

\[
t_\epsilon = t_b(\varepsilon, \tau) = c_\epsilon |\tau|^{-1/3} \varepsilon^{2/3}, \quad c_\epsilon = \left( \frac{\pi}{4} \right)^{1/3} |\mu_j|^{-1/3}.
\]

(34)

Then \( t_\epsilon \leq t_\nu/2 \) and, by (32), \( |\frac{t_\epsilon}{2\pi} (\lambda_j(t_\epsilon) - \gamma_j t_\nu^2) | \leq \frac{3\pi}{16} < \frac{\pi}{4} \). Applying the estimate \( |\sin y| \geq \frac{2}{\pi} |y| \) for \( |y| \leq \pi/2 \) and using the lower estimate (32), we obtain

\[
\left| \sin \left( \frac{1}{2} t_\epsilon e^{-2(\lambda_j(t_\epsilon) - \gamma_j t_\nu^2)} \right) \right| \geq \frac{|\tau|}{\pi \varepsilon^2} |\lambda_j(t_\epsilon) - \gamma_j t_\nu^2| \geq \frac{|\tau||\mu_j|}{2\pi \varepsilon^2} t_\nu^3 = \frac{1}{8}.
\]

Together with (33), this yields \( \frac{1}{4} \varepsilon^2 (t_\nu^2 + \varepsilon^2)^{-s/2} \leq \tilde{C}(\tau) \varepsilon \) for all sufficiently small \( \varepsilon \). By (34), this implies

\[
\frac{1}{4} \left( \frac{(\varepsilon|\tau|)^{s/3-1}}{c_\epsilon^2 + (\varepsilon|\tau|)^2/3} \right) \leq \frac{\tilde{C}(\tau)}{|\tau|}
\]

(35)

for all sufficiently small \( \varepsilon > 0 \). But estimate (35) is not true for large \( |\tau| \) and \( \varepsilon = |\tau|^{-1} \) since \( \lim_{\tau \to \infty} \tilde{C}(\tau)/|\tau| = 0 \). This contradiction completes the proof.

The following statement confirms the sharpness of Theorems 3.5 and 3.6.

**Theorem 3.10:** Suppose that \( N_0 = 0 \) and \( N^{(q)} \neq 0 \) for some \( q \in \{1, \ldots, p\} \). Let \( s \geq 2 \). Then there does not exist a positive function \( C(\tau) \) such that \( \lim_{\tau \to -\infty} C(\tau)/|\tau|^{1/2} = 0 \) and estimate (25) holds for all \( \tau \in \mathbb{R} \) and all sufficiently small \( |t| \) and \( \varepsilon > 0 \).

**Proof:** By Remark 3, the conditions \( N_0 = 0 \) and \( N^{(q)} \neq 0 \) for some \( q \in \{1, \ldots, p\} \) mean that \( \mu_j = 0 \) for all \( l = 1, \ldots, n \) and \( \nu_j \neq 0 \) at least for one \( j \). Then for sufficiently small \( t_\nu \) relations (30) hold.

We prove by contradiction. Suppose the opposite. Then, similarly to the proof of Theorem 3.9, we conclude that there exists a positive function \( \tilde{C}(\tau) \) such that \( \lim_{\tau \to -\infty} \tilde{C}(\tau)/|\tau|^{1/2} = 0 \) and (33) holds for all sufficiently small \( |t| \) and \( \varepsilon \).

Let \( \tau \neq 0 \), and let \( \varepsilon \leq \varepsilon_1 |\tau|^{1/2} \), where \( \varepsilon_1 = \frac{1}{2} \pi^{-1/2} |\nu_j|^{1/2} t_\nu^2 \). We put

\[
t_1 = t_1(\varepsilon, \tau) = c_1 |\tau|^{-1/4} \varepsilon^{1/2}, \quad c_1 = \left( \frac{\pi}{4} \right)^{1/4} |\nu_j|^{-1/4}.
\]

(36)

Then \( t_1 \leq t_\nu/2 \) and, by (30), \( |\frac{t_1}{2\pi} (\lambda_j(t_1) - \gamma_j t_\nu^2) | \leq \frac{3\pi}{16} < \frac{\pi}{4} \). Applying the estimate \( |\sin y| \geq \frac{2}{\pi} |y| \) for \( |y| \leq \pi/2 \) and using the lower estimate (30), we obtain

\[
\left| \sin \left( \frac{1}{2} t_1 e^{-2(\lambda_j(t_1) - \gamma_j t_\nu^2)} \right) \right| \geq \frac{|\tau|}{\pi \varepsilon^2} |\lambda_j(t_1) - \gamma_j t_\nu^2| \geq \frac{|\tau||\nu_j|}{2\pi \varepsilon^2} t_\nu^4 = \frac{1}{8}.
\]

Combining this with (33), we have \( \frac{1}{4} \varepsilon^2 (t_\nu^2 + \varepsilon^2)^{-s/2} \leq \tilde{C}(\tau) \varepsilon \) for all sufficiently small \( \varepsilon \). By (36), this is equivalent to

\[
\frac{1}{4} \left( \frac{(\varepsilon|\tau|)^{-s/3-1}}{c_1^2 + (\varepsilon|\tau|)^2/3} \right) \leq \frac{\tilde{C}(\tau)}{|\tau|^{1/2}}
\]

(37)

for all sufficiently small \( \varepsilon > 0 \). But estimate (37) is not true for large \( |\tau| \) and \( \varepsilon = |\tau|^{-1/2} \) since \( \lim_{\tau \to -\infty} \tilde{C}(\tau)/|\tau|^{1/2} = 0 \). This contradiction completes the proof.
4. Approximation of the sandwiched operator exponential

4.1. The operator family $A(t) = M^*\hat{A}(t)M$

Let $\hat{H}$ be yet another separable Hilbert space. Let $X(t) = \hat{X}_0 + t\hat{X}_1 : \hat{H} \to \hat{H}$ be a family of operators of the same form as $X(t)$, and suppose that $X(t)$ satisfies the assumptions of Section 2.1. Let $M : \hat{H} \to \hat{H}$ be an isomorphism. Suppose that $MDomX_0 = Dom\hat{X}_0$, $X(t) = \hat{X}(t)M$, and then also $X_0 = \hat{X}_0M$, $X_1 = \hat{X}_1M$. In $\hat{H}$, we consider the family of self-adjoint operators $\hat{A}(t) = \hat{X}(t)^*\hat{X}(t)$. Then, obviously,

$$A(t) = M^*\hat{A}(t)M. \quad (38)$$

In what follows, all the objects corresponding to the family $\hat{A}(t)$ are marked by the sign “$\hat{}$”. Note that $\hat{\mathfrak{F}} = M\mathfrak{F}$ and $\hat{\mathfrak{F}}_s = \mathfrak{F}_s$.

In $\hat{H}$ we consider the positive definite operator $Q := (MM^*)^{-1}$. Let $Q_{\hat{\mathfrak{F}}}$ be the block of $Q$ in the subspace $\hat{\mathfrak{F}}$, i.e. $Q_{\hat{\mathfrak{F}}} = \hat{P}Q|_{\hat{\mathfrak{F}}}$. Obviously, $Q_{\hat{\mathfrak{F}}}$ is an isomorphism in $\hat{\mathfrak{F}}$.

Condition 2.4 implies that for $\hat{A}(t)$ we have $\hat{A}(t) \geq \tilde{c}_e t^2 I$, $\tilde{c}_e = c_e\|M\|^{-2}$, $|t| \leq t^0$.

According to [16, Proposition 1.2], the orthogonal projection $P$ of $\hat{\mathfrak{F}}$ onto $\mathfrak{F}$ and the orthogonal projection $\hat{P}$ of $\hat{\mathfrak{F}}$ onto $\hat{\mathfrak{F}}$ satisfy the following relation:

$$P = M^{-1}(Q_{\hat{\mathfrak{F}}})^{-1}\hat{P}(M^{*})^{-1}. \quad (39)$$

Let $\hat{S} : \hat{\mathfrak{F}} \to \mathfrak{F}$ be the spectral germ of $\hat{A}(t)$ at $t = 0$, and let $S$ be the germ of $A(t)$. The following identity was obtained in [9, Chapter 1, Section 1.5]:

$$S = PM^*\hat{S}M|_{\mathfrak{F}}. \quad (40)$$

4.2. The operators $\hat{Z}_Q$ and $\hat{N}_Q$

For the operator family $\hat{A}(t)$, we introduce the operator $\hat{Z}_Q$ acting in $\hat{H}$ and taking an element $\hat{u} \in \hat{H}$ to the solution $\hat{\psi}_Q$ of the problem $\hat{X}_0^*(\hat{X}_0\hat{\psi}_Q + \hat{X}_1\hat{\omega}) = 0$, $Q\hat{\psi}_Q \perp \hat{\mathfrak{F}}$, where $\hat{\omega} = \hat{P}\hat{u}$. According to [10, §6], the operator $Z$ for $A(t)$ and the operator $\hat{Z}_Q$ introduced above satisfy

$$\hat{Z}_Q = MZM^{-1}\hat{P}. \quad (41)$$

Next, we put $\hat{N}_Q := \hat{Z}_Q^*\hat{X}_1\hat{P} + (\hat{R}\hat{P})^*\hat{X}_1\hat{Z}_Q$. According to [10, §6], the operator $N$ for $A(t)$ and the operator $\hat{N}_Q$ satisfy

$$\hat{N}_Q = \hat{P}(M^{*})^{-1}NM^{-1}\hat{P}. \quad (42)$$

Since $N = N_0 + N_s$, we have $\hat{N}_Q = \hat{N}_{0,Q} + \hat{N}_{s,Q}$, where

$$\hat{N}_{0,Q} = \hat{P}(M^{*})^{-1}N_0M^{-1}\hat{P}, \quad \hat{N}_{s,Q} = \hat{P}(M^{*})^{-1}N_sM^{-1}\hat{P}. \quad (43)$$

The following lemma was proved in [26, Lemma 5.1].

**Lemma 4.1 ([26]):** The relation $N = 0$ is equivalent to the relation $\hat{N}_Q = 0$. The relation $N_0 = 0$ is equivalent to the relation $\hat{N}_{0,Q} = 0$.

4.3. The operators $\hat{Z}_{2,Q}$, $\hat{R}_{2,Q}$, and $\hat{N}_{1,Q}$

Let $\hat{u} \in \hat{H}$ and let $\hat{\phi}_Q = \hat{\phi}_Q(\hat{u}) \in Dom\hat{X}_0$ be a (weak) solution of the equation

$$\hat{X}_0^*(\hat{X}_0\hat{\phi}_Q + \hat{X}_1\hat{Z}_Q\hat{\omega}) = -\hat{X}_1^*\hat{R}\hat{\omega} + Q(Q_{\hat{\mathfrak{F}}})^{-1}\hat{P}\hat{X}_1^*\hat{R}\hat{\omega}, \quad Q\hat{\phi}_Q \perp \hat{\mathfrak{F}},$$

where $\hat{\omega} = \hat{P}\hat{u}$. Clearly, the right-hand side of this equation belongs to $\hat{\mathfrak{F}}^\perp = Ran\hat{X}_0^*$, thereby the solvability condition is satisfied. We define an operator $\hat{Z}_{2,Q} : \hat{H} \to \hat{H}$ by the formula $\hat{Z}_{2,Q}\hat{u} = \hat{\phi}_Q(\hat{u})$. 
Now, we introduce an operator \( \hat{R}_{2,Q} : \hat{\mathcal{M}} \to \mathfrak{H}_s \) by \( \hat{R}_{2,Q} = \hat{X}_0 \hat{Z}_{2,Q} + \hat{X}_1 \hat{Z}_Q \). Finally, we define the operator \( \hat{N}^{0}_{1,Q} \): \[ \hat{N}^{0}_{1,Q} = \hat{Z}_2^{*} \hat{X}_1^{*} \hat{R} \hat{P} + (\hat{R} \hat{P})^{*} \hat{X}_1 \hat{Z}_{2,Q} + \hat{R}_{2,Q} \hat{R}_{2,Q}^{*} \hat{P}. \]

According to [13, Section 6.3], \( \hat{Z}_{2,Q} = MZ_2 M^{-1} \hat{P}, R_2 = \hat{R}_{2,Q} M|_{\hat{\mathcal{M}}}, \hat{R}_{2,Q} = R_2 M^{-1}|_{\hat{\mathcal{M}}} \), \[ \hat{N}^{0}_{1,Q} = \hat{P}(M^{*})^{-1} N^{0}_{1} M^{-1} \hat{P}. \] (44)

### 4.4. Relations between the operators and the coefficients of the power series expansions

Now, we describe the relations between the coefficients of the power series expansions (7) and (8) and the operators \( \hat{S} \) and \( Q_{\hat{\mathcal{M}}} \). (See [10, Sections 1.6, 1.7].) Denote \( \zeta_i := M\omega_i \in \hat{\mathcal{M}}, l = 1, \ldots, n \). Then relations (9) and (39), (40) show that \[ \hat{S}\zeta_i = \gamma \hat{Q}_{\hat{\mathcal{M}}} \zeta_i, \quad l = 1, \ldots, n. \] (45)

The set \( \zeta_1, \ldots, \zeta_n \) forms a basis in \( \hat{\mathcal{M}} \) orthonormal with the weight \( Q_{\hat{\mathcal{M}}} : (Q_{\hat{\mathcal{M}}} \zeta_i, \zeta_j) = \delta_{ij}, l, j = 1, \ldots, n. \)

The operators \( \hat{N}_{0,Q} \) and \( \hat{N}_{s,Q} \) can be described in terms of the coefficients of the expansions (7) and (8); cf. (12). We put \( \zeta_i := M\omega_i \in \hat{\mathcal{M}}, l = 1, \ldots, n \). Then

\[ \hat{N}_{0,Q} = \sum_{k=1}^{n} \mu_k (\cdot, Q_{\hat{\mathcal{M}}} \zeta_k) Q_{\hat{\mathcal{M}}} \zeta_k, \quad \hat{N}_{s,Q} = \sum_{k=1}^{n} \gamma_k (\cdot, Q_{\hat{\mathcal{M}}} \zeta_k) Q_{\hat{\mathcal{M}}} \zeta_k + (\cdot, Q_{\hat{\mathcal{M}}} \zeta_k) Q_{\hat{\mathcal{M}}} \zeta_k. \] (46)

Now, we return to the notation of Section 2.6. Recall that the different eigenvalues of the germ \( S \) are denoted by \( \gamma^j, j = 1, \ldots, p \), and the corresponding eigenspaces by \( \mathfrak{N}_j \). The set of the vectors \( \omega_i, l = i, \ldots, i + k_j - 1 \), where \( i = i(j) = k_1 + \ldots + k_{j-1} + 1 \), forms an orthonormal basis in \( \mathfrak{N}_j \). Then the same numbers \( \gamma^j, j = 1, \ldots, p \), are the different eigenvalues of the problem (45) and \( M\mathfrak{N}_j =: \hat{\mathfrak{N}}_{j,Q} \) are the corresponding eigenspaces. The vectors \( \zeta_i = M\omega_i, l = i, \ldots, i + k_j - 1 \), form a basis in \( \hat{\mathfrak{N}}_{j,Q} \) (orthonormal with the weight \( Q_{\hat{\mathcal{M}}} \)). By \( \mathcal{P}_j \) we denote the “skew” projection of \( \hat{\mathfrak{N}} \) onto \( \hat{\mathfrak{N}}_{j,Q} \) that is orthogonal with respect to the inner product \( (Q_{\hat{\mathcal{M}}} \cdot, \cdot) \), i.e. \( \mathcal{P}_j = \sum_{l=1}^{i+k_j-1} (\cdot, Q_{\hat{\mathcal{M}}} \zeta_l) \zeta_l \). It is easily seen that \( \mathcal{P}_j = MP_j M^{-1} \hat{P} \). Using (14), (42), and (43), it is easy to check that

\[ \hat{N}_{0,Q} = \sum_{j=1}^{p} \mathcal{P}_j^{*} \hat{N}_Q \mathcal{P}_j, \quad \hat{N}_{s,Q} = \sum_{1 \leq l \leq \mathcal{P}_j, j \neq l} \mathcal{P}_l^{*} \hat{N}_Q \mathcal{P}_j. \] (47)

Next, we find a relationship between the eigenvalues and eigenvectors of problem (17) and the operator \( \hat{N}_Q \). Let \( \gamma^q \) be the \( q \)th eigenvalue of problem (45) of multiplicity \( k_q \). Then from (17) and (42) and the obvious identity \( MP_q = \hat{P}_{q,Q} \mathcal{P}_q \), where \( \hat{P}_{q,Q} \) is the orthogonal projection of \( \hat{\mathfrak{N}} \) onto \( \hat{\mathfrak{N}}_{q,Q} \), it is seen that

\[ \hat{P}_{q,Q} \hat{N}_Q \zeta_i = \mu_i Q_{\hat{\mathcal{M}}_{q,Q}} \zeta_i, \quad l = i(q), \ldots, i(q) + k_q - 1, \] (48)

where \( Q_{\hat{\mathcal{M}}_{q,Q}} = \hat{P}_{q,Q} \mathfrak{M}_{q,Q} \). Recall that the different eigenvalues of problem (17) are denoted by \( \mu^j_q, q' = 1, \ldots, p'(q), \) and the corresponding eigenspaces by \( \mathfrak{N}_{q',q} \). Then the same numbers \( \mu^j_{q',q} q' = 1, \ldots, p'(q), \) are different eigenvalues of problem (48), and \( M\mathfrak{N}_{q',q} =: \hat{\mathfrak{N}}_{q',q,Q} \) are the corresponding eigenspaces.
Finally, we connect the eigenvalues and eigenvectors of problem (19) and the operator
\[ N^{(q', q)}_Q = \tilde{P}_{q', q} Q (N_{1, Q}^0 - Y_Q - Y_Q^*) \mid \hat{N}_{q', q}^0 + N_{0, Q}^{(q', q)}, \]
where \( Y_Q := \frac{1}{\epsilon} \hat{Z}_Q Q \hat{Z}_Q Q^{-1} \hat{S} \) and \( N^{(q', q)}_{0, Q} \) is the operator in \( \hat{N}_{q', q}^0 Q \) generated by the form
\[ \hat{N}^{(q', q)}_{0, Q} \{ \cdot, \cdot \} = \sum_{j \in \{1, \ldots, p\}, \, j \neq q} \frac{(\tilde{P}_{j, Q} (MM^*) \tilde{P}_{j, Q} \tilde{N}_{Q}, \tilde{N}_{Q}^*)}{\gamma^2_j - \gamma^2_j}, \]
and \( \tilde{P}_{q', q} \) is the orthogonal projection onto \( \hat{N}_{q', q}^0 Q \). From (19), (40)–(42), (44), and the identities \( MP_j = \tilde{P}_{j, Q} M P_j, \tilde{P}_{j, Q} M (I - P_j) = 0, \) \( j = 1, \ldots, p, MP_{q', q} = \tilde{P}_{q', q} Q M P_{q', q} \), it is seen that
\[ \hat{N}^{(q', q)}_{Q} \zeta_l = \nu_l Q \hat{N}_{q', q}^0 \zeta_l, \quad l = i', \ldots, i' + k_{q', q} - 1. \] (49)
Here \( i' = i' \( q', q \) = i(q) + k_{1, q} + \cdots + k_{q' - 1, q} \) and \( Q \hat{N}_{q', q}^0 \equiv \tilde{P}_{q', q} Q | \hat{N}_{q', q}^0 \).

Remark 5: Let \( \hat{N}_{0, Q} = 0 \). By (46), this condition is equivalent to the relations \( \mu_l = 0 \) for all \( l = 1, \ldots, n \). In this case, we have \( \Theta_{l, q} = \Theta_q, q = 1, \ldots, p \). Then we shall write \( N^{(q, q)}_Q \) instead of \( \hat{N}_{0, Q}^{(q, q)} \).

Remark 6: Let \( N^{(q', q)}_Q \neq 0 \) for some \( q \) and \( q' \). Then, by (49), \( \nu_l \neq 0 \) for some \( l = i' \( q', q \), \ldots, i' \( q', q \) + k_{q', q} - 1 \). From (19) it directly follows that \( N^{(q', q)}_Q \neq 0 \).

4.5. **Approximation of the sandwiched operator exponential**

In this section, we find approximation for the operator exponential \( e^{-ir \infty^{-2}A(t)} \) of the family (38) in terms of the germ \( \hat{S} \) of \( \hat{A}(t) \) and the isomorphism \( M \). It is convenient to border the exponential by appropriate factors.

Denote \( M_0 := (Q \hat{S})^{-1/2} \). The following estimates were proved in [26, Lemma 5.3]:
\[
\|Me^{-irA(t)}M^{-1}P - M_0 e^{-ir t^2 M_0 \hat{S} M_0 M_0^{-1}P}\| \leq \|M\|^2 \|M^{-1}\|^2 \|e^{-ir A(t)}P - e^{-ir t^2 SP}P\|, \]
\[
\|e^{-ir A(t)}P - e^{-ir t^2 SP}P\| \leq \|M\|^2 \|M^{-1}\|^2 \|Me^{-ir A(t)}M^{-1}P - M_0 e^{-ir t^2 SP}M_0 M_0^{-1}P\|. \]

Theorems 3.4–3.6, Lemma 4.1, and inequality (50) directly imply the following results.

**Theorem 4.2** ([22]): Under the assumptions of Section 4.1 for \( \tau \in \mathbb{R}, \epsilon > 0, \) and \( |t| \leq \tau^0 \) we have
\[
\|Me^{-ir \tau \epsilon^{-2} A(t)}M^{-1}P - M_0 e^{-ir \tau \epsilon^{-2} t^2 M_0 \hat{S} M_0 M_0^{-1}P}\| \epsilon^3 (t^2 + \epsilon^2)^{-3/2} \leq \|M\|^2 \|M^{-1}\|^2 (C_1 + C_2 |\tau|) \epsilon. \]

**Theorem 4.3:** Suppose that the assumptions of Section 4.1 are satisfied. Suppose that \( \hat{N}_Q = 0 \). Then for \( \tau \in \mathbb{R}, \epsilon > 0, \) and \( |t| \leq \tau^0 \) we have
\[
\|Me^{-ir \tau \epsilon^{-2} A(t)}M^{-1}P - M_0 e^{-ir \tau \epsilon^{-2} t^2 M_0 \hat{S} M_0 M_0^{-1}P}\| \epsilon^2 (t^2 + \epsilon^2)^{-1} \leq \|M\|^2 \|M^{-1}\|^2 (C_1 + C_4 |\tau|^{1/2}) \epsilon. \]
Theorem 4.4: Suppose that the assumptions of Section 4.1 and Condition 2.4 are satisfied. Suppose that \( \hat{\mathcal{N}}_{0,Q} = 0 \). Then for \( \tau \in \mathbb{R} \), \( \epsilon > 0 \), and \( |t| \leq t^0 \) we have
\[
\|M e^{-i \tau \epsilon^2 A(t)} M^{-1} \hat{\mathcal{P}} - M_0 e^{-i \tau \epsilon^2 t^2 M_0 \hat{\mathcal{S}} M_0^{-1} \hat{\mathcal{P}}} \epsilon^2 (t^2 + \epsilon^2)^{-1} \leq \|M\| \|M^{-1}\| (C_5 + C_6 |\tau|^{1/2}) \epsilon.
\]

Theorem 4.2 was proved in [22, Theorem 3.2].

Remark 7: Theorems 4.3 and 4.4 improve the results of Theorems 5.8 and 5.9 from [26] with respect to \( \tau \).

4.6. The sharpness of the results

Theorems 3.7–3.10, Lemma 4.1, Remark 6, and inequality (51) directly imply the following statements.

Theorem 4.5 ([26]): Suppose that \( \hat{\mathcal{N}}_{0,Q} \neq 0 \). Let \( \tau \neq 0 \) and \( 0 \leq s < 3 \). Then there does not exist a constant \( C(\tau) > 0 \) such that estimate (52) holds for all sufficiently small \( |t| \) and \( \epsilon \).

Theorem 4.6: Let \( \hat{\mathcal{N}}_{0,Q} = 0 \) and \( \hat{\mathcal{N}}_{Q}^{(q)} \neq 0 \) for some \( q \in \{1, \ldots, p\} \). Let \( \tau \neq 0 \) and \( 0 \leq s < 2 \). Then there does not exist a constant \( C(\tau) > 0 \) such that estimate (52) holds for all sufficiently small \( |t| \) and \( \epsilon \).

Theorem 4.7: Suppose that \( \hat{\mathcal{N}}_{0,Q} = 0 \). Let \( s \geq 3 \). Then there does not exist a positive function \( C(\tau) \) such that \( \lim_{\tau \to \infty} C(\tau)/|\tau| = 0 \) and estimate (52) holds for all \( \tau \in \mathbb{R} \) and for all sufficiently small \( |t| \) and \( \epsilon > 0 \).

Theorem 4.8: Suppose that \( \hat{\mathcal{N}}_{0,Q} = 0 \) and \( \hat{\mathcal{N}}_{Q}^{(q)} \neq 0 \) for some \( q \in \{1, \ldots, p\} \). Let \( s \geq 2 \). Then there does not exist a positive function \( C(\tau) \) such that \( \lim_{\tau \to \infty} C(\tau)/|\tau|^{1/2} = 0 \) and estimate (52) holds for all \( \tau \in \mathbb{R} \) and for all sufficiently small \( |t| \) and \( \epsilon > 0 \).

Chapter 2. Periodic differential operators in \( L_2(\mathbb{R}^d; \mathbb{C}^n) \)

5. The class of periodic differential operators

5.1. Preliminaries: lattices and the Gelfand transformation

Let \( \Gamma \) be a lattice in \( \mathbb{R}^d \) generated by the basis \( \mathbf{a}_1, \ldots, \mathbf{a}_d \), i.e. \( \Gamma = \{ \mathbf{a} \in \mathbb{R}^d : \mathbf{a} = \sum_{j=1}^d n_j \mathbf{a}_j, n_j \in \mathbb{Z} \} \), and let \( \Omega \) be the elementary cell of this lattice: \( \Omega := \{ \mathbf{x} \in \mathbb{R}^d : \mathbf{x} = \sum_{j=1}^d \xi_j \mathbf{a}_j, 0 < \xi_j < 1 \} \). The basis \( \mathbf{b}_1, \ldots, \mathbf{b}_d \) dual to \( \mathbf{a}_1, \ldots, \mathbf{a}_d \) is defined by the relations \( \langle \mathbf{b}_i, \mathbf{a}_j \rangle = 2\pi \delta_{ij} \). This basis generates the lattice \( \Gamma \) dual to \( \Gamma \). Denote by \( \overline{\Omega} \) the central Brillouin zone of \( \Gamma \):
\[
\overline{\Omega} = \left\{ \mathbf{k} \in \mathbb{R}^d : |\mathbf{k}| < |\mathbf{k} - \mathbf{b}|, 0 \neq \mathbf{b} \in \Gamma \right\}.
\]

Denote \( |\Omega| = \text{meas } \Omega \), \( |\overline{\Omega}| = \text{meas } \overline{\Omega} \). Note that \( |\Omega| |\overline{\Omega}| = (2\pi)^d \). Let \( r_0 \) be the radius of the ball inscribed in \( \text{clos } \overline{\Omega} \). We have \( 2r_0 = \min |\mathbf{b}|, 0 \neq \mathbf{b} \in \Gamma \).
With the lattice \( \Gamma \), we associate the discrete Fourier transformation \( \{ \hat{u}_b \} \rightarrow u: u(x) = |\Omega|^{-1/2} \sum_{b \in \Gamma} \hat{u}_b e^{i(b \cdot x)} \), which is a unitary mapping of \( L_2(\Gamma; \mathbb{C}^n) \) onto \( L_2(\Omega; \mathbb{C}^n) \). By \( \tilde{H}_0^\sigma(\Omega; \mathbb{C}^n) \) we denote the subspace of functions from \( H^\sigma(\Omega; \mathbb{C}^n) \) whose \( \Gamma \)-periodic extension to \( \mathbb{R}^d \) belongs to \( H^\sigma_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^n) \). We have

\[
\int_\Omega |(D + k)u|^2 \, dx = \sum_{b \in \Gamma} |b + k|^2 |\hat{u}_b|^2, \quad u \in \tilde{H}_1(\Omega; \mathbb{C}^n), \quad k \in \mathbb{R}^d,
\]

and convergence of the series in the right-hand side of (54) is equivalent to the inclusion \( u \in \tilde{H}_1(\Omega; \mathbb{C}^n) \). From (53) and (54) it follows that

\[
\int_\Omega |(D + k)u|^2 \, dx \geq \sum_{b \in \Gamma} |k|^2 |\hat{u}_b|^2 = |k|^2 \int_\Omega |u|^2 \, dx, \quad u \in \tilde{H}_1(\Omega; \mathbb{C}^n), \quad k \in \tilde{\Omega}.
\]

Initially, the Gelfand transformation \( \mathcal{V} \) is defined on the functions from the Schwartz class \( \psi \in \mathcal{S}(%(\mathbb{R}^d, \mathbb{C}^n)) \) by the formula

\[
\tilde{\psi}(k, x) = (\mathcal{V} \psi)(k, x) = |\tilde{\Omega}|^{-1/2} \sum_{a \in \Gamma} e^{-i(k \cdot x + a)} \psi(x + a), \quad x \in \Omega, \quad k \in \tilde{\Omega},
\]

and extends by continuity up to a unitary mapping: \( \mathcal{V}: L_2(\mathbb{R}^d; \mathbb{C}^n) \rightarrow \int_{\tilde{\Omega}} \oplus L_2(\Omega; \mathbb{C}^n) \, dk =: \mathcal{K} \).

### 5.2. Factorized second order operators \( \mathcal{A} \)

Let \( b(D) = \sum_{l=1}^d b_l D_l \), where \( b_l \) are constant \((m \times n)\)-matrices (in general, with complex entries). It is assumed that \( m \geq n \). Consider the symbol \( b(\xi) = \sum_{l=1}^d b_l \xi, \xi \in \mathbb{R}^d \). Suppose that \( \text{rank} b(\xi) = n \), \( 0 \neq \xi \in \mathbb{R}^d \). This condition is equivalent to the inequalities

\[
\alpha_0 1_n \leq b(\theta)^* b(\theta) \leq \alpha_1 1_n, \quad \theta \in \mathbb{R}^{d-1}, \quad 0 < \alpha_0 \leq \alpha_1 < \infty,
\]

with some positive constants \( \alpha_0, \alpha_1 \). Let \( f(x) \) be a \( \Gamma \)-periodic \((n \times n)\)-matrix-valued function and \( h(x) \) be a \( \Gamma \)-periodic \((m \times m)\)-matrix-valued function such that

\[
f, f^{-1} \in L_\infty(\mathbb{R}^d); \quad h, h^{-1} \in L_\infty(\mathbb{R}^d).
\]

Consider the closed operator \( \mathcal{X}: L_2(\mathbb{R}^d; \mathbb{C}^n) \rightarrow L_2(\mathbb{R}^d; \mathbb{C}^m) \) given by \( \mathcal{X} = bh(D)f \) on the domain \( \text{Dom} \mathcal{X} = \{ u \in L_2(\mathbb{R}^d; \mathbb{C}^m) : f u \in H^1(\mathbb{R}^d; \mathbb{C}^m) \} \). The self-adjoint operator \( \mathcal{A} = \mathcal{X}^* \mathcal{X} \) in \( L_2(\mathbb{R}^d; \mathbb{C}^n) \) is generated by the closed quadratic form \( a(u, u) = \| \mathcal{X} u \|^2_{L_2(\mathbb{R}^d)} \), \( u \in \text{Dom} \mathcal{X} \). Formally, we have

\[
\mathcal{A} = f(x)^* b(D)^* g(x) b(D) f(x),
\]

where \( g(x) := h(x)^* h(x) \). Using the Fourier transformation, and (56) and (57), it is easily seen that

\[
\alpha_0 \| g^{-1} \|_{L_\infty}^1 \| D(f(u)) \|^2_{L_2(\mathbb{R}^d)} \leq a[u, u] \leq \alpha_1 \| g \|_{L_\infty} \| D(f(u)) \|^2_{L_2(\mathbb{R}^d)}, \quad u \in \text{Dom} \mathcal{X}.
\]

### 5.3. The operators \( \mathcal{A}(k) \)

We put

\[
\mathcal{H} = L_2(\Omega; \mathbb{C}^n), \quad \mathcal{H}_* = L_2(\Omega; \mathbb{C}^m)
\]

and consider the closed operator \( \mathcal{X}(k): \mathcal{H} \rightarrow \mathcal{H}_* \) depending on the parameter \( k \in \mathbb{R}^d \) and given by \( \mathcal{X}(k) = bh(D + k)f \) on the domain \( \text{Dom} \mathcal{X}(k) = \{ u \in \mathcal{H} : f u \in \tilde{H}_1(\Omega; \mathbb{C}^m) \} =: \mathcal{D} \). The self-adjoint operator \( \mathcal{A}(k) = \mathcal{X}(k)^* \mathcal{X}(k): \mathcal{H} \rightarrow \mathcal{H} \) is generated by the quadratic form \( a(k)[u, u] = \| \mathcal{X}(k) u \|^2_{\mathcal{H}_*} \),
From (61) it follows that
\[ m = \alpha_0 \| g^{-1} \|_{L^\infty}^2 \| (D + k)f \|_{L^2(\Omega)}^2 \leq a(k)[u, u] \leq \alpha_1 \| g \|_{L^\infty} \| (D + k)f \|_{L^2(\Omega)}^2, \quad u \in \mathcal{D}. \] (60)

From (55) and the lower estimate (60) it follows that
\[ a(k) \geq c_s |k|^2 I, \quad k \in \tilde{\Omega}, \quad c_s = \alpha_0 \| f^{-1} \|_{L^\infty}^2 \| g^{-1} \|_{L^\infty}^{-1}. \] (61)

We put \( \mathfrak{N} := \text{Ker} \mathcal{A}(0) = \text{Ker} \lambda(0) \). Relations (60) with \( k = 0 \) show that
\[ \mathfrak{N} = \{ u \in L_2(\Omega; \mathbb{C}^n) : f u = c \in \mathbb{C}^n \}, \quad \dim \mathfrak{N} = n. \] (62)

Let \( E_j(k), \ j \in \mathbb{N}, \) be the consecutive eigenvalues of the operator \( \mathcal{A}(k) \) (the band functions): \( E_1(k) \leq E_2(k) \leq \cdots \leq E_j(k) \leq \cdots, \ k \in \mathbb{R}^d \). The band functions \( E_j(k) \) are continuous and \( \tilde{\Gamma} \)-periodic. From (61) it follows that \( E_j(k) \geq c_s |k|^2, \ k \in \text{clos} \tilde{\Omega}, \ j = 1, \ldots, n \). In [9, Chapter 2, Section 2.2], it was shown that \( E_{n+1}(k) \geq c_r r_0^2 \) and \( E_{n+1}(0) \geq 4c_r r_0^2 \).

### 5.4. The direct integral for the operator \( \mathcal{A} \)

Under the Gelfand transformation \( \mathcal{U} \), the operator \( \mathcal{A} \) expands in the direct integral of the operators \( \mathcal{A}(k) \):
\[ \mathcal{U} \mathcal{A} \mathcal{U}^{-1} = \int_{\Omega} \oplus \mathcal{A}(k) \, dk. \] (63)

This means the following. If \( v \in \text{Dom} \lambda \), then \( \tilde{v}(k, \cdot) \in \mathcal{D} \) for a.e. \( k \in \tilde{\Omega} \) and
\[ a[v, v] = \int_{\Omega} a(k)[\tilde{v}(k, \cdot), \tilde{v}(k, \cdot)] \, dk. \] (64)

Conversely, if \( \tilde{v} \in \mathcal{K} \) satisfies \( \tilde{v}(k, \cdot) \in \mathcal{D} \) for a.e. \( k \in \tilde{\Omega} \) and the integral in (64) is finite, then \( v \in \text{Dom} \lambda \) and (64) is valid.

### 5.5. Incorporation of the operators \( \mathcal{A}(k) \) in the abstract scheme

For \( d > 1 \), the operators \( \mathcal{A}(k) \) depend on the multidimensional parameter \( k \). According to [9, Chapter 2], we introduce the one-dimensional parameter \( t = |k| \). We shall apply the method described in Chapter 1. Now, all constructions will depend on the additional parameter \( \theta = k / |k| \in \mathbb{S}^{d-1} \) and we need to make our estimates uniform with respect to \( \theta \). The spaces \( \mathcal{F} \) and \( \mathcal{F}_s \) are defined by (59). We put \( X(t) = X(t; \theta) := \mathcal{X}(t; \theta) \). Then \( X(t; \theta) = X_0 + tX_1(\theta) \), where \( X_0 = h(x)b(D)f(x) \), \( \text{Dom} X_0 = \mathcal{D} \), and \( X_1(\theta) \) is the bounded operator of multiplication by the matrix \( h(x)b(\theta)f(x) \). Next, we put \( A(t) = A(t; \theta) := \mathcal{A}(t; \theta) \). The kernel \( \mathcal{N} = \text{Ker} X_0 \) is described by (62). As was shown in [9, Chapter 2, §3], the distance \( d^0 \) from the point \( \lambda_0 = 0 \) to the rest of the spectrum of the operator \( \mathcal{A}(0) \) satisfies \( d^0 \geq 4c_r r_0^2 \). The condition \( n \leq n_s = \dim \text{Ker} X_0^s \) is also fulfilled. Moreover, either \( n_s = n \) (if \( m = n \)), or \( n_s = \infty \) (if \( m > n \)).
In Section 2.1, it was required to fix a number \( \delta \in (0, d^0/8) \). Since \( d^0 \geq 4c_\ast r_0^2 \), we choose
\[
\delta = \frac{1}{4}c_\ast r_0^2 = \frac{1}{4} \alpha_0 \|f^{-1}\|_{L^\infty}^{-1}\|g^{-1}\|_{L^\infty}^{-1}r_0^2.
\] (65)

Note that by (56) and (57), we have
\[
\|X_1(\theta)\| \leq \alpha_1^{1/2} \|h\|_{L^\infty} \|f\|_{L^\infty}, \quad \theta \in \mathbb{S}^{d-1}.
\] (66)

We put (see (6))
\[
l^0 = \delta^{1/2} \alpha_1^{-1/2}\|h\|_{L^\infty}^{-1}\|f\|_{L^\infty}^{-1} = \frac{r_0}{2} \alpha_0^{1/2} \alpha_1^{-1/2}(\|h\|_{L^\infty} \|h^{-1}\|_{L^\infty} \|f\|_{L^\infty} \|f^{-1}\|_{L^\infty})^{-1}.
\] (67)

Note that \( l^0 \leq r_0/2 \). Thus the ball \(|k| \leq l^0\) lies inside \( \tilde{\Omega} \). It is important that \( c_\ast, \delta, l^0 \) (see (61), (65) and (67)) do not depend on \( \theta \). By (61), Condition 2.4 is fulfilled. The germ \( S(\theta) \) of the operator \( A(t; \theta) \) is nondegenerate uniformly in \( \theta \); we have \( S(\theta) \geq c_\ast l\tilde{\Omega} \) (cf. (13)).

6. The effective characteristics of the operator \( \hat{A} = b(D)^*g(x)b(D) \)

6.1. The operator \( A(t; \theta) \) in the case where \( f = 1_n \)

The operator \( A(t; \theta) \) in the case where \( f = 1_n \) plays a special role. In this case, we agree to mark all the associated objects by hat “\( \hat{\cdot} \)”. Then for the operator
\[
\hat{A} = b(D)^*g(x)b(D)
\] (68)

the family \( \hat{A}(k) \) is denoted by \( \hat{\Lambda}(t; \theta) \). The kernel (62) takes the form
\[
\hat{\zeta} = \{ u \in L_2(\Omega; \mathbb{C}^n) : u = c \in \mathbb{C}^n \},
\] (69)

i.e. \( \hat{\zeta} \) consists of constant vector-valued functions. The orthogonal projection \( \hat{P} \) of the space \( L_2(\Omega; \mathbb{C}^n) \) onto the subspace (69) is the operator of averaging over the cell:
\[
\hat{P}u = |\Omega|^{-1} \int_\Omega u(x) \, dx.
\] (70)

According to [9, Chapter 3, §1], the spectral germ \( \hat{S}(\theta) : \hat{\zeta} \to \hat{\zeta} \) of the family \( \hat{\Lambda}(t; \theta) \) is represented as \( \hat{S}(\theta) = b(\theta)^*g^0b(\theta), \theta \in \mathbb{S}^{d-1} \), where \( g^0 \) is the so-called effective matrix. The constant \((m \times m)\)-matrix \( g^0 \) is defined as follows. Let \( \Lambda \in \hat{H}^1(\Omega) \) be a \( \Gamma \)-periodic \((n \times m)\)-matrix-valued function satisfying the equation
\[
b(D)^*g(x)(b(D)\Lambda(x) + 1_m) = 0, \quad \int_\Omega \Lambda(x) \, dx = 0.
\] (71)

The effective matrix \( g^0 \) can be described in terms of the matrix \( \Lambda(x) \):
\[
g^0 = |\Omega|^{-1} \int_\Omega \tilde{g}(x) \, dx,
\] (72)
\[
\tilde{g}(x) := g(x)(b(D)\Lambda(x) + 1_m).
\] (73)

It turns out that the matrix \( g^0 \) is positive definite. Consider the symbol
\[
\hat{S}(k) := t^{2}\hat{S}(\theta) = b(k)^*g^0b(k), \quad k \in \mathbb{R}^d.
\] (74)

Expression (74) is the symbol of the DO
\[
\hat{A}^0 = b(D)^*g^0b(D)
\] (75)

acting in \( L_2(\mathbb{R}^d; \mathbb{C}^n) \) and called the effective operator for the operator \( \hat{A} \).
Let \( \hat{\mathcal{A}}^0(k) \) be the operator family in \( L_2(\Omega; \mathbb{C}^n) \) corresponding to operator (75). Then \( \hat{\mathcal{A}}^0(k) \) is given by \( \hat{\mathcal{A}}^0(k) = b(D + k)^*g^0b(D + k) \) with periodic boundary conditions. Taking into account (70) and (74), we have

\[
\hat{S}(k)\hat{P} = \hat{\mathcal{A}}^0(k)\hat{P}.
\]

### 6.2. Properties of the effective matrix

The following properties of \( g^0 \) were checked in [9, Chapter 3, Theorem 1.5].

**Proposition 6.1** ([9]): The effective matrix satisfies the following estimates:

\[
g \leq g^0 \leq \bar{g},
\]

where \( \bar{g} := |\Omega|^{-1} \int_{\Omega} \bar{g}(x) \, dx \) and \( g := (|\Omega|^{-1} \int_{\Omega} g(x)^{-1} \, dx)^{-1} \). If \( m = n \), then \( g^0 = \bar{g} \).

For specific DOs, estimates (77) are known as the Voigt–Reuss bracketing. Now, we distinguish the cases where one of the inequalities in (77) becomes an identity. The following statements were obtained in [9, Chapter 3, Propositions 1.6, 1.7].

**Proposition 6.2** ([9]): The identity \( g^0 = \bar{g} \) is equivalent to the relations

\[
b(D)^*g_k(x) = 0, \quad k = 1, \ldots, m,
\]

where \( g_k(x) \), \( k = 1, \ldots, m \), are the columns of the matrix \( g(x) \).

**Proposition 6.3** ([9]): The identity \( g^0 = g \) is equivalent to the representations

\[
l_k(x) = l_k^0 + b(D)w_k(x), \quad l_k^0 \in \mathbb{C}^m, \quad w_k \in \hat{H}^1(\Omega; \mathbb{C}^n), \quad k = 1, \ldots, m,
\]

where \( l_k(x) \), \( k = 1, \ldots, m \), are the columns of the matrix \( g(x)^{-1} \).

### 7. The operator \( \mathcal{A}(k) \). Application of the scheme of Section 4

#### 7.1. The operator \( \mathcal{A}(k) \)

We apply the scheme of Section 3 to study the operator \( \mathcal{A}(k) = f^*\hat{\mathcal{A}}(k)f \). Now, \( \hat{S} = \hat{\mathcal{S}} = L_2(\Omega; \mathbb{C}^n) \), \( \hat{S}_n = L_2(\Omega; \mathbb{C}^m) \), the role of \( A(t) \) is played by \( A(t; \theta) = \mathcal{A}(k) \), the role of \( \hat{A}(t) \) is played by \( \hat{A}(t; \theta) = \hat{\mathcal{A}}(k) \). The isomorphism \( M \) is the operator of multiplication by the matrix-valued function \( f(x) \). The operator \( Q \) is the operator of multiplication by the matrix-valued function \( Q(x) = (f(x)f(x)^*)^{-1} \). The block of \( Q \) in the subspace \( \mathcal{S} \) (see (69)) is the operator of multiplication by the constant matrix \( Q = (f^*)^{-1} = |\Omega|^{-1} \int_{\Omega} f(x)f(x)^* \, dx \). Next, \( M_0 \) is the operator of multiplication by the constant matrix

\[
f_0 = (Q)^{-1/2} = (f^*)^{-1/2}.
\]

Note that \( |f_0| \leq \|f\|_{L_\infty} \), \( |f_0|^{-1} \leq \|f^{-1}\|_{L_\infty} \).

In \( L_2(\mathbb{R}^d; \mathbb{C}^n) \), we define the operator

\[
\mathcal{A}_n := f_0\mathcal{A}^0f_0 = f_0b(D)^*g^0b(D)f_0.
\]

Let \( \mathcal{A}_n(k) \) be the corresponding family of operators in \( L_2(\Omega; \mathbb{C}^n) \). Then \( \mathcal{A}_n(k) = f_0\mathcal{A}_n^0(k)f_0 \). By (69) and (76),

\[
f_0\hat{S}(k)f_0\hat{P} = \mathcal{A}_n^0(k)\hat{P}.
\]
7.2. The analytic branches of eigenvalues and eigenvectors

According to (40), the spectral germ $S(\theta)$ of the operator $A(t; \theta)$ acting in the subspace $\mathcal{H}$ (see (62)) is represented as $S(\theta) = P \phi b(\theta)^*g^0 b(\theta) f |_{\mathcal{H}}$, where $P$ is the orthogonal projection of $L_2(\Omega; C^n)$ onto $\mathcal{H}$.

The analytic (in $t$) branches of the eigenvalues $\lambda_i(t, \theta)$ and the eigenvectors $\varphi_i(t, \theta)$ of the operator $A(t; \theta)$ admit the power series expansions of the form (7) and (8) with the coefficients depending on $\theta$:

$$\lambda_i(t, \theta) = \gamma_i(\theta)t^2 + \mu_i(\theta)t^3 + v_i(\theta)t^4 + \cdots, \quad l = 1, \ldots, n, \quad (83)$$

$$\varphi_i(t, \theta) = \omega_i(\theta) + t\psi_i^{(1)}(\theta) + \cdots, \quad l = 1, \ldots, n. \quad (84)$$

The vectors $\omega_1(\theta), \ldots, \omega_n(\theta)$ form an orthonormal basis in the subspace $\mathcal{H}$, and the vectors $\xi_i(\theta) = f\omega_i(\theta), l = 1, \ldots, n,$ form a basis in $\mathcal{H}$ (see (69)) orthonormal with the weight $\mathcal{Q}$. The numbers $\gamma_i(\theta)$ and the elements $\omega_i(\theta)$ are eigenvalues and eigenvectors of the spectral germ $S(\theta)$. According to (45),

$$b(\theta)^*g^0 b(\theta)\xi_i(\theta) = \gamma_i(\theta)\mathcal{Q}\xi_i(\theta), \quad l = 1, \ldots, n. \quad (85)$$

7.3. The operator $\hat{N}_Q(\theta)$

We need to describe the operators $\hat{Z}_Q(\theta)$ and $\hat{N}_Q$ which now depend on $\theta$ (see Section 4.2). Let $\Lambda_Q(x)$ be a $\Gamma$-periodic solution of the problem

$$b(D)^*g(x)(b(D)\Lambda_Q(x) + 1_m) = 0, \quad \int_{\Omega} Q(x)\Lambda_Q(x) \, dx = 0.$$ 

Clearly, $\Lambda_Q(x) = \Lambda(x) - (\mathcal{Q})^{-1}(\mathcal{Q}\Lambda)$. As shown in [7, §5], the operators $\hat{Z}_Q(\theta)$ and $\hat{N}_Q(\theta)$ take the form $\hat{Z}_Q(\theta) = \Lambda_Q b(\theta)\hat{P}$,

$$\hat{N}_Q(\theta) = b(\theta)^*L_Q(\theta)b(\theta)\hat{P},$$

$$L_Q(\theta) := |\Omega|^{-1}\int_{\Omega} (\Lambda_Q(x)^*b(\theta)^*g(x) + \tilde{g}(x)^*b(\theta)\Lambda_Q(x)) \, dx. \quad (86)$$

Some sufficient conditions where $\hat{N}_Q(\theta) = 0$ were distinguished in [7, §§4,5].

Proposition 7.1 ([7]): Suppose that at least one of the following conditions is fulfilled:

1°. $A = f(x)^*D^*g(x)Df(x)$, where $g(x)$ is a symmetric matrix with real entries.
2°. Relations (78) are satisfied, i.e. $g^0 = \tilde{g}$.
3°. Relations (79) are satisfied, i.e. $g^0 = \tilde{g}$ (if $m = n$, this is the case) and $f = 1_n$.

Then $\hat{N}_Q(\theta) = 0$ for any $\theta \in S^{d-1}$.

Recall (see Section 4.2) that $\hat{N}_Q(\theta) = \hat{N}_{0,Q}(\theta) + \hat{N}_{s,Q}(\theta)$. By (46),

$$\hat{N}_{0,Q}(\theta) = \sum_{i=1}^n \mu_i(\theta)(\mathcal{Q}^0(\theta)\xi_i(\theta))_{L_2(\Omega)}(\tilde{\mathcal{Q}}\xi_i(\theta)).$$

We have $(\mathcal{N}_Q(\theta)\xi_i(\theta), \xi_j(\theta))_{L_2(\Omega)} = (\mathcal{N}_{0,Q}(\theta)\xi_i(\theta), \xi_j(\theta))_{L_2(\Omega)} = \mu_j(\theta), l = 1, \ldots, n.$

In [7, Proposition 5.2], the following proposition was proved.

Proposition 7.2 ([7]): Suppose that the matrices $b(\theta), g(x)$, and $Q(x)$ have real entries. Suppose that in the expansions (84) the “embryos” $\omega_i(\theta), l = 1, \ldots, n$, can be chosen so that the vectors $\xi_i(\theta) = f\omega_i(\theta)$ are real. Then in (83) we have $\mu_i(\theta) = 0, l = 1, \ldots, n$, i.e. $\hat{N}_{0,Q}(\theta) = 0$ for any $\theta \in S^{d-1}$. 
We put possible to choose the vectors $\hat{\zeta}(\theta)$ is orthogonal with respect to the inner product with the weight $Q$. We arrive at the following corollary.

**Corollary 7.3:** Suppose that the matrices $b(\theta), g(x),$ and $Q(x)$ have real entries and the spectrum of the generalized spectral problem (85) is simple. Then $\hat{N}_{0,Q}(\theta) = 0$ for any $\theta \in S^{d-1}$.

However, as is seen from [26, Example 8.7], [23, Section 14.3], in the “real” case it is not always possible to choose the vectors $\hat{\zeta}(\theta)$ to be real. It may happen that $\hat{N}_{0,Q}(\theta) \neq 0$ at some isolated points $\theta$.

### 7.4. The operators $\hat{Z}_{2,Q}(\theta), \hat{R}_{2,Q}(\theta), \hat{N}_{1,Q}^0(\theta)$

We need to describe the operators $\hat{Z}_{2,Q}, \hat{R}_{2,Q}, \hat{N}_{1,Q}^0$ (which in the abstract terms are defined in Section 4.3). Let $\Lambda_{Q,j}^{(2)}(x)$ be a $\Gamma$-periodic solution of the problem

$$b(D)^*g(x)(b(D)\Lambda_{Q,j}^{(2)}(x) + b_i\Lambda_Q(x)) = -b_i^*\hat{g}(x) + Q(x)(\hat{Q})^{-1}b_i^*g^0,$$

$$\int_{\Omega} Q(x)\Lambda_{Q,j}^{(2)}(x) \, dx = 0.$$

We put $\Lambda_{Q}^{(2)}(x; \theta) := \sum_{j = 1}^d \Lambda_{Q,j}^{(2)}(x)\theta_j$. In [14, Section 8.4], it was shown that

$$\hat{Z}_{2,Q}(\theta) = \Lambda_{Q}^{(2)}(x; \theta)b(\theta)\hat{P}, \quad \hat{R}_{2,Q}(\theta) = h(x)(b(D)\Lambda_{Q}^{(2)}(x; \theta) + b(\theta)\Lambda_Q(x))b(\theta).$$

Finally, in [14, Section 8.5] the following representation was obtained:

$$\hat{N}_{1,Q}^0(\theta) = b(\theta)^*L_{2,Q}(\theta)b(\theta)^*\hat{P},$$

$$L_{2,Q}(\theta) := |\Omega|^{-1} \int_{\Omega} (\Lambda_{Q}^{(2)}(x; \theta)^*b(\theta)^*\hat{g}(x) + \hat{g}(x)^*b(\theta)^*\Lambda_{Q}^{(2)}(x; \theta)) \, dx$$

$$+ |\Omega|^{-1} \int_{\Omega} (b(D)\Lambda_{Q}^{(2)}(x; \theta) + b(\theta)\Lambda_Q(x))^*g(x)(b(D)\Lambda_{Q}^{(2)}(x; \theta) + b(\theta)\Lambda_Q(x)) \, dx.$$

### 7.5. Multiplicities of the eigenvalues of the germ

Considerations of this section concern the case where $n \geq 2$. Now, we return to the notation of Section 2.6. In general, the number $p(\theta)$ of different eigenvalues $\gamma_1(\theta), \ldots, \gamma_{p(\theta)}(\theta)$ of $S(\theta)$ (or of problem (85)) and their multiplicities $k_1(\theta), \ldots, k_{p(\theta)}(\theta)$ depend on the parameter $\theta \in S^{d-1}$. For a fixed $\theta$ denote by $N_j(\theta)$ the eigenspace of the germ $S(\theta)$ corresponding to the eigenvalue $\gamma_j(\theta)$. Then $\hat{N}_j(\theta) = \hat{N}_j,Q(\theta)$ is the eigenspace of problem (85) corresponding to the same eigenvalue $\gamma_j(\theta)$. We introduce the notation $P_j(\theta)$ for the “skew” projection of $L_2(\Omega; \mathbb{C}^n)$ onto the subspace $\hat{N}_j,\hat{Q}(\theta); P_j(\theta)$ is orthogonal with respect to the inner product with the weight $\hat{Q}$. By (47),

$$\hat{N}_{0,Q}(\theta) = \sum_{j = 1}^{p(\theta)} P_j(\theta)^*\hat{N}_j(\theta)P_j(\theta), \quad \hat{N}_{s,Q}(\theta) = \sum_{1 \leq j \leq p(\theta); j \neq l} P_j(\theta)^*\hat{N}_j(\theta)P_l(\theta). \quad (87)$$
7.6. The coefficients $v_l(\theta), l = 1, \ldots, n$

According to (17), the numbers $\mu_l(\theta)$ and the elements $\omega_l(\theta), l = i(q, \theta), \ldots, i(q, \theta) + k_q(\theta) - 1$, are eigenvalues and eigenvectors of the operator $P_q(\theta)N(\theta)|_q(\theta)$. Then, by (48), we have

$$\tilde{P}_{q,Q}(\theta)\tilde{N}_Q(\theta)\xi_l(\theta) = \mu_l(\theta)\tilde{P}_{q,Q}(\theta)\tilde{N}_Q(\theta), \quad l = i(q, \theta), \ldots, i(q, \theta) + k_q(\theta) - 1,$$

where $\tilde{P}_{q,Q}(\theta)$ is the orthogonal projection onto $\tilde{N}_Q(\theta)$.

The number $p'(q, \theta)$ of different eigenvalues $\mu_{1,q}(\theta), \ldots, \mu_{p'(q,\theta),q}(\theta)$ of the operator $P_q(\theta)N(\theta)|_q(\theta)$ and their multiplicities $k_{1,q}(\theta), \ldots, k_{p'(q,\theta),q}(\theta)$ depend on the parameter $\theta \in S^{d-1}$. For a fixed $\theta$, we denote by $\tilde{N}_{q',q}(\theta)$ the eigenspace corresponding to the eigenvalue $\mu_{q',q}(\theta)$. Then $f\tilde{N}_{q',q}(\theta) = \tilde{N}_{q',q}(\theta)$ is the eigenspace of problem (88) corresponding to the same eigenvalue $\mu_{q',q}(\theta)$.

Finally, according to (49), the numbers $v_l(\theta)$ and the elements $\xi_l(\theta), l = i'(q', q, \theta), \ldots, i'(q', q, \theta) + k_{q',q}(\theta) - 1$, where $i'(q', q, \theta) = i(q, \theta) + k_{1,q}(\theta) + \cdots + k_{q'-1,q}(\theta)$, are the eigenvalues and the eigenvectors of the following generalized spectral problem:

$$\tilde{N}_Q^{(q',q)}(\theta)\xi_l(\theta) = v_l(\theta)\tilde{P}_{q',q,Q}(\theta)\tilde{N}_Q(\theta), \quad l = i'(q', q, \theta), \ldots, i'(q', q, \theta) + k_q(\theta) - 1,$$

where

$$\tilde{N}_Q^{(q',q)}(\theta) := \tilde{P}_{q',q,Q}(\theta) \left( \tilde{N}_Q(\theta) - Y_Q(\theta) - Y_Q(\theta)^* \right) \tilde{N}_Q(\theta),$$

and

$$Y_Q(\theta) := \frac{1}{2} \tilde{Z}_Q(\theta)\tilde{Q}_Q(\theta)\tilde{Q}_Q(\theta)^{-1}\tilde{Q}_Q(\theta)\tilde{P} + \tilde{P}_{q',q,Q}(\theta)$$

is the orthogonal projection onto $\tilde{N}_{q',q}(\theta)$.

Note that in the case where $\tilde{N}_Q(\theta) = 0$ we have $\tilde{N}_{q',q}(\theta) = \tilde{N}_Q(\theta), q = 1, \ldots, p(\theta)$. Then we shall write $\tilde{N}_Q^{(q)}(\theta)$ instead of $\tilde{N}_Q^{(1,q)}(\theta)$.

8. Approximations for the sandwiched operator $e^{-irr^{-2}A(k)}$

8.1. The general case

Consider the operator $\mathcal{H}_0 = -\Delta$ in $L_2(\mathbb{R}^d; \mathbb{C}^n)$. Under the Gelfand transformation, the operator $\mathcal{H}_0$ expands in the direct integral of the operators $\mathcal{H}_0(k)$ acting in $L_2(\Omega; \mathbb{C}^n)$ and given by $|D + k|^2$ with periodic boundary conditions. Denote

$$\mathcal{R}(k, \varepsilon) := \varepsilon^2(\mathcal{H}_0(k) + \varepsilon^2I)^{-1}.$$ 

Obviously,

$$\mathcal{R}(k, \varepsilon)^{s/2}\tilde{P} = \varepsilon^s(t^2 + \varepsilon^2)^{-s/2}\tilde{P}, \quad s > 0.$$ 

Note that for $|k| > t_0$ we have

$$\|\mathcal{R}(k, \varepsilon)^{s/2}\tilde{P}\|_{L_2(\Omega)\rightarrow L_2(\Omega)} \leq (t_0)^{-s}\varepsilon^s, \quad \varepsilon > 0, \ k \in \tilde{\Omega}, \ |k| > t_0.$$ 

Next, using the Fourier series decomposition, we see that

$$\|\mathcal{R}(k, \varepsilon)^{s/2}(I - \tilde{P})\|_{L_2(\Omega)\rightarrow L_2(\Omega)} = \sup_{0 \neq b \in \tilde{\Omega}} \varepsilon^s(|b + k|^2 + \varepsilon^2)^{-s/2} \leq r_0^{-s}\varepsilon^s, \quad \varepsilon > 0, \ k \in \tilde{\Omega}. $$
Denote

$$J(k, \varepsilon; \tau) := fe^{-ir\varepsilon^2 A(k)}f^{-1} - f_0 e^{-ir\varepsilon^2 A_0(k)}f_0^{-1}. \quad (93)$$

We shall apply theorems of Section 4.5 to the operator $A(t; \theta) = A(k)$. In doing so, we may trace the dependence of the constants in estimates on the problem data. Note that $\varepsilon, \delta$, and $t_0^0$ do not depend on $\theta$ (see (61), (65), (67)). According to (66) the norm $\|X_1(\theta)\|$ can be replaced by $\alpha^{1/2} \|g\|_{L_\infty}^{1/2}\|f\|_{L_\infty}$. Hence, the constants in Theorems 4.2 and 4.3 (applied to the operator $A(k)$) will be independent of $\theta$. They depend only on $\alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|f\|_{L_\infty}, \|f^{-1}\|_{L_\infty}$, and $r_0$.

Applying Theorem 4.2 and taking (82), (90)–(92) into account, we arrive at the following statement proved before in [22, Theorem 8.1].

**Theorem 8.1 ([22]):** For $\tau \in \mathbb{R}, \varepsilon > 0$, and $k \in \hat{\Omega}$, we have

$$\|J(k, \varepsilon; \tau)R(k, \varepsilon)^{3/2}\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_1(1 + |\tau|)\varepsilon,$$

where $C_1$ depends only on $\alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|f\|_{L_\infty}, \|f^{-1}\|_{L_\infty}$, and $r_0$.

### 8.2. Improvement of the general result

Now, we apply Theorem 4.3 assuming that $\hat{N}_Q(\theta) = 0$ for any $\theta \in \mathbb{S}^{d-1}$. Taking (82) and (90)–(92) into account, we obtain the following result.

**Theorem 8.2:** Let $\hat{N}_Q(\theta)$ be the operator defined by (86). Suppose that $\hat{N}_Q(\theta) = 0$ for any $\theta \in \mathbb{S}^{d-1}$. Then for $\tau \in \mathbb{R}, \varepsilon > 0$, and $k \in \hat{\Omega}$, we have

$$\|J(k, \varepsilon; \tau)R(k, \varepsilon)^{3/2}\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_2(1 + |\tau|^{1/2})\varepsilon,$$

where $C_2$ depends only on $\alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|f\|_{L_\infty}, \|f^{-1}\|_{L_\infty}$, and $r_0$.

Now, we reject the assumptions of Theorem 8.2, but we assume instead that $\hat{N}_{0,Q}(\theta) = 0$ for any $\theta$. We would like to apply Theorem 4.4. However, there is an additional difficulty: the multiplicities of the eigenvalues of the operator $S(\theta)$ may change at some points $\theta$ (they are also the eigenvalues of the generalized spectral problem (85)). Near such points the distance between some pair of different eigenvalues tends to zero and we are not able to choose the parameters $\varepsilon_j, t_{j0}^0$ to be independent on $\theta$. Therefore, we are forced to impose additional conditions. We have to take care only about those eigenvalues for which the corresponding term in the second formula in (87) is not zero. Now, it is more convenient to use the initial enumeration of the eigenvalues $\gamma_1(\theta), \ldots, \gamma_n(\theta)$ of $S(\theta)$ (each eigenvalue is repeated according to its multiplicity); we agree to enumerate them in the nondecreasing order: $\gamma_1(\theta) \leq \gamma_2(\theta) \leq \ldots \leq \gamma_n(\theta)$. For each $\theta$, let $P^{(k)}(\theta)$ be the “skew” projection (orthogonal with the weight $Q$) of $L_2(\Omega; \mathbb{C}^n)$ onto the eigenspace of problem (85) corresponding to the eigenvalue $\gamma_k(\theta)$. Clearly, for each $\theta$ the operator $P^{(k)}(\theta)$ coincides with one of the projections $P_j(\theta)$ introduced in Section 7.5 (but the number $j$ may depend on $\theta$).

**Condition 8.3:**

1°. $\hat{N}_{0,Q}(\theta) = 0$ for any $\theta \in \mathbb{S}^{d-1}$.

2°. For any pair of indices $(k, r), 1 \leq k, r \leq n, k \neq r$, such that $\gamma_k(\theta_0) = \gamma_r(\theta_0)$ for some $\theta_0 \in \mathbb{S}^{d-1}$, we have $(P^{(k)}(\theta))^* \hat{N}_Q(\theta)P^{(r)}(\theta) = 0$ for any $\theta \in \mathbb{S}^{d-1}$.

Condition 8.3(2°) can be reformulated: we assume that for the “blocks” $(P^{(k)}(\theta))^* \hat{N}_Q(\theta)P^{(r)}(\theta)$ of the operator $\hat{N}_Q(\theta)$ that are not identically zero, the corresponding branches of the eigenvalues $\gamma_k(\theta)$ and $\gamma_r(\theta)$ do not intersect.

Obviously, Condition 8.3 is ensured by the following more restrictive condition.
Remark 8: 1°. Assumption 8.4 of Condition 8.4 is a fortiori satisfied, if the spectrum of problem (85) is simple for any \( \theta \in S^{d-1} \).

2°. From Corollary 7.3 it follows that Condition 8.4 is satisfied if the matrices \( b(\theta), g(x), \) and \( Q(x) \) have real entries, and the spectrum of problem (85) is simple for any \( \theta \in S^{d-1} \).

So, suppose that Condition 8.3 is satisfied and put

\[ K := \{(k, r) : 1 \leq k, r \leq n, k \neq r, (\mathcal{P}^{(k)}(\theta))^* N_Q(\theta) \mathcal{P}^{(r)}(\theta) \neq 0\}. \]

Denote \( c_{kr}^0(\theta) := \min\{c_{kr}(\theta), n^{-1}|\gamma_k(\theta) - \gamma_r(\theta)|\}, (k, r) \in K \).

Since \( S(\theta) \) depends on \( \theta \in S^{d-1} \) continuously, then \( \gamma_j(\theta) \) are continuous on \( S^{d-1} \). By Condition 8.3(2°), for \( (k, r) \in K \) we have

\[ |\gamma_k(\theta) - \gamma_r(\theta)| > 0 \quad \text{for any} \quad \theta \in S^{d-1}, \]

whence \( c_{kr}^0 := \min_{\theta \in S^{d-1}} c_{kr}^0(\theta) > 0 \) for \( (k, r) \in K \). We put

\[ c^0 := \min_{(k, r) \in K} c_{kr}^0. \quad (94) \]

Clearly, the number (94) is a realization of (23) chosen independently of \( \theta \). Under Condition 8.3, the number \( t^{00} \) subject to (22) also can be chosen independently of \( \theta \in S^{d-1} \). Taking (65) and (66) into account, we put

\[ t^{00} = (8\beta_2)^{-1} r_0 \alpha_1^{-3/2} \alpha_0^{1/2} \|g\|_{L^{3/2}} \|g^{-1}\|_{L^{1/2}} \|f\|_{L^{3}} \|f^{-1}\|_{L^{\infty}} c^0. \]

(The condition \( t^{00} \leq t^0 \) is valid automatically, since \( c^0 \leq \|S(\theta)\| \leq \alpha_1 \|g\|_{L^\infty} \|f\|_{L^\infty}^2 \).

Applying Theorem 4.4, we deduce the following result.

Theorem 8.5: Suppose that Condition 8.3 (or more restrictive Condition 8.4) is satisfied. Then for \( \tau \in \mathbb{R}, \varepsilon > 0, \) and \( k \in \tilde{\Omega} \) we have

\[ \|J(k, \varepsilon; \tau) R(k, \varepsilon)\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq C_3 (1 + |\tau|^{1/2}) \varepsilon, \]

where the constant \( C_3 \) depends only on \( \alpha_0, \alpha_1, \|g\|_{L^\infty}, \|g^{-1}\|_{L^\infty}, \|f\|_{L^\infty}, \|f^{-1}\|_{L^{\infty}}, r_0, \) and also on \( n \) and the number \( c^0 \).

8.3. The sharpness of the results

Application of Theorems 4.5 and 4.6 allows us to confirm that the results of Theorems 8.1, 8.2 and 8.5 are sharp with respect to the smoothing operator.

Theorem 8.6 ([26]): Suppose that \( \tilde{N}_{0, Q}(\theta_0) \neq 0 \) for some \( \theta_0 \in S^{d-1} \). Let \( \tau \neq 0 \) and \( 0 \leq s < 3 \). Then there does not exist a constant \( C(\tau) > 0 \) such that the estimate

\[ \|(fe^{-ir\varepsilon^2 A(k)}f^{-1} - f_0e^{-ir\varepsilon^2 A^0(k)}f_0^{-1}) R(k, \varepsilon)^{s/2}\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq C(\tau) \varepsilon \quad (95) \]

holds for almost every \( k \in \tilde{\Omega} \) and sufficiently small \( \varepsilon > 0 \).
Theorem 8.7: Suppose that \( \hat{N}_{0,Q}(\theta) = 0 \) for any \( \theta \in S^{d-1} \) and \( \hat{N}^{(q)}_{Q}(\theta_0) \neq 0 \) for some \( \theta_0 \in S^{d-1} \) and some \( q \in \{1, \ldots, p(\theta_0)\} \). Let \( \tau \neq 0 \) and \( 0 < \varepsilon < 2 \). Then there does not exist a constant \( C(\varepsilon) > 0 \) such that estimate (95) holds for almost every \( k \in \Omega \) and sufficiently small \( \varepsilon > 0 \).

Theorem 8.6 was proved in [26, Theorem 11.7]. Theorem 8.7 is proved with the help of Theorem 4.6 in a similar way as [26, Theorem 11.7]. Application of Theorem 4.7 allows us to confirm that the result of Theorem 8.1 is sharp with respect to time.

Theorem 8.8: Suppose that \( \hat{N}_{0,Q}(\theta_0) \neq 0 \) for some \( \theta_0 \in S^{d-1} \). Let \( s \geq 3 \). Then there does not exist a positive function \( C(\varepsilon) \) such that \( \lim_{\tau \to \infty} C(\varepsilon) = 0 \) and estimate (95) holds for all \( \tau \in \mathbb{R} \), almost every \( k \in \Omega \), and sufficiently small \( \varepsilon > 0 \).

Proof: We prove by contradiction. Suppose that for some \( s \geq 3 \) there exists a function \( C(\varepsilon) > 0 \) such that \( \lim_{\tau \to \infty} C(\varepsilon) = 0 \) and estimate (95) holds for almost every \( k \in \Omega \) and sufficiently small \( \varepsilon > 0 \). By (90) and (92), it follows that there exists a function \( \tilde{C}(\varepsilon) > 0 \) such that \( \lim_{\tau \to \infty} \tilde{C}(\varepsilon) = 0 \) and the estimate

\[
\left\| \left( f e^{-i\varepsilon^{-2} A(k)} f^{-1} - f_{0} e^{-i\varepsilon^{-2} A^{0}(k)} f_{0}^{-1} \right) F_{\Omega} \right\|_{L_{2}(\Omega)} \leq C_{1} \left\| k \right\|_{\Omega} \leq t^{0},
\]

(see (11)), we conclude that there exists a function \( \tilde{C}(\varepsilon) > 0 \) such that \( \lim_{\tau \to \infty} \tilde{C}(\varepsilon) = 0 \) and the estimate

\[
\left\| f e^{-i\varepsilon^{-2} A(k)} f^{0} \tilde{Q} - f_{0} e^{-i\varepsilon^{-2} A^{0}(k)} f_{0}^{-1} \tilde{P} \right\|_{L_{2}(\Omega)} \leq \tilde{C}(\varepsilon) \varepsilon^{s-2} \leq C(\varepsilon) \tilde{C}(\varepsilon)
\]

holds for almost every \( k \in \Omega \) in the ball \( \left\| k \right\|_{\Omega} \leq t^{0} \) and sufficiently small \( \varepsilon > 0 \).

For fixed \( \tau \) and \( \varepsilon \), the operator under the norm sign in (98) is continuous with respect to \( k \) in the ball \( \left\| k \right\|_{\Omega} \leq t^{0} \) (see [26, Lemma 11.8]). Hence, estimate (98) holds for all \( k \) in this ball, in particular, for \( k = \tau \theta_0 \) if \( t \leq t^{0} \). Applying inequality (97) and the identity \( P f^{0} \tilde{Q} = f^{-1} \tilde{P} \) once again, we obtain the estimate

\[
\left\| \left( f e^{-i\varepsilon^{-2} A(\tau \theta_0)} f^{-1} - f_{0} e^{-i\varepsilon^{-2} A^{0}(\tau \theta_0)} f_{0}^{-1} \right) F_{\Omega} \right\|_{L_{2}(\Omega)} \leq C(\varepsilon) \tilde{C}(\varepsilon)\left( \varepsilon^{2} \right)^{-s/2} \leq \tilde{C}(\varepsilon) \varepsilon^{s-2}
\]

for all \( t \leq t^{0} \) and sufficiently small \( \varepsilon > 0 \), where \( \tilde{C}(\varepsilon) > 0 \) and \( \lim_{\tau \to \infty} \tilde{C}(\varepsilon) = 0 \).

In the abstract terms, estimate (99) corresponds to (52). Since it is assumed that \( \hat{N}_{0,Q}(\theta_0) \neq 0 \), applying Theorem 4.7, we arrive at a contradiction.

Similarly, application of Theorem 4.8 allows us to confirm the sharpness of Theorems 8.2 and 8.5.

Theorem 8.9: Suppose that \( \hat{N}_{0,Q}(\theta) = 0 \) for any \( \theta \in S^{d-1} \) and \( \hat{N}^{(q)}_{Q}(\theta_0) \neq 0 \) for some \( \theta_0 \in S^{d-1} \) and some \( q \in \{1, \ldots, p(\theta_0)\} \). Let \( s \geq 2 \). Then there does not exist a positive function \( C(\varepsilon) \) such that \( \lim_{\tau \to \infty} C(\varepsilon) = 0 \) and estimate (95) holds for all \( \tau \in \mathbb{R} \), almost every \( k \in \Omega \), and sufficiently small \( \varepsilon > 0 \).
Chapter 3. Homogenization for the Schrödinger-type equations

9. Homogenization of the sandwiched operator $e^{-irA_\varepsilon}$

If $\psi(x)$ is a $\Gamma$-periodic measurable function in $\mathbb{R}^d$, we denote $\psi^s(x) := \psi(e^{-1}x)$, $\varepsilon > 0$. Our main object is the operator $A_\varepsilon$ acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and formally given by

$$A_\varepsilon := (f^s(x))^* b(D)^* g^s(x)b(D)f^s(x).$$

(100)

The precise definition is given in terms of the corresponding quadratic form (cf. Section 5.2).

Let $T_\varepsilon$ be the unitary scaling transformation in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ defined by $(T_\varepsilon u)(x) = \varepsilon^{d/2} u(\varepsilon x)$, $\varepsilon > 0$. Then $A_\varepsilon = \varepsilon^{-2} T_\varepsilon^* A T_\varepsilon$. Hence,

$$e^{-irA_\varepsilon} = T_\varepsilon^* e^{-ir\varepsilon^{-2}A} T_\varepsilon.$$  

(101)

Applying the scaling transformation to the resolvent of $H_0 = -\Delta$, we obtain

$$R(\varepsilon) := \varepsilon^2(H_0 + \varepsilon^2 I)^{-1} = T_\varepsilon(H_0 + I)^{-1} T_\varepsilon^*.$$  

(102)

Finally, if $\psi(x)$ is a $\Gamma$-periodic function, then $[\psi^s] = T_\varepsilon^*[\psi] T_\varepsilon$.

Let $A^0$ be defined by (81). Using the relations of the form (101) (for the operators $A_\varepsilon$ and $A^0$) and (102), we obtain

$$\left(\begin{array}{l}
(f^s e^{-irA_\varepsilon} f^s)^{-1} - f_0 e^{-irA^0 f_0^{-1}} (H_0 + I)^{-s/2} \\
= T_\varepsilon^* (f_0 e^{-ir\varepsilon^{-2}A f_0^{-1}} - f_0 e^{-ir\varepsilon^{-2}A^0 f_0^{-1}}) R(\varepsilon)^{s/2} T_\varepsilon, \quad \varepsilon > 0.
\end{array}\right)$$  

(103)

Next, the operator $R(\varepsilon)$ expands in the direct integral of the operators (89): $R(\varepsilon) = \mathcal{U}^{-1} (f_{\Omega} \oplus R(k, \varepsilon) \, dk) \mathcal{U}$. Recall also notation (93). By decomposition (63) for $A$ and $A^0$, we have

$$\left(\begin{array}{l}
\| (f_0 e^{-ir\varepsilon^{-2}A f_0^{-1}} - f_0 e^{-ir\varepsilon^{-2}A^0 f_0^{-1}}) R(\varepsilon)^{s/2} \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \\
= \text{ess sup} \| f(k, \varepsilon; \tau) R(k, \varepsilon)^{s/2} \|_{L_2(\Omega) \rightarrow L_2(\Omega)}. 
\end{array}\right)$$  

(104)

From Theorem 8.1 and identities (103) and (104), we deduce the following result proved before in [22, Theorem 12.4].

**Theorem 9.1** ([22]): Let $A_\varepsilon$ and $A^0$ be the operators defined by (100) and (81), respectively. Then for $0 \leq s \leq 3$, and $r \in \mathbb{R}$, $\varepsilon > 0$ we have

$$\| f^s e^{-irA_\varepsilon} (f^s)^{-1} - f_0 e^{-irA^0 f_0^{-1}} \|_{H^s(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \mathcal{C}_1(s)(1 + |\tau|)^{s/3} \varepsilon^{s/3},$$

where $\mathcal{C}_1(s) = (2 \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty})^{1-s/3} \mathcal{C}_1^{s/3}$. The constant $\mathcal{C}_1$ depends only on $\alpha_0, \alpha_1$, $\| g \|_{L_\infty}$, $\| g^{-1} \|_{L_\infty}$, $\| f \|_{L_\infty}$, $\| f^{-1} \|_{L_\infty}$, and $r_0$.

This result can be improved under some additional assumptions.

**Theorem 9.2:** Suppose that the assumptions of Theorem 9.1 are satisfied. Suppose that the operator $\hat{N}_Q(\theta)$ defined by (86) is equal to zero: $\hat{N}_Q(\theta) = 0$ for any $\theta \in \mathbb{S}^{d-1}$; or suppose that Condition 8.3 (or more restrictive Condition 8.4) is satisfied. Then for $0 \leq s \leq 2$, and $r \in \mathbb{R}$, $\varepsilon > 0$ we have

$$\| f^s e^{-irA_\varepsilon} (f^s)^{-1} - f_0 e^{-irA^0 f_0^{-1}} \|_{H^s(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \mathcal{C}_2(s)(1 + |\tau|^{1/2})^{s/2} \varepsilon^{s/2},$$  

(105)

where $\mathcal{C}_2(s) = (2 \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty})^{1-s/2} \mathcal{C}_2^{s/2}$. If $\hat{N}_Q(\theta) = 0$ for any $\theta \in \mathbb{S}^{d-1}$, then the constant $\mathcal{C}_2$ depends only on $\alpha_0, \alpha_1$, $\| g \|_{L_\infty}$, $\| g^{-1} \|_{L_\infty}$, $\| f \|_{L_\infty}$, $\| f^{-1} \|_{L_\infty}$, and $r_0$. Under Condition 8.3, the constant $\mathcal{C}_2$ depends on the same parameters and also on $n$ and the number $c^0$. 
Proof: Applying Theorem 8.2 (or Theorem 8.5) and using (103) and (104), we obtain
\[ \| (f^e e^{-irA_0}(f^e)^{-1} - f_0 e^{-irA_0 f_0^{-1}}) (H_0 + I)^{-1} \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_2 (1 + |r|^{1/2}) \varepsilon. \tag{106} \]

Obviously,
\[ \| (f^e e^{-irA_0}(f^e)^{-1} - f_0 e^{-irA_0 f_0^{-1}}) \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq 2 \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty}. \tag{107} \]

Interpolating between (107) and (106), for \( 0 \leq s \leq 2 \) we obtain
\[ \| (f^e e^{-irA_0}(f^e)^{-1} - f_0 e^{-irA_0 f_0^{-1}}) (H_0 + I)^{-s/2} \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq 2^{1-s/2} C_2^{s/2} (1 + |r|^{1/2})^{s/2} \varepsilon^{s/2}. \tag{108} \]

The operator \((H_0 + I)^{s/2}\) is an isometric isomorphism of \(H^s(\mathbb{R}^d; \mathbb{C}^n)\) onto \(L_2(\mathbb{R}^d; \mathbb{C}^n)\). Therefore, (108) is equivalent to (105). \(\square\)

Theorem 9.2 improve the results of Theorems 13.8 and 13.10 from [26] with respect to dependence of the estimates on \( \tau \).

Application of Theorems 8.6 and 8.7 allows us to confirm that the results of Theorems 9.1 and 9.2 are sharp with respect to the type of the operator norm.

**Theorem 9.3 ([26]):** Suppose that \( \hat{N}_{0,Q}(\theta_0) \neq 0 \) for some \( \theta_0 \in S^{d-1} \). Let \( \tau \neq 0 \) and \( 0 \leq s < 3 \). Then there does not exist a constant \( C(\tau) > 0 \) such that the estimate
\[ \| f^e e^{-irA_0}(f^e)^{-1} - f_0 e^{-irA_0 f_0^{-1}} \|_{H^s(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(\tau) \varepsilon \]
holds for all sufficiently small \( \varepsilon > 0 \).

**Theorem 9.4:** Suppose that \( \hat{N}_{0,Q}(\theta) = 0 \) for any \( \theta \in S^{d-1} \) and \( \hat{\mathcal{N}}_Q^{(q)}(\theta_0) \neq 0 \) for some \( \theta_0 \in S^{d-1} \) and some \( q \in \{1, \ldots, p(\theta_0)\} \). Let \( \tau \neq 0 \) and \( 0 \leq s < 2 \). Then there does not exist a constant \( C(\tau) > 0 \) such that estimate (109) holds for all sufficiently small \( \varepsilon > 0 \).

Theorem 9.3 was proved in [26, Theorem 13.12].

Application of Theorem 8.8 allows us to confirm that the result of Theorem 9.1 is sharp with respect to dependence of the estimates on time.

**Theorem 9.5:** Suppose that \( \hat{N}_{0,Q}(\theta_0) \neq 0 \) for some \( \theta_0 \in S^{d-1} \). Let \( s \geq 3 \). Then there does not exist a positive function \( C(\tau) \) such that \( \lim_{\tau \to \infty} C(\tau)/|\tau| = 0 \) and estimate (109) holds for all \( \tau \in \mathbb{R} \) and all sufficiently small \( \varepsilon > 0 \).

Similarly, application of Theorem 8.9 allows us to confirm that the result of Theorem 9.2 is sharp.

**Theorem 9.6:** Suppose that \( \hat{N}_{0,Q}(\theta_0) = 0 \) for any \( \theta \in S^{d-1} \) and \( \hat{\mathcal{N}}_Q^{(q)}(\theta_0) \neq 0 \) for some \( \theta_0 \in S^{d-1} \) and some \( q \in \{1, \ldots, p(\theta_0)\} \). Let \( s \geq 2 \). Then there does not exist a positive function \( C(\tau) \) such that \( \lim_{\tau \to \infty} C(\tau)/|\tau|^{1/2} = 0 \) and estimate (109) holds for all \( \tau \in \mathbb{R} \) and all sufficiently small \( \varepsilon > 0 \).
10. Homogenization of the Cauchy problem

Let \( u_\varepsilon(x, \tau), x \in \mathbb{R}^d, \tau \in \mathbb{R} \), be the solution of the Cauchy problem

\[
\begin{cases}
\frac{i}{\varepsilon} \frac{\partial u_\varepsilon(x, \tau)}{\partial \tau} = (f^\varepsilon(x))^* b(D)^* g^\varepsilon(x) b(D) f^\varepsilon(x) u_\varepsilon(x, \tau) + (f^\varepsilon(x))^{-1} F(x, \tau), \\
\tau)
\end{cases}
\]

(110)

where \( \phi \in L_2(\mathbb{R}^d; \mathbb{C}^n), F \in L_{1,loc}(\mathbb{R}; L_2(\mathbb{R}^d; \mathbb{C}^n)) \). The solution can be represented as

\[
\begin{aligned}
\quad u_\varepsilon(\cdot, \tau) = e^{-i\tau A_\varepsilon(f^\varepsilon)^{-1} \phi - i \int_0^\tau e^{-i(\tau - \tilde{\tau}) A_\varepsilon(f^\varepsilon)^{-1} F(\cdot, \tilde{\tau})} \, d\tilde{\tau}.
\end{aligned}
\]

Let \( u_0(x, \tau), x \in \mathbb{R}^d, \tau \in \mathbb{R} \), be the solution of the homogenized problem

\[
\begin{cases}
\frac{i}{\varepsilon} \frac{\partial u_0(x, \tau)}{\partial \tau} = f_0 b(D)^* g^0 b(D) f_0 u_0(x, \tau) + f_0^{-1} F(x, \tau), \\
0)
\begin{cases}
\quad f_0 u_0(x, 0) = \phi(x).
\end{cases}
\end{aligned}
\]

Then

\[
\begin{aligned}
\quad u_0(\cdot, \tau) = e^{-i\tau A_0 f_0^{-1} \phi - i \int_0^\tau e^{-i(\tau - \tilde{\tau}) A_0 f_0^{-1} F(\cdot, \tilde{\tau})} \, d\tilde{\tau}.
\end{aligned}
\]

The following result is deduced from Theorem 9.1 (it has been proved before in [22, Theorem 14.5]).

**Theorem 10.1** ([22]): Let \( u_\varepsilon \) be the solution of problem (110), and let \( u_0 \) be the solution of problem (111).

1°. If \( \phi \in H^s(\mathbb{R}^d; \mathbb{C}^n), F \in L_{1,loc}(\mathbb{R}; H^s(\mathbb{R}^d; \mathbb{C}^n)) \), where \( 0 \leq s \leq 3 \), then for \( \tau \in \mathbb{R} \) and \( \varepsilon > 0 \) we have

\[
\|f^\varepsilon u_\varepsilon(\cdot, \tau) - f_0 u_0(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq C_1(s) \varepsilon^{s/3}(1 + |\tau|)^{s/3} \left( \|\phi\|_{H^s(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^s(\mathbb{R}^d))} \right).
\]

Under the additional assumption that \( F \in L_p(\mathbb{R}^d; H^s(\mathbb{R}^d; \mathbb{C}^n)) \), where \( p \in [1, \infty) \), for \( \tau = \pm \varepsilon^{-\alpha}, 0 < \varepsilon \leq 1, 0 < \alpha < s(s + 3/p')^{-1} \) we have

\[
\|f^\varepsilon u_\varepsilon(\cdot, \pm \varepsilon^{-\alpha}) - f_0 u_0(\cdot, \pm \varepsilon^{-\alpha})\|_{L^2(\mathbb{R}^d)} \leq 2^{s/3} C_1(s) \varepsilon^{s(1-\alpha)/3} \left( \|\phi\|_{H^s(\mathbb{R}^d)} + \varepsilon^{-\alpha/p'} \|F\|_{L^p(\mathbb{R}^d; H^s(\mathbb{R}^d))} \right).
\]

The constant \( C_1(s) \) is defined in Theorem 9.1. Here \( p^{-1} + (p')^{-1} = 1 \).

2°. If \( \phi \in L_2(\mathbb{R}^d; \mathbb{C}^n) \) and \( F \in L_{1,loc}(\mathbb{R}; L_2(\mathbb{R}^d; \mathbb{C}^n)) \), then

\[
\lim_{\varepsilon \to 0} \|f^\varepsilon u_\varepsilon(\cdot, \tau) - f_0 u_0(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} = 0, \quad \tau \in \mathbb{R}.
\]

Under the additional assumption that \( F \in L_1(\mathbb{R}^d; L_2(\mathbb{R}^d; \mathbb{C}^n)) \), for \( 0 < \alpha < 1 \) we have

\[
\|f^\varepsilon u_\varepsilon(\cdot, \pm \varepsilon^{-\alpha}) - f_0 u_0(\cdot, \pm \varepsilon^{-\alpha})\|_{L^2(\mathbb{R}^d)} \to 0, \text{ as } \varepsilon \to 0.
\]

The result of Theorem 10.1 can be refined under some additional assumptions. Application of Theorem 9.2 yields the following result.
Theorem 10.2: Suppose that the assumptions of Theorem 10.1 are satisfied. Let \( \hat{N}_Q(\theta) \) be the operator defined by (86). Suppose that \( \hat{N}_Q(\theta) = 0 \) for any \( \theta \in S^{d-1} \) or Condition 8.3 (or more restrictive Condition 8.4) is satisfied.

1°. If \( \phi \in H^s(\mathbb{R}^d; C^n) \), \( F \in L_{1,0,\text{loc}}(\mathbb{R}; H^s(\mathbb{R}^d; C^n)) \), where \( 0 \leq s \leq 2 \), then for \( \tau \in \mathbb{R} \) and \( \varepsilon > 0 \) we have

\[
\| f^\varepsilon u_\varepsilon (\cdot, \tau) - f_0 u_0 (\cdot, \tau) \|_{L^2(\mathbb{R}^d)} \\
\leq C_2(s) \varepsilon^{s/2} (1 + |\tau|^{1/2})^{s/2} \left( \| \phi \|_{H^s(\mathbb{R}^d)} + \| F \|_{L^1((0,\varepsilon); H^s(\mathbb{R}^d))} \right).
\]

Under the additional assumption that \( F \in L_p(\mathbb{R}^d; H^s(\mathbb{R}^d, C^n)) \), where \( p \in [1, \infty] \), for \( \tau = \pm \varepsilon^{-\alpha} \), \( 0 < \varepsilon \leq 1 \), and \( 0 < \alpha < 2s + 4/p' - 1 \) we have

\[
\| f^\varepsilon u_\varepsilon (\cdot, \pm \varepsilon^{-\alpha}) - f_0 u_0 (\cdot, \pm \varepsilon^{-\alpha}) \|_{L^2(\mathbb{R}^d)} \\
\leq 2^{s/2} C_2(s) \varepsilon^{s(1-\alpha/2)/2} \left( \| \phi \|_{H^s(\mathbb{R}^d)} + \varepsilon^{-\alpha/p'} \| F \|_{L^p(\mathbb{R}^d; H^s(\mathbb{R}^d))} \right).
\]

The constant \( C_2(s) \) is defined in Theorem 9.2. Here \( p^{-1} + (p')^{-1} = 1 \).

°. If \( \phi \in L_2(\mathbb{R}^d; C^n) \) and \( F \in L_1(\mathbb{R}^d; L_2(\mathbb{R}^d, C^n)) \), for \( 0 < \alpha < 2 \) we have \( \| f^\varepsilon u_\varepsilon (\cdot, \pm \varepsilon^{-\alpha}) - f_0 u_0 (\cdot, \pm \varepsilon^{-\alpha}) \|_{L^2(\mathbb{R}^d)} \to 0 \), as \( \varepsilon \to 0 \).

11. Applications of the general results

11.1. The Schrödinger-type equation with the operator \( \hat{A}_e = - \hat{\nabla} g^e \hat{\nabla} \)

Consider the scalar elliptic operator

\[
\hat{A} = - \text{div} g(x) \nabla = D^* g(x) D
\]
acting in \( L_2(\mathbb{R}^d) \), \( d \geq 1 \), which is a particular case of the operator (68). In this case \( n = m = d \), \( b(D) = D \).

The effective matrix \( g^0 \) is defined in the standard way. Let \( \psi_j \in \tilde{H}^1(\Omega) \) be a (weak) \( \Gamma \)-periodic solution of the problem

\[
\text{div} g(x)(\nabla \psi_j(x) + e_j) = 0, \quad \int_{\Omega} \psi_j(x) \, dx = 0.
\]

Here \( e_1, \ldots, e_d \) is the standard orthonormal basis in \( \mathbb{R}^d \). The matrix \( \tilde{g}(x) \) is the matrix with the columns \( \tilde{g}_j(x) := g(x)(\nabla \psi_j(x) + e_j) \), \( j = 1, \ldots, d \). Then \( g^0 = |\Omega|^{-1} \int_{\Omega} \tilde{g}(x) \, dx \).

Now \( f = 1_n, Q = 1_n \), and \( f_0 = 1_n \). If \( g(x) \) is a symmetric matrix with real entries, then, by Proposition 7.1(1°), \( \tilde{N}_1(\theta) =: \hat{N}(\theta) = 0 \) for any \( \theta \in S^{d-1} \). If \( g(x) \) is a Hermitian matrix with complex entries, then, in general, \( \hat{N}(\theta) \) is not zero. Since \( n = 1 \), then \( \hat{N}(\theta) \) is the operator of multiplication by \( \hat{\mu}(\theta) \), where \( \hat{\mu}(\theta) \) is the coefficient in the expansion for the first eigenvalue \( \hat{\lambda}(t, \theta) = \hat{\gamma}(\theta)t^2 + \hat{\mu}(\theta)t^3 + \hat{\nu}(\theta)t^4 + \ldots \) of the operator \( \hat{A}(k) \). A calculation (see [7, Section 10.3]) shows that

\[
\hat{N}(\theta) = \hat{\mu}(\theta) = -i \sum_{j,l,r=1}^d (a_{jlr} - a_{jlr}^n) \theta_j \theta_l \theta_r,
\]

\[
a_{jlr} = |\Omega|^{-1} \int_{\Omega} \psi_j(x)^n g(x)(\nabla \psi_l(x) + e_l), e_r \, dx, \quad j, l, r = 1, \ldots, d.
\]

An example of the operator (112) with \( \hat{\mu}(\theta) \neq 0 \) can be found in [7, Section 10.4].
Next, let $\phi_{jl}(x)$ be a $\Gamma$-periodic solution of the problem

$$-
abla \cdot \nabla \phi_{jl}(x) - \psi_j(x)e_l = \delta_{ij} - \tilde{\eta}_{jl}(x), \quad \int_\Omega \phi_{jl}(x) \, dx = 0. \quad (114)$$

The operator $\hat{N}_1^{(1,1)}(\theta) := \hat{N}^{(1,1)}(\theta)$ is the operator of multiplication by $\hat{v}(\theta)$. A calculation (see [14, Section 14.5]) shows that

$$\hat{v}(\theta) = \sum_{p, q, r = 1}^d (\alpha_{pqr} - (\psi_p^* \psi_q) \delta_{ij}) \theta_p \theta_q \theta_r,$$

$$\alpha_{pqr} = |\Omega|^{-1} \int_\Omega (\tilde{g}_{lp}(x) \phi_{qr}(x) + \tilde{\eta}_{pq}(x) \phi_{pl}(x)) \, dx$$

$$+ |\Omega|^{-1} \int_\Omega (\nabla \phi_{qr}(x) - \psi_q(x)e_r), \nabla \phi_{pl}(x) - \psi_p(x)e_l) \, dx,$$

$$p, q, l, r = 1, \ldots, d. \quad (115)$$

The form (115) was studied in [30] in the case where the matrix $g(x)$ has real entries; it was proved that the form (115) is negative semidefinite. The following simple lemma is true (see also [30, Remark 3.3]).

**Lemma 11.1:** Let $d = 1$ and $\hat{A} = -\frac{d}{dx} g(x) \frac{d}{dx}$. If $g \neq \text{const}$, then $\hat{v}(-1) = \hat{v}(1) < 0$.

**Proof:** The problem (113) now takes the form $\frac{d}{dx} g(x) \left(\frac{d}{dx} \psi_1(x) + 1\right) = 0$, $\overline{\psi_1} = 0$. Then $\frac{d}{dx} \psi_1(x) = g(g(x))^{-1} - 1$. Since $g(x) \neq \text{const}$, then $g(g(x))^{-1} - 1 \neq 0$, whence $\psi_1 \neq 0$. Next, $\tilde{g}(x) = g = g^0$ and Equation (114) takes the form $\frac{d}{dx} g(x) \left(\frac{d}{dx} \phi_{11}(x) - \psi_1(x)\right) = 0$, $\overline{\phi_{11}} = 0$. Then $\frac{d}{dx} \phi_{11}(x) - \psi_1(x) = 0$. It is easy to check that $\alpha_{1111}$ in (115) is equal to zero: $\alpha_{1111} = 0$. Since $\overline{\psi_1^2} g^0 \neq 0$, then $\hat{v}(-1) = \hat{v}(1) < 0$.  

Consider the Cauchy problem (110) with the operator $\hat{A}_\epsilon = -\nabla \cdot g^\epsilon(x) \nabla$. We can apply Theorem 10.1 in the general case and Theorem 10.2 in the “real” case. These results are sharp with respect to smoothness of the initial data and with respect to dependence of the estimates on time.

### 11.2. The nonstationary Schrödinger equation with a singular potential

(See [9, Chapter 6, Section 1.1].) In the space $L^2(\mathbb{R}^d)$, $d \geq 1$, we consider the operator $\mathcal{H} = D^* \tilde{g}(x) D + V(x)$, where a symmetric $(d \times d)$-matrix-valued function $\tilde{g}(x)$ with real entries and a real-valued potential $V(x)$ are $\Gamma$-periodic and satisfy

$$\tilde{g}(x) > 0, \quad \tilde{g}, \tilde{g}^{-1} \in L_\infty; \quad V \in L_q(\Omega), \quad q > d/2 \text{ for } d \geq 2, \quad q = 1 \text{ for } d = 1.$$

Adding an appropriate constant to $V(x)$, we may assume that the point $\lambda = 0$ is the bottom of the spectrum of $\mathcal{H}$. Then $\mathcal{H}$ can be written in the factorized form:

$$\mathcal{H} = \omega^{-1} D^* \tilde{g}^2 D \omega^{-1}, \quad (116)$$

where $\omega(x)$ is a positive $\Gamma$-periodic solution of the equation $D^* \tilde{g}(x) D \omega(x) + V(x) \omega(x) = 0$, $\int_\Omega \omega^2(x) \, dx = |\Omega|$. Therefore, the operator (116) is a particular case of the operator (58). In this case $n = 1, m = d, b(D) = D, g = \omega^2 \tilde{g}, f = \omega^{-1}$. 

---

**Note:** This text is a snippet from a larger document, focusing on mathematical analysis, particularly on the nonstationary Schrödinger equation with a singular potential. The content provided covers the formulation of a problem involving the Laplace operator, its boundary conditions, and the study of solutions under certain conditions. The text also delves into the properties of a specific operator, its factorization, and its application in the context of the Schrödinger equation. This excerpt is part of a broader discussion on the existence and uniqueness of solutions to such equations.
Let \( g^0 \) be the effective matrix for the operator (112) (with \( g = \omega^2 \hat{g} \)). Now \( Q(x) = \omega^2(x) \), and, by the normalization condition for \( \omega \), we have \( Q = 1 \) and \( f_0 = (Q)^{-1/2} = 1 \). Therefore, the operator (81) takes the form \( \mathcal{H}_\epsilon = D^* \hat{g}^\epsilon D \).

Now we consider the operator

\[
\mathcal{H}_\epsilon = (\omega^\epsilon)^{-1} D^* (\omega^\epsilon)^2 \hat{g}^\epsilon D (\omega^\epsilon)^{-1}.
\]

In the initial form, the operator (117) can be written as \( \mathcal{H}_\epsilon = D^* \hat{g}^\epsilon D + \epsilon^{-2} V^\epsilon \). Note that this expression contains a large factor \( \epsilon^{-2} \) at the rapidly oscillating potential \( V^\epsilon \).

Let us consider the Cauchy problem (110) with the operator (117). By Proposition 7.1(7.1), \( \hat{N}_Q(\theta) = 0 \) for any \( \theta \in S^{d-1} \). We can apply Theorem 10.2. This result is sharp with respect to smoothness of the initial data and with respect to dependence of the estimates on time.

**Remark 9:** It is also possible to consider the Cauchy problem for the magnetic Schrödinger equation with a small magnetic potential, using an appropriate factorization for the magnetic Schrödinger operator; see [26, Section 15.4]. In this case, we do not have improvement of the general results.

### 11.3. The nonstationary two-dimensional Pauli equation

(See [7, Chapter 4, §12, Section 12.3].) Let the magnetic potential \( A = \{A_1, A_2\} \) be a \( \Gamma \)-periodic real vector-valued function in \( \mathbb{R}^2 \) such that \( A \in L_p(\Omega; \mathbb{C}^2) \), \( p > 2 \). By the gauge transformation, we may assume that \( \text{div} A = 0 \), \( \int_{\Omega} A(x) \, dx = 0 \). Under these conditions there exists a (unique) \( \Gamma \)-periodic real-valued function \( \varphi \) such that \( \nabla \varphi = \{A_2, -A_1\}, \int_{\Omega} \varphi(x) \, dx = 0 \).

In \( L_2(\mathbb{R}^2; \mathbb{C}^2) \), we consider the Pauli operator

\[
P = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix}, \quad P_+ = \omega_+ \partial_+ \omega_+^2 \partial_- \omega_-, \quad P_- = \omega_- \partial_- \omega_-^2 \partial_+ \omega_+, \tag{118}
\]

where \( \omega_{\pm}(x) = e^{\pm \varphi(x)} \) and \( \partial_{\pm} = D_1 \pm i D_2 \). If the potential \( A \) is sufficiently smooth, then the blocks \( P_{\pm} \) of the operator (118) are of the form \( P_{\pm} = (D - A)^2 \pm B \), where \( B = \partial_1 A_2 - \partial_2 A_1 \) is the strength of the magnetic field.

The operator (118) can be written as \( P = f_x b_x(D) g_x b_x(D) f_{x,0} \), where

\[
b_x(D) = \begin{pmatrix} 0 & \partial_- \\ \partial_+ & 0 \end{pmatrix}, \quad f_x(x) = \begin{pmatrix} \omega_+(x) & 0 \\ 0 & \omega_-(x) \end{pmatrix}, \quad g_x(x) = \begin{pmatrix} \omega_+^2(x) & 0 \\ 0 & \omega_-^2(x) \end{pmatrix}.
\]

The operator \( p \) is of the form (58) with \( m = n = d = 2 \), \( b(D) = b_x(D), f(x) = f_x(x), g(x) = g_x(x) \).

Since \( m = n \), the effective matrix is equal to \( g_x = g_x = \text{diag}(g_0^0, g_0^0), g_{\pm} = \omega_{\pm}^2 \). The matrix \( Q_x = f_x^{-2} = g_x^{-1} \) plays the role of \( Q \). Then \( \Omega_x = \text{diag}(g_0^{0,0}), (g_0^{0,0})^{-1} \). The role of \( f_0 \) is played by \( f_{x,0} = \text{diag}(g_0^{0,1/2}, g_0^{0,1/2}) \). The operator (81) now takes the form

\[
P_{x,0} = f_{x,0} b_x(D) g_x^0 b_x(D) f_{x,0} = \begin{pmatrix} -\gamma \Delta & 0 \\ 0 & -\gamma \Delta \end{pmatrix}.
\]

Here \( \gamma = g_0^{0,0} = |\Omega| \|\omega_+\|_{L^2(\Omega)}^2 \|\omega_-\|_{L^2(\Omega)}^{-2} \).

Now we describe the operator \( \hat{N}_{Q_x}(\theta) \) that plays the role of \( \hat{N}_{Q}(\theta) \) for \( P \). Let \( w_{\pm}(x) \) be the \( \Gamma \)-periodic solutions of the problems \( \partial_{\pm} w_{\pm}(x) = g_x^0 \omega_{\pm}^2(x) - 1, \int_{\Omega} w_{\pm}(x) \, dx = 0 \). Then \( \hat{N}_{Q_x}(\theta) = \text{diag}(\hat{N}_{Q_x^-}(\theta), \hat{N}_{Q_x^+}(\theta)), \hat{N}_{Q_x^+}(\theta) = -2\gamma (\theta_1 \text{Re} \omega_{\pm}^2 w_{\pm} + \theta_2 \text{Im} \omega_{\pm}^2 w_{\pm}), \theta \in S^1 \).

An example of the
operator $P$ with $\tilde{N}_{Q, \varepsilon}(\theta) \neq 0$ can be found in [26, Example 16.2]. Now, we consider the operator

$$
P_{\varepsilon} = f_{\varepsilon} b_{\varepsilon} (D) g_{\varepsilon} b_{\varepsilon} (D) f_{\varepsilon} = \begin{pmatrix} P_{-\varepsilon} & 0 \\ 0 & P_{+\varepsilon} \end{pmatrix},
$$

(119)

where $P_{-\varepsilon} = \omega_{\varepsilon}^2 \partial_+ (\omega_{\varepsilon}^2)^2 \partial_- \omega_{\varepsilon}^2$ and $P_{+\varepsilon} = \omega_{\varepsilon}^2 \partial_- (\omega_{\varepsilon}^2)^2 \partial_+ \omega_{\varepsilon}^2$. If the potential $A$ is sufficiently smooth, then the blocks of the operator (119) are of the form $P_{\pm\varepsilon} = (D - \varepsilon^{-1} A_{\varepsilon})^2 \pm \varepsilon^{-2} B_{\varepsilon}$.

Let us consider the Cauchy problem (110) with the operator (119). We can apply Theorem 10.1. In general, this result is sharp with respect to smoothness of the initial data and with respect to dependence of the estimates on time.

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