A REFINEMENT OF HARDY INEQUALITY VIA SUPERQUADRATIC FUNCTION

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Abstract. A refinement of the Hardy inequality has been presented by use of superquadratic function.

1. Introduction

The classical Hardy inequality asserts that if $p > 1$ and $f$ is a non-negative $p$-integrable function on $(0, \infty)$, then
\[
\int_0^\infty \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^p \, dx \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty f(x)^p \, dx.
\] (1)

Let $p > 1$ and let $F : (0, \infty) \to B(H)^+$ be a weakly measurable mapping such that
\[
\int_0^\infty F(x)^p \, dx \in B(H).
\]

Assume that the real valued function $f$ is defined on $(0, \infty)$ by $f(t) = \langle F(t)\eta, \eta \rangle$. Then $f$ is $p$-integrable, since
\[
\int_0^\infty f(t)^p \, dt = \int_0^\infty \langle F(t)\eta, \eta \rangle^p \, dt
\]
\[
\leq \int_0^\infty \langle F(t)^p \eta, \eta \rangle \, dt \quad \text{by (1)}
\]
\[
= \left\langle \int_0^\infty F(t)^p \, dt \eta, \eta \right\rangle.
\]

The classical Hardy inequality (1) now implies that
\[
\int_0^\infty \left( \frac{1}{x} \int_0^x \langle F(t)\eta, \eta \rangle \, dt \right)^p \, dx \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty \langle F(t)\eta, \eta \rangle^p \, dt.
\] (2)

In the case where $p \in (1, 2]$, Hansen [3] proved that a stronger form of (2) holds true:
\[
\int_0^\infty \left( \frac{1}{x} \int_0^x F(t) \, dt \right)^p \, dx \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty F(t)^p \, dt.
\] (3)

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However, if \( p > 2 \), the inequality (3) is not valid in general, see [3]. In this paper, utilizing the notion of the superquadratic functions, we give an improvement of the Hardy inequality (1) for \( p \geq 2 \). Furthermore, our result will provide some difference counterpart to Hardy inequality.

2. Preliminaries

Superquadratic functions have been introduced as a modification of convex functions in [1]. A function \( f : [0, \infty) \rightarrow \mathbb{R} \) is said to be superquadratic whenever for all \( a \geq 0 \) there exists a constant \( C_a \in \mathbb{R} \) such that

\[
 f(b) \geq f(a) + C_a(b - a) + f(|b - a|) \tag{4}
\]

for all \( b \geq 0 \). If such \( f \) is positive, then it is convex too and a sharper Jensen inequality holds true: For every probability measure \( \mu \) on \( \Omega \) and every \( \mu \)-integrable function \( \phi \) on \( \Omega \), if \( f \) is superquadratic, then

\[
 f\left( \int_{\Omega} \phi(t) d\mu(t) \right) \leq \int_{\Omega} f(\phi(t)) d\mu(t) - \int_{\Omega} f\left( |\phi(t) - \int_{\Omega} \phi(s) d\mu(s)| \right) d\mu(t). \tag{5}
\]

Assume that \( B(\mathcal{H}) \) is the \( C^* \)-algebra of all bounded linear operators on a Hilbert space \( \mathcal{H} \) and \( I \) is the identity operator. An operator extension of (5) has been presented in [8]:

**Theorem A.** If \( f : [0, \infty) \rightarrow \mathbb{R} \) is a continuous superquadratic function, then

\[
 f(\langle A\eta, \eta \rangle) \leq \langle f(A)\eta, \eta \rangle - \langle f(\|A - \langle A\eta, \eta \rangle\|)\eta, \eta \rangle \tag{6}
\]

for every positive operator \( A \) and every unit vector \( \eta \in \mathcal{H} \).

Theorem A provides a refinement of the well-known inequality (see [10])

\[
 g(\langle A\eta, \eta \rangle) \leq \langle g(A)\eta, \eta \rangle, \tag{7}
\]

which holds for every continuous convex function \( g \). Moreover, a generalization of (6) has been shown in [7]:

**Theorem B.** Let \( \Phi \) be a unital positive linear mapping on \( B(\mathcal{H}) \). If \( f : [0, \infty) \rightarrow \mathbb{R} \) is a continuous superquadratic function, then

\[
 f(\langle \Phi(A)\eta, \eta \rangle) \leq \langle \Phi(f(A))\eta, \eta \rangle - \langle \Phi (f (\|A - \langle \Phi(A)\eta, \eta \rangle\|)) \eta, \eta \rangle. \tag{8}
\]

for every positive operator \( A \) and every unit vector \( \eta \in \mathcal{H} \).

3. Refinement of Hardy inequality

**Lemma 3.1.** Let \( p \geq 2 \). If the mappings \( G : (0, \infty) \rightarrow B(\mathcal{H})^+ \) is weakly measurable such that

\[
 \int_0^\infty G(x)x^p \frac{dx}{x} \in B(\mathcal{H}),
\]
then
\[ \int_0^\infty \left( \frac{1}{x} \int_0^x \langle G(t) \eta, \eta \rangle dt \right)^p \frac{dx}{x} \leq \int_0^\infty \langle G(t)^p \eta, \eta \rangle \frac{dt}{t} \]
\[ - \int_0^\infty \frac{1}{x} \int_0^x \left\langle G(t) - \frac{1}{x} \int_0^x G(s) ds \eta, \eta \right\rangle^p \eta, \eta \rangle dt \frac{dx}{x}. \]

Proof. Let \( \mathcal{A} \) be the \( C^* \)-algebra of all weakly measurable mappings \( F : [0, x] \to \mathcal{B}(\mathcal{H}) \) with \( \int_0^x F(t) dt \in \mathcal{B}(\mathcal{H}) \). Define the unital positive linear mapping \( \Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) by
\[ \Phi(F) = \frac{1}{x} \int_0^x F(t) dt, \quad \forall F \in \mathcal{A}. \]

If \( p \geq 2 \), then the function \( f : (0, \infty) \to (0, \infty) \) defined by \( f(t) = t^p \) is superquadratic. If \( \eta \in \mathcal{H} \) is a unit vector, then Theorem B implies that
\[ \left\langle \frac{1}{x} \int_0^x G(t) dt \eta, \eta \right\rangle^p \leq \left\langle \frac{1}{x} \int_0^x G(t)^p dt \eta, \eta \right\rangle \]
\[ - \left\langle \frac{1}{x} \int_0^x \left| G(t) - \frac{1}{x} \int_0^x G(s) ds \eta, \eta \right\rangle^p dt \eta, \eta \right\rangle \]
for every \( x > 0 \). Multiplying both sides by \( \frac{1}{x} \) and integrating over \( (0, \infty) \) we obtain
\[ \int_0^\infty \left\langle \frac{1}{x} \int_0^x G(t) dt \eta, \eta \right\rangle^p \frac{dx}{x} \leq \int_0^\infty \left\langle \frac{1}{x} \int_0^x G(t)^p dt \eta, \eta \right\rangle \frac{dx}{x} \]
\[ - \int_0^\infty \left\langle \frac{1}{x} \int_0^x \left| G(t) - \frac{1}{x} \int_0^x G(s) ds \eta, \eta \right\rangle^p dt \eta, \eta \right\rangle \frac{dx}{x}. \]

Noting that
\[ \int_0^\infty \left\langle \frac{1}{x} \int_0^x G(t)^p dt \eta, \eta \right\rangle \frac{dx}{x} = \int_0^\infty \frac{1}{x} \int_0^x \langle G(t)^p \eta, \eta \rangle \frac{dx}{x} \]
\[ = \int_0^\infty \langle G(t)^p \eta, \eta \rangle \int_t^\infty \frac{1}{x^2} dx \ dt \]
\[ = \int_0^\infty \langle G(t)^p \eta, \eta \rangle \frac{dt}{t}, \]
we get from \((9)\) that
\[ \int_0^\infty \left\langle \frac{1}{x} \int_0^x G(t) dt \eta, \eta \right\rangle^p \frac{dx}{x} \leq \int_0^\infty \langle G(t)^p \eta, \eta \rangle \frac{dt}{t} \]
\[ - \int_0^\infty \frac{1}{x} \int_0^x \left\langle G(t) - \frac{1}{x} \int_0^x G(s) ds \eta, \eta \right\rangle^p \eta, \eta \rangle dt \frac{dx}{x}. \]

\[ \square \]
Theorem 3.2. Let \( p \geq 2 \). If the mapping \( F : (0, \infty) \to \mathcal{B}(\mathcal{H})^+ \) is weakly measurable such that
\[
\int_0^\infty F(x)^p dx \in \mathcal{B}(\mathcal{H}),
\]
then
\[
\int_0^\infty \left\langle \frac{1}{x} \int_0^x F(t) dt, \eta, \eta \right\rangle^p dx \leq \left( \frac{p}{p-1} \right) \int_0^\infty \left\langle F(x)^p, \eta, \eta \right\rangle dx
\]
\[
- \left( \frac{p}{p-1} \right)^{p-2} \left\langle \int_0^\infty \frac{1}{x} \int_0^x x^\frac{1}{p} t^{\frac{1}{p}} \left| x^\frac{1}{p} t^{\frac{1}{p}} F(t) \right| dt \eta, \eta \right\rangle \int_0^x t^\frac{1}{p} t^\frac{1}{p} F(t) - \left( \frac{p-1}{p} \right)^p \left\langle 1, \int_0^x F(r) dr, \eta, \eta \right\rangle \right| dt dx \eta, \eta \rangle.
\]
for every unit vector \( \eta \in \mathcal{H} \).

Proof. We use Lemma 3.1 and proceed as argument applied in [3, Theorem 2.3]. Put \( G(t) = F(t^{\frac{p}{p-1}}) t^{\frac{1}{p-1}} \) so that \( G \) is weakly measurable. Applying Lemma 3.1 to \( G \) we get
\[
\int_0^\infty \left\langle \frac{1}{x} \int_0^x F(t^{\frac{p}{p-1}}) t^{\frac{1}{p-1}} dt, \eta, \eta \right\rangle^p dx \leq \int_0^\infty \left\langle F(t^{\frac{p}{p-1}}) t^{\frac{1}{p-1}}, \eta, \eta \right\rangle \frac{dt}{t}
\]
\[
- \int_0^\infty \frac{1}{x} \int_0^x \left\langle F(t^{\frac{p}{p-1}}) t^{\frac{1}{p-1}} - \frac{1}{x} \int_0^x F(s^{\frac{p}{p-1}}) s^{\frac{1}{p-1}} ds, \eta, \eta \right\rangle^p dx \frac{dx}{x}.
\]

Let we use the symbol \( I \leq II - III \) for (11). With substituting \( y = t^{\frac{p}{p-1}} \) and \( dy = \frac{p}{p-1} t^{-\frac{1}{p-1}} dt \) we obtain
\[
I = \int_0^\infty \left\langle \frac{1}{x} \int_0^x F(t^{\frac{p}{p-1}}) t^{\frac{1}{p-1}} dt, \eta, \eta \right\rangle^p dx \frac{dx}{x} = \left( \frac{p-1}{p} \right)^p \int_0^\infty \left\langle \frac{1}{x} \int_0^x F(y) dy, \eta, \eta \right\rangle^p dx \frac{dx}{x}
\]
(12)

and
\[
II = \int_0^\infty \left\langle F(t^{\frac{p}{p-1}}) t^{\frac{1}{p-1}}, \eta, \eta \right\rangle \frac{dt}{t} = \int_0^\infty \left\langle F(y)^p, \eta, \eta \right\rangle dt.
\]

Moreover, using substituting \( r = s^{\frac{p}{p-1}} \) and \( dr = \frac{p}{p-1} s^{\frac{1}{p-1}} ds \) we get
\[
III = \int_0^\infty \frac{1}{x} \int_0^x \left\langle F(t^{\frac{p}{p-1}}) t^{\frac{1}{p-1}} - \frac{1}{x} \int_0^x F(s^{\frac{p}{p-1}}) s^{\frac{1}{p-1}} ds, \eta, \eta \right\rangle^p dx \frac{dx}{x}
\]
\[
= \frac{p-1}{p} \int_0^\infty \frac{1}{x} \int_0^x F(y)^{\frac{1}{p}} - \frac{p-1}{p} \left\langle \frac{1}{x} \int_0^x F(r) dr, \eta, \eta \right\rangle^p dy \frac{dy}{x}.
\]
(13)
Applying the substitution $z = x^{\frac{p}{p-1}}$ and $\frac{dz}{z} = \frac{p}{p-1} \frac{dx}{x}$ in (12) and (13) respectively, we can write

$$I = \left(\frac{p-1}{p}\right)^p \int_0^\infty \left\langle \frac{1}{x} \int_0^x F(y) dy \eta \right\rangle \left(\frac{p-1}{p}\right)^p \int_0^\infty \left\langle \frac{1}{z} \int_0^z F(y) dy \eta \right\rangle dz,$$

and

$$III = \int_0^\infty \frac{1}{x} \int_0^\frac{x}{x^{\frac{p}{p-1}}} \left\langle F(y) y^p - \frac{p-1}{p} \left\langle \frac{1}{x} \int_0^x F(y) dr \eta \right\rangle \right\rangle dy \left\langle \frac{1}{z} \int_0^z F(y) dr \eta \right\rangle dy dz,$$

$$= \left(\frac{p-1}{p}\right)^2 \int_0^\infty \frac{1}{x} \int_0^x \left\langle F(y) y^p - \frac{p-1}{p} \left\langle \frac{1}{x} \int_0^x F(y) dr \eta \right\rangle \right\rangle dy dz,$$

$$= \left(\frac{p-1}{p}\right)^2 \left\langle \int_0^\infty \frac{1}{x} \int_0^x \left\langle F(y) y^p - \frac{p-1}{p} \left\langle \frac{1}{x} \int_0^x F(y) dr \eta \right\rangle \right\rangle dy dz \right\rangle.$$

Assume that $f$ is a positive $p$-integrable ($p \geq 2$) function on $(0, \infty)$. Consider the mapping $F : (0, \infty) \to \mathcal{B}(\mathcal{H})^+$ defined by $F(t) = f(t)I$. Theorem 3.2 then gives the following refinement of the classical Hardy inequality (1).

**Corollary 3.3.** If $p \geq 2$ and $f : (0, \infty) \to (0, \infty)$ is a $p$-integrable function, then

$$\int_0^\infty \left\langle \frac{1}{x} \int_0^x f(t) dt \right\rangle^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx - \left(\frac{p}{p-1}\right)^{p-2} \int_0^\infty \frac{1}{x} \int_0^x \frac{1}{x} \frac{1}{t^p} \left\langle f(t) - \frac{p-1}{p} \frac{1}{x} \int_0^x f(r) dr \right\rangle^p dt dx.$$

We give an example to show that Corollary 3.3 really gives an improvement of the classical Hardy inequality (1). The calculations in the next example has been done by the Mathematica software.

**Example 3.4.** Put $p = 2$ and assume that $f(t) = \frac{1}{t+1}$ so that $f$ is square-integrable. Then

$$\int_0^\infty \left(\frac{1}{x} \int_0^x \frac{1}{t+1} dt \right)^2 = \frac{\pi^2}{3}, \quad \int_0^\infty \frac{1}{(x+1)^2} dx = 1,$$

$$\int_0^\infty \frac{1}{x} \int_0^x \frac{1}{t+1} \left(\frac{1}{x^2 t^2} + \frac{1}{x t^2} - \frac{1}{2x^2} \int_0^x \frac{1}{r+1} dr \right)^2 dt dx = 2 - \frac{\pi^2}{6},$$

and Corollary 3.3 gives

$$\frac{\pi^2}{3} \leq 4 - (2 - \frac{\pi^2}{6}),$$

while the Hardy inequality (1) gives $\frac{\pi^2}{3} \leq 4.$
Remark 3.5. It will be helpful to point out that the power function \( f(t) = -t^p \) is superquadratic for \( 1 < p \leq 2 \). A same argument as in Theorem 3.2 will provide a difference counterpart to the Hardy inequality. With assumption as in Theorem 3.2 except \( 1 < p \leq 2 \), we obtain

\[
\left( \frac{p}{p-1} \right)^p \int_0^\infty \langle F(x)^p \eta, \eta \rangle \, dx \\
\leq \int_0^\infty \left( \frac{1}{x} \int_0^x F(t) dt \, \eta, \eta \right)^p dx \\
+ \left( \frac{p}{p-1} \right)^{p-2} \int_0^\infty \frac{1}{x} \int_0^x x^\frac{1}{p} t^ \frac{1}{p} \left| x^\frac{1}{p} t^ \frac{1}{p} F(t) - \frac{p-1}{p} \left( \frac{1}{x} \int_0^x F(r) \, dr \, \eta, \eta \right) \right|^p dt \, \eta, \eta \rangle.
\]

4. **External Jensen inequality for superquadratic functions**

**Theorem 4.1.** Let \( f : [0, \infty) \to \mathbb{R} \) be a superquadratic function. Let \( x, y \in \mathcal{H} \) with \( \|x\|^2 - \|y\|^2 = 1 \). If \( A, B \) are two positive operators, then

\[
f(\langle Ax, x \rangle - \langle By, y \rangle) \geq \|x\|^2 f \left( \left\langle A \frac{x}{\|x\|}, \frac{x}{\|x\|} \right\rangle \right) - \langle f(B)y, y \rangle + f \left( B - \frac{1}{\|y\|^2} \langle By, y \rangle \right) y, y \\
+ f \left( \|y\|^2 \left( \frac{1}{\|x\|^2} \langle Ax, x \rangle - \frac{1}{\|y\|^2} \langle By, y \rangle \right) \right) \\
+ \|y\|^2 f \left( \frac{1}{\|x\|^2} \langle Ax, x \rangle - \frac{1}{\|y\|^2} \langle By, y \rangle \right),
\]

provided that \( \langle Ax, x \rangle - \langle By, y \rangle \geq 0 \).

\[
f \left( \left\langle A \frac{x}{\|x\|}, \frac{x}{\|x\|} \right\rangle \right) = f \left( \frac{1}{\|x\|^2} \langle Ax, x \rangle - \langle By, y \rangle \right) + \frac{\|y\|^2}{\|x\|^2} \left\langle B \frac{y}{\|y\|}, \frac{y}{\|y\|} \right\rangle \\
\leq \frac{1}{\|x\|^2} f(\langle Ax, x \rangle - \langle By, y \rangle) + \frac{\|y\|^2}{\|x\|^2} f \left( \left\langle B \frac{y}{\|y\|}, \frac{y}{\|y\|} \right\rangle \right) \tag{15} \\
- \frac{1}{\|x\|^2} f \left( \frac{\|y\|^2}{\|x\|^2} \langle Ax, x \rangle - \langle By, y \rangle - \frac{1}{\|y\|^2} \langle By, y \rangle \right) \\
- \|y\|^2 f \left( \frac{1}{\|x\|^2} \langle Ax, x \rangle - \langle By, y \rangle - \frac{1}{\|y\|^2} \langle By, y \rangle \right).
\]

Since \( f \) is superquadratic, it follows from (6) that

\[
f \left( \left\langle B \frac{y}{\|y\|}, \frac{y}{\|y\|} \right\rangle \right) \leq \frac{1}{\|y\|^2} \langle f(B)y, y \rangle - \frac{1}{\|y\|^2} f \left( B - \frac{1}{\|y\|^2} \langle By, y \rangle \right) y, y \tag{16}
\]
Multiplying both sides of (15) by $\|x\|^2$ and using (16) we get

$$\|x\|^2 f \left( \left\langle \frac{Ax}{\|x\|}, \frac{x}{\|x\|} \right\rangle \right) \leq f(\langle Ax, x \rangle - \langle By, y \rangle) + f \left( B - \frac{1}{\|y\|^2} \langle By, y \rangle \right) y, y$$

$$- f \left( \|y\|^2 \left\langle \frac{1}{\|x\|^2} (Ax, x) - \frac{1}{\|y\|^2} (By, y) \right\rangle \right)$$

$$- \|y\|^2 f \left( \left\langle \frac{1}{\|x\|^2} (Ax, x) - \frac{1}{\|y\|^2} (By, y) \right\rangle \right),$$

which concludes the result.

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