Hopping dynamics for localized Lyapunov vectors in many-hard-disk systems

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The dynamical instability and many-body nature play essential roles in the justification of a statistical treatment for deterministic dynamical systems. The dynamical instability is described as a rapid expansion of the difference between two nearby trajectories, namely the Lyapunov vector, and the system is called chaotic if at least one exponential rate (Lyapunov exponent) of divergence or contraction of the amplitude of Lyapunov vector is positive. The Lyapunov exponent $\lambda$ is defined for each independent direction of the phase space, so the chaotic properties of many-body systems can be characterized by an ordered set of Lyapunov exponents, the so-called Lyapunov spectrum $\{\lambda^{(1)}, \lambda^{(2)}, \ldots\}$ where $\lambda^{(1)} \geq \lambda^{(2)} \geq \ldots$. The Lyapunov spectrum is connected to the contraction rate of the phase space volume (roughly speaking, the dissipation rate) through the sum of all the Lyapunov exponents, allowing the calculation of transport coefficients from the Lyapunov spectra $\sum_{i} \lambda^{(i)}$. The conjugate pairing rule for Lyapunov spectra reduces the calculation of the sum of all the Lyapunov exponents to the sum of just one pair of Lyapunov exponents $\lambda_{1,2}$. The set of all positive Lyapunov exponents specifies the natural invariant measure $\mathcal{F}$, which is used to calculate various quantities using periodic orbit theory and led to the first form of the fluctuation theorem. On the other hand, recently, much attention has been paid to individual Lyapunov exponents and their Lyapunov vectors for many-body systems. As each Lyapunov exponent has dimensions of inverse time, the Lyapunov spectrum can be regarded as a time-scale spectrum. From this point of view, Lyapunov exponents with small absolute values are connected to large and macroscopic time-scale behavior of many-body systems, and in this region the wave-like structure of Lyapunov vectors, known as the Lyapunov modes, is observed. The Lyapunov mode is a reflection of a collective movement (phonon mode) of many-body systems, and comes from dynamical conservation laws and translational invariance. On the other hand, large Lyapunov exponents are dominated by small and microscopic time-scale movement, and in this region the spatially localized behavior of Lyapunov vector, the so-called Lyapunov localization, appears.

For the largest Lyapunov exponent of many-body chaotic systems, analytical calculations have been attempted. The time-scale separation in many-body systems is crucial to extract a macroscopic dynamics from microscopic many-body dynamics, and the Lyapunov spectrum allows us to discuss it dynamically. As one of the features of Lyapunov vectors for many-body systems, the Lyapunov localization appears as a behavior in which the Lyapunov vector components for a few particles are significantly larger than the other components. Moreover, the localized region of a Lyapunov vector moves as a function of time. The magnitude of the localization of each Lyapunov vector can be measured quantitatively by the Lyapunov localization spectrum, which is defined as a set of exponential functions of entropy-like quantities for the normalized amplitudes of the Lyapunov vector components. We have previously reported that the Lyapunov localization spectra show a bending behavior at low density, and its connection with kinetic theory properties (e.g. the Krylov relation for the largest Lyapunov exponent, and the mean free time being inversely-proportional to density, etc.) is discussed for many-hard-disk systems. However, the Lyapunov localization spectrum requires taking a time-average and characterizes only the static localization of the Lyapunov vectors. The physical meaning of the movement of the localized regions of Lyapunov vectors is not clearly understood.

The principal aim of this paper is to discuss dynamically the movement of the localized region of the Lyapunov vectors in many-hard-disk systems at low density. To discuss this problem we use the fact shown in Ref. at low density only two particle components of the normalized Lyapunov vector for large Lyapunov exponents have a non-zero value. This is due to the short-range of particle interactions in a many-hard-disk system. The movement of the localized region of the Lyapunov vector appears to be a series of jumps or hops, so we can introduce a hopping rate to describe the dynamics. To simplify the problem, in this paper we consider quasi-one-dimensional systems, in which the system width is so narrow that disks always remain in the same order

I. INTRODUCTION
We show that this hopping rate depends on the hopping distance, and is a decreasing function of the hopping distance. This implies that there is a spatial correlation among localized regions of Lyapunov vectors.

We explain the hopping-distance dependence of the hopping rate in many-hard-disk systems in two ways. In the first approach we use a simple model expressed as an accumulation of bricks. Here, the hopping of the localized region of the Lyapunov vectors is expressed as a change in the position of the highest brick site. This model is a one-dimensional version of the so-called clock model, which has been used to calculate Lyapunov exponents for many-hard-disk systems \[31, 32, 33\]. In this paper we demonstrate that this model can reproduce the largest Lyapunov exponent for quasi-one-dimensional hard-disk systems. As the second approach to the hopping behavior of the localized Lyapunov vectors, we propose an analytical method to calculate the hopping rate from the sum of probability distributions for the brick height configurations between two separated highest brick sites. Using this analytical approach we can also discuss the relation between the hopping rate and the Lyapunov exponent. Hopping rates calculated by these two approaches are in good agreement with the ones for quasi-one-dimensional hard-disk systems.

The outline of this paper is as follows. In Sec. II we introduce the quasi-one-dimensional hard-disk system, and show the localized behavior of the Lyapunov vector corresponding to the largest Lyapunov exponent at low density. In Sec. III we introduce the hopping rate of the localized region of the Lyapunov vectors, and show the hopping-distance dependence for many-hard-disk systems. In Sec. IV we discuss the hopping rate using a brick accumulation model. In Sec. V we propose an analytical expression for the hopping rate. Section VI is our conclusion and some remarks. In Appendix A we discuss the technical details of the calculation of the hopping rate. In Appendix B we give a microscopic derivation of the brick accumulation model.

II. QUASI-ONE-DIMENSIONAL SYSTEM AND LOCALIZATION OF LYAPUNOV VECTORS

The system which we consider in this paper is a quasi-one-dimensional system consisting of \( N \) hard-disks in periodic boundary conditions. All of the particles are identical with radius \( R \) and mass \( M \), and the shape of the system is rectangular with the length \( L_x \) and the width \( L_y \) satisfying the inequality \( 2R < L_y < 4R \). The schematic illustration of the system is given in Fig. 1, in which we number particles \( 1, 2, \ldots, N \) from the left to right in this system. For the actual numerical results shown in this paper, we used: the radius of a particle \( R = 1 \), the mass of a particle \( M = 1 \), the total energy of the system \( E = N \), the system width \( L_y = 2R(1 + 10^{-6}) \), and the system length \( L_x = N L_y (1 + d) \) with the constant \( d \) controlling the density \( \rho \equiv N \pi R^2 / (L_x L_y) \). In the quasi-one-dimensional system, the particle interactions are restricted to nearest-neighbor particles only, so particles remain in the same order. These features require less calculation effort and a simpler representation of results for quasi-one-dimensional system compared with fully two-dimensional systems. The quasi-one-dimensional system has already been used to investigate the localized behavior of Lyapunov vectors \[28, 31\], the wave-like structure of Lyapunov vectors \[18, 19, 21\] and the transition between quasi-one-dimensional and fully two-dimensional systems \[34\].

The dynamics of Lyapunov vectors in many-hard-disk systems is well established, and readers should refer to the references, for example, Ref. \[41\], for more detailed discussions. In many-hard-disk systems the dynamics is separated into a free-flight part and a collision part, and the free-flight part of the dynamics is integrable. This property allow us to express the dynamical evolution as a simple multiplication of time-evolutional matrices for the free-flight dynamics and the collision dynamics, leading to a fast and more accurate numerical simulation than for soft-core interaction models. For numerical calculations of the Lyapunov vectors shown in this paper, we use the algorithm developed by G. Benettin, et al \[37\] and I. Shimada and T. Nagashima \[38\] (also see Refs. \[39, 40\]). This algorithm is characterized by intermittent (e.g. after every collision) re-orthogonalization and renormalization of Lyapunov vectors, preventing a divergence of the amplitude of the Lyapunov vectors.

In this paper we use the notation \( \delta \mathbf{\Gamma}^{(n)}(t) = (\delta \Gamma_{1}^{(n)}(t), \delta \Gamma_{2}^{(n)}(t), \ldots, \delta \Gamma_{N}^{(n)}(t)) \) for the Lyapunov vector corresponding to the \( n \)-th Lyapunov exponent \( \lambda^{(n)} \) at time \( t \). Here, \( \delta \Gamma_{j}^{(n)}(t) \) is the Lyapunov vector component contributed by the \( j \)-th particle in the \( n \)-th Lyapunov exponent at time \( t \). To express the localized behavior of the Lyapunov vectors we introduce the quantity \( \gamma_{j}^{(n)}(t) \) as

\[
\gamma_{j}^{(n)}(t) = \frac{|\delta \Gamma_{j}^{(n)}(t)|^2}{\sum_{k=1}^{N} |\delta \Gamma_{k}^{(n)}(t)|^2},
\]

which is the normalized amplitude of the Lyapunov vector component for the \( j \)-th particle for the \( n \)-th Lyapunov exponent \( \lambda^{(n)} \) at time \( t \). The localized behavior of Lyapunov vector, namely the Lyapunov localization, is the phenomenon where only a few of the \( \gamma_{j}^{(n)}(t) \), \( j = 1, 2, \ldots, N \) have a non-zero value at any time \( t \).

Fig. 2 is an example of the Lyapunov localization. It is a graph of the normalized amplitude \( \gamma_{j}^{(1)} \) of the Lyapunov vector components for the \( j \)-th particle corresponding to the largest Lyapunov exponent \( \lambda^{(1)} \) as a function of the particle index \( j \) and the collision number \( n_{c} \) (\( \approx t / \tau \) with time \( t \) and the mean free time \( \tau \)). The system is
a quasi-one-dimensional system consisting of 50 hard-disk particles and with $d = 10^3$ (the density $\rho \approx 7.85 \times 10^{-6}$). From this figure we recognize that the non-zero components of the Lyapunov vector are concentrated at two nearest-neighbor particles. This characteristic is shown for hard-disk systems at low density and for large Lyapunov exponents in general, using the Lyapunov localization spectrum [28, 34].

Moreover, we observe that such localized regions of Lyapunov vector components move with time in discrete jumps or hops, in Fig. 2. This characteristic hopping has already been shown in Ref. [28] and that such spatial hopping of the non-zero components of $\gamma^{(n)}(t)$ are caused by some particle-particle collisions. However, not every collision causes a hopping of the localized Lyapunov vector. In this sense, particle collisions themselves are not sufficient to explain the hopping movement of the Lyapunov localization.

The Lyapunov localization appears in the Lyapunov vectors corresponding to large Lyapunov exponents in general, but in this paper for simplicity we consider only the Lyapunov vector $\delta \Gamma^{(1)}$ corresponding to the largest Lyapunov exponent $\lambda^{(1)}$.

III. HOPPING RATE OF LOCALIZED LYAPUNOV VECTORS

An advantage of the quasi-one-dimensional system in investigations of Lyapunov localization is that the movement of particles in the transverse direction are suppressed, and roughly speaking, the particle sequence corresponds to the particle's position. Noting this feature, in this section we describe the hopping behavior of spatially localized Lyapunov vectors as the hopping of particle indices whose $\gamma^{(n)}_j$ defined by Eq. (1) have non-zero values.

As shown in Refs. [28, 34], the particle indices with non-zero $\gamma^{(1)}_j$ are a pair of nearest-neighbor particles in the low density limit, and we can introduce the hopping distance $h$ at each collision as the change of particle indices. In this definition the hopping distance $h$ is an integer number satisfying the inequality $-[N/2] \leq h \leq [N/2]$, where $[x]$ is the integer part of the real number $x$. It should also be noted that we take the particle index $j$ as the one equivalent to the index $j \pm N$, because of periodic boundary conditions, so the hopping distance $h \pm N$ is equivalent to $h$. More technical details of the calculation of the hopping distance $h$ are given in Appendix A. Using this hopping distance $h$, in numerical simulations, we count the number $N_T(h)$ of hops with hopping distance $h$ in a time-interval $T$, and we define the hopping rate $P_N(h)$ as a function proportional to $N_T(h)$ as $T \to \infty$. In this paper we use the hopping rate $P_N(h)/P_N(1) = \lim_{T \to \infty} N_T(h)/N_T(1)$ normalized by $P_N(1)$. From the reflection symmetry of the quasi-one-dimensional system in the longitudinal direction, the hopping rate $P_N(h)$ must be symmetric, namely $P_N(h) = P_N(-h)$. We use this hopping rate to quantitatively discuss the hopping dynamics for localized Lyapunov vectors.

Figure 3 shows log-log plots of the hopping rates $P_N(h)/P_N(1)$ normalized by $P_N(1)$ as a function of $|h|$ in a quasi-one-dimensional hard-disk system with $d = 10^3$ at different numbers of particles: $N = 25$ (circles), 50 (triangles), and 75 (squares). Noting the symmetric property $P_N(-h) = P_N(h)$, we use $[P_N(h) - P_N(-h)]/P_N(1)$ as error bars in Fig. 3. It is clear from Fig. 3 that the hopping rate $P_N(|h|)$ decreases as $|h|$ increases. This implies that there is a spatial correlation among the localized regions of a Lyapunov vector, rather than a random hopping.

The turn-up in the tail of the normalized hopping rate $P_N(h)/P_N(1)$ in Fig. 3 can be explained as an effect of periodic boundary conditions. Under periodic boundary conditions the hopping distance $h + jN$ for any integer $j$ is observed as the hopping distance $h$ in the quasi-one-dimensional system consisting of $N$ hard disks. Therefore, using the hopping rate $P_\infty(h)$ for the thermodynamics limit ($N \to \infty$ at a fixed density) for $h = 0, \pm 1, \pm 2, \cdots, \pm \infty$, the hopping rate $P_N(h)$ for a finite system should be represented as

$$P_N(h) = \sum_{j=-\infty}^{\infty} P_\infty(h + jN) \quad (2)$$

for $h = \pm [N/2], \pm [N/2] + 1, \cdots, [N/2]$. The terms on the right-hand side of Eq. (2) with $j \neq 0$ cause the turn-up in the tail of the hopping rate $P_N(h)$ for a finite system.

In this sense we can explain the $N$-dependence of the hopping rate $P_N(h)$ using an $N$-independent asymptotic distribution $P_\infty(h)$. To make this explanation convincing we fitted the numerical data for $P_N(h)/P_N(1)$ shown in Fig. 3 to the function $f(h) \equiv \alpha |h|^\beta + \sum_{j=1}^{\infty} ([h + jN]^\beta + |h - jN|^\beta)$ assuming that the decay of $P_\infty(h) = \alpha |h|^\beta$ with fitting parameters $\alpha$ and $\beta$, and neglecting the higher order small terms for $|j| \geq 3$. Here, the fitting lines are dotted for $N = 25$, the broken for $N = 50$ and solid for $N = 75$. Note that these sets ($\alpha, \beta$) of fitting parameter values, summarized in Table I are almost independent.
TABLE I: Fitting parameters for the power function $P_\infty(h) = \alpha |h|^\beta$ for quasi-one-dimensional systems of different numbers of particles $N$ with $d = 10^3$. Notice that the coefficients $\alpha$ and $\beta$ are essentially independent of $N$.

| $N$ | $\alpha$ | $\beta$ |
|-----|----------|---------|
| 25  | 0.788    | 1.67    |
| 50  | 0.788    | 1.72    |
| 75  | 0.753    | 1.71    |

FIG. 4: The brick accumulation model expressing the dynamics of the amplitude of the Lyapunov vector component for each particle in a quasi-one-dimensional system.

of the number of particles $N$. As shown in Fig. 3 these fitting lines nicely reproduce the values of the numerical data, especially the turn-ups.

We can also calculate the hopping rate $P_N(h)$ at different densities, as long as the density is low enough so that only a few of the normalized Lyapunov vector component amplitudes $\{\gamma_j^{(n)}\}$ have non-zero values. In Appendix A we show that the normalized hopping rate $P_N(h)/P_N(1)$ is almost density-independent at least for $10^3 < d < 10^5$, (namely in the density region $7.85 \times 10^{-6} < \rho < 7.85 \times 10^{-4}$). However, a subtle increase of the normalized hopping rate as the density decreases is recognizable in this low density region.

IV. BRICK ACCUMULATION MODEL

As shown in Sec. III the hopping rate of the localized region of the Lyapunov vector is correlated spatially. In this section we explain this characteristic using a simple one-dimensional model, which we call the brick accumulation model.

A schematic illustration of the brick accumulation model, or simply the brick model, is given in Fig. 4. It is a one-dimensional model with $N$ sites in the horizontal direction. The bricks are dropped at random and occupy a pair of neighboring sites. Each brick has width two and the height 1 and they accumulate at sites without overlap. The dynamics of the brick model is described using the brick height $K_j(n)$ at the $j$-th site after $n$ bricks have been dropped. The total brick height $K_j(n)$ takes on integer values and its dynamics is expressed as follows. If the $n$-th brick is dropped on sites $j_n$ and $j_n + 1$ then

$$K_i(n) = \begin{cases} \max \{K_{j_n}(n-1), K_{j_n+1}(n-1)\} + 1 & \text{if } l \in \{j_n, j_n + 1\} \\ K_i(n-1) & \text{if } l \notin \{j_n, j_n + 1\} \end{cases}$$

(note that the particle index $N + 1$ is equivalent to 1). Here the site number $j_n$ is chosen randomly on $[1, N]$ for each $n$.

One may notice that the brick accumulation model is a one-dimensional version of the so-called clock model 31, 32, 33, which has been used to calculate the Lyapunov exponents for many-hard-disk systems. In the clock model, the brick height $K_j(n)$ is usually called the clock value of the $j$-th particle after the $n$-th particle-particle collision, and the dynamics is expressed as an adjustment of the clock values of colliding particles. However we use the name brick accumulation model for the one-dimensional version of the clock model, because in a simple image of brick accumulations, it is easily to visualize the configuration of brick heights which is essential for the analytical approach to the hopping rate discussed in the next section IV, although this image is applicable only for the one-dimensional case. The image of brick accumulations also helps us to easily recognize the similarity of this model to ballistic aggregation models, whose scaling properties have been studied analytically and numerically 32, 33.

In the brick model described by the brick height $K_j(n)$, the site number $j$ corresponds to the particle index, and the number $n$ of dropped bricks corresponds to the collision number $n_t$ for the quasi-one-dimensional hard-disk system. Moreover, the brick height $K_j(n)$ itself is connected to the Lyapunov vector component amplitude $\delta F^{(1)}(t)$ by

$$\left| \delta F^{(1)}(\tau n_t) \right| \sim \left| \delta F^{(1)}(0) \right| \exp \{-K_j(n_t) \ln \rho\}$$

for the particle index $j$ and the collision number $n_t$ with the mean free time $\tau$ in the asymptotic limit of low density $\rho \to 0$. The derivation of the relation from microscopic dynamics for hard-particle systems is given in Appendix B.

From the relation between the brick height and the Lyapunov vector, the amplitude of the localized Lyapunov vector components with the highest brick height have much larger values than the other components because of the huge factor $-\ln \rho$ appearing in the exponent of Eq. 4 at low density $\rho << 1$. Therefore, to a good approximation, the normalized amplitude $\gamma_j^{(n)}$ given by Eq. 1 has non-zero components only for particles corresponding to these largest amplitudes, as shown in Fig. 2. The combination of Eqs. 3 and 4 also explains why the amplitudes of the localized components of the Lyapunov vectors for nearest-neighbor particles take almost the same value, thus appearing as a flat top in the localized regions in Fig. 2. After all, the localized region of the Lyapunov vector represented in Eq. 4 is given as the site indices whose brick height $K_j(n_t)$ take the maximum value after the $n$-th brick is dropped, and such sites with the highest brick height can be calculated using the brick model dynamics 40 only, without referring to the hard-disk dynamics. Based on these features we can calculate the hopping distance $h$ for the brick accumulation model (see the end of Appendix A for more detail of the hopping distance for the brick model), and therefore the hopping rate $P_N(h)$, similarly to the quasi-one-dimensional hard-disk system.

Figure 4 shows the normalized hopping rates $P_N(h)/P_N(1)$ for the brick model with 50 sites (cir-
cles) and for a quasi-one-dimensional hard-disk system with \( d = 10^5 \) (squares) as a function of the absolute value of hopping distance \(|h|\) for \( N = 50\) as log-log plots. The error bars are given by \( |P_N(h) - P_N(-h)|/P_N(1)\). The broken line is the analytical expression for the hopping rate discussed in Sec. II.

**TABLE II:** Particle number \((N)\) dependences of the ratio \( P_N(0)/P_N(1)\) of the hopping rates at the hopping distances \( h = 0 \) and 1 in the quasi-one-dimensional hard-disk systems with \( d = 10^5 \) and in the brick accumulation model.

| \( N \)  | Hard-disk model \( P_N(0)/P_N(1) \) | Brick model \( P_N(0)/P_N(1) \) |
|---------|-----------------|-----------------|
| 25      | 13.5            | 17.1            |
| 50      | 26.8            | 37.9            |
| 75      | 39.6            | 58.8            |
| 100     | 52.1            | 79.7            |

FIG. 5: Normalized hopping rates \( P_N(h)/P_N(1)\) for the brick model (circles) and the quasi-one-dimensional system with \( d = 10^5 \) (squares) as a function of the absolute value of hopping distance \(|h|\) for \( N = 50\) as log-log plots. The error bars are given by \( |P_N(h) - P_N(-h)|/P_N(1)\). The broken line is the analytical expression for the hopping rate discussed in Sec. II.

FIG. 6: The sum \( \sum_{j=1}^{N} \mathcal{K}_j(n) \) of brick heights \( \mathcal{K}_j(n) \) as a function of the number \( n \) of dropped bricks in the brick accumulation model with \( N = 50\). Inset: The same graph but enlarged in the much smaller \( n \) interval [1000, 1050]. The broken line (which is almost on the data points of the main graph) is a fit of numerical data to a linear function.

Before finishing this section, we discuss one more property of the brick accumulation model, which will use in the next section. Fig. 6 is the sum \( \sum_{j=1}^{N} \mathcal{K}_j(n) \) of brick heights as a function of \( n \). Note that this sum must increase at a speed of more than 2 per dropped brick, because accumulated bricks capture spaces below them which cannot be occupied by further dropped bricks. It is clear from this figure that this sum increases linearly, and a fit of the data to the function \( \sum_{j=1}^{N} \mathcal{K}_j(n) = \alpha + \beta n \) with fitting parameters \( \alpha \) and \( \beta \) leads to the values \( \alpha \approx 0.00 \) and \( \beta \approx 3.99 \) [47]. (Note that data points in the main graph of Fig. 6 look exactly like this fit because of the large scale. In the inset to Fig. 6 we show the graph of \( \sum_{j=1}^{N} \mathcal{K}_j(n) \) as a function of \( n \) on a much smaller scale, to show its fluctuating behavior.) Therefore, on average, each dropped brick adds a contribution of 4 to the sum \( \sum_{j=1}^{N} \mathcal{K}_j(n) \), meaning that each brick occupies not only its own 2 spaces but also captures 2 empty spaces below it.

It is important to note that from Eq. (4) the brick height \( \mathcal{K}_j(n) \) dominates the exponential growth rate of the Lyapunov vector component amplitude. Using this feature of the brick heights, their sum \( \sum_{j=1}^{N} \mathcal{K}_j(n) \) is connected to the largest Lyapunov exponent \( \lambda^{(1)} \) as

\[
\lambda^{(1)} \sim - \lim_{n \to +\infty} \frac{1}{n \tau} \frac{1}{N} \sum_{j=1}^{N} \mathcal{K}_j(n) \ln \rho
\]

with the mean free time \( \tau \) and the density \( \rho \) in the asymptotic limit of low density. [More detailed discussion for the formula (5) is given in Appendix B.] As a numerical check of the formula (5) we show in Fig. 7 the largest Lyapunov exponent \( \lambda^{(1)} \) as a function of the density \( \rho \) in a quasi-one-dimensional hard-disk systems and in the brick accumulation model using the formula (5) with \( N = 50\).

Here, to calculate the largest Lyapunov exponent from Eq. (5) we used the relation \( \sum_{j=1}^{N} \mathcal{K}_j(n) \approx 4n \) as shown in Fig. 6 and the values of the mean free time \( \tau \) and the density \( \rho \) of the quasi-one-dimensional system whose Lyapunov exponents are plotted in Fig. 7 and the data points are connected by a dashed line for ease of visibility. Figure 7 shows that Eq. (5) reproduces successfully the values of the largest Lyapunov exponent for the quasi-one-dimensional hard-disk systems, not only in the limit of low density but also at relatively high density such as \( \rho < 0.3 \). It may be noted that a linear dependence of the sum \( \sum_{j=1}^{N} \mathcal{K}_j(n) \) of brick heights with respect to \( n \)
is necessary to get a finite value of the largest Lyapunov exponent $\lambda^{(1)}$ by Eq. (5).

V. ANALYTICAL EXPRESSION FOR THE HOPPING RATE

In this section we discuss another approach to the hopping rate of the localized region of the Lyapunov vectors. This approach is inspired by some of the characteristics of the brick accumulation model discussed in Sec. IV, although we greatly simplified the brick model by omitting some other aspects of the model. The advantage of this approach is that we can get an analytical expression for the hopping rate, while to a good approximation it still reproduces the hopping rate for hard-disk systems. Another important point in this approach is that it connects the dynamics of the localized region of the Lyapunov vectors with a static property, the probability distribution of brick-height differences between nearest-neighbor sites.

In the brick accumulation model, a hop of the highest brick site occurs when two (non-nearest-neighbor) sites have the same brick height. Noting this characteristic we consider the probability distribution $\tilde{P}_∞(h)$ under the constraint that the two sites $\mu$ and $\mu + h$ have the highest brick height. We require that there is no other highest site between the two highest sites $\mu$ and $\mu + h$ for $|h| \geq 2$:

$$K_1(n) < K_\mu(n) = K_{\mu+h}(n),$$

$$\text{in } l = \begin{cases} 
\mu + 1, \mu + 2, \cdots, \mu + h - 1 & \text{for } h \geq 2 \\
\mu + h + 1, \mu + h + 3, \cdots, \mu - 1 & \text{for } h \leq -2,
\end{cases}$$

Using this probability distribution $\tilde{P}_∞(h)$, we can estimate the hopping rate $\tilde{P}_∞(h)$ in the thermodynamic limit as the one proportional to $\tilde{P}_∞(h)$: $P_∞(h) \propto \tilde{P}_∞(h)$. Then we can calculate the hopping rate $P_N(h)$ for a finite size system using Eq. (2), apart from a constant factor.

To calculate the probability distribution $\tilde{P}_∞(h)$ we introduce the distribution $\Lambda(k)$ of brick-height differences $k$ between nearest-neighbor sites. These two distribution functions are connected by

$$\tilde{P}_∞(h) = \sum_{k_1} \sum_{k_2} \cdots \sum_{k_{|h|-1}} \Lambda(-k_1)\Lambda(-k_2)\cdots\Lambda(-k_{|h|-1})\Lambda\left(\sum_{j=1}^{k_{|h|-1}} k_j\right),$$

In Eq. (7), $-k_i$ is a brick-height difference of nearest neighbor sites ($l = 1, 2, \cdots, |h| - 1$). The probability distribution for the specific brick-height configuration with $k_l$, $l = 1, 2, \cdots, |h| - 1$ is given by $\Lambda(-k_1)\Lambda(-k_2)\cdots\Lambda(-k_{|h|-1})\Lambda(\sum_{j=1}^{k_{|h|-1}} k_j)$, and the hopping rate $\tilde{P}_∞(h)$ is calculated as the summation of this probability distribution over possible values of $k_l$, $l = 1, 2, \cdots, |h| - 1$.

In Eq. (7), the lower bounds of the sums on the right-hand side of Eq. (6) come from the condition $\sum_{j=1}^{k_{|h|-1}} k_j$ for a finite configuration with $\sum_{j=1}^{k_{|h|-1}} k_j$ for a finite configuration with $\sum_{j=1}^{k_{|h|-1}} k_j$, and the hopping rate $\tilde{P}_∞(h)$ is an even function of $h$, namely $\tilde{P}_∞(-h) = \tilde{P}_∞(h)$.

Now, for simplicity, we assume that the distribution function $\Lambda(k)$ can be expressed as an exponential function

$$\Lambda(k) = W \exp\{-\eta|k|\}$$

with constants $W$ and $\eta$ ($> 0$). (We will discuss the validity of this assumption later.) Inserting Eq. (8) into Eq. (7), and replacing the sums over $k_1$ in Eq. (7) with the ones over $k'_1 \equiv k_1 + 1$, we obtain

$$\tilde{P}_∞(h) = \begin{cases} 
W^2 \Upsilon \sum_{k=0}^{+\infty} \Upsilon^k & \text{for } |h| = 2 \\
W^{\left|h\right|} \Upsilon \sum_{k_1=0}^{+\infty} \Upsilon^{k_1} \sum_{k_2=-k_1}^{+\infty} \Upsilon^{k_2 \theta(k_2)} \sum_{k_3=-k_2-k_1}^{+\infty} \Upsilon^{k_3 \theta(k_3)} \cdots \sum_{k_{|h|-1}=-k_{|h|-2}}^{+\infty} \Upsilon^{k_{|h|-1} \theta(k_{|h|-1})} & \text{for } |h| \geq 3
\end{cases}$$

where $\Upsilon$ is defined by

$$\Upsilon \equiv \exp\{-2\eta\}$$

and $\theta(x)$ is the Heaviside function taking the value 1 for $x > 0$ and the value 0 for $x \leq 0$. The summations
appearing in Eq. (9) can be carried out successively, and we obtain

\[ \tilde{P}_\infty(h) = \frac{W(h)}{(1 - \gamma)^{|h| - 1}} \Omega(|h|) \tag{11} \]

where the function \( \Omega(k) \) of \( k \) is can be written as

\[ \Omega(2) = 1, \tag{12} \]
\[ \Omega(3) = 1 + \gamma, \tag{13} \]
\[ \Omega(4) = 1 + 3\gamma + \gamma^2, \tag{14} \]
\[ \Omega(5) = (1 + \gamma)(1 + 5\gamma + \gamma^2), \tag{15} \]
\[ \Omega(6) = 1 + 10\gamma + 20\gamma^2 + 10\gamma^3 + \gamma^4, \tag{16} \]
\[ \Omega(7) = (1 + \gamma)(1 + 14\gamma + 36\gamma^2 + 14\gamma^3 + \gamma^4), \tag{17} \]

and so on. Eq. (11), with Eqs. (12), (13), (14), (15), (16), etc., gives an analytical expression for the hopping rate, for example, using the relation \( P_\infty(h)/P_\infty(1) = \tilde{P}_\infty(h)/\tilde{P}_\infty(1) \).

The coefficients \( W \) and \( \eta \) appearing in Eq. (5) can be determined from the two sum rules:

\[ \sum_{k = -\infty}^{+\infty} \Lambda(k) = 1, \tag{18} \]
\[ \sum_{k = -\infty}^{+\infty} |k|\Lambda(k) = \Delta K, \tag{19} \]

where \( \Delta K \) is the mean value of the absolute value of the brick height difference between nearest-neighbor sites. The first condition (13) is the normalization of the probability distribution \( \Lambda(k) \) which leads to

\[ W = \frac{\exp(\eta) - 1}{\exp(\eta) + 1}. \tag{20} \]

Using Eq. (20) the second condition (19) gives

\[ \eta = \ln \left\{ \frac{1 + \sqrt{1 + \Delta K^2}}{\Delta K} \right\}, \tag{21} \]

satisfying the inequality \( \eta > 0 \). Inserting Eq. (21) into Eqs. (10) and (20) we obtain

\[ \gamma = \left( \frac{\Delta K}{1 + \sqrt{1 + \Delta K^2}} \right)^2 \tag{22} \]
\[ W = \frac{1 + \sqrt{1 + \Delta K^2} - \Delta K}{1 + \sqrt{1 + \Delta K^2} + \Delta K}. \tag{23} \]

From Eqs. (22) and (23), there is only one parameter \( \Delta K \) remaining to determine the hopping rate using Eq. (11).

To estimate the value of \( \Delta K \), we use a property of the brick accumulation model discussed at the end of Sec. (14). Previously, we showed an approximate relation \( \sum_{j=1}^{N} K_j(n) \approx 4n \) for the sum of brick heights, which means that in the brick accumulation model each dropped brick gives a mean contribution of 4 to this sum. This contribution consists of a contribution of 2 as the space occupied by a brick itself and another 2 as empty space below the brick which can now not be occupied by other bricks. This implies that the averaged brick-height difference between nearest-neighbor sites is about 2, so that

\[ \Delta K \approx 2. \tag{24} \]

We use this value to calculate the hopping rate based on Eq. (11). One may notice that from the relation \( \sum_{j=1}^{N} K_j(n) \approx (\Delta K + 2)n \) and the formula (5) we obtain

\[ \Delta K \sim \frac{N\tau}{\ln \rho} \gamma(1) - 2 \tag{25} \]

in the limit of low density. From the relation between the parameter \( \Delta K \) specifying the hopping rate and the largest Lyapunov exponent \( \gamma(1) \), the hopping rate of the localized region of the Lyapunov vectors is connected to the largest Lyapunov exponent.

Before comparing the hopping rate based on the analytical expression (11) with the ones for hard-disk systems and the brick accumulation model, we note that this analytical expression for the hopping rate may not be appropriate for small hopping distances \( |h| \). In particular, it does not give the correct value of \( P_N(1) \), because in the brick model the hopping distance \( h = \pm 1 \) does not appear from separated sites with the same brick height, the assumption used to derive Eq. (11). Therefore it is not appropriate to calculate the hopping rate normalized by \( P_N(1) \) from this analytical expression and to compare it with the numerical results. The hopping rate \( P_N(\pm 2) \) from Eq. (11) may also be problematic, because in the brick model the hopping distance \( h = \pm 2 \) occurs when 4 consecutive sites have the highest brick height, while to derive Eq. (11) we assumed that \( h = \pm 2 \) occurs when non-consecutive separate sites \( \mu \) and \( \mu + h \) have the same highest brick height. Based on these considerations we do not calculate the value \( P_N(\pm 1) \) from the analytical approach in this section, and plot the hopping rate so that \( P_N(5) \) by Eq. (11) gives the same value as that from the brick model.

In Fig. 5 for the normalized hopping rate \( P_N(h)/P_N(1) \), we plotted the function \( \Psi(h) \equiv \tilde{P}_N(h)\tilde{P}_N(5)/[\tilde{P}_N(5)\tilde{P}_N(1)] \) using the value \( P_N(5)/P_N(1) \) of the brick model with \( P_N(h) = P_N(\infty)/P_N(h) \). Using the analytical expression \( \tilde{P}_\infty(h) \) given by Eq. (11) for \( |h| \geq 2 \). Note \( \Psi(5)/P_N(1) \) so that \( \Psi(h) \) coincides with \( P_N(h)/P_N(1) \) of the brick model at \( |h| = 5 \).

Here, the value of the hopping rate values are given at integer values of \( h \), but we connect them with a broken line for ease of visibility. Figure 4 shows that the analytical expression (11) for the hopping rate reproduces the hopping rate for the brick accumulation model as well as the quasi-one-dimensional hard-disk system to a good approximation. It may be noted that for this plot we used the first two dominant terms on the right-hand side of Eq. (2), and part of the small deviation of the hopping rate in the tail between the brick model and the analytical expression should come from the omission of higher order terms in Eq. (2).

We notice that the approach in this section is simple enough to get an analytical expression for the hopping rate but it is not completely consistent with the
brick accumulation model discussed in the previous section. Previously, we have mentioned the irrelevance of Eq. (11) as a description of a small hopping rate in the brick model. Actually, in the approach of this section we omitted the essential characteristic of the bricks as components of the brick accumulation model, except for the property (21), and treated the model components as blocks (or half bricks). As another example, we show in Fig. 2 the numerical result for the distribution $\Lambda(k)$ of brick-height differences $k$ between nearest-neighbor sites in the brick accumulation model with $N = 50$. Here, $\Lambda(k)$ is normalized by $\sum_{k=-2N}^{2N} \Lambda(k) = 1$, instead of Eq. (18), because we cannot calculate $\Lambda(k)$ in $|k| \to +\infty$ numerically. In this figure we added the exponential distribution $\mathcal{S}$ using Eqs. (22), (23) and (24). Figure 9 shows that the probability distribution $\Lambda(k)$ does not coincide with an exponential distribution $\mathcal{S}$, although it may be justified as a first approximation. (On the other hand, the numerical evaluation of $\Delta K$ from the distribution $\Lambda(k)$ as a numerical result in Fig. 9 gives the value 1.99, then Eq. (24) is still justified.) On another point, in the brick model the highest brick sites appear as a pair of nearest-neighbor sites, but we did not take into account this characteristic in the analytical approach in this section. Despite the omission of characteristics of the brick accumulation model, the analytical approach in this section reproduces the hopping rate for many-hard-disk systems reasonably well, and it suggests that this approach still keeps enough of the essential characteristics that describe the dynamics of the Lyapunov localization.

VI. CONCLUSION AND REMARKS

In this paper we discussed the dynamics of the spatially localized region of the Lyapunov vector corresponding to the largest Lyapunov exponent in a quasi-one-dimensional hard-disk system. To discuss the dynamics of the localized region of the Lyapunov vector we introduced a hopping rate for the localized region, and showed that the hopping rate decreases as the absolute value of hopping distance increases. This hopping-distance dependence of the hopping rate was explained quantitatively in two ways: a brick accumulation model and an analytical approach. In the brick accumulation model, the hopping behavior of the localized Lyapunov vectors was explained as the movement of the highest position in the brick accumulations. It was shown that using this brick model we can calculate the largest Lyapunov exponent for quasi-one-dimensional hard-disk systems successfully. On the other hand, in the analytical approach the hopping rate was calculated from probability distributions for brick height differences of nearest neighbor sites via multiple summations over possible configurations that connect two separated highest sites. The result is related to the largest Lyapunov exponent. Both of the approaches successfully reproduced the hopping-distance dependence of the hopping rate for the localized Lyapunov vectors of quasi-one-dimensional hard-disk systems.

As a remark, it may be useful to mention a previous conjecture for the dynamics of the localized region of Lyapunov vectors corresponding to largest Lyapunov exponent. Before we started this work, there had been a view that the origin of the hopping behavior of localized Lyapunov vectors was:

\textbf{Conjecture} The localized region of the Lyapunov vector hops to the position of a new grazing collision.

This conjecture was suggested from the fact that a change of Lyapunov vectors in particle collisions may be dominated by the factor $1/(\sigma \cdot \Delta p)$ in the collision dynamics for Lyapunov vector (see Appendix B), where $\sigma$ is the normalized collision vector and $\Delta p$ is the momentum difference of the colliding particles before the collision. In this argument, if two particles collide at a small angle (a grazing collision) with a small value of $\sigma \cdot \Delta p$, then the Lyapunov vector can change significantly and the position of the localized region may move. However, this scenario cannot be correct for the following reasons. First, this conjecture implies that as the position at which a grazing collision occurs is random, the hopping distance should also be random. This contradicts our numerical result that the hopping rate decreases with increasing hopping distance, as shown in Fig. 9. Second, we show in Fig. 10 the distribution $D(\sigma \cdot \Delta p)$ of $\sigma \cdot \Delta p$ in general collisions (dashed line), and the distribution of collisions causing hopping of the localized Lyapunov vector (solid line) is independent of collision parameters like collision angles, the momentum difference of colliding particles, and also the quantity $\sigma \cdot \Delta p$. This also suggests that the conjecture for the origin of the hopping of the localized region of the Lyapunov vector cannot be justified. On the other hand, the numerical evaluation of $\Delta K$ may be justified as a first approximation. (On the other hand, the numerical evaluation of $\Delta K$ from the distribution $\Lambda(k)$ as a numerical result in Fig. 9 gives the value 1.99, then Eq. (24) is still justified.) On another point, in the brick model the highest brick sites appear as a pair of nearest-neighbor sites, but we did not take into account this characteristic in the analytical approach in this section. Despite the omission of characteristics of the brick accumulation model, the analytical approach in this section reproduces the hopping rate for many-hard-disk systems reasonably well, and it suggests that this approach still keeps enough of the essential characteristics that describe the dynamics of the Lyapunov localization.

FIG. 9: The distribution $\Lambda(k)$ of the brick-height differences $k$ between nearest-neighbor sites as a function of $|k|$ in the brick accumulation model with $N = 50$ as a linear-log plot. Here, $\Lambda(k)$ is normalized by $\sum_{k=-2N}^{2N} \Lambda(k) = 1$. The distribution $\Lambda(k)$ is an even function of $k$, and the error bars in this graph are given by $|\Lambda(k) - \Lambda(-k)|$. The line is an exponential function used to obtain the analytical expression for the hopping rate given by Eqs. (21), (22), (23) and (24).

FIG. 10: The distribution $D(\sigma \cdot \Delta p)$ of the quantity $\sigma \cdot \Delta p$ with the normalized collision vector $\sigma$ and the momentum difference $\Delta p$ of colliding particles just before the collision; for the general case (dashed line) and for the case in which non-zero hopping of a localized Lyapunov vector occurs. The system is a quasi-one-dimensional system with $d = 10^3$ and $N = 25$. 

As a remark, it may be useful to mention a previous conjecture for the dynamics of the localized region of Lyapunov vectors corresponding to largest Lyapunov exponent. Before we started this work, there had been a view that the origin of the hopping behavior of localized Lyapunov vectors was:

\textbf{Conjecture} The localized region of the Lyapunov vector hops to the position of a new grazing collision.

This conjecture was suggested from the fact that a change of Lyapunov vectors in particle collisions may be dominated by the factor $1/(\sigma \cdot \Delta p)$ in the collision dynamics for Lyapunov vector (see Appendix B), where $\sigma$ is the normalized collision vector and $\Delta p$ is the momentum difference of colliding particles before the collision. In this argument, if two particles collide at a small angle (a grazing collision) with a small value of $\sigma \cdot \Delta p$, then the Lyapunov vector can change significantly and the position of the localized region may move. However, this scenario cannot be correct for the following reasons. First, this conjecture implies that as the position at which a grazing collision occurs is random, the hopping distance should also be random. This contradicts our numerical result that the hopping rate decreases with increasing hopping distance, as shown in Fig. 9. Second, we show in Fig. 10 the distribution $D(\sigma \cdot \Delta p)$ of $\sigma \cdot \Delta p$ in general collisions (dashed line), and the distribution of collisions causing hopping of the localized Lyapunov vector (solid line) is independent of collision parameters like collision angles, the momentum difference of colliding particles, and also the quantity $\sigma \cdot \Delta p$. This also suggests that the conjecture for the origin of the hopping of the localized region of the Lyapunov vector cannot be justified. On the other hand, the numerical evaluation of $\Delta K$ may be justified as a first approximation. (On the other hand, the numerical evaluation of $\Delta K$ from the distribution $\Lambda(k)$ as a numerical result in Fig. 9 gives the value 1.99, then Eq. (24) is still justified.) On another point, in the brick model the highest brick sites appear as a pair of nearest-neighbor sites, but we did not take into account this characteristic in the analytical approach in this section. Despite the omission of characteristics of the brick accumulation model, the analytical approach in this section reproduces the hopping rate for many-hard-disk systems reasonably well, and it suggests that this approach still keeps enough of the essential characteristics that describe the dynamics of the Lyapunov localization.
hand, this collision parameter independence of the hopping behavior in the brick model cannot explain why the distribution $D(\sigma - \Delta P)$ is different from the general case and the hopping case in Fig. 11. This is an open problem.

As another open problem, the fits in Fig. 8 suggest that the hopping-distance dependence of the hopping rate in the thermodynamic limit seems to be a power law: $P_{\infty}(h) \sim h^\beta$ with $\beta \approx 1.7$. It remains to be determined whether this power behavior of the hopping rate can be justified analytically in the brick accumulation model or the analytical approach discussed in Sec. 1.

In this paper we showed that the hopping rate of the localized region of a Lyapunov vector, and the largest Lyapunov exponent, are well described by the brick accumulation model. Then, one may ask a more direct numerical check of the justification of the brick accumulation model in quasi-one-dimensional hard-disk systems, for example, to check Eq. (4) or to observe numerically an actual brick configuration like that presented in Fig. 4. However, such a check of the brick model is not trivial for the following reasons. First, the brick accumulation model is only justified in the limit of low density, but numerical simulations have to be at some finite density. This effect appears, for example, as gradual changes of the Lyapunov vector component amplitudes, as shown in Fig. 11, which do not appear in the brick accumulation model. Second, even if we could simulate at an extremely low density in which such finite density effects can be neglected (although the case presented in this paper is not at such a low density), the factor $-\ln \rho$ in Eq. (4) may be too large for an actual numerical calculation. Finally, to calculate Lyapunov vectors in this paper we used the algorithm developed by Benettin et al. 37, 40. This algorithm includes intermittent renormalizations of Lyapunov vectors, to prevent a divergence of the amplitudes of Lyapunov vectors in numerical calculations, but the brick accumulation model does not have such a normalization procedure in its dynamics. This difference makes a direct numerical check of Eq. (4) difficult in hard-disk systems. Different from Eq. (4) itself, the localized region of the Lyapunov vectors, which is required to calculate the hopping rate, is given simply by particle indices with the largest Lyapunov vector component amplitude, which is not influenced by such a difference of normalization procedure in calculations of Lyapunov vectors.

In this paper, in order to introduce the hopping rate used the property of quasi-one-dimensional systems, that the order of particles is an invariant. In this sense, it is not trivial to generalize our argument to fully two-(or three-) dimensional systems. An effective way to describe the dynamics for the localized region of the Lyapunov vectors for higher spatial dimensions remains an important future problem. Related to this it should still be noted that it is known that the clock model version of the brick accumulation model itself can be easily generalized to any spatial dimensional case, although in higher dimension we do not have the concept of the accumulation of bricks, as we do for the quasi-one-dimensional case.

Finally, one should notice that the brick accumulation model (more generally the clock model) used in this paper has been justified for hard-disk (or hard-sphere) systems only (at least so far). On the other hand, the Lyapunov localization is observed not only in many-hard-disk systems but also in a wide variety of many-body chaotic systems, such as the Kuramoto-Sivashinsky model, a random matrix model, map systems, coupled nonlinear oscillators, etc. It should be an important future problem to develop approaches to the dynamics of Lyapunov localization in this wider class of systems.

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APPENDIX A: HOPPING RATE OF A LOCALIZED LYAPUNOV VECTOR

In this appendix we give the detailed definition of the hopping rate for the localized region of the Lyapunov vectors, which is used in this paper.

As suggested in Ref. 28 44, the normalized Lyapunov vectors corresponding to the large Lyapunov exponents have non-zero components for only two particles in the low density limit, and changes of the non-zero components are caused by particle-particle collisions. Applying this characteristic of the Lyapunov vector to the quasi-one-dimensional system with periodic boundary conditions, we can introduce a hopping distance $h[n]$ at the $n$-th particle-particle collision as

$$h[n] = j_{n+1} - j_n - N \min \left\{ \frac{j_{n+1} - j_n}{N} \right\}$$

in the low density limit. Here $\{j_n, j_{n+1}\}$ are the non-zero components before the $n$-th collision and $\{j_{n+1}, j_{n+1}+1\}$ are the non-zero components just after the $n$-th collision. The function $\min\{x\}$ is the closest integer to the real number $x$. We assume that changes in the position of the localized region of the Lyapunov vector are negligible during the free-flight interval, so $\{j_{n+1}, j_{n+1}+1\}$ can also be interpreted as the set of the particle indices
FIG. 11: The normalized amplitude $\gamma_j^{(1)}$ of the Lyapunov vector component $\delta \Gamma_j^{(1)}$ corresponding to the largest Lyapunov exponent $\lambda_j^{(1)}$ as a function of the collision number $n$ and the particle index $j$ in a quasi-one-dimensional system with $d = 10^4$ and $N = 50$. Here, the collision number interval shown in this graph is $[k, k + 68]$ with $k = 3600448$. On the base of this graph is a contour plot of $\gamma_j^{(1)}$ at the level 0.2. Three hops of the localized region of the Lyapunov vector are visible in this time interval; The first two hops are sharp and the last one is gradually.

whose Lyapunov vector components take non-zero values just before the $(n+1)$-th collision. We count the number of times $N_T(h)$ that we see a hop of size $h$ in a time-interval $T$ where $-\lfloor N/2 \rfloor \leq h \leq \lfloor N/2 \rfloor$. The normalized hopping rate $P_N(h)$ can be introduced as $P_N(h) = \lim_{T \to \infty} N_T(h)/N_T(1)$.

However, in actual numerical simulations, the particle density $\rho$ is always finite, and non-zero Lyapunov components of more than two particles can often be seen, at least down to a density $\rho \approx 10^{-5}$ which is the low density limit of the numerical simulations in this paper. This makes the above definition (A1) of the hopping distance $h^{[n]}$ impractical. To explain this point concretely we show Fig. 11, which is a graph of the normalized Lyapunov vector component amplitude $\gamma_j^{(1)}$ defined by Eq. (1) as a function of the collision number $n$ and the particle index $j$ in a quasi-one-dimensional system with $N = 50$ and $d = 10^4$, (namely a density $\rho \approx 7.85 \times 10^{-5}$). In this figure we can recognize three types of hops of the localized region of the Lyapunov vector. The first two hops keep the non-zero Lyapunov vector components of almost two particles sharp enough to apply the definition (A1) of the hopping distance, but the third hop occurs gradually so that no clear hopping time can be determined. Note that the localized region of Lyapunov vector component amplitudes in the three dimensional plot (11) always has a flat top with a width of two-particles even for the third hop in Fig. 11.

In this paper, for concrete calculations, we define the localized region of the Lyapunov vector components as the particle indices $j$ for which $\gamma_j^{(n)} > 0.2$ (Here, we use the similarity between particle positions and particle indices given in Fig. 11 for the quasi-one-dimensional system.) In Fig. 12 this localized region is approximately given by the region surrounded by the contour lines (level 0.2) on the base of the graph. We introduce the quantity $l_n$ as the number of particles satisfying the inequality $\gamma_j^{(n)} > 0.2$ just before the $n$-th collision. Note that $0 \leq \gamma_j^{(n)} \leq 1$ by definition (11) of $\gamma_j^{(n)}$, so $l_n$ cannot be larger than 5. If $l_n$ is always 2 as in the low density limit, then we can use the hopping distance definition (A1), but in numerical simulations at finite density $l_n > 2$ can happen as shown in Fig. 11. The problem then is how do we define the hopping distance $h^{[n]}$ at the $n$-th collision, when $l_n > 2$.

The definitions of the hopping distance for each case are categorized as follows.

Case (a): $l_n = 2 \to l_{n+1} = 2$ Here only two particle indices are in the localized region before and after the collision, and they are always nearest-neighbors, so the hopping distance $h^{[n]}$ is given by Eq. (A1).

Case (b1): $l_n = 2 \to l_{n+1} = 3$ Here we assume that the localized regions of the Lyapunov vector are given by $(j_n, j_{n+1})$, $(j_{n+1}, j_{n+1}+1, j_{n+1}+2)$ and $j_{n+1} = j_n$ or $j_{n+1} = j_n - 1$. We take the value of the hopping distance to be $1$ (-1) for $h^{[n]}$ where $j_{n+1} = j_n$ ($j_{n+1} = j_n - 1$).

Case (b2): $l_n = 3 \to l_{n+1} = 2$ In this case we assume that the localized regions of the Lyapunov vector are given by $(j_n, j_{n+1}, j_{n+2})$, $(j_{n+1}, j_{n+1}+1)$ and $j_{n+1} = j_n$ or $j_{n+1} = j_n + 1$, and the hopping distance is 0 in both cases.

Case (b3): $l_n = 3 \to l_{n+1} = 3$ In this case we assume that the localized regions of the Lyapunov vector are given by $(j_n, j_{n+1}+1, j_{n+1}+2)$ and $j_{n+1} = j_n + h^{[n]}$ with the hopping distance $h^{[n]}$. We take into account the case $h^{[n]} = -1, 0$ or 1 only.

Case (c1): $l_n = 2 \to l_{n+1} = 4$ In this case we assume that the localized regions of the Lyapunov vector are given by $(j_n, j_{n+1}+1)$ and $(j_{n+1}, j_{n+1}+1)$ satisfying $|j_n - j_{n+1}| \geq 2$. The hopping distance is defined by Eq. (A1) using these $j_{n+1}$ and $j_n$.

Case (c2): $l_n = 4 \to l_{n+1} = 2$ In this case we assume that the localized regions of the Lyapunov vector are given by $(j_n, j_{n+1})$, $(j_{n+1}, j_{n+1}+1)$ and $(j_{n+1}, j_{n+1}+1)$ satisfying $|j_n - j_{n+1}| \geq 2$ and $|j_n - j_{n+1}| \leq 1$. The hopping distance $h^{[n]}$ is $h^{[n]} = j_{n+1} - j_n = -1, 0$ or 1.

Case (c3): $l_n = 4 \to l_{n+1} = 4$ In this case we consider only the case in which the localized regions of Lyapunov vector are given by $(j_n, j_{n+1})$, $(j_{n+1}, j_{n+1})$. 

FIG. 12: Schematic illustrations of the hopping types for the localized region of the Lyapunov vector in quasi-one-dimensional systems at low density. The contours represent the level $\gamma_j^{(1)} = 0.2$. (These are the contours on the base of Fig. 11) The vertical dotted line is collision number $n$ at which the hopping distance of the localized Lyapunov vector is to be determined. The seven illustrations in this figure indicate: (a) the case of $(l_n, l_{n+1}) = (2, 2)$, (b1) the case of $(l_n, l_{n+1}) = (2, 3)$, (b2) the case of $(l_n, l_{n+1}) = (3, 2)$, (b3) the case of $(l_n, l_{n+1}) = (3, 3)$, (c1) the case of $(l_n, l_{n+1}) = (2, 4)$, (c2) the case of $(l_n, l_{n+1}) = (4, 2)$, and (c3) the case of $(l_n, l_{n+1}) = (4, 4)$, where $l_n$ is the number of particles in the localized region of Lyapunov vector just before the $n$-th collision. The numbers in the right-bottom of each illustration give the possible values of the hopping distance.
The error bars are given by \(|t - |t\}| \times (1 + d) L_0^2\).

The hopping distance takes the value 0 in this case.

\{j_{n+1}, j_{n+1} + 1\} and \{j'_{n+1}, j'_{n+1} + 1\} satisfying

\(j_{n+1} = j_n, j'_{n+1} = j'_n\) and \(|j_n - j'_n| \geq 2\). The hopping distance takes the value 0 in this case.

(For each case above, the corresponding schematic illustration is shown in Fig. 12.) Here, we use periodic boundary conditions for the particle index, so that the localized regions \(\{N, 1\}, \{N, 1, 2\}\) and \(\{N - 1, N, 1\}\) should be translated to \(\{0, 1\}, \{0, 1, 2\}\) and \(\{-1, 0, 1\}\), respectively, in the above definition of the hopping distance. In the examples shown in Fig. 11 the first two hops of the localized Lyapunov vector can be described as case (a), and the third hop is described as cases (c1) and (c2). Note that asymmetric definitions of hopping distances \(h_n^{(b)}\) between cases (c1) and (c2) [and similarly between cases (b1) and (b2)] are required so that we can interpret the hopping distance of the third non-zero hop in Fig. 11 as only 2 in spite of it involving both cases (c1) and (c2). Using the above hopping distance we count the number \(N_T(h)\) of hops of distance \(h\) in time interval \(T\), and introduce the normalized hopping rate as \(P_N(h)/P_N(1) = \lim_{T \to \infty} N_T(h)/N_T(1)\) for \(h = \{-N/2\}, \{-N/2 + 1\}, \ldots, \{N/2\}\). Notice that there are possibilities apart from those shown above, but it is observed that in numerical simulations the probabilities of these are extremely small (for example, more than 96 percent of all hops could be categorized this way).

In Fig. 13 we show the normalized hopping rate \(P_N(h)/P_N(1)\) in a quasi-one-dimensional system of 50 hard-disks for \(d = 10^3\) (circles), 10⁴ (triangles) and 10⁶ (squares). The hopping rate is almost density-independent in this low density range, although it may be very slightly larger at the lowest density for \(|h| = 2, 3, \ldots\).

We also calculate the hopping rate for the brick accumulation model explained in Sec. 4 in a similar way. In the brick accumulation model we can introduce the localized region of the Lyapunov vectors as the site (particle) indices whose brick height are highest. For the brick model, cases (b1), (b2), and (b3) above cannot happen, and only cases (a), (c1), (c2) and (c3) above are taken into account in the numerical calculations. It may be noted that in the brick model, case (a) above can happen only when the hopping distance is \(-1, 0\) or 1, and Eq. (41) alone is not enough to calculate the hopping distance. This is another reason to take into account the case where \(l_n > 2\) in the calculation of the hopping distance.

**APPENDIX B: CLOCK MODEL FOR MANY-HARD-PARTICLE SYSTEMS**

Here we give an extension of the derivation of the clock model for the Lyapunov vector dynamics in many-hard-particle systems in the limit of low density. We also derive the formula (5) for the largest Lyapunov exponent from the clock model. The one-dimensional version of the clock model is the brick accumulation model used in this paper. In particular, we clarify the assumptions needed to justify the use of this model to discuss the Lyapunov localization. For the basic idea of the clock model, see, for example, Refs. [22, 32]. However, note that in this appendix we use notation and assumptions that are a little different from these references, so that the derived clock model is consistent with the discussions in this paper and able to be compared with the numerical results of Ref. [28].

We consider a \(D\)-dimensional system with \(N\) hard-disks (or hard-spheres, etc.) with identical radius \(R\) and mass \(M\). We assume that there is no external field in the system so that the dynamics is simply free-flights, and collisions between two particles. We put \(\delta q_j (q_j)\) as the spatial part of the Lyapunov vector component (the spatial coordinate) of the \(j\)-th particle, and \(\delta p_j (p_j)\) as the momentum part of the Lyapunov vector component (the momentum) of the \(j\)-th particle.

We take \(t = t_n\) to be the time of the \(n\)-th collision which involves particles \(j_n\) and \(k_n\), and define \(\tau_n\) to be

\[\tau_n = \frac{t_n - t_{n-1}}{M},\]

so that the \(n\)-th free flight time is given by \(\tau_n M\). The free flight part of dynamics of the Lyapunov vector is represented as

\[\delta q_j(t_n^-) = \delta q_j(t_{n-1}^+) + \tau_n \delta p_j(t_{n-1}^-),\]

\[\delta p_j(t_n^-) = \delta p_j(t_{n-1}^+).\]

Here, the argument \(t_n^\pm\) refers to the limit of the quantity before the \(n\)-th collision (−) or after the \(n\)-th collision (+). On the other hand, the change in the Lyapunov vector in particle-particle collisions is represented as

\[\delta q_{j_n}(t_n^+) = \delta q_{j_n}(t_n^-) + \Theta[n] \delta q_{k_n,j_n}(t_n^-),\]

\[\delta q_{k_n}(t_n^+) = \delta q_{k_n}(t_n^-) - \Theta[n] \delta q_{k_n,j_n}(t_n^-),\]

\[\delta q_l(t_n^+) = \delta q_l(t_n^-), \quad \text{for } l \notin \{j_n, k_n\},\]

\[\delta p_{j_n}(t_n^+) = \delta p_{j_n}(t_n^-) + \Theta[n] \delta p_{k_n,j_n}(t_n^-) + Q[n] \delta q_{k_n,j_n}(t_n^-),\]

\[\delta p_{k_n}(t_n^+) = \delta p_{k_n}(t_n^-) - \Theta[n] \delta p_{k_n,j_n}(t_n^-) - Q[n] \delta q_{k_n,j_n}(t_n^-),\]

\[\delta p_l(t_n^+) = \delta p_l(t_n^-), \quad \text{for } l \notin \{j_n, k_n\},\]

where \(\delta q_{k_n,j_n} = \delta q_{k_n} - \delta q_{j_n}, \delta p_{k_n,j_n} = \delta p_{k_n} - \delta p_{j_n}\), and
\( \Theta^{[n]} \) and \( Q^{[n]} \) are defined by
\[
\Theta^{[n]} = \sigma^{[n]} \sigma^{[n]^T},
\]
\[
Q^{[n]} = \frac{\sigma^{[n]^T} \Delta p^{[n]} \sigma^{[n]^T}}{2R} \left( I + \frac{\sigma^{[n]} \Delta p^{[n]^T} \sigma^{[n]^T}}{\sigma^{[n]^T} \Delta p^{[n]} \sigma^{[n]^T}} \right) \tag{B10}
\]
with
\[
\sigma^{[n]} = \frac{q_{k,n}(t^n_n) - q_{n}(t^n_n)}{|q_{k,n}(t^n_n) - q_{n}(t^n_n)|},
\]
\[
\Delta p^{[n]} = p_{k,n}(t^n_n) - p_n(t^n_n) \tag{B13}
\]
and the \((DN) \times (DN)\) identity matrix \( I \). Note that in this appendix we introduce all vectors as column vectors, so for example, \( \sigma^{[n]^T} \Delta p^{[n]} \sigma^{[n]^T} \) is a scalar and \( \sigma^{[n]} \Delta p^{[n]^T} \sigma^{[n]} \) is a matrix where \( T \) is the transpose. For later use we note that
\[
\delta p_{j,n}(t^n_n) + \delta p_{k,n}(t^n_n) = \delta p_{j,n}(t^n_n) + \delta p_{k,n}(t^n_n) \tag{B14}
\]
which can be derived from Eqs. \( B37 \) and \( B38 \). For the derivation of Eqs. \( B54 \), \( B55 \), \( B56 \), \( B57 \), \( B58 \) and \( B59 \) for the Lyapunov vector dynamics, for example, see Ref. 11.

We consider a low density case, in which the free flight time \( \tau_n M \) is large (as the free flight time is inversely proportional to the density). This justifies our first approximation for the Lyapunov vector:
\[
\delta q_j(t^n_n) \sim \tau_n \delta p_j(t^n_{n-1}) \tag{B15}
\]
in the limit of low density, the term containing \( \tau_n \) is much larger than the other term on the right-hand side of Eq. \( B22 \). This asymptotic relation \( B35 \) leads to
\[
\delta q_{k,n,j}(t^n_n) \sim \tau_n \delta p_{k,n,j}(t^n_{n-1}) \tag{B16}
\]
from the definition of \( \delta q_{k,n,j}(t^n_n) \) and \( \delta p_{k,n,j}(t^n_{n-1}) \). Using the relation \( B16 \), Eqs. \( B37 \), \( B38 \) and \( B39 \) can be rewritten as
\[
\delta p_{j,n}(t^n_n) \sim \delta p_{j,n}(t^n_{n-1}) + \Theta^{[n]} \delta p_{k,n,j}(t^n_{n-1}) + \tau_n Q^{[n]} \delta p_{k,n,j}(t^n_{n-1}) \tag{B17}
\]
\[
\delta p_{k,n}(t^n_n) \sim \delta p_{k,n}(t^n_{n-1}) - \Theta^{[n]} \delta p_{k,n,j}(t^n_{n-1}) - \tau_n Q^{[n]} \delta p_{k,n,j}(t^n_{n-1}) \tag{B18}
\]
\[
\delta p_{j,n}(t^n_n) = \delta p_{j,n}(t^n_{n-1}), \text{ for } l \notin \{j, k_n\}, \tag{B19}
\]
where we have used Eq. \( B38 \). Note that the spatial part of the Lyapunov vector does not appear in the dynamics described in \( B17 \) and \( B18 \). \( B19 \) anymore. The first and second terms on the right-hand side of \( B17 \) and \( B18 \) are negligible compared with the third term because of the large value of \( \tau_n \), so we obtain
\[
\delta p_{j,n}(t^n_n) \sim \tau_n Q^{[n]} \delta p_{k,n,j}(t^n_{n-1}), \tag{B20}
\]
\[
\delta p_{k,n}(t^n_n) \sim -\tau_n Q^{[n]} \delta p_{k,n,j}(t^n_{n-1}), \tag{B21}
\]
which lead to
\[
\delta p_{j,n}(t^n_n) + \delta p_{k,n}(t^n_n) \sim 0. \tag{B22}
\]
On the other hand, for the dynamics given by \( B54 \), \( B55 \) and \( B56 \) for the spatial part of Lyapunov vector we obtain
\[
\delta q_{j,n}(t^n_n) \sim \tau_n \left[ \delta p_{j,n}(t^n_{n-1}) + \Theta^{[n]} \delta p_{k,n,j}(t^n_{n-1}) \right] \tag{B23}
\]
\[
\delta q_{k,n}(t^n_n) \sim \tau_n \left[ \delta p_{k,n}(t^n_{n-1}) - \Theta^{[n]} \delta p_{k,n,j}(t^n_{n-1}) \right] \tag{B24}
\]
\[
\delta q_j(t^n_n) \sim \tau_n \delta p_j(t^n_{n-1}), \text{ for } l \notin \{j, k_n\}, \tag{B25}
\]
using Eqs. \( B15 \) and \( B16 \). Now we note
\[
\delta p_{j,n}(t^n_{n-1}) = \frac{\delta p_{j,n}(t^n_{n-1}) + \delta p_{k,n}(t^n_{n-1}) + \delta p_{k,n}(t^n_{n-1})}{2} + \frac{\delta p_{j,n}(t^n_{n-1}) - \delta p_{k,n}(t^n_{n-1})}{2} \tag{B26}
\]
where we used Eqs. \( B18 \), \( B19 \) and \( B22 \). Similarly we have
\[
\Omega^{[n]} \sim \frac{-\tau_n}{2} + \Theta^{[n]}, \tag{B30}
\]
The asymptotic equations \( B19 \), \( B20 \), \( B21 \), \( B25 \), \( B26 \) and \( B29 \) give the Lyapunov vector dynamics in the limit of low density. It should also be noted that the assumptions used to derive this dynamics breaks some conservation laws in the original dynamics, for example, the quantity \( \sum_{j=1}^N \delta p_j(t) \) is conserved in the original dynamics \( B13 \), \( B17 \), \( B18 \) and \( B19 \) but this cannot be guaranteed exactly in the low density dynamics \( B19 \), \( B20 \) and \( B21 \).

Now we consider the Lyapunov vector \( \delta \Gamma \) corresponding to a positive Lyapunov exponent. The positivity of the Lyapunov exponent means that the amplitude \( |\delta \Gamma| \) of Lyapunov vector diverges exponentially in time. The dynamics given by \( B19 \), \( B20 \), \( B21 \), \( B25 \), \( B26 \) and \( B29 \) shows that the divergence of \( \delta \Gamma \) must come from the Lyapunov vector components corresponding to colliding particles, as the other Lyapunov vector components diverge at most linearly. For this reason we neglect the change of the Lyapunov vector components for non-colliding particles. Under this assumption the Lyapunov vector dynamics for the Lyapunov vector component \( \delta \Gamma_j \equiv \delta q_j / \delta p_j \) for the \( j \)-th particle is summarized as
\[
\delta \Gamma_j(t^n_n) \sim -\delta \Gamma_k(t^n_n) \sim \tau_n \delta \Xi^{[n-1]} \tag{B31}
\]
\[
\delta \Gamma_l(t^n_n) \sim \delta \Gamma_l(t^n_{n-1}), \text{ for } l \notin \{j, k_n\}, \tag{B32}
\]
where \( \delta \Xi^{[n-1]} \) is defined by
\[
\delta \Xi^{[n-1]} \equiv \left( \Omega^{[n]} \delta p_{k_n,n_n}(t_{n-1}^+) / Q^{[n]} \delta p_{k_n,n_n}(t_{n-1}^-) \right). \tag{B33}
\]

It is essential to note that the Lyapunov vector components \( \delta \Gamma_j(t_n^+) \) and \( \delta \Gamma_k(t_n^+) \) for the colliding particles have the same amplitude, and \( \delta \Xi^{[n-1]} \) is independent of \( \tau_n \).

We assume that the ratio
\[
\mu = |\delta q_j| / |\delta p_j| \tag{B34}
\]

between the amplitudes of the spatial and momentum parts of the Lyapunov vector for the \( j \)-th particle, are independent of the particle index \( j \). Ref. suggests that the ratio \( \mu \) need not be of order 1 in general. From Eq. (B34) we have
\[
|\delta p_j| \approx \frac{1}{\sqrt{1 + \mu^2}} |\delta \Gamma_j|. \tag{B35}
\]

Using this assumption we estimate the magnitude of the vector \( \delta \Xi^{[n-1]} \) as
\[
|\delta \Xi^{[n-1]}| = \sqrt{|\Omega^{[n]} \delta p_{k,n_n}(t_{n-1}^-)|^2 + |Q^{[n]} \delta p_{k_n,n_n}(t_{n-1}^+)|^2} = \max \left\{ |\delta p_{j_n}(t_{n-1}^-)|, |\delta p_{k_n}(t_{n-1}^+)| \right\} \times \sqrt{|\Omega^{[n]} e^{[n]}|^2 + |Q^{[n]} e^{[n]}|^2} \approx \max \left\{ |\delta \Gamma_{j_n}(t_{n-1}^-)|, |\delta \Gamma_{k_n}(t_{n-1}^+)| \right\} \times \sqrt{|\Omega^{[n]} e^{[n]}|^2 + |Q^{[n]} e^{[n]}|^2} \tag{B36}
\]

where \( e^{[n]} \) is defined by
\[
e^{[n]} = \frac{\delta p_{k_n}(t_{n-1}^+) - \delta p_{j_n}(t_{n-1}^-)}{\max \left\{ |\delta p_{k_n}(t_{n-1}^+)|, |\delta p_{j_n}(t_{n-1}^-)| \right\}}, \tag{B37}
\]

and satisfies the inequality
\[
|e^{[n]}| \leq \frac{|\delta p_{k_n}(t_{n-1}^+)| + |\delta p_{j_n}(t_{n-1}^-)|}{\max \left\{ |\delta p_{k_n}(t_{n-1}^+)|, |\delta p_{j_n}(t_{n-1}^-)| \right\}} \leq 2. \tag{B38}
\]

From Eq. (B30) we can estimate the magnitudes \( |\delta \Gamma_{j_n}(t_{n-1}^+)| \) and \( |\delta \Gamma_{k_n}(t_{n-1}^+)\) through \( |\delta \Gamma_{j_n}(t_{n-1}^+) = |\delta \Gamma_{k_n}(t_{n-1}^+) = |\delta \Xi^{[n-1]}| \).

Now, we introduce the clock value \( \mathcal{K}_j(n) \) of the \( j \)-th particle just after the \( n \)-th collision as
\[
\mathcal{K}_j(n) \equiv \frac{1}{\ln \rho} \ln |\delta \Gamma_j(t_{n-1}^+)|, \tag{B39}
\]

or equivalently,
\[
|\delta \Gamma_j(t_{n-1}^+)| = \left( \frac{1}{\rho} \right)^{\mathcal{K}_j(n)} |\delta \Gamma_j(0)| \tag{B40}
\]

leading to Eq. (A). Here \( \rho \) is the particle density whose value is \( 0 < \rho < 1 \), and \( \delta \Gamma_j(0) \) is the Lyapunov vector component for the \( j \)-th particle at the initial time. We choose the initial Lyapunov vector \( \delta \Gamma_j(0) \) so that its components \( \delta \Gamma_1(0), \delta \Gamma_2(0), \cdots, \delta \Gamma_{N-1}(0) \) and \( \delta \Gamma_N(0) \) have the same order of magnitude, and a larger clock value means larger magnitude of the Lyapunov vector component, \( |\delta \Gamma_j(t_n)| \).

We have already used the fact that the free flight time increases as the density \( \rho \) decreases. To write down the clock value more meaningfully, we use the specific relation between the free flight time and the density, namely that the free-flight time \( \tau_n M \) is approximately inversely proportional to the density \( \rho \) at low density:
\[
\tau_n \sim s_n / \rho \tag{B41}
\]

where \( s_n \) is independent of the density. Using Eq. (B41), the expression (B39) for the clock value \( \mathcal{K}_j(n) \) can be rewritten as
\[
\mathcal{K}_j(n) \sim \begin{cases} 
\max \{ \mathcal{K}_{j_n}(n - 1), \mathcal{K}_{k_n}(n - 1) \} + 1 + \Delta \Phi^{[n]} & \text{for } l = j_n \text{ or } l = k_n \\
\mathcal{K}_{j_n}(n - 1) & \text{for } l \notin \{j_n, k_n\},
\end{cases} \tag{B42}
\]

where \( \Delta \Phi^{[n]} \) is defined by
\[
\Delta \Phi^{[n]} = -\frac{1}{\ln \rho} \ln \left( s_n \sqrt{|\Omega^{[n]} e^{[n]}|^2 + |Q^{[n]} e^{[n]}|^2} / (1 + \mu^2) \right). \tag{B43}
\]

and where we have used (B31), (B36), (B39) and \( |\delta \Gamma_{j_n}(0)| \approx |\delta \Gamma_{k_n}(0)| \). Our final assumption to derive the clock model is that
\[
\lim_{\rho \to 0} \Delta \Phi^{[n]} = 0. \tag{B44}
\]

To justify the assumption (B44) note that the right-hand side of Eq. (B43) for the quantity \( \Delta \Phi^{[n]} \) has a factor \( 1/\ln \rho \) which goes to zero in the limit as \( \rho \to 0 \), and \( s_n \), \( Q^{[n]} \) and \( \Omega^{[n]} \) are almost independent of the density \( \rho \), and the magnitude of the vector \( e^{[n]} \) is finite even in the limit \( \rho \to 0 \) from the inequality (B38). Eqs. (B42) and (B43) lead to the clock dynamics
\[
\mathcal{K}_l(n) \sim \begin{cases} 
\max \{ \mathcal{K}_{j_n}(n - 1), \mathcal{K}_{k_n}(n - 1) \} + 1 & \text{for } l = j_n \text{ or } l = k_n \\
\mathcal{K}_{j_n}(n - 1) & \text{for } l \notin \{j_n, k_n\},
\end{cases} \tag{B45}
\]

in the low density limit, which is closed by the clock value \( \mathcal{K}_j(n) \) only. From Eq. (B45), the dynamics for the clock model is expressed as (i) the clock value is changed only when the corresponding particle collides, and (ii) the clock values of colliding particles are tuned to the same value given by 1 plus the larger of the two clock values of the particles just before the collision.

In the quasi-one-dimensional system with periodic boundary conditions, particle indices of the colliding particles can be taken so that \( k_n = j_n + 1 \) (note that index \( N + 1 \) is equivalent to 1). Therefore we obtain the dynamics (A) for the brick accumulation model explained in Sec. (A) as the one-dimensional version of the clock model. Moreover, in the one-dimensional case, \( \mathcal{K}_j(n) \)
can be interpreted as the brick height of the $j$-site just after the $n$-th brick is dropped.

Finally we derive the formula for the largest Lyapunov exponent from the clock value $K_j(n)$. In the limit of low density $\rho \ll 1$, the amplitude $|\delta \Gamma(t_n^\tau)|$ of the Lyapunov vector can be approximated by

$$|\delta \Gamma(t_n^\tau)| = \sqrt{\sum_{j=1}^N |\delta \Gamma_j(t_n^\tau)|^2}$$

$$\sim \alpha_n \left(\frac{1}{\rho}\right) \delta^{[\max]}(n) |\delta \Gamma_j(0)|$$  \hspace{1cm} (B46)

noting that the sum $\sum_{j=1}^N |\delta \Gamma_j(t_n^\tau)|^2$ is dominated by the Lyapunov vector component amplitude $|\delta \Gamma_j(t_n^\tau)|$ with the largest clock value $\delta^{[\max]}(n)$:

$$\delta^{[\max]}(n) \equiv \max \{K_1(n), K_2(n), \ldots, K_N(n)\}$$  \hspace{1cm} (B47)

at $n$, because of the huge factor $1/\rho$. Here, $\alpha_n$ is the number of particle with the largest clock value $\delta^{[\max]}(n)$, and we have also used our assumption that $|\delta \Gamma_j(0)|$ is almost independent of the particle index $j$. Now we assume the approximate relation

$$\frac{\delta^{[\max]}(n)}{n} \sim \lim_{n \to \infty} \frac{1}{nN} \sum_{j=1}^N K_j(n)$$  \hspace{1cm} (B48)

in the limit of large $n$. (We have checked this relation numerically for the brick accumulation model.) Using the relations (B46), (B48) and $t \sim n\tau$ we obtain

$$\lambda \equiv \lim_{t \to \infty} \frac{1}{t} \ln |\delta \Gamma(t)|$$

$$\sim - \lim_{n \to \infty} \frac{1}{n\tau} \delta^{[\max]}(n) \ln \rho$$

$$\sim - \lim_{n \to \infty} \frac{1}{n\tau} \sum_{j=1}^N K_j(n) \ln \rho$$  \hspace{1cm} (B49)

for the Lyapunov exponent $\lambda$ corresponding to the Lyapunov vector $\delta \Gamma$. Therefore we obtain the Eq. (14), which is independent of the system shape and the number of spatial dimensions.

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[46] Ref. [28] suggests that the ratio \( \mu \) depends on the density \( \rho \), and it can be \( \mu \gg 1 \) for a Lyapunov vector corresponding to a positive Lyapunov exponent at low density. Therefore the assumption \[ B44 \] is not obvious.
[47] In the actual simulation of the brick accumulation model we discarded the brick accumulations after the first 100\( N \) bricks have been dropped. In this sense, the sum \( \sum_{j=1}^{N} K_j(n) \) is not zero at \( n = 0 \). This is the reason why we added the fitting parameter \( \alpha \) in the fitting function \( \sum_{j=1}^{N} K_j(n) = \alpha + \beta n \) in Fig. 6 for consistency, although it is almost zero in this case.
[48] More exactly, the distribution \( D(\sigma \cdot \Delta p) \) for non-zero hopping cases in Fig. 10 was calculated as follows. First we chose the case satisfying the conditions: (i) the particle indices satisfying \( \gamma_j^{(1)} > 0.2 \) are given by \( \{j_n, j_n + 1\} \) just before the \( n \)-th collision and \( \{j_{n+1}, j_{n+1} + 1\} \) just after the \( n \)-th collision, such that \( j_n \neq j_{n+1} \), and (ii) the \( j_{n+1} \)-th particle and \( (j_{n+1}+1) \)-th particle collide at the \( n \)-th collision. (Here, if the particle indices for the localized region of the Lyapunov vector satisfy \( \gamma_j^{(1)} > 0.2 \) \( \{1, N\} \), then the particle indices pair \( \{j_n, j_n + 1\} \) or \( \{j_{n+1}, j_{n+1} + 1\} \) above should be taken as \( \{0, 1\} \), based on periodic boundary conditions in the longitudinal direction of the quasi-one-dimensional system.) Under this circumstance we calculate the distribution \( D(\sigma \cdot \Delta p) \) from the value of \( \sigma \cdot \Delta p \) in the \( n \)-th collision between these \( j_{n+1} \)-th and \( (j_{n+1}+1) \)-th particles.
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