FINITE-DIMENSIONAL ALGEBRAS ARE
$(m > 2)$-CALABI-YAU-TILTED

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Abstract. We observe that over an algebraically closed field, any finite-dimensional
algebra is the endomorphism algebra of an $m$-cluster-tilting object in a triangulated
$m$-Calabi-Yau category, where $m$ is any integer greater than 2.

1. Introduction

Cluster categories were introduced by Buan, Marsh, Reineke, Reiten and Todorov [8]
as a means to model the combinatorics of cluster algebras with acyclic skew-symmetric
exchange matrices within the framework of quiver representations (see also the work of
Caldero, Chapoton and Schiffler [12] for the case of $A_n$ quivers).

The cluster category of an acyclic quiver is the orbit category of the bounded derived
category of its path algebra with respect to the autoequivalence $F = \nu \Sigma^{-2}$ where $\nu$
denotes the Serre functor and $\Sigma$ is the suspension functor. By a result of Keller [20],
the cluster category is a triangulated 2-Calabi-Yau category. Particular role is played
by the 2-cluster-tilting objects within this category (the precise definitions will be given
in Section 2.1 below) which model the clusters in the corresponding cluster algebra.

As already shown in [20], by replacing $\Sigma^{-2}$ by $\Sigma^{-m}$ for $m > 2$ and considering the
orbit category with respect to the autoequivalence $\nu \Sigma^{-m}$, one gets an $m$-Calabi-Yau
triangulated category with $m$-cluster-tilting objects. The categories obtained in this way,
called $m$-cluster categories, were the subject of many investigations, see [5, 26, 28, 29].

The endomorphism algebras of 2-cluster-tilting objects in cluster categories are known
as cluster-tilted algebras and they possess many remarkable representation-theoretic and
homological properties [4, 9, 10, 23]. More generally, consider an $m$-Calabi-Yau-tilted
algebra, i.e. an algebra $A = \text{End}_C(T)$ where $C$ is a $K$-linear, triangulated, Hom-finite,
$m$-Calabi-Yau category over a field $K$ and $T$ is an $m$-cluster-tilting object in $C$ for some
positive integer $m$. Keller and Reiten have shown the following results in the case $m = 2$
(for the first two points, see Sections 2 and 3 of [24] and for the third one, see [21, §2]):

- $A$ is Gorenstein of dimension at most $m - 1$ (i.e. $\text{id}_A A \leq m - 1$ and $\text{pd}_A DA \leq m - 1$, where $D(-) = \text{Hom}_K(-, K)$);
- The stable category of Cohen-Macaulay $A$-modules is $(m + 1)$-Calabi-Yau;
- If $K$ is algebraically closed, $C$ is algebraic and $A \cong KQ$ for an acyclic quiver $Q$,
  then $C$ is triangle equivalent to the $m$-cluster category of $Q$.

In addition, they have shown that these results hold also in the case $m > 2$ provided
that one imposes an additional condition on the $m$-cluster-tilting object $T$ stated in

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terms of the vanishing of some of its negative extensions, namely

\[ \text{Hom}_\mathcal{C}(T, \Sigma^{-i}T) = 0 \text{ for any } 0 < i < m - 1, \]

see [24, §4]. Note that by the \( m \)-Calabi-Yau property of \( \mathcal{C} \) this condition is equivalent to the vanishing of the positive extensions \( \text{Hom}_\mathcal{C}(T, \Sigma^iT) \) for all \( m < i < 2m - 1 \), whereas these extensions for \( 0 < i < m \) always vanish since \( T \) is \( m \)-cluster-tilting.

In the terminology of [6], the condition (\( \star \)) means that \( T \) is \( (m-2) \)-corigid, and in [6, Theorem B] Beligiannis presents a more refined result connecting the corigidity property of an \( m \)-cluster-tilting object with the Gorenstein property of its endomorphism algebra and the Calabi-Yau property of its stable category of Cohen-Macaulay modules. We note also that a condition analogous to (\( \star \)), stated for \( m \)-cluster-tilting subcategories inside bounded derived categories of modules and abbreviated “vosnex”, appears in the works of Amiot-Oppermann [3, Definition 4.9] and Iyama-Oppermann [13, Notation 3.5].

Whereas for \( m = 2 \) the condition (\( \star \)) is empty and hence automatically holds, this is no longer the case for \( m \geq 3 \). In particular, there are examples of triangulated \( m \)-Calabi-Yau categories \( \mathcal{C} \) and \( m \)-cluster-tilting objects \( T \in \mathcal{C} \) for which the vosnex condition (\( \star \)) does not hold and moreover the endomorphism algebra \( \text{End}_\mathcal{C}(T) \) is a path algebra of a connected acyclic quiver, see Iyama and Yoshino [19, Theorem 9.3] for \( m = 3 \) and [19, Theorem 10.2] for \( m \) odd, and also [24, Example 4.3]. Another example, where the algebra \( \text{End}_\mathcal{C}(T) \) is not Gorenstein, is given in [23, Example 5.3].

The purpose of this note is to extend this class of examples by showing that given \( m \geq 3 \), any finite-dimensional algebra over an algebraically closed field is the endomorphism algebra of an \( m \)-cluster-tilting object in an \( m \)-Calabi-Yau triangulated category. Hence, without any further assumptions on \( \mathcal{C} \) and \( T \), one cannot say too much about the algebras \( \text{End}_\mathcal{C}(T) \).

To this end we invoke the construction of generalized cluster categories due to Amiot [1] in the case \( m = 2 \) and generalized by Guo [17] to the case \( m > 2 \). This construction produces an \( m \)-Calabi-Yau triangulated category with an \( m \)-cluster-tilting object from any \( \text{dg} \)-algebra which is homologically smooth, \( (m+1) \)-Calabi-Yau and satisfies additional finiteness conditions. A rich source of such \( \text{dg} \)-algebras is provided by the deformed Calabi-Yau completions defined and investigated by Keller [22].

Given a basic finite-dimensional algebra \( A \), we choose a \( \text{dg} \)-algebra \( B \) with two properties; firstly, the underlying graded algebra of \( B \) is the path algebra of a graded quiver whose arrows are concentrated in degrees 0 and -1, and secondly, \( H^0(B) \cong A \). Such \( \text{dg} \)-algebra can be constructed from any presentation of \( A \) as a quotient of a path algebra of a quiver by an ideal generated by a finite sequence of elements. Conversely, any such \( \text{dg} \)-algebra arises in this way.

It turns out that for any \( m \geq 2 \), the \( (m+1) \)-Calabi-Yau completion of \( B \) is a Ginzburg \( \text{dg} \)-algebra \( \Gamma \) of a graded quiver with homogeneous superpotential of degree \( 2 - m \) which can be written explicitly in terms of the quiver and the sequence of elements. In the case \( m = 2 \), the zeroth homology \( H^0(\Gamma) \) is a split extension of \( A \), whereas when \( m > 2 \) it is isomorphic to \( A \), hence \( \Gamma \) satisfies the finiteness conditions required in the construction of [1, 17] and thus gives rise to a Hom-finite \( m \)-Calabi-Yau triangulated category with an \( m \)-cluster-tilting object whose endomorphism algebra is isomorphic to \( A \).

The \( m \)-cluster-tilting object we get almost never satisfies the vosnex condition (\( \star \)). More precisely, that condition holds if and only if \( A \) is the path algebra of an acyclic
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quiver and \(B\) is chosen such that \(B = A\). Moreover, the flexibility in the choice of the dg-algebra \(B\) allows to construct, for certain algebras \(A\) and any \(m > 2\), inequivalent \(m\)-Calabi-Yau categories \(\mathcal{C} \neq \mathcal{C}'\) with \(m\)-cluster-tilting objects \(T \in \mathcal{C}\) and \(T' \in \mathcal{C}'\) such that \(\text{End}_\mathcal{C}(T) \cong \text{End}_\mathcal{C}(T') \cong A\).

2. Recollections

2.1. Notations. We recall the definitions of Calabi-Yau triangulated categories and cluster-tilting objects. Throughout, we fix a field \(K\).

**Definition 2.1.** Let \(\mathcal{C}\) be a \(K\)-linear triangulated category with suspension functor \(\Sigma\).

(a) We say that \(\mathcal{C}\) is Hom-finite if the spaces \(\text{Hom}_\mathcal{C}(X, Y)\) are finite-dimensional over \(K\) for any \(X, Y \in \mathcal{C}\).

(b) Let \(m \in \mathbb{Z}\). We say that \(\mathcal{C}\) is \(m\)-Calabi-Yau if \(\mathcal{C}\) is Hom-finite and there exist functorial isomorphisms

\[
\text{Hom}_\mathcal{C}(X, Y) \cong D \text{Hom}_\mathcal{C}(Y, \Sigma^m X)
\]

for any \(X, Y \in \mathcal{C}\), where \(D\) denotes the duality \(D(\cdot) = \text{Hom}_K(\cdot, K)\).

For an object \(X\) in an additive category \(\mathcal{C}\), denote by \(\text{add } X\) the full subcategory of \(\mathcal{C}\) whose objects are finite direct sums of direct summands of \(X\). For a subcategory \(\mathcal{Y}\) of \(\mathcal{C}\), let \(\perp \mathcal{Y} = \{X \in \mathcal{C} : \text{Hom}_\mathcal{C}(X, Y) = 0\text{ for any } Y \in \mathcal{Y}\}\).

**Definition 2.2.** Let \(\mathcal{C}\) be a triangulated \(m\)-Calabi-Yau category for some integer \(m \geq 1\). An object \(T\) of \(\mathcal{C}\) is \(m\)-cluster-tilting if:

(i) \(\text{Hom}_\mathcal{C}(T, \Sigma^i T) = 0\) for any \(0 < i < m\); and

(ii) if \(X \in \mathcal{C}\) is such that \(\text{Hom}_\mathcal{C}(X, \Sigma^i T) = 0\) for any \(0 < i < m\), then \(X \in \text{add } T\).

Equivalently, \(\text{add } T = \bigcap_{0<i<m} \perp \Sigma^i(\text{add } T)\).

**Definition 2.3.** Let \(m \geq 1\). A \(K\)-algebra \(\Lambda\) is \(m\)-Calabi-Yau-tilted (\(m\)-CY-tilted for short) if there exist a triangulated \(m\)-Calabi-Yau category \(\mathcal{C}\) and an \(m\)-cluster-tilting object \(T\) of \(\mathcal{C}\) such that \(\Lambda \cong \text{End}_\mathcal{C}(T)\).

2.2. Generalized cluster categories. In this section we briefly review the construction of triangulated \(m\)-Calabi-Yau categories with \(m\)-cluster-tilting object due to Amiot [1] (for the case \(m = 2\)) and its generalization by Guo [17] (for the case \(m > 2\)).

Let \(\Gamma\) be a differential graded (dg) algebra over a field \(K\). Denote by \(\mathcal{D}(\Gamma)\) its derived category and by \(\text{per } \Gamma\) its smallest full triangulated subcategory containing \(\Gamma\) and closed under taking direct summands. Let \(\mathcal{D}_{\text{id}}(\Gamma)\) denote the full subcategory of \(\mathcal{D}(\Gamma)\) whose objects are those of \(\mathcal{D}(\Gamma)\) with finite-dimensional total homology.

**Definition 2.4.** \(\Gamma\) is said to be homologically smooth if \(\Gamma \in \text{per } \Gamma^e\), where \(\Gamma^e = \Gamma^{op} \otimes_K \Gamma\).

Let \(\Gamma\) be homologically smooth and let \(\Omega = \text{RHom}_{\Gamma^e}(\Gamma, \Gamma^e)\). By definition, \(\Omega \in \mathcal{D}((\Gamma^e)^{op})\). We can view \(\Omega\) as an object of \(\mathcal{D}(\Gamma^e)\) via restriction of scalars along the morphism \(\tau : \Gamma^e \xrightarrow{\sim} (\Gamma^e)^{op}\) given by \(\tau(x \otimes y) = y \otimes x\). By [22, Lemma 3.4] (see also [21, Lemma 4.1] and [17, Lemma 2.1]) one has \(\mathcal{D}_{\text{id}}(\Gamma) \subseteq \text{per } \Gamma\) and

\[
\text{Hom}_{\mathcal{D}(\Gamma)}(L \otimes_\Gamma \Omega, M) \cong D \text{Hom}_{\mathcal{D}(\Gamma)}(M, L)
\]

for any \(L \in \mathcal{D}(\Gamma), M \in \mathcal{D}_{\text{id}}(\Gamma)\).
If $\Gamma$ is homologically smooth and bimodule $m$-Calabi-Yau then the triangulated category $\mathcal{D}_{id}(\Gamma)$ is $m$-Calabi-Yau. More precisely, (2.1) yields functorial isomorphisms

$\text{Hom}_{\mathcal{D}(\Gamma)}(\Sigma^{-m}L, M) \cong D\text{Hom}_{\mathcal{D}(\Gamma)}(M, L)$

for any $L \in \mathcal{D}(\Gamma)$, $M \in \mathcal{D}_{id}(\Gamma)$.

Theorem 2.6 ([11, §2], [17, §2]). Let $m \geq 1$ and let $\Gamma$ be a dg-algebra satisfying the following conditions:

(i) $\Gamma$ is homologically smooth;
(ii) $H^0(\Gamma) = 0$ for any $i > 0$;
(iii) $\dim_K H^0(\Gamma) < \infty$;
(iv) $\Gamma$ is bimodule $(m + 1)$-Calabi-Yau.

Consider the triangulated category $\mathcal{C}_\Gamma = \text{per } \Gamma / \mathcal{D}_{id}(\Gamma)$. Then:

(a) $\mathcal{C}_\Gamma$ is Hom-finite and $m$-Calabi-Yau.
(b) For any $i \in \mathbb{Z}$, set $D^{\leq i} = \{ X \in \mathcal{D}(\Gamma) : H^p(X) = 0 \text{ for all } p > i \}$ and let $\mathcal{F} = D^{\leq 0} \cap D^{\leq -m} \cap \text{per } \Gamma$. The restriction of the canonical projection $\pi : \text{per } \Gamma \to \mathcal{C}_\Gamma$ to $\mathcal{F}$ induces an equivalence of $K$-linear categories $\mathcal{F} \overset{\sim}{\to} \mathcal{C}_\Gamma$.
(c) The image $\pi \Gamma$ of $\Gamma$ in $\mathcal{C}_\Gamma$ is an $m$-cluster-tilting object and $\text{End}_{\mathcal{C}_\Gamma}(\pi \Gamma) \cong H^0(\Gamma)$.

The object $\pi \Gamma$ occurring in part (c) of the theorem is called the canonical $m$-cluster-tilting object in $\mathcal{C}_\Gamma$. The statement of the next lemma concerning negative extension groups of the canonical $m$-cluster-tilting object is implicit in the proof of [17, Corollary 3.4].

Lemma 2.7. Let $\Gamma$ be a dg-algebra satisfying the conditions of Theorem 2.6 and let $T = \pi \Gamma$ be the canonical $m$-cluster-tilting object in $\mathcal{C}_\Gamma$. Then $\text{Hom}_{\mathcal{C}_\Gamma}(T, \Sigma^{-i}T) \cong H^{-i}(\Gamma)$ for any $0 \leq i \leq m - 1$.

Proof. As observed in [17], the objects $\Gamma, \Sigma \Gamma, \ldots, \Sigma^{m-1} \Gamma$ are in the fundamental domain $\mathcal{F}$. Therefore, for any $0 \leq i \leq m - 1$,

$\text{Hom}_{\mathcal{C}_\Gamma}(\pi \Gamma, \Sigma^{-i} \pi \Gamma) \cong \text{Hom}_{\mathcal{C}_\Gamma}(\Sigma^i \pi \Gamma, \pi \Gamma) = \text{Hom}_{\mathcal{C}_\Gamma}(\pi \Sigma^i \Gamma, \pi \Gamma) \cong \text{Hom}_{\mathcal{D}(\Gamma)}(\Sigma^i \Gamma, \Gamma) \cong \text{Hom}_{\mathcal{D}_m(\Gamma)}(\Gamma, \Sigma^{-i} \Gamma) \cong H^{-i}(\Gamma)$.

2.3. Ginzburg dg-algebras. Ginzburg dg-algebras were introduced by Ginzburg in [10]. We recall their definition in the case of a graded quiver with homogeneous superpotential and quote the result of Keller [22] that they are homologically smooth and bimodule Calabi-Yau (see also the paper [27] by Van den Bergh). We note that a graded version of quivers with potentials (and their mutations) has also been introduced in [2] and [13], however in these papers one still implicitly considers Ginzburg dg-algebras which are bimodule 3-Calabi-Yau. In contrast, in the setting described below the degree of the superpotential affects the Calabi-Yau dimension of its Ginzburg dg-algebra.

A quiver is a finite directed graph. More precisely, it is a quadruple $Q = (Q_0, Q_1, s, t)$, where $Q_0$ and $Q_1$ are finite sets (of vertices and arrows, respectively) and $s, t : Q_1 \to Q_0$
are functions specifying for each arrow its starting and terminating vertex, respectively. A quiver $Q$ is graded if we are given a grading $|\cdot|: Q_1 \to \mathbb{Z}$.

A path $p$ in $Q$ is a sequence of arrows $\alpha_1 \alpha_2 \ldots \alpha_n$ such that $s(\alpha_{i+1}) = t(\alpha_i)$ for all $1 \leq i < n$. For a path $p$ we denote by $s(p)$ its starting vertex $s(\alpha_1)$ and by $t(p)$ its terminating vertex $t(\alpha_n)$. A path $p$ is a cycle if it starts and ends at the same vertex, i.e. $s(p) = t(p)$. Any vertex $v \in Q_0$ gives rise to a cycle $c_v$ of length zero with $s(c_v) = t(c_v) = v$.

The path algebra $KQ$ has a basis consisting of the paths of $Q$, and the product of two paths $p$ and $q$ is their concatenation $pq$ if $s(q) = t(p)$ and zero otherwise. The path algebra is graded if $Q$ is graded, with the degree of a path being the sum of the degrees of its arrows. The degree of a homogeneous element $x$ of $KQ$ will be denoted by $|x|$.

Consider the $K$-bilinear map $[-,-]: KQ \times KQ \to KQ$ whose value on a pair of homogeneous elements $x, y$ is given by their supercommutator $[x, y] = xy - (-1)^{|x||y|}yx$. Denote by $[KQ,KQ]$ the linear subspace of $KQ$ spanned by all the supercommutators. The quotient $KQ/[KQ,KQ]$ has a basis consisting of cycles considered up to cyclic permutation “with signs”.

**Definition 2.8.** A superpotential on $Q$ is a homogeneous element in $KQ/[KQ,KQ]$.

Any arrow $\alpha \in Q_1$ gives rise to a linear map $\partial_\alpha : KQ/[KQ,KQ] \to KQ$ (called cyclic derivative with respect to $\alpha$) whose value on any cycle $c$ is given by

$$\partial_\alpha c = (-1)^{|\alpha|} \sum_{p = uvw} (-1)^{|\alpha|+|\alpha|} v u$$

where the sum runs over all possible decompositions $p = uvw$ with $u, v$ paths of length $\geq 0$. More explicitly, if $p = \alpha_1 \alpha_2 \ldots \alpha_n$ has degree $w$, then

$$\partial_\alpha (\alpha_1 \alpha_2 \ldots \alpha_n) = (-1)^{|\alpha|} \sum_{\alpha_1 \alpha_2 \ldots \alpha_n} (-1)^{|w-1|(|\alpha_1|+\ldots+|\alpha_{\ell-1}|)\alpha_{\ell+1} \ldots \alpha_n \alpha_1 \ldots \alpha_{\ell-1}}$$

is homogeneous of degree $w - |\alpha|$.

**Definition 2.9.** Let $Q$ be a graded quiver and let $m \in \mathbb{Z}$. Let $\bar{Q}$ be the graded quiver whose vertices are those of $Q$ and its set of arrows consists of

- the arrows of $Q$ (with their degree unchanged);
- an arrow $\alpha^* : j \to i$ of degree $1 - m - |\alpha|$ for each arrow $\alpha : i \to j$ of $Q$;
- a loop $t_i : i \to i$ of degree $-m$ for each vertex $i \in Q_0$.

Let $W$ be a superpotential on $Q$ of degree $2 - m$. The Ginzburg dg-algebra of $(Q, W)$, denoted $\Gamma_{m+1}(Q, W)$, is the dg-algebra whose underlying graded algebra is the path algebra $K\bar{Q}$ and the differential $d$ is defined by its action on the generators as

- $d(\alpha) = 0$ and $d(\alpha^*) = \partial_\alpha W$ for each $\alpha \in Q_1$;
- $d(t_i) = e_i \sum_{\alpha \in Q_1} [\alpha, \alpha^*] e_i$ for each $i \in Q_0$.

**Remark 2.10.** Note that for each $\alpha \in Q_1$, the element $\partial_\alpha W$ is homogeneous of degree $2 - m - |\alpha|$ and the supercommutator $[\alpha, \alpha^*]$ is homogeneous of degree $1 - m$, hence the definition of the differential $d$ makes sense. Moreover, as $[\alpha, \alpha^*] = \alpha \alpha^* - (-1)^{|\alpha||\alpha^*|}\alpha^*\alpha$ for any $\alpha \in Q_1$, by using the sign conventions in (2.2) one verifies that $d^2(t_i) = 0$ for any $i \in Q_0$, so that $d$ is indeed a differential. Note also that in [22] the differential of $t_i$ is $(-1)^{m-1}t_i$ times the one given here, but of course by replacing each $t_i$ by $(-1)^{m-1}t_i$ one sees that the dg-algebras are isomorphic.
Theorem 2.11 ([22, Theorem 6.3]). Let $m \in \mathbb{Z}$ and let $W$ be a superpotential of degree $2 - m$ on a graded quiver $Q$. Then $\Gamma_{m+1}(Q,W)$ is homologically smooth and bimodule $(m+1)$-Calabi-Yau.

We record two useful observations. Let $Q$ be a graded quiver, $m \in \mathbb{Z}$ an integer and $W$ a homogeneous superpotential on $Q$ of degree $2 - m$.

Lemma 2.12. Suppose that $\alpha: i \to j$ is an arrow in $Q$ such that no term of $W$ contains $\alpha$. Define a graded quiver $Q'$ by $Q'_0 = Q_0$ and $Q'_1 = Q_1 \setminus \{\alpha\} \cup \{\alpha^*\}$ where $\alpha^*: j \to i$ has degree $1 - m - |\alpha|$. Then $W$ can be naturally viewed as a superpotential $W'$ on $Q'$ and $\Gamma_{m+1}(Q,W) \cong \Gamma_{m+1}(Q',W')$.

Proof. Since no term of $W$ contains $\alpha$, we can view $W$ as an element $W'$ in the path algebra $KQ'$. The graded quivers $Q'$ and $Q$ are isomorphic by the map $\varphi: \overline{Q'} \to \overline{Q}$ sending $(\alpha^*)^*$ to $\alpha$ and fixing all other arrows. Moreover, since $\partial_\alpha W = 0 = \partial_\alpha W'$ and $\partial_\beta W = \partial_\beta W'$ for any $\beta \in Q_1 \cap Q'_1$, the map $\varphi$ induces an isomorphism of the Ginzburg dg-algebras $\Gamma_{m+1}(Q',W') \cong \Gamma_{m+1}(Q,W)$. \hfill $\square$

For a subset of arrows $\Omega \subseteq Q_1$, let $Q_\Omega$ be the subquiver of $Q$ with $(Q_\Omega)_0 = Q_0$ and $(Q_\Omega)_1 = \Omega$. For the definition of Calabi-Yau completion, see [22, §4].

Lemma 2.13. Suppose that $\Omega \subseteq Q_1$ is a set of arrows such that $W = \sum_{\beta \in Q_1 \setminus \Omega} \beta \omega_\beta$ with $\omega_\beta \in KQ_\Omega$ for each $\beta \notin \Omega$. Consider the subquiver $Q'$ of $\overline{Q}$ defined by $Q'_0 = Q_0$ and $Q'_1 = \Omega \cup \{\beta^*: \beta \in Q_1 \setminus \Omega\}$. Let $d$ be the differential on $\Gamma_{m+1}(Q,W)$. Then:

(a) $d(KQ') \subseteq KQ'$, hence $B = (KQ',d)$ is a sub-dg-algebra of $\Gamma_{m+1}(Q,W)$.
(b) $\Gamma_{m+1}(Q,W)$ is isomorphic to the $(m + 1)$-Calabi-Yau completion of $B$.

Proof. The first claim holds since $d(\alpha) = 0$ for any $\alpha \in \Omega \subseteq Q_1$ and $d(\beta^*) = \partial_\beta W = (-1)^{|\beta|} \beta \omega_\beta \in KQ_\Omega \subseteq KQ'$ for any $\beta \notin \Omega$. This argument also shows that the differential $d$ on $KQ'$ satisfies the condition in [22, §3.6] via the filtration $\varnothing \subseteq \Omega \subseteq Q'$ of the set of arrows. Hence we can use [22, Proposition 6.6] to compute the $(m + 1)$-Calabi-Yau completion of $B$ and get that it is isomorphic to $(K\overline{Q'},d')$ with the differential $d'$ given on the generators by

$$d'(\alpha) = \partial_\alpha W', \quad d'((\beta^*)^*) = \partial_\beta W' = 0, \quad d'(\alpha^*) = \partial_\alpha W', \quad d'(\beta^*) = \partial(\beta^*), W'$$

and $d(t_i) = (-1)^{m+1} e_i (\sum_{\gamma \in Q'_1 \setminus \Omega} [\gamma, \gamma^*]) e_i$, where $W' \in K\overline{Q'}$ is the element

$$W' = \sum_{\alpha \in \Omega} (-1)^{|\alpha|} \alpha^* d(\alpha) + \sum_{\beta \in Q_1 \setminus \Omega} (-1)^{|\beta|} (\beta^*)^* d(\beta^*) = \sum_{\beta \in Q_1 \setminus \Omega} (-1)^{m-1} (\beta^*)^* \omega_\beta.$$

Finally, the isomorphism $\varphi: K\overline{Q} \to K\overline{Q'}$ defined on the generators by

$$\varphi(\gamma) = \begin{cases} (-1)^{m-1} \gamma & \text{if } \gamma = \alpha^* \text{ for some } \alpha \in Q_1, \\ \beta & \text{if } \gamma = (\beta^*)^* \text{ for some } \beta \in Q_1 \setminus \Omega, \\ \gamma & \text{otherwise} \end{cases}$$

induces an isomorphism $(K\overline{Q'},d') \cong \Gamma_{m+1}(Q,W)$ of dg-algebras. \hfill $\square$
2.4. Non-positively graded Ginzburg dg-algebras. In this section we restrict attention to Ginzburg dg-algebras which are concentrated in non-positive degrees and quote the construction of Guo [17] (generalizing that of Amiot in [1] for the case \( m = 2 \) of the generalized cluster category associated to a quiver with superpotential.

Let \( Q \) be a graded quiver and let \( w \in \mathbb{Z} \). Denote by \( Q^{(w)} \) the subquiver of \( Q \) consisting of the arrows of degree \( w \), in other words, \( Q^{(w)}_0 = Q_0 \) and \( Q^{(w)}_1 = \{ \alpha \in Q_1 : |\alpha| = w \} \).

Now fix a graded quiver \( Q \) and let \( m \in \mathbb{Z} \). Let \( Q \) denote the quiver constructed in Definition 2.9. Fix a homogeneous superpotential \( W \) on \( Q \) of degree \( 2 - m \) and let \( \Gamma = \Gamma_{m+1}(Q, W) \) be the Ginzburg dg-algebra. We immediately observe:

**Remark 2.14.** \( \Gamma^i = 0 \) for any \( i > 0 \) if and only if all the arrows of \( Q \) have non-positive degrees. This condition implies that \( H^0(\Gamma) = 0 \) for any \( i > 0 \).

**Lemma 2.15.** Assume that all the arrows of \( Q \) have non-positive degrees. Then \( H^0(\Gamma) \cong KQ^{(0)}/(da : \alpha \in Q_1^{-1}) \).

**Proof.** Since there are no arrows of \( Q \) of positive degree, the graded piece \( \Gamma^0 \) equals the path algebra of \( Q^{(0)} \) and the graded piece \( \Gamma^{-1} \) is spanned by the elements of the form \( uav \) where \( u, v \in KQ^{(0)} \) and \( \alpha \in Q_1^{-1} \). As \( d(uav) = u(da)v \), the claim follows. \( \square \)

**Lemma 2.16.** The following conditions are equivalent:

(a) Each arrow of \( Q \) has non-positive degree;
(b) \( m \geq 0 \) and for any \( \alpha \in Q_1 \) one has \( 1 - m \leq |\alpha| \) and \( |\alpha| \leq 0 \).

**Proof.** If \( m < 0 \) then the each of the loops \( t_i \) in \( Q \) has positive degree \( -m \). In addition, each other arrow of \( Q \) is either \( \alpha \) or \( \alpha^* \) for some arrow \( \alpha \in Q_1 \), and its degree is \( |\alpha| \) or \( 1 - m - |\alpha| \), respectively. These observations imply the statement of the lemma. \( \square \)

In the next two examples we discuss the cases \( m = 0 \) and \( m = 1 \).

**Example 2.17.** Assume that \( m = 0 \). Lemma 2.10 implies that the arrows of \( Q \) have non-positive degrees if and only if \( Q \) has no arrows. In this case \( Q \) is a disjoint union of graded quivers of the form

\[
\bullet \circlearrowright t
\]

with \( |t| = 0 \) and \( dt = 0 \). Hence \( \Gamma \) is concentrated in degree 0 and it is a finite direct product of polynomial rings \( k[t] \). The algebra \( k[t] \) is 1-Calabi-Yau, see [21, §4.2]

**Example 2.18.** Assume that \( m = 1 \). Lemma 2.10 implies that the arrows of \( Q \) have non-positive degrees if and only if all the arrows of \( Q \) are in degree 0, so we can regard \( Q \) as an ungraded quiver. Since the superpotential on \( Q \) is homogeneous of degree 1, it must vanish. In this case all the arrows \( \alpha \) and \( \alpha^* \) of \( Q \) are in degree 0 and the differential of each loop \( t_i \), whose degree is \( -1 \), is given by \( d(t_i) = e_i(\sum_{\alpha \in Q_1}[\alpha, \alpha^*])e_i \). By Lemma 2.10 \( H^0(\Gamma) \) is isomorphic to the preprojective algebra of the quiver \( Q \).

When \( m \geq 2 \), the next lemma shows that we may assume that \( Q^{(1-m)} \) has no arrows.

**Lemma 2.19.** Assume that \( m \geq 2 \) and that \( 1 - m \leq |\alpha| \leq 0 \) for any \( \alpha \in Q_1 \). Then there exist a graded quiver \( Q' \) with \( 2 - m \leq |\alpha'| \leq 0 \) for each \( \alpha' \in Q'_1 \) and a homogeneous superpotential \( W' \) on \( Q' \) of degree \( 2 - m \) such that \( \Gamma_{m+1}(Q, W) \cong \Gamma_{m+1}(Q', W') \).
Proof. We define the graded quiver $Q'$ as a subquiver of $\overline{Q}$; we set $Q'_0 = Q_0$ and

$$Q'_1(w) = \begin{cases} Q_1(0) \cup \{\alpha^* : \alpha \in Q_1^{(1-m)}\} & \text{if } w = 0, \\
Q_1(w) & \text{if } 1 - m < w < 0,
\end{cases}$$

with $Q'_1(w)$ empty for any other $w \in \mathbb{Z}$. Then $2 - m \leq |\alpha'| \leq 0$ for any arrow $\alpha' \in Q'_1$ by construction. Moreover, since the superpotential $W$ is of degree $2 - m$ and all the arrows of $Q$ have non-positive degrees, no term of $W$ can contain any arrows of $Q^{(1-m)}$. The result now follows by iterated application of Lemma 2.12 for each of the arrows in $Q^{(1-m)}$. $\square$

In particular, when $m = 2$ one can always reduce to the classical setting of an ungraded quiver with potential.

Corollary 2.20. Let $Q$ be a graded quiver such that $-1 \leq |\alpha| \leq 0$ for any $\alpha \in Q_1$ and let $W$ be a homogeneous superpotential on $Q$ of degree 0. Then there exist a quiver $Q'$ concentrated in degree 0 and a superpotential $W'$ on $Q'$ such that $\Gamma_3(Q, W) \cong \Gamma_3(Q', W')$.

The next lemma generalizes [25, Lemma 2.11].

Lemma 2.21. Assume that $m \geq 2$ and $2 - m \leq |\alpha| \leq 0$ for any $\alpha \in Q_1$. Then

$$H^0(\Gamma_{m+1}(Q, W)) \cong KQ(0)/\partial_\alpha W : \alpha \in Q_1^{(2-m)}).$$

Proof. Observe that since the arrows of $Q$ have non-positive degrees and $W$ is of degree $2 - m$, the cyclic derivative $\partial_\alpha W$ with respect to any arrow $\alpha$ of degree $2 - m$ lies in the path algebra of $Q(0)$ and the quotient in the right hand side makes sense.

The arrows of $\overline{Q}$ have non-positive degrees by Lemma 2.10 hence we may apply Lemma 2.15 to deduce that $H^0(\Gamma_{m+1}(Q, W)) \cong K\overline{Q}(0)/(d\alpha : \alpha \in \overline{Q}^{(-1)}_1)$. Observe that $\overline{Q}(0) = Q(0)$ since $Q_1^{(1-m)}$ is empty by assumption and $\overline{Q}^{(-1)}_1 = Q^{(-1)}_1 \cup \{\alpha^* : \alpha \in Q_1^{(2-m)}\}$ with $d(\alpha) = 0$ and $d(\alpha^*) = \partial_\alpha W$ for any $\alpha \in Q_1$, hence the claim follows. $\square$

Theorem 2.22 ([17, Theorem 3.3]). Let $m \geq 1$ be an integer, let $Q$ be a graded quiver, let $W$ be a superpotential on $Q$ of degree $2 - m$ and denote by $\Gamma = \Gamma_{m+1}(Q, W)$ the Ginzburg dg-algebra. Assume that:

(i) $1 - m \leq |\alpha| \leq 0$ for all $\alpha \in Q_1$;
(ii) $H^0(\Gamma)$ is finite-dimensional.

Then the triangulated category $\mathcal{C}(Q, W) = \text{per} \Gamma/\mathcal{D}^b(\Gamma)$ is Hom-finite and $m$-Calabi-Yau. Moreover, the image of $\Gamma$ in $\mathcal{C}(Q, W)$ is an $m$-cluster-tilting object whose endomorphism algebra is isomorphic to $H^0(\Gamma)$.

Proof. The condition (i) implies that $H^i(\Gamma) = 0$ for all $i > 0$. Now Theorem 2.11 and condition (ii) imply that $\Gamma$ satisfies the conditions of Theorem 2.10 and the result is now a consequence of that theorem. $\square$

3. The construction

3.1. Superpotentials from quivers with relations. In this section we construct, given a quiver $Q$, a finite sequence $R$ of relations on $Q$ and an integer $m \geq 2$, a graded quiver with homogeneous superpotential of degree $2 - m$. The construction generalizes
that of Keller in [22, §6.9] for the case where \( m = 2 \) and \( KQ/(R) \) has global dimension 2. The idea of adding, for each relation, an arrow in the opposite direction appears already in the description of relation-extension algebras by Assem, Brüstle and Schiffler [1].

**Definition 3.1.** Let \( Q \) be a quiver and let \( \tau \) be the ideal of \( KQ \) generated by all the arrows. A relation on \( Q \) is an element of \( e_i e_j \) for some \( i, j \in Q_0 \). In other words, a relation is a linear combination of paths of positive lengths starting at \( i \) and ending at \( j \).

We start with some preparations concerning split extensions of algebras which will be needed for the case \( m = 2 \).

**Definition 3.2.** An algebra \( \tilde{A} \) is a split extension of an algebra \( A \) if there exist algebra homomorphisms \( \iota: A \to \tilde{A} \) and \( \pi: \tilde{A} \to A \) such that \( \pi \iota = id_A \).

Let \( Q \) be a quiver and let \( \tilde{Q} \) be a quiver such that \( \tilde{Q}_0 = Q_0 \) and \( Q_1 \subseteq \tilde{Q}_1 \) (in other words, \( \tilde{Q} \) is obtained from \( Q \) by adding arrows). Then the path algebra \( K\tilde{Q} \) is a split extension of the path algebra \( KQ \). Indeed, there are algebra homomorphisms

\[
KQ \xrightarrow{\iota_{Q,\tilde{Q}}} K\tilde{Q} \xrightarrow{\pi_{Q,\tilde{Q}}} KQ
\]

whose values on the generators are given by

\[
\iota_{Q,\tilde{Q}}(\alpha) = \alpha \quad (\alpha \in Q_1) \quad \text{and} \quad \pi_{Q,\tilde{Q}}(\alpha) = \begin{cases} 
\alpha & \text{if } \alpha \in Q_1, \\
0 & \text{if } \alpha \in \tilde{Q}_1 \setminus Q_1.
\end{cases}
\]

Denote by \( \iota' \) the ideal of \( K\tilde{Q} \) generated by the arrows in the set \( \tilde{Q}_1 \setminus Q_1 \). Let \( R \) be a set of relations in \( KQ \) and let \( \tilde{R} \) be a set of relations in \( K\tilde{Q} \) such that \( R \subseteq \tilde{R} \) (via the natural embedding \( \iota_{Q,\tilde{Q}}: KQ \to K\tilde{Q} \)).

**Lemma 3.3.** Assume that \( \tilde{R} \setminus R \subseteq \iota' \). Then the algebra \( K\tilde{Q}/(\tilde{R}) \) is a split extension of the algebra \( KQ/(R) \).

**Proof.** Consider the composition \( KQ \xrightarrow{\iota_{Q,\tilde{Q}}} K\tilde{Q} \to K\tilde{Q}/(\tilde{R}) \). Since \( R \subseteq \tilde{R} \), the image of any relation \( \rho \in R \) vanishes and we get an algebra homomorphism \( \iota: KQ/(R) \to K\tilde{Q}/(\tilde{R}) \). Consider now the composition \( K\tilde{Q} \xrightarrow{\pi_{Q,\tilde{Q}}} KQ \to KQ/(R) \) and let \( \rho \in \tilde{R} \). If \( \rho \in R \), then its image obviously vanishes. Otherwise, \( \rho \not\in R \) and our assumption that \( R \setminus R \subseteq \iota' \) implies that \( \pi_{Q,\tilde{Q}}(\rho) = 0 \), so its image vanishes as well. Hence we get a well defined algebra homomorphism \( \pi: K\tilde{Q}/(\tilde{R}) \to KQ/R \). The composition \( \pi \iota \) maps the image of any arrow \( \alpha \in \tilde{Q} \) in \( KQ/(R) \) to itself, therefore it is the identity on \( KQ/(R) \).

Any quiver with a finite sequence of relations gives rise to a dg-algebra whose underlying graded algebra is the path algebra of a graded quiver with arrows concentrated in degrees 0 and -1. The details are given in the construction below.

**Construction 3.4.** Let \((Q,R)\) be a pair where \( Q \) is a quiver and \( R = \bigcup_{i,j \in Q_0} R_{i,j} \), where each \( R_{i,j} \) is a finite sequence of relations inside \( e_i e_j \) and \( R \) is the concatenation of these sequences (there may be repetitions inside each sequence \( R_{i,j} \) and moreover the zero element can appear inside several such sequences). For a relation \( \rho \in R_{i,j} \), set \( s(\rho) = i \) and \( t(\rho) = j \).
We define a graded quiver $Q'$ as follows:

- The set of vertices of $Q'$ equals that of $Q$;
- The set of arrows consists of
  - the arrows of $Q$, with their degree set to 0;
  - an arrow $\eta_\rho: s(\rho) \rightarrow t(\rho)$ of degree $-1$ for each relation $\rho \in R$;
and denote by $B(Q, R)$ the dg-algebra whose underlying graded algebra is the path algebra $KQ$ with the differential acting on the generators by

- $d(\alpha) = 0$ for any $\alpha \in Q_1$;
- $d(\eta_\rho) = \rho$ for any $\rho \in R$.

Obviously, $B(Q, R)$ is concentrated in non-positive degrees and $H^0(B(Q, R))$ is isomorphic to $KQ/(R)$, but in general $B(Q, R)$ is not quasi-isomorphic to its zeroth homology.

**Remark 3.5.** If $B = (KQ', d)$ is any dg-algebra whose underlying graded algebra is the path algebra of a graded quiver with arrows concentrated in degrees 0 and $-1$ and the image of the differential $d$ lies in the ideal generated by the arrows, then $B \cong B(Q, R)$ for some $(Q, R)$. Indeed, we can take $Q = Q'(0)$ and for each $i, j \in Q_0$ let $R_{i,j}$ be the list of $d(\alpha)$ where $\alpha$ runs over the arrows in $Q_1^{(-1)}$ starting at $i$ and ending at $j$.

It turns out (see Lemma 3.10 below) that for any $m \geq 2$, the $(m + 1)$-Calabi-Yau completion of $B(Q, R)$ is a Ginzburg dg-algebra of a graded quiver with homogeneous superpotential of degree $2 - m$ whose construction is described below.

**Construction 3.6.** Let $(Q, R, m)$ be a triple where $(Q, R)$ is as in Construction 3.4 and $m \geq 2$ is an integer. We construct a graded quiver $\tilde{Q}$ with homogeneous superpotential $W$ of degree $2 - m$ as follows.

- The set of vertices of $\tilde{Q}$ equals that of $Q$;
- The set of arrows of $\tilde{Q}$ consists of
  - the arrows of $Q$, with their degree set to 0;
  - an arrow $\varepsilon_\rho: t(\rho) \rightarrow s(\rho)$ of degree $2 - m$ for each relation $\rho \in R$;
- The superpotential $W$ is the image of the element $\sum_{\rho \in R} \varepsilon_\rho \rho$ in $K\tilde{Q}/[K\tilde{Q}, K\tilde{Q}]$.

We denote by $\Gamma(Q, R, m)$ the Ginzburg dg-algebra $\Gamma_{m+1}(\tilde{Q}, W)$.

**Remark 3.7.** The quiver with superpotential $(\tilde{Q}, W)$ depends on the particular choice of the sequence $R$ and not only on the two-sided ideal $(R)$ it generates in $KQ$, as we shall see in Example 3.30.

For the rest of this section, we fix a triple $(Q, R, m)$ where $Q$ is a quiver, $R$ is a finite sequence of relations on $Q$ and $m \geq 2$. We denote by $(\tilde{Q}, W)$ the graded quiver with superpotential of degree $2 - m$ associated to $(Q, R, m)$ as in Construction 3.6 and by $\Gamma = \Gamma(Q, R, m) = \Gamma_{m+1}(\tilde{Q}, W)$ its Ginzburg dg-algebra. We denote the elements of $R$ by $\rho_1, \rho_2, \ldots, \rho_n$ and write $|R| = n$. For simplicity, we denote by $\varepsilon_k$ the arrow $\varepsilon_{\rho_k}$ of degree $2 - m$ in $\tilde{Q}$ corresponding to the relation $\rho_k$, so that $W$ is the image of $\sum_{k=1}^n \varepsilon_k \rho_k$ modulo $[K\tilde{Q}, K\tilde{Q}]$. Similarly, denote by $\eta_k$ the arrow $\eta_{\rho_k}$ and let $B = B(Q, R)$ be the dg-algebra of Construction 3.4.

We start by describing the graded quiver $\overline{Q}$ underlying $\Gamma$ occurring in Definition 2.9.
Lemma 3.8. The arrows of the graded quiver $\overline{Q}$, their degrees and their differentials are as given in Table 1.

Proof. The description of the arrows and their degrees is evident from Definition 2.9. For the differentials, note that since none of the arrows $\varepsilon_k$ occur in any $\rho \in R$, we have

$$\partial_{\varepsilon_k} W = \partial_{\varepsilon_k} \left( \sum_{\ell=1}^{n} \varepsilon_{\ell} \rho_{\ell} \right) = \partial_{\varepsilon_k} (\varepsilon_k \rho_k) = (-1)^{|\varepsilon_k|} \rho_k = (-1)^m \rho_k$$

for any $1 \leq k \leq n$.

Lemma 3.9. Consider the algebras $A = KQ/(R)$ and $\tilde{A} = H^0(\Gamma)$.

(a) If $m > 2$ then $\tilde{A} \cong A$.

(b) If $m = 2$ then $\tilde{A}$ is a split extension of $A$.

Proof. By Lemma 2.21, $\tilde{A} \cong K\tilde{Q}^{(0)}/\langle \partial_{\varepsilon} W : \alpha \in \tilde{Q}^{(2-m)} \rangle$.

If $m > 2$ then $\tilde{Q}^{(0)} = Q$, the arrows of $\tilde{Q}^{(2-m)}$ are $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ and $\partial_{\varepsilon_k} W = (-1)^m \rho_k$ according to Table 1. Hence $\tilde{A} \cong KQ/(\rho_1, \rho_2, \ldots, \rho_n)$. This shows part (a).

If $m = 2$ then all the arrows of the quiver $\tilde{Q}$ have degree 0, and we can think of $\tilde{Q}$ as an ungraded quiver consisting of the arrows of $Q$ and the arrows $\varepsilon_k$ for $1 \leq k \leq n$. In other words, $\tilde{Q} = Q^{(0)}$ and $Q \subseteq \tilde{Q}$ with $\tilde{Q} \setminus Q = \{ \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \}$. Moreover, $\tilde{A} = K\tilde{Q}/(R)$ for $R = \{ \partial_{\varepsilon_k} W : \alpha \in Q_1 \}$. Now $R \subseteq \tilde{R}$ since $\partial_{\varepsilon_k} W = \rho_k$ for each $1 \leq k \leq n$. In addition, for any $\alpha \in Q_1$, the element $\partial_{\alpha} W = \sum_{k=1}^{n} \partial_{\alpha} (\varepsilon_k \rho_k)$ lies in the ideal generated by $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ in $\tilde{Q}$. Part (b) now follows from Lemma 3.3.

The next two lemmas relate the dg-algebra $B$ and the Ginzburg dg-algebra $\Gamma$. For a similar result in the case $m = 2$, see [22] [6.7 and Proposition 6.8].

Lemma 3.10. $\Gamma$ is the $(m+1)$-Calabi-Yau completion of $B$.

Proof. This is a consequence of Lemma 2.3. Indeed, we may take $\Omega = Q_1$ so that $Q_1 \setminus \Omega = \{ \varepsilon_1, \ldots, \varepsilon_n \}$ and the superpotential $W = \sum_{k=1}^{n} \varepsilon_k \rho_k$ has the required form. We only note that the differential of $\Gamma$ restricts to the path algebra of the quiver with arrows $Q_1 \cup \{ \varepsilon_1^*, \ldots, \varepsilon_n^* \}$ and the resulting dg-algebra is isomorphic to $B$ by mapping each arrow $\varepsilon_k^*$ to $(-1)^m \rho_k$ and sending each $\alpha \in Q_1$ to itself.

Lemma 3.11. $H^{-i}(\Gamma) \cong H^{-i}(B)$ for any $i < m - 2$. 

| arrow | degree | differential |
|-------|--------|--------------|
| $\varepsilon_k$ | 0 | 0 $(\alpha \in Q_1)$ |
| $\varepsilon_k$ | -1 | $(-1)^m \rho_k$ $(1 \leq k \leq n)$ |
| $\alpha$ | $2 - m$ | 0 $(1 \leq k \leq n)$ |
| $t_i$ | $-m$ | $d(t_i)$ $(i \in Q_0)$ |

Table 1. The arrows of the graded quiver $\overline{Q}$ and their differentials.
Proof. From Table 1 we see that \( \Gamma^{-1} \cong B^{-i} \) for any \( i < m - 2 \), hence \( H^{-i}(\Gamma) \cong H^{-i}(B) \) for any \( i < m - 3 \). The graded piece \( \Gamma^{2-m} \) can be decomposed as \( \Gamma^{2-m} \cong B^{2-m} \oplus F \), where the space \( F \) is spanned by the elements of the form \( u \varepsilon_k v \) where \( 1 \leq k \leq n \) and \( u, v \) are paths in \( Q \). Since the differential of each such element vanishes, one has \( d(F) = 0 \), hence \( d(\Gamma^{2-m}) \cong d(B^{2-m}) \) and therefore \( H^{3-m}(\Gamma) \cong H^{3-m}(B) \) as well.

Lemma 3.12. If \( R \) is empty then \( H^{-i}(\Gamma) = 0 \) for any \( 1 \leq i \leq m - 2 \).

Proof. If \( R \) is empty, then by Lemma 3.8 the arrows in the graded quiver \( \Gamma \) have degrees \( 0, 1 - m \) or \(-m\). Hence the graded piece \( \Gamma^{-i} \) vanishes for each \( 1 \leq i \leq m - 2 \) and the claim follows.

The next lemma provides a partial converse to Lemma 3.12.

Lemma 3.13. If \( m > 2 \) and \( R \subseteq k^2 \), then \( \dim_K H^{2-m}(\Gamma) \geq |R| \).

Proof. The graded piece \( \Gamma^{2-m} \) is spanned by two types of elements:

1. \( u_1 \varepsilon_{k_1} \cdots u_m \varepsilon_{k_{m-1}} u_m \), where \( u_1, u_2, \ldots, u_m \) are paths in \( Q \) and \( 1 \leq k_j \leq n \) for each \( 1 \leq j \leq m - 2 \);
2. \( u \varepsilon_k v \), where \( u, v \) are paths in \( Q \) and \( 1 \leq k \leq n \);

As a \( K \)-vector space, we may thus decompose \( \Gamma^{2-m} \) into a direct sum \( E \oplus E' \), where \( E \) is the \( n \)-dimensional subspace \( E = \{ \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \cdots + \lambda_n \varepsilon_n : (\lambda_1, \lambda_2, \ldots, \lambda_n) \in K^n \} \) and \( E' \) is spanned by all the elements of type (1) and those of type (2) such that at least one of \( u, v \) has positive length.

We will show that \( d(\Gamma^{1-m}) \subseteq E' \) and hence no non-zero element in \( E \) lies in the image of the differential \( d \) acting on the graded piece \( \Gamma^{1-m} \). Since \( d \) vanishes on \( E \), this will yield an \( n \)-dimensional subspace inside \( H^{2-m}(\Gamma) \).

If \( m > 3 \), the graded piece \( \Gamma^{1-m} \) is spanned by three types of elements:

1. \( u_1 \varepsilon_{k_1} \varepsilon_{k_2} \varepsilon_{k_{m-1}} u_m \), where \( u_1, u_2, \ldots, u_m \) are paths in \( Q \) and \( 1 \leq k_j \leq n \) for each \( 1 \leq j \leq m - 1 \);
2. \( u \varepsilon_k v \) and \( u \varepsilon_k v \), where \( u, v, w \) are paths in \( Q \) and \( 1 \leq k, l \leq n \);
3. \( u \alpha \varepsilon^* v \), where \( u, v \) are paths in \( Q \) and \( \alpha \in Q_1 \).

If \( m = 3 \), in addition to the elements above there is a fourth type

4. \( u \varepsilon_k v \), where \( u, v \) are paths in \( Q \) and \( 1 \leq k, l \leq n \).

It suffices to prove that the differential of any of these elements belongs to \( E' \). This is clear for the elements of the type (1) since

\[
d(u_1 \varepsilon_{k_1} \varepsilon_{k_2} \varepsilon_{k_{m-1}} u_m) = \sum_{j=1}^{m-1} \pm u_1 \varepsilon_{k_1} \varepsilon_{k_2} \varepsilon_{k_{j-1}} u_j \varepsilon_{k_j} u_{j+1} \varepsilon_{k_{j+1}} \cdots \varepsilon_{k_{m-1}} u_m
\]

and none of the arrows \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \) can appear in the right hand side. This is also clear for the elements of type (1) since their differential vanishes.

Consider an element of type (2). Then \( d(u \varepsilon_k v \varepsilon^* w) = u \varepsilon_k v \varepsilon^* \varepsilon w \), hence the differential is spanned by elements of the form \( u' \varepsilon_k v' \) where \( u', v' \) are paths in \( Q \) and \( v' \) has positive length. The case of \( d(u \varepsilon_k v \varepsilon^* w) \) is similar.

Consider an element of type (3). Then \( d(u \alpha \varepsilon^* v) = u(\partial_u W)v = \sum_{j=1}^{n} u \varepsilon_{k_j} v(\varepsilon_{k_{j+1}} v) \). Our assumption that \( R \subseteq k^2 \) implies that each \( \varepsilon_{k_j} v(\varepsilon_{k_{j+1}} v) \) is a linear combination of terms \( u'' \varepsilon_k v'' \) where at least one of the paths \( u'', v'' \) has positive length.

□
Corollary 3.14. If $m > 2$ and $R \subseteq \mathfrak{r}^2$, then $H^{2-m}(\Gamma) = 0$ if and only if $R$ is empty.

Consider the triangulated category $C_{(Q,R,m)} = \text{per}\Gamma(Q,R,m)/D\text{id}(\Gamma(Q,R,m))$.

Theorem 3.15. Let $Q$ be a quiver, let $R$ be a finite sequence of relations on $Q$ such that the algebra $A = KQ/(R)$ is finite-dimensional and let $m > 2$ be an integer. Then:

(a) The category $C = C_{(Q,R,m)}$ is Hom-finite and $m$-Calabi-Yau.

(b) The image $T$ of $\Gamma(Q,R,m)$ in $C$ is an $m$-cluster-tilting object with $\text{End}_C(T) \cong A$.

(c) If $R \subseteq \mathfrak{r}^2$ then $\dim K\text{Hom}_C(T, \Sigma^{-(m-2)}T) \geq |R|$.

Proof. Recall that $\Gamma(Q,R,m) = \Gamma_{m+1}(\hat{Q},W)$ where $(\hat{Q},W)$ is the graded quiver with superpotential of degree $2 - m$ associated to the triple $(Q,R,m)$ as in Construction 3.6. We claim that $(\hat{Q},W)$ satisfies the conditions of Theorem 2.22. Indeed, condition (i) holds since the degree of any arrow in $\hat{Q}$ is either 0 or $2 - m$, and condition (ii) holds since by Lemma 3.9 $H^0(\Gamma) \cong A$ and the algebra $A$ is assumed to be finite-dimensional.

Hence, by Theorem 2.22 the triangulated category $C = C_{(Q,R,m)} = C_{(Q,W)}$ is Hom-finite, $m$-Calabi-Yau and the image $T$ of $\Gamma = \Gamma(Q,R,m)$ under the canonical projection $\Gamma \to C$ is an $m$-cluster-tilting object whose endomorphism algebra is $\text{End}_C(T) \cong H^0(\Gamma) \cong A$. The last assertion follows from Lemma 2.7 and Lemma 3.13. $\Box$

Remark 3.16. The $m$-Calabi-Yau category $C$ of Theorem 3.15 depends on $Q$ and $R$ and not only on the algebra $A$. There exist quivers $Q$ and sequences of relations $R$ and $R'$ on $Q$ such that the algebras $KQ/(R)$ and $KQ/(R')$ are finite-dimensional and isomorphic but for any $m > 2$ the categories $C_{(Q,R,m)}$ and $C_{(Q,R',m)}$ are not equivalent, see Example 3.30 below.

Remark 3.17. Keep the notations and assumptions of Theorem 3.15 and write $A$ as $A = \bigoplus_{\Gamma \in Q_0} P_\Gamma$ where $P_\Gamma = e_\Gamma A$ are the indecomposable projective right $A$-modules. The decomposition $\Gamma = \bigoplus_{\Gamma \in Q_0} e_\Gamma \Gamma$ for $\Gamma = \Gamma(Q,R,m)$ induces a decomposition $T = \bigoplus_{\Gamma \in Q_0} T_\Gamma$ with $T_i$ being the image of $e_i \Gamma$ under the canonical projection to $C$. Hence for any finitely generated projective $A$-module $P = \bigoplus_{\Gamma \in Q_0} P_\Gamma$ such that $e_i > 0$ for all $i \in Q_0$ there exists an $m$-cluster-tilting object $T_P = \bigoplus_{\Gamma \in Q_0} T_\Gamma$ in $C$ with $\text{End}_C(T_P) \cong \text{End}_A(P)$.

The next result shows that the $m$-cluster category of any acyclic quiver can be realized as a category of the form $C(Q,R,m)$. Recall that a quiver is acyclic if it has no cycles of positive length. We refer to [17] Corollary 3.4 for a related result.

Proposition 3.18. Let $Q$ be a quiver, let $R$ be a finite sequence of relations on $Q$ such that $R \subseteq \mathfrak{r}^2$ and the algebra $KQ/(R)$ is finite-dimensional, and let $m > 2$ be an integer. Then the following conditions are equivalent, where $C$ denotes the $m$-Calabi-Yau category $C_{(Q,R,m)}$ and $T$ is the canonical $m$-cluster-tilting object in $C$.

(a) The quiver $Q$ is acyclic and the sequence $R$ is empty;

(b) The dg-algebra $B(Q,R)$ is concentrated in degree 0 and has finite total dimension;

(c) $\text{Hom}_C(T, \Sigma^{-i}T) = 0$ for any $0 < i < m - 1$;

(d) $\text{Hom}_C(T, \Sigma^{-(m-2)}T) = 0$.

Moreover, if any of these equivalent conditions holds and the field $K$ is algebraically closed, then $C$ is triangle equivalent to the $m$-cluster category of $Q$. 
Proof. The equivalence of (a) and (b) is clear. Let \( \Gamma = \Gamma(Q, R, m) \). For the implication (a) \( \Rightarrow (c) \), note that if \( Q \) is acyclic and \( R \) is empty then \( \text{Hom}_C(T, \Sigma^{-i}T) \cong H^{-i}(\Gamma) = 0 \) for any \( 0 < i < m-1 \) by Lemma 2.7 and Lemma 3.12. The implication (a) \( \Rightarrow (d) \) is clear. For the implication (a) \( \Rightarrow (m) \), note that by Lemma 2.7 and Lemma 3.13
\[
\dim_K \text{Hom}_C(T, \Sigma^{-(m-2)}T) = \dim_K H^{2-m}(\Gamma) \geq |R|,
\]
hence if \( \text{Hom}_C(T, \Sigma^{-(m-2)}T) \) vanishes \( R \) must be empty and then \( Q \) is acyclic by our assumption that \( KQ/(R) \) is finite-dimensional.

If any of these conditions holds, then \( C \) is an algebraic Hom-finite, \( m \)-Calabi-Yau triangulated category with an \( m \)-cluster-tilting object \( T \) such that \( \text{End}_C(T) \cong KQ \) for an acyclic quiver \( Q \) and \( \text{Hom}_C(T, \Sigma^{-i}T) = 0 \) for any \( 0 < i < m-1 \). By the characterization of higher cluster categories of Keller and Reiten [24, Theorem 4.2]) if \( K \) is algebraically closed, then \( C \) is triangle equivalent to the \( m \)-cluster-category of the quiver \( Q \). \( \Box \)

3.2. Systems of relations. Let \( Q \) be a quiver and let \( r \) be the two-sided ideal of \( KQ \) generated by the arrows of \( Q \). An ideal \( I \) of \( KQ \) is admissible if there exists some \( N \geq 2 \) such that \( r^N \subseteq I \subseteq r^2 \).

Definition 3.19 ([7]). Let \( I \) be an ideal of \( KQ \). A system of relations for \( I \) is a set \( R \) of relations such that \( R \), but no proper subset of it, generates \( I \) as a two-sided ideal.

The following statement is well-known.

Lemma 3.20. If \( I \) is admissible then there exists a finite system of relations for \( I \).

Proof. By assumption, there is some \( N \geq 2 \) such that \( r^N \subseteq I \). The algebra \( KQ/r^N \) is finite-dimensional, as it is spanned by all the paths of \( Q \) of length smaller than \( N \). Therefore the space \( I/r^N \) is also finite-dimensional. For each \( i, j \in Q_0 \), choose a basis of \( e_i(I/r^N)e_j \) and choose a set \( R_{i,j} \) inside \( e_iIe_j \) whose image modulo \( r^N \) equals that basis. Then \( I = (R) \) for the finite set \( R \) given by
\[
R = \{ \text{the paths of length } N \text{ in } Q \} \cup \bigcup_{i,j \in Q_0} R_{i,j}.
\]

If \( R \) is not a system of relations for \( I \) then there is a proper subset \( R' \) of \( R \) such that \( I = (R') \). In this way we can repeatedly remove elements and still have a set generating \( I \). Since \( R \) is finite, this process must terminate and we eventually end with a system of relations for \( I \). \( \Box \)

Under some conditions, another approach to the construction of systems of relations involves lifting of basis elements of the space \( I/(I+rI) \), see the discussion in [11, §7].

Lemma 3.21. If \( I \) is admissible, then \( I/(I+rI) \) is finite-dimensional.

Proof. Let \( N \geq 2 \) be such that \( r^N \subseteq I \). Then \( r^{N+1} \subseteq (I+rI) \) and we have an inclusion and a surjection
\[
KQ/r^{N+1} \supseteq I/r^{N+1} \twoheadrightarrow I/(I+rI).
\]
The claim now follows since the quotient \( KQ/r^{N+1} \) is finite-dimensional. \( \Box \)

Lemma 3.22. Let \( I \) be an ideal of \( KQ \) and let \( R \) be a set of relations inside \( I \).

(a) If \( I = (R) \) then the image of \( R \) modulo \( I+rI \) spans the vector space \( I/(I+rI) \).
(b) Assume that the ideal \((R)\) is admissible and the image of \(R\) modulo \(I\alpha + \lambda I\) spans the vector space \(I/(I\alpha + \lambda I)\). Then \(I = (R)\).

(c) Assume that the ideal \((R)\) is admissible and the image of \(R\) modulo \(I\alpha + \lambda I\) is a basis of the vector space \(I/(I\alpha + \lambda I)\). Then \(R\) is a system of relations for \(I\).

**Proof.** For part (a), observe that since each \(\rho \in R\) is a relation, multiplying it from the left or from the right by an element of the form \(\sum_{i \in Q_0} \lambda_i e_i\) gives a scalar multiple of \(\rho\).

For part (b), we slightly modify the argument in [9, Lemma 3.6]. Let \(N \geq 2\) such that \(r^N \subseteq (R)\) and let \(x \in I\). By assumption, we can write

\[
(3.1) \quad x = \lambda_1 \rho_1 + \cdots + \lambda_n \rho_n + x_1' r_1' + \cdots + x_k' r_k' + x_1'' + \cdots + x_l''
\]

with scalars \(\lambda_1, \ldots, \lambda_n \in K\), relations \(\rho_1, \ldots, \rho_n \in R\) and \(x_1', \ldots, x_k', x_1'', \ldots, x_l'' \in I\), with \(r_1', \ldots, r_k', r_1'', \ldots, r_l'' \in \ell\). Writing each of the elements \(x_1', \ldots, x_k', x_1'', \ldots, x_l''\) using (3.1) and repeating this process \(N\) times, we conclude that \(x = \rho + r\) where \(\rho \in (R)\) and \(r \in r^N\), hence \(x \in (R)\).

Finally, part (c) follows from (a) and (b). Moreover, \(R\) is finite by Lemma 3.21. \(\square\)

**Example 3.23.** The assumption in parts (b) and (c) that the ideal \((R)\) is admissible cannot be dropped. For example, consider the quiver \(Q\) given by

\[
\begin{array}{ccc}
\alpha & \circ & \circ \\
+ & \times & \beta
\end{array}
\]

and let \(I = (\alpha^2 - \beta \alpha \beta, \beta^2 - \alpha \beta \alpha, \alpha^2 \beta)\). One can check that \(r^5 \subseteq I \subseteq r^2\) hence the ideal \(I\) is admissible. The 8-dimensional algebra \(KQ/I\) is an algebra of quaternion type in the sense of Erdmann [14]. When \(K\) is algebraically closed of characteristic 2, this algebra is isomorphic to the group algebra of the quaternion group.

The image of \(\alpha^2 \beta\) in \(I\alpha + \lambda I\) vanishes, as the following calculation shows:

\[
\alpha^2 \beta = (\alpha^2 - \beta \alpha \beta)\beta + \beta \alpha (\beta^2 - \alpha \beta \alpha) + \beta \alpha^2 \beta \alpha \in \ell \alpha + \lambda I,
\]

hence \(I/(I\alpha + \lambda I)\) is spanned by the images of the elements \(\alpha^2 - \beta \alpha \beta\) and \(\beta^2 - \alpha \beta \alpha\). Nevertheless, the ideal \(I' = (\alpha^2 - \beta \alpha \beta, \beta^2 - \alpha \beta \alpha)\) is not equal to \(I\). Indeed, by letting \(\alpha\) and \(\beta\) act on \(K\) as the identity we get a one-dimensional module over \(KQ/I'\) with a non-zero action of \(\alpha^2 \beta\), hence \(\alpha^2 \beta \notin I'\).

### 3.3. Finite-dimensional algebras are \((m > 2)\)-CY-tilted.

In this section we assume that the field \(K\) is algebraically closed. A finite-dimensional algebra \(A\) over \(K\) is called **basic** if \(A_A \cong P_1 \oplus \cdots \oplus P_r\) where \(P_1, \ldots, P_r\) are representatives of the isomorphism classes of the indecomposable projective right \(A\)-modules.

**Theorem 3.24** (Gabriel). Let \(A\) be a basic, finite-dimensional algebra over \(K\). Then there exist a quiver \(Q\) and an admissible ideal \(I\) of \(KQ\) such that \(A \cong KQ/I\).

**Proof.** See [15, §4.3]. \(\square\)

Let \(A\) be a basic, finite-dimensional algebra. By Theorem 3.24 we can write \(A = KQ/I\) for a quiver \(Q\) and an admissible ideal \(I\) of \(Q\). We denote by \(S_i\) the simple \(A\)-module corresponding to a vertex \(i \in Q_0\) and consider the \(A\)-module \(S = \bigoplus_{i \in Q_0} S_i\).

**Lemma 3.25.** If \(R\) is a system of relations for \(I\) then \(|R| \geq \dim_K \text{Ext}_A^2(S, S)\).
Proof. We have $|R| \geq \dim_K I/(I + I) = \dim_K \text{Ext}_A^2(S, S)$ where the left inequality is a consequence of Lemma 3.22(a) and the right equality is [7, Corollary 1.1]. □

Note that Example 3.28 shows that the inequality in Lemma 3.25 can be strict.

**Theorem 3.26.** Let $A$ be a basic finite-dimensional algebra. Then the set of pairs $(Q, R)$ consisting of a quiver $Q$ and a sequence of relations $R \subseteq \mathfrak{r}^2$ such that $A \cong KQ/(R)$ is not empty. For any such pair $(Q, R)$ and any integer $m > 0$, the triangulated category $\mathcal{C} = \mathcal{C}(Q, R, m)$ is Hom-finite, m-Calabi-Yau and its canonical $m$-cluster-tilting object $T$ satisfies $\text{End}_\mathcal{C}(T) \cong A$ and $\dim_K \text{Hom}_\mathcal{C}(T, \Sigma^-(m-2)T) \geq \dim_K \text{Ext}_A^2(S, S)$.

Proof. The first claim is a consequence of Theorem 3.24 and Lemma 3.20. The second claim is a consequence of Theorem 3.15 and Lemma 3.25. □

**Corollary 3.27.** A finite-dimensional algebra over an algebraically closed field is $m$-CY-tilted for any $m > 2$.

Proof. We can write $A \cong \text{End}_A(P)$ for a finite-dimensional, basic algebra $A$ and a finitely generated projective $A$-module $P$ containing as direct summands all the indecomposable projective $A$-modules. Hence the claim is a consequence of Theorem 3.26 and Remark 3.17. □

**Remark 3.28.** If $\text{gldim} A \geq 2$, then for any of the $m$-Calabi-Yau categories $C$ of Theorem 3.26, the small negative extension $\text{Hom}_C(T, \Sigma^-(m-2)T)$ of the canonical $m$-cluster-tilting object $T$ cannot vanish. Had it vanished, Proposition 3.18 would then imply that $A$ is the path algebra of an acyclic quiver, a contradiction.

3.4. **Examples.** Our first example is similar in spirit to [17, Example 3.5].

**Example 3.29.** Consider the algebra $A = KQ/I$ where $Q$ is the left quiver

and $I = (R)$ for the system of relations $R = \{\alpha \beta - \gamma \delta\}$. The algebra $A$ has global dimension 2, hence it cannot be 2-CY-tilted by [23, Corollary 2.1].

Let $B = B(Q, R)$ be the dg-algebra of Construction 3.2. Its graded quiver is shown in the middle; the arrows $\alpha, \beta, \gamma$ and $\delta$ have degree 0 while $\eta$ has degree $-1$ and $d(\eta) = \alpha \beta - \gamma \delta$. Observe that the dg-algebra $B$ is quasi-isomorphic to $A \cong \text{H}^0(B)$.

Let $m \geq 2$ and let $(\bar{Q}, W)$ be the graded quiver with homogeneous superpotential of degree $2 - m$ of Construction 3.6. The graded quiver $\bar{Q}$ is shown on the right; the degrees of the arrows $\alpha, \beta, \gamma, \delta$ are 0, that of $\varepsilon$ is $2 - m$ and the superpotential is $W = \varepsilon(\alpha \beta - \gamma \delta)$.

The Ginzburg dg-algebra $\Gamma = \Gamma_{m+1}(\bar{Q}, W)$ is the $(m+1)$-Calabi-Yau completion of $B$ (Lemma 3.10). Since $B$ is quasi-isomorphic to $A$ and $A$ is derived equivalent to the path algebra of the Dynkin quiver $D_4$, the Morita invariance of Calabi-Yau completions [22, Proposition 4.2] implies that $\Gamma$ is Morita equivalent to $\Gamma_{m+1}(D_4, 0)$, hence by [17, Corollary 3.4] (or Proposition 3.18) the generalized $m$-cluster category $\mathcal{C}_\Gamma$ is triangle equivalent to the $m$-cluster category of type $D_4$. 

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If \( m = 2 \) then \( H^0(\Gamma) \cong K\tilde{Q}/(\tilde{R}) \), where \( \tilde{R} = \{ \alpha\beta - \gamma\delta, \varepsilon\alpha, \beta\varepsilon, \varepsilon\gamma, \delta\varepsilon \} \). This is a cluster-tilted algebra [9] of type \( D_4 \), which is the relation-extension [4] of the tilted algebra \( A \).

If \( m > 2 \) then \( H^0(\Gamma) \cong A \) and \( A \) is \( m \)-CY-tilted by Theorem 3.13.

**Example 3.30.** Let \( K \) be algebraically closed and consider the algebra \( A = K \) whose quiver \( Q \) is \( \bullet \). Consider the two sequences of relations \( R = \{ \} \) and \( R' = \{ 0 \} \) on \( Q \).

Let \( m > 2 \). The Ginzburg dg-algebras \( \Gamma = \Gamma(Q, R, m) \) and \( \Gamma' = \Gamma(Q, R', m) \) are given by the following graded quivers with differentials

\[
\begin{align*}
|t| &= -m \\
\text{d}(t) &= 0 \\
\end{align*}
\]

A basis for the graded piece \( \Gamma^n \) is given by \( (\varepsilon^*)^i \) if \( 0 < i < m - 2 \); \((\varepsilon^*)^{m-2}, \varepsilon\) if \( i = m - 2 \); \((\varepsilon^*)^{m-1}, \varepsilon\varepsilon^*, \varepsilon^2 \) if \( i = m - 1 \) and \( m > 3 \); and \((\varepsilon^*)^{m-1}, \varepsilon\varepsilon^*, \varepsilon^2 \) if \( i = m - 1 \) and \( m = 3 \), hence one computes

\[
\dim_K H^{-i}(\Gamma) = \begin{cases} 
1 & \text{if } i = 0, \\
0 & \text{if } 0 < i < m,
\end{cases} \\
\dim_K H^{-i}(\Gamma') = \begin{cases} 
1 & \text{if } 0 \leq i < m - 2, \\
2 & \text{if } i = m - 2, \\
2 + \delta_{3,m} & \text{if } i = m - 1.
\end{cases}
\]

Let \( C = C(Q, R, m) \) and \( C' = C(Q, R', m') \) be the corresponding \( m \)-Calabi-Yau categories and let \( T \in C \) and \( T' \in C' \) be the canonical \( m \)-cluster-tilting objects. We have \( \text{End}_C(T) \cong \text{End}_{C'}(T') \cong K \).

By Proposition 3.18, the category \( C \) is triangle equivalent to the \( m \)-cluster category of \( Q \). Hence \( m \)-cluster-tilting objects in \( C \) are the indecomposable objects \( \Sigma^j T \) for \( 0 \leq j < m \). Since \( \text{Hom}_C(T, \Sigma^{-i} T) = 0 \) for any \( 0 < i < m \), this remains true if we replace \( T \) by any of the other \( m \)-cluster-tilting objects \( \Sigma^j T \) in \( C \). On the other hand, \( T' \) is an \( m \)-cluster-tilting object in \( C' \) with \( \text{Hom}_{C'}(T', \Sigma^{-i} T') \neq 0 \) for any \( 0 < i < m \) by (3.2), hence \( C' \) cannot be triangle equivalent to \( C \).

The previous example can be generalized as follows. A *Dynkin quiver* is a quiver obtained by orienting the edges of a Dynkin diagram of type \( A_n \) (\( n \geq 1 \)), \( D_n \) (\( n \geq 4 \)) or \( E_n \) (\( n = 6, 7, 8 \)).

**Proposition 3.31.** Let \( Q \) be a Dynkin quiver and let \( m > 2 \). There exists an \( m \)-Calabi-Yau triangulated category \( C' \) with an \( m \)-cluster-tilting object \( T' \) such that \( \text{End}_{C'}(T') \cong K Q \) but \( C' \) is not triangle equivalent to the \( m \)-cluster category of \( Q \).

**Proof.** Let \( C \) be the \( m \)-cluster category of \( Q \). Since \( Q \) is Dynkin, the category \( C \) has only finitely many indecomposable objects, hence the number of \( m \)-cluster-tilting objects \( T \) in \( C \) such that \( \text{End}_C(T) \cong K Q \) is finite. Therefore there exists an integer \( n \) such that \( \dim_K \text{Hom}_C(T, \Sigma^{-(m-2)} T) < n \) for any such \( T \).

Let \( R = \{ 0, 0, \ldots, 0 \} \) be a sequence consisting of \( n \) zero elements (it does not matter which starting and ending vertex we assign to each zero element) and let \( C' = C(Q,R,m) \). By Theorem 3.13, the category \( C' \) is Hom-finite and \( m \)-Calabi-Yau with an \( m \)-cluster-tilting object \( T' \) satisfying \( \text{End}_{C'}(T') \cong K Q \) and \( \dim_K \text{Hom}_{C'}(T', \Sigma^{-(m-2)} T') \geq n \).
If $F: \mathcal{C} \to \mathcal{C}$ were a triangulated equivalence, then $T = FT'$ would be an $m$-cluster-tilting object in $\mathcal{C}$ with $\text{End}_\mathcal{C}(T) \cong KQ$ and $\dim_K \text{Hom}_\mathcal{C}(T, \Sigma^{-(m-2)}T) \geq n$, a contradiction. □

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