Cohomological Hall algebras, vertex algebras and instantons

Miroslav Rapčák, Yan Soibelman, Yaping Yang, Gufang Zhao

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Abstract

We define an action of the (double of) Cohomological Hall algebra of Kontsevich and Soibelman on the cohomology of the moduli space of spiked instantons of Nekrasov. We identify this action with the one of the affine Yangian of \( \mathfrak{gl}(1) \). Based on that we derive the vertex algebra at the corner \( W_{r_1, r_2, r_3} \) of Gaiotto and Rapčák. We conjecture that our approach works for a big class of Calabi-Yau categories, including those associated with toric Calabi-Yau 3-folds.

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1 Introduction

1.1 Mathematical motivation

Nakajima’s construction of an action of the infinite Heisenberg algebra on the (equivariant) cohomology of the moduli space of $U(1)$-instantons on $\mathbb{C}^2$ (see [58]) is an archetypical example of description of a Lie algebra via its action by correspondences on the cohomology of a moduli space. This branch of geometric representation theory has long history.

Nakajima’s result can be also interpreted as a geometric description (or definition if one prefers) of the $\mathcal{W}$-algebra of the affine Lie algebra $\widehat{\mathfrak{gl}}(1)$.

The case of $\mathcal{W}$-algebras $\mathcal{W}_r := \mathcal{W}(\mathfrak{gl}(r))$ was discussed recently in several papers in relation to the proof of the AGT conjecture. The closest to our point of view are [56, 79].
Main motivation for our project is the growing role in QFT of the notion of Cohomological Hall algebra (COHA for short) introduced in 2010 in [50]. Originally, it was considered by the authors as the mathematical incarnation of the notion of multiparticle BPS algebra. The idea of the algebra structure on closed BPS states goes back to Harvey and Moore (see e.g. [40]). Notice that differently from the original expectations of [40] the associative algebra structure proposed in [50] exists only on multiparticle BPS states and does not depend on the central charge of the stability structure. In order to derive the Lie algebra structure on single-particle BPS states one has to do more work (see the original conjecture in [50] and further developments in [15]). Then COHA “looks like” the universal enveloping algebra of this Lie algebra.

COHA was introduced originally in the framework of Quillen-smooth associative algebras with potential. It was mentioned in [50] (see e.g. Section 3.5 in the loc. cit.) that it can be defined in fact for a sufficiently general class of 3-dimensional Calabi-Yau categories (3CY categories for short) and an appropriate cohomology theory of constructible dg-stacks. Roughly, COHA is given by an algebra structure on the cohomology of the stack of objects of a 3CY-category with coefficients in the sheaf of vanishing cycles of its potential. Motivic Donaldson-Thomas invariants (DT-invariants for short) introduced in [51] and revisited in [50] can be defined in terms of the virtual Poincaré polynomial (a.k.a Serre polynomial) of COHA. Relation to DT-invariants explains the initial interest of physicists in COHA. Surprisingly the algebra structure of COHA was not seriously used in physics.

In this paper we illustrate the general hope that some (maybe all?) quantum algebras which appear in the interplay between 4d gauge theories and 2d CFTs come from appropriate Cohomological Hall algebras. This class of algebras includes as special cases different versions of Yangians as well as W-algebras. If one keeps in mind that COHA is related to factorization algebras (see [50]) then its role in the String Theory and Gauge Theory seems to be universal (see [13]).

More precisely, COHA acts naturally on the cohomology (or K-theory for the K-theoretical version) of the moduli spaces of stable configurations of geometric and algebraic objects which appear in the Gauge Theory, very much in the spirit of Nakajima’s seminal paper mentioned at the beginning. The relation between instanton partition functions in 4d gauge theories and D-branes on Calabi-Yau 3-folds is the natural framework in which the representation theory of COHA should appear (cf. [82, 84]). In this paper we will illustrate this idea in the case of spiked instantons introduced by Nekrasov (see [66, 65, 67, 64, 68, 69]). Mathematically, we will follow the approach to the representation theory of COHA proposed in [82] as well as its generalizations.

From the point of view of BPS algebras we will consider the action of the BPS algebra of D0-branes in $\mathbf{C}^3$ on the equivariant cohomology of the moduli space of stable framed D0−D4 states (generalization to stable framed D0−D2−D4 is also possible for more general Calabi-Yau manifolds). More precisely, we consider only D4 branes which correspond to coherent sheaves on $\mathbf{C}^3$ supported on the divisor $\sum_{1 \leq i \leq 3} r_i \mathbf{C}^2_i$. Here $r_i \geq 0$ are given integers, and $\mathbf{C}^2_i, 1 \leq i \leq 3$ are the coordinate planes $\{z_i = 0\} \subset \mathbf{C}^3$, where the vector space $\mathbf{C}^3$ is endowed with the standard coordinates $(z_1, z_2, z_3)$. Equivalently, we have a toric divisor $z_0 z_1^{r_1} z_2^{r_2} z_3^{r_3} = 0$ in $\mathbf{CP}^4$ intersected with the plane $z_0 = 1$.

From this point of view it is natural to study more general toric diagrams and corresponding local Calabi-Yau 3-folds. More generally one can start with an arbitrary dimer model, provided the 2-dimensional faces are “colored” by non-negative integers $r_i$. These in-

\footnote{We use the term “Cohomological” even in the case when we are talking about versions for K-theory or any other generalized cohomology theory.}
tegers correspond to the ranks of gauge groups. In particular one can hope for the following result.

**Conjecture 1.1.1** With a toric Calabi-Yau 3-fold $X$ and a collection of non-negative integers $r_1, \ldots, r_k$ (ranks) assigned to the 2-dimensional faces of the toric diagram of $X$, one can associate a VOA $W_{X, r_1, \ldots, r_k}$ which we will call the $W$-algebra associated to $X$ and the ranks $r_i$, $1 \leq i \leq k$.

If $X = \mathbb{C}^3$ and $D = \sum_{1 \leq i \leq 3} r_i \mathbb{C}^2$, then $W_{\mathbb{C}^3, r_1, r_2, r_3} := W_{r_1, r_2, r_3}$ coincides with the vertex algebra $Y_{r_1, r_2, r_3}$ defined in [30]. It was later used in [75], [76] as a building block for construction of more complicated vertex algebras associated with toric CY 3-folds via the gluing procedure proposed in the loc. cit. This gluing algorithm corresponds to the mathematical notion of conformal extension of VOAs (see for example [21]).

The vertex algebra $W_{r_1, r_2, r_3}$ is isomorphic to the quotient of the vertex algebra $W_{1+\infty}$ by the 2-sided ideal which corresponds to the curve

$$\sum_{1 \leq i \leq 3} \frac{r_i}{\lambda_i} = 1$$

in the space of triples $(\lambda_1, \lambda_2, \lambda_3)$ such that

$$\sum_{1 \leq i \leq 3} \frac{1}{\lambda_i} = 0$$

used in [73], [74] to parametrize $W_{1+\infty}$ algebras. For $r_1 = r, r_2 = r_3 = 0$ the quotient is isomorphic to the $W$-algebra $W_r := W(\mathfrak{gl}(r))$.

There are at least two ways to construct of $W_{X, r_1, \ldots, r_k}$ for general toric $X$. We have already mentioned the approach of [75] which is analogous to the topological vertex formula in the sense that one constructs vertex algebras associated to the colored toric diagrams starting with the basic one for $\mathbb{C}^3$. The idea of gluing complicated vertex algebras from the basic ones was discussed in the case of toric surfaces for example in [16] and [25].

Alternatively one can construct $W_{X, r_1, \ldots, r_k}$ starting with the Cohomological Hall algebras $H_X$ of the Calabi-Yau 3-fold $X$. In that case $W_{X, r_1, \ldots, r_k}$ is defined as the commutant of the subalgebra of screening operators acting on a highest weight $H_X$-module. In this paper we use this approach in the case of $W_{r_1, r_2, r_3}$.

Speculating further one can hope that there exists a vertex algebra associated to a general ind-constructible locally ind-Artin three-dimensional Calabi-Yau category endowed with a stability structure and a framing (see [51], [82] for the background material).

**Conjecture 1.1.2** With any ind-constructible locally ind-Artin 3CY-category endowed with a stability structure one can associate a class of vertex algebras, parametrized by rays in $\mathbb{C} = \mathbb{R}^2$ and a choice of framing object.

The above discussion naturally leads to a question about different versions of the AGT conjecture. The conventional one proved for $W_r$ corresponds to the case $X = \mathbb{C}^3, r_1 = r, r_2 = r_3 = 0$.

From the perspective of COHA the appearance of the moduli space of instantons in the conventional AGT conjecture is due to the “dimensional reduction” proposed in [50], Section 4.8 (see also a detailed discussion in [14]). Recall that the dimensional reduction allows one to replace COHA associated with a 3-dimensional Calabi-Yau category by a simpler algebra associated with a 2-dimensional Calabi-Yau category.
Remark 1.1.3 We would like to make a comment about the terminology and clarify the confusion in the literature. In [79] and subsequent papers of the same authors they used the term “Cohomological Hall algebra” for a special case of general COHA. The former is the “dimensional reduction” of the one introduced in [50]. In [87] the same object is called preprojective COHA.

In order to avoid the confusion and some conflict in terminology we propose to call the COHA introduced in the foundational paper [50] by 3d COHA, while its special case considered in [79, 87] will be called 2d COHA. The terminology is justified by the observation that in [50] the authors defined COHA in the framework of 3-dimensional Calabi-Yau categories, while in [79] the authors deal with a special case of a 2-dimensional Calabi-Yau category (2CY-category for short).

In the current paper we will often call the 3d COHA simply by COHA keeping the term 2d COHA for the special case considered in [79, 87].

We also remark that although in this paper we use several techniques of [79, 87], where the 2d COHA was utilized, they should not be sufficient in general, especially for non-toric CY 3-folds. In those cases the 3d COHA will be necessary to use.

1.2 Remark on the moduli spaces of Nekrasov instantons

Hopefully there is a generalization of COHA to a class of 4-dimensional Calabi-Yau varieties. The main example should be worked out in the case of $\mathbb{C}^4$. It will allow us to accommodate more general gauge theories introduced by Nekrasov (see [66, 65, 67, 64, 68, 69]). That story involves non-holomorphic ADHM-type relations. In the current paper we discuss a special case related to the moduli spaces of spiked instantons. Then we deal with $\mathbb{C}^3$ instead of $\mathbb{C}^4$ and all the relations are holomorphic.

It is not clear at the moment how to define an analog of COHA for an appropriate class of 4-dimensional Calabi-Yau categories. The framework [66, 65, 67, 64, 68, 69] is special, since one can still define the perfect obstruction theory. It gives one a hope that there is a 4-dimensional COHA defined in terms of the cohomology with coefficients in a constructible sheaf on the moduli stack of objects of a 4-dimensional Calabi-Yau category.

Let us briefly discuss the origin of the perfect obstruction theory in Nekrasov’s story. Recall that the $qq$-characters of Nekrasov (see [66, 65, 67, 64, 68, 69]) are defined as integrals of the generating functions of Chern classes of some natural vector bundles on the moduli spaces of solutions of generalized ADHM equations introduced in the loc. cit. An important part of the story is that one can obtain a real virtual fundamental class (in fact of odd dimension).

A more general set generalizing the one of [66, 65, 67, 64, 68, 69] should be the 4-dimensional non-compact toric Calabi-Yau manifold with a toric subscheme. More precisely, the manifold considered in the loc.cit. is $\mathbb{C}^4$, and the subscheme is given by a collection of six coordinate planes $\mathbb{C}^2_{ij}$, $1 \leq i, j \leq 4$ with multiplicities $r_{ij} \in \mathbb{Z}_{\geq 0}$.

Existence of the virtual fundamental class over which one should perform the integration is not obvious. For a 4-dimensional CY category endowed with a stability condition, the moduli space of stable (or polystable) objects of the heart of a $t$-structure does not have to support a perfect obstruction theory. Indeed besides of the group $Ext^1(E, E)$ which gives the tangent space $T_E$ at an object $E$ one has also groups $Ext^i(E, E), i \geq 2$. Differently from the case of 3CY categories those groups do not cancel each other in the virtual tangent space, since now we have the Serre duality in dimension 4 rather than 3. Notice that by Serre duality, the vector space $Ext^2(E, E)$ carries a non-degenerate complex-valued bilinear form $(\cdot, \cdot)$.
Let us choose a real vector subspace $\text{Ext}^2_{\mathbb{R}}(E, E) \subset \text{Ext}^2(E, E)$ such that the restriction of $(\cdot, \cdot)$ to $\text{Ext}^2_{\mathbb{R}}(E, E)$ is positive. Then we can obtain a perfect obstruction theory associated with the vector space $\text{Ext}^0(E, E) \oplus \text{Ext}^1(E, E) \oplus \text{Ext}^2_{\mathbb{R}}(E, E) \oplus (\text{Ext}^2_{\mathbb{R}}(E, E))^*$. The corresponding real vector bundle $(\text{Ext}^*(E, E), (\cdot, \cdot))$ over the moduli space of (poly)stable objects $E$ should be oriented for the virtual integration purposes. There is a condition on $\det(\text{Ext}^*(E, E), (\cdot, \cdot))$ which imposes topological restrictions on the underlying moduli space. Similarly to the notion of orientation data in [51], the determinant itself is a tensor square, but the choice of square root is an additional piece of data.

Remark 1.2.1 Technically, the above considerations hold only for compact Calabi-Yau manifolds (or CY categories with the compact space of (poly)stable objects). This is not the case for $\mathbb{C}^4$ or any toric Calabi-Yau 4-fold. In such cases one can use the equivariant version of the above considerations and observe that the set of the fixed points of the natural action of the torus on the moduli space of Nekrasov instantons on $\mathbb{C}^4$ is compact. When there are toric divisors consisting of coordinate planes, one can use the $(\mathbb{C}^*)^3$-action on $\mathbb{C}^4$ obtained from the natural $(\mathbb{C}^*)^4$-action by imposing the condition that the product of weights of the action is equal to 1. This ensures that the standard holomorphic Calabi-Yau form on $\mathbb{C}^4$ is preserved. In the end one obtains a real equivariant virtual fundamental class which can be used for equivariant integration as in [66, 65, 67, 64, 68, 69].

1.3 Motivations from physics

Let us rephrase the above discussion in a slightly more physical language. We start by reviewing the physics of the BPS algebra [40, 50] associated to Calabi-Yau 3-folds and its relation to VOAs. Motivated by the standard Alday-Gaiotto-Tachikawa [4, 90] setup, we propose a generalization of the AGT correspondence for spiked instanton configurations of [66, 65, 67, 64] restricted to branes in toric Calabi-Yau 3-folds. We finish the introduction by some speculations related to more general configurations.

1.3.1 BPS algebra and VOA

A rich class of examples of BPS algebras from [40] should arise from the compactification of type IIA string theory on a Calabi-Yau 3-fold $X$, i.e. superstrings in $\mathbb{R}^4 \times X$. BPS particles are then associated to $D6 - D4 - D2 - D0$ branes wrapping compact complex cycles inside $X$ and spanning one extra direction inside $\mathbb{R}^4$. The corresponding BPS algebra is an algebra capturing dynamics of such BPS particles. The physically motivated notion of BPS algebra was mathematically formalized in [50] in the notion of the Cohomological Hall algebra.

Turning on a fixed background of $D6$ and $D4$ branes (possibly including branes supported on non-compact cycles), one can look at the subalgebra of BPS particles associated to $D2 - D0$ branes supported on compact cycles. The corresponding configuration can be studied from two different perspectives. First, from the perspective of $D6$ and $D4$ branes, the compact $D2 - D0$ branes introduce a gauge-field flux and correspond to non-trivial instanton configurations. COHA is then expected to relate configurations of different instanton numbers. Secondly, from the perspective of the compact $D2 - D0$ branes, the moduli space of instantons has a description in terms of the moduli space of vacua of a quiver quantum mechanics. COHA then relates vacua of theories of different ranks.

From the above perspective, it is natural to expect that the equivariant cohomology of the moduli space of instantons of the $5d$ and $7d$ theories on $D4$- and $D6$-branes carries the...
structure of a representation of an appropriate COHA (or its subalgebra associated to the dynamics of $D2 - D0$ branes).

In this paper, we will be mostly interested in configurations concerning non-compact $D4$-branes (see e.g. [5, 45, 71]). Lifting the type IIA brane setup to M-theory by introducing an extra M-theory circle, $D4$-branes lift to $M5$-branes wrapping also the extra circle. This lift leads to the standard configuration of the $2d - 4d$ correspondence associated to $M5$-branes wrapping a complex two-dimensional variety $M_4$ and an extra Riemann surface $M_2 = S^1 \times \mathbb{R}$. Roughly speaking, compactification of the $M$-brane theory on $M_2$ leads to a four-dimensional gauge theory supported on $M_4$ whereas compactification on $M_4$ is expected to lead to a two-dimensional CFT for compact $M_4$ or a chiral algebra for non-compact $M_4$. The duality between these two theories is known as the AGT correspondence, $2d - 4d$ correspondence or the BPS/CFT correspondence [4, 25, 30, 65]. The corresponding VOA[$M_4$] appears as an algebra of chiral operators in the 2$d$ CFT, generating symmetries of the theory and extending the Virasoro algebra generating conformal transformations.

From the 2$d$ perspective, the BPS algebra of $D0$- and $D2$-branes gives rise to operators on the Hilbert space of the theory that can be identified with the equivariant cohomology of the moduli space of instantons associated to the divisor $M_4$. It is natural to expect that COHA acting on the equivariant cohomology of the moduli space actually leads to the vertex operator algebra VOA[$M_4$]. The equivariant cohomology of the moduli space of instantons can be then identified with a generic module for VOA[$M_4$]. The Calabi-Yau perspective and the corresponding COHA then provides a way to universally study VOA[$M_4$] for a large class of complex surfaces $M_4$ associated to different divisors in a Calabi-Yau 3-fold. Let us now discuss some examples starting with the well-known story of $M_4 = \mathbb{C}^2$ in the $\Omega$-background.

1.3.2 Standard AGT setup

The simplest example of the above setup is the configuration of Alday-Gaiotto-Tachikawa [4, 90] relating the Nekrasov partition function [63] of a $U(r)$ gauge theory on $M_4 = \mathbb{C}^2$ in the $\Omega$-background with conformal blocks of the $W_r$ algebra$^2$ on $M_2$. This configuration can be simply embedded inside our setup by considering $r$ M5-branes wrapping $\mathbb{C}^2_{x_1, x_2}$ inside the Calabi-Yau 3-fold $\mathbb{C}^3_{x_1, x_2, x_3}$ in the presence of a $B$-field with equivariant parameters $\epsilon_1$, $\epsilon_2$, $\epsilon_3 = -\epsilon_1 - \epsilon_2$ associated to the rotations of the $\mathbb{C}_{x_i}$ planes.

A key step in the proof of the AGT correspondence is the construction of the action of $W_r$ on the equivariant cohomology of the moduli space of instantons on $\mathbb{C}^2$ with equivariant parameters $\epsilon_1, \epsilon_2$ [50, 79]. The moduli space has an alternative description in terms of representations of the ADHM quiver [1, 19, 20]. Physically, the ADHM quiver can be viewed as a quiver of the $U(n)$ gauge theory on $n$ $D0$-branes bound to the stack of $r$ $D4$-branes. The ADHM moduli can be then identified with the moduli space of vacua of such a supersymmetric quiver quantum mechanics. The dual perspective in terms of a type IIB configuration (to be described below) from [50, 75, 79] indeed associates the algebra $W_r$ to such a simple divisor.

$^2$We use the notation $W_r = \mathcal{W}(\hat{gl}(r))$, i.e. the $\mathcal{W}$-algebra associated to the principal embedding of $\hat{sl}(2)$ inside $gl(r)$, instead of $W_r = \mathcal{W}(\hat{sl}(r))$ used in some of the literature. These two differ by a factor of $\hat{gl}(1)$.
orthogonal directions with discussed above. Singularities of the torus fibration correspond to \((p,q)\) the directions in which the corresponding to the M-theory circle degeneration of the fibers in the \(\mu\) degenerates at \(\mu_1 = 0, \mu_2 > 0\), the \(t_2\) action degenerates at \(\mu_2 = 0\) and \(\mu_1 > 0\) and finally \(t_1 + t_2\) degenerates at \(\mu_1 = \mu_2 < 0\). The degeneration of the fibers in the \(\mu_1, \mu_2\) plane is shown in the figure \(\text{3}\) on the left.

From the dual point of view, one gets a type IIB theory on \(\mathbb{R}^8 \times T^2\) with one of the cycle \(S_1 \subset T^2\) coming from the toric fibration of the Calabi-Yau 3-fold and the other cycle corresponding to the M-theory circle \(S_2 \subset T^2\) from the lift of the type IIA configuration discussed above. Singularities of the torus fibration correspond to \((p,q)\)-branes spanning orthogonal directions with \(p\) and \(q\) labeling the degenerating circle. The geometry of the

1.3.3 The \(C^3\) example

The configuration above has a natural generalization from the three-dimensional perspective. One can consider three stacks of \(r_1, r_2\) and \(r_3\) M5-branes supported on \(C^2_{x_1,x_2}, C^2_{x_2,x_3}\), and \(C^2_{x_2,x_3}\) inside \(C^3_{x_1,x_2,x_3}\), i.e., a configuration associated to the divisor \(r_1 C^2_{x_2,x_3} + r_2 C^2_{x_2,x_3} + r_3 C^2_{x_1,x_2}\) with \(r_i \in \mathbb{Z}_{\geq 0}\). Compactifying on the extra Riemann surface shared by all the branes, one obtains a triple of four-dimensional \(U(r_i)\) gauge theories supported at the three irreducible components of the divisor, namely on \(C^2_{x_1,x_2}, C^2_{x_1,x_3}\), and \(C^2_{x_2,x_3}\), mutually interacting along their intersections \(C_{x_1}, C_{x_2}, C_{x_3}\) via bi-fundamental 2d fields. This setup can be identified with a restriction of the more general spiked-instanton setup of M5-branes intersecting inside \(C^4\) from \([60, 65, 67]\). The corresponding moduli space of instantons has a quiver description from the figure \(\text{4}\) that can be again derived as the moduli space of vacua for \(D0\)-branes bound to \(D4\)-branes from the figure \(\text{4}\) as shown in \([64]\). In the presence of a single stack of \(M4\)-branes, e.g., \(r_1 = r_2 = 0\), the quiver reduces to the standard ADHM quiver with only two loops and a single framing node of rank \(r_3\).

The above M-theory setup can be related along the lines of \([52, 70]\) to the configuration from \([50]\) using the duality between the M-theory on a torus and type IIB string theory in the presence of a web of \((p,q)\)-branes. In the example at hand, \(C^3 = \mathbb{R}^6\) endowed with the standard symplectic structure has the natural Hamiltonian action of \(T^2 = U(1)^3\), whose moment map realizes \(C^3\) as a singular Lagrangian \(T^3\)-fibration over the first octant in \(\mathbb{R}^3\). The action of the 2-dimensional subtorus \(T^2 \subset T^3\) preserving the canonical bundle is generated by the following rotations \((e^{it_1} z_1, e^{it_2} z_2, e^{it_3} z_3)\) and \((z_1, e^{it_2} z_2, e^{it_2} z_3)\). The moment map of this \(T^2\) action from \(C^3\) to \(\mathbb{R}^3\) is given by \(\mu_1 = |z_1|^2 - |z_2|^2\) and \(\mu_2 = |z_2|^2 - |z_3|^2\). The directions in which the \(T^2\) torus fibration, when projected to \(\mathbb{R}^2\), are as follows. The \(t_1\) action degenerates for \(z_1 = z_3 = 0\), corresponding to the \(\mu_1 = 0, \mu_2 > 0\), the \(t_2\) action degenerates at \(\mu_2 = 0\) and \(\mu_1 > 0\) and finally \(t_1 + t_2\) degenerates at \(\mu_1 = \mu_2 < 0\). The degeneration of the fibers in the \(\mu_1, \mu_2\) plane is shown in the figure \(\text{3}\) on the left.
Calabi-Yau 3-fold thus maps to a web of \((p,q)\)-branes. \(M5\)-branes associated to the faces in the toric diagram map to \(D3\)-branes attached to \((p,q)\)-branes from the three corners as shown in the figure 2. This is exactly the setup of \[36\] that identified a three-parameter family of VOAs \(W_{r_1,r_2,r_3}\) as an algebra of local operators at a two-dimensional junction of interfaces in the four-dimensional theory coming from the low-energy dynamics of the \(D3\)-branes.

Note that \(W_{r_1,r_2,r_3}\) becomes the standard \(W_i\) algebra if two of the remaining parameters \(r_j = 0\) for \(j \neq i\) vanish. Since the quiver description of spiked instantons reduces to the standard ADHM quiver in this case, the duality explains the standard AGT correspondence. Motivated by the above duality and this observation, it is natural to expect that \(W_{r_1,r_2,r_3}\) should act on the equivariant cohomology of the moduli space of spiked instantons in a greater generality. Most of the paper is devoted to the proof of this proposal.

### 1.3.4 Divisors in toric CY 3-folds

The example of \(\mathbb{C}^3\) has a natural generalization for an arbitrary toric Calabi-Yau 3-fold given by a toric diagram specifying loci where the torus cycles degenerate. The two simplest examples are shown in the figure 3 and correspond to the bundles \(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1\) and \(\mathcal{O}(-2) \oplus \mathcal{O} \to \mathbb{P}^1\) respectively. From the type IIB perspective, the toric diagram can be again identified with a nontrivial web of \((p,q)\)-branes. \(M5\)-branes associated to each face in the diagram map to stacks of \(D3\)-branes attached to the \((p,q)\)-branes from various corners. Vertex operator algebras associated to such configurations were identified in \[76\] with various extensions of tensor products of \(W_{r_1,r_2,r_3}\) associated to each trivalent junction of the diagram by bi-fundamental fields associated to internal lines of the diagram.

The free field realization of \(W_{r_1,r_2,r_3}\) and their modules from \[21\] \[76\] gives an explicit realization of the bi-fundamental fields in terms of exponential vertex operators of the free boson. This generalizes the well-known constructions of lattice vertex operator algebras \[6\] \[29\] from the \(W_{0,0,1} = \widehat{gl}(1)\) case to an arbitrary \(W_{r_1,r_2,r_3}\) with the necessity to include contour integrals of screening charges along the lines of \[15\] \[27\].

\[3\]Note also closely related gluing at the level of affine Yangians \[32\] \[33\], quantum toroidal algebras \[26\] or minimal models \[39\].
The duality reviewed above suggests a generalization of the AGT correspondence for divisors in toric Calabi-Yau 3-folds. The gauge theories supported on smooth components of the divisor $D$ fixed under the toric action and interacting along the intersection of the smooth components correspond to $(p, q)$-web VOAs.

Based on the analysis of the vacuum character of the glued algebras from [75], we can conjecture that the Drinfeld double of the COHA associated to the last two configurations in figure 3 can be identified with shifted affine Yangians of $\hat{gl}(1\mid 1)$ and $\hat{gl}(2)$ respectively. Shifts are determined by the intersection number of the corresponding divisor $D$ with the $P^1$ associated to the internal line. In particular for the last two examples from figure 3, we get:

$$\#(D \cap P^1) = r_2 + r_4 - r_1 - r_3, \quad \#(D \cap P^1) = r_2 + r_4 - 2r_3,$$

(1)

where the divisors are

$$r_2(\text{Fiber over the north pole of } P^1) + r_4(\text{Fiber over the south pole of } P^1) + r_1(O(-1) \rightarrow P^1) + r_3(O(-1) \rightarrow P^1),$$

(2)

in the case $O(-1) \oplus O(-1)$ (middle) and analogously for the case $O(-2) \oplus O$ (right)

$$r_2(\text{Fiber over the north pole of } P^1) + r_4(\text{Fiber over the south pole of } P^1) + r_1(O(-2) \rightarrow P^1) + r_3(O \rightarrow P^1).$$

(3)

Similarly, one can consider $(p, q)$-webs of $n$ D5-branes ending from the left and $m$ D5-branes ending from the right of a sequence of $(n, 1)$-branes for varying integer $n$ such that the $(p, q)$ charges are conserved at each vertex. These configurations are expected to lead to shifted Yangians of $\hat{gl}(n\mid m)$ with shifts determined by the intersection numbers of $D$ with various $P^1$’s associated to internal lines. The Yangians associated to more complicated web-diagrams are highly unexplored.

Let us comment on the overlap of our setup with the one from the second section of [25]. Their configurations associated to the type $A$ quiver can be easily realized inside a toric Calabi-Yau threefold. One can simply start with a single vertex corresponding to $\mathbb{C}^3$.

Note that the situation $\#(D \cap P^1) = 1$ corresponds to the same configuration discussed in [32, 33]. From our perspective, we expect their supersymmetric Yangian to be the shifted Yangian of $\hat{gl}(1\mid 1)$. 
with a single stack of M5-branes turned on. Attaching extra legs to the corresponding toric diagram under different angles (similarly as we did in the examples from the figure 3), one gets a toric diagram of some Calabi-Yau 3-fold with a divisor of a single smooth component. Different angles then correspond to different parameters of gluing from \([25]\). Depending on the angle of the extra legs, the corresponding VOAs are different extensions of the tensor product of \(W_r\) algebras. Note also that the affine Yangians that we conjectured to appear in the two examples above are in contradiction with the conjecture of \([25]\) that expects both configurations to give rise to different shifts of the Yangian of \(\widehat{\mathfrak{gl}}(2)\). On the other hand, character calculations from \([75]\) suggest an appearance of the Yangian of \(\widehat{\mathfrak{gl}}(1|1)\) in the \(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1\) case and the Yangian of \(\widehat{\mathfrak{gl}}(2)\) in the \(\mathcal{O}(-2) \oplus \mathcal{O} \to \mathbb{P}^1\) case. The shifts are determined by the intersection number of the corresponding \(\mathbb{P}^1\) with the divisor. It would be interesting to explore this issue further.

1.3.5 General case

The appearance of VOAs related to the COHA is expected to have a generalization also for divisors inside non-toric Calabi-Yau 3-folds extending the story of \([25]\). Even more general configuration that can be still embedded inside M-theory is associated to M5-branes supported on complex two-dimensional cycles inside a Calabi-Yau four-fold with the spiked instantons of \([66, 65, 67, 64]\) corresponding to the simplest example of \(C^4\). Investigation of such general configurations can be viewed as a long term goal in the subject.

Note also the existence of an alternative construction of \(W\)-algebras in related gauge theory setups from \([46, 47, 48]\). Furthermore, there exists (most probably) closely related work started by \([17, 55, 2, 3]\) and continued by many others, constructing intertwining operators for the quantum toroidal algebras associated to toric diagrams. The precise relation of these two directions is unclear to us and deserves further investigation.

1.4 Contents of the paper

Let us briefly discuss the contents of the paper.

In \(\S\ 2\) we discuss the relation of COHA in our main example of the quiver with potential and its relation to torsion sheaves on \(C^3\). We also introduce the equivariant and spherical versions of COHA. In \(\S\ 3\) we discuss a particular framing of our quiver with potential as well as the corresponding framed stable representations. In \(\S\ 4\) after a more technical reminder on COHA, we state our main result, and discuss the strategy we use to prove the main result. This part can also be considered as an over view of the rest of the paper. In \(\S\ 5\) we show that the COHA naturally acts on the cohomology of the moduli space of stable framed representations.

In \(\S\ 6\) we define a Cartan algebra, which acts on the COHA and the cohomology of the moduli space of stable framed representations via characteristic classes of the tautological bundles. The COHA, with the Cartan algebra added, admits a Drinfeld coproduct. In \(\S\ 7\) we prove that the Drinfeld double associated to the coproduct from \(\S\ 6\) acts on the cohomology of the moduli space of stable framed representations. In particular, the Drinfeld double is isomorphic to the affine Yangian \(Y_{h_1,h_2,h_3}(\widehat{\mathfrak{gl}}(1))\)

In \(\S\ 8\) we define a central extension of the double COHA, and define a more interesting coproduct on the centrally extended algebra. This coproduct geometrically comes from hyperbolic localizations on the moduli space of stable framed representations, therefore has the usual \(R\)-matrix formalism as in \([56]\). Finally, after recalling on the VOA at the corner
in §9, we prove, in §10 that the action of the centrally extended COHA on the cohomology of the moduli space of stable framed representations gives rise to the VOA. Moreover, the coproduct from §8 gives the free field realizations of the VOA.

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2 COHA and torsion sheaves

We will work over the field of complex numbers $\mathbb{C}$, although many results hold over any field of characteristic zero.

2.1 Equivaraint COHA, spherical COHA and the dimensional reduction

Technical details and formal definition of COHA will be recalled in §4.1 in detail. Here we just briefly summarize a few facts.

We are going to use the following notation. For a quiver $Q$ with the set of vertices $I$ we denote by $\tilde{Q}$ the double quiver obtained from $Q$ by adding an opposite arrow $a^*$ for any arrow $a$ of $Q$. We denote by $\check{Q}$ the triple quiver obtained from $\tilde{Q}$ by adding a new loop $l_i$ for each vertex $i \in I$ of $Q$. A potential is an element $W$ of the vector space $\mathbb{C}Q/\langle [CQ, CQ] \rangle$, where $CQ$ is the path algebra of the quiver $Q$. In other words $W$ is a cyclically invariant non-commutative polynomial in arrows of $Q$.

Let $W$ be a potential for $\check{Q}$ given by the formula $W = \sum_{a,i} l_i[a, a^*]$, where the summation is taken over all vertices $i$ and all arrows $a$ of $Q$.

Example 2.1.1 Let $Q = J$ be the Jordan quiver (one vertex and one loop). The triple quiver $Q_3 := \tilde{J}$ has three loops $B_1, B_2, B_3$. The above potential has the form $W_3 = B_3[B_1, B_2]$ (plus cyclic permutations, which we always skip in the notation).

It is well-known that for any quiver $Q$ the pair $(\check{Q}, W)$ gives rise to a 3CY-category $C_{(\check{Q}, W)}$ which has a t-structure with the heart given by finite-dimensional representations of the Jacobi algebra $\mathbb{C}Q/\langle \partial W \rangle$ (i.e. we consider the quotient of the path algebra by the 2-sided ideal $\langle \partial W \rangle$ generated by cyclic derivatives of $W$).

E.g., in the above example of the pair $(Q_3, W_3)$, the t-structure in question has the heart given by the category of finite-dimensional representations of the polynomial algebra $\mathbb{C}[B_1, B_2, B_3]$, i.e. it is the category $\text{Tors}(\mathbb{C}^3)_0$ of torsion sheaves on $\mathbb{C}^3$ with zero-
dimensional support. Considering \((\mathbb{C}^*)^3\)-equivariant torsion sheaves which correspond to cyclic modules we see that they are enumerated by 3d partitions.

The dimensional reduction (see [50], Section 4.8) associates with the 3CY-category \(\mathcal{C}_{(Q,W)}\) a 2CY-category \(\mathcal{C}_{(\hat{Q},\hat{W})}\) which has a \(t\)-structure with the heart given by finite-dimensional representations of the preprojective algebra \(\Pi_Q = \mathbb{C}\hat{Q}/(\sum_a [a,a^*])\). Intuitively, the relation between \(\mathcal{C}_{(Q,W)}\) and \(\mathcal{C}_{(\hat{Q},\hat{W})}\) can be thought of as a relation between two Calabi-Yau manifolds: a 3-dimensional one and a 2-dimensional one, so that the former is the product of the latter by an affine line. More generally, one can upgrade a (triangulated \(A_\infty\)) category of homological dimension 2 to a 3CY category by the categorical analog of the geometric construction of taking the total space of canonical bundle.

More generally, the 3d critical COHA was defined in [50] such as follows:
\[
\mathcal{H}^{(Q,W)} = \bigoplus_{\gamma \in \mathbb{Z}_{\geq 0}^d} \mathcal{H}_\gamma^{(Q,W)} = \bigoplus_{\gamma \in \mathbb{Z}_{\geq 0}^d} \left( \mathcal{H}^*_{\text{c.c.}}(M_\gamma, W_\gamma) \right)^\vee.
\]

Here \(M_\gamma\) is the algebraic variety of finite-dimensional representations of the quiver \(Q\) of dimension \(\gamma = (\gamma_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^d\) in the coordinate space \(\prod_{i \in I} \mathbb{C}^{\gamma_i}\), gauge group \(G_\gamma := \prod_{i \in I} GL_{\gamma_i}\), where \(GL_m := GL(m, \mathbb{C})\) is the general linear group, acts naturally by changing the basis, \(W_\gamma := \text{Tr}(W) : M_\gamma \to \mathbb{C}\), and the dual is taken to the compactly supported equivariant cohomology of \(M_\gamma\) with coefficients in the sheaf of vanishing cycles of the regular \(G_\gamma\)-invariant function \(W_\gamma\). Alternatively we can work with the rapid decay cohomology or any other cohomology theory (see [50]).

**Remark 2.1.2**

1) Notice that one can consider representations of quivers with potential in any abelian category, e.g. in the category of coherent sheaves on a smooth algebraic variety. Although in that case the arising category is not necessary 3-dimensional Calabi-Yau, the dimensional reduction still works.

2) We can (and will) consider equivariant versions \(\mathcal{H}^{(Q,W),T_{m+1}}\) of COHA with respect to the torus \(T_{m+1} := (\mathbb{C}^*)^{m+1}\) rescaling the arrows of \(Q\) with arbitrary weights. Here \((m+1)\) is the number of arrows of \(Q\). This version of COHA was also introduced in [50].

We will also consider a \(T_m\)-equivariant version \(\mathcal{H}^{(Q,W),T_m}\) of 3d COHA. Graded components \(\mathcal{H}^{(Q,W),T_m}_{t_{\gamma}}\) are the equivariant Borel-Moore homology with respect to the Cartesian product of the torus \(T_m := (\mathbb{C}^*)^m\) and the group \(G_\gamma\). The weights \(\varepsilon_i\) of the \(T_m\)-action are chosen in such a way that \(\prod_i \varepsilon_i = 1\) (Calabi-Yau condition).

3) There are many papers in which the action of COHA can be computed explicitly, see e.g. [52, 47].

The spherical COHA is defined as the subalgebra \(\mathcal{SH}^{(Q,W)} \subset \mathcal{H}^{(Q,W)}\) generated by representations with dimension vectors \(\varepsilon_i = (0,\ldots,1,\ldots,0), 1 \leq i \leq |I|\). There is an obvious equivariant version of the spherical COHA.

### 2.2 2-dimensional COHA, algebra SH\(^c\) and torsion sheaves on \(\mathbb{C}^2\)

Technical details of the 2d COHA will be discussed in [72] in detail. Here we recall few basics facts in order to make a comparison with the 3d COHA and torsion sheaves.

In [79] the authors defined an associative algebra \(\mathcal{SH}^c\) over the field of rational functions in one variable \(\mathbb{C}(k)\). It was proved in the loc. cit. that this algebra is \(\mathbb{Z}\)-graded, countably

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5 Recall that we call it 3d COHA, or simply COHA.
The stack of finite-dimensional representations of the algebra $C$ of scheme $Hilb^3$ product is defined in term of correspondences in the usual Nakajima (or Hall algebra) style. The equivariant Borel-Moore homology by adding infinitely many central elements computed in Proposition 7 from [50]. In the next subsection we consider the spherical COHA $SH_{Q_3,W_3}$ which is isomorphic to $SH^>$. It was shown in [79] (see Corollary 6.4) that after extension of scalars to the field of rational functions in two variables. For the spherical COHA the corresponding motivic DT-series coincides with the above infinite product.

The algebra $SH^>$ is closely related to the 2d COHA $H(Coh_{T_2,0}(C^2))$ of the category $Coh_{T_2,0}(C^2)$ of $T_2 := (C^*)^2$-equivariant coherent sheaves on $C^2$ which have zero-dimensional support. In particular, as graded vector space the 2d COHA is isomorphic to $\oplus_{d \geq 0} H_{BM}(Coh_{T_2,0}(C^2,d))$. Here $Coh_{T_2,0}(C^2,d)$ denotes the stack of equivariant coherent sheaves with 0-dimensional support having degree $d$, and $H_{BM}$ denotes the Borel-Moore homology of this stack. Since our stack is the quotient stack, its (co)homology is defined as the equivariant (co)homology of the corresponding algebraic variety.\footnote{More accurately, our BM-homology is defined as the dual to the critical compactly supported cohomology from [50], so they should be called critical Borel-Moore homology.}

The algebra $SH^{<,0}$ is obtained from the graded commutative algebra $H_{BM}^{GL(\infty) \times T_2}(pt)$ of equivariant Borel-Moore homology by adding infinitely many central elements $c = (c_1, c_2, ...)$ (see loc. cit. for the details).

The associative product on the 2d COHA was defined in [79] by observing that the stack $Coh_{T_2,0}(C^2,d)$ is isomorphic to the stack of pairs of commuting $d \times d$ matrices. The algebra product is defined in term of correspondences in the usual Nakajima (or Hall algebra) style.

It is known that the algebra structure on the 2d COHA $H(Coh(C^2)_0)$ (no equivariance conditions) agrees with the one obtained as a result of the dimensional reduction of $H^{Q_3,W_3}$ (see Appendix to [77]). Similar result holds for equivariant and spherical versions.

The spherical 2d COHA $SH(Coh_{T_2,0}(C^2)) \subset H(Coh_{T_2,0}(C^2))$ is the subalgebra generated by the equivariant Borel-Moore homology of $1 \times 1$ matrices. Both algebras, 2d COHA and spherical 2d COHA are the dimensional reductions along one of the loops of the corresponding versions of 3d COHA in the sense of [50]. It was shown in [79] (see Corollary 6.4) that after extension of scalars, the spherical Hall algebra $SH(Coh(C^2)_0)$ and the algebra $SH^>$ become isomorphic.

The structure of the algebra $SH^c$ was studied in many papers, both mathematical and physical (to mention just a few: [49, 60, 86, 113]). Furthermore the $W$-algebra $W_r$ is closely related (see [79]) to the algebra $SH^c$. From the point of view of the above discussion, it becomes clear that $W_r$ is the “dimensional reduction” of some “3-dimensional VOA”. This agrees with the previously made remark that $W_r \simeq W_{r,t_2=0, t_3=0}$.

### 2.3 3d COHA and torsion sheaves on $C^3$

The stack of finite-dimensional representations of the algebra $C[x_1, x_2]$ contains the Hilbert scheme $Hilb(C^2) = \sqcup_{n \geq 0} Hilb_n(C^2)$ consisting of cyclic finite-dimensional representations of $C[x_1, x_2]$. As we know since Nakajima’s work this Hilbert scheme is isomorphic to the moduli space of stable framed representations of a certain quiver, and there is a similar
description for the rank $r$ torsion free sheaves $F$ on $\mathbb{P}^2$ which are framed at the line $\mathbb{P}^1_\infty$ at infinity via an isomorphism $F_{|\mathbb{P}^1_\infty} \simeq \mathcal{O}_\mathbb{P}^r$ (see [59]).

It is known (see e.g. [50]) that the 2d COHA $\mathcal{H}(\text{Coh}_{\mathcal{T}_2,0}(\mathbb{C}^2))$ is isomorphic to the positive part of the affine Yangian $\mathcal{Y}_{h_1,h_2,h_3}(\mathfrak{gl}(1))$. More conceptual approach to the quiver Yangians of [50] based on 3d COHA was proposed in [13].

Furthermore, for each pair $(r,n) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ one can define the moduli space $\mathfrak{M}_r(n)$ of rank $r$ framed torsion-free sheaves on $\mathbb{P}^2$ which have the second Chern class $n$. Let $V_r = \oplus_n H^0_{BM}(\mathfrak{M}_r(n))$ be the equivariant Borel-Moore homology of the disjoint union of such moduli spaces. Then the 2d COHA $\mathcal{H}(\text{Coh}_{\mathcal{T}_2,0}(\mathbb{C}^2))$ acts faithfully on $V_r$ by correspondences. This gives a generalization of the classical results of Nakajima (see [59]).

One of our aims is to explain the 3-dimensional avatar of this observation. First we define a spherical 3d COHA for a general quiver with potential $(Q,W)$ in the obvious way: it is the subalgebra $S\mathcal{H}^{Q,W} \subset \mathcal{H}^{Q,W}$ generated by 1-dimensional representations. The equivariant version is defined similarly.

We will be mostly interested in the special case of the quiver with potential $(Q_3,W_3)$ (see Example 1.2.1). The corresponding 3d COHA is $\mathcal{H}^{(Q_3,W_3)} = \mathcal{H}(\text{Coh}(\mathbb{C}^3)_0)$, i.e. it is the COHA of the category of 0-dimensional torsion sheaves on $\mathbb{C}^3$. Then we have the corresponding spherical 3d COHA $S\mathcal{H}(\text{Coh}(\mathbb{C}^3)_0)$ as well as the equivariant versions $S\mathcal{H}(\text{Coh}_{\mathcal{T}_3,0}(\mathbb{C}^3)) \subset \mathcal{H}(\text{Coh}_{\mathcal{T}_3,0}(\mathbb{C}^3))$ which are COHA and spherical COHA of the category of $\mathcal{T}_3 = (\mathbb{C}^*)^3$-equivariant 0-dimensional torsion sheaves on $\mathbb{C}^3$.

In general we will denote by $\text{Coh}(X)_0$ the category of sheaves on a toric Calabi-Yau 3-fold which have a 0-dimensional support. We will be mostly interested in the case $X = \mathbb{C}^3$.

**Proposition 2.3.1** The category $\text{Coh}(\mathbb{C}^3)_0$ is equivalent to the category of finite-dimensional modules over the Jacobi algebra $CQ_3/(\partial W_3)$, where $CQ_3$ is the path algebra of the quiver $Q_3$ (see Example 2.1.1).

**Proof.** It follows from the observation that the corresponding Jacobi algebra is isomorphic to the algebra of polynomials $C[B_1,B_2,B_3]$. $\blacksquare$

Then COHA $\mathcal{H}(\text{Coh}_{\mathcal{T}_3,0}(\mathbb{C}^3)) = \oplus_{d \geq 0} H^0_{BM}(\text{Coh}_{\mathcal{T}_3,0}(\mathbb{C}^3,d))$, where $\text{Coh}_{\mathcal{T}_3,0}(\mathbb{C}^3,d)$ is the stack of degree $d$ equivariant sheaves on $\mathbb{C}^3$ which have 0-dimensional support. This COHA is isomorphic to the equivariant version of COHA $\mathcal{H}^{(Q_3,W_3)}$.

### 3 Framed quiver and its stable representations

#### 3.1 Framed quiver

Let $(Q_3,W_3)$ be the quiver with potential considered in Example 2.1.1. That is, $Q_3$ is the quiver with one vertex labelled by $0$, and three loops $B_1,B_2,B_3$. The potential is $W_3 = B_3[B_1,B_2]$ (plus cyclic permutations). In this subsection we are going to define the framed quiver with potential $(Q^f_3,W^f_3)$. The framed quiver $Q^f_3$ is obtained by adding to $Q_3$ three new vertices (framing vertices) with three pairs of opposite arrows.

More precisely, the quiver $Q^f_3$ is defined such as follows:

- The set of vertices is $\{0,1,2,3\}$, where the vertices $1,2,3$ are framing vertices.
- There are two types of arrows.
1. We have three loops $B_1, B_2, B_3$ at the vertex 0. Hence $(0, B_1, B_2, B_3)$ is the quiver $Q_3$ considered previously.

2. We also have “framing” arrows $I_{12}: 3 \to 0$, $J_{12}: 0 \to 3$; $I_{13}: 2 \to 0$, $J_{13}: 0 \to 2$; $I_{23}: 1 \to 0$, $J_{23}: 0 \to 1$.

We define the framing potential $W_3^{fr}$ by the formula

$$W_3^{fr} := B_3([B_1, B_2] + I_{12}J_{12}) + B_2([B_1, B_3] + I_{13}J_{13}) + B_1([B_2, B_3] + I_{23}J_{23})$$

$$= W_3 + B_1I_{23}J_{23} + B_2I_{13}J_{13} + B_3I_{12}J_{12}.$$

In order to define a representation of $(Q_3^{fr}, W_3^{fr})$ let us fix a dimension vector $(n, r_1, r_2, r_3)$ at the vertices $\{0, 1, 2, 3\}$ respectively. Then for a representation of $Q_3^{fr}$ of this dimension we have the linear maps

$$B_i : \mathbb{C}^n \to \mathbb{C}^n, i \in \{1, 2, 3\}$$

$$I_{12} : \mathbb{C}^{r_3} \to \mathbb{C}^n, J_{12} : \mathbb{C}^n \to \mathbb{C}^{r_3};$$

$$I_{13} : \mathbb{C}^{r_2} \to \mathbb{C}^n, J_{13} : \mathbb{C}^n \to \mathbb{C}^{r_2};$$

$$I_{23} : \mathbb{C}^{r_1} \to \mathbb{C}^n, J_{23} : \mathbb{C}^n \to \mathbb{C}^{r_1}.$$

We will use the shorthand notation $\vec{r} := (r_1, r_2, r_3)$, and $\mathbb{C}^{\vec{r}} := \mathbb{C}^{r_1} \oplus \mathbb{C}^{r_2} \oplus \mathbb{C}^{r_3}$. The space of representations of $Q_3^{fr}$ with the dimension vector $(n, \vec{r})$ is

$$\mathcal{M}_{\vec{r}}(n) := \{(B_i, I_{ab}, J_{ab}) | i \in \mathbb{Z}, ab \in \mathbb{Z}\}$$

$$= \text{Hom}(\mathbb{C}^n, \mathbb{C}^n)^3 \oplus \text{Hom}(\mathbb{C}^n, \mathbb{C}^{\vec{r}}) \oplus \text{Hom}(\mathbb{C}^{\vec{r}}, \mathbb{C}^n),$$

where $\mathbb{Z} := \{1, 2, 3\}$, $\mathbb{Z} := \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

The groups $GL_n(\mathbb{C})$ and $GL_{\vec{r}} := GL_{r_1}(\mathbb{C}) \times GL_{r_2}(\mathbb{C}) \times GL_{r_3}(\mathbb{C})$ act on $\mathcal{M}_{\vec{r}}(n)$ by conjugation. Let $T_2$ be the two dimensional torus. We think $T_2$ as a subtorus in $(\mathbb{C}^*)^3$ given by the equation $t_1t_2t_3 = 1$. Then, $T_2$ acts on $\mathcal{M}_{\vec{r}}(n)$ by

$$(t_1, t_2, t_3) \cdot \{(B_i, I_{ab}, J_{ab}) = (t_1B_1, t_2B_2, t_3B_3, t_1I_{12}J_{12}, t_1t_2J_{12}, t_1t_3I_{13}, t_1I_{23}, t_2t_3J_{23})$$

where $t_1t_2t_3 = 1$, and $\{(B_i, I_{ab}, J_{ab}) \in \mathcal{M}_{\vec{r}}(n)$. Under this $T_2$-action, the potential $W_3^{fr}$ is invariant.

Let us introduce the “moment maps” $\mu_{ab} : \mathcal{M}_{\vec{r}}(n) \to gl_n$, for $ab \in \mathbb{Z}$ given by

$$\mu_{12} = [B_1, B_2] + I_{12}J_{12}, \quad \mu_{13} = [B_1, B_3] + I_{13}J_{13}, \quad \mu_{23} = [B_2, B_3] + I_{23}J_{23}.$$

Then $W_3^{fr} = B_3\mu_{12} + B_2\mu_{13} + B_1\mu_{23}$. 

Figure 4: Quiver for the spiked-instanton configuration.
Lemma 3.1.1 For \( 1 \leq a, b, c \leq 3 \), and \( a, b, c \) are distinct, we have
\[
\frac{\partial W^{fr}_3}{\partial B_a} = \mu_{bc}, \quad \frac{\partial W^{fr}_3}{\partial I_{ab}} = J_{ab}B_c, \quad \frac{\partial W^{fr}_3}{\partial J_{ab}} = B_cI_{ab}.
\]

Proof. We have
\[
\frac{\partial W^{fr}_3}{\partial B_1} = \frac{\partial (B_3\mu_{12} + B_2\mu_{13} + B_1\mu_{23})}{\partial B_1} = [B_2, B_3] + [B_3, B_2] + \mu_{23} = \mu_{23}.
\]

The other identities follow from a straightforward calculation. The lemma is proven. \( \blacksquare \)

3.2 Stable representations of the framed quiver

We say (cf. \([65]\)) that a representation \((B_i, I_{ab}, J_{ab})_{i \in 3, ab \in \mathcal{E}}\) belonging to \(\mathcal{M}_F(n)\) is stable if the following condition is satisfied:

\[
\mathcal{C}(B_1, B_2, B_3)I_{12}(\mathcal{C}^\mathcal{E}) + \mathcal{C}(B_1, B_2, B_3)I_{13}(\mathcal{C}^\mathcal{E}) + \mathcal{C}(B_1, B_2, B_3)I_{23}(\mathcal{C}^\mathcal{E}) = \mathcal{C}^n, \quad (4)
\]

where \(\mathcal{C}(B_1, B_2, B_3)\) is the ring of non-commutative polynomials in the variables \(B_1, B_2, B_3\). In other words, a representation is stable, if the non-commutative polynomials in \(B_1, B_2, B_3\) applied to the image of \(I_{12}, I_{13}, I_{23}\) generate \(\mathcal{C}^n\).

Example 3.2.1 If \(\bar{r} = (0, 0, 1)\), the stability condition \((4)\) becomes \(\mathcal{C}(B_1, B_2, B_3)I_{12}(\mathcal{C}^1) = \mathcal{C}^n\), which means \(I_{12}\) is a cyclic vector.

One can also impose a stronger stability condition by saying that the representation is stable if the following condition is satisfied:

\[
\mathcal{C}(B_1, B_2)I_{12}(\mathcal{C}^\mathcal{E}) + \mathcal{C}(B_1, B_3)I_{13}(\mathcal{C}^\mathcal{E}) + \mathcal{C}(B_2, B_3)I_{23}(\mathcal{C}^\mathcal{E}) = \mathcal{C}^n. \quad (5)
\]

Note that \((5)\) implies \((4)\).

Proposition 3.2.2 Assume \((B_i, I_{ab}, J_{ab})_{i \in 3, ab \in \mathcal{E}}\) is in the critical locus of \(W^{fr}_3\). That is, \((B_i, I_{ab}, J_{ab})\) satisfies the following equations
\[
\mu_{12} = [B_1, B_2] + I_{12}J_{12} = 0, \quad \mu_{13} = [B_1, B_3] + I_{13}J_{13} = 0,
\]
\[
\mu_{23} = [B_2, B_3] + I_{23}J_{23} = 0,
\]
\[
J_{12}B_3 = 0, \quad J_{13}B_2 = 0, \quad J_{23}B_1 = 0,
\]
\[
B_3I_{12}, \quad B_2I_{13}, \quad B_1I_{23} = 0.
\]

Then, the condition \((4)\) is equivalent to \((5)\).

Proof. We start with a representation \((B_i, I_{ab}, J_{ab}) \in \mathcal{M}_F(n)\), which is stable under condition \((4)\):

\[
\mathcal{C}^n = \mathcal{C}(B_1, B_2, B_3)I_{12}(\mathcal{C}^\mathcal{E}) + \mathcal{C}(B_1, B_2, B_3)I_{13}(\mathcal{C}^\mathcal{E}) + \mathcal{C}(B_1, B_2, B_3)I_{23}(\mathcal{C}^\mathcal{E}).
\]
Assume it satisfies the equations \( B_3 I_{12} = 0, B_1 I_{23} = 0, B_2 I_{13} = 0 \). As \( B_3 I_{12} = 0 \), we have

\[
C(B_1, B_2, B_3) I_{12}(C^3) = C(B_1, B_2) I_{12}(C^3) + C(B_1, B_2, B_3) [B_1, B_3] C(B_1, B_2, B_3) I_{12}(C^3) + C(B_1, B_2, B_3) [B_2, B_3] C(B_1, B_2, B_3) I_{12}(C^3)
\]

As a consequence, the condition \( 4 \) can be simplified to

\[
C^n = C(B_1, B_2) I_{12}(C^3) + C(B_1, B_2, B_3) I_{13}(C^3) + C(B_1, B_2, B_3) I_{23}(C^3).
\]

As a consequence, the condition \( 4 \) can be simplified to

\[
C^n = C(B_1, B_2) I_{12}(C^3) + C(B_1, B_2, B_3) I_{13}(C^3) + C(B_1, B_2, B_3) I_{23}(C^3).
\]

Moreover, for any homogenous \( f(B_1, B_2, B_3) \in C(B_1, B_2, B_3) \), such that

\[
f(B_1, B_2, B_3) I_{12}(C^3) = f_1(B_1, B_2) I_{12}(C^3) + g_1(B_1, B_2, B_3) I_{13}(C^3) + h_1(B_1, B_2, B_3) I_{23}(C^3),
\]

we could assume \( \deg_{B_1}(f) > \deg_{B_1}(g_1), \deg_{B_3}(f) > \deg_{B_3}(h_1) \). Similarly, we have

\[
g_1(B_1, B_2, B_3) I_{13}(C^3) = f_2(B_1, B_2, B_3) I_{12}(C^3) + g_2(B_1, B_3) I_{13}(C^3) + h_2(B_1, B_2, B_3) I_{23}(C^3),
\]

with \( \deg_{B_1}(f) > \deg_{B_1}(f_2) \).

One can thus iteratively get

\[
C^n = C(B_1, B_2) I_{12}(C^3) + C(B_1, B_2, B_3) I_{13}(C^3) + C(B_1, B_2, B_3) I_{23}(C^3).
\]

Repeatedly using the equations of the critical locus of \( W_3^{fr} \), we can also get rid of \( B_1 \) from the last factor. This shows the two conditions are equivalent. The proposition is proven. ■

**Remark 3.2.3** Alternative proof of Proposition 3.2.2 follow from the non-holomorphic generalization of ADHM equations proposed by Nekrasov. The corresponding discussion of stability can be found in [66], Section 3.4. Roughly, the idea is to study critical points of the function \( f = Tr(\sum M_i M_i^*) \), where \( M_i \) are matrices in the representations of the left hand sides of the relations from Proposition 3.2.2 and \( M_i^* \) are their Hermitian conjugates. One shows that if at a critical point of \( f \) the stability assumption holds, then it is (up to the gauge group action) a point of absolute minimum of \( f \). On the one hand at such points all \( M_i = 0 \), i.e. the equations from the Proposition are satisfied. On the other hand they are stable representations, since they are points of absolute minimum. The details can be found in the loc.cit. in a bigger generality.

Let

\[
\mathcal{M}_f(n)^{st} = \left\{ (B_i, I_{ab}, J_{ab}) \in \mathcal{M}_f(n) \left| \sum_{ab \in \mathfrak{g}} C(B_1, B_2, B_3) I_{ab}(C^{r_{ab}}) = C^n \right. \right\}
\]

be the stable locus of \( \mathcal{M}_f(n) \). It consists of the representations of \( (Q_3^{fr}, W_3^{fr}) \) of dimension vector \( (n, \bar{r}) \) that satisfy the equation \( 4 \). The quotient \( \mathcal{M}_f(n)^{st} / GL_n \) is denoted by \( \mathfrak{M}_f(n) \). We call this moduli space of solutions to the equation \( 4 \) the moduli space of spiked instantons.
Remark 3.2.4 The above representations can be thought of as generalizations of stable framed representations of the quiver with potential discussed in [82]. In fact in the loc. cit. the general notion of stable framed object in the framework of triangulated categories endowed with stability structure is proposed.

The difference of the previous discussion with [82] is that here we consider representations of COHA for the quiver \((Q_3, W_3)\) in the cohomology of the moduli space of stable representations of the quiver with potential \((Q_3^{fr}, W_3^{fr})\). Hence the relations for the stable framed representations are given by another potential. The rest of the property to be stable framed is the same as in the loc. cit. Namely, we consider representations of the Jacobi algebra \(C \langle B_1, B_2, B_3 \rangle / \langle \partial W_3^{fr} \rangle\) in the \(n\)-dimensional vector space \(V \simeq \mathbb{C}^n\) such that there is no proper subrepresentation which contains images of all maps \(I_{ab}\). In other words

\[
\sum_{ab \in \overline{3}} C(B_1, B_2, B_3) I_{ab}(E_{ab}) = V,
\]

where \(E_{ab} \simeq \mathbb{C}^{r_{ab}}\) are framing vector spaces.

Similarly to [82] one can prove the following result.

Proposition 3.2.5 The group of automorphisms of a stable framed representation is trivial. Moreover, stable framed representations naturally form an algebraic variety, called the moduli space of stable framed representations.

Remark 3.2.6 Recall the geometric interpretation of the moduli of stable representations of \((Q_3^{fr}, W_3^{fr})\) in the case when \(r_2 = r_3 = 0\), and hence \(i_k = 0, j_k = 0, k = 2, 3,\) and \(r_1 = 1\). It is the moduli space of torsion sheaves on \(\mathbb{C}^3\) such that their restriction to any plane \(x_3 = c\) is a cyclic module over the algebra \(\mathbb{C}[B_1, B_2, B_3 = c] = \mathbb{C}[B_1, B_2]\). Such a module is the same as a point of the Hilbert scheme \(\text{Hilb}(\mathbb{C}^2)\), but it is better to think that \(\mathbb{C}^2\) is placed in \(\mathbb{C}^3\) as a plane \(x_3 = c\).

Generally, if \(r_1 = r \geq 1\) we obtain the moduli space of rank \(r\) instantons on \(\mathbb{CP}^2\), equivalently, torsion-free rank \(r\) sheaves on \(\mathbb{CP}^2\) endowed with an isomorphism with \(O^r\) on the line \(l_\infty \simeq \mathbb{CP}^1 = \mathbb{CP}^2 - \mathbb{C}^2\) at infinity.

In the subsequent paper we plan to study an analogous geometric description for an arbitrary triple \((r_1, r_2, r_3)\).

4 Reminder on COHA for \((Q_3, W_3)\) and main result

4.1 Reminder on the 3d COHA

In this section, we focus on the quiver \(Q_3\) with potential \(W_3\) and discuss in detail the critical 3d COHA defined in [50] of this quiver with potential. The general setup is in [2.1]. As we have already mentioned, to save the terminology we will call “critical 3d COHA” simply COHA, in case if does not lead to a confusion. Nevertheless in this section we will often keep the adjective “critical” in order to make it easier for the reader to compare our exposition with the foundational paper [50].

Recall that the quiver \(Q_3\) has only one vertex, and three loops \(B_1, B_2, B_3\). The potential \(W_3\) is given by \(W_3 = B_1[B_2, B_3]\).
For any $n \in \mathbb{N}$, let $V$ be an $n$-dimensional vector space. Then the space of representations of $Q_3$ with dimension vector $n$ consists of triple of $n \times n$ matrices:
\[
\{(B_1, B_2, B_3) \mid B_i \in \mathfrak{g}_{\mathbb{N}}^3, i \in \mathbb{N}\} = \mathfrak{g}_{\mathbb{N}}^3.
\]
The group $\text{GL}_n$ acts on the representation space $\mathfrak{g}_{\mathbb{N}}^3$ via simultaneous conjugation.
Denote by $\text{tr}(W_3)_n$ the trace of $W_3$ on $\mathfrak{g}_{\mathbb{N}}^3$. Note that we have
\[
\text{tr}(W_3)_n = \text{tr}(B_1[B_2, B_3]) = \text{tr}(B_2[B_3, B_1]) = \text{tr}(B_3[B_1, B_2]).
\]
Let $\text{Crit}(\text{tr}(W_3)_n)$ be the critical locus of $\text{tr}(W_3)_n$. It is easy to see that the following holds.

**Lemma 4.1.1**
\[
\text{Crit}(\text{tr}(W_3)_n) = \{(B_1, B_2, B_3) \in \mathfrak{g}_{\mathbb{N}}^3 \mid [B_1, B_2] = 0, [B_2, B_3] = 0, [B_1, B_3] = 0\}.
\]

Let $D^b(X)$ be the derived category of constructible sheaves of $\mathbb{Q}$-vector spaces on a variety $X$ and $\mathbb{D}$ be the Verdier duality functor for $D^b(X)$. Denote by $H_*(X) \overset{\mathbb{D}}{\rightarrow} H_*(X)^{\mathbb{D}}$ the Verdier dual of the compactly supported cohomology $H_*(X)$ of $X$. Let $p : X \rightarrow \text{pt}$ be the structure map of $X$. We then have $H^*_c(X)^{\mathbb{D}} = \mathbb{D}_{\text{pt}}(p_! \mathbb{Q}_X)$. If $X$ carries a $G$-action, we denote by $H^*_c(X)^{\mathbb{D}}$ the corresponding equivariant cohomology of $X$.

Let $\varphi(W_3)_n$ be the vanishing cycle complex on $\mathfrak{g}_{\mathbb{N}}^3$ with support on $\text{Crit}(\text{tr}(W_3)_n)$. We have an isomorphism
\[
H^*_c(\text{GL}_n)(\mathfrak{g}_{\mathbb{N}}^3, \varphi(W_3)_n)^{\mathbb{D}} \cong H^*_c(\text{GL}_n)(\text{Crit}(\text{tr}(W_3)_n), \varphi(W_3)_n)^{\mathbb{D}}.
\]

Consider the graded vector space
\[
\mathcal{H}^{(Q_3, W_3)} := \bigoplus_{n \in \mathbb{N}} H^{(Q_3, W_3)}(n) = \bigoplus_{n \in \mathbb{N}} H^*_c(\text{GL}_n)(\mathfrak{g}_{\mathbb{N}}^3, \varphi(W_3)_n)^{\mathbb{D}}.
\]

For $n_1, n_2 \in \mathbb{N}$ such that $n = n_1 + n_2$, let $V_1 \subset V$ be a subspace of $V$ of dimension $n_1$. We write $G_n := \text{GL}_n$ for short. Let $P \subset \text{GL}_n$ be the parabolic subgroup preserving the subspace $V_1$ and $L := G_{n_1} \times G_{n_2}$ be the Levi subgroup of $P$. We have the corresponding Lie algebras
\[
\text{Lie}(G_n) = \mathfrak{g}_{\mathbb{N}}^3, \quad \text{Lie}(P) = \mathfrak{p}, \quad \text{Lie}(L) = \mathfrak{l} = \mathfrak{g}_{\mathbb{N}}^3 \times \mathfrak{g}_{\mathbb{N}}^3.
\]
Set $\mathfrak{p}^3_{n_1, n_2} = \{(B_i)_{i \in \mathbb{N}} \in \mathfrak{g}_{\mathbb{N}}^3 \mid B_i(V_1) \subset V_1, \text{ for all } i \in \mathbb{N}\}$. We have the following correspondence of $L$-varieties.
\[
\mathfrak{g}_{\mathbb{N}}^3 \times \mathfrak{g}_{\mathbb{N}}^3 \xrightarrow{p} \mathfrak{p}^3_{n_1, n_2} \xrightarrow{\eta} \mathfrak{g}_{n_1 + n_2}^3,
\]
where $p$ is the natural projection induced from $\mathfrak{p} \rightarrow \mathfrak{l}$, and $\eta$ is the embedding. The trace functions $\text{tr}(W_3)_{n_1}$ on $\mathfrak{g}_{n_1}^3$ induce a function $\text{tr}(W_3)_{n_1} \boxplus \text{tr}(W_3)_{n_2}$ on the product $\mathfrak{g}_{n_1}^3 \times \mathfrak{g}_{n_2}^3$. We define $\text{tr}(W_3)_{n_1, n_2}$ on $\mathfrak{p}^3_{n_1, n_2}$ to be
\[
\text{tr}(W_3)_{n_1, n_2} := p^*(\text{tr}(W_3)_{n_1}) \boxplus \text{tr}(W_3)_{n_2}) = \eta^*(\text{tr}(W_3)_{n_1 + n_2}).
\]
Note that we have $p^{-1}(\text{Crit}(\text{tr}(W_3)_{n_1})) \times \text{Crit}(\text{tr}(W_3)_{n_2})) \nsubseteq \eta^{-1}(\text{Crit}(\text{tr}(W_3)_{n_1 + n_2}))$. Indeed,
\[
p^{-1}(\text{Crit}(\text{tr}(W_3)_{n_1})) \times \text{Crit}(\text{tr}(W_3)_{n_2})) = \{(B_i)_{i \in \mathbb{N}} \in \mathfrak{p}^3_{n_1, n_2} \mid [\text{pr}(B_i), \text{pr}(B_j)] = 0, i, j \in \mathbb{N}\}
\]
while $\eta^{-1}(\text{Crit}(\text{tr}(W_3)_{n_1 + n_2})) = \{(B_i)_{i \in \mathbb{N}} \in \mathfrak{p}^3_{n_1, n_2} \mid [B_i, B_j] = 0, i, j \in \mathbb{N}\}$. 

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By Lemma 4.1.1 we have \( \text{Crit}(\text{tr}(W_3)_n) \subseteq (\text{tr}(W_3)_n)^{-1}(0) \). Therefore, we have a Thom-Sebastiani isomorphism \[^{[50]} \text{Definition 27}\].

The multiplication \( m^{\text{crit}} : \mathcal{H}^{(Q_3,W_3)}(n_1) \otimes \mathcal{H}^{(Q_3,W_3)}(n_2) \to \mathcal{H}^{(Q_3,W_3)}(n) \) is the composition of the following \[^{[50]} \].

1. The Thom-Sebastiani isomorphism

\[
\mathcal{H}^{(Q_3,W_3)}(n_1) \otimes \mathcal{H}^{(Q_3,W_3)}(n_2) \cong H^*_{c,L}(I_{n_1,n_2}, \varphi(W_3)_{n_1} \boxplus \text{tr}(W_3)_{n_2})^\vee.
\]

2. Using the fact that \( p^3 \) is an affine bundle over \( I_{n_1,n_2} \) with fiber \( n_{n_1,n_2}^3 \), and \( \text{tr}(W_3)_{n_1,n_2} \) is the pullback of \( \text{tr}(W_3)_n \) along \( p^3 \), we have

\[
p^*: H^*_{c,L}(I_{n_1,n_2}, \varphi(W_3)_{n_1} \boxplus \text{tr}(W_3)_{n_2})^\vee \cong H^*_{c,L}(p^3, \varphi(W_3)_{n_1,n_2})^\vee.
\]

3. Using the fact \( \text{tr}(W_3)_{n_1,n_2} \) is the restriction of \( \text{tr}(W_3)_n \) to \( p^3 \), we have

\[
\eta_* : H^*_{c,L}(p^3, \varphi(W_3)_{n_1,n_2})^\vee \to H^*_{c,L}(g_{n_1}^3, \varphi(W_3)_n)^\vee.
\]

4. Pushforwarding along \( G \times p : g_{n_1}^3 \to g_{n_1}^3, (g, m) \mapsto gm^{-1} \), we obtain

\[
H^*_{c,L}(g_{n_1}^3, \varphi(W_3)_n)^\vee \cong H^*_{c,P}(g_{n_1}^3, \varphi(W_3)_n)^\vee \cong H^*_{c,G}(G \times p : g_{n_1}^3, \varphi(W_3)_n)^\vee
\]

\[
\to H^*_{c,G}(g_{n_1}^3, \varphi(W_3)_n)^\vee = \mathcal{H}^{(Q_3,W_3)}(n).
\]

It was proved in \[^{[50]} \] that the multiplication \( m^{\text{crit}} \) is associative.

**Definition 4.1.2** \[^{[50]} \text{§7.6}\] The COHA of the quiver with potential \( (Q_3, W_3) \) is defined as an associative algebra given by the graded vector space

\[
\mathcal{H}^{(Q_3,W_3)} : = \bigoplus_{n \in \mathbb{N}} H^*_{c, GL_n}(g_{n_1}^3, \varphi(W_3)_n)^\vee,
\]

endowed with the multiplication \( m^{\text{crit}} \) described above.

### 4.2 Main result

Let \( \text{tr} W_3^{fr} \) be the trace function on \( M_{fr}(n) \) of the quiver with potential \( (Q_3^{fr}, W_3^{fr}) \). Let \( \varphi_{W_3^{fr}} \) be the vanishing cycle complex on \( M_{fr}(n) \) with support on the critical locus of \( \text{tr} W_3^{fr} \).

Recall the group \( GL_n \times GL_{fr} \times T_2 \) acts on \( M_{fr}(n) \). Consider the equivariant cohomology

\[
V_{r_1, r_2, r_3}(n) := H^*_{c, GL_n} \times GL_{fr} \times T_2 (M_{r_1, r_2, r_3}(n)^{\text{fr}}, \varphi_{W_3^{fr}})^\vee
\]

of \( M_{r_1, r_2, r_3}(n)^{\text{fr}} \) with values in \( \varphi_{W_3^{fr}} \), and

\[
V_{r_1, r_2, r_3} := \bigoplus_{n \in \mathbb{N}} V_{r_1, r_2, r_3}(n).
\]

The main theorem of this paper is the following.
Theorem 4.2.1. The vertex operator algebra $\mathcal{W}_{r_1,r_2,r_3}$ acts on $V_{r_1,r_2,r_3}$.

We prove Theorem 4.2.1 through the following steps.

Step 1: We show the 3d COHA introduced in [50], denoted by $\mathcal{H}(Q_3,W_3)$, associated to the quiver with potential $(Q_3,W_3)$ acts on $V_{r_1,r_2,r_3}$. This action is naturally obtained via a geometric construction, which is similar to that of the Hall multiplication of the 3d COHA in [50]. This is done in §5.

Step 2: We extend the action in Step 1 to an action of the Drinfeld double $D(\mathcal{H}(Q_3,W_3)) \cong \mathcal{H}(Q_3,W_3) \otimes H^0 \otimes \mathcal{H}(Q_3,W_3)$ of the spherical 3d COHA $\mathcal{H}(Q_3,W_3)$. The Drinfeld double $D(\mathcal{H}(Q_3,W_3))$ will be shown to be isomorphic to the affine Yangian $Y_{h_1,h_2,h_3}(\mathfrak{gl}(1))$. This is done in §6 and §7 by constructing a Drinfeld coproduct on $D(\mathcal{H}(Q_3,W_3))$ and checking its compatibility with certain restriction map $V_{r_1,r_2,r_3} \to (V_{1,0,0})^{\otimes r_1} \otimes (V_{0,1,0})^{\otimes r_2} \otimes (V_{0,0,1})^{\otimes r_3}$ (Proposition 6.4.1).

Step 3: We introduce a central extension of $D(\mathcal{H}(Q_3,W_3)) \cong Y_{h_1,h_2,h_3}(\mathfrak{gl}(1))$ by a polynomial ring with infinitely many variables $C[c_i^{(1)},c_i^{(2)},c_i^{(3)} : i \geq 0]$. Denote this central extension by $\mathcal{SH}^\mathbb{F}$. We construct a coproduct $\Delta^\mathbb{F}$ (different than the Drinfeld coproduct) on $\mathcal{SH}^\mathbb{F}$, and deduce a free field realization of $\mathcal{SH}^\mathbb{F}$, i.e., an action on the Fock space $(V_{1,0,0})^{\otimes r_1} \otimes (V_{0,1,0})^{\otimes r_2} \otimes (V_{0,0,1})^{\otimes r_3}$. Furthermore, this free field realization is compatible with the hyperbolic restriction (defined in §8.3) of the moduli of spiked instantons. That is, we have the following diagram

\[
\begin{array}{ccc}
\mathcal{SH}^\mathbb{F} & \xrightarrow{(\Delta^\mathbb{F})^{r_1+r_2+r_3}} & (\mathcal{SH}^\mathbb{F})^{\otimes (r_1+r_2+r_3)} \\
\downarrow & & \downarrow \\
V_{r_1,r_2,r_3} & \xrightarrow{\text{hyperbolic restriction}} & (V_{1,0,0})^{\otimes r_1} \otimes (V_{0,1,0})^{\otimes r_2} \otimes (V_{0,0,1})^{\otimes r_3}
\end{array}
\]

Step 4: We show that the action of $\mathcal{SH}^\mathbb{F}$ on $V_{r_1,r_2,r_3}$ factors through $\mathcal{W}_{r_1,r_2,r_3}$. This is done by proving that the action on the free field realization factors through the screening operators from Definition 9.2.1. This will be done in §10.

Remark 4.2.2. As we mentioned in the introduction, examples illustrating our main result should be derived from toric Calabi-Yau 3-folds or from dimer models (a.k.a. brane tilings in physics [44, 22, 12, 72, 77]). Recall the framework for the latter. In general a dimer model can be understood as a bipartite graph with vertices colored in two colors $B$ (black) and $W$ (white) on the 2-dimensional torus $(S^1)^2$ such that the complement is a disjoint union of finitely many contractible subsets (faces). With such geometric data one can associate a quiver with potential. Vertices of the quiver are in bijection with faces, edges correspond to the edges of the dual graph, and the orientation is chosen in such a way that if an edge connects two adjacent faces then the vertex $B$ is on the right and $W$ is on the left. The potential is a sum (with signs) of paths about vertices. Framing ranks $r_1,...,r_k$ needed for the notion of framed representation is an additional piece of data.

The relation of dimer models, toric Calabi-Yau 3-folds and stable framed representations of quivers is known in many classes of examples (see e.g. [77]). E.g. in the case of the resolved conifold (see [83]) dimer models parametrize stable framed representations of the corresponding Klebanov-Witten quiver with potential. These representations are fixed points for the action of the two-dimensional torus $T_2$. By localization theorem they generate the
equivariant cohomology of the moduli spaces of stable framed representations. Our main result claims that it is acted by the corresponding equivariant COHA, its spherical version and their doubles.

5 Action on the cohomology of the moduli of stable framed representations

5.1 Action of the 3d COHA in the non-equivariant case

The general construction of representations of COHA from stable framed objects can be found in [82, Section 4]. In this section, we deal with the special stable framed object $M_{r}(n)$ of the quiver with potential $(Q_{3}, W_{3})$. We prove the following

Theorem 5.1.1 The algebra $\mathcal{H}(Q_{3}, W_{3})$ acts on

$$V_{r_{1}, r_{2}, r_{3}} = \bigoplus_{n \in \mathbb{N}} H_{c, GL_{n} \times GL_{r}(M_{st}(n), \varphi_{W_{fr}})}^{*},$$

for any $(r_{1}, r_{2}, r_{3}) \in \mathbb{N}^{3}$.

Proof. The notations are the same as before. For $n_{1}, n_{2} \in \mathbb{N}$ such that $n = n_{1} + n_{2}$. Let $V = \mathbb{C}^{n}$, $E_{r_{i}} = \mathbb{C}^{r_{i}}, i = 1, 2, 3$ be vector spaces. We fix subspaces $V_{1} \subset V$, and $\{0\} \subset E_{r_{i}}, i = 1, 2, 3$, with $\dim(V_{1}) = n_{1}$. We have the short exact sequences of vector spaces

$$0 \rightarrow V_{1} \rightarrow V \rightarrow V_{2} \rightarrow 0, \text{ and } 0 \rightarrow \{0\} \rightarrow E_{r_{i}} \rightarrow E_{r_{i}} \rightarrow 0.$$

Define the following subvariety of $M_{r}(n)$.

$$Z_{p} := \{(B_{i}, I_{ab}, J_{ab}) \in M_{r}(n) \mid B_{i}(V_{1}) \subset V_{1}, J_{ab}(V_{1}) = 0, i \in \mathbb{N}, ab \in \mathbb{N}\}.$$

For an element $(B_{i}, I_{ab}, J_{ab})_{i \in \mathbb{N}, ab \in \mathbb{N}} \in Z_{p}$, as $B_{i}, I_{ab}, J_{ab}$ preserves the subspaces $V_{1} \subset V$, and $\{0\} \subset E_{r_{i}}$, we have the restriction map on the subspace $V_{1}$

$$(B_{i})_{1} : V_{1} \rightarrow V_{1},$$

and $(I_{ab})_{1} : \{0\} \rightarrow V_{1}, (J_{ab})_{1} : V_{1} \rightarrow \{0\}$ are zero maps.

We also have the induced maps on the corresponding quotients.

$$(B_{i})_{2} : V_{2} \rightarrow V_{2}, \quad I_{ab} : E_{r_{i}} \rightarrow V_{2}, \quad J_{ab} : V_{2} \rightarrow E_{r_{i}},$$

where $\{a, b, c\} = \{1, 2, 3\}$. This gives rise to the following correspondence

$$\mathfrak{g}_{n_{1}}^{3} \times M_{r}(n) \xrightarrow{p} Z_{p} \xrightarrow{\eta} M_{r}(n) \quad (8)$$

The maps in (8) are given by

$$p : (B_{i}, I_{ab}, J_{ab}) \mapsto (B_{i})_{1}, \quad \left( (B_{i})_{2}, I_{ab}, J_{ab} \right),$$

$$\eta : (B_{i}, I_{ab}, J_{ab}) \mapsto (B_{i}, I_{ab}, J_{ab}) \text{ is the natural inclusion.}$$
Taking into account the stability conditions, the correspondence \( Z_p \) induces the following correspondence of \( L \)-varieties.

\[
\begin{align*}
\mathfrak{g}^{\mathfrak{l}}_{\mathfrak{n}_1} \times M_r(n_2)^{st} & \xrightarrow{p^{-1}} \mathfrak{g}^{\mathfrak{l}}_{\mathfrak{n}_1} \times M_r(n_2)^{st} \\
\mathfrak{g}^{\mathfrak{l}}_{\mathfrak{n}_1} \times M_r(n_2) & \xrightarrow{p} Z_p^{st} \\
& \xrightarrow{\eta^{st}} M_r(n)^{st} \\
& \xrightarrow{\eta} M_r(n)
\end{align*}
\]

Here we define \( Z_p^{st} := Z_p \cap M_r(n)^{st} \). Explicitly, \( Z_p^{st} \) consists of

\[
\{(B_i, I_{ab}, J_{ab}) \in \mathfrak{g}^{\mathfrak{l}}_{\mathfrak{n}_1} \times M_r(n) | B_i(V_1) \subset V_1, J_{ab}(V_1) = 0, \sum_{\mathfrak{n} \in \mathfrak{g}} C(B_1, B_2, B_3) Im(I_{ab}) = V \}\.
\]

For an element \((B_i, I_{ab}, J_{ab}) \in Z_p^{st}\), the stability condition \( \sum_{\mathfrak{n} \in \mathfrak{g}} C(B_1, B_2, B_3) Im(I_{ab}) = V \) implies that

\[
\sum_{\mathfrak{n} \in \mathfrak{g}} C((B_1)_1, (B_2)_2, (B_3)_2) Im(I_{ab}) = V_2
\]

under the projection \( p \). Therefore, the projection \( p^{st} : Z_p^{st} \to \mathfrak{g}^{\mathfrak{l}}_{\mathfrak{n}_1} \times M_r(n)^{st} \) in (9) is well-defined.

Note that \( p : Z_p \to \mathfrak{g}^{\mathfrak{l}}_{\mathfrak{n}_1} \times M_r(n) \) is an affine bundle, and we have an open embedding \( Z_p^{st} \subset p^{-1}(\mathfrak{g}^{\mathfrak{l}}_{\mathfrak{n}_1} \times M_r(n)^{st}) \).

**Lemma 5.1.2** Assume \( X \) is a \( G \)-variety, with \( f : X \to C \) a \( G \)-invariant function on \( X \). Let \( \varphi_f \) be the vanishing cycle sheaf for \( f \). Let \( j : U \subset X \) be an open embedding such that \( G \) acts on \( U \). Denote by \( f_U : U \to C \) be the restriction of \( f \) on \( U \). Then we have the natural homomorphism of graded vector spaces:

\[
H^*_{c,G}(X, \varphi_f)^\vee \to H^*_{c,G}(U, \varphi_{f_U})^\vee.
\]

**Proof.** By definition, we have \( H^*_{c}(X, \varphi_f) := pX|\psi_f[-1](Q_X) \), where \( p_X : X \to pt \) is the projection to a point. As \( j \) is an open embedding, we have \( j^* = j^+ \). This gives the following isomorphism

\[
H^*_c(U, \varphi_{f_U}) := pU|\psi_{f_U}[-1](Q_U) = pX|j_!\psi_{f_U}[-1]j^*Q_X = pX|j_!j^!\psi_f[-1]Q_X,
\]

where the last isomorphism uses the fact \( \psi_{f_U}j^+ \cong j^*\psi_f \) (See [13] Page 10)). The natural map \( j_!j^! \to id \) induces \( H^*_c(U, \varphi_{f_U}) \to H^*_c(X, \varphi_f) \). The desired map is obtained by taking the Verdier duality. This completes the proof. \( \blacksquare \)

We now construct an action map

\[
a^{\text{crit}} : \mathcal{H}_{(Q_3,W_3)}(n_1) \otimes V_{r_1,r_2,r_3}(n_2) \to V_{r_1,r_2,r_3}(n).
\]

as the composition of the following morphisms.

1. The Thom-Sebastiani isomorphism

\[
\mathcal{H}_{(Q_3,W_3)}(n_1) \otimes V_{r_1,r_2,r_3}(n_2) \cong H^*_{c,L \times GL_r}(\mathfrak{g}^{\mathfrak{l}}_{\mathfrak{n}_1} \times M_r(n)^{st}, \varphi(W_3)_{n_1} \oplus (W_4^r)_{n_2})^\vee.
\]
2. Using the fact that $p^{-1}(\mathfrak{gl}^3_{n_1} \times \mathcal{M}_{\mathbb{F}}(n_2)^{st})$ is an affine bundle over $\mathfrak{gl}^3_{n_1} \times \mathcal{M}_{\mathbb{F}}(n_2)^{st}$, and $\text{tr}(W^3_{fr})_{n_1,n_2}$ is the pullback of $\text{tr}(W^3_{fr})_{n_1} \oplus \text{tr}(W^3_{fr})_{n_2}$, we have
\[ p^* : H^*_{c,L \times \text{GL}_p}(\mathfrak{gl}^3_{n_1} \times \mathcal{M}_{\mathbb{F}}(n_2)^{st}, \varphi(W^3_{fr})_{n_1,n_2})^\vee \]
\[ \cong H^*_{c,L \times \text{GL}_p}(p^{-1}(\mathfrak{gl}^3_{n_1} \times \mathcal{M}_{\mathbb{F}}(n_2)^{st}), \varphi(W^3_{fr})_{n_1,n_2})^\vee \]

3. By Lemma 5.1.2, let $j : Z^st_p \subset p^{-1}(\mathfrak{gl}^3_{n_1} \times \mathcal{M}_{\mathbb{F}}(n_2)^{st})$ be the open embedding, then we have a natural map by restriction
\[ j^* : H^*_{c,L \times \text{GL}_p}(p^{-1}(\mathfrak{gl}^3_{n_1} \times \mathcal{M}_{\mathbb{F}}(n_2)^{st}), \varphi(W^3_{fr})_{n_1,n_2})^\vee \to H^*_{c,L \times \text{GL}_p}(Z^st_p, \varphi(W^3_{fr})_{n_1,n_2})^\vee \]

4. Using the fact $\text{tr}(W^3_{fr})_{n_1,n_2}$ is the restriction of $\text{tr}(W^3_{fr})_{n}$ on $\mathcal{M}_{\mathbb{F}}(n)^{st}$ to $Z^st_p$. We have
\[ \eta^st : H^*_{c,L \times \text{GL}_p}(Z^st_p, \varphi(W^3_{fr})_{n_1,n_2})^\vee \to H^*_{c,L \times \text{GL}_p}(\mathcal{M}_{\mathbb{F}}(n)^{st}, \varphi(W^3_{fr})_{n})^\vee. \]

5. Pushforward along $G \times_p \mathcal{M}_{\mathbb{F}}(n)^{st} \to \mathcal{M}_{\mathbb{F}}(n)^{st}, (g, m) \mapsto gmg^{-1},$ we get
\[ H^*_{c,L \times \text{GL}_p}(\mathcal{M}_{\mathbb{F}}(n)^{st}, \varphi(W^3_{fr})_{n})^\vee \equiv H^*_{c,G \times \text{GL}_p}(\mathcal{M}_{\mathbb{F}}(n)^{st}, \varphi(W^3_{fr})_{n})^\vee \]
\[ \cong H^*_{c,G \times \text{GL}_p}(G \times_p \mathcal{M}_{\mathbb{F}}(n)^{st}, \varphi(W^3_{fr})_{n})^\vee \to H^*_{c,G \times \text{GL}_p}(\mathcal{M}_{\mathbb{F}}(n)^{st}, \varphi(W^3_{fr})_{n})^\vee. \]

The map $\varphi^\text{crit}$ gives rise to an action of $\mathcal{H}^{(Q_3,W_3)}$ on $V_{r_1,r_2,r_3}$ by a similar argument as the proof of associativity of the product on $\mathcal{H}^{(Q_3,W_3)}$ given in [30].

This completes the proof. \( \blacksquare \)

### 5.2 Action of the equivariant COHA

The 2-dimensional torus $T_2$ acts on $\mathfrak{gl}^3_{n}$ by
\[ (t_1, t_2) \cdot (B_1, B_2, B_3) = (t_1B_1, t_2B_2, (t_1t_2)^{-1}B_3). \] (11)

Under the action, the potential $W_3 = \text{tr}(B_3[B_1, B_2])$ is invariant. Recall in [3] we also have the action of $T_2$ on $\mathcal{M}_{\mathbb{F}}(n)$ by
\[ (t_1, t_2) \cdot \{B_i, I_{ab}, J_{ab}\} = (t_1B_1, t_2B_2, t_3B_3, I_{12}, t_1t_2J_{12}, I_{13}, t_1t_3J_{13}, I_{23}, t_2t_3J_{23}), \]
where $t_1t_2t_3 = 1$. Under this action, the potential $W^3_{fr}$ is invariant.

We can encode the $T_2$ action in the definition of $\mathcal{H}^{(Q_3,W_3)}$, and its modules by considering
\[ \mathcal{H}^{(Q_3,W_3)}_{T_2} := \bigoplus_{n \in \mathbb{N}} H^*_{c, \text{GL}_n \times T_2}(\mathfrak{gl}^3_{n}, \varphi(W_3))^\vee \]
and $V_{r_1,r_2,r_3} := \bigoplus_{n \in \mathbb{N}} H^*_{c, \text{GL}_n \times \text{GL}_p \times T_2}(\mathcal{M}_{\mathbb{F}}(n)^{st}, \varphi(W^3_{fr}))^\vee$.

This gives a $T_2$-equivariant version of COHA and the module over it.

In fact it is obtained by the restriction to the subtorus $T_2$ of the full $T_3$-version. Let us formulate this result as well.
Proposition 5.2.1 For any \((r_1, r_2, r_3), r_k \in \mathbb{Z}_{>0}, k = 1, 2, 3\) the equivariant cohomology \(V_{r_1, r_2, r_3} = \bigoplus_{n \in \mathbb{N}} H^*_e \text{GL}_n \times \text{GL}_r \times T_3 (M_F(n)^{st}, \varphi_W^{st})^\vee\) forms a module over the equivariant COHA \(\mathcal{H}(Q_3, W_3)^{T_3} \simeq \mathcal{H}(\text{Coh}_{T_3}(\mathbb{C}^3)^0)\).

Remark 5.2.2 More precisely, there are two natural actions by correspondences of the equivariant COHA on \(V_{r_1, r_2, r_3}\). One is given by “creation operators”, while the other one is given by “annihilation operators” (see [82]). Combining together these two actions one can define the action of an associative algebra denoted by \(D(\mathcal{H}(Q_3, W_3)^{T_3}) := D(\mathcal{H}(\text{Coh}_{T_3}(\mathbb{C}^3)^0),\) which we will call the (equivariant) double COHA (see [82]).

Instead of constructing the equivariant double by the Nakajima-style construction mentioned in the above Remark, we are going to utilize a more familiar approach by introducing a Hopf algebra structure on the equivariant spherical COHA and making the corresponding Drinfeld double. More precisely, let \(SH_{Q_3, W_3}^{T_2}\) be the spherical subalgebra of \(\mathcal{H}(Q_3, W_3)^{T_2}\) generated by elements in \(\mathcal{H}(Q_3, W_3)^{T_2}(1) = H^*_e \text{GL}_3 \times T_2(\mathfrak{gl}_3^0, \mathbb{C})^\vee\). Let \(D(SH_{Q_3, W_3}^{T_2})\) be the Drinfeld double of \(\mathcal{H}(Q_3, W_3)^{T_2}\). In Sections 3 and 7 we spell out the action of this double \(D(SH_{Q_3, W_3}^{T_2})\) on \(V_{r_1, r_2, r_3}\), and identify it with the affine Yangian of \(\mathfrak{gl}(1)\).

6 The extended COHA and Drinfeld coproduct

In this section we introduce an extended 3d COHA \(\mathcal{H}^0 \times \mathcal{H}(Q_3, W_3)\) by adding to \(\mathcal{H}(Q_3, W_3)\) a polynomial algebra \(\mathcal{H}^0\) in infinitely many variables. We call \(\mathcal{H}^0\) the Cartan algebra, as it will be the commutative subalgebra in the triangular decomposition of the double COHA \(D(SH_{Q_3, W_3}^{T_2}) = \bigotimes_{n \in \mathbb{N}} SH_{Q_3, W_3}^{T_2}\). Furthermore, we extend the action of \(\mathcal{H}(Q_3, W_3)\) on \(V_{r_1, r_2, r_3}\) to an action of the extended COHA \(\mathcal{H}^0 \times \mathcal{H}(Q_3, W_3)\). In particular, the action of \(\mathcal{H}^0\) is obtained using the tautological bundles on \(M_F(n)^{st}\).

6.1 The tautological bundles

Let \(\mathcal{H}^0\) be a polynomial algebra in the generators \(\{\psi_j \mid j \in \mathbb{N}\}\). Consider the generating function

\[
\psi(z) := 1 - (h_1 h_2 h_3) \sum_{j \geq 0} \psi_j z^{-j-1} \in \mathcal{H}^0[[z^{-1}]].
\] (12)

Recall the moduli space of stable representations of \((Q_3^{fr}, W_3^{fr})\): \(M_{fr}^{st}(n)\). Let \(\mathcal{V}_n\) be the \(\text{GL}_n\)-equivariant vector bundle

\[
\mathcal{V}_n := M_{fr}^{st}(n) \times \mathbb{C}^n
\]
on \(M_{fr}^{st}\), where \(\mathbb{C}^n\) is the standard representation of \(\text{GL}_n\), and \(\text{GL}_n\) acts on \(M_{fr}^{st}(n) \times \mathbb{C}^n\) diagonally. We think \(\mathcal{V}_n\) as an element in the equivariant K-theory \(K_{\text{GL}_n}(M_{fr}^{st}(n)) \cong K(M_{fr}^{st}(n)/\text{GL}_n)\). The collection \(\mathcal{V} := \{\mathcal{V}_n\}_{n \geq 0}\) can be thought of as a single bundle on \(M_{fr}^{st}\).

For \(i = 1, 2, 3\), let \(\mathcal{E}_i := M_{fr}^{st} \times \mathbb{C}^{r_i}\) be the trivial \(\mathbb{C}^{r_i}\) bundles on \(M_{fr}^{st}\).

In this section, we use the torus \(T_2\)-action as in Section 5.2. For integers \((m_1, m_2, m_3) \in \mathbb{Z}^3\), we define a \(T_2\)-module structure on \(\mathbb{C}\) by

\[
(t_1, t_2, t_3) \cdot v := t_1^{m_1} t_2^{m_2} t_3^{m_3} v, \quad (t_1, t_2, t_3) \in T_3 \text{ with } t_1 t_2 t_3 = 1, v \in \mathbb{C},
\]

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Proof. For any Lemma 6.1.2 the stability condition on that. Recall the following correspondence from (9), which is used to define the action of \( d \) on \( V \). The ADHM equations \( F \) obtained from the alternating sum of the above complex.

The morphisms are given by

\[
d_1 = \begin{bmatrix}
B_3 \\
-B_2 \\
B_1 \\
J_{23} \\
-J_{13} \\
-J_{12}
\end{bmatrix}
\quad d_2 = \begin{bmatrix}
B_2 & B_3 & 0 & I_{23} & 0 & 0 \\
-B_1 & 0 & B_3 & 0 & I_{13} & 0 \\
0 & -B_1 & -B_2 & 0 & 0 & I_{12} \\
J_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & J_{13} & 0 & 0 & 0 & 0 \\
0 & 0 & J_{23} & 0 & 0 & 0
\end{bmatrix},
\]

and \( d_3 = [B_1 \ B_2 \ B_3 \ I_{12} \ I_{13} \ I_{23}] \).

**Lemma 6.1.1** The ADHM equations

\[
[B_a, B_b] + I_{ab} J_{ab} = 0, B_a I_{bc} = 0, J_{ab} B_c = 0, \quad \text{where} \quad \{a, b, c\} = \{1, 2, 3\}
\]

imply \( d_2 \circ d_1 = 0 \), and \( d_3 \circ d_2 = 0 \).

**Lemma 6.1.2** The stability condition \((4)\) implies the map \( d_3 \) is surjective.

**Proof.** For any \( v \) in \( C^n \), by \((4)\), we have

\[
v = f_1(B_1, B_2, B_3) I_{12}(v_3) + f_2(B_1, B_2, B_3) I_{13}(v_2) + f_3(B_1, B_2, B_3) I_{23}(v_1),
\]

for some polynomials \( f_1, f_2, f_3 \) and \( v_i \) in \( C^n, i = 1, 2, 3 \). Therefore, \( v \) in \( \text{Im}(d_3) \). This implies that \( d_3 \) is surjective. \( \blacksquare \)

Note that the map \( d_1 \) is not injective in general.

Consider the following tautological element in \( K_{\text{GL}^n} \times \text{GL}_r \times \mathbb{T}_2(\mathcal{M}_{\text{st}}^{\eta^P}(n)) \), which is obtained from the alternating sum of the above complex.

\[
\mathcal{F}(\mathcal{V}_n, \mathcal{E}_p) := (q_1 - q_1^{-1} + q_2 - q_2^{-1} + q_3 - q_3^{-1}) \mathcal{V}_n + (q_1^{-1} - 1) \mathcal{E}_{r_1} + (q_2^{-1} - 1) \mathcal{E}_{r_2} + (q_3^{-1} - 1) \mathcal{E}_{r_3}
\]

Recall the following correspondence from \((9)\), which is used to define the action of \( \mathcal{H}(Q_3, W_3) \) on \( V_{r_1, r_2, r_3} \), where \( n = n_1 + n_2 \):

\[
\mathfrak{g}^{\mathfrak{t}_{n_1}}_n \times \mathcal{M}_{\text{st}}^{\eta^P}(n_2) \xrightarrow{p'^*} Z^* \xrightarrow{\eta'^*} \mathcal{M}_{\text{st}}^{\eta^P}(n_2)
\]

Let \( \mathcal{V}_{n_1} := \mathfrak{g}^{\mathfrak{t}_{n_1}}_n \times C^{n_1} \) be the \( GL_{n_1} \)-equivariant vector bundle on \( \mathfrak{g}^{\mathfrak{t}_{n_1}}_n \). The following lemma can be proved by a straightforward calculation.

**Lemma 6.1.3** On \( K_{\text{GL}_{n_1}} \times \text{GL}_{n_2} \times \text{GL}_r \times \mathbb{T}_2(\mathbb{Z}^{\eta^P}_p) \), we have the following equality:

\[
(p'^*)^* \mathcal{F}(\mathcal{V}_n, \mathcal{E}_p) - (1 \boxtimes \mathcal{F}(\mathcal{V}_{n_2}, \mathcal{E}_p)) = \mathcal{F}(\mathcal{V}_{n_1} \boxtimes 1).
\]

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6.2 The extended 3d COHA

Let $X$ be a complex algebraic variety with a $G$ action. Let $\mathcal{L} \to X$ be a $G$-equivariant line bundle on $X$, and $f$ a $G$-invariant regular function on $X$. Denote by $s : X \hookrightarrow \mathcal{L}$ the zero section of $\mathcal{L}$. We define the first Chern class of $\mathcal{L}$ as

$$c_1(\mathcal{L}) := s^*s_*(1) \in H_G(X, \varphi f)^\vee.$$

By the splitting principle, we can define the Chern roots of an equivariant vector bundle. Let $\lambda_1, \cdots, \lambda_n$ be the Chern roots of an equivariant vector bundle $\mathcal{V}$. Its Chern polynomial is denoted by

$$\lambda_{-1/z}(\mathcal{V}) = \prod_{i=1}^n (1 - \lambda_i/z) \in H_G(X, \varphi f)^\vee[[z^{-1}]].$$

We follow the notations of Section 6.1. Let $\lambda_1, \cdots, \lambda_n$ be the Chern roots of $\mathcal{V}_n$, and $\mu_1^{(i)}, \cdots, \mu_{r_i}^{(i)}$ be the Chern roots of $\mathcal{E}_r$, $i = 1, 2, 3$.

We define an $\mathcal{H}_0$-action on the 3d COHA $\mathcal{H}^{(Q_3, W_3)}$ as follows. For any $f \in \mathcal{H}^{(Q_3, W_3)}(n)$ in the degree $n$ piece of $\mathcal{H}^{(Q_3, W_3)}$, and the generating series $\psi(z)$ as in (12), the action of $\psi(z)$ on $f$ is defined as

$$\psi(z) \cdot f := \psi(z)f\psi(z)^{-1}$$

$$= f \prod_{t=1}^n \frac{(z - \lambda_t - h_1)(z - \lambda_t - h_2)(z - \lambda_t - h_3)}{(z - \lambda_t + h_1)(z - \lambda_t + h_2)(z - \lambda_t + h_3)} \in \mathcal{H}^{(Q_3, W_3)}(n)[[z^{-1}]].$$

In particular, the action of $\mathcal{H}_0$ on $\mathcal{H}^{(Q_3, W_3)}$ preserves the degree.

**Definition 6.2.1** We define the extended 3d COHA as

$$\mathcal{H}_0 \ltimes \mathcal{H}^{(Q_3, W_3)}.$$

We now define an $\mathcal{H}_0$-action on $V_{r_1, r_2, r_3} = \bigoplus_{n \in \mathbb{N}} V_{r_1, r_2, r_3}(n)$ as follows. For any $m \in V_{r_1, r_2, r_3}(n)$, and $\psi(z)$ as in (12),

$$\psi(z) \cdot m := \lambda_{-1/z}(\mathcal{F}(\mathcal{V}_n, \mathcal{E}_r))m$$

$$= \left( \prod_{a=1}^{r_1} \frac{z - \mu_a^{(1)}}{z - \mu_a^{(1)}} \prod_{b=1}^{r_2} \frac{z - \mu_b^{(2)}}{z - \mu_b^{(2)}} \prod_{b=1}^{r_3} \frac{z - \mu_c^{(3)}}{z - \mu_c^{(3)}} \cdot \prod_{d=1}^n \frac{z - \lambda_d - h_1}{z - \lambda_d + h_1} \frac{z - \lambda_d - h_2}{z - \lambda_d + h_2} \frac{z - \lambda_d - h_3}{z - \lambda_d + h_3} \right) m \in V_{r_1, r_2, r_3}(n)[[z^{-1}]].$$

**Proposition 6.2.2** The actions of $\mathcal{H}^{(Q_3, W_3)}$ and $\mathcal{H}_0$ on $V_{r_1, r_2, r_3}$ can be extended to an action of the extended 3d COHA $\mathcal{H}_0 \ltimes \mathcal{H}^{(Q_3, W_3)}$ on $V_{r_1, r_2, r_3}$.

**Proof.** For any $f \in \mathcal{H}^{(Q_3, W_3)}(n_1)$, and $m \in V_{r_1, r_2, r_3}(n_2)$, with $n = n_1 + n_2$, we need to show the following equality in $V_{r_1, r_2, r_3}(n)$:

$$\psi(z) \left( f(\psi(z)^{-1}m) \right) = (\psi(z) \cdot f)m.$$
By \textsuperscript{[13]}, it suffices to show
\[
\psi_n(z) \left( f(\psi_{n_2}(z)^{-1} m) \right) = \left( f \prod_{t=1}^{n_1} (z - \lambda_t - h_1)(z - \lambda_t - h_2)(z - \lambda_t - h_3)(z - \lambda_t + h_1)(z - \lambda_t + h_2)(z - \lambda_t + h_3) \right) (m), \tag{15}
\]
where we write a subscript \(n\) in \(\psi_n(z) \cdot v\) to emphasize the action of \(\psi(z) \in H^0[[z^{-1}]]\) on the element \(v \in V_{r_1, r_2, r_3}(n)\). Both sides of \textsuperscript{(15)} are elements in \(V_{r_1, r_2, r_3}(n_1 + n_2) = V_{r_1, r_2, r_3}(n)\).

We have the following correspondence from \textsuperscript{(9)}:
\[
\begin{array}{c}
gl^3_{\mathfrak{h}_{\mathfrak{c}}} \times \mathcal{M}^s_{\mathfrak{c}}(n_2) \xrightarrow{\psi^s} \mathcal{M}^s_{\mathfrak{c}}(n) \\
\end{array}
\]

By the action of \(H(Q_3, W_3)\) (used in the first equality) and the projection formula (used in the third equality as follows), the left hand side of \textsuperscript{(15)} is
\[
\psi_n(z) \left( f(\psi_{n_2}(z)^{-1} m) \right) = \left( f \prod_{t=1}^{n_1} (z - \lambda_t - h_1)(z - \lambda_t - h_2)(z - \lambda_t - h_3)(z - \lambda_t + h_1)(z - \lambda_t + h_2)(z - \lambda_t + h_3) \right) (m),
\]
where the second last equality follows from Lemma \textsuperscript{6.1.3}. This completes the proof. \(\blacksquare\)

### 6.3 The Drinfeld coproduct

Let \(S\mathcal{H}^{\geq 0} := \mathcal{H}^0 \times S\mathcal{H}(Q_3, W_3)\) be the extended spherical COHA. In this section, we describe the Drinfeld coproduct on \(S\mathcal{H}^{\geq 0}\)
\[
\Delta : S\mathcal{H}^{\geq 0} \to (S\mathcal{H}^{\geq 0}) \hat\otimes (S\mathcal{H}^{\geq 0})
\]
following \textsuperscript{[88]}. The difference with \textsuperscript{[88]} is that the Cohomological Hall algebra \(H(Q_3, W_3)\) in the current paper is associated to a quiver with potential. For completeness we add the detailed formulas.

Similarly to \(\S 6.1\), we have the \(GL_n\)-equivariant vector bundle \(V_n := \mathfrak{g}l^3_n \times C^n\) on \(\mathfrak{g}l^3_n\), where \(C^n\) is the standard representation of \(GL_n\), and \(GL_n\) acts on \(\mathfrak{g}l^3_n \times C^n\) diagonally. Let \(\{\lambda_1, \lambda_2, \cdot \cdot \cdot, \lambda_n\}\) be the \(GL_n\)-equivariant Chern roots of \(V_n\).

Let \(A = \{1, \cdot \cdot \cdot , n\}, B = \{n + 1, \cdot \cdot \cdot , n + m\}\), and let \(h_1 + h_2 + h_3 = 0\). Set
\[
\text{fac}(\lambda_A | \lambda_B) := \prod_{s \in A} \prod_{t \in B} \frac{(\lambda_s - \lambda_t - h_1)(\lambda_s - \lambda_t - h_2)(\lambda_s - \lambda_t - h_3)}{\lambda_s - \lambda_t}.
\]

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The coproduct $\Delta$ on $\mathcal{SH}^{\geq 0}$ is determined as follows (see [SS] for details). For $\psi(z) \in \mathcal{H}^0[[z^{-1}]]$, and $P(\lambda_1, \cdots, \lambda_n) \in \mathcal{SH}^{(Q_3, W_3)}(n)$, we define

$$\Delta(\psi(z)) = \psi(z) \otimes \psi(z),$$

$$\Delta(P(\lambda_1, \cdots, \lambda_n)) = \sum_{a+b=n, A=[a, a+1, n]} \prod_{t \in B} \frac{\psi(\lambda_t)P(\lambda_A \otimes \lambda_B)}{\text{fac}(\lambda_B | \lambda_A)}$$

$$= \sum_{a+b=n} \prod_{t \in [a+1, n]} \prod_{t \in [a+1, n]} \psi(\lambda_t)P(\lambda)[1, a] \otimes \lambda_{[a+1, n]} \prod_{t \in [a+1, n]} (\lambda_t - \lambda_a - h_1)(\lambda_t - \lambda_a - h_2)(\lambda_t - \lambda_a - h_3)$$

By the same reason as in [SS], this formula defines a coproduct on $\mathcal{SH}^{\geq 0}$.

In particular, when $n = 1$, we have

$$\mathcal{H}^{(Q_3, W_3)}\mathcal{SH}(1) = H^*_{c, GL_1} \times T_2(\mathfrak{gl}^1, C)^\vee \cong C[h_1, h_2] \otimes C[\lambda].$$

In this case, for any $P(\lambda) \in \mathcal{H}^{(Q_3, W_3)}\mathcal{SH}(1)$, the Drinfeld coproduct is given by

$$\Delta(P(\lambda)) = \psi(\lambda) \otimes P(\lambda) + P(\lambda) \otimes 1.$$

### 6.4 The action of the extended 3d COHA

Let $\vec{r} = (r_1', r_2', r_3') \in \mathbb{Z}^3_{\geq 0}, \vec{r''} = (r_1'', r_2'', r_3'') \in \mathbb{Z}^3_{\geq 0}$. We denote the sum $\vec{r} + \vec{r''}$ by $\vec{r} = (r_1, r_2, r_3)$, and let $|\vec{r}| = r_1 + r_2 + r_3$. We write $n = n' + n'' \in \mathbb{Z}_{\geq 0}$, for any $n', n'' \in \mathbb{Z}_{\geq 0}$.

**Proposition 6.4.1** Let $\mathcal{SH}^{\geq 0} = \mathcal{H}^0 \ltimes \mathcal{SH}^{(Q_3, W_3)}$ be the extended spherical COHA.

1. There is a map of vector spaces $l : V_{\vec{r}} \to V_{\vec{r}'} \otimes V_{\vec{r}'},$ which is an isomorphism up to localization, such that

$$l(\alpha \bullet x) = \Delta(\alpha) \cdot l(x),$$

where $\alpha \in \mathcal{SH}^{\geq 0}$, $x \in V_{\vec{r}}$ and $\Delta : \mathcal{SH}^{\geq 0} \to (\mathcal{SH}^{\geq 0}) \otimes (\mathcal{SH}^{\geq 0})$ is the Drinfeld coproduct of $\mathcal{SH}^{\geq 0}$.

2. The two maps

$$\Delta^{\otimes |\vec{r}|} : \mathcal{SH}^{\geq 0} \to (\mathcal{SH}^{\geq 0}) \otimes (\mathcal{SH}^{\geq 0})$$

and

$$l^{\otimes |\vec{r}|} : V_{\vec{r}} \to (V_{1,0,0})^{\otimes r_1} \otimes (V_{0,1,0})^{\otimes r_2} \otimes (V_{0,0,1})^{\otimes r_3}$$

intertwine the actions of $\mathcal{SH}^{\geq 0}$ on $V_{\vec{r}}$ and $(\mathcal{SH}^{\geq 0}) \otimes (\mathcal{SH}^{\geq 0})$ on $(V_{1,0,0})^{\otimes r_1} \otimes (V_{0,1,0})^{\otimes r_2} \otimes (V_{0,0,1})^{\otimes r_3}$.

**Proposition 6.4.1** can be summarized into the following two commutative diagrams.

$$\begin{array}{ccc}
\mathcal{SH}^{\geq 0} \otimes V_{\vec{r}} & \to & V_{\vec{r}} \\
\Delta \times l & \downarrow & l \\
(\mathcal{SH}^{\geq 0} \otimes \mathcal{SH}^{\geq 0}) \otimes (V_{\vec{r}'} \otimes V_{\vec{r}'}) & \to & V_{\vec{r}'} \otimes V_{\vec{r}'}
\end{array}$$

and

$$\begin{array}{ccc}
\mathcal{SH}^{\geq 0} \otimes V_{\vec{r}} & \to & V_{\vec{r}} \\
\Delta^{\otimes |\vec{r}|} \times l^{\otimes |\vec{r}|} & \downarrow & l^{\otimes |\vec{r}|} \\
(\mathcal{SH}^{\geq 0}) \otimes (V_{1,0,0})^{\otimes r_1} \otimes (V_{0,1,0})^{\otimes r_2} \otimes (V_{0,0,1})^{\otimes r_3} & \to & ((V_{1,0,0})^{\otimes r_1} \otimes (V_{0,1,0})^{\otimes r_2} \otimes (V_{0,0,1})^{\otimes r_3})
\end{array}$$

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In the rest of this section we construct the map \( l : V \to V \otimes V \), in Proposition \[6.4.1\]. The proof of Proposition \[6.4.1\] will be given in Appendix \[3\].

### 6.4.1 A correspondence

Using the same notation as before, let \( V = \mathbb{C}^n \), \( E_{r_i} = \mathbb{C}^{r_i}, \ i = 1, 2, 3 \). We fix subspaces \( V_1 \subset V \), and \( (E_{r_i})_1 \subset E_{r_i}, \ i = 1, 2, 3 \), such that

\[
\dim(V_1) = n_1, \quad \dim(E_{r_i}) = r_i', \ i = 1, 2, 3.
\]

We have the short exact sequences of vector spaces

\[
0 \to V_1 \to V \to V_2 \to 0, \quad \text{and} \quad 0 \to (E_{r_i})_1 \to E_{r_i} \to (E_{r_i})_2 \to 0.
\]

Define the following subvariety of \( \mathcal{M}_r(n) \).

\[
Z_{(r', r'')} (n', n'') := \{ (B_i, I_{ab}, J_{ab}) \in \mathcal{M}_r(n) \mid B_i(V_1) \subset V_1, I_{ab}((E_{r_i})_1) \subset V_1, J_{ab}(V_1) \subset (E_{r_i})_1, i \in \mathbb{N}, \ \{ a, b, c \} = \{ 1, 2, 3 \} \}.
\]

For any element \( (B_i, I_{ab}, J_{ab}) \in Z_{(r', r'')} (n', n'') \), as \( B_i, I_{ab}, J_{ab} \) preserves the subspaces \( V_1 \subset V \), and \( (E_{r_i})_1 \subset E_{r_i} \), we have the restricted maps of the corresponding subspaces

\[
(B_1)_1 : V_1 \to V_1, \quad (I_{ab})_1 : (E_{r_i})_1 \to V_1, \quad (J_{ab})_1 : V_1 \to (E_{r_i})_1.
\]

We also have the induced maps on the corresponding quotients.

\[
(B_2)_2 : V_2 \to V_2, \quad (I_{ab})_2 : (E_{r_i})_2 \to V_2, \quad (J_{ab})_2 : V_2 \to (E_{r_i})_2.
\]

It gives rise to the following correspondence

\[
\mathcal{M}_r(n') \times \mathcal{M}_r(n'') \xrightarrow{p} Z_{(r', r'')} (n', n'') \xrightarrow{\eta} \mathcal{M}_r(n)
\]

The maps in (16) are given by

\[
p : (B_i, I_{ab}, J_{ab}) \mapsto \big((B_1)_1, (I_{ab})_1, (J_{ab})_1\big), \quad (B_2)_2, (I_{ab})_2, (J_{ab})_2\big),
\]

\[
\eta : (B_i, I_{ab}, J_{ab}) \mapsto (B_i, I_{ab}, J_{ab}) \text{ is the natural inclusion.}
\]

**Example 6.4.2** When \( r' = 0 \), we have \( r'' = r \). In this case, \( (E_{r_i})_1 = \{ 0 \} \), and \( Z_{(r', r'')} (n', n'') \) coincides with \( Z_p \) in the correspondence \[9\].

In general, we fix a flag \( F_1 \subset F_2 \subset V \) of linear subspaces of \( V \), with \( \dim(F_1) = n' \), and \( \dim(F_2) = n' + n'' \). Let \( F_3 := V/F_2 \) be the quotient with \( \dim(F_3) = n''' \). Therefore, \( \dim(V) = n = n' + n'' + n''' \), and we have the exact sequence:

\[
0 \to F_1 \to F_2 \to V \to F_3 \to 0
\]

Similarly, we fix a flag \( (E_{r_i})_1 \subset (E_{r_i})_2 \subset E_{r_i} \) of linear subspaces of \( E_{r_i} \), with \( \dim((E_{r_i})_1) = r_i' \), and \( \dim((E_{r_i})_2) = r_i' + r_i'' \). We have the exact sequence

\[
0 \to (E_{r_i})_1 \to (E_{r_i})_2 \to E_{r_i} \to (E_{r_i})_3 \to 0.
\]
Define the following subvariety of $\mathcal{M}_F(n)$.

$$F_{(\varphi, \varphi', \varphi'')} (n', n'', n''') := \{(B_i, I_{ab}, J_{ab}) \in \mathcal{M}_F(n) \mid i \in \{1, 2, 3\}, B_i(F_1) \subset F_1, I_{ab}((E_{r_i})_1) \subset F_1, J_{ab}(F_1) \subset (E_{r_i})_1, B_i(F_2) \subset F_2, I_{ab}((E_{r_i})_2) \subset F_2, J_{ab}(F_2) \subset (E_{r_i})_2\}$$

We have the commutative diagram

$$\begin{array}{ccc}
\mathcal{M}_F(n) & \xrightarrow{\cup_1} & F_{(\varphi, \varphi', \varphi'')} (n', n'', n''') \\
p_{1,23}^{-1} \downarrow & & \downarrow p_{1,23}^{-1} \\
\mathcal{M}_F(n') \times Z_{(\varphi, \varphi', \varphi'')} (n', n'', n''') & \xrightarrow{p} & Z_{(\varphi, \varphi', \varphi'')} (n', n'') \times \mathcal{M}_F(n''') \\
p_{23}^{-1} \downarrow & & \downarrow p_{12}^{-1} \\
\mathcal{M}_F(n') \times \mathcal{M}_F(n'') \times \mathcal{M}_F(n''') & \xrightarrow{p} & \mathcal{M}_F(n') \times \mathcal{M}_F(n'') \times \mathcal{M}_F(n''')
\end{array}$$

where $p_{1,23}^{-1}$ is obtained by restriction to the subspace $F_1$, and projection to the quotient $V/F_1$. The map $p_{23}^{-1}$ is defined using the short exact sequence

$$0 \to F_2/F_1 \to V/F_1 \to F_3 \to 0.$$

The map $p_{12}^{-1}$ is obtained by restriction to the subspace $F_2$, and projection to the quotient $F_3$. The map $p_{23}^{-1}$ is defined using the short exact sequence

$$0 \to F_1 \to F_2 \to F_2/F_1 \to 0.$$

### 6.4.2 The definition

**Lemma 6.4.3**

1. **Notations are as in [16].** We have

$$p^{-1}(\mathcal{M}_F(n')^{st} \times \mathcal{M}_F(n'')^{st}) \subset \eta^{-1}(\mathcal{M}_F(n)^{st}) \subset p^{-1}(\mathcal{M}_F(n') \times \mathcal{M}_F(n'')^{st}).$$

2. **Notations are as in [17].** We have

$$p^{-1}(\mathcal{M}_F(n')^{st} \times \mathcal{M}_F(n'')^{st} \times \mathcal{M}_F(n''')^{st}) \subset \eta^{-1}(\mathcal{M}_F(n)^{st})$$

**Proof.** We show the statement (1). For any element $(B_i, I_{ab}, J_{ab}) \in \mathcal{M}_F(n')^{st} \times \mathcal{M}_F(n'')^{st}$, by definition,

$$C((\tilde{B})_1) Im((\tilde{I})_1) = V_1,$$

$$C((\tilde{B})_2) Im((\tilde{I})_2) = V_2.$$

Consider the following commutative diagram

$$\begin{array}{ccc}
0 & \xrightarrow{0} & V_1 & \xrightarrow{0} & V_2 & \xrightarrow{0} & 0 \\
\downarrow (\tilde{I}_1) & & \downarrow \bar{F} & & \downarrow (\tilde{I}_2) & & \\
0 & \xrightarrow{0} & \bar{E}_1 & \xrightarrow{0} & \bar{E} & \xrightarrow{0} & \bar{E}_2 & \xrightarrow{0}
\end{array}$$
where $\vec{E} = (E_{r_1}, E_{r_2}, E_{r_3})$, $\vec{E}_1 = ((E_{r_1})_1, (E_{r_2})_1, (E_{r_3})_1)$, and $\vec{E}_2$ is the corresponding quotient.

For any $v \in V$, let $v_2 \in V_2$ be the image of $v$ under $\pi$. By Definition 6.4.5, there exists a function $f$, and a vector $\vec{e}_2 \in \vec{E}_2$ such that $f((\vec{B})_2)(\vec{I})_2(\vec{e}_2) = v_2$.

Let $\vec{c} \in \vec{E}$ be any lifting of the vector $\vec{e}_2 \in \vec{E}_2$. Consider the element

$$f(\vec{B})\vec{I}(\vec{c}) \in V.$$

We have $\pi(f(\vec{B})\vec{I}(\vec{c})) = f((\vec{B})_2)\vec{I}_2(\vec{e}_2) = v_2$ by the commutativity of diagram (29). Therefore, there exists an element $v_1 \in V_1$, such that $v = v_1 + f(\vec{B})\vec{I}(\vec{c})$. By condition (18), there exists a function $g$, and a vector $\vec{e}_1 \in \vec{E}_1$ such that $g((\vec{B})_1)(\vec{I})_1(\vec{e}_1) = v_1$. Therefore,

$$v = g((\vec{B})_1)(\vec{I})_1(\vec{e}_1) + f(\vec{B})\vec{I}(\vec{c}) = g((\vec{B})_1)(\vec{I})_1(\vec{e}_1) + f(\vec{B})\vec{I}(\vec{c}).$$

This implies the inclusion $p^{-1}(\mathcal{M}_{\vec{r}}(n')^{st} \times \mathcal{M}_{\vec{r}'}(n'')^{st}) \subset \eta^{-1}(\mathcal{M}_{\vec{r}}(n)^{st})$.

The inclusion $\eta^{-1}(\mathcal{M}_{\vec{r}}(n)^{st}) \subset p^{-1}(\mathcal{M}_{\vec{r}}(n') \times \mathcal{M}_{\vec{r}'}(n'')^{st})$ is clear. Indeed, by projecting to the quotient spaces $\vec{E}_2, V_2$, the condition $C(\vec{B})\text{Im}(\vec{I}) = V$ implies $C(\vec{B})\text{Im}(\vec{I}_2) = V_2$.

The assertion (2) follows from a similar argument. This completes the proof. 

\textbf{Definition 6.4.4}

1. When $\vec{E}_1 = 0$, and $V_1 \neq 0$, we have $\mathcal{M}_{\vec{r}'}(n')^{st} = \emptyset$, as $C(\vec{B})\text{Im}(\vec{I}_1) = \{0\} \neq V_1$. In this case, we define $Z(\vec{r}, \vec{r}')^{st}(n', n'')^{st} := \eta^{-1}(\mathcal{M}_{\vec{r}}(n)^{st})$.

Similarly, we define $F(\vec{r}, \vec{r}', \vec{r}'')^{st}(n', n'', n''')^{st} := p^{-1}(\mathcal{M}_{\vec{r}}(n') \times \mathcal{M}_{\vec{r}'}(n'')^{st} \times \mathcal{M}_{\vec{r}''}(n''')^{st}) \cap \eta^{-1}(\mathcal{M}_{\vec{r}}(n)^{st})$.

2. Otherwise, we define $Z(\vec{r}, \vec{r}')^{st}(n', n'')^{st} := p^{-1}(\mathcal{M}_{\vec{r}}(n')^{st} \times \mathcal{M}_{\vec{r}'}(n'')^{st})$.

Similarly, we define $F(\vec{r}, \vec{r}', \vec{r}'')^{st}(n', n'', n''')^{st} := p^{-1}(\mathcal{M}_{\vec{r}}(n')^{st} \times \mathcal{M}_{\vec{r}'}(n'')^{st} \times \mathcal{M}_{\vec{r}''}(n''')^{st})$.

This gives rise to the following correspondence of $L$-varieties.

\begin{equation}
\begin{array}{ccc}
\mathcal{M}_{\vec{r}}(n')^{st} \times \mathcal{M}_{\vec{r}'}(n'')^{st} & \overset{p^{st}}{\longrightarrow} & Z(\vec{r}, \vec{r}')^{st}(n', n'')^{st} \\
\downarrow & & \downarrow \\
\mathcal{M}_{\vec{r}}(n') \times \mathcal{M}_{\vec{r}'}(n'') & \overset{p}{\longrightarrow} & Z(\vec{r}, \vec{r}')^{st}(n', n'') \\
\downarrow & & \downarrow \\
\mathcal{M}_{\vec{r}}(n') & \overset{\eta^{st}}{\longrightarrow} & \mathcal{M}_{\vec{r}}(n)
\end{array}
\end{equation}

\textbf{Definition 6.4.5} Passing to the localization of $V_{\vec{r}}$ and $V_{\vec{r}'} \otimes V_{\vec{r}''}$, we define the map $l : V_{\vec{r}} \to V_{\vec{r}'} \otimes V_{\vec{r}''}$ in Proposition 6.4.4 as the inverse of the following morphism:

$$\eta^{st} \circ (p^{st})^* : V_{\vec{r}'} \otimes V_{\vec{r}''} \to V_{\vec{r}}.$$

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6.5 Nakajima-type operators

Let $\mathcal{M}_r(n)$ be the quotient $\mathcal{M}_r(n)^{st} / \text{GL}_n$. Consider the subvariety $C^+_r(n-1,n) \subset \mathcal{M}_r(n-1) \times \mathcal{M}_r(n)$ consisting of the pairs $(V_2,V) \in \mathcal{M}_r(n-1) \times \mathcal{M}_r(n)$, such that $V_2$ is a quotient representation of $V$. There is a tautological line bundle $L$ on $C^+_r(n-1,n)$ which classifies the one-dimensional sub-representation $V_1 \subset V$.

To be more explicitly, recall in (9), we have the correspondence

$$\text{gl}_1^3 \times \mathcal{M}_r(n-1)^{st} \xrightarrow{p^{st}} Z^st_p \xrightarrow{\eta^{st}} \mathcal{M}_r(n)^{st}$$

where

$$Z^st_p = \{(B_i, I_{ab}, J_{ab}) \in \mathcal{M}_r(n) \mid B_i(V_1) \subset V_1, J_{ab}(V_1) = 0, \sum_{ab \in \mathbb{I}} C(B_1, B_2, B_3) \text{Im}(I_{ab}) = V\}.$$

The subvariety $C^+_r(n-1,n)$ is obtained by the quotient $Z^st_p / P$.

We have the following diagram

$$
\begin{array}{ccc}
C^+_r(n-1,n) & \subset & \mathcal{M}_r(n-1) \\
\mathcal{M}_r(n-1) & \xrightarrow{p^1} & \mathcal{M}_r(n).
\end{array}
$$

Denote by $p_1 : C^+_r(n-1,n) \to \mathcal{M}_r(n-1), p_2 : C^+_r(n-1,n) \to \mathcal{M}_r(n)$ the composition of the inclusion with the natural projections.

We also have the following commutative diagram

$$
\begin{array}{ccc}
G \times _P (\text{gl}_1^3 \times \mathcal{M}_r(n-1)^{st}) & \xleftarrow{p^{st}} & G \times _P Z^st_p \\
\mathcal{M}_r(n-1) & \xrightarrow{p^1} & C^+_r(n-1,n) \\
& & \xrightarrow{p^2} \mathcal{M}_r(n)
\end{array}
$$

where the vertical maps are obtained by quotient by $G$. Let $L$ be the tautological line bundle on $C^+_r(n-1,n)$. Let $\Psi(f(c_1(L))) \in \text{End}(V_{r_1,r_2,r_3})$ be the raising operation given by convolution with $f(c_1(L))$. In other words, let $\alpha \in V_{r_1,r_2,r_3}(n-1) \cong H_{G \times \mathbb{C}^2} \mathcal{M}_r(n-1), \varphi(W_{r,\alpha})$, 

$$\Psi(f(c_1(L)))(\alpha) := p_{2\times}(f(c_1(L)) \cdot p^1(\alpha)) \in V_{r_1,r_2,r_3}(n).$$

The action of the spherical COHA $\mathcal{SH}^{(Q_3, W_3)}$ on $V_{r_1,r_2,r_3}$ can be described in terms of the above Nakajima operators.

**Proposition 6.5.1** For any $f(\lambda) \in \mathbb{C}[\lambda]$, view $f(\lambda) \in \mathcal{SH}^{(Q_3, W_3)}(1) \cong \mathbb{C}[\lambda]$, we have the equality

$$\Psi(f(c_1(L))) = \Phi(f(\lambda))$$

in $\text{End}(V_{r_1,r_2,r_3})$, where $\Phi$ is the action of the 3d spherical COHA $\mathcal{SH}^{(Q_3, W_3)}$.

**Proof:** By the same proof as in [56] Theorem 5.6. □
7 The Drinfeld double of COHA and its action

We show the extended spherical COHA $\mathcal{SH}^{\geq 0}$ is isomorphic to the Borel subalgebra of the affine Yangian $Y_{h_1, h_2, h_3}(\widehat{\mathfrak{gl}(1)})$. When the framed quiver variety has a 2d description, the action of $Y_{h_1, h_2, h_3}(\widehat{\mathfrak{gl}(1)})$ on its cohomology has been known in literature. In this special case, the action of $Y_{h_1, h_2, h_3}(\widehat{\mathfrak{gl}(1)})$ is compatible with the action of $\mathcal{SH}^{\geq 0}$ constructed in Section 6. In the current section, we prove that for a general framed quiver variety (possibly without a 2d description) the entire Yangian $Y_{h_1, h_2, h_3}(\widehat{\mathfrak{gl}(1)})$ acts on its cohomology.

7.1 The affine Yangian

We review some facts of the affine Yangian of $\widehat{\mathfrak{gl}(1)}$ (see [85] for details). Let $h_1, h_2, h_3$ be formal parameters satisfying $h_1 + h_2 + h_3 = 0$. The affine Yangian $Y_{h_1, h_2, h_3}(\widehat{\mathfrak{gl}(1)})$ is an associative algebra, generated by the variables $\{e_j, f_j, \psi_j \mid j \in \mathbb{N}\}$ with the following defining relations.

$$[\psi_i, \psi_j] = 0, \quad (Y0)$$
$$[e_{i+3}, e_j] - 3[e_{i+2}, e_{j+1}] + 3[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}] + \sigma_2([e_{i+1}, e_j] - [e_i, e_{j+1}]) = -\sigma_3(e_i, e_j). \quad (Y1)$$
$$[f_{i+3}, f_j] - 3[f_{i+2}, f_{j+1}] + 3[f_{i+1}, f_{j+2}] - [f_i, f_{j+3}] + \sigma_2([f_{i+1}, f_j] - [f_i, f_{j+1}]) = \sigma_3(f_i, f_j). \quad (Y2)$$
$$[e_i, f_j] = \psi_{i+j} \quad (Y3)$$
$$[\psi_{i+3}, e_j] - 3[\psi_{i+2}, e_{j+1}] + 3[\psi_{i+1}, e_{j+2}] - [\psi_i, e_{j+3}] + \sigma_2([\psi_{i+1}, e_j] - [\psi_i, e_{j+1}]) = -\sigma_3(\psi_i, e_j). \quad (Y4)$$
$$[\psi_0, e_j] = 0, [\psi_1, e_j] = 0, [\psi_2, e_j] = 2e_j, \quad (Y4')$$
$$[\psi_{i+3}, f_j] - 3[\psi_{i+2}, f_{j+1}] + 3[\psi_{i+1}, f_{j+2}] - [\psi_i, f_{j+3}] + \sigma_2([\psi_{i+1}, f_j] - [\psi_i, f_{j+1}]) = \sigma_3(\psi_i, f_j) \quad (Y5)$$
$$[\psi_0, f_j] = 0, [\psi_1, f_j] = 0, [\psi_2, f_j] = -2f_j, \quad (Y5')$$
$$\text{Sym}_{e_3}[e_i, [e_{i+2}, e_{i+1}]] = 0, \quad \text{Sym}_{e_3}[f_{i+1}, [f_{i+2}, f_{i+1}]] = 0. \quad (Y6)$$

where $\sigma_2 = h_1 h_2 + h_2 h_3 + h_1 h_3$, and $\sigma_3 = h_1 h_2 h_3$.

Let $Y^0$, $Y^+$ be the subalgebra generated by $\{f_j\}$, $\{\psi_j\}$ and $\{e_j\}$ respectively. Let $Y^{\geq 0}$, $Y^{\leq 0}$ be the subalgebras generated by $Y^0, Y^+$, and $Y^-, Y^0$ respectively. The following properties can be found in [85] Proposition 1.4.

1. $Y^0$ is a polynomial algebra in the generators $\{\psi_j\}$.
2. $Y^-$ and $Y^+$ are the algebras generated by $\{f_j\}$ and $\{e_j\}$ with the defining relations $(Y2), (Y6)$, and $(Y1), (Y6)$.
3. $Y^{\leq 0}$ and $Y^{\geq 0}$ are the algebras generated by $\{\psi_j, f_j\}$ and $\{\psi, e_j\}$ with the defining relations $(Y0), (Y2), (Y5), (Y5'), (Y6)$, and $(Y0), (Y1), (Y4), (Y4'), (Y6)$.
4. Multiplication induces an isomorphism of vector spaces

$$m: Y^- \otimes Y^0 \otimes Y^+ \to Y_{h_1, h_2, h_3}(\widehat{\mathfrak{gl}(1)}).$$
We compare the Borel subalgebra $Y^* \geq 0$ of the affine Yangian $Y_{h_1,h_2,h_3}(\hat{\mathfrak{gl}(1)})$ with the extended spherical COHA $\mathcal{SH}^* \geq 0$. Thanks to [79, 85], when two of the three coordinates of $r' = (r_1, r_2, r_3)$ are zero, the entire affine Yangian $Y_{h_1,h_2,h_3}(\hat{\mathfrak{gl}(1)})$ has an action on $V_{r_1,r_2,r_3}$. For instance, let $r_1 = r_2 = 0, r_3 = 1$. Using the dimension reduction in [113], we have an isomorphism $V_{0,0,1} \cong \bigoplus_{n \in \mathbb{N}} H^*(\text{Hilb}^n(\mathbb{C}^2), \mathbb{C})$, which carries an affine Yangian action [6, 79, 85].

**Theorem 7.1.1**

1. There is an algebra homomorphism
   \[
   \Psi : Y^+ \rightarrow \mathcal{H}(Q_3, W_3)(1) \cong \mathbb{C}[\lambda]
   \]
   Furthermore, it induces an isomorphism $Y^+ \cong \mathcal{H}(Q_3, W_3)$.

2. The isomorphism in (1) extends to an isomorphism $Y^* \cong \mathcal{SH}^* \geq 0$. In particular, $Y^* \geq 0$ acts on $V_{r_1,r_2,r_3}$ for any $(r_1, r_2, r_3) \in \mathbb{N}^3$.

3. The isomorphism from (2) intertwines the Drinfeld coproduct on $Y^* \geq 0$ and that on $\mathcal{SH}^* \geq 0$ from § 6.3.

4. When $r_a = r_b = 0$ and $r_c = r$, such that \{a, b, c\} = \{1, 2, 3\}, the action of $Y^* \geq 0$ on $V_r$ from (2) is compatible with the actions constructed in [79, 85].

In Section 6 we have constructed an action of $\mathcal{SH}^* \geq 0$ on $V_{r_1,r_2,r_3}$ for any $(r_1, r_2, r_3) \in \mathbb{N}^3$. The isomorphism in Theorem 7.1.1(2) gives an action of $Y^* \geq 0$ on $V_{r_1,r_2,r_3}$ for any $(r_1, r_2, r_3) \in \mathbb{N}^3$. Theorem 7.1.1(4) shows, in the case when two of the three coordinates in $r' = (r_1, r_2, r_3)$ are zero, that this is compatible with the actions in [79, 85]. Using this result, we will show in Section 7.3 a stronger statement that the whole Yangian $Y_{h_1,h_2,h_3}(\hat{\mathfrak{gl}(1)})$ acts on $V_{r_1,r_2,r_3}$ for any $(r_1, r_2, r_3) \in \mathbb{N}^3$.

We prove Theorem 7.1.1(1) and (4) in [72] and Appendix A. The statement (3) follows directly by comparing the Drinfeld coproduct formulas on $Y^* \geq 0$ and $\mathcal{SH}^* \geq 0$. We now assume Theorem 7.1.1(1), and prove Theorem 7.1.1(2).

**Proof of Theorem 7.1.1(2).** We need to show that the action of $\mathcal{H}^0$ on the spherical COHA [13] is compatible with the relation (Y4) of $Y_{h_1,h_2,h_3}(\hat{\mathfrak{gl}(1)})$.

It is shown in [85] Proposition 1.5 that the defining relation (Y4) of $Y_{h_1,h_2,h_3}(\hat{\mathfrak{gl}(1)})$ is equivalent to
\[
P(z, \sigma^+)\psi(z)e_j + P(\sigma^+, z)e_j \otimes \psi(z) = 0, j \in \mathbb{N},
\]
where $\sigma^+$ is the shifting operators
\[
\sigma^+ : Y^* \rightarrow Y^* \text{ determined by } \psi_j \mapsto \psi_j, e_j \mapsto e_{j+1}.
\]
and $P(z, w) := (z - w + h_1)(z - w + h_2)(z - w + h_3)$. In our case, we take $f$ in [13] to be $e_j = \lambda^j \in \mathcal{H}(Q_3, W_3)(1) \cong \mathbb{C}[\lambda]$. Then, the relation (13) becomes
\[
\psi(z)\lambda^j\psi(z)^{-1} = \frac{(z - \lambda - h_1)(z - \lambda - h_2)(z - \lambda - h_3)}{(z - \lambda + h_1)(z - \lambda + h_2)(z - \lambda + h_3)}\lambda^j
\]
\[
= \frac{(z - \sigma^+ - h_1)(z - \sigma^+ - h_2)(z - \sigma^+ - h_3)}{(z - \sigma^+ + h_1)(z - \sigma^+ + h_2)(z - \sigma^+ + h_3)}\lambda^j
\]
\[
= -\frac{P(\sigma^+, z)}{P(z, \sigma^+)}\lambda^j,
\]
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which is equivalent to $P(z, \sigma^+)\psi(z)\lambda^j = -P(\sigma^+, z)\lambda^j\psi(z)$. This completes the proof of Theorem 7.1.1(2). □

7.2 Review of the 2d COHA

Let $J$ be the Jordan quiver. Recall the details of the 2d COHA of $J$ a.k.a preprojective COHA (see §78 [79, 86]). In this section, we recall the definition and a few facts of the 2d COHA which has been discussed briefly in §2. Consider the commuting variety

$$C_n = \{B_1, B_2 \in \mathfrak{gl}_n^2 \mid [B_1, B_2] = 0\}.$$

The two dimensional torus $T_2$ acts on $C_n$ by

$$(t_1, t_2) \cdot (B_1, B_2) = (t_1B_1, t_2B_2). \quad (23)$$

This action gives rise to the quotient stack. The 2d COHA is the cohomology of this stack endowed with the natural Hall multiplication (see §79 [86]). As we have already mentioned the 2d COHA of any quiver is the dimensional reduction of a 3d COHA of the triple quiver endowed with the cubic potential. By definition, the 2d COHA of $J$ as a graded vector space is isomorphic to

$$H_{B_3=0}^{(Q_3, W_3), T_2}(C_n) := \bigoplus_{n \in \mathbb{N}} H_{BM}^{\mathfrak{gl}_n \times T_2}(C_n),$$

Here the notation stresses two facts. First, the 2d COHA is the dimensional reduction to the plane $B_3 = 0$ of the 3d COHA $H^{(Q_3, W_3)}$ and second, equivariance with respect to the torus $T_2$.

The above-mentioned associative product coincides with the one obtained via the dimensional reduction from the $T_2$-equivariant 3d COHA (see Appendix of [77] and [87]).

**Remark 7.2.1** There is no special notation for a 2d COHA. In the case of the Jordan quiver $J$ the 2d COHA was denoted by $C'$ in §78 [79]. The notation $P(B_{BM}, Q)$ was used in §80 for a general quiver $Q$. Since all of those are just special cases of a more general notion of 3d COHA $H^{(Q, W)}$, we use the notation which also explains how the two are related in the case of the quiver $J$.

The obvious inclusion $C_n \subset \mathfrak{gl}_n \times \mathfrak{gl}_n$ gives an $H_{BM}^{\mathfrak{gl}_n \times T_2}$-module homomorphism

$$(H_{B_3=0}^{(Q_3, W_3), T_2})_n \rightarrow H_{BM}^{\mathfrak{gl}_n \times T_2}(\mathfrak{gl}_n \times \mathfrak{gl}_n),$$

which gives an isomorphism of the $R$-algebra $\bigoplus_{n \geq 0} (H_{B_3=0}^{(Q_3, W_3), T_2})_n$ with its image. More precisely, we have a surjective $R$-algebra homomorphism of $(H_{B_3=0}^{(Q_3, W_3), T_2})_n$ onto its image. This map was conjectured to be an isomorphism in §80 Conjecture 4.4. The conjecture was later proved in [81] Proposition 4.6]. We will identify these algebras and call either of them the 2-dimensional COHA of the quiver $J$.

The relation between the 2d COHA of $J$ and the 3d COHA of $(Q_3, W_3)$ can be made very explicit in the following way.

**Theorem 7.2.2** ([77] Appendix [87] Theorem 4.1) Assume the $T_2$-action on $C_n$ is given by (23), and on $\mathfrak{gl}_n^2$ by (11). There is an isomorphism of $\mathbb{N}$-graded associative algebras $\Xi : H_{B_3=0}^{(Q_3, W_3), T_2} \rightarrow H^{(Q_3, W_3), T_2}$ whose restriction to the degree-$n$ piece is

$$\Xi : H_{B_3=0}^{(Q_3, W_3), T_2} \rightarrow (H^{(Q_3, W_3), T_2})_n, \quad f \mapsto (-1)^{j(2)} f.$$
As a consequence, Theorem 5.1.1 implies the following.

**Corollary 7.2.3** The 2d COHA $\mathcal{H}^{(Q_3,W_3),T_2}$ acts on $V_{r_1,r_2,r_3}$, for any $(r_1,r_2,r_3) \in \mathbb{N}^3$.

The 2d COHA $\mathcal{H}^{(Q_3,W_3),T_2}$ has an explicit shuffle description (see [79 §4.4] and also [SH]) which we now recall. It can be used to compare the affine Yangian with the 3d COHA in Appendix A. The shuffle algebra $\text{Sh}$ is an $\mathbb{N}$-graded $\mathbb{C}[h_1,h_2]$-algebra. As a $\mathbb{C}[h_1,h_2]$-module, we have

$$\text{Sh} = \bigoplus_{n \in \mathbb{N}} \text{Sh}_n = \bigoplus_{n \in \mathbb{N}} \mathbb{C}[h_1,h_2][\lambda_1,\ldots,\lambda_n]^{S_n}.$$  

For any $n$ and $m \in \mathbb{N}$, we consider $\text{Sh}_n \otimes \text{Sh}_m$ as a subspace of $\mathbb{C}[h_1,h_2][\lambda_1,\ldots,\lambda_{n+m}]$ by sending $\lambda'_s$ to $\lambda_s$, and $\lambda''_t$ to $\lambda_{t+n}$. Set:

$$\text{fac} := \prod_{s=1}^{n} \prod_{t=1}^{m} \frac{(\lambda'_s - \lambda''_t - h_1)(\lambda'_s - \lambda''_t - h_2)(\lambda'_s - \lambda''_t - h_3)}{\lambda'_s - \lambda''_t}$$

(24)

where $h_1 + h_2 + h_3 = 0$.

Let $\text{Sh}(n,m)$ be the shuffle of $n,m$. The multiplication of $f_1(\lambda_1,\ldots,\lambda_n) \in \text{Sh}_n$ and $f_2(\lambda_1,\ldots,\lambda_m) \in \text{Sh}_m$ is defined to be

$$\sum_{\sigma \in \text{Sh}(n,m)} \sigma(f_1 \cdot f_2 \cdot \text{fac}) \in \mathbb{C}[h_1,h_2][\lambda_1,\ldots,\lambda_{n+m}]^{S_{n+m}}.$$  

(25)

**Theorem 7.2.4** [79, Theorem 4.7] There is exists a unique $\mathbb{C}[h_1,h_2]$-algebra embedding $\text{Sh}^{(Q_3,W_3),T_2} \rightarrow \text{Sh}$, where $\text{Sh}^{(Q_3,W_3),T_2}$ is the spherical subalgebra of $\mathcal{H}^{(Q_3,W_3),T_2}$ generated by the first graded component $(\mathcal{H}^{(Q_3,W_3),T_2})_1$.

We remark that in [79] the algebra $\text{Sh}^{(Q_3,W_3),T_2}$ is denote by $\text{SC}$.

In Appendix A, we prove Theorem 7.1.1 (1) by constructing an algebra homomorphism from $Y_{h_1,h_2,h_3}(\widehat{\mathfrak{gl}(1)})$ to $\text{Sh}$. This will induce an algebra isomorphism $Y_{h_1,h_2,h_3}(\widehat{\mathfrak{gl}(1)}) \cong \text{SH}^{(Q_3,W_3),T_2}$. The latter is isomorphic to $\text{SH}^{(Q_3,W_3),T_2}$ by Theorem 7.2.2. This proves Theorem 7.1.1 (1).

When $r_a = r_b = 0$ and $r_c = r$, such that $\{a,b,c\} = \{1,2,3\}$. The action of the positive subalgebra $Y^+$ on $V_r$ is defined via Nakajima raising operators. Theorem 7.1.1 (4) follows from Proposition 6.5.1.

### 7.3 Action of the whole algebra $Y_{h_1,h_2,h_3}(\widehat{\mathfrak{gl}(1)})$

In this subsection, we show the “creation operators” and “annihilation operators” of the 3d COHA can be glued to give rise to an action of the double COHA $D(\text{SH}^{(Q_3,W_3)})$ on $V_{r_1,r_2,r_3}$ for any $(r_1,r_2,r_3) \in \mathbb{N}$. This is shown using the affine Yangian action on the 2d framed quiver varieties and the compatibility result in Proposition 6.4.1.

On each free boson space $V_{1,0,0}, V_{0,1,0}, V_{0,0,1}$, we have constructed an action of the 3d COHA $\text{SH}^{\geq 0} \cong Y^{\geq 0}$. By Proposition 6.5.1, this action of $\text{SH}^{(Q_3,W_3)}$ can be described in terms of Nakajima raising operators. Similarly, we could define an action of $\text{SH}^{\leq 0} \cong Y^{\leq 0}$.
using the lowering operators. More explicitly, as notations in [6,5] let \( \beta \in V_{r_1,r_2,r_3}((n) \cong H_{c,G,W_2}(\Omega(n),\varphi_{(W/r_n)}))' \), the convolution \( p_1(\tilde{f}(c_1(\mathcal{L}) \cdot p_2^*(\beta))) \in V_{r_1,r_2,r_3}((n-1) \) gives an action of \( \tilde{f}((\lambda) \in SH^{(Q_3,W_3)}(1) = \mathbb{C}[\lambda] \) on \( V_{r_1,r_2,r_3} \), for any \( r_1,r_2,r_3 \in \mathbb{N} \)

Theorem 7.1.1 together with Proposition 6.4.1 implies the following. Let \( Y^{\geq 0} \otimes Y^{\leq 0} \) be the free non-commutative tensor product of the algebras \( Y^{\geq 0} \) and \( Y^{\leq 0} \).

**Corollary 7.3.1** The action of \( Y^{\geq 0} \otimes Y^{\leq 0} \) on \( V_{r_1,r_2,r_3} \) factors through \( Y_{h_1,h_2,h_3}(\mathfrak{gl}_1) \).

\[
\begin{array}{ccc}
Y^{\geq 0} \otimes Y^{\leq 0} & \longrightarrow \text{End}(V_{r_1,r_2,r_3}) \\
& \downarrow \\
Y_{h_1,h_2,h_3}(\mathfrak{gl}_1)
\end{array}
\]

**Proof.** It is known that the action of \( Y^{\geq 0} \otimes Y^{\leq 0} \) on each free boson space \( V_{0,0,0}, V_{0,1,0}, V_{0,0,1} \) factors through \( Y_{h_1,h_2,h_3}(\mathfrak{gl}(1)) \), and the coproduct \( \Delta : Y^{\geq 0} \otimes Y^{\leq 0} \rightarrow Y_{h_1,h_2,h_3}(\mathfrak{gl}(1)) \otimes Y_{h_1,h_2,h_3}(\mathfrak{gl}(1)) \) factors through \( Y_{h_1,h_2,h_3}(\mathfrak{gl}(1)) \). Up to localization, the map (see Definition 6.4.5)

\[
\phi : V_{r_1,r_2,r_3} \rightarrow (V_{0,0,0})^{\otimes r_1} \otimes (V_{0,1,0})^{\otimes r_2} \otimes (V_{0,0,1})^{\otimes r_3}
\]

is an isomorphism. Therefore, the action of \( Y^{\geq 0} \otimes Y^{\leq 0} \) on \( V_{r_1,r_2,r_3} \) also factors through \( Y_{h_1,h_2,h_3}(\mathfrak{gl}(1)) \). This completes the proof. \( \blacksquare \)

As a consequence, the actions of \( SH^{\geq 0} \) and \( SH^{\leq 0} \) on \( V_{r_1,r_2,r_3} \) satisfy the commutation relations of the Drinfeld double \( D(SH^{(Q_3,W_3)}) \cong Y_{h_1,h_2,h_3}(\mathfrak{gl}(1)) \). Thus, we have an action of \( D(SH^{(Q_3,W_3)}) \) on \( V_{r_1,r_2,r_3} \), for any \( r_1,r_2,r_3 \in \mathbb{N} \).

**Remark 7.3.2** Closely related to the affine Yangian, there is an algebra called the deformed current algebra, which deforms the enveloping algebra of the double current algebra \( \mathfrak{gl}_N \otimes \mathbb{C}[u,v] \). Let \( D_{\epsilon,\delta}(\mathfrak{gl}_N) \) denote the deformed current algebra. It has been conjectured by K. Costello [13, Section 1.8] that \( D_{\epsilon,\delta}(\mathfrak{gl}_N) \) should act on the equivariant Donaldson-Thomas theory of \( \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_N \), where \( \mathbb{C}^2/\mathbb{Z}_N \) is the minimal resolution of \( \mathbb{C}^2/\mathbb{Z}_N \). Furthermore, the 3d COHA associated to the 3-fold \( \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_N \) embeds into \( D_{\epsilon,\delta}(\mathfrak{gl}_N) \), such that the action of \( D_{\epsilon,\delta}(\mathfrak{gl}_N) \) restricts to the natural action of the 3d COHA on the same space. From the physics perspective, [12] probes the configuration of a single D6-brane wrapping the whole \( \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_N \). This configuration is expected to lead to a “vacuum” module of the involved affine Yangian (or the deformed double current algebra).

In particular, when \( N = 1 \), the expectation of [13, Section 1.8] is that \( D_{\epsilon,\delta}(\mathfrak{gl}_1) \) acts on \( \bigoplus_{n \in \mathbb{N}} H^*_c(\text{Hilb}_n(\mathbb{C}^3), \varphi_{W_3^{fr}}) \). Note that \( \text{Hilb}_n(\mathbb{C}^3) \) is obtained from the representations \( M_{0,0,1}(n) \) of the quiver with potential \( (Q_3^{fr}, W_3^{fr}) \) using the stability condition [4] without imposing the relations

\[
B_3I_{12} = 0, B_1I_{23} = 0, B_2I_{13} = 0,
\]

i.e. considering the moduli space associated to a single D6-brane. This configuration is expected to lead to the MacMahon module of the affine Yangian of \( \mathfrak{gl}_1 \) or the corresponding deformed double current algebra. We remark that the space \( \bigoplus_{n \in \mathbb{N}} H^*_c(\text{Hilb}_n(\mathbb{C}^3), \varphi_{W_3^{fr}}) \) is different from the space \( V_{r_1,r_2,r_3} \) considered in the present paper.

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8 Free field realization

In this section, we first introduce a central extension $SH^c$ of the affine Yangian by a polynomial ring $C_{c_1(1),c_1(2),c_1(3)}: l \geq 0$. This algebra $SH^c$ also acts on $V_{r_1,r_2,r_3}$ such that the central elements $\vec{c} = \{c_1(1),c_1(2),c_1(3) : l \geq 0\}$ action depends on the framing vector $\vec{r} = (r_1,r_2,r_3)$. We construct a coproduct $\Delta^c$ (different from the Drinfeld coproduct) on $SH^c$. When $c_1(2) = c_1(3) = 0$, the algebra $SH\{c_1(1),0,0\}$ and the coproduct $\Delta^c(1,0,0)$ coincide with the algebra $SH^c$ and coproduct $\Delta$ introduced by Schiffmann–Vasserot [79].

Recall the quotient $M_{s,\ell}(n) / GL_n$ is denoted by $M_{s,\ell}(n)$. Geometrically, for any 1-dimensional subgroup $A$ of the maximal torus of $GL_{r_1} \times GL_{r_2} \times GL_{r_3}$, the fixed point locus $(M_{s,\ell})^A$ is a product of framed quiver varieties of the form $\prod_{k} M_{s,\ell}$ with $\sum_k r_k = \vec{r}$.

For any decomposition $\vec{r} = \vec{r}^f + \vec{r}^h$, by choosing appropriate torus $A$ in $GL_{r_1} \times GL_{r_2} \times GL_{r_3}$, the fixed point locus $(M_{s,\ell})^A$ is isomorphic to $M_{s,\ell^h} \times M_{s,\ell^f}$. In §8.3, we will recall the hyperbolic restriction functor, and show that it commutes with the vanishing cycle functor (see also, e.g., [41 Proposition 5.4.1.2]). Therefore, we have the hyperbolic restriction $h : V_{\vec{r}} \to V_{\vec{r}^f} \otimes V_{\vec{r}^h}$.

Taking the subtorus $A \subset GL_{r_1} \times GL_{r_2} \times GL_{r_3}$ to be generic enough, we get a hyperbolic restriction

$$h : V_{r_1,r_2,r_3} \to (V_{1,0,0})^{\otimes r_1} \otimes (V_{0,1,0})^{\otimes r_2} \otimes (V_{0,0,1})^{\otimes r_3},$$

where the target is the Fock space of $(r_1 + r_2 + r_3)$-free Bosons.

We show the actions of $SH^c$ on $V_{r_1,r_2,r_3}$ and on $(V_{1,0,0})^{\otimes r_1} \otimes (V_{0,1,0})^{\otimes r_2} \otimes (V_{0,0,1})^{\otimes r_3}$ (via the coproduct $\Delta^c$) are compatible with the hyperbolic restriction map.

8.1 A central extension

Now we introduce the central extended algebra $SH^c$. This is similar to its 2d counterpart from [79].

Denote by $A_F$ the ring of symmetric polynomials in infinitely many variables with coefficients in $F = C(h_1,h_2)$. It has a basis given by $\{J_{\lambda} | \lambda$ is a partition$\}$, where $J_{\lambda}$ is the integral form of the Jack polynomial associated to $\lambda$, and to the parameter $-h_1/h_2$.

Let $D_{0,l} \in End(A_F)$ be the operator of multiplication by the power-sum function $p_l$. Let $D_{0,l} \in End(A_F)$ be the Sekiguchi operators, which form a family of commuting differential operators whose joint spectrum is given by the Jack polynomials $\{J_{\lambda} | \lambda$ is a partition$\}$.

Let $SH^{0}$ be the unital subalgebra of $End(A_F)$ generated by $\{D_{0,l} | l \geq 1\}$. For $l \geq 0$, we set $D_{1,l} = [D_{0,l+1},D_{0,l}]$. Let $SH^c$ be the unital subalgebra of $SH^{0}$ generated by $\{D_{1,l} | l \geq 0\}$, and $SH^0$ the subalgebra of $SH^{0}$ generated by $\{D_{0,l} | l \geq 1\}$. It is known that $SH^c = F[D_{0,0},D_{0,1},\cdots]$ is the polynomial algebra and $SH^{c0} = SH^c \otimes SH^0$ (see [79], Section 1.7).

Let $SH^c$ be the opposite algebra of $SH^c$ whose generator corresponding to $D_{1,l}$ is denoted by $D_{-1,l}$. The algebra $SH$ is generated by $SH^c,SH^0,SH^c$ modulo a certain set of relations involving the commutators of elements from $SH^c$ and $SH^c$. In particular, $SH$ has a triangular decomposition $SH = SH^c \otimes SH^0 \otimes SH^c$.

In the setup of the present work, we use the following normalization of generators of $SH^c,SH^c,SH^c$ by scaling the generators $D_{k,l}$ via appropriate powers of $h_1,h_2$ according to their geometric actions on the quiver varieties:

$$f_{0,l} := h_1^{l+1}D_{0,l+1}, \quad f_{1,l} := h_1^{l-1}h_2^{-1}D_{1,l} \quad \text{and} \quad f_{-1,l} := h_1^{l}D_{-1,l}. \quad (26)$$
Let $GL_{r_3} \times GL_{r_2} \times GL_{r_3}$ be the general linear group at the framing vertices of $Q_3^{fr}$. Denote by $T_{r_1, r_2, r_3} \subset GL_{r_1} \times GL_{r_2} \times GL_{r_3}$ its maximal torus, then

$$H_{T_{r_1, r_2, r_3} \times T_2}(pt) = C[h_1, h_2, \mu_1^{(1)}, \ldots, \mu_l^{(1)}, \mu_2^{(2)}, \ldots, \mu_l^{(2)}, \mu_3^{(3)}, \ldots, \mu_r^{(3)}].$$

Let $K_{r_1, r_2, r_3} = C[h_1, h_2, \mu_1^{(1)}, \ldots, \mu_{r_1}^{(1)}, \mu_1^{(2)}, \ldots, \mu_{r_2}^{(2)}, \mu_1^{(3)}, \ldots, \mu_{r_3}^{(3)}]$ be the fractional field of $H_{T_{r_1, r_2, r_3} \times T_2}(pt)$.

We introduce a new family $c := (c_1^{(1)}, c_2^{(2)}, c_3^{(3)}) = (c_0, c_0, c_0, c_1, c_1, c_1, c_1, \cdots)$ of formal parameters, which roughly speaking comes from $\lim_{(r_1, r_2, r_3) \to \infty} H_{T_{r_1, r_2, r_3}}(pt)$. Now set

$$F^c = C[h_1, h_2][c_1^{(1)}, c_2^{(2)}, c_3^{(3)} : l \geq 0], \quad SH^c, 0 = F^c[f_{0, l} : l \geq 0].$$

**Definition 8.1.1** Let $SH^c$ be the $F$-algebra generated by $SH^c$, $SH^c$, and $SH^c$, modulo the following set the relations

\[
\begin{align*}
&c_i^{(1)}, c_i^{(2)}, c_i^{(3)} \text{ are central, } l \geq 0, \\
&[f_{0, l}, f_{1, k}] = f_{1, l+k}, l \geq 1, \\
&[f_{0, l}, f_{-1, k}] = -f_{-1, l+k}, l \geq 1, \\
&[f_{1, l}, f_{-1, k}] = G_{l+k}, l, k \geq 0,
\end{align*}
\]

where $f_{0, 0} = 0$, and the elements $G_{k+l}$ are determined through the formula

\[
1 - h_1h_2h_3 \sum_{l \geq 0} G_{il}t^{l+1} = \exp\left(\sum_{l \geq 0} (-1)^{l+1} c_i^{(1)} \phi_i(t, h_1)\right) \exp\left(\sum_{l \geq 0} (-1)^{l+1} c_i^{(2)} \phi_i(t, h_2)\right)
\]

\[
\exp\left(\sum_{l \geq 0} (-1)^{l+1} c_i^{(3)} \phi_i(t, h_3)\right) \exp\left(\sum_{l \geq 0} f_{0, l} \Phi_i(t)\right).
\]

Here the functions $\phi_i(t, h)$ and $\Phi_i(t)$ are formal power series in $t$ depending on $h_1, h_2, h_3$. They are given by the following formulas.

\[
\exp\left(\sum_{l \geq 0} (-1)^{l+1} a^l \phi_i(t, h_i)\right) = \frac{1 + t(a-h_i)}{1 + ta},
\]

\[
\exp\left(\sum_{l \geq 0} (-1)^{l+1} a^l \Phi_i(t)\right) = \frac{(1 + t(a-h_1))(1 + t(a-h_2))(1 + t(a-h_3))}{(1 + t(a+h_1))(1 + t(a+h_2))(1 + t(a+h_3))},
\]

where $a$ is any element in $C$, and we impose the condition that $h_1 + h_2 + h_3 = 0$.

The algebra $SH^c$ from [79], which is generated by $SH^c$, $SH^c$, together with a family of central elements $c = \{c_0, c_1, c_2, \cdots\}$, modulo a certain set of relations involving the commutators $[D_{-1, k}, D_{1, l}]$ [79], is a specialization of the algebra $SH^c$ above with

$c_i^{(2)} = c_i^{(3)} = 0$

Similarly to [79], for any $r = (r_1, r_2, r_3)$, there is an algebra $SH^{r_1, r_2, r_3}$ obtained by the specialization of $SH^c \otimes K_{r_1, r_2, r_3}$ with

\[
\begin{align*}
&c_i^{(1)} = r_1, \quad c_i^{(2)} = r_2, \quad c_i^{(3)} = r_3, \\
&c_i^{(k)} = p_i(\mu_1^{(k)}, \mu_2^{(k)}, \cdots, \mu_{r_k}^{(k)}), k = 1, 2, 3.
\end{align*}
\]

The central elements $\{c_i^{(1)}, i \geq 1\}$ of $SH^c$ correspond to the central elements $\{c_i h_i, i \geq 1\}$ of $SH^c$ in [79], and $c_0^{(3)}$ of $SH^c$ corresponds to $\frac{a_3}{a_1}$ of $SH^c$ in [79].
This specialization is determined using the following formula

$$\exp\left(\sum_{l \geq 0}(-1)^{l+1} p_l(\mu^{(1)}_a)\phi_l(t,v)\right) \exp\left(\sum_{l \geq 0}(-1)^{l+1} p_l(\mu^{(2)}_a)\phi_l(t,v)\right) \cdot \exp\left(\sum_{l \geq 0}(-1)^{l+1} p_l(\mu^{(3)}_a)\phi_l(t,v)\right)$$

$$= \prod_{a=1}^{r_1} \frac{z - \mu^{(1)}_a + \mu_1}{z - \mu^{(1)}_a} \prod_{a=1}^{r_2} \frac{z - \mu^{(2)}_a + \mu_2}{z - \mu^{(2)}_a} \prod_{a=1}^{r_3} \frac{z - \mu^{(3)}_a + \mu_3}{z - \mu^{(3)}_a},$$

where $z = -1/t$, and $\mu^{(1)}_1, \ldots, \mu^{(3)}_3$ are the Chern roots of $\mathcal{E}_{r_k}$, $k = 1, 2, 3$.

**Proposition 8.1.2**

1. There is an algebra homomorphism $\text{SH}^F_{\mathbb{C}} \to Y_{h_1,h_2,h_3}(\mathfrak{gl}(1))$, given by

   $$f_{1,1} \mapsto e_l, \ f_{-1,1} \mapsto f_l, \ G_l \mapsto \psi_l, \ c_i^{(k)} \mapsto 0, \ k = 1, 2, 3.$$ 

2. The algebra $\text{SH}^F_{\mathbb{C}}$ acts on $V_{r_1, r_2, r_3}$ through the specialization $\text{SH}^F_{r_1, r_2, r_3}$ action on $V_{r_1, r_2, r_3}$, for any $\mathbf{r} = (r_1, r_2, r_3) \in \mathbb{Z}^3_{\geq 0}$.

**Proof.** For (1): By [3] Corollary 6.4], there is an algebra isomorphism of $\text{SH}^F_{\mathbb{C}}$ and the spherical 2d COHA $\text{SH}^F_{t_{n=0}, t^2}$. The latter is in turn isomorphic to $Y^+$ by Theorem 7.1.1.

Thus, we have the isomorphisms of algebras

$$\text{SH}^F_{\mathbb{C}} \cong Y^+, \ f_{1,1} \mapsto e_l \text{ and } \text{SH}^F_{\mathbb{C}} \cong Y^-, \ f_{-1,1} \mapsto f_l.$$ 

Define a natural map $\text{SH}^F_{\mathbb{C}} \mapsto Y_{h_1,h_2,h_3}(\mathfrak{gl}(1))$ by $G_l \mapsto \psi_l, \ c_i^{(k)} \mapsto 0$. It is straightforward to check that the above assignments define an algebra homomorphism $\text{SH}^F_{\mathbb{C}} \to Y_{h_1,h_2,h_3}(\mathfrak{gl}(1)).$

For (2): Using the isomorphisms $\text{SH}^F_{\mathbb{C}} \cong Y^+, \ \text{SH}^F_{\mathbb{C}} \cong Y^+$, we deduce that the algebras $\text{SH}^F_{\mathbb{C}}$ act on $V_{r_1, r_2, r_3}$. We require the central elements $\{c_i^{(k)} \mid i \in \mathbb{Z}_{\geq 0}, k = 1, 2, 3\}$ act by the formula $p_i(\mu^{(1)}_1, \mu^{(2)}_2, \ldots, \mu^{(k)}_{r_k})$. The series $1 - h_1 h_2 h_3 \sum_{i \geq 0} G_l^{i+1}$ acts by $\lambda_i(\mathcal{F}(\mathcal{V}_{n, \mathcal{E}_{\mathbf{r}}}))$, which is given by the formula (14). In particular, the action of $f_{0,1} \in \text{SH}^F_{\mathbb{C}, 0}$ is determined by the formula

$$\exp\left(\sum_{l \geq 0} f_{0,1} \phi_l(t)\right) \cdot m = \prod_{d=1}^{n} \frac{z - \lambda_d - h_1 z - \lambda_d - h_2 z - \lambda_d - h_3}{z - \lambda_d + h_1 z - \lambda_d + h_2 z - \lambda_d + h_3} \cdot m,$$

for $m \in V_{r_1, r_2, r_3}$, $z = -1/t$, and $\lambda_1, \ldots, \lambda_d$ are the Chern roots of the tautological bundle $\mathcal{V}$. It is straightforward to check that the above assignments give an action of $\text{SH}^F_{\mathbb{C}}$ on $V_{r_1, r_2, r_3}$. The action obviously factors through the algebra $\text{SH}^F_{r_1, r_2, r_3}$.]

**Remark 8.1.3**

1. The action of $\text{SH}^F_{\mathbb{C}}$ on $V_{r_1, r_2, r_3}$ does not factor through the quotient map $\text{SH}^F \to Y_{h_1,h_2,h_3}(\mathfrak{gl}(1))$, as the actions of the central elements $c_i^{(k)}$ are not trivial. Instead, the action of $\text{SH}^F_{\mathbb{C}}$ factors through the specialization $\text{SH}^F_{r_1, r_2, r_3}$ via $c_i^{(k)} = p_i(\mu^{(1)}_1, \mu^{(2)}_2, \ldots, \mu^{(k)}_{r_k})$. 

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8.2 The coproduct of \( \text{SH}^\varepsilon \)

Introduce the following elements of \( \text{SH} \)

\[
\text{Lemma 8.2.1}
\]

The following lemma is analogous to [79, Proposition 1.40], and the proof is similar.

\[
\text{Proposition 8.2.2}
\]

The algebra product is uniquely determined by the following formulas.

To compare with the notation in the current paper, we have

\[
\text{up to localization. In Appendix C, we prove the following.}
\]

\[
\text{When } c^{(1)}, c^{(2)} = 0, \text{ the Heisenberg subalgebra of } \text{SH}^\varepsilon \text{ is generated by } \{b_{-l}, b_l, b_0, E_0 | l \geq 0 \}. \]

To compare with the notation in the current paper, we have

\[
\lambda := \frac{b_{-1}}{b_1}, \quad B_{-1} = \frac{b_{-1}}{b_1}, \quad B_1 = \frac{b_1}{b_1}, \quad G_0 = \frac{E_0}{b_1b_2}, \quad B_0 := G_1 = \frac{E_1}{b_2} = \frac{b_0}{b_1}.
\]
Lemma 8.3.1

Let \( \bar{r} = r + r' \), and write \( C_r = C^{r_1} \oplus C^{r_2} \). Let \( A \subseteq \text{GL}_r \) be a 1-dimensional subtorus with coordinate \( t \) so that the eigenvalues on \( C_{r_1}^{t} \) are \( t \) and the eigenvalues on \( C_{r_2}^{t} \) are 1. Then, taking direct sum of framed representations defines an isomorphism \( \mathfrak{M}_{r_1} \times \mathfrak{M}_{r_2} \cong \mathfrak{M}_{r_1}^{A} \), with their tautological bundles satisfying \( V_1 \oplus V_2 \cong V \) and \( E_{r_1} \oplus E_{r_2} \cong E_{r} \). Moreover, the potential function \( \text{tr} W \) on \( \mathfrak{M}_r \), when restricted to the fixed point set \( \mathfrak{M}_r^{A} \), agrees with the potential function \( \text{tr} W \oplus \text{tr} W \) on the product \( \mathfrak{M}_{r_1} \times \mathfrak{M}_{r_2} \) under this isomorphism.

The attracting set \( \mathcal{A}_{\mathfrak{M}_r} \), consisting of the subset of \( \mathfrak{M}_r \) that has a limit as \( t \to 0 \), can be identified as a quotient of the total space of the affine bundle \( \text{Hom}(V_1, V_2)^{\oplus} \oplus \text{Hom}(E_{r_1}, E_{r_2}) \oplus \text{Hom}(V_1, E_{r_2}) \). In particular, the potential function \( \text{tr} W \) on \( \mathfrak{M}_r \), when restricted to \( \mathcal{A}_{\mathfrak{M}_r} \), is equal to the pull-back of \( \text{tr} W \oplus \text{tr} W \) on \( \mathfrak{M}_r^{A} \), as the trace of an endomorphism only depends on the diagonal.

We have the natural correspondence

\[
\mathfrak{M}_r^{A} \xrightarrow{\pi} \mathcal{A}_{\mathfrak{M}_r} \xrightarrow{\eta} \mathfrak{M}_r.
\]

**Lemma 8.3.1**

1. The variety \( \mathfrak{M}_r(n) \) is smooth;

2. \( \mathcal{A}_{\mathfrak{M}_r(n)} \) is an affine bundle on \( \mathfrak{M}_r(n)^{A} \), whose rank is equal to the codimension of \( \mathcal{A}_{\mathfrak{M}_r(n)} \) in \( \mathfrak{M}_r(n) \).

**Proof.** Let \( (B_i, I_{jk}, J_{jk}) \) be the representative of an isomorphism class of representations in \( \mathfrak{M}_r(n) \). We need to prove that its automorphism group is trivial (cf. Proposition 3.2.5). We have the following 3 vector subspaces on \( C^n \),

\[
V_1 = \{ f(B_2, B_3)I_{23}(C^{r_i}) \mid f \text{ a non-commutative polynomial in 2 variables} \}
\]

and similarly \( V_2 \) and \( V_3 \). The point \( (B_i, I_{jk}, J_{jk}) \) being stable means that \( C^n = V_1 + V_2 + V_3 \).

Let \( g : C^n \to C^n \) be an automorphism of \( (B_i, I_{jk}, J_{jk}) \), hence commutes with \( B_i \) and \( I_{jk} \). In particular, it is closed on the subspaces \( V_l \) for \( l = 1, 2, 3 \). By definition of \( V_1 \), it is also closed under \( B_2, B_3 \), hence, we have a representation \( (B_2, B_3, I_{23}) \) on the vector space \( V_1 \) of the standard framed double loop quiver which is furthermore stable in the usual sense (e.g., in [38 § 3.2]). Therefore, by the standard results about stability conditions (see e.g., [38 Lemma 3.2.3.]) \( g|_{V_1} \) is trivial. Similarly, \( g|_{V_2} \) and \( g|_{V_3} \) are both trivial. As \( C^n = V_1 + V_2 + V_3 \), this implies that \( g \) is the identity on \( C^n \). \( \blacksquare \)

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Proposition 8.3.2  
1. We have a natural isomorphism \( p_\ast \circ \eta^1 \circ \varphi_W C_{\mathcal{M}_F} \cong \varphi_W C_{\mathcal{M}_A}[2d] \), where \( d = \text{codim } \mathcal{M}_A \).

2. This natural isomorphism induces a map \( h : H^*_{c,T}(\mathcal{M}_F, \varphi)^{\vee} \to H^*_{c,T}(\mathcal{M}_A \times \mathcal{M}_F, \varphi)^{\vee} \).

Now we prove Proposition 8.3.2. The proof essentially follows the same argument as in [8.3] pp.29-31. For completeness, we include the details.

Lemma 8.3.3  
The map \( p_\ast \eta^1 \) intertwines the vanishing cycle functors on \( \mathcal{M}_F(n) \) and \( \mathcal{M}_F(n)^A \)

\[ \varphi_W p_\ast \eta^1 \cong p_\ast \eta^1 \varphi_W. \]

Moreover, \( p_\ast \eta^1 C_{\mathcal{M}_F(n)}[\dim \mathcal{M}_F(n)] \cong C_{\mathcal{M}_F(n)^A}[\dim \mathcal{M}_F(n)^A]. \)

Proof.  
For simplicity, here in the proof we denote \( \mathcal{M}_F(n) \) by \( \mathcal{X} \), with the potential function \( f = \text{tr } W: \mathcal{X} \to C \). It restricts to functions on \( A_\mathcal{X} \) and \( \mathcal{X}^A \) denoted by \( f_A \) and \( f_A \), with the zero locus denoted by \( X, A_\mathcal{X} \) and \( X^A \) respectively. We have the diagram

\[
\begin{array}{ccc}
\mathcal{X}^A & \xrightarrow{p_X} & A_\mathcal{X} \\
\downarrow r_A & & \downarrow r_X \\
X & \xrightarrow{j_X} & \mathcal{X}
\end{array}
\]

We first show that the hyperbolic localization commutes with the nearby cycle functor

\[ p_X j_X^! \Psi_f = \Psi_f^p \Psi_f. \]

Let \( c : \hat{C}^* \to C \) be the universal cover of \( C^* \subseteq C \), and we have the fiber diagram

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{c}_X} & \hat{C}^* \\
\downarrow c & & \downarrow c \\
\mathcal{X} & \xrightarrow{c} & C.
\end{array}
\]

Similarly we have \( \hat{c}_A : \hat{A}_\mathcal{X} \to A_\mathcal{X} \) and \( \hat{c}_A : \hat{X}^A \to \mathcal{X}^A \). Recall that \( \Psi_f = r_A^\ast \hat{c}_X \ast \hat{c}_\mathcal{X}^\ast \), and similar for \( \Psi_f^p \). We need to show that

1. \( p_{\hat{X}_A} j_{\hat{X}}^! \hat{c}_{\mathcal{X}}^\ast = c_A^* p_{A_\mathcal{X}} j_{\mathcal{X}}^! \hat{c}_\mathcal{X}^\ast; \)
2. \( c_A^* p_{\hat{X}_A} j_{\hat{X}}^! = p_{A_\mathcal{X}} j_{\mathcal{X}}^! c_{\mathcal{X}}^\ast; \)
3. \( p_{\hat{X}_A} j_{\hat{X}}^! r_A^\ast = r_A^\ast p_{A_\mathcal{X}} j_{\mathcal{X}}^! \).

By Lemma 8.3.1 \( p_{\hat{X}_A} j_{\hat{X}}^! \hat{c}_{\mathcal{X}}^\ast \) and \( p_{\hat{X}_A} j_{\hat{X}}^! \hat{c}_{\mathcal{X}}^\ast \) differ only by a homological degree shifting. We also have

\[ p_{\hat{X}_A} j_{\hat{X}}^! \hat{c}_{\mathcal{X}}^\ast = p_{\hat{X}} c_A^* j_{\hat{X}}^! \hat{c}_\mathcal{X}^\ast \]

which in turn differs from \( c_A^* p_{\hat{X}_A} j_{\hat{X}}^! \) by the same shifting as above. This proves (1). The other two are proved in a similar way: (3) is proven using diagram (28), and replacing \( p_{\hat{X}_A} j_{\hat{X}}^! \) by \( p_{\hat{X}_A} j_{\hat{X}}^! \) and a homological shifting; (2) is straightforward.
The algebra in [36]. Generic modules of parameter families of vertex operator algebras

In this section, we review the free field representation and some properties of a class of one-

Vertex operator algebra

According to Lemma 8.3.1, \( A_X \rightarrow \mathfrak{X} \) is a regular embedding, and \( A_X \rightarrow \mathfrak{X}^A \) is an affine bundle. Hence, \( p_\ast \eta \mathfrak{C}_{\mathfrak{M}_r(n)} \) is \( \mathfrak{C}_{\mathfrak{M}_r(n)} \) up to a homological shifting. Moreover, the codimension of \( A_X \rightarrow \mathfrak{X} \) is equal to the dimension of the fibers of \( A_X \rightarrow \mathfrak{X}^A \), hence the shifting is the codimension of \( \mathfrak{X}^A \) in \( \mathfrak{X} \). Therefore, \( p_\ast \eta \mathfrak{C}_{\mathfrak{M}_r(n)}[\dim \mathfrak{X}] \cong \mathfrak{C}_{\mathfrak{M}_r(n)} \).  

Proposition 8.3.2 (1) follows directly by combining the two parts in this lemma.

Now we explain the definition of the map \( h \). We have the natural adjunction \( \eta \eta^! \rightarrow \text{id} \). Applying to \( \varphi_\mathfrak{W} \mathfrak{C}_{\mathfrak{M}_r} \), we get \( \eta \eta^! \varphi_\mathfrak{W} \mathfrak{C}_{\mathfrak{M}_r} \rightarrow \varphi_\mathfrak{W} \mathfrak{C}_{\mathfrak{M}_r} \). Apply to it the functor \( \text{D} \circ p_{\mathfrak{M}_r} \), we get

\[
H^+_{\mathfrak{M}_r}^+ (\mathfrak{M}_r, \varphi)^\vee \cong \text{D}p_{\mathfrak{M}_r} \varphi_\mathfrak{W} \mathfrak{C}_{\mathfrak{M}_r} \rightarrow \text{D}p_{\mathfrak{M}_r} \eta \eta^! \varphi_\mathfrak{W} \mathfrak{C}_{\mathfrak{M}_r} \cong \text{D}p_{\mathfrak{M}_r} \eta \eta^! \varphi_\mathfrak{W} \mathfrak{C}_{\mathfrak{M}_r} \cong \text{D}p_{\mathfrak{M}_r} \eta \eta^! \varphi_\mathfrak{W} \mathfrak{C}_{\mathfrak{M}_r} [d] \cong H^+_{\mathfrak{M}_r}^+ (\mathfrak{M}_r, \varphi)^\vee.
\]

This proves Proposition 8.3.2 (2).

In Appendix C we prove that \( h \) defined in this way satisfies Proposition 8.2.3.

9 Vertex operator algebra \( \mathcal{W}_{r_1, r_2, r_3} \)

In this section, we review the free field representation and some properties of a class of one-parameter families of vertex operator algebras \( \mathcal{W}_{r_1, r_2, r_3} \) for \( r_i \in \mathbb{Z}_{\geq 0}, i = 1, 2, 3 \), introduced in [36]. Generic modules of \( \mathcal{W}_{r_1, r_2, r_3} \) can be identified with the modules \( V_{r_1, r_2, r_3} \) of the algebra \( \mathfrak{SH}^c \) as shown by comparing the free field representation with the one coming from the coproduct of \( \mathfrak{SH}^c \).

9.1 \( \mathcal{W}_{r_1, r_2, r_3} \) as extensions of the Virasoro algebra

The algebra \( \mathcal{W}_{r_1, r_2, r_3} \) to be defined in the next section is an extension of the Virasoro algebra generated by modes of fields \( W_1, W_2, \ldots, W_n \) of conformal weight 1, 3, \ldots, \( n \) for some integer \( n \),

\[
W_i(z) = \sum_l W_{i, l} z^{-l-i}, \tag{30}
\]
where the modes of $W_2$ satisfy the Virasoro algebra

$$[W_{2,m}, W_{2,n}] = (m - n)W_{2,m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{31}$$

for $c$ a central extension called the central charge and $W_i$ for $i \neq 2$ are primary fields of conformal weight $i$, i.e. their modes have the following commutation relations with the Virasoro generators

$$[W_{2,n}, W_{i,m}] = ((i - 1)n - m)W_{i,m+n}. \tag{32}$$

The generators $W_{i,t}$ are endomorphisms on a vector space spanned by $W_{i,1}, -l_1 \ldots W_{i,t}, -l_t |0\rangle$, $l_i \geq i$, $t \geq 0 \tag{33}$

for $|0\rangle$ the vacuum state, i.e. a state annihilated by all $W_{i,-l_1}$ with $l_i < i$. For general values of parameters $r_1, r_2, r_3$, the algebra $W_{r_1,r_2,r_3}$ is not freely generated and one needs to work modulo modes of some composite fields to satisfy Jacobi identities.

It is also convenient to introduce the derivative of fields and the operation of normal ordered product. Modes of derivatives of a field $O$ are defined by

$$\left(\partial^n O\right)_m = (-1)^n \prod_{i=0}^{n-1} (h_O + m + i)O_m \tag{34}$$

where $h_O$ is the conformal dimension ($[W_{2,0}, \cdot]$ eigenvalue) of the field $O$. On the other hand, modes of the normal ordered product are given by

$$\left(O^1O^2\right)_m = \sum_{n \leq -h_O} O^1_n O^2_{m-n} + \sum_{n > -h_O} O^2_{m-n} O^1_n. \tag{35}$$

9.2 Definition of $W_{r_1,r_2,r_3}$

Let $H$ be the Heisenberg algebra generated by modes $b_i \in \mathbb{Z}$. We define $W_{r_1,r_2,r_3}$ as a subalgebra of a tensor product of $m = r_1 + r_2 + r_3$ Heisenberg algebras. The algebra $W_{r_1,r_2,r_3}$ can be defined according to [7, 54, 76] as an intersection of kernels of vertex operators of the form

$$V[\alpha](z) = T_\alpha \exp \left[\alpha \sum_{n<0} \frac{b_n}{n} z^{-n}\right] \exp \left[\alpha \sum_{n>0} \frac{b_n}{n} z^{-n}\right], \tag{36}$$

associated to these $m$ Heisenberg algebras, where $T_\alpha$ sends the vacuum $|0\rangle$ to a more general highest weight vector $|\alpha\rangle$ satisfying $b_0|\alpha\rangle = \alpha|\alpha\rangle$.

Definition 9.2.1 Let $\mathcal{H}^{\otimes m}$ be a tensor product of $m$ Heisenberg vertex operator algebras generated by fields $b^{(3)}_i(z), b^{(2)}_j(z), b^{(1)}_k(z)$ for $i = 1, \ldots, r_3$, $j = r_3 + 1, \ldots, r_3 + r_2$, $k = r_3 + r_2 + 1, \ldots, r_3 + r_2 + r_1$ of conformal weight 1 together with commutation relations

$$[b^{(a)}_i b^{(b)}_j] = -\frac{h_a}{h_1 h_2 h_3} \delta_{a,0} \delta_{i+j,0}. \tag{37}$$
Let us consider a set of \( m - 1 \) screening currents given by products of vertex operators of the form:

\[
S_{i}^{33}(z) = V_{i}^{(1)}[-h_{1}(z) \otimes V_{i+1}^{(1)}[h_{1}(z)], \quad i \in [1, r_{3} - 1] \\
S_{i}^{32}(z) = V_{i}^{(1)}[-h_{2}(z) \otimes V_{i+1}^{(2)}[h_{3}(z)] \\
S_{i}^{22}(z) = V_{i}^{(2)}[-h_{3}(z) \otimes V_{i+1}^{(2)}[h_{3}(z)], \quad i \in [r_{3} + 1, r_{3} + r_{2} - 1] \\
S_{i}^{21+3}(z) = V_{i}^{(2)}[-h_{1}(z) \otimes V_{i+1}^{(3)}[h_{2}(z)] \\
S_{i}^{11}(z) = V_{i}^{(3)}[-h_{2}(z) \otimes V_{i+1}^{(3)}[h_{2}(z)], \quad i \in [r_{3} + r_{2} + 1, r_{3} + r_{2} + r_{1} - 1] 
\]

where \( V_{i}^{(\kappa)}[a] \) is a vertex operator for \( b_{i}^{(\kappa)} \). The algebra \( \mathcal{W}_{r_{1}, r_{2}, r_{3}} \) is then defined as a subalgebra of \( \mathcal{H}^{\otimes m} \) given by an intersection of kernels of the zero modes of the above screening currents \( \oint dz S \).

Let us now give a few important examples.

Example 9.2.2 The algebra \( \mathcal{W}_{0,0,1} \) can be simply identified with the Heisenberg algebra \( \mathcal{H} \).

Example 9.2.3 The algebra \( \mathcal{W}_{0,0,2} \) can be identified with the Virasoro algebra tensored with the Heisenberg algebra \( \text{Vir} \times \mathcal{H} \).

Example 9.2.4 More generally, the algebra \( \mathcal{W}_{0,0,r} \) is the well known \( \mathcal{W}_{r} \)-algebra \([24, 18]\). This algebra is closely related to the specialization \( \mathcal{SH}^{(r)} \) introduced by Schiffmann-Vasserot in \([79]\).

Example 9.2.5 \( \mathcal{W}_{0,1,1} \) can be identified with a quotient of the well studied algebra \( \mathcal{W}_{3} \times \mathcal{H} \) at central charge \( c = -2 \) from \([87]\). We will return to this example later.

Apart from the free field realization, there exist two more (conjecturally equivalent) definitions of the algebra \( \mathcal{W}_{r_{1}, r_{2}, r_{3}} \). It was argued in \([75]\) that \( \mathcal{W}_{r_{1}, r_{2}, r_{3}} \) can be viewed as a quotient of a family of algebras \( \mathcal{W}_{1,\infty}[\lambda_{1}, \lambda_{2}, \lambda_{3}] \) depending on three complex parameters \( \lambda_{i} \in \mathbb{C} \) subject to the constraint\[^{9}\]

\[
\frac{1}{\lambda_{1}} + \frac{1}{\lambda_{2}} + \frac{1}{\lambda_{3}} = 0 \quad (43)
\]

in the same way as \( \mathcal{SH}^{(r_{1}, r_{2}, r_{3})} \) are specializations of \( \mathcal{SH}^{c} \). The algebra \( \mathcal{W}_{1,\infty}[\lambda_{1}, \lambda_{2}, \lambda_{3}] \) was recently constructed in \([53]\) based on the previous work of \([33, 31, 73]\). It was argued\[^{10}\] in \([74, 75]\) that the algebra \( \mathcal{W}_{1,\infty}[\lambda_{1}, \lambda_{2}, \lambda_{3}] \) contains an ideal whenever the parameters are specialized to

\[
\frac{r_{1}}{\lambda_{1}} + \frac{r_{2}}{\lambda_{2}} + \frac{r_{3}}{\lambda_{3}} = 1. \quad (44)
\]

Conjecturally, the algebra \( \mathcal{W}_{r_{1}, r_{2}, r_{3}} \) can be defined as a quotient of \( \mathcal{W}_{1,\infty}[\lambda_{1}, \lambda_{2}, \lambda_{3}] \) by such an ideal and depends on the continuous parameter \( \Psi = -\frac{\lambda_{1}}{\lambda_{2}} = -\frac{\lambda_{2}}{\lambda_{3}} \).

\[^{9}\]The parameters \( \lambda_{i} \) can be expressed in terms of the parameters above as \( \lambda_{i} = (r_{1}h_{1} + r_{2}h_{2} + r_{3}h_{3})/h_{i} \).

\[^{10}\]See also \([53]\) and references therein for discussion of special classes of such truncations.
The original definition of $\mathcal{W}_{r_1, r_2, r_3}[\Psi]$ from [36] was given in terms of a combination of the quantum Drinfeld-Sokolov reduction [23] and the BRST coset [44] of super Kac-Moody algebras. We refer reader to [36, 75] for detailed discussion.

The definition of the algebra is invariant under the mutual permutation of the parameters $r_i$ and $\hbar_i$. In [36], this triality symmetry was a non-trivial consequence of the S-duality of boundary conditions. From the spiked instanton configuration point of view, this triality is a trivial consequence of relabeling of the coordinates $z_i$ for $i = 1, 2, 3$ associated to the three $C \subset C^3$.

Note also that one might consider different ordering of the free bosons in the free field realizations above. Each ordering gives a different realization of the same algebra. For the purpose of this work, we use the ordering corresponding to the screening charges above.

9.3 $\mathcal{W}_{0,0,2}$ example in detail

The simplest non-trivial example $\mathcal{W}_{0,0,2}$ is freely generated by the Heisenberg field $W_1$ and the Virasoro field $W_2$. For completeness, let us write down explicitly their free field realization in terms of a pair of free bosons $b_1^{(3)}$ and $b_2^{(3)}$:

$$W_1 = b_1^{(3)} + b_2^{(3)},$$
$$W_2 = -\frac{h_1 h_2}{4}(\tilde{b}, \tilde{b}) - \frac{h_3}{2} \partial \tilde{b}$$

where we have introduced

$$\tilde{b} = b_1^{(3)} - b_2^{(3)}.\quad (47)$$

9.4 $\mathcal{W}_{0,1,1}$ example in detail

Let us further discuss $\mathcal{W}_{0,1,1}$ that plays an important role in the comparison of the geometric action on the equivariant cohomology on the moduli space of spiked instantons with generic modules of $\mathcal{W}_{r_1, r_2, r_3}$. The algebra $\mathcal{W}_{0,1,1}$ is generated by modes of the Heisenberg algebra field $W_1$, the stress-energy tensor $W_2$ and a primary field of spin three $W_3$ with the following commutation relation

$$[W_{3,m}, W_{3,n}] = \frac{4}{3} (m-n) \sum_{k=-\infty}^{\infty} W_{2,m+n-k} W_{2,k} - \frac{6}{5} m (m^2 - 1)(m^2 - 4) \delta_{m,-n}$$
$$-\sqrt{6} (m-n)(m+2)(2m+3) W_{2,m+n}.\quad (48)$$

The algebra $\mathcal{W}_{0,1,1}$ is not generated freely and there exists a null field

$$X = (W_3 W_3) - 128(W_2(W_2 W_2)) - 76(\partial W_2 \partial W_2) - 112(\partial^2 W_2 W_2) + \frac{32}{3} \partial^4 W_2$$

whose modes need to be factored out. This combination will be automatically zero in the free field realization of the VOA as one can easily check.

The full vertex operator algebra decomposes into the subalgebra of positive, negative and zero modes. Let us now describe how to induce the modules of interest from the one

\footnote{Note that modes of $W_1$ commute with the modes of $W_2$. One can recover the standard commutation relations by adding a multiple of normal ordered product ($W_1 W_1$) to $W_2$.}
dimensional modules for the algebra of positive and zero modes. Let us consider a vector $|u, h, w\rangle$ annihilated by all the positive modes and being an eigenstate of the zero modes

$$W_{i,n}|w_1, w_2, w_3\rangle = 0, \quad \text{for } n > 0, i = 1, 2, 3 \quad (50)$$

$$W_{i,0}|w_1, w_2, w_3\rangle = w_i|w_1, w_2, w_3\rangle \quad (51)$$

Note also that the algebra of zero modes is commutative. The existence of the null state $X$ puts a constraint on the weights $u, h, w$ since $X_0|w_1, w_2, w_3\rangle = 0$. Acting by $X_0$ on the state $|w_1, w_2, w_3\rangle$ leads to the constraint (52)

$$w_3^2 = 16w_2^2(8w_2 + 1).$$

Having a triple of numbers $w_1, w_2, w_3$ satisfying the above condition, one can consider a module of the full algebra generated by an action of negative modes on the vector $|w_1, w_2, w_3\rangle$ and quotient by the negative modes of $X$.

Using the free field realization in terms of a pair of free bosons, one can induce the above modules from the vertex operators (Fock modules) of the free fields. Let us consider two Heisenberg algebras generated by fields $b^{(2)}_1$ and $b^{(3)}_2$ normalized as above. The fields $W_1, W_2, W_3$ can then be then realized as

$$W_1 = b^{(2)}_1 + b^{(3)}_2 \quad (53)$$

$$W_2 = \frac{1}{2} \left( \hat{b}^2 + \partial \hat{b} \right) \quad (54)$$

$$W_3 = 4(\hat{b}\hat{b}) + 6(\hat{b}\partial \hat{b}) + \partial^2 \hat{b} \quad (55)$$

where now we have introduced

$$\tilde{b} = h_3b^{(2)}_1 - h_2b^{(3)}_2 \quad (56)$$

having trivial commutation relations with $W_1$.

Generic modules can be induced using the free field realization by an action of the positive modes of $W_1, W_2, W_3$ on the highest weight state

$$|q_1, q_2\rangle = V^{(3)}_1(q_1)(0)|0\rangle \otimes V^{(2)}_1(q_2)(0)|0\rangle \quad (57)$$

One can in particular compute the action of the zero modes to establish the connection with the parameters $w_1, w_2, w_3$ above. One finds

$$w_1 = -\frac{q_1}{h_1 h_3} - \frac{q_2}{h_1 h_2} \quad (58)$$

$$w_2 = \frac{q(q + 1)}{2} \quad (59)$$

$$w_3 = -2q(q + 1)(2q + 1) \quad (60)$$

where we have introduced $q = (q_1 - q_2)/h_1$.

\[\text{Footnotes:}\]

12 Highest weight representations of this form (and similarly for a general $W_{r_1, r_2, r_3}$) are parametrized by the spectrum of the Zhu algebra [11, 91, 53] that turn out to be commutative in the case of $W_{r_1, r_2, r_3}$. We expect the representation theory associated to more general toric Calabi-Yau 3-folds to be more complicated.

13 For simplicity, we again decouple $W_1$ as in the Virasoro case above.

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10 The action of $\mathcal{W}_{r_1,r_2,r_3}$ on framed quiver varieties

In this section, we show the following.

**Theorem 10.0.1** The action of $SH^\varepsilon$ on $V_{r_1,r_2,r_3}$, constructed in §8.1, factors through the vertex operator algebra $\mathcal{W}_{r_1,r_2,r_3}$.

In particular, this finishes Step 4 in proof of Theorem 4.2.1, hence complete the proof of the main theorem.

To show the Theorem 10.0.1 we first compare the free field realization of $SH^\varepsilon$ coming from the coproduct $\Delta^\varepsilon$ in §8.2 and the one coming from the kernel of screening charges. This implies the action of $SH^\varepsilon$ on $(V_{1,0,0})^{\otimes r_1} \otimes (V_{0,1,0})^{\otimes r_2} \otimes (V_{0,0,1})^{\otimes r_3}$ factors through $\mathcal{W}_{r_1,r_2,r_3}$. Recall by [27], we have the hyperbolic restriction map

$$h : V_{r_1,r_2,r_3} \rightarrow (V_{1,0,0})^{\otimes r_1} \otimes (V_{0,1,0})^{\otimes r_2} \otimes (V_{0,0,1})^{\otimes r_3},$$

which is compatible with the coproduct $\Delta^\varepsilon$ by Proposition 8.2.3. Furthermore, $h$ is an isomorphism after localization. This implies the action of $\mathcal{W}_{r_1,r_2,r_3}$ on $V_{r_1,r_2,r_3}$.

**10.1 Action on $V_{0,0,1}$**

The Heisenberg elements $B_i$ of the algebra $SH^\varepsilon$ acting on $V_{0,0,1}$ can be identified with the Heisenberg VOA generators $b_i^{(3)}$. Moreover, the action of $f_{0,1} \in SH^\varepsilon$ on such module can be identified according to [79] (8.46) as

$$f_{0,1} \mapsto \frac{h_1^2 h_2^2}{2} \sum_{k,l \geq 1} \left( b_{-l-k}^{(3)} b_k^{(3)} + b_{l-k}^{(3)} b_{l+k}^{(3)} \right)$$

$$+ \frac{h_1 h_2 h_3}{2} \sum_{l \geq 1} (l-1) b_{-l}^{(3)} b_l^{(3)} + \mu_1^{(3)} h_1 h_2 \sum_{l \geq 1} b_{-l}^{(3)} b_l^{(3)},$$

where $\mu_1^{(3)}$ is the Chern root of the line bundle $\mathcal{E}_{0,0,1}$ on $\mathcal{M}_{(0,0,1)}$ (see §6.1).

Similarly for the free bosons in other directions, one get analogous expression with the parameters $h_i$ permuted.

The algebra $SH^\varepsilon$ is generated by the Heisenberg elements $\{B_i\}$ and $f_{0,1}$. Therefore, the action of $SH^\varepsilon$ on $V_{0,0,1}$ factors through the Heisenberg algebra. Similarly for the actions on $V_{0,1,0}, V_{0,0,1}$. Let $m = r_1 + r_2 + r_3 \in \mathbb{Z}_{>0}$. One can compose the coproduct $(\Delta^\varepsilon)^m : SH^\varepsilon \rightarrow (SH^\varepsilon)^{\otimes m}$ with the maps $SH^\varepsilon$ to the three Heisenberg algebras with $r_i SH^\varepsilon$ factors mapped in the $i$’th way. Thus, the action of $SH^\varepsilon$ on $(V_{1,0,0})^{\otimes r_1} \otimes (V_{0,1,0})^{\otimes r_2} \otimes (V_{0,0,1})^{\otimes r_3}$ factors through the tensor $\mathcal{H}^{\otimes m}$.

**10.2 Comparison for $V_{0,0,2}$**

We have the hyperbolic restriction map $h : V_{0,0,2} \rightarrow V_{0,0,1} \otimes V_{0,0,1}$. By the computation in [79] §8.8, we have the following.

**Proposition 10.2.1** The action of $SH^{0,0,2}$ on $V_{0,0,1} \otimes V_{0,0,1}$ factors through $\mathcal{W}_{0,0,2}$.

The algebra $SH^\varepsilon$ is generated by $\{B_i\}$ and $f_{0,1}$. It suffices to show the actions of $\{B_i\}$ and $f_{0,1}$ on $V_{0,0,1} \otimes V_{0,0,1}$ factor through $\mathcal{W}_{0,0,2}$. This is done in [79] Lemma 8.20.
10.3 Comparison for $V_{0,1,1}$

We have the hyperbolic restriction map $h : V_{0,1,1} \to V_{0,1,0} \otimes V_{0,0,1}$.

**Proposition 10.3.1** The action of $SH^{(0,1,1)}$ on $V_{0,1,0} \otimes V_{0,0,1}$ factors through $W_{0,1,1}$.

**Proof.** It suffices to consider the generators $\{B_i\}$ and $f_{0,1}$ of $SH^\mathbb{Z}$. The action of $B_i$ is obvious since we can identify

$$W_1 = b_1^{(2)} + b_2^{(3)}.$$  

(63)

The action of $f_{0,1}$ can be checked by expressing the free field realization of $f_{0,1}$ in terms of modes of the freely realized $W_i$. This can be seen along the lines of [79, §8]. From the coproduct and the expression (62), the free field realization of $f_{0,1}$ is

$$f_{0,1} = \frac{h_1 h_2 h_3}{2} \sum_{l \geq 1} (l - 1)b_{-l}^{(2)}b_l^{(2)} + \frac{h_1^2 h_3^2}{2} \sum_{k,l \geq 1} (b_{-l-k}^{(2)}b_l^{(2)} + b_{-l-k}^{(2)}b_{l+k}^{(2)})
+ \frac{h_1 h_2 h_3}{2} \sum_{l \geq 1} (l - 1)b_{-l}^{(3)}b_l^{(3)} + \frac{h_1^2 h_2^2}{2} \sum_{k,l \geq 1} (b_{-l-k}^{(3)}b_l^{(3)} + b_{-l-k}^{(3)}b_{l+k}^{(3)})
+ \mu_1^{(2)} h_1 h_3 \sum_{l \geq 1} b_{-l}^{(2)}b_l^{(2)} + \mu_1^{(3)} h_1 h_2 \sum_{l \geq 1} b_{-l}^{(3)}b_l^{(3)} + h_1 h_2 h_3 \sum_{l \geq 1} b_{-l}^{(2)}b_l^{(3)}.$$  

(64)

Using $\mu_1^{(2)} = h_1 h_3 b_0^{(2)}$ and $\mu_1^{(3)} = h_1 h_2 b_0^{(3)} + h_3$, we can rewrite the expression as

$$f_{0,1} = \frac{h_1 h_2 h_3}{4} \sum_{l = -\infty}^{\infty} |l| : (b_{-l}^{(2)}b_l^{(2)} + b_{-l}^{(3)}b_l^{(3)}) : + \frac{h_1 h_2 h_3}{4} \sum_{l = -\infty}^{\infty} : (b_{-l}^{(3)}b_l^{(3)} - b_{-l}^{(2)}b_l^{(2)}) : + \sum_{k,l = -\infty}^{\infty} : \left( \frac{h_1^2 h_3^2}{6} b_{-l-k}^{(2)}b_l^{(2)} + \frac{h_1^2 h_2^2}{6} b_{-l-k}^{(3)}b_l^{(3)} \right) : + h_1 h_2 h_3 \sum_{l \geq 1} b_{-l}^{(2)}b_l^{(3)} + c$$

Let us show that this expression can be written in terms of modes of $W_1, W_2, W_3$ from [58]. Similarly as in the Virasoro case, one can easily get rid of the term containing the absolute value by subtracting a $h_1 h_2 h_3/4$ multiple of

$$\sum_{l = -\infty}^{\infty} |l| : W_{1, -l}W_{1,l} := 2 \sum_{l = -\infty}^{\infty} |l| : b_{-l}^{(2)}b_l^{(3)} : + \sum_{l = -\infty}^{\infty} |l| : b_{-l}^{(2)}b_l^{(2)} + b_{-l}^{(3)}b_l^{(3)} :.$$  

(65)

The term containing absolute value cancels and the tail of the twisted coproduct combine with the first term on the right hand side into an expression of the form of a zero mode of some vertex operator. In total, one gets

$$\frac{h_1 h_2 h_3}{4} \sum_{l \in \mathbb{Z}} : (b_{-l}^{(2)}b_l^{(2)} - b_{-l}^{(3)}b_l^{(3)}) : - \frac{h_1 h_2 h_3}{2} \sum_{l = -\infty}^{\infty} b_{-l}^{(2)}b_l^{(3)}
+ \sum_{k,l \in \mathbb{Z}} : \left( \frac{h_1^2 h_3^2}{6} b_{-l-k}^{(2)}b_l^{(2)} + \frac{h_1^2 h_2^2}{6} b_{-l-k}^{(3)}b_l^{(3)} \right) : + c.$$  

(66)

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Note now that subtracting the zero mode of the combination

\[
\frac{1}{24} (h_3 - h_2)W_3 + h_2 h_3 (W_1, W_2) + \frac{h_2^2 h_3^2}{6} (W_1, (W_1, W_1))
\]

(67)

\[
- \frac{h_2^2 h_3^2}{2 (h_2 - h_3)} (W_1, \partial W_1) + \frac{h_2^2 + 2 h_1 h_2 + 2 h_3^2}{4 (h_2 - h_3)} \partial W_2
\]

(68)

cancels the cubic term together with the sum containing \( l \). One is left with

\[
\frac{h_1 h_2 h_3}{4} \sum_{l \in \mathbb{Z}} \left( b^{(2)}_{-l} b^{(2)}_{l} - b^{(3)}_{-l} b^{(3)}_{l} \right) + \frac{h_1 h_2 h_3}{2} \frac{h_2 + h_3}{h_2 - h_3} \sum_{l=\infty}^{\infty} b^{(3)}_{-l} b^{(2)}_{l} + c
\]

(69)

Finally, subtracting

\[
\frac{1}{h_1 (h_3 - h_2)} W_{2,0} - \frac{h_1^2 + 2 h_1 h_2 + 2 h_3^2}{4 h_1 (h_2 - h_3)} (W_1 W_1),
\]

(70)

e one can get rid of all the terms containing \( b^{(j)} \). This completes the proof. \( \square \)

10.4 Comparison for \( V_{r_1, r_2, r_3} \)

To summarize, we have proven that the action of \( \text{SH}^{(0,0,2)} \) on \( V_{0,0,1} \otimes V_{0,0,1} \) factors through \( W_{0,0,2} \) and the action of \( \text{SH}^{(0,1,1)} \) on \( V_{0,1,0} \otimes V_{0,0,1} \) factors through \( W_{0,1,1} \). Analogous proposition holds also for the other combinations \( W_{0,2,0}, W_{2,0,0}, W_{1,1,0} \) simply by permuting the parameters \( h_i \). Keeping in mind the definition of \( W_{r_1, r_2, r_3} \) in terms of an intersection of screening-charges kernels, the consequence of the above is the action of \( \text{SH}^{(r_1, r_2, r_3)} \) on \( (V_{1,0,0})^{\otimes r_1} \otimes (V_{0,1,0})^{\otimes r_2} \otimes (V_{0,0,1})^{\otimes r_3} \) factors through \( W_{r_1, r_2, r_3} \). The desired action of \( W_{r_1, r_2, r_3} \) on \( V_{r_1, r_2, r_3} \) is then obtained using the hyperbolic restriction \( h : V_{r_1, r_2, r_3} \rightarrow (V_{1,0,0})^{\otimes r_1} \otimes (V_{0,1,0})^{\otimes r_2} \otimes (V_{0,0,1})^{\otimes r_3} \). This finishes the proof of Theorem 10.0.1.

A The proof of Theorem 7.1.1

In this section, we prove Theorem 7.1.1 (1). Let \( Y^+ \) be the positive part of the affine Yangian \( \hat{\mathcal{Y}}_{h_1, h_2, h_3} (\mathfrak{gl}(1)) \), and \( \text{Sh} \) be the shuffle algebra associated to the 2d COHA \( \mathcal{H}_{Q_2 = 0} \) in \( \mathfrak{t}_2 \). By Theorem 7.2.2 and Theorem 7.2.4 it suffices to show there is an algebra homomorphism from \( Y^+ \) to the shuffle algebra \( \text{Sh} \). We now check the assignment

\[
\Psi : Y^+ \rightarrow \text{Sh}, \quad e_r \mapsto \lambda' \in \mathfrak{sh}(1) \cong \mathbb{C}[\lambda]
\]

preserves the relations \( \text{Y1} \) and \( \text{Y6} \).

Let \( \lambda_{12} = \lambda_1 - \lambda_2 \), and let

\[
\text{fac}(\lambda_{12}) := \frac{(\lambda_{12} - h_1)(\lambda_{12} - h_2)(\lambda_{12} - h_3)}{\lambda_{12}}.
\]

\[
\text{fac}'(\lambda_{12}) := \text{fac}(\lambda_{21}) = \frac{(\lambda_{12} + h_1)(\lambda_{12} + h_2)(\lambda_{12} + h_3)}{\lambda_{12}}.
\]

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Let $\sigma_2 := h_1 h_2 + h_2 h_3 + h_1 h_3$. Under the condition $h_1 + h_2 + h_3 = 0$, we have the following identities.

\begin{align}
(X + h_1)(X + h_2)(X + h_3) &- (X - h_1)(X - h_2)(X - h_3) = 2h_1 h_2 h_3. \tag{71} \\
(X + h_1)(X + h_2)(X + h_3) + (X - h_1)(X - h_2)(X - h_3) = 2X^3 + 2\sigma_2 X. \tag{72}
\end{align}

Therefore,

\[
\text{fac}(\lambda_{12}) - \text{fac}'(\lambda_{12}) = \text{fac}(\lambda_{21}) = \frac{-2h_1 h_2 h_3}{\lambda_{12}},
\]
\[
\text{fac}(\lambda_{12}) + \text{fac}'(\lambda_{12}) = \text{fac}(\lambda_{12}) + \text{fac}'(\lambda_{21}) = 2\frac{\lambda_{12}^3 + \sigma_2 \lambda_{12}}{\lambda_{12}}.
\]

Let $R = \mathbb{C}[h_1, h_2]$. By the shuffle formula \cite{Shuffle}, the multiplication of $\mathbf{Sh}$ is given by

\[
R[\lambda_1] \otimes R[\lambda_2] \rightarrow R[\lambda_1, \lambda_2]^{\otimes^2},
\]
\[
(f(\lambda_1), g(\lambda_2)) \mapsto f(\lambda_1)g(\lambda_2) \text{fac}(\lambda_{12}) + f(\lambda_2)g(\lambda_1) \text{fac}'(\lambda_{12})
\]

Therefore, for any $a, b \in \mathbb{N}$, $\lambda^a \lambda^b = \lambda_1^a \lambda_3^b \text{fac}(\lambda_{12}) + \lambda_1^b \lambda_2^a \text{fac}'(\lambda_{12})$. This gives that

\[
\lambda^a \lambda^b - \lambda^b \lambda^a = -2\frac{h_1 h_2 h_3}{\lambda_{12}} \left( \lambda_1^a \lambda_2^b - \lambda_1^b \lambda_2^a \right)
\]
\[
\lambda^a \lambda^b + \lambda^b \lambda^a = \left( \lambda_1^a \lambda_2^b + \lambda_1^b \lambda_2^a \right) \frac{2\lambda_{12}^3 + \sigma_2 \lambda_{12}}{\lambda_{12}} \tag{73}
\]

Using (71), we now compute

\[
\Psi([e_{i+3}, e_j] - 3[e_{i+2}, e_{j+1}] + 3[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}] + \sigma_2([e_{i+1}, e_j] - [e_i, e_{j+1}]))
\]
\[
= -2\frac{h_1 h_2 h_3}{\lambda_{12}} \left( \lambda_1^i \lambda_2^j - \lambda_1^j \lambda_2^i - 3(\lambda_1^{i+2} \lambda_2^{j+1} - \lambda_1^{j+1} \lambda_2^{i+2}) + 3(\lambda_1^{i+1} \lambda_2^{j+2} - \lambda_1^{j+2} \lambda_2^{i+1})
\right.
\]
\[
- (\lambda_1^i \lambda_2^{j+3} - \lambda_1^{i+3} \lambda_2^j) + \sigma_2((\lambda_1^{i+3} \lambda_2^j - \lambda_1^j \lambda_2^{i+3}) - (\lambda_1^i \lambda_2^{j+1} - \lambda_1^{j+1} \lambda_2^i))
\]
\[
= -2\frac{h_1 h_2 h_3}{\lambda_{12}} \left( \lambda_1^i \lambda_2^j (\lambda_3^{i+3} + \sigma_2 \lambda_{21}) - \lambda_1^j \lambda_2^i (\lambda_3^{j+3} + \sigma_2 \lambda_{21}) \right)
\]
\[
= -2\frac{h_1 h_2 h_3}{\lambda_{12}} (\lambda_1^i \lambda_2^j + \lambda_1^j \lambda_2^i) (\lambda_3^{i+3} + \sigma_2 \lambda_{21})
\]

By (73), the above is the same as

\[
-h_1 h_2 h_3 (\lambda^i \lambda^j + \lambda^j \lambda^i) = \Psi(-\sigma_4\{e_i, e_j\}).
\]

Therefore, the assignment $\Psi$ preserves the relation (Y1).

By the shuffle formula \cite{Shuffle}, the multiplication of $\mathbf{Sh}$ is given by

\[
R[\lambda_1] \otimes R[\lambda_2, \lambda_3]^{\otimes^2} \rightarrow R[\lambda_1, \lambda_2, \lambda_3]^{\otimes^3},
\]
\[
(f(\lambda_1), g(\lambda_2, \lambda_3)) \mapsto f(\lambda_1)g(\lambda_2, \lambda_3) \text{fac}(\lambda_{12}) \text{fac}(\lambda_{13})
\]
\[
+ f(\lambda_2)g(\lambda_1, \lambda_3) \text{fac}(\lambda_{21}) \text{fac}(\lambda_{23}) + f(\lambda_3)g(\lambda_2, \lambda_1) \text{fac}(\lambda_{32}) \text{fac}(\lambda_{31}),
\]

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\[ R[\lambda_1, \lambda_2]^{\oplus 2} \otimes R[\lambda_3] \to R[\lambda_1, \lambda_2, \lambda_3]^{\oplus 3}, \]
\[ (f(\lambda_1, \lambda_2), g(\lambda_3)) \to f(\lambda_1, \lambda_2)g(\lambda_3) \text{ fac}(\lambda_{13}) \text{ fac}(\lambda_{23}) + f(\lambda_3, \lambda_2)g(\lambda_1) \text{ fac}(\lambda_{31}) \text{ fac}(\lambda_{21}) + f(\lambda_1, \lambda_3)g(\lambda_2) \text{ fac}(\lambda_{12}) \text{ fac}(\lambda_{32}). \]

Therefore, we have
\[
e_c \ast [e_a, e_b] = -2\sigma_3 \lambda_1^c \ast \left( \frac{\lambda_3^b \lambda_3^b - \lambda_2^b \lambda_2^b}{\lambda_{23}} \right) \]
\[= -2\sigma_3 \frac{(\lambda_3^a \lambda_3^b - \lambda_2^a \lambda_2^b)\lambda_3^c}{\lambda_{23}} \text{ fac}(\lambda_{12}) \text{ fac}(\lambda_{13}) + 2\sigma_3 \frac{(\lambda_3^a \lambda_2^b - \lambda_1^a \lambda_2^b)\lambda_2^c}{\lambda_{13}} \text{ fac}(\lambda_{21}) \text{ fac}(\lambda_{23}) - 2\sigma_3 \frac{(\lambda_3^a \lambda_1^b - \lambda_2^a \lambda_1^b)\lambda_1^c}{\lambda_{21}} \text{ fac}(\lambda_{32}) \text{ fac}(\lambda_{31}), \]

and
\[
[e_a, e_b] \ast e_c = -2\sigma_3 \frac{(\lambda_2^a \lambda_2^b - \lambda_1^a \lambda_2^b)\lambda_3^c}{\lambda_{12}} \ast \lambda_3^c \]
\[= -2\sigma_3 \frac{(\lambda_3^a \lambda_3^b - \lambda_2^a \lambda_3^b)\lambda_3^c}{\lambda_{12}} \text{ fac}(\lambda_{13}) \text{ fac}(\lambda_{23}) - 2\sigma_3 \frac{(\lambda_3^a \lambda_2^b - \lambda_1^a \lambda_2^b)\lambda_3^c}{\lambda_{32}} \text{ fac}(\lambda_{31}) \text{ fac}(\lambda_{21}) - 2\sigma_3 \frac{(\lambda_3^a \lambda_1^b - \lambda_2^a \lambda_1^b)\lambda_3^c}{\lambda_{31}} \text{ fac}(\lambda_{12}) \text{ fac}(\lambda_{32}). \]

We compute
\[
[e_c, [e_a, e_b]] = e_c \ast [e_a, e_b] - [e_a, e_b] \ast e_c \]
\[= -2\sigma_3 \frac{(\lambda_3^a \lambda_3^b - \lambda_2^a \lambda_3^b)\lambda_3^c}{\lambda_{23}} \text{ fac}(\lambda_{12}) \text{ fac}(\lambda_{13}) - 2\sigma_3 \frac{(\lambda_3^a \lambda_2^b - \lambda_1^a \lambda_2^b)\lambda_3^c}{\lambda_{13}} \text{ fac}(\lambda_{21}) \text{ fac}(\lambda_{23}) - 2\sigma_3 \frac{(\lambda_3^a \lambda_1^b - \lambda_2^a \lambda_1^b)\lambda_3^c}{\lambda_{21}} \text{ fac}(\lambda_{32}) \text{ fac}(\lambda_{31}) + 2\sigma_3 \frac{(\lambda_3^a \lambda_3^b - \lambda_2^a \lambda_3^b)\lambda_3^c}{\lambda_{12}} \text{ fac}(\lambda_{13}) \text{ fac}(\lambda_{32}) + 2\sigma_3 \frac{(\lambda_3^a \lambda_2^b - \lambda_1^a \lambda_2^b)\lambda_3^c}{\lambda_{32}} \text{ fac}(\lambda_{31}) \text{ fac}(\lambda_{12}) + 2\sigma_3 \frac{(\lambda_3^a \lambda_1^b - \lambda_2^a \lambda_1^b)\lambda_3^c}{\lambda_{31}} \text{ fac}(\lambda_{12}) \text{ fac}(\lambda_{32}) - 2\sigma_3 \frac{(\lambda_3^a \lambda_3^b - \lambda_2^a \lambda_3^b)\lambda_3^c}{\lambda_{23}} \left( \frac{\text{ fac}(\lambda_{12}) \text{ fac}(\lambda_{13})}{\lambda_{23}} + \frac{\text{ fac}(\lambda_{21}) \text{ fac}(\lambda_{31})}{\lambda_{32}} \right) - 2\sigma_3 \frac{(\lambda_3^a \lambda_2^b - \lambda_1^a \lambda_2^b)\lambda_3^c}{\lambda_{13}} \left( \frac{\text{ fac}(\lambda_{21}) \text{ fac}(\lambda_{23})}{\lambda_{13}} + \frac{\text{ fac}(\lambda_{12}) \text{ fac}(\lambda_{32})}{\lambda_{31}} \right) - 2\sigma_3 \frac{(\lambda_3^a \lambda_1^b - \lambda_2^a \lambda_1^b)\lambda_3^c}{\lambda_{21}} \left( \frac{\text{ fac}(\lambda_{32}) \text{ fac}(\lambda_{31})}{\lambda_{21}} + \frac{\text{ fac}(\lambda_{32}) \text{ fac}(\lambda_{12})}{\lambda_{12}} \right). \]

Plug the above formula into the following
\[ \text{Sym}_{\mathfrak{S}_3}[e_{i_1}, [e_{i_2}, e_{i_3+1}]] = [e_{i_1}, [e_{i_2}, e_{i_3+1}]] + [e_{i_2}, e_{i_3+1}] + [e_{i_1}, [e_{i_2}, e_{i_3+1}]] + [e_{i_1}, [e_{i_2}, e_{i_3+1}]] + [e_{i_1}, [e_{i_3}, e_{i_2+1}]] + [e_{i_2}, e_{i_3+1}] + [e_{i_3}, [e_{i_2}, e_{i_2+1}]] + [e_{i_3}, [e_{i_2}, e_{i_2+1}]] \quad (74) \]
The term in (74) involving \( \lambda_1^i \lambda_2^j \lambda_3^k \) is

\[
-2\sigma_3 \lambda_1^i \lambda_2^j \lambda_3^k \left( -\text{fac}(\lambda_{12}) \text{fac}(\lambda_{13}) + \text{fac}(\lambda_{21}) \text{fac}(\lambda_{31}) - \text{fac}(\lambda_{21}) \text{fac}(\lambda_{23}) + \text{fac}(\lambda_{12}) \text{fac}(\lambda_{32}) - \text{fac}(\lambda_{32}) \text{fac}(\lambda_{31}) + \text{fac}(\lambda_{23}) \text{fac}(\lambda_{13}) \right) = 0
\]

By symmetry, all other terms involving \( \lambda_1^a \lambda_2^b \lambda_3^c \), for \( \{a, b, c\} = \{1, 2, 3\} \), in (74) are zero. Therefore, \( \Psi(\text{Sym}_3[e_{i_1}, [e_{i_2}, e_{i_3}+1]]) = 0 \). This completes the proof. ■

B The proof of Proposition 6.4.1

In this section, we prove Proposition 6.4.1.

Lemma B.0.1 Notations as in § 6.4 let \( \mathcal{V}_n \) and \( \mathcal{E}_\phi \) be the tautological bundles on \( \mathcal{M}_\phi(n) \). We have

1. \( p^*(\mathcal{V}_{n'}) \otimes \mathcal{V}_{n''} = \eta^* \mathcal{V}_n \);
2. \( p^*(\mathcal{E}_{\phi'}) \otimes \mathcal{E}_{\phi''} = \eta^* \mathcal{E}_{\phi} \);
3. Consequently, for \( \psi(z) \in \mathcal{H}^0 \), and for all \( x \in V_{r_1, r_2, r_3} \), we have

\[
\Delta(\psi(z)) \cdot l(x) = l(\psi(z) \cdot x)
\]

Proof. By the definition of \( l \), it suffices to show that

\[
\eta_x^{st} \circ (p^{st})^*(\Delta(\psi(z)) \cdot l(x)) = \psi(z) \cdot x.
\]

By definition \( \Delta(\psi(z)) = \psi(z) \otimes \psi(z) \) and \( \psi(z) \cdot x = \lambda_{-1/z}(\mathcal{F}_{\phi, \phi}) \cdot x \). Therefore,

\[
\eta_x^{st} \circ (p^{st})^*(\Delta(\psi(z)) \cdot l(x)) = \eta_x^{st} \circ (p^{st})^*(\psi(z) \otimes \psi(z) \cdot l(x)) = \eta_x^{st} \circ (p^{st})^*(\lambda_{-1/z}(\mathcal{F}_{n', \phi'} \otimes \mathcal{F}_{n'', \phi''}) \cdot l(x)) = \lambda_{-1/z}(\mathcal{F}_{\phi, \phi}) \cdot \eta_x^{st}(p^{st})^*(l(x)) = \psi(z) \cdot l^{-1}(x) = \psi(z) \cdot x.
\]

This completes the proof. ■

We now prove Proposition 6.4.1. Lemma B.0.1 implies Proposition 6.4.1 when \( \alpha \) is an element in \( \mathcal{H}^0 \). We will now focus on the case when \( \alpha \in \mathcal{S}\mathcal{H}^{(Q_3, W_3)} \). The proof below is similar to the proof of associativity of the Hall multiplication.

Similar to the correspondence \( Z(\mathcal{F}, \mathcal{F})_r(n', n'') \), we also have \( Z(\mathcal{F}, \mathcal{F})_r(n' + 1, n'') \) and \( Z(\mathcal{F}, \mathcal{F})_r(n', n'' + 1) \). The diagram (17) and correspondence (21) induce the following di-
The maps in the above diagram are schematically represented as follows. Here the numbers $1, n', n''$ in the rectangle are the sizes of the corresponding matrices, which also schematically represents the subquotients in (75).

and

and

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We have two ways from the lower left corner \((\mathfrak{gl}_1)^3 \times \mathcal{M}_{\alpha}^f(n)\) to the upper right corner \((\mathcal{M}_{\alpha}^f(n'+1) \times \mathcal{M}_{\alpha}^f(n'')) \cup (\mathcal{M}_{\alpha}^f(n') \times \mathcal{M}_{\alpha}^f(n'+1))\). The corresponding maps
\[
H^*_c((\mathfrak{gl}_1)^3 \times \mathcal{M}_{\alpha}^f(n); \varphi_{\alpha})^\vee \to (V_{\alpha}(n'+1) \otimes V_{\alpha}(n'')) \bigoplus (V_{\alpha}(n') \otimes V_{\alpha}(n'+1))
\]
will simply be referred to as Way 1 and Way 2.

Way 1: using the bottom horizontal and right vertical correspondences of (75) (76), and follow the standard procedure as in §5.

Way 2: using the left vertical and top horizontal correspondences of (75) (76), and follow the standard procedure as in §6.

For any \(\alpha \in \mathcal{SH}(Q^3, W_3)\), \(x \in V_{\alpha}\), clearly Way 1 applied to \((\alpha \otimes x)\) gives \(l(\alpha \bullet x)\) by (75), (76). We claim that Way 2 applied to \((\alpha \otimes x)\) gives \(\Delta(\alpha) \bullet l(x)\). Then, convolutions on the level of critical cohomology of diagrams (75), (76) lead to \(l(\alpha \bullet x) = \Delta(\alpha) \bullet l(x)\). This in turn implies Proposition 6.4.1.

In the rest of this section we present the proof in relative details.

**Lemma B.0.2** Notations are as follows.

\[
\begin{align*}
(\mathfrak{gl}_1)^3 \times \mathcal{M}_{\alpha}^f(n) & \xrightarrow{p_1} Z_{(0, \alpha)}(1, n)^{st} \xrightarrow{\eta_1} \mathcal{M}_{\alpha}^f(1 + n) \\
\mathcal{M}_{\alpha}^f(n) \times (\mathfrak{gl}_1)^3 & \xrightarrow{p_2} Z_{(\bar{\alpha}, 0)}(n, 1)^{st} \xrightarrow{\eta_2} \mathcal{M}_{\alpha}^f(1 + n)
\end{align*}
\]

(77)

For \(\alpha(\lambda) \in \mathcal{H}^{Q_3, W_3}(1) = \mathbb{C}[\lambda]\), and \(x \in V_{r_1, r_2, r_3}(n)\), we have the equality
\[
(\eta_1)_\ast (p_1)\ast (\alpha(\lambda), x) = (\eta_2)_\ast (\psi(\lambda) \cdot (p_2)\ast (x, \alpha(\lambda)))
\]

**Proof.** Let \(V\) (resp. \(E_{r_i}, i = 1, 2, 3\)) be the tautological dimension \(n\) (resp. dimension \(r_i, i = 1, 2, 3\)) bundle on \(\mathcal{M}_{r_1, r_2, r_3}(n)\) and \(V_1\) is the tautological line bundle on \(\mathcal{M}_{0,0,0}(1)\).

The extension \(\eta_2\) in (77) consists of
\[
\{(B_i, I_{ab}, J_{ab})_{i \in \mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{a}_3} \in \mathcal{M}_F(1 + n) \mid B_i(V_1) \subset V_1, J_{ab}(V_1) = 0\},
\]
and the extension \(\eta_1\) consists of
\[
\{(B_i, I_{ab}, J_{ab})_{i \in \mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{a}_3} \in \mathcal{M}_F(1 + n) \mid B_i(V) \subset V, I_{ab}(E_{r_i}) \subset V, a \neq b \neq c\}.
\]

Therefore, the difference of \(\eta_2\) and \(\eta_1\) is
\[
\left(\text{Hom}(V_1, V) \oplus \text{Hom}(V_1, E_{r_1}) \oplus \text{Hom}(V_1, E_{r_2}) \oplus \text{Hom}(V_1, E_{r_3})\right)
\]
\[
- \left(\text{Hom}(V, V_1) \oplus \text{Hom}(E_{r_1}, V_1) \oplus \text{Hom}(E_{r_2}, V_1) \oplus \text{Hom}(E_{r_3}, V_1)\right)
\]

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Let $\lambda$ be the Chern root of $V_1$, $\lambda_1, \cdots, \lambda_n$ the Chern roots of $V$, and $\mu_1, \cdots, \mu_{r_1}$ the Chern roots of $E_{r_1}$, $i = 1, 2, 3$. Taking into account the torus action in $\mathcal{E}_{r_1}$ we have

$$eu\bigg(\text{Hom}(V_1, q_1 V) \oplus \text{Hom}(V_1, q_2 V) \oplus \text{Hom}(V_1, q_3 V)\bigg)$$

$$- \text{Hom}(V, q_1 V_1) \oplus \text{Hom}(V, q_2 V_1) \oplus \text{Hom}(V, q_3 V_1)\bigg) = eu(V_1^* \otimes q_1 V_1)eu(V_1^* \otimes q_2 V_1)eu(V_1^* \otimes q_3 V_1)$$

$$= \prod_{i=1}^{n} \frac{\lambda_i - \lambda + h_1}{\lambda - \lambda_i} \prod_{i=1}^{n} \frac{\lambda_i - \lambda + h_2}{\lambda - \lambda_i} \prod_{i=1}^{n} \frac{\lambda_i - \lambda + h_3}{\lambda - \lambda_i}$$

$$= (-1)^{3n} \prod_{i=1}^{n} \frac{\lambda - \lambda_i - h_1}{\lambda - \lambda_i} \prod_{i=1}^{n} \frac{\lambda - \lambda_i - h_2}{\lambda - \lambda_i} \prod_{i=1}^{n} \frac{\lambda - \lambda_i - h_3}{\lambda - \lambda_i}$$

$$eu\bigg(\text{Hom}(V_1, q_2 q_3 E_{r_1}) - \text{Hom}(E_{r_1}, V_1)\bigg) = eu\bigg(V_1^* \otimes q_2 q_3 E_{r_1} - E_{r_1}^* \otimes V_1\bigg)$$

$$= \prod_{i=1}^{r_1} \frac{\mu_i - \lambda - h_1}{\lambda - \mu_i} = (-1)^{r_1} \prod_{i=1}^{r_1} \frac{\lambda - \mu_i + h_1}{\lambda - \mu_i}.$$

Similarly, we have

$$eu\bigg(\text{Hom}(V_1, q_1 q_3 E_{r_2}) - \text{Hom}(E_{r_2}, V_1)\bigg) = (-1)^{r_2} \prod_{i=1}^{r_2} \frac{\lambda - \mu_i + h_2}{\lambda - \mu_i},$$

$$eu\bigg(\text{Hom}(V_1, q_1 q_2 E_{r_3}) - \text{Hom}(E_{r_3}, V_1)\bigg) = (-1)^{r_3} \prod_{i=1}^{r_3} \frac{\lambda - \mu_i + h_3}{\lambda - \mu_i}.$$

Therefore,

$$eu\bigg((\text{Hom}(V_1, V) \oplus \text{Hom}(V_1, E_{r_1}) \oplus \text{Hom}(V_1, E_{r_2}) \oplus \text{Hom}(V_1, E_{r_3}))$$

$$- (\text{Hom}(V, V_1) \oplus \text{Hom}(E_{r_1}, V_1) \oplus \text{Hom}(E_{r_2}, V_1) \oplus \text{Hom}(E_{r_3}, V_1))\bigg)$$

$$= (-1)^{3n + r_1 + r_2 + r_3} \left( \prod_{a=1}^{r_1} \frac{\lambda - \mu_a + h_1}{\lambda - \mu_a} \prod_{b=1}^{r_2} \frac{\lambda - \mu_b + h_2}{\lambda - \mu_b} \prod_{c=1}^{r_3} \frac{\lambda - \mu_c + h_3}{\lambda - \mu_c} \right)$$

$$\cdot \prod_{i=1}^{n} \frac{\lambda - \lambda_i - h_1}{\lambda - \lambda_i} \prod_{i=1}^{n} \frac{\lambda - \lambda_i - h_2}{\lambda - \lambda_i} \prod_{i=1}^{n} \frac{\lambda - \lambda_i - h_3}{\lambda - \lambda_i}.$$

Recall the $\mathcal{H}^0$-action on $x \in V_{r_1, r_2, r_3}(n)$ is given by

$$\psi(z) \cdot x = \left( \prod_{a=1}^{r_1} \frac{z - \mu_a + h_1}{z - \mu_a} \prod_{b=1}^{r_2} \frac{z - \mu_b + h_2}{z - \mu_b} \prod_{c=1}^{r_3} \frac{z - \mu_c + h_3}{z - \mu_c} \right) x \in V_{r_1, r_2, r_3}(n).$$

The formula (78) coincides (up to a sign $(-1)^{3n + r_1 + r_2 + r_3}$) with $\psi(\lambda) \cdot x$ replacing $z$ by $\lambda$.

This completes the proof.}$
Lemma B.0.3  Way 2 gives $\Delta(\alpha) \bullet l(x)$.

Proof. Assume $\alpha = f(\lambda)$ is an element in $\mathcal{H}^{(Q_3, W_3)}(1) = \mathbb{C}[\lambda]$, for some polynomial $f$. By definition of the Drinfeld coproduct, we have

$$\Delta(\alpha) = \psi(\lambda) \otimes \alpha + \alpha \otimes 1$$

$$= 1 \otimes \alpha - (h_1 h_2 h_3) \sum_{j \geq 0} \psi_j \otimes \lambda^{j+1} \alpha + \alpha \otimes 1 \in Y^{\geq 0} \otimes Y^{\geq 0}.$$

For $\alpha \in \mathcal{H}^{(Q_3, W_3)}(1)$, and $x_1 \in V_{\vec{r}}, x_2 \in V_{\vec{r}'}$, applying pullback and pushforward via the following correspondence,

$$\begin{align*}
(gl_1)^3 \times M_{\vec{r}}(n') & \xrightarrow{Z_{\vec{0}, \vec{r}'}(1, n') st} M_{\vec{r'}}(n'') \times \left( M_{\vec{r}}(n' + 1) \times M_{\vec{r}''}(n'') \right) 
\end{align*}$$

we obtain $(\alpha \otimes 1) \bullet (x_1 \otimes x_2) = (\alpha \bullet x_1) \otimes x_2$.

We now use the following correspondence

$$\begin{align*}
(gl_1)^3 \times M_{\vec{r}}(n') \times M_{\vec{r}'}(n'') & \xrightarrow{M_{\vec{r}}(n') \times Z_{\vec{0}, \vec{r}'}(1, n') st} \left( M_{\vec{r}'}(n') \times M_{\vec{r}''}(n' + 1) \right) 
\end{align*}$$

Notice that we need to switch $(gl_1)^3$ with $M_{\vec{r}}(n')$ in order to apply pullback and pushforward to the above correspondence. By Lemma B.0.2, we obtain $(\psi(\lambda) \otimes \alpha) \bullet (x_1 \otimes x_2)$.

Therefore, Way 2 gives $\Delta(\alpha) \bullet l(x)$. This completes the proof. ■

C  More on the coproduct

In this section we prove Proposition 8.2.3 and Proposition 8.2.2. The outline of the proofs is as follows:

1. Prove the primitivity of the Heisenberg operator $B_1$ using the definition of the hyperbolic localization.

2. Take the opposite $A$-action on $M_{\vec{r}}$, and the comparison of these two $A$-actions defines an $R$-matrix. This $R$-matrix defines a subalgebra of $\text{End}(\oplus V_{\vec{r}})$ (a generalization of the Maulik-Okounkov Yangian in this 3d setting). Step 1, together with the geometric interpretations of the central and Cartan elements shows that $SH^\vec{r}$ has a natural map to this subalgebra.

3. When two of the three coordinates of $\vec{r} = (r_1, r_2, r_3)$ are zero, the hyperbolic localization $h : V_{\vec{r}} \to V_{\vec{r}} \otimes V_{\vec{r}'}$ under the dimensional reduction, agrees with the one from Nakajima and hence the stable envelope of Maulik-Okounkov.

4. Observe that the map $\Delta^\vec{r} : SH^\vec{r} \to (SH^\vec{r})^{\otimes 2}$ is uniquely determined by the fact that, when applied to modules $V_{\vec{r}}$ with two of the three coordinates of $\vec{r} = (r_1, r_2, r_3)$ being zero, it agrees with $[56, 61, 79]$. This in particular proves both Proposition 8.2.2 and Proposition 8.2.3.

These will be done respectively in the rest of this section. Through out this section, for simplicity we ignore homological degree shiftings.
C.0.1 The primitivity of $B_1$

Consider the following commutative diagram

$$
\begin{array}{c}
\mathcal{M}_r(n) \xrightarrow{\eta_1} A_{\mathcal{M}_r(n)} & \xrightarrow{p_1} \mathcal{M}_r(n)^A & \xrightarrow{p_1} \mathcal{M}_r(n) (n_1) \times \mathcal{M}_r(n_2) \\
\xrightarrow{q_1} & \xrightarrow{p_2} & \xrightarrow{q_3} \\
C_r(n, n + 1) \xrightarrow{q_2} A & \xrightarrow{b_2} C_r(n, n + 1)^A & \xrightarrow{b_3} \mathcal{M}_r(n_1 + 1) \times \mathcal{M}_r(n_2 + 1) \\
\xrightarrow{b_2} & \xrightarrow{p_3} & \xrightarrow{b_3} \\
\mathcal{M}_r(n + 1) \xrightarrow{\eta_2} A_{\mathcal{M}_r(n+1)} & \xrightarrow{p_3} \mathcal{M}_r(n + 1)^A & \xrightarrow{p_3} \mathcal{M}_r(n + 1) (n_1 + 1) \times \mathcal{M}_r(n_2 + 1) \\
\end{array}
$$

(79)

The right two squares in the diagram are not Cartesian. Nevertheless, the maps $p_2$ and $p_3$ factorize with $p_2''$ and $p_3''$ affine bundles, and the squares with $p_1, q_2, p_2', q_3$ and $p_2, b_2, p_3', b_3'$ both Cartesian. We remark that $p_2''$ is identity on the component with $(n_2 + 1)$ and $p_3''$ is identity on the component with $(n_1 + 1)$.

Note also that $q_2' : H^s(\mathcal{M}_r(n), \varphi) \to H^s(C_r(n, n + 1), \varphi)$ and similarly $q_3'$ are well-defined only on the correspondence where dimensions differ by 1, and hence, the potential function on $C_r(n, n + 1)$ is the pullback of the potential function on $\mathcal{M}_r(n)$.

Composing $D_{p_{3\mathcal{M}_r(n+1)}}(id \to \eta_3\eta_3')$, and $(id \to b_1'\eta_3')$, we get a commutative diagram

$$
\begin{array}{c}
D_{p_{3\mathcal{M}_r(n+1)}}(id \to b_1'\eta_3') \\
\xrightarrow{\eta_1} \xrightarrow{\eta_2} \xrightarrow{\eta_3'} \xrightarrow{\eta_3} \\
D_{p_{3\mathcal{M}_r(n+1)}}(id \to \eta_3\eta_3')b_1'\eta_3' \xrightarrow{\eta_1} C_{\mathcal{M}_r(n+1)} \\
\end{array}
$$

We have $p_{3\mathcal{M}_r(n+1)}\eta_3 = p_{3\mathcal{M}_r(n+1)}\eta_3'$. We compare $p_{3\mathcal{M}_r(n+1)}\eta_3'\eta_3'(id \to b_1'\eta_3')$ and $p_{3\mathcal{M}_r(n+1)}\eta_3'(id \to b_3'\eta_3')$. As we have

$$
b_3' b_3^* p_3^* \eta_3' = b_3^* b_3' p_3^* \eta_3' \Rightarrow b_3' b_3^* p_3^* \eta_3' = b_3' b_3^* p_3^* \eta_3' = b_3' b_3^* p_3^* \eta_3',
$$

we get a natural transform $(p_{3\mathcal{M}_r(n+1)}\eta_3'(id \to b_1'\eta_3')) \to (p_{3\mathcal{M}_r(n+1)}\eta_3'(id \to b_1'\eta_3'))$, which therefore gives a commutative diagram

$$
\begin{array}{c}
D_{p_{3\mathcal{M}_r(n+1)}}(id \to b_1'\eta_3') \xrightarrow{\eta_3'} \xrightarrow{\eta_3'} \xrightarrow{\eta_3'} \xrightarrow{\eta_3'} \\
\xrightarrow{\eta_1} \xrightarrow{\eta_2} \xrightarrow{\eta_3'} \xrightarrow{\eta_3'} \\
D_{p_{3\mathcal{M}_r(n+1)}}(id \to b_3'\eta_3') C_{\mathcal{M}_r(n+1)} \xrightarrow{\eta_1} \xrightarrow{\eta_2} \xrightarrow{\eta_3'} C_{\mathcal{M}_r(n+1)} \\
\end{array}
$$

This proves the commutativity of hyperbolic restrictions and pushforwards in the diagram (79).
Similarly, composing $\mathcal{D}\eta_1\eta_1^i$, id $\rightarrow \eta_1\eta_1^i$, and $(q_1^i q_1^i \rightarrow \text{id})$, we get a commutative square

\[
\begin{array}{c}
\mathcal{D}\eta_1\eta_1^i (q_1^i q_1^i \varphi) C_{3\mathcal{R}(n)}(T) \\
\downarrow \\
\mathcal{D}\eta_1\eta_1^i \eta_1 \eta_1^i \varphi C_{3\mathcal{R}(n)}(T) \\
\uparrow \\
\mathcal{D}\eta_1\eta_1^i (q_1^i q_1^i \varphi) C_{3\mathcal{R}(n)}(T)
\end{array}
\]

We have $p_{3\mathcal{R}(n)}(\eta_1^i) = p_{3\mathcal{R}(n),i} p_1$. Now we compare $p_{1+i}(q_1^i q_1^i \rightarrow \text{id})$ and $(q_3^i q_3^i \rightarrow \text{id}) p_1 \eta_1^i$. As we have

\[
q_3^i q_3^i i^i \eta_1^i \rightarrow q_3^i q_3^i i^i \eta_1^i = q_3^i q_3^i i^i \eta_1^i = p_{1+i} q_1^i q_1^i \eta_1^i = p_{1+i} q_1^i q_1^i \eta_1^i = p_{1+i} q_1^i q_1^i \eta_1^i,
\]

we get a natural transform $(p_{1+i}(q_1^i q_1^i \rightarrow \text{id})) \rightarrow ((q_3^i q_3^i \rightarrow \text{id}) p_1 \eta_1^i)$, which therefore gives a commutative diagram

\[
\begin{array}{c}
\mathcal{D}\eta_1\eta_1^i (q_1^i q_1^i \varphi) C_{3\mathcal{R}(n)}(T) \\
\downarrow \\
\mathcal{D}\eta_1\eta_1^i (q_3^i q_3^i \varphi) C_{3\mathcal{R}(n)}(T) \\
\uparrow \\
\mathcal{D}\eta_1\eta_1^i (q_1^i q_1^i \varphi) C_{3\mathcal{R}(n)}(T)
\end{array}
\]

This proves the commutativity of hyperbolic restrictions and pullbacks in the diagram (79).

\section*{C.0.2 The $R$-matrices}

Using the correspondence

\[
\mathcal{M}_\rho \overset{\varphi}{\longrightarrow} A_{3\mathcal{R}(n)} \overset{\eta}{\longrightarrow} \mathcal{M}_\rho^t,
\]

we have constructed the hyperbolic restriction map

\[
h : V_{r_1,r_2,r_3} \rightarrow V_{r_1',r_2',r_3'}, \quad V_{r_1',r_2',r_3'} \oplus V_{r_1'',r_2'',r_3''},
\]

where $(r_1, r_2, r_3) = (r_1', r_2', r_3') + (r_1'', r_2'', r_3'')$. Note that the map $h$ depends on the torus $A$ action $t \rightarrow A(t)$, $t \in \mathbb{C}^*$. Consider the opposite action of $A$, that is, the action given by $t \rightarrow A(t^{-1})$, $t \in \mathbb{C}^*$. This opposite action gives rise to another hyperbolic restriction map

\[
h^{op} : V_{r_1,r_2,r_3} \rightarrow V_{r_1',r_2',r_3'} \oplus V_{r_1'',r_2'',r_3''},
\]

which is an isomorphism after localization by the localization theorem in equivariant cohomology.

Similarly to [56], we define the R-matrix to be

\[
h^{op} \circ h^{-1} : V_{r_1,r_2,r_3} \otimes V_{r_1',r_2',r_3'} \rightarrow (V_{r_1,r_2',r_3'} \otimes V_{r_1',r_2'',r_3''}).
\]

The geometrically defined R-matrix gives rise to a Yangian $\mathcal{Y}(Q_{3\mathcal{R}(n),W_{3\mathcal{R}(n)}})$ in the 3d case using the RTT=TTR formalism (see [56] Section 5.2) for details). In particular, $\mathcal{Y}(Q_{3\mathcal{R}(n),W_{3\mathcal{R}(n)}})$ is a subalgebra

\[
\mathcal{Y}(Q_{3\mathcal{R}(n),W_{3\mathcal{R}(n)}}) \subset \prod_{i=1}^{n} \text{End}(F_i(u_1) \otimes F_2(u_2) \cdots \otimes F_n(u_n)),
\]

where $F_i$'s are the representations $V_{r_1,r_2,r_3}$, for some $(r_1, r_2, r_3) \in \mathbb{Z}_{\geq 0}$, and $F_i(u) = F_i \otimes \mathbb{C}[u]$.  

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Lemma C.0.1 There is an algebra homomorphism $\text{SH}^\vec{r} \to \mathcal{Y}_{(Q^{r_i}_i, W^{r_j}_j)}$, such that the following diagram commutes.

$$
\begin{array}{c}
\text{SH}^\vec{r} \longrightarrow \text{End}(\oplus_r V_r) \\
\downarrow \\
\mathcal{Y}_{(Q^{r_i}_i, W^{r_j}_j)}
\end{array}
$$

Proof. The algebra $\text{SH}^\vec{r}$ is generated by $B_1, B_{-1}$ and $\text{SH}^{\vec{0}}$. To show $\text{SH}^\vec{r}$ maps to $\mathcal{Y}_{(Q^{r_i}_i, W^{r_j}_j)}$, we only need to show the actions of $B_1, B_{-1}$ and $\text{SH}^{\vec{0}}$ on $V_r$ come from the Yangian, for any $\vec{r} \in \mathbb{Z}_{\geq 0}^3$. By Section [6] the action of $\text{SH}^{\vec{0}}$ is given by the Chern classes of the tautological bundles, therefore, it lies in the Yangian. By step 1, the element $\vec{r}$ is primitive, thus, it suffice to consider its action on $V_r$, for $r \in \mathbb{Z}_{\geq 0}$. In this 2d case, it is known that $B_1$ is in the image of the map $\mathcal{Y}_{(Q^{r_i}_i, W^{r_j}_j)} \to \text{End}_{r \in \mathbb{Z}_{\geq 0}}(V_r)$. Similar argument holds for $B_{-1}$. ■

Note that in particular, the generators of $\text{SH}^\vec{r}$ are closed under the comultiplication on $\mathcal{Y}_{(Q^{r_i}_i, W^{r_j}_j)}$, which in turn is induced by $h$. Denote this comultiplication by $\Delta^h$. In § C.0.4 we will show $\Delta^h$ agrees with $\Delta^\vec{r}$ from Proposition 8.2.2

C.0.3 Dimensional reduction

Now we work with $\vec{r} = (r_1, r_2, r_3)$ such that two of the three coordinates are zero. Without loss of generality, assume $r_1 = r \neq 0, r_2 = r_3 = 0$. In this section, write $X = \{(B_2, B_3, I_{23}, J_{23}) \in \mathcal{M}_r(n) \mid C(B_2, B_3)/I_{23}(C^{23}) = C^{n_1}/GL_{n_1}$, and hence by the definition of stability conditions we have $\mathcal{M}_r(n) = X \times \mathfrak{gl}_n$, endowed with a projection $\pi_X : X \times \mathfrak{gl}_n \to X$. Following the notations as in [14], here we write $Z \subset X$ be the locus consisting of orbits of $(B_2, B_3, I_{23}, J_{23})$ where $tr_B(B_{ij}, I_{ij}, J_{ij})k, i,j = 1,2,3 = 0$ for all $B_1 \in \mathfrak{gl}_n$, with the natural embedding $i_Z : Z \to X$. Note that $Z$ is isomorphic to the Nakajima quiver variety of dimension $n$ and framing $r$.

Let $A \subset GL_r$ be such that $\mathcal{M}_r(n) = \mathcal{M}_r(n') \times \mathcal{M}_r(n'')$. We focus on a component labeled by $n' + n'' = n$, and denote the fixed points loci and the attracting loci by $(X \times \mathfrak{gl}_n)^A \cong X^A \times 1, X^A, Z^A, A_X \times \mathfrak{gl}_n \cong A_X \times p, A_X, A_Z$ respectively. We have the following diagram.

![Diagram](image_url)
Here we introduce the variety

\[ Y = \{(c, x, x^*) \mid c \in p, x \in I, x^* \in I \mid c_1 = [x, x^*]\}, \]

where \( c_1 \in I \) is the projection of \( c \in p \) under the natural map \( p \to I \). Under \( p \cong n \oplus I \), we decompose \( c \) as \( c = c_1 + c_n \), where \( c_n \) is the corresponding element in \( n \). Note that we have the isomorphism \( Y \cong n \times I^2, (c, x, x^*) \mapsto (c_n, x, x^*) \). The maps are given by \( q_X' : A_X \to Y, (x, x^*) \mapsto ([x, x^*], x, x^*) \), \( d_Y : Y \to X^A, (c, x, x^*) \mapsto (x, x^*) \), and \( i_Y : Z^A \to Y, (x, x^*) \mapsto (0, x, x^*) \). It is straightforward to check that the square with \( i_Y, q_X', q_Z, \) and \( i_{A_Z} \) is Cartesian. The maps \( c, d \) are natural projections. Therefore, the square with \( c, \pi_{X,A}^*, q_X, \) and \( \pi_{A_X} \) is Cartesian; the two left squares are both Cartesian.

The following identities are easy to prove:

1. \( q_{Z!} j_Z^* i_Z^* = i_X^! q_Z^! j_Z^! \)
2. \( q_X \times_{\mathfrak{gl}_n} j_X^! X \times_{\mathfrak{gl}_n} \pi_X^* = d d^* \pi_{X,A}^* X \times_{\mathfrak{gl}_n} j_X^! \)
3. \( i_{Z!}^! \pi_X^! = i_{Z!}^! d_Y^! q_Y^* = i_Y^! d_Y^! q_Y^* \)
4. \( \pi_{X,A}^* X \times_{\mathfrak{gl}_n} j_X^! = \pi_X^! q_X^! j_X^! \)
5. \( i_{Z!}^* \pi_{X,A}^* X \times_{\mathfrak{gl}_n} j_X^! = i_{Z!}^* \pi_X^! q_X^! j_X^! \)

We have the dimensional reduction \( \pi_X^! \pi_{X,Z, A}^* i_Z^* = \pi_X^! \varphi_{tr} W \pi_A^* \) and \( \pi_X^! \pi_{X,Z, A}^* i_Z^* = \pi_X^! \varphi_{tr} W \pi_A^* \).

We write \( \mathfrak{g} = \mathfrak{gl}_n \) for short. We have the natural isomorphisms of functors

\[ q_X^! j_X^! \pi_X^! \varphi_{tr} \pi_X^* = \pi_X^! \varphi_{tr} q_X \times_{\mathfrak{gl}_n} j_X^! \pi_X^* = \pi_X^! \varphi \pi_X^* q_X \times_{\mathfrak{gl}_n} j_X^! \pi_X^* \]

and

\[ q_X^! j_X^! \pi_X^! \varphi_{tr} \pi_X^* = \pi_X^! \varphi_{tr} q_X \times_{\mathfrak{gl}_n} j_X^! \pi_X^* \]

with the equalities given by (4), (2, (5), (1), (3) respectively.

**Lemma C.0.2** The following diagram is commutative.

\[ \begin{array}{ccc}
q_X^! j_X^! \pi_X^! & \xrightarrow{\cong} & \pi_X^! \varphi_{tr} \pi_{X,A}^* \pi_X^! j_X^! \\
\downarrow & & \downarrow \\
q_X^! j_X^! \pi_X^! & \xrightarrow{\cong} & \pi_X^! \varphi_{tr} \pi_{X,A}^* i_Z^* q_X^! j_X^! \\
\end{array} \quad (80) \]

with the vertical arrows the dimensional reductions, and horizontal arrows given as above.

**Proof.** We follow a similar strategy as in [14 § A1], and by the same construction as in loc. cit. we have \( i_Z, \pi_{A_Z}, \) and \( \pi_{Z,A} \), with the natural transforms \( \pi_X^! i_Z^* \varphi_{tr} W \pi_{X,A}^* \) and \( \pi_X^! i_Z^* \pi_{X,A}^* \).
and \( \pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot \phi \cdot w \cdot \pi_X \) being isomorphisms. Starting with \( q_X \cdot j_X^* \) composed with \( \pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot \phi \cdot w \cdot \pi_X^* \) using (4) we get

\[
q_X \cdot j_X^* \cdot \pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot \phi \cdot w \cdot \pi_X \\
\pi_X \cdot \pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot \phi \cdot w \cdot \pi_X^*
\]

which is a commutative square of natural isomorphisms. Then by [14, Lemma A.4] and Lemma \[8.3.3\] we get a natural isomorphism

\[
\pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot w \cdot \pi_X \cdot q_X \cdot \pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot \phi \cdot w \cdot \pi_X, \tag{81}
\]

which composes to

\[
\pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot w \cdot \pi_X \cdot q_X \cdot \pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot \phi \cdot w \cdot \pi_X \tag{82}
\]

Similarly, starting with \( \pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot w \cdot \pi_X \cdot q_X \cdot \pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot \phi \cdot w \cdot \pi_X \) composed with \( q_X \cdot j_X^* \) and use (2) we get a commutative diagram

\[
\pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot w \cdot \pi_X \cdot q_X \cdot \pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot \phi \cdot w \cdot \pi_X \tag{83}
\]

Now using the analogues of (3), (1), and (5), we get a natural map

\[
\pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot w \cdot \pi_X \cdot q_X \cdot \pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot \phi \cdot w \cdot \pi_X.
\]

Its composition with the right vertical map in [83] gives the same map as (82), due to the naturality of the transforms \( \text{id} \rightarrow \tilde{Z} \cdot \tilde{A} \) and \( \phi \cdot w \rightarrow \text{id} \).

Composing \( Dp_X \cdot \text{id} \rightarrow j_X \cdot j_X^* \), and \( \pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot \phi \cdot w \cdot \pi_X \), we get the commutative square

\[
Dp_X \cdot \pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot \phi \cdot w \cdot \pi_X \\
Dp_X \cdot j_X \cdot \pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot \phi \cdot w \cdot \pi_X
\]

where the left vertical arrow gives the map of Braverman-Finkelberg-Nakajima [10], and the right vertical arrow gives \( h \). Applying the natural isomorphism of functors \( p_X \cdot j_X \cdot j_X^* = p_X \cdot q_X \cdot j_X^* \), we get the commutative square

\[
Dp_X \cdot j_X \cdot j_X^* \cdot \pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot \phi \cdot w \cdot \pi_X \\
Dp_X \cdot j_X \cdot j_X^* \cdot \pi_X \cdot \tilde{Z} \cdot \tilde{A} \cdot \phi \cdot w \cdot \pi_X
\]

Combining this with Lemma \[C.0.2\] this shows that the coproduct \( h \) defined here is compatible with \[50\] \[61\] \[79\] under the dimensional reduction.
C.0.4 Concluding the proofs

By the definition of the central extended algebras, for each \( k = 1, 2, 3 \), there is a specialization map \( \text{SH}^\varepsilon \to \text{SH}^{c(k)} \), with \( \text{SH}^{c(k)} \) isomorphic to the algebra \( \text{SH}^\varepsilon \) defined in [79]. The direct sum of these three specialization maps embeds the algebra \( \text{SH}^\varepsilon \) into \( \bigoplus_{k=1,2,3} \text{SH}^{c(k)} \). Moreover, this embedding fits into the upper part of the following diagram.

\[
\begin{array}{ccc}
\bigoplus_{i \in \mathbb{N}, k=1,2,3} C\{c_i^{(k)}\} & \xrightarrow{\delta} & \text{SH}^\varepsilon \\
\downarrow & & \downarrow \Delta^\varepsilon \\
\bigoplus_{i \in \mathbb{N}, k=1,2,3} C\{c_i^{(k)}\} & \xrightarrow{\oplus_k \Delta^{c(k)}} & \oplus_{k=1,2,3} \text{SH}^{c(k)} \\
& & \downarrow \left(\text{SH}^{c(k)}\right)^{\otimes 2} \\
& & \oplus_{k=1,2,3} \left(\text{SH}^{c(k)}\right)^{\otimes 2} \\
\end{array}
\]

In other words, \( \text{SH}^\varepsilon \) is characterized as the universal central extension of \( \text{SH} \), which specializes into \( \text{SH}^{c(k)} \) for each \( k = 1, 2, 3 \).

From the formula of \( \Delta^\varepsilon \) on the generators of \( \text{SH}^\varepsilon \) in Proposition 8.2.2 specializes to \( \Delta^{c(k)} : \text{SH}^{c(k)} \to \left(\text{SH}^{c(k)}\right)^{\otimes 2} \) by specializing \( c^{(k')} \mapsto 0 \), for \( 1 \leq k \neq k' \leq 3 \). These three maps \( \Delta^{c(k)} \) are well-defined algebra homomorphisms, which agree on the specialization \( \text{SH} \).

Therefore, by the aforementioned universal property, \( \Delta^\varepsilon \) given in Proposition 8.2.2 is a well-defined algebra homomorphism. By the same universal property, \( \Delta^\varepsilon \) is coassociative. This in particular proves Proposition 8.2.2.

To prove Proposition 8.2.3, it suffices to prove the commutativity of the left part of the following diagram of the actions of \( \text{SH}^\varepsilon \).

\[
\begin{array}{ccc}
\text{End}(\oplus \tau V) & \xrightarrow{h} & \text{End}(W) \\
\downarrow \Delta^\varepsilon & & \downarrow h_W \\
\text{SH}^\varepsilon & \xrightarrow{\text{End}(\oplus \tau V) \otimes \text{End}(\oplus \tau V)} & \text{End}(W) \otimes \text{End}(W)
\end{array}
\]

where \( W := (\oplus \tau_1 V_{\tau_1,0,0}) \bigoplus (\oplus \tau_3 V_{0,0,0}) \bigoplus (\oplus \tau_2 V_{0,0,0}) \). By definition, \( \Delta^h \) from \( \boxed{\text{C.0.2}} \) automatically makes the entire diagram commutative. Therefore we need to show that \( \Delta^\varepsilon \) agrees with \( \Delta^h \). By \( \boxed{\text{C.0.3}} \) \( \Delta^\varepsilon \) makes the lower square (containing \( \Delta^\varepsilon \) and \( h_W \)) commutative. Thus, \( \Delta^\varepsilon \) and \( \Delta^h \) agree when restricting on \( \text{SH}^{c(k)} \), \( k = 1, 2, 3 \). By the same argument as above, \( \Delta^\varepsilon \) is determined by the three specializations \( \Delta^{c(k)} \), \( k = 1, 2, 3 \). Therefore, \( \Delta^\varepsilon \) and \( \Delta^h \) agree.
References

[1] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Y. I. Manin, Construction of Instantons, Phys. Lett. A 65, 185 (1978). doi:10.1016/0375-9601(78)90141-X

[2] H. Awata, B. Feigin, A. Hoshino, M. Kanai, J. Shiraiishi and S. Yanagida, Notes on Ding-Iohara algebra and AGT conjecture, arXiv:1106.4088 [math-ph].

[3] H. Awata, B. Feigin and J. Shiraiishi, Quantum Algebraic Approach to Refined Topological Vertex, JHEP 1203, 041 (2012) doi:10.1007/JHEP03(2012)041 [arXiv:1112.6074 [hep-th]].

[4] L. Alday, D. Gaiotto, Y. Tachikawa, Liouville Correlation Functions from Four-dimensional Gauge Theories, arXiv:0906.3219. Lett. Math. Phys. Vol. 91, Issue 2, pp 167–197.

[5] M. Aganagic, D. Jafferis and N. Saulina, Branes, black holes and topological strings on toric Calabi-Yau manifolds, JHEP 0612, 018 (2006) doi:10.1088/1126-6708/2006/12/018 [hep-th/0512245].

[6] N. Arbesfeld, O. Schiffmann, A presentation of the deformed $W_{1+\infty}$ algebra, Symmetries, integrable systems and representations, 1–13, Springer Proc. Math. Stat. 40 (2013), Arxiv1209.0429.

[7] M. Bershtein, B. Feigin and G. Merzon, Plane partitions with a “pit”: generating functions and representation theory, Sel. Math. New Ser. 24(1) (2018) 21-62, Arxiv1512.08779

[8] J-E. Bourgine, Y. Matsuo, H. Zhang, Holomorphic field realization of $\mathfrak{SH}_c$ and quantum geometry of quiver gauge theories, arXiv:1512.02492.

[9] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the monster, Proc. Nat. Acad. Sci. 83, 3068 (1986). doi:10.1073/pnas.83.10.3068

[10] A. Braverman, M. Finkelberg, H. Nakajima, Instanton moduli spaces and $W$-algebras. Astérisque No. 385 (2016), vii+128 pp.

[11] D. Brungs and W. Nahm, The Associative algebras of conformal field theory, Lett. Math. Phys. 47, 379 (1999) doi:10.1023/A:1007525300192 [hep-th/9811239].

[12] K. Costello, Holography and Koszul duality: the example of the M2 brane. Arxiv1705.02500

[13] K. Costello, O. Gwilliam, Factorization algebras in Quantum Field Theory, vol. 1. Cambridge University Press, 2017

[14] B. Davison, The critical COHA of a quiver with potential, preprint, (2015). The Quarterly Journal of Mathematics, Volume 68, Issue 2, 1 June 2017, Pages 635-703. Arxiv1311.7172

[15] B. Davison, BPS algebras and character varieties, talk at the Fields Institute, November 2016.
[16] M. Dedushenko, S. Gukov, P. Putrov, *Vertex algebras and 4-manifold invariants*, arXiv:1705.01645.

[17] J. Ding and K. Iohara, *Generalization and deformation of Drinfeld quantum affine algebras*, Lett. Math. Phys. 41, 181 (1997). doi:10.1023/A:1007341410987

[18] V. S. Dotsenko and V. A. Fateev, *Conformal Algebra and Multipoint Correlation Functions in Two-Dimensional Statistical Models*, Nucl. Phys. B 240, 312 (1984). doi:10.1016/0550-3213(84)90269-4

[19] M. R. Douglas, *Branes within branes*, hep-th/9512077. Strings, Branes and Dualities, pp 267–275

[20] M. R. Douglas, *Gauge fields and D-branes*, J. Geom. Phys. 28, 255 (1998) doi:10.1016/S0393-0440(97)00024-7 [hep-th/9604198].

[21] B. Feigin, *Extensions of vertex algebras. Constructions and applications*, Russian Math. Surveys, 72:4, 2017, p. 707-763.

[22] V. Futorny, D. Grantcharov, and L. E. Ramirez, *Irreducible generic gelfand-tsetlin modules of gl(n)*, arXiv:1409.8413

[23] B. Feigin and E. Frenkel, *Quantization of the Drinfeld-Sokolov reduction*, Phys. Lett. B 246, 75 (1990). doi:10.1016/0370-2693(90)91310-8

[24] B. Feigin and E. Frenkel, *Integrals of motion and quantum groups*” Lect. Notes Math. 1620, 349 (1996) doi:10.1007/BFb0094794 [hep-th/9310022].

[25] B. Feigin, S. Gukov, *VOA[M4]*, arXiv:1806.02470

[26] B. Feigin, M. Jimbo, T. Miwa and E. Mukhin, *Branching rules for quantum toroidal gl_n*, Adv. Math. 300, 229 (2016) doi:10.1016/j.aim.2016.03.019 [arXiv:1309.2147 [math.QA]].

[27] G. Felder, *BRST Approach to Minimal Models*, Nucl. Phys. B 317, 215 (1989) Erratum: [Nucl. Phys. B 324, 548 (1989)]. doi:10.1016/0550-3213(89)90481-1, 10.1016/0550-3213(89)90568-3

[28] S. Franco, A. Hanany, K. D. Kennaway, D. Vegh and B. Wecht, *Brane dimers and quiver gauge theories*, JHEP 0601, 096 (2006) doi:10.1088/1126-6708/2006/01/096 [hep-th/0504110].

[29] I. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras And The Monster*, BOSTON, USA: ACADEMIC (1988) 508 P. (PURE AND APPLIED MATHEMATICS, 134)

[30] A. Gadde, S. Gukov and P. Putrov, *Fivebranes and 4-manifolds*, arXiv:1306.4320 [hep-th].

[31] M. R. Gaberdiel and R. Gopakumar, *Triality in Minimal Model Holography*, JHEP 1207, 127 (2012) doi:10.1007/JHEP07(2012)127 [arXiv:1205.2472 [hep-th]].

[32] M. R. Gaberdiel, W. Li, C. Peng and H. Zhang, *The supersymmetric affine Yangian*, JHEP 1805 (2018) 200 doi:10.1007/JHEP05(2018)200 [arXiv:1711.07449 [hep-th]].
[33] M. R. Gaberdiel, W. Li and C. Peng, *Twin-plane-partitions and N = 2 affine Yangian*, arXiv:1807.11304 [hep-th].

[34] M. Gaberdiel, R. Gopakumar, W. Li, C. Peng, *Higher spins and Yangian symmetries*, arXiv:1702:05100.

[35] D. Gaiotto and J. Lamy-Poirier, *Irregular Singularities in the H^+_3 WZW Model*, arXiv:1301.5342 [hep-th].

[36] D. Gaiotto, M. Rapčák, *Vertex Algebras at the Corner*, Arxiv1703.00982.

[37] D. Gaiotto and J. Teschner, *Irregular singularities in Liouville theory and Argyres-Douglas type gauge theories, I*, JHEP **1212**, 050 (2012) doi:10.1007/JHEP12(2012)050 [arXiv:1203.1052 [hep-th]].

[38] V. Ginzburg, *Lectures on Nakajima’s Quiver Varieties*, (Grenoble, 2008).

[39] K. Harada and Y. Matsuo, *Plane Partition Realization of (Web of) W-algebra Minimal Models*, arXiv:1810.08512 [hep-th].

[40] J. Harvey, G. Moore, *On the algebras of BPS states*, arXiv:hep-th/9609017.

[41] A. Hanany and K. D. Kennaway, *Dimer models and toric diagrams*, hep-th/0503149.

[42] A. Hanany and D. Vegh, *Quivers, tilings, branes and rhombi*, JHEP **0710**, 029 (2007) doi:10.1088/1126-6708/2007/10/029 [hep-th/0511063].

[43] K. Hornfeck, *W-algebras of negative rank* Phys. Lett. B **343**, 94 (1995) doi:10.1016/0370-2693(94)01442-F [hep-th/9410013].

[44] S. Hwang and H. Rhedin, *The BRST Formulation of G/H WZNW models* Nucl. Phys. B **406**, 165 (1993) doi:10.1016/0550-3213(93)90165-L [hep-th/9305174].

[45] D. Jafferis, *Crystals and intersecting branes* hep-th/0607032.

[46] T. Kimura, *Double quantization of Seiberg-Witten geometry and W-algebras*, arXiv:1612.07590.

[47] T. Kimura, V. Pestun, *Quiver elliptic W-algebras*, arXiv:1608.04651.

[48] T. Kimura, V. Pestun, *Quiver W-algebras*, arXiv:1512.08333.

[49] P. Koroteev, *A-type Quiver Varieties and ADHM Moduli Spaces* arXiv:1805.00986 [math.AG].

[50] M. Kontsevich, Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants*, Commun. Number Theory Phys. **5** (2011), no. 2, 231–352. MR2851153

[51] M. Kontsevich, Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435.

[52] N. C. Leung and C. Vafa, *Branes and toric geometry* Adv. Theor. Math. Phys. **2**, 91 (1998) doi:10.4310/ATMP.1998.v2.n1.a4 [hep-th/9711013].
[53] A. R. Linshaw, *Universal two-parameter $W_\infty$-algebra and vertex algebras of type $W(2,3,\ldots,N)$* Arxiv1710.02275 [math.RT].

[54] A. Litvinov, L. Spodyneiko, *On W algebras commuting with a set of screenings* JHEP **1611**, 138 (2016) doi:10.1007/JHEP11(2016)138 [Arxiv1609.06271 [hep-th]].

[55] K. Miki, *A $(q,\gamma)$-analog of the $W_{1+\infty}$ algebra*, Journal of Mathematical Physics 48 (Dec., 2007) 123520123520.

[56] D. Maulik, A. Okounkov, *Quantum groups and quantum cohomology*, arXiv:1211.1287. to appear in Asterisque.

[57] S. Mozgovoy, M. Reineke, *On the non-commutative Donaldson-Thomas invariants arising from brane tilings*, arXiv:0809.0117.

[58] H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. of Math. (2) 145 (1997), no. 2, 379–388.

[59] H. Nakajima, *Lectures on Hilbert Schemes of Points on Surfaces*, AMS, University Lecture Series, vol. 18, 1999.

[60] H. Nakajima, *Quiver varieties and finite dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), no. 1, 145–238, MR1808477

[61] H. Nakajima, *Lectures on perverse sheaves on instanton moduli spaces*, IAS/Park City Mathematics Series 2015.

[62] A. Negut, *AGT relations for sheaves on surfaces*, arXiv:1711.00390 [math.AG].

[63] N. A. Nekrasov, *Seiberg-Witten prepotential from instanton counting* Adv. Theor. Math. Phys. **7**, no. 5, 831 (2003) doi:10.4310/ATMP.2003.v7.n5.a4 [hep-th/0206161].

[64] N. Nekrasov, N. Prabhakar, *Spiked instantons from intersecting D-branes*, Arxiv1611.03478

[65] N. Nekrasov, *BPS/CFT correspondence: non-perturbative Dyson-Schwinger equations and qq-characters*, arXiv:1512.05388.

[66] N. Nekrasov, *BPS/CFT correspondence II: Instantons at crossroads, moduli and compactness theorem*, Arxiv1608.07272

[67] N. Nekrasov, *BPS/CFT Correspondence III: Gauge Origami partition function and qq-characters*, arXiv:1701.00189.

[68] N. Nekrasov, *BPS/CFT correspondence IV: sigma models and defects in gauge theory*, arXiv:1711.11011.

[69] N. Nekrasov, *BPS/CFT correspondence V: BPZ and KZ equations from qq-characters*, 1711.11582.

[70] N. Nekrasov and E. Witten, *The Omega Deformation, Branes, Integrability, and Liouville Theory* JHEP **1009**, 092 (2010) doi:10.1007/JHEP09(2010)092 [arXiv:1002.0888 [hep-th]].

70
[71] T. Nishinaka, S. Yamaguchi, Y. Yoshida, Two-dimensional crystal melting and $D4 - D2 - D0$ on toric Calabi-Yau singularities, ArXiv:1304.6724.

[72] H. Ooguri and M. Yamazaki, Crystal Melting and Toric Calabi-Yau Manifolds, Commun. Math. Phys. 292, 179 (2009) doi:10.1007/s00220-009-0836-y [arXiv:0811.2801 [hep-th]].

[73] T. Procházka, Exploring $W_{\infty}$ in the quadratic basis JHEP 1509, 116 (2015) 10.1007/JHEP09(2015)116, Arxiv:1411.7697

[74] T. Procházka, $W$-symmetry, topological vertex and affine Yangian JHEP 1610, 077 (2016) 10.1007/JHEP10(2016)077, Arxiv1512.07178.

[75] T. Procházka, M. Rapčák, Webs of $W$-algebras, Arxiv1711.06888.

[76] T. Procházka and M. Rapčák, $W$-algebra Modules, Free Fields, and Gukov-Witten Defects, arxiv:1808.08837 [hep-th].

[77] J. Ren, Y. Soibelman, Cohomological Hall algebras, semiclassical bases and Donaldson-Thomas invariants for 2-dimensional Calabi-Yau categories (with an appendix by Ben Davison) Algebra, geometry, and physics in the 21st century, 261–293, Progr. Math., 324, Birkhuser/Springer, Cham, 2017. ArXiv1508.06068.

[78] O. Schiffmann, E. Vasserot, The elliptic Hall algebra and the K-theory of the Hilbert scheme of $\mathbb{A}^2$. Duke Math. J. 162 (2013), no. 2, 279-366. MR3018956

[79] O. Schiffmann, E. Vasserot, Cherednik algebras, $W$-algebras and the equivariant cohomology of the moduli space of instantons on $\mathbb{A}^2$. Publ. Math. Inst. Hautes Etudes Sci. 118 (2013), 213-342.

[80] O. Schiffmann, E. Vasserot, On cohomological Hall algebras of quivers: Yangians, arXiv:1705.07491.

[81] O. Schiffmann, E. Vasserot, On cohomological Hall algebras of quivers: generators. Arxiv1705.07488.

[82] Y. Soibelman, Remarks on Cohomological Hall algebras and their representations, Arbeitstagung Bonn 2013, 355–385, Progr. Math., 319, Birkhuser/Springer, Cham, 2016. Arxiv1404.1606.

[83] B. Szendroi, Non-commutative Donaldson-Thomas invariants and the conifold Geom. Topol. 12 (2008), no. 2, 1171–1202. ArXiv0705.3419.

[84] B. Szendroi, Nekrasov’s partition function and refined Donaldson-Thomas theory: the rank one case, SIGMA Symmetry Integrability Geom. Methods Appl. 8 (2012), Paper 088, 16 pp. ArXiv1210.5181.

[85] A. Tsymbaliuk, The affine Yangian of $\mathfrak{gl}_1$ revisited. Advances in Mathematics, 304 (2017), 583–645. Arxiv1404.5240

[86] Y. Yang and G. Zhao, The cohomological Hall algebra of a preprojective algebra. Proc. Lond. Math. Soc. 116, 1029–1074. Arxiv1407.7994

71
[87] Y. Yang and G. Zhao, *On two cohomological Hall algebras*, Proc. Roy. Soc. Edinburgh Sect. A., to appear. Arxiv1604.01477

[88] Y. Yang and G. Zhao, *Cohomological Hall algebras and affine quantum groups*, Selecta Math., Vol. 24, Issue 2 (2018), pp. 1093–1119. Arxiv1604.01865

[89] W. Q. Wang, *Classification of irreducible modules of $W_3$ algebra with $c = -2$*, Commun. Math. Phys. 195, 113 (1998). doi:10.1007/s002200050382

[90] N. Wyllard, *$A(N-1)$ conformal Toda field theory correlation functions from conformal $N=2$ $SU(N)$ quiver gauge theories*, JHEP 0911, 002 (2009) doi:10.1088/1126-6708/2009/11/002 [arXiv:0907.2189 [hep-th]].

[91] Y. Zhu, *Modular Invariance of Characters of Vertex Operator Algebras*, J. Amer. Math. Soc. 9 (1996), pp. 237302

**The authors:**

M. Rapčák- Perimeter Institute for Theoretical Physics, 31 Caroline St N, Waterloo, ON N2L 2Y5, Canada, mrapcak@perimeterinstitute.ca

Y. Soibelman- Department of Mathematics, Kansas State University, Manhattan, KS 66506, USA, soibel@math.ksu.edu

Y. Yang- School of Mathematics and Statistics, The University of Melbourne, 813 Swanston Street, Parkville VIC 3010, Australia, yaping.yang1@unimelb.edu.au

G. Zhao- Institute of Science and Technology Austria, Am Campus, 1, Klosterneuburg 3400, Austria. current address: School of Mathematics and Statistics, The University of Melbourne, 813 Swanston Street, Parkville VIC 3010, Australia, gufangz@unimelb.edu.au