Two Particle Physics Models With Spontaneous $CP$
Violation From Gauge Theory On Discrete Group

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Abstract

Based on the differential calculus and the gauge theory on discrete groups, we reconstruct two physics models with spontaneous $CP$ violation: (1) The Georgi-Glashow-Lee model with two Higgs triplets; (2) The Weinberg-Salam-Branco model with three Higgs doublets and the natural flavor conservation (NFC). We focus on the Lagrangian terms containing the Higgs particles and show that with an appropriate choice of the discrete groups, we can obtain the physically meaningful Yukawa couplings and the Higgs potentials which lead to the spontaneous $CP$ violation consequently.

1 Introduction

There were many mechanisms to explain the $CP$ nonconservation after the theoretical prediction raised by Lee, Oheeme and Yang as early as 1956 and the experimental support, $K_L^0 \rightarrow \pi^+\pi^-$ observed by Chrirtenson, Cronin, Fitch and Turlay in 1964 [1]. But it still remains an open question for none of the explanation could match all the experimental data fairly well. Among these mechanisms, the spontaneous $CP$ violation (SCPV) first suggested by T.D. Lee [1] has drawn much attention because it seems natural and elegant, the total Lagrangian is assumed to satisfy invariance under $CP$ and $T$ transformation, renormalizability and invariance under certain weak- electromagnetic gauge transformations. Lee also generalized a more realistic physical model at that time, Georgi-Glashow Model to a version with SCPV. In this model, two Higgs triplets were introduced to provide SCPV and satisfy other physical condition. Having considered the natural flavor conservation (NFC), G.Branco gave a "minimal" model based on the standard $SU(2) \times U(1)$ gauge theory with three Higgs doublets [2], the $CP$ nonconservation is entirely due to Higgs-boson exchange, not in the manner of Kobayashi-Maskawa [3]. Recently, Y-L Wu proposed a $SU(2)_L \times U(1)_Y$ gauge theory with two Higgs doublets which has both the SCPV and NFC, the Glashow-Weinberg criterion for NFC is sufficient but not necessary [4].

In these mechanisms, the Higgs particles play a key role to violate $CP$. But we did not understand the origin of Higgs particles very well. The development of non-commutative geometry (NCG) has made a great progress toward the understanding for the origin of Higgs particles [5-9]. According to NCG, the Higgs fields are gauge fields with respect to discrete symmetry, the Higgs potential can be viewed as Yang-Mills term in general gauge field theory and can be determined with less ambiguity. Moreover, the non-commutative Yang-Mills term together with fermions can reproduce many particle physics models with some constraints in free parameters. Lots of efforts have been made along this way.[10-18]

This paper is one of our series of work on developing the gauge theory on discrete group. We start with the NCG and take the discrete group to be $Z_2 \times Z_2$ and $Z_2 \times Z_2 \times Z_2$.
respectively, then construct physics models with SCPV. Following the same way in \cite{15,18}, we get the Yukawa coupling terms, Higgs kinetic terms as well as Higgs potential which is important in SCPV. The paper is arranged as follows: In section 2 we give a brief introduction of gauge theory on discrete groups. In section 3 we reconstruct the Georgi-Glashow-Lee model in the spirit of NCG. In section 4 we take the discrete group to be $Z_2 \times Z_2 \times Z_2$ and obtain Weinberg-Salam-Branco three Higgs model with SCPV, while the NFC are ensured naturally. At section 5 we end with some discussions and remarks.

2 The Gauge theory on the Discrete Group $Z_2 \times Z_2$

2.1 Differential Calculus on Discrete Groups $G$

Let $G$ be a discrete group of size $N_G$, its elements are \{e, $g_1$, $g_2$, \ldots, $g_{N_G-1}$\}, and $\mathcal{A}$ the algebra of all complex valued functions on $G$. The right and left multiplication on $G$ induce natural automorphisms of $\mathcal{A}$, $R_g$ and $L_g$ respectively, 

\[(R_h f)(g) = f(g \cdot h), (L_h f)(g) = f(h \cdot g)\]  
(2.1)

The basis $\partial_i$, $(i = 1, \ldots, N_G - 1)$ of the left invariant vector space $\mathcal{F}$ on $\mathcal{A}$ are defined as

\[\partial_i f = f - R_i f, \quad \forall f \in \mathcal{A},\]  
(2.2)

$\partial_i$ also satisfies

\[\partial_i \partial_j = \sum_k C^{k}_{ij} \partial_k, \quad C^{k}_{ij} = \delta^{k}_{i} + \delta^{k}_{j} - \delta^{k}_{i} \cdot \delta^{k}_{j}\]  
(2.3)

where $i, j, \ldots, (i \cdot j)$ denote $g_i, g_j, \ldots, (g_i \cdot g_j)$ respectively. The Haar integral, which remains invariant under group action, is introduced as a complex valued linear functional on $\mathcal{A}$ as,

\[\int_{G} f = \frac{1}{N_G} \sum_{g \in G} f(g).\]  
(2.4)

Having chosen the basis of $\mathcal{F}$ we can introduce the dual basis of $\mathcal{F}^*$ consisting of one forms, $\chi^i$, satisfy

\[\chi^i(\partial_j) = \delta^i_j.\]  
(2.5)

There exists exactly one linear operator $d,d : \Omega^n \to \Omega^{n+1}$, which is nilpotent, $d^2 = 0$, satisfies the graded Leibniz rule and for every $f \in \mathcal{A}$ and every vector field $v$, $df(v) = v(f)$, provided that $\chi^i$ satisfy the following two conditions \cite{8}

\[\chi^i f = (R_i f)\chi^i, \quad \forall f \in \mathcal{A},\]  
\[d\chi^i = -\sum_{j,k} C^{i}_{jk} \chi^j \otimes \chi^k.\]  
(2.6)

To construct a physical model we must define the involution operator $\ast$ on $\mathcal{A}$, which agrees with the complex conjugation on $\mathcal{A}$, takes the assumption that $(\chi^g)^* = -\chi^{g^{-1}}$, and (graded) commutes with $d$, i.e. $d(\omega^*) = (-1)^{deg} (d\omega)^*$.

2.2 Gauge Theory on Discrete Group $G$

Using the differential calculus introduced in last section, we can construct the generalized gauge theory on finite groups. The gauge transformations group is often taken to be the group of unitary elements of zero-forms $\tilde{\mathcal{A}}$,

\[\mathcal{H} = U(\tilde{\mathcal{A}}) = a \in \tilde{\mathcal{A}} : aa^* = a^*a = 1\]  
(2.7)
Like the usual gauge theory, the $d+\phi$ is gauge covariant which requires the transformation of gauge field one form $\phi$ as,

$$\phi \rightarrow H\phi H^{-1} + HdH^{-1}.$$  \hfill (2.8)  

If we write $\phi = \sum_g \phi_g \chi^g$, the coefficients $\phi_g$ transform as

$$\phi_g \rightarrow H\phi_g (R_g H^{-1}) + H \partial_g H^{-1}.$$  \hfill (2.9)  

It is convenient to introduce a new field $\Phi_g = 1 - \phi_g$, then (2.10) is equivalent to

$$\Phi_g \rightarrow H\Phi_g (R_g H^{-1}).$$  \hfill (2.10)  

The extended anti-hermitian condition $\phi^* = -\phi$ results in the following relations on its coefficients $\phi_g$ as well as $\Phi_g$,

$$\phi^*_g = R_g (\phi_g^{-1}), \quad \Phi^*_g = R_g (\Phi_g^{-1})$$ \hfill (2.11)  

It can be easily shown that the curvature two form $F = d\phi + \phi \otimes \phi$ is gauge covariant and can be written in terms of its coefficients

$$F = \sum_{g,h} F_{gh} \chi^g \otimes \chi^h.$$  \hfill (2.12)  

$$F_{gh} = \Phi_g R_g (\Phi_h) - \Phi_{h,g}.$$  \hfill (2.13)  

In order to construct the Lagrangian of this gauge theory on discrete groups we need to introduce a metric on the forms. Let us first define the metric $\eta$ as a bilinear form on the bimodule $\Omega^1$ valued in the algebra $A$,

$$\eta : \Omega^1 \otimes \Omega^1 \rightarrow A$$  \hfill (2.14)  

$$< \chi^g, \chi^h > = \eta^{gh}.$$  \hfill (2.15)  

The gauge invariance requires that $\eta^{gh} \sim \delta^{gh-1}$ \cite{10}. The metric on the two forms becomes,

$$< \chi^g \otimes \chi^h, \chi^p \otimes \chi^q > = \alpha \eta^{gh} \eta^{pq} + \beta \eta^{gp} \eta^{hq} + \gamma \eta^{gq} \eta^{hp}.$$  \hfill (2.16)  

where the term proportional to $\gamma$ is only appeared when $G$ is commutative \cite{8}. In the next section, we will get a constrain relation of $\alpha, \beta$ and $\gamma$ if the spontaneous $CP$ violation is considered. Then the most general Yang-Mills action is given,

$$L = - \int_G < F, \overline{F} >$$  \hfill (2.17)  

where we have used the involution relations

$$\overline{F} = (\chi^g)^* \otimes (\chi^p)^* F_{pq}^\dagger, \quad (\chi^g)^* = -\chi^{g^{-1}}.$$  \hfill (2.18)  

2.3 Gauge Field Theory on $M^4 \times G$

To deal with the model building in particle physics, we must include fermions and their Yukawa couplings to Higgs. A detailed construction of the gauge theory on discrete groups coupled to the fermions has given in \cite{10} \cite{11}. Here we only cite some conclusion which is useful to the model building in the next section. $M^4$ is the four-dimension space-time.
We defined the exterior derivative operators $d_M$ and $d_G$ on $M^4$ and $G$ respectively, i.e. $d_M f = \partial_\mu f dx^\mu$, $d_G f = \partial_g f x^g$, then $d_{M\times G} = d_M + d_G$ can be extended to the covariant derivative

\[
D_{M\times G} = d_{M\times G} + A
\]

\[
d_{M\times G} f = \partial_\mu f dx^\mu + \partial_g f x^g
\]

\[
A = ig A_\mu dx^\mu + \frac{1}{4} \partial_g \chi x^g
\]

(2.19)

The generalized curative two form:

\[
F = dA + A \otimes A
\]

\[
= \frac{ig}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu + \frac{1}{4} F_{gh} dx^\mu \otimes \chi^g + \frac{1}{\mu^2} F_{gh} \chi^g \otimes \chi^h
\]

(2.20)

where

\[
F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu + ig [A_\mu, A_\nu]
\]

\[
F_{gh} = \partial_\mu \Phi_g + ig A_\mu \phi_g - ig \phi_g A_\mu
\]

(2.21)

\[
F_{gh} = \Phi_g R_g (\Phi_h) - \Phi_h R_g.
\]

In next section we will find that $F_{\mu\nu}$ is the ordinary gauge field intense, $F_{gh}$ will lead to the kinetic term of Higgs, and $F_{gh}$ lead to the Higgs potential.

The "local gauge invariant" fermion field Lagrangian on $M_4 \times G$ is written as,

\[
\mathcal{L} = \bar{\psi}(x,g)[i\gamma^\mu (\partial_\mu + A_\mu(x,g))] + \mu \sum_h (\partial_\mu + \frac{1}{\mu} \phi_h(x,g) R_h) |\psi(x,g)
\]

(2.22)

3 The Georgi-Glashow-Lee Model on the Discrete Group $Z_2 \times Z_2$

This model is a generalization of the Georgi-Glashow model given by T.D. Lee. The basic gauge group of the weak and electromagnetic interaction is $SO_3$. There is a triplet of spin 1 gauge field $W_\mu$ and at least eight quark-like hadron fields grouped into two triplets $\psi$, $\psi'$, as well as two singlets $\chi$, $\chi'$.

These fields can be expressed by the physical states $p^+, n^0, \lambda^0, q^-$ and $p^{+'}, n^{0'}, \lambda^{0'}, q^{-'},$ but here we focused on two $SO_3$ triplets of spin 0 Hermitian field $\Phi_R, \Phi_I$, which play important roles in $CP$ spontaneous violation of this model. Under the time reversal transformation $T$,

\[
T \bar{\Phi}_R T^{-1} = +\bar{\Phi}_R, \quad \text{but} \quad T \bar{\Phi}_I T^{-1} = -\bar{\Phi}_I
\]

(3.23)

The total Lagrangian density can be written as

\[
\mathcal{L} = \mathcal{L}(W, \psi) + \mathcal{L}(\bar{\Phi}_R, \bar{\psi}) + \mathcal{L}(\bar{\Phi}_I, \bar{\psi}) + \mathcal{L}(W, \bar{\Phi})
\]

(3.24)

in which $\mathcal{L}(W, \psi)$ contain massive free fermion terms and their coupling with gauge fields $W$, and

\[
\mathcal{L}(\bar{\Phi}_R, \bar{\psi}) = -(g_R \bar{\psi} \bar{I} \psi + g_R' \bar{\psi}' \bar{I} \psi') \cdot \bar{\Phi}_R
\]

(3.25)

\[
\mathcal{L}(\bar{\Phi}_I, \bar{\psi}) = -i(g_I \bar{\psi} \bar{I} \psi + g_I' \bar{\psi}' \bar{I} \psi') \cdot \bar{\Phi}_I
\]

(3.26)

where $g_R$ and $g_I$ are real by hermiticity, and $\bar{I}$ is the $3 \times 3$ matrix representation of $SO_3$ generators. For simplicity, we omitted all other $CP$ invariant and $SO_3$ invariant couplings between $\Phi_R$ and the hadron fields. We also assume $\Phi_R$ not directly interacting with the singlet fields $\chi$ and $\chi'$.

\[
\mathcal{L}(W, \bar{\Phi}) = -\frac{1}{4} \bar{W}_{\mu\nu}^2 - \frac{1}{2} \partial_\mu \bar{\Phi}_R^2 - \frac{1}{2} \partial_\mu \bar{\Phi}_I^2 - V(\Phi)
\]

(3.27)

where

\[
\bar{W}_{\mu\nu} = \partial_\mu \bar{W}_{\nu} - \partial_\nu \bar{W}_{\mu} + e(\bar{W}_\mu \times \bar{W}_{\nu})
\]

(3.28)
\[ \partial_i \Phi_i = (\partial_\mu + e \vec{W}_\mu \times \vec{\Phi}) \quad (i = R, I) \]  

\[ V(\Phi) = -\frac{1}{2} \lambda_R \Phi_R^2 - \frac{1}{2} \lambda_I \Phi_I^2 + \frac{1}{4} A_I (\Phi_I^2)^2 + \frac{1}{4} A_I (\Phi_I^2)^2 - \frac{1}{4} C \Phi_R^2 \Phi_I^2 \]  

in which the constants \( \lambda_R, \lambda_I, \ldots, C \) are real by Hermiticity. As in [4], to insure that \( V(\Phi) \) has a minimum, we assume  

\[ A_R > 0, \quad A_I > 0 \quad \text{and} \quad A_I A_R > \frac{1}{4} (B - C)^2 \]  

In order that the minimum of \( V(\Phi) \) occurs at the values of \( \Phi_R \) and \( \Phi_I \) given by  

\[ < \Phi_R > = \bar{\rho}_R \neq 0, \quad < \Phi_I > = \bar{\rho}_I \neq 0 \]  

we assume  

\[ A_I \lambda_R > \frac{1}{2} (B - C) \lambda_I, \quad A_R \lambda_I > \frac{1}{2} (B - C) \lambda_R \]  

In addition we require \( C > 0 \) so that \( \bar{\rho}_R \parallel \bar{\rho}_I \).

Now we apply the gauge theory on discrete group discussed in last section to the case of \( G = Z_2 \times Z_2 \). \( G \) is isomorphism to Klein group \( K_4 \) and has four elements, denoted by \( \{ e, a, b, ab \} \) or the tensor product forms of \( Z_2 = \{ e, r \}, r^2 = e, \)  

\[ e = e_1 \otimes e_2, \quad a = e_1 \otimes r_2, \]

\[ b = r_1 \otimes e_2, \quad ab = r_1 \otimes r_2. \]  

The multiplication \( * \) is defined as  

\[ g * h = g_1 h_1 \otimes g_2 h_2, \quad g = g_1 \otimes g_2, \quad h = h_1 \otimes h_2 \]  

There are three steps in our approach to reconstruct the physical model. First we give a classification for all the fields on \( M_4 \times G \) according to their transformation characteristics with respect to the discrete symmetry, more precisely, we can define the right displacement \( R_g \) as follows,  

\[ R_g f = g_1 g_2 f g_2^{-1} g_1^{-1}, \quad g = g_1 \otimes g_2 \]  

If we take the \( g_1 = (CPT)^2, g_2 = CPCP = PCPC \), where \( C, P, T \) denote electrical charge conjugation, parity, and time reversal transformations respectively, we get a classification of the fields on \( M_4 \times G \). Then we apply the differential calculus and gauge theory on \( M_4 \times G \) to this case, and choose a consistent arrangement for those fields. Finally we do calculation straightforward and compare our results with the physical model we discussed.

Following this way, we arrange the fermion fields as,

\[ \psi(x, e) = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \psi(x, a) = -\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \]  

\[ \psi(x, b) = -\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \psi(x, ab) = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \]  

and the gauge fields as,  

\[ W_\mu(x, h) = W_\mu(x) \]  

We also take the following arrangement of Higgs fields,  

\[ \Phi_a(x, h) = \begin{pmatrix} 0 & g_R \vec{I} \cdot \vec{\Phi}_R \\ g_R \vec{I} \cdot \vec{\Phi}_R & 0 \end{pmatrix}, \quad \Phi_b(x, h) = \begin{pmatrix} 0 & -ig_I \vec{I} \cdot \vec{\Phi}_I \\ ig_I \vec{I} \cdot \vec{\Phi}_I & 0 \end{pmatrix} \]  

\[ \Phi_c(x, h) = 0, \quad \Phi_{ab}(x, h) = 0 \]
where $h$ is any element of $G$. It can be checked that these arrangements are consistent and satisfy the extended anti-Hermitian condition (2.11). According to the equations (2.22), the Yukawa coupling terms are,

$$L_{Yukawa} \propto \bar{\psi}(x,g)\phi_h(x,g)R_h\psi(x,g)$$

$$\propto \bar{\psi}(x,g)\phi_a(x,g)R_a\psi(x,g) + \bar{\psi}(x,g)\phi_b(x,g)R_b\psi(x,g)$$

$$\propto -g_R \bar{\psi}_L \vec{\Phi}_R \psi_R - g_R \bar{\psi}_R \vec{\Phi}_L \psi_L - ig_I \bar{\psi}_L \vec{\Phi}_I \psi_R + ig_I \bar{\psi}_R \vec{\Phi}_I \psi_L$$

$$\propto -g_R \bar{\psi}_L \vec{\Phi}_R \psi - ig_I \bar{\psi}_I \vec{\Phi}_I \gamma_5 \psi$$

(3.40)

where $\gamma_5 \psi_R = +\psi_R$, $\gamma_5 \psi_L = -\psi_L$ have been used. For another triplet $\psi'$, the similar procedure can be done. As mentioned in previous section, $F_{\mu\nu}$ lead to the Higgs kinetic terms which is exactly the equation (3.29). It can be easily calculated because on $M_4 \times G$ we have required the ordinary gauge fields keep the same.

Now we calculate the Higgs potential which is more complicated. The only nontrivial curvatures are,

$$F_{aa} = \Phi_a \cdot R_a \Phi_a - 1, \quad F_{bb} = \Phi_b \cdot R_b \Phi_b - 1,$$

$$F_{ab} = \Phi_a \cdot R_a \Phi_b, \quad F_{ba} = \Phi_b \cdot R_b \Phi_a. \quad (3.41)$$

If we take the matrix representation of the $SO_3$ generators $\vec{I}$ and rewrite $\vec{I} \cdot \vec{\Phi}_R$ and $\vec{I} \cdot \vec{\Phi}_I$ as follows,

$$\vec{I} \cdot \vec{\Phi}_R = \begin{pmatrix} 0 & -i\Phi_{R3} & i\Phi_{R2} \\ i\Phi_{R3} & 0 & -i\Phi_{R1} \\ -i\Phi_{R2} & i\Phi_{R1} & 0 \end{pmatrix}, \quad \vec{I} \cdot \vec{\Phi}_I = \begin{pmatrix} 0 & -i\Phi_{I3} & i\Phi_{I2} \\ i\Phi_{I3} & 0 & -i\Phi_{I1} \\ -i\Phi_{I2} & i\Phi_{I1} & 0 \end{pmatrix}, \quad (3.42)$$

we have

$$Tr[(\vec{I} \cdot \vec{\Phi}_R)^2] = 2\vec{\Phi}_i^2$$

$$Tr[(\vec{I} \cdot \vec{\Phi}_I)^4] = 2(\vec{\Phi}_i^2)^2 \quad i = R, I$$

$$Tr[((\vec{I} \cdot \vec{\Phi}_R)(\vec{I} \cdot \vec{\Phi}_I))^2] = 2(\vec{\Phi}_R \cdot \vec{\Phi}_I)^2$$

$$Tr[(\vec{I} \cdot \vec{\Phi}_R)^2(\vec{I} \cdot \vec{\Phi}_I)^2] = (\vec{\Phi}_R \cdot \vec{\Phi}_R)^2 + (\vec{\Phi}_I \cdot \vec{\Phi}_I)^2$$

(3.43)

With the help of these identities and the equation (2.17), the "Yang-Mills term" can be calculated straightforwardly,

$$< F, \bar{F} > = Tr[ \alpha(E_a^2F_{aa} + 2E_aE_bF_{ab}F_{ba} + E_b^2F_{bb})$$

$$\quad + \beta(E_a^2F_{aa} + 2E_aE_bF_{ab}F_{ba} + E_b^2F_{bb}) + \gamma(E_a^2F_{aa} + 2E_aE_bF_{ab}F_{ba} + E_b^2F_{bb})$$

$$\quad + \gamma(E_a^2F_{aa} + 2E_aE_bF_{ab}F_{ba} + E_b^2F_{bb}) + 2(\gamma - \beta)E_aE_bF_{ab}^2 + 2\alpha E_aE_bF_{aa}F_{bb}]$$

(3.44)

Then we have the following expression of Higgs potential similar with the one in [I], i.e (3.30),

$$V(\vec{\Phi}_R, \vec{\Phi}_I) = 2(\alpha + \beta + \gamma)E_a^2g_R^4(\vec{\Phi}_R^2)^2 + 2(\alpha + \beta + \gamma)E_b^2g_I^4(\vec{\Phi}_I^2)^2$$

$$-4g_R^2[(\alpha + \beta + \gamma)E_a + \alpha E_aE_b] \vec{\Phi}_R^2 - 4g_I^2[(\alpha + \beta + \gamma)E_b + \alpha E_aE_b] \vec{\Phi}_I^2$$

$$+ [4(\gamma - \beta) + 2\alpha]E_aE_bg_R^2g_I^2(\vec{\Phi}_I \cdot \vec{\Phi}_R)^2 + 2\alpha E_aE_bg_R^2g_I^2(\vec{\Phi}_R \cdot \vec{\Phi}_I)^2$$

(3.45)

If we absorbed the Yukawa coupling constants $g_R, g_I$ into the Higgs fields, and require that,

$$E_a > 0, \quad E_b > 0$$

$$\alpha > 0, \quad \alpha + \beta + \gamma > 0, \quad \beta - \gamma > \frac{\alpha}{2}$$

(3.46) (3.47)

The condition (3.31)-(3.33) are satisfied automatically.
4  Weinberg-Branco Model on the Discrete Group $Z_2 \times Z_2 \times Z_2$

It was shown that if one imposed NFC in the context of the standard $SU_2 \times U_1$ gauge theory, then at least three Higgs doublets are needed in order to violate $CP$. In a three-Higgs toy model of SCPV was constructed via the gauge theory on $Z_2 \times Z_2 \times Z_2$, which allow each Higgs couple to all the quarks but ensure the NFC by imposing some constrains on the Yukawa coupling constant matrices. Here we choose another approach to ensure the NFC, i.e preventing the third Higgs doublet $\Phi_3$ from coupling to quarks. As in \cite{2}, we take the element of third $Z_2$ group to be a reflection symmetry $R' : \Phi_3 \rightarrow -\Phi_3$ while all other fields remain unchanged. The first $Z_2$ group is still generated by $\theta = (CPT)^2$, the second $Z_2$ group is generated by $R : \Phi_2 \rightarrow -\Phi_2$, $D_R' \rightarrow -D_R'$ with all the other fields unchanged.

Following the method in last section, we construct the Weinberg-Salam-Branco model on the discrete group $\theta \times R \times R' = g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7$

$$\begin{align*}
g_0 &= e_1 \otimes e_2 \otimes e_3, & g_1 &= r_1 \otimes e_2 \otimes e_3, \\
g_2 &= e_1 \otimes r_2 \otimes e_3, & g_3 &= e_1 \otimes e_2 \otimes r_3, \\
g_4 &= r_1 \otimes r_2 \otimes e_3, & g_5 &= r_1 \otimes e_2 \otimes r_3, \\
g_6 &= e_1 \otimes r_2 \otimes r_3, & g_7 &= r_1 \otimes r_2 \otimes r_3.
\end{align*}$$

(4.48)

where $e_i$ denotes identity, $r_i$ denotes the transformation of each discrete group respectively.

The multiplication $\ast$ is defined as

$$g \ast h = g_1 h_1 \otimes g_2 h_2 \otimes g_3 h_3, \quad g = g_1 \otimes g_2 \otimes g_3, \quad h = h_1 \otimes h_2 \otimes h_3$$

(4.49)

According to their transformation properties of the symmetry $g, R$ and $R'$, i.e $R g f = g_1 g_2 g_3 f g_3^{-1} g_2^{-1} g_1^{-1}$, we take the following arrangement of the fermion fields, gauge fields and Higgs fields, (To compare with the model in \cite{2}, we only write down the quark fields. The inclusion of lepton fields is straightforward and given in \cite{15, 18}.)

$$\psi(x, g) = \left( \begin{array}{c} L(x, g) \\ R(x, g) \end{array} \right), \quad L(x, g) = \left( \begin{array}{c} U(x, g) \\ D(x, g) \end{array} \right)_L, \quad R(x, g) = \left( \begin{array}{c} U(x, g) \\ D(x, g) \end{array} \right)_R,$$

(4.50)

where $U = (u, c, t), \quad D = (d, s, b)$, with the left-handed components $U^i_L, D^i_L$ forming $SU_2$ doublets $(U^i_L, D^i_L)$, while the right-handed components are singlets.

$$U^i(x, g) = +U^i(x), \quad D^i_R(x, g) = +D^i_R(x), \quad \text{if } g = g_0, g_2, g_3, g_6,

-D^i_R(x), \quad \text{if } g = g_1, g_4, g_5, g_7.$$  

and $D^i_L(x, g)$ is arranged as same as $U^i(x, g)$. and the gauge fields as,

$$A_\mu(x, g) = A_\mu(x), \quad \text{for } g \in G$$

(4.52)

The Higgs fields,

$$\Phi_{g_1}(x, h) = \left( \begin{array}{c} 0 \\ \Phi_1(x) \end{array} \right), \quad \text{for } g \in G$$

(4.53)

$$\Phi_{g_3}(x, g) = \left( \begin{array}{c} 0 \\ \Phi_2(x) \end{array} \right), \quad \text{if } g = g_0, g_1, g_3, g_5$$

$$or \quad \left( \begin{array}{c} 0 \\ \Phi_2(x) \end{array} \right), \quad \text{if } g = g_2, g_4, g_6, g_7$$

(4.54)
\[
\Phi_{g_1}(x, g) = \begin{pmatrix}
0 & \Phi_3(x) \\
\Phi_1^*(x) & 0
\end{pmatrix}, \quad \text{if } g = g_0, g_1, g_2, g_4
\]

or \[
\Phi_{g_6}(x, g) = \Phi_3(x), \quad \text{if } g = g_3, g_5, g_6, g_7
\]

where \[
\Phi_n(x) = \begin{pmatrix}
\phi_n^{0*} & \phi_n^+ \\
-\phi_n^{++*} & \phi_n^0
\end{pmatrix} \otimes I_i \cdot \begin{pmatrix}
\Gamma_n^U & \Gamma_n^D
\end{pmatrix}
\] (4.57)

\(\Gamma_n^U, \Gamma_n^D\) are the coefficients of Yukawa couplings.

The above arrangement of the quarks and Higgs fields differ from the arrangement in [13], because we took the second and the third \(Z_2\) to be the \(R\) and \(R'\) symmetry respectively. This modification is important to prevent the third Higgs particle from coupling to the quarks, the Higgs \(\Phi_1\) from coupling to \(D_R^i\) as well as the Higgs \(\Phi_2\) from coupling to \(U_R^i\). We have

\[
\mathcal{L}_{\text{Yukawa}} = \sum \sum \psi(x, g) \phi_h(x, g) R_h \psi(x, g)
= - \sum_{n=1,2} \Gamma_n^U (\bar{U}_L \phi_n^0 - \bar{D}_L \phi_n^0) U_R + \Gamma_n^D (\bar{U}_L \phi_n^+ + \bar{D}_L \phi_n^0) \bar{D}_R + h.c.
\]

(4.58)

where \(L^i = (\bar{U}_L^i, \bar{D}_L^i), \pi_n = (\phi_n^+, \phi_n^0)^T, \) and \(\bar{\pi}_1 = i\sigma_2\pi_1.\)

Thus we obtained the Yukawa interactions discussed in [2]. On the other hand, as we expected, this modification does not impact on the Higgs kinetic terms and the Higgs potential, which still contain three Higgs as in [13]

\[
V(\Phi_1, \Phi_2, \Phi_3) = a Tr[\Phi_1^2 - 1] + E_2(\Phi_2^2 - 1) + E_3(\Phi_3^2 - 1)]^2
+ \beta [E_2^2 Tr(\Phi_2^2 - 1)]^2 + E_3^2 Tr(\Phi_3^2 - 1)]^2
+ 2E_1E_2 Tr(\Phi_1 \Phi_2^2 + 2E_1E_3 Tr(\Phi_1 \Phi_3^2)
+ 2E_2E_3 Tr(\Phi_2 \Phi_3^2)]
+ \gamma [E_1^2 Tr(\Phi_1^2 - 1)]^2 + E_2^2 Tr(\Phi_2^2 - 1)]^2 + E_3^2 Tr(\Phi_3^2 - 1)]^2
+ E_1E_2 Tr(\Phi_1 \Phi_2^2 + 2E_1E_3 Tr(\Phi_1 \Phi_3^2)
+ E_2E_3 Tr(\Phi_2 \Phi_3^2)]
+ E_3E_4 Tr(\Phi_3 \Phi_2^2 + 2E_1E_3 Tr(\Phi_1 \Phi_3^2)
+ E_2E_3 Tr(\Phi_2 \Phi_3^2)]
\]

(4.59)

It is very important that the third Higgs isn’t cancelled from the Higgs potential, for at least three Higgs are needed to violate \(C P\) in the Weinberg-Salam-Branco model.

\[
\begin{align*}
Tr(\Phi_m \Phi_m^\dagger) &= 2 Tr(\Gamma^U_m \Gamma^U_m + \Gamma^D_m \Gamma^D_m)(\pi_m^\dagger \pi_m) \\
Tr(\Phi_m \Phi_m^\dagger \Phi_m \Phi_m^\dagger) &= 2 Tr(\Gamma^U_m \Gamma^U_m + \Gamma^D_m)(\pi_m^\dagger \pi_m)(\pi_m^\dagger \pi_m) \\
Tr(\Phi_m \Phi_m^\dagger \Phi_m \Phi_m^\dagger) &= 2 Tr(\Gamma^U_m \Gamma^U_m + \Gamma^D_m)(\pi_m^\dagger \pi_m)(\pi_m^\dagger \pi_m) \\
Tr(\Phi_m \Phi_m^\dagger \Phi_m \Phi_m^\dagger) &= 2 Tr(\Gamma^U_m \Gamma^U_m + \Gamma^D_m)(\pi_m^\dagger \pi_m)(\pi_m^\dagger \pi_m) \\
Tr(\Phi_m \Phi_m^\dagger \Phi_m \Phi_m^\dagger) &= 2 Tr(\Gamma^U_m \Gamma^U_m + \Gamma^D_m)(\pi_m^\dagger \pi_m)(\pi_m^\dagger \pi_m) \\
\end{align*}
\]

(4.60)
With the help of these identities, we obtained

\[ V(\pi_1, \pi_2, \pi_3) = \sum_{n=1}^{3} [a_{nm}(\pi_n^\dagger \pi_n)^2 \pi_n + m_n(\pi_n^\dagger \pi_n)] + \sum_{n<m} \{ a_{nm}(\pi_n^\dagger \pi_n)(\pi_m^\dagger \pi_m) + b_{nm}(\pi_n^\dagger \pi_m)(\pi_n^\dagger \pi_n) + \sum_{n<m} \{ c_{nm}(\pi_n^\dagger \pi_m)(\pi_n^\dagger \pi_n) + h.c. \} \} + \sum_{n<m} \{ a_{nm}(\pi_n^\dagger \pi_n)(\pi_n^\dagger \pi_n) + b_{nm}(\pi_n^\dagger \pi_m)(\pi_n^\dagger \pi_n) \} + \sum_{n<m} \{ a_{nm}(\pi_n^\dagger \pi_m)(\pi_n^\dagger \pi_m) + b_{nm}(\pi_n^\dagger \pi_m)(\pi_n^\dagger \pi_n) \} + \sum_{n<m} \{ a_{nm}(\pi_n^\dagger \pi_m)(\pi_n^\dagger \pi_m) + b_{nm}(\pi_n^\dagger \pi_m)(\pi_n^\dagger \pi_m) + \sum_{n<m} \{ c_{nm}(\pi_n^\dagger \pi_m)(\pi_n^\dagger \pi_m) + h.c. \} \} \] (4.61)

where

\[ c_{nm} = 2\gamma E_nE_mTr(\Gamma_U^U\Gamma_U^U\Gamma_U^U + \Gamma_U^D\Gamma_U^D\Gamma_U^D) \] (4.62)

If we assume the minimum to be at \( \pi_i = \frac{1}{\sqrt{2}}(0, \nu_i e^{i\theta_i})^T \) and set \( \theta_2 = 0 \), we can classify the solutions of stationarity condition \[17] as in \[4\]

(1) \( CP \)- conserving solution:

\[ \theta_1 = \frac{1}{2} n\pi, \quad \theta_3 = \frac{1}{2} m\pi, \quad m, n \in \mathbb{Z} \] (4.63)

(2) \( CP \)- violating solution:

\[ \cos 2\theta_1 = \frac{1}{2} \left( \frac{d_{13}d_{23}}{d_{12}^2} - \frac{d_{23}}{d_{13}} - \frac{d_{13}}{d_{23}} \right), \quad \cos 2\theta_3 = \frac{1}{2} \left( \frac{d_{13}d_{12}}{d_{23}^2} - \frac{d_{12}}{d_{13}} - \frac{d_{13}}{d_{12}} \right) \] (4.64)

where \( d_{nm} = c_{nm}v_n^2v_m^2 \), \( c_{nm} \) are given by (4.62).

5 conclusion and remark

We have reconstructed Georgi-Glashow-Lee model and Weinberg-Salam-Branco model with SCPV, based on the differential calculus and gauge theory on discrete group. We showed that in these models, all the Lagrangian terms containing Higgs particles are related as a whole by the gauge theory with respect to certain discrete symmetry, which can determine the Yukawa couplings, the Higgs kinetic terms and the Higgs potential with less ambiguity. In the first model to insure that \( V(\Phi) \) has a minimum and the minimum occurs at the vacuum expected value of \( \Phi_R, \Phi_I \), the metric parameters must satisfy some condition. In the second model, to insure NFC we take the two \( Z_2 \) symmetries to be the \( R \) and \( R' \), find this modification also lead to SCPV. Thus we established a correlation between the physics model and the nontrivial geometry structure behind it. The constraints of NCG on the physics models are to be investigated further.

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