Citation: Nguyen, Hoang, Nguyen, Tuan, Abdel-Wahab, M., Bordas, S. P. A., Nguyen-Xuan, H. and Vo, Thuc (2017) A refined quasi-3D isogeometric analysis for functionally graded microplates based on the modified couple stress theory. Computer Methods in Applied Mechanics and Engineering, 313. pp. 904-940. ISSN 0045-7825

Published by: Elsevier

URL: http://dx.doi.org/10.1016/j.cma.2016.10.002 <http://dx.doi.org/10.1016/j.cma.2016.10.002>

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Highlights

• An efficient NUBRS-based formulation is proposed to deal with the size effects of small-scale functionally graded plates.

• A novel seventh-order quasi-3D plate theory with only four unknowns requiring $C^1$-continuity is used to sufficiently describe the shear deformation and stretching effects through plate’s thickness.

• A modified couple stress theory with only one material length scale parameter, which requires second-order derivatives of the unknowns, efficiently captures the size effects of the microplates.

• The reliability and validity of the proposed method are illustrated by a number of convergence and comparison results including benchmark numerical examples.

• Effects of material length scale parameter, material index and plate’s aspect ratio on the mechanical behaviours of microplates are investigated.
A Refined Quasi-3D Isogeometric Analysis for Functionally Graded Microplates based on the Modified Couple Stress Theory

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Abstract

The isogeometric analysis associated with a novel quasi-3D shear deformation theory is proposed to investigate size-dependent behaviours of functionally graded microplates. The modified couple stress theory with only one material length scale parameter is employed to effectively capture the size-dependent effects within the microplates. Meanwhile, the quasi-3D theory which is constructed from a novel seventh-order shear deformation refined plate theory with four unknowns is able to consider both shear deformations and thickness stretching effect without requiring shear correction factors. The NURBS-based isogeometric analysis is integrated to exactly describe the geometry and approximately calculate the unknown fields with higher-order derivative and continuity requirements. The proposed approach is successfully applied to study the static bending, free vibration and buckling responses of rectangular and circular functionally graded microplates with various types of boundary conditions in which some benchmark numerical examples are presented. A number of investigations are also conducted to illustrate the effects of the material length scale, material index, and aspect ratios on the responses of the microplates.

Keywords: Isogeometric analysis, Functionally graded microplates, Modified couple stress theory, Refined plate theory, Quasi-3D theory.

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Preprint submitted to Computer Methods in Applied Mechanics and Engineering October 14, 2016
1. Introduction

Functionally graded materials (FGMs) are composite materials formed of two or more constituent phases in which material properties vary smoothly within the structure. Consequently, FGMs avoid high interlaminar shear stresses, stress concentration and delamination phenomena which are often cited as shortcomings of laminated composite materials. A FGM consisting of ceramic and metal possesses higher thermal resistance and better ductility which are inherited from the ceramic and metal phases, respectively. Owing to these striking features, FGMs are applicable to various fields of engineering including aerospace, nuclear power, chemistry and bio-engineering. FGMs have also been widely studied for various types of structures such as beams [1–4], plates [5–8], and shells [9–11].

Recent advances in technology lead to new industrial fields in which small-scale elements are involved. Such elements have been applied in micro- and nano-electro-mechanical systems [12, 13], actuators [14], space and bio-engineering [15]. These applications encourage new research area that focuses on investigating and predicting the behaviours of such micro structures. A number of approaches have been employed to analyse the characteristics of small-scale structures both experimentally and numerically [16, 17]. Indeed, typical structural sizes range from a few to dozens of polycrystalline grains only, such that the actual local grain morphology has a strong influence on the global structural behaviour [18]. One approach is to handle grains explicitly and represent each of them in the model. This leads to large computational demands because of the lack of scale separation. Such constitutive models must be able to account for size effects which are characteristic of small-scale structures. This is confirmed following a number of theoretical and experimental studies of Fleck et al. [19], Stolken and Evans [20], and Lam et al. [21]. From an experimental observation from bending test of epoxy polymeric microbeams, Lam et al. [21] point out that the bending rigidity increases 2.4 times as a result of the reduction of the beam thickness from 115 \( \mu m \) to 20 \( \mu m \).

In order to take into account the size effects, a number theories have been developed including nonlocal elasticity theory [22], strain gradient theory [23], and modified couple stress theory [24]. It is worth noting that the classical elasticity is fundamentally founded by the introduction of the Hooke’s Law in which the force and the change in displacement are linearly related via the stiffness of the component where the forces are applied. This physical principle governs the linearly elastic behaviour of materials. Aiming at a more general description of materials’ responses, Mindlin and Tiersten [25] and Mindlin [26] developed higher-order theories of elasticity. Based on the employment of deformation metrics, those theories can be classified into two categories: strain gradient and couple stress theories. With regard to the strain gradient theories, this concept was first developed by Fleck and Hutchinson [23] in which Mindlin’s theory [26] was extended. There are two components, which are classified using the second-order deformation tensor that include stretch gradient tensor and rotation gradient tensor [27]. Within the concept of couple stress theory, both strain and curvature jointly govern the strength of the solid. In addition, while the antisymmetric part of the second-order deformation gradients represent rotation gradients, the symmetric part is neglected. Based on the initial ideas of the couple stress theory, a number of attempts
have been conducted to further develop such concepts that are applicable to size-dependent problems. Yang et al. [24] proposed the equilibrium of moment of couples which was an additional equilibrium relation that forces the couple stress tensor to be symmetric. Therefore, the deformation energy is only influenced by the symmetric part of the rotation gradient and the symmetric part of the displacement gradient. In addition, instead of using two material length scale parameters as needed in the classical couple stress theory, this modified couple stress theory (MCST) requires only one material length scale parameter to construct the constitutive relation. Park and Gao [28] utilised the principle of minimum total potential energy to develop a variational formulation of the MCST. This method not only derives the equilibrium equations but also forms the introduction of boundary conditions which are not available in Yang’s theory [24]. Possessing those beneficial characteristics in which a symmetric couple stress included and only one material length scale parameter involved, the MCST is considered to have advantages over other size-dependent theories such as classical couple stress theory and nonlocal theory.

In recent years, the MCST has been applied to study various behaviours at small-scales. For beam analysis, static bending, buckling, and vibration analysis have been solved using Euler-Bernoulli [29], Timoshenko [30–33], and higher-order beam theories [34]. The MCST was also applied to small-scale plate analyses in several ways, Tsiatas [35] initially employed it to investigate the static bending response of isotropic Kirchhoff microplates. Yin et al. [36] investigated the vibration behaviour of Kirchhoff microplates using the standard separation of variables to derive the closed-form solution for natural frequencies. Bending and vibration behaviours of Mindlin microplates were studied by Ma et al. [37] in which the thickness stretching effect was also taken into account. With regard to small-scale functionally graded (FG) structures, a number of investigations have been conducted for FG microbeams and microplates using the MCST. Static bending and buckling of FG microbeams were studied by Simsek et al. [38] and Nateghi et al. [39]. Thai and his colleagues utilised the Navier’s approach to deriving solutions for FG microplates in which Kirchhoff, Mindlin and sinusoidal plate theories were used [40–42]. Ke et al. [43, 44] employed the p-version of the Ritz method and different quadrature method to solve free vibration and bending, buckling problems of the rectangular and annular FG Mindlin microplates, respectively. Using the MCST, a refined plate theory was utilised to predict the bending, buckling, and vibration behaviours of FG microplates by He et al. [45] following the Navier approach. Reddy et al. [46–49] studied the nonlinear behaviour of small-scale FG microplates for different geometries based on finite element method (FEM) with eleven-unknown C⁰ element formulation. Most of these efforts followed either analytical approaches being able to solve specific problem for a limit set of boundary conditions or C⁰ FEMs with high number of unknowns which are computationally expensive.

A large proportion of the studies in small-scale FG structures employs the classical plate theory (CPT) and the first-order shear deformation theory (FSDT). However, the CPT (or the Kirchhoff-Love theory), which neglects shear deformation, provides acceptable solutions for thin plates (i.e. length-to-thickness ratios are larger than 20) only. The FSDT (or Reissner-Mindlin plate theory), which accounts for transverse shear effects, is applicable for both thin and moderately thick plates [50–52]. The shortcomings of the FSDT include inac-
curate distribution of transverse shear strain/stress and violation of traction free boundary conditions at the top and bottom surfaces. For this reason, shear correction factors are required to adjust the transverse shear stress distribution. However, these factors vary from problem to problem and one may find it difficult to choose appropriate values in general. In order to avoid using shear correction factors, the third-order shear deformation theory (TSDT) \[53\], higher-order shear deformation theory (HSDT) \[54\], sinusoidal shear deformation theory (SSDT) \[55\], and refined plate theories (RPT) \[56\] have been developed yielding more accurate and robust results. The RPT was initially proposed by Senthilnathan et al. \[56\] by employing four independent unknowns which is one less than that of TSDT. Shimpi et al. \[57–59\] then further developed the RPT for isotropic and orthotropic plates by using only two variables. However, HSDT and RPT require the \(C^1\)-continuity of the generalised displacements which cause significant challenge to derive the second derivative of deflection in the framework of finite element analysis (FEA) with \(C^0\) elements. In order to overcome these continuity issues, some \(C^0\) approximations \[60\] and Hermite interpolation functions with \(C^1\) elements \[8\] which involve adding extra variables of derivative of displacement can be adopted.

Recently, a new numerical method so-called Isogeometric Analysis (IGA) which is able to deal with \(C^1\)-continuity problem without using any additional variables or Hermite interpolation function has been introduced by Hughes and his co-workers \[61\]. This method bridges the existing gap between computer-aided design (CAD) and the fields of FEA. The essential idea of the IGA is that the basis functions, commonly the non-uniform rational B-splines (NURBS), which are employed to exactly describe the geometry domain will also be used for approximations of the unknown fields. In addition, these basic functions are high smoothness and able to tailor the continuity order easily through the domain \[61, 62\]. With these striking features, the NURBS-based IGA appears to be a potential approach in dealing with the \(C^1\) HSST and RPT problems \[63\]. One may find the guidance on computer implementation of IGA in the literature \[64–66\]. IGA has been widely implemented in a number of linearly and non-linearly mechanical and thermal problems such as static, free vibration, and buckling of laminated composite and FG plates with various plate theories including layerwise \[67\], FSDT \[68–70\], HSDT \[71–73\], and RPT \[63\]. The IGA is also applicable for the analysis of the shell structures \[74–76\]. However, as far as authors are aware, there is no work published on the analysis of small-scale plates based on the MCST and NURBS basis functions.

In this study, the bending, free vibration and buckling behaviours of FG microplates based on the MCST and four-variable refined plate and quasi-3D theories are investigated using IGA. While the MCST is employed to capture the small-scale effects, the displacement fields of those microplates are expressed based on a novel seventh-order refined plate theory and quasi-3D theory. The mechanical behaviour of FG microplates is then analysed by this IGA in which NURBS functions are simultaneously used to exactly describe the geometry and construct the basis functions of the approximations. It is worth mentioning that even though NURBS may not fully perform their ability to describe geometry exactly since the plate’s domains investigated in this study are not highly complex, NURBS far outweigh traditional FEA in the higher-order derivative of approximations which are essentially required.
in the proposed RPT and MCST.

The outline of this study is as follows. The next section presents theories which are applicable for analysis of FG microplates including MCST, RPT and quasi-3D theory. In addition, a brief note on FGMs is also introduced in this section. Section focuses on the IGA and NURBS-based formulation of the quasi-3D theory. The numerical examples which cover static bending, free vibration and buckling analysis of rectangular and circular FG microplates with various boundary conditions are provided in Section 4. Finally, conclusions are given in Section 5.

2. A novel theory for FG microplates

In this section, a brief review on the formulation of the MCST with only one material length scale that accounts for size-dependent effects is presented. It is followed by the definitions required to describe FGMs of which the studied microplates are made. The displacement field of these plates is then derived based on the four-variable refined plate theory and quasi-3D plate theory where a novel seventh-order shear deformation theory is proposed.

2.1. Modified couple stress theory

According to the MCST which is proposed by Yang et al. [24], the strain energy density \( w \) for linear isotropic material is a quadratic function of generalised strains

\[
  w = \frac{1}{2} \lambda (\text{tr}\varepsilon)^2 + \mu (\varepsilon : \varepsilon + \ell^2 \chi : \chi),
\]

where \( \lambda \) and \( \mu \) are Lamé’s constants, \( \mu \) is also known as shear modulus which is often denoted as \( G \), \( \ell \) represents material length scale parameter and the strain tensor \( \varepsilon \) and symmetric curvature tensor \( \chi \) are defined by

\[
  \varepsilon = \frac{1}{2} \left[ \nabla u + (\nabla u)^T \right],
\]

\[
  \chi = \frac{1}{2} \left[ \nabla \theta + (\nabla \theta)^T \right],
\]

where \( u \) is the displacement vector and the rotation vector \( \theta \) is given by

\[
  \theta = \frac{1}{2} \text{curl}(u).
\]

The strain energy \( U \) stored in a deformed elastic body is then defined as

\[
  U = \int_V \! w dV = \int_V \! (\sigma : \varepsilon + m : \chi) dV,
\]
where $\sigma$ and $m$ are the symmetric stress tensor and the deviatoric part of the symmetric couple stress tensor, respectively. These components, $\sigma$ and $m$, which are conjugated to the deformation measures $\varepsilon$ and $\chi$, respectively, are given as

\begin{align}
\sigma (\varepsilon) &= \lambda (tr \varepsilon) I + 2\mu \varepsilon, \quad (5a) \\
m (\chi) &= 2\mu \ell^2 \chi, \quad (5b)
\end{align}

where $I$ is the identity tensor. Apparently, only one material length scale needed and the deviatoric couple stress tensor $m$ is also symmetric, from which the MCST is formed.

2.2. Functionally graded material

The model configuration of a FGM which is made of metal and ceramic is illustrated in Fig. 1. There are several homogeneous models that are employed to estimate the effective properties of the FGMs. According to the rule of mixtures, the corresponding effective properties of these FGMs can be expressed as follows

\begin{align}
P_e &= P_m V_m + P_c V_c, \quad (6)
\end{align}

where $P_m$ and $P_c$ are the material properties of the metallic and ceramic phases, respectively, including the Young’s modulus $E$, the density $\rho$ and the Poisson’s ratio $\nu$. Meanwhile, $V_m$ and $V_c$ represent the volume fraction of metal and ceramic phases, respectively, which are defined as follows [8]

\begin{align}
V_c (z) &= \left( \frac{1}{2} + \frac{z}{h} \right)^n, \quad V_m = 1 - V_c, \quad -\frac{h}{2} \leq z \leq \frac{h}{2}, \quad (7)
\end{align}

where $n$ is the material index. This equation implies a smooth variation in material properties governed by the material index $n$. As can be inferred from Eq. (7), $n = 0$ leads to a homogeneous ceramic material while a fully metallic material is obtained as $n$ approaches $+\infty$.

Nevertheless, the rule of mixtures fails to describe the interaction between the material phases [77, 78]. Therefore, the Mori-Tanaka scheme [79, 80] was developed which integrates the effective bulk modulus $K_e$ and shear modulus $G_e$ are given by

\begin{align}
\frac{K_e - K_m}{K_e - K_m} &= \frac{V_c}{1 + V_m \frac{K_e - K_m}{K_m + 2G_m}}, \quad \frac{G_e - G_m}{G_e - G_m} = \frac{V_c}{1 + V_m \frac{G_e - G_m}{G_m + f_1}}, \quad (8)
\end{align}

where

\begin{align}
f_1 &= \frac{G_m (9K_m + 8G_m)}{6 (K_m + 2G_m)}. \quad (9)
\end{align}

The effective Young’s modulus $E_e$ and Poisson’s ratio $\nu_e$ are then defined as

\begin{align}
E_e &= \frac{9K_e G_e}{3K_e + G_e}, \quad \nu_e = \frac{3K_e - 2G_e}{2 (3K_e + G_e)}. \quad (10)
\end{align}

The variation in effective Young’s modulus of Al/Al$_2$O$_3$ estimated by the rule of mixtures and Mori-Tanaka scheme is depicted in Fig. 2. As can be seen, the effective material property of the FGM varies continuously from the metal-rich surface at the bottom to the ceramic-rich surface at the top of the plate.
2.3. A novel seventh-order shear deformation plate theory

With regard to the plate theories, the third-order shear deformation model initially proposed by Reddy [53] is widely considered as a reliable theory in which no shear correction factor is required. In Reddy’s theory, the displacement field, for \( z \in [-h/2; h/2] \), is defined as

\[
\begin{align*}
  u(x, y, z) &= u_0(x, y) + z\beta_x(x, y) + g(z) \left( \beta_x(x, y) + w_{,x}(x, y) \right), \quad (11a) \\
  v(x, y, z) &= v_0(x, y) + z\beta_y(x, y) + g(z) \left( \beta_y(x, y) + w_{,y}(x, y) \right), \quad (11b) \\
  w(x, y, z) &= w_0(x, y), \quad (11c)
\end{align*}
\]

where the comma notation \( (,x) \) indicates a derivative with respect to the spatial variable, \( g(z) = -4z^3/(3h^2) \) [53] and the variables \( u_0 = [u_0 \ v_0]^T \), \( w_0 \), and \( \beta = [\beta_x \ \beta_y]^T \) are the membrane displacements, the transverse deflection of the mid-plane surface, and the rotations, respectively. By making further assumptions, \( w_0 = w_b + w_s, \beta_x = -w_{b,x}, \beta_y = -w_{b,y} \), to Reddy’s theory which contains five unknowns, Senthilnathan [56] proposed the four-variable refined plate theory which can be expressed in the generalised form as

\[
\begin{align*}
  u(x, y, z) &= u_0(x, y) - zw_{b,x}(x, y) + f(z) w_{s,x}(x, y), \quad (12a) \\
  v(x, y, z) &= v_0(x, y) - zw_{b,y}(x, y) + f(z) w_{s,y}(x, y), \quad (12b) \\
  w(x, y, z) &= w_b(x, y) + \phi(z) w_s(x, y). \quad (12c)
\end{align*}
\]

where \( w_b \) and \( w_s \) represent bending and shear components of transverse displacement, respectively. The function \( g : z \rightarrow g(z) = f(z) - z \) is employed to describe the distribution of transverse shear strains and stresses through the plate’s thickness. It is necessary to have the first derivative of \( f \) satisfies the tangential zero value at \( z = \pm h/2 \) such that the traction-free condition at top and bottom surfaces is met. Consequently, the shear correction factor is no longer required for higher-order shear deformation theory and refined plate theory.

It should be noted that both the higher-order shear deformation theory and refined plate theory fail to capture the thickness stretching effect of normal deformation \( (\varepsilon_z \neq 0) \) due to the constant deflection through the plate thickness which can be inferred from Eq. (12c). In order to bypass this shortcoming, a number of theories which consider the thickness stretching effect have been developed [81–83]. Zenkour [84, 85] proposed the four-variable quasi-3D plate theory accounting for both transverse shear and normal deformations which can be alternatively expressed as follows

\[
\begin{align*}
  u(x, y, z) &= u_0(x, y) - zw_{b,x}(x, y) + f(z) w_{s,x}(x, y), \quad (13a) \\
  v(x, y, z) &= v_0(x, y) - zw_{b,y}(x, y) + f(z) w_{s,y}(x, y), \quad (13b) \\
  w(x, y, z) &= w_b(x, y) + \phi(z) w_s(x, y). \quad (13c)
\end{align*}
\]

As can be observed, this quasi-3D model has a similar form to that of the four-variable refined plate theory shown in Eq. (12). Indeed, the displacement field based on the refined
plate theory can be readily obtained by simplifying those of quasi-3D theory in which \( f \) and \( \phi \) are replaced by \( g \) and 1, respectively. In addition, this formulation of quasi-3D displacement field requires less number of unknowns than that of existing theories [86–88].

It is worth noting that although, by the theoretical material models, the neutral plane of the functionally graded plate would not perfectly coincide with its mid-plane, the assumption of their coincidence which is widely used is applied in the above displacement fields.

A number of distribution functions, \( f \) and \( \phi \), are available for FGM plates based on higher-order shear deformation theory, refined plate theory, and quasi-3D plate theory. One may find the general framework to construct such polynomial functions in the recent work of Nguyen et al. [89]. In this study, a novel seventh-order function of \( f \) and its corresponding function \( \phi \) are proposed for the four-variable refined plate theory and quasi-3D theory. The function \( f \) which represents the nonlinear distribution of the transverse shear strains and stresses is carefully chosen to satisfy the traction-free boundary conditions, therefore no shear correction factor is required for this refined plate theory. In addition, the function’s coefficients are obtained by conducting optimisation procedure in which the minimisation of the differences between the outcome results and the existing analytical solutions is considered as objective functions and the coefficients play roles of design variables. The proposed functions of \( f \) and \( \phi \) are presented in Table 1 and Fig. 3 along with others existing in the literature.

According to the displacement field and the strain-displacement relation, which are presented in Eqs. (13) and (2a), respectively, the following strain expressions can be obtained as

\[
\begin{align*}
\epsilon(x, y, z) &= \epsilon_0 + z \kappa_b + f(z) \kappa_s, \\
\gamma(x, y, z) &= [f'(z) + \phi(z)] \epsilon_s,
\end{align*}
\]

where

\[
\begin{align*}
\epsilon &= \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}, & \epsilon_0 &= \begin{bmatrix} u_{0,x} \\ v_{0,y} \\ u_{0,y} + v_{0,x} \end{bmatrix}, & \kappa_b &= \begin{bmatrix} w_{b,xx} \\ w_{b,yy} \\ 2w_{b,xy} \end{bmatrix}, & \kappa_s &= \begin{bmatrix} w_{s,xx} \\ w_{s,yy} \\ 2w_{s,xy} \end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
\gamma &= \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}, & \epsilon_s &= \begin{bmatrix} w_{s,x} \\ w_{s,y} \end{bmatrix}, & \varepsilon_z &= \phi'(z) w_s.
\end{align*}
\]

Using Eqs. (13), (3), and (2b), the rotation vector and the curvature vector are expressed as

\[
\begin{align*}
\theta &= \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2w_{b,y} - (f' - \phi) w_{s,y} \\ -2w_{b,x} + (f' - \phi) w_{s,x} \\ v_{0,x} - u_{0,y} \end{bmatrix}, \\
\chi &= \begin{bmatrix} \chi_b \\ \chi_s \\ \chi_{zz} \end{bmatrix} = \begin{bmatrix} \chi_{b0} \\ \chi_{s0} \\ 0 \end{bmatrix} + \begin{bmatrix} f' \chi_{b1} \\ f'' \chi_{s2} \\ 0 \end{bmatrix} + \begin{bmatrix} \phi \chi_{b3} \\ \phi' \chi_{s4} \\ 0 \end{bmatrix},
\end{align*}
\]

8
where
\[
\chi_b = \begin{bmatrix}
\chi_{xx} \\
\chi_{yy} \\
\chi_{xy}
\end{bmatrix},
\chi_{b0} = \frac{1}{2} \begin{bmatrix}
w_{b,xy} \\
w_{b,yy} - w_{b,xx}
\end{bmatrix},
\chi_{b1} = \frac{1}{4} \begin{bmatrix}
-2w_{s,xy} \\
2w_{s,xy} \\
w_{s,yy} - w_{s,xx}
\end{bmatrix},
\chi_{b2} = \frac{1}{4} \begin{bmatrix}
v_{0,xx} - w_{0,xy} \\
v_{0,yy} - w_{0,xy}
\end{bmatrix},
\chi_{b3} = \frac{1}{4} \begin{bmatrix}
2w_{s,yy} - w_{s,xx}
\end{bmatrix},
\chi_{s} = \begin{bmatrix}
\chi_{xz} \\
\chi_{yz}
\end{bmatrix},
\chi_{s0} = \frac{1}{4} \begin{bmatrix}
v_{0,xx} - w_{0,xy} \\
v_{0,yy} - w_{0,xy}
\end{bmatrix},
\chi_{s1} = \frac{1}{4} \begin{bmatrix}
-2w_{s,y} \\
w_{s,x}
\end{bmatrix},
\chi_{s2} = \frac{1}{4} \begin{bmatrix}
2w_{s,y} - w_{s,x}
\end{bmatrix},
\chi_{s3} = \frac{1}{4} \begin{bmatrix}
-2w_{s,y}
\end{bmatrix}.
\]

According to Eq. (5), the constitutive relations for classical and modified couple stress theories can be presented in an explicit form as
\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\sigma_{xy} \\
\tau_{xz} \\
\tau_{yz}
\end{bmatrix} = \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\
Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\
Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & Q_{66} & 0 & 0 \\
0 & 0 & 0 & 0 & Q_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & Q_{44}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\varepsilon_{xy} \\
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix},
\]
\[
m_{ij} = 2G_e e^2 \chi_{ij},
\]
where, for the proposed quasi-3D theory ($\varepsilon_z \neq 0$), $Q_{ij}$ are the three-dimensional elastic constants which write
\[
Q_{11} = Q_{22} = Q_{33} = \frac{(1 - \nu_e) E_e}{(1 - 2\nu_e)(1 + \nu_e)},
Q_{12} = Q_{13} = Q_{23} = \frac{\nu_e E_e}{(1 - 2\nu_e)(1 + \nu_e)},
Q_{44} = Q_{55} = Q_{66} = \frac{E_e}{2(1 + \nu_e)},
\]
meanwhile, for the proposed refined plate theory ($\varepsilon_z = 0$), $Q_{ij}$ are reduced plane-stress elastic constants and are expressed as
\[
Q_{11} = Q_{22} = \frac{E_e}{1 - \nu_e^2},
Q_{12} = Q_{21} = \frac{E_e \nu_e}{1 - \nu_e^2},
Q_{44} = Q_{55} = Q_{66} = \frac{E_e}{2(1 + \nu_e)},
\]
and the shear modulus $G_e = \frac{E_e}{2(1 + \nu_e)}$.

In this study, the weak form of the static bending, vibration, and buckling problems are derived using the Hamilton’s principle and weak formulation. One can find details on those
well-known procedures in the literature [63, 90, 91]. Firstly, the weak form of the static bending of the couple-stress-based microplates subjected to transverse load $q_0$ can be expressed in the following compact form

$$
\int_{\Omega} \delta \varepsilon^T_b D^b \varepsilon_b d\Omega + \int_{\Omega} \delta \varepsilon^T_s D^s \varepsilon_s d\Omega + \int_{\Omega} \delta (\chi^c_b)^T D^c \chi^c_b d\Omega + \int_{\Omega} \delta (\chi^c_s)^T D^c \chi^c_s d\Omega \\
= \int_{\Omega} \left[ \delta w_b + \phi \left( \frac{h}{2} \right) \delta w_s \right] q_0 d\Omega ,
$$

where the strain tensors and material matrices in the first two terms in Eq. (21) related to classical elastic theory are

$$
\varepsilon^b_b = \begin{bmatrix} \varepsilon_0 \\ \kappa_b \\ w_s \end{bmatrix}, \quad \varepsilon^s_s = \begin{bmatrix} w_{s,x} \\ w_{s,y} \end{bmatrix}, \quad D^b = \begin{bmatrix} A & B & E & X \\ B & D & F & Y^b \\ E & F & H & Y^s \\ X & Y^b & Y^s & Z_{33} \end{bmatrix},
$$

in which the material matrices are calculated by

$$(A, B, D, E, F, H) = \int_{-h/2}^{h/2} [1, z, z^2, f(z), zf(z), f^2(z)] \bar{Q} dz, \quad (23a)$$

$$(X, Y^b, Y^s) = \int_{-h/2}^{h/2} [\phi'(z), z\phi'(z), f(z)\phi'(z)] \bar{Q} dz, \quad (23b)$$

$$Z_{33} = \int_{-h/2}^{h/2} [\phi'(z)]^2 Q_{33} dz, \quad (23c)$$

$$D^s = \int_{-h/2}^{h/2} [f'(z) + \phi(z)]^2 \tilde{Q} dz, \quad (23d)$$

$$\bar{Q} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} Q_{13} \\ Q_{23} \\ 0 \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} Q_{14} & 0 \\ 0 & Q_{55} \end{bmatrix}, \quad (23e)$$

and where the curvature tensors and material matrices in the third and fourth terms in Eq. (21) representing the couple stress theory are

$$\chi^c_b = \begin{bmatrix} \chi_{b0} \\ \chi_{b1} \\ \chi_{b3} \end{bmatrix}, \quad \chi^c_s = \begin{bmatrix} \chi_{s0} \\ \chi_{s2} \\ \chi_{s4} \end{bmatrix}, \quad D^c_b = \begin{bmatrix} A^c & B^c & E^c \\ B^c & D^c & F^c \\ E^c & F^c & H^c \end{bmatrix}, \quad D^c_s = \begin{bmatrix} X^c & Y^c & T^c \\ Y^c & Z^c & V^c \\ T^c & V^c & W^c \end{bmatrix}, \quad (24)$$
in which the material matrices are be defined as

\[
(A^c, B^c, D^c, E^c, F^c, H^c) = \int_{-h/2}^{h/2} \left(1, f'(z), [f'(z)]^2, \phi(z), f'(z) \phi(z), [\phi(z)]^2 \right) \tilde{G} dz, \quad (25a)
\]

\[
(X^c, Y^c, Z^c, T^c, V^c, W^c) = \int_{-h/2}^{h/2} \left(1, f''(z), [f''(z)]^2, \phi'(z), f''(z) \phi'(z), [\phi'(z)]^2 \right) \tilde{G} dz,
\]

where

\[
\tilde{G} = 2G_c \ell^2 \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad \tilde{G} = 2G_c \ell^2 \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}. \quad (26)
\]

The weak form of the free vibration of the couple-stress-based microplates is briefly expressed as

\[
\int \delta \tilde{e}_b^T D^b \delta e_b d\Omega + \int \delta \tilde{e}_s^T D^s \delta e_s d\Omega + \int \delta (\chi_b^c)^T D_b^c \chi_b^c d\Omega + \int \delta (\chi_s^c)^T D_s^c \chi_s^c d\Omega = \int \delta \tilde{u}^T \tilde{m} \tilde{u} d\Omega,
\]

where \( \tilde{u} = [u_0, w_{b,x}, w_{s,x}, v_0, w_{b,y}, w_{s,y}, w_b, w_s]^T \), and the mass matrix \( \tilde{m} \) is defined by

\[
\tilde{m} = \begin{bmatrix}
I_0 & 0 & 0 \\
0 & I_0 & 0 \\
0 & 0 & I_1
\end{bmatrix}
\]

in which \( I_0 = \begin{bmatrix} I_1 & I_2 & I_4 \\ I_2 & I_3 & I_5 \\ I_4 & I_5 & I_6 \end{bmatrix} \), \( I_1 = \begin{bmatrix} I_1 & I_7 & 0 \\ I_7 & I_8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \).

\[
(1, I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8) = \int_{-h/2}^{h/2} \rho [1, z, z^2, f(z), z f(z), f^2(z), \phi(z), \phi^2(z)] dz. \quad (28b)
\]

For buckling analysis, the weak form of the couple-stress-based microplates subjected to in-plane loading is of the form

\[
\int \delta \tilde{e}_b^T D^b \delta e_b d\Omega + \int \delta \tilde{e}_s^T D^s \delta e_s d\Omega + \int \delta (\chi_b^c)^T D_b^c \chi_b^c d\Omega + \int \delta (\chi_s^c)^T D_s^c \chi_s^c d\Omega
\]

\[
+ \int \nabla^T \delta [w_b + \phi(0) w_s] N_0 \nabla [w_b + \phi(0) w_s] d\Omega = 0
\]

where \( \nabla^T = [\partial/\partial x \ \partial/\partial y]^T \) and \( N_0 = \begin{bmatrix} N_x^0 & N_y^0 \\ N_{xy}^0 & N_{yy}^0 \end{bmatrix} \) are the transpose of gradient operator and matrix of pre-buckling loads, respectively.
3. FG microplate formulation based on NURBS basis functions

In this section, a brief review on NURBS which serves as the basis functions of IGA will be presented. It is followed by a novel NURBS-based formulation for couple-stress microplate bending, free vibration and buckling that rely upon the refined plate theory and the quasi-3D theory.

3.1. B-splines and NURBS basis functions

The starting point to express NURBS basis functions is a non-decreasing knot vector \( \Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\} \) where the \( i^{th} \) knot \( \xi_i \in \mathbb{R} \), \( n \) represents the number of basis functions, and \( p \) denotes the polynomial order. The knot vectors can be either uniform if the knots are equally spaced in the parameter space or open if its first and last knot values are repeated \( p + 1 \) times. The knot spans which are bounded by knots define element domains.

The B-spline basis functions that are constructed by the Cox-de Boor recursion formula, starting with the zeroth order \((p = 0)\) are defined by [61, 62]

\[
N_{i,0}(\xi) = \begin{cases} 
1 & \text{if } \xi_i \leq \xi < \xi_{i+1}, \\
0 & \text{otherwise},
\end{cases}
\]  

\[
N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi), \text{for } p \geq 1,
\]

in which the fraction \( 0/0 \) is defined as zero. While the basis functions are smooth, e.g. \( C^{\infty} \) continuity, within this domain, they are \( C^{p-k} \) continuous across the knots, where \( k \) is the multiplicity of the knot. Therefore, for \( p \geq 2 \), the basis functions are of \( C^1 \) continuous at each knot with single multiplicity (single knot) and at the boundary of the knot span. Two-dimensional B-splines are obtained by introducing a second knot vector \( H = \{\eta_1, \eta_2, \ldots, \eta_{m+q+1}\} \), where \( m \) and \( q \) are the number of basis functions and the polynomial in \( \eta \) direction, respectively, and using the tensor product of \( \Xi \) and \( H \) in the parametric dimensions yielding

\[
N_A(\xi, \eta) = N_{i,p}(\xi) M_{j,q}(\eta).
\]

For illustration purposes, Fig. 4 depicts the one- and two-dimensional B-spline basis functions which are generated from the knot vector \( \Xi = \{0, 0, 0, 0, 1/5, 2/5, 3/5, 3/5, 4/5, 1, 1, 1\} \) and its combination with the knot vector \( H = \{0, 0, 0, 1/4, 1/2, 3/4, 1, 1, 1\} \), respectively.

The non-uniform rational B-splines (NURBS) basis functions are then further defined by providing an additional weight \( \zeta_A \) to each control point given by [62]

\[
R_A(\xi, \eta) = \frac{N_A \zeta_A}{\sum_A N_A(\xi, \eta) \zeta_A}.
\]

It is noted that B-splines basis function are special cases of NURBS function. Indeed, if all the individual weights corresponding the control points are assigned an equal constant, the NURBS function degenerates to a B-spline function.
3.2. A novel NURBS-based formulation of modified couple stress theory

By using NURBS basis functions, the displacement variables $u$ of a microplate can be approximately calculated as follows

$$u^h(\xi, \eta) = \sum_{A=1}^{n \times m} R_A(\xi, \eta)q_A,$$  \hspace{1cm} (33)

where $n \times m$ is the number of basis functions and $q_A = [u_{0A} \ v_{0A} \ w_{0A} \ w_s A]^{T}$ denotes the vector of degrees of freedom associated with the control point A. By substituting the approximations Eq. (33) into the strain-displacement relations Eq. (15), the in-plane and shear strains can be obtained

$$[\varepsilon_0^T \ \kappa_0^T \ \kappa_s^T] = \sum_{A=1}^{n \times m} \left[ (B_A^m)^T \ (B_A^{b1})^T \ (B_A^{b2})^T \ (B_A^s)^T \right]^{T} q_A,$$  \hspace{1cm} (34)

where

$$B_A^m = \begin{bmatrix} R_{A,x} & 0 & 0 & 0 \\ 0 & R_{A,y} & 0 & 0 \\ R_{A,y} & R_{A,x} & 0 & 0 \end{bmatrix}, \quad B_A^{b1} = \begin{bmatrix} 0 & 0 & R_{A,xx} & 0 \\ 0 & 0 & R_{A,yy} & 0 \\ 0 & 0 & 2R_{A,xy} & 0 \end{bmatrix},$$  \hspace{1cm} (35)

$$B_A^{b2} = \begin{bmatrix} 0 & 0 & 0 & R_{A,xx} \\ 0 & 0 & 0 & R_{A,yy} \\ 0 & 0 & 2R_{A,xy} & 0 \end{bmatrix}, \quad B_A^s = \begin{bmatrix} 0 & 0 & 0 & R_{A,x} \\ 0 & 0 & 0 & R_{A,y} \end{bmatrix},$$

and the curvatures are obtained by substituting Eq. (33) into Eq. (17) :

$$[\chi_0^T \ \chi_1^T \ \chi_2^T \ \chi_3^T \ \chi_s^T]^T = \sum_{A=1}^{n \times m} \left[ \tilde{B}_A^{b0} \ (\tilde{B}_A^{b1})^T \ (\tilde{B}_A^{b2})^T \ (\tilde{B}_A^s)^T \right]^T q_A,$$  \hspace{1cm} (36)

where

$$\tilde{B}_A^{b0} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 2R_{A,xy} & 0 \\ 0 & 0 & -2R_{A,xy} & 0 \\ 0 & 0 & -R_{A,xx} + R_{A,yy} & 0 \end{bmatrix}, \quad \tilde{B}_A^{b1} = \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 & -2R_{A,xy} \\ 0 & 0 & 0 & 2R_{A,xy} \\ 0 & 0 & 0 & (R_{A,xx} - R_{A,yy}) \end{bmatrix},$$

$$\tilde{B}_A^{b2} = \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 & 2R_{A,xy} \\ 0 & 0 & 0 & -2R_{A,xy} \\ 0 & 0 & 0 & (R_{A,xx} + R_{A,yy}) \end{bmatrix}, \quad \tilde{B}_A^{s0} = \frac{1}{4} \begin{bmatrix} -R_{A,xy} & R_{A,xx} & 0 & 0 \\ -R_{A,yy} & R_{A,xy} & 0 & 0 \end{bmatrix},$$

$$\tilde{B}_A^{s2} = \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 & -R_{A,y} \\ 0 & 0 & 0 & R_{A,x} \end{bmatrix}, \quad \tilde{B}_A^{s4} = \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 & R_{A,y} \\ 0 & 0 & 0 & -R_{A,x} \end{bmatrix}. $$

Substituting Eqs. (34) and (36) into Eqs. (21), (27), and (29), the matrix form of the global equilibrium equations for static bending, free vibration, and buckling can be
respectively written as follows

\[ Kq = F, \]  
\[ (K - \omega^2 M) q = 0, \]  
\[ (K - \lambda_{cr} K_g) q = 0, \]

where the global stiffness matrix \( K \) is the summation of the stiffness matrices corresponding to the classical theory \( K_s \) and the couple stress theory \( K_c \), i.e. \( K = K_s + K_c \). These matrices are calculated as follows

\[
K_s = \int_\Omega \left( \begin{bmatrix} B^{11} & B^{12} & B^{13} \\ B^{21} & B^{22} & B^{23} \\ B^{31} & B^{32} & B^{33} \end{bmatrix} \right)^T \left( \begin{bmatrix} A & B & E & X \\ B & D & F & Y^b \\ E & F & H & Y^s \\ X & Y & Z & 3s \end{bmatrix} \right) \left( \begin{bmatrix} B^{11} \\ B^{12} \\ B^{13} \\ B^{22} \\ B^{23} \\ B^{33} \end{bmatrix} \right) + (B^s)^T D^s B^s \ d\Omega, \tag{39a}
\]

\[
K_c = \int_\Omega \left( \begin{bmatrix} \tilde{B}^{11} & \tilde{B}^{12} & \tilde{B}^{13} \\ \tilde{B}^{21} & \tilde{B}^{22} & \tilde{B}^{23} \\ \tilde{B}^{31} & \tilde{B}^{32} & \tilde{B}^{33} \end{bmatrix} \right)^T \left( \begin{bmatrix} A^c & B^c & E^c & X^c \\ B^c & D^c & F^c & Y^c \\ E^c & F^c & H^c & Z^c \\ X^c & Y^c & T^c & V^c \end{bmatrix} \right) \left( \begin{bmatrix} \tilde{B}^{11} \\ \tilde{B}^{12} \\ \tilde{B}^{13} \\ \tilde{B}^{22} \\ \tilde{B}^{23} \\ \tilde{B}^{33} \end{bmatrix} \right) + (\tilde{B}^s)^T D^s \tilde{B}^s \ d\Omega, \tag{39b}
\]

in which \( B_A^z = \begin{bmatrix} 0 & 0 & 0 & R_A \end{bmatrix} \). The load vector \( F \) is given by

\[ F = \int_\Omega q_0 R d\Omega, \tag{40} \]

where \( R = \begin{bmatrix} 0 & 0 & R_A & \phi \left( \frac{h}{2} \right) R_A \end{bmatrix}^T \), the global mass matrix is computed by

\[ M = \int_\Omega \tilde{R}^T \tilde{m} \tilde{R} d\Omega, \tag{41} \]

in which \( \tilde{R} = \{ \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \}^T \), where

\[
R_1 = \begin{bmatrix} R_A & 0 & 0 & 0 \\ 0 & 0 & -R_{A,x} & 0 \\ 0 & 0 & 0 & R_{A,x} \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & R_A & 0 & 0 \\ 0 & 0 & -R_{A,y} & 0 \\ 0 & 0 & 0 & R_{A,y} \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & 0 & R_A & 0 \\ 0 & 0 & 0 & R_A \end{bmatrix}, \tag{42}
\]

the geometric stiffness matrix is given as

\[ K_g = \int_\Omega (B^g)^T N_0 B^g d\Omega, \tag{43} \]

where

\[ B^g = \begin{bmatrix} 0 & 0 & R_{A,x} & \phi \left( 0 \right) R_{A,x} \\ 0 & 0 & R_{A,y} & \phi \left( 0 \right) R_{A,y} \end{bmatrix}, \tag{44} \]
and $\omega$ and $\lambda_{cr}$ represent the natural frequency and the critical buckling value, respectively.

As can be observed from Eq. (39), by introducing the distribution function $f$, the four-variable refined plate theory and quasi-3D theory do not require any shear correction factor, which is usually needed if the first-order shear deformation theory is applied, to describe the transverse shear stresses satisfying traction-free conditions. In addition, the expressions of $\mathbf{B}$ and $\tilde{\mathbf{B}}$ matrices in Eqs. (35) and (37) show the employment of the second-order derivatives of the approximation functions $R_A$. Consequently, $C^1$ continuous approximations are required. This requirement may cause difficulties to finite element analysis which can be solved by using the mixed interpolation of tensorial components (MITC) or increasing the degrees of freedom to transform the $C^1$ problems to $C^0$ ones [46, 92, 93]. Apparently, these approaches results in higher number of variables and larger computational cost. However, within the platform of isogeometric analysis in which NURBS basis functions are employed, the $C^1$-continuity requirement is naturally satisfied for $p \geq 2$ since the basis functions are $C^{p-1}$ continuous across knot spans, i.e. elements. Therefore, the NURBS-based IGA would be a prominent numerical approach to deal with the proposed four-unknown $C^1$ quasi-3D refined plate theory and modified couple stress theory.

4. Numerical examples and discussion

In this section, convergence and verification studies are conducted to demonstrate the accuracy of the novel approaches presented in Section 2 and 3. In order to illustrate the efficiency of IGA approach in dealing with the MCST, this section is then continued by the computational analysis of FG rectangular and circular microplates with various types of boundary conditions for static bending, free vibration and buckling problems. In these investigations, the FG microplates made of a mixture of metal and ceramic whose material properties are presented in Table 2 are used. Throughout the numerical examples, unless otherwise specified, the material length scale $\ell$ is chosen as $17.6 \times 10^{-6}$m which was suggested by Lam et al. [21]. There are two types of boundary conditions considered

Simply supported (S)

$v_0 = w_b = w_s = 0 \quad$ at $x = 0, a$

$u_0 = w_b = w_s = 0 \quad$ at $y = 0, b$

Clamped (C)

$u_0 = v_0 = w_b = w_s = 0$ and $w_{b,x} = w_{b,y} = w_{s,x} = w_{s,y} = 0$

It should be noted that, within a IGA approach, while the homogeneous boundary conditions corresponding to the displacement itself, e.g. $u_0, v_0, w_b, w_s$, are easily treated in a similar way to the traditional finite element method, those require the first derivative of the displacement components, e.g. $w_{b,x}, w_{b,y}, w_{s,x}, w_{s,y}$, can be enforced by assigning zero values to all displacements of control points which are directly related to clamped edges and their adjacent points [91, 94].

4.1. Convergence and verification studies

In order to evaluate the convergence and reliability of the approaches proposed in Sections 2 and 3, the MCST-based size-dependent analysis of homogeneous fully simply-supported
(SSSS) square microplate which is shown in Fig. 5 is conducted using RPT model. Moderately thick plates \((a/h = 20)\) with four different values of material length scale ratio \((\ell/h = 0, 0.2, 0.6, 1)\) are investigated. For each case, eight different finite element meshes are analysed to study the convergence rate of the proposed IGA approach. As can be seen from Table 3, while the fast convergence of the analysis for polynomial order \(p = 3\) and \(p = 4\) is obtained, solutions using quadratic polynomial \(p = 2\) experience relatively slower convergence rate toward analytical solutions reported by Thai and Kim [40]. This agrees well with the expectation in which the higher polynomial functions give better solutions in terms of accuracy and convergence rate. Fig. 6 presents the convergence study with the relative error of non-dimensional central deflection of homogeneous square microplates with respect to the analytical solutions [40]. Based on the convergence study, the cubic \((p = 3)\) NURBS element mesh of \(11 \times 11\) is relatively sufficient for all analysis cases. Therefore, this mesh whose geometry is shown in Fig. 5 will be used throughout the next examples unless otherwise specified.

Further investigation on the accuracy of the proposed method is conducted using FG plates made of alumina and aluminum (Al/Al\(_2\)O\(_3\)). In this case, without considering couple stress effects, the proposed RPT \((\varepsilon_z = 0)\) and quasi-3D \((\varepsilon_z \neq 0)\) theories are applied to analyses of SSSS square plates using the rule of mixtures. The plates are subjected to uniformly and sinusoidally distributed loads which are defined as \(q_0\) and \(q_0 \sin \left(\frac{\pi x}{a}\right) \sin \left(\frac{\pi y}{a}\right)\), respectively. As can be observed in Table 4, the present results are in good agreement with those available in published works using various 2-D and quasi-3D theories. It should be noted that some numerical results generated from the proposed IGA approach using the distribution functions from other existing works [53, 84, 91] are also presented in the Table 4. The above investigations confirm the validity and reliability of the proposed approaches.

4.2. Static bending analysis

In this section, the static bending analysis of FG microplates based on the MCST will be investigated. The SSSS square microplates are assumed to follow the rule of mixtures. The aspect ratio \(a/h\), material length scale ratio \(\ell/h\), and material index \(n\) are taken into account. Table 5 presents the comparison of non-dimensional central deflection of an SSSS square plate with those of Thai and Kim [40]. While the results generated from the proposed RPT theory are in very good agreement, quasi-3D theory yields slightly different responses in terms of displacement. This is attributed to the consideration of the thickness stretching effect in the quasi-3D theory.

The bending responses of fully-clamped (CCCC) square Al/Al\(_2\)O\(_3\) microplates under sinusoidally and uniformly distributed loads are further studied and presented in Table 6. It is noted that since no study on the static behaviours of CCCC microplates using MCST is reported in the literature, the results are compared with those generated from Reddy’s HSDT model [53] using the proposed IGA approach. As can be seen, the results based on Reddy’s model are in excellent agreement with the proposed RPT-based solutions. The effects of material index \(n\) and material length scale \(\ell\) on the central displacement of a CCCC square Al/Al\(_2\)O\(_3\) plate are depicted in Fig. 7 in which data are generated by the
proposed RPT and quasi-3D theories for three different ratios $\ell/h$ of 0, 0.4 and 1.0. As can be observed, an increase in the material index $n$ leads to a rise of the plate’s central deflection due to the decrease in the plate’s stiffness. On the contrary, the growth of the material length scale ratio $\ell/h$ is followed by a decline in the displacement. In other words, for specific material length scale $\ell$, the thinner the microplate, the higher plate’s stiffness. It can be further observed that the discrepancy in terms of central deflection by the proposed RPT and quasi-3D is significantly decreased as $\ell/h$ increases and vanishes when $\ell/h = 1.0$.

Fig. 8 depicts the deformed configurations of the Al/Al$_2$O$_3$ square microplates with various boundary conditions subjected to a sinusoidally distributed load in which $a/h, \ell/h$ and $n$ are equal to 5, 0.4 and 10, respectively. It should be noted that the deformed shapes of the microplates are scaled up for illustration purposes.

4.3. Free vibration analysis

In this part, the free vibration analysis of FG microplates based on a MCST is discussed. The proposed quasi-3D model is initially tested for linear elastic SSSS Al/ZrO$_2$-1 plates with various theories taking into account the normal shear deformation. It can be seen that the results given in Table 7 agree well with other published works. The proposed RPT and quasi-3D models using IGA are further tested for homogeneous square microplates. The results shown in Table 8 are compared with analytical solutions generated from CPT by Yin et al. [36] and TSDT by Thai and Kim [40]. As can be observed, while the proposed RPT model yields slight discrepancy with respect to Yin et al.’s [36] due to their CPT assumption neglecting shear deformations, it shows excellent agreement with Thai and Kim [40], especially as plates become thinner, i.e. $a/h$ ratio is relatively large. Due to the consideration of the thickness stretching effects, the proposed quasi-3D model gives slightly different results in comparison with other theories which based on assumption of $\varepsilon_z = 0$.

Table 9 presents non-dimensional natural frequency of SSSS Al/Al$_2$O$_3$ square microplates. The results are compared with those of Thai and Kim [40] in which an analytical approach based on TSDT model is employed. Thick ($a/h = 5$), moderately thick ($a/h = 20$) and thin ($a/h = 100$) microplates are considered. The results reveal good agreement between the RPT and TSDT [40], especially when the material length scale ratio $\ell/h$ is small, e.g. 0 or 0.2. On the contrary, the discrepancy becomes larger as $\ell/h$ gets closer to 1. However, this phenomenon just happens for thick plates and tends to be less pronounced as the plates become thinner. Meanwhile, the quasi-3D gives slightly different results compared to that of RPT model. A general observation from Table 9 reveals that the higher material length scale ratio is chosen, the larger the natural frequencies of the plates the plate’s stiffness increases. Fig. 9 presents the variation of the normalised natural frequency of CCCC Al/Al$_2$O$_3$ square microplate with respect to the material length scale parameter ratio $\ell/h$, plate’s aspect ratio $a/h$ with different values of material index $n$. Fig. 10 provides a closer look at the effects of material index $n$ and material length scale ratio $\ell/h$ on the plate’s natural frequencies which are computed using the proposed RPT and quasi-3D models.

Similar to the previous case of bending analysis, the plate’s stiffness decreases as a result of rising in material index $n$ and decreasing material length scale ratio $\ell/h$ which leads to a decrease in natural frequency of the plate. In addition, the discrepancy in terms of frequency
results predicted by the proposed RPT and quasi-3D becomes less significant as $\ell/h$ gets bigger and almost vanish difference when $\ell/h = 1.0$. The first six natural frequencies of Al/Al$_2$O$_3$ square microplates with different types of boundary conditions are given in Table 10 in which the results are generated for $n = 1$ and $\ell/h = 0.2$. The present quasi-3D results show good agreement with those of Zenkour’s quasi-3D model [84] using the proposed IGA approach. The first six mode shapes corresponding to the quasi-3D vibration analysis of CCCC microplates with $a/h = 10$ are presented in Fig. 11.

In the next step, free vibration of circular plates whose geometry configuration and mesh are shown in Fig. 12 will be investigated. Since there is no publication on the vibration behaviours of FG circular microplates based on the MCST, the investigation of circular plates in this study can serve as benchmark examples. Table 11 presents the fundamental natural frequencies which are firstly tested for homogeneous plates. Natural frequencies of plates without considering size-dependent effects ($l = 0$) are compared with results reported by Mohammadi et al. [95] and Nguyen et al. [91]. The results show very good agreement between the theories, especially proposed RPT and Nguyen et al.’s [91] which also uses another polynomial-based RPT model. Table 12 presents the first six natural frequencies of Al/Al$_2$O$_3$ circular plates with simple and clamped supports. The plate’s thickness $h$ and material index $n$ are set as 0.2$R$ and 1, respectively. As in the previous case of square microplates, Zenkour’s quasi-3D model [84] using the proposed IGA approach is added for reference purpose since no study can be found for this problem in the literature. The first six vibration mode shapes of clamped plates with $\ell/h = 0.2$ using quasi-3D model are given in Fig. 13.

4.4. Buckling analysis

The buckling behaviour of square and circular FG microplates is discussed. In order to verify the proposed method and models in dealing with buckling analysis, the critical buckling load of SSSS FG microplates bearing biaxial loads is firstly calculated. The results are compared with analytical solutions based on the CPT and the FSDT by Thai and Choi [41] and refined plate theory by He et al. [45] for which material properties are $E_1 = 14.4$GPa, $\rho_1 = 12.2 \times 10^3$kg/m$^3$, $E_2 = 1.44$GPa, $\rho_2 = 1.22 \times 10^3$kg/m$^3$, $\nu_1 = \nu_2 = 0.38$. As can be seen in Table 13 although there is discrepancy for the case of thick microplates ($a/h = 5$), the results predicted by proposed RPT theory are in good agreement with those calculated by FSDT [41] and RPT [45] as plates become thinner. On the other hand, while CPT-based solutions are significantly different compared to that of FSDT and RPTs due to the ignorance of shear deformations, especially for thick plates, the proposed quasi-3D theory which takes into account normal deformation yields similar results with respect to those of shear deformable theories of FSDT and RPTs.

Table 14 presents the biaxial buckling analysis results of Al/Al$_2$O$_3$ square microplates. The results are calculated based on the proposed RPT and quasi-3D theories which can serve as benchmark examples for future references since no result exists in the literature. The first six non-dimensional biaxial buckling loads of Al/Al$_2$O$_3$ square plates are reported in Table 15 for $n = 10$ and $\ell/h = 0.2$. While the results generated from the proposed quasi-3D and IGA-based Zenkour’s quasi-3D theories [84] are relatively close to each other for both types
of boundary conditions, the RPT’s show a clear discrepancy to the other theories, especially for the CCCC plates. This is due to the consideration of normal deformation of the quasi-3D theories. For \( a/h = 5 \), the first six buckling mode shapes of CCCC plates based on the proposed quasi-3D theory which are scaled up for illustration purposes are presented in Fig. 14.

Finally, this section ends with a number of investigations on buckling of circular FG microplates. Table 16 presents the results of the critical buckling loads of CCCC Al/ZrO\(_2\)-2 circular plates without considering couple stress effects. It should be noted that, for this particular attempt of comparison purpose, the material volume fractions are defined as \( V_m = (0.5 - z/h)^n \) and \( V_c = 1 - V_m \) [63, 96–98]. The comparison reveals that the results generated from proposed RPT are in good agreement with those of other shear deformation theories even with relatively thick and thick plates. The proposed quasi-3D approach yields slightly different results in all cases.

Table 17 presents the non-dimensional critical buckling loads of Al/Al\(_2\)O\(_3\) circular microplates with various boundary conditions based on the proposed RPT and quasi-3D theories. The difference between the two theories is relatively small for both boundary conditions. The effects of the material index \( n \) and material length scale ratio \( \ell/h \) on the critical buckling loads of simple and clamped supports of Al/Al\(_2\)O\(_3\) circular microplates with \( h/R = 0.2 \) are illustrated in Fig. 15. Fig. 16 presents the variation of the normalised critical buckling loads of simply-supported Al/Al\(_2\)O\(_3\) circular microplates with respect to the material length scale ratio \( \ell/h \) and plate’s aspect ratio \( a/h \) with different value of material index \( n \).

The first six buckling loads of Al/Al\(_2\)O\(_3\) circular microplates with various aspect ratios for \( n = 1 \) and \( \ell/h = 0.6 \) along with the results generated from Zenkour’s quasi-3D theory [84] using proposed IGA approach are reported in Table 18. The first six buckling mode shapes corresponding to simply-supported circular microplates for \( h/R = 0.2 \) based on proposed quasi-3D theory are presented in Fig. 17.

5. Conclusions

In this study, a novel computational approach, based on the modified couple stress theory, the four-variable refined plate theory and the quasi-3D theory, and the NURBS-based isogeometric analysis, has been presented to investigate the static bending, free vibration, and buckling of functionally graded microplates with various geometries and boundary conditions. Within this proposed approach, the mathematical model governing the behaviour of the plates is constructed based on the modified couple stress theory with only one material length scale which efficiently accounts for the size dependency in small-scale structures. The novel seventh-order refined plate theory and quasi-3D theory with only four unknowns are also presented. This proposed quasi-3D approach not only considers shear deformations but also is able to accurately capture the thickness stretching effect which is neglected by other classical, higher-order shear deformation, and refined plate theories. The NURBS-based isogeometric analysis is used to exactly describe the geometry and approximately construct the unknown field in which the higher-order continuity requirement of the proposed kinematical and constitutive theories are readily satisfied. A number of investigations including novel
benchmark problems which have never been reported in the literature have confirmed the validity and efficiency of the proposed approach. The results also reveal that increase of the material length scale parameter ratio rises the microplate’s stiffness which results in a decrease in the central displacement and an increase in the natural frequency and buckling load. On the contrary, an increase in the material index leads to the opposite effect in which the microplate’s stiffness decreases. For all cases considered, the numerical results that are generated by the proposed refined plate theory and quasi-3D theory are slightly different due to the consideration of the thickness stretching effect. However, these discrepancies become less pronounced as the microplate becomes thinner.

Acknowledgement

The first and last authors gratefully acknowledge the financial support from the Northumbria University via the Researcher Development Framework.

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graded rectangular plates, Composite Structures 113 (2014) 89 – 107. doi:http://dx.doi.org/10.1016/j.compstruct.2014.03.006
Table 1: Various forms of distribution function used for HSDT, RPT, and quasi-3D theories

| Theory       | $\varepsilon_z$ | $f(z)$                                    | $\phi_z$ |
|--------------|-----------------|-------------------------------------------|-----------|
| HSDT [53]    | $= 0$           | $z - \frac{4}{3} \frac{z^3}{h^2}$        | -         |
| RPT [63]     | $= 0$           | $\arctan \left( \sin \left( \frac{\pi}{h} z \right) \right)$ | -         |
| Quasi-3D [84]| $\neq 0$        | $h \sinh \left( \frac{z}{h} \right) - \frac{4}{3h^2} \cosh \left( \frac{1}{2} \right)$ $\frac{1}{12} f'(z)$ |         |
| Quasi-3D [88]| $\neq 0$        | $\frac{h}{2} \sin \left( \frac{\pi z}{h} \right) - z$ $f'(z) + 1$ |         |
| Quasi-3D [91]| $\neq 0$        | $\frac{16}{h} z - \frac{96}{5h^3} z^3 + \frac{288}{25h^5} z^5$ $\frac{1}{8} f'(z)$ |         |
| Present RPT  | $= 0$           | $-8z + \frac{10z^3}{h^2} + \frac{6z^5}{5h^4} + \frac{8z^7}{7h^6}$ | -         |
| Present quasi-3D | $\neq 0$   | $-8z + \frac{10z^3}{h^2} + \frac{6z^5}{5h^4} + \frac{8z^7}{7h^6}$ $\frac{3}{20} f'(z)$ |         |
Table 2: Material properties of FG plates

| Property | Material | Al | Al\(_2\)O\(_3\) | ZrO\(_2\)-1 | ZrO\(_2\)-2 |
|----------|----------|----|---------------|-------------|-------------|
| E (GPa)  |          | 70 | 380           | 200         | 151         |
| \(\nu\)  |          | 0.3| 0.3           | 0.3         | 0.3         |
| \(\rho\) (kg/m\(^3\)) | | 2707 | 3800 | 3000 | 3000 |
Table 3: Convergence of non-dimensional central deflections $\bar{w} = \frac{10Eh^3}{q_0L^4} w (a/2, a/2, 0)$ of SSSS homogeneous square plate subjected to sinusoidally distributed load ($a/h = 20$)

| $\ell/h$ | $p$ | Element Mesh | Analytical [40] |
|---------|-----|--------------|-----------------|
| 0       |     | 3×3 5×5 7×7 9×9 11×11 13×13 15×15 17×17 | 0.2842 |
| 2       | 0.2734 0.2799 0.2819 0.2828 0.2833 0.2835 0.2837 0.2838 |
| 3       | 0.2823 0.2841 0.2842 0.2842 0.2842 0.2842 0.2842 0.2842 |
| 4       | 0.2843 0.2842 0.2842 0.2842 0.2842 0.2842 0.2842 0.2842 |
| 0.2     |     | 3×3 5×5 7×7 9×9 11×11 13×13 15×15 17×17 | 0.2430 |
| 2       | 0.2346 0.2397 0.2413 0.2420 0.2424 0.2426 0.2427 0.2428 |
| 3       | 0.2415 0.2430 0.2431 0.2431 0.2431 0.2431 0.2431 0.2431 |
| 4       | 0.2432 0.2431 0.2431 0.2431 0.2431 0.2431 0.2431 0.2431 |
| 0.6     |     | 3×3 5×5 7×7 9×9 11×11 13×13 15×15 17×17 | 0.1124 |
| 2       | 0.1098 0.1115 0.1121 0.1123 0.1125 0.1125 0.1126 0.1126 |
| 3       | 0.1120 0.1127 0.1127 0.1127 0.1127 0.1127 0.1127 0.1127 |
| 4       | 0.1127 0.1127 0.1127 0.1127 0.1127 0.1127 0.1127 0.1127 |
| 1       |     | 3×3 5×5 7×7 9×9 11×11 13×13 15×15 17×17 | 0.0542 |
| 2       | 0.0532 0.0539 0.0541 0.0542 0.0543 0.0543 0.0543 0.0543 |
| 3       | 0.0541 0.0544 0.0544 0.0544 0.0544 0.0544 0.0544 0.0544 |
| 4       | 0.0544 0.0544 0.0544 0.0544 0.0544 0.1127 0.1127 0.1127 |
Table 4: Comparison of non-dimensional deflections $\bar{w} = \frac{10E_c \, h^3}{q_0 L^4} \, w(a/2, b/2, 0)$ of SSSS Al/Al$_2$O$_3$ square plates (rule of mixtures scheme)

| n  | Theory                      | Theory                      | Theory                      | Theory                      |
|----|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
|    | $\varepsilon_z = 0$         | $\varepsilon_z \neq 0$     | $\varepsilon_z = 0$         | $\varepsilon_z \neq 0$     |
|    | $a/h = 4$                   | $a/h = 10$                  | $a/h = 4$                   | $a/h = 10$                  |
|    | $a/h = 100$                 | $a/h = 100$                 | $a/h = 100$                 | $a/h = 100$                 |

**Sinusoidally distributed load**

1 Zenkour [84] - - - 0.6828 0.5592 0.5624
Neves et al. [86] - - - 0.6997 0.5845 0.5624
Mantari and Soares [99] - - - 0.693 0.569 0.545
Carrera et al. [81] 0.7289 0.5890 0.5625 0.7171 0.5691 0.5457
Akavci and Tanrikulu [100] 0.7282 0.5889 0.5625 0.6908 0.5691 0.5457
IGA-Reddy [53] 0.7284 0.5889 0.5625 - - -
IGA-Zenkour [84] 0.7284 0.5889 0.5625 0.6828 0.5592 0.5459
IGA-Nguyen et al. [91] 0.7275 0.5888 0.5625 0.6880 0.5640 0.5422
IGA-Present† 0.7284 0.5889 0.5625 0.6935 0.5691 0.5460

4 Zenkour [84] - - - 1.1001 0.8404 0.7933
Neves et al. [86] - - - 1.1178 0.8750 0.8286
Mantari and Soares [99] - - - 1.105 0.838 0.793
Carrera et al. [81] 1.1673 0.8828 0.8286 1.1585 0.8821 0.8286
Akavci and Tanrikulu [100] 1.1613 0.8818 0.8287 1.0983 0.8417 0.7925
IGA-Reddy [53] 1.1598 0.8815 0.8287 - - -
IGA-Zenkour [84] 1.1599 0.8815 0.8287 1.1001 0.8404 0.7933
IGA-Nguyen et al. [91] 1.1625 0.8820 0.8287 1.0931 0.8363 0.7875
IGA-Present† 1.1590 0.8813 0.8287 1.0868 0.8392 0.7933

10 Zenkour [84] - - - 1.3391 0.9806 0.9140
Neves et al. [86] - - - 1.1178 0.8750 0.8286
Mantari and Soares [99] - - - 1.308 0.972 0.911
Carrera et al. [81] 1.3925 1.0090 0.9361 1.3745 1.0072 0.9361
Akavci and Tanrikulu [100] 1.3917 1.0089 0.9362 1.3352 0.9818 0.9141
IGA-Reddy [53] 1.3908 1.0087 0.9362 - - -
IGA-Zenkour [84] 1.3908 1.0087 0.9362 1.0611 0.8822 0.8641
IGA-Nguyen et al. [91] 1.3908 1.0087 0.9362 1.0701 0.8898 0.8582
IGA-Present† 1.3902 1.0086 0.9362 1.3116 0.9748 0.9132

**Uniformly distributed load**

1 Zenkour [101] - 0.9287 - - - -
Akavci and Tanrikulu [100] - 0.9288 - - 0.8977 -
IGA-Reddy [53] 1.1319 0.9288 0.8904 - - -
IGA-Zenkour [84] 1.1319 0.9288 0.8904 1.0611 0.8822 0.8641
IGA-Nguyen et al. [91] 1.1308 0.9286 0.8904 1.0701 0.8898 0.8582
IGA-Present† 1.1319 0.9288 0.8904 1.0788 0.8978 0.8642

4 Zenkour [101] - 1.3890 - - - -
Akavci and Tanrikulu [100] - 1.3888 - - 1.3259 -
IGA-Reddy [53] 1.7941 1.3884 1.3116 - - -
IGA-Zenkour [84] 1.7942 1.3884 1.3116 1.7020 1.3238 1.2556
IGA-Nguyen et al. [91] 1.7983 1.3892 1.3116 1.6919 1.3174 1.2464
IGA-Present† 1.7928 1.3882 1.3116 1.6827 1.3223 1.2556

10 Zenkour [101] - 1.5876 - - - -
Akavci and Tanrikulu [100] - 1.5875 - - 1.5453 -
IGA-Reddy [53] 2.1442 1.5872 1.4818 - - -
IGA-Zenkour [84] 2.1442 1.5872 1.4818 2.0656 1.5433 1.4466
IGA-Nguyen et al. [91] 2.1454 1.5876 1.4818 2.0466 1.5362 1.4397
IGA-Present† 2.1432 1.5870 1.4818 2.0244 1.5347 1.4454

†Proposed RPT and quasi-3D models are used for the case of $\varepsilon_z = 0$ and $\varepsilon_z \neq 0$, respectively.
Table 5: Non-dimensional deflection \( \bar{w} = \frac{10E_c h^3}{q_0 L^4} w(a/2, b/2, 0) \) of SSSS Al/Al\(_2\)O\(_3\) square microplates subjected to sinusoidally distributed load (rule of mixtures scheme)

| \(a/h\) | \(\ell/h\) | \(n = 0\) | \(n = 1\) | \(n = 10\) |
|--------|--------|--------|--------|--------|
|        | RPT    | Quasi-3D | TSDT [40] | RPT    | Quasi-3D | TSDT [40] | RPT    | Quasi-3D | TSDT [40] |
| 5      |        |        |        |        |        |        |        |        |        |
| 0      | 0.3433 | 0.3360 | 0.3433 | 0.6688 | 0.6401 | 0.6688 | 1.2271 | 1.1663 | 1.2276 |
| 0.2    | 0.2898 | 0.2853 | 0.2875 | 0.5505 | 0.5321 | 0.5468 | 0.7140 | 0.7043 | 0.6908 |
| 0.4    | 0.1975 | 0.1965 | 0.1934 | 0.3601 | 0.3537 | 0.3535 | 0.4694 | 0.4711 | 0.4514 |
| 0.6    | 0.1292 | 0.1296 | 0.1251 | 0.2288 | 0.2274 | 0.2224 | 0.3174 | 0.3220 | 0.3052 |
| 0.8    | 0.0871 | 0.0879 | 0.0838 | 0.1517 | 0.1520 | 0.1464 | 0.2242 | 0.2289 | 0.2158 |
| 1      | 0.0614 | 0.0623 | 0.0588 | 0.1060 | 0.1069 | 0.1017 | 0.2242 | 0.2289 | 0.2158 |
| 20     |        |        |        |        |        |        |        |        |        |
| 0      | 0.2842 | 0.2836 | 0.2842 | 0.5689 | 0.5516 | 0.5689 | 0.9537 | 0.9280 | 0.9538 |
| 0.2    | 0.2431 | 0.2427 | 0.2430 | 0.4739 | 0.4619 | 0.4737 | 0.8313 | 0.8120 | 0.8303 |
| 0.4    | 0.1695 | 0.1694 | 0.1693 | 0.3157 | 0.3105 | 0.3153 | 0.6001 | 0.5906 | 0.5986 |
| 0.6    | 0.1127 | 0.1127 | 0.1124 | 0.2029 | 0.2008 | 0.2025 | 0.4102 | 0.4061 | 0.4090 |
| 0.8    | 0.0767 | 0.0768 | 0.0765 | 0.1352 | 0.1343 | 0.1349 | 0.2842 | 0.2825 | 0.2834 |
| 1      | 0.0544 | 0.0544 | 0.0542 | 0.0947 | 0.0943 | 0.0944 | 0.2038 | 0.2031 | 0.2033 |
| 100    |        |        |        |        |        |        |        |        |        |
| 0      | 0.2804 | 0.2803 | 0.2804 | 0.5625 | 0.5460 | 0.5625 | 0.9362 | 0.9132 | 0.9362 |
| 0.2    | 0.2401 | 0.2400 | 0.2401 | 0.4689 | 0.4574 | 0.4689 | 0.8176 | 0.8001 | 0.8176 |
| 0.4    | 0.1677 | 0.1677 | 0.1677 | 0.3128 | 0.3076 | 0.3128 | 0.5925 | 0.5833 | 0.5925 |
| 0.6    | 0.1116 | 0.1116 | 0.1116 | 0.2012 | 0.1990 | 0.2011 | 0.4062 | 0.4018 | 0.4061 |
| 0.8    | 0.0760 | 0.0760 | 0.0760 | 0.1341 | 0.1332 | 0.1341 | 0.2820 | 0.2799 | 0.2820 |
| 1      | 0.0539 | 0.0539 | 0.0539 | 0.0939 | 0.0934 | 0.0939 | 0.2024 | 0.2014 | 0.2024 |
Table 6: Non-dimensional deflection $\bar{w} = \frac{10E_c h^3}{q_0 L^4} w(a/2, b/2, 0)$ of CCCC Al/Al$_2$O$_3$ square microplates (rule of mixtures scheme)

| $a/h$ | $\ell/h$ | $n = 0$ | $n = 1$ | $n = 10$ |
|-------|----------|---------|---------|---------|
|       |          | RPT Quasi-3D IGA-Reddy | RPT Quasi-3D IGA-Reddy | RPT Quasi-3D IGA-Reddy |
| Sinusoidally distributed load |
| 5     | 0        | 0.1601 0.1359 0.1601 | 0.3021 0.2554 0.3020 | 0.6111 0.4740 0.6113 |
| 0.2   | 0.1378   | 0.1197 0.1377 | 0.2555 0.2214 0.2554 | 0.5178 0.4199 0.5183 |
| 0.4   | 0.0974   | 0.0883 0.0973 | 0.1751 0.1586 0.1751 | 0.3568 0.3132 0.3575 |
| 0.6   | 0.0655   | 0.0616 0.0655 | 0.1151 0.1081 0.1151 | 0.2358 0.2204 0.2364 |
| 0.8   | 0.0450   | 0.0435 0.0449 | 0.0779 0.0752 0.0779 | 0.1602 0.1559 0.1606 |
| 1     | 0.0321   | 0.0316 0.0320 | 0.0551 0.0542 0.0551 | 0.1136 0.1134 0.1139 |
| 20    | 0.1035   | 0.0950 0.1035 | 0.2065 0.1863 0.2065 | 0.3505 0.3150 0.3505 |
| 0.2   | 0.0919   | 0.0849 0.0919 | 0.1797 0.1638 0.1797 | 0.3150 0.2857 0.3151 |
| 0.4   | 0.0688   | 0.0645 0.0688 | 0.1294 0.1204 0.1294 | 0.2419 0.2237 0.2420 |
| 0.6   | 0.0485   | 0.0462 0.0485 | 0.0882 0.0837 0.0882 | 0.1746 0.1647 0.1747 |
| 0.8   | 0.0343   | 0.0331 0.0343 | 0.0611 0.0587 0.0611 | 0.1258 0.1204 0.1258 |
| 1     | 0.0250   | 0.0243 0.0250 | 0.0438 0.0425 0.0438 | 0.0926 0.0896 0.0926 |
| 100   | 0.0999   | 0.0955 0.0999 | 0.2003 0.1872 0.2003 | 0.3336 0.3131 0.3336 |
| 0.2   | 0.0889   | 0.0853 0.0889 | 0.1746 0.1643 0.1746 | 0.3013 0.2842 0.3013 |
| 0.4   | 0.0668   | 0.0646 0.0668 | 0.1262 0.1204 0.1262 | 0.2336 0.2228 0.2336 |
| 0.6   | 0.0473   | 0.0461 0.0473 | 0.0863 0.0835 0.0863 | 0.1701 0.1640 0.1701 |
| 0.8   | 0.0336   | 0.0329 0.0336 | 0.0599 0.0584 0.0599 | 0.1232 0.1199 0.1233 |
| 1     | 0.0244   | 0.0241 0.0244 | 0.0430 0.0422 0.0430 | 0.0910 0.0891 0.0910 |
| Uniformly distributed load |
| 5     | 0.2239   | 0.1860 0.2238 | 0.4220 0.3500 0.4219 | 0.8557 0.6408 0.8559 |
| 0.2   | 0.1924   | 0.1641 0.1924 | 0.3566 0.3040 0.3565 | 0.7233 0.5701 0.7239 |
| 0.4   | 0.1358   | 0.1215 0.1358 | 0.2443 0.2186 0.2442 | 0.4976 0.4286 0.4985 |
| 0.6   | 0.0914   | 0.0851 0.0913 | 0.1606 0.1495 0.1605 | 0.3288 0.3034 0.3295 |
| 0.8   | 0.0627   | 0.0602 0.0627 | 0.1087 0.1041 0.1087 | 0.2234 0.2154 0.2239 |
| 1     | 0.0447   | 0.0439 0.0447 | 0.0769 0.0752 0.0769 | 0.1584 0.1570 0.1588 |
| 20    | 0.1436   | 0.1300 0.1436 | 0.2864 0.2553 0.2864 | 0.4863 0.4307 0.4863 |
| 0.2   | 0.1275   | 0.1164 0.1275 | 0.2493 0.2248 0.2493 | 0.4372 0.3912 0.4373 |
| 0.4   | 0.0956   | 0.0887 0.0956 | 0.1797 0.1657 0.1797 | 0.3360 0.3072 0.3361 |
| 0.6   | 0.0674   | 0.0637 0.0674 | 0.1227 0.1156 0.1227 | 0.2428 0.2269 0.2429 |
| 0.8   | 0.0478   | 0.0457 0.0478 | 0.0850 0.0813 0.0850 | 0.1750 0.1664 0.1751 |
| 1     | 0.0348   | 0.0336 0.0348 | 0.0610 0.0589 0.0610 | 0.1289 0.1240 0.1289 |
| 100   | 0.1384   | 0.1314 0.1384 | 0.2775 0.2577 0.2775 | 0.4622 0.4310 0.4622 |
| 0.2   | 0.1232   | 0.1174 0.1232 | 0.2421 0.2265 0.2421 | 0.4176 0.3915 0.4176 |
| 0.4   | 0.0927   | 0.0891 0.0927 | 0.1751 0.1664 0.1751 | 0.3242 0.3075 0.3242 |
| 0.6   | 0.0657   | 0.0637 0.0657 | 0.1199 0.1155 0.1199 | 0.2363 0.2268 0.2363 |
| 0.8   | 0.0467   | 0.0456 0.0467 | 0.0833 0.0810 0.0833 | 0.1713 0.1661 0.1713 |
| 1     | 0.0340   | 0.0334 0.0340 | 0.0598 0.0586 0.0598 | 0.1266 0.1236 0.1266 |
Table 7: Comparison of non-dimensional natural frequencies $\bar{\omega} = \frac{\omega^2}{h} \sqrt{\frac{\rho_m}{E_m}}$ of SSSS Al/ZrO$_2$-1 plates (Mori-Tanaka scheme)

| Theory                      | $n = 1$ |         |         | $n = 2$ | $n = 3$ | $n = 5$ |
|-----------------------------|---------|---------|---------|---------|---------|---------|
|                             | $a/h = 5$ | $a/h = 10$ | $a/h = 20$ |         |         |         |
| Vel and Batra [102]         | 5.4806  | 5.9609  | 6.1076  | 5.5023  | 5.5285  | 5.5632  |
| Matsunaga [103]             | 5.7123  | 6.1932  | 6.3390  | 5.6599  | 5.6757  | 5.7020  |
| Neves et al. [86]           | 5.4825  | 5.9600  | 6.1200  | 5.4950  | 5.5300  | 5.5625  |
| Belabed et al. [104]        | 5.4800  | 5.9700  | 6.1200  | 5.5025  | 5.5350  | 5.5625  |
| Alijani and Amabili [105]   | 5.4796  | 5.9578  | 6.1040  | 5.4919  | 5.5279  | 5.5633  |
| Akavci and Tanrikulu [100]  | 5.4829  | 5.9676  | 6.1160  | 5.5064  | 5.5388  | 5.5644  |
| Present                     | 5.5172  | 6.0023  | 6.1505  | 5.5324  | 5.5642  | 5.5886  |
Table 8: Comparison of non-dimensional natural frequencies $\bar{\omega} = \frac{\omega^2}{\bar{h}} \sqrt{\frac{\rho}{E}}$ of SSSS homogeneous microplates

| $a/h$ | Theory       | $\ell/h$                  |
|-------|--------------|---------------------------|
|       |              | 0   | 0.2 | 0.4 | 0.6 | 0.8 | 1   |
| 5     | CPT [36]     | 5.9734 | 6.4556 | 7.7239 | 9.4673 | 11.4713 | 13.6213 |
|       | TSDT [40]    | 5.2813 | 5.7699 | 7.0330 | 8.7389 | 10.6766 | 12.7408 |
|       | RPT (Present)| 5.2813 | 5.7496 | 6.9667 | 8.6191 | 9.8943 | 9.9791 |
|       | Quasi-3D (Present) | 5.3090 | 5.7622 | 6.9438 | 8.5509 | 9.8943 | 9.9791 |
| 20    | CPT [36]     | 5.9734 | 6.4556 | 7.7239 | 9.4673 | 11.4713 | 13.6213 |
|       | TSDT [40]    | 5.9199 | 6.4027 | 7.6708 | 9.4116 | 11.4108 | 13.5545 |
|       | RPT (Present)| 5.9199 | 6.4009 | 7.6646 | 9.4005 | 11.3945 | 13.5330 |
|       | Quasi-3D (Present) | 5.9235 | 6.4030 | 7.6633 | 9.3952 | 11.3854 | 13.5202 |
| 100   | CPT [36]     | 5.9734 | 6.4556 | 7.7239 | 9.4673 | 11.4713 | 13.6213 |
|       | TSDT [40]    | 5.9712 | 6.4535 | 7.7217 | 9.4651 | 11.4689 | 13.6186 |
|       | RPT (Present)| 5.9712 | 6.4534 | 7.7215 | 9.4646 | 11.4682 | 13.6178 |
|       | Quasi-3D (Present) | 5.9723 | 6.4544 | 7.7222 | 9.4650 | 11.4683 | 13.6177 |
Table 9: Non-dimensional natural frequency $\bar{\omega} = \omega^2 \frac{a^2}{h} \sqrt{\frac{\rho_c}{E_c}}$ of SSSS Al/Al$_2$O$_3$ square plates (rule of mixtures scheme)

| $a/h$ | $\ell/h$ | $n = 0$ | $n = 1$ | $n = 10$ |
|-------|---------|--------|--------|--------|
|       | RPT | Quasi-3D | TSDT [40] | RPT | Quasi-3D | TSDT [40] | RPT | Quasi-3D | TSDT [40] |
| 5     | 0   | 5.2813 | 5.3090 | 5.2813 | 4.0781 | 4.1521 | 4.0781 | 3.2519 | 3.3126 | 3.2514 |
|       | 0.2 | 5.7496 | 5.7622 | 5.7699 | 4.4959 | 4.5542 | 4.5094 | 3.5312 | 3.5740 | 3.5548 |
|       | 0.4 | 6.9667 | 6.9438 | 7.0330 | 5.5620 | 5.5865 | 5.6071 | 4.2584 | 4.2627 | 4.3200 |
|       | 0.6 | 8.6191 | 8.5509 | 8.7389 | 6.9822 | 6.9681 | 7.0662 | 5.2471 | 5.2115 | 5.3335 |
|       | 0.8 | 9.8943 | 9.8943 | 10.6766 | 8.2313 | 8.2313 | 8.7058 | 5.8571 | 5.8571 | 6.4759 |
|       | 1   | 9.9791 | 9.9791 | 12.7408 | 8.3019 | 8.3019 | 10.4397 | 5.9073 | 5.9073 | 7.6895 |
| 20    | 0   | 5.9199 | 5.9235 | 5.9199 | 4.5228 | 4.5919 | 4.5228 | 3.7623 | 3.8129 | 3.7622 |
|       | 0.2 | 6.4009 | 6.4030 | 6.4027 | 4.9556 | 5.0179 | 4.9568 | 4.0299 | 4.0761 | 4.0323 |
|       | 0.4 | 7.6646 | 7.6633 | 7.6708 | 6.0714 | 6.1203 | 6.0756 | 4.7428 | 4.7794 | 4.7488 |
|       | 0.6 | 9.4005 | 9.3952 | 9.4116 | 7.5739 | 7.6107 | 7.5817 | 5.7369 | 5.7640 | 5.7453 |
|       | 0.8 | 11.3945 | 11.3854 | 11.4108 | 9.2768 | 9.3042 | 9.2887 | 6.8914 | 6.9106 | 6.9013 |
|       | 1   | 13.5330 | 13.5202 | 13.5545 | 11.0882 | 11.1082 | 11.1042 | 8.1384 | 8.1510 | 8.1494 |
| 100   | 0   | 5.9712 | 5.9723 | 5.9712 | 4.5579 | 4.6263 | 4.5579 | 3.8058 | 3.8533 | 3.8058 |
|       | 0.2 | 6.4534 | 6.4544 | 6.4535 | 4.9922 | 5.0546 | 4.9922 | 4.0724 | 4.1168 | 4.0725 |
|       | 0.4 | 7.7215 | 7.7222 | 7.7217 | 6.1124 | 6.1635 | 6.1126 | 4.7837 | 4.8215 | 4.7840 |
|       | 0.6 | 9.4646 | 9.4650 | 9.4651 | 7.6220 | 7.6630 | 7.6224 | 5.7778 | 5.8090 | 5.7782 |
|       | 0.8 | 11.4682 | 11.4683 | 11.4689 | 9.3339 | 9.3673 | 9.3344 | 6.9341 | 6.9600 | 6.9345 |
|       | 1   | 13.6178 | 13.6177 | 13.6186 | 11.1554 | 11.1832 | 11.1560 | 8.1842 | 8.2060 | 8.1846 |
Table 10: The first six non-dimensional natural frequencies $\bar{\omega} = \frac{\omega^2}{a^2 h} \sqrt{\frac{\rho m}{E_m}}$ of Al/Al$_2$O$_3$ square plates (Mori-Tanaka scheme)

| BC     | $a/h$ | Theory            | Mode          |
|--------|-------|-------------------|---------------|
|        |       |                   | 1  | 2   | 3   | 4   | 5   | 6   |
| SSSS   | 5     | IGA-Zenkour       | 7.9366 | 13.8049 | 13.8049 | 17.3259 | 17.3259 | 19.5422 |
|        |       | Quasi-3D (Present)| 7.8883 | 13.8049 | 13.8049 | 17.2045 | 17.2045 | 19.5422 |
|        |       | RPT (Present)     | 7.7844 | 13.8049 | 13.8049 | 16.9943 | 16.9943 | 19.5422 |
|        | 10    | IGA-Zenkour       | 8.5940 | 20.3230 | 20.3230 | 27.5893 | 27.5893 | 31.7475 |
|        |       | Quasi-3D (Present)| 8.5607 | 20.2031 | 20.2031 | 27.5893 | 27.5893 | 31.5547 |
|        |       | RPT (Present)     | 8.4401 | 19.9188 | 19.9188 | 27.5893 | 27.5893 | 31.1386 |
|        | 100   | IGA-Zenkour       | 8.8399 | 21.7851 | 21.7851 | 35.3333 | 43.0764 | 43.0764 |
|        |       | Quasi-3D (Present)| 8.8390 | 21.7802 | 21.7802 | 35.3221 | 43.0573 | 43.0573 |
|        |       | RPT (Present)     | 8.7127 | 21.4598 | 21.4598 | 34.8171 | 42.6074 | 42.6074 |
| CCCC   | 5     | IGA-Zenkour       | 12.7213 | 22.6661 | 22.6661 | 27.9021 | 27.9021 | 31.1450 |
|        |       | Quasi-3D (Present)| 13.1029 | 22.9300 | 22.9300 | 27.8791 | 27.8791 | 31.3005 |
|        |       | RPT (Present)     | 12.1531 | 22.0683 | 22.0683 | 26.3182 | 26.3182 | 30.5398 |
|        | 10    | IGA-Zenkour       | 15.2379 | 28.9665 | 28.9665 | 41.2221 | 47.8649 | 47.9221 |
|        |       | Quasi-3D (Present)| 15.4413 | 29.3267 | 29.3267 | 41.5955 | 48.2657 | 48.4504 |
|        |       | RPT (Present)     | 14.3638 | 27.8276 | 27.8276 | 39.9305 | 46.0435 | 46.4772 |
|        | 100   | IGA-Zenkour       | 16.0795 | 32.6473 | 32.6473 | 48.5912 | 58.0858 | 58.3951 |
|        |       | Quasi-3D (Present)| 16.0540 | 32.6082 | 32.6082 | 48.5477 | 58.0353 | 58.3402 |
|        |       | RPT (Present)     | 15.5212 | 31.5708 | 31.5708 | 47.0903 | 56.2566 | 56.5207 |
Table 11: The first six non-dimensional natural frequencies $\bar{\omega} = \omega R^2 \sqrt{\frac{\rho h}{D}}$ of homogeneous circular microplates

| $\ell/h$ | Theory       | Mode | 1      | 2      | 3      | 4      | 5      | 6      |
|---------|--------------|------|--------|--------|--------|--------|--------|--------|
|         | Simple support |      |        |        |        |        |        |        |        |
| 0       | Mohammadi et al. [95] | 4.9345 | 13.8981 | 25.5132 | 29.7198 | 39.9571 | 48.4788 |
|         | Nguyen et al. [91] | 4.9304 | 13.8578 | 25.4798 | 29.5390 | 39.6331 | 48.0046 |
|         | RPT (Present) | 4.9304 | 13.8591 | 25.4799 | 29.5456 | 39.6518 | 48.0402 |
|         | Quasi-3D (Present) | 4.9385 | 13.8701 | 25.4983 | 29.5691 | 39.6881 | 48.0906 |
| 0.2     | RPT (Present) | 4.9925 | 14.5095 | 26.5426 | 31.1786 | 41.8855 | 50.8427 |
|         | Quasi-3D (Present) | 5.0024 | 14.5206 | 26.5613 | 31.1981 | 41.9148 | 50.8808 |
| 0.4     | RPT (Present) | 5.1213 | 16.2743 | 29.4369 | 35.5996 | 47.7406 | 58.5547 |
|         | Quasi-3D (Present) | 5.1365 | 16.2857 | 29.4529 | 35.6092 | 47.7542 | 58.5621 |
| 0.6     | RPT (Present) | 5.2422 | 18.8078 | 33.6370 | 41.9096 | 55.8684 | 69.6631 |
|         | Quasi-3D (Present) | 5.2649 | 18.8192 | 33.6497 | 41.9065 | 55.8623 | 69.6631 |
| 0.8     | RPT (Present) | 5.3324 | 21.8342 | 38.6993 | 49.3881 | 65.3972 | 82.8177 |
|         | Quasi-3D (Present) | 5.3642 | 21.8450 | 38.7080 | 49.3713 | 65.3694 | 82.7497 |
| 1       | RPT (Present) | 5.3954 | 25.1769 | 44.3292 | 57.5842 | 75.8223 | 97.1772 |
|         | Quasi-3D (Present) | 5.4379 | 25.1869 | 44.3335 | 57.5538 | 75.7721 | 97.0720 |
|         | Clamped support |      |        |        |        |        |        |        |        |
| 0       | Mohammadi et al. [95] | 10.2158 | 21.2604 | 34.8772 | 39.7706 | 51.0295 | 60.8290 |
|         | Nguyen et al. [91] | 10.1839 | 21.1433 | 34.5892 | 39.3624 | 50.4385 | 59.9580 |
|         | RPT (Present) | 10.1842 | 21.1459 | 34.5885 | 39.3832 | 50.4865 | 60.0416 |
|         | Quasi-3D (Present) | 10.4466 | 21.6458 | 35.2774 | 40.2833 | 51.5045 | 61.3186 |
| 0.2     | RPT (Present) | 10.8087 | 22.4449 | 36.9361 | 41.8103 | 53.5802 | 63.7743 |
|         | Quasi-3D (Present) | 11.0612 | 22.9236 | 37.1800 | 42.6664 | 54.5550 | 64.9738 |
| 0.4     | RPT (Present) | 12.4963 | 25.9527 | 41.2953 | 48.3406 | 61.7419 | 73.9729 |
|         | Quasi-3D (Present) | 12.7255 | 26.3811 | 41.9956 | 49.0934 | 62.6149 | 74.9922 |
| 0.6     | RPT (Present) | 14.8997 | 30.9242 | 48.3696 | 57.5674 | 73.1139 | 88.5334 |
|         | Quasi-3D (Present) | 15.0927 | 31.2968 | 48.9732 | 58.2073 | 73.8637 | 89.3634 |
| 0.8     | RPT (Present) | 17.7051 | 36.7699 | 56.8123 | 68.4011 | 86.4224 | 106.6689 |
|         | Quasi-3D (Present) | 17.8850 | 37.0938 | 57.3265 | 68.9439 | 87.0506 | 106.3411 |
| 1       | RPT (Present) | 20.7715 | 43.1360 | 66.1009 | 80.1939 | 100.9243 | 124.3039 |
|         | Quasi-3D (Present) | 20.9330 | 43.4206 | 66.5388 | 80.6591 | 101.4425 | 124.8498 |

$D = \frac{Eh^3}{12(1-\nu^2)}$
Table 12: The first six non-dimensional natural frequencies $\bar{\omega} = \omega R^2 \sqrt{\frac{\rho_c h}{D_c}}$ of Al/Al$_2$O$_3$ circular microplates (Mori-Tanaka scheme)

| $\ell/h$ | Theory       | Mode | 1             | 2             | 3             | 4             | 5             | 6             |
|---------|--------------|------|---------------|---------------|---------------|---------------|---------------|---------------|
|         | Simple support                      |     |               |               |               |               |               |               |
| 0       | IGA-Zenkour   | 3.4629 | 8.9254 | 8.9254 | 12.9890 | 12.9890 | 15.4780 |
|         | Quasi-3D (Present) | 3.4132 | 8.8258 | 8.8258 | 12.9916 | 12.9916 | 15.3331 |
|         | RPT (Present)  | 3.3572 | 8.6722 | 8.6722 | 12.9121 | 12.9121 | 15.0490 |
| 0.2     | IGA-Zenkour   | 3.5224 | 9.3961 | 9.3961 | 13.1894 | 13.1894 | 16.2612 |
|         | Quasi-3D (Present) | 3.4675 | 9.2928 | 9.2928 | 13.1928 | 13.1928 | 16.1022 |
|         | RPT (Present)  | 3.4118 | 9.1595 | 9.1595 | 13.1037 | 13.1037 | 15.8706 |
| 0.4     | IGA-Zenkour   | 3.6392 | 10.6411 | 10.6411 | 13.3115 | 13.3115 | 18.2644 |
|         | Quasi-3D (Present) | 3.5698 | 10.5288 | 10.5288 | 13.3143 | 13.3143 | 18.1203 |
|         | RPT (Present)  | 3.5114 | 10.4415 | 10.4415 | 13.2326 | 13.2326 | 18.0361 |
| 0.6     | IGA-Zenkour   | 3.7577 | 12.2959 | 12.2959 | 13.5601 | 13.5601 | 18.4956 |
|         | Quasi-3D (Present) | 3.6687 | 12.1873 | 12.1873 | 13.5445 | 13.5445 | 18.4939 |
|         | RPT (Present)  | 3.6018 | 12.1378 | 12.1378 | 13.4872 | 13.4872 | 18.4475 |
| 0.8     | IGA-Zenkour   | 3.8632 | 13.2220 | 13.2220 | 14.8766 | 14.8766 | 18.6662 |
|         | Quasi-3D (Present) | 3.7546 | 13.2030 | 13.2030 | 14.7552 | 14.7552 | 18.6655 |
|         | RPT (Present)  | 3.6721 | 13.1245 | 13.1245 | 14.7032 | 14.7032 | 18.6192 |
| 1       | IGA-Zenkour   | 3.9548 | 13.4288 | 13.4288 | 17.1268 | 17.1268 | 18.8623 |
|         | Quasi-3D (Present) | 3.8303 | 13.4243 | 13.4243 | 16.9751 | 16.9751 | 18.8618 |
|         | RPT (Present)  | 3.9557 | 13.5531 | 13.5531 | 17.4735 | 17.4735 | 18.8633 |

| $\ell/h$ | Theory       | Mode | 1             | 2             | 3             | 4             | 5             | 6             |
|---------|--------------|------|---------------|---------------|---------------|---------------|---------------|---------------|
|         | Clamp support                        |     |               |               |               |               |               |               |
| 0       | IGA-Zenkour   | 6.7718 | 12.9968 | 12.9968 | 19.8267 | 19.8608 | 21.9673 |
|         | Quasi-3D (Present) | 6.8745 | 13.1770 | 13.1770 | 20.0202 | 20.0388 | 22.2480 |
|         | RPT (Present)  | 6.3384 | 12.4133 | 12.4133 | 19.0879 | 19.1363 | 20.9877 |
| 0.2     | IGA-Zenkour   | 7.2163 | 13.9289 | 13.9289 | 21.1653 | 21.5889 | 23.7182 |
|         | Quasi-3D (Present) | 7.3195 | 14.0950 | 14.0950 | 21.3095 | 21.7503 | 23.9885 |
|         | RPT (Present)  | 6.8208 | 13.4172 | 13.4172 | 20.5401 | 20.9799 | 22.8902 |
| 0.4     | IGA-Zenkour   | 8.4091 | 16.4035 | 16.4035 | 24.6620 | 25.9011 | 25.9011 |
|         | Quasi-3D (Present) | 8.5121 | 16.5390 | 16.5390 | 24.7273 | 25.8504 | 25.8504 |
|         | RPT (Present)  | 8.0909 | 16.0391 | 16.0391 | 24.2526 | 24.3345 | 24.3345 |
| 0.6     | IGA-Zenkour   | 10.0860 | 19.8463 | 19.8463 | 25.9424 | 25.9424 | 28.6525 |
|         | Quasi-3D (Present) | 10.1880 | 19.9431 | 19.9431 | 25.8948 | 25.8948 | 28.6525 |
|         | RPT (Present)  | 9.8463 | 19.6321 | 19.6321 | 24.3655 | 24.3655 | 28.6525 |
| 0.8     | IGA-Zenkour   | 12.0447 | 23.8379 | 23.8379 | 25.9920 | 25.9920 | 28.9380 |
|         | Quasi-3D (Present) | 12.1441 | 23.8850 | 23.8850 | 25.9507 | 25.9507 | 28.9380 |
|         | RPT (Present)  | 11.8741 | 23.7016 | 23.7016 | 24.4627 | 24.4627 | 28.9380 |
| 1       | IGA-Zenkour   | 14.1674 | 26.0476 | 26.0476 | 28.1435 | 28.1435 | 29.3010 |
|         | Quasi-3D (Present) | 14.2615 | 25.9953 | 25.9953 | 28.1492 | 28.1492 | 29.3010 |
|         | RPT (Present)  | 14.0568 | 24.4150 | 24.4150 | 28.2217 | 28.2217 | 29.3010 |

$\dfrac{1}{D_c} = \dfrac{E_c h^3}{12(1 - \nu_c^2)}$
Table 13: Comparison of non-dimensional critical buckling loads $\bar{P}_{cr} = \frac{P_{cr}a^2}{E_2h^3}$ of square FG microplates (rule of mixtures scheme)

| $\ell/h$ | Theory | $a/h = 5$ | $a/h = 10$ | $a/h = 20$ |
|---------|---------|-----------|------------|------------|
|         |         | $n = 0$  | $n = 1$  | $n = 10$  | $n = 0$  | $n = 1$  | $n = 10$  |
| 0       | CPT     | 19.2255  | 8.2145    | 3.8359    | 19.2255  | 8.2145    | 3.8359    |
|         | FSDT    | 15.3228  | 6.8576    | 2.9979    | 18.0746  | 7.8273    | 3.5853    |
|         | RPT     | 15.3322  | 6.8611    | 2.7672    | 18.0754  | 7.8276    | 3.4969    |
|         | RPT (Present) | 15.3321  | 6.8610    | 2.7702    | 18.0756  | 7.8277    | 3.4982    |
|         | Quasi-3D (Present) | 15.3629  | 7.3905    | 3.0118    | 18.1561  | 8.5396    | 3.8921    |
| 0.2     | CPT     | 22.0863  | 9.7879    | 4.3560    | 22.0863  | 9.7879    | 4.3560    |
|         | FSDT    | 17.6150  | 8.1715    | 3.4076    | 20.7607  | 9.3581    | 4.0246    |
|         | RPT     | 18.0422  | 8.3399    | 3.3619    | 20.9025  | 9.3767    | 4.0513    |
|         | RPT (Present) | 17.8878  | 8.2820    | 3.2917    | 20.8497  | 9.4358    | 4.0710    |
|         | Quasi-3D (Present) | 17.7286  | 8.153    | 3.4728    | 20.8583  | 10.0344   | 4.3958    |
| 0.4     | CPT     | 30.685   | 14.5082   | 5.9164    | 30.685   | 14.5082   | 5.9164    |
|         | FSDT    | 24.2899  | 11.9922   | 4.6013    | 28.748   | 13.7742   | 5.5151    |
|         | RPT     | 26.1539  | 12.7754   | 5.0407    | 29.3735  | 14.0232   | 5.6631    |
|         | RPT (Present) | 25.5457  | 12.5322   | 4.8371    | 29.1700  | 13.9459   | 5.5925    |
|         | Quasi-3D (Present) | 24.8060  | 12.6741   | 4.8557    | 28.9624  | 14.5168   | 5.9066    |
| 0.6     | CPT     | 44.9723  | 22.3753   | 8.5171    | 44.9723  | 22.3753   | 8.5171    |
|         | FSDT    | 34.7856  | 17.9838   | 6.4804    | 41.8271  | 21.0597   | 7.8802    |
|         | RPT     | 39.6393  | 20.1658   | 7.0001    | 43.4732  | 21.7657   | 8.2906    |
|         | RPT (Present) | 38.2867  | 19.5858   | 7.3772    | 43.0329  | 21.5846   | 8.1871    |
|         | Quasi-3D (Present) | 36.5415  | 19.2256   | 7.1597    | 42.4620  | 21.9814   | 8.4246    |
| 0.8     | CPT     | 64.9976  | 33.3892   | 12.1581   | 64.9976  | 33.3892   | 12.1581   |
|         | FSDT    | 48.2915  | 25.6654   | 8.9020    | 59.6657  | 30.9928   | 11.1065   |
|         | RPT     | 58.4862  | 30.5105   | 11.3322   | 63.1958  | 32.6036   | 11.9349   |
|         | RPT (Present) | 56.0961  | 29.4240   | 10.9005   | 62.4358  | 32.2693   | 11.8036   |
|         | Quasi-3D (Present) | 52.8623  | 28.3151   | 10.8343   | 61.3467  | 32.4199   | 11.9498   |
| 1       | CPT     | 90.7444  | 47.5499   | 16.8303   | 90.7444  | 47.5499   | 16.8303   |
|         | FSDT    | 63.8913  | 34.4981   | 11.7042   | 81.8269  | 43.3274   | 15.1152   |
|         | RPT     | 82.9388  | 43.8094   | 15.9522   | 88.5416  | 46.5372   | 16.6033   |
|         | RPT (Present) | 78.9675  | 42.0388   | 15.4071   | 87.3775  | 45.9981   | 16.4431   |
|         | Quasi-3D (Present) | 73.6925  | 39.8872   | 14.5287   | 85.6043  | 45.8223   | 16.4819   |
Table 14: Non-dimensional critical buckling load $\bar{P}_{cr} = \frac{P_{cr}a^2}{D_m}$ of Al/Al$_2$O$_3$ square microplates (Mori-Tanaka scheme)

| BC  | $a/h$ | $t/h$ | $n = 0$ |  | $n = 1$ |  | $n = 10$ |  |
|-----|-------|-------|---------|---|---------|---|---------|---|
|     |       |       | RPT     | Quasi-3D | RPT     | Quasi-3D | RPT     | Quasi-3D |
| SSSS| 5     | 0     | 87.4747 | 86.5475 | 35.0795 | 35.6610 | 21.7236 | 21.7189 |
|     | 0.2   | 103.6359 | 101.8797 | 42.3816 | 42.6185 | 25.4909 | 25.2307 |
|     | 0.4   | 152.0215 | 147.6312 | 64.2668 | 63.3902 | 36.7873 | 35.7417 |
|     | 0.6   | 232.4351 | 223.1633 | 100.6957 | 97.7189 | 55.6034 | 53.1891 |
|     | 0.8   | 344.7280 | 327.6683 | 151.6424 | 145.2904 | 81.9336 | 77.4961 |
|     | 1     | 488.8378 | 460.3907 | 217.0988 | 205.8260 | 115.7764 | 108.5941 |
|     | 20    | 105.6668 | 105.6221 | 42.0033 | 43.4065 | 27.5182 | 27.8578 |
|     | 0.2   | 123.5339 | 123.4102 | 49.9504 | 51.3272 | 31.5521 | 31.8705 |
|     | 0.4   | 177.1295 | 176.7715 | 73.7907 | 75.0881 | 43.6520 | 43.9085 |
|     | 0.6   | 266.4425 | 265.6972 | 113.5226 | 114.6856 | 63.8152 | 63.9707 |
|     | 0.8   | 391.4648 | 390.1733 | 169.1451 | 170.1141 | 92.0401 | 92.0558 |
|     | 1     | 552.1931 | 550.1813 | 240.6580 | 241.3659 | 128.3267 | 128.1618 |
|     | 100   | 107.0958 | 107.1271 | 42.5423 | 44.0165 | 27.9978 | 28.3566 |
|     | 0.2   | 125.0926 | 125.1206 | 50.5370 | 52.0102 | 32.0486 | 32.4065 |
|     | 0.4   | 179.0825 | 179.1009 | 74.5212 | 75.9913 | 44.2009 | 44.5563 |
|     | 0.6   | 269.0653 | 269.0682 | 114.4947 | 115.9596 | 64.4546 | 64.8061 |
|     | 0.8   | 395.0406 | 395.0223 | 170.4576 | 171.9154 | 92.8097 | 93.1557 |
|     | 1     | 557.0082 | 556.9633 | 242.4098 | 243.8584 | 129.2661 | 129.6052 |
| CCCC | 5     | 0     | 178.2578 | 188.3478 | 72.2150 | 78.0421 | 42.1220 | 45.4860 |
|     | 0.2   | 206.9297 | 215.7331 | 85.4028 | 90.6731 | 49.1101 | 52.1032 |
|     | 0.4   | 292.6287 | 295.5433 | 124.8630 | 127.4320 | 69.9984 | 71.1001 |
|     | 0.6   | 435.0030 | 424.8837 | 190.5274 | 187.0650 | 104.7011 | 101.8084 |
|     | 0.8   | 633.8611 | 601.6084 | 282.3839 | 268.7548 | 153.1709 | 144.0664 |
|     | 1     | 899.1171 | 823.8983 | 400.4310 | 371.8457 | 92.8097 | 93.1557 |
|     | 20    | 273.9507 | 288.6976 | 109.0237 | 117.3906 | 70.8926 | 75.2177 |
|     | 0.2   | 308.7803 | 323.9291 | 124.5481 | 133.0578 | 78.8262 | 83.2085 |
|     | 0.4   | 413.0833 | 429.0109 | 171.0330 | 179.8054 | 102.5575 | 107.0378 |
|     | 0.6   | 586.6304 | 603.0536 | 248.3764 | 257.2793 | 141.9833 | 146.4815 |
|     | 0.8   | 829.3607 | 845.5938 | 356.5558 | 365.3387 | 197.0586 | 201.4295 |
|     | 1     | 1141.2885 | 1156.5929 | 495.5793 | 503.9582 | 267.7793 | 271.8584 |
|     | 100   | 283.7646 | 292.2008 | 112.7294 | 119.3117 | 74.1644 | 77.0743 |
|     | 0.2   | 319.1326 | 327.9091 | 128.4414 | 135.1344 | 82.1289 | 85.1055 |
|     | 0.4   | 425.0550 | 434.6081 | 175.4917 | 182.4322 | 105.9838 | 109.1182 |
|     | 0.6   | 601.2918 | 611.6685 | 253.7742 | 260.9696 | 145.6726 | 148.9836 |
|     | 0.8   | 847.7694 | 858.8291 | 363.2623 | 370.6627 | 201.1735 | 204.6383 |
|     | 1     | 1164.4998 | 1176.0787 | 503.9643 | 511.5167 | 272.4865 | 276.0730 |

$D_m = \frac{E_m h^3}{12(1 - \nu_m^2)}$
Table 15: The first six non-dimensional buckling loads $P = \frac{Pa^2}{Dm}$ of Al/Al$_2$O$_3$ square microplates, $n = 10$, $\ell/h = 0.2$ (Mori-Tanaka scheme)

| BC   | $a/h$ | Theory               | Mode | 1     | 2     | 3     | 4     | 5     | 6     |
|------|-------|----------------------|------|-------|-------|-------|-------|-------|-------|
|      |       |                      |      |       |       |       |       |       |       |
| SSSS | 5     | IGA-Zenkour          | 25.4064 | 47.0158 | 47.0158 | 62.7947 | 66.4706 | 66.4706 |
|      |       | Quasi-3D (Present)   | 25.2307 | 46.2828 | 46.2828 | 61.4689 | 64.8256 | 64.8256 |
|      |       | RPT (Present)        | 25.4909 | 47.9306 | 47.9306 | 64.7895 | 68.7456 | 68.7456 |
|      | 10    | IGA-Zenkour          | 30.3554 | 67.5835 | 67.5835 | 101.6321 | 115.5866 | 115.5866 |
|      |       | Quasi-3D (Present)   | 30.2890 | 67.2254 | 67.2254 | 100.9300 | 114.4706 | 114.4706 |
|      |       | RPT (Present)        | 30.1012 | 67.3570 | 67.3570 | 101.9693 | 116.1512 | 116.1512 |
|      | 20    | IGA-Zenkour          | 31.8815 | 76.0319 | 76.0319 | 121.4313 | 143.5095 | 143.5095 |
|      |       | Quasi-3D (Present)   | 31.8705 | 75.8988 | 75.8988 | 121.1662 | 143.0021 | 143.0021 |
|      |       | RPT (Present)        | 31.5521 | 75.2628 | 75.2628 | 120.4131 | 142.2468 | 142.2468 |
|      | 100   | IGA-Zenkour          | 32.3844 | 79.0722 | 79.0722 | 129.2976 | 155.1773 | 155.1773 |
|      |       | Quasi-3D (Present)   | 32.4065 | 79.1198 | 79.1198 | 129.3662 | 155.2457 | 155.2457 |
|      |       | RPT (Present)        | 32.0486 | 78.2296 | 78.2296 | 127.9519 | 153.4678 | 153.4678 |
| CCCC | 5     | IGA-Zenkour          | 50.6557 | 66.0350 | 66.0350 | 78.0114 | 80.2597 | 84.0819 |
|      |       | Quasi-3D (Present)   | 52.1032 | 66.3156 | 66.3156 | 77.2179 | 78.9774 | 84.1043 |
|      |       | RPT (Present)        | 49.1101 | 66.7907 | 66.7907 | 80.3088 | 82.9464 | 85.5400 |
|      | 10    | IGA-Zenkour          | 74.8665 | 114.2196 | 114.2196 | 146.4170 | 160.1781 | 172.6816 |
|      |       | Quasi-3D (Present)   | 75.4661 | 115.7598 | 115.7598 | 147.8696 | 161.3725 | 174.2442 |
|      |       | RPT (Present)        | 70.1001 | 110.6293 | 110.6293 | 144.4919 | 158.8413 | 167.4729 |
|      | 20    | IGA-Zenkour          | 83.8952 | 140.5084 | 140.5084 | 191.9601 | 219.4306 | 239.0973 |
|      |       | Quasi-3D (Present)   | 83.2085 | 140.4610 | 140.4610 | 192.6237 | 220.3776 | 239.1104 |
|      |       | RPT (Present)        | 78.8262 | 133.9531 | 133.9531 | 185.0964 | 211.8925 | 227.4324 |
|      | 100   | IGA-Zenkour          | 85.2475 | 149.0723 | 149.0723 | 210.8794 | 246.4373 | 267.8293 |
|      |       | Quasi-3D (Present)   | 85.1055 | 148.8861 | 148.8861 | 210.7133 | 246.2408 | 267.4391 |
|      |       | RPT (Present)        | 82.1289 | 143.8128 | 143.8128 | 203.9109 | 237.9639 | 257.7571 |

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Table 16: Comparison of non-dimensional critical buckling loads $P_{cr} = \frac{P_{cr}R^2}{D_m}$ of CCCC Al/ZrO$_2$-2 circular plates (rule of mixtures scheme)

| $n$ | Theory | $h/R$ | 0.1 | 0.2 | 0.25 | 0.3 |
|-----|--------|-------|-----|-----|------|-----|
| 0   | TSDT [96] | 14.089 | 12.574 | 11.638 | 10.670 |
|     | UTSDT [97] | 14.089 | 12.575 | 11.639 | 10.670 |
|     | TSDT [98] | 14.1089 | 12.7281 | 11.8143 | 10.8666 |
|     | RPT [63] | 14.0932 | 12.5776 | 11.6409 | 10.6719 |
|     | RPT (Present) | 14.1093 | 12.5777 | 11.6409 | 10.6719 |
|     | Quasi-3D (Present) | 14.8264 | 13.4557 | 12.4564 | 11.3775 |
| 0.5 | TSDT [96] | 19.411 | 17.311 | 16.013 | 14.672 |
|     | UTSDT [97] | 19.413 | 17.310 | 16.012 | 14.672 |
|     | TSDT [98] | 19.4391 | 17.3327 | 16.0334 | 14.6910 |
|     | RPT [63] | 19.5663 | 17.5180 | 16.2506 | 14.9381 |
|     | RPT (Present) | 19.4169 | 17.3133 | 16.0153 | 14.6740 |
|     | Quasi-3D (Present) | 20.5166 | 18.6074 | 17.2206 | 15.7222 |
| 2   | TSDT [96] | 23.074 | 20.803 | 19.377 | 17.882 |
|     | UTSDT [97] | 23.075 | 20.805 | 19.378 | 17.881 |
|     | TSDT [98] | 23.1062 | 20.8319 | 19.4033 | 17.9060 |
|     | RPT [63] | 23.2592 | 21.0569 | 19.6687 | 18.2099 |
|     | RPT (Present) | 23.0809 | 20.8088 | 19.3812 | 17.8848 |
|     | Quasi-3D (Present) | 24.4332 | 22.3510 | 20.8035 | 19.1161 |
| 5   | TSDT [96] | 25.439 | 22.971 | 21.414 | 19.780 |
|     | UTSDT [97] | 25.442 | 22.969 | 21.412 | 19.778 |
|     | TSDT [98] | 25.4743 | 22.9992 | 21.4407 | 19.8043 |
|     | RPT [63] | 25.6418 | 23.2426 | 21.7268 | 20.1313 |
|     | RPT (Present) | 25.4469 | 22.9742 | 21.4168 | 19.7813 |
|     | Quasi-3D (Present) | 26.8812 | 24.6195 | 22.9303 | 21.0878 |
| 10  | TSDT [96] | 27.133 | 24.423 | 22.725 | 20.948 |
|     | UTSDT [97] | 27.131 | 24.422 | 22.725 | 20.949 |
|     | TSDT [98] | 27.1684 | 24.4542 | 22.7536 | 20.9750 |
|     | RPT [63] | 27.3429 | 24.6994 | 23.0389 | 21.2986 |
|     | RPT (Present) | 27.1395 | 24.4287 | 22.7297 | 20.9524 |
|     | Quasi-3D (Present) | 28.6197 | 26.1483 | 24.3140 | 22.3196 |
Table 17: Non-dimensional critical buckling loads $P_{cr} = \frac{P_{cr}R^2}{D_m}$ of Al/Al$_2$O$_3$ circular microplates (Mori-Tanaka scheme)

| $h/R$ | $\ell/h$ | $n = 0$ | | $n = 1$ | | $n = 10$ | |
|-------|----------|---------|----------|---------|----------|----------|
|       |          | RPT     | Quasi-3D | RPT     | Quasi-3D | RPT     | Quasi-3D |
| Simple support | | |
| 0.1   | 0        | 22.5182 | 22.6953  | 9.5368  | 9.7960   | 5.9574  | 6.0309   |
| 0.2   | 23.0489  | 23.2627  | 9.8179  | 10.0793 | 6.0876   | 6.1617   |
| 0.4   | 24.1022  | 24.4172  | 10.3474 | 10.6292 | 6.3543   | 6.4360   |
| 0.6   | 25.0682  | 25.5337  | 10.8125 | 11.1391 | 6.6130   | 6.7163   |
| 0.8   | 25.7985  | 26.4629  | 11.1543 | 11.5512 | 6.8182   | 6.9619   |
| 1     | 26.3291  | 27.2420  | 11.3981 | 11.8897 | 6.9718   | 7.1752   |
| 0.2   | 0        | 21.7456  | 22.0263  | 9.2059  | 9.4510   | 5.6942  | 5.7510   |
| 0.4   | 22.2179  | 22.5928  | 9.4873  | 9.7327  | 5.8318   | 5.8833   |
| 0.6   | 23.2739  | 23.7059  | 9.9893  | 10.2520 | 6.1019   | 6.1509   |
| 0.8   | 24.9581  | 25.8386  | 10.8003 | 11.2074 | 6.5891   | 6.7170   |
| 1     | 25.5480  | 26.7897  | 11.0777 | 11.6254 | 6.7647   | 6.9866   |
| 0.3   | 0        | 20.5707  | 20.8606  | 8.7033  | 8.9036   | 5.3041  | 5.3287   |
| 0.2   | 21.0993  | 21.4128  | 8.9860  | 9.1777  | 5.4526   | 5.4621   |
| 0.4   | 22.0854  | 22.4669  | 9.4810  | 9.6686  | 5.7409   | 5.7276   |
| 0.6   | 23.0439  | 23.5641  | 9.9511  | 10.1693 | 6.0352   | 6.0281   |
| 0.8   | 23.8746  | 24.6485  | 10.3517 | 10.6610 | 6.2927   | 6.3431   |
| 1     | 24.5707  | 25.7228  | 10.6823 | 11.1443 | 6.5055   | 6.6623   |
| Clamped support | | |
| 0.1   | 0        | 76.5059  | 80.4859  | 30.4539 | 32.7398  | 19.7743 | 20.9413  |
| 0.2   | 86.3696  | 90.4095  | 34.8524 | 37.1578 | 22.0235  | 23.1967  |
| 0.4   | 115.9595 | 120.0466 | 48.0474 | 50.3612 | 28.7616  | 29.9236  |
| 0.6   | 165.2737 | 169.2564 | 70.0379 | 72.3004 | 39.9728  | 41.0821  |
| 0.8   | 234.3115 | 237.9906 | 100.8235 | 102.9597 | 55.6486 | 56.6598 |
| 1     | 323.0727 | 326.2293 | 140.4042 | 142.3316 | 75.7879 | 76.6551  |
| 0.2   | 0        | 68.2782  | 73.0452  | 27.3245 | 29.8617  | 17.1442 | 18.6240  |
| 0.2   | 77.6521  | 82.3859  | 31.5460 | 34.0702 | 19.3412  | 20.7979  |
| 0.4   | 105.7687 | 110.0650 | 44.2098 | 46.5462 | 25.9122  | 27.1925  |
| 0.6   | 152.6199 | 155.7501 | 65.3154 | 67.1544 | 36.8218  | 37.7110  |
| 0.8   | 218.1997 | 219.2330 | 94.8629 | 95.8167 | 52.0495  | 52.3297  |
| 1     | 302.5060 | 300.3159 | 132.8523 | 132.4584 | 71.5905 | 71.0373  |
| 0.3   | 0        | 57.9331  | 61.7636  | 23.3438 | 25.4648  | 14.0464 | 15.3047  |
| 0.2   | 66.6302  | 70.2136  | 27.3083 | 29.3268 | 16.1359  | 17.3213  |
| 0.4   | 92.7032  | 95.0207  | 39.2004 | 40.6571 | 22.3875  | 23.1508  |
| 0.6   | 136.1160 | 135.6444 | 59.0182 | 59.2280 | 32.7704  | 32.6634  |
| 0.8   | 196.8399 | 191.6493 | 86.7603 | 84.8742 | 47.2643  | 45.8312  |
| 1     | 274.8619 | 262.6208 | 122.4264 | 117.4429 | 65.8628 | 62.6312  |
Table 18: The first six non-dimensional buckling loads $\bar{P} = \frac{PR^2}{D_m}$ of Al/Al$_2$O$_3$ circular microplates, $n = 1$, $\ell/h = 0.6$ (Mori-Tanaka scheme)

| BC          | $h/R$ | Theory       | Mode    | 1  | 2  | 3  | 4  | 5  | 6  |
|-------------|-------|--------------|---------|----|----|----|----|----|----|
| Simple support | 0.1   | IGA-Zenkour  | $11.4689$ | 54.8031 | 54.8031 | 95.3143 | 123.2488 | 129.6396 |
|             |      | Quasi-3D (Present) | $11.1391$ | 54.0884 | 54.0884 | 93.7842 | 122.0854 | 128.1298 |
|             |      | RPT (Present) | $10.8125$ | 53.6215 | 53.6215 | 93.3944 | 122.3489 | 128.3281 |
|             | 0.2   | IGA-Zenkour  | $11.3113$ | 50.7663 | 50.7663 | 83.1176 | 107.1987 | 110.9343 |
|             |      | Quasi-3D (Present) | $10.7545$ | 49.5358 | 49.5358 | 80.6606 | 104.8026 | 108.1436 |
|             |      | RPT (Present) | $10.4423$ | 50.2378 | 50.2378 | 83.3889 | 109.1766 | 112.9355 |
|             | 0.3   | IGA-Zenkour  | $10.8014$ | 45.2015 | 45.2015 | 69.1649 | 89.4125 | 90.9795 |
|             |      | Quasi-3D (Present) | $10.1693$ | 43.6917 | 43.6917 | 66.3080 | 86.1911 | 87.4685 |
|             |      | RPT (Present) | $9.9511$  | 45.6739 | 45.6739 | 71.5215 | 93.8041 | 95.5819 |
| Clamped support | 0.1   | IGA-Zenkour  | $72.6509$ | 126.0657 | 126.0657 | 172.3464 | 205.4419 | 222.4846 |
|             |      | Quasi-3D (Present) | $72.3004$ | 125.7477 | 125.7477 | 171.5372 | 205.0589 | 221.0537 |
|             |      | RPT (Present) | $70.0379$ | 123.1185 | 123.1185 | 169.2944 | 203.3425 | 218.6139 |
|             | 0.2   | IGA-Zenkour  | $66.8949$ | 108.9270 | 108.9270 | 140.4481 | 166.8569 | 175.2599 |
|             |      | Quasi-3D (Present) | $67.1544$ | 108.3685 | 108.3685 | 138.0488 | 164.4947 | 172.3137 |
|             |      | RPT (Present) | $65.3154$ | 109.4813 | 109.4813 | 143.4150 | 171.7706 | 179.5962 |
|             | 0.3   | IGA-Zenkour  | $58.9576$ | 90.1444 | 90.1444 | 110.5155 | 131.5270 | 134.2574 |
|             |      | Quasi-3D (Present) | $59.2280$ | 88.4285 | 88.4285 | 106.6289 | 127.0395 | 129.6436 |
|             |      | RPT (Present) | $59.0182$ | 93.6789 | 93.6789 | 117.0824 | 140.3857 | 143.1361 |
Figure 1: The functionally graded microplate model.
Figure 2: The effective modulus of Al/Al$_2$O$_3$ plates according to the rule of mixtures scheme (in solid lines) and Mori-Tanaka scheme (in dash lines).
Figure 3: Distribution function \( f(z) \).
(a) Cubic basis functions corresponding to $\Xi = \{0, 0, 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5}, 1, 1, 1\}$.

(b) Bivariate B-spline basis functions.

Figure 4: One- and two-dimensional B-spline basis functions.
(a) Geometric configuration.

(b) Control point net and 11×11 cubic elements.

Figure 5: Geometry and element mesh of a square microplate.
Figure 6: Relative error of non-dimensional central deflection of homogeneous square microplates.
Figure 7: Effects of material index $n$ and material length scale ratio $\ell/h$ on the central deflection of CCCC Al/Al$_2$O$_3$ square microplates subjected to uniformly distributed load, $a/h = 5$ (rule of mixtures scheme).
Figure 8: Deformed configuration of Al/Al$_2$O$_3$ square microplates.
Figure 9: Variation of natural frequency of CCCC Al/Al₂O₃ square microplates (rule of mixtures scheme).
Figure 10: Effects of material index $n$ and material length scale ratio $\ell/h$ on the natural frequency of CCCC Al/Al$_2$O$_3$ square microplates, $a/h = 5$ (rule of mixtures scheme).
Figure 11: The first six free vibration mode shapes of Al/Al2O3 CCCC square microplates.

(a) $\bar{\omega}_1 = 15.4413$.

(b) $\bar{\omega}_2 = 29.3267$.

(c) $\bar{\omega}_3 = 29.3267$.

(d) $\bar{\omega}_4 = 41.5955$.

(e) $\bar{\omega}_5 = 48.2657$.

(f) $\bar{\omega}_6 = 48.4504$. 
Figure 12: Geometry and element mesh of a circular microplate.

(a) Geometric configuration.

(b) Control point net and 11×11 cubic elements.
Figure 13: The first six free vibration mode shapes of clamped Al/Al₂O₃ circular microplates.

(a) $\bar{\omega}_1 = 7.3195$.
(b) $\bar{\omega}_2 = 14.0950$.
(c) $\bar{\omega}_3 = 14.0950$.
(d) $\bar{\omega}_4 = 21.3095$.
(e) $\bar{\omega}_5 = 21.7503$.
(f) $\bar{\omega}_6 = 23.9885$. 
Figure 14: The first six buckling mode shapes of Al/Al$_2$O$_3$ CCCC square microplates.

(a) $\bar{P}_1 = 52.1032$.

(b) $\bar{P}_2 = 66.3156$.

(c) $\bar{P}_3 = 66.3156$.

(d) $\bar{P}_4 = 77.2179$.

(e) $\bar{P}_5 = 78.9774$.

(f) $\bar{P}_6 = 84.1043$. 

Figure 14: The first six buckling mode shapes of Al/Al$_2$O$_3$ CCCC square microplates.
Figure 15: Effects of material index $n$ and material length scale ratio $\ell/h$ on the critical buckling loads of Al/$\text{Al}_2\text{O}_3$ circular microplates, $h/R = 0.2$ (Mori-Tanaka scheme).
Figure 16: Variation of the critical buckling loads of Al/Al₂O₃ circular microplates (Mori-Tanaka scheme).
Figure 17: The first six buckling mode shapes of simply-supported Al/Al$_2$O$_3$ circular microplates.

(a) $\bar{P}_1 = 10.7545$.

(b) $\bar{P}_2 = 49.5358$.

(c) $\bar{P}_3 = 49.5358$.

(d) $\bar{P}_4 = 80.6606$.

(e) $\bar{P}_5 = 104.8026$.

(f) $\bar{P}_6 = 108.1436$. 