A quantum channel with additive minimum output entropy

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We give a direct proof of the additivity of the minimum output entropy of a particular quantum channel which breaks the multiplicativity conjecture. This yields additivity of the classical capacity of this channel, a result obtained by a different method in [10]. Our proof relies heavily upon certain concavity properties of the output entropy which are of independent interest.

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INTRODUCTION

A number of important issues of quantum information theory would be greatly clarified if several resources and parameters were proved to be additive. However, the proof of additivity of such resources as the minimum output entropy of a quantum memoryless channel and its classical capacity remains in general an open problem, see e.g. [8]. Recently Shor [4] provided a new insight into how several additivity–type properties are related to each other. He proved that: (i) additivity of the minimum output entropy of a quantum channel, (ii) additivity of the classical capacity of a quantum channel, (iii) additivity of the entanglement of formation, and (iv) strong superadditivity of the entanglement of formation are equivalent in the sense that if one of them holds for all channels then the others also hold for all channels.

In this paper we study the additivity of the minimum output entropy for a channel which is particularly interesting because it breaks a closely related multiplicativity property [2]. For this channel the additivity of the classical capacity and of the minimum output entropy are equivalent, which allows us to derive an alternative proof of the result in [10], where additivity of its capacity was established. The problem of additivity of the minimum output entropy is interesting and important in its own right (it is straightforward, addresses a fundamental geometric feature of a channel and may provide insight into more complicated channel properties).

In this paper, the key observation that ensures the additivity is that the output entropy of the product channel exhibits specific concavity properties as a function of the Schmidt coefficients of the input pure state. It is our hope that a similar mechanism might be responsible for the additivity of the minimum output entropy in other interesting cases.

THE ADDITIVITY CONJECTURE

A channel $\Phi$ in the finite dimensional Hilbert space $\mathcal{H} \simeq \mathbb{C}^d$ is a linear trace-preserving completely positive map of the $*$–algebra of complex $d \times d$–matrices. A state is a density matrix $\rho$, that is Hermitian matrix such that $\rho \geq 0$, $\text{Tr}\rho = 1$. The minimum output entropy of the channel is defined as

$$h(\Phi) := \min_{\rho} S(\Phi(\rho)), \quad (1)$$

where the minimization is over all possible input states of the channel. Here $S(\sigma) = -\sigma \log \sigma$ is the von Neumann entropy of the channel output matrix $\sigma = \Phi(\rho)$. The additivity problem for the minimum output entropy is to prove that

$$h(\Phi_1 \otimes \Phi_2) = h(\Phi_1) + h(\Phi_2), \quad (2)$$
where $\Phi_1, \Phi_2$ are two channels in $\mathcal{H}_1, \mathcal{H}_2$ respectively, $\otimes$ denotes tensor product.

A channel $\Phi$ is covariant, if there are unitary representations $U_g, V_g$ of a group $G$ such that

$$\Phi(U_g \rho U_g^*) = V_g \Phi(\rho) V_g^*; \quad g \in G. \tag{3}$$

If both representations are irreducible, then we call the channel irreducibly covariant. In this case there is a simple formula

$$\bar{C}(\Phi) = \log d - h(\Phi),$$

relating the Holevo capacity $\bar{C}(\Phi)$ of the channel with $h(\Phi)$. Since the tensor product of irreducibly covariant channels (with respect to possibly different groups $G_1, G_2$) is again irreducibly covariant (with respect to the group $G_1 \times G_2$), it follows that if (2) holds for two such channels, then

$$\bar{C}(\Phi_1 \otimes \Phi_2) = \bar{C}(\Phi_1) + \bar{C}(\Phi_2). \tag{5}$$

Notice that this does not follow from the result of [4] which asserts that if (2) holds for all channels, then (5) also holds for all channels. In the latter case also $\bar{C}(\Phi_1 \otimes \cdots \otimes \Phi_n) = \bar{C}(\Phi_1) + \cdots + \bar{C}(\Phi_n)$, which implies that $\bar{C}(\Phi)$ is equal to the classical capacity of the channel $\Phi$ (see [5] for more detail).

The concavity of the von Neumann entropy implies that the minimization in (1) can be restricted to pure input states, since the latter correspond to the extreme points of the convex set of input states. Hence, we can equivalently write the minimum output entropies in the form

$$h(\Phi) = \min_{|\psi\rangle \in \mathcal{H}} S(\Phi(|\psi\rangle \langle \psi|)); \tag{6}$$

$$h(\Phi_1 \otimes \Phi_2) = \min_{|\psi_{12}\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2} S((\Phi_1 \otimes \Phi_2)(|\psi_{12}\rangle \langle \psi_{12}|)). \tag{7}$$

Here $|\psi_{12}\rangle \langle \psi_{12}|$ is a pure state of a bipartite system with the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_i \simeq \mathbb{C}^{d_i}$ for $i = 1, 2$. In order to prove (2), it is sufficient to show that the minimum in (7) is attained on unentangled vectors $|\psi_{12}\rangle$. Consider the Schmidt decomposition

$$|\psi_{12}\rangle = \sum_{\alpha} \sqrt{\lambda_\alpha} |\alpha; 1\rangle |\alpha; 2\rangle,$$

where $d = \min\{d_1, d_2\}$, $\{|\alpha; j\rangle\}$ is an orthonormal basis in $\mathcal{H}_j; j = 1, 2$, and $\lambda = (\lambda_1, \ldots, \lambda_d)$ is the vector of the Schmidt coefficients. The state $|\psi_{12}\rangle \langle \psi_{12}|$ can then be expressed as

$$|\psi_{12}\rangle \langle \psi_{12}| = \sum_{\alpha, \beta} \sqrt{\lambda_\alpha \lambda_\beta} |\alpha; 1\rangle \langle \beta; 1| \otimes |\alpha; 2\rangle \langle \beta; 2|. \tag{9}$$

The Schmidt coefficients form a probability distribution

$$\lambda_\alpha \geq 0; \quad \sum_{\alpha} \lambda_\alpha = 1, \tag{10}$$

thus the vector $\lambda$ varies in the $(d - 1)$-dimensional simplex $\Sigma_d$, defined by these constraints. Extreme points (vertices) of $\Sigma_d$ correspond precisely to unentangled vectors $|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. The proof of (2) becomes straightforward if we can prove that for every choice of the bases, the function

$$\Delta \in \Sigma_d \mapsto S(M(\Delta))$$

is concave, where $M(\Delta)$ is the operation defined by (4).
attains its minimum at the vertices of \( \Sigma_d \). Here \( S(M(\lambda)) \) is the von Neumann entropy of the channel matrix

\[
M(\lambda) := (\Phi_1 \otimes \Phi_2) (|\psi_{12}\rangle\langle\psi_{12}|) = \sum_{\alpha,\beta=1}^{d} \sqrt{\lambda_{\alpha,\beta}} \Phi_1(|\alpha; 1\rangle\langle\beta; 1|) \otimes \Phi_2(|\alpha; 2\rangle\langle\beta; 2|). \tag{12}
\]

Two special properties of a function can guarantee this: one is convexity, and another is Shur convexity (see the Appendix). Both of them appear useful in consideration of the particular channel we pass to.

**THE CHANNEL**

The channel considered in this paper was introduced in [2]. It is defined by its action on \( d \times d \) matrices \( \mu \) as follows:

\[
\Phi(\mu) = \frac{1}{d-1} (I - \text{tr}(\mu) - \mu^T)
\]

where \( \mu^T \) denotes the transpose of the matrix \( \mu \), and \( I \) is the unit matrix in \( \mathcal{H} \approx \mathbb{C}^d \). It is easy to see that the map \( \Phi \) is linear and trace-preserving. For the proof of complete positivity see [2]. Moreover, \( \Phi \) is irreducibly covariant since for any arbitrary unitary transformation \( U \)

\[
\Phi(U\mu U^*) = \bar{U}\Phi(\mu)\bar{U}^*, \tag{14}
\]

hence the relation (4) holds for this channel.

Our aim will be to prove the additivity relation

\[
h(\Phi \otimes \Phi) = 2h(\Phi), \tag{15}
\]

for the channel (13). For \( d = 2 \), (13) is a unital qubit channel, for which property (15) follows from [3]. For \( d \geq 3 \), (15) can be deduced from additivity of the Holevo capacity [5], established in [10], by a different method. Here we provide a direct proof based on the idea described at the end of the previous section.

For a pure state \( \rho = |\psi\rangle\langle\psi| \) the channel output is given by

\[
\Phi(|\psi\rangle\langle\psi|) = \frac{1}{d-1} (I - \overline{|\psi\rangle\langle\psi|}), \tag{16}
\]

where the entries of vector \( \overline{|\psi\rangle} \) are complex conjugates of the corresponding entries of vector \( |\psi\rangle \). The matrix \( \Phi(|\psi\rangle\langle\psi|) \) has a non-degenerate eigenvalue equal to 0 and an eigenvalue \( 1/(d-1) \) which is \( (d-1) \)-fold degenerate. The von Neumann entropy \( S(\Phi(|\psi\rangle\langle\psi|)) \) is obviously the same for all pure states, and so

\[
h(\Phi) = \log(d-1). \tag{17}
\]

As argued in the previous section, in order to prove (15), it is sufficient to show that the minimum in

\[
h(\Phi \otimes \Phi) = \min_{|\psi_{12}\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d}} S((\Phi \otimes \Phi)(|\psi_{12}\rangle\langle\psi_{12}|))
\]

is attained on entangled vectors \( |\psi_{12}\rangle \). Consider the Schmidt decomposition [5] of \( |\psi_{12}\rangle \). Owing to the property (14), we can choose for \( \{ |\alpha; j\rangle \}_{j=1,2} \), the canonical basis in \( \mathbb{C}^d \). As it was shown in the previous section, it suffices to check that \( S(M(\lambda)) \) attains its minimum at the vertices of \( \Sigma_d \). Here \( M(\lambda) \) is the matrix defined in (12) for the channel under consideration:

\[
M(\lambda) = \sum_{\alpha,\beta=1}^{d} \sqrt{\lambda_{\alpha,\beta}} \Phi(|\alpha\rangle\langle\beta|) \otimes \Phi(|\alpha\rangle\langle\beta|),
\]
where by (13)
\[ \Phi(|\alpha\rangle\langle\beta|) = \frac{1}{d-1} (\delta_{\alpha\beta} I - |\beta\rangle\langle\alpha|), \]
owing to the fact that $|\alpha\rangle$ and $|\beta\rangle$ are real.

Using (8) and the completeness relations:
\[ I = \sum_{\alpha=1}^{d} |\alpha\rangle\langle\alpha|, \quad I \otimes I = \sum_{\alpha,\beta=1}^{d} |\alpha\beta\rangle\langle\alpha\beta|, \]
we obtain
\[ M(\lambda) = \frac{1}{(d-1)^2} \left[ \sum_{\alpha,\beta=1}^{d} |\alpha\beta\rangle\langle\alpha\beta| (1 - \lambda_{\alpha} - \lambda_{\beta}) + \sum_{\alpha,\beta=1}^{d} \sqrt{\lambda_{\alpha}\lambda_{\beta}} |\alpha\rangle\langle\beta| \right]. \quad (18) \]

In order to find the eigenvalues of $M(\lambda)$, it is instructive to first study the secular equation of a more general $n \times n$ matrix:
\[ A = \sum_{j=1}^{n} \mu_j |j\rangle\langle j| + \sum_{j,k=1}^{n} \sqrt{\eta_j \eta_k} |j\rangle\langle k|. \]
Matrix $A$ gives $(\Phi \otimes \Phi)(|\psi_{12}\rangle\langle\psi_{12}|)$ for a particular choice of the parameters $\mu_j$ and $\eta_j$ [see eq. (21) below]. It has the form:
\[
\begin{pmatrix}
\mu_1 + \eta_1 & \sqrt{\eta_1 \eta_2} & \cdots & \cdots & \sqrt{\eta_1 \eta_n} \\
\sqrt{\eta_2 \eta_1} & \mu_2 + \eta_2 & \sqrt{\eta_2 \eta_3} & \cdots & \cdots & \sqrt{\eta_2 \eta_n} \\
\vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\
\sqrt{\eta_n \eta_1} & \cdots & \cdots & \cdots & \mu_n + \eta_n 
\end{pmatrix}. \quad (19) \]
The secular equation $\det(A - \gamma I) = 0$ can be written as
\[ F(\gamma) = 0, \quad (20) \]
where
\[ F(\gamma) = \prod_j (\mu_j - \gamma) \left[ 1 + \frac{\eta_1}{\mu_1 - \gamma} + \cdots + \frac{\eta_n}{\mu_n - \gamma} \right]. \]
Solving eq. (20) would be in general non-trivial. However, representing the matrix $[(d-1)^2 M(\lambda)]$ in the form (19) results in a convenient expression for $F(\gamma)$. This allows us to identify many of the eigenvalues of $(d-1)^2 M(\lambda)$. More precisely, we identify $j$ with a pair $(\alpha, \beta)$ and obtain
\[ \mu_j \equiv \mu_{\alpha\beta} = 1 - \lambda_{\alpha} - \lambda_{\beta}; \quad \eta_j \equiv \eta_{\alpha\beta} = \lambda_{\alpha} \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \ldots, d. \quad (21) \]
Therefore,
\[ F(\gamma) = \prod_{\alpha,\beta=1}^{d} (1 - \lambda_{\alpha} - \lambda_{\beta} - \gamma) \left[ 1 + \sum_{\alpha',\beta'=1}^{d} \frac{\lambda_{\alpha'}\delta_{\alpha'\beta'}}{(1 - \lambda_{\alpha'} - \lambda_{\beta'} - \gamma)} \right] \]
\[ = \prod_{\alpha,\beta=1}^{d} (1 - \lambda_{\alpha} - \lambda_{\beta} - \gamma) \left[ \prod_{\alpha'=1}^{d} (1 - 2\lambda_{\alpha'} - \gamma) \left[ 1 + \sum_{\alpha''=1}^{d} \frac{\lambda_{\alpha''}}{(1 - 2\lambda_{\alpha''} - \gamma)} \right] \right]. \quad (22) \]
Eq. (20) yields the following equations:

\[
(1 - \lambda_\alpha - \lambda_\beta - \gamma) = 0, \quad \alpha \neq \beta, \quad \alpha, \beta = 1, 2, \ldots, d,
\]

where \(\lambda_\alpha, \lambda_\beta\) denote the Schmidt coefficients (8). Equation (23) implies that there are \(d(d-1)\) eigenvalues of the form

\[
\gamma = 1 - \lambda_\alpha - \lambda_\beta, \quad \alpha \neq \beta \quad \alpha, \beta = 1, \ldots, d.
\]

The roots of the equation

\[
\prod_{\alpha=1}^{d} (1 - 2\lambda_\alpha - \gamma) \left\{ 1 + \sum_{\alpha'=1}^{d} \frac{\lambda_{\alpha'}}{(1 - 2\lambda_{\alpha'} - \gamma)} \right\} = 0
\]

give the remaining \(d\) eigenvalues of the matrix \([(d-1)^2 M(\Delta)]\).

For the case \(d = 3\) the roots of (25) can be explicitly evaluated. This is done in the next section. The case of arbitrary \(d > 3\) is discussed in sections that follow. Note that the sum of all eigenvalues of \([(d-1)^2 M(\Delta)]\) equals

\[
\text{tr} [(d-1)^2 M(\Delta)] = (d-1)^2 \text{Tr} M(\Delta) = (d-1)^2,
\]

since \(M(\Delta)\) is a density matrix acting in \(\mathbb{C}^d^2\).

**EIGENVALUES FOR \(d = 3\)**

For \(d = 3\), there are \(d(d-1) = 6\) eigenvalues of the matrix \([(d-1)^2 M(\Delta)] = 4M(\Delta)\), which are given by (24). The sum of these eigenvalues is:

\[
\sum_{\alpha, \beta=1 \atop \alpha \neq \beta}^{3} (1 - \lambda_\alpha - \lambda_\beta) = 2 \left[ 3 - 2(\lambda_1 + \lambda_2 + \lambda_3) \right] = 2
\]

since

\[
\lambda_1 + \lambda_2 + \lambda_3 = 1.
\]

The remaining three eigenvalues of \(4M(\Delta)\) are given by the roots of the equation

\[
\prod_{\alpha=1}^{3} (1 - 2\lambda_\alpha - \gamma) \left\{ 1 + \sum_{\alpha'=1}^{3} \frac{\lambda_{\alpha'}}{(1 - 2\lambda_{\alpha'} - \gamma)} \right\} = 0.
\]

Since the sum of all the eigenvalues is equal to \((d-1)^2 = 4\), these remaining three eigenvalues sum up to 
\(4 - 2 = 2\). Using (20), we can cast (27) as:

\[
\gamma^3 + a_2 \gamma^2 + a_1 \gamma + a_0 = 0
\]

where

\[
a_0 = -4\lambda_1 \lambda_2 \lambda_3 \quad a_1 = 1 \quad a_2 = -2.
\]
The three roots of (28) are given by

\[ \tilde{\gamma}_1 := -\frac{a_2}{3} + (T_1 + T_2), \]

\[ \tilde{\gamma}_2 := -\frac{a_2}{3} - \frac{1}{2} (T_1 + T_2) + \frac{1}{2} i\sqrt{3} (T_1 - T_2), \]

\[ \tilde{\gamma}_3 := -\frac{a_2}{3} - \frac{1}{2} (T_1 + T_2) - \frac{1}{2} i\sqrt{3} (T_1 - T_2). \]  

(30)

Here

\[ T_1 := \left[ R + \sqrt{D} \right]^{1/3} \quad \text{and} \quad T_2 := \left[ R - \sqrt{D} \right]^{1/3}, \]

and

\[ R = \frac{1}{54} (9a_1a_2 - 27a_0 - 2a_2^3), \quad D = Q^2 + R^2, \quad Q := \frac{1}{9} (3a_1 - a_2^2). \]  

(31)

Thus, the matrix \( M(\Delta) \) has six eigenvalues of the form \( (1/4)(1 - \lambda_\alpha - \lambda_\beta) \), where \( \alpha, \beta = 1, 2, 3 \) and \( \alpha \neq \beta \), and three eigenvalues \( \gamma_1, \gamma_2 \) and \( \gamma_3 \), with \( \gamma_i := \tilde{\gamma}_i/4 \). The output entropy \( S(M(\Delta)) \) can be expressed as the sum:

\[ S(M(\Delta)) = S_1(\Delta) + S_2(\Delta). \]

Here

\[ S_1(\Delta) = -\sum_{\alpha, \beta=1}^{3} \frac{1}{4} (1 - \lambda_\alpha - \lambda_\beta) \log \left[ \frac{1}{4} (1 - \lambda_\alpha - \lambda_\beta) \right] \]

\[ = -\frac{1}{2} \sum_{\alpha=1}^{3} \lambda_\alpha \log \lambda_\alpha = \frac{1}{2} H(\Delta) + 1 \]  

(32)

where \( H(\Delta) = -\sum_{\alpha=1}^{d} \lambda_\alpha \log \lambda_\alpha \) denotes the Shannon entropy of \( \Delta \), and

\[ S_2(\Delta) = -\sum_{i=1}^{3} \gamma_i \log \gamma_i. \]

Since \( H(\Delta) \) is a concave function of \( \Delta = (\lambda_1, \lambda_2, \lambda_3) \), so is \( S_1(\Delta) \). Hence \( S_1(\Delta) \) attains its minimum at the vertices of \( \Sigma_3 \).

Let us now evaluate the summand \( S_2(\Delta) \). Substituting the values of \( a_0, a_1 \) and \( a_2 \) from (29) into (31), we get

\[ R = -\frac{1}{27} + 2t, \quad Q = -\frac{1}{9}, \quad D = -4t(\frac{1}{27} - t) \leq 0. \]

Here \( t = \lambda_1\lambda_2\lambda_3, 0 \leq t \leq 1/27 \). Hence, we can write

\[ R + \sqrt{D} = R + i\sqrt{|D|} = re^{i\theta}, \]

where \( r = \sqrt{R^2 + |D|} = 1/27 \) and \( \theta = \arctan(\sqrt{|D|}/R), 0 \leq \theta \leq \pi \), so that

\[ \tan \theta = \frac{\sqrt{t(1/27 - t)}}{t - 1/54}. \]
Considering the sign of this expression we find that \( t = 0 \) corresponds to \( \theta = \pi \), while \( t = 1/27 \) to \( \theta = 0 \). In terms of \( \theta \) the eigenvalues \( \gamma_k \), \( k = 1, 2, 3 \) can now be expressed as:

\[
\gamma_k = \frac{1}{6} \left[ 1 + \cos \left( \frac{\theta}{3} - \frac{2\pi(k-1)}{3} \right) \right] = \frac{1}{3} \cos^2 \left( \frac{\theta}{6} - \frac{2\pi(k-1)}{6} \right).
\]

Hence,

\[
S_2(\lambda) = -\sum_{k=1}^{3} \frac{1}{3} \cos^2 \left( \frac{\theta}{6} - \frac{2\pi(k-1)}{6} \right) \log \left[ \frac{1}{3} \cos^2 \left( \frac{\theta}{6} - \frac{2\pi(k-1)}{6} \right) \right].
\]  (33)

An argument similar to Lemma 3 of [6] shows that the RHS of (33) has a global minimum, equal to 1, at \( \theta = \pi \), corresponding to \( t = \lambda_1\lambda_2\lambda_3 = 0 \). Hence, \( S_2(\lambda) \) attains its minimal value 1 at every point of the boundary \( \partial\Sigma_3 \), in particular at its vertices: \( \lambda_i = 1, \lambda_j = 0 \) for \( j \neq i, i = 1, 2, 3 \). The summand \( S_1(\lambda) \), given by (32), also attains its minimum, equal to 1, at the vertices. Therefore the sum \( S(M(\lambda)) \) attains its minimum, equal to 2, at the vertices of \( \Sigma_3 \). Hence, \( h(\Phi \otimes \Phi) = 2 \), and the additivity (15) holds, as \( h(\Phi) = 1 \) by (17).

We conjecture that the entropy \( S(M(\lambda)) \) as a function of \( \lambda \) is concave. This is supported by a 3D-plot of \( S(M(\lambda)) \) as a function of two independent Schmidt coefficients \( \lambda_1 \) and \( \lambda_2 \); here \( \lambda_i \geq 0 \) for \( i = 1, 2 \) and \( \lambda_1 + \lambda_2 \leq 1 \). See Figure 1 below.

![FIG. 1: The entropy \( S(M(\lambda)) \) as a function of two independent Schmidt coefficients \( \lambda_1 \) and \( \lambda_2 \).](image)

However \( S_2(\lambda) \) is not concave as can be seen e.g. by taking \( \lambda_2 = \lambda_1, \lambda_3 = 1 - 2\lambda_1 \). See Figure 2.

**MINIMUM OUTPUT ENTROPY IN \( d > 3 \) DIMENSIONS**

In a previous section we found that the matrix \([(d-1)^2M(\lambda)]\), where \( M(\lambda) \) is the output density matrix of the channel \( \Phi \otimes \Phi \) and is given by (32), has \( d(d-1) \) eigenvalues of the form

\[
(1 - \lambda_\alpha - \lambda_\beta), \quad \text{with} \quad \alpha \neq \beta, \ \alpha, \beta = 1, 2, \ldots, d,
\]  (34)
and the remaining $d$ eigenvalues are given by the roots $\gamma_1, \ldots, \gamma_d$ of (25). Hence, the matrix $M(\lambda)$ has $d(d-1)$ eigenvalues of the form

$$e_{\alpha\beta} := \frac{1}{(d-1)^2} \left(1 - \lambda_\alpha - \lambda_\beta\right), \quad \alpha \neq \beta, \quad \alpha, \beta = 1, 2, \ldots, d,$$

and $d$ eigenvalues of the form

$$g_i := \frac{\gamma_i}{(d-1)^2}, \quad i = 1, 2, \ldots, d.$$

Note that the $\gamma_i$'s and $g_i$’s are functions of $\lambda \in \Sigma_d$. Accordingly, we write the von Neumann entropy of the output density matrix as a sum

$$S(M(\lambda)) = S_1(\lambda) + S_2(\lambda)$$

where

$$S_1(\lambda) := - \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \sum_{\beta \neq \alpha} e_{\alpha\beta} \log e_{\alpha\beta}, \quad S_2(\lambda) := - \sum_{i=1}^{d} g_i \log g_i.$$  \hspace{1cm} (35)

Note that

$$\sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} e_{\alpha\beta} = \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \frac{1 - \lambda_\alpha - \lambda_\beta}{(d-1)^2} = \frac{d-2}{d-1}. \hspace{1cm} (37)$$

Define the following variables:

$$\tilde{e}_{\alpha\beta} (\equiv \tilde{e}_{\alpha\beta}(\lambda)) := \frac{d-1}{d-2} e_{\alpha\beta} = \frac{1}{(d-1)(d-2)} \sum_{1 \leq \delta \leq d, \atop \delta \neq \alpha, \beta} \lambda_\delta, \quad \alpha \neq \beta, \quad \alpha, \beta = 1, \ldots, d.$$ \hspace{1cm} (38)

$$\tilde{g}_i (\equiv \tilde{g}_i(\lambda)) := (d-1) g_i = \frac{1}{(d-1)} \gamma_i, \quad i = 1, \ldots, d. \hspace{1cm} (39)$$
For $d \geq 3$ we have $\tilde{e}_{\alpha \beta} \geq 0$, and from (37) it follows that
\[
\sum_{\alpha = 1}^{d} \sum_{\beta \neq \alpha}^{d} \tilde{e}_{\alpha \beta} = \frac{d - 1}{d - 2} \sum_{\alpha = 1}^{d} \sum_{\beta \neq \alpha}^{d} e_{\alpha \beta} = 1, \quad \sum_{i=1}^{d} \tilde{g}_i = (d - 1) \left(1 - \frac{d - 2}{d - 1}\right) = 1.
\]

Hence, $\tilde{e} := \{\tilde{e}_{\alpha \beta} | \alpha \neq \beta, \alpha, \beta = 1, \ldots, d\}$ and $\tilde{g} = \{\tilde{g}_i | i = 1, \ldots, d\}$ are probability distributions. In terms of these variables
\[
S_1(\tilde{\Delta}) = \frac{d - 2}{d - 1} H(\tilde{e}) - \frac{d - 2}{d - 1} \log \left(\frac{d - 2}{d - 1}\right),
\]
where $H(\tilde{e})$ denotes the Shannon entropy of $\tilde{e}$. In view of (36)
\[
S_2(\lambda) = \frac{1}{(d - 1)} H(\tilde{g}) + \frac{1}{d - 1} \log (d - 1).
\]

From (40) it follows that $S_1(\tilde{\Delta})$ in (36) is a concave function of the variables $\tilde{e}_{\alpha \beta}$. These variables are affine functions of the Schmidt coefficients $\lambda_1, \ldots, \lambda_d$. Hence, $S_1$ is a concave function of $\lambda$ and attains its minimum at the vertices of $\Sigma_d$, defined by the constraints (10).

Let us now analyze $S_2(\lambda)$. We wish to prove the following:

**Theorem.** The function $S_2$ is Schur-concave in $\lambda \in \Sigma_d$, i.e., $\lambda < \lambda' \implies S_2(\lambda) \geq S_2(\lambda')$, where $<$ denotes the majorization order (see the Appendix).

Since every $\lambda \in \Sigma_d$ is majorized by the vertices of $\Sigma_d$, this will imply that $S_2(\lambda)$ also attains its minimum at the vertices. Thus $S(\lambda) = S_1(\lambda) + S_2(\lambda)$ is minimized at the vertices, which correspond to unentangled states. As was observed, this implies the additivity.

**PROOF OF THE THEOREM**

We will use the quite interesting observation made in [7], that the Shannon entropy $H(x)$ is a monotonically increasing function of the elementary symmetric polynomials $s_k(x_1, x_2, \ldots, x_d)$, $k = 0, \ldots, d$, in the variables $x = (x_1, x_2, \ldots, x_d)$. The latter are defined by equations (65) of the Appendix. Hence the Shannon entropy $H(\tilde{g})$ in (41) is a monotonically increasing function of the symmetric polynomials
\[
\tilde{s}_k(\lambda) := s_k(\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_d) = \frac{1}{(d - 1)^k} s_k(\gamma_1, \gamma_2, \ldots, \gamma_d), \quad k = 0, \ldots, d.
\]

Therefore, to prove the Theorem it is sufficient to prove that the functions $\tilde{s}_k(\lambda)$ are Schur concave in $\lambda \in \Sigma_d$. Here the variables $\tilde{g}_i$ are given by (69), and the variables $\gamma_i$ are the roots of eq. (25). Define the variables
\[
\nu_\alpha := 1 - 2\lambda_\alpha, \quad \alpha = 1, 2, \ldots, d.
\]

Note that $-1 \leq \nu_\alpha \leq 1$, owing to the inequality $0 \leq \lambda_\alpha \leq 1$. Moreover,
\[
\sum_{\alpha=1}^{d} \nu_\alpha = d - 2,
\]
since $\sum_{\alpha=1}^{d} \lambda_\alpha = 1$. In terms of the variables $\nu_\alpha$, (26) can be expressed as
\[
\prod_{\alpha=1}^{d} (\nu_\alpha - \gamma) \left\{1 + \frac{1}{2} \sum_{\alpha'=1}^{d} \frac{1 - \nu_{\alpha'}}{(\nu_{\alpha'} - \gamma)}\right\} = 0.
\]
Since the roots $\gamma_1, \ldots, \gamma_d$ of (25) are identified, trivially, as the zeroes of the product $(\gamma_1 - \gamma)(\gamma_2 - \gamma) \cdots (\gamma_d - \gamma)$, equation (43) can be expressed in terms of these roots as follows:

$$\sum_{k=0}^{d} \gamma^k (-1)^k s_{d-k}(\gamma_1, \gamma_2, \ldots, \gamma_d) = 0. \quad (44)$$

In terms of the elementary symmetric polynomials $s_i$, of the variables $\nu_1, \nu_2, \ldots, \nu_d$, (43) can be rewritten as

$$\sum_{k=0}^{d} \gamma^k (-1)^k s_{d-k}(\nu_1, \nu_2, \ldots, \nu_d) + \sum_{k=0}^{d-1} \gamma^k (-1)^k \sum_{l=1}^{d} s_{d-1-k}(\nu_1, \ldots, \nu_l \ldots, \nu_d) \frac{(1 - \nu_l)}{2} = 0, \quad (45)$$

where the symbol $\nu_l$ means that the variable $\nu_l$ has been omitted from the arguments of the corresponding polynomial. Equating the LHS of (44) with the LHS of (45) yields, for each $0 \leq k \leq d - 1$:

$$s_{d-k}(\gamma_1, \gamma_2, \ldots, \gamma_d) = s_{d-k}(\nu_1, \nu_2, \ldots, \nu_d) + \sum_{l=1}^{d} s_{d-1-k}(\nu_1, \ldots, \nu_l \ldots, \nu_d) \frac{(1 - \nu_l)}{2}. \quad (46)$$

Note that in (46), values $s_{d-k}(\gamma_1, \gamma_2, \ldots, \gamma_d)$ are expressed in terms of values of elementary symmetric polynomials in the variables $\nu_1, \nu_2, \ldots, \nu_d$ (which are themselves linear functions of the Schmidt coefficients $\lambda_1, \ldots, \lambda_d$).

Our aim is to prove that $s_{k}(\lambda)$ is Schur concave in the Schmidt coefficients $\lambda_1, \ldots, \lambda_d$. Eq. (42) implies that this amounts to proving Schur concavity of $s_{d-k}(\gamma_1, \gamma_2, \ldots, \gamma_d)$ as a function of $\lambda_1, \ldots, \lambda_d$, for all $0 \leq k \leq d$.

The functions

$$\Phi_k(\nu_1, \ldots, \nu_d) := s_{d-k}(\nu_1, \ldots, \nu_d) + \sum_{l=1}^{d} s_{d-1-k}(\nu_1, \ldots, \nu_l \ldots, \nu_d) \frac{(1 - \nu_l)}{2} \equiv \text{RHS of (46)} \quad (47)$$

are symmetric in the variables $\nu_1, \nu_2, \ldots, \nu_d$, and hence in the variables $\lambda_1, \ldots, \lambda_d$. By eq. (44) (see the Appendix) it remains to prove

$$(\lambda_i - \lambda_j)(\frac{\partial \Phi_k}{\partial \lambda_i} - \frac{\partial \Phi_k}{\partial \lambda_j}) \equiv (\nu_i - \nu_j)(\frac{\partial \Phi_k}{\partial \nu_i} - \frac{\partial \Phi_k}{\partial \nu_j}) \leq 0, \quad \forall 1 \leq i, j \leq d. \quad (48)$$

By (43) we have

$$\frac{\partial \Phi_k}{\partial \nu_i} = \frac{\partial}{\partial \nu_i} s_{d-k}(\nu_1, \ldots, \nu_d) + \frac{\partial}{\partial \nu_i} \sum_{l=1}^{d} s_{d-1-k}(\nu_1, \ldots, \nu_l \ldots, \nu_d) \frac{(1 - \nu_l)}{2}$$

$$= s_{d-1-k}(\nu_1, \ldots, \nu_l \ldots, \nu_d) + \sum_{l=1 \atop l \neq i}^{d} s_{d-1-k}(\nu_1, \ldots, \nu_l \ldots, \nu_d) \frac{(1 - \nu_l)}{2}$$

$$- \frac{1}{2} s_{d-1-k}(\nu_1, \ldots, \nu_i \ldots, \nu_d). \quad (49)$$

Therefore,

$$\left(\frac{\partial \Phi_k}{\partial \nu_i} - \frac{\partial \Phi_k}{\partial \nu_j}\right)(\nu_1, \ldots, \nu_d) = s_{d-1-k}(\nu_1, \ldots, \nu_l \ldots, \nu_d) - s_{d-1-k}(\nu_1, \ldots, \nu_j \ldots, \nu_d)$$

$$+ \sum_{l=1 \atop l \neq i}^{d} s_{d-1-k}(\nu_1, \ldots, \nu_l \ldots, \nu_d) \frac{(1 - \nu_l)}{2} - \sum_{l=1 \atop l \neq j}^{d} s_{d-1-k}(\nu_1, \ldots, \nu_l \ldots, \nu_d) \frac{(1 - \nu_l)}{2}$$

$$- \frac{1}{2} \left[ s_{d-1-k}(\nu_1, \ldots, \nu_i \ldots, \nu_d) - s_{d-1-k}(\nu_1, \ldots, \nu_j \ldots, \nu_d) \right]. \quad (50)$$
Using \( \frac{\partial \Phi_k}{\partial \nu_i} - \frac{\partial \Phi_k}{\partial \nu_j} \) we get

\[
\left( \frac{\partial \Phi_k}{\partial \nu_i} - \frac{\partial \Phi_k}{\partial \nu_j} \right)(\nu_1, \ldots, \nu_d) = \frac{1}{2}(\nu_j - \nu_i)s_{d-k-2}(\nu_1, \ldots, \nu_{i-1}, \nu_i, \ldots, \nu_{j-1}, \nu_j, \ldots, \nu_d) + \frac{(\nu_i - \nu_j)}{2}s_{d-k-2}(\nu_1, \ldots, \nu_{i-1}, \nu_i, \ldots, \nu_{j-1}, \nu_j, \ldots, \nu_d)
+ \sum_{l \neq i, j}^d \frac{(1 - \nu_l)}{2}[s_{d-k-2}(\nu_1, \ldots, \nu_{i-1}, \nu_i, \ldots, \nu_{j-1}, \nu_j, \ldots, \nu_d) - s_{d-k-2}(\nu_1, \ldots, \nu_{i-1}, \nu_i, \ldots, \nu_{j-1}, \nu_j, \ldots, \nu_d)]
= \sum_{l \neq i, j}^d \frac{(1 - \nu_l)}{2}[s_{d-k-2}(\nu_1, \ldots, \nu_{i-1}, \nu_i, \ldots, \nu_{j-1}, \nu_j, \ldots, \nu_d) - s_{d-k-2}(\nu_1, \ldots, \nu_{i-1}, \nu_i, \ldots, \nu_{j-1}, \nu_j, \ldots, \nu_d)]
= \sum_{l \neq i, j}^d \frac{(1 - \nu_l)}{2}(\nu_j - \nu_i)s_{d-k-3}(\nu_1, \ldots, \nu_{i-1}, \nu_i, \ldots, \nu_{j-1}, \nu_j, \ldots, \nu_d). \tag{51}
\]

Substituting (51) in (48) we obtain that Schur concavity holds if and only if

\[
\sum_{l \neq i, j}^d (1 - \nu_l)s_{d-k-3}(\nu_1, \ldots, \nu_i, \ldots, \nu_{j-1}, \nu_j, \ldots, \nu_d) \geq 0, \quad \forall \ 1 \leq i, j \leq d. \tag{52}
\]

The variables \( \nu_i \) and \( \nu_j \) do not appear in (52). Owing to symmetry, without loss of generality, we can choose \( i = d - 1 \) and \( j = d \). Then omitting \( \nu_{d-1} \) and \( \nu_d \) results in replacing (52) by

\[
\sum_{l \neq i, j}^{d-2} (1 - \nu_l)s_{d-k-3}(\nu_1, \ldots, \nu_{i-1}, \nu_i, \ldots, \nu_{d-2}) \geq 0.
\]

By setting \( n = d - 2 \), we can express the condition for Schur concavity by the following lemma.

**Lemma.** The functions \( \Phi_k \) defined in (47), are Schur concave in the Schmidt coefficients \( \lambda_1, \ldots, \lambda_d \) if

\[
\sum_{l=1}^n (1 - \nu_l)s_{n-k-1}(\nu_1, \ldots, \nu_{i-1}, \nu_i, \ldots, \nu_{n}) \geq 0, \quad 0 \leq k \leq d - 3, \tag{53}
\]

where the variables \( \nu_i := 1 - 2\lambda_i, \ 1 \leq i \leq n \), satisfy

\[
-1 \leq \nu_i \leq 1, \quad \sum_{l=1}^n \nu_l \geq n - 2. \tag{54}
\]

**Note:** The constraints (54) follow from the relations: \( \lambda_l \geq 0 \forall \ l, \) and \( \sum_{l=1}^n \lambda_l = \sum_{l=1}^{d-2} \lambda_l \leq 1. \)

**Proof of the Lemma.**

The constraints (54) imply that at most one of the variables \( \nu_1, \ldots, \nu_n \) can be negative. Note that \( (1 - \nu_l) \) is always nonnegative since \( \nu_l \leq 1 \). Thus if all \( \nu_1, \ldots, \nu_n \geq 0 \), (53) obviously holds. Hence, we need to prove (53) only in the case in which one, and only one, of the variables \( \nu_1, \ldots, \nu_n \) is negative.

To establish the latter fact, we first prove the inequality

\[
\sum_{l=1}^n \frac{1 - \nu_l}{\nu_l} \leq 0, \quad \text{or} \quad \sum_{l=1}^n \frac{\lambda_l}{1 - 2\lambda_l} \leq 0. \tag{55}
\]
Without loss of generality we can choose \( \nu_1 < 0 \) and \( \nu_l > 0 \) for all \( l = 2, 3, \ldots, n \). Hence, \( \lambda_1 > 1/2 \) and \( \lambda_l < 1/2 \) for all \( l = 2, 3, \ldots, n \). Write:

\[
\text{LHS of } (55) = \frac{\lambda_1}{1 - 2\lambda_1} + \sum_{l=2}^{n} \frac{\lambda_l}{1 - 2\lambda_l} := T_1 + T_2.
\]

Note that \( T_1 \leq 0 \) since \( \lambda_1 > 1/2 \). The function

\[
f(\lambda_i) := \frac{\lambda_i}{1 - 2\lambda_i}, \quad 0 \leq \lambda_i < 1/2,
\]

is convex. Hence, \( T_2 (\lambda_2, \cdots, \lambda_n) \), as a sum of convex functions, is convex on the simplex defined by

\[
\lambda_2 + \cdots + \lambda_n \leq 1 - \lambda_1, \quad 0 \leq \lambda_i < 1/2, \quad i = 2, \ldots, n,
\]

with fixed \( \lambda_i > 1/2 \).

Hence, \( T_2 \) achieves its maximum on the vertices of the simplex. One vertex is \((0, \cdots, 0)\), for which \( T_2 = 0 \), and hence \( T_1 + T_2 < 0 \). Other vertices are obtained by permutations from \((1 - \lambda_1, 0, \cdots, 0)\) and give

\[
T_2 = \frac{1 - \lambda_1}{1 - 2(1 - \lambda_1)} = -\frac{1 - \lambda_1}{1 - 2\lambda_1}.
\]

Thus the maximal value of \( T_1 + T_2 \) is

\[
\frac{\lambda_1}{1 - 2\lambda_1} - \frac{1 - \lambda_1}{1 - 2\lambda_1} = \frac{1 - 2\lambda_1}{1 - 2\lambda_1} = -1
\]

which proves (55).

To prove (53), using the definition (56) of elementary symmetric polynomials, we write:

\[
s_{n-k-1}(\nu_1, \ldots, \nu_l \ldots, \nu_n) = \frac{c_n}{(k-1)!} \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_k=1}^{n} \frac{1}{\nu_{j_1} \cdots \nu_{j_k} \nu_l}.
\]

Here \( c_n := \nu_1, \ldots, \nu_l, \ldots, \nu_n < 0 \). Hence, the required inequality (55) becomes

\[
\sum_{l=1}^{n} \sum_{j_1 \neq l}^{n} \sum_{j_2 \neq l,j_1}^{n} \cdots \sum_{j_k \neq l,j_1 \cdots j_{k-1}}^{n} \frac{1 - \nu_l}{\nu_{j_1} \cdots \nu_{j_k} \nu_l} \leq 0.
\]

Once again, without loss of generality we can choose \( \nu_1 < 0 \) and \( \nu_l > 0 \) for all \( l = 2, 3, \ldots, n \). Then

\[
\text{LHS of } (56) = \sum_{j_1=2}^{n} \sum_{j_2=2}^{n} \cdots \sum_{j_k=2}^{n} \frac{1 - \nu_l}{\nu_{j_1} \cdots \nu_{j_k} \nu_l} \left[ \frac{1 - \nu_l}{\nu_l} + \sum_{r=1}^{k} \frac{1 - \nu_r}{\nu_l} + \sum_{l=2}^{n} \frac{1 - \nu_l}{\nu_l} \right]
\]

Equation (57) can be derived as follows: Let

\[
T(l, j_1, j_2, \ldots, j_k) := \frac{1 - \nu_l}{\nu_{j_1} \cdots \nu_{j_k} \nu_l},
\]

with \( l, j_1, j_2, \ldots, j_k \in \{1, 2, \ldots, n\} \) and \( l, j_1, j_2, \ldots, j_k \) all different.
Without loss of generality we can choose $\nu_1 < 0$ and $\nu_l > 0$ for all $l = 2, 3, \ldots, n$. Then

\[
\text{LHS of (59)} = \sum_{l=1}^{n} \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_{k-1}=1}^{n} T(l, j_1, j_2, \ldots, j_k)
\]

\[
= \sum_{l=2}^{n} \sum_{j_1=2}^{n} \sum_{j_2=2}^{n} \cdots \sum_{j_{k-1}=2}^{n} T(1, j_1, j_2, \ldots, j_k) + \sum_{l=2}^{n} \sum_{j_2=2}^{n} \cdots \sum_{j_{k-1}=2}^{n} T(l, j_1, j_2, \ldots, j_k)
\]

\[
+ \sum_{l=2}^{n} \sum_{j_1=2}^{n} \cdots \sum_{j_{k-1}=2}^{n} T(l, j_1, \ldots, j_{k-1}, 1) + \sum_{l=2}^{n} \sum_{j_1=2}^{n} \cdots \sum_{j_{k-1}=2}^{n} T(l, j_1, \ldots, j_k).
\]

(59)

Now,

\[
\sum_{l=2}^{n} \sum_{j_1=2}^{n} \cdots \sum_{j_{k-1}=2}^{n} T(l, j_1, \ldots, j_{k-1}, 1, j_{i+1}, \ldots, j_k)
\]

\[
= \sum_{l=2}^{n} \sum_{j_1=2}^{n} \cdots \sum_{j_{k-1}=2}^{n} \frac{1 - \nu_l}{\nu_1 \nu_{j_1} \cdots \nu_{j_{i-1}} \nu_{j_{i+1}} \cdots \nu_{j_k}}
\]

\[
= \sum_{j_1=2}^{n} \cdots \sum_{j_{k-1}=2}^{n} \frac{1 - \nu_{j_{i+1}}}{\nu_{j_1} \cdots \nu_{j_k}} \left( \frac{1 - \nu_{j_{i+1}}}{\nu_1} \right).
\]

(60)

In the second last line on the RHS of (60), we have changed the dummy variable from $l$ to $j_i$. Hence,

\[
\text{RHS of (59)} = \sum_{j_1=2}^{n} \cdots \sum_{j_{k-1}=2}^{n} \frac{1}{\nu_{j_1} \cdots \nu_{j_k}} \left[ \frac{1 - \nu_1}{\nu_1} + \sum_{r=1}^{k} \frac{1 - \nu_r}{\nu_r} + \sum_{l=2}^{n} \frac{1 - \nu_l}{\nu_l} \right]
\]

\[
= \text{RHS of (59)}.
\]

(61)

From (59) it follows that for given $j_1, j_2, \ldots, j_k$, with $2 \leq j_r \leq n$ for $r = 1, 2, \ldots, k$, and $j_m \neq j_k$ for all $m \neq k$:

\[
\sum_{l=2}^{n} \frac{1 - \nu_1}{\nu_l} + \sum_{r=1}^{k} \frac{1 - \nu_r}{\nu_r} + 1 - \nu_1 \leq 0.
\]
Hence,

\[
\sum_{l=2}^{n} \frac{1 - \nu_l}{\nu_l} \leq - \left[ \sum_{r=1}^{k} \frac{1 - \nu_{jr}}{\nu_{jr}} + \frac{1 - \nu_1}{\nu_1} \right]
\]  \quad (62)

Substituting (62) on the RHS of (57) yields

\[
\text{RHS of (57)} \leq \sum_{j=2}^{n} \sum_{j=2}^{n} \sum_{j=2}^{n} \frac{1}{\nu_{j_1} \cdots \nu_{j_k}} \left[ \sum_{r=1}^{k} (1 - \nu_{jr}) \left( \frac{1}{\nu_1} - \frac{1}{\nu_{jr}} \right) \right]
\]

\[
= \sum_{j=2}^{n} \sum_{j=2}^{n} \sum_{j=2}^{n} \frac{1}{\nu_{j_1} \cdots \nu_{j_k}} \left[ \sum_{r=1}^{k} (1 - \nu_{jr}) \left( \nu_{jr} - \nu_1 \nu_{jr} \right) \right] \leq 0.
\]  \quad (63)

This proves (60) and hence (53) for \( n \geq 3 \) and all \( k = 1, 2, \ldots, n-2 \).

**APPENDIX**

A real-valued function \( \Phi \) on \( \mathbb{R}^n \) is said to be *Schur concave* (see [11]) if:

\[ x \prec y \implies \Phi(x) \geq \Phi(y). \]

Here the symbol \( x \prec y \) means that \( x = (x_1, x_2, \ldots, x_n) \) is *majorized* by \( y = (y_1, y_2, \ldots, y_n) \) in the following sense: Let \( x^\downarrow \) be the vector obtained by rearranging the coordinates of \( x \) in decreasing order:

\[ x^\downarrow = (x_\downarrow^1, x_\downarrow^2, \ldots, x_\downarrow^n) \quad \text{means} \quad x_\downarrow^1 \geq x_\downarrow^2 \geq \ldots \geq x_\downarrow^n. \]

For \( x, y \in \mathbb{R}^n \), we say that \( x \) is majorized by \( y \) and write \( x \prec y \) if

\[ \sum_{j=1}^{k} x_j^\downarrow \leq \sum_{j=1}^{k} y_j^\downarrow, \quad 1 \leq k \leq n, \]

and

\[ \sum_{j=1}^{n} x_j^\downarrow = \sum_{j=1}^{n} y_j^\downarrow. \]

In the simplex \( \Sigma_d \), defined by the constraints (10), the minimal point is \( (1/d, \ldots, 1/d) \) (the baricenter of \( \Sigma_d \)), and the maximal points are the permutations of \( (1, 0, \ldots, 0) \) (the vertices).

A differentiable function \( \Phi(x_1, x_2, \ldots, x_n) \) is Schur concave if and only if:

1. \( \Phi \) is symmetric
2. \[(x_i - x_j) \left( \frac{\partial \Phi}{\partial x_i} - \frac{\partial \Phi}{\partial x_j} \right) \geq 0, \quad \forall 1 \leq i, j \leq n. \]  \quad (64)
The $l^{th}$ elementary symmetric polynomial $s_l$ in the variables $x_1, x_2, \ldots, x_n$ is defined as

\[
\begin{align*}
  s_0(x_1, x_2, \ldots, x_n) &= 1, \\
  s_l(x_1, x_2, \ldots, x_n) &= \sum_{1 \leq i_1 < i_2 \cdots < i_l \leq d} x_{i_1}x_{i_2}\cdots x_{i_l} \quad \text{for } l = 1, 2, \ldots, n. 
\end{align*}
\]

We shall use the following identities:

\[
\frac{\partial}{\partial x_j} s_k(x_1, x_2, \ldots, x_n) = s_{k-1}(x_1, \ldots, \hat{x}_j, \ldots, x_n) \tag{66}
\]

and

\[
s_k(x_1, \ldots, \hat{x}_i, \ldots, x_n) - s_k(x_1, \ldots, \hat{x}_j, \ldots, x_n) = (x_j - x_i)s_{k-1}(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n) \tag{67}
\]

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