On the Resummation of the $\alpha \ln^2 z$ Terms for QED Corrections to Deep-Inelastic $ep$ Scattering and $e^+e^-$ Annihilation

J. Blümlein, S. Riemersma

DESY–Zeuthen
Platanenallee 6, D–15735 Zeuthen, Germany

A. Vogt

Institut für Theoretische Physik, Universität Würzburg
Am Hubland, D–97074 Würzburg, Germany

Abstract

The resummation of the $\alpha \ln^2(z)$ non-singlet contributions is performed for initial state QED corrections. As examples, the effect of the resummation on neutral-current deep-inelastic scattering and the $e^+e^- \rightarrow \mu^+\mu^-$ scattering cross section near the $Z^0$-peak is investigated.
1 Introduction

The non-singlet splitting functions of QCD are known to behave as $\alpha_{s}^{l+1} \ln^{2l}(z)$ for small values of $z$, the momentum fraction determining the corresponding radiator function. A similar behaviour is observed also in QED\[3, 4\] These terms may potentially yield large contributions to the radiative corrections. In an approach based on the systematic evaluation of the Feynman diagrams at a fixed order in the coupling constant, the contributions of $O[\alpha_{s}^{l+1} \ln^{2l}(z)]$ emerge from a wide class of terms, see for example \[3, 5\]. Therefore the all-order resummation of these terms cannot be carried out by direct diagram calculations but is performed by solving so-called infrared evolution equations \[1\].

In the present paper we calculate the contribution of the small-$z$ resummed terms to the initial state radiative corrections for deep-inelastic ep scattering (DIS). We compare these corrections with those resummed by the non-singlet Altarelli–Parisi equation in QED, $\propto \alpha^{l} \ln^{l}(Q^{2}/m_{e}^{2})$. We also evaluate the contribution of these terms to the initial state corrections to $e^{+}e^{-} \rightarrow \mu^{+}\mu^{-}$ at the $Z^{0}$-peak.

2 Basic Relations

The evolution of the non-singlet electron structure function $D(z, Q^{2})$ is governed by

$$\frac{\partial D(z, Q^{2})}{\partial \ln Q^{2}} = P[z, \alpha(Q^{2})] \otimes D(z, Q^{2}),$$

where $\otimes$ denotes the Mellin convolution

$$A(z) \otimes B(z) \equiv \int_{0}^{1} \int_{0}^{1} dz_{1}dz_{2} A(z_{1})B(z_{2})\delta(z - z_{1}z_{2}).$$

The splitting function $P[z, \alpha(Q^{2})]$ can be represented by the series

$$P[z, \alpha(Q^{2})] = \sum_{k=1}^{\infty} a^{k}(Q^{2})P_{k}(z),$$

with $a(Q^{2}) = \alpha(Q^{2})/(4\pi)$. In leading order, the evolution of the QED coupling constant $a(Q^{2})$ is described by

$$\frac{\partial a(Q^{2})}{\partial \ln Q^{2}} = \frac{4}{3}a^{2}(Q^{2}),$$

yielding

$$a(Q^{2}) = \frac{a(m_{e}^{2})}{1 - \frac{4}{3}a(m_{e}^{2}) \log \left(\frac{Q^{2}}{m_{e}^{2}}\right)}.$$

Here we have considered only the electron threshold in the evolution. For the solution of eq. (4), we use the first-order splitting function

$$P_{1}(z) = 2 \left(\frac{1 + z^{2}}{1 - z}\right)_{+}.$$  

\[1\]A first application to QED was discussed in \[3\], considering forward $e^{+}e^{-} \rightarrow \mu^{+}\mu^{-}$ annihilation in the high energy limit.
For the higher order contributions in $a(Q^2)$, we account for the leading terms as $z \to 0$, which are $\propto a^{l+1} \ln^2(z)$. The latter terms are obtained in resummed form in Mellin space by

\[ \mathcal{M}[P_{z \to 0}](N, a) \equiv \int_0^1 dz \; z^{N-1} P_{z \to 0}(z) \equiv -\frac{1}{2} \Gamma_{z \to 0}^{-}(N, a) = \frac{1}{8\pi^2} f_0^-(N, a). \]  

(7)

$f_0^-(N, a)$ is the solution to the equation

\[ f_0^-(N, a) = 16\pi^2 \frac{a}{N} + 8 \frac{a}{N^2} f_V^+(N, a) + \frac{1}{8\pi^2} \left[ f_0^-(N, a) \right]^2, \]  

and $f_V^+(N, a)$ obeys

\[ f_V^+(N, a) = 16\pi^2 \frac{a}{N} + \frac{1}{8\pi^2} \left[ f_V^+(N, a) \right]^2. \]  

(9)

Here the coefficients in eqs. (8,9), originally given for $SU(N)$ in ref. [1], were adjusted to the case of QED, see ref. [4].

For the resummed anomalous dimension, one finally obtains

\[ \Gamma_{z \to 0}^{-, \text{QED}}(N, a) = -N \left\{ 1 - \frac{8a}{N^2} \left[ 1 - 2\sqrt{1 - \frac{8a}{N^2}} \right] \right\}. \]  

(10)

$\Gamma_{z \to 0}^{-, \text{QED}}(N, a)$ can be represented in terms of a Taylor series in $a$ by

\[ \Gamma_{z \to 0}^{-, \text{QED}}(N, a) = \sum_{k=0}^{\infty} c_k a^{k+1} \ln^2(z), \quad c_k = \frac{p_k}{(2k)!}. \]  

(11)

The term $2a/N$ corresponds to the small-$z$ contribution of eq. (6). The resummed small-$z$ part of the splitting function $P(z, a)$ is obtained transforming eq. (11) back to $z$-space,

\[ P_{z \to 0}(z, a) = \sum_{k=0}^{\infty} c_k a^{k+1} \ln^2(z), \quad c_k = \frac{p_k}{(2k)!}. \]  

(12)

The numerical values of the first coefficients $c_k$ are listed in Table 1. We use

\[ D(z, Q_0^2 = m_e^2) = \delta(1-z) \]  

as the initial condition for the solution of eq. (11). For the splitting functions $P_k(z)$ in eq. (3),

\[ \int_0^1 dz P_k(z) = 0 \]  

(14)

holds due to fermion number conservation. For the resummed kernel $P_{z \to 0}(z, a)$, eq. (12), the integral condition eq. (14) is not obeyed \textit{a priori} but has to be restored. In the subsequent treatment we will subtract the term $p_{k-1} \delta(1-z)$ in $O(a^k)$.

As outlined in refs. [1, 6] for the resummation of the small-$x$ terms for different processes in QCD, less singular terms can be as important as the leading singular terms. In QED, the $O(\alpha^2 \ln(z) \ln(Q^2/m_e^2))$ terms are known for $e^+ e^-$ annihilation [3]. From the different contributions, all terms but the well-known term due to the vacuum polarization function cancel. In $O(\alpha^2)$ the respective correction is

\[ -12 \frac{a^2}{N^3} \left( 1 - \frac{2}{9} N \right). \]

In this order, the coefficient of the less singular term is much smaller than that of the leading term.
Table 1: Coefficients of the expansion of the small-$z$ resummation $P_{z\to 0}(z,a) = \sum_{k=0}^{\infty} c_k a^{k+1} \ln^{2k}(z)$.

3 Non-Singlet QED Radiative Corrections to Deeply Inelastic $ep$ Scattering

The Born cross section for neutral-current deep-inelastic $ep$ scattering is given by

$$\frac{d^2\sigma_{NC}^{B}}{dx\,dy} = \frac{2\pi\alpha^2 S}{Q^4} \left[ Y_+ F_2(x, Q^2) + Y_- x F_3(x, Q^2) \right],$$  \hspace{1cm} (15)$$

with $Y_{\pm} = 1 \pm (1 \pm y)^2$, $x$ and $y$ are the Bjorken variables, $S$ is the cm energy squared, $Q^2 = xyS$ and

$$F_2(x, Q^2) = F_2(x, Q^2) + 2|Q_e|(v_e + \lambda a_e)\chi(Q^2)G_2(x, Q^2) + 4(v_e^2 + a_e^2 + 2\lambda v_e a_e)\chi^2(Q^2)H_2(x, Q^2) \hspace{1cm} (16)$$

$$xF_3(x, Q^2) = -2 \text{sign}(Q_e) \left\{ Q_e (a_e + \lambda v_e)\chi(Q^2)xG_3(x, Q^2) + [2v_e a_e + \lambda(v_e^2 + a_e^2)]\chi^2(Q^2)xH_3(x, Q^2) \right\},$$  \hspace{1cm} (17)$$

with $Q_e = -1$ for electron and $Q_e = 1$ for positron scattering. $\lambda = \xi \text{sign}(Q_e)$ denotes the lepton polarization, $v_e = 1 - 4\sin^2\theta_W$, $a_e = 1$, $\theta_W$ the weak mixing angle, and

$$\chi(Q^2) = \frac{G_F}{\sqrt{2}} \frac{M_Z^2}{8\pi\alpha^2 Q^2 + M_Z^2} \frac{Q^2}{\sqrt{2}}.$$  \hspace{1cm} (18)$$

$G_F$ is the Fermi constant and $M_Z$ the mass of the $Z^0$-boson. The neutral-current structure functions in eqs. (16,17) are described in the parton model by

$$F_2(x, Q^2) = x \sum_{i=1}^{N_f} e_i^2 \left[q_i(x, Q^2) + \overline{q}_i(x, Q^2)\right],$$  \hspace{1cm} (19)$$

$$G_2(x, Q^2) = x \sum_{i=1}^{N_f} |e_i|v_i \left[q_i(x, Q^2) + \overline{q}_i(x, Q^2)\right],$$  \hspace{1cm} (20)$$
The case of leptonic variables \[7, 8\]. Here the shifted quantities \( \hat{x}, \hat{y}, \hat{z} \), with \( \hat{v}_i = 1 - 4|e_i|\sin^2 \theta_W \), \( a_i = 1 \), \( N_f \) the number of flavours, and \( q_i, \bar{q}_i \) denote the quark and antiquark densities, respectively.

The contributions to eq. (26) are taken into account up to order terms we add the solution of eq. (1) in the soft limit where

\[
D_{1, 2, 3} = \left[ a_{1, 2, 3} \right] \cdot \frac{1}{k!} \cdot \zeta^k(Q^2) \cdot \mathcal{J}(x, y, z)
\]

receives contributions from the iteration of the non-singlet kernel \( R_1(z) = P_1(z)|_{z < 1} \), which are obtained by

\[
D_{AP}(z, Q^2) = \sum_{k=1}^{\infty} \frac{1}{k!} \cdot \zeta^k(Q^2) \cdot \mathcal{J}(x, y, z)
\]

where

\[
\zeta(Q^2) = -\frac{3}{2} \ln \left[ 1 - \frac{4}{3} a_0 \ln \left( \frac{Q^2}{m_e^2} \right) \right].
\]

In the subsequent numerical calculation, we evaluate the initial-state radiative corrections for the case of leptonic variables \[7, 8\]. Here the shifted quantities \( \hat{x}, \hat{y}, \hat{z} \) and the threshold \( z_0 \) are

\[
\hat{x} = \frac{xy}{1}, \quad \hat{y} = \frac{z + y - 1}{z}, \quad \hat{z} = zS, \quad z_0 = \frac{1 - y}{1 - xy}.
\]

The contributions to eq. (26) are taken into account up to \( k = 3 \) completely\(^2\). For the higher-order terms we add the solution of eq. (1) in the soft limit

\[
D_{AP}(z, Q^2)|_{(4)} = 2\zeta(1 - z)^{2k-1} \cdot \frac{\exp[\zeta(\frac{3}{2} - 2\gamma_E)]}{\Gamma(1 + 2\zeta)} \cdot \frac{2\zeta}{1 - z} - [3 + 4 \ln(1 - z)] \cdot \frac{\zeta^2}{1 - z}
\]

\[
- \left[ 4 \ln^2(1 - z) + 6 \ln(1 - z) + \frac{9}{4} \cdot \frac{2\pi^2}{3} \right] \cdot \frac{\zeta^3}{1 - z}.
\]

The contribution of the small-\( z \) resummed terms to \( D(z, Q^2) \) is

\[
D_{z \to 0}(z, Q^2) = \sum_{k=1}^{\infty} c_k \int_{m_\bar{e}^2}^{Q^2} \frac{dq^2}{q^2} a^{k+1}(q^2) \ln^{2k}(z).
\]

\(^2\) Analytic expressions for the convolutions of \( R_1(z) \) are easily obtained, see \[8\] and \[9\] for explicit expressions.
In Figures 1a–c, we show the contributions to the initial state QED corrections to $d^2\sigma_{NC}^R/dxdy$ in the kinematic range of HERA starting with the terms in $O(\alpha^2)$ to allow for a better comparison. The first order corrections are well-known, see refs. [13, 2]. We compare the small-$z$ resummed terms to those obtained by iterating the kernel $R_1(z)$. The small-$z$ resummed terms are negative and contribute only for large values of $y$. There they diminish the positive leading order corrections $O[^2\alpha l\ln l(Q^2/m_e^2)]$ significantly. These corrections are therefore relevant and need to be considered in the case of high $y$ measurements, such as the determination of the structure function $F_L(x,Q^2)$ in the small-$x$ range. For larger values of $x$, the small-$z$ resummed corrections contribute only for highest values of $y$.

4 \(\alpha\ln^2(z)\) QED corrections to the $Z^0$ peak

A second important application of the small-$z$ resummation concerns its possible effect upon the $e^+e^- \rightarrow \mu^+\mu^-$ cross section near the $Z^0$-peak. The implications would be quite profound were the resummation-improved cross section to have a measurable impact on the total cross section or upon the position of the $Z^0$-peak or width on the order of an MeV.

The QED corrections up to $O(\alpha^2)$ were calculated in [3]. We consider the initial state corrections which are calculated accounting for the contributions to $O(\alpha^2)$ and soft-photon exponentiated terms using the code ZFITTER [11]. The small-$z$ resummed terms (12) are taken into account for the contributions higher than second order by

\[
\sigma_{z\rightarrow 0} = 2 \int_0^1 dz [\Theta(z - z_0)\sigma_B(zs) - \sigma_B(s)] R_{z\rightarrow 0}(z, s),
\]

where $z_0 = s'/s$, $s'$ being the cm energy entering the annihilation, and $\sigma_B(s)$ the Born cross section. The radiator $R_{z\rightarrow 0}(z, s)$ is given by

\[
R_{z\rightarrow 0}(z, s) = \int_{m^2_e}^s \frac{ds'}{s'} \sum_{k=3}^{\infty} c_k a^{k+1}(s') \ln^{2k}(z).
\]

The factor of 2 enters in eq. (31) because of the initial state radiation from both the electron and the positron line. A series of cuts on $s'$ has been made and the results are listed in Table 2. The parameters of the calculation are $M_{Z} = 91.1887$ GeV, $\Gamma_{Z} = 2.4974$ GeV, and $\sin^2\theta_W = 0.2319$. The small-$z$ contribution is six orders of magnitude down from the cross section containing the standard QED corrections. A measurement of this effect is clearly out of the question.

| $E_{\mu_{min}}$ (GeV) | 5   | 10  | 20  | 40  |
|------------------------|-----|-----|-----|-----|
| $\sigma_R$ (nb)        | 1.4723 | 1.4713 | 1.4702 | 1.4674 |
| $\sigma_{z\rightarrow 0}$ (nb) | 1.05341 $10^{-6}$ | 1.13476 $10^{-6}$ | 1.11480 $10^{-6}$ | 1.11465 $10^{-6}$ |

Table 2: Dependence of the cross section $\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ at the $Z^0$-peak from the minimum energy of the final state muons. $\sigma_R$ : scattering cross section including the initial-state QED corrections to $O(\alpha^2)$ and soft-photon exponentiation; $\sigma_{z\rightarrow 0}$ : small-$z$ resummed contributions beyond $O(\alpha^2)$, eq. (31).
We also have compared the maximum cross section of ZFITTER as a function of the cm energy with and without the small-$z$ resummation. We find the difference to be smaller than 40 eV, widely independent of the $z$-cut. Here the effects of the small-$z$ contributions beyond second order are much smaller than the experimental resolution.

5 Conclusions

The resummation of the $O(\alpha \ln^2(z))$ non-singlet contributions was performed for initial state QED corrections. As examples, we investigated the effects of the resummation for two processes: neutral-current deep-inelastic scattering and $e^+e^- \rightarrow \mu^+\mu^-$ scattering near the $Z^0$-peak. The influence upon the DIS results is particularly strong in the low-$x$, high-$y$ region. In this region, the small-$z$ corrections negate a sizeable portion of the $O(\alpha^2 \ln^2(Q^2/m^2_e))$ and higher-order contributions. The effect diminishes as $x \rightarrow 1$ but still remains important near $y \approx 1$. The incorporation of these corrections is therefore important in analyses of deep-inelastic data in the high $y$ range.

The small-$z$ resummation, on the other hand, has no visible effect upon the $e^+e^- \rightarrow \mu^+\mu^-$ cross section near the $Z^0$ peak. It contributes to the cross section at a level of $10^{-6}$ only. Correspondingly the shift in the peak cross section is negligibly small.

Acknowledgements: For discussions we would like to thank W. van Neerven and D. Bardin. This work was supported in part by the EC Network ‘Human Capital and Mobility’ under contract No. CHRX–CT923–0004 and by the German Federal Ministry for Research and Technology (BMBF) under contract No. 05 7WZ91P (0).

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Figure 1: Second and higher order initial state radiative corrections to deep inelastic $ep$ scattering at HERA, $\sqrt{s} = 314$ GeV. Dashed lines: contribution due to the resummed small–$z$ terms; dash–dotted lines: contribution due to the solution of the Altarelli–Parisi equation with the leading order NS–evolution kernel accounting for the complete $O(\alpha^2 L^2)$ and $O(\alpha^3 L^3)$ terms and the soft-photon exponentiation beyond $O(\alpha^3)$; full lines: resulting correction. a: $x = 0.0001$, b: $x = 0.01$, and c: $x = 0.5$. 
\[ \frac{d\sigma_{\text{res, QED}}^{\text{NS}}(O(\alpha^{1+n}))}{d\sigma_0^{\text{NC}}} - 1 \]

\[ x = 0.01 \]
\[ \frac{d\sigma_{\text{res}}^{\text{NS, QED}}(O(\alpha^{1+n}))}{d\sigma_0^{\text{NC}}} - 1 \]

\[ x = 0.5 \]