Biaxially symmetric solutions to 4D higher spin gravity

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Abstract

We review some aspects of biaxially symmetric solutions to Vasiliev’s equations in four-dimensional spacetime with a negative cosmological constant. The solutions, which activate bosonic fields of all spins, are constructed using gauge functions, projectors and deformed oscillators. The deformation parameters, which are formally gauge invariant, are related to generalized electric and magnetic charges in asymptotic weak-field regions. Alternatively, the solutions can be characterized in a dual fashion using 0-form charges which are higher spin Casimir invariants built from combinations of curvatures and all their derivatives that are constant on shell and well-defined everywhere.

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1. Introduction

Vasiliev’s equations \[3, 4\] (see \[5, 6, 8, 9\] for reviews) provide a classical description of a large class of higher spin gravities: these are generally covariant gauge theories with (anti-)de Sitter vacua and perturbative spectra consisting of massless symmetric tensors and tensor–spinors of all possible ranks forming unirreps of underlying non-Abelian higher spin algebras. Vasiliev’s framework for higher spin gravities, which at present is the only known fully nonlinear such framework, exhibits a level of complexity in between that of gravity and string field theory: the spectra and algebras are given by direct-product squares of singletons \[13, 7\], that is, conformal fields on the spacetime boundary. For suitable gauge algebras, field contents and
couplings, these models are thus candidates for consistent truncations of string field theory in a tensionless limit in the presence of a finite cosmological constant down to the first Regge trajectory, and by now there is indeed strong evidence that such models correspond holographically to weakly coupled boundary theories, or rather, truncations thereof down to the sector of bilinear composites. Vasiliev’s theory thus opens windows to the AdS/CFT correspondence, so far explored in both four [14, 15] and three bulk dimensions [16, 17] (see more references in [17]), in the weak/weak coupling regimes, which has triggered the recent increase of interest in higher spin gravity.

The aim of this brief review is to present some exact classical solutions that go beyond the aforementioned perturbative results. The study of classical solutions facilitates the understanding of a number of interesting aspects of the theory: first, Vasiliev’s equations are given in terms of locally defined fields living on charts, whereas a globally defined formulation must take into account a (choice of) structure group containing the transition functions used to glue together such field configurations across chart boundaries, as well as conditions on the fields at the boundary of the base manifold. Both of these data in general require a splitting of the 1-form connection and its gauge parameters into a Yang–Mills-like subsector and a frame-like one containing a generalized soldering form [31]. Both of them can be used to construct observables, but of different geometric meaning. Second, given such a geometric formulation, it can be used to construct super-selection sectors of the moduli space perturbatively as well as non-perturbatively [1]. Finally, as HS gravity exhibits rather special integrability properties, one may hope to actually be able to eventually provide an exact description of a classical moduli space with observables that can be (deformation-)quantized using geometric methods possibly along the lines of [11].

In this work, we shall review some recently-found classical exact solutions [1], that possess at least two commuting Killing vectors, in the case of Vasiliev’s twistor formulation in four dimensions; for exact solutions in lower dimensions, see [18–20]. In higher dimensions, the on-shell projection cannot be solved identically by means of the twistor transform. Thus, whether one works with twistor [12] or vector-oscillator [4, 7, 8, 10] formulations, there remain non-trivial internal constraints to be solved, and so far the only known exact solutions in dimensions higher than 4 are the maximally symmetric spacetime vacua.

The plan of this review is as follows: we continue by first presenting Vasiliev’s equations, stressing their special features in the four-dimensional case. We then proceed by discussing exact solutions.

2. Bosonic Vasiliev’s equations in four dimensions

Fundamental fields and kinematics. The basic variables of Vasiliev’s formulation of higher spin gravity are differential forms on \( \mathbb{C} \), a non-commutative symplectic manifold with symplectic structure \( \Omega \), that we shall refer to as the correspondence space. Locally, \( \mathbb{C} \) is the product of a phase spacetime, containing the ordinary (commutative) spacetime, and internal directions, and it is endowed with an associative product \( \ast \) (containing the standard wedge-product among cotangent-space basis elements) and exterior derivative \( \hat{d} \) obeying

\[
\hat{d}(f \ast \hat{g}) = (\hat{d}f) \ast \hat{g} + (-1)^{\deg(f)} f \ast (\hat{d}\hat{g}),
\]

where \( \hat{f} \) and \( \hat{g} \) are differential forms on \( \mathbb{C} \). The differential forms take their values in a unital associative algebra \( A \) whose product is assumed to be contained in \( \ast \) as well, i.e. the
differential forms are elements of the associative differential algebra $\mathcal{A} \otimes \Omega(\mathfrak{C})$, assumed to admit a Hermitian conjugation operation obeying
\[(\hat{f} \star \hat{g})^\dagger = \hat{g}^\dagger \star \hat{f}^\dagger, \quad (\hat{d} \hat{f})^\dagger = \hat{d}^\dagger (\hat{f})^\dagger.\] (2.2)

The field equations admit truncations to fields valued in the subalgebra $\mathcal{A} \subset \mathfrak{A}$, that is, to elements of $\mathcal{A} \otimes \Omega(\mathfrak{C})$, which is what we shall assume from now on. The correspondence space we shall be dealing with can be assumed to take the factorized form
\[\mathfrak{E} \cong \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}\] (2.3)
on local coordinate charts, where

- $\mathcal{Y}$ and $\mathcal{Z}$ are two four-dimensional real symplectic manifolds with coordinates $Y^\mu_a = (Y^\mu_a)^\dagger$ and $Z^\mu_a = (Z^\mu_a)^\dagger$ and with two forms $\Omega_{\mathfrak{Y}} = \frac{1}{2} dY^\mu_a dY^\nu_a$ and $\Omega_{\mathfrak{Z}} = -\frac{1}{2} dZ^\mu_a dZ^\nu_a$, with non-trivial commutation relations normalized as follows:
\[[Y^\mu_a, Y^\nu_b] = 2iC_{\mu\nu}^{ab}, \quad [Z^\mu_a, Z^\nu_b] = -2iC_{\mu\nu}^{ab}, \quad C_{\mu\nu}^{ab} := \begin{pmatrix} \epsilon_{a\beta} & 0 \\ 0 & \epsilon_{a\beta} \end{pmatrix}.\] (2.4)
The $\mathfrak{sp}(4, \mathbb{C})$-quartets can be split into $\mathfrak{sl}(2, \mathbb{C})$-doublets as $Y^\mu = (\gamma^\mu, \bar{\gamma}^\mu)$, with $\bar{\gamma}^\mu = (\gamma^\mu)^\dagger$, idem for $Z^\mu$, non-trivial commutation relations
\[[Y^\mu, Y^\nu] = 2i\epsilon_{\mu\nu}, \quad [z^\nu, z^\mu]_* = -2i\epsilon_{\mu\nu},\] (2.5)
together with their Hermitian conjugates.

- $\mathcal{X}$ is a spacetime manifold$^6$ coordinatized with $x^\mu$.

The fundamental fields are a locally defined 0-form $\hat{\Phi}$, a locally defined 1-form $\hat{A}$ and a globally defined complex 2-form $(\hat{J}, \hat{\bar{J}})$. These master fields obey the reality conditions
\[(\hat{\Phi}, \hat{A}, \hat{J}, \hat{\bar{J}})^\dagger = (\pi(\hat{\Phi}), -\hat{A}, -\hat{J}, -\hat{\bar{J}}).\] (2.6)

In bosonic models, they also obey the projections
\[\pi(\hat{\Phi}, \hat{A}) = (\hat{\Phi}, \hat{A}), \quad \pi(\hat{J}, \hat{\bar{J}}) = \hat{J}, \hat{\bar{J}} = (\hat{J}, \hat{\bar{J}}),\] (2.7)
where $\pi$ and $\bar{\pi}$ are the involutive automorphisms defined by $d\pi = \pi d, d\bar{\pi} = \bar{\pi} d$ and
\[\pi(x^\mu; y^\mu, \bar{y}^\mu, c^\mu, \bar{c}^\mu) = (x^\mu; -y^\mu, \bar{y}^\mu, -c^\mu, \bar{c}^\mu), \quad \bar{\pi}(\hat{f} \star \hat{g}) = \pi(\hat{f}) \star \pi(\hat{g}).\] (2.8)
\[\bar{\pi}(x^\mu; y^\mu, \bar{y}^\mu, c^\mu, \bar{c}^\mu) = (x^\mu; y^\mu, -\bar{y}^\mu, \bar{c}^\mu, -\bar{c}^\mu), \quad \bar{\pi}(\hat{f} \star \hat{g}) = \bar{\pi}(\hat{f}) \star \bar{\pi}(\hat{g}).\] (2.9)

In minimal bosonic models, the master fields obey the stronger projections
\[\tau(\hat{\Phi}, \hat{A}, \hat{J}, \hat{\bar{J}}) = (\pi(\hat{\Phi}), -\hat{A}, -\hat{J}, -\hat{\bar{J}}),\] (2.10)
where $\tau$ is the graded anti-automorphism defined by $d\tau = \tau d$ and
\[\tau(x^\mu; Y^\mu_a, Z^\mu_a) = (x^\mu; iY^\mu_a, -iZ^\mu_a), \quad \tau(\hat{f} \star \hat{g}) = (\tau(\hat{f}) \star \pi(\hat{g})).\] (2.11)

$^6$ The elements of $\Omega(\mathfrak{C})$ are thus composite operators given by functions of local non-commutative coordinates on $\mathfrak{E}$, which can be presented via their symbols comprising the expansion coefficients with respect to different bases, that correspond to choices of different ordering prescriptions; for further details, see for example appendix B in [1].

$^7$ This choice of $\mathfrak{C}$ corresponds to a truncation of a more general correspondence space locally admitting the factorization $\mathfrak{E} = \mathcal{Y} \times \mathcal{Z} \times \mathcal{K} \times \bar{\mathcal{E}}$ where $\mathcal{K}$ is coordinatized by the outer Kleinian elements $k$ and $\bar{k}$ obeying $k^2 = \bar{k}^2 = 1$, $\{k, \bar{k}\}_a = 0$, $k^2 = \bar{k}^2 = 0$ and having trivial commutators with all coordinates except for $\{k, y\}_a = 0$, $\{k, \bar{y}\}_a = 0$ and their Hermitian conjugates (see for example [5]); and $\mathfrak{E}$ is a universal non-commutative manifold that one may take to be $T^* \mathcal{X}$, where $\mathcal{X}$ is a universal commutative manifold containing spacetime coordinatized with $x^\mu$. The system on $T^* \mathcal{X}$ can be projected onto $\mathcal{X}$ (see [1] for more details), which is what we assume in this paper.
and obeying $r^2 = \pi \bar{\pi}$. The perturbative spectra of the bosonic and minimal bosonic models consist of real Fronsdal fields of integer and even-integer spins, respectively, with each spin occurring in the spectrum with multiplicity 1.

The automorphisms $\pi$ and $\bar{\pi}$ are inner and generated via the adjoint action of inner Kleinian operators as follows:

$$\pi(f) = \tilde{k} \ast \tilde{f} \ast \tilde{k}, \quad \bar{\pi}(f) = \tilde{k} \ast \tilde{f} \ast \tilde{k},$$

where $k_\gamma$ and $k_\zeta$ are the inner Kleinians [2], obeying

$$k_\gamma \ast k_\gamma = 1, \quad [k_\gamma, y_\alpha] = 0, \quad [k_\gamma, z_\alpha] = 0,$$

$idem$ for $k_\zeta$ upon exchanging $y$ and $z$ and for their Hermitian conjugates. The inner Kleinians are given in Weyl order by

$$k_\gamma = 2\pi \delta^2(y), \quad k_\zeta = 2\pi \delta^2(z),$$

$idem$ for $k_\zeta$ and $k_\zeta$ [2] (for more details on Kleinians in various orderings, see appendix B in [1]).

An explicit, integral realization of the $\ast$-product among functions of $(Y, Z)$ oscillators (corresponding to normal ordering with respect to the creation/annihilation operators $\hat{y}(y_u + z_\alpha, -iy_u + iz_\alpha)$) is

$$\tilde{f}_1 \ast \tilde{f}_2 = \int_R \frac{d^4Ud^4V}{(2\pi)^8} e^{i(\epsilon^*_u y_u + \epsilon^*_u z_\alpha)} f_1(y + \epsilon_u + \epsilon_u; z + u, \zeta - u) \tilde{f}_2(y + v, \zeta + v; z - v, \zeta + v),$$

where $(u, v)$ and $(\epsilon_u, \epsilon_u)$ are treated as real and independent variables.

The factorization property (2.13), which holds in all orders, is crucial for the separation of twistor-space variables that we shall use in the solution ansätze below.

### Field equations and deformed oscillators.

The unfolded equations of motion of the four-dimensional bosonic higher spin gravities that we shall study can be written as

$$\Omega_{\gamma}^{\nu} \ast \tilde{D}\tilde{\Phi} = 0, \quad \Omega_{\gamma}^{\nu} \ast (\tilde{F}(\tilde{\Phi}) - \tilde{F}(\tilde{\Phi}) \ast \tilde{F}(\tilde{\Phi}) \ast \tilde{F}(\tilde{\Phi})) = 0,$$

with Yang–Mills-like curvatures $\tilde{F} := \tilde{\partial}\tilde{A} + \tilde{A} \ast \tilde{A}$ and $D\tilde{\Phi} := \tilde{\partial}\tilde{\Phi} + [\tilde{A}, \tilde{\Phi}]_\pi$, where $[\tilde{F}, \tilde{G}]_\pi := \tilde{F} \ast \tilde{G} - (-1)^{\text{deg}(\tilde{F})\text{deg}(\tilde{G})}\tilde{G} \ast \pi(\tilde{F})$ for $\tilde{F}, \tilde{G} \in \Omega(\mathcal{C})$. The interaction ambiguities $\tilde{F}$ and $\tilde{F} = (F)_{\tilde{F}}$ are given by

$$\tilde{F}(\tilde{\Phi}) = \sum_{n=0}^{\infty} f_{2n+1} (\tilde{F}(\tilde{\Phi})) (\tilde{F}(\tilde{\Phi}) \ast \tilde{F}(\tilde{\Phi}))^{n} \ast \tilde{F},$$

where $f_{2n+1}$ are complex-valued 0-form charges obeying

$$\tilde{d}f_{2n+1} = 0,$$

as we shall describe in more detail below. Integrability requires the algebraic constraints

$$\tilde{F}(\tilde{\Phi}, \tilde{\Lambda}) = (\tilde{\Phi}, \tilde{\Lambda}) \ast \tilde{F}, \quad \tilde{F}(\tilde{\Phi}, \tilde{\Lambda}) = (\tilde{\Phi}, \tilde{\Lambda}) \ast \tilde{F},$$

modulo terms that are annihilated by $\Omega_{\gamma}^{\nu} \ast$. In other words, equations (2.17), (2.18) and (2.20) are compatible with $\tilde{D}^2 = 0$ modulo equation (2.21), hence defining a universal (i.e. valid on
any $\mathcal{X}$ quasi-free associative differential algebra. Factoring out perturbative redefinitions of $\hat{\Theta}$, the ambiguity residing in $\mathcal{F}$ reduces down to [5, 31]

$$\mathcal{F} = B \star \hat{\Theta}, \quad B = \exp_\epsilon (i\theta [\hat{\Theta} \star \pi (\hat{\Theta})]) ,$$

(2.22)

$$\theta [\hat{\Theta} \star \pi (\hat{\Theta})] = \sum_{n=0}^\infty \theta_{2n}[\hat{\Theta} \star \pi (\hat{\Theta})](\hat{\Theta} \star \pi (\hat{\Theta}))^{*n},$$

(2.23)

which breaks parity except in the following two cases [15]:

Type-A model (scalar) : $\theta = 0, \quad P(\hat{\Theta}, \hat{\bar{A}}, \hat{\bar{J}}) = (\hat{\bar{\Theta}}, \hat{\bar{A}}, \hat{\bar{J}})$,

(2.24)

Type-B model (pseudo-scalar) : $\theta = \frac{\pi}{2}, \quad P(\hat{\Theta}, \hat{\bar{A}}, \hat{\bar{J}}) = (-\hat{\bar{\Theta}}, \hat{\bar{A}}, -\hat{\bar{J}})$,

(2.25)

where the parity operation is the automorphism of $\Omega (\mathcal{C})$ defined by

$$P(x^\mu, y^\mu, \tilde{z}^\mu, \tilde{\zeta}^\mu) = (x^\mu, \tilde{y}^\mu, y^\mu, -\tilde{z}^\mu, -\tilde{\zeta}^\mu), \quad \hat{d}P = P\hat{d}.$$

(2.26)

The gauge transformations read

$$\delta_\epsilon \hat{\Theta} = -[\epsilon, \hat{\Theta}]_\pi, \quad \delta_\epsilon \hat{\bar{A}} = \hat{D}\epsilon, \quad \delta_\epsilon \hat{\bar{J}} = 0,$$

(2.27)

with $\hat{D}\epsilon := \hat{D}\epsilon + [\hat{\bar{A}}, \epsilon]_\pi$, and where $\epsilon$ is subject to the same kinematic conditions as $\hat{A}$. In globally defined formulations (see [1, 31] for more details), transition functions $T_{ij}$ (where the indices $I, I'$ denote charts and are understood in most of this paper) glue together the locally defined configurations $(\hat{\Theta}_I, \hat{A}_I, \hat{\bar{J}}_I)$ as follows:

$$\hat{\Theta}_I = (\hat{T}_I)^{-1} \star \hat{\Phi}_I \star \pi (\hat{T}_I'), \quad \hat{\bar{A}}_I = (\hat{T}_I)^{-1} \star (\hat{A}_I + \hat{\bar{d}}) \star \hat{T}_I', \quad \hat{\bar{J}}_I = \hat{T}_I'.

(2.28)

The projection implied by the $\star$-multiplication by $\Omega_0^2$ can be solved locally on $\mathcal{C}$ by taking the master fields to be forms on $\mathcal{X} \times \mathcal{Z}$ valued in the algebra $\Omega^{(0)}(\mathcal{Y})$ of 0-forms on $\mathcal{Y}$. Thus

$$\hat{\bar{A}} = \hat{\bar{U}} + \hat{\bar{V}},$$

(2.29)

where $\hat{\bar{U}} = dx^\mu \hat{U}_\mu(x; Z; Y), \quad \hat{\bar{V}} = dZ^\nu \hat{V}_\nu(x; Z; Y) = dv^\mu \hat{V}_\mu(x; Z; Y) + dz^\mu \hat{V}_\mu(x; Z; Y)$, and the algebraic constraints (2.21) admit the solution

$$\hat{\bar{J}} = - \frac{i}{4} dv^\mu \wedge dz^\mu \hat{\kappa}, \quad \hat{\bar{J}} = - \frac{i}{4} d\tilde{z}^\mu \wedge d\tilde{z}^\mu \hat{\kappa}.$$

(2.30)

In order to find exact solutions, it is convenient to cast the remaining differential constraints into Vasiliev’s original deformed-oscillator format:

$$d\hat{\bar{U}} + \hat{\bar{U}} \star \hat{\bar{U}} = 0, \quad d\hat{\Phi} + \hat{\bar{U}} \star \hat{\Phi} = \hat{\Phi} \star \pi (\hat{\bar{U}}) = 0,$$

(2.31)

$$d\hat{\bar{\kappa}}_x + [\hat{\bar{\kappa}}, \hat{\bar{\kappa}}_x] = 0, \quad \hat{\bar{\kappa}} = \hat{\bar{U}} \star \hat{\bar{U}},$$

(2.32)

$$\hat{\bar{\kappa}} \star \hat{\Phi} + \hat{\Phi} \star \pi (\hat{\bar{\kappa}}) = 0, \quad \hat{\bar{\kappa}} \star \hat{\bar{\kappa}} = \hat{\Phi} \star \pi (\hat{\bar{\kappa}}) = 0,$$

(2.33)

$$[\hat{\bar{\kappa}}, \hat{\bar{\kappa}}]_\pi = -2i\epsilon_{\mu \rho} (1 - B \star \hat{\Phi} \star \kappa), \quad [\hat{\bar{\kappa}}, \hat{\bar{\kappa}}]_\pi = -2i\epsilon_{\mu \rho} (1 - B \star \hat{\Phi} \star \kappa),$$

(2.34)

$$[\hat{\bar{\kappa}}, \hat{\bar{\kappa}}]_\pi = 0,$$

(2.35)

where we have defined $d = dv^\mu \partial_\mu$ and

$$\hat{\bar{\kappa}} = \hat{\bar{\kappa}} + 2i\hat{\bar{V}} = (\hat{\bar{\kappa}} - 2i\hat{\bar{V}}, -\tilde{z}_a + 2i\hat{\bar{V}}).$$

(2.36)

The integrability of the system implies the gauge transformations

$$\delta_\epsilon \hat{\Phi} = -[\epsilon, \hat{\Phi}]_\pi, \quad \delta_\epsilon \hat{\bar{\kappa}} = -[\epsilon, \hat{\bar{\kappa}}]_\pi, \quad \delta_\epsilon \hat{\bar{\kappa}} = d\epsilon + [\hat{\bar{\kappa}}, \hat{\bar{\kappa}}]_\pi.$$

(2.37)
Manifest Lorentz-invariance and component fields. Manifest local Lorentz covariance can be achieved by means of the field redefinition \([5, 26, 31]\)

\[
\hat{W} := \hat{U} - \hat{K}, \quad \hat{K} := \frac{1}{4i}(\omega^{\alpha\beta} \hat{M}_{\alpha\beta} + \bar{\omega}^{\alpha\beta} \hat{\bar{M}}_{\alpha\beta}),
\]

(2.38)

where \((\omega^{\alpha\beta}, \bar{\omega}^{\alpha\beta})\) is the canonical Lorentz connection, and

\[
\hat{M}_{\alpha\beta} := \hat{M}_{\alpha\beta}^{(0)} + \hat{M}_{\alpha\beta}^{(S)}, \quad \hat{\bar{M}}_{\alpha\beta} := \hat{\bar{M}}_{\alpha\beta}^{(0)} + \hat{\bar{M}}_{\alpha\beta}^{(S)},
\]

(2.39)

are the full Lorentz generators, consisting of the internal part

\[
\hat{M}_{\alpha\beta}^{(0)} := \gamma(\alpha \ast \gamma(\beta) - z(\alpha \ast z(\beta)), \quad \hat{\bar{M}}_{\alpha\beta}^{(0)} := \bar{\gamma}(\alpha \ast \bar{\gamma}(\beta) - \bar{z}(\alpha \ast \bar{z}(\beta)),
\]

(2.40)

rotating the spinor indices carried by \((\hat{S}_{\alpha}, \hat{\bar{S}}_{\alpha})\). As a result, the master equations read

\[
\nabla \hat{W} + \hat{W} \ast \nabla + \frac{1}{4i}(r^{\alpha\beta} \hat{M}_{\alpha\beta} + \bar{r}^{\alpha\beta} \hat{\bar{M}}_{\alpha\beta}) = 0, \quad \nabla \hat{\Phi} + \hat{W} \ast \hat{\Phi} - \hat{\Phi} \ast \pi(\hat{W}) = 0,
\]

(2.42)

\[
\nabla \hat{S}_{\alpha} + \hat{W} \ast \hat{S}_{\alpha} - \hat{\bar{S}}_{\alpha} \ast \hat{W} = 0, \quad \nabla \hat{\bar{S}}_{\alpha} + \hat{W} \ast \hat{\bar{S}}_{\alpha} - \hat{S}_{\alpha} \ast \hat{W} = 0
\]

(2.43)

\[
[\hat{S}_{\alpha}, \hat{\bar{S}}_{\beta}]_{\ast} = -2ie_{\alpha\beta}(1 - \mathcal{B} \ast \hat{\Phi} \ast \hat{\pi}), \quad [\hat{\bar{S}}_{\alpha}, \hat{S}_{\beta}]_{\ast} = -2ie_{\alpha\beta}(1 - \mathcal{B} \ast \hat{\Phi} \ast \hat{\pi})
\]

(2.45)

where \(r^{\alpha\beta} := d\omega^{\alpha\beta} + \omega^{\alpha\gamma} \omega_{\gamma\beta}\) and \(\bar{r}^{\alpha\beta} := d\bar{\omega}^{\alpha\beta} + \bar{\omega}^{\alpha\gamma} \omega_{\gamma\beta}\), and

\[
\nabla \hat{W} := \nabla \hat{\Phi} := \frac{1}{4i}[\omega^{\alpha\beta} \hat{M}_{\alpha\beta} + \bar{\omega}^{\alpha\beta} \hat{\bar{M}}_{\alpha\beta}, \hat{W}],
\]

(2.47)

\[
[\hat{S}_{\alpha}, \hat{\bar{S}}_{\beta}]_{\ast} = \frac{1}{4i}[\omega^{\alpha\beta} \hat{M}_{\alpha\beta} + \bar{\omega}^{\alpha\beta} \hat{\bar{M}}_{\alpha\beta}, \hat{\Phi}],
\]

(2.48)

\[
[\hat{\bar{S}}_{\alpha}, \hat{S}_{\beta}]_{\ast} = \frac{1}{4i}[\omega^{\alpha\beta} \hat{M}_{\alpha\beta} + \bar{\omega}^{\alpha\beta} \hat{\bar{M}}_{\alpha\beta}, \hat{\bar{\Phi}}]
\]

(2.49)

\[
[\hat{\bar{S}}_{\alpha}, \hat{S}_{\beta}]_{\ast} = \frac{1}{4i}[\omega^{\alpha\beta} \hat{M}_{\alpha\beta} + \bar{\omega}^{\alpha\beta} \hat{\bar{M}}_{\alpha\beta}, \hat{\bar{\Phi}}]
\]

(2.50)

Besides their manifest local Lorentz symmetry, these equations are by construction also left invariant under the local shift-symmetry with parameter \((\zeta^{\alpha\beta}, \bar{\zeta}^{\alpha\beta}) = dx^{\alpha\beta}(\zeta_{\mu}^{\alpha\beta}, \bar{\zeta}_{\mu}^{\alpha\beta})\) acting such that

\[
\delta_{\zeta}(\hat{U}, \hat{\Phi}, \hat{S}_{\alpha}, \hat{\bar{S}}_{\alpha}) = 0, \quad \delta_{\zeta}(\omega^{\alpha\beta}, \bar{\omega}^{\alpha\beta}) = (\zeta^{\alpha\beta}, \bar{\zeta}^{\alpha\beta}) \Rightarrow \delta_{\zeta} \hat{W} = -\frac{1}{4i}(\zeta^{\alpha\beta} \hat{M}_{\alpha\beta} + \bar{\zeta}^{\alpha\beta} \hat{\bar{M}}_{\alpha\beta}).
\]

(2.51)

The canonical Lorentz connection can be embedded into the full theory by using the aforementioned shift-symmetry to impose

\[
\frac{\partial^2}{\partial y^\mu \partial y^\nu} \hat{W} \bigg|_{y=Z=0} = 0, \quad \frac{\partial^2}{\partial \bar{y}^\mu \partial \bar{y}^\nu} \hat{W} \bigg|_{y=Z=0} = 0.
\]

(2.52)

For the projection of equations (2.42)–(2.46) to manifestly generally covariant equations of motion for dynamical component fields in four-dimensional spacetime \(\mathcal{X}_d\), see the appendix D in [1]. In essence, after choosing a manifestly \(\hat{S}p(4; \mathbb{R})_{\text{diag}}\)-invariant ordering
scheme, eliminating the auxiliary fields related to the unfolded description on $X_0$ and $\mathcal{Z}$, and fixing suitable physical gauges (such as the universal twistor gauge condition $\hat{x}_\alpha \overline{\nabla}_a = 0$ and generalized holonomic gauges on $W_\mu$), there remains a set of dynamical fields consisting of a physical scalar field

$$\phi \equiv C := \Phi|_{Y=Z=0}, \quad (2.53)$$

which together with the self-dual Weyl tensors $C_{\alpha}(\mathcal{Z}, (s = 1))$ make up the generating function $(s \geq 0)$

$$C := \hat{\Phi}|_{Z=0, Y=0}, \quad C_{\alpha}(\mathcal{Z}) := \left. \frac{\partial^{2s}}{\partial \alpha_1 \ldots \partial \alpha_{2s}} C \right|_{Y=0}, \quad (2.54)$$

and a tower of manifestly Lorentz-covariant, symmetric and doubly traceless tensor gauge fields, or Fronsdal tensors, given by $(s \geq 1)$

$$\phi_{\mu(s)} := \left. \frac{\partial^{2s}}{\partial \alpha_1 \ldots \partial \alpha_{2s}} W_{\mu(s)} \right|_{Y=0}, \quad (2.55)$$

where $x^\mu$ are local coordinates on $X$ and

$$W := \hat{W}|_{Z=0} = \left( \hat{U} - \frac{1}{4i} (\omega^{\alpha (\cdot)} (\epsilon_{\alpha (\cdot) \beta} + \tilde{S}_{\alpha} \star \tilde{S}_{\beta}) + \tilde{\alpha}^{\alpha (\cdot)} (\epsilon_{\alpha (\cdot) \beta} + \tilde{S}_{\alpha} \star \tilde{S}_{\beta})) \right)|_{Z=0}. \quad (2.56)$$

### 3. Exact solutions

#### 3.1. Gauge function method and moduli space

Equations (2.31) and (2.32) can be solved (on a chart $\mathcal{C}_I$) by [27]

$$\hat{U}_I = \hat{L}_I^{-1} \star \hat{d} \hat{L}_I, \quad \hat{\Phi}_I = \hat{L}_I^{-1} \star \hat{\Phi}' \star \hat{\pi}(\hat{L}_I), \quad \hat{S}_{l(I)} = \hat{L}_I^{-1} \star \hat{S}_{l} \star \hat{L}_I, \quad (3.1)$$

where $\hat{L}_I(X, Y, Z)$ is a gauge function, assumed to obey

$$\hat{L}_I|_{X=Y=Z=0} = 1, \quad (3.2)$$

and $(\hat{\Phi}', \hat{S}_{l})$ are integration constants for the 0-forms on $X_0$ given by

$$(\hat{\Phi}', \hat{S}_{l}) = (\Theta, \bar{S}_{l})|_{X=0} \quad (3.3)$$

and obeying the remaining twistor-space equations

$$\hat{S}_{\alpha} \star \hat{\Phi}' + \hat{\Phi}' \star \hat{\pi}(\hat{S}_{\alpha}) = 0, \quad \hat{\Sigma}_\alpha \star \hat{\Phi}' + \hat{\Phi}' \star \hat{\pi}(\hat{S}_\alpha) = 0 \quad (3.4)$$

$$[\hat{S}_{\alpha}, \hat{\Sigma}_\rho],_\alpha = -2i\epsilon_{\alpha (\cdot)} (1 - B \star \hat{\Phi}' \star \kappa), \quad [\hat{\Sigma}_\rho, \hat{S}_\alpha],_\rho = -2i\epsilon_{\alpha (\cdot)} (1 - B \star \hat{\Phi}' \star \kappa) \quad (3.5)$$

$$[\hat{S}_{\alpha}, \hat{S}_{\rho}],_\alpha = 0. \quad (3.6)$$

Given a solution to these equations, the generating functions of the Weyl tensors and of the gauge fields respectively take the form $C_I = (\hat{L}_I^{-1} \star \hat{\Phi}' \star \hat{\pi}(\hat{L}_I))|_{Z=0, Y=0}$ and

$$W_I = \hat{L}_I^{-1} \left[ \hat{d} - \frac{1}{4i} (\omega^{\alpha (\cdot)} (\epsilon_{\alpha (\cdot) \beta} + \tilde{S}_{\alpha} \star \tilde{S}_{\beta}) + \tilde{\alpha}^{\alpha (\cdot)} (\epsilon_{\alpha (\cdot) \beta} + \tilde{S}_{\alpha} \star \tilde{S}_{\beta})) \right] \hat{L}_I \bigg|_{Z=0} \quad (3.7)$$

subject to (2.52), which serves to determine $(\omega^{\alpha (\cdot)} , \tilde{\alpha}^{\alpha (\cdot)})$.

In the following (omitting again the chart index $I$), we shall work with Gaussian gauge functions with a factorized form

$$\hat{L}(x|Y, Z) = L(x|Y) \star \hat{L}(x|Z), \quad (3.8)$$
realized as \(\phi\)-exponentials of bilinears in \(Y^a\) and \(Z^a\), respectively. The \(Y\)-dependent factor reconstructs spacetime and will be chosen such that the flat connection \(\Omega^{(0)} := L^{-1} \phi dL\) describes \(\text{AdS}_4\). One may choose \(L\) to be manifestly Lorentz covariant leading to \([21, 28, 29]\)

\[
L = \exp \left(4i\xi x^a P_a \right) = \frac{2h}{1 + h} \left[ \exp \left( \frac{4ix^a P_a}{1 + h} \right)_\text{Weyl} \right], \quad x^2 < 1, \quad x^2 := x^a x_a, \tag{3.9}
\]

\[
\xi := (1 - h^2)^{-\frac{1}{2}} \tanh^{-1} \sqrt{1 - x^2}, \quad h := \sqrt{1 - x^2}. \tag{3.10}
\]

The corresponding vacuum connection \(\Omega^{(0)}\) consists of the \(\text{AdS}_4\) vierbein \(e_{(0)}^{a\alpha} = -h^{-2}(\sigma^a)^{\alpha\beta} dx_\beta\) and the Lorentz connection \(\omega_{(0)}^{\alpha\beta} = -h^{-2}(\sigma^a)^{\alpha\beta} dx_\beta\), corresponding to presenting the metric in stereographic coordinates as \(ds^2_{(0)} = 4(1 - x^2)^{-2}\). On the other hand, allowing for a non-trivial \(Z\)-dependent factor fixes different gauges that may be helpful for studying the nature of certain singularities arising in some solutions that we shall review here. In order to illustrate how this issue arises, we shall begin with the trivial choice \(L(x|Z) = 1\).

A particular class of solutions, containing the exact solutions that we shall review here, admits perturbative expansions

\[
\Phi' = \sum_{n=0}^{\infty} \Phi'^{(n)}, \quad \hat{S}_a = \sum_{n=0}^{\infty} \hat{S}_a^{(n)} = Z_a - 2i \sum_{n=0}^{\infty} \hat{V}_a^{(n)}, \tag{3.11}
\]

where \((\hat{S}_a^{(n)}, \hat{V}_a^{(n)})\) are of the \(n\)th order in the integration constant \(\Phi'(Y) = \Phi'(Y, Z)|_{Z=0}\), and \(\hat{S}_a^{(0)}\) is a flat connection in twistor space obeying \([\hat{S}_a^{(0)}, \hat{V}_a^{(0)}]_z = -2i C_{a\beta\gamma}\).

Depending on the boundary conditions on \(\hat{S}_a^{(0)}\) in twistor space, there are various natural approaches to solving these equations: if the boundary conditions are chosen such that there exists a gauge where \(\hat{V}_a^{(0)} = 0\), one may adapt the perturbative scheme (see for example [1], appendix D) to the case at hand. The solutions we shall discuss in this paper are of the form (3.11) but are obtained by solving the deformed-oscillator problem (3.4)–(3.6) using separation of variables in \(Y \times Z\) space and the non-perturbative method of [19, 21] adapted to the present case in [1]. This method also encompasses non-trivial flat connections \(\hat{V}_a^{(0)}\), essentially by activating Fock-space projectors in the space of functions on \(Y \times Z\). The resulting solutions appear naturally in gauges that differ radically from the aforementioned universal twistor gauge in the sense that the space of residual symmetries is not isomorphic to \(\mathfrak{hs}(4)\) or its non-minimal extension, as we shall discuss below.

The space of solutions can thus be coordinatized by the following moduli (for a more detailed discussion on (iii) and (iv), see [11, 31]):

(i) local degrees of freedom contained in \(\Phi'(Y)\);
(ii) boundary degrees of freedom contained in \(\hat{L}|_{\partial C}\) where \(\partial C\) in particular contains the boundary of its four-dimensional spacetime sub-manifold;
(iii) monodromies and projectors contained in flat connections \(\hat{V}_a^{(0)}\) on \(Z \times Y\) and \(\hat{U}^{(0)}\) on \(T^* X\); 
(iv) windings contained in the transition functions \(\hat{T}_I^Z\) between charts of the correspondence space.

In what follows, we shall mainly activate (i), (ii) and to some extent (iii), while we shall briefly discuss the possibility (iv) that more than one chart is required in reference to one particular family of solutions.

8 The metric remains well defined for \(x^2 > 1\) such that the regions \(x^2 < 1\) and \(x^2 > 1\) together yield a single cover of \(\text{AdS}_4\). For relations to global embedding coordinates and global spherically symmetric coordinates, see appendix A in [1].
3.2. Solutions with spherical, cylindrical and biaxial symmetry

Six infinite families of exact solutions admitting at least two commuting Killing vectors have been found in [1] by extending the projector ansatz used in [2] and combining it with the gauge function method. All of them can be obtained by solving the internal Z-space equations via the expansions

\[ \tilde{\Phi} = \sum_n v_n P_n(Y) \ast \kappa_n, \]  
\[ \tilde{S}_a = z_a - 2i \sum_n P_n(Y) \ast (V_a)_n(z), \quad \tilde{S}_\bar{a} = \bar{z}_a - 2i \sum_n P_n(Y) \ast (\bar{V}_\bar{a})_n(\bar{z}), \]  

where \( \pi_n((V_a)_n) = -(V_a)_n, \pi_n((\bar{V}_\bar{a})_n) = -(\bar{V}_\bar{a})_n \). \( v_n \) are \( a \ priori \) complex constant deformation parameters and \( P_n = \pi \tilde{\pi}(P_n) \) are projectors labelled by the (discrete) occupation numbers \( n := (n_1, n_2) \) and assumed to obey

\[ P_n \ast P_m = \delta_{nm} P_n, \]  

and to form a set that is invariant under the operations \( \pi, \hat{\tau} \) and \( \ast \)-multiplication by \( \kappa_n \tilde{\kappa}_n \), such that

\[ \pi(P_n) =: P_{\pi(n)}, \quad (P_n)^\dagger =: P_{\dagger(n)}, \quad \tau(P_n) =: P_{\tau(n)}, \]  

\[ P_n \ast \kappa_n \tilde{\kappa}_n =: \kappa_n P_n, \]  

with \( \pi^2(n) = \hat{\tau}^2(n) = n \) and \( (\kappa_n)^2 = 1 \). The reality conditions fix the real or imaginary nature of the deformation parameters (which depends on \( n \)) [1].

This solution space\(^9\) forms an associative subalgebra of the \( \ast \)-product algebra [1]. Defining

\[ (\Sigma^n_w)_n := Z_w - 2i(V_w)_n, \quad B \equiv \exp i\theta[\tilde{\Phi} \ast \pi(\tilde{\Phi})] =: \sum_n P_n \ast B_n, \]  

and using the factorization property (2.13), one can show that (i) the orthogonality of projectors splits the internal equations (3.4)–(3.6) into separate reduced deformed-oscillator problems for every \( n \); (ii) assuming \( v_n = \text{const} \) and \( \pi_n((\Sigma^n_w)_n) = -(\Sigma^n_w)_n \) solves (3.4) identically; and (iii) the holomorphicity of the reduced deformed oscillators \((\Sigma^n_w)_n \) solves (3.6) identically. We are therefore left with

\[ [\Sigma^n_a, \Sigma^n_{\bar{a}}]_s = -2i\epsilon_{\alpha\beta}(1 - v_n B_n \kappa_n), \]  
\[ [\bar{\Sigma}_a^n, \bar{\Sigma}_\bar{a}^n]_s = -2i\epsilon_{\alpha\beta}(1 - \kappa_n \bar{v}_n \bar{B}_n \bar{\kappa}_\bar{n}), \]  

which are defined modulo the residual holomorphic gauge transformations

\[ \delta_{\alpha} \Sigma^n_a = [\Sigma^n_a, \epsilon^n_\alpha]_s, \quad \delta_{\alpha} \bar{\Sigma}^n_{\bar{a}} = [\bar{\Sigma}^n_{\bar{a}}, \bar{\epsilon}^n_\alpha]_s, \]  
\[ \tilde{\partial}_\alpha \epsilon^n_\alpha = 0, \quad \bar{\partial}_\alpha \bar{\epsilon}^n_\alpha = 0. \]  

\(^9\) More generally, one can consider expansions over generalized, non-diagonal projectors \( P_{nm} \sim [n|m] \) (in this notation \( P_n := P_{nn} \)). However, the latter lie along gauge orbits that can be reached from the diagonal solutions here considered [1] (barring subtleties related to the admissibility of the corresponding gauge transformations). Moreover, while one may also allow for Z-dependent coefficients \( \Phi_n(Z) \) for the expansion of the Weyl 0-form (3.12) and for non-holomorphic coefficients \((V_w)_n(Z) \) in (3.13), it is possible to show perturbatively in the initial datum \( v_n := \Phi_n|_{Z=0} \) that one can always land on the forms (3.12) and (3.13) via a partial gauge fixing (see [1] for details).
Reduced deformed oscillators. These equations can be solved exactly by adapting the \( \circ \)-product method of [19], later refined in [21] (see also [22]). A crucial difference with respect to the solutions found in those papers is that the deformation terms on the rhs of (3.18) and (3.19) are distributional on \( Z \), admitting the limit representation

\[
2\pi \delta^2(z) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} e^{-i z \epsilon w} = \sigma_k z, \quad \text{where } \sigma \text{ is a sign and a spin frame } u^\sigma_a \text{ (in the terminology of [25], see also [1]) has been introduced (}u^a_{\mu \nu} = 1\text{) in order to split } z^\pm := u^\pm a z_\sigma, \quad w^\pm := z^\pm w^\pm, \quad [z^-, z^+] = -2i. \text{ Clearly, such splitting also breaks the manifest } SL(2; \mathbb{C}) \text{ covariance. Correspondingly, splitting}
\]

\[
\Sigma^n(z) = u^{-} \Sigma^{n+}(z) - u^{+} \Sigma^{-}(z), \quad [\Sigma^{n-}, \Sigma^{n+}] = -2i(1 - B_n v_n k_z),
\]

and representing \((\Sigma^n_a(\zeta), \tilde{\Sigma}^n_a(\tilde{\zeta}))\) by the generalized Laplace transforms

\[
\Sigma^{\pm n} \equiv u^{\pm a} \Sigma^{n\pm}(z) = 4z^\pm \int_{-1}^{1} \frac{dt}{(t + 1)^2} f^{\pm n}_n(t) e^{i \eta \kappa n t + i w},
\]

where \((\sigma_n)^2 = 1\) can be chosen independently for each \( n \), one can show that (3.18) and (3.19) are solved provided the transforms \( f^{n\pm}_n(t) \) obey the integral equations

\[
(f^{n-}_n \circ f^{n+}_n)(t) = \delta(t - 1) - \sigma_n B_n v_n, \quad (3.24)
\]

defines a commutative and associative product on the space of functions on the unit interval. Note that, different from the Lorentz-invariant case [21, 22], (3.24) constrains the \( \circ \)-product of two functions and not the square of a single one, and as a consequence its solution space is parameterized by an undetermined function. One can show however that the latter is a gauge artefact (i.e., can be completely accounted for by the residual gauge symmetry (3.20), see [1]). One natural gauge choice is to work with symmetric solutions\(^{11}\) \( f^+_n = f^-_n = f_n \), and we shall therefore drop the \( \pm \) referring to the spin frame henceforth whenever not necessary. The solution in the holomorphic sector reads

\[
\Sigma^n_a = z^a \left( 1 - 2i \int_{-1}^{1} \frac{dt}{(t + 1)^2} j_n(t) e^{i \eta \kappa n t + i w} \right),
\]

\[
\begin{align*}
    j_n(t) &= q_n(t) - 2 \sum_{k=0}^{\infty} \theta_0 k \left[ 1 - \frac{1 + (-1)^k}{2} \right] \left( 1 - \sqrt{1 - \frac{\sigma_n B_n v_n}{1 + k}} \right) p_k(t), \\
    q_n(t) &= -\frac{\sigma_n B_n v_n}{4} \left[ \frac{1}{2} + 2 \left( \frac{\sigma_n B_n v_n}{2} \log \frac{1}{2} \right) \right] \log \frac{1}{2}, \quad p_k(t) \equiv \left( \frac{-1}{k!} \right)^k \delta^{(k)}(t),
\end{align*}
\]

where \( \sigma_n \in \{ \pm 1 \} \) and \( \theta_0 k \in [0, 1] \). The second term in \( j_n(t) \) corresponds to the contribution of \( \circ \)-product projectors \( p_k \circ p_l = \delta_{kl} p_k \) that activate a flat yet non-trivial part of the Z-space connection \( V^\alpha_{\mu \nu} \) that survives in the limit \( v_n \to 0 \) and receives \( v_n \)-dependent corrections (see [22, 1] for more details). Such Z-space vacua are parameterized via the discrete parameters \( \theta_0 k \), that therefore represent true independent moduli along with the continuous deformation

---

\(^{10}\) Whether or not \( \kappa_z \) is singular depends on the ordering prescription one is working with. Indeed, as shown in [1], while singular in Weyl ordering, \( \kappa_z \) assumes a regular (Gaussian) form in normal ordering, and one can in fact solve for \((\Sigma^n_a, \tilde{\Sigma}^n_a)\) in normal ordering and then recast the so-obtained expressions of the deformed oscillators in Weyl ordering. The two procedures lead to the same result.

\(^{11}\) Interestingly, the transformation to the most-asymmetric gauge choice, in which one sets, say, \( f^+_n = 1 \) and hence \( f^-_n = 1 - \sigma_n B_n v_n \), also shows that in the latter the linear-in-\( v_n \) correction in the deformed oscillators is actually exact (as happens in the gauge used in [2]).
parameters $v_n$. Note that, independent of the values of $\theta_{n,k}$, the branch-cut in (3.27) can be chosen such that the internal connection is analytic for $\text{Re}(\sigma_5 B_n v_n) < 1$, where also the particular solution can be shown to be real analytic [21]. We shall only examine in the following, the case where all $\theta_{n,k} = 0$. Finally, we note that the singularity (both in the $t$-measure and in the exponent) at $t = -1$ in the integral representation of the reduced deformed oscillators $\Sigma^\alpha_n$ in (3.26) is precisely what reproduces the delta-function-like source $\kappa_i$ [1] from their $\star$-commutator. As a result, there is an algebraic singularity in $Z$-space in $\Sigma^\alpha_n$, viz $u^{\alpha\beta} \Sigma^\alpha_n \sim 1/z^\alpha$ as $z^\alpha \sim 0$ [1]. Thus, in order to compute $\star$-products involving the $\Sigma^\alpha_n$, it is assumed that these are defined by exchanging the order of auxiliary integrals, such that one first performs the integrals over twistor spaces and then the $t$-integrals.

**Projector algebras.** It is possible to build the rank-1 projectors $P_{n_1, n_2}$, on which the ansatz (3.12)–(3.13) is based, starting from two commuting $\mathfrak{sp}(4, \mathbb{C})$ elements, which we shall denote $K^{(\pm)} = \frac{1}{8} \kappa_i^{(\pm)} Y^2 \star Y_2 \frac{1}{2} (w_2 \pm w_1)$, generating $\mathfrak{g} \equiv \mathfrak{so}(2)_{(+) \oplus \mathfrak{so}(2)_{(-)} \subset \mathfrak{sp}(4, \mathbb{C})$. They obey ($n_i \in \mathbb{Z} + \frac{1}{2}$)

$$P_{n_1, n_2} \star P_{n'_1, n'_2} = \delta_{n_1, n'_1} \delta_{n_2, n'_2} P_{n_1, n_2}, \quad (w_i - n_i) \star P_{n_1, n_2} = 0,$$  
(3.29)

and admit the Weyl-ordered integral presentation ($e_i := n_i/|n_i|$)

$$P_{n_1, n_2} = 4(-1)\sum_{|n_i|=1} \oint_{C(e_i)} \frac{ds_1}{2 \pi i} \frac{(s_1 + 1)^{n_1 + 1} - 1}{2} \oint_{C(e_2)} \frac{ds_2}{2 \pi i} \frac{(s_2 + 1)^{n_2 + 1} - 1}{2} e^{-2 \sum_{i} s_i w_i}$$  
(3.30)

$$= 4(-1)^{|n_1|-1} e^{-2(w_1+w_2)} L_{n_1 - \frac{1}{2}} (4w_1) L_{n_2 - \frac{1}{2}} (4w_2),$$  
(3.31)

where $C(e_i)$ are small contours encircling $e_i$. With the exception of the (anti-)ground-state projectors ($q = e_1 e_2$) $P_{e_1 e_2} (K_{(q)})$, which depend only on $K_{(+)}$ or only on $K_{(-)}$ and are therefore symmetric under the entire centralizer $\mathfrak{g}_{\mathfrak{sp}(4, \mathbb{C})}(K_{(q)})$, these projectors depend on both $K_{(+)}$ and $K_{(-)}$ and are hence $\mathfrak{g}$-invariant. We therefore refer to the latter and to the solutions built on them as being biaxially symmetric (or axisymmetric). In order for the exponential in (3.30) to give rise to a projector under $\star$-product it is crucial that the $\mathfrak{sp}(4, \mathbb{C})$ matrix $K^{(\pm)}_{ab}$ satisfies

$$(K_{(q)})_{a}^{\hat{\beta}} (K_{(q)})_{\hat{\beta} c} = -C_{a \beta \gamma}.$$  
(3.32)

This constraint leaves, as possible choices of two such commuting generators that are inequivalent up to $\mathfrak{sp}(4, \mathbb{R})$ rotations, the following Cartan pairs:

$$(E, J), \quad (J, iB), \quad (iB, iP),$$  
(3.33)

where $E := P_0 = M_{00}$ is the AdS energy, $J := M_{12}$ is a spin, $B := M_{03}$ is a boost and $P := P_1 = M_{01}$ is a translation. Each pair gives rise to two families of solutions that are distinguished by the choice of the Cartan generator that the ground-state projector depends on, and which we will refer to in the following as principal Cartan generator. The resulting six families can be therefore labelled as

$$\mathcal{M}_E(E, J), \quad \mathcal{M}_J(E, J); \quad \mathcal{M}_J(J, B), \quad \mathcal{M}_B(J, B); \quad \mathcal{M}_B(B, P), \quad \mathcal{M}_P(B, P).$$  
(3.34)

When $K_{(+)}$ is principal, i.e. for the family $\mathcal{M}_{K_{(a)}}(K_{(+)}, K_{(-)})$, the corresponding projectors have Cartan-eigenvalues such that $|K_{(+)}| > |K_{(-)}|$, while the opposite inequality $|K_{(+)}| < |K_{(-)}|$ holds when $K_{(-)}$ is principal.

---

12 We denote with $M_{ab}$ ($A, B = 0, 1, 2, 3$) the $\mathfrak{so}(3, 2)$ generators, that admit a realization as bilinears in $\gamma$ as $M_{ab} = -\frac{1}{4} (\Gamma_{ab})_{\gamma \delta} \gamma^\epsilon \star \gamma^\delta$ (where $(\Gamma_{ab})_{\gamma \delta}$ are Dirac’s gamma matrices) and can be split into Lorentz rotations $M_{\alpha \beta}$ and AdS translations $P_a$, $a, b = 0, ..., 3$. We refer the reader to appendix A in [1] for our AdS$_4$ and spinor conventions.
Each family contains a subset of solutions that possess the same symmetry under \( \varepsilon_{sp(4,\mathbb{R})}(K_{(q)}) \) of the ground state, i.e. one of the two \( \mathfrak{so}(2) \) symmetries enhances either \( \mathfrak{so}(3) \) or \( \mathfrak{so}(2,1) \). Such symmetry-enhanced solutions result from summing all axisymmetric projectors with a fixed eigenvalue of the principal Cartan generator in such a way that the dependence on the other Cartan generator drops out and one is left with the enhanced projectors \( (n = \pm 1, \pm 2, \ldots) \)

\[
P_n(K_{(q)}) = \sum_{n_1 + q n_2 = n \ , \ \varepsilon_1 \varepsilon_2 = q} P_{n_1,n_2} = 4(-1)^{n-1} \sqrt{2} e^{-4K_{(2)}} L_n^{(1)}(8K_{(q)}) \tag{3.35}
\]

or

\[
2(-1)^{n-1} \frac{1}{2} \oint \frac{d\eta}{2i} \left( \frac{n + 1}{n - 1} \right)^n e^{-4K_{(q)}} \tag{3.36}
\]

that only depend on the principal Cartan generator.

Thus, for each family the ansatz (3.12) for the Weyl 0-form corresponds to choosing functions on \( \mathcal{Y} \) that can be diagonalized over bases of eigenstates \( |n\rangle \) of the Cartan pairs (aside from the \( \star \)-multiplication with \( K_\gamma \)). These solution spaces are coordinatized by deformation parameters \( \nu_n \) representing the eigenvalues of the Weyl 0-form master field in the aforementioned bases. As we shall soon see, the principal Cartan generator not only determines the symmetry of the ground-state solution (and of the symmetry-enhanced subset of solutions), but also determines the spacetime behaviour of the Weyl tensors.

It is interesting also to note that the \( \mathcal{M}_E(E,J) \) family of solutions is based on projectors on scalar and spinor singleton states [29], i.e. non-polynomial elements that are enveloping-algebra realizations of the states of boundary conformal scalar and spinor fields.

While the integral and non-integral presentations of the projectors given in equations (3.30)–(3.31) and (3.35)–(3.36) are equivalent as long as \( \star \)-products among projectors with quantum numbers of the same sign are considered, the integral presentation ensures the orthogonality conditions in (3.29) (simply via a change of variable, see appendix F in [1]) while preserving associativity also in the case that \( \varepsilon_1 \varepsilon_2 = -1 \) or \( \varepsilon_2 \varepsilon_3 = -1 \), whereas the non-integral one gives rise to divergencies. The integral presentation (3.30) and (3.35) is therefore required whenever reality or kinematical conditions (such as the minimal model truncation) force projectors with opposite quantum numbers in the expansions of the master fields.

**Weyl 0-form master field.** Using the gauge function (3.8), the \( L \)-rotation of the Weyl 0-form master field in (3.1) gets the following form:

\[
\tilde{\Phi}(x|Y, Z) = \sum_n \nu_n P_n^L(Y) \star \kappa_n , \tag{3.37}
\]

where we use the notation \( P_n^L(Y) \equiv L^{-1}(x|Y) \star P_n(Y) \star L(x|Y) \). The conjugation with \( L \) induces an \( x \)-dependent rotation \( \left( K_{(q)}^{L(\alpha)} \right)_{\alpha \beta} (x) = L_{\alpha \gamma}^\beta (x) L_{\gamma \delta}^\epsilon (x) (K_{(q)})_{\epsilon \delta} \) of the Cartan matrices in the exponent of (3.30) or (3.36) that preserves the conditions \( K^{(+) \epsilon}_{(+) \delta} = 0 \) and \( (K_{(q)}^{L(\alpha)})_{\alpha \beta} = -C_{\alpha \beta} \). Each \( (K_{(q)}^{L(\alpha)})_{\alpha \beta} \) is a complexified \( \text{AdS}_4 \) global symmetry parameter satisfying \( D^{(0)}K_{(4)}^{L(\alpha)} \bigg|_{\alpha \beta} = 0 \) and admitting the \( SL(2,\mathbb{C}) \)-decomposition [23]

\[
K_{\alpha \beta}^{L(\alpha)} = \begin{pmatrix}
L_{\alpha \beta}^L & \nu_{\alpha \beta}^L \\
\bar{\nu}_{\alpha \beta}^L & \bar{L}_{\alpha \beta}^L
\end{pmatrix} , \quad \nu_{\alpha \beta}^L = \bar{v}_{\beta \alpha}^L , \tag{3.38}
\]

yielding a complexified \( \text{AdS}_4 \) Killing vector \( v_{\alpha \beta}^L (x) = \bar{v}^L_{\beta \alpha} (x) \) and the self-dual and anti-self-dual components \( x_{\alpha \beta}^L (x) \) and \( \bar{x}_{\alpha \beta}^L (x) \), respectively, of the corresponding Killing 2-form \( x_{\mu \nu}^{(L)} := \nabla_\mu v_{\nu}^L \).
Taking the $\cdot$-product with $\kappa$, and restricting our attention, for simplicity, to the symmetry-enhanced projectors$^{13}$ (3.36), the resulting Weyl 0-form reads

$$\hat{\Phi} = \frac{2}{\sqrt{(\kappa^{L}_{(\eta)})^2}} \sum_{n=\pm 1, \pm 2, \ldots} (-1)^n \frac{1}{\eta} v_n \int_{C(\nu)} \frac{d\eta}{2\pi i \eta (\eta + 1)^n} \left\{ \frac{1}{n} \left[ \bar{y}^{\mu} \left( \kappa^{L}_{(\eta)} \right)_{\alpha \beta}^{-1} \bar{y}^{\gamma} + \frac{1}{2} \bar{y}^{\mu} \left( \hat{\kappa}^{L}_{(\eta)} \right)_{\alpha \beta}^{-1} \bar{y}^{\gamma} + iy^{\mu} \bar{y}^{\nu} \left( \kappa^{L}_{(\eta)} \right)_{\alpha \beta}^{-1} \left( v^{L}_{(\eta)} \right)_{\gamma}^{\beta} \right] \right\},$$

(3.39)

where we denote (suppressing all the other labels) $\kappa := \frac{1}{2} \kappa^{\mu \beta} \kappa_{\alpha \beta}$ and where $\kappa^{\alpha \beta}_{\eta} := -\kappa_{\alpha \beta} / \kappa^2$ and $\varepsilon := n/|n|$. Note the dependence on the inverse square root of $(\kappa^{L}_{(\eta)})^2$ appearing in the prefactor and in the exponent (through $(\kappa^{L}_{(\eta)})_{\eta}^{-1}$). For the solutions based on $\pi$-odd principal Cartan generators ($E$ and $iP$), for which the $x$-independent $K^{(\eta)}_{(\eta)}$ matrix is off-diagonal and $(\kappa^{L}_{(\eta)})_{\alpha \beta} = 0$, the $x$-independent Weyl master 0-form $\hat{\Phi}$ has a delta-function-like behaviour in twistor space. The latter is thus softened by the spacetime dependence introduced via the gauge function, and in particular, $\sqrt{(\kappa^{L}_{(\eta)})^2}$ appears as the parameter of a limit representation of the delta function. From (3.39) it is also clear that the principal Cartan generator $K^{(\eta)}_{(\eta)}$ specifies, essentially through the determinant of its upper diagonal block $(\kappa^{L}_{(\eta)})^2$, the spacetime behaviour and the singularities of every spin-$s$ Weyl tensor. Note however that the singular behaviour of the individual Weyl tensors is not a higher-spin-invariant statement, and that even a singular, delta-function-like behaviour of the master field in some spacetime point can be an artefact of the chosen ordering prescription$^{14}$ (which is also a consequence of the fact that the fibre space that the master fields are valued in is infinite dimensional). We shall later examine this issue at the level of certain higher-spin-invariant quantities.

Let us now specialize (3.39) to the case of solutions based on the spherically symmetric projectors $P_n(E)$. One can show that, starting from $K_{(1)} = E$ (i.e., $K^{(\eta)}_{(\eta)} = (\Gamma_0)_{\alpha \beta}$), the rotation with the gauge function (3.38) gives rise to non-vanishing diagonal blocks with determinant $(\kappa^{L})^2 = -\kappa^2$, where $r$ is the radial coordinate in the AdS$_4$ spherical coordinate system. The reality conditions on the master fields in this case require that $v_n = \bar{v}^{\mu} \mu_\nu$, where $\mu_\nu \in \mathbb{R}$. For any fixed projector $P_n(E)$, expanding in $y$ the Weyl 0-form and performing the auxiliary integration yields the physical scalar ($s = 0$) and an infinite tower of spherically symmetric type-D $^{24, 25}$ Weyl tensors of spin $s \geq 1$ of the form (up to real $n$-dependent numerical factors)

$$C^{(n)}_{\alpha(2s)} \sim \frac{\bar{e}^{\nu-1} \mu_\rho}{r^{s+1}} (\bar{u}^{\mu} \bar{u}^\nu)^\rho_{\nu}^{\nu} \bar{u}_{(\alpha(2s))},$$

(4.40)

where $(\bar{u}_\mu, \bar{u}_\nu)$ are eigenspinors of $\kappa^{L}_{\eta}$ generating a spin frame at every spacetime point where $(\kappa^{L})^2 \neq 0$ (and are only dependent on the angular variables $(\theta, \phi)$) [1]. As first noted in the case $n = 1$ in [2], the spin-2 Weyl tensor coincides with that of an AdS$_4$-Schwarzschild black hole of mass $\mu_\nu$, which here appears together with infinitely many partners of all integer spins. However, examining individual Weyl tensors only makes sense asymptotically (i.e. for $r \rightarrow \infty$), where they are all small and as a result fields of different spin are weakly coupled. In strong-field regions, and in particular in the proximity of the apparent singularity in $r = 0$, the higher spin symmetry is fully realized and one should rather examine higher-spin-invariant quantities.

There are a few observations that one can make from (4.40). First, we note that the Weyl 0-form components are real for $n$ odd (i.e. for solutions built on projectors $P_n(E)$ over

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$^{13}$ See [1] for details of the Weyl 0-form in the axisymmetric case.

$^{14}$ See, for example, the comment in footnote 10.
combinations of states belonging to the scalar singleton representation and imaginary for \( n \) even (i.e., for solutions based on spinor singleton projectors). On the linearized higher spin equations in the parity preserving case, the gauge-field curvatures are equated to the above-derived Weyl 0-form components up to an extra factor \( b = 1 \) (type-A model) or \( b = i \) (type-B model). Therefore, one can in this sense regard the deformation parameters of the solutions based on the scalar singleton as generalized electric charges (or generalized masses) in the type-A model and generalized magnetic charges (or generalized NUT charges) in the type-B model, and, conversely, those of the solutions based on the spinor singleton as magnetic-like charges in the type-A model and electric-like charges in the type-B model, and the two models seem to be connected via a generalized electric/magnetic duality.

Second, any solution based on a single projector is activated by a single deformation parameter that switches on the entire tower of Weyl tensors of all spins (and, in fact, interestingly enough the Didenko–Vasiliev solution, based on \( \mathcal{P}_l(E) \) alone, appears to be extremal [21]). On the other hand, building the solutions over a generic expansion in projectors opens up the possibility of diagonalizing \( \Phi \) with respect to the spin-\( s \) instead of the occupation number \( n \), thus having different deformation parameters \( \mathcal{M}_s \) in front of the spin-\( s \) Weyl tensors that are sums of the \( v_\nu \) with spin-dependent coefficients obtained from the contour integration,

\[
\mathcal{M}_s = N_s \sum_{n} (-1)^{n-\frac{1+4s}{2}} v_\nu \oint_{C(\epsilon)} \frac{d\eta}{2\pi i\eta^{s+1}} \left( \frac{\eta + 1}{\eta - 1} \right)^n, \tag{3.41}
\]

where \( N_s \) is an \( s \)-dependent normalization coefficient.

While identifying \( \mathcal{M}_s \) with a spin-\( s \) asymptotic charge may be very tempting, it is complicated by the fact that these solutions have been obtained in a gauge in which the 1-form field strengths are nonlinear in \( v_\nu \) [1]: as a consequence, the Weyl 0-form components (3.40) may differ from the linearized spin-\( s \) gauge-field curvatures asymptotically by nonlinear lower spin constructs that may give contributions of the same order in the limit \( r \to \infty \) and modify the proposed asymptotic charge by nonlinear terms in the \( \mathcal{M}_s \) parameters. An additional complication arises from the fact that the gauge we work with is not the standard, universal twistor gauge of the perturbative analysis mentioned at the end of section 2, as we shall comment more on in the following section.

Finally, specializing (3.39) to projectors \( \mathcal{P}_n(J) \) that only depend on \( K_{(-)} = J \) (i.e., \( K_{(-)}^{\alpha\beta} = (\Gamma_{12})_{\alpha\beta} \)), one can check that the rotation with the gauge function (3.8) modifies the already non-vanishing determinant of the diagonal blocks as (\( \chi^L = (\hat{\chi}^L)^2 = 1 + r^2 \sin^2 \theta \)), giving rise to a tower of Weyl 0-form components exhibiting cylindrical symmetry of the form

\[
C^{(n)}_{\nu(2s)} \sim \frac{i^{n+s+1}\mu_n}{\left(1 + r^2 \sin^2 \theta \right)^{-s}} \, \left( \bar{u}^\nu \bar{u}^\nu \right)^s_{\nu(2s)}. \tag{3.42}
\]

Note that such Weyl tensors do not blow up anywhere and do not vanish at spatial infinity (they are constant along the \( z \)-axis, with a behaviour similar to that of the Melvin solution in general relativity [32]). Moreover, since the Killing 2-form is imaginary, for every fixed \( n \) the electric/magnetic type of the type-D Weyl tensors flips according to whether the spin is even/odd, for \( n \) odd, vice versa for \( n \) even. These solutions are \( \mathfrak{so}(2)_J \oplus \mathfrak{so}(2, 1)_{E,M,\alpha,\beta} \)-symmetric, and are built on the spacelike AdS Killing vector \( \partial / \partial \phi \) in the same way as the spherically symmetric ones are based on the timelike vector \( \partial / \partial t \), i.e., \( \mathfrak{so}(2, 1) \) is the stability subalgebra of \( \partial / \partial \phi \). In other words, here the roles of \( E \) and \( J \) are exchanged, with respect to the rotationally invariant case, and the corresponding solutions are based on projectors onto combination of states belonging to non-unitary analogues of the (anti-)super-singleton of fixed \( J \) and vanishing energy.
Deformed oscillators and gauge fields. We now turn to examining the spacetime-dependent deformed oscillators. Defining \( \hat{S}_{\pm}^\pm := u^{k\alpha} \hat{S}_{\alpha}^x \), one has

\[
\hat{S}_{\pm}^\pm := (L)^{-1} \star \hat{S}_{\pm}^\pm \star L = \varepsilon_{\pm} - 2i \sum_n p_n^L \star v_n^\pm,
\]

where we recall that \( V_{n\pm} \) denotes the second term in (3.26), and the comment made below (3.26) concerning how to compute \( \star \)-products involving this element, given its algebraic singularity at \( z^\pm \approx 0 \). As explicitly shown in [1], the \( \star \)-product of the latter with the \( L \)-rotated, \( x \)-dependent projector \( P_n^L \) pushes such singular points outside the integration interval for generic values of \( x \). In short, this happens because the \( \star \)-product with \( P_n^L \) gives rise to a Gaussian determinant depending on \( \chi_{\mu}^{\alpha} \) that effectively shifts the potential singularities in the integral representation of \( \hat{S}_{\pm}^\pm \) to the zeros of

\[
(t+1)^2 - i\sigma_n (t^2-1) \chi_{\alpha \beta}^{\pm} (u_n^\alpha u_n^\beta + u_n^\alpha u_n^\beta) + (\chi_{\alpha}^{\pm})^2 (t-1)^2
\]

\[
= (t+1)^2 + 2i\sigma_n e_2 \Theta (t^2 - 1) \cos \theta - \Theta^2 (t-1)^2
\]

\[
= ((1 + i\sigma_n e_2 \Theta) (t+1) - 2i\sigma_n e_2 \Theta)^2 + 2i\sigma_n e_2 \Theta (t^2 - 1) \cos \theta - 1,
\]

and this same shift takes place at the denominator in the exponent. Above, we have denoted \( \Theta^2 := - (\chi_{\alpha}^{\pm})^2 \), and the factor of \( \cos \theta \) (where \( \theta \) is the polar angle in spherical coordinates) enters via the contraction of the \( x \)-dependent eigenspinors of \( (\chi_{\alpha}^{\pm})_\nu^\beta \) with the rigid spin frame \( (u_n^\alpha, u_n^\beta) \). The crux of the matter is that at any point \( x \) in which \( \Theta \) is real and non-vanishing, the singularity in the deformed oscillators acquires an imaginary part and is therefore pushed out of the integration domain, provided that \( \theta \neq \pi/2 \). This is what happens, in particular, for the spherically symmetric solutions.

Recalling that the singularities in the deformed oscillators are inherited by the gauge fields via (3.7), it is important to clarify at this point whether this singular behaviour at the equator is physical or not. However, the fact that it comes from a point-wise non-collinearity of two spin frames suggests that it should be pure gauge. Indeed, one can modify the gauge function (3.8) as follows:

\[
\hat{L}_{(K)}(x|Y, Z) = L(x|Y) \star \hat{L}_{(K)}(x|Z),
\]

with a non-trivial \( Z \)-dependent factor \( \hat{L}_{(K)} : R^4 \rightarrow SL(2; \mathbb{C})/CSL(2; \mathbb{C})(K^L) \) (where \( CSL(2; \mathbb{C})(K^L) \) is the centralizer of the principal Cartan generator, that we shall here simply denote as \( K \) that aligns the spin frame of \( Z \) with the spin frame of \( Y \) generated by the eigenspinors of \( (\chi_{\alpha})_\nu^\beta \) (hence the labelling with the relevant global symmetry parameter \( K \)). Substituting \( L \) with \( \hat{L}_{(K)} \) in (3.43), one obtains

\[
\hat{S}_{(K)}^\pm = (\hat{L}_{(K)})^{-1} \star \host{S}_{\pm}^\pm \star \hat{L}_{(K)} = \hat{z}_{(K)}^\pm - 2i \sum_n p_n^L \star \hat{v}_{(K)}^\pm,
\]

where now \( \hat{z}_{(K)}^\pm := (\hat{L}_{(K)})^{-1} \star \hat{z}_{(K)}^\pm \star \hat{L}_{(K)} = \hat{a}_{(K)}^\alpha u_n^\alpha \), where we denote with \( (\hat{a}_{(K)}^\alpha, \hat{u}_n^\alpha)_{(K)} \) the \( x^i \)-eigenspinors, and

\[
\hat{v}_{(K)}^\pm = 2i\hat{z}_{(K)}^\pm \int_{-1}^1 \frac{dt}{(t+1)^2} t^2 e^{i\hat{a}_{(K)}^\alpha u_n^\alpha}.
\]

After one computes the \( \star \)-product of the latter with the projectors, recalling the prescription in the discussion following (3.26), the shift (3.44) takes place, but the alignment between the spin frames now sets cos \( \theta = 1 \). This has two consequences.

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15 To be precise, this is the shift after the contour-integration (entering through the projector) has been carried out. If one keeps the contour integral, some of the terms in (3.44) will also contain powers of the auxiliary integration variables \( x_1, x_2 \) or \( \eta \) appearing in (3.30) and (3.36). See [1] for details.

16 See appendix E in [1] for the detailed form of this change of twistor basis implemented by \( \hat{L}_{(K)} \).
(1) The deformed oscillators $\tilde{S}_{(k)}^{\pm}$ are real-analytic in $Y$ and $Z$ if $K = E$ and $\Theta > 0$ or if $K = J$ and $\sigma_{a}e_{2} > 0$ (in which case $-i\Theta \geq 1$). On the other hand, if $K = J$ and $\sigma_{a}e_{2} < 0$, then there remains a singularity at a distinct $t \in [0, 1]$ for all the allowed values of $\Theta$. Since both the pre-factor and the argument of the exponent blow up at this $t$-value, one may perform the integral by means of analytical continuation in the $t$-independent quantity in the exponent, resulting in that $\tilde{S}_{(k)}^{\pm}$ is analytic in $Y$ and $Z$ for any $x$ except at a proper subspace of twistor space (including $Z = Y = 0$). Whether this is an acceptable solution, and in particular whether there exists a gauge in which spacetime gauge fields can be extracted from it remains to be examined. Finally, if $K = iB$ and $K = iP$, then $\Theta$ is indefinite and there is a finite region of spacetime in which $\Theta$ is real and hence $\tilde{S}_{(k)}^{\pm}$ is real-analytic in $Y$ and $Z$.

(2) $L_{(k)}$ implements a different gauge choice on the deformed oscillators (and consequently on the gauge fields). The standard gauge choice of the perturbative expansion—\footnote{Fixing this gauge leaves the (minimal) bosonic higher spin algebra $\mathfrak{hs}_{1}$ (4) ($\mathfrak{hs}(4)$) as residual local symmetry algebra.}—the aforementioned universal twistor gauge, $z^{a}V_{a} = 0$—is disrupted as soon as a non-trivial $L_{(k)}$ is introduced, since the latter rotates any $Z_{a}$ in the master fields in a $K$-dependent way. While this does not affect any consideration based on HS-invariant quantities, it complicates the extraction of asymptotic charges—especially in view of the fact that $L_{(k)}$ does not trivialize at the boundary of spacetime. Moreover, due to this gauge choice, a precise comparison of our solution based on $P_{1}(E)$ with the Didenko–Vasiliev solution \footnote{In particular, while (3.40) coincides with the Weyl 0-forms obtained in [2] for the case $n = 1$ treated there, the deformed oscillators in [2] reduce to $Z_{a}$ asymptotically, while here (in the spherically symmetric case) $\tilde{S}_{(k)}^{\pm} \rightarrow \tilde{Z}_{(k)}^{\pm}$ for $r \rightarrow \infty$ (see (3.48)).} still remains an open problem\footnote{Note that, due to the appearance of such modified oscillators, even choosing $L(z) = I$, the solution would not be in the universal twistor gauge. The additional complication that $L(z)I$ introduces is that even the asymptotics of the spherically symmetric solution are not in the universal twistor gauge (and reduce to $AdS_{4}$ only on the submanifold $Z = 0$).}.

In the simplest spherically symmetric case (i.e. for the solution based on the ground-state projector $P_{1}(E)$), the deformed oscillators can be written as

\[ \tilde{S}_{(k)}^{\pm} = z^{a} + 8P_{1}(E^{\pm})\tilde{a}^{\pm} \int_{-1}^{1} \frac{dt}{(t + 1 + iaz(t - 1))^{2}} j_{1}(t) e^{i(z(t - 1) - a^{\alpha}y_{\alpha})}, \]  

(3.48)

where $\tilde{a}^{\pm} := \tilde{a}\mp i\tilde{a}$ and $a_{\alpha} := z_{\alpha} + i(\kappa_{a}^{\beta}y_{\beta} + \nu_{a}^{\beta}\tilde{\gamma}_{\beta})$ coincide with the modified oscillators\footnote{In fact, the Didenko–Vasiliev solution [2], which is given in the most-asymmetric gauge discussed in footnote no 7, exhibits a related delta-function-like non-analyticity in twistor space in the limit $r \rightarrow 0$.} of [2], obeying $z_{\alpha} \ast P_{1}(E) = a_{\alpha}P_{1}(E)$. Note that, as anticipated, the internal connection for the spherically symmetric case is non-analytic only at $r = 0$, as the form of the Weyl tensors (3.40) suggests\footnote{As shown in [1], since the $V_{a}$ self-replicate under the $\ast$-product, the gauge fields inherit the spacetime behaviour from the deformed oscillators, and are therefore regular for generic spacetime points.}

Once the deformed oscillators have been obtained, the generating functions of the gauge fields follow by computing their $\ast$-product as in (3.7). As shown in [1], since the $V_{a}$ self-replicate under the $\ast$-product, the gauge fields inherit the spacetime behaviour from the deformed oscillators, and are therefore regular for generic spacetime points.

We refer the reader to [1] for the deformed oscillators in the general case and for the explicit form of the gauge fields generating functions in the general spherically symmetric case.

\textit{Classical observables.} In order to provide a gauge-invariant characterization of exact solutions that remains valid in strong-coupling regions where the weak-field expansion breaks
down, it is useful to develop a formalism for classical observables. These are functionals of the locally defined master fields and transition functions, that are defined globally in generalized spacetimes carrying various higher spin geometric structures [31]. There are several globally defined formulations, or phases, of the theory, based on different unbroken gauge groups, or structure groups. In what follows, we shall mainly focus on 0-form charges, which are observables in the unbroken phase and do not break any gauge symmetries. The basic such observables are Wilson loops in commuting sub-manifolds of $X$. These loops can be decorated with insertions of 0-form composites that transform as adjoint elements [31]. In the case of trivial monodromy, these can be contracted down to a single point $x$ resulting in 0-form charges given by the generating function

$$I(\sigma, k, \bar{k}; \lambda, \bar{\lambda}) = \widehat{T}_{\mathbb{R}}[\kappa \bar{\kappa}^{-}\exp(\kappa \Upsilon \bar{\Upsilon} + \bar{\kappa} \Upsilon \bar{\Upsilon} \bar{\kappa}) \ast (\hat{\Phi} \ast \bar{\Phi})^{k} \ast (\Phi \ast \bar{\Phi})^{\bar{k}}],$$

(3.49)

where $\widehat{T}_{\mathbb{R}}$ is the chiral trace defined by

$$\widehat{T}_{\mathbb{R}}[\Omega(Y, Z)] = \int_{\mathbb{R}} \frac{d^4Y d^4Z}{(2\pi)^4} \Omega(Y, Z),$$

(3.50)

with $(y, z)$ treated as real and independent variables; $(\sigma, k, \bar{k})$ are natural numbers defined modulo $(\sigma, k, \bar{k}) \sim (\sigma \pm 2, k, \bar{k}) \sim (\sigma, k \pm 2, \bar{k} \mp 2) \sim (\sigma \pm 1, k \pm 1, \bar{k} \mp 1)$ and $(\lambda, \bar{\lambda})$ are commuting spinors. The 0-form charges are manifestly higher spin gauge invariant and hence defined globally on any base manifold; it follows that

$$\text{d}I(\sigma, k, \bar{k}; \lambda, \bar{\lambda}) = 0,$$

(3.51)

modulo the equations of motion. The trace operation that defines them is also cyclic and independent of ordering prescriptions modulo possible boundary terms in twistor space. In what follows, we shall mainly be concerned with $I(\sigma, k, \bar{k}) := I(\sigma, k, \bar{k}; 0, 0)$, and in particular with the super-traces

$$I_{2N} := I(1, 2N, 0) = \widehat{T}_{\mathbb{R}}[\kappa \bar{\kappa}^{-} \ast (\hat{\Phi} \ast \pi(\hat{\Phi}))^{Nk}].$$

(3.52)

Inserting the general expression of the Weyl 0-form (3.37) and using that $\kappa_{y} \ast \kappa_{y} = 1$ and the orthogonality and idempotency of the projectors, one obtains

$$I_{2N} := \widehat{T}_{\mathbb{R}}[(\hat{\Phi} \ast \pi(\hat{\Phi}))^{Nk} \ast \bar{\kappa} \bar{\kappa}] = \widehat{T}_{\mathbb{R}}[(\hat{\Phi} \ast \pi(\hat{\Phi}))^{Nk} \ast \bar{\kappa} \bar{\kappa}] = \sum_{n \in \mathbb{Z}+\frac{1}{2}} v_{n}^{2N} P_{n}|_{y=0},$$

(3.53)

for the axisymmetric projectors and analogously, substituting the double index $n$ with the single index $n = \pm 1, \pm 2, \ldots$ everywhere, for the symmetry-enhanced projectors $P_{n}$. From the forms (3.31) and (3.35) (equivalently (3.30) and (3.36)) of the projectors, it thus follows that

$$I_{2N}(K_{(+)}, K_{(-)}) = 4 \sum_{n \in \mathbb{Z}+\frac{1}{2}} (-1)^{n+1} v_{n}^{2N},$$

(3.54)

for the axisymmetric solutions based on a given Cartan pair $(K_{(+)}, K_{(-)})$, and

$$I_{2N}(K_{(0)}) = 4 \sum_{n = \pm 1, \pm 2, \ldots} (-1)^{n-1} |n| v_{n}^{2N},$$

(3.55)

for the symmetry-enhanced ones, where we recall that the relation between $(n_{1}, n_{2})$ and $n$ is $n := q n_{1} + n_{2}$. Thus, the 0-form invariants $I_{2N}$ extract, in general, a linear combination of powers of the deformation parameters $v_{n}$ that characterize every solution, and that can be thought of as the

\[\text{See [1, 31] for certain $p$-form charges that may play an important role in the characterization of solutions.}\]
eigenvalues of the expansion of the solution on the basis of projectors. For solutions based on a single projector (such as, for example, the BPS solution of [2]), these local invariants capture (even powers of) the unique deformation parameter sitting in front of the spin-2 Weyl tensor as well as of its higher and lower spin partners, formally resembling the ADM mass.

Interestingly, (3.55) is not divergent for any choice of (finite) deformation parameters, at least as long as the examined solution is based on finitely many projectors. This means that, for instance, although the rotationally invariant Weyl curvatures (3.40) asymptotically resemble those of a collection of ‘higher spin Schwarzschild black holes’, the apparent singularity in \( r = 0 \) (i.e., in the strong-curvature region, where the pure spin-2 curvature invariants are no longer good observables) of the individual Weyl tensors does not actually lead to divergent higher-spin-invariant 0-form charges.\(^{22}\)

We defer the interesting issues of the physical significance and evaluation of other relevant invariants to a future publication.

4. Conclusions

In this paper, we have reviewed some properties of six infinite families of exact solutions to Vasiliev’s four-dimensional higher spin field equations, as well as the method through which they have been obtained. The latter is a combination of the gauge-function method, previously used for other exact solutions [21, 22, 18], with an internal ansatz generalizing that of [2], based on the separation of the dependence of the master fields on \( Y \) and \( Z \) twistor variables. The resulting solutions are organized in three pairs, each pair characterized by a biaxial isometry group \( \mathfrak{so}(2) \oplus \mathfrak{so}(2) \) embedded into \( \mathfrak{sp}(4; \mathbb{C}) \) in three inequivalent ways. One of the families contains a subset of solutions in which one of the two \( \mathfrak{so}(2) \) enhances \( \mathfrak{so}(3) \), while in the remaining families the enhanced symmetry algebra is \( \mathfrak{so}(2, 1) \). In all of our solutions, all spins are activated for generic choices of deformation parameters.

The study of the non-perturbative regime of higher spin gravity is nowadays of extreme interest both in its own right, for uncovering the physics of higher spin fields, and for its relevance in testing the proposed holographic duality [14]. In this sense, it is especially interesting to understand the extent to which the singular family \( \mathcal{M}_E(E, J) \) can be thought of as a higher spin generalization of black holes. To this purpose, it is crucial to carry out a more detailed study of whether, for instance, their singularities are physical and not gauge artefacts, and whether these solutions possess an event horizon. These questions can in principle be addressed by analysing either the propagation of small fluctuations over the backgrounds or suitable higher-spin-invariant line elements as discussed in [31]. The answers may have important surprises in store, since the deviations from Einstein gravity in the strong-curvature region, as discussed above, may be radical, essentially due to the non-locality of interactions induced by the unbroken higher spin symmetry. To probe this region, it may be necessary to extend the usual tools of differential geometry to the higher spin context, since standard concepts based on spin-2 field constructs alone, such as the relativistic interval, are not higher spin invariant.

We have also reviewed the evaluation of certain 0-form charges [21, 22, 30, 31] on our solutions. They are a set of functionals of the 0-form master fields, defined via the trace of the \( \star \)-product algebra, that are conserved on the field equations and provide useful instruments for distinguishing gauge-inequivalent solutions and for characterizing them physically even

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\(^{22}\) The regularity of these observables in \( r = 0 \) may be traced back to the formal insensitiveness of the trace operation to the choice of ordering prescription. Indeed, at \( r = 0 \) the Weyl-ordered symbol of this master field is a distribution in twistor space [1]. However, by moving to normal ordering the resulting symbol becomes a regular, Gaussian function. In this sense, the spacetime singularities may be resolved at the level of master fields living in correspondence space.
in strong-field regions. As the non-locality on $\mathcal{T}$ of the star-product is mapped via the field equations to spacetime non-locality, the 0-form charges hide their higher derivative nature into the $\star$-products between master fields, and this facilitates their evaluation. We find that certain 0-form charges involving the spacetime curvatures are well defined on our solutions, and amount to linear combinations of powers of the squared deformation parameters $\mu^2_n$, that therefore characterize the various field configurations in a gauge-independent way. Interestingly enough, all these invariants are finite everywhere (unless the solution under consideration is based on infinitely many projectors and the eigenvalues $\mu_n$ are not too small).

The solutions of four-dimensional higher spin gravity that we have presented here are different from the BTZ-like ones of the three-dimensional Chern–Simons versions of the theory. The latter have recently been examined in some detail, for example in [20], and indeed share several features of the BTZ black-hole solution in AdS gravity. The main difference stems from the fact that the four-dimensional dynamics involves higher spin fields that are not topological. These fields may hence support localizable solutions possibly with curvature singularities, features that are absent from the BTZ-like higher spin black-hole solutions constructed in three dimensions (though we would still like to draw attention to the aforementioned fact that as far as the 0-form charges here calculated are concerned, there are no signs of curvature singularities in the solutions considered here). There exists, however, a slightly simpler three-dimensional analogue of the four-dimensional Vasiliev equations, due to Prokushkin and Vasiliev [19], that describes local degrees of freedom. These equations thus furnish a natural starting point for further development of the various higher spin gauge-invariant methods proposed in [31] for examining exact solutions in theories with local degrees of freedom.

The study of exact solutions in higher spin gravity, in both three and four dimensions, also prompts the tackling of some crucial issues that have remained poorly explored so far: a global description of the solutions (and correspondingly adapted choices of gauge functions), the characterization of boundary conditions and super-selection sectors within the framework of the unfolded equations formulated on correspondence spaces (see [1] for more details), the extraction of asymptotic charges and symmetries, and the description of possible global degrees of freedom carried by the $\mathcal{Z}$-dependence are but a few.

Finally, it would also be interesting to extend the ansatz here presented to Kerr-like solutions and to study whether or not it is possible to generalize it to the construction of multi-soliton configurations.

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