Set-optimization meets variational inequalities

Giovanni P. Crespi∗ Carola Schrage†

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Abstract

We study necessary and sufficient conditions to attain solutions of set-optimization problems in terms of variational inequalities of Stampacchia and Minty type. The notion of a solution we deal with has been introduced in [17], for convex set-valued objective functions. To define the set-valued variational inequality, we introduce a set-valued directional derivative and we relate it to the Dini derivatives of a family of linearly scalarized problems. The optimality conditions are given by Stampacchia and Minty type Variational inequalities, defined both by the set valued directional derivative and by the Dini derivatives of the scalarizations. The main results allow to obtain known variational characterizations for vector valued optimization problems.

1 Introduction

Since the seminal papers by F. Giannessi (see [9, 10]), variational inequalities have been applied to obtain necessary and sufficient optimality conditions in vector optimization. In [17] a new approach to study set-valued problems has been applied to have a fresh look to vector optimization. Indeed, it turns out that vector optimization can be treated as a special case of set-valued optimization. The aim of this paper is to provide some variational characterization of (convex) set-valued optimization. Following the approach known as set-optimization we mean to introduce set-valued variational inequalities, both of Stampachia and Minty type, by means of Dini-type derivatives (see e.g. [14]). Under suitable assumptions (e.g. lower semi-continuity type assumptions), we can prove equivalence between solutions of the variational inequalities and solution of a (primitive) set-optimization problem, as introduced in [17] and deepened in [20]. To prove the main results we need also to deal with scalarization problems. However, while in the vector case this might only be a technical need, we prove that eventually the set-valued variational inequalities and their scalar counterparts provides different insight on the problem. Some relevant information on the solution of the set-optimization problem are provided only through the scalar version of the inequality. The special case of vector optimization is finally studied, to recover classical results stated in [4, 25].

The paper is organized as follows. Section 2 is devoted to preliminar results on set-optimization that will be used throughout the paper. The concept of solution to a set-optimization problem and the Dini-type derivatives are presented and some properties are proved. Section 3

∗University of Valle d’Aosta, Department of Economics and Political Sciences, Loc. Grand Chemin 73-75, 11020 Saint-Christophe, Aosta, Italy. g.crespi@univda.it
†University of Valle d’Aosta, Department of Economics and Political Sciences, Loc. Grand Chemin 73-75, 11020 Saint-Christophe, Aosta, Italy. carolaschrage@gmail.com
presents the main results. As the notion of solution relays on two properties, we develop two different webs of relations between our variational inequalities and the set-optimization. The first one provides a variational characterization of the notion of infimizer, while the second one is devoted to characterize the notion of minimality. Finally, Section 4 applies the previous results to vector optimization. The notion of epigraphical extension of a single valued function is introduced to obtain a set-optimization problem equivalent to a vector one. The relations proved for the convex case in this paper reproduce those already known for the vector case between optimization and variational inequalities. We leaves as an open question, for further research, whether convexity can be relaxed, as it holds indeed for vector valued functions.

2 Preliminaries

2.1 Setting and operations with sets

Let \( Z \) be a locally convex Hausdorff space with the dual space \( Z^* \). The set \( \mathcal{U} \) is the set of all closed, convex and balanced 0 neighborhoods in \( Z \), a 0–neighborhood base of \( Z \). By \( \text{cl} \, A \), \( \text{co} \, A \) and \( \text{int} \, A \), we denote the closed or convex hull of a set \( A \subseteq Z \) and the topological interior of \( A \), respectively. The conical hull of a set \( A \) is \( \text{cone} \, A = \{ ta | a \in A, \, 0 < t \} \).

The recession cone of a nonempty closed convex set \( A \subseteq Z \) is given by

\[
0^+ A = \{ z \in Z | A + \{ z \} \subseteq A \},
\]

a closed convex cone, [26, p.6]. By definition, \( 0^+ \emptyset = \emptyset \) is assumed.

\( Z \) is ordered through a closed convex cone \( C \) with \( 0 \in C \) and nontrivial negative dual cone

\[
C^- = \{ z^* \in Z^* | \forall c \in C : z^*(c) \leq 0 \},
\]

\( C^- \setminus \{0\} \neq \emptyset \) by setting

\[
z_1 \leq z_2 \iff z_2 + C \subseteq z_1 + C
\]

for all \( z_1, z_2 \in Z \). This ordering is extended to the power set of \( Z \) (compare [12] and the references therein) by setting

\[
A_1 \preceq A_2 \iff A_2 + C \subseteq A_1 + C
\]

for all \( A_1, A_2 \subseteq Z \).

We introduce the subset

\[
\mathcal{G}(Z, C) = \{ A \subseteq Z | A = \text{cl} \, \text{co} \, (A + C) \}
\]

which is an order complete lattice and \( A_1 \preceq A_2 \) is equivalent to \( A_1 \supseteq A_2 \) whenever \( A_1, A_2 \in \mathcal{G}(Z, C) \). For any subset \( A \subseteq \mathcal{G}(Z, C) \), supremum and infimum of \( A \) in \( \mathcal{G}(Z, C) \) are given by

\[
\inf A = \text{cl} \, \text{co} \bigcup_{A \in A} A; \quad \sup A = \bigcap_{A \in A} A.
\]

When \( A = \emptyset \), then we agree on \( \inf A = \emptyset \) and \( \sup A = Z \). Especially, \( \mathcal{G}(Z, C) \) possesses a greatest and smallest element \( \inf \mathcal{G}(Z, C) = Z \) and \( \sup \mathcal{G}(Z, C) = \emptyset \).
The Minkowsky sum and multiplication with negative reals need to be slightly adjusted to provide operators on \( G(Z, C) \). We define

\[
\forall A, B \in G(Z, C) : \quad A \oplus B = \text{cl} \{a + b \in Z \mid a \in A, b \in B\}; \quad (2.3)
\]

\[
\forall A \in G(Z, C), \forall 0 < t : \quad t \cdot A = \{ta \in Z \mid a \in A\}; \quad (2.4)
\]

\[
\forall A \in G(Z, C) : \quad 0 \cdot A = C. \quad (2.5)
\]

Especially, \( 0 \cdot \emptyset = 0 \cdot Z = C \) and \( \emptyset \) dominates the addition in the sense that \( A \oplus \emptyset = \emptyset \) is true for all \( A \in G(Z, C) \). Moreover, \( A \oplus C = A \) is satisfied for all \( A \in G(Z, C) \), thus \( C \) is the neutral element with respect to the addition.

As a consequence,

\[
\forall A \subseteq G(Z, C), \forall B \in G(Z, C) : \quad B \oplus \inf A = \inf \{B \oplus A \mid A \in A\}, \quad (2.6)
\]

or, equivalently, the inf–residual

\[
A \rightarrow B = \inf \{M \in G(Z, C) \mid A \preceq B \oplus M\} \quad (2.7)
\]

exists for all \( A, B \in G(Z, C) \). It holds

\[
A \rightarrow B = \{z \in Z \mid B + \{z\} \subseteq A\}; \quad (2.8)
\]

\[
A \preceq B \oplus (A \rightarrow B), \quad (2.9)
\]

compare [13, Theorem 2.1].

Overall, the structure of \( G^\oplus = (G(Z, C), \oplus, \cdot, C, \preceq) \) is that of an inf–residuated conlinear space, compare also [6], [7], [8], [21] for a more thorough study of this structure.

Historically, it is interesting to note that R. Dedekind [5] introduced the residuation concept and used it in order to construct the real numbers. The construction above is in this line of ideas, but in a rather abstract setting.

**Example 2.1** Let us consider \( Z = \mathbb{R} \), \( C = \mathbb{R}_{+} \). Then \( G(Z, C) = \{[r, +\infty) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\emptyset\} \), and \( G^\oplus \) can be identified (with respect to the algebraic and order structures which turn \( G(\mathbb{R}, \mathbb{R}_{+}) \) into an ordered conlinear space and a complete lattice admitting an inf-residuation) with \( \mathbb{R} = \mathbb{R} \cup \{\pm\infty\} \) using the 'inf-addition' \( \oplus' \) (see [13], [22]). The inf-residuation on \( \mathbb{R} \) is given by

\[
r \rightarrow s = \inf \{t \in \mathbb{R} \mid r \leq s + t\}
\]

for all \( r, s \in \mathbb{R} \), compare [13] for further details.

Each element of \( G^\oplus \) is closed and convex and \( A = A + C \), hence by a separation argument we can prove

\[
\forall A \in G^\oplus : \quad A = \bigcap_{z^* \in C^- \setminus \{0\}} \{z \in Z \mid -\sigma(z^*|A) \leq -z^*(z)\}, \quad (2.10)
\]

where \( \sigma(z^*|A) = \sup \{z^*(z) \mid z \in A\} \) is the support function of \( A \) at \( z^* \). Especially, \( A = \emptyset \) if and only if there exists \( z^* \in C^- \setminus \{0\} \) such that \( -\sigma(z^*|A) = +\infty \), or equivalently if the same holds true for all \( z^* \in C^- \setminus \{0\} \).
Lemma 2.2 [24, Proposition 3.5] Let $A \subseteq G^\wedge$ be a set, then
\[
\inf_A = \bigcap_{z^* \in C^- \setminus \{0\}} \{ z \in Z | \inf \{ -\sigma(z^*|A) | A \in A \} \leq z^*(z) \} \quad (2.11)
\]
\[
\forall z^* \in C^- \setminus \{0\} : -\sigma(z^*|A) = \inf \{ -\sigma(z^*|A) | A \in A \}. \quad (2.12)
\]

Lemma 2.3 [13, Proposition 5.20] Let $A, B \in G^\wedge$, then
\[
A \smash \supseteq B = \bigcap_{z^* \in C^- \setminus \{0\}} \{ z \in Z | (-\sigma(z^*|A)) - (-\sigma(z^*|B)) \leq z^*(z) \}; \quad (2.13)
\]
\[
\forall z^* \in C^- \setminus \{0\} : (-\sigma(z^*|A)) - (-\sigma(z^*|B)) \leq -\sigma(z^*|A \smash \supseteq B). \quad (2.14)
\]

Example 2.4 Let $Z = \mathbb{R}^2$ and $C = \text{cl cone} (0,1)^T$, $B = \{(x,y) \in \mathbb{R}^2 | -1 \leq x \leq 1, 0 \leq y \}$ and $A = C$. Then
\[
A \smash \supseteq B = \{ z \in Z | 1 \leq (-1,0)^T z, 1 \leq (1,0)^T z, 0 \leq (0,1)^T z \}
\]
and $(-\sigma(z^*|A)) - (-\sigma(z^*|B)) \in \mathbb{R}$ is satisfied for all $z^* \in M^*$ while $A \smash \supseteq B = \emptyset$, thus $-\sigma(A \smash \supseteq B) = +\infty$. The difference of the scalarizations and the scalarization of the difference do not coincide.

The following rules will be used frequently later on.

Lemma 2.5 Let $A, B, D \in G^\wedge$, $0 < s$ and $t \in (0,1)$ be given, then
(a) $s(A \smash \supseteq B) = sA \smash \supseteq sB$;
(b) $(tA \oplus (1-t)B) - D \preceq t(A \smash \supseteq D) \oplus (1-t)(B - D)$;
(c) $A \smash \supseteq D \preceq (A \smash \supseteq B) \oplus (B - D)$;
(d) If $A \neq \emptyset$, then $0^+ A = (A \smash \supseteq A)$.

Proof.
(a) It holds $z \in (A \smash \supseteq B)$, if and only if $B + \{z\} \subseteq A$ or equivalently $sA \preceq sB + \{sz\}$.
(b) As $D \in G^\wedge$ is assumed, $tD \oplus (1-t)D = D$. Let $z_A \in A \smash \supseteq D$ and $z_B \in B - D$ be given, then $(tA \oplus (1-t)B) \preceq D + (tz_A + (1-t)z_B)$ is satisfied.
(c) The inclusion is true, if and only if
\[
A \preceq (A \smash \supseteq B) \oplus (B - D) \oplus D.
\]
As we know that $B \preceq (B - D) \oplus D$ and $A \preceq (A \smash \supseteq B) \oplus B$, this inclusion is true.
(d) This is immediate from the definition of $0^+ A$.

Lemma 2.5 (d) suggests that, if needed, we can use the recession cone of a set as 0–element in certain inequalities. It is notable that for any $A \in G^\wedge$, either $A = \emptyset$, or $0^+ A \preceq C$. In order to implement these remarks in the sequel, we will use the following properties of recession cones.
Lemma 2.6 Let $A \in G^\Delta$ be a nonempty set, then

$$0^+ A = \{ z \in Z \mid \forall z^* \in C^- \setminus \{0\} : -\sigma(z^*|A) = -\infty \lor 0 \leq -z^*(z) \}.$$  

Especially, for all $A \in G^\Delta$, either $A = \emptyset$, or

$$0^+ A = \bigcap_{z^* \in C^- \setminus \{0\}} \{ z \in Z \mid 0 \leq -z^*(z) \}.$$  \hspace{1cm} (2.15)

**Proof.** Assume $z \notin 0^+ A$, then either $A = \emptyset$ or there exists a $z^* \in Z^*$ such that $\sigma(z^*|A) < z^*(a + z)$ is satisfied for some $a \in A$. As $z^*(a + z) \leq \sigma(z^*|A) + z^*(z)$, this implies $-z^*(z) < 0$ and $-\sigma(z^*|A) \neq -\infty$ and therefore $z^* \in C^- \setminus \{0\}$. On the other hand, assume $z \in 0^+ A$, then $A$ is nonempty and $A + \{ z \} \subseteq A$, hence for all $z^* \in Z^*$ it holds $\sigma(z^*|A + \{ z \}) \leq \sigma(z^*|A)$, hence $\sigma(z^*|A) + z^*(z) \leq \sigma(z^*|A)$. This implies that either $-\sigma(z^*|A) = -\infty$ or $0 \leq -z^*(z)$ is true for all $z^* \in Z^*$ and thus especially for $z^* \in C^- \setminus \{0\}$.

If $A = Z$, then $-\sigma(z^*|Z) = -\infty \notin \mathbb{R}$ is satisfied for all $z^* \in C^- \setminus \{0\}$, hence formula (2.15) is true with $0^+ Z = Z$. Hence let $A \neq Z$ or $\emptyset$, then $-\sigma(z^*|A) \notin \mathbb{R}$ implies $-\sigma(z^*|A) = -\infty$ and the statement is proved.

\[ \square \]

Lemma 2.7 Let $A \in G^\Delta$ be a nonempty set, then

$$\{ z^* \in Z^* \mid -\sigma(z^*|A) \in \mathbb{R} \} \subseteq (0^+ A)^-. $$

**Proof.** Assume $-\sigma(z^*|A) \in \mathbb{R}$ and $A + z \subseteq A$. Then

$$-\sigma(z^*|A) \leq -\sigma(z^*|A + z) = -\sigma(z^*|A) - z^*(z)$$

implies $0 \leq -z^*(z)$, in other words $z^* \in (0^+ A)^-$.

\[ \square \]

Lemma 2.8 Let $A, B \in G^\Delta$ be nonempty, then

$$0^+ (A \oplus B) = \text{cl co} (0^+ A \cup 0^+ B) = 0^+ A \oplus 0^+ B; $$

$$A \preceq B \Rightarrow 0^+ A \preceq 0^+ B.$$  

**Proof.** Assume $A + z_A \subseteq A$ and $B + z_B \subseteq B$, then for all $a \in A$ and all $b \in B$ it holds

$$a + b + (z_A + z_B) \in A \oplus B$$

and as both $0^+ A$ and $0^+ B$ are convex cones, for all $t \in [0, 1]$ it holds

$$ta + (1-t)b + (z_A + z_B) \in A \oplus B.$$  

If $z \in A \oplus B$, then for all $U \in \mathcal{U}$ there exist $a \in A$, $b \in B$ and $t \in [0, 1]$ with $ta + (1-t)b \in \{ z \} + U$, such that

$$ta + (1-t)b + (z_A + z_B) \in \{ z + (z_A + z_B) \} + U,$$  

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and hence \( z + (z_A + z_B) \in A \oplus B \), proving \( 0^+ A + 0^+ B \subseteq 0^+ (A \oplus B) \). As \( A \oplus B \) is a closed convex set, the recession cone is a closed convex cone, so
\[
0^+ A \oplus 0^+ B = \text{cl} \text{co} (0^+ A + 0^+ B) \subseteq 0^+ (A \oplus B).
\]
Since \( 0 \in 0^+ A \cap 0^+ B \) implies \( 0^+ A \cup 0^+ B \subseteq 0^+ A \oplus 0^+ B \), also \( \text{cl} \text{co} (0^+ A \cup 0^+ B) \subseteq 0^+ A \oplus 0^+ B \) holds true. On the other hand, if \( z_A \in 0^+ A \) and \( z_B \in 0^+ B \) are given, then \( z_A + z_B \in \text{co} (0^+ A \cup 0^+ B) \), hence \( \text{cl} \text{co} (0^+ A \cup 0^+ B) \supseteq 0^+ A \oplus 0^+ B \) proves equality.

Finally, let \( A \preceq B \) be satisfied, \( B + \{z\} \subseteq B \) and \( a + z \notin A \) for some \( a \in A \). Then there exists a neighborhood \( U \in \mathcal{U} \) such that \( \{a + z\} + U \cap A = \emptyset \), as \( A \) is closed and thus there exists \( t \in (0, 1) \) such that
\[
t \left( b + \frac{1}{t} z \right) + (1 - t)a = a + z + t(b - a) \in \{a + z\} + U.
\]
But since \( A \) is convex and \( 0^+ B \) is a cone, this implies
\[
t \left( b + \frac{1}{t} z \right) + (1 - t)a \in \text{co} (B + A) \subseteq A,
\]
a contradiction. \( \square \)

Moreover, we can remark that for any set \( A \in \mathcal{G}^\Delta \) the following properties hold true.

(i) \( 0^+ A \oplus 0^+ \emptyset = 0^+ (A \oplus \emptyset) \);

(ii) \( 0^+ A \preceq 0^+ \emptyset \).

On the contrary, \( 0^+ A \oplus 0^+ \emptyset \preceq 0^+ A \cup 0^+ \emptyset \) can be proven if and only if \( A = \emptyset \).

**Lemma 2.9** If \( A \rightarrow B \neq \emptyset \), then
\[
0^+ (A \rightarrow B) \preceq 0^+ A \preceq 0^+ B.
\]
If additionally \( B \neq \emptyset \), then we also get
\[
0^+ (A \rightarrow B) = 0^+ A.
\]

**Proof.** Assume \( A \rightarrow B \neq \emptyset \). If \( B = \emptyset \), then \( A \rightarrow B = Z \) and the first equation is immediate. Hence let \( B \neq \emptyset \). Then \( \emptyset \neq B \oplus (A \rightarrow B) \subseteq A \) and because \( A \) is closed and convex by assumption, we can apply lemma 2.8 to prove
\[
0^+ B \cup 0^+ (A \rightarrow B) \subseteq 0^+ (B \oplus (A \rightarrow B)) \subseteq 0^+ A.
\]
On the other hand, if \( B + \{z\} \subseteq A \), that is \( z \in A \rightarrow B \), then for all \( z_0 \in 0^+ A \) it holds \( B + \{z + z_0\} \subseteq A \), hence \( 0^+ A \subseteq 0^+ (A \rightarrow B) \). \( \square \)
2.2 Set valued functions

Let $X$ be linear space. A function $f: X \to \mathcal{G}^\Delta$ is called convex when
\[
\forall x_1, x_2 \in X, \forall t \in (0, 1): f(t x_1 + (1 - t)x_2) \preceq tf(x_1) \oplus (1 - t)f(x_2). \tag{2.16}
\]

It is an easy exercise (see, for instance, [12]) to show that $f$ is convex if and only if the set
\[
\text{graph } f = \{(x, z) \in X \times Z: z \in f(x)\}
\]
is convex. A $\mathcal{G}^\Delta$-valued function $f$ is called positively homogeneous when
\[
\forall 0 < t, \forall x \in X: f(tx) \preceq tf(x),
\]
and it is called sublinear if it is positively homogeneous and convex. It can be shown that $f$ is sublinear if and only if graph $f$ is a convex cone. Compare also [2, Definition 2.1.1.] on above definitions.

The (effective) domain of a function $f: X \to \mathcal{G}^\Delta$ is the set $\text{dom } f = \{x \in X | f(x) \neq \emptyset\}$. Since $\emptyset$ is the supremum of $\mathcal{G}^\Delta$, the previous notion of domain of a set valued function extends the scalar notion of effective domain. The image set of a subset $A \subseteq X$ through $f$ is denoted by
\[
f[A] = \{f(x) \in \mathcal{G}^\Delta | x \in A\} \subseteq \mathcal{G}^\Delta.
\]

Notice that $f[A]$ is a subset of $\mathcal{P}(Z)$ rather then a subset of $Z$, while $\text{inf } f[A] = \text{cl co } \bigcup_{a \in A} f(a)$ is an element of $\mathcal{P}(Z)$, hence a subset of $Z$.

**Lemma 2.10** Let $f: X \to \mathcal{G}^\Delta$ be a convex function, $x_0 \in \text{dom } f$. If $x \in \text{dom } f$, then $t \mapsto 0^+(f(x + t(x_0 - x)))$ is constant on $(0, 1)$ and $0^+(f(x + t(x_0 - x))) \preceq 0^+f(x) \cup 0^+f(x_0)$ is satisfied for all $t \in (0, 1)$.

**Proof.** Let $t \in [0, 1]$ and denote $x_t = x + t(x_0 - x)$. By convexity of $f$, for any $z_0 \in 0^+f(x_0)$ and $z \in 0^+f(x)$, $z_t = tz + (1 - t)z_0 \in 0^+f(x_t)$ is satisfied. Since both recession cones contain $0$, especially we have $z_0 + 0 \in 0^+f(x_t)$ and $z + 0 \in 0^+f(x_t)$. Therefore
\[
0^+f(x_t) \supseteq 0^+f(x_0) \cup 0^+f(x).
\]

Moreover let $0 < s < t < 1$ be given. By replacing $x$ with $x_t$ in above argument,
\[
0^+f(x_s) \supseteq 0^+f(x_0) \cup 0^+f(x_t) = 0^+f(x_t)
\]
and replacing $x_0$ by $x_s$ implies
\[
0^+f(x_t) \supseteq 0^+f(x_s) \cup 0^+f(x) = 0^+f(x_s),
\]
hence $0^+f(x_s) = 0^+f(x_t)$ is proven for all $s, t \in (0, 1)$. \hfill $\Box$

Given a function $f: X \to \mathcal{G}^\Delta$, the family of extended real-valued functions $\varphi_{f,x^*}: X \to \mathbb{R} \cup \{\pm \infty\}$ defined by
\[
\varphi_{f,x^*}(x) = \inf \{-z^*(z) | z \in f(x)\}, \quad z^* \in C^- \setminus \{0\}
\]
is the family of (linear) scalarizations for $f$. Some properties of $f$ are inherited by its scalarizations and vice versa. For instance, $f$ is convex if and only if $\varphi_{f,x^*}$ is convex for
each \( z^* \in C^- \setminus \{0\} \). In turn, convexity of \( \varphi_{f,z^*} \) is equivalent to convexity of the function \( f_{z^*} : X \to G^\delta \) given by

\[
f_{z^*}(x) = \{ z \in Z \mid \varphi_{f,z^*}(x) \leq -z^*(z) \}.
\]

Moreover, a standard separation argument shows

\[
\forall x \in X : f(x) = \bigcap_{z^* \in C^- \setminus \{0\}} f_{z^*}(x).
\]

**Remark 2.11** The function \( f_{z^*} : X \to G^\delta \) is actually nothing but the function mapping \( x \) to the sublevel set \( L_{z^*}^\leq (-\varphi_{f,z^*}(x)) \) of \( z^* \) to the level \( -\varphi_{f,z^*}(x) \), that is for all \( z^* \in C^- \setminus \{0\} \) and all \( x \in X \) it holds

\[
f_{z^*}(x) = L_{z^*}^\leq (\sigma(z^*|f(x))) = \{ z \in Z \mid z^*(z) \leq -\varphi_{f,z^*}(x) \} \quad (2.17)
\]

and either \( f_{z^*}(x) \in \{0, Z\} \), or it is a closed affine half space with a supporting point \( z \in f_{z^*}(x) \) such that \( \varphi_{f,z^*}(x) = -z^*(z) \). If \( f(x) \neq \emptyset \), then either \( f_{z^*}(x) = Z \), or \( \varphi_{f,z^*}(x) \in \mathbb{R} \), thus

\[
\forall x \in X : f(x) = \emptyset \lor f(x) = \bigcap_{z^* \in C^- \setminus \{0\}; \varphi_{f,z^*}(x) \in \mathbb{R}} f_{z^*}(x).
\]

**Definition 2.12** (a) Let \( \varphi : X \to \overline{\mathbb{R}} \) be a function, \( x_0 \in X \). Then \( \varphi \) is said to be lower semicontinuous (l.s.c.) at \( x_0 \), iff

\[
\forall r \in \mathbb{R} : r < \varphi_{f,z^*}(x_0) \Rightarrow \exists U \in \mathcal{U} : \forall u \in U : r < \varphi_{f,z^*}(x_0 + u).
\]

(b) Let \( f : X \to G^\delta \) be a function, \( M^* \subseteq C^- \setminus \{0\} \). Then \( f \) is said \( M^* \)-lower semicontinuous (\( M^*-\text{l.s.c.} \)) at \( x_0 \), iff \( \varphi_{f,z^*} \) is l.s.c. at \( x_0 \) for all \( z^* \in M^* \).

(c) Let \( f : X \to G^\delta \) be a function. If

\[
f(x) \leq \liminf_{u \to 0} f(x + u) = \bigcap_{U \in \mathcal{U}} \text{cl} \, \text{co} \left( \bigcup_{u \in U} f(x + u) \right)
\]

is satisfied, then \( f \) is lattice lower semicontinuous (lattice l.s.c.) at \( x \).

(d) A function \( f : X \to G^\delta \) is closed if and only if it is lattice l.s.c. everywhere.

In [18], it has been proven that if \( f \) is \( C^- \setminus \{0\}\)-l.s.c. at \( x \), then it is also lattice l.s.c. at \( x \). One can show that \( f \) is closed if and only if graph \( f \subseteq X \times Z \) is a closed set with respect to the product topology, see [20, Proposition 2.34].

Notice that none of the above continuity concepts coincide with those used in e.g. [1, 2, 11], compare [18] for a detailed study of continuity concepts for set valued functions.

**Remark 2.13** For notational simplicity we set the restriction of a set valued function \( f : X \to G^\delta \) to a segment with end points \( x_0, x \in X \) as \( f_{x_0,x} : \mathbb{R} \to G^\delta \), given by

\[
f_{x_0,x}(t) = \begin{cases} f(x_0 + t(x - x_0)), & \text{if } t \in [0, 1]; \\ \emptyset, & \text{elsewhere}. \end{cases}
\]
This is equivalent to the restriction of a scalar valued function \( \varphi : X \to \mathbb{R} \) to the same segment, defined by
\[
\varphi_{x_0,x}(t) = \begin{cases} 
\varphi(x_t), & \text{if } t \in [0,1]; \\
+\infty, & \text{elsewhere.} 
\end{cases}
\]

Setting \( x_t = x_0 + t(x - x_0) \) for all \( t \in \mathbb{R} \), the scalarization of the restricted function \( f_{x_0,x} \) is equal to the restriction of the scalarization of \( f \) for all \( z^* \in C^- \setminus \{0\} \).

If \( f \) is convex, \( x_0, x_t \in \text{dom} \ f \) for some \( t \in (0,1) \), then \( (\varphi_{f,x^*})_{x_0,x} \) is lower semicontinuous on \( (0,t) \), hence \( f_{x_0,x} \) is lattice l.s.c. on \( (0,t) \).

For the sequel we will need the following notion, as introduced in [14].

**Definition 2.14** Let \( f : X \to G^\triangle \) be a function and \( M \subseteq X \). We define the inf-translation of \( f \) by \( M \) to be the function \( \hat{f} \left( \cdot ; M \right) : X \to G^\triangle \) given by
\[
\hat{f} (x; M) = \inf f [M + \{X\}] = \text{cl co} \bigcup_{m \in M} f (m + x).
\]

The function \( \hat{f} (\cdot ; M) \) is nothing but the canonical extension of \( f \) at \( M + x \) as defined in [17].

The following properties will be used in the proofs of the main results.

**Lemma 2.15** [14, Lemma 5.8 (b)] Let \( f : X \to G^\triangle \) be a convex function, \( M \subseteq X \), then \( \hat{f} (\cdot ; M) : X \to G^\triangle \) is a convex function.

**Lemma 2.16** Let \( f : X \to G^\triangle \) be a function, \( z^* \in C^- \setminus \{0\} \) and \( M \subseteq X \) a nonempty set. Then
\[
\forall x \in X : \inf \varphi_{f,z^*} [M + x] = \varphi_{\hat{f}(\cdot ; M),z^*} (x).
\]

By defining \( \hat{\varphi}_{f,z^*} (x; M) = \inf \varphi_{f,z^*} [M + x] \), it holds
\[
\forall x \in X : \hat{\varphi}_{f,z^*} (x; M) = \varphi_{\hat{f}(\cdot ; M),z^*} (x),
\]
the operations of taking the inf translation of a function and taking its scalarization commute.

**Proof.** The statement is an easy consequence of Lemma 2.2. \( \square \)

**Lemma 2.17** Let \( f : X \to G^\triangle \) be a function, \( M \subseteq X \) a nonempty set, then the domain of \( \hat{f} (\cdot ; M) : X \to G^\triangle \) is the set
\[
\text{dom} \hat{f} (\cdot ; M) = \bigcup_{m \in M} \text{dom} f + \{-m\}.
\]

**Proof.** Since \( x \in \text{dom} \hat{f} (\cdot ; M) \) if and only if \( \inf f [M + \{x\}] \neq \emptyset \), that is there exists \( m \in M \) such that \( f(m + x) \neq \emptyset \). Therefore, \( x \in \text{dom} \hat{f} (\cdot ; M) \) if and only if \( m + x \in \text{dom} f \) for some \( m \in M \), in other words \( x \in \bigcup_{m \in M} \text{dom} f + \{-m\} \). \( \square \)
Lemma 2.18 Let $f : X \to \mathcal{G}^\Delta$ be a convex function, $M \subseteq X$ a nonempty set and $z^* \in C^- \setminus \{0\}$. If either of the following conditions is satisfied, then the restriction of $\hat{f}(\cdot; co M)$ to the segment $[0, x]$ is $C^- \setminus \{0\}$-l.s.c. in 0 for all $x \in X$.

(a) $\hat{f}(0; M) = \inf f [X]$;

(b) $0 \in \operatorname{int} \bigcup_{m \in co M} (\operatorname{dom} f + \{-m\})$;

(c) $(\varphi_{f,z^*})_{m,x} : X \to \overline{\mathbb{R}}$ is continuous in 0 for all $m \in co M$, $x \in X$ and all $z^* \in C^- \setminus \{0\}$.

PROOF.

(a) If $\hat{f}(0; M) = \inf f [X]$, then $\varphi_{f(\cdot; co M),z^*}(0) = \inf \varphi_{f(\cdot; co M),z^*}[X]$ is true for all $z^* \in C^- \setminus \{0\}$, hence each scalarization $\varphi_{f(\cdot; co M),z^*}$ is l.s.c. in 0 and therefore $\hat{f}(\cdot; co M)$ is $C^- \setminus \{0\}$-l.s.c in 0.

(b) By Lemma 2.17, $\bigcup_{m \in co M} (\operatorname{dom} f + \{-m\})$ is the domain of $\hat{f}(\cdot; co M)$ and by Lemma 2.15, $\hat{f}(\cdot; co M)$ is convex. This is true, if and only if each scalarization of $\hat{f}(\cdot; co M)$ is convex, which are given by $(\hat{\varphi}_{f,z^*})(\cdot; co M)$, compare Lemma 2.16. If $0 \in \operatorname{int} \bigcup_{m \in co M} (\operatorname{dom} f + \{-m\})$ is assumed, then the restriction of each scalarization $\varphi_{f,z^*}(\cdot; co M)$ to $[x_0, x]$ is l.s.c. in 0, as $\operatorname{dom} \hat{f}(\cdot; co M) = \operatorname{dom} (\hat{\varphi}_{f,z^*})(\cdot; co M)$.

(c) Let $(\varphi_{f,z^*})_{m,x} : X \to \overline{\mathbb{R}}$ be continuous in 0 for all $m \in co M$ and all $x \in X$. In this case,

$$
\limsup_{t \downarrow 0}(\varphi_{f(\cdot; co M),z^*})_{0,x}(t) = \limsup_{t \downarrow 0} \inf_{m \in co M} (\varphi_{f,z^*})_{m,x}(t) \\
\leq \inf_{m \in co M} \limsup_{t \downarrow 0} (\varphi_{f,z^*})_{m,x}(t) \\
= \inf_{m \in co M} (\varphi_{f,z^*})_{m,x}(0) \\
= \hat{\varphi}_{f,z^*}(0; co M).
$$

Hence for each $z^* \in C^- \setminus \{0\}$, the restriction of $\varphi_{f,z^*}(\cdot; co M)$ to $[0, x]$ is convex and u.s.c. in 0, thus especially l.s.c. in 0, too. □

In this framework, we are interested to study the problem

$$
\text{minimize } f(x) \quad \text{subject to } x \in X \quad \text{(P)}
$$

where $f$ is a $\mathcal{G}^\Delta$-valued function. Following [17], to solve (P) means to look for the infimum in $\mathcal{G}^\Delta$ as introduced in Formula (2.2), and for subsets of $X$ where the infimum is attained. This approach is different from most other approaches in set optimization, see for example [19, Definition 14.2], [15], [16] and the references therein.

More formally, we introduce a solution concept based on Definitions 2.19 and 2.22.

Definition 2.19 Let $f : X \to \mathcal{G}^\Delta$. A subset $M \subseteq X$ is called an infimizer of $f$ if

$$
\inf \{ f(m) \mid m \in M \} = \inf \{ f(x) \mid x \in X \}.
$$
According to the definition of \( \hat{f} (\cdot; M) : X \to \mathcal{G}^\triangle \), it follows easily that
\[
\forall M \neq \emptyset : \inf \left\{ \hat{f} (x; M) \mid x \in X \right\} = \inf \left\{ f(x) \mid x \in X \right\}
\]
and \( M \) is an infimizer of \( f \), if and only if \( \{0\} \) is an infimizer of \( \hat{f} (\cdot; M) : X \to \mathcal{G}^\triangle \),
\[
\hat{f} (0; M) = \inf \left\{ \hat{f} (x; M) \mid x \in X \right\} \iff \inf \left\{ f(m) \mid m \in M \right\} = \inf \left\{ f(x) \mid x \in X \right\}.
\]

**Corollary 2.20** [14, Proposition 5.9] Let \( f : X \to \mathcal{G}^\triangle \) be a convex function, \( M \subseteq X \), then the following are equivalent.

(a) \( M \) is an infimizer of \( f \);
(b) \( \{0\} \) is an infimizer of \( \hat{f} (\cdot; M) \);
(c) \( \{0\} \) is an infimizer of \( \hat{f} (\cdot; \co M) \) and \( \hat{f} (0; M) = \hat{f} (0; \co M) \).

**Lemma 2.21** Let \( f : X \to \mathcal{G}^\triangle \) and \( x_0 \in \text{dom } f \). Then the following are equivalent

(a) \( f(x_0) = \inf f[X] \);
(b) \( \forall x \in X, \forall z^* \in C^- \setminus \{0\} : \varphi_{f,z^*}(x_0) \leq \varphi_{f,z^*}(x) \);
(c) \( \forall x \in X, \forall z^* \in C^- \setminus \{0\} : \varphi_{f,z^*}(x_0) \sim \varphi_{f,z^*}(x) \leq 0 \);
(d) \( \forall x \in X : 0 \in f(x_0) \sim f(x) \).

Each of those conditions implies

(f) \( \forall x \in X : 0^+ f(x_0) \preceq f(x) \sim f(x_0) \).

**Proof.** The equivalence between (a), (b), (c) and (e) is immediate, by Lemma 2.3 (c) and (d) are equivalent and by Lemma 2.6, (e) implies (f). \( \square \)

The property \( 0 \in f(x_0) \sim f(x) \) is equivalent to \( f(x_0) \sim f(x) \preceq C \). The equivalence between \( f(x_0) \preceq f(x) \) and \( f(x_0) \sim f(x) \preceq C \) is the \( \mathcal{G}^\triangle \) valued version of the trivial scalar result that \( a \preceq b \) is equivalent to \( a - b \preceq 0 \). On the other hand, \( f(x_0) \preceq f(x) \) implies \( 0^+ f(x_0) \preceq f(x) \sim f(x_0) \), which can be read as the \( \mathcal{G}^\triangle \) valued version of \( 0 \preceq (b-a) \). Notice however that \( 0^+ f(x_0) \) is not necessarily equal to \( C \), the neutral element in \( \mathcal{G}^\triangle \), but \( 0^+ f(x_0) \preceq C \), whenever \( f(x_0) \neq \emptyset \).

The notion of infimizer can be trivial, as \( \text{dom } f \) is always an infimizer of \( f \). Therefore further requirements are usually assumed, as additional conditions on the values \( f(x) \) for \( x \in M \), e.g. to be minimal in some sense. See [14, 17, 20] for further motivations and corresponding concepts.

**Definition 2.22** Let \( f : X \to \mathcal{G}^\triangle \) be a function. An element \( x_0 \in X \) is called a minimizer of \( f \), iff \( f(x_0) \) is minimal in \( f[X] \),
\[
\forall x \in X : f(x) \preceq f(x_0) \implies f(x) = f(x_0).
\]

The set of all minimal elements of \( f[X] \) is denoted by \( \text{Min } f[X] \).
If $x_0$ is a minimizer of a convex (set valued) function $f$, then $f(x) = f(x_0)$ is satisfied if and only if $f$ is constant on the set $\{x_t \in X \mid x_t = x_0 + t(x - x_0), t \in [0,1]\}$.

Some properties on the scalarizes functions $\varphi_{f,z^*}$ of $f$ can be related to the existence of a minimizer.

**Lemma 2.23** Let $f : X \to G^\Delta$ be a function, $x_0 \in \text{dom } f$. Then the following are equivalent

(a) $f(x_0) \in \text{Min } f[X]$;

(b) $f(x) \neq f(x_0) \Rightarrow \exists z^* \in C^- \setminus \{0\} : \varphi_{f,z^*}(x_0) < \varphi_{f,z^*}(x)$;

(c) $f(x) \neq f(x_0) \Rightarrow \exists z^* \in C^- \setminus \{0\} : \varphi_{f,z^*}(x) \neq -\infty \land \varphi_{f,z^*}^{-}(x_0)-\varphi_{f,z^*}^{-}(x) < 0$;

(d) $f(x) \neq f(x_0) \Rightarrow \exists z^* \in C^- \setminus \{0\} : 0 < \varphi_{f,z^*}(x)-\varphi_{f,z^*}^{-}(x_0)$;

(e) $f(x) \neq f(x_0) \Rightarrow 0 \notin f(x)-f(x_0)$.

**Proof.** Equivalences from (a) through (d) is immediate and by Lemma 2.3, (d) and (e) are equivalent. \[\square\]

**Definition 2.24 (Solution)** [17] Let $f : X \to G^\Delta$ be a function. An infimizer of $f$ consisting of only minimizers is called a solution to the optimization problem $(P)$.

**Example 2.25** Let $f : \mathbb{R} \to G^\Delta(\mathbb{R}^2, \mathbb{R}^2_+)$ be given as

$$f(x) = \{(-x, -x)\} \oplus \mathbb{R}^2_+.$$  

Then $\mathbb{N} \subseteq \mathbb{R}$ as well as any intervall $(x, +\infty) \subseteq \mathbb{R}$ are infimizers of $f$. However, $\text{Min } f[\mathbb{R}] = \emptyset$ hence no solution of $f$ exists.

In [14] the concept of $z^*$-minimizers was introduced, defining $x_0 \in X$ as a $z^*$-minimizer of $f : X \to G^\Delta$, iff $x_0$ is a minimizer of $\varphi_{f,z^*} : X \to \mathbb{R}$. In fact, this concept is independent from the one we are investigating. The following Example 2.26(a) due to F. Heyde proves that a solution in the sense of Definition 2.24 does not need to be a $z^*$-solution, while Example 2.26(b) provides a counterexample to the revers implication.

**Example 2.26** (a) Let $X = Z = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. The (closed and convex) function $f : X \to G^\Delta$ is defined as follows

$$f(x) = \begin{cases} 
\{z \in -x_1 + x_2 \leq z_1, -x_1 - x_2 \leq z_2, x_1 \leq z_1 + z_2\}, & \text{if } 0 \leq x_1; \\
\emptyset, & \text{else.}
\end{cases}$$

Then each $x_0 \in \text{dom } f$ is minimal and $M = \{x \in X \mid 0 < x_1, x_2\}$ is a solution of $(P)$, while no $x \in M$ is a $z^*$-solution for any $z^* \in C^- \setminus \{0\}$.  

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(b) Let $X = \mathbb{R}$, $Z = \mathbb{R}^2$ and $C = \mathbb{R}_+^2$. The (closed and convex) function $f : X \to \mathcal{G}^\triangle$ is defined as follows:

$$f(x) = \begin{cases} \{ z \in Z \mid \frac{1}{z_1} \leq z_2 \}, & \text{if } 0 = x; \\ \{ z \in Z \mid 0 \leq z_1, z_2 \}, & \text{if } 1 = x; \\ x f(1) \oplus (1-x) f(0), & \text{if } 0 \leq x \leq 1; \\ \emptyset, & \text{else.} \end{cases}$$

Then each $x_0 \in \text{dom } f$ is $z^*-$minimal with respect to $z^* \in \{(0,-1)^T, (-1,0)^T\}$, but the only minimizer of $f$ is $x = 1$ and $M = \{1\}$ is the only solution of (P).

### 2.3 Directional derivatives

The notion of variational inequalities related to an optimization problem involves the concept of directional derivatives.

Although we apply the following definition to convex functions $f : X \to \mathcal{G}^\triangle$, it extends the well known concept of (lower) Dini derivatives to functions mapping to any inf–residuated image space.

We stress that this approach allows to extend the Dini derivative of scalar valued functions to extended real valued functions (see e.g. [14, 23]), compare Example 2.32 below.

**Definition 2.27** Let $f : X \to \mathcal{G}^\triangle$ be a convex function, $x, u \in X$, then the directional derivative of $f$ at $x$ along direction $u$ is defined as

$$f'(x, u) = \liminf_{t \downarrow 0} \frac{1}{t} \left( f(x + tu) - f(x) \right) = \bigcap_{0 < t_0} \text{cl co} \bigcup_{t \in (0, t_0)} \frac{1}{t} \left( f(x + tu) - f(x) \right).$$

For convex (set valued) functions, the differential quotient is monotone.

**Proposition 2.28** Let $f : X \to \mathcal{G}^\triangle$ be a convex function, $x_0 \in X$ and $g : (0, +\infty) \to \mathcal{G}^\triangle$ be given by $g(t) = \frac{1}{t} \left( f(x + tu) - f(x) \right)$. Then for all $0 < s \leq t$ it holds $g(s) \preceq g(t)$.

**Proof.** Let $z_t \in g(t)$ and $0 < s < t$ be satisfied, then it exists an $r \in (0, 1)$ such that $s = rt$ and $f(x + su) \preceq (1-r)f(x) \oplus rf(x + tu)$. Thus,

$$f(x + su) - f(x) \preceq r(f(x + tu) - f(x)),$$

which in turn implies that

$$\frac{1}{s} \left( f(x + su) - f(x) \right) \preceq \frac{r}{rt} \left( f(x + tu) - f(x) \right),$$

as desired. □

The following result extends a well known property for convex single valued functions.
Proposition 2.29 Let \( f : X \to \mathcal{G}^\triangle \) be a convex function, \( x \in \text{dom} \ f \) and \( u \in X \). Then

\[
f'(x, u) = \inf_{0 < t} \frac{1}{t} \left( f(x + tu) - f(x) \right),
\]

\( f'(x, 0) = 0^+ f(x) \) and the function \( u \mapsto f'(x, u) \) is sublinear as a function from \( X \) to \( \mathcal{G}^\triangle(Z, 0^+ f(x)) \).

Proof. The first statement comes immediately from Proposition 2.28.

For all \( x \in X \), \( f'(x, 0) = \inf \frac{1}{t} (f(x) - f(x)) \) and thus \( f'(x, 0) = 0^+ f(x) \) whenever \( x \in \text{dom} f \) and \( f'(x, 0) = Z, \) else.

By definition, for all \( 0 < s, u \in X \) it holds

\[
f'(x, su) = s \cdot \inf_{0 < t \leq t_0} \frac{1}{t} \left( f(x + tsu) - f(x) \right) = sf'(x, u).
\]

Let \( x, u_1, u_2 \in X \) and \( s \in (0, 1) \) be assumed, then by Proposition 2.28 for all \( 0 < t_0 \) it holds

\[
f'(x, su_1 + (1 - s)u_2) = \inf_{0 < t \leq t_0} \frac{1}{t} \left( f(s(x + tu_1) + (1 - s)(x + tu_2)) - f(x) \right).
\]

By Lemma 2.5 (b), this implies

\[
f'(x, su_1 + (1 - s)u_2) \leq \inf_{0 < t \leq t_0} \frac{1}{t} \left( s \left( f(x + t(u_1)) - f(x) \right) \right) \oplus (1 - s) \left( f(x + t(u_2)) - f(x) \right)
\]

and as \( \mathcal{G}^\triangle \) is inf–residuated and again by Proposition 2.28,

\[
f'(x, su_1 + (1 - s)u_2) \leq \frac{1}{t_0} \left( s(\left( f(x + tu_1) - f(x) \right) \right) \oplus (1 - s) \left( f(x + tu_2) - f(x) \right)
\]

\[
= s \frac{1}{t_0} \left( (\left( f(x + tu_1) - f(x) \right) \right) \oplus (1 - s)f'(x, u_2).
\]

But, as this is true for all \( 0 < t_0 \) and \( \mathcal{G}^\triangle \) is inf–residuated,

\[
f'(x, su_1 + (1 - s)u_2) \leq sf'(x, u_1) \oplus (1 - s)f'(x, u_2)
\]

is satisfied. \( \square \)

Remark 2.30 Since the differential quotients form a decreasing net of convex sets, their union is convex. Therefore

\[
f'(x, u) = \text{cl} \bigcup_{t > 0} \frac{1}{t} \left( f(x + tu) - f(x) \right)
\]

whenever \( f : X \to \mathcal{G}^\triangle \) is a convex function, \( x, u \in X \), the convex hull can be dropped.
Remark 2.31 Let \( f : X \to \mathcal{G}^\triangle \) be a convex function, \( x_0 \in \text{dom } f \) and \( x \in X \). If \( f'(x_0, x - x_0) \neq \emptyset \), then \([0, t_0] \subseteq \text{dom } f_{x_0,x} \) is true for some \( t_0 \in (0, 1) \) and for all \( t \in (0, t_0) \) it holds

\[
0^+ f'(x_0, x - x_0) \subseteq 0^+ f(x_t) \not\subseteq 0^+ f(x_0).
\]

Indeed, as \( f \) is convex, \( 0^+ f(x_t) \) is constant on the set \((0, t_0)\) and \( 0^+ f(x_t) \not\subseteq 0^+ f(x_0) \). Also,

\[
f'(x_0, x - x_0) \subseteq \frac{1}{t} \left( f(x_t) - f(x_0) \right)
\]

and both sets are convex, hence \( 0^+ f'(x_0, x - x_0) \not\subseteq 0^+ f(x_0) \) by Lemma 2.9.

Example 2.32 Let \( \varphi : X \to \overline{\mathbb{R}} \) be a convex function, \( f : X \to \mathcal{G}^\triangle(\mathbb{R}, \mathbb{R}_+) \) the set valued extension of \( \varphi \). If \( \varphi : X \to \overline{\mathbb{R}} \) is proper, \( x \in \text{dom } \varphi \), then \( f'(x,u) \) coincides with the upper Dedekind cut of the classic directional derivative of \( \varphi \), while in general,

\[
f'(x,u) = \left( \inf_{0 < t} \frac{1}{t} \left( \varphi(x + tu) - \varphi(x) \right) \right) + \mathbb{R}_+.
\]

Especially, if \( \varphi(x) = +\infty \), then \( f'(x,u) = \mathbb{R} \) for all \( u \in X \), while if \( x \in \text{dom } \varphi \) and \( \varphi(x) = -\infty \), then a careful case study provides

\[
f'(x,u) = \begin{cases} \mathbb{R}, & \text{if } u \in \text{cone (dom } \varphi - \{ x \}) \\ \emptyset, & \text{else.} \end{cases}
\]

Setting

\[
\varphi'(x,u) = \inf_{0 < t} \frac{1}{t} \left( \varphi(x + tu) - \varphi(x) \right)
\]

for all \( x,u \in X \), then this is an extension from the known definition to the case where \( \varphi \) can be improper or \( x \not\in \text{dom } \varphi \).

If \( f(x) = f_{z^*}(x) \) for all \( x \in X \) and \( f \) is a convex function, then the scalarization of the derivative \( \varphi'_{f_{z^*}(x),z^*}(u) \) is equal to the derivative of the scalarization, \( \varphi'_{f,x}(x,u) \) for all \( x,u \in X \). In general only the following inequality can be proven.

\[
\forall z^* \in C^-(\mathbb{R} \setminus \{ 0 \}), \forall x,u \in X : \quad \varphi'_{f_{z^*}(x),z^*}(u) \leq \varphi'_{f(x,u),z^*}(u)
\]

Example 2.33 Let \( f : \mathbb{R} \to \mathcal{G}^\triangle(\mathbb{R}, \{ 0 \}) \) be defined as \( f(x) = \left( -\sqrt{1-x^2}, \sqrt{1-x^2} \right) \), whenever \( x \in [-1,1] \) and \( f(x) = \emptyset \), else. Then \( f'(0,0) + z \notin f(t) \) for any \( t \neq 0 \), so \( f'(0,0) = \emptyset \). On the other hand, \( \varphi_{f,s}(x) = -|s| \cdot \sqrt{1-x^2} \) for all \( s \neq 0 \) and thus \( \varphi'_{f,s}(x,u) = -|s| \cdot \frac{x}{\sqrt{1-x^2}} \cdot u \)

for all \( x \in (-1,1) \), especially \( \varphi'_{f,s}(0,0) = 0 \) for all \( s \neq 0 \). Hence,

\[
\emptyset = f'(0,u) \subsetneq \bigcap_{z^* \in \{ 0 \}} f'_{z^*}(0,u) = \{ 0 \}
\]

Proposition 2.34 Let \( f : X \to \mathcal{G}^\triangle \) be a convex function, \( x,u \in X \). Then

\[
\bigcap_{z^* \in C^- \setminus \{ 0 \}} f'_{z^*}(x,u) \not\subseteq f'(x,u); \quad \forall z^* \in C^- \setminus \{ 0 \} \colon \quad \varphi'_{f_{z^*}(x),z^*}(u) \leq \varphi'_{f(x,u),z^*}(u).
\]
PROOF. By definition and Lemmas 2.3 and 2.2,

\[
f'(x, u) = \text{cl co} \bigcup_{0 < t, z^* \in C^- \setminus \{0\}} \bigcap_{z^* \in C^- \setminus \{0\}} \left\{ z \in Z \mid \frac{1}{t} \left( \varphi_{f,z^*}(x + tu) - \varphi_{f,z^*}(x) \right) \leq -z^*(z) \right\}
\]

\[
\subseteq \bigcap_{z^* \in C^- \setminus \{0\}} \text{cl co} \bigcup_{0 < t} \left\{ z \in Z \mid \frac{1}{t} \left( \varphi_{f,z^*}(x + tu) - \varphi_{f,z^*}(x) \right) \leq -z^*(z) \right\}
\]

\[
= \bigcap_{z^* \in C^- \setminus \{0\}} \left\{ z \in Z \mid \inf_{0 < t} \frac{1}{t} \left( \varphi_{f,z^*}(x + tu) - \varphi_{f,z^*}(x) \right) \leq -z^*(z) \right\}
\]

hence the inclusion is proven, implying the inequality as well. \qed

In the sequel, some results require equality in at least one of the inequalities given in Proposition 2.34. By strong regularity, we refer to condition

\[
\forall z^* \in C^- \setminus \{0\} : \quad \varphi_{f',(x,)z^*}(u) = \varphi'_{f,z^*}(x,u)
\]

(SR)

and by weak regularity to the following condition.

\[
f'(x,u) = \bigcap_{z^* \in C^- \setminus \{0\}} f'_{z^*}(x,u)
\]

(WR)

Clearly, (SR) implies (WR).

Remark 2.35 Let \( f : X \to \mathcal{G} \) be a convex function. It is easy to see that if \( x \notin \text{dom} \ f \), then \( f'(x,u) = Z \) and \( \varphi'_{f,z^*}(x,u) = -\infty \) are satisfied for all \( u \in X \) and all \( z^* \in C^- \setminus \{0\} \).

On the other hand, if \( x \in \text{dom} \ f \), then \( \text{dom} \ \varphi'_{f,z^*}(x, \cdot) = \text{cone} \{ \text{dom} \ f + \{ -x \} \} \cup \{0\} \) is true for all \( z^* \in C^- \setminus \{0\} \) and the derivative is sublinear. Hence, \( \varphi'_{f,z^*}(x,u) = -\infty \) implies either \( \varphi_{f,z^*}(x) = -\infty \), or \( \varphi'_{f,z^*}(x,-u) = +\infty \).

Especially, \( \text{dom} f'(x, \cdot) \subseteq \text{dom} \varphi'_{f,z^*}(x, \cdot) \) is always satisfied, hence if \( \varphi_{f,z^*}(x) \in \mathbb{R} \), then either \( x - tu \notin \text{dom} \ f \) is satisfied for all \( 0 < t \), or \( -\infty < \varphi'_{f,z^*}(x,u) \leq \varphi_{f,(x,\cdot)z^*}(u) \).

3 Main Results

As our solution concept involves both attainment of the infimum in a set and minimality in each element of this set, we need suitable inequalities for each of these properties.

The solution of a variational inequality is usually a singleton, while the infimizer of (P) is a set. However, Corollary 2.20 allows to characterize an infimizer \( M \) by proving \( \hat{f}(0; M) = \inf f \{ X \} \), or in other words \( \{0\} \) is a single valued infimizer of the optimization problem

\[
\text{minimize } \hat{f}(x; M) \text{ subject to } x \in X. \quad \text{(P(M))}
\]

Given a single valued convex function \( \varphi : X \to \mathcal{R} \), a solution to a variational inequality of Stampacchia type is a point \( x_0 \in X \) such that \( 0 \leq \varphi'(x_0, x - x_0) \) for all \( x \in X \). According to our setting, a most natural extension of this property is given in the following definition.
Definition 3.1 Let $f : X \to \mathcal{G}^\triangledown$ be a convex function, $x_0 \in \text{dom } f$. Then $x_0$ solves the strict set valued Stampacchia inequality, if and only if

$$\forall x \in X : \ 0^+ f(x_0) \preceq f'(x_0, x - x_0). \quad (SVI_I)$$

However, it turns out that, in the set–valued case, infimizers (and minimizers) are often characterized more adequately if a scalar type of variational inequalities is considered, instead of considering a set–valued variational inequality.

Definition 3.2 Let $f : X \to \mathcal{G}^\triangledown$ be a convex function, $x_0 \in \text{dom } f$. Then $x_0$ solves the strict scalarized Stampacchia inequality, if and only if

$$\forall x \in X, \forall z^* \in C^- \setminus \{0\} : \ \varphi_{f,z^*}(x_0) = -\infty \lor 0 \leq \varphi'_{f,z^*}(x_0, x - x_0). \quad (svi_I)$$

About the relation of scalarized an set valued variational inequalities, throughout the paper it shall be clear that while one type implies the other without further assumptions, for the revers implication either one of the regularity assumption (SR) or (WR) will be needed.

Proposition 3.3 Let $f : X \to \mathcal{G}^\triangledown$ be a convex function, $x_0 \in \text{dom } f$. If $x_0$ solves $(svi_I)$, then it also solves $(SVI_I)$. If additionally the strong regularity condition (SR) is satisfied, then the revers implication is true as well.

Proof. By Lemma 2.6 and Proposition 2.34, $(svi_I)$ implies

$$0^+ f(x_0) \preceq \bigcap_{z^* \in C^- \setminus \{0\}} f'_{z^*}(x_0, x - x_0) \preceq f'(x_0, x - x_0).$$

On the other hand, by Lemma 2.6, (SR) combined with $(SVI_I)$ implies $(svi_I)$. \hfill \square

Theorem 3.4 Let $f : X \to \mathcal{G}^\triangledown$ be a convex function, $x_0 \in \text{dom } f$. Then $x_0$ solves $(svi_I)$, if and only if $f(x_0) = \inf f[X]$.

Proof. By Lemma 2.21, $f(x_0) = \inf f[X]$ is true, if and only if

$$\forall x \in X, \forall z^* \in C^- \setminus \{0\} : \ \varphi_{f,z^*}(x_0) = -\infty \lor 0 \leq \varphi'_{f,z^*}(x) - \varphi'_{f,z^*}(x_0),$$

which immediately implies $(svi_I)$, the opposite implication being true, as $f$, thus $\varphi_{f,z^*}$ is convex. \hfill \square

Remark 3.5 According to Proposition 3.3, the set valued variational inequality $(SVI_I)$ is a necessary condition for $\{x_0\}$ to be an infimizer of $f$. Under the regularity condition (SR) it is also a sufficient condition.

Given a single valued convex function $\varphi : X \to \mathbb{R}$, a solution to a variational inequality of Minty type is a point $x_0 \in X$ such that $\varphi'(x, x_0 - x) \leq 0$ for all $x \in X$. 

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Definition 3.6 Let \( f : X \rightarrow \mathcal{G}^\triangledown \) be a convex function, \( x_0 \in \text{dom } f \). Then \( x_0 \) solves the strict set valued Minty inequality, if and only if

\[
\forall x \in X : \quad f'(x, x_0 - x) \leq 0^+ f(x_0). \tag{MVI}_f
\]

Equivalently, \( x_0 \) is a solution to the strict set valued Minty inequality, if and only if

\[
\forall x \in X : \quad 0 \in f'(x, x_0 - x). \tag{mvi}_f
\]

The previous definition can be related to the following family of a scalar Minty inequalities.

Definition 3.7 Let \( f : X \rightarrow \mathcal{G}^\triangledown \) be a convex function, \( x_0 \in \text{dom } f \). Then \( x_0 \) solves the strict scalarized Minty inequality, if and only if

\[
\forall x \in X, \forall z^* \in C^- \setminus \{0\} : \quad \phi'_{f,z^*}(x, x_0 - x) \leq 0. \tag{mvi}_f
\]

Proposition 3.8 Let \( f : X \rightarrow \mathcal{G}^\triangledown \) be a convex function, \( x_0 \in \text{dom } f \). If \( x_0 \) solves \((MVI)_f\), then it also solves \((mvi)_f\). If additionally the regularity condition \((WR)\) is satisfied, the revers implication is true.

Proof. If \( x_0 \) solves \((MVI)_f\), then Proposition 2.34 implies \((mvi)_f\). On the other hand, assuming \((mvi)_f\) and the regularity condition \((WR)\), then \( 0 \in f'(x, x_0 - x) \) is satisfied for all \( x \in X \), in other words \((MVI)_f\). \( \square \)

Theorem 3.9 Let \( f : X \rightarrow \mathcal{G}^\triangledown \) be a convex function, \( x_0 \in \text{dom } f \). Then \( f(x_0) = \inf f[X] \), if and only if \( x_0 \) solves \((MVI)_f\) and for all \( x \in X \) the function \( f_{x_0,x} : [0, 1] \rightarrow \mathcal{G}^\triangledown \) is lattice l.s.c. in 0.

On the other hand, if \( x_0 \) solves \((mvi)_f\) and for all \( x \in X \) the function \( f_{x_0,x} \) is C\(^-\) \( \setminus \{0\} \)-l.s.c. in 0, then \( f(x_0) = \inf f[X] \).

Proof. By Lemma 2.21, \( f(x_0) = \inf f[X] \), iff \( 0 \in f(x_0) - f(x) \) for all \( x \in X \), hence by the monotonicity of the differential quotient (see Proposition 2.28)

\[
f'(x, x_0 - x) \leq f(x_0) - f(x) \leq 0^+ f(x_0)
\]

is satisfied, proving \((MVI)_f\). When \( f(x_0) \leq f(x) \) is assumed,

\[
f(x_0) \leq \bigcap_{t_0 \in (0,1)} \text{cl co } \bigcup_{t \in (0,t_0)} f_{x_0,x}(t)
\]

is satisfied, hence \( f_{x_0,x} \) is lattice l.s.c. in 0 for all \( x \in X \). On the other hand, property \((MVI)_f\) combined with convexity of \( f \) implies

\[
\forall x \in X, \forall s, t \in (0, 1] : \quad s < t \Rightarrow f(x_s) \leq f(x_t),
\]

hence if \( f_{x_0,x} \) is lattice l.s.c. in 0, then

\[
\forall x \in X : \quad f(x_0) = \inf f_{x_0,x} [0, 1] \leq f(x),
\]

and \( f(x_0) = \inf f[X] \) is proven, if the same holds true for all \( x \in X \).

The proof of the last implication goes along the same lines. \( \square \)

Recall that if \( f_{x_0,x} : [0, 1] \rightarrow \mathcal{G}^\triangledown \) is C\(^-\) \( \setminus \{0\} \)-l.s.c. at 0 for all \( x \in X \), then each such function is also lattice l.s.c. in 0. In this case, \((MVI)_f\) and \((mvi)_f\) are equivalent.
Remark 3.10  The previous results are summarized in the following scheme of relations.

Applying the previous relations and the inf–translations we can finally get a variational characterization of a set $M$ to be an infimizer of $f$.

**Corollary 3.11** Let $f : X \to \mathcal{G}^\triangle$ be a convex function, $M \subseteq X$ a set with $M \cap \text{dom } f \neq \emptyset$ and

$$\hat{f}(0; M) = \hat{f}(0; \text{co } M).$$

Then $M$ is an infimizer of $f$, if and only if $(\text{svi}_1)$ is satisfied at 0 for $\hat{f}(\cdot; \text{co } M)$. In this case, $\hat{f}(\cdot; \text{co } M)$ is $C^- \setminus \{0\}$–l.s.c. at 0 and $(\text{MVI}_1)$ (and $(\text{mvi}_1)$) is satisfied at 0 for $\hat{f}(\cdot; \text{co } M)$.

On the other hand, if $(\text{MVI}_1)$ (or $(\text{mvi}_1)$) is satisfied at 0 for $\hat{f}(\cdot; \text{co } M)$ and one of the conditions in Lemma 2.18 is satisfied, then $\hat{f}(\cdot; \text{co } M)$ is $C^- \setminus \{0\}$–l.s.c. at 0 and $M$ is an infimizer of $f$.

In the remainder of this section, we deal with variational inequalities to characterize the notion of minimizers. The variational inequalities of Stampacchia, as well as Minty, type are presented both in a set valued and a scalar(ized) formulation.

**Definition 3.12** Let $f : X \to \mathcal{G}^\triangle$ be a convex function, $x_0 \in \text{dom } f$. Then $x_0$ solves the set valued Stampacchia inequality, if and only if

$$f(x_0) = Z \lor \forall x \in \text{dom } f : f(x) \neq f(x_0) \Rightarrow 0 \notin f'(x_0, x - x_0). \quad (\text{SVI}_M)$$

**Remark 3.13** In $(\text{SVI}_M)$, the condition $0 \notin f'(x_0, x - x_0)$ provides a set valued version of the property $\varphi'(x_0, x - x_0) \notin 0$ for scalar convex functions. The same inequality can be expressed also by the condition

$$f(x_0) = Z \lor \forall x \in \text{dom } f : f(x) \neq f(x_0) \Rightarrow f'(x_0, x - x_0) \cap -0^+f(x_0) = \emptyset. \quad (3.1)$$

However, since the image space $\mathcal{G}^\triangle$ is not totally ordered, there is a notable difference between these and the condition $f'(x_0, x - x_0) \subset 0^+f(x_0)$.

**Definition 3.14** Let $f : X \to \mathcal{G}^\triangle$ be a convex function, $x_0 \in \text{dom } f$. Then $x_0$ solves the scalarized Stampacchia inequality, if and only if

$$f(x_0) = Z \lor \forall x \in \text{dom } f : f(x) \neq f(x_0) \Rightarrow \exists z^* \in C^- \setminus \{0\} : 0 < \varphi'_{f, z^*}(x_0, x - x_0). \quad (\text{svi}_M)$$
Property \( (svi_M) \) implies
\[
\begin{aligned}
\forall x \in \text{dom} \ f & : \ f(x_0) \neq f(x) \ \Rightarrow \\
\exists z^* \in C^- \setminus \{0\} & : \ -\infty = \varphi_{f,z^*}(x_0) < \varphi_{f,z^*}(x) \ \lor \ 0 < \varphi'_{f,z^*}(x_0, x - x_0).
\end{aligned}
\tag{3.2}
\]

If additionally \( f_{x_0,x} : \mathbb{R} \to \mathbb{R} \) is \( C^- \setminus \{0\} \)-l.s.c. in 1 for all \( x \in X \), then both properties are equivalent.

**Proposition 3.15** Let \( f : X \to G^\triangle \) be a convex function, \( x_0 \in \text{dom} \ f \). If \( x_0 \) solves \( (svi_M) \), then it also solves \( (SVI_M) \). If additionally the regularity condition \( (WR) \) is satisfied, then \( x_0 \) solves \( (SVI_M) \), if and only if it solves \( (svi_M) \).

**Proof.** By Proposition 2.34, \( (svi_M) \) implies \( (SVI_M) \). On the other hand, \( (SVI_M) \) combined with the regularity condition \( (WR) \) implies \( (svi_M) \). \( \square \)

For the sake of completeness, we quote [14, Proposition 5.5], where it is proven that, if \( \text{dom} \ f \neq \emptyset \), then
\[
f_{z^*}(x) = Z \ \lor \ \forall x \in X : \ 0 \leq (\varphi_{f,z^*})'(x_0, x - x_0)
\]
is equivalent to \( f_{z^*}(x_0) = \inf f_{z^*} [X] \). However, as it has already been shown in Example 2.26, this concept of optimality is not equivalent to that investigated in this paper.

**Theorem 3.16** Let \( f : X \to G^\triangle \) be a convex function and \( x_0 \in \text{dom} \ f \). If \( x_0 \) solves \( (SVI_M) \) or \( (3.2) \), then \( f(x) \in \text{Min} f [X] \).

**Proof.** Let \( x_0 \) be a solution of \( (SVI_M) \), then
\[
f(x) \neq f(x_0) \ \Rightarrow \ 0 \notin f(x) - f(x_0)
\]
is immediate, hence by Lemma 2.23 \( x_0 \) is a minimizer of \( f \). Assuming property \( (3.2) \) to be satisfied, then
\[
f(x) \neq f(x_0) \ \Rightarrow \ \exists z^* \in C^- \setminus \{0\} : \ 0 < \varphi_{f,z^*}(x) - \varphi_{f,z^*}(x_0)
\]
is satisfied for all \( x \in \text{dom} \ f \), by Lemma 2.23 implying \( f(x_0) \in \text{Min} f [X] \). \( \square \)

In a similar way, we approach the Minty type inequalities.

**Definition 3.17** Let \( f : X \to G^\triangle \) be a convex function, \( x_0 \in \text{dom} \ f \). Then \( x_0 \) solves the set valued Minty inequality, if and only if
\[
f(x) \neq f(x_0) \ \Rightarrow \ 0^+ f(x) \neq f^*(x, x_0 - x). \tag{MVI_M}
\]

Again, \( (MVI_M) \) can be interpreted as the set valued version of the scalar Minty variational inequality, given by
\[
\varphi(x) \neq \varphi(x_0) \ \Rightarrow \ 0 \nsubseteq \varphi'(x, x_0 - x),
\]
but there is a significant difference to the condition \( 0^+ f(x) \subset f^*(x, x_0 - x) \), as \( G^\triangle \) is not totally ordered.
**Definition 3.18** Let \( f : X \to G^\triangle \) be a convex function, \( x_0 \in \text{dom} f \). Then \( x_0 \) solves the scalarized Minty inequality, if and only if

\[
f(x) \neq f(x_0) \implies \exists z^* \in C^- \setminus \{0\} : \; \varphi_{f,z^*}(x) \neq -\infty \land \varphi'_{f,z^*}(x,x_0-x) < 0. \quad (\text{mvi}_M)
\]

**Proposition 3.19** Let \( f : X \to G^\triangle \) be a convex function, \( x_0 \in \text{dom} f \). If \( x_0 \) solves (MVI\(_M\)), then it also solves (mvi\(_M\)). If additionally the regularity condition \((\text{SR})\) is satisfied, then \( x_0 \) solves (MVI\(_M\)), if and only if it solves (mvi\(_M\)).

**Proof.** If \( x_0 \) solves (MVI\(_M\)), then Proposition 2.34 implies (mvi\(_I\)). On the other hand, assuming (mvi\(_M\)) and the regularity condition \((\text{SR})\), then for all \( x \in X \) with \( f(x) \neq f(x_0) \) there is an element \( z \in f'(x,x_0-x) \setminus 0^+f(x) \) (compare Lemma 2.6 and Remark 2.35), in other words (MVI\(_M\)) is satisfied.

**Proposition 3.20** Let \( f : X \to G^\triangle \) be a convex function and \( x_0 \in \text{dom} f \). Then \( x_0 \) solves (mvi\(_M\)), if and only if for all \( x \in \text{dom} f \)

\[
f(x) \neq f(x_0) \implies \inf f_{x_0,x} (0,1) \preccurlyeq f(x) \land \inf f_{x_0,x} (0,1) \neq f(x). \quad (3.3)
\]

**Proof.** Let \( x_0 \) be a solution of (mvi\(_M\)). This is equivalent to state that for each \( x \in \text{dom} f \) with \( f(x) \neq f(x_0) \) there exists a \( z^* \in C^- \setminus \{0\} \) and a \( t \in (0,1) \) such that \( \varphi_{f,z^*}(x_t) - \varphi_{f,z^*}(x) < 0 \) and \( \varphi_{f,z^*}(x) \neq -\infty \), or equivalently \( \varphi_{f,z^*}(x_t) < \varphi_{f,z^*}(x) \).

In this case, (3.3) is immediate, as

\[
\inf f_{x_0,x} (0,1) \preccurlyeq \bigcap_{t_0 \in (0,1)} \text{cl} \bigcup_{t \in (t_0,1)} f_{x_0,x} (t) \preccurlyeq f(x)
\]

by convexity and \( \inf f_{x_0,x} (0,1) \preccurlyeq f(x_t) \), hence strict inclusion is satisfied. On the other hand, (3.3) implies that, if \( f(x) \neq f(x_0) \), then there exists an \( t \in (0,1) \) and \( z^* \in C^- \setminus \{0\} \) such that \( \varphi_{f,z^*}(x_t) < \varphi_{f,z^*}(x) \), hence \( \varphi_{f,z^*}(x) \neq -\infty \) and \( \varphi'_{f,z^*}(x,x_0-x) < 0 \) are satisfied, as the scalarization \( \varphi_{f,z^*} : X \to \mathbb{R} \) is convex.

**Theorem 3.21** Let \( f : X \to G^\triangle \) be a convex function and \( x_0 \in \text{dom} f \). If \( f(x_0) \in \text{Min} f[X] \), then \( x_0 \) solves (mvi\(_M\)). If \( x_0 \) solves

\[
f(x) \neq f(x_0) \implies \exists z^* \in M^* : \; \varphi_{f,z^*}(x) \neq -\infty \land \varphi'_{f,z^*}(x,x_0-x) < 0 \quad (3.4)
\]

where \( M^* \subseteq C^- \setminus \{0\} \) is a finite set and \( f_{x_0,x} \) is \( M^* \)-l.s.c. at \( 0 \), then \( f(x_0) \in \text{Min} f[X] \).

**Proof.** Let \( f(x_0) \in \text{Min} f[X] \) be assumed, then by Lemma 2.23

\[
f(x) \neq f(x_0) \implies \exists z^* \in C^- \setminus \{0\} : \; \varphi_{f,z^*}(x) \neq -\infty \land \varphi'_{f,z^*}(x,x_0-x) < 0.
\]

As the differential quotient in decreasing, this implies (mvi\(_M\)). On the other hand, let (3.4) be satisfied and let \( (\varphi_{f,z^*})_{x,x_0} : [0,1] \to \mathbb{R} \) be l.s.c. at \( 0 \) for all \( z^* \in M^* \), then \( f(x) \neq f(x_0) \) and
the convexity and lower semicontinuity of the scalarizations implies that there exist \( z^* \in M^* \) and \( t \in [0, 1) \) such that
\[
\inf (\varphi_{f,z^*})_{x_0,x} [0, 1] = \varphi_{f,z^*}(x_t) < \varphi_{f,z^*}(x).
\]
Now either \( f(x_t) = f(x_0) \) and \( f(x) \not\equiv f(x_0) \), or there exist \( t_1 \in [0, t) \) and \( z_1^* \in M^* \setminus \{z^*\} \) such that
\[
\inf (\varphi_{f,z_1^*})_{x_0,x} [0, 1] = \varphi_{f,z_1^*}(x_{t_1}) < \varphi_{f,z_1^*}(x_t) \leq \varphi_{f,z_1^*}(x).
\]
Especially,
\[
\varphi_{f,z_1^*}(x_t) = -\infty \lor 0 \leq \varphi_{f,z_1^*}(x_{t_1}, x_0 - x)
\]
\[
\varphi_{f,z_1^*}(x) \not\equiv -\infty \land \varphi_{f,z_1^*}(x, x_0 - x) < 0
\]
are satisfied. As \( M^* \) is finite, there exists \( t_0 \in [0, 1) \) such that
\[
\exists z_0^* \in M^* : \inf (\varphi_{f,z_0^*})_{x_0,x} [0, 1] = \varphi_{f,z_0^*}(x_{t_0}) < \varphi_{f,z_0^*}(x); \\
\forall z^* \in M^* : 0 \leq \varphi_{f,z^*}(x_{t_0}, x_0 - x) \lor \varphi_{f,z^*}(x_{t_0}) = -\infty.
\]
Hence especially \( f(x_{t_0}) = f(x_0) \) and \( f(x) \not\equiv f(x_0) \). \qed

**Remark 3.22** The previous results can be summarized in the following scheme of relations.

\[\text{Scheme of relations}\]

---

4 Application to vector optimization

In this section, we consider a vector valued function \( \psi : S \subseteq X \to Z \) and its epigraphical extension \( f = \psi^C : X \to \mathcal{G}^\Delta \), defined by \( f(x) = \psi(x) + C \), whenever \( x \in S \) and \( f(x) = \emptyset \), elsewhere. In the sequel, we refer only to \( \text{dom } f \), which is of course equal to \( S \), which is the effective domain of \( \psi \).

The function \( \psi \) is called \( C \)-convex, when for all \( x_1, x_2 \in S \) and all \( t \in (0, 1) \) it holds
\[
(1 - t)\psi(x_1) + t\psi(x_2) \subseteq \psi(x_1 + t(x_2 - x_1)) + C,
\]
or equivalently when graph \( f = \text{epi } \psi = \{(x, z) \in X \times Z \mid z \in \psi(x) + C\} \) is a convex set, compare \([19, \text{Definition 14.6}]\).

**Lemma 4.1** Let \( \psi : S \subseteq X \to Z \) be a \( C \)-convex function, \( x_0, x \in S \), then for all \( t \in (0, 1) \) it holds
\[
\frac{1}{t} \left( f(x_0 + t(x - x_0)) - f(x_0) \right) = \frac{1}{t} \left( \psi(x_0 + t(x - x_0)) - \psi(x_0) \right) + C,
\]
\[
\frac{1}{t} \left( \psi(x_0 + t(x - x_0)) - \psi(x_0) \right) \text{ is decreasing as } t \text{ converges to } 0 \text{ and (SR) is satisfied.}
\]
Proof. By definition, \( f(x_t) = \psi(x_t) + C \), as \( \psi \) is \( C \)-convex and \( x_0, x \in S \). Hence,

\[
\forall t \in (0, 1) : \quad \left( z \in \frac{1}{t} \left( f(x_t) - f(x_0) \right) \right) \iff \psi(x_0) + tz \in \psi(x_t) + C,
\]

or equivalently \( z \in \frac{1}{t} (\psi(x_0 + t(x - x_0)) - \psi(x_0)) + C \). By Proposition 2.28, the differential quotient is decreasing as \( t \) converges to 0 and by Lemma 2.2,

\[
-\sigma(z^* f'(x_0, x - x_0)) = \inf \left\{ -\sigma(z^*) \left( \frac{1}{t} \left( f(x_0 + t(x - x_0)) - f(x_0) \right) \right) \mid 0 < t \right\}
\]

for all \( z^* \in C^- \setminus \{0\} \). But it is easy to see that \( \varphi_{f, z^*}(x) = -z^* \psi(x) \) is satisfied for all \( z^* \in C^- \setminus \{0\} \) and all \( x \in S \), hence

\[
-\sigma(z^*) \left( \frac{1}{t} \left( f(x_0 + t(x - x_0)) - f(x_0) \right) \right) = -\frac{1}{t} (z^* \psi(x_0 + t(x - x_0)) - z^* \psi(x_0)),
\]

for all \( z^* \in C^- \setminus \{0\} \), proving the statement. \( \square \)

We introduce the set of infinite elements \( Z_{\infty} = \{z_{\infty} \mid z \in Z\} \). The element \( z_{\infty} \) will be interpreted as infinite element in direction \( z \), in other words

\[
z_{\infty} = \lim_{t \uparrow \infty} tz.
\]

It holds \( z_{\infty} = y_{\infty} \), if and only if \( y = \lambda z \) for some \( 0 < \lambda \) and \( 0_{\infty} = 0 \in Z \). For any \( z^* \in Z^* \) and \( z \in Z \), we define \( z^*(z_{\infty}) = \lim_{t \uparrow + \infty} z^*(tz) \). Especially, \( z^*(z_{\infty}) \in \mathbb{R} \) is satisfied, if and only if \( z^*(z_{\infty}) = z^*(z) = 0 \).

For a subset \( S \subseteq Z \), \( S_{\infty} \) denotes the set of all \( z_{\infty} \in Z_{\infty} \) with \( z \in S \setminus \{0\} \).

The space \( \tilde{Z} = Z \cup Z_{\infty} \) can be endowed with a topology defined by local bases of neighborhoods as follows. To any element \( z \in Z \), the set \( U(z) = U + \{z\} \) is a local base of neighborhoods in \( \tilde{Z} \). To any element \( z \in Z \setminus \{0\} \), the set

\[
U(z_{\infty}) = \{(tz) + \text{cone}(U + \{z\}) \cup (U + \{z\})_{\infty} \mid 0 < t, U \in U(z) \}
\]

is a local base of neighborhoods of \( z_{\infty} \). Especially, if \( K \subseteq Z \) is an open cone with \( z \in K \) and \( y \in Z \), then \( \{y\} + K \cup K_{\infty} \) is a neighborhood of \( z_{\infty} \), compare [3].

**Lemma 4.2** Let \( z \in Z \) be given and define \( (z_{\infty} + C) = \lim_{t \uparrow \infty} \inf(tz) + C \), then \( z_{\infty} + C = \lim_{t \uparrow \infty} \sup(tz) + C \) is satisfied. If \( z \notin -C \), then \( z_{\infty} + C = \sup_{0 < t} \{tz\} + C = 0 \), and if \( z \in -C \), then \( z_{\infty} + C = \inf_{0 < t} \{tz\} + C \). Especially, \( z_{\infty} + C = C \), if \( z \in C \cap -C \) and \( z_{\infty} + C = Z \), if for all \( z^* \in C^- \setminus \{0\} \) it holds \(-z^*(z) < 0 \).

**Proof.** By definition, \( z_{\infty} + C = \bigcap_{0 < t_0} \text{cl} \co \bigcup_{t_0 \leq t} \{tz\} + C \). Let \( z \in -C \), then \( \bigcap_{t_0 \leq t} \{tz\} + C = \{t_0 z\} + C \) and we claim \( \text{cl} \co \bigcup_{0 < t_0} \bigcap_{t_0 \leq t} \{tz\} + C \) = \( \text{cl} \co \bigcup_{0 < t_0} \{t_0 z\} + C \), or equivalently

\[
\lim_{t \uparrow \infty} \sup_{t \uparrow \infty} \{tz\} + C = \inf_{t \uparrow \infty} \{tz\} + C.
\]

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But since $\inf_{t>0} \{tz\} + C \preceq \liminf_{t \uparrow \infty} \{tz\} + C \preceq \limsup_{t \uparrow \infty} \{tz\} + C$ always holds true, this implies

$$z_{\infty} + C = \inf_{t>0} \{tz\} + C = \limsup_{t \uparrow \infty} \{tz\} + C.$$ 

On the other hand, let $z \notin -C$ be assumed, then $0 < -z^*(z)$ is satisfied for some $z^* \in C^- \setminus \{0\}$. Thus,

$$-\sigma(z^*)\text{cl co} \bigcup_{t_0 \leq t} \{\{tz\} + C\} = -z^*(t_0 z)$$

converges to $+\infty$ as $t_0$ converges to $+\infty$, hence $z_{\infty} + C = \emptyset$. But, since

$$\emptyset = \liminf_{t \uparrow \infty} \{\{tz\} + C\} \preceq \limsup_{t \uparrow \infty} \{\{tz\} + C\} \preceq \emptyset$$

it is proven that

$$z_{\infty} + C = \sup_{t>0} \{\{tz\} + C\} = \limsup_{t \uparrow \infty} \{\{tz\} + C\}.$$ 

Finally, recall that by Lemma 2.2 for $z \in -C$ it holds

$$z_{\infty} + C = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{ y \in Z \mid \inf_{0 < t} -z^*(tz) \leq -z^*(y) \right\}.$$

Hence if $z \in C \cap -C$, it is immediate that

$$z_{\infty} + C = \bigcap_{z^* \in C^- \setminus \{0\}} \{y \in Z \mid 0 \leq -z^*(y)\} = C,$$

while if for all $z^* \in C^- \setminus \{0\}$ it is assumed that $-z^*(z) < 0$ holds true, then $z_{\infty} + C = Z$.

In [3], a Dini directional derivative of $\psi : S \subseteq X \to Z$ at $x_0 \in S$ in direction $(x - x_0)$ with $x \in S$ has been defined as

$$\psi'(x_0, x - x_0) = \text{Lim sup}_{t \downarrow 0} \left\{ \frac{1}{t} \left( \psi(x_0 + t(x - x_0)) - \psi(x_0) \right) \right\} \subseteq \tilde{Z}$$

where $\text{Lim sup}_{t \downarrow 0} \{z_t\} = \\hat{\tilde{z}} \in \tilde{Z} \mid \exists \{z_{t_i}\}_{i \in \mathbb{N}} \subseteq \{z_t\}_{0 < t}, z_{tn} \to \hat{\tilde{z}}$ is the outer Painlevé-Kuratowski limit in $\tilde{Z}$ of a net $\{z_t\}_{t \downarrow 0} \subseteq Z$.

**Lemma 4.3** Let $\psi : S \subseteq X \to Z$ be a $C$-convex function, $f(x) = \psi(x)$ for all $x \in X$ and $x_0, x \in S$.

(a) If $z \in \psi'(x_0, x - x_0) \cap Z$, then $z + C = f'(x_0, x - x_0)$ and for all $z^* \in C^- \setminus \{0\}$ it holds

$$\varphi_{f^*, z^*}(x_0, x - x_0) = -z^*(z);$$

(b) If $z_{\infty} \in \psi'(x_0, x - x_0) \cap Z_{\infty}$, then $z \in -C$ and $z_{\infty} + C \subseteq 0^+ f(x_0, x - x_0);$

(c) If $\psi'(x_0, x - x_0) \cap Z \neq \emptyset$ and $z_{\infty} \in \psi'(x_0, x - x_0) \cap Z_{\infty}$, then $z \in C \cap -C$. 

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Proof.

(a) By definition, $z \in \psi'(x_0, x-x_0) \cap Z$ is satisfied, iff there is a decreasing sequence $\{t_i\}_{i \in \mathbb{N}} \subseteq \mathbb{R}_+$ such that $\frac{1}{t_i} \left( \psi(x_0 + t_i(x-x_0)) - \psi(x_0) \right)$ converges to $z$ as $i$ converges to $+\infty$. But this implies
\[
\forall z^* \in C^- \setminus \{0\} : -z^*(z) \leq \varphi_{z^*}'(x_0, x-x_0),
\]
hence $\{z\} + C \supseteq f'(x_0, x-x_0)$. On the other hand,
\[
z \in \text{cl} \bigcup_{0 < t} \left( \frac{1}{t} \left( \psi(x_0 + t(x-x_0)) - \psi(x_0) \right) + C \right) = f'(x_0, x-x_0).
\]

(b) Assume to the contrary that $z_{\infty} \in \psi'(x_0, x-x_0)$ and $z \notin -C$. Then there exist a $U \in U$ such that cone $(U + \{z\}) \cap -C = \emptyset$ and a subsequence $z_i = \frac{1}{t_i} \left( \psi(x_0 + t_i(x-x_0)) - \psi(x_0) \right)$, $i \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ there exists a $i_0 \in \mathbb{N}$ such that for all $i_0 \leq i$ it holds $z_i \in \{nz\} + \text{cone} \ (U + \{z\})$, especially $(\{z_1\} + (-C)) \cap (\{nz\} + \text{cone} \ (U + \{z\})) \neq \emptyset$ for all $n \in \mathbb{N}$. However, choosing $n$ sufficiently large, $nz - z_1 \in \text{cone} \ (U + \{z\})$ is satisfied, implying $\emptyset \neq -C \cap (\{nz-z_1\} + \text{cone} \ (U + \{z\})) \subseteq -C \cap \text{cone} \ (U + \{z\}) = \emptyset$, a contradiction.

(c) Especially by (a), $y \in \psi'(x_0, x-x_0) \cap Z$ is a lower bound of the set
\[
\left\{ \frac{1}{t} \left( \psi(x_0 + t(x-x_0)) - \psi(x_0) \right) \mid 0 < t \right\},
\]
hence if $z_{\infty} \in \psi'(x_0, x-x_0)$, then $\forall z^* \in C^- \setminus \{0\} : -z^*(y) \leq -z^*(z_{\infty})$, hence by (b) $z \in C \cap -C$.

More generally, we remark that taking the limit over a net of singletons and adding the ordering cone does not commute.

Example 4.4 Let $Z = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ be given, $\{z_i\}_{0 < t} \subseteq Z$ a subset of $Z$ with $z_i = (-t, -t^2)$. Then $\{z_i\}_{0 < t}$ is decreasing as $t$ converges to $+\infty$ and $\text{Lim sup}_{t \uparrow +\infty} \{z_i\} = (0, -1)$. However,
\[
\text{Lim sup}_{t \uparrow +\infty} \{z_i\} + C = \{z = (z_1, z_2) \in Z \mid 0 \leq z_1 \} \subsetneq \lim_{t \uparrow +\infty} \{z_i\} + C = Z.
\]

Proposition 4.5 [3] If $Z$ has finite dimension, then $\tilde{Z}$ is compact.

By Proposition 4.5, if $Z$ has finite dimension, then for a $C$-convex function $\psi : S \subseteq X \to Z$, $x_0, x \in S$ it holds
\[
\emptyset \neq \psi'(x_0, x-x_0) \subseteq Z \cup (-C)_\infty,
\]
so the derivative of $\psi$ at $x_0$ in the direction $x-x_0$ is nonempty and each element of $\psi'(x_0, x-x_0)$ is either finite (i.e. an element of $Z$), or an element of $(-C)_\infty$, (that is an infinite element of $\tilde{Z}$ which is "less or equal" than $0 \in Z$ ).
The set of all efficient elements of $\psi [X]$ is given by

$$\text{Eff} \psi [X] = \{ z \in \psi [X] : \forall y \in \psi [X] : z \in y + C \Rightarrow z \in y + (-C \cap C) \}. \quad (\text{Eff})$$

and $x_0 \in \text{dom} f$ is an efficient solution, iff $\psi(x_0) \in \text{Eff} \psi [X]$. An element $x_0 \in \text{dom} f$ is a minimizer of $f$ if and only if it is an efficient solution to $\psi$. Moreover,

$$\bigcup_{f(x) \in \text{Min} f[X]} f(x) = \text{Eff} \psi [X] + C \quad (4.1)$$

and a solution to $(P)$ exists if and only if $\text{cl} \text{co} (\text{Eff} \psi [X] + C) = \text{cl} \text{co} (\psi [X] + C)$.

In the sequel, we only focus on the characterization of minimizers of $\psi$ in order to get conditions on efficient solutions of $\psi$. In this setting, we do not get any results about infimizer but those already obtained in Section 3, as the inf–translation $(\psi^C)_M : X \rightarrow G^0$ is in general not the extension of a vector valued function.

**Corollary 4.6** Let $\psi : S \subseteq X \rightarrow Z$ be a $C$–convex function, $x_0 \in S$ and $f(x) = \psi^C(x)$ for all $x \in X$. Then $(SVI_M)$, $(svi_M)$ and $(3.2)$ are equivalent. Especially, if for all $x \in S$ with $\psi(x) \neq \psi(x_0)$ there exists $z \in Z$ such that $z \in \psi'(0,x-x_0) \setminus -C$, then $\psi(x_0) \in \text{Eff} \psi [X]$.

**Proof.** The first part of the statement is Proposition 3.15, as by Lemma 4.1, (SR), hence especially (WR) is satisfied. The existence of $z \in Z$ with $z \in \psi'(0,x-x_0) \setminus -C$ implies the existence of a $z^* \in C^- \setminus \{0\}$ with $0 < \varphi_{f,z^*}^f(x_0,x-x_0)$, compare Lemma 4.3(a). Thus in this case, $(3.2)$ is satisfied, proving the statement.

**Corollary 4.7** Let $\psi : S \subseteq X \rightarrow Z$ be a $C$–convex function, $x_0 \in S$ and $f(x) = \psi^C(x)$ for all $x \in X$. Then $x_0$ solves $(MVI_M)$, if and only if it solves $(mvi_M)$. Moreover, $(MVI_M)$ is equivalent to

$$(x \in S, t \in (0,1), \psi(x_t) \neq \psi(x_0)) \Rightarrow \exists z^* \in C^- \setminus \{0\} : (-z^*)^f(x_t,x_0-x) < 0. \quad (4.2)$$

**Proof.** Again, the first part of the statement is true as (SR) is guaranteed by Lemma 4.1 (compare Proposition 3.8). As $f(x) = \psi^C(x)$ for all $x \in X$ is assumed, $\varphi_{f,z^*}^f(x) \neq -\infty$ is always true for all $z^* \in C^- \setminus \{0\}$. It is left to prove the implication of $(mvi_M)$ under the assumption of $(4.2)$.

Let $x \in S$ and $\psi(x_t) \neq \psi(x_0)$ be assumed for some $t \in (0,1)$. Then by convexity of $\varphi_{f,z^*}^f : X \rightarrow \mathbb{R}$, $(-z^*)^f(x_t,x_0-x) < 0$ implies $(-z^*)^f(x_0,x-x_0) < 0$. On the other hand, if $\psi(x_t) = \psi(x_0)$ is satisfied for all $t \in (0,1)$, then also by convexity of the scalarizations

$$-z^* \psi(x_0) = \liminf_{t \downarrow 0} (-z^* \psi(x_t)) \leq -z^* \psi(x)$$

is satisfied for all $z^* \in C^- \setminus \{0\}$. Especially, $\psi(x) \neq \psi(x_0)$ implies

$$\exists z^* \in C^- \setminus \{0\} : -z^* (x_0) = -z^* \psi(x_t) < -z^* \psi(x),$$

hence $\varphi_{f,z^*}^f(x,x_0-x) = -\infty < 0$. \qed
Remark 4.8 As \((-z^* \psi)'(x, \cdot) : X \rightarrow \overline{\mathbb{R}}\) is sublinear, if \(\psi : S \subseteq X \rightarrow Z\) is \(C\)-convex, \(x_0, x \in S\) implies \((-z^* \psi)'(x_1, x_0 - x) \in \overline{\mathbb{R}}\) for all \(z^* \in C^- \setminus \{0\}\) and all \(t \in (0, 1)\). In this case, \(z_{\infty} \in \psi'(x_1, x_0 - x)\) implies \(z \in C \cap C^-\).

Indeed, under the given assumptions, \(-z^* \psi(x_1) \in \overline{\mathbb{R}}\) is true for all \(t \in (0, 1)\), hence
\[
0 = (-z^* \psi)'(x_1, 0) \leq (-z^* \psi)'(x_1, x - x_0) + z^* \psi'(x_1, x_0 - x)
\]
and \((-z^* \psi)'(x_1, x_0 - x) = -\infty\) implies \((-z^* \psi)'(x_1, x - x_0) = +\infty\). But as the domain of \((-z^* \psi)'(x_1, \cdot)\) is given by \(\text{dom}(-z^* \psi)'(x_1, \cdot) = \text{cone}(S + \{x_1\})\), this is a contradiction. By Lemma 4.3(b), \(z_{\infty} \in \psi'(x_1, x_0 - x)\) implies \(z \in -C\). Assuming \(z \notin C\) would imply the existence of a \(z^* \in C^- \setminus \{0\}\) such that \(\psi'(x_1, x_0 - x) = -\infty\), a contradiction.

Proposition 4.9 Let \(\psi : S \subseteq X \rightarrow Z\) be a \(C\)-convex function, \(x_0 \in S\) and \(f(x) = \psi^C(x)\) for all \(x \in X\). If \(x \in S\) and \(t \in (0, 1)\) imply
\[
\psi(x_1) \neq \psi(x_0) \quad \Rightarrow \quad \psi'(x_1, x_0 - x) \notin (C \cup C_{\infty}),
\]
then \(x_0\) solves \((MVI_M)\) and
\[
\psi(x_1) \neq \psi(x_0) \quad \Rightarrow \quad \psi'(x_1, x_0 - x) \subseteq (C \cap -C)_{\infty} \cup (Z \setminus C).
\]

Proof. Under the given assumptions, let \(\psi(x_1) \neq \psi(x_0)\) be true, then \(\psi'(x_1, x_0 - x) \neq \emptyset\) and especially,
\[
\psi'(x_1, x_0 - x) \cap ((-C)_{\infty} \cap C) \neq \emptyset.
\]
Thus if \(z \in \psi'(x_1, x_0 - x) \cap (Z \setminus C)\), then there is a \(z^* \in C^- \setminus \{0\}\) satisfying \(\varphi_{f,z^*}'(x_1, x_0 - x) < 0\).

On the other hand, if \(z_{\infty} \in \psi'(x_1, x_0 - x) \cap ((-C)_{\infty} \cap C_{\infty})\), then \(\varphi_{f,z^*}'(x_1, x_0 - x) = -\infty\) is satisfied for some \(z^* \in C^- \setminus \{0\}\), a contradiction. Hence
\[
\emptyset \neq \psi'(x_1, x_0 - x) \subseteq ((-C)_{\infty} \cap C_{\infty}) \cup Z
\]
and thus by assumption
\[
\emptyset \neq \psi'(x_1, x_0 - x) \cap (Z \setminus C).
\]
But this implies
\[
\forall z \in \psi'(x_1, x_0 - x) \cap (Z \setminus C) : \quad \emptyset \neq \psi'(x_1, x_0 - x) \cap Z \subseteq \{z\} + (C \cap -C) \subseteq Z \setminus C,
\]
implying the existence of a \(z^* \in C^- \setminus \{0\}\) satisfying \(\varphi_{f,z^*}'(x_1, x_0 - x) < 0\), hence \((mvi_M)\) and therefore \((MVI_M)\) is satisfied.

Theorem 4.10 Let \(\psi : S \subseteq X \rightarrow Z\) be a \(C\)-convex function, \(x_0 \in S\) and \(f(x) = \psi^C(x)\) for all \(x \in X\). If \(f_{x_0,x} : C^- \setminus \{0\}\) is.c. in \(0\) for all \(x \in X\) and \(C\) is polyhedral, then \(x_0\) solves \((MVI_M)\), if and only if \(\psi(x_0) \in \text{Eff}\psi[X]\).

Proof. If \(C\) is polyhedral, then so is \(C^-\), that is it exists a finite set \(M^* = \{m_1, ..., m_n\} \in C^- \setminus \{0\}\) such that
\[
C = \bigcap_{i=1}^n \{z \in Z \mid 0 \leq -m_i^*(z)\}.
\]
Also, for all \( z^* \in C^- \setminus \{0\} \), \( z^* \in \text{cone co} M^* \) and for all \( z \in Z \) and all \( z^* \in C^- \setminus \{0\} \), if
\[
z^* = \sum_{i=1}^{n} t_i m_i^*, \quad 0 \leq t_1, \ldots, t_n,
\]
then \( -z^*(z) = -\sum_{i=1}^{n} t_i m_i^*(z) \). Let \( (-z^* \psi)'(x, x_0 - x) < 0 \) be satisfied for some \( z^* = \sum_{i=1}^{n} t_i m_i^* \in C^- \setminus \{0\} \) and \( x_0 \in S \). Then there exists \( 0 < \bar{s} \) such that (for all \( s \in (0, \bar{s}) \))
\[
-z^*(\frac{1}{s} (\psi(x + s(x_0 - x)) - \psi(x))) < 0,
\]
hence there exists at least one \( i \in \{1, \ldots, n\} \) such that
\[
-m_i^*(\frac{1}{s} (\psi(x_t + s(x_0 - x)) - \psi(x_t))) < 0,
\]
implying \( (-m_i^* \psi)'(x, x_0 - x) < 0 \). Especially in this case, \( mvi_M \) implies (3.4), thus they are equivalent. Moreover, by Corollary 4.7, \( mvi_M \) and \( MVI_M \) are equivalent. As \( \psi(x_0) \in \text{Eff} \psi[X] \) is satisfied, if and only if \( f(x_0) \in \text{Min} f[X] \), Theorem 3.21 proves the statement.

Finally, we can prove as special case the following Minty variational principle for vector valued functions, which can be found in e.g.\([4, 25]\).

**Corollary 4.11** Let \( Z = \mathbb{R}^m \) and \( C = \mathbb{R}^m_{+} \). Let \( \psi : S \subseteq X \rightarrow Z \) be a \( C \)-convex function \( x_0 \in S \) and \( f(x) = \psi^C(x) \) for all \( x \in X \). If \( f_{x_0} \) is \( C^- \setminus \{0\} \)-l.s.c. in \( 0 \) for all \( x \in X \), then \( \psi(x_0) \in \text{Eff} \psi[X] \) is satisfied if and only if \( x \in S \) and \( t \in (0, 1) \) imply
\[
\psi(x_t) \neq \psi(x_0) \implies \psi'(x_t, x_0 - x) \subseteq Z \setminus C.
\]
Especially in this case, \( \psi'(x_t, x_0 - x) \subseteq Z \) is single valued.

**Proof.** By Proposition 4.5, \( \psi'(x_t, x_0 - x) \neq \emptyset \) is satisfied under the given assumptions and \( C \) is polyhedral and pointed. Especially, \( C \cap -C = \{0\} \), thus \( \emptyset \neq \psi'(x_t, x_0 - x) \subseteq Z \) is true for all \( x \in S \) and all \( t \in (0, 1) \) and \( \psi'(x_t, x_0 - x) \) is single valued. Hence, \( \psi'(x_t, x_0 - x) \subseteq Z \setminus C \) is equivalent to \( \psi'(x_t, x_0 - x) \notin (C \cap C_{\infty}) \) (compare Lemma 4.3), and under the given assumptions \( MVI_M \) is satisfied (compare Proposition 4.9). But by Theorem 4.10, \( MVI_M \) is equivalent to \( \psi(x_0) \in \text{Eff} \psi[X] \). On the other hand, by Corollary 4.7, \( MVI_M \) is equivalent to (4.2), implying
\[
t \in (0, 1), \; \psi(x_t) \neq \psi(x_0) \implies \psi'(x_t, x_0 - x) \setminus C \neq \emptyset,
\]
which in turn implies
\[
t \in (0, 1), \; \psi(x_t) \neq \psi(x_0) \implies \psi'(x_t, x_0 - x) \subseteq Z \setminus C,
\]
as proposed. \( \square \)

The last formula is a version of a vector valued Minty variational inequality for efficient solutions. In fact it states that if \( \psi : S \subseteq X \rightarrow \mathbb{R}^m \) is \( \mathbb{R}^m_{+} \)-convex and each \( \psi^C_{x_0, x} \) is \( \mathbb{R}^m \setminus \{0\} \)-l.s.c. at \( 0 \) (or \( \psi \) is \( \mathbb{R}^m \setminus \{0\} \)-l.s.c. at \( x_0 \)), then \( x_0 \in \text{Eff} \psi[X] \) if and only if \( x_0 \) solves the following
\[
\forall x \in S, \; 1 \in \text{int dom} \psi^C_{x_0, x} : \; \psi(x_0) \neq \psi(x) \implies \psi'(x, x_0 - x) \subseteq Z \setminus C.
\]
In this case, \( \psi'(x, x_0 - x) \subseteq Z \) is single valued.
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