An integrable (2+1)-dimensional Camassa-Holm hierarchy with peakon solutions

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Abstract
In this letter, we propose a (2+1)-dimensional generalized Camassa-Holm (2dgCH) hierarchy with both quadratic and cubic nonlinearity. The Lax representation and peakon solutions for the 2dgCH system are derived.

Keywords: Camassa-Holm (CH) equation, peakon, Lax representation

1. Introduction
In recent years, the Camassa–Holm (CH) equation [1]

\[ m_t - au_x + 2mu_x + m_xu = 0, \quad m = u - u_{xx}, \]  

(1)

has attracted a great deal of attention in the theory of integrable systems and solitons. This equation was derived as a model for the propagation of shallow water waves over a flat bed [1, 11]. In the literature, this equation was implied in the paper of Fuchssteiner and Fokas on hereditary symmetries as a very special case [2]. Since the work of Camassa and Holm [1], various remarkable studies on this equation have been developed [6–14]. The most remarkable feature of the CH equation (1) is that it admits peaked soliton (peakon) solutions in the case of \( \alpha = 0 \) [1, 3]. A peakon is a weak solution in some Sobolev space with a corner at its crest. The stability and interaction of peakons were discussed in several references [9–14].

In addition to the CH equation being an integrable model with peakon solutions, other integrable peakon models have been found, including the Degasperis–Procesi (DP) equation [15] whose Lax pair, bi-Hamiltonian formulation and peakon solutions were discovered in [16, 17], the cubic nonlinear peakon equations [6, 18–20], and a generalized CH equation (gCH) with both quadratic and cubic nonlinearity [4, 5, 21]

\[ m_t = \frac{1}{2} k_1 \left( m (u^2 - u_x^2) \right)_x + \frac{1}{2} k_2 (2mu_x + m_xu), \]

\[ m = u - u_{xx}, \]

(2)

where \( k_1 \) and \( k_2 \) are two arbitrary constants. Through some appropriate rescaling, equation (2) could be transformed to the one in the papers of Fokas and Fuchssteiner [4, 5], where it was derived from the motion of a two-dimensional, inviscid, incompressible fluid over a flat bottom. In [21], the Lax pair, bi-Hamiltonian structure, peakons, weak kinks, kink-peakon interactional and smooth soliton solutions of equation (2) are presented.

It is an interesting task to study the (2+1)-dimensional generalizations of the peakon equations. For example, in [22, 23] the authors provided a (2+1)-dimensional extension of the CH hierarchy, and they further studied the hodograph transformations and peakon solutions for their (2+1)-dimensional CH equation. In this paper, we generalize the gCH equation (2) to the whole integrable hierarchies in (1+1) and (2+1)-dimensions. We show that the gCH hierarchies admit Lax representations and construct a relation between the gCH hierarchies in (1+1) and (2+1)-dimensions. Moreover, we derive the single-peakon solution and the multi-peakon dynamic system for the (2+1)-dimensional gCH equation.
This paper is organized as follows. In section 2, we review the CH hierarchies in (1+1) and (2+1)-dimensions. In section 3, we present the gCH hierarchies in (1+1) and (2+1)-dimensions. In particular, we give their Lax representations. In section 4, we derive the peakon solutions to the (2+1)-dimensional gCH equation. Conclusions are drawn in section 5.

2. Overviews

In this section, we review the (1+1) and (2+1)-dimensional CH hierarchies presented in [8, 22, 23]. The new results we find are a relation between the CH hierarchies in (1+1) and (2 +1)-dimensions and isospectral Lax representations for the CH hierarchies.

2.1. The CH hierarchies in (1+1) and (2+1)-dimensions

Let us consider the Lenard operators pair [1]

\[ J = \partial_t m + m \partial_x, \quad K = \frac{1}{2} (\partial_x^3 - \partial_t). \]  

(3)

The Lenard gradients \( b_{-k} \) are defined recursively by

\[ Kb_{-k} = Jb_{-k+1}, \quad Kb_0 = 0, \quad k \in \mathbb{Z}^+. \]  

(4)

Taking an initial value \( b_0 = -\frac{1}{2} \), one may generate the negative CH hierarchy [8]

\[ \begin{align*}
   m_{-j} &= Jb_{-j}, \\
   Kb_{-j} &= Jb_{-j+1}, \quad 1 \leq j \leq n.
\end{align*} \]  

(5)

For \( n = 1 \), (5) becomes

\[ \begin{align*}
   m_{-1} &= (mb_{-1})_x + mb_{-1,x}, \\
   \frac{1}{2} (b_{-1,xxx} - b_{-1,x}) &= -\frac{1}{2} m_y,
\end{align*} \]  

(6)

which is nothing but the CH equation (1) with \( \alpha = 0 \) [1]. For \( n = 2 \), we arrive at

\[ \begin{align*}
   m_{-1} &= (mb_{-2})_x + mb_{-2,x}, \\
   \frac{1}{2} (b_{-2,xxx} - b_{-2,x}) &= (mb_{-1})_x + mb_{-1,x}, \\
   \frac{1}{2} (b_{-1,xxx} - b_{-1,x}) &= -\frac{1}{2} m_y.
\end{align*} \]  

(7)

In what follows, we call equation (7) the 2nd CH equation. For the general \( n \), we refer to (5) as the \( n \)th CH equation.

In [22, 23], the authors proposed a (2+1)-dimensional CH equation

\[ \begin{align*}
   m_t &= (mb_{-2})_x + mb_{-2,x}, \\
   \frac{1}{2} (b_{-2,xxx} - b_{-2,x}) &= m_y.
\end{align*} \]  

(8)

In general, a (2+1)-dimensional generalization of the CH hierarchy could be written as [22, 23]

\[ \begin{align*}
   m_{t,x} &= Jb_{-n}, \\
   Kb_{-j} &= Jb_{-j+1}, \quad 3 \leq j \leq n.
\end{align*} \]  

(9)

In [22, 23], the authors also studied the hodograph transformations and the peakon solutions of the (2 +1)-dimensional CH equation.

2.2. Lax representation

Let

\[ U = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} + \lambda m & 0 \end{pmatrix}, \quad V^{(-n)} = -\frac{1}{2} U + \sum_{i+j=n, 0 \leq a \leq n} \lambda^{-i} \tilde{V}^{(-j)}, \]  

(10)

where

\[ \tilde{V}^{(-j)} = \begin{pmatrix} -\frac{1}{2} b_{-j,x} & b_{-j} + \frac{1}{2} - \frac{1}{2 \lambda} \\ m b_{-j} + \frac{1}{2} \lambda - \frac{1}{2} b_{-j,xx} & \frac{1}{2} b_{-j,x} + \frac{1}{2} \end{pmatrix}. \]  

(11)

\( \lambda \) is the eigenparameter and \( b_j \) is defined through equation (4).

By a direct calculation, we obtain the following result.

**Proposition 1.** The \( n \)th CH equation (5) admits the Lax representation

\[ U_{-n} = V^{(-n)} + \left[ U, V^{(-n)} \right] = 0, \]  

(12)

where the Lax pair \( U \) and \( V^{(-n)} \) given by (10).

As \( n = 1 \), we recover the Lax pair of the well-known CH equation (1) with \( \alpha = 0 \) [1]

\[ U = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} + \lambda m & 0 \end{pmatrix}, \quad V^{(-1)} = \begin{pmatrix} -\frac{1}{2} b_{-1,x} & b_{-1} - \frac{1}{2 \lambda} \\ m b_{-1} - \frac{1}{2} b_{-1,xx} + \frac{1}{2} \end{pmatrix}. \]  

(13)
As \( n = 2 \), we obtain the Lax pair of the 2nd CH equation (7)

\[
U = \begin{pmatrix} 1 & 0 \\ \frac{1}{4} \lambda + \lambda m & 0 \end{pmatrix},
\]

\[
V^{-2} = \begin{pmatrix} \frac{1}{2} b_{-2,x} - \frac{1}{2} b_{-2} & b_{-1} - \frac{1}{2} \lambda \\ \frac{1}{4} b_{-2} - \frac{1}{2} m - \frac{1}{8} \lambda & \frac{1}{2} b_{-1,x} + \frac{1}{2} \lambda \\ \frac{1}{4} b_{-1} + \frac{1}{2} \lambda & \frac{1}{2} b_{-1,x} + \frac{1}{2} \lambda \end{pmatrix}.
\]

Proposition 2. Let \( t_{-1} = y \), \( t_{-2} = t \). Let \( m(x, y, t) \) be a compatible solution of the CH equation (6) and the 2nd CH equation (7). Then \( m(x, y, t) \) provides a special solution to the (2+1)-dimensional CH equation (8). In general, if \( m(x, t_{-1}, t_{-n}) \) is a compatible solution of the CH equation (6) and the \( n \)th CH equation (5), then the (2+1)-dimensional CH hierarchy (9) has a special solution \( m(x, t_{-1}, t_{-n}) \).

Remark 1. Based on proposition 2, we may construct the algebraic-geometric solution of the (2+1)-dimensional CH hierarchy with the method developed in [8, 27, 28]. We will consider this topic in another publication.

3. The gCH hierarchies in (1+1)- and (2+1)-dimensions

Let us first introduce a pair of Lenard operators [21]

\[
J = k_1 \partial_x m \partial_x^{-1} m \partial_x + \frac{1}{2} k_2 (\partial_x m + m \partial_x),
\]

\[
K = \partial_x - \partial_x^3,
\]

and define the Lenard gradients \( b_{-1} \) recursively by

\[
K b_{-k} = J b_{-k+1}, \quad K b_0 = 0, \quad k \in \mathbb{Z}^+.
\]

We define a gCH hierarchy in (1+1)-dimension as follows

\[
\begin{align*}
J_{-n} &= J b_{-n}, \\
K b_{-j} &= J b_{-j+1}, \quad 2 \leq j \leq n, \\
K b_{-1} &= m_x, \\
m &= b_{-1} - b_{-1,xx},
\end{align*}
\]

which is nothing but the gCH equation (2). For \( n = 2 \), equation (17) is cast into the 2nd gCH equation in the gCH hierarchy (17)

\[
\begin{align*}
J_{-2} &= k_1 \left[ m \partial_x^{-1} m \partial_x^{-1} m \right], \\
J_{-1} &= k_2 \left[ 2 m \partial_x^{-1} m \partial_x^{-1} m + m \partial_x \right], \\
b_{-2,xx} &= b_{-2,xxx} = \frac{1}{2} k_1 [m \partial_x^{-1} m \partial_x^{-1} m], \\
b_{-2,x} &= b_{-2,xx} = \frac{1}{2} k_2 [2 m \partial_x^{-1} m \partial_x^{-1} m + m \partial_x],
\end{align*}
\]

For the general case \( n \geq 2 \), we refer to (17) as the \( n \)th gCH equation.

Similar to the (2+1)-dimensional generalization of the CH equation, we extend the (1+1)-dimensional gCH equation (2) to the (2+1)-dimensional system as follows:

\[
\begin{align*}
m_{-1} &= k_1 \left[ m \partial_x^{-1} m \right], \\
&\quad + \frac{1}{2} k_2 \left[ 2 m \partial_x^{-1} m + m \partial_x \right], \\
m_{y} &= b_{-2,x} - b_{-2,xx},
\end{align*}
\]

Furthermore, we may define the (2+1)-dimensional gCH hierarchy in the following form:

\[
\begin{align*}
m_{-n} &= J b_{-n}, \\
K b_{-j} &= J b_{-j+1}, \quad 3 \leq j \leq n, \\
m_{x} &= K b_{-2},
\end{align*}
\]

In particular, as \( k_1 = 0 \) and \( k_2 = 2 \), our (2+1)-dimensional gCH hierarchy (21) is reduced to the (2+1)-dimensional CH hierarchy (9).
Let us now show that the gCH hierarchies admit Lax representations. Let
\[ U = \frac{1}{2} \left( \begin{array}{cc} -1 & \lambda m \\ -k_1 \lambda m - k_2 \lambda & 1 \end{array} \right), \]
\[ V^{(-n)} = U + \sum_{0 \leq j \leq n-1} \lambda^{-2j} \tilde{V}^{(-n-j)}, \]  
where
\[ \tilde{V}^{(-j)} = \frac{1}{2} \left( \begin{array}{cc} A & B \\ C & -A \end{array} \right), \]
with
\[ A = \lambda^{-2} + k_1 \partial^{-2} b_{-j,x} + \frac{1}{2} k_2 (b_{-j} - b_{-j,x}) - 1, \]
\[ B = -\lambda^{-1} \left( m - b_{-j} + b_{-j,x} \right) + \lambda m \left( -k_1 \partial^{-2} b_{-j,x} - \frac{1}{2} k_2 b_{-j} + 1 \right), \]
\[ C = \lambda^{-1} \left[ k_1 (m + b_{-j,x} - b_{-j}) + k_2 \right] - \lambda (k_1 m + k_2) \]
\[ \times \left( -k_1 \partial^{-2} b_{-j,x} - \frac{1}{2} k_2 b_{-j} + 1 \right). \]

Direct calculations lead to the following proposition.

**Proposition 3.** The gCH hierarchy (17) possesses the Lax representation

\[ U_{tn} = V^{(-n)} - \left[ U, V^{(-n)} \right] = 0, \]

with the Lax pair \( U \) and \( V^{(-n)} \) given by (22).

In particular, the Lax pair of the gCH equation (18) is given by

\[ U = \frac{1}{2} \left( \begin{array}{cc} -1 & \lambda m \\ -k_1 \lambda m - k_2 \lambda & 1 \end{array} \right), \]
\[ V^{(-1)} = U + \frac{1}{2} \left( \begin{array}{cc} A_1 & B_1 \\ C_1 & -A_1 \end{array} \right), \]
with
\[ A_1 = \lambda^{-2} + \frac{1}{2} k_1 \left( b_{-2} - b_{-1,x} \right) + \frac{1}{2} k_2 (b_{-2} - b_{-1,x}), \]
\[ B_1 = -\lambda^{-1} (b_{-1} - b_{-1,x}) - \lambda m \left[ k_1 (b_{-2} - b_{-1,x}) + k_2 b_{-1} \right], \]
\[ C_1 = \lambda^{-1} \left[ k_1 (b_{-1} + b_{-1,x}) + k_2 \right] + \frac{1}{2} \lambda \left[ k_1^2 m (b_{-2} - b_{-1,x}) + k_1 \right] \]
\[ \times \left( k_1 b_{-2} + b_{-2,x} - b_{-1,x} \right) + k_1^2 b_{-1} \].

The Lax pair of the 2nd gCH equation (19) is given by

\[ U = \frac{1}{2} \left( \begin{array}{cc} -1 & \lambda m \\ -k_1 \lambda m - k_2 \lambda & 1 \end{array} \right), \]
\[ V^{(-2)} = U + \frac{1}{2} \left( \begin{array}{cc} A_2 & B_2 \\ C_2 & -A_2 \end{array} \right), \]
where \( A_2, B_2, C_2 \) are defined by (23) and (24).

One may easily check the following results.

**Proposition 4.** Let \( t_1 = y, t_2 = t \). Let \( m(x, y, t) \) be a compatible solution of the gCH equation (18) and the 2nd gCH equation (19). Then \( m(x, y, t) \) provides a special solution to (2+1)-dimensional gCH equation (20). In general, if \( m(x, t_{-1}, t_{-n}) \) is a compatible solution of the gCH equation (18) and the \( n \)th gCH equation (17), then the (2+1)-dimensional gCH hierarchy (21) has a special solution \( m(x, t_{-1}, t_{-n}) \).

### 4. Peakon solutions to the 2dgCH Equation (20)

Assume that the single-peakon solution of the (2+1)-dimensional gCH equation (20) is given in the form of

\[ b_{-2} = p(y, t) e^{-k_{-2} y}, \quad m = 2r(y, t) \delta(x - q(y, t)), \]

where \( p(y, t), q(y, t) \) and \( r(y, t) \) are to be determined. Substituting (28) into (20) and integrating against the test function with support around the peak, we finally arrive at

\[ \begin{cases} r_y = n = 0, \\ q_y = -\frac{\partial}{r}, \\ q_t = -\frac{3}{2} k_1 rp - \frac{1}{2} k_2 p, \end{cases} \]

which yields

\[ \begin{cases} r = c, \\ q = F \left( y + \left( \frac{1}{3} k_1 c^2 + \frac{1}{2} k_2 c \right) t \right), \\ p = -c q, \end{cases} \]

where \( c \) is an arbitrary constant and \( F \) is an arbitrary smooth function. Thus, the single-peakon solution of equation (20) is given by

\[ \begin{aligned} b_{-2} &= -cF \left( y + \left( \frac{1}{3} k_1 c^2 + \frac{1}{2} k_2 c \right) t \right) \\ & \quad \times e^{-\frac{1}{k_2} F \left( y + \left( \frac{1}{3} k_1 c^2 + \frac{1}{2} k_2 c \right) t \right)}, \end{aligned} \]

\[ m = 2cd \left( x - F \left( y + \left( \frac{1}{3} k_1 c^2 + \frac{1}{2} k_2 c \right) t \right) \right). \]

As \( k_1 = 0, k_2 = 2 \), we recover the single-peakon solution of the (2+1)-dimensional CH equation proposed in [22].

In particular, if we take \( F \left( y + \left( \frac{1}{3} k_1 c^2 + \frac{1}{2} k_2 c \right) t \right) = y + \left( \frac{1}{3} k_1 c^2 + \frac{1}{2} k_2 c \right) t \), then the single-peakon solution of equation (20) becomes

\[ \begin{aligned} b_{-2} &= -c e^{-\frac{1}{k_2} F \left( y - \left( \frac{1}{3} k_1 c^2 + \frac{1}{2} k_2 c \right) t \right)}, \\ m &= 2cd \left( x - y - \left( \frac{1}{3} k_1 c^2 + \frac{1}{2} k_2 c \right) t \right). \end{aligned} \]

See figure 1 for the graph of the single-peakon solution \( b_{-2}(x, y, t) \) at \( t = 0 \). If we take \( F \left( y + \left( \frac{1}{3} k_1 c^2 + \frac{1}{2} k_2 c \right) t \right) = \left( y + \left( \frac{1}{3} k_1 c^2 + \frac{1}{2} k_2 c \right) t \right)^2 \), then the single-peakon solution (31)
In general, let us suppose that the $N$-peakon has the following form

\[
b_{-2} = \sum_{j=1}^{N} p_j(y, t) e^{-|y-q_j(y, t)|},
\]

\[m = 2 \sum_{j=1}^{N} r_j(y, t) \delta \left(x - q_j(y, t)\right). \tag{34}\]

Similar to the cases of one-peakon but with a lengthy calculation, we are able to obtain the following $N$-peakon dynamical system

\[
r_{j,y} = 0,
\]

\[
r_{j,t} = -\frac{1}{2} k_2 r_j \sum_{k=1}^{N} p_k \text{sgn} \left(q_j - q_k\right) e^{-|y-q_k|},
\]

\[
p_j = -r_j q_{j,y},
\]

\[
q_{j,t} = \frac{1}{6} k_1 r_j p_j - \frac{1}{2} k_2 \sum_{k=1}^{N} p_k e^{-|y-q_k|}
\]

\[+ \frac{1}{6} k_1 \sum_{j,k=1}^{N} r_j r_k \text{sgn} \left(q_j - q_i\right) \text{sgn} \left(q_j - q_k\right) + 1)
\]

\[\times e^{-|y-q_j|} - |y-q_k|. \tag{35}\]

**5. Conclusion**

In this paper, we have extended the $g$CH equation to the hierarchies in (1+1)-dimensions and (2+1)-dimensions. We first show the $g$CH hierarchies admit Lax representation. Then we show that the (2+1)-dimensional $g$CH equation possesses a single peakon solution as well as multi-peakon solutions. Other topics, such as smooth soliton solutions, cuspons, peakon stability, and algebra-geometric solutions, remain to be developed.

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