Tkachenko Oscillations and the Compressibility of a Rotating Bose Gas

M. Cozzini, L. P. Pitaevskii, and S. Stringari

Dipartimento di Fisica, Università di Trento and BEC-INFN, I-38050 Povo, Italy
Kapitza Institute for Physical Problems, ul. Kosygina 2, 117334 Moscow, Russia

(Dated: March 22, 2022)

The elastic oscillations of the vortex lattice of a cold Bose gas (Tkachenko modes) are shown to play a crucial role in the saturation of the compressibility sum rule, as a consequence of the hybridization with the longitudinal degrees of freedom. The presence of the vortex lattice is responsible for a $q^2$ behavior of the static structure factor at small wavevectors $q$, which implies the absence of long range order in 2D configurations at zero temperature. Sum rules are used to calculate the Tkachenko frequency in the presence of harmonic trapping. Results are derived in the Thomas-Fermi regime and compared with experiments as well as with previous theoretical estimates.

PACS numbers: 03.75.Lm, 03.75.Kk, 32.80.Lg

Tkachenko Oscillations and the Compressibility of a Rotating Bose Gas

In 1966 Tkachenko developed the theory of the elastic oscillations of a vortical lattice in incompressible superfluids, predicting the dispersion law $\omega_T = \sqrt{\hbar \Omega/4m} q$ where $m$ is the mass of the fluid particles, $q$ is the wavevector of the wave and $\Omega$ is the angular velocity of the fluid, related to the number $n_c$ of vortices per unit surface by the relation $n_c = 2\Omega m/h$. The frequency $\omega_T$ is calculated in the reference frame rotating with angular velocity $\Omega$. The Tkachenko modes are peculiar of superfluids, where the formation of singly quantized vortices in the rotating liquid gives rise to regular crystalline structures (Abrikosov lattice), stationary in the rotating frame. The theory of the Tkachenko modes was later developed by Baym, who generalized the hydrodynamic theory of superfluids taking into account the elasticity of the vortex lattice. The effects of the compressibility on the Tkachenko modes were discussed in details by Sonin. The compressibility changes dramatically the dispersion law at small wavevectors $q$. In particular, when $q \ll \Omega/c$, where $c$ is the sound velocity, the dispersion relation is no longer linear, but becomes quadratic in $q$ as a result of the hybridization with the sound waves.

The availability of vortex lattices in rapidly rotating Bose-Einstein condensates has stimulated new theoretical studies of the Tkachenko oscillations in harmonically trapped atomic gases. In particular in Ref. 2, the discretized Tkachenko frequencies have been calculated taking into account the finite and inhomogeneous nature of the system. In Ref. 3 the corresponding values of $q$ have been employed within an improved dispersion law which includes the effects of compressibility. Fully numerical calculations of the Tkachenko waves in harmonically trapped gases have been also recently carried out using Gross-Pitaevskii theory. The first experimental observation of the Tkachenko modes in harmonically trapped gases has been recently reported by the group of JILA. These modes represent the low energy counterpart of the hydrodynamic modes already investigated experimentally in the presence of the vortex lattice, in good agreement with theory.

The main purpose of the present work is to show that, despite their elastic nature, the Tkachenko oscillations can be naturally excited using density perturbations, thereby opening new perspectives of experimental investigation. In particular we will show that, in the homogeneous case, these modes exhaust the compressibility sum rule in the limit of long wavelengths. In harmonically trapped configurations the sum rule approach will be used to calculate the frequency of the lowest azimuthally symmetric Tkachenko mode and explicit results will be derived in the Thomas-Fermi regime.

Let us start our discussion by evaluating the density response function of a uniform gas containing a vortex lattice rotating in the $x$-$y$ plane with angular velocity $\Omega = \Omega e_z$ at zero temperature, where $e_z$ is the unit vector along the $z$-direction. In a compressible fluid the condition of uniformity can be fulfilled by adding the harmonic potential $m\Omega^2(x^2 + y^2)/2$, which compensates the centrifugal effect produced by the rotation. This corresponds, in the rotating frame, to using the Hamiltonian

$$H = \sum_{k=1}^{N} \left( \frac{p - m\Omega \wedge r}{2m} \right)^2 + \frac{1}{2} \sum_{i \neq j} g \delta (r_i - r_j),$$

where $N$ is the number of atoms, $p$ is the momentum and $g$ is the coupling constant of the 2-body interaction. The density response is easily evaluated using the coupled hydrodynamic-elastic formalism characterized, in addition to the hydrodynamic energy functional

$$E_{hd} = \int d\mathbf{r} \left[ \frac{m}{2} (\mathbf{v} - \Omega \wedge \mathbf{r})^2 n + \frac{1}{2} gn^2 \right]$$

where $\mathbf{v}$ is the velocity field in the laboratory frame and $n$ is the density, by the elastic term

$$E_{el} = \int d\mathbf{r} \left\{ 2C_1 (\nabla \cdot \mathbf{e})^2 + \right.$$}

$$+ C_2 \left[ \left( \frac{\partial e_x}{\partial x} - \frac{\partial e_y}{\partial y} \right)^2 + \left( \frac{\partial e_x}{\partial y} + \frac{\partial e_y}{\partial x} \right)^2 \right] \right\}$$

where $C_1$ and $C_2$ are elastic constants.
sensitive to the deformation of the lattice through the vortex displacement field $\epsilon$ and characterized by the elastic parameters $C_1$ and $C_2$. At equilibrium one has $\mathbf{v} = \Omega \wedge \mathbf{r}$ and $\epsilon = 0$. The dynamic response function takes the form

$$\chi(q, \omega) = -N \frac{q^2 \omega^2/m - \omega^2 \omega^2/mc^2}{[\omega^2 - \omega^2]/2 - \omega^2}$$

(4)

with $\eta \to 0^+$, where $\omega_+$ and $\omega_-$ are, respectively, the upper and lower branches of the energy spectrum. Expression (4) holds in the macroscopic regime $q \ll (\hbar/m\Omega)^{-1/2}$ corresponding to wavelengths larger than the average distance between vortices. The general expression for $\omega_k$ as a function of $C_1$ and $C_2$ has been derived in Ref. [7]. Here we report the results in the Thomas-Fermi regime $\hbar\Omega \ll mc^2$, corresponding to the condition that the size $\hbar/\Omega/c$ of the vortex cores is much smaller than $\sqrt{\hbar/m\Omega}$. In the Thomas-Fermi regime one has $C_2 = -C_1 = \hbar\Omega/8$. In this case the upper branch follows the dispersion law $\omega_k^2 = 4\Omega^2 + c^2q^2$ and exhibits a gap at $q = 0$. Conversely the low frequency branch, hereafter called Tkachenko branch ($\omega_\pm \equiv \omega_T$), obeys the gapless law

$$\omega_T^2 = \frac{\hbar\Omega}{4m} \frac{c^2q^4}{4\Omega^2 + c^2q^2}.$$  

(5)

For large $q$ Eq. (5) reproduces the original Tkachenko law $\sqrt{\hbar\Omega/4m}$, while for small $q$ it exhibits the quadratic behavior $\omega_T = \sqrt{\hbar/16m\Omega} q^2$. The transition between the $q^2$ and $q$ dependence takes place at values $q \sim \Omega/\varepsilon$ which, in trapped condensates, can be significantly larger than the inverse of the radial size of the system. This suggests that the effects of compressibility, characterizing the $q^2$ dependence, play a crucial role in the Tkachenko modes of a trapped gas, as explicitly pointed out in Ref. [7].

Starting from the density response function (4) one can easily calculate the energy weighted and the inverse energy weighted sum rules relative to the density operator $\rho_q = \sum_{k=1}^N e^{-iqx}$. These are given, respectively, by $m_1(\rho_q) = \sum_n |\langle n |\rho_q| 0 \rangle|^2 E_{n0}$ and $m_{-1}(\rho_q) = \sum_n |n|\rho_q| 0 \rangle|^2 E_{n0}$, where $\sum_n$ is the sum over all the excited states and $E_{n0} = E_n - E_0$ is the difference between the eigenenergies of the excited state $|n\rangle$ and of the initial configuration $|0\rangle$ containing the vortex lattice. The relation between these sum rules and the asymptotic behavior of the density response function is given by $\chi(q, \omega) \to -2m_1(\rho_q)/(\hbar\omega)^2$ and $\chi(q, 0) = 2m_{-1}(\rho_q)$. In the first case one recovers the model independent $f$-sum rule $m_1(\rho_q) = N(\hbar\omega)^2/2m$ which, at small $q$, is exhausted by the high energy branch $\omega_+$. By taking the $\omega \to 0$ limit of $\chi(q, \omega)$ one instead finds the result $m_{-1}(\rho_q) = N/2mc^2$ also known as the compressibility sum rule. The Tkachenko branch plays a crucial role in satisfying the latter sum rule. In fact, because of the gap, the high energy branch $\omega_+$ contributes to $m_{-1}(\rho_q)$ only through terms of order $q^2$.

It is also worth noticing that the static structure factor $S(q) = N^{-1} \sum_{n} |\langle n |\rho_q| 0 \rangle|^2$ is deeply affected by the rotation of the gas and behaves like $q^2$, differently from what happens in non rotating interacting fluids where it is linear in $q$. By using the zero temperature relationship $NS(q) = (\hbar/\pi)^3 \int_0^{\infty} \text{Im} \chi(q) \omega \, dq$ one finds the result

$$S(q) \sim \frac{\hbar q^2}{4N} \left(1 + \sqrt{\frac{\hbar\Omega}{4m\omega}}\right)$$

(6)

when $q \to 0$, where the second term in the parenthesis is the contribution of the Tkachenko branch [13]. The corresponding suppression of the density fluctuations results in a dramatic enhancement of the fluctuations of the phase of the order parameter, which destroy long range order. This can be easily seen using the uncertainty principle inequality $2S(q)/(2n_q + 1) \geq n_q$, where $n_q$ is the particle occupation number and $n_q = \text{Bose-Einstein}$ condensate fraction. Since $S(q) \to q^2$ one finds that $n_q$ diverges at least like $1/q^2$ at low $q$, thereby ruling out Bose-Einstein condensation in 2D even at zero temperature [15].

In the following we will use the sum rule technique to evaluate the Tkachenko frequency through the ratio

$$\frac{(\hbar\omega_T)^2}{m_{-1}(F)} = \frac{\sum_n |\langle n |F| 0 \rangle|^2 E_{n0}}{\sum_n |\langle n |F| 0 \rangle|^2 E_{n0}^{-1}}$$

(7)

between the energy weighted and inverse energy weighted sum rules relative to the excitation operator $F$. Our final goal is to derive explicit results in the presence of harmonic trapping. Eq. (7) provides a rigorous upper bound to the frequency of the lowest energy mode excited by $F$. The proper choice of the operator is a crucial step in the calculation. For example, in the uniform case it would not be appropriate to use the density operator $\rho_q$ for $F$ since, as already pointed out, the $f$-sum rule $m_1(\rho_q)$ is exhausted by the upper branch at small $q$ and the ratio (7) would not coincide with the Tkachenko frequency.

The general strategy for the identification of the excitation operator is suggested by the fact that the Tkachenko modes have zero energy in the hydrodynamic approximation, where elasticity effects are ignored. So one should look for excitation operators $F$ whose energy weighted sum rule $m_1(F) = \langle |F|, [H, F]| 0 \rangle/2$ vanishes when evaluated in the hydrodynamic approximation.

In a uniform system this condition is satisfied by the non local choice $F = \rho_q - i(q/2m) \sum_{k=1}^N e^{-iqx} (p_y - m\Omega x)$. In fact the corresponding double commutator takes the form $[F, [H, F]] = (\hbar^2q^4/4m^3\Omega^2) \sum_{k=1}^N (p_y - m\Omega x)^2$ and its expectation value identically vanishes if one uses the hydrodynamic prescription $p = m\mathbf{v}(\mathbf{r})$ with the equilibrium condition $\mathbf{v} = \Omega \wedge \mathbf{r}$. The elastic contribution to $m_1(F)$ can be conveniently calculated applying to the equilibrium configuration the unitary transformation $U = e^{i\theta S}$, where $S = (F + F^\dagger)/2$ and $\theta$ is a small parameter. Due to the presence of the non local transverse
Hamiltonian (1) should be replaced by rotating gas, where the non-interacting part of the exactly the Tkachenko dispersion law (5).

To obtain \( m_{-1}(F) \) we calculate the static response \( \chi(F) \) to the perturbation \(-\lambda F + \text{h.c.} \) using the dispersion law (5) and setting \( q \) is the Thomas-Fermi radial size of the gas [18] and \( \perp \) is the unit vector along the transverse momentum operator and \( mR \) is fixed by the compressibility of the gas.

For larger values of \( q \), or for incompressible fluids, the static response is instead determined by the transverse dynamic profile which produces the vortex displacement \( \delta \omega \) of the lowest Tkachenko mode. Here \( \delta \omega \) is the Thomas-Fermi radial size of the gas [18] and \( \alpha \) is a dimensionless parameter characterizing the discretization of the normal modes. The value \( \alpha = 5.45 \) for the lowest azimuthally symmetric mode was extracted from the result of Ref. [19] holding in the incompressible regime where the frequency is linear in \( q \). Since in a trapped gas the relevant excitations are affected by the compressibility of the gas, it is not obvious that the above estimate is enough accurate and it is hence important to have more precise calculation of the frequency of the lowest Tkachenko mode, taking into account the finite size, the inhomogeneity as well as the compressibility of the gas.

For small values of \( q \) the leading contribution arises from the density component \( \rho_q \) in the Tkachenko operator and is fixed by the compressibility of the gas. For larger values of \( q \), or for incompressible fluids, the static response is instead determined by the transverse current term \( e^{-iqx}\rho_q \) in \( F \). In general one finds the result \( m_{-1}(F) = (N/8\Omega m^2)(4\Omega^2 + c^2 q^2) \).

Using the Thomas-Fermi equilibration assumptions one can write the frequency in the form \( \omega = 2 \mu/(\hbar m) \) and analogously for \( \lambda \).

Given by the second order term of the expansion which can be calculated using the elastic energy change \( E \).

One finds the result \( m_{-1}(F) = \hbar^2/(\hbar \mu) \langle P^2 r^2 \rangle / \langle r^2 \rangle \).

The above results hold for any choice of the functions \( P \) and \( Q \) satisfying the condition [3]. We have written \( P \) as a polynomial expression of the form \( P = \sum_s p_s (\mathbf{r} \cdot \mathbf{R}_1)^s \) and analogously for \( Q \). Using the Thomas-Fermi equilibrium profile \( n = (\mu + (\omega^2 - \Omega^2)r^2 + \omega^2 z^2)/2 \), where \( \mu = m(\omega^2 - \Omega^2)R_1^2/2 \) is the chemical potential, all the integrals involved in \( m_{-1}(F) \) and \( m_{-1}(F) \) are analytical and one can write the frequency in the form

\[
\omega_T^2 = \frac{\hbar \Omega}{4m R_1^2} \frac{1}{f(\Omega/\omega_\perp)} .
\]

We have determined the frequency of the lowest mode by minimizing the function \( f \) with respect to the coefficients \( p_s \) at fixed \( \Omega/\omega_\perp \). In practice good convergence to the exact value of \( \omega_T \) is already assured for \( s = 3 \). The results are reported in Fig. 1. When \( \Omega \ll \omega_\perp \)
function $f$ approaches a constant value. When $\Omega \to \omega_\perp$ it instead vanishes like $1 - \Omega^2/\omega_\perp^2$. Notice, however, that if $\Omega$ is too close to $\omega_\perp$, the validity of the present calculation, based on the Thomas-Fermi approximation, breaks down due to the diluteness of the gas produced by the rotation. In Fig. 1 we also show the prediction obtained using Eq. (3) with $q = \alpha/R_\perp$ and $\alpha = 5.45$. Using the relation $mc^2 = \mu$ it is immediate to verify that also in this case the dispersion law can be factorized in the form (11) with $f(x) = (1 - x^2)/[a^{-1}(1 - x^2) + b^{-1}x^2]$ and $a = \alpha^2 = 29.7$, $b = \alpha^4/8 = 110.3$. The results of the sum rule approach instead correspond to $a = 31.3$ and $b = 75.8$ [20]. The relative difference between the two predictions is more and more pronounced as $\Omega \to \omega_\perp$ pointing out the inadequacy of the choice $q = 5.45/R_\perp$ when $R_\perp$ is large and $q$ becomes small. Fig. 1 also shows that the sum rule prediction is systematically closer to the experimental values and that the deviations of the measured Tkachenko frequencies from the Thomas-Fermi values become important only for values of $\Omega$ very close to $\omega_\perp$, where quantum Hall effects should be taken into account [7, 15, 21].

In Fig. 2 we report the shape of the density deformation [3] associated with the Tkachenko oscillation. A density perturbation of this form, produced by a proper change of the trapping potential, should result in a significant excitation of the Tkachenko mode. This density change differs from the scaling deformation associated with the radial breathing hydrodynamic mode. Actually the breathing and the Tkachenko modes are orthogonal as confirmed by the fact that the density variation [3] does not result in any change of the average square radius: $\int dr^2 \delta n = 0$. In Fig. 2 (inset) we also show the amplitude of the corresponding vortex lattice deformation, obtained from the relation $e = \delta n/\omega T$. Similar shapes have been obtained in the theoretical calculations of Refs. [2, 3] and observed experimentally in Refs. [21, 22].

In conclusion we have shown the occurrence of important compressional features exhibited by the Tkachenko oscillations in rotating Bose gases. Sum rules have permitted to provide accurate estimates of the frequency of the lowest mode allowing for a detailed comparison with experiments.

Useful discussions with Gordon Baym, Jean Dalibard and Peter Engels are acknowledged.

[1] V. K. Tkachenko, Zh. Eksp. Teor. Fiz. 50, 1573 (1966) [Sov. Phys. JETP 23, 1049 (1966)].
[2] G. Baym and E. Chandler, J. Low Temp. Phys. 50, 57 (1983); 62, 119 (1986).
[3] E. B. Sonin, Rev. Mod. Phys. 59, 87 (1987).
[4] K. W. Madison et al., Phys. Rev. Lett. 84, 806 (2000); J. R. Abo-Shaeer et al., Science 292, 476 (2001).
[5] P. C. Haljan et al., Phys. Rev. Lett. 87, 210403 (2001).
[6] J. R. Anglin and M. Crescimanno, cond-mat/0210063.
[7] G. Baym, cond-mat/0305294.
[8] T. P. Simula et al., cond-mat/0307130.
[9] I. Coddington et al., Phys. Rev. Lett. 91, 100402 (2003).
[10] M. Cozzini and S. Stringari, Phys. Rev. A 67, 041602(R) (2003).
[11] D. Pines and P. Nozières, The Theory of Quantum Liquids, Vol. 1 (Benjamin, New York, 1966).
[12] L. P. Pitaevskii and S. Stringari, Bose-Einstein Condensation (Oxford University Press, Oxford, 2003).
[13] Result (5) holds in the $\hbar \Omega \ll mc^2$ limit. In the general case one has $S(q) \to (h^2q^2/4m\Omega)(1 + \sqrt{2c_2/mc^2})$.
[14] L. P. Pitaevskii and S. Stringari, J. Low Temp. Phys. 85, 377 (1991).
[15] The absence of Bose-Einstein condensation in rotating vortical fluids was proven by J. Sinova et al., Phys. Rev. Lett. 89, 030403 (2002) in the $\hbar^2 \gg mc^2$ quantum Hall regime; see also G. Baym, cond-mat/0308342.
[16] F. Chevy and S. Stringari, Phys. Rev. A 68, 053601 (2003).
[17] The first term in the hydrodynamic functional [2] should be correspondingly replaced by $\int d^3r n [\nu^2/2 + m(\omega_\perp^2 r^2 + \omega_z^2 z^2)]/2 - \Omega \cdot \langle r \wedge \nu r \rangle$.
[18] In 3D one has $R_\perp = a_\perp [15N\omega_\perp/\alpha_\perp \omega_\perp (1 - \Omega^2/\omega_\perp^2)]^{1/3}$, where $a$ is the scattering length and $a_\perp = \sqrt{\hbar/m \omega_\perp}$.
[19] Notice that Eq. (5) can also be obtained from Eqs. (4), [10] imposing the condition $\delta \nu = (\Omega/2D^2) \wedge (g \delta n/m)$, which characterizes a class of zero energy hydrodynamic solutions. This points out explicitly the relation between these special hydrodynamic solutions and the Tkachenko modes [13].
[20] These values are obtained evaluating the integrals with 3D Thomas-Fermi density profiles. If one uses 2D Thomas-Fermi profiles one finds $a_{2D} = 29.5$ and $b_{2D} = 92.8$. Notice that the value $a_{2D}$ is very close to the value $a = \alpha^2 = 29.7$ extracted from the 2D calculation of Ref. [2].
[21] V. Schweikhard et al., cond-mat/0308582.