On the skew-spectral distribution of randomly oriented graphs

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Abstract

The randomly oriented graph $G_{n,p}^\sigma$ is an Erdős-Rényi random graph $G_{n,p}$ with a random orientation $\sigma$, which assigns to each edge a direction so that $G_{n,p}^\sigma$ becomes a directed graph. Denote by $S_n$ the skew-adjacency matrix of $G_{n,p}^\sigma$. Under some mild assumptions, it is proved in this paper that, the spectral distribution of $S_n$ (under some normalization) converges to the standard semicircular law almost surely as $n \to \infty$. It is worth mentioning that our result does not require finite moments of the entries of the underlying random matrix.

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1 Introduction

Let $G$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and $G^\sigma$ be an oriented graph of $G$ with the orientation $\sigma$, which assigns to each edge of $G$ a direction so that $G^\sigma$ becomes a directed graph. The skew-adjacency matrix $S(G^\sigma) = (s_{ij}) \in \mathbb{R}^{n \times n}$ is a real skew-symmetric matrix, where $s_{ij} = 1$ and $s_{ji} = -1$ if $(v_i, v_j)$ is an arc of $G^\sigma$, otherwise $s_{ij} = s_{ji} = 0$. The well-known Erdős-Rényi random graph model $G_{n,p}$ is a probability space [6], which consists of all simple graphs with vertex set $V$ where each of the possible $\binom{n}{2} = n(n-1)/2$ edges occurs independently with probability $p = p(n)$. For a random graph $G_{n,p} \in \mathcal{G}_{n,p}$, the randomly oriented graph $G_{n,p}^\sigma$ is obtained by orienting every edge $(v_i, v_j)$ $(i < j)$ in $G_{n,p}$ as $(v_i, v_j)$ with probability $q = q(n)$ and the other way with probability $1 - q$ independently of each other. Here, the superscript $\sigma = \sigma(q)$ indicates the orientation.

The above randomly oriented graph model was first studied in [17] and a similar model based on the lattice structure (instead of $G_{n,p}$) was discussed in [13]. The question of whether the existences of directed paths between
various pairs of vertices are positively or negatively correlated has attracted some research attention recently; see e.g. [1, 2, 15]. Diclique structure has been studied in [20]. In this paper, we shall explore this model from a spectral perspective. Basically, we determine the limit spectral distribution of the random matrix underlying the randomly oriented graph. A semicircular law reminiscent of Wigner’s famous semicircular law [23] is obtained by the moment approach (see Theorem 1 below). We mention that there is recent increased interest in the spectral properties of oriented graphs in classical graph theory, see e.g. [8, 10, 14, 19].

As is customary, we say that a graph property \( P \) holds almost surely (a.s., for short) for \( G_{n,p} \) if the probability that \( G_{n,p} \in \mathcal{G}_{n,p} \) has the property \( P \) tends to one as \( n \to \infty \). We will also use the standard Landau’s asymptotic notations such as \( o, O, \sim \) etc. Let \( 1_E \) be the indicator of the event \( E \) and \( i = \sqrt{-1} \) be the imaginary unit.

## 2 The results

In this section, we characterize the spectral properties for the skew-adjacency matrices of randomly oriented graphs.

Recall that a square matrix \( M = (m_{ij}) \) is said to be skew-symmetric if \( m_{ij} = -m_{ji} \) for all \( i \) and \( j \). It is evident that the skew-adjacency matrix \( S_n := S(G^n_{n,p}) = (s_{ij}) \in \mathbb{R}^{n \times n} \) of the randomly oriented graph \( G^n_{n,p} \) is a skew-symmetric random matrix such that the upper-triangular elements \( s_{ij} \) (\( i < j \)) are i.i.d. random variables satisfying

\[
P(s_{ij} = 1) = pq, \quad P(s_{ij} = -1) = p(1 - q) \quad \text{and} \quad P(s_{ij} = 0) = 1 - p.
\]

Hence, the eigenvalues of \( S_n \) are all purely imaginary numbers. We assume the eigenvalues are \( i\lambda_1, i\lambda_2, \ldots, i\lambda_n \), where all \( \lambda_i \in \mathbb{R} \).

Let \( Y_n \in \mathbb{R}^{n \times n} \) be a skew-symmetric matrix whose elements above the diagonal are 1 and those below the diagonal are \(-1\). Define a quantity

\[
r = r(p, q) = \sqrt{(1 + p(1 - 2q))^2pq + (1 - p(1 - 2q))^2p(1 - q)}
\]

and a normalized matrix

\[
X_n = \frac{-iS_n - ip(1 - 2q)Y_n}{r}.
\]  

It is straightforward to check that \( X_n = (x_{ij}) \in \mathbb{C}^{n \times n} \) is a Hermitian matrix with the diagonal elements \( x_{ii} = 0 \) and the upper-triangular elements \( x_{ij} \) (\( i < j \)) being i.i.d. random variables satisfying mean \( \mathbb{E}(x_{ij}) = 0 \) and variance \( \mathbb{V}(x_{ij}) = \mathbb{E}(x_{ij}^2) = 1 \).
In general, for a Hermitian matrix $M \in \mathbb{C}^{n \times n}$ with eigenvalues $\mu_1(M), \mu_2(M), \cdots, \mu_n(M)$, the empirical spectral distribution of $M$ is defined by

$$F_M(x) = \frac{1}{n} \cdot \#\{\mu_i(M)|\mu_i(M) \leq x, i = 1, 2, \cdots, n\},$$

where $\#\{\cdots\}$ means the cardinality of a set.

**Theorem 1.** Suppose that $nr^2 \to \infty$ and $p(1 - 2q) \to 0$ as $n \to \infty$. Then

$$\lim_{n \to \infty} F_{n^{-1/2}X_n}(x) = F(x) \text{ a.s.}$$

i.e., with probability 1, the empirical spectral distribution of the matrix $n^{-1/2}X_n$ converges weakly to a distribution $F(x)$ as $n$ tends to infinity, where $F(x)$ has the density

$$f(x) = \frac{1}{2\pi \sqrt{4 - x^2}} 1_{|x| \leq 2}.$$

Before presenting the proof of Theorem 1, we first give a couple of remarks.

**Remark 1.** The above function $F(x)$ follows the standard semicircular distribution according to Wigner. However, Theorem 1 extends the classical result of Wigner [23]. To see this, set $q = 1/2$. The assumptions in Theorem 1 reduce to $np \to \infty$. It is easy to check that $r = \sqrt{p}$ and $|E(x_{12}^{k+2})| = 1/p^{k/2}$ if $k$ is even. Hence, if $p = o(1)$, the condition in Wigner’s semicircular law that $E(|x_{12}|^k) < \infty$ for any $k \in \mathbb{N}$ is violated (see e.g. [9, 23]). In the more recent study of spectral convergence results for Hermitian random matrices, it is common to assume finite lower-order moments (e.g. fourth-order or eighth-order moments) of the elements of the underlying matrices [4, 5, 7, 11, 12, 18]. Therefore, our result does not fit in these frames either.

**Remark 2.** Apart from Theorem 1, we can also derive an estimate for the eigenvalues $i\lambda_1, i\lambda_2, \cdots, i\lambda_n$ of the matrix $S_n$. Note that the eigenvalues of $Y_n$ are $\mu_i(Y_n) = i \cot(\pi(2i-1)/2n)$ for $i = 1, 2, \cdots, n$. It follows from Theorem 2.12 in [3] that $\rho(n^{-1/2}X_n) \to 2$ a.s., where $\rho(\cdot)$ stands for the spectral radius. By [11], we have

$$-iS_n \frac{r}{\sqrt{n}} = \frac{X_n}{\sqrt{n}} + \frac{ip(1 - 2q)Y_n}{r\sqrt{n}}.$$

If we arrange the eigenvalues of a Hermitian matrix $M \in \mathbb{C}^{n \times n}$ as $\hat{\mu}_1(M) \geq \hat{\mu}_2(M) \geq \cdots \geq \hat{\mu}_n(M)$, then the Weyl’s inequality [22] implies that for all
\[ \hat{\mu}_n \left( \frac{X_n}{\sqrt{n}} \right) + \hat{\mu}_i \left( \frac{ip(1 - 2q)Y_n}{r\sqrt{n}} \right) \leq \hat{\mu}_i \left( \frac{-iS_n}{r\sqrt{n}} \right) \]
\[ \leq \hat{\mu}_1 \left( \frac{X_n}{\sqrt{n}} \right) + \hat{\mu}_i \left( \frac{ip(1 - 2q)Y_n}{r\sqrt{n}} \right). \]

Putting the above comments together, we obtain
\[ r\sqrt{n} \left( -2 + p(2q - 1) \cot \left( \frac{\pi(2i - 1)}{2n} \right) + o(1) \right) \leq \hat{\mu}_i(-iS_n) \]
\[ \leq r\sqrt{n} \left( 2 + p(2q - 1) \cot \left( \frac{\pi(2i - 1)}{2n} \right) + o(1) \right) \quad \text{a.s.} \quad (2) \]
when \( q \geq 1/2, \) and
\[ r\sqrt{n} \left( -2 + p(2q - 1) \cot \left( \frac{\pi(2n - 2i + 1)}{2n} \right) + o(1) \right) \leq \hat{\mu}_i(-iS_n) \]
\[ \leq r\sqrt{n} \left( 2 + p(2q - 1) \cot \left( \frac{\pi(2n - 2i + 1)}{2n} \right) + o(1) \right) \quad \text{a.s.} \quad (3) \]
when \( q < 1/2. \) Since \( \hat{\mu}_i(-iS_n) \) is the \( i \)-th largest values in the collection \{\( \lambda_1, \lambda_2, \cdots, \lambda_n \)\} by our notation, the estimates for the eigenvalues of \( S_n \) readily follow from (2) and (3).

Now comes the proof of Theorem 1.

**Proof of Theorem 1.** By the moment approach, it suffices to show that the moments of the empirical spectral distribution converge to the corresponding moments of the semicircular law almost surely (see e.g. [3]). That is,
\[ \lim_{n \to \infty} \int x^k dF_{n-1/2X_n}(x) = \int x^k dF(x) \quad \text{a.s.} \quad (4) \]
for each \( k \in \mathbb{N}. \)

First note that under the assumptions of Theorem 1, it can be checked that
\[ E(x^k_{12}) \sim \begin{cases} \frac{1}{r^x} & k \equiv 0 \pmod{4} \\ \frac{1}{r^x+1} & k \equiv 1 \pmod{4} \\ \frac{1}{r^x+2} & k \equiv 2 \pmod{4} \\ \frac{1}{r^x} & k \equiv 3 \pmod{4} \end{cases} \quad (5) \]
for any \( k \in \mathbb{N} \) and \( k > 1. \) Recall that \( x_{ij} \) \((1 \leq i < j \leq n)\) are independently and identically distributed as \( x_{12}, \) and \( x_{ij} = -x_{ji} \) for all \( i \) and \( j. \) To show (4), we consider the following two scenarios according to whether \( k \) is odd or even.
(A) \( k \) is odd. Fix a \( k = 2t + 1 \) with \( t \in \mathbb{N} \cup \{0\} \). By symmetry, we have \( \int_2^0 x^k f(x) dx = 0 \). On the other hand, the integral on the left-hand side of (4) yields

\[
\int x^k dF_{n-1/2 X_n}(x) = \frac{1}{n} \mathbb{E} \left( \text{Trace} \left( \frac{X_n^k}{\sqrt{n^k}} \right) \right) = \frac{1}{n^{1+k/2}} \mathbb{E}(\text{Trace}(X_n^k))
\]

where each summand in (6) can be viewed as a closed walk of length \( k \) following the arcs \((v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_k}, v_{i_1})\) in the complete graph \( G = K_n \) of order \( n \). Clearly, each such walk contains an edge, say \( \{v_i, v_j\} \), that the total number \( n_{ij} \) of times that arcs \((v_i, v_j)\) and \((v_j, v_i)\) are traveled during this walk is odd. Given a closed walk of length \( k \), denote by \( \Omega \) the set of edges in it as described above. Thus, we have \( \Omega \neq \emptyset \). Now consider the following two cases: (A1) there exists \( \{v_i, v_j\} \in \Omega \) such that \( n_{ij} = 1 \); (A2) \( n_{i'j'} \geq 3 \) for all \( \{v_{i'}, v_{j'}\} \in \Omega \).

For (A1), note that the variables in the summands in (6) are independent and \( \mathbb{E}(x_{ij}) = 0 \). Therefore, such walks contribute zero to the right-hand side of (6).

For (A2), let \( m \) denote the number of distinct vertices in a closed walk of length \( k \). Hence, \( m \) is less than or equal to the number of distinct vertices in a closed walk of length \( 2t \), in which each edge (in either direction) appears even times. Clearly, \( m \leq t + 1 \) (the equality \( m = t + 1 \) is attained by a walk in which each arc and the one of opposite direction are traveled exactly once, respectively, and all edges in the walk form a tree). Therefore, these walks will contribute

\[
\frac{1}{n^{1+k/2}} \sum_{m=1}^{t+1} \sum_{\#(i_1, i_2, \ldots, i_k) = m} |\mathbb{E}(x_{i_1i_2i_3i_4 \ldots i_k})| 
\]

\[
\leq \frac{1}{n^{3/2+t}} \sum_{m=1}^{t+1} n^m m^k \left( \frac{1}{r} \right)^{k-2(m-1)} 
\]

\[
\leq \frac{1}{n^{3/2+t}} (t+1)t^{t+1} (t+1)^k \left( \frac{1}{r} \right)^{2t+2t} 
\]

\[
= \frac{(t+1)^{k+1}}{n^{1/2r}},
\]

where the first inequality is due to (5), (6) and the fact that the number of such closed walks is at most \( n^m m^k \). Consequently, combining (A1) and (A2), it follows from (6) that

\[
\int x^k dF_{n-1/2 X_n}(x) = O \left( \frac{1}{n^{1/2r}} \right) \to 0
\]
as \( n \to \infty \), by our assumptions. We complete the proof of (1) for odd \( k \).

**(B) \( k \) is even.** Fix a \( k = 2t \) with \( t \in \mathbb{N} \cup \{0\} \). We have

\[
\int_{-2}^{2} x^k f(x) \, dx = \frac{1}{2\pi} \int_{-2}^{2} x^k \sqrt{4-x^2} \, dx = \frac{1}{\pi} \int_{0}^{2} x^{2t} \sqrt{4-x^2} \, dx
\]

\[
= \frac{2^{2t+1}}{\pi} \int_{0}^{1} y^{t-1/2} (1-y)^{1/2} \, dy
\]

\[
= \frac{2^{2t+1}}{\pi} \frac{\Gamma(t+1/2)\Gamma(3/2)}{\Gamma(t+2)}
\]

\[
= \frac{1}{t+1} \binom{2t}{t}.
\]

(7)

Given a closed walk of length \( k \) in \( K_n \), we still set \( m \) as the number of distinct vertices in it. To analyze the terms in (6), we consider the following three cases: (B1) there exists an edge, say \( \{v_i, v_j\} \), in the closed walk such that the total number of times that arcs \( (v_i, v_j) \) and \( (v_j, v_i) \) are traveled during this walk is odd; (B2) no such \( \{v_i, v_j\} \) exists, and \( m \leq t \); (B3) no such \( \{v_i, v_j\} \) exists, and \( m = t+1 \). Note that if each edge (in either direction) of the closed walk appears even times, we have \( m \leq t+1 \). The equality holds if and only if each arc and the one of opposite direction are traveled exactly once, respectively, and all edges in the walk form a tree.

For (B1), we argue similarly as in (A1) and know that the contribution to the right-hand side of (6) is zero.

For (B2), an analogous derivation as in (A2) reveals that the contribution to the right-hand side of (6) amounts to

\[
\frac{1}{n^{1+k/2}} \sum_{m=1}^{t} \sum_{\#\{i_1, i_2, \ldots, i_k\} = m} |E(x_{i_1}x_{i_2}x_{i_3} \cdots x_{i_k}x_1)|
\]

\[
\leq \frac{1}{n^{1+t}} \sum_{m=1}^{t} n^m m^k \left( \frac{1}{r} \right)^{k-2(m-1)}
\]

\[
\leq \frac{1}{n^{1+t}} \cdot t \cdot n^t \cdot t^k \cdot \left( \frac{1}{r} \right)^{2t-2(t-1)}
\]

\[
= \frac{t^{k+1}}{nr^2}.
\]

For (B3), noting that \( E(x_{12}x_{21}) = -E(x_{12}^2) = 1 \) and the independence of the variables, we obtain that each term \( E(x_{i_1, i_2, i_3, \ldots, i_k}) \) in (6) equals 1. Recall that a combinatorial result [5] Lemma 2.4] says that the number of the closed walks of length \( 2t \) on \( t+1 \) vertices, which satisfy that each each arc and the one of opposite direction both appear exactly once, and
all edges in the walk form a tree, equals $\frac{1}{t+1}(\binom{2t}{t})(t+1)!$. Since there are $\binom{n}{t+1}$ choices of a set of $t+1$ vertices, we conclude that the contribution to the left-hand side of (6) amounts to 
$$\frac{1}{n^{1+k/2}} \cdot \frac{1}{t+1} \binom{2t}{t}(t+1)! \cdot \binom{n}{t+1} = \frac{n(n-1)\cdots(n-t)}{n^{1+t}} \cdot \frac{1}{t+1} \binom{2t}{t}.$$ 

Finally, combining (B1), (B2) and (B3), it follows from (6) that 
$$\int x^k dF_{n^{1/2}X_n}(x) = O\left(\frac{1}{n^{k^2}}\right) + \frac{n(n-1)\cdots(n-t)}{n^{1+t}} \cdot \frac{1}{t+1} \binom{2t}{t} \rightarrow \frac{1}{t+1} \binom{2t}{t},$$

as $n \to \infty$, by our assumptions. In view of (7), the proof of (4) for even $k$ is complete. \(\square\)

To conclude the paper, we simulate the randomly oriented graph model and computed the eigenvalue distribution for the matrix $n^{-1/2} X_n$ (see Figure 1). The simulation results show a perfect agreement with our theoretical prediction. For future work, it would be interesting to explore some other properties (see e.g. [16, 21]) in the setting of randomly oriented graphs.

![Figure 1](image-url) 

Figure 1: Limiting skew-spectral distribution for $G_{\sigma(q)}$ with $n = 1000$, $p = 0.1$ and $q = 0.5$. (a) Histogram of the spectrum of $n^{-1/2} X_n$. A solid line shows the semicircular distribution for comparison. (b) The average spectral density for $n^{-1/2} X_n$ over 500 graphs.

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