Slowly Varying Regression under Sparsity

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Abstract

We consider the problem of parameter estimation in slowly varying regression models with sparsity constraints. We formulate the problem as a mixed integer optimization problem and demonstrate that it can be reformulated exactly as a binary convex optimization problem through a novel exact relaxation. The relaxation utilizes a new equality on Moore-Penrose inverses that convexifies the non-convex objective function while coinciding with the original objective on all feasible binary points. This allows us to solve the problem significantly more efficiently and to provable optimality using a cutting plane-type algorithm. We develop a highly optimized implementation of such algorithm, which substantially improves upon the asymptotic computational complexity of a straightforward implementation. We further develop a heuristic method that is guaranteed to produce a feasible solution and, as we empirically illustrate, generates high quality warm-start solutions for the binary optimization problem. We show, on both synthetic and real-world datasets, that the resulting algorithm outperforms competing formulations in comparable times across a variety of metrics including out-of-sample predictive performance, support recovery accuracy, and false positive rate. The algorithm enables us to train models with 10,000s of parameters, is robust to noise, and able to effectively capture the underlying slowly changing support of the data generating process.

Keywords: Slowly-Varying Regression, Sparsity, Time-Varying Models, Spatially-Varying Models, Mixed Integer Optimization, Binary Convex Optimization, Cutting Plane Algorithm

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1. Introduction

Consider a multiple regression problem with $N$ cases having outcomes $y^1, \ldots, y^T$, where $y^t \in \mathbb{R}^N$ for $t \in [T] := \{1, \ldots, T\}$, and features $X^1, \ldots, X^T$, where $X^t \in \mathbb{R}^{N \times D}$ for $t \in [T]$. The idea of slowly varying regression is to assume that the regression coefficients and relevant features (i.e., features that correspond to nonzero coefficients) have to change slowly between pairs of regressions $(s, t) \in [T] \times [T]$ that are considered similar. Two of the most prominent applications include temporally varying regression and spatially varying regression. In the temporal case, the $T$ regressions are scattered over $T$ consecutive time periods, and regressions between two consecutive time periods are considered to be similar. In the spatial case, the $N$ regressions are conducted over $T$ spatial areas, some of which are adjacent to each other, and it is common to assume that regressions in adjacent areas have to be similar.

In this paper, we consider the general problem of slowly varying regression with sparsity constraints, whereby the $T$ regressions are conducted over a graph $G$ with vertices $V$ of size $|V| = T$. For $v, w \in V$, the edge $(v, w)$ is in the set of edges $E$ if and only if $v$ and $w$ are considered to be similar. Then, the slowly varying regression problem with sparsity constraints can be formulated as below:

$$\min_{\beta^1, \ldots, \beta^T} \sum_{t=1}^T \|y^t - X^t\beta^t\|^2_2 + \lambda_\beta \sum_{t=1}^T \|\beta^t\|^2_2 + \lambda_\delta \sum_{(s, t) \in E} \|\beta^t - \beta^s\|^2_2$$

subject to

$$|\text{Supp}(\beta^t)| \leq K_L, \quad \forall t \in [T],$$

$$\left| \bigcup_{t=1}^T \text{Supp}(\beta^t) \right| \leq K_G,$$

$$\sum_{(s, t) \in E} |\text{Supp}(\beta^t) \triangle \text{Supp}(\beta^s)| \leq K_C,$$

where $\text{Supp}(\beta)$ denotes the set that corresponds to the support of vector $\beta$ and $S_1 \triangle S_2$ denotes the symmetric difference of sets $S_1, S_2$. The objective function (1) penalizes both the least-squares loss of the $T$ regressions and the $l_2$ coefficient distance between regressions that are similar with magnitude $\lambda_\delta$. We also introduce a further $l_2$ regularization term of magnitude $\lambda_\beta$ for robustness purposes (Bertsimas and Copenhaver, 2018). There are three types of constraints on the regression coefficients $\beta^t$:

- **Local Sparsity**: Each regression can have at most $K_L$ relevant features (constraint (2)).

- **Global Sparsity**: There can be at most $K_G$ relevant features across all $T$ regressions (constraint (3)).

- **Sparsely Varying Support**: There can be a difference of at most $K_C$ relevant features among similar regressions $s, t$ across all pairs of similar regressions (constraint (4)).

The parameters $K_L, K_G, K_C$ have to be consistent, i.e., they need to satisfy $K_L \leq K_G \leq D$ and $K_C \leq 2K_LT$. 

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This exact formulation is generally considered infeasible beyond toy scales ($D \leq 10^2$, $T \leq 10$) due to the combinatorial complexity of the sparsity constraints. Therefore, many authors have proposed various relaxations in order to solve this problem, including fused lasso (Tibshirani et al., 2005) and sum-of-norms regularization (Ohlsson et al., 2010). Our key contribution in this paper is to show that this general problem can be reformulated as a binary convex optimization problem, which then can be solved efficiently using a cutting plane-type algorithm. This reformulation is primarily enabled by an exact smooth relaxation of the solution under sparsity constraints, which, to the best of our knowledge, has not appeared in prior literature.

1.1 Relevant Literature

Much of the work in slowly varying regression has focused on the problem of *fused lasso*, where the square loss objective is augmented by a $\ell_1$ penalty term on the difference between the regression coefficients $\|\beta^t - \beta^s\|_1$ among similar pairs to control for total variation and account for pairwise similarity. Since the original work of Tibshirani et al. (2005), there have been many works focusing on fast and accurate algorithms to solve the fused lasso problem (Wytock et al., 2014; Tibshirani and Taylor, 2011) or extending the formulation to different settings (Bleakley and Vert, 2011; Rojas and Wahlberg, 2014). Our work significantly differs with this line of work by utilizing a $\ell_2$ penalty term on the difference between the regression coefficients $\|\beta^t - \beta^s\|_2^2$ instead, and further considers sparsity in the coefficients to limit the number of relevant features that each regression can have.

There are also various other works that utilize different convex regularizers to penalize variation in coefficients and support, including sum-of-norms regularization (Ohlsson et al., 2010) and total variation regularization (Wytock, 2014). (Rojas and Wahlberg, 2014) focus on change point detection using the fused lasso.

Another stream of related work in the application of spatially varying regression are spatially varying coefficient (SVC) models. Instead of imposing a strict constraint on the degree of variability, SVC models focus on identifying the heterogeneity in coefficient estimates varying across space. Notable methods include the spatial expansion method (Casetti, 1972), geographically weighted regression (Brunsdon et al., 1996), and Bayesian SVC models (Besag et al., 1991).

As far as the sparse regression problem is concerned, a recent line of work has focused on solving the actual sparse regression formulation, despite it being known to be NP-hard (Natarajan, 1995), primarily motivated by the tremendous progress in the field of mixed-integer optimization (MIO). In particular, since the work of Bertsimas et al. (2016), who cast sparse regression as an MIO problem and solve problems with $D \sim 10^3$ features to optimality, using both standard MIO techniques and tailored first-order methods as warm starts, more advanced approaches have emerged. These include both exact methods, such as the cutting plane approach of Bertsimas and Van Parys (2020), which scales to problems with $D \sim 10^5$ features, and the custom nonlinear branch and bound framework of Hazimeh et al. (2020), which scales to problems with $D \sim 10^7$ features, as well as heuristic solution methods, e.g., the subgradient method of Bertsimas et al. (2020), and the method of Hazimeh and Mazumder (2018) that combines coordinate descent with local search.
1.2 Contributions & Outline

Our key contributions can be summarized as follows:

- We formulate the problem of slowly varying regression under sparsity as a mixed-integer optimization problem by imposing constraints which ensure that the learned model is indeed sparse, as well as slowly and sparsely varying.

- We demonstrate that the problem can be reformulated exactly as a binary convex optimization problem through an original exact relaxation of the original problem. The proposed relaxation relies upon a novel equality on Moore-Penrose inverses that convexifies the non-convex objective function while coinciding with the original objective on all feasible binary points. We study the properties of our proposed relaxation and develop intuition on why it works.

- Leveraging the convexity of the reformulated problem, we propose a cutting plane-type algorithm that enables us to solve the binary convex optimization problem at hand to optimality and, by exploiting the structure of the problem, we discuss how to efficiently implement the proposed algorithm. Our highly optimized implementation substantially improves upon the asymptotic computational complexity of a straightforward implementation.

- We develop a heuristic stepwise method which is guaranteed to produce a feasible solution and, as we empirically illustrate, computes high-quality solutions to the binary convex optimization problem. The obtained solution can then be used as warm-start for the cutting plane algorithm.

- We empirically evaluate our proposed method on both synthetic and real-world data. In synthetic experiments, we demonstrate the resulting algorithm outperforms competing formulations in comparable times across a variety of metrics while being scalable, robust to noise, and can effectively capture the slowly changing support of the data generating process. In real-world experiments, we show an increased out-of-sample predictive power, and illustrate how the resulting sparse and slowly varying model can provide insights into the problem at hand.

The rest of the paper is organized as follows. In Section 2, we introduce a mixed-integer optimization (MIO) formulation of the problem (1)–(4). In Section 3, we prove the equality that allows us to construct an exact relaxation for the MIO formulation, so that we can reformulate the problem as a pure binary convex optimization problem. In Section 4, we briefly explore the properties of such relaxation and how it could be extended to general quadratic models. In Section 5, we show how the resulting problem can be solved to optimality using a cutting plane-type algorithm. In Section 6, we develop a heuristic algorithm that enables us to compute high quality solutions, which can be used as warm starts for the cutting plane algorithm to further improve its scalability. Finally, in Sections 7 and 8, we empirically evaluate our proposed approach using both synthetic and real-world datasets.
2. An MIO Formulation

In this section, we develop a mixed-integer optimization (MIO) formulation for the problem defined in (1)–(4).

Local Sparsity. First, we introduce binary variables $z_t^d$ encoding the support of the coefficients $\beta_t^d$, $\forall t$, as

$$z_t^d = 0 \implies \beta_t^d = 0, \quad \forall t \in [T], d \in [D].$$

Then, the requirement that the number of nonzero coefficients at each vertex is less than $K_L$ can be expressed as:

$$\sum_{d=1}^{D} z_t^d \leq K_L, \quad \forall t \in [T]. \quad (5)$$

Global Sparsity. Similarly, we introduce binary variables encoding the union of supports over vertices. We require that $s_d$ is set to 1 if $z_t^d$ is set to 1 at least once over all vertices, i.e.,

$$s_d \geq z_t^d, \quad \forall t \in [T], d \in [D]. \quad (6)$$

Then,

$$\sum_{d=1}^{D} s_d \leq K_G. \quad (7)$$

Sparsely Varying Support. To be able to capture the sparsely varying support requirement, we introduce another set of binary variables

$$w^{t,s}_d = 0 \implies \|\beta_t^d\|_0 = \|\beta_s^d\|_0 \implies z_t^d = z_s^d, \quad \forall (s,t) \in E, d \in [D].$$

This can be rewritten as

$$w^{t,s}_d \geq z_t^d - z_s^d \quad \text{and} \quad w^{t,s}_d \geq z_s^d - z_t^d, \quad \forall (s,t) \in E, d \in [D]. \quad (8)$$

We then require that

$$\sum_{(s,t) \in E} \sum_{d=1}^{D} w^{t,s}_d \leq K_C. \quad (9)$$
**Overall Formulation.** With these helper binary variables and with constraints (5)-(9), we can now rewrite the original problem defined in (1)-(4) as follows:

\[
\min_{z, s, w} \min_{\beta} \sum_{t=1}^{T} \| y^t - X^t Z^t \beta^t \|_2^2 + \lambda_{\beta} \sum_{t=1}^{T} \| Z^t \beta^t \|_2^2 + \lambda_{\delta} \sum_{(s,t) \in E} \| Z^t \beta^t - Z^s \beta^s \|_2^2
\]

\[\text{s.t.} \sum_{d=1}^{D} z^t_d \leq K_L, \quad \forall t \in [T], \tag{11}\]

\[s_d \geq z^t_d, \quad \forall t \in [T], d \in [D], \tag{12}\]

\[\sum_{d=1}^{D} s_d \leq K_G, \tag{13}\]

\[w^t_{d,s} \geq z^t_d - z^s_d, \quad \forall (s,t) \in E, d \in [D], \tag{14}\]

\[w^t_{d,s} \geq z^s_d - z^t_d, \quad \forall (s,t) \in E, d \in [D], \tag{15}\]

\[\sum_{(s,t) \in E} \sum_{d=1}^{D} w^t_{d,s} \leq K_C, \tag{16}\]

where \( Z^t = \text{Diag}(z_1^t, \ldots, z_D^t) \) are diagonal binary matrices of \( z \) variables. For convenience, we denote the optimization problem over \( z, s, w \) as the outer optimization problem, while the optimization over \( \beta \) as the inner optimization problem.

**3. The Binary Convex Reformulation**

In this section, we reformulate the mixed-integer optimization problem defined in (10)-(16) as a pure-binary convex optimization problem.

First, we note the following lemma:

**Lemma 1** The MIO optimization problem defined in (10)-(16) can be written as:

\[
\min_{z, s, w} \min_{\beta} \text{c}(z, \beta) := \beta^\top (Z(M + \lambda_{\beta} I)Z)\beta - \mu^\top \beta
\]

where \( \beta = (\beta^1, \ldots, \beta^T) \), \( Z = \text{Diag}(z^1, \ldots, z^T) \), and \( Z \times S \times W \) is the polyhedral feasible set as defined by the constraints (11)-(16). \( M \in \mathbb{R}^{TD \times TD} \) and \( \mu = (\mu^1, \ldots, \mu^T) \) are defined as follows:

\[
M^t_{i,j}^{s} = \begin{cases} 
\sum_n (X^t_{n,i})^2 + \lambda_{\beta} + d^t \lambda_{\delta}, & \text{if } s = t \text{ and } i = j, \\
\sum_n X^t_{n,i} X^t_{n,j}, & \text{if } s = t \text{ and } i \neq j, \\
-\lambda_{\delta}, & \text{if } (s,t) \in E \text{ and } i = j, \\
0, & \text{otherwise},
\end{cases}
\]

\[\mu^t = (X^t)^\top y^t,\]

where \( d^t \) denotes the degree of vertex \( t \). Furthermore, \( M \) is a positive semi-definite matrix.
Proof This follows immediately from rearrangement of the variables.

With this formulation, we can solve the inner problem easily using the first order condition and reduce the problem to a binary optimization problem:

**Lemma 2** Denote \( \beta^*(z) = \arg \min_\beta c(z, \beta) \). Then, we have
\[
\beta^*(z) = (Z(M + \lambda_\beta I)Z)^\dagger Z\mu,
\]
where \( \dagger \) denotes the Moore-Penrose Pseudo-inverse. \(^1\) Furthermore, it holds that
\[
\min_{z,s,w \in Z \times S \times W} c(z, \beta) = \min_{z,s,w \in Z \times S \times W} -\frac{\mu^\top \beta^*(z)}{2}.
\]

**Proof** As \( M \) is positive semi-definite, to solve the inner problem we only need to derive the first order condition, which is:
\[
\frac{\partial c(z, \beta)}{\partial \beta} = (Z(M + \lambda_\beta I)Z)\beta - Z\mu = 0.
\]
\( Z \) is a rank \( \leq TK_L < TD \) matrix, so \( (Z(M + \lambda_\beta I)Z) \) is rank-deficient, and thus to solve this first-order condition, we can utilize the Moore-Penrose pseudo-inverse to write:
\[
\beta^*(z) = (Z(M + \lambda_\beta I)Z)^\dagger Z\mu.
\]

The second assertion follows immediately from substituting the first order equality \( (Z(M + \lambda_\beta I)Z)\beta^*(z) = Z\mu \) into the objective expression.

However, the resulting problem:
\[
\min_{z,s,w \in Z \times S \times W} -\frac{\mu^\top \beta^*(z)}{2}
\]
is neither convex nor differentiable in \( z \), when \( \beta^*(z) = (Z(M + \lambda_\beta I)Z)^\dagger Z\mu \), making the problem intractable.

However, we have the following key observation: we only care about \( \beta^*(z) \) for binary vectors \( z \). Therefore, we proceed to consider exact convex relaxations of \( (Z(M + \lambda_\beta I)Z)^\dagger Z \) such that it agrees with \( (Z(M + \lambda_\beta I)Z)^\dagger Z \) on all binary points \( z \). Specifically, we prove the following proposition, which allows us to convexify the expression above:

**Proposition 1** Let \( M \) be a positive semi-definite matrix. Then we have, for \( z \in \{0, 1\}^{TD} \), \( Z = \text{Diag}(z^1, \cdots, z^T) \), and \( \lambda_\beta > 0 \):
\[
(Z(M + \lambda_\beta I)Z)^\dagger Z = (\lambda_\beta I + ZM)^{-1} Z.
\]

**Proof** We prove this in two steps. First, we establish the following relation for the pseudoinverse:

---

1. The Moore-Penrose pseudoinverse \( A^\dagger \in \mathbb{R}^{n \times m} \) of \( A \in \mathbb{R}^{m \times n} \) is the unique matrix that satisfies the four conditions: 1. \( A^\dagger A A^\dagger = A^\dagger \), 2. \( A A^\dagger A = A \), 3. \( (AA^\dagger)^* = AA^\dagger \), 4. \( (A^\dagger A)^* = A^\dagger A \), where \( * \) is the Hermitian operator with \( A^*_{ij} = \overline{A_{ji}} \).
Lemma 3

\[(Z(M + \lambda_\beta I)Z)^\dagger = (\lambda_\beta I + ZMZ)^{-1} - \lambda_\beta (I - Z).\]  \hspace{1cm} (19)

**Proof** We verify that the expression on the right satisfies the definition of a Moore-Penrose pseudoinverse for \(Z(M + \lambda_\beta I)Z\). The Moore-Penrose pseudoinverse \(A^\dagger\) is the unique matrix that satisfies:

1. \(AA^\dagger A = A\)
2. \(A^\dagger AA^\dagger = A^\dagger\)
3. \((AA^\dagger)^* = AA^\dagger\)
4. \((A^\dagger A)^* = A^\dagger A\)

Here, \(^*\) represents the Hermitian conjugate. The assertions follow immediately if we have \(A^\dagger A = AA^\dagger = I\), which we next prove:

\[
A^\dagger A = (\lambda_\beta I + ZMZ)^{-1} - \lambda_\beta (I - Z))Z(M + \lambda_\beta I)Z
\]

\[
= (\lambda_\beta I + ZMZ)^{-1} (ZMZ + \lambda_\beta Z)
\]

\[
= (\lambda_\beta I + ZMZ)^{-1} (ZMZ + \lambda_\beta I - \lambda_\beta (I - Z))
\]

\[
= (I - (\lambda_\beta I + ZMZ)^{-1} \lambda_\beta (I - Z))
\]

\[
= (I - \left( \frac{1}{\lambda_\beta} (I - Z(\lambda_\beta I + MZ)^{-1} MZ) \right) \lambda_\beta (I - Z))
\]

\[
= I,
\]

where on the second last line we utilized the binomial inverse theorem (Henderson and Scarle, 1981). Here, we have \(Z^2 = Z\) as \(Z\) is a binary diagonal matrix. The case for \(AA^\dagger\) is identical.

Then, we note the following equivalence:

**Lemma 4**

\[(\lambda_\beta I + ZMZ)^{-1} Z = (\lambda_\beta I + ZM)^{-1} Z.\]  \hspace{1cm} (20)

**Proof** By the binomial inverse theorem (Henderson and Scarle, 1981), we have:

\[
(\lambda_\beta I + ZMZ)^{-1} Z = \frac{1}{\lambda_\beta} (I - Z(\lambda_\beta I + MZ)^{-1} MZ)Z = \frac{1}{\lambda_\beta} (Z - Z(\lambda_\beta I + MZ)^{-1} MZ).
\]

Similarly, we have:

\[
(\lambda_\beta I + ZM)^{-1} Z = \frac{1}{\lambda_\beta} (I - Z(\lambda_\beta I + MZ)^{-1} M)Z = \frac{1}{\lambda_\beta} (Z - Z(\lambda_\beta I + MZ)^{-1} MZ).
\]

This proves the statement required.
We now prove the final desired statement, utilizing Lemmas 3 and 4:

\[
(Z(M + \lambda \beta I)Z)^\dagger Z = ((\lambda \beta I + ZMZ)^{-1} - \lambda \beta (I - Z))Z = (\lambda \beta I + ZMZ)^{-1}Z = (\lambda \beta I + ZM)^{-1}Z.
\]

Finally, we prove that, using the reformulation in Proposition 1, the problem becomes convex in \(z\):

**Theorem 1** Let \(M, \mu\) be defined in Lemma 1, and \(\lambda \beta > 0\). Then, the optimization problem in (10)—(16) is equivalent to the following convex optimization problem:

\[
\min_{z, s, w \in Z \times S \times W} -\mu^\top \beta^*(z) \frac{1}{2},
\]

where \(\beta^*(z) = (\lambda \beta I + ZM)^{-1}Z\mu\).

**Proof** The equivalence of the two optimization problems follows immediately from Lemma 2 and Proposition 1. We proceed to prove that \(f(z) = -\frac{1}{2} \mu^\top \beta^*(z)\) is convex in \(z\). Denote the element-wise products \((\beta \cdot M)_{ij} = \beta_i M_{ij}\) and \((\beta \cdot \mu) = \beta_i \mu_i\). Then, by direct calculation, the Hessian of \(f(z)\) in the direction of \(\beta\) can be calculated as:

\[
\beta^\top \frac{\partial f(z)}{\partial z} \beta^\top = \mu^\top (\lambda \beta I + ZM)^{-1}(\beta \cdot M)(\lambda \beta I + ZM)^{-1}(\mu \cdot \beta) - \mu^\top (\lambda \beta I + ZM)^{-1}(\beta \cdot M)(\lambda \beta I + ZM)^{-1}((\beta \cdot M)(\lambda \beta I + ZM)^{-1}Z\mu)
= \mu^\top (\lambda \beta I + ZM)^{-1}(\beta \cdot M)(\lambda \beta I + ZM)^{-1}(\beta \cdot (I - M(\lambda \beta I + ZM)^{-1}Z)\mu)
= \frac{1}{\lambda \beta} \mu^\top (\lambda \beta I + ZM)^{-1}(\beta \cdot M)(\lambda \beta I + ZM)^{-1}(\beta \cdot (\lambda \beta I + MZ)^{-1}\mu)
= \frac{1}{\lambda \beta} (\beta \cdot (\lambda \beta I + MZ)^{-1}\mu)^\top M(I + ZM)^{-1}(\beta \cdot (\lambda \beta I + MZ)^{-1}\mu)
\geq 0.
\]

The final expression is a quadratic form and \(M(I + ZM)^{-1}\) is positive semidefinite, as \(M\) is positive semi-definite. Since this inequality holds for any \(\beta\), the Hessian matrix of \(f(z)\) is positive semidefinite and hence \(f(z)\) is convex in \(z\).

**Theorem 1** shows that the original problem as shown in (10)—(16) can be reformulated into a binary convex optimization problem over \(z, s, w\), which is amenable to a cutting plane-type algorithm.

We point out that the key ingredient that enabled such convex relaxation, Proposition 1, is by no means obvious. There are infinitely many relaxations one could construct that
match exactly the binary points of \((Z(M + \lambda I)Z)\dagger Z\). In fact, an arguably more natural construction of a relaxation is the following equality (that can be easily shown using Lemma 3):

\[
(Z(M + \lambda I)Z)\dagger Z = (\lambda I + ZMZ)^{-1}Z.
\] (22)

However, such a relaxation, unlike the one shown in Proposition 1, results in a non-convex reformulation of the problem as stated in (10)–(16), making it significantly more difficult to solve.

4. Discussion of the Relaxation

As shown in Section 3, the key observation that enabled us to create the convex reformulation is Proposition 1. In this section, we discuss why the relaxation works and illustrate it intuitively.

For simplicity, we consider the case where we have \(T = D = 1\), and a single binary variable \(z\). Then by Theorem 1, the objective function for the optimization problem defined in (10)–(16) has the form:

\[
f_1(z) = -(z(m + \lambda I)z)\dagger I^2 z = \begin{cases} 0, & z = 0, \\ -\frac{\mu^2}{z(m + \lambda I)}, & z \neq 0. \end{cases}
\]

Proposition 1 then reads, for all \(m > 0\), and \(z \in \{0, 1\}^\times\):

\[
(z(m + \lambda I)z)\dagger I = \frac{z}{\lambda I + mz}.
\]

So the objective function after reformulation has the form:

\[
f_2(z) = -\frac{\mu^2 z}{\lambda I + mz}.
\]
While the other natural relaxation we can construct, as defined in Equation (22) gives the following objective function:

\[ f_3(z) = -\frac{\mu^2 z}{\lambda \beta + mz^2}. \]

In Figure 1 we plot \( f_1(z), f_2(z), f_3(z) \) for \( m = 19, \mu = 1, \) and \( \lambda \beta = 1 \). First we observe that in one dimension, the pseudoinverse is a discontinuous and non-convex function that follows a \(-\frac{1}{z}\) type curve everywhere except for \( z = 0 \), where it takes the value of 0. This clearly reflects the difficulty to solve the sparse problem as formulated in the standard way.

We then observe that both \( f_2(z) \) and \( f_3(z) \) agree with \( f_1(z) \) when \( z \in \{0, 1\} \), and therefore \( f_2(z), f_3(z) \) are both valid relaxations of the discontinuous function \( f_1(z) \) on the binary values of \( z \). However, we clearly see that \( f_3(z) \) is not a convex function in \( z \), while \( f_2(z) \) is. This illustrates how the carefully chosen relaxation enables efficient convex algorithms to be utilized.

We finally note that the relaxation utilized in this paper, \( f_2(z) \), is by no means the only smooth relaxation possible for the discontinuous function \( f_1(z) \). In fact, it is not too hard to show, by standard results on composition of convex functions that the following set of functions are convex for all \( 0 < a < 1 \):

\[ f_a(z) = -\frac{\mu^2 z^a}{\lambda \beta + mz^a}, \]

and we believe that this could be a fruitful direction in future research to develop even more effective convex relaxations to this problem.

5. An Exact Cutting Plane Algorithm

In this section, we propose a cutting plane-type algorithm that solves Problem (21) to optimality. The proposed Algorithm 1 is based on the outer approximation method by Duran and Grossmann (1986), which iteratively tightens a piecewise linear lower approximation of the objective function. Algorithm 1 provides pseudocode for the proposed approach.

Recall from Theorem 1 that the objective function \( c(z, \beta) \) is indeed convex in \( z \) and can, in fact, be written as function only of the binary variables \( z \) by solving the inner problem to optimality, i.e.,

\[
\min_{\beta} c(z, \beta) := c(z) = -\frac{\mu^T \beta^*(z)}{2} = -\frac{1}{2} \mu^T (\lambda \beta I + ZM)^{-1} Z \mu. \tag{23}
\]

Algorithm 1 also requires the computation of the gradient of the cost function \( \nabla_z c(z) \) at every binary point \( z \) it visits. We therefore aim to differentiate the loss function \( c(z) \) with respect to the diagonal entries of the matrix \( Z = \text{Diag}(z^1, \ldots, z^T) \). The partial derivative with respect to component \( z^t_d \) can be computed numerically using finite differences as

\[
\frac{\partial c(z)}{\partial z^t_d} = \frac{c(z) - c(z - \varepsilon e^t_d)}{\varepsilon}, \tag{24}
\]

where \( e^t_d \) denotes the basis vector with 1 in position \((t, d)\) and 0’s elsewhere and \( \varepsilon \) is a sufficiently small constant. Nevertheless, such an approach would be highly impractical, as it
Algorithm 1: Cutting Plane Algorithm

\textbf{Input:} Data \((X^t, y^t)_{t=1}^T\), similarity graph \(G\), sparsity parameters \((K_L, K_G, K_C)\), regularization weights \((\lambda_\beta, \lambda_\delta)\).

\textbf{Output:} Learned coefficients \(\beta^*\).

\begin{itemize}
  \item \(
  \triangleright\) Find warm start using Algorithm 2:
  \(
  \beta^{(0)} \leftarrow \text{find\_start}\left((X^t, y^t)_{t=1}^T, G, (K_L, K_G, K_C), (\lambda_\beta, \lambda_\delta)\right)
  \)
  \end{itemize}

\begin{itemize}
  \item Compute corresponding binary variables:
  \(
  (z^{(0)}, s^{(0)}, w^{(0)}) \leftarrow \text{find\_binaries}(\beta^{(0)})
  \)
  \end{itemize}

\begin{itemize}
  \item \((i, \eta^{(0)}) \leftarrow (0, 0)\)
  \end{itemize}

\begin{itemize}
  \item Cutting plane iterations:
  \begin{itemize}
    \item \text{while} \(c(z^{(i)}) > \eta^{(i)}\) \text{do}
    \begin{itemize}
      \item \((z, s, w, \eta)_{(i+1)} \leftarrow \arg\min_{\eta \in \mathbb{R}^+, z, s, w \in \mathbb{Z} \times S \times W} \eta \geq c(z^{(\tau)}) + \nabla_z c(z^{(\tau)})^\top (z - z^{(\tau)}), \forall \tau \in [i]
      \end{itemize}
    \end{itemize}
    \item \(i \leftarrow i + 1\)
  \end{itemize}
\end{itemize}

\begin{itemize}
  \item \text{Estimate coefficients using Theorem 1:}
  \begin{itemize}
    \item \(\beta^* \leftarrow \beta^*(z^{(i)})\)
    \item \text{return } \beta^*
  \end{itemize}
\end{itemize}

would require \(TD\) evaluations of the cost function (23). Instead, we utilize the chain rule to compute the gradient in closed form, as shown below:

**Lemma 5** Let \(K = K(z) := (\lambda_\beta I + ZM)\) and let \(E_d^t\) denote a \(TD \times TD\) matrix, with 1 at position \((t, d), (t, d)\) and 0’s elsewhere. Then we have:

\[
\frac{\partial c(z)}{\partial z_d} = \frac{1}{2} \mu^\top K^{-1} (E_d^t M K^{-1} Z - E_d^t) \mu.
\]

**Proof** We begin by differentiating matrix \(K\) with respect to \(Z\)’s diagonal component \(z_d^t\):

\[
\frac{\partial K}{\partial z_d} = \frac{\partial K(z)}{\partial z_d} = \frac{\partial (\lambda_\beta I + ZM)}{\partial z_d} = E_d^t M.
\]

The partial derivative of the inverse of \(K\) is then given by

\[
\frac{\partial K^{-1}}{\partial z_d} = -K^{-1} \frac{\partial K}{\partial z_d} K^{-1} = -K^{-1} E_d^t M K^{-1}.
\]
Finally, we have
\[
\frac{\partial c(z)}{\partial z^t_d} = \frac{\partial (-\frac{1}{2} \mu^\top K^{-1} Z \mu)}{\partial z^t_d} = \frac{1}{2} \mu^\top \left( \frac{\partial K^{-1}}{\partial z^t_d} Z + K^{-1} \frac{\partial Z}{\partial z^t_d} \right) \mu
\]
\[
= -\frac{1}{2} \mu^\top \left( -K^{-1} E_d^t MK^{-1} Z + K^{-1} E_d^t \right) \mu
\]
\[
= \frac{1}{2} \mu^\top K^{-1} \left( E_d^t MK^{-1} Z - E_d^t \right) \mu.
\]

(27)

We next discuss the computational complexity of the cut generation for Algorithm 1. Recall that the cut generation process requires the evaluation of the cost function \( c(z) \) and its gradient \( \nabla_z c(z) \).

**Lemma 6** Let \( z \) be a feasible binary vector for Problem (21). Then, the cost function \( c(z) \) and its gradient \( \nabla_z c(z) \) can be evaluated in \( O(T^3K^2 + T^2K^3 + T^2KL) \) operations.

**Proof** We first introduce some notation: given any vector (matrix) \( a \) \( (A) \) and a binary vector \( z \) (feasible for Problem (21)), \( a_z \) \( (A_z; \text{ or } A_{z \cdot}) \) is formed by selecting all entries \((t, d)\) of vector \( a \) (all rows \((t, d)\) or all columns \((t, d)\) of matrix \( A \), respectively) for which \( z^t_d = 1 \). Accordingly, the subscript \( z^c \) selects the entries/rows/columns for which \( z^t_d = 0 \).

**Cost function evaluation.** Define \( K = (\lambda_\beta I + ZM) \). Then given a feasible binary vector \( z \), the cost function \( c(z) \) is \(-\frac{1}{2} \mu^\top K^{-1} Z \mu \). To evaluate this equation, we first need to invert matrix \( K \). The size of matrix \( K \) is \( TD \times TD \), so a naive implementation would require \( O(T^3D^3) \) operations. We can reduce the complexity of the inversion by exploiting the structure of the matrix as follows:

- We reorder the rows and columns of matrix \( K \) so that it takes the form:

\[
\tilde{K} := \begin{bmatrix}
\lambda_\beta I + M_{z, z} & M_{z, z^c} \\
0 & \lambda_\beta I
\end{bmatrix},
\]

where \( M_{z, z} \in \mathbb{R}^{TK_l \times TK_l} \) and \( M_{z, z^c} \in \mathbb{R}^{TK_l \times T(D-K_l)} \). We then similarly reorder \( \mu \) and \( Z \) to \( \tilde{\mu} = [\mu_z, \mu_{z^c}] \) and \( \tilde{Z} = \text{Diag}(1_z, 0_{z^c}) \). Note that the reordering does not change the objective value, and therefore the objective function is now

\[
-\frac{1}{2} \tilde{\mu}^\top \tilde{K}^{-1} \tilde{Z} \tilde{\mu}.
\]

- We perform blockwise inversion, which gives

\[
\tilde{K}^{-1} = \begin{bmatrix}
(\lambda_\beta I + M_{z, z})^{-1} & \frac{1}{\lambda_\beta} (\lambda_\beta I + M_{z, z})^{-1} M_{z, z^c} \\
0 & \frac{1}{\lambda_\beta} I
\end{bmatrix}.
\]

(28)
Since \( Z \) has zeros on the diagonals for all \( z^c \) columns, we thus have
\[
\tilde{K}^{-1} \tilde{Z} = \begin{bmatrix} (\lambda_\beta I + M_{z^i, z^i})^{-1} & 0 \\ 0 & 0 \end{bmatrix}
\]
and therefore the objective function can be now written as
\[
-\frac{1}{2} \tilde{\mu}^\top \tilde{K}^{-1} \tilde{Z} \tilde{\mu} = \frac{1}{2} \tilde{\mu}_z^\top \left( \lambda_\beta I + M_{z^i, z^i} \right)^{-1} \tilde{\mu}_z.
\]
Noting that the matrix \( (\lambda_\beta I + M_{z^i, z^i}) \) has block tri-diagonal structure, with \( T \) blocks of size \( K_L \times K_L \) each, its inverse \( (\lambda_\beta I + M_{z^i, z^i})^{-1} \) can be computed by recursive application of blockwise inversion in \( O \left( T^2 K_L^2 (T + K_L) \right) \) operations.

- The remaining operations to evaluate the objective are the vector-matrix multiplications with \( \mu_z \), which require \( O \left( T^2 K_L^2 \right) \) operations.

**Gradient evaluation.** As far as the gradient evaluation is concerned, we compute each of the \( TD \) entries of the gradient as per Lemma 5:
\[
\frac{\partial c(z)}{\partial z_d} = \frac{1}{2} \mu_z^\top K^{-1} \left( E_{d}^t MK^{-1}Z - E_{d}^t \right) \mu.
\]
We use \( v^i \in \mathbb{R}^{TD} \) to denote auxiliary vectors. We work as follows:

- We compute the multiplication \( K^{-1} Z \mu \). Noting that \( K^{-1} Z \) selects the columns \((t, d)\) of \( K^{-1} \) for which \( z_{d}^i = 1 \) and sets the remaining columns to 0, and observing that \( K_{z^i, z^i}^{-1} = 0 \) from Equation 28, we in fact only need to compute
\[
v_0^z = K_{z^i, z^i}^{-1} \mu_z.
\]
The remaining entries of \( v^0 \in \mathbb{R}^{TD} \), i.e., \( v_{z^c}^0 \), are set to 0. This only needs to be performed once, independently of which entry of the gradient is being computed, and \( K^{-1} \) is readily available from the evaluation of the cost function. Thus, the complexity is \( O \left( T^2 K_L^2 \right) \) operations.

- We compute part of the multiplication \( \mu^\top K^{-1} \), namely,
\[
v_1^z = \left( \mu_z^\top K_{z^i, z^i}^{-1} \right)^\top,
\]
which requires \( O \left( T^2 K_L^2 \right) \) operations.
To compute the remaining entries of \( v^1 \in \mathbb{R}^{TD} \), i.e., \( v_{z^c}^1 \), we again reorder \( \mu \) to be \( \tilde{\mu} := [\mu_z, \mu_{z^c}] \) similar to above, and then use the formula indicated in Equation (28). Put together, we have
\[
v^1 = \begin{bmatrix} \mu_z^\top (\lambda_\beta I + M_{z^i, z^i})^{-1} - \frac{1}{\lambda_\beta} \mu_z^\top (\lambda_\beta I + M_{z^i, z^i})^{-1} M_{z^i, z^c} + \mu_{z^c}^\top \\ \frac{1}{\lambda_\beta} \left( v_z^1 M_{z^i, z^c} + \mu_{z^c}^\top \right) \end{bmatrix},
\]
which requires \( O \left( T^2 K_L D \right) \) operations.
The above steps only need to be performed once, independently of which entry of the gradient is being computed. Overall, the complexity is \( O \left( T^2 K_L D \right) \) operations.
For each \((t, d)\), we compute the multiplication \((E_{d}^{t}M)(K^{-1}Z\mu)\). Noting that the multiplication \(E_{d}^{t}M\) yields a matrix that is nonzero only at row \((t, d)\), and since the result is multiplied with the vector \(K^{-1}Z\mu\), we implement the multiplication as
\[
v_{(t,d)}^{2} = M_{(t,d),z}v_{z}^{0}.
\]
This requires \(O(T^2DK_L)\) operations in total across all \((t, d)\).

For each \((t, d)\), we compute the multiplication \((\mu^{\top}K^{-1})(E_{d}^{t}MK^{-1}Z\mu)\). We first note that \(E_{d}^{t} = E_{d}^{t}E_{d}^{t}\) and therefore the multiplication can be rewritten as:
\[
(\mu^{\top}K^{-1}E_{d}^{t})(E_{d}^{t}MK^{-1}Z\mu) = (\mu^{\top}K^{-1}E_{d}^{t})(E_{d}^{t}v^{2})
\]
Therefore, the first term selects the \((t, d)\) column of \(\mu^{\top}K^{-1}\) and the second term selects the \((t, d)\) row of \(v^{2}\). Using this fact, along with Equation (30), we can now go back and calculate the final product
\[
v_{(t,d)}^{3} = (\mu^{\top}K^{-1}E_{d}^{t})(E_{d}^{t}MK^{-1}Z\mu) = v_{(t,d)}^{1} \cdot v_{(t,d)}^{2}.
\]
The complexity is \(O(TD)\) operations in total across all \((t, d)\).

For each \((t, d)\), we compute the multiplication \(\mu^{\top}K^{-1}E_{d}^{t}\mu\) as
\[
v_{(t,d)}^{4} = v_{(t,d)}^{1} \cdot \mu_{(t,d)} + 1(z_{d} = 0) \frac{(\mu_{(t,d)})^{2}}{\lambda_{g}}.
\]
This requires \(O(TD)\) operations in total across all \((t, d)\).

For each \((t, d)\), we compute \((O(TD)\) operations in total across all \((t, d)\)) the corresponding entry of the gradient as
\[
\frac{\partial c(z)}{\partial z_{d}^{t}} = \frac{v_{(t,d)}^{3} - v_{(t,d)}^{4}}{2}.
\]

After completing the steps outlined above, we have the ingredients to compute all entries of the gradient \(\nabla_{z}c(z)\). In total, the cost is \(O(T^2DK_L)\) operations.

**Cut generation.** All told, the complexity of the entire cut generation process is
\[
O(T^2K_L[K_L(T + K_L) + D])
\]
Finally, Theorem 2 asserts that Algorithm 1 converges to the optimal value of Problem (21) within a finite number of iterations. Intuitively, finite termination is guaranteed since the feasible set is finite and the outer-approximation process of Algorithm 1 never visits a point twice. As a sidenote, we also remark that we need not solve a new binary optimization problem at each iteration of Algorithm 1 by integrating the entire algorithm within a single branch-and-bound tree, as proposed by Quesada and Grossmann (1992), using lazy constraint callbacks.
Theorem 2 Assume Problem (21) is feasible. Then, in a finite number of iterations, Algorithm 1 terminates and returns an optimal solution.

Proof From Theorem 1, we have that \( f(z) = -\frac{1}{2}z^\top z \), where \( z^* = (\lambda I + ZM)^{-1} Z \mu \) is convex in \( z \). We can conclude on termination and convergence of the outer-approximation cutting plane algorithm (described in Algorithm 1) by application of the classic result from Fletcher and Leyffer (1994).

6. A Heuristic Stepwise Algorithm

In this section, we develop a heuristic algorithm for solving the MIO formulation defined by equations (1)-(4).

The idea of the proposed algorithm is that we fit separate sparse regressions per vertex in \( G \), each satisfying only the local sparsity constraint (2). We refer to the set of features that have been selected in at least one of the separate sparse regressions as the global support \( S \). Now, since the sparse regressions were fit separately, the global sparsity constraint (3) and the sparsely varying support constraint (4) need not be satisfied. Thus, as long as any of those constraints is violated, we iteratively remove features from \( S \) and refit separate regressions per vertex in \( G \), allowing only \( K_L \) among the features from the updated \( S \) to be selected in each vertex. Once constraints (3) and (4) are both satisfied, we compute the binary vector \( z \) that encodes the overall support (across all vertices in \( G \)). Finally, we use Theorem 1 to estimate the associated coefficients \( \beta^*(z) \) (i.e., for given support \( z \), we only need to solve the inner problem to get the associated coefficients). The proposed approach is presented in Algorithm 2.

We next show that Algorithm 2 will eventually yield a feasible solution for Problem (1)-(4).

Theorem 3 Assume Problem (1)-(4) is feasible. Then, in a finite number of iterations, Algorithm 2 terminates and provides a feasible solution.

Proof The termination condition of this algorithm guarantees that, at termination, the solution must be a feasible solution for Problem (1)-(4). Therefore, we only need to prove that Algorithm 2 terminates in a finite number of iterations.

After the first stage, where we fit separate sparse regressions per vertex in \( G \), \( S \) consists of at most \( \min\{TK_L, D\} \) features. Since, after each iteration, we remove one feature from \( S \), the global sparsity constraint (3) is guaranteed to be satisfied after at most \( \min\{TK_L, D\} - K_G \) iterations. Similarly, and assuming exactly \( K_L \) features are selected in each vertex, after at most \( \min\{TK_L, D\} - (K_L + \frac{K_C}{2}) \) iterations, all separately fit sparse regressions will be constrained to include the same set of \( K_L + \frac{K_C}{2} \) features and hence any pair of similar regressions will differ in at most \( K_C \) features. Alternatively, if we allow less than \( K_L \) features in each vertex, then, after at most \( \min\{TK_L, D\} - K_C \) iterations, any pair of similar regressions will differ in at most \( K_C \) features. Therefore, the algorithm terminates in at most:

\[
\max \{ \min\{TK_s, D\} - K_G, \min\{TK_L, D\} - (K_L + \frac{K_C}{2}), \min\{TK_L, D\} - K_C \}
\]
Algorithm 2: Stepwise Algorithm

**Input:** Data \((X^t, y^t)_{t=1}^T\), similarity graph \(G\), sparsity parameters \((K_L, K_G, K_C)\), regularization weights \((\lambda_\beta, \lambda_\delta)\).

**Output:** Learned coefficients \(\hat{\beta}\).

\[\triangleright\] Fit separate sparse regressions per vertex in \(G\) (using a heuristic algorithm):

\[
\text{for } t \in [T] \text{ do} \\
\text{ } \beta^t \leftarrow \arg\min_{\beta \in \mathbb{R}^D} \|y^t - X^t\beta\|_2^2 + \lambda_\beta \|\beta\|_2^2 \text{ s.t. } |\text{Supp}(\beta)| \leq K_L
\]

\[
S \leftarrow \bigcup_{t=1}^T \text{Supp}(\beta^t)
\]

\[\triangleright\] As long as constraints on global support not satisfied:

\[\text{while } |S| > K_G \text{ or } \sum_{(s,t) \in E} |\text{Supp}(\beta^s) \triangle \text{Supp}(\beta^t)| > K_C \text{ do} \]

\[
\triangleright\text{ Evaluate loss function after removing each feature } j \text{ that is in current support:} \\
\text{for } j \in S \text{ do} \\
\text{ } S_{-j} \leftarrow S \setminus \{j\} \\
\text{ } L_{-j} \leftarrow \sum_{t \in [T]} \|y^t - \sum_{d \in S_{-j}} X^t_d \beta^t_d\|_2^2 + \lambda_\beta \sum_{t \in [T], d \in S_{-j}} (\beta^t_d)^2 + \lambda_\delta \sum_{(s,t) \in E, d \in S_{-j}} (\beta^t_d - \beta^s_d)^2
\]

\[
\text{end for}
\]

\[\triangleright\text{ Find feature whose removal achieves the minimum loss:} \\
\text{ } j^* \leftarrow \arg\min_{j \in S} L_{-j}
\]

\[
S \leftarrow S_{-j^*}.
\]

\[\triangleright\text{ Find vertices in which feature } j^* \text{ is used:} \\
\text{for } t \in [T] \text{ do} \\
\text{ } \text{if } |\beta^t_{j^*}| > 0 \text{ then} \\
\text{ } \triangleright \text{ Randomly select an adjacent vertex:} \\
\text{ } s \leftarrow \text{sample}(\{s : (s,t) \in E\})
\text{ end if} \\
\text{end for}
\]

\[\triangleright\text{ Add this feature to the support of } t: \\
\text{ } S^t \leftarrow (\text{Supp}(\beta^t) \cup \{j^*\}) \setminus \{j^*\}
\]

\[\triangleright\text{ Refit regularized regression in vertex } t \text{ using features in } S^t: \\
\beta^t \leftarrow (X^t_{S_t})^T X^t_{S_t} + \lambda_\beta I)^{-1} (X^t_{S_t})^T y^t
\]

\[
\text{end if}
\]

\[\text{end while}
\]

\[\triangleright\text{ Find final support:} \\
\text{for } t \in [T], d \in [D] \text{ do} \\
\text{ } z^t_d \leftarrow 1_{(|\beta^t_d| > 0)}
\]

\[
\text{end for}
\]

\[\triangleright\text{ Estimate coefficients using Theorem 1:} \\
\beta \leftarrow \beta^*(z)
\]

return \(\hat{\beta}\)
iterations, which is finite. Therefore, algorithm 2 terminates and returns a feasible solution of the original problem.

We note that Algorithm 2 is not guaranteed to converge to the optimal solution. Nevertheless, as we empirically show, it is able to find high-quality solutions, which can be used as warm starts for Algorithm 1. To speed up Algorithm 2, we approximately solve the sparse regression problem at each vertex using computationally efficient algorithms. Such examples include the dual sub-gradient algorithm of Bertsimas et al. (2020), which solves the Boolean relaxation of the sparse regression formulation, or the greedy stepwise procedure of Efroymson (1966), which, in fact, inspired the backward elimination approach to the sparse and slowly varying regression problem that our proposed Algorithm 2 takes.

7. Experiments on Synthetic Datasets

In this section, we test our proposed sparse slowly varying regression framework on synthetic data.

**Algorithms and software.** We first give the implementation details of the algorithms which we compare in our experiments. Note that the following algorithms and software, as well as their corresponding tuning processes, were used in both the synthetic experiments, presented in this section, and the real-world experiments, presented in Section 8. For a fair comparison, we implement all algorithms in Julia.

- **Sparse regression:** We fit a single (static) sparse regression model across all vertices. Note that, as a result, this approach uses \( N' = NT \) data points to train \( D \) parameters (since the same set of parameters is estimated across all vertices). We solve the boolean relaxation of the sparse regression formulation (as per Bertsimas et al. (2020)) using the `SubsetSelection.jl` Julia package (available at https://github.com/jeanpauphilet/SubsetSelection.jl). We refer to this approach as static.

- **Sum-of-norms regularization:** We fit a slowly varying regression model in which penalize the sum across all pairs of adjacent vertices of the \( \ell_1 \) difference between the corresponding coefficients. As discussed in Ohlsson et al. (2010), we use the \( \ell_1 \) norm to encourage sparsely-changing coefficients. The resulting learning problem can be formulated as a linear optimization problem, which we solve using the JuMP modeling language for mathematical optimization (Dunning et al., 2017) and the CPLEX commercial solver. We refer to this approach as sum-norms.

- **Sum-of-norms and lasso regularization:** We expand the sum-norms approach with an \( \ell_1 \) penalty on the coefficients to add robustness and -hopefully- encourage some level of sparsity. We again formulate the resulting learning problem as a linear optimization problem, which we solve using JuMP and CPLEX. We refer to this approach as sum-norms-lasso.

- **Sparse and slowly varying regression via the heuristic stepwise algorithm:** We implement Algorithm 2 using the `SubsetSelection.jl` Julia package to fit separate sparse
regressions per vertex. Note that, as we explain next, this approach is part of our proposed cutting plane algorithm, so we generally do not distinguish between the two. We refer to this approach as stepwise.

- Sparse and slowly varying regression via the exact cutting plane algorithm: We implement Algorithm 1 using JuMP and CPLEX. To keep computational times short, we impose a time limit of 5 minutes for the cutting plane algorithm, at which point the branch-and-bound process stops and returns the best solution found so far. As part of this approach, we use stepwise to find an initial warm-start solution. We refer to this approach as cutplane.

Each of the above models is hyper-parameter tuned using holdout validation; we exhaustively search a grid of 7 values for the regularization weight $\lambda_\beta$ (starting from a relatively large value of $N$ -or $N'$ for static- and iteratively decreasing it by a factor of 3) and 3 values for the slowly varying penalty $\lambda_\delta$ (selected in a similar fashion).

Concerning the remaining hyperparameters of cutplane and stepwise, in the synthetic experiments, we assume that the parameters $K_L, K_G, K_C$ are known. In real-world problems, this need not be the case and, in fact, in the real-world experiments presented in Section 8, we do explore a few different values for each of those hyperparameters as well. This is one argument commonly used against exact (cardinality-constrained) sparse regression formulations: the sparsity parameters might not be known, in which case they need to be cross-validated hence resulting in a dramatic increase in the required computational effort. Although, in many cases, such parameters are determined by the application, Kenney et al. (2018) address such concerns by proposing efficient cross validation strategies.

**Data generation methodology.** We generate a matrix of coefficients $\beta \in \mathbb{R}^{T \times D}$. Each element $\beta^t \in \mathbb{R}^D$ is the vector of coefficients of the regression at vertex $t \in [T]$. To generate $\beta$, we control the following parameters:

1. $K_L \in \mathbb{Z}^+$: the local sparsity, i.e., the sparsity of each element $\beta^t$ for $t \in [T]$. Namely, we impose $||\beta^t||_0 \leq K_L$.
2. $K_G \in \mathbb{Z}^+$: the global sparsity, as detailed in Equation (3).
3. $K_C \in \mathbb{Z}^+$: the number of changes in support, as detailed in Equation (4).
4. $\sigma_v \in \mathbb{R}^+$: the maximum allowed magnitude of change in coefficients between similar vertices; we draw the actual change in coefficients at any vertex uniformly at random from $[-\sigma_v, +\sigma_v]$.
5. $\xi \in \mathbb{R}^+$: the signal-to-noise ratio for the noise added to the outcome variable.
6. $E \in \mathbb{Z}^+$: number of edges in the similarity graph.

Then, the generation of $\beta$ is as follows:

- In the temporally varying case, we consider, without loss of generality, node $t$, $t \in [T-1]$, to be adjacent to node $t+1$. Moreover, we do not explicitly determine the global sparsity parameter $K_G$ - it can be indirectly controlled as function of $K_L$ and $K_C$. 

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SLOWLY VARYING REGRESSION UNDER SPARSITY
We start by generating an initial vector of coefficients $\beta^1$, satisfying the local sparsity constraint, uniformly at random from $\{-1,1\}^{K_L}$. Then, we traverse the similarity graph, which, in this case, is a chain, generate $\beta^{t+1}, t \in [T-1]$, by perturbing $\beta^t$ according to the desired $\sigma_v$.

The desired number of changes in support is performed by randomly replacing features which originally were in the support, with features which were not, at randomly selected vertices from $[T]$.

- In the spatially varying case, we control all six parameters. We generate a random Erdos-Renyi graph $G$ with $E$ edges. We randomly choose the global support $S$ according to the desired value of $K_G$, i.e., $|S| = K_G$.

For each connected component $C$ of $G$, we generate an initial vector of coefficients $\beta^C$, satisfying the local sparsity constraint, uniformly at random from $\{-1,1\}^{K_L}$ (note that we allow only features from the global support to be selected). Then, for each vertex $t \in C$, we construct $\beta^t$ by perturbing $\beta^C$ according to the desired $\sigma_v$.

The desired number of changes in support is performed by randomly replacing features which originally were in the support, with features which were not, at randomly selected vertices from the global support $S$.

We then create the design matrix $X \in \mathbb{R}^{N \times T \times D}$ by first generating two random matrices $X^a \in \mathbb{R}^{N \times T \times D}$ and $X^b \in \mathbb{R}^{N \times T \times D}$ with i.i.d. Gaussian entries with mean 0 and standard deviation 1. Then, we introduce a correlation across time according to a parameter $\rho_t$, as $X^a_{i::,t+1::} = X^a_{i::,t+1::} + \rho_t X^a_{i::,t::}, t \in [T-1]$, and a correlation across features according to a parameter $\rho_d$, as $X^b_{i::,d+1::} = X^b_{i::,d+1::} + \rho_d X^b_{i::,d::}, d \in [D-1]$. Finally, we set $X = X^a + X^b$.

The outcome vectors $Y \in \mathbb{R}^{N \times T}$ are created by applying $\beta$ on $X$ and adding i.i.d. noise drawn from a normal distribution $\mathcal{N}(0, \sigma^2)$ to each entry in $Y$, where $\sigma^2$ is selected to satisfy

$$\xi^2 = \frac{\sum_{t \in T} \|X^t \beta^t\|^2}{\sigma^2}$$

according to the desired signal-to-noise ratio $\xi$.

Our task is to estimate $\beta$ and make out-of-sample predictions for unseen data $X_{test} \in \mathbb{R}^{N_{test} \times T \times D}$ and $Y_{test} \in \mathbb{R}^{N_{test} \times T}$, generated according to the same process as $X$ and $Y$. We benchmark on 4 metrics: the mean absolute error (MAE) on the coefficients of $\beta$, the out-of-sample $R^2$, the percentage of support recovered, and the percentage of false positives in the estimated support. In our first experiment, we compare the scalability of cutplane with that of static as function of the problem size $(N, T, D, K_L, K_G, K_C)$. We independently generate 3 datasets and report the mean of the results. We then perform four additional experiments, in each of which we vary one of the key dimensions controlling the ground truth coefficients $\beta$ and the data $X$ and $Y$. For each of these experiments, we independently generate 10 datasets and report both the mean and standard deviation of each metric.

### 7.1 Experiment 1: Scalability with respect to the problem size

The purpose of this experiment is to test the scalability of the proposed framework for increasing problem size. We thus compare our implementation of cutplane with static, which, as shown in Bertsimas et al. (2020), very efficiently solves the boolean relaxation of the standard sparse regression formulation.
We set $\sigma_v = 0.25$, $\xi = 2$, $\rho_t = \rho_d = 0.6$, and experiment for increasing problem size. In particular, in each regime defined by $(T, D, K_L, K_G, K_C)$, we increase the number of data points from 300 through 10,000. Moreover, we assume that the similarity graph has temporal structure, namely, it is a chain connecting each vertex $t \in [T - 1]$ with vertex $t + 1$. We train the models on $X$ and $Y$, and measure the out-of-sample $R^2$ on $X_{\text{test}}$ and $Y_{\text{test}}$, the MAE in the estimated coefficients, and the computational time it takes to refit the best model for each method on the entire training set (i.e., after the validation process is completed).

| $(T, D, K_L, K_G, K_C)$ | N   | Out-of-sample $R^2$ | MAE in coefficients | Computational time (in sec) |
|-------------------------|-----|----------------------|---------------------|-----------------------------|
|                         |     | static | cutplane | static | cutplane | static | cutplane |
| (10,30,3,4,1)           | 300 | 0.724  | 0.798    | 0.02   | 0.004    | 0.685  | 320.117  |
|                         | 1000| 0.734  | 0.799    | 0.021  | 0.003    | 1.703  | 308.439  |
|                         | 3000| 0.741  | 0.8      | 0.02   | 0.001    | 4.162  | 318.498  |
|                         | 5000| 0.732  | 0.8      | 0.02   | 0.001    | 8.232  | 336.584  |
|                         | 10000| 0.72  | 0.8      | 0.021  | 0.001    | 16.339 | 361.817  |
| (20,50,5,7,2)           | 300 | 0.755  | 0.797    | 0.017  | 0.006    | 1.773  | 309.488  |
|                         | 1000| 0.753  | 0.797    | 0.019  | 0.003    | 4.741  | 324.749  |
|                         | 3000| 0.753  | 0.799    | 0.018  | 0.002    | 15.606 | 338.236  |
|                         | 5000| 0.754  | 0.799    | 0.018  | 0.002    | 23.03  | 336.409  |
|                         | 10000| 0.745 | 0.797    | 0.019  | 0.002    | 46.422 | 366.951  |
| (30,100,10,13,3)        | 300 | 0.761  | 0.792    | 0.017  | 0.008    | 3.881  | 312.534  |
|                         | 1000| 0.758  | 0.798    | 0.018  | 0.004    | 13.706 | 342.804  |
|                         | 3000| 0.761  | 0.799    | 0.017  | 0.002    | 43.644 | 352.674  |
|                         | 5000| 0.766  | 0.799    | 0.016  | 0.002    | 72.275 | 375.242  |
|                         | 10000| 0.763 | 0.797    | 0.016  | 0.001    | 151.336| 445.456  |
| (40,250,20,24,4)        | 300 | 0.762  | 0.785    | 0.013  | 0.008    | 9.377  | 443.668  |
|                         | 1000| 0.766  | 0.795    | 0.013  | 0.005    | 30.989 | 380.846  |
|                         | 3000| 0.766  | 0.798    | 0.013  | 0.003    | 112.481| 441.335  |
|                         | 5000| 0.764  | 0.799    | 0.013  | 0.002    | 187.283| 524.504  |
|                         | 10000| 0.767 | 0.799    | 0.013  | 0.002    | 260.068| 745.591  |
| (50,600,30,35,5)        | 1000| 0.767  | 0.794    | 0.008  | 0.003    | 93.545 | 1504.994 |
|                         | 3000| 0.767  | 0.798    | 0.008  | 0.002    | 227.448| 1624.862 |
|                         | 5000| 0.767  | 0.799    | 0.008  | 0.002    | 303.374| 1756.476 |
|                         | 10000| 0.769 | 0.799    | 0.008  | 0.001    | 433.352| 2272.335 |

Table 1: Results for experiment 1.

The results, presented in Table 1, show that, in all cases, cutplane has an edge over static, in terms of both its out-of-sample predictive power and its estimation accuracy. Most importantly, we see that cutplane enables us to compute more accurate solutions in reasonable times, i.e., the computational overhead is not prohibitive. For example, using cutplane, we estimate 30,000 parameters $(T = 50, D = 600)$ with 10,000 data points in minutes.

We remark that, in our experiments, cutplane usually terminates due to the 5 minute time limit (the additional computational time shown in Table 1 is due to stepwise). In such cases, although cutplane does find high-quality solutions, which are associated with a suboptimality gap that can be directly derived from the branch-and-bound tree, the solutions are not provably optimal. This is not a surprising observation; as discussed in Bertsimas et al. (2020), the computational time required to solve the sparse regression problem to provable optimality is highly dependent on the regularization weight $\lambda_\beta$. As $\lambda_\beta \to 0$, it might take a huge amount of time to solve the problem to provable optimality (although the optimal
solution is usually attained fast), whereas when $\lambda_\beta \to \infty$ the problem can be solved trivially (with the optimal solution being $\beta = 0$).

### 7.2 Experiment 2: Number of changes in support $K_C$

In this experiment, we set $N = 300, T = 20, D = 50, K_L = 5, \sigma_v = 0.1, \rho_t = \rho_d = 0.6, \xi = 2$, and vary the number of changes $K_C$ in the support of the ground truth coefficients $\beta$. We again assume that the similarity graph has temporal structure. We train all models (we now include sum-norms and sum-norms-lasso in the comparison) on $X$ and $Y$, measure their out-of-sample $R^2$ on $X_{test}$ and $Y_{test}$, and evaluate the three other metrics (described previously) using the ground truth coefficients $\beta$. We present the results in Figure 2.

Figure 2a shows that methods incorporating a slowly varying component (sum-norms, sum-norms-lasso, and cutplane) significantly outperform the static sparse method (static) in terms of out-of-sample $R^2$, with a slight, but statistically significant advantage for our cutplane (Figure 2b). Figure 2c shows that as the number of changes in support increases, static is the only method not able to fully recover the support of $\beta$, going below 60% for $K_C > 15$. However, only cutplane is also able to detect the truth and nothing but the truth, with 0% false positives, while the false positive rate for sum-norms and sum-norms-lasso is at $\approx 90\%$ and $\approx 70\%$, respectively, because of its lack of sparsity, as demonstrated in Figure 2d. Last but not least, we can see in 2e that cutplane significantly outperforms the other methods in terms of MAE in the coefficients of $\beta$, with the static being the most affected with number of changes in support.

### 7.3 Experiment 3: Magnitude of changes in coefficients $\sigma_v$

In this experiment, we set $N = 300, T = 20, D = 50, K_L = 5, K_C = 2, \rho_t = \rho_d = 0.6, \xi = 2$. We study the impact of the magnitude of changes in the coefficients $\sigma_v$. We assume that the similarity graph has temporal structure. We get the results in Figure 3. We make the following observations:

- cutplane slightly outperforms sum-norms and sum-norms-lasso, and significantly outperforms static in terms of out-of-sample $R^2$ (Figure 3a).

- All methods achieve above 98% true support recovery for $\beta$. The only method that perfectly recovers the entire support is sum-norms; this, however, is due to the fact that the estimated model is fully-dense, i.e., it puts a nonzero coefficient on all features (Figure 3b).

- cutplane outperforms static in terms of false positives in the support, while sum-norms and sum-norms-lasso fail to identify non-relevant dimensions, with false positive rates at $\approx 90\%$ and $\approx 70\%$, respectively (Figure 3c).

- cutplane very significantly outperforms the 3 other methods in terms of MAE in coefficients (between 3 and 10 times smaller) (Figure 3d).
Slowly Varying Regression under Sparsity

(a) Out-of-Sample R² vs. Number of Changes in Support.

(b) Out-of-Sample R² vs. Number of Changes in Support (Zoomed).

(c) Percentage of Support Recovered vs. Number of Changes in Support.

(d) Percentage of False Positives vs. Number of Changes in Support.

(e) Mean Absolute Error in Coefficient vs. Number of Changes in Support.

Figure 2: Results for experiment 2.

7.4 Experiment 4: Signal-to-noise ratio ξ

In this experiment, we set $N = 300, T = 20, D = 50, K_L = 5, K_C = 2, \sigma_v = 0.1, \rho_t = \rho_d = 0.6$, and vary the signal-to-noise ratio $\xi$. We assume that the similarity graph has temporal
structure. Figure 4 shows the robustness of the cutplane against noise. It also confirms the findings of experiments 2 and 3 and allows to conclude that cutplane is more accurate, in terms of out-of-sample $R^2$, and even more so, in terms of MAE in coefficients, than both the state-of-the-art sum-norms and static models. Moreover, cutplane is able not only to recover the supports of the regression coefficients almost perfectly, but it also does so without false positives.

7.5 Experiment 5: Number of edges $E$ in the similarity graph

In this experiment, we set $N = 300, T = 20, D = 50, K_L = 5, K_C = 4, \sigma_v = 0.1, \rho_t = \rho_d = 0.6$. We examine how the number of edges $E$ in the similarity graph affects the performance of each method. Note that we no longer assume that the similarity graph has temporal structure; we generate the graph according to the process described in the spatially varying case of the data generation methodology. The results are shown in Figure 5. cutplane is almost insensitive to the graph structure and performs well, in terms of all metrics, regardless of the number of edges in the similarity graph. All slowly varying methods provide accurate predictions that are not affected by the number of edges (Figure 5a). Again, cutplane is the only method that recovers the truth (Figure 5b) and nothing but the truth (Figure 5c), and achieves the lowest MAE in the estimated coefficients (Figure 5d). Finally, we remark that,
as the number of edges increases, the graph is more likely to be fully connected and, due to our data generating process, the coefficient in all vertices become similar; this explains why static eventually seems to catch up with the slowly varying methods.

8. Experiments on Real-World Datasets

In this section, we study the performance of our proposed sparse slowly varying regression framework on real-world data. The algorithms and software that we use are exactly as in Section 7.

8.1 Appliances energy prediction case study

In this experiment, we focus on a real-world case study concerned with appliances energy prediction (Candanedo et al., 2017). The dataset is publicly available at the University of California Irvine (UCI) Machine Learning repository, at https://archive.ics.uci.edu/ml/datasets/Appliances+energy+prediction.

Each observation in the dataset is a vector of measurements made by a wireless sensor network in a low energy building. The features include the temperature and humidity conditions in various rooms in the building, the weather conditions in the nearest weather
station, and a couple of noise variables. The goal is to predict the energy consumption of the building’s appliances energy use. Measurements are taken every 10 minutes over a 4.5-month period.

We preprocess the dataset as follows. We construct the similarity graph by assigning a vertex to each hour of the day, so that $T = 24$. To capture the temporal structure in the problem, the graph is a chain, i.e., vertex $t \in [T - 1]$ is considered adjacent to vertex $t + 1$. For each day $d$ in the data, we create 6 data points per vertex $t \in [T]$, by collecting all 6 measurements that were taken at hour $t$ and during day $d$. For example, for $t = 15$, we collect the measurements taken at 3pm, 3:10pm, ..., 3:50pm, across all days in the data. By doing so, we get a total of $N = 822$ data points per vertex. Each data point consists of $D = 26$ features, which we standardize.

We split the dataset into training (70% of the data) and testing set (30%), and hyperparameter tune each of the models (static, sum-norms, sum-norms-lasso, stepwise, and cutplane) in the exact same way as described in Section 7. Notice that we now also run stepwise decoupled from cutplane to showcase the improvement (if any) that we get over the warm-start solution. For static, stepwise, and cutplane, we present results separately for two different sets of sparsity parameters. The main difference with the synthetic experiments is that the true underlying $\beta$ is now unknown, so we can only benchmark these models based
on the quality of the predictions, i.e, the out-of-sample $R^2$. The results are presented in Table 2.

| Method       | Parameters | Out-of-sample $R^2$ | Computational time (in sec) |
|--------------|------------|---------------------|-----------------------------|
| static       | $K_L$ $K_G$ $K_C$ | 0.069 | 1.182 |
| sum-norms    | 26 26 - | 0.15 | 125.396 |
| sum-norms-lasso | 25 25 - | 0.15 | 126.915 |
| stepwise     | 3 6 24 | 0.068 | 2.911 |
| cutplane     | 3 6 24 | 0.077 | 326.745 |
|              | 12 15 24 | 0.151 | 309.795 |

Table 2: Benchmark of regression methods for the Energy experiment (temporal).

Table 2 shows an improvement in terms of $R^2$ for the same sparsity over non-slowly varying sparse regression, and a significant improvement in terms of sparsity compared to existing slowly varying methods (sum-norms and sum-norms-lasso), going from 26 and 25 features selected, respectively, to 12 for the same level of accuracy.

### 8.2 Housing price prediction case study

In our second real-world case study, we explore the application of our framework to the task housing price prediction, in Ames, Iowa (De Cock, 2011). This dataset is also publicly available at [http://jse.amstat.org/v19n3/decock/DataDocumentation.txt](http://jse.amstat.org/v19n3/decock/DataDocumentation.txt). Contrary to the previous benchmark, which had temporal structure, this task is spatial and the components are slowly varying from location to location and not from time to time.

The original dataset contains 2,930 observations and a large number of features (23 nominal, 23 ordinal, 14 discrete, and 20 continuous) involved in assessing home values. The sales took place in Ames, Iowa, from 2006 to 2010. The goal is to predict the price at which the house was sold.

We preprocess the dataset as follows. We first drop features with missing values in over 1% of the observations; then, we drop any observation that still has missing features. We use one-hot encoding for the nominal features, integer encoding for ordinal and discrete variables, and we standardize the data. Each data point consists of $D = 199$ features. The dataset contains information on the neighborhood where each house is located, so we could have used these neighborhoods as the vertices in the similarity graph. Nevertheless, such an approach leads to highly imbalanced vertices in terms of the number of data points that fall therein (due to the fact that many sales were performed at some neighborhoods and very few at others). To address this issue, we cluster the neighborhoods into larger groups, while requiring that neighborhoods that fall into the same group be adjacent and that the number of data points that fall into each group be relatively balanced. Then, we construct the similarity graph by adding an edge between groups of neighborhoods that are adjacent. In the end, we obtain $T = 7$ groups of neighborhoods, each with at least $N = 352$ data points (for simplicity, we randomly select exactly $N = 352$ data points in each group). The similarity graph consists of $E = 8$ edges.
We work exactly as in the first real-world case study: we split the dataset into training (70% of the data) and testing set (30%), and hyper-parameter tune each of the models (static, sum-norms, sum-norms-lasso, stepwise, and cutplane) as described in Section 7. The results are presented in Table 3, whereas Figure 6 provides a visualization of the proposed learned model.

| Method            | Parameters K_L K_G K_C | Out-of-sample $R^2$ | Computational time (in sec) |
|-------------------|------------------------|----------------------|-----------------------------|
| static            | 30 30 -                | 0.853                | 19.186                      |
|                   | 50 50 -                | 0.856                | 49.965                      |
| sum-norms         | 198 198 -              | 0.736                | 279.341                     |
| sum-norms-lasso   | 62 68 -                | 0.865                | 290.054                     |
| stepwise          | 30 40 60               | 0.837                | 156.524                     |
|                   | 50 60 100              | 0.868                | 429.251                     |
| cutplane          | 30 40 60               | 0.864                | 457.372                     |
|                   | 50 60 100              | **0.876**            | 729.254                     |

Table 3: Benchmark of regression methods for the Housing experiment (spatial).

Table 3 shows that cutplane outperforms all its competitors in terms of out-of-sample $R^2$, with 0.876 for a local sparsity of 50. We notice an edge over both the non-slowly varying sparse regression, in terms of $R^2$, and the non-sparse slowly varying methods (sum-norms and sum-norms-lasso), in terms of $R^2$ and sparsity. Last but not least, this experiment shows that the cutting plane method does add substantial improvement to the stepwise heuristic.

| Neighborhood cluster name | $\beta_1$: lot area | $\beta_2$: land slope | $\beta_3$: year built | $\beta_4$: heating quality and condition |
|---------------------------|----------------------|------------------------|------------------------|-------------------------------------------|
| NN                        | 0.118                | -0.023                 | 0.087                  | 0                                         |
| N                         | 0.064                | 0.001                  | 0.124                  | 0                                         |
| NE                        | 0.054                | 0.019                  | 0.154                  | 0                                         |
| E                         | 0.052                | 0                      | 0.171                  | 0.027                                     |
| S                         | 0.075                | 0                      | 0.133                  | 0.08                                      |
| W                         | 0.002                | 0                      | 0.120                  | 0.030                                     |
| NW                        | 0.074                | 0                      | 0.126                  | 0                                         |

Table 4: Selected learned coefficients in each neighborhood cluster obtained from the best cutplane solution (as per Table 3).

In Figure 6, we show a map of Ames, IA, along with the neighborhood clusters that we used in our experiment, and, in Table 4, we show a collection of (selected) learned coefficients obtained from the best cutplane solution (as per Table 3). The coefficients shown correspond to the following features: lot area (feature 1, continuous), land slope (feature 2, ordinal), year built (feature 3, discrete), and heating quality and condition (feature 4, ordinal). For example, we observe that, as we move to the southeast, the year a house was built becomes increasingly more important; the corresponding regressor starts at 0.087 in the northernmost neighborhoods and increases to 0.133 in the south and 0.171 in the east. The land slope seems to be relevant in the north and northeast neighborhoods, as the corresponding regressor is set to zero elsewhere; on the other hand, the regressor that corresponds to heating quality and condition takes nonzero values in the southernmost
9. Conclusion

In this paper, we considered the sparse slowly varying regression problem and, after rigorously stating it as a mixed integer optimization problem, we showed that it can be reformulated exactly as a binary convex optimization problem. The reformulation is possible thanks to a novel relaxation that we proved, which utilizes a new equality on Moore-Penrose inverses that convexifies the non-convex objective function while coinciding with the original objective on all feasible binary points. We demonstrated that the resulting binary convex reformulation can be solved efficiently and to provable optimality using a cutting plane-type algorithm and developed a highly optimized implementation of such algorithm. We further developed a heuristic algorithm that is guaranteed to produce a feasible solution and, as we empirically
illustrated, generates high quality solutions that can be fed to the cutting plane algorithm as warm-starts.

In our synthetic data experiments, we demonstrated the superior performance of the proposed formulation compared to competing algorithms on a variety of metrics including out-of-sample predictive performance, support recovery, and robustness to noise. The algorithm is able to capture the underlying slowly changing support of the data while scaling to large-scale datasets of up to 10,000s of parameters.

In our real-world data experiments, we further confirmed the increased out-of-sample predictive power on two real-world datasets that are slowly varying in temporal and spatial dimensions respectively. We then illustrated how the resulting sparse and slowly varying model can effectively be interpreted and hence provide insights into the problem at hand.

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