Effect of space-time dispersion on the propagation of electromagnetic waves in photonic crystals

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Abstract
We study the influence of the space and time dispersion on the frequency dependence of the wave vectors of electromagnetic waves propagating in three-dimensional photonic crystals. Two types of structures are considered: media with weak periodic modulation of the permittivity, and photonic crystals composed of the periodically arranged identical resonant dielectric particles. It is shown that in these systems, in contrast to electrons in solid crystals, different types of excitations exist. For example, a peculiar kind of polaritons arises in the photonic crystals due to the interaction of the electromagnetic field, eigenoscillations of the dielectric medium, and Debye resonance. The widths of the transparency zones and of the band gaps have been calculated as functions of the frequency and of the parameters of the media. It is shown that in the photonic crystals with dispersion, the number of transparency bands is larger than in non-dispersive systems, and the width of the gaps in the frequency spectrum of photons depends on the wave vector. The interaction of different types of waves deforms the Brillouin zone, so that it may not have a plane boundary (for example, a sphere), in which case the classical Bragg condition does not hold.

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1 Introduction

The propagation of waves in periodic structures is a topic of profound importance, both for understanding the general properties of matter-radiation interactions, and for the ever-increasing number of practical applications (for example,
photonic crystals) [1], [2], [3]. In spite of the close similarity to the quantum-mechanical problem of electrons in a crystal lattice, the propagation of classical waves in photonic crystals exhibits many rather unusual distinctive features. The most dramatic example of such a difference is the effect of time and space dispersion, which has no analogue in the motion of conduction electrons. While the propagation of electromagnetic waves in homogeneous dispersive media has been thoroughly studied [4], the effect of dispersion on the radiation propagating in periodic structures still remains poorly understood. The exceptions are periodic multiple quantum well structures (periodic arrays of dielectric layers), for which the polariton spectrum and the optical response have been studied in considerable detail [5], [6], [7], [8]. In the recent publication [9], the exciton-polariton susceptibility matrix was calculated, taking into account the well-well interaction, and the polariton dispersion curves for an infinite periodic array of quantum wells have been computed. Some interesting polariton-induced effects were discovered. However, the one-dimensionality of the system considered prevented the investigation of important effects inherent only in dispersive media of higher dimensionality. In the present paper, we deal with the linear wave equation of the most general type and study the spectral properties of dispersive periodic dielectric media, namely, the effect of the time and space dispersion on the frequency dependence of the wave vector. The methods employed (small perturbation and local perturbation methods) impose no restrictions on the dimensionality of the system. As formulated, the problem is closely related to the calculation of the energy of an electron as a function of the wave vector in periodic crystals. However, in contrast to the later case, the solution of the corresponding dispersion equation for electromagnetic waves can have several branches, i.e. different types of waves can propagate. The interaction of these waves is the crucial factor in the formation of the photonic spectrum, and gives rise to rather unusual effects. In particular, it deforms the Brillouin zone, so that it may not have a plane boundary (for example, a sphere), in which case the classical Bragg condition is violated.

We consider the most general form of the wave equation

\[ \hat{H}(\frac{\partial}{i\hat{\partial}r}, \frac{\partial}{i\hat{\partial}t})E + \hat{h}(\vec{r}, \frac{\partial}{i\hat{\partial}r}, \frac{\partial}{i\hat{\partial}t})E = 0, \]  

where \( \hat{H} \) and \( \hat{h} \) are arbitrary functions of \( \frac{\partial}{i\hat{\partial}r} \) and \( \frac{\partial}{i\hat{\partial}t} \), and \( \hat{h} \) is a periodic function of \( \vec{r} \) with periods \( \vec{r}_n = n_1\vec{r}_1 + n_2\vec{r}_2 + n_3\vec{r}_3 \), where \( \vec{r}_1, \vec{r}_2, \vec{r}_3 \) define the unit cell of the photonic crystal, and \( n_1, n_2, n_3 \) are integers. In non-dispersive media, Eq. (1) becomes a classical second-order wave equation; space-time dispersion corresponds to higher-order derivatives with respect to the coordinates and the time in Eq. (1).

According to the Bloch theorem, the solution of this equation has the form
\[ E(\vec{r}, t) = E(\vec{r}) e^{i(k\vec{r} - \omega(\vec{k}) t)} , \]

where \( E(\vec{r}) \) is a periodic function with periods \( \vec{r}_n \).

Our goal is to investigate the spectrum of waves propagating in the periodic system, i.e. to find the dependences, \( \omega_l(\vec{k}) \), of the frequency \( \omega \) on the wave vector \( \vec{k} \) (the subscript \( l \) indicates different types of waves). The periodic functions \( E(\vec{r}) \) and \( \hat{h}(\vec{r}, \frac{\partial}{\partial \vec{r}}, \omega) \) can be expanded in Fourier series,

\[ E(\vec{r}) = \sum_n E_n e^{i\vec{k}_n \cdot \vec{r}} , \]

(3)

\[ \hat{h}(\vec{r}, \frac{\partial}{\partial \vec{r}}, \omega) = \sum_{n'} e^{i\vec{k}_n \cdot \vec{r}} h_{n'}(\frac{\partial}{\partial \vec{r}}, \omega) , \]

(4)

where \( \vec{k}_n \) are the reciprocal lattice vectors multiplied by \( 2\pi \).

Substituting Eq. (2) and (3) in Eq. (1) and setting the coefficients of each \( e^{i\vec{r}_m \cdot \vec{r}} \) equal to zero, we obtain the following infinite system of equations,

\[ H_m(\vec{\kappa}_m, \omega) E_m + \sum_n h_{m-n}(\vec{\kappa}_n, \omega) E_n = 0 \]

(5)

with \( \vec{\kappa}_m = \vec{k} + \vec{k}_m \).

This system of equations has a nontrivial solutions, \( E_n \), if its determinant vanishes,

\[ D(\vec{\kappa}_m, \omega) = 0 \]

(6)

Equation (6) is the dispersion equation that determines the function \( \omega(\vec{k}) \). From Eq. (2), it follows that \( \omega(\vec{k}) \) is a periodic function of \( \vec{k} \) with periods \( \vec{k}_n \) [10], [11]. To avoid problems associated with the infinite rank of the determinant in Eq. (6), we consider two limiting cases: a weak periodic modulation that corresponds to the nearly-free electron approximation, and a crystal, for which the method of local perturbation is applicable (an analog of the tight-binding approximation).

2 Weak periodic modulation

In this section, we assume that the amplitude of the periodic modulation of the medium is small, so that

\[ \left| \hat{h}\left(\vec{r}, \frac{\partial}{\partial \vec{r}}, \frac{\partial}{\partial t}\right) E \right| \ll \left| \hat{H}\left(\frac{\partial}{\partial \vec{r}}, \frac{\partial}{\partial t}\right) E \right| , \]

(7)
and the term $\hat{h}(r, \frac{\partial}{\partial r}, \frac{\partial}{\partial t})E$ in Eq. (1) can be considered as a small perturbation.

When $\hat{h} = 0$ (zero-order approximation) all coefficients $h_{m-n}$ in Eq. (5) are zero, $E_m = 0$, except for $m = 0$ ($E_0 = \text{const}$), and the dispersion equation for the homogeneous medium takes the form

$$H \left( \vec{k}, \omega \right) = 0 \quad (8)$$

Note that only real solutions $\omega_l(\vec{k})$ correspond to propagating waves. We denote the number of such waves by $L$. To find coefficients $E_n$ to first order of the small $\left| \hat{h} \right|$, there is no need to deal with the exact dispersion equation, Eq. (6). Instead, we set to zero all $E_n$ with $n \neq 0$ in the second term in Eq. (5) and obtain

$$E_n = -\frac{h_n(\vec{k}_n, \omega)}{H(\vec{k}_n, \omega)} E_0; \quad (9)$$

$$\delta \omega_l = \frac{1}{\frac{dH(\vec{k}, \omega_l)}{d\omega}} \sum_n \frac{[h_n(k_n, \omega_l)]^2}{H(k_n, \omega_l)}. \quad (10)$$

It follows from Eq. (7) that $|E_m| << |E_0|$ except for the case when there exists some $m = \nu$ for which $H(\vec{k}_\nu, \omega(\vec{k}_\nu))$ is small. In this instance $E_\nu$ in Eq. (5) is not small compared to $E_0$, and one has to keep in Eq. (5) also terms containing $E_\nu$. This yields

$$H \left( \vec{k}, \omega \right) E_0 + h_{-\nu} (\vec{\omega}_{-\nu}, \omega) E_\nu \ = \ 0 \quad (11)$$

$$H \left( \vec{k}, \omega \right) E_\nu + h_{\nu} (\vec{\omega}_{\nu}, \omega) E_0 \ = \ 0 \quad (12)$$

The condition for nontrivial solutions of Eqs. (11), (12) to exist leads to the following dispersion equation in the periodically modulated medium in the degenerate case

$$H \left( \vec{k}, \omega \right) H \left( \vec{\omega}_\nu, \omega \right) - h_{\nu} \left( \vec{\omega}_\nu, \omega \right) h_{-\nu} (\vec{\omega}_{-\nu}, \omega) = 0. \quad (13)$$

To solve the dispersion equation Eq. (13) we write $H \left( \vec{k}, \omega \right)$ in the form

$$H \left( \vec{k}, \omega \right) = f \left( \vec{k}, \omega \right) \prod_{l=1}^{L} \left[ \omega - \omega_l (\vec{k}) \right], \quad (14)$$
where function $f$ has no poles on the real axis. Substituting Eq. (14) into Eq. (13), we obtain

$$
\prod_{l=1}^{L} \left[ \omega - \omega_l \left( \vec{k} \right) \right] \prod_{m=1}^{L} \left[ \omega - \omega_m \left( \vec{\kappa}_\nu \right) \right] - \frac{h_\nu \left( \vec{k}, \omega \right) h_{-\nu} \left( \vec{k}, \omega \right)}{f \left( \vec{k}, \omega \right) f \left( \vec{\zeta}_\nu, \omega \right)} = 0.
$$

(15)

If we write the solutions of Eq. (15) as

$$
\omega \left( \vec{k} \right) = \omega_l \left( \vec{k} \right) + \delta \omega_l \left( \vec{k} \right), \quad (l = 1, ..., L)
$$

(16)

and assume that for all non-interacting waves

$$
| \omega_l \left( \vec{k} \right) - \omega_i \left( \vec{\kappa}_\nu \right) | >> \delta \omega_l \left( \vec{k} \right),
$$

(17)

the corrections $\delta \omega_l \left( \vec{k} \right)$ can be found by successive approximations. This procedure yields:

$$
\delta \omega_l = \frac{\left| h_\nu \left( \vec{k}_\nu, \omega_l \left( \vec{k} \right) \right) \right|^2}{f \left( \vec{k}, \omega_l \left( \vec{k} \right) \right) f \left( \vec{k}_\nu, \omega_l \left( \vec{k}_\nu \right) \right) \prod_{m \neq l}^{L} \left[ \omega_l \left( \vec{k} \right) - \omega_m \left( \vec{k} \right) \right] \prod_{m \neq l}^{L} \left[ \omega_l \left( \vec{k} \right) - \omega_m \left( \vec{\kappa}_\nu \right) \right]}
$$

(18)

It is important to note that this correction is of second order in the small perturbation $h$.

The situation is different when for some $l = n$, there exists $\vec{k}$ for which $\omega_n(\vec{k})$ is close to $\omega_n(\vec{\kappa}_\nu)$, and inequality Eq. (17) is violated. Then, the following approximate equation for the dispersion curves $\omega_n(\vec{k})$ can be derived from Eq. (15):

$$
\left[ \omega - \omega_n \left( \vec{k} \right) \right] \left[ \omega - \omega_n \left( \vec{\kappa}_\nu \right) \right] - \eta_n^2 \left( \vec{k}, \vec{\kappa}_\nu \right) = 0
$$

(19)

$$
\eta_n^2 \left( \vec{k}, \vec{\kappa}_\nu \right) = \frac{\left| h_\nu \left( \vec{k}_\nu, \omega_n \left( \vec{k} \right) \right) \right|^2}{f_n \left( \vec{k}, \omega_n \left( \vec{k} \right) \right) f_n \left( \vec{\zeta}_\nu, \omega_n \left( \vec{\zeta}_\nu \right) \right) \prod_{l=1}^{L} \left[ \omega_n - \omega_l \left( \vec{\zeta}_\nu \right) \right] \prod_{l=1}^{L} \left[ \omega_n - \omega_l \left( \vec{k} \right) \right]}
$$

(20)

The two solutions of the quadratic equation Eq. (19) are:

$$
\omega_{n \pm} \left( \vec{k} \right) = \frac{\omega_n \left( \vec{k}_\nu \right) + \omega_n \left( \vec{k} \right)}{2} \pm \sqrt{\frac{\left( \omega_n \left( \vec{k} \right) - \omega_n \left( \vec{\kappa}_\nu \right) \right)^2}{4} + \eta_n^2 \left( \vec{k}, \vec{\kappa}_\nu \right)}
$$

(21)

Note that for propagating waves, $\eta_n$ is real, i.e. $\eta_n^2 > 0$. If
\[ \omega_n(\vec{k}) = \omega_n(\vec{k}_\nu), \]  
(22)

then obviously

\[ \omega_{n \pm}(\vec{k}) = \omega_n(\vec{k}) \pm \eta_n(\vec{k}, \vec{k}_\nu), \]  
(23)

which means that there is a gap, \( \Delta \omega_n \), in the frequency spectrum

\[ \Delta \omega_n(\vec{k}, \vec{k}_\nu) = \omega_{n+}(\vec{k}, \vec{k}_\nu) - \omega_{n-}(\vec{k}, \vec{k}_\nu) = 2\eta_n(\vec{k}, \vec{k}_\nu) \]  
(24)

A similar gap exists in the energy spectrum of electrons in a periodic potential [10], with the following important difference: the width of the gap in the frequency spectrum of photons, Eq. (24), depends on the wave vector \( \vec{k} \).

Equation (22) defines the surface that is the boundary of the Brillouin zone of the photon. For an isotropic medium, the frequency \( \omega_n \) depends on the modulus of the wave vector, and the solution of Eq. (22) is \( k^2 = k^2_\nu \). Hence, the equation for the Brillouin zone boundary takes the form

\[ 2\vec{k}\vec{k}_\nu = -\vec{k}^2_\nu, \]  
(25)

which is the equation of a plane in \( \vec{k} \)-space. Eq. (25) coincides with the well-known Bragg condition.

Much more unusual is the photonic spectrum when two roots of Eq. (13) that belong to different branches (say, \( \omega_n(\vec{k}) \) and \( \omega_q(\vec{k}) \)) are close to each other, i.e. \( \omega_n(\vec{k}) \simeq \omega_q(\vec{k}_\nu) \). Then, the solution of the dispersion equation Eq. (13) is given by Eq. (21) with \( \omega_n(\vec{k}_\nu) \) replaced by \( \omega_q(\vec{k}_\nu) \) and \( \eta_n(\vec{k}, \vec{k}_\nu) \) substituted by

\[ \eta_{nq}^2(\vec{k}, \vec{k}_\nu) = \frac{|h_{\nu}(\vec{k}_\nu, \omega_n(\vec{k}))|^2}{f_1(\vec{k}, \omega_n(\vec{k})) f_2(\vec{k}_\nu, \omega_n(\vec{k})) \prod_{l \neq n}^{L} \left[ \omega_l(\vec{k}) - \omega_n \right] \prod_{l \neq q}^{L} \left[ \omega_l(\vec{k}_\nu) - \omega_q \right]} \]  
(26)

To exhibit the boundary of the Brillouin zone, we consider small values of the wave vector, so that

\[ \omega_i(\vec{k}) = \omega_{i0} + \beta_i k^2; \quad i = n, q \]  
(27)

\[ \beta_i = \frac{\partial \omega_i(0)}{\partial (k^2)} \]  
(28)
The equality $\omega_n(\vec{k}) = \omega_q(\vec{k}_\nu)$ yields the following equation for the desired surface in $\vec{k}$–space:

$$\left(\vec{k} - \vec{k}_R\right)^2 = R^2,$$

(29)

with

$$\vec{k}_R = \frac{\beta_1}{\beta_2 - \beta_1} \vec{k}_\nu$$

$$R^2 = \frac{\beta_1 \beta_2}{(\beta_2 - \beta_1)^2 k^2_\nu} + \frac{\omega_2(0) - \omega_1(0)}{\beta_2 - \beta_1}$$

Equation (29) is the equations of a sphere with the center at $\vec{k}_R$ and having radius $R$. The spherical shape of the Brillouin zone originates from the interaction of different types of waves ($n$ and $q$), and has no precedent in the theory of electrons in periodic lattices. In the absence of interactions, Eq. (29) loses its physical meaning, and $R^2$ becomes negative.

It is interesting to point out another distinctive feature of photonic periodic structures. In contrast to electrons, in electrodynamics the radiation (or scattering) problem exists along with the eigenvalues problem considered above. In this case, the frequency, $\omega$, is an external parameter, and the dependence of the wave vector on the frequency, $\vec{k}(\omega)$, is to be found. This dependence can be obtained from the solution of the eigenvalues problem, $\omega(\vec{k})$. From Eqs. (16) and (23) it follows that

$$\vec{k}(\omega) = \vec{k}_0(\omega) + \delta \vec{k}(\omega).$$

(30)

When $|\delta \vec{k}| << |\vec{k}_0|$,

$$\delta \omega = \vec{v}_g \delta \vec{k},$$

(31)

where $\vec{v}_g = \frac{\partial \omega}{\partial \vec{k}}$ is the group velocity and $\theta$ is the angle between $\vec{v}_g$ and $\vec{k}_0$.

3 Periodic array of resonant particles

In this section, we consider a photonic crystal composed of periodically arranged identical dielectric particles having refractive index $n$, and study the spectrum of excitations (eigenwaves) associated with a resonance in a single particle. Such a resonance exists when the particle size, $d$, is of order of the wavelength inside the particle, $\lambda = \frac{2\pi c}{n\omega}$, (Mi resonances, for example). This system is an analog of the tight-binding model in the solid state theory [12], and presents an opposite limiting case to that considered in the previous section. Indeed, while in a medium with weak periodic modulation, the band gap is small compared to the conduction
band, the gaps in the spectrum of the photonic crystal studied in this section are much wider than the passing zones.

In what follows, we consider the frequency range in which the wavelength of the radiation in the medium (having refractive index \( n_0 \)) is larger than the typical size of the particles

\[
d \ll \frac{2\pi c}{n_0 \omega}
\]  

(32)

In this case, the problem can be handled by means of the local perturbation method (LPM) [13], [14], [15], [16] which is based on the assumption that the (unknown) field is independent of the coordinates inside each particle. In contrast to the Born approximation, which considers only weak scattering, LPM leaves room for arbitrarily large amplitudes of scattered fields, and does not rule out the existence of resonances that take place when \( d \approx \frac{2\pi c}{n_0 \omega} \).

Equation (1) now takes the form

\[
\hat{H} \left( \frac{\partial}{\partial \vec{r} \cdot \omega}, \frac{\partial}{\partial \omega} \right) E + \sum_{p=-\infty}^{\infty} \hat{U} \left( \vec{r} - \vec{r}_p, \frac{\partial}{\partial \omega} \right) E = 0,
\]  

(33)

where the spatial-time dispersion of the homogeneous host medium is incorporated in \( \hat{H} \). The operator \( \hat{U} \) is non-zero only inside the particles and depends on their shape, position, and dielectric properties. For the sake of simplicity, we shall assume that the dielectric particles possess only time dispersion.

Since \( \hat{U} \) in Eq. (33) is a periodic function of \( \vec{r} \), the solution of Eq. (33) can be written in the form of Eq. (2). If the inequality (32) holds, the LPM approximation is applicable, implying that

\[
U \left( \vec{r} - \vec{r}_p, \omega \right) E \left( \vec{r} \right) e^{ik\vec{r}} \approx U \left( \vec{r} - \vec{r}_p, \omega \right) E \left( \vec{r}_p \right) e^{ik\vec{r}_p}
\]  

(34)

where \( E \left( \vec{r} \right) \) is a periodic function. If we assume that one of the particles is located at the origin, \( \vec{r} = 0 \), then \( E \left( \vec{r}_p \right) = E \left( 0 \right) \). Substitution of Eq. (34) into Eq. (33) yields:

\[
\hat{H} \left( \frac{\partial}{\partial \vec{r} \cdot \omega} \right) E \left( \vec{r} \right) + E \left( 0 \right) \sum_{p=-\infty}^{\infty} U \left( \vec{r} - \vec{r}_p, \omega \right) e^{ik\vec{r}_p} = 0.
\]  

(35)

Equation (35) can be rewritten in integral form,

\[
E \left( \vec{r} \right) + E \left( 0 \right) \sum_{p} \int G \left( \vec{r} - \vec{r}_p, \omega \right) U \left( \vec{r} - \vec{r}_p, \omega \right) d\vec{r}_p = 0,
\]  

(36)

where the Green function of the homogeneous \( (U = 0) \) host medium, \( G \), is defined by the equation

\[
\hat{H} \left( \frac{\partial}{\partial \vec{r} \cdot \omega} \right) G \left( \vec{r}, \vec{r} \right) = \delta \left( \vec{r} - \vec{r} \right)
\]  

(37)
Changing Eq. (37) to spatial Fourier transforms, we obtain

\[ G(\vec{r}', \vec{r}) = \frac{1}{(2\pi)^3} \int \frac{e^{i \vec{k} \cdot (\vec{r}' - \vec{r})}}{H(\vec{k}, \omega)} d\vec{k} \quad (38) \]

In an isotropic medium, \( H(\vec{k}, \omega) = H(k^2, \omega) \), the integral in Eq. (38) can be calculated explicitly:

\[ G(\vec{r}' - \vec{r}) = \frac{1}{4\pi |\vec{r}' - \vec{r}|} \sum_{l=1}^{L} e^{-|\zeta_l||\vec{r}' - \vec{r}|} H'(\zeta_l^2, \omega) \quad (39) \]

where \( \zeta_l^2 \) are roots of the equation

\[ H(-\zeta_l^2, \omega) = 0 \quad (40) \]

and \( H' = \frac{dH}{d(-\zeta_l^2)} \).

By substituting \( \vec{r}' = 0 \) in Eq. (36) we obtain the following dispersion equation for \( \omega (\bar{k}) \)

\[ 1 + \int G(\vec{r}, \omega) U(\vec{r}, \omega) d\vec{r}' + \int U(\vec{r}, \omega) d\vec{r}' G(\vec{r}', \omega) e^{i\bar{k}\vec{r}'} \sum_{n \neq 0} G(\vec{r}, \omega) e^{i\bar{k}\vec{r}_n} = 0 \quad (41) \]

It is apparent that the second term in Eq. (41) corresponds to the resonance frequencies of an isolated single particle, while the third term gives the corrections due to the multiple scattering (interactions). This term is small compared to the second one, therefore, we seek the solutions of Eq. (41) of the form

\[ \omega_p = \omega_{0p} + \delta \omega_p, \quad (42) \]

where \( \omega_{0p} \) are the solutions of the "zero-order" (without interactions) equation

\[ 1 + \int G(\vec{r}, \omega) U(\vec{r}, \omega) d\vec{r}' = 0, \quad (43) \]

and \( \delta \omega_p << \omega_{0p} \).

Since the summands in Eq. (39) decrease exponentially with increasing \( |\vec{r}_n| \), we may keep only the term with the maximal \( \zeta_l \) (denote it as \( \zeta_m \)). Then, for \( \delta \omega_p \), we find

\[ \delta \omega_p = \left\{ -\frac{d}{d\omega} H(-\zeta_m^2) \int G(\vec{r}, \omega) d\vec{r}' \sum_{n \neq 0} \frac{\exp(-\zeta_m |\vec{r}_n|)}{|\vec{r}_n|} e^{i\bar{k}\vec{r}_n} \right\}_{\omega = \omega_{0p}} \quad (44) \]

For the same reason, in the sum in Eq. (44), we take into account only the terms corresponding to the interaction with nearest neighbors. For a cubic crystal having side \( b \), this yields
\[
\delta \omega_p = -A (\cos k_x b + \cos k_y b + \cos k_z b),
\]
(45)

\[
A = -\frac{\int U (\vec{r}) \, d\vec{r} e^{-\zeta_m b}}{4\pi b \frac{d}{a_\omega} H' (-\zeta_m^2) \int G (\vec{r}, \omega) U (\vec{r}, \omega) \, d\vec{r}}
\]
(46)

The width of the conduction band is equal to \(6A \ll \omega_{p0}\). To carry out the integration in Eq. (44), we consider the example of a homogeneous medium with time dispersion. Then the operator \(\hat{H}\) in (35) is the Helmholtz operator

\[
\hat{H} = \Delta + \frac{\omega^2}{c^2} \varepsilon(\omega),
\]
(47)

where \(\varepsilon(\omega)\) is the dielectric permittivity of the host medium, in which the periodic grating of small particles (local perturbations) is located. For definiteness, we assume that the dielectric constants of both host medium and particles are of the Drude form:

\[
\varepsilon_0 (\omega) = 1 + \frac{\Omega^2}{\omega_0^2 - \omega^2};
\]
(48)

\[
\varepsilon (\omega) = 1 + \frac{\Omega_1^2}{(\omega_0')^2 - \omega^2};
\]
(49)

The dispersion equation Eq. (43) has two solutions:

\[
\omega_{1,2} = \frac{\Omega_1^2 + (\omega_0')^2 + \omega_s^2}{2} \pm \sqrt{\left(\frac{\Omega_1^2 + (\omega_0')^2 + \omega_s^2}{2}\right)^2 - \omega_s^2 (\omega_0')^2}
\]
(50)

where

\[
\omega_s^2 = \frac{4\pi c^2}{S}
\]

is the frequency of the "geometrical" resonance related to the finite size, \(a\,_{\text{r}}\) of the local perturbation (Debye resonance of the single particle); \(S = \int \frac{\hat{f}(\vec{r}) d\vec{r}}{r} a^2\). Note that the number of solutions depends on the explicit form of the functions \(\varepsilon_0(\omega)\) and \(\varepsilon(\omega)\), and can be larger than two.

Multiple scattering between neighboring particles leads to the broadening of the discrete levels \(\omega_{1,2}\) into passing zones of finite width. Frequencies \(\omega_{1,2} = \omega_{01,02} + \delta \omega_{1,2}\) correspond to a peculiar kind of polaritons that arise in photonic crystals with time dispersion due to the interaction of the electromagnetic field \(\omega\), eigenoscillations of the dielectric medium \(\omega_0\), and Debye resonance \(\omega_s\).

For the width of the interaction-induced passing zones, \(\Delta \omega_{1,2}\), Eq. (45) yields

\[
\Delta \omega_{1,2} \approx 6 \frac{V e^{-\zeta_m b} \omega_{1,2}}{S b},
\]
(51)
\[ V = \int f(\vec{r}) \, d\vec{r}. \] (52)

Terms independent on \( \vec{k} \) (small shifts of \( \omega_{1,2} \)) are omitted in Eq. (51). Recall that for the exponent in the Green function to be real (stability condition), it is necessary that
\[ 1 - \frac{\Omega^2}{\omega_{1,2}^2} < 0. \] (53)

One can see from (51) that the zone width is maximal at \( \omega_i = \Omega \) and is equal \( \Delta \omega_i = \frac{a}{b} \omega_i \). For \( \frac{\Omega^2}{\omega_{1,2}^2} \) increasing, the width decreases exponentially, and at \( \frac{\Omega^2}{\omega_{1,2}^2} \gg 1 \), the zone structure disappears. When the spacing between levels, \( |\omega_1 - \omega_2| \), is of order of \( \omega_s \), the width of the passing zone is \( \frac{1}{b} \gg 1 \) times smaller than the size of the band gap.

To conclude, time-spatial dispersion causes important differences in the spectrum of photons in periodic dielectric systems from that of electrons in solid state crystals. It gives rise to different types of waves, changes the number and structure of passing zones and band gaps, and dramatically deforms the Brillouin zones.

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