The Chern–Ricci flow on primary Hopf surfaces

Gregory Edwards

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Abstract
The Hopf surfaces provide a family of minimal non-Kähler surfaces of class VII on which little is known about the Chern–Ricci flow. We use a construction of Gauduchon–Ornea for locally conformally Kähler metrics on primary Hopf surfaces of class 1 to study solutions of the Chern–Ricci flow. These solutions reach a volume collapsing singularity in finite time, and we show that the metric tensor satisfies a uniform upper bound, supporting the conjecture that the Gromov-Hausdorff limit is isometric to a round $S^1$. Uniform $C^{1+\beta}$ estimates are also established for the potential. Previous results had only been known for the simplest examples of Hopf surfaces.

1 Introduction
The Chern–Ricci flow is a parabolic flow of Hermitian metrics first studied by Gill [10] and later introduced in greater generality by Tosatti–Weinkove [39]. We say $g(t)$ is a solution to the Chern–Ricci flow starting from a Hermitian metric $g_0$ if

\[
\begin{aligned}
\frac{\partial}{\partial t} g &= -\text{Ric}^{Ch}(g) \\
 g(0) &= g_0
\end{aligned}
\]

(1.1)

where $\text{Ric}^{Ch}$ is the Chern–Ricci tensor of $g$ defined by

\[\text{Ric}^{Ch}_{i\overline{j}} = -\partial_i \partial_{\overline{j}} \log \det g.\]

If the associated $(1, 1)$-form, $\omega_0 = \sqrt{-1}(g_0)_{i\overline{j}} dz^i \wedge d\overline{z}^j$ is closed, then $g_0$ is a Kähler metric and the Chern–Ricci tensor is equal to the usual Ricci tensor. Thus the Chern–Ricci flow yields the same solution as the well known Kähler–Ricci flow [3,4,6,18–25,27,31,34–36,40,42]. Other flows of Hermitian metrics have also been proposed and studied [17,28–30,43,44].

One direction of interest introduced in [38] is to classify the behavior of the Chern–Ricci flow of Gauduchon metrics on complex surfaces. On complex surfaces, a Gauduchon metric is a Hermitian metric whose associated $(1, 1)$-form satisfies $\overline{\partial} \omega_0 = 0$. A well known result
of Gauduchon states that every Hermitian metric lies in the conformal class of a Gauduchon metric \[7\]. Furthermore any Hermitian metric in the \(\frac{\partial}{\partial \bar{\partial}}\)-class, 
\[ \mathcal{H}_{\omega_0} = \{ \omega | \omega = \omega_0 + \sqrt{-1} \frac{\partial}{\partial \bar{\partial}} \psi > 0 \text{ for } \psi \in C^\infty(M) \}, \]
is also Gauduchon. On surfaces, the Gauduchon condition is preserved by the Chern–Ricci flow \[39\] and the Chern–Ricci flow of Gauduchon metrics on complex surfaces has been studied in several contexts \[11,38,39,41\].

For surfaces which are not minimal (i.e. those which have exceptional divisors) and with Kodaira dimension not equal to \(-\infty\), the flow reaches a finite time non-collapsing singularity at which time it contracts finitely many disjoint exceptional curves in the Gromov-Hausdorff topology, up to a condition on the \(\frac{\partial}{\partial \bar{\partial}}\)-class of the limiting form \[38,39\], generalizing results for the Kähler–Ricci flow \[23,24,26\].

By the Enriques–Kodaira classification of complex surfaces \[1\], all minimal non-Kähler surfaces can be classified into the following families:

(i) Kodaira surfaces,
(ii) Minimal non-Kähler properly elliptic surfaces,
(iii) Inoue surfaces,
(iv) Hopf surfaces
(v) Minimal surfaces of class VII with \(b_2(M) > 0\),

where Kodaira surfaces are minimal surfaces with \(b_1(M)\) odd and Kodaira dimension 0; Inoue surfaces are those with universal cover \(\mathbb{C} \times H\) where \(H\) is the upper half plane; Hopf surfaces are those with universal cover \(\mathbb{C}^2 \setminus \{0\}\); and surfaces of class VII are surfaces with \(b_1(M) = 1\) and Kodaira dimension \(-\infty\). By \[2,14,16,32\], a class VII surface with \(b_2(M) = 0\) must be either a Hopf or Inoue surface.

Solutions of the Chern–Ricci flow have been studied in several of the cases above: On manifolds with vanishing first Bott–Chern class—in any dimension—the flow converges smoothly to a Chern–Ricci flat Hermitian metric \[10\] using the uniform \(C^0\)-estimate of \[37\]; on minimal non-Kähler elliptic surfaces the normalized Chern-Ricci flow converges in the Gromov–Hausdorff topology to an orbifold Kähler–Einstein metric on a Riemann surface \[41\]; and on Inoue surfaces, after a conformal change to the initial metric, the Chern-Ricci flow converges in the Gromov–Hausdorff topology to a round \(S^1\) up to scaling \[5\]. The surfaces of type (v) are not yet classified except for the case \(b_2(M) = 1\) \[33\] and one long-term goal of study for the Chern–Ricci flow is to provide new topological or geometric information about Class VII surfaces in general.

On Hopf surfaces, the flow always reaches a finite time singularity at which time the volume goes to zero \[39\]. Beyond this, little is currently known about the Chern–Ricci flow on Hopf surfaces in any generality. The round metric on \(S^3 \times S^1\) admits a compatible complex structure as a Hopf surface, and the Chern–Ricci flow of this metric has an explicit maximal solution \[39\]. The solution becomes extinct at time \(T = \frac{1}{2}\), and \((M, g(t))\) converges in the Gromov–Hausdorff topology to a round \(S^1\) up to a scaling factor \[38\]. Moreover, if the initial metric is in the same \(\frac{\partial}{\partial \bar{\partial}}\)-class as the round metric, then the solution satisfies an upper bound and the potential converges in \(C^{1+\beta}\) for every \(\beta \in (0, 1)\) \[39\].

The primary Hopf surfaces of class 1, as defined in \[12\], form a large class of Hopf surfaces. These are defined as the quotients \(M = M_{\alpha, \beta} = (\mathbb{C}^2 \setminus \{0\})/\sim\) by the action \((z_1, z_2) \mapsto (\alpha z_1, \beta z_2)\) for \(\alpha, \beta \in \mathbb{C}\), with \(1 < |\alpha| \leq |\beta|\). All primary Hopf surfaces\(^1\) are

\(^1\) The primary Hopf surfaces consist of both those of class 1, and those of class 0 which are defined as quotients of \(\mathbb{C}^2 \setminus \{0\}\) of the form \((z_1, z_2) \mapsto (\beta^m z_1 + \lambda z_2^m, \beta z_2)\) for some positive integer \(m\) and \(\beta, \lambda \in \mathbb{C}\) with \(1 < |\beta|\) and \(\lambda \neq 0\).
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diffeomorphic to $S^3 \times S^1$, and all Hopf surfaces are finitely covered by a primary Hopf surface [13,14]. In particular, since the second Betti number vanishes, it is clear these surfaces do not admit any Kähler metric.

While $M$ is never Kähler, one can construct Hermitian metrics on $M$ which are locally conformally Kähler. The existence of such metrics was first proved by LeBrun (see [9]), for $|\alpha|, |\beta|$ distinct but sufficiently close, and explicit examples were constructed by Gauduchon–Ornea [9]. These particular metrics are of interest because they provide examples of Hermitian metrics on these surfaces. It is a difficult problem in general to give explicit Hermitian metrics on Hopf surfaces, particularly ones for which $\alpha \neq \beta$.

These LCK metrics are constructed as follows: We define a function, $\Phi_1(|z_1|, |z_2|)$, on $\mathbb{C}^2\{0\}$ which satisfies the relation

$$|z_1|^2 \Phi_1 - 2k_1 + |z_2|^2 \Phi_1 - 2k_2 = 1,$$

(1.2)

where

$$k_1 = \frac{\log |\alpha|}{\log |\alpha| + \log |\beta|} \leq k_2 = \frac{\log |\beta|}{\log |\alpha| + \log |\beta|}.$$  

Indeed for any real constants $a, b$, not both zero,

$$s \mapsto a^2 s^{-2k_1} + b^2 s^{-2k_2}$$

is continuous and strictly decreasing from $\infty$ to 0 for $s > 0$, and hence there is a unique value for which $a^2 s^{-2k_1} + b^2 s^{-2k_2} = 1$.

While $\Phi_1$ is a well defined function on $\mathbb{C}^2\{0\}$, it does not define a function on $M$. However, the $(1, 1)$-form

$$\hat{\omega} = \frac{\sqrt{-1} \partial \bar{\partial} \Phi}{\Phi}$$

(1.3)

is well defined and positive definite (see Remark 2.2), and hence it defines a Hermitian metric on $M$. These Hermitian metrics are never Kähler, but they are locally conformally Kähler (LCK), and satisfy

$$d\hat{\omega} = \hat{\omega} \wedge \theta$$

(1.4)

for a closed, real 1-form given by

$$\theta = \frac{d \Phi}{\Phi}.$$  

(1.5)

The existence of a closed 1-form satisfying (1.4) is equivalent to the Hermitian metric being LCK [8,15,45].

As a special case of (1.3), when $|\alpha| = |\beta|$, we recover $\Phi = r^2 = |z_1|^2 + |z_2|^2$, and $(M, \hat{\omega})$ is isometric to the round metric on $S^3 \times S^1$. We call this the round metric on the standard Hopf surface, and $\omega(t) = \hat{\omega} - t \text{Ric}_{\hat{\omega}}$ provides an explicit maximal solution to the Chern–Ricci flow (1.1) on $M \times [0, \frac{1}{2})$ converging in the Gromov–Hausdorff topology to a round $S^1$ [38,39].

The LCK condition (1.4) is not preserved under the flow, even for the round metric on the standard Hopf surface. However, we show that these initial metrics are Gauduchon (see Corollary (2.5)), and therefore any metric in their $\partial\bar{\partial}$-class is also Gauduchon. Since the Chern–Ricci flow preserves the Gauduchon condition it is of interest to study solutions starting from the $\partial\bar{\partial}$-class of the LCK metrics on non-standard primary Hopf surfaces defined above.
Our main theorem is the following:

**Theorem 1.1** Let $\hat{\omega}$ be the LCK metric constructed above, and $\omega_0 = \hat{\omega} + \sqrt{-1} \partial \bar{\partial} \psi$ for some smooth plurisubharmonic function $\psi$ with $g_0$ the associated Hermitian metric. Then a maximal solution to the Chern-Ricci flow (1.1) exists on the time interval $[0, \frac{1}{2})$ and there is a uniform constant $C > 0$, independent of $t$, such that

$$\omega(t) \leq C \hat{\omega}.$$ 

on $M \times [0, \frac{1}{2})$.

Unlike the case for the round metric on the standard Hopf surface, we do not obtain an explicit solution to the Chern–Ricci flow from any initial starting metric. Indeed, it seems such solutions are very difficult to find explicitly, and consequently there are difficulties at present in controlling the Gromov-Hausdorff limit of the solutions.

From the bound of the trace we also obtain the following result on the convergence of the potential.

**Corollary 1.2** Set $\omega(t) = \hat{\omega} - t \text{Ric}(\hat{\omega}) + \sqrt{-1} \partial \bar{\partial} \psi$ with $\psi$ normalized to satisfy equation (3.1) below. Then as $t \to T^-$, $\psi(t)$ converges subsequentially to a function $\psi(T)$ in $C^{1+\beta}$ for every $\beta \in (0, 1)$.

This follows from the fact that by the estimates in Lemma 3.1 below, after passing to subsequence, $\psi(t)$ converges pointwise to a function $\psi(T)$ as $t \to T^-$, and by Theorem 1.1, $|\Delta_{\hat{\omega}} \psi|$ is uniformly bounded, and so $\|\psi(t)\|_{C^{1+\beta}}$ is uniformly bounded for any $\beta \in (0, 1)$. It follows that, after passing to subsequence, $\psi(t) \to \psi(T)$ in $C^{1+\beta}$ as $t \to T^-$ for any $\beta \in (0, 1)$.

**Outline**

The outline of the rest of the paper is as follows. In Sect. 2, we establish some geometric properties of the LCK metrics defined above and show that they satisfy the Gauduchon condition. In Sect. 3, we formulate the Chern–Ricci flow as a parabolic complex Monge–Ampère equation, and recall the uniform estimate on the potential, and an upper bound on its time derivative. In Sect. 4, we bound the trace of the evolving metric with respect to the LCK metric and complete the proof of Theorem 1.1.

**2 Geometry of the locally conformally Kähler metrics**

In order to compute various geometric quantities related to the LCK metrics on non-standard Hopf surfaces we first compute the form of the metric in coordinates. We define the following (1,1)-form related to (1.5),

$$\Theta = \sqrt{-1} \theta^{(1,0)} \wedge \theta^{(0,1)} = \sqrt{-1} \partial \Phi \wedge \bar{\partial} \Phi.$$ 

Clearly, $\Theta$ is closed, non-negative, real, and of rank one. To obtain the components of the metric, we proceed as follows using $\Phi_i$ as a shorthand for $\partial_i \Phi$, $\Phi_{ij} = \partial_i \partial_j \Phi$, etc.:

First, differentiating (1.2),

$$\Phi_i = z_i \Phi^{1-2k} Z^{-1}$$ 

(2.2)
where
\[ Z = 2(k_1|z_1|^2\Phi^{-2k_1} + k_2|z_2|^2\Phi^{-2k_2}). \] (2.3)

Note that \( Z \) descends to a well defined function on \( M \) which satisfies
\[ 2k_1 \leq Z \leq 2k_2. \] (2.4)

We compute
\[ Z_i = 2\bar{z}_i k_i \Phi^{-2k_i} - 4\bar{z}_i \Phi^{-2k_i} Z^{-1} \sum_a |z_a|^2 k_a^2 \Phi^{-2k_a}, \]
so that
\[ \Phi_{ij} = \delta_{ij} \frac{\Phi^{1-2k_i}}{Z} + (1 - 2k_i - 2k_j)\bar{z}_i z_j \frac{\Phi^{1-2k_i-2k_j}}{Z^2} + 4\bar{z}_i z_j \frac{\Phi^{1-2k_i-2k_j}}{Z^3} \sum_a |z_a|^2 k_a^2 \Phi^{-2k_a}, \]
and we have
\[ \Phi_{ij} = \delta_{ij} \frac{\Phi^{1-2k_i}}{Z} + \left( 1 - 2k_i - 2k_j + \frac{4}{Z} \sum_a k_a^2 |z_a|^2 \Phi^{-2k_a} \right) \frac{\Phi_i \Phi_j}{\Phi}. \] (2.5)

Next, we compute the determinant of \( \hat{g} \).

**Proposition 2.1** The determinant of \( \hat{g} \) is given by the following identity,
\[ \det(\hat{g}) = \frac{1}{\Phi^2 Z^3}. \]

**Proof** The proof is contained in Gauduchon–Ornea [9]. We provide it here, adapted to our slightly different conventions, for convenience. We first compute the individual components of the complex Hessian of \( \Phi \), for instance:
\[ \Phi_{1\bar{T}} = \frac{\Phi^{1-2k_1}}{Z} + \left( 1 - 4k_1 + \frac{4}{Z} (k_1^2 |z_1|^2 \Phi^{-2k_1} + k_2^2 |z_2|^2 \Phi^{-2k_2}) \right) \frac{|z_1|^2 \Phi^{1-4k_1}}{Z^2} \]
\[ = \frac{\Phi^{1-2k_1}}{Z^3} \left( Z^2 - (1 - 4k_1)Z|z_1|^2 \Phi^{-2k_1} + 4k_1^2 |z_1|^4 \Phi^{-4k_1} + 4k_2^2 |z_1|^2 |z_2|^2 \Phi^{-2} \right), \]
and similarly we obtain
\[ \Phi_{2\bar{T}} = \frac{2\Phi^{1-2k_2}}{Z^3} \left( k_2 |z_2|^4 \Phi^{-4k_2} + 2k_2^2 |z_1|^4 \Phi^{-4k_1} + k_1 (1 + 2k_1) |z_1|^2 |z_2|^2 \Phi^{-2} \right), \]
\[ \Phi_{1\bar{z}} = \frac{2\bar{z}_1 z_2 \Phi^{-1}}{Z^3} (k_1 - k_2) \left( k_1 |z_1|^2 \Phi^{-2k_1} - k_2 |z_2|^2 \Phi^{-2k_2} \right). \]

Then we find the determinant of the matrix,
\[ A = \begin{bmatrix} \Phi_{1\bar{T}} & \Phi_{1\bar{z}} \\ \Phi_{2\bar{T}} & \Phi_{2\bar{z}} \end{bmatrix}, \]
We use the identity
\[ \det A = \frac{8}{Z^6} \left( k_1^3 |z_1|^8 \Phi^{-8k_1} + k_2^3 |z_2|^8 \Phi^{-8k_2} + 3k_1k_2 |z_1|^4 |z_2|^4 \Phi^4 \\
+ k_1^2 (1 + 2k_2) |z_1|^6 |z_2|^2 \Phi^{-4k_1-2} + k_2 (1 + 2k_1) |z_1|^2 |z_2|^6 \Phi^{-4k_2-2} \right) \]
\[ = \frac{1}{Z^3}, \]
and therefore we have
\[ \det(\hat{\gamma}) = \frac{1}{\Phi^2 Z^3} \]
which was claimed. \(\square\)

Remark 2.2 From the calculations above, it follows that \(\hat{\gamma}_{ij}\) has strictly positive determinant and, by inspection, has strictly positive trace. Since \(\dim C(M) = 2\), it follows that \(\hat{\gamma}_{ij}\) defines a positive definite Hermitian metric.

Next, we have the following geometric identity.

Proposition 2.3 The following equality holds for the trace of \(\Theta\):
\[ \text{tr}_\omega \Theta = 1. \]

As an immediate and crucial consequence is the following Corollary, obtained from the calculation of the trace and non-negativity of \(\Theta\).

Corollary 2.4 We have the inequality of \((1, 1)\)-forms:
\[ \Theta \leq \hat{\omega}. \]

Proof of Proposition 2.3 From (2.1)
\[ \Theta_{ij} = \frac{\Phi_i \Phi_j}{\Phi^2}. \]

We use the identity
\[ \text{tr}_\hat{\omega} \Theta = 2 \frac{\hat{\omega} \wedge \Theta}{\hat{\omega}^2} = \Phi^2 Z^3 \left( \Phi^{-3} \Phi_{2\bar{\omega}} \Phi_\bar{\omega} - \Phi^{-3} \Phi_{1\bar{\omega}} \Phi_\bar{\omega} - \Phi^{-3} \Phi_{1\bar{\omega}} \Phi_\bar{\omega} + \Phi^{-3} \Phi_{1\bar{\omega}} \Phi_\bar{\omega} \right), \]
and then using the calculations above
\[ = \frac{Z^3}{\Phi} \left( \Phi^{1-2k_1} \frac{2|z_1|^2}{Z^5} (k_2 |z_2|^4 \Phi^{-4k_2} + 2k_1^2 |z_1|^4 \Phi^{-4k_1} + k_1 (1 + 2k_1) |z_1|^2 |z_2|^2 \Phi^{-2}) \right) \]
\[ + \frac{2|z_2|^2}{Z^5} \Phi^{1-2k_2} (k_1 |z_1|^4 \Phi^{-4k_1} + 2k_2^2 |z_2|^4 \Phi^{-4k_2} + k_2 (1 + 2k_2) |z_1|^2 |z_2|^2 \Phi^{-2}) \]
\[ - \frac{4 |z_1|^2 |z_2|^2}{Z^5} \Phi^{-1} (k_1 - k_2) (k_1 |z_1|^2 \Phi^{-2k_1} - k_2 |z_2|^2 \Phi^{-2k_2}) \]
\[ = \frac{4}{Z^2} \left( k_1 |z_1|^6 \Phi^{-6k_1} + k_2 |z_2|^6 \Phi^{-6k_2} + k_2 (k_1 + 1) |z_1|^2 |z_2|^4 \Phi^{-2-2k_2} \right. \]
\[ \left. + k_1 (1 + k_2) |u|^4 |v|^2 \Phi^{-2-2k_1} \right) = 1 \]
where we have used (1.2) in the last line. □

Proposition 2.3 also allows us to obtain the following result for the LCK metrics.

**Corollary 2.5** The metrics \( \hat{\omega} \) satisfy the Gauduchon condition.

**Proof** Indeed

\[
\partial \overline{\partial} \hat{\omega} = \partial \overline{\partial} \left( \sqrt{-1} \frac{\partial \overline{\partial} \phi}{\phi} \right) \\
= \sqrt{-1} \left( -\frac{\partial \overline{\partial} \phi \wedge \partial \overline{\partial} \phi}{\phi^2} + 2 \frac{\partial \phi \wedge \overline{\partial} \phi \wedge \partial \overline{\partial} \phi}{\phi^3} \right) \\
= \frac{1}{\sqrt{-1}} (-\hat{\omega}^2 + 2\Theta \wedge \hat{\omega}) \\
= \frac{1}{\sqrt{-1}} (\text{tr} \hat{\omega} \Theta - 1) \hat{\omega}^2 = 0
\]

which proves the claim. □

Let us now define another metric which will be useful for our purposes:

\[
\chi_{i\overline{j}} = \Phi^{-2k_i} \delta_{ij}.
\]

One can check that \( \chi \) transforms in the correct way to define a Hermitian metric. On the standard Hopf surface this is equal to the round metric, but otherwise it is distinct from \( \hat{\omega} \).

The benefit of introducing the new metric is that

\[
\det(\chi) = \frac{1}{\Phi^2},
\]

and so its Chern-Ricci form is given by:

\[
\text{Ric}(\chi) = 2\sqrt{-1} \partial \overline{\partial} \log \phi \\
= 2 \frac{\partial \overline{\partial} \phi}{\phi} - 2 \sqrt{-1} \frac{\partial \phi \wedge \overline{\partial} \phi}{\phi^2} \\
= 2\hat{\omega} - 2\Theta \geq 0
\]

using Corollary 2.4 to obtain the inequality.

### 3 The Chern–Ricci flow

Let \( \omega(t) \) be the solution to the Chern–Ricci flow (1.1) starting from \( \omega(0) = \hat{\omega} + \sqrt{-1} \partial \overline{\partial} \psi \) for a smooth plurisubharmonic function \( \psi \). Then, we can write the solution as

\[
\omega(t) = \hat{\omega} - t(2\hat{\omega} - 2\Theta + 3\sqrt{-1} \partial \overline{\partial} \log Z) + \sqrt{-1} \partial \overline{\partial} \psi(t)
\]

where \( \psi(t) \) solves the parabolic complex Monge-Ampère equation

\[
\begin{cases}
\psi = \log \frac{(\hat{\omega} - t(2\hat{\omega} - 2\Theta + 3\sqrt{-1} \partial \overline{\partial} \log Z) + \sqrt{-1} \partial \overline{\partial} \psi)^2}{\hat{\omega}^2} \\
\psi(0) = \psi.
\end{cases}
\]
But since \( \log Z \) is a globally defined smooth function, we can write

\[
\omega(t) = \hat{\omega} - 2t(\hat{\omega} - \Theta) + \sqrt{-1} \delta \bar{\delta} \psi(t)
\]

by setting

\[
\varphi(t) = \psi(t) - 3t \log Z.
\] (3.2)

Then since \( \hat{\omega}^2 = \frac{1}{Z} \chi^2 \), \( \varphi \) satisfies the equation

\[
\begin{cases}
\dot{\varphi} = \log \left( \frac{\hat{\omega} - 2t(\hat{\omega} - \Theta) + \sqrt{-1} \delta \bar{\delta} \psi}{\hat{\omega}^2} \right) - 3 \log Z = \log \left( \frac{\hat{\omega} - 2t(\hat{\omega} - \Theta) + \sqrt{-1} \delta \bar{\delta} \psi}{\chi^2} \right) \\
\varphi(0) = \psi.
\end{cases}
\] (3.3)

We define the family of reference metrics,

\[
\omega_t = (1 - 2t) \hat{\omega} + 2t \Theta,
\]

so that

\[
\omega(t) = \omega_t + \sqrt{-1} \delta \bar{\delta} \varphi,
\]

and note that

\[
\frac{\partial}{\partial t} \omega_t = -2 \hat{\omega} + 2 \Theta = -\text{Ric}^\chi (\chi).
\]

From Tosatti–Weinkove \[39\] we have that the Chern–Ricci flow exists on a maximal time interval \([0, T)\) where \( T \) depends only on the \( \delta \bar{\delta} \)-class of \( \omega_t \), and for Gauduchon metrics on complex surfaces, \( T \) is given explicitly by

\[
T = \sup \{ t \mid \int_M \omega_t^2 > 0, \text{ and } \int_D \omega_t > 0 \text{ for all irreducible divisors } D \text{ with } D^2 < 0 \}.
\]

It follows that \( T = \frac{1}{2} \), since \( M \) has no divisors with \( D^2 < 0 \) and

\[
\omega_t^2 = (1 - 2t)^2 \hat{\omega}^2 + 4t(1 - 2t) \hat{\omega} \wedge \Theta
\]

\[
= (1 - 2t) \hat{\omega}^2
\]

since \( 2 \hat{\omega} \wedge \Theta = (\text{tr}_{\omega} \Theta) \hat{\omega}^2 \).

We have the following estimates on the potential for the solutions.

**Lemma 3.1** There exists a uniform constant \( C > 0 \) such that for \( t \in [0, T) \),

(i) \( |\psi(t)| + \dot{\psi}(t) \leq C \)

(ii) \( |\varphi(t)| + \dot{\varphi}(t) \leq C \)

**Proof** The proof of part (i) is standard and is contained in Tosatti–Weinkove \[39\] (in the Kähler setting the proof is due to Tian–Zhang \[35\]), we include it here for convenience. We use \( \Delta = \Delta_\omega \) for the Laplacian with respect to \( g(t) \). Applying the maximum principle to \( (\psi - At) \) for a constant \( A > 0 \), we have that at a point of maximum with \( t > 0 \)

\[
0 \leq \frac{\partial}{\partial t} (\psi - At) \leq \log \frac{\omega_t^\rho}{\omega^\rho} - A < 0
\]

if \( A \) is chosen sufficiently large. Hence the maximum occurs at \( t = 0 \), and therefore we have the upper bound on \( \psi \). The lower bound follows a similar argument.
To obtain the upper bound for $\dot{\psi}$, we apply the maximum principle to

$$Q = t\dot{\psi} - \psi - 2t$$

so that

$$\left( \frac{\partial}{\partial t} - \Delta \right) Q = -\text{tr}_\omega \hat{\omega} < 0.$$ 

By the maximum principle,

$$\sup_M Q(\cdot, 0) \geq \sup_M Q(\cdot, t)$$

and it follows that $\dot{\psi}$ is uniformly bounded from above.

Part (ii) follows from part (i) and (3.2) since $\log Z$ is bounded. \qed

### 4 Bound of the metric along the Chern–Ricci flow

Since $\hat{\omega}$ is controlled by $\chi$ is suffices to bound $\text{tr}_\chi \omega(t)$. Let $\hat{g}_{i\bar{j}}$ be the Hermitian metric associated to $\hat{\omega}$, and $g_{i\bar{j}}$ the metric associated to $\omega(t)$. We often use the Hermitian metrics and their associated (1, 1)-forms interchangeably.

Let us fix the notation that $\text{Ric}$ will denote the Chern–Ricci tensor of $\chi$, and so

$$2\hat{g} = \text{Ric} + 2\Theta.$$  

(4.1)

First, we note that

$$\left( \frac{\partial}{\partial t} - \Delta_\omega \right) \text{tr}_\chi \omega = -g^{i\bar{j}} \partial_i \partial_{\bar{j}} \chi^{k\bar{l}} g_{k\bar{l}} + \chi^{k\bar{l}} g^{i\bar{j}} \left( \partial_k \partial_{\bar{j}} \hat{g}_{i\bar{j}} - \partial_i \partial_{\bar{j}} \hat{g}_{i\bar{j}} \right)$$

$$- 2 \text{Re} \left( g^{i\bar{j}} \partial_i \chi^{k\bar{l}} \partial_{\bar{j}} g_{k\bar{l}} \right) - \chi^{k\bar{l}} g^{i\bar{j}} g_{i\bar{j}} \partial_k g_{p\bar{q}} \partial_{\bar{q}} g_{i\bar{j}}.$$ 

Indeed, since

$$\frac{\partial}{\partial t} \text{tr}_\chi \omega(t) = \chi^{k\bar{l}} \partial_k \partial_{\bar{l}} \log \det g$$

$$= -\chi^{k\bar{l}} g^{i\bar{j}} g_{i\bar{j}} \partial_k g_{p\bar{q}} \partial_{\bar{q}} g_{i\bar{j}} + \chi^{k\bar{l}} g^{i\bar{j}} \partial_k \partial_{\bar{l}} g_{i\bar{j}},$$

and

$$\Delta_\omega \text{tr}_\chi \omega(t) = g^{i\bar{j}} \partial_i \partial_{\bar{j}} \left( \chi^{k\bar{l}} g_{k\bar{l}} \right)$$

$$= g^{i\bar{j}} \partial_i \partial_{\bar{j}} \chi^{k\bar{l}} g_{k\bar{l}} + g^{i\bar{j}} \chi^{k\bar{l}} \partial_i \partial_{\bar{j}} g_{k\bar{l}} + 2 \text{Re} \left( g^{i\bar{j}} \partial_i \chi^{k\bar{l}} \partial_{\bar{j}} g_{i\bar{j}} \right),$$

the difference gives

$$\left( \frac{\partial}{\partial t} - \Delta_\omega \right) \text{tr}_\chi \omega = -g^{i\bar{j}} \partial_i \partial_{\bar{j}} \chi^{k\bar{l}} g_{k\bar{l}} + \chi^{k\bar{l}} g^{i\bar{j}} \left( \partial_k \partial_{\bar{j}} g_{i\bar{j}} - \partial_i \partial_{\bar{j}} g_{i\bar{j}} \right)$$

$$- 2 \text{Re} \left( g^{i\bar{j}} \partial_i \chi^{k\bar{l}} \partial_{\bar{j}} g_{k\bar{l}} \right) - \chi^{k\bar{l}} g^{i\bar{j}} g_{i\bar{j}} \partial_k g_{p\bar{q}} \partial_{\bar{q}} g_{i\bar{j}}.$$ 

Then

$$\partial_k \partial_{\bar{j}} g_{i\bar{j}} - \partial_i \partial_{\bar{j}} g_{k\bar{l}} = \partial_k \partial_{\bar{j}} \left( \hat{g}_{i\bar{j}} - 2 \text{Ric}_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi \right) - \partial_i \partial_{\bar{j}} \left( \hat{g}_{k\bar{l}} - 2 \text{Ric}_{k\bar{l}} + \partial_k \partial_{\bar{l}} \varphi \right).$$
but since $\text{Ric}_{ij}$ is the Chern–Ricci tensor of $\chi$, it satisfies
\[
\partial_k \partial_l \text{Ric}_{ij} = \partial_i \partial_j \text{Ric}_{kl},
\]
and therefore
\[
\partial_k \partial_l g_{ij} - \partial_i \partial_j g_{kl} = \partial_k \partial_l \hat{g}_{ij} - \partial_i \partial_j \hat{g}_{kl}.
\]
(4.2)
We obtain the equality claimed above.

We now estimate the four terms above in succession.

**Lemma 4.1** There is a uniform constant $C_M > 0$, depending only on $M$, such that:

(i) $-g^{ij} \partial_i \partial_j \chi k l g_{kl} \leq -C_M^{-1} (\text{tr}_g \text{Ric}) \text{tr}_g g - g^{ij} g_{kl} \partial_i \chi^{kl} \partial_j \chi^{rs}$

(ii) $\chi^{kl} g^{ij} (\partial_k \partial_l \hat{g}_{ij} - \partial_i \partial_j \hat{g}_{kl}) \leq C_M \text{tr}_g \text{Ric} + C_M \text{tr}_g \Theta$

(iii) $-2 \text{Re}(g^{ij} \partial_i \chi^{kl} \partial_j g_{kl}) \leq C_M \text{tr}_g \Theta + \chi^{kl} g^{pq} \partial_k g_{pq} \partial_l \hat{g}_{ij} + g^{ij} g_{kl} \partial_i \chi^{kl} \partial_j \chi^{rs}$

From the stated estimates we obtain the following Corollary.

**Corollary 4.2** There is a uniform constant $C_M > 0$ depending only on the geometry of $M$ such that
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \text{tr}_g \omega \leq -\text{tr}_g \text{Ric} \left( C_M^{-1} \text{tr}_g g - C_M \right) + C_M \text{tr}_g \Theta.
\]

**Proof of Lemma 4.1** To prove part (i), we use that $\chi^{kl} = \Phi^{2ki} \delta^{kl}$, so that
\[
\partial_i \chi^{kl} = \partial_i \left( \Phi^{2ki} \delta^{kl} \right) = 2k_i \Phi^{2ki} \delta^{kl} \phi_i = 2k_l \chi^{kl} \phi_i, \quad (4.3)
\]
and
\[
\partial_i \partial_j \chi^{kl} = \partial_i \partial_j \left( \Phi^{2ki} \delta^{kl} \phi_i \phi_j \right) = 2k_i \Phi^{2ki} \delta^{kl} \left( \phi_i \phi_j + (2k_l - 1) \phi_i \phi_j \right)
\]
\[
= 2k_l \chi^{kl} \left( \delta_{ij} + (2k_l - 1) \theta_{ij} \right)
\]
and by (4.1),
\[
\partial_i \partial_j \chi^{kl} = k_l \chi^{kl} \text{Ric}_{ij} + 4k_l^2 \chi^{kl} \theta_{ij}.
\]
Finally, we note that by (4.3)
\[
\chi_{rl} \partial_i \chi^{kl} \partial_j \chi^{rs} = 4k_l^2 \chi^{kl} \theta_{ij},
\]
and then we obtain the inequality in part (i) provided $C_M > \frac{2}{k_l}$.

Next, for the claim in part (ii), we can take $C_M > 0$ to be a constant large enough that
\[
\chi^{kl} \partial_k \partial_l \hat{g}_{ij} - \chi^{kl} \partial_i \partial_j \hat{g}_{kl} \leq C_M \hat{g}_{ij}.
\]
Since $\chi$ and $\hat{g}$ are fixed, it is clear that the constant depends only on the geometry of $M$. Now, using (4.1),
\[
g^{ij} \left( \chi^{kl} \partial_k \partial_l \hat{g}_{ij} - \chi^{kl} \partial_i \partial_j \hat{g}_{kl} \right) \leq C_M \text{tr}_g \text{Ric} + C_M \text{tr}_g \Theta
\]
which proves part (ii).
Finally, moving on to part (iii), we write

\[-2 \text{Re} \left( g^{ij} \partial_i \chi^{kl} \partial_j g_{kl} \right) = -2 \text{Re} \left( g^{ij} \partial_i \chi^{kl} \partial_j g_{kl} \right) - 2 \text{Re} \left( g^{ij} \partial_i \chi^{kl} \left( \partial_j g_{kl} - \partial_l g_{kj} \right) \right).\]

For the second term, we have

\[\partial_j g_{kl} - \partial_l g_{kj} = \partial_j \hat{g}_{kl} - \partial_l \hat{g}_{kj},\]

(see the argument preceding (4.2)). Then we use

\[\hat{g}_{ij} = \frac{\Phi_{ij}}{\Phi},\]

to obtain

\[\partial_j \hat{g}_{kl} = \frac{\Phi_{kl}}{\Phi} = \frac{\Phi_{kl} \Phi_{ij}}{\phi^2},\]

so that

\[\partial_j \hat{g}_{kl} - \partial_l \hat{g}_{kj} = \frac{\Phi_{ij} \Phi_{kl} - \Phi_{kl} \Phi_{ij}}{\phi^2},\]

and using (4.3)

\[\partial_i \chi^{kl} \left( \partial_j \hat{g}_{kl} - \partial_l \hat{g}_{kj} \right) = 2k_l \chi^{kl} \left( \hat{g}_{kij} - \hat{g}_{klj} \right),\]

and therefore

\[-2 \text{Re} \left( g^{ij} \partial_i \chi^{kl} \partial_j g_{kl} \right) \leq 2g^{ij} \chi^{kl} \hat{g}_{kij} \Theta_{ij} \leq C_M g^{ij} \Theta_{ij} \quad (4.4)\]

after taking \( C_M > 0 \) large enough that \( 4 \hat{g} \leq C_M \chi \). Again, the constant here depends only on \( M \).

Now,

\[-2 \text{Re} \left( g^{ij} \partial_i \chi^{kl} \partial_j g_{kl} \right) \leq -2 \text{Re} \left( g^{ij} \chi^{kl} g^{kl} \left( \chi_{pq} \partial_q \partial_i \chi^{uq} \right) \partial_l g_{kl} \right)\]

\[\leq g^{ij} \chi^{kl} \partial_i \chi^{kl} \partial_j \chi^{ir} + g^{ij} \chi^{kl} \partial_i \chi^{kl} \partial_j \chi^{ir} + g^{ij} \chi^{kl} \partial_i \chi^{kl} \partial_j \chi^{ir},\]

and then, combining with (4.4), we have

\[-2 \text{Re} \left( g^{ij} \partial_i \chi^{kl} \partial_j g_{kl} \right) \leq C_M \text{tr}_g \Theta + g^{ij} \chi^{kl} \partial_i \chi^{kl} \partial_j \chi^{ir} + g^{ij} \chi^{kl} \partial_i \chi^{kl} \partial_j \chi^{ir},\]

which proves the claim in part (iii). \( \square \)

**Remark 4.3** The presence of the \( C_M \text{tr}_g \Theta \) term in Corollary 4.2 introduces difficulties in applying the maximum principle argument. These difficulties are dealt with in the final step of proving Theorem 1.1.

Finally, we prove the main Theorem.
Proof of Theorem 1.1} Applying the previous Corollary, we arrive at

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \text{tr}_g \omega \leq - \text{tr}_g \text{Ric} \left( C_M^{-1} \text{tr}_g g - C_M \right) + C_M \text{tr}_g \Theta
\]

Now, for large constants \( A, B > 0 \) to be fixed later, define

\[
Q = \text{tr}_g \omega - A \phi - A(1 - 2t)(\log(1 - 2t) - 1) - Bt.
\]

Then we have the evolution inequality,

\[
\left( \frac{\partial}{\partial t} - \Delta \right) Q \leq - \text{tr}_g \text{Ric} \left( C_M^{-1} \text{tr}_g \omega - C_M \right) + C_M \text{tr}_g \Theta - (A - 1) \omega \omega_t
\]

Next, by the arithmetic-geometric mean inequality,

\[
\omega_t \geq (1 - 2t) \omega + 2t \Theta \geq A^{-1} \left( \frac{(1 - 2t)^2 \chi^2}{\omega^2} \right)^{\frac{1}{2}}
\]

provided \( A \) is taken sufficiently large. Furthermore, by Corollary 2.4

\[
\omega_t = (1 - 2t) \hat{\omega} + 2t \Theta \geq \Theta
\]

for all \( t \geq 0 \), and so we may fix \( A = A(C_M) \) large enough that

\[
C_M \text{tr}_g \Theta - (A - 1) \omega \omega_t \leq 0.
\]

Now, we have

\[
\left( \frac{\partial}{\partial t} - \Delta \right) Q \leq - \text{tr}_g \text{Ric} \left( C_M^{-1} \text{tr}_g \omega - C_M \right)
\]

\[
+ A \log \left( \frac{(1 - 2t)^2 \chi^2}{\omega^2} \right) - A^{-1} \left( \frac{(1 - 2t)^2 \chi^2}{\omega^2} \right)^{\frac{1}{2}} + 2A - B
\]

\[
\leq - \text{tr}_g \text{Ric} \left( C_M^{-1} \text{tr}_g \omega - C_M \right),
\]

for \( B = B(A) > 0 \) sufficiently large since \( (A \log s - A^{-1}s^{\frac{1}{2}} + 2A) \) is bounded from above for \( s > 0 \). It then follows that if \( Q \) achieves a maximum with \( t_0 > 0 \), then at that point

\[
0 \leq - \text{tr}_g \text{Ric} \left( C_M^{-1} \text{tr}_g \omega - C_M \right),
\]

but then since \( \text{Ric} \) is non-negative, it follows that at the point of maximum

\[
\text{tr}_g \omega \leq C_M^2.
\]

Finally, since \( \phi, Bt \), and \( (1 - 2t) \log(1 - 2t) \) are all bounded, we obtain that \( Q \) is bounded above on \( M \times [0, \frac{1}{2}) \), and therefore

\[
\text{tr}_g \omega \leq C
\]

for a uniform constant \( C > 0 \), which completes the proof. \( \square \)
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References

1. Barth, W., Hulek, K., Peters, C., van de Ven, A.: Compact Complex Surfaces, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge/A Series of Modern Surveys in Mathematics, vol. 4, 2 edn. Springer, Berlin (2004)
2. Bogomolov, F.A.: Surfaces of class VII₀ and affine geometry. Izv. Akad. Nauk SSSR Ser. Mat. 46(4), 710–761 (1982)
3. Cao, H.D.: Deformation of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds. Invent. Math 81(2), 359–372 (1985)
4. Chen, X.X., Wang, B.: Kähler–Ricci flow on Fano manifolds (I). J. Eur. Math. Soc. (JEMS) 14(6), 2001–2038 (2012)
5. Fang, S., Tosatti, V., Weinkove, B., Zheng, T.: Inoue surfaces and the Chern–Ricci flow. J. Funct. Anal. 271(11), 3162–3185 (2016)
6. Feldman, M., Ilmanen, T., Knopf, D.: Rotationally symmetric shrinking and expanding gradient Kähler–Ricci solitons. J. Differ. Geom. 65(2), 169–209 (2003)
7. Gauduchon, P.: Le théorème de l’excentricité nulle. C. R. Acad. Sci. Paris 285, 387–390 (1977)
8. Gauduchon, P.: La 1-forme de torsion d’une variété hermitienne compacte. Math. Ann. 267, 495–518 (1984)
9. Gauduchon, P., Ornea, L.: Locally conformally Kähler metrics on Hopf surfaces. Ann. Inst. Fourier (Grenoble) 48(4), 1107–1127 (1998)
10. Gill, M.: Convergence of the parabolic complex Monge–Ampère equation on compact Hermitian manifolds. Commun. Anal. Geom. 19(2), 277–303 (2011)
11. Gill, M., Smith, D.J.: The behavior of Chern scalar curvature under Chern–Ricci flow. Proc. Am. Math. Soc. 143(11), 4875–83 (2013)
12. Harvey, R., Blaine Lawson, H.: An intrinsic characterization of Kähler manifolds. Invent. Math. 74(2), 169–198 (1983)
13. Kodaira, K.: Complex structures on $S^1 \times S^3$. Proc. Natl. Acad. Sci. USA 55, 240–243 (1966)
14. Kodaira, K.: On the structure of compact complex analytic surfaces. II. Am. J. Math. 88(3), 682–721 (1966)
15. Lee, H.C.: A kind of even dimensional differential geometry and its application to exterior calculus. Am. J. Math. 65(3), 433–438 (1943)
16. Li, J., Yau, S.-T., Zheng, F.: On projectively flat Hermitian manifolds. Commun. Anal. Geom. 2, 103–109 (1994)
17. Liu, K., Yang, X.: Geometry of Hermitian manifolds. Int. J. Math. 23(6), 1250055 (2012)
18. Phong, D.H., Song, J., Sturm, J., Weinkove, B.: The Kähler–Ricci flow and the $\bar{\partial}$ operator on vector fields. J. Differ. Geom. 81(3), 631–647 (2009)
19. Phong, D.H., Sturm, J.: On stability and the convergence of the Kähler–Ricci flow. J. Differ. Geom. 72(1), 149–168 (2006)
20. Song, J., Tian, G.: The Kähler–Ricci flow on surfaces of positive Kodaira dimension. Invent. Math. 170(3), 609–653 (2007)
21. Song, J., Tian, G.: Canonical measures and Kähler–Ricci flow. J. Am. Math. Soc 25, 303–353 (2012)
22. Song, J., Tian, G.: The Kähler–Ricci flow through singularities. Invent. Math. 207(2), 519–595 (2017)
23. Song, J., Weinkove, B.: The Kähler–Ricci flow on Herzbruch surfaces. J. Reine Angew. 659, 141–168 (2011)
24. Song, J., Weinkove, B.: Contracting exceptional divisors by the Kähler–Ricci flow. Duke Math. J. 162(2), 367–415 (2011)
25. Song, J., Weinkove, B.: An introduction to the Kähler–Ricci flow. In: Boucksom, S., Eyssidieux, P., Guedj, V. (eds.), An Introduction to the Kähler–Ricci Flow, Lecture Notes in Math., vol. 2086, pp. 89–188. Springer, Heidelberg (2013)
26. Song, J., Weinkove, B.: Contracting exceptional divisors by the Kähler–Ricci flow. II. Proc. Lond. Math. Soc. 108(6), 1529–1561 (2014)
27. Song, J., Yuan, Y.: Metric flips with Calabi ansatz. Geom. Funct. Anal. 22(2), 240–265 (2012)
28. Streets, J., Tian, G.: A parabolic flow of pluriclosed metrics. Int. Math. Res. Not. IMRN 16, 3101–3133 (2010)
29. Streets, J., Tian, G.: Hermitian curvature flow. J. Eur. Math. Soc. (JEMS) 13(3), 601–634 (2011)
30. Streets, J., Tian, G.: Regularity results for pluriclosed flow. Geom. Topol. 17(4), 2389–2429 (2013)
31. Székelyhidi, G.: The Kähler–Ricci flow and K-polystability. Am. J. Math. 132(4), 1077–1090 (2010)
32. Teleman, A.: Projectively flat surfaces and Bogomolov’s theorem on class $\text{vi}i_0$-surfaces. Int. J. Math. 5, 253–264 (1994)
33. Teleman, A.: Donaldson theory on non-Kählerian surfaces and class VII surfaces with $b_2 = 1$. Invent. Math. 162(3), 493–521 (2005)
34. Tian, G.: New results and problems on the Kähler–Ricci flow. Astérisque No. 322, 71–92 (2008)
35. Tian, G., Zhang, Z.: On the Kähler–Ricci flow on projective manifolds of general type. Chi. Ann. Math. 27(2), 179–192 (2006)
36. Tosatti, V.: Kähler–Ricci flow on stable Fano manifolds. J. Reine. Angew. Math. 640, 2010 (2010)
37. Tosatti, V., Weinkove, B.: The complex Monge–Ampère equation on compact Hermitian manifolds. J. Am. Math. Soc. 23(4), 1187–1195 (2010)
38. Tosatti, V., Weinkove, B.: The Chern–Ricci flow on complex surfaces. Compos. Math. 149(12), 2101–2138 (2013)
39. Tosatti, V., Weinkove, B.: On the evolution of a Hermitian metric by its Chern–Ricci form. J. Differ. Geom. 99(1), 125–163 (2015)
40. Tosatti, V., Weinkove, B., Yang, X.: Kähler-Ricci flow, Ricci-flat metrics and collapsing limits. Am. J. Math. 140(3) (2014)
41. Tosatti, V., Weinkove, B., Yang, X.: Collapsing of the Chern–Ricci flow on elliptic surfaces. Math. Ann. 362(3–4), 1223–1271 (2015)
42. Tsuji, H.: Existence and degeneration of Kähler–Einstein metrics on minimal algebraic varieties of general type. Math. Ann. 281, 123–133 (1988)
43. Ustinovskiy, Y.: Hermitian curvature flow on complex homogeneous manifolds. arXiv:1706.07023
44. Ustinovskiy, Y.: The Hermitian curvature flow on manifolds with non-negative Griffiths curvature. arXiv:1604.04813
45. Vaisman, I.: Some curvature properties of locally conformally Kähler manifolds. Trans. Am. Math. Soc. 259(2), 439–447 (1980)

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