LOJASIEWICZ EXPONENT OF A SURFACE: AN INTRINSIC VIEW

E. BILGIN, G. KAYA, AND M. TOSUN

Abstract. For a surface $X$ with an ADE-type singularity, we establish a relation between the elements of the local ring $\mathcal{O}_{X,0}$ and the Lojasiewicz exponent $L_0(X)$ and we give an estimate of $L_0(X)$ when $X$ has a rational singularity of multiplicity 3.

Contents

1. Introduction
2. Integrally closed ideals in the local ring of a rational singularity
3. Lojasiewicz exponent and length of an ideal
4. Lojasiewicz exponent of rational singularities of higher multiplicity

References

1. Introduction

Let $F : \mathbb{C}^N \rightarrow \mathbb{C}$ be an analytic function such that the origin is an isolated singularity of $X = F^{-1}(0)$. The Lojasiewicz exponent $L_0(X)$ at $0 \in \mathbb{C}^N$ is defined as the infimum of the elements in the set

$\{ \theta > 0 \mid \exists U \subset \mathbb{C}^N$ and $\exists c \in \mathbb{R}_+ \text{ such that } \|z\|^\theta \leq c \cdot \|\nabla F(z)\|$ for all $z \in U \}$

Here $\|z\| = \max_i \{ |z_i| \}$ with $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ and $\nabla F = (\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_N}) : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^N, 0)$ is the gradient of $F$. The inequality in $1.1$ is called the Lojasiewicz gradient inequality. As the name suggests, the first use of an inequality of this nature is due to Lojasiewicz in $[13, 14]$. The Lojasiewicz exponent of $X$ is defined as the Lojasiewicz exponent of $\nabla F$. It is conjectured in $[20]$ that $L_0(X)$ is a topological invariant. We know that the conjecture is true for the following class of hypersurfaces because the weights are a topological invariant $[27, 15]$.

Theorem 1.1. $[3]$ Let $F : \mathbb{C}^N \rightarrow \mathbb{C}$ be a weighted homogeneous polynomial with weight $w = (w_1, \ldots, w_N)$ and of degree $d$. Then

$L_0(F) \leq \max_{i=1}^N \{ w_i - 1 \}$

where the equality holds when $N = 3$.

In particular, $L_0(X)$ such that $X$ is a surface with a rational singularity of multiplicity 2 at the origin, called ADE-type singularity can be computed directly by this formula.

| ADE-type singularity | $(w_1, w_2, w_3)$ | $d$ | $L_0(X)$ |
|----------------------|------------------|-----|----------|
| $A_n$, $(n = 2k)$: $z_1^2 + z_2^2 + z_3^{n-1} = 0$ | $(2, 2k + 1, 2k + 1)$ | $4k + 2$ | $n$ |
| $A_n$, $(n = 2k + 1)$: $z_1^2 + z_2^2 + z_3^{n+1} = 0$ | $(1, k + 1, k + 1)$ | $2k + 2$ | $n$ |
| $D_n$: $z_1^n + z_2^2 + z_3^{n-1} = 0$ | $(2, n - 2, n - 1)$ | $2(n - 1)$ | $n - 2$ |
| $E_6$: $z_1^3 + z_2^3 + z_3^2 = 0$ | $(3, 4, 6)$ | $12$ | $3$ |
| $E_7$: $z_1^3 + z_2^3 + z_3^2z_2 = 0$ | $(4, 6, 9)$ | $18$ | $4$ |
| $E_8$: $z_1^3 + z_2^3 + z_3^2 = 0$ | $(6, 10, 15)$ | $30$ | $4$ |

Table 1. Lojasiewicz exponent of ADE-singularities

2000 Mathematics Subject Classification. 58K20, 32S25.

This work is supported by the projects 113F293 and 18F320 under the programs of the Scientific and Technological Research Council of Turkey and the first author is also supported by the project 16.504.001 of Galatasaray University.
Definition 2.3. Let $\nu \in \mathbb{N}$ be the order of a positive divisor $D$. In fact, for $[3, 12]$, any element $\nu \in \mathbb{N}$ of a special ideal in $O_X$ has a rational singularity of multiplicity $m$. The Lojasiewicz exponent of a rational singularity of multiplicity $m$ is bounded by the length of a special ideal in $O_{X,0}$.

Proposition 1.3. Let $X$ be a reduced equidimensional complex analytic space. If this set is empty we say that $\mathcal{L}_f(I) = \infty$. Note that $\mathcal{L}_f(I) < \infty$ when $I \subset \sqrt{J}$.

Proposition 1.3. With preceding notation, we have $\mathcal{L}_f(I) \in \mathbb{Q}_+$. In Section 3, we also relate $\mathcal{L}_f(I)$ with the length $\ell(I)$ of the ideal in $O_{X,0}$ in order to study the Lojasiewicz exponent of $X$ by the local data of the singular point instead of the ambient data in $(\mathbb{C}^N, 0)$. When $X$ has a rational singularity of multiplicity $m \geq 3$ we study the Lojasiewicz exponent of the mapping $F = (f_1, \ldots, f_k) : \mathbb{C}^N \to \mathbb{C}^k$. In Section 4, we compute $\mathcal{L}_0(X)$ when $m = 3$ at the origin. We then conjecture that Lojasiewicz exponent of a rational singularity of multiplicity $m$ is bounded by the length of a special ideal in $O_{X,0}$.

2. Integrally closed ideals in the local ring of a rational singularity

Let $X$ be a germ of a normal surface in $\mathbb{C}^N$ with a singularity at $0$ and $O_{X,0}$ be its local ring. An element $g \in O_{X,0}$ is said to be integral on an ideal $I \subset O_{X,0}$ if it satisfies an equation $g^n + a_1 g^{n-1} + \ldots + a_n = 0$ with $a_i \in I^r$ for all $i = 1, \ldots, n$. Denote by $I$ the set of all elements in $O_{X,0}$ which are integral over $I$: it is an ideal and called the integral closure of $I$ in $O_{X,0}$. We have $I \subseteq I$. When $I = I$ we say that $I$ is an integrally closed ideal in $O_{X,0}$.

Let $\pi : \tilde{X} \to X$ be the minimal resolution of $X$. It is well known that $E := \pi^{-1}(0)$ is normal crossing and each irreducible component $E_i$ of $E = E_1 \cup \ldots \cup E_n$ is a rational curve.

Theorem 2.1. The product of integrally closed ideals in $O_{X,0}$ is an integrally closed ideal in $O_{X,0}$. An ideal $I$ in $O_{X,0}$ is called $M$-primary if $M = \sqrt{I}$ and $I \subseteq \sqrt{I}$. Let $S(I)$ be the set of $M$-primary integrally closed ideals $I$ in $O_{X,0}$ such that the pullback $I\mathcal{O}_{\tilde{X}}$ of $I$ by $\pi$ is principal; equivalently, $I\mathcal{O}_{\tilde{X}} = \mathcal{O}(-D)$ where $D$ is a positive divisor supported on $E$. The set $S(I)$ is a semigroup with respect to the product of ideals. The elements in $S(I)$ can be studied using their associated positive divisors as follows: By $[12]$, any element $h \in O_{X,0}$ defines a positive divisor $D_h$ supported on $E$ such that $\pi^*(h) = D_h + T_h$ where $T_h$ is the strict transform of $h$ by $\pi$. If we denote by $\nu_{E_i}(h)$ the vanishing order of the divisor $\pi^*(h)$ along $E_i$, we have $D_h = \sum_{i=1}^n \nu_{E_i}(h) E_i$ with $\nu_{E_i}(h) \geq 1$ for all $i$ since $E$ is connected. Hence $(D_h \cdot E_i) \leq 0$ for all $i$.

Theorem 2.2. There exists a bijection between the semigroups $S(I)$ and $E(\pi)$.

In fact, for $I, J \in S(I)$ we have $I \mathcal{O}_{X,0} = \mathcal{O}(-D_I - D_J)$ with $D_I + D_J \in E(\pi)$. Conversely, to each positive divisor $D$ supported on $E$ such that $D \cdot E \leq 0$ for all $i$, we associate an ideal $D_I$ in $O_{X,0}$ defined as the stalk at $0$ of $\mathcal{O}_X(-D)$. Moreover, $E(\pi)$ is a partially ordered set by $\leq$ defined as follows: For any $D, D' \in E(\pi)$ we say that $D \leq D'$ whenever $\nu_{E_i}(D) \leq \nu_{E_i}(D')$ for all $i = 1, \ldots, n$.

Definition 2.3. Let $I \in S(I)$. An element $g \in I$ is called generic in $I$ if $\nu_{E_i}(g) \leq \nu_{E_i}(h)$ for all $h \in I$ and for all $i = 1, \ldots, n$.

The order $\nu_{E_i}(I)$ of an ideal $I$ is defined as $\nu_{E_i}(I) := \inf \{ \nu_{E_i}(h) \mid h \in I \}$. 

Put $X := \{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid F(z_1, \ldots, z_N) = 0\}$. Throughout this work, we will assume that $X$ is a germ of a normal surface in $\mathbb{C}^N$ with a rational singularity of multiplicity $m$ at the origin. In the next section, we introduce some properties of these singularities. In Section 3, we give some nice properties for the Lojasiewicz exponent of an ideal in the local ring $O_{X,0}$. 

Theorem 2.1. The product of integrally closed ideals in $O_{X,0}$ is an integrally closed ideal in $O_{X,0}$. An ideal $I$ in $O_{X,0}$ is called $M$-primary if $M = \sqrt{I}$ and $I \subseteq \sqrt{I}$. Let $S(I)$ be the set of $M$-primary integrally closed ideals $I$ in $O_{X,0}$ such that the pullback $I\mathcal{O}_{\tilde{X}}$ of $I$ by $\pi$ is principal; equivalently, $I\mathcal{O}_{\tilde{X}} = \mathcal{O}(-D)$ where $D$ is a positive divisor supported on $E$. The set $S(I)$ is a semigroup with respect to the product of ideals. The elements in $S(I)$ can be studied using their associated positive divisors as follows: By $[12]$, any element $h \in O_{X,0}$ defines a positive divisor $D_h$ supported on $E$ such that $\pi^*(h) = D_h + T_h$ where $T_h$ is the strict transform of $h$ by $\pi$. If we denote by $\nu_{E_i}(h)$ the vanishing order of the divisor $\pi^*(h)$ along $E_i$, we have $D_h = \sum_{i=1}^n \nu_{E_i}(h) E_i$ with $\nu_{E_i}(h) \geq 1$ for all $i$ since $E$ is connected. Hence $(D_h \cdot E_i) \leq 0$ for all $i$. Let $E(\pi)$ be the set of the positive divisors $D_h$ supported on $E$ for all $h \in O_{X,0}$. The set $E(\pi)$ is a semigroup with respect to the addition, called the semigroup of Lipman associated with $\pi$.

Theorem 2.2. There exists a bijection between the semigroups $S(I)$ and $E(\pi)$.

In fact, for $I, J \in S(I)$ we have $I \mathcal{O}_{X,0} = \mathcal{O}(-D_I - D_J)$ with $D_I + D_J \in E(\pi)$. Conversely, to each positive divisor $D$ supported on $E$ such that $D \cdot E \leq 0$ for all $i$, we associate an ideal $D_I$ in $O_{X,0}$ defined as the stalk at $0$ of $\mathcal{O}_X(-D)$. Moreover, $E(\pi)$ is a partially ordered set by $\leq$ defined as follows: For any $D, D' \in E(\pi)$ we say that $D \leq D'$ whenever $\nu_{E_i}(D) \leq \nu_{E_i}(D')$ for all $i = 1, \ldots, n$.
Hence, for a generic element \( g \) in \( I \), we have \( \nu_{E_i}(I) = \nu_{E_i}(g) \), so \( D_I = D_g \). This says that any ideal \( I \) in \( \mathcal{S}(I) \) corresponds to an element \( D_I \in \mathcal{E}(\pi) \) through its generic element. The smallest element in \( \mathcal{E}(\pi) \) is called the Artin divisor \( D_{\pi} \) of \( I \) and corresponds to (generic element of) the maximal ideal \( M \) in \( \mathcal{O}_{X,\pi} \).

We replace \( \text{Proposition 3.1.} \)

In particular, we have:

\[
\text{Corollary 3.3.}
\]

By \([9, 5]\), the number \( L \) of \( E_i \) is locally principal in \( \pi \), we fix an ordering on the irreducible components of \( E \). Let \( M(E) = (e_{ij}) \) denote the intersection matrix of \( E \) where \( e_{ij} = (E_i \cdot E_j) \) for all \( 1 \leq i, j \leq n \); it is a negative definite symmetric matrix. If \( h \) is the generic element of an ideal \( I \in \mathcal{S}(I) \) its associated divisor \( D_h = \sum_{i=1}^n \nu_{E_i}(h)F_i \) in \( \mathcal{E}(\pi) \) satisfies the equality:

\[
\mathcal{M}(E) \cdot (\nu_{E_1}(h), \nu_{E_2}(h), \ldots, \nu_{E_n}(h))^t = (y_1, y_2, \ldots, y_n)^t
\]

where \( y_i < 0 \) for all \( i \) except at least one \( y_i \), \( i \in \{1, \ldots, n\} \). Let us denote by \( \delta_i \) the column matrix with coefficients 0 everywhere except in the \( i \)-th row, where the entry is \(-1\). Consider the system \( \mathcal{M}(E) \cdot (m_1, m_2, \ldots, m_n)^t = \delta_i \). Put \( F_i = \sum_{j=1}^n m_{ij}E_j \). We have \( m_{ij} \in \mathbb{Q}^+ \) for all \( i, j \). Write \( F_i \) as \( F_i = k_iF_i' \) for (smallest) \( k_i \in \mathbb{Q}^+ \) such that the coordinates of the \( F_i \) are positive integers and relatively prime. By construction, each \( F_i \) belongs to \( \mathcal{E}(\pi) \) and each element in \( \mathcal{E}(\pi) \) can be written as a linear combination of \( F_1, \ldots, F_n \) with coefficients in \( \mathbb{Q}^+ \). Put \( \mathcal{G}(\pi) := \{F_1, \ldots, F_n\} \). An element of \( \mathcal{G}(\pi) \) is called a \( \mathbb{Q} \)-generator for \( \mathcal{E}(\pi) \).

3. Lojasiewicz exponent and length of an ideal

Consider two ideals \( I, J \in \mathcal{O}_{X,\pi} \). By \([10, 4]\), the Lojasiewicz exponent of \( I \) with respect to \( J \) is given by

\[
L(I) = \min \{ \frac{a}{b} \mid a, b \in \mathbb{Z}_{\geq 1}, b \cdot D_M \geq b \cdot D_I \}
\]

In particular, we have:

**Proposition 3.1.** Let \( I \in \mathcal{S}(I) \). Then

\[
L_0(I) := L_M(I) = \min \{ \frac{a}{b} \mid a, b \in \mathbb{Z}_{\geq 1}, a \cdot D_M \geq b \cdot D_I \}
\]

**Proof.** We replace \( J \) by \( M \) in Definition \([12]\) and put \( T_b = D_I \) since \( I \in \mathcal{S}(I) \) is integrally closed. Then we rewrite the inclusion at the right hand side in terms of the associated divisors through their generic elements. Since \( I \cdot \mathcal{O}_X \) is locally principal in \( \pi \), we have \( \mathcal{O}_X(-D_M) \supseteq \mathcal{O}_X(-D_I) \); so \( D_I \geq D_M \). The fact \( D_M = b \cdot D_I \) gives the inequality. \( \square \)

**Proposition 3.2.** (See \([22]\), p.289) Let \( I \in \mathcal{S}(I) \). Then

\[
L_0(I) = \max_{i=1}^n \left\{ \frac{\nu_{E_i}(D_I)}{\nu_{E_i}(D_M)} \right\}
\]

In particular, we have \( L_0(M) = 1 \).

**Corollary 3.3.** Let \( I, J \in \mathcal{S}(I) \). Then we have \( L_J(I) \leq \frac{L_0(I)}{L_0(J)} \).

**Proof.** Let \( L_0(I) = \max_{i=1}^n \left\{ \frac{\nu_{E_i}(D_I)}{\nu_{E_i}(D_M)} \right\} = \frac{\nu_{E_i}(D_I)}{\nu_{E_i}(D_M)} \). In this case

\[
\frac{L_0(I)}{L_0(J)} = \max_{i=1}^n \left\{ \frac{\nu_{E_i}(D_I)}{\nu_{E_i}(D_M)} \right\} = \max_{i=1}^n \left\{ \frac{\nu_{E_i}(D_M)\nu_{E_i}(D_M)}{\nu_{E_i}(D_I)\nu_{E_i}(D_I)} \right\} = \max_{i=1}^n \left\{ \frac{\nu_{E_i}(D_I)}{\nu_{E_i}(D_J)} \right\} = L_J(I).
\]

**Proposition 3.4.** For \( I, J \in \mathcal{S}(I) \) we have

\[
L_J(I) \geq \frac{(D_I \cdot D_J)}{(D_I \cdot D_J)}
\]

**Proof.** By \([9, 5]\), the number \(-(D_I \cdot D_J)\) equals the multiplicity \( e(I) \) of the ideal \( I \) in \( \mathcal{O}_{X,\pi} \). By \([21]\), we have \( L_I(I) \geq \frac{e(I)}{e(I)\cdot e(J)} \). Here \( e(I \mid J) \) represents the mixed multiplicity of the ideals \( I, J \in \mathcal{S}(I) \) which is given by

\[
e_1(I \mid J) = \frac{1}{2} (e(IJ) - e(I) - e(J))
\]

Since \( e(IJ) = -(D_I + D_J) \cdot (D_I + D_J) \), we have \( e_1(I \mid J) = -(D_I \cdot D_J) \). \( \square \)
Proposition 3.5. For $I_1, I_2, J \in S(I)$, we have

$$L_J(I_1 I_2) = L_J(I_1) + L_J(I_2)$$

Proof. From the previous discussion we have:

$$L_J(I_1 I_2) = \max_{i=1}^{n} \frac{\nu_E(D_{I_1 I_2})}{\nu_E(D_J)}$$

$$= \max_{i=1}^{n} \frac{\nu_E(D_{I_1} + D_{I_2})}{\nu_E(D_J)}$$

$$= \max_{i=1}^{n} \frac{\nu_E(D_{I_1}) + \nu_E(D_{I_2})}{\nu_E(D_J)}$$

$$= \max_{i=1}^{n} \frac{\nu_E(D_{I_1})}{\nu_E(D_J)} + \max_{i=1}^{n} \frac{\nu_E(D_{I_2})}{\nu_E(D_J)}$$

$$= L_J(I_1) + L_J(I_2)$$

\[ \square \]

Corollary 3.6. For any $k \in \mathbb{N}^*$ and $I \in S(I)$ we have

$$L_0(I^k) = k \cdot L_0(I)$$

Remark 3.7. When $X$ is of an ADE-type singularity, Table 2 shows that there exists an element $I \in S(I)$ such that $L_0(X) = L_0(I)$. For an $A_n$-type singularity, using Theorem 1.1 we get $L_0(A_n) = n$. Note that the Artin divisor $D_M$ is reduced, means $\nu_{D_M}(E_i) = 1$ for all $i$; so, by Proposition 3.8, the biggest coefficient $\nu_D(E)$ in each $D_I \in \mathcal{E}(\pi)$ gives the Lojasiewicz exponent of the corresponding $I \in S(I)$. In particular, $L_0(I) = n$ for the ideal $I$ corresponding to the $\mathbb{Q}$-generator $D_I = (1, 2, 3, \ldots, n-1, n)$ of $\mathcal{E}(\pi)$.

Length of an ideal. The length (or co-length) of an ideal $I$ in a ring $R$, denoted by $\ell(R/I)$, is the dimension of $R/I$ over the field $k$. Since $X$ has a rational singularity, each $\mathcal{M}$-primary ideal in $\mathcal{O}_{X,0}$ has finite length and the length of $I$ in $\mathcal{O}_{X,0}$ is $\dim_{\mathbb{C}}(\mathcal{O}_{X,0}/I)$. Obviously, $\mathcal{M}$ has length 1. It is easier to compute the length of an ideal in $S(I)$ using its associated divisor in $\mathcal{E}(\pi)$.

Theorem 3.8. (20, Remark 3.2) Let $I \in S(I)$. Then

$$\ell(I) = \frac{-(D_I \cdot D_J) - \sum_{i=1}^{n} \nu_E(D_I)(w_i - 2)}{2}$$

where $w_i = -E_i^2$ for all $i$.

In the sequel, we use the notations $\ell(D_I)$ and $\ell(I)$ equivalently and call the length of the ideal $I$.

Proposition 3.9. With preceding notation, we have $L_M(I) \leq \ell(I)$ for every $I \in S(I)$.

Proof. It results from the fact that we have $M^p \subseteq I$ if $\ell(I) = p$ for an ideal $I$ (see Tables 2).

Remark 3.10. The length defines the map $\ell : S(I) \rightarrow \mathbb{R}$ and we have $\ell(I \cdot J) = \ell(D_I) + \ell(D_J)$. For each $p \in \mathbb{N}^*$, there may not exist an ideal of length $p$ in $\mathcal{O}_{X,0}$ and, if exists, there are a finite number of ideal of length $p$ (see Table 2).

Let $D_I$ be an element in $\mathcal{E}(\pi)$. Consider the components $E_i$ of $E$ such that $(D_I \cdot E_i) < 0$. Let us reindex these components as $F_1, \ldots, F_k$. We have $k \leq n$ where $n$ is the total number of the irreducible components of $E$. Consider the set

$$E - \{F_1, \ldots, F_k\} = \prod_{j=1}^{s} \mathcal{E}^j$$

with $j = 1, \ldots, s$. Each sub-configuration $\mathcal{E}^j$ is called Tjurina component of $E$ with respect to $D_I$ and all elements of $\mathcal{E}(\pi)$ can be constructed by one of the process given in the following theorem:

Proposition 3.11. (23) Let $I, J \in S(I)$. Then we have $D_I = D_J + D'$ for some positive divisor $D'$.

(i) If $D' = Z(\mathcal{E}^j)$ we have $\ell(I) = \ell(J) + 1$.

(ii) If $D' = E_{i_0}$ such that $E_{i_0}$ is attached only to the vertices $E_j$ of $E$ such that $(D_I \cdot E_j) < 0$, we have $\ell(I) = \ell(J) - (D_J \cdot D') + 1$. 

Consider the case where $X$ is an $E_r$-type singularity. The integral closure of the Jacobian ideal $\mathcal{J} = \langle z_3, z_2^2, z_1^3 \rangle$ is $\mathcal{J} = (z_1^2, z_2^2, z_3)$. Note that it is hard to compute the integral closure of an ideal if it is not a monomial ideal. The length $\ell(\mathcal{J})$ is 6. We have $\mathcal{J} \subset \mathcal{M}$ but $\mathcal{M}^\circ \not\subset \mathcal{J}$. More precisely, we get $\mathcal{M}^{t+1} \subset \mathcal{J}$, for $t \in \mathbb{N}$, so $\mathcal{L}_0(\mathcal{J}) = 4$. Furthermore, by [6, 17], we know that the generic element $p$ of $\mathcal{J}$ corresponds to the divisor $D_p = (5, 10, 15, 12, 9, 6, 3, 8) \in \mathcal{E}(\pi)$ according to the ordering in $E$ taken as

$$E_1, E_2, E_3, E_4, E_5, E_6, E_7$$

$$E_8$$

We obtain $\ell(\mathcal{J}) = \ell(D_p) = 4$. As Table 2 shows, we have $\mathcal{L}_0(X) = \ell(D_p)$ for $E_r$-type singularities.

| Some $D_j$'s in $\mathcal{E}(\pi)$ for $E_r$-type | $\ell(\mathcal{J})$ | $\mathcal{L}_0(\mathcal{J})$ | Some $D_j$'s in $\mathcal{E}(\pi)$ for $E_7$-type | $\ell(\mathcal{J})$ | $\mathcal{L}_0(\mathcal{J})$ | Some $D_j$'s in $\mathcal{E}(\pi)$ for $E_6$-type | $\ell(\mathcal{J})$ | $\mathcal{L}_0(\mathcal{J})$ | Some $D_j$'s in $\mathcal{E}(\pi)$ for $E_5$-type | $\ell(\mathcal{J})$ | $\mathcal{L}_0(\mathcal{J})$ |
|-----------------------------------------------|-----------------|-----------------|-----------------------------------------------|-----------------|-----------------|-----------------------------------------------|-----------------|-----------------|-----------------------------------------------|-----------------|-----------------|
| $\{1, 2, 3, 2, 1\}^*$                         | 1               | 1               | $\{2, 3, 4, 5\}^*$                             | 1               | 1               | $\{4, 5, 6, 5\}^*$                             | 1               | 1               | $\{1, 2, 3, 3, 2, 1\}^*$                       | 1               | 1               |
| $\{2, 3, 4, 2, 2\}^*$                         | 2               | 2               | $\{2, 4, 6, 5\}^*$                             | 2               | 2               | $\{6, 9, 13, 9\}^*$                            | 2               | 2               | $\{2, 3, 3, 2, 1\}^*$                         | 2               | 2               |
| $\{3, 4, 6, 4, 3\}^*$                         | 4               | 4               | $\{3, 6, 9, 9\}^*$                             | 4               | 4               | $\{6, 12, 18, 15, 12, 9\}^*$                  | 4               | 4               | $\{3, 4, 2, 1\}^*$                            | 4               | 4               |
| $\{3, 6, 9, 3, 5\}^*$                         | 7               | 7               | $\{4, 12, 12, 9, 6\}^*$                        | 7               | 7               | $\{8, 14, 20, 16, 12, 9\}^*$                  | 7               | 7               | $\{3, 6, 9, 3, 5\}^*$                         | 7               | 7               |
| $\{5, 10, 12, 8, 4, 6\}^*$                   | 15              | 15              | $\{5, 9, 12, 9, 6, 3\}^*$                      | 15              | 15              | $\{8, 14, 20, 16, 12, 9\}^*$                  | 15              | 15              | $\{5, 10, 12, 8, 4, 6\}^*$                    | 15              | 15              |
| $\{4, 8, 10, 15, 5, 6\}^*$                   | 15              | 15              | $\{4, 12, 10, 7, 4, 6\}^*$                     | 15              | 15              | $\{8, 16, 24, 20, 15, 10, 5\}^*$              | 15              | 15              | $\{4, 8, 10, 15, 5, 6\}^*$                    | 15              | 15              |
| $\{6, 12, 18, 15, 10, 5, 9\}^*$              | 15              | 15              | $\{10, 20, 30, 24, 18, 12, 6, 15\}^*$          | 15              | 15              | $\{6, 12, 18, 15, 10, 5, 9\}^*$              | 15              | 15              | $\{6, 12, 18, 15, 10, 5, 9\}^*$              | 15              | 15              |

Table 2. The $Q$-generators are represented by $^*$ in each case.

Remark 3.14. Let $X = V(I)$. The generic element $p$ of the integral closure of the ideal $\mathcal{J} + I$ defines a curve, called the polar curve of $X$ [20]. We have $\pi^*(p) = D_p + \mathcal{T}_p$ and the strict transform $\mathcal{T}_p$ of $p$ by $\pi$ intersects the irreducible components $E_j$'s with $(D_p \cdot E_j) < 0$. These intersection points of $\mathcal{T}_p$ and $E$ give the base points of $p$. In the case of ADE-singularities and the rational singularities with reduced Artin cycle, the base points of $p$ are described precisely in [6, 17, 18]. It is still an open problem for other classes of rational singularities. Here we relate $\ell(D_p)$ and $\mathcal{L}_0(X)$ and, for RTP-singularities, we find the candidate divisors for $D_p$.

When $X$ is an $A_n$-type singularity, using [18], we know that the strict transform $\mathcal{T}_p$ of the polar curve passes through the intersection point of two irreducible components in the middle of $E$ when with $n = 2k$ and $\mathcal{T}_p$ intersects the unique irreducible component which is in the middle of $E$ when $n2k + 1$. Hence

$$D_p = (1, 2, 3, \ldots, k - 1, k, k - 1, \ldots, 3, 2, 1), \quad D_p = (1, 2, 3, \ldots, k + 1, k, \ldots, 3, 2, 1)$$

for $n = 2k$ and $n = 2k + 1$ respectively. Here the ordering of the irreducible components of $E$ is taken as

$$E_1, E_2, E_3, E_4, \ldots, E_{n-1}, E_n$$

When $X$ is a $D_n$-type singularity, using [17, 5] we get:

$$D_p = (k, 2k, 2k - 1, \ldots, 2, 1, k), \quad D_p = (k, 2k, 2k - 1, 2k - 2, \ldots, 4, 3, 2, k), \ (k \geq 2)$$

for $n = 2k$ and $n = 2k + 1$ respectively such that the ordering in $E$ is taken as

$$E_1, E_2, E_3, \ldots, E_{n-1}, E_n$$
Consequently, when $X$ is $A_n$-type or $D_n$-type, we have $\ell(D_p) = \frac{n}{2}$ for $n$ even, $\ell(D_p) = \frac{n+1}{2}$ for $n$ odd. In the cases of $E_6$-type and $E_7$-type singularities, we get the divisors $D_p$ as given in Table 2 with respect to the following orderings in $E$

$$E_1 \ E_2 \ E_3 \ E_4 \ E_5 \ \text{and} \ \ E_6 \ E_7$$

**Observation 3.15.** If $X$ is of ADE-type then

$$L_0(X) \leq \frac{\text{mult}_{0}(X)}{\text{mult}_{0}(X) - 1} \cdot L_0(D_p)$$

We can also replace $L_0(D_p)$ in the inequality by $\ell(D_p)$.

Recall that, for ADE-singularities, the Milnor number $\mu(X) = \text{dim}_C (\mathcal{O}_{C,N}^t \otimes N)$ of $X$ equals the Tjurina number $\tau(X) = \text{dim}_C \left( \frac{\mathcal{O}_{C,N}^t}{\mathcal{J}_N} \right)$ which is the dimension of the base space of a semi-universal deformation of $X$ [16, 25].

### 4. Lojasiewicz Exponent of Rational Singularities of Higher Multiplicity

Assume that $X \subset \mathbb{C}^N$ is the germ of a surface with a rational singularity at the origin of multiplicity $m > 2$. By [3], the multiplicity $m$ equals $N - 1$ and, by [25], $X$ is defined by $q := \frac{(N-1)(N-2)}{2}$ equations. In other words, there exists $q$ germs of holomorphic functions $f_i : \mathbb{C}^N \rightarrow \mathbb{C}$ so that the fiber over 0 of the application $F : (f_1, f_2, \ldots, f_q) : \mathbb{C}^N \rightarrow \mathbb{C}^q$ is the surface $X = \{ z \in \mathbb{C}^N \mid f_1(z) = \ldots = f_q(z) = 0 \}$ with multiplicity $N - 1$ at 0. Let $g_1, \ldots, g_q$ be the determinants of $(N - 2) \times (N - 2)$ minors of the Jacobian matrix $(\frac{\partial f_i}{\partial z_j})_{i,j}$ where $s := (N-2)(N-3)$. The ideal $\mathcal{J} = \langle g_1, \ldots, g_q \rangle$ is called the Jacobian ideal. Consider the map

$$G := (g_1, \ldots, g_q) : \mathbb{C}^N \rightarrow \mathbb{C}^s$$

As in the case where $q = 1$, the Lojasiewicz exponent of $X$ is defined as the smallest element of the set

$$\{ \theta > 0 \mid \exists U \subset \mathbb{C}^N \text{ and } \exists c \in \mathbb{R}_+ \text{ such that } \|z\|^\theta \leq c \cdot \|G(z)\| \text{ for all } z \in U \}$$

that is, $L_0(X) = L_0(\mathcal{J})$. Now let us restrict our attention on a special class of rational singularities.

For this, recall that a map $F = (f_1, \ldots, f_s) : \mathbb{C}^N \rightarrow \mathbb{C}^s$ is called quasi-homogeneous if there exists $w \in (\mathbb{R}_+ - \{0\})^N$ and $d \in (\mathbb{R}_+ - \{0\})^s$ such that, for each $i$, we have

$$f_i(\lambda^{w_1}z_1, \lambda^{w_2}z_2, \ldots, \lambda^{w_N}z_N) = \lambda^d f_i(z_1, z_2, \ldots, z_N)$$

where $w = (w_1, \ldots, w_N)$ and $d = (d_1, d_2, \ldots, d_s) = (d(f_1), \ldots, d(f_s))$.

**Example 4.1.** Let $F = (f_1, f_2, f_3) : \mathbb{C}^4 \rightarrow \mathbb{C}^3$ defines a rational singularity of multiplicity 3, called RTP-singularities. By [23, 2], they are defined by the equations presented in the following table, so each of them is quasi-homogeneous.

| RTP-type | Equations | RTP-type | Equations | RTP-type | Equations |
|----------|-----------|----------|-----------|----------|-----------|
| $A_{k-1}, k \geq 2, m \geq 1$ | $x w + y w - y^m w = 0$ | $C_{k-1, k \geq 1}$ | $x z - y^m w = 0$ | $D_{k-1}$ | $x w - y^m w = 0$ |
| $B_{k-1}, n = 2k > 3$ | $x z - y^{2k+1} - y^{2k} z = 0$ | $B_{k-1}, n = 2k + 1 \geq 3$ | $u^3 - x y^{k+1} - y^{k+2} = 0$ | $F_{k-1}$ | $x z - y^m w = 0$ |
| $H_n, n = 3k$ | $x^2 - x w y + x w - y^3 = 0$ | $H_n, n = 3k + 1$ | $u^3 - x y^k - y^k w = 0$ | $H_n$ | $x^2 - x w y - y^3 = 0$ |
| $E_{6, 0}$ | $x^2 - x y^2 = 0$ | $E_{6, 0}$ | $x^3 - x w y = 0$ | $E_{7, 0}$ | $x^2 + x w = 0$ |

**Table 3.** The equations defining RTP-singularities.
Proposition 4.2. \[\text{If } X \text{ is an RTP-singularity then } G = (g_1, \ldots, g_{18}) : \mathbb{C}^4 \to \mathbb{C}^{18} \text{ is quasi-homogeneous with weight } w = (w_1, w_2, w_3, w_4) \text{ and with quasi-degree } d = (d(g_1), \ldots, d(g_{18})) \in \mathbb{Z}^{18} \text{ and the Lojasiewicz exponent } L_0(X) \text{ is bounded as } \]
\[
\frac{\min\{d(g_1), \ldots, d(g_{18})\}}{\min\{w_1, w_2, w_3, w_4\}} \leq L_0(X) \leq \frac{\max\{d(g_1), \ldots, d(g_{18})\}}{\min\{w_1, w_2, w_3, w_4\}}.
\]

The possible values of the Lojasiewicz exponent of $L_0(X)$ can be computed using the explicit equations and represented in the table below.

| RTP | weights | minw | mind | maxd |
|-----|---------|------|------|-------|
| $A_1$, $A_2$ | $(m_1, 1, k, l)$ | 1 | $2m$ | $2k + l - 1$ |
| $B_{2l-1}$ | $(2l-1, 2, 2k+1, 2l)$ | 2 | $4k + 1$ for $l < k + 1$, $4k + 2$ for $l \geq k + 1$ | $4k + 2$ for $k = l$, $4k + l$ for $l < k$, $6l$ for $l \geq k + 1$ |
| $B_{2l}$ | $(2l-2, 2k+1, 2l-1)$ for $l < k+1$, $(2k, 2k, k, k, k, k)$ for $l \geq k + 1$ | 2 | $4l - 3$ for $l < k + 1$, $4l - 4$ for $l = k$, $4l - 5$ for $l > k$ | |
| $D_{2l-1}$ | $(4, 3, 3k, 3l, 3l)$ | 3 | $2l - 1$ | $2l - 1$ |
| $D_{2l}$ | $(3k - 3, 3l - 3, 3l - 1)$ | 3 | $3k - 4$ | $8, 14, 12k - 12$ for $k > 4$ |
| $E_{6,0}$ | $(5, 6, 8, 10)$ | 5 | 20 | 36 |
| $E_{7,0}$ | $(1, 6, 10, 14)$ | 5 | 16 | 30 |

Table 4. $L_0(X)$ for RTP-singularities

The following theorem gives a nice upper bound on $L_0(X)$ by showing that the upper bound we obtained for the ADE-singularities is also valid for the RTP-singularities.

Theorem 4.3. Let $X$ be a surface with an RTP-type singularity. Then the inequality \[(4.2)\]

$$L_0(X) \leq \frac{\text{mult}_0(X)}{\text{mult}_0(X) - 1} L_0(D_p)$$

holds.

Proof. Let $X$ be of $E_{6,0}$-type singularity. We have $X = V(I)$ with $I := f_1 f_2 f_3$. Consider the ideal $\mathcal{J} = \langle J_{ij}(z_1 z_2), J_{ij}(z_1 z_3), J_{ij}(z_2 z_3), J_{ij}(z_2 z_4), J_{ij}(z_3 z_4) \rangle$

with $i, j = 1, 2, 3$ and $i \neq j$ where $J_{ij}(z_1 z_2) = \frac{\partial f_i}{\partial z_1} \frac{\partial f_j}{\partial z_2} - \frac{\partial f_i}{\partial z_2} \frac{\partial f_j}{\partial z_1}$. A generating set for $\mathcal{J}$ is

\[
2z^2 + yw, \quad 6x^2 z + 17w^2, \quad xzw, \quad xyw, \quad x^2 w, \quad yzw,
\]

\[
3x^2 y - 17zw, \quad 19g^3 w + 4x^2 w, \quad y^3 z + x^2 z + 2w^2, \quad y^4 + x^2 y - 4zw, \quad x^4 y^2 w^2 yzw
\]

The length of the ideal $\mathcal{J}$ is 17. It is hard to compute the integral closure of that ideal. However, we can still use the formula given in [3, 1] in order to get an estimation on $L_0(\mathcal{J})$. If for some $a, b \in \mathbb{Z}_{\geq 1}$ we have $M^a \subseteq \mathcal{J}^b$, then this implies that $L_0(\mathcal{J}) \leq \frac{a}{b}$. We obtain $M^5 \subseteq \mathcal{J}$ and the best estimation we get is the inclusion $M^{4t+1} \subseteq \mathcal{J}^t$ with $t \in \mathbb{N}^*$, so we can say $L_0(X) = L_0(\mathcal{J}) \leq 4$. We also conclude that $D_p$ is among the divisors with length $\leq 17$. In Table 4, we give all possible $L_0(D_p)$ for the divisors $D$ with lengths $\leq 17$ and we see that the inequality $\leq 4$ holds.

Now, consider the surface $X \subset \mathbb{C}^4$ of $E_{6,0}$-type singularity. A generating set of the Jacobian ideal $\mathcal{J}$ is

\[
w^2, \quad z w, \quad y^2 w - 2w^2, \quad x y w, \quad x^2 w - 4y z w, \quad z^3, \quad x z^2, \quad x y z, \quad 2x^2 z - 4y z^2 - 3w^2, \quad y^3 + 2z^2 + y w, \quad x y^2, \quad x^2 y - 2y^2 z - 4z w, \quad y^2 z^2. \quad x^4
\]

The length $\ell(\mathcal{J})$ equals 16. Again we proceed with $\mathcal{J}$ instead of the integral closure of $\mathcal{J}$ and we obtain $M^5 \subseteq \mathcal{J}$ but $M^4 \not\subseteq \mathcal{J}$. More precisely, we have $M^{6t} \subseteq \mathcal{J}^{2t}$ with $t \in \mathbb{N}^*$. Hence $L_0(X) = L_0(\mathcal{J}) \leq 4$ and we again have the inequality.
Table 5. For $E_{0,0}$ and $E_{0,7}$ singularities

| RTP | $u$ with $\ell(J) \leq u$ | $j$ with $\mathcal{L}_0(J) \leq j$ |
|-----|----------------|----------------|
| $D_{k-2}$ | $k = f + m \geq 1$ | $k = f$ with $k \geq f \geq m \geq 1$ |
| $E_{k-2}$ | $5f + k \neq 2$ for $f \geq k + 1$, $k + 2f + 2$ for $f \leq k + 1$ | $k = f + 1$ |
| $C_{k-1, k+1}$ | $k = f + 8$ | $k = 2$ |
| $D_{k-1}$ | $k = 13$ | $k = 3$ |
| $P_{k-1}$ | $k = 16$ | $k = 3$ |
| $D_{k-1}$ | $6k + 2$ | $2k$ |
| $P_{k-1}$ | $6k + 4$ | $2k$ |
| $P_{k-1}$ | $6k + 6$ | $2k + 1$ |
| $E_{0,7}$ | $16$ | $4$ |
| $P_{0,7}$ | $17$ | $5$ |
| $P_{7,0}$ | $17$ | $5$ |

Table 6. $\ell(J)$ and $\mathcal{L}_0(J)$ for RTP-singularities respectively

The computation above gives a nice upper bound on $\mathcal{L}_0(X)$ which permits also to determine the approximate location of $D_p$.

**Conjecture 4.4.** The inequality in Theorem 4.3 is true for a rational singularity of higher multiplicity.

**References**

[1] S. Altinok, M. Tosun, *Generators for the semigroup of Lipman*, Bull. of the Brazilian Math. Soc., New Series, 39, (2008), 123-135.

[2] A. Altintas-Sharland, G. Cevik, M. Tosun, *Nonisolated forms of rational triple singularities*, Rocky Mountain J. Math. 46-2, (2016), 357-388.

[3] M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. 88 (1966), 129-136.

[4] C. Bivias-Ausina and S. Encinas, *Lojasiewicz exponents and resolution of singularities*, Arch. Math. 93-3 (2009), 225-234.

[5] R. Boudil, *General elements of an M-primal ideal on a normal surface*, Séminaire-Congrès, 10, (2005), 11-20.

[6] G. Gonzalez-Springberg, *Résolution de Nash des points doubles rationnels*, Ann. de l’Institut Fourier, 32-2, (1982), 111-178.

[7] A. Haraux and T. S. Pham, *On the Lojasiewicz exponent of quasi-homogeneous functions*, J. of Sing., vol. 11 (2015), 52-66.

[8] T. Krasiński, G. Oleksiak and A. Płoski, *The Lojasiewicz exponent of an isolated weighted homogeneous surface singularity*, Proc. of the Amer. Math. Soc. 137-1, (2009), 3387-3397.
D.T. Lê and B. Teissier, "Variétés polaires locales et classes de Chern des variétés singulières," Ann. of Math. 114, (1981), 457-491.

M. Lejeune-Jalabert and B. Teissier, "Clôture intégrale des idéaux et équisingularité, avec 7 compléments," Ann. Fac. des Sci. de Toulouse, V.XVII, 4, (2008), 781-859.

D.T. Lê and M. Tosun, "Combinatorics of rational surface singularities," Comm. Math. Helv. (2004), 582-604.

J. Lipman, "Rational singularities with applications to algebraic surfaces and unique factorization," Publ. Math. l'IHES, 36, (1969), 195-279.

S. Lojasiewicz, "Une propriété topologique des sous ensembles analytiques réels," Colloques internationaux du CNRS, 117, (1963), 87-89.

O. Saeki, "Topological invariance of weights for weighted homogeneous isolated singularities in \(\mathbb{C}^3\)," Proc. Amer. Math. Soc. 103-3 (1988), 905-909.

K. Salto, "Quasihomogene isolierte Singularitaten von Hyperflächen," Inv. Math. 14 (1971), 123-142.

J. Snoussi, "Base points of polar curves on a surface of type \(z^n = f(x, y)\)," Kodai Math. J., 28-1, (2005), 31-46.

M. Spivakovsky, "Sandwiched singularities and desingularization of surfaces by normalized Nash transformations," Ann. of Math. 131 (1990), 411-491.

C. Huneke and I. Swanson, "Integral Closure of Ideals, Rings, and Modules," London Math. Soc. 336, (2006).

B. Teissier, "Variétés polaires I," Invent. Math. 40 (1977), 267-292.

B. Teissier, "Some resonances of Lojasiewicz Inequalities," Wiad. Math. 48-2, (2012), 271-284.

B. Teissier, "Cycles évanescentes, sections planes et conditions de Whitney," Astérisque 7-8, Soc. Math. de France, (1973), 285-362.

G. N. Tjurina, "Absolute isolation of rational singularities, and triple rational points" (Russian) Funkc. Anal. Prilozen. 2-4, (1968), 70-81.

M. Tosun, "Tjurina components and rational cycles for rational singularities," Turkish J. of Math. 23-3, (1999), 361-374.

J. Wahl, "Equations defining rational singularities," Ann. Sci. École Normale Sup. 10-2, (1977), 231-263.

K.-i. Watanabe and K. Yoshida, "Hilbert-Kunz multiplicity, McKay correspondence and good ideals in two-dimensional rational singularities," Manuscripta Math. 104, (2001), 275-294.

Yau, S. S. T., "Topological types and multiplicities of isolated quasi-homogeneous surface singularities," Bull. Amer. Math. Soc. 19 (1988), 447-454.

Mathematics Group, Middle East Technical University, Northern Cyprus Campus 99738 Kalkanlı, Güzelyurt, TRNC, via Mersin 10, Turkey
Email address: bemel@metu.edu.tr

Department of Mathematics, Galatasaray University, Ortaköy 34357, İstanbul, Turkey
Email address: kagulay@gmail.com

Department of Mathematics, Galatasaray University, Ortaköy 34357, İstanbul, Turkey
Email address: mtosun@gsu.edu.tr