STABILIZATION OF HYPERBOLIC EQUATIONS WITH MIXED BOUNDARY CONDITIONS

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Abstract. This paper is devoted to study decay properties of solutions to hyperbolic equations in a bounded domain with two types of dissipative mechanisms, i.e. either with a small boundary or an internal damping. Both of the equations are equipped with the mixed boundary conditions. When the Geometric Control Condition on the dissipative region is not satisfied, we show that sufficiently smooth solutions to the equations decay logarithmically, under sharp regularity assumptions on the coefficients, the damping and the boundary of the domain involved in the equations. Our decay results rely on an analysis of the size of resolvent operators for hyperbolic equations on the imaginary axis. To derive this kind of resolvent estimates, we employ global Carleman estimates for elliptic equations with mixed boundary conditions.

1. Motivation and formulation of the problem. In this paper, we will consider the longtime behavior of solutions to damped hyperbolic equations in a bounded domain. When the energy of the solution tends to zero as time tends to infinity, we will analyze what the explicit decay rate of the solution is. In the literature, most of the works are concerned with three types of decay rates for the solutions of evolution equations, i.e. exponential decay, polynomial decay and logarithmic decay. Consider the following abstract Cauchy problem on a Hilbert space $X$:

$$
\frac{dx(t)}{dt} = Ax(t), \quad t \geq 0, \quad x(0) = x_0, \quad x \in X \quad (1.1)
$$

where $A$ generates a $C_0$ semigroup $T(t) = e^{At}$ on $X$. The solution of (1.1) is then given by $x(t) = T(t)x_0$. Assume that $i\mathbb{R} \subset \rho(A)$, where $\rho(A)$ is the resolvent set of $A$. We recall the following characterization on decay rates: (i) The solution of (1.1) decays exponentially if and only if $\sup \{|| (A - i\beta I)^{-1} ||_{L(X)} | \beta \in \mathbb{R} \} < \infty$ (see [13, 22]); (ii) The solution of (1.1) decays polynomially if and only if $\sup \beta^{-1} || (A - i\beta I)^{-1} ||_{L(X)} \leq L$ for some $l, L > 0$ and all $|\beta| \geq 1$ (see [5, 18]); (iii) The solution of (1.1) decays logarithmically if and only if there exists a constant $C > 0$ such that $\forall \beta \in \mathbb{R}, || (A - i\beta I)^{-1} ||_{L(X)} \leq Ce^{C|\beta|}$ (see [9, 16, 17]). We refer to [3, 8] and

2010 Mathematics Subject Classification. Primary: 93B05; Secondary: 93B07, 35B37.
Key words and phrases. Logarithmic decay, hyperbolic equations, Neumann-Robin boundary condition, global Carleman estimate, resolvent estimate.

This work was supported by the NSF grants 11231007 and 11322110, the Foundation for the Author of National Excellent Doctoral Dissertation of China (No.201213), Innovative Research Team (No.IRT1273) and the National Basic Research Program of China (973 Program) under grant 2011CB808002.
references therein for the recent progress on characterization of decay rates for the solutions of abstract evolution equations.

The aim of this paper is to derive the explicit decay rates of the solutions to hyperbolic equations with a small boundary or internal damping term. This work was motivated by the above mentioned references, especially reference [9] for logarithmic stabilization of wave equations with Zaremba boundary conditions. More precisely, our problem can be formulated as follows.

Let \( \Omega \subset \mathbb{R}^n (n \in \mathbb{N}) \) be a bounded domain with boundary \( \partial \Omega \) of class \( C^2 \), \( \Gamma \) be a smooth hypersurface of \( \partial \Omega \) which splits the boundary into two non-empty open sets \( \Gamma_0, \Gamma_1 \) so that \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma \). Let \( \omega \) be a given nonempty open subset of \( \Omega \) such that \( \overline{\omega} \subseteq \Omega \). Denote by \( \overline{\Omega} \) the closure of \( \Omega \), and \( \bar{c} \) the complex conjugate of \( c \in \mathbb{C} \).

Let \( a^{jk}(\cdot) \in C^1(\overline{\Omega}; \mathbb{R}) \) be a fixed function satisfying

\[
a^{jk}(x) = a^{kj}(x), \quad \forall \ x \in \overline{\Omega}, \ j, k = 1, 2, \cdots, n, \tag{1.2}
\]

and for some constant \( s_0 > 0 \),

\[
\sum_{j,k=1}^{n} a^{jk}(x)\xi^j \xi^k \geq s_0 |\xi|^2, \quad \forall (x, \xi) \in \overline{\Omega} \times \mathbb{C}^n, \quad \xi = (\xi^1, \cdots, \xi^n). \tag{1.3}
\]

We are interested in the decay property for solution to the following damped hyperbolic equation:

\[
\begin{cases}
u_{tt} - \sum_{j,k=1}^{n} (a^{jk} u_{x_j})_{x_k} = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\
\sum_{j,k=1}^{n} a^{jk} u_{x_j} \nu_{x_k} + a(x)u_t = 0 & \text{on } \mathbb{R}^+ \times \Gamma_1, \\
(u(0), u_t(0)) = (u^0, u^1) & \text{in } \Omega,
\end{cases} \tag{1.4}
\]

with a velocity feedback on \( \Gamma_1 \). In (1.4), \( a = a(x) \) is a non-negative bounded function, \( a(x) \geq 0 \) and \( a(x) \) is supported in \( \Gamma_1 \). We impose any one of the following boundary conditions on \( \Gamma_0 \):

\[
\begin{align*}
u|_{\Gamma_0} &= 0 & \text{Dirichlet type or,} \\
\left( \sum_{j,k=1}^{n} a^{jk} u_{x_j} \nu_{x_k} + p(x)u \right)|_{\Gamma_0} &= 0 & \text{Neumann-Robin type,} \tag{1.5}
\end{align*}
\]

where \( p(x) \geq 0 \) and \( \nu = (\nu_{x_1}, \cdots, \nu_{x_n}) \) stands for the unit outward normal vector of \( \Omega \) at \( x = (x_1, \cdots, x_n) \in \partial \Omega \).

Also, we will consider the decay properties of the following hyperbolic equation with a locally internal damping:

\[
\begin{cases}
u_{tt} - \sum_{j,k=1}^{n} (a^{jk} u_{x_j})_{x_k} + d(x)u_t = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\
(u(0), u_t(0)) = (u^0, u^1) & \text{in } \Omega,
\end{cases} \tag{1.6}
\]

where \( d = d(x) \) is a non-negative bounded function, \( d(x) \geq 0 \) and \( d(x) \) supported in \( \omega \). Equation (1.6) is equipped with the following boundary mixed boundary
where $p(x) \geq 0$ and $G_0$, $G_1$ are two open subsets of $\partial \Omega$ satisfy $G_0 \cup G_1 = \partial \Omega$ and $G_0 \cap G_1 = \emptyset$. In particular, if $G_1 = \emptyset$, then (1.7) is the Dirichlet boundary condition $u|_{\partial \Omega} = 0$, which acts on the whole boundary. Similarly, if $G_0 = \emptyset$, (1.7) is the Neumann-Robin boundary condition which acts on the whole boundary.

There are numerous studies on the decay properties of hyperbolic equations. For exponential decay of the solutions of wave equations, Bardos-Lebeau-Rauch [2] show that the energy of the solutions decays exponentially if and only if the Geometric Control Condition is satisfied. We refer to [15, 24, 23, 27] and references therein for more exponential decay results for the wave equations with Dirichlet or Neumann boundary conditions; and [6, 7, 20] for polynomial decay for the wave equations with Dirichlet boundary condition under special geometries. When the Geometric Control Condition is not satisfied, the logarithmic decay results obtained in [4, 16, 17] depend on the logarithmic decay results on the boundary.

We remark that the authors do need these strong regularity. Naturally, we want to know whether the logarithmic decay results still hold under some sharp regularity on the coefficients $a_j^{jk}(\cdot)$, the damping $a(\cdot)$ and the boundary $\partial M$. In this respect, we refer to [1] for the Dirichlet boundary stabilization of wave equation and [10, 11] for both the boundary stabilization and the internal stabilization of hyperbolic equations by imposing the Neumann boundary conditions. However, to the best of our knowledge, there are no references addressing the stabilization of hyperbolic equations with Neumann-Robin boundary conditions even for $C^\infty$-regularity of the coefficients, the damping and the
boundary. Moreover, as we mentioned before, most of very interesting logarithmic decay results were given in [9, 17] and the references therein for the hyperbolic equation under the regularity assumption that the coefficients \( a^{jk}(\cdot) \) and \( a(\cdot) \) are \( C^\infty \)-smooth, and the damping at least is \( C^s(s > \frac{1}{2}) \)-smooth. In this paper, we shall develop an approach based on two global Carleman estimates to prove the boundary stabilization of system (1.4) equipped with boundary condition (1.5), and the internal stabilization of system (1.6) equipped with boundary conditions (1.7), under sharp regularity of the coefficients \( a^{jk}(\cdot) \in C^1(\Omega) \), the damping \( a(\cdot) \in L^\infty(\partial \Omega; \mathbb{R}^+) \), \( d(\cdot) \in L^\infty(\Omega; \mathbb{R}^+) \) and the boundary \( \partial \Omega \) is \( C^2 \)-smooth.

2. Statement of the main results. To begin with, we assume that \( a(\cdot) \) is a bounded real valued functions satisfying

\[
a(x) \geq 0 \text{ on } \partial \Omega \text{ and } \Gamma^1 \triangleq \{ x \in \Gamma_1 \mid a(x) > a_0 > 0 \text{ a.e. } \} \neq \emptyset,
\]

where \( a_0 \in \mathbb{R} \). As to the hypersurface \( \Gamma \) of the \( \partial \Omega \), we have the following assumption:

\[
a(x) \to 0 \text{ a.e. as } x \to \Gamma.
\]

Also, we assume that \( d(\cdot) \) is a bounded real valued functions satisfying

\[
d(x) \geq 0 \text{ in } \Omega \text{ and } \omega^0 \triangleq \{ x \in \omega \mid d(x) > d_0 > 0 \text{ a.e. } \} \neq \emptyset,
\]

for some \( d_0 \in \mathbb{R} \).

Put \( H \triangleq H^1(\Omega) \times L^2(\Omega) \). Write an element \( U = (u, v) \in H \). Then, we define an unbounded operator \( A_1 : D(A_1) \subset H \to H \) by

\[
\begin{align*}
D(A_1) & \triangleq \left\{ (u, v) \in H^2(\Omega) \times H^1(\Omega), u|_{\Gamma_0} = 0 \text{ and } \left( \sum_{j,k=1}^n a^{jk}u_{x_j} \nu_{x_k} + av \right)|_{\Gamma_1} = 0 \right\}, \\
A_1 U & = \left( v, \sum_{j,k=1}^n (a^{jk}(x)u_{x_j})_{x_k} \right)
\end{align*}
\]

As for Neumann-Robin boundary condition on \( \Gamma_0 \), we define \( A_2 : D(A_2) \subset H \to H \) by

\[
\begin{align*}
D(A_2) & \triangleq \left\{ (u, v) \in H^2(\Omega) \times H^1(\Omega), \left( \sum_{j,k=1}^n a^{jk}u_{x_j} \nu_{x_k} + pu \right)|_{\Gamma_0} = 0 \\
& \quad \text{and } \left( \sum_{j,k=1}^n a^{jk}u_{x_j} \nu_{x_k} + av \right)|_{\Gamma_1} = 0 \right\}, \\
A_2 U & = \left( v, \sum_{j,k=1}^n (a^{jk}(x)u_{x_j})_{x_k} \right)
\end{align*}
\]

with \( p = p(x) \geq 0 \).

It is easy to show that \( A_j \) \( (j = 1, 2) \) generate \( C_0 \)-semigroups \( \{ e^{tA_j} \}_{t \in \mathbb{R}^+} \) \((j = 1, 2)\) on \( H \). Therefore, system (1.4) is well-posed in \( H \). For the equation (1.4) with \( u|_{\Gamma_0} = 0 \) boundary condition, we define its energy by

\[
E_1(t) = \frac{1}{2} \int_{\Omega} \left[ \sum_{j,k=1}^n a^{jk}u_{x_j} \nu_{x_k} + |u_t|^2 \right] dx.
\]
For the equation (1.4) with the boundary condition $\left(\sum_{j,k=1}^{n} a^{jk} u_{x_{j}} u_{x_{k}} + p(x) u\right)|_{\Gamma_{0}} = 0$, we define its energy by

$$E_{2}(t) = \frac{1}{2} \int_{\Omega} \left[ \sum_{j,k=1}^{n} a^{jk} u_{x_{j}} u_{x_{k}} + |u_{t}|^{2} \right] + \frac{1}{2} \int_{\Gamma_{0}} p(x)|u|^{2} dx.$$

By (2.1)–(2.2), for $t_{2} > t_{1} > 0$, it is easy to show that

$$E_{1}(t_{2}) - E_{1}(t_{1}) = E_{2}(t_{2}) - E_{2}(t_{1}) = -\int_{t_{1}}^{t_{2}} \int_{\Gamma} a(x)|u_{t}|^{2} dx dt \leq 0.$$

Hence, the energy of every solution to (1.4) with the boundary condition (1.5) is non-increasing. Following the well-known unique continuation property for solutions of the wave equation, it is easy to show that there are no nonzero solutions of (1.4) which conserve energy. Hence, by using LaSalle's invariance principle ([12, p. 18]), we conclude that the energy of every solution of (1.4) tends to zero as $t$ tends to infinity, without any geometric conditions on the domain $\Omega$.

For system (1.6) equipped with boundary conditions (1.7), we define an unbounded operator $B: D(B) \subset H \to H$ by

$$D(B) \triangleq \{(u, v) : (u, v) \in H^{2}(\Omega) \times H^{1}(\Omega), \ u|_{\Gamma_{0}} = 0$$

$$\text{and } \left(\sum_{j,k=1}^{n} a^{jk} u_{x_{j}} u_{x_{k}} + p(x) u\right)|_{\Gamma_{1}} = 0\};$$

$$BU = \left(v, \sum_{j,k=1}^{n} a^{jk}(x) u_{x_{j}} u_{x_{k}} - d(x) v\right).$$

Also, it is easy to check that $B$ generate $C_{0}$-semigroups $\{e^{tB}\}_{t \in \mathbb{R}^{+}}$ on $H$. Thus, system (1.6) is well-posed in $H$ and the following energy of every solution of (1.6) with the boundary condition (1.7) is non-increasing as well:

$$E_{3}(t) = \frac{1}{2} \int_{\Omega} \left[ \sum_{j,k=1}^{n} a^{jk} u_{x_{j}} u_{x_{k}} + |u_{t}|^{2} \right] + \frac{1}{2} \int_{\Gamma_{1}} p(x)|u|^{2} dx.$$

Throughout this paper, we will use $C = C\left(\Omega, ~ \Gamma_{1}^{*}, \sum_{j,k=1}^{n} ||a^{jk}||_{C^{1}(\Omega)}, s_{0}\right)$ and $C^{*} = C^{*}\left(\Omega, ~ \omega^{*}, \sum_{j,k=1}^{n} ||a^{jk}||_{C^{1}(\Omega)}, s_{0}\right)$ to denote generic positive constants which may vary from line to line (unless otherwise stated). Our main results can be stated as follows.

**Theorem 2.1.** Let $a^{jk}(\cdot) \in C^{1}(\overline{\Omega}; \mathbb{R})$ satisfy (1.2)–(1.3). $(i)$ Let (2.1) and (2.2) hold. Then the solutions $e^{tA_{j}}(u^{0}, u^{1}) \equiv (u, u_{t}) \in C(\mathbb{R}; D(A_{j})) \cap C^{1}(\mathbb{R}; H) (j = 1, 2)$ of system (1.4) with the boundary condition (1.5) satisfy

$$||e^{tA_{j}}(u^{0}, u^{1})||_{H} \leq \frac{C}{\ln(2 + t)} ||(u^{0}, u^{1})||_{D(A_{j})}, \quad \forall (u^{0}, u^{1}) \in D(A_{j}), \forall t > 0.$$

(2.4)
Under the assumption of Theorem 2.1, it holds that

\[ \text{Theorem 2.2.} \]

If we impose the boundary condition\text{Remark 2.3.} \quad p \text{ equations with boundary damping discussed in [10, 17].}

\[ i.e. \quad p \text{ obtained via the size of the resolvent operator } R(\lambda, A_j) = (A_j - \lambda I)^{-1} \quad (j = 1, 2) \]

\[ \text{of } A_j \text{ and } R(\lambda, B) = (B - \lambda I)^{-1} \text{ of } B \text{ on the imaginary axis. Denote by } \rho(A_j) \text{ and } \rho(B) \text{ the resolvent sets of } A_j \text{ and } B, \text{ respectively. We have the following result.} \]

\[ \text{Theorem 2.2. Under the assumption of Theorem 2.1, it holds that } i\mathbb{R} \subset \rho(A_j) \quad (j = 1, 2) \text{ and } i\mathbb{R} \subset \rho(B), \text{ and there exist two constants } C > 0 \text{ and } C^* > 0 \text{ such that} \]

\[ \begin{align*}
(\text{i}) \quad \forall \beta \in \mathbb{R}, \quad ||(A_j - i\beta I)^{-1}||_{\mathcal{L}(H)} & \leq C e^{C|\beta|}, \quad j = 1, 2. \\
(\text{ii}) \quad \forall \beta \in \mathbb{R}, \quad ||(B - i\beta I)^{-1}||_{\mathcal{L}(H)} & \leq C^* e^{C^*|\beta|}.
\end{align*} \]

We have the following several remarks.

\[ \text{Remark 2.1. If we impose the boundary condition } u = 0 \text{ on } \Gamma_0, \text{ this boundary condition together with the boundary condition on } \Gamma_1 \text{ presented in (1.4) is the so-called Zaremba boundary conditions. In this respect, we refer to [9] for the related logarithmic decay of system (1.4).} \]

\[ \text{Remark 2.2. If we impose the boundary condition } \sum_{j,k=1}^{n} a^{jk}u_{x_j}v_{x_k} = 0 \text{ on } \Gamma_0, \]

\[ \text{i.e. } p(x) = 0 \text{ in (1.5), we can obtain the stabilization result for the hyperbolic equations with boundary damping discussed in [10, 17].} \]

\[ \text{Remark 2.3. If we impose the boundary condition } \sum_{j,k=1}^{n} a^{jk}u_{x_j}v_{x_k} = 0 \text{ on } \partial\Omega, \]

\[ \text{i.e. } p(x) = 0 \text{ in (1.5), we can obtain the stabilization result for the hyperbolic equations with internal damping discussed in [11]. Similarly, if we impose zero Dirichlet boundary condition, i.e. } u = 0 \text{ on } \partial\Omega, \text{ we can obtain the same stabilization result for the hyperbolic equations with internal damping discussed in [16].} \]

The rest of this paper is organized as follows. In Section 3, we shall prove two global Carleman estimates for elliptic equations with mixed boundary conditions, via which, two interpolation inequalities for elliptic equations are presented in Section 4. Section 5 is devoted to the proof of our main results.

3. Global Carleman estimates for elliptic equations with mixed boundary conditions. In this section, we shall establish two global Carleman estimates for the elliptic equations with mixed boundary conditions. Denote \[ Q = (-2, 2) \times \Omega, \quad \Sigma_1 = (-2, 2) \times \Gamma_1, \quad \Sigma_0 = (-2, 2) \times \Gamma_0. \]

Let us consider the following elliptic equation:

\[
\begin{aligned}
\left\{ \begin{array}{ll}
z_{ss} + \sum_{j,k=1}^{n} \left( a^{jk} z_{x_j} \right)_{x_k} = z^0 & \text{in } Q, \\
\sum_{j,k=1}^{n} a^{jk} z_{x_j} v_{x_k} - i a(x) z_s = a(x) z^1 & \text{on } \Sigma_1,
\end{array} \right.
\end{aligned}
\]

\[ \quad \text{(3.1)} \]
where \( z^0 \in L^2(X) \), \( z^1 \in L^2(\Sigma_1) \). As to the boundary \( \Gamma_0 \), we impose any one of the following boundary conditions:

\[
\begin{align*}
\|z\|_{\Sigma_0} &= 0 \quad \text{Dirichlet type,} \\
\left( \sum_{j,k=1}^{n} a^{jk} z_{x_j} \nu_{x_k} + p(x)z \right)_{\Sigma_0} &= 0 \quad \text{Neumann-Robin type.}
\end{align*}
\] (3.2)

Also, we will consider the global Carleman estimate for the following elliptic equation:

\[
\begin{align*}
z_{ss} + \sum_{j,k=1}^{n} (a^{jk} z_{x_j})_{x_k} + p(x)z = z^0 \quad \text{in } Q,
\end{align*}
\] (3.3)

where \( z^0 \in L^2(X) \) and system (3.3) is equipped with the boundary conditions

\[
\begin{cases}
\|z\|_{G_0} = 0 \quad \text{Dirichlet type,} \\
\left( \sum_{j,k=1}^{n} a^{jk} z_{x_j} \nu_{x_k} + p(x)z \right)_{G_1} = 0 \quad \text{Neumann-Robin type.}
\end{cases}
\] (3.4)

To present our global Carleman results for systems (3.1) and (3.3), we first recall the following known results.

**Lemma 3.1.** ([14, Lemma 2.3]) Let \( \tilde{\Gamma} \) be an arbitrary relatively nonempty open set of \( \partial \Omega \). Then there exists a function \( \psi^1 \in C^2(\Omega; \mathbb{R}) \) such that

\[
\begin{align*}
\psi^1 &> 0 \quad \text{in } \Omega, \\
|\nabla \psi^1| &> 0 \quad \text{in } \overline{\Omega}, \\
\psi^1 &= 0 \quad \text{on } \partial \Omega \setminus \tilde{\Gamma}, \\
\sum_{j,k=1}^{n} a^{jk} \psi^1_{x_j} \nu_{x_k} &\leq 0 \quad \text{on } \partial \Omega \setminus \tilde{\Gamma}.
\end{align*}
\] (3.5)

**Lemma 3.2.** ([14, Lemma 2.1]) Let \( \omega_0 \) be an arbitrary fixed sub-domain of \( \Omega \) such that \( \omega_0 \subset \omega_* \). Then there exists a function \( \psi^2 \in C^2(\Omega; \mathbb{R}) \) such that

\[
\begin{align*}
\psi^2 &> 0 \quad \text{in } \Omega, \\
\psi^2 &= 0 \quad \text{on } \partial \Omega, \\
|\nabla \psi|^2 &> 0 \quad \text{in } \overline{\Omega} \setminus \omega_0.
\end{align*}
\] (3.6)

For any \( \mu > \ln 2 \), put

\[
b \triangleq \sqrt{1 + \frac{1}{\mu} \ln(2 + e^\mu)}, \quad b_0 \triangleq \sqrt{b^2 - \frac{1}{\mu} \ln \left( \frac{1 + e^\mu}{e^\mu} \right)}.
\] (3.7)

It is easy to check that \( 1 < b_0 < b \leq 2 \). For parameter \( \lambda, \mu > 1 \), assume that there exists a function \( \psi^* \in C^2(\overline{\Omega}) \) (which will be given later), we define the weight functions \( \theta \) and \( \phi \) as follow:

\[
\begin{align*}
\theta &= e^\ell, \quad \ell = \lambda \phi, \quad \phi = e^\mu \psi, \quad \psi(s, x) = \frac{\psi^*(x)}{\|\psi^*\|_{L^\infty(\Omega)}} + b^2 - s^2.
\end{align*}
\] (3.8)

We have the following results.

**Theorem 3.1.** Let \( a^{jk}(\cdot) \) satisfy (1.2)–(1.3). Let \( z \in H^2(Q; C) \) satisfy (3.1)–(3.2), with \( z(-b, \cdot) = z(b, \cdot) = 0 \) in \( \Omega \). Then there is a constant \( \mu_0 > 0 \) such that for all \( \mu \geq \mu_0 \), one can find three constants \( C = C(\mu) > 0, \ C^* = C^*(\mu) > 0 \) and
\( \lambda_0 = \lambda_0(\mu) \) so that for any \( \lambda \geq \lambda_0 \), the following holds.

\[
\lambda \mu^2 \int_{-b}^{b} \int_{\Omega} \theta^2 \phi(|\nabla z|^2 + |z_s|^2 + \lambda^2 \mu^2 \phi^2 |z|^2) \, dx \, ds \\
\leq C \left\{ ||\theta z'||^2_{L^2(Q)} + e^{C\lambda}||z||^2_{L^2(\Sigma_{1})} + e^{C\lambda} \int_{b}^{b} \int_{\Gamma_{t}^2} (|z_s|^2 + |z|^2) \, dx \, ds \right\}.
\]  

(3.9)

**Theorem 3.2.** Let \( a^{jk}(\cdot) \) satisfy (1.2)–(1.3). Let \( z \in H^2(Q; \mathbb{C}) \) satisfy (3.3) and (3.4), with \( z(-b, \cdot) = z(b, \cdot) = 0 \) in \( \Omega \). Then there is a constant \( \mu_0 > 0 \) such that for all \( \mu \geq \mu_0 \), one can find three constants \( C = C(\mu) > 0 \), \( C^* = C^*(\mu) > 0 \) and \( \lambda_0 = \lambda_0(\mu) \) so that for any \( \lambda \geq \lambda_0 \), the following holds:

\[
\lambda \mu^2 \int_{-b}^{b} \int_{\Omega} \theta^2 \phi(|\nabla z|^2 + |z_s|^2 + \lambda^2 \mu^2 \phi^2 |z|^2) \, dx \, ds \\
\leq C^* \left\{ ||\theta z'||^2_{L^2(Q)} + e^{C^*\lambda} \int_{b}^{b} \int_{\omega^*} (|z_s|^2 + |z|^2) \, dx \, ds \right\}.
\]  

(3.10)

The proof of Theorem 3.1 will be based on the following weighted inequality, which can be obtained immediately from [11, Theorem 2.1].

**Lemma 3.3.** Let \( a^{jk}(\cdot) \) satisfy (1.2)–(1.3), and let \( z \in C^2((-b, b) \times \Omega; \mathbb{C}) \). Then, there is a constant \( \mu_1 > 0 \) such that for all \( \mu \geq \mu_1 \), one can find two constants \( C = C(\mu) > 0 \) and \( \lambda_1 = \lambda_1(\mu) \) so that for all \( z \in H^1((-b, b) \times \Omega) \) with \( z(-b, \cdot) = z(b, \cdot) = 0 \) in \( \Omega \), and \( z_{ss} + \sum_{j,k=1}^{n} (a^{jk} z_{x_j}) z_{x_k} = f \) (in \( (-b, b) \times \Omega \), in the sense of distribution) with \( f \in L^2((-b, b) \times \Omega) \), and for all \( \lambda \geq \lambda_1 \), it holds that

\[
\lambda \mu^2 \int_{-b}^{b} \int_{\Omega} \theta^2 \phi(|\nabla \psi|^2 + |\nabla z|^2 + \lambda^2 \mu^2 \phi^2 |\nabla \psi|^2 |z|^2) \, dx \, ds \\
\leq C \left\{ \int_{-b}^{b} \int_{\Omega} \theta^2 |f|^2 \, dx \, ds + \int_{-b}^{b} \int_{\partial \Omega} \sum_{k=1}^{n} V^k \cdot \nu_{x_k} \, dx \, ds \\
+ \lambda \mu \int_{-b}^{b} \int_{\Omega} \theta^2 \phi(|z_s|^2 + |\nabla z|^2 + \lambda^2 \mu^2 \phi^2 |z|^2) \, dx \, ds \right\},
\]  

(3.11)

where

\[
A = \ell^2_s + \ell_{ss} + \sum_{j,k=1}^{n} a^{jk} \ell_{x_j} \ell_{x_k} + \sum_{j,k=1}^{n} (a^{jk} \ell_{x_j}) x_k + \sum_{j,k=1}^{n} a^{jk} x_k \ell_{x_j}, \quad v = \theta z,
\]

\[
V^k = \sum_{j,j',k'} a^{jk} \ell_{x_j} (v_{x_j} \nu_s + \nu_{x_j} v_s) + 2(\ell_{ss} + a^{jk'} \ell_{x_j} x_k) a^{jk} (v_{x_j} \nu_s + \nu_{x_j} v_s) \\
- 2a^{jk} x_k \ell_{x_j} |v_{x_j}|^2 + 2a^{jk} A \ell_{x_j} |v|^2 + \left( 2a^{jk'} \ell_{x_j} - a^{jk} a^{jk'} \right) \ell_{x_j} (v_{x_j} \nu_{x_j} + \nu_{x_j} v_{x_j}).
\]

(3.12)

**Remark 3.1.** We refer to [26, Theorem 3.2] for more information on the pointwise weighted identity for a more general partial differential operator of second order than that appeared in Lemma 3.3. Such kind of weighted identities are quite useful for the study of control theory and inverse problems for various kinds of partial differential equations (see [26] and the references therein).
Proof of Lemma 3.3. Integrating inequality (3.17) in [11, Theorem 3.1] on \((-b, b) \times \Omega\), noting that \(z(-b, x) = z(b, x) = 0\), we obtain the desired result immediately.

Finally, we recall the following known identity.

**Lemma 3.4.** ([10, Lemma 2.3]) Let \(b^{jk} \in C^2(\mathbb{R}^n; \mathbb{R})\) satisfy \(b^{jk} = b^{kj}(j, k = 1, \ldots, n)\), and \(g \triangleq (g^1, \ldots, g^n) : \mathbb{R}_s \times \mathbb{R}^n_\nu \to \mathbb{R}^n\) be a vector field of class \(C^1\). Then for any \(z \in C^2(\mathbb{R}_s \times \mathbb{R}^n_\nu; \Omega)\), it holds

\[
\begin{aligned}
- \sum_{k=1}^{n} \left[ (g \cdot \nabla z) \sum_{j=1}^{n} b^{jk} z_{x_j} + (g \cdot \nabla z) \sum_{j=1}^{n} b^{jk} z_{x_j} - g^k \left( |z_s|^2 + \sum_{i,l=1}^{n} b^{il} z_{x_i} z_{x_l} \right) \right]_{x_k} \\
= - \left[ z_{ss} + \sum_{j,k=1}^{n} (b^{jk} z_{x_j})_{x_k} \right] g \cdot \nabla z + \left[ z_{ss} - \sum_{j,k=1}^{n} (b^{jk} z_{x_j})_{x_k} \right] g \cdot \nabla z \\
+ (z_s g \cdot \nabla z + z_s g \cdot \nabla z)_s - (z_s g_s \cdot \nabla z + z_s g_s \cdot \nabla z) \\
+ (\nabla \cdot g)|z_s|^2 - 2 \sum_{j,k,l=1}^{n} b^{jk} z_{x_j} z_{x_k} \frac{\partial g^l}{\partial x_k} + \sum_{j,k=1}^{n} z_{x_j} z_{x_k} \nabla \cdot (b^{jk} g).
\end{aligned}
\]

(3.13)

### 3.1. Proof of Theorem 3.1

In this subsection, we will prove Theorem 3.1. Our proof is based on Lemma 3.3. The key point is to estimate the boundary terms in Lemma 3.3 with mixed boundary conditions. The proof is divided into several steps.

**Step 1.** Note that in system (4.8) we need to consider the problem with inhomogeneous Neumann-type boundary. As in [10, 11], we shall choose two weight functions, one is positive while another one is negative with respect to the space domain \(\Omega\), such that some undesired boundary terms vanish on \(\partial \Omega\). More precisely, we choose other weight functions \(\tilde{\theta}\) and \(\tilde{\phi}\) as follows

\[
\begin{aligned}
\tilde{\theta} = \tilde{\epsilon}, \quad \tilde{\epsilon} = \lambda \tilde{\phi}, \quad \tilde{\phi} = e^{\mu \tilde{\psi}}, \quad \tilde{\psi} = -\frac{\psi^*(x)}{||\psi^*||_{L^\infty(\Omega)}} + b^2 - s^2
\end{aligned}
\]

(3.14)

where \(b\) is given by (3.7) and \(\psi^*\) will be given later, it is easy to check that

\[
\begin{aligned}
\tilde{\psi} \leq \psi, \quad 0 < \tilde{\phi} \leq \phi, \quad 0 < \tilde{\theta} \leq \theta.
\end{aligned}
\]

(3.15)

Now, we choose \(\psi^* = \psi^1\), which is given by (3.5), and we apply Lemma 3.3 to (3.1). By (3.15), and noting that \(z(-b, x) = z(b, x) = 0\), there is a constant \(\mu_1 > 0\) such that for all \(\mu \geq \mu_1\), one can find a \(\lambda = \lambda_1(\mu)\) so that for all \(\lambda \geq \lambda_1\):

\[
\begin{aligned}
&\lambda \mu^2 \int_{-b}^{b} \int_{\Omega} \theta^2 \phi |\nabla \psi|^2 \left( |z_s|^2 + |\nabla z|^2 + \lambda^2 \mu^2 \phi^2 |\nabla \psi|^2 |z|^2 \right) dxds \\
\leq & \ C \left\{ ||\theta^0||_{L^2(Q)}^2 + \int_{-b}^{b} \int_{\partial \Omega} \sum_{k=1}^{n} (V^k + \tilde{V}^k) \cdot \nu_{x_k} dxds \right. \\
& \left. + \lambda \mu \int_{-b}^{b} \int_{\Omega} \theta^2 \phi \left( |z_s|^2 + |\nabla z|^2 + \lambda^2 \mu^2 \phi^2 |z|^2 \right) \right\},
\end{aligned}
\]

(3.16)
where $\tilde{V}^k$ has the same form with $V^k$, only replace $\ell, A$ by $\tilde{\ell}, \tilde{A}$.

**Step 2.** Let us estimate the boundary term $\int_{-b}^{b} \int_{\partial\Omega} \sum_{k=1}^{n} (V^k + \tilde{V}^k) \cdot \nu_{x_k} dxds$.

By (3.5), (3.8) and (3.14), we have

$$
\begin{cases}
\phi = \tilde{\phi}, \ell = \tilde{\ell}, \theta = \tilde{\theta}, \ell_s = \lambda \mu \psi_s = \lambda \mu \tilde{\psi}_s = \tilde{\ell}_s, \quad &\text{on } \partial\Omega \setminus \Gamma^*_1, \\
\sum_{j,k=1}^{n} a^{jk}(\ell_{x_j} + \tilde{\ell}_{x_j}) \nu_{x_k} = \lambda \mu \phi \sum_{j,k=1}^{n} a^{jk}(\psi_{x_j} + \tilde{\psi}_{x_j}) \nu_{x_k} = 0, \quad &\text{on } \partial\Omega \setminus \Gamma^*_1.
\end{cases}
$$

However, note that

$$
\int_{-b}^{b} \int_{\partial\Omega} \sum_{k=1}^{n} (V^k + \tilde{V}^k) \cdot \nu_{x_k} dxds = \int_{-b}^{b} \left( \int_{\partial\Omega \setminus \Gamma^*_1} + \int_{\Gamma^*_1} \right) \sum_{k=1}^{n} (V^k + \tilde{V}^k) \cdot \nu_{x_k} dxds.
$$

(3.17)

By Lemma 3.1 we know that $\sum_{j,k=1}^{n} a^{jk} \psi_{x_j} \nu_{x_k} \leq 0$ on $\partial\Omega \setminus \Gamma^*_1$. In case of $\sum_{j,k=1}^{n} a^{jk} \psi_{x_j} \nu_{x_k} = 0$ on $\partial\Omega \setminus \Gamma^*_1$ in Lemma 3.1, if $z|_{\Gamma_0} = 0$, by (3.12), (3.17) and (2.1)–(2.2), we have

$$
\int_{-b}^{b} \int_{\partial\Omega \setminus \Gamma^*_1} \sum_{k=1}^{n} (V^k + \tilde{V}^k) \cdot \nu_{x_k} dxds = 0,
$$

(3.19)

and if $\left( \sum_{j,k=1}^{n} a^{jk} \nu_{x_k} + p z \right)|_{\Gamma_0} = 0$, by (3.12), (3.17) and (2.1)–(2.2), we have

$$
\int_{-b}^{b} \int_{\partial\Omega \setminus \Gamma^*_1} \sum_{k=1}^{n} (V^k + \tilde{V}^k) \cdot \nu_{x_k} dxds = \int_{-b}^{b} \int_{\Gamma_0} \sum_{k=1}^{n} (V^k + \tilde{V}^k) \cdot \nu_{x_k} dxds
$$

$$
= -4 \int_{-b}^{b} \int_{\Gamma_0} \theta^2 p(x) \left( \ell_{ss} + 2 \lambda \mu \phi \sum_{j',k'=1}^{n} a^{j'k'} \nu_{x_{j'}} \nu_{x_{k'}} \right) |z|^2 dxds \leq 0.
$$

(3.20)

On the other hand, in case of $\sum_{j,k=1}^{n} a^{jk} \psi_{x_j} \nu_{x_k} < 0$ on $\partial\Omega \setminus \Gamma^*_1$ in Lemma 3.1, by (3.12), (3.17) and (2.1)–(2.2), noting that $v = \theta z$, $\tilde{v} = \tilde{\theta} z$, if $z|_{\Gamma_0} = 0$, a short calculation shows that

$$
\int_{-b}^{b} \int_{\partial\Omega \setminus \Gamma^*_1} \sum_{k=1}^{n} (V^k + \tilde{V}^k) \cdot \nu_{x_k} dxds = \int_{-b}^{b} \int_{\Gamma_0 \setminus (\Gamma_1 \setminus \Gamma^*_1)} \sum_{k=1}^{n} (V^k + \tilde{V}^k) \cdot \nu_{x_k} dxds
$$

$$
\leq C \lambda^2 \mu^2 \int_{-b}^{b} \int_{\Gamma_0 \setminus (\Gamma_1 \setminus \Gamma^*_1)} \theta^2 \phi^2 |z|^2 dxds
$$

(3.21)

and if we impose Neumann-Robin boundary condition $\left( \sum_{j,k=1}^{n} a^{jk} \nu_{x_k} + p z \right)|_{\Gamma_0} = 0$ in (3.2), by (3.12), (3.17) and (2.1)–(2.2), note that $z(-b, x) = z(b, x) = 0$ and
$p(x) \geq 0$, we have
\[
\int_{-b}^{b} \int_{\partial \Omega \setminus \Gamma_1} \sum_{k=1}^{n} (V^k + \tilde{V}^k) \cdot \nu_{x_k} \, dxds = \int_{-b}^{b} \int_{\partial \Omega \setminus \partial \Omega \setminus (\Gamma_1 \setminus \Gamma_1)} \sum_{k=1}^{n} (V^k + \tilde{V}^k) \cdot \nu_{x_k} \, dxds 
\]
\[
\leq -4 \int_{-b}^{b} \int_{\Gamma_0} \theta^2 p(x) \left( \ell_{ss} + 2 \lambda \mu^2 \phi \sum_{j,k=1}^{n} a^{j} \phi \psi_{x_j} \psi_{x_k} \right) |z|^2 \, dxds
\]
\[
+ C \lambda^2 \mu^2 \int_{-b}^{b} \int_{\partial \Omega \setminus \Gamma_1} \theta^2 \phi^2 |z|^2 \, dxds \leq C \lambda^2 \mu^2 \int_{-b}^{b} \int_{\partial \Omega \setminus \Gamma_1} \theta^2 \phi^2 |z|^2 \, dxds.
\]
(3.22)

**Step 3.** Let us estimate $\lambda^2 \mu^2 \int_{-b}^{b} \int_{\partial \Omega \setminus \Gamma_1} \theta^2 \phi^2 |z|^2 \, dxds$ in case of \[
\sum_{j,k=1}^{n} a^{j,k} \psi_{x_j} \nu_{x_k} < 0 \text{ on } \partial \Omega \setminus \Gamma_1.
\]
For this purpose, we will integrate $\lambda \mu \theta^2 \psi \cdot (3.1) + \lambda \mu \theta^2 \phi \cdot (3.1)$ on $(-b,b) \times \Omega$, noting that $z(-b,x) = z(b,x) = 0$ and $a^{j,k} = a^{k,j}$, using integrating by parts, we get
\[
\int_{-b}^{b} \int_{\Omega} \lambda \mu \theta^2 \phi (\tau z^0 + z \overline{\tau}) \, dxds
\]
\[
= \int_{-b}^{b} \int_{\Omega} \left[ -2 \lambda \mu \theta^2 \phi |z|^2 + (\lambda \mu \theta^2 \phi)_{ss} |z|^2 \right] \, dxds 
\]
\[
+ \int_{-b}^{b} \int_{\Omega} \sum_{j,k=1}^{n} \left[ -2 \lambda \mu \theta^2 \phi a^{j,k} \psi_{x_j} \psi_{x_k} + \left(a^{j,k} (\lambda \mu \theta^2 \phi)_{x_k}\right)_{x_j} |z|^2 \right] \, dxds 
\]
\[
+ \int_{-b}^{b} \int_{\partial \Omega} \sum_{j,k=1}^{n} \left[ -a^{j,k} (\lambda \mu \theta^2 \phi)_{x_j} |z|^2 + \lambda \mu \theta^2 \phi a^{j,k} (z_{x_j} \tau + \overline{z}_{x_j} \tau) \right] \nu_{x_k} \, dxds.
\]
(3.23)

Next, by (3.8), it is easy to check that
\[
- \int_{-b}^{b} \int_{\partial \Omega} \sum_{j,k=1}^{n} a^{j,k} (\lambda \mu \theta^2 \phi)_{x_j} \nu_{x_k} |z|^2 \, dxds
\]
\[
= \int_{-b}^{b} \int_{\partial \Omega} \left( - \sum_{j,k=1}^{n} a^{j,k} \psi_{x_j} \nu_{x_k} \right) \theta^2 \left[ \lambda^2 \mu^2 \phi^2 + O_{\mu}(\lambda) \right] |z|^2 \, dxds.
\]
(3.24)

Further, if $z|_{\Gamma_0} = 0$ in (3.2), by (2.1) and (2.2), we have
\[
\int_{-b}^{b} \int_{\partial \Omega} \lambda \mu \theta^2 \phi \sum_{j,k=1}^{n} a^{j,k} (z_{x_j} \tau + \overline{z}_{x_j} \tau) \nu_{x_k} \, dxds = \int_{-b}^{b} \int_{\Gamma_1} \lambda \mu \theta^2 \phi a(x) (z^1 \tau + \overline{z^1} \tau) \, dxds
\]
(3.25)

and if \[
\left( \sum_{j,k=1}^{n} a^{j,k} z_{x_j} \nu_{x_k} + p(x) z \right) \bigg|_{\Gamma_0} = 0 \text{ in } (3.2), \text{ by } (2.1)-(2.2), \text{ we have}
\]
\[
\int_{-b}^{b} \int_{\partial \Omega} \lambda \mu \theta^2 \phi \sum_{j,k=1}^{n} a^{j,k} (z_{x_j} \tau + \overline{z}_{x_j} \tau) \nu_{x_k} \, dxds
\]
\[
= -2 \lambda \mu \int_{-b}^{b} \int_{\Gamma_0} \theta^2 \phi p(x) |z|^2 \, dxds + \int_{-b}^{b} \int_{\Gamma_1} \lambda \mu \theta^2 \phi a(x) (z^1 \tau + \overline{z^1} \tau) \, dxds.
\]
(3.26)
Combining (3.23) and (3.24)–(3.26), noting that \(\sum_{j,k=1}^{n} a^{jk} \psi_{x_j} \nu_{x_k} < 0\) on \(\partial \Omega \setminus \Gamma_1^*\), there is a \(\mu_2 > 0\) such that for any \(\mu \geq \mu_2\), one can find a \(\lambda_2 > 0\) such that for any \(\lambda \geq \lambda_2\), we have

\[
\lambda^2 \mu^2 \int_{-b}^{b} \int_{\partial \Omega \setminus \Gamma_1^*} \theta^2 \phi^2 |z|^2 dxds 
\leq C \left[ \|\theta z^1\|^2_{L^2(\Sigma_1)} + \lambda^2 \mu^2 \int_{-b}^{b} \int_{\Gamma_1^*} \theta^2 \phi^2 |z|^2 dxds 
+ \|\theta z^0\|^2_{L^2(Q)} + \lambda \mu \int_{-b}^{b} \int_{\Gamma_1^*} \theta^2 \phi \left( |z_x|^2 + |\nabla z|^2 + \lambda^2 \mu^2 \phi^2 |z|^2 \right) dxds \right].
\]

(3.27)

Therefore, combining (3.16), (3.18)–(3.22) and (3.27), noting that \(|\nabla \psi| > 0\) in \(\Omega\), \(v = \theta z\), \(\tilde{v} = \theta z\), and \(z\) satisfy the boundary condition \(\sum_{j,k=1}^{n} a^{jk} z_{x_j} \nu_{x_k} = ia(x) z_x + a(x) z^1\) on \(\Gamma_1^*\), by an elementary calculation, one can find a constant \(\mu_3 > 0\) such that for all \(\mu \geq \mu_3\), there is a \(\lambda_3 = \lambda_3(\mu)\) so that for all \(\lambda \geq \lambda_3\):

\[
\lambda \mu^2 \int_{-b}^{b} \int_{\Omega} \phi \left( |z_x|^2 + |\nabla z|^2 + \lambda^2 \mu^2 \phi^2 |z|^2 \right) dxds \leq C \|\theta z^0\|^2_{L^2(Q)} 
+ C e^{\lambda \gamma} \left[ \|z^1\|^2_{L^2(\Sigma_1)} + \int_{-b}^{b} \int_{\Gamma_1^*} (|z_x|^2 + |z|^2) dxds \right] + C \int_{-b}^{b} \int_{\Gamma_1^*} |\nabla z|^2 dxds.
\]

(3.28)

where we have used the following facts:

\[
\int_{-b}^{b} \int_{\Gamma_1^*} \sum_{k=1}^{n} (V^k + \tilde{V}^k) \cdot \nu_{x_k} dxds 
\leq C \int_{-b}^{b} \int_{\Gamma_1^*} |\nabla z|^2 dxds + C e^{\lambda \gamma} \int_{-b}^{b} \int_{\Gamma_1^*} (|z_x|^2 + |z|^2 + |z^1|^2) dxds.
\]

Step 4. Let us estimate \(\int_{-b}^{b} \int_{\Gamma_1^*} |\nabla z|^2 dxds\).

First, we choose a \(g_0 \in C^1(\overline{\Omega}; \mathbb{R}^n)\) such that \(g_0 = \nu\) on \(\partial \Omega\). By Lemma 3.1 there exists a real value function, that we denote \(\rho\) here, such that \(\rho \in C^2(\overline{\Omega}; [0,1])\), \(\rho > 0\) in \(\Omega\) and \(\rho = 0\) on \(\partial \Omega \setminus \Gamma_1\). Let \(g = g_0 \rho\). Integrating (4.3) (in Lemma 3.4) in \((-b, b) \times \Omega\), with \(b^{jk}\) replaced by \(a^{jk}\), using integration by parts, and noting \(z(-b, x) = z(b, x) = 0\), by (3.1), (1.3) and (2.1)–(2.2), we have

\[
\int_{-b}^{b} \int_{\partial \Omega} \rho \left( |z_x|^2 + \sum_{j,l=1}^{n} a^{jl} z_{x_j} \bar{z}_{x_l} \right) dxds 
\leq C \int_{-b}^{b} \int_{\Omega} (|z^0|^2 + |z_x|^2 + |\nabla z|^2) dxds
\]

(3.29)

\[
+ \int_{-b}^{b} \int_{\Gamma_1^*} a(x) \left[ (g \cdot \nabla \bar{z})(iz_x + z^1) + (g \cdot \nabla z)(iz_x + z^1) \right] dxds.
\]
By (2.1) and (3.29), we conclude that
\[
\int_{-b}^{b} \int_{\Gamma_1^b} |\nabla z|^2 \, dx \, ds
\leq C \left[ ||z^0||_{L^2(Q)}^2 + ||z^1||_{L^2(\Sigma_1)}^2 + \int_{-b}^{b} \int_{\Omega} (|z_s|^2 + |\nabla z|^2) \, dx \, ds + \int_{-b}^{b} \int_{\Gamma_1^b} |z_s|^2 \, dx \, ds \right].
\] (3.30)

Combining (3.28) and (3.30), there is a \( \mu_0 \triangleq \max\{\mu_1, \mu_2, \mu_3\} > 0 \) such that for all \( \mu \geq \mu_0 \), one can find a \( \lambda_0 = \lambda_0(\mu) \) so that for any \( \lambda \geq \lambda_0 \), the inequality (3.9) holds.

3.2. Proof of Theorem 3.2. The proof of Theorem 3.2 is divided into several steps.

**Step 1.** First, we choose \( \psi^* = \psi^2 \), which is given by (3.6). Then, by (3.6), (3.8) and (3.14), we have
\[
\begin{align*}
\phi = \tilde{\phi}, \quad \ell = \tilde{\ell}, \quad \theta = \tilde{\theta}, \quad \ell_s = \lambda \mu \phi \tilde{\psi}_s = \lambda \mu \phi \tilde{\psi}_s = \tilde{\ell}_s, & \quad \text{on } \partial \Omega, \\
\sum_{j,k=1}^{n} a^{jk}(\ell_{x_j} + \tilde{\ell}_{x_j}) \nu_{x_k} = \lambda \mu \phi \sum_{j,k=1}^{n} a^{jk}(\psi_{x_j} + \tilde{\psi}_{x_j}) \nu_{x_k} = 0, & \quad \text{on } \partial \Omega.
\end{align*}
\] (3.31)

By Lemma 3.2, we know that \( \sum_{j,k=1}^{n} a^{jk} \psi_{x_j} \nu_{x_k} \leq 0 \) on \( \partial \Omega \). Thus, proceeding exactly the same analysis on the boundary terms as in Subsection 3.1, in case of \( \sum_{j,k=1}^{n} a^{jk} \psi_{x_j} \nu_{x_k} = 0 \) on \( \partial \Omega \), by (3.4), (3.12) and (3.31), we have
\[
\int_{-b}^{b} \int_{\partial \Omega} (V^k + \tilde{V}^k) \, dx \, ds \leq 0
\] (3.32)
and in case of \( \sum_{j,k=1}^{n} a^{jk} \psi_{x_j} \nu_{x_k} < 0 \) on \( \partial \Omega \), we have
\[
\int_{-b}^{b} \int_{\partial \Omega} \sum_{k=1}^{n} (V^k + \tilde{V}^k) \cdot \nu_{x_k} \, dx \, ds
= \int_{-b}^{b} \int_{\partial \Omega} \sum_{k=1}^{n} (V^k + \tilde{V}^k) \cdot \nu_{x_k} \, dx \, ds \leq C^* \lambda^2 \mu^2 \int_{-b}^{b} \int_{G_1} \theta^2 |\phi^2| |z|^2 \, dx \, ds.
\] (3.33)

**Step 2.** Let us estimate \( \lambda^2 \mu^2 \int_{-b}^{b} \int_{G_1} \theta^2 |\phi^2| |z|^2 \, dx \, ds \). For this purpose, we will integrate \( \lambda \mu \theta^2 \phi \bar{z} \cdot (3.3) + \lambda \mu \theta^2 \phi z \cdot (3.3) \) on \((-b, b) \times \Omega \), noting that \( z(-b, x) = z(b, x) = 0 \) and \( a^{jk} = a^{kj} \), by (3.4), (3.8) and using integrating by parts, we get
\[
\int_{-b}^{b} \int_{\Omega} \lambda \mu \theta^2 \phi (z \bar{z}^0 + z \bar{z}^0) \, dx \, ds - \int_{-b}^{b} \int_{\Omega} \lambda \mu \theta^2 \phi d(x) \left[ i(\bar{z} z_s - z \bar{z}_s) \right] \, dx \, ds
= \int_{-b}^{b} \int_{\Omega} \left[ -2 \lambda \mu \theta^2 |z_s|^2 + (\lambda \mu \theta^2 |z_s|^2 |z|^2) \right] \, dx \, ds.
\]
\[
+ \int_{-b}^{b} \int_{\Omega} \sum_{j,k=1}^{n} \left[ -2\lambda \mu \theta^2 \phi a^{jk} \varepsilon_{x_j} z_{x_k} + (a^{jk}(\lambda \mu \theta^2 \phi)_{x_j}) z^2 \right] dxds \\
+ \int_{-b}^{b} \int_{G_1} \sum_{j,k=1}^{n} \lambda \mu \theta^2 \phi \left[ -a^{jk} \psi_{x_j} \nu_{x_k} (2\lambda \mu \phi + \mu) - 2p(x) \right] z^2 dxds.
\] (3.34)

Noting that \( \sum_{j,k=1}^{n} a^{jk} \psi_{x_j} \nu_{x_k} < 0 \) on \( \partial \Omega \), there is a \( \mu_1 > 0 \) such that for any \( \mu \geq \mu_1 \), one can find a \( \lambda_1 > 0 \) such that for any \( \lambda \geq \lambda_1 \), we have
\[
\int_{-b}^{b} \int_{\partial \Omega} \sum_{k=1}^{n} (V^k + \tilde{V}^k) \cdot \nu_{x_k} dxds \leq C^* \lambda^2 \mu^2 \int_{-b}^{b} \int_{G_1} \theta^2 \phi^2 |z|^2 dxds \\
\leq C^* \left[ \|\theta z^0\|_{L^2(\Omega)}^2 + \mu \int_{-b}^{b} \int_{\Omega} \theta^2 \phi \left( |z_s|^2 + |\nabla z|^2 + \lambda^2 \mu^2 \phi^2 |z|^2 \right) dxds \right].
\] (3.35)

Therefore, by (3.16), (3.32) and (3.35), we have
\[
\lambda \mu^2 \int_{-b}^{b} \int_{\Omega} \theta^2 \phi (|\nabla \psi|^2 + |\nabla z|^2 + \lambda^2 \mu^2 \phi^2 |\nabla \psi|^2 |z|^2) dxds \\
\leq C^* \left( \|\theta z^0\|_{L^2(\Omega)}^2 + \mu \int_{-b}^{b} \int_{\Omega} \theta^2 \phi \left( |z_s|^2 + |\nabla z|^2 + \lambda^2 \mu^2 \phi^2 |z|^2 \right) dxds \right).
\] (3.36)

Next, recalling (3.6) in Lemma 3.2, it is easy to see that
\[
\text{Left hand side of } (3.36) \geq c_0 \lambda \mu^2 \int_{-b}^{b} \int_{\Omega \setminus \omega_0} \theta^2 \phi (|\nabla \psi|^2 + |\nabla z|^2 + \lambda^2 \mu^2 \phi^2 |z|^2) dxds \\
- C^* \lambda \mu^2 \int_{-b}^{b} \int_{\omega_0} \theta^2 \phi (|\nabla \psi|^2 + |\nabla z|^2 + \lambda^2 \mu^2 \phi^2 |z|^2) dxds,
\] for some positive constant \( c_0 \). Now, by (3.36)–(3.37), there is a \( \lambda_2 > 1 \) such that
\[
\lambda \mu^2 \int_{-b}^{b} \int_{\omega_0} \theta^2 \phi (|\nabla \psi|^2 + |z_s|^2 + \lambda^2 \mu^2 \phi^2 |z|^2) dxds \\
\leq C^* \left( \|\theta z^0\|_{L^2(\Omega)}^2 + \lambda \mu^2 \int_{-b}^{b} \int_{\omega_0} \theta^2 \phi (|\nabla z|^2 + |z_s|^2 + \lambda^2 \mu^2 \phi^2 |z|^2) dxds \right).
\] (3.38)

**Step 3.** Let us estimate \( \|\lambda \mu^2 \int_{-b}^{b} \int_{\omega_0} \theta^2 \phi |\nabla \psi|^2 dxds \| \).

Now, we choose a cut-off function \( \zeta \in C^2(\omega^*; [0, 1]) \) such that \( \zeta (x) = 1 \) on \( \omega_0 \). We will integrate \[ \zeta \lambda \mu \theta^2 \psi_x \cdot (3.3) + \zeta \lambda \mu \theta^2 \psi_z \cdot (3.3) \] on \( (-b, b) \times \Omega \), using integrations by parts, by (1.3), noting that \( z(-b, x) = z(b, x) \equiv 0 \) and \( \sum_{j,k=1}^{n} a^{jk} \psi_{x_j} \nu_{x_k} \leq 0 \), we get
\[
\lambda \mu^2 \int_{-b}^{b} \int_{\omega_0} \theta^2 \phi |\nabla \psi|^2 dxds \leq C^* \left( \|\theta z^0\|_{L^2(\Omega)}^2 + c C^* \lambda \int_{-b}^{b} \int_{\omega^*} (|z_s|^2 + |z|^2) dxds \right).
\] (3.39)
Finally, combining (3.38) and (3.39), we obtain the desired result immediately.

4. Two interpolation inequalities for elliptic equations. In this section, by means of the global Carleman estimates, we shall derive two interpolation inequalities for the elliptic equations with mixed boundary conditions.

Let us consider the following elliptic equation:

\[
\begin{aligned}
  w_{ss} + \sum_{j,k=1}^{n} (a^{jk} w_{x_j})_{x_k} &= w^0 \quad \text{in } X = (-2, 2) \times \Omega, \\
  w &= 0 \quad \text{or} \quad \sum_{j,k=1}^{n} a^{jk} w_{x_j} \nu_{x_k} + p(x)w = 0 \quad \text{on } \Sigma_0 = (-2, 2) \times \Gamma_0, \\
  \sum_{j,k=1}^{n} a^{jk} w_{x_j} \nu_{x_k} - ia(x) w_s &= a(x)w^1 \quad \text{on } \Sigma_1 = (-2, 2) \times \Gamma_1.
\end{aligned}
\]

(4.1)

where \( p(x) \geq 0 \) and \( w^0 \in L^2(X), \ w^1 \in L^2(\Sigma_1) \).

Also, we will consider the interpolation inequality for the following elliptic equation:

\[
\begin{aligned}
  w_{ss} + \sum_{j,k=1}^{n} (a^{jk} w_{x_j})_{x_k} + id(x) w_s &= w^0 \quad \text{in } X = (-2, 2) \times \Omega, \\
  w &= 0 \quad \text{on } \Sigma^0 = (-2, 2) \times G_0 \quad (4.2) \\
  \sum_{j,k=1}^{n} a^{jk} w_{x_j} \nu_{x_k} + p(x)w &= 0 \quad \text{on } \Sigma^1 = (-2, 2) \times G_1.
\end{aligned}
\]

Denote

\[
X = (-2, 2) \times \Omega, \ Y = (-1, 1) \times \Omega, \ Z^* = (-2, 2) \times \Gamma_1^*, \ X^* = (-2, 2) \times \omega^*. \quad (4.3)
\]

We have the following interpolation inequalities.

**Theorem 4.1.** Under the assumptions in Theorem 2.1, there exist constants \( C > 0, C^* > 0 \) such that, for any \( \varepsilon > 0 \), the following hold.

(i) The solution \( w \) of system (4.1) satisfies

\[
||w||_{H^1(Y)} \leq Ce^{C/\varepsilon} \left[ ||w^0||_{L^2(Q)} + ||w^1||_{L^2(\Sigma_1)} + ||w_s||_{L^2(Z^*)} + ||w_s||_{L^2(Z^*)} \right] + Ce^{-2/\varepsilon} ||w||_{H^1(Q)}.
\]

(4.4)

(ii) The solution \( w \) of system (4.2) satisfies

\[
||w||_{H^1(Y)} \leq Ce^{C/\varepsilon} \left[ ||w^0||_{L^2(Q)} + ||w||_{L^2(X^*)} + ||w_s||_{L^2(X^*)} \right] + Ce^{-2/\varepsilon} ||w||_{H^1(Q)}.
\]

(4.5)

**Remark 4.1.** Similar interpolation inequality (4.5) for the real valued elliptic equation with Robin boundary condition can be found in [19].

**Proof.** We borrow some ideas from [21]. Note however that there is no boundary condition for \( w \) at \( s = \pm 2 \). Therefore, we need to introduce a cut-off function \( \varphi = \varphi(s) \in C^0_0(-b,b) \subset C^0_0(\mathbb{R}) \) such that
It holds that
\[
\begin{cases}
0 \leq \varphi(s) \leq 1 & |s| < b, \\
\varphi(s) = 1, & |s| \leq b_0
\end{cases}
\] (4.6)
where \(b\) and \(b_0\) were given by (3.7), and it is easy to see that \(1 < b_0 < b \leq 2\).

Next, we put
\[z = \varphi w.\] (4.7)

Then, noting that \(\varphi\) does not depend on \(x\), by (4.1), it follows
\[
\begin{align*}
z_{ss} + \sum_{j,k=1}^n (a^{jk} z_{x,j})_{x,k} &= \varphi_{ss} w + 2\varphi_s w_s + \varphi w^0 & \text{in } Q, \\
z &= 0 & \text{or} & \sum_{j,k=1}^n a^{jk} z_{x,j} \nu_{x,k} + p(x) z = 0 & \text{on } \Sigma_0 \\
\sum_{j,k=1}^n a^{jk} z_{x,j} \nu_{x,k} - ia(x) z_s &= -ia(x) \varphi_s w + a(x) \varphi w^1 & \text{on } \Sigma_1.
\end{align*}
\] (4.8)

Now, by applying Theorem 3.1 to system (4.8), we have
\[
\lambda \mu^2 \int_{-b}^b \int_{\Omega} \theta^2 \phi(|\nabla z|^2 + |z_s|^2 + \lambda^2 \mu^2 \phi^2 |z|^2) dx ds
\leq C \left\{ \int_{-b}^b \int_{\Omega} \theta^2 |\varphi_{ss} w + 2\varphi_s w_s + \varphi w^0|^2 dx ds + e^{C \lambda} ||w^1||_L^2(\Sigma_1) \right\}
+ e^{C \lambda} \int_{-b}^b \int_{\Gamma^1_1} (|z|^2 + |z_s|^2) dx ds.
\] (4.9)

However, recalling (3.8) for the definition of \(\phi\) and by (3.7), it is easy to see that
\[
\begin{cases}
\phi(s, \cdot) \geq 2 + e^\mu, & \text{for any } s \text{ satisfying } |s| \leq 1, \\
\phi(s, \cdot) \leq 1 + e^\mu, & \text{for any } s \text{ satisfying } b_0 \leq |s| \leq b.
\end{cases}
\] (4.10)

Denoting \(c_0 = 2 + e^\mu > 1\), and note that \(b_0 \in (1, b)\). By using (4.6) and (4.10), note that \(z = \varphi w\), we get
\[
\begin{align*}
\lambda e^{2\lambda c_0} \int_{-1}^1 \int_{\Omega} ((|\nabla w|^2 + |w_s|^2 + |w|^2) dx ds
\leq C e^{C \lambda} \left\{ \int_{-2}^2 \int_{\Omega} |w^0|^2 dx ds + \int_{-2}^2 \int_{\Gamma_1} |w^1|^2 dx ds + \int_{-2}^2 \int_{\Gamma_1} (|w|^2 + |w_s|^2) dx ds \right\}
+ C e^{2\lambda (c_0 - 1)} \int_{(-b, -b_0) \cup (b_0, b)} \int_{\Omega} (|w|^2 + |w_s|^2) dx ds.
\end{align*}
\] (4.11)

By (4.11), one concludes that there exists an \(\varepsilon_1 > 0\) such that for every \(\varepsilon \in (0, \varepsilon_1]\), it holds
\[
||w||_{H^1(Y)} \leq C e^{C/\varepsilon} \left[ ||w^0||_{L^2(Q)} + ||w^1||_{L^2(\Sigma_1)} + ||w||_{L^2(\Sigma')} + ||w_s||_{L^2(\Sigma')} \right]
+ C e^{-2/\varepsilon} ||w||_{H^1(Q)},
\] (4.12)
which yields the desired result (4.4). The second inequality (4.5) in Theorem 4.1 is derived in a similar way. \[\square\]
5. Proofs of Theorems 2.1 and 2.2. In this section, we shall give the proofs of our main results.

Proof of Theorem 2.2. First, fix \( F = (f^0, f^1) \in H \) and \( U = (u^0, u^1) \in D(A_j) \) \((j = 1, 2)\). It is easy to see that the following equation \((A_j - i\beta I)U = F\) is equivalent to

\[
\begin{cases}
\sum_{j,k=1}^n (a^{jk} u^0_{x_j})_{x_k} + \beta^2 u^0 = i\beta f^0 + f^1 & \text{in } \Omega, \\
u_0 = 0 \text{ or } \sum_{j,k=1}^n a^{jk} u^0_{x_j} \nu_{x_k} + p(x) u^0 = 0 & \text{on } \Gamma_0, \\
\sum_{j,k=1}^n a^{jk} u^0_{x_j} \nu_{x_k} + i\alpha u^0 = -af^0 & \text{on } \Gamma_1, \\
u^1 = f^0 + i\beta u^0 & \text{in } \Omega.
\end{cases}
\]  

(5.1)

Put \( v = e^{-\beta s} u^0 \).

(5.2)

It is easy check that \( v \) satisfies the following equation:

\[
\begin{cases}
v_{ss} + \sum_{j,k=1}^n (a^{jk} v_{x_j})_{x_k} = (i\beta f^0 + f^1)e^{-\beta s} & \text{in } \mathbb{R} \times \Omega, \\
v = 0 \text{ or } \sum_{j,k=1}^n a^{jk} v_{x_j} \nu_{x_k} + p(x) v = 0 & \text{on } \mathbb{R} \times \Gamma_0, \\
\sum_{j,k=1}^n a^{jk} v_{x_j} \nu_{x_k} - iav_s = -af^0 e^{-\beta s} & \text{on } \mathbb{R} \times \Gamma_1.
\end{cases}
\]  

(5.3)

Now, by (5.2) and (4.3), we have the following estimates.

\[
\begin{cases}
||u^0||_{H^1(\Omega)} \leq Ce^{C|\beta|} ||v||_{H^1(\Omega)}, & ||v||_{H^1(\Omega)} \leq C(||\beta| + 1)e^{C|\beta|} ||u^0||_{H^1(\Omega)}, \\
||v||_{L^2(\Omega)} \leq Ce^{C|\beta|} ||u^0||_{L^2(\Gamma_1)}, & ||v||_{L^2(\Omega)} \leq C|\beta|e^{C|\beta|} ||u^0||_{L^2(\Gamma_1)}.
\end{cases}
\]  

(5.4)

Now, applying Theorem 4.1 to (5.3), and by (5.4), we have

\[
||u^0||_{H^1(\Omega)} \leq Ce^{C|\beta|} \left[ ||f^0||_{H^1(\Omega)} + ||f^1||_{L^2(\Omega)} + ||u^0||_{L^2(\Gamma_1)} \right].
\]  

(5.5)

On the other hand, multiplying (5.1) by \( \pi^0 \) and integrating it on \( \Omega \), by (2.1)–(2.2), it follows that

\[
\int_{\Omega} (i\beta f^0 + f^1) \pi^0 \, dx = \beta^2 ||u^0||_{L^2(\Omega)}^2 - \sum_{j,k=1}^n \int_{\Omega} a^{jk} u^0_{x_j} \pi^0_{x_k} \, dx \\
- \int_{\Gamma_1} (i\beta a u^0 + af^0) \pi^0 \, dx + \int_{\Gamma_0} \sum_{j,k=1}^n a^{jk} u^0_{x_j} \nu_{x_k} \pi^0 \, dx.
\]  

(5.6)
However, it is easy to check that
\[
\int_{\Gamma_0} \sum_{j,k=1}^n a^{jk} u^j_{x_k} \nu^k dx = \begin{cases} 0, & \text{if } u^0|_{\Gamma_0} = 0, \\ -\int_{\Gamma_0} p(x)|u^0|^2 dx, & \text{if } \left( \sum_{j,k=1}^n a^{jk} u^j_{x_k} p(x) u^0 \right)|_{\Gamma_0} = 0. \end{cases}
\]  
(5.7)

Taking the imaginary part in both sides of (5.6) and by (5.7), we conclude that
\[
|\beta| \int_{\Gamma_1} a|u^0|^2 dx \leq C \left[ ||(i\beta f^0 + f^1)||_{L^2(\Omega)} ||u^0||_{L^2(\Omega)} + ||f^0||_{H^1(\Omega)} ||u^0||_{H^1(\Omega)} \right].
\]  
(5.8)

Hence, combining (5.5) and (5.8), we have
\[
||u^0||_{H^1(\Omega)} \leq Ce^{C|\beta|} \left[ ||f^0||_{H^1(\Omega)} + ||f^1||_{L^2(\Omega)} \right].
\]  
(5.9)

Recalling that \(u^1 = f^0 + i\beta u^0\), it follows
\[
||u^1||_{L^2(\Omega)} \leq ||f^0||_{L^2(\Omega)} + ||\beta|| ||u^0||_{L^2(\Omega)} \leq Ce^{C|\beta|} \left[ ||f^0||_{H^1(\Omega)} + ||f^1||_{L^2(\Omega)} \right].
\]  
(5.10)

By (5.9)–(5.10), we know that \(A_j - i\beta I\) is injective. Therefore \(A_j - i\beta I\) is bi-injective from \(D(A_j)\) to \(H\) (\(j = 1, 2\)). Moreover,
\[
||A_j - i\beta I||^{-1}_{L(H)} \leq Ce^{C|\beta|}, \quad j = 1, 2.
\]
This gives the first result of Theorem 2.2.

The analysis for the internal damping case can be derived in a similar way. Here, we only give some explanation on the relationship of the spectral equation with the elliptic equation we considered in (4.2). For fixed \(F = (f^0, f^1) \in H\) and \(U = (u^0, u^1) \in D(B)\). It is easy to see that the following equation \((B - i\beta I)U = F\) is equivalent to
\[
\begin{align*}
\sum_{j,k=1}^n (a^{jk} u^j_{x_k})_{x_k} + \beta^2 u^0 - i\beta du^0 &= [d(x) + i\beta] f^0 + f^1 \quad \text{in } \Omega, \\
u_{x_k} &= 0 \quad \text{on } G_0, \\
\sum_{j,k=1}^n a^{jk} u^j_{x_k} p(x) u^0 &= 0 \quad \text{on } G_1, \\
u^1 &= f^0 + i\beta u^0 \quad \text{in } \Omega.
\end{align*}
\]  
(5.11)

By setting the same transform \(v = e^{-\beta s} u^0\), it is easy check that \(v\) satisfies the following equation:
\[
\begin{align*}
v_{ss} + \sum_{j,k=1}^n (a^{jk} v_{x_j})_{x_k} + id(x)v_s &= [d(x) + i\beta] f^0 + f^1 \quad \text{in } \mathbb{R} \times \Omega, \\
v &= 0 \quad \text{on } \mathbb{R} \times G_0, \\
\sum_{j,k=1}^n a^{jk} v_{x_j} u_{x_k} + p(x)v &= 0 \quad \text{on } \mathbb{R} \times G_1.
\end{align*}
\]  
(5.12)

Then, by using Theorem 4.1 and the multiplier technique used in (5.6), a short calculation will yield the second result in Theorem 2.2 directly. \(\square\)
Now, we will give a brief proof of the decay rates of hyperbolic equations.

**Proof of Theorem 2.1.** As we said before that once suitable resolvent estimates are established, the existing abstract semi-group results can be adopted to yield the desired energy decay rate. Therefore, based on Theorem 2.2, for any positive integer \(k\), by using an abstract theorem in [6, Théorème 3], we conclude that there exists a constant \(C > 0\) such that

\[
\| e^{tA_j} (I - A_j)^{-k} U \|_H \leq \left( \frac{C}{\ln(2 + t)} \right)^k \| U \|_H, \forall t \geq 0, U \in H, k \geq 2, k \in \mathbb{N}, j = 1, 2
\]

that is

\[
\| e^{tA_j} \|_{\mathcal{L}(D(A_j^k), H)} \leq \left( \frac{C}{\ln(2 + t)} \right)^k, \forall t \geq 0, k \geq 2, k \in \mathbb{N}, j = 1, 2. \tag{5.14}
\]

On the other hand, for any \(s \in [0, 1]\) and \(k \geq 2\), by definition \(D(A_j^{sk})\) is the interpolate space of order \(s\) between \(D(A_j^0) = H\) and \(D(A_j^k) (j = 1, 2)\).

Note also that

\[
\| e^{tA_j} U \|_{\mathcal{L}(H, H)} \leq C
\]

Therefore, for any \(s \in [0, 1]\), by applying the Calderon-Lions interpolation theorem as explained in [6, 25], we have that

\[
\| e^{tA_j} \|_{\mathcal{L}(D(A_j^{sk}), H)} \leq C^{1-s} \left( \frac{C}{\ln(2 + t)} \right)^k, \forall t \geq 0, k \geq 2, k \in \mathbb{N}, j = 1, 2.
\]

Taking \(k = 2\) and \(s = \frac{1}{2}\), then

\[
\| e^{tA_j} \|_{\mathcal{L}(D(A_j^k), H)} \leq \frac{C}{\ln(2 + t)}, \forall t \geq 0, j = 1, 2. \tag{5.16}
\]

which gives the desired decay rate result (i) in Theorem 2.1. The second decay property in Theorem 2.1 is derived in a similar way.

**Acknowledgments.** The author would like to thank the anonymous referees for their helpful comments.

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Received July 2014; revised January 2015.

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