A Mathematical Model of Economic Population Dynamics in a Country That Has Optimal Zakat Management

M Subhan*
Mathematics Department, Universitas Negeri Padang, Indonesia

*13subhan@fmipa.unp.ac.id

Abstract. Zakat is the main tools against two issues in Islamic economy: economic justice and helping the poor. However, no government of Islamic countries can solve the economic disparity today. A mathematical model could give some understanding about this phenomenon. The goal of this research is to obtain a mathematical model that can describe the dynamic of economic group population. The research is theoretical based on relevance references. From the analytical and numerical simulation, we conclude that well-manage zakat and full commitment of the wealthy can achieve wealth equilibrium that represents minimum poverty.

1. Introduction
Zakat, the third Islamic pillar, may be the first pillar in Islamic economy system. Zakat is the main tools for two main sensitive issues in Islam: economic justice and providing aid for the poor. Zakat is also a base for wealth distribution and has the power to reduce poverty and other social problems [1]. The obligation for zakat is written in holy Quran more than 30 times, where 28 of them is related to obigation of pray. Some hadiths also mention the obligation. However, most of Muslim majority countries are among the poor countries in the world and for other nations the gap between the wealthy and the poor is still very deep [2].

The government’s role in collecting and redistributing zakat is very important. Prophet Muhammad as Islam leader is given order to collect zakat, while the amil manage the distribution [3]. It indicates that zakat is not managed by individual, but by a government institution since era of Osmania caliphates. Today, we can see zakat management by government in Saudi Arabia, Malaysia, Libya, Pakistan and Sudan. In other countries like Kuwait, Jordan, Iraq, Oman, Qatar, Bahrain, Bangladesh, and Indonesia, zakat is managed by some non-governmental institution and mostly by individuals.

The potential of zakat, by some researchers, may achieve 3-3.6% of Gross Domestic Product (GDP) every year in countries that have Muslim majority. Indonesia that has largest muslim population in the world have potential of zakat up to 2% of its GDP [4]. This number is large enough to overcome issues that we mention before. If zakat is distributed among various categories of recipients, the cumulative result of this annual redistribution of wealth will substantially reduce unemployment, expand investment base, eliminate poverty and extreme disparity of wealth between rich and poor in a few years [5]. But, the goals only can be achieved if the system of collection and distribution of zakat is credible, cost-effective, loophole-free and supported by strong accountability mechanisms to assure its integrity. Today, we observe that the management of zakat is not fully integrated and also the awareness of the zakat-obligatory is low in the Islam majority countries. That makes poverty and rich-poor gap still major problems in such countries.
Increasing the role of government in collecting and distributing zakat will be effective and efficient if the problem is comprehensively understood. A mathematical model can provide some better view of the problem. In this research, we build a deterministic model about the dynamic of economic population. The main concern is population of zakat-obligatory (wealthy) and zakat-receiver (poor).

2. Method
This research is theoretical research. Using relevance theories from corresponded references, we build a concept that mathematically valid in form of a mathematical model. We began the research by studying population dynamic, economic aspect and zakat. In next step, we build a deterministic mathematical model, then analyse the model, and interpreting the result. We also include a numerical simulation of the model.

3. Results

3.1. The initial model
The population will divide into four compartments, which is:

- \( X \): the number of individuals that have zakat obligation and pay it.
- \( Y \): the number of individuals that have zakat obligation and neglect it.
- \( Z \): the number of individuals that receive zakat.
- \( W \): the number of individuals that neither zakat-obligatory nor zakat-receiver; the ‘neutral’ population.

Total population is \( N \) where \( N = X + Y + Z + W \)

Some assumptions that are used in this model are:

a. The population is in the region that has integrated zakat management (government institution) with orientation to weak economy empowerment.

b. Every wealthy individual cannot be poor immediately, but may become ‘neutral population.

c. The change from individuals obeying to neglecting zakat happen when they interacted one to another, and also the contrary.

d. Zakat-receiver will have possibility of positive economic increasing. Even though some of the portion of the zakat goes to consumption, some portion are used in productive investment [6].

e. The change from zakat-obligatory to ‘neutral or ‘neutral’ to zakat-receiver are caused by economic recession. On the contrary, from zakat-receiver to ‘neutral or ‘neutral’ to zakat-obligatory is affected by economic growth.

f. Total population is constant because birth rate equals to death rate and effect of migration is ignored.

g. The rate and probability of change from zakat-obligatory that paying into neglecting zakat is constant, and also the contrary.

Based on the assumptions, we have some parameters in this model:

Parameter:

a. Rate of change from zakat-obligatory who neglect zakat to pay zakat affected by interaction among them is \( \beta \). The value is positive if the zakat payers have better influence, and negative if the contrary.

b. Proportion of zakat payer to non-payer for new wealth is \( \rho \).

c. Rate of change from zakat-obligatory who neglect zakat to ‘neutral’ compartment affected by penalty from the government when they caught breaking the rule is \( \delta \).

d. Rate of change from zakat-receiver to ‘neutral’ compartment affected by optimal use of zakat is \( \alpha \).

e. Birth rate and also death rate is represented by \( \mu \).
f. Rate of change of each population affected by economic growth, equal for all populations, is $\varepsilon$. Some research shows that growth can reduce poverty [7,8], but others show that growth also increases poverty [9].

g. Rate of business failures is $\gamma$.

From the assumptions, we build the first initial model as follows:

$$
\frac{dX}{dt} = \varepsilon W + \beta \left( \frac{X}{N} \right) Y - (\gamma + \mu)X
$$

$$
\frac{dY}{dt} = \alpha (1 - \rho) W - \beta \left( \frac{X}{N} \right) Y - (\delta + \gamma + \mu)Y
$$

$$
\frac{dZ}{dt} = \gamma W - (\varepsilon + \alpha + \mu) Z
$$

$$
\frac{dW}{dt} = \mu N + \gamma (X + Y - W) + \delta Y + (\varepsilon + \alpha) Z - (\varepsilon + \mu)W
$$

(1)

Since the population is constant, we can simplify the model into just three equations. Using these new scaling variables

$$
P = \frac{X}{N}, U = \frac{Y}{N}, R = \frac{Z}{N}, S = \frac{W}{N}
$$

we have $S = 1 - P - U - R$. Our concern is three variables $P$ (proportion of zakat-payer population), $U$ (proportion of not paying zakat population), and $R$ (proportion of zakat-receiver). So, the new model is:

$$
\frac{dP}{dt} = \varepsilon (1 - P - U - R) + \beta PU - (\gamma + \mu)P
$$

$$
\frac{dU}{dt} = \alpha (1 - \rho) (1 - P - U - R) - \beta PU - (\delta + \gamma + \mu)U
$$

$$
\frac{dR}{dt} = \gamma (1 - P - U - R) - (\varepsilon + \alpha + \mu) R
$$

In this model, all variables are feasible on $[0,1]$. We validate the model on boundaries, and we have all $\frac{dP}{dt}, \frac{dU}{dt}, \frac{dR}{dt}$ equals zero when the value of the variable is zero, and all $\frac{dP}{dt}, \frac{dU}{dt}, \frac{dR}{dt}$ less than zero when the value of the variable is one. We can conclude that the model is well-defined and we can use the model.

3.2. Analyse of the model in case of full obedience

Since zakat has been mandatory, in sharia law countries, the population of non-paying zakat is almost zero. We have $U = 0, P = 1$ and $\delta = 0$ for this situation and our model will be as follows

$$
\frac{dP}{dt} = \alpha (1 - P - R) - (\gamma + \mu)P
$$

$$
\frac{dR}{dt} = \gamma (1 - P - R) - (\varepsilon + \alpha + \mu) R
$$

This system of ordinary differential equation has one fixed point

$$
P = \frac{\alpha (\varepsilon + \alpha + \mu)}{(\alpha + \varepsilon)(\gamma + \varepsilon + \mu) + \varepsilon \mu + (\gamma + \mu)^2}
$$

$$
R = \frac{\gamma (\gamma + \mu)}{(\alpha + \varepsilon)(\gamma + \varepsilon + \mu) + \varepsilon \mu + (\gamma + \mu)^2}
$$

Linearizing the system around the fixed point we have the Jacobian matrix:

$$
\begin{bmatrix}
-(\gamma + \varepsilon + \mu) & -\varepsilon \\
-\gamma & -(\gamma + \alpha + \varepsilon + \mu)
\end{bmatrix}
$$

The eigenvalues of the matrix are

$$
-\frac{1}{2} (2\gamma + \alpha + 2\varepsilon + 2\mu) \pm \frac{1}{2} \sqrt{\alpha^2 + 4\varepsilon \gamma}
$$
When \( \varepsilon > 0 \), both eigenvalues are negative in their real parts. So, the fixed point is stable asymptotic.

If we consider to the reducing poverty effort, then the threshold value of \( dR \) is when

\[
\frac{S}{R} = \frac{\alpha + \varepsilon + \mu}{\gamma}
\]

It means that proportion of zakat-receiver (poor) will decrease when ratio between \( S \) and \( R \) is below \( \frac{\alpha + \varepsilon + \mu}{\gamma} \) and will increase when the ratio is above \( \frac{\alpha + \varepsilon + \mu}{\gamma} \).

3.3. Numerical simulation

Using the following set of parameter values

\[
\alpha = 0.000273973/\text{day}, \quad \gamma = 0.000342466/\text{day}, \quad \varepsilon = 0.001369863/\text{day}, \quad \mu = 0.000547945/\text{day},
\]

we have the fixed point \((0.5709187875, 0.05798397965)\) which represent proportion of zakat-payer (wealthy) and zakat-receiver (poor) are respectively around 57% and 5.8%. Figures 1 and 2 show the trajectory of the dynamical system confirm our analysis before.

![Figure 1](image1.png)  
**Figure 1.** Trajectory of \( R \) over 100 years when annual economic growth is 5% for initial value \( P(0) = 0.05, R(0) = 0.15 \) and \( P(0) = 0.02, R(0) = 0.7 \).

![Figure 2](image2.png)  
**Figure 2.** Trajectory of \( P \) over 100 years when annual economic growth is 5% for initial value \( P(0) = 0.05, R(0) = 0.15 \) and \( P(0) = 0.02, R(0) = 0.7 \).

When the economic growth rather low, halving the \( \varepsilon \), the dynamic’s behaviour quite the same, but the proportion of the wealth and poor become about 38% and 11% respectively. The trajectories in Figures 3 and 4 shows that the fixed point also stable.
4. Conclusion
The model that we formed shows that if zakat is collected well and managed optimally, then the country will achieve its wealth equilibrium that represents the minimum numbers of poor population. Even though the poverty cannot be diminished, the numbers can be keep low. Model analysis and numerical simulation, for case fully obedience of paying zakat, confirm that zakat can contribute positively in poverty alleviation. For other case, we still need to analyze the model to have additional conclusion.

References
[1] Kusuma D B W and Sukmana R 2010 The power of zakah in poverty alleviation. Proc. 7th International Conference The Tawhidi Epistemology
[2] Kuzudisli A 2017 Fight against poverty from the Islamic point of view: the wealth distribution and shareEconomics World Vol. 519-15
[3] Dogarawa, AB 2008 Islamic social welfare and the role of zakah, MPRA Paper No. 23192A
[4] Aisyah M 2014 The role of zakah and binary economics in poverty reduction. J. Bisnis dan Management Vol 42.
[5] Quraishi M A 1999 The Institution of Zakat and Its Economic Impact on SocietyProc. of the Second Harvard University Forum on Islamic Financepp.77-81
[6] Abdelbaki H H 2013 The Impact of Zakat on Poverty and Income Inequality in Bahrain Review of Integrative Bussiness and Economy Research Vol 21 pp 133-154
[7] Mahat N I and Warokka A 2013 Investigation on zakat as an indicator for moslem countries’ economic growth. J. Global Business Advancement Vol. 61
[8] Sboui F2012Effects of growth and inequality on poverty in Tunisia. Region et Developpement 3557-80