Matter Enhancement of T Violation in Neutrino Oscillation

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– Abstract –

We study the matter enhancement of T violation in neutrino oscillation with three generations. The magnitude of T violation is proportional to Jarlskog factor $J$. Recently, the elegant relation, $(\Delta m_{12}^2)(\Delta m_{23}^2)(\Delta m_{31}^2)J_m = \Delta_{12}\Delta_{23}\Delta_{31}J$, was derived, where $\Delta_{ij} = \Delta m_{ij}^2/2E$ and subscript $m$ implies the quantities in matter. Using this relation, we reconsider how $J_m$ changes as a function of the matter potential $a$ under the approximation $|\Delta m_{12}^2| \ll |\Delta m_{13}^2|$. We show that the number of maxima for $J_m$ depends on the magnitude of $\sin^2 2\theta_{13}$ and there are two maxima considering the constraint on $\sin^2 2\theta_{13}$ from the CHOOZ experiment. One maximum of $J_m$ at $a = O(\Delta_{12})$ is given by $J/\sin 2\theta_{12}$, which leads to the large enhancement of $J_m$ in the case of the SMA MSW solution. The other maximum at $a = O(\Delta_{13})$ is $|\Delta_{12}/\Delta_{13}|J/\sin 2\theta_{13}$, and the enhancement is possible, if $\sin 2\theta_{13}$ is small enough. These maximal values are consistent with the results obtained by other methods.

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1 Introduction

Solar neutrino experiments have been observing a $\nu_e$ deficit for a long time [1] and the ratio of $\nu_\mu/\nu_e$ in atmospheric neutrino has implied a $\nu_\mu$ deficit [2], which are explained by $\nu_e-\nu_\mu$ oscillation and $\nu_\mu-\nu_\tau$ oscillation, respectively. These experiments provide strong evidence that there exist masses and mixings in the lepton sector with three generations [3].

Long baseline experiments [4] and neutrino factories [5] are operated, or planned, in order to obtain more convincing evidence for neutrino oscillation. Furthermore it could also be possible to observe CP and T violations.

As the neutrinos pass through the earth in these experiments, matter effects must be considered. It has been studied in the context of long baseline experiments [6], and in the context of a neutrino factory [7]. T violation is different from CP violation in matter and it is pointed out that it is easy to calculate T violation compared with CP violation for neutrino oscillation in matter [8]. The T violating part in matter, $\Delta P_T = P(\nu_\alpha \to \nu_\beta) - P(\nu_\beta \to \nu_\alpha)$, $(\alpha, \beta = e, \mu, \tau)$ is proportional to Jarlskog factor $J_m$ [9] of the lepton sector, unlike the CP violating part. The dependence of $J_m$ on the matter potential $a = \sqrt{2}G_FN_e$ is investigated in other works [10, 11].

Recently, Harrison and Scott [12] derived the relation

$$ (\Delta_m)_{12}(\Delta_m)_{23}(\Delta_m)_{31}J_m = \Delta_{12}\Delta_{23}\Delta_{31}J, $$

where $\Delta_{ij} = \Delta m^2_{ij}/(2E)$ and the quantities with the subscript $m$ are those in matter. The inverse of $J_m$ is the square root of a quartic function of $a$. This means that $J_m$ has either one or two local maxima as a function of $a$.

In this letter, we present both the exact and approximate form of $J_m$ as a function of $a$ using the above relation. It is shown that the number of resonant maxima of $J_m$ depends on the magnitude of $\sin^22\theta_{13}$. Taking account of the constraint on $\sin^22\theta_{13}$ from the CHOOZ experiment [13], we show that there exist two maxima. We also estimate the maximal values of $J_m$ in the cases of small mixing angle (SMA) and large mixing angle (LMA) MSW solutions [14].

2 T Violation in Neutrino Oscillation

We review T violation in three-neutrino oscillations and state the strategy of this letter. In vacuum, flavor eigenstates $\nu_\alpha(\alpha = e, \mu, \tau)$ are related to mass eigenstates $\nu_i(i = 1, 2, 3)$, which have the mass eigenvalues $m_i$, by the unitary transformation,

$$ \nu_\alpha = U_{\alpha i}\nu_i, $$

where $U_{\alpha i}$ is the Maki-Nakagawa-Sakata matrix [3]. The T violating part, $\Delta P_T(\nu_\alpha \to \nu_\beta) \equiv P(\nu_\alpha \to \nu_\beta) - P(\nu_\beta \to \nu_\alpha)$, in three generation after traveling a distance $L$ is calculated as

$$ \Delta P_T = 16J \sin \frac{\Delta_{12}L}{2} \sin \frac{\Delta_{23}L}{2} \sin \frac{\Delta_{31}L}{2}, $$

1
where
\[ J \equiv \text{Im}[U_{\alpha i}U_{\beta j}^* U_{\alpha i}^* U_{\beta j}]. \]  
(4)

In order to obtain the T violating part in matter, we only have to replace \( \Delta_{ij} \rightarrow (\Delta_m)_{ij} \), \( U_{\alpha i} \rightarrow (U_m)_{\alpha i} \), hence, \( J \rightarrow J_m \).

We would like to study the case where large \( \Delta P_T \) is realized. In eq. (3), \( \Delta P_T \) is a product of \( J_m \) and trigonometric functions. In the following calculation, we focus on the matter effect of \( J_m \) which does not depend on \( L \) and determine the maxima of \( J_m \).

As seen in eq. (4), \( J \) consists of the product of \( U_{\alpha i} \). It is complicated to calculate \( J_m \) directly from \( (U_m)_{\alpha i} \), which diagonalizes the matter-modified Hamiltonian \( H_m \), although the numerical calculation has been performed [10]. However, it is possible to calculate \( J_m \) without direct calculation of \( (U_m)_{\alpha i} \) from the relation
\[
(\Delta_m)_{12}(\Delta_m)_{23}(\Delta_m)_{31} J_m = \Delta_{12}\Delta_{23}\Delta_{31} J, \tag{5}
\]
derived by Harrison and Scott [12]. Since the right hand-side of eq. (5) is a constant which does not depend on the matter effect, \( J_m \) is inversely proportional to a triple product of \( (\Delta_m)_{ij} \). Therefore, we study the function of the matter potential \( a \) such as
\[
f(a) \equiv [(\Delta_m)_{12}(\Delta_m)_{23}(\Delta_m)_{31}]^2, \tag{6}
\]
and determine the minima of \( f(a) \).

### 3 Triple Product of Mass Square Differences

In this section, we study the matter effect on \( f(a) \). Harrison and Scott [12] suggest that \( f(a) \) is a quartic function of the matter potential \( a \) and in principle its coefficients can be written by the parameters \( \Delta m^2_{ij} \) and \( U_{\alpha i} \) in vacuum, although it is complicated in practice. We present the exact form of \( f(a) \) in relatively simple form by introducing new parameters. The coefficients of \( f(a) \) is further simplified under the approximation \( |\Delta m^2_{12}| \ll |\Delta m^2_{33}| \).

First, let us note that \( (\Delta_m)_{ij} \) included in \( f(a) \) are rewritten by the eigenvalues \( (\lambda_m)_i \) of the matter-modified Hamiltonian \( H_m \) as \( (\Delta_m)_{ij} = (\lambda_m)_j - (\lambda_m)_i \). The eigenvalues \( (\lambda_m)_i \) are the solutions of the equation for \( t \),
\[
\det(H_m - t) = (\lambda_1 - t)(\lambda_2 - t)(\lambda_3 - t) + a(t - \delta_2)(t - \delta_3) = 0, \tag{7}
\]
where \( \delta_i (i = 2, 3) \) and \( \lambda_i (i = 1, 2, 3) \) are the eigenvalues of the 2×2 submatrix \( H_{ij} (i, j = 2, 3) \) and 3×3 matrix \( H \) in vacuum. After a calculation, \( f(a) \) is expressed as a quartic function of \( a \):
\[
f(a) = f_4 a^4 + f_3 a^3 + f_2 a^2 + f_1 a + f_0, \tag{8}
\]
where the coefficients \( f_i \ (i = 0, \ldots, 4) \) are presented by \( \lambda_i \) and \( \delta_i \) in the following.
The coefficients $f_4$ and $f_0$ are

$$f_4 = (\delta_2 - \delta_3)^2,$$
$$f_0 = \{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_3)\}^2.$$  \hfill (10)  

By definition (6), $f(a)$ is semi-positive definite, hence, $f_4, f_0 \geq 0$ must be satisfied taking account of the limit $a \to \infty$ and $a \to 0$. The relations (10) and (11) are consistent with these conditions.

The other coefficients are

$$f_3 = 2[(\delta_2 - \lambda_1)(\delta_2 - \lambda_2)(\delta_2 - \lambda_3) + (\delta_3 - \lambda_1)(\delta_3 - \lambda_2)(\delta_3 - \lambda_3)]$$
$$-2(\delta_2 - \delta_3)^2[(\delta_2 - \lambda_1) + (\delta_3 - \lambda_1) + \text{cyclic of } \lambda_i],$$  \hfill (12)  

$$f_2 = [(\delta_2 - \lambda_1)(\delta_3 - \lambda_2) + \text{cyclic of } \lambda_i]^2$$
$$-6[(\delta_2 - \lambda_1)(\delta_2 - \lambda_2)[(\delta_3 - \lambda_1) + (\delta_3 - \lambda_2)](\delta_3 - \lambda_3) + \text{cyclic of } \lambda_i],$$  \hfill (13)  

$$f_1 = 4[(\delta_2 - \lambda_2)(\delta_3 - \lambda_2)(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_1) + \text{cyclic of } \lambda_i]$$
$$+ 2(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_3)[(\delta_2 - \lambda_1)(\delta_3 - \lambda_2) + \text{cyclic of } \lambda_i],$$  \hfill (14)  

which are relatively simple compared with the case where we don’t introduce new parameters $\delta_i$. In section 5, we present the figures using these coefficients.

Next, let us show that these coefficients are further simplified under the approximation $|\Delta_{12}| \ll |\Delta_{13}|$. As $\delta_i$ are the eigenvalues of submatrix

\[
\begin{pmatrix}
H_{22} & H_{23} \\
H_{32} & H_{33}
\end{pmatrix}
= \lambda_1 \mathbf{1} + \Delta_{13} \begin{pmatrix}
|U_{\mu 3}|^2 & U_{\mu 3}U_{\tau 3}^* \\
U_{\tau 3}U_{\mu 3}^* & |U_{\tau 3}|^2
\end{pmatrix} + \Delta_{12} \begin{pmatrix}
|U_{\mu 2}|^2 & U_{\mu 2}U_{\tau 2}^* \\
U_{\tau 2}U_{\mu 2}^* & |U_{\tau 2}|^2
\end{pmatrix},
\]  \hfill (15)  

where $\mathbf{1}$ is the unit matrix, they are approximated by

$$\delta_2 = \lambda_1 + \frac{|U_{e 1}|^2 \Delta_{12}}{1 - |U_{e 3}|^2}, \quad \delta_3 = \lambda_1 + (1 - |U_{e 3}|^2)\Delta_{13} + \frac{|U_{e 2}|^2 |U_{e 3}|^2 \Delta_{12}}{1 - |U_{e 3}|^2},$$  \hfill (16)  

up to the first order of $\Delta_{12}$ using the unitarity condition. Substituting eq. (16) for $\delta_i$ in eqs. (10)~(14) and taking the standard parameterization, $U_{e 1} = c_{12}c_{13}$, $U_{e 2} = s_{12}c_{13}$, $U_{e 3} = s_{13}e^{-i\delta}$, where $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$, the coefficients are calculated as

$$f_4 \simeq c_{13}^4(\Delta_{13})^2, \quad f_3 \simeq -2c_{13}^4 \cos 2\theta_{13}(\Delta_{13})^3, \quad f_2 \simeq c_{13}^4(\Delta_{13})^4, \quad f_1 \simeq -2c_{13}^4 \cos 2\theta_{13}\Delta_{12}(\Delta_{13})^4, \quad f_0 \simeq (\Delta_{12})^2(\Delta_{13})^4,$$  \hfill (17)  

at the leading order. Note that the order of $\Delta_{12}$ for $f_i$ is important when we determine the minima of $f(a)$. $f_1$ is the first order of $\Delta_{12}$ and $f_2, f_3, f_4$ are the zeroth order. Its difference determines the magnitude of $a$ for each minima.

4 Matter Enhancement of the Jarlskog Factor

In this section, we calculate the minima of $f(a)$ using the coefficients (17) in order to determine the maxima of $J_m$. First, we show that the number of minima depends
on the magnitude of $\sin^2 2\theta_{13}$, and that there are two minima taking account of the constraint on $\sin^2 2\theta_{13}$ from the CHOOZ experiment. Second, we estimate the maximal values of $J_m$ and the energies of the neutrino at maxima in the cases of the SMA and LMA MSW solutions.

Let us start with differentiating $f(a)$ in terms of $a$:

$$f(a)' = 4f_4a^3 + 3f_3a^2 + 2f_2a + f_1 = 0.$$ (18)

Since only $f_1$ is $O(\Delta_{12})$ in $f_i(i = 1, 2, 3, 4)$ from eq. (17), in the limit of $\Delta_{12} \to 0$, eq. (18) reduces to

$$a(4f_4a^2 + 3f_3a + 2f_2) = 0.$$ (19)

Hence, there exists a solution at $a = 0$ in this limit. This means that a solution at $a = O(\Delta_{12})$ exists for $\Delta_{12} \neq 0$.

On the other hand, whether another minimum exists or not is determined by the discriminant $D$ of the quadratic equation in the parenthesis of eq. (19),

$$D = 9f_3^2 - 32f_4f_2 = 4c_{13}^8(\Delta_{13})^{12}(1 - 9\sin^2 2\theta_{13}).$$ (20)

If $\sin^2 2\theta_{13} > 1/9$, there exists only one minimum at $a = O(\Delta_{12})$ as Fig. 1 (a). If $\sin^2 2\theta_{13} < 1/9$, then there exists another minimum at $a = O(\Delta_{13})$ as Fig. 1 (b). The restriction of the CHOOZ experiment, $\sin^2 2\theta_{13} \leq 0.10$ [13], is included in the case of Fig. 1 (b).

![Diagram](image)

Fig. 1. $f(a)$ has two(one) local minima for $D > 0$ ($D < 0$). The CHOOZ experiment favors $D > 0$ and the figure (b).

(I) The maximal value of $J_m$ at $a = O(\Delta_{12})$

The solution of eq. (18) at $a = O(\Delta_{12})$ is

$$a = \frac{\cos 2\theta_{12}}{\cos^2 \theta_{13}}\Delta_{12}.$$ (21)

The minimal value is

$$f(a) = \sin^2 2\theta_{12}(\Delta_{12})^2(\Delta_{13})^4,$$ (22)
and thus, from eq. (5), the maximum of the ratio is given by
\[
\frac{J_m}{J} = \frac{1}{\sin 2\theta_{12}},
\] (23)
which is consistent with other works [10, 11]. This means that \(J_m\) is largely enhanced in the case of the SMA MSW solution.

We estimate \(J_m/J\) in two MSW solutions:
\[
\frac{J_m}{J} = \begin{cases} 
  12, & \text{for SMA MSW}, \\
  1.1, & \text{for LMA MSW}, 
\end{cases}
\] (24)
where we use \(\sin^2 2\theta_{12} = 7.2 \times 10^{-3}\) (SMA MSW), 0.79 (LMA MSW) [15].

The neutrino energy corresponding to the maximum of \(J_m/J\), from eq. (21), is
\[
E = \frac{\cos 2\theta_{12} \Delta m^2_{12}}{2\sqrt{2} G_F N_e c^2_{13}},
\] (25)
where \(N_e\) is the electron number density: \(N_e = 8.2 \times 10^{23}\text{cm}^{-3}\) in the earth’s crust. Substituting the experimental data, it is obtained as
\[
E = \begin{cases} 
  25 \text{ MeV}, & \text{for SMA MSW}, \\
  62 \text{ MeV}, & \text{for LMA MSW}, 
\end{cases}
\] (26)
where we use \(\Delta m^2_{12} = 5.0 \times 10^{-6}\) (SMA MSW), \(2.7 \times 10^{-5}\) (LMA MSW), and \(\sin^2 2\theta_{13} = 0.10\) (the upper limit of the CHOOZ experiment).

**II** The maximal value of \(J_m\) at \(a = O(\Delta_{13})\)

The other solutions of eq. (18) at \(a = O(\Delta_{13})\) are
\[
a = \frac{1}{4} \left( 3 \cos 2\theta_{13} \pm \sqrt{1 - 9 \sin^2 2\theta_{13}} \right) \Delta_{13},
\] (27)
where the sign + for \(\cos 2\theta_{13} \geq 0\) and the sign − for \(\cos 2\theta_{13} \leq 0\). The minimal value is
\[
f(a) = \frac{c^4_{13} (\Delta_{13})^6}{32} \left[ 4 - 3(1 - 3 \sin^2 2\theta_{13})^2 - \cos 2\theta_{13}(1 - 9 \sin^2 2\theta_{13})^2 \right]
\] (28)
and the maximum of the ratio is given by
\[
\frac{J_m}{J} = \frac{\Delta_{12}}{\Delta_{13}} \frac{1}{c^2_{13}} \frac{4 \sqrt{2}}{\sqrt{4 - 3(1 - 3 \sin^2 2\theta_{13})^2 - \cos 2\theta_{13}(1 - 9 \sin^2 2\theta_{13})^2}}.
\] (29)
Because of the suppression factor \(|\Delta_{12}/\Delta_{13}|\), the enhancement of \(J_m\) is small compared with the case (I).

Furthermore, we can obtain more simple forms for eq. (27) and eq. (29) under the approximation \(9 \sin^2 2\theta_{13} \ll 1\), although this approximation is not justified near the
upper limit, $\sin^2 2\theta_{13} \simeq 0.10$, of the CHOOZ experiment. In this case, the value of $a$ for the maximum of $J_m$ is

$$a = \left(1 - \frac{3}{2} \sin^2 2\theta_{13}\right) \Delta_{13}$$

(30)

and the ratio is

$$\frac{J_m}{J} = \left|\frac{\Delta_{12}}{\Delta_{13}}\right| \frac{1}{\sin 2\theta_{13}}.$$  \hspace{1cm} (31)

It is understood from this result that the enhancement of $J_m$ is not always realized because of the suppression factor $|\Delta_{12}/\Delta_{13}|$. However, we still have an enhancement for small $\sin 2\theta_{13}$. For example, at $\sin^2 2\theta_{13} = 4.0 \times 10^{-6}$ which corresponds to $\sin \theta_{13} = 1.0 \times 10^{-3}$ and which is much smaller than the present upper limit, the maximum of the ratio is given by

$$\frac{J_m}{J} = \begin{cases} 0.78, & \text{for SMA MSW,} \\ 4.2, & \text{for LMA MSW,} \end{cases}$$

(32)

where we use the experimental values for $\Delta m^2_{23} = 3.2 \times 10^{-3}\text{eV}^2$. Thus, $J_m$ for the LMA MSW solution has an enhancement which is several times as large as $J$ in this example.

The neutrino energy corresponding to the maximum of $J_m$ is calculated from eq. (30),

$$E = \left(1 - \frac{3}{2} \sin^2 2\theta_{13}\right) \frac{\Delta m^2_{13}}{2\sqrt{2}G_FN_e}.$$  \hspace{1cm} (33)

Substituting the same experimental data as before we obtain

$$E = 16 \text{ GeV}.$$  \hspace{1cm} (34)

We summarize the above two maxima of $J_m$ in Table 1.

| $a$ | $J_m/J$ and $E$ | SMA MSW | LMA MSW |
|-----|-----------------|---------|---------|
| $O(\Delta_{12})$ | $J_m/J$ | $1/\sin 2\theta_{12}$ | 12 | 1.1 |
| $E$ | $\cos 2\theta_{12} \Delta m^2_{12}/(2\sqrt{2}G_FN_e c^2_{13})$ | 24MeV | 60MeV |
| $O(\Delta_{13})$ | $J_m/J$ | $\Delta m^2_{12}/(\Delta m^2_{13} \sin 2\theta_{13})$ | 0.78 | 4.2 |
| $E$ | $[1 -(3/2)\sin^2 2\theta_{13}] \Delta m^2_{13}/(2\sqrt{2}G_FN_e)$ | 16GeV | 16GeV |

Table 1: The maxima of $J_m/J$ and the neutrino energies $E$. Input parameters for vacuum are the same as in the text, and $\sin^2 2\theta_{12} = 4.0 \times 10^{-6}$ is chosen.

5 Numerical Estimation of the Ratio $J_m/J$

In this section, we numerically study the dependence of the ratio $J_m/J$ on the neutrino energy $E$ using eqs. (9)–(15). First, we illustrate the magnitude of maximum for $J_m$ taking account of $\sin^2 2\theta_{12}$ and $\Delta m^2_{12}$ given by two MSW solutions and the
constraint on $\sin^2 2\theta_{13}$ from the CHOOZ experiment. Second, we study the effect of the signs of $\Delta_{12}$ and/or $\Delta_{13}$. In the following calculation, input parameters are the same as in the previous section and we restrict to the range $0 < \theta_{ij} \leq \pi/4$ for simplicity.

Let us show the energy $E$ dependence of $J_m/J$ in the cases of the SMA and LMA MSW solutions with $\sin \theta_{13} = |U_{e3}| = 0.16$ and $|U_{e3}| = 1.0 \times 10^{-3}$ in Fig. 2 (a)∼(d) 1.

Fig. 2. The neutrino energy $E$ dependence of $J_m/J$ in the cases of the SMA and LMA MSW solutions with $|U_{e3}| = 0.16$ and $|U_{e3}| = 1.0 \times 10^{-3}$. The symbol + denotes the maxima determined in the previous section.

Comparing Fig. 2 (a) with (b) (or Fig. 2 (c) with (d)), we conclude that the SMA MSW solution has larger enhancement than the LMA MSW solution has for the maximum of $J_m/J$ at $E = O(10\text{MeV})$. The enhancement, $J_m/J > 1$, occurs in the wide energy region around this maximum and the values calculated numerically almost coincide with the results (24) and (26) obtained approximately. Next, comparing Fig. 2 (a) and (c) (or Fig. 2 (b) and (d)), we conclude that if $\sin \theta_{13}$ is small enough, the enhancement for the maximum of $J_m/J$ at $E = O(10\text{GeV})$ is possible although the energy region is small.

Next, we study the cases where $\Delta m_{12}^2$ and/or $\Delta m_{13}^2$ is negative. Since we have implicitly assumed that both $\Delta m_{12}^2$ and $\Delta m_{13}^2$ are positive until now, two maxima appear in “neutrino” oscillation. However, exactly speaking, whether the maxima appear in “neutrino” oscillation or “anti-neutrino” oscillation depends on the signs of

1The energy dependence of $J_m/J$ for a LMA MSW solution with $|U_{e3}| = 0.090$ (corresponding to Fig. 2(b)) is shown in ref.[10].

7
In order to examine such cases, we numerically calculate $J_m/J$ in the cases where $\Delta m_{12}^2$ and $\Delta m_{13}^2$ respectively, are positive and/or negative, and show the results for the SMA MSW solution as an example in Fig. 3.

Comparing Fig. 3 (a) with (b) (or Fig. 3 (c) with (d)), we conclude that the appearance of the maximum for $J_m/J$ at $E = O(10\text{MeV})$ depends on the sign of $\Delta m_{12}^2$. Although the maximum appears in “neutrino” oscillation in the case of $\Delta m_{12}^2 > 0$, it appears in “anti-neutrino” oscillation in the case of $\Delta m_{12}^2 < 0$. Comparing Fig. 3 (a) with (c) (or Fig. 3 (b) with (d)), we conclude that the appearance of the maximum for $J_m/J$ at $E = O(10\text{GeV})$ depends on the sign of $\Delta m_{13}^2$. The maximum appears in “neutrino” oscillation in the case of $\Delta m_{13}^2 > 0$. On the other hand, it appears in “anti-neutrino” oscillation in the case of $\Delta m_{13}^2 < 0$.

These differences originate from the fact that the matter potential $a$ for the maxima of $J_m$ (see eqs. (21) and (27)) is proportional to $\Delta m_{ij}^2$. If $\Delta m_{ij}^2$ is negative, then $a$ is also negative and $J_m$ has the maximum not in “neutrino” oscillation but in “anti-neutrino” oscillation. This is because the matter-modified Hamiltonian for anti-neutrino is obtained by replacing $a \to -a$. 

Fig. 3. The dependence of $J_m$ on the sign of $\Delta m_{ij}^2$. (a) and (c) are for $\Delta m_{12}^2 > 0$, (b) and (d) for $\Delta m_{12}^2 < 0$. (a) and (b) are for $\Delta m_{13}^2 > 0$, (c) and (d) for $\Delta m_{13}^2 < 0$. The other conditions are the same as in Fig. 2(a).
6 Summary and Discussions

In this letter, we have studied matter modified Jarlskog factor \( J_m \) which appears in \( T \) violation for the lepton sector in neutrino oscillation. It was shown \([12]\) that the inverse of \( J_m \) is proportional to the square root of a quartic polynomial of matter potential \( a \) from the relation

\[
(\Delta m_{12})^2 (\Delta m_{23}) (\Delta m_{31}) J_m = \Delta_{12} \Delta_{23} \Delta_{31} J.
\]

We have presented the exact form of this polynomial with parameters in vacuum and have reconsidered the matter enhancement of \( J_m \) under the approximation \(|\Delta m_{21}| \ll |\Delta m_{31}|\).

We show that \( J_m \) has \( i \) one maximum at \( a = O(\Delta_{12}) \) in the case of \( \sin^2 2\theta_{13} \geq 1/9 \) and \( ii \) two maxima at \( a = O(\Delta_{12}) \) and \( a = O(\Delta_{13}) \) in the case of \( \sin^2 2\theta_{13} < 1/9 \). Considering the constraint on \( \sin^2 2\theta_{13} \) from the CHOOZ experiment, we conclude that the case \( ii \) is realized.

One maximum of \( J_m \) at \( a = (\cos 2\theta_{12}/\cos^2 \theta_{13}) \Delta_{12} \) is given by \( J/J \sin 2\theta_{12} \). \( J_m/J \) is roughly estimated as 12 for the SMA MSW, thus large enhancement is realized.

The other maximum at \( a = (1 - 3/2 \sin^2 2\theta_{13}) \Delta_{13} \) is given by \(|\Delta_{12}/\Delta_{13}| J/J \sin 2\theta_{13} \) for \( \sin^2 2\theta_{13} \ll 1/9 \). If \( \theta_{13} \) is small enough, the ratio \( J_m/J \) is enhanced. We have roughly estimated \( J_m/J \) as 4.2 for the LMA MSW solution at \( \sin^2 2\theta_{13} = 4 \times 10^{-6} \).

In the case of \( \sin^2 2\theta_{13} = 4 \times 10^{-6} \), our results agree with the results obtained by a different method \([11]\). Our results are also applicable around \( \sin^2 2\theta_{13} \simeq 0.10 \) which is the upper limit from the CHOOZ experiment.

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