New indications on the Higgs boson mass from lattice simulations

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Abstract

The ‘triviality’ of $\Phi_4^4$ has been traditionally interpreted within perturbation theory where the prediction for the Higgs boson mass depends on the magnitude of the ultraviolet cutoff $\Lambda$. This approach crucially assumes that the vacuum field and its quantum fluctuations rescale in the same way. The results of the present lattice simulation, confirming previous numerical indications, show that this assumption is not true. As a consequence, large values of the Higgs mass $m_H$ can coexist with the limit $\Lambda \to \infty$. As an example, by extrapolating to the Standard Model our results obtained in the Ising limit of the one-component theory, one can obtain a value as large as $m_H = 760 \pm 21$ GeV, independently of $\Lambda$. 
The ‘triviality’ of $\Phi^4$ theories in 3+1 space-time dimensions \cite{1} is generally interpreted within perturbation theory. In this interpretation, these theories represent just an effective description, valid only up to some cutoff scale $\Lambda$. Without a cutoff, the argument goes, there would be no scalar self-interactions and without them no symmetry breaking.

This conventional view extends to any number of scalar field components and, when used in the Standard Model, leads to predict that the Higgs boson mass squared, $m_H^2$, is proportional to $g_R v_R^2$, where $v_R$ is the known weak scale (246 GeV) and $g_R \sim 1/\ln \Lambda$ is the renormalized scalar self-coupling. Therefore, the ratio $m_H/v_R$ would be a cutoff-dependent quantity that becomes smaller and smaller when $\Lambda$ is made larger and larger.

By accepting the validity of this picture, there are important phenomenological implications. For instance, a precise measurement of $m_H$, say $m_H = 760 \pm 21$ GeV, would constrain the possible values of $\Lambda$ to be smaller than about 2 TeV.

In an alternative approach \cite{2,3}, however, this conclusion is not true. The crucial point is that the ‘Higgs condensate’ and its quantum fluctuations undergo different rescalings when changing the ultraviolet cutoff. Therefore, the relation between $m_H$ and the physical $v_R$ is not the same as in perturbation theory.

To better clarify the issue, we observe that, beyond perturbation theory, in a broken-symmetry phase, there are two different definitions of the field rescaling. There is a rescaling of the ‘condensate’, say $Z \equiv Z_\varphi$, and a rescaling of the fluctuations, say $Z \equiv Z_{\text{prop}}$.

To this end, let us consider a one-component scalar theory and introduce the bare expectation value $v_B = \langle \Phi_{\text{latt}} \rangle$ associated with the ‘lattice’ field as defined at the cutoff scale. By $Z \equiv Z_\varphi$ we mean the rescaling that is needed to obtain the physical vacuum field $v_R = v_B/\sqrt{Z_\varphi}$. By physical, we mean that the quadratic shape of the effective potential $V_{\text{eff}} = V_{\text{eff}}(\varphi_R)$, evaluated at $\varphi_R = \pm v_R$, is precisely given by $m_H^2$. Since the second derivative of the effective potential is the zero-four-momentum two-point function, this standard definition is equivalent to define $Z_\varphi$ as

$$Z_\varphi = m_H^2 \chi_2(0)$$

where $\chi_2(0)$ is the zero-momentum susceptibility.

On the other hand, $Z \equiv Z_{\text{prop}}$ is determined from the residue of the connected propagator on its mass shell. Assuming ‘triviality’ and the Källen-Lehmann representation for the shifted quantum field, one predicts $Z_{\text{prop}} \rightarrow 1$ when approaching the continuum theory.
Now, in the standard approach one assumes $Z_{\phi} = Z_{\text{prop}}$ (up to small perturbative corrections). On the other hand, in a different interpretation of triviality \[2, 3\], although $Z_{\text{prop}} \to 1$, as in leading-order perturbation theory, $Z_{\phi} \sim \ln \Lambda$ is fully non perturbative and diverges in the continuum limit.

In this case, differently from perturbation theory, in order to obtain $v_R$ from the bare $v_B$ one has to apply a non-trivial correction. As a consequence, $m_H$ and $v_R$ scale uniformly in the continuum limit. From a phenomenological point of view, assuming to know the value of $v_R$, a measurement of $m_H$ does not provide any information on the magnitude of $\Lambda$ since the ratio $C = m_H/v_R$ is a cutoff-independent quantity. Moreover, in this approach, the quantity $C$ does not represent the measure of any observable interaction.

The difference between $Z_{\phi}$ and $Z_{\text{prop}}$ has an important physical meaning, being a distinctive feature of the Bose condensation phenomenon \[4\]. In gaussian-like approximations to the effective potential, one finds $m_H/v_R = 2\pi \sqrt{2\zeta}$, with $0 < \zeta \leq 2$ \[4\], $\zeta$ being a cutoff-independent number determined by the quadratic shape of the effective potential $V_{\text{eff}}(\phi_R)$ at $\phi_R = 0$. For instance, $\zeta = 1$ corresponds to the classically scale-invariant case or ‘Coleman-Weinberg regime’.

To check the alternative picture of Refs. \[2, 3\] against the generally accepted point of view, one can run numerical simulations of the theory. In this respect, we observe that numerical evidence for different cutoff dependencies of $Z_{\phi}$ and $Z_{\text{prop}}$ has already been reported in Refs. \[5, 6, 7\]. In those calculations, performed in the Ising limit of the one-component theory, one was fitting the lattice data for the connected propagator to the (lattice version of the) two-parameter form

$$G_{\text{fit}}(p) = \frac{Z_{\text{prop}}}{p^2 + m_{\text{latt}}^2}$$

(2)

After computing the zero-momentum susceptibility $\chi_{\text{latt}}$, it was possible to compare the value of $Z_{\phi} \equiv m_{\text{latt}}^2 \chi_{\text{latt}}$ with the fitted $Z_{\text{prop}}$, both in the symmetric and broken phases. While no difference was found in the symmetric phase, $Z_{\phi}$ and $Z_{\text{prop}}$ were found to be sizeably different in the broken phase. In particular, $Z_{\text{prop}}$ was very slowly varying and steadily approaching unity from below in the continuum limit. $Z_{\phi}$, on the other hand, was found to rapidly increase above unity in the same limit.

A possible objection to this strategy is that the two-parameter form Eq.(2), although providing a good description of the lattice data, neglects higher-order corrections to the structure of the propagator. As a consequence, one might object that the extraction of
TABLE I: We compare our determinations of $\langle |\phi| \rangle$ and $\chi_{\text{latt}}$ for given $\kappa$ with corresponding determinations found in the literature (Ref. [10]). In the algorithm column, 'S-W' stands for the Swendsen-Wang algorithm [8], while 'W' stands for the Wolff algorithm [9]. 'Ksweeps' stands for sweeps multiplied by $10^3$.

| $\kappa$ | lattice | algorithm | Ksweeps | $\langle |\phi| \rangle$ | $\chi_{\text{latt}}$ |
|----------|---------|-----------|---------|----------------|----------------|
| 0.077    | 32$^4$  | S-W       | 3500    | 0.38951(1)    | 18.21(4)      |
| 0.077    | 16$^4$  | Ref. [10] | 10000   | 0.38947(2)    | 18.18(2)      |
| 0.076    | 20$^4$  | W         | 400     | 0.30165(8)    | 37.59(31)     |
| 0.076    | 20$^4$  | Ref. [10] | 7500    | 0.30158(2)    | 37.85(6)      |

the various parameters is affected in an uncontrolled way (even though the fitted $Z_{\text{prop}}$ was found [5, 6] in good agreement with its perturbative prediction).

For this reason, we have decided to change strategy by performing a new set of lattice calculations. Rather than studying the propagator, we have addressed the model-independent lattice measurement of the susceptibility. In this way, assuming the mass values from perturbation theory, one can obtain a precise determination of $Z_\phi$ that can be compared with the perturbative predictions. Our results, will be presented in the following.

For our simulations we have considered again the Ising limit of a one-component $\Phi^4_4$ theory. Traditionally, this has been considered as a convenient laboratory to obtain non-perturbative information on the theory and corresponds to the lattice action

$$S_{\text{ising}} = -\kappa \sum_x \sum_\mu \{ \phi(x + \hat{e}_\mu) \phi(x) + \phi(x - \hat{e}_\mu) \phi(x) \}$$

(3)

where $\phi(x) = \pm 1$. In an infinite lattice, the broken phase is found for $\kappa > 0.07475$.

We performed Monte-Carlo simulations of this Ising action using the Swendsen-Wang [8] and Wolff [9] cluster algorithms to compute the zero-momentum susceptibility

$$\chi_{\text{latt}} = L^4[\langle |\phi|^2 \rangle - \langle |\phi| \rangle^2]$$

(4)

As a check of the validity of our algorithms, we show in Table II a comparison with previous determinations of $\chi_{\text{latt}}$ obtained by other authors.

To compare our results with perturbation theory, we have adopted the Lüscher-Weisz
TABLE II: The details of the lattice simulations for each \( \kappa \) corresponding to \( m_{\text{input}} \). In the algorithm column, ’S-W’ stands for the Swendsen-Wang algorithm \[8\], while ’W’ stands for the Wolff algorithm \[9\]. ’Ksweeps’ stands for sweeps multiplied by \( 10^3 \).

| \( m_{\text{input}} \) | \( \kappa \) | lattice | algorithm | Ksweeps | \( \chi_{\text{latt}} \) |
|-----------------|-----------|---------|----------|---------|----------------|
| 0.4             | 0.0759    | 32\(^4\) | S-W      | 1750    | 41.714 (0.132) |
| 0.4             | 0.0759    | 48\(^4\) | W        | 60      | 41.948 (0.927) |
| 0.3             | 0.0754    | 32\(^4\) | S-W      | 345     | 87.449 (0.758) |
| 0.3             | 0.0754    | 48\(^4\) | W        | 406     | 87.821 (0.555) |
| 0.2             | 0.0751    | 48\(^4\) | W        | 27      | 203.828 (3.058) |
| 0.2             | 0.0751    | 52\(^4\) | W        | 48      | 201.191 (6.140) |
| 0.2             | 0.0751    | 60\(^4\) | W        | 7       | 202.398 (8.614) |
| 0.1             | 0.0749    | 68\(^4\) | W        | 24      | 1125.444 (36.365) |
| 0.1             | 0.0749    | 72\(^4\) | W        | 8       | 1140.880 (39.025) |

scheme \[11\] where the prediction for the ratio \( m_H/v_R \) can be expressed as

\[
\left[ \frac{m_H}{v_R} \right]_{\text{LW}} \equiv \sqrt{\frac{g_R}{3}} \tag{5}
\]

Assuming the values of \( g_R \) reported in the second column of Table 3 of Ref. \[11\], the ratio in Eq.\( \text{(5)} \) becomes smaller and smaller when approaching the continuum limit.

As anticipated, to check the consistency of this prediction, we shall adopt the perturbative input values for the mass and denote by \( m_{\text{input}} \) the value of the parameter \( m_R \) reported in the first column of Table 3 in Ref. \[11\] for any value of \( \kappa \) (the Ising limit corresponding to the value of the other parameter \( \bar{\lambda} = 1 \)). In this way, computing the susceptibility on the lattice, we shall compare the quantity

\[
Z_\varphi \equiv 2\kappa m_{\text{input}}^2 \chi_{\text{latt}} \tag{6}
\]

with the perturbative prediction for \( Z_{\text{LW}} \equiv 2\kappa Z_R \) where \( Z_R \) is defined in the third column of Table 3 in Ref. \[11\].

Our lattice results for \( \chi_{\text{latt}} \) are reported in Table II for the different values of \( \kappa \) corresponding to \( m_{\text{input}} = 0.4, 0.3, 0.2, 0.1 \). In Table II we have also indicated the algorithm used for upgrading the lattice configurations and the number of sweeps at each value of \( \kappa \) and
TABLE III: The values of $g_R$ and $Z_{LW}$ for each $m_{\text{input}}$ as given in Table 3 of Ref. [11]. $Z_\phi$ is defined in Eq. (6). The errors quoted on $Z_\phi$ are only due to the statistical uncertainty of $\chi_{\text{latt}}$ (see Table II).

| $m_{\text{input}}$ | $g_R$  | $Z_{LW}$ | $Z_\phi$ |
|---------------------|--------|----------|----------|
| 0.4                 | 27 (2) | 0.929 (14) | 1.019 (23) |
| 0.3                 | 24 (2) | 0.932 (14) | 1.192 (8) |
| 0.2                 | 20 (1) | 0.938 (12) | 1.216 (52) |
| 0.1                 | 16.4 (9) | 0.944 (11) | 1.709 (58) |

lattice size. In the case of the Wolff algorithm the number of sweeps is the number of Wolff sweeps multiplied by the ratio between the average cluster size and the lattice volume. We used different lattice sizes at each value of $\kappa$ to have a check of the finite-size effects. The statistical errors have been estimated using the jackknife.

We have reported in Table III the corresponding entries for $Z_\phi$, $Z_{LW}$ and $g_R$. As one can see, the two $Z$’s are sizeably different and the discrepancy becomes larger and larger when approaching the continuum limit, precisely the same trend found in Refs. [5, 6]. This confirms that, approaching the continuum limit, the rescaling of the ‘Higgs condensate’ cannot be described in perturbation theory.

Now, if zero-momentum quantities rescale differently from the perturbative predictions, one may wonder about the relation between $m_H$ and $v_R$, when this is rescaled through $Z \equiv Z_\phi$ rather than through the perturbative $Z \equiv Z_{LW}$. In this case, one finds the alternative relation

$$\frac{m_H}{v_R} = \sqrt{\frac{g_R}{3} \frac{Z_\phi}{Z_{LW}}} = C$$

obtained by replacing $Z_{LW} \rightarrow Z_\phi$ in Ref. [11] but correcting for the perturbative $Z_{LW}$ introduced in the Lüscher and Weisz approach.

According to the picture of Refs. [2, 3], one expects $Z_\phi \sim \ln \Lambda$ to compensate the $1/\ln \Lambda$ from $g_R$ so that $C$ should be a cutoff-independent constant. To this end, one can check the values of $Z_\phi$, $Z_{LW}$ and $g_R$ in our Table III. We find that $C$ is a constant, to a good approximation, $C = 3.087 \pm 0.084$. As an example, this value, when combined with the Standard Model value $v_R = 246$ GeV, would yield a Higgs mass $m_H = 760 \pm 21$ GeV independently of the ultraviolet cutoff $\Lambda \sim \pi/a$ (see Fig. 1).
FIG. 1: The values of $m_H$ as defined through Eq. (11) versus $m_{\text{input}} = a m_R$. The error band corresponds to a one standard deviation error in the determination of $m_H$ through a fit with a constant function.

Notice that this value is not a prediction for the mass of the Higgs boson, neither in the one-component theory nor in the Standard Model. In fact, the uncertainty is dominated by the statistical error in $\chi_{\text{latt}}$ at any value of $\kappa$ and neglects any theoretical uncertainty associated with approaching the critical line in the Ising limit. Traditionally, the Ising limit corresponds to the maximal value of $m_H/v_R$, as determined from the perturbative trend Eq. (5) with $g_R/3 \sim A/\ln \Lambda$. However, our simulation show that one is faced with the more general scenario Eq. (11) where $Z_\varphi \sim B \ln \Lambda$ so that $m_H/v_R \sim \sqrt{AB}$.

In this sense, the implications of our results for the Standard Model are mainly of ‘qualitative’ nature and amount to the statement that the value of the Higgs boson mass, in units of 246 GeV, does not depend on the magnitude of the ultraviolet cutoff. However,
as a consequence of our results, the whole issue of the upper bounds on the Higgs mass is affected suggesting the need of more extensive studies of the critical line to compare the possible values of $C = \sqrt{AB}$ with the value $C_{\text{Ising}} \simeq \pi$ obtained in the Ising limit. This should also be performed in the O(4)-symmetric case which, after all, is the one relevant for the Standard Model. Independently of this more refined analysis, it is also true that a value as large as $m_H = 760 \pm 21$ GeV, would be in good agreement with a recent phenomenological analysis of radiative corrections \[12\] that points toward substantially larger Higgs masses than previously obtained through global fits to Standard Model observables.

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