Decoherence of quantum registers

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Abstract

We consider decoherence of quantum registers, which consist of the qubits sited approximately periodically in space. The sites of the qubits are permitted to have a small random variance. We derive the explicit conditions under which the qubits can be assumed decohering independently. In other circumstances, the qubits are decohered cooperatively. We describe two kinds of collective decoherence. In each case, a scheme is proposed for reducing the collective decoherence. The schemes operate by encoding the input states of the qubits into some "subdecoherent" states.

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1 Introduction

Quantum computation has become an active field since Shor discovered that quantum computers could solve the problem of finding factors of a large number in a time which is a polynomial function of the length (number of the bits) of the number [1,2]. However, there are some obstacles to realize quantum computation. The main one is decoherence of the qubits caused by the interaction with the environment [3-6]. Unruh analyzed decoherence in quantum memory with the assumption that the qubits are decohered independently [3]. To reduce this kind of decoherence, Shor proposed a subtle strategy called quantum error correction which could restore useful information from the decohered states [7]. Many quantum error-correcting codes have since been discovered to correct quantum errors occurring during the store of the information or during the gate operations [8-24]. Apart from the independent decoherence, there are other circumstances. The qubits may be decohered collectively. The collective decoherence has some new features, which make the strategy for reducing this kind of decoherence is much different from the quantum error correction schemes [25-27].

In this paper, we consider a practical model of the quantum register. The register consists of the qubits sited approximately periodically in space. But the sites of the qubits are permitted to have a small random variance. This small disorder may be due to the limited manufacture precision or caused by the thermal variation of the qubits. Starting from a general decoherence model of the quantum register, we obtain its exact solution. Then we discuss in which circumstances the qubits can be regarded decohering independently. From the cooperative decoherence to the independent decoherence, the small disorder of the sites of the qubits plays an important role. The independent decoherence is an ideal case. There is another ideal case, i.e., the collective decoherence. We derive two kinds of collective decoherence. The first case has been discussed in [25] and [26]. In this case the qubits lie in the coherent length of the environment.
The second case of the collective decoherence is new. It results from the approximate periodicity of the register. The existing quantum error correction schemes are not suitable for reducing the collective decoherence. So in each case of the collective decoherence, we propose an alternate decoherence-reducing strategy, which exploits the new feature of the collective decoherence.

The paper is arranged as follows: The general decoherence model of the quantum register is described and solved in Sec. 1. In Sec. 2, we derive the explicit conditions under which the qubits can be assumed decohering independently. There are two circumstances. Section 3 describes two kinds of collective decoherence and the corresponding decoherence-reducing strategy.

2 The decoherence model of quantum registers and its exact solution

For a practical quantum register, it is reasonable to assume that the qubits are arranged approximately periodically in space. So the coordinate of the \( \overrightarrow{r}_l \) qubit can be expressed as \( \overrightarrow{r}_l = \overrightarrow{R}_l + \overrightarrow{\delta}_l \), where \( \overrightarrow{R}_l \) is a rigorous periodical function of \( \overrightarrow{r}_l \) with a lattice constant \( d \). (For simplicity we assume the lattice constants are same along different directions.) \( \overrightarrow{\delta}_l \) is a small random variable, which satisfies \( \langle \overrightarrow{\delta}_l \rangle = 0 \), and \( \sqrt{\langle \overrightarrow{\delta}_l \cdot \overrightarrow{\delta}_l \rangle} = \delta \). Generally \( \delta << d \).

Decoherence of the qubits is caused by the coupling with the environment. The noise field in the environment may be a radiative field (such as two-level atoms in a cavity [28]) or a phonon field (such as the trapped ions [29]). Here we consider the decoherence by its narrow meaning, i.e., we only consider the dephasing process. The loss of the energy is not included. The qubits can always be described by the Pauli operators \( \overrightarrow{\sigma}_l \) and the environment is modelled by a bath of oscillators. The total Hamiltonian describing the dephasing process takes the form

\[
H = \hbar \left[ \omega_0 \sum_{\overrightarrow{r}_l} \sigma_+^z_l + \sum_{\overrightarrow{k}} \omega_{\overrightarrow{k}} a_{\overrightarrow{k}}^+ a_{\overrightarrow{k}} + \sum_{\overrightarrow{k}, \overrightarrow{l}} \left( g_{\overrightarrow{k} \overrightarrow{l}} a_{\overrightarrow{k}}^+ + g^*_{\overrightarrow{k} \overrightarrow{l}} a_{\overrightarrow{k}} \right) \sigma^z_{\overrightarrow{l}} \right],
\]
where $a_{\vec{k}}$ is the annihilation operator of the bath mode $\vec{k}$ and $g_{\vec{k} \vec{l}}$ is the coupling coefficient. If the mode functions of the noise field are plane waves, $g_{\vec{k} \vec{l}}$ can be expressed as

$$g_{\vec{k} \vec{l}} = g_{\vec{k}} e^{-i \vec{k} \cdot \vec{r}''}.$$ (2)

In the following we assume Eq. (2) holds.

We solve the decoherence model in the interaction picture. The interaction Hamiltonian is

$$H = \hbar \sum_{\vec{k}, \vec{l}} \left[ g_{\vec{k} \vec{l}} a_{\vec{k}} e^{-i \omega_{\vec{k}} t} + g_{\vec{k} \vec{l}}^{*} a_{\vec{k}}^{+} e^{i \omega_{\vec{k}} t} \right].$$ (3)

In Ref. [25], the time evolution operator is expressed as $U(t) = \exp \left[ -i \frac{\hbar}{\hbar} \int_{0}^{t} H_{I} (t') dt' \right]$. But this expression is not correct since $[H_{I}(t), H_{I}(t')] \neq 0$. In fact it is not difficult to verify that the evolution operator corresponding the Hamiltonian (3) has the form

$$U(t) = \exp \left\{ \sum_{\vec{k}, \vec{l}} \left( \xi_{\vec{k} \vec{l}}^{*} (t) a_{\vec{k}}^{+} - \xi_{\vec{k} \vec{l}} (t) a_{\vec{k}} \right) \sigma_{\vec{l}}^{z} \right\} e^{i f(t)},$$ (4)

where

$$\xi_{\vec{k} \vec{l}} (t) = \frac{g_{\vec{k} \vec{l}} \left( 1 - e^{-i \omega_{\vec{k}} t} \right)}{\omega_{\vec{k}}}.$$ (5)

and

$$f(t) = \sum_{\vec{k}} \left\{ \frac{\omega_{\vec{k}} t - \sin \left( \frac{\omega_{\vec{k}} t}{2} \right)}{\omega_{\vec{k}}^{2}} \left( \sum_{\vec{l}} \left| g_{\vec{k} \vec{l}} \sigma_{\vec{l}}^{z} \right|^{2} \right) \right\}.$$ (6)

In the following, we will see the factor $e^{i f(t)}$ in Eq. (4) missed by Ref. [25] results in the Lamb phase shift, which plays an important role in the collective decoherence.

The time evolution of the register is completely determined by the operator $U(t)$. To see this, let $\rho_{i_{\vec{l}} j_{\vec{l}}} = \langle i_{\vec{l}} | j_{\vec{l}} \rangle$ and

$$\rho_{\{i_{\vec{l}} j_{\vec{l}}\}} = \rho_{i_{1}, j_{1}} \otimes \rho_{i_{2}, j_{2}} \otimes \cdots \otimes \rho_{i_{L}, j_{L}},$$ (7)
where $i \rightarrow l = \pm 1, j \rightarrow l = \pm 1$, and $|\pm 1\rangle$ are two eigenstates of the operator $\sigma^z$. $L = L_1 L_2 L_3$ is the total number of the qubits. With this notation, the initial density $\rho_s(0)$ of the register can be expanded into

$$\rho_s(0) = \sum_{\{i \rightarrow l, j \rightarrow l\}} c_{\{i \rightarrow l, j \rightarrow l\}} \rho_{\{i \rightarrow l, j \rightarrow l\}}.$$  \hfill (8)

The environment is supposed in the thermal equilibrium. So its initial density in the coherent representation has the form [30]

$$\rho_{env}(0) = \prod_{\vec{k}} \int d^2\alpha_{\vec{k}} \frac{1}{\pi \langle N_{\omega_{\vec{k}}} \rangle} \exp \left( -\frac{|\alpha_{\vec{k}}|^2}{\langle N_{\omega_{\vec{k}}} \rangle} \right) |\alpha_{\vec{k}}\rangle \langle \alpha_{\vec{k}}|,$$  \hfill (9)

where $\langle N_{\omega_{\vec{k}}} \rangle$ is the mean photon or phonon number of the mode $\vec{k}$

$$\langle N_{\omega_{\vec{k}}} \rangle = \frac{1}{\exp \left( \frac{\hbar \omega_{\vec{k}}}{k_B T} \right) - 1}.$$  \hfill (10)

When the operator (4) acts on the coherent state $|\alpha_{\vec{k}}\rangle$, it only generates a displacement. So with this evolution operator, the reduced density of the register at time $t$ can easily be obtained. We have

$$\rho_s(t) = \sum_{\{i \rightarrow l, j \rightarrow l\}} c_{\{i \rightarrow l, j \rightarrow l\}} \rho_{\{i \rightarrow l, j \rightarrow l\}} \exp \left[ -\eta_{\{i \rightarrow l, j \rightarrow l\}}(t) + i\phi_{\{i \rightarrow l, j \rightarrow l\}}(t) \right],$$  \hfill (11)

where the phase damping factor

$$\eta_{\{i \rightarrow l, j \rightarrow l\}}(t) = \sum_{\vec{k}} |g_{\vec{k}}|^2 \coth \left( \frac{\hbar \omega_{\vec{k}}}{2k_B T} \right) \frac{1 - \cos \left( \omega_{\vec{k}} t \right)}{\omega_{\vec{k}}^2} \lambda_{\vec{k}},$$  \hfill (12)

and the Lamb phase shift

$$\phi_{\{i \rightarrow l, j \rightarrow l\}}(t) = \sum_{\vec{k}} |g_{\vec{k}}|^2 \frac{\omega_{\vec{k}} t - \sin \left( \omega_{\vec{k}} t \right)}{\omega_{\vec{k}}^2} \lambda_{\vec{k}}.$$  \hfill (13)
In Eqs. (12) and (13), $\lambda_{1\rightarrow k}$ and $\lambda_{2\rightarrow k}$ are defined as follows

$$
\lambda_{1\rightarrow k} = \left| \sum_{l'} (i_{\tau} - j_{\tau}) e^{i \vec{k} \cdot \vec{r}_{\tau}} \right|^2,
$$

(14)

$$
\lambda_{2\rightarrow k} = \left| \sum_{l'} i_{\tau} e^{i \vec{k} \cdot \vec{r}_{\tau}} \right|^2 - \left| \sum_{l'} j_{\tau} e^{i \vec{k} \cdot \vec{r}_{\tau}} \right|^2.
$$

(15)

In the derivation of Eqs. (12) and (13), the decomposition (2) of the coupling coefficient has been used.

It is convenient to use the state fidelity to describe the decoherence. For a pure input state $|\Psi (0)\rangle$, the fidelity is defined as

$$
F = \langle \Psi (0) | \rho_s (t) | \Psi (0) \rangle.
$$

(16)

Suppose the input state of the register is pure and expressed as $|\Psi (0)\rangle = \sum_{\{i_{\tau}\}} c_{\{i_{\tau}\}} |\{i_{\tau}\}\rangle$, then from Eq. (11) the fidelity is

$$
F = \sum_{\{i_{\tau}, j_{\tau}\}} \left| c_{\{i_{\tau}\}} \right|^2 \left| c_{\{j_{\tau}\}} \right|^2 \exp \left[ -\eta_{\{i_{\tau}, j_{\tau}\}} (t) + i\phi_{\{i_{\tau}, j_{\tau}\}} (t) \right].
$$

(17)

From this expression, we see that the phase damping and the Lamb phase shift all contribute to the decoherence of the state. Eq. (13) reveals that the phase shift increases with time approximately linearly. So with a sufficient large $t$ the phase shift will play an important role. In the next section we will show the factor $\lambda_{2\rightarrow k}$ reduces to zero for the independent decoherence. Therefore, the Lamb phase shift only contributes to the cooperative decoherence.

The two factors $\lambda_{1\rightarrow k}$ and $\lambda_{2\rightarrow k}$ defined by (14) and (15) are important in determining whether the qubits are decohered independently or collectively. We discuss this problem in the following two sections.
3 Independent decoherence

We first look at the phase damping. Eq. (12) can be rewritten as

\[
\eta_{\{i \rightarrow j \rightarrow r \}}(t) = x \sum_{k} h_1(ω_{\rightarrow k}) \lambda_{1 \rightarrow k},
\]

where \( h_1(ω_{\rightarrow k}) \) is a normalized distribution which satisfies \( \sum_{k} h_1(ω_{\rightarrow k}) = 1 \). \( x \) is the normalization constant

\[
x = \sum_{k} |g_{\rightarrow k}|^2 \coth \left( \frac{\hbar ω_{\rightarrow k}}{2k_B T} \right) \frac{1 - \cos (ω_{\rightarrow k} t)}{ω_{\rightarrow k}^2}
\]

The expression of \( h_1(ω_{\rightarrow k}) \) is given by comparing (18) with (12). Its explicit form depends on the coupling coefficient \( |g_{\rightarrow k}|^2 \), whereas the latter is determined by the specific characteristics of the physical system. But here we take a simplification. The distribution \( h_1(ω_{\rightarrow k}) \) is approximately characterized by its mean value \( ω_1 \) and variance \( \Delta ω_1 \). Generally, \( \Delta ω_1 < ω_1 \). The same simplification can be taken for the Lamb phase shift, which is expressed as the mean value of \( λ_{2 \rightarrow k} \) under the distribution \( h_2(ω_{\rightarrow k}) \). \( h_2(ω_{\rightarrow k}) \) is characterized by \( ω_2 \) and \( \Delta ω_2 \). In the following we use the four parameters \( ω_1, \Delta ω_1, ω_2, \Delta ω_2 \) to discuss the decoherence behavior of the register. First we show that the cooperative coupling with the environment can yield independent decoherence of the qubits in certain circumstances. There are two cases.

**Case 1** \( \frac{ω_1}{v}, \frac{ω_2}{v} \geq π \).

In the above condition, \( v \) indicates the velocity of the noise field and \( \delta \) is the variance of sites of the qubits. Under this condition, for the effective mode \( \vec{k} \) (a mode \( \vec{k} \) is called effective if in Eqs. (12) and (13) it has sufficient contributions to the summation.), \( γ_{\vec{r}} = i \vec{r} e^{i \vec{k} \cdot \vec{r}} \) becomes a random variable which satisfies

\[
\langle γ_{\vec{r}} \rangle = 0, \quad \langle |γ_{\vec{r}}|^2 \rangle = 1.
\]

Obviously, the variables \( γ_{\vec{r}} \) are independent of each other. So the mean \( λ_{2 \rightarrow k} \) becomes

\[
\langle λ_{2 \rightarrow k} \rangle = L - L = 0
\]
where $L$ is the total number of the qubits. Similarly, if $i \vec{r} \neq j \vec{r}$, $\gamma \vec{r} = \frac{i \vec{r} - j \vec{r}}{2} e^{i \vec{k} \cdot \vec{r}} \vec{r}$ is also a random variable satisfying Eq. (20). Suppose $L_0$ is the number of the pairs $(i \vec{r}, j \vec{r})$ with $i \vec{r} \neq j \vec{r}$, the mean $\lambda_{1 \vec{k}}$ is thus simplified to

$$\langle \lambda_{1 \vec{k}} \rangle = 4L_0 = \sum_{\vec{r}} (i \vec{r} - j \vec{r})^2. \quad (20)$$

With (21) and (22), the phase damping and the Lamb phase shift become, respectively,

$$\eta_{\{i \vec{r}, j \vec{r}\}} (t) = x \sum_{\vec{r}} (i \vec{r} - j \vec{r})^2, \quad (21)$$

$$\phi_{\{i \vec{r}, j \vec{r}\}} (t) = 0. \quad (22)$$

This is just the result of the independent decoherence, which is obtained in [25] and [26] under the assumption that the qubits interact with different environments. Here we see, provided the disorder in the register is sufficiently large, the qubits will be decohered independently, even if they couple with the same environment. In the independent decoherence, the Lamb phase shift reduces to zero.

### Case 2 $\frac{\Delta \omega_{1 \vec{r}} d}{v} >> 1, \frac{\Delta \omega_{2 \vec{r}} d}{v} >> 1$.

In the above condition, $d$ indicates the lattice constant. If $\vec{l}_1 \neq \vec{l}_2$, let $\vec{k} \cdot (\vec{r}_{\vec{l}_1} - \vec{r}_{\vec{l}_2}) = \frac{sd}{v} \omega_{\vec{k} \vec{r}}$, where $s$ and $d$ have the same order of magnitude. Under the distribution $h_1 (\omega_{\vec{k} \vec{r}})$ or $h_2 (\omega_{\vec{k} \vec{r}})$, the following mean value

$$\left\langle e^{i \vec{k} \cdot (\vec{r}_{\vec{l}_1} - \vec{r}_{\vec{l}_2})} \right\rangle = \langle e^{i \frac{sd}{v} \omega_{\vec{k} \vec{r}}} \rangle \quad (23)$$

is a Fourier transformation of the weight function. Suppose $\Delta \omega_i$ is the variance of the distribution, which can be approximated by a Gaussian function, hence we have

$$\langle e^{i \frac{sd}{v} \omega_{\vec{k} \vec{r}}} \rangle \sim \exp \left[ - \left( \frac{\Delta \omega_i sd}{v} \right)^2 \right] \sim 0. \quad (24)$$

So after summation over the mode $\vec{k}$, only the non-variation terms, such as $(i \vec{r} - j \vec{r})^2$, in $\lambda_{1 \vec{k}}$ and $\lambda_{2 \vec{k}}$ have contributions to the result. We then obtain
Eqs. (23) and (24) again. Therefore, in this case the qubits are also decohered independently.

To reduce the independent decoherence, many kinds of quantum error correction schemes have been proposed. However, the two conditions for the independent decoherence are not always satisfied in practice. In the next section we discuss other circumstances.

4 Collective decoherence

The independent decoherence is an ideal case. In this section, we discuss another ideal case, the collective decoherence. This requires that the disorder in the register should be small, i.e., the variance $\delta$ should satisfy $\frac{\Delta \omega_i}{\omega_i} \ll \pi$, and $\frac{\Delta \omega_d}{\omega_d} \ll \pi$. (We have assumed $\Delta \omega_i \leq \overline{\omega}_i$ ($i = 1, 2$).) Under this condition, for the effective $\vec{k}$, we approximately have $\vec{k} \cdot \vec{r} \approx \vec{k} \cdot \vec{R}$ where $\vec{R}$ is a rigorous periodical function of $\vec{l}$. There are two circumstances which can result in the collective decoherence.

Case 1 $\frac{\Delta \omega_d}{\omega_d} \ll \pi$, $\frac{\Delta \omega_d}{\omega_d} \ll \pi$.

In this case, two adjacent qubits lie in the coherent length of the environment. We call two adjacent qubits a qubit-pair. Suppose there are $L$ qubit-pairs (so $2L$ qubits) in the register. The two qubits in the $\vec{l}$ qubit-pair are indicated by $\vec{l}$ and $\vec{l}'$, respectively. Then, for the effective $\vec{k}$, the factor $\lambda_{1\vec{k}}$ approximately becomes

$$\lambda_{1\vec{k}} \approx \left| \sum_{\vec{l}} \left( i_{\vec{l}} + i_{\vec{l}'} - j_{\vec{l}} - j_{\vec{l}'} \right) e^{i\vec{k} \cdot \vec{R}} \right|^2. \quad (25)$$

$\lambda_{2\vec{k}}$ has a similar expression. Eq. (27) reveals the collective decoherence of the two qubits in a qubit-pair. In the collective decoherence, the decoherence rate is sensitive to the type of the input states. The states which undergo no or reduced decoherence are called ”subdecoherent” states.

The existing quantum error schemes are not suitable for reducing the collective decoherence. Fortunately, for the collective decoherence, there is a simpler
decoherence-reducing strategy. The input states of $L$ qubits can be encoded into the "subdecoherent" states of $L$ qubit-pairs by the following encoding

$$
|−1\rangle \rightarrow |−1, 1\rangle,
|1\rangle \rightarrow |1, −1\rangle.
$$

Because of Eq. (27) and a similar equation of $\lambda_2 k$, the encoded states obviously undergo no phase damping and Lamb phase shift. So the coherence is preserved. The encoding (28) has been mentioned in [25] and extensively discussed in [27]. It can be simply fulfilled by the quantum controlled-NOT gates.

**Case 2** $\frac{\Delta \omega_1 L d}{v} << \pi$, $\frac{\Delta \omega_2 L d}{v} << \pi$.

In this case, the lattice constant $d$ and the effective wave length $\frac{2\pi v}{\omega_i}$ of the noise field have the same order of magnitude. So the adjacent qubits do not lie in the coherent length of the environment. But the distribution functions $h_1 (\omega_k)$ and $h_2 (\omega_k)$ have a peak and the width of the peak is small so that $L d << \frac{\pi v}{2\Delta \omega_i}$ ($i = 1, 2$). (For simplicity, here we consider the one-dimensional register. The discussion of the three-dimensional circumstances is very similar.) From Eq. (18), the phase damping factor is thus simplified to

$$
\eta_{\{i, j\}}(t) = x \lambda_1 \bar{k}_1 = x \sum_l (i_l - j_l) e^{i \bar{k}_1 R_l}^2,
$$

where $\bar{k}_1 = \frac{2\pi}{v}$ and $x$ is given by Eq. (19). Similarly, the Lamb phase shift $\phi_{\{i, j\}}(t) \propto \lambda_2 \bar{k}_2$. We assume $\bar{k}_2 \approx \bar{k}_1 = \bar{k}$.

Eq. (29) suggests that the qubits are decohered collectively. This results from the periodicity of the sites $R_l$. Similar to the case 1, the collective decoherence described by Eq. (29) can also be reduced by pairing the qubits. But this time the qubit-pairs do not consist of two adjacent qubits. Since $\frac{L d}{2\pi} \sim 1$, there exist round numbers $m$ and $n$ to satisfy $|m \frac{L d}{2\pi} - n| << 1$, where $m$ is chosen as small as possible. Hence we have $e^{i \bar{k}(R_l + m - R_l)} \approx (-1)^n$. So the $l$ qubit and the $l + m$ qubit can be put into a pair. The state of the qubits can be transformed into the "subdecoherent" state of the qubit-pairs by the following encoding

$$
|−1\rangle \rightarrow |−1, (−1)^n\rangle,
|1\rangle \rightarrow |1, (−1)^{n+1}\rangle.
$$
Obviously the decoherence of the encoded state is reduced.

In the above we require $\frac{\Delta \omega_i L_d}{v} \ll \pi$. This condition is too strong and in fact it is not necessary. It is clear that the decoherence-reducing strategy described in the above paragraph still works if $m \frac{\Delta k d}{\pi} \ll 1$, where $m$ is a small round number. So for reducing this kind of decoherence, we only need $\frac{\Delta \omega_i m d}{v} \ll \pi$ ($i = 1, 2$).

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