A Simpler Eulerian Variational Principle
for Barotropic Fluids

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Abstract

The variational principle of barotropic Eulerian fluid dynamics is known to be quite cumbersome containing as much as eleven independent functions. This is much more than the the four functions (density and velocity) appearing in the Eulerian equations of motion. This fact may have discouraged applications of the variational method. In this paper a four function Eulerian variational principle is suggested and the implications are discussed briefly.

1 Introduction

The motion of a fluid is usually described in two ways. In the Lagrangian approach we follow the trajectory of each fluid particle. While in the Eulerian approach we study the evolution of the velocity and density fields.

Variational principles have been developed in both cases. A variational principle in terms of Lagrangian variables has been described by Eckart [1] and Bretherton [2]. While a variational principle in term of the Eulerian variables is given by Herivel [3], Serrin [4], Lin [5] and Seliger & Whitham [6].

The Eulerian approach appears to be much more appealing involving measurable quantities such as velocity and density fields instead of the trajectories of unseen particles of the Lagrangian approach.

However, regretfully the variational principle of the Eulerian flow appears to be much more cumbersome than the Lagrangian case, containing quite a few ”Lagrange multipliers” and ”potentials”. In fact the total number of independent functions in this formulation according to the approach suggested by Herivel [3], Serrin [4] and Lin [5] is eleven which exceeds by many the four functions of velocity and density appearing in the Eulerian equations of a barotropic flow.

The variational principle of the Lagrangian approach is on the other hand simple and straightforward to implement. Bretherton [2] has suggested to use the Lagrangian variational principle with Eulerian variables and especially constructed
"Eulerian displacements" for which the equations of motion are derived. This procedure appears to be inconvenient for stability and numerical calculations.

Instead I intend to develop in this paper a simpler variational principle in terms of Eulerian variables. This will be done by rearranging the terms in the original variational principle and using a simple trick to be discussed below. The result will be a variational principle in terms of four Eulerian independent functions which is the appropriate number.

The plan of this paper is as follows: first I review the original Eulerian variational principle and the equations derived from it. Then I derive a new variational principle and discuss its implications.

2 The Eulerian Approach to the Barotropic Flow

In the Eulerian description of barotropic fluid dynamics we seek a solution for the velocity \( \vec{v} = \vec{v}(x^k, t) \) and density \( \rho = \rho(x^k, t) \) fields of a given flow. Those fields depend on the space coordinates \( \vec{r} = (x^k) = (x, y, z) \) \([k, l, m, n = 1, 2, 3]\) and on the time variable \( t \). The fields are obtained by solving the Euler equations:

\[
\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} + \nabla (h + \Phi) = 0 \tag{1}
\]

in which \( \nabla = \frac{\partial}{\partial x} \). The potential \( \Phi = \Phi(x^k, t) \) is a given function of coordinates, while \( h = h(\rho) \) is the specific enthalpy which is a given function of \( \rho \) usually designated as the equation of state. In addition the density and velocity fields must satisfy the continuity equation:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0. \tag{2}
\]

Thus we have to solve four equations in order to obtain four unknown fields. This can be done when supplemented the appropriate boundary conditions.

2.1 The Classical Eulerian Variational Principle

Researchers (Herivel [3], Serrin [4], Lin [5]) seeking a variational principle from which the Euler and continuity equations (equation (1) and equation (2)) can be derived, arrived at the action \( A \) given in terms of the Lagrangian \( L \) as:

\[
A = \int_{t_0}^{t_1} L dt \quad \text{with}
\]

\[
L = \int_V \left[ \frac{1}{2} \vec{v}^2 - \varepsilon(\rho) - \Phi \right] \rho d^3 x
+ \int_V \left[ \nu \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right) - \rho \vec{\alpha} \cdot \frac{D \vec{\beta}}{Dt} \right] d^3 x. \tag{3}
\]

In which \( \varepsilon(\rho) \) is the specific internal energy connected to the specific enthalpy by the equation:

\[
h = \frac{\partial (p \varepsilon)}{\partial p}. \tag{4}
\]
The volume element is given by $d^3x$ and the operator $\frac{D}{Dt}$ is defined such that: $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}$. The variation principle contains in addition to the desired four functions $\rho, \vec{v}$ also the seven "potentials" $\nu, \vec{\alpha}, \vec{\beta}$ reaching to a total of eleven variational variables. Varying the above action with respect to those variables such that the variations vanish for $\vec{r}$ and $t$ sufficiently large. We see that in order to have $\delta A$ vanish for otherwise arbitrary $(\delta \vec{v}, \delta \rho, \delta \nu, \delta \vec{\alpha}, \delta \vec{\beta})$, the following equations must be satisfied:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \quad (5)$$

$$\frac{\partial (\rho \vec{\alpha}_i)}{\partial t} + \vec{\nabla} \cdot (\rho \vec{\alpha}_i \vec{v}) = 0 \quad (6)$$

$$\vec{v} = \vec{\alpha} \cdot \vec{\nabla} \vec{\beta} + \vec{\nabla} \nu \quad (7)$$

$$\frac{D\vec{\beta}}{Dt} = 0 \quad (8)$$

$$\frac{D \nu}{Dt} = \frac{1}{2} \vec{v}^2 - h - \Phi. \quad (9)$$

Combining equation (8) and equation (5) we arrive at the simpler equation:

$$\frac{D\vec{\alpha}}{Dt} = 0. \quad (10)$$

Calculating the expression $\frac{D\vec{v}}{Dt}$ using equations (7,8,10,9) we arrive at the equations of Euler:

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla}(h + \Phi). \quad (11)$$

Thus the functions $\vec{v}$ and $\rho$ are extrema of the action $A$ if they satisfy the Euler and continuity equations.

2.2 The Variational Principle of Seliger & Witham

Seliger & Witham have proposed to take $\vec{\alpha} = (\alpha, 0, 0)$ and $\vec{\beta} = (\beta, 0, 0)$, in this way one obtains a variational principle with only seven functions. Rewriting equations (12,13,14,15,16) for this case we obtain the following set of equations:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \quad (12)$$

$$\vec{v} = \alpha \vec{\nabla} \vec{\beta} + \vec{\nabla} \nu \quad (13)$$

$$\frac{D\alpha}{Dt} = 0 \quad (14)$$

$$\frac{D\beta}{Dt} = 0 \quad (15)$$

$$\frac{D \nu}{Dt} = \frac{1}{2} \vec{v}^2 - h - \Phi. \quad (16)$$

We see that equations (12,16) remain unchanged. The velocity $\vec{v}$ is given now by the Clebsch representation (equation (13) ) and the entire set of equations are designated...
as Clebsch’s transformed equations of hydrodynamics (Eckart [1], Lamb [7] p. 248). Bretherton [2] have quoted a remark by H. K. Moffat concerning the possibility of the function $\nu$ being not single valued. According to Moffat [8] the helicity integral:

$$H = \int \vec{\nabla} \times \vec{v} \cdot d^3x$$

which is a measure of the knottedness of the vortex lines, must be zero if $\nu$ is single valued which is clearly not true in general. This can be shown as follows: suppose that $\nu$ is single valued than by inserting equation (13) into equation (17) and integrating by parts we obtain:

$$H = \int \vec{\nabla} \nu \cdot \vec{\nabla} \rho$$

If we choose a volume made of closed vortex filaments such that $\vec{\nabla} \times \vec{v} \cdot dS = 0$ than clearly $H = 0$. However, as Bretherton [2] noticed the equations of motion being local are unaffected by global properties such as the non single-valuedness of $\nu$.

An analogue from classical mechanics may make things even clearer. A few will object to describe the two dimensional motion of a particle moving under the influence of the potential $V(R)$ (where $R$ is the radial coordinate), by the coordinates $R, \phi$. Where $\phi$ is the azimuthal angel which is not single valued. This form is found to be more convenient than the single valued Cartesian $x, y$ representation.

3 The Reduced Variational Principle

Although Seliger & Witham (1968) have managed to reduce the Eulerian variational principle from eleven to seven functions. The number of variational variables is still too much, since the Eulerian equations contain only four unknown functions. I intend to suggest a solution to this problem here.

First let us rewrite the Lagrangian of Seliger & Witham (equation (3) with $\alpha$ and $\beta$ being scalars):

$$L = \int_V \left[ \frac{1}{2} \rho \dot{v}^2 - \varepsilon(\rho) - \Phi \right] \rho d^3x$$

Next we rearrange terms:

$$L = \int_V \left[ -\rho \frac{\partial \beta}{\partial t} + \nu \frac{\partial \rho}{\partial t} \right] d^3x$$

Furthermore, we introduce the identities:

$$\nu \frac{\partial \rho}{\partial t} = \frac{\partial (\nu \rho)}{\partial t} - \rho \frac{\partial \nu}{\partial t}$$

$$\nu \vec{\nabla} \cdot (\rho \vec{v}) = \vec{\nabla} \cdot (\nu \rho \vec{v}) - \rho \vec{v} \cdot \vec{\nabla} \nu.$$
Inserting the above identities into equation (20) and rearranging terms again we have:

\[ L = -\int_V \left[ \alpha \frac{\partial \beta}{\partial t} + \frac{\partial \nu}{\partial t} \right] \rho d^3 x \]

\[ + \int_V \left[ \frac{1}{2} \rho \tilde{v}^2 - \rho \tilde{v} \cdot (\alpha \tilde{\nabla} \beta + \tilde{\nabla} \nu) \right] d^3 x - \int_V [\varepsilon(\rho) + \Phi] \rho d^3 x \]

\[ + \int_V \frac{\partial (\nu \rho)}{\partial t} d^3 x + \int_V \tilde{\nabla} \cdot (\nu \rho \tilde{v}) d^3 x. \]  

(22)

Now since:

\[ \frac{1}{2} \tilde{v}^2 - \tilde{v} \cdot (\alpha \tilde{\nabla} \beta + \tilde{\nabla} \nu) = \frac{1}{2} (\tilde{v} - \alpha \tilde{\nabla} \beta - \tilde{\nabla} \nu)^2 - \frac{1}{2} (\alpha \tilde{\nabla} \beta + \tilde{\nabla} \nu)^2 \]  

(23)

We finally obtain:

\[ L = L_r + L_v \]  

(24)

in which:

\[ L_r = \int_V \left[ -\left( \alpha \frac{\partial \beta}{\partial t} + \frac{\partial \nu}{\partial t} \right) - \frac{1}{2} (\alpha \tilde{\nabla} \beta + \tilde{\nabla} \nu)^2 - \varepsilon(\rho) - \Phi \right] \rho d^3 x + \frac{\partial}{\partial t} \int_V \nu \rho d^3 x. \]  

(25)

And

\[ L_v = \int_V \frac{1}{2} (\alpha \tilde{\nabla} \beta + \tilde{\nabla} \nu - \tilde{v})^2 \rho d^3 x + \int_V \tilde{\nabla} \cdot (\nu \rho \tilde{v}) d^3 x. \]  

(26)

The Lagrangian \( L \) is dissected into a part depending on the fluid velocity \( \tilde{v} \) which is denoted as \( L_v \) and the remaining part which does not depend on \( \tilde{v} \) which is denoted \( L_r \). Let us look at the action:

\[ A_r = \int_{t_0}^{t_1} L_r dt. \]  

(27)

This action is a functional of the four variables: \( \alpha, \beta, \nu, \rho \). Note that the last term of \( L_r \) is a full time differential and thus does not contribute to the equations of motion. In order to simplify the variational calculations we introduce the notations:

\[ \tilde{u} \equiv \alpha \tilde{\nabla} \beta + \tilde{\nabla} \nu \quad \text{and} \quad \tilde{D} \equiv \frac{\partial}{\partial t} + \tilde{u} \cdot \tilde{\nabla}. \]  

(28)

Using the above notations and assuming that the arbitrary variations \( (\delta \rho, \delta \nu, \delta \alpha, \delta \beta) \) vanish for \( \tilde{r} \) and \( t \) sufficiently large we obtain that \( \delta A_r = 0 \) only if the following equations are satisfied:

\[ \frac{\partial \rho}{\partial t} + \tilde{\nabla} \cdot (\rho \tilde{u}) = 0 \]  

(29)

\[ \frac{\partial (\rho \alpha)}{\partial t} + \tilde{\nabla} \cdot (\rho \tilde{u} \alpha) = 0 \]  

(30)

\[ \frac{\tilde{D} \beta}{\tilde{D} t} = 0 \]  

(31)

\[ \frac{\tilde{D} \nu}{\tilde{D} t} = \frac{1}{2} \tilde{u}^2 - h - \Phi. \]  

(32)
Combining equation (29) and equation (30) we arrive at the following set of equations:

\[
\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \quad (33)
\]

\[
\frac{\bar{D} \alpha}{Dt} = 0 \quad (34)
\]

\[
\frac{\bar{D} \beta}{Dt} = 0 \quad (35)
\]

\[
\frac{\bar{D} \nu}{Dt} = \frac{1}{2} \vec{u}^2 - h - \Phi. \quad (36)
\]

If we take \( \vec{u} \) to be a Clebsch representation of some velocity field \( \vec{v} = \vec{u} \), then the above equations become the Clebsch transformed equations of fluid motion (equations 12, 14, 15, 16). Thus we have achieved a variational principle that does not contain \( \vec{v} \) as a variational variable. This situation in which only the potentials appear in the variational principle but not the physical velocity field \( \vec{v} \) itself is familiar from other branches of physics and will be discussed below.

It remains to discuss the Lagrangian \( L_v \) given in equation (26), using the theorem of Gauss and the definition (28) of \( \vec{u} \) we arrive at the form:

\[
L_v = \int_V \frac{1}{2} (\vec{u} - \vec{v})^2 \rho d^3 x + \int \nu \rho \vec{v} \cdot d\vec{S}. \quad (37)
\]

If we take \( \vec{u} \) to be a Clebsch representation of \( \vec{v} \) than the first part of \( L_v \) vanishes. Further more the surface integral remaining vanish for both fluids contained in a vessel in which \( \vec{v} \) is parallel to the vessels surface and for astrophysical flows in which the density vanishes on the surface of the flow. However, in the case of helical flows \( \nu \) is not single-valued and one must add in addition to the physical surface a "cut". Thus one obtains:

\[
L_v = \int_{\text{cut}} [\nu] \rho \vec{v} \cdot d\vec{S} \quad (38)
\]

where \([\nu]\) represents the discontinuity of \( \nu \) across the "cut". We conclude that for helical flows \( L_v \) does not vanish altogether. And thus for those flows the new Lagrangian \( L_r \) is not equal to the Seliger & Witham Lagrangian \( L \), the difference being a surface integral over the "cut".

### 3.1 A Comment Regarding Esthetics

It was already noted that the last term of \( L_r \) given in equation (25) does not contribute to equations of motion. It seem esthetically plausible to omit this term altogether thus obtaining:

\[
L_s = \int_V \left[ -\left( \alpha \frac{\partial \beta}{\partial t} + \alpha \frac{\partial \nu}{\partial t} \right) - \frac{1}{2} \left( \alpha \vec{\nabla} \beta + \vec{\nabla} \nu \right)^2 - \varepsilon (\rho) - \Phi \right] \rho d^3 x. \quad (39)
\]

The Lagrangian is quadratic in the spatial Clebsch representation \( \alpha \vec{\nabla} \beta + \vec{\nabla} \nu \), and linear in the temporal "Clebsch representation": \( \alpha \frac{\partial \beta}{\partial t} + \frac{\partial \nu}{\partial t} \).

\(^1\)Compare the transition from equation (1) to equation (10).
3.2 Potentials as Lagrangian Variables

The Lagrangian $L_s$ contains variables $(\alpha, \beta, \nu)$ that have no physical meaning outside the context of the Clebsch representation. This is also the situation in electromagnetics in which the Lagrangian (see for example Goldstein (1980) p. 582) is given by:

$$L_{em} = \int_V \left[ \frac{1}{2} \left( \frac{\partial \vec{A}^2}{\partial t} + 2 \vec{\nabla} A_0 \cdot \frac{\partial \vec{A}}{\partial t} + \vec{\nabla} A_0^2 - \vec{\nabla} \times \vec{A}^2 \right) + e(\vec{j} \cdot \vec{A} - \rho A_0) \right] d^3x. \quad (40)$$

In which $\vec{A}, A_0$ are the vector and scalar electromagnetic potentials, $e$ is the charge of the electron, $\vec{j}$ is the current and $\rho$ is the charge density. Only after the problem is solved in terms of the potentials $\vec{A}, A_0$ the physical electric $\vec{E}$ and magnetic $\vec{B}$ fields can be obtained through the relations:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} A_0 \quad \vec{B} = \vec{\nabla} \times \vec{A} \quad (41)$$

which are analogue to the Clebsch representation given by equation (13) in the fluid dynamical case.

3.3 Linearity in Time Derivatives

Another odd characteristic of the Lagrangian $L_s$ given by equation (39) is that it contains only linear terms in time derivatives. This is unlike the generic case in classical mechanics in which terms quadratic in time derivatives appear in the kinetical energy part the Lagrangian. For this situation I have been able to find an analogue from quantum mechanics. The Schroedinger Lagrangian is given by:

$$L_{sch} = \int \left[ -i\hbar \frac{\partial \psi}{\partial t} \psi^* + \frac{\hbar^2}{2m} \vec{\nabla} \psi \cdot \vec{\nabla} \psi^* + V \psi \psi^* \right] d^3x. \quad (42)$$

in which $\hbar = \frac{h}{2\pi}$ is the Planck Constant, $i = \sqrt{-1}$, $\psi$ is the wave function and $\psi^*$ is its complex conjugate. $L_{sch}$ is linear in time derivatives and in this respect resembles $L_s$.

4 Conclusions

Although a compact variational principle have been obtained in equation (39) it remains to see the possible implications of this expression. One possible utility is to obtain better numerical algorithms for solving fluid flow problems based on the above variational principle (see Yahalom [10]). Another may be to study the stability of certain stationary flows in the spirit of the works by Katz, Inagaki & Yahalom [11] and Yahalom, Katz & Inagaki [12].

\[2\text{Not the matter density!}\]
Further More, it is also desirable to have a variational principle for incompressible flows which should contain only three functions since the density is not a variable in this case. A similar reduction in degrees of freedom should be obtained for two dimensional flows in which the velocity has only two components. The above list of extensions of the Eulerian variational principles is of course not comprehensive. And the undertakings mentioned are left for future papers.

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