Blowup for the Euler and Euler-Poisson Equations with Repulsive Forces

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Abstract

In this paper, we study the blowup of the N-dim Euler or Euler-Poisson equations with repulsive forces, in radial symmetry. We provide a novel integration method to show that the non-trivial classical solutions \((\rho, V)\), with compact support in \([0, R]\), where \(R > 0\) is a positive constant and in the sense which \(\rho(t, r) = 0\) and \(V(t, r) = 0\) for \(r \geq R\), under the initial condition

\[
H_0 = \int_0^R rV_0 dr > 0, \tag{1}
\]

blow up on or before the finite time \(T = R^3/(2H_0)\) for pressureless fluids or \(\gamma > 1\).

The main contribution of this article provides the blowup results of the Euler \((\delta = 0)\) or Euler-Poisson \((\delta = 1)\) equations with repulsive forces, and with pressure \((\gamma > 1)\), as the previous blowup papers ([1], [2], [3] and [4]) cannot handle the systems with the pressure term, for \(C^1\) solutions.

Key Words: Euler Equations, Euler-Poisson Equations, Integration Method, Blowup, Repulsive Forces, With Pressure, \(C^1\) Solutions, No-Slip Condition

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1 Introduction

The isentropic Euler ($\delta = 0$) or Euler-Poisson ($\delta = \pm 1$) equations can be written in the following form:

$$
\begin{cases}
\rho_t + \nabla \cdot (\rho u) = 0, \\
\rho [u_t + (u \cdot \nabla) u] + \nabla P = \rho \nabla \Phi, \\
\Delta \Phi(t, x) = \delta \alpha(N) \rho,
\end{cases}
$$

(2)

where $\alpha(N)$ is a constant related to the unit ball in $\mathbb{R}^N$: $\alpha(1) = 1$, $\alpha(2) = 2\pi$ and $\alpha(3) = 4\pi$. And as usual, $\rho = \rho(t, x) \geq 0$ and $u = u(t, x) \in \mathbb{R}^N$ are the density and the velocity respectively. $P = P(\rho)$ is the pressure function. The $\gamma$-law can be applied on the pressure term $P(\rho)$, i.e.

$$
P(\rho) = K \rho^\gamma,
$$

(3)

which is a common hypothesis. If the parameter is set as $K > 0$, we call the system with pressure; if $K = 0$, we call it pressureless. The constant $\gamma = c_P/c_v \geq 1$, where $c_P$, $c_v$ are the specific heats per unit mass under constant pressure and constant volume respectively, is the ratio of the specific heats, that is, the adiabatic exponent in the equation (3). In particular, the fluid is called isothermal if $\gamma = 1$. If $K > 0$, we call the system with pressure; if $K = 0$, we call it pressureless.

In the above systems, the self-gravitational potential field $\Phi = \Phi(t, x)$ is determined by the density $\rho$ itself, through the Poisson equation (23).

When $\delta = -1$, the system can model fluids that are self-gravitating, such as gaseous stars. In addition, the evolution of the simple cosmology can be modelled by the dust distribution without pressure term. This describes the stellar systems of collisionless and gravitational $n$-body systems [9]. And the pressureless Euler-Poisson equations can be derived from the Vlasov-Poisson-Boltzmann model with the zero mean free path [9]. For $N = 3$ and $\delta = -1$, the equations (2) are the classical (non-relativistic) descriptions of a galaxy in astrophysics. See [7] and [8], for details about the systems.

When $\delta = 1$, the system is the compressible Euler-Poisson equations with repulsive forces. The equation (23) is the Poisson equation through which the potential with repulsive forces is determined by the density distribution of the electrons. In this case, the system can be viewed as a
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On the other hand, the Poisson equation (2) can be solved as

$$\Phi(t, x) = \delta \int_{\mathbb{R}^N} G(x - y) \rho(t, y) dy,$$

where $G$ is Green’s function for the Poisson equation in the $N$-dimensional spaces defined by

$$G(x) = \begin{cases} 
|x|, & N = 1; \\
\log |x|, & N = 2; \\
\frac{-1}{|x|^{N-2}}, & N \geq 3.
\end{cases}$$

Usually, the Euler-Poisson equations can be rewritten in the scalar form:

$$\begin{cases} 
\frac{\partial \rho}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial \rho}{\partial x_k} + \rho \sum_{k=1}^{N} \frac{\partial u_k}{\partial x_k} = 0, \\
\rho \left( \frac{\partial V}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial V}{\partial x_k} \right) + \frac{\partial P}{\partial x_i} = \rho \frac{\partial \Phi}{\partial x_i}, \text{ for } i = 1, 2, \ldots N.
\end{cases}$$

For the construction of the analytical solutions for the systems, interested readers should refer to [11], [12], [13], [14] and [15]. The results for local existence theories can be found in [16], [17] and [18]. The analysis of stabilities for the systems may be referred to [19], [20], [21], [1], [2], [3], [22], [13], [23], [24], [4] and [25].

We seek the radial symmetry solutions

$$\rho(t, \bar{x}) = \rho(t, r) \text{ and } \bar{u} = \frac{\bar{x}}{r} V(t, r) =: \frac{\bar{x}}{r} V,$$

with the radius $r = \left( \sum_{i=1}^{N} x_i^2 \right)^{1/2}$.

For the solutions in spherical symmetry, the Poisson equation (2) is transformed to

$$r^{N-1} \Phi_{rr} (t, x) + (N - 1) r^{N-2} \Phi_r = \alpha(N) \delta \rho r^{N-1},$$

$$\Phi_r = \frac{\alpha(N) \delta}{r^{N-1}} \int_0^r \rho(t, s) s^{N-1} ds.$$

By standard computation, the Euler-Poisson equations in radial symmetry can be written in the following form:

$$\begin{cases} 
\rho_t + V \rho_r + \rho V_r + \frac{N-1}{r} \rho V = 0, \\
\rho (V_t + VV_r) + P_r(\rho) = \rho \Phi_r (\rho).
\end{cases}$$
Historically, Makino, Ukai and Kawashima initially defined the tame solutions \([1]\) for outside the compact of the solutions

\[ V_t + VV_r = 0. \]  

(11)

Following this, Makino and Perthame considered the tame solutions for the system with gravitational forces \([2]\). After that Perthame discovered the blowup results for 3-dimensional pressureless system with repulsive forces \([3]\) (\(\delta = 1\)). In short, all the results above rely on the solutions with radial symmetry:

\[ V_t + VV_r = \frac{\alpha(N)\delta}{r^{N-1}} \int_0^r \rho(t,s)s^{N-1}ds. \]  

(12)

And the Emden ordinary differential equations were deduced on the boundary point of the solutions with compact support:

\[ \frac{D^2 R}{Dt^2} = \frac{\delta M}{R^{N-1}}, \quad R(0, R_0) = R_0 \geq 0, \quad \dot{R}(0, R_0) = 0, \]  

(13)

where \(\frac{dR}{dt} := V\) and \(M\) is the mass of the solutions, along the characteristic curve. They showed the blowup results for the \(C^1\) solutions of the system \([10]\).

Recently, Chae and Tadmor \([4]\) showed the finite time blowup, for the pressureless Euler-Poisson equations with attractive forces (\(\delta = -1\)), under the initial condition,

\[ S := \{ a \in R^N \mid \rho_0(a) > 0, \quad \Omega_0(a) = 0, \quad \nabla \cdot u(0, x(0) < 0) \neq \phi, \} \]  

(14)

where \(\Omega\) is the rescaled vorticity matrix \((\Omega_{ij}) = \frac{1}{2}(\partial_i u_j^0 - \partial_j u_i^0)\) with the notation \(u = (u^1, u^2, \ldots, u^N)\) in their paper and some point \(x_0\).

They use the analysis of spectral dynamics to show the Racatti differential inequality,

\[ \frac{D \text{div } u}{Dt} \leq -\frac{1}{N} (\text{div } u)^2. \]  

(15)

The solution for the inequality \([15]\) blows up on or before \(T = -N/\nabla \cdot u(0, x_0(0))\).

However, their method cannot be applied to the system with repulsive forces to obtain the similar blowup result.

On the other hand, in \([24]\), we have the blowup results if the solutions with compact support under the condition,

\[ 2 \int_{\Omega(t)} (\rho |u|^2 + 2P)dx < M^2 - \epsilon, \]  

(16)
where $M$ is the mass of the solution.

In this article, the alternative approach is adopted to show that there is no global existence of $C^1$ solutions for the system, (4) ($\delta = 0$ or $\delta = 1$), with compact support without the condition (14). We notice that the conditions in our result are different from the works of Engerlberg et al [26].

**Theorem 1** Consider the $N$-dimensional Euler ($\delta = 0$) or Euler-Poisson equations with repulsive forces ($\delta = 1$) [2]. The non-trivial classical solutions $(\rho, V)$, in radial symmetry, with compact support in $[0, R]$, where $R > 0$ is a positive constant (which $\rho(t,r) = 0$ and $V(t,r) = 0$ for $r \geq R$) and the initial velocity such that:

$$H_0 = \int_0^R rV_0 dr > 0,$$

(17)

blow up on or before the finite time $T = R^3/(2H_0)$, for pressureless fluids ($K = 0$) or $\gamma > 1$.

The solutions $(\rho, u)$ may lose their regularity, for example the velocity function $V \in C^0$ only or the shock waves appear on or before the finite time $T$.

## 2 Integration Method

In this section, we present the proof of Theorem 1. The technique of the proof was selected simply to deduce the partial differential equations to the Racatti equation, to show the blowup result. However, we note our integration method is novel to the studies of blowup for this kind of the systems.

**Proof.** In general, we show that the $\rho(t, x(t; x))$ preserves its positive nature as the mass equation (11) can be converted to be

$$\frac{D\rho}{Dt} + \rho \nabla \cdot u = 0,$$

(18)

with the material derivative,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (u \cdot \nabla).$$

(19)

We integrate the equation (18):

$$\rho(t, x) = \rho_0(x_0(0, x_0)) \exp \left( - \int_0^t \nabla \cdot u(t, x(t; 0, x_0)) dt \right) \geq 0,$$

(20)
for \( \rho_0(x_0(0, x_0)) \geq 0 \), along the characteristic curve.

We use the momentum equation (10) with the non-trivial solutions in radial symmetry, \( \rho_0 \neq 0 \), to have:

\[
V_t + V V_r + K \gamma \rho^{\gamma - 2} \rho_r = \Phi_r, 
\]

\[
V_t + \frac{\partial}{\partial r} \left( \frac{1}{2} V^2 \right) + K \gamma \rho^{\gamma - 2} \rho_r = \Phi_r, 
\]

\[
rV_t + r \frac{\partial}{\partial r} \left( \frac{1}{2} V^2 \right) + K \gamma r \rho^{\gamma - 2} \rho_r = r \Phi_r, 
\]

with multiplying \( r \) on the both sides.

We take integration with respect to \( r \), to the above equation, for \( \gamma > 1 \) or \( K \geq 0 \):

\[
\int_0^R rV_t dr + \int_0^R r \frac{d}{dr} \left( \frac{1}{2} V^2 \right) + \int_0^R K \gamma \rho^{\gamma - 2} \rho_r dr = \int_0^R r \Phi_r dr, 
\]

\[
\int_0^R rV_t dr + \int_0^R r \frac{d}{dr} \left( \frac{1}{2} V^2 \right) + \int_0^R K \gamma r \rho^{\gamma - 1} dr = \int_0^R \left[ \alpha(N) \delta r - \int_0^r \rho(t, s) s^{N - 1} ds \right] dr, 
\]

\[
\int_0^R rV_t dr + \int_0^R r \frac{d}{dr} \left( \frac{1}{2} V^2 \right) + \int_0^R K \gamma r \rho^{\gamma - 1} \geq 0, 
\]

for \( \delta \geq 0 \).

It follows with integration by part:

\[
\int_0^R rV_t dr - \frac{1}{2} \int_0^R V^2 dr + \frac{1}{2} \left[ RV(t, R)^2 - 0 \cdot V(t, 0)^2 \right] - \int_0^R K \gamma \rho^{\gamma - 1} dr + \frac{K \gamma}{\gamma - 1} \left[ R \rho^{\gamma - 1}(t, R) - 0 \cdot \rho^{\gamma - 1}(t, 0) \right] \geq 0. 
\]

(27)

The above inequality with the boundary compact condition of \( V(t, R) = 0 \) and \( \rho(t, R) = 0 \), becomes

\[
\int_0^R rV_t dr - \frac{1}{2} \int_0^R V^2 dr - \int_0^R K \gamma \rho^{\gamma - 1} dr = 0. 
\]

(28)

As \( r \) and \( t \) are independent variables and \( V \) is \( C^1 \) in the domain [0, \( R \)] in the assumption of the theorem, we may change the differentiation and the integration as the following:

\[
\frac{d}{dt} \int_0^R rV dr - \frac{1}{2} \int_0^R V^2 dr - \int_0^R K \gamma \rho^{\gamma - 1} dr \geq 0, 
\]

(29)

\[
\frac{d}{dt} \int_0^R V dr^2 - \frac{1}{2} \int_0^R \frac{1}{2r} V^2 dr^2 \geq \int_0^R K \gamma \rho^{\gamma - 1} dr \geq 0, 
\]

(30)

for \( \gamma > 1 \) or \( K = 0 \).

For the non-trivial initial condition \( \rho_0 \geq 0 \), we have the following differential inequality:

\[
\frac{d}{dt} \int_0^R V dr^2 - \frac{1}{2} \int_0^R \frac{1}{2r} V^2 dr^2 \geq 0, 
\]

(31)
\[
\frac{d}{dt} \int_0^R V^2 dr \geq \frac{1}{2R} \int_0^R V^2 dr^2 \geq \frac{1}{2} \int_0^R V^2 dr^2, 
\]
\[
\frac{d}{dt} \int_0^R V^2 dr \geq \frac{1}{2R} \int_0^R V^2 dr^2. 
\]

By denoting
\[
H := H(t) = \int_0^R rV dr = \frac{1}{2} \int_0^R V dr^2, 
\]
and with the Cauchy-Schwarz inequality,
\[
\left| \int_0^R V \cdot 1 \, dr \right| \leq \left( \int_0^R V^2 dr^2 \right)^{1/2} \left( \int_0^R 1 \, dr^2 \right)^{1/2}, 
\]
\[
\frac{\left| \int_0^R V^2 dr \right|}{R} \leq \left( \int_0^R V^2 dr^2 \right)^{1/2}, 
\]
\[
\frac{4H^2}{R^2} \leq \int_0^R V^2 dr^2, 
\]
\[
\frac{2H^2}{R^3} \leq \frac{1}{2R} \int_0^R V^2 dr^2, 
\]
the inequality \((32)\) becomes
\[
\frac{d}{dt} H \geq \frac{1}{2R} \int_0^R V^2 dr^2 \geq \frac{2H^2}{R^3}, 
\]
\[
\frac{d}{dt} H \geq \frac{2H^2}{R^3}. 
\]

With the initial condition: \(H_0 = \int_0^R rV_0 dr > 0\), we can obtain
\[
H \geq -\frac{R^3 H_0}{2H_0 t - R^3}. 
\]

Therefore, the solutions blow up on or before the finite time \(T = R^3/(2H_0)\).

This completes the proof. ■

**Remark 2** For controlled experiments in engineering, fluids are kept in a fixed ball solid container with a radial \(R\). Therefore, it requires the compact support condition for \(t \geq 0\),
\[
\rho(t, r) = 0 \quad \text{and} \quad V(t, r) = 0, 
\]
with \(r \geq R\). This corresponding condition is called no-slip condition (solid boundary condition) \([27]\) and \([28]\).

On the other hand, in computing simulations, the systems are usually coupled with the similar
boundary conditions for real applications. Therefore, the condition for compact support (no-slip condition) is reasonable in modeling. But for free boundary problems, fluids may not be bounded by a fixed volume for all time. Therefore, further research is needed to study the corresponding result in future works.

**Remark 3** It is still an open question whether or not there exists time-local $C^1$-solution with compact support for any initial condition with compact support. On the other hand, if the global solutions with compact support whose radii expand unboundedly as time tends to infinity, the discussion of this paper can offer no information about this case.

**Remark 4** This article has shed new light on situations with the pressure term. In particular, it provides the blowup results of the Euler ($\delta = 0$) or Euler-Poisson ($\delta = 1$) equations with repulsive forces, and with pressure ($\gamma > 1$). This is the main contribution of the article, as the previous blowup papers ([1], [3], [4], and [5]) cannot handle the systems with the pressure term, for $C^1$ solutions. A further refinement for the non-radial symmetry is expected in future studies.

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