A description of the neutralino observables in terms of projectors

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Abstract

Applying Jarlskog’s treatment of the CKM matrix, to the neutralino mass matrix in MSSM for real soft gaugino SUSY breaking and $\mu$-parameters, we construct explicit analytical expressions for the four projectors which acting on any neutralino state project out the mass eigenstates. Analytical expressions for the neutralino mass eigenvalues in terms of the various SUSY parameters, are also given. It is shown that these projectors and mass eigenvalues are sufficient to describe any physical observable involving neutralinos, to any order of perturbation theory. As an example, the $e^-e^+ \rightarrow \tilde{\chi}^0_i \tilde{\chi}^0_j$ cross section at tree level is given in terms of these projectors. The expected magnitude of their various matrix elements in plausible SUSY scenarios is also discussed.

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1 Introduction

If the Minimal Supersymmetric Extension of the Standard Model (MSSM) is realized in nature, then some at least of the four possible neutralinos, should be among the lightest new particles to exist \[1\]. This is particularly true if R-parity is conserved, so that the lightest neutralino is absolutely stable and probably describes the Dark Matter.

Since the four neutralinos are generally mixed, their study is complicated, and it thus necessitates the discovery of efficient physically motivated ways to describe it. At the tree level, this mixing depends on the SUSY breaking gaugino masses \(M_1, M_2\), and the \(\mu\) Higgs sector parameter. These parameters could be real or complex depending on whether the new (beyond the Standard Model) interactions contained in MSSM respect the CP-symmetry or not. There already exist many papers where observables related to neutralinos have been calculated in terms of this mixing \[2, 3, 4, 5\].

The main purpose of the present paper it to show that for real \((M_1, M_2, \mu)\), all possible neutralino observables can be described in terms of the matrix elements of four neutralino projections operators (one for each neutralino), and the neutralino masses. For all of them, explicit analytical expressions are then derived. Since these projection matrix elements and masses are themselves physically observable; a considerable economy in the description is thereby achieved.

In order to state our results and establish our notation, we first note that the neutralino mass term in the minimal SUSY Lagrangian is given by \[1\]

\[
\mathcal{L}_m = -\frac{1}{2} \psi_L^0 \mathcal{C} Y \psi_L^0 + \text{h.c.},
\]

where \(\mathcal{C} = i\gamma^2 \gamma^0\) in \[4\] is the usual Dirac charge conjugation matrix, while the L-components of all neutralino fields are described in the "weak basis" by the column vector

\[
\psi_L^0 \equiv \begin{pmatrix} \tilde{B}_L \\ \tilde{W}^3_L \\ \tilde{H}^0_{1L} \\ \tilde{H}^0_{2L} \end{pmatrix}.
\]

The mass-matrix \(Y\) is of course symmetric and given by

\[
Y = \begin{pmatrix}
M_1 & 0 & -m_Z s_W c_\beta & m_Z s_W s_\beta \\
0 & M_2 & m_Z c_W c_\beta & -m_Z c_W s_\beta \\
-m_Z s_W c_\beta & m_Z c_W c_\beta & 0 & -\mu \\
m_Z s_W s_\beta & -m_Z c_W s_\beta & -\mu & 0 \\
\end{pmatrix},
\]

where \(s_\beta \equiv \sin \beta\), \(c_\beta \equiv \cos \beta\). The only neutralino fields entering the complete MSSM Lagrangian are the components of \(\psi^0_{\alpha L}\) in \[4\] and their hermitian conjugates.

If \((M_1, M_2, \mu)\) are real, then the symmetric matrix \(Y\) in \[3\] can be diagonalized.
through the real orthogonal transformation $U^0$ giving

$$U^{0*}YU^0 = \begin{pmatrix}
\tilde{m}_{\tilde{\chi}_1^0} & 0 & 0 & 0 \\
0 & \tilde{m}_{\tilde{\chi}_2^0} & 0 & 0 \\
0 & 0 & \tilde{m}_{\tilde{\chi}_3^0} & 0 \\
0 & 0 & 0 & \tilde{m}_{\tilde{\chi}_4^0}
\end{pmatrix},$$

(4)

where the real eigenvalues $\tilde{m}_{\tilde{\chi}_j^0}$ can be of either sign and have been ordered so that

$$|\tilde{m}_{\tilde{\chi}_1^0}| \leq |\tilde{m}_{\tilde{\chi}_2^0}| \leq |\tilde{m}_{\tilde{\chi}_3^0}| \leq |\tilde{m}_{\tilde{\chi}_4^0}| .$$

(5)

We call $\tilde{m}_{\tilde{\chi}_j^0}$ the "signed" neutralino masses, while the physical (positive) neutralino masses $m_{\tilde{\chi}_j^0} > 0$ are related to them by

$$\tilde{m}_{\tilde{\chi}_j^0} = \eta_j m_{\tilde{\chi}_j^0} \quad \text{with} \quad \eta_j = \pm 1 .$$

(6)

As it is well known, the signs $\eta_j$ in (6) determine the CP-eigenvalues of the physical neutralino fields $\tilde{\chi}_j^0$ ($j = 1 - 4$). Their L-parts are related to the "weak basis" L-neutralino fields $\Psi_{\alpha L}^0$ ($\alpha = 1 - 4$), by

$$\Psi_{\alpha L}^0 = \sum_{j=1}^{4} U_{\alpha j}^0 \tilde{\chi}_j^0 L \tilde{\eta}_j ,$$

(7)

where we define $(\tilde{\eta}_j = 1 \text{ or } i)$ depending on whether $(\eta_j = 1 \text{ or } \, -1)$. Thus, $\eta_j = \tilde{\eta}_j^2$.

In $U_{\alpha j}^0$ at (7), the first index $\alpha$ counts the weak basis neutralino fields, while the second index $j$ refers the mass eigenstate neutralinos. Using it, we then remark that the projector matrix to the $j$-th neutralino state is given by

$$P_j = U^0 E_j U^{0*} ,$$

(8)

where the matrix elements of the four-by-four basic matrices $E_j$ are $(E_j)_{ik} \equiv \delta_{ij} \delta_{jk}$, so that the weak basis matrix elements of $P_j$ are

$$P_{j\alpha\beta} = U_{\alpha j}^0 U_{\beta j}^0 .$$

As expected, the usual projector relations

$$P_j P_i = P_j \delta_{ji} , \quad Tr P_j = 1 , \quad P_j = P_j^T = P_j^\dagger ,$$

$$Y = \sum_{j=1}^{4} \eta_j m_{\tilde{\chi}_j^0} P_j .$$

(9)

are satisfied by (8).

As already stated, in this paper we first construct explicit expressions for the projectors to the neutralino mass eigenstates. This is done in analogy to the Jarlskog’s treatment

\footnote{Here of course, there is no summation over $j$.}
of the CKM Matrix and it is presented in Section 2 [8]. In the first version of these expressions, the projectors are written in terms of the \((M_1, M_2, \mu)\) parameters and the neutralino masses and CP eigenvalues. But versions are also given in which the projectors are expressed in terms of \((M_1, M_2, \mu)\) and e.g. the masses of only the two lightest, or even only the very lightest neutralino. Phenomenologically these later expressions may be more useful in situations where only one or two of the lightest neutralino masses are known [4, 5].

In the same Section 2 we also give an analytic solution of the characteristic equation for \(Y\), which determines the signed masses \(\tilde{m}_{\tilde{\chi}^0_j}\) in terms of \((M_1, M_2, \mu)\). In principle this neutralino-mass solution should be equivalent to the one presented in [7]. Nevertheless we give it here since its form is somewhat different, and because it is very useful for constructing the aforementioned projectors.

In Section 3 we then show that all neutralino propagators that can possibly appear in MSSM, are completely expressed in terms of the above projectors and the signed neutralino masses. This then leads to the conclusion that all neutralino information contained in any cross section involving either virtual or external neutralinos, is fully described in terms of the neutralino projectors and signed neutralino masses. As an example, we give the tree level formulae for the \(e^-e^+ \to \tilde{\chi}_i^0\tilde{\chi}_j^0\) cross section expressed this way. In the same Section we also discuss the numerical expectations for the matrix elements of the neutralino projectors, in various SUSY scenarios. A summary of the conclusions is given in Section 4.

2 The neutralino Projectors and masses.

The characteristic equation for the neutralino mass matrix generated by \(Y\) in [8] is

\[
x^4 - Ax^3 + Bx^2 - Cx + D = 0,
\]

with

\[
A = M_1 + M_2,
B = -m_Z^2 - \mu^2 + M_1 M_2,
C = -M_1(\mu^2 + m_Z^2 c_W^2) - M_2(\mu^2 + m_Z^2 s_W^2) + \mu^2 m_Z^2 s_{2\beta},
D = -M_1 M_2 \mu^2 + \mu^2 m_Z^2 (M_1 c_W^2 + M_2 s_W^2) s_{2\beta},
\]

where \(s_{2\beta} \equiv 2s_\beta c_\beta\)

There should exist four real solutions of (10) determining the signed masses \(\tilde{m}_{\tilde{\chi}^0_j}\), and thereby \(\eta_j\) and \(m_{\tilde{\chi}^0_j}\); compare [8]. To determine them we first construct one real root of the auxiliary cubic equation [8]

\[
\theta^3 + a\theta + b = 0,
\]

\[4\]
a \equiv \frac{1}{4} \left( -\frac{AC}{4} + \frac{B^2}{12} + D \right), \\
b \equiv \frac{1}{4} \left( -\frac{A^2 D}{16} + \frac{ABC}{48} - \frac{B^3}{216} + \frac{BD}{6} - \frac{C^2}{16} \right). \tag{12}

Depending on the signs of \(a\) and \(\Delta\) defined as

\[ \Delta \equiv \frac{b^2}{4} + \frac{a^3}{27}, \tag{13} \]

the needed single real root of (12) is constructed as:

- if \(\Delta \leq 0, \ a < 0\), then
  \[ \cos(3\phi) \equiv -\frac{b}{2} \left( \frac{3}{|a|} \right)^{3/2}, \quad \theta = 2 \sqrt{\frac{|a|}{3}} \cos \left( \phi + \frac{2n\pi}{3} \right) \tag{14} \]
  where any of the three choices \(n = 1, 2, 3\) may be used.

- if \(\Delta > 0, \ a < 0\), then
  \[ \cosh(3\phi) = \frac{|b|}{2} \left( \frac{3}{|a|} \right)^{3/2}, \quad \theta = -2 \ \text{Sign}(b) \sqrt{\frac{|a|}{3}} \cosh(\phi) \tag{15} \]

- if \(\Delta > 0, \ a > 0\), then
  \[ \sinh(3\phi) = -\frac{b}{2} \left( \frac{3}{|a|} \right)^{3/2}, \quad \theta = 2 \sqrt{\frac{|a|}{3}} \sinh(\phi) \tag{16} \]

We should remark at this point, that in all MSSM case studies we are aware of, the situation \((\Delta \leq 0, \ a < 0)\) is met, which indicates that (14) is probably the most useful case.

Using \(\theta\) and defining also

\[ E = \frac{1}{4} \left( A^2 - \frac{8B}{3} + 16\theta \right)^{1/2}, \quad F = \frac{1}{4E} \left( C - \frac{AB}{6} - 2A\theta \right) \tag{17} \]

we obtain the four signed neutralino masses from

\[ \tilde{m}_{\tilde{\chi}_j^0} \equiv \eta_j m_{\tilde{\chi}_j^0} = \frac{1}{2} \left\{ \left( \frac{A}{2} - 2E \right) \pm \sqrt{\left( \frac{A}{2} - 2E \right)^2 - 4 \left( \frac{B}{6} + 2\theta + F \right)} \right\}, \]

\[ = \frac{1}{2} \left\{ \left( \frac{A}{2} + 2E \right) \pm \sqrt{\left( \frac{A}{2} + 2E \right)^2 - 4 \left( \frac{B}{6} + 2\theta - F \right)} \right\}. \tag{18} \]
The general expressions for the parameters \((A - F)\) do not allow us to order equations (18), so that (19) is satisfied in any model. This can efficiently be done in specific models though. As such we choose to consider the thirteen benchmark SUGRA scenarios of [9], which are more or less consistent with what is presently known from LEP, \(b \to s \gamma\), \(g_\mu - 2\) and cosmology. As the scale in which to apply relations (18) (as well as the formulae for the neutralino projectors below) we take \(2^3 Q = \sqrt{\tilde{m}_{i_1} \tilde{m}_{i_2}}\), in agreement with the one chosen in Table 3 of [9]. This allows us to check that in all cases the absolute values of \(\tilde{m}_{\chi^0_1}, \tilde{m}_{\chi^0_2}, \tilde{m}_{\chi^0_3}, \tilde{m}_{\chi^0_4}\) agree with those quoted there. For all the above scenarios we then find that \(\tilde{m}_{\chi^0_3}\) is negative, while all other “signed” masses are positive. In Table 1 we present them together with the \((M_1, M_2, \mu)\)-parameters at the scale \(Q\), for seven of the benchmark scenarios of [9], which are characterised by neutralinos that are sufficiently light to be producible in the future Colliders. At the end of the next Section, we come back to the consideration of these scenarios.

Table 1: Parameters at the scale \(Q = \sqrt{\tilde{m}_{i_1} \tilde{m}_{i_2}}\), for some SUGRA scenarios of [9]. (Dimensions in GeV.)

| Scenario | \(Q\) | \(M_2\) | \(M_1\) | \(\mu\) | \(\tan \beta\) | \(\tilde{m}_{\chi^0_1}\) | \(\tilde{m}_{\chi^0_2}\) | \(\tilde{m}_{\chi^0_3}\) | \(\tilde{m}_{\chi^0_4}\) |
|----------|------|------|------|------|------|------|------|------|------|
| A        | 1050 | 480  | 254  | 768  | 5    | 252  | 467  | -770 | 785  |
| B        | 726  | 317  | 165  | 520  | 10   | 163  | 303  | -524 | 540  |
| C        | 937  | 421  | 221  | -662 | 10   | 221  | 414  | -667 | 674  |
| D        | 1129 | 245  | 125  | 255  | 10   | 117  | 197  | -262 | 318  |
| E        | 687  | 299  | 161  | 485  | 20   | 159  | 286  | -490 | 505  |
| F        | 650  | 279  | 146  | 454  | 35   | 144  | 266  | -460 | 475  |
| G        | 826  | 361  | 187  | 560  | 45   | 186  | 349  | -565 | 578  |

We next turn to the projectors to the neutralino \(\tilde{\chi}_j^0\) mass-eigenstate, defined in (8). In the most interesting case that all neutralino masses are different from each other, these projectors may be written, in analogy to the CKM treatment of Jarlskog [3], as:

\[
P_1 = \frac{(\tilde{m}_{\chi^0_1} - Y)(\tilde{m}_{\chi^0_2} - Y)(\tilde{m}_{\chi^0_2} - Y)}{(\tilde{m}_{\chi^0_2} - \tilde{m}_{\chi^0_1})(\tilde{m}_{\chi^0_1} - \tilde{m}_{\chi^0_1})(\tilde{m}_{\chi^0_2} - \tilde{m}_{\chi^0_1})},
\]

\[
P_2 = \frac{(\tilde{m}_{\chi^0_1} - Y)(\tilde{m}_{\chi^0_2} - Y)(Y - \tilde{m}_{\chi^0_1})}{(\tilde{m}_{\chi^0_2} - \tilde{m}_{\chi^0_1})(\tilde{m}_{\chi^0_1} - \tilde{m}_{\chi^0_1})(\tilde{m}_{\chi^0_2} - \tilde{m}_{\chi^0_1})},
\]

\[
P_3 = \frac{(\tilde{m}_{\chi^0_1} - Y)(Y - \tilde{m}_{\chi^0_2})(Y - \tilde{m}_{\chi^0_2})}{(\tilde{m}_{\chi^0_2} - \tilde{m}_{\chi^0_1})(\tilde{m}_{\chi^0_1} - \tilde{m}_{\chi^0_1})(\tilde{m}_{\chi^0_2} - \tilde{m}_{\chi^0_1})},
\]

\(^2\text{Here} \tilde{m}_{i_1}\text{ and } \tilde{m}_{i_2}\text{ are the masses of the two stop quarks.}\)

\(^3\text{If needed, the generalization to the improbable case of neutralino mass degeneracy can easily be done following the instructions in [8].}\)
\[ P_4 = \frac{(Y - \tilde{m}_{\chi_0})(Y - \tilde{m}_{\chi_0})(Y - \tilde{m}_{\chi_0})}{(\tilde{m}_{\chi_0} - \tilde{m}_{\chi_0})(\tilde{m}_{\chi_0} - \tilde{m}_{\chi_0})(\tilde{m}_{\chi_0} - \tilde{m}_{\chi_0})}, \quad (19) \]

where \( Y \) is given in (3).

In (19), the projectors are written in terms of the matrix \( Y \) and all the "signed" neutralino masses. By using the explicit form of \( Y \), and the theory of characteristic equations (17), a more useful expression is obtained, which, apart from \( Y \)-matrix elements, involves only the corresponding signed neutralino mass. To present them we define (\( i = 1 - 4 \)),

\[ P_i = \frac{\tilde{P}_i}{\Delta_i}, \quad (20) \]

with

\[ \Delta_i \equiv -3\tilde{m}_{\chi_i}^4 + 2m_{\chi_i}^3 A - \tilde{m}_{\chi_i}^2 B + D, \quad (21) \]

where \( A, B, D \) are given in (11). The matrix elements for \( \tilde{P}_i \) (\( i = 1, 4 \)) then read

\[
\begin{align*}
\tilde{P}_{i11} &= -\tilde{m}_{\chi_i}^2 M_1 + \tilde{m}_{\chi_i}^2 (M_1 M_2 - m_2^2 s_\omega^2) + \tilde{m}_{\chi_i} [M_1 \mu^2 + m_2^2 (M - \mu s_\omega^2 s_\beta)] + D, \\
\tilde{P}_{i12} &= -\tilde{m}_{\chi_i}^2 M_2 + \tilde{m}_{\chi_i}^2 (M_1 M_2 - m_2^2 c_\omega^2) + \tilde{m}_{\chi_i} [M_2 \mu^2 + m_2^2 (M - \mu c_\omega^2 s_\beta)] + D, \\
\tilde{P}_{i13} &= -\tilde{m}_{\chi_i}^2 (\mu^2 + m_2^2 c_\omega^2) + \tilde{m}_{\chi_i} [\mu^2 (M_1 + M_2) + m_2^2 (-\mu s_\beta + M c_\omega^2)] + D, \\
\tilde{P}_{i14} &= -\tilde{m}_{\chi_i}^2 (\mu^2 + m_2^2 s_\omega^2) + \tilde{m}_{\chi_i} [\mu^2 (M_1 + M_2) + m_2^2 (-\mu s_\beta + M s_\omega^2)] + D, \\
\tilde{P}_{i21} &= \tilde{m}_{\chi_i}^2 m_2 s_\omega c_{\omega D} (\tilde{m}_{\chi_i} + \mu s_\beta), \\
\tilde{P}_{i22} &= \tilde{m}_{\chi_i}^2 m_2 s_\omega c_{\omega D} (\tilde{m}_{\chi_i} + \mu c_\beta), \\
\tilde{P}_{i23} &= -\tilde{m}_{\chi_i}^2 M_1 c_\omega s_{\omega D} (\tilde{m}_{\chi_i} c_\beta + \mu s_\beta), \\
\tilde{P}_{i24} &= -\tilde{m}_{\chi_i}^2 M_1 s_\omega s_{\omega D} (\tilde{m}_{\chi_i} s_\beta + \mu c_\beta), \\
\tilde{P}_{i31} &= \tilde{m}_{\chi_i} [\tilde{m}_{\chi_i}^2 \mu + \tilde{m}_{\chi_i}^2 (-\mu (M_1 + M_2) + m_2^2 s_\omega^2)] + \mu M_1 M_2 - M m_2^2 s_\omega^2 s_\beta, \quad (22)
\end{align*}
\]

where \( M \equiv M_1 c_\omega s_{\omega D} + M_2 s_\omega s_{\omega D} \). As seen from (21, 22), each of these projector expression only involves the corresponding neutralino mass \( \tilde{m}_{\chi_i} \) and the \( (M_1, M_2, \mu) \)-parameters. We have verified that these expressions satisfy the constraints (9).

As advocated in 4, 5, when the neutralinos will start being discovered, it would be interesting to consider situations where only \( \tilde{\chi}_1^0 \), or \( \tilde{\chi}_1^0 \) and \( \tilde{\chi}_2^0 \), have been seen. In such a case it would be advantageous to have projector expressions, in which only the lightest one or two neutralino masses will be contained.

Thus if we assume that \( m_{\chi_1^0}, m_{\chi_2^0} \) as well as the the signs \( \eta_1, \eta_2 \) are already known, then the expressions (14) for \( A, B, C, D \) and their connection to the \( Y \)-eigenvalues determine.

\footnote{Notice that the requirement \( |\tilde{m}_{\chi_i}| \leq |\tilde{m}_{\chi_i}| \), fully determines the identification of the solutions.}
so that the projectors in (20, 22) can be expressed in terms of just the signed masses of the two lightest neutralinos, and an independent combination of $M_1, M_2, \mu$ [4].

Furthermore, by using the equations in Appendix B of [4], we can also express all projector elements in (20, 22) in terms of just $\tilde{m}_{\chi^0_i}$ and the needed two independent combinations of the relevant SUSY parameters. Thus, the projector formalism can be easily adapted to the various situations that might appear during the neutralino searching, [4, 5].

To avoid any possible confusion, we should emphasize at this point that the above formalism for the neutralino masses and projectors is mathematically (and numerically) equivalent to the approach of [7].

3 The neutralino Observables in MSSM.

Since the only neutralino fields entering the MSSM Lagrangian are the "weak-basis" fields appearing in (2) and their hermitian conjugates; it follows that the only neutralino propagators that can contribute in any possible diagram are just

\[
\langle 0 | \Psi_{\alpha L}(x) \vec{\psi}_{\beta L}(y) | 0 \rangle, \quad \langle 0 | \Psi_{\alpha L}^\dagger(x) \vec{\psi}_{\beta L}^\dagger(y) | 0 \rangle, \quad \langle 0 | \Psi_{\alpha L}(x) \Psi_{\beta L}^\dagger(y) | 0 \rangle, \quad \langle 0 | \Psi_{\alpha L}^\dagger(x) \Psi_{\beta L}(y) | 0 \rangle, \tag{24}
\]

where $(\alpha, \beta)$ count the weak-basis fields.

Denoting then the elementary propagator functions as

\[
\Delta_F(x-y; m) = i \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{1}{k^2 - m^2 + i\epsilon},
\]

\[
S_F^{(1)}(x-y; m) = i \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{k}{k^2 - m^2 + i\epsilon}, \tag{25}
\]

we find that the weak-basis neutralino propagators of (24) are given at lowest order by

\[
\langle 0 | \Psi_{\alpha L}^0(x) \Psi_{\beta L}^\dagger(y) | 0 \rangle = - \langle 0 | \Psi_{\alpha L}^\dagger(x) \Psi_{\beta L}^0(y) | 0 \rangle
\]

\[= - \sum_{j=1}^4 \eta_j m_{\tilde{\chi}^0_j} \left( P_{j\alpha\beta} \Delta_F(x-y; m_{\tilde{\chi}^0_j}) C \frac{1 - \gamma_5}{2} \right), \tag{26}
\]

\[
\langle 0 | \Psi_{\alpha L}^0(x) \Psi_{\beta L}^\dagger(y) | 0 \rangle = \sum_{j=1}^4 P_{j\alpha\beta} S_F^{(1)}(x-y; m_{\tilde{\chi}^0_j}) \frac{1 - \gamma_5}{2}, \tag{27}
\]

where the usual charge conjugation Dirac matrix $C$ has been given immediately after (1). Notice that (26) only involves the scalar propagator function in (23). As it is seen from (26, 27), the $\eta_j$ signs always appear multiplied by the corresponding neutralino masses. 

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Therefore, any neutralino exchange contribution in any SUSY diagram, is fully described by the neutralino signed masses and the projectors we have already constructed. The Cutkosky rules then guarantee, that the observable contribution from any external neutralino can also always be described in terms of its signed mass and projector. Therefore, the signed masses and projectors contain all observable neutralino contributions to any order of perturbation theory.

As an example of this result, we give below the tree level differential cross section for $e^{-}e^{+} \rightarrow \tilde{\chi}_1^{0}\tilde{\chi}_j^{0}$. The contributions to this process arise from $s$-channel $Z$-exchange, and $t$- and $u$-channel $\tilde{e}_L$ and $\tilde{e}_R$ exchanges. The differential cross section may then be written as

$$
\frac{d\sigma(e^{-}e^{+} \rightarrow \tilde{\chi}_1^{0}\tilde{\chi}_j^{0})}{dt} = \frac{\alpha^2 \pi}{4s^2(1 + \delta_j)} \left[ \Sigma_Z + \Sigma_{\tilde{e}_L} + \Sigma_{\tilde{e}_R} + \Sigma_{Z\tilde{e}_L} + \Sigma_{Z\tilde{e}_R} \right],
$$

with the r.h.s. representing respectively the $Z$-, $\tilde{e}_L$- and $\tilde{e}_R$-square contributions, as well as the $Z\tilde{e}_L$- and $Z\tilde{e}_R$-interferences. These can be written as

$$\Sigma_Z = \frac{g_{ve}^2 + g_{ae}^2}{2s_W^4c_W^4(s - m_Z^2)^2} \left[ P_{i33}P_{j33} + P_{i44}P_{j44} - 2P_{i34}P_{j34} \right]
\cdot \left[ (t - m_\tilde{\chi}_0^2)(t - m_{\tilde{\chi}_j}^2) + (u - m_\tilde{\chi}_0^2)(u - m_{\tilde{\chi}_j}^2) - 2s_jm_{\tilde{\chi}_0}m_{\tilde{\chi}_j} \right],
$$

$$\Sigma_{\tilde{e}_L} = \frac{1}{4s_W^4c_W^4}[c_W^2P_{i22} + s_W^2P_{i11} + 2s_Wc_WP_{i12}][c_W^2P_{j22} + s_W^2P_{j11} + 2s_Wc_WP_{j12}]
\cdot \left[ (t - m_{\tilde{\chi}_0}^2)(t - m_{\tilde{\chi}_j}^2) + (u - m_{\tilde{\chi}_0}^2)(u - m_{\tilde{\chi}_j}^2)
- 2s_jm_{\tilde{\chi}_0}m_{\tilde{\chi}_j} \right],$$

$$\Sigma_{\tilde{e}_R} = \frac{4P_{i11}P_{j11}}{c_W^4} \left[ (t - m_{\tilde{\chi}_0}^2)(t - m_{\tilde{\chi}_j}^2) + (u - m_{\tilde{\chi}_0}^2)(u - m_{\tilde{\chi}_j}^2)
- 2s_jm_{\tilde{\chi}_0}m_{\tilde{\chi}_j} \right],$$

$$\Sigma_{Z\tilde{e}_L} = -\frac{(g_{ve} + g_{ae})}{2s_W^4c_W^4(s - m_Z^2)} \left[ c_W^2(P_{i33}P_{j23} - P_{i24}P_{j24}) + s_W^2(P_{i13}P_{j13} - P_{i14}P_{j14})
+ c_Ws_W(P_{i34}P_{j23} - P_{i24}P_{j24}) \right]
\cdot \left[ (t - m_{\tilde{\chi}_0}^2)(t - m_{\tilde{\chi}_j}^2) - s_jm_{\tilde{\chi}_0}m_{\tilde{\chi}_j}
- (u - m_{\tilde{\chi}_0}^2)(u - m_{\tilde{\chi}_j}^2) - s_jm_{\tilde{\chi}_0}m_{\tilde{\chi}_j} \right],$$

$$\Sigma_{Z\tilde{e}_R} = \frac{2(g_{ve} - g_{ae})}{s_W^2c_W^4(s - m_Z^2)} \left[ P_{i13}P_{j13} - P_{i14}P_{j14} \right]
\cdot \left[ (t - m_{\tilde{\chi}_0}^2)(t - m_{\tilde{\chi}_j}^2) - s_jm_{\tilde{\chi}_0}m_{\tilde{\chi}_j}
- (u - m_{\tilde{\chi}_0}^2)(u - m_{\tilde{\chi}_j}^2) - s_jm_{\tilde{\chi}_0}m_{\tilde{\chi}_j} \right].$$

\(^5\)To the extend that we neglect the electron mass, there is never any $\tilde{e}_L\tilde{e}_R$-interference; neither any Higgsino-$e\tilde{e}$ coupling.
where \( g_{ve} = -0.5 + 2s^2_V \) and \( g_{ae} = -0.5 \) are the vector and axial \( Z e e \)-couplings.

These results have already been presented in [3]. The only new thing here is that they have been expressed in terms of the neutralino projectors. Notice that, since the neutral gauginos do not couple to the gauge bosons in MSSM, the \( Z \)-square contribution in (29) only depends on the Higgsino-Higgsino matrix elements of the \( \tilde{\chi}_i^0 \) and \( \tilde{\chi}_j^0 \) projectors. Similarly, the \( \tilde{e}_R \)-square contribution in (31) only depends on the Bino-Bino elements of the same projectors. For the \( \tilde{e}_L \)-square contribution in (30) though, the non-vanishing of both the Bino and Wino couplings allow the appearance of the Bino-Bino, Wino-Wino and Bino-Wino matrix elements of \( P_i \) and \( P_j \). The \( Z \tilde{e}_R \)-interference only depends on the Bino-Higgsino matrix elements of \( P_i \), \( P_j \); while in the \( Z \tilde{e}_L \)-interference case, the Wino-Higgsino matrix elements of the same projectors also appear; compare (32, 33).

We next turn to the numerical expectations for the various matrix elements of the neutralino projectors. These can easily be constructed in any model with real \((M_1, M_2, \mu)\) parameters, by using either the set of equations (19), or the set (20, 22) and the results of Section 2. An obvious remark we should nevertheless probably make, is that each such set of equations has to be used at a definite scale. This means that in case two neutralinos have very different masses, we would have to use the appropriately scaled different values for the \((M_1, M_2, \mu)\) parameters, in order to determine their projectors at their physical masses.

As an example of such projectors, we turn again to the supergravity inspired benchmark scenarios of [9]. For the same scenarios as in Table 1, we give in Table 2 the implied matrix elements of the four neutralino projectors. In these scenarios \( \tan \beta \) is varying between 5 and 45. We note that the results in Tables 1 and 2 are at the same scale as the corresponding benchmark parameters appearing in Table 3 of [3], and may be combined for studying various SUSY appearances.

Following [3], we discuss separately below the scenarios (A,C,D,G,I,L) and E respectively.

- **A,C,D,G,I,L.**

  The projector for the lightest neutralino \( \tilde{\chi}_1^0 \) in all these cases, is such that the \( P_{111} \) element is much larger than all the rest. Thus, \( \tilde{\chi}_1^0 \) is always almost a pure Bino.

  The next neutralino \( \tilde{\chi}_2^0 \) is dominated by the Wino, but the admixture from the two Higgsinos is not absolutely negligible; compare \( P_{222} \) with \( P_{223} \) and \( P_{224} \). In fact this admixture tends to increase with the ordering of the scenarios appearing in Tables 1 and 2.

  The next two neutralinos tend to consist of roughly equal mixtures of the two Higgsinos. In fact, from the signs of \( P_{j34} \) in Table 2, we see that for the A, C, G, I and L scenarios, \( \tilde{\chi}_3^0 \sim (\tilde{H}_1^0 + \tilde{H}_2^0)/\sqrt{2} \); while for D, (which is the only \( \mu < 0 \) case) \( \tilde{\chi}_3^0 \sim (\tilde{H}_1^0 - \tilde{H}_2^0)/\sqrt{2} \). In all cases, \( \eta_3 = -1 \).

6Other related work may be found in [11].

7This of course implies that an appropriate adjustment of all these parameters to the scale of the specific physical process considered, may be needed.
For $\tilde{\chi}_0^4$, we approximately have $\tilde{\chi}_0^4 \sim (\tilde{H}_0^0 - \tilde{H}_2^0)/\sqrt{2}$ for A, C, G, I, L, and $\tilde{\chi}_0^4 \sim (\tilde{H}_1^0 + \tilde{H}_2^0)/\sqrt{2}$ for D. But the admixture from the Wino is somewhat larger in this case.

- E

This is one of the two “focus-point” scenarios in [9], corresponding to $m_0 \gg m_{1/2}$ and $\tan \beta = 10$. Again $\tilde{\chi}_1^0$ is mainly a Bino, but the admixture from the two Higgsinos is not negligible; compare $P_{113}$ and $P_{114}$ from Table 2.

The most important contribution to $\tilde{\chi}_2^0$ comes from the Wino; but the contributions from the Higgsinos are also quite important and even the Bino is non-negligible. Compare $P_{222}$, $P_{233}$, $P_{244}$, $P_{212}$, $P_{213}$, $P_{214}$ and $P_{211}$. A similar situation appears also for $\tilde{\chi}_4^0$, but with a smaller Bino and a larger Higgsino contributions.

Finally for the third neutralino, we again have $\tilde{\chi}_3^0 \sim (\tilde{H}_1^0 + \tilde{H}_2^0)/\sqrt{2}$ with $\eta_3 = -1$

Before concluding the discussion of the $P_{j\alpha\beta}$ matrix elements in the various scenarios of [9] on the basis of Table 2, we should emphasize that they have been calculated at the rather large scale $Q$ given in the second column of Table 1. These numbers give then, only a rough indication of the situation. In order to get the corresponding matrix elements affecting the actual production of a specific neutralino, we should of course determine $P_{j\alpha\beta}$ by substituting in (20, 22) the values of $M_1$, $M_2$ and $\mu$ at the appropriate scale of the physical situation involved.

We next turn to the discussion of the implication of these results to the $e^-e^+ \rightarrow \tilde{\chi}_i^0\tilde{\chi}_j^0$ cross section presented in (28-33). As an example, we concentrate to the $\tilde{\chi}_1^0\tilde{\chi}_2^0$ production in the scenarios A,C,D,G,I,L. Since then, $\tilde{\chi}_1^0$ is almost a pure Bino, and $\tilde{\chi}_2^0$ a pure Wino, the dominant contribution arises from terms involving a $P_{111}P_{222}$ product, which can only come from the $\tilde{e}_L$-square contribution in (30). Correspondingly, the $\tilde{\chi}_2^0\tilde{\chi}_2^0$ production mainly arises from $\tilde{e}_L$ exchanged described by (33).

In the same scenarios, the $\tilde{\chi}_1^0\tilde{\chi}_3^0$ or $\tilde{\chi}_1^0\tilde{\chi}_4^0$ productions are strongly suppressed, since $\tilde{\chi}_3^0$ and $\tilde{\chi}_4^0$ are almost pure Higgsinos, and there is no Feynman diagram inducing a Bino-Higgsino production. Because of this, in (23, 33) there is never any contribution involving products of the form $P_{111}P_{333}$, $P_{111}P_{344}$, $P_{111}P_{334}$ or $P_{111}P_{434}$, $P_{111}P_{433}$, $P_{111}P_{444}$.

4 Summary

Restricting to real values of the $(M_1, M_2)$ SUSY breaking parameters and of the SUSY scalar sector parameter $\mu$; we have constructed explicit expressions for the neutralino projector matrices and the signed neutralino mass eigenvalues.

We have then shown that these quantities are sufficient for expressing any physical observable related to neutralinos. In fact, the cross section for any process involving a $\tilde{\chi}_j^0$ neutralino in the final state, will just be proportional to matrix elements of its projector
and may also depend on its signed mass $\tilde{m}_{\chi_j}^0 = \eta_j m_{\chi_j}^0$. These quantities contain therefore, all physically relevant information concerning the specific neutralino.

If more than one external neutralinos are involved in a process, then in a physical observable, each neutralino will be described by its own projector matrix elements and signed mass.

Moreover, the projectors and signed neutralino masses, are also sufficient to describe any virtual neutralino exchange contribution in MSSM, to any order in perturbation theory.

These facts, together with their easy construction from the parameters in the SUSY Lagrangian, should make the use of projectors a rewarding approach for calculating neutralino processes.

To convince the reader for that, we have presented the $e^- e^+ \rightarrow \tilde{\chi}_i^0 \tilde{\chi}_j^0$ differential cross section in terms of the neutralino projector matrix elements. As an orientation on their expected magnitude, we have also given their values at the scale $Q = \sqrt{m_{\tilde{t}_1} m_{\tilde{t}_2}}$, for all SUGRA benchmark scenarios of [9], which can accommodate that at least some of the neutralinos will be within the reach of LHC or the future Colliders.

On the basis of these we could certainly claim that the use of the projectors strongly emphasizes the physical origin of the various terms in neutralino related observables.

Finally a word of caution should be added. The claim that the contribution of any neutralino can be completely described by the corresponding projector matrix and signed mass, is only made for real ($M_1, M_2, \mu$)-parameters in the MSSM Lagrangian. For complex ($M_1, M_2, \mu$), additional information is needed. The investigation of this problem is beyond our present scope.
Table 2: The Matrix elements of the neutralino projectors at the scale $Q = \sqrt{m_t m_{\tilde{t}_2}}$, for the SUGRA scenarios of Table 1, [3].

| $\chi^0_j$ | $P_{j11}$ | $P_{j22}$ | $P_{j33}$ | $P_{j44}$ | $P_{j12}$ | $P_{j13}$ | $P_{j14}$ | $P_{j23}$ | $P_{j24}$ | $P_{j34}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| A | $\chi_1^0$ | 0.994 | 0 | 0.005 | 0.001 | -0.016 | 0.068 | -0.034 | -0.001 | 0.001 | -0.002 |
|   | $\chi_2^0$ | 0.001 | 0.953 | 0.03 | 0.016 | 0.032 | -0.006 | 0.004 | -0.17 | 0.123 | -0.022 |
|   | $\chi_3^0$ | 0.002 | 0.001 | 0.497 | 0.501 | -0.001 | -0.017 | -0.017 | 0.025 | 0.025 | 0.499 |
|   | $\chi_4^0$ | 0.005 | 0.046 | 0.468 | 0.482 | -0.014 | -0.046 | 0.047 | 0.146 | -0.148 | -0.475 |
| C | $\chi_1^0$ | 0.988 | 0.001 | 0.01 | 0.001 | -0.025 | 0.099 | -0.04 | -0.003 | 0.001 | -0.004 |
|   | $\chi_2^0$ | 0.003 | 0.923 | 0.052 | 0.022 | 0.052 | -0.012 | 0.008 | -0.219 | 0.141 | -0.304 |
|   | $\chi_3^0$ | 0.002 | 0.004 | 0.492 | 0.502 | -0.002 | -0.028 | -0.029 | 0.042 | 0.043 | 0.497 |
|   | $\chi_4^0$ | 0.008 | 0.072 | 0.445 | 0.474 | -0.024 | -0.059 | 0.061 | 0.18 | -0.185 | -0.46 |
| D | $\chi_1^0$ | 0.994 | 0 | 0.005 | 0 | -0.004 | -0.072 | -0.017 | 0 | 0 | 0.001 |
|   | $\chi_2^0$ | 0 | 0.957 | 0.032 | 0.01 | 0.018 | 0.003 | 0.002 | 0.175 | 0.098 | 0.018 |
|   | $\chi_3^0$ | 0.001 | 0.003 | 0.495 | 0.501 | -0.002 | 0.027 | -0.027 | -0.04 | 0.040 | -0.498 |
|   | $\chi_4^0$ | 0.004 | 0.039 | 0.468 | 0.489 | -0.012 | 0.042 | 0.043 | -0.136 | -0.139 | 0.478 |
| E | $\chi_1^0$ | 0.903 | 0.010 | 0.067 | 0.019 | -0.098 | 0.246 | -0.132 | -0.026 | 0.014 | -0.036 |
|   | $\chi_2^0$ | 0.071 | 0.536 | 0.237 | 0.155 | 0.197 | -0.13 | 0.105 | -0.357 | 0.288 | -0.192 |
|   | $\chi_3^0$ | 0.005 | 0.010 | 0.476 | 0.509 | -0.007 | -0.05 | -0.053 | 0.07 | 0.071 | 0.492 |
|   | $\chi_4^0$ | 0.019 | 0.444 | 0.221 | 0.316 | -0.091 | -0.065 | 0.077 | 0.313 | -0.375 | -0.265 |
| G | $\chi_1^0$ | 0.986 | 0.001 | 0.011 | 0.002 | -0.026 | 0.106 | -0.04 | -0.003 | 0.001 | -0.004 |
|   | $\chi_2^0$ | 0.003 | 0.917 | 0.058 | 0.022 | 0.054 | -0.014 | 0.008 | -0.23 | 0.143 | -0.036 |
|   | $\chi_3^0$ | 0.002 | 0.005 | 0.491 | 0.502 | -0.003 | -0.032 | 0.032 | 0.048 | 0.048 | 0.497 |
|   | $\chi_4^0$ | 0.008 | 0.078 | 0.44 | 0.474 | -0.026 | -0.061 | 0.063 | 0.185 | -0.192 | -0.457 |
| I | $\chi_1^0$ | 0.985 | 0.001 | 0.013 | 0.001 | -0.025 | 0.112 | -0.038 | -0.003 | 0.001 | -0.004 |
|   | $\chi_2^0$ | 0.003 | 0.911 | 0.063 | 0.023 | 0.056 | -0.015 | 0.009 | -0.24 | 0.145 | -0.038 |
|   | $\chi_3^0$ | 0.002 | 0.006 | 0.49 | 0.502 | -0.004 | -0.035 | -0.035 | 0.052 | 0.053 | 0.496 |
|   | $\chi_4^0$ | 0.008 | 0.038 | 0.435 | 0.473 | -0.027 | -0.062 | 0.065 | 0.19 | -0.198 | -0.454 |
| L | $\chi_1^0$ | 0.991 | 0 | 0.008 | 0.001 | -0.015 | 0.09 | -0.032 | -0.001 | 0 | -0.003 |
|   | $\chi_2^0$ | 0.002 | 0.929 | 0.05 | 0.02 | 0.038 | -0.009 | 0.006 | -0.215 | 0.137 | -0.032 |
|   | $\chi_3^0$ | 0.002 | 0.004 | 0.493 | 0.502 | -0.002 | -0.028 | -0.029 | 0.042 | 0.042 | 0.497 |
|   | $\chi_4^0$ | 0.006 | 0.068 | 0.449 | 0.477 | -0.021 | -0.053 | 0.055 | 0.174 | -0.179 | -0.463 |
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