Optimal error estimate of an accurate second-order scheme for Volterra integrodifferential equations with tempered multi-term kernels

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Abstract
In this paper, we investigate and analyze numerical solutions for the Volterra integrodifferential equations with tempered multi-term kernels. Firstly, we derive some regularity estimates of the exact solution. Then a temporal-discrete scheme is established by employing Crank-Nicolson technique and product integration (PI) rule for discretizations of the time derivative and tempered-type fractional integral terms, respectively, from which, nonuniform meshes are applied to overcome the singular behavior of the exact solution at $t = 0$. Based on deduced regularity conditions, we prove that the proposed scheme is unconditionally stable and possesses accurately temporal second-order convergence in $L_2$-norm. Numerical examples confirm the effectiveness of the proposed method.

Keywords Volterra integrodifferential equations · Tempered multi-term kernels · Accurate second order · Stability · Optimal error estimate

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1 Introduction

In this work, we consider the following nonlocal evolution equation with tempered multi-term kernels

$$\frac{\partial u}{\partial t} + Au + \sum_{j=1}^{m}(\beta_j * B_j u)(t) = f(t), \quad t > 0, \quad 1 \leq m < \infty,$$

$$u(0) = u_0,$$

(1.1)

in which $A$ is a self-adjoint positive-definite linear operator, not necessarily bounded operator, with compact inverse, defined on a dense subspace $D(A)$ of the Hilbert space $H$, and $B_j$ is the self-adjoint linear operator in $H$ such that $D(B_j) \supset D(A)$, $1 \leq j \leq m$. Letting

$$\|w\|_q = \|A^{q/2}w\| = (A^q w, w)^{1/2},$$

where $(\cdot, \cdot)$ is the inner product in $H$ with the norm $\| \cdot \|$, we suppose that in this paper the operator $A$ dominates $B_j$ in the sense that [8]

$$|(B_j w, v)| \leq C \|w\|_q \|v\|_{\vartheta}, \quad \vartheta = 0, 1, 2, \quad \vartheta + q = 2.$$

(1.2)

Define the convolution

$$(\beta_j * \varphi)(t) := \int_{0}^{t} \beta_j(t - s)\varphi(s)ds, \quad 1 \leq j \leq m, \quad t > 0,$$

see [16, 20], where the tempered singular kernels

$$\beta_j(t) = \frac{e^{-\kappa t}{\alpha_j}^{-1}}{\Gamma(\alpha_j)}, \quad \kappa \geq 0, \quad 0 < \alpha_j < 1,$$

(1.3)

and $\Gamma(\cdot)$ denotes the Euler’s Gamma function, see [16]. Problems of type (1.1) with the single-term kernel ($m = 1, \kappa = 0$) can be considered as the model arising in heat transfer theory with memory, population dynamics, and viscoelastic materials, see [4, 6] and references therein; with $m = 1$ and $\kappa > 0$, (1.1) is called the tempered fractional integro-differential equation, in which the tempered fractional integral of Brownian motion, called tempered fractional Brownian motion, can show semi-long-range correlation, and then the increment of this process is called tempered fractional Gaussian noise, which affords a useful new stochastic model for the wind speed data, see [18]. Furthermore, when $A = 0$, problem (1.1) with self-adjoint boundary conditions appears in a linear model for heat flow in a rectangular, orthotropic material with the memory [2, 5, 10], from which the axes of orthotropy are parallel to the edges of the rectangle.

Due to its broad and practical applications, many scholars have carried out a series of theoretical and numerical studies regarding the problem of type (1.1), e.g., in terms
of theoretical research with \( A = 0 \) and \( f = 0 \), Hannsgen and Wheeler [5] first studied the asymptotic behavior as \( t \to \infty \) of solutions of (1.1), combined with a resolvent formula, which gave the long-time estimate of solutions with the weight function. Then based on [5], Noren [13] proved the long-time estimate of solutions without the weight function by giving variable hypothesis of kernel functions \( \beta_j(t) \). Subsequently, Noren [13] derived that \( u'(t) \) is integrable uniformly with respect to the parameters of solutions under some assumptions regarding kernel functions \( \beta_j(t) \). With the maturity of theoretical studies, numerical researches have gradually developed. Xu proved the weighted \( L^1 \) asymptotic stability of the numerical solutions by temporal backward Euler (BE) method and first-order convolution quadrature rule. Hereafter, Xu considered the weighted \( L^1 \) asymptotic convergence of the numerical solutions by same temporal approximation. After that, in order to improve accuracy to the second order for time, Xu proved the weighted \( L^1 \) asymptotic stability [23] and weighted \( L^1 \) asymptotic convergence [24] by second-order backward differentiation formula (BDF) and second-order convolution quadrature, from which, temporal convergence can not reach accurate second order due to the singular behavior of the exact solution at the initial point \( t = 0 \). Besides, some studies of (1.1) with \( m = 2 \) were considered recently, see [1, 17], which only achieved temporal first-order accuracy; also, there has been a lot of excellent work on similar multi-term problems, see [7, 9, 25]. Thus, the limitations of existing methods motivate us to carry out the following research. The major aim of this work is to consider an accurate second-order numerical scheme for Volterra integrodifferential equations with tempered-type multi-term singular kernels. This time-discrete scheme is constructed by the Crank-Nicolson method and PI rule on the nonuniform meshes. In addition, the main contribution of this work can be summarized in the following aspects:

\[(i)\] Based on certain suitable assumptions, we derive the regularity estimates of the exact solution of problem (1.1);  
\[(ii)\] We employ the graded meshes in time to establish the discrete scheme, which can compensate for the singular behaviour of the exact solutions at \( t = 0 \);  
\[(iii)\] The energy method is utilized to deduce the stability and convergence of the proposed scheme;  
\[(iv)\] Provided numerical examples satisfying the regularity substantiate the theoretical analyses, which can reflect and illustrate the accurate second-order accuracy for time by choosing the grading index \( \gamma \geq 2/(1 + \alpha) \), \( \alpha = \min_{1 \leq j \leq m} \{\alpha_j\} \).

The rest of this article is arranged as follows. In Sect. 2, the regularity of exact solutions is deduced on the \( L^2 \) space. Section 3 gives some notations and formulates a time-discrete scheme. Then the stability and convergence of proposed scheme are proved in Sect. 4. Section 5 provides several numerical examples to verify our analysis. Finally, some concluding remarks are presented in Sect. 6.

Throughout this paper, \( C \) denotes a generic positive constant that is independent of the space-time step sizes and may be not necessarily the same on each occurrence.
2 Regularity of exact solutions

For the further analysis, denoting

\[ v(t) = e^{kt} u(t), \quad g(t) = e^{kt} f(t), \quad t > 0, \quad (2.1) \]

then \((1.1)\) is converted to

\[ \frac{\partial v}{\partial t} + Av + \sum_{j=1}^{m} (\omega_{a_j} * B_j v)(t) - \kappa v = g(t), \quad t > 0, \quad m \geq 1, \quad (2.2) \]

where the Abel kernel \( \omega_{a_j}(t) = \frac{\omega_{a_j}^{\alpha_j}}{\Gamma(\alpha_j)} \) with \( 1 \leq j \leq m \).

Below, we shall give some regularity estimates of the exact solution of \((1.1)\). That is, we wish to present that

\[ t\|Av''(t)\| + \|v''(t)\| + \|Av'(t)\| \leq Ct^{\alpha_j-1}, \quad \alpha_j = \min_{1 \leq j \leq m} \{\alpha_j\}, \quad (2.3) \]

\[ t\|Au''(t)\| + \|u''(t)\| + \|Au'(t)\| \leq Ce^{-\kappa t}t^{\alpha_j-1}, \quad t \to 0^+. \]

Later \( \|\cdot\|_{L^p} \) denotes the norm on the Sobolev space \( L^p(\Omega) \) with \( 1 < p \leq \infty \); besides, Lipschitz domain \( \Omega \subseteq \mathbb{R}^d, d \geq 1 \). In fact, before establishing the regularity estimates of \( u \), we only need to consider \( v \) defined in \((2.2)\). By splitting the solution of \((2.2)\) into \( v(t) = v_1(t) + v_2(t) \), which implies that for \( t > 0 \), we shall further solve

\[ v_1'(t) + Av_1(t) = g(t), \quad v_1(0) = u_0, \quad (2.4) \]

and

\[ v_2'(t) + Av_2(t) = -\sum_{j=1}^{m} \int_{0}^{t} \omega_{a_j}(t - s) B_j v(s) ds + \kappa v(t), \quad v_2(0) = 0, \quad (2.5) \]

respectively. Then, we rewrite the solution of \((2.4)\) as

\[ v_1(t) = E(t)u_0 + \int_{0}^{t} E(t - s)g(s) ds, \quad (2.6) \]

where \( E(t) = \exp(-At) \) is the analytic semigroup in space \( L^p(\Omega) \), generated by \(-A\) (cf. [15]), such that

\[ \|E(t)w\|_{L^p} + t\|AE(t)w\|_{L^p} + t^2\|A^2E(t)w\|_{L^p} \leq C\|w\|_{L^p}, \quad t \in (0, T) \quad (2.7) \]

for any \( w \in L^p(\Omega) \). Then similar to \([8, 12]\), we introduce the following key lemmas.
Lemma 1 Assume that \( p \in (1, \infty) \) and \( \varphi(t) \in L_p(\Omega), t > 0 \). Suppose that \( \| \varphi(t) \|_{L_p} \leq Ct^{\alpha - 1} \) and \( \| \varphi'(t) \|_{L_p} \leq Ct^{\alpha - 2} \) with \( \alpha = \min_{1 \leq j \leq m} \{ \alpha_j \} \). Then it holds that

\[
\left\| \int_0^t A E(t - s) \varphi(s) ds \right\|_{L_p} \leq Ct^{\alpha - 1}, \quad \alpha \in (0, 1).
\]

**Proof** The desired result can be proved similarly by [8, Lemma 5.1]. □

Lemma 2 Let \( p \in (1, \infty) \) and \( \varphi(t) \in L_p(\Omega) \) for \( t > 0 \), then

\[
\left\| \int_0^t A E(t - s) \int_0^s \omega_{\alpha_j}(s - \sigma) \varphi(\sigma) d\sigma ds \right\|_{L_p} \leq C \int_0^t \omega_{\alpha_j}(t - \sigma) \| \varphi(\sigma) \|_{L_p} d\sigma.
\]

**Proof** Exchanging the order of integration and swapping variables, we have

\[
\int_0^t A E(t - s) \int_0^s \omega_{\alpha_j}(s - \sigma) \varphi(\sigma) d\sigma ds = \int_0^t \left[ \int_0^{t - \sigma} A E(t - \sigma - \vartheta) \omega_{\alpha_j}(\vartheta) d\vartheta \right] \varphi(\sigma) d\sigma.
\]

Then an application of Lemma 1 completes the proof. □

Lemma 3 Suppose that \( \| Au_0 - g(0) \| + \| g(t) \| \leq C \) and \( \| (Au_0 - g(0)) E'(t) \| + \| g'(t) \| + t \| g''(t) \| \leq Ct^{\alpha - 1} \) with \( \alpha = \min_{1 \leq j \leq m} \{ \alpha_j \} \). Then \( v_1(t) \) denoted by (2.4) satisfies

\[
\| Av_1(t) \| + \| v_1'(t) \| \leq C, \quad \| Av_1'(t) \| + \| v_1''(t) \| \leq Ct^{\alpha - 1}.
\]

**Proof** We can finish the proof analogously by [12, Lemma 2.3]. □

Below we derive the regularity estimate with respect to \( v(t) \).

**Theorem 1** Based on the assumptions of Lemma 3, then the solution \( v(t) \) of (2.2) satisfies that

\[
\| v'(t) \| + \| Av(t) \| \leq C \| u_0 \|, \quad 0 < t \leq T.
\]

**Proof** By (2.5)-(2.7) and the assumption (1.2) (A dominates \( B_j \)), then \( v_2(t) \) given by (2.5) satisfies

\[
\| v'_2(t) \| \leq \sum_{j=1}^m \left\| \int_0^t \omega_{\alpha_j}(t - s) (B_j A^{-1}) Av(s) ds \right\|
\]

\[
+ \| Av_2(t) \| + \kappa \left( \| v_1(t) \| + \| v_2(t) \| \right)
\]

\[
\leq \sum_{j=1}^m \left[ \int_0^t \omega_{\alpha_j}(t - s) \| Av(s) \| ds + \| Av_2(t) \| 
\right.
\]

\[
+ \kappa \left( \| u_0 \| + \int_0^t \| g(s) \| ds \right) + \kappa \int_0^t \| v'_2(s) \| ds.
\]
Then using Duhamel’s principle and (2.3), we deduce

$$v_2(t) = - \int_0^t E(t-s) \sum_{j=1}^m \int_0^s \omega_{\alpha_j} (s-\sigma) B_j v(\sigma) d\sigma ds$$

$$- \kappa \int_0^t E(t-s) v(s) ds := v_{21}+v_{22}. \tag{2.8}$$

From Lemma 2 and Lemma 3, we yield

$$\|Av_{21}\| \leq \left\| \int_0^t AE(t-s) \sum_{j=1}^m \int_0^s \omega_{\alpha_j} (s-\sigma) (B_j A^{-1}) Av(\sigma) d\sigma ds \right\|$$

$$\leq C \sum_{j=1}^m \int_0^t \omega_{\alpha_j} (t-\sigma) (\|Av_1(\sigma)\| + \|Av_2(\sigma)\|) d\sigma$$

$$\leq C \sum_{j=1}^m \left( t^{\alpha_j} + \int_0^t \omega_{\alpha_j} (t-\sigma) \|Av_2(\sigma)\| d\sigma \right),$$

and (2.7) gives

$$\|Av_{22}\| \leq \kappa \left\| \int_0^t E(t-s) Av(s) ds \right\| \leq C \int_0^t \|Av(s)\| ds.$$

Therefore, we have

$$\|Av_2(t)\| \leq C \left( t^{\alpha} + \int_0^t \|Av(s)\| ds \right) + \int_0^t \omega_{\alpha} (t-\sigma) \|Av_2(\sigma)\| d\sigma,$$

from which, an application of Grönwall’s lemma (see [3, Lemma 1]) yields

$$\|Av_2(t)\| \leq C \left( t^{\alpha} + \int_0^t \|Av(s)\| ds \right),$$

which follows from Lemma 3 that $\|Av(t)\| \leq C + C \int_0^t \|Av(s)\| ds$, then we have $\|Av(t)\| \leq C$ and $\|Av_2(t)\| \leq Ct^{\alpha}$. Thus, we can get

$$\|v_2'(t)\| \leq C \left( \sum_{j=1}^m t^{\alpha_j} + \|u_0\| \right) + \kappa \int_0^t \|v_2'(s)\| ds,$$

then Grönwall’s lemma and Lemma 3 gives

$$\|v'(t)\| \leq \|v_1'(t)\| + \|v_2'(t)\| \leq C \|u_0\|, \tag{2.9}$$

which completes the proof. \qed
We will give other estimates of \( v(t) \) regarding the second time derivative.

**Theorem 2** Based on the assumptions of Theorem 1. Then it follows that

\[
\|v''(t)\| + \|A v'(t)\| \leq C t^{\alpha-1} \|u_0\|_2, \quad \alpha = \min_{1 \leq j \leq m} \{\alpha_j\}, \quad t \in (0, T].
\]

**Proof** Differentiate (2.5) to get

\[
v''_2(t) + A v'_2(t) = \kappa v'(t) - \sum_{j=1}^{m} \omega_{\alpha_j}(t) B_j u_0 - \sum_{j=1}^{m} \int_{0}^{t} \omega_{\alpha_j}(s) B_j \frac{\partial v(t-s)}{\partial t} ds,
\]

(2.10)

where using (2.4)-(2.5), we have \( v'_2(0) = v_1(0) = u_0 \), thus Duhamel’s principle gives

\[
v'_2(t) = - \left( E(t)u_0 - \kappa \int_{0}^{t} E(t-s)(v'_1(s) + v'_2(s)) ds \right)
- \sum_{j=1}^{m} \int_{0}^{t} E(t-s) \omega_{\alpha_j}(s)(B_j A^{-1}) A u_0 ds
- \sum_{j=1}^{m} \int_{0}^{t} E(t-s) \int_{0}^{s} \omega_{\alpha_j}(s-\sigma) B_j v'_2(\sigma) d\sigma ds
- \sum_{j=1}^{m} \int_{0}^{t} E(t-s) \int_{0}^{s} \omega_{\alpha_j}(s-\sigma) B_j v'_1(\sigma) d\sigma ds
:= R_{21} + R_{22} + R_{23} + R_{24}.
\]

Then, each term of (2.11) will be estimated. First, Lemma 3 and (2.7) lead to

\[
\|R_{21}\|_2 \leq C \|Au_0\| + C \int_{0}^{t} \|E(s)(v'_1(t-s) + v'_2(t-s))\| ds
\leq C \left( \|u_0\|_2 + \sum_{j=1}^{m} t^{\alpha_j} + \int_{0}^{t} \|v'_2(s)\| ds \right),
\]

(2.12)

besides, Lemmas 1-3 and assumption (1.2) give

\[
\|R_{22}\|_2 \leq \sum_{j=1}^{m} \left| \int_{0}^{t} A E(t-s) \omega_{\alpha_j}(s)(B_j A^{-1}) A u_0 ds \right|
\leq C \sum_{j=1}^{m} t^{\alpha_j} \|Au_0\| \leq C t^{\alpha-1} \|u_0\|_2.
\]

(2.13)
and
\[
\|R_{23}\|_2 \leq \sum_{j=1}^{m} \left\| \int_{0}^{t} AE(t-s) \int_{0}^{s} \omega_{a_j}(s-\sigma)(B_j A^{-1}) Av_2'(\sigma) d\sigma \right\|
\]
\[
\leq C \sum_{j=1}^{m} \int_{0}^{t} \omega_{a_j}(t-\sigma) \|Av_2'(\sigma)\| d\sigma \leq C \int_{0}^{t} \omega_{a}(t-\sigma) \|Av_2'(\sigma)\| d\sigma.
\]
(2.14)

Then for the estimate of \(\|R_{24}\|_2\), we first utilize (2.6) to obtain
\[
Av_1'(t) = E'(t)(Au_0 - g(0)) + \int_{0}^{t} AE(t-s)g'(s) ds.
\]
(2.15)

Hence, we have
\[
AR_{24} = -\sum_{j=1}^{m} \int_{0}^{t} AE(t-s)(B_j A^{-1}) G_j(s) ds
\]
\[
- \sum_{j=1}^{m} \int_{0}^{t} AE(t-s) \int_{0}^{s} \omega_{a_j}(s-\sigma) H_j(\sigma) d\sigma ds,
\]
(2.16)

from which, \(H_j(t) = \int_{0}^{t} (B_j A^{-1}) AE(t-\theta)g'(\theta) d\theta\) and
\[
G_j(t) = \int_{0}^{t} \omega_{a_j}(s) AE(t-s)(Au_0 - g(0)) ds.
\]
(2.17)

Then, Lemma 1 can be applied to (2.16) by satisfying two conditions. Firstly, we have \(\|G_j(t)\| \leq Ct^{a_j-1} \|Au_0 - g(0)\| \leq Ct^{a_j-1}\). Further we estimate the derivative of \(G_j(t)\). By writing
\[
G_j(t) = \left( \int_{0}^{t/2} + \int_{t/2}^{t} \right) \omega_{a_j}(s) AE(t-s)(Au_0 - g(0)) ds := (G_1)_j(t) + (G_2)_j(t).
\]
Next we employ \(\frac{d[AE(t-s)]}{dt} = -A^2 E(t-s)\) to get
\[
(G_1)'_j(t) = \omega_{a_j}(t/2) AE(t/2)(Au_0 - g(0)) - \int_{0}^{t/2} \omega_{a_j}(s) A^2 E(t-s)(Au_0 - g(0)) ds,
\]
thus we have
\[
\| (G_1)'_j(t) \| = Ct^{a_j-2} \|Au_0 - g(0)\| \leq Ct^{a_j-2}.
\]
(2.18)
Further, by applying $\frac{dE(t-s)}{ds} = AE(t-s)$ and integrating by parts, we obtain
\[(G_2)_j(t) = \omega_{\alpha_j}(s)E(t-s)\bigg|_{t/2}^t (Au_0 - g(0))ds - \int_{t/2}^t \omega'_{\alpha_j}(s)E(t-s)(Au_0 - g(0))ds,\]
therefore (2.7) gives $\| (G_2)_j(t) \| \leq Ct^{\alpha_j-1}$. Then, differentiating $(G_2)_j(t)$ and similar to the estimate of $\| (G_1)'_j(t) \|$, we can yield that
\[
\| (G_2)'_j(t) \| = Ct^{\alpha_j-2} \| Au_0 - g(0) \| \leq Ct^{\alpha_j-2}. \tag{2.19}
\]
In view of (2.18) and (2.19), we use Lemmas 1 and 2 to deduce
\[
\| R_{24} \|_2 \leq \sum_{j=1}^m t^{\alpha_j-1} + C \sum_{j=1}^m \int_0^t (t-s)^{\alpha_j-1} \| H_j(s) \| ds
\leq C \sum_{j=1}^m \int_0^t (t-s)^{\alpha_j-1} s^{\alpha_j-1} ds
\leq C \left( t^{\alpha_j-1} + \sum_{j=1}^m t^{\alpha_j+\alpha_j-1} \right) \leq Ct^{\alpha_j-1}. \tag{2.20}
\]
Using (2.11)-(2.14) and (2.20), then
\[
\| Av'_2(t) \| \leq C \left( t^{\alpha_j-1} \| u_0 \|_2 + \int_0^t \omega_\alpha(t-\sigma) \| Av'_2(\sigma) \| d\sigma \right). \tag{2.21}
\]
from which using Grönwall’s lemma (see [3, Lemma 1]), we get $\| Av'_2(t) \| \leq Ct^{\alpha_j-1} \| u_0 \|_2$. From (2.10) and above analyses, we have
\[
\| v'_2(t) \| \leq C \| Av'_2(t) \| + \kappa \| v'(t) \| + \sum_{j=1}^m \omega_{\alpha_j}(t) \| u_0 \|_2
+ \sum_{j=1}^m \int_0^t \omega_{\alpha_j}(t-s) \| (B_j A^{-1}) Av'(s) \| ds
\leq Ct^{\alpha_j-1} \| u_0 \|_2, \tag{2.22}
\]
which combines Lemma 3, we finish the proof. \hfill \Box

According to the relation between $v(t)$ and $u(t)$, we give the following theorems.

**Theorem 3** Under the assumptions of Theorem 1. Then the solution of (1.1) satisfies that
\[
\| u'(t) \| + \| Au(t) \| \leq Ce^{-\kappa t} \| u_0 \|, \quad 0 < t \leq T.
\]
Proof By Theorem 1 and $v(t) = e^{\kappa t} u(t)$, we have

$$\|u'(t) + \kappa u(t)\| + \|Au(t)\| \leq C\|u_0\|e^{-\kappa t},$$

which follows from the triangle inequality that

$$\|u'(t)\| + \|Au(t)\| \leq e^{-\kappa t}\left(C\|u_0\| + \kappa \|v(t)\|\right),$$

which yields the desired result. 

Theorem 4 Under the assumptions of Theorem 2, then for $0 < t < T$, the solution of (1.1) satisfies

$$\|u''(t)\| + \|Au'(t)\| \leq Ce^{-\kappa t}t^{\alpha - 1}\|u_0\|_2, \quad \alpha \in \min \{\min_{1 \leq j \leq m} \{\alpha_j\}\}.$$

Proof By Theorem 2, Theorem 3 and the triangle inequality, we get

$$\|Au'(t)\| \leq e^{-\kappa t}\|Av'(t)\| + \kappa \|Au(t)\| \leq Ce^{-\kappa t}t^{\alpha - 1}\|u_0\|_2,$$

and

$$\|u''(t)\| \leq e^{-\kappa t}\|v''(t)\| + \kappa^2 e^{-\kappa t}\|v(t)\| + 2\kappa\|u'(t)\| \leq Ce^{-\kappa t}t^{\alpha - 1}\|u_0\|_2.$$

We complete the proof.

Moreover, to show the regularity result (2.3), we need to derive that $\|Av''(t)\| \leq Ct^{\alpha - 2}$, which can be proved by the procedure that was utilized to present that $\|Au'(t)\| \leq Ct^{\alpha - 1}$, based on certain changes and appropriate assumptions. These naturally deduce that $\|Au''(t)\| \leq Ce^{-\kappa t}t^{\alpha - 2}$ from previous analyses.

3 Establishment of numerical scheme

Before formulating the time-discrete scheme, we first introduce some helpful notations.

3.1 Some notations

In this section, we shall give some symbols in order to construct our method. First, we present the temporal levels $0 = t_0 < t_1 < t_2 < \cdots$ and define that $k_n = t_n - t_{n-1}$ and $t_{n-1/2} = \frac{1}{2}(t_n + t_{n-1})$ for $n \geq 1$. Moreover, we define that $\mathcal{V}_k = \{\mathcal{V}^n|0 \leq n \leq N\}$ and that

$$\delta_t \mathcal{V}^{n-\frac{1}{2}} = \frac{1}{k_n}(\mathcal{V}^n - \mathcal{V}^{n-1}), \quad \mathcal{V}^{n-\frac{1}{2}} = \frac{1}{2}(\mathcal{V}^n + \mathcal{V}^{n-1}), \quad 1 \leq n \leq N,$$
and we also define the piecewise constant approximation as follows

$$
\overline{v}(t) = \begin{cases} 
v^1, & t_0 < t < t_1, \\
v^{n-1/2}, & t_{n-1} < t < t_n, \quad n \geq 2.
\end{cases}
$$

(3.1)

Then in view of above results, for approximating the integral term given in (2.2), we utilize the PI rule to denote the discrete fractional integral (see [11])

$$
I^{(\alpha_j)}v^{n-1/2} = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} I^{(\alpha_j)}\overline{v}(t)dt = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_0^t w_{\alpha_j}(t - \vartheta)\overline{v}(\vartheta)d\vartheta dt
$$

$$
= w^{(j)}v^1k_1 + \sum_{p=2}^{n} w^{(j)}v^{p-1/2}k_p, \quad 1 \leq j \leq m,
$$

(3.2)

where

$$
w^{(j)} = \frac{1}{k_n k_p} \int_{t_{n-1}}^{t_n} \int_{t_{p-1}}^{\min\{t, t_p\}} \omega_{\alpha_j}(t - \vartheta)d\vartheta dt > 0.
$$

(3.3)

Further, for \(1 \leq p \leq n - 1\), we obtain

$$
w^{(j)} = \frac{1}{k_n k_p} \int_{t_{n-1}}^{t_n} \int_{t_{p-1}}^{t_p} \omega_{\alpha_j}(t - \vartheta)d\vartheta dt = \frac{\lambda^{(j)}_{n, p} - \lambda^{(j)}_{n-1, p}}{k_n k_p \Gamma(\alpha_j + 2)},
$$

(3.4)

from which, \(\lambda^{(j)}_{n, p} = (t_n - t_{p-1})^{\alpha_j + 1} - (t_n - t_p)^{\alpha_j + 1}\), and for \(n \geq 1\), then

$$
w^{(j)} = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \omega_{\alpha_j}(t - \vartheta)d\vartheta dt = \frac{k_n^{\alpha_j - 1}}{\Gamma(\alpha_j + 2)}.
$$

(3.5)

Besides, for the inhomogeneous term in (2.2), we denote

$$
g^{n-1/2} \approx \frac{1}{k_n} \int_{t_{n-1}}^{t_n} g(t)dt, \quad n \geq 1,
$$

(3.6)

from which, we suppose that

$$
\left\| g^{1/2}k_1 - \int_{t_0}^{t_1} g(t)dt \right\| \leq C \int_{t_0}^{t_1} t\|g'(t)\|dt,
$$

(3.7)

and that

$$
\left\| g^{n-1/2}k_n - \int_{t_{n-1}}^{t_n} g(t)dt \right\| \leq C k_n^2 \int_{t_{n-1}}^{t_n} \|g''(t)\|dt, \quad n \geq 2.
$$

(3.8)
Also, e.g., $g^{n-\frac{1}{2}} = \frac{1}{2} [g(t_{n-1}) + g(t_n)]$ or $g^{n-\frac{1}{2}} = g(t_{n-1/2})$ are permissible.

Then, to overcome the singular behaviour of the exact solution at $t = 0$, we give some hypotheses of non-uniform meshes [11] that

$$k_n \leq C_\gamma k \min \left\{1, t_n^{1-1/\gamma}\right\}, \quad n \geq 1, \quad \gamma \geq 1,$$

(3.9)

where $C_\gamma$ does not depend on $k$, and that

$$t_1 = k_1 \geq c_\gamma k^{\gamma}, \quad t_n \leq C_\gamma t_{n-1}, \quad n \geq 2,$$

(3.10)

and that (a more rigid hypothesis)

$$0 \leq k_{n+1} - k_n \leq C_\gamma k^2 \min \left\{1, t_n^{1-2/\gamma}\right\}, \quad n \geq 2.$$  

(3.11)

Thence, for $0 \leq t \leq T < \infty$, the case satisfying the hypotheses (3.9)-(3.11) is

$$t_n = (nk)^\gamma, \quad k = T^{1/\gamma} / N, \quad 0 \leq n \leq N.$$  

(3.12)

### 3.2 Time discrete scheme

First, we integrate (2.2) from $t = t_{n-1}$ to $t = t_n$ and multiply the term $\frac{1}{k_n}$, then

$$\delta_t v^{n-\frac{1}{2}} + \frac{1}{k_n} \int_{t_{n-1}}^{t_n} A v(t) dt + \frac{1}{k_n} \sum_{j=1}^{m} \int_{t_{n-1}}^{t_n} (\omega_{\alpha_j} \ast B_j v)(t) dt - \frac{\kappa}{k_n} \int_{t_{n-1}}^{t_n} v(t) dt = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} g(t) dt,$$

(3.13)

which follows from (3.1)-(3.8) that

$$\delta_t v^{\frac{1}{2}} + A v^{\frac{1}{2}} + \sum_{j=1}^{m} I^{(\alpha_j)} B_j v^{\frac{1}{2}} - \kappa v^{\frac{1}{2}} = g^{\frac{1}{2}} + c^{1/2},$$  

(3.14)

and for $n \geq 2$ that

$$\delta_t v^{n-\frac{1}{2}} + A v^{n-\frac{1}{2}} + \sum_{j=1}^{m} I^{(\alpha_j)} B_j v^{n-\frac{1}{2}} - \kappa v^{n-\frac{1}{2}} = g^{n-\frac{1}{2}} + c^{n-1/2},$$  

(3.15)
where $v^n = v(t_n)$, $v^{n-1}_2 = \frac{v^n + v^{n-1}}{2}$ and $R^{n-1/2}_s = \sum_{s=1}^{4} R^{n-1/2}_s$ for $n \geq 1$, and

\[
R^{1/2}_1 = Av^1 - \frac{1}{k_1} \int_{t_0}^{t_1} Av(t)dt,
\]

\[
R^{n-1/2}_1 = Av^{n-1}_2 - \frac{1}{k_n} \int_{t_{n-1}}^{t_n} Av(t)dt, \quad n \geq 2,
\]

\[
R^{n-1/2}_2 = \sum_{j=1}^{m} \left( f(\alpha_j) B_j v^{n-1}_2 - \frac{1}{k_n} \int_{t_{n-1}}^{t_n} (\omega_{\alpha_j} * B_j v)(t)dt \right), \quad n, m \geq 1,
\]

\[
R^{1/2}_3 = -\kappa \left( v^1 - \frac{1}{k_1} \int_{t_0}^{t_1} v(t)dt \right),
\]

\[
R^{n-1/2}_3 = -\kappa \left( v^{n-1}_2 - \frac{1}{k_n} \int_{t_{n-1}}^{t_n} v(t)dt \right), \quad n \geq 2,
\]

\[
R^{n-1/2}_4 = g^{n-1}_2 - \frac{1}{k_n} \int_{t_{n-1}}^{t_n} g(t)dt, \quad n \geq 1.
\]

Then omitting the truncation errors $R^{n-1/2}_s$ and replacing $v^n$ with its numerical approximation $V^n$, then time discrete scheme is yielded as follows

\[
\delta_t V^{1/2}_n + AV^1 + \sum_{j=1}^{m} f(\alpha_j) B_j V^{1/2}_n - \kappa V^1 = g^{1/2}_n,
\]

\[
\delta_t V^{n-1/2} + AV^{n-1/2} + \sum_{j=1}^{m} f(\alpha_j) B_j V^{n-1/2} - \kappa V^{n-1/2} = g^{n-1/2}_n, \quad 2 \leq n \leq N,
\]

\[
V^0 = u_0.
\]

4 Analysis of time discrete scheme

In this section, we shall present the stability and convergence of the proposed time-discrete scheme (3.17)-(3.19).

4.1 Stability

Here, we first establish the stability result of the numerical scheme. Then the following theorem is obtained.

**Theorem 5** Assume that the numerical solution $V^n$ is denoted by (3.17) and (3.18) for $1 \leq n \leq N$. Let $U^n$ be the approximate solution of $u(t_n)$ given in (1.1). Then for
$T < \infty$ and $0 \leq \kappa < \infty$, it holds that

$$\max_{1 \leq n \leq N} \|V^n\| \leq C(T) \left( \|V^0\| + \sum_{n=1}^{N} k_n \left\| g^{n-1/2} \right\| \right),$$

and that

$$\max_{1 \leq n \leq N} \|U^n\| \leq C(T) \left( \|U^0\| + \sum_{n=1}^{N} k_n e^{\kappa t_{n-1/2}} \left\| f^{n-1/2} \right\| \right).$$

**Proof** By taking the inner product of (3.17)-(3.18) with $k_1 V^1$ and $k_n V^{n-\frac{1}{2}}$ correspondingly, and summing (3.8) for $n$ from 2 to $N$, we obtain that

$$k_1 (\delta_t V^{\frac{1}{2}}, V^1) + k_1 (AV^1, V^1) + k_1 \sum_{j=1}^{m} \left( I^{(\alpha_j)} B_j V^{1/2}, V^1 \right) - \kappa k_1 (V^1, V^1) = k_1 \left( g^{\frac{1}{2}}, V^1 \right),$$

and that

$$\sum_{n=2}^{N} k_n (\delta_t V^{n-\frac{1}{2}}, V^{n-\frac{1}{2}}) + \sum_{n=2}^{N} k_n (AV^{n-\frac{1}{2}}, V^{n-\frac{1}{2}}) - \kappa \sum_{n=2}^{N} k_n (V^{n-\frac{1}{2}}, V^{n-\frac{1}{2}})$$

$$+ \sum_{j=1}^{m} \sum_{n=2}^{N} k_n \left( I^{(\alpha_j)} B_j V^{n-1/2}, V^{n-1/2} \right) = \sum_{n=2}^{N} k_n \left( g^{n-\frac{1}{2}}, V^{n-\frac{1}{2}} \right).$$

Adding above two formulae and applying the positiveness of operator $A$, we have

$$k_1 (\delta_t V^{\frac{1}{2}}, V^1) + \sum_{n=2}^{N} k_n (\delta_t V^{n-\frac{1}{2}}, V^{n-\frac{1}{2}}) - \kappa \left( k_1 \|V^1\|^2 + \sum_{n=2}^{N} k_n \|V^{n-\frac{1}{2}}\|^2 \right)$$

$$+ \sum_{j=1}^{m} \left\{ k_1 \left( I^{(\alpha_j)} B_j V^{\frac{1}{2}}, V^1 \right) + \sum_{n=2}^{N} k_n \left( I^{(\alpha_j)} B_j V^{n-\frac{1}{2}}, V^{n-\frac{1}{2}} \right) \right\}$$

$$= k_1 \left( g^{\frac{1}{2}}, V^1 \right) + \sum_{n=2}^{N} k_n \left( g^{n-\frac{1}{2}}, V^{n-\frac{1}{2}} \right).$$

Let $B_j$ be the positive-definite operator and then from [12, Lemma 3.1], we get

$$k_1 \left( I^{(\alpha_j)} B_j V^{\frac{1}{2}}, V^1 \right) + \sum_{n=2}^{N} k_n \left( I^{(\alpha_j)} B_j V^{n-\frac{1}{2}}, V^{n-\frac{1}{2}} \right) \geq 0.$$  

$\mathbb{Q}$ Springer
Therefore, using the Cauchy-Schwarz inequality, then (4.3) turns into

\[
\begin{align*}
&k_1 \left( \delta_t V_{1/2}^1, V^1 \right) + \sum_{n=2}^N k_n \left( \delta_t V_{n-1/2}^n, V_{n-1/2}^n \right) - \kappa \left( k_1 \| V^1 \|^2 + \sum_{n=2}^N k_n \| V_{n-1/2}^n \|^2 \right) \\
&\quad \leq k_1 \left\{ G_{1/2}^1 \right\} \left\{ V^1 \right\} + \sum_{n=2}^N k_n \left\{ G_{n-1/2}^n \right\} \left\{ V_{n-1/2}^n \right\}.
\end{align*}
\]

(4.5)

Then it is easy to yield

\[
\left( \delta_t V_{1/2}^1, V^1 \right) \geq \frac{\| V^1 \|^2 - \| V^0 \|^2}{2k_1}, \quad \left( \delta_t V_{n-1/2}^n, V_{n-1/2}^n \right) = \frac{\| V^n \|^2 - \| V^{n-1} \|^2}{2k_n}.
\]

(4.6)

Thus, we further obtain

\[
\begin{align*}
\left\{ V^N \right\}^2 - \left\{ V^0 \right\}^2 &\leq k_1 \left( \kappa \left\{ V^1 \right\} + \left\{ G_{1/2}^1 \right\} \right) \left\{ V^1 \right\} \\
&\quad + \sum_{n=2}^N k_n \left( \kappa \left\{ V_{n-1/2}^n \right\} + \left\{ G_{n-1/2}^n \right\} \right) \left\{ V_{n-1/2}^n \right\}.
\end{align*}
\]

(4.7)

Next, choosing a suitable \( M \) such that \( \| V^M \| = \max_{0 \leq n \leq N} \| V^n \| \), then

\[
\begin{align*}
\left\{ V^M \right\}^2 &\leq \left\{ V^0 \right\}^2 + 2k_1 \left( \kappa \left\{ V^1 \right\} + \left\{ G_{1/2}^1 \right\} \right) \left\{ V^1 \right\} \\
&\quad + 2 \sum_{n=2}^M k_n \left( \kappa \left\{ V_{n-1/2}^n \right\} + \left\{ G_{n-1/2}^n \right\} \right) \left\{ V_{n-1/2}^n \right\} \\
&\leq \left\{ V^0 \right\} \left\{ V^M \right\} + 2k_1 \left( \kappa \left\{ V^1 \right\} + \left\{ G_{1/2}^1 \right\} \right) \left\{ V^M \right\} \\
&\quad + 2 \sum_{n=2}^M k_n \left( \kappa \left\{ V_{n-1/2}^n \right\} + \left\{ G_{n-1/2}^n \right\} \right) \left\{ V^M \right\}.
\end{align*}
\]

(4.8)

Consequently,

\[
\begin{align*}
\left\{ V^N \right\} &\leq \left\{ V^M \right\} \\
&\leq \left\{ V^0 \right\} + 2k_1 \left( \kappa \left\{ V^1 \right\} + \left\{ G_{1/2}^1 \right\} \right) \\
&\quad + 2 \sum_{n=2}^N k_n \left( \kappa \left\{ V_{n-1/2}^n \right\} + \left\{ G_{n-1/2}^n \right\} \right).
\end{align*}
\]
\[ \leq \kappa k_N \| V^N \| + 2\kappa \sum_{n=1}^{N-1} (k_n + k_{n+1}) \| V^n \| \\
+ \left( \| V^0 \| + 2 \sum_{n=1}^{N} k_n \| g^{n-\frac{1}{2}} \| \right). \quad (4.9) \]

Then an application of Grönwall’s lemma (see [19]) yields

\[ \| V^N \| \leq C \sum_{n=1}^{N-1} (k_n + k_{n+1}) \left( \| V^0 \| + 2 \sum_{n=1}^{N} k_n \| g^{n-\frac{1}{2}} \| \right) \]
\[ \leq C(T) \left( \| V^0 \| + \sum_{n=1}^{N} k_n \| g^{n-\frac{1}{2}} \| \right), \quad (4.10) \]

which can finish the proof by using (2.1). \[\square\]

### 4.2 Convergence

Based on above analyses, we shall deduce the convergence of the scheme by the energy argument. We first define

\[ \rho^n = v(t_n) - V^n, \quad e^n = u(t_n) - U^n, \quad 0 \leq n \leq N. \quad (4.11) \]

Then we subtract (3.17)-(3.18) from (3.14)-(3.15) to get error equations as follows

\[ \delta_t \rho^{\frac{1}{2}} + A \rho^1 + \sum_{j=1}^{m} I^{(\alpha_j)} B_j \rho^{\frac{1}{2}} - \kappa \rho^1 = R^{\frac{1}{2}}, \quad (4.12) \]
\[ \delta_t \rho^{n-\frac{1}{2}} + A \rho^{n-\frac{1}{2}} + \sum_{j=1}^{m} I^{(\alpha_j)} B_j \rho^{n-\frac{1}{2}} - \kappa \rho^{n-\frac{1}{2}} = R^{n-\frac{1}{2}} \quad (4.13) \]

with \( 2 \leq n \leq N \) and \( \rho^0 = 0 \), where \( R^{n-\frac{1}{2}} \) is given in (3.16).

In order to further derive the convergence, we shall introduce several auxiliary lemmas. At first, from [11, Corollary 3.4], we can yield the following two lemmas.

**Lemma 4** Suppose that \( t_n \) satisfies the assumptions (3.9)-(3.11) and the exact solution satisfies the regularity condition (2.3), then for \( 1 \leq m < \infty, \gamma \geq 1 \) and \( \alpha = \min_{1 \leq j \leq m} \{ \alpha_j \} \), we have

\[ \sum_{n=1}^{N} k_n \left\| R^{n-\frac{1}{2}} \right\| \leq C_{\alpha, \gamma, T, m} \times \Xi(k, \alpha, \gamma, T), \]

\[ \Xi \text{ Springer} \]
where
\[ \Xi(k, \alpha, \gamma, T) := \begin{cases} 
  k^\gamma(\alpha+1), & \text{if } 1 \leq \gamma < 2/(\alpha + 1), \\
  k^2 \log(t_N/t_1), & \text{if } \gamma = 2/(\alpha + 1), \\
  k^2, & \text{if } \gamma > 2/(\alpha + 1). 
\end{cases} \]

**Lemma 5** Assuming that \( t_n \) satisfies the assumptions (3.9)-(3.11) and the modified source term satisfies \( t \|g'(t)\| + t^2 \|g''(t)\| \leq Ct^\alpha \), then for \( \alpha = \min_{1 \leq j \leq m} \{\alpha_j\} \) and \( \gamma \geq 1 \), it holds that
\[
\sum_{n=1}^{N} k_n \left\| \mathcal{R}_n^{n-\frac{1}{2}} \right\| \leq C_{\alpha, \gamma} \times \Xi(k, \alpha, \gamma, T).
\]

Furthermore, we will estimate the remaining error terms in (3.16) based on certain reasonable conditions. Firstly, we give the following result.

**Lemma 6** Assume that assumptions (3.9)-(3.11) hold and the exact solution satisfies the regularity (2.3). Then for \( \gamma \geq 1 \) and \( \alpha = \min_{1 \leq j \leq m} \{\alpha_j\} \), it holds that
\[
\sum_{n=1}^{N} k_n \left\| \mathcal{R}_n^{n-\frac{1}{2}} \right\| \leq C_{\alpha, \gamma} \times \Xi(k, \alpha, \gamma, T).
\]

**Proof** First, when \( n = 1 \), we use the Taylor expansion with integral remainder to obtain
\[
k_1 \mathcal{R}_1^{\frac{1}{2}} = \int_0^{k_1} \int_0^{k_1} Av'(\xi)d\xi dt = \int_0^{k_1} \int_0^{\xi} Av'(\xi)dt d\xi = \int_0^{k_1} \xi Av'(\xi)d\xi. \tag{4.14}
\]
Similarly, we also have
\[
v(t) - v(t_{n-1/2}) = (t - t_{n-1/2})v'(t_{n-1/2}) + \int_{t_{n-1/2}}^{t} (t - \xi)v''(\xi)d\xi, \tag{4.15}
\]
and
\[
v^{n-\frac{1}{2}} - v(t_{n-\frac{1}{2}}) = \frac{1}{2} \left[ \int_{t_{n-1}}^{t_{n-1/2}} (\xi - t_{n-1})v''(\xi)d\xi - \int_{t_{n-1/2}}^{t_n} (\xi - t_n)v''(\xi)d\xi \right]. \tag{4.16}
\]
Then for $n \geq 2$, we yield
\[
k_n \mathcal{R}_1^{n-\frac{1}{2}} = \int_{t_{n-1}}^{t_n} A \left[ v^{n-\frac{1}{2}} - v(t_{n-1}/2) \right] d\zeta + \int_{t_{n-1}}^{t_n} A \left[ v(t_{n-1}/2) - v(t) \right] d\zeta
\]
\[
= k_n \left[ \int_{t_{n-1}}^{t_{n-1}/2} (\zeta - t_{n-1}) A v''(\zeta) d\zeta - \int_{t_{n-1}}^{t_n} (\zeta - t_n) A v''(\zeta) d\zeta \right]
\]
\[
- \frac{A v'(t_{n-1}/2)}{2} (t - t_{n-1}/2)^2 \bigg|_{t_{n-1}}^{t_n} - \int_{t_{n-1}}^{t_n} \int_{t_{n-1}/2}^{t} (t - \zeta) A v''(\zeta) d\zeta dt.
\]
(4.17)

Thus, using (4.14), (4.17), regularity condition (2.3) and (3.10), we obtain
\[
\sum_{n=1}^{N} k_n \left\| \mathcal{R}_1^{n-\frac{1}{2}} \right\| \leq \int_{0}^{k_1} \zeta \| A v'(\zeta) \| d\zeta + \sum_{n=2}^{N} k_n \int_{t_{n-1}}^{t_n} \| A v''(\zeta) \| d\zeta
\]
\[
\leq C_{\alpha} \left( \int_{0}^{k_1} \zeta^{\alpha} d\zeta + \sum_{n=2}^{N} k_n^{3} t_{n-2}^{\alpha} \right)
\]
\[
\leq C_{\alpha, \gamma} \left( k_{\gamma(\alpha+1)} + k^2 \sum_{n=2}^{N} t_{n-2}^{\alpha-2/\gamma} k_n \right),
\]
from which,
\[
\sum_{n=2}^{N} t_{n-2}^{\alpha-2/\gamma} k_n \leq C \int_{k_1}^{t_N} \zeta^{\alpha-2/\gamma} d\zeta \leq C \times \begin{cases} \frac{k_{\gamma(\alpha+1)-2}}{2/\gamma - (\alpha+1)}, & \text{if } \gamma < \frac{2}{\alpha+1}, \\ \log(t_N/t_1), & \text{if } \gamma = \frac{2}{\alpha+1}, \\ \frac{t_{n-2}^{\alpha+1-2/\gamma}}{(\alpha+1)-2/\gamma}, & \text{if } \gamma > \frac{2}{\alpha+1}. \end{cases}
\]
(4.18)

This completes the proof. \(\square\)

Then, after a similar analysis, the following result holds.

**Lemma 7** Suppose that assumptions (3.9)-(3.11) are valid and the exact solution satisfies the regularity (2.3). For $\gamma \geq 1$, $\kappa \geq 0$ and $\alpha = \min_{1 \leq j \leq m} \{ \alpha_j \}$, it holds that
\[
\sum_{n=1}^{N} k_n \left\| \mathcal{R}_3^{n-\frac{1}{2}} \right\| \leq C_{\alpha, \kappa, \gamma, T} \times k^{2}.
\]

**Proof** For $n = 1$, using the Taylor expansion with integral remainder, we get
\[
k_1 \mathcal{R}_3^{\frac{1}{2}} = -\kappa \int_{t_1}^{k_1} v'(\zeta) d\zeta dt = -\kappa \int_{0}^{k_1} \zeta v'(\zeta) d\zeta.
\]
(4.19)
Then for \( n \geq 2 \), we employ (4.15) and (4.16) to obtain

\[
k_n R^{n-\frac{1}{2}} = \int_{t_{n-1}}^{t_n} \left[ v_{n-\frac{1}{2}} - v(t_{n-1/2}) \right] d\zeta + \int_{t_{n-1}}^{t_n} \left[ v(t_{n-1/2}) - v(t) \right] d\zeta
\]

\[
= -\frac{\kappa k_n}{2} \left[ \int_{t_{n-1}}^{t_n-1/2} (\zeta - t_{n-1}) v''(\zeta) d\zeta \right] - \int_{t_{n-1}}^{t_n} (\zeta - t_n) v''(\zeta) d\zeta
\]

\[
+ \frac{\kappa}{2} \left( t_{n-1/2} - t_{n-1} \right)^2 \left[ \int_{t_{n-1}}^{t_n} (t - \zeta) v''(\zeta) d\zeta dt \right].
\]

(4.20)

Next, applying (4.19), (4.20), Theorem 1 and Theorem 2, we have

\[
\sum_{n=1}^{N} k_n \left\| R^{n-\frac{1}{2}} \right\| \leq \kappa \int_{0}^{k_1} \| v'(\zeta) \| d\zeta + \kappa \sum_{n=2}^{N} k_n^2 \int_{t_{n-1}}^{t_n} \| v''(\zeta) \| d\zeta
\]

\[
\leq C_{\alpha,\kappa} \left( \int_{0}^{k_1} \zeta d\zeta + \sum_{n=2}^{N} k_n^2 (t_n^\alpha - t_{n-1}^\alpha) \right)
\]

\[
\leq C_{\alpha,\kappa,\gamma} \left( k_1^2 + k^2 \sum_{n=2}^{N} (t_n^\alpha - t_{n-1}^\alpha) \right),
\]

which finishes the proof.

Based on above analyses, we can yield the following convergence result.

**Theorem 6** Let \( V^n \) denoted by (3.17) and (3.18) and \( U^n \) be the approximate solution of and \( v(t_n) \) and \( u(t_n) \), respectively. For \( T < \infty, \alpha = \min_{1 \leq j \leq m} \{ \alpha_j \}, 1 \leq m \leq \infty, \gamma \geq 1 \) and \( 0 \leq \kappa < \infty \), then it holds that

\[
\max_{1 \leq n \leq N} \| V^n - v(t_n) \| \leq C_{\alpha,\kappa,\gamma,T,m} \times \Xi(k, \alpha, \gamma, T),
\]

\[
\max_{1 \leq n \leq N} \| U^n - u(t_n) \| \leq C_{\alpha,\kappa,\gamma,T,m} \times \Xi(k, \alpha, \gamma, T),
\]

where the notation \( \Xi(k, \alpha, \gamma, T) \) is denoted in Lemma 4.

**Proof** In view of (4.11)-(4.13) and Theorem 5, we get

\[
\max_{1 \leq n \leq N} \| \rho^n \| \leq C(T) \left( \| \rho^0 \| + \sum_{n=1}^{N} k_n \| R^{n-1/2} \| \right),
\]

\[
\max_{1 \leq n \leq N} \| e^n \| \leq C(T) \left( \| e^0 \| + \sum_{n=1}^{N} k_n \| R^{n-1/2} \| \right),
\]

from which \( \| \rho^0 \| = \| e^0 \| = 0 \). Then utilizing the triangle inequality, (2.1) and Lemmas 4-7, the proof is completed.
5 Numerical experiment

In this section, in order to further show the time convergence of proposed scheme, we formulate the fully discrete scheme by the time semidiscrete scheme (3.17)-(3.19) and a standard spatial finite difference method, with the spatial step size $h = \frac{L}{M} (M \in \mathbb{Z}^+)$ is the number of spatial partitions). Below we choose the parameters $T = L = 1$ and denote the $L_2$-norm error

$$\mathcal{E}_{CN}(N, M) = \max_{1 \leq n \leq N} \| U^n - u(t_n) \|,$$

and the temporal convergence rate

$$\text{rate}^k_{CN} = \log_2 \left( \frac{\mathcal{E}_{CN}(N, M)}{\mathcal{E}_{CN}(2N, M)} \right).$$

In addition, we define the following notations for illustrating the numerical stability of the proposed scheme,

$$U^* = \max_{1 \leq n \leq N} \| U^n \|, \quad U^{**} = \sum_{n=1}^{N} k_n \| U^n \|.$$

Example 1 Here, we first consider the one-dimensional case ($d = 1$) of (1.1) with the parameter $m = 2$. Set the operators $A = B_1 = B_2 = -\frac{\partial^2}{\partial x_1^2}$ over the domain $\Omega = (0, L)$ with the homogeneous Dirichlet boundary conditions (DBCs). To meet the regularity assumption (2.3), the exact solution of (1.1) is given via

$$u(x_1, t) = \left( t^{1+\alpha_1} + t^{1+\alpha_2} \right) e^{-\kappa t} \sin \pi x_1,$$

thus the initial condition $u_0(x_1) = 0$ and the source term $f(x_1, t)$ can be computed accordingly.

In Table 1, we list the $L_2$-norm errors and time convergence rates by fixing $\kappa = 1$, $\alpha_1 = 0.2$, $\alpha_2 = 0.8$ and $M = 256$, from which, we discuss three cases regarding

| $N$ | $\mathcal{E}_{CN}$ | $\text{rate}^k_{CN}$ | $\mathcal{E}_{CN}$ | $\text{rate}^k_{CN}$ | $\mathcal{E}_{CN}$ | $\text{rate}^k_{CN}$ |
|-----|-----------------|----------------|----------------|----------------|----------------|----------------|
| 16  | 1.2098e-02     | *             | 1.1501e-03    | *             | 8.3435e-04    | *             |
| 32  | 4.9207e-03     | 1.30          | 2.9352e-04    | 1.97          | 2.2642e-04    | 1.88          |
| 64  | 2.0432e-03     | 1.27          | 7.9232e-05    | 1.89          | 5.6305e-05    | 2.01          |
| 128 | 8.6517e-04     | 1.24          | 2.1719e-05    | 1.87          | 1.0427e-05    | 2.43          |
| 256 | 3.7105e-04     | 1.22          | 5.9279e-06    | 1.87          | 2.0922e-06    | 2.32          |
grading index $\gamma$, i.e., $\gamma = 1$, $\gamma = \frac{2}{\alpha + 1}$ and $\gamma = \frac{2}{\alpha + 1} + 1$, respectively. It can be seen clearly from Table 1 that time convergence rates of proposed scheme reach the order $1 + \min\{\alpha_1, \alpha_2\}$ with uniform temporal step sizes, and second-order accuracy for time can be obtained with nonuniform temporal step sizes ($\gamma \geq \frac{2}{\alpha + 1}$). These are consistent with Theorem 6. Below, we only consider the cases with the optimal grading index $\gamma = \frac{2}{\alpha + 1}$.

Table 2 shows the $L_2$-norm errors and time convergence rates when $\kappa = 2$, $\gamma = \frac{2}{\alpha + 1}$ and $M = 512$, from which we present three cases with different $\alpha_1$ and $\alpha_2$, including (i) $\alpha_1 < \frac{1}{2} < \alpha_2$, (ii) $\alpha_1, \alpha_2 < \frac{1}{2}$ and (iii) $\alpha_1, \alpha_2 > \frac{1}{2}$. Then, the numerical results in Table 2 demonstrate that uniform temporal second-order accuracy can be yielded under three situations.

Besides, in Table 3, fixed $\alpha_1 = \alpha_2 = 0.5$, $\gamma = \frac{2}{\alpha + 1}$ and $M = 512$, we discuss three cases about different tempered parameter $\kappa$, involving $\kappa = 0.2$, $\kappa = 1$ and $\kappa = 5$, respectively, from which the results approximate second-order convergence for time. These validate our theoretical analysis and illustrate the effectiveness of proposed scheme.

Then, in Table 4, fixed $\kappa = 1$, $\gamma = \frac{2}{\alpha + 1}$ and $M = 256$, we list two types of numerical solution $U^*$ and $U^{**}$. Table 4 show that as $N$ increases gradually, the value of $U^*$ gradually stabilizes and finally remains unchanged, and the value of $U^{**}$ has

| $N$ | $\alpha_1 = 0.15, \alpha_2 = 0.85$ | $\alpha_1 = 0.10, \alpha_2 = 0.20$ | $\alpha_1 = 0.80, \alpha_2 = 0.90$ |
|-----|----------------------------------|----------------------------------|----------------------------------|
| $E_{CN}$ | $rate_{CN}$ | $E_{CN}$ | $rate_{CN}$ | $E_{CN}$ | $rate_{CN}$ |
| 8   | 4.8275e-03 | *   | 8.0324e-03 | *   | 4.7639e-03 | *   |
| 16  | 1.1850e-03 | 2.03 | 1.9166e-03 | 2.07 | 8.8628e-04 | 2.43 |
| 32  | 3.0304e-04 | 1.97 | 4.7020e-04 | 2.03 | 1.6001e-04 | 2.47 |
| 64  | 8.1431e-05 | 1.90 | 1.2227e-05 | 1.94 | 3.3122e-05 | 2.27 |
| 128 | 2.2048e-05 | 1.88 | 3.2098e-05 | 1.93 | 8.0217e-06 | 2.05 |

| $N$ | $\alpha_1 = 0.15, \alpha_2 = 0.85$ | $\alpha_1 = 0.10, \alpha_2 = 0.20$ | $\alpha_1 = 0.80, \alpha_2 = 0.90$ |
|-----|----------------------------------|----------------------------------|----------------------------------|
| $E_{CN}$ | $rate_{CN}$ | $E_{CN}$ | $rate_{CN}$ | $E_{CN}$ | $rate_{CN}$ |
| 8   | 8.1009e-03 | *   | 7.6310e-03 | *   | 5.5947e-03 | *   |
| 16  | 1.7039e-03 | 2.25 | 1.6545e-03 | 2.20 | 1.4190e-03 | 1.98 |
| 32  | 3.9316e-04 | 2.12 | 3.8782e-04 | 2.09 | 3.6162e-04 | 1.97 |
| 64  | 1.0427e-04 | 1.91 | 1.0361e-04 | 1.90 | 1.0037e-04 | 1.85 |
| 128 | 2.8791e-05 | 1.86 | 2.8701e-05 | 1.85 | 2.8255e-05 | 1.83 |
| 256 | 8.0558e-06 | 1.84 | 8.0397e-06 | 1.84 | 7.9604e-06 | 1.83 |
maintained a non-increasing trend. These results demonstrate the numerical stability of proposed scheme in the time direction. Furthermore, the same phenomenon is exhibited in Table 5, by fixing some different parameters.

Example 2 In this example, we consider the two-dimensional (2D) case \((d = 2)\) of (1.1) with \(m = 2\). Let \(A\) be the 2D Laplace operator, and the operators \(B_1 = -\frac{\partial^2}{\partial x_1^2}\) and \(B_2 = -\frac{\partial^2}{\partial x_2^2}\) over the domain \(\Omega = (0, L) \times (0, L)\) with the homogeneous DBCs. In order to satisfy the regularity assumption (2.3), we present the exact solution of (1.1) as follows

\[
u(x_1, x_2, t) = \left(t^{1+\min(\alpha_1, \alpha_2)} + 1\right) e^{-\kappa t} \sin \pi x_1 \sin \pi x_2,
\]

then the initial condition

\[
u_0(x_1, x_2) = \sin \pi x_1 \sin \pi x_2,
\]

and the source term \(f(x_1, x_2, t)\) can be calculated correspondingly.

### Table 5: Example 1: Numerical solutions when \(\alpha_1 = \alpha_2 = 0.5, \gamma = \frac{2}{\alpha+1}\) and \(M = 32\)

| \(N\) | \(U^*\) | \(U^{**}\) | \(U^*\) | \(U^{**}\) | \(U^*\) | \(U^{**}\) |
|------|---------|---------|---------|---------|---------|---------|
| 32   | 2.5016e−01 | 1.0454e−01 | 1.5173e−01 | 7.3025e−02 | 9.2033e−02 | 5.1659e−02 |
| 64   | 2.5018e−01 | 1.0231e−01 | 1.5175e−01 | 7.1730e−02 | 9.2040e−02 | 5.0921e−02 |
| 128  | 2.5019e−01 | 1.0119e−01 | 1.5175e−01 | 7.1080e−02 | 9.2042e−02 | 5.0548e−02 |
| 256  | 2.5019e−01 | 1.0063e−01 | 1.5175e−01 | 7.0753e−02 | 9.2043e−02 | 5.0360e−02 |
| 512  | 2.5019e−01 | 1.0035e−01 | 1.5175e−01 | 7.0590e−02 | 9.2043e−02 | 5.0266e−02 |
| 1024 | 2.5019e−01 | 1.0021e−01 | 1.5175e−01 | 7.0509e−02 | 9.2043e−02 | 5.0219e−02 |
| 2048 | 2.5019e−01 | 1.0014e−01 | 1.5175e−01 | 7.0468e−02 | 9.2043e−02 | 5.0195e−02 |
Here, in Table 6 we present the $L_2$-norm errors and time convergence rates when fixing $\kappa = 2$, $\gamma = \frac{2}{\alpha+1}$ and $M = 70$. The numerical results incarnate time second-order convergence by selecting three kinds of values of $\alpha_1$ and $\alpha_2$, when $N$ increases gradually.

Fixing $\alpha_1 = 0.3$, $\alpha_2 = 0.6$, $\gamma = \frac{2}{\alpha+1} + \frac{1}{2}$ and $M = 80$, the numerical results from Table 7 approximate second order in the time direction, with different tempered parameter $\kappa$ ($\kappa = 0, 0.2, 1, 5$), which is in accordance with the theory.

### 6 Concluding remarks

In this work, we have considered and analyzed numerical solutions for Volterra integrodifferential equations with tempered multi-term kernels. First, we deduced certain regularity estimates of the exact solution of (1.1). Then under graded meshes, we applied the Crank-Nicolson method and PI rule to construct a time discrete scheme. Based on the regularity assumptions, we proved the unconditional stability and accurate second-order convergence for time by the energy argument. Finally, numerical experiments verified our theoretical results.

Moreover, the regularity of exact solution of (1.1) can be extended to a semilinear case (source term $f(t)$ replaced by $f(t, u(t))$) by certain appropriate assumptions; also, theoretical results about new numerical scheme could be similarly yielded by adding a Lipschitz condition $|f(t, u_1) - f(t, u_2)| \leq L|u_1 - u_2|$.

### Table 6

| $\kappa = 2$ | $L_2$-norm errors and time convergence rates when $\kappa = 2$, $\gamma = \frac{2}{\alpha+1}$ and $M = 70$ |
|-------------|--------------------------------------------------|
| $\alpha_1 = 0.10$, $\alpha_2 = 0.90$ | $\alpha_1 = 0.15$, $\alpha_2 = 0.20$ | $\alpha_1 = 0.80$, $\alpha_2 = 0.75$ |
| $N$ | $\mathcal{E}_{CN}$ | rate$^{k}_{CN}$ | $\mathcal{E}_{CN}$ | rate$^{k}_{CN}$ | $\mathcal{E}_{CN}$ | rate$^{k}_{CN}$ |
|-----|----------------|---------------|----------------|---------------|----------------|---------------|
| 12  | 1.3009e−02 | *             | 1.8510e−02 | *             | 2.8330e−03 | *             |
| 24  | 3.8048e−03 | 1.77          | 5.6062e−03 | 1.74          | 8.7227e−04 | 1.70          |
| 48  | 1.0148e−03 | 1.91          | 1.5145e−03 | 1.88          | 2.4016e−04 | 1.86          |
| 96  | 2.6995e−04 | 1.91          | 4.0689e−04 | 1.90          | 6.2739e−05 | 1.94          |

### Table 7

| $\kappa = 0$ | $L_2$-norm errors and time convergence rates when $\alpha_1 = 0.3$, $\alpha_2 = 0.6$, $\gamma = \frac{2}{\alpha+1} + \frac{1}{2}$ and $M = 80$ |
|-------------|--------------------------------------------------|
| $N$ | $\mathcal{E}_{CN}$ | rate$^{k}_{CN}$ | $\mathcal{E}_{CN}$ | rate$^{k}_{CN}$ | $\mathcal{E}_{CN}$ | rate$^{k}_{CN}$ | $\mathcal{E}_{CN}$ | rate$^{k}_{CN}$ |
|-----|----------------|---------------|----------------|---------------|----------------|---------------|----------------|---------------|
| 4   | 2.3908e−02 | *             | 2.3711e−02 | *             | 2.2942e−02 | *             | 1.9567e−02 | *             |
| 8   | 5.8707e−03 | 2.03          | 5.8643e−03 | 2.01          | 5.8393e−03 | 1.97          | 5.7221e−03 | 1.77          |
| 16  | 1.3915e−03 | 2.08          | 1.3904e−03 | 2.08          | 1.3858e−03 | 2.07          | 1.3634e−03 | 2.07          |
| 32  | 3.2342e−04 | 2.10          | 3.2318e−04 | 2.10          | 3.2219e−04 | 2.10          | 3.1738e−04 | 2.10          |
| 64  | 1.0394e−04 | 1.64          | 8.5550e−05 | 1.92          | 6.6420e−05 | 2.28          | 6.5801e−05 | 2.27          |
Note that in this paper, we only consider the discrete scheme and numerical analysis in the time direction. In our future work, the fully discrete scheme can be considered combined with some high-precision spatial-discrete techniques, such as local discontinuous Galerkin method, finite difference method, orthogonal spline collocation method, compatible wavelet, etc.

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Declarations

Conflict of interest The authors declare no competing interests.

References

1. Cao, Y., Nikan, O., Avazzadeh, Z.: A localized meshless technique for solving 2D nonlinear integro-differential equation with multi-term kernels. Appl. Numer. Math (2022)
2. Carslaw, H.S., Jaeger, J.C.: Conduction of heat in solids, 2nd edn. Charendon Press, Oxford (1959)
3. Chen, C., Thomée, V., Wahlbin, L.B.: Finite element approximation of a parabolic integro-differential equation with a weakly singular kernel. Math. Comput. 58, 587–602 (1992)
4. Friedman, A., Shinbrot, M.: Volterra integral equations in Banach space. Trans. Amer. Math. Soc. 126, 31–179 (1967)
5. Hannsgen, K.B., Wheeler, R.L.: Uniform $L^1$ behavior in classes of integrodifferential equations with completely monotonic kernels. SIAM J. Math. Anal. 15, 579–594 (1984)
6. Heard, M.L.: An abstract parabolic Volterra integrodifferential equation. SIAM J. Math. Anal. 13, 81–105 (1982)
7. Jin, B., Lazarov, R., Liu, Y., Zhou, Z.: The Galerkin finite element method for a multi-term time-fractional diffusion equation. J. Comput. Phys. 281, 825–843 (2015)
8. Larsson, S., Thomée, V., Wahlbin, L.: Numerical solution of parabolic integro-differential equations by the discontinuous Galerkin method. Math. Comput. 67, 45–71 (1998)
9. Liu, F., Meerschaert, M.M., McGough, R.J., et al.: Numerical methods for solving the multi-term time-fractional wave-diffusion equation. Fract. Calcul. Appl. Anal. 16, 9–25 (2013)
10. MacCamy, R.C.: An integro-differential equation with application in heat flow. Quart. Appl. Math. 35, 1–19 (1977)
11. McLean, W., Mustapha, K.: A second-order accurate numerical method for a fractional wave equation. Numer. Math. 105, 481–510 (2007)
12. Mustapha, K., Mustapha, H.: A second-order accurate numerical method for a semilinear integro-differential equation with a weakly singular kernel. IMA J. Numer. Anal. 30, 555–578 (2010)
13. Noren, R.: Uniform $L^1$ behavior in classes of integro-differential equations with convex kernels. J. Integ. Equ. Appl. 1, 385–369 (1988)
14. Noren, R.: Uniform $L^1$ behavior in a class of linear Volterra equations. Quart. Appl. Math. 47, 547–554 (1989)
15. Pazy, A.: Semigroups of linear operators and applications to partial differential equations. Springer, New York (1983)
16. Podlubny, I.: Fractional differential equations. Academic Press, San Diego (1999)
17. Qiu, W., Xu, D., Guo, J.: Numerical solution of the fourth-order partial integro-differential equation with multi-term kernels by the Sinc-collocation method based on the double exponential transformation. Appl. Math. Comput. 392, 125693 (2021)
18. Sabzikar, F., Meerschaert, M.M., Chen, J.: Tempered fractional calculus. J. Comput. Phys. 293, 14–28 (2015)
19. Sloan, I.H., Thomée, V.: Time discretization of an integro-differential equation of parabolic type. SIAM J. Numer. Anal. 23, 1052–1061 (1986)
20. Xu, D.: The long-time global behavior of time discretization for fractional order Volterra equations. Calcolo 35, 93–116 (1998)
21. Xu, D.: The time discretization in classes of integro-differential equations with completely monotonic kernels: Weighted asymptotic stability. Sci. China Math. 56, 395–424 (2013)
22. Xu, D.: The time discretization in classes of integro-differential equations with completely monotonic kernels: weighted asymptotic convergence. Numer. Methods Part. Differ. Equ. 32, 896–935 (2016)
23. Xu, D.: Numerical asymptotic stability for the integro-differential equations with the multi-term kernels. Appl. Math. Comput. 309, 107–132 (2017)
24. Xu, D.: Second-order difference approximations for Volterra equations with the completely monotonic kernels. Numer. Algorithms 81, 1003–1041 (2019)
25. Zhou, J., Xu, D., Chen, H.: A weak Galerkin finite element method for multi-term time-fractional diffusion equations. E. Asian J. Appl. Math. 8, 181–193 (2018)

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