Algebro-geometric approach to a fermion self-consistent field theory on coset space $\frac{SU_{m+n}}{S(U_m \times U_n)}$

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Abstract

The integrability-condition method is regarded as a mathematical tool to describe the symmetry of collective sub-manifold. We here adopt the particle–hole representation. In the conventional time-dependent (TD) self-consistent field (SCF) theory, we take the one-form linearly composed of the TD SCF Hamiltonian and the infinitesimal generator induced by the collective-variable differential of canonical transformation on a group. Standing on the differential geometrical viewpoint, we introduce a Lagrange-like manner familiar to fluid dynamics to describe collective coordinate systems. We construct a geometric equation, noticing the structure of coset space $\frac{SU_{m+n}}{S(U_m \times U_n)}$. To develop a perturbative method with the use of the collective variables, we aim at constructing a new fermion SCF theory, i.e., renewal of TD Hartree-Fock (TDHF) theory by using the canonicity condition under the existence of invariant subspace in the whole HF space. This is due to a natural consequence of the maximally decoupled theory because there exists an invariant subspace, if the invariance principle of Schrödinger equation is realized. The integrability condition of the TDHF equation determining a collective sub-manifold is studied, standing again on the differential geometric viewpoint. A geometric equation works well over a wide range of physics beyond the random phase approximation.

Keywords: Particle-hole representation; TDHF theory; $U(N)$ Lie algebra; Lagrange-like manner; Integrability condition; Geometric equation; Coset space $\frac{SU_{m+n}}{S(U_m \times U_n)}$; Random phase approximation

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1 Short history of theory for nuclear collective motion

According to Yamamura-Kuriyama (YK) [1], we view simply a short history for the model and the theory: In the Bohr-Mottelson (BM) model [2], a liquid drop model is taken for the collective motion and the independent particle motion is described by the shell model. However, microscopically, the constituents of the liquid drop are nucleons themselves, which move in a nucleus independently. In the year 1960, Arvieu-Veneroni, Baranger and Marumori proposed, independently, a collective-motion theory for spherical even-mass nuclei [3] called the quasi-particle random phase approximation (Q-RPA). There exist the two types of the fundamental correlations, that is, a short-range correlation and a long-range one [4]. The former is expressed in terms of the pairing interaction and generates a superconducting mode. The excited states are classified by a seniority coupling scheme and well described in terms of the quasi-particles given by the Hartree-Bogoliubov (HB) theory [5]. The latter is expressed in terms of the particle and hole operators and give rise to the collective motions related to the density fluctuations around the equilibrium state. The particle-hole random phase approximation (ph-RPA) is a method for collective motions such as rotational and vibrational motions around the equilibrium state.

The Q- and ph-RPA theories are the systematic methods for treating phenomena of coexistence of both the correlations. However, these theories are essentially a harmonic oscillator approximation. They can be extended to take into account of the nonlinear terms contained in the equations of motion. To solve such a problem, the boson expansion theory based on the HB theory (abbreviated as BEHB) has been proposed by Belyaev and Zelevinsky [6], and Marumori, Yamamura and Tokunaga [7]. The essence of the BEHB theory is to express the fermion-pairs obeying the SO(2N) algebra with N single-particle states as the functions of the boson operators, to estimate the deviation of the fermion-pairs from the pure boson character. Keeping the original idea, up to now, the various BEHBs have been proposed. These methods are classified into mainly the two classes: the first class: the boson representation is constructed to reproduce the Lie algebra which the fermion-pairs obey; the second class: in so-called the Marumori-type, let the state vectors in the fermion Fock space correspond to the state vectors in the boson Fock space by the one-to-one mapping and the boson representation is constructed to achieve the coincidence of the transition matrix-values of any physical quantity for between the boson-state vectors and the vectors in the original fermion states. In the former, there are Holstein-Primakoff (HP)-type, Schwinger-type and Dyson-type. The HP expresses the fermion-pairs in terms of an infinite series of the boson operators, the other two do that in terms of the finite series ones. The algebraic structure governing the fermion-pairs and the BEHB theories have been investigated. The boson operators proposed by Providencia and Weneser and Marshellk [8] are based on the boson representation of particle-hole pairs forming SU(N) algebra. Fukutome, Yamamura and one of the present authors S. N. [9, 10, 11], found the fermions to obey the SO(2N+1), SO(2N+2) and U(N+1) Lie algebras, respectively. The BEHB theory formulated by the Schwinger type boson representation has been greatly developed by Fukutome and S. N. However, the BEHB theories themselves do not contain any scheme under which the collective degree of freedom is selected from the whole degrees of freedom.

Meanwhile, there exists another approach to the microscopic theory of the collective motion, that is, the time dependent Hartree-Fock (TDHF) or time dependent Hartree-Bogoliubov (TDHB) theory. At an early stage, the idea of the TDHF theory was proposed by Nogami [12] and soon later by Marumori [3] for the small amplitude vibrational motions. With the help of this method, we determine the time dependence of any physical quantity, function of the density matrix. The frequency of the small fluctuation around the static HF field and the equation for the frequency are the same forms as those given by the RPA theory. The RPA
theory is quantal and the frequency given by this method means the excitation energy of the first excited state. Then, the RPA theory is a possible quantization of the TDHF theory in the small amplitude limit. In fact, as proved by Malshalek-Horzwarth [13], the BEHB theory reduces to the TDHB theory at any order, under the replacement of the boson operators by the classical canonical variables. Using the technique analogous to the canonical transformation in the classical mechanics, it is expected to obtain a scheme for choosing the collective degree of freedom in the TDHF theory. Historically, there is a stream, whose origin is the cranking model given by Inglis [14]. The basic standpoint of this model is a kind of the adiabatic perturbation theories. It starts from the assumption that the speed of the collective motion is much slower than that of any other non-collective motions. The TDHF theory with the adiabatic treatment (ATDHF) was presented by Shono-Tanaka and Thouless-Valatin, respectively, at the early stage of the study of such an adiabatic treatment [15] [16].

At the middle of 1970, the TDHF theory was revived not only for the study of anharmonic vibration but also for studies of heavy ion-reaction and nuclear fission. These problems have a common feature from the viewpoint of the large amplitude collective motions. In this new situation, the ATDHF theory was developed mainly by Baranger-Veneroni [17], Brink [18] and Goeke-Reinhard and Mukherjee-Pal [19]. Especially, the most important point of the ATDHF theory by Villars [20] is the introduction of the concept of collective path into the phase space. A collective motion corresponds to a trajectory in the phase space which moves along the collective path. Along the same spirit, Holtzwarth-Yukawa [21], Rowe-Bassermann [22], and Marumori, Maskawa, Sakata and Kuriyama gave the TDHF theory so called the maximal decoupling method [23]. This theory is formulated in a canonical form framework. The various techniques of the classical mechanics are useful and the canonical quantization is expected. By solving the equation, we obtain the corrections whose order is higher than the RPA order. The collective sub-manifold is, in some sense, a possible extension from the collective path. This theory has a potentiality to give not only the collective modes but also the intrinsic modes.

In the BM model the degrees of freedom of collective motion and independent-particle one are overestimated. This problem has been inquired from the microscopical theoretic viewpoint. See our “exact canonical momenta approach” [24]. At an early stage of the study, Marumori, Yukawa and Tanaka [25] and Tomonaga [26, 27] proposed independently the remarkable theories which, however, are, in certain sense, kinematical because the collective motion is given a priori. In this concern, there exist three points to be solved dynamically: i) to determine the microscopic structure of the collective motion, which is the ensemble of the individual particles motion, in relation to the dynamics under consideration, ii) to determine independent-particle motions which should be orthogonal to the collective motion and iii) to give a coupling between these two types of motions. The TDHF theory in the canonical form enables us to select the collective motion in relation to the dynamics. However, it gives us no scheme to take into account the effect of the quasi-fermion, because the TD Slater determinant (S-det) contains only variables to represent the collective motion. Then, for example, an odd-particle system cannot be described. Along the same spirit as the TDHF split, YK extended the TDHF theory to that on a fermion coherent state constructed on the TD S-det. The coherent state contains not only the usual canonical variables but also the Grassmann variables. Candlin [28], Berezin [29] and Casalbuoni obtained a classical image of the fermions by regarding Grassmann variables as canonical [30]. The constraints governing the variables to remove the overestimated degrees of freedom were decided under a physical consideration. With the use of Dirac’s canonical theory [31] for a constrained system, the TDHF theory was successfully solved by YK, for a unified description of the collective and independent-particle motions in the classical mechanics [32].
2 Symmetry of the evolution equation

The historical evolution is summarized in order to present the optimal coordinate system to describe the group manifold itself, based on the Lie algebra in which pairs of the finite-dimensional fermion obey, or to do dynamics on the manifold. The boson operator in the BEHB is the operator arising from the coordinate system of the tangent spaces on the manifold in the fermion Fock space. The BEHB itself does not contain any scheme under which the collective degree of freedom can be selected from the whole degrees of freedom. While, approaches to collective motions by the TDHF theory suggest that the coordinate system on which the collective motions can be described is deeply related not only to the global symmetry of the finite-dimensional (FD) group manifold itself but is also behind the local symmetry besides the Hamiltonian \[33\]. Therefore the various collective motions are understood by taking the local symmetry into account. The local symmetry may have close connection with an infinite-dimensional (ID) Lie algebra. The TDHF leads to nonlinear dynamics owing to the SCF character. However there have not been enough attempts to manifestly understand the collective motions in relation to the local symmetry. Then from the viewpoint of the symmetry of the evolution equations in nonlinear problems, we should study an algebro-geometric structure toward unified understanding of both the motions. The first theme of our study is to investigate the curvature equation as a geometric equation to extract the collective sub-manifolds out of the TDHF manifold. We show that the zero-curvature equation in a particle-hole frame (PHF) leads to the nonlinear RPA theory which is the natural extension of the usual RPA. We denote the RPA/QRPA simply as RPA. We had started from a question whether soliton equations exist or not in the TDHF manifold, in spite of the difference that the former is described in terms of the infinite degrees of freedom and the latter uses finite ones. We had met with AKNS formulation in the Inverse Scattering Transform (IST) method \[34\] and the differential geometric approach on group manifold developed by Sattinger \[35\]. The essential point in the geometric viewpoint has attracted us:AKNS stands on group manifold \(SL(2)\) and then integrable system can be explained by the zero-curvature (integrability condition) of connection on the corresponding Lie group of the system.

In various approaches to the collective motions, those from the viewpoint of curvature were scarce. In studying the maximal decoupling method by Marumori and YK, we had an image for the large amplitude collective motions: If a collective sub-manifold is the collection of collective paths, the infinitesimal condition to switch from a path to another is nothing but an integrability condition for the sub-manifold with respect to time \(t\) describing the trajectory under the SCF Hamiltonian and the parameters specifying any point on the sub-manifold. However, the trajectory is unable to exactly remain on the manifold. The curvature is able to work as a criterion of the effectiveness of the collective sub-manifold. The RPA is considered as an integrability condition under a linear approximation so that the idea existing behind the theory can be contained in the idea of the curvature. From the wide viewpoint of the symmetry, the RPA must be extended to any point on the manifold because an equilibrium state which we select as a starting point must be equivalent to other points on the manifold among one another. The RPA was introduced as a linear approximation to treat excited states around the equilibrium ground state which is essentially a harmonic oscillator approximation. The amplitude of oscillation becomes larger and then an anharmonicity appears so that we treat the anharmonicity by taking into account the nonlinear terms in the equation of motion. We present a set of equations defining the curvature of collective sub-manifold which become a geometric equation to treat the anharmonicity called the formal RPA. It is useful to understand the algebro-geometric meanings of the large amplitude collective motions. The solution procedure by the perturbative method \[36\] suggests to study an ID Lie algebra working behind it.

To develop a perturbative method, we study the relation between the method extracting
collective motions in the SCF method and the \(\tau\)-functional method (\(\tau\)-FM) \[37\](a) constructing integrable equations in solitons, Dickey \[38\]. This is the second theme. The relation between the \(\tau\)-function and the coherent state has been pointed out first by D’Ariano and Rasetti \[39\] for an ID harmonic charged Fermi gas \[10\]. If we stand on the observation, we assert that the SCF method presents the theoretical scheme for integrable sub-dynamics on a certain ID fermion Fock space. Up to now, however, it has been insufficiently investigated the relation between the SCF method in the finite-dimensional (FD) fermion system and the \(\tau\)-FM in the ID fermion system because the description of dynamical fermion systems by them have looked very differently. We investigate the relation between the collective sub-manifold and various subgroup orbits in the SCF manifold. Using the TDHF theory on \(U(N)\) group, we study the relation between both the methods, according to an essential point of the picture in the conventional SCF method. To tackle this problem we have to solve the following problems: first, how to imbed the FD fermion system into the ID one and to rebuild the TDHF theory on it; second, to make sure that any algebraic mechanism working behind the particle and collective motions, and any relation between the collective variable and the spectral parameter in soliton theory remains present; last, to select from the SCF Hamiltonian various subgroup orbits and to generate a collective sub-manifold of them and how to relate the above to the formal RPA in the first theme.

We obtain a unified aspect for both the methods. The HF theory is made by a variational method to optimize the energy expectation value by S-det and to obtain a variational equation for orbitals in S-det \[11\]. The particle-hole pair operators of fermion with \(N\) single-particle states are closed under the Lie multiplication and forms the basis of Lie algebra \(u_N\) \[12\]. It generates the Thouless transformation \[43\] which induces a representation (rep) of the corresponding \(U(N)\) group. The \(U(N)\) canonical transformation changes the S-det with \(m\) particles into another S-det. Any S-det is obtained by such a transformation of a given reference S-det (Thouless theorem). The Thouless transform provides an exact generator coordinate (GC) rep of the fermion state vectors. The GC is a \(U(N)\) group and generating wave function (WF) is an independent-particle WF. This is the generalized coherent state rep (GCS rep) \[44\].

In soliton theory on a group, transformation group to cover the solution for the soliton equation is the ID Lie group whose infinitesimal generator of the corresponding Lie algebra is expressed as infinite-order differential operator of the associative affine Kac-Moody (KM) algebra. The operator is represented in terms of the ID fermion. A space of complex polynomial algebra is realized in terms of the ID fermion \(F\). The soliton equation becomes nothing else than the differential equation defining the group orbit of the highest weight vector in the ID Fock space \(F\). The generating WF, GCS rep is just the S-det (Thouless theorem) and provides a key to elucidate the interrelation of the HF WF to the \(\tau\)-function in soliton theory \[37\](a).

In abstract fermion Fock spaces, we find the common features in the SCF method and the \(\tau\)-FM: Each solution space is described as the ID Grassmannian (Gr) that is the group orbit of the corresponding vacuum state. The former implicitly explains the Plücker relation in terms of not bilinear differential equations defining finite-dimensional Gr but physical concept of quasi-particle and vacuum and mathematical language of coset space-variable. The various boson expansion methods are built on the Plücker relation to hold the Gr. The latter asserts that the soliton equations are nothing else the bilinear differential equations. This fact gives the boson rep of the Plücker relation. We study them and show both the methods stand on the common feature to be the Plücker relation or the bilinear differential equation defining the Gr.

Meanwhile, we observe the two different points between both the methods: (1) The former is built on the FD Lie algebra but the latter on the ID one. (2) The former has the SCF Hamiltonian consisting of the fermion one-body operator, derived from the functional derivative of expectation value of fermion Hamiltonian by ground-state WF but the latter introduces
artificially the one-body type fermion Hamiltonian as the boson mapping operator from the states on fermion Fock space to the corresponding ones on \( \tau \)-functional space (\( \tau \)-FS).

Overcoming the difference due to the dimension of fermion, we aim at having the close connection between the concept of mean-field (MF) potential and the gauge of fermion inherent in the SCF method and to making a role of the loop group \([45]\) clear. Through the observation, we make the ID fermion operators with finite-dimension by the Laurent expansion in terms of a circle \( S^1 \). Then with the use of affine KM algebra owing to the idea of Dirac’s electron-positron theory, we rebuilt a TDHF theory in \( F_\infty \). The TDHF result in a gauge theory of fermions and the collective motion (motion of MF potential) appears as the motion of fermion gauges with a common factor. The physical concept of particle-hole and vacuum in the SCF method on \( S^1 \) connects to the Plücker relations according to the idea of Dirac, saying, the algebraic mechanism extracting various sub-group orbits consisting of loop path out of the TDHF manifold is just the Hirota’s bilinear form \([46]\) which is \( su_N(\in sl_N) \) reduction to \( gl_N \) in the \( \tau \)-FM. For the \( sl_2 \) case, see Lepowsky-Wilson \([47]\). As a result, it is shown that the algebraic structure of ID fermion system is also realizable in the finite-dimension. In such a constructive way, the roles of the soliton equation (Plücker relation) and the TDHF equation are made clear and we also understand a non-dispersive property of the Gr and the SCF dynamics through the gauge of interacting ID fermions. Thus we have a simple unified aspect for both the methods.

We derive an algebraic mechanism arising the concept of the particle and collective motions and induce a close connection between the collective variable and the spectral parameter, but we clear the relation more explicitly. For the last problem, further the research must be made.

As the last theme, basing on the above viewpoint of the TDHF theory on circle \( S^1 \), we show the algebraic mechanism bringing both the motions. The mechanism can be elucidated from the following points: first, the \( su_N \)-condition for HF Hamiltonian; second, the vacuum state (highest weight vector) according to the idea of Dirac; and last, the phase of fermion gauge is separated into the particle mode and the collective one. We propose a new theory for unified description of both the motions, beyond the static HF equation and the RPA equation in the usual manner. This theory simply and clearly elucidates not only the collective motion as that of the MF potential but also the symmetry breaking and the occurrence of the collective motion due to the recovery of the symmetry.

Tajiri has suggested significant problems to be inquired about why soliton solution for classical wave equation shows fermion-like behavior in quantum dynamics and about what symmetry is hidden in soliton equation \([48]\). This is a very interesting problem. As we start from a quasi-classical dynamics in the TDHF and do not notice the fermion systems behind them, we inquire why their solution space is Grassmannian. However, we have not been concerned with this problem yet but gave a few observations to future problems. The anti-commutation relations of fermions are regarded as quantal orthogonal conditions among the canonical coordinate variables described by the Grassmann number \([32]\). The tangent space has no norm as like as that of bosons, since the Grassmann number has only anti-commutative property but no measure as a classical number. On the contrary, \( SO(2N+1) \) theory \([49, 50, 9]\) describes all degrees of freedom with respect to pair and unpaired fermions with the use of a classical number. Then we inquire how both systems of boson and fermion to relate with each other from algebro-geometric viewpoint. We, further, inquire whether the above questions are concerned with the fermion-like and boson-like behaviors of solitons suggested by Tajiri-Watanabe \([48]\). In which the idea of nonlinear superposition principle so called imbricate series \([51]\) is similar to that of GC method in the SCF method. The SCF method and soliton theory have some similar features. However, the relation between them has been insufficiently investigated yet. The explicit approaches from the viewpoint of local symmetry are not sufficient in the SCF method.
3 Group theoretical preliminaries

Let \( c_\alpha \) and \( c_\alpha^\dagger \) (\( \alpha = 1, \cdots, N \)) be the annihilation-creation operators of the fermion. Owing to the anti-commutation relations among the fermion operators, the fermion operators \( E_{\alpha\beta} = c_\alpha^\dagger c_\beta \) forming the \( U(N) \) Lie algebra \( \{ E_{\alpha\beta}, E_{\gamma\delta} \} = \delta_{\gamma\beta} E_{\alpha\delta} - \delta_{\alpha\delta} E_{\gamma\beta} \) generates a canonical transformation \( U(G) \) (Thouless transformation \[43\]) specified by a matrix \( g \) belonging to a \( U(N) \) unitary group. We decompose the creation operator \( [c_\alpha^\dagger, c_\beta] = [\hat{c}_\alpha, \hat{c}_\beta] \) and the annihilation operator \( [c_\alpha, c_\beta^\dagger] = [\hat{a}_\alpha, \hat{b}_\beta] \). Then, the canonical transformation is expressed as

\[
\begin{align*}
[d, \bar{d}] &= U(g)[\hat{c}, \hat{c}^\dagger]U^{-1}(g) = \bar{c}^\dagger \bar{c}, \\
\hat{g}g^\dagger &= g^\dagger g = 1_N,
\end{align*}
\]

where \( \bar{c} \equiv (\bar{a}, \bar{b}) = (\bar{a}_\alpha, \bar{b}_\beta) \) and \( \bar{d} \equiv (\bar{d}_a, \bar{d}_b) = (\bar{d}_a, \bar{d}_b) \) are row vectors and \( \hat{g} = (\hat{a}_{ab}) \) is the HF density matrix given in detail in (A.14). The HF energy functional is defined in (A.15) and the Fock operator \( H_{\text{HF}} \) is given as

\[
E_{\text{HF}} = \frac{1}{2} \sum_{\alpha\beta} \gamma_{\alpha\beta} c_\alpha^\dagger c_\beta,
\]

where \( Q = \frac{1}{2} \sum_{\alpha\beta} \gamma_{\alpha\beta} Q_{\alpha\beta} \) and \( F_{\alpha\beta} = \sum_{\gamma\delta} \gamma_{\gamma\delta} F_{\alpha\beta} \). The set \( (F) = (F_{\alpha\beta}) \) is a particle-hole vacuum expectation value of the Lie operator, expressed, using the dummy index convention to take summation over the repeated index, as

\[
\begin{align*}
F_{\alpha\beta} &= \langle \bar{a}_\alpha | \bar{d}_\beta \rangle, \\
F_{\alpha\beta} &= \langle \bar{a}_\alpha | \bar{d}_\beta \rangle, \\
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F_{\alpha\beta} &= \langle \bar{a}_\alpha | \bar{d}_\beta \rangle,
\end{align*}
\]
4 Integrability condition and collective sub-manifold

We consider an evolution equation \( \partial_t u = K(u) \) for \( u(x,t) \). \( K(u) \) is an operator acting on a function of \( u \) dependent on \( x \) and is expressed as a polynomial of differential to \( u \) with respect to \( x \). Regarding \( \partial_t u = K(u) \) as an equation to give an infinitesimal transformation of function \( u \), we search for a symmetry included in the evolution equation. We introduce another evolution equation, \( \partial_s u(x,t,s) = \hat{K}(u(x,t,s)) \) for which we should search. \( \hat{K} \) also consists of the polynomial operator of differential. This does not necessarily mean \( [K, \hat{K}] = 0 \) to seek for the required symmetry of the present subject, e.g., rotational symmetry. The infinitesimal condition for the existence of the symmetry appears as the well-known integrability condition \( \partial_s K(u(x,t,s)) = \partial_t \hat{K}(u(x,t,s)) \) \( \text{[37](b)} \).

The \textit{maximal decoupling} method by Marumori based on the invariance principle of Schrödinger equation and the canonicity condition \( \text{[23]} \) are considered to adopt the integrability condition into parametrized symmetries. This method is seen as a description of the symmetries of the collective submanifold with respect to time and collective variables, in which the canonicity condition makes the collective variables roles of the orthogonal coordinate of a system.

In differential geometrical approach to nonlinear problem, integrability condition is regarded as zero curvature of connection on the corresponding Lie group of system. The nonlinear evolution equations, famous KdV and sine/sinh-Gordan equations etc., come from the well-known Lax equation \( \text{[54]} \) arisen as zero curvature \( \text{[35]} \). These soliton equations describe motions of tangent space of local gauge fields dependent on time \( t \) and space \( x \): equations of Lie-valued-function arising from the integrability condition of the gauge field with respect to \( t \) and \( x \). We also get a two-dimensional soliton solution \( \text{[55]} \), i.e., \textit{dromion} of the Davey-Stewartson equation \( \text{[56, 57, 58, 59]} \). On the other hand, in the SCF method, the corresponding Lie groups are the unitary transformation groups of their orthonormal bases dependent on \( t \) but not on \( x \). Although at this point the construction of means for both the dynamical systems are different from each other, the RPA theory also describes a motion of \textit{tangent space} on the group manifold.

We aim at concept of curvature unfamiliar in the nuclear physics. The reason why is the following: let us consider a description of motions of systems on a group manifold. An arbitrary state of the system induced by a transitive group action corresponds to any one point in the full group parameter space and its time evolution is given by an integral curve. In the whole rep space adopted, assume the existence of \( 2\mu \) parameters specifying a proper subspace in which the original motion of the system is well defined; saying, the existence of certain symmetry. Suppose we start from a given point on a space consisting of time and \( 2\mu \) parameters, and end at the same point again along a closed curve. Then we have in general value of the group parameter different from the one at an initial point on the subspace. We search for some quantity characterizing such a difference. To achieve our aim, standing on the differential geometrical viewpoint, we introduce a sort of Lagrange manner familiar to fluid dynamics to describe collective coordinate systems and take a one-form \( \Omega \) which is linearly composed of the MF Hamiltonian and infinitesimal generators induced by collective variable differentials of certain canonical transformation. The integrability conditions of our system read the zero curvature. We study curvature equations of the TDHF equation to determine collective sub-manifolds from group theoretical viewpoint. The non-zero curvature with respect to time and collective variable are shown to be gradient of the expectation value of the residual Hamiltonian along the direction of the collective coordinate. The set of expectation value of the zero-curvature equation for vacuum state function is shown to be nothing but the set of equations of motion, imposing restrictions of \textit{weak} canonical commutation relations. We study the relation between Marumori’s ≪maximal decoupled≫ theory and YK and ours. We discuss the role of the non zero-curvature equation not appearing in the former theory. We investigate the nonlinear time evolution equation arising from the
zero-curvature equation. We find that the expression for equation in a quasi PHF is nothing but the formal RPA equation and show it to be regarded as the nonlinear RPA theory. We give a solution procedure by the power expansion with respect to the collective variables. Our methods are constructed manifesting itself the structure of group under consideration in order to make easy to understand physical characters at any point on the group manifold.

4.1 Lagrange-like manner

The $U(N)$ WF $|\phi(g)\rangle$ is constructed by a transitive action of $U(N)$ canonical transformation $U^{-1}(g)$ on $|0\rangle$; $|\phi(g)\rangle=U^{-1}(g)|0\rangle, g\in U(N)$. In the conventional TDHF, TD WF $|\phi(\tilde{g})\rangle$ is given through the group $U(N)$ parameters $\hat{a}, \hat{b}, \hat{b}$ and $\hat{a}$. They characterize the TD SC mean HF field $F$ whose dynamical change induces the collective motions of many fermion systems. As in the TDHF by Marumori et al. [23] and YK [32], we introduce the following TD $U(N)$ canonical transformation (TR) to derive collective motions within the TDHF framework:

$$U(\tilde{g})=U[\tilde{g}(\hat{\Lambda}(t), \hat{\Lambda}^*(t))], \quad \tilde{g}(\hat{\Lambda}(t), \hat{\Lambda}^*(t)) = \begin{bmatrix} \hat{a}(\hat{\Lambda}(t), \hat{\Lambda}^*(t)) & \hat{b}(\hat{\Lambda}(t), \hat{\Lambda}^*(t)) \\ \hat{b}(\hat{\Lambda}(t), \hat{\Lambda}^*(t)) & \hat{a}(\hat{\Lambda}(t), \hat{\Lambda}^*(t)) \end{bmatrix},$$

(4.1)

where a set of TD complex functions $(\hat{\Lambda}(t), \hat{\Lambda}^*(t)) = (\hat{\Lambda}_\nu(t), \hat{\Lambda}_\nu^*(t)) ; \nu = 1, \cdots, \mu$ associated with the collective motions specifies the group parameters $\tilde{g}$. The number $\mu$ is assumed to be much smaller than the degree of the $U(N)$ Lie algebra, which means there exist only a few collective degrees of freedom. The $U(\tilde{g})$ is a natural extension of the method in the simple TDHF case to the TDHF on the 2$\mu$-dimensional (2$\mu$D) collective sub-manifold [60]. For our aim, taking a Lagrange-like manner, it is convenient to introduce complex parameters $\{\Lambda, \Lambda^*\}$ which are regarded as local coordinates to specify any point of 2$\mu$D collective sub-manifold. Instead of $(\hat{\Lambda}(t), \hat{\Lambda}^*(t))$, we use the functions of $\{\Lambda, \Lambda^*\}$ and $t$,

$$\tilde{\Lambda}_\nu(t) = \hat{\Lambda}_\nu(\Lambda, \Lambda^*; t), \quad \tilde{\Lambda}_\nu^*(t) = \hat{\Lambda}_\nu^*(\Lambda, \Lambda^*; t).$$

(4.2)

This means that the set $\{\Lambda, \Lambda^*\} \in \mathbb{C}^3$ is mapped into the set $\{\Lambda, \Lambda^*\} \in \mathbb{C}^2$, i.e., $\{\Lambda, \Lambda^*\} \to \{\tilde{\Lambda}, \tilde{\Lambda}^*\}$ through the functions $\{\tilde{\Lambda}, \tilde{\Lambda}^*\}$. This means that one is considering at $t$ a coordinate set $S_t$ that depends on $t$ and is labeled by the set of pairs $\{\Lambda, \Lambda^*\}$. At $t'$ the set $S_{t'}$ is still labeled by the same set of pairs $\{\Lambda, \Lambda^*\}$. However, the state which is labeled by the pair $(\Lambda, \Lambda^*)$ at $t$ is different from the state which is labeled by the same pair $(\Lambda, \Lambda^*)$ at $t'$. This manner seems very analogous to the one founded by Lagrange in the fluid dynamics. An invariant subspace labeled by the parameters $\{\Lambda, \Lambda^*\}$ is being assumed. The collective subspace becomes defined only when such a TR as the one which has been formulated is considered, but this is not enough. Now comes the definition of collective subspace. The invariant subspace is the collective subspace if the evolution of the physical system determined by the TDHF theory is such that its coordinates in $S_t$, namely $(\Lambda, \Lambda^*)$, do not change with $t$. Using (4.2), $U(N)$ canonical TR is rewritten as $U(\tilde{g})=U[g(\Lambda, \Lambda^*, t)] \in U(N)$. Notice that the functional form $\tilde{g}(\hat{\Lambda}(t), \hat{\Lambda}^*(t))$ in (4.1) changes into another functional form $g(\Lambda, \Lambda^*, t)$ due to the Lagrange-like manner. This enables us to take a one-form $\Omega$, composed of the infinitesimal generators induced by time and collective variable differentials $(\partial_t, \partial_\Lambda, \partial_{\Lambda^*})$ of canonical TR $U[g(\Lambda, \Lambda^*, t)]$. Introducing the one-form $\Omega$, we search for collective path and collective Hamiltonian separated from other remaining degrees of freedom.

4.2 Integrability conditions

Following YK [32], we define Lie-algebra-valued infinitesimal generators for collective Hamiltonian $H_c$ and collective coordinates $(O^c_\nu, O^c_\nu) (\nu = 1, \cdots, \mu)$ in collective sub-manifolds as follows:
\[ H_c \stackrel{d}{=} (i\hbar \partial_t U(g))U^{-1}(g), \quad \text{(4.3)} \]
\[ O^\Lambda_{\nu} \stackrel{d}{=} (i\partial_{\Lambda^\nu} U(g))U^{-1}(g), \quad O_\nu \stackrel{d}{=} (i\partial_{\Lambda^\nu} U(g))U^{-1}(g), \quad \text{(4.4)} \]

We abbreviate \( g(\Lambda, \Lambda^\nu, t) \) simply as \( g \). In the TDHF theory, the Lie-algebra-valued infinitesimal generators are expressed by the trace form as

\[ \text{generators are expressed by the trace form as} \]
\[ \Omega \text{ linearly composed of the infinitesimal generators (4.3) and (4.4):} \]
\[ \text{well known as integrability conditions is useful. For this aim, we take the following one-form} \]
\[ \text{where} \]
\[ \text{with the aid of the one-form} \Omega, \text{the integrability conditions of the system read} \]
\[ C \stackrel{d}{=} d\Omega - \Omega \wedge \Omega = 0, \quad \text{(4.9)} \]

The direct derivation of (4.6) using (4.3) and (4.4) is given in detail in Appendix A. We, however, for the moment, remove the factor \( e^\Omega \) in (4.5) and (4.6). Multiplying \( \phi(g) \) on the both sides of (4.3) and (4.4), we get a set of equations on the \( U(N) \) Lie algebra:

\[ D_t|\phi(g)| \stackrel{d}{=} (i\hbar \partial_t - H_c)|\phi(g)| = 0, \]
\[ D_{\Lambda^\nu}|\phi(g)| \stackrel{d}{=} (\partial_{\Lambda^\nu} + iO^\Lambda_{\nu})|\phi(g)| = 0, \quad \text{We regard these equations (4.7) as partial differential equations for} |\phi(g)|. \]

In order to discuss the conditions for that the differential equation (4.7) can be solved, the mathematical method well known as integrability conditions is useful. For this aim, we take the following one-form \( \Omega \) linearly composed of the infinitesimal generators (4.3) and (4.4):

\[ \Omega = H_c \cdot dt + \hbar O^\Lambda_{\nu} \cdot d\Lambda^\nu + \hbar O_\nu \cdot d\Lambda^\nu_{\nu}. \quad \text{(4.8)} \]

With the aid of the one-form \( \Omega \), the integrability conditions of the system read

\[ C \stackrel{d}{=} d\Omega - \Omega \wedge \Omega = 0, \quad \text{(4.9)} \]

where \( d \) and \( \wedge \) denote the exterior differentiation and the exterior product, respectively. From the differential geometrical viewpoint, the quantity \( C \) defined in the above means the curvature of a connection. Then the integrability conditions is interpreted as the vanishing of the curvature of the connection \( (D_t, D_{\Lambda^\nu}, D_{\Lambda^\nu}^\nu) \). The detailed structure of the curvature is calculated to be

\[ C = C_{t, \Lambda^\nu} d\Lambda^\nu \wedge dt + C_{t, \Lambda^\nu} d\Lambda^\nu_{\nu} \wedge dt + C_{\Lambda^\nu, \Lambda^\nu_{\nu}} d\Lambda^\nu_{\nu} \wedge d\Lambda^\nu_{\nu} + \frac{1}{2} C_{\Lambda^\nu, \Lambda^\nu_{\nu}} d\Lambda^\nu_{\nu} \wedge d\Lambda^\nu_{\nu}, \]

where

\[ C_{t, \Lambda^\nu} \stackrel{d}{=} [D_t, D_{\Lambda^\nu}] = i\hbar \partial_t O^\nu_{\nu} - i\partial_{\Lambda^\nu} H_c + [O^\nu_{\nu}, H_c], \]
\[ C_{t, \Lambda^\nu_{\nu}} \stackrel{d}{=} [D_t, D_{\Lambda^\nu_{\nu}}] = i\hbar \partial_t O_{\nu_{\nu}} - i\partial_{\Lambda^\nu} H_c + [O_{\nu_{\nu}}, H_c], \]
\[ C_{\Lambda^\nu, \Lambda^\nu_{\nu}} \stackrel{d}{=} [D_{\Lambda^\nu}, D_{\Lambda^\nu_{\nu}}] = i\partial_{\Lambda^\nu} O_{\nu_{\nu}} - i\partial_{\Lambda^\nu} O^\nu_{\nu} + [O_{\nu_{\nu}}, O^\nu_{\nu}], \]
\[ C_{\Lambda^\nu_{\nu}, \Lambda^\nu} \stackrel{d}{=} [D_{\Lambda^\nu_{\nu}}, D_{\Lambda^\nu}] = i\partial_{\Lambda^\nu_{\nu}} O_{\nu_{\nu}} - i\partial_{\Lambda^\nu_{\nu}} O^\nu_{\nu} + [O_{\nu_{\nu}}, O^\nu_{\nu}], \quad \text{(4.10)} \]

The vanishing of curvature \( C \) means \( C_{\cdots \nu} = 0 \). For basic study of differential geometry, see the famous textbooks [61] and [62].
Finally using the expressions of \((4.5)\) and \((4.6)\), we get the following set of Lie-algebra-valued equations as the integrability conditions of the partial differential equations \((4.7)\).

\[
C_{t,\Lambda_{\nu}} = [\hat{c}^\dagger, \hat{c}]C_{t,\Lambda_{\nu}} \left[ \frac{\partial}{\partial c} \right], \quad C_{t,\Lambda'_{\nu}} = [\hat{c}^\dagger, \hat{c}]C_{t,\Lambda'_{\nu}} \left[ \frac{\partial}{\partial c} \right],
\]

\[
C_{\Lambda_{\nu},\Lambda'_{\nu}} = [\hat{c}^\dagger, \hat{c}]C_{\Lambda_{\nu},\Lambda'_{\nu}} \left[ \frac{\partial}{\partial c} \right], \quad C_{\Lambda_{\nu},\Lambda'_{\nu}} = [\hat{c}^\dagger, \hat{c}]C_{\Lambda_{\nu},\Lambda'_{\nu}} \left[ \frac{\partial}{\partial c} \right], \quad C_{\Lambda'_{\nu},\Lambda'_{\nu}} = [\hat{c}^\dagger, \hat{c}]C_{\Lambda'_{\nu},\Lambda'_{\nu}} \left[ \frac{\partial}{\partial c} \right],
\]

where

\[
C_{t,\Lambda_{\nu}} = i\hbar\partial_t \theta_{\nu} - i\partial\Lambda_{\nu} \mathcal{F}_c + [\theta_{\nu}, \mathcal{F}_c], \quad C_{t,\Lambda'_{\nu}} = i\hbar\partial_t \theta_{\nu} - i\partial\Lambda'_{\nu} \mathcal{F}_c + [\theta_{\nu}, \mathcal{F}_c],
\]

\[
C_{\Lambda_{\nu},\Lambda'_{\nu}} = i\partial\Lambda_{\nu} \theta_{\nu} - i\partial\Lambda'_{\nu} \theta_{\nu}' + [\theta_{\nu}, \theta'_{\nu}], \quad C_{\Lambda'_{\nu},\Lambda'_{\nu}} = i\partial\Lambda'_{\nu} \theta_{\nu} - i\partial\Lambda'_{\nu} \theta_{\nu}' + [\theta_{\nu}, \theta'_{\nu}].
\]

The quantities \(\mathcal{F}_c, \theta_{\nu}\) and \(\theta_{\nu}'\) are defined through partial differential equations,

\[
i\hbar\partial_t g = \mathcal{F}_c g,
\]

\[
i\partial\Lambda_{\nu} g = \theta_1^\dagger g, \quad i\partial\Lambda'_{\nu} g = \theta_\nu g.
\]

The quantities \(C_{\bullet,\bullet}\) are naturally regarded as the curvature of the connection on the group manifold. The reason becomes clear if we take the following procedure quite parallel with the above one: Starting from \((4.13)\) and \((4.14)\), we are led to a set of partial differential equations on the \(U(N)\) Lie group \(g\) as

\[
D_t g \overset{d}{=} (i\hbar\partial_t - \mathcal{F}_c) g = 0, \quad D_{\Lambda_{\nu}} g \overset{d}{=} (\partial_{\Lambda_{\nu}} + i\partial_\nu) g = 0, \quad D_{\Lambda'_{\nu}} g \overset{d}{=} (\partial_{\Lambda'_{\nu}} + i\partial_{\nu}') g = 0.
\]

The curvature \(C_{\bullet,\bullet} \left[ \overset{d}{=} [D_{\bullet}, D_{\bullet}] \right]\) of the connection \((D_t, D_{\Lambda_{\nu}}, D_{\Lambda'_{\nu}})\) is shown to be equivalent to the quantity \(C_{\bullet,\bullet}\) in \((4.12)\). The above set of the Lie-algebra-valued equations \((4.11)\) evidently leads us to putting all the curvatures \(C_{\bullet,\bullet}\) equal to be zero.

On the other hand, TDHF Hamiltonian \((3.3)\), i.e., Hamiltonian on \(U(N)\) WF space, is represented in the same form as \((4.3)\)

\[
H_{\text{HF}} = (i\partial U(g')) U^{-1}(g'),
\]

where \(g'\) is any point on the \(U(N)\) group manifold. The RHS of \((4.15)\) is transformed into the same form as \((4.3)\). This fact leads us to the well-known TDHF equation,

\[
i\hbar\partial_t g' = \mathcal{F} g'.
\]

The TDHF Hamiltonian is decomposed into two components at a reference point \(g' = g\) on the group manifold as

\[
H_{\text{HF}} |_{U(g)=U(g')} = H_c + H_{\text{res}}, \quad \mathcal{F} |_{g'=g} = \mathcal{F}_c + \mathcal{F}_{\text{res}},
\]

where the second parts \(H_{\text{res}}\) and \(\mathcal{F}_{\text{res}}\) mean residual components extracted out of a well-defined collective sub-manifold for which we must search now. For our aim, let us introduce another curvatures \(C'_{t,\Lambda_{\nu}}\) and \(C'_{\Lambda_{\nu},\Lambda'_{\nu}}\) with the same forms as those in \((4.11)\) and \((4.12)\), in which instead of \(\mathcal{F}_c\), it is replaced by \(\mathcal{F} |_{g'=g} (= \mathcal{F}_c + \mathcal{F}_{\text{res}})\). The vacuum expectation values of another Lie-algebra-valued curvatures are easily calculated as

\[
\langle C_{t,\Lambda_{\nu}} \rangle = \langle C_{t,\Lambda'_{\nu}} \rangle = 0, \quad \langle C'_{t,\Lambda_{\nu}} \rangle = -i\partial\Lambda \langle H_{\text{res}} \rangle g, \quad \langle C'_{\Lambda_{\nu},\Lambda'_{\nu}} \rangle = -i\partial\Lambda' \langle H_{\text{res}} \rangle g,
\]

\[
\langle H_{\text{res}} \rangle = -\text{Tr} \left[ g \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \\ 0 & -1 \\ \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & -1 \end{array} \right] \right], \quad \langle \mathcal{F} g' - i\partial\Lambda g' g' \rangle,
\]

The first Eq. of \((4.19)\) is derived through the relations \(i\hbar\partial_\nu = i\partial\Lambda_{\nu} \mathcal{F}_c\) and \(i\hbar\partial'_{\nu} = i\partial\Lambda'_{\nu} \mathcal{F}_c\) which are obtained by \((4.13)\) and \((4.14)\). Eqs. \((4.19)\) and \((4.20)\) mean that values of \(C'_{t,\Lambda_{\nu}}\) and \(C'_{\Lambda_{\nu},\Lambda'_{\nu}}\) are the gradient of energy of the residual Hamiltonian in 2mD manifold. Suppose the existence of
a well-defined collective sub-manifold. Then it is not so wrong to deduce the following result: The energy of the residual Hamiltonian has almost constant value on the collective sub-manifold,

\[ \delta_g \langle H_{\text{res}} \rangle_g \approx 0, \quad \partial_{\Lambda^c} \langle H_{\text{res}} \rangle_g \approx 0 \quad \text{and} \quad \partial_{\Lambda^*} \langle H_{\text{res}} \rangle_g \approx 0, \]  

(4.21)

where \( \delta_g \) denotes the \( g \)-variation, regarding \( g \) as functions of \((\Lambda, \Lambda^*)\) and \( t \). It may be achieved if we should determine \( g \) (collective path) and \( F_c \) (collective Hamiltonian) through the auxiliary quantity \((\theta, \theta^t)\) so as to satisfy \( H_c + \text{const.} = H_{\text{HF}} \) as far as possible. Then putting \( F_c = F \) in (4.12), we seek for \( g \) and \( F_c \) satisfying

\[ C_t, \Lambda_g \approx 0 \quad \text{and} \quad C_t, \Lambda^*_g \approx 0, \quad C_{\Lambda^c, \Lambda_g} = 0 \quad \text{and} \quad C_{\Lambda^c, \Lambda^*_g} = 0. \]  

(4.22)

The set of the equations \( C_{\bullet, \bullet} = 0 \) makes an essential role to determine the collective sub-manifold in the TDHF method. The set of the Eqs. (4.22) and (4.15) becomes our geometric equation for describing the collective motions, under the restrictions given later, i.e., (4.32).

### 4.3 Correspondence of Lagrange-like manner to the usual one

If we hope to describe collective motions through TD and complex variables \((\tilde{\Lambda}(t), \Lambda^*(t))\), we must know explicit forms of \( \tilde{\Lambda} \) and \( \Lambda^* \) in (4.2) in terms of \((\Lambda, \Lambda^*)\) and \( t \). For this aim, it is necessary to discuss the correspondence of the Lagrange-like manner to the usual one.

We define the Lie-algebra-valued infinitesimal generators of collective sub-manifolds as

\[
\begin{align*}
\dot{O}_\nu^1 & = (i\partial_{\Lambda^c} U(\hat{g})) U^{-1}(\hat{g}), \\
\dot{O}_\nu & = (i\partial_{\Lambda^c} U(\hat{g})) U^{-1}(\hat{g}),
\end{align*}
\]

(4.23)

whose form is the same as the one in (4.4). In order to guarantee the variables \( \tilde{\Lambda}_\nu = (\tilde{\Lambda}_\nu(t)) \) and \( \Lambda^*_\nu(t) \) to be canonical, following Marumori [23] and YK [32], we set up the following expectation values with use of the \( U(N) \) WF \( |\phi(\hat{g})\rangle\):

\[
\begin{align*}
\langle \phi(\hat{g}) | i\partial_{\Lambda^c} | \phi(\hat{g})\rangle & = \langle \phi(\hat{g}) | \dot{O}_\nu^1 | \phi(\hat{g})\rangle = i\Lambda^*_\nu, \\
\langle \phi(\hat{g}) | i\partial_{\Lambda^c} | \phi(\hat{g})\rangle & = \langle \phi(\hat{g}) | \dot{O}_\nu | \phi(\hat{g})\rangle = -i\Lambda_\nu.
\end{align*}
\]

(4.24)

The above relation leads us to the week canonical commutation relation

\[
\begin{align*}
\langle \phi(\hat{g}) | [\dot{O}_\nu^1, \dot{O}_\nu] | \phi(\hat{g})\rangle & = \delta_\nu\nu', \\
\langle \phi(\hat{g}) | [\dot{O}_\nu^1, \dot{O}_\nu] | \phi(\hat{g})\rangle & = 0, \\
\langle \phi(\hat{g}) | [\dot{O}_\nu, \dot{O}_\nu^1] | \phi(\hat{g})\rangle & = 0,
\end{align*}
\]

(4.25)

proof of which is, shown by Marumori and YK [23, 32], due to the integrability conditions.

Using (4.7), the collective Hamiltonian \( H_c \) and the infinitesimal generators \( O_\nu^1 \) and \( O_\nu \) in the Lagrange-like manner are expressed by \( \dot{O}_\nu^1 \) and \( \dot{O}_\nu \) in the usual manner as follows:

\[
\begin{align*}
H_c & = \hbar \partial_t \tilde{\Lambda} \dot{\tilde{\Lambda}}^* + \hbar \partial_t \Lambda \dot{\Lambda}^*, \\
O_\nu^1 & = \partial_{\Lambda^c} \tilde{\Lambda}_\nu^* \dot{\tilde{\Lambda}}_\nu + \partial_{\Lambda^c} \Lambda^*_\nu \dot{\Lambda}_\nu, \\
O_\nu & = \partial_{\Lambda^c} \Lambda_\nu \dot{\Lambda}^*_\nu + \partial_{\Lambda^c} \Lambda^*_\nu \dot{\Lambda}_\nu.
\end{align*}
\]

(4.26)

Substituting (4.26) into (4.10), it is carried out to evaluate the expectation value of Lie-algebra-valued curvature \( C_{\bullet, \bullet} \) by the \( U(N) \) WF \( |\phi(\tilde{\Lambda}(t), \Lambda^*(t))\rangle = |\phi(\Lambda, \Lambda^*, t)\rangle \). A weak integrability condition requiring the expectation value \( \langle \phi(\hat{g}) | C_{\bullet, \bullet} | \phi(\hat{g})\rangle = 0 \) yields partial differential equations with aid of the quasi particle-hole vacuum property, \( d|\phi(g)\rangle = 0 \):

\[
\begin{align*}
\partial_{\Lambda^c} \Lambda_\nu \partial_t \Lambda^*_\nu - \partial_{\Lambda^c} \Lambda^*_\nu \partial_t \Lambda_\nu & = -\text{Tr} \{ \mathcal{Q}(g) [\theta^1_\nu, F_c / \hbar] \}, \\
\partial_{\Lambda^c} \Lambda^*_\nu \partial_t \Lambda^*_\nu - \partial_{\Lambda^c} \Lambda^*_\nu \partial_t \Lambda^*_\nu & = -\text{Tr} \{ \mathcal{Q}(g) [\theta^1_\nu, F_c / \hbar] \}.
\end{align*}
\]

(4.27)
\[
\begin{align*}
\partial_{\lambda'}\tilde{A}_{\nu'}\partial_{\lambda'}\tilde{A}_{\nu'} - \partial_{\lambda'}\tilde{A}_{\nu'}\partial_{\lambda'}\tilde{A}_{\nu'} &= \text{Tr}\{\mathcal{Q}(g)[\theta_{\nu'}, \theta_{\nu'}]\}, \\
\partial_{\lambda_{\nu'}}\tilde{A}_{\mu'}\partial_{\lambda_{\nu'}}\tilde{A}_{\mu'} - \partial_{\lambda_{\nu'}}\tilde{A}_{\mu'}\partial_{\lambda_{\nu'}}\tilde{A}_{\mu'} &= \text{Tr}\{\mathcal{Q}(g)[\theta_{\nu'}, \theta_{\nu'}]\}, \\
\partial_{\lambda'}\tilde{A}_{\nu'}\partial_{\lambda_{\nu'}}\tilde{A}_{\mu'} - \partial_{\lambda_{\nu'}}\tilde{A}_{\nu'}\partial_{\lambda'}\tilde{A}_{\mu'} &= \text{Tr}\{\mathcal{Q}(g)[\theta_{\nu'}, \theta_{\nu'}]\},
\end{align*}
\]

(4.28)

where another \(U(N)\) HF density matrix \(\mathcal{Q}(g)\) is defined as

\[
\mathcal{Q}(g) = g \begin{bmatrix} I_m & 0 \\ 0 & -1^{-N-m} \end{bmatrix} g^\dagger \equiv \begin{bmatrix} \hat{\mathcal{Q}} & \hat{\mathcal{Q}} \end{bmatrix}, \quad \hat{\mathcal{Q}}^1 = \mathcal{Q}, \quad \hat{\mathcal{Q}}^2 = 1_N, \quad (4.29)
\]

using the expression for \(g\), \(\hat{\Lambda}, \hat{\Lambda}^\dagger\), the density matrix \(\mathcal{Q}\) is explicitly expressed as

\[
\mathcal{Q} = \begin{bmatrix} C(\xi)e^\nu - S(\xi)e^{-\nu} \end{bmatrix} \begin{bmatrix} 1_m & 0 \\ 0 & -1^{-N-m} \end{bmatrix} \begin{bmatrix} e^{-\nu}C(\xi) & e^{-\nu}S(\xi) \\ e^{-\nu}S(\xi) & e^{-\nu}C(\xi) \end{bmatrix} = 2Q-1_N \equiv \begin{bmatrix} 2\hat{Q} & 2\hat{Q} \\ 2\hat{Q} & 2\hat{Q} \end{bmatrix}. \quad (4.30)
\]

The parameters involved in \(g\) are functions of the complex variables \((\Lambda, \Lambda^\dagger)\) and \(t\). In the derivation of \((4.27)\) and \((4.28)\), we have used the transformation property \((4.5)\) and \((4.6)\) and the differential formulae are expressed as

\[
\langle \phi | i\partial_{\lambda_{\nu'}}\hat{Q}_\mu | \phi \rangle = -\delta_{\nu'\nu}, \quad \langle \phi | i\partial_{\lambda_{\nu'}}\hat{Q}_{\mu | \nu} | \phi \rangle = \delta_{\nu'\nu}, \quad \langle \phi | i\partial_{\lambda_{\nu'}}\hat{Q}_{\mu | \nu} | \phi \rangle = 0, \quad \langle \phi | i\partial_{\lambda_{\nu'}}\hat{Q}_{\nu | \mu} | \phi \rangle = 0. \quad (4.31)
\]

This is the consequence of canonicity condition and \textit{weak} canonical commutation relation.

Through the above procedure, finally, we get the correspondence of Lagrange-like manner to the usual one. We have no unknown quantities in the RHS of \((4.27)\) and \((4.28)\), if we could completely solve the geometric equations. Then we become able to know in principle the explicit forms of the functions \(\hat{\Lambda}\) and \(\hat{\Lambda}^\dagger\) in terms of \((\Lambda, \Lambda^\dagger)\) and \(t\) by solving the set of partial differential equations \((4.27)\) and \((4.28)\). However we should take enough notice of the roles different from each other made by \((4.27)\) and \((4.28)\), respectively, to construct the solutions. It turns out that the LHS in \((4.28)\) has a close connection with Lagrange bracket often appeared in analytical dynamics. Since we have set up from the outset the canonicity conditions to guarantee the complex variables \((\hat{\Lambda}, \hat{\Lambda}^\dagger)\) in the usual manner being canonical, the functions \(\hat{\Lambda}\) and \(\hat{\Lambda}^\dagger\) in \((4.2)\) can be interpreted as functions giving a canonical transformation from \((\hat{\Lambda}, \hat{\Lambda}^\dagger)\) to another complex variables \((\Lambda, \Lambda^\dagger)\) in the Lagrange-like manner. From this interpretation, we see that requirement of the canonical invariance imposes the following restrictions on the RHS of \((4.28)\):

\[
\text{Tr}\{\mathcal{Q}(g)[\theta_{\nu'}, \theta_{\nu'}]\} = \delta_{\nu'\nu}, \quad \text{Tr}\{\mathcal{Q}(g)[\theta_{\nu'}, \theta_{\nu'}]\} = 0, \quad \text{Tr}\{\mathcal{Q}(g)[\theta_{\nu'}, \theta_{\nu'}]\} = 0. \quad (4.32)
\]

It is quite self-evident that, combining \((4.28)\) with \((4.32)\), we get the Lagrange bracket for the canonical transformation from \((\hat{\Lambda}, \hat{\Lambda}^\dagger)\) to \((\Lambda, \Lambda^\dagger)\). From the correspondence arguments, it is reasonable to add the restrictions \((4.32)\) to our geometric equations in order to describe the collective motion in terms of the canonical coordinate variables.

We have studied the integrability conditions of the TDHF equation to determine the collective sub-manifolds from the group theoretical viewpoint. Our idea lies in the adoption of the Lagrange-like manner to describe the collective coordinates. It should be noted that the variables are nothing but the parameters to describe the symmetry of the TDHF equation. Introducing the one-form, we gave the integrability conditions, the vanishing of the curvatures of the connection, expressed as the Lie-algebra-valued equations. The TDHF Hamiltonian \(H_{\text{HF}}\) is decomposed into a collective Hamiltonian \(H_c\) and a residual one \(H_{\text{res}}\). To search for \textit{well-defined} collective sub-manifold, we demand that the expectation value of the curvature is minimized to satisfy \(H_{\text{res}}\ll\text{const.}/H_c+\text{const.} = H_{\text{HF}}\) as far as possible. We also require the Lagrange bracket for the usual and Lagrange-like variables. Our geometric equation together with the requirement describes the collective motion of a system. It is expected to work well over a wide range of physics beyond the \(U(N)\) RPA as the small amplitude limit, with certain boundary condition.
4.4 Geometric equation in quasi PHF

To extract the collective sub-manifolds, we demand the zero curvature of the connection on the TDHF manifold. We transform the geometric equation into the equation in the quasi PHF. The TDHF Hamiltonian in the quasi PHF is expressed as

\[ U^{-1}(g) H_{\text{HF}} U(g) = [\hat{d}, \hat{d}] \mathcal{F} \left[ \frac{\hat{d}}{\hat{d}} \right] = [\hat{c}^\dagger, \hat{c}] g^\dagger \mathcal{F}^g \left[ \frac{\hat{c}}{\hat{c}} \right], \quad g^\dagger \mathcal{F} g = \mathcal{F}_o. \]  

(4.33)

The infinitesimal generators of collective sub-manifolds and integrability conditions for \( \nu = 1, \cdots, \mu \), expressed as Lie-algebra-valued equation, are rewritten in the quasi PHF as,

\[ H_c = [\hat{c}^\dagger, \hat{c}] \mathcal{F}_{o-c}, \quad \mathcal{F}_{o-c} = \left[ \hat{F}_{o-c} - \hat{F}_{o-c} \right], \]  

(4.34)

\[ O_\nu = [\hat{c}^\dagger, \hat{c}] \theta_{o-\nu} \left[ \frac{\hat{c}}{\hat{c}} \right], \quad (\theta_{o-\nu} \equiv g^\dagger \theta_{o-\nu} g), \quad O_\nu = [\hat{c}^\dagger, \hat{c}] \theta_{o-\nu} \left[ \frac{\hat{c}}{\hat{c}} \right], \]  

(4.35)

\[ U^{-1}(g) C_{t, \Lambda_o} U(g) = [\hat{c}^\dagger, \hat{c}] C_{o-t, \Lambda_o} \left[ \frac{\hat{c}}{\hat{c}} \right] = U^{-1}(g) C_{t, \Lambda_o} \left[ \frac{\hat{c}}{\hat{c}} \right] = 0, \]  

(4.36)

\[ U^{-1}(g) C_{o, \Lambda_o} U(g) = [\hat{c}^\dagger, \hat{c}] C_{o-o, \Lambda_o} \left[ \frac{\hat{c}}{\hat{c}} \right] = U^{-1}(g) C_{o, \Lambda_o} \left[ \frac{\hat{c}}{\hat{c}} \right] = 0, \]  

(4.37)

where all the curvatures \( C_{o-o} \) should be vanished. The \( \mathcal{F}_{o-c} \), \( \theta_{o-\nu} \) and \( \theta_{o-\nu} \) satisfy the partial differential equations on the \( U(N) \) Lie group manifold \( g \),

\[ -i \hbar \partial_\nu g^\dagger = \mathcal{F}_{o-c} g^\dagger, \quad -i \partial_\nu g^\dagger = \theta_{o-\nu} g^\dagger, \quad -i \partial_{o-\nu} g^\dagger = \theta_{o-\nu} g^\dagger. \]  

(4.38)

The TDHF Hamiltonian in a quasi PHF is decomposed into collective and residual ones as

\[ U^{-1}(g) H_{\text{HF}} U(g) = \mathcal{H}_c + \mathcal{H}_{\text{res}}, \quad \mathcal{F}_o = \mathcal{F}_{o-c} + \mathcal{F}_{o-\text{res}}, \]  

(4.39)

at a reference point \( g \) on \( U(N) \) group. The curvatures \( C_{t, \Lambda_o} \) and \( C_{o, \Lambda_o} \) introduced previously have the same forms as those in (4.36) and (4.37) except that \( \mathcal{F}_{o-c} \) is replaced by \( \mathcal{F}_o \). Then the corresponding curvatures \( C_{o-t, \Lambda_o} \) and \( C_{o-t, \Lambda_o}^\ast \) are also decomposed into as,

\[ C_{o-t, \Lambda_o} = C_{o-t, \Lambda_o} + C_{o-t, \Lambda_o}^{\text{res}}, \quad C_{o-t, \Lambda_o}^\ast = C_{o-t, \Lambda_o}^\ast + C_{o-t, \Lambda_o}^{\text{res}}. \]  

(4.40)

The collective curvatures \( C_{o-t, \Lambda_o} \) and \( C_{o-t, \Lambda_o}^\ast \) arising from \( \mathcal{F}_{o-c} \) are given in the same forms as the ones in (4.37). The residual ones \( C_{o-t, \Lambda_o}^{\text{res}} \) and \( C_{o-t, \Lambda_o}^{\text{res}} \) arising from \( \mathcal{F}_{o-\text{res}} \) are defined as

\[ C_{o-t, \Lambda_o}^{\text{res}} = -i \partial_{\Lambda_o} \mathcal{F}_{o-\text{res}} - \theta_{o-\nu} \mathcal{F}_{o-\text{res}} \mathcal{F}_{o-\text{res}}, \quad C_{o-t, \Lambda_o}^{\text{res}} = -i \partial_{\Lambda_o} \mathcal{F}_{o-\text{res}} - \theta_{o-\nu} \mathcal{F}_{o-\text{res}} \mathcal{F}_{o-\text{res}}. \]  

(4.41)

Using (4.38) and (4.39), Lie-algebra-valued forms of curvatures are calculated to be

\[ C_{t, \Lambda_o}^{\text{res}} = -i \partial_{\Lambda_o} <H_{\text{res}}, \mathcal{F}_{o-\text{res}}|g>, \quad C_{t, \Lambda_o}^{\text{res}} = -i \partial_{\Lambda_o} <H_{\text{res}}, \mathcal{F}_{o-\text{res}}|g>. \]  

(4.42)

Supposing there exist a well-defined collective sub-manifold satisfying (4.38), we should demand that the following curvature are made equal to zero:
\[
C_{o-t,A\nu} = C_{o-t,A\nu}^c = 0, \quad C_{o-A\nu,A\nu} = 0, \quad C_{o-A\nu,A\nu}^c = 0, \quad C_{o-A\nu,A\nu}^c = 0, \quad C_{o-A\nu,A\nu}^c = 0, \quad (4.43)
\]

the first equation of which lead us to the Lie-algebra-valued relations,
\[
C_{t,A\nu} = C_{t,A\nu}^{res} = -i\partial_{A\nu} <H_{res}>_g \quad \text{and} \quad C_{t,A\nu} = C_{t,A\nu}^{res} = -i\partial_{A\nu} <H_{res}>_g . \quad (4.44)
\]

Then the curvatures \(C_{t,A\nu}^{res}\) and \(C_{t,A\nu}^{res}\) are regarded as the gradient of quantum-mechanical potentials due to the existence of the residual Hamiltonian \(H_{res}\) on the collective sub-manifolds. The potentials become almost flat on the collective sub-manifolds, \(H_{HT}=H_c+\text{const.}\), if the proper subspace determined is almost invariant subspace of full TDHF Hamiltonian. This collective subspace is the almost degenerate eigenstate of the residual Hamiltonian. The residual curvature at a point on the invariant subspace is extremely small. The way of extracting collective sub-manifolds out of the TDHF manifold is possible by the minimization of the residual curvature. Therefore, a deeper insight into \(4.44\) becomes necessary.

Finally, restrictions to assure the Lagrange bracket for the usual collective variables and Lagrange-like ones are transformed into the forms represented in quasi PHF as
\[
\text{Tr}\left\{\begin{bmatrix} 1 & m \\ 0 & -1 \end{bmatrix} \left[\theta_{\nu\nu} - \theta_{\nu\nu}^1\right]\right\} = \delta_{\nu\nu^*}, \quad \text{Tr}\left\{\begin{bmatrix} 1 & m \\ 0 & -1 \end{bmatrix} \left[\theta_{\nu\nu} - \theta_{\nu\nu}^1\right]\right\} = \text{Tr}\left\{\begin{bmatrix} 1 & m \\ 0 & -1 \end{bmatrix} \left[\theta_{\nu\nu} - \theta_{\nu\nu}^1\right]\right\} = 0. \quad (4.45)
\]

### 4.5 Transform of Lagrange-like picture to picture in quasi PHF

We discuss how the Lagrange-like picture is transformed into the picture in the quasi PHF. We regard any point on the collective sub-manifold as an initial point (initial value). Suppose a time evolution of system with various initial value. Then we have a relation between the Lagrange-like way and the usual one by through a similar way to the previous one as
\[
\mathcal{F}_{o-c} = \hbar \partial_t \hat{A}_\nu \hat{\theta}_{o-\nu} + \hbar \partial_t \hat{\Lambda}_\nu \hat{\theta}_{o-\nu}, \quad (4.46)
\]
\[
\theta_{o-\nu} = \partial_{\nu\nu} \hat{A}_\nu \hat{\theta}_{o-\nu} + \partial_{\nu\nu} \hat{\Lambda}_\nu \hat{\theta}_{o-\nu}, \quad \theta_{o-\nu} = \partial_{\nu\nu} \hat{A}_\nu \hat{\theta}_{o-\nu} + \partial_{\nu\nu} \hat{\Lambda}_\nu \hat{\theta}_{o-\nu}, \quad (4.47)
\]
in which the transformation functions are set up by the initial conditions,
\[
\hat{\Lambda}_\nu(t)|_{t=0} = \hat{\Lambda}_\nu(\Lambda, \Lambda^* , t)|_{t=0} = \Lambda_\nu, \quad \partial_{\Lambda_\nu} \hat{\Lambda}_\nu|_{t=0} = \delta_{\nu\nu}, \quad \partial_{\Lambda_\nu^*} \hat{\Lambda}_\nu|_{t=0} = \delta_{\nu\nu^*}
\]
\[
\hat{\Lambda}_\nu^*(t)|_{t=0} = \hat{\Lambda}_\nu^*(\Lambda, \Lambda^* , t)|_{t=0} = \Lambda_\nu^*, \quad \partial_{\Lambda_\nu} \hat{\Lambda}_\nu^*|_{t=0} = 0, \quad \partial_{\Lambda_\nu^*} \hat{\Lambda}_\nu^*|_{t=0} = 0. \quad (4.48)
\]
to guarantee both the pictures to coincide at \(t=0\). The Hamiltonian \(\mathcal{F}_{o-c}\) is expressed as
\[
\mathcal{F}_{o-c} = i\hbar v_\nu(\Lambda, \Lambda^* , t) \theta_{o-\nu} + \hbar v_\nu^*(\Lambda, \Lambda^* , t) \hat{\theta}_{o-\nu}. \quad (4.49)
\]

Expansion coefficients \((v_\nu, v_\nu^*)\) are interpreted as velocity field in the Lagrange-like manner. Substituting \((4.47)\) into \((4.39)\) and comparing with \((4.46)\), we get the relations
\[
\hat{\Lambda}_\nu = \partial_{\nu\nu} \Lambda_\nu = v_\nu \partial_{\nu\nu} \hat{\Lambda}_\nu + v_\nu \partial_{\nu\nu} \hat{\Lambda}_\nu, \quad \hat{\Lambda}_\nu^* = \partial_{\nu\nu} \Lambda_\nu^* = v_\nu \partial_{\nu\nu^*} \hat{\Lambda}_\nu^* + v_\nu \partial_{\nu\nu^*} \hat{\Lambda}_\nu^*, \quad (4.50)
\]
from which the initial conditions of the velocity fields are given as
\[
\hat{\Lambda}_\nu(t)|_{t=0} = \partial_{\nu\nu} \Lambda_\nu|_{t=0} = v_\nu(\Lambda, \Lambda^* , t)|_{t=0}, \quad \hat{\Lambda}_\nu^*(t)|_{t=0} = \partial_{\nu\nu^*} \Lambda_\nu^*|_{t=0} = v_\nu^*(\Lambda, \Lambda^* , t)|_{t=0}. \quad (4.51)
\]

Then we obtain the correspondence of the time derivatives of the collective coordinates in the usual manner to the velocity fields in the Lagrange-like one.

Finally we impose the canonicity conditions in the usual manner,
\[
\langle \phi(\hat{c})|\hat{O}_\nu|\phi(\hat{c})\rangle = i\hat{A}_\nu, \quad \langle \phi(\hat{c})|\hat{O}_\nu|\phi(\hat{c})\rangle = -i\hat{\Lambda}_\nu, \quad (4.52)
\]
which leads to the weak canonical commutation relation with the aid of \((4.43)\) and \((4.52)\).
5 On the validity of \(\ll\)maximally decoupled\(\gg\) theory

The our basic concept lies in the invariance principle of Schrödinger equation, and the TDHF equation is solved under the canonicity condition and vanishing of non-collective dangerous terms. We have no justification on the validity of maximally decoupled method, we must give a criterion how it extract the collective sub-manifolds effectively out of the TDHF manifold. To establish the criterion, we express the collective Hamiltonian \(\mathcal{H}_{0-c}\) and the residual one \(\mathcal{H}_{0-res}\) in the same form as those of the TDHF Hamiltonian \(\mathcal{H}_o\) \((4.33)\). We also represent the quantities \(\theta_{o,\nu}(\nu=1, \cdots , \mu), C_{o-t,\Lambda_n}^{res}\) and \(C_{o-t,\Lambda_n^*}^{res}\) in the form consisting of \(m \times m, (N-m) \times (N-m)\) block matrices, respectively as follows:

\[
\theta_{o,\nu} = \begin{bmatrix} \xi_o & \varphi_o \\ \psi_o & \zeta_o \end{bmatrix}, \quad C_{o-t,\Lambda_n}^{res} = \begin{bmatrix} C_{\xi,\varphi}^{res} & C_{\varphi,\zeta}^{res} \\ C_{\psi,\zeta}^{res} & C_{\zeta,\xi}^{res} \end{bmatrix}, \quad C_{o-t,\Lambda_n^*}^{res} = \begin{bmatrix} C_{\xi,\varphi}^{res} & C_{\varphi,\zeta}^{res} \\ C_{\zeta,\xi}^{res} & C_{\xi,\varphi}^{res} \end{bmatrix}.
\]

Substitution of the explicit form of \(\mathcal{H}_{0-res}\) and \((5.1)\) into \((4.1)\) yields

\[
C_{\xi,\varphi}^{res} = -i \partial_{\Lambda_v} \hat{F}_{0-res} - [\xi_o, \hat{F}_{0-res}] - \varphi_o, \hat{F}_{0-res} + \hat{F}_{0-res} \psi_o, \nu, \\
C_{\psi,\zeta}^{res} = -i \partial_{\Lambda_v} \hat{F}_{0-res} - \zeta_o, \hat{F}_{0-res} + \hat{F}_{0-res} \xi_o, \nu - \psi_o, \nu, \hat{F}_{0-res} + \hat{F}_{0-res} \varphi_o, \nu, \\
C_{\varphi,\zeta}^{res} = -i \partial_{\Lambda_v} \hat{F}_{0-res} - \zeta_o, \hat{F}_{0-res} + \hat{F}_{0-res} \xi_o, \nu - \varphi_o, \hat{F}_{0-res} + \hat{F}_{0-res} \varphi_o, \nu, \\
C_{\zeta,\xi}^{res} = -i \partial_{\Lambda_v} \hat{F}_{0-res} - \zeta_o, \hat{F}_{0-res} - \psi_o, \nu, \hat{F}_{0-res} + \hat{F}_{0-res} \varphi_o, \nu.
\]

Another \(\theta_{o,\nu}^\dagger\) in \((4.46)\) and \((4.47)\) is expressed in the same form as the one given in \((5.1)\). Substituting this expression for \(\theta_{o,\nu}^\dagger\) into \((4.46)\) and using \((4.39)\), we obtain the relations

\[
\hat{F}_{0-res} = \hat{F}_0 - h \partial_{\Lambda_v} \hat{A}_v \varphi_o, \nu - h \partial_{\Lambda_v} \hat{A}_v \psi_o, \nu, \hat{A}_v \varphi_o, \nu, \\
\hat{F}_{0-res} = \hat{F}_0 - h \partial_{\Lambda_v} \hat{A}_v \psi_o, \nu - h \partial_{\Lambda_v} \hat{A}_v \varphi_o, \nu, \hat{A}_v \psi_o, \nu,
\]

together with their c.c. Further using \((5.2)\), we can get the relations

\[
\text{Tr} \xi_{o,\nu} = i \Lambda_v^\dagger, \quad \text{Tr} \xi_{o,\nu}^\dagger = -i \Lambda_v.
\]

The way of extracting the collective sub-manifolds out of the full TDHF manifold is made possible by the minimization of the residual curvature. This is achieved if we require at least the expectation values of the residual curvatures be minimized as far as possible,

\[
\langle \phi(\hat{g}) \rangle_{\mathcal{H}_{0,\Lambda_n}} | \phi(\hat{g}) \rangle = \text{Tr} C_{\xi,\psi}^{res} \approx 0, \quad \langle \phi(\hat{g}) \rangle_{\mathcal{H}_{0,\Lambda_n^*}} | \phi(\hat{g}) \rangle = \text{Tr} C_{\xi,\psi}^{res} \approx 0.
\]

\(C_{\xi,\psi}^{res}\) and \(C_{\xi,\psi}^{res}\) are given in the same way as the one in \((4.36)\). We adopt the condition of the stationary HF method \((YK)\) \[32\]. The so-called dangerous term in \(\mathcal{H}_{0,\Lambda_n}\) must vanish,

\[
\hat{F}_{0-res} = 0, \quad \hat{F}_{0-res} = 0.
\]

With the aid of \((5.3)\) and \((5.4)\), \((5.6)\) and \((5.5)\) are rewritten as

\[
\hat{F}_0 = h \hat{A}_v \varphi_o, \nu + h \hat{A}_v \varphi_o, \nu, \hat{A}_v \varphi_o, \nu, \\
\hat{F}_0 = h \hat{A}_v \varphi_o, \nu + h \hat{A}_v \varphi_o, \nu.
\]

\[
\partial_{\Lambda_v} \text{Tr} \hat{F}_{0-res} = i \partial_{\Lambda_v} \text{Tr} \hat{F}_0 + \partial_{\Lambda_v} (h \hat{A}_v \hat{A}_v^\dagger - h \hat{A}_v \hat{A}_v^\dagger) \approx 0, \\
\partial_{\Lambda_v^*} \text{Tr} \hat{F}_{0-res} = i \partial_{\Lambda_v^*} \text{Tr} \hat{F}_0 + \partial_{\Lambda_v^*} (h \hat{A}_v \hat{A}_v^\dagger - h \hat{A}_v \hat{A}_v^\dagger) \approx 0.
\]
5.1 Geometric equation in fluctuating quasi PHF

The functional form \( g(\Lambda, \Lambda^*, t) \) is divided into the stationary and fluctuating components, \( g = g_0 \hat{g} \), product of stationary \( g_0 \) and fluctuating \( \hat{g}(\Lambda, \Lambda^*, t) \). The \( g_0 \) satisfies the HF eigenvalue equation.

A density matrix \( \mathcal{Q}(\Lambda, \Lambda^*, t) \) is decomposed as \( \mathcal{Q} = g_0 \hat{\mathcal{Q}} g_0^\dagger \). The fluctuating \( \hat{\theta}_\nu \) and \( \hat{\tilde{\theta}}_\nu \) are given as \( \hat{\theta}_\nu = \hat{g}_\nu^\dagger \theta_\nu g_0 \) and \( \hat{\tilde{\theta}}_\nu = \hat{g}_\nu^\dagger \tilde{\theta}_\nu g_0 \). Under the decomposition \( g = g_0 \hat{g} \), zero curvatures (5.13) are transformed to

\[
\begin{align*}
  i\hbar \partial_t \hat{\tilde{\theta}}^\dagger_\nu - i\partial_{\Lambda^\nu} \hat{\tilde{\theta}}^\dagger_\nu + [\hat{\tilde{\theta}}^\dagger_\nu, \hat{\tilde{\theta}}^\dagger_\nu] = 0, \\
  i\partial_{\Lambda^\nu} \hat{\theta}^\dagger_\nu - i\partial_{\Lambda^\nu} \hat{\tilde{\theta}}^\dagger_\nu = 0.
\end{align*}
\]

Putting \( \hat{\tilde{\theta}}_\nu = \hat{\tilde{\theta}}_\nu \) into (5.9), we look for collective-path \( \hat{g} \) and collective-Hamiltonian \( \hat{F}_c \) under the minimization of the residual curvature arising from residual Hamiltonian \( \hat{F}_c \).

Next, for our convenience of further discussion, we also introduce modified fluctuating auxiliary quantities with the same forms as those in the previous section, through

\[
\begin{align*}
  \hat{\tilde{\theta}}^\dagger_{\nu - \nu'} = \hat{g}^\dagger \hat{\tilde{\theta}}^\dagger_{\nu - \nu'} \hat{g}, \\
  \hat{\theta}_{\nu - \nu'} = \hat{g}^\dagger \hat{\theta}_{\nu - \nu'} \hat{g}.
\end{align*}
\]

We rewrite a set of equations (5.9), (5.10) and (5.11) in terms of the above quantities as

\[
\begin{align*}
  i\hbar \partial_t \hat{\tilde{\theta}}^\dagger_{\nu - \nu'} - i\hat{g}^\dagger (\partial_{\Lambda^\nu} \hat{\tilde{\theta}}^\dagger_\nu) \hat{g} = 0, \\
  i\partial_{\Lambda^\nu} \hat{\theta}_{\nu - \nu'} - i\hat{g}^\dagger (\partial_{\Lambda^\nu} \hat{\tilde{\theta}}^\dagger_\nu) \hat{g} = 0,
\end{align*}
\]

\[
\begin{align*}
  i\partial_{\Lambda^\nu} \hat{\theta}_{\nu - \nu'} - i\partial_{\Lambda^\nu} \hat{\tilde{\theta}}^\dagger_{\nu - \nu'} = 0, \\
  i\partial_{\Lambda^\nu} \hat{\theta}_{\nu - \nu'} - i\partial_{\Lambda^\nu} \hat{\tilde{\theta}}^\dagger_{\nu - \nu'} = 0.
\end{align*}
\]

In the derivation of Eqs. (5.15) and (5.16), we have used (5.12) and (5.13), respectively. While, \( \hat{\tilde{\theta}}^\dagger_{\nu - \nu'} = \left[ \hat{g}^\dagger \hat{\tilde{\theta}}^\dagger_{\nu - \nu'} \hat{g} \right] \) obeys \(-i\partial_{\Lambda^\nu} \hat{g}^\dagger \hat{\theta}_{\nu - \nu'} \hat{g} \) by which (5.7) is changed to the equation of path for collective motion. Using the rep. (5.1) of \( g \), we get the partial differential equations

\[
\begin{align*}
  \xi_{\theta, \nu'} = -i(\partial_{\Lambda^\nu} \hat{a}^\dagger \hat{a} + \partial_{\Lambda^\nu} \hat{b}^\dagger \hat{b}), \\
  \varphi_{\theta, \nu'} = -i(\partial_{\Lambda^\nu} \hat{a}^\dagger \hat{b} + \partial_{\Lambda^\nu} \hat{b}^\dagger \hat{a}), \\
  \psi_{\theta, \nu'} = -i(\partial_{\Lambda^\nu} \hat{b}^\dagger \hat{a} + \partial_{\Lambda^\nu} \hat{a}^\dagger \hat{b}).
\end{align*}
\]

Substituting \( \hat{\tilde{\theta}} = \hat{g}^\dagger \hat{F}_c \hat{g} \) and (5.17) into (5.7), we obtain for \( \hat{F}_o \) and \( \hat{F}_a \) approximately as

\[
\begin{align*}
  \hat{a}(\hat{F}_a + \hat{F}_b) + \hat{b}(\hat{F}_a + \hat{F}_b) + \hat{h}_a \hat{b} - \hat{h}_a \hat{a} + \hat{h}_b \hat{a} - \hat{h}_b \hat{b} = 0,
\end{align*}
\]

\[
\begin{align*}
  \hat{a}(\hat{F}_a + \hat{F}_b) - \hat{h}_a \hat{b} + \hat{h}_b \hat{a} = 0,
\end{align*}
\]

\[
\begin{align*}
  \hat{a}(\hat{F}_a + \hat{F}_b) - \hat{h}_a \hat{b} + \hat{h}_b \hat{a} = 0,
\end{align*}
\]

\[
\begin{align*}
  \hat{a}(\hat{F}_a + \hat{F}_b) + \hat{b}(\hat{F}_a + \hat{F}_b) = 0.
\end{align*}
\]
where we have assumed the anti-commutativity $\partial_{\Lambda^\nu} \hat{a}^i = -\hat{a}^i \partial_{\Lambda^\nu}$, and $\partial_{\Lambda^\nu} \hat{b}^i = -\hat{b}^i \partial_{\Lambda^\nu}$.

From (3.3), we have the differential formulas for $Q$ as follows:

$$\frac{\partial Q}{\partial a} = \begin{bmatrix} \hat{a}^i \\ \hat{b}^i \end{bmatrix} \begin{bmatrix} \hat{a}^i \\ \hat{b}^i \end{bmatrix} = \begin{bmatrix} a^i \\ b^i \end{bmatrix}, \quad \frac{\partial Q}{\partial b} = \begin{bmatrix} \hat{b}^i \\ \hat{a}^i \end{bmatrix} \begin{bmatrix} \hat{b}^i \\ \hat{a}^i \end{bmatrix} = \begin{bmatrix} 0 \\ b^i \\ 0 \end{bmatrix}. \quad (5.20)$$

Let us denote the HF energy functional (3.3) simply as $\langle H \rangle_g$. Then we prove that the relations

$$\partial_{\hat{a}} \langle H \rangle_g = \hat{F} \hat{a} + \hat{F} \hat{b}, \quad \partial_{\hat{b}} \langle H \rangle_g = \hat{F} \hat{a} + \hat{F} \hat{b}, \quad (5.21)$$

and that show, through which the TDHF equation is rewritten as

$$i\hbar \dot{g} = \begin{bmatrix} \partial_{\hat{a}} \\ \partial_{\hat{b}} \\ \partial_{\hat{a}} \end{bmatrix} \langle H \rangle_g, \quad \text{where} \quad \partial_{\hat{a}} \langle H \rangle_g = \hat{F} \hat{a} + \hat{F} \hat{b}, \quad \partial_{\hat{b}} \langle H \rangle_g = \hat{F} \hat{a} + \hat{F} \hat{b}. \quad (5.22)$$

Using the relation similar to (5.21), Eqs. (5.18) and (5.19) are reduced respectively to

$$\dot{\hat{a}} = \left\{ \left( \hat{F} \hat{a} + \hat{F} \hat{b} \right) - \hbar \left( i\hat{A}_{\nu} \partial_{\Lambda_{\nu}} \hat{b} + i\hat{A}_{\nu}^{*} \partial_{\Lambda_{\nu}} \hat{a} \right) \right\} + \hat{b} = 0, \quad (5.23)$$

$$\dot{\hat{a}} = \left\{ \left( \hat{F} \hat{a} + \hat{F} \hat{b} \right) - \hbar \left( i\hat{A}_{\nu} \partial_{\Lambda_{\nu}} \hat{b} + i\hat{A}_{\nu}^{*} \partial_{\Lambda_{\nu}} \hat{a} \right) \right\} + \hat{b} = 0. \quad (5.24)$$

As a way of satisfying (5.23), we adopt the following partial differential equations:

$$\partial_{\hat{a}} \langle H \rangle_g - \hbar \left( i\hat{A}_{\nu} \partial_{\Lambda_{\nu}} \hat{b} + i\hat{A}_{\nu}^{*} \partial_{\Lambda_{\nu}} \hat{a} \right) = 0, \quad \partial_{\hat{b}} \langle H \rangle_g - \hbar \left( i\hat{A}_{\nu} \partial_{\Lambda_{\nu}} \hat{b} + i\hat{A}_{\nu}^{*} \partial_{\Lambda_{\nu}} \hat{a} \right) = 0 \quad \text{c.c.} \quad (5.25)$$

$$\partial_{\hat{a}} \langle H \rangle_g - \hbar \left( i\hat{A}_{\nu} \partial_{\Lambda_{\nu}} \hat{b} + i\hat{A}_{\nu}^{*} \partial_{\Lambda_{\nu}} \hat{a} \right) = 0, \quad \partial_{\hat{b}} \langle H \rangle_g - \hbar \left( i\hat{A}_{\nu} \partial_{\Lambda_{\nu}} \hat{b} + i\hat{A}_{\nu}^{*} \partial_{\Lambda_{\nu}} \hat{a} \right) = 0 \quad \text{c.c.} \quad (5.26)$$

From (4.30) and (4.33), we have $\langle \phi(\bar{g}) | C_{\Lambda_{\nu}} | \phi(\bar{g}) \rangle = 0$ and $\langle \phi(\bar{g}) | C_{\Lambda_{\nu}^{*}} | \phi(\bar{g}) \rangle = 0$ in which $F_c$ is replaced by $F_o$. Using the transformation property of the differentials

$$\partial_{\Lambda_{\nu}} = \partial_{\Lambda_{\nu}} \hat{a} \partial_{\Lambda_{\nu}} + \partial_{\Lambda_{\nu}} \hat{A}_{\nu} \partial_{\Lambda_{\nu}}^{*}, \quad \partial_{\Lambda_{\nu}^{*}} = \partial_{\Lambda_{\nu}} \hat{A}_{\nu} \partial_{\Lambda_{\nu}} + \partial_{\Lambda_{\nu}} \hat{A}_{\nu} \partial_{\Lambda_{\nu}}^{*}, \quad (5.27)$$

and differential formulas for expectation values of Hamiltonian $H$ and HF one $H_{HF}$

$$\partial_{\Lambda_{\nu}} \langle H \rangle_g = -\text{Tr}[\partial_{\Lambda_{\nu}} \hat{Q}(\bar{g}) \hat{F}], \quad \partial_{\Lambda_{\nu}} \langle H_{HF} \rangle_g = -\partial_{\Lambda_{\nu}} \text{Tr} \hat{F}_o = \partial_{\Lambda_{\nu}} \langle H \rangle_g - \text{Tr}[\hat{Q}(\bar{g}) \partial_{\Lambda_{\nu}} \hat{F}] \quad (5.28)$$

Due to (5.21), (5.22), (5.4) and $\Lambda_{\nu} \text{Tr} \hat{F}_o \approx -\partial_{\Lambda_{\nu}} \langle H \rangle_g$, the invariance principle of Schrödinger equation and canonicity condition leads to the canonical forms of equations of motion

$$i\hbar \hat{A}_{\nu} = -\partial_{\Lambda_{\nu}} \langle H \rangle_g, \quad i\hbar \hat{A}_{\nu}^{*} = -\partial_{\Lambda_{\nu}^{*}} \langle H \rangle_g. \quad (5.29)$$

The structure of (5.23) shows that it becomes the equation of path for collective motion under the substitution of (5.29). Then it realizes a construction of the equation of path in the TDHF [23 33 32]. Eqs. (5.25) and (5.29) determine the behavior of maximally decoupled collective motions in the TDHF. It is a renewal of the TDHF equation by using the canonicity condition under the existence of invariant subspace in the TDHF. This is due to a natural consequence of the maximally decoupled theory because there exists an invariant subspace, if the invariance principle of Schrödinger equation is realized. Then we can investigate the validity of maximally decoupled theory with the use of the condition. The reason why the condition occurs in our theory, which did not appear in the maximally decoupled theory. Since the theory has no such terms, the condition is trivially fulfilled. This is the essential difference between the maximally decoupled theory and the ours. We stress that our theory has been formulated by manifesting the group structure which makes the present work applicable to the $SO(2N+1)$ group [31].
6 Nonlinear RPA arising from zero-curvature equation

The integrability condition of TDHF equation determining a collective sub-manifold has been studied based on the differential geometric viewpoint. A geometric equation works well over a wide range of physics beyond the random phase approximation (RPA). A linearly approximated solution of TDHF equation becomes the RPA. Suppose we solved the geometric equation by expanding it in power series of collective variables \((\Lambda_\nu, \Lambda^*_\nu) \ (\nu=1, \cdot, \mu; \mu \ll N^2)\). The geometric equation has a RPA solution at the lowest power of the collective variables. The fluctuating density matrix \(\tilde{\mathcal{Q}}\) and HF matrix \(\tilde{F}\) in the quasi PHF are expressed as follows:

\[
\tilde{\mathcal{Q}}(\tilde{g}) = g_0^\dagger \mathcal{Q}(g) g_0, \quad \tilde{F} = g_0^\dagger \mathcal{F} g_0, \quad \tilde{F} = \begin{bmatrix} \tilde{\varepsilon}_0 + \tilde{f} & \tilde{f} \\ \tilde{f} & \tilde{\varepsilon}_0 - \tilde{f} \end{bmatrix}, \quad \mathcal{F}_0 g_0 = \begin{bmatrix} \varepsilon_0 & 0 \\ 0 & \varepsilon_0 \end{bmatrix},
\]

(6.1)

where \(\mathcal{F}_0\) is a stationary HF matrix and \(\varepsilon_0\) and \(\tilde{\varepsilon}_0\) denote the hole-energy and the particle-energy. The \(\tilde{f}, \tilde{f}, \tilde{f}\) and \(f\) are given later. A part of the geometric equation (5.11) is rewritten as

\[
\mathrm{Tr} \left\{ \begin{bmatrix} 1_m & 0 \\ 0 & -1_{N-m} \end{bmatrix} \left[ \partial_{\tilde{\nu}-\nu}, \partial_{\tilde{\nu}-\nu}^\dagger \right] \right\} = \delta_{\nu\nu'},
\]

(6.2)

and the first of (6.1), we have the relation between \(\mathcal{Q}(g)\) and \(\tilde{\mathcal{Q}}(\tilde{g})\) used later as

\[
\mathcal{Q}(g) \equiv \begin{bmatrix} \tilde{\mathcal{Q}}(g) & \tilde{\mathcal{Q}}^0(g) \\ \tilde{\mathcal{Q}}^0(g) & \tilde{\mathcal{Q}}(g) \end{bmatrix} = \mathcal{Q}_0 + 2g_0 \begin{bmatrix} \tilde{\mathcal{Q}}(g) & \tilde{\mathcal{Q}}^0(g) \\ \tilde{\mathcal{Q}}^0(g) & \tilde{\mathcal{Q}}(g) \end{bmatrix} g_0^\dagger, \quad \mathcal{Q}_0 \equiv g_0 \begin{bmatrix} 1_m & 0 \\ 0 & -1_{N-m} \end{bmatrix} g_0^\dagger = \begin{bmatrix} \tilde{\mathcal{Q}}_0 & \tilde{\mathcal{Q}}_0 \end{bmatrix}.
\]

(6.3)

In order to investigate the matrix-valued nonlinear time evolution equation (6.15) arising from the zero curvature one, we give here the \(\partial_{\Lambda_\nu}\) and \(\partial_{\Lambda^*_\nu}\) differential forms of the TDHF density matrix and collective Hamiltonian. First, using (6.3) and \(\tilde{g}^\dagger \tilde{g} = g^\dagger g = 1_N\), we have

\[
\partial_{\Lambda_\nu} \tilde{\mathcal{Q}}(\tilde{g}) = \partial_{\Lambda_\nu} \tilde{g} \begin{bmatrix} 1_m & 0 \\ 0 & -1_{N-m} \end{bmatrix} \tilde{g}^\dagger + \tilde{g} \begin{bmatrix} 1_m & 0 \\ 0 & -1_{N-m} \end{bmatrix} \partial_{\Lambda_\nu} \tilde{g}^\dagger
\]

(6.4)

\[
= \partial_{\Lambda_\nu} \tilde{g} \tilde{g}^\dagger \tilde{g} \begin{bmatrix} 1_m & 0 \\ 0 & -1_{N-m} \end{bmatrix} \tilde{g}^\dagger + \tilde{g} \begin{bmatrix} 1_m & 0 \\ 0 & -1_{N-m} \end{bmatrix} \partial_{\Lambda_\nu} \tilde{g} \tilde{g}^\dagger = -i(\tilde{g}\tilde{g}^\dagger) \partial_{\nu} \tilde{g} \begin{bmatrix} 1_m & 0 \\ 0 & -1_{N-m} \end{bmatrix} \tilde{g}^\dagger + i\tilde{g} \begin{bmatrix} 1_m & 0 \\ 0 & -1_{N-m} \end{bmatrix} \tilde{g}^\dagger \partial_{\nu} \tilde{g} \tilde{g}^\dagger,
\]

where we have used \(i\partial_{\Lambda_\nu} \tilde{g} = \partial_{\nu} \tilde{g}\). Next, by using (5.11), (6.4) is transformed into

\[
\partial_{\Lambda_\nu} \tilde{\mathcal{Q}}(\tilde{g}) = -i \tilde{g} \begin{bmatrix} 1_m & 0 \\ 0 & -1_{N-m} \end{bmatrix} \tilde{g}^\dagger.
\]

(6.5)

From (6.1) and (6.3), we have

\[
\partial_{\Lambda_\nu} \tilde{\mathcal{Q}}(\tilde{g}) = 2\partial_{\Lambda_\nu} \tilde{\mathcal{Q}}(\tilde{g}) = 2g_0 \begin{bmatrix} \partial_{\Lambda_\nu} \tilde{\mathcal{Q}}(\tilde{g}) & \partial_{\Lambda_\nu} \tilde{\mathcal{Q}}(\tilde{g}) \\ \partial_{\Lambda_\nu} \tilde{\mathcal{Q}}(\tilde{g}) & \partial_{\Lambda_\nu} \tilde{\mathcal{Q}}(\tilde{g}) \end{bmatrix} g_0^\dagger, \quad g_0 \equiv \begin{bmatrix} \tilde{a}_0 & \tilde{b}_0 \\ \tilde{b}_0 & \tilde{a}_0 \end{bmatrix}.
\]

(6.6)

Let us substitute the explicit reps for \(\tilde{g} \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} \) and \(\partial_{\nu} \tilde{g} \begin{bmatrix} \xi_{\nu-\nu} & \varphi_{\nu-\nu} \\ \psi_{\nu-\nu} & \zeta_{\nu-\nu} \end{bmatrix} \) into the RHS of (6.5) and combine it with (6.6). Then, we obtain the final \(\partial_{\Lambda_\nu}\) differential form of the TDHF \((U(N))\) density matrix as follows:

\[
\begin{bmatrix} \partial_{\Lambda_\nu} \tilde{\mathcal{Q}}(\tilde{g}) & \partial_{\Lambda_\nu} \tilde{\mathcal{Q}}(\tilde{g}) \\ \partial_{\Lambda_\nu} \tilde{\mathcal{Q}}(\tilde{g}) & \partial_{\Lambda_\nu} \tilde{\mathcal{Q}}(\tilde{g}) \end{bmatrix} = g_0 \begin{bmatrix} i(\tilde{b} \psi_{\nu-\nu} \tilde{a} - \tilde{a} \varphi_{\nu-\nu} \tilde{b}^\dagger), & -i(\tilde{a} \psi_{\nu-\nu} \tilde{a}^\dagger - \tilde{b} \varphi_{\nu-\nu} \tilde{b}^\dagger) \\ i(\tilde{a} \psi_{\nu-\nu} \tilde{a}^\dagger - \tilde{b} \varphi_{\nu-\nu} \tilde{b}^\dagger), & -i(\tilde{a} \psi_{\nu-\nu} \tilde{a}^\dagger - \tilde{b} \varphi_{\nu-\nu} \tilde{b}^\dagger) \end{bmatrix} g_0^\dagger.
\]

(6.7)
\( \partial_{\Lambda^\ast} \) differentiation of the \( U(N) \) density matrix is also made analogously to the above case. The explicit form of (6.3) without constant matrices is given as
\[
\begin{bmatrix}
\langle E^\ast y | E^\ast y \rangle - g_0 \\
\langle E^\ast y | E^\ast y \rangle - g_0 \end{bmatrix} = g_0 \begin{bmatrix}
\tilde{a}_0(\tilde{Q}^\dagger \tilde{a}_0^\dagger + \tilde{Q}^\dagger \tilde{b}_0^\dagger) + b_0(\tilde{Q}_0^\dagger + \tilde{Q}_0^\dagger), \\
\tilde{a}_0(\tilde{Q}^\dagger \tilde{a}_0^\dagger + \tilde{Q}^\dagger \tilde{b}_0^\dagger) + b_0(\tilde{Q}_0^\dagger + \tilde{Q}_0^\dagger)
\end{bmatrix}.
\]

From (6.1), we have \( \tilde{F} = \int \frac{d^2 f}{f} \) (except diagonal \( \tilde{e}_0 \) and \( \tilde{e}_0 \) terms) and \( \tilde{F} = g_0 \tilde{F} g_0^\dagger \).

Substituting (6.8) without constant matrices into (3.4), we have
\[
\begin{bmatrix}
f_{ab} = [ab|cd] \langle E^\ast y | g_0 \rangle, \\
f_{ai} = [ai|bj] \langle E^\ast y | g_0 \rangle
\end{bmatrix}
\begin{bmatrix}
\tilde{a}_i(\tilde{Q}^\dagger \tilde{a}_i^\dagger + \tilde{Q}^\dagger \tilde{b}_i^\dagger) + b_i(\tilde{Q}_i^\dagger + \tilde{Q}_i^\dagger), \\
\tilde{a}_i(\tilde{Q}^\dagger \tilde{a}_i^\dagger + \tilde{Q}^\dagger \tilde{b}_i^\dagger) + b_i(\tilde{Q}_i^\dagger + \tilde{Q}_i^\dagger)
\end{bmatrix}
\begin{bmatrix}
f_{ij} = [ij|kl] \langle E^\ast y | g_0 \rangle
\end{bmatrix},
\]
\[
\begin{bmatrix}
f_{ab} = [ab|cd] \langle \tilde{Q}_0^\dagger \tilde{a}_0^\dagger + \tilde{Q}_0^\dagger \tilde{b}_0^\dagger, \\
f_{ai} = [ai|bj] \langle \tilde{Q}_0^\dagger \tilde{a}_i^\dagger + \tilde{Q}_0^\dagger \tilde{b}_i^\dagger \rangle
\end{bmatrix}.
\]

Then the components of \( g_0 \tilde{F} g_0^\dagger \) become linear functionals of \( \tilde{Q}(\tilde{y}), \tilde{Q}(\tilde{y}), \tilde{Q}(\tilde{y}) \) and \( \tilde{Q}(\tilde{y}) \). Using (6.7), we easily calculate the \( \partial_{\Lambda^\ast} \) differential of, for example, \( f \) and \( f \) as follows:
\[
\partial_{\Lambda^\ast} f_{ab} = [ab|cd] \langle \tilde{Q}_0^\dagger \tilde{a}_0^\dagger + \tilde{Q}_0^\dagger \tilde{b}_0^\dagger, \\
\partial_{\Lambda^\ast} f_{ai} = [ai|bj] \langle \tilde{Q}_0^\dagger \tilde{a}_i^\dagger + \tilde{Q}_0^\dagger \tilde{b}_i^\dagger \rangle
\]
\[
\begin{bmatrix}
\partial_{\Lambda^\ast} f_{ij} = [ij|kl] \langle \tilde{Q}_0^\dagger \tilde{a}_i^\dagger + \tilde{Q}_0^\dagger \tilde{b}_i^\dagger \rangle
\end{bmatrix}.
\]

The (D) etc. have the same form as the one in (6.4) and new (D) etc. are also defined by
\[
\langle D \rangle = \|(ij|D|k|l)\|, \quad \langle F \rangle = \|(ij|F|k|l)\|, \quad \langle \tilde{F} \rangle = \|(ij|\tilde{F}|k|l)\|.
\]

Using the simplified notations \( a_0 \) for \( (\tilde{a}_0, \tilde{a}_0) \) and \( b_0 \) for \( (\tilde{b}_0, \tilde{b}_0) \), we define \( \langle ij|D|kl \rangle \) etc. as
\[
\begin{bmatrix}
\langle ij|D|kl \rangle = \langle ij|D|k|l \rangle \tilde{a}_k^\dagger \tilde{a}_l^\dagger - \langle ij|\tilde{D}|k|l \rangle \tilde{b}_k^\dagger \tilde{b}_l^\dagger,
\langle ij|F|k|l \rangle = \langle ij|F|k|l \rangle \tilde{a}_k^\dagger \tilde{a}_l^\dagger - \langle ij|\tilde{F}|k|l \rangle \tilde{b}_k^\dagger \tilde{b}_l^\dagger,
\langle ij|\tilde{F}|k|l \rangle = \langle ij|\tilde{F}|k|l \rangle \tilde{a}_k^\dagger \tilde{a}_l^\dagger - \langle ij|\tilde{D}|k|l \rangle \tilde{b}_k^\dagger \tilde{b}_l^\dagger,
\end{bmatrix}
\]
\[
\begin{bmatrix}
\langle ij|D|kl \rangle = -[a_ia^\dagger_j|a_ia^\dagger_j] - \langle ij|D|k|l \rangle \tilde{a}_k^\dagger \tilde{a}_l^\dagger [a_ia^\dagger_j|a_ia^\dagger_j] + \langle ij|F|k|l \rangle \tilde{a}_k^\dagger \tilde{a}_l^\dagger [a_ia^\dagger_j|a^\dagger_ja^\dagger_i] - \langle ij|\tilde{F}|k|l \rangle \tilde{a}_k^\dagger \tilde{a}_l^\dagger [a_ia^\dagger_j|a^\dagger_ja^\dagger_i],
\end{bmatrix}
\]

Under the minimization of the residual curvature, putting \( \tilde{F} = \tilde{F}_c \), we get
\[
\partial_{\Lambda^\ast} \tilde{F}_c = i \begin{bmatrix}
-\langle F \rangle \psi_{\nu-} - \langle F \rangle \varphi_{\nu-} - \langle D \rangle^* \psi_{\nu-} - \langle D \rangle^* \varphi_{\nu-},
\langle D \rangle \psi_{\nu-} + \langle D \rangle \varphi_{\nu-}^*,
\end{bmatrix}
\]

\[
\begin{bmatrix}
[\langle D \rangle \psi_{\nu-} + \langle D \rangle \varphi_{\nu-}^* \langle F \rangle^* \psi_{\nu-} + \langle F \rangle^* \varphi_{\nu-}^* \end{bmatrix}.
\]

Derive a new equation analogous to the \( U(N) \) RPA equation. To do this, we decompose the fluctuating pair mode amplitude \( \tilde{g} \to \tilde{g} \tilde{g}(\tilde{z}, \tilde{z}) \). We finally get the following matrix-valued equations:
\[
\tilde{g}^\dagger(\tilde{z}, \tilde{z}), \psi_{\nu-} + \varphi_{\nu-}^*,
\]
\[
\begin{bmatrix}
ih \partial_t \xi_{\nu-} + \tilde{[\xi_0, \xi_{\nu-}]} + \varphi_{\nu-} + \tilde{[\xi_0, \xi_{\nu-}]} + \psi_{\nu-} - \langle D \rangle^* \psi_{\nu-} - \langle D \rangle^* \varphi_{\nu-},
\end{bmatrix}
\]
\[
\begin{bmatrix}
i \partial_t \varphi_{\nu-} - \tilde{[\xi_0, \varphi_{\nu-}]} - \psi_{\nu-} + \langle D \rangle \varphi_{\nu-}^* + \tilde{[\xi_0, \varphi_{\nu-}]} + \psi_{\nu-} + \langle D \rangle \varphi_{\nu-}^* + \tilde{[\xi_0, \varphi_{\nu-}]} \cdot \tilde{g}(\tilde{z}, \tilde{z}) = 0.
\end{bmatrix}
\]
with the modified new matrices \( \{ \mathbf{F} \} \) etc. defined through
\[
\{ \mathbf{F} \} = \{ \{ ij \} F | kl \} , \quad \{ \mathbf{F} \} = \{ \{ ij \} | T F | kl \} , \quad \{ \mathbf{D} \} = \{ \{ ij \} D | kl \} , \quad \{ \mathbf{D} \} = \{ \{ ij \} | D T F | kl \} ,
\]
whose matrix elements are given by
\[
\begin{align*}
- \{ ij | F | kl \} &= \| b^*_i a_{j'} (i' j') | D | kl \| - \| a^*_i \hat{b}_{j'} (i' j') | \mathbf{D} | kl \|^* \\
\{ ij | \bar{F} | kl \} &= \| \hat{a}_{i'} \hat{b}_{j'} (i' j') | D | kl \|^* - \| \hat{b}_{i'} \hat{a}_{j'} (i' j') | \mathbf{D} | kl \|, \\
\{ ij | D | kl \} &= \| \hat{a}_{i'} \hat{b}_{j'} (i' j') | D | kl \|^* - \| \hat{b}_{i'} \hat{a}_{j'} (i' j') | \mathbf{D} | kl \|, \\
- \{ ij | \bar{D} | kl \} &= \| \hat{b}_{i'} \hat{b}_{j'} (i' j') | D | kl \|^* - \| \hat{a}_{i'} \hat{a}_{j'} (i' j') | \mathbf{D} | kl \|.
\end{align*}
\]
By making the block off-diagonal matrices of the LHS of the equation vanish, we get
\[
\begin{align*}
\text{i} \hbar \partial_t \psi_{\alpha-\nu} &= \{ \hat{\xi}_0, \psi_{\alpha-\nu} \} + \{ \mathbf{D} \} \psi_{\alpha-\nu} - \{ \mathbf{D} \} \varphi_{\alpha-\nu}, \\
\text{i} \hbar \partial_t \varphi_{\alpha-\nu} &= -\{ \hat{\xi}_0, \varphi_{\alpha-\nu} \} + \{ \mathbf{D} \}^* \psi_{\alpha-\nu} + \{ \mathbf{D} \}^* \varphi_{\alpha-\nu}.
\end{align*}
\]
Due to the vanishing of the block diagonal matrices in the LHS of (6.15), we obtain
\[
\text{i} \hbar \partial_t \xi_{\alpha-\nu} = -\{ \hat{\xi}_0, \xi_{\alpha-\nu} \} + \{ \mathbf{F} \} \psi_{\alpha-\nu} + \{ \mathbf{F} \} \varphi_{\alpha-\nu}.
\]
Eqs. (6.19) and (6.20) are very similar to TD equation (6.3) and have matrices \( \{ \mathbf{D} \} \) etc. with \((\Lambda, \Lambda^*, t)\)-dependence. Starting from the lowest solution of \( \psi_{\alpha-\nu}, \varphi_{\alpha-\nu} \) and \( \xi_{\alpha-\nu} \), we proceed to the next leading solution of \( \psi_{\alpha-\nu} \) and \( \varphi_{\alpha-\nu} \). Using these, \( \xi_{\alpha-\nu} \) with the same power is obtained. Then, we determine time dependence of the amplitudes corresponding to each power iteratively. It works well over a wide range of physics beyond the \( SO(2N) \) RPA.

Finally, following Rajeev (65), we show the existence of symplectic 2-form \( \omega \). Using \( \frac{SO(2N)}{U(N)} \) coset variable \( q = ba^{-1} = -q^T \), \( SO(2N) \) (HB) density matrix \( \mathcal{R}(G) \) is expressed as
\[
\mathcal{R}(G) = G \begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix} G^t \begin{bmatrix} 2R(G) - 1_N & -2K^* (G) \\ 2K^* (G) & -2R^* (G) + 1_N \end{bmatrix} = \begin{bmatrix} q^t q (1_N + q^t q)^{-1} \\ -q (1_N + q^t q)^{-1} \end{bmatrix}.
\]
The two-form \( \omega \) is given as
\[
\omega = -\frac{i}{8} \text{Tr} \{ (d \mathcal{R}(G))^3 \} = -\frac{i}{8} \text{Tr} \{ (d \mathcal{R}(G))^3 \mathcal{R}(G)^2 \} , \quad d\omega = -d\omega = 0, \quad \text{(closed form)}.
\]
\[
\omega(U, V) = -\frac{i}{8} \text{Tr} \left[ \begin{bmatrix} 1_N & 0 \\ 0 & -1_N \end{bmatrix} [U, V] \right] = \frac{i}{8} \text{Tr} [u^t v - v^t u], \quad U \equiv \begin{bmatrix} 0 & u \\ u^t & 0 \end{bmatrix} \text{ and } V \equiv \begin{bmatrix} 0 & v \\ v^t & 0 \end{bmatrix},
\]
which is a symplectic form and makes it possible to practice a geometric quantization on the FD- and ID- Grassmannians (66, 67).
7 Geometric equation with $\frac{SU_{m+n}}{S(U_m \times U_n)}$ structure

Here we construct alternative geometric equation, noticing the special structure of the coset space $\frac{SU_{m+n}}{S(U_m \times U_n)}$. Composing linearly of the same kind as the infinitesimal generators (7.5) and (7.6), we adopt the following one-form $\Omega_o$ and the integrability condition:

$$\Omega_o = \lambda^l \mathcal{F}_o \cdot dt + \psi^l \cdot d\Lambda^\nu + \phi^l \cdot d\Lambda^\nu_\ast,$$

$$d\Omega_o = 0.$$  \hspace{1cm} (7.1)

where the expressions for $\mathcal{F}_o$, $\phi^l$ and $\psi^l$ are given as

$$\mathcal{F}_o = \left[ \begin{array}{cc} \mathcal{F}_o^x & \mathcal{F}_o^y \\ \mathcal{F}_o^x & \mathcal{F}_o^y \\ \end{array} \right], \quad \phi^l = \left[ \begin{array}{c} \xi_0 \\ \psi_0 \\ \end{array} \right], \quad \psi^l = \left[ \begin{array}{c} \xi^l \\ \psi^l \\ \end{array} \right]. \hspace{1cm} (7.2)

Under the conditions $\mathcal{F}_o = -\mathcal{F}_o^\dagger$, $\phi^l = -\phi^l$, and $n = N - m$, $\Omega_o$ is divided into as follows:

$$\Omega_o = \lambda^l \mathcal{F}_o \cdot dt + \psi^l \cdot d\Lambda^\nu + \phi^l \cdot d\Lambda^\nu_\ast,$$

$$d\Omega_o = 0.$$  \hspace{1cm} (7.3)

where the $\lambda_{o-m}$ and $\lambda_{o-n}$ are expressed as

$$\lambda_{o-m} = \frac{1}{n} \mathcal{F}_o \cdot dt + \xi_{o-\nu} \cdot d\Lambda_\nu + \xi_{o-\nu} \cdot d\Lambda^\nu_\ast,$$

$$\lambda_{o-n} = \frac{1}{n} \mathcal{F}_o \cdot dt + \zeta_{o-\nu} \cdot d\Lambda_\nu + \zeta_{o-\nu} \cdot d\Lambda^\nu_\ast.$$  \hspace{1cm} (7.4)

and according to Casaliero–Mignan–Sciuto (CMS) [68], we adopt the $\lambda$-modified $\hat{\Omega}_o$ given as

$$\hat{\Omega}_o = \lambda^l \mathcal{F}_o \cdot dt + \lambda^l \psi^l \cdot d\Lambda^\nu + \lambda^l \phi^l \cdot d\Lambda^\nu_\ast.$$  \hspace{1cm} (7.5)

Further, the integrability condition is calculated as

$$d\hat{\Omega}_o = -\partial \hat{\Omega}_o \wedge d\Lambda_\nu - \partial \hat{\Omega}_o \wedge d\Lambda^\nu_\ast - \partial \hat{\Omega}_o \wedge d\Lambda^\nu_\ast,$$

$$= \lambda^l \mathcal{F}_o \cdot dt + \lambda^l \psi^l \cdot d\Lambda^\nu + \lambda^l \phi^l \cdot d\Lambda^\nu_\ast.$$

$$= \lambda^l \mathcal{F}_o \cdot dt + \lambda^l \psi^l \cdot d\Lambda^\nu + \lambda^l \phi^l \cdot d\Lambda^\nu_\ast.$$  \hspace{1cm} (7.6)

in which $d\hat{\Omega}_o$ is given as

$$d\hat{\Omega}_o = -\partial \hat{\Omega}_o \wedge d\Lambda_\nu - \partial \hat{\Omega}_o \wedge d\Lambda^\nu_\ast - \partial \hat{\Omega}_o \wedge d\Lambda^\nu_\ast,$$

$$= \lambda^l \mathcal{F}_o \cdot dt + \lambda^l \psi^l \cdot d\Lambda^\nu + \lambda^l \phi^l \cdot d\Lambda^\nu_\ast.$$  \hspace{1cm} (7.7)

On the other hand, the term to equalize with $d\hat{\Omega}_o$ is computed as

$$d\hat{\Omega}_o \wedge \hat{\Omega}_o \wedge \hat{\Omega}_o,$$

$$= \lambda^l \mathcal{F}_o \cdot dt + \lambda^l \psi^l \cdot d\Lambda^\nu + \lambda^l \phi^l \cdot d\Lambda^\nu_\ast.$$  \hspace{1cm} (7.8)

$$+ \lambda^l \mathcal{F}_o \cdot dt + \lambda^l \psi^l \cdot d\Lambda^\nu + \lambda^l \phi^l \cdot d\Lambda^\nu_\ast.$$
Equating the \( \lambda \) term and the \( \frac{1}{\hbar} \) term in both sides of Eqs. (7.7) and (7.8), we get the relations

\[
\begin{align*}
\text{\( \lambda \)-term:} & \quad \partial_{\lambda_{\nu}} \tilde{F}_{\nu} = \zeta_{o-\nu} \tilde{F}_{\nu} - \tilde{F}_{\nu} \zeta_{o-\nu}, \quad \partial_{\lambda_{\nu}} \tilde{F}_{\nu} = \zeta_{o-\nu} \tilde{F}_{\nu} - \tilde{F}_{\nu} \zeta_{o-\nu}, \\
\text{\( \frac{1}{\hbar} \)-term:} & \quad \hbar \partial_{\psi_{o-\nu}} = \tilde{F}_{\nu} \psi_{o-\nu} - \psi_{o-\nu} \tilde{F}_{\nu}, \quad \hbar \partial_{\psi_{o-\nu}} = \tilde{F}_{\nu} \psi_{o-\nu} - \psi_{o-\nu} \tilde{F}_{\nu}, \\
& \quad \partial_{\lambda_{\nu}} \psi_{o-\nu} = \psi_{o-\nu} \zeta_{o-\nu} + \zeta_{o-\nu} \psi_{o-\nu}, \quad \partial_{\lambda_{\nu}} \psi_{o-\nu} = \psi_{o-\nu} \zeta_{o-\nu} + \zeta_{o-\nu} \psi_{o-\nu}, \\
& \quad \partial_{\lambda_{\nu}} \psi_{o-\nu} = \psi_{o-\nu} \zeta_{o-\nu} + \zeta_{o-\nu} \psi_{o-\nu}. 
\end{align*}
\]

Using \( \varphi_{o-\nu} = -\psi_{o-\nu} \), from Eqs. of second and third lines in (7.10), we acquire \( \psi_{o-\nu} = -\zeta_{o-\nu} \).

In (7.6), we have

\[
d\tilde{\Omega}_{o-\nu} = -\partial_{t_{\tilde{\Omega}_{o-\nu}}} \tilde{\Omega}_{o-\nu} \wedge dt - \partial_{\lambda_{\nu}} \tilde{\Omega}_{o-\nu} \wedge dt - \partial_{\lambda_{\nu}} \tilde{\Omega}_{o-\nu} \\
= -\left( \partial_{\lambda_{\nu}} \frac{1}{\hbar} \tilde{F}_{\nu} \cdot dt + \partial_{\lambda_{\nu}} \zeta_{o-\nu} \cdot dt + \partial_{\lambda_{\nu}} \zeta_{o-\nu} \cdot \frac{1}{\hbar} \tilde{F}_{\nu} \right) \wedge dt \\
- \left( \partial_{\lambda_{\nu}} \frac{1}{\hbar} \tilde{F}_{\nu} \cdot dt + \partial_{\lambda_{\nu}} \zeta_{o-\nu} \cdot \frac{1}{\hbar} \tilde{F}_{\nu} \right) \wedge dt \\
- \left( \partial_{\lambda_{\nu}} \frac{1}{\hbar} \tilde{F}_{\nu} \cdot dt + \partial_{\lambda_{\nu}} \zeta_{o-\nu} \cdot \frac{1}{\hbar} \tilde{F}_{\nu} \right) \wedge dt. 
\]

(7.11)

The terms to equalize with \( d\tilde{\Omega}_{o-\nu} \) are computed as

\[
d\tilde{\Omega}_{o-\nu} = \frac{\lambda_{\nu}}{\hbar} \tilde{F}_{\nu} \cdot dt + \frac{1}{\hbar} \tilde{F}_{\nu} \cdot dt + \zeta_{o-\nu} \cdot \frac{1}{\hbar} \tilde{F}_{\nu} \cdot dt + \frac{1}{\hbar} \tilde{F}_{\nu} \cdot dt + \frac{1}{\hbar} \tilde{F}_{\nu} \cdot dt \\
= -\left( \frac{1}{\hbar} \tilde{F}_{\nu} \psi_{o-\nu} \cdot dt - \psi_{o-\nu} \frac{1}{\hbar} \tilde{F}_{\nu} \right) \wedge dt \quad \frac{1}{\hbar} \tilde{F}_{\nu} \psi_{o-\nu} \cdot dt - \psi_{o-\nu} \frac{1}{\hbar} \tilde{F}_{\nu} \right) \wedge dt \\
+ \left( \frac{1}{\hbar} \tilde{F}_{\nu} \psi_{o-\nu} \cdot dt - \psi_{o-\nu} \frac{1}{\hbar} \tilde{F}_{\nu} \right) \wedge dt \\
+ \left( \frac{1}{\hbar} \tilde{F}_{\nu} \psi_{o-\nu} \cdot dt - \psi_{o-\nu} \frac{1}{\hbar} \tilde{F}_{\nu} \right) \wedge dt. 
\]

(7.12)

Here we should discard the \( \frac{1}{\hbar} \) term. Equating both sides of (7.11) and (7.12), we get

\[
\begin{align*}
\text{\( \lambda \)-term:} & \quad \partial_{\lambda_{\nu}} \tilde{F}_{\nu} = \tilde{F}_{\nu} \psi_{o-\nu} - \psi_{o-\nu} \tilde{F}_{\nu}, \quad \partial_{\lambda_{\nu}} \tilde{F}_{\nu} = \tilde{F}_{\nu} \psi_{o-\nu} - \psi_{o-\nu} \tilde{F}_{\nu}, \\
\text{\( \frac{1}{\hbar} \)-term:} & \quad \hbar \partial_{\psi_{o-\nu}} = \tilde{F}_{\nu} \psi_{o-\nu} - \psi_{o-\nu} \tilde{F}_{\nu}, \quad \hbar \partial_{\psi_{o-\nu}} = \tilde{F}_{\nu} \psi_{o-\nu} - \psi_{o-\nu} \tilde{F}_{\nu}, \\
& \quad \partial_{\lambda_{\nu}} \psi_{o-\nu} = \zeta_{o-\nu} \cdot \nu_{o-\nu}, \quad \partial_{\lambda_{\nu}} \psi_{o-\nu} = \zeta_{o-\nu} \cdot \nu_{o-\nu}, \quad \partial_{\lambda_{\nu}} \psi_{o-\nu} = \zeta_{o-\nu} \cdot \nu_{o-\nu}, \\
& \quad \partial_{\lambda_{\nu}} \psi_{o-\nu} = \zeta_{o-\nu} \cdot \nu_{o-\nu}. 
\end{align*}
\]

(7.13)

Following CMS [68], under \( h_{m}, k_{m} \in U_{m} \) and \( h_{n}, k_{n} \in U_{n} \), we assume

\[
\tilde{F}_{o} = h_{n} \tilde{F}_{0} h_{m}, \quad \tilde{F}_{o} = \begin{bmatrix} \cdots & \tilde{F}_{0} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}_{m \times n}, \quad \psi_{o-\nu} = \nu_{o-\nu} \psi_{o-\nu} \psi_{o-\nu}, \\
(\tilde{F}_{o})_{ij} = \delta_{ij} \tilde{F}_{0,ij}, \quad \partial_{\lambda_{\nu} \psi_{o-\nu}} h_{n} \tilde{F}_{0} h_{m} = 0, \quad (\psi_{o-\nu})_{ij} = \delta_{ij} \tilde{F}_{0,ij}, \quad \hbar \partial_{\psi_{o-\nu}} h_{n} \tilde{F}_{0} h_{m} = 0. 
\]

(15)

As also suggested by CMS [68], Eqs. (7.9) and (7.10) imply

\[
\partial_{\lambda_{\nu}} (\psi_{o-\nu} \psi_{o-\nu})^{T} = 0, \quad \psi_{o-\nu} \psi_{o-\nu} = c_{\psi \psi_{o-\nu}}, \quad \hbar \partial_{\psi_{o-\nu}} (\psi_{o-\nu} \psi_{o-\nu})^{T} = 0, \quad \psi_{o-\nu} \psi_{o-\nu} = c_{\psi \psi_{o-\nu}}. 
\]

(16)

which is proved as
Further putting we have reasonably put We should discard the \( U(7.13) \) and \( U(7.14) \), under the transformation

\[
0 = \partial_{\Lambda^*} \text{Tr} \left( \hat{\mathcal{F}}^\dagger_o \hat{\mathcal{F}}_o \right)^T = \text{Tr} \left( \partial_{\Lambda^*} \hat{\mathcal{F}}^T_o \hat{\mathcal{F}}^{*\dagger}_o + \hat{\mathcal{F}}^T_o \partial_{\Lambda^*} \hat{\mathcal{F}}^*_o \right)
= \text{Tr} \left( \left( \hat{\mathcal{F}}^T_o \zeta_{\nu o} \xi^{\dagger}_{\nu o} - \xi^{\dagger}_{\nu o} \hat{\mathcal{F}}^T_o \right) \hat{\mathcal{F}}^{*\dagger}_o \right) = \text{Tr} \left( \left( \hat{\mathcal{F}}_o \hat{\mathcal{F}}^T_o - \hat{\mathcal{F}}_o \hat{\mathcal{F}}^*_o \right) \left( \zeta_{\nu o} \xi^{\dagger}_{\nu o} \right) \right)
\]

\[
0 = \hbar \partial_{\nu \xi} \text{Tr} \left( \psi^{\dagger}_{\nu o} \psi_{\nu o} \right)^T = \text{Tr} \left( \hbar \partial_{\xi} \psi^{\dagger}_{\nu o} \psi_{\nu o} + \psi^{\dagger}_{\nu o} \hbar \partial_{\xi} \psi_{\nu o} \right)
= \text{Tr} \left( \psi^{\dagger}_{\nu o} \left( \hat{\mathcal{F}}^*_o \psi_{\nu o} \right) - \left( \hat{\mathcal{F}}^T_o \psi_{\nu o} \right) \psi^{\dagger}_{\nu o} \right)
= \text{Tr} \left( \psi^{\dagger}_{\nu o} \psi_{\nu o} \left( \hat{\mathcal{F}}_o + \hat{\mathcal{F}}^*_o \right) \right) - \text{Tr} \left( \psi^{\dagger}_{\nu o} \psi_{\nu o} \left( \hat{\mathcal{F}}_o + \hat{\mathcal{F}}^*_o \right) \right) \quad \text{(Due to Tr(\hat{\mathcal{F}}_o + \hat{\mathcal{F}}^*_o) = 0)}
\]

Further putting \( \zeta_{\nu o} = 0 \) and substituting it into \( (7.9) \) and \( (7.10) \), we have the relations

\[
\partial_{\Lambda^*} \hat{\mathcal{F}}_o = -\hat{\mathcal{F}}_o \xi_{\nu o}, \quad \text{i.e.,} \quad \xi_{\nu o} = -\hbar m^2 \partial_{\Lambda^*} \hat{\mathcal{F}}_o = \partial_{\Lambda^*} \psi_{\nu o} \xi_{\nu o}.
\] (7.18)

Lastly, in \( (7.6) \), we have

\[
d\hat{\Omega}_{o-m} = -\partial_{\xi} \hat{\Omega}_{o-m} \delta t + \partial_{\xi} \hat{\Omega}_{o-m} \delta \Lambda_{\nu} - \partial_{\Lambda^*} \hat{\Omega}_{o-m} \delta \Lambda_{\nu}^* = -\left( \partial_{\xi} \hat{\mathcal{F}}_o \right) \delta t + \partial_{\xi} \hat{\mathcal{F}}_o \delta \Lambda_{\nu} - \partial_{\Lambda^*} \hat{\mathcal{F}}_o \delta \Lambda_{\nu}^* \quad \text{(7.19)}
\]

While, the term to equalize with \( d\hat{\Omega}_{o-m} \) is computed as

\[
\hat{\Omega}_{o-m} \sim \hat{\Omega}_{o-m} - \hat{\mathcal{F}}_o \delta t + \xi_{\nu o} \delta \Lambda_{\nu} + \xi^*_{\nu o} \delta \Lambda_{\nu}^* - \hat{\mathcal{F}}_o \delta \Lambda_{\nu}^* + \partial_{\xi} \hat{\mathcal{F}}_o \delta \Lambda_{\nu} + \partial_{\Lambda^*} \hat{\mathcal{F}}_o \delta \Lambda_{\nu}^* \quad \text{(7.20)}
\]

We should discard the \( \frac{1}{\hbar} \) term. Equating both sides of \( (7.19) \) and \( (7.20) \), we get

\[
\partial_{\Lambda^*} \hat{\mathcal{F}}_o - \hbar \partial_{\Lambda^*} \xi_{\nu o} = \left( \xi_{\nu o} \hat{\mathcal{F}}_o + \hat{\mathcal{F}}^*_o \psi_{\nu o} - \psi^{\dagger}_{\nu o} \hat{\mathcal{F}}_o \right)
\] (7.21)

\[
\partial_{\Lambda^*} \xi_{\nu o} = \xi_{\nu o} \delta t + \xi^{\dagger}_{\nu o} \delta \Lambda_{\nu}^* - \xi^{\dagger}_{\nu o} \delta \Lambda_{\nu}^* - \partial_{\Lambda^*} \xi_{\nu o} - \hat{\mathcal{F}}_o \psi_{\nu o} \psi_{\nu o} \hat{\mathcal{F}}_o \quad \text{(7.22)}
\]

By exploiting the invariance of the integrability condition \( (7.6) \), especially by noticing Eqs. \( (7.13) \) and \( (7.14) \), under the transformation \( U = U(\Lambda, \Lambda^*, t) \in S(U_m \times U_n) \),

\[
\begin{bmatrix}
\hat{\Omega}_{o-m} & 0 \\
0 & \hat{\Omega}_{o-n}
\end{bmatrix} \Rightarrow U^{-1} \left[
\begin{bmatrix}
\hat{\Omega}_{o-m} & 0 \\
0 & \hat{\Omega}_{o-n}
\end{bmatrix}
\right] U - U dU, \quad \hat{\Omega}_{o} \Rightarrow U^{-1} \hat{\Omega}_{o} U,
\]

we have reasonably put

\[
\hat{\mathcal{F}}_o = \hat{\mathcal{F}}_0 h_m, \quad \xi_{\nu o} = 0, \quad \psi_{\nu o} = k_n \psi^{0}_{\nu o}.
\] (7.24)

According to CMS \[\text{CMS}\], we take
Taking the idea of construction of $(1+1)$D model with $SU(\alpha)$ and one can determine $T$ which to the Lipkin-Meshkov-Glick model\cite{69} will be published elsewhere. The present particle-extended the model to the $(1+2)$ where $\partial T$ With the aid of the relation (7.13) and the expression for $\hat{\psi}^\dagger$ from which, thus we have

\[\begin{bmatrix} S & -T^\dagger \\ T & 0 \end{bmatrix} \begin{bmatrix} \psi_{o-\nu,1} \\ \psi_{o-\nu,2} \end{bmatrix} = \begin{bmatrix} S & -T^\dagger \\ T & 0 \end{bmatrix} \begin{bmatrix} S \psi_{o-\nu,1} - T^\dagger \psi_{o-\nu,2} - i\alpha \psi_{o-\nu,1} \\ T \psi_{o-\nu,1} - i\alpha \psi_{o-\nu,2} \end{bmatrix}, \]

(7.26)

from which we have the explicit expressions for $T$ and $S$ as

\[T = (\hbar \partial T + i\alpha) \psi_{o-\nu,1}\psi_{o-\nu,1}^{-1}, \]

\[S = \{ (\hbar \partial T + i\alpha) \psi_{o-\nu,1} + T \psi_{o-\nu,2} \} \psi_{o-\nu,1}^{-1} \]

\[= (\psi_{o-\nu,1})^{-1}(i\alpha \psi_{o-\nu,1} - \psi_{o-\nu,2}\psi_{o-\nu,2}^{-1}) \psi_{o-\nu,1} + \psi_{o-\nu,1} \hbar \partial T \psi_{o-\nu,1} + \hbar \partial T \psi_{o-\nu,2} \psi_{o-\nu,2}^{-1} \psi_{o-\nu,1}. \]

(7.27)

The unknown field $\alpha$ can be determined through Eq. (7.27) by imposing the condition $\text{Tr} \Omega_o = \text{Tr} (\Omega_{o-m} + \Omega_{o-n}) = 0$ given in (7.3) which reads

\[\begin{align*}
\text{Tr} \Omega_o &= \text{Tr} (\Omega_{o-m} + \Omega_{o-n}) = 0, \\
\text{Tr} S &= \left( \alpha \left\{ \text{Tr} m - (\psi_i^{\dagger}, \psi_{o-\nu,1} - \psi_i^{\dagger}) \psi_{o-\nu,2}^{-1} \psi_{o-\nu,2} + (\psi_{o-\nu,1} \psi_{o-\nu,2} - (\psi_i^{\dagger}, \psi_{o-\nu,1} \psi_{o-\nu,2}) \hbar \partial T \psi_{o-\nu,1} + \hbar \partial T \psi_{o-\nu,2} \psi_{o-\nu,2}^{-1} \psi_{o-\nu,1}. \right\}
\end{align*} \]

(7.28)

and one can determine $\alpha$ as

\[\alpha = i \left( \frac{\text{Tr} (\psi_i^{\dagger}, \psi_{o-\nu,1} \hbar \partial T \psi_{o-\nu,1} - \text{Tr} (\psi_i^{\dagger}, \psi_{o-\nu,1} \hbar \partial T \psi_{o-\nu,2} \psi_{o-\nu,2}^{-1} \psi_{o-\nu,2})}{2m - \text{Tr} \left[ (\psi_{o-\nu,1} \psi_{o-\nu,2} - (\psi_i^{\dagger}, \psi_{o-\nu,1} \psi_{o-\nu,2}) \hbar \partial T \psi_{o-\nu,1} + \hbar \partial T \psi_{o-\nu,2} \psi_{o-\nu,2}^{-1} \psi_{o-\nu,1}. \right]}. \right) \]

(7.29)

Using $\xi_{o-\nu} = -\hbar m \Omega_{o}, h_m, \tilde{F}_o = \tilde{F}_0 h_m = \tilde{F}_{0,1}, \psi_{o-\nu} = k_n \psi_{o-\nu}$ and $\tilde{F}_o = i\alpha \cdot \mathbb{I}$, (7.27) is changed as

\[i\alpha \cdot \mathbb{I} + \hbar \partial (h_n^{-1} \partial \nu \cdot h_n) = h_n \tilde{F}_{0,1} \psi_{o-\nu} - \psi_{o-\nu} \cdot h_n \tilde{F}_0 h_m = \tilde{F}_{0,1} \psi_{o-\nu,1} - \psi_{o-\nu,1} \tilde{F}_{0,1}. \]

(7.30)

With the aid of the relation (7.13) and the expression for $\tilde{F}_o$, (7.25), we obtain the differential operation $\partial \nu \cdot F_o$ in the following forms:

\[\begin{bmatrix} \partial \nu \cdot S - \partial \nu \cdot T^\dagger \\ \partial \nu \cdot T \end{bmatrix} \begin{bmatrix} \psi_{o-\nu,1} \psi_{o-\nu,2} \end{bmatrix} = \begin{bmatrix} \tilde{F}_{0,1} \psi_{o-\nu,1} - \psi_{o-\nu,1} \tilde{F}_{0,1}^\dagger \\ \psi_{o-\nu,2} - \psi_{o-\nu,2} \tilde{F}_{0,1}^\dagger \\ \psi_{o-\nu,1} \tilde{F}_{0,1}^\dagger - \psi_{o-\nu,1} \tilde{F}_{0,1}^\dagger \\ \psi_{o-\nu,2} - \psi_{o-\nu,2} \tilde{F}_{0,1}^\dagger \end{bmatrix}, \]

(7.31)

from which, thus we have

\[\partial \nu \cdot S = \tilde{F}_{0,1} \psi_{o-\nu,1} - \psi_{o-\nu,1} \tilde{F}_{0,1}^\dagger, \quad \partial \nu \cdot T = -\psi_{o-\nu,2} \tilde{F}_{0,1}^\dagger. \]

(7.32)

Taking the idea of construction of $(1+1)$D model with $SU(m,n)$ structure by CMS \cite{68}, we have extended the model to the $(1+2)$D one. We have successfully derived the geometric equation with $SU(m,n)$ structure to get the classical equation of motion for integrable system an application of which to the Lipkin-Meshkov-Glick model \cite{69} will be published elsewhere. The present particle-hole formalism for the geometric equation is also applicable to the superconducting formalism on the coset space $SO(2N)_{U(N)}$ for the paired mode \cite{52} and on the coset space its extension $SO(2N+2)_{U(N+1)}$ for both the paired and unpaired modes \cite{9, 11, 50}. This work will be possible in the near future.
8 Bilinear differential equation in SCF method

The fermion operators \(E_{\alpha \beta}^\ast = c_{\beta}^\dagger c_{\alpha}\) span the \(U(N)\) Lie algebra \([E_{\alpha \beta}^\ast , E_{\gamma \delta}^\ast ] = \delta_{\alpha \gamma}E_{\beta \delta}^\ast - \delta_{\alpha \delta}E_{\beta \gamma}^\ast\) and generate a canonical transformation \(U(g)(= e^{\gamma a_{\sigma}^\dagger c_{\sigma}}; \gamma^\ast = -\gamma)\) specified by a \(U(N)\) matrix \(g(=e^{\gamma})\) as

\[U(g)c_{\alpha}^\dagger U^{-1}(g) = c_{\beta}g_{\alpha \beta}, \quad g^g = g, \quad g^g = 1_N \quad \text{and} \quad c_{\alpha}(0) = 0. \quad (8.1)\]

An \(m\) particle S-det \(|\phi_m\rangle = c_{\dagger}^m \cdots c_{\dagger}^1|0\rangle\) is an exterior product of \(m\) single-particle state (simple state). Due to the Thouless theorem [13], different S-det is produced as

\[U(g)|\phi_m\rangle = (c_{\dagger}^g)^m \cdots (c_{\dagger}^g)|0\rangle \quad \text{and} \quad U(g)|0\rangle = |0\rangle. \quad (8.2)\]

The set of all the simple states of unit modulus together with the equivalence relation identifying distinct sets only in phase with the same state, constitute a manifold \(\text{Gr}_m\) (group orbit). Any simple state \(|\phi_m\rangle\) defines a decomposition of single-particle Hilbert spaces into sub-Hilbert spaces of occupied and unoccupied states [70]. Thus the \(\text{Gr}_m\) corresponds to a coset space

\[\text{Gr}_m \sim \frac{U(m+)}{(m!)^2 U(m)}, \quad N = m+n. \quad (8.3)\]

Following Fukutome [11, 42], for the coset variable \(p\) defined later, we express it as \(U(g)|\phi_m\rangle = \langle \phi_m|U(g_k g_0)|\phi_m\rangle e^{p \alpha a_{\sigma} c_{\sigma}}\), using the relations \(g = g_k g_0\), \(\langle \phi_m|U(g_k g_0)|\phi_m\rangle = [\det(1 + p^h g)\] \(p^v u \quad \text{and} \quad \sum_{p=0}^{\text{Mmax}} \sum_{1 \leq a_1 \cdots a_n \leq N} \{ \mathcal{A}(p_{1a_1} \cdots p_{aa_p}) \} \text{det} v \quad \text{and} \quad \sum_{p=0}^{\text{Mmax}} \sum_{1 \leq a_1 \cdots a_n \leq N} \{ \mathcal{A}(p_{1a_1} \cdots p_{aa_p}) \} \} \) specified by a coset variable \(p\).

We give the Plücker coordinate studied in the textbook of Miller and Sturmfels [71], which has played important roles of algebraic construction of soliton theory by Sato [72].

\[U(g)|\phi_m\rangle = \sum_{n \geq a_m} \cdot \cdot \cdot a_1 \cdot v_{a_{m-1}, a_1} c_{a_1} \cdots c_{a_1} |0\rangle, \quad v_{a_{m-1}, a_1} = \text{det} \begin{bmatrix} g_{a_{1,1}} & \cdots & g_{a_{1,m}} \\ \vdots & \vdots & \vdots \\ g_{a_{m-1,1}} & \cdots & g_{a_{m,m}} \end{bmatrix}. \quad (8.6)\]

We easily find that the Plücker coordinate \(v_{a_{m-1}, a_1}\) has a relation

\[\sum_{i=1}^{m} (-1)^i v_{a_{m-1}, \cdot \cdot \cdot, a_i} \cdot v_{a_{m-1}, \cdot \cdot \cdot, a_1} = \text{Plücker relation}. \quad (8.7)\]

The indices denote the distinct sets \(1 \leq a_1 \cdots a_{m-1} \leq N\) and \(1 \leq \beta_1 \cdots \beta_{m+1} \leq N\). The Plücker relation is equivalent to a bilinear identity equation

\[\sum_{n=1}^{N} c_{n}^i U(g)|\phi_m\rangle \otimes c_{a} U(g)|\phi_m\rangle = \sum_{n=1}^{N} U(g)|c_{n}^i \phi_m\rangle \otimes U(g)c_{a} |\phi_m\rangle = 0, \quad (8.8)\]

and analogous to the Hirota form. Following the regular rep of \(SO(2N+1)\) group by Fukutome [73] and introducing a phase variable \(\tau = i \ln \text{det} v\), it is shown that the Lie commutation relation is satisfied by the differential operators for particle-hole pairs given below

\[e_{ia}^* = \frac{\partial}{\partial p_j a_{j}} - \frac{\partial}{\partial p_{j a_{j}}}, \quad e_{ia} = \frac{\partial}{\partial p_{j a_{j}}} + \frac{\partial}{\partial p_j a_{j}} - \frac{\partial}{\partial p_{j a_{j}}}, \quad (8.9)\]

which obey

\[e_{ia}^* \Phi_{m,m}(p,p^s,\tau) = p_{ia} \Phi_{m,m}(p,p^s,\tau), \quad e_{ia} \Phi_{m,m}(p,p^s,\tau) = 0, \quad (8.10)\]

\[e_{ij}^* \Phi_{m,m}(p,p^s,\tau) = 0, \quad e_{ij} \Phi_{m,m}(p,p^s,\tau) = \delta_{\alpha\beta} \Phi_{m,m}(p,p^s,\tau), \quad (8.10)\]

and \([e_{ia}^*, p_{jb}] = -p_{ia} p_{ja}\). The vacuum function \(\Phi_{m,m}(p,p^s,\tau)\) is defined as

\[\Phi_{m,m}(p,p^s,\tau) = v_{m-1}^\ast (g g_0) = \left[ \det(1 + p^h g) \right]^{-\frac{1}{2}} e^{i\tau}. \quad (8.11)\]

The present framework becomes the dual of regular rep by Fukutome. In both the SCF theory and soliton theory on a group, we find common features that \(\text{Gr}_m\) is identical with solution space of bilinear differential equation. The solution space of differential equation becomes an integral surface, subspace in the \(\text{Gr}_m\) on which the residual coset variables are remained to be constant.
9 Summary and further perspective

By KN we have clarified the relation of concept of particle-hole and collective motion in the TDHF to that of soliton theory from the loop group. The subgroup orbits made of loop-group paths exist innumerably in $\mathbf{Gr}_m$. To develop a perturbative method with the use of the collective variables, using the ID fermion, we have aimed at constructing the SCF theory on affine KM algebra along the soliton theory. The ID fermion operator has been introduced through the Laurent expansion of finite-dimensional (FD) fermion operators with respect to the degrees of freedom of fermions related to the mean-field (MF) potential. $\mathbf{Gr}_m$ is identified with $\mathbf{Gr}_\infty$ which is affiliated with the manifold obtained by reduction of $\mathfrak{gl}_\infty$ to $\mathfrak{su}_N$. The extraction of the subgroup orbits from $\mathbf{Gr}_m$ is equivalent to the construction of differential equation (Hirota equation) for the $\mathfrak{su}_N$ reduced KP hierarchy. The SCF theory on $F_\infty$ results in the gauge theory of fermion. The collective motion due to quantal fluctuations of the SC MF potential is attributed to the motion of the gauge of fermion. A common factor explains interference among fermions.

The concept of particle-hole and collective motions is regarded as the compatible condition for particle-hole and collective modes. The SCF theory on $F_\infty$ gives us a new algebraic method for the understanding of the fermions. Prescribing the fermion to form a pair by absorbing the change of gauge, the SCF Hamiltonian made of only $H_{F_\infty, HF}$ is induced. Through the compatible condition for particle and collective modes and a special choice of the Laurent expansion, the fermion gauge arises. We have the expressions for pair operators of the ID fermions in terms of the Laurent spectral numbers. Though TDHF theory on $F_\infty$ describes the dynamics on the real fermion-harmonic oscillators, soliton theory does on the complex fermion-harmonic oscillators. This gives a problem on the relation of the present theory to the resonating (Res) MF one and a task how to construct the ID boson variable from the group parameter.

We start with $\mathfrak{su}_N$ algebra consisting of particle-hole (p-h) component and $\mathfrak{u}_N$ algebra including particle-particle (p-p) and hole-hole (h-h) ones for the state $|\phi\rangle$. Then we have $U(N)$ group orbit. We must notice the equivalence relation which identifies the states different from each other in phases dependent only on diagonal components in h-h types, with the same state. Under the equivalence relation, we ought to treat $SU(N)$ group orbit. However the HF Hamiltonian has value on $\mathfrak{u}_N$ but not on $\mathfrak{su}_N$. To describe the dynamics on $SU(N)$ group orbit, we must remove extra components not satisfying $\mathfrak{su}_N$ from the fully parametrized HF Hamiltonian. With the help of the equivalence relation we ought to take diagonal components in p-p and h-h types into account to assign them. These quantities can play a role of fermion gauge phase as the canonical transformation describes. The gauge-phase is separated into a term which comes from single pair of fermions and a term coming from the particle-number operator, on the Lie algebra through which the fermion pairs are governed. The former relates to the particle mode and the latter to the collective one. Removing the above superfluous components of the HF Hamiltonian, assignment of them to both the modes turns out to bring the concept of particle-hole and collective motions. The usual TDHF theory has not a complete scheme to treat separately both the motions. The TDHF equation on $S^1$, however, has such a scheme. It provides not only manifest and algebraic understanding of the motions but also a scheme to describe a large amplitude collective motion. As for the particle-hole and collective motions, assuming a time-periodicity of motions, we can derive a new unified equation for both the motions beyond the HF and RPA ones. The new TDHF theory on $S^1$ is constructed on a collection of various subgroup orbits consisting of loop paths in the $\mathbf{Gr}_m$ of the FD fermion Fock space and is shown to be built on the ID $\mathbf{Gr}$ of the ID fermion Fock space $F_\infty$. If we choose a broken-symmetry vacuum on $S^1$, the vacuum state is able to deform by through the shift operators associated with collective modes so that $\tau$-function in soliton theory is also deformed.
We find an algebraic mechanism for appearance of the collective motion induced by the TD MF potential. This gives an algebraic understanding of physical concepts of the symmetry breaking and the successive occurrence of the collective motion due to the recovery of the symmetry.

A multi-circle TDHF theory is exciting but has a problem how the Plücker relation on multi-circle is constructed. It is related to a multi-soliton theory \[82, 83\]. The ID algebraic approaches have been proposed using the Bethe ansatz (BA) WF \[84\], Lipkin-Meshkov-Glick (LMG)-model \[85, 69\] and pairing theory \[86, 87\]. Rowe has showed that a number-projected WF satisfies some recursion relations and expressed it in a determinantal form \[86\] with completely anti-symmetric Schur function in the theory of group character by Littlewood \[88\] and MacDonald \[89\]. The WF is described by an ID Lie algebra \[90, 91\]. The algebra is constructed by the power series expansion of the FD Lie algebra with respect to the parameters involved in Hamiltonian. See \[90\](b) which has given an ID affine Lie algebra \(\hat{su}(2)\) and an exactly solvable pairing Hamiltonian. It has also shown the conditions for solving the eigenvalue problem using the BA method. It is a very exciting problem to compare the ID affine Lie algebra \(\hat{su}(2)\) with the ID affine Lie algebra \(\hat{sl}(2,\mathbb{C})\) of p-h LMG-model. They have the \(\hat{su}(2)\) and \(\hat{sl}(2,\mathbb{C})\) invariant Casimir operators which provide almost the pairing and the LMG Hamiltonians. This will be solved and presented elsewhere in the near future. Further one has possible generalization and initial value problem for solution of Sine-Gordon equation, Mansfield \[92\]. This suggests an infinitesimal form of the corresponding nonlinear algebra generated by the building blocks of the BA WF \[93\]. We obtain a collective sub-manifold decided by the SCF Hamiltonian of LMG-model. Then we acquire a clue to build the relation between the methods using Gaudin model \[94, 93\], Oritzu exactly-solvable model \[95\]. Oritzu exactly-solvable model \[95\] and further using the minimal matrix product state and the geminal WF \[96\]. Finally we give some remarks: Daboul has shown that the dynamical symmetry of hydrogen atom leads in a natural way to the ID algebra, twisted affine KM algebras of \(\hat{so}(4)\) and \(\hat{so}(3,1)\) \[97\] based on a suggestive paper by Goddard and Oliv \[98\].

We have studied the curvature equation along the idea of Lax unfamiliar to the conventional treatments of fermion system and adopted the geometric equation to truncate the collective sub-manifolds out of the TDHF manifold. We show below the following subjects to be studied:

(I) The expectation values of the zero curvatures for the state function become the set of equations of motion in the analytical mechanics, imposing the weak orthogonal conditions among the infinitesimal generators, i.e., the equation for the tangent vector fields on the group sub-manifold consisting of the collective variables. The expectation values of the non-zero curvatures become the gradients of potential, which arises from the existence of the residual Hamiltonian, along the collective variables. These quantities can be expected to give a criterion how the collective sub-manifolds is effectively truncated, if we might understand such the treatments.

(II) The expression for the zero curvature conditions is nothing but the formal RPA equation imposed by the weak orthogonal conditions. The formal RPA equation has a simple geometrical interpretation: relative vector fields made of the SCF Hamiltonian around each point on an integral curve also constitute the solutions for the formal RPA equation around the same point which is in turn a fixed point. It means that the formal RPA equation is a natural extension of the usual RPA equation for small-amplitude quantal fluctuations around the ground state to that at any point on the collective sub-manifold which should be studied. Moreover, the enveloping curve, made of the solutions of the formal RPA at each point on an integral curve, becomes another integral curve. The integrability condition is just the infinitesimal condition to transfer a solution to another one for the evolution equation under consideration. Then the usual treatment of RPA equation for small amplitude around the ground state becomes nothing but a method of determining an infinitesimal transformation of symmetry under the assump-
tion that the fluctuating fields are composed of only normal-modes. See Klein-Walet-Dang [99]. At the beginning, their descriptions of dynamical fermion systems in both the methods had looked very different manners at first glance. In the abstract fermion Fock spaces, each solution space in both the methods belongs to the corresponding $\text{Gr}$. There is a difference of the FD and ID fermion systems. Overcoming the difference, we have aimed at clearing a close connection between the concepts of MF potential and gauge of fermion making a role of the loop group.

(1) The Plücker relations on the coset variables becomes analogous to the Hirota’s bilinear form [46]. The SCF method has been devoted to a construction of boson-coordinate systems rather than soliton solution by the $\tau$-FM. Both the methods are equivalent to each other, if we stand on the viewpoint of the Plücker relation or the bilinear identity equation defining the $\text{Gr}$.

(2) The ID fermion operators are introduced through the Laurent expansion of the FD fermions with respect to the degrees of freedom of fermion related to the MF potential. Inversely, the collectivity of the MF potential is attributed to the gauge of interacting ID fermion. The construction of the fermion operator is contained in that of Clifford algebra. This fact permits us to introduce the affine $\text{KM}$ algebra. It means that the usual perturbative method in terms of the collective variables with time periodicity has implicitly stood on the reduction to $su_N$. The TDHF theory becomes the gauge theory of fermion and the collective motion appears as the motion of fermion gauge with a common factor. The physical concept of particle-hole and vacuum in the SCF method dependent on $S^1$ connects to the Plücker relation. The algebraic treatment of extracting sub-group orbit consisting of loop paths out of $\text{Gr}_m$ is just the formation of the Hirota equation for the $su_N$ reduced KP hierarchy in the soliton theory. The present framework gives a manifest structure of the gauge theory of fermion inherent in the SCF method and provides a new algebraic tool for the microscopic understandings of the fermion systems.

(3) Through the investigation of physical meanings for the ID shift operators and the conditions of reduction to $sl_N$ from the loop group viewpoint [100], it is induced that there is the close connection between the collective variables and the spectral parameter in soliton theory and that the algebraic mechanism bringing the physical concept of particle and collective motions arises from the reduction from $u_N$ to $su_N$ for the ID HF Hamiltonian.

(4) Though the TDHF theory describes a dynamics on the real fermion-harmonic oscillator, the soliton theory does on the complex fermion-harmonic oscillator. This remark gives important tasks to extend the TDHF theory on the real space affine $\hat{su}_N$ to the complex space affine $\hat{sl}_N$ and to understand the concept of particle energy and boson energy. They are itemized as follows:

(i) The algebraic mechanism can be derived from the three important elements: first, $su_N$ condition for HF Hamiltonian; second, the vacuum state according to the idea of Dirac; and last, the phase of fermion gauge can be separated into the particle and collective modes.

(ii) From this mechanism, a new theory for unified description of both the motions can be presented. Introducing values with a time periodicity, a unified theory of both the motions beyond the static HF equation and RPA one can be obtained. With the help of the affine $\text{KM}$ algebra, the theory provides the algebraic mechanism to elucidate the physical concepts clearly: collective motion induced by a TD MF potential and the symmetry breaking of fermion systems and the successive occurrence of the collective motion due to the recovery of the symmetry in which circle $S^1$ takes an active role in causing the resonance (interference) between fermions.

(iii) The theory gives a toy model to clear algebraic structure among the original fermion field, vacuum field defined in the SCF potential and bosonic field associated with Laurent spectra.

(iv) To solve the new equation, we must study how to extract the various subgroup orbits satisfying the Plücker relation and how to determine the solution in the $su_N \subset sl_N$ hierarchy.

We attempt to construct an optimal coordinate system on group manifold. Then the relation of boson expansion method for FD fermion system to $\tau$-FM for ID one is investigated to clarify
the algebro-geometric structure of integrable system. Some physical concepts and mathematical methods work well in the ID system. The SCF method based on the global symmetry is much improved, if we notice a local symmetry of the systems. The generator coordinate (GC) method gives the superposition principle on nonlinear space. Standing on the local symmetry of the system behind the global symmetry in the FD system, we have reconstructed the GC and the nonlinear superposition methods. We have many problems related to further perspective. They are itemized as follows:

1. To study path integral method (PIM) formal RPA from the viewpoint of symmetry:
   In PIM, the RPA equation is described as the fluctuational mode with time periodicity of the Jacobi field around a classical path on the phase space. Then we intend to illustrate the relation between both the methods from the viewpoint of symmetry.

2. To extend the particle-hole formalism for \((1+2\mu)\)-dimensional \(SU_m+SU_n\) model, Caseller–Megna-Sciuto to a superconducting paired formalism on the \(SO(2N)\) coset space:
   We have successfully derived the \(SU_m+SU_n\) geometric equation to get the classical equation of motion for an integrable system. The particle-hole formalism is applicable to the superconducting paired formalism on the \(SO(2N)\) and \(SO(2N+2)\) coset spaces. We will study the relation of the present model with the \(\sigma\)-model on the Gr.

3. To study the hiddenness behind the gauge of state-function and formation of fermion pair:
   For forming a pair we give classifications of Laurent spectra in the ID fermion. The Laurent coefficient of \(n\)-soliton solution and \(\tau\)-function for affine \(\hat{su}(N)\) have been given in.

4. To clarify the relation between the spectral parameter and the collective variables:
   A spectral parameter of the iso-spectral equation in soliton theory and collective variable in the SCF method, though being seen different aspect, work as scaling parameters on \(S^1\). The former relates to the scaling parameters in description by analytical continuation. The latter makes a role of the deformation parameter of loop paths in \(Gr_m\).

5. To study the relation between collective motions in SCF theory and Far Fields:
   In the SCF method, we have not known approaches from the viewpoint of Far Fields. How do we bring in the idea of the Far Fields to relate it to the adiabatic one which is introduced to separate a collective motion from the others? The theory bases on that the speed of collective motion is much slower than that of any other non-collective motions.

6. To study why soliton solution for classical wave equation shows fermion-like behavior in quantum dynamics and what symmetry is hidden in soliton equation:
   The nonlinear Schrödinger equation as a classical image of corresponding bose field has multi-soliton solutions and appears as solutions for non-scattering potential. Nogami used a multi-component nonlinear Schrödinger equation instead of TDHF equation.

7. To study another expression of the TDHB equation using the Kähler coset space:
   To describe the classical motion on the coset manifold, we start from the local equation of motion which becomes a Riccati-type equation. We get a general solution of the TD Riccati-HB equation for coset variables. We obtain the Harish-Chandra decomposition for the \(SO(2N)\) matrix based on the nonlinear Möbius transformation.

8. To study the relation between the nonlinear superposition principle in soliton theory and the Res MF theory in SCF method and the algebraic (Alg) MF one:
   What relation does exist between the construction of exact solutions based on the idea of imbricate series in soliton equation and Res- and Alg- MF theories? Such the idea bases on the group integration on the solution space of soliton equation. The GC method in SCF method stands on the group integration. Further we discuss on the ordinary MF theory related to the Alg MF theory based on the coadjoint orbit leading to the non-degenerate symplectic form.
Appendix

A Derivation of (4.3) and (4.4) and density matrix

We give a decomposition of the generator \( U(g) \), i.e., \( U(N) \) canonical transformation \( U^{-1}(g)U^{-1}(g) = U(N) = U(N) \). We express \( g(\Lambda, \Lambda^t, t) \) as \( g \). For our aim, we set up variable \( \xi, (N - m) \times m \) matrix \( (\xi_{ia}) \), \( a = 1, 2, \ldots, m \) (occupied state) and \( i = m + 1, \ldots, N \) (unoccupied state) and variable \( v = \xi - v^t \), \( m \times m \) matrix \( (v_{ab}) \), \( a, b = 1, 2, \ldots, m \) and variable \( v^s = -v^t \), \( m \times m \) matrix \( (v_{ij}) \), \( i, j = m + 1, \ldots, N \), respectively. Further, we decompose the creation operator \([\hat{c}^\dagger, c]^\dagger\) as \([\hat{c}^\dagger, c]^\dagger\) and the annihilation operator \([c, a]\) as \([c, a] = [\hat{c}, \hat{c}]\) and prepare the following two operators \( \hat{c}, \hat{c} \) and prepare the following two operators \( \hat{c}, \hat{c} \):

\[
\hat{\Xi} = [\hat{c}, \hat{c}] \xi, \quad \hat{\Upsilon} = [\hat{c}, \hat{c}] T [\xi, \xi] \xi, \quad \gamma \equiv e^{\xi} e^{\xi^t}, \quad g_{\xi} = e^{\xi} e^{\xi^t}, \quad g_{\xi} = e^{\xi} e^{\xi^t}.
\]

where the triangular matrix functions \( S(\xi), C(\xi), \tilde{C}(\xi) \) are given by Fukutome \( 11, 12 \) as,

\[
S(\xi) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k + 1)!} \xi(\xi^t)^k = \sqrt{\frac{\xi^t \xi}{\xi^t \xi}}; \\
C(\xi) = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1}{(2k)!} (\xi^t)^k = \cos \sqrt{\frac{\xi^t \xi}{\xi^t \xi}} = C(\xi), \\
\tilde{C}(\xi) = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1}{(2k)!} (\xi^t)^k = \cos \sqrt{\frac{\xi^t \xi}{\xi^t \xi}} = \tilde{C}(\xi).
\]

which hold the properties analogous to the usual triangular functions

\[
C^2(\xi) + S^2(\xi) S(\xi) = 1, \quad C^2(\xi) + S(\xi) S(\xi) = 1, \quad S(\xi) C(\xi) = \tilde{C}(\xi) S(\xi).
\]

Then using the matrices \( g_{\xi} \) and \( g_{\gamma} \), the matrix \( g \) is given as

\[
g = g_{\xi} g_{\gamma} = e^{\xi} e^{\xi^t} = \begin{bmatrix} C(\xi) & -S(\xi) \end{bmatrix} \begin{bmatrix} \xi^t & 0 \end{bmatrix} = \begin{bmatrix} C(\xi) e^{\xi^t} - S(\xi) e^{\xi^t} \end{bmatrix} = \begin{bmatrix} \hat{a} & \hat{b} \end{bmatrix}.
\]

From \( A.1 \), we get,

\[
U(g_{\xi})\hat{c}^\dagger_a U^{-1}(g_{\xi}) = \hat{c}^\dagger_b [C(\xi)]_{ba} + \hat{c}^\dagger_d [S(\xi)]_{da}, \quad U(g_{\xi}) \hat{c}^\dagger_a U^{-1}(g_{\xi}) = \hat{c}^\dagger_j [C(\xi)]_{ji} - \hat{c}^\dagger_a [S(\xi)]_{ai}, \quad U(g_{\xi}) \hat{c}^\dagger_a U^{-1}(g_{\xi}) = \hat{c}^\dagger_i [S(\xi)]_{ia},
\]

\[
U(g_{\gamma}) \hat{c}^\dagger_a U^{-1}(g_{\gamma}) = \hat{c}^\dagger_a e^{\xi^t} [S(\xi)]_{ba}, \quad U(g_{\gamma}) \hat{c}^\dagger_a U^{-1}(g_{\gamma}) = \hat{c}^\dagger_i [S(\xi)]_{ia},
\]

Finally, from \( A.5 \), we obtain the following formula:

\[
[\hat{d}^\dagger, \hat{d}^\dagger] = U(g_{\xi}) U(g_{\gamma}) [\hat{c}^\dagger, \hat{c}^\dagger] U^{-1}(g_{\xi}) U^{-1}(g_{\gamma}) = U(g_{\xi}) [\hat{c}^\dagger, \hat{c}^\dagger] U^{-1}(g_{\xi}) g_{\gamma} = [\hat{c}^\dagger, \hat{c}^\dagger] g,
\]

\[
\begin{bmatrix} \hat{d}^\dagger \hat{d}^\dagger \end{bmatrix} = U(g_{\xi}) U(g_{\gamma}) [\hat{c}^\dagger, \hat{c}^\dagger] U^{-1}(g_{\xi}) U^{-1}(g_{\gamma}) [\hat{c}^\dagger, \hat{c}^\dagger] g.
\]

(A. 6)
We demand that the canonical transformation
\[ \varphi \psi = \psi \varphi \]
be expressed in terms of \( \varphi \) and \( \psi \) as
\[ \varphi = \varphi(\psi) \quad \text{and} \quad \psi = \psi(\varphi) \]
Similarly, the differential \( \partial_t e^\Xi \) is also computed as
\[ \partial_t e^\Xi = \partial_t e^{\Xi_L} - e^{\Xi_L} \partial_t e^\Xi, \quad \text{due to} \quad [\hat{\Xi}, \partial_t \hat{\Xi}] = 0. \]
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