Co-Hopfian virtually free groups and elementary equivalence

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1 | INTRODUCTION

A group is said to be \textit{virtually free} if it has a free subgroup of finite index. In what follows, all virtually free groups are assumed to be finitely generated. A group $G$ is \textit{co-Hopfian} if every injective endomorphism of $G$ is an automorphism. This paper is concerned with the classification of co-Hopfian virtually free groups up to elementary equivalence. Notable examples of co-Hopfian virtually free groups are $\text{GL}_2(\mathbb{Z})$ (which is isomorphic to the amalgamated product $D_4 *_{D_2} D_6$ where $D_n$ denotes the dihedral group of order $2n$) and $S_{n+1} *_{S_n} S_{n+1}$ where $S_n$ denotes the symmetric group on $n \geq 2$ elements (see [12] for a characterization of co-Hopfian groups among virtually free groups). Recall that non-abelian–free groups are elementarily equivalent by the famous work of Sela [17] and Kharlampovich–Myasnikov [9], but free groups are far from being co-Hopfian, and it is natural to expect that co-Hopfian virtually free groups behave very differently from free groups from a model-theoretic point of view; it is indeed the case, as shown by
the following theorem (see paragraph 2.1.2 for a definition of ∀∃-equivalence and elementary equivalence).

**Theorem 1.1.** Let $G$ and $G'$ be two co-Hopfian virtually free groups. The following three assertions are equivalent.

1. $G$ and $G'$ are ∀∃-equivalent.
2. $G$ and $G'$ are elementarily equivalent.
3. $G$ and $G'$ are isomorphic.

It is worth pointing out that this result is not an immediate consequence of the classification of virtually free groups up to ∀∃-equivalence established in [2]. In particular, it is not true that two ∀∃-equivalent virtually free groups embed into each other. For instance, $G = \text{GL}_2(\mathbb{Z}) \cong D_4 \ast D_6$ and $G' = \langle G, t \mid [t, D_2] = 1 \rangle$ are ∀∃-equivalent but $G'$ does not embed into $G$ since $G$ is co-Hopfian and $G, G'$ are not isomorphic.

We also consider homogeneity. Recall that a group $G$ is *homogeneous* if two tuples of elements that are indistinguishable by means of first-order formulae are in the same orbit under the action of the group of automorphisms of $G$ (see paragraph 2.1.3 for a formal definition). Perin and Sklinos [15], and independently Ould Houcine [13], proved that free groups are homogeneous (and even ∀∃-homogeneous, see 2.1.3). In [1], we proved that virtually free groups satisfy a weaker property, which we called almost-homogeneity. We also proved that virtually free groups are not ∀∃-homogeneous in general, and conjectured that they are not homogeneous in general. However, our next result shows that co-Hopfian virtually free groups are ∀∃-homogeneous.

**Theorem 1.2.** Co-Hopfian virtually free groups are ∀∃-homogeneous.

Last, we consider the class of virtually free groups $G$ that are co-Hopfian and such that $\text{Out}(G)$ is finite. As an example, $\text{GL}_2(\mathbb{Z})$ satisfies these two conditions. We prove the following results (see Section 2.1 for a definition of ∃-equivalence, ∃-homogeneity and prime groups).

**Theorem 1.3.** Let $G$ and $G'$ be two co-Hopfian virtually free groups with finite outer automorphism groups. The following three assertions are equivalent.

1. $G$ and $G'$ are ∃-equivalent.
2. $G$ and $G'$ are elementarily equivalent.
3. $G$ and $G'$ are isomorphic.

**Theorem 1.4.** Let $G$ be a co-Hopfian virtually free groups with $\text{Out}(G)$ finite. Then $G$ is ∃-homogeneous and prime.

2 | PRELIMINARIES

2.1 | Model theory

For detailed background, the reader may, for instance, consult [11].
2.1.1 First-order formulae

The language of groups uses the following symbols: the quantifiers $\forall$ and $\exists$, the logical connectors $\land$, $\lor$, $\Rightarrow$, the equality and inequality relations $=$ and $\neq$, the symbols $1$ (standing for the identity element), $^{-1}$ (standing for the inverse), $\cdot$ (standing for the group multiplication), parentheses (and ), and variables $x$, $y$, $g$, $z$ ..., which are to be interpreted as elements of a group. The terms are words in the variables, their inverses, and the identity element (for instance, $x \cdot y \cdot x^{-1} \cdot y^{-1}$ is a term). For convenience, we omit group multiplication. A first-order formula is made from terms iteratively: One can first make atomic formulae by comparing two terms by means of the symbols $=$ and $\neq$ (for instance, $x\cdot y\cdot x^{-1}\cdot y^{-1} = 1$ is an atomic formula), then one can use logical connectors and quantifiers to make new formulae from old formulae, for instance, $\exists x((x \neq 1) \land (\forall y(x\cdot y\cdot x^{-1}\cdot y^{-1} = 1)))$.

We sometimes drop parentheses when there is no ambiguity. A variable is free if it is not bound by any quantifier $\forall$ or $\exists$. A sentence is a formula without free variables. Given a formula $\varphi(x_1, \ldots, x_n)$, a group $G$ and a tuple $(g_1, \ldots, g_n) \in G^n$, one says that $G$ satisfies $\varphi(g_1, \ldots, g_n)$ if this statement is true in the usual sense when the variables are interpreted as elements of $G$. An existential formula is a formula of the form $\varphi(x) : \exists y \theta(x, y)$ where $\theta(x, y)$ is a finite disjunction of conjunctions of equations and inequations in the variables of the tuples $x, y$, that is, a string of symbols of the form $\bigvee_{i=1}^p \bigwedge_{j=1}^{q_i} w_{i,j}(x, y) \varepsilon_i 1$, where each $\varepsilon_i$ denotes $=$ or $\neq$, $p$ and $q_i$ are integers and $w_{i,j}$ is a reduced word in the variables of $x$ and $y$ and their inverses. Similarly, a $\forall\exists$-formula is a formula of the form $\varphi(x) : \forall y \exists z \theta(x, y, z)$ where $\theta(x, y, z)$ is a finite disjunction of conjunctions of equations and inequations in the variables of the tuples $x, y, z$.

2.1.2 Elementary equivalence

Two groups $G$ and $G'$ are said to be elementarily equivalent, denoted $G \equiv G'$, if they satisfy the same first-order sentences. We say that $G$ and $G'$ are existentially equivalent, denoted $G \equiv \exists G'$, if they satisfy the same existential sentences. We define similarly the notion of $\forall\exists$-equivalence, denoted $\equiv_{\forall\exists}$.

2.1.3 Homogeneity

Let $G$ be a group. We say that two $n$-tuples $u$ and $v$ of elements of $G$ have the same type if, for every first-order formula $\varphi(x)$ with $n$ free variables, $G$ satisfies $\varphi(u)$ if and only if $G$ satisfies $\varphi(v)$. Similarly, we say that $u$ and $v$ have the same existential type (respectively $\forall\exists$-type) if, for every $\exists$-formula (respectively $\forall\exists$-formula) $\varphi(x)$ with $n$ free variables, $G$ satisfies $\varphi(u)$ if and only if $G$ satisfies $\varphi(v)$. The group $G$ is said to be homogeneous (respectively, $\exists$-homogeneous and $\forall\exists$-homogeneous) if for any two $n$-tuples $u$ and $v$ having the same type (respectively, $\exists$-type and $\forall\exists$-type), there exists an automorphism $\sigma$ of $G$ mapping $u$ to $v$.

2.1.4 Prime models

A map $\varphi : G \to G'$ between two groups $G$ and $G'$ is said to be elementary if the following condition holds: For every first-order formula $\theta(x)$ with $n$ free variables in the language of groups, and for
every $n$-tuple $u \in G^n$, $G$ satisfies $\theta(u)$ if and only if $G'$ satisfies $\theta(u)$. In particular, $\varphi$ is a morphism and is injective. The group $G$ is **prime** if for every group $G'$ that is elementarily equivalent to $G$, there exists an elementary embedding $\varphi : G \to G'$.

### 2.2 Tree of cylinders

Let $G$ be a finitely generated group acting on a tree $T$. If $v$ is a vertex of $T$, we denote by $G_v$ the stabilizer of $v$. Similarly we denote by $G_e$ the stabilizer of an edge $e$. A subgroup $H$ of $G$ is said to be **elliptic** for this action if it fixes a point of $T$, that is, if there exists a vertex $v$ of $T$ such that $H \subset G_v$. The **deformation space** of a simplicial $G$-tree $T$, introduced by Forester in [6], is the set of $G$-trees $T'$ that have the following property for every subgroup $H$ of $G$: $H$ is elliptic for the action of $G$ on $T$ if and only if $H$ is elliptic for the action of $G$ on $T'$. Forester proved in [6] that any two trees in the same deformation space are connected by a finite sequence of simple operations called elementary expansions and collapses.

Now let us fix an integer $k \geq 1$, and let $\Delta$ be a splitting of $G$ over finite groups of order $k$. Let $\Delta$ denote the Bass–Serret tree of $\Delta$. In [7], Guirardel and Levitt associate with $T$ a tree $T_c$, called the tree of cylinders of $T$. This tree is canonical in the sense that it only depends on the deformation space of $T$ (which can be proved by checking that elementary expansions and collapses do not change $T_c$). We summarize below the construction of the tree of cylinders $T_c$.

First, we define an equivalence relation $\sim$ on the set of edges of $T$: We declare two edges $e$ and $e'$ to be equivalent if $G_e = G_{e'}$. Since all edge stabilizers have the same order, the union of all edges in the equivalence class of an edge $e$ is a subtree $Y_e$, called a cylinder of $T$. In other words, $Y_e$ is the subset of $T$ pointwise fixed by the edge group $G_e$. Two distinct cylinders meet in at most one point. The **tree of cylinders** $T_c$ of $T$ is the bipartite tree with set of vertices $V_0(T_c) \sqcup V_1(T_c)$ such that $V_0(T_c)$ is the set of vertices $x$ of $T$ which belong to at least two cylinders, $V_1(T_c)$ is the set of cylinders $Y_e$ of $T$, and there is an edge $e = (x, Y_e)$ between $x$ and $Y_e$ in $T_c$ if and only if $x \in Y_e$. If $Y_e$ belongs to $V_1(T_c)$, the vertex group $G_{Y_e}$ is the global stabilizer of $Y_e$ in $T$, that is, the normalizer of $G_e$ in $G$ (see below).

**Lemma 2.1.** The global stabilizer of $Y_e$ in $G$ coincides with $N_G(G_e)$.

**Proof.** If $g$ belongs to $\text{Stab}(Y_e)$, then there exists an edge $e' \in Y_e$ such that $ge = e'$, that is, $gG_eg^{-1} = G_{e'}$. In addition, $G_{e'} = G_e$ since $e'$ belongs to the same cylinder as $e$, so $gG_eg^{-1} = G_e$. Conversely, if $g$ belongs to $N_G(G_e)$, then $G_{e'}^g = G_{ge} = G_e$, that is, $ge$ and $e$ are in the same cylinder. □

The lemma below follows immediately from the previous lemma and from the fact that a bounded subset in a tree admits a center, which is preserved by every element that preserves this bounded subset.

**Lemma 2.2.** Assume that $Y_e$ has bounded diameter in $T_k$. Then $N_G(G_e)$ is elliptic in $T_k$.

The stabilizer of the edge $e = (x, Y_e)$ is $G_e = G_x \cap G_{Y_e} = N_{G_x}(G_e)$. Note that the inclusion $G_e \subset G_x$ may be strict. As a consequence, $T$ and $T_c$ do not belong to the same deformation space in general. Note that $T_c$ may be trivial even if $T$ is not.
2.3 An equivalence relation

Let $G$ be a finitely generated virtually free group. Since $G$ has a free subgroup $H$ of finite index, it also has a normal free subgroup of finite index, namely the intersection of the conjugates of $H$. Hence, there is a short exact sequence

$$1 \to F_n \to G \xrightarrow{\pi} A \to 1,$$

where $F_n$ denotes a free group of rank $n$ and $A$ denotes a finite group. It follows that there exists a bound on the order of finite subgroups of $G$: Indeed, the surjection $\pi$ is injective on finite subgroups of $G$ since its kernel $\ker(\pi) = i(F_n)$ is torsion-free, and hence the order of a finite subgroup of $G$ is bounded by $K = |A|$. In fact, the following stronger result is true: $G$ has only finitely many conjugacy classes of finite subgroups. This result follows easily from Linnell's theorem recalled in the first paragraph of Section 3. We will use this result several times in this paper, notably in the proof of Lemma 2.4 below. More generally, it is worth noting that every hyperbolic group has only finitely many conjugacy classes of finite subgroups (see, for instance, [4]).

Given an element $g$ in a group $G$, we write $\text{ad}(g)$ for the inner automorphism $x \mapsto gxg^{-1}$.

**Definition 2.3.** Let $G$ be a virtually free group. Let $G'$ be a group. We say that two homomorphisms $\phi, \phi' : G \to G'$ are equivalent, denoted by $\phi \sim \phi'$, if for every finite subgroup $H$ of $G$, there exists an element $g' \in G'$ such that $\phi$ and $\phi'$ coincide on $H$ up to conjugacy by $g'$, that is, $\phi'|_H = \text{ad}(g') \circ \phi|_H$.

The following lemma shows that the previous equivalence relation on $\text{Hom}(G, G')$ can be expressed using an existential formula.

**Lemma 2.4.** Let $G$ be a finitely generated virtually free group, and let $\{s_1, \ldots, s_n\}$ be a generating set of $G$. Let $G'$ be a group. There exists an existential formula $\psi_G(x_1, \ldots, x_{2n})$ with $2n$ free variables such that, for every morphisms $\phi, \phi' \in \text{Hom}(G, G')$, the following assertions are equivalent:

1. $\phi$ and $\phi'$ are equivalent in the sense of Definition 2.3;
2. $G'$ satisfies $\psi_G(\phi(s_1), \ldots, \phi(s_n), \phi'(s_1), \ldots, \phi'(s_n))$.

**Proof.** As recalled above, $G$ has only finitely many conjugacy classes of finite subgroups. Let $H_1, \ldots, H_r$ be finite subgroups of $G$ such that any finite subgroup of $G$ is conjugate to some $H_i$. For every $1 \leq i \leq r$, let $h_{i,1}, \ldots, h_{i,k_i}$ denote the elements of $H_i$. For every $1 \leq i \leq r$ and $1 \leq j \leq k_i$, there exists a word $w_{i,j}(x_1, \ldots, x_n)$ in $n$ variables such that $h_{i,j} = w_{i,j}(s_1, \ldots, s_n)$. Define

$$\psi_G(x_1, \ldots, x_{2n}) : \exists y_1 \ldots \exists y_r \bigwedge_{i=1}^r \bigwedge_{j=1}^{k_i} w_{i,j}(x_1, \ldots, x_n) = y_i w_{i,j}(x_{n+1}, \ldots, x_{2n}) y_i^{-1}.$$

Since $\phi(h_{i,j}) = w_{i,j}(\phi(s_1), \ldots, \phi(s_n))$ and $\phi'(h_{i,j}) = w_{i,j}(\phi'(s_1), \ldots, \phi'(s_n))$ for $1 \leq i \leq r$ and $1 \leq j \leq k_i$, the tuple $(\phi(s_1), \ldots, \phi(s_n), \phi'(s_1), \ldots, \phi'(s_n))$ satisfies $\psi_G(x_1, \ldots, x_{2n})$ if and only if the homomorphisms $\phi$ and $\phi'$ coincide up to conjugacy on every finite subgroup of $G$. \hfill $\Box$

**Definition 2.5.** Let $G$ be a virtually free group. We say that $G$ is rigid if every endomorphism $\phi : G \to G$ such that $\phi \sim \text{id}_G$ is an automorphism.
In Section 4, we shall prove that co-Hopfian virtually free groups are rigid.

3 A PROPERTY OF VIRTUALLY FREE GROUPS

Let $G$ be a finitely generated group. Under the hypothesis that there exists a constant $K$ such that every finite subgroup of $G$ has order at most $K$, Linnell proved in [10] that $G$ splits as a finite graph of groups with finite edge groups and all of whose vertex groups are finite or one-ended. According to the paragraph at the beginning of the previous section, Linnell’s result applies to finitely generated virtually free groups, so let us suppose now that $G$ is a finitely generated virtually free group. Let $\Delta$ be a splitting of $G$ given by Linnell’s theorem and let $G_v$ be a vertex group of this splitting. We claim that this group is necessarily finite. Indeed, $G_v$ is virtually free as a subgroup of a virtually free group, and $G_v$ is finitely generated as a vertex group in a splitting of $G$ over finite groups. It follows that $G_v$ is quasi-isometric to a free group $F_n$ (a free subgroup of finite index in $G_v$). But the number of ends is preserved under quasi-isometry, and hence either $G_v$ is finite (if $n = 0$), or $G_v$ has two ends (if $n = 1$), or $G_v$ has infinitely many ends (if $n \geq 2$). Since we know by Linnell’s theorem that $G_v$ is either finite or one-ended, the last two cases are excluded, and so $n = 0$ and $G_v$ is finite. Therefore, a finitely generated group $G$ is virtually free if and only if it splits as a finite graph of finite groups (and in particular, $G$ has only finitely many conjugacy classes of finite subgroups, as recalled in the previous section). Such a splitting is called a Stallings splitting (or tree) of $G$. A Stallings tree $T$ of $G$ is said to be reduced if for every edge $e = [v, w]$ such that $G_e = G_v$, the vertices $v$ and $w$ are in the same orbits; in other words, if $G_e = G_v$ then the endpoints of $e$ must be identified in the quotient graph $T/G$. A vertex $v$ of $T$ is called redundant if it has degree 2 and if the two edges $e$ and $e'$ incident to $v$ satisfy $G_e = G_{e'} = G_v$. The tree $T$ is called non-redundant if every vertex is non-redundant (note that reduced implies non-redundant).

A Stallings splitting is not unique in general, but the conjugacy classes of finite vertex groups are the same in all reduced Stallings splittings of $G$. The Stallings deformation space of $G$, denoted by $D(G)$, is the set of Stallings trees of $G$ up to equivariant isomorphism (where trees are viewed as simplicial graphs).

The following result is well known, see, for instance, Lemmas 2.20 and 2.22 in [5], and Definition 2.19 in [5] (definition of an isomorphism of graphs of groups).

**Proposition 3.1.** Let $T$ and $T'$ be two Stallings trees of $G$. The following two assertions are equivalent.

1. The quotient graphs of groups $T/G$ and $T'/G$ are isomorphic.
2. There exist an automorphism $\sigma$ of $G$ and a $\sigma$-equivariant isomorphism $f : T' \to T$ of simplicial graphs.

The automorphism group $\text{Aut}(G)$ acts naturally on the deformation space $D(G)$, as follows: If $T$ is a Stallings tree, let $\varphi$ denote the corresponding action of $G$ on $T$, that is a morphism from $G$ to the group of automorphisms of $T$ viewed as a simplicial graph. For $\sigma \in \text{Aut}(G)$, we define the Stallings tree $\sigma \cdot T$, simply denoted by $T^\sigma$, by precomposing the action $\varphi$ by $\sigma$. In the proposition above, the $\sigma$-equivariant isomorphism $f : T' \to T$ is the same as an equivariant isomorphism from $T'$ to $T^\sigma$, and hence $T' = T^\sigma$ since we consider elements in $D(G)$ up to equivariant isomorphism. The following proposition claims that $D(G)$ is cocompact under the action of $\text{Aut}(G)$. We refer the reader to [1, Proposition 2.9].
Proposition 3.2. Let $G$ be a virtually free group. There exist finitely many trees $S_1, \ldots, S_n$ in $D(G)$ such that, for every non-redundant tree $T \in D(G)$, there exist an automorphism $\sigma$ of $G$ and an integer $1 \leq \ell \leq n$ such that $T = S_\ell^\sigma$.

The following proposition plays an important role in the proofs of our results. Note that when $G'$ is a torsion-free hyperbolic group and $G$ is a one-ended finitely generated group, a similar statement was proved by Sela in [18]. This result was generalized by Reinfeldt and Weidmann in [16] without assuming torsion-freeness. The main point of the proposition below is that $G$ is not one-ended (except if it is finite).

Proposition 3.3. Let $G$ and $G'$ be two finitely generated virtually free groups. There exists a finite subset $F$ of $G \setminus \{1\}$ such that, for every non-injective homomorphism $\phi : G \to G'$, there exists an automorphism $\sigma \in \text{Aut}(G)$ such that $\ker(\phi \circ \sigma) \cap F \neq \emptyset$.

Proof. Let $\Delta$ and $\Delta'$ be two Stallings splittings of $G$ and $G'$, respectively. Let $T$ and $T'$ denote their Bass–Serre trees. Let $H_1, \ldots, H_r$ be finite subgroups of $G$ such that any finite subgroup of $G$ is conjugate to $H_i$ for some $1 \leq i \leq r$.

Let $\phi : G \to G'$ be a non-injective homomorphism. As a first step, we build a $\phi$-equivariant map $f : T \to T'$. Let $v_1, \ldots, v_n$ be some representatives of the orbits of vertices for the action of $G$ on the Bass–Serre tree $T$ of $\Delta$. For every $1 \leq k \leq n$, $\phi(G_{v_k})$ is finite, and thus it fixes a vertex $v'_k \in T'$ (which is not necessarily unique). Set $f(v_k) = v'_k$ (note that we could just as well choose another fixed point of $\phi(G_{v_k})$ in the definition of $f$, which would change $f$ without changing the fact that $f$ is a $\phi$-equivariant map as desired). Then, define $f$ on each vertex of $T$ by $\phi$-equivariance. It remains to define $f$ on the edges of $T$: If $e$ is an edge of $T$, with endpoints $v_1$ and $v_2$, there exists a unique path $e'$ from $f(v_1)$ to $f(v_2)$ in $T'$; we define $f(e) = e'$.

If $\phi$ is not injective on the vertex groups of $T$, then $\phi$ is not injective on $H_i$ for some $1 \leq i \leq r$. From now on, let us assume that $\phi$ is injective on the vertex groups of $T$.

Note that $f$ sends an edge of $T$ to a path of $T'$. Up to subdivising the edges of $T$, one can assume that $f$ sends an edge to an edge or a vertex of $T'$. Moreover, note that $f$ is not an isometry: Indeed, there is a non-trivial element $g \in G$ such that $\phi(g) = 1$, and hence $f(gv) = \phi(g)f(v) = f(v)$, and $gv$ is distinct from $v$, otherwise $g$ would belong to $G_v$, contradicting the assumption that $\phi$ is injective on the vertex groups of the tree $T$. As a consequence, $f$ maps an edge of $T$ to a point, or folds two edges.

Case 1. If $f$ maps the edge $e = [v, w]$ of $T$ to a point in $T'$, we collapse $e$ in $T$, as well as all its translates under the action of $G$. Collapsing $e$ gives rise to a new $G$-tree $T_1$ with a new vertex $x$ labelled by $G_x = \langle G_v, G_w \rangle$ if $v$ and $w$ are not in the same orbit, or $G_x = \langle G_v, g \rangle$ if $w = g v$.

Case 2. Suppose that $f$ folds some pair of edges, as pictured below.

![Diagram](image)

We fold $e$ and $e'$ together in $T$, as well as all their translates under the action of $G$. Folding $e$ and $e'$ gives rise to a new $G$-tree $T_2$ with a new vertex $x$ labelled by $G_x = \langle G_w, G_w' \rangle$ if $w$ and $w'$ are not in the same orbit, or $G_x = \langle G_w, g \rangle$ if $w' = g w$. 
The map \( f : T \to T' \) factors through the quotient map \( \pi_1 : T \to T_1 \). Let \( f_1 : T_1 \to T' \) be the map such that \( f = f_1 \circ \pi_1 \). If \( T_1 \) belongs to the Stallings deformation space \( \mathcal{D}(G) \), then the same argument as above shows that \( f_1 \) is not an isometry, and one can perform another collapsing or folding of edges. We get a sequence \( T \to T_1 \to T_2 \to \cdots \). Then, observe that \( T \) has only finitely many orbits of edges under the action of \( G \), which implies that one can perform only finitely many collapsing or folding of edges. Hence the previous sequence of trees is necessarily finite. Let \( T_{k+1} \) be the last tree in the sequence, with \( k \geq 0 \). Note that \( T_{k+1} \) does not belong to the Stallings deformation space, otherwise one can perform one more collapsing or folding. Therefore, the last collapsing or folding in the sequence, namely, \( T_k \to T_{k+1} \), gives rise to an infinite vertex group. More precisely, one of the following holds, where \( N \) denotes the maximal order of an element of \( G' \) of finite order:

- either there is an edge \([v, w]\) in \( T_k \) such that \( \langle G_v, G_w \rangle \) is infinite and \( \phi \) kills the \( N! \)th power of any element of \( \langle G_v, G_w \rangle \) of infinite order,
- or there exist two edges \([v, w]\) and \([v, w']\) such that \( w, w' \) are not in the same orbit, \( \langle G_w, G_{w'} \rangle \) is infinite and \( \phi \) kills the \( N! \)th power of any element of \( \langle G_w, G_{w'} \rangle \) of infinite order,
- or there exist two edges \([v, w]\) and \([v, w']\) such that \( w' = g w \), \( \langle G_w, g \rangle \) is infinite and \( \phi \) kills the \( N! \)th power of any element of \( \langle G_w, g \rangle \) of infinite order.

Hence, one can associate to \( T_k \) a finite set of elements of \( G \) of infinite order such that \( \phi \) kills an element of this finite set.

Now, up to forgetting the possibly redundant vertices of \( T_k \), one can assume that \( T_k \) is non-redundant. By Proposition 3.2, there exist an automorphism \( \sigma \) of \( G \) and an integer \( 1 \leq \ell \leq n \) such that \( T_k = S^{\sigma}_{\ell} \).

As a conclusion, one can associate to every tree \( S_{\ell} \), with \( 1 \leq \ell \leq n \), a finite set \( F_{\ell} \) of elements of \( G \) of infinite order such that for any non-injective morphism \( \phi : G \to G' \), there exists \( \sigma \in \text{Aut}(G) \) such that \( \phi \circ \sigma \) kills an element of \( F_{\ell} \) for some \( 1 \leq \ell \leq n \) or an element of \( H_i \) for some \( 1 \leq i \leq r \). Last, define \( F = F_1 \cup \cdots \cup F_n \cup H_1 \cup \cdots \cup H_r \).

Before stating the next proposition, let us define a subgroup of \( \text{Aut}(G) \).

**Definition 3.4.** We denote by \( \text{Aut}_0(G) \) the subgroup of \( \text{Aut}(G) \) defined as follows:

\[
\text{Aut}_0(G) = \{ \phi \in \text{Aut}(G) \mid \phi \sim \text{id}_G \}.
\]

The following lemma is straightforward since a virtually free group \( G \) has only finitely many conjugacy classes of finite subgroups.

**Lemma 3.5.** The subgroup \( \text{Aut}_0(G) \) has finite index in \( \text{Aut}(G) \).

Then, write \( \text{Aut}(G) = \sigma_1 \circ \text{Aut}_0(G) \cup \cdots \cup \sigma_N \circ \text{Aut}_0(G) \) where \( N = [\text{Aut}(G) : \text{Aut}_0(G)] \), and observe that Proposition 3.3 remains true if \( \text{Aut}(G) \) and \( F \) are replaced by \( \text{Aut}_0(G) \) and \( \sigma_1(F) \cup \cdots \cup \sigma_N(F) \), respectively. Hence the following result follows immediately from Proposition 3.3.

**Proposition 3.6.** Let \( G \) and \( G' \) be two finitely generated virtually free groups. There exists a finite subset \( F \) of \( G \setminus \{1\} \) such that, for every non-injective homomorphism \( \phi : G \to G' \), there exists an automorphism \( \sigma \in \text{Aut}_0(G) \) such that \( \ker(\phi \circ \sigma) \cap F \neq \emptyset \).
Remark 3.7. The reason why we define this subgroup Aut₀(G) lies in the fact that φ ◦ σ and φ are equivalent in the sense of Definition 2.3 when σ belongs to Aut₀(G). This observation will be very useful later on.

4 | CO-HOPFIAN VIRTUALLY FREE GROUPS ARE RIGID

4.1 | Preliminary lemma

Lemma 4.1. Let G be a finitely generated virtually free group. Suppose that G is infinite. Let Δ be a reduced Stallings splitting of G. Let k denote the smallest order of an edge group of Δ. Denote by Δ_k the splitting of G obtained from Δ by collapsing each edge e such that |G_e| > k. Let φ be an endomorphism of G such that φ ~ id_G. If v is a vertex of Δ_k, then φ(G_v) is contained in a conjugate of G_v.

Proof. First, we will prove that φ(G_v) fixes a vertex v' in the Bass–Serre tree T of Δ_k. Note that the vertex v of Δ_k was created by collapsing to a point a subgraph Δ_v of Δ all of whose edge groups have order > k, and that G_v is the fundamental group of Δ_v.

First of all, suppose that the graph Δ_v has no edge, that is, that Δ_v is a point. In that case G_v is finite, and since φ ~ id_G we have φ(G_v) = gG_vg⁻¹ for some g ∈ G.

Then, suppose that the graph Δ_v has at least one edge. Let T_v be the Bass–Serre tree of Δ_v, and let x be a vertex in T_v. The group G_x has order > k and φ is injective on G_x (since φ ~ id_G), so φ(G_x) has order > k. As a finite group φ(G_v) fixes a vertex in T_v, and this vertex is unique since edge groups of T have order k. Now, let x, y be two adjacent vertices in T_v, let x' ∈ T_v be the unique vertex fixed by φ(G_x) and let y' ∈ T_v be the unique vertex fixed by φ(G_y). Note that the edge group G_x ∩ G_y has order > k. Therefore, the group φ(G_x) ∩ φ(G_y) has order > k (again because φ is injective on finite groups). It follows that the vertices x' and y' coincide, as otherwise the path [x', y'] in T_v would be fixed by a finite group of order > k, contradicting the definition of Δ_k. In conclusion, there exists a vertex v' in T_v that is fixed by φ(G_x) for any vertex x ∈ T_v, and moreover v' is the unique vertex fixed by φ(G_x) for any x ∈ T_v. Now, if g is an element of G_v and x is any point of T_v, φ(G_gx) fixes v'. But φ(G_gx) = φ(gG_xg⁻¹) = φ(g)φ(G_x)φ(g)⁻¹, and thus φ(G_x) fixes φ(g)⁻¹v'. By uniqueness we can conclude that φ(g)⁻¹v' = v' and therefore φ(g) fixes v' for every g ∈ G, which proves that φ(G_v) fixes v'.

It remains to prove that v' is a translate of the vertex v. Let x ∈ T_v. By the previous paragraph, we know that v'' is the unique vertex fixed by φ(G_x) in T. But G_x is finite and thus φ(G_x) = gG_xg⁻¹ for some element g ∈ G since φ ~ id_G. Moreover, v is the unique vertex fixed by G_x in T, because |G_x| > k. In consequence, v' = gv and φ(G_v) is contained in G_v^g. □

4.2 | Characterization of co-Hopfian virtually free groups

Let G be a virtually free group and let Δ be a Stallings–Dunwoody splitting of G. We denote by Δ_k the splitting of G obtained from Δ by collapsing each edge whose stabilizer has order > k. We denote by T_k the Bass–Serre tree of Δ_k. In his PhD thesis [12], Moioli gave a complete characterization of virtually free groups that are co-Hopfian. Here below are two versions of this characterization: The first one is geometric (see Theorem 4.2 below) and the second one is a purely group theoretical criterion expressed in terms of the normalizers of the edge groups (see Theorem 4.3 below).
Theorem 4.2 (Moioli). Let $G$ be a virtually free group, and let $\Delta$ be a Stallings splitting of $G$. Then $G$ is co-Hopfian if and only if the following condition holds: For every integer $k$, and for every edge $e$ of $T_k$ such that $|G_e| = k$, the cylinder $Y_e$ of $e$ has bounded diameter in $T_k$.

Theorem 4.3 (Moioli). Let $G$ be a virtually free group, and let $\Delta$ be a Stallings splitting of $G$. For every edge $e$ of $\Delta$, let $\Delta_e$ be the graph of groups obtained by collapsing each edge different from $e$ in $\Delta$. Then $G$ is co-Hopfian if and only if, for every edge $e$ of $\Delta$, the following conditions hold.

- If $\Delta_e$ is a splitting of the form $A *_C B$, then $N_A(C) = C$ or $N_B(C) = C$.
- If $\Delta_e$ is a splitting of the form $A *_{\alpha} A'$ where $\alpha : C \to C'$ is an isomorphism between two finite subgroups of $A$, then $C$ and $C'$ are non-conjugate in $A$, and $N_A(C) = C$ or $N_A(C') = C'$.

Remark 4.4. A subgroup of a co-Hopfian group is not co-Hopfian in general. However, it follows from the previous theorem that, if $\Lambda$ is a subgraph of $\Delta$ (with the same notations as above), then the fundamental group of $\Lambda$ is co-Hopfian.

4.3 Co-Hopfian virtually free groups are rigid

In this subsection, we shall prove that co-Hopfian virtually free groups are rigid in the sense of Definition 2.5. In other words, we shall prove that every endomorphism of a co-Hopfian virtually free group $G$ that is equivalent to $id_G$ (that is, that coincides with an inner automorphism on each finite subgroup of $G$) is an automorphism of $G$. First, let us prove a lemma.

Lemma 4.5. Let $G$ be a co-Hopfian virtually free group. Let $\Delta$ be a Stallings splitting of $G$. Let $k$ denote the smallest order of an edge group of $\Delta$. Denote by $\Delta_k$ the splitting of $G$ obtained from $\Delta$ by collapsing each edge $e$ such that $|G_e| > k$. Let $\phi$ be an endomorphism of $G$ satisfying the following two properties:

1. $\phi \sim id_G$;
2. for each vertex group $v$ of $\Delta_k$, $\phi$ is injective on $G_v$.

Then $\phi$ is an automorphism.

Proof. Let $T_k$ denote the Bass–Serre tree of $\Delta_k$, and let $T_c$ be its tree of cylinders. Recall that the tree $T_c$ is bipartite: Its set of vertices is $V_0(T_c) \sqcup V_1(T_c)$ where $V_0(T_c)$ is the set of vertices $v$ of $T_k$ that belong to at least two cylinders and $V_1(T_c)$ is the set of cylinders $Y_e$ of $T_k$. We refer the reader to Subsection 2.2 for the exact definition of $T_c$.

As a first step, let us define a $\phi$-equivariant map $f : T_k \to T_k$. Let $v_1, \ldots, v_n$ be some representatives of the orbits of vertices of $T_k$. By Lemma 4.1, for every $1 \leq i \leq n$, there exists an element $g_i \in G$ such that $\phi(G_{v_i}) \subset G^{g_i}_{v_i}$. By Remark 4.4, $G_{v_i}$ is co-Hopfian (as every vertex group of $\Delta_k$ corresponds to a subgraph of $\Delta$), and by assumption $\phi$ is injective on $G_{v_i}$, and hence $\phi(G_{v_i}) = G^{g_i}_{v_i}$. Set $f(v_i) = g_i v_i$. Then, we define $f$ on every vertex of $T_k$ by $\phi$-equivariance, so that $\phi(G_v) = G^{f(v)}_v$ for every vertex $v$ of $T_k$. Next, we define $f$ on the edges of $T_k$ in the following way: If $e$ is an edge of $T_k$, with endpoints $v$ and $w$, there exists a unique path $e'$ from $f(v)$ to $f(w)$ in $T_k$, and we let $f(e) = e'$.

Then, the map $f$ induces a $\phi$-equivariant map $f_c : T_c \to T_c$. Indeed, for each cylinder $Y_e = \text{Fix}(G_e) \subset T_k$, the image $f(Y_e)$ is contained in $\text{Fix}(\phi(G_e))$ of $T_k$, which is a cylinder since $\phi(G_e)$ is...
conjugate to $G_e$. If $v \in T_k$ belongs to two cylinders, so does $f(v)$. This allows us to define $f_c$ on vertices of $T_c$, by sending $v \in V_0(T_c)$ to $f(v) \in V_0(T_c)$ and $Y \in V_1(T_c)$ to $f(Y) \in V_1(T_c)$. If $(v,Y)$ is an edge of $T_c$, then $f_v(v)$ and $f_y(Y)$ are adjacent in $T_c$.

We shall prove that $f_c$ does not fold any pair of edges and, therefore, that $f_c$ is injective. Assume toward a contradiction that there exist a vertex $v$ of $T_c$, and two distinct vertices $w$ and $w'$ adjacent to $v$ such that $f_{v}(w) = f_{v}(w')$.

First, assume that $v$ is not a cylinder. Since $T_c$ is bipartite, $w$ and $w'$ are two cylinders, associated with two edges $e$ and $e'$ of $T_k$. Since $f_{v}(w) = f_{v}(w')$, we have $\phi(G_e) = \phi(G_{e'})$ by definition of $f_c$. But $\phi$ is injective on $G_v$ by assumption, and $G_e, G_{e'}$ are two distinct subgroups of $G_v$ (by definition of a cylinder). This is a contradiction.

Now, assume that $v = Y_e$ is a cylinder. Hence $w$ and $w'$ are two vertices of $T_k$. Since $f_{v}(w) = f_{v}(w')$, we have $f(w) = f(w')$. By definition of the map $f$, there exists an element $g \in G$ such that $w' = gw$. We have $G_{w'} = gG_wg^{-1}$ and thus $\phi(G_{w'}) = \phi(g)\phi(G_w)\phi(g)^{-1}$. But $\phi(G_{w'}) = G_{f(w')} = G_{f(w)} = \phi(G_w)$, and therefore $\phi(g)$ belongs to $\phi(G_w)$, so one can assume that $\phi(g) = 1$ up to multiplying $g$ by an element of $G_w$. Now, observe that $\phi$ is injective on $G_v = N_G(G_e)$: Indeed, by Theorem 4.2, $Y_e$ has bounded diameter in $T_k$, and hence $N_G(G_e)$ is elliptic in $T_k$ by Lemma 2.2; it follows that $\phi$ is injective on $N_G(G_v)$, as $\phi$ is injective on the vertex groups of $T_k$ by assumption. Therefore $g$ does not belong to $G_v = N_G(G_e)$ since $\phi(g) = 1$. Then observe that $G_e$ is contained in $G_w$ and in $G_{w'}$, and that $gG_eg^{-1}$ is contained in $gG_wg^{-1} = G_{w'}$. The edge groups $G_e$ and $gG_eg^{-1}$ are distinct since $g$ does not lie in $N_G(G_e)$, but $\phi(G_e) = \phi(gG_eg^{-1})$ since $\phi(g) = 1$. This contradicts the injectivity of $\phi$ on $G_{w'}$.

Hence, $f_c$ is injective. It follows that $\phi$ is injective. Indeed, let $g$ be an element of $G$ such that $\phi(g) = 1$. Then $f_{c}(gv) = f_{c}(v)$ for each vertex $v$ of $T_c$, so $gv = v$ for each vertex $v$ of $T_c$. But $\phi$ is injective on vertex groups of $T_c$, so $g = 1$.

Last, $G$ being co-Hopfian, $\phi$ is an automorphism of $G$.

\begin{proof}
Let $\Delta$ be a Stallings splitting of $G$. Let $k$ denote the smallest order of an edge group of $\Delta$. Denote by $\Delta_k$ the splitting of $G$ obtained from $\Delta$ by collapsing each edge $e$ such that $|G_e| > k$.

Let $\phi$ be an endomorphism of $G$ such that $\phi \sim \text{id}_G$. Assume toward a contradiction that $\phi$ is not injective. Then, by Lemma 4.5, there exists a vertex $v$ of $\Delta_k$ such that $\phi$ is not injective on $G_v$. Moreover, by Lemma 4.1, there exists an element $g \in G$ such that $\phi(G_v)$ is contained in $G_v^g$. As a consequence, $\text{ad}(g^{-1})\circ\phi$ is a non-injective endomorphism of $G_v$ that coincides with a conjugation on every finite subgroup of $G_v$.

If $G_v$ is finite, we get a contradiction since $\phi$ is injective on finite subgroups of $G$. Otherwise, the group $G_v$ splits as a non-trivial tree of finite groups $\Delta_v$, and the smallest order of an edge group of $\Delta_v$ is strictly greater than $k$, by definition of $\Delta$. Then, we repeat the previous operation. Since there are only finitely many orders of edge groups of $\Delta$, we get a contradiction after finitely many iterations.

Hence, every endomorphism $\phi$ of $G$ such that $\phi \sim \text{id}_G$ is injective. Since $G$ is co-Hopfian by assumption, $\phi$ is an automorphism.
\end{proof}
5 | PROOFS OF THE MAIN RESULTS

5.1 | Elementary equivalence

We shall prove the following result.

**Theorem 5.1.** Let $G$ and $G'$ be two co-Hopfian virtually free groups. The following three assertions are equivalent.

1. $G$ and $G'$ are $\forall \exists$-equivalent.
2. $G$ and $G'$ are elementarily equivalent.
3. $G$ and $G'$ are isomorphic.

This theorem is an immediate consequence of the following proposition, together with the fact that co-Hopfian virtually free groups are rigid (see Proposition 4.6).

**Proposition 5.2.** Let $G$ and $G'$ be two virtually free groups. Suppose that $G$ is rigid, and that $G$ and $G'$ are $\forall \exists$-equivalent. Then $G$ embeds into $G'$.

**Proof.** By Proposition 3.6, there exists a finite subset $F = \{g_1, \ldots, g_k\}$ of $G \setminus \{1\}$ such that for every non-injective homomorphism $\phi : G \to G'$, there exists an automorphism $\sigma \in \text{Aut}_0(G)$ such that $\phi \circ \sigma(g_i) = 1$ for some $1 \leq i \leq k$. Let us fix a finite presentation $G = \langle s_1, \ldots, s_n \mid \Sigma(s_1, \ldots, s_n) = 1 \rangle$, where $\Sigma(x_1, \ldots, x_n) = 1$ is a finite system of equations in the variables $x_1, \ldots, x_n$. For every integer $1 \leq i \leq k$, the element $g_i$ can be written as a word $w_i(s_1, \ldots, s_n)$ in the generators $s_1, \ldots, s_n$.

Suppose toward a contradiction that $G$ does not embed into $G'$. Then, for every morphism $\phi : G \to G'$ (which is automatically non-injective), there exists an automorphism $\sigma \in \text{Aut}_0(G)$ such that $\phi \circ \sigma(g_i) = 1$ for some $1 \leq i \leq k$. Observe that $\phi' := \phi \circ \sigma$ and $\phi$ are equivalent in the sense of Definition 2.3, and hence $G'$ satisfies the existential formula $\psi_G(\phi(s_1), \ldots, \phi(s_n), \phi'(s_1), \ldots, \phi'(s_n))$ given by Lemma 2.4. Then, observe that there is a one-to-one correspondence between the set of morphisms from $G$ to $G'$ and the set of solutions in $G'^n$ of the system of equations $\Sigma(x_1, \ldots, x_n) = 1$. Therefore, we can write a $\forall \exists$-sentence $\mu$ (see below) that is satisfied by $G'$, and whose meaning is ‘for every morphism $\phi : G \to G'$, there is a morphism $\phi' : G \to G'$ such that $\phi' \sim \phi$ and $\phi'(g_i) = 1$ for some $1 \leq i \leq k$.

$$\mu : \forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_n \left( \begin{array}{l} 
\Sigma(x_1, \ldots, x_n) = 1 \Rightarrow \left( \begin{array}{l} 
\Sigma(y_1, \ldots, y_n) = 1 \\
\land \psi_G(x_1, \ldots, x_n, y_1, \ldots, y_n) \\
\land \bigvee_{1 \leq i \leq k} w_i(y_1, \ldots, y_n) = 1
\end{array} \right) \right) \right).$$

This $\forall \exists$-sentence is satisfied by $G'$. But $G$ and $G'$ have the same $\forall \exists$-theory, so $\mu$ is satisfied by $G$ as well. The interpretation of $\mu$ in $G$ is ‘for every morphism $\phi : G \to G$, there is a morphism $\phi' : G \to G$ such that $\phi' \sim \phi$ and $\phi'(g_i) = 1$ for some $1 \leq i \leq k$'. Now, take for $\phi$ the identity of $G$. Since $G$ is assumed to be rigid, the endomorphism $\phi'$ is an automorphism, which contradicts the equality $\phi'(g_i) = 1$. Therefore $G$ embeds into $G'$.

$\square$
5.2 Homogeneity

In this subsection, we shall prove that co-Hopfian virtually free groups are $\forall \exists$-homogeneous. First, we need some preliminary results.

**Definition 5.3.** Let $G$ be a virtually free group. We say that an endomorphism $\phi$ of $G$ is a class-permuting endomorphism if there exists an integer $n \geq 1$ such that $\phi^n \sim \text{id}_G$ (in the sense of Definition 2.3).

**Remark 5.4.** The terminology is motivated by the fact that an endomorphism $\phi$ is class-permuting if and only if it induces a permutation of the set of conjugacy classes of (maximal) finite subgroups of $G$ (see Lemma 5.5 below).

If $G$ is a co-Hopfian virtually free group, every class-permuting endomorphism of $G$ is an automorphism, by Proposition 4.6. It is not completely obvious from Definition 5.3 that being a class-permuting endomorphism is expressible via a first-order sentence. As a first step, we need to reformulate this definition.

**Lemma 5.5.** Let $G$ be a virtually free group. An endomorphism $\phi$ of $G$ is class-permuting if and only if the following two conditions hold.

1. If $A$ is a maximal finite subgroup, then $\phi(A)$ is a maximal finite subgroup.
2. If $A$ and $B$ are two maximal finite subgroups, then $\phi(A)$ and $\phi(B)$ are conjugate if and only if $A$ and $B$ are conjugate.

**Proof.** Let $E$ denote the set of conjugacy classes of maximal finite subgroups of $G$. Suppose that the two conditions above hold. By the first condition, $\phi$ induces a well-defined map from $E$ to $E$. By the second condition this map is injective, and hence it is bijective since $E$ is finite. Therefore, there exists an integer $m \geq 1$ such that $\phi^m$ maps every maximal finite subgroup $A$ to a conjugate of $A$. Then, there is a nonzero multiple $n$ of $m$ such that $\phi^n$ is equivalent to $\text{id}_G$. Conversely, it is not hard to see that every class-permuting endomorphism of $G$ satisfies the two conditions above. □

Using the previous lemma, we shall prove that being class-permuting can be expressed via a universal formula. We shall need a characterization of maximal finite subgroups in a virtually free group.

**Lemma 5.6.** Let $A$ be a finite subgroup of a virtually free group $G$. The following two conditions are equivalent.

1. $A$ is a maximal finite subgroup of $G$.
2. For every element $g \in G \setminus A$ of finite order, there exists an element $a \in A$ such that $ga$ has infinite order.

**Proof.** Let $A$ be a maximal finite subgroup of $G$, and let us prove that the second condition above is satisfied. Let $\Delta$ be a reduced Stallings splitting of $G$, and let $T$ be its Bass-Serre tree. Since $A$ is finite, it is contained in a vertex group $G_v$ of $T$. Since $A$ is maximal among finite subgroups of $G$, we have $A = G_v$. If $A$ is the unique maximal finite subgroup of $G$, that is, if every finite
subgroup of $G$ is contained in $A$, then every element of $G$ of finite order belongs to $A$ and the second condition is obvious. Otherwise, let $g$ be an element of $G \setminus A$ of finite order. The element $g$ belongs to a vertex group $G_v$ of $T$, with $v \neq w$. Let $e$ denote the path between $v$ and $w$ in $T$. Note that $g$ does not belong to $G_e$. Moreover, since $\Delta$ is reduced, and since $G$ is not a finite extension of a free group, there exists an element $a \in A = G_v$ such that $a$ does not belong to $G_e$. The element $ga$ has infinite order: Indeed, if $ga$ had finite order, then the subgroup $\langle g, a \rangle$ would be elliptic in $T$ (by a well-known lemma of Serre), which is not possible since $\text{Fix}(a) \cap \text{Fix}(g) = \emptyset$.

Conversely, let us prove the contrapositive of $(2) \Rightarrow (1)$. Let $A$ be a finite subgroup of $G$ that is not maximal. First, let us prove that $A$ is not a vertex group of $T$. Suppose toward a contradiction that $A = G_v$ for some vertex $v \in T$. Since $A$ is not maximal, it is contained in a finite group $A'$ with $|A'| > |A|$. The finite group $A'$ fixes a vertex $w \in T$ with $w \neq v$, and thus $A$ fixes the path joining $v$ to $w$. Let $v_0 = v, v_1, \ldots, v_n = w$ be the sequence of vertices on this path, with $n \geq 1$. Let $i$ be the largest integer such that $G_{v_i} = A$. By definition of $i$, the vertex group $G_{v_{i+1}}$ contains $A$ strictly. Set $x = v_i$ and $y = v_{i+1}$. The edge $e = [x, y]$ satisfies $G_e = G_x$ but $y$ is not in the orbit of $x$ since $|G_x| < |G_y|$, which contradicts the fact that $T$ is reduced. In conclusion, $A$ is not a vertex group. Let $v$ be a vertex of $T$ fixed by $A$. There exists an element $g$ in $G_v \setminus A$; this element has finite order, and $ga$ has finite order for every $a \in A$. □

The following lemma shows that being class-permuting can be expressed by means of a universal formula.

Lemma 5.7. Let $G$ be a virtually free group, and let $\{s_1, \ldots, s_n\}$ be a generating set for $G$. There exists a universal formula $\exists(x_1, \ldots, x_n)$ with $n$ free variables such that, for every endomorphism $\phi$ of $G$, the tuple $(\phi(s_1), \ldots, \phi(s_n))$ satisfies $\exists(x_1, \ldots, x_n)$ in $G$ if and only if $\phi$ is class-permuting.

Proof. Let $A_1, \ldots, A_r$ be a collection on representatives of the conjugacy classes of maximal finite subgroups of $G$. By virtue of Lemma 5.5, we just have to check that the following two conditions are expressible via a universal formula:

1. for every $1 \leq i \leq r$, $\phi(A_i)$ is a maximal finite subgroup,
2. and for every $1 \leq i \neq j \leq r$, $\phi(A_i)$ and $\phi(A_j)$ are not conjugate.

The second condition is clearly a universal condition (in natural language: ‘for every $g \in G$, $\phi(A_i)$ and $g\phi(A_j)g^{-1}$ are distinct’). It remains to prove that the first condition is universal. Let $N$ denote the maximal order of a finite subgroup of $G$. By Lemma 5.6, the first condition is equivalent to the following: For every element $g \in G \setminus \phi(A_i)$ of finite order (that is, such that $g^{|N|} = 1$), there exists an element $h \in \phi(A_i)$ such that $gh$ has infinite order (that is, such that $(gh)^{|N|} \neq 1$). Again, this statement is expressible by a universal formula (indeed, the statement about the existence of $h \in \phi(A_i)$ such that $gh$ has infinite order does not require an existential disjunction quantifier since we just have to write a finite disjunction of inequalities). □

We are ready to prove the main result of this subsection.

Theorem 5.8. Co-Hopfian virtually free groups are $\forall \exists$-homogeneous.

Proof. Let $G$ be a co-Hopfian virtually free group, and let $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$ be two $k$-tuples of elements of $G$ having the same $\forall \exists$-type. We shall prove that there exists a class-permuting endomorphism $\phi$ of $G$ mapping $u$ to $v$. 
Fix a finite presentation $G = \langle s_1, \ldots, s_n \mid \Sigma(s_1, \ldots, s_n) = 1 \rangle$, where $\Sigma(x_1, \ldots, x_n) = 1$ is a finite system of equations in the variables $x_1, \ldots, x_n$. For every $1 \leq i \leq k$, the element $u_i$ can be written as a word $w_i(s_1, \ldots, s_n)$. We can write a $\exists \forall$-formula $\mu(u)$ (see below) that is satisfied by $G$, and whose meaning is ‘there exists a class-permuting endomorphism $\phi$ of $G$ that maps $u$ to $u'$’ (note that this statement is obviously true since we can take $\phi = \text{id}_G$). In the following formula, $\vartheta(x_1, \ldots, x_n)$ denotes the universal formula given by Lemma 5.7.

$$\mu(u) : \exists x_1 \ldots \exists x_n \Sigma(x_1, \ldots, x_n) = 1 \wedge u_i = w_i(x_1, \ldots, x_n) \wedge \vartheta(x_1, \ldots, x_n).$$

Since $u$ and $v$ have the same $\exists \forall$-type (as they have the same $\forall \exists$-type), the formula $\mu(v)$ is satisfied by $G$ as well. Let $g_1, \ldots, g_k$ be the elements of $G$ given by the interpretation of $\mu(v)$ in $G$. We can define an endomorphism $\phi$ of $G$ mapping $s_i$ to $g_i$ for every $1 \leq i \leq k$. This endomorphism maps $u$ to $v$ and it is class-permuting thanks to the previous lemma. By definition, there exists an integer $m \geq 1$ such that $\phi^m$ is equivalent to $\text{id}_G$, and hence $\phi^m$ is an automorphism of $G$ by Proposition 4.6. Thus $\phi$ is an automorphism of $G$. □

5.3 | Prime models

Recall that a group $G$ is prime if it elementary embeds in every group $G'$ that is elementarily equivalent to $G$. In this subsection, we consider co-Hopfian virtually free groups with finite outer automorphism group. We shall see that these groups are prime and $\exists$-homogeneous.

In [14], Pettet gave a characterization of virtually free groups that have finite outer automorphism group. Note that this class is different from the class of co-Hopfian virtually free groups, as shown by the following examples.

Example 5.9. Here is an example of a co-Hopfian virtually free group with infinitely many outer automorphisms. Let $A$, $B$ and $C$ be three groups isomorphic to the symmetric group $\mathfrak{S}_3$. Let $a$, $b$, $c$ be elements of order 2 in $A$, $B$, $C$ respectively. Define $H = \langle a \rangle \times (B \ast_B C)$ and $G = A \ast_{\langle a \rangle} H$. In other words, $G$ is the fundamental group of the following graph of groups:

- $\langle a \rangle$
- $\langle a \rangle \times \langle b = c \rangle$
- $A$
- $\langle a \rangle \times B$
- $\langle a \rangle \times C$

We easily see that $N_A(\langle a \rangle) = \langle a \rangle$ and $N_{\langle a \rangle \times C}(\langle a \rangle \times \langle c \rangle) = \langle a \rangle \times \langle c \rangle$, and hence $G$ is co-Hopfian by Theorem 4.3. On the other hand, Out($G$) is infinite. Indeed, if $h \in H$ is an element of infinite order, the Dehn twist $\phi_h$ (defined by $\phi_h(x) = x$ if $x \in A$ and $\phi_h(x) = h x h^{-1}$ if $x \in H$) has infinite order in Out($G$).

Example 5.10. Here is an example of a virtually free group with only finitely many outer automorphisms, and which is not co-Hopfian. Let $G = \mathbb{Z} / 3\mathbb{Z} \ast \mathbb{Z} / 3\mathbb{Z} \simeq \text{PSL}_2(\mathbb{Z})$. As a free product, $G$ is not co-Hopfian. But Out($G$) is finite by [14]. More generally, Guirardel and Levitt proved in [8] (Theorem 7.14) that a hyperbolic group $G$ has an infinite outer automorphism group if and only if $G$ splits over a $\mathbb{Z}_{\text{max}}$-subgroup (that is, a virtually cyclic subgroup with infinite center which is maximal for inclusion among virtually cyclic subgroups with infinite center). Therefore, if
$G = A \ast_C B$ with $A$ and $B$ finite then $\text{Out}(G)$ is finite. For instance, $G = \mathbb{Z}/4\mathbb{Z} \ast_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} \simeq \text{SL}_2(\mathbb{Z})$ is not co-Hopfian but $\text{Out}(G)$ is finite.

**Example 5.11.** $\text{GL}_2(\mathbb{Z})$ is co-Hopfian and it has only finitely many outer automorphisms.

The following definition was introduced by Ould Houcine in [13, Definition 1.4].

**Definition 5.12.** A group $G$ is said to be **strongly co-Hopfian** if there exists a finite set $F \subset G \setminus \{1\}$ such that for every endomorphism $\phi$ of $G$, if $\ker(\phi) \cap F = \emptyset$ then $\phi$ is an automorphism.

In [13, Lemma 3.5], Ould Houcine observed that being strongly co-Hopfian has interesting model-theoretic consequences.

**Lemma 5.13.** Let $G$ be a finitely presented group. If $G$ is strongly co-Hopfian, then $G$ is prime and $\exists$-homogeneous.

Examples of strongly co-Hopfian groups include torsion-free hyperbolic groups that do not split non-trivially over $\mathbb{Z}$ or as a free product (see [18]), $\text{Out}(F_n)$, $\text{Aut}(F_n)$ and the mapping-class group $\text{MCG}(\Sigma_g)$ of a connected closed orientable surface of genus $g$ sufficiently large (as observed in [3]). Therefore, all these groups are prime and $\exists$-homogeneous.

**Proposition 5.14.** Let $G$ be a co-Hopfian virtually free group with finite outer automorphism group. Then $G$ is strongly co-Hopfian. As a consequence, $G$ is prime and $\exists$-homogeneous.

**Proof.** Let $F$ be the finite subset of $G \setminus \{1\}$ given by Proposition 3.3. By assumption, the group $\text{Inn}(G)$ of inner automorphisms of $G$ has finite index in $\text{Aut}(G)$. Write $\text{Aut}(G) = \bigcup_{1 \leq i \leq \ell} \sigma_i \circ \text{Int}(G)$ and set $F' = \bigcup_{1 \leq i \leq \ell} \sigma_i(F)$. By Proposition 3.3, every endomorphism $\phi$ of $G$ such that $\ker(\phi) \cap F' = \emptyset$ is injective, and hence $\phi$ is an automorphism since $G$ is co-Hopfian. 

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