Center of the Yangian double in type A

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ABSTRACT. We prove the R-matrix and Drinfeld presentations of the Yangian double in type A are isomorphic. The central elements of the completed Yangian double in type A at the critical level are constructed. The images of these elements under a Harish-Chandra-type homomorphism are calculated by applying a version of the Poincaré-Birkhoff-Witt theorem for the R-matrix presentation. These images coincide with the eigenvalues of the central elements in the Wakimoto modules.

1. Introduction

During the last few decades, quantum groups introduced by Drinfeld [7] and Jimbo [15] have been studied extensively and played important roles in many branches of mathematics and physics. The Yangian $Y_h(g)$ is one of the most important examples in the theory of quantum groups. The Hopf algebra first appeared in the framework of the inverse scattering method introduced by Faddeev, Sklyanin and their colleagues, see [8], [9], [10]. As is well-known, the Yangian $Y_h(g)$ is a canonical deformation of the universal enveloping algebra $U(g[t])$. The Yangian double, which is the quantum double [6] of the Yangian $Y_h(g)$, has nice applications in massive field theory [11], [19], [24]. In these works, the Yangian double $DY_h(g)$ without central extension was discussed for $g = sl_2$ [18]. As for the general case, Iohara [14] defined the Yangian double $DY_h(g)$ with a central extension for $g = gl_n, sl_n$.

The purpose of this paper is to give explicit formulas for generating central elements of the completed Yangian double $DY_h(gl_n)_{cr}$ at the critical level (Theorem 4.4). The formulas are given with the help of the RLL-presentation of $DY_h(sl_n)$. We use a version of the Poincaré-Birkhoff-Witt theorem for this presentation to introduce an analog of the Harish-Chandra homomorphism and calculate the image of the central elements, and these central elements are constructed as certain higher Casimir operators (cf. [2]). Moreover, we construct an isomorphism between the R-matrix and Drinfeld realization of the Yangian double $DY_h(gl_n)$, i.e. the Yangian analog of the

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Ding-Frenkel isomorphism in quantum affine algebras [5]. Then we apply
the isomorphism to calculate the eigenvalues of the central elements in the
Wakimoto modules over $\text{DY}_h(\mathfrak{gl}_n)_{cr}$. As an application of Theorem 4.4, we
give explicit formulas for invariants of the vacuum module $V_h(\mathfrak{gl}_n)$ over the
Yangian double. The invariants are obtained by the action of the central
elements on the vacuum vector.

The center of the quantum affine algebra at the critical level has been
studied extensively [4, 11]. It is thus expected that similar results hold
for the Yangian double, and we intend to give detailed computation for the
image of Harish-Chandra homomorphism. The Yangian double in type A
at the critical level admits a quantum vertex algebra structure [16]. In the
BCD cases we expect there is a similar quantum vertex algebra structure as
results of this paper indicate.

The layout of the paper is as follows. In section 2 we give deta-
iled construction of the Yangian double $\text{DY}_h(\mathfrak{gl}_n)$ in the FRT-formulism and study
the quantum root generators. This result should be known to the experts
but we include the details as we could not find it in the literature, though
the similar construction is given for type BCD in [17]. In section 3 we pass
it down to the Drinfeld realization of the double Yangian $\text{DY}_h(\mathfrak{sl}_n)$ using
a different method from that of the quantum affine algebra. The center of
the completed algebra $\text{DY}_h(\mathfrak{gl}_n)$ at the critical level is discussed in section
4. In this section, we use a slightly different definition of $\text{DY}_h(\mathfrak{gl}_n)$ by nor-
malizing the Yang R-matrix. In section 5 we define the Harish-Chandra
homomorphism for the quantum algebra $\text{DY}_h(\mathfrak{gl}_n)_{cr}$, and then in section 6
we determine the eigenvalues of the Harish-Chandra operator at the crit-
ical level and find that the eigenvalues are also expressed as some shifted
elementary symmetric functions (cf. [20]).

2. Drinfeld generators of the Yangian double

Let $V = \mathbb{C}^n$, and $P$ the permutation operator of $V \otimes V$ such that
$$Pv \otimes w = w \otimes v, \quad v, w \in V.$$

Let $h$ be a parameter, and let $\bar{R}(u)$ be Yang’s R-matrix:
$$\bar{R}(u) = I + \frac{h}{u}P \in \text{End}(V \otimes V).$$
(2.1)

where $\bar{R}(u)$ is expanded in negative powers of $u$ or positive powers of $h$. The
matrix $\bar{R}(x)$ satisfies the Yang-Baxter equation on $V^{\otimes 3}$ with the unitary
condition:
$$\bar{R}_{12}(u-v)\bar{R}_{13}(u)\bar{R}_{23}(v) = \bar{R}_{23}(v)\bar{R}_{13}(u)\bar{R}_{12}(u-v),$$
(2.2)
$$\bar{R}_{12}(u)\bar{R}_{21}(-u) = \frac{u^2 - h^2}{u^2}I.$$  
(2.3)

Here for any $A = \sum a_{(1)} \otimes a_{(2)} \in \text{End}(V^{\otimes 2})$, one denotes that $A_{12} = A \otimes I, A_{23} = I \otimes A$, and $A_{13} = \sum a_{(1)} \otimes I \otimes a_{(2)} \in \text{End}(V^{\otimes 3})$. 

Let $\mathcal{A}$ be the ring $\mathbb{C}[[h]]$ in the $h$-adic topology.

**Definition 2.1.** The Yangian double $\text{DY}_h(\mathfrak{gl}_n)$ is the associative algebra over $\mathcal{A}$ generated by $\{l_{ij}^{(k)} | 1 \leq i, j \leq n, \ k \in \mathbb{Z}\}$ and $c$. The relations are defined in terms of the generating series

\[
l_{ij}^+(u) = \delta_{ij} - h \sum_{k \in \mathbb{Z} \geq 0} l_{ij}^{(k)} u^{-k-1},
\]

\[
l_{ij}^-(u) = \delta_{ij} + h \sum_{k \in \mathbb{Z} < 0} l_{ij}^{(k)} u^{-k-1},
\]

and $l_{ij}^\pm(u)$ are in the matrix form:

\[
L^\pm(u) = (l_{ij}^\pm(u))_{1 \leq i, j \leq n}.
\]

The defining relations are given as follows.

1. $[L^\pm(u), c] = 0$, \hspace{1cm} (2.4)
2. $\bar{R}(u - v)L_1^+(u)L_2^+(v) = L_2^+(v)L_1^+(u)\bar{R}(u - v)$, \hspace{1cm} (2.5)
3. $\bar{R}(u - v - \frac{1}{2}hc)L_1^+(u)L_2^-(v) = L_2^-(v)L_1^+(u)\bar{R}(u - v + \frac{1}{2}hc)$, \hspace{1cm} (2.6)

where the $R$-matrix is viewed as power series in $u^{-1}$ (via appropriate $h$-adic topology) and we have adopted the convention

\[
L_1^\pm(u) = L^\pm(u) \otimes id,
\]

\[
L_2^\pm(u) = id \otimes L^\pm(u).
\]

The algebra $\text{DY}_h(\mathfrak{gl}_n)$ has a Hopf algebra structure given by

\[
\Delta(L^\pm(u)) = \sum_{k=1}^{n} L_{kj}^\pm(u \pm \frac{1}{4}hc_2) \otimes L_{ik}^\pm(u \pm \frac{1}{4}hc_1),
\]

\[
\Delta(c) = c \otimes 1 + 1 \otimes c,
\]

\[
\varepsilon(L^\pm(u)) = I, \quad \varepsilon(c) = 0,
\]

\[
S(L^\pm(u)) = [L^\pm(u)]^{-1}, \quad S(c) = -c,
\]

where $c_1 = c \otimes 1$ and $c_2 = 1 \otimes c$. 
Definition 2.2. For $1 \leq p, q \leq n$, we define the submatrices $L^\pm_{p,q}(u)$ of $L^\pm(u)$ as follows.

$$p = q, \quad L^\pm_{p,p}(u) = (l_{ij}(u))_{1 \leq i,j \leq p}$$  \hspace{1cm} \quad (2.7)$$

$$p < q, \quad L^\pm_{p,q}(u) = \begin{pmatrix}
    l^\pm_{11}(u) & \cdots & l^\pm_{1,p-1}(u) & l^\pm_{1,q}(u) \\
    \vdots           & \ddots & \vdots           & \vdots           \\
    l^\pm_{p-1,1}(u) & \cdots & l^\pm_{p-1,p-1}(u) & l^\pm_{p-1,q}(u) \\
    l^\pm_{p,1}(u)   & \cdots & l^\pm_{p,p-1}(u) & l^\pm_{p,q}(u)
\end{pmatrix}$$ \hspace{1cm} (2.8)$$

$$p > q, \quad L^\pm_{p,q}(u) = \begin{pmatrix}
    l^\pm_{11}(u) & \cdots & l^\pm_{1,q-1}(u) & l^\pm_{1,q}(u) \\
    \vdots           & \ddots & \vdots           & \vdots           \\
    l^\pm_{q-1,1}(u) & \cdots & l^\pm_{q-1,q-1}(u) & l^\pm_{q-1,q}(u) \\
    l^\pm_{p,1}(u)   & \cdots & l^\pm_{p,q-1}(u) & l^\pm_{p,q}(u)
\end{pmatrix}$$ \hspace{1cm} (2.9)$$

Then we can give the Gauss decomposition of $L^\pm(u)$ in the following result.

Theorem 2.3. $L^\pm(u)$ have the following unique decompositions:

$L^\pm(u) = F^\pm(u)H^\pm(u)E^\pm(u)$

$$= \begin{pmatrix}
    1 & \cdots & 0 \\
    \vdots & \ddots & \vdots \\
    f_{n,1}^\pm(u) & \cdots & f_{n,n-1}^\pm(u) & 1
\end{pmatrix}
\begin{pmatrix}
    f_{2,1}^\pm(u) & \cdots & 0 \\
    \vdots & \ddots & \vdots \\
    f_{n,1}^\pm(u) & \cdots & f_{n,n-1}^\pm(u) & 1
\end{pmatrix}
\begin{pmatrix}
    1 & \cdots & 0 \\
    \vdots & \ddots & \vdots \\
    e_{1,2}^\pm(u) & \cdots & e_{1,n}^\pm(u) \\
    \vdots & \ddots & \vdots \\
    e_{n-1,n}^\pm(u) & \cdots & 1
\end{pmatrix}$$

To prove this theorem, we only need to show that each component $f^\pm_{p,q}(u)$, $k_p^\pm(u)$ or $e^\pm_{p,q}(u)$ is well-defined in $DY_h(\mathfrak{gl}_n)[[u^\pm 1]]$. In fact, we have the following explicit formulas of these elements in terms of quantum minors. Then the elements are well-defined since the algebra $DY_h(\mathfrak{gl}_n)$ is $h$-adically completed.

The following lemma is proved in [14]. Recall that the (column) quantum determinant of a matrix $M = (m_{ij}(x))_{n \times n}$ is defined by

$$qdet(M) = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) m_{\sigma_1,1}(x)m_{\sigma_2,2}(x + h) \cdots m_{\sigma_n,n}(x + (n - 1)h).$$
Lemma 2.4. We have the following expansions for the quantum determinants
\[ \text{qdet}L^+(u) = k_1^+(u)k_2^+(u + h) \cdots k_n^+(u + (n - 1)h). \]  
(2.10)

Now set
\[ X_i^-(u) = f_{i+1,i}^+(u + \frac{1}{4} h) - f_{i+1,i}^-((u - \frac{1}{4} h), \]
\[ X_i^+(u) = e_{i+1,i}^+(u - \frac{1}{4} h) - e_{i+1,i}^-(u + \frac{1}{4} h). \]

To decompose \( \text{DY}_h(\mathfrak{g}_n) \) into a product of two subalgebras, we introduce the following currents:
\[ H_i^\pm(u) = k_{i+1}^\pm(u + \frac{1}{2} hi)k_i^\pm(u + \frac{1}{2} hi)^{-1}, \]
\[ K_i^\pm(u) = \prod_{i=1}^n k_i^\pm(u + (i - \frac{n+1}{2})h), \]
\[ E_i(u) = \frac{1}{h}X_i^+(u + \frac{1}{2} hi), F_i(u) = \frac{1}{h}X_i^-(u + \frac{1}{2} hi). \]

We define \( \text{DY}_h(\mathfrak{sl}_n) \) to be the subalgebra of \( \text{DY}_h(\mathfrak{g}_n) \) generated by \( H_i^\pm(u), E_i(u), F_i(u) \) and \( c \). Note that \( \text{DY}_h(\mathfrak{sl}_n) \) inherits the Hopf algebra structure (cf. [3]) from that of \( \text{DY}_h(\mathfrak{g}_n) \). The Heisenberg subalgebra of \( \text{DY}_h(\mathfrak{g}_n) \) generated by \( K_i^\pm(u) \) commutes with each element of \( \text{DY}_h(\mathfrak{sl}_n) \). Next we will write down the relations between the currents \( k_i^\pm(u), X_i^+(u) \) and \( X_i^-(u) \).

Theorem 2.5. The following relations hold in the algebra \( \text{DY}_h(\mathfrak{g}_n) \):
\[ k_i^\pm(u)k_j^\pm(v) = k_j^\pm(v)k_i^\pm(u) \]
\[ \frac{u_- - v_+ + h}{u_- - v_+}k_i^+(u)k_i^-(v) = \frac{u_+ - v_- + h}{u_+ - v_-}k_i^-(v)k_i^+(u) \]
\[ k_i^+(u)k_j^-(v) = k_j^-(v)k_i^+(u)(i < j) \]
\[ \frac{(u_+ - v_-)^2}{(u_+ - v_-)^2 - h^2}k_i^+(u)k_i^-(v) = \frac{(u_- - v_+)^2}{(u_- - v_+)^2 - h^2}k_i^-(v)k_i^+(u)(i < j) \]
\[ k_i^\pm(u)^{-1}X_i^+(v)k_i^\pm(u) = \frac{u_+ - v_- + h}{u_+ - v_-}X_i^+(v) \]
\[ k_i^\pm(u)X_i^-(v)k_i^\pm(u)^{-1} = \frac{u_+ - v_- + h}{u_+ - v_-}X_i^-(v) \]
\[ k_{i+1}^\pm(u)X_i^+(v)k_{i+1}^\pm(u)^{-1} = \frac{u_+ - v_- + h}{u_+ - v_-}X_i^+(v) \]
\[ k_{i+1}^\pm(u)X_i^-(v)k_{i+1}^\pm(u)^{-1} = \frac{u_+ - v_- + h}{u_+ - v_-}X_i^-(v) \]
\[ k_j^\pm(u)^{-1}X_i^+(v)k_j^\pm(u) = X_i^+(v), k_j^\pm(u)X_i^-(v)k_j^\pm(u)^{-1} = X_i^-(v) \quad (j \neq i, i+1) \]
\[ (u - v \mp h)X_i^+(u)X_i^+(v) = (u - v \mp h)X_i^+(v)X_i^+(u) \]
\[ (u - v + h)X_i^+(u)X_i^+(v) = (u - v)X_i^+(v)X_i^+(u) \]
\[ (u - v)X_i^-(u)X_i^+(v) = (u - v + h)X_i^+(v)X_i^-(u) \]
The defining relations (2.5) and (2.6) imply the following relations:

\[ X_i^\pm(u_1)X_i^\pm(u_2)X_j^\pm(v) - 2X_i^\pm(u_1)X_j^\pm(v)X_i^\pm(u_2) + X_j^\pm(v)X_i^\pm(u_1)X_i^\pm(u_2) + \{u_1 \leftrightarrow u_2\} = 0 \quad \text{if } |i - j| = 1 \]

\[ X_i^\pm(u)X_j^\pm(v) = X_j^\pm(v)X_i^\pm(u) \quad \text{if } |i - j| > 1 \]

\[ [X_i^+(u), X_j^-(v)] = \hbar \delta_{ij} \left( \delta(u_-, v_+)k_{i+1}^+(u_-)k_i^+(u_-) - \delta(u_+, v_-)k_{i+1}^-(v_-)k_i^-(v_-) \right) \]

\[ k_i^-(v_-)^{-1} \] where \( \delta(u - v) = \sum_{k \in \mathbb{Z}} u^{-k-1} v^k \).

**Remark.** The relations in Theorem 2.5 are similar to those from Iohara’s paper [14], though Iohara’s paper did not contain its proof. We also note that the R-matrix \( \tilde{R}(u) \) in our paper is a bit different from the R-matrix \( R(u) \) used in [14].

**Proof.** Note that

\[ X_i^+(u) = e_{i,i+1}^+(u_-) - e_{i,i+1}^-(u_+), \quad X_i^-(u) = f_{i,i+1}^+(u_+) - f_{i,i+1}^-(u_-), \]

(2.11) where \( u_\pm = u \pm \frac{1}{2} \hbar c \). In the rest of the paper, we will use \( f_i^\pm(u) \) to denote \( f_{i,i+1}^\pm(u) \) and \( e_i^\pm(u) \) to denote \( e_{i,i+1}^\pm(u) \). Since \( k_i^\pm(u) \) are invertible, we can express the elements \( e_i^\pm(u), f_j^\pm(u) \) and \( k_i^\pm(u)(i < j) \) in terms of the matrix coefficients of \( L^\pm(u) \) in a unique manner.

We can express the inverse of the matrix \( L^\pm(u) \) as follows:

\[
L^\pm(u)^{-1} = \begin{pmatrix}
1 & \cdots & 0 \\
-e_i^+(u) & \ddots & \cdots \\
\vdots & \ddots & -e_{n-1}^\pm(u) & 1 \\
1 & -f_1^\pm(u) & \cdots & * \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & -f_{n-1}^\pm(u) & 1
\end{pmatrix}
\]  

(2.12)

The defining relations (2.5) and (2.6) imply the following relations:

\[
L_1^\pm(v)^{-1}\tilde{R}_{21}(u - v)L_2^\pm(u) = L_2^\pm(u)\tilde{R}_{21}(u - v)L_1^\pm(v)^{-1}
\]

(2.13)

\[
L_1^-(v)^{-1}\tilde{R}_{21}(u_- - v_+)L_2^\pm(u) = L_2^\pm(u)\tilde{R}_{21}(u_+ - v_-)L_1^-(v)^{-1}
\]

(2.14)

\[
\frac{(u_+ - v_-)^2}{(u_+ - v_-)^2 - \hbar^2}\tilde{R}_{21}(u_+ - v_-)L_2^\pm(u)L_1^\pm(v) = \frac{(u_- - v_+)^2}{(u_- - v_+)^2 - \hbar^2}L_1^\pm(v)
\]

(2.15)

\[ L_2^\pm(u)\tilde{R}_{21}(u_- - v_+) \]
\[
\frac{(u_+ - v_-)^2}{(u_+ - v_-)^2 - k^2} \tilde{R}_21(u_+ - v_-) L_2^-(u) = \frac{(u_- - v_+)^2}{(u_- - v_+)^2 - k^2} L_2^-(u) \]
(2.16)

\[\tilde{R}_21(u_- + v_+) L_1^+(v)^{-1} \]

\[
L_2^+(u)^{-1} L_1^+(v)^{-1} \tilde{R}_21(u - v) = \tilde{R}_21(u - v) L_1^+(v)^{-1} L_2^+(u)^{-1} \]
(2.17)

\[
L_2^+(u)^{-1} L_1^-(v)^{-1} \tilde{R}_21(u_- - v_+) = \tilde{R}_21(u_+ - v_-) L_1^-(v)^{-1} L_2^+(u)^{-1} \]
(2.18)

First we consider the case \(n = 2\). By definition, we have the following formulas:

\[
L^\pm(u) = \left[ \begin{array}{ccc}
\frac{k^\pm_1(u)}{f^\pm_1(u)} & \frac{k^\pm_1(u)e^\pm_1(u)}{k^\pm_2(u)} & 0 \\
0 & \frac{k^\pm_2(u)}{f^\pm_1(u)} & 0 \\
0 & 0 & 0
\end{array} \right]
\]
(2.19)

\[
L^\pm(u)^{-1} = \left[ \begin{array}{ccc}
\frac{u^2 + 1}{u^2} & 0 & 0 \\
0 & \frac{1}{u^2} & 0 \\
0 & 0 & \frac{1}{u^2}
\end{array} \right]
\]
(2.20)

\[
\tilde{R}_21(u) = \tilde{R}(u) = \left[ \begin{array}{ccc}
\frac{u + h}{u} & 0 & 0 \\
0 & \frac{1}{u} & 0 \\
0 & \frac{h}{u} & 1
\end{array} \right]
\]
(2.21)

\[
L^\pm_1(v)^{-1} = \left[ \begin{array}{ccc}
0 & 0 & \frac{- e^\pm_1(v)k^\pm_2(v)^{-1}}{k^\pm_2(v)} \\
\frac{e^\pm_1(v)k^\pm_2(v)^{-1}}{k^\pm_2(v)} & 0 & 0 \\
0 & \frac{k^\pm_2(v)}{k^\pm_2(v)} & 0
\end{array} \right]
\]
(2.22)

\[
L^\pm_2(v)^{-1} = \left[ \begin{array}{ccc}
\frac{e^\pm_1(u)k^\pm_2(u)^{-1}}{k^\pm_2(u)} & 0 & 0 \\
0 & \frac{k^\pm_2(u)}{k^\pm_2(u)} & 0 \\
0 & 0 & \frac{- e^\pm_1(u)k^\pm_2(u)^{-1}}{k^\pm_2(u)}
\end{array} \right]
\]
(2.23)

From (2.20) and (2.13)-(2.19), we get all the relations among \(k^\pm_1(u), k^\pm_2(u)\).

\[
k^\pm_1(u)k^\pm_1(v) = k^\pm_1(v)k^\pm_1(u)
\]
(2.24)

\[
k^\pm_2(u)k^\pm_2(v) = k^\pm_2(v)k^\pm_2(u)
\]
(2.25)

\[
\frac{u_+ - v_- + h}{u_+ - v_-} k^\pm_1(u)k^\pm_1(v) = \frac{u_+ - v_- + h}{u_+ - v_-} k^\pm_1(v)k^\pm_1(u)
\]
(2.26)

\[
\frac{u_+ - v_- + h}{u_+ - v_-} k^\pm_2(u)k^\pm_2(v) = \frac{u_+ - v_- + h}{u_+ - v_-} k^\pm_2(v)k^\pm_2(u)
\]
(2.27)

\[
k^\pm_1(u)k^\pm_2(v) = k^\pm_2(v)k^\pm_1(u)
\]
(2.28)
Next we write down the relations between $k_1^\pm (u)$ and $e_1^\pm (u)$ or $f_1^\pm (u)$.

\[
\frac{u - v + h}{u - v} k_1^\pm (u) k_1^\pm (v) k_1^\pm (u) = \frac{h}{u - v} f_1^\pm (u) k_1^\pm (v) k_1^\pm (u) + \frac{h}{u - v} k_1^\pm (v) k_1^\pm (u) e_1^\pm (u)
\]  

(2.31)

\[
\frac{u - v - h}{u - v} k_1^\pm (u) k_1^\pm (v) e_1^\pm (v) = k_1^\pm (v) e_1^\pm (v) k_1^\pm (u) + \frac{h}{u - v} k_1^\pm (v) k_1^\pm (u) e_1^\pm (u)
\]  

(2.32)

\[
\frac{u - v - h}{u - v} f_1^\pm (v) k_1^\pm (v) k_1^\pm (u) = \frac{h}{u - v} f_1^\pm (u) k_1^\pm (v) k_1^\pm (v) + \frac{h}{u - v} k_1^\pm (v) k_1^\pm (v) e_1^\pm (u)
\]  

(2.33)

Thus

\[
k_1^\pm (u)^{-1} e_1^\pm (v) k_1^\pm (u) = \frac{u - v + h}{u - v} e_1^\pm (v) - \frac{h}{u - v} e_1^\pm (u)
\]  

(2.35)

\[
k_1^\pm (u) f_1^\pm (v) k_1^\pm (u)^{-1} = \frac{u - v + h}{u - v} f_1^\pm (v) - \frac{h}{u - v} f_1^\pm (u)
\]  

(2.36)

\[
k_1^\pm (v)^{-1} e_1^\pm (v) k_1^\pm (u) = \frac{u - v - h}{u - v} e_1^\pm (v) - \frac{h}{u - v} e_1^\pm (u)
\]  

(2.37)

\[
k_1^\pm (u) f_1^\pm (v) k_1^\pm (v)^{-1} = \frac{u - v - h}{u - v} f_1^\pm (v) - \frac{h}{u - v} f_1^\pm (u)
\]  

(2.38)

Similarly we have

\[
k_1^\pm (u)^{-1} e_1^\pm (v) k_1^\pm (u) = \frac{u - v + h}{u - v} e_1^\pm (v) - \frac{h}{u - v} e_1^\pm (u)
\]  

(2.39)

\[
k_1^\pm (u) f_1^\pm (v) k_1^\pm (v)^{-1} = \frac{u - v + h}{u - v} f_1^\pm (v) - \frac{h}{u - v} f_1^\pm (u)
\]  

(2.40)

Then we have

\[
k_1^\pm (u)^{-1} e_1^\pm (v) k_1^\pm (u) = \frac{u - v + h}{u - v} e_1^\pm (v) - \frac{h}{u - v} e_1^\pm (u)
\]  

(2.41)

\[
k_1^\pm (u) f_1^\pm (v) k_1^\pm (u)^{-1} = \frac{u - v + h}{u - v} f_1^\pm (v) - \frac{h}{u - v} f_1^\pm (u).
\]  

(2.42)
From (2.37) and (2.47), we get
\[ k_1^\pm(u)X_1^1(v)k_1^\pm(u) = \frac{u_+ - v + h}{u_+ - v}X_1^1(v), \tag{2.43} \]
\[ k_1^\pm(u)X_1^{-1}(v)k_1^\pm(u)^{-1} = \frac{u_+ - v + h}{u_+ - v}X_1^{-1}(v). \tag{2.44} \]

Then we obtain the relations among \( f_1^\pm(u), f_2^\pm(v) \) and \( e_1^\pm(u), e_2^\pm(v) \).
\[ k_1^\pm(u)e_1^\pm(v)k_1^\pm(u) = k_1^\pm(v)e_1^\pm(v)k_1^\pm(u)e_1^\pm(u), \tag{2.45} \]
\[ f_1^\pm(v)k_1^\pm(v)f_1^\pm(u)k_1^\pm(u) = f_1^\pm(u)k_1^\pm(u)f_1^\pm(v)k_1^\pm(v), \tag{2.46} \]
\[ \frac{u_+ - v_+ + h}{u_+ - v_+}k_1^\pm(u)e_1^\pm(u)k_1^\pm(v)e_1^\pm(v) = \frac{u_+ - v_+ + h}{u_+ - v_+}k_1^\pm(v)e_1^\pm(v)k_1^\pm(u)e_1^\pm(u), \tag{2.47} \]
\[ \frac{u_+ - v_+ + h}{u_+ - v_+}f_1^\pm(u)k_1^\pm(u)f_1^\pm(v)k_1^\pm(v) = \frac{u_+ - v_+ + h}{u_+ - v_+}f_1^\pm(v)k_1^\pm(v)f_1^\pm(u)k_1^\pm(u). \tag{2.48} \]

From (2.35) and (2.36), we get
\[ \frac{u - v + h}{u - v}e_1^\pm(v)e_1^\pm(u) - \frac{h}{u - v}e_1^\pm(u)e_1^\pm(v) = \frac{v - u + h}{v - u}e_1^\pm(u)e_1^\pm(v) - \frac{h}{v - u}e_1^\pm(v)e_1^\pm(v). \tag{2.49} \]

From (2.37) and (2.47), we get
\[ \frac{u_+ - v_+ + h}{u_+ - v_+}e_1^\pm(v)e_1^\pm(u) - \frac{h}{u_+ - v_+}e_1^\pm(u)e_1^\pm(v) = \frac{v_+ - u_+ + h}{v_+ - u_+}e_1^\pm(u)e_1^\pm(v) - \frac{h}{v_+ - u_+}e_1^\pm(v)e_1^\pm(v). \tag{2.50} \]

Similarly, we can prove
\[ \frac{u_+ - v_+ + h}{u_+ - v_+}e_1^\pm(v)e_1^\pm(u) - \frac{h}{u_+ - v_+}e_1^\pm(u)e_1^\pm(v) = \frac{v_+ - u_+ + h}{v_+ - u_+}e_1^\pm(u)e_1^\pm(v) - \frac{h}{v_+ - u_+}e_1^\pm(v)e_1^\pm(v). \tag{2.51} \]

Therefore,
\[ \frac{u_+ - v_+ + h}{u_+ - v_+}e_1^\pm(v)e_1^\pm(u) - \frac{h}{u_+ - v_+}e_1^\pm(u)e_1^\pm(v) = \frac{v_+ - u_+ + h}{v_+ - u_+}e_1^\pm(u)e_1^\pm(v) - \frac{h}{v_+ - u_+}e_1^\pm(v)e_1^\pm(v). \tag{2.52} \]
From (2.36) and (2.46), we get
\[
\frac{u - v + h}{u - v} f_1^\pm(u) f_1^\pm(v) - \frac{h}{u - v} f_1^\pm(u) f_1^\pm(u) = \frac{v - u + h}{v - u} f_1^\pm(v) f_1^\pm(u) - \frac{h}{v - u} f_1^\pm(v) f_1^\pm(v). \tag{2.53}
\]

From (2.38) and (2.48), we get
\[
\frac{u - v + h}{u - v} f_1^+(u) f_1^-(v) - \frac{h}{u - v} f_1^+(u) f_1^+(u) = \frac{v - u + h}{v - u} f_1^-(v) f_1^+(u) - \frac{h}{v - u} f_1^-(v) f_1^-(v). \tag{2.54}
\]

Similarly, we can prove
\[
\frac{u - v + h}{u - v} f_1^-(u) f_1^+(v) - \frac{h}{u - v} f_1^-(u) f_1^-(u) = \frac{v - u + h}{v - u} f_1^+(v) f_1^-(u) - \frac{h}{v - u} f_1^+(v) f_1^+(v). \tag{2.55}
\]

Hence,
\[
\frac{u - v + h}{u - v} f_1^+(u) f_1^+(v) - \frac{h}{u - v} f_1^+(u) f_1^+(u) = \frac{v - u + h}{v - u} f_1^+(v) f_1^+(u) - \frac{h}{v - u} f_1^+(v) f_1^+(v). \tag{2.56}
\]

From (2.49) and (2.52), we get
\[
(u - v - h) X_1^+(u) X_1^+(v) = (u - v + h) X_1^+(v) X_1^+(u). \tag{2.57}
\]

From (2.53) and (2.56), we get
\[
(u - v + h) X_1^-(u) X_1^-(v) = (u - v - h) X_1^-(v) X_1^-(u). \tag{2.58}
\]

The relations between $f_1^\pm(u)$, $e_1^\pm(u)$ and $k_2^\pm(u)$ are:
\[
- \frac{u - v + h}{u - v} e_1^\pm(u) k_2^\pm(u)^{-1} k_2^\pm(v)^{-1} = -k_2^\pm(v)^{-1} e_1^\pm(u) k_2^\pm(u)^{-1} - \frac{h}{u - v} e_1^\pm(v) k_2^\pm(v)^{-1} k_2^\pm(u)^{-1}, \tag{2.59}
\]
\[
- \frac{u - v + h}{u - v} k_2^\pm(v)^{-1} k_2^\pm(u)^{-1} f_1^\pm(u) = -k_2^\pm(u)^{-1} f_1^\pm(u) k_2^\pm(v)^{-1} - \frac{h}{u - v} k_2^\pm(u)^{-1} k_2^\pm(v)^{-1} f_1^\pm(v), \tag{2.60}
\]
From (2.64) and (2.70), we get
\[ -\frac{u_- - v_+ + h}{u_- - v_+} e_1^+(u)k_2^+(u)^{-1}k_2^-(v) = -k_2^-(v)^{-1}e_1^+(u)k_2^+(u)^{-1} - \frac{h}{u_+ - v_-} e_1^-(v)k_2^-(v)^{-1}k_2^+(u)^{-1}, \] (2.61)

\[ -\frac{u_+ - v_- + h}{u_+ - v_-} k_2^-(v)^{-1}k_2^+(u)^{-1}f_1^+(u) = -k_2^+(u)^{-1}f_1^+(u)k_2^-(v)^{-1} - \frac{h}{u_- - v_+} k_2^+(u)^{-1}k_2^-(v)^{-1}f_1^-(v). \] (2.62)

Then
\[ k_2^+(v)^{-1}e_1^+(u)k_2^+(v) = \frac{u - v + h}{u - v} e_1^+(u) - \frac{h}{u - v} e_1^+(v), \] (2.63)

\[ k_2^+(v)f_1^+(u)k_2^+(v)^{-1} = \frac{u - v + h}{u - v} f_1^+(u) - \frac{h}{u - v} f_1^+(v), \] (2.64)

\[ k_2^-(v)^{-1}e_1^+(u)k_2^+(v) = \frac{u_+ - v_- + h}{u_+ - v_-} e_1^+(u) - \frac{h}{u_+ - v_-} e_1^+(v), \] (2.65)

\[ k_2^-(v)f_1^+(u)k_2^+(v)^{-1} = \frac{u_+ - v_- + h}{u_+ - v_-} f_1^+(u) - \frac{h}{u_+ - v_-} f_1^+(v). \] (2.66)

Similarly, we can prove
\[ k_2^+(v)^{-1}e_1^-(u)k_2^-(v) = \frac{u_+ - v_+ + h}{u_+ - v_+} e_1^-(u) - \frac{h}{u_+ - v_+} e_1^+(v), \] (2.67)

\[ k_2^+(v)f_1^-^+(u)k_2^-(v)^{-1} = \frac{u_+ - v_+ + h}{u_+ - v_+} f_1^-(u) - \frac{h}{u_+ - v_+} f_1^+(v). \] (2.68)

So we have
\[ k_2^+(v)^{-1}e_1^+(u)k_2^+(v) = \frac{u_+ - v_+ + h}{u_+ - v_+} e_1^+(u) - \frac{h}{u_+ - v_+} e_1^+(v), \] (2.69)

\[ k_2^+(v)f_1^+(u)k_2^+(v)^{-1} = \frac{u_+ - v_+ + h}{u_+ - v_+} f_1^+(u) - \frac{h}{u_+ - v_+} f_1^+(v). \] (2.70)

From (2.63) and (2.69), we get
\[ k_2^+(u)^{-1}X_1^+(v)k_2^+(u) = \frac{u_+ - v + h}{u_+ - v} X_1^+(v). \] (2.71)

From (2.61) and (2.70), we get
\[ k_2^+(u)X_1^+(v)k_2^+(u)^{-1} = \frac{u_+ - v + h}{u_+ - v} X_1^+(v). \] (2.72)

The relations between \( k_1^\pm(u), k_2^\pm(u), e_1^\pm(u), f_1^\pm(u) \) are:
\[
\frac{h}{u - v} (k_2^\pm(v) + f_1^\pm(v)k_1^\pm(u)e_1^\pm(v))k_1^\pm(u) = \frac{h}{u - v} (k_2^\pm(u) + f_1^\pm(u)k_1^\pm(u)e_1^\pm(u))k_1^\pm(v) + k_1^\pm(u)e_1^\pm(u) f_1^\pm(v) k_1^\pm(u),
\] (2.73)
\[
\frac{h}{u_+ - v_-}(k_2^-(v) + f_1^-(v)k_1^+(v)e_1^+(v))k_1^+(u) + f_1^-(v)k_1^-(v)k_1^+(u)e_1^+(u) \\
= \frac{h}{u_- - v_+}(k_2^+(u) + f_1^+(u)k_1^+(u)e_1^+(u))k_1^-(v) + k_1^+(u)e_1^+(u)f_1^-(v)k_1^-(v).
\]

Therefore,
\[
\frac{h}{u - v}(k_2^+(v)k_1^+(u) - k_2^-(u)k_1^+(v)) \\
= \frac{h}{u - v}f_1^+(u)k_1^+(u)e_1^+(u)k_1^+(v) + k_1^+(u)e_1^+(u)f_1^+(v)k_1^+(v) \\
- f_1^+(v)(\frac{h}{u - v}k_1^+(v)e_1^+(u)k_1^+(v) + k_1^+(v)k_1^+(u)e_1^+(u)) \\
= \frac{h}{u - v}f_1^+(u)k_1^+(u)e_1^+(u)k_1^+(v) + k_1^+(u)e_1^+(u)f_1^+(v)k_1^+(v) - \frac{u - v + h}{u - v}f_1^+(u)k_1^+(u)e_1^+(u)k_1^+(v) \\
= \frac{h}{u - v}f_1^+(u)k_1^+(u)e_1^+(u)k_1^+(v) - \frac{u - v + h}{u - v}f_1^+(v)k_1^+(u)e_1^+(u)k_1^+(v) \\
+ k_1^+(u)e_1^+(u)f_1^+(v)k_1^+(v) \\
= -k_1^+(u)f_1^+(v)e_1^+(u)k_1^+(v) + k_1^+(u)k_1^+(v)e_1^+(u)f_1^+(v)k_1^+(v) \\
= k_1^+(u)e_1^+(u), f_1^+(v)]k_1^+(v).
\]

So
\[
[e_1^+(u), f_1^+(v)] = \frac{h}{u - v}(k_2^+(v)k_1^+(v) - k_2^+(u)k_1^+(u)).
\]

Similarly, we can prove
\[
[e_1^+(u), f_1^+(v)] = \frac{h}{u_+ - v_+}k_2^+(v)k_1^+(v)^{-1} - \frac{h}{u_- - v_-}k_2^+(u)k_1^+(u)^{-1}.
\]

Here the denominators are power series in \(\frac{v}{u}\) and \(\frac{u}{v}\) respectively. Since
\[
X_1^+(u) = e_1^+(u_+) - e_1^+(u_-), X_1^-(v) = f_1^+(v_+) - f_1^-(v_-),
\]
\[
[X_1^+(u), X_1^-(v)] = h\{\delta(u_- - v_+)k_2^+(u_-)k_1^+(u_-)^{-1} \\
+ \delta(u_+ - v_-)k_2^+(v_-)k_1^+(v_-)^{-1}\}.
\]

Now we consider the case \(n = 3\). In this section, we will denote \(\vec{R}(u)\) by \(R_0(u)\) for referring dimension \(n\). Let us restrict (2.5) and (2.6) to \(e_{ij} \otimes e_{kl}, i, j, k, l \leq 2\), then we get
\[
\vec{R}_2(u - v)J_2^+(u)J_2^-(v) = J_2^+(v)J_2^+(u)\vec{R}_2(u - v),
\]
\[
\vec{R}_2(u_+ - v_+)J_2^+(u)J_2^-(v) = J_2^+(v)J_2^+(u)\vec{R}_2(u_+ - v_-),
\]
where we have denoted
\[
J^\pm(u) = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
k_1^\pm(u) & 0 \\
0 & k_2^\pm(u)
\end{pmatrix}
\begin{pmatrix}
1 & e_1^\pm(u) \\
0 & 1
\end{pmatrix}.
\]
Thus we derive the same relations as in the case \( n = 2 \). Similarly consider (2.5) and (2.6) and restrict them to \( e_{ij} \otimes e_{kl}, 2 \leq i, j, k, l \leq 3 \), then we have

\[
\begin{align*}
\tilde{R}_2(u - v)\tilde{J}_1^+(u)\tilde{J}_2^+(v) &= \tilde{J}_2^+(v)\tilde{J}_1^+(u)\tilde{R}_2(u - v), \\
\tilde{R}_2(u_+ - v_+)\tilde{J}_1^+(u)\tilde{J}_2^-(v) &= \tilde{J}_2^-(v)\tilde{J}_1^+(u)\tilde{R}_2(u_+ - v_+)
\end{align*}
\]

(2.81) (2.82)

where we have denoted

\[
\tilde{J}_1^\pm(u) = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
f_{23,1}(u) & f_{23,2}(u) & 1
\end{pmatrix}
\begin{pmatrix}
k_1^\pm(u) & 0 & 0 \\
k_2^\pm(u) & 0 & k_3^\pm(u) \\
0 & k_3^\pm(u) & 1
\end{pmatrix}
\begin{pmatrix}
e_1^\pm(u) & e_1^\pm(1) \\
e_2^\pm(u) & e_2^\pm(1) \\
e_3^\pm(1)
\end{pmatrix}
\]

(2.83)

It is also the same as in \( n = 2 \) case. Thus we only need to verify the relations between \( k_3^\pm(u), f_{13}^\pm(u), e_3^\pm(u) \) and \( k_3^\pm(u), f_{23}^\pm(u), e_2^\pm(u) \). The other relations can be deduced from the results for \( n = 2 \) case.

By the Gauss decomposition of \( L^\pm(u) \), we have

\[
L^\pm(u) = \begin{pmatrix}
k_1^\pm(u) & k_2^\pm(u) & k_3^\pm(u) \\
k_1^\pm(u) & k_2^\pm(u) & k_3^\pm(u) \\
k_1^\pm(u) & k_2^\pm(u) & k_3^\pm(u)
\end{pmatrix}
\begin{pmatrix}
e_1^\pm(1) & e_1^\pm(1) & e_1^\pm(1) \\
e_2^\pm(1) & e_2^\pm(1) & e_2^\pm(1) \\
e_3^\pm(1)
\end{pmatrix}
\]

(2.84)

Let \( x^\pm = k_3^\pm(v)^{-1}(-f_{23}^\pm(v) + f_{23}^\pm(v)f_{13}^\pm(v)), y^\pm = (-e_{23}^\pm(v) + e_{23}^\pm(v)e_{23}^\pm(v)) k_3^\pm(v)^{-1} \), then

\[
L_1^\pm(v)^{-1} = \begin{pmatrix}
* & * & y^\pm I \\
* & k_3^\pm(v)^{-1}f_{23}^\pm(v) & -e_{23}^\pm(v)k_3^\pm(v)^{-1}I \\
x^\pm I & -k_3^\pm(v)^{-1}f_{23}^\pm(v)I & k_3^\pm(v)^{-1}I
\end{pmatrix}
\]

(2.86)

Now we start to check the relations between \( f_{23}^\pm(u) \) and \( f_{23}^\pm(u) \), and the relations between \( e_{23}^\pm(u) \) and \( e_{23}^\pm(u) \).

From (2.14), (2.15), (2.16) and (2.86), we have

\[
\frac{h}{u - v} x^\pm k_1^\pm(u) - \frac{u - v + h}{u - v} k_3^\pm(v)^{-1}f_{23}^\pm(v)f_{13}^\pm(u)k_1^\pm(u)
\]

(2.87)

\[
+ \frac{h}{u - v} k_3^\pm(v)^{-1}f_{23,1}^\pm(u)k_1^\pm(u) = -f_{13}^\pm(u)k_1^\pm(u)k_3^\pm(v)^{-1}f_{23}^\pm(v)
\]

(2.88)

We multiply (2.87) by \( k_3^\pm(v) \) on the left side, and \( k_1^\pm(u)^{-1} \) on the right side, and (2.88) by \( k_3^\pm(v) \) on the left, and by \( k_1^\pm(u)^{-1} \) on the right.
From (2.93)-(2.94), we get

\[ (u - v)f_1^\pm(u)f_2^\mp(v) = (u - v + h)f_2^\mp(v)f_1^\pm(u) + h(f_{3,1}^\pm(v) - f_{3,1}^\mp(u) - f_2^\mp(v)f_1^\pm(v)), \quad (2.89) \]

\[ (u - v)f_1^\pm(u)f_2^\mp(v) = (u - v + h)f_2^\mp(v)f_1^\pm(u) + h(f_{3,1}^\pm(v) - f_{3,1}^\mp(u) - f_2^\mp(v)f_1^\pm(v)). \quad (2.90) \]

Similarly, we can prove

\[ (u_+ - v_+)f_1^\mp(u)f_2^\mp(v) = (u_+ - v_+ + h)f_2^\mp(v)f_1^\mp(u) + h(f_{3,1}^\mp(u) - f_{3,1}^\mp(u) - f_2^\mp(v)f_1^\mp(v)). \quad (2.91) \]

So we have

\[ (u_+ - v_\pm)f_1^\mp(u)f_2^\mp(v) = (u_+ - v_\pm + h)f_2^\mp(v)f_1^\mp(u) + h(f_{3,1}^\mp(v) - f_{3,1}^\mp(u) - f_2^\mp(v)f_1^\mp(v)). \quad (2.92) \]

From (2.89), we get

\[ (u - v)f_1^\pm(u)f_2^\mp(v_\pm) = (u - v + h)f_2^\mp(v_\pm)f_1^\pm(u_\pm) + h(f_{3,1}^\pm(v_\pm) - f_{3,1}^\pm(u_\pm) - f_2^\mp(v_\pm)f_1^\pm(v_\pm)). \quad (2.93) \]

From (2.90), we get

\[ (u - v)f_1^\pm(u)f_2^\mp(v_\pm) = (u - v + h)f_2^\mp(v_\pm)f_1^\pm(u_\pm) + h(f_{3,1}^\mp(v_\pm) - f_{3,1}^\pm(u_\pm) - f_2^\mp(v_\pm)f_1^\mp(v_\mp)). \quad (2.94) \]

From (2.93)-(2.94), we get

\[ (u - v)X_1^-(u)X_2^-(v) = X_2^-(v)X_1^-(u)(u - v + h). \quad (2.95) \]

As for \( e_1^\mp(u), e_2^\mp(u) \), we have

\[ -e_2^\mp(v)k_3^\pm(v)^{-1}k_1^\pm(u)e_1^\pm(u) = \frac{h}{u - v}k_1^\pm(u)y^\pm \]

\[ -\frac{u - v}{u - v + h}k_1^\pm(u)e_1^\pm(u)e_2^\pm(v)k_3^\pm(v)^{-1} + \frac{h}{u - v}k_1^\pm(u)e_1^\pm(u)k_3^\pm(v)^{-1} \quad (2.96) \]

\[ -e_2^\pm(v)k_3^\pm(v)^{-1}k_1^\pm(u)e_1^\pm(u) = \frac{h}{u_+ - v_1}k_1^\pm(u)y^- \]

\[ -\frac{u_+ - v_+ + h}{u_+ - v_-}k_1^\pm(u)e_1^\pm(u)e_2^\pm(v)k_3^\pm(v)^{-1} + \frac{h}{u_+ - v_-}k_1^\pm(u)e_1^\pm(u)k_3^\pm(v)^{-1} \quad (2.97) \]
Then we get
\[
(u - v)e_2^+(v_{\pm})e_1^+(u_{\pm}) = (u - v + h)e_2^+(u_{\pm})e_2^+(v_{\pm}) + h(e_1^+(v_{\pm}) - e_{1,3}^+(u_{\pm}) - e_1^+(v_{\pm})e_2^+(v_{\pm}))
\] (2.98)
\[
(u - v)e_2^+(v_{\pm})e_1^+(u_{\pm}) = (u - v + h)e_1^+(u_{\pm})e_2^+(v_{\pm}) + h(e_1^+(v_{\pm}) - e_{1,3}^+(u_{\pm}) - e_1^+(v_{\pm})e_2^+(v_{\pm}))
\] (2.99)

From (2.98)-(2.99), we get
\[
(u - v)X_2^+(v)X_1^+(u) = X_1^+(u)X_2^+(v)(u - v + h).
\] (2.100)

From (2.95) and (u - v + h)X_1^-(u)X_1^- (v) = X_1^- (v)X_1^- (u)(u - v - h), we get
\[
X_1^- (u_1)X_1^- (u_2)X_2^-(v) - 2X_1^- (u_1)X_2^-(v)X_1^- (u_2)
+ X_2^-(v)X_1^- (u_1)X_1^- (u_2) + \{u_1 \leftrightarrow u_2\} = 0.
\] (2.101)

From and (u - v - h)X_1^+(u)X_1^+(v) = X_1^+(v)X_1^+(u)(u - v + h), we get
\[
X_1^+(u_1)X_1^+(u_2)X_2^+(v) - 2X_1^+(u_1)X_2^+(v)X_1^+(u_2)
+ X_2^+(v)X_1^+(u_1)X_1^+(u_2) + \{u_1 \leftrightarrow u_2\} = 0.
\] (2.102)

Thus we have proved all the relations for n = 3. Now we proceed to the proof for general n. First of all, we restrict (2.5) and (2.6) to e_{ij} \otimes e_{kl}, 1 \leq i, j, k, l \leq n - 1, then we get
\[
\tilde{R}_{n-1}(u - v)J_1^\pm(u)J_2^\pm(v) = J_2^\pm(v)J_1^\pm(u)\tilde{R}_{n-1}(u - v) \quad \text{(2.103)}
\]
\[
\tilde{R}_{n-1}(u_+ - v_+)J_1^\pm(u)J_2^\pm(v) = J_2^\pm(v)J_1^\pm(u)\tilde{R}_{n-1}(u_+ - v_-) \quad \text{(2.104)}
\]
\[
J^\pm(u) = \begin{pmatrix}
1 & \cdots & 0 \\
\cdots & \ddots & \cdots \\
0 & \cdots & k_{n-1}^\pm(u)
\end{pmatrix}
\begin{pmatrix}
e_1^\pm(u) & \cdots & 0 \\
\cdots & \ddots & \cdots \\
0 & \cdots & k_{n-1}^\pm(u)
\end{pmatrix}
\text{(2.105)}
\]

Similarly restricting (2.5) and (2.6) to e_{ij} \otimes e_{kl}, 2 \leq i, j, k, l \leq n, we derive that
\[
\tilde{R}_{n-1}(u - v)J_1^\pm(u)J_2^\pm(v) = J_2^\pm(v)J_1^\pm(u)\tilde{R}_{n-1}(u - v) \quad \text{(2.106)}
\]
\[
\tilde{R}_{n-1}(u_+ - v_+)J_1^\pm(u)J_2^\pm(v) = J_2^\pm(v)J_1^\pm(u)\tilde{R}_{n-1}(u_+ - v_-) \quad \text{(2.107)}
\]
\[ f^\pm(u) = \begin{pmatrix} 1 & 0 & 0 \\ e_2^\pm(u) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_n^\pm(u) \end{pmatrix} \begin{pmatrix} k_1^\pm(u) & \cdots & 0 \\ 0 & \cdots & k_n^\pm(u) \\ \cdots & \cdots & \cdots \end{pmatrix} \quad (2.108) \]

By induction, we only need to check the relations between \( e_1^\pm(u), k_1^\pm(u), f_1^\pm(u) \) and \( e_n^\pm(u), k_n^\pm(u), f_n^\pm(u) \). We now use the formulas (2.14), (2.15) and (2.17). First we write down the matrices \( L^\pm(u) \) and their inverse \( L^\pm(u)^{-1} \),

\[ L^\pm(u) = \begin{pmatrix} k_1^\pm(u) & k_1^\pm(v)e_1^\pm(u) & \cdots \\ f_1^\pm(u)k_1^\pm(u) & \cdots & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix} \quad (2.109) \]

and

\[ L^\pm(u)^{-1} = \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & -e_n^\pm(u)k_n^\pm(u)^{-1} \\ \cdots & \cdots & k_n^\pm(u) \end{pmatrix} \quad (2.110) \]

From (2.14), (2.15) and (2.17), we get

\[ k_1^\pm(u)k_n^\pm(v) = k_n^\pm(v)k_1^\pm(u) \quad (2.111) \]
\[ k_1^\pm(u)f_{n-1}^\pm(v) = f_{n-1}^\pm(v)k_1^\pm(u) \quad (2.112) \]
\[ k_1^\pm(u)e_{n-1}^\pm(v) = e_{n-1}^\pm(v)k_1^\pm(u) \quad (2.113) \]
\[ k_n^\pm(u)f_1^\pm(v) = f_1^\pm(v)k_n^\pm(u) \quad (2.114) \]
\[ k_n^\pm(u)e_1^\pm(v) = e_1^\pm(v)k_n^\pm(u) \quad (2.115) \]
\[ k_1^\pm(u)k_1^\pm(v) = k_1^\pm(v)k_1^\pm(u) \quad (2.116) \]

\[ \frac{(u_+-v_-)^2}{(u_+-v_-)^2 - h^2}k_1^\pm(u)k_n^\pm(v) = \frac{(u_+-v_-)^2}{(u_+-v_-)^2 - h^2}k_1^\pm(v)k_1^\pm(u) \quad (2.117) \]
\[ k_n^\pm(u)f_1^\pm(v) = f_1^\pm(v)k_n^\pm(u) \quad (2.118) \]
\[ k_n^\pm(u)e_1^\pm(v) = e_1^\pm(v)k_n^\pm(u) \quad (2.119) \]
\[ k_1^\pm(u)f_{n-1}^\pm(v) = f_{n-1}^\pm(v)k_1^\pm(u) \quad (2.120) \]
Corollary 2.6. The following relations hold in the algebra $DY_h(sl_n)$:

\[
[ H_i^\pm(u), H_j^\pm(v) ] = 0,
\]

\[
(u_\mp - v_\mp + hB_{ij})(u_\mp - v_\mp - hB_{ij})H_i^\pm(u)H_j^\mp(v) = \]
\[
( u_\mp - v_\mp - hB_{ij})(u_\mp - v_\mp + hB_{ij})H_i^\mp(v)H_j^\pm(u),
\]

\[
H_i^\pm(u)^{-1} E_j(v) H_i^\pm(u) = \frac{u_\pm - v + hB_{ij}}{u_\pm - v + hB_{ij}} E_j(v),
\]

\[
H_i^\pm(u) F_j(v) H_i^\pm(u)^{-1} = \frac{u_\mp - v - hB_{ij}}{u_\mp - v + hB_{ij}} F_j(v),
\]

\[
( u - v - hB_{ij}) E_i(u) E_j(v) = ( u - v + hB_{ij}) E_j(v) E_i(u),
\]

\[
( u - v + hB_{ij}) F_i(u) F_j(v) = ( u - v - hB_{ij}) F_j(v) F_i(u),
\]

From (2.113) and (2.121), we get

\[
k_i^\pm(u)^{-1} X_{n-1}^+(v) k_i^\pm(u) = X_{n-1}^+(v).
\]

From (2.112) and (2.120), we get

\[
k_i^\pm(u) X_{n-1}^-(v) k_i^\pm(u)^{-1} = X_{n-1}^-(v).
\]

From (2.122) and (2.124), we get

\[
X_i^+(u) X_{n-1}^+(v) = X_{n-1}^+(v) X_i^+(u).
\]

From (2.128) and (2.129), we get

\[
X_i^-(u) X_{n-1}^-(v) = X_{n-1}^-(v) X_i^-(u).
\]

This completes the proof of all the relations in the general case. \qed

Corollary 2.6. The following relations hold in the algebra $DY_h(sl_n)$:
Case 1:

From (2.130), (2.131) and (2.132), we get
\[
\sum_{\sigma \in \mathbb{S}_m} [E_i(u_{\sigma(1)}), [E_i(u_{\sigma(2)}), \ldots, [E_i(u_{\sigma(m)}), E_j(v)]] = 0 \quad (i \neq j),
\]
\[
\sum_{\sigma \in \mathbb{S}_m} [F_i(u_{\sigma(1)}), [F_i(u_{\sigma(2)}), \ldots, [F_i(u_{\sigma(m)}), F_j(v)]] = 0 \quad (i \neq j),
\]
\[
[E_i(u), F_j(v)] = \frac{1}{h} \delta_{ij} \{\delta(u_- - v_+)H_i^+(v) - \delta(u_+ - v_-)H_i^-(v)\}.
\]

Here we set \(B_{ij} = \frac{1}{2} a_{ij}\), where \(A = (a_{ij})\) is the Cartan matrix of the Lie algebra \(\mathfrak{sl}_n\).

**Proof.** Since \(k_i^\pm(u)k_j^\pm(v) = k_j^\pm(v)k_i^\pm(u)\), we have
\[
k_i^\pm(u)k_j^\pm(v)^{-1} = k_j^\pm(v)^{-1}k_i^\pm(u),
\]
\[
k_i^\pm(u)^{-1}k_j^\pm(v) = k_j^\pm(v)k_i^\pm(u)^{-1},
\]
\[
k_i^\pm(u)^{-1}k_j^\pm(v)^{-1} = k_j^\pm(v)^{-1}k_i^\pm(u)^{-1},
\]
so we get \([H_i^\pm(u), H_j^\pm(v)] = 0\). Next, we prove that
\[
(u_- - v_+ + hB_{ij})(u_- - v_- - hB_{ij})H_i^\pm(u)H_j^\mp(v) = (u_- - v_- - hB_{ij})(u_- - v_+ + hB_{ij})H_i^\mp(v)H_j^\pm(u),
\]

**Case 1:** \(i = j\), \(B_{ij} = 1\)

\[
\frac{u_- - v_+ + h}{u_- - v_-} k_{i+1}^+(u + \frac{1}{2} hi)k_{i+1}^-(v + \frac{1}{2} hi) = \frac{u_+ - v_- + h}{u_+ - v_+} k_{i+1}^-(v + \frac{1}{2} hi)k_{i+1}^+(u + \frac{1}{2} hi),
\]
\[
\frac{u_- - v_+ + h}{u_- - v_-} k_i^+(u + \frac{1}{2} hi)k_i^-(v + \frac{1}{2} hi)^{-1} = \frac{u_+ - v_- + h}{u_+ - v_+} k_i^-(v + \frac{1}{2} hi)^{-1}k_i^+(u + \frac{1}{2} hi),
\]
\[
\frac{(v_+ - u_-)^2}{(v_+ - u_-)^2 - h^2} k_{i+1}^+(u + \frac{1}{2} hi)k_{i+1}^-(v + \frac{1}{2} hi)^{-1} = \frac{(v_- - u_+)^2}{(v_- - u_+)^2 - h^2} k_i^-(v + \frac{1}{2} hi)^{-1}k_i^+(u + \frac{1}{2} hi).
\]

From (2.130), (2.131) and (2.132), we get
\[
(u_- - v_+ + h)(u_+ - v_- - h)H_i^\pm(u)H_j^\mp(v) = (u_- - v_- - h)(u_+ - v_- + h)H_j^\mp(v)H_i^\pm(u).
\]

Swapping \(u\) and \(v\), we get
\[
(u_+ - v_- + h)(u_- - v_+ - h)H_i^\mp(u)H_j^\pm(v) = (u_+ - v_- - h)(u_- - v_+ + h)H_j^\mp(v)H_i^\pm(u).
\]
Thus, 
\[(u_\mp - v_\mp + h)(u_\pm - v_\mp + h)H^\pm_i(u)H^\mp_i(v)\]
\[= (u_\mp - v_\mp - h)(u_\pm - v_\mp + h)H^\mp_i(v)H^\mp_i(u).\]

Case 2: \( j = i + 1 \) or \( j = i - 1 \), \( B_{ij} = -\frac{1}{2} \)

First, we assume \( j = i + 1 \),

\[
\frac{u_+ - v_- + \frac{1}{2}h}{u_+ - v_- - \frac{1}{2}h}k^+_{i+1}(u + \frac{1}{2}hi)k^-_{i+1}(v + \frac{1}{2}hi + \frac{1}{2}h)^{-1}
\begin{equation}
(2.133)
\end{equation}
\]
\[= \frac{u_- - v_+ + \frac{1}{2}h}{u_- - v_+ - \frac{1}{2}h}k^-_{i+1}(v + \frac{1}{2}hi + \frac{1}{2}h)^{-1}k^+_{i+1}(u + \frac{1}{2}hi).\]

From \((2.133)\), we get
\[
(u_- - v_- - \frac{1}{2}h)(u_+ - v_- + \frac{1}{2}h)H^\mp_i(u)H^\pm_{i+1}(v)
\]
\[= (u_- - v_- + \frac{1}{2}h)(u_+ - v_- - \frac{1}{2}h)H^\pm_{i+1}(v)H^\mp_i(u).\]

Similarly, we can prove
\[
(u_+ - v_- - \frac{1}{2}h)(u_- + v_+ + \frac{1}{2}h)H^\mp_i(u)H^\pm_{i+1}(v)
\]
\[= (u_+ - v_- + \frac{1}{2}h)(u_- - v_+ - \frac{1}{2}h)H^\pm_{i+1}(v)H^\mp_i(u).\]

So we have
\[
(u_\mp - v_\pm - \frac{1}{2}h)(u_\pm + v_\mp + \frac{1}{2}h)H^\pm_i(u)H^\mp_{i+1}(v)
\]
\[= (u_\mp - v_\pm + \frac{1}{2}h)(u_\pm - v_\mp - \frac{1}{2}h)H^\mp_{i+1}(v)H^\pm_i(u).\]

Similarly, we can prove
\[
(u_\mp - v_\pm - \frac{1}{2}h)(u_\pm - v_\mp + \frac{1}{2}h)H^\pm_i(u)H^\mp_{i-1}(v)
\]
\[= (u_\mp - v_\pm + \frac{1}{2}h)(u_\pm - v_\mp - \frac{1}{2}h)H^\mp_{i-1}(v)H^\pm_i(u).\]

Case 3: \( |i - j| > 1 \), \( B_{ij} = 0 \)

Suppose \( j < i \), then we have \( j < j + 1 < i < i + 1 \). It follows that \( H^\pm_i(u)H^\mp_j(v) = H^\mp_j(v)H^\pm_i(u) \).

Combining the above three cases, we derive the following relation
\[
(u_\mp - v_\pm + hB_{ij})(u_\pm - v_\mp - hB_{ij})H^\pm_i(u)H^\mp_j(v)
\]
\[= (u_\mp - v_\pm - hB_{ij})(u_\pm - v_\mp + hB_{ij})H^\mp_j(v)H^\pm_i(u).\]

Next, we prove that
\[
H^\pm_i(u)^{-1}E_j(v)H^\pm_i(u) = \frac{u_\pm - v - hB_{ij}}{u_\pm - v + hB_{ij}}E_j(v).
\]
Case 1: \( i = j, B_{ij} = 1 \)

\[
k_{i+1}^\pm (u + \frac{1}{2}hi)^{-1}E_i(v)k_{i+1}^\pm (u + \frac{1}{2}hi) = \frac{u_\pm - v - h}{u_\pm - v}E_i(v), \quad (2.134)
\]

\[
k_i^\pm (u + \frac{1}{2}hi)E_i(v)k_i^\pm (u + \frac{1}{2}hi)^{-1} = \frac{u_\pm - v}{u_\pm - v + h}E_i(v), \quad (2.135)
\]

From (2.134) and (2.135), we get

\[
H_i^\pm (u)^{-1}E_i(v)H_i^\pm (u) = \frac{u_\pm - v - h}{u_\pm - v + h}E_i(v).
\]

Case 2: \( j = i + 1, B_{ij} = \frac{1}{2} \)

\[
k_{i+1}^\pm (u + \frac{1}{2}hi)^{-1}E_{i+1}(v)k_{i+1}^\pm (u + \frac{1}{2}hi) = \frac{u_\pm - v + \frac{1}{2}h}{u_\pm - v - \frac{1}{2}h}E_{i+1}(v), \quad (2.136)
\]

\[
k_i^\pm (u + \frac{1}{2}hi)E_{i+1}(v)k_i^\pm (u + \frac{1}{2}hi)^{-1} = E_{i+1}(v), \quad (2.137)
\]

From (2.136) and (2.137), we get

\[
H_i^\pm (u)^{-1}E_{i+1}(v)H_i^\pm (u) = \frac{u_\pm - v + \frac{1}{2}h}{u_\pm - v - \frac{1}{2}h}E_{i+1}(v).
\]

Case 3: \( j = i - 1, B_{ij} = \frac{1}{2} \)

\[
k_{i-1}^\pm (u + \frac{1}{2}hi)^{-1}E_{i-1}(v)k_{i-1}^\pm (u + \frac{1}{2}hi) = E_{i-1}(v), \quad (2.138)
\]

\[
k_i^\pm (u + \frac{1}{2}hi)E_{i-1}(v)k_i^\pm (u + \frac{1}{2}hi)^{-1} = \frac{u_\pm - v + \frac{1}{2}h}{u_\pm - v - \frac{1}{2}h}E_{i-1}(v), \quad (2.139)
\]

From (2.138) and (2.139), we get

\[
H_i^\pm (u)^{-1}E_{i-1}(v)H_i^\pm (u) = \frac{u_\pm - v + \frac{1}{2}h}{u_\pm - v - \frac{1}{2}h}E_{i-1}(v).
\]

Case 4: \( |i - j| > 1, B_{ij} = 0 \)

\[
k_{i+1}^\pm (u + \frac{1}{2}hi)^{-1}E_j(v)k_{i+1}^\pm (u + \frac{1}{2}hi) = E_j(v), \quad (2.140)
\]

\[
k_i^\pm (u + \frac{1}{2}hi)E_j(v)k_i^\pm (u + \frac{1}{2}hi)^{-1} = E_j(v), \quad (2.141)
\]

From (2.140) and (2.141), we get

\[
H_i^\pm (u)^{-1}E_j(v)H_i^\pm (u) = E_j(v).
\]

Combining the above four cases, we derive the following relation

\[
H_i^\pm (u)^{-1}E_j(v)H_i^\pm (u) = \frac{u_\pm - v - hB_{ij}}{u_\pm - v + hB_{ij}}E_j(v).
\]

Similarly, we can prove

\[
H_i^\pm (u)F_j(v)H_i^\pm (u)^{-1} = \frac{u_\pm - v - hB_{ij}}{u_\pm - v + hB_{ij}}F_j(v).
\]
From the following three relations:

\[(u - v - h)X_i^+ (u)X_i^+ (v) = (u - v + h)X_i^+ (v)X_i^+ (u),\]
\[(u - v + h)X_i^+ (u)X_{i+1}^+ (v) = (u - v)X_{i+1}^+ (v)X_i^+ (u),\]
\[X_i^+ (u)X_j^+ (v) = X_j^+ (v)X_i^+ (u) \quad \text{if } |i - j| > 1,\]

we get

\[(u - v - hB_{ij})E_i (u)E_j (v) = (u - v + hB_{ij})E_j (v)E_i (u).\]

Similarly, we can prove

\[(u - v + hB_{ij})F_i (u)F_j (v) = (u - v - hB_{ij})F_j (v)F_i (u).\]

From the following two relations:

\[X_i^+ (u_1)X_i^+ (u_2)X_j^+ (v) - 2X_i^+ (u_1)X_j^+ (v)X_i^+ (u_2) + X_i^+ (v)X_i^+ (u_1)X_i^+ (u_2)\]
\[+ \{u_1 \leftrightarrow u_2\} = 0 \quad \text{if } |i - j| = 1,\]
\[X_i^+ (u)X_j^+ (v) = X_j^+ (v)X_i^+ (u) \quad \text{if } |i - j| > 1,\]

we get

\[\sum_{\sigma \in \mathcal{S}_m} [E_i (u_{\sigma(1)}), [E_i (u_{\sigma(2)}), \ldots, [E_i (u_{\sigma(m)}), E_j (v)] \ldots] = 0 \quad i \neq j, m = 1 - a_{ij}.\]

Similarly, we can prove

\[\sum_{\sigma \in \mathcal{S}_m} [F_i (u_{\sigma(1)}), [F_i (u_{\sigma(2)}), \ldots, [F_i (u_{\sigma(m)}), F_j (v)] \ldots] = 0 \quad i \neq j, m = 1 - a_{ij}.\]

From

\[[X_i^+ (u), X_j^- (v)]\]
\[= h\delta_{ij} \{ \delta(u_+ - v_+)k_{i+1}^+(u_-)k_i^+(u_-)^{-1} - \delta(u_+ - v_-)k_{i+1}^-(v_-)k_i^-(v_-)^{-1} \},\]

we get

\[[E_i (u), F_j (v)] = \frac{1}{h}\delta_{ij} \{ \delta(u_- - v_+)H_i^+ (u_-) - \delta(u_+ - v_-)H_i^- (v_-) \}.\]

\[\square\]

Remark. Let \(f (u)\) be a scalar function, and \(R(u) = f (u)\hat{R}(u)\), Corollary 2.6 still holds if we change the normalization of R-matrix in Definition 2.1 to \(R(u)\). The relations in Corollary 2.6 were announced in Iohara’s paper [14], while we provide a complete proof in this section.
3. Drinfeld realization of the Yangian double

In this section, we will describe the Drinfeld realization of $DY_h(\mathfrak{gl}_n)$ and $DY_h(\mathfrak{sl}_n)$. First of all, we define a natural ascending filtration on the Yangian double $DY_h(\mathfrak{gl}_n)$ by setting $\deg l^{(r)}_{ij} = r - 1$, $\deg l^{(-r)}_{ij} = -r$ for all $r \geq 1$ and $\deg c = \deg h = 0$. Denote by $\bar{l}^{(r)}_{ij}$ the image of $l^{(r)}_{ij}$ in the $(r-1)$ (or $(-r)$)-th component of the associated graded algebra $\text{gr} DY_h(\mathfrak{gl}_n)$. Let $\mathfrak{gl}_n$ be the central extension $\mathfrak{gl}_n[x, x^{-1}] \oplus \mathbb{C} K$ defined by the commutation relations

$$[E_{ij}[r], E_{kl}[s]] = \delta_{kj}E_{il}[r + s] - \delta_{il}E_{kj}[r + s] + r\delta_{kj}\delta_{il}\delta_{r-s}K,$$

and the element $K$ is central. One then has the following result for the graded algebras.

**Proposition 3.1.** The mapping

$$E_{ij}[r - 1] \mapsto \bar{l}^{(r)}_{ij}, \quad E_{ij}[-r] \mapsto \bar{l}^{(-r)}_{ij}, \quad K \mapsto \bar{c}, \quad h \mapsto \bar{h} \quad (3.1)$$

defines an isomorphism

$$U(\mathfrak{gl}_n[x, x^{-1}] \oplus \mathbb{C} K)[[h]] \twoheadrightarrow \text{gr} DY_h(\mathfrak{gl}_n), \quad (3.2)$$

where $K$ is the central element.

**Proof.** Using the expansion

$$\frac{1}{u - v} = u^{-1} + u^{-2}v + u^{-3}v^2 + \cdots,$$

and taking the coefficient at $u^{-r}v^{-s}$ ($r, s \geq 1$) and keeping the highest degree terms on both sides of the relation $(2.6)$ gives that

$$[\bar{l}^{(r)}_{ij}, \bar{l}^{(s)}_{kl}] = \left\{ \begin{array}{ll}
\delta_{kj}\bar{l}^{(r-s-1)}_{il} - \delta_{il}\bar{l}^{(r-s-1)}_{kj} + (r - 1)\delta_{kj}\delta_{il}\delta_{r,s+1}\bar{c}, & r \leq s \\
\delta_{kj}\bar{l}^{(r-s)}_{il} - \delta_{il}\bar{l}^{(r-s)}_{kj} + (r - 1)\delta_{kj}\delta_{il}\delta_{r,s+1}\bar{c}, & r > s
\end{array} \right. \quad (3.3)$$

Similarly, the coefficients at $u^{-r}v^{-s}$ ($r, s \geq 1$) and $u^r v^s$ ($r, s \geq 1$) of $(2.5)$ imply that

$$[\bar{l}^{(r)}_{ij}, \bar{l}^{(s)}_{kl}] = \delta_{kj}\bar{l}^{(r+s-2)}_{il} - \delta_{il}\bar{l}^{(r+s-2)}_{kj}, \quad (3.4)$$
$$[\bar{l}^{(-r)}_{ij}, \bar{l}^{(-s)}_{kl}] = \delta_{kj}\bar{l}^{(-r-s)}_{il} - \delta_{il}\bar{l}^{(-r-s)}_{kj}. \quad (3.5)$$

It follows that the map $(3.2)$ is a surjective homomorphism. Note that the ordered monomials in the generators $l^{(r)}_{ij}, l^{(-r)}_{ij}$ and $c$ form a topological basis of the algebra $DY_h(\mathfrak{gl}_n)$, see [16]. This implies that the map is also injective. \hfill \Box

Next we introduce the $ij$-th quasideterminant of a matrix. Let $A = [a_{ij}]$ be an $N \times N$ matrix over a ring with 1. Delete the $i$-th row and $j$-th column of $A$, we obtain a submatrix of $A$. We will denote it by $A^{ij}$. If the matrix
$A^{ij}$ is invertible, we define the $ij$-th quasideterminant of $A$ by the following formula

$$|A|_{ij} = a_{ij} - r_i^j (A^{ij})^{-1} c_j^i,$$

where $r_i^j$ is the row matrix obtained from the $i$-th row of $A$ by deleting the element $a_{ij}$, and $c_j^i$ is the column matrix obtained from the $j$-th column of $A$ by deleting the element $a_{ij}$; see [12]. We also denote the quasideterminant $|A|_{ij}$ by boxing the entry $a_{ij}$,

$$|A|_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{iN} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{Nj} & \cdots & a_{NN} \end{vmatrix}. $$

Assume the Gauss decomposition of the generator matrix $L^\pm(u)$ is $L^\pm(u) = F^\pm(u) H^\pm(u) E^\pm(u)$, then we have the well-known formulas for the entries of the matrices $H^\pm(u), E^\pm(u)$ and $F^\pm(u)$; see e.g. [21 Sec. 1.11].

**Proposition 3.2.** For $i = 1, \ldots, n$,

$$k^\pm_i(u) = qdet L^\pm_{i,i-1}(u - (i-1)h) qdet L^\pm_{i-1,i}(u - (i-1)h)^{-1}$$

For $1 \leq i < j \leq n$,

$$e^\pm_{ij}(u) = qdet L^\pm_{i,j}(u - (i-1)h)^{-1} qdet L^\pm_{i,j}(u - (i-1)h)$$

and

$$f^\pm_{ji}(u) = qdet L^\pm_{j,i}(u - (i-1)h)^{-1} qdet L^\pm_{j,i}(u - (i-1)h).$$
The coefficients of the matrices are referred to as the Gaussian generators, explicitly
\[ e_{ij}^{(r)}(u) = h \sum_{r=1}^{\infty} e_{ij}^{(-r)} u^{-r}, f_{ji}^{(r)}(u) = h \sum_{r=1}^{\infty} f_{ji}^{(-r)} u^{-r}, k_{ij}^{(r)}(u) = 1 + h \sum_{r=1}^{\infty} k_{ij}^{(-r)} u^{-r}, \]
\[ e_{ij}^{-}(u) = -h \sum_{r=1}^{\infty} e_{ij}^{(-r)} u^{r-1}, f_{ji}^{-}(u) = -h \sum_{r=1}^{\infty} f_{ji}^{(-r)} u^{r-1}, k_{ij}^{-}(u) = 1 - h \sum_{r=1}^{\infty} k_{ij}^{(-r)} u^{r-1}. \]

Now we are in a position to give the Drinfeld realization of $DY_h(gl_n)$ and $DY_h(sl_n)$.

**Theorem 3.1.** The Yangian double $DY_h(gl_n)$ is topologically generated by the coefficients of the series $k_{i}^{\pm}(u)(i = 1, \ldots, n)$, $e_{i}^{\pm}(u)$ and $f_{j}^{\pm}(u)(j = 1, \ldots, n - 1)$ and the central element $c$ subject to the defining relations in Theorem 2.5, where the indices run through all admissible values.

**Proof.** Let $\tilde{DY}_h(gl_n)$ be the algebra with generators and relations as in the statement of the theorem. Theorem 2.5 implies that there is a homomorphism $\phi : \tilde{DY}_h(gl_n) \to DY_h(gl_n)$ which takes the generators $k_{i}^{(r)}$, $e_{i}^{(r)}$, $f_{j}^{(r)}$ and $c$ of $\tilde{DY}_h(gl_n)$ to the corresponding elements of $DY_h(gl_n)$. To prove the surjectivity of the map $\phi$, we only need to show that $e_{i,n}^{\pm}(u)$ and $f_{n,1}^{\pm}(u)$ are generated by $k_{i}^{\pm}(u), e_{i}^{\pm}(u)$ and $f_{j}^{\pm}(u)$ in the algebra $DY_h(gl_n)$ where $i = 1, \ldots, n, j = 1, \ldots, n - 1$. Since all other elements $e_{i,j}^{\pm}(u)$ and $f_{j,i}^{\pm}(u)$ are generated by $k_{i}^{\pm}(u), e_{j}^{\pm}(u)$ and $f_{j}^{\pm}(u)$ by induction. From (2.14) and (2.15), we can get the relations between $e_{i,n-1}^{\pm}(u)$ and $e_{n-1,n}^{\pm}(u)$, and the relations between $f_{n-1,1}^{\pm}(u)$ and $f_{n,n-1}^{\pm}(u)$, which also contain $e_{i,n}^{\pm}(u)$ and $f_{n}^{\pm}(u)$. These formulas are similar to (2.87), (2.88), (2.96) and (2.97) in the case when $n = 3$. It follows that $e_{1,n}^{\pm}(u)$ and $f_{n,1}^{\pm}(u)$ are generated by $k_{i}^{\pm}(u), e_{i}^{\pm}(u)$ and $f_{j}^{\pm}(u)$. Thus we have proved that $\phi$ is surjective. Next we show that $\phi$ is injective. We start by showing that the set of monomials in $\quad k_{i}^{(r)}, \quad i = 1, \ldots, n, r \in \mathbb{Z}^\times,$
and
$\quad e_{ij}^{(r)}, f_{ji}^{(r)}, \quad 1 \leq i < j \leq n, r \in \mathbb{Z}^\times,$
and $c$ taken in some fixed order is linearly independent in the Yangian double $DY_h(gl_n)$. Applying Proposition 3.1 to the matrix $T^+(u)$, we deduce that the images of the elements $k_{i}^{(r)}, e_{ij}^{(r)}$ and $f_{ji}^{(r)}(r \geq 1)$ in the $(r - 1)$-th component of the graded algebra $\text{gr}DY_h(gl_n)$ under the isomorphism (3.2) respectively correspond to the elements $E_{ii} x^{-r-1}, E_{ij} x^{-r-1}$ and $E_{ji} x^{-r-1}$. Similarly, the images of the elements $k_{i}^{(-r)}, e_{ij}^{(-r)}$ and $f_{ji}^{(-r)}(r \geq 1)$ in the $(-r)$-th component of the graded algebra $\text{gr}DY_h(gl_n)$ under the isomorphism (3.2) respectively correspond to the elements $E_{ii} x^{-r}, E_{ij} x^{-r}$ and $E_{ji} x^{-r}$.
$E_{ji}x^{-r}$. Hence the claim follows from the Poincaré-Birkhoff–Witt theorem for $U(\mathfrak{g}_n[x, x^{-1}] \oplus \mathbb{C}K)$.

For any $1 \leq i < j \leq n$, $r \in \mathbb{Z}^+$, define elements $e_{ij}^{(r)}$ and $f_{ij}^{(r)}$ of $\widetilde{\mathcal{DY}}_h(\mathfrak{g}_n)$ inductively by the relations $e_{i,i+1}^{(r)} = e_i^{(r)}$, $f_{i,i+1}^{(r)} = f_i^{(r)}$ and

$$e_{i,j+1}^{(r)} = [e_{ij}^{(r)}, e_j^{(1)}], \quad f_{j+1,i}^{(r)} = [f_j^{(1)}, f_{ji}^{(r)}]$$

for $j > i, r \in \mathbb{Z}^+$. Obviously, these relations are consistent with those in $\mathcal{DY}_h(\mathfrak{g}_n)$. The injectivity of $\phi$ will follow if we prove that the algebra $\widetilde{\mathcal{DY}}_h(\mathfrak{g}_n)$ is spanned by the monomials in $k_i^{(r)}$, $e_{ij}^{(r)}$, $f_{ji}^{(r)}$ and $c$ taken in some fixed order. To see this, we introduce some notations. Denote by $\hat{\mathcal{E}}$, $\hat{\mathcal{F}}$ and $\hat{\mathcal{H}}$ the subalgebras of $\widetilde{\mathcal{DY}}_h(\mathfrak{g}_n)$ respectively generated by all elements of the form $e_i^{(r)}$, $f_i^{(r)}$ and $k_i^{(r)}$. Denote by $\check{\mathcal{E}}^+$, $\check{\mathcal{F}}^+$ and $\check{\mathcal{H}}^+$ the subalgebras of $\widetilde{\mathcal{DY}}_h(\mathfrak{g}_n)$ respectively generated by all elements of the form $e_i^{(r)}$, $f_i^{(r)}$ and $k_i^{(r)}$ with $r > 0$. Denote by $\check{\mathcal{E}}^-$, $\check{\mathcal{F}}^-$ and $\check{\mathcal{H}}^-$ the subalgebras of $\widetilde{\mathcal{DY}}_h(\mathfrak{g}_n)$ respectively generated by all elements of the form $e_i^{(r)}$, $f_i^{(r)}$ and $k_i^{(r)}$ with $r < 0$. Define an ascending filtration on $\check{\mathcal{E}}^-$ by setting $\deg e_i^{(r)} = -r, \deg h = 0$. Denote by $\text{gr} \check{\mathcal{E}}^-$ the corresponding graded algebra. Let $\check{e}_{ij}^{(r)}$ be the image of $e_{ij}^{(r)}$ in the $(-r)$-th component of the graded algebra $\text{gr} \check{\mathcal{E}}^-$. Note that the algebra $\check{\mathcal{E}}^+$ is spanned by the monomials in the elements $e_{ij}^{(r)}(r > 0)$ taken in some fixed order; see [21] Sec. 3.1]. Similarly, the desired spanning property of the algebra $\check{\mathcal{E}}^-$ follows from the relations

$$[\check{e}_{ij}^{(r)}, \check{e}_{kl}^{(-s)}] = \delta_{ij}\check{e}_{il}^{(-r-s)} - \delta_{ij}\check{e}_{kj}^{(-r-s)}. \quad (3.9)$$

We can use the same method in [21] Sec. 3.1 to prove (3.9). In addition, we can swap $e_{ij}^{(r)}$ and $e_{kl}^{(-s)}$ ($r, s > 0$) by the relations between $e_{ij}^{(r)}(u)$ and $e_{kl}^{(-s)}(v)$. These relations can be obtained from (2.52), (2.99), (2.124) and

$$e_{i,j+1}^{(r)} = [e_{ij}^{(r)}, e_j^{(1)}]$$

by induction. For example, if we want to swap $e_{13}^{(r)}$ and $e_2^{(-s)}$, write $e_{13}^{(r)}$ in the form $[e_{12}^{(r)}, e_{21}^{(1)}]$ firstly. Then we swap $e_{12}^{(r)}$ and $e_{21}^{(-s)}$, $e_1^{(1)}$ and $e_2^{(-s)}$ with the relations between $e_{12}^{(r)}(u)$ and $e_{21}^{(-s)}(v)$, $e_{21}^{(r)}(u)$ and $e_2^{(-s)}(v)$ respectively. After these steps, we can write the product $e_{13}^{(r)}e_2^{(-s)}$ or $e_2^{(-s)}e_{13}^{(r)}$ as the ordered monomials in the elements $e_{ij}^{(r)}$ and $e_{ij}^{(-r)}$. Therefore, the algebra $\check{\mathcal{E}}$ is spanned by the ordered monomials in the elements $e_{ij}^{(r)}$ and $e_{ij}^{(-r)}$ taken in some fixed order. The same is true for $\check{\mathcal{F}}$. Observe that the ordered monomials in the $k_i^{(r)}$ and the $k_i^{(-r)}$ span $\check{\mathcal{H}}$. Moreover, the defining relations of $\widetilde{\mathcal{DY}}_h(\mathfrak{g}_n)$ implies that the multiplication map

$$\check{\mathcal{F}} \otimes \check{\mathcal{H}} \otimes \check{\mathcal{E}} \otimes \mathbb{C}c \to \widetilde{\mathcal{DY}}_h(\mathfrak{g}_n) \quad (3.10)$$

is surjective. Here $A \otimes B$ denotes the topological tensor product of the algebras $A$ and $B$, which is the $h$-adic completion of $A \otimes_{\mathbb{C}[h]} B$. Therefore,
if we let the elements of \( \hat{F} \) precede the elements of \( \hat{H} \), and the latter precede the elements of \( \hat{E} \), \( c \) included in the ordering in an arbitrary way, then the ordered monomials in the set of elements \( k_i^{(r)}, e_{ij}^{(r)}, f_{ji}^{(r)} \) and \( c \) with \( r \in \mathbb{Z}^\times \) span \( \text{span} \hat{\mathfrak{Y}}_h(\mathfrak{gl}_n) \). It follows that \( \phi \) is injective.

\[ \square \]

**Corollary 3.2.** The Yangian double \( \text{DY}_h(\mathfrak{sl}_n) \) is topologically generated by the Drinfeld generators \( \{ h_{il}, e_{il}, f_{il} \mid i = 1, \cdots, n - 1, l \in \mathbb{Z} \} \) and the central element \( c \) subject to the defining relations in Corollary 2.6, where the Drinfeld currents are defined as follows:

\[
H_i^+(u) = 1 + h \sum_{l \geq 0} h_{il} u^{-l-1}, \quad H_i^-(u) = 1 - h \sum_{l < 0} h_{il} u^{-l-1},
\]

\[
E_i(u) = \sum_{l \in \mathbb{Z}} e_{il} u^{-l-1}, \quad F_i(u) = \sum_{l \in \mathbb{Z}} f_{il} u^{-l-1}.
\]

**Proof.** Let \( \hat{\text{DY}}_h(\mathfrak{sl}_n) \) be the algebra with generators and relations as in the statement of the corollary. Corollary 2.6 implies that there is a surjective homomorphism \( \varphi : \hat{\text{DY}}_h(\mathfrak{sl}_n) \to \text{DY}_h(\mathfrak{sl}_n) \) which takes the generators \( h_{il}, e_{il}, f_{il} \) and \( c \) of \( \hat{\text{DY}}_h(\mathfrak{sl}_n) \) to the corresponding elements of \( \text{DY}_h(\mathfrak{sl}_n) \). Denote by \( \mathcal{E}, \mathcal{F} \) and \( \mathcal{H} \) the subalgebras of \( \text{DY}_h(\mathfrak{gl}_n) \) respectively generated by all elements of the form \( e_{ij}^{(r)}, f_{ji}^{(r)} \) and \( k_i^{(r)} \). By the decomposition

\[
\text{DY}_h(\mathfrak{sl}_n) = \mathcal{E} \otimes (\text{DY}_h(\mathfrak{sl}_n) \cap \mathcal{H}) \otimes \mathcal{F} \otimes \mathcal{A} c, \tag{3.11}
\]

the corresponding arguments of the proof of Theorem 3.1 gives that \( \varphi \) is also injective. This completes the proof of the corollary. \[ \square \]
4. The center of the Yangian double at the critical level

In this section, we will construct central elements of the Yangian double at the critical level. First we need to normalize the Yang R-matrix to satisfy the crossing symmetry condition. Let us define \( R(u) = f(u) \bar{R}(u) \), where \( \bar{R}(u) \) is Yang’s R-matrix (2.1), and \( f(u) = 1 + \sum_{k=1}^{\infty} f_k u^{-k} \) is the function of \( u, h \) defined by the power series expansion and satisfies the functional equation

\[
    f(u - nh) = \frac{u^2 - h^2}{u^2} f(u). \tag{4.1}
\]

Clearly the coefficients \( f_k \) are rational functions in \( h \) uniquely determined by (4.1). It can be seen that

\[
    f(u) = \prod_{k=1}^{\infty} (1 - \frac{h^2}{(u + knh)^2}). \tag{4.2}
\]

Now we modify the defining relations of the Yangian double \( \text{DY}_h(\mathfrak{g}l_n) \) as follows:

\[
    [L^\pm(u), c] = 0, \tag{4.3}
\]

\[
    R(u - v)L^\pm_1(u)L^\pm_2(v) = L^\pm_2(v)L^\pm_1(u)R(u - v), \tag{4.4}
\]

\[
    R(u - v - \frac{1}{2}hc)L^\pm_1(u)L^\pm_2(v) = L^\pm_2(v)L^\pm_1(u)R(u - v - \frac{1}{2}hc), \tag{4.5}
\]

In the rest of the paper, the Yangian double defined here is still denoted by \( \text{DY}_h(\mathfrak{g}l_n) \).

Denote by \( \overline{\text{DY}}_h(\mathfrak{g}l_n)_{cr} \) the Yangian double at the critical level \( c = -n \), which is the quotient of \( \text{DY}_h(\mathfrak{g}l_n) \) modulo the ideal generated by the relation \( c = -n \). Define its completion \( \overline{\text{DY}}_h(\mathfrak{g}l_n)_{cr} \) as the inverse limit

\[
    \overline{\text{DY}}_h(\mathfrak{g}l_n)_{cr} = \lim_{\leftarrow} \overline{\text{DY}}_h(\mathfrak{g}l_n)_{cr}/J_p, \quad p > 0
\]

where \( J_p \) is the left ideal of \( \overline{\text{DY}}_h(\mathfrak{g}l_n)_{cr} \) generated by all elements \( l^r_{ij} \) with \( r \geq p \).

Let \( A[\mathfrak{S}_k] \) be the group algebra of the symmetry group \( \mathfrak{S}_k \), which naturally acts on \( (\mathbb{C}^n)^\otimes k \) by permutation. Let \( A_k \) be the antisymmetrizer

\[
    A_k = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (\text{sgn } \sigma) \sigma, \tag{4.6}
\]

where \( \text{sgn } \sigma \) is the sign of the permutation \( \sigma \).

For each \( k = 1, \ldots, n \), introduce the Laurent series \( \ell_k(u) \) in \( u \) by

\[
    \ell_k(u) = \text{tr}_{1\cdots k} A_k L^-_1(u_1) \cdots L^-_k(u_k) \cdot L^+_k(u_k + \frac{1}{2}hn)^{-1} \cdots L^+_1(u_1 + \frac{1}{2}hn)^{-1}, \tag{4.7}
\]

where \( u_i = u + (i - 1)h \) and the partial trace is taken over all \( k \) copies of \( \text{End} \mathbb{C}^n \) in \( (\text{End} \mathbb{C}^n)^\otimes k \otimes \overline{\text{DY}}_h(\mathfrak{g}l_n)_{cr} \), thus the coefficients of \( \ell_k(u) \) belong to \( \overline{\text{DY}}_h(\mathfrak{g}l_n)_{cr} \).
The following famous lemma of Jucys is well-known, see [21].

**Lemma 4.1.** One has that

\[
\prod_{1 \leq i < j \leq k} R_{ij}(u_i - u_j) = k! A_k,
\]

where the product is taken in the lexicographical order on the pairs \((i, j)\).

Using Lemma 4.1 and the RTT relations (4.4), we can easily get the following lemma.

**Lemma 4.2.** One has that

\[
A_k L_1(u_1) \cdots L_k(u_k) = L_k(u_k) \cdots L_1(u_1) A_k
\]

(4.9)

\[
A_k L_k^+(u_k + \frac{1}{2} \hbar n)^{-1} \cdots L_1^+(u_1 + \frac{1}{2} \hbar n)^{-1}
\]

\[
= L_1^+(u_1 + \frac{1}{2} \hbar n)^{-1} \cdots L_k^+(u_k + \frac{1}{2} \hbar n)^{-1} A_k
\]

(4.10)

The following condition (crossing symmetry) is important for our later discussion.

**Lemma 4.3.** The R-matrix \( R(u) \) satisfies the crossing symmetry relations:

\[
(R_{12}(u)^{-1})^{t_2} R_{12}(u - \hbar n)^{t_2} = I,
\]

(4.11)

\[
R_{12}(u - \hbar n)^{t_1} (R_{12}(u)^{-1})^{t_1} = I.
\]

(4.12)

**Proof.** Set \( Q = P^{t_1} = P^{t_2} \), it is easy to check that

\[
Q^2 = nQ.
\]

then we have

\[
\bar{R}(-u)^{t_2} \bar{R}(u - \hbar n)^{t_2} = (I - \frac{\hbar}{u} Q) (I + \frac{\hbar}{u - \hbar n} Q) = I.
\]

It follows that

\[
(R_{12}(u)^{-1})^{t_2} R_{12}(u - \hbar n)^{t_2} = \frac{f(u - \hbar n)}{f(u)} \frac{u^2}{u^2 - \hbar^2} \bar{R}(-u)^{t_2} \bar{R}(u - \hbar n)^{t_2} = I.
\]

We can prove the second crossing symmetry relation similarly. \(\square\)

**Theorem 4.4.** The coefficients of \( \ell_k(u) \) belong to the center of the completed Yangian double at the critical level \( DY_h(gl_n)_{cr} \) for all \( k = 1, \cdots, n \).

**Proof.** Consider the tensor product \( End \mathbb{C}^n \otimes (End \mathbb{C}^n)^{\otimes k} \otimes DY_h(gl_n)_{cr} \) and label the first copy of \( End \mathbb{C}^n \) by 0. It suffices to show that \( \ell_k(u) \)
commutes with $L^+_0(z)$. From the defining relation [4.3], we get

$$L^+_0(z)L^-_k(u_k) \cdots L^-_1(u_1) = R_{0k}(z-u_k + \frac{1}{2}hn)^{-1} \cdots R_{01}(z-u_1 + \frac{1}{2}hn)^{-1}$$

$$R_{01}(z-u_1 + \frac{1}{2}hn) \cdots R_{0k}(z-u_k + \frac{1}{2}hn)L^+_0(z)L^-_k(u_k) \cdots L^-_1(u_1)$$

$$= R_{0k}(z-u_k + \frac{1}{2}hn)^{-1} \cdots R_{01}(z-u_1 + \frac{1}{2}hn)^{-1}L^-_k(u_k) \cdots L^-_1(u_1)L^+_0(z)$$

$$R_{01}(z-u_1 - \frac{1}{2}hn) \cdots R_{0k}(z-u_k - \frac{1}{2}hn).$$

Relation [4.4] gives

$$L^+_0(z)R_{0a}(z-u_a - \frac{1}{2}hn)L^+_a(u_a + \frac{1}{2}hn)^{-1}$$

$$= L^+_a(u_a + \frac{1}{2}hn)^{-1}R_{0a}(z-u_a - \frac{1}{2}hn)L^+_0(z)$$

for $a = 1, \ldots, k$. Therefore,

$$L^+_0(z)R_{01}(z-u_1 - \frac{1}{2}hn) \cdots R_{0k}(z-u_k - \frac{1}{2}hn)L^+_1(u_1 + \frac{1}{2}hn)^{-1} \cdots$$

$$L^+_k(u_k + \frac{1}{2}hn)^{-1} = L^+_1(u_1 + \frac{1}{2}hn)^{-1} \cdots L^+_k(u_k + \frac{1}{2}hn)^{-1}R_{01}(z-u_1 - \frac{1}{2}hn) \cdots$$

$$R_{0k}(z-u_k - \frac{1}{2}hn)L^+_0(z).$$

To conclude $L^+_0(z)L^+_k(u) = L^+_k(u)L^+_0(z)$, it is enough to prove

$$tr_{1,\ldots,k} R_{0k}(z-u_k + \frac{1}{2}hn)^{-1} \cdots R_{01}(z-u_1 + \frac{1}{2}hn)^{-1}L^-_k(u_k) \cdots L^-_1(u_1)$$

$$L^+_1(u_1 + \frac{1}{2}hn)^{-1} \cdots L^+_k(u_k + \frac{1}{2}hn)^{-1}R_{01}(z-u_1 - \frac{1}{2}hn) \cdots$$

$$R_{0k}(z-u_k - \frac{1}{2}hn)A_k = \ell_k(u).$$

The R-matrix $R(u)$ satisfies the Yang-Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)$$

Therefore, Lemma [4.5] implies

$$R_{01}(z-u_1 - \frac{1}{2}hn) \cdots R_{0k}(z-u_k - \frac{1}{2}hn)A_k$$

$$= A_kR_{0k}(z-u_k - \frac{1}{2}hn) \cdots R_{01}(z-u_1 - \frac{1}{2}hn)$$

$$R_{0k}(z-u_k + \frac{1}{2}hn)^{-1} \cdots R_{01}(z-u_1 + \frac{1}{2}hn)^{-1}A_k$$

$$= A_kR_{01}(z-u_1 + \frac{1}{2}hn)^{-1} \cdots R_{0k}(z-u_k + \frac{1}{2}hn)^{-1}$$

(4.17)
Using (4.17), (4.18) and Lemma 4.2 we can write (4.16) in the form

\[
\text{tr}_{1,\ldots,k} R_{0k}(z-u_k + \frac{1}{2}hn)^{-1} \cdots R_{01}(z-u_1 + \frac{1}{2}hn)^{-1} A_k L_1^-(u_1) \cdots L_k^-(u_k)
\]
\[
L_k^+(u_k + \frac{1}{2}hn)^{-1} \cdots L_1^+(u_1 + \frac{1}{2}hn)^{-1} R_{0k}(z-u_k - \frac{1}{2}hn)^{-1} \cdots R_{01}(z-u_1 - \frac{1}{2}hn).
\]

(4.19)

Now replace \( A_k \) with \((A_k)^2\) and move one copy of \( A_k \) to the left with help of (4.18) and the other copy to the right. Then we get

\[
\text{tr}_{1,\ldots,k} A_k R_{01}(z-u_1 + \frac{1}{2}hn)^{-1} \cdots R_{0k}(z-u_k + \frac{1}{2}hn)^{-1} L_k^- (u_k) \cdots L_1^- (u_1)
\]
\[
L_1^+(u_1 + \frac{1}{2}hn)^{-1} \cdots L_k^+(u_k + \frac{1}{2}hn)^{-1} R_{01}(z-u_1 - \frac{1}{2}hn)^{-1} \cdots R_{0k}(z-u_k - \frac{1}{2}hn) A_k.
\]

Next we use the cyclic property of trace to move the left copy of \( A_k \) to the right-most position and replace \((A_k)^2\) with \( A_k \). After these transformations, we obtain the following expression

\[
\text{tr}_{1,\ldots,k} R_{01}(z-u_1 + \frac{1}{2}hn)^{-1} \cdots R_{0k}(z-u_k + \frac{1}{2}hn)^{-1} A_k L_1^- (u_1) \cdots L_k^- (u_k)
\]
\[
L_k^+(u_k + \frac{1}{2}hn)^{-1} \cdots L_1^+(u_1 + \frac{1}{2}hn)^{-1} R_{0k}(z-u_k - \frac{1}{2}hn)^{-1} \cdots R_{01}(z-u_1 - \frac{1}{2}hn).
\]

(4.20)

Now we let

\[
X = R_{01}(z-u_1 + \frac{1}{2}hn)^{-1} \cdots R_{0k}(z-u_k + \frac{1}{2}hn)^{-1} A_k L_1^- (u_1) \cdots L_k^- (u_k)
\]
\[
L_k^+(u_k + \frac{1}{2}hn)^{-1} \cdots L_1^+(u_1 + \frac{1}{2}hn)^{-1},
\]
\[
Y = R_{0k}(z-u_k - \frac{1}{2}hn) \cdots R_{01}(z-u_1 - \frac{1}{2}hn).
\]

Since \((A(I \otimes B))^{t_2} = (I \otimes B)^{t_2} A^{t_2}\) for all \( A \in (\text{End} \mathbb{C}^n)^{\otimes 2}\), we have

\[
X^{t_1 \cdots t_k} = L^{t_1 \cdots t_k}(R_{01}(z-u_1 + \frac{1}{2}hn)^{-1})^{t_1} \cdots (R_{0k}(z-u_k + \frac{1}{2}hn)^{-1})^{t_k},
\]

(4.21)

\[
Y^{t_1 \cdots t_k} = (R_{0k}(z-u_k - \frac{1}{2}hn))^{t_k} \cdots (R_{01}(z-u_1 - \frac{1}{2}hn))^{t_1},
\]

(4.22)

where \( L = A_k L_1^-(u_1) \cdots L_k^-(u_k)L_k^+(u_k + \frac{1}{2}hn)^{-1} \cdots L_1^+(u_1 + \frac{1}{2}hn)^{-1} \). The first crossing symmetry relation implies

\[
(R_{0i}(z-u_i + \frac{1}{2}hn)^{-1})^{t_i} R_{0i}(z-u_i - \frac{1}{2}hn)^{t_i} = \text{id}.
\]
By using the property $tr_{1,\ldots,k}XY = tr_{1,\ldots,k}X^{t_1\ldots t_k}Y^{t_1\ldots t_k}$, we have

$$tr_{1,\ldots,k}XY = tr_{1,\ldots,k} L^{t_1\ldots t_k}(R_{01}(z - u_1 + \frac{1}{2}hn)^{-1})^{t_1} \cdots (R_{0k}(z - u_k + \frac{1}{2}hn)^{-1})^{t_k} (R_{0k}(z - u_k - \frac{1}{2}hn)^{t_k} \cdots (R_{01}(z - u_1 - \frac{1}{2}hn))^{t_1} = tr_{1,\ldots,k} L^{t_1\ldots t_k} = tr_{1,\ldots,k} L$$

(4.23)

which coincides with $\ell_k(u)$ as defined in (4.17). Thus, $\ell_k(u)$ commutes with $L^+_0(z)$. Similarly, we can prove $L^-_0(z)\ell_k(u) = \ell_k(u)L_0^-(z)$ by using the second crossing symmetry relation. Hence, the coefficients of $\ell_k(u)$ belong to the center of the completed Yangian double at the critical level $\text{DY}_h(\mathfrak{gl}_n)_{cr}$.

**Remark.** The theorem is similar to the special case of [16, Thm. 4.4] with $\mu = (1^n)$.

For the spectral parameter dependent matrix $B(u) = [B(u)_{ij}]$, set $u_i = u + (i - 1)h$, the quantum minor $B(u)_{b_1\ldots b_n}$ is defined by

$$A_nB_1(u_1)B_2(u_2)\cdots B_n(u_n) = \sum_{a_1b_1} e_{a_1b_1} \otimes e_{a_2b_2} \cdots \otimes e_{a_nb_n} \otimes B(u)_{a_1\ldots a_n}^{b_1\cdots b_n}$$

(4.24)

The explicit formula for the quantum minors $L^\pm(u)_{a_1\ldots a_k}$ can be written immediately from the definition. For $a_1 < \cdots < a_k$, it is of Sklyanin determinant type:

$$L^\pm(u)_{a_1\ldots a_k} = \sum_{\sigma \in \mathfrak{S}_k} sgn(\sigma) l^\pm_{a_{\sigma(1)}b_1}(u) \cdots l^\pm_{a_{\sigma(k)}b_k}(u + (k - 1)h)$$

(4.25)

and then for $\tau \in \mathfrak{S}_k$

$$L^\pm(u)_{a_{\tau(1)}\ldots a_{\tau(k)}} = sgn(\tau) L^\pm(u)_{a_1\ldots a_k}.$$  

(4.26)

Similarly for $b_1 < \cdots < b_k$, we have

$$L^\pm(u)_{b_1\ldots b_k} = \sum_{\sigma \in \mathfrak{S}_k} sgn(\sigma) l^\pm_{a_{\sigma(k)}b_1}(u + (k - 1)h) \cdots l^\pm_{a_{\sigma(1)}b_{\sigma(1)}}(u)$$

(4.27)

and then for any $\tau \in \mathfrak{S}_k$ we have

$$L^\pm(u)_{b_{\tau(1)}\ldots b_{\tau(k)}} = sgn(\tau) L^\pm(u)_{a_1\ldots a_k}.$$  

(4.28)

It is easy to see that the quantum determinant $qdet L^\pm(u)$ is just the quantum minor $L^\pm(u)_{1\ldots n}$. Moreover, the quantum minor is zero if two top or two bottom indices are equal.

**Theorem 4.5.** The coefficients of the quantum determinant $qdet L^\pm(u)$ belong to the center of the Yangian double $\text{DY}_h(\mathfrak{gl}_n)_{-k}$ at arbitrary level $-k$. 


PROOF. Introduce the product \( R(v_0, v_1, \cdots, v_n) = \prod_{0 \leq a < b \leq n} R_{ab}(v_a - v_b) \), where the \( v_a \) are variables and the product is taken in the lexicographical order on the pairs \((a, b)\). From the defining relations (4.4) and (4.5), we have

\[
R(z + \frac{1}{2}hk, u_1, \cdots, u_n)L_0^+(z)L_1^-(u_1)L_2^-(u_2) \cdots L_n^-(u_n) = L_n^-(u_n) \cdots L_2^-(u_2)L_1^-(u_1)L_0^+(z)R(z - \frac{1}{2}hk, u_1, \cdots, u_n). \tag{4.29}
\]

Since \( R(u) = f(u)\bar{R}(u), \prod_{1 \leq a < b \leq n} \bar{R}_{ab}(u_a - u_b) = n! A_n \), and

\[
A_nL_1^-(u_1)L_2^-(u_2) \cdots L_n^-(u_n) = L_n^-(u_n) \cdots L_2^-(u_2)L_1^-(u_1)A_n = A_nqdetL^-(u),
\]

by canceling common factors on both sides of the equality we get

\[
\prod_{a=1}^{n} f(z + \frac{1}{2}hk - u_a) \prod_{a=1}^{\rightarrow} \bar{R}_{0a}(z + \frac{1}{2}hk - u_a)A_nL_0^+(z)qdetL^-(u) = qdetL^-(u)L_0^+(z)A_n \prod_{a=1}^{\leftarrow} \bar{R}_{0a}(z - \frac{1}{2}hk - u_a) \prod_{a=1}^{n} f(z - \frac{1}{2}hk - u_a). \tag{4.30}
\]

Observe that

\[
\prod_{a=1}^{\rightarrow} \bar{R}_{0a}(u_0 - u_a)A_n = A_n \prod_{a=1}^{\leftarrow} \bar{R}_{0a}(u_0 - u_a) = A_n(1 + \frac{h}{u_0 - u_1}).
\]

Indeed, by the first equality, it suffices to verify the second equality on the basis vectors of the form \( e_i \otimes e_i \otimes e_1 \otimes \cdots \otimes e_{i-1} \otimes e_{i+1} \otimes \cdots \otimes e_n \) for \( i = 1, \cdots, n \). The verification is trivial, so we omit it here. Since \( f(u-nh) = \frac{u^2-h^2}{u^2} \), we get

\[
F(u) = \frac{u^2-h^2}{u^2} F(u),
\]

where \( F(u) = f(u)f(u - h) \cdots f(u - (n - 1)h) \). The series \( F(u) \) is uniquely determined by this relation. Therefore, we have

\[
F(u) = f(u)f(u - h) \cdots f(u - (n - 1)h) = (1 + hu^{-1})^{-1}.
\]

It follows that

\[
\prod_{a=1}^{n} \frac{f(z - \frac{1}{2}hk - u_a)}{f(z + \frac{1}{2}hk - u_a)} = \frac{1 + \frac{h}{z+\frac{1}{2}hk-u}}{1 + \frac{h}{z-\frac{1}{2}hk-u}}.
\]

From the above relation and (4.30), we can conclude that \( L_0^+(z)qdetL^-(u) = qdetL^-(u)L_0^+(z) \). Using the unitarity property of the R-matrix \( \bar{R}(u) \):

\[
\bar{R}(-u) = \frac{u^2-h^2}{u^2} \bar{R}_{21}(u)^{-1},
\]

the relation \( L_0^+(z)qdetL^+(u) = qdetL^+(u)L_0^+(z) \) can be checked by a similar argument. The remaining two relations \( L_0^+(z)qdetL^\pm(u) = qdetL^\pm(u)L_0^+(z) \) are similarly obtained by using the defining relations (4.4). \( \square \)
Remark. The proof of Theorem 4.5 is a little bit different from those in [16 Prop. 2.8].

**Theorem 4.6.** The coefficients of $qdetL^\pm(u)$ are algebraically independent and generate the center of $\text{DY}_h(\mathfrak{gl}_n)_{cr}$.

**Proof.** Let $qdetL^+(u) = 1 + h \sum_{r \geq 1} d_r^+ u^{-r}$, $qdetL^-(u) = 1 - h \sum_{r \geq 1} d_r^- u^{-r}$. Introduce the filtration on the Yangian double at the critical level $\text{DY}(\mathfrak{gl}_n)_{cr}$ by $\deg l_{ij}^r = -r$ for all $r \geq 1$ and $\deg h = 0$. Consider the central extension $\hat{\mathfrak{gl}}_n = \mathfrak{gl}[t, t^{-1}] \oplus \mathbb{C}K$ defined by the commutation relations

$$[E_{ij}[r], E_{kl}[s]] = \delta_{kj}E_{il}[r + s] - \delta_{il}E_{kj}[r + s] + r\delta_{r,-s}K(\delta_{kj}\delta_{il} - \delta_{ij}\delta_{kl}).$$

As in [16 Prop. 2.1], the corresponding graded algebra $\text{gr} \text{DY}(\mathfrak{gl}_n)_{cr}$ is isomorphic to $U(\mathfrak{gl}_n)_{cr}[[h]]$, where $U(\mathfrak{gl}_n)_{cr}$ is the quotient of $U(\mathfrak{gl}_n)$ modulo the ideal generated by the relation $K = -n$. Now we derived from the definition of quantum determinant that the coefficient $d_r^+$ of $qdetL^+(u)$ has the form $d_r^+ = l_{11}^r + \cdots + l_{nn}^r$ plus terms of degree less than $r - 1$. Therefore, the image of $d_r^+$ in the $(r - 1)$-th component of $\text{gr} \text{DY}(\mathfrak{gl}_n)_{cr}$ coincides with $I r^{-1}$ where $I = E_{11} + \cdots + E_{nn}$. Similarly, the image of $d_r^-$ in the $(-r)$-th component of $\text{gr} \text{DY}(\mathfrak{gl}_n)_{cr}$ coincides with $I r^r$. The elements $I r^r (r \in \mathbb{Z})$ are algebraically independent, so are the elements $d_r^+, d_r^-$. The elements $I r^r (r \in \mathbb{Z})$ generate the center of $U(\mathfrak{gl}_n)_{cr}$. This implies that $d_1^+, d_2^+, \cdots, d_1^-, d_2^-, \cdots$ generate the center of $\text{DY}_h(\mathfrak{gl}_n)_{cr}$.

As a corollary of Theorem 4.4, we describe the invariants of the vacuum module over the Yangian double explicitly. First of all, we define the vacuum module at the critical level $V_h(\mathfrak{gl}_n)$ to be the quotient of $\text{DY}_h(\mathfrak{gl}_n)_{cr}$ by the left ideal generated by all the elements $I r_{ij}$ with $r \geq 0$. The module $V_h(\mathfrak{gl}_n)$ is generated by the vector $1$ (the image of $1 \in \text{DY}_h(\mathfrak{gl}_n)_{cr}$ in the quotient) such that

$$L^+(z)1 = I1,$$

where $I$ is the identity matrix. The subspace of invariants of $V_h(\mathfrak{gl}_n)$ is defined by

$$\mathfrak{Z}_h(\mathfrak{gl}_n) = \{ v \in V_h(\mathfrak{gl}_n) \mid L^+(z)v = Iv \}.$$

For $k = 1, \cdots, n$, introduce the series $\tilde{e}_k(u)$ with coefficients in $V_h(\mathfrak{gl}_n)$ by

$$\tilde{e}_k(u) = tr_{1, \cdots, k} A_k L_1^- (u_1) \cdots L_k^- (u_k).$$

**Corollary 4.7.** All coefficients of the series $\tilde{e}_k(u)$ with $k = 1, \cdots, n$ belong to the subspace of invariants $\mathfrak{Z}_h(\mathfrak{gl}_n)$. Moreover, the coefficients of all series $\tilde{e}_k(u)$ commute with each other.
PROOF. By Theorem 4.4, we have \( L^+(z) \ell_k(u) = \ell_k(u) L^+(z) \). Note that \( \ell_k(u) 1 = \ell_k(u) 1 \), so the first part of the corollary follows by the application of both sides of \( L^+(z) \ell_k(u) = \ell_k(u) L^+(z) \) to the vector \( 1 \in V_h(\mathfrak{gl}_n) \). We can prove the second part by applying both sides of the equality \( \ell_k(u) \ell_m(v) = \ell_m(v) \ell_k(u) \) to the vector \( 1 \). For the left hand side, we get
\[
\ell_k(u) \ell_m(v) 1 = \ell_k(u) \ell_m(v) 1 = \ell_m(v) \ell_k(u) 1 = \ell_m(v) \ell_k(u) 1.
\]
Similarly, for the right hand side we have
\[
\ell_m(v) \ell_k(u) 1 = \ell_m(v) \ell_k(u) 1.
\]
Therefore, \( \bar{\ell}_k(u) \ell_m(v) = \bar{\ell}_m(v) \ell_k(u) \).

\( \square \)

Remark. The series \( \ell_k(u) \) coincides with \( T^+_\mu(u) \) with \( \mu = (1^n) \), see [16, Cor. 4.6].

By calculating the trace in (4.7), we can write \( \ell_k(u) \) in terms of the entries of \( L^-(u) = [l^-(ij)(u)] \) and \( \bar{L}^+(u) := L^+(u)^{-1} = [\bar{l}^+(ij)(u)] \).

Proposition 4.1. For \( k = 1, \ldots, n \), we have
\[
\ell_k(u) = \sum_{j_1, \ldots, j_k} \sum_{1 < i_2 < \cdots < i_k} \sum_{\sigma \in S_k} \text{sgn}(\sigma) l^+_\sigma(j_1) \cdots l^+_\sigma(j_k) (u + (k - 1)h)
\]
and
\[
\ell_k(u) = \sum_{j_1, \cdots, j_k} \sum_{1 < i_2 < \cdots < i_k} \sum_{\sigma \in S_k} \text{sgn}(\sigma) l^-_\sigma(j_1) \cdots l^-_\sigma(j_k) (u + (k - 1)h)
\]

Proof. Using (4.17), we regard
\[
A_k L^+_1 (u_1) \cdots L^-_k (u_k) \bar{L}^+_k (u_k + \frac{1}{2}hn) \cdots \bar{L}^+_1 (u_1 + \frac{1}{2}hn)
\]
with \( u_i = u + (i - 1)h \) as an operator in the vector space \((\mathbb{C}^n)^\otimes k\). Since the operator is divisible on the right by \( A_k \), its trace equals to \( k! \) times the sum of the diagonal matrix elements corresponding to basis vectors of the form \( e_{i_1} \otimes \cdots \otimes e_{i_k} \) with \( i_1 < \cdots < i_k \). Note that \( \text{sgn}(\sigma) = \text{sgn}(\sigma^{-1}) \), and for any \( \sigma \in S_k \),
\[
\sigma(e_{i_1} \otimes \cdots \otimes e_{i_k}) = e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(k)}},
\]
(4.31) immediately follows from (4.6). Similarly, we can prove (4.32) by using the basis vectors \( e_{i_k} \otimes \cdots \otimes e_{i_1} \) with the same condition \( i_1 < \cdots < i_k \) on the indices.

We have the following well-known expression for \( \ell_n(u) \) (cf. [16]).
Corollary 4.8.
\[ \ell_n(u) = qdetL^-(u)(qdetL^+(u + \frac{1}{2}hn))^{-1}. \] (4.33)

Proof. The definition of the quantum determinant implies
\[ A_nL_n^+ (u_n)^{-1} \cdots L_1^+ (u_1)^{-1} = (qdetL^+(u))^{-1} A_n. \] (4.34)
Replacing \( u \) by \( u + \frac{1}{2}hn \) and using \( tr_{1,\ldots,n}A_n = 1 \) we obtain the desired formula (4.33) from (4.7).

Lemma 4.9. The entries of the inverse matrix \( L^+(u)^{-1} \) are given by
\[ [L^+(u)^{-1}]_{ij} = (-1)^{j-i}(qdetL^+(u - (n-1)h))^{-1}L^+(u - (n-1)h)^{1: \ldots: n}_{1: \ldots: n}, \]
where the hat means omitting the indices.

Proof. By the definition of the quantum determinant, we have
\[ A_nL_1^+(u_1) \cdots L_{n-1}^+(u_{n-1}) = A_n qdetL^+(u)L_n^+(u_n)^{-1}, \]
where \( u_i = u + (i-1)h \). Now apply both sides to the basis vector \( e_1 \otimes \cdots \otimes \hat{e}_i \otimes \cdots \otimes e_n \otimes e_j \) and replace \( u \) by \( u - (n-1)h \), we get the desired formula (4.35). \( \square \)
5. Harish-Chandra homomorphisms

We discuss the Poincaré-Birkhoff-Witt theorem for the Yangian double. Define a total ordering \( \prec \) on the set of generators as follows. First, each generator \( l_{ij}^{(r)} (r < 0) \) precedes any generator \( l_{km}^{(s)} (s \geq 0) \). Furthermore, \( l_{ij}^{(r)} \prec l_{km}^{(s)} (r, s < 0) \) if and only if the triple \((j - i, i, r)\) precedes \((m - k, k, s)\) in the lexicographical order. Finally, we set \( l_{ij}^{(r)} \prec l_{km}^{(s)} (r, s \geq 0) \) if and only if the triple \((i - j, i, r)\) precedes \((k - m, k, s)\) in the lexicographical order. Consider the Yangian double \( DY_h(gl_n)_{cr} \) at the critical level \( c = -n \). By PBW theorem for the Yangian double, any element \( x \in DY_h(gl_n)_{cr} \) can be written as a unique linear combination of ordered monomials in the generators \( l_{ij}^{(k)} \). Let \( DY^0 \) be the subspace of \( DY_h(gl_n)_{cr} \) spanned by those monomials which do not contain any generators \( l_{ij}^{(k)} \) with \( i \neq j \). Denote by \( \theta \) the projection from \( DY_h(gl_n)_{cr} \) to \( DY^0 \), and let \( x_0 = \theta(x) \). Then we extend \( \theta \) by continuity to get the projection \( \theta : DY_h(gl_n)_{cr} \rightarrow \widetilde{DY}^0 \), where \( \widetilde{DY}^0 \) is the corresponding completed vector space of \( DY^0 \).

Let \( \Pi_c(n) \) be the algebra of polynomials in independent variables \( l_i^k \) with \( i = 1, \ldots, n \) and \( k \in \mathbb{Z} \). The mapping \( l_{ij}^{(k)} \mapsto l_i^k \) extends to an isomorphism of vector spaces \( \eta : DY^0 \rightarrow \Pi_c(n) \). Define the completion \( \tilde{\Pi}_c(n) \) as the inverse limit

\[
\tilde{\Pi}_c(n) = \lim_{\rightarrow} \Pi_c(n)/I_p, \quad p > 0
\]

where \( I_p \) is the ideal of \( \Pi_c(n) \) generated by all elements \( l_i^k \) with \( k \geq p \). We can extend \( \eta \) to an isomorphism of the respective completed vector spaces \( \eta : \widetilde{DY}^0 \rightarrow \tilde{\Pi}_c(n) \). Let \( \chi = \eta \circ \theta \), then \( \chi \) is a linear map \( \chi : DY_h(gl_n)_{cr} \rightarrow \tilde{\Pi}_c(n) \). In the following proposition, we give an analogue of the Harish-Chandra homomorphism for the Yangian double at the critical level.

**Proposition 5.1.** The restriction of the map \( \chi \) to the center \( Z_c(gl_n) \) of the algebra \( DY_h(gl_n)_{cr} \) is a homomorphism of commutative algebras.

**Proof.** For \( x, y \in Z_c(gl_n) \), set \( x_0 = \chi(x) \) and \( y_0 = \chi(y) \). By the PBW theorem for the Yangian double, we can write \( y \) as a (possibly infinite) linear combination of ordered monomials in the generators \( l_{ij}^{(r)} \). Suppose that

\[
m = \prod_{a} l_{i_a,j_a}^{(r_a)} \prod_{b} l_{i_b,j_b}^{(r_b)} \quad (r_a < 0, r_b \geq 0)
\]

is an ordered monomial which occurs in the linear combination. From the defining relations (3.4) and (3.5), we get

\[
[l_{ij}^{(0)}, l_{km}^{(v)}] = \delta_{kj} l_{km}^{(v)} - \delta_{im} l_{kj}^{(v)}.
\]

(5.1)
Therefore,

\[ [l_{ii}^{(0)}, l_{km}^{(0)}(v)] = \begin{cases} l_{km}^{\pm}(v) & k = i, m \neq i \\ -l_{km}^{\pm}(v) & k \neq i, m = i \\ 0 & \text{otherwise} \end{cases} \] (5.2)

Since \([l_{ii}^{(0)}, m] = [l_{ii}^{(0)}, \prod_a l_{ia_ja}^{(r_a)} \prod_b l_{ib_jb}^{(r_b)}] = 0\) for \(i = 1, \ldots, n\), we have

\[ \sum_a (i_a - j_a) + \sum_b (i_b - j_b) = 0, \] (5.3)

which is implied by (5.2). Suppose that \(m \in \text{Ker} \chi\). From (5.3), we have \(i_a > j_a\) for some \(a\) or \(i_b > j_b\) for some \(b\). Since \(x\) is in the center, we have

\[ xm = \prod_a l_{ia_ja}^{(r_a)} x \prod_b l_{ib_jb}^{(r_b)}. \]

We can write \(xm\) as a linear combination of ordered monomials by using the defining relations (3.4). Due to the property that \(i_a > j_a\) for some \(a\) or \(i_b > j_b\) for some \(b\), we derive that \(xm \in \text{Ker} \chi\). Thus only \(\chi(xy_0)\) can give a nonzero contribution to the image \(\chi(x)\), and \(xy_0\) has an expression of the form

\[ \prod_a l_{ia_ja}^{(r_a)} x \prod_b l_{ib_jb}^{(r_b)}. \]

If \(p\) is an ordered monomial which occurs in the linear combination representing \(x\) and \(\chi(p) = 0\), then we conclude that \(\chi(\prod_a l_{ia_ja}^{(r_a)} p \prod_b l_{ib_jb}^{(r_b)}) = 0\) by applying property (5.3) to the monomial \(p\). Finally, by (3.4) we can swap any two generators \(l_{ii}^{(r)}\) and \(l_{jj}^{(s)}\) with \(r, s \geq 0\) (resp. \(r, s < 0\)) modulo \(\text{Ker} \chi\) within any monomial of the form \(\prod_a l_{ia_ja}^{(r_a)} \prod_b l_{ib_jb}^{(r_b)}\). This implies that \(\chi(xy) = x_0y_0 = \chi(x)\chi(y)\). \(\square\)

Now we will find the image of the Harish-Chandra map for \(\ell_k(u)\) defined in Theorem 4.4. Introduce the following series whose coefficients are the generators of the algebra \(\Pi_c(n)\)

\[ l_i^+(u) = 1 - h \sum_{k \in \mathbb{Z}_{>0}} l_i^k u^{-k-1}, \quad l_i^-(u) = 1 + h \sum_{k \in \mathbb{Z}_{<0}} l_i^k u^{-k-1} \]

and for \(i = 1, \ldots, n\) set

\[ \lambda_i(u) = \frac{l_i^- u l_i^+(u - h + \frac{1}{2}hn) \cdots l_{i-1}^- u (u - (i - 1)h + \frac{1}{2}hn)}{l_i^+(u + \frac{1}{2}hn) \cdots l_i^+(u - (i - 1)h + \frac{1}{2}hn)}. \]

Note that \(\lambda_i(u)\) a Laurent series in \(u\) with coefficients in the completed algebra \(\bar{\Pi}_c(n)\).
Theorem 5.1. For each $k = 1, \cdots, n$ the image of the series $\ell_k(u)$ under the Harish-Chandra homomorphism is given by

$$\chi : \ell_k(u) \mapsto \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1}(u)\lambda_{i_2}(u + h) \cdots \lambda_{i_k}(u + (k - 1)h).$$

Proof. We can express $[L^+(u)^{-1}]_{ij}$ in terms of quantum minors by using Lemma 4.9. From the definition of the quantum determinant, we have

$$L^+(u)^{1 \cdots \hat{j} \cdots n}_{1 \cdots \hat{i} \cdots n} = \text{sgn}(\omega)L^+(u)^{n \cdots \hat{j} \cdots 1}_{n \cdots \hat{i} \cdots 1},$$

where $\omega \in \mathfrak{S}_{n-1}$ reverses the order of the lower indices. Now expand this quantum minor by (4.25) and use formula (4.31) for $\ell_k(u)$, we see that a nonzero contribution to the image $\chi(\ell_k(u))$ can only come from the summands in (4.31) with $i_{\sigma(1)} \leq j_1 \leq i_1$. It follows that $\sigma(1) = 1$ and $i_1 = j_1$. Similarly, the defining relations of $\text{DY}_h(\mathfrak{gl}_n)$ imply that $\sigma(2) = 2$ and $i_2 = j_2$, etc. Thus a nonzero contribution only comes from the summands with $\sigma = 1$ and $i_a = j_a$ for all $a = 1, \cdots, k$. After apply Lemma 4.9 and formulas (4.25) again, we conclude that the images of the quantum minors are given by

$$q\text{det}L^+(u) \mapsto l_1^+(u + (n - 1)h) \cdots l_n^+(u)$$

and

$$L^+(u)^{1 \cdots \hat{i} \cdots n}_{1 \cdots \hat{i} \cdots n} \mapsto l_1^+(u + (n - 2)h) \cdots l_{i-1}^+(u + (n - i)h)l_i^+(u + (n - i - 1)h) \cdots l_n^+(u).$$

This finishes the calculation of the Harish-Chandra image of $\ell_k(u)$.

Remark. Note that Molev and Mukhin’s paper [23] constructed an analogue of the Harish-Chandra homomorphism for the Yangian $Y(\mathfrak{gl}_n)$. Here we extend this result to the Yangian double at the critical level $\text{DY}_h(\mathfrak{gl}_n)_{cr}$ using a different method.
6. Eigenvalues in Wakimoto modules

We recall the construction of $h$-deformed Wakimoto modules over $\text{DY}_h(sl_n)$ at the critical level. It can be realized in the bosonic Fock space by an explicit action of the generator series $H_i^\pm(u), E_i(u), F_i(u)$ as in [13]. We reproduce this construction as follows.

Introduce the following set of $n^2 - 1$ Heisenberg algebras with generators $a_{ij}^+(1 \leq i \leq n-1), b_{ij}^+, c_{ij}^+$ (1 $\leq i < j \leq n$) with $n \in \mathbb{Z} - \{0\}$ and $p_{ij}$, $q_{ij}$ ($1 \leq i \leq n-1), p_{ij}, q_{ij}, p_{ij}, q_{ij}$ ($1 \leq i < j \leq n$).

\[
[a_{ij}^+, a_{kl}^-] = (k + g)B_{ij}n\delta_{n+m,0}, \quad [p_{ij}, q_{kl}] = (k + g)B_{ij},
\]

(6.1)

\[
[b_{ij}^+, b_{kl}^-] = -n\delta_{i,j}\delta_{j,k}\delta_{n+m,0}, \quad [p_{ij}, q_{ij}] = -\delta_{i,j}\delta_{j,k},
\]

(6.2)

\[
[c_{ij}^+, c_{kl}^-] = n\delta_{i,j}\delta_{j,k}\delta_{n+m,0}, \quad [p_{ij}, q_{ij}] = \delta_{i,j}\delta_{j,k},
\]

(6.3)

where $g = n$ is the dual Coxeter number for the Cartan matrix of type $A_{n-1}, B_{ij} = \frac{1}{2}a_{ij}$, where $a_{ij}$ are entries of the Cartan matrix of the type $A_{n-1}$.

The Fock space $F_h(n)$ corresponding to the above Heisenberg algebras can be defined as follows. Let $|0\rangle$ be the vacuum state defined by

\[
[a_{ij}^+, |0\rangle] = [b_{ij}^+, |0\rangle] = [c_{ij}^+, |0\rangle] = 0 \quad (n > 0),
\]

\[
p_{ij} |0\rangle = q_{ij} |0\rangle = 0.
\]

For $X = a_i, b_{ij}, c_{ij}$, let us now define

\[
X(u; A, B) = \sum_{n>0} \frac{X_n}{n} (u + Ah)^n - \sum_{n>0} \frac{X_n}{n} (u + Bh)^{-n} + \log(u + Bh)p_X + q_X,
\]

\[
X_+(u; B) = -\sum_{n>0} X_n (u + Bh)^{-n} + \log(u + Bh)p_X,
\]

\[
X_-(u; A) = \sum_{n>0} X_n (u + Ah)^n + q_X,
\]

\[
X(u; A) = X(u; A, A), X(u) = X(u; 0).
\]

For $b_{ij}, c_{ij}$, define

\[
\hat{X}_\pm(u) = \mp(X_\pm(u; -\frac{1}{2}) - X_\pm(u; \frac{1}{2})).
\]

For the bosonic fields $a_i(u; A, B)$, define

\[
\hat{a}_i^+(u) = a_i^+(u; 0) - a_i^+(u; k + g),
\]

\[
\hat{a}_i^-(u) = \frac{1}{k + g} \sum_{j,l=1}^{n-1} (B^{-1})^{jl}(a_j^+(u; B_{ij}) - a_j^-(u; -B_{ij})).
\]
The \((h\text{-deformed})\) Wakimoto module over \(DY_h(sl_n)\) at the critical level is realized by defining the action of the series \(H^\pm_i(u), E_i(u), F_i(u)\) in the Fock space \(F_h(n)\). Here we only give the formulas for the action of \(H^\pm_i(u)\) and \(E_i(u)\).

\[
H^\pm_i(u) =: \exp\left\{ \sum_{l=1}^i \hat{b}_\pm^{i+1}(u \pm \frac{1}{2}(k + l - 1)h) \right\} - \sum_{l=1}^{i-1} \hat{b}_\pm^l(u \pm \frac{1}{2}(k + l)h) + \hat{a}_\pm^i(u \mp \frac{1}{4}kh) \tag{6.4} \]

and

\[
E_i(u) = -\frac{1}{h} \sum_{m=1}^i : \exp\{ (b + c)^m i (u + \frac{1}{2}(m - 1)h) \} : \times \left\{ \exp(\hat{b}_+^{m,i+1}(u + \frac{1}{2}(m - 1)h) - (b + c)^m,i+1(u + \frac{1}{2}(m - 2)h)) \right\} \times \left\{ \exp(\hat{b}_-^{l,i+1}(u + \frac{1}{2}(l - 1)h) - \hat{b}_+^{l}(u + \frac{1}{2}lh)) \right\}.
\]

where \((b + c)^m i (u) = b^i (u) + c^i (u)\). We define a normal ordering in the product so that the coefficients \(b_n^m\) with \(n < 0\) or \(q_m^p\) should be placed to the left of the coefficients \(b_n^m\) with \(n > 0\) or \(p_n^m\). A similar rule applies to the coefficients of \(c^m i (u)\). By Theorem 4.5, the coefficients of the quantum determinants are central elements in the algebra \(DY_h(gl_n)_{cr}\). Therefore, we can extend the irreducible Wakimoto modules to \(DY_h(gl_n)_{cr}\) by specifying the eigenvalues \(K^\pm(u)\) of \(qdetL^\pm(u)\). Lemma 2.4 implies the following conditions

\[
k_1^\pm(u)k_2^\pm(u + h) \cdots k_n^\pm(u + (n - 1)h) \rightarrow K^\pm(u),
\]

where \(K^-(u)\) and \(K^+(u)\) are power series in \(u\) and \(u^{-1}\), respectively. Hence, we can use relation (6.4) to define the action of the coefficients of all series \(k_i^\pm(u)\) in the space \(F_h(n)\). For any \(X \in DY_h(gl_n)_{cr}\) we will denote \(\langle 0 \mid X \mid 0 \rangle\) by the coefficient of \(\mid 0 \rangle \) in the expansion of \(X \mid 0 \rangle\). Moreover, a relation of the form \(\langle 0 \mid X = \langle 0 \mid d\) for a constant \(d\) means that \(\langle 0 \mid XY \mid 0 \rangle = d\langle 0 \mid X \mid 0 \rangle\) for any element \(Y \in DY_h(gl_n)_{cr}\). Next we will use this notation to parameterize the corresponding modules over \(DY_h(gl_n)_{cr}\) by the power series \(\varphi_i^+(u)\) and \(\varphi_i^-(u)\) in \(u^{-1}\) and \(u\), respectively, such that

\[
k_i^+(u) \mid 0 \rangle = \varphi_i^+(u) \mid 0 \rangle \quad \text{and} \quad k_i^-(u) \mid 0 \rangle = \varphi_i^-(u) \mid 0 \rangle \tag{6.6}
\]
for all \(i = 1, \ldots, n\). The series \(x_i^+(u)\) satisfy the following relations

\[
x_{i+1}^+(u)x_i^+(u)^{-1} = \exp(\hat{a}_i^+(u + \frac{n - 2i}{4}h)), \\
x_{i+1}^-(u)x_i^-(u)^{-1} = \exp(\hat{a}_i^-(u - \frac{n + 2i}{4}h))
\]

for all \(i = 1, \ldots, n - 1\). Since \(\hat{a}_i^+(u) = 0\), the series \(x_i^+(u)\) is the same for each \(i\). We will denote them by \(x^+(u)\) for convenience.

**Theorem 6.1.** Given an irreducible Wakimoto module over \(DY_{h}(\mathfrak{g}l_n)_{cr}\) with the parameters \(x^+(u)\) and \(x_i^-(u)\), we have the following formulas for the eigenvalues of the series \(\ell_k(u)\) in the module

\[
\ell_k(u) \mapsto \sum_{1 \leq i_1 < \cdots < i_k \leq n} \Lambda_{i_1}(u)\Lambda_{i_2}(u + h)\cdots\Lambda_{i_k}(u + (k - 1)h), \quad k = 1, \ldots, n,
\]

where

\[
\Lambda_i(u) = x_i^-(u)x_i^+(u + \frac{1}{2}hn)^{-1}, \quad i = 1, \ldots, n.
\]

**Proof.** Any irreducible Wakimoto module is generated by the vacuum vector \(|0\rangle\) over \(DY_{h}(\mathfrak{g}l_n)_{cr}\). So the eigenvalues of the series \(\ell_k(u)\) can be found by calculating the series \((0 \mid \ell_k(u) \mid 0)\). It follows from (6.5) that \(E_i(u) \mid 0\rangle\) is a power series in \(u\) for all \(i\). Therefore, the definition of \(E_i(u)\) implies \(e_{i,i+1}^+(u) \mid 0\rangle = 0\) for \(i = 1, \ldots, n - 1\). We have the following relations for the action of the generators \(l_{ij}^{(0)}\) on the vacuum vector \(|0\rangle\):

\[
l_{i,i+1}^{(0)} \mid 0\rangle = 0 \quad \text{and} \quad l_{ii}^{(0)} \mid 0\rangle = \frac{1}{h}(c_i - c_{i-1}) \mid 0\rangle, \quad (6.7)
\]

where \(c_0 = 0\) and \(c_i(i \geq 1)\) denotes the coefficient of \(u^{-1}\) in \(x^+(u)\cdots x^+(u + (i - 1)h)\). Indeed, (2.10) and (6.6) imply that \(L^+(u)_{i:i+1} \mid 0\rangle\) is a scalar power series in \(u^{-1}\). Using (1.25) to expand the quantum minor and taking the coefficient of \(u^{-1}\) we get that \((l_{11}^{(0)} + \cdots + l_{ii}^{(0)}) \mid 0\rangle\) is a scalar multiple of the vacuum vector \(|0\rangle\). Indeed, we have \(L^+(u)_{1:i} = k^+_i(u)\cdots k^+_i(u + (i - 1)h)\). Therefore, \(-h(l_{11}^{(0)} + \cdots + l_{ii}^{(0)}) \mid 0\rangle = c_i \mid 0\rangle\), which implies the second relation in (6.7). Now use the relation \(L^+(u)_{1:i-1,i} \mid 0\rangle = 0\) implied by (3.7). Taking the coefficient of \(u^{-1}\), we get \(l_{i,i+1}^{(0)} \mid 0\rangle = 0\). As a next step, we will prove

\[
e_{i,j}^+(u) \mid 0\rangle = 0 \quad \text{for all} \quad i < j. \quad (6.8)
\]

(5.1) implies that

\[
[l_{j+1,i}^{(0)}, l_{km}^{(0)}(u)] = 0 \quad \text{for all} \quad j > k, m \quad (6.9)
\]

and

\[
[l_{j+1,i}^{(0)}, l_{j,i}^{(0)}(u)] = l_{j+1,i}^{(0)}(u) \quad \text{for all} \quad j > i. \quad (6.10)
\]
we get
\[ u_{i,j}^{0}(u) = e_{i,j}^{-}(u) \] for all \( j > i \)
(6.11)
and we get (6.3) from (6.7) by induction. Therefore, we have
\[ L^{+}(u)_{1-i-1} = 0 \] for all \( i < j \).
(6.12)
Indeed, use (4.25) and (4.28) to expand the quantum minor we get
\[ e_{i,i}^{-} = 1 \] for all \( j > i \) and
(6.13)
where the sum is taken over the permutations \( \sigma \) of the set \( \{1, \cdots, i-1, j\} \).
Since
\[ L^{+}(u)_{1-i} = \sum_{\sigma} \text{sgn}(\sigma) l^{+}_{\sigma(i),i}(u) \cdots l^{+}_{\sigma(1),1}(u+(i-1)h) \] (6.14)
is a scalar power series in \( u^{-1} \), it follows that
\[ l^{+}_{j,i}(u) = 0 \] for all \( j > i \)
(6.15)
and \( l^{+}_{i,i}(u) = 0 \) is a scalar power series in \( u^{-1} \) by induction on \( j-i \). Furthermore, (3.7) and formulas (6.6) imply that
\[ l^{+}_{i,i}(u) = u^{-1}(u) (6.16) \]
for all \( i = 1, \cdots, n \). Similarly, we have
\[ \langle 0 | l^{-}_{j,i}(u) = 0 \] for all \( j > i \)
(6.17)
and \( \langle 0 | l^{-}_{i,i}(u) = 0 | x_{i}^{-}(u) \] for \( i = 1, \cdots, n \).
We observe that \( \langle 0 | E_{i}(u) \) is a power series in \( u^{-1} \). Indeed, this fact follows from (6.5) by using the relations
\[ \langle 0 | a_{i}^{j} = 0 | b_{i}^{j} = 0 | c_{i}^{j} = 0 \] for all \( n < 0 \).
(6.18)
An extra step is to apply the commutation relations (6.2) and (6.3) which implies that
\[ \exp(q_{b}^{ij}) \cdot u^{p} = u^{p} \cdot \exp(q_{b}^{ij}) \cdot u \]
and
\[ \exp(q_{c}^{ij}) \cdot u^{p} = u^{p} \cdot \exp(q_{c}^{ij}) \cdot u^{-1} \]
Hence, these relations and the definition of \( E_{i}(u) \) imply the relation \( \langle 0 | e_{i,i+1}^{-}(u) = 0 \) for \( i = 1, \cdots, n-1 \). The rest of the arguments is basically the same with some adjustments. For instance, we use the expansion
\[ \sum_{\sigma} \text{sgn}(\sigma) l^{+}_{\sigma(1),1}(u) \cdots l^{+}_{\sigma(i),i}(u+(i-1)h) \] (6.19)
to calculate the coefficient of \( u^{0} \) in the power series \( \langle 0 | L^{-}(u)_{1-i} \) Similarly, we get
\[ \langle 0 | l^{-}_{i+1,i} = 0 \] and
\[ \langle 0 | l^{-}_{i,i} = -\frac{1}{h}(d_{i} - d_{i-1})(0 \] (6.20)
where $d_0 = 0$, $d_i (i \geq 1)$ denotes the coefficient of $u^0$ in $\varphi_i^{-1}(u) \cdots \varphi_1^{-1}(u + (i - 1)h)$. Finally, we use the relations
\[
\langle 0 \mid \sum_{\sigma} \text{sgn}(\sigma) l_{\alpha(1),1}^{-}(u) \cdots l_{\alpha(j),i}^{-}(u + (i - 1)h) \rangle = 0
\]
summed over permutations $\sigma$ of the set $\{1, \cdots, i - 1, j\}$, and the expansion \eqref{6.19} instead of \eqref{6.13} and \eqref{6.14}. With the use of relations \eqref{6.15}, \eqref{6.16} and \eqref{6.17}, we can conclude that the eigenvalue $\langle 0 \mid l_k(u) \mid 0 \rangle$ coincides with the image of the series $\ell_k(u)$ under the Harish-Chandra homomorphism calculated in Theorem 5.1 for the specialization
\[
l_i^+(u) = \varphi_i^+(u) \quad \text{and} \quad l_i^-(u) = \varphi_i^-(u)
\]
for $i = 1, \cdots, n$. This completes the proof by setting $\lambda_i(u)$ to $\Lambda_i(u)$.

\[\square\]

Remark. The calculations in Sections 4, 5 and 6 are similar to those for the quantum affine algebras $U_q(\hat{\mathfrak{g}l}_n)$. The results for the double Yangian can be viewed as a counterpart of those for the quantum affine algebras.

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