TOTALLY UMBILICAL HYPERSURFACES OF Spin$^c$ MANIFOLDS CARRYING SPECIAL SPINOR FIELDS

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ABSTRACT. Under some dimension restrictions, we prove that totally umbilical hypersurfaces of Spin$^c$ manifolds carrying a parallel, real or imaginary Killing spinor are of constant mean curvature. This extends to the Spin$^c$ case the result of O. Kowalski stating that, every totally umbilical hypersurface of an Einstein manifold of dimension greater or equal to 3 is of constant mean curvature. As an application, we prove that there are no extrinsic hyperspheres in complete Riemannian Spin manifolds of non-constant sectional curvature carrying a parallel, Killing or imaginary Killing spinor.

1. Introduction

Using classical submanifold techniques, a lot of results on the geometry of totally umbilical submanifolds (and other special hypersurfaces) in ambient manifolds of special geometries were obtained [7, 12, 8, 9, 10, 11, 38, 43, 44, 45]. As one example, O. Kowalski [31] used the Codazzi-Mainardi equation to prove the following elementary and well-known result:

Theorem 1.1. Every totally umbilical connected hypersurface of an Einstein manifold of dimension greater or equal to 3 is of constant mean curvature.

Examples of ambient Riemannian Einstein manifolds $(\tilde{M}^{m+1}, \tilde{g})$ of dimension $m + 1 \geq 3$ are Riemannian Spin manifolds carrying an $\alpha$-Killing spinor ($\alpha \in \mathbb{C}$), i.e., a spinor field $\psi$ satisfying the equation

$$\tilde{\nabla}_X \psi = \alpha X \cdot \psi,$$

for any vector $X$ tangent to $\tilde{M}$, where $\tilde{\nabla}$ denotes the spinorial Levi-Civita connection on the spinor bundle and $\cdot$ the Clifford multiplication, compare Section 2.

For Spin manifolds it is known that the Killing constant $\alpha$ has to be zero (parallel spinor), a nonzero real constant (real Killing spinor) or a nonzero purely imaginary constant (imaginary Killing spinor) [14]. When $\alpha$ is real, such spinors characterize the limiting case in the Friedrich’s and Hijazi’s inequalities which provide a lower bound for the eigenvalues of the Dirac operator involving the infimum of the scalar

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2010 Mathematics Subject Classification. 53C27, 53C25, 53C42.

Key words and phrases. Totally umbilical hypersurfaces, constant mean curvature, Spin$^c$ manifolds with special spinor fields, differential forms, extrinsic hypersphere.

Acknowledgement: This work was initiated during the stay of both authors at the ‘Centre International de Rencontres Mathématiques’ in Luminy, Marseille-France (Research in Pairs program). Then, it was continued during the stay of both authors at CIRM (Centro Internazionale per la Ricerca Matematica) in Italy (Research in Pairs program). The authors gratefully acknowledge the support of both centers. The second author thanks the Institute of Mathematics of the University of Freiburg for its continuous support and hospitality during many research stays. The first author was supported by the Juniorprofessurenprogramm Baden-Württemberg.
curvature or the first eigenvalue of the Yamabe operator [16, 22, 23]. Moreover, the existence of $\alpha$-Killing spinors leads to restrictions on the geometry and topology of the manifold. In fact besides being Einstein (and even Ricci-flat when $\alpha = 0$), $\tilde{M}$ is automatically compact if $\alpha$ is real and noncompact if $\alpha$ is purely imaginary. Complete simply connected Spin manifolds with real, parallel or imaginary Killing spinors have been classified by Wang [46], Bär [1] and Baum [3, 4, 5] and the existence is glued to the holonomy of the manifold. This classification gives, in some dimensions, other examples than the most symmetric ones as Euclidean space, the sphere or the hyperbolic space. These examples are relevant to physicists in general relativity where the Dirac operator plays a central role.

Techniques from Spin geometry have been successfully used to produce striking advances in extrinsic geometry (see e.g. the study of CMC or minimal surfaces in homogeneous 3-spaces which arise in Thurston’s classification of 3-dimensional geometries and Alexandrov-type theorems as in [27, 21, 26, 24, 2]). It is remarkable that, in many extrinsic results, Spin geometrical tools - in particular special/natural spinor fields and the Dirac operator - have played a central role and inspired further research directions.

When shifting from the from the classical Spin geometry to Spin$^c$ geometry, the situation is more general and many obstacles appear since the Spin$^c$ structure will not only depend on the geometry of the manifold but also on the connection (and hence the curvature) of the auxiliary line bundle associated with the fixed Spin$^c$ structure. From a physical point of view, spinors model fermions while Spin$^c$-spinors can be interpreted as fermions coupled to an electromagnetic field. Transferring the idea to use spinorial methods in the study of submanifolds to the Spin$^c$ world, allows us to cover more ambient geometric structures (CR structures, Kähler and Sasaki structures). Indeed, O. Hijazi, S. Montiel and F. Urbano constructed on Kähler-Einstein manifolds with positive scalar curvature [25], Spin$^c$ structures carrying Kählerian Killing spinors. The restriction of these spinors to minimal Lagrangian submanifolds provides topological and geometric restrictions on these submanifolds (see [41, 37] for other applications of the use of Spin$^c$ geometry in extrinsic geometry). Equation (1) on Spin$^c$ manifolds has been studied by A. Moroianu [35] when $\alpha$ is real and by the authors [19] when $\alpha$ is purely imaginary. In fact, a complete simply connected manifold has a parallel Spin$^c$ spinor if and only if it is isometric to the Riemannian product between a simply connected Kähler manifold (with its canonical or anti-canonical Spin$^c$ structure) and a simply connected Spin manifold carrying a parallel spinor. The only simply connected Spin$^c$ manifolds admitting real non-parallel Killing spinors other than the Spin manifolds are the non-Einstein Sasakian manifolds endowed with their canonical or anti-canonical Spin$^c$ structure. Beside that, complete Spin$^c$ manifolds with imaginary Killing spinors are isometric to a special warped product of a Spin$^c$ manifold with a parallel spinor with $\mathbb{R}$. These classification results, stated above, show that, in contrast to the Spin case, Spin$^c$ manifolds carrying an $\alpha$-Killing spinor are not in general Einstein manifolds. For this reason, we start by extending Theorem 1.1 to ambient Riemannian Spin$^c$ manifolds carrying an $\alpha$-Killing spinor. In fact, our main result is:

**Theorem 1.2.** Every connected totally umbilical hypersurface $M^m$ of a Riemannian Spin$^c$ manifold $\tilde{M}^{m+1}$ with $m+1 \geq 5$ carrying a real (including parallel) Killing
spinor or of dimension \(m+1 \geq 3\) carrying an imaginary Killing spinor is of constant mean curvature.

Here, we recall that if \(\tilde{M}\) is already Spin, Theorem 1.2 is just a special case of Theorem 1.1 because in this case \(\tilde{M}\) is Einstein. Using the classification of Killing spinors on \(\text{Spin}^c\) manifolds cited above we thus in particular obtained:

**Corollary 1.3.** Every connected totally umbilical hypersurface \(M^m\) of a Kähler or a Sasakian manifold has constant mean curvature.

For Kähler manifolds the last statement is known from [12, Thm. 4.2] and we merely give a spinorial proof here. The counterpart for Sasakian manifolds was not known before to our best knowledge.

We will show by counterexamples that Theorem 1.2 is sharp in the sense that it fails if the ambient \(\text{Spin}^c\) manifold is of dimension 3 or 4 carrying a parallel or real Killing spinor. The proof of Theorem 1.2 relies on two families of differential forms naturally associated to the spinor obtained by the restriction of the \(\alpha\)-Killing spinor to the hypersurface \(M\). These differential forms and their exterior derivatives involve the mean curvature \(H\) of the isometric immersion and hence allow to deduce that \(H\) is constant. Dependent on whether \(\alpha\) is real or imaginary, the proof of Theorem 1.2 differs in these cases and is carried out separately (see Section 5).

As further applications of Theorem 1.2, we give some no-existence results of extrinsic hyperspheres in some special complete \(\text{Spin}\) manifolds.

**Theorem 1.4.** There are no extrinsic hyperspheres in

(i) complete manifolds with holonomy \(G_2\) and \(\text{Spin}(7)\).

(ii) complete simply connected 3-Sasakian manifold of dimension \(4m+3\) which is not of constant curvature

(iii) complete simply connected Sasakian Einstein manifold of dimension \(4m+3\), \(m \geq 2\) which is not 3-Sasakian

(iv) compact Sasakian-Einstein manifold of dimension \(2m+1\), \(m \geq 2\) which are not locally symmetric.

(v) homogeneous warped product \(\tilde{M} = N \times_f \mathbb{R}\) where \(f(t) = e^{4\mu t}\) (\(\mu \in \mathbb{R}^*\)) and \(N\) a complete Riemannian \(\text{Spin}\) manifold with a parallel spinor and of non-constant sectional curvature.

Note that (i) was already obtained in [28] and we give here just a spinorial proof. Moreover, Theorem 1.4 is a particular case of the more general Theorem 6.3. In fact, we prove that there are no extrinsic hyperspheres in Riemannian \(\text{Spin}\) manifolds of non-constant sectional curvature and carrying an \(\alpha\)-Killing spinor field.

2. Preliminaries

In this section, we briefly review some basic facts about \(\text{Spin}^c\) structures on oriented Riemannian manifolds and their hypersurfaces [17, 33, 14, 2, 37, 6].

2.1. Hypersurfaces and induced \(\text{Spin}^c\) structures.

**\(\text{Spin}^c\) structures on manifolds:** Let \((\tilde{M}^{m+1}, \tilde{g})\) be a Riemannian \(\text{Spin}^c\) manifold of dimension \(m+1 \geq 3\) without boundary. On such a manifold, we have a Hermitian complex vector bundle \(\Sigma\tilde{M}\) endowed with a natural scalar product \(\langle \cdot, \cdot \rangle\) and with a connection \(\tilde{\nabla}\) which parallelizes the metric. We denote by \(\Re \langle \cdot, \cdot \rangle\) the real part
of the scalar product $(.,.)$. This complex vector bundle, called the $\text{Spin}^c$ bundle, is endowed with a Clifford multiplication denoted by $\cdot$, $T\tilde{M} \rightarrow \text{End}_C(\Sigma\tilde{M})$, such that at every point $x \in \tilde{M}$, defines an irreducible representation of the corresponding Clifford algebra. Hence, the complex rank of $\Sigma\tilde{M}$ is $2^{\frac{|\tilde{M}|+1}{2}}$. The Clifford multiplication can be extended to exterior products of the tangent bundle and to differential forms, such that $(v_1 \wedge \ldots \wedge v_k) \cdot \varphi := v_1 \cdot \ldots \cdot v_k \cdot \varphi$ if the $v_i$'s are mutually orthogonal and such that $\psi^2 \cdot \psi := \psi \cdot \psi$ for all vector fields $v_i$, $v$ and spinors $\psi$ and where $\cdot$ denotes the isomorphism $TM \rightarrow T^*\tilde{M}$ induced by the metric.

Given a $\text{Spin}^c$ structure on $(\tilde{M}^{m+1}, g)$, one can prove that the determinant line bundle $\text{det}(\Sigma\tilde{M})$ has a root of index $2^{|\tilde{M}|+1}-1$. We denote by $L$ this root line bundle over $\tilde{M}$ and it is called the auxiliary line bundle associated with the $\text{Spin}^c$ structure. Locally, a Spin structure always exists. We denote by $\Sigma'\tilde{M}$ the (possibly globally non-existent) spinor bundle. Moreover, the square root of the auxiliary line bundle $\tilde{L}$ always exists locally. But, $\Sigma\tilde{M} = \Sigma'\tilde{M} \otimes \tilde{L}^\frac{1}{2}$ exists globally. This essentially means that, while the spinor bundle and $\tilde{L}^\frac{1}{2}$ may not exist globally, their tensor product (the $\text{Spin}^c$ bundle) is defined globally. Thus, the connection $\tilde{\nabla}$ on $\Sigma\tilde{M}$ is the twisted connection of the one on the spinor bundle (coming from the Levi-Civita connection) and a fixed connection on $\tilde{L}$.

We may now define the Dirac operator $\tilde{D}$ acting on the space of smooth sections of $\Sigma\tilde{M}$ by the composition of the metric connection and the Clifford multiplication. In local coordinates this reads as

$$\tilde{D} = \sum_{j=1}^{m+1} e_j \cdot \tilde{\nabla} e_j,$$

where $\{e_1, \ldots, e_{m+1}\}$ is a local oriented orthonormal tangent frame. It is a first order elliptic operator, formally self-adjoint with respect to the $L^2$-scalar product and satisfies the Schrödinger-Lichnerowicz formula

$$\tilde{D}^2 = \tilde{\nabla}^* \tilde{\nabla} + \frac{1}{4} \text{scal} + \frac{i}{2} \Omega,$$

where $\tilde{\nabla}^*$ is the adjoint of $\tilde{\nabla}$ with respect to the $L^2$-scalar product, $\tilde{\text{scal}}$ is the scalar curvature of $\tilde{M}$, $i\tilde{\Omega}$ is the curvature of the auxiliary line bundle $\tilde{L}$ associated with the fixed connection ($\tilde{\Omega}$ is a real 2-form on $\tilde{M}$) and $\tilde{\Omega}$ is the extension of the Clifford multiplication to differential forms. For any $X \in \Gamma(T\tilde{M})$ and any spinor field $\psi \in \Gamma(\Sigma\tilde{M})$, the Ricci identity is given by

$$\sum_{k=1}^{m+1} e_k \cdot \tilde{\mathcal{R}}(e_k, X) \psi = \frac{1}{2} \tilde{\mathcal{R}}(X) \psi - \frac{i}{2} (X, \tilde{\Omega}) \cdot \psi,$$

where $\tilde{\mathcal{R}}$ is the Ricci curvature of $(\tilde{M}^{m+1}, g)$ and $\tilde{\mathcal{R}}$ is the curvature tensor of the spinorial connection $\tilde{\nabla}$.

When $m$ is even, the complex volume form $\tilde{\omega}_\psi := i^{\frac{m+2}{2}} e_1 \cdot \ldots \cdot e_{m+1}$ acts on $\Sigma\tilde{M}$ as the identity, i.e., $\tilde{\omega}_\psi \psi = \psi$ for any spinor field $\psi \in \Gamma(\Sigma\tilde{M})$. Besides, if $m$ is odd, we have $\tilde{\omega}_\psi \tilde{\psi} = 1$. We denote by $\Sigma^+\tilde{M}$ the eigebundles corresponding to the eigenvalues $\pm 1$, hence $\Sigma\tilde{M} = \Sigma^+\tilde{M} \oplus \Sigma^-\tilde{M}$ and a spinor field $\psi$ can be written as $\psi = \psi^+ + \psi^-$. The conjugate $\bar{\psi}$ of $\psi$ is defined by $\bar{\psi} = \psi^+ - \psi^-$. 
As pointed out, the Clifford multiplication can be extended to differential forms and one sees that
\[ \langle \delta \cdot \psi, \psi \rangle = (-1)^{\frac{k(k+1)}{2}} \langle \delta \cdot \psi, \psi \rangle \]
for any \( k \)-form \( \delta \) and a spinor field \( \psi \in \Gamma(\Sigma \widetilde{M}) \). This directly implies that for mutually orthogonal vector fields \( v_1, \ldots, v_k \) we have
\[ \langle v_1 \cdot v_2 \cdot \ldots \cdot v_k \cdot \psi, \psi \rangle \in \begin{cases} \mathbb{R} & \text{for } k \equiv 0, 3 \mod 4 \\ i\mathbb{R} & \text{for } k \equiv 1, 2 \mod 4. \end{cases} \]

**Spin\( ^c \) structures on hypersurfaces:** The following can be e.g. found in [36]. Any Spin\( ^c \) structure on \((\widetilde{M}^{m+1}, g)\) induces a Spin\( ^c \) structure on an oriented hypersurface \((M^m, g)\) of dimension \( m \geq 2 \), and we have
\[ \Sigma M \simeq \begin{cases} \Sigma \widetilde{M}_M & \text{if } m \text{ is even}, \\ \Sigma^+ \widetilde{M}_M & \text{if } m \text{ is odd}. \end{cases} \]

Furthermore Clifford multiplication by a vector field \( X \), tangent to \( M \), is given by
\[ X \cdot_M \varphi = (X \cdot \nu \cdot \psi)|_M, \]
where \( \psi \in \Gamma(\Sigma \widetilde{M}) \) (or \( \psi \in \Gamma(\Sigma^+ \widetilde{M}) \) if \( m \) is odd), \( \varphi \) is the restriction of \( \psi \) to \( M \), \( \nu \cdot \psi \) the Clifford multiplication on \( \widetilde{M} \) and \( \nu \) is the unit normal vector field of \( M \) in \( \widetilde{M} \). Also, when \( m \) is odd, we obtain \( \Sigma M \simeq \Sigma^- \widetilde{M}_M \). With this identification, the Clifford multiplication is given by \( X \cdot_M \varphi = -(X \cdot \nu \cdot \psi)|_M. \) In particular, we have \( \Sigma M \simeq \Sigma M \oplus \Sigma M \).

Moreover, the corresponding auxiliary line bundle \( L \) on \( M \) is the restriction to \( M \) of \( \widetilde{L} \) and the curvature 2-form \( i\Omega \) on \( L \) is given by \( i\Omega = i\tilde{\Omega}|_M \). For every \( \psi \in \Gamma(\Sigma \widetilde{M}) \) (\( \psi \in \Gamma(\Sigma^+ \widetilde{M}) \) if \( m \) is odd), the real 2-forms \( \Omega \) and \( \tilde{\Omega} \) are related by
\[ (\tilde{\Omega} \cdot \psi)|_M = \Omega \cdot_M \varphi - (\nu \cdot \tilde{\Omega}) \cdot_M \varphi. \]

We denote by \( \nabla \) the Spin\( ^c \) connection on \( \Sigma M \). Then, for all \( X \in \Gamma(TM) \), we have the Spin\( ^c \) Gauss formula:
\[ (\tilde{\nabla} X \psi)|_M = \nabla_X \varphi + \frac{1}{2} \Pi X \cdot_M \varphi, \]
where \( \Pi \) denotes the Weingarten map of the hypersurface. Denoting by \( D \) the Dirac operator on \( M \) and by the same symbol any spinor and its restriction to \( M \), we have
\[ D \varphi = \frac{m}{2} H \varphi - \nu \cdot D \varphi - \nabla^*_\nu \varphi, \]
where \( H = \frac{1}{m} \text{tr}(\Pi) \) denotes the mean curvature and \( D^M = D \) if \( m \) is even and \( D^M = D \oplus (-D) \) if \( m \) is odd.

### 3. Totally umbilical hypersurfaces of Spin\( ^c \) manifolds carrying an \( \alpha \)-Killing spinor

Let \((\widetilde{M}^{m+1}, g)\) be a Riemannian Spin\( ^c \) manifold with an \( \alpha \)-Killing spinor \( \psi \) of Killing constant \( \alpha \in \mathbb{C} \). It is known that for \( m \geq 1 \), the Killing constant \( \alpha \) has to
be purely real or purely imaginary [19, Theorem 1.1]. Moreover, if $\alpha$ is real, then $
abla \psi$ has constant norm since, for any $X \in \Gamma(TM)$, we have

$$X(|\psi|^2) = 2\Re(\nabla X \psi, \psi) = 2\alpha \Re(X \cdot \psi, \psi) = 0.$$  

(6)

Hence, real Killing spinors have no zeros. When $\alpha$ is purely imaginary, the function $|\psi|$ is a non-constant and nowhere vanishing function [3, 19]. In this case, the set of zeros of $\psi$ is discrete [19, 40, 34, 32]. Using the definition (1) of an $\alpha$-Killing spinor $\psi$, we have

$$\bar{D}\psi = \sum_{j=1}^{m+1} e_j \cdot \nabla_{e_j} \psi = (m+1)\alpha \psi \quad \text{and} \quad \bar{D}^2\psi = (m+1)^2 \alpha^2 \psi.$$

Then, the Schrödinger-Lichnerowicz formula (2) on $\tilde{M}$ gives

$$m(m+1)\alpha^2 \psi = \frac{\text{scal}}{4} \psi + \frac{1}{2} \Omega \cdot \psi.$$

From now on we assume that $(M, g)$ is an oriented totally umbilical hypersurface of $(\tilde{M}, \tilde{g})$. Totally umbilical means $II X = HX$ for all $X \in \Gamma(TM)$. Note that this implies $(\nabla_Y II)(X) = dH(Y)X$ for all $X, Y \in \Gamma(TM)$.

We choose the local orthonormal frame $e_i$ on $\tilde{M}$ such that $\{e_1, \ldots, e_m\}$ is a local orthonormal frame of $\tilde{M}$, $\nabla e_i = 0$ and that $e_{m+1} = \nu$ a unit normal vector to $M$. The Ricci identity (3) on $\tilde{M}$ for $X = \nu$ applied to the $\alpha$-Killing spinor $\psi$ reads

$$\frac{\tilde{R}(\nu, \nu)}{2} \nu \cdot \psi + \frac{1}{2} \sum_{j=1}^{m} \tilde{R}(\nu, e_j) e_j \cdot \psi - \frac{i}{2} (\nu, \tilde{\Omega}) \cdot \psi = 2m\alpha^2 \nu \cdot \psi.$$  

(7)

where we also used the calculation $\bar{R}(e_k, \nu) \psi = 2\alpha^2 e_k \cdot \nu \cdot \psi$. Now, the Codazzi-Mainardi equation [39, Prop. 33] gives that

$$\tilde{g}(\bar{R}(X, Y)U, \nu) = g(\nabla_X II)(Y, U) - g((\nabla_Y II)(X), U) = dH(X)(Y, U) - dH(Y)(X, U)$$

for all $X, Y, U \in \Gamma(TM)$. Hence,

$$\tilde{R}(X, \nu) = \sum_{l=1}^{m} \tilde{g}(\bar{R}(X, e_l) e_l, \nu) = (m-1)dH(X).$$

Replacing this in Equation (7) and taking then the Clifford multiplication by $\nu$, we obtain for $\varphi = \psi|_M$ that

$$-\frac{1}{2} \tilde{R}(\nu, \nu) \varphi - \frac{m-1}{2} dH \cdot M \varphi + \frac{i}{2} (\nu, \tilde{\Omega}) \cdot M \varphi = -2m\alpha^2 \varphi.$$  

(8)

**Lemma 3.1.** Assume that $(M, g)$ is a totally umbilical oriented hypersurface of $(\tilde{M}, \tilde{g})$ carrying an $\alpha$-Killing spinor $\psi$. Then, for $\varphi = \psi|_M$ we have

1. **The Schrödinger-Lichnerowicz formula on $M$:**

$$m(m-1) \left( \alpha^2 + \frac{H^2}{4} \right) \varphi + \frac{m-1}{2} dH \cdot M \varphi = \frac{\text{scal}}{4} \varphi + \frac{i}{2} \Omega \cdot M \varphi.$$  

(9)
(2) The Ricci identity on $M$:

$$\frac{1}{2} (\text{Ric}(X) - i(X_\ast \Omega^M)) \cdot \nabla M \varphi = - \frac{1}{2} dH \cdot \nabla M \varphi - \frac{m}{2} dH(X) \varphi + \frac{(m-1)}{2} H^2 \nabla M \varphi + 2(m-1) \alpha^2 \nabla M \varphi.$$  \hfill (10)

Proof. Using Equation (5) we have:

$$\nabla e_i \varphi = \tilde{\nabla} e_i \varphi - \frac{H}{2} e_i \cdot \nabla M \varphi = \alpha e_i \cdot \varphi - \frac{H}{2} e_i \cdot \nabla M \varphi$$ \hfill (11)

and

$$\nabla e_j \nabla e_i \varphi = \nabla e_j \left( \alpha e_i \cdot \varphi - \frac{H}{2} e_i \cdot \nabla M \varphi \right) = \alpha \nabla e_j (e_i \cdot \varphi) - \frac{dH(e_j)}{2} e_i \cdot \nabla M \varphi - \frac{H}{2} e_i \cdot \nabla M \nabla e_j \varphi$$

$$= \alpha \nabla e_j (e_i \cdot \varphi) - \frac{H}{2} e_i \cdot \nabla M \varphi - \frac{dH(e_j)}{2} e_i \cdot \nabla M \varphi$$

$$= \alpha^2 e_i \cdot e_j \cdot \varphi + \alpha H \delta_{ij} \cdot \varphi - \frac{H}{2} e_i \cdot e_j \cdot \varphi - \frac{dH(e_j)}{2} e_i \cdot \nabla M \varphi$$

$$= \left( \alpha^2 + \frac{H^2}{4} \right) e_i \cdot e_j \cdot \varphi - \frac{dH(e_j)}{2} e_i \cdot \nabla M \varphi. \hfill (12)$$

Now, we calculate

$$-\nabla^\ast \nabla \varphi = \sum_{i=1}^m \nabla e_i \nabla e_i \varphi = -m \left( \alpha^2 + \frac{H^2}{4} \right) \varphi - \frac{1}{2} dH \cdot \nabla M \varphi,$$

and for the Dirac operator on the hypersurface we obtain

$$D \varphi = \sum_{i=1}^m e_i \cdot M \left( \alpha e_i \cdot \varphi - \frac{H}{2} e_i \cdot \nabla M \varphi \right) = m \alpha \nabla M \varphi + m \frac{H}{2} \varphi.$$

Hence, we have

$$D^2 \varphi = \sum_{i=1}^m e_i \cdot M \nabla e_i \left( m \alpha \nabla M \varphi + m \frac{H}{2} \varphi \right)$$

$$= m \alpha \sum_{i=1}^m e_i \cdot M \nabla e_i (\nu \cdot \varphi) + \frac{m}{2} dH \cdot M \varphi + \frac{mH}{2} D \varphi$$

$$= m^2 \alpha^2 \varphi + \frac{m}{2} dH \cdot M \varphi + \frac{m^2}{4} H^2 \varphi.$$
\[ = 2 \left( \alpha^2 + \frac{H^2}{4} \right) e_i \cdot e_j \cdot \varphi - \frac{dH(e_j)}{2} e_i \cdot M \varphi + \frac{dH(e_i)}{2} e_j \cdot M \varphi. \]

Hence this implies that
\[ \sum_{j=1}^{m} e_j \cdot M R_{e_j, e_i, \varphi} = 2(m - 1) \left( \alpha^2 + \frac{H^2}{4} \right) e_i \cdot M \cdot \varphi - \frac{dH(e_i)}{2} e_i \cdot M \varphi - \frac{m}{2} dH(e_i) \varphi. \]

The last identity together with the Ricci identity (3) on \( M \) can be written as:
\[ \frac{1}{2} \left( \text{Ric}(X) - i(X_{\alpha} M) \right) M \varphi = -\frac{1}{2} dH \cdot M X \cdot M \varphi - \frac{m}{2} dH(X) \varphi \]
\[ + \frac{(m - 1)}{2} H^2 X \cdot M \varphi + 2(m - 1) \alpha^2 X \cdot M \varphi. \]

\[ \square \]

4. Differential forms on the totally umbilical hypersurface \( M \) build from the \( \alpha \)-Killing spinor

In this section, we consider again that \( (\widetilde{M}, \tilde{g}) \) is a totally umbilical oriented hypersurface of \( (\widetilde{M}, \tilde{g}) \) carrying an \( \alpha \)-Killing spinor \( \psi \).

**Lemma 4.1.** Let \( \xi \) be the vector field on \( M \) defined by \( g(X, \xi) = -i(X \cdot M \varphi, \varphi) \). Then on \( M \) we have
\[ \xi \cdot \Omega^M = (m - 1)|\varphi|^2 dH, \]
\[ dH(\xi) = 0. \]

**Proof.** We recall that the Killing constant \( \alpha \) for \( m \geq 2 \) is either purely real or purely imaginary. Thus, for \( m \geq 1 \) we have \((m - 1)\alpha^2 \in \mathbb{R}\). Then the real part of the scalar product with \( \varphi \) of the Ricci identity (10) together with (4) gives
\[ -\frac{i}{2} (X \cdot \Omega^M, M \varphi, \varphi) = \left( \frac{1}{2} dH(X) - \frac{m}{2} dH(X) \right) |\varphi|^2 = \frac{(1 - m)}{2} dH(X) |\varphi|^2. \]

Since \((\xi \cdot \Omega^M)(X) = -(X \cdot \Omega^M)(\xi) = -g(X, \Omega^M, \xi)\), we obtain (13) and hence (14).

Next we define differential forms on \( M \) depending on whether \( \alpha \) is real or imaginary. The first of these forms has been introduced in [20].

**Lemma 4.2.** On the totally umbilical hypersurface \( M \) of \( \widetilde{M} \), we define differential \( p \)-forms \( \omega_p \) by
\[ \omega_p(e_1, e_2, \ldots, e_p) := \langle e_1 \wedge e_2 \wedge \ldots \wedge e_p \rangle M \varphi, \varphi \rangle. \]

If the Killing constant \( \alpha \) is real, we have for all \( p \geq 1 \),
\[ d\omega_p = \frac{1}{2} H (1 - (-1)^p) \omega_{p+1}, \]
\[ dH \wedge \omega_{2p} = 0. \]

**Proof.** Using \([e_i, e_j] = 0\), Equations (11) and (4), we have
\[ (p + 1) d\omega_p(e_1, e_2, \ldots, e_p, e_{p+1}) \]
\[ = \sum_{j=1}^{p+1} (-1)^{j-1} e_j (\omega_p(e_1, e_2, \ldots, \hat{e}_j, \ldots, e_p, e_{p+1})) \]
Proof. With an analog calculation as in the last lemma and using 
If the Killing constant \( \alpha \) is in \( \mathbb{R} \), we have for \( p \geq 1 \),
\[
\begin{align*}
\frac{1}{p+1}d\omega_{p-1} &= 0, \\
\frac{1}{p+1}d\omega_{2k-1} &= H\omega_{2k}.
\end{align*}
\]
Differentiating the last equality we obtain \( dH \wedge \omega_{2k} = 0 \) for any \( k \geq 1 \).

Lemma 4.3. On the totally umbilical oriented hypersurface \( M \) of \( \tilde{M} \), we define differential \( p \)-forms \( \eta_p \) by
\[
\eta_p(e_1, e_2, \ldots, e_p) := \langle (e_1 \wedge e_2 \wedge \cdots \wedge e_p) \cdot M \varphi, \nu \varphi \rangle.
\]
If the Killing constant \( \alpha \) is in \( \mathbb{R} \) \( \setminus \{0\} \), we have for \( p \geq 1 \),
\[
\begin{align*}
d\eta_p &= -\frac{1}{2}H(1 + (-1)^p)\eta_{p+1}, \\
dH \wedge \eta_{2p-1} &= 0.
\end{align*}
\]

Proof. With an analog calculation as in the last lemma and using \( \alpha \in \mathbb{R} \) we obtain
\[
(p + 1)d\eta_p(e_1, e_2, \ldots, e_p, e_{p+1})
\]
\[
\begin{align*}
&= \sum_{j=1}^{p+1} (-1)^{j-1} \langle e_1 \cdot M \cdots \hat{e}_j \cdot M \cdots e_{p+1} \cdot M \left( \alpha e_j \cdot \varphi - \frac{1}{2}He_j \cdot M \varphi \right), \nu \cdot \varphi \rangle \\
&\quad + \sum_{j=1}^{p+1} (-1)^{j-1} \langle e_1 \cdot M \cdots \hat{e}_j \cdot M \cdots e_{p+1} \cdot M \varphi, \alpha e_j \cdot \varphi - \frac{1}{2}He_j \cdot M \varphi \rangle \\
&= -\frac{1}{2}H \sum_{j=1}^{p+1} (-1)^{j-1} \langle e_1 \cdot M \cdots \hat{e}_j \cdot M \cdots e_{p+1} \cdot M \varphi, \nu \cdot \nabla e_j \varphi - \frac{1}{2}He_j \cdot M \nu \cdot \varphi \rangle \\
&\quad - \frac{1}{2}H \sum_{j=1}^{p+1} (-1)^{j-1} \langle e_1 \cdot M \cdots \hat{e}_j \cdot M \cdots e_{p+1} \cdot M \varphi, e_j \cdot M \nu \cdot \varphi \rangle \\
&= -(p + 1)\frac{1}{2}H \left((-1)^p + 1\right)\eta_{p+1}(e_1, e_2, \ldots, e_p, e_{p+1}),
\end{align*}
\]
and, thus, for all $p \geq 1$

\[
\begin{aligned}
d\eta_{2p-1} &= 0, \\
d\eta_{2p} &= -H\eta_{2p-1}.
\end{aligned}
\]

Differentiating the last equality then again gives $dH \wedge \eta_{2p-1} = 0$ for any $p \geq 1$. □

5. Proof of the main result: Theorem 1.2

The goal of this section is to prove Theorem 1.2. If $\tilde{M}$ is spin, $\Omega^M = 0$ and the statement follows directly from (14). For the general Spin$^c$ case we split the proof into the two cases:

Case 1: The $\alpha$-Killing spinor $\psi$ is a real Killing spinor ($\alpha \in \mathbb{R}$)

Case 2: The $\alpha$-Killing spinor $\psi$ is an imaginary Killing spinor ($\alpha \in i\mathbb{R} \setminus \{0\}$).

First we note, that the hypersurfaces in Section 5 is not assumed to be orientable. But since all our calculations are local, we at least have locally always an induced Spin$^c$ structure as in Section 2 and can use all the spinorial formula from above.

**Proof of Theorem 1.2 for Case 1.** We prove this by contradiction. In fact, assume that $dH$ is not identically zero. Then, there is a point $x \in M$ and a neighborhood $U$ of $x$ where $\text{grad}_g H$ is nonzero. Hence, we find a local orthonormal frame $(e_1, \ldots, e_{m-1}, Z = \frac{\text{grad}_g H}{|\text{grad}_g H|})$ of $TU$. Then, $(e_1, \ldots, e_{m-1}, Z, \nu)$ is a local orthonormal frame of $\tilde{M}$ on $U$. Note that then $dH^M = \text{grad}_g^M$.

First we prove the claim for $m > 4$: From Equation (16), it is clear that for $2k \leq m-1$ and for each subset $i_1, \ldots, i_{2k}$ of $\{1, \ldots, m-1\}$, we have

\[\omega_{2k}(e_{i_1}, \ldots, e_{i_{2k}}) = 0.\]

Thus the spinors in \(\{\varphi \cup \{e_{i_1} \cdot_M e_{i_2} \cdot_M \varphi\}\}\), $\varphi$ is a real Killing spinor ($\alpha \in \mathbb{R}$), and \(\varphi = \psi|_x \in \Sigma_x\tilde{M}\), are mutually orthogonal. Hence they span a complex vector subspace of $\Sigma_x\tilde{M}$ of complex dimension

\[\binom{m-1}{0} + \binom{m-1}{2} + \cdots + \binom{m-1}{2l} = 2^{m-2}.\]

Since $\dim(\Sigma_xM) = 2\left\lceil \frac{m}{2} \right\rceil$, we obtain $2\left\lceil \frac{m}{2} \right\rceil \geq 2^{m-2}$. It follows that $\left\lceil \frac{m}{2} \right\rceil \geq m - 2$, so $m \leq 4$, which is a contradiction and finishes the proof for $m > 4$.

Let now $m = 4$. In dimension 4 the spinor bundle splits into positive and negative spinors $\Sigma\tilde{M}|_M \cong \Sigma M \cong \Sigma^+ M \oplus \Sigma^- M$, both $\Sigma^\pm M$ have $\mathbb{C}^2$-fibers, and we have $\varphi := \psi|_M = \varphi^+ + \varphi^-$ with $\varphi_{\pm} \in \Gamma(\Sigma^\pm M)$. Moreover, $e_i : \Gamma(\Sigma^\pm M) \to \Gamma(\Sigma^\mp M)$ and $\overline{\varphi} = -e_1 \cdot_M e_2 \cdot_M e_3 \cdot_M Z \cdot_M \overline{\varphi}$.

Using Equation (16) we have

\[0 = (dH \wedge \omega_2)(Z, e_2, e_3) = dH(Z)\omega_2(e_2, e_3) = dH(Z)e_2 \cdot_M e_3 \cdot_M \varphi, \overline{\varphi} \]

\[= (Z \cdot_M e_2 \cdot_M e_3 \cdot_M \varphi, dH \cdot_M \varphi) = -\langle e_1 \cdot_M e_2 \cdot_M e_3 \cdot_M Z \cdot_M \varphi, dH \cdot_M \varphi \rangle = \langle \overline{\varphi}, dH \cdot_M e_1 \cdot_M \varphi \rangle\]

and analogously $(dH \cdot_M e_i \cdot_M \varphi, \overline{\varphi}) = 0$ for $i = 1, 2, 3$.

Let $\xi$ be as defined in Lemma 4.1. Then (14) implies that $\xi$ is in the span of $\{e_1, e_2, e_3\}$. Taking the Clifford multiplication with $dH \cdot_M$ in the Ricci identity

\[\text{Ric}(dH \cdot_M e_1 \cdot_M \varphi, \overline{\varphi}) = 0.\]
Hence, Equation (10) for $X = \xi$ and then the imaginary part of the scalar multiplication with $\overline{\varphi}$, we obtain
\[ 0 = \langle \xi, \Omega^M, Z \rangle \langle \varphi, \overline{\varphi} \rangle. \]
Together with (13) this implies $|dH|_U^2 \langle \varphi, \overline{\varphi} \rangle = 0$. Since $\varphi \neq 0$ for a real Killing spinor and $dH|_U \neq 0$ by assumption, we obtain $\langle \varphi, \overline{\varphi} \rangle = 0$. Let $X \in \Gamma(TM)$ with $|X| = 1$. Using $\nu \cdot \Gamma(\Sigma^\pm M) \to \Gamma(\Sigma^\pm M)$, see [18, p. 31], we calculate
\[ \nabla_X \overline{\varphi} = -\alpha X \cdot \overline{\varphi}. \]
Differentiating $\langle \varphi, \overline{\varphi} \rangle = 0$ and using Equation (5) we then obtain
\[ \alpha(X \cdot \varphi, \overline{\varphi}) - \frac{1}{2}H(X \cdot_M \varphi, \overline{\varphi}) + \frac{i}{2}H(\varphi, X \cdot_M \overline{\varphi}) - \alpha(\varphi, X \cdot \overline{\varphi}) = 0. \]
Hence, we have
\[ 2\alpha(X \cdot \varphi, \overline{\varphi}) = H(X \cdot_M \varphi, \overline{\varphi}). \] (18)
Let also $e_4 := Z$. We calculate using $\nabla_{e_j} e_i = 0$ (and hence $\nabla_{e_j} e_i = H\delta_{ij} \nu$) that
\[ e_j \langle e_i \cdot \varphi, \overline{\varphi} \rangle = \langle \nabla_{e_j} (e_i \cdot \varphi), \overline{\varphi} \rangle + \langle e_i \cdot \varphi, \nabla_{e_j} \overline{\varphi} \rangle \]
\[ = \langle H\delta_{ij} \nu \cdot \varphi + e_i \cdot \nabla_{e_j} \varphi, \overline{\varphi} \rangle + \langle e_i \cdot \varphi, -\alpha e_j \cdot \overline{\varphi} \rangle \]
\[ = \langle H\delta_{ij} \nu \cdot \varphi, \overline{\varphi} \rangle + \langle \alpha e_i, e_j \cdot \varphi, \overline{\varphi} \rangle - \langle e_i \cdot \varphi, \alpha e_j \cdot \overline{\varphi} \rangle \]
\[ = H\delta_{ij} \langle \nu \cdot \varphi, \overline{\varphi} \rangle. \] (19)
and
\[ e_j \langle e_i \cdot_M \varphi, \overline{\varphi} \rangle = \langle e_i \cdot_M \nabla_{e_j} \varphi, \overline{\varphi} \rangle + \langle e_i \cdot_M \varphi, \nabla_{e_j} \overline{\varphi} \rangle \]
\[ = \langle \alpha e_i \cdot_M e_j \cdot \varphi - \frac{1}{2}H e_i \cdot_M e_j \cdot_M \varphi, \overline{\varphi} \rangle + \langle e_i \cdot_M \varphi, -\alpha e_j \cdot \overline{\varphi} - \frac{1}{2}H e_j \cdot_M \varphi \rangle \]
\[ = -H \langle e_i \cdot_M e_j \cdot_M \varphi, \overline{\varphi} \rangle. \] (20)
By (4) and $\overline{\varphi} = -e_{1 \cdot M} e_{2 \cdot M} e_{3 \cdot M} Z \cdot_M \varphi$, the left hand sides of both of the equations (19) and (20) are real. On the other hand the right hand side of (19) is imaginary and the one of (20) is imaginary for $i \neq j$ and 0 for $i = j$. Hence, all sides have to be zero. Using this when differentiating (18) for $X = e_i$ in direction of $Z$, we obtain
\[ Z(H)\langle e_i \cdot_M \varphi, \overline{\varphi} \rangle = 0 \]
for all $i = 1, \ldots, 4$.
Hence, $\langle e_i \cdot_M \varphi, \overline{\varphi} \rangle = 0$ and thus
\[ \Re\langle e_i \cdot_M \varphi_+, \varphi_- \rangle = 0 \]
for all $i = 1, \ldots, 4$. (21)
We note that in dimension 4 every non-zero element $\psi \in \Sigma_+ M|_y$ for $y \in U$ gives rise to a real basis $e_i \cdot M \psi$ of $\Sigma_+ M|_y$ with respect to the scalar product $\langle \cdot, \cdot \rangle := \Re\langle \cdot, \cdot \rangle$. Hence, Equation (21) implies that at each point $y \in U$, either $\varphi_+ = 0$ or $\varphi_- = 0$ is perpendicular to the four dimensional real vector space $\Sigma_+ M|_y$ w.r.t this real scalar product, i.e, $\varphi_- = 0$.
Since $|\varphi|^2 = |\varphi_+|^2 + |\varphi_-|^2$ is of constant norm by (6), we obtain that $\varphi_+ = 0$ or $\varphi_- = 0$ on all of $U$. Assume that $\varphi_- = 0$ on $U$ (the other case is analogous), then
\[ 0 = \nabla_X \varphi_- = \alpha X \cdot \varphi_+ + \frac{1}{2}H X \cdot_M \varphi_. \]
The real part of the scalar product of the last identity with $X \cdot_M \varphi_+$ gives
\[ \frac{1}{2}H |X|^2 |\varphi_+|^2 = 0. \]
Since $\varphi$ is non-zero, $\varphi_+$ has no zeros on $U$ and we get that $H = 0$ on $U$. Thus, $dH = 0$ on $U$ which gives the contradiction. \[ \square \]
**Proof of Theorem 1.2 for Case 2.** Assume that \( dH \) is not identically zero. Then, there is a point \( x \in \tilde{M} \) and a neighborhood \( U \) of \( x \) where \( \nabla_{g} H \) is nonzero. Hence, we find a local orthonormal frame \( (e_1, \ldots, e_{m-1}, Z) = \frac{\nabla_{g} H}{|\nabla_{g} H|^{2}} \) of \( TU \). Then, we have with \( (e_1, \ldots, e_{m-1}, Z, \nu) \) again a local orthonormal frame of \( \tilde{M} \) on \( U \).

On all of \( U \) we have by Equation (17) that \( dH \wedge \eta_{1} = 0 \). Then with \( dH(e_i) = 0 \) we obtain

\[
0 = dH \wedge \eta_{1} \left( \frac{\nabla_{g} H}{|\nabla_{g} H|^{2}}, e_{i} \right) = \eta_{1}(e_{i}) = -(e_{i}, \varphi, \varphi)
\]

for all \( 1 \leq i \leq m-1 \) which will used in following without any further comment.

We consider three different subcases: First assume that \( \langle dH \cdot \varphi, \varphi \rangle = \langle \nu \cdot \varphi, \varphi \rangle = 0 \) on \( U \). Note that for all \( X \in \Gamma(TM) \) the vector \( V \), defined on \( \tilde{M} \) by \( \tilde{g}(V,X) := i(X\varphi, \varphi) \), vanishes on \( U \), see [40, 3, 4, 5, 19]. From [3, 40] we have \( \nabla_{X} V = 2\alpha|\varphi|^{2}X \) for all \( X \in \Gamma(T\tilde{M}) \). Since \( V \equiv 0 \), this implies that \( \varphi \equiv 0 \) on \( U \). This gives a contradiction in the first case.

Second let \( \langle dH \cdot \varphi, \varphi \rangle = 0 \) and let \( \langle \nu \cdot \varphi, \varphi \rangle \) be nonzero on a possibly smaller \( U \). In particular, we can make \( U \) small enough such that \( \varphi \) has no zeros on \( U \). Then, the imaginary part of the scalar product of Equation (8) with \( \nu \cdot \varphi \) gives

\[
\text{Ric}(\nu, \nu) = 4m\alpha^{2}.
\]

Reinserting into Equation (8) gives

\[
m - \frac{1}{2}dH \cdot_{M} \varphi = \frac{1}{2} i(\nu, \tilde{\Omega}) \cdot_{M} \varphi,
\]

and hence

\[
m - \frac{1}{2} \langle dH \cdot_{M} \varphi, Z \cdot_{M} \varphi \rangle = \frac{1}{2} \left( \sum_{i=1}^{m-1} \tilde{\Omega}(\nu, e_{i}) \langle e_{i} \cdot_{M} \varphi, Z \cdot_{M} \varphi \rangle + \tilde{\Omega}(\nu, Z)|Z \cdot_{M} \varphi|^{2} \right).
\]

Taking the imaginary part of the last equality implies \( \tilde{\Omega}(\nu, Z)|\varphi|^{2} = 0 \). Since \( \varphi \) has no zeros, we obtain \( \tilde{\Omega}(Z, \nu) = 0 \). The real part of the scalar product of Equation (9) with \( e_{i} \cdot \varphi \) then gives

\[
m - \frac{1}{2} \langle dH \cdot \nu \cdot \varphi, e_{i} \cdot \varphi \rangle
\]

\[
= - \frac{1}{2} \text{Im} \left( \sum_{j<k} \Omega^{M}(e_{j}, e_{k}) \langle e_{j} \cdot e_{k} \cdot \varphi, e_{i} \cdot \varphi \rangle + \sum_{j} \Omega^{M}(e_{j}, Z) \langle e_{j} \cdot Z \cdot \varphi, e_{i} \cdot \varphi \rangle \right) = 0,
\]

where the last equality uses Equation (4) and \( \langle Z \cdot \varphi, \varphi \rangle = \langle e_{i} \cdot \varphi, \varphi \rangle = 0 \). Thus, taking the scalar product of Equation (22) with \( e_{i} \cdot \varphi \) implies

\[
0 = \sum_{j=1}^{m-1} \tilde{\Omega}(\nu, e_{j}) \langle e_{j} \cdot_{M} \varphi, e_{i} \cdot \varphi \rangle.
\]

By taking the imaginary part we obtain \( \tilde{\Omega}(\nu, e_{i}) = 0 \) and hence \( \nu \cdot \tilde{\Omega} = 0 \). Reinserting in Equation (22) implies \( dH = 0 \) which gives the contradiction in the second case.
The third case covers the remaining possibility that \( \langle dH \cdot \nu, \varphi \rangle \) is nonzero at a point in \( U \). The next calculations will be carried out at this point. Taking the real part of the scalar product of Equation (9) with \( e_i \cdot \varphi \) gives
\[
\frac{m-1}{2} \langle dH \cdot \nu, e_i \cdot \varphi \rangle = \frac{1}{2} \Omega^M (e_i, Z) \langle Z, \varphi \rangle.
\]
(23)

On the other hand taking the scalar product of the Ricci identity (10) for \( X = e_i \) with \( \nu \cdot \varphi \) gives
\[
-\frac{1}{2} \text{Ric} (e_i, Z) \langle Z \cdot \varphi, \varphi \rangle + \frac{1}{2} \Omega^M (e_i, Z) \langle Z, \varphi, \varphi \rangle = -\frac{1}{2} \langle dH \cdot \nu, e_i \cdot \varphi \rangle.
\]

The imaginary part of the last identity gives Ric \( \langle e_i, Z \rangle = 0 \). Reinserting this into the above equation and using Equation (23) implies \( \langle dH \cdot \nu, e_i \cdot \varphi \rangle = \Omega^M (e_i, Z) = 0 \).

Taking again the scalar product of the Ricci identity (10) but this time for \( X = Z \) then gives
\[
\frac{1}{2} \text{Ric} (Z, Z) Z \cdot_M \varphi = \frac{1}{2} \left[ dH |\varphi|^2 + \left( \frac{1}{2} (m-1) H^2 - 2(m-1) \alpha^2 \right) \right] Z \cdot_M \varphi.
\]

The real part of the scalar product with of the last identity with \( \varphi \) gives that \( \frac{1}{2} \left[ dH |\varphi|^2 = 0 \). But \( \varphi \) has no zeros on \( U \) and \( m > 1 \), so \( dH = 0 \) which gives the desired contradiction for the remaining case.

The following example shows that the dimension constraint for Case 1 in Theorem 1.2 is necessary.

**Example 5.1.** There exist totally umbilical connected hypersurfaces with non-constant mean curvature in Riemannian Spin\(^c\) manifolds of dimension 3 or 4 and carrying parallel or real Killing spinors:

*Dimension 3:* The product of the canonical Spin\(^c\) structure on \( \mathbb{S}^2 \) with the Spin structure on \( \mathbb{R} \) defines a Spin\(^c\) structure on the manifold \( \tilde{M} = \mathbb{S}^2 \times \mathbb{R} \) [35]. This Spin\(^c\) structure carries a parallel spinor [35]. Totally umbilical hypersurfaces (which are not totally geodesic) of \( \mathbb{S}^2 \times \mathbb{R} \) have been classified in [42]. Moreover, they are not of constant mean curvature [42, Remark 10]. We point out that \( \tilde{M} = \mathbb{S}^2 \times \mathbb{R} \) is Spin but does not carry a real or parallel Killing Spin spinor.

*Dimension 4:* The Spin\(^c\) manifold \( \tilde{M} = \mathbb{S}^2 \times \mathbb{H}^2 \) carries a parallel spinor for the product of the canonical Spin\(^c\) structure on \( \mathbb{S}^2 \) with the canonical Spin\(^c\) structure on \( \mathbb{H}^2 \). In [29], the author classified totally umbilical hypersurfaces of \( \mathbb{S}^2 \times \mathbb{H}^2 \) (see [29, Theorem 4.5.3]) and showed that these hypersurfaces are not of constant mean curvature in general. We also point out that \( \tilde{M} = \mathbb{S}^2 \times \mathbb{H}^2 \) is Spin but does not carry a real or parallel Killing Spin spinor.

### 6. Extrinsic Hyperspheres in Riemannian Spin Manifolds

In this section, we give some additional information if the ambient manifold carrying a Killing spinor is already spin. As a first corollary, we get:

**Corollary 6.1.** Let \( M^m \hookrightarrow \tilde{M}^{m+1} \) be a totally umbilical isometric immersion. Assume that \( \tilde{M} \) is a Spin manifold with a Killing spinor \( \psi \) of Killing constant \( \alpha \) (could be zero, real or purely imaginary). Then, \( M \) is Einstein with scalar curvature \( \text{scal} = m(m-1)(H^2 + 4\alpha^2) \).
Proof of Corollary 6.1. From Thm 1.2, $H$ is constant. By the Ricci identity (10), we have
\[
\frac{1}{2} \text{Ric}(e_j),_M \phi = 2(m - 1) \left[ \alpha^2 + \frac{H^2}{4} \right] e_j, M \phi.
\]
This means that $M$ is Einstein with constant scalar curvature $\text{scal} = m(m - 1)(H^2 + 4\alpha^2)$. \hfill \Box

For later use, we recall here Koiso’s Theorem.

**Theorem 6.2.** [30, Thm. B] Let $M$ be a totally umbilical Einstein hypersurface in a complete Einstein manifold $(\tilde{M}, \tilde{g})$. Then the only possible cases are:

1. $g$ has positive Ricci curvature. Then $g$ and $\tilde{g}$ have constant sectional curvature.
2. $\tilde{g}$ has negative Ricci curvature. If $\tilde{M}$ is compact or homogeneous, then $g$ and $\tilde{g}$ have constant sectional curvature.
3. $g$ and $\tilde{g}$ have zero Ricci curvature. If $\tilde{M}$ is simply connected, then $\tilde{M} = (M, g) \times \mathbb{R}$ where $M$ is totally geodesic hypersurface in $\tilde{M}$ which contains $\tilde{M}$.

An important special case of totally umbilical hypersurfaces with constant mean curvature are totally geodesic hypersurfaces (when the mean curvature $H$ is zero). The other cases are called extrinsic hypersphere (when the mean curvature is a nonzero constant).

From Theorem 6.2 and Corollary 6.1, we deduce the following result:

**Theorem 6.3.** Let $\tilde{M}$ be a complete Riemannian Spin manifolds of non-constant sectional curvature that carry an $\alpha$-Killing spinor. If $\alpha \in i \mathbb{R} \setminus \{0\}$, we assume moreover that $\tilde{M}$ is homogeneous. Then, there are no extrinsic hyperspheres in $\tilde{M}$.

**Proof.** Assume that $M$ is an extrinsic hypersphere ($H \neq 0$) in a Riemannian Spin manifold with an $\alpha$-Killing spinor. By Corollary 6.1, $M$ is Einstein with scalar curvature $m(m - 1)(H^2 + 4\alpha^2)$. If $H^2 + 4\alpha^2 > 0$, the Ricci curvature of $M$ is positive. If $H^2 + 4\alpha^2 \leq 0$, then $\alpha \in i \mathbb{R} \setminus \{0\}$ and hence the Ricci curvature of $\tilde{M}$ is negative and hence, in both cases, we have by Koiso’s theorem 6.2 that $\tilde{g}$ is of constant sectional curvature, which is a contradiction. \hfill \Box

Theorem 1.4 is a particular case of Theorem 6.3 as is easily seen as follows: All the manifolds appearing in this Theorem are Spin, complete, with $\alpha$-Killing spinor and of non-constant sectional curvature (see [15] and [13, Prop 3.1]).

One can add further examples for Theorem 6.3, such as 6-dimensional nearly Kähler manifolds which are not Kähler and of non-constant sectional curvature and 7-dimensional weak $G_2$ manifolds of non-constant sectional curvature.

The completeness assumptions in Theorem 6.3 is necessary not only because we want to use Koiso’s theorem but also because otherwise, every manifold is an extrinsic hypersphere in its (non-complete) metric cone.
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