Higher matching complexes of complete graphs and complete bipartite graphs

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Abstract

For \( r \geq 1 \), the \( r \)-matching complex of a graph \( G \), denoted \( M_r(G) \), is a simplicial complex whose faces are the subsets \( H \subseteq E(G) \) of the edge set of \( G \) such that the degree of any vertex in the induced subgraph \( G[H] \) is at most \( r \). In this article, we give a closed form formula for the homotopy type of the \( (n-2) \)-matching complex of complete graph on \( n \) vertices. We also prove that the \( (n-1) \)-matching complex of complete bipartite graph \( K_{n,n} \) is homotopy equivalent to a sphere of dimension \( (n-1)^2 - 1 \).

Keywords: matching complex, higher matching complex, bounded degree complex, discrete Morse theory.

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1 Introduction

A matching in a graph \( G \) is a subset \( H \subseteq E(G) \) of the edge set of \( G \) such that the degree of any vertex in the induced subgraph \( G[H] \) is at most 1. The matching complex of \( G \), denoted \( M_1(G) \), is a simplicial complex whose vertices are the edges of \( G \) and faces are all the matchings in \( G \). Matching complexes have emerged from developments in many fields, and have shown to be combinatorial objects with rich and subtle topological structure. In particular, these complexes play a fundamental role in the study of Quillen order complexes associated to posets of (elementary abelian) primary subgroups of a given finite group. Subsequently, and starting from the work of Bouc [2], the study of matching complexes has become a very active and fruitful area of research on its own. Likewise, Chessboard complexes, i.e., matching complexes of complete bipartite graphs, arise in Garst’s work [5] as coset complexes associated to symmetric groups, and also play a role in Vrećica-Zivaljević’s analysis of halving hyperplanes (see [19]). For more on these complexes, interested reader is referred to Wach’s survey paper [20, Section 2].

Reiner and Roberts [12] generalized the concept of matching complexes by defining bounded degree complexes. Let \( G \) be a graph and \( V(G) = \{1, 2, \ldots, n\} = [n] \) be the vertex set of \( G \). For \( \vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \) a sequence of non-negative integers, the bounded degree complex, denoted \( BD^{\vec{\lambda}}(G) \), is a simplicial complex whose simplicies are all subgraphs \( H \subseteq E(G) \) such that the degree of vertex \( i \) in \( H \) is at most \( \lambda_i \) for each \( i \in [n] \). Jonsson [9] further studied these complexes and derived a lower bound for the connectivity of \( BD^{\vec{\lambda}}(G) \). When \( \lambda_i = r \) for all \( i \in [n] \), the bounded degree complex \( BD^{(r, \ldots, r)}(G) \) is called the \( r \)-matching complex of \( G \) and is denoted by \( M_r(G) \) (see Figure 1 for example).

There are only a few classes of graphs for which the exact homotopy type of higher matching complexes are known. For example, for trees [14, 15], wheel graphs [18], and cycle graphs [10, 14], \( M_r(G) \) has the homotopy type of wedge of spheres for each \( r \geq 1 \). However, for complete graphs,
the homotopy type of these complexes are a bit more mysterious. For example, Shareshian and Wachs [13] showed that the integral homology groups of 1-matching complexes of complete graphs are not torsion-free in many cases. For 2 ≤ r < n, Jonsson [9, Theorem 12.8] computed a lower bound for the connectivity degree (see Definition 2) of $M_r(K_n)$ and observed that the integral homology groups of higher matching complexes of complete graphs can also have torsion (see [4 Table 12.2]). Due to this, in general, it is extremely difficult to determine the explicit homotopy type of higher matching complexes of complete graphs.

In this paper, we determine the homotopy type of $M_{n-2}(K_n)$. In particular, we prove the following.

**Theorem 1.1.** For $n \geq 3$, the complex $M_{n-2}(K_n)$ is homotopy equivalent to a wedge of spheres. More precisely,

$$M_{n-2}(K_n) \simeq \bigvee_{n-1} S^t,$$

where $t = \binom{n-1}{2} - 1$.

Another class of graphs whose matching complexes have been studied extensively is complete bipartite graphs $K_{m,n}$. In [21], Ziegler showed that $M_1(K_{m,n})$ is shellable whenever $n \geq 2m - 1$. Shareshian and Wachs [13] studied the integral homology groups of $M_1(K_{m,n})$ and showed that they are not torsion-free in many cases. In [12 Theorem 3.3], Reiner and Roberts computed the rational homology groups of higher matching complexes of complete bipartite graphs, also known as chessboard complexes with multiplicities. Combining [12 Proposition 2.4], [12 Proposition 3.2] and [3 Theorem 3.9] one gets that the complex $M_{n-1}(K_{m,n})$ is stably homotopy equivalent (see [17 Section 7.1]) to a sphere of dimension $(n-1)^2 - 1$. Here, we strengthen this result by showing that these complexes are homotopy equivalent. More precisely, we have the following result.

**Theorem 1.2.** For $n \geq 2$, the $(n-1)$-matching complex of complete bipartite graph $K_{n,n}$ is homotopy equivalent to $(n-1)^2 - 1$-dimensional sphere.

This article is organized as follows: In Section 2, we present some definitions and tools which are crucial for this article. Section 3 is dedicated towards the study of $(n-2)$-matching complex of $K_n$. In Section 4, we prove Theorem 1.2. Finally, in Section 5, we outline some open problems.

## 2 Preliminaries

A graph is an ordered pair $G = (V, E)$ where $V$ is called the set of vertices and $E$ is the set of (not necessarily all) cardinality-2 subsets of $V$, called the set of edges of $G$. The vertices $v_1, v_2 \in V$ are said to be adjacent, if $\{v_1, v_2\} \in E$. The number of vertices adjacent to a vertex $v$ in $G$ is called degree of $v$ in $G$, denoted $\deg_G(v)$. A vertex $v$ is said to be adjacent to an edge $e$, if $v$ is an end point of $e$, i.e., $e = (v, w)$.

A graph $H$ with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is called a subgraph of the graph $G$. For a nonempty subset $H$ of $E(G)$, the induced subgraph $G[H]$, is the subgraph of $G$ with $V(G[H]) = V(G)$ and edges $E(G[H]) = H$.

For $n \geq 1$, the complete graph, denoted $K_n$, is a graph with vertex set $V(K_n) = \{1, \ldots, n\}$ and edge set $E(K_n) = \{(i, j) : 1 \leq i < j \leq n\}$. For $m, n \geq 1$, the complete bipartite graph, denoted $K_{m,n}$, is a graph with vertex set $V(K_{m,n}) = \{a_1, \ldots, a_m\} \sqcup \{b_1, \ldots, b_n\}$ and edge set $E(K_{m,n}) = \{(a_i, b_j) : 1 \leq i \leq m, 1 \leq j \leq n\}$.
**Definition 1.** An *(abstract) simplicial complex* \( \mathcal{K} \) on a finite set \( X \) is a collection of subsets of \( X \) such that

(i) \( \emptyset \in \mathcal{K} \), and

(ii) if \( \sigma \in \mathcal{K} \) and \( \tau \subseteq \sigma \), then \( \tau \in \mathcal{K} \).  

The elements of \( \mathcal{K} \) are called *simplices* of \( \mathcal{K} \). Inclusion-wise maximal simplices of \( \mathcal{K} \) are called *facets* of \( \mathcal{K} \). If \( \sigma \in \mathcal{K} \) and \( |\sigma| = k + 1 \), then \( \sigma \) is said to be *\( k \)-dimensional*, denoted as \( \dim(\sigma) = k \) (here, \( |\sigma| \) denotes the cardinality of \( \sigma \) as a set). A complex is called *pure* if all facets are of same dimension. Further, if \( \sigma \in \mathcal{K} \) and \( \tau \subseteq \sigma \) then \( \tau \) is called a *face* of \( \sigma \) and if \( \tau \neq \sigma \) then \( \tau \) is called a *proper face* of \( \sigma \). The set of 0-dimensional simplices of \( \mathcal{K} \) is denoted by \( V(\mathcal{K}) \), and its elements are called *vertices* of \( \mathcal{K} \). A *subcomplex* of a simplicial complex \( \mathcal{K} \) is a simplicial complex whose simplices are contained in \( \mathcal{K} \). For \( s \geq 0 \), the *\( k \)-skeleton* of a simplicial complex \( \mathcal{K} \), denoted \( \mathcal{K}^{(s)} \), is the collection of all those simplices of \( \mathcal{K} \) whose dimension is at most \( s \). In this article, we do not distinguish between an abstract simplicial complex and its geometric realization. Therefore, a simplicial complex will be considered as a topological space, whenever needed.

For \( j \geq 0 \), simplicial complex \( \mathcal{K} \) is called *\( j \)-connected* if, for all \( d \in \{0, \ldots, j\} \), every continuous map \( f : S^d \to \mathcal{K} \) has a continuous extension \( g : B^{d+1} \to \mathcal{K} \). Here, \( S^d \) and \( B^d \) denote the \( d \)-dimensional sphere and closed ball respectively. By convention, \( \mathcal{K} \) is \((-1)\)-connected if it is nonempty.

**Definition 2.** The *connectivity degree* of a simplicial complex \( \mathcal{K} \) is the largest integer \( j \) such that \( \mathcal{K} \) is \( j \)-connected (\(+\infty\) if \( \mathcal{K} \) is \( j \)-connected for all \( j \)). The *shifted connectivity degree* of \( \mathcal{K} \) is obtained by adding one to the connectivity degree.

**Definition 3.** For \( k \geq 1 \), a *\( k \)-matching* of a graph \( G \) is a subset of edges \( H \subseteq E(G) \) such that any vertex \( v \in G[H] \) has degree at most \( k \). The *\( k \)-matching complex* of a graph \( G \), denoted \( M_k(G) \), is a simplicial complex whose vertices are the edges of \( G \) and faces are given by \( k \)-matchings of \( G \).

**Example:** Figure 1 consists of graph \( G \) and \( M_2(G) \). The complex \( M_2(G) \) consists of 3 maximal simplices, namely \( \{e_1, e_2, e_4\}, \{e_1, e_3, e_4\} \) and \( \{e_2, e_3, e_4\} \).

![Figure 1: 2-matching complex of a graph G](image-url)
2.1 Tools from discrete Morse theory

We now discuss some tools needed from discrete Morse theory. The classical reference for this is [4]. However, here we closely follow [10] for notations and definitions.

**Definition 4 ([10, Definition 11.1]).** A partial matching on a poset \((P, <)\) is a subset \(M \subseteq P \times P\) such that

(i) \((a, b) \in M\) implies \(a < b\); i.e., \(a < b\) and no \(c\) satisfies \(a < c < b\), and

(ii) each \(a \in P\) belong to at most one element in \(M\).

Note that, \(M\) is a partial matching on a poset \(P\) if and only if there exists \(A \subset P\) and an injective map \(\mu : A \to P \setminus A\) such that \(\mu(a) > a\) for all \(a \in A\).

An acyclic matching is a partial matching \(M\) on the poset \(P\) such that there does not exist a cycle

\[\mu(a_1) > a_1 < \mu(a_2) > a_2 < \mu(a_3) > a_3 \ldots \mu(a_t) > a_t < \mu(a_1), t \geq 2.\]

For an acyclic partial matching on \(P\), those elements of \(P\) which do not belong to the matching are called critical.

The main result of discrete Morse theory is the following.

**Theorem 2.1 ([10, Theorem 11.13]).** Let \(K\) be a simplicial complex and \(M\) be an acyclic matching on the face poset of \(K\). Let \(c_i\) denote the number of critical \(i\)-dimensional cells of \(K\) with respect to the matching \(M\). Then \(K\) is homotopy equivalent to a cell complex \(K_c\) with \(c_i\) cells of dimension \(i\) for each \(i \geq 0\), plus a single 0-dimensional cell in the case where the empty set is also paired in the matching.

The following can be inferred from Theorem 2.1

**Corollary 2.2.** If an acyclic matching on the face poset of \(K\) has critical cells only in a fixed dimension \(i\), then \(K\) is homotopy equivalent to a wedge of \(i\)-dimensional spheres.

In this article, by matching on a simplicial complex \(K\), we will mean that the matching is on the face poset of \(K\).

2.2 Morse matching induced by a sequence of vertices

Let \(K\) be a simplicial complex and \(N_x = \{\sigma \in K : \sigma \setminus \{x\}, \sigma \cup \{x\} \in K\}\) be a subcomplex of \(K\), where \(x \in V(K)\). Define a matching on \(K\) using \(x\) as follows:

\[M_x = \{\sigma \setminus \{x\}, \sigma \cup \{x\} : \sigma \setminus \{x\}, \sigma \cup \{x\} \in K\}.\]

Note that the condition \(\sigma \setminus \{x\} \in K\), for \(N_x\) and \(M_x\) above, is superfluous since \(K\) is a simplicial complex. However, this is not the case when we define an element matching on a subset of a simplicial complex (for instance, see Proposition 2.3).

**Definition 5.** Matching \(M_x\), as defined above, is called an element matching on \(K\) using vertex \(x\).

The following result tells us that an element matching is always acyclic.
Lemma 2.3 ([11] Lemma 3.2). The matching $M_x$ is an acyclic matching on $K$ and perfect acyclic matching on $N_x$.

To obtain an acyclic matching on a simplicial complex $K$, the next result tells us that one can define a sequence of element matchings on $K$ using its vertices.

Proposition 2.4 ([7] Proposition 3.1). Let $K_1$ be a simplicial complex and $x_1, x_2, \ldots, x_n$ be vertices of $K_1$. Then, $\bigcup_{i=1}^{n} M_{x_i}$ is an acyclic matching on $K_1$, where $M_{x_i} = \{(\sigma \setminus \{x_i\}, \sigma \cup \{x_i\}) : \sigma \setminus \{x_i\}, \sigma \cup \{x_i\} \in K_i\}$ and $K_{i+1} = K_i \setminus \{\sigma : (\sigma \setminus \{x_i\}, \sigma \cup \{x_i\}) \in M_{x_i}\}$ for $i \in \{1, \ldots, n\}$.

Note that the above result is a particular case of a more general result called Cluster Lemma, which has been re-discovered by many authors over and over (see for instance, [9] Lemma 4.2 or [8] Lemma 4.1). Proposition 2.4 will be used repeatedly in this article.

3 Proof of Theorem 1.1

The set of vertices of $K_n$ will be denoted by $[n]$ and the notation $G + \{i, j\}$ will mean that add the edge $\{i, j\}$ to $G$ if it is not already there. Similarly, $G - \{i, j\}$ will mean that delete the edge $\{i, j\}$ from $G$ if it is available.

The aim of this section is to determine the homotopy type of $M_{n-2}(K_n)$ for $n \geq 3$. We do so by defining a sequence of element matchings on $M_{n-2}(K_n)$ in $n-1$ steps.

Step 1: Let $A_{1,1} = M_{n-2}(K_n)$. We first use the vertex $\{1, 2\}$ of $A_{1,1}$ for element matching. Define,

\[
M_{1,2} = \{(G, G + \{1, 2\}) : \{1, 2\} \notin E(G) \text{ and } G, G + \{1, 2\} \in A_{1,1}\},
\]

\[
N_{1,2} = \{G \in A_{1,1} : (G - \{1, 2\}, G + \{1, 2\}) \in M_{1,2}\}, \text{ and}
\]

\[
A_{1,2} = A_{1,1} \setminus N_{1,2}.
\]

Using Lemma 2.3 we get that $M_{1,2}$ is an acyclic matching on $A_{1,1}$ with $A_{1,2}$ as the set of the critical cells.

Claim 1. $A_{1,2} = \{G \in A_{1,1} : \{1, 2\} \notin E(G), \deg_G(2) < n - 2, \deg_G(1) = n - 2\} \cup \{G \in A_{1,1} : \{1, 2\} \notin E(G), \deg_G(2) = n - 2\}$.

Proof of Claim 1. For simplicity of notations, let

\[
B_{1,2} = \{G \in A_{1,1} : \{1, 2\} \notin E(G), \deg_G(2) < n - 2, \deg_G(1) = n - 2\}, \text{ and}
\]

\[
C_{1,2} = \{G \in A_{1,1} : \{1, 2\} \notin E(G), \deg_G(2) = n - 2\}. \tag{3.1}
\]

Clearly, $B_{1,2}$ are $C_{1,2}$ are disjoint sets. Further, if $G \in B_{1,2} \cup C_{1,2}$, then $\{1, 2\} \notin E(G)$ and $\deg_G(1) = n - 2$ or $\deg_G(2) = n - 2$. This implies that $G + \{1, 2\} \notin A_{1,1}$, and hence $G \notin N_{1,2}$. Therefore, $B_{1,2} \cup C_{1,2} \subseteq A_{1,2}$.

Now consider $G \in A_{1,2}$. If $\{1, 2\} \notin E(G)$, then clearly $(G - \{1, 2\}, G) \in M_{1,2}$ which is a contradiction. Further, if $\{1, 2\} \notin E(G)$, $\deg_G(1) < n - 2$ and $\deg_G(2) < n - 2$ then also $G + \{1, 2\} \in N_{1,2}$. Therefore, if $G \in A_{1,2}$ then $\{1, 2\} \notin E(G)$ and, either $\deg_G(2) = n - 2$ or $\deg_G(1) = n - 2$ whenever $\deg_G(2) < n - 2$. \qed

\]

5
We now extend the matching $M_{1,2}$ on $A_{1,1}$ by defining a sequence of element matchings on $A_{1,2}$ using vertices $\{1, i\}$ for each $i \in \{3, \ldots, n\}$. For $i \in \{3, \ldots, n\}$, define

$$M_{1,i} = \{ (G, G + \{1, i\}) : \{1, i\} \notin E(G) \text{ and } G, G + \{1, i\} \in A_{1,i-1}\},$$

$$N_{1,i} = \{ G \in A_{1,i-1} : (G - \{1, i\}, G + \{1, i\}) \in M_{1,i}\}, \text{ and}$$

$$A_{1,i} = A_{1,i-1} \setminus N_{1,i}.$$  

Using Proposition 2.4 we get that $\bigcup_{2 \leq i \leq n} M_{1,i}$ is an acyclic matching on $A_{1,1}$ with $A_{1,1}$ as the set of the critical cells. We now analyze cells of $A_{1,2}$.

**Claim 2.** $A_{1,2} = B_{1,2} \cup \{ G \in A_{1,1} : \{1, i\} \notin E(G) \text{ and } \deg_G(i) = n - 2 \text{ for each } i \in \{2, \ldots, n\}\}.$

**Proof of Claim.** Define

$$C_1 = \{ G \in A_{1,1} : \{1, i\} \notin E(G) \text{ and } \deg_G(i) = n - 2 \text{ for each } i \in \{2, \ldots, n\}\}. \tag{3.2}$$

Clearly, if $G \in C_1$ then, for each $i \in \{3, \ldots, n\}$, $\deg_{G + \{1, i\}}(i) = n - 1$ which implies that $G + \{1, i\} \notin A_{1,2}$. This gives us that $G \notin N_{1,i}$ for any $i \in \{3, \ldots, n\}$, thereby showing that $G \in A_{1,2}$. If $G \in B_{1,2}$ then $\deg_G(2) < n - 2$ and $\deg_G(1) = n - 3$ for each $i \in \{3, \ldots, n\}$. This gives us that $G - \{1, i\} \notin A_{1,2}$ for any $i \in \{3, \ldots, n\}$. Therefore, $B_{1,2} \cup C_1 \subseteq A_{1,2}$.

Now consider $G \in A_{1,2,1} \subseteq A_{1,2}$. If $\deg_G(2) = n - 2$ then clearly $G \in B_{1,2}$. Let $\deg_G(2) = n - 2$ and $G \notin C_1$, i.e. $G \in A_{1,2} \setminus C_1$. Then, either $\{1, i\} \in E(G)$ for some $i \in \{2, \ldots, n\}$ or $\deg_G(j) < n - 2$ for some $j \in \{2, \ldots, n\}$. Let $i_0 = \min\{i : \{1, i\} \in E(G) \text{ or } \deg_G(i) < n - 2\}$. Since $G \in A_{1,2}$, $i_0 \geq 2$. In this case, it is easy to see that $G \in N_{1,i_0}$ which contradicts the assumption that $G \in A_{1,2}$. \qed

Our idea to define matchings in each coming steps will be similar to this step. In step $i$, we are going to use vertices $\{i, j\}$ for $j \in \{i + 1, \ldots, n\}$. To make our writing easier in the next step, we observe the following.

**Proposition 3.1.** Let $B_{1,2}$ and $C_1$ be the sets as defined in Equation (3.1) and Equation (3.2) respectively. Then,

1. $|C_1| = 1$ and if $G \in C_1$ then $|E(G)| = \binom{n-1}{2}$.

2. If $G \in C_1$ then $G - \{i, j\}, G + \{i, j\} \notin B_{1,2}$ for any $2 \leq i < j \leq n$.

**Proof.** If $G \in C_1$ then vertex 1 is isolated and any other two vertices are joined by an edge, i.e. $G$ is disjoint union of an isolated vertex $\{1\}$ and a complete graph on vertex set $\{2, \ldots, n\}$. This proves the first part.

Now consider $G \in C_1$. Since $\{i, j\} \in G$ for each $2 \leq i < j \leq n$, $G + \{i, j\} = G \notin B_{1,2}$. Further, $\deg_{G - \{i, j\}}(1) = 0$ for all $2 \leq i < j \leq n$, which implies that $G - \{i, j\} \notin B_{1,2}$ for any $2 \leq i < j \leq n$. \qed

From Proposition 3.1 it is clear that $C_1$ is not going to play any role in any element matching using vertices $\{i, j\}$ where $2 \leq i < j \leq n$. Therefore, for the time being it is enough to proceed with set $B_{1,2}$.

**Step 2:** Let $A_{2,2} = B_{1,2} = A_{1,2} \setminus C_1$. We now define a sequence of element matchings on $A_{2,2}$
using vertices \( \{2, i\} \) for each \( i \in \{3, \ldots, n\} \). For \( i \in \{3, \ldots, n\} \), define

\[
M_{2,i} = \{(G, G + \{2, i\}) : \{2, i\} \notin E(G) \text{ and } G, G + \{2, i\} \in A_{2,i-1}\},
\]

\[
N_{2,i} = \{G \in A_{2,i-1} : (G - \{2, i\}, G + \{2, i\}) \in M_{2,i}\}, \text{ and}
\]

\[
A_{2,i} = A_{2,i-1} \setminus N_{2,i}.
\] (3.3)

**Proposition 3.2.** For \( i \in \{2, \ldots, n\} \), let \( A_{2,i} \) be as defined above.

1. \( A_{2,3} = B_{2,3} \sqcup C_{2,3} \), where

\[
B_{2,3} = \{G \in A_{2,2} : \{2, 3\} \notin E(G), \text{ deg}_{G}(3) < n - 2, \text{ deg}_{G}(2) = n - 3\}, \text{ and}
\]

\[
C_{2,3} = \{G \in A_{2,2} : \{2, 3\} \notin E(G), \text{ deg}_{G}(3) = n - 2\}.
\] (3.4)

2. \( A_{2,n} = B_{2,3} \sqcup C_{2}, \) where

\[
C = \{G \in A_{2,2} : \{2, i\} \notin E(G) \text{ and } \text{deg}_{E}(i) = n - 2 \text{ for each } i \in \{3, \ldots, n\}\}.
\] (3.5)

3. \(|C| = 1 \text{ and if } G \in C \text{ then } |E(G)| = \binom{n-2}{2} \).

4. \( B_{2,3} = \emptyset \text{ if and only if } n = 3 \).

5. If \( G \in C \) and \( B_{2,3} \neq \emptyset \), then \( G - \{i, j\}, G + \{i, j\} \notin B_{2,3} \) for any \( 3 \leq i < j \leq n \).

**Proof.** 1. Clearly, \( B_{2,3} \) are \( C_{2,3} \) are disjoint sets. Further, if \( G \in B_{2,3} \sqcup C_{2,3} \), then \( \{2, 3\} \notin E(G) \) and \( \text{deg}_{G}(2) = n - 3 \) or \( \text{deg}_{G}(3) = n - 2 \). This implies that \( G + \{2, 3\} \notin A_{2,2} \), and hence \( G \notin N_{2,3} \). Therefore, \( B_{2,3} \sqcup C_{2,3} \subseteq A_{2,3} \).

Now consider \( G \in A_{2,3} \). If \( \{2, 3\} \in E(G) \), then clearly \( (G - \{2, 3\}, G) \in M_{2,3} \) which is a contradiction. Further, if \( \{2, 3\} \notin E(G) \) and \( \text{deg}_{G}(2) < n - 3 \) as well as \( \text{deg}_{G}(3) < n - 2 \) then also \( G + \{2, 3\} \in N_{2,3} \). Therefore, if \( G \in A_{2,3} \) then \( \{2, 3\} \notin E(G) \) and either \( \text{deg}_{G}(3) = n - 2 \) or \( \text{deg}_{G}(2) = n - 3 \) whenever \( \text{deg}_{G}(3) < n - 2 \).

2. If \( G \in C \), then, for each \( i \in \{3, 4, \ldots, n\} \), \( \text{deg}_{G + \{2, i\}}(i) = n - 1 \) which implies that \( G + \{2, i\} \notin A_{2,2} \). This gives us that \( G \notin N_{2,i} \) for any \( i \in \{3, \ldots, n\} \), thereby showing that \( G \notin A_{2,n} \). If \( G \in B_{2,3} \) then \( \{2, 3\} \notin E(G) \), \( \text{deg}_{G}(3) < n - 2 \) and \( \text{deg}_{G - \{2, i\}}(2) < n - 3 \) for each \( i \in \{4, \ldots, n\} \) implying that \( G - \{2, i\} \notin A_{2,3} \) for any \( i \in \{4, \ldots, n\} \). Therefore, \( B_{2,3} \sqcup C \subseteq A_{2,n} \).

Let \( G \in A_{2,n} \subseteq A_{2,3} \). If \( \text{deg}_{G}(3) < n - 2 \) then clearly \( G \in B_{2,3} \). Let \( \text{deg}_{G}(3) = n - 2 \) and \( G \notin C \), i.e. \( G \notin A_{2,3} \sqcup C \). Then, either \( \{2, i\} \in E(G) \) for some \( i \in \{4, \ldots, n\} \) or \( \text{deg}_{G}(j) < n - 2 \) for some \( j \in \{4, \ldots, n\} \). Let \( i_0 = \min\{i : \{2, i\} \in E(G) \text{ or } \text{deg}_{G}(i) < n - 2\} \). It is easy to see that \( G \in N_{2,i_0} \) which contradicts the assumption that \( G \in A_{2,n} \).

3. If \( G \in C \) then it is easy to see that \( G \) is disjoint union of an isolated vertex \( \{2\} \) and a complete graph on vertex set \( \{1, 3, 4, \ldots, n\} \).

4. To prove this, we just need to look at the definition of \( B_{2,3} \) in expanded form.

\[
B_{2,3} = \{G \in A_{2,2} : \{2, 3\} \notin E(G), \text{ deg}_{G}(3) < n - 2, \text{ deg}_{G}(2) = n - 3\}
\]

\[
= \{G \in M_{n-2}(K_n) : \{1, 2\}, \{2, 3\} \notin E(G), \text{ deg}_{G}(1) = n - 2, \text{ deg}_{G}(2) = n - 3, \text{ deg}_{G}(3) < n - 2\}.
\] (3.6)

Equation (3.6) clearly implies the result.
5. Let $G \in \mathcal{C}_2$ and $\mathcal{B}_{2,3} \neq \emptyset$. Since $\{i, j\} \in G$ for each $3 \leq i < j \leq n$, $G + \{i, j\} = G \notin \mathcal{B}_{1,2}$. Further, $\deg_{G-\{i,j\}}(2) = 0$ for all $3 \leq i < j \leq n$, which implies that $G - \{i, j\} \in \mathcal{B}_{2,3}$ for some $3 \leq i < j \leq n$ only when $n = 3$, which contradicts the fact that $\mathcal{B}_{2,3} \neq \emptyset$.

We now move to step $k$, where $2 < k < n$. Inductively, let

$$
\mathcal{B}_{k-1,k} = \{G \in M_{n-2}(K_n) : \{i, i+1\} \notin E(G) \text{ for any } i \in [k-1], \deg_G(1) = n-2, \\
\deg_G(i) = n-3 \text{ for each } i \in \{2, \ldots, k-1\}, \deg_G(k) < n-2\}.
$$

Compare Equation (3.7) with Equation (3.6) when $k = 3$.

**Step k:** Let $\mathcal{A}_{k,k} = \mathcal{B}_{k-1,k}$. As is step 2, here also we define a sequence of element matchings on $\mathcal{A}_{k,k}$ using vertices $\{k, i\}$ for each $i \in \{k + 1, \ldots, n\}$. For $i \in \{k + 1, \ldots, n\}$, define

$$
M_{k,i} = \{(G, G + \{k, i\}) : \{k, i\} \notin E(G) \text{ and } G, G + \{k, i\} \in \mathcal{A}_{k,i-1}\},
$$

$$
N_{k,i} = \{G \in \mathcal{A}_{k,i-1} : (G - \{k, i\}, G + \{k, i\}) \in M_{k,i}\}, \text{ and}
$$

$$
\mathcal{A}_{k,i} = \mathcal{A}_{k,i-1} \setminus N_{k,i}.
$$

The following result analyses the set of critical cells after this step.

**Proposition 3.3.** For $i \in \{k, \ldots, n\}$, let $\mathcal{A}_{k,i}$ be as defined above.

1. $\mathcal{A}_{k,k+1} = \mathcal{B}_{k,k+1} \cup \mathcal{C}_{k,k+1}$, where

$$
\mathcal{B}_{k,k+1} = \{G \in \mathcal{A}_{k,k} : \{k, k+1\} \notin E(G), \deg_G(k+1) < n-2, \deg_G(k) = n-3\}, \text{ and}
$$

$$
\mathcal{C}_{k,k+1} = \{G \in \mathcal{A}_{k,k} : \{k, k+1\} \notin E(G), \deg_G(k+1) = n-2\}.
$$

2. $\mathcal{A}_{k,n} = \mathcal{B}_{k,k+1} \cup \mathcal{C}_k$, where

$$
\mathcal{C}_k = \{G \in \mathcal{A}_{k,k} : \{k, i\} \notin E(G) \text{ and } \deg_G(i) = n-2 \text{ for each } i \in \{k + 1, \ldots, n\}\}.
$$

3. $|\mathcal{C}_k| = 1$ and if $G \in \mathcal{C}_k$ then $|E(G)| = \binom{n-1}{2}$.

4. $\mathcal{B}_{k,k+1} = \emptyset$ if and only if $n = k + 1$.

5. If $G \in \mathcal{C}_k$ and $\mathcal{B}_{k,k+1} \neq \emptyset$, then $G - \{i, j\}, G + \{i, j\} \notin \mathcal{B}_{k,k+1}$ for any $k + 1 \leq i < j \leq n$.

**Proof.** Proof of parts (1), (2), (4) and (5) is similar as in the proof of Proposition 3.2. To prove part (3), let $G \in \mathcal{C}_k$ and $F$ denote the set of edges $\{\{i, i+1\} : i \in [k-1]\} \cup \{\{k, j\} : j \in \{k+1, \ldots, n\}\}$. By definition of $\mathcal{C}_k$, $E(G) \cap F = \emptyset$. Further, $\deg_G(i) = n-2$ for each $i \in \{1, k+1, \ldots, n\}$ and $\deg_G(j) = n-3$ for each $j \in \{2, \ldots, k-1\}$ imply that $E(G) = E(K_n) \setminus F$. Therefore, $\mathcal{C}_k$ contains exactly one graph and $|E(G)| = |E(K_n)| - |F| = \binom{n}{2} - (n-1) = \binom{n-1}{2}$.

**Proof of Theorem 2.1** Using Proposition 2.3, we get that $\bigcup_{1 \leq i \leq n-1, \ i+1 \leq j \leq n} M_{i,j}$ is an acyclic matching on $\mathcal{A}_{1,1} = M_{n-2}(K_n)$ with $\mathcal{C} = \bigcup_{i \in [n-1]} \mathcal{C}_i$ as the set of the critical cells. From Proposition 3.1(1), Proposition 3.2(3) and Proposition 3.3(3), we have $|\mathcal{C}| = n - 1$ and each graph in $\mathcal{C}$ has exactly $\binom{n-1}{2}$ edges. Therefore, Corollary 2.2 implies that $M_{n-2}(K_n)$ is homotopy equivalent to a wedge of $(n-1)$ spheres of dimension $\binom{n-1}{2} - 1$. 

8
For $2 \leq d \leq n - 1$, Jonsson [9, Theorem 12.8] obtained a connectivity bound for $M_d(K_n)$ and stated that “we do not believe that the derived bound is actually equal to the connectivity degree”. Here, we compare the findings of Theorem 1.1 with Jonsson’s result and show that the bound given by him is actually sharp for $d = n - 2$.

**Theorem 3.4.** [9, Theorem 12.8] Let $d \geq 2$ and $n \geq d + 1$. Write $n = (d + 4)k + r$, where $d + 1 \leq r \leq 2d + 4$. Then $M_d(K_n)$ is $(\lceil \nu_n^d \rceil - 1)$-connected, where

$$
\nu_n^d = \frac{(d^2 + 3d - 1)n}{2(d + 4)} - \frac{\epsilon_d(r)}{2} - 1, \quad \text{and}
$$

$$
\epsilon_d(r) = \frac{3r}{d + 4}
\begin{cases}
1 & \text{if } r = d + 1; \\
2 & \text{if } d + 2 \leq r \leq d + 3; \\
3 & \text{if } d + 4 \leq r \leq 2d + 3; \\
4 & \text{if } r = 2d + 4.
\end{cases}
$$

It is easy to see that $\nu_n^{n-2} = (\frac{n-1}{2}) - 1$. Therefore, Theorem 1.1 implies the following.

**Remark 3.5.** The connectivity bound for $M_{n-2}(K_n)$ given in Theorem 3.4 is sharp.

4 Higher matching complexes of complete bipartite graphs

It is easy to see that, for $m > r \geq n \geq 1$, the complex $M_r(K_{m,n})$ is the join of $n$ copies of the $(r - 1)$-skeleton of an $(m - 1)$-dimensional simplex. Therefore, from [1, Lemma 2.5], $M_r(K_{m,n})$ is homotopy equivalent to a wedge of spheres whenever $m > r \geq n$. In this section, we determine the homotopy type of $M_{n-1}(K_{n,n})$. We first fix some notations.

$$
V(K_{m,n}) = \{a_i : i \in [m]\} \cup \{b_j : j \in [n]\}, \quad \text{and}
$$

$$
E(K_{m,n}) = \{\{a_i, b_j\} : i \in [m], j \in [n]\}. \quad (4.1)
$$

**Proof of Theorem 3.2** We prove this by defining a sequence of element matchings on $M_{n-1}(K_{n,n})$ in $(n - 1)$-steps.

**Step 1:** Let $\mathcal{H}_{1,0} = M_{n-1}(K_{n,n})$. For $1 \leq i \leq n$, define

$$
M_{1,i} = \{(G, G + \{a_1, b_i\}) : \{a_1, b_i\} \notin E(G) \text{ and } G, G + \{a_1, b_i\} \in \mathcal{H}_{1,i-1}\},
$$

$$
N_{1,i} = \{G \in \mathcal{H}_{1,i-1} : (G - \{a_1, b_i\}, G + \{a_1, b_i\}) \in M_{1,i}\}, \quad \text{and}
$$

$$
\mathcal{H}_{1,i} = \mathcal{H}_{1,i-1} \setminus N_{1,i}.
$$

**Claim 3.** $\mathcal{H}_{1,n} = \{G \in \mathcal{H}_{1,0} : \{a_1, b_1\} \notin E(G), \ deg_G(b_1) < n - 1 \text{ and } deg_G(a_1) = n - 1\}$.

**Proof of Claim 3** Observe that, if $G \in \mathcal{H}_{1,1}$ then $\{a_1, b_1\} \notin E(G)$ and either $deg_G(a_1) = n - 1$ or $deg_G(b_1) = n - 1$. Define,

$$
\mathcal{J}_{1,1} = \{G \in \mathcal{H}_{1,1} : deg_G(b_1) = n - 1\}
$$

$$
= \{G \in \mathcal{H}_{1,0} : \{a_1, b_1\} \notin E(G), \ deg_G(b_1) = n - 1\} \quad \text{and}
$$

$$
\mathcal{J}_{1,1} = \{G \in \mathcal{H}_{1,1} : deg_G(b_1) < n - 1, \ deg_G(a_1) = n - 1\}
$$

$$
= \{G \in \mathcal{H}_{1,0} : \{a_1, b_1\} \notin E(G), \ deg_G(b_1) < n - 1, \ deg_G(a_1) = n - 1\}. \quad (4.2)
$$

\footnote{The join of two simplicial complexes $K_1$ and $K_2$ is a simplicial complex whose simplices are disjoint union of simplices of $K_1$ and of $K_2$.}
Clearly $\mathcal{H}_{1,1} = \mathcal{J}_{1,1} \sqcup \mathcal{J}_{1,1}$. Further if $G \in \mathcal{J}_{1,1}$, then for each $i \in \{2, \ldots, n\}$, $\{a_i, b_i\} \in E(G)$ and $G - \{a_i, b_i\} \notin \mathcal{H}_{1,1}$. Therefore $\mathcal{J}_{1,n} \subseteq \mathcal{H}_{1,n}$. We now show that $\mathcal{H}_{1,n} \subseteq \mathcal{J}_{1,1}$. Let $H \in \mathcal{H}_{1,n} \setminus \mathcal{J}_{1,1}$, i.e., $\deg_H(b_1) = n - 1$. Let $t = \min\{i \in \{2, \ldots, n\} : \text{either } \{a_i, b_i\} \in E(H) \text{ or } \deg_H(b_i) < n - 1\}$. Observe that $t$ exists because, if $\{a_1, b_1\} \notin E(H)$ and $\deg_H(b_1) = n - 1$ for each $i \in [n]$, then there exists $j \in \{2, \ldots, n\}$ such that $\deg_H(a_j) > n - 1$ which contradicts the fact that $H \in M_{n-1}(K_{n,n})$. Now, it is easy to see that $H \in \mathcal{J}_{1,1}$. Therefore $\mathcal{H}_{1,n} = \mathcal{J}_{1,1}$.

We now move to step $k$, where $1 < k < n$. At step $k - 1$ we defined a sequence of elements matchings using vertices $\{a_k, b_1\} < \cdots < \{a_k, b_n\}$. Inductively, let the set of critical cells after step $k - 1$ be

$$\mathcal{H}_{k-1,n} = \{G \in \mathcal{H}_{1,0} : \{a_i, b_1\} \notin E(G), \deg_G(a_i) = n - 1, \forall i \in [k-1], \deg_G(b_1) < n-k+1\}. \tag{4.3}$$

Compare Equation (4.3) with Claim 3 for $k = 2$.

**Step k**: Let $\mathcal{H}_{k,0} = \mathcal{H}_{k-1,n}$. Define a sequence of elements matchings on $\mathcal{H}_{k,0}$ using vertices $\{a_k, b_1\} < \cdots < \{a_k, b_n\}$. For $1 \leq j \leq n$, define

$$M_{k,j} = \{(G, G + \{a_k, b_j\}) : \{a_k, b_j\} \notin E(G) \text{ and } G, G + \{a_k, b_j\} \in \mathcal{H}_{k,j-1}\},$$

$$N_{k,j} = \{G \in \mathcal{H}_{k,j-1} : (G - \{a_k, b_j\}, G + \{a_k, b_j\}) \in M_{k,j}\},$$

$$\mathcal{H}_{k,j} = \mathcal{H}_{k,j-1} \setminus N_{k,j}.$$

Since $k < n$, using similar arguments as in the proof of Claim 3 we get that

$$\mathcal{H}_{k,n} = \{G \in \mathcal{H}_{k,0} : \{a_k, b_1\} \notin E(G), \deg_G(b_1) < n - k \text{ and } \deg_G(a_k) = n - 1\}.$$

After step $n - 1$, we have that

$$\bigcup_{1 \leq i \leq n-1} \bigcup_{1 \leq j \leq n} M_{i,j} \text{ is an acyclic matching on } M_{n-1}(K_{n,n}) \text{ and the set of critical cells is}$$

$$\mathcal{H}_{n-1,n} = \{G \in \mathcal{H}_{n-2,n} : \{a_{n-1}, b_1\} \notin E(G), \deg_G(b_1) < 1 \text{ and } \deg_G(a_{n-1}) = n - 1\}$$

$$= \{G \in M_{n-1}(K_{n,n}) : \{a_i, b_1\} \notin E(G), \deg_G(a_i) = n - 1, \forall i \in [n-1], \deg_G(b_1) = 0\}.$$

It is easy to see that the set $\mathcal{H}_{n-1,n}$ contains exactly one element which is isomorphic to the complete bipartite graph $K_{n-1,n-1}$ and two isolated vertices namely $a_n$ and $b_1$. Corollary 2.2 thus implies that

$$M_{n-1}(K_{n,n}) \simeq S^{(n-1)^2-1}.$$

This completes the proof of Theorem 1.2 □

5 Concluding remarks

In this section, we list a few interesting open problems.
5.1 Complexes of graphs with bounded domination number

For a graph \( G \), a set \( S \subseteq V(G) \) is called a dominating set of \( G \), if for each \( v \in V(G) \setminus S \) there exists \( s \in S \) such that \( \{s, v\} \in E(G) \). The domination number of \( G \) is defined to be the cardinality of the minimum dominating set, i.e.,

\[
\text{dom}(G) = \min\{i : \text{there exists a dominating set of } G \text{ of cardinality } i\}.
\]

In [6], González and Hoekstra-Mendoza studied the complexes of graphs on \( n \) vertices with domination number at least \( \gamma \), denoted as \( D_{n,\gamma} \). When we fix \( n \) and vary \( \gamma \), we get the following filtration

\[
\emptyset = D_{n,n} \subset D_{n,n-1} \subset D_{n,n-2} \subset \cdots \subset D_{n,2} \subset D_{n,1} = \Delta \left( \begin{array}{c} n \\end{array} \right) - 1
\]

It is easy to observe that \( D_{n,n-1} \) is disjoint union \( \left( \begin{array}{c} n \\end{array} \right) \) vertices. Therefore, the “first” non-trivial cases are \( D_{n,2} \) and \( D_{n,n-2} \). González and Hoekstra-Mendoza [6] showed that the complex \( D_{n,n-2} \) is homotopy equivalent to a wedge of 2-dimensional spheres.

Observe that, \( G \in M_{n-2}(K_n) \) if and only if \( \text{dom}(G) \geq 2 \), i.e., \( G \in D_{n,2} \). Therefore, Theorem [11] gives a closed form formula for the homotopy type of \( D_{n,2} \) and settles one more spot in Equation (5.1). This raises the following question.

**Question 1.** For \( n > \gamma \geq 1 \), is \( D_{n,\gamma} \) homotopy equivalent to a wedge of spheres?

Here, empty wedge represents a contractible space. Looking at the homotopy type of \( D_{n,n-2} \) and \( D_{n,2} \), one might be tempted to guess that \( D_{n,\gamma} \) is homotopy equivalent to a wedge of equi-dimension spheres. But that is not the case in general, for instance, using SageMath [16] one can see that

\[
\tilde{H}_i(D_{6,3}; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z}^{115}, & \text{if } i = 4; \\
\mathbb{Z}^{24}, & \text{if } i = 5; \\
0, & \text{otherwise.}
\end{cases}
\]

Which implies that \( D_{6,3} \) is not homotopic to a wedge of equi-dimensional spheres. Here \( \tilde{H}_i(\mathcal{X}; \mathbb{Z}) \) denotes the reduced \( i \)th homology group of simplicial complex \( \mathcal{X} \) with integer coefficients.

5.2 Homotopical depth

A pure simplicial complex \( L \) is called homotopically Cohen-Macaulay (CM) if link of any simplex \( \sigma \) in \( L \), denoted as \( \text{lk}_L(\sigma) \), is \( (\dim(\text{lk}_L(\sigma)) - 1) \)-connected. The homotopical depth of a simplicial complex \( \mathcal{X} \) (not necessarily pure) is the largest \( k \) such that the \( k \)-skeleton of \( \mathcal{X} \), denoted as \( \mathcal{X}^{(k)} \), is pure and homotopically CM.

It is easy to see that the homotopical depth of a pure simplicial complex \( \mathcal{X} \) is at most the shifted connectivity degree of \( \mathcal{X} \). The homotopical depth of \( M_1(K_n) \) is known to be equal to the shifted connectivity degree of \( M_1(K_n) \) which is \( \left\lceil \frac{n-4}{3} \right\rceil \), see [9] Corollary 11.13. In [9] Proposition 12.11, Jonsson showed that the homotopical depth of \( M_2(K_n) \) is at least \( \left\lceil \frac{3n-7}{4} \right\rceil \). In this direction, we strongly believe that the following is true.

**Conjecture 1.** The homotopical depth of \( M_{n-2}(K_n) \) and \( M_{n-1}(K_{n,n}) \) is equal to the respective shifted connectivity degree.
The purity of $M_{n-2}(K_n)\binom{(n-1)}{2}$ and $M_{n-1}(K_{n,n})\binom{(n-1)^2}{1}$ comes from the following observation.

**Proposition 5.1.** If $G \in M_{n-2}(K_n)$ and $H \in M_{n-1}(K_{n,n})$ are facets in respective complexes then $|E(G)| \geq \binom{(n-1)}{2}$ and $|E(H)| \geq (n-1)^2$.

**Proof.** We prove the result for $H \in M_{n-1}(K_{n,n})$. Proof for $G \in M_{n-2}(K_n)$ will follow using similar arguments. Let $H \in M_{n-1}(K_{n,n})$ and $|E(H)| < (n-1)^2$. Let $A = \{a_i : i \in [n]\}$ and $B = \{b_i : b \in [n]\}$ be the partition of vertices of $K_{n,n}$ as in Equation (4.1). To show that $H$ is not a facet, we need to find two non-adjacent vertices (one from each partition of $V(K_{n,n})$) with degree less than $n-1$. Suppose that there does not exist such pair, i.e., $H$ is facet. Let $C \subseteq A$ and $D \subseteq B$ such that $\text{deg}_H(x) < n-1$ if and only if $x \in C \cup D$. Since $|E(H)| < (n-1)^2$, $C \neq \emptyset \neq D$. Moreover, there does not exist a non-adjacent desired pair implies that $C \cup D$ form a complete bipartite subgraph of $H$. Assuming that $|C| = c$, and $|D| = d$, we count the number of edges of $H$.

$$
|E(H)| \geq (n-1)(n-c) + cd + (c-1)(n-d) = n^2 - n - cn + c + cd - cd + cn - n + d = n^2 - 2n + c + d.
$$

Our assumption thus implies that $n^2 - 2n + c + d < (n-1)^2 = n^2 - 2n + 1$. Thus $c + d < 1$ and this contradicts the fact that $C \neq \emptyset \neq D$. \qed

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