Free Gap Information from the Differentially Private Sparse Vector and Noisy Max Mechanisms

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ABSTRACT

Noisy Max and Sparse Vector are selection algorithms for differential privacy and serve as building blocks for more complex algorithms. In this paper we show that both algorithms can release additional information for free (i.e., at no additional privacy cost). Noisy Max is used to return the approximate maximizer among a set of queries. We show that it can also release for free the noisy gap between the approximate maximizer and runner-up. Sparse Vector is used to return a set of queries that are approximately larger than a fixed threshold. We show that it can adaptively control its privacy budget (use less budget for queries that are likely to be much larger than the threshold) and simultaneously release for free a noisy gap between the selected queries and the threshold. It has long been suspected that Sparse Vector can release additional information, but prior attempts had incorrect proofs. Our version is proved using randomness alignment, a proof template framework borrowed from the program verification literature. We show how the free extra information in both mechanisms can be used to improve the utility of differentially private algorithms.

1. INTRODUCTION

Industry and government agencies are increasingly adopting differential privacy [17] to protect the confidentiality of users who provide data. Current and planned major applications include data gathering by Google [21], Apple [14], and Microsoft [13]; database querying by Uber [27], and publication of population statistics at the U.S. Census Bureau [33, 9, 24, 1].

The accuracy of differentially private data releases is very important in these applications. One way to improve accuracy is to increase the value of the privacy parameter \( \epsilon \), known as the privacy loss budget, as it provides a tradeoff between an algorithm’s utility and its privacy protections. However, values of \( \epsilon \) that are deemed too high can subject a company to criticisms of not providing enough privacy [40]. For this reason, researchers invest significant effort in tuning algorithms [11, 55, 28, 1, 38, 22] and privacy analyses [6, 97, 38, 20] to provide better utility while using smaller privacy budgets.

Differentially private algorithms are built out of smaller components called mechanisms [35]. Popular mechanisms include the Laplace Mechanism [17], Geometric Mechanism [23], Noisy Max [19] and Sparse Vector [19, 32]. As we will explain in this paper, the latter two mechanisms, Noisy Max and Sparse Vector, inadvertently throw away information that is useful for designing accurate algorithms. Our contribution is to present novel variants of these mechanisms that provide more functionality at the same privacy cost (under pure differential privacy).

Given a set of queries, Noisy Max returns the identity (not value) of the query that is likely to have the largest value — it adds noise to each query answer and returns the index of the query with the largest noisy value. Meanwhile, Sparse Vector takes a stream of queries and a predefined public threshold \( T \). It tries to return the identities of the first \( k \) queries that are probably larger than the threshold. To do so, it adds noise to the threshold. Then, as it sequentially processes each query, it outputs “\( \top \)” or “\( \bot \)”, depending on whether the noisy value of the current query is larger or smaller than the noisy threshold. The mechanism terminates after \( k \) “\( \top \)” outputs.

In recent work [43], using program verification tools, Wang et al. showed that Sparse Vector can provide additional information at no additional cost to privacy. That is, when Sparse Vector returns “\( \top \)” for a query, it can also return the gap between its noisy value and the noisy threshold [1]. We refer to their algorithm as Sparse-Vector-with-Gap.

Inspired by this program verification work, we propose many novel variations of Sparse Vector and Noisy Max. For Sparse Vector, we show that in addition to releasing this gap information, even stronger improvements are possible — we present an adaptive version that can answer more queries than before by controlling how much privacy budget it uses to answer each query. The intuition is that we would like to spend less of our privacy budget for queries that are probably much larger than the threshold (compared to queries that are probably closer to the threshold). A careful accounting of the privacy impact shows that this is possible and our experiments confirm that Adaptive-Sparse-Vector-with-Gap can answer many more queries than the prior versions [33, 9, 24] at the same privacy cost.

For Noisy Max, we show that it too inadvertently throws away information. Specifically, at no additional cost to privacy, it can release an estimate of the gap between the largest and second largest queries (we call the resulting mechanism Noisy-Max-with-Gap). We then generalize this result to Noisy Top-K — showing that one can release an estimate of the identities of the \( k \) largest queries and, at no extra privacy cost, release noisy estimates of the pairwise gaps (dif-

\footnote{This was a surprising result given the number of incorrect attempts at improving Sparse Vector based on flawed manual proofs [32] and shows the power of automated program verification techniques.}
The extra information that these mechanisms can provide opens up new directions in the construction of differentially private algorithms. We present some of these applications in this paper. For instance, one common task is to select the approximate top-$k$ queries and then use additional privacy loss budget to provide estimates of their values. We show how the noisy gap information from our top-$k$ mechanism can be used inside such an algorithm to provide more accuracy for this task (we give similar results for Sparse Vector).

We prove our results using the alignment of random variables technique \cite{11, 43, 44}, which is based on the following question: if we change the input to a program, how must we change its random variables so that output remains the same? This technique was used in several human-readable \cite{11, 32, 43} and machine-readable \cite{44, 43} proofs. However, it is also often used incorrectly (as discussed by Lyu et al. \cite{32}). Thus a secondary contribution of our work is to lay out the precise steps and conditions that must be checked and to provide helpful lemmas that ensure these conditions are met. The resulting proof template simplifies the process of proving the correctness of such mechanisms. In addition to using it to prove the correctness of our mechanisms, we also use it to provide the first human-readable proof of Sparse-Vector-with-Gap.

To summarize, our contributions are as follows:

- We provide a simplified template for writing correctness proofs for intricate differentially private algorithms.
- Using this technique, we propose and prove the correctness of three new mechanisms: (1) Adaptive-Sparse-Vector-with-Gap, (2) Noisy-Max-with-Gap and (3) Noisy-Top-K-with-Gap. These algorithms improve on the original versions of Sparse Vector, Noisy Max, and Noisy Top-K (i.e. they provide more information at the same privacy cost). We also provide the first human-readable proof of (non-adaptive) Sparse-Vector-with-Gap, which previously only had a machine-readable proof \cite{43}.
- We demonstrate some of the uses of the gap information that is provided by these new mechanisms. When an algorithm needs to use Noisy Max or Sparse Vector to select some queries and then measure them (i.e., obtain their noisy answers), we show how the gap information from our versions can be used to improve the accuracy of the noisy measurements. We also show how the gap information in Sparse Vector can be used to estimate the confidence that a query’s true answer really is larger than the threshold.
- We empirically evaluate these new mechanisms on a variety of datasets to demonstrate their improved utility.

In Section 2 we discuss related work. We present background and notation in Section 3. We present simplified proof templates for randomness alignment in Section 4. We present novel variants of Noisy Max in Section 5 and novel variants of Sparse Vector in Section 6. We present experiments in Section 7, key proofs in Section 8 and conclusions in Section 9. The rest of our proofs can be found in the Appendix.

2. RELATED WORK

Selection algorithms, such as Exponential Mechanism \cite{44, 39}, Sparse Vector \cite{19, 32}, and Noisy Max \cite{19} are used to select a set of items (typically queries) from a much larger set. They have applications in hyperparameter tuning \cite{11, 31}, iterative construction of microdata \cite{25}, feature selection \cite{42}, frequent itemset mining \cite{6}, exploring a privacy/accuracy tradeoff \cite{30}, data pre-processing \cite{12}, etc.

Various generalizations have been proposed \cite{30, 9, 42, 59, 10, 31, Liu and Talwar \cite{31} and Raskhodnikova and Smith \cite{39} extend the exponential mechanism for arbitrary sensitivity queries. Beimel et al. \cite{5} and Thakurta and Smith \cite{42} use the propose-test-release framework \cite{16} to find a gap between the best and second best queries and, if the gap is large enough, release the identity of the best query. These two algorithms rely on a relaxation of differential privacy called approximate ($\epsilon$, $\delta$)-differential privacy \cite{14} and can fail to return an answer (in which case they return $\perp$). Our algorithms work with pure $\epsilon$-differential privacy. Chaudhuri et al. \cite{10} also proposed a large margin mechanism (with approximate differential privacy) which finds a large gap separating top queries from the rest and returns one of them.

There have also been unsuccessful attempts to generalize selection algorithms such as the Sparse Vector (incorrect versions are catalogued by Lyu et al. \cite{32}), which has sparked innovations in program verification for differential privacy (e.g., \cite{4, 5, 43}). One technique, known as Randomness Alignment \cite{44} is a simplification of probabilistic coupling \cite{4}, and shows promise for helping automation of privacy proofs \cite{44, 3}. It considers what changes need to be made to random variables in order to make two executions of a program, with different inputs, produce the same output. Such ideas have appeared in handwritten proofs \cite{11, 19, 32} and were mis-used in incorrect proofs \cite{32}. In this paper, we provide a technique called Randomness Alignment Templates which simplifies the manual construction of correct human-readable proofs (e.g., \cite{11, 32}) by removing a lot of boilerplate from the proof.

3. NOTATION AND BACKGROUND

In this paper, we use the following notation. $D$ and $D'$ refer to databases. We use the notation $D \sim D'$ to represent adjacent databases. $M$ denotes a randomized algorithm whose input is a database. $\Omega$ denotes the range of $M$ and $\omega \in \Omega$ denotes a specific output of $M$. We use $E \subseteq \Omega$ to denote a set of possible outputs. Because $M$ is randomized, it also relies on a random noise vector $H \in \mathbb{R}^\infty$ (which usually consists of independent zero-mean Laplace random variables). This noise sequence is infinite, but of course $M$ will only use a finite-length prefix of $H$. When we need to draw attention to the noise, we use the notation $M(D, H)$ to indicate the execution of $M$ with database $D$ and randomness coming from $H$. Otherwise we use the notation $M(D)$. Define $S_{M, D, E} = \{H \mid M(D, H) \in E\}$ to be the set of noise vectors that allow $M$, on input $D$, to produce an output in the set $E \subseteq \Omega$. To avoid overburdening the notation, the notion of adjacency depends on the application. Some papers define it as $D$ can be obtained from $D'$ by modifying one record \cite{17} or by adding/deleting one record \cite{14}.
we write $S_{D,E}$ for $S_{M,D,E}$ and $S_{D'}$ for $S_{M,D',E}$ when $M$ is clear from the context. When $E$ consists of a single point $\omega$, we write these sets as $S_{D,\omega}$ and $S_{D',\omega}$. This notation is summarized in the table below.

| Table 1: Notation |
|-------------------|
| Symbol            | Meaning                                      |
| $M$               | privacy mechanism                            |
| $D, D'$           | database                                     |
| $D \sim D'$       | $D$ is adjacent to $D'$                      |
| $H = (\eta_1, \eta_2, \ldots)$ | input noise vector                           |
| $\Omega$          | the space of all output of $M$               |
| $\omega$          | a possible output; $\omega \in \Omega$       |
| $E$               | a set of possible outputs; $E \subseteq \Omega$ |
| $S_{D,E} = S_{M,D,E}$ | $\{ H \mid M(D, H) \in E \}$              |
| $S_{D,\omega} = S_{M,D,\omega}$ | $\{ H \mid M(D, H) = \omega \}$            |

3.1 Formal Privacy

Differential privacy \cite{17,14,19} is currently the gold standard for releasing privacy-preserving information about a database. It has a parameter $\epsilon > 0$ known as the privacy loss budget. The smaller it is, the more privacy is provided. Differential privacy bounds the effect of one record on the output of the algorithm (for small $\epsilon$, the probability of any output is barely affected by any person’s record).

**Definition 1 (Pure Differential Privacy \cite{14}).** Given an $\epsilon > 0$, a randomized algorithm $M$ with output space $\Omega$ satisfies (pure) $\epsilon$-differential privacy if for all $E \subseteq \Omega$ and all pairs of adjacent databases $D \sim D'$, the following holds:

$$\mathbb{P}(M(D, H) \in E) \leq e^\epsilon \mathbb{P}(M(D', H) \in E)$$

where the probability is only over the randomness of $H$.

With the notation in Table 1, the differential privacy condition from Equation 1 is $\mathbb{P}(S_{D,E}) \leq e^\epsilon \mathbb{P}(S_{D',E})$.

Differential privacy enjoys the following nice properties:

- **Resilience to Post-Processing.** If we apply an algorithm $A$ to the output of an $\epsilon$-differentially private algorithm $M$, then the composite algorithm $A \circ M$ still satisfies $\epsilon$-differential privacy. In other words, privacy is not reduced by post-processing.

- **Composition.** If $M_1, M_2, \ldots, M_k$ satisfy differential privacy with privacy loss budgets $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$, the algorithm that runs all of them and releases their outputs satisfies ($\sum \epsilon_i$)-differential privacy.

Many differentially private algorithms take advantage of the Laplace mechanism \cite{54}, which provides a noisy answer to a vector-valued query $q$ based on its $L_1$ *global sensitivity* $\Delta_q$, defined as follows:

**Definition 2 (L$_1$ Global Sensitivity \cite{19}).** The *global sensitivity* of a query $q$ is $\Delta_q = \sup_{D \sim D'} \| q(D) - q(D') \|_1$.

**Theorem 1 (Laplace Mechanism \cite{17}).** Given a privacy loss budget $\epsilon$, consider the mechanism that returns $q(D) + H$, where $H$ is a vector of independent random samples from the Laplace($\Delta_q/\epsilon$) distribution with mean 0 and scale parameter $\Delta_q/\epsilon$. This Laplace mechanism satisfies $\epsilon$-differential privacy.

If the output of $q$ only consists of multiples of some number $\gamma$, then one can use the discrete Laplace mechanism:

**Theorem 2 (Discrete Laplace \cite{23}).** Let $q$ be a vector-valued query, where each component of $q(D)$ is always a multiple of $\gamma$. Given a privacy loss budget $\epsilon$, consider the mechanism that returns $q(D) + H$, where $H$ is a vector of independent random samples from the $\text{DiscLaplace}($$\Delta_q/\epsilon$$)$ distribution having probability mass function $f(k) = \frac{\gamma}{\Delta_q/\epsilon} \frac{e^{-\frac{k\gamma}{\Delta_q/\epsilon}}}{\sum_{k=0}^{\infty} e^{-\frac{k\gamma}{\Delta_q/\epsilon}}}$ for $k = 0, \pm\gamma, \pm2\gamma, \ldots$. This Discrete Laplace mechanism satisfies $\epsilon$-differential privacy.

4. RANDOMNESS ALIGNMENT

To establish that the algorithms we propose are differentially private, we use an idea called *randomness alignment* that previously had been used to prove the privacy of a variety of sophisticated algorithms \cite{19,32,11} and incorporated into verification/synthesis tools \cite{44,43,3}. While powerful, this technique is also easy to use incorrectly \cite{32}, as there are many technical conditions that need to be checked. In this section, we present results (namely Lemma 1) that significantly simplify this process and make it easy to prove the correctness of our proposed algorithms.

In general, to prove $\epsilon$-differential privacy for an algorithm $M$, one needs to show $\mathbb{P}(M(D) \in E) \leq e^\epsilon \mathbb{P}(M(D') \in E)$ for all pairs of adjacent databases $D \sim D'$ and sets of possible outputs $E \subseteq \Omega$. In our notation, this inequality is represented as $\mathbb{P}(S_{D,E}) \leq e^\epsilon \mathbb{P}(S_{D',E})$. Establishing such inequalities is often done with the help of a function $\phi_{D,D',\omega}$ called a *randomness alignment* (there is a function $\phi_{D,D'}$ for every pair $D \sim D'$, that maps noise vectors $H$ into noise vectors $H'$ so that $M(D', H')$ produces the same output as $M(D, H)$).

**Definition 3 (Randomness Alignment).** Let $M$ be randomized algorithm. Let $D \sim D'$ be a pair of adjacent databases. A randomness alignment is a function $\phi_{D,D',\omega} : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ such that for all $H$ on which $M(D, H)$ terminates, we have $M(D, H) = M(D', \phi_{D,D',\omega}(H))$.

**Example 1.** Let $D$ be a database that records the salary of every person, which is guaranteed to be between 0 and 100. Let $q(D)$ be the sum of the salaries in $D$. The sensitivity of $q$ is thus 100. Let $H = (\eta_1, \eta_2, \ldots)$ be a vector of independent Laplace(100/\epsilon) random variables. The Laplace mechanism outputs $q(D) + \eta_1$ (and ignores the remaining variables in $H$). For every pair of adjacent databases $D \sim D'$, one can define the corresponding randomness alignment $\phi_{D,D',\omega}(H) = H' = (\eta_1', \eta_2', \ldots)$, where $\eta_i' = \eta_i + q(D) - q(D')$ and $\eta_i' = \eta_i$ for $i > 1$. Note that $q(D) + \eta_1 = q(D') + \eta_1'$, so the output of $M$ remains the same.

In practice, $\phi_{D,D',\omega}$ is constructed locally (piece by piece) as follows. For each possible output $\omega \in \Omega$, one defines a function $\phi_{D,D',\omega}$ that maps noise vectors $H$ into noise vectors $H'$ with the following properties: if $M(D, H) = \omega$ then $M(D', H') = \omega$ (that is, $\phi_{D,D',\omega}$ only cares about what it takes to produce the specific output $\omega$). We obtain our randomness alignment $\phi_{D,D'}$ in the obvious way by piecing together the $\phi_{D,D',\omega}$ as follows: $\phi_{D,D'}(H) = \phi_{D,D',\omega^*}(H)$, where $\omega^*$ is the output of $M(D, H)$. Formally,

**Definition 4 (Local Alignment).** Let $M$ be randomized algorithm. Let $D \sim D'$ be a pair of adjacent databases and $\omega$ a
possible output of $M$. A local alignment for $M$ is a function $\phi_{D',D',\omega} \colon S_{D'} \to S_{D'}$ (see notation in Table 1) such that for all $H \in S_{D'}$, we have $M(D, H) = M(D', \phi_{D',D',\omega}(H))$.

Example 2. Continuing the setup from Example 1, consider the mechanism $M_1$ that, on input $D$, outputs $T$ if $q(D) + \eta_1 \geq 10,000$ (i.e., if the noisy total salary is at least 10,000) and $\perp$ if $q(D) + \eta_1 < 10,000$. Let $D'$ be a database that differs from $D$ in the presence/absence of one record. Consider the local alignments $\phi_{D',D',\top}$ and $\phi_{D',D',\perp}$ defined as follows. $\phi_{D',D',\top}(H) = (\eta_1', \eta_2', \ldots)$ where $\eta_1' = \eta_1 + 100$ and $\eta_i = \eta_i$ for $i > 1$; and $\phi_{D',D',\perp}(H) = H' = (\eta_1', \eta_2', \ldots)$ where $\eta_1' = \eta_1 - 100$ and $\eta_i' = \eta_i$ for $i > 1$. Clearly, if $M_1(D, H) = \top$ then $M_1(D', H') = \top$ and if $M_1(D, H) = \perp$ then $M_1(D', H') = \perp$. We piece these two local alignments together to create a randomness alignment $\phi_{D',D'}(H) = H^* = (\eta_1', \eta_2', \ldots)$ where:

$$\eta_1^* = \begin{cases} 
\eta_1' + 1 & \text{if } M(D, H) = \top \\
\eta_1' - 1 & \text{if } M(D, H) = \perp
\end{cases}$$

$$\eta_i^* = \eta_i' \text{ for } i > 1$$

4.1 Special Properties of Alignments

Not all alignments can be used to prove differential privacy. In this section we discuss some additional properties that help prove differential privacy.

We first make two mild assumptions about the mechanism $M$: (1) it terminates with probability one and (2) based on the output of $M$, we can determine how many random variables it used. The vast majority of differentially private algorithms in the literature satisfy these properties.

We next define two properties of alignments: whether they are acyclic and what their cost is.

Definition 5 (Acyclic). Let $M$ be a randomized algorithm. Let $\phi_{D',D',\omega}$ be a local alignment for $M$. For any $H = (\eta_1, \eta_2, \ldots)$, let $H' = (\eta_1', \eta_2', \ldots)$ denote $\phi_{D',D',\omega}(H)$. We say that $\phi_{D',D',\omega}$ is acyclic if there exists a permutation $\pi$ and piecewise differentiable functions $\psi^{(j)}_{D',D',\omega}$ such that:

$$\eta_{\pi(1)}' = \eta_{\pi(1)} + \text{number that only depends on } D', \omega$$

$$\eta_{\pi(j)}' = \eta_{\pi(j)} + \psi^{(j)}_{D',D',\omega}(\eta_{\pi(1)}, \ldots, \eta_{\pi(j-1)}) \text{ for } j \geq 2$$

Essentially, a local alignment $\phi_{D',D',\omega}$ is acyclic if there is some ordering of the variables so that $\eta_j'$ is the sum of $\eta_j$ and a function of the variables that came earlier in the ordering. The local alignments $\phi_{D',D',\top}$ and $\phi_{D',D',\perp}$ from Example 2 are both acyclic (in general, each local alignment function is allowed to have its own specific ordering and differentiable functions $\psi^{(j)}_{D',D',\omega}$). The pieced-together randomness alignment $\phi_{D',D'}$ itself need not be acyclic.

Definition 6 (Alignment Cost). Let $M$ be a randomized algorithm that uses $H$ as its source of randomness. Let $\phi_{D',D',\omega}$ be a local alignment for $M$. For any $H = (\eta_1, \eta_2, \ldots)$, let $H' = (\eta_1', \eta_2', \ldots)$ denote $\phi_{D',D',\omega}(H)$. If each $\eta_i$ is generated independently from a Laplace($\alpha_i$) or DiscLaplace($\alpha_i$) distribution then the cost of $\phi_{D',D',\omega}$ is defined as:

$$\text{cost}(\phi_{D',D',\omega}) = \sum_i |\eta_i' - \eta_i|/\alpha_i$$

The following lemma uses those properties to establish that $M$ satisfies $\epsilon$-differential privacy.

Lemma 1. Let $M$ be an algorithm with input randomness $H = (\eta_1, \eta_2, \ldots)$. If the following conditions are satisfied, then $M$ satisfies $\epsilon$-differential privacy.

i. $M$ terminates with probability 1.

ii. The number of random variables used by $M$ can be determined from its output.

iii. Each $\eta_i$ is generated independently from a Laplace($\alpha_i$) or DiscLaplace($\alpha_i$) distribution.

iv. For every $D \sim D'$ and $\omega$ there exists a local alignment $\phi_{D',D',\omega}$ that is acyclic with cost($\phi_{D',D',\omega}$) ≤ $\epsilon$.

v. For every $D \sim D'$ the number of distinct local alignments is countable. That is, the set $\{\phi_{D',D',\omega} \mid \omega \in \Omega\}$ is countable (i.e., for many choices of $\omega$ we get the same exact alignment function).

We defer the proof to Section 8.

Example 3. Consider the randomness alignment $\phi_{D,D',\omega}$ from Example 1. We can define all of the local alignments $\phi_{D,D',\omega}$ to be the same function: $\phi_{D,D',\omega}(H) = \phi_{D,D'}(H)$. Clearly cost($\phi_{D,D',\omega}$) = $\sum_{i=0}^{\infty} |\eta_i' - \eta_i| = \sum_i |\eta_i(D') - \eta_i(D)| \leq \sum_i 100 \\ 100 = 100 = \epsilon$. For Example 2, there are two acyclic local alignments $\phi_{D,D',\top}$ and $\phi_{D,D',\perp}$; both have cost = $100 - \epsilon/100 = \epsilon$. The other conditions in Lemma 1 are trivial to check. Thus both mechanisms satisfy $\epsilon$-differential privacy by Lemma 1.

4.2 Warmup: Sparse Vector with Gap

As a warmup demonstration of the power of Lemma 1, we provide the first human-readable proof of Sparse-Vector-with-Gap [43]. The mechanism of Wang et al. [43] is an im-

Algorithm 1: Sparse Vector (with Gap [43]) For the original SVT [10,32], remove the box on Line 8

input : $q$ : a list of queries of global sensitivity 1

$D$ : database; $\epsilon$ : privacy budget; $T$ : threshold

$k$ : bound on above-$T$ answers to output

1 function SparseVector($q$, $D$, $T$, $k$) :$
2 \epsilon_0 \leftarrow \epsilon/2$; $\epsilon_1 \leftarrow \epsilon/(4k)$; count $\leftarrow 0$
3 $\eta \leftarrow \text{Lap}(1/\epsilon_0)$
4 $T \leftarrow T + \eta$
5 foreach $i \in \{1, \ldots, \text{len}(q)\}$ do
6 $\eta_i \leftarrow \text{Lap}(1/\epsilon_1)$
7 if $q_i(D) + \eta_i - \tilde{T} \geq 0$ then
8 output: $T$, gap = ($q_i(D) + \eta_i - \tilde{T}$)
9 count $\leftarrow$ count + 1
10 else
11 output: $\perp$
12 if count $\geq k$ then break

provenement over the original Sparse Vector Technique (SVT)
To create a local alignment for each query, we can use a mechanism whose input is a sequence of queries with sensitivity 1. It estimates the identities of the first k queries whose values are over a public threshold T. The original mechanism adds noise to the threshold and to each query. Then as it processes the queries in order, it returns T if the noisy query is above the noisy threshold and ⊥ if it is below. It stops after outputting T for k times. Wang et al. [43] showed that in addition to outputting T, it can also output at no extra cost to privacy an estimate of the gap between the noisy query (for which T was produced) and the noisy threshold. Both versions of the mechanism are shown in Algorithm 1, with the changes proposed by Wang et al. [43] appearing in the box on line 8 (note it re-uses the same random variables used by M). Let κ = |I| and ν = |I|. If κ ≤ 2ν, there are countably many distinct databases, D, H, with the difference in noisy values between the largest and second largest queries. This extra information comes at no additional cost to privacy (meaning that the original Noisy Max mechanism threw away useful information). We then generalize this result to a noisy top-k mechanism and show how the free gap information can be used to improve algorithms that use this building block. Specifically, when an algorithm asks for the identities of the top-k queries and then asks for the noisy values of these queries (i.e., measurements), we show that the free gap information can be combined with the measurements to improve their accuracy.

5. IMPROVING NOISY MAX

In this section, we present novel variations of the Noisy Max mechanism [19]. Given a list of queries with sensitivity 1 of the queries, there are at most 2k above-the-threshold queries answered so |I| ≤ k. Thus all local alignments have cost ≤ ε.

5.1 Noisy-Max-with-Gap

Both the original Noisy Max mechanism and our improved Noisy-Max-with-Gap are shown in Algorithm 2 (for the original version, remove the boxes). The function arg max2 in the pseudocode returns the index of the largest and second largest elements. Both mechanisms take in n queries, each having sensitivity 1. Both mechanisms add the same amount of Laplace noise to each query. The original mechanism then returns the index of the query with the largest noisy value. Our improved version returns this index along with the difference in noisy values between the largest and second largest queries. Our claim is that both mechanisms have exactly the same privacy cost, yet Noisy-Max-with-Gap has more utility because it provides more information.

Algorithm 2: Noisy-Max-with-Gap (for the original Noisy Max, remove the boxes)

input: q: a list of n queries of global sensitivity 1
D: database, ε: privacy budget
1 function NoisyMax(q, D, ε):
2 foreach i ∈ {1, ..., n} do
3 \( \eta_i \leftarrow \text{Lap}(2/\epsilon) \)
4 \( \hat{q}_i \leftarrow q_i(D) + \eta_i \)
5 best, runnerup \leftarrow \text{arg max2}(\hat{q}_1, ..., \hat{q}_n) \)
6 gap \leftarrow \hat{\eta}_{\text{best}} - \hat{\eta}_{\text{runnerup}} \)
7 return best, gap
The original Noisy Max mechanism satisfies ε-differential privacy. In the special case that all the \( q_i \) are counting queries\(^3\), then it satisfies \( \varepsilon/2 \)-differential privacy. We will show the same properties for Noisy-Max-with-Gap. However, first it is important to discuss the difference between the theoretical analysis of Noisy Max\(^19\) and its practical implementation on finite-precision computers.

**Implementation issues.** The analysis of the original Noisy Max mechanism assumed the use of true Laplace noise (i.e., a continuous distribution) so that ties are impossible between the largest and second largest noisy queries\(^19\). On finite precision computers, ties are possible (breaking the privacy proof\(^19\)) and furthermore, common implementations of Laplace noise are known to be insecure\(^36\). One way to generate a secure approximation to the Laplace distribution on finite-precision computers is to select a small base \( \gamma \) (e.g., a negative power of two), round a query answer to the nearest multiple of \( \gamma \), and hence the privacy guarantees will also fail to the approximation\(^6\). Thus this is an upper bound on the probability that the differential privacy guarantees will fail. Typically, one would expect \( \gamma \) to be close to machine epsilon (e.g., \( \approx 2^{-52} \)) so the probability of a tie is negligible. In this section we will also analyze our algorithms under the assumption of continuous noise, and hence the privacy guarantees will also fail with this negligible probability \( \delta \) (formally, this known as approximate \((\varepsilon, \delta)\)-differential privacy\(^15\)).

**Local alignment.** To prove the privacy of Algorithm\(^2\), we need to create local alignment functions. Note that our mechanism only uses \( n \) random variables. Let \( H = (\eta_1, \eta_2, \ldots, \eta_n) \) where \( \eta_i \) is the noise that should be added to the \( i \)th query. We view the output \( \omega = (j_\omega, g_\omega) \) as a pair where \( j_\omega \) is the index of the noisy maximizer and \( g_\omega \) is the noisy gap between the maximizer and the runner-up. As discussed in the implementation issues, we will base our analysis on continuous noise (so that there are no ties when selecting the maximizer). Thus the noisy gap is strictly positive (i.e., \( g_\omega > 0 \)). Let \( \mathcal{I} = \{ j_\omega \} \) be the singleton set containing the maximizer and set \( \mathcal{I} = \{ 1, \ldots, n \} \setminus \mathcal{I} \) (i.e., the first \( n \) integers with \( j_\omega \) removed). Let \( D \sim D' \) be any adjacent pair of databases and let \( q_i \) (resp., \( q'_i \)) be the value of the \( i \)th query when evaluated on \( D \) (resp., \( D' \)). For \( H \in S_{D, \omega} \) define a local alignment \( \phi_{D, D', \omega}(H) = H' = (\eta'_1, \eta'_2, \ldots) \) as

\[
\eta'_i = \begin{cases} 
\eta_i & i \neq j_\omega \\
\eta_i + q_i - g'_i + \max_{q' \in \mathcal{I}} (q' + \eta_i) - \max_{q' \in \mathcal{I}} (q + \eta_i) & i = j_\omega 
\end{cases}
\]

The main idea behind this alignment is that we don’t want to change the noise added to the losing queries. This means that with database \( D \) and noise \( H \), the value of the runner up query \( q' \) is \( q' \) while with database \( D' \) and noise \( H' \) it is \( \max_{q' \in \mathcal{I}} (q' + \eta'_i) \), which is the same as \( \max_{q' \in \mathcal{I}} (q' + \eta_i) \) due to the first line of Equation\(^3\). Under \( D' \) we only want to change the noise of query \( j_\omega \) so it remains the maximizer and so that its noisy gap with the second best query (i.e., \( q'_\omega + \eta'_\omega - \max_{q' \in \mathcal{I}} (q' + \eta_i) \)) is equal to the noisy gap under \( D \) (i.e., \( q_\omega + \eta_\omega - \max_{q' \in \mathcal{I}} (q' + \eta_i) \)). Solving for \( \eta'_\omega \) results in the second line of Equation\(^3\). Please note \( \max_{q' \in \mathcal{I}} (q' + \eta_i) - \max_{q' \in \mathcal{I}} (q' + \eta_i) \) is equal to \( \max_{q' \in \mathcal{I}} (q' + \eta_i) - \max_{q' \in \mathcal{I}} (q' + \eta_i) \leq 1, \) so the runner-up value changes by at most 1. The \( \phi_{D, D', \omega} \) we defined are clearly acyclic. The following lemma establishes that these functions are local alignments, finds their costs, and shows Noisy-Max-with-Gap has the same exact privacy properties as the original Noisy Max mechanism.

**Lemma 4.** Let \( M \) be the Noisy-Max-with-Gap algorithm. For all \( D \sim D' \) and \( \omega \), the functions \( \phi_{D, D', \omega} \) defined above are local alignments for \( M \). The cost of each \( \phi_{D, D', \omega} \) is at most \( \epsilon \) and Noisy-Max-with-Gap satisfies \( \epsilon \)-differential privacy. Additionally, if all of the queries are counting queries the privacy properties are stronger: the cost of each \( \phi_{D, D', \omega} \) is bounded by \( \epsilon/2 \) and Noisy-Max-with-Gap satisfies \( (\epsilon/2) \)-differential privacy.

The proof is in the Appendix.

5.2 Noisy-Top-K-with-Gap

The Noisy Max mechanism can easily be extended to a top-\( k \) version: use \( k \) times as much noise and return the indices of the top \( k \) noisy queries. We show that free noisy gap information can be included as well. Let \( \text{arg max}_{k+1} \) be the function that returns the indices of the \( k+1 \) largest elements (in decreasing order). The following mechanism, which we call Noisy-Top-K-with-Gap, releases the estimated indices of the top-\( k \) queries along with noisy gaps between them.

**Algorithm 3:** Noisy-Top-K-with-Gap

\begin{algorithm}
\begin{algorithmic}
\State \textbf{input:} \( q \): a list of \( n \) queries of global sensitivity 1
\State \( D \): database, \( k \): # of indices, \( \epsilon \): privacy budget
\Function{NoisyTopK}{\( q, D, k, \epsilon \)}
\ForEach{\( i \in \{1, \ldots, n\} \)}
\State \( \eta_i \leftarrow \text{Lap}(2k/\epsilon) \)
\State \( q_i \leftarrow q_i(D) + \eta_i \)
\EndFor
\State \( \{j_1, \ldots, j_{k+1}\} \leftarrow \text{arg max}_{k+1}(\tilde{q}_1, \ldots, \tilde{q}_n) \)
\ForEach{\( i \in \{1, \ldots, k\} \)}
\State \( g_i \leftarrow \tilde{q}_{j_i} - \tilde{q}_{j_{i+1}} \quad \text{// \ ith gap} \)
\EndFor
\State \Return{\( \{j_1, q_1, \ldots, j_k, q_k\} \) \quad // \ ith gap}
\EndFunction
\end{algorithmic}
\end{algorithm}

We note that keeping the noisy gaps hidden does not decrease the privacy cost (same as with Noisy-Max-with-Gap). Furthermore, this algorithm gives estimates of the pairwise gaps between any pair of the \( k \) queries it selects. For example, suppose we are interested in estimating the gap between the \( a \)th largest and \( b \)th largest queries (where \( a < b \leq k \)). This is equal to \( \sum_{i=a}^{b-1} g_i \) because: \( \sum_{i=a}^{b-1} g_i = \sum_{i=a}^{b-1} g_i = \sum_{i=a}^{b-1} g_i = \tilde{q}_{j_a} - \tilde{q}_{j_b} \) and hence its variance is \( \text{Var}(q_{j_a} - q_{j_b}) = 16k^2/\epsilon^2 \). We first prove that the mechanism satisfies \( \epsilon \)-differential privacy and then in Section\(^5.3\) we show how to use this gap information.

**Local alignment.** We prove privacy by creating a local alignment for each possible pair \( D \sim D' \) and output \( \omega \). First
let $H = (\eta_1, \eta_2, \ldots, \eta_n)$ where $\eta_i$ is the noise that should be added to the $i$th query (note that the algorithm only uses $n$ random variables). As discussed in the implementation notes at the beginning of this section, we analyze the case of continuous distributions, so that the probability of ties among the top-$k$ noisy queries is 0. Thus each gap is positive: $q_i > 0$.

Let $I_k = \{j_1, \ldots, j_k\}$ and $I_n = \{1, \ldots, n\} \setminus I_k$. That is, $I_k$ is the index set of the $k$ largest queries and $I_n$ is the index set of all other queries. For $H \in S_{D,\omega}$ define

$$
\phi_{D,D',\omega}(H) = H' = (\eta_1, \eta_2, \ldots)
$$

Note that it is of the same form as the local alignment function (3); the only difference is that $I_n$ now has $k$ elements instead of 1. The idea behind this local alignment is similar: we want to keep the noise of the losing queries the same (when the input is $D$ or its neighbor $D'$). But, for each of the $k$ selected queries, we want to modify its noise to account for how much the query changes ($q_i - q_j$) as we go from $D$ to $D'$ and how much the value of the $k + 1$th noisy query (i.e., the value of the best losing noisy query) changes, which is $\max(q_i + \eta) - \max(q_j + \eta)$.

LEMMA 5. Let $M$ be the Noisy-Top-K-with-Gap mechanism. For all $D \sim D'$ and $\omega$, the functions $\phi_{D,D',\omega}$ defined above are local alignments for $M$. Furthermore the alignment cost is at most $\epsilon$ and Noisy-Top-K-with-Gap satisfies $\epsilon$-differential privacy. Additionally, if all of the queries are counting queries the privacy properties are stronger: the alignment cost is bounded by $\epsilon/2$ and Noisy-Top-K-with-Gap satisfies $(\epsilon/2)$-differential privacy.

The proof is in the Appendix.

5.3 Utilizing Gap Information

Let us consider one scenario that takes advantage of the gap information. Suppose a data analyst is interested in the accuracy of these measurements. Noisy-Top-K-with-Gap and postprocessing to improve the accuracy of these measurements. Let $\alpha_i = q_i(D) + \xi_i$ be the noisy measurements, where $\xi_i$ is a Laplace$(2k/\epsilon)$ random variable. Let $q_1, \ldots, q_k$ be the top-$k$ queries that were returned by Algorithm 3 and let $g_i = q_i(D) - q_{i+1}(D) + \eta_i - \eta_{i+1}$ be the gap variables returned by the mechanism, where each $\eta_i$ is a Laplace$(4k/\epsilon)$ random variable (recall the mechanism was run with a privacy budget of $\epsilon/2$). Let $q = [q_1(D), \ldots, q_k(D)]^T$, $\alpha = [\alpha_1, \ldots, \alpha_k]^T$ and $g = [g_1, \ldots, g_{k-1}]^T$. Our goal is then to find the best linear unbiased estimate (BLUE) $\beta$ of $q$ in terms of the measurements $\alpha$ and gap information $g$.

\[ X = \frac{1}{5k} \begin{pmatrix} 1 + 4k & 1 & \cdots & 1 \\ 1 & 1 + 4k & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 + 4k \end{pmatrix} \]

\[ Y = \frac{1}{5k} \begin{pmatrix} [k - 1, k - 2, \ldots, 1] & [0, 0, \ldots, 0] \\ [k - 1, k - 2, \ldots, 1] & [k, k, \ldots, 0] \\ \vdots & \vdots & \ddots & \vdots \\ [k - 1, k - 2, \ldots, 1] & [k, k, \ldots, k] \end{pmatrix} \]

For proof, see the Appendix.

Even though this is a matrix multiplication, it is easy to see that it translates into the following algorithm that is linear in $k$:

1. Compute $\alpha_1 = \sum_{i=1}^k \alpha_i$ and $p = \sum_{i=1}^{k-1}(k - i)q_i$.
2. Set $p_0 = 0$. For $i = 1, \ldots, k - 1$ compute the prefix sum $p_i = \sum_{j=1}^i q_i = p_{i-1} + q_i$.
3. For $i = 1, \ldots, k$, set $\beta_i = (\alpha + 4k\alpha_i + p - kp_{i-1})/5k$.

Now, each $\beta_i$ is an estimate of the value of $q_i(D)$. How does it compare to the direct measurement $\alpha_i$ (which has variance $8k^2/\epsilon^2$)? The following result compares the expected error of using only the direct measurements (i.e., $\alpha$ only).

COROLLARY 1. For all $i = 1, \ldots, k$, we have

\[ \frac{E((\beta_i - q_i(D))^2)}{E((\alpha_i - q_i(D))^2)} = \frac{\text{Var}(\beta_i)}{\text{Var}(\alpha_i)} = 4k + 1 \frac{4k^2}{5k}. \]

For proof, see the Appendix. Since $\lim_{k \to \infty} \frac{4k^2}{5k} = \frac{4}{5}$, we see that the free gap information helps achieve an improvement of up to 20%. Our experiments in Section 7 confirm this theoretical result.

6. IMPROVING SPARSE VECTOR

In this section we return to the Sparse Vector technique. We propose a novel variant that can answer more queries than both the original Sparse Vector and the Sparse-Vector-with-Gap of Wang et al. [43]. We also discuss how the free gap information can be used.

6.1 Adaptive-Sparse-Vector-with-Gap

The Sparse Vector techniques are designed to solve the following problem in a privacy-preserving way: given a stream of queries (with sensitivity 1), find the first $k$ queries whose answers are larger than a public threshold $T$. This is done by adding noise to the queries and threshold and finding the first $k$ queries whose noisy answers exceed the noisy threshold. Sometimes this procedure creates a feeling of regret – if these $k$ queries are much larger than the threshold, we could have used more noise (hence consumed less privacy budget) to achieve the same result. In this section, we show that Sparse Vector can be made adaptive – so that it will probably use more noise (less privacy budget) for the larger queries. This means if the first $k$ queries are very large, it will still have privacy budget left over to find additional queries that are likely to be over the threshold. Our Adaptive Sparse Vector is shown in Algorithm 6.
Algorithm 4: Adaptive-Sparse-Vector-with-Gap

\textbf{input}: \( q \): a list of queries of global sensitivity 1
\( D \): database, \( \epsilon \): privacy budget, \( T \): threshold
\( k \): adaptive SVT can answer up to \( 2k \) queries.

\begin{algorithmic}[1]
\Function{AdaptiveSparseVector}{\( q, D, T, k, \epsilon \)}
\State \( \epsilon_0 \leftarrow \epsilon/2; \quad \epsilon_1 \leftarrow \epsilon/(8k); \quad \epsilon_2 \leftarrow \epsilon/(4k); \quad \sigma \leftarrow 2\sqrt{2\epsilon_1} \)
\State \( \widetilde{T} \leftarrow T + \eta \)
\State \( \text{cost} \leftarrow \epsilon_0 \)
\ForEach{\( i \in \{1, \ldots, \text{len}(q)\} \)}
\State \( \xi_i \leftarrow \text{Lap}(1/\epsilon_1); \quad \eta_i \leftarrow \text{Lap}(1/\epsilon_2) \)
\If{\( q_i(D) + \xi_i - \widetilde{T} \geq \sigma \)}
\State \textbf{output}: \( \top, q_i(D) + \xi_i - \widetilde{T}, \text{bud}_\text{used} = 2\epsilon_1 \)
\State \text{cost} \leftarrow \text{cost} + 2\epsilon_1 \)
\ElseIf{\( q_i(D) + \eta_i - \widetilde{T} \geq 0 \)}
\State \textbf{output}: \( \top, q_i(D) + \eta_i - \widetilde{T}, \text{bud}_\text{used} = 2\epsilon_2 \)
\Else
\State \textbf{output}: \( \bot, \text{bud}_\text{used} = 0 \)
\EndIf
\EndFor
\If{\text{cost} > \epsilon - 2\epsilon_2} \text{break} \EndIf
\EndFunction
\end{algorithmic}

The main idea behind this algorithm is that, given a target privacy budget \( \epsilon \) and a \( k \), the algorithm will create three noise scale parameters: \( \epsilon_0, \epsilon_1, \epsilon_2 \) (with \( \epsilon_1 < \epsilon_2 \)) that are used as follows. First, the algorithm adds Laplace(1/\( \epsilon_0 \)) noise to the threshold and consumes \( \epsilon_0 \) of the privacy budget. Then, when a query comes in, the algorithm first adds a lot of noise (i.e., Laplace(1/\( \epsilon_1 \))) to the query. The first “if” branch checks if this value is much larger than the noisy threshold (i.e., checks if the gap is \( \geq \sigma \) for some \( \sigma \)). If so, then it outputs the following three items: (1) the noisy threshold, and (3) the amount of privacy budget used (which is 2\( \epsilon_1 \)). The use of alignments will show that failing this “if” branch consumes no privacy budget. If the first “if” branch fails, then the algorithm adds more moderate noise to the query answer (i.e., Laplace(1/\( \epsilon_2 \))). If this noisy value is larger than the noisy threshold, the algorithm outputs: (2') the noisy gap, and (3') the amount of privacy budget consumed (i.e., 2\( \epsilon_2 \)). If this “if” condition also fails, then the algorithm outputs: (1') \( \bot \) and (2') the privacy budget consumed (0 in this case).

To summarize, for each query, if the top branch succeeds then the privacy budget consumed is 2\( \epsilon_1 \), or if the middle branch succeeds, the privacy cost is 2\( \epsilon_2 \), and if the bottom branch succeeds, there is no additional privacy cost. These properties can be easily seen by focusing on the local alignment - if \( M(D, H) \) produces a certain output, how much does \( H \) need to change to get a noise vector \( H' \) so that \( M(D', H') \) returns the same exact output.

\textbf{Local alignment}. To create a local alignment for each pair \( D \sim D' \), let \( H = (\eta_1, \xi_1, \eta_2, \xi_2, \ldots) \), where \( \eta \) is the noise added to the threshold \( T \), and \( \xi \) (resp. \( \eta \)) is the noise that should be added to the \( i \)-th query \( q_i \) in Line 6 (resp. Line 11), if execution ever reaches that point. We view the output \( \omega = (\omega_1, \ldots, \omega_j) \) as a variable-length sequence where each \( \omega \) is either \( \bot \) or a nonnegative gap (we omit the \( \top \) as it is redundant), together with a tag \( \omega \in \{0, 2\epsilon_1, 2\epsilon_2\} \) indicating which branch \( \omega \) is from (and the privacy budget consumed to output \( \omega \)). Let \( I_S = \{i \mid \text{tag}(\omega_i) = 2\epsilon_1\} \) and \( J_S = \{i \mid \text{tag}(\omega_i) = 2\epsilon_2\} \). That is, \( I_S \) is the set of indexes where the output is a gap from the top branch, and \( J_S \) is the set of indexes where the output is a gap from the middle branch. For \( H \in \mathbb{SD}_\omega \) define \( \phi_{D,D',\omega}(H) = H' = (\eta', \xi', \xi', \eta', \eta', \ldots) \) where
\[
\eta' = \eta + 1,
\]
\[
\xi' = \begin{cases} 
(\xi_i + 1 + q_i - q_i'), & \text{if } i \in I_S \\
(\xi_i, \eta_i + 1 + q_i - q_i'), & \text{otherwise}
\end{cases}
\]
in other words, we add 1 to the noise that was added to the threshold (thus if the noisy \( q(D) \) failed a specific branch, the noisy \( q(D') \) will continue to fail it because of the higher noisy threshold). If a noisy \( q(D) \) succeeded in a specific branch, we adjust the query’s noise so that the noisy version of \( q(D') \) will succeed in that same branch.

\textbf{Lemma 6}. Let \( M \) be the Adaptive-Sparse-Vector-with-Gap mechanism. For all \( D \sim D' \) and \( \omega \), the functions \( \phi_{D,D',\omega} \) defined above are local alignments for \( M \). Furthermore the alignment cost of each \( \phi_{D,D',\omega} \) is at most \( \epsilon \) and Adaptive-Sparse-Vector-with-Gap satisfies \( \epsilon \)-differential privacy.

The proof can be found in the Appendix. Clearly this algorithm can be easily extended with multiple additional “if” branches. For simplicity we do not include such variations.

In our setting, \( \epsilon_1 = \epsilon_2/2 \) so, theoretically, if queries are very far from the threshold, our adaptive version of Sparse Vector will be able to find twice as many of them as the non-adaptive version.

6.2 Utilizing Gap Information

When Sparse-Vector-with-Gap or Adaptive-Sparse-Vector-with-Gap returns a gap \( \gamma_i \) for a query \( q_i \), we can add it to the public threshold \( T \). This means \( \gamma_i + T \) is an estimate of the value of \( q_i(D) \). We can ask two questions: how can we improve the accuracy of this estimate and how can we be confident that the true answer \( q_i(D) \) is really larger than the threshold \( T \)?

\textbf{Lower Confidence Interval}. Recall that the randomness in the gap in Sparse-Vector-with-Gap (Algorithm [1]) is of the form \( \eta - \eta \) where \( \eta \) and \( \eta \) are independent zero mean Laplace variables with scale \( 1/\epsilon_0 \) and \( 1/\epsilon_1 \). The random variable \( \eta - \eta \) has the following lower tail bound:

\textbf{Lemma 7}. For any \( t \geq 0 \) we have
\[
\Pr(\eta - \eta \geq t) = \begin{cases} 
\frac{2e^{-t\epsilon_0} - e^{-t\epsilon_1}}{1 - (2e^{0.5\epsilon_0/2\epsilon_1})e^{-0.5t\epsilon_1}}, & \epsilon_0 \neq \epsilon_1 \\
1 - (2e^{0.5\epsilon_0/2\epsilon_1})e^{-0.5t\epsilon_1}, & \epsilon_0 = \epsilon_1
\end{cases}
\]

For proof see the Appendix. For any confidence level, say 95%, we can use this result to find a number \( t_{95} \) such that \( \Pr(\eta - \eta \geq t_{95}) = .95 \). This is a lower confidence bound, so that the true value \( q_i(D) \) is \( \geq t_{95} \) with probability 0.95.

\textbf{Improving accuracy}. To improve accuracy, one can split the privacy budget \( \epsilon \) in half. The first half, \( \epsilon_1 \equiv \epsilon/2 \) can be
used to run Sparse-Vector-with-Gap (or Adaptive-Sparse-Vector-with-Gap) and the second half \( \epsilon_2 \equiv \epsilon/2 \) can be used to provide an independent noisy measurement of the selected queries (i.e., if we selected \( k \) queries, we add \( \text{Laplace}(k/\epsilon_2) \) noise to each one). Suppose the selected queries are \( q_1, \ldots, q_k \), the noisy gaps are \( \gamma_1, \ldots, \gamma_k \) and the independent noisy measurements are \( \alpha_1, \ldots, \alpha_k \).

The noisy estimates can be combined together with the gaps to get improved estimates \( \beta_i \) of \( q_i(D) \) in the standard way (inverse-weighting by variance):

\[
\beta_i = \left( \frac{\alpha_i}{\text{Var}(\alpha_i)} + \frac{\gamma_i + E}{\text{Var}(\gamma_i)} \right) \left( \frac{1}{\text{Var}(\alpha_i)} + \frac{1}{\text{Var}(\gamma_i)} \right).
\]

As shown in [32], the optimal budget allocation between threshold noise and query noises within SVT (and therefore also Sparse-Vector-with-Gap) is the ratio \( 1 : (2k)^{3/2} \). Under this setting, we have \( \text{Var}(\gamma_i) = 8(1 + (2k)^{3/2})^3/\epsilon^2 \). Also, we know \( \text{Var}(\alpha_i) = 8k^2/\epsilon^2 \). Therefore,

\[
\frac{E(|\beta_i - q_i|^2)}{E(|\alpha_i - q_i|^2)} = \frac{\text{Var}(\beta_i)}{\text{Var}(\alpha_i)} = \frac{(1 + \sqrt[3]{4k^3})^3}{(1 + \sqrt[3]{4k^3})^3 + k^2} < 1.
\]

Since \( \lim_{k \to \infty} \frac{(1 + \sqrt[3]{4k^3})^3}{(1 + \sqrt[3]{4k^3})^3 + k^2} = \frac{4}{3} \), the improvement in accuracy approaches 20% as \( k \) increases. Our experiments confirm this improvement.

7. EXPERIMENTS

We now evaluate the algorithms proposed in this paper.

7.1 Datasets

For evaluation, we used the two real datasets from [32]: BMP-POS, Kosarak and a synthetic dataset T4010D100K created by the generator from the IBM Almaden Quest research group. These datasets are collections of transactions (each transaction is a set of items). In our experiments, the queries correspond to the counts of each item (i.e., how many transactions contained item #237?). The statistics of the datasets are listed below.

| Dataset       | # of Records | # of Unique Items |
|---------------|--------------|-------------------|
| BMS-POS       | 515,597      | 1,657             |
| Kosarak       | 990,002      | 41,270            |
| T4010D100K    | 100,000      | 942               |

7.2 Gap Information + Postprocessing

The first set of experiments is to measure how gap information can help us improve estimates in selected queries. We use the setup of Sections 5.3 and 6.2. That is, a data analyst splits the privacy budget \( \epsilon \) in half. She uses the first half to select \( k \) queries (using Noisy-Top-K-with-Gap or Sparse-Vector-with-Gap) and then uses the remaining privacy budget to obtain independent noisy measurements of each selected query.

If one were unaware that gap information came for free, one would just use those noisy measurements as estimates for the query answers. The error of this approach is the gap-free baseline. However, since the gap information does come for free, we can use the postprocessing described in Sections 5.3 and 6.2 to improve accuracy (we call this latter approach Sparse-Vector-with-Gap with Measures and Noisy-Top-K-with-Gap with Measures).

We first evaluate the percent improvement in mean squared error (MSE) of the postprocessing approach compared to the gap-free baseline and compare this improvement to our theoretical analysis. As discussed in Section 6.2, we set the budget allocation ratio within the Sparse-Vector-with-Gap algorithm (i.e., the budget allocation between the threshold and queries in Algorithm 1) to be \( 1 : (2k)^{3/2} \) - such a ratio is recommended in [32] for the original Sparse Vector.

Our theoretical analysis in Sections 5.3 and 6.2 suggested that the improvements can reach up to 20% as \( k \) increases. This is confirmed in Figures 1a, 1b, 1c for Sparse-Vector-with-Gap and Figures 1d, 1e, 1f for our top-k algorithm. These figures plot the theoretical and empirical percent improvement in MSE as a function of \( k \) and show the power of the free gap information.

We also generated corresponding plots where \( k \) is held fixed and the total privacy budget \( \epsilon \) is varied. For Sparse-Vector-with-Gap, Figures 2a, 2b, and 2c confirm that this improvement is stable for different \( \epsilon \) values. For our Top-k algorithm, Figures 2d, 2e, 2f confirm that this improvement is also stable for different values of \( \epsilon \).

7.3 Adaptive Sparse Vector with Gap

In this subsection, we present the evaluation of our novel Adaptive-Sparse-Vector-with-Gap to show that it can answer more above-threshold queries than Sparse Vector and Sparse-Vector-with-Gap at the same privacy cost. First note that Sparse Vector and Sparse-Vector-with-Gap both answer exactly the same amount of queries, so we only need to compare Adaptive-Sparse-Vector-with-Gap to the original Sparse Vector [19, 32].

In both algorithms, the budget allocation between the threshold noise and query noise (e.g., \( \epsilon_0 \) and \( \epsilon_1 \) in Algorithm 1) is set according to the ratio \( 1 : (2k)^{3/2} \) instead of \( 1 : 1 \), following recommendations for SVT by Lyu et. al. [32] (the privacy properties of our algorithms do not change). The threshold is set to be the 95% quantile in each dataset and all reported numbers are averaged over 20,000 runs.

Number of queries answered. We first compare the number of queries answered by each algorithm as the parameter \( k \) is varied from 2 to 25 with a privacy budget of \( \epsilon = 0.7 \) (results for other settings of the total privacy budget are similar). The results are shown in Figure 3a, 3b, and 3c. In each of these bar graphs, the left (blue) bar is the number of answers returned by Sparse Vector and the right bar is the number of answers returned by Adaptive-Sparse-Vector-with-Gap. This right bar is broken down into two components: the number of queries returned from the top “if” branch (corresponding to queries that were significantly larger than the threshold even after a lot of noise was added) and the number of queries returned from the middle “if” branch. Queries returned from the top branch of Adaptive-Sparse-Vector-with-Gap have less privacy cost than the queries returned by Sparse Vector. Queries returned from the middle branch of Adaptive-Sparse-Vector-with-Gap have the same privacy cost as in Sparse Vector.

In the case of the BMS-POS and T40100K datasets, we see that most queries are answered in the top branch of Adaptive-Sparse-Vector-with-Gap, meaning that the above-
Figure 1: Improvement percentage of Mean Square Error for Sparse Vector with Measures and Noisy Top-K with Measures under different datasets and settings. Privacy budget $\epsilon = 0.7$ and $x$-axis: $k$.

Figure 2: Improvement percentage of Mean Square Error for Sparse-Vector-with-Gap with Measures and Noisy Top-K with Measures under different datasets and settings. $k$ is set to 10 and $x$-axis: $\epsilon$. 
threshold queries are generally large (much larger than the threshold). Since Adaptive-Sparse-Vector-with-Gap uses more noise in the top branch, it uses less privacy budget to answer those queries and uses the remaining budget to provide additional answers (up to an average of 15 more answers when \( k \) was set to 25).

In the kosarak dataset, we see that roughly half the queries are returned in the top branch and half in the middle branch of Adaptive-Sparse-Vector-with-Gap. This suggests that many of the above-threshold queries were actually close to the threshold and so there should be less benefit in adaptivity. Figure 3b confirms this. Still, the adaptive algorithm is able to answer more queries than the original.

**Precision.** Although the adaptive algorithm can answer more above-threshold queries than the original, one can still ask the question of whether the returned queries really are above the threshold. Thus we can look at the precision of the returned results (the fraction of returned queries that are actually above the threshold). One would expect that the precision of Adaptive-Sparse-Vector-with-Gap should be less than that of Sparse Vector, because the adaptive version can use more noise when processing queries. In Figures 3d, 3e, and 3f we compare the precision of the two algorithms. Generally we see very little difference in precision — when returning at least 25 above-threshold answers, the adaptive algorithm has less than one additional false positive (on average).

In this section, we prove Lemma 1 which was used to establish the privacy properties of the algorithms we proposed. The proof of the lemma requires a more general theorem for working with randomness alignment functions. We explicitly list all of the conditions needed for the sake of reference (many prior works had incorrect proofs because they did not have such a list to follow).

In the general setting, the method of randomness alignment requires the following steps.

1. For each pair of adjacent databases \( D \sim D' \) and \( \omega \in \Omega \), define a randomness alignment \( \phi_{D,D',\omega} \) or local alignment functions \( \phi_{D,D',\omega} : S_D,\omega \rightarrow S_{D'},\omega \) (see notation in Table 1). In the case of local alignments this involves proving that if \( M(D,H) = \omega \) then \( M(D',\omega) = \omega \).
2. Show that \( \phi_{D,D'} \) (or all the \( \phi_{D,D',\omega} \)) is one-to-one (it does not need to be onto). That is, if we know \( D, D', \omega \) and we are given the value \( \phi_{D,D'}(H) \) (or \( \phi_{D,D',\omega}(H) \)), we can obtain the value \( H \).
3. For each pair of adjacent databases \( D \sim D' \), bound the alignment cost of \( \phi_{D,D'} \) (\( \phi_{D,D'} \) is either given or constructed by piecing together the local alignments). Bounding the alignment cost means the following: If \( f \) is the density (or probability mass) function of \( H \), find a constant \( a \) such that \( f(H)/f(\phi_{D,D'}(H)) \leq a \) for all \( H \) (except a set of measure 0). In the case of local alignments, one can instead show the following. For all \( \omega \), and adjacent \( D \sim D' \) the ratio \( f(H)/f(\phi_{D,D',\omega}(H)) \leq a \) for all \( H \) (except on a set of measure 0).
4. Bound the change-of-variables cost of \( \phi_{D,D'} \) (only necessary when \( H \) is not discrete). One must show that the Jacobian of \( \phi_{D,D'} \), defined as \( J_{\phi_{D,D'}} = \frac{\partial \phi_{D,D'}}{\partial H} \), exists.

8. **GENERAL RANDOMNESS ALIGNMENT AND PROOF OF LEMMA 1**

![Figure 3: Results for Adaptive-Sparse-Vector-with-Gap under different k's. Privacy budget \( \epsilon = 0.7 \) and x-axis: \( k \).](image-url)
(i.e. \(\phi_{D,D'}\) is differentiable) and is continuous except on a set of measure 0. Furthermore, for all pairs \(D \sim D'\), show the quantity \([\det J_{\phi_{D,D'}}]\) is lower bounded by some constant \(b > 0\). If \(\phi_{D,D'}\) is constructed by piecing together local alignments \(\phi_{D,D',\omega}\) then this is equivalent to showing the following (i) \([\det J_{\phi_{D,D',\omega}}]\) is lower bounded by some constant \(b > 0\) for every \(D \sim D'\) and \(\omega\) and (ii) for each \(D \sim D'\), the set \(\Omega\) can be partitioned into countably many disjoint measurable sets \(\Omega = \bigcup_i \Omega_i\) such that whenever \(\omega\) and \(\omega'\) are in the same partition, then \(\phi_{D,D',\omega}\) and \(\phi_{D,D',\omega'}\) are the same function. Note that this last condition (ii) is equivalent to requiring that the local alignments must be defined without using the axiom of choice (since non-measurable sets are not constructible) and for each \(D \sim D'\), the number of distinct local alignments is countable. That is, the set \(\{\phi_{D,D',\omega} \mid \omega \in \Omega\}\) is countable (i.e., for many choices of \(\omega\) we get the same exact alignment function).

**Theorem 4.** Let \(M\) be a randomized algorithm that terminates with probability 1 and suppose the number of random variables used by \(M\) can be determined from its output. If, for all pairs of adjacent databases \(D \sim D'\), there exist randomness alignment functions \(\phi_{D,D'}\) (or local alignment functions \(\phi_{D,D',\omega}\) for \(\omega \in \Omega\)) that satisfy conditions 1 through 4 above, then \(M\) satisfies \(\ln(a/b)\)-differential privacy.

Proof. We need to show that for all \(D \sim D'\) and \(E \subseteq \Omega\),

\[
\mathbb{P}(S_{D,E}) \leq (a/b) \mathbb{P}(S_{D',E}).
\]

First we note that if we have a randomness alignment \(\phi_{D,D'}\), we can define corresponding local alignment functions as follows \(\phi_{D,D',\omega}(H) = \phi_{D,E}(H)\) (in other words, they are all the same). The conditions on local alignments are a superset of the conditions on randomness alignments, so for the rest of the proof we work with the \(\phi_{D,D',\omega}\).

Let \(\phi_1, \phi_2, \ldots\) be the distinct local alignment functions (there are countably many of them by Condition 4). Let \(E_i = \{\omega \in E \mid \phi_{D,D',\omega} = \phi_i\}\). By Conditions 1 and 2 we have that for each \(\omega \in E_i, \phi_i\) is one-to-one on \(S_{D,\omega}\) and \(\phi_i(S_{D,\omega}) \subseteq S_{D',\omega}\). Note that \(S_{D,E} = \cup_i E_i, S_{D,\omega} = \cup_i \omega_i, S_{D',E} = \cup_i E_i; S_{D',\omega}\). Furthermore, the sets \(S_{D,\omega}\) are pairwise disjoint for different \(\omega\) and the sets \(S_{D',\omega}\) are pairwise disjoint for different \(\omega\). It follows that \(\phi_i\) is one-to-one on \(S_{D,E_i}\) and \(\phi_i(S_{D,E_i}) \subseteq S_{D',\omega_i}\). Thus for any \(H' \in \phi_i(S_{D,E_i})\) there exists \(H \in S_{D,E_i}\) such that \(H = \phi_i^{-1}(H')\). By Conditions 3 and 4, we have \(f(H') = \frac{f(\phi_i^{-1}(H'))}{|\det J_{\phi_i}|} \leq a\) for all \(H \in S_{D,E_i}\) and \(\{\det J_{\phi_i}\} \geq b\) (except on a set of measure 0). Then the following is true:

\[
\mathbb{P}(S_{D,E_i}) = \int_{S_{D,E_i}} f(H) \, dH
\]

\[
= \int_{\phi_i(S_{D,E_i})} f(\phi_i^{-1}(H')) \frac{1}{|\det J_{\phi_i}|} \, dH'
\]

\[
\leq \int_{\phi_i(S_{D,E_i})} a f(H') \, dH' = a \int_{\phi_i(S_{D,E_i})} f(H') \, dH'
\]

\[
\leq \frac{a}{b} \int_{S_{D',E_i}} f(H') \, dH' = \frac{a}{b} \mathbb{P}(S_{D',E_i}).
\]

The second equation is the change of variables formula in calculus. The last inequality follows from the containment \(\phi_i(S_{D,E_i}) \subseteq S_{D',E_i}\) and the fact that the density \(f\) is non-negative. In the case that \(H\) is discrete, simply replace the density \(f\) with a probability mass function, change the integral into a summation, ignore the Jacobian term and set \(b = 1\). Finally, since \(E = \bigcup_i E_i\) and \(E_i \cap E_j = \emptyset\) for \(i \neq j\), we conclude that

\[
\mathbb{P}(S_{D,E}) = \sum_i \mathbb{P}(S_{D,E_i}) \leq \frac{a}{b} \sum_i \mathbb{P}(S_{D',E_i}) = \frac{a}{b} \mathbb{P}(S_{D',E}).
\]

We now present the proof of Lemma 1.

**Proof.** Let \(\phi_{D,D',\omega}(H) = H' = (\eta_{i,1}, \eta_{i,2}, \ldots)\). By acyclicity there is some permutation \(\pi\) under which \(\eta_{i,1} = \eta_{i,1}' - c\) where \(c\) is some constant depending on \(D \sim D'\) and \(\omega\). Thus \(\eta_{i,1}\) is uniquely determined by \(H'\). Now (as an induction hypothesis) assume \(\eta_{i,1}, \ldots, \eta_{i,j-1}\) are uniquely determined by \(H'\) for some \(j > 1\), then \(\eta_{i,j} = \eta_{i,j} - \psi_{i,1}(\eta_{i,1}, \ldots, \eta_{i,j-1})\), so \(\eta_{i,j}\) is also uniquely determined by \(H'\). Thus by strong induction \(H\) is uniquely determined by \(H'\), i.e., \(\phi_{D,D',\omega}\) is one-to-one. It is easy to see that with this ordering, \(J_{\phi_{D,D',\omega}}\) is an upper triangular matrix with 1’s on the diagonal. Since permuting variables doesn’t change \([\det J_{\phi_{D,D',\omega}}]\), we have \([\det J_{\phi_{D,D',\omega}}]\) = 1 since that is the determinant of upper triangular matrices. Furthermore, since

\[
\ln f(H) = \left(\prod_i \frac{\epsilon_i}{2}\right) \exp \left(-\sum_i \epsilon_i |\eta_i|\right),
\]

we have

\[
\ln \frac{f(H)}{f(\phi_{D,E}(H))} = \sum_i \epsilon_i (|\eta_i| - |\eta_i'|) \leq \sum_i \epsilon_i |\eta_i' - \eta_i| \leq \epsilon.
\]

The first inequality is by triangular inequality. The Lemma now follows by Theorem 4.

**9. CONCLUSIONS AND FUTURE WORK**

In this paper we introduced the Adaptive Sparse Vector with Gap, Noisy Max with Gap, and Noisy Top-K with Gap mechanisms, which were based on the observation that the classical Sparse Vector and Noisy Max mechanisms could release additional information at no cost to privacy. The construction and proof of these mechanisms was based on a simplified proof template we provided. We also provided applications of this free gap information.

Future directions include using this technique to design additional mechanisms as well as finding new applications for these mechanisms in fine-tuning the accuracy of data release algorithms that use differential privacy.

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**10. REFERENCES**

[1] M. Abadi, A. Chu, I. Goodfellow, H. B. McMahan, I. Mironov, K. Talwar, and L. Zhang. Deep learning with differential privacy. In *Proceedings of the 2016 ACM SIGSAC Conference on Computer and Communications Security*, pages 308–318. ACM, 2016.
[2] J. M. Abowd. The us census bureau adopts differential privacy. In Proceedings of the 24th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining, pages 2867–2867. ACM, 2018.
[3] A. Albarghouthi and J. Hsu. Synthesizing coupling proofs of differential privacy. Proceedings of the ACM on Programming Languages, 2(POPL):58, 2017.
[4] G. Bartho, M. Gaboardi, B. Gregoire, J. Hsu, and P.-Y. Strub. Proving differential privacy via probabilistic couplings. In IEEE Symposium on Logic in Computer Science (LICS), 2016.
[5] A. Beimel, K. Nissim, and U. Stemmer. Private learning and sanitization: Pure vs. approximate differential privacy. Theory of Computing, 12(1):1–61, 2016.
[6] R. Bhaskar, S. Luxman, A. Smith, and A. Thakurta. Discovering frequent patterns in sensitive data. In Proceedings of the 16th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, 2010.
[7] A. Bittau, U. Erlingsson, P. Maniatis, I. Mironov, A. Raghunathan, D. Lie, M. Rudominer, U. Kode, J. Timnes, and B. Seefeld. Prochlo: Strong privacy for analytics in the crowd. In Proceedings of the 26th Symposium on Operating Systems Principles, SOSP ’17, 2017.
[8] M. Bun and T. Steinke. Concentrated differential privacy: Simplifications, extensions, and lower bounds. In Proceedings of the 14th International Conference on Theory of Cryptography - Volume 9985, 2016.
[9] U. S. C. Bureau. On the map: Longitudinal employer-household dynamics. https://lehd.ces.census.gov/applications/help/onthemap.html#!confidentiality_protection
[10] K. Chaudhuri, D. Hsu, and S. Song. The large margin mechanism for differentially private maximization. In Proceedings of the 27th International Conference on Neural Information Processing Systems - Volume 1, 2014.
[11] K. Chaudhuri, C. Monteleoni, and A. D. Sarwate. Differentially private empirical risk minimization. Journal of Machine Learning Research, 12(Mar):1069–1109, 2011.
[12] Y. Chen, A. Machanavajjhala, J. P. Reiter, and A. F. Barrientos. Differentially private regression diagnostics. In IEEE 16th International Conference on Data Mining (ICDM), 2016.
[13] B. Ding, J. Kulkarni, and S. Yekhanin. Collecting telemetry data privately. In Advances in Neural Information Processing Systems (NIPS), 2017.
[14] C. Dwork. Differential privacy. In Proceedings of the 33rd International Conference on Automata, Languages and Programming - Volume Part II, ICALP’06, pages 1–12, Berlin, Heidelberg, 2006. Springer-Verlag.
[15] C. Dwork, K. Kenthapadi, F. McSherry, I. Mironov, and M. Naor. Our data, ourselves: Privacy via distributed noise generation. In Annual International Conference on the Theory and Applications of Cryptographic Techniques, pages 486–503. Springer, 2006.
[16] C. Dwork and J. Lei. Differential privacy and robust statistics. In Proceedings of the forty-first annual ACM symposium on Theory of computing, pages 371–380. ACM, 2009.
[17] C. Dwork, F. McSherry, K. Nissim, and A. Smith. Calibrating noise to sensitivity in private data analysis. In Theory of cryptography conference, pages 265–284. Springer, 2006.
[18] C. Dwork, M. Naor, O. Reingold, G. N. Rothblum, and S. Vadhan. On the complexity of differentially private data release: efficient algorithms and hardness results. In Proceedings of the forty-first annual ACM symposium on Theory of computing, pages 381–390. ACM, 2009.
[19] C. Dwork and A. Roth. The algorithmic foundations of differential privacy. Foundations and Trends in Theoretical Computer Science, 9(34):211–407, 2014.
[20] ´U. Erlingsson, V. Feldman, I. Mironov, A. Raghunathan, K. Talwar, and A. Thakurta. Amplification by shuffling: From local to central differential privacy via anonymity. In Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019, 2019.
[21] ´U. Erlingsson, V. Pihur, and A. Korolova. Rappor: Randomized aggregatable privacy-preserving ordinal response. In Proceedings of the 2014 ACM SIGSAC conference on computer and communications security, pages 1054–1067. ACM, 2014.
[22] M. Fanaeepour and B. I. P. Rubinstein. Histogramming privately ever after: Differentially-private data-dependent error bound optimisation. In Proceedings of the 34th International Conference on Data Engineering, ICDE. IEEE, 2018.
[23] A. Ghosh, T. Roughgarden, and M. Sundararajan. Universally utility-maximizing privacy mechanisms. In STOC, pages 351–360, 2009.
[24] S. Haney, A. Machanavajjhala, J. M. Abowd, M. Graham, M. Kutzbach, and L. Vilhuber. Utility cost of formal privacy for releasing national employer-employee statistics. In Proceedings of the 2017 ACM International Conference on Management of Data, SIGMOD ’17, 2017.
[25] M. Hardt, K. Ligett, and F. McSherry. A simple and practical algorithm for differentially private data release. In NIPS, 2012.
[26] M. Hardt and G. N. Rothblum. A multiplicative weights mechanism for privacy-preserving data analysis. In Proceedings of the 2010 IEEE 51st Annual Symposium on Foundations of Computer Science, FOCS ’10, pages 61–70, Washington, DC, USA, 2010. IEEE Computer Society.
[27] N. Johnson, J. P. Near, and D. Song. Towards practical differential privacy for sql queries. Proc. VLDB Endow., 11(5), 2018.
[28] I. Kotsogiannis, A. Machanavajjhala, M. Hay, and G. Miklau. Pythia: Data dependent differentially private algorithm selection. In Proceedings of the 2017 ACM International Conference on Management of Data, SIGMOD ’17, 2017.
[29] E. Lehmann and G. Casella. Theory of Point Estimation. Springer Verlag, 1998.
[30] K. Ligett, S. Neel, A. Roth, B. Waggoner, and S. Z.
[31] J. Liu and K. Talwar. Private selection from private candidates. *arXiv preprint arXiv:1811.07971*, 2018.

[32] M. Lyu, D. Su, and N. Li. Understanding the sparse vector technique for differential privacy. *Proceedings of the VLDB Endowment*, 10(6):637–648, 2017.

[33] A. Machanavajjhala, D. Kifer, J. Abowd, J. Gehrke, and L. Vilhuber. Privacy: From theory to practice on the map. In *Proceedings of the IEEE International Conference on Data Engineering (ICDE)*, pages 277–286, 2008.

[34] F. McSherry and K. Talwar. Mechanism design via differential privacy. In *Proceedings of the 48th Annual IEEE Symposium on Foundations of Computer Science*, pages 94–103, 2007.

[35] F. D. McSherry. Privacy integrated queries: An extensible platform for privacy-preserving data analysis. In *Proceedings of the 2009 ACM SIGMOD International Conference on Management of Data*, pages 19–30, 2009.

[36] I. Mironov. On significance of the least significant bits for differential privacy. In *Proceedings of the 2012 ACM Conference on Computer and Communications Security (CCS)*, 2012.

[37] I. Mironov. Rényi differential privacy. In *30th IEEE Computer Security Foundations Symposium, CSF*, 2017.

[38] N. Papernot, S. Song, I. Mironov, A. Raghunathan, K. Talwar, and I. far Erlingsson. Scalable private learning with pate. In *International Conference on Learning Representations (ICLR)*, 2018.

[39] S. Raskhodnikova and A. D. Smith. Lipschitz extensions for node-private graph statistics and the generalized exponential mechanism. In *FOCS*, pages 495–504. IEEE Computer Society, 2016.

[40] J. Tang, A. Korolova, X. Bai, X. Wang, and X. Wang. Privacy loss in apple’s implementation of differential privacy. In *3rd Workshop on the Theory and Practice of Differential Privacy at CCS*, 2017.

[41] A. D. P. Team. Learning with privacy at scale. *Apple Machine Learning Journal*, 1(8), 2017.

[42] A. G. Thakurta and A. Smith. Differentially private feature selection via stability arguments, and the robustness of the lasso. In *Proceedings of the 26th Annual Conference on Learning Theory*, 2013.

[43] Y. Wang, Z. Ding, G. Wang, D. Kifer, and D. Zhang. Proving differential privacy with shadow execution. *arXiv preprint arXiv:1903.12254*, 2019.

[44] D. Zhang and D. Kifer. Lightdp: Towards automating differential privacy proofs. In *ACM Symposium on Principles of Programming Languages (POPL)*, pages 888–901, 2017.

[45] D. Zhang, R. McKenna, I. Kotsogiannis, M. Hay, A. Machanavajjhala, and G. Miklau. Ektelo: A framework for defining differentially-private computations. In *Proceedings of the 2018 International Conference on Management of Data, SIGMOD ’18*, 2018.
Proof of Lemma 2

Proof. First we show that for any adjacent pair \( D \sim D' \) there are countably many local alignment functions \( \delta_{D,D'} \). First, \( \delta_{D,D'} \omega \) only depends on \( \omega \) through \( \ell_\omega \) (the set of queries whose noisy values were larger than the noisy threshold). Since \( |\ell_\omega| \leq k \), there are only countably many possibilities for \( \ell_\omega \) and hence there are countably many distinct \( \delta_{D,D'} \omega \).

Pick a pair from \( D \sim D' \) and an \( \omega = (\omega_1, \ldots, \omega_j) \) for some \( j \). For a given \( H = (\eta_1, \eta_2, \ldots) \) such that \( M(D, H) = \omega \), let \( H' = (\eta_1', \eta_2', \ldots) = \delta_{D,D'} \omega (H) \).

Let \( M(D', H') = \omega' = (w_1', \ldots, w_i') \). Our goal is to show \( \omega' = \omega \) (since that will mean \( \delta_{D,D'} \omega \) is a local alignment). Choose an \( i \leq \min(s,t) \).

- If \( i \in \ell_\omega \) (i.e. \( M(D, H) \) returned \( T \) and a gap for query \( q_i \)), then by (2) we have
  \[
  (q_i(D') + \eta_i) - (T + \eta) = (q_i(D') + \eta_i + 1 + q_i(D)) - (T + \eta + 1) = (q_i(D) - \eta_i) - (T + \eta) = w_i \geq 0.
  \]
  Therefore, \( w_i' = w_i \geq 0 \) (since the noisy value of the query remains over the noisy threshold with the same gap).
- If \( i \notin \ell_\omega \) (i.e. \( M(D, H) \) returned \( \perp \) for query \( q_i \)), then by (2) we have
  \[
  (q_i(D') + \eta_i) - (T + \eta) = (q_i(D') + \eta_i) - (T + \eta + 1) \leq (q_i(D) + \eta_i) - (T + 1) = 0.
  \]
  Hence \( w_i' = w_i = \perp \).

Therefore for all \( 1 \leq i \leq \min(s,t) \), we have \( w_i' = w_i \). That is, every \( \omega' \) is a prefix of \( \omega \) or vice versa. Let \( q \) be the vector of queries passed to the algorithm and let \( \text{len}(q) \) be the number of queries it contains (which can be finite or infinity). By the termination condition of Sparse Vector we have two possibilities.

- \( s = \text{len}(q) \): in this case there are fewer than \( k \) above-threshold answers in the first \( s \) queries, and we must have \( t = \text{len}(q) \) too because \( M(D', H') \) will also run through all the queries (it cannot stop until it answers \( k \) above threshold queries or hits the end of the query sequence).
- \( s < \text{len}(q) \): in this case \( q_s \) is the above-threshold query to reach the bound \( k \) and we must also have \( t = s \).

Thus \( t = s \) and \( \omega' = \omega \). This shows that \( \delta_{D,D'} \omega \) really is a local alignment function from \( S_D \omega \) to \( S_{D'} \omega \).

A.2 Probability of Ties Among \( n \) Queries with the Discrete Laplace Distribution

Let \( \gamma \) be the base of the discrete laplace distribution. We will first consider the probability of a tie between two queries and then use the union bound over all pairs of queries.

Suppose \( \eta_1 \) and \( \eta_2 \) are two i.i.d zero mean discrete laplace random variables with scale \( 1/\epsilon \) and base \( \gamma \). Without loss of generality, let \( q_1 - q_2 = m \gamma \geq 0 \). Then the probability that \( q_1 + \eta_1 = q_2 + \eta_2 \) is:

\[
\Pr(q_1 + \eta_1 = q_2 + \eta_2) = \sum_{\ell \in \mathbb{Z}} \Pr(\eta_1 = \gamma \ell) \Pr(\eta_2 = (\ell + m) \gamma) = (1 - e^{-\gamma \epsilon})^2 \sum_{\ell \in \mathbb{Z}} e^{-(\gamma \epsilon) \ell} e^{-(\gamma \epsilon) (\ell + m)} = \frac{(1 - e^{-\gamma \epsilon})^2}{(1 + e^{-\gamma \epsilon})^2} \sum_{\ell = -m}^{\infty} e^{\gamma \epsilon \ell} e^{\gamma \epsilon (\ell + m)} + \sum_{\ell = 1}^{\infty} e^{-\gamma \epsilon \ell} e^{-\gamma \epsilon (\ell + m)} = \frac{(1 - e^{-\gamma \epsilon})^2}{(1 + e^{-\gamma \epsilon})^2} \frac{1}{1 - e^{-2 \gamma \epsilon}} + \frac{e^{-2 \gamma \epsilon}}{1 - e^{-2 \gamma \epsilon}} \leq (1 - e^{-\gamma \epsilon})^2 \frac{1}{1 - e^{-2 \gamma \epsilon}} = \frac{1}{1 - e^{-2 \gamma \epsilon}} = (1 + \gamma e^{e^{-\gamma \epsilon} m}) \leq \gamma e + (1 - e^{-\gamma \epsilon})^2 e^{-\gamma \epsilon} m \leq \gamma \epsilon e + (1 - e^{-\gamma \epsilon})^2 e^{-\gamma \epsilon} m = \gamma \epsilon (1 + \gamma e^{e^{-\gamma \epsilon} m}) \leq \gamma \epsilon (1 + e^{-1}) \text{ (since } e^{-x} \text{ maximized at } x = 1)\]

Since there are \( n \) queries, we can conservatively estimate the probability of a tie as the probability that any pair of \( n \) items has a tie. Using the union bound, we get the probability of a tie is at most \( n^2 \epsilon \). In floating point, we expect a Laplace distribution to be implemented using a Discrete Laplace with \( \gamma \) being close to machine epsilon, which for double-precision floating point numbers is around \( 2^{-52} \).

A.3 Proof of Lemma 4

To prove this lemma, we need some intermediate results.

First we show that the \( \delta_{D,D'} \omega \) are alignments and that there are countably many of them.

**Lemma 8.** Let \( M \) be the Noisy-Max-with-Gap algorithm. For all \( D \sim D' \) and \( \omega \), the functions \( \delta_{D,D'} \omega \) defined in Equation [3] are local alignments for \( M \). Furthermore, for every pair \( D \sim D' \), there are countably many distinct \( \delta_{D,D'} \omega \).

Proof. First note that \( \delta_{D,D'} \omega \) only depends on \( \omega = (j, \omega) \) through \( \ell_\omega = (j, \omega) \) (the index of the largest query). There are \( n \) queries and therefore \( n \) distinct \( \delta_{D,D'} \omega \).

Pick an adjacent pair \( D \sim D' \) and an \( \omega = (j, \omega) \). For a given \( H = (\eta_1, \eta_2, \ldots) \) such that \( M(D, H) = \omega \), let \( H' = (\eta_1', \eta_2', \ldots) = \delta_{D,D'} \omega (H) \). Next, we show that \( M(D', H') = \omega \) from [3] we have

\[
q_s + \eta_s - \max_{i \in \mathbb{Z}} (q_i' + \eta_i') = (q_s + \eta_s + \max_{i \in \mathbb{Z}} (q_i + \eta_i) - \max_{i \in \mathbb{Z}} (q_i + \eta_i)) - \max_{i \in \mathbb{Z}} (q_i' + \eta_i) = q_s + \eta_s - \max_{i \in \mathbb{Z}} (q_i + \eta_i) = \omega > 0.
\]

Thus \( M(D', H') = (j, \omega) = \omega \).
To establish the alignment cost, we need the following lemma and definition.

**Lemma 9.** Let \((x_1, \ldots, x_m), (x'_1, \ldots, x'_m) \in \mathbb{R}^m\) be such that \(\forall i, |x_i - x'_i| \leq 1\). Then \(\max_i (x_i) - \max_j (x'_j) \leq 1\).

**Proof.** Let \(s\) be an index that maximizes \(x_s\) and let \(t\) be an index that maximizes \(x'_t\). Without loss of generality, assume \(x_s \geq x'_t\). Then \(x_s \geq x'_t \geq x'_s \geq x_s - 1\). Hence 
\[
|x_s - x'_t| = x_s - x'_t \leq x_s - (x_s - 1) = 1.
\]

**Definition 7 (Monotonicity).** A sequence of numerical queries \(q = (q_1, q_2, \ldots)\) is monotonic if for all pair of adjacent databases \(D \sim D'\) we have either \(\forall i : q_i(D) \leq q_i(D')\) or \(\forall i : q_i(D) \geq q_i(D')\).

Counting queries have the monotonic property.

**Lemma 10.** Algorithm \(\mathcal{A}\) satisfies \(\epsilon\)-differential privacy. If \(q\) is monotonic, then Algorithm \(\mathcal{A}\) satisfies \(\epsilon/2\)-differential privacy.

**Proof.** Let \(M\) be the algorithm. The \(\phi_{D, D', \omega}\) are local alignments by Lemma \(\mathcal{S}\). Clearly \(M\) terminates with probability 1 (Condition (i) of the Lemma \(\mathcal{S}\)). The number of random variables used by \(M\) is \(\log(q)\). Hence Condition (ii) is satisfied. Condition (iii) is trivial and Condition (iv) follows from Lemma \(\mathcal{S}\). The only thing left to show is that Condition (iv) holds. Clearly the local alignments are acyclic. To find their cost, we use the \(\epsilon\) defined in Algorithm \(\mathcal{A}\).

\[
\sum_{i=1}^{\infty} |q'_i - q_i| \leq \frac{\epsilon}{2} \sum_{i=1}^{\infty} (|q'_i - q_i| + |q'_i - q_i|) \leq \frac{\epsilon}{2} (1 + 1) = \epsilon
\]

where \(|q'_i - q_i| \leq 1\) by sensitivity and \(\max_{i \in I_\omega} (q'_i + q_i) - \max_{i \in I_\omega} (q'_i + q_i) \leq 1\) by Lemma \(\mathcal{A}\) (applied to \(x_i = q_i + q_i\) and \(x'_i = q'_i + q'_i\)). When \(q\) is monotonic, then

- either \(\forall i : q_i \leq q'_i\) in which case \(q'_i - q_i \in [-1, 0]\) and \(\max_{i \in I_\omega} (q'_i + q_i) - \max_{i \in I_\omega} (q'_i + q_i) \in [0, 1]\),
- or \(\forall i : q_i \geq q'_i\) in which case \(q_i - q'_i \in [0, 1]\), and \(\max_{i \in I_\omega} (q'_i + q_i) - \max_{i \in I_\omega} (q'_i + q_i) \in [-1, 0]\).

In both cases we have \(q_i - q'_i \leq \max_{i \in I_\omega} (q'_i + q_i) - \max_{i \in I_\omega} (q'_i + q_i) \leq 1\) and thus 
\[
\text{cost}(\phi_{D, D', \omega}) \leq \frac{\epsilon}{2}.
\]

Lemma \(\mathcal{A}\) then follows immediately from Lemmas \(\mathcal{A}\) and \(\mathcal{\mathcal{S}}\).

**A.4 Proof of Lemma \(\mathcal{A}\)**

We prove the Lemma using the following two results

**Lemma 11.** Let \(M\) be the Noisy-Top-K-with-Gap algorithm. For all \(D \sim D'\) and \(\omega\), the functions \(\phi_{D, D', \omega}\) defined above are local alignments for \(M\). Furthermore, for every pair \(D \sim D'\), there are countably many distinct \(\phi_{D, D', \omega}\).

**Proof.** First note that \(\phi_{D, D', \omega}\) only depends on \(\omega\) through \(I_\omega\) and \(J_\omega\) (the sets of queries whose noisy values were larger than the noisy threshold). Since \(|I_\omega| \leq k\) and \(|J_\omega| \leq k\), there are only countably many possibilities for \(I_\omega\) and \(J_\omega\), hence there are countably many distinct \(\phi_{D, D', \omega}\).

Pick an adjacent pair \(D \sim D'\) and an \(\omega = (\omega_1, \omega_2, \ldots)\). For a given \(H = (\eta_1, \eta_2, \ldots)\) such that \(M(D, H) = \omega\), let \(H' = (\eta'_1, \eta'_2, \ldots) = \phi_{D, D', \omega}(H)\). Next, we show that \(M(D', H') = \omega\). Since \(\phi_{D, D', \omega}\) is identity on components \(i \notin I_\omega\), we have \(m_\omega := \max_{i \in I_\omega} (q_i + \eta_i) = \max_{i \in I_\omega} (q_i + \eta_i')\) and 
\[
(m'_\omega := \max_{i \in I_\omega} (q_i' + \eta_i') = \max_{i \in I_\omega} (q_i' + \eta_i').
\]

Therefore,
\[
(q'_i + \eta_i) - \max_{i \in I_\omega} (q_i' + \eta_i) = (q_i + \eta_i) - m_\omega = q_i - \epsilon > 0
\]

and for \(1 \leq i < k\),
\[
(q'_i + \eta_i)(q_{i+1} + \eta_{i+1}) = (q_i + \eta_i) - m_\omega = (q_i + \eta_i) - (q_{i+1} + \eta_{i+1}) = g_i > 0.
\]

Thus \(M(D', H') = \omega\).

**Lemma 12.** Algorithm \(\mathcal{A}\) satisfies \(\epsilon\)-differential privacy. If \(q\) is monotonic (Definition \(\mathcal{\mathcal{S}}\)), then Algorithm \(\mathcal{A}\) satisfies \(\epsilon/2\)-differential privacy.

**Proof.** Again, the only thing nontrivial is to show that Condition (iv) of Lemma \(\mathcal{S}\) holds. The local alignments are acyclic (any permutation that puts \(I_\omega\) before \(J_\omega\) does the trick). To find their cost, we use the \(\epsilon\) defined in Algorithm \(\mathcal{A}\).

\[
\sum_{i=1}^{\infty} |q'_i - q_i| \leq \frac{\epsilon}{2k} \sum_{i=1}^{\infty} (|q'_i - q_i| + |q'_i - q_i|) \leq \frac{\epsilon}{2k} (1 + 1) = \epsilon
\]

where \(|q'_i - q_i| \leq 1\) by sensitivity. By Lemma \(\mathcal{S}\) (applied to \(x_i = q_i + q_i\) and \(x'_i = q'_i + q'_i\)). When \(q\) is monotonic, then

- either \(\forall i : q_i \leq q'_i\) in which case \(q'_i - q_i \in [-1, 0]\) and \(\max_{i \in I_\omega} (q'_i + q_i) - \max_{i \in I_\omega} (q'_i + q_i) \in [0, 1]\),
- or \(\forall i : q_i \geq q'_i\) in which case \(q_i - q'_i \in [0, 1]\), and \(\max_{i \in I_\omega} (q'_i + q_i) - \max_{i \in I_\omega} (q'_i + q_i) \in [-1, 0]\).

In both cases we have \(q_i - q'_i + \max_{i \in I_\omega} (q'_i + q_i) - \max_{i \in I_\omega} (q'_i + q_i) \in [-1, 1]\) so \(q_i - q'_i + \max_{i \in I_\omega} (q'_i + q_i) - \max_{i \in I_\omega} (q'_i + q_i) \leq 1\) and thus 
\[
\text{cost}(\phi_{D, D', \omega}) \leq \epsilon/2.
\]

Lemma \(\mathcal{A}\) then follows immediately from Lemmas \(\mathcal{A}\) and \(\mathcal{\mathcal{S}}\).

**A.5 Proof of Lemma \(\mathcal{A}\)**

We prove the Lemma using the following two results

**Lemma 13.** Let \(M\) be the Adaptive-Sparse-Vector-with-Gap algorithm. For all \(D \sim D'\) and \(\omega\), the functions \(\phi_{D, D', \omega}\) defined above are local alignments for \(M\). Furthermore, for every pair \(D \sim D'\), there are countably many distinct \(\phi_{D, D', \omega}\).

**Proof.** First note that \(\phi_{D, D', \omega}\) only depends on \(\omega\) through \(I_\omega\) and \(J_\omega\) (the sets of queries whose noisy values were larger than the noisy threshold). Since \(|I_\omega| \leq k\) and \(|J_\omega| \leq k\), there are only countably many possibilities for \(I_\omega\) and \(J_\omega\), hence there are countably many distinct \(\phi_{D, D', \omega}\).
\textbf{Lemma 14.} Algorithm [3] satisfies $\epsilon$-differential privacy.

\textit{Proof.} Again, the only thing nontrivial is to show that Condition (iv) of Lemma 1 holds. The local alignments are clearly acyclic. To find their cost, we use the $\epsilon$-dition (iv) of Lemma 1 holds. The local alignments are defined in Algorithm 4. From (5) we have

$$\mathbf{E}(\xi^2 - \xi) = \mathbf{E}(\xi^2) - \mathbf{E}(\xi) = \lambda$$

Then the first inequality is due to the sensitivity restriction: $|q_i - q_j| \leq 1 \implies q'_i - 1 \leq q_i$. Hence $w'_i = w_i$. Therefore, either $\omega'$ is a prefix of $\omega$ or vice versa. By a similar argument on the termination condition as before, we must have $s = t$ and thus $\omega' = \omega$. \hfill \square

\section{A.6 Proof of Theorem 3 (BLUE)}

\textit{Proof.} Recall that $\alpha_i = q_i + \xi_i$ and $q_i = q_i - q_{i+1} + \eta_i - \eta_{i+1}$ where $\xi_i$ and $\eta_i$ are independent Laplacian random variables with scale $2k/\epsilon$ and $4k/\epsilon$ respectively. Let

$$q = \begin{bmatrix} q_1 \\ \vdots \\ q_k \end{bmatrix}, \xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_k \end{bmatrix}, \eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_k \end{bmatrix}, \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix}, \gamma = \begin{bmatrix} g_1 \\ \vdots \\ g_{k-1} \end{bmatrix},$$

then $\alpha = q + \xi$ and $\gamma = N(q + \eta)$ where

\begin{align*}
N &= \begin{bmatrix} 1 & -1 & \cdots & -1 \\
-1 & 1 & \cdots & -1 \\
\vdots & \ddots & \ddots & \vdots \\
-1 & \cdots & 1 & -1 \end{bmatrix}_{(k-1) \times k} \\
\end{align*}

Our goal is then to find the best linear unbiased estimate (BLUE) $\beta$ of $q$ in terms of $\alpha$ and $\gamma$. In other words, we need to find a $k \times k$ matrix $X$ and a $k \times (k-1)$ matrix $Y$ such that $\beta = X\alpha + Y\gamma$ with $E(||\beta - q||^2)$ as small as possible. Unbiasedness dictates that $\forall q, E(\beta) = Xq + YNq = q$. Therefore $X + YN = I_k$ and thus $X = I_k - YN$, hence $\beta = (I_k - YN)\alpha + Y\gamma = \alpha - Y(N\alpha - \gamma)$. Let

$$\theta = N\alpha - \gamma = N(\xi - \eta) = \begin{bmatrix} \xi_1 - \xi_2 - \eta_1 + \eta_2 \\
\vdots \\
\xi_{k-1} - \xi_k - \eta_{k-1} + \eta_k \end{bmatrix}$$

then we have $\beta - q = \alpha - q - Y\theta = \xi - \eta$. Therefore, finding the BLUE is equivalent to solving the optimization problem $Y = \arg \min \Phi$ where

$$\Phi = E(||\xi - Y\theta||^2) = E((\xi - Y\theta)^T(\xi - Y\theta)) = E(\xi^2 - \xi^2 - \theta^T Y^T \xi + \theta^T Y^T Y \theta)$$

Taking the partial derivatives of $\Phi$ w.r.t. $Y$, we have

$$\frac{\partial \Phi}{\partial Y} = E(0 - \theta^T - \theta^T + Y(\theta^T (\theta^T + \theta \theta^T)))$$

By setting $\frac{\partial \Phi}{\partial Y} = 0$ we have

$$YE(\theta \theta^T) = E(\theta \theta^T) \implies Y = E(\theta \theta^T)E(\theta \theta^T)^{-1}.$$ 

Recall that $(\theta \theta^T)_{ij} = \xi_i(\xi_j - \xi_{j+1} - \eta_j + \eta_{j+1})$ have,

$$E((\xi^2 - \xi^2 - \theta^T Y^T \xi + \theta^T Y^T Y \theta)$$

Thus

$$E((\xi^2 + \theta^T Y^T \xi + \theta^T Y^T Y \theta) = 10 \lambda \ i = j,$$

$$E(-\xi^2 + \theta^T Y^T \xi - 5 \lambda \ i = j + 1$$

$$E(-\xi^2 + \theta^T Y^T \xi - 5 \lambda \ i = j - 1$$

Hence

$$E(\alpha^T) = \lambda N^T$$

$$E(\gamma^T) = \lambda B$$

Therefore, $Y = E(\theta \theta^T)E(\theta \theta^T)^{-1} = N^T B^{-1}$. Let

$$Y = \frac{1}{5k} \begin{bmatrix} k-1 & k-2 & \cdots & 1 \\
1 & k-2 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
k & k & \cdots & k \end{bmatrix}.$$
It can be directly verified that $YB = N_T$, thus above is the formula for $Y$. Hence

$$X = I_k - YN = \frac{1}{5k} \begin{bmatrix} 1 + 4k & 1 & \cdots & 1 \\ 1 & 1 + 4k & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 + 4k \end{bmatrix}.$$  

\[\square\]

A.7 Proof of Corollary \[1\]

Recall that $\alpha_i = q_i + \xi_i$ and $g_i = q_i - q_{i+1} + \eta_i - \eta_{i+1}$. Let $\text{Var}(\xi_i) = 8k^2/\epsilon = \lambda$ and $\text{Var}(\eta_i) = 32k^2/\epsilon = 4\lambda$. From the matrices $X$ and $Y$ in Theorem \[3\] we have that $\beta_i = \frac{X_i + Y_i}{5k}$ where

$$x_i = \alpha_1 + \cdots + (4k + 1)\alpha_i + \cdots + \alpha_k$$

and

$$y_i = -g_1 - 2g_2 - \cdots - (i - 1)g_{i-1} + (k - i)g_i + \cdots + 2g_{k-2} + g_{k-1}$$

$$= -(q_1 + \eta_1) - (q_2 + \eta_2) - \cdots - (q_{i-1} + \eta_{i-1}) + (k - 1)(q_i + \eta_i) - (q_{i+1} + \eta_{i+1}) - \cdots - (q_k + \eta_k).$$

Therefore

$$\text{Var}(x_i) = (k - 1)\lambda + (4k + 1)^2\lambda = (16k^2 + 9k)\lambda$$

$$\text{Var}(y_i) = (k - 1)4\lambda + (k - 1)^24\lambda = (4k^2 - 4k)\lambda$$

and thus

$$\text{Var}(\beta_i) = \frac{\text{Var}(x_i) + \text{Var}(y_i)}{25k^2} = \frac{(20k^2 + 5k)}{25k^2} \lambda = \frac{(4k + 1)}{5k} \lambda.$$  

A.8 Proof of Lemma \[7\]

The density function of $\eta - \eta$ is

$$f_{\eta - \eta}(z) = \int_{-\infty}^{\infty} f_{\eta}(x) f_{\eta}(x - z) \, dx$$

$$= \frac{\epsilon_0\epsilon_1}{4} \int_{-\infty}^{\infty} e^{-\epsilon_1|z|} e^{-\epsilon_0|x-z|} \, dx.$$  

First consider the case $\epsilon_0 \neq \epsilon_1$. When $z \geq 0$, we have

$$f_{\eta_1 - \eta}(z) = \frac{\epsilon_0\epsilon_1}{4} \int_{-\infty}^{\infty} e^{-\epsilon_1|z|} e^{-\epsilon_0|x-z|} \, dx$$

$$= \int_{-\infty}^{0} e^{-\epsilon_1z} e^{\epsilon_0(x-z)} \, dx + \int_{0}^{\infty} e^{-\epsilon_1z} e^{\epsilon_0(x-z)} \, dx$$

$$= \frac{\epsilon_0\epsilon_1}{4} \left( \frac{e^{-\epsilon_0z}}{\epsilon_0 + \epsilon_1} + \frac{e^{-\epsilon_1z}}{\epsilon_0 - \epsilon_1} + \frac{e^{-\epsilon_1z}}{\epsilon_0 + \epsilon_1} \right)$$

$$= \frac{\epsilon_0\epsilon_1(e^{-\epsilon_1z} - \epsilon_1 e^{-\epsilon_0z})}{2(\epsilon_0^2 - \epsilon_1^2)}.$$  

Thus by symmetry we have for all $z \in \mathbb{R}$

$$f_{\eta_1 - \eta}(z) = \frac{\epsilon_0\epsilon_1(e^{-\epsilon_1|z|} - \epsilon_1 e^{-\epsilon_0|z|})}{2(\epsilon_0^2 - \epsilon_1^2)}$$

and

$$P(\eta_1 - \eta \geq -t) = \int_{-t}^{\infty} f_{\eta_1 - \eta}(z) \, dz = \int_{-t}^{0} f_{\eta_1 - \eta}(z) \, dz + 1$$

$$= 1 - \frac{\epsilon_0^2 e^{-\epsilon_1t} - \epsilon_1^2 e^{-\epsilon_0t}}{2(\epsilon_0^2 - \epsilon_1^2)}.$$  

Now if $\epsilon_0 = \epsilon_1$, then by similar computations we have

$$f_{\eta_1 - \eta}(z) = \left( \frac{\epsilon_0}{4} + \frac{\epsilon_0^2|z|}{4} \right) e^{-\epsilon_0|z|}$$

and

$$P(\eta_1 - \eta \geq -t) = 1 - \left( \frac{2 + \epsilon_0t}{4} \right) e^{-\epsilon_0t}.$$