Research Article

VL Reciprocal Status Index and Co-Index of Graphs

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For the vertex set \( V(G) \) of a graph \( G \), the sum of reciprocals of the breadth (distances) between the vertex \( v \in V(G) \) and whole other remaining vertices of \( G \) is called reciprocal status of \( v \). In this study, first of all, we introduced the \( VL \) reciprocal status index and \( VL \) reciprocal status co-index of a graph \( G \). Later, we exposed some sharp bounds for these indices. Furthermore, we determined the \( VL \) reciprocal status (co-)index over some standard graphs. Finally, we presented correlations between \( VL \) reciprocal status index and some properties of Butane derivatives via a table and illustrated with a figure.

1. Introduction and Preliminaries

Topological indices are known as special graph invariants, and they are mainly used to calculate QSAR (quantitative structure-activity relationship) and QSPR (quantitative structure-property relationship) studies, cf. [15, 16]. Although the Wiener index ([17]) is the oldest topological index, later on some distance-based topological indices have defined as hyper Wiener index [13], Harary index [4, 8], and so on. The main idea in the study of topological indices is whether they are useful for identifying chemical properties. In general, octane isomers are the exceptional data for these studies.

If we suppose \( G \) is connected having order \( n \) also having size \( m \), then it is quite well known that, while the vertex and edge sets are shown by \( V(G) \) and \( E(G) \), respectively, an edge connecting (joining) the vertices \( u_1 \) and \( u_2 \) is shown by \( u_1u_2 \), and the degree \( d_u \) of \( v \in E(G) \) is actually the number of edges incident to \( u \). On the contrary, \( d(u_1, u_2) \), which notates the distance between any two vertices \( u_1 \) and \( u_2 \), is actually the length of the shortest path connecting \( u_1 \) and \( u_2 \), and the maximum breadth between any pair of vertices of \( G \) is defined as the diameter \( diam(G) \). For unmentioned graph theoretic terminologies, we may refer [1]. The following reminders will be needed for the construction of results in this paper:

In [2], for \( G \), the \( VL \) index has recently been exposed by

\[
VL(G) = \frac{1}{2} \sum_{uv \in E(G)} \left[d_u + d_f + 4\right],
\]

(1)

where \( d_u = d(u_1) + d(u_2) - 2 \) and \( d_f = d(u_1) \cdot d(u_2) - 2 \). According to the same reference, the \( VL \) index indicates a nice output in the meaning of polychlorinated biphenyl and octane isomers. Actually, \( VL(G) \) index can also be written as

\[
VL(G) = \frac{1}{2} \sum_{u_1u_2 \in E(G)} \left[d(u_1) + d(u_2) + d(u_1) \cdot d(u_2)\right].
\]

(2)

We will develop some new type of \( VL \) indices, see Definition 1 below.

For the vertex \( u_1 \), Harary [3] defined a parameter called status (or, equivalently and transmission), which is the sum of \( u_1 \)'s breadths to remaining vertices of \( G \), and that is shown by

\[
\sigma(u_1) = \sum_{u_2 \in V(G)} d(u_1, u_2).
\]

(3)
The reciprocal status [11] of $u_i \in V(G)$ is defined by the sum of reciprocals of $u_i$’s breadths (distances) to remaining vertices in $G$. This is shown by

$$rs(u_i) = \sum_{u_j \in V(G)} \frac{1}{d(u_i, u_j)}$$  \hspace{1cm} (4)

As it mentioned at the beginning, the oldest index, which is namely the Wiener index [17], is given by

$$W(G) = \sum_{\{u_i, u_j\} \in V(G)} d(u_i, u_j) = \frac{1}{2} \sum_{u_i \in V(G)} \sigma(u_i).$$  \hspace{1cm} (5)

In fact, it is also called as total status [3]. In literature knowledge, there are some studies about the different types of transmission topological indices, see, for instance, [5–7, 9, 10, 12, 14]. Additionally, the status and reciprocal status having base (topological) co-indices have recently been introduced in [12], and also, explicit formulae for them via order and size of $G$ are obtained.

Based on the construction of different types of reciprocal status-based topological indices and co-indices until now, in here, we present two new topological based indices, namely, VL reciprocal status index and VL reciprocal status co-index of a connected graph $G$.

Therefore, the following definition plays a key role for this study:

**Definition 1.** Let us consider a connected graph $G$. Therefore, the VL reciprocal status index over $G$ is defined by

$$VLRS(G) = \frac{1}{2} \sum_{u_i, u_j \in E(G)} [rs(u_i) + rs(u_j) + rs(u_i) \cdot rs(u_j)],$$  \hspace{1cm} (6)

whereas the VL reciprocal status co-index is defined by

$$VLRS\Gamma(G) = \frac{1}{2} \sum_{u_i, u_j \notin E(G)} [rs(u_i) + rs(u_j) + rs(u_i) \cdot rs(u_j)].$$  \hspace{1cm} (7)

For an example of $VLRS(G)$ and $VLRS\Gamma(G)$ indices, one may see the graph in Figure 1. Clearly, $VLRS(G) = 83.625$ and $VLRS\Gamma(G) = 17$.

This paper is organised as in the following. After this introductory part, we will present some “nice” bounds for the VL reciprocal status index, see equation (6) in Section 2. In Section 3, we will state and prove some results about VL reciprocal status index of standard graphs such as complete $K_n$, complete bipartite $K_{p,q}$, path $P_n$, cycle $C_n$, wheel $W_{n+1}$, windmill $F_k$, and star graph $S_n$. In Section 4 and 5, in parallel to Sections 2 and 3, we will compute some nice bounds for the VL reciprocal status co-index defined in equation (7) and present some results about this new co-index of standard graphs depicted above. In Section 6, the correlation of the VL reciprocal status index with some properties of Butane derivatives will be presented by means of a table and figure. Section 7 shows the importance of these new indices in terms of heavy atomic count and some future ideas.

2. Bounds for VL Reciprocal Status Index

In Section 2, we present some lower and upper bounds for $VLRS$ defined in equation (6) and then characterize the strictness conditions of these bounds.

The following is needed to obtain our first main result in this paper:

**Lemma 1.** Assume $G$ is connected owning $n$ vertices whereas $di am(G) = D$. Thus,

$$\frac{1}{8} \sum_{u \in V(G)} 2s + t + (n - 1)(s + n + 3) \leq VLRS(G) \leq \frac{1}{2} \sum_{u \in V(G)} p(2 + p) + (1 + p)qs + q^2t,$$

where $s = d(u) + d(v)$, $t = d(u) \cdot d(v)$, $p = (n - 1/D)$, and $q = 1 - (1/D)$. Both equalities hold if $di am(G) = 2$.

**Proof.** We first note that, for any $u \in V(G)$, there exist $d_u$ vertices having distance one from $u$ and the remaining $n - 1 - d_u$ vertices having distances at least 2.

Upper bound: for $u \in V(G)$,

$$rs(u) \geq d_u + \frac{1}{D} (n - 1 - d_u) = \frac{n - 1}{D} + d_u \left(1 - \frac{1}{D}\right),$$

which implies that

$$VLRS(G) = \frac{1}{2} \sum_{u \in V(G)} [rs(u) + rs(v) + rs(u) \cdot rs(v)] \leq \frac{1}{2} \sum_{u \in V(G)} p(2 + p) + (1 + p)qs + q^2t,$$

where $s = d_u + d_v$, $t = d_u \cdot d_v$, $p = (n - 1/D)$, and $q = 1 - (1/D)$.

Lower bound: again, for $u \in V(G)$, there is

$$rs(u) \geq d_u + \frac{1}{2} (n - 1 - d_u) = \frac{1}{2} [d_u + (n - 1)].$$

Therefore,

$$VLRS(G) = \frac{1}{2} \sum_{u \in V(G)} [rs(u) + rs(v) + rs(u) \cdot rs(v)] \geq \frac{1}{8} \sum_{u \in V(G)} 2s + t + (n - 1)(s + n + 3),$$

**Figure 1:** A graph with five vertices.
where \( s = d_u + d_v, \ t = d_u \cdot d_v, \ p = (n - 1/D), \) and \( q = 1 - (1/D) \) as required.

For the equality: suppose the diameter of \( G \) is given as 1 or 2. Thus, equality holds. For the converse part, let \( \text{di am} \geq 3 \), and let us assume that (Text translation failed), where \( s = d_u + d_v, \ t = d_u \cdot d_v, \ p = (n - 1/D), \) and \( q = 1 - (1/D) \). Thus, there exists one pair of vertices (at least) \( u_1, u_2 \) having \( d(u_1, u_2) \geq 3 \). So,

\[
rs(u_1) \leq d_{u_1} + \frac{1}{3} + \frac{1}{2}(n - 2 - d_{u_1}) = \frac{n}{2} + \frac{d_{u_1}}{2} 
\]  

(13)

Similarly, for \( u_2 \), we get \( rs(u_2) \leq (n/2) - (2/3) + (d_{u_2}/2) \) but for remaining whole vertices \( u \in V(G) \), it is obtained \( rs(u) \leq (n/2) - (1/2) + (d_{u}/2) \).

Now, our aim is to reach a contradiction by assuming \( \text{di am} \geq 3 \). So, let us separate the partition \( E(G) \) into three sets \( E_1, E_2, \) and \( E_3 \) such that

\[
VLRS(G) = \frac{1}{2} \sum_{uv \in E(G)} [rs(u) + rs(v) + rs(u).rs(v)] = \frac{1}{2} \sum_{u_1 \in E_1} [rs(u_1) + rs(v) + rs(u_1).rs(v)] 
\]

\[
+ \frac{1}{2} \sum_{u_2 \in E_2} [rs(u_2) + rs(v) + rs(u_2).rs(v)] 
\]

\[
+ \frac{1}{2} \sum_{u \in E_3} [rs(u) + rs(v) + rs(u).rs(v)] > \frac{1}{8} \sum_{uv \in E(G)} 2s + t + (n - 1)(s + n + 3), 
\]

which gives a contradiction. Further, we get \( \text{di am}(G) \leq 2 \). \( \square \)

By means of direct application of Lemma 1, we can give our first main result after expanding the sums:

**Theorem 1.** Suppose \( G \) is connected, having total \( n \) vertices, and finally, suppose \( \text{di am}(G) = D \). Thus,

\[
\frac{1}{8} \left[ \left( n + 1 \right) \|d\|^2 + 2A(d) \cdot d + (n - 1)(n + 3) \right] \leq VLRS(G), 
\]

(17)

and

\[
\frac{m}{8} \left[ \left( \delta + n \right) \left( \delta + n + 2 \right) - 3 \right] \leq VLRS(G) \leq \frac{m}{2} \left[ p \left( 2 + p \right) + 2q \Delta \left( 1 + p \right) + q^2 \Delta^2 \right], 
\]

(19)

lower bound, and \( d_u + d_v \leq 2\Delta \) but \( d_u \cdot d_v \leq \Delta^2 \) in the upper bound of Theorem 2, we get the required result. \( \square \)

**Corollary 1.** Let \( G \) be as in Theorem 2 having \( m \) edges. Let \( \delta \) and \( \Delta \) be the min. and max. degrees of the elements in \( V(G) \), respectively. After that,

\[
\frac{m}{8} \left[ \left( r + n \right) \left( r + n + 2 \right) - 3 \right] \leq VLRS(G) \leq \frac{m}{2} \left[ p \left( 2 + p \right) + 2r \left( q \left( 1 + p \right) + q^2 r^2 \right) \right], 
\]

(20)

\[
E_1 = \left\{ u_i.v | rs(u_i) \leq \frac{n}{2} - \frac{1}{3} + \frac{d_{u_i}}{2}, \right. \and \left. rs(v) \leq \frac{n}{2} - \frac{1}{3} + \frac{d_v}{2} \right\}, 
\]

(14)

\[
E_2 = \left\{ u_i.v | rs(u_i) \leq \frac{n}{2} - \frac{1}{3} + \frac{d_{u_i}}{2}, \right. \and \left. rs(v) \leq \frac{n}{2} - \frac{1}{3} + \frac{d_v}{2} \right\}, 
\]

(15)

Clearly, \( |E_1| = d_{u_1}, |E_2| = d_{u_2}, \) and \( |E_3| = m - d_{u_1} - d_{u_2} \). Therefore,
where $p = (n - 1/D)$ and $q = 1 - (1/D)$. Moreover, equalities in above hold if $D \leq 2$.

3. VL Reciprocal Status Index of Some Standard Graphs

As we mentioned in Section 1, we will determine VL reciprocal status index defined in equation (6) for some special graphs.

In the following, the first result of this section is about complete graphs.

Proposition 1. For a complete graph $K_n$ on $n$ vertices and $m$ edges, $\text{VLRS}(K_n) = (m(n^2 - 1)/2)$.

Proof. It is known that $rs(u) = n - 1$ for any $u \in V(K_n)$. Henceforth, by equation (6), we get the result.

\[ \text{VLRS}(K_n) = \frac{pq}{8} [2(p + q) + pq + (p + q - 1)(q + p + q + 3)] \]

\[ = \frac{pq}{8} [2(p + q) + pq + (p + q - 1)(2p + 2q + 3)]. \] \hspace{1cm} (22)

Hence, we get the result on $\text{VLRS}(K_{pq})$ as required.

Now, we can investigate the situation for cycle graphs as follows.

\[ \text{VLRS}(C_n) = \begin{cases} \frac{n}{2} \left[ \frac{4}{n} + 4 \sum_{i=1}^{(n-2)/2} \frac{1}{i} \right] + \left( \frac{2}{n} + 2 \sum_{i=1}^{(n-2)/2} \frac{1}{i} \right)^2, & \text{if } n \text{ is even,} \\ 2n \left[ \sum_{i=1}^{(n-2)/2} \frac{1}{i} \left( 1 + \sum_{i=1}^{(n-2)/2} \frac{1}{i} \right) \right], & \text{if } n \text{ is odd.} \end{cases} \] \hspace{1cm} (23)

Proof

Case (i): if $n$ is an even number, then for any vertex $u \in V(C_n)$, we have

\[ rs(u) = 2 \left[ 1 + \frac{1}{2} + \cdots + \frac{1}{(n-2)/2} \right] + \frac{1}{(n/2)2} \]

\[ = \frac{2}{n} + 2 \sum_{i=1}^{(n-2)/2} \frac{1}{i}. \] \hspace{1cm} (24)

Therefore, by equation (6), we obtain

\[ \text{VLRS}(C_n) = \frac{n}{2} \left[ \frac{4}{n} + 4 \sum_{i=1}^{(n-2)/2} \frac{1}{i} \right] + \left( \frac{2}{n} + 2 \sum_{i=1}^{(n-2)/2} \frac{1}{i} \right)^2, \]

\hspace{1cm} (25)

while $n$ is even.

The next proposition exposes the result for VLRS on complete bipartite graphs.

Proposition 2. For a complete bipartite graph $K_{p,q}$, we have

\[ \text{VLRS}(K_{p,q}) = \frac{pq}{8} [2(p + q) + pq + (p + q - 1)(2p + 2q + 3)]. \] \hspace{1cm} (21)

Proof. One can partition $V(K_{p,q})$ into two different subsets $V_1$ and $V_2$ with the condition every single edge $uv$ of $K_{p,q}$ and we let $u \in V_1$ and $v \in V_2$. Therefore, $d_u = q$ and $d_v = p$ such that $|V_1| = p$ and $|V_2| = q$. Clearly, $K_{p,q}$ has $n = p + q$ vertices and $m = pq$ edges, and also, the diameter $diam(K_{p,q}) \leq 2$. Hence, by the equality part of Theorem 2, it is achieved.

Proposition 3. For a cycle graph $C_n$ on $n \geq 3$ vertices,

\[ \text{VLRS}(C_n) = 2n \left[ \sum_{i=1}^{(n-2)/2} \frac{1}{i} \left( 1 + \sum_{i=1}^{(n-2)/2} \frac{1}{i} \right) \right]. \] \hspace{1cm} (27)

Case (ii): if $n$ is an odd number, then for $u \in V(C_n)$, we get

\[ rs(u) = 2 \left[ 1 + \frac{1}{2} + \cdots + \frac{1}{(n-2)/2} \right] = 2 \sum_{i=1}^{(n-2)/2} \frac{1}{i}. \] \hspace{1cm} (26)

and then, by equation (6), we reach to

\[ \text{VLRS}(C_n) = 2n \left[ \sum_{i=1}^{(n-2)/2} \frac{1}{i} \left( 1 + \sum_{i=1}^{(n-2)/2} \frac{1}{i} \right) \right], \]

as required.
Another result in special graphs can be presented for a path as in the following.

**Proposition 4.** For a path graph $P_n$ on $n$ vertices, we have

\[
VLRS(P_n) = \frac{1}{2} \left[ \frac{n}{n-1} + 2 \sum_{i=1}^{n-1} \frac{n-1}{i} + \sum_{i=1}^{n-1} \frac{n-1}{i} \right] + \frac{1}{2} \sum_{i=2}^{n-2} \left[ \frac{n}{i(n-i)} + 2 \left( \sum_{j=1}^{i-1} \frac{n-1}{j} + \sum_{j=1}^{i-1} \frac{n-1}{j} \right) \right] + \left( \sum_{j=1}^{i-1} \frac{n-1}{j} + \sum_{j=1}^{i-1} \frac{n-1}{j} \right).
\]  

(28)

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices of $P_n$, where $v_i$ is connected to $v_{i+1}$ for $i = 1, 2, \ldots, n-1$. Therefore, for each $1 \leq i \leq n$, we get

\[
rs(v_i) = \sum_{j=1}^{i-1} \frac{1}{j} rs(v_j) \text{ and } rs(v_{i+1}) = \sum_{j=1}^{i-1} \frac{1}{j} + \sum_{j=n-i}^{n-1} \frac{1}{j}.
\]

(29)

(2 \leq i \leq n - 1).

Therefore,

\[
VLRS(P_n) = \frac{1}{2} \sum_{uv \in E(P_n)} \left[ rs(u) + rs(v) + rs(u)rs(v) \right]
\]

\[
= \frac{1}{2} \left[ rs(v_1) + rs(v_2) + rs(v_1) \cdot rs(v_2) \right] + \frac{1}{2} \sum_{i=2}^{n-2} \left[ rs(v_i) + rs(v_{i+1}) + rs(v_i) \cdot rs(v_{i+1}) \right] + \left[ rs(v_{n-1}) + rs(v_n) + rs(v_{n-1}) \cdot rs(v_n) \right],
\]

which implies the required result on $VLRS(P_n)$. □

Remaining three propositions will be clarified, $VLRS$ indices for wheel, windmill, and star graphs, respectively.

**Proposition 5.** Let us consider a wheel graph $W_{k+1}$ with $(k \geq 3)$. Then,

\[
VLRS(W_{k+1}) = \frac{1}{8} \sum_{uv \in E(W_{k+1})} 2[d_u + d_v] + d_u \cdot d_v + k[d_u + d_v + k + 4]
\]

\[
= \frac{1}{8} \sum_{uv \in E_1} 2[d_u + d_v] + d_u \cdot d_v + k[d_u + d_v + k + 4]
\]

\[
+ \frac{1}{8} \sum_{uv \in E_2} 2[d_u + d_v] + d_u \cdot d_v + k[d_u + d_v + k + 4]
\]

\[
= \frac{k}{8} (3k^2 + 22k + 27).
\]

(31)

**Proof.** If we partition the edge set $E(W_{k+1})$ into two sets $E_1 = \{ uv | d_u = k, d_v = 3 \}$ and $E_2 = \{ uv | d_u = 3, d_v = k \}$, then it is easy to see that $|E_1| = |E_2| = k$ such that $\text{di am}(W_{k+1}) = 2$. Therefore, by the equality part of Theorem 2,

\[
VLRS(W_{k+1}) = \frac{k}{8} (3k^2 + 22k + 27).
\]

□
Proposition 6. For a windmill graph $F_k$ with $k \geq 2$, we have
\[ \text{VLRS}(F_k) = \frac{k}{2}(5k^2 + 14k + 5). \] (33)

Proof. In the proof, we will follow the same way as in the proof of Proposition 3. So, let us partition $E(F_k)$ into subsets $E_1 = \{uv | d_u = 2k, d_v = 2\}$ and $E_2 = \{uv | d_u = 2, d_v = 2\}$ which gives $|E_1| = 2k$ and $|E_2| = k$. Also, $\text{di am}(F_k) = 2$. After that, by considering the equality part of Theorem 2, we obtain the result on $\text{VLRS}(F_k)$. \( \square \)

Since a star graph is actually a tree on $n$ nodes and one node’s degree is $n - 1$ and the remaining $n - 1$ vertices having degree one, so next result easily follows by Theorem 2:

Proposition 7. For a star graph $S_n$, $\text{VLRS}(S_n) = (m/4)(n^2 + 2n - 2)$.

4. Bounds for the VL Reciprocal Status Co-Index

Now, we present some bounds for VL reciprocal status co-index of graphs defined in equation (7) and characterize the equality conditions on them.

\[
\frac{1}{16} [n(n - 1) - 2m] [(\delta + n)(\delta + n + 2) - 3] \leq \text{VLRS}(G) \text{ and } \text{VLRS}(G) \leq \frac{1}{4} [n(9) - 2m] p (2 + p) + 2q(1 + p) + q^2\Delta^2. \]

where $p = (n - 1)/(2D)$ and $q = 1 - (1/D)$.

Proof. For $u \in V(G)$, we know that $\delta \leq d_u \leq \Delta$ which implies $2\delta \leq d_u + d_v \leq 2\Delta$. It is also known that the graph $G$ under these assuming conditions has $n(n - 1)/2 - m$ pair of nonadjacent vertices. Substituting $d_u + d_v \geq 2\delta$ and $d_u \cdot d_v \geq \delta^2$ in the lower bound and $d_u + d_v \leq 2\Delta$ and $d(u) \cdot d(v) \leq \Delta^2$ in the upper bound of Theorem 2, we achieve the result. \( \square \)

Corollary 4. Let $G$ be connected $r$-regular and has $n$ vertices, and also, let $\text{di am}(G) = D$. Then,
\[
\frac{m}{8} [(r + n)(r + n + 2) - 3] \leq \text{VLRS}(G)
\leq \frac{m}{2} [p (2 + p) + 2rq(1 + p) + r^2q^2].
\] (36)

where $p = (n - 1)/(2D)$ and $q = 1 - (1/D)$. Equalities hold if $\text{di am}(G) = D$.

Proof. \( \forall u \in V(G) \), and substituting $d_u = r$ in Theorem 2, we get the result. \( \square \)

Proof of the next result is very similar as in the proof of Theorem 2 and, therefore, will be skipped.

Theorem 2. Let $G$ be connected and has $n$ vertices and also $\text{di am}(G) = D$. Thus,
\[
\frac{1}{8} \sum_{uv \in E(G)} 2s + t + (n - 1)(s + n + 3) \leq \text{VLRS}(G)
\leq \frac{1}{2} \sum_{uv \in E(G)} p(2 + p) + (1 + p)qs + q^2t,
\]
where $s = d_u + d_v$, $t = d_u \cdot d_v$, $p = (n - 1)/(2D)$, and $q = 1 - (1/D)$. Moreover, equality holds on both sides if $D \leq 2$.

The following consequences are as important as Theorem 2

Corollary 3. Assume that $G$ is connected and has $n$ vertices and $m$ edges and also $\text{di am}(G) = D$. Assume also that $\delta$ and $\Delta$ is the min. and max. degrees of vertices in $V(G)$, respectively. Therefore,

\[
\frac{1}{16} [n(n - 1) - 2m] [(\delta + n)(\delta + n + 2) - 3] \leq \text{VLRS}(G) \text{ and } \text{VLRS}(G) \leq \frac{1}{4} [n(9) - 2m] p (2 + p) + 2q(1 + p) + q^2\Delta^2. \] (35)

5. VL Reciprocal Status Co-Index of Some Standard Graphs

With a similar approach as in Section 3, we will determine $\text{VL}$ reciprocal status co-index given in equation (7) of some special graph structures such as complete, complete bipartite, cycle, path, wheel, and windmill graphs.

Since the details of the following proposition are clear by considering the definitions of complete graphs and $\text{VL}$ reciprocal status co-index, the proof will be omitted.

Proposition 8. For $K_m$, $\text{VLRS}(K_m) = 0$.

Proposition 9. For $K_{p,q}$, there exists
\[
\text{VLRS}(K_{p,q}) = \frac{p(p - 1)}{4} \left[ 2q + p - 1 + \left( q + \frac{p - 1}{2} \right)^2 \right] + \frac{q(q - 1)}{4} \left[ 2p + q - 1 + \left( p + \frac{q - 1}{2} \right)^2 \right].
\] (37)

Proof. For $|V_1| = p$ and $|V_2| = q$, let us assume $V_1$ and $V_2$ are the partite sets of $G(K_{p,q})$ with every single edge of $K_{p,q}$
owning one end in $V_1$ and other end in $V_2$. If $u \in V_1$, then $rs(u) = q + (1/2)(p - 1)$, and also, if $u \in V_2$, then $rs(u) = p + (1/2)(q - 1)$. Therefore, for $u, v \in V_1$, $rs(u) + rs(v) = 2q + (p - 1)$, and for $u, v \in V_2, rs(u) + rs(v) = 2p + (q - 1)$. Hence, by equation (7), we obtain

$$\text{VLRS}(G) = \frac{1}{2} \sum_{u,v \in V_1} [rs(u) + rs(v) + rs(u)rs(v)] + \frac{1}{2} \sum_{u,v \in V_2} [rs(u) + rs(v) + rs(u)rs(v)]$$

$$= \frac{p(p - 1)}{4} \left[2q + p - 1 + \left(q + \frac{p - 1}{2}\right)^2\right] + \frac{q(q - 1)}{4} \left[2p + q - 1 + \left(p + \frac{q - 1}{2}\right)^2\right].$$

as required.

\[ \Box \]

**Proposition 10.** For a cycle graph $C_n$ on $n \geq 3$ vertices, we have

$$\text{VLRS}(C_n) = \begin{cases} \frac{n(n - 1) - 2n}{4} \left[\frac{4}{n} + 4 \sum_{i=1}^{\lfloor (n-2)/2 \rfloor} \frac{1}{i} + \left(\frac{2}{n} + 2 \sum_{i=1}^{\lfloor (n-2)/2 \rfloor} \frac{1}{i}\right)^2\right], & \text{if } n \text{ is even}, \\ \frac{n(n - 1) - 2n}{4} \left[\frac{(n-2)/2}{1} \left(1 + \sum_{i=1}^{\lfloor (n-2)/2 \rfloor} \frac{1}{i}\right)\right], & \text{if } n \text{ is odd.} \end{cases}$$

**Proof**

Case (i): as obtained in Proposition 3 in Case (i), if $n$ is an even number, for any vertex $u$ of $C_n$, we get $rs(u) = (2/n) + 2 \sum_{i=1}^{\lfloor (n-2)/2 \rfloor} (1/i)$. Therefore,

$$\text{VLRS}(C_n) = \frac{1}{2} \sum_{uv \notin E(C_n)} [rs(u) + rs(v) + rs(u)rs(v)]$$

$$= \frac{n(n - 1) - 2n}{4} \left[\frac{4}{n} + 4 \sum_{i=1}^{\lfloor (n-2)/2 \rfloor} \frac{1}{i} + \left(\frac{2}{n} + 2 \sum_{i=1}^{\lfloor (n-2)/2 \rfloor} \frac{1}{i}\right)^2\right].$$

Case (ii): similarly, as we obtained in Proposition 3 in Case (ii), if $n$ is odd, then for $u \in V(C_n)$, we have $rs(u) = 2 \sum_{i=1}^{\lfloor (n-2)/2 \rfloor} (1/i)$. Therefore, we get

$$\text{VLRS}(C_n) = \frac{1}{2} \sum_{uv \notin E(C_n)} [rs(u) + rs(v) + rs(u)rs(v)]$$

$$= [n(n - 1) - 2n] \left[\sum_{i=1}^{\lfloor (n-2)/2 \rfloor} \frac{1}{i} + \left(1 + \sum_{i=1}^{\lfloor (n-2)/2 \rfloor} \frac{1}{i}\right)\right].$$

\[ \Box \]

**Proposition 11.** For a wheel graph $W_{k+1}$ ($k \geq 4$),

$$\text{VLRS}(W_{k+1}) = \frac{1}{16} (k^2 - 3k)(k^2 + 10k + 21).$$

**Proof.** It is known that the nonadjacent pairs of vertices of the wheel graph $W_{k+1}$ have degree 3, and there are actually
total \( k(k + 1)/2 - 2k \) pairs of nonadjacent vertices. Also, \( \text{di am}(W_{k+1}) = 2 \). Therefore, by the equality part of Theorem 2,

\[
\text{VLRS}(W_{k+1}) = \frac{1}{8} \sum_{uv \notin E(W_{k+1})} 2[d(u) + d(v)] + d(u) \cdot d(v) + k[d(u) + d(v) + k + 4]
\]

\[
= \frac{1}{16} (k^2 - 3k)(k^2 + 10k + 21).
\]

Table 1: Correlation of VLRS index and some properties of Butane derivatives.

| Name of the compound       | Surface tension | Complexity | Heavy atomic count | VLRS  |
|----------------------------|-----------------|------------|--------------------|-------|
| 1,4-Butanedithiol          | 31.1            | 17.5       | 6                  | 37.9902 |
| 2-Butanone                 | 22.9            | 38.5       | 5                  | 28.3333 |
| 1,3-Butanediol             | 34.9            | 28.7       | 6                  | 42.0138 |
| Butane dinitrile           | 40.7            | 92         | 6                  | 80.0276 |
| Butanediamide              | 53              | 96.6       | 8                  | 79.8916 |
| Butane-1-sulfonamide       | 41.9            | 133        | 8                  | 85.0402 |
| 1-Butanethiol              | 24.8            | 13.1       | 5                  | 25.1528 |
| 1,4-Diaminobutane          | 35.8            | 17.5       | 6                  | 37.9902 |
| Butane-1,4-disulfonic acid | 77.9            | 266        | 12                 | 191.8427 |
| Butyraldehyde              | 23.1            | 24.8       | 5                  | 25.1528 |
| 2,3-Butanedione             | 27.3            | 71.5       | 6                  | 46.6667 |
| Butanedihydrazide          | 59              | 119        | 10                 | 137.2034 |
| 1-Butanesulfonyl chloride  | 36.4            | 133        | 8                  | 85.0402 |
Hence, the result.

**Proposition 12.** For a windmill graph $F_k$ with $(k \geq 2)$, 
$\nabla_{VLRS}(F_k) = (1/16)(4k^2 - 4k)(4k^2 + 16k + 12)$.

$$\nabla_{VLRS}(F_k) = \frac{1}{8} \sum_{uv \in E(F_k)} 2[d(u) + d(v)] + d(u) \cdot d(v) + 2k[d(u) + d(v) + 2k + 4]$$

$$= \frac{1}{16}(4k^2 - 4k)(4k^2 + 16k + 12).$$

as required.

6. Correlation of $VL$ Reciprocal Status Index and Some Properties of Butane Derivatives

QSPR studies have taken attention from both mathematicians and chemists since it opens new ways to new inventions. In particular, obtaining the properties of compounds without any big effort is time and money saving. Here, we see that $VL$ reciprocal status index $VLRS$ has a nice correlation including all three physical properties surface tension, complexity, and heavy atomic count, see Table 1.

Correlation coefficient value of $VLRS$ with surface tension, complexity, and heavy atomic count is 0.95, 0.98, and 0.97, respectively, see Figure 2.

Using the data of Table 1, the scatter plot between the surface tension, complexity, heavy atomic count, and $VLRS$ index of Butane derivatives is shown in Figure 2.

7. Conclusion

In the meaning of invariants over graphs, graph theoretical indices are used for the computation QSAR and QSPR. A large number indices have been exposed in literature which some of them have placed the application areas such as model physical and molecules in chemistry and pharmacy fields. By introducing the $VL$ reciprocal status index and $VL$ reciprocal status co-index of connected graphs as new indices in mathematical part of chemistry, in here, we stated and proved some upper and lower bounds on these new indices. Further, $VLRS$ index and $VLRS$ co-index of certain standard graphs are obtained. As an important conclusion, it is observed that $VLRS$ index has a nice correlation on heavy atomic count 0.98.

Motivating this work, in forthcoming papers, one may sketch the results related to $VL$ status (co-)index in terms of different parameters and indices. Also, it may be planned to design on the results about $VL$ status index and co-index of some transmission regular graphs, nanostructures, and certain Archimedean lattice as well as may be planned to create a platform to study different structured $VL$ indices in provisions of chemical/biological aspects. We are sure this paper will be useful into QSPR/QSAR studies.

Proof. It is already known that the nonadjacent pairs of vertices of the windmill graph $F_k$ have degrees 2, and there are total $(2k(2k + 1)/2) - 3k$ such pairs in $F_k$. Further, $\text{di am}(F_k) = 2$. Therefore, again by the equality part of Theorem 2,

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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