Scalar Field in Noncommutative Curved Space Time

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We study the issue of complex scalar field theories in noncommutative curved space time (NCCST) with a new star-product. In this paper, the equation of motion of scalar field and the canonical energy-momentum tensor of scalar field in static noncommutative curved space time (SNCCST) will be found. The most important point is the assumption of the noncommutative parameter ($\theta$) be $x^\mu$-independent, the noncommutativity with time attended (TNC) and the metric tensor is time independent.

Keywords: Non-commutative Geometry, Non-commutative QED

I. INTRODUCTION

Since Newton the concept of space time has gone through various changes. All stages, however, had in common the notion of a continues linear space. Today we formulate fundamental laws of physics, field theories, gauge field theories, and the theory of gravity on differentiable manifolds. That a change in the concept of space for very short distances might be necessary was already anticipated by Riemann. There are indications today that at very short distances we might have to go beyond differential manifolds. This is only one of several arguments that we have to expect some changes in physics for very short distances. Other arguments are based on the singularity problem in quantum field theory and the fact theory of gravity is non re-normalizable when quantized. Why not try an algebraic concept of space time that could guide us to changes in our present formulation of laws of physics? This is different from the discovery of quantum mechanics. There physics data forced us to introduce the concept of noncommutativity.

The concepts of noncommutative subjects were born many years ago, where the idea of noncommutative coordinates is almost as old as quantum mechanics. There are many approaches to noncommutative geometry and its use in physics. The operator algebra and C*-algebra one, the deformation quantization one, the quantum group one, and the matrix algebra (fuzzy geometry) one. Most of these approaches focus on free or interacting QFTs on the Moyal-Weyl deformed or $\kappa$-deformed Minkowski space time. The some physicists prefer to work in the noncommutative field theory, because the NCFT has many ambiguous problems such as UV/IR divergence and causality[1–5].

We briefly introduce the new star-product. By deforming the ordinary Moyal-Wyle star-product ($*$-product), we propose a new star-product ($\triangleright$-product) which takes into consideration the missing terms cited above which generate gravitational terms to the order $\theta^2$. In Ref. [6], the authors have shown that we can formulate of field theory on noncommutative curved space time with replacing of operators variables. We can replace the noncommutative flat space time coordinates variables [$\hat{x}^\mu$, $\hat{x}^\nu$] = $i\theta^{\mu\nu}$ with noncommutative curved space time coordinates variables $\hat{X}^\mu = \hat{x}^\mu + \frac{\theta^{\mu\nu}}{2\sqrt{-g}} \partial^\nu \hat{R}_{\alpha\beta}^{\mu\nu}$ where $\hat{R}_{\alpha\beta}^{\mu\nu}$ stands for the Riemann’s curvature tensor and $\sqrt{-g}$ is determinant of metric, where they are functions in noncommutative coordinates. This product is a nonassociative case, in contrast of the Moyal-Wyle $*$-product but, with certain conditions, would be an associative product. For two any smooth functions, $A$ and $B$, we have $\hat{A}(\hat{X})\hat{B}(\hat{X}) = A(x) e^{\frac{i\theta^{\mu\nu}}{2\sqrt{-g}} \partial^\nu \hat{R}_{\alpha\beta}^{\mu\nu}} B(x)$ where $\hat{\cdot}$ stands for the identity of vector spaces. By considering of $\Delta^\mu = r_2^{\mu\nu\rho\sigma} \partial^\nu \hat{R}_{\alpha\beta}^{\mu\nu}$ and $\triangleright \equiv e^{\Delta^\mu \partial^\nu (\hat{\Sigma} \otimes \hat{\Sigma}) + \frac{\theta^{\mu\nu}}{2\sqrt{-g}} \partial^\nu \hat{R}_{\alpha\beta}^{\mu\nu}}$ we have $(A \triangleright B)(\hat{x}) = A e^{\frac{i\theta^{\mu\nu}}{2\sqrt{-g}} \partial^\nu \hat{R}_{\alpha\beta}^{\mu\nu}} B(x) + \Delta^\mu \partial^\nu (A \triangleright B)(x)$ when all of functions to fall of faster than $| \hat{\cdot} |^{-\frac{1}{2}(d-1)}$, so we can remove the last term and this implies that the new star product will be an associative. In continue, we would like to introduce a new symbol $S_\ast(A_1, A_2, ..., A_n)$ where the $S_\ast$ takes the total symmetrical structures of $A_1,...,A_n$.

II. CONSTRUCTION OF ACTION AND SEARCH OF THE EQUATION OF MOTION OF FIELDS

We start by showing how to construct an action for scalar field (or vector scalar field) in the TNC with consideration of the metric tensor be time independent. If one directly follows the general rule of transforming usual theories in
noncommutative ones by replacing product of fields by star product [2, 3] and we believe that these changes should be done on the Lagrangian density. If the metric tensor will be time independent, so the Riemann’s curvature tensor is time independent, and the last term in $\triangleright$-product ($\frac{\alpha_{\mu} g_{\rho \sigma}}{2\sqrt{-g}} \partial_{\sigma} R_{\alpha \mu \rho \gamma} (\mathbb{S}_A \otimes \mathbb{S}_B)$) can be dropped out so, we can write an associative noncommutative quantum field theory. In fact, $s_{Cm}(\mathcal{L}_{Cm}) \rightarrow s_{Nc}(\mathcal{L}_{Nc}^{Sym})$ or

$$S = \int d^4x \sqrt{-g} \mathcal{L}_{S_{Nc}^{Sym}}$$

The classical Lagrangian density for scalar field in general space time is

$$\mathcal{L} = \frac{1}{2} \sqrt{-g} g^{\mu \nu} \partial_{\mu} \Phi(x) \partial_{\nu} \Phi(x) - m^2 \Phi(x)^2 - \xi \mathcal{R}(x) \Phi(x)^2$$

where $\Phi$ is the scalar field, $m$ is mass of field, $\nabla_{\mu}$ is the covariant derivative where for the scalar field we have $\nabla_{\mu} \Phi = \partial_{\mu} \Phi$ and the coupling between the field and gravitational field represented by the term $\xi \mathcal{R}(x)$ where $\xi$ is a numerical factor and $\mathcal{R}(x)$ is the Ricci’s scalar curvature. But because of the absence of quantum gravity certainly here, after we take $\xi = 0$. In the noncommutative curved space time with $\triangleright$-product the action is more complicated due to taking into account symmetric ordering.

We are in particular interested in the deformation of the canonical action

$$s = \int d^4x \left( \frac{1}{2} \sqrt{-g} \left( S_{c}(\nabla_{\mu} \Phi, g^{\mu \nu}, \nabla_{\nu} \Phi) - m^2 \Phi(x)^2 \right) \right)$$

(2)

we should take the new covariant derivative ($\nabla_{\mu} \ast$) because the curve space time is consideration and we know $\nabla_{\mu} \ast A^\alpha = \partial_{\mu} A^\alpha + \Gamma_{\rho \mu}^\alpha \ast A^\rho$. If we have the spatial noncommutativity, the symmetric ordering can be choice

$$S_{c}(\nabla_{\mu} \Phi, g^{\mu \nu}, \nabla_{\nu} \Phi)$$

(3)

but in the case of TNC the metric tensor does not participate in the star-product and the earlier star product ($\triangleright$-product) will be an associative (only if the $\Delta$ will be a constant). By using these tools, we can deform the classical Lagrangian into TNC, $2\mathcal{L}_{\ast} = g^{\mu \nu} S_{\ast}(\nabla_{\mu} \Phi, \nabla_{\nu} \Phi) - m^2 S_{\ast}(\Phi, \Phi)$ so we can deform the classical action to the global expression

$$s = \int d^4x \left( \frac{1}{2} \sqrt{-g} \left( g^{\mu \nu} S_{\ast}(\nabla_{\mu} \Phi, \nabla_{\nu} \Phi) - m^2 \Phi(x)^2 \right) \right)$$

(4)

where

$$\Phi_{\alpha} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \end{pmatrix}$$

(5)

For the research of the equation of motion of field we can write the principle of the least action $\frac{\delta s}{\delta \Phi_{\alpha}(z)} = 0$, namely,

$$\frac{\delta s}{\delta \Phi_{\alpha}(z)} = \frac{\delta}{\delta \Phi_{\alpha}(z)} \int d^4x \sqrt{-g} \left( g^{\mu \nu} (\nabla_{\mu} \Phi_{\alpha} \ast \nabla_{\nu} \Phi_{\alpha}) - m^2 \Phi_{\alpha} \ast \Phi_{\alpha} \right)$$

(6)

but we know $\nabla_{\mu} \ast (A \ast B_i) = \partial_{\mu} A \ast B_i + A \ast \nabla_{\mu} \ast B_i$ so we get to

$$\partial_{\nu} (\sqrt{-gg^{\mu \nu}} \ast \nabla_{\mu} \Phi_{\alpha}) - [\Gamma_{\nu \lambda}^\mu, \sqrt{-gg^{\mu \lambda}} \ast \nabla_{\mu} \Phi_{\alpha}] + m^2 \nabla_{\nu} \Phi_{\alpha} \ast \Phi_{\alpha} = 0$$

(7)

$$\partial_{\nu} (\nabla_{\mu} \Phi_{\alpha} \ast (\sqrt{-gg^{\mu \nu}})) - [\Gamma_{\nu \lambda}^\mu, \nabla_{\mu} \Phi_{\alpha} \ast (\sqrt{-gg^{\mu \lambda}})] + m^2 \Phi_{\alpha} \ast \sqrt{-g} = 0$$

(8)

For real scalar field we get to

$$\partial_{\nu} (\nabla_{\mu} \Phi \ast \sqrt{-gg^{\mu \nu}}) - [\Gamma_{\nu \lambda}^\mu, \{ \nabla_{\mu} \Phi, \sqrt{-gg^{\mu \lambda}} \}] + m^2 \{ \Phi \ast \sqrt{-g} \} = 0$$

(9)
III. THE FORMAL CANONICAL ENERGY-MOMENTUM TENSOR

It is useful to study the derivative the different pieces of the action with respect to $g^{\mu\nu}$. In view of the physical interpretations to be given later we introduce the new tensor $T^{\mu\nu}$ via

$$-2\frac{\partial}{\partial \sqrt{-g}} \frac{\delta s}{\delta g^{\mu\nu}} = T^{\mu\nu}$$

as the field symmetric energy-momentum tensor. Consider the action of field for $\theta^{0i} \neq 0$. In this case, we can remove all $\star$’s related to metric tensor, because we do not search the momentum of metric [7]. However, variation with respect to $g_{\mu\nu}$ for 4 gives

$$\delta s = \int d^d x(\delta \sqrt{-g} \ L_\star + \sqrt{-g} \ \delta L_\star) ,$$

(10)

We first perform the variation of $\sqrt{-g}$ with respect to $\delta g_{\mu\nu}$. For this we write $\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$ [8].

$$\delta s = \int d^d x(\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \ L_\star + \sqrt{-g} \ \delta L_\star) ,$$

(11)

consider now the variation of Lagrangian density, we get to

$$T_{\eta\kappa} = 2g_{\eta\kappa} L_\star - (\nabla_\eta \Phi^a \star \nabla_\kappa \Phi_a + \nabla_\kappa \Phi^a \star \nabla_\eta \Phi_a)$$

(12)

and we see that $T_{\mu\nu} = T_{\nu\mu}$.

Discussion

We consider a new star-product in noncommutative geometry developed in a static curved space time and we write an action for the complex scalar field theory in SNCCST and we study the equation of motion of fields. We are lead to the new equation of motion of field which it is reduced to the equation of motion of scalar field in noncommutative flat space time when the Riemann’s curvature tensor will be zero and $\theta$ be $x^\mu$-independent. Additionally, we construct the typical symmetric energy-momentum tensor, from general way in static noncommutative curved space time.

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