THE ALEXANDER POLYNOMIAL AT PRIME POWERS

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Abstract. When $K$ is a knot and $p \gg 0$ is a prime, we discuss a finite set whose cardinality is $\Delta_K(p^n)$, the value of the Alexander polynomial of $K$ at $p^n$.

1. Introduction

This paper and its sequel explores a literal interpretation of the standard analogy between the Frobenius in the absolute Galois group of a finite field, and the generator of the fundamental group of a circle.

We define an “$F$-field” on a manifold $M$ to be a locally constant sheaf of algebraically closed fields of positive characteristic. The best example is the $F$-field $k$ on a circle whose fiber at a base point is an algebraic closure $\bar{k}$ of $\mathbb{F}_p$, and whose monodromy around the circle is the $p$th power map — we write $S^1_{\mathbb{F}_p}$ to indicate the circle along with such an $F$-field. Every other good example is pulled back from this one along a map $f : M \to S^1_{\mathbb{F}_p}$.

The pullback of $k$ along $f$ is a sheaf of rings, and we can consider sheaves of modules over it. For example, the category of locally free $k$-modules of finite rank on $S^1_{\mathbb{F}_p}$ is equivalent to the category of finite-dimensional $\mathbb{F}_p$-vector spaces. In this paper we will prove the following:

Proposition. If $M$ is the complement of a knot in $S^3$, $p$ is a sufficiently large prime, and $f : M \to S^1_{\mathbb{F}_p}$ has degree $n$ on $H_1(M)$, then the set of isomorphism classes of invertible $f^*k$-modules is a finite commutative group of cardinality $\Delta_K(p^n)$.

In the Proposition, $\Delta_K(x) = c_0 + c_1 x + \cdots + c_n x^n$ denotes the classical Alexander polynomial of $K$, with Alexander’s normalization that $c_0$ should be a positive integer. The primes $p$ that must be excluded are the divisors of $c_0$.

The Proposition gives a combinatorial interpretation of the Alexander polynomial that seems to be new, though it can be proved by unwinding the sheaf-theoretic definitions until one arrives at the cokernel of Alexander’s matrix, evaluated at $x = p^n$. We will express this “unwinding of definitions” as a brief development of theory in §2, and give the proof in §3.

A sequel paper will make a more detailed study of the influence of $F$-fields on categories of constructible sheaves and other categories associated to symplectic manifolds. The Proposition perhaps gives some motivation for doing so.

In contrast with knot complements and most other 3-manifolds, when $M$ is two-dimensional the locally constant sheaves of $f^*k$-modules come in positive-dimensional moduli over the ground field $k$. Local systems over a nontrivial $F$-field are much more stable than traditional local systems, in the GIT sense that their automorphism groups are finite. When $M$ is symplectic, one can study a Fukaya category whose objects are Lagrangian submanifolds $L \subset M$ equipped with locally constant sheaves of $(f^*k)|_L$-modules. The sequel will develop some examples related to Deligne-Lusztig varieties.
1.1. **Notation.** Throughout, \( p \) is a prime number. We let \( \mathbb{F}_p \) denote the finite field with \( p \) elements, and fix an algebraic closure \( k \) of \( \mathbb{F}_p \) once and for all. If \( q \) is a power of \( p \), we write \( \mathbb{F}_q \) for the subfield of \( k \) with \( q \) elements, fixed by the \( q \)th-power automorphism of \( k \).

2. **\( F \)-fields and local systems**

For each prime power \( q \) we fix an oriented circle \( S^1_{\mathbb{F}_q} \), endowed with a base point. For definiteness, we put \( S^1_{\mathbb{F}_q} = \mathbb{R} / \log(q) \mathbb{Z} \), oriented in the positive direction and with the base point at the coset of 0. We call these the “reference circles.”

We endow each reference circle with a sheaf of rings \( k = k_{\mathbb{F}_q} \). It is locally constant with fiber \( k \) (as in §1.1), and the monodromy from 0 to \( \log(q) \) is given by \( a \mapsto a^q \). More formally, define the \( \acute{e}tale \) space of \( k \) to be the quotient of \( \mathbb{R} \times k \) by the equivalence relation

\[
(2.0.1) \quad k = (\mathbb{R} \times k) / \sim \quad (t, x) \sim (t + \log(q), x^q)
\]

2.1. **\( F \)-fields.** We define an \( F \)-field of characteristic \( p \) on a space \( X \) to be a continuous map \( \mathfrak{f}_X : X \to S^1_{\mathbb{F}_p} \). When \( q = p^r \) is a power of \( p \) we have a canonical \( F \)-field \( S^1_{\mathbb{F}_q} \to S^1_{\mathbb{F}_p} \) sending \( t + \log(q) \mathbb{Z} \) to \( t + \log(p) \mathbb{Z} \).

When \( X \) carries an \( F \)-field \( \mathfrak{f} : X \to S^1_{\mathbb{F}_p} \), it carries a locally constant sheaf of rings \( \mathfrak{f}^* k \) as well, the pullback of (2.0.1) along \( \mathfrak{f} \). If the \( F \)-field is clear from context we will sometimes write \( k_X \) or just \( k \) in place of \( \mathfrak{f}^* k \). We write \( \text{Loc}(X, \mathfrak{f}^* k) \) for the category of locally free sheaves of \( \mathfrak{f}^* k \)-modules on \( X \). It is an abelian category but it is not usually \( k \)-linear. For example, \( \text{Loc}(S^1_{\mathbb{F}_q}, k) \) is equivalent to the category of finite-dimensional \( \mathbb{F}_q \)-modules. The equivalence is given by the global sections functor \( L \mapsto \Gamma(S^1_{\mathbb{F}_q}, L) \), whose \( \mathbb{F}_q \)-module structure comes from \( \Gamma(S^1_{\mathbb{F}_q}, k) \cong \mathbb{F}_q \).

2.2. **Variants of \( k \).** Any automorphism at all can be repurposed as a locally constant sheaf on a circle; e.g. we could replace \( k \) with any perfect ring \( R \) along with its \( p \)th power automorphism. If we let \( \mathring{R} \) denote the corresponding sheaf of rings on \( S^1_{\mathbb{F}_p} \), then studying \( \text{Loc}_{\mathring{n}}(X, \mathfrak{f}^* \mathring{R}) \) as \( R \) runs through perfect \( k \)-algebras gives a moduli functor that is represented by a perfect Deligne-Mumford stack. We will study these and similar moduli problems in the sequel paper — in the case of a knot complement these moduli stack s are zero-dimensional.

Here are three other significant examples of a different flavor:

2.2.1. **Isocrystals.** Let \( \overline{\mathbb{Q}}_p \) be the maximal unramified extension of \( \mathbb{Q}_p \), whose residue field is \( k \). The Frobenius automorphism of \( k \) lifts to a field automorphism of \( \overline{\mathbb{Q}}_p \) that we denote by \( \sigma \). Then \( \overline{\mathbb{Q}}_p \) is the fiber of a sheaf of rings on \( S^1_{\mathbb{F}_p} \) whose monodromy is \( \sigma \), we denote it by \( \overline{\mathbb{Q}}_p^{\infty} \). A locally constant sheaf of \( \overline{\mathbb{Q}}_p^{\infty} \)-modules is the same data as an isocrystal, studied in [Dieu].

2.2.2. **The Tate motive.** The action of \( \pi_1(S^1_{\mathbb{F}_p}) \) on the multiplicative group \( k^* \) of \( k \) corresponds to a local system of abelian groups that we denote by \( k^* \). There is a closely related local system of \( \mathbb{Z}[1/p] \)-modules, which we denote by \( \mathbb{Z}[1/p](1) \) — the fiber is \( \mathbb{Z}[1/p] \) and the generator of \( \pi_1(S^1_{\mathbb{F}_p}) \) acts by multiplication by \( p \). The two sheaves are related by

\[
(\text{sheaf hom})(\mathbb{Z}[1/p](1), \text{const. sheaf with fiber } k^*) \cong k^*
\]
2.2.3. Finite Chevalley groups. If $G$ is an algebraic group over $k$, and $\sigma : G \to G$ is the Frobenius isogeny coming from an $F_q$-rational structure on $G$, then $\sigma$ induces an automorphism on $k$-points $G(k) \to G(k)$. We denote the corresponding locally constant sheaf of groups on $S_{F_q}^1$ by $(G(k), \sigma)$. The groupoid of $(G(k), \sigma)$-torsors over $S_{F_q}^1$ is equivalent to the classifying groupoid of torsors over the finite group $G(k)^\sigma$.

There is a similar construction for the Suzuki or Ree isogenies of $G = Sp_4$ when $p = 2$ and $G = G_2$ when $p = 3$ — the square of one of these isogenies is a Frobenius map for an $F_q$-rational structure. Thus they again induce bijections on $k$-points and it is natural to regard $(G, \sigma)$ as a sheaf of groups on a reference circle of circumference $\log(\sqrt{q})$.

2.3. Framings. We write $\text{Loc}_n(X, \mathfrak{f}^q k)$ for the groupoid of locally free sheaves of rank $n$ in $\text{Loc}(X, \mathfrak{f}^q k)^\cong$ and all isomorphisms between them. There are two ways to present the groupoid as a quotient:

2.3.1. Point framings. Let $x \in X$ be a base point and let $E \in \text{Loc}_n(X, \mathfrak{f}^q k)$. A point framing of $E$ at $x$ is a $(\mathfrak{f}^q k)_x$-basis in $E_x$. Note $(\mathfrak{f}^q k)_x$ is canonically identified with $k_{\mathfrak{f}(x)}$. The set of isomorphism classes of point-framed local systems of rank $n$ is in bijection with the set of homomorphisms

\[
\rho : \pi_1(X, x) \to \text{GL}_n(k_{\mathfrak{f}(x)}) \rtimes \pi_1(S_{F_q}^1)
\]

that commute with the projections to $\pi_1(S_{F_q}^1)$. In the semidirect product in (2.3.1), the distinguished generator of $\pi_1(S_{F_q}^1)$ acts on $\text{GL}_n(k_{\mathfrak{f}(x)})$ by raising each matrix entry to the $p$th power.

If we write $\rho(\gamma) = \rho_1(\gamma) \rtimes \mathfrak{f}(\gamma)$, the bijection sends $\rho$ to the locally constant sheaf of abelian groups whose fiber above $x$ is $k^\mathfrak{f} X, x$, and whose monodromy around the loop $\gamma$ acts on the vector $(v_1, \ldots, v_n)$ by

\[
\rho_1(\gamma)(v_1^{\mathfrak{f}(\gamma)}, \ldots, v_n^{\mathfrak{f}(\gamma)}).
\]

The groupoid $\text{Loc}_n(X, \mathfrak{f}^q k)$ is equivalent to the quotient of the set of (2.3.1) by the $\text{GL}_n(k_{\mathfrak{f}(x)})$-conjugation action.

2.3.2. Section framings. With $q = p^r$, we shall call the data of a map $s : S_{F_q}^1 \to X$ commuting with the projections to $S_{F_q}^1$, a $\nu$-sheeted multisection of the $F$-field $\mathfrak{f}$. Each $E \in \text{Loc}_n(X, \mathfrak{f}^q k)$ restricts to a locally constant sheaf on such a multisection, or equivalently an $n$-dimensional $F_q$-vector space, $\Gamma(s, E)$. A section framing of $E$ at $x$ is an $F_q$-basis for this vector space.

A section framing induces a point framing, at the image of the base point of $S_{F_q}^1$ under $s$. Using $g$ to denote the distinguished generator of $\pi_1(S_{F_q}^1)$, we say that homomorphism (2.3.1) preserves the section-framing if it carries $s$ to $1 \rtimes g^\nu$. These homomorphisms are in bijection with the set of section-framed local systems. A change of basis in $\Gamma(s, E)$ corresponds to conjugating $\rho$ by an element in the finite group $\text{GL}_n(k_{\mathfrak{f}(x)}) \subset \text{GL}_n(k_{\mathfrak{f}(x)})$. In particular this shows that $\text{Loc}_n(X, \mathfrak{f}^q k)$ has finite isotropy groups whenever $X$ is connected and the $F$-field is nontrivial (i.e. whenever $\pi_1(X) \to \pi_1(S_{F_q}^1)$ is nonzero).

2.4. Invertible modules. When $n = 1$, the homomorphisms (2.3.1) do not necessarily factor through an abelian quotient of $\pi_1(X, x)$. Nevertheless the tensor product over $\mathfrak{f}^q k$ endows the set of isomorphism classes in $\text{Loc}_1(X, \mathfrak{f}^q k)$ with the structure of commutative group. The groupoid $\text{Loc}_1(X, \mathfrak{f}^q k)$ has a symmetric monoidal structure — it is a commutative 2-group.
Indeed, if we regard the abelian group GL$_1(k_{ij}(x))$ as a $\pi_1(X, x)$-module through the homomorphism $\pi_1(X, x) \rightarrow \pi_1(S^{3}_F, p)$, and $\rho_1 : \pi_1(X, x) \rightarrow$ GL$_1(k_{ij}(x))$ is a 1-cocycle on $\pi_1(X, x)$ with coefficients in this module, then $\rho_1 \times f$ is a homomorphism of the form (2.3.1). This is a bijection between such cocycles and point-framed local systems. If we write $Z^1 := Z^1(\pi_1(X, x), \text{GL}_1(k_{ij}(x)))$ for this group of cocycles, the commutative 2-group structure on Loc$_1(X, f^*k)$ can be encoded by the two-term chain complex

$$\text{GL}_1(k_{ij}(x)) \rightarrow Z^1$$

where the differential is the usual differential in group cohomology. If $\pi_1(X, x) \rightarrow \pi_1(S^{1}_F, p)$ is nonzero, the kernel is a finite group (it is GL$_1$ of a finite subfield of $k_{ij}(x)$).

3. PROOF OF THE PROPOSITION

Fix a diagram $D$ for a knot with $v$ crossing points and $v + 2$ regions. Let us label the regions $0, \ldots, v+1$, and orient $K$. Then there is a Dehn presentation of $\pi_1(S^{3} - K)$ with $v + 1$ generators and $v$ relations [Alex, §8]. The generators $g_i$ correspond to regions $1, \ldots, v + 1$ of $D$, and we put $g_0 = 1$. Each crossing gives a relation between the generators associated to the four regions incident with it, as in the following diagram,

$$g_jg_k^{-1}g_lg_m^{-1} = 1 \quad \frac{j}{m} \quad \frac{k}{\ell}$$

(In [Alex], knot diagrams are drawn with two dots on the left side of the underpass crossing — we have put them in the diagram above as well.)

The “index” of a region defined by Alexander [Alex, Fig. 2] gives a homomorphism

$$I : \pi_1(S^{3} - K) \rightarrow \mathbb{Z}$$

A choice of prime $p$ and prime power $q = p^v$ turns the index homomorphism into an $F$-field $f_1$ by identifying it with a homotopy class of maps $S^{3} - K \rightarrow S^{1}_F$. A rank one point-framed local system of $f^*k$-modules on $S^{3} - K$ is completely specified by a family of scalars $z_j \in \text{GL}_1(k)$, one for each region. In terms of these scalars, the action of $g_j$ (and its inverse, recorded for convenience) on $x \in k$ is given by

$$g_j(x) = x^{q^{l(j)}}z_j \quad \text{(and } g_j^{-1}(x) = x^{q^{-l(j)}}z_j^{-1})$$

The scalars $z_j$ must obey $z_0 = 1$ and an additional relation for each crossing incident with regions $j, k, \ell, m$ as above, which reduces to

$$z_jz_k^{-q}z_\ell z_m = 1 \quad \text{or} \quad z_jz_k^{-q}z_\ell z_m^{-1} = 1$$

according to whether the crossing is left-handed or right-handed, respectively. An element $y \in \text{GL}_1(k)$ acts on the point-framing by sending $(z_j)_{j=1}^{v+1}$ to $(y^{l(j)}-1)_{j=1}^{v+1}z_j$. Thus, the group of isomorphism classes of objects in Loc$_1(X, f^*k)$ is isomorphic to the middle cohomology of the chain complex

$$\text{GL}_1(k) \rightarrow \text{GL}_1(k)^{v+1} \rightarrow \text{GL}_1(k)^v$$

where the first and second differentials are $y \mapsto (y^{q^{l(j)-1}})_{j=1}^{v+1}$ and $(z_j)_{j=1}^{v+1} \mapsto (z_jz_k^{-q}z_\ell z_m^{q}z_m^{-1})$. 

\[\text{(3.0.1)}\]
Consider the $v \times (v + 1)$-matrix whose rows are indexed by the crossings, whose columns are indexed by non-null regions, and whose $(c, j)$-entry is indicated by the following diagram if $j$ is incident with $c$, and is otherwise 0:

\[
\begin{array}{c|c}
1 & -1 \\
-x & x \\
\end{array}
\begin{array}{c|c}
1 & -1 \\
-x^{-1} & x^{-1} \\
\end{array}
\]

Let $D'$ denote the diagram obtained from $D$ by switching the sense of over and under at every crossing. If we multiply each row corresponding to a right-hand crossing by $x$, we obtain the usual Alexander matrix (i.e. the coefficient matrix of the system of equations [Alex, Eq. 3.3]) for the diagram $D'$, with the column corresponding to the null region left off. In particular $A(x)$ is elementary equivalent to the usual Alexander matrix for both $D'$ and (using the mirror invariance of the Alexander invariants) $D$.

By evaluating $A(x)$ at $x = q = p^ν$, we obtain a matrix $A(q) \in \mathbb{Z}[1/p]^{v \times (v + 1)}$. Then (3.0.1) is obtained by taking Hom($C, GL_1(k)$), where $C$ is a complex of free $\mathbb{Z}[1/p]$-modules of the form

\[
\mathbb{Z}[1/p]^v \xrightarrow{A(q)} \mathbb{Z}[1/p]^{v+1} \rightarrow \mathbb{Z}[1/p]
\]

In particular, the order of the middle cohomology of (3.0.1) is equal to the order of the middle cohomology of this complex, which is equal to the order of the torsion subgroup of the cokernel of $A(q)$. This in turn is equal to the prime-to-$p$ part of the greatest common divisor of all the $v \times v$-minors of $A(q)$. If $p$ does not divide the constant term of $\Delta_K(x)$, that divisor is just $\Delta_K(q)$.

4. Plausible generalizations

It is natural to consider the size of $\text{Loc}_1(M; \mathfrak{f}^*k)$ “as an orbifold”, i.e. to weight each isomorphism class by the reciprocal of the order of its automorphism group. In the case of a knot complement these automorphism groups are not very sensitive to $\mathfrak{f}$ so that the orbifold count is $\Delta_K(q)/(q - 1)$. In the case of a more general 3-manifold both the kernel and cokernel of (2.4.1) are more irregular as $\mathfrak{f}$ varies, but the orbifold count is likely to have a clean relation to the multivariable Alexander polynomial.

One can study local systems of modules over $\mathfrak{f}^*\mathbb{Q}_p$. By restricting such a local system $L$ to a meridian of the knot, one gets a rank one isocrystal whose slope is a discrete invariant of $L$. The set of $L$ of a fixed slope make a $p$-adic analytic manifold (in fact a torsor for a commutative $p$-adic analytic group) whose dimension over $\mathbb{Q}_p$ is the degree of the Alexander polynomial. I suspect that these $p$-adic manifolds carry natural measures, perhaps up to powers of $p$, of volume $\Delta_K(p^n)$.

One might obtain interesting invariants by counting nonabelian local systems of $\mathfrak{f}^*k$-modules, or more generally torsors for the sheaves of groups $\mathfrak{f}^*(G, \sigma)$. The problem of doing so can be expressed as the problem of solving high-degree equations in the entries of a matrix in $G(k)$. For example (using the Wirtinger presentation of $\pi_1(S^3 - K)$ in place of the Dehn presentation) a $\mathfrak{f}^*(G, \sigma)$-torsor on the complement of a trefoil is determined by an element $g \in G(k)$ subject to the equation

\[g\sigma^2(g) = \sigma(g)\]
and taken up to the conjugation action by the finite group $G^\sigma$. But even in the simplest examples I have not been able to solve these equations directly when $G$ is not commutative.

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