DEFORMATIONS OF EUCLIDEAN SUPERSYMMETRIES

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Abstract

We consider quantum supergroups that arise in non-anticommutative deformations of $N=(\frac{1}{2}, \frac{1}{2})$ and $N=(1, 1)$ four-dimensional Euclidean supersymmetric theories. Twist operators in the corresponding deformed algebras of superfields contain left spinor generators. We show that non-anticommutative $\star$-products of superfields transform covariantly in the deformed supersymmetries. This covariance guarantees the invariance of deformed superfield actions of models involving $\star$-products of superfields.

Key words: Supersymmetry, superspace, deformation, twist

1 Introduction

The simplest type of the space-time noncommutativity is connected with the relation

$$C_{\star}^{mn} \equiv \hat{x}^m \star \hat{x}^n - \hat{x}^n \star \hat{x}^m - i\vartheta^{mn} = 0$$

(1.1)

for the coordinate operators $\hat{x}^m$, where $\vartheta^{mn}$ ($m, n = 0, 1, 2, 3$) is some constant tensor specifying the deformation of the commutative four-dimensional coordinates $x^m$ (see, e.g. [1, 2]). The noncommutative algebra of fields $\hat{f}(\hat{x})$ on this deformed space-time is formally analogous to the Weyl algebra on the quantized phase space $\hat{x}, \hat{p}$, $[\hat{p}, \hat{x}] = i\hbar$. The noncommutative algebra $A_{\star}$ is defined as the algebra of formal polynomials in $\hat{x}^m$ factored over quadratic relation (1.1). The Weyl ordering of the operator field involves a decomposition in terms of completely symmetrized monomials $\hat{x}^{(m_1 \star \ldots \star m_n)}$

$$\hat{f}(\hat{x}) = \sum_{n=0}^{\infty} c_{m_1 \ldots m_n} \hat{x}^{(m_1 \star \ldots \star m_n)},$$

(1.2)

where $c_{m_1 \ldots m_n}$ are numerical coefficients that are symmetric in $m_1, \ldots, m_n$. We can use the correspondence of this ordered operator function and an ordinary smooth function $f(x)$

$$w[\hat{f}(\hat{x})] = f(x) = \sum_{n=0}^{\infty} c_{m_1 \ldots m_n} x^{m_1} \ldots x^{m_n},$$

(1.3)
which we call the commutative image of $\hat{f}(\hat{x})$. The map inverse to the operator representation is denoted by $w^{-1}[f(x)] = \hat{f}(\hat{x})$.

A realization of this noncommutative algebra $A_\star$ that is popular in field theory can be defined on smooth functions $f(x)$ and $g(x)$ of commutative coordinates using the following pseudolocal representation of the noncommutative product:

$$w[\hat{f} \star \hat{g}] = f \star g = fe^P g = fg + \frac{i}{2} \vartheta^{mn} \partial_m f \partial_n g - \frac{1}{8} \vartheta^{mn} \vartheta^{pq} \partial_m \partial_p f \partial_n \partial_q g + \ldots,$$

$$fPg = \frac{i}{2} \vartheta^{mn} \partial_m f \partial_n g. \quad (1.4)$$

where $\partial_m = \partial / \partial x_m$. All products of the functions and their partial derivatives in the right-hand side are commutative. The formula $(\hat{f} \star \hat{g})(\hat{x}) = w^{-1}[(f \star g)(x)]$ allows constructing the ordered decomposition of the noncommutative product of operator functions via the power series of products of smooth functions. Basic computational relation (1.1) is identically satisfied in representation (1.4)

$$x^m \star x^n = x^m x^n + \frac{i}{2} \vartheta^{mn}. \quad (1.5)$$

Very many results of investigations in noncommutative field theory are obtained exactly in this convenient field theory representation using arbitrary-order derivatives of local fields, although all properties of the theory can be reformulated in the operator representation.

Let us consider a simple $\vartheta$-noncommutative interaction of the real scalar field $\phi(x)$ in the noncommutative algebra $A_\star$

$$S_\star = \int d^4x L_\star(\phi) = \frac{1}{2} \int d^4x (\eta^{mn} \partial_m \phi \star \partial_n \phi - M^2 \phi \star \phi - \lambda \phi^4), \quad (1.6)$$

where $M$ and $\lambda$ are the mass and coupling constant, respectively, $\eta^{mn}$ is the Minkowski space metric, and $\phi^4 = \phi \star \phi \star \phi \star \phi$. In representation (1.4), quadratic terms contain the standard undeformed free action and additional vanishing integrals of total derivatives. The nonlinear interaction depends manifestly on $\vartheta_{mn}$, and is therefore not invariant with respect to the standard Lorentz transformation. Despite this breaking of the Lorentz invariance, all models of the $\vartheta$-noncommutative field theory use the following selection rule: Basic noncommutative (primary) fields transform as representations of the Poincaré group, and interactions of these fields are constructed on the basis of algebra $A_\star$. With the $\star$-products being noncovariant with respect to the Lorentz group, this rule has no simple interpretation in the framework of usual symmetries.

As shown in [7]-[12], the selection rule follows from the invariance of the noncommutative field theory under the $t$-deformed quantum Poincaré group involving the twist operator

$$\mathcal{F} = \exp(\frac{i}{2} \vartheta^{mn} \partial_m \otimes \partial_n). \quad (1.7)$$

Note that other variants of quantum-group deformations of the Poincaré group were previously been considered [3]-[5]. In particular, various forms of the Drinfeld twist operator [6] were widely used in these investigations. Nevertheless, the corresponding field models have not been studied in as much detail as the models based on relation (1.1).
In sect. 2, we consider the \( t \)-deformed Lorentz transformations for the \( \star \)-product of primary fields using local relations between the differential operators on commutative and noncommutative algebras in [10, 11].

Deformations of supersymmetric theories are characterized by a Poisson bracket \( APB \) on the superfields \( A \) and \( B \), where the operator \( P \) is a general quadratic form in terms of derivatives with respect to the even and odd coordinates in the superspace [13, 14]. Important classes of nilpotent \( Q \)-deformations of the Euclidean supersymmetries were found in [15] for the case of \( N=\left(\frac{1}{2}, \frac{1}{2}\right) \), \( D=4 \) supersymmetry and in [16, 17] for the \( N=\left(1, 1\right) \), \( D=4 \) supersymmetry. The deformation operators \( P \) for these models are constructed from the left spinor generators of supersymmetries \( Q \). They preserve superfield constraints for the undeformed supersymmetry representations, for instance, the chirality or Grassmann analyticity constraints. The \( Q \)-deformed superfield theories [15]-[19] use \( \star \)-products of undeformed primary superfields in the pseudolocal representation. Deformed theories are not invariant under the action of generators of the basic Euclidean supersymmetry that do not commute with \( P \).

The interpretation of deformed Euclidean supersymmetries in the framework of the Hopf algebras was introduced in [21]. Section 3 is devoted to discussing of the twist-deformed \( N=\left(\frac{1}{2}, \frac{1}{2}\right) \) supersymmetry. The non-anticommutative \( \star \)-product in the deformed superalgebra is defined on supercommutative superfields in the ordinary superspace. Primary superfields of this model are transformed as representations of \( N=\left(\frac{1}{2}, \frac{1}{2}\right) \) supersymmetry realized by the first-order differential operators. The twist operator \( \mathcal{F} \) for the nilpotent \( Q \)-deformation determines the coproduct in the \( t \)-deformed \( N=\left(\frac{1}{2}, \frac{1}{2}\right) \) supersymmetry and also the \( \star \)-product in the corresponding noncommutative algebra of superfields. We formulate the local covariance principle in this algebra: primary superfields \( A, B \) and their \( \star \)-product \( A \star B \) have the same transformations in the \( t \)-deformed supersymmetry. Generators of the deformed transformations in the operator representation are uniquely defined by the twist operator and the corresponding undeformed supersymmetry generator. For instance, the deformed generators of the right supersymmetry transformations contain deformation constants and the second-order Grassmann derivatives in the operator representation, although the Lie superalgebra remains undeformed. We show that the deformed superfield actions using the \( \star \)-product preserve the invariance under the \( t \)-deformed supersymmetry.

In sect. 4, we consider the twist operator defining the coproduct in the \( t \)-deformed \( N=\left(1, 1\right) \) supersymmetry and the corresponding \( \star \)-product in the non-anticommutative algebra of harmonic superfields. The deformed superfield actions constructed in the \( N=\left(1, 1\right) \) harmonic superspace [18]-[20] partially violate the standard supersymmetry, but these superfield actions are invariant under the transformations of the \( t \)-deformed \( N=\left(1, 1\right) \) supersymmetry. It is notable that the quadratic superfield terms of the action also preserve the ordinary supersymmetry.

It should be remarked that the quantum-group deformations of the supersymmetry with more complex superfield geometry were previously considered [23], but we do not discuss these models in our work. The \( t \)-deformed supersymmetries in the superfield theories were briefly described in [22].
2 Deformed Poincaré group

We review the basic applications of the $t$-deformed Poincaré symmetry [7]-[11] in the noncommutative field theory. The corresponding twist operator $\mathcal{F} = \exp(\mathcal{P})$ (1.7) acts on tensor products of functions

$$\mathcal{F} \circ f \otimes g = f \otimes g + \mathcal{P} \circ f \otimes g + \frac{1}{2} \mathcal{P}^2 \circ f \otimes g + \ldots,$$

$$\mathcal{P} \circ f \otimes g = i \frac{1}{2} \theta^{mn} \partial_m f \otimes \partial_n g.$$ (2.1)

The rigorous definition of noncommutative product (1.4) is related to the operator $\mathcal{F}$ and the analogous map $\mu * = \mu \circ \mathcal{F}$ defines the product in the algebra $A \star$. The twist operator is analogous to the pseudolocal operator $e^P$ (1.4) in the field theory constructions, and the tensor product can be treated as a nonlocal product of ordinary fields in different points $f(x_1)g(x_2)$.

We consider a local representation of the generators of the Poincaré group

$$P_m = \partial_m, \quad M_{mn} = x_n \partial_m - x_m \partial_n$$ (2.3)

and the corresponding infinitesimal transformations of the scalar field

$$\delta_c \phi = -c^m P_m \phi = -P_c \phi, \quad \delta_\omega \phi = -\frac{1}{2} \omega^{mn} M_{mn} \phi = -M_\omega \phi,$$ (2.4)

where $c_m, \omega^{mn}$ are the infinitesimal parameters of translations and Lorentz transformations. A finite transformation of the form of the scalar function (active Poincaré transformation) can be represented as

$$\phi'(x) = \phi(\tilde{x}) = e^{-M_\omega} e^{-P_c} \phi(x), \quad \tilde{x}^m = e^{-M_\omega} (x^m - c^m).$$ (2.5)

The transformation operator $e^{-M_\omega} e^{-P_c}$ belongs to the universal enveloping bialgebra of functions of generators $U(P_m, M_{mn})$, where the associative product of generators and coproduct $\Delta: U \to U \otimes U$ are defined. The coproduct acts trivially on unity and the generators of $U(P_m, M_{mn})$

$$\Delta(1) = 1 \otimes 1, \quad \Delta(P_c) = P_c \otimes 1 + 1 \otimes P_c, \quad \Delta(M_\omega) = M_\omega \otimes 1 + 1 \otimes M_\omega,$$ (2.6)

and the action of $\Delta$ on functions of generators is defined accordingly, for instance,

$$\Delta(M_\omega M_\omega) = \Delta(M_\omega) \Delta(M_\omega), \quad \Delta(e^{-M_\omega}) = e^{-M_\omega} \otimes e^{-M_\omega}.$$

The coproduct gives the action of generators and their functions on the tensor product of fields

$$\Delta(M_\omega) \circ f \otimes g = M_\omega f \otimes g + f \otimes M_\omega g.$$ (2.7)
Thus, the standard Leibniz rule for the infinitesimal Lorentz transformation in the commutative algebra follows from the formula for $\Delta(M_\omega)$

$$\delta_\omega(fg) = \mu \circ \delta_\omega(f \otimes g) = (\delta_\omega f)g + f\delta_\omega g.$$  \hspace{1cm} \tag{2.8}

This rule is postulated in the commutative field theory, but deformations of the coproduct and the corresponding transformation laws of the products of functions are possible in the noncommutative case.

It is evident that the noncommutative product transforms noncovariantly in the ordinary Lorentz group

$$\delta_\omega(f \ast g) = (\delta_\omega f) \ast g + (f \ast \delta_\omega g) \neq -M_\omega(f \ast g).$$  \hspace{1cm} \tag{2.9}

By definition, the $t$-deformation of the Poincaré group: $U(P_m, M_{mn}) \to U_t(P_m, M_{mn})$ does not change the Lie algebra of generators, and we can therefore use the standard representation (2.3) in $U_t$. The coproduct in $U_t(P_m, M_{mn})$ is deformed for generators $M_{mn}$

$$\Delta_t(P_c) = \exp(-\mathcal{P})\Delta(P_c)\exp(\mathcal{P}) = \Delta(P_c),$$   

$$\Delta_t(M_\omega) = \exp(-\mathcal{P})\Delta(M_\omega)\exp(\mathcal{P}) = \Delta(M_\omega)$$  

$$+ \frac{i}{2} \omega^{mn}\partial_{mn} P_n \otimes P^s - \frac{i}{2} \omega^{mn} \partial_{rn} P^r \otimes P_m.$$  \hspace{1cm} \tag{2.10}

The deformed transformations of fields in the noncommutative algebra $A_*$ was described in details in the recent works of the Munich group [10, 11]. These authors constructed a map between differential operators on the commutative and noncommutative algebras of functions. Let $\xi = \xi^m(x)\partial_m$ be the first-order differential operator on $A(R^4)$ containing an arbitrary function $\xi^m(x)$; for instance, the infinitesimal Lorentz transformation is defined by the function $\xi^m = \omega^{mn}x_n$. One can construct the corresponding operator $\hat{X}_\xi$ on the noncommutative algebra $A_*$ satisfying the simple relation

$$(\xi f) = \mu \circ (\xi^m \otimes \partial_m)(1 \otimes f) = \mu_* \exp(-\mathcal{P})(\xi^m \otimes \partial_m)(1 \otimes f) = \xi^m \ast \partial_m f$$   

$$- \frac{i}{2} \theta^{rs} \partial_r \xi^m \ast \partial_s \partial_m f - \frac{1}{8} \theta^{rs} \theta^{pq} \partial_r \partial_p \xi^m \ast \partial_s \partial_q \partial_m f + O(\theta^3) = (\hat{X}_\xi \ast f).$$  \hspace{1cm} \tag{2.11}

Note that in the general case $\hat{X}_\xi$ contains derivatives of arbitrary orders and acts, by definition, on any noncommutative functions or their commutative images.

The local generators $P_m$ and $M_{mn}$ (2.3) acting on commutative images $f(x)$ correspond to the following operators on the noncommutative functions $\hat{f}(\hat{x})$ in algebra $A_*$:

$$\partial_m f = (\hat{P}_m \ast f), \quad (\hat{P}_m \ast \hat{f}) = w^{-1}(\hat{P}_m \ast f),$$   

$$(M_{mn} f) = (\hat{M}_{mn} \ast f), \quad (\hat{M}_{mn} \ast \hat{f}) = w^{-1}(\hat{M}_{mn} \ast f).$$  \hspace{1cm} \tag{2.12}

We note that the translation generator has the standard form, and the generator of the Lorentz transformation in the operator representation contains second-order derivatives

$$\hat{M}_{mn} = x_n \ast \partial_m - x_m \ast \partial_n + \frac{1}{2} \eta_{mr} \theta^{rs} \partial_s \partial_n - \frac{1}{2} \eta_{mr} \theta^{rs} \partial_r \partial_m,$$   

$$\hat{M}_\omega = \frac{1}{2} \omega^{mn} \hat{M}_{mn} = \omega^{mn} x_n \ast \partial_m + \frac{1}{2} \omega^{mn} \eta_{mr} \theta^{rs} \partial_s \partial_n.$$  \hspace{1cm} \tag{2.13}
It is natural to use operators $\hat{P}_m = \partial_m$ and $\hat{M}_{mn}$ in the operator representation of the noncommutative algebra $A_*$ on functions $f(\textbf{x})$, for instance,

$$
(\hat{M}_\omega \star \hat{x}^m \star \hat{x}^n) = \eta_{sr}[\omega^{ms} \hat{x}^r \star \hat{x}^n + \omega^{ns} \hat{x}^m \star \hat{x}^r - \frac{i}{2}\omega^{ms}\partial^m + \frac{i}{2}\omega^{ns}\partial^n].
$$

(2.14)

It is easy to verify that the quantity $C^m_{*\nu}$ (1.1) is covariant with respect to the transformations of the deformed Lorentz group

$$(\hat{M}_\omega \star C^m_{*\nu}) = \omega^{ns}\eta_{sr}C^n_{*\nu} + \omega^{ns}\eta_{sr}C^m_{*\nu}.
$$

(2.15)

The anticommutator $A^m_{*\nu} = \hat{x}^m \star \hat{x}^n + \hat{x}^n \star \hat{x}^m$ also transforms covariantly in $U_l(P_m, M_{mn})$.

As follows from relations (2.12), the operators $\hat{P}_m$ and $\hat{M}_{mn}$ form the Lie algebra (and the corresponding associative algebra) isomorphic to the standard algebra with the generators $P_m$ and $M_{mn}$

$$(\hat{M}_{mn} \star \hat{M}_{rs} \star \hat{f}) = (M_{mn}M_{rs}f), \quad (\hat{P}_s \star \hat{M}_{mn} \star \hat{f}) = (P_sM_{mn}f).
$$

(2.16)

Coproduct (2.10) acts on tensor products of functions as

$$
-\frac{1}{2}\omega^{mn}\Delta_t(M_{mn}) \circ f \otimes g = (\delta_\omega f) \otimes g + f \otimes (\delta_\omega g)
$$

$$
+\frac{1}{2}(\omega^{mn}\nu_{ns} - \eta^{mn}\nu_{ns})\partial_m f \otimes \partial^ng.
$$

(2.17)

We apply the map $\mu_*$ (2.2) to this relation and obtain the formula for the covariant action of the deformed Lorentz transformation in the algebra $A_*$

$$
\delta_\omega \star (f \star g) = -\frac{1}{2}\omega^{mn}\mu_* \circ \Delta_t(M_{mn}) \circ f \otimes g = (\delta_\omega f) \star g + f \star (\delta_\omega g)
$$

$$
+\frac{1}{2}(\omega^{mn}\nu_{ns} - \eta^{mn}\nu_{ns})\partial_m f \star \partial^ng = -M_\omega(f \star g).
$$

(2.18)

The last relation can be verified using the pseudolocal formula (1.4). Thus, the noncommutative product of scalar fields transforms as the scalar in $U_l(P_m, M_{mn})$. It should be noted that the corresponding finite transformations of $f \star g$ in $U_l(P_m, M_{mn})$ have a form analogous to the transformation (2.5) of the primary scalar fields

$$
(f \star g)'(x) = (e^{-M_\omega}e^{-P_c}f \star g)(x).
$$

(2.19)

In this pseudolocal representation, we consider the active transformations of all field objects in the fixed point and do not discuss the dual transformations of the noncommutative product of fields together with the coordinate transformations.

Applying the map $\mu$ to (2.17) yields a noncovariant action of $U_l(P_m, M_{mn})$ on the commutative product of fields

$$
\delta_\omega \star (fg) = -\frac{1}{2}\omega^{mn}\mu \circ \Delta_t(M_{mn}) \circ f \otimes g = (\delta_\omega f)g + f(\delta_\omega g)
$$

$$
+\frac{1}{2}(\omega^{mn}\nu_{ns} - \eta^{mn}\nu_{ns})\partial_m f \partial^ng \neq -\frac{1}{2}\omega^{mn}M_{mn}(fg).
$$

(2.20)

Four-dimensional integrals of the $\star$-products of fields (1.6) are invariant under the $U_l(P_m, M_{mn})$ symmetry. The quadratic (free) terms also have the standard Poincaré invariance.

\footnote{The action of the dual quantum Poincaré group was discussed in refs.[7, 12].}
3 Deformation of $N=(\frac{1}{2}, \frac{1}{2})$ supersymmetry

In this section, we consider the simplest Euclidean supersymmetry SUSY($\frac{1}{2}, \frac{1}{2}$) generalizing the ISO(4) symmetry of the space $R^4$. We use the chiral coordinates $z^M = (y_m, \theta^a, \bar{\theta}^a)$ in the Euclidean $N=(\frac{1}{2}, \frac{1}{2})$ superspace, where $m = 1, 2, 3, 4$, $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$ are the vector and spinor indices of the group SU(2)$_L \times$SU(2)$_R$. Note that these coordinates are pseudoreal with respect to special conjugation [16]

$$(y_m)^* = y_m, \quad (\theta^a)^* = \varepsilon_{\alpha\beta} \theta^\beta, \quad (\bar{\theta}^a)^* = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\theta}^\dot{\beta}, \quad (AB)^* = B^* A^*$$  \hspace{1cm} (3.1)

for any superfields $A(z)$ and $B(z)$. For instance, one can use the reality condition for the even Euclidean chiral superfield: $[\phi(y, \theta)]^* = \phi(y, \theta)$. The central and right 4D coordinates can be expressed via the chiral coordinates

$$x_m = y_m - i\theta \sigma_m \bar{\theta}, \quad \bar{x}_m = y_m - 2i\theta \sigma_m \bar{\theta},$$  \hspace{1cm} (3.2)

where $(\sigma_m)_{\alpha\dot{\alpha}}$ are the Weyl matrices of the group SO(4). Generators of the supergroup SUSY($\frac{1}{2}, \frac{1}{2}$) have the following form:

$$L_\alpha^\beta = L_\alpha^\beta(y) + L_\alpha^\beta(\theta) = \frac{1}{4} (\sigma_m \sigma_n)_{\alpha\beta}^\gamma (y_m \partial_n - y_n \partial_m) + \theta^\beta \partial_\alpha - \frac{1}{2} \delta^\beta_\gamma \partial_{\gamma},$$

$$R_{\dot{\alpha}}^\dot{\beta} = R_{\dot{\alpha}}^\dot{\beta}(y) + R_{\dot{\alpha}}^\dot{\beta}(\bar{\theta}) = \frac{1}{4} (\sigma_m \sigma_n)_{\dot{\alpha}\dot{\beta}}^\gamma (y_m \partial_n - y_n \partial_m) + \bar{\theta}^\dot{\beta} \partial_{\dot{\alpha}} - \frac{1}{2} \delta^\dot{\beta}_{\dot{\gamma}} \partial_{\dot{\gamma}},$$

$$O = \theta^\alpha \partial_\alpha - \bar{\theta}^\dot{\alpha} \partial_{\dot{\alpha}}, \quad Q_\alpha = \partial_\alpha, \quad \bar{Q}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} - 2i \theta^\alpha \partial_{\alpha\dot{\alpha}}, \quad P_m = \partial_m,$$  \hspace{1cm} (3.3)

where $(\sigma_m)_{\dot{\alpha}}^{\alpha\dot{\alpha}}$ and $(\sigma_m)_{\alpha\dot{\alpha}}^{\alpha\dot{\alpha}}$ are the partial derivatives in the chiral coordinates

$$\partial_m y_n = \delta_{mn}, \quad \partial_{\alpha\dot{\alpha}} = (\sigma_m)_{\alpha\dot{\alpha}} \partial_m, \quad \partial_\alpha \theta^\beta = \delta^\beta_\alpha, \quad \partial_{\dot{\alpha}} \bar{\theta}^\dot{\beta} = \delta^\dot{\beta}_{\dot{\alpha}}.$$  \hspace{1cm} (3.4)

The generators $L_\alpha^\beta, R_{\dot{\alpha}}^\dot{\beta}$ and $O$ correspond to the automorphism group SU(2)$_L \times$SU(2)$_R \times$O(1,1).

Let us consider even combinations of the supersymmetry generators and the corresponding transformation parameters $c_m, \lambda_\alpha^\beta, R_{\dot{\alpha}}^\dot{\beta}, a, \epsilon^\alpha$ and $\epsilon^{\dot{\alpha}}$:

$$P_\epsilon = c_m P_m, \quad aO, \quad L_\lambda = \lambda_\alpha^\beta L_\alpha^\beta, \quad R_\rho = R_{\dot{\alpha}}^\dot{\beta} R_{\dot{\alpha}}^\dot{\beta}, \quad Q_\epsilon = \epsilon^\alpha Q_\alpha, \quad \bar{Q}_{\dot{\epsilon}} = \epsilon^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}.$$  \hspace{1cm} (3.5)

In studying the deformations, it is convenient to divide the operators of SUSY($\frac{1}{2}, \frac{1}{2}$) transformations into two sets

$$\delta A = (\delta_g + \delta_G) A = -(g + G) A, \quad g = P_\epsilon + R_\rho + Q_\epsilon, \quad G = L_\lambda + aO + \bar{Q}_{\dot{\epsilon}}.$$  \hspace{1cm} (3.6)

We let $S(4|2,2)$ and $C(4|2,0)$ denote the standard algebras of general and chiral superfields, respectively. The product in these algebras is supercommutative for superfields $A(z)$ and $B(z)$ with the fixed $Z_2$ parities $p(A)$ and $p(B)$

$$AB = (-1)^{p(A)p(B)} BA.$$  \hspace{1cm} (3.7)

\textsuperscript{2}We use the following conventions for the antisymmetric symbols:

$$\varepsilon_{\alpha\beta\bar{\varepsilon}^{\beta\gamma}} = \delta^\gamma_\alpha, \quad \varepsilon_{\dot{\alpha}\dot{\beta}\bar{\varepsilon}^{\dot{\beta}\dot{\gamma}}} = \delta^\dot{\gamma}_{\dot{\alpha}}.$$
This product is defined formally by the bilinear map $\mu$

$$\mu \circ A \otimes B = A(z)B(z). \quad (3.8)$$

In field theory, the tensor product corresponds to a nonlocal product of superfields $A(z_1)B(z_2)$ defined on independent sets of coordinates, and $\mu$ can then be interpreted as providing an identification of $z_1$ and $z_2$.

The coproduct is trivial for the generators of SUSY($\frac{1}{2}, \frac{1}{2}$)

$$\Delta(g) = g \otimes 1 + 1 \otimes g, \quad \Delta(G) = G \otimes 1 + 1 \otimes G. \quad (3.9)$$

This coproduct defines the action of the supersymmetry on the tensor product of superfields

$$\delta(A \otimes B) = -\Delta(g + G)(A \otimes B) = \delta A \otimes B + A \otimes \delta B \quad (3.10)$$

and yields the standard Leibniz rule for supersymmetry transformations on the local product of superfields

$$\delta(AB) = \mu \circ \delta(A \otimes B) = (\delta A)B + A\delta B. \quad (3.11)$$

A non-anticommutative deformation of the coordinates of the Euclidean $N=(\frac{1}{2}, \frac{1}{2})$ superspace $\hat{z} = (y_m, \hat{\theta}^\alpha, \bar{\theta}\dot{\alpha})$ was considered in [15]. The Clifford coordinate of the deformed superspace $\hat{\theta}^\alpha$ satisfies the simple relations

$$T^{\alpha\beta} = \hat{\theta}^\alpha \star \hat{\theta}^\beta + \hat{\theta}^\beta \star \hat{\theta}^\alpha - C^{\alpha\beta} = 0, \quad (3.12)$$

where $C^{\alpha\beta}$ are some constants, and the coordinates $y_m, \bar{\theta}\dot{\alpha}$ remain undeformed

$$[\hat{\theta}^\alpha, y_m] = [\bar{\theta}\dot{\alpha}, y_m] = [y_m, y_n] = 0, \quad \{\hat{\theta}^\alpha, \bar{\theta}\dot{\alpha}\} = \{\bar{\theta}\dot{\alpha}, \hat{\theta}^\beta\} = 0. \quad (3.13)$$

The canonical decomposition of the ordered operator superfield $\hat{A}(\hat{z})$ is based on the assumption of antisymmetrization in $\hat{\theta}^\alpha$

$$\hat{A}(y, \hat{\theta}, \bar{\theta}) = r(y, \bar{\theta}) + \hat{\theta}^\alpha \chi_\alpha(y, \bar{\theta}) + \varepsilon_{\alpha\beta} \hat{\theta}^\alpha \star \hat{\theta}^\beta s(y, \bar{\theta}). \quad (3.14)$$

Non-anticommutative deformations of $S(4|2, 2)$ and $C(4|2, 0)$ are denoted by $S_\star(4|2, 2)$ and $C_\star(4|2, 0)$.

In the pseudolocal representation, the noncommutative superfield $\hat{A}(\hat{z})$ corresponds to the supercommutative image $A(z)$ in the undeformed superspace

$$w[\hat{A}(y, \hat{\theta}, \bar{\theta})] = A(z) = r(y, \bar{\theta}) + \theta^\alpha \chi_\alpha(y, \bar{\theta}) + \varepsilon_{\alpha\beta} \theta^\alpha \theta^\beta s(y, \bar{\theta}). \quad (3.15)$$

The corresponding $\star$-product of superfields $A(z)$ and $B(z)$ contains the left supersymmetry generators $Q_\alpha$

$$w[\hat{A} \star \hat{B}] = (A \star B)(z) = Ae^P B = AB - \frac{1}{2}(-1)^{p(A)} C^{\alpha\beta} Q_\alpha A Q_\beta B$$

$$-\frac{1}{32} C^{\alpha\beta} C_{\alpha\beta} Q^2 A Q^2 B, \quad APB = -\frac{1}{2}(-1)^{p(A)} C^{\alpha\beta} Q_\alpha A Q_\beta B, \quad P^3 = 0, \quad (3.16)$$
where $P$ is the nilpotent bidifferential operator and $Q^2 = \varepsilon^{\alpha\beta}Q_\alpha Q_\beta$. In the right-hand side of this formula, all products of superfields and their derivatives are supercommutative. Using decompositions of these products in powers of $\theta^\alpha$, we can easily construct the ordered decomposition of the operator product $(A \star B)(\bar{z}) = w^{-1}[(A \star B)(\bar{z})]$ in terms of $\bar{\theta}^\alpha$. Relation (3.12) is satisfied automatically in representation (3.16).

It is evident that the noncommutative algebra $S_\ast(4|2, 2)$ is noncovariant under transformations of the undeformed supersymmetry $\delta_G$ (3.6), for instance,

$$\delta_\epsilon(A \star B) = - (e^\epsilon \bar{Q}_\alpha A) \star B - A \star e^\epsilon \bar{Q}_\alpha B \neq - e^\epsilon \bar{Q}_\alpha (A \star B),$$

although the operator $\delta_g$ acts covariantly

$$\delta_g(A \star B) = (-gA) \star B - A \star (gB) = -g(A \star B).$$

The twist operator $F = \exp(\mathcal{P})$ in this superspace was considered in [21] (see, also [24, 25])

$$\mathcal{P} = -\frac{1}{2} C^{\alpha\beta} Q_\alpha \otimes Q_\beta, \quad \mathcal{P}(A \otimes B) = -\frac{1}{2}(-1)^{p(A)} C^{\alpha\beta} Q_\alpha A \otimes Q_\beta B,$$

$$F(A \otimes B) = A \otimes B - \frac{1}{2}(-1)^{p(A)} C^{\alpha\beta} Q_\alpha A \otimes Q_\beta B - \frac{1}{32} C^{\alpha\beta} C_{\alpha\beta} Q^2 A \otimes Q^2 B. \quad (3.19)$$

The operator $\mathcal{P}$ is real with respect to the Hermitian conjugation including the map (3.1) on the supersymmetry generators $Q^*_\alpha = Q^{\alpha}$ and the transposition

$$\mathcal{P}^* = -\frac{1}{2} (C^{\alpha\beta})^* Q^*_\beta \otimes Q^*_\alpha = \mathcal{P},$$

if the condition $(C^{\alpha\beta})^* = \varepsilon_{\alpha\rho} \varepsilon_{\beta\sigma} C^{\rho\sigma}$ is satisfied. The reality of the Poisson bracket follows from this property

$$(A P B)^* = \mu \circ \mathcal{P}^*(B^* \otimes A^*) = B^* P A^*. \quad (3.21)$$

The bilinear map $\mu_\ast$ in $S_\ast(4|2, 2)$ can be defined via the twist operator

$$A \star B \equiv \mu_\ast \circ A \otimes B = \mu \circ \exp(\mathcal{P}) A \otimes B. \quad (3.22)$$

By analogy with the map between differential operators on the commutative and noncommutative algebras of functions (2.11), the map of the $k$-th order differential operator $D = \xi^{M_1 \ldots M_k}(z) \partial_{M_k} \ldots \partial_{M_1}$ on the supercommutative algebra $S(4|2, 2)$ to the differential operator $\hat{X}_D$ on $S_\ast(4|2, 2)$ can be easily defined

$$(DA) = \mu \circ \xi^{M_1 \ldots M_k} \otimes \partial_{M_k} \ldots \partial_{M_1} A = \mu_\ast \circ \exp(-\mathcal{P}) \xi^{M_1 \ldots M_k} \otimes \partial_{M_k} \ldots \partial_{M_1} A$$

$$= \xi^{M_1 \ldots M_k} \ast \partial_{M_k} \ldots \partial_{M_1} A + \frac{1}{2} (-1)^{p(D)} C^{\alpha\beta} Q_\alpha \xi^{M_1 \ldots M_k} \ast \partial_{M_k} \ldots \partial_{M_1} Q_\beta A$$

$$- \frac{1}{32} C^{\alpha\beta} C_{\alpha\beta} Q^2 \xi^{M_1 \ldots M_k} \ast \partial_{M_k} \ldots \partial_{M_1} Q^2 A = (\hat{X}_D \ast A), \quad (3.23)$$

where $p(D)$ is the $Z_2$ parity of the operator $D$. The operator $\hat{X}_D$ includes derivatives of the orders $k, k+1$ and $k+2$. The differential operators on $S(4|2, 2)$ form an associative algebra,
and the map \( D \to \hat{X}_D \) generates the isomorphic algebra of the differential operators on \( S_*(4|2,2) \)
\[
(D_1 D_2 A) = (\hat{X}_{D_1} \star \hat{X}_{D_2} \star A).
\]

(3.24)

In general, the image \( \hat{X}_\xi \) for the first-order operator \( \xi = \xi^M(z) \partial_M \) contains terms with derivatives of the first, second and third orders in \( S_*(4|2,2) \)
\[
(\hat{\partial}_N \star A) = \partial_N A, \quad (\hat{\partial}_N \star z^N) = \delta^N_M, \\
(\hat{\bar{X}}_\xi \star A) = (\xi A) = \mu_* \circ \exp(-\mathcal{P}) (\xi^M(z) \otimes \partial_M) (1 \otimes A) = \xi^M(z) \star \partial_M A \\
+ \frac{1}{2} (-1)^{p(\xi)} C^\alpha_\beta Q_\alpha \xi^M(z) \star \partial_M Q_\beta A - \frac{1}{32} C^\alpha_\beta C^\beta_\gamma Q_\alpha Q_\beta Q_\gamma A. \quad (3.25)
\]

The noncommutative images of the operators \( \hat{Q}_\alpha \) and \( \hat{L}_\alpha^\beta \) (3.3) are the second-order differential operators in \( S_*(4|2,2) \):
\[
(\hat{\bar{Q}}_\alpha \star A) = (\hat{\bar{Q}}_\alpha - 2i \theta^\alpha \partial_{\alpha \dot{a}} + i C^\alpha_\beta \partial_{\alpha \dot{a}} Q_\beta) \star A = \hat{Q}_\alpha A, \\
(\hat{\bar{L}}_\alpha^\beta \star A) = (\hat{\bar{L}}_\alpha^\beta - \frac{1}{2} C^\gamma_\beta Q_\gamma Q_\alpha - 1) \star A = \hat{L}_\alpha^\beta A, \quad (3.26)
\]

while the operators \( P_m, R_\alpha^\beta \) and \( O \) preserve their form under this map, for instance,
\[
(O A) = (\theta^\alpha \partial_\alpha - \bar{\theta}^\alpha \bar{\partial}_\alpha) \star A. \quad (3.27)
\]

The additional term in \( \hat{\bar{O}} \) vanishes, \( C^\alpha_\beta Q_\alpha Q_\beta = 0 \).

Expression (3.23) includes the differentiation and the \( \star \)-product in the pseudolocal representation; however, this formula allows defining the action of the corresponding operator \( (\hat{X}_D \star A) = w^{-1}[([\hat{X}_D \star A]) \] on non-anticommutative superfields in an arbitrary representation of the algebra \( S_*(4|2,2) \). In the operator representation, it is easy to verify that the quantity \( T^\alpha_\beta \) (3.12) is covariant under the action of the deformed generators
\[
(\hat{\bar{Q}}_\alpha \star T^\alpha_\beta) = 0, \quad (\hat{\bar{L}}_\alpha^\beta \star T^\alpha_\beta) = \delta^\alpha_\sigma T^{\rho\beta} + \delta^\beta_\sigma T^{\alpha\rho} - \delta^\rho_\sigma T^{\alpha\beta}. \quad (3.28)
\]

According to relation (3.24), the deformed generators on \( S_*(4|2,2) \) form the Lie superalgebra isomorphic to the undeformed Lie superalgebra of generators (3.3).

The coproduct \( \Delta_t(G) = e^{-\mathcal{P}} \Delta(G) e^\mathcal{P} \) in the deformed supersymmetry \( \text{SUSY}_t(\frac{1}{2}, \frac{1}{2}) \) is changed on some of the generators
\[
\Delta_t(\hat{Q}_\ell) = \hat{Q}_\ell \otimes 1 + 1 \otimes \hat{Q}_\ell + i e^{\hat{Q}_\ell} C^\alpha_\beta (\partial_{\alpha \dot{a}} \otimes Q_\beta - Q_\alpha \otimes \partial_{\beta \dot{a}}), \quad (3.29)
\]
\[
\Delta_t(\hat{L}_\lambda) = \hat{L}_\lambda \otimes 1 + 1 \otimes \hat{L}_\lambda + \frac{i}{2} C^{\rho\sigma}(\lambda^\rho_\alpha Q_\alpha \otimes Q_\sigma + \lambda^\rho_\sigma Q_\rho \otimes Q_\alpha), \quad (3.30)
\]
\[
\Delta_t(O) = O \otimes 1 + 1 \otimes O - C^\alpha_\beta Q_\alpha \otimes Q_\beta. \quad (3.31)
\]

The coproduct is not deformed on the generators \( R_\alpha^\beta, P_m \) and \( Q_\alpha \).

By definition, the action of \( \text{SUSY}_t(\frac{1}{2}, \frac{1}{2}) \) on a primary superfield \( \hat{A} \) in an arbitrary representation is generated by the undeformed supersymmetry transformations of the supercommutative image \( A(z) \)
\[
\hat{\delta} \star A = -(g + G)A = -(\hat{g} + \hat{G}) \star A, \quad (3.32)
\]
\[
\hat{\delta} \star \hat{A} = w^{-1}[\hat{\delta} \star A] = -(\hat{g} + \hat{G}) \star \hat{A},
\]
where relations between operators and superfields in different representations are used.

The deformed coproduct $\Delta_t(G)$ in SUSY$_t(\frac{1}{2}, \frac{1}{2})$ determines transformations of the noncommutative product in the algebra $S_+(4|2,2)$

$$\hat{\delta}_t \star (A \star B) = -\mu \circ \Delta_t(Q_{\hat{\epsilon}})A \otimes B = -(Q_{\hat{\epsilon}}A) \star B - A \star Q_{\hat{\epsilon}}B$$

$$-i\epsilon^\alpha C^{\alpha\beta}[(1-p(A))\partial_{\alpha\beta}A \star Q_\beta B - Q_\alpha A \star \partial_{\beta\alpha}B], \quad (3.33)$$

$$\hat{\delta}_\lambda \star (A \star B) = -\mu \circ \Delta_t(L_\lambda)A \otimes B = -(L_\lambda A) \star B - A \star L_\lambda B$$

$$-\frac{1}{2}(1-p(A))C^{\alpha}\lambda_\rho Q_\alpha A \star Q_\sigma B + \lambda_\rho Q_\sigma A \star Q_\alpha B, \quad (3.34)$$

$$\hat{\delta}_a \star (A \star B) = -a(\partial_\lambda B) - aA \star (QB) + a(1-p(A))C^{\alpha\beta}Q_\alpha A \star Q_\beta B. \quad (3.35)$$

The appearance of terms with $C^{\alpha\beta}$ in the transformations of $A \star B$ can be treated as a deformation of the Leibniz rules for $\hat{\delta}_t, \hat{\delta}_\lambda$ and $\hat{\delta}_a$. It is not difficult to show that these relations yield the covariance of the noncommutative product

$$\hat{\delta}_G \star (A \star B) = -G(A \star B), \quad (3.36)$$

which transforms similarly to primary superfields $A$ and $B$ (3.32) in SUSY$_t(\frac{1}{2}, \frac{1}{2})$. For instance, the formula $\hat{\delta}_t \star (A \star B) = -Q_{\hat{\epsilon}}(A \star B)$ is derived from eq.(3.33) using the relations

$$-Q_{\hat{\epsilon}}(A \star B) = -[Q_{\hat{\epsilon}}, (Ae^P B)] = -(Q_{\hat{\epsilon}}A)e^P B - A(Q_{\hat{\epsilon}}, e^P)B = A(\partial_\epsilon B),$$

$$-A(\partial_\epsilon B)B = -i\epsilon^\alpha C^{\alpha\beta}[(1-p(A))\partial_{\alpha\beta}A \star Q_\beta B - Q_\alpha A \star \partial_{\beta\alpha}B]. \quad (3.37)$$

We note that superfield $AB$ is a noncovariant quantity in SUSY$_t(\frac{1}{2}, \frac{1}{2})$. For example, it is easy to define noncovariant actions of the operators $\hat{\delta}_t$ and $\hat{\delta}_\lambda$ on the ordinary product of even chiral superfields

$$\hat{\delta}_t \star (\phi_1 \phi_2) \equiv -\mu \circ \Delta_t(Q_{\hat{\epsilon}})\phi_1 \otimes \phi_2 = -Q_{\hat{\epsilon}}(\phi_1 \phi_2) - i\epsilon^\alpha C^{\alpha\beta}(\partial_{\alpha\beta}\phi_1 Q_\beta \phi_2)$$

$$-Q_\alpha \phi_1 \partial_{\beta\alpha} \phi_2 = \hat{\delta}_t(a_1 a_2) + \theta^\alpha \hat{\delta}_t(a_1 \psi_{2\alpha} + a_2 \psi_{1\alpha}) + O(\theta^2),$$

$$\hat{\delta}_\lambda \star (\phi_1 \phi_2) \equiv -\mu \circ \Delta_t(L_\lambda)\phi_1 \otimes \phi_2 = -\lambda_\beta \partial_\beta \phi_1 \phi_2$$

$$-\frac{1}{2}C^{\alpha\beta}(\lambda_\rho Q_\alpha \phi_1 Q_\rho \phi) = \hat{\delta}_\lambda(a_1 a_2) + O(\theta). \quad (3.38)$$

The first terms in these formulas coincide with the transformations of the undeformed supersymmetry. Using the $\theta$-decomposition of these superfield formulas

$$\phi_i = a_i + \theta^\alpha \psi_{i\alpha} + \theta^2 f_i, \quad (3.39)$$

$$Q_\alpha \phi_i = \psi_{i\alpha} + 2\theta_{\alpha} f_i, \quad Q^2 \phi_i = -4f_i,$$

one can obtain the deformed transformations of component fields, for instance,

$$\hat{\delta}_t(a_1 a_2) = -i\epsilon^\alpha C^{\alpha\beta}(\partial_{\alpha\beta}a_1 \psi_{2\beta} - \psi_{2\alpha} \partial_{\beta\alpha}a_1),$$

$$\hat{\delta}_\lambda(a_1 a_2) = -\lambda_\rho L_\alpha (y) (a_1 a_2) - \frac{1}{2}C^{\alpha\beta}(\lambda_\rho \psi_{1\alpha} \psi_{2\beta} + \lambda_\alpha \psi_{1\beta} \psi_{2\alpha}). \quad (3.40)$$
The expansion of the $\star$-product of two chiral superfields in $\theta^\alpha$ depends on the constants $C^{\alpha\beta}$

\begin{align}
\Phi_{12} &= \phi_1 \star \phi_2 = B + \theta^\alpha \Psi_\alpha + \theta^2 F, \\
B &= a_1 a_2 - \frac{1}{2} C^{\alpha\beta} \psi_1 \psi_2 - \frac{1}{2} C^{\alpha\beta} C_{\alpha\beta} f_1 f_2, \\
\Psi_\alpha &= a_1 \psi_{2\alpha} + a_2 \psi_{1\alpha} - C_{\alpha\beta} (f_1 \psi_2^\beta - f_2 \psi_1^\beta), \\
F &= a_1 f_2 + a_2 f_1 - \frac{1}{2} \psi_1^\alpha \psi_{2\alpha}.
\end{align}

(3.41)

These relations generate the deformed tensor calculus for the product of the chiral component multiplets. The transformations of the composed components (3.41) in the deformed supersymmetry are completely analogous to the transformations of the primary components $a_i, \psi_{\alpha i}$ and $f_i$

\begin{align}
\hat{\delta}_i B &= 0, \quad \hat{\delta}_i \Psi_\alpha = -2i \bar{\epsilon}^\alpha \partial_{\alpha a} B, \quad \hat{\delta}_i F = -i \bar{\epsilon}^\alpha \partial_{\alpha a} \Psi_\alpha, \\
\hat{\delta}_\lambda B &= -\lambda_\beta L_\beta (y) B, \quad \hat{\delta}_\lambda \Psi_\gamma = \lambda_\gamma \Psi_\alpha - \lambda_\alpha L_\alpha (y) \Psi_\gamma, \quad \hat{\delta}_\lambda F = -\lambda_\alpha L_\alpha (y) F.
\end{align}

(3.42)

(3.43)

These transformations are compatible with the noncovariant transformations of the products of components (3.40).

The non-anticommutative deformation of the Euclidean model with an arbitrary number of chiral (antichiral) superfields and gauge superfields $V(z)$ involves $\star$-products of these superfields in the superfield action [15]. Each term of the $\star$-polynomial decomposition of this action is separately invariant with respect to the transformations of SUSY $t_i(\frac{1}{2}, \frac{1}{2})$, and the quadratic terms are also invariant under the ordinary supersymmetry.

## 4 Deformed $\mathcal{N}=(1, 1)$ supersymmetry

Nilpotent deformations of the Euclidean $\mathcal{N}=(1, 1)$ supersymmetry were considered in the framework of the harmonic-superspace formalism [16, 17] using the $\text{SU}(2)/\text{U}(1)$ harmonics $u_i^\pm$ and the chiral superspace coordinates

\begin{align}
\eta^{\mu \nu} = (y_m, \theta_k^\alpha, \bar{\theta}^{\dot{\alpha} k}), \quad y_m &= x_m + i \bar{\theta}_m \sigma_m \tilde{\theta}^k, \\
\bar{y}_m &= y_m, \quad \bar{\theta}_k^\alpha = \theta_\alpha^k, \quad \bar{\theta}^{\dot{\alpha} k} = -\bar{\theta}_{\dot{\alpha} k}
\end{align}

(4.1)

where $x_m$ are the central 4D coordinates. Standard conjugation of these coordinates changes positions of all spinor indices

\begin{align}
\bar{y}_m &= y_m, \quad \bar{\theta}_k^\alpha = \theta_\alpha^k, \quad \bar{\theta}^{\dot{\alpha} k} = -\bar{\theta}_{\dot{\alpha} k}
\end{align}

(4.2)

and in particular preserves the invariance under the $\text{SU}(2)$ automorphisms acting on index $k$. In the same superspace, the alternative pseudoconjugation can be defined as [16]

\begin{align}
(y_m)^* &= y_m, \quad (\theta_k^\alpha)^* = \theta_{ak}, \quad (\bar{\theta}^{\dot{\alpha} k})^* = \bar{\theta}_{\dot{\alpha} k},
\end{align}

(4.3)

which does not change the position of the $\text{SU}(2)$-index $k$. The spinor derivatives $D_\alpha^k$ and $\bar{D}_{\dot{\alpha} k}$ in these coordinates are given by

\begin{align}
D_\alpha^k &= \partial_\alpha^k + 2 i \bar{\theta}^{\dot{\alpha} k} \partial_{\alpha \dot{\alpha}}, \quad \bar{D}_{\dot{\alpha} k} = \bar{\partial}_{\dot{\alpha} k}.
\end{align}

(4.4)
The even part of the harmonic superspace \( R^1 \times S^2 \) has a dimension 4+2; it is convenient to use separate symbols for the left and right odd dimensions of the Grassmann coordinates of the general superspace (4,4), the chiral superspace (4,0), and the analytic superspace (2,2), respectively. We use symbol \( S(4,2|4,4) \) for the supercommutative algebra of general harmonic superfields and \( S_*(4,2|4,4) \) for the corresponding non-anticommutative algebra.

We use the following representation of the SUSY(1,1) supersymmetry generators as the differential operators on the algebra \( S(4,2|4,4) \):

\[
T^k_l = -\theta^i_l \partial^k_\alpha + \frac{i}{2} \delta^k_\alpha \theta^\alpha_j \partial^i_j + \bar{\theta}^{\dot{a}k} \bar{\partial}_{\dot{a}l} - \frac{1}{2} \delta^k_\alpha \bar{\theta}^{\dot{a}j} \bar{\partial}_{\dot{a}j} - u^+_l \partial^{T_k}_l + \frac{1}{2} \delta^k_\alpha u^+_j \partial^{T_j},
\]

\[
L_\alpha^\beta = L_\alpha^\beta(y) + \theta^\alpha_k \partial^\beta_\alpha - \frac{1}{2} \delta^\beta_\alpha \theta^\alpha_k \partial^\gamma_\gamma, \quad R_\alpha^\beta = R_\alpha^\beta(y) + \bar{\theta}^{\dot{a}k} \bar{\partial}_{\dot{a}k} - \frac{1}{2} \delta^\beta_\alpha \bar{\theta}^{\dot{a}j} \bar{\partial}_{\dot{a}j},
\]

\[
O = \theta^\alpha_k \partial^\beta_\alpha - \bar{\theta}^{\dot{a}k} \bar{\partial}_{\dot{a}k}, \quad Q_\alpha^k = \partial^\beta_\alpha, \quad \bar{Q}_{\dot{a}k} = \bar{\partial}_{\dot{a}k} = 2i \theta^\alpha_k \partial_{\alpha\dot{a}}, \quad P_m = \partial_m, \quad (4.5)
\]

where \( L_\alpha^\beta(y) \) and \( R_\alpha^\beta(y) \) are defined in (3.3) and the partial derivatives satisfy the relations

\[
\partial_m y_n = \delta_{mn}, \quad \partial^\beta_\alpha \theta^\alpha_k = \delta^\beta_k \delta^\alpha_\alpha, \quad \bar{\partial}_{\dot{a}k} \bar{\theta}^{\dot{a}k} = \delta^\beta_{\dot{a}k} \delta^\alpha_{\dot{a}k}, \quad \partial^{T_1}_l u^+_l = \delta^\beta_\alpha.
\]

To study deformations, it is convenient to separate the SUSY(1,1) transformations into two parts

\[
\delta_A = -gA, \quad \delta_G A = -GA, \quad (4.7)
\]

using the following combinations of generators and the corresponding parameters:

\[
g = P_\epsilon + R_\rho + Q_\epsilon, \quad G = T_u + L_\lambda + \bar{Q}_\xi + aO, \quad (4.8)
\]

\[
P_\epsilon = \epsilon_m P_m, \quad T_u = u^k_l T^l_k, \quad L_\lambda = \lambda_\beta L_\alpha^\beta, \quad R_\rho = \rho_\beta \rho_\alpha^\beta, \quad Q_\epsilon = \epsilon_k^\alpha Q_k^\alpha, \quad \bar{Q}_\xi = \bar{\epsilon}^\alpha \bar{Q}_\alpha^\lambda.
\]

The analytic coordinates of the harmonic superspace \( (x_A, \theta^{\pm}, \bar{\theta}^{\pm}) \) can be defined using the harmonic projections of the Grassmann coordinates \( \theta^{\pm\alpha} = u^\alpha_k \theta^{\alpha k} \) and \( \bar{\theta}^{\pm\dot{a}} = u^{\dot{a}}_k \bar{\theta}^{\dot{a} k} \). The corresponding representation of the spinor and harmonic derivatives can be found in [16, 18]

\[
D^+_\alpha = \partial_{-\alpha}, \quad D^-_\alpha = -\partial_{+\alpha} + 2i \bar{\theta}^{-\dot{a}} \partial_{\alpha\dot{a}},
\]

\[
D^+_\dot{a} = \partial_{-\dot{a}}, \quad D^-_\dot{a} = -\partial_{+\dot{a}} - 2i \theta^{\alpha} \partial_{\alpha\dot{a}},
\]

\[
D^+_{-\alpha} = \partial^{++} - 2i \theta^{\alpha} \bar{\theta}^{\dot{a}+} \partial_{\alpha\dot{a}} + \theta^{\alpha} \partial_{-\alpha} + \theta^{\dot{a}+} \bar{\partial}_{-\dot{a}}, \quad (4.9)
\]

where \( \partial_{\pm\alpha} \equiv \partial/\partial \theta^{\pm\alpha} \), \( \partial_{\pm\dot{a}} \equiv \partial/\partial \bar{\theta}^{\pm\dot{a}} \).

The \( N=(1,1) \) twist operator \( \mathcal{F} = \exp(\mathcal{P}) \) contains the nilpotent operator

\[
\mathcal{P} = -\frac{1}{2} C_{kl}^{\alpha\beta} Q^k_\alpha \otimes Q^l_\beta, \quad \mathcal{P}^5 = 0, \quad (4.11)
\]

where \( C_{kl}^{\alpha\beta} \) are the deformation constants. The operator \( \mathcal{P} \) is Hermitian under the conjugation constructed from the transposition and conjugation (4.2)

\[
\overline{\mathcal{P}} = -\frac{1}{2} C_{kl}^{\alpha\beta} Q^l_\beta \otimes Q^k_\alpha = \mathcal{P}, \quad \overline{Q^k_\alpha} = \varepsilon^{\alpha\rho} \varepsilon_{kj} Q^j_\rho, \quad (4.12)
\]
if the conditions $\overline{C}_{kl}^{\alpha\beta} = \varepsilon_{\alpha\rho} \varepsilon_{\beta\sigma} \varepsilon^{ki} \varepsilon^{lj} C_{ij}^{\rho\sigma}$ are satisfied.

The action of the operator $\mathcal{P}$ on the tensor product of superfields $A$ and $B$ is compatible with the $Z_2$ grading

$$\mathcal{P} A \otimes B = -\frac{1}{2}(-1)^{p(A)} C_{kl}^{\alpha\beta} Q_k^\alpha A \otimes Q_l^\beta B.$$  

(4.13)

A non-anticommutative product in the corresponding deformed algebra $S_\ast(4,2|4,4)$ can be defined using the equivalent formulas

$$A \ast B = A \exp(P) B = \mu \circ \exp(\mathcal{P}) A \otimes B = \mu_\ast \circ A \otimes B$$  

(4.14)

where $\mu$ and $\mu_\ast$ are the bilinear maps for $S(4,2|4,4)$ and $S_\ast(4,2|4,4)$, and $P$ is the bidifferential operator in [16, 17]

$$A \mathcal{P} B = -\frac{1}{2}(-1)^{p(A)} C_{kl}^{\alpha\beta} Q_k^\alpha A Q_l^\beta B = \mu \circ \mathcal{P} A \otimes B.$$  

(4.15)

The non-anticommutative algebras of the $N=1,1$ chiral or analytic superfields are defined as subalgebras of $S_\ast(4,2|4,4)$ using the superfield constraints

$$\hat{D}_{\hat{\alpha}k} B = 0, \quad \text{or} \quad (\hat{D}_\hat{\alpha}^+, \hat{D}_{\hat{\alpha}}^+) \Lambda = 0,$$

(4.16)

which are preserved by the deformation operator $\exp(P)$.

By analogy with eq. (3.23), each differential operator on the algebra $S(4,2|4,4)$ corresponds to the operator on the noncommutative algebra $S_\ast(4,2|4,4)$, for instance,

$$\hat{Q}_{\hat{\alpha}k} \ast A = \hat{Q}_{\hat{\alpha}k} A,$$

$$\hat{Q}_{\hat{\alpha}k} = \hat{\partial}_{\hat{\alpha}k} - 2i \theta_k^\hat{\alpha} \ast \hat{\partial}_{\hat{\alpha}} - C_{kl}^{\alpha\beta} \hat{\partial}_l^\beta \hat{\partial}_{\hat{\alpha}\hat{\beta}}.$$  

(4.17)

In the twist-deformed supersymmetry SUSY$_t(1,1)$, we can use the standard representation of generators (4.5) on the primary superfields.

The deformed coproduct in SUSY$_t(1,1)$, $\Delta_t(\mathcal{G}) = e^{-P} \Delta(\mathcal{G}) e^P$, can be easily calculated on the following generators:

$$\Delta_t(\hat{Q}_\hat{\epsilon}) = \hat{Q}_\hat{\epsilon} \otimes 1 + 1 \otimes \hat{Q}_\hat{\epsilon} + i \varepsilon^{\hat{\alpha}k} C_{k\hat{j}}^{\alpha\beta} \hat{\partial}_{\hat{\alpha}\hat{\beta}} \otimes Q_j^\alpha \otimes \hat{\partial}_{\hat{\alpha}\hat{\beta}},$$

$$\Delta_t(T_u) = T_u \otimes 1 + 1 \otimes T_u - \frac{1}{2} u_k^l C_{k\hat{j}}^{\alpha\beta} Q_k^\alpha \otimes Q_l^\beta - \frac{1}{2} u_k^l C_{k\hat{j}}^{\alpha\beta} Q_k^\alpha \otimes Q_{\hat{j}}^\beta,$$

$$\Delta_t(L_\lambda) = L_\lambda \otimes 1 + 1 \otimes L_\lambda + \frac{1}{2} \lambda_{\hat{k}}^l C_{\hat{k}l}^{\alpha\beta} Q_k^\alpha \otimes Q_{\hat{j}}^\beta + \frac{1}{2} \lambda_{\hat{k}}^l C_{\hat{k}l}^{\alpha\beta} Q_k^\alpha \otimes Q_{\hat{j}}^\beta,$$

$$\Delta_t(\mathcal{O}) = \mathcal{O} \otimes 1 + 1 \otimes \mathcal{O} - C_{kl}^{\alpha\beta} Q_k^\alpha \otimes Q_l^\beta.$$  

(4.18)

The deformation of the coproduct vanishes for the operator $g = Q_\hat{\epsilon} + P_\epsilon + R_\rho^\gamma$: $e^{-P} \Delta(g) e^P = 1 \otimes g + g \otimes 1$.

The noncommutative $\ast$-products of the $N=1,1$ superfields preserve the covariance under the deformed transformations of SUSY$_t(1,1)$

$$\hat{\delta}_\mathcal{G} \ast (A \ast B) = -\mu_\ast \circ \Delta_t(\mathcal{G}) A \otimes B = -G(A \ast B).$$  

(4.19)
The deformed Leibniz rules for the $\ast$-products are derived from formulas (4.18), for instance,
\[ \hat{\delta}_t \ast (A \ast B) = -\mu_\ast \circ \Delta_t(\hat{Q}_t)A \otimes B = -(\hat{Q}_t A) \ast B + A \ast (\hat{Q}_t B) \]
\[ -i(-1)^{p(A)}C_{kj}^{\alpha \beta} \epsilon^{\dot{\alpha} \dot{\beta}} \partial_{\dot{\alpha} \dot{\beta}} A \ast Q_\beta^j B + iC_{ik}^{\alpha \beta} \epsilon^{\dot{\alpha} \dot{k}} Q_\alpha^i A \ast \partial_{\dot{\beta} \dot{\alpha}} B = -\hat{Q}_t(A \ast B). \] 
(4.20)

The transformation \( \hat{\delta}_t \) acts noncovariantly on the supercommutative product of superfields
\[ \hat{\delta}_t \ast (AB) = -\mu \circ \Delta_t(\hat{Q}_t)A \otimes B = -\hat{Q}_t(AB) \]
\[ -i(-1)^{p(A)}C_{kj}^{\alpha \beta} \epsilon^{\dot{\alpha} \dot{\beta}} \partial_{\dot{\alpha} \dot{\beta}} A \otimes Q_\beta^j B + iC_{ik}^{\alpha \beta} \epsilon^{\dot{\alpha} \dot{k}} Q_\alpha^i A \partial_{\dot{\beta} \dot{\alpha}}. \] 
(4.21)

It is easy to define the deformation of transformations for products of the \( N=(1,1) \) component fields using the corresponding Grassmann expansions of the superfield transformations.

In the special case of the singlet deformation \([18, 19]\), the twist operator contains the parameter \( I \) and the \( SU(2) \times SU(2)_L \) invariant constant tensor
\[ C_{kl}^{\alpha \beta} = 2I \epsilon^{\alpha \beta} \epsilon_{kl} \Rightarrow P_s = -IQ_\alpha^i \otimes Q_\alpha^i. \] 
(4.22)

The deformation corresponding to the operator \( P_s \) vanishes on the generators of the \( SU(2) \) and \( SU(2)_L \) transformations and remains only for the \( N=(0,1) \) and \( O(1,1) \) generators
\[ \Delta_t(\hat{Q}_{\dot{\alpha} \dot{k}}) = \hat{Q}_{\dot{\alpha} \dot{k}} \otimes 1 + 1 \otimes \hat{Q}_{\dot{\alpha} \dot{k}} + 2iI \partial_{\dot{\alpha} \dot{\alpha}} \otimes Q_\alpha^k - 2iIQ_\alpha^k \otimes \partial_{\dot{\alpha} \dot{\alpha}}, \]
\[ \Delta_t(O) = O \otimes 1 + 1 \otimes O - 2IQ_\alpha^k \otimes Q_\alpha^k. \] 
(4.23)

The degenerate \( N=(1, \frac{1}{2}) \) deformation in \([16]\) corresponds to using the twist operator \( \mathcal{F}_{\text{deg}} = \exp(P_{\text{deg}}) \), where the operator
\[ P_{\text{deg}} = -\frac{1}{2}C_{\alpha \beta} Q_\alpha^2 \otimes Q_\beta^2 \] 
(4.24)
is Hermitian under the alternative pseudoconjugation (4.3)
\[ (Q_\alpha^k)^* = Q^{\dot{\alpha} \dot{k}}, \quad (\hat{Q}_\alpha^k)^* = \hat{Q}_\alpha^k, \quad (C_{\alpha \beta})^* = C_{\beta \alpha}. \] 
(4.25)

In this case, the coproduct \( \Delta_t \) is deformed on the generators \( \tilde{Q}_{\dot{\alpha} \dot{2}}, L_{\dot{\alpha} \dot{2}}, O, \) and \( T_{\dot{1} \dot{1}} \).

In \([26]\), we considered the deformation of \( S(4, 2|4, 4) \) acting simultaneously in the chiral and antichiral sectors of the superspace and using the pseudoconjugation (4.3). This deformation corresponds to the twist operator \( \hat{\mathcal{F}} = \exp(\hat{P}) \)
\[ \hat{P} = -\frac{1}{2}C^{\alpha \beta} Q_\alpha^2 \otimes Q_\beta^2 - \frac{1}{2}C^{\dot{\alpha} \dot{\beta}} \hat{Q}_{\dot{\alpha} \dot{1}} \otimes \hat{Q}_{\dot{\beta} \dot{1}} - B^{\alpha \dot{\alpha}}(Q_\alpha^2 \otimes \hat{Q}_{\dot{\alpha} \dot{1}} + \hat{Q}_{\dot{\alpha} \dot{1}} \otimes Q_\alpha^2). \] 
(4.26)

It is interesting that an analogous twist operator can be used to deform the \( N=2, D=(3, 1) \) superspace on the basis of the Minkowski space. The Hermitian symmetry of \( \hat{\mathcal{F}} \) is possible in this case if the alternative conjugation is used in the central coordinates of the \( N=2, D=(3, 1) \) superspace
\[ (x^m)^\dagger = x^m, \quad (\theta^a_k)^\dagger = \bar{\theta}_k^a, \quad (\bar{\theta}_k^a)^\dagger = \theta^a_k. \] 
(4.27)

which breaks down the automorphism group \( SU(2) \) but is compatible with covariant conjugation of spinors in the group \( SL(2, C) \). Operator (4.26) deforms the coproduct on

\[ ^3\text{We note that the alternative and usual conjugations act identically on the SU(2) invariant quantities.} \]
the generators of $\text{SL}(2,\mathbb{C})$, $U(2)$, $Q^1_\alpha$ and $\tilde{Q}_{\alpha 2}$.

The differential operators ($\partial_m, D^k_\alpha, D_{\alpha k}, \partial/\partial u^\pm_k$) satisfy the standard Leibniz rules for all nilpotent deformations considered.

We consider the primary analytic superfields of the hypermultiplet $q^+$, $\tilde{q}^+$ and the gauge multiplet $V^{++}$ [16, 17], which have the following noncommutative gauge transformations:

$$\delta_A V^{++} = D^{++} A + [V^{++}, A], \quad \delta_A q^+ = [q^+, A].$$

(4.28)

where $\Lambda$ is the superfield gauge parameter. These superfields and their $\star$-products transform covariantly in the deformed supersymmetry $\text{SUSY}_{t}(1,1)$

$$\hat{\delta}_G \star [V^{++}(z, u_1) \star V^{++}(z, u_2)] = -G[V^{++}(z, u_1) \star V^{++}(z, u_2)],$$

(4.29)

$$\hat{\delta}_G \star (V^{++} \star q^+) = -G(V^{++} \star q^+),$$

(4.30)

where the generators $G$ are given by eq. (4.8) or by the equivalent relations in the analytic coordinates.

The gauge action of $V^{++}$ is defined in the full or chiral superspaces [16, 17, 18]. In an arbitrary gauge, this action is invariant under the SUSY$_t(1,1)$ transformations. In the analytic superspace, the superfield action of the hypermultiplets contains the integral with the measure $d^4 x_A(D^-)^4$, and the simple example of the analytic density has the form

$$L^{++}_4 = \tilde{q}^+ \star (D^{++} q^+ + [V^{++}, q^+]) + \lambda q^+ \star q^+ \star \tilde{q}^+. \quad (4.31)$$

The deformed transformations of this superfield density are covariant in an arbitrary gauge of $V^{++}$

$$(\hat{\delta}_c + \hat{\delta}_u + \hat{\delta}_l + \hat{\delta}_a) \star L^{++}_4 = (\tilde{\epsilon}^\alpha k \tilde{Q}_{\alpha k} + u^l_k T^k_l + l^2_\beta L^{3}_\alpha + aO)L^{++}_4, \quad (4.32)$$

and the analytic-superspace integral of these variations vanishes. All superfield actions using the $\star$-products in the non-anticommutative harmonic superspace [18, 19, 20] are invariant with respect to the transformations of the deformed supersymmetry SUSY$_t(1,1)$. The free quadratic terms of these theories also preserve the undeformed $N = (1,1)$ supersymmetry.

The gauge superfield in the WZ-gauge defines the component fields of the vector multiplet

$$V^{++}_{WZ} = (\theta^+)^2 \bar{\phi}(x_\lambda) + (\theta^+)^2 \phi(x_\lambda) + 2(\theta^+ \sigma_m \bar{\theta}^+) A_m(x_\lambda) + 4(\bar{\theta}^+)^2 \theta^{+ \alpha} u^k_\alpha \Psi^k_{\alpha}(x_\lambda) + 4(\theta^+)^2 \bar{\theta}^+ u^k_\alpha \Psi^k_{\alpha}(x_\lambda) + 3(\theta^+)^2 u^k_\alpha u^l_\beta D^{kl}(x_\lambda). \quad (4.33)$$

The SUSY$_t(1,1)$ transformations of the quantity $V^{++}_{WZ}$ contain the standard terms with the supersymmetry generators (4.8) complemented by the composite gauge transformations

$$(\hat{\delta}_c + \hat{\delta}_e) \star V^{++}_{WZ} = -(Q_c + \tilde{Q}_e) V^{++}_{WZ} - D^{++}(\Lambda_c + \Lambda_e) - [V^{++}, (\Lambda_c + \Lambda_e)]_*, \quad (4.34)$$
where the $N=(1,1)$ supersymmetry generators are considered in the analytic coordinates

\begin{align}
Q^k_\alpha &= -u^+k \partial_{+\alpha} - u^{-k} \partial_{-\alpha} + 2iu^{-k} \bar{\theta}^+ \partial_{\alpha \dot{a}}, \\
Q_{\dot{a}k} &= u^+_{\dot{a}} \partial_{+\dot{a}} + u^-_{\dot{a}} \bar{\partial}_{-\dot{a}} + 2iu^-_{\dot{a}} \theta^+ \partial_{\alpha \dot{a}}.
\end{align}

(4.35)

The composite parameters of the $N=(1,1)$ transformations in the WZ-gauge are given by

\begin{align}
\Lambda_\epsilon &= 2\epsilon^{-\alpha} [\theta^+ \bar{\phi} + \bar{\theta}^+ \hat{\alpha} A_{\alpha \dot{a}} + u^-_{\dot{a}} (\hat{\theta}^+) 2\Psi^l_{\alpha} + 2u^+_{\dot{a}} \bar{\theta}^+ \bar{\Psi}^\dagger_{\dot{a}} + u^-_{\dot{a}} \theta^+ (\hat{\theta}^+) 2D^j_{\dot{a}}], \\
\Lambda_{\bar{\epsilon}} &= 2\bar{\epsilon}^{-\dot{a}} [\bar{\theta}^+ \phi + \theta^+ \hat{a} A_{\dot{a} \alpha} + u^-_{\alpha} (\hat{\bar{\theta}}^+) 2\Psi^l_{\dot{a}} + 2u^+_{\alpha} \bar{\theta}^+ \bar{\Psi}^\dagger_{\alpha} + u^-_{\alpha} \theta^+ (\hat{\bar{\theta}}^+) 2D^j_{\alpha}],
\end{align}

(4.36)

where $\epsilon^{-\alpha} = \epsilon^{\alpha k} u^-_{\dot{a}}$, $\bar{\epsilon}^{-\dot{a}} = \bar{\epsilon}^{\dot{a} k} u^-_{\alpha}$. The deformed transformations of the vector-multiplet components are determined from the Grassmann expansion of the transformations of $V^+_{WZ}$, and $V^+_{WZ}$ in (4.34). These transformations contain nonlinear gauge terms. The SUSY $t(1,1)$ transformations for the hypermultiplets are compatible with the transformation of $WZ$ by virtue of the gauge symmetry.

\begin{equation}
(\hat{\epsilon}_t + \hat{\bar{\epsilon}}_t) \ast q^+ = - (Q_t + \bar{Q}_t) q^+ + [q^+, (\Lambda_t + \Lambda_{\bar{t}})]_*.
\end{equation}

(4.37)

Additional terms with $\Lambda_t$ and $\Lambda_{\bar{t}}$ do not violate the SUSY $t(1,1)$ invariance of the action by virtue of the gauge symmetry.

The singlet $D$-deformation of the $N=(1,1)$ supersymmetry was considered in [16, 17]. This deformation corresponds to the alternative singlet twist operator constructed using the spinor derivatives $D^k_\alpha$

\begin{equation}
(A \ast B)_D = \mu \circ \exp(P_D) A \otimes B, \quad P_D = -JD^k_\alpha \otimes D^\alpha_k,
\end{equation}

(4.38)

where $J$ is some constant. The Leibniz rules for the noncommutative product are now deformed for the operator $\bar{D}_{\dot{a}k}$

\begin{align}
\bar{D}_{\dot{a}k}(A \ast B)_D &= (\bar{D}_{\dot{a}k} A \ast B)_D + (-1)^p(A \ast \bar{D}_{\dot{a}k} B)_D \\
&- 2iJ (-1)^p \partial_{\alpha \dot{a}} A \ast D^\alpha_k B - 2iJD^\alpha_k A \ast \partial_{\alpha \dot{a}} B
\end{align}

(4.39)

and for the $O(1,1)$ generator, but this deformation vanishes for the remaining generators of SUSY(1,1) and the spinor derivative $D^k_\alpha$.

The singlet operator $P_D$ (4.38) does not preserve chirality but does preserve anti-chirality and Grassmann analyticity. The $D$-deformation is interesting for the general objects of the superfield geometry in the $N= (1,1)$ gauge theory [17].

5 Conclusions

We have analyzed the $t$-deformations of the Euclidean $N=(\frac{1}{2}, \frac{1}{2})$ and $N=(1,1)$ supersymmetries using the twist operators depending on the left supersymmetry generators. In
this approach, the coproduct in the universal enveloping supersymmetry algebra is deformed, and the Lie superalgebra remains undeformed. In the pseudolocal representation, the deformed supersymmetry is covariantly realized on the noncommutative $\star$-products of primary fields, while the $t$-supersymmetry transformations of the supercommutative product of superfields depend on the deformation parameters. A map of the differential operators on the ordinary superspace to the differential operators in an arbitrary representation of the deformed noncommutative superfields was constructed. In this representation, the part of the $t$-supersymmetry generators is realized by the second-order differential operators, and it therefore becomes evident that the corresponding transformations of the $\star$-product do not satisfy the usual Leibniz rule.

The covariance of the $\star$-product and the invariance of the superfield action using this product are the main principles of the superfield formalism of the $t$-deformed theories. The bilinear free terms of the deformed action are also invariant under the usual supersymmetry. The deformation constants of the non-anticommutative superfield theories violate some initial (super)symmetries; however, these constants are compatible with the deformed supersymmetries. The invariance of the superfield formalism under the $t$-supersymmetry should be used to study the renormalizability of the deformed supersymmetric theories. The formalism of deformations can be regarded as an interesting analog of the spontaneous (super)symmetry breaking mechanism if the deformation does not destroy the good quantum properties of the supersymmetric field theories.

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### References

[1] M.R. Douglas, N.A. Nekrasov, Rev. Mod. Phys. **73** (2001) 977 ; R.J. Szabo, Phys. Rep. **378** (2003) 207.

[2] N. Seiberg and E. Witten, JHEP **9909** (1999) 032.

[3] P. Podleś, S.L. Woronowicz, Comm. Math. Phys. **178** (1996) 61.

[4] O. Ogievetsky, W.B. Schmidke, J. Wess, B. Zumino, Commun.Math.Phys. **150** (1992) 495.

[5] J. Lukierski, H. Ruegg, V.N. Tolstoy, A. Nowicki, J. Phys. **A 27** (1994) 2389.

[6] V.G. Drinfeld, Leningrad Math. J. **1** (1990) 1419.

[7] R. Oeckl, Nucl. Phys. B **581** (2000) 559.

[8] M. Chaichian, P.P. Kulish, K. Nishijima, A. Tureanu, Phys. Lett. **B 604** (2004) 98.
[9] M. Chaichian, P. Presnajder, A. Tureanu, Phys. Rev. Lett. 94 (2005) 151602.

[10] J. Wess, Deformed coordinate spaces. Derivatives, hep-th/0408080.

[11] P. Aschieri, C. Bluhmann, M. Dimitrijević, F. Meyer, P. Schupp, J. Wess, Class. Quant. Grav. 22 (2005) 3511.

[12] P. Kosiński, P. Maślanka, Lorentz-invariant interpretation of noncommutative space-time – global version, hep-th/0408100.

[13] S. Ferrara and M.A. Lledó, JHEP 0005 (2000) 008.

[14] D. Klemm, S. Penati and L. Tamassia, Class. Quant. Grav. 20 (2003) 2905.

[15] N. Seiberg, JHEP 0306 (2003) 010.

[16] E. Ivanov, O. Lechtenfeld, B. Zupnik, JHEP 0402 (2004) 012.

[17] S. Ferrara, E. Sokatchev, Phys. Lett. B 579 (2004) 226.

[18] S. Ferrara, E. Ivanov, O. Lechtenfeld, E. Sokatchev, B. Zupnik, Nucl. Phys. B704 (2005) 154.

[19] E. Ivanov, O. Lechtenfeld, B. Zupnik, Nucl. Phys. B 707 (2005) 69.

[20] A. De Castro, E. Ivanov, O. Lechtenfeld, L. Quevedo, Non-singlet deformations of the N=(1,1) gauge multiplet in harmonic superspace, hep-th/0510013.

[21] Y. Kobayashi, S. Sasaki, Int. J. Mod. Phys. A 20 (2005) 7175; hep-th/0410164.

[22] B.M. Zupnik, Twist-deformed supersymmetries in non-anticommutative superspaces, Phys. Lett. B 627 (2005) 208, hep-th/0506043.

[23] P. Kosiński, J. Lukierski, P. Maślanka, J. Sobczyk, J. Phys 27A (1994) 6827; J. Math. Phys. 37 (1996) 3041.

[24] P.P. Kulish, Contemp. Math., 391 (2006) 213.

[25] M. Ihl, C. Sämann, JHEP 0601 (2006) 065, hep-th/0506057.

[26] E.A. Ivanov, B.M. Zupnik, TMF 142 (2005) 235 [Theor. Math. Phys. 142 (2005) 197].