Ln(3) and Black Hole Entropy

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Abstract

We review an idea that uses details of the quasinormal mode spectrum of a black hole to obtain the Bekenstein-Hawking entropy of $A/4$ in Loop Quantum Gravity. We further comment on a recent proposal concerning the quasinormal mode spectrum of rotating black holes. We conclude by remarking on a recent proposal to include supersymmetry.

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I. INTRODUCTION

Ever since Bekenstein and Hawking argued that a black hole has an entropy $S$ which is a quarter of its horizon area $A$ it has been a challenge to theoretical physicists to explain what this is an entropy of. In recent years candidate theories of quantum gravity have provided concrete realizations of quantum mechanical microstates. Both String theory and Loop Quantum Gravity have given derivations for the Bekenstein-Hawking entropy.

In this article we will describe how one can fix a free parameter, known as the Immirzi parameter, to obtain the exact Bekenstein-Hawking result for the entropy in Loop Quantum Gravity. We will do this using an input from classical relativity, namely the quasinormal mode spectrum of black holes. The first three sections of this paper deal with this argument. The following sections then deal with rotating black holes and also discuss a new insight into the problem that relies on supersymmetry.

II. QUASINORMAL MODES IN THE HIGH DAMPING LIMIT

The reaction of a black hole to perturbations is dominated by an infinite discrete set of damped oscillation called quasinormal modes. One remarkable feature of the quasinormal mode spectrum for non-rotating black holes is that the real part of the frequency approaches the non-zero value $\omega_\infty = \ln 3/8\pi M$ as the damping increases (see figure 1).

![Figure 1: The position of the quasinormal mode frequencies in the complex $\omega$-plane. For higher damping the real part of the frequency becomes approaches the limiting value of $\ln 3/8\pi M$.](image)

A non-rotating black hole has thus associated with it one distinguished frequency. One
might then ask if there is a special process in the underlying quantum theory that gives rise to this particular frequency. We will come back to this question after having taken a closer look at Loop Quantum Gravity.

### III. FACTS FROM LOOP QUANTUM GRAVITY

In Loop Quantum Gravity a surface is thought to acquire area through the punctures of spin network edges. Spin networks form the basis of the Hilbert space of LQG. They consist of graphs whose edges are labelled by representations of the gauge group of the theory. In our case this is SU(2) (or SO(3)). The labels are thus (half-)integers. For a puncture with label $j$ the area is

$$A(j) = 8\pi l_p^2 \gamma \sqrt{j(j+1)}.$$  

Here $l_p^2$ is the Planck length. The free parameter $\gamma$ is the so-called Immirzi parameter.

One can think of the horizon area of a black hole to be the result of a large number of spin network edges puncturing the horizon surface. The Hilbert space of the theory living on the boundary is increased by each puncture. If the edge has a label $j$ the dimension of the Hilbert space increases by a factor of $2j + 1$. The statistically most important contribution comes from the lowest possible spin $j_{\text{min}}$. The dimension of the Hilbert space is thus

$$(2j_{\text{min}} + 1)^N,$$  

where $N$ is the number of punctures. Since the area due to one puncture is $A(j_{\text{min}})$ (see formula (1)) this number $N$ is given by $A/A(j_{\text{min}})$. The entropy of the black hole is thus

$$S = \frac{A}{8\pi l_p^2 \gamma \sqrt{j_{\text{min}}(j_{\text{min}} + 1)}} \ln(2j_{\text{min}} + 1).$$  

The entropy is proportional to the area but the result so far is not satisfactory since it contains the free parameter $\gamma$.

### IV. FIXING THE IMMIRZI PARAMETER

When discussing the high damping limit of the quasinormal mode frequencies we asked the question whether there could exist a quantum mechanical process that could give rise to the the limiting frequency $\omega_\infty$. Given the picture developed in the last section such a process
is easy to imagine. It is the appearance or disappearance of a puncture of the horizon (see figure 2a).

FIG. 2: a The proposed quantum transition giving rise to the emission of a quantum of energy $\hbar \omega_{\infty}$. b The emerging “It from trit” picture.

As Hod[8] pointed out, the emission of a quanta of frequency $\omega_{\infty}$ would change the area of a non-rotating black hole by $4 \ln 3 \ell_p^2$. This has to be equated with the change in area $A(j_{\min})$ due to one puncture. This equation fixes the Immirzi parameter and gives for the entropy

$$S = \frac{A}{4\ell_p^2} \frac{\ln(2j_{\min} + 1)}{\ln 3}.$$  \hfill (4)

We see that we obtain exactly the Bekenstein-Hawking result if $j_{\min} = 1$. This can be achieved if one chooses SO(3) as the gauge group instead of SU(2)[11]. Wheelers[12] “It from bit” gets thus modified to “It from trit” (see figure 2b)[20].

V. ROTATING BLACK HOLES

In the last section we have used quasinormal modes to learn something about the quantum mechanics of non-rotating black holes. A new argument by Hod[13] tries to go the other way. This time the quantum mechanics of black holes is used to gain insight into the quasinormal mode spectrum of a rotating black hole. The starting point is the first law of black hole mechanics

$$\Delta M = T \Delta S + \Omega \Delta J.$$  \hfill (5)

Using the expressions in the Kerr metric for the temperature $T = \sqrt{M^2 - a^2}/4\pi Mr_+$ and the angular velocity $\Omega = a/(r_+^2 + a^2)$, where $r_+ = M \pm \sqrt{M^2 - a^2}$ are the horizon radii, together with the identification $\Delta M = \hbar \text{Re}(\omega)$ and $\Delta S = 4 \ln(3)$ one obtains an expression
for $\text{Re}(\omega)$ from equation (5):

$$\text{Re}(\omega) = \frac{\ln(3)}{\pi} \frac{(M^2 - a^2)^{1/2}}{M(M + (M^2 - a^2)^{1/2})} + \frac{2a}{(M + (M^2 - a^2)^{1/2})^2 + a^2} \quad (6)$$

We thus obtain a prediction for how the frequency $\text{Re}(\omega)$ changes as a function of the rotation parameter $a = J/M$.

To check the validity of this prediction Hod compared it to numerical data obtained by Onozawa [14][21], who used numerical techniques developed by Leaver [15]. The graph obtained is reproduced in figure 3 using our own numerical data.

![Figure 3: Comparing Hod’s prediction with numerical data for the $n = 8$ and $l = m = 2$ modes. The dashed line is the theoretical prediction. The curve is matched well. The biggest discrepancy occurs at the origin, which is to be expected since we are looking at small damping and the $\omega$ is assumes its limiting value for large damping.

The two graphs match well. The largest deviation occurs at the origin. This is to be expected since we are looking at low damping.

It is assumed that the situation improves for larger damping. This is not the case though for the largest damping we have tested. Figure 4a shows quasinormal modes for $n = 79$. Figure 4b then shows the $\text{Re}(\omega)(a)$ plot, this time including all the values of $m$. None of the graphs shows a good matching with the theoretical prediction.

VI. SUPERSYMMETRY

In section IV we saw that we could get perfect agreement of the loop quantum gravity calculation with the Bekenstein-Hawking result provided we change the gauge group from
The Kerr quasinormal modes for higher damping. We have $n = 79$, $l = 2$, $m = -2, \ldots, 2$, and $a \approx 0.14$.

The function $\text{Re}(\omega)(a)$. There is no good agreement for any of the curves.

SU(2) to SO(3). This step is problematic since it is not clear how to include fermions in such a theory. Using supersymmetric spin networks Ling and Zhang were able to circumvent this problem. For supersymmetric spin networks the area spectrum is $A(j) = 8\pi \gamma l_j^2 \sqrt{j(j + 1/2)}$ and the dimension of the representation spaces is $4j + 1$. Using the same arguments as above Ling and Zhang obtain the following result for the entropy:

$$S = \frac{A \ln(4j_{\text{min}} + 1)}{4 \ln(3)}$$

Exact agreement is thus obtained for $j_{\text{min}} = 1/2$.

VII. CONCLUSIONS

We have seen how the classical theory of quasinormal modes could be used to fix an ambiguity in Loop Quantum Gravity. A more sophisticated treatment of black holes in Loop Quantum Gravity will be required to see how real this connection is.

The recent results by Hod concerning rotating black holes are interesting but the numerical data is ambiguous. More numerical data on highly damped Kerr modes is required to settle this question.

The appearance of supersymmetry is surprising. The deeper significance will only become clear when we better understand the dynamic processes underlying the proposed connection.

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[19] This fact was first observed by Hod [8] using numerical data by Nollert [9] and it has been recently proved rigorously by Motl [10].

[20] I thank Jonathan Oppenheim for this quote.

[21] Since this talk was given our knowledge of quasinormal modes for the Kerr black hole has improved dramatically. The asymptotic behavior has been exhaustively investigated numerically by Berti et. al. [17, 18]. The results contained in these references show clearly that the prediction made by Hod is not valid.