Projecting Lipschitz Functions Onto Spaces of Polynomials

Petr Hájek and Tommaso Russo

Abstract. The Banach space $\mathcal{P}(^2X)$ of 2-homogeneous polynomials on the Banach space $X$ can be naturally embedded in the Banach space $\text{Lip}_0(B_X)$ of real-valued Lipschitz functions on $B_X$ that vanish at 0. We investigate whether $\mathcal{P}(^2X)$ is a complemented subspace of $\text{Lip}_0(B_X)$. This line of research can be considered as a polynomial counterpart to a classical result by Joram Lindenstrauss, asserting that $\mathcal{P}(^1X) = X^*$ is complemented in $\text{Lip}_0(B_X)$ for every Banach space $X$. Our main result asserts that $\mathcal{P}(^2X)$ is not complemented in $\text{Lip}_0(B_X)$ for every Banach space $X$ with non-trivial type.

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1. Introduction

Given a pointed metric space $(M, 0_M)$, $\text{Lip}_0(M)$ denotes the Banach space of all scalar-valued Lipschitz functions on $M$ that vanish on $0_M$. The study of Banach spaces of Lipschitz functions, and even more of their canonical preduals, the Lipschitz-free spaces, has been one of the most active fields of research within Banach space theory in the last two decades. We refer, e.g., to [1, 3, 27, 28] and [14, 15, 60, 61] for recent results and additional references on Lipschitz-free spaces and spaces of Lipschitz functions, respectively. One of the earliest results in this area is the extremely useful result by Joram Lindenstrauss [44], according to which $X^*$ is 1-complemented in $\text{Lip}_0(X)$, for every Banach space $X$ (see [13, Chapter 7] for more details and applications).

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Our main goal in this paper is to study possible extensions of the said result to higher order polynomials, in particular to the space \( \mathcal{P}(2X) \) of 2-homogeneous polynomials on \( X \). This is an instance of a well-studied line of research aiming at obtaining polynomial versions of linear results, [6–9, 21, 22, 50, 56]. When trying to approach the said question, we face the obvious problem that non-zero polynomials of degree at least 2 on a Banach space \( X \) are not Lipschitz functions. However, the restriction of a polynomial on \( X \) to \( B_X \) is a Lipschitz function and such restriction defines an isomorphic embedding of \( \mathcal{P}(2X) \) in \( \text{Lip}_0(B_X) \) (see Sect. 2.2 for details). Moreover, \( \text{Lip}_0(B_X) \) is isomorphic to \( \text{Lip}_0(X) \) for every Banach space \( X \), [38]. Therefore, the search for the polynomial counterpart to Lindenstrauss’ result leads us to the following problem.

Problem 1.1. Identify \( \mathcal{P}(2X) \) with a subspace of \( \text{Lip}_0(B_X) \) by restricting a polynomial on the Banach space \( X \) to \( B_X \). Is \( \mathcal{P}(2X) \) a complemented subspace of \( \text{Lip}_0(B_X) \)?

Let us mention in passing that a positive answer to the above problem would also give information on approximation properties for Banach spaces of Lipschitz functions. For example, if \( \mathcal{P}(2\ell_2) \) were a complemented subspace of \( \text{Lip}_0(\ell_2) \), then \( \text{Lip}_0(\ell_2) \) would fail the approximation property. This is because the approximation property passes to complemented subspaces, while \( \mathcal{P}(2\ell_2) \) fails the approximation property [24, p. 173] (see also [19]). Let us also notice that, while approximation properties of Lipschitz-free Banach spaces have been widely investigated, [4, 27–29, 41, 51], the study of approximation properties of spaces of Lipschitz functions seems to be a largely unexplored topic, [36]. For example, it is apparently unknown if \( \text{Lip}_0(\ell_2) \) has the approximation property.

The main result of our paper, Theorem B below, answers Problem 1.1 in the negative for a rather large class of Banach spaces, including all spaces with non-trivial type, in particular all super-reflexive Banach spaces. The simplest space which is not covered by our results is \( c_0 \). As it is perhaps to be expected, our main theorem will be derived from its quantitative counterpart for Euclidean spaces, that reads as follows. (The precise definitions of the spaces \( \mathcal{P}(2E_n) \), \( \text{Lip}_0(B_{E_n}) \), and \( C^1_0(B_{E_n}) \) will be given in Sect. 2.)

**Theorem A.** Let \( E_n \) denote the \( n \)-dimensional Euclidean space \((\mathbb{R}^n, \|\cdot\|_2)\). If \( Q \) is any projection from \( C^1_0(B_{E_n}) \) onto \( \mathcal{P}(2E_n) \), then

\[
\|Q\| \geq C \left( n - 2\sqrt{2} \right)^{1/5},
\]

where \( C := \frac{2}{5} \left( \frac{\sqrt{2} - 1}{3} \cdot \frac{3}{\pi} \right)^{1/5} \).

In particular, the same estimate holds for every projection from \( \text{Lip}_0(B_{E_n}) \) onto \( \mathcal{P}(2E_n) \).

The main reason why we stated the theorem in the stronger form concerning \( C^1_0(B_{E_n}) \) is that the averaging argument that we shall need (see Sect. 2.3) is simpler to describe in \( C^1_0(B_{E_n}) \) rather than in the bigger space
Lip_0(B_{E_n}). In this context, it is perhaps worth observing that C_0^1(B_F) fails to be complemented in Lip_0(B_F), for every finite-dimensional Banach space F. Indeed C_0^1([-1,1]) is not complemented in Lip_0([-1,1]), since the former is isometric to C([-1,1]) and the latter to L_\infty([-1,1]). By ‘bootstrap’ from this observation, the claim for the general finite-dimensional Banach space F readily follows.

The study of projections onto a finite-dimensional subspace of a Banach space is a very classical and wide topic and Theorem A can be considered as one more result in the area. By the classical Kadets–Snobar theorem [37], every n-dimensional subspace of a Banach space is \sqrt{n}-complemented and this is asymptotically sharp [39]. As a non-exhaustive list of results on projection constants, let us quote [11,26,31,42], or the monographs [59,62]. The particular case of projections onto spaces of polynomials is also very well studied (particularly in the trigonometric case), for its connections with harmonic ([62, III.B], [30,49]) and numerical [25,57,58] analysis. Let us mention for example the classical Lozinski–Kharshiladze theorem [47] (see, e.g., [62, III.B.22], or [40, pp. 150–155]) on the projection constant of the space T_\infty^n of trigonometric polynomials of degree at most n on the 1-dimensional torus \mathbb{T}, with the sup-norm. The result claims in particular that if P_n is any projection from C(T) to T_\infty^n, then \|P_n\| \geq \frac{4}{\pi} \log(n+1) + o(1). There are however three crucial differences between these classical results and our paper: we consider algebraic rather than trigonometric polynomials; we are interested in the asymptotic behaviour when the dimension of the space grows, as opposed to letting the degree of the polynomials grow; and finally we consider the Lipschitz norm instead of the sup-norm.

Having Theorem A at our disposal, we can formulate the main result of our paper, that in particular answers in the negative Problem 1.1. Prior to the statement, we need to recall one definition. A Banach space X is said to contain uniformly complemented (\ell^2_n)_{n=1}^\infty if there are a constant C and a sequence \( (F_n)_{n=1}^\infty \) of subspaces of X such that, for each n, F_n is C-isomorphic to \ell^2_n and C-complemented in X.

**Theorem B.** If X contains uniformly complemented (\ell^2_n)_{n=1}^\infty, then \mathcal{P}(^2X) is not complemented in Lip_0(B_X). In particular:

(i) If X has non-trivial type, \mathcal{P}(^2X) is not complemented in Lip_0(B_X),
(ii) \mathcal{P}(^2\ell_2) is not complemented in Lip_0(B_{\ell_2}).

As we will see in Sect. 4, the first clause of Theorem B is a purely formal consequence of Theorem A. In the same section we will also extend the result to the spaces \mathcal{P}(^nX) of n-homogeneous polynomials on X and \mathcal{P}^n(X) of polynomials of degree at most n (Theorem 4.3). For information on type and cotype we refer to [2,18,48]; here we just recall that X has non-trivial type if it has type p, for some p > 1. (ii) is, of course, consequence of (i), since Hilbert spaces have type 2. More generally Theorem B applies to every super-reflexive Banach space, as super-reflexive spaces have non-trivial type, [52]; on the other hand, there are non-reflexive spaces of type 2, [35].

The statement in (i) follows from the first part of the theorem and a deep result of Figiel and Tomczak-Jaegermann, [23]. Indeed, it is proved in
that if the Banach space $X$ has non-trivial type, there is a constant $C$ such that, for every $\varepsilon > 0$ and $n \in \mathbb{N}$, $X$ contains a $C$-complemented subspace that is $(1 + \varepsilon)$-isomorphic to $\ell^2_n$. The result can also be found in the said monographs, [18, Theorem 19.3], [48, Theorem 15.10]; a different proof was given in [12].

At this stage, one might even be led to conjecture that $P(2X)$ is complemented in $\text{Lip}_0(B_X)$, for no infinite-dimensional Banach space $X$ (plainly, $P(2X)$ is complemented in $\text{Lip}_0(B_X)$ for finite-dimensional $X$). However this has long been known to be false. Indeed, back in 1976, Aron and Schottensloher [10] proved that $P(k\ell_1)$ is isomorphic to $\ell_\infty$, hence even injective (for each $k \geq 1$). More generally, Arias and Farmer [5] proved that $P(kX)$ is isomorphic to $\ell_\infty$ whenever $X$ is a separable $L_1$-space; in particular, $P(kL_1)$ is complemented in $\text{Lip}_0(B_{L_1})$. Since the result in [5] is stated in a somewhat different form for $L_p$-spaces, $1 < p < \infty$, we decided to sketch the proof of the said result in Proposition 4.1.

Our paper is organised as follows: Sect. 2 collects basic definitions and ancillary results that we need, while the main part of the paper is Sect. 3, where Theorem A is proved. Finally, in Sect. 4 we prove Theorem B and Proposition 4.1; we also mention extensions of our results to higher order polynomials.

2. Preliminary Material

Our notation is standard, e.g., as in [2]. We denote by $B_X$ the closed unit ball of a Banach space $X$. The notation $(E_n, |\cdot|)$ indicates the $n$-dimensional Euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$; when the dimension is not important, we just indicate by $(E, |\cdot|)$, or $E$, a finite-dimensional Euclidean space. Below we gather the definitions and some basic properties of the objects we shall consider in our paper.

2.1. Spaces of Lipschitz Functions

Let $(M, d, 0_M)$ be a pointed metric space, namely a metric space $(M, d)$ with a distinguished point $0_M \in M$. The vector space $\text{Lip}_0(M)$ comprising all Lipschitz functions $f : M \to \mathbb{R}$ such that $f(0_M) = 0$ becomes a Banach space when endowed with the norm given by the best Lipschitz constant

$$
\|f\|_{\text{Lip}_0} := \text{Lip}(f) := \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x \neq y \in M \right\}.
$$

In our paper we will only consider the cases when $M$ is a Banach space $X$ or its closed unit ball $B_X$. In either case, the distinguished point will be the origin of the Banach space and it will be denoted simply by 0. Let us recall also here that, for every Banach space $X$, the spaces $\text{Lip}_0(X)$ and $\text{Lip}_0(B_X)$ are isomorphic, [38, Corollary 3.3].

The open unit ball of the Euclidean space $E$ will be denoted $B_E^o$. We denote by $\nabla f$ the gradient of a differentiable function $f : B_E^o \to \mathbb{R}$. Let

$$
\mathcal{C}_0^1(B_E) := \left\{ f : B_E \to \mathbb{R} : f \in C(B_E) \cap C^1(B_E^o), f(0) = 0, \nabla f \text{ is uniformly continuous on } B_E^o \right\}.
$$
By uniform continuity, the function $\nabla f$ is bounded on $B^o_E$ and it admits a (unique) extension, that we also denote $\nabla f$, to $B_E$. Plainly, $\|f\|_{\text{Lip}}_0 = \|\nabla f\|_\infty$, for every $f \in C^1_0(B_E)$; in particular, $C^1_0(B_E) \subseteq \text{Lip}_0(B_E)$. Moreover, it is a standard result in calculus that $C^1_0(B_E)$ is a closed subspace of $\text{Lip}_0(B_E)$.

2.2. Polynomials

In this section we shall briefly revise basic results on polynomials; we refer to [32, § 1.1] for further details. We denote by $L^n(X)$ the vector space of all $n$-linear forms $M : X \times \cdots \times X \to \mathbb{R}$ such that

$$
\|M\|_L := \sup_{x_j \in B_X \atop j = 1, \ldots, n} |M(x_1, \ldots, x_n)| < \infty.
$$

$L^n(X)$ is a Banach space when endowed with the norm $\|\cdot\|_L$. $L^s(n)X$ denotes the closed subspace of $L^n(X)$ comprising all symmetric $n$-linear forms.

A function $P : X \to \mathbb{R}$ is a $n$-homogeneous polynomial if there is $M \in L^n(X)$ such that $P(x) = M(x, \ldots, x)$ ($x \in X$). The vector space $P^n(X)$ of all $n$-homogeneous polynomials is turned into a Banach space when furnished with the norm

$$
\|P\|_P := \sup_{x \in B_X} |P(x)|.
$$

By a standard symmetrisation argument, for every $P \in P^n(X)$ there is a unique symmetric $n$-linear form $\tilde{P} \in L^s(n)X$ such that $P(x) = \tilde{P}(x, \ldots, x)$. Moreover, the form $\tilde{P}$ satisfies

$$
\|P\|_P \leq \|\tilde{P}\|_L \leq \frac{n^n}{n!} \|P\|_P. \tag{2.1}
$$

Quite importantly, however, in case of a Hilbert space $H$ one has $\|P\|_P = \|P\|_L$ for every $P \in P^n(H)$, [32, Theorem 1.16]. These results imply that $P^n(X)$ and $L^s(n)X$ are isomorphic Banach spaces for every Banach space $X$ and that they are indeed isometric when $X = H$ is a Hilbert space.

We shall next discuss the crucial fact mentioned in the Introduction that $P^n(X)$ naturally embeds into $\text{Lip}_0(B_X)$, for every Banach space $X$.

**Fact 2.1.** Let $X$ be any Banach space and $P \in P^n(X)$. Then

$$
n\|P\|_P \leq \|P\|_{B_X, \text{Lip}_0} \leq n \|\tilde{P}\|_L.
$$

**Proof.** For fixed $x \in B_X$, the function $[-1,1] \ni t \mapsto P(tx) = t^n P(x)$ has Lipschitz constant equal to $n \cdot |P(x)|$. Therefore, $\text{Lip}(P\mid_{B_X}) \geq n \cdot |P(x)|$ and the former inequality follows. For the latter, fix $x, y \in B_X$ and write
\[ P(x) - P(y) = \tilde{P}(x, \ldots, x) - \tilde{P}(y, \ldots, y) \]
\[ = \sum_{k=1}^{n} \tilde{P}(x, \ldots, x, y, \ldots, y) - \tilde{P}(x, \ldots, x, y, \ldots, y) \]
\[ = \sum_{k=1}^{n} \tilde{P}(x, \ldots, x, y - y, \ldots, y) \text{ for } k\text{-many} \]
\[ \text{and } (k-1)\text{-many} \]

Thus, \(|P(x) - P(y)| \leq \sum_{k=1}^{n} \| \tilde{P} \|_{\mathcal{L}} \| x \|^{k-1} \| y \|^{n-k} \| x - y \| \leq n \| \tilde{P} \|_{\mathcal{L}} \| x - y \|. \]

Therefore, it follows from Fact 2.1 and (2.1) that the map \( P \mapsto P\upharpoonright_{B_X} \) defines an isomorphic embedding of \( \mathcal{P}(n) \mathcal{X} \) into \( \text{Lip}_0(B_X) \). In the case of a Hilbert space, we have \( n\|P\|_{\mathcal{P}} = \|P\upharpoonright_{B_X}\|_{\text{Lip}_0} \); hence, such an embedding is additionally a multiple of an isometric one. Throughout the paper, when we consider \( \mathcal{P}(n) \mathcal{X} \) as a subspace of \( \text{Lip}_0(B_X) \) this embedding will be the one under consideration.

### 2.3. Averaging and Invariant Projections

It is a standard and useful fact that projections of minimal norm tend to respect the symmetries of the Banach spaces they act on (see, e.g., [53–55]).

In this part, we shall recall the version of this result that we need; we also sketch a proof, for the sake of completeness.

Let us denote by \((E, |\cdot|)\) the Euclidean space \((\mathbb{R}^n, \|\cdot\|_2)\) and by \(O_n\) its orthogonal group \([16, \S\ 1.1]\). Since \(O_n\) is a compact topological group, we can select a normalised Haar measure, which we denote \(\mu\), on it \([33, \text{Chapter 4}]\).

Given a projection \(Q : C_0^1(B_E) \to \mathcal{P}(2E)\), we shall consider the projection \(\tilde{Q} : C_0^1(B_E) \to \mathcal{P}(2E)\) defined by

\[ \tilde{Q}(f) := \int_{\omega \in O_n} Q(f \circ \omega) \circ \omega^{-1} \, d\mu(\omega) \quad (f \in C_0^1(B_E)). \]

We first observe that \(\tilde{Q}(f)\) is well defined as a Bochner integral of the function \(\omega \mapsto Q(f \circ \omega) \circ \omega^{-1}\), from \(O_n\) to \(\mathcal{P}(2E)\). Indeed, it suffices to check that the said function is continuous.

**Fact 2.2.** The map \(\omega \mapsto Q(f \circ \omega) \circ \omega^{-1}\) is continuous.

**Proof.** We only need to show that the map \(\omega \mapsto f \circ \omega\), from \(O_n\) to \(C_0^1(B_E)\), is continuous at the identity matrix \(I\). For every \(x \in B_E\) we have

\[ |\nabla(f \circ \omega)(x) - \nabla f(x)| = |\nabla f(\omega x)\omega - \nabla f(x)| \]
\[ \leq |\nabla f(\omega x)\omega - \nabla f(\omega x)| + |\nabla f(\omega x) - \nabla f(x)| \]
\[ \leq \|\nabla f\|_{\infty} \cdot \|\omega - I\| + m_{\nabla f}(\|\omega - I\|), \]

where \(m_{\nabla f}\) denotes the modulus of continuity\(^1\) of \(\nabla f\). Thus, as \(\omega \to I\),

\[ \|\nabla(f \circ \omega - f)\|_{\infty} \leq \|\nabla f\|_{\infty} \cdot \|\omega - I\| + m_{\nabla f}(\|\omega - I\|) \to 0. \]

\(^1\)We don’t use the standard notation \(\omega\) for a modulus of continuity as it would conflict with \(\omega \in O_n\).
Once we know that $\tilde{Q}$ is well defined, it is straightforward to verify the following properties (see [62, Theorem III.B.13]). Property (i) claims that $\tilde{Q}$ is invariant under $O_n$.

**Lemma 2.3.** [55] $\tilde{Q}$ is a projection from $C^1_0(B_E)$ onto $P(2E)$. Moreover,

(i) $\tilde{Q}(f \circ \omega) = \tilde{Q}(f) \circ \omega$ for each $f \in C^1_0(B_E)$ and $\omega \in O_n$;

(ii) $\|\tilde{Q}\| \leq \|Q\|$.

Moreover, recall that Bochner integrals commute with bounded linear operators. Thus for every bounded linear operator $T: P(2E) \to Y$ one has

$$T(\tilde{Q}f) = \int_{\omega \in O_n} T(Q(f \circ \omega) \circ \omega^{-1}) \, d\mu(\omega) \quad (f \in C^1_0(B_E)).$$

In particular, this also yields the pointwise formula

$$\tilde{Q}(f)(x) = \int_{\omega \in O_n} Q(f \circ \omega)(\omega^{-1}x) \, d\mu(\omega) \quad (x \in B_E).$$

Similarly as the average of a projection, we can also define the average of a function $f \in C^1_0(B_E)$ to be the function

$$\tilde{f} := \int_{\omega \in O_n} f \circ \omega \, d\mu(\omega).$$

As before, we see that $\tilde{f}$ is well defined as a Bochner integral. Hence $\tilde{f} \in C^1_0(B_E)$ and

$$\nabla \tilde{f} = \int_{\omega \in O_n} \nabla (f \circ \omega) \, d\mu(\omega).$$

Moreover, we easily see that $Lip(\tilde{f}) \leq Lip(f)$ and that $\tilde{f}$ is invariant under $O_n$, in the sense that $\tilde{f} \circ \omega = \tilde{f}$ for every $\omega \in O_n$. Finally, $\tilde{f} = f$ if $f$ is invariant under $O_n$.

### 3. Proof of Theorem A

This section is dedicated to the proof of the finite-dimensional result Theorem A. Before entering the details of the argument, we shall sketch here the the main steps of the proof and the very idea behind it. We will construct a $C^1$ function $\psi: B_{\mathbb{R}^2} \to \mathbb{R}$ (hence Lipschitz) which is equal to zero on a neighbourhood of the coordinate axes and whose average over all rotations of the plane is close to a ‘large’ multiple of the polynomial $N_2 := (\|\cdot\|_2)^2$. Such a function can be considered as the 2-dimensional analogue of the function $\varrho: [-1, 1] \to \mathbb{R}$ defined by $\varrho(x) := (|x| - 2\varepsilon)^2 1_{[2\varepsilon, 1]}(|x|) \, (x \in [-1, 1])$; actually, the radial behaviour of $\psi$ will be determined by $\varrho$.

We will then define a $C^1$ function $\Psi: B_{\mathbb{R}^n} \to \mathbb{R}$ by means of $\psi$. We shall show that the Lipschitz constant of $\Psi$ is bounded uniformly in $n$ (but depending on $\varepsilon$), while the projection of $\Psi$ has a Lipschitz constant that grows with $n$. 
Proof of Theorem A. For the sake of simplicity, we shall just denote by $E$ the $n$-dimensional Euclidean space $E_n = (\mathbb{R}^n,\|\cdot\|_2)$ and $\|\cdot\|$ its norm. Let

$$K_n := \inf \left\{ \|Q\| : Q \text{ is a projection from } C_0^1(B_E) \text{ onto } \mathcal{P}(2E) \right\}.$$ 

Since $\mathcal{P}(2E)$ is a finite-dimensional Banach space, a standard compactness argument shows that $K_n$ is attained, [34, Theorem 3] (see, e.g., the argument in [20, Lemma 5.17.(iii)], or [43, p. 12]). Therefore, we can pick a projection $Q$ from $C_0^1(B_E)$ onto $\mathcal{P}(2E)$ such that $\|Q\| = K_n$. Moreover, Lemma 2.3 allows us to assume additionally that $\|Q\| = K_n$. Moreover, we are now ready to start the first step of the proof.

**Step 1. A function in the plane.**

We shall define a $C^1$ function $\psi: B_{\mathbb{R}^2} \to \mathbb{R}$ with some symmetries and with the property that, if $\psi(x, y) \neq 0$, then $|x|, |y| \geq \varepsilon$. The function $\psi$ will be defined in polar coordinates, as follows.

Fix a parameter $\varepsilon > 0$ (whose value will be chosen at the end of the argument in Step 5). Consider the functions $\varrho: [0, 1] \to \mathbb{R}$ and $\tau_0: [0, \pi/2] \to \mathbb{R}$ defined by

$$\varrho(r) := \begin{cases} (r - 2\varepsilon)^2 & r \geq 2\varepsilon \\ 0 & r < 2\varepsilon \end{cases} \quad (r \in [0, 1])$$

$$\tau_0(\theta) := \max \left\{ \frac{\pi}{12} - \left| \theta - \frac{\pi}{4} \right|, 0 \right\} \quad (\theta \in [0, \pi/2]).$$

Since $\tau_0$ is not $C^1$, we approximate it by a smooth function that shares some of its properties. More precisely, we fix $\delta > 0$ with $\delta < \pi/72$ and we pick a $C^1$ function $\tau: [0, \pi/2] \to \mathbb{R}$ such that

(i) $\tau \geq 0$ and $\tau_0 - \delta \leq \tau \leq \tau_0$,

(ii) $\tau$ is symmetric with respect to $\theta = \pi/4$,

(iii) $\text{Lip}(\tau) \leq 1$.

Such a function $\tau$ can be easily obtained by a standard convolution argument. The function $\psi$ is the function whose expression in polar coordinates in the first quadrant is given by $\varrho \cdot \tau$. More precisely, we set

$$\psi(x, y) := \varrho \left( \sqrt{x^2 + y^2} \right) \cdot \tau \left( \text{arctg} \left| \frac{x}{y} \right| \right) \quad ((x, y) \in B_{\mathbb{R}^2}).$$

**Fact 3.1.** The function $\psi: B_{\mathbb{R}^2} \to \mathbb{R}$ has the following properties:

(i) $\psi(x, y) = \psi(|x|, |y|) = \psi(y, x),$

(ii) $\psi(x, y) \neq 0$ only when $|x|, |y| \geq \varepsilon$,

(iii) $\psi \in C_0^1(B_{\mathbb{R}^2})$ and $\text{Lip}(\psi) \leq 2$.

**Proof of Fact 3.1.** (i) The fact that $\psi(x, y) = \psi(y, x)$ is consequence of the symmetry of $\tau$ with respect to $\theta = \pi/4$. The equality $\psi(x, y) = \psi(|x|, |y|)$ is obvious by definition.

(ii) If $\psi(x, y) \neq 0$, then $x^2 + y^2 \geq 4\varepsilon^2$ and $\frac{1}{\sqrt{3}}|x| \leq |y| \leq \sqrt{3}|x|$. Substituting $|y| \leq \sqrt{3}|x|$ into the first inequality yields $|x| \geq \varepsilon$; similarly, one gets $|y| \geq \varepsilon$. 


(iii) The fact that $\psi$ is $C^1$ on $B_{\mathbb{R}^2}$ is clear (the smoothness in the points of the two axes follows, e.g., from (ii)). Thus, it suffices to show that $|\nabla \psi(x, y)| \leq 2$ for each $(x, y) \in B_{\mathbb{R}^2}$. Via the expression for the gradient in polar coordinates, we get

$$
|\nabla \psi(x, y)|^2 = \left( \frac{1}{x^2 + y^2} \left( \sqrt{x^2 + y^2} \cdot \frac{x}{y} \right) \right)^2 \\
+ \frac{1}{x^2 + y^2} \left( \sqrt{x^2 + y^2} \cdot \frac{y}{x} \right)^2 \\
\leq (2 \cdot \pi/12)^2 + \frac{1}{x^2 + y^2} \left( \sqrt{x^2 + y^2} \right)^2 \leq \pi^2/36 + 1 \leq 4.
$$

□

**Step 2.** The functions $\Psi$ and $\Psi_d$.

We shall now pass to defining functions on $B_E$. Fix indices $1 \leq i < j \leq n$ and define $\psi_{ij} : B_E \to \mathbb{R}$ by

$$
\psi_{ij}(x) := \psi(x_i, x_j) \quad (x = (x_1, \ldots, x_n) \in B_E).
$$

Next, we set $d := \lfloor n/\sqrt{2} \rfloor$ and we define the $C^1$ functions

$$
\Psi := \sum_{i<j \leq n} \psi_{ij} \quad \text{and} \quad \Psi_d := \sum_{i<j \leq d} \psi_{ij}.
$$

The proof will now consist in showing that the Lipschitz constants of $\Psi$ and $\Psi_d$ can be bounded uniformly in $n$ (but depending on $\varepsilon$). On the other hand, the Lipschitz constants of $Q(\Psi)$ and $Q(\Psi_d)$ grow with $n$. This latter fact will be shown in the subsequent steps, while we now show the former clause. The main reason behind it is that only ‘few’ of the functions $\psi_{ij}$ can be nonzero at a given point, since only ‘few’ coordinates of $x \in B_E$ can be larger than $\varepsilon$ in absolute value.

**Fact 3.2.** $\text{Lip}(\Psi), \text{Lip}(\Psi_d) \leq 1/\varepsilon^4$.

**Proof of Fact 3.2.** Fix any $x \in B_E$, any $\xi > 0$ and let $I := \{i = 1, \ldots, n : |x_i| \geq \frac{\varepsilon}{1+\xi}\}$. Observe that $|I| \leq \left( \frac{1+\xi}{\varepsilon} \right)^2$. Indeed,

$$
1 \geq \sum_{i=1}^{n} x_i^2 \geq \sum_{i \in I} \left( \frac{\varepsilon}{1+\xi} \right)^2 = \left( \frac{\varepsilon}{1+\xi} \right)^2 \cdot |I|.
$$

Now pick any $y \in B_E$ with $\|x - y\| \leq \frac{\varepsilon \xi}{1+\xi}$ and assume that $\psi_{ij}(y) \neq 0$. Fact 3.1 yields $|y_i|, |y_j| \geq \varepsilon$, whence $|x_i|, |x_j| \geq \frac{\varepsilon}{1+\xi}$. Therefore, $i, j \in I$; in other words, in the ball $B(x, \frac{\varepsilon \xi}{1+\xi})$ only the functions $\psi_{ij}$ with $i, j \in I$ are different from 0. Consequently, $\Psi$ is locally the sum of at most $\frac{1}{2} \cdot \left( \frac{1+\xi}{\varepsilon} \right)^4$ functions whose Lipschitz constant is at most 2. It follows that $\Psi$ is $\left( \frac{1+\xi}{\varepsilon} \right)^4$-Lipschitz and letting $\xi \to 0$ proves the claim for $\Psi$.

The argument for $\Psi_d$ is identical. □
Step 3. Computation of $Q(\Psi)$ and $Q(\Psi_d)$ and estimate of \(\max \{ \text{Lip}(Q(\Psi)) , \text{Lip}(Q(\Psi_d)) \} \).

We start by giving formulas for the 2-homogeneous polynomials $Q(\Psi)$ and $Q(\Psi_d)$ exploiting the symmetries of $\Psi$ and $\Psi_d$ and the invariance of $Q$. Since $\psi_{ij}$ is a rotation of $\psi_{12}$, it suffices to compute $Q(\psi_{12})$.

Let $(a_{ij})_{i,j=1}^n$ be scalars with $a_{ij} = a_{ji}$ and such that

$$Q(\psi_{12})(x) = \sum_{i,j=1}^n a_{ij} \cdot x_i x_j.$$

Fix $k = 1, \ldots, n$ and let $\omega_k$ be the reflection $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{k-1}, -x_k, x_{k+1}, \ldots, x_n)$. Since $\psi_{12}$ is invariant under $\omega_k$, the same is true for $Q(\psi_{12})$, namely

$$\sum_{i,j=1}^n a_{ij} \cdot x_i x_j = \sum_{i,j=1}^n a_{ij} \cdot x_i x_j + a_{kk} \cdot x_k^2 - 2 \sum_{i=1, i \neq k}^n a_{ik} \cdot x_i x_k.$$

Therefore, $a_{ik} = 0$, whenever $i \neq k$. Since $k = 1, \ldots, n$ was arbitrary, it follows that $a_{ij} = 0$ for distinct $i, j = 1, \ldots, n$; hence $Q(\psi_{12})(x) = \sum_{i=1}^n a_{ii} \cdot x_i^2$.

Next, the invariance of $\psi_{12}$ under the reflection $(x_1, \ldots, x_n) \mapsto (x_2, x_1, x_3, \ldots, x_n)$ implies that $a_{11} = a_{22}$. Similarly, the invariance of $\psi_{12}$ under the reflection that permutes the coordinates $i$ and $j$ ($i, j = 3, \ldots, n$) yields that $a_{33} = \cdots = a_{nn}$.

In conclusion, there are scalars $\alpha$ and $\beta$ such that

$$Q(\psi_{12})(x) = \alpha \left( x_1^2 + x_2^2 \right) + \beta \left( x_3^2 + \cdots + x_n^2 \right). \quad (3.1)$$

This also implies

$$Q(\psi_{ij})(x) = \alpha \left( x_i^2 + x_j^2 \right) + \beta \sum_{\ell=1, \ell \neq i,j}^n x_{\ell}^2.$$

Consequently, we can now compute

$$Q(\Psi)(x) = \sum_{i<j \leq n} \left[ \alpha \left( x_i^2 + x_j^2 \right) + \beta \sum_{\ell=1, \ell \neq i,j}^n x_{\ell}^2 \right] = \left[ \alpha(n-1) + \beta \frac{(n-1)(n-2)}{2} \right] (x_1^2 + \cdots + x_n^2).$$

Letting $N \in \mathcal{P}(^2E)$ be the square of the norm, i.e., the polynomial $N(x) := x_1^2 + \cdots + x_n^2$, we rewrite the above as

$$Q(\Psi) = \left[ \alpha(n-1) + \beta \frac{(n-1)(n-2)}{2} \right] \cdot N. \quad (3.2)$$
Similarly, we compute $Q(\Psi_d)$.

$$Q(\Psi_d)(x) = \sum_{i \neq j \leq d} \alpha(x_i^2 + x_j^2) + \beta \sum_{\ell \neq i,j} x_\ell^2$$

$$= \left[ \alpha(d-1) + \beta \frac{(d-1)(d-2)}{2} \right] (x_1^2 + \cdots + x_d^2)$$

$$+ \beta \frac{d(d-1)}{2} (x_{d+1}^2 + \cdots + x_n^2).$$

As before, we can set $N_d(x) := x_1^2 + \cdots + x_d^2$ and obtain

$$Q(\Psi_d) = \left[ \alpha(d-1) + \beta \frac{(d-1)(d-2)}{2} \right] \cdot N_d$$

$$+ \beta \frac{d(d-1)}{2} (N - N_d). \quad (3.3)$$

Since the polynomials $N$ and $N_d$ are 2-Lipschitz on $B_E$, the equations (3.2) and (3.3) give

$$\text{Lip}(Q(\Psi)) = 2 \left| \alpha(n-1) + \beta \frac{(n-1)(n-2)}{2} \right| \quad (3.4)$$

$$\text{Lip}(Q(\Psi_d)) \geq 2 \left| \alpha(d-1) + \beta \frac{(d-1)(d-2)}{2} \right|. \quad (3.5)$$

Note that in the second estimate we used the fact that $\text{Lip}(Q(\Psi_d)) \geq \text{Lip}((Q(\Psi_d)|_{\mathbb{R}^d})$.

We are now in a position to give a better estimate that only depends on $\alpha$ (and not on $\beta$).

**Claim 3.3.** The polynomials $Q(\Psi)$ and $Q(\Psi_d)$ satisfy

$$\max \{ \text{Lip}(Q(\Psi)), \text{Lip}(Q(\Psi_d)) \} \geq 2 \frac{2}{3} \left( \sqrt{2} - \frac{1}{2} \right) |\alpha| \left( n - 2\sqrt{2} \right). \quad (3.6)$$

**Proof of Claim 3.3.** We fix a parameter $\lambda > 0$, whose value will be determined later, and we distinguish two cases.

- In case $|\beta| \geq \frac{\lambda |\alpha|}{n-2}$, from (3.4) we have

$$\text{Lip}(Q(\Psi)) = 2 \left| \alpha(n-1) + \beta \frac{(n-1)(n-2)}{2} \right|$$

$$\geq 2|\beta| \frac{(n-1)(n-2)}{2} - 2|\alpha|(n-1)$$

$$\geq \lambda |\alpha|(n-1) - 2|\alpha|(n-1) = (\lambda - 2)|\alpha|(n-1).$$
In case $|\beta| \leq \frac{|\alpha|}{n-2}$, on the other hand, we use (3.5) and we obtain
\[
\text{Lip}(Q(\Psi_d)) \geq 2 \left| \alpha(d-1) + \beta \frac{(d-1)(d-2)}{2} \right| \\
\geq 2|\alpha|(d-1) - 2|\beta| \frac{(d-1)(d-2)}{2} \\
\geq 2|\alpha|(d-1) - \lambda|\alpha|(d-1) \frac{d-2}{n-2} \\
\geq 2|\alpha|(d-1) - \lambda|\alpha|(d-1) \frac{1}{\sqrt{2}} \\
\geq |\alpha|(d-1)(2 - \lambda/\sqrt{2}) \\
\geq \left( \sqrt{2} - \lambda/2 \right) |\alpha| \left( n - 2\sqrt{2} \right).
\]

Here, we used the inequalities $\frac{d-2}{n-2} \geq 1/\sqrt{2}$ and $d - 1 \geq \frac{n-2\sqrt{2}}{2}$ that follow readily from our previous choice $d := \lfloor n/\sqrt{2} \rfloor$.

The optimal choice for $\lambda$, namely $\lambda := \frac{2}{3} (\sqrt{2} + 2)$ yields $\lambda - 2 = \sqrt{2} - \lambda/2 = \frac{2}{3} (\sqrt{2} - 1)$. Thus, we have:

- In case $|\beta| \geq \frac{|\alpha|}{n-2}$,
  \[
  \text{Lip}(Q(\Psi)) \geq \frac{2}{3} \left( \sqrt{2} - 1 \right) |\alpha|(n-1) \\
  \geq \frac{2}{3} \left( \sqrt{2} - 1 \right) |\alpha| \left( n - 2\sqrt{2} \right). 
  \]

- In case $|\beta| \leq \frac{|\alpha|}{n-2}$,
  \[
  \text{Lip}(Q(\Psi_d)) \geq \frac{2}{3} \left( \sqrt{2} - 1 \right) |\alpha| \left( n - 2\sqrt{2} \right). 
  \]

Therefore, the estimate (3.6) is proved. \qed

**Step 4. Estimate of $\alpha$.**

The estimate (3.6) that we obtained in Claim 3.3 in the previous step is still not sufficient, since we have no information on the parameter $\alpha$. It is thus the purpose of this step to handle this and to give a lower bound for $\alpha$. To wit, we prove here the following claim.

**Claim 3.4.** Assume that $2\varepsilon K_n \leq 1$. Then, the parameter $\alpha$ that appears in (3.6) satisfies
\[
\alpha \geq \left( \frac{\pi}{72} - \delta \right) \cdot (1 - 2\varepsilon K_n) \geq 0.
\] (3.7)

The proof is based on the fact, mentioned at the beginning of the present section, that the average of $\psi$ under rotations of the plane is close to a multiple of the square of the norm of $\mathbb{R}^2$. Therefore, its projection onto the set of 2-homogeneous polynomials shares the same property.
Proof of Claim 3.4. We identify the group $SO_2$ with the subgroup of $O_n$ of rotations of span$\{e_1, e_2\}$. Accordingly, for $\omega \in SO_2$, we also write $\omega$ to denote the rotation $\omega \oplus 1_{n-2} \in O_n$ defined by $\omega \oplus 1_{n-2}(x_1, \ldots, x_n) = (\omega(x_1, x_2), x_3, \ldots, x_n)$. We also denote by $\mu_2$ the Haar measure of $SO_2$.

Let $\psi$ be the average of $\psi_{12}$ under the group $SO_2$ (see Sect. 2.3), i.e., let

$$\tilde{\psi} := \int_{\omega \in SO_2} \psi_{12} \circ \omega \, d\mu_2(\omega).$$

Then for every $x \in B_E$, we have

$$\tilde{\psi}(x) = \int_{\omega \in SO_2} \psi_{12}(\omega(x_1, x_2), x_3, \ldots, x_n) \, d\mu_2(\omega)$$

$$= \int_{\omega \in SO_2} \psi(\omega(x_1, x_2)) \, d\mu_2(\omega)$$

$$= 4 \cdot \int_{\theta \in [0, \pi/2]} \varrho \left( \sqrt{x_1^2 + x_2^2} \right) \cdot \frac{\tau(\theta)}{2\pi} \, d\theta$$

$$= \varrho \left( \sqrt{x_1^2 + x_2^2} \right) \cdot \eta,$$

where we defined $\eta := 4 \cdot \int_{[0, \pi/2]} \tau(\theta) \frac{d\theta}{2\pi}$.

Letting, as before, $N_2 \in \mathcal{P}(2E)$ be the polynomial $N_2(x) = x_1^2 + x_2^2$ and recalling that $\varrho(r) := (r - 2\epsilon)^2 \cdot 1_{[2\epsilon, 1]}(r)$, the above yields

$$\left( \eta N_2 - \tilde{\psi} \right)(x) = \eta \cdot \begin{cases} x_1^2 + x_2^2 & \text{if } x_1^2 + x_2^2 \leq 4\epsilon^2 \\ 4\epsilon \sqrt{x_1^2 + x_2^2} - 4\epsilon^2 & \text{elsewhere}. \end{cases}$$

Consequently,

$$\text{Lip} \left( \eta N_2 - \tilde{\psi} \right) \leq 4\epsilon \eta. \quad (3.8)$$

Moreover, since $\tau_0 - \delta \leq \tau \leq \tau_0$ and $4 \cdot \int_{[0, \pi/2]} \tau_0(\theta) \frac{d\theta}{2\pi} = \frac{\pi}{72}$, we have the estimate

$$0 < \frac{\pi}{72} - \delta \leq \eta \leq \frac{\pi}{72}. \quad (3.9)$$

On the other hand, using the facts that Bochner integrals commute with linear operators, that $Q$ is invariant under $O_n$ (hence, under its subgroup $SO_2$) and that $Q(\psi_{12})$ is invariant under $SO_2$ by (3.1), we derive

$$Q(\tilde{\psi}) = \int_{\omega \in SO_2} Q(\psi_{12} \circ \omega) \, d\mu_2(\omega)$$

$$= \int_{\omega \in SO_2} Q(\psi_{12}) \circ \omega \, d\mu_2(\omega)$$

$$= \int_{\omega \in SO_2} Q(\psi_{12}) \, d\mu_2(\omega) = Q(\psi_{12}).$$

Hence, using again $Q\psi_{12} = \alpha N_2 + \beta(N - N_2)$ from (3.1), we obtain

$$Q \left( \eta N_2 - \tilde{\psi} \right) = Q(\eta N_2 - \psi_{12}) = (\eta - \alpha) N_2 + \beta(N - N_2),$$
whence
\[
\text{Lip} \left( Q \left( \eta N_2 - \tilde{\psi} \right) \right) \geq \text{Lip} \left( Q \left( \eta N_2 - \tilde{\psi} \right) \mid_{\mathbb{R}^2} \right) = \text{Lip} \left( (\eta - \alpha) N_2 \right) = 2|\alpha - \eta|.
\]  
(3.10)

Finally, combining (3.8) with (3.10) yields
\[
2|\alpha - \eta| \leq \text{Lip} \left( Q \left( \eta N_2 - \tilde{\psi} \right) \right) \\
\leq K_n \cdot \text{Lip} \left( \eta N_2 - \tilde{\psi} \right) \\
\leq 4\varepsilon \eta \cdot K_n.
\]

Therefore, \( \alpha \geq \eta (1 - 2\varepsilon K_n) \geq 0 \) whence, using (3.9), the conclusion follows. \( \square \)

**Step 5.** *Plugging estimates together.*
In this step, we shall combine the estimates obtained in Fact 3.2 and Claim 3.3 with the lower bound on \( \alpha \) from Claim 3.4 and we shall conclude the proof.

From Fact 3.2 and Claim 3.3 we deduce
\[
\frac{2}{3} \left( \sqrt{2} - 1 \right) |\alpha| \left( n - 2\sqrt{2} \right) \leq \max \{ \|Q(\Psi)\|_{\text{Lip}_0}, \|Q(\Psi_d)\|_{\text{Lip}_0} \} \\
\leq K_n \cdot \max \{ \|\Psi\|_{\text{Lip}_0}, \|\Psi_d\|_{\text{Lip}_0} \} \leq K_n \cdot \frac{1}{\varepsilon^4}.
\]  
(3.11)

We now set (the reason behind such a choice will be clear in a few lines)
\[
c := \frac{2}{3} \left( \sqrt{2} - 1 \right) \frac{\pi}{72} \quad \text{and} \quad \varepsilon := \frac{2^{1/5}}{c^{1/5} \left( n - 2\sqrt{2} \right)^{1/5}}
\]
and we distinguish two cases, depending on whether \( 1 - 2\varepsilon K_n \geq 0 \) or not.

- If \( 2\varepsilon K_n \leq 1 \), we see from Claim 3.4 that \( \alpha \geq 0 \). Therefore, we can plug the estimate (3.7) for \( \alpha \) in (3.11) and we obtain
\[
\frac{1}{\varepsilon^4} \cdot K_n \geq \frac{2}{3} \left( \sqrt{2} - 1 \right) \cdot \left( \frac{\pi}{72} - \delta \right) \cdot (1 - 2\varepsilon K_n) \left( n - 2\sqrt{2} \right).
\]  
(3.12)
We can now let \( \delta \to 0 \) and obtain
\[
\frac{1}{\varepsilon^4} \cdot K_n \geq \frac{2}{3} \left( \sqrt{2} - 1 \right) \cdot \frac{\pi}{72} (1 - 2\varepsilon K_n) \left( n - 2\sqrt{2} \right) \\
= c \cdot (1 - 2\varepsilon K_n) \left( n - 2\sqrt{2} \right),
\]
which we rewrite as follows:
\[
c \cdot \left( n - 2\sqrt{2} \right) \leq \left[ \frac{1}{\varepsilon^4} + 2\varepsilon c \left( n - 2\sqrt{2} \right) \right] K_n.
\]
The above choice of \( \varepsilon \) minimises the value of the square parenthesis above and gives the value
\[
\left[ \frac{1}{\varepsilon^4} + 2\varepsilon c \left( n - 2\sqrt{2} \right) \right] = c^{4/5} \frac{5}{2^{4/5}} \left( n - 2\sqrt{2} \right)^{4/5}.
\]
Therefore, we have
\[ K_n \geq \frac{2^{4/5}}{5} c^{1/5} (n - 2\sqrt{2})^{1/5} = C (n - 2\sqrt{2})^{1/5}. \]
Here,
\[ C := \frac{2^{4/5}}{5} c^{1/5} = \frac{2}{5} \left( \frac{\sqrt{2} - 1}{3} \cdot \frac{\pi}{72} \right)^{1/5}, \]
hence we obtained the inequality claimed in Theorem A.

- In the case \( 2\varepsilon K_n \geq 1 \), then
\[ K_n \geq \frac{1}{2\varepsilon} = \frac{1}{2^{6/5}} c^{1/5} (n - 2\sqrt{2})^{1/5} \geq C (n - 2\sqrt{2})^{1/5}, \]
since \( \frac{1}{2^{6/5}} \geq \frac{2^{4/5}}{5} \). Consequently, also in this case, the desired estimate is valid.

\[ \square \]

4. Infinite-Dimensional Results

The present section is dedicated to our infinite-dimensional results. We shall start with the proof of our main result, Theorem B. We then briefly mention the situation for separable \( \mathcal{L}_1 \)-spaces. Finally we conclude the section discussing how to extend our results to higher order polynomials.

**Proof of Theorem B.** As we already observed in the Introduction, (i) follows from [23] and the first part of the theorem, while (ii) is a particular case of (i). We now prove the first clause.

Let \( X \) be a Banach space that contains uniformly complemented \( (\ell_2^n)_{n=1}^\infty \); towards a contradiction, assume that there exists a projection \( Q \) from \( \operatorname{Lip}_0(B_X) \) onto \( \mathcal{P}(2X) \). By definition, we may pick a sequence \( (F_n)_{n=1}^\infty \) of \( C \)-complemented subspaces of \( X \) such that \( F_n \) is \( C \)-isomorphic to \( \ell_2^n \) (\( n \in \mathbb{N} \)). Let \( P_n \) be projections from \( X \) onto \( F_n \) with \( \|P_n\| \leq C \) and let \( T_n : \ell_2^n \to F_n \) be isomorphisms with \( \|T_n\| \leq 1 \) and \( \|T_n^{-1}\| \leq C \), for each \( n \in \mathbb{N} \).

We define a projection \( Q_n \) from \( \operatorname{Lip}_0(C^2 \cdot B_{\ell_2^n}) \) onto \( \mathcal{P}(2\ell_2^n) \) by the following formula.

\[
Q_n(f) := (Q(f \circ T_n^{-1} \circ (P_n |_{B_X})) |_{F_n} \circ T_n \quad (f \in \operatorname{Lip}_0(C^2 \cdot B_{\ell_2^n})).
\]

\[
\begin{array}{cccc}
\operatorname{Lip}_0(C^2 \cdot B_{\ell_2^n}) & \xrightarrow{\left( T_n^{-1} \circ (P_n |_{B_X}) \right)^{\dagger}} & \operatorname{Lip}_0(B_X) \\
\downarrow Q_n & & \downarrow Q \\
\mathcal{P}(2\ell_2^n) & \xleftarrow{T_n^\dagger} & \mathcal{P}(2F_n) & \xleftarrow{\downarrow F_n} & \mathcal{P}(2X)
\end{array}
\]

Here, \( T_n^\dagger \) is defined by \( T_n^\dagger g := g \circ T_n \) (and similarly for \( (T_n^{-1} \circ (P_n |_{B_X}))^\dagger \)). Notice that in this case we consider \( \mathcal{P}(2\ell_2^n) \) as a subspace of \( \operatorname{Lip}_0(C^2 \cdot B_{\ell_2^n}) \) via the embedding \( P \mapsto P |_{C^2 \cdot B_{\ell_2^n}} \).

Since \( \|T_n^{-1} \circ P_n\| \leq C^2 \), \( f \circ T_n^{-1} \circ (P_n |_{B_X}) \) is indeed a Lipschitz function on \( B_X \), when \( f \in \operatorname{Lip}_0(C^2 \cdot B_{\ell_2^n}) \). Thus, \( Q(f \circ T_n^{-1} \circ (P_n |_{B_X})) \) is a polynomial
on $X$ whose Lipschitz constant on $B_X$ is at most $C^2 \|Q\| \cdot \text{Lip}(f)$. Hence, its Lipschitz constant on $C^2 \cdot B_X$ is bounded by $C^4 \|Q\| \cdot \text{Lip}(f)$. Consequently, we obtain that $\|Q_n\| \leq C^4 \|Q\|$. Moreover, it is easy to realise that $Q_n$ is a projection from $\text{Lip}_0(C^2 \cdot B_{\ell^2_2})$ onto $\mathcal{P}(\ell^2_{2})$.

By scaling, we also obtain projections from $\text{Lip}_0(B_{\ell^2_2})$ onto $\mathcal{P}(\ell^2_{2})$ with norms at most $C^4 \|Q\|$, for every $n \in \mathbb{N}$. However, for large $n$, this contradicts Theorem A and concludes the proof. \hfill $\Box$

We already mentioned in the Introduction that there is no hope to extend the conclusion of Theorem B to every infinite-dimensional Banach space. The first result in this direction was that $\mathcal{P}(^k\ell_1)$ is isomorphic to $\ell_\infty$, [10]. Its proof goes as follows: one first shows the rather straightforward fact that $\mathcal{L}(^k\ell_1)$ is isometric to $\ell_\infty$. Then, by the Polarisation formula, $\mathcal{L}^*(^k\ell_1)$ is complemented in $\mathcal{L}(^k\ell_1)$ (recall that $\mathcal{L}^*(X)$ is isomorphic to $\mathcal{P}(X)$ for every $X$). The conclusion follows from Lindenstrauss’ result that complemented subspaces of $\ell_\infty$ are isomorphic to it, [45]. Alternatively, one could directly use that $\mathcal{L}^*(X)$ is isomorphic to $\mathcal{L}(X)$ when $X^2$ is isomorphic to $X$, [17].

**Proposition 4.1.** [5] If $X$ is a separable $\mathcal{L}_1$-space, $\mathcal{P}(^kX)$ is isomorphic to $\ell_\infty$, for every $k \geq 1$. Hence, $\mathcal{P}(^kX)$ is complemented in $\text{Lip}_0(B_X)$.

Combining this with the comments below on general polynomials, it also follows that $\mathcal{P}^k(X)$ is isomorphic to $\ell_\infty$. We refer to [46] for the definition of the $\mathcal{L}_1$-spaces.

**Proof.** As before, it suffices to embed $\mathcal{L}(^kX)$ as a complemented subspace of $\ell_\infty$ and appeal to [45]. Since $X$ is a separable $\mathcal{L}_1$-space, there are two uniformly bounded sequences $(\varphi_j)_{j=1}^\infty$ and $(\psi_j)_{j=1}^\infty$ of linear maps $\varphi_j: X \to \ell^1_1$, $\psi_j: \ell^1_1 \to X$ such that $\varphi_j \circ \psi_j = id$ and $(\psi_j(\ell^1_1))_{j=1}^\infty$ is an increasing sequence whose union is dense in $X$ ([46, Proposition II.5.9]). Define a bounded linear map $\Psi: \mathcal{L}(^kX) \to \left(\sum_{j=1}^\infty \mathcal{L}(^k\ell^1_1)\right)_\infty$ by $\Psi(M):= (M \circ \psi_j)_{j=1}^\infty$; notice that $\left(\sum_{j=1}^\infty \mathcal{L}(^k\ell^1_1)\right)_\infty$ is isometric to $\ell_\infty$.

We also define $\Phi: \left(\sum_{j=1}^\infty \mathcal{L}(^k\ell^1_1)\right)_\infty \to \mathcal{L}(^kX)$ by $(M_j)_{j=1}^\infty \mapsto \lim_{U} M_j \circ \varphi_j$ where $U$ is a free ultrafilter on $\mathbb{N}$ and the limit is in the pointwise topology. Plainly, $\Phi$ is a bounded linear map. Moreover, it is easy to see that $\Phi \circ \Psi$ is the identity of $\mathcal{L}(^kX)$ [5, Lemma 2.3]. Hence, $\mathcal{L}(^kX)$ embeds as a complemented subspace of $\ell_\infty$, as desired. \hfill $\Box$

**Remark 4.2.** Let $X$ be a separable $\mathcal{L}_p$-space, $1 < p < \infty$. The same argument as above shows that $\mathcal{L}(^kX)$ embeds as a complemented subspace of $\left(\sum_{j=1}^\infty \mathcal{L}(^k\ell^1_p)\right)_\infty$, which is complemented in $\mathcal{L}(^k\ell_p)$. Then, one shows that $\mathcal{L}(^k\ell_p)$ embeds as a complemented subspace of $\mathcal{L}(^kX)$ [5, Lemma 2.4] and invokes Pelczyński’s decomposition method to conclude that $\mathcal{L}(^kX)$ is isomorphic to $\mathcal{L}(^k\ell_p)$, [5, Theorem 2.1].

Let us also point out the similarity between this argument and some techniques from [14]. This is also true for the statement of some their results, e.g., [14, Theorem 3.3].
In conclusion to our article, we shall mention how to extend our results to $k$-homogeneous polynomials ($k \geq 2$) and general polynomials. For the definition of $k$-homogeneous polynomials we refer to Sect. 2.2. A polynomial of degree at most $k$ on $X$ is a function $P: X \to \mathbb{R}$ of the form $P = \sum_{j=0}^{k} P_j$, where $P_j \in \mathcal{P}^j(X)$. The collection of all polynomials of degree at most $k$ is denoted by $\mathcal{P}^k(X)$ and it is a Banach space under the same norm $\|P\|_\mathcal{P} := \sup_{x \in B_X} |P(x)|$. We denote by $\mathcal{P}_0^k(X)$ the closed subspace comprising polynomials that vanish at 0, namely those where $P_0 = 0$ (of course, such restriction is necessary in order to embed into $\text{Lip}_0(B_X)$).

It is an important fact that $\mathcal{P}^k(X) = \mathcal{P}^0(X) \oplus \mathcal{P}^1(X) \oplus \cdots \oplus \mathcal{P}^k(X)$ via the natural decomposition $P = \sum_{j=0}^{k} P_j \mapsto (P_j)_{j=0}^k$. Moreover, the norm of the projection $P \mapsto P_j$ does not depend on $X$. (This can be shown via Vandermonde matrices as in [32, Fact 1.1.42], or via the Polarisation formula, [32, Lemma 1.1.47].) In particular, the map $P \mapsto P|_{B_X}$ also defines an isomorphic embedding of $\mathcal{P}_0^k(X)$ into $\text{Lip}_0(B_X)$.

Since $\mathcal{P}(2^X)$ is complemented in $\mathcal{P}_0^k(X)$, a formal consequence of Theorem B is that $\mathcal{P}_0^k(X)$ is not complemented in $\text{Lip}_0(B_X)$, whenever $X$ is a Banach space as in Theorem B. For $\mathcal{P}(k^X)$ the argument is more complicated; the idea is to embed $\mathcal{P}(2^X)$ as a complemented subspace of $\mathcal{P}(k^X)$ [10, Proposition 5.3] and mimic the proof of Theorem A. However, for us it will be more convenient to embed $\mathcal{P}(2^X)$ as a complemented subspace of $\mathcal{P}(k^X \oplus \mathbb{R})$. This depends on the canonical isomorphism $H$ between $\mathcal{P}^k(X)$ and $\mathcal{P}(k^X \oplus \mathbb{R})$, given by homogenisation.

More precisely, let $\zeta: X \oplus \mathbb{R} \to \mathbb{R}$ be the functional $\zeta(x,t) := t$. The action of $H$ on $P = \sum_{j=0}^{k} P_j \in \mathcal{P}^k(X)$ is given by $P \mapsto h \cdot P := \sum_{j=0}^{k} \zeta^{k-j} P_j$. Conversely, every $P \in \mathcal{P}(k^X \oplus \mathbb{R})$ has the form $P = \sum_{j=0}^{k} \zeta^{k-j} P_j$, with $P_j \in \mathcal{P}^j(X)$, and $H^{-1}(P) = \sum_{j=0}^{k} P_j$ (roughly speaking, $H^{-1}(P)(x) = P(x,1)$). Moreover, the isomorphism constant between $\mathcal{P}^k(X)$ and $\mathcal{P}(k^X \oplus \mathbb{R})$ depends on $k$, but not on $X$.

**Theorem 4.3.** Let $X$ be a Banach space that contains uniformly complemented $(\ell^2_n)_{n=1}^\infty$. Then, for every $k \geq 2$, $\mathcal{P}(k^X)$ and $\mathcal{P}_0^k(X)$ are not complemented in $\text{Lip}_0(B_X)$.

**Proof.** We only need to show the assertion concerning $\mathcal{P}(k^X)$.

Assume that $X$ is a Banach space that contains uniformly complemented $(\ell^2_n)_{n=1}^\infty$ and, towards a contradiction, assume that $\mathcal{P}(k^X)$ is complemented in $\text{Lip}_0(B_X)$. The same diagram chasing argument as in the proof of Theorem B shows that there are projections from $\text{Lip}_0(B_{\ell^2_n})$ onto $\mathcal{P}(k^\ell^2_n)$ whose norms are bounded uniformly in $n \in \mathbb{N}$. We let $F_n := \ell^2_n \oplus_{\infty} \mathbb{R}$. Since $F_n$ is $\sqrt{2}$-isomorphic to $\ell^2_n$, it also follows that $\mathcal{P}(k^F_n)$ is complemented in $\text{Lip}_0(B_{F_n})$, uniformly in $n$.

Next, we exploit the isomorphism $H$. $\mathcal{P}(2^\ell^2_n)$ is complemented in $\mathcal{P}(k^\ell^2_n)$, uniformly in $n$, and the latter space is isomorphic to $\mathcal{P}(k^F_n)$, uniformly in $n$ as well. Therefore, the image of $\mathcal{P}(2^\ell^2_n)$ under $H$ is complemented in $\text{Lip}_0(B_{F_n})$, uniformly in $n$. However, we shall show that this is not the case.
To simplify the notation, we let $E := \ell^2_2$ and $F := F_n = E \oplus \mathbb{R}$. Moreover, let
\[ \mathcal{P} := H(\mathcal{P}(^2E)) = \{ P \in \mathcal{P}(^kF) : P = \zeta^{k-2} \cdot P_2, \text{ for some } P_2 \in \mathcal{P}(^2E) \}. \]
Recall that $\zeta$ was defined on $F$ by $\zeta(x, t) := t \cdot (x \in E, t \in \mathbb{R})$. Heuristically, $\mathcal{P}$ consists of polynomials of the form $t^{k-2}P_2(x)$, for some 2-homogeneous polynomial $P_2$ on $E$.

Our goal will be to show that if $Q$ is any projection from $\text{Lip}_0(B_F)$ onto $\mathcal{P}$, then the norm of $Q$ diverges with $n$, which leads to the desired contradiction. As in Theorem A, we actually consider the restriction of $Q$ to $\mathcal{C}_0(B_F)$ and, by averaging, we assume that $Q$ is invariant under $O_n$, the group of rotations of the Euclidean space $E$. We then follow the proof of Theorem A, but multiplying the Lipschitz functions there by $\zeta^{k-2}$.

Consider the functions $\varphi_{ij} := \psi_{ij} \cdot \zeta^{k-2}$, $\Phi := \Psi \cdot \zeta^{k-2}$, and $\Phi_d := \Psi_d \cdot \zeta^{k-2}$. The presence of the factor $\zeta^{k-2}$ now leads to an extra $k/2$ factor in Fact 3.2, namely we have $Lip(\Phi), Lip(\Phi_d) \leq k/2\varepsilon^4$ (indeed, each $p_{ij}$ is $k$-Lipschitz). The invariance argument in Step 3 proceeds identically to give that $Q(\Phi) = (Q(\Psi)) \cdot \zeta^{k-2}$ and analogously for $\Phi_d$. Hence, Claim 3.3 remains true with the same estimate. In the estimate of $\alpha$, the unique difference is an extra $(k - 1)$ factor; in other words, (3.7) becomes
\[ \alpha \geq \left( \frac{\pi}{72} - \delta \right) \cdot (1 - 2\varepsilon K_n) \cdot (k - 1). \] (4.1)

Finally, when putting estimates together in Step 5, we obtain an inequality stronger than (3.12), since $k/2 \leq k - 1$. Consequently, we actually even obtain the same lower bound on $\|Q\|$ as the one in Theorem A. We omit further details.

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Petr Hájek and Tommaso Russo
Department of Mathematics, Faculty of Electrical Engineering
Czech Technical University in Prague
Technická 2
16627 Prague 6
Czech Republic
e-mail: russo@math.cas.cz ; russotom@fel.cvut.cz

Petr Hájek
e-mail: hajek@math.cas.cz
Tommaso Russo
Institute of Mathematics
Czech Academy of Sciences
Žitná 25
11567 Prague 1
Czech Republic

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