Tensor-induced B modes with no temperature fluctuations

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The recent indications for a tensor-to-scalar ratio \( r \approx 0.2 \) from BICEP2 measurements of the cosmic microwave background (CMB) B-mode polarization present some tension with upper limits \( r \lesssim 0.1 \) from measurements of CMB temperature fluctuations. Here we point out that tensor perturbations can induce B modes in the CMB polarization without inducing \textit{any} temperature fluctuations nor E-mode polarization whatsoever, but only, at the expense of violating the Copernican principle. We present this mathematical possibility as a new ingredient for the model-builder’s toolkit in case the tension between B modes and temperature fluctuations cannot be resolved with more conventional ideas.

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The BICEP2 experiment has received considerable attention for their claimed detection \( \hat{1} \) of a large-angle curl component (B modes) \( \hat{2, 3} \) in the cosmic microwave background (CMB) polarization. The interpretation is that these B modes are due to gravitational waves produced during the time of inflation \( \hat{4, 5} \). However, the tensor-to-scalar ratio \( r \approx 0.2 \) indicated by BICEP2 CMB polarization is in \( \sim 2\sigma \) conflict with upper bounds \( r \lesssim 0.1 \) from measurements of CMB temperature fluctuations \( \hat{6, 7} \). The CMB upper limits can be relieved if the scalar spectral index is allowed to run \( \hat{8} \), but the running required is unusually high for single-field slow-roll models. It may be worthwhile to explore explanations for this disagreement between the two values of \( r \) inferred in case the tension continues after the dust of the initial detection has settled.

Here we show that it is possible for a tensor fluctuation to produce a B mode in the CMB polarization without producing \textit{any} temperature fluctuation nor E-mode polarization whatsoever. The argument is relatively straightforward given the total-angular-momentum (TAM) formalism we have developed in recent work \( \hat{9} \). The only catch is that, as we show, the tensor fluctuation required to do this violates the Copernican principle; i.e., it implies that we occupy a preferred location in the Universe.

We begin by reviewing how a polarization map is decomposed into E and B modes. The linear polarization in any direction \( \hat{u} \) on the sky is represented by Stokes parameters \( Q(\hat{u}) \) and \( U(\hat{u}) \). However, these quantities are components of a symmetric trace-free (STF) \( 2 \times 2 \) tensor, or equivalently, spin-2 field, that lives on the surface of the sky, the two-sphere. The polarization can therefore be expanded in terms of tensor spherical harmonics, a complete set of basis functions for STF \( 2 \times 2 \) tensors on the two-sphere. The linear polarization is specified at any point by the two Stokes parameters, and so two sets, E (for the curl-free part of the polarization field) and B (for the curl component) are required to provide a complete basis. The E and B tensor spherical harmonics can be written \( \hat{2} \)

\[ Y_{(\ell m)AB}(\hat{n}) = \sqrt{\frac{\ell!}{(\ell + 2)!}} \left( -\nabla_A \nabla_B + \frac{1}{2} g_{AB} \nabla_C \nabla_C \right) Y_{(\ell m)}(\hat{n}) \]

\[ \equiv - \sqrt{\frac{(\ell - 2)!}{2(\ell + 2)!}} W_{AB}^E Y_{(\ell m)}(\hat{n}), \]

\[ Y_{(\ell m)AB}(\hat{n}) = \sqrt{\frac{(\ell - 2)!}{2(\ell + 2)!}} \left( \epsilon_B^C \nabla_C \nabla_A + \epsilon_A^C \nabla_C \nabla_B \right) Y_{(\ell m)}(\hat{n}) \]

\[ \equiv - \sqrt{\frac{(\ell - 2)!}{2(\ell + 2)!}} W_{AB}^B Y_{(\ell m)}(\hat{n}), \] (1)

where here \( \{A, B\} = \{\theta, \phi\} \), and \( \nabla_A \) is a covariant derivative on the two-sphere, with metric \( g_{AB} = \text{diag}(1, \sin^2 \theta) \) and antisymmetric tensor \( \epsilon_{AB} \). The last equality in each line defines the two orthogonal tensor-valued derivative operators \( W_{AB}^E \) and \( W_{AB}^B \) (written down in another form in Eq. (90) of Ref. \( \hat{9} \)) that when applied to the usual scalar spherical harmonics \( Y_{(\ell m)}(\hat{n}) \) give rise to the E and B tensor spherical harmonics. Thus, the two sets of tensor spherical harmonics can be obtained by applying two orthogonal STF \( 2 \times 2 \) derivative operators to the scalar spherical harmonics. The presence of the antisymmetric tensor in the definition \( Y_{(\ell m)AB}(\hat{n}) \) indicates that the B mode and E mode have opposite parities for fixed \( \ell \) and \( m \). It is important to note that the derivative operators \( W_{AB}^E \) and \( W_{AB}^B \) commute with the total-angular-momentum operator and its third component. The tensor spherical harmonics are thus eigenstates of the total angular momentum with quantum numbers \( \ell \) and \( m \).

We will now develop an analogous decomposition of \textit{three}-dimensional transverse-traceless tensor fields, following the treatment of Ref. \( \hat{9} \). Before doing so, however, we review the standard treatment, in which the
transverse-traceless tensor field is decomposed,

\[ h_{ab}(x) = \sum_{k,s} h_s(k) \tilde{e}^{s*}_{ab}(k) e^{ik \cdot x}, \]  

(2)

into Fourier components \( h_s(k) \) of wavevector \( k \) and polarization \( s \) which can be + or \( \times \), and \( \tilde{e}^{s*}_{ab}(k) \) are polarization tensors. Here \( a, b, c, \ldots \) are three-dimensional spatial indices unlike two-dimensional indices \( A, B, C, \ldots \). This decomposition into Fourier modes can be done because the plane waves \( \tilde{e}^{s*}_{ab}(k) e^{ik \cdot x} \) constitute a complete orthonormal set of basis functions for a transverse-traceless tensor field. Power spectra \( P_h(k) \) for these tensor perturbations (or gravitational waves) are given by,

\[ \langle h_s(k) h^{s*}_{s'}(k') \rangle = \delta_{ss'}(2\pi)^3 \delta_D(k-k') \frac{P_h(k)}{4}, \]  

(3)

In Ref. \[9\] we showed that transverse-traceless tensor fields can alternatively be expanded in transverse-traceless tensor TAM waves. To see how this works, we begin by noting that a scalar function \( \phi(x) \) in three dimensions can be expanded,

\[ \phi(x) = \int \frac{d^3k}{(2\pi)^3} \tilde{\phi}(k) \Psi^k(x), \]  

(4)

where

\[ \tilde{\phi}(k) = \int d^3x \phi(x) \left[ \Psi^k(x) \right]^*. \]

(5)

in terms of scalar TAM waves

\[ \Psi^k_{(\ell m)}(x) \equiv j_\ell(kr) Y_{(\ell m)}(\hat{n}), \]  

(6)

where \( j_\ell(x) \) is the spherical Bessel function of the first kind. These TAM waves are eigenfunctions of the three-dimensional Laplacian \( \nabla^2 \) with eigenvalue \(-k^2\) and also eigenfunctions of the total angular momentum and its third component with quantum numbers \( \ell \) and \( m \) respectively. These scalar TAM waves constitute a complete orthonormal basis for scalar functions. The orthonormality relation for the basis functions is

\[ \int d^3x \left[ \Psi^k_{(\ell m)}(x) \right]^* \Psi^{k'}_{(\ell' m')} (x) = \frac{\delta_{\ell\ell'} \delta_{mm'}}{(2\pi)^3} \frac{(2\pi)^3}{k^2} \delta_D(k-k'), \]

(7)

where \( \delta_{ij} \) is the Kronecker delta. Completeness is demonstrated by

\[ \sum_{\ell m} \int \frac{k^2 dk}{(2\pi)^3} \left[ 4\pi i^\ell \Psi^k_{(\ell m)}(x) \right]^* \left[ 4\pi i^\ell \Psi^k_{(\ell m)}(x') \right] = \delta_D(x-x'). \]

(8)

Just as the E/B tensor spherical harmonics were obtained by applying appropriately defined tensor-valued derivative operators, transverse-traceless-tensor TAM waves can be obtained by applying transverse-traceless-tensor–valued derivative operators to the scalar TAM waves. The appropriate derivative operators are

\[ T^{B}_{ab}(k) = K_{c(a} M_{b)c} + M_{c(a} K_{b)c} + 2D_{c(a} K_{b)c}, \]

\[ T^{E}_{ab}(k) = M_{c(a} M_{b)c} - K_{c(a} K_{b)c} + 2D_{c(a} M_{b)c}, \]

(9)

where we define operators,

\[ D_a \equiv \frac{i}{k} \nabla_a, \quad K_a \equiv -iL_a, \quad M_a \equiv \epsilon_{abc} D^b K^c, \]

(10)

for the subspace of \( \nabla^2 = -k^2 \), and \( L_a = -i \epsilon_{abc} x^b \nabla^c \) is the orbital-angular-momentum operator. The two tensor operators \( T_{ab} \) and \( T^{E}_{ab} \) are orthogonal. They are also both transverse and traceless, and they both commute with the Laplacian and with the total angular momentum and its third component. Thus, the transverse-traceless-tensor TAM waves,

\[ \Psi^{B}_{(\ell m)ab}(x) = -\sqrt{\frac{(\ell-2)!}{2(\ell+2)!}} T^{B}_{ab} \Psi^{k}_{(\ell m)}(x), \]

\[ \Psi^{E}_{(\ell m)ab}(x) = -\sqrt{\frac{(\ell-2)!}{2(\ell+2)!}} T^{E}_{ab} \Psi^{k}_{(\ell m)}(x), \]

(11)

constitute a complete basis for transverse-traceless tensor fields in three dimensions. Note that \( T^{E}_{ab} \) is even under a parity inversion, while \( T^{B}_{ab} \) is odd. The E and B tensor TAM waves thus have opposite parity for the same \( \ell \) and \( m \).

Thus, the most general three-dimensional transverse-traceless tensor field can be written,

\[ h_{ab}(x) = \sum_{\ell m} \int \frac{k^2 dk}{(2\pi)^3} 4\pi i^\ell \left[ h^{E}_{\ell m}(k) \Psi^{E}_{(\ell m)ab}(x) \right. \]

\[ \left. + h^{B}_{\ell m}(k) \Psi^{B}_{(\ell m)ab}(x) \right]. \]

(12)

An alternative set of basis TAM waves can be derived in which the TAM waves have fixed helicity. These helicity \( s = \pm 2 \) TAM waves will be useful below and are related to the E/B TAM waves through \( \Psi^{s = \pm 2,k}_{(\ell m)ab}(x) = [\Psi^{E,k}_{(\ell m)ab}(x) \pm i \Psi^{B,k}_{(\ell m)ab}(x)]/\sqrt{2} \). They allow the most general transverse-traceless tensor field to be expanded,

\[ h_{ab}(x) = \sum_{\ell m s = \pm 2} \int \frac{k^2 dk}{(2\pi)^3} 4\pi i^\ell h^{s}_{\ell m}(k) \Psi^{s,k}_{(\ell m)ab}(x). \]

(13)

In Ref. \[8\] it was proved for scalar fluctuations that scalar TAM waves with quantum numbers \( \ell m \) contribute only to CMB (and any other observable) power spectra of the same \( \ell \). This is a consequence of the fact that the TAM waves and the scalar/vector/tensor spherical harmonics transform as representations of the rotation group of order \( \ell \). The same will be true for tensor TAM waves as well: i.e., tensor TAM waves of order \( \ell \) will contribute only to observable power spectra of multipole moment \( \ell \). More importantly for our purposes, though,
a similar statement applies to the E/B decomposition. The argument is based on the parity transformation, i.e., the dependence of the basis functions under a flip of the displacement vector \( \mathbf{x} \rightarrow -\mathbf{x} \) with respect to the location of the observer. The E-mode tensor TAM wave, scalar spherical harmonic, and E-mode tensor spherical harmonic all have parity \((-1)^{\ell}\), while the B-mode tensor TAM wave and B-mode tensor spherical harmonic transform as \((-1)^{\ell+1}\). Since the physics (Thomson scattering and radiative transfer) involved in the generation of CMB fluctuations is parity conserving, the B-mode component of the tensor field gives rise to B-mode polarization but contributes nothing to the temperature fluctuation nor the E-mode polarization. The converse is true for the E-mode component of the tensor field. As an obvious corollary, if B-mode TAM-wave components have larger power than the E-mode components do, the observed B-mode polarization power spectrum may be larger than would be inferred from the temperature and E-mode polarization power spectra.

As we now demonstrate, an excess of B-mode power over E-mode power in the TAM-wave expansion requires a violation of the Copernican principle. This is because the decomposition of the tensor field into E and B modes is not translation invariant. This can be seen from the following physics argument: CMB polarization is generated by Thomson scattering of temperature fluctuations. B-mode polarization therefore requires that there be a temperature fluctuation at the surface of last scatter, even though, as argued above, a tensor B-mode TAM wave produces no temperature fluctuation for an observer at the origin. Thus, the decomposition into E- and B-mode tensor TAM waves is not translationally invariant.

More explicitly, one can show that a B-mode TAM wave defined with respect to a given origin has nonzero functional overlap with an E-mode TAM wave defined with respect to a different origin. To avoid technical complications, here we demonstrate for spin-one, transverse vector field; the statement is also true for a transverse-traceless tensor field. The vector analogue of Eq. (11) is simply

\[
\Psi^B_{(\ell m)a}(\mathbf{x}) = [\ell(\ell + 1)]^{-1/2} K_a(\mathbf{x}) \Psi^B_{(\ell m)}(\mathbf{x}),
\]

\[
\Psi^E_{(\ell m)a}(\mathbf{x}) = [\ell(\ell + 1)]^{-1/2} M_a(\mathbf{x}) \Psi^E_{(\ell m)}(\mathbf{x}),
\]

where the vector operators \( K_a \) and \( M_a \) explicitly depend on the choice of coordinate origin. We can compute the overlap between E- and B-mode TAM waves defined with respect to two different coordinate origins separated by a constant vector \( \mathbf{d} \). Taking identical \( k, \ell \) and \( m \), for example,

\[
\int d^3x \left[ \Psi^B_{(\ell m)a}(\mathbf{x} + \mathbf{d}) \right]^* \Psi^E_{(\ell m)a}(\mathbf{x})
\]

\[
\propto \int d^3x \left[ K_a(\mathbf{x} + \mathbf{d}) \Psi^B_{(\ell m)}(\mathbf{x} + \mathbf{d}) \right]^* M_a(\mathbf{x}) \Psi^E_{(\ell m)}(\mathbf{x})
\]

\[
= - \int d^3x \left[ \Psi^E_{(\ell m)}(\mathbf{x} + \mathbf{d}) \right]^* \left[ K_a(\mathbf{x} + \mathbf{d}) M_a(\mathbf{x}) \Psi^B_{(\ell m)}(\mathbf{x}) \right].
\]

(15)

Now the two operators act successively on the scalar TAM wave because of the anti-Hermitian condition \([K_a(\mathbf{x})]^{\dagger} = -K_a(\mathbf{x})\). Note that \( K_a(\mathbf{x} + \mathbf{d}) M_a(\mathbf{x}) \) does not vanish if \( \mathbf{d} \neq 0 \), while \( K_a(\mathbf{x}) M_a(\mathbf{x}) = 0 \) is true. Simple algebra gives,

\[
K_a(\mathbf{x} + \mathbf{d}) M_a(\mathbf{x}) = (K_a(\mathbf{x}) - \epsilon_{abc} \mathbf{d}^b \nabla^c) M_a(\mathbf{x}).
\]

\[
= -\epsilon_{abc} d^b \nabla^c M_a(\mathbf{x}) = \mathbf{i} k d^b K_b(\mathbf{x}).
\]

(16)

If \( \mathbf{d} \) is along the z-direction so that \( d^b L_z \), we have

\[
\int d^3x \left[ \Psi^B_{(\ell m)a}(\mathbf{x} + \mathbf{d}) \right]^* \Psi^E_{(\ell m)a}(\mathbf{x})
\]

\[
\propto - \int d^3x \left[ \Psi^E_{(\ell m)}(\mathbf{x} + \mathbf{d}) \right]^* k dL_z \Psi^B_{(\ell m)}(\mathbf{x})
\]

\[
= -m k d \int d^3x \left[ \Psi^E_{(\ell m)}(\mathbf{x} + \mathbf{d}) \right]^* \Psi^B_{(\ell m)}(\mathbf{x}).
\]

(17)

The overlap between two scalar TAM waves of shifted coordinate origins is in general nonzero. This implies that even if the power of E-mode TAM waves might vanish when the E/B-decomposition is performed with respect to one observer, the decomposition yields nonzero E-mode power with respect to another observer at a different location in the Universe.

We now show in a different way that statistical homogeneity requires the power in E and B TAM transverse-traceless tensor TAM waves to waves to be the same. We first relate the power spectra of TAM-wave coefficients to those of Fourier coefficients. Plane-wave tensor states of polarization \( \pm \) and \( \times \) can be added to construct helicity plane waves with Fourier amplitudes \( h_s(\mathbf{k}) \) labelled by the wave vector \( \mathbf{k} \) and the helicity \( s = \pm 2 \). If statistical homogeneity (or translation invariance) is respected, then \( \langle h_s(\mathbf{k}) h_s^*(\mathbf{k}') \rangle \propto \delta_{ss'} \delta_D(k - k') \) (it is not required to be proportional to \( \delta_{ss'} \) though). On the other hand, in terms of the TAM-wave coefficient,

\[
\left\langle h^\alpha_{\ell m}(\mathbf{k}) h_{\ell' m'}^\beta(\mathbf{k}') \right\rangle = P_{\alpha\beta}(k) \delta_{\ell\ell'} \delta_{mm'} \frac{(2\pi)^3}{k^2} \delta_D(k - k').
\]

(18)

with \( \alpha, \beta = \{E, B\} \). We have assumed diagonalization with respect to angular-momentum quantum numbers to preserve statistical isotropy. Note that parity invariance would require \( P_{EB}(k) = 0 \). Transforming to the helicity
part of r value for...be useful should the...uture fluctuations or E-mode polarization, may prove to...a larger B-mode signal, without inducing any tempera-
tiveness significant to warrant abandoning the Copernican
t. We therefore conclude that the Copernican principle re-
we find the power spectra for Fourier components,

\[ P_{ss'}(k) = \left( \begin{array}{cc} P_{22}(k) & P_{2,-2}(k) \\ P_{-2,2}(k) & P_{-2,-2}(k) \end{array} \right), \]  

with

\[ P_{22}(k) = \frac{1}{2} \left[ P_{EE}(k) + P_{BB}(k) \right] - \Im m P_{EB}(k), \]
\[ P_{-2,-2}(k) = \frac{1}{2} \left[ P_{EE}(k) + P_{BB}(k) \right] + \Im m P_{EB}(k), \]
\[ P_{2,-2}(k) = [P_{-2,2}(k)]^* = \frac{1}{2} \left[ P_{EE}(k) - P_{BB}(k) \right] + i \Im m P_{EB}(k). \]  

Now using the relation,

\[ h_s(k) = \sum_{\ell m} \left[ -s Y_{\ell m}(k) \right]^* h_{\ell m}^s(k), \]

we find the power spectra for Fourier components,

\[ \langle h_s(k) h_{s'}(k') \rangle^* = P_{s's'}(k) \frac{(2\pi)^3}{k^2} \delta_{D}(k-k'), \]
\[ \times \sum_{\ell m} \left[ -s Y_{\ell m}(k) \right]^* s' Y_{\ell m}(k). \]  

Only for \( s = s' \) does the summation over \( \ell, m \)
add up to \( \delta_D(k-k') \), so that the power spectrum
\[ \langle h_s(k) h_{s'}(k') \rangle^* \propto \delta_{D}(k-k') \delta_{D}(k-k')/k^2 = \delta_{D}(k-k'). \]

We therefore conclude that the Copernican principle re-
quires \( P_{2,-2}(k) = 0 \); i.e. \( P_{EE}(k) = P_{BB}(k) \) and the real part of \( P_{EB}(k) \) vanishes, \( \text{whether or not} \) the two-point statistics are parity invariant.

To conclude, the \( \sim 2\sigma \) tension between the BICEP2
value for \( r \) and CMB upper limits is probably not suf-
ciently significant to warrant abandoning the Copernican
principle. Still, the possibility discussed here to produce
a larger B-mode signal, without inducing any tempera-
ture fluctuations or E-mode polarization, may prove to
be useful should the \( \sim 2\sigma \) disagreement continue to be
established with greater significance. And perhaps this
may be useful in an explanation that does not involve
a violation of the Copernican principle. For example,
perhaps a theory in which the amplitudes of the E- and
B-mode tensor TAM waves have a highly non-Gaussian
distribution. There may then be an \( \text{apparent} \) violation
of the Copernican principle in our observable Universe that
renders the B-mode TAM-wave amplitude larger, rela-
tive to the E-mode TAM-wave amplitude, than would
be expected with a Gaussian distribution. We leave the
construction of a model to implement this idea to future
work.

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