On the Tchebychev Vector Field in the Relative Differential Geometry

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Abstract

In this paper we deal with relative normalizations of hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$. Considering a relative normalization $\bar{y}$ of a hypersurface $\Phi$ we decompose the corresponding Tchebychev vector $\bar{T}$ in two components, one parallel to the Tchebychev vector $\bar{T}_{EUK}$ of the Euclidean normalization $\bar{\xi}$ and one parallel to the orthogonal projection $\bar{y}_T$ of $\bar{y}$ in the tangent hyperplane of $\Phi$. We use this decomposition to investigate some properties of $\Phi$, which concern its Gaussian curvature, the support function, the Tchebychev vector field etc.

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1 Introduction

To set the stage for this work the classical notation of relative differential geometry is briefly presented; for this purpose the paper [6] is used as general reference.

In the Euclidean space $\mathbb{R}^{n+1}$ let $\Phi: \vec{x} = \vec{x}(u^i), (u^i) := (u^1, \ldots, u^n) \in U \subset \mathbb{R}^n$ be an injective $C^r$-immersion with Gaussian curvature $K \neq 0 \quad \forall (u^i) \in U$. A $C^s$-mapping $\bar{y}: U \rightarrow \mathbb{R}^{n+1} (r > s \geq 1)$ is called a relative $C^s$-normalization if

$$\bar{y}(P) \notin T_P \Phi, \quad \bar{y}_{ij}(P) \in T_P \Phi \quad (i = 1, \ldots, n) \tag{1}$$

at every point $P \in \Phi$, where $T_P \Phi$ is the tangent vector space of $\Phi$ at $P$.

The covector $\bar{X}$ of the tangent hyperplane is defined by

$$\langle \bar{X}, \bar{x}_{ij} \rangle = 0 \quad (i = 1, \ldots, n) \quad \text{and} \quad \langle \bar{X}, \bar{y} \rangle = 1, \tag{2}$$

1Partial derivatives of a function $f$ are denoted by $f_i := \frac{\partial f}{\partial u^i}, \ f_{ij} := \frac{\partial^2 f}{\partial u^i \partial u^j}$
where \( \langle \cdot , \cdot \rangle \) denotes the standard scalar product in \( \mathbb{R}^{n+1} \). The relative metric \( G \) on \( U \) is introduced by
\[
G_{ij} = \langle \bar{X}, \bar{x}_{/ij} \rangle.
\]
The support function of the relative normalization \( \bar{y} \) is defined by
\[
q := \langle \bar{\xi}, \bar{y} \rangle : U \rightarrow \mathbb{R}, \quad q \in C^s(U),
\]
where \( \bar{\xi} : U \rightarrow \mathbb{R}^{n+1} \) is the Euclidean normalization of \( \Phi \). Because of (1) the support function \( q \) never vanishes on \( U \). Furthermore, from (2) it follows the relation
\[
\bar{X} = q^{-1} \bar{\xi}.
\]
On account of (3) and (4), we obtain
\[
G_{ij} = q^{-1} h_{ij},
\]
where \( h_{ij} \) are the components of the second fundamental form \( \text{II} \) of \( \Phi \). We mention that given a support function \( q \), the relative normalization \( \bar{y} \) is uniquely determined and possesses the following parametrization (see [3, p. 197])
\[
\bar{y} = -h^{ij} q_{/i} \bar{x}_{/j} + q \bar{\xi},
\]
where \( h^{ij} \) are the components of the inverse tensor of \( h_{ij} \).

Let \( Q \) be a definite quadratic form. For a \( C^r(U) \)-function \( f \) we denote by \( \nabla^Q f \) the first Beltrami differential operator, by \( \triangle^Q f \) the second Beltrami differential operator and by \( \nabla^Q_i f \) the covariant derivative of \( f \) with respect to \( Q \).

We consider the components
\[
A_{ijk} := \langle \bar{X}, \nabla^G_k \nabla^G_j \bar{x}_{/i} \rangle
\]
of the symmetric Darboux tensor. Then the Tchebychev vector field \( \bar{T} \), which corresponds to the relative normalization \( \bar{y} \), is defined by
\[
\bar{T} = \frac{1}{n} A_{ijk} G^{jk} G^{im} \bar{x}_{/m}.
\]
We mention that the relation
\[
\frac{\triangle^G \bar{x}}{n} = \bar{T} + \bar{y}.
\]
holds [2]. The relative shape operator \( B \) has the components \( B_i^j : U \rightarrow \mathbb{R} \), defined by
\[
\bar{y}_{/i} = -B_i^j \bar{x}_{/j}.
\]
For the relative mean curvature \( H := \frac{1}{n} B_i^j \) we have according to [3]
\[
H = q H_I + \frac{1}{n} \left[ \triangle^\text{II} (\ln q) + \frac{2-n}{4} \nabla^\text{II} (\ln q) \right],
\]
where \( H_I \) is the Euclidean mean curvature of \( \Phi \).

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2From now on we follow similar notation for the inverse of a given tensor.
2 A decomposition of the Tchebychev vector field

Let $I = g_{ij} du^i du^j$ be the first fundamental form and $III = e_{ij} du^i du^j$ the third fundamental form of $\Phi$. Taking into account the Weingarten equations

$$\bar{x}_i = -h_{ij} g^{jk} \bar{x}_k,$$

and the relations (8), we obtain

$$\nabla^{II}(f, \bar{\xi}) = -\nabla^{I}(f, \bar{x}) = q^{-1} \nabla^{G}(f, \bar{\xi}),$$

(9)

$$\nabla^{II}(f, \bar{x}) = -\nabla^{III}(f, \bar{\xi}) = q^{-1} \nabla^{G}(f, \bar{\xi}).$$

(10)

We firstly compute the vectors $\triangle^{II}\bar{x}$ and $\triangle^{II}\bar{\xi}$. To this end we consider the components $T^{i}_{jk} = I^{i}_{\Gamma^{j}k} - II^{i}_{\Gamma^{j}k}$ of the difference tensor of the Levi-Civita connections with respect to $I$ and $II$. It is known that the relations

$$T^{i}_{jk} = -\frac{1}{2} \tilde{h}^{km} \nabla^{I}_{m} h_{ij},$$

$$\nabla^{II}_{j} \bar{x}_i = T^{i}_{jk} \bar{x}_j + h_{ij} \bar{\xi},$$

$$T^{i}_{im} = -\frac{K_{/m}}{2K},$$

hold on $U$ (see [1] p. 22 and [3] p. 197). Using them and the Mainardi-Codazzi equations $\nabla^{I}_{m} h_{ij} - \nabla^{I}_{j} h_{im} = 0$ we find

$$-2h^{ij} T^{k}_{ij} = h^{ik} \tilde{h}^{km} \nabla^{I}_{m} h_{ij} = h^{km} h^{ij} \nabla^{I}_{j} h_{im} = -2h^{km} T^{i}_{im} = \frac{h^{km}}{K} K_{/m},$$

so that, by a direct computation, we arrive at

$$\triangle^{II}\bar{x} = -\frac{1}{2K} \nabla^{II}(K, \bar{x}) + n \bar{\xi}. \quad (11)$$

Following similar computation and taking account of the relations (for $n = 2$ see [1]),

$$\nabla^{II}_{j} \bar{\xi}_i = -T^{k}_{ij} \bar{\xi}_k - e_{ij} \bar{\xi},$$

$$H_{I} = \frac{e_{ij} h^{ij}}{n},$$

we find

$$\triangle^{II}\bar{\xi} = \frac{1}{2K} \nabla^{II}(K, \bar{\xi}) - n H_{I} \bar{\xi}. \quad (12)$$

Remark 1 Relations (11) and (12) for the case $n = 2$ were proved in [2] with sign convention such that $\triangle = -\frac{\partial^{2}}{\partial x_{2}^{2}} - \frac{\partial^{2}}{\partial y_{2}^{2}}$ for the metric $ds^{2} = dx^{2} + dy^{2}$. 

3
Continuing our considerations, we may compute the second Beltrami differential operator $\Delta^G f$ for a $C^r(U)$-function $f$. It is known ([3, p. 196]) that between the Levi-Civita connections with respect to $G$ and $II$ the relation

$$G_{\gamma k}^i - II_{\gamma k}^i = \delta^k_i q_j + \delta^k_j q_i - h_{ij} h^{km} q_{/m}$$

holds. By using them and (5), we obtain the relation

$$\Delta^G f = q \Delta^{II} f - \frac{n-2}{2} \nabla^{II} (q, f).$$

(13)

We apply now (13) to $\bar{x}$ and make use of (6), (10) and (11). Thus we find

$$\Delta^G \bar{x} = \frac{q}{2K} \nabla^{III} (K, \tilde{\xi}) - \frac{n+2}{2} \nabla^{II} (q, \tilde{\xi}) + n \tilde{y}.$$  

(14)

Similarly, by applying (13) to $\tilde{\xi}$ and by using (9) and (12) we get

$$\Delta^G \tilde{\xi} = \frac{-q}{2K} \nabla^{I} (K, \bar{x}) + \frac{n-2}{2} \nabla^{I} (q, \bar{x}) - n q H_1 \tilde{\xi},$$

(15)

Taking into account (7) and (14) we obtain for the Tchebychev vector field $\bar{T}$ of $\Phi$, which corresponds to the support function $q$,

$$\bar{T} = \frac{q}{2nK} \nabla^{III} (K, \tilde{\xi}) - \frac{n+2}{2n} \nabla^{II} (q, \tilde{\xi}).$$

(16)

In the case of the Euclidean normalization ($q = 1$) it is $\bar{y} = \tilde{\xi}$, whereupon we find for the corresponding Tchebychev vector field

$$\bar{T}_{EUK} = \frac{1}{2nK} \nabla^{III} (K, \tilde{\xi}).$$

(17)

Introducing the tangent vector field

$$\bar{Q} := \frac{1}{2nq} \nabla^{III} (q, \tilde{\xi})$$

(18)

of $\Phi$ and inserting this, as well as (17), in (16), we get

$$\bar{T} = q \bar{T}_{EUK} - q (n + 2) \bar{Q}.$$  

(19)

Similarly, we obtain from (14)

$$\Delta^G \bar{x} = n q \left[ \bar{T}_{EUK} + (n-2) \bar{Q} + \tilde{\xi} \right]$$  

and from (6), (10)

$$\bar{y} = q (2n \bar{Q} + \tilde{\xi}),$$

(20)

(21)

i.e. the vector $\bar{Q}$ is parallel to the orthogonal projection $\bar{y}_r$ of the relative normalization $\bar{y}$ in the the tangent vector space $T_P \Phi$ of $\Phi$ at $P$. 

4
From (19) and (21) we see that the vector field \( \frac{1}{q} [2n \bar{T} + (n + 2) \bar{\xi}] \) is independent of the relative normalization and equals \( 2n \bar{T}_\text{EUK} + (n + 2) \bar{\xi} \).

Finally, taking into account (10) and (16), we can write the Tchebychev vector field \( \bar{T} \) as gradient (see [2, p. 243])

\[
\bar{T} = \nabla^G \left( \ln \varphi(u^i), \bar{x} \right),
\]

where

\[
\varphi(u^i) = |K|^\alpha |q|^{\frac{n+2}{2n}}.
\]

Consequently, \( \bar{T} \) is irrotational with respect to the relative metric \( G \).

3 Relatively normalized surfaces by \( ^{\alpha} \bar{y} \)

We consider now the relative normalizations \( ^{\alpha} \bar{y} : U \rightarrow \mathbb{R}^n \), which are introduced by F. Manhart [3], and, on account of (13), are defined by the support functions

\[
^{\alpha} q := |K|^\alpha, \quad \alpha \in \mathbb{R}.
\]

Denoting by \( ^{\alpha} \bar{Q} \) the corresponding vector field and taking into account (17) and (18), we find

\[
^{\alpha} \bar{Q} = \alpha \bar{T}_\text{EUK}.
\]

Conversely, if the vector fields \( \bar{Q} \) and \( \bar{T}_\text{EUK} \) are such that \( \bar{Q} = \alpha \bar{T}_\text{EUK} \) for \( \alpha \in \mathbb{R} \), it turns out that

\[
q = \lambda |K|^\alpha,
\]

where \( \lambda \in \mathbb{R}^* := \mathbb{R} - \{0\} \) is an arbitrary constant. Consequently, we have the following

**Proposition 2.** The vector fields \( \bar{Q} \) and \( \bar{T}_\text{EUK} \) satisfy the relation \( \bar{Q} = \alpha \bar{T}_\text{EUK} \) for a constant \( \alpha \in \mathbb{R} \), if and only if the support function has the form \( q = \lambda |K|^\alpha \), where \( \lambda \in \mathbb{R}^* \) is an arbitrary constant.

We denote by \( ^{\alpha} G \) the relative metric and by \( ^{\alpha} \bar{T} \) the Tchebychev vector with respect to the relative normalization \( ^{\alpha} \bar{y} \). Then from (19), (20), (21) and (25) we have

\[
^{\alpha} \bar{y} = ^{\alpha} q [2n \bar{T}_\text{EUK} + \bar{\xi}],
\]

\[
^{\alpha} \bar{T} = ^{\alpha} q [1 - \alpha(n + 2)] \bar{T}_\text{EUK},
\]

\[
\triangle^{\alpha} G \bar{x} = n \left( ^{\alpha} q \left\{ [1 + \alpha (n - 2)] \bar{T}_\text{EUK} + \bar{\xi} \right\} \right),
\]

while for the function \( \varphi(u^i) \) in (23) we find

\[
^{\alpha} \varphi(u^i) = |K|^{\frac{\alpha(n+2)-1}{2n}}.
\]

We note that the formulae (27)–(28) remain invariant if the support function \( q \) has the form (26).

In the one-parameter family of relative normalizations \( ^{\alpha} \bar{y} \), which are determined by the support functions \( ^{\alpha} q \), among other relative normalizations
• the Euclidean normalization (when $\alpha = 0$) and
• the equiaffine normalization (when $\alpha = 1/(n + 2)$)

are contained. Furthermore we find

$$^{(0)}\bar{Q} = 0, \quad ^{(0)}\bar{y} = \bar{\xi}, \quad ^{(0)}\bar{T} = \bar{T}_{EUK},$$

$$^{(\frac{1}{n+2})}\bar{Q} = \frac{1}{n+2}\bar{T}_{EUK}, \quad ^{(\frac{1}{n+2})}\bar{y} = n^{\frac{n+2}{n}}\sqrt{|K|} \left[\frac{2n}{n+2}\bar{T}_{EUK} + \bar{\xi}\right], \quad ^{(\frac{1}{n+2})}\bar{T} = 0.$$  

4 Applications

4.1. In this paragraph, using the vector fields $\bar{T}, \bar{\Delta G \bar{x}}$ and $\bar{\Delta G \bar{\xi}}$, we find necessary and sufficient conditions for the Gaussian curvature $K$ of $\Phi$ to be constant or for the support function $q$ to be of the form (26).

A. From (17) and (18) we have

$$\langle \bar{T}_{EUK}, d\bar{\xi} \rangle = \frac{1}{2nK}dK,$$

$$\langle \bar{Q}, d\bar{\xi} \rangle = \frac{1}{2nq}dq,$$

so that, from (20) we obtain

$$2\langle \bar{\Delta G \bar{x}}, d\bar{\xi} \rangle = q d\left(\ln(|K| \cdot |q|^{n-2})\right).$$  \tag{29}$$

Hence, we have

**Proposition 3**  (a) When $n = 2$, it holds $\langle \bar{\Delta G \bar{x}}, d\bar{\xi} \rangle = 0$ at every point $P \in \Phi$ if and only if $K = const$.

(b) When $n \geq 3$, the relation $\langle \bar{\Delta G \bar{x}}, d\bar{\xi} \rangle = 0$ holds at every point $P \in \Phi$ if and only if the support function has the form $q = \lambda |K|^\frac{1}{n-2}$, where $\lambda$ is an arbitrary not vanishing constant.

B. From (10) and (22) it follows

$$\langle \bar{T}, d\bar{\xi} \rangle = -q d\ln \varphi.$$

Obviously, we have $\langle \bar{T}, d\bar{\xi} \rangle = 0$ if and only if the function $\varphi(u^i)$, which is defined in (23), is constant, or if and only if $|K| \cdot |q|^{-(n+2)} = const.$, i.e. the support function has the form

$$q = \lambda |K|^\frac{1}{n-2}$$

where $\lambda \in \mathbb{R}^*$ is an arbitrary constant. So we have
Proposition 4  The following properties are equivalent:
(a) The function $\varphi(u^i) = |K|^{\frac{n-2}{n}} \cdot |q|^{\frac{n+2}{n}}$ is constant.
(b) $\langle \bar{T}, d\xi \rangle = 0$ at every point $P \in \Phi$.
(c) The support function has the form $q = \lambda |K|^{\frac{1}{1-n}}$, where $\lambda \in \mathbb{R}^*$ is an arbitrary constant.

C. From (15) we have

$$2\langle \Delta^G \xi, d\bar{x} \rangle = q d(\ln(|K|^{-1} \cdot |q|^{n-2})).$$

Consequently, we obtain

Proposition 5  (a) When $n = 2$, it holds $\langle \Delta^G \xi, d\bar{x} \rangle = 0$ at every point $P \in \Phi$ if and only if $K = \text{const}$.
(b) When $n \geq 3$, the relation $\langle \Delta^G \xi, d\bar{x} \rangle = 0$ holds at every point $P \in \Phi$ if and only if the support function has the form $q = \lambda |K|^{\frac{1}{1-n}}$, where $\lambda \in \mathbb{R}^*$ is an arbitrary constant.

D. From (29) and (30) it follows

$$\langle \Delta^G \bar{x}, d\xi \rangle + \langle \Delta^G \xi, d\bar{x} \rangle = (n - 2) dq,$$

$$\langle \Delta^G \bar{x}, d\xi \rangle - \langle \Delta^G \xi, d\bar{x} \rangle = \frac{q}{K} dK,$$

which lead to

Proposition 6  (a) Let $n = 2$. For every relative normalization the relation $\langle \Delta^G \bar{x}, d\xi \rangle + \langle \Delta^G \xi, d\bar{x} \rangle = 0$ holds at every point $P \in \Phi$.
(b) For $n \neq 2$ the relation $\langle \Delta^G \bar{x}, d\xi \rangle + \langle \Delta^G \xi, d\bar{x} \rangle = 0$ holds at every point $P \in \Phi$ if and only if the support function $q$ is constant.
(c) The relation $\langle \Delta^G \bar{x}, d\xi \rangle - \langle \Delta^G \xi, d\bar{x} \rangle = 0$ holds at every point $P \in \Phi$ if and only if the Gaussian curvature $K$ is constant.

4.2. Let $\bar{y}_i$, $i = 1, 2$, be two relative normalizations of $\Phi$. We denote by $q_i, G_i$ and $\bar{T}_i$ the corresponding support functions, relative metrics and Tchebychev vector fields, respectively.

A. From (21) it turns out

$$\langle \bar{y}_1, \bar{y}_2 \rangle = \nabla^{III}(q_1, q_2) + q_1 q_2,$$

so that: The relative normalizations $\bar{y}_1$ and $\bar{y}_2$ are orthogonal if and only if the corresponding support functions satisfy the relation

$$\nabla^{III}(\ln |q_1|, \ln |q_2|) = -1.$$

B. On account of (13), we have:

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Proposition 7 The vector fields $\vec{Q}_1$ and $\vec{Q}_2$ satisfy the relation $\vec{Q}_2 = \alpha \vec{Q}_1$ for a constant $\alpha \in \mathbb{R}$, if and only if the corresponding support functions $q_1$ and $q_2$ satisfy the relation $q_2 = \lambda q_1^\alpha$, where $\lambda \in \mathbb{R}^*$ is an arbitrary constant.

For the corresponding relative normalizations and Tchebychev vector fields we find

\[
\vec{y}_2 = \lambda q_1^{\alpha-1} \left[ \vec{y}_1 + (\alpha - 1) \nabla^{III}(q_1, \xi) \right],
\]

\[
\vec{T}_2 = \lambda q_1^{\alpha-1} \left[ \vec{T}_1 - \frac{(\alpha - 1)(n + 2)}{2n} \nabla^{III}(q_1, \xi) \right].
\]

C. We study now the case of two relative normalizations $\vec{y}_i, i = 1, 2$, for which there is a constant $\alpha \in \mathbb{R}$, such that the corresponding Tchebychev vector fields satisfy the relation $\vec{T}_2 = \alpha \vec{T}_1$. Taking into account (19), we see that the last relation is equivalent to

\[
(q_2 - \alpha q_1) \nabla^{III}(\ln |K|, \xi) - (n + 2) \nabla^{III}(q_2 - \alpha q_1, \xi) = 0.
\] (31)

For $q_2 \neq \alpha q_1$ it follows that

\[
|q_2 - \alpha q_1| = \lambda q_{AFF},
\] (32)

where $q_{AFF} := |K|^{\frac{1}{n+2}}$ is the support function of the equiaffine normalization and $\lambda$ is an arbitrary positive constant. For $q_2 = \alpha q_1$ the relation (32) is still valid (for $\lambda = 0$).

We denote by $\vec{y}_{AFF}$ the equiaffine relative normalization. Then, by using (19) and (21), it turns out that (32) holds, if and only if the relative normalizations $\vec{y}_i, i = 1, 2$, satisfy the relation

\[
\vec{y}_2 = \alpha \vec{y}_1 + \mu \vec{y}_{AFF},
\]

for $\mu = \varepsilon \lambda$, where $\varepsilon = \text{sign}(q_2 - \alpha q_1)$. So we have the result:

Proposition 8 The following properties are equivalent:

(a) The Tchebychev vector fields $\vec{T}_1$ and $\vec{T}_2$ satisfy the relation $\vec{T}_2 = \alpha \vec{T}_1$, where $\alpha \in \mathbb{R}$.

(b) The support functions $q_1$ and $q_2$ satisfy the relation $|q_2 - \alpha q_1| = \lambda q_{AFF}$, where $\lambda$ is an arbitrary non-negative constant.

(c) The relative normalizations $\vec{y}_1$ and $\vec{y}_2$ satisfy the relation $\vec{y}_2 = \alpha \vec{y}_1 + \mu \vec{y}_{AFF}$, where $\mu$ is an arbitrary constant.

Remark 9 Given a relative normalization $\vec{y}_1$, a $C^\alpha$-mapping $\vec{y}_2 : U \rightarrow \mathbb{R}^{n+1}$ satisfying the relation $\vec{y}_2 = \alpha \vec{y}_1 + \mu \vec{y}_{AFF}$, where $\alpha, \mu \in \mathbb{R}$, is a relative normalization if and only if $\alpha q_1 + \mu q_{AFF} \neq 0$.

From Proposition 8 we obtain the
Corollary 10  (a) Let $\bar{y}_1$ be a relative normalization of $\Phi$ and $\bar{T}_1$ be the corresponding Tchebychev vector field. Then all relative normalizations of the one-parameter family
\begin{equation*}
\{ \bar{y} / \bar{y} = \bar{y}_1 + \mu \bar{y}_{AFF}, \mu \in \mathbb{R}, \mu \neq -q_1^{-1} \}
\end{equation*}
have $\bar{T}_1$ as common corresponding Tchebychev vector field.
(b) All relative normalizations of the one-parameter family
\begin{equation*}
\{ \bar{y} / \bar{y} = \bar{\xi} + \mu \bar{y}_{AFF}, \mu \in \mathbb{R}, \mu \neq -q_1^{-1} \}
\end{equation*}
have $\bar{T}_{EUK}$ as common corresponding Tchebychev vector field.

As immediate consequences of (7), (8) it follows:

Proposition 11  If the Tchebychev vector fields $\bar{T}_1$ and $\bar{T}_2$ satisfy the relation $\bar{T}_2 = \alpha \bar{T}_1$ for $\alpha \in \mathbb{R}$, then we have
(a) The vector fields $\Delta G_i \bar{x}, i = 1, 2$, satisfy the relation
\begin{equation*}
\Delta G_2 \bar{x} = \alpha \Delta G_1 \bar{x} + \frac{\mu}{n} \bar{y}_{AFF},
\end{equation*}
where $\mu$ is an arbitrary constant.
(b) The corresponding relative mean curvatures $H_i$ of the normalizations $\bar{y}_i$, $i = 1, 2$, satisfy the relation
\begin{equation*}
H_2 = \alpha H_1 + \mu H_{AFF},
\end{equation*}
where $H_{AFF}$ is the equiaffine mean curvature and $\mu$ is an arbitrary constant.

D. Finally, from (15) and (20) we have
\begin{equation*}
\frac{\Delta G_1 \bar{x}}{q_1} - \frac{\Delta G_2 \bar{x}}{q_2} = \frac{n-2}{2} \nabla^{III} \left( \ln \left| \frac{q_1}{q_2} \right|, \bar{\xi} \right),
\end{equation*}
\begin{equation*}
\frac{\bar{T}_1}{q_1} - \frac{\bar{T}_2}{q_2} = -(n+2) \left( \bar{q}_1 - \bar{q}_2 \right) = \frac{n+2}{2} \nabla^{III} \left( \ln \left| \frac{q_1}{q_2} \right|, \bar{\xi} \right),
\end{equation*}
\begin{equation*}
\frac{\Delta G_1 \bar{\xi}}{q_1} - \frac{\Delta G_2 \bar{\xi}}{q_2} = \frac{n-2}{2} \nabla^{I} \left( \ln \left| \frac{q_1}{q_2} \right|, \bar{x} \right),
\end{equation*}
so that we conclude

Proposition 12  (a) When $n = 2$, the vector fields $q^{-1} \Delta G \bar{x}$ and $q^{-1} \Delta G \bar{\xi}$ are independent of the normalization.
(b) When $n \geq 3$ the following properties are equivalent:
(i) $\langle \frac{\Delta G_1 \bar{x}}{q_1} - \frac{\Delta G_2 \bar{x}}{q_2}, d\bar{\xi} \rangle = 0$ at every point $P \in \Phi$.
(ii) $\langle \frac{\bar{T}_1}{q_1} - \frac{\bar{T}_2}{q_2}, d\bar{\xi} \rangle = 0$ at every point $P \in \Phi$.
(iii) $\langle \frac{\Delta G_1 \bar{\xi}}{q_1} - \frac{\Delta G_2 \bar{\xi}}{q_2}, d\bar{x} \rangle = 0$ at every point $P \in \Phi$.
(iv) $q_1 = \lambda q_2$, where $\lambda$ is an arbitrary non-vanishing constant.
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