The Absolute continuity of convolution products of orbital measures in exceptional symmetric spaces

K. E. Hare and Jimmy He

Abstract. Let $G$ be a non-compact group, let $K$ be the compact subgroup fixed by a Cartan involution and assume $G/K$ is an exceptional, symmetric space, one of Cartan type $E$, $F$ or $G$. We find the minimal integer, $L(G)$, such that any convolution product of $L(G)$ continuous, $K$-bi-invariant measures on $G$ is absolutely continuous with respect to Haar measure. Further, any product of $L(G)$ double cosets has non-empty interior. The number $L(G)$ is either 2 or 3, depending on the Cartan type, and in most cases is strictly less than the rank of $G$.

1. Introduction

In [18] Ragozin showed that if $G$ is a connected simple Lie group and $K$ a compact, connected subgroup fixed by a Cartan involution of $G$, then the convolution of any $\dim G/K$, continuous, $K$-bi-invariant measures on $G$ is absolutely continuous with respect to the Haar measure on $G$. This was improved by Graczyk and Sawyer in [8], who showed that when $G$ is non-compact and $n = \text{rank } G/K$, then the convolution product of any $n+1$ such measures is absolutely continuous. They conjectured that $n+1$ was sharp with this property. This conjecture was shown to be false, in general, for the symmetric spaces of classical Cartan type in [11] (where it was shown that rank $G/K$ suffices except when the restricted root system is type $A_n$). Here we prove the conjecture is also false for the symmetric spaces of exceptional Cartan type (Types $E_I-I_X$, $F_I$, $F_{II}$ and $G$), except when the restricted root system is type $BC_1$ or $A_2$. In fact, the sharp answer for the exceptional symmetric spaces (as detailed in Cor. 1) is always either two or three, depending on the type, even though the rank can be as much as 8.

A special example of $K$-bi-invariant continuous measures on $G$ are the orbital measures
\[ \nu_z = m_K * \delta_z * m_K, \]
where $m_K$ is the Haar measure on $K$. These singular probability measures are supported on the double cosets $KzK$ in $G$. One consequence of our result is that any product of three (and often two) double cosets has non-empty interior in $G$. 

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Questions about the convolution of orbital measures are also of interest because of the connection with the product of spherical functions, c.f. [6, 7].

To establish our result, we actually study an equivalent absolute continuity problem for measures on \( g \), the Lie algebra of \( G \). The Cartan involution gives rise to a decomposition of the Lie algebra as \( g = \mathfrak{t} \oplus \mathfrak{p} \) where \( \mathfrak{t} \) is the Lie algebra of \( K \). By the \( K \)-invariant measures on \( p \), we mean those satisfying \( \mu(E) = \mu(\text{Ad}(k)E) \) for all \( k \in K \) and Borel sets \( E \). An example of a \( K \)-invariant, singular probability measure on \( p \) is the orbital measure \( \mu_Z \) defined for \( Z \in p \) by

\[
\int_p f \, d\mu_Z = \int_K f(\text{Ad}(k)Z) \, dm_K(k)
\]

for any continuous, compactly supported function \( f \) on \( p \). This measure is supported on the \( \text{Ad}(K) \)-orbit of \( Z \), denoted

\[
O_Z = \{ \text{Ad}(k)Z : k \in K \}.
\]

It was shown in [1] that \( \mu_{Z_1} \ast \cdots \ast \mu_{Z_k} \) is absolutely continuous with respect to Lebesgue measure on \( p \) if and only if \( \nu_{z_1} \ast \cdots \ast \nu_{z_k} \) is absolutely continuous on \( G \) when \( z_j = \exp Z_j \), and the former is the problem we will actually study.

A compact, connected simple Lie group \( G \) can be viewed as the symmetric space \( (G \times G)/G \). The absolute continuity problem has been studied for orbital measures in this setting, as well, with the sharp answers found for the exceptional Lie groups in [13]. The arguments used there (as well as for the classical Lie groups/algebras in [10, 12]) were based upon harmonic analysis methods not generally available in symmetric spaces. Instead, here we rely heavily upon a combinatorial condition for absolute continuity that was discovered by Wright in [20] for compact Lie algebras, and was extended to symmetric spaces in [11]. Another key idea in our work is an embedding argument based on the Freudenthal magic square construction of particular exceptional symmetric spaces. This embedding idea was inspired, in part, by an embedding argument used in [9] for particular classical symmetric spaces.

In [19] Ricci and Stein proved that if a convolution product of orbital measures is absolutely continuous, then its density function is actually in \( L^{1+\varepsilon} \) for some \( \varepsilon > 0 \). It would be interesting to know the size of \( \varepsilon \), however our arguments do not yield any information about this. For other approaches to the study of sums of adjoint orbits, we refer the reader to [5] and [16], for example.

2. Notation and Statement of main results

2.1. Notation and Terminology. Let \( G \) be a non-compact, simple, connected Lie group with Lie algebra \( g \) and let \( \Theta \) be an involution of \( G \). We assume that \( K = \{ g \in G : \Theta(g) = g \} \) is compact and connected. The quotient space, \( G/K \), is called a symmetric space.

The map \( \Theta \) induces an involution of \( g \), denoted \( \theta \). We put

\[
\mathfrak{t} = \{ X \in g \mid \theta(X) = X \} \quad \text{and} \quad \mathfrak{p} = \{ X \in g \mid \theta(X) = -X \},
\]

the \( \pm 1 \) eigenspaces of \( \theta \), respectively. The decomposition \( g = \mathfrak{t} \oplus \mathfrak{p} \) is called the Cartan decomposition of the Lie algebra \( g \). We fix a maximal abelian (as a subalgebra of \( g \)) subspace \( \mathfrak{a} \) of \( \mathfrak{p} \). It is known that \( \text{Ad}(k) : \mathfrak{p} \rightarrow \mathfrak{p} \) whenever \( k \in K \) where \( \text{Ad}(\cdot) \) denotes the adjoint action of \( G \) on \( g \). If we put \( A = \exp \mathfrak{a} \), then \( G = KA K \), hence every double coset contains an element of \( A \). Similarly, every \( \text{Ad}(K) \)-orbit
contains an element of \( a \), so in studying the orbital measures \( \nu_x \) or \( \mu_X \) there is no loss in assuming \( x \in A \) and \( X \in a \).

By the absolute root system, we mean the root system of \( \mathfrak{g} \). By the restricted roots of the symmetric space we mean the restrictions to \( a \) of the roots that are non-zero on \( a \). This set, called the restricted root system, will be denoted \( \Phi \). The vector spaces

\[
\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} : [H, X] = \alpha (H) X \text{ for all } H \in a \},
\]

for \( \alpha \in \Phi \), are known as the restricted root spaces. We remark that these need not be one dimensional and that \( \theta (\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha} \). The space \( p \) can also be described as

\[
p = sp \{ X - \theta X : X \in \mathfrak{g}_\alpha, \ \alpha \in \Phi^+ \} \oplus a
\]

where by \( sp \) we mean the real span. We put

\[
X^+ = X + \theta X \text{ and } X^- = X - \theta X.
\]

Of course, \( X^+ \in \mathfrak{k} \).

Throughout the paper, we assume \( G/K \) is an irreducible, Riemannian, globally symmetric space of Type III, whose absolute root system is exceptional. (We call these ‘exceptional symmetric spaces’). They have Cartan classifications \( EI - IX \), \( FI \), \( FII \) and \( G \). In the appendix, we summarize basic information about these symmetric spaces, including the absolute and restricted root systems, and the rank and dimension of \( G/K \). This information is taken from \( [3] \) and \( [15] \).

Given \( Z \in a \), we let

\[
\Phi_Z = \{ \alpha \in \Phi : \alpha (Z) = 0 \}
\]

be the set of annihilating (restricted) roots of \( Z \) and let

\[
\mathcal{N}_Z = sp \{ X^- : X \in \mathfrak{g}_\alpha, \ \alpha \notin \Phi_Z \} \subseteq p.
\]

The set \( \Phi_Z \) is itself a root system that is \( \mathbb{R} \)-closed (meaning \( sp \Phi_Z \cap \Phi = \Phi_Z \)) and is a proper root subsystem provided \( Z \neq 0 \). By the type of \( Z \) we mean the Lie type of \( \Phi_Z \).

The annihilating root subsystems, \( \Phi_Z \), and the associated spaces, \( \mathcal{N}_Z \), are of fundamental importance in studying orbits and orbital measures. Indeed, the tangent space to the \( Ad(K) \)-orbit of \( Z \) is \( T_Z (O_Z) = \mathcal{N}_Z \). In particular, \( \dim O_Z = \dim \mathcal{N}_Z \), so \( O_Z \) is a proper submanifold of \( p \) and hence of measure zero. It has positive dimension if \( Z \neq 0 \), in which case we call the orbit \( O_Z \) and the orbital measure \( \mu_Z \) non-trivial. More generally, if \( X \in O_Z \), say \( X = Ad(k)Z \), then \( T_X (O_Z) = Ad(k)T_Z (O_Z) = Ad(k)\mathcal{N}_Z \).

Similar statements hold for the orbital measures on \( G \) and double cosets. In particular, we call an orbital measure \( \nu_z \) non-trivial if \( z = \exp Z \) where \( \Phi_Z \neq \Phi \).

Equivalently, \( z \) is not in the normalizer of \( K \) in \( G \).

2.2. Main Result. Here is the statement of our main result, whose proof will be the content of the remainder of the paper. Note that when we say an orbital measure is absolutely continuous (or singular) we mean either with respect to Haar measure on \( G \) or Lebesgue measure on \( p \), depending on whether the orbital measure is on \( G \) or \( p \).

**Theorem 1.** Let \( G/K \) be an exceptional symmetric space. For each non-zero \( X \in a \) there is an integer \( L = L_X \) such that \( \mu^L_X \) and \( \nu^L_{\exp X} \) is absolutely continuous.
if $L \geq L_X$ and singular otherwise. Moreover, $L_X = 2$ other than in the following cases when $L_X = 3$:

| Cartan type of $G/K$ | $EI$ | $EIV$ | $EV$ | $EVII$ | $EVIII$ |
|----------------------|------|-------|------|--------|---------|
| Type of $X$          | $D_5, A_5$ | $A_1$ | $E_6, D_6$ | $C_2$ | $E_7$ |

Let $L(G) = 3$ if $G/K$ is one of type $EI$, $EIV$, $EV$, $EVII$ and $EVIII$ and let $L(G) = 2$ otherwise. Then any convolution product of $L(G)$ non-trivial orbital measures on $G$ or $p$ is absolutely continuous.

We continue to use the same definition of $L(G)$ in the corollary.

**Corollary 1.**

1. Any product of $L(G)$, non-trivial double cosets and any sum of $L(G)$, non-trivial $Ad(K)$-orbits has non-empty interior in $G$ and $p$ respectively.
2. Any convolution product of $L(G)$, $K$-bi-invariant, continuous measures on $G$ or $K$-invariant, continuous measures on $p$ is absolutely continuous and this is sharp.

**Proof.** The first statement follows from the fact that $\mu_{Z_1} \ast \cdots \ast \mu_{Z_t}$ is absolutely continuous with respect to Lebesgue measure on $p$ if and only if the sum $O_{Z_1} + \cdots + O_{Z_t}$ has non-empty interior if and only if $K \exp Z_1 K \cdots K \exp Z_t K$ has non-empty interior, see [11].

Furthermore, in [18] it was shown that if every product of $L$ non-trivial double cosets has non-empty interior, then any convolution product of $L K$-bi-invariant, continuous measures on $G$ is absolutely continuous. By [1], this is equivalent to any product of $L K$-invariant continuous measures on $p$ being absolutely continuous.

The sharpness of $L(G)$ is immediate from the theorem.

**Remark 1.** We note that the ranks of the symmetric spaces for which $L(G) = 2$ vary from 1-4, while those with $L(G) = 3$ range from 2-8.

In section 3, we will prove the absolute continuity for $L \geq L_X$. In section four we prove singularity for $L < L_X$. It will suffice to work with orbital measures on $p$ because of the equivalence result of Anchouche and Gupta [1].

3. **Proof of Absolute continuity**

3.1. **Criteria for Absolute continuity.** There are two main ideas we use to establish absolute continuity. The first is a combinatorial argument based on the annihilating root systems.

Note that by the **rank of a root system** we mean the dimension of the Euclidean space it spans. An annihilating root system $\Phi_X$, for $X \neq 0$, always has proper rank. By the **dimension of a root system** $\Phi_0$ we mean

$$\dim \Phi_0 = \dim \text{sp}\{E_\alpha : E_\alpha \in g_\alpha, \alpha \in \Phi_0\},$$

that is, the cardinality of $\Phi_0^+$ counted by dimension of the corresponding restricted root spaces, equivalently, the sum of the multiplicities of the restricted positive roots in $\Phi_0$.

**Theorem 2.** [11], [20] Assume $G/K$ is a symmetric space with restricted root system $\Phi$ and Weyl group $W$. Suppose $X_1, \ldots, X_m \in a$ and assume

$$\dim \Phi - \dim \Psi \geq \sum_{i=1}^{m} \left( \dim \Phi_{X_i} - \min_{\sigma \in W} \dim(\Phi_{X_i} \cap \sigma(\Psi)) \right)$$

\[\text{(3.1)}\]
for all \(\mathbb{R}\)-closed, root subsystems \(\Psi \subseteq \Phi\) of co-rank 1. Then \(\mu_{X_1} \ast \cdots \ast \mu_{X_m}\) is absolutely continuous.

An easy calculation shows the following.

**Corollary 2.** Suppose \(X_1, \ldots, X_m \in \mathfrak{a}\) and assume that for each \(X = X_i\),

\[
(m - 1) (\dim \Phi - \dim \Psi) - 1 \geq m \left( \dim \Phi_X - \min_{\sigma \in \mathcal{W}} (\dim (\Phi_X \cap \sigma(\Psi))) \right)
\]

for all \(\mathbb{R}\)-closed, root subsystems \(\Psi \subseteq \Phi\) of co-rank 1. Then \(\mu_{X_1} \ast \cdots \ast \mu_{X_m}\) is absolutely continuous.

**Remark 2.** We note that if (3.2) holds for some \(m\), then it holds for all greater integers. We also note that if the inequality holds for \(X\), then it also holds for any \(Y \in \mathfrak{a}\) with \(\Phi_Y \subseteq \Phi_X\) since \(\dim \Phi_Y - \dim (\Phi_Y \cap \Psi) \leq \dim \Phi_X - \dim (\Phi_X \cap \Psi)\).

The second result is a geometric characterization of absolute continuity. It will be used both to prove absolute continuity (in this section) and singularity in section 4.

**Theorem 3.** Suppose \(Z_1, \ldots, Z_m \in \mathfrak{a}\). Then \(\mu_{Z_1} \ast \cdots \ast \mu_{Z_m}\) is absolutely continuous if and only if there is some \(k_1 = 1d, k_2, \ldots, k_m \in K\) (equivalently, for a.e. \((k_1, \ldots, k_m) \in K^m\)) such that

\[
\dim \Phi \cap \sigma(\Psi) = \dim \Phi_X \cap \sigma(\Psi) = m \dim (\Phi_X \cap \Psi).
\]

The nature of the proof of the sufficiency part of the theorem will differ depending on the type of the exceptional symmetric spaces. There are three cases to consider; these comprise subsections 3.2, 3.3, 3.4.

### 3.2. The symmetric spaces whose restricted root spaces all have dimension one [Types \(EI, EV, EVIII, FI\) and \(G\)]

We will first prove the absolute continuity of \(\mu^L_{\Phi_X}(\mathcal{X})\) for the exceptional symmetric spaces in which all the restricted root spaces have dimension one. These are the symmetric spaces in which the absolute and restricted root systems coincide, the symmetric spaces \(EI, EV, EVIII, FI, G\). Their (absolute and restricted) root systems are types \(E_6, E_7, E_8, F_4, G_2\), respectively.

**Proposition 1.** Let \(G/K\) be any of the symmetric spaces \(EI, EV, EVIII, FI\), and \(G\). Then inequality (3.2) holds with \(m = L_X\), as defined in Theorem 3, for any non-zero \(X \in \mathfrak{a}\) and all \(\Phi\) of co-rank 1.

**Proof.** We note, first, that (3.2) is equivalent to the statement

\[
(m - 1) \dim \mathcal{N}_\Phi - 1 \geq m (\dim (\Phi_X \cap \mathcal{N}_\Phi))
\]

for all \(\mathbb{R}\)-closed, root subsystems \(\Psi \subseteq \Phi\) of co-rank 1.

Assume \(\Phi\) has rank \(n\) and base \(\{\alpha_1, \ldots, \alpha_n\}\). Up to Weyl conjugacy, a base for \(\Psi\) consists of all but one \(\alpha_j\). Hence if \(\{\lambda_1, \ldots, \lambda_n\}\) are the fundamental dominant weights, then there exists \(j\) such that \(\Phi(\sigma(\Psi)) = \{\alpha \in \Phi : (\alpha, \lambda_j) \neq 0\}\) for a suitable Weyl conjugate. Replacing \(\Phi_X\) by \(\Phi_{\sigma(\Psi)}\), there is no loss in assuming \(\Phi \setminus \Psi = \{\alpha \in \Phi : (\alpha, \lambda_j) \neq 0\}\), thus our task is to prove

\[
(m - 1) \dim \mathcal{N}_{\Phi_j} - 1 \geq m (\dim (\Phi_X \cap \mathcal{N}_{\Psi_j}))
\]

whenever \(\mathcal{N}_{\Phi_j} = \text{sp}(E_{\alpha}^- : (\alpha, \lambda_j) \neq 0\} and \(X'\) is Weyl conjugate to \(X\).
This follows from the calculations done in [13] and [14], as we will now explain. Put
\[ X_1 = \{ \alpha \in \Phi^+ : (\alpha, \lambda_j) \neq 0 \} \]
and
\[ B_1 = B_1(\Phi_X) = \{ \alpha \in \Phi_X^+ : (\alpha, \lambda_j) \neq 0 \}. \]
Thus \( \dim N_{\Phi_j} = |X_1| \) and \( \dim \Phi_X \cap N_{\Phi_j} = |B_1| \), so (3.3) is equivalent to proving
\[ (m - 1)|X_1| - m|B_1| \geq 1 \]
for all \( j \) and all \( X' \) Weyl conjugate to \( X \).

It is shown in [14] Section 2.2] that \( m = 2 \) satisfies (3.3), for any \( j \), when the root system is type \( G_2 \) or \( F_4 \), and similarly \( m = 3 \) works when the root system is type \( E_6 \), \( E_7 \) or \( E_8 \).

More refined calculations were done in [13] to show that the inequality (3.5) actually holds with \( m = 2 \), except when \( \Phi_X \) is of type \( D_5 \) or \( A_5 \) in the root system \( E_6 \), type \( E_6 \) or \( D_6 \) in the root system \( E_7 \), or type \( E_7 \) in the root system \( E_8 \). \( \square \)

To summarize:

**Corollary 3.** If \( G/K \) is one of the symmetric spaces \( EI, EV, EVIII, FI, \) and \( G \), then \( \mu_X^{L_X} \) is absolutely continuous for any non-zero \( X \in \mathfrak{a} \). Furthermore, any convolution of two non-trivial orbital measures in the symmetric spaces of type \( FI \) or \( G \), or convolution of three orbital measures in types \( EI, EV \) and \( EVIII \) is absolutely continuous.

### 3.3. An Embedding argument [Types \( EII, EVI \) and \( EIX \)]

Motivated by ideas in [9], it was shown in [11] that if \( G_1/K_1 \) suitably embeds into \( G_2/K_2 \), then absolute continuity information for \( G_2/K_2 \) can be obtained from that for \( G_1/K_1 \). In this section we will show that under mild technical conditions it is possible to turn a suitable embedding of \( (g_1, \mathfrak{t}_1) \) into \( (g_2, \mathfrak{t}_2) \) into an embedding of the Lie groups’ universal covers. This will be useful for us because (as we will see) absolute continuity questions for orbital measures are unchanged if \( G \) is replaced by its universal cover.

We will then apply this embedding argument to deduce the desired absolute continuity properties for the symmetric space of Cartan types \( EII, EVI \) and \( EIX \) from those of the symmetric space \( FI \) where we have already seen in the previous subsection that \( L_X = 2 \) holds.

We first recall what ‘suitably embeds’ means at the group level and the embedding theorem that was proven in [11].

**Definition 1.** Let \( G_1/K_1 \) and \( G_2/K_2 \) be two symmetric spaces. We say that \( G_1/K_1 \) is **embedded** into \( G_2/K_2 \) if there is a mapping \( \mathcal{I} : G_1 \to G_2 \) satisfying the following properties.

1. \( \mathcal{I} \) is a group isomorphism into \( G_2 \).
2. \( \mathcal{I} \) restricted to \( A_1 \) is a topological group isomorphism onto \( A_2 \).
3. \( \mathcal{I} \) maps \( K_1 \) into \( K_2 \).

**Theorem 4.** [11] Suppose \( G_1/K_1 \) is embedded into \( G_2/K_2 \) with the mapping \( \mathcal{I} : G_1 \to G_2 \). Let \( X_1, \ldots, X_m \in \mathfrak{a}_1 \). If \( \mu_{X_1} \ast \cdots \ast \mu_{X_m} \) is absolutely continuous on \( \mathfrak{p}_1 \) and \( \exp(Z_j) = \mathcal{I}(\exp X_j) \), then \( \mu_{Z_1} \ast \cdots \ast \mu_{Z_m} \) is absolutely continuous on \( \mathfrak{p}_2 \).

We now turn to embeddings of Lie algebras.
For each Lie group \( G \) and therefore is contained in such that \( G \) is simply connected. As \( \Theta \) induces an involution on any quotient \( k \to \) topological group isomorphism. □

Because \( \phi \) is a bijective local diffeomorphism, which means it is a global diffeomorphism, and so a map \( \tilde{A} \) follows from the fact that \( G \) has finite center. As \( \phi \) is an open map and thus \( \pi \) descends to a unique involution \( \Theta \)

We claim \( \tilde{A} \) embeds \( G_1/N \) into \( G_2 \) in the sense of Definition 1.

(1) \( \tilde{I} \) is a group homomorphism into \( G_2 \) and clearly it is injective.

(2) Since \( K_i \) is generated by \( exp(\mathfrak{t}_i) \) and \( \iota \) maps \( \mathfrak{t}_1 \) into \( \mathfrak{t}_2 \), \( \tilde{I} \) maps \( K_1 \) into \( K_2 \) and \( \tilde{I} \) maps \( \pi(K_1) \) into \( K_2 \).

(3) Similarly, \( \tilde{I} \) maps \( \pi(A_1) \cong A_1 \) into \( A_2 \). It is surjective since for any \( exp(Y) \in A_2 \), \( \tilde{I}(exp(\iota^{-1}(Y))) = exp(Y) \) and hence is a bijection from \( \pi(A_1) \cong A_1 \) to \( A_2 \). Because \( \varphi : p_1 \times K_1 \to G_1 \) given by \( \varphi(X, k) = (exp X)k \) is a diffeomorphism (VI 1.1)), it is an open map and thus \( A_1 = \varphi(a_1, e) \) is closed being the image of a closed set. As \( \pi \) is an open map, \( \pi(A_1) \) is also closed.

Thus \( \pi(A_1) \) and \( A_2 \) are Lie groups with the subspace topology and \( \tilde{I} \) is a bijective Lie group homomorphism between the two. Hence \( \tilde{I} : \pi(A_1) \to A_2 \) is a bijective local diffeomorphism, which means it is a global diffeomorphism, and so a topological group isomorphism. □
To apply this result, we will make use of the Freudenthal magic square construction for models of the exceptional Lie algebras/groups. The construction we will use is due to Vinberg [17]. We give a brief overview here, as explained in Barton and Sudbery [2].

The method we use is called the Cayley-Dickson construction. We begin with $F = \mathbb{R}, \mathbb{C}$ or the quaternions $\mathbb{H}$, and define a multiplication on $F \times F$ by $(x_1, y_1)(x_2, y_2) = (x_1x_2 + \varepsilon y_1y_2, y_1x_2 + y_2x_1)$, with conjugation given by $(x, y) = (x, -y)$ and quadratic form $N(x, y) = (x, y)(x, y)$. Here $\varepsilon = \pm 1$, with $\varepsilon = -1$ giving a division algebra and $\varepsilon = 1$ giving a split algebra. We will write $F \oplus lF$ to denote the composition algebra that arises from the Cayley-Dickson construction starting with $F$ and the new imaginary unit $l$ where $l^2 = \varepsilon$. Of course, when $\varepsilon = -1$ and $F = \mathbb{R}$, we have $l = i$ and $F \oplus lF = \mathbb{C}$. We can similarly obtain $\mathbb{H}$ and the octonions, $\mathbb{O}$, in this fashion.

Given unital composition algebras $C_j$ over $\mathbb{R}$, we let $A_3(C_1 \otimes C_2)$ denote the $3 \times 3$ matrices with entries in $C_1 \otimes C_2$ that are skew-Hermitian and let $A'_3(C_1 \otimes C_2)$ denote the subset of trace zero matrices. We denote by $\text{Der}(C_j)$ the derivations on $C_j$.

Put
\begin{equation}
V_3(C_1, C_2) = A'_3(C_1 \otimes C_2) \oplus \text{Der}(C_1) \oplus \text{Der}(C_2).
\end{equation}

With the definitions given below for the Lie bracket, $V_3(C_1, C_2)$ is a Lie algebra.

1. $\text{Der}(C_1) \oplus \text{Der}(C_2)$ is a Lie subalgebra with $[\text{Der}(C_1), \text{Der}(C_2)] = 0$ and the Lie bracket within each subalgebra is the standard Lie bracket, $[X, Y] = XY - YX$.

2. For $D \in \text{Der}(C_1) \oplus \text{Der}(C_2)$ and $A \in A'_3(C_1 \otimes C_2)$ we put $[D, A] = D(A)$, where $D(A)$ is defined to be the matrix whose entries are obtained by applying $D$ to the corresponding entry in $A$.

3. For $A, B \in A'_3(C_1 \otimes C_2)$ we have
\[ [A, B] = AB - BA - \frac{\text{tr}(AB - BA)}{3} I + \frac{1}{3} \sum_{i,j} D_{A_{ij}, B_{ij}} \]

where
\[ D_{u_1 \otimes v_1, u_2 \otimes v_2} = \langle v_1, u_2 \rangle D_{u_1, u_2} + \langle u_1, u_2 \rangle D_{v_1, v_2} \]

and
\[ D_{x, y} = [L_x, L_y] + [L_x, R_y] + [R_x, R_y]. \]

Here $L$ and $R$ denote left and right multiplication and the brackets are the standard brackets for operators on a vector space.

This Lie algebra is relevant for us because with appropriate choices of $C_1$ and $C_2$ it gives rise to various exceptional symmetric spaces. In particular, if $C_1 = \mathbb{H} \oplus l\mathbb{H}$ is the split octonions and $C_2$ is a division algebra, then we obtain the exceptional symmetric spaces as indicated below:

| Cartan type of $V_3(C_1, C_2)$ | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{H}$ | $\mathbb{O}$ |
|-------------------------------|-----------|-----------|-----------|-----------|
| $V_3(\mathbb{C}, \mathbb{C})$ | $F'I$     | $E'I$     | $EIV$     | $EIX$     |

We will need to understand the Cartan decomposition of these algebras. Towards this end, consider a composition algebra $C_1 = F \oplus lF$. By $F'$ we mean the subspace of $F$ orthogonal to $R$. 
For each linear, symmetric (and traceless if $F = \mathbb{H}$) map $S : F' \to F'$, define $G_S$ by

$$G_S(a) = l(Sa) \quad \text{for } a \in F' \quad \text{and} \quad G_S(l) = 0,$$

and extend $G_S$ to $C_1$ by linearity and the derivation property. It can be checked that $G_S$ is a derivation on $C_1$ and that if $\lfloor \cdot, \cdot \rfloor$ denotes the usual bracket on $Der(C_1)$, then $[G_S, G_T] = l^2[S, T]$, where we denote by $\overline{D}$ the unique extension of $D$ to a derivation on $C_1$ that sends $l$ to 0.

The derivations $E_a$ and $F_a$ mentioned in the next theorem are defined in [2] for each $a \in F'$, but as the specific details of these will not be important for us we have not given the definitions here.

**Theorem 6.** [2] Section 7] Let $C_1 = F \oplus lF$ be a split algebra and $C_2$ be a division algebra. Let $V_3(C_1, C_2)$ denote the Lie algebra as in [3,7]. The Cartan decomposition $V_3(C_1, C_2) = \mathfrak{t} \oplus \mathfrak{p}$ is given by

$$\mathfrak{t} = A_3'(F \otimes C_2) \oplus Der_0(C_1) \oplus Der(C_2)$$

and

$$\mathfrak{p} = A_3'(lF \otimes C_2) \oplus Der_1(C_1),$$

where

$$Der_0(C_1) = \text{sp}\{D \in Der(F), a \in F'\}$$

and

$$Der_1(C_1) = \text{sp}\{F_a, G_S : a \in F', S \text{ linear, symmetric}\}.$$

Using this decomposition, we can find a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$. This is the content of the next proposition.

**Proposition 3.** Let $V_3(C_1, C_2) = \mathfrak{t} \oplus \mathfrak{p}$ be a Lie algebra obtained from the Vinberg construction, with Cartan decomposition as given above and $C_1 = \mathbb{H} \oplus \mathbb{H}$. Define the following elements in $A_3'(l\mathbb{H} \otimes C_2)$:

$$H_1 = \begin{bmatrix} l \otimes 1 & 0 & 0 \\ 0 & -l \otimes 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & l \otimes 1 & 0 \\ 0 & 0 & -l \otimes 1 \end{bmatrix}.$$

A maximal abelian subalgebra is spanned by $H_1$, $H_2$ and $G_{S_1}$, $G_{S_2}$, where $S_1, S_2$ span the diagonal traceless operators $\mathbb{H}' \to \mathbb{H}'$.

**Proof.** First, observe that $D_{1,1} = 0$ since $L_1, R_1$ are both commutative and $D_{1,1}x = [L_1, R_1]x = l(ax) - (lx)l = 0$. It is clear that $H_1H_2 - H_2H_1 = 0$ so we have that $[H_1, H_2] = 0$.

As $G_{S_1}l = 0$ we have

$$[G_{S_1}, H_j] = G_{S_1}(H_j) = 0 \quad \text{for } j = 1, 2.$$

Also, $[G_{S_1}, G_{S_2}] = l^2[S_1, S_2] = 0$ since both $S_1, S_2$ were chosen to be diagonal. Thus the given vectors span an abelian subalgebra.

Maximality follows because the maximal abelian subalgebra has dimension equal to the rank of its associated symmetric space.

Having explained the Vinberg construction, we will now show there are embeddings of the Lie algebras and then appeal to Theorem [3] to obtain suitable embeddings at the group level. This could be done in more generality, but it is the case $F = \mathbb{H}$ that is of interest to us.
Proposition 4. Let \( C_0 = \mathbb{H} \oplus \mathbb{H} \) be the split octonions and let \( C_1 \) denote one of \( \mathbb{C}, \mathbb{H} \) or \( \mathbb{O} \). Then \( V_3(C_0, \mathbb{R}) \) embeds into \( V_3(C_0, C_1) \).

Proof. Let \( t_0 \oplus p_0 \) and \( t_1 \oplus p_1 \) denote the Cartan decompositions of \( V_3(C_0, \mathbb{R}) \) and \( V_3(C_0, C_1) \) respectively, and let \( a_0 \) and \( a_1 \) be associated the maximal abelian subalgebras described in Prop. 3. We note that \( V_3(C_0, \mathbb{R}) = A'_3(C_0 \otimes \mathbb{R}) \oplus \text{Der}(C_0) \) as \( \mathbb{R} \) has no non-trivial derivations.

Define the map \( \iota : V_3(C_0, \mathbb{R}) \to V_3(C_0, C_1) \) by

\[
\iota(A + D) = A + D.
\]

Here we have the natural inclusion \( A'_3(C_0 \otimes \mathbb{R}) \to A'_3(C_0 \otimes C_1) \), which is just an extension of scalars, and the identity map on \( \text{Der}(C_0) \).

One can easily check that \( \iota \) is a Lie algebra homomorphism. Furthermore, \( \iota \) is clearly an injective map into \( V_3(C_0, C_1) \). Since \( \iota(H_i) = H_i \) and \( \iota(G_{S_j}) = G_{S_j} \) (with the natural identifications), it is clear from Prop. 3 that \( \iota : a_0 \to a_1 \) bijectively. Finally, \( \iota \) maps \( A'_3(\mathbb{H} \otimes \mathbb{R}) \) into \( A'_3(\mathbb{H} \otimes C_1) \), \( \text{Der}_0(C_0) \) to itself, hence it maps \( t_0 \) into \( t_1 \).

Corollary 4. We have the embeddings of the symmetric space of Cartan type \( FI \) into \( EII, EVI \) and \( EIX \).

Proof. Let \( G \) be the non-compact simple Lie group with Lie algebra \( V_3(C_0, \mathbb{R}) \) and let \( \tilde{G} \) denote its simply connected universal cover. The centre of \( \tilde{G} \) is the fundamental group of \( G \), namely \( \mathbb{Z}_2 \) [4], thus Theorem 5 lifts the embeddings of the Vinberg construction demonstrated in the previous proposition to the group level.

Corollary 5. The convolution of any two orbital measures in \( EII, EVI \) or \( EIX \) is absolutely continuous.

Proof. We first remark that the problem of the absolute continuity of the product of orbital measures is unaffected by the choice of \( G \) with a given Lie algebra \( \mathfrak{g} \). This is because any such group \( G \) is isomorphic to \( \tilde{G}/Z \) where \( \tilde{G} \) is the common universal cover and \( Z \) is a subset of the centre of \( G \). Further, \( K \simeq \tilde{K}/Z \). As the adjoint action is trivial on the centre, the adjoint action on \( K \) and \( \tilde{K} \) coincide.

As shown in Prop. 11 the convolution of any two orbital measures in \( FI \) is absolute continuous. Thus the corollary holds because of the embedding Theorem 5.

3.4. The remaining exceptional symmetric spaces [Types \( EIII, EIV, EVII, FII \)]. For the remaining exceptional symmetric spaces we use ad-hoc arguments to prove the required absolute continuity results.

Type \( FII \): Here the restricted root space is type \( BC_1 \). Any non-zero \( X \in \mathfrak{a} \) is regular, meaning \( \Phi_X \) is empty, and it is known that the convolution of any two regular elements is absolutely continuous [6].

Type \( EIV \): The restricted root space is type \( A_2 \). All non-zero \( X \) are regular, except \( X \) of type \( A_1 \), and for the regular \( X \), \( \mu_X^2 \) is absolutely continuous. As the rank of \( EIV \) is 2, any convolution product of three non-trivial orbital measures is absolutely continuous according to [8].

Type \( EIII \): The restricted root space is type \( BC_2 \), with the roots \( \alpha_1 \pm \alpha_2 \) having multiplicity 6, the roots \( \alpha_j \) having multiplicity 8 and the roots \( 2\alpha_j \) having
multiplicity 1. The co-rank one (equivalently, rank one in this case) closed root subsystems are either \( \{\alpha_1 + \alpha_2\}, \{\alpha_1 - \alpha_2\} \) or \( \{\alpha_j, 2\alpha_j\} \). With this information it is simple to check that (3.2) is satisfied with \( m = 2 \) for all \( \Phi_X, X \neq 0 \). Consequently any product of two \( K \)-invariant, continuous orbital measures is absolutely continuous.

Type EVII: In this case, the restricted root system is type \( C_3 \), with the short roots having multiplicity 8, the long roots multiplicity 1, and \( \dim \Phi = 51 \). The co-rank one, closed root subsystems \( \Psi \) are type \( C_2 \), \( A_2 \) or \( A_1 \times C_1 \).

If \( X \) is of type \( C_2 \), then \( \dim \Phi_X = 18 \). The intersection of any such \( \Phi_X \) with any \( \Psi \) of type \( A_2 \) must contain a short root and hence the intersection has dimension at least 8. It easily follows from (3.2) that \( \mu_X^3 \) is absolutely continuous. If \( X \) is of type \( A_1 \times C_1 \), then \( \dim \Phi_X = 9 \) and it is even easier to see that in this case \( \mu_X^2 \) is absolutely continuous. If \( X \) is type \( A_1 \) or \( C_1 \), then \( \Phi_X \) is contained in the set of annihilating roots of an \( X' \) of type \( A_1 \times C_1 \) and again it follows that \( \mu_X^2 \) is absolutely continuous.

Finally, if \( X \) is type \( A_2 \), then \( \dim \Phi_X = 24 \) and (3.2) is clearly satisfied with \( m = 3 \). It follows that the convolution of any 3 non-trivial orbital measures on \( p \) is absolutely continuous. Thus \( L(G) = 3 \) suffices.

However, to complete the proof of the absolute continuity claims of Theorem 1 we must show that if \( X \) is type \( A_2 \), then \( \mu_X^3 \) is absolutely continuous. For this we will use the geometric criteria for absolute continuity given in Theorem 3.

The absolute root system of the symmetric space EVII is type \( E_7 \) and we can take as a base the roots \( \frac{1}{2}(e_1 + e_8 - (e_2 - \cdots - e_7), e_1 + e_2, e_1 - e_2, e_3 - e_2, \ldots, e_6 - e_5) \). One can see from the Satake diagram (c.f., [1], p. 534] that \( e_j|_\alpha = 0 \) for \( j = 1, 2, 3, 4 \) and that the roots \( e_5 \pm e_6 \) and \( e_7 - e_8 \) are distinct non-zero elements of \( \alpha^* \). We have \( \theta(e_j) = e_j \) for \( j = 1, 2, 3, 4 \),

\[
\theta(e_5 \pm e_6) = -(e_5 \pm e_6), \quad \theta(e_7 - e_8) = -(e_7 - e_8),
\]

\[
\theta(e_j \pm e_i) = -e_j \pm e_i \text{ for } j = 5, 6, i = 1, 2, 3, 4
\]

and

\[
\theta \left( \frac{1}{2}(e_8 - e_7 + e_6 \pm e_5 + \sum_{j=1}^4 s_j e_j) \right) = -\frac{1}{2} \left( e_8 - e_7 + e_6 \pm e_5 - \sum_{j=1}^4 s_j e_j \right)
\]

(he \( s_j = \pm 1 \) and \( \prod s_j = -1 \)).

Up to Weyl conjugacy, the set of annihilating roots of an element \( X \) of type \( A_2 \) consists of the restricted roots \( \{ \frac{1}{2}(e_8 - e_7 + e_6 \pm e_5), e_5 \} \), each of which has multiplicity 8. The non-annihilating roots are the restricted roots \( \{ \frac{1}{2}(e_8 - e_7 - e_6 \pm e_5), e_6, e_5 \pm e_6, e_8 - e_7 \} \), with the latter three having multiplicity one. The restricted root spaces of the non-annihilating roots of multiplicity 8 are

\[
\mathfrak{g}_{\frac{1}{2}}(e_8 - e_7 - e_6 \pm e_5) = sp \left\{ E_{\beta}^{-} : \beta = \frac{1}{2}(e_8 - e_7 - e_6 \pm e_5 + \sum_{j=1}^4 s_j e_j) \right\} \quad \text{and}
\]

\[
\mathfrak{g}_{e_5} = sp \{ E_{e_6 \pm e_j}^{-} : j = 1, \ldots, 4 \}.
\]
Let
\[ F_1 = E_{e_5 + e_6} + E_{-e_5 + e_6} = E_{e_5 + e_6}^+, \]
\[ F_2 = E_{e_5 - e_6} + E_{e_6 - e_5} = E_{e_5 - e_6}^+, \]
\[ F_3 = E_{e_5 - e_7} + E_{e_7 - e_5} = E_{e_5 - e_7}^+. \]
Routine calculations show that if \( F = c_1 F_1 + c_2 F_2 + c_3 F_3 \in K \), then
\[ ad(F) E_{e_5 \pm e_j}^- = (c_2 - c_1) E_{e_5 \pm e_j}^- \text{ for } j = 1, 2, 3, 4. \]
If \( \beta^+ = \frac{1}{2}(e_8 - e_7 + e_6 - e_5 + \sum_{j=1}^{4} s_j e_j) \), \( \beta^- = \frac{1}{2}(e_8 - e_7 - e_6 + e_5 + \sum_{j=1}^{4} s_j e_j) \), and
\[ \gamma^+ = \frac{1}{2}(e_8 - e_7 + e_6 + e_5 + \sum_{j=1}^{4} s_j e_j), \quad \gamma^- = \frac{1}{2}(e_8 - e_7 - e_6 + e_5 + \sum_{j=1}^{4} s_j e_j), \]
then \( \beta^- \mid_{\alpha}, \gamma^- \mid_{\alpha} \) are non-annihilating roots and
\[ ad(F) E_{\beta^-}^- = (c_2 - c_3) E_{\beta^+}^- \text{ and } ad(F) E_{\gamma^-}^- = (c_1 - c_3) E_{\gamma^+}^- . \]
Furthermore, if \( c_j \neq 0 \), then
\[ sp(ad(F) \{ E_{e_5 \pm e_6}^-, E_{e_5 - e_7}^- \} ) = \alpha. \]
Thus, provided we choose the \( c_j \) non-zero and distinct, then
\[ \mathcal{N}_X + ad(F) \mathcal{N}_X = p. \]
Now consider \( f_t = \exp t F \in K \) for small \( t > 0 \). For any \( Y, \ Ad(f_t)(Y) = Y + t \cdot ad(F)(Y) + S_t(Y) \) where \( \| S_t \| \leq Ct^2 \) for a constant \( C \) independent of \( Y \) and \( t \). Hence, for all \( t > 0 \),
\[ sp\{ \mathcal{N}_X, Ad(f_t)\mathcal{N}_X \} = sp \left\{ \mathcal{N}_X, (ad(F) + \frac{1}{t} S_t)\mathcal{N}_X \right\} \subseteq p \]
where \( \| S_t \| \leq Ct \). But since \( sp\{ \mathcal{N}_X, ad(F)\mathcal{N}_X \} = p \), a linear algebra argument implies that the same is true for \( sp\{ \mathcal{N}_X, Ad(f_t)\mathcal{N}_X \} \) for small enough \( t > 0 \). Thus Theorem 3 implies \( \mu_X^2 \) is absolutely continuous when \( X \) is type \( A_2 \).
That completes the absolute continuity part of Theorem 1.

4. Singularity

We now turn to proving that \( \mu_X^2 \) is singular to Lebesgue measure in the cases where we have claimed \( L_X = 3 \). (Of course, if \( L_X = 2 \), then that is sharp as all orbits have measure zero and hence all orbital measures are singular.)

When \( X \) is type \( D_5 \) in \( EI \), type \( E_6 \) or \( D_6 \) in \( EV \), or type \( E_7 \) in \( EVIII \), this is very easy: Simply note that for all of these elements \( X \), \( \dim \mathcal{N}_X < \frac{1}{2} \dim p \) and hence inequality (4.3) of Theorem 3 with \( m = 2 \), must necessarily fail.

For the remaining cases, we will make use of the following condition that guarantees singularity. It was originally established in [20] for compact Lie algebras.

**Proposition 5.** Let \( X_1, \ldots, X_m \in a \) and assume that for \( j = 1, \ldots, m, \sigma_j(\mathcal{N}_{X_j}) \cap \mathcal{N}_{\Psi} \), are pairwise disjoint for some closed, co-rank one root subsystem \( \Psi \) of \( \Phi \) and \( \sigma_j \in W \). Then \( \mu_{X_1} \cdots \mu_{X_m} \) is singular to Lebesgue measure.
Proof. The proof in the symmetric space case is very similar to that given in [20]. Given \( \Psi \), choose \( Z \in a \) satisfying \( \Phi_Z = \Psi \). For each \( j \), let \( K_j = \{ k \in K : \text{Ad}(k)X_j = X_j \} \) and let \( p_j = \{ Y \in p : [Y, X_j] = 0 \} \). Define \( M_j \) to be the \( K_j \)-orbit of \( Z \). Then \( M_j \) is a submanifold of \( O_Z \) and is contained in \( p_j \). The tangent space to \( M_j \) at \( Z \) is
\[
T_Z(M_j) = \{ Y \in p : [Y, Z] \neq 0 \} \cap p_j = N_Z \cap p_j.
\]
Note that \((T_Z(M_j))^\perp \) in \( T_Z(O_Z) \) is equal to \( N_Z \cap N_{X_j} \). By [20] Lemma 2.5, \( \bigcap_{j=1}^m \text{Ad}(k_j)M_j \) is non-empty for \((k_1, \ldots, k_m)\) near the identity and hence the same is true for the larger space \( \bigcap_{j=1}^m \text{Ad}(k_j)p_j \). But then standard Hilbert space arguments (c.f., [11] Lemma 4) imply that
\[
\text{sp}\left\{ \text{Ad}(k_j)N_{X_j} : j = 1, \ldots, m \right\} \neq p
\]
for all \((k_1, \ldots, k_m)\) near the identity. By Theorem 3 this implies \( \mu_{X_1} \ast \cdots \ast \mu_{X_m} \) is singular. \( \square \)

We will use this criteria for the remaining cases.

1. \( X \) type \( A_1 \) in \( EIIV \) where \( \Phi \) is type \( A_2 \): By passing to a Weyl conjugate, we can assume \( \Phi_X = \{ e_1 - e_2 \} \) and \( \Phi_{\sigma(X)} = \{ e_2 - e_3 \} \). Put \( \Psi = \{ e_1 - e_3 \} \).

Then \( N_X \cap N_\Psi = \{ e_1 - e_3 \} \), while \( N_{\sigma(X)} \cap N_\Psi = \{ e_1 - e_2 \} \).

2. \( X \) type \( C_2 \) in \( EVII \) where \( \Phi \) is type \( C_3 \): Here we can assume \( \Phi_X = \{ e_1 \pm e_2, 2e_2, 2e_1 \}, \Phi_{\sigma(X)} = \{ e_2 \pm e_3, 2e_2, 2e_3 \} \), and take \( \Psi = \{ e_1 \pm e_3, 2e_1, 2e_3 \} \).

3. \( X \) type \( A_5 \) in \( EI \) where \( \Phi \) is type \( E_6 \): This was shown to be singular using the criterion of Theorem 3 for the Lie algebra setting in [13] 4.3]. The same argument applies here.

This completes the singularity argument and hence the proof of Theorem 4.
### Basic facts about non-compact irreducible Riemannian symmetric spaces

In the column labelled $g$, the number in the bracket is the signature of the Killing form, $\dim p - \dim \mathfrak{t}$.

|   | $g$     | $\mathfrak{t}$ | Absolute Root System | Restricted Root System | Dimension of $G/K$ | Rank of $G/K$ | Multiplicities |
|---|---------|---------------|----------------------|------------------------|-------------------|--------------|----------------|
| $E_{I}$ | $\varepsilon_6(6)$ | $\mathfrak{sp}(4)$ | $E_6$ | $E_6$ | 42 | 6 | All: 1 |
| $E_{II}$ | $\varepsilon_6(2)$ | $\mathfrak{su}(6) \oplus \mathfrak{su}(2)$ | $E_6$ | $F_4$ | 40 | 4 | Not Needed |
| $E_{III}$ | $\varepsilon_6(-14)$ | $\mathfrak{so}(10) \oplus \mathfrak{so}(2)$ | $E_6$ | $BC_2$ | 32 | 2 | $\varepsilon_i \pm \varepsilon_j$: 6, $\varepsilon_i$: 8, $2\varepsilon_i$: 1 |
| $E_{IV}$ | $\varepsilon_6(-26)$ | $\mathfrak{f}(4)$ | $E_6$ | $A_2$ | 26 | 2 | All: 8 |
| $E_{V}$ | $\varepsilon_7(7)$ | $\mathfrak{su}(8)$ | $E_7$ | $E_7$ | 70 | 7 | All: 1 |
| $E_{VI}$ | $\varepsilon_7(-5)$ | $\mathfrak{so}(12) \oplus \mathfrak{su}(2)$ | $E_7$ | $F_4$ | 64 | 4 | Not Needed |
| $E_{VII}$ | $\varepsilon_7(-25)$ | $\mathfrak{e}(6) \oplus \mathfrak{so}(2)$ | $E_7$ | $C_3$ | 54 | 3 | $\varepsilon_i \pm \varepsilon_j$: 8, $2\varepsilon_i$: 1 |
| $E_{VIII}$ | $\varepsilon_8(8)$ | $\mathfrak{so}(16)$ | $E_8$ | $E_8$ | 128 | 8 | All: 1 |
| $E_{IX}$ | $\varepsilon_8(-24)$ | $\mathfrak{e}(7) \oplus \mathfrak{su}(2)$ | $E_8$ | $F_4$ | 112 | 4 | Not Needed |
| $F_{I}$ | $\mathfrak{f}_4(4)$ | $\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$ | $F_4$ | $F_4$ | 28 | 4 | All: 1 |
| $F_{II}$ | $\mathfrak{f}_4(-20)$ | $\mathfrak{so}(9)$ | $F_4$ | $BC_1$ | 16 | 1 | $\varepsilon_i$: 8, $2\varepsilon_i$: 7 |
| $G$ | $\mathfrak{g}_2(2)$ | $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ | $G_2$ | $G_2$ | 8 | 2 | All: 1 |
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DEPT. OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONT., CANADA, N2L 3G1
E-mail address: kehare@uwaterloo.ca

DEPT. OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONT., CANADA, N2L 3G1
E-mail address: jimmy.he@uwaterloo.ca