Quantum Clifford-Hopf Algebras for Even Dimensions

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Abstract

In this paper we study the quantum Clifford-Hopf algebras $\mathcal{CH}_q(D)$ for even dimensions $D$ and obtain their intertwiner $R$–matrices, which are elliptic solutions to the Yang-Baxter equation. In the trigonometric limit of these new algebras we find the possibility to connect with extended supersymmetry. We also analyze the corresponding spin chain hamiltonian, which leads to Suzuki’s generalized $XY$ model.
1. Introduction

The quantum group structure plays an important role in the study of two dimensional integrable models because \( R \)-matrices intertwining between different irreps of a quantum group provide solutions to the Yang-Baxter equation. Two important families of integrable models are the 6-vertex and 8-vertex solutions to the Yang-Baxter equation [1]. Whereas the 6-vertex solutions are intertwiners \( R \)-matrices for \( \hat{U}_q(sl(2)) \), a quantum group interpretation for the elliptic 8-vertex family is not yet known.

Nevertheless, the 8-vertex regime is well understood for the particular class of solutions to the Yang-Baxter equation satisfying the free-fermion condition [2]

\[
R_{00}^{10}(u)R_{11}^{01}(u) + R_{01}^{10}(u)R_{10}^{01}(u) = R_{00}^{11}(u)R_{00}^{11}(u) + R_{01}^{01}(u)R_{10}^{10}(u) \quad (1)
\]

Indeed, a quantum group like structure has been found recently for the most general free fermionic elliptic 8-vertex model in a magnetic field. The matrix of its Boltzmann weights [3, 4] acts as intertwiner for the afzinization of a quantum Hopf deformation of the Clifford algebra in two dimensions, noted \( \hat{CH}_q(2) \) [5].

A major interest of the free fermionic solutions to the Yang-Baxter equation is in their connection, in the 6-vertex limit \( (R_{00}^{11}(u) = R_{11}^{00}(u) = 0) \), with \( N = 2 \) supersymmetric integrable models. The free fermionic 6-vertex solutions are given by the \( R \)-matrix intertwiners between nilpotent irreps of the Hopf algebra \( \hat{U}_\epsilon(sl(2)) \), with \( \epsilon^4 = 1 \) (the nilpotent irreps are a special case of the cyclic representations that enlarge the representation theory of \( \hat{U}_\epsilon(sl(2)) \) when \( \epsilon \) is a root of unity). In the trigonometric limit the \( R \)-matrix for \( CH_q(2) \) becomes that for \( \hat{U}_\epsilon(sl(2)), \epsilon^4 = 1 \).

In this article we construct the quantum Clifford-Hopf algebras \( CH_q(D) \) for even dimensions \( D \geq 2 \), generalizing the results in [5]. This general case is interesting because it yields one of the rare examples of elliptic \( \hat{R} \)-matrices. The \( R \)-matrices we find admit several spectral parameters, due to the structure of \( \hat{CH}_q(D) \) as a Drinfeld twist [6] of the tensor product of several copies of \( \hat{CH}_q(2) \). The possibility to connect with extended supersymmetry in the trigonometric limit of \( CH_q(D) \), and a related supersymmetric integrable model are analyzed in sect.3. Finally, in sect.4, we study the spin chain hamiltonian associated to these algebras. The model obtained represents several \( XY \) Heisenberg chains in an external magnetic field [7] coupled among them in a simple way. Though the coupling is simple it can be an starting point to get a quantum group structure for more complicated models built through the coupling of two \( XY \) or \( XX \) models (Bariev model [8], 1-dimensional Hubbard model). The last part of this section is devoted to showing
the equivalence of this model—under some restrictions—with a generalized XY model proposed by M. Suzuki in relation with the 2-dimensional dimer problem [9].

2. The quantum Clifford algebra

A Clifford algebra \( C(\eta) \) related to a quadratic form or metric \( \eta \) is the associative algebra generated by the elements \( \{\Gamma_{\mu}\}_{\mu=1}^{D} \), which satisfy

\[
\{\Gamma_{\mu},\Gamma_{\nu}\} = 2\eta_{\mu\nu}1, \quad \mu, \nu = 1, \ldots, D
\]

The quantum Clifford-Hopf algebra \( CH_q(D) \) is a generalization and quantum deformation of \( C(\eta) \), generated by elements \( \Gamma_{\mu}, \Gamma_{D+1} \) (the analog of \( \gamma_{5} \) for the Dirac matrices) and new central elements \( E_{\mu} \) (\( \mu = 1, \ldots, D \)) verifying

\[
\Gamma_{\mu}^{2} = \frac{q^{E_{\mu}} - q^{-E_{\mu}}}{q - q^{-1}}, \quad \Gamma_{D+1}^{2} = 1
\]

\[
\{\Gamma_{\mu},\Gamma_{\nu}\} = 0, \quad \mu \neq \nu
\]

\[
\{\Gamma_{\mu},\Gamma_{D+1}\} = 0
\]

\[
[E_{\mu},\Gamma_{\nu}] = [E_{\mu},\Gamma_{D+1}] = [E_{\mu},E_{\nu}] = 0 \quad \forall \mu, \nu
\]

The charges \( E_{\mu} \) result from elevating the components of the metric \( \eta \) from numbers to operators. The generator \( \Gamma_{D+1} \) will plays a similar role to \((-1)^{F}\), with \( F \) the fermion number operator. Although for the standard Clifford algebra \( D \) represents the dimension of the space-time, in our case \( D \) is only a parameter labeling (3). The algebra \( CH_q(D) \) is a Hopf algebra with the following comultiplication \( \Delta \), antipode \( S \) and counit \( \epsilon \)

\[
\Delta(E_{\mu}) = E_{\mu} \otimes 1 + 1 \otimes E_{\mu}, \quad S(E_{\mu}) = -E_{\mu}, \quad \epsilon(E_{\mu}) = 0
\]

\[
\Delta(\Gamma_{\mu}) = q^{E_{\mu}/2}\Gamma_{D+1} \otimes \Gamma_{\mu} + \Gamma_{\mu} \otimes q^{-E_{\mu}/2}, \quad S(\Gamma_{\mu}) = \Gamma_{\mu}\Gamma_{D+1}, \quad \epsilon(\Gamma_{\mu}) = 0
\]

\[
\Delta(\Gamma_{D+1}) = \Gamma_{D+1} \otimes \Gamma_{D+1}, \quad S(\Gamma_{D+1}) = \Gamma_{D+1}, \quad \epsilon(\Gamma_{D+1}) = 1
\]

The irreducible representations of \( CH_q(D) \) are in one to one correspondence with those of the Clifford algebra \( C(\eta) \) for all possible signatures of the metric \( \eta \), in \( D \) (\( D \) even) or \( D+1 \) (\( D \) odd) dimensions respectively. They are labelled by complex parameters \( \{\lambda_{\mu}\}_{\mu=1}^{D} \), the eigenvalues of the Casimir operators \( K_{\mu} = q^{E_{\mu}} \). From now on we restrict ourselves to the case \( D \) even, \( D = 2M \).

The irreps of \( CH_q(2M) \) are isomorphic to the tensor product of \( M \) \( CH_q(2) \) irreps, being their dimension \( 2^M \). Thus, a basis for \( CH_q(2M) \) can be obtained from the \( CH_q(2)^{\otimes M} \)
generators as follows \((\gamma_\alpha, E_\alpha (\alpha = 1, 2), \gamma_3 \in CH_q(2))\):

\[
\begin{align*}
\Gamma_{2(n-1)+\alpha} & = \gamma_3 \otimes \cdots \otimes \gamma_3 \otimes \gamma_\alpha \otimes 1 \otimes \cdots \otimes 1 \\
E_{2(n-1)+\alpha} & = 1 \otimes \cdots \otimes 1 \otimes E_\alpha \otimes 1 \otimes \cdots \otimes 1 \\
\Gamma_{D+1} & = \gamma_3 \otimes \cdots \otimes \gamma_3
\end{align*}
\]

(5)

The Hopf algebra \(CH_q(2M)\) is related to the tensor product \(CH_q(2)^{\otimes M}\) by a Drinfeld twist \(B\)[6]

\[
\Delta_{CH_q(2M)}(g) = B \Delta_{CH_q(2)^{\otimes M}}(a) B^{-1} \quad \forall g \in CH_q(2M)
\]

(6)

where the operator \(B \in CH_q(2)^{\otimes M} \otimes CH_q(2)^{\otimes M}\) acting on the tensor product of two \(CH_q(2M)\) irreps is defined by

\[
B = (-1)^{F \ast F}
\]

\[
F \ast F = \sum_{1 \leq j < i \leq M} (1 \otimes \cdots \otimes f^{\otimes i} \otimes \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes f^{\otimes j} \otimes \cdots \otimes 1)
\]

with \(f = 0\) (boson), 1 (fermion) the fermion number for the two vectors in a \(CH_q(2)\) irrep. The reason to introduce the operator \(B\) in formula (6) is that the comultiplication in \(CH_q(2)^{\otimes M}\) treats each factor \(CH_q(2)\) separately. This can be represented by a twist between the \(CH_q(2)\) pieces of a \(CH_q(2M)\) irrep. Since one of the vectors in a \(CH_q(2)\) irrep behaves as a fermion, this twist has the effect of introducing some signs that we represent by the operator \(B\) (fig.1).

Next we introduce a sort of affinization of the Hopf algebra \(CH_q(D)\). The generators of this new algebra \(\widehat{CH_q}(D)\) are \(\Gamma_\mu^{(i)}, E_\mu^{(i)} (i = 0, 1)\) and \(\Gamma_{D+1}\) verifying (3) and (4) for each value of \(i\). We impose also that the anticommutator \(\{\Gamma_\mu^{(1)}, \Gamma_\nu^{(2)}\}\) belong to the center of \(\widehat{CH_q}(D)\) \(\forall \mu, \nu\).

Let’s give now the explicit realization of \(\widehat{CH_q}(2)\). It is an useful example, and it will provide us with the building blocks for any \(D\). A two-dimensional irrep \(\pi_\xi\) of \(\widehat{CH_q}(2)\) is labelled by \(\xi = (z, \lambda_1, \lambda_2) \in C^3\) and reads as follows

\[
\begin{align*}
\pi_\xi(\gamma_1^{(0)}) & = (\frac{\lambda_1^{1} - \lambda_1}{q - q^{-1}})^{1/2} \begin{pmatrix} 0 & z^{-1} \\ z & 0 \end{pmatrix}, \quad \pi_\xi(\gamma_1^{(1)}) = (\frac{\lambda_1^{1} - \lambda_1}{q - q^{-1}})^{1/2} \begin{pmatrix} 0 & z \\ z^{-1} & 0 \end{pmatrix} \\
\pi_\xi(\gamma_2^{(0)}) & = (\frac{\lambda_2^{1} - \lambda_2}{q - q^{-1}})^{1/2} \begin{pmatrix} 0 & -iz^{-1} \\ iz & 0 \end{pmatrix}, \quad \pi_\xi(\gamma_2^{(1)}) = (\frac{\lambda_2^{1} - \lambda_2}{q - q^{-1}})^{1/2} \begin{pmatrix} 0 & -iz \\ iz^{-1} & 0 \end{pmatrix}
\end{align*}
\]

(8)
\[ \pi_\xi(\gamma_3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \pi_\xi(q^{E^{(0)}}) = \lambda_1^{-1} \quad , \quad \pi_\xi(q^{E^{(1)}}) = \lambda_1 \quad , \quad \pi_\xi(q^{E^{(2)}}) = \lambda_2^{-1} \quad , \quad \pi_\xi(q^{E^{(1)}}) = \lambda_2 \]

For the affine \( \widetilde{CH}_q(2M) \) we can define a straightforward generalization of the expression (5). It allows to introduce \( M \) different affinization parameters \( \{z_n\}_{n=1}^M \), one for each \( \widetilde{CH}_q(2) \) piece

\[
\Gamma^{(i)}_{2(n-1)+\alpha} = \gamma_3 \otimes \cdots \otimes \gamma_3 \otimes \gamma^{(i)}_\alpha \otimes 1 \otimes \cdots \otimes 1 \quad n = 1, \ldots, M; \quad \alpha = 1, 2; \quad i = 0, 1
\]

\[
E^{(i)}_{2(n-1)+\alpha} = 1 \otimes \cdots \otimes 1 \otimes E^{(i)}_\alpha \otimes 1 \otimes \cdots \otimes 1
\]

(9)

The intertwiner \( R \)-matrix for two irreps with labels \( \xi = \{z_n, \lambda_{2n-1}, \lambda_{2n}\}_{n=1}^M \) is defined by the condition

\[
R_{\xi_1 \xi_2} \Delta_{\xi_1 \xi_2}(g) = \Delta_{\xi_2 \xi_1}(g) R_{\xi_1 \xi_2} \quad \forall g \in \widetilde{CH}_q(2M)
\]

(10)

with \( \Delta_{\xi_1 \xi_2} = \pi_{\xi_1} \otimes \pi_{\xi_2}(\Delta) \). Since (6) remains true for any element \( g \in \widetilde{CH}_q(2M) \), the intertwiner \( R \)-matrix between two irreps (which furthermore satisfies the Yang-Baxter equation) is given by

\[
R_{\widetilde{CH}_q(2M)}(u_1, \ldots, u_M) = B \ R_{\widetilde{CH}_q(2) \otimes M}(u_1, \ldots, u_M) \ B^{-1}
\]

(11)

\[
R_{\widetilde{CH}_q(2) \otimes M}(u_1, \ldots, u_M) = R^{(1)}_{\widetilde{CH}_q(2)}(u_1) \ldots R^{(M)}_{\widetilde{CH}_q(2)}(u_M)
\]

The matrices \( R^{(n)}_{\widetilde{CH}_q(2)} = R^{(n)\xi_1}_{\xi_2^2} \) (\( \xi^{(n)} = (z_n, \lambda_{2n-1}, \lambda_{2n}) \)) are the \( \widetilde{CH}_q(2) \) intertwiners

\[
R^{00}_{00} = 1 - e(u_n)e_1e_2 \quad , \quad R^{11}_{01} = e(u_n) - e_1e_2
\]

\[
R^{01}_{01} = e_1 - e(u_n)e_2 \quad , \quad R^{01}_{10} = e_2 - e(u_n)e_1
\]

\[
R^{01}_{01} = R^{10}_{10} = (e_1sn_1)^{1/2}(e_2sn_2)^{1/2}(1 - e(u_n))/sn(u_n/2)
\]

\[
R^{11}_{00} = R^{10}_{10} = -ik(e_1sn_1)^{1/2}(e_2sn_2)^{1/2}(1 + e(u_n))sn(u_n/2)
\]

(12)

where \( e(u_n) = cn(u_n) + isn(u_n) \) is the elliptic exponential of modulus \( k_n \), \( e_i = e(\psi_n^i) \), \( sn_i = sn(\psi_n^i) \) \( (i = 1, 2) \) and \( u_n, \psi_n^i \) are elliptic angles depending on the labels \( \xi^{(n)}_i \) (see ref.[5] for details).

There is a constraint on the irrep labels so that (12) be indeed their intertwiner

\[
\frac{2(\lambda_{2n-1} - \lambda_{2n})}{(1 - \lambda_{2n-1}^2)(1 - \lambda_{2n}^2)^{1/2}(z_n^2 - z_n^{-2})} = k_n \quad , \quad n = 1, \ldots, M
\]

(13)
All the $R^{(n)}_{CH_q(2)}$ matrices are independent and commute among them. It’s remarkable that the spectral curve (13) of irreps that admit an intertwiner is parametrized by $M$ independent elliptic moduli $k_n$. Indeed, some of them can be in the elliptic regime and others in the trigonometric ($k = 0$). The matrix $R_{CH_q(2M)}$ can be thought of as the scattering matrix for objects composed of $M$ different kinds of particles. There is real interaction when two equal particles scatter from each other, given by $R^{(n)}_{CH_q(2)}$; otherwise there is only a sign coming from their statistics and represented by the operator $B$ (fig 2).

Finally, note that the $R$–matrix (12) coincides with the Boltzmann weights for the most general 8-vertex free fermionic solution to the Yang-Baxter equation in non zero magnetic field [3, 4].

3. Extended supersymmetry

In order to analyze the connection of $CH_q(2M)$ with supersymmetry algebras, we will study the limit in which the $R$–matrix (12) becomes trigonometric. Let us consider first the case $D = 2$ in detail. This case turns out to be related to an $N = 2$ (2 supersymmetry charges) integrable Ginzburg-Landau model. We shall also give an heuristic motivation for the construction of the Hopf algebra $\widehat{CH}_q(2)$ based on its trigonometric 6-vertex limit.

The 6-vertex free fermionic solutions are given by the intertwiner $R$–matrix between nilpotent irreps of $U_\epsilon(\widehat{sl}(2)), \epsilon^4 = 1 \quad (\Rightarrow \epsilon = i)$ [10]. In a $U_\epsilon=\iota(\widehat{sl}(2))$ nilpotent irrep the values of the special Casimirs are $Q_\pm^2 = 0 \quad (Q_\pm = S_\pm \epsilon^{\pm H/2})$ and $K^2 = \lambda^2$ arbitrary ($K = \epsilon^H$); namely, they are the highest weight case of the cyclic irreps. Furthermore when $\epsilon^4 = 1$ the anticommutator $\{Q_+, Q_-\}$ also belongs to the center, suggesting the connection with a Clifford algebra through the mixing of the positive and negative root generators $Q_\pm$. The total fermion number is conserved in the 6-vertex solutions to the Yang-Baxter equation, but it is not in the elliptic regime. Hence a non trivial mixing is needed to represent the elliptic regime. The Hopf algebra $CH_q(2)$ assigns different central elements $[E_1]_q, [E_2]_q$ to the square of the generators $\gamma_1, \gamma_2$ respectively, in such a way that the mixing can only be undone (trigonometric limit) when $E_1 = E_2 = E$. It implies $k = 0$ in (13). For the affine $\widehat{CH}_q(2)$ this limit leads to $U_\epsilon=\iota(\widehat{sl}(2))$ (this statement is only rigorous for the affine case): i.e. $R_{CH_q(2)}$ becomes the $R$–matrix intertwiner for $U_\epsilon=\iota(\widehat{sl}(2))$, provide the labels of the two algebras are related by $\lambda = q^E$.

Using the generators $Q_\pm, \overline{Q}_\pm \in U_\epsilon(\widehat{sl}(2))$, we can define an $N = 2$ supersymmetry
algebra with topological extension $T_{\pm}$ [11, 12]

$$Q_\pm^2 = \overline{Q}_\pm^2 = \{Q_\pm, \overline{Q}_\pm\} = 0$$

$$\{Q_\pm, \overline{Q}_\pm\} = 2T_\pm, \quad |T_\pm| = |E_q|$$

$$\{Q_+, Q_-\} = 2m z^2, \quad \{\overline{Q}_+, \overline{Q}_-\} = 2m z^{-2}$$

(14)

satisfying the Bogomolnyi bound $|T_\pm| = m$. The free fermionic condition (1) ensures the $N = 2$ invariance of the $R$-matrix. Moreover, the $N = 2$ part of the scattering matrix for the solitons of the Ginzburg-Landau superpotential $W = X^{n+1}/(n+1) - \beta X$ [13] is given by $R$-matrices of $U_q(gl(1,1))$ with $q^{2n} = 1$ [14], or equivalently by those of $U_{\epsilon=i}(sl(2))$ between nilpotent irreps with labels $\lambda = \hat{q}$ [15].

The Ginzburg-Landau models have a particular importance in the context of $N = 2$ supersymmetry, since they allow to classify a wide variety of $N = 2$ superconformal field theories [16]. Of great interest are the relevant perturbations of these theories giving massive integrable models, as happens for the superpotential $W(X) = X^{n+1}/(n+1) - \beta X$.

We would like now to make plausible in this context why the supersymmetry algebra (14) has a non-trivial comultiplication. In a $N = 2$ Ginzburg-Landau model, the superpotential enters explicitly in the SUSY commutators through

$$\{Q_+, \overline{Q}_+\} = \Delta W, \quad \{Q_-, \overline{Q}_-\} = \Delta W^*$$

(15)

$$\Delta W = W(X^j) - W(X^i)$$

with $X(-\infty) = X^i$, $X(\infty) = X^j$ and $X^i, X^j$ minima of $W$. Let’s call $K_{(i,j+l)}$ the soliton going from $X^i$ to $X^j$, where $l = j - i$. It is straightforward to see that $\Delta W$ depends on both $l$ and $i$. Naively, the dependence in $i$ was not expected since all the solitons with the same $l$ are equivalent. For the superpotential proposed it is possible to obtain a supersymmetric algebra without this dependence, at the price of reabsorbing it in a non-trivial quantum group comultiplication

$$\Delta(Q_{\pm}) = q^{\pm E} \gamma_3 \otimes Q_\pm + Q_\pm \otimes 1$$

$$\Delta(\overline{Q}_{\pm}) = q^{\mp E} \gamma_3 \otimes \overline{Q}_\pm + \overline{Q}_\pm \otimes 1$$

(16)

On the other hand, it is worth noting the relation of (16) with the fermion number of the solitons. In the solitonic sectors, the fermion number operator acquires a fractional constant piece due to the interaction of the fermionic degrees of freedom with the solitonic
background. The fractional piece of the fermion number in a soliton sector $K_{(i,j)}$, is given by [17, 18]
\[
f = -\frac{1}{2\pi} \left( \text{Im} \ln W''(X) \right) |^{X_i = s} = \frac{s}{n} \quad s = 1, \ldots, n - 1
\] (17)
The relation with $CH_q(2)$ labels is $q^E = e^{i\pi s/n}$. Therefore $q^{±Eγ^3}$ in (16) would be the analog of $e^{±iπF}$, with $F$ the fermion number operator. This interpretation fails for $Δ(\overline{Q}_±)$, where the signs are interchanged, leading in fact to a quantum group structure instead to a Lie superalgebra.

Let us return to building extended supersymmetry algebras from the general $CH_q(2M)$, in the same sense as above. The trigonometric limit of $CH_q(2M)$ is obtained as an independent trigonometric limit in each $CH_q(2)$ piece. Then the affine Hopf algebra $CH_q(2M)$ becomes in essence the anticommuting tensor product of $MU_{ε=1}(sl(2))$ factors, each with its own spectral parameter. Imposing that the eigenvalues of all the central charges $E_i$ and the spectral parameters $z_i (i = 1, \ldots, M)$ coincide, we get $M$ copies of the same structure (14), $\{Q_{±}^{(i)}, Q_{±}^{(i)}, T_{±}^{(i)} = T_{±}\}_{i=1}^M$. Therefore we find an $N = 2M$ supersymmetry algebra with $M$ topological charges. Indeed, the dimension of a $CH_q(2)$ irrep is $2^M$ as is needed to saturate the Bogomolnyi bound $|T_{±}^{(i)}| = |T_{±}| = m$.

Besides, we have seen that the $CH_q(2M)$ irreps can be thought of as collections of $M$ independent solitons $CH_q(2)$. Let us consider the more general trigonometric limit with equal values of the central charges $E_i$, but arbitrary spectral parameters $z_i (i = 1, \ldots, M)$. Then the charges
\[
Q_T^{±} = \sum_{i=1}^M Q_{±}^{(i)} , \quad \overline{Q}_T^{±} = \sum_{i=1}^M \overline{Q}_{±}^{(i)}
\] (18)
verify the commutation relations of $N = 2$ supersymmetry (14). In fact, (14) is satisfied even if we allow different central charges $E_i$. However, in this case the comultiplication doesn’t preserve the expression (18) of $Q_T^{±}, \overline{Q}_T^{±}$.

4. Generalized XY spin chains

The quantum group structure plays an important role in 2-dimensional statistical models, since $R$–matrix intertwiners provide systematic solutions to the integrability condition, the Yang-Baxter equation. In this way integrable models can be built associated to a quantum group, allowing to connect integrability with an underlying symmetry principle. As noted above, the intertwiner $R$–matrix for the Clifford-Hopf algebra $CH_q(2)$
reproduces the 8-vertex free fermion model in magnetic field. In this section we will analyze the model defined by the algebras $\hat{CH}_q(D)$ for general $D = 2M$. Following the transfer matrix method, the study of a 2-dimensional statistical model is equivalent to that of its corresponding spin chain. The L-site hamiltonian for a periodic chain defined by the $\hat{CH}_q (2M)$ Hopf algebras is given by (provided that $R(0) = 1$)

\[
H = \sum_{j=1}^{L} i \frac{\partial}{\partial u} R_{j,j+1}(u) \big|_{u=0} \\
H = \sum_{j=1}^{L} \sum_{n=1}^{M} \{(J^n_x \sigma^n_{x,j} \sigma^n_{x,j+1} + J^n_y \sigma^n_{y,j} \sigma^n_{y,j+1}) \sigma^n_{z,j+1} \sigma^n_{z,j+1} + h^n \sigma^n_{z,j}\}
\]

where $\sigma^n_a$ ($a = x, y, z$, $n = 1, \ldots, M$) are $M$ sets of Pauli matrices, and the constants $J^n_x, J^n_y, h^n$ depend on the quantum labels of the irreps whose intertwiner is $R$

\[
J^n_x = 1 + \Gamma^n, \quad J^n_y = 1 - \Gamma^n \quad n = 1, \ldots, M \\
\Gamma^n = k_n s \psi^n \\
h^n = 2c_n \psi^n
\]

The requirement $R(0) = 1$ implies $\psi_1^n = \psi_2^n = \psi^n$.

The hamiltonian (19) can be diagonalized through a Jordan-Wigner transformation and its excitations behave as free fermions (massless when $J^n_x = J^n_y$ massive otherwise). This model provides $M$ groups of Pauli matrices $\sigma^n_{a,j}$ ($a = x, y, z$) for each site $j$ on the chain, so it behaves as having $M$ layers with an $XY$ model defined in each layer. The factors $(\sigma_{z,j+1}^n \sigma_{z,j}^1 \ldots \sigma_{z,j}^{k-1} + \sigma_{z,j}^1 \ldots \sigma_{z,j+k-1}^1)$ make the fermionic excitations on different layers anticommute. Thus the algebra $CH_q(2M)$ provides a way to put different non-interacting fermions in a chain with a quantum group interpretation.

When $M = 1$, $H$ reduces to the hamiltonian of an $XY$ Heisenberg chain in an external magnetic field $h$, that is the spin chain associated with the 8-vertex free fermion model \[7\]

\[
H = \sum_{j=1}^{L} \{J_x \sigma_{x,j} \sigma_{x,j+1} + J_y \sigma_{y,j} \sigma_{y,j+1} + h \sigma_{z,j}\}
\]

The aim of this section is to show that the model above is equivalent under some restrictions to the generalized integrable $XY$ chain proposed and solved in ref. \[9\],

\[
\tilde{H} = - \sum_{k=1}^{K} \sum_{j=1}^{L'} (\tilde{J}_x^k \sigma_{x,j} \sigma_{x,j+k} + \tilde{J}_y^k \sigma_{y,j} \sigma_{y,j+k}) \sigma_{z,j+1} \ldots \sigma_{z,j+k-1} + h \sum_{j=1}^{L'} \sigma_{z,j}
\]
finding in this way a quantum group structure for this integrable model. The Hamiltonian (22) can also be diagonalized with a Jordan-Wigner transformation and its quasi-particles behaves as free fermions. The main application of the generalized XY model is the problem of covering a surface with horizontal and vertical dimers. Indeed, the ground state of \( \tilde{H} \) for a particular choice of parameters reproduces the two-dimensional pure dimer problem \([9]\), first solved in terms of a Pfaffian \([19]\).

To see the relation between \( H \) and \( \tilde{H} \), let us choose identical XY models on each layer of the former chain

\[
J^n_x = J_x, \quad J^n_y = J_y, \quad h^n = h \quad n = 1, \ldots, M
\]  

(23)

and rearrange the spin labels to form a single-layer chain

\[
\sigma^n_{a,j} = \sigma_{a,j+n} \quad n = 1, \ldots, M; \ a = x, y, z
\]  

(24)

Then the Hamiltonians \( H \) and \( \tilde{H} \) coincide if we set in the latter

\[
\tilde{J}_x^k = -J_x \delta_{M,k}, \quad \tilde{J}_y^k = -J_y \delta_{M,k} \quad k = 1, \ldots, K
\]  

(25)

The general \( \tilde{H} \) (22) is obtained by adding Hamiltonians \( H^{(M)} \) derived from \( CH_q(2M) \) R-matrices. The fact that this sum is also solvable relies on setting equal parameters in each \( H^{(M)} \) (this is the same condition that leads to \( N = 2M \) supersymmetry in the trigonometric limit of \( CH_q(2M) \)). Therefore, the affine quantum Clifford Hopf algebras \( CH_q(2M) \) encode the hidden quantum group for the generalized XY spin chain (22).

5. Comments

We have studied the quantum Clifford algebras \( CH_q(2M) \) in connection with extended supersymmetry and with statistical integrable models.

It is worth noting that the Hamiltonian derived from \( CH_q(4) \) in the trigonometric regime and without magnetic field, is the limiting case \( U \to \infty \) of the two layer chain \([8]\):

\[
H = -\frac{1}{2} \sum_{j=1}^{L} \left\{ (\sigma_x^j \sigma_x^{j+1} + \sigma_y^j \sigma_y^{j+1})(1 - U \tau_z^{j+1}) + (\tau_x^j \tau_x^{j+1} + \tau_y^j \tau_y^{j+1})(1 - U \sigma_z^j) \right\}
\]  

(26)

The coupling between the two layers in this model implies real interaction, so the excitations are not free fermions, and the ground state presents spontaneous magnetization (if \( U \neq 0, \infty \)). It still can be solved by Bethe Ansatz techniques, but a R-matrix interpretation for it is not known. The algebra \( CH_q(4) \) gives us a simple way of coupling
two $XY$ models. Perhaps it would be possible to twist (may be in a way related to a quantum deformation proposed recently for the Clifford algebras [20]) and break the full set of generators to a shorter set giving a quantum group structure for this model.

We have built extended supersymmetric algebras from the $\hat{CH}_q(2M)$ generators in the trigonometric limit. The Clifford Hopf algebras can be thought of as elliptic generalizations of supersymmetry (the anticommutators of charges that give the momentum $P$ and $\overline{P}$ get deformed in the elliptic case, but are still central elements). It would be interesting to analyze what deformation of the Poincaré group one gets in such a way.

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\[
\Delta_{CH_q(4)}(g)((a,i) \otimes (b,j)) = \Delta_{CH_q(2)^{\otimes 2}}(g)
\]

Figure 1: Graphical representation of the expression (6) for \( CH_q(4) \). \((a, i)\) denote the vectors in a \( CH_q(2)^{\otimes 2} \) irrep, the index \( a \) corresponding to the first \( CH_q(2) \) and \( i \) to the second.

\[
R_{CH_q(4)} = \begin{array}{cc}
\bullet & \\
\bullet & \\
\bullet & \\
\bullet & \\
\end{array}
\]

Figure 2: Graphical representation of the \( CH_q(4) \) \( R \)-matrix.