THE SHAFAREVICH CONJECTURE FOR HYPER SURFACES IN ABELIAN VARIETIES

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ABSTRACT. Faltings proved that there are finitely many abelian varieties of genus \( g \) of a number field \( K \), with good reduction outside a finite set of primes \( S \). Fixing one of these abelian varieties \( A \), we prove that there are finitely many smooth hypersurfaces in \( A \), with good reduction outside \( S \), representing a given ample class in the Neron-Severi group of \( A \), up to translation, as long as the dimension of \( A \) is at least 4. Our approach builds on the approach of [10] which studies \( p \)-adic variations of Hodge structure to turn finiteness results for \( p \)-adic Galois representations into geometric finiteness statements. A key new ingredient is an approach to proving big monodromy for the variations of Hodge structure arising from the middle cohomology of these hypersurfaces using the Tannakian theory of sheaf convolution on abelian varieties.

1. Introduction

Fix a number field \( K \) with ring of integers \( \mathcal{O}_K \), and let \( S \) be a finite set of primes of \( \mathcal{O}_K \). Fix an abelian variety \( A \), defined over \( K \), with good reduction at all primes outside \( S \). We say \( H \) has good reduction at \( p \) if the closure of \( H \) in the unique smooth projective model of \( A \) over \( \mathcal{O}_K[1/S] \) is smooth at \( p \). Our main result is the following.

**Theorem 1.1** (Theorem 11.4). Suppose \( \dim A \geq 4 \). Fix an ample class \( \phi \) in the Neron-Severi group of \( A \). There are only finitely many smooth hypersurfaces \( H \subseteq A \) representing \( \phi \), with good reduction outside \( S \), up to translation.

If we fix a Picard class \( \psi \), rather than a Neron-Severi class, this theorem becomes a finiteness result for a Diophantine equation, in principle concrete. (Because there are finitely many Neron-Severi classes in each Picard class up to translation, this is equivalent to the original statement). The hypersurfaces in a given Picard class form a projective space, and the singular ones form a divisor as soon as \( \psi \) is very ample, by a classical result (e.g. [58, Theorem 1.18]) which uses the fact that \( A \) is not ruled by projective spaces. Thus, the singular hypersurfaces are the vanishing locus of some discriminant polynomial \( \Delta(x_1, \ldots, x_n) \) in the coordinates of that projective space. In this case, Theorem 11.4 shows that there are finitely many solutions of the equation \( \Delta(x_1, \ldots, x_N) = u \) with coordinates in \( \mathcal{O}_K[1/S] \), for \( u \) an \( S \)-unit in \( \mathcal{O}_K \). The reverse implication holds if we assume that there are finitely many solutions for each \( u \) in some system of coset representatives of \( S \)-units mod \( \deg \Delta \)th powers of \( S \)-units.

For \( \dim A = 3 \) there are additional combinatorial difficulties, leading to a more complicated result. Let \( a(i) \) be the sequence

\[ 1, 5, 20, 76, 285, 1065, \ldots \]
satisfying
\[ a(1) = 1, a(2) = 5, a(i + 2) = 4a(i + 1) + 1 - a(i) \]

Let \( d(i) \) be the sequence
\[ d(i) = \left( \frac{a(i) + a(i + 1)}{a(i)} \right) \]

so that
\[ d(1) = 6, d(2) = 53130, d(3) = 216182590635135019896, d(4) = 2.5 \times 10^{79}, \ldots \]

**Theorem 1.2** (Theorem 11.5). Suppose \( \dim A = 3 \). Fix an ample class \( \phi \) in the Neron-Severi group of \( A \). Assume that the intersection number \( \phi \cdot \phi \cdot \phi \) is not divisible by \( d(i) \) for any \( i \geq 2 \). There are only finitely many smooth hypersurfaces \( H \subseteq A \) representing \( \phi \), with good reduction outside \( S \), up to translation.

Since \( a(i) \) increases exponentially, \( d(i) \) increases superexponentially. Because of this rapid rate of increase, and because \( d(2) \) is already large, a very small proportion of possible intersection numbers are ruled out by this.

If \( \dim A = 2 \), then hypersurfaces in \( A \) are curves, and our result follows from the Shafarevich conjecture for curves.

Our result is analogous to the Shafarevich conjecture for curves, now a theorem of Faltings [19]: there are only finitely many isomorphism classes of curve of fixed genus \( g \), defined over \( K \), and having good reduction outside \( S \). Similar results are now known for various families of varieties: abelian varieties [19], K3 surfaces [2 and 56], del Pezzo surfaces [52], flag varieties [28], complete intersections of Hodge level at most 1 [27], surfaces fibered smoothly over a curve [26], and Fano threefolds [29].

Javanpeykar and Loughran have suggested that the Shafarevich conjecture should hold in broad generality (see for example [27, Conj. 1.4]; the present result is further evidence in this direction. They show that the Lang–Vojta conjecture implies the Shafarevich conjecture for certain families of complete intersections [27, Thm. 1.5]; their argument uses the hyperbolicity of a certain moduli space. One expects the implication to hold for still more general families of varieties: for any family that gives rise to a locally injective period map, Griffiths transversality and the geometry of period domains imply that the base must be hyperbolic, and the argument of Javanpeykar–Loughran applies. Indeed, in our proof we use a big monodromy statement that may be seen as a strong form of injectivity of the period map.

To understand the relationship between our work and previous work, it is helpful to compare and contrast with two previous finiteness theorems, both due to Faltings, involving abelian varieties. The first is the Shafarevich conjecture for abelian varieties [19], i.e. the result that there are only finitely many isomorphism classes of abelian varieties of dimension \( n \) over \( K \) with good reduction outside \( S \). The second is the result, in [20], that the \( K \)-rational points in any closed subvariety \( Z \) of an abelian variety defined over \( K \) lie in finitely many translates of abelian subvarieties contained in \( Z \).

Both of these have been very useful for proving further arithmetic finiteness theorems. [19] was applied, using the natural maps from the moduli space of curves, certain moduli spaces of K3 surfaces, and moduli spaces of complete intersections
of Hodge level 1 to the moduli space of abelian varieties, to prove most of the Shafarevich-type statements discussed above. Similarly, finiteness results for points on curves over number fields of fixed degree are proven using [20] and the maps from symmetric powers of a curve to the Jacobian abelian variety.

There does not seem to be any logical relation between our work and these two finiteness theorems. There is no reason to believe that there exists a nonconstant map from the moduli space of smooth hypersurfaces \( H \subseteq A \) to any moduli space of abelian varieties for general \( d \) and \( A \), except when \( \dim A = 2 \) and one can take the Jacobian of \( H \). Thus, our result does not seem to follow from [19]. There does exist a map from the moduli space of hypersurfaces to an abelian variety - in fact \( A \) - by sending each hypersurface to its Picard class, but this is surjective so [20] is not helpful. Instead, this map can be used to reduce the finiteness problem to the moduli space of smooth hypersurfaces in a given Picard class, which is an open subset of projective space. Because an open subset of projective space does not have a nonconstant map to any abelian variety, [20] cannot be applied at this point.

Instead, our main result seems to be synergistic with prior finiteness results in abelian varieties. Faltings proved that there only finitely many abelian varieties \( A \) of a given dimension with good reduction outside \( S \). One can check that each of these abelian varieties has only finitely many ample Neron-Severi classes of a given intersection number, up to automorphism. We have proven that each of these ample classes contains only finitely many smooth hypersurfaces with good reduction outside \( S \), up to translation. Finally Faltings proved that each of these hypersurfaces contains only finitely many \( K \)-rational points, outside of finitely many translates of abelian subvarieties.

The present work relies on a study of period maps and Galois representations applicable to cohomology in arbitrary degree [40].

The paper [40] introduces general machinery to bound integral points on a variety \( X \), assuming one can find a family over \( X \) whose cohomology has big monodromy. Suppose \( Y \to X \) is a smooth proper family of varieties, extending to a smooth proper model \( Y \to X \) over \( O_{K,S} \). Then for every integral point \( x \in X(O_{K,S}) \), the étale cohomology of the fiber \( Y_x \) gives rise to a global Galois representation

\[
\rho_x : \text{Gal}_K \to \text{Aut}(H^i_{\text{ét}}(Y_x, \mathbb{Q}_p)).
\]

A lemma of Faltings shows that there are only finitely many possibilities for \( \rho_x \), up to semisimplification. In various settings, it is possible to show that the representation \( \rho_x \) varies \( p \)-adically in \( x \), and deduce that the \( S \)-integral points \( X(O_{K,S}) \) are not Zariski dense in \( X \).

A key input to the methods of [40] is control on the image of the monodromy representation

\[
\pi_1(X, x_0) \to \text{Aut}(H^i_{\text{sing}}(Y_{x_0})).
\]

In order to show that a certain period map has big image, we need to know that the Zariski closure of the image of monodromy is “big” in a certain sense. In particular, in the case studied in this paper the image of monodromy is sufficiently big if its Zariski closure includes one of the classical groups \( \text{SL}_N, \text{Sp}_N, \text{SO}_N \). Because this is the sufficient condition we use in our argument, one can think of “bigness” in terms of classical groups, but the precise definition in [40] is substantially more flexible, which might prove useful elsewhere. For example, Theorem 8.21 requires that the
monodromy group be “strongly $c$-balanced” in the sense of Definition 7.5 as well as two numerical conditions that are more easily satisfied when the monodromy group is larger.

For the monodromy groups of the universal family of hypersurfaces in $A$, there are multiple geometric and topological arguments that could demonstrate that the monodromy contains a classical group. This would be sufficient to prove Zariski nondensity of the integral points. Then, one hopes to pass from Zariski nondensity of $\mathcal{X}(\mathcal{O}_{K,S})$ to finiteness by an inductive argument. (This idea was suggested in [40, Sec. 10.2].) Specifically, if $\mathcal{X}(\mathcal{O}_{K,S})$ is infinite, we may take $Z$ the Zariski closure of $\mathcal{X}(\mathcal{O}_{K,S})$ in $X$, and repeat the argument with $Z$ in place of $X$. However, this requires us to prove large monodromy, not just over $X$, but for every positive-dimensional subvariety $Z \subseteq X$. In our case, we may assume only that $Z$ carries some smooth family of hypersurfaces in $A$ which is not a translate of the constant family.

For most families of varieties, such as hypersurfaces in projective space, this problem would seem totally insurmountable. Either we know almost nothing about the monodromy groups of arbitrary subvarieties, or, as in the universal family of abelian varieties, we can construct explicit subfamilies with too-small monodromy, e.g. low-dimensional Shimura subvarieties. However, working with a family of subvarieties of a fixed abelian variety $A$ provides us with a way out. The inverse image of $H$ under the multiplication-by-$\ell^n$ map $A \to A$ has good reduction everywhere $H$ does, except possibly at $\ell$, and we can run the argument with its étale cohomology. The $\ell^n$-torsion points $A[\ell^n]$ act on this inverse image, and thus on its cohomology; this action splits the cohomology into a sum of eigenspaces, each with its own monodromy representation. It suffices for our purposes to show that one of these representations has big monodromy.

This additional freedom allows a new type of argument, based on the Tannakian theory of sheaf convolution developed by Krämer and Weissauer [38]. They defined a group, the “Tannakian monodromy group”, associated to a subvariety in an abelian variety (and in fact to much more general objects). Its definition is subtler than the definition of the usual monodromy group, but it is a better tool to work with because it depends only on a single hypersurface in the family, whose geometry can be controlled, rather than an arbitrary family of hypersurfaces, whose geometry is far murkier. We prove a group-theoretic relationship between the usual monodromy group of a typical $A[\ell^n]$-eigenspace in the cohomology of a family of hypersurfaces and the Tannakian monodromy group of a typical member of the family of hypersurfaces. One can think of this as analogous to the relationship between the monodromy groups of the generic horizontal and vertical fibers of a family of varieties over (an open subset of) a product $X_1 \times X_2$. Using purely geometric arguments, we show that the Tannakian monodromy group contains a classical group, and then using this relationship, we show that the usual monodromy group does as well.

We believe the problem of proving big monodromy for the restriction of a local system to an arbitrary subvariety to be very difficult without this Tannakian method, but owing to its arithmetic applications, it would be very interesting to look for new examples where this can be established by a different method. Roughly speaking, one looks for a variety $X$ of dimension at least two and a smooth family $Y \to X$ such that for all pointed curves $(Z, z) \subseteq X$, the Zariski closure of the image
of monodromy

$$\pi_1(Z, z) \to \text{Aut } H^1(Y_z)$$

is large. Here “large” means, ideally, that the Zariski closure contains a classical group, but should at least mean that the dimension of the Zariski closure is greater than some constant. Ideally one wants an example where nothing is known already about the the integral points on $X$ - for instance $X$ should not map to the moduli space of abelian varieties or to a hyperbolic curve. At the very least, one should avoid trivial examples, like $X$ a product of curves and $Y$ a fiber product of families over those curves, where the monodromy over any curve is necessarily at least as large as the monodromy over one of the original families. If the variety $X$ has multiple natural smooth families $Y_i \to X$, it suffices to prove that for each $Z, z$ there exists one $Y_i$ with big monodromy.

The methods of this paper can likely be applied to many different classes of subvarieties of abelian varieties, beyond hypersurfaces. To make this generalization, the additional inputs needed are a result giving some control on the Tannakian monodromy group associated to the subvariety and the verification of a certain inequality involving the Hodge numbers and this group (see Lemma 10.1).

1.1. Outline of the proof. The argument of [40] derives bounds on $X(\mathcal{O}_{K,S})$ from a family $f: Y \to X$, through a study of various cohomology objects $R^i f_*(-)$ on $X$. The étale local system $R^i f_*(\mathbb{Q}_p)$ gives rise to the global Galois representations to which Faltings’s lemma is applied; a filtered $\mathbb{F}$-isocrystal coming from crystalline cohomology is used to study the $p$-adic variation of these Galois representations; and a variation of Hodge structure allows one to relate a $p$-adic period map to topological monodromy. The method allows one to conclude that $X(\mathcal{O}_{K,S})$ is not Zariski dense in $X$.

In the present setting, we will need to apply these results to $R^i f_*(L)$, where $L$ is a nontrivial local system on $Y$. To this end, we introduce the notion of Hodge–Deligne system on $X$ (Definition 6.2), a variant of Deligne’s definition of a motive [14]. A Hodge–Deligne system on $X$ consists of a singular local system, an étale local system, a vector bundle with filtration and integrable connection, and a filtered $\mathbb{F}$-isocrystal, equipped with the usual comparison isomorphisms.

For technical reasons (to be explained below), we will assume $K = \mathbb{Q}$. There is no harm in this: if $X$ is a variety over $K$, then its Weil restriction $\text{Res}^K_{\mathbb{Q}} X$ is a variety over $\mathbb{Q}$; we can also arrange that integral points of $X$ correspond to integral points of $\text{Res}^K_{\mathbb{Q}} X$.

We will work with Hodge–Deligne systems that are pure of some fixed weight, with integral Frobenius eigenvalues; by Faltings’s lemma (Lemma 6.43), there are only finitely many possibilities for the semisimplification of the Galois representation

$$\rho_x: \text{Gal}_K \to \text{Aut } H^k(Y_x, \mathbb{Q}_p),$$

as $x$ varies over $X(\mathcal{O}_{K,S})$.

As in [40], we want to show that the fibers of the map

$$x \mapsto \rho_x^{ss}$$

are not Zariski dense. To do this, we consider the map that takes a $p$-adic point $x \in X(\mathbb{Q}_p)$ to a local Galois representation. By $p$-adic Hodge theory, the local
Galois representation
\[
\rho_{x,p}: \text{Gal}_{\mathbb{Q}_p} \to \text{Aut} H^k(Y_x, \mathbb{Q}_p)
\]
determines the filtered \(\phi\)-module
\[
(V_x, \phi_x, F_x) = H^k_{\text{cris}}(Y_x).
\]

The global semisimplification \(\rho_x \mapsto \rho^{ss}_x\) causes substantial technical difficulties; so much so that, in addition to our main argument in Section 8, we include an alternative, simpler version under the additional assumption that every relevant representation \(\rho_x\) is semisimple, in Section 9. For this sketch, to illustrate ideas, let us make the same assumption; that is, let us imagine that every global representation \(\rho_x\) is semisimple. Then there are literally only finitely many possibilities for the isomorphism class \(\rho_x\), so (restricting to the local Galois representation and applying the crystalline Dieudonné functor) there are only finitely many possibilities for the filtered \(\phi\)-module \((V_x, \phi_x, F_x)\), as \(x\) ranges over all integral points. In this simplified setting, we need only show that
\[
\{ x \in X(\mathbb{Q}_p) \mid (V_x, \phi_x, F_x) \cong (V_0, \phi_0, F_0) \}
\]
is contained in a positive-codimension algebraic subset of \(X\).

We recall from [40, §3] some facts about the variation of \((V_x, \phi_x, F_x)\) with \(x\). For \(x\) in a fixed mod-\(p\) residue disk \(\Omega\), the pair \((V_x, \phi_x)\) is constant: these spaces are canonically identified with the crystalline cohomology of the mod-\(p\) reduction of \(Y_x\). The filtration \(F_x\) varies with \(x\). The assignment \(x \mapsto F_x\) defines a \(p\)-adic period map
\[
\Phi_p: \Omega \to \mathcal{H}
\]
to a certain flag variety. Isomorphism classes of triple \((V_x, \phi_x, F_x)\) correspond to orbits of the Frobenius centralizer \(Z(\phi)\) on \(\mathcal{H}\).

Thus we want to control \(\Phi_p^{-1}(Z)\), where \(Z\) is an orbit of the Frobenius centralizer. We’ll have the result we want if we can prove precise versions of the following two conditions.

(a) The Frobenius centralizer is small.

(b) The image of \(\Phi_p\) is not contained in a small algebraic set.

In fact, since we don’t know that the global Galois representations are semisimple, we need a stronger form of (a)

(a’) (See Lemma 8.20) Fix a \(\phi\)-module \((V, \phi)\) and a semisimple global Galois representation \(\rho^{ss}\). Consider all global Galois representations \(\rho\) whose semisimplification is \(\rho^{ss}\), and such that \(D_{\text{cris}}(\rho|_{\mathbb{Q}_p}) \cong (V, \phi, F)\), for some filtration \(F\) on \(V\).

The \(F\) that arise in this way all lie in a subvariety \(Z \subseteq \mathcal{H}\) of low dimension.

Once we have items (a’) and (b) we know that \(X(\mathbb{Z}[1/S])\) is contained in \(\Phi_p^{-1}(Z)\). A \(p\)-adic version of the Bakker–Tsimerman transcendence theorem (Theorem 7.3) will imply that \(X(\mathbb{Z}[1/S])\) is not Zariski dense.

Condition (a’) comes from two ingredients. First, the semilinearity of Frobenius gives an upper bound on its centralizer (Lemma 6.32). Second, the possible subrepresentations of a global Galois representation are constrained by purity (Lemma 8.2), which restricts the structure of local Galois representations coming from global \(\rho\) having a given semisimplification. It is this latter result that requires us to work
over \( \mathbb{Q} \) (or at least a number field that has no CM subfield). As mentioned above, we can always pass to this situation by restriction of scalars.

Condition (b) is a question about the monodromy of the variation of Hodge structure given by \( V \). As mentioned above, we only need a very weak lower bound on the Zariski closure of the monodromy group. We will show that this group is strongly \( \epsilon \)-balanced (Definition 7.5; see Corollary 5.9 and Lemma 7.6 for precise statements). The technical difficulty in Corollary 5.9 is that it applies uniformly to any family of hypersurfaces in an abelian variety; this will allow us to apply Noetherian induction to smaller and smaller subvarieties of \( X \).

It is now crucial that, in our case, \( Y \) is a subvariety of \( A \times X \), with the map \( f \) the restriction of the projection map \( A \times X \) to \( X \). The abelian variety \( A \) has many rank one local systems \( L \), each of which we can pull back to \( Y \), push forward to \( X \), and apply this machinery to. These local systems are associated to characters of the fundamental group \( \pi_1(A) \).

For the Noetherian induction to work, it suffices to have, for each subvariety of \( X \), a local system \( L \) such that \( R^{n-1}f_!(L) \) has big monodromy in our sense. (There are some additional technical conditions that we suppress here to focus on the main difficulty.) In fact we will show big monodromy for almost all rank one local systems \( L \) in a precise sense. To do this, it is necessary to have a framework in which the vector spaces \( R^{n-1}f_!(L)_x \) for different local systems \( L \) can be studied all at once. This is accomplished by the Tannakian theory of sheaf convolution [38].

The fundamental objects of the Tannakian theory of sheaf convolution are perverse sheaves. The fundamental perverse sheaf for us is the constant sheaf on \( Y_x \), pushed forward to \( A \), and placed in degree \( 1 - n \). The vector space \( R^{n-1}f_!(L)_x \) can be recovered from this by applying the functor \( K \mapsto H^0(A,K \otimes L) \). The theory of [38] views (a slight modification of) the category of perverse sheaves on \( A \) as the category of representations of a certain group, where the functors \( K \mapsto H^0(A,K \otimes L) \) are all isomorphic to the functor taking a representation to the underlying vector space. The image of this group on the representation associated to a perverse sheaf \( K \) is the Tannakian monodromy group.

We show that, for \( Z \) a subvariety of \( X \), the Tannakian monodromy group of the constant sheaf on \( Y_x \) contains a classical group for some \( x \in Z \), and if the family \( Y \) over \( Z \) is not equal to a translate of the constant family, then for almost all \( L \), the monodromy group of \( R^{n-1}f_!(L)_x \) contains a classical group. Because translations of the constant family produce only finitely many hypersurfaces up to translation, we can stop the induction there. Hence to make the argument work, it suffices to show for all \( x \) that the Tannakian monodromy group of the constant sheaf on \( Y_x \) contains a classical group. In other words, we must show this for all smooth hypersurfaces \( H \).

To show that the Tannakian monodromy monodromy groups contains a classical group, we use recent results of Krämer [35 36], to reduce to a small number of cases - essentially, the simple algebraic groups acting by their minuscule representations - and then some intricate but elementary combinatorics involving Hodge numbers to eliminate the non-classical cases.

1.2. Sheaf convolution and uniform big monodromy. Given an abelian variety \( A \) over an algebraically closed field, Krämer and Weissauer [37] construct a Tannakian category as a quotient of the category of perverse sheaves on \( A \). A perverse sheaf \( N \) on \( A \) is said to be negligible if its Euler characteristic is zero;
the negligible sheaves form a thick subcategory, and the sheaf convolution category is defined as the quotient of the category of all perverse sheaves by the negligible sheaves. The convolution of two perverse sheaves has negligible perverse homology in nonzero degrees; in other words, it is “perverse up to negligible sheaves,” and convolution defines a tensor structure on this category.

One original motivation for this construction was the Schottky problem [38]. Given a principally polarized abelian variety $A$ (say of dimension $g$, with theta divisor $\Theta$), one wants to know whether $A$ is isomorphic to a Jacobian, say $\text{Jac} C$. In this case, $\Theta$ would be the $(g - 1)$-st convolution power of $C$. Informally, the role of the Tannakian formalism here is to determine whether $\Theta$ is “a $(g - 1)$-st convolution power of something.”

An alternate motivation for the sheaf convolution theory comes from work of Katz. This time, one works with an abelian variety $A$ over a finite field $F_q$. A perverse sheaf $K$ on $A$ has a trace function $f_K$ on $A(F_q)$. Associated to a character $\chi$ of $A(F_q)$ is the character sum $\sum_{x \in A(F_q)} f_K(x) \chi(x)$. Katz showed (in unpublished work analogous to [31]) that the distribution of $\sum_{x \in A(F_q)} f_K(x) \chi(x)$, viewed as a random variable for uniformly random $\chi$, converges to a distribution determined by the Tannakian group, in the limit as $q$ goes to $\infty$. More precisely, the distribution is like the trace of a random element in the maximal compact subgroup of the Tannakian group. To gain some intuition for this, note that given representations $V_1, V_2$, we have $\text{tr}(g, V_1 \otimes V_2) = \text{tr}(g, V_1) \text{tr}(g, V_2)$; that is, taking the tensor product of representations has the effect of multiplying the traces. For the character sums $\sum_{x \in A(F_q)} f_K(x) \chi(x)$, convolution has the same effect:

$$\sum_{x \in A(F_q)} (f_{K_1} * f_{K_2})(x) \chi(x) = \left( \sum_{x \in A(F_q)} f_{K_1}(x) \chi(x) \right) \left( \sum_{x \in A(F_q)} f_{K_2}(x) \chi(x) \right).$$

In other words, convolution of these functions $f_K$ has a similar effect on this sum as tensor product of the representations $V$ has on the trace. It stands to reason that a framework where perverse sheaves correspond to representations, and convolution of sheaves correspond to tensor product of representations, would have relevance to the distribution of the trace. In particular, this should be plausible if one is familiar with Deligne’s equidistribution theorem [13, Theorem 3.5.3], whose proof is similar to the argument Katz uses to establish the relationship between the distribution and the Tannakian group [31, Corollary 7.4].

For non-algebraically closed fields, such as finite fields, we can construct a Tannakian category in almost the same way as Krämer and Weissauer did - again defining negligible sheaves as those with zero Euler characteristic. The key facts, like that the convolution of two perverse sheaves has negligible perverse homology in nonzero degrees, hold over the base field once checked over its algebraic closure.

To relate these two categories, it is convenient to restrict attention to geometrically semisimple perverse sheaves on $A_k$, and to perverse sheaves on $A_{\overline{\mathbb{F}_q}}$ which are summands of the pullback from $A_k$ to $A_{\overline{\mathbb{F}_q}}$. Having done this, we obtain an exact sequence of pro-algebraic groups

$$1 \to G_{\overline{\mathbb{F}_q}} \to G_k \to \text{Gal}_k \to 1$$

where $G_k$ is the Tannakian group of a suitable category of perverse sheaves on $A_k$, $G_{\overline{\mathbb{F}_q}}$ is the Tannakian group of a suitable category of perverse sheaves on $A_{\overline{\mathbb{F}_q}}$, and
Gal\textsubscript{k} is the Tannakian group of the category of \(\ell\)-adic Gal(\(\overline{k}/k\))-representations. We think of this as a close analogue of the usual exact sequence
\[
1 \to \pi_1^{\text{geom}}(X) \to \pi_1^{\text{arith}}(X) \to \text{Gal}(k/k) \to 1
\]
for a variety \(X\) over a field \(k\).

Just like this usual exact sequence, it often has splittings. In our case, splittings arise from local systems \(L\) on \(A\) defined over \(k\), as their cohomology is then a Galois representation, on which Gal\textsubscript{k} acts, and we can check that this action factors through the Tannakian group \(G_k\), giving the splitting.

Fix now a subvariety \(Z\) of the universal family of hypersurfaces in an abelian variety \(A\). Let \(k\) be the field of functions on the generic point of \(Z\). Let \(H\) be the universal hypersurface in \(A\), defined over \(k\). Let \(K\) be the constant sheaf on \(H\), pushed forward to \(A\), placed in degree \(1 - n\) - our perverse sheaf of interest.

Associated to \(k\) is a representation of \(G_k\). The action of \(G_k\) on this representation is a purely geometric object. By geometric methods, we will show that the image of \(G_k\) acting on this representation contains \(SL_N\), \(SO_N\), or \(Sp_N\) as a normal subgroup.

So the image of \(G_k\) on the representation associated to \(k\) contains the same classical group as a normal subgroup. Because the action of Gal\textsubscript{k} in this setting matches the action of the fundamental group, it will suffice for our induction step to show that the action of Gal\textsubscript{k} on the representation associated to \(k\) contains the same classical group as a normal subgroup.

To do this, we construct a battery of tests, each consisting of representations of the normalizer of the classical group, such that any subgroup of the normalizer contains the classical group if and only if it has no invariants on any of these representations. Associated to each of these representations is a perverse sheaf on \(A_k\). We prove a lemma showing that the action of Gal\textsubscript{k} on the cohomology of a perverse sheaf, defined using a generic local system \(L\), has invariants if and only if the perverse sheaf has a very special form. Then we check that the relevant perverse sheaves do not have this special form unless the family of hypersurfaces over \(Z\) is constant, up to translation by a section of \(A\).

Next we describe how we check that the image of the \(G_k\) action on the representation associated to a smooth hypersurface in \(A\) contains a classical group acting by the standard representation as a normal subgroup. This proceeds in two steps. The first step uses very general geometric arguments using deep machinery, and shows that the commutator of the identity component of this image group is a simple algebraic group acting by a minuscule representation. (Recall that a minuscule representation is one where the eigenvalues of the maximal torus action are conjugate under the Weyl group.) The second step eliminates all such pairs of a group and a representation except the standard representations of the classical groups. The first step uses sophisticated machinery (beyond the perverse sheaves and sheaf convolution needed to formulate the statement) but is relatively general and conceptual, while the second uses no additional machinery (except a bit of Hodge theory) but involves an intricate combinatorial argument.

For the first step, we apply results of Krämer that study the characteristic cycle of a perverse sheaf. This is a fundamental invariant of any perverse sheaf on a smooth variety, defined as an algebraic cycle on the cotangent bundle of that variety. For abelian varieties, the cotangent bundle is a trivial vector bundle, which gives it some interesting additional structure - for instance, it naturally maps to a vector space the same dimension as the variety. By examining how the characteristic cycle of a perverse sheaf changes when it is convolved with another perverse
sheaf, Krämer was able to relate the Tannakian group to the characteristic cycle. In particular, he gave criteria for the commutator subgroup of the identity component to be a simple group, and for the representation of it to be minuscule. The fact that our hypersurface is smooth makes its characteristic cycle relatively simple - it is simply the conormal bundle to the hypersurface. This makes Krämer’s miniscu-
leness criterion straightforward to check, but to check simplicity we must make a modification of Krämer’s argument. The reason for this is that Krämer, motivated by the theta divisor and the Schottky problem, assumed that a hypersurface in \( A \) was invariant under the inversion map, while we do not wish to assume this.

For the second step, the exceptional groups and spin groups are not too hard to eliminate, as they only occur for representations of very specific dimensions. The Tannakian dimension in our setting is the topological Euler characteristic of the hypersurface, which we have an explicit formula for. Comparing these, we can see that the problematic cases only occur for curves in an abelian surface, which we have assumed away by taking the dimension of \( A \) at least 4. The only remaining case, except for the good classical cases, is the case of a special linear group acting by a wedge power representation. Here the dimension of the representation is a binomial coefficient, which we cannot immediately rule out. However, we are eventually able to rule it out using a more sophisticated numerical invariant, the Hodge numbers. If the Tannakian group acts on the representation associated to \( H \) by the \( k \)’th wedge power of an \( m \)-dimensional representation, we might except that the Hodge structure on the cohomology of \( H \), or the cohomology of \( H \) twisted by a rank one local system, is the \( k \)’th wedge power of an \( m \)-dimensional Hodge structure. This would place some restrictions on the Hodge numbers. We don’t prove this, but instead prove a \( p \)-adic Hodge-theory analogue, using the Gal\(_k\) action discussed earlier. On the other hand, we can calculate the Hodge numbers of the cohomology of \( H \) twisted by a rank one local system using the Hirzebruch-Riemann-Roch formula. Working this out gives a complicated set of combinatorial relations between the Hodge numbers of the original \( m \)-dimensional Hodge structure. By a lengthy combinatorial argument, we find all solutions of these relations, noting in particular that they occur only for abelian varieties of dimension less than four. This completes the proof.

1.3. Outline of the paper. In Section 2 we compute Euler characteristics of hypersurfaces in an abelian variety, as well as of vector bundles on those hypersurfaces, and use them to calculate the Hodge-Tate weights of the arithmetic local systems we will study in the rest of the paper. After this preamble, the argument proceeds in three parts.

First, we use the sheaf convolution formalism to prove a big monodromy result for families of hypersurfaces. In Section 3 we introduce the sheaf convolution category, a Tannakian category of perverse sheaves on an abelian variety. In Section 4 we investigate the Tannakian group of a hypersurface; we show in many cases that this group must be as big as possible. In Section 5 we relate the Tannakian group to the geometric monodromy group, which gives the big monodromy statement we need.

Sections 6-9 explain how to deduce non-density of integral points, following the strategy in [40]. Section 6 contains some technical preliminaries. We introduce the notion of Hodge–Deligne system, which is closely related to Deligne’s “system of realizations” of a motive, although we include only the realizations that are relevant for our argument. We discuss “\( H^0 \)-algebras”, roughly, algebra objects
in the category of Artin motives with rational coefficients, which we need to express the semilinearity of Frobenius. We also recall some facts from the theory of not-necessarily-connected reductive groups. Section 7 relates the big monodromy statement from Section 5 to the $p$-adic period map, via the theorem of Bakker and Tsimerman [3]. In Section 8, we deduce the non-density of integral points. The argument here involves combinatorics on reductive groups, analogous to [40, §11]. We conclude with Theorem 8.21, which is analogous to Lemma 4.2, Prop. 5.3, and Thm. 10.1 in [10]. The main technical difficulty comes from semisimplification of the global representation; in Section 9 we present a much simpler argument, assuming all the global representations that arise are semisimple.

Finally, we wrap up the proof. In Section 10, we verify the two numerical conditions in the hypotheses of Theorem 8.21. The proof of our main theorem occupies Section 11.

Appendices A and B contain some purely combinatorial calculations involving Eulerian numbers. We prove Prop. 4.11, which is used to show that the representation of the Tannakian group associated to a smooth hypersurface is not the wedge power of a smaller-dimensional representation — the last remaining case where the Tannakian group could be too small.

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Appendix B. Collected inequalities involving Eulerian numbers

References
2. Euler characteristics and Hodge numbers

In this section, we fix an abelian variety $A$ and a hypersurface $H$, and compute various Euler characteristics of $H$ - its arithmetic Euler characteristic, its topological Euler characteristic, and the Euler characteristics of the wedge powers of its cotangent sheaf. Using these, we will calculate the dimension, and Hodge numbers, of the cohomology of $H$ with coefficients in a rank one lisse sheaf. These Hodge numbers will be used at multiple points in our later arguments.

Lemma 2.1. Let $L$ be a line bundle on $A$.

$$\chi(A, L) = c_1(L)^n / n!$$

Proof. By Hirzebruch-Riemann-Roch, the Euler characteristic of the coherent sheaf $L$ is the integral of its Chern character against the Todd class. By definition, the Chern character of $L$ is $e^{c_1(L)} = \sum_{k=0}^{n} c_1(L)^k / k!$. Because the tangent bundle of $A$ is trivial, its Todd class is 1. Integrating is equivalent to taking the degree $n$ term, which is $c_1(L)^n / n!$. \hfill \Box

Lemma 2.2. Let $H$ be a smooth hypersurface in an abelian variety $A$ of dimension $n$. The arithmetic Euler characteristic of $H$ is $(-1)^{n-1}[H]^{n-1}/n!$.

Proof. Using the exact sequence $0 \to \mathcal{O}_A(-H) \to \mathcal{O}_A \to \mathcal{O}_H \to 0$, we observe that

$$\chi(H, \mathcal{O}_H) = \chi(A, \mathcal{O}_A) - \chi(A, \mathcal{O}_A(-H)) = 0 - (-[H])^n / n! = (-1)^{n-1}[H]^{n-1}/n!$$

by Lemma 2.1. \hfill \Box

Lemma 2.3. Let $H$ be a smooth hypersurface in an abelian variety $A$ of dimension $n$. The topological Euler characteristic of $H$ is $(-1)^{n-1}[H]^n$.

Proof. The topological Euler characteristic of $H$ is the top Chern class of the tangent bundle of $H$. Using the exact sequence $0 \to \mathcal{O}(-[H]) \to \Omega^1_A \to \Omega^1_H \to 0$, and the fact that all Chern classes of $\Omega^1_A$ vanish, we see that the top Chern class of $\Omega^1_H$ is $[H]^n$. \hfill \Box

Motivated by Lemma 2.2, we define the degree $d$ of $H$ to be $\frac{[H]^n}{n!}$, which is always a positive integer for $H$ an ample hypersurface.

The Hodge numbers of $H$ can be computed in terms of Eulerian numbers. For a general reference on Eulerian numbers in combinatorics, see [45, Chap. 1].

Lemma 2.4. Let $H$ be a smooth hypersurface in an abelian variety $A$ of dimension $n$. We have

$$(1) \quad \chi(H, \Omega^1_H) = (-1)^{n-1-i} dA(n, i)$$

where $A(n, i)$ is the Eulerian number.

Proof. Let $\mathcal{L} = \mathcal{O}_A(H)$. We have the exact sequence

$$0 \to \mathcal{L}^{-1} \to \mathcal{O}_A \to \mathcal{O}_H \to 0.$$ 

There is also an exact sequence for the cotangent bundle:

$$0 \to N^*_H/A \to \Omega^1_A|_H \to \Omega^1_H \to 0,$$

which because $A$ is an abelian variety, so $\Omega^1_A$ is trivial, reduces (noncanonically) to

$$0 \to \mathcal{L}^{-1}|_H \to \mathcal{O}^n_H \to \Omega^1_H \to 0.$$
Passing to wedge powers gives
\[ 0 \to \mathcal{L}^{-1} \otimes \Omega_{H}^{i-1} \to \Omega_{A|H}^{i} \to \Omega_{H}^{i} \to 0, \]
or, noncanonically,
\[ 0 \to \mathcal{L}^{-1} \otimes \Omega_{H}^{i-1} \to \mathcal{O}_{H}^{(n)} \to \Omega_{H}^{i} \to 0. \]

In what follows we use freely
\[ \chi(\mathcal{L}^r) = r^n \chi(\mathcal{L}). \]
For every line bundle \( E \) on \( A \), we have
\[ \chi(E|_H) = \chi(E) - \chi(E \otimes \mathcal{L}^{-1}). \]
In particular, we have
\[ \chi(\mathcal{L}^r|_H) = (r^n - (r - 1)^n) \chi(\mathcal{L}) \]
and (combining this with (2))
\[ \chi(\mathcal{L}^r \otimes \Omega_{H}^{i}) = \binom{n}{i}(r^n - (r - 1)^n) \chi(\mathcal{L}) - \chi(\mathcal{L}^{r-1} \otimes \Omega_{H}^{i-1}). \]
We deduce that
\[ \chi(\mathcal{L}^r \otimes \Omega_{H}^{i}) = \sum_{j=0}^{i} \binom{n}{i-j} (-1)^j ((r-j)^n - (r-j-1)^n) \chi(\mathcal{L}). \]
In particular, setting \( r = 0 \), we get
\[ \chi(H, \Omega_{H}^{i}) = (-1)^n \sum_{j=0}^{i} \binom{n}{i-j} (-1)^j (j^n - (j+1)^n) \chi(\mathcal{L}) = (-1)^n \sum_{j=0}^{i+1} (-1)^j \left( \binom{n}{i-j} + \binom{n}{i-j+1} \right) j^n \chi(\mathcal{L}) = (-1)^n \sum_{j=0}^{i+1} (-1)^j \binom{n+1}{i+1-j} j^n \chi(\mathcal{L}) \]
We now use the combinatorial identity ([45, Cor. 1.3])
\[ (-1)^n \sum_{j=0}^{i+1} (-1)^j \binom{n+1}{i+1-j} j^n = (-1)^{n-1-i} A(n, i) \]
and Lemma 2.1
\[ \chi(\mathcal{L}) = d \]
to derive (1). □

We will work with lisse rank one sheaves on an abelian variety. It will be convenient to parametrize them by representations of the fundamental group.

**Definition 2.5.** Let \( A \) be an abelian variety over a field \( k \). Fix a character \( \chi : \pi_1^{et}(A_F) \to \mathbb{G}_m^\times \). We define the character sheaf \( \mathcal{L}_{\chi} \) to be the unique rank one sheaf on \( A_F^n \) whose monodromy representation is \( \chi \).

We also have a canonical way to descend these sheaves to \( A_k \):

**Definition 2.6.** Let \( A \) be an abelian variety over a field \( k \). Let \( \chi \) be a character of \( \pi_1^{et}(A_F) \) that is Gal(\( \overline{k}/k \))-invariant. We define the character sheaf \( \mathcal{L}_{\chi} \) to be the unique lisse rank one sheaf on \( A_k \) whose monodromy restricts to \( \chi \) on \( \pi_1^{et}(A_F) \) and whose stalk at the identity has trivial Galois action.
Lemma 2.7. Let $H$ be a smooth hypersurface defined over a field $k$. Let $\chi : \pi_1(A_k) \to \mathbb{Q}_p^\times$ be a finite-order character such that $H^i(X_{k'}, \mathcal{L}_\chi) = 0$ for $i \neq n - 1$.

Then the Hodge-Tate weights of the $\text{Gal}(\overline{k}|k')$ action on $H^{n-1}(X_{k'}, \mathcal{L}_\chi)$ are $0, \ldots, n - 1$ where the multiplicity of the weight $q$ is $dA(n, q)$.

Proof. As a finite order character of $\pi_1(A)$, $\chi$ factors through $A[m]$ for some $m$. Let $X'$ be the inverse image of $X$ under the multiplication-by-$m$ map of $A$. Then we can express $H^*(H_{k'}, \mathcal{L}_\chi)$ as the part of the étale cohomology of $H^*(H_{k'}^*, \mathbb{Q}_p)$ where $A[m]$ acts by the character $\chi$. Applying $p$-adic Hodge theory, we see that the dimension of the Hodge Tate weight $q$ subspace of $H^{p+q}(H_{k'}, \mathcal{L}_\chi)$ is equal to the dimension of the subspace of $H^p(H', \Omega^q_{X'})$ on which $A[m]$ acts by the character $\chi$. By descent, this is the dimension of $H^p(H, \Omega^q_X \otimes L)$ for a torsion line bundle $L$ on $X$. Because $H^{p+q}(H_{k'}, \mathcal{L}_\chi)$ vanishes for $p \neq n - 1 - q$, the dimension of $H^p(X, \Omega^q_X \otimes L)$ for $p = n - 1 - q$ is equal to $(-1)^{n-1-q}$ times the Euler characteristic $\chi(X, \Omega^q_X \otimes L) = \chi(X, \Omega^q_X)$ because, as $L$ is torsion, its Chern class vanishes. By Lemma 2.4 this Euler characteristic is $(-1)^{n-1-q}dA(n, q)$, so the dimension and Hodge-Tate multiplicity are both $dA(n, q)$. \qed
3. Sheaf convolution over a field

A Tannakian category over $\mathbb{Q}_\ell$ is a rigid symmetric monoidal $\mathbb{Q}_\ell$-linear abelian category with an exact tensor functor to the category of vector spaces over $\mathbb{Q}_\ell$. The point of these conditions is that Tannakian categories are necessarily equivalent to the category of representations of some pro-algebraic group (the group of automorphisms of the functor), together with the underlying vector space functor. Thus, associated to each object is some representation of this pro-algebraic group. For such a representation $V$, we refer to the image of the Tannakian group inside $GL(V)$ as the Tannakian monodromy group.

Kri"amer and Weissauer \[37\] constructed a Tannakian category as a quotient of the category of perverse sheaves on an abelian variety over an algebraically closed field (initially of characteristic zero, but Weissauer \[61\] later extended it to characteristic $p$), where the tensor operation is sheaf convolution. We will use the Tannakian monodromy groups from their theory to control usual monodromy groups.

In this section, we check that these Tannakian monodromy groups behave similarly to the usual monodromy groups with respect to the distinction between the geometric and arithmetic monodromy groups. In the setting of the \'{e}tale fundamental group, we can define both geometric and arithmetic monodromy groups, with the geometric a normal subgroup of the arithmetic. We will check that the same works here. The Tannakian group of the category defined by Kri"amer and Weissauer will function as the geometric group, and we will define a Tannakian category of perverse sheaves over a non-algebraically closed field whose Tannakian monodromy groups will function as the arithmetic groups. We will verify that the geometric groups are normal subgroups of the arithmetic groups.

Our construction of the Tannakian category over a non-algebraically closed field will follow a version of the strategy of Kri"amer and Weissauer, and thus will also serve as a very brief review of our construction.

Let $A$ be an abelian variety over a field $k$ of characteristic zero. Let $D^b_c(A, \mathbb{Q}_\ell)$ be the derived category of bounded complexes of $\ell$-adic sheaves on $A$ with constructible cohomology. Define a sheaf convolution functor $*: D^b_c(A, \mathbb{Q}_\ell) \times D^b_c(A, \mathbb{Q}_\ell) \rightarrow D^b_c(A, \mathbb{Q}_\ell)$ that sends complexes $K_1, K_2$ to

$$K_1 * K_2 = a_* (K_1 \boxtimes K_2)$$

for $a: A \times A \rightarrow A$ the group law.

Lemma 3.1. $(D^b_c(A, \mathbb{Q}_\ell), *)$ is a rigid symmetric monoidal category, where the unit object is the skyscraper sheaf at $0$, and the dual of a complex $K$ is

$$K^\vee = [-1]^* D K$$

where $D$ is Verdier duality and $[-1]: A \rightarrow A$ is the inversion map.

Proof. These were proved in \[59, \S 2.1\] (the symmetric monoidality and unit) and \[60, Proposition\] (the rigidity and description of the dual). These results are stated in the case where $k$ is an algebraically closed field, but they proceed without modification in the general case. \[\square\]

Let $\mathcal{P}$ be the category of perverse sheaves on $A$ with $\mathbb{Q}_\ell$-coefficients. Let $\mathcal{N}$ be the subcategory of perverse sheaves with Euler characteristic zero. Let $D^b(\mathcal{N})$ be the category of complexes in $D^b_c(A, \mathbb{Q}_\ell)$ whose perverse homology objects lie in $\mathcal{N}$. 
The Tannakian category will be constructed by combining this rigid symmetric monoidal structure with the abelian structure on the category of perverse sheaves. This requires modifying the category of perverse sheaves slightly because it is not quite stable under convolution. Instead one verifies that it is stable under convolution “up to” \( N \), i.e. that the convolution of two perverse sheaves has all perverse homology objects in nonzero degrees lying in \( N \). This lets us give \( \mathcal{P}/N \) the structure of a rigid symmetric monoidal \( \mathbb{Q}_l \)-linear abelian category.

**Lemma 3.2.**  
(1) Perverse sheaves on \( A \) have nonnegative Euler characters.  
(2) \( N \) is a thick subcategory of \( \mathcal{P} \) (i.e. it is stable under taking subobjects, quotients, and extensions).  
(3) \( \mathcal{D}^b(N) \) is a thick subcategory of \( \mathcal{D}^b_c(A, \mathbb{Q}_l) \) (i.e. for any distinguished triangle with two objects in \( \mathcal{D}^b(N) \), the third one is in \( \mathcal{D}^b(N) \) as well.)  
(4) For \( K_1, K_2 \in \mathcal{D}^b_c(A, \mathbb{Q}_l) \), if \( K_1 \) or \( K_2 \) lies in \( \mathcal{D}^b(N) \), then \( K_1 \ast K_2 \) lies in \( \mathcal{D}^b(N) \).  
(5) For \( K_1, K_2 \in \mathcal{P}, p^\mathcal{H}^i(K_1 \ast K_2) \in N \) if \( i \neq 0 \).

**Proof.** It suffices to check all these statements after passing to \( A\mathbb{F} \), where they were checked by Krämer and Weissauer.

Indeed, for (1) it suffices to pass to \( A\mathbb{F} \) because Euler characteristics are preserved by base change, and the result then follows from [22 Corollary 1.4]. (2) follows immediately from (1) because Euler characteristic is additive in exact sequences, so if the middle of a short exact sequence has Euler characteristic zero, then the sides must have Euler characteristic zero, and vice versa. (3) follows from (2) because each perverse homology object of the third member of the distinguished triangle is an extension of a subobject of a perverse homology object of the first member by a quotient of a perverse homology object of a second member.

For (4) and (5), it suffices to pass to \( A\mathbb{F} \) because sheaf convolution is preserved by base change. Furthermore it suffices to handle the case where \( K_1 \) and \( K_2 \) are irreducible perverse sheaves, and thus in particular are semisimple. Then (4) follows from [37 Prop 10.1 and preceding paragraph]. For (5), this follows from [37 Lemma 13.1].

**Lemma 3.3.**  
(1) Convolution descends to a functor \( \mathcal{D}^b_c(A, \mathbb{Q}_l)/\mathcal{D}^b(N) \times \mathcal{D}^b_c(A, \mathbb{Q}_l)/\mathcal{D}^b(N) \rightarrow \mathcal{D}^b_c(A, \mathbb{Q}_l)/\mathcal{D}^b(N) \).  
(2) The essential image of \( \mathcal{P} \) in \( \mathcal{D}^b_c(A, \mathbb{Q}_l)/\mathcal{D}^b(N) \) is equivalent to \( \mathcal{P}/N \).  
(3) The essential image of \( \mathcal{P} \) in \( \mathcal{D}^b_c(A, \mathbb{Q}_l)/\mathcal{D}^b(N) \) is stable under convolution.  
(4) \( (\mathcal{P}/N, \ast) \) is a rigid symmetric monoidal \( \mathbb{Q}_l \)-linear abelian category.

**Proof.** (1) follows from Lemma 3.2(4).  
(2) follows from Lemma 3.2(2) and [24 Proposition 3.6.1].  
(3) follows from Lemma 3.2(5) because using the \( t \)-structure we see that any object \( K_1 \ast K_2 \) in \( \mathcal{D}^b_c(A, \mathbb{Q}_l) \) satisfying \( p^\mathcal{H}^i(K_1 \ast K_2) \in N \) if \( i \neq 0 \) can be reduced to an object of \( \mathcal{P} \) by taking two mapping cones with objects of \( \mathcal{D}^b(N) \).

For (4), we first check that the rigid symmetric monoidal structure of Lemma 3.1 descends from \( \mathcal{D}^b_c(A, \mathbb{Q}_l) \) to \( \mathcal{D}^b_c(A, \mathbb{Q}_l)/\mathcal{D}^b(N) \) (because isomorphisms and commutative diagrams are preserved by taking quotients) and that it restricts to the full subcategory \( \mathcal{P}/N \). Again, isomorphisms and diagrams restrict to full subcategories, so the only things to check are that the identity object lies in \( \mathcal{P}/N \), the dual of any object in \( \mathcal{P}/N \) lies in \( \mathcal{P}/N \), and the convolution of any two objects in...
\( \mathcal{P}/\mathcal{N} \) lies in \( \mathcal{P}/\mathcal{N} \). The first is because the skyscraper sheaf is perverse, the second because Verdier duality preserves perverse sheaves, and the last is part (3).

For \( \chi \) a character of \( \pi_{\text{et}}^1(A_{\overline{k}}) \), let \( \mathcal{P}^\chi \) be the subcategory of perverse sheaves \( K \) with \( H^i(A_{\overline{k}}, Q \otimes L_\chi) = 0 \) for all \( Q \) a subquotient of \( K \) defined over \( \overline{k} \) and all \( i \neq 0 \), and let \( \mathcal{N}^\chi = \mathcal{P}^\chi \cap \mathcal{N} \).

**Lemma 3.4.**

1. The essential image of \( \mathcal{P}^\chi \) in \( \mathcal{P}/\mathcal{N} \) is equivalent to \( \mathcal{P}^\chi/\mathcal{N}^\chi \).
2. \( \mathcal{P}^\chi/\mathcal{N}^\chi \) contains the unit and is stable under convolution and duality.
3. \( K \mapsto \) \( H^0(A_{\overline{k}}, K \otimes L_\chi) \) is an exact tensor functor from \( \mathcal{P}^\chi/\mathcal{N}^\chi \) to \( \mathbb{Q}_\ell \)-vector spaces.
4. The category \( \mathcal{P}^\chi/\mathcal{N}^\chi \), convolution, and the functor \( H^0(A_{\overline{k}}, K \otimes L_\chi) \) are a rigid symmetric monoidal \( \mathbb{Q}_\ell \)-linear abelian category with an exact tensor functor to \( \mathbb{Q}_\ell \)-vector spaces.

**Proof.**

(1) follows from [37, Lemma 12.3] and the fact that \( \mathcal{P}^\chi \), by construction, is a thick subcategory.

The claims in (2) may be checked after passing to an algebraically closed field. To check that it contains the unit, we must check that the skyscraper sheaf at zero has cohomology only in degree zero, which is obvious. To check that it is closed under duality, it suffices to observe that
\[
H^i(A_{\overline{k}}, [-1]^*DQ \otimes L_\chi) = H^i(A_{\overline{k}}, D(Q \otimes L_\chi)) = H^i(A_{\overline{k}}, Q \otimes L_\chi^*)
\]
so if one vanishes for all \( i \neq 0 \) the other does. That it is closed under convolution is checked in [37, Theorem 13.2].

The claims in (3) may be checked after passing to an algebraically closed field, where they are proved in [37, Theorem 13.2]. Specifically, [37, Proposition 4.1] reduces this to the case where \( \chi \) is trivial. In this case, exactness follows from the long exact sequence of cohomology, which reduces to a short exact sequence because higher and lower cohomology groups vanish, and tensorness follows from the Künneth formula, which gives
\[
H^0(A_{\overline{k}}, \ast_{\text{et}}(K_1 \boxtimes K_2)) = H^0(A_{\overline{k}} \times A_{\overline{k}}, K_1 \boxtimes K_2) = H^0(A_{\overline{k}}, K_1) \otimes H^0(A_{\overline{k}}, K_2)
\]
again using the vanishing of higher and lower cohomology.

(4) It is rigid symmetric monoidal by part (2) and Lemma 3.3(4). It is \( \mathbb{Q}_\ell \)-linear abelian because it is the quotient of a \( \mathbb{Q}_\ell \)-linear abelian category by a thick subcategory. The functor is an exact tensor functor by part (3).

□

Any two fiber functors of the same Tannakian category are equivalent, and give rise to equivalent Tannakian groups, so it will not matter which \( \chi \) we pick. Moreover, by [37, Theorem 1.1], such \( \chi \) exists for any \( K \). So we can refer to the “Tannakian group” of an object without fixing a choice of \( \chi \), as we can always make a choice and the answer is independent of our choice.

For \( A \) an abelian variety over a field \( k \) with algebraic closure \( \overline{k} \), we say that a perverse sheaf \( K \) on \( A \) is geometrically semisimple if its pullback to \( A_{\overline{k}} \) is a sum of irreducible perverse sheaves. If \( k = \overline{k} \) then semisimple and geometrically semisimple are equivalent.
Lemma 3.5. Let $K_1$ and $K_2$ be geometrically semisimple perverse sheaves on $A$. Then $\text{Hom}_{\mathcal{P}/\mathcal{N}}(K_1, K_2)$ is the quotient of the space of homorphisms $K_1 \to K_2$ by the subspace of homomorphisms factoring through an element of $\mathcal{N}$.

Proof. Without loss of generality, we may assume that $K_1$ and $K_2$ are indecomposable. Because $K_1$ is semisimple when pulled back to $\kbar$, the set of isomorphism classes of irreducible components of this pullback must form a single $\text{Gal}(\kbar/k)$-orbit, as otherwise the splitting into orbits would be defined over $k$ and make them fail to be indecomposable. Thus, either all irreducible components of $K_1$ are in $\mathcal{N}$ or none of them are. The same is true for $K_2$.

If all irreducible components of $K_1$ or $K_2$ are in $\mathcal{N}$, then $K_1$ or $K_2$ is in $\mathcal{N}$, so maps in the quotient category are zero and all maps factor through elements of $\mathcal{N}$, and the statement holds.

Thus, we may assume that no irreducible components of $K_1$ and $K_2$ are in $\mathcal{N}$. By definition, $\text{Hom}_{\mathcal{P}/\mathcal{N}}(K_1, K_2)$ is the limit of $\text{Hom}(K_1', K_2')$ where $K_1'$ is a subobject of $K_1$ whose quotient lies in $\mathcal{N}$ and $K_2'$ is a quotient of $K_2$ by an object in $\mathcal{N}$. By assumption, we must have $K_1' = K_1$ and $K_2' = K_2$, and similarly no nonzero map from $K_1$ to $K_2$ factors through an object in $\mathcal{N}$, so the statement holds in this case as well. □

Lemma 3.6. (1) The full subcategory of $\mathcal{P}^\chi/\mathcal{N}^\chi$ consisting of geometrically semisimple perverse sheaves is a Tannakian subcategory of $\mathcal{P}^\chi/\mathcal{N}^\chi$.

(2) The full subcategory of $\mathcal{P}_k^\chi/\mathcal{N}_k^\chi$ consisting of summands of the pullbacks to $\mathcal{A}_k^\chi$ of geometrically semisimple elements of $\mathcal{P}^\chi/\mathcal{N}^\chi$ on $\mathcal{A}_k$ is a Tannakian subcategory of $\mathcal{P}_k^\chi/\mathcal{N}_k^\chi$.

Proof. To prove part (1), we must check that this subcategory contains the unit, and is closed under kernels, cokernels, direct sums, convolution, duals. For the unit, this is clear because the skyscraper sheaf is geometrically semisimple. For kernels and cokernels, by Lemma 3.5 it suffices to check that kernels and cokernels of maps between geometrically semisimple sheaves are geometrically semisimple, which is clear. That sums of geometrically semisimple sheaves are semisimple is obvious, and that duals of geometrically semisimple sheaves are geometrically semisimple follows from double-duality. For convolution, this follows from Kashiwara’s conjecture, proven in [16] and [25].

To prove part (2), the arguments are similar. Because the pullbacks from $\mathcal{A}_k$ to $\mathcal{A}_k^\chi$ of geometrically semisimple perverse sheaves are (geometrically) semisimple, their summands are as well. Hence any kernel or cokernel of a map in $\mathcal{P}_k^\chi/\mathcal{N}_k^\chi$ between them is a kernel or cokernel of an honest map, hence a subobject or quotient, thus (by semisimplicity) a summand, and hence is also a summand of the pullback of the original geometrically semisimple perverse sheaf. A direct sum of summands is a summand of the direct sum, a convolution of summands is a summand of the convolution, and a dual of a summand is a summand of the dual, so the remaining three properties proceed by the same argument. □

Fix an abelian variety $\mathcal{A}$ over $k$ and a character $\chi$ of $\pi_1^{\text{com}}(\mathcal{A})$.

Let $G_k$ be the Tannakian fundamental group of the full subcategory of $\mathcal{P}^\chi/\mathcal{N}^\chi$ consisting of geometrically semisimple perverse sheaves.
Let $G_k'$ be the Tannakian fundamental group of the full subcategory of $\mathcal{P}^{\chi}/\mathcal{N}^{\chi}$ consisting of summands of the pullbacks to $A_k$ of geometrically semisimple perverse sheaves on $A_k$.

Let $\Gal_k$ be the Tannakian group of the category of $\ell$-adic Galois representations over $k$.

**Lemma 3.7.** The group $G_k'$ is a normal subgroup of $G_k$, with quotient $\Gal_k$.

*Proof.* There is a functor from the Tannakian category of Galois representations over $k$ to the geometrically semisimple objects of $\mathcal{P}^{\chi}/\mathcal{N}^{\chi}$ that sends a Galois representation to the corresponding skyscraper sheaf at the identity.

There is a functor from geometrically semisimple perverse sheaves on $A_k$ to summands of pullbacks of geometrically semisimple perverse sheaves to $A_k$, given by pullback to $A_k$.

Because these functors are both exact tensor functors, they define maps $G_k' \to \Gal_k$. We wish to show that this is an exact sequence of groups, i.e. $G_k'$ is a normal subgroup of $G_k$ whose quotient is $\Gal_k$. To do this, we check the criteria of [17, Theorem A.1].

First, to check that $G_k' \to \Gal_k$ is surjective, it suffices to check that the functor from Galois representations to skyscraper sheaves at the origin is full, and that a subquotient of a skyscraper sheaf at the origin is a skyscraper sheaf at the origin. These are both easy to check.

Second, to check that $G_k' \to G_k$ is a closed immersion, we must check that every representation of $G_k'$ is a subquotient of a pullback from $G_k$ to $G_k$ representation of $G_k$. This is automatic, as the Tannakian category of representations of $G_k$ is defined to consist of perverse sheaves that are summands of pullbacks of perverse, geometrically semisimple sheaves on $A_k$ that lie in $\mathcal{P}^{\chi}$, which by definition are representations of $G_k$, and because summands are an example of subquotients.

Third we must check that a perverse sheaf on $A_k$ is a skyscraper sheaf at the origin if and only if is trivial when pulled back to $A_k$. This is obvious.

Fourth, we must check that given a geometrically semisimple perverse sheaf on $A_k$, its maximal trivial subobject over $A_k$ (i.e. the maximal sub-perverse sheaf that is a skyscraper sheaf at the origin) is a subobject over $A_k$. By duality, it is equivalent to check this with quotient objects, where the maximal trivial quotient is simply the stalk at zero of the zeroth homology and hence is certainly defined over $k$.

The fifth condition is simply the second condition with “subquotient” replaced with “subobject”. This follows again because summands are subobjects. □

It is likely possible to prove the analogous theorem, without the “geometrically semisimple” conditions in the definitions of the key Tannakian categories, by a similar but more complicated argument. However, this additional level of generality is not needed for our paper, and so we did not pursue this.

Using the fact that $G_k'$ is a normal subgroup of $G_k$, we will show that the Galois action on our fiber functor normalizes the Tannakian monodromy. First, we need a definition to make this Galois action well-defined:

**Lemma 3.8.** Let $A$ be an abelian variety over a field $k$. Let $\chi$ be a character of $\pi_1^t(A_k)$ that is $\Gal(k/k)$-invariant. Let $\mathcal{L}_\chi$ be the associated character sheaf.

Let $K$ be a geometrically semisimple perverse sheaf on $A$ such that $H^i_c(A_k, A \otimes \mathcal{L}_\chi)$ vanishes for $i \neq 0$. Then

Let $G_k'$ be the Tannakian fundamental group of the full subcategory of $\mathcal{P}^{\chi}/\mathcal{N}^{\chi}$ consisting of summands of the pullbacks to $A_k$ of geometrically semisimple perverse sheaves on $A_k$. Let $\Gal_k$ be the Tannakian group of the category of $\ell$-adic Galois representations over $k$.

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Let $K$ be a geometrically semisimple perverse sheaf on $A$ such that $H^i_c(A_k, A \otimes \mathcal{L}_\chi)$ vanishes for $i \neq 0$. Then
(1) The action of $\text{Gal} ( \mathbb{F} | k )$ on $H^0_c ( A, A \otimes L_\chi )$ normalizes the geometric Tannakian monodromy group of $K$.

(2) The action of $\text{Gal} ( \mathbb{F} | k )$ on $H^0_c ( A, A \otimes L_\chi )$ normalizes the commutator subgroup of the identity component of the geometric Tannakian monodromy group of $K$.

Proof. For (1), note that the Galois group acts by automorphisms of the fiber functor $H^i_c ( A, A \otimes L_\chi )$ of the arithmetic Tannakian category, giving a map $\text{Gal}_k \to G_k$. A trivial calculation involving skyscraper sheaves at the origin shows that this map splits the exact sequence of Lemma 3.7; hence $\text{Gal}_k$ normalizes the geometric Tannakian group. Here it is crucial that, because the category of representations of $G_k$ is a full Tannakian subcategory of $P^\Lambda / N^\Lambda$, Tannakian groups measured by this category are equal to the usual Tannakian groups.

(2) follows from (1) because the commutator subgroup of the identity component is a characteristic subgroup. $\blacksquare$
4. The Tannakian Group of a Hypersurface

Let $A$ be an abelian variety of dimension $n \geq 2$ over the complex numbers and let $H \subset A$ be a smooth hypersurface. Let $i : H \to A$ be the inclusion. Let $G_H$ be the Tannakian group of the perverse sheaf $i_*\mathbb{Q}[n-1]$. Let $d = [H]^n/n!$, which by Lemma 2.2 is an integer, equal to $(-1)^{n-1}$ times the arithmetic Euler characteristic of $H$.

Let $N = (n!)^d = [H]^n$. Recall from the introduction that $a(i)$ is the sequence $1, 5, 20, 76, 285, 1065, \ldots$ satisfying

$$a(1) = 1, a(2) = 5, a(i + 2) = 4a(i + 1) + 1 - a(i)$$

satisfying

$$a(1) = 1, a(2) = 5, a(i + 2) = 4a(i + 1) + 1 - a(i)$$

(OEIS A061278).

Theorem 4.1. Assume that $H$ is not equal to the translate of $H$ by any nontrivial point of $A$. Assume also that none of the following hold:

1. $n = 2$ and $d = \binom{2k-1}{k}$ for some $k > 2$.
2. $n = 2$ and $d = 2^r$ for some $r > 1$.
3. $n = 2$ and $d = 28$.
4. $n = 3$ and $d = a(i)^2 + a(i+1)/6$ for some $i \geq 2$.

Then $G_H$ contains as a normal subgroup either $\text{SL}_N$, $\text{Sp}_N$, or $\text{SO}_N$. If $H$ is not equal to any translate of $[-1]^nH$ then the $\text{SL}_N$ case holds. If $H$ is equal to such a translate, then if $n$ is even $\text{Sp}_N$ holds and if $n$ is odd $\text{SO}_N$ holds.

Remark 4.2. In the exceptional cases, there are only a few other possibilities for $G_H$. Suppose $G_H$ does not contain $\text{SL}_N$, $\text{Sp}_N$, or $\text{SO}_N$. Then in case (1), $G_H$ contains as a normal subgroup $\text{SL}_2$, acting by the representation $\wedge^r$. In case (2), $G$ contains as a normal subgroup some $SO$ acting by a $2^{r+1}$ dimensional spin representation. In case (3) it contains $E_7$ acting by its 56-dimensional representation. In (4) it contains as a normal subgroup $\text{SL}_{a+b}$ acting by the representation $\wedge^b$.

The proof occupies the remainder of this section.

Lemma 4.3. The dimension of the standard representation of $G_H$ is $N$.

Proof. By construction, the dimension of the representation associated to any object in the Tannakian category is the Euler characteristic of the corresponding perverse sheaf, which is $(-1)^{n-1}$ times the topological Euler characteristic of $H$. This now follows from Lemma 2.3. □

Lemma 4.4. Assume that $H$ is not equal to the translate of $H$ by any nontrivial point of $A$. Then the Lie algebra of $G_H$ acts irreducibly on the distinguished representation of $G_H$.

Proof. This is [35, Corollary 1.6]. □

Lemma 4.5. Assume that $H$ is not equal to the translate of $H$ by any nontrivial point of $A$. Then the distinguished representation of $G_H$ is a minuscule representation of the Lie algebra of $G_H$.

Proof. Because $H$ is smooth, the characteristic cycle of $i_*\mathbb{Q}[n]$ is simply the conormal bundle of $H$ with multiplicity 1, hence has a single irreducible component, with multiplicity one. The result then follows from [35, Corollary 1.9] □
Lemma 4.6. Assume $H$ is not translation-invariant by any nontrivial point of $A$. Then the identity component of $G$ is simple modulo center.

Proof. We follow the argument of [36, Theorem 6.2.1], with minor modifications. That theorem is not directly applicable because it assumes that $H$ is symmetric (i.e. stable under inversion), which is not necessarily true here. Thus we restate the proof, which we can also simplify somewhat because our assumption that $H$ is smooth is stronger than the analogous assumption in [36]. First, we review some notation and terminology from [36].

The proof relies on the notion of the characteristic cycle of a perverse sheaf. Classically, the characteristic cycle of a perverse sheaf on a variety $X$ are negligible [36, Definition 1.2.2]. He defines clean under the Gauss map is not dense, and a cycle negligible Gauss map. Krämer considers an irreducible component $n$ of the degree of the Gauss map on that cycle. All such subvarieties are automatically the conormal bundle to an irreducible subvariety of $X$, of arbitrary dimension, their support.

For $A$ an abelian variety, because $T^*A$ is a trivial bundle, we can express it as a product $A \times H^0(A, \Omega_A^1)$. The projection onto the second factor is called the Gauss map. Krämer considers an irreducible component negligible if its image under the Gauss map is not dense, and a cycle clean if none of its components are negligible [36, Definition 1.2.2]. He defines $cc(K)$ for a perverse sheaf $K$ to be the usual characteristic cycle but ignoring any negligible components, making it automatically clean [36, Definition 2.1.1].

The degree of a cycle is the degree of the Gauss map restricted to that cycle. It is manifestly a sum over components of the degree of the Gauss map on that component, which vanishes if and only if the component is negligible.

Clean cycles are determined by their restriction to any open set in $H^0(A, \Omega_A^1)$. In particular, given two such cycles $\Lambda_1, \Lambda_2$, because

$$\dim \Lambda_1 = \dim \Lambda_2 = n = \dim H^0(A, \Omega_A^1)$$

one can find an open set over which both $\Lambda_1$ and $\Lambda_2$ are finite. The fiber product $\Lambda_1 \times_U \Lambda_2$ then maps to $U$ by the obvious projection and to $A$ by composing the two projections $\Lambda_1 \to A, \Lambda_2 \to A$ with the multiplication map $A \times A \to A$. Hence $\Lambda_1 \times_U \Lambda_2$ maps to $A \times U$. Its image has a unique clean extension to $A \times H^0(A, \Omega_A^1)$. Krämer defines this $\Lambda_1 \circ \Lambda_2$ to be this extension [36, Example 1.3.2].

A key property of this convolution product is that $\deg(\Lambda_1 \circ \Lambda_2) = \deg(\Lambda_1) \deg(\Lambda_2)$; as a consequence, if $\Lambda_1, \Lambda_2$ are clean and nonzero, then $\Lambda_1 \circ \Lambda_2$ is nonzero as well.

We are now ready to begin the argument.

Assume for contradiction that the identity component of $G$ is not simple modulo its center. Then its Lie algebra is not simple modulo its center. By [36, Proposition 6.1.1], it follows from this that there exists $m \in \mathbb{N}$ and effective clean cycles $\Lambda_1, \Lambda_2$ on $A$, with $\deg(\Lambda_i) > 1$, such that

$$[m]_c cc(i_* \mathbb{Q}[n]) = \Lambda_1 \circ \Lambda_2$$

where $[m]$ is the multiplication-by-$n$ map. Because $H$ is smooth, the characteristic cycle of $i_* \mathbb{Q}[n]$ is simply the conormal bundle $\Lambda_H$ of $H$, which is irreducible. Its degree is $d \cdot n!$, because the degree of the Gauss map of the conormal bundle to $H$ is the sum of the multiplicities of vanishing of a general 1-form on $H$, which is the Euler characteristic of $H$, which is $d \cdot n!$. In particular, this degree is nonzero, so $cc(i_* \mathbb{Q}[n]) = \Lambda_H$. 


Because $H$ is not translation-invariant, the map from $H$ to its image under $[m]$ is generically one-to-one, and so $[m]_\ast \Lambda_H$ is an irreducible cycle with multiplicity one in the cotangent bundle of $A$. This implies $\Lambda_1$ and $\Lambda_2$ are irreducible: If not, say if $\Lambda_1 = \Lambda_a^1 \circ \Lambda_2$ and $\Lambda_b^1 \circ \Lambda_2$ nonzero, contradicting irreducibility.

It follows that $\Lambda_1$ and $\Lambda_2$ must be the conormal bundles $\Lambda_{Z_1}$ and $\Lambda_{Z_2}$ of varieties $Z_1$ and $Z_2$. Because $\deg(\Lambda_i) > 1$, neither $Z_1$ nor $Z_2$ can be a point, as the conormal bundle to a point is simply an affine space, and its Gauss map is an isomorphism, and thus has degree 1.

Let $Y$ be the image of $H$ under $[m]$. By [36, Lemma 5.2.2] there is a dominant rational maps from $Y$ to $Z_1$ (say), and thus a dominant rational map from $H$ to $Z_1$. Because $H$ is smooth and $Z_1$ is a subvariety of an abelian variety, this dominant rational map automatically extends to a surjective morphism [11, Theorem 4.4.1]. Moreover, by the Lefschetz hyperplane theorem, $A$ is the Albanese of $H$, so the surjective morphism $f_1 : H \to Z_1 \subseteq A$ extends to a homomorphism $g_1 : A \to A$, giving a commutative diagram

$$
\begin{array}{ccc}
H & \xrightarrow{f_1} & Z_1 \\
\downarrow & & \downarrow \\
A & \xrightarrow{g_1} & A
\end{array}
$$

Let $B_1$ be the image of $g_1$. Because $f_1$ is surjective, $Z_1 \subseteq B_1$.

If $Z_1 = B_1$ then $Z_1$ is an abelian variety, and the conormal bundle to any nontrivial abelian subvariety of $A$ has Gauss map degree 0, contradicting $\deg(\Lambda_i) > 1$. Otherwise, by commutativity of the diagram, $H \subseteq g_1^\ast Z_1$. Because $H$ is a hypersurface, it is a maximal proper subvariety of $A$, so $H = g_1^\ast Z_1$. This contradicts ampleness of $H$ unless $g_1$ is finite, and contradicts $H$ not being translation-invariant unless $g_1$ is an isomorphism. This means $Z_1$ and $H$ are isomorphic as subvarieties of an abelian variety. Thus $\deg(\Lambda_{Z_1}) = \deg(T^\ast_{Z_1} Z_1) = \deg(T^\ast_H H)$ and so because $\deg(\Lambda_{Z_1}) \deg(\Lambda_{Z_2}) = \deg(T^\ast_H H)$, we have $\deg(\Lambda_{Z_1}) = 1$, contradicting $\deg(\Lambda_i) > 1$.

Because we have a contradiction in every case, we have shown that $G$ is simple modulo its center.

**Lemma 4.7.** Assume that $H$ is not equal to the translate of $H$ by any nontrivial point of $A$. Then the commutator subgroup of the identity component of $G_H$ is one of the following:

1. $\text{SL}_N, \text{Sp}_N$ or $\text{SO}_N$ acting by their standard representation.
2. $\text{SO}_m$ acting by one of its spin representations, or $E_6$ or $E_7$ acting by their lowest-dimensional representation.
3. $\text{SL}_m$ acting by the representation $\wedge^k$ for some $2 \leq k \leq m/2$.

**Proof.** It follows from the Lemma 4.6 that the commutator subgroup of the identity component of $G_H$ is a simple Lie group. Furthermore, from Lemmas 4.4 and 4.5 its standard representation must be irreducible and minuscule. But the above is an exhaustive list of minuscule representations of simple Lie groups (see e.g. [35, p. 7]).

□
Lemma 4.8. Assume that $G_H$ contains as a normal subgroup one of $SL_N$, $Sp_N$, or $SO_N$. Then it contains $SL_N$ only if $H$ is not equal to any translate of $[-1]^*H$, it contains $Sp_N$ only if $n$ is even, and it contains $SO_N$ only if $n$ is odd.

Proof. This is essentially the argument of [38, Lemma 2.1] in a different setting.

Note first that the standard representation of any subgroup of $GL_N$ which contains $Sp_N$ or $SO_N$ as a normal subgroup is equal to the tensor product of its dual representation with a one-dimensional representation, since it is contained in the normalizer $GSp_N$ or $GO_N$ respectively. Conversely, if $N > 2$ then the standard representation of any subgroup of $GL_N$ which contains $SL_N$ as a normal subgroup is not equal to the tensor product of its dual representation with any one-dimensional representation.

Translating into the language of the Tannakian category, we see that under this assumption, $G_H$ contains $SL_N$ as a normal subgroup if, and only if, the perverse sheaf $i_* Q[n-1]$ is not isomorphic, up to negligible factors, to the convolution of its dual $[-1]^* Di_* Q[n-1] = [-1]^* i_* Q[n-1]$ with any perverse sheaf corresponding to a one-dimensional representation. Now perverse sheaves corresponding to a one-dimensional representation are always skyscraper sheaves [37, Proposition 10.1], and convolution with a skyscraper sheaf is equivalent to translation, so it is equivalent to say that $i_* Q[n-1]$ is not isomorphic, up to negligible factors, to any translate of $[-1]^* i_* Q[n-1]$. Because $i_* Q[n-1]$ and $[-1]^* i_* Q[n-1]$ are both irreducible perverse sheaves, there can be no negligible factors, and so this happens if and only if they are isomorphism. Because $i_* Q[n-1]$ and $[-1]^* i_* Q[n-1]$ are each constant sheaves on their support, they are isomorphic if and only if their supports are equal, which happens exactly when $H$ is equal to a translate of $[-1]^* H$. This handles the $SL_N$ case.

To distinguish $Sp_N$ and $SO_N$, we observe that the second tensor power of the standard representations of both $GSP_N$ and $GO_N$ contain a one-dimensional summand, but for $GSP_N$ it is contained in the $\wedge^2$ part while in $GO_N$ it is contained in the $Sym^2$ part. More specifically this one-dimensional summand is the same one-dimensional representation from the previous argument, so it is equal to the skyscraper sheaf $\delta_a$ supported at a point $a$ such that $H = a - H$. The second tensor power of the standard representation corresponds to the convolution

$$(i_* Q[n-1]) * (i_* Q[n-1]) = (mult \circ (i \times i))_* Q[2n-2].$$

The relevant skyscraper sheaf must equal the degree 0 part of the stalk of $(mult \circ (i \times i))_* Q[2n-2]$ at $a$, which is the shift by $2n-2$ of the cohomology of the fiber over $a$ of the map $mult \circ (i \times i) : H \times H \to A$. Because $H = a - H$, this fiber is a diagonally embedded copy of $H$, of dimension $n-1$, so the degree 0 part of the shift by $2n-2$ of its cohomology is its top degree cohomology. The involution switching the two copies of $(i_* Q[n-1])$ acts on this cohomology group by its action on the top cohomology of this fiber, which is trivial, twisted by the standard $(-1)^{n-1}$ from the Koszul sign rule for the tensor product of two complexes. Hence if $n$ is odd, that involution fixes this skyscraper sheaf, so the skyscraper sheaf lies in $Sym^n$, and the normal subgroup must be $SO_N$, and if $n$ is even, the involution acts by $(-1)$, so the skyscraper sheaf lies in $\wedge^2$, and the normal subgroup must be $Sp_N$. 

□
To prove the main theorem, it remains to give a complete list of $n, d$ for which $G_H$ can contain as a normal subgroup one of the groups in Lemma 4.7 cases (2) or (3).

Case (2) is relatively easy as for these groups the dimensions have a special form.

**Lemma 4.9.** Let $A, H$, and $G_H$ be as in the setting of Theorem 4.7 so that the commutator subgroup of the identity component of $G_H$ is one of the groups listed in Lemma 4.7. Suppose it is one of the groups in case (2): $SO_m$ acting by a spin representation, or $E_6$ or $E_7$ acting by the standard representation. Then $n = 2$ and $d$ is equal to $28$ or equal to $2r$ for some $r > 1$.

**Proof.** The spin representations have dimension a power of 2, and have exceptional isomorphisms to $SL_2, Sp_4$, or $SL_4$ as long as their dimensions are $\leq 4$. For a power of two greater than four to equal $(n!d)$, we must have $n = 2$ and $d$ a power of 2 greater than two. The standard representations of $E_6$ and $E_7$ have dimension 27 and 56 respectively. We can never have $(n!d) = 27$ and we can only have $(n!d) = 56$ if $n = 2$ and $d = 28$. $\square$

The remainder of the section is devoted to restricting the case of wedge powers. We will obtain further numerical obstructions by introducing a Hodge torus into our Tannakian group. The action of the Hodge torus is obtained using the Galois action on $H^{n-1}(X_T, \mathcal{L}_\chi)$ and $p$-adic Hodge theory, and thus relies on Lemma 3.8 and our earlier construction of a Tannakian category of perverse sheaves over a non-algebraically closed field. An alternate approach should be possible, using classical Hodge theory and a Tannakian category of mixed Hodge modules on $A_C$, as suggested by [37] Example 5.2, but we did not take this approach as we found the Galois action useful elsewhere in the argument.

**Lemma 4.10.** Let $H$ be a hypersurface of “degree” $d$ on an abelian variety $A$ of dimension $n$. Suppose that the commutator subgroup of the identity component of $G_H$ is $\wedge^k SL_m$. Then there exists a function $m_H$ from the integers to the natural numbers and an integer $s$ such that

$$
\sum_{m_H(i) = m} \prod_i \binom{m_H(i)}{m_S(i)} = dA(n, q).
$$

(3)

Here we use the convention that $A(n, q) = 0$ unless $0 \leq q < n$.

**Proof.** By our assumption on the Tannakian group $G_H$ and Lemma 3.8(2), it follows that the Galois action on $H^{n-1}(X_T, \mathcal{L}_\chi)$ lies in the normalizer of $SL_m$. This normalizer is $GL_m/\mu_k$ if $k \neq m/2$ and $GL_m/\mu_k \times (Z/2)$ if $k = m/2$.

The category of Galois representations generated by $H^{n-1}(X_T, \mathcal{L}_\chi)$ under tensor products, direct sums, subobjects, and quotients is a Tannakian category isomorphic to the category of representations of the Zariski closure of the image of $\text{Gal}(\bar{k}/k')$ acting on $H^{n-1}(X_T, \mathcal{L}_\chi)$. Because $H^{n-1}(X_T, \mathcal{L}_\chi)$ is Hodge-Tate and Hodge-Tate representations are closed under all these operations, all representations in this category are Hodge-Tate, and so the functor to a graded vector space provided by $p$-adic Hodge theory is an exact tensor functor. Because the category of graded vector spaces is isomorphic to the category of representations of $G_m$, by
Tannakian duality there is a map from $\mathbb{G}_m$ to the Zariski closure of the image of \( \text{Gal}(k'|k) \), whose weights are the Hodge-Tate weights of \( H^{n-1}(X_{\mathbb{F}}, L_\chi) \).

Thus we have a map from $\mathbb{G}_m$ to $GL_m/\mu_k$ (or its semidirect product with $\mathbb{Z}/2$). Because $\mathbb{G}_m$ is connected, this defines a map $\mathbb{G}_m \to GL_m/\mu_k$. Its weight multiplicities on the representation $\wedge^k$ must equal the Hodge-Tate multiplicities of $H^{n-1}(X_{\mathbb{F}}, L_\chi)$.

Homomorphisms $\mathbb{G}_m \to GL_m$ are parameterized by their weights, an $m$-tuple of integers with $w_1 \leq \cdots \leq w_m$. We can parameterize homomorphisms $\mathbb{G}_m \to GL_m/\mu_k$ by tuples $w_1, \ldots, w_m, s$ of integers $w_1 \leq \cdots \leq w_m$ and $0 \leq s \leq m-1$, where the “weights” of the standard representation are $w_1 - \frac{s}{k}, \ldots, w_m - \frac{s}{k}$.

Then the weights of the representation $\wedge^k$ are exactly $(\sum_{i \in S} w_i) - s$ for all $S \subseteq \{1, \ldots, m\}$ with $|S| = k$.

It follows that the multiplicity of the weight $q$ inside $\wedge^k$ of the standard representation is the number of subsets $S \subseteq \{1, \ldots, m\}$ with $|S| = k$ and $\sum_{i \in S} w_i = s + q$.

To calculate this, let $m_H(j)$ be the number of $i$ such that $w_i = j$. For $S \subseteq \{1, \ldots, m\}$, let $m_S(j)$ be the number of $i \in S$ such that $w_i = j$. Then $|S| = \sum_j m_S(j)$ and $\sum_{i \in S} w_i = \sum_j m_S(j)$. Furthermore the number of sets $S$ attaining a given function $m_S$ is $\prod_k \binom{m_H(j)}{m_S(i)}$.

The stated identities then follow from Lemma 2.7.

To complete the argument, we have the following purely combinatorial proposition, which is proven in Appendix A. Recall from the introduction

**Proposition 4.11** (Appendix A). Suppose that there exists a natural number $k$, function $m_H$ from the integers to the natural numbers and an integer $s$ such that $1 \leq k \leq -1 + \sum_i m_H(i)$ and satisfying Equation (3) for all $q \in \mathbb{Z}$. Then we have one of the cases

1. $m = 4$ and $k = 2$
2. $m = 2$ and $d = \binom{2k-1}{k}$ for some $k > 2$.
3. $n = 3$ and $d = \binom{a(i)+a(i+1)}{a(i)}/6$ for some $i \geq 2$. (Here $a(i)$ is defined by $a(1) = 1, a(2) = 5, a(i+2) = 4a(i+1) + 1 - a(i)$.)

We are now ready to prove the main theorem of this section:

**Proof of Theorem 4.1**. The commutator subgroup of the identity component of $G_H$ must be one of the groups listed in the three cases of Lemma 4.7. In case (1), we know immediately that $G_H$ contains as a normal subgroup one of $SL_N$, $Sp_N$ or $SO_N$, and we conclude by Lemma 4.8.

We rule out case (2) by Lemma 4.9 and case (3) by Proposition 4.11. Note that Proposition 4.11 does not rule out the case $m = 4, k = 2$, but $\wedge^2 SL_4$ is $SO_6$ so $G_H$ contains $SO_6$ as a normal subgroup in this case.

\[\square\]
5. Big monodromy from big Tannakian monodromy

Let $X$ be a variety, $A$ an abelian variety, and $Y \subseteq X \times A$ a family over $X$ of smooth hypersurfaces in $A$, with $X$, $A$, $Y$ all defined over $\mathbb{C}$. Let $n = \dim A$, so that $n-1$ is the relative dimension of $Y$ over $X$. Let $\eta$ be the generic point of $X$ and $\overline{\eta}$ the geometric generic point. Let $f : Y \to X$ and $g : Y \to A$ be the projections.

Let $i : Y_\eta \to A_\eta$ be the inclusion, and let $K$ be the perverse sheaf $K = i_* \mathbb{Q}_l[n-1]$ on $A_\eta$. Let $G$ be the Tannakian group of $K$. Let $G^*$ be the commutator subgroup of the identity component of $G$.

**Lemma 5.1.** Assume that $G^*$ is a simple algebraic group acting by an irreducible representation, and that $Y$ is not equal to a constant family of hypersurfaces translated by a section of $A$.

Let $K'$ be an irreducible perverse sheaf on $A_\eta$ in the Tannakian category generated by $K$. Assume that $K'$ is a pullback from $A_C$ to $A_\eta$ of a perverse sheaf on $A_C$. Then $G^*$ acts trivially on the irreducible representation of $G$ corresponding to $K'$.

**Proof.** Let $pr_1, pr_2 : A_{\eta \times 0} \to A_\eta$ be the two natural projection maps. The Tannakian monodromy groups of $pr_1^*K$ and $pr_2^*K$ are both $G$, so the Tannakian monodromy group of $pr_1^*K + pr_2^*K$ is a subgroup of $G \times G$ whose projection onto each factor is surjective. By Goursat’s lemma, there are normal subgroups $H_1, H_2$ in $G$ and an isomorphism $a : (G/H_1) \to (G/H_2)$ such that the Tannakian group of $pr_1^*K + pr_2^*K$ is

$$\{(g_1, g_2) \in G \times G | a(g_1) = g_2 \mod H_2\}.$$ 

Note that $G$ has a unique factor in its Jordan-Hölder decomposition which is a nonabelian connected simple group. Hence this factor appears either in $H_1$ and $H_2$ or in $G/H_1$ and $G/H_2$.

In the first case, we must have $G^* \subseteq H_1$ and $G^* \subseteq H_2$. This is because, if the nonabelian connected simple factor appears in $H_i$, then it must appear in $H_i \cap G^*$ as $H_i/(H_i \cap G^*) \subseteq G/G^*$ which, modulo scalars, is contained in the outer automorphism group of $G^*$ and thus is virtually abelian and cannot contain a nonabelian connected simple factor. Furthermore $H_i \cap G^*$ is a normal subgroup of $G^*$, and since it cannot be a finite group, it must be $G^*$.

Now, using the fact that $G^* \subseteq H_1$ and $G^* \subseteq H_2$, we will show that $G^*$ acts trivially on the irreducible representation corresponding to $K'$. To do this, observe that $pr_1^*K'$ and $pr_2^*K'$ are isomorphic. These correspond to two representations of the Tannakian group of $pr_1^*K + pr_2^*K$ that factor through the projection onto the first and second factors respectively. Because $G^*$ lies in $H_1$ and $H_2$, the Tannakian group of $pr_1^*K + pr_2^*K$ contains two copies of $G^*$. The first copy of $G^*$ acts trivially on $pr_2^*K'$, so it must act trivially on $pr_1^*K'$, so $G^*$ acts trivially on $K'$, as desired.

In the second case, $H_1$ and $H_2$ must both be contained in the scalars. To see this, because the scalars are the centralizers of $G^*$, it suffices to show that the image of $H_i$ in the automorphisms of $G^*$ vanishes. Equivalently, we must show that the image of $H_i$ in the automorphisms of the Lie algebra $G^*$ vanishes. This automorphism group is an extension of the finite outer automorphism group of $G^*$ by $G^*$ mod its center. Because the image of $H_i$ in the automorphism group is normalized by $G^*$, it either contains $G^*$ or is finite, and it cannot contain $G^*$, so it is finite. Because it is finite and normalized by $G^*$, it commutes with $G^*$. Because the Lie algebra of $G^*$ is an irreducible representation of $G^*$, this forces the image of $H_i$ to act as scalars. But there are no nontrivial scalar automorphisms of a nonabelian Lie algebra, as
they would never preserve any equation \([x, y] = z\), and so the image of \(H_1\) is trivial, as desired.

Now, using the fact that \(H_1\) and \(H_2\) are both contained in the scalars, we will derive a contradiction. Thus the Tannakian group \(pr_1^* K + pr_2^* K\) is contained in the set \(\{(g_1, g_2) \in G^2 | a(g_1) = g_2\}\) for some automorphism \(a\) of \(G\) mod scalars. Let us first check that the automorphism \(a\) is inner. To do this, let \(\eta_m\) be the geometric generic point of \(X^m\). Let \(pr_1, \ldots, pr_m\) be the projections to \(\eta\). For each \(i\) we have a homomorphism \(\rho_i\) from the Tannakian group of \(\bigoplus_{i=1}^m pr_i^* K\) to the Tannakian group of \(pr_i^* K\) modulo scalars. Then because for any pair \(i, j\) there is a projection \(X^m \to X^2\) onto the \(i\)th and \(j\)th copies of \(X\), \(pr_i^* K + pr_j^* K\) is isomorphic to \(pr_i^* K + pr_j^* K\), so its Tannakian group is equal to the Tannakian group of \(pr_i^* K + pr_j^* K\). Hence for any \(i\) and \(j\), \(\rho_i\) and \(\rho_j\) are equal up to an automorphism of \(G\) mod scalars. Hence if \(m\) is greater than the order of the outer automorphism group of \(G\) mod scalars, there are \(i\) and \(j\) such that \(\rho_i\) equals \(\rho_j\) up to an inner automorphism. So in fact the maps from the Tannakian group of \(pr_i^* K + pr_j^* K\) to the Tannakian groups of \(pr_i^* K\) and \(pr_j^* K\) are equal, mod scalars up to an inner automorphism, which we can ignore because the Tannakian group is well-defined only up to inner automorphisms in the first place.

Now the representation associated to \(pr_1^* K \boxtimes [-1]^m Dpr_2^* K\) is the standard representation of the first \(G\) tensored with the dual of the standard representation of the second \(G\). Because the Tannakian group consists of pairs \((g_1, g_2)\) where \(g_2 = \lambda g_1\) for a scalar \(\lambda\), this tensor product contains a one-dimensional subrepresentation \(\chi\). (Viewing this tensor product as the space of endomorphisms of the standard representation, the one-dimensional subrepresentation consists of scalar endomorphisms). Because \(\chi\) admits a nontrivial map to \(\text{std}_1 \otimes \text{std}_2\), we have a natural map \(\chi \otimes \text{std}_2 \to \text{std}_1\), which must be an isomorphism because both sides are irreducible. Any one-dimensional representation of the Tannakian group must be a skyscraper sheaf \(\delta_x\) [37 Proposition 10.1], so we have an isomorphism \(K_2 \ast \delta_x = K_1\) for some \(x \in A(\eta \times \eta)\). Considering the support, we see that the translation of \(Y_m\) by \(x\) is \(Y_n\). Spreading out this identity and then specializing \(\eta_2\) to a sufficiently general point, we see that \(Y\) is generically the translation of a constant variety by a section \(x\) of \(A\). We can extend this section to some open set, and then \(Y\) over some open set will be the translation of a constant variety by a section, and then because \(Y\) is a smooth proper family this will be true globally, contradicting the assumption. □

**Lemma 5.2.** Assume that \(G^*\) is a simple algebraic group acting by an irreducible representation, and that \(Y\) is not equal to a constant family of hypersurfaces translated by a section of \(A\).

Let \(c\) be a positive integer, and let \(i_1, \ldots, i_c\) be the inclusions of \(A\) into \(A^c\) that send \(A\) to one of the \(c\) coordinate axes.

Then the Tannakian monodromy group of \(\bigoplus_{j=1}^c i_{j*} K\) contains \((G^*)^c\), acting by the sum of \(c\) copies of the standard representation of \(G^*\), as a normal subgroup, and this normal subgroup acts trivially on any representation of the Tannakian monodromy group corresponding to a perverse sheaf that is pulled back from \(A^c_\eta\) to \(A^c_\eta\).

**Proof.** Because \(i_{j*}^c\) has Tannakian group \(G\) for all \(j\), the Tannakian group of \(\bigoplus_{j=1}^c i_{j*} K\) is contained in \(G^c\). Let us check that its Tannakian group is actually equal to \(G^c\). To do this, it suffices to show that each irreducible representation
of $G^c$ remains irreducible when restricted to the Tannakian group, and that nonisomorphic irreducible representations remain nonisomorphic. (This implication follows from [15] Proposition 2.21(a) where the first hypothesis is automatic and for the second hypothesis, because $G$ is reductive, we may restrict to irreducible representations.) Each irreducible representation of $G^c$ is the tensor product of an ordered tuple of $c$ irreducible representations $V_1, \ldots, V_c$ of $G$. Each of these irreducible representations $V_j$ of $G$ corresponds to an irreducible perverse sheaf $K_j$ on one of the copies of $A$, and the associated perverse sheaf on $A^c$ is

$$i_1^*K_1 \ast i_2^*K_2 \ast \cdots \ast i_c^*K_c = K_1 \boxtimes K_2 \boxtimes \cdots \boxtimes K_c.$$ 

Because the external tensor product of irreducible perverse sheaves is irreducible and perverse, the representation remains irreducible. Similarly, the external tensor product of two tuples of irreducible perverse sheaves, not elementwise isomorphic to each other, will remain nonisomorphic.

Because $G^c$ is the Tannakian group of $\bigoplus_{j=1}^c i_j^* K$, it contains $(G^*)^c$ as a normal subgroup.

Now let us check that any perverse sheaf corresponding to an irreducible representation which is a pullback of a perverse sheaf from $A^c_\eta$ to $A^\eta_\ell$ is trivial on $(G^*)^c$. Again, such a perverse sheaf is an external tensor product $K_1 \boxtimes K_2 \boxtimes \cdots \boxtimes K_c$ of perverse sheaves $K_j$ on $A^\eta_\ell$ corresponding to irreducible representations $V_j$ of $G$, and the only way it can be a pullback is if all these perverse sheaves $K_j$ are individually pullbacks, since we can recover the individual perverse sheaves by restricting to horizontal or vertical fibers and taking irreducible components. But then by Lemma 5.1, $G^*$ acts trivially on all the $V_j$, so $(G^*)^c$ acts trivially on $V_1 \otimes \cdots \otimes V_c$. \hfill $\square$

We say that the dual torus of $\pi_1(A)$ is the set of characters $\pi_1(A) \to \mathbb{C}^\times$. Fixing an isomorphism $\mathbb{Q}_\ell^\times$ and $\mathbb{C}$, the dual torus is equal to the set of characters $\pi_1(A) \to \mathbb{Q}_\ell^\times$, and thus contains the set of continuous characters $\pi_1^G(A) \to \mathbb{Q}_\ell^\times$ as a subgroup.

We call the set of characters of $\pi_1(A)$ trivial on the fundamental group of a nontrivial abelian subvariety of $A$ a proper subtorus of the dual torus of $\pi_1(A)$.

Note that if a continuous character of $\pi_1^G(A)$, when viewed as a character of $\pi_1(A)$, is trivial on the topological fundamental group of an abelian subvariety of $A$, then it is trivial on the étale fundamental group of that abelian subvariety, because the topological fundamental group is dense in the étale fundamental group.

**Lemma 5.3.** Let $K \in D^b(A_{\mathbb{C}(\eta)}, \mathbb{Q}_\ell)$ be a perverse sheaf of geometric origin. If no irreducible component of $K$ is a pullback from $A_{\mathbb{C}}$, then for all characters $\chi : \pi_1^G(A) \to \mathbb{Q}_\ell^\times$ outside a finite set of torsion translates of proper subtori of the dual torus to $\pi_1(A)$, we have $H^i(A_{\mathbb{C}(\eta)}, K \otimes L_\chi) = 0$ for $i \neq 0$ and \(H^0(A_{\mathbb{C}(\eta)}, K \otimes L_\chi) \text{Gal}(\mathbb{C}(\eta)/\mathbb{C}(\eta)) = 0\).

**Proof.** The first claim is a statement of the generic vanishing theorem of Kramer and Weissauer [37] Lemma 11.2. (This is stated for varieties over $\mathbb{C}$ and singular cohomology, but we may embed $\mathbb{C}(\eta)$ into $\mathbb{C}$ and then base change from the étale to the analytic site).

The second claim follows from the same theorem, but indirectly. By restricting to an open subset of $X$, we may assume $X$ is smooth. Let $m$ be the dimension of $X$. We may spread $K$ out (using the fact that it is of geometric origin) to a sheaf
$K'$ over $A \times X$ such that $K'[m]$ is perverse. Let $\pi : A \times X \to A$ and $\rho : A \times X \to X$ be the projections.

We will prove the second claim by contradiction. We will first assume that $\left( H^0(A_{C(\eta)}, K \otimes L_\chi) \right)^{\text{Gal}(C(\eta) \backslash C(\eta))} \neq 0$ for a particular $\chi$ such that $H^i(A_{C(\eta)}, K \otimes L_\chi) = 0$ for $i \neq 0$, and derive some conclusions from this. We will then define a finite set of torsion translates of proper subtori of the dual torus, assume that this nonvanishing holds for some $\chi$ outside their union, and derive a contradiction from that.

Let us first see how to interpret the nonvanishing of monodromy invariants in terms of the perverse sheaf $K'$. This will essentially be the usual observation that sheaves with monodromy invariants have global sections, and thus have nonzero $H^0$. Additional care must be taken because these is again a finite set.

The stalk of $\rho_*(K' \otimes L_\chi)$ at the generic point is the complex $H^*(A_{C(\eta)}, K \otimes L_\chi)$. By the assumption that this cohomology group vanishes in degree $0$, the stalk of $\rho_*(K' \otimes L_\chi)$ at the generic point is supported in degree 0. There is some open subset of $Y$ over which $\rho_*(K' \otimes L_\chi)$ remains a lisse sheaf in degree 0, and the Galois action matches the monodromy action of that open subset. By the decomposition theorem, $\rho_*(K' \otimes L_\chi)$ is a sum of shifts of irreducible perverse sheaves. In particular, this monodromy action is semisimple.

Now we assume that the Galois invariants are nonzero. It follows that the monodromy invariants are nonzero, and thus, by semisimplicity, there is a rank one monodromy-invariant summand. Equivalently, there is a summand of $\rho_*(K' \otimes L_\chi)$, restricted to this open set, that is isomorphic to the constant sheaf $\mathbb{Q}_\ell$. Because $\rho_*(K' \otimes L_\chi)$ is a sum of shifts of irreducible perverse sheaves, this irreducible summand $\mathbb{Q}_\ell$ on an open set must extend to a shift of an irreducible perverse sheaf on the whole space. Because $Y$ is smooth, the unique irreducible extension of the constant sheaf $\mathbb{Q}_\ell$ from an open subset to all of $X$ is $\mathbb{Q}_\ell$, which is a perverse sheaf shifted by $m$. (In general, it would be the IC sheaf of $X$.)

Because $H^0(X, \mathbb{Q}_\ell) \neq 0$, and $\mathbb{Q}_\ell$ is a summand of $\rho_*(K' \otimes L_\chi)$, it follows that $H^0(X, \rho_*(K' \otimes L_\chi)) \neq 0$.

Now that we have interpreted the existence of nontrivial monodromy invariants cohomologically, we can re-express the cohomology group in terms of sheaves on $A$, which will enable us to understand its dependence on $\chi$ using the generic vanishing theorem. It follows from the Leray spectral sequence that

$$0 \neq H^0(X, R\rho_*(K' \otimes L_\chi)) = H^0(A \times X, K' \otimes L_\chi) = H^0(A, \pi_*K' \otimes L_\chi).$$

For the last equality, we need to know $\pi_*(K' \otimes L_\chi) = (\pi_*K') \otimes L_\chi$. This is a local question, and working étale-locally, $L_\chi$ is trivial, so it is automatic.

Now we choose our finite union of torsion translates of subtori. We apply the generic vanishing theorem to every perverse sheaf in sight. For all $j \in \mathbb{Z}$ with $\mathcal{H}^j(\pi_*K') \neq 0$, we take from [37, Lemma 11.2] a finite set of torsion translates of subtori such that $\mathcal{H}^j(A, \mathcal{H}^j(\pi_*K') \otimes L_\chi) = 0$ for all $\chi$ not in this set and all $i \neq 0$. Because there are only finitely many $j$ where $\mathcal{H}^j(\pi_*K') \neq 0$, the union of all of these is again a finite set.

Assume that $\left( H^0(A_{C(\eta)}, K \otimes L_\chi) \right)^{\text{Gal}(C(\eta) \backslash C(\eta))} \neq 0$ for a particular $\chi$ not in this set. The vanishing of $H^i(A, \mathcal{H}^j(\pi_*K') \otimes L_\chi)$ for all $j$ and all $i \neq 0$ forces the
spectral sequence
\[ H^i(A, p^* H^j(\pi_+ K') \otimes L_\chi) \rightarrow H^{i+j}(A, \pi_+ K' \otimes L_\chi) \]
to degenerate, giving
\[ 0 \neq H^0(A, \pi_+ K' \otimes L_\chi) = H^0(A, p^* H^0(\pi_+ K')L_\chi). \]

In particular, we can conclude that
\[ p^* H^0(\pi_+ K') \neq 0. \]

We will now derive a contradiction from this simpler statement, which notably is independent of \( \chi \).

Note first that because \( K'[m] \) is perverse, and \( \pi \) has fibers of dimension at most \( m \), we have \( p^* H^i(\pi_+ K') = 0 \) for \( i < 0 \) by [33, 4.2.4]. Hence there is a natural map
\[ p^* H^0(\pi_+ K') \rightarrow \pi_+ K' \]
arising from the perverse \( t \)-structure. Because \( p^* H^{2m}(\pi_+ K') \) is nonzero, this map must be nonzero. Applying adjunction, we obtain a nonzero map \( \pi^* p^* H^0(\pi_+ K')|m| \rightarrow K' \), and and shifting by \( m \), a nonzero map
\[ \pi^* p^* H^0(\pi_+ K')|m| \rightarrow K'[m]. \]

Because this is a nonzero map between perverse sheaves, some irreducible component of the source is equal to some irreducible component of the target. This cannot happen because by assumption no irreducible component of \( K \) is a pullback from \( A_C \), giving a contradiction. \( \square \)

**Lemma 5.4.** Let \( G^* \) be a simple algebraic group with a standard representation, and let \( N(G^*) \) be its normalizer inside the group of automorphisms of this representation. Then there is a finite list of irreducible representations of \( N(G^*)^c \), such that a reductive subgroup of \( N(G^*)^c \) contains \((G^*)^c\) if and only if it has no invariants on any of these representations.

**Proof.** Let us first handle the case where \( c = 1 \). Let \( H \) be a reductive subgroup of \( N(G^*) \). By Larsen’s alternative, if \( H \) acts irreducibly on the Lie algebra of \( G^* \) then \( H \) is either finite mod scalars or contains \( G^* \). (To prove this, note that the Lie algebra of \( N(G^*)^c \) splits into the Lie algebra of \( G^* \) and scalars. The Lie algebra of \( H \) is an \( H \)-invariant subspace of this, so if the Lie algebra of \( G^* \) is irreducible as a representation of \( H \), the Lie algebra of \( H \) either contains the Lie algebra of \( G^* \) or is contained in the scalars. In the first case, \( H \) contains \( G^* \), and in the second case, \( H \) is finite modulo scalars. This is a slight variant of a statement due to Larsen in unpublished preprint work, and later published by Katz [33, Theorem 2.2.2].)

If \( H \) is finite mod scalars, then by the Jordan-Schur theorem \( H \) contains a normal abelian subgroup of index \( \leq f(N) \) for some explicit function \( f \). Thus all irreducible representations of \( H \) have dimension \( \leq f(N) \). Let \( V \) be some fixed irreducible representation of \( N(G^*) \) of dimension \( > f(N) \) that remains irreducible when restricted to \( G^* \). If \( H \) acts irreducibly on the Lie algebra of \( G^* \) and on \( V \) then \( H \) is finite. To check that \( H \) acts irreducibly on \( V \), it is sufficient to check that it acts nontrivially on every irreducible subrepresentation of \( V \otimes V^\vee / \mathbb{C} \), where we mod out by the trivial subrepresentation \( \mathbb{C} \) arising from scalars. The same works for the adjoint representation.

Based on this, we take our finite list to be all irreducible subrepresentations of \((V \otimes V^\vee) / \mathbb{C}\) and \((\text{ad } G^* \otimes \text{ad } G^*\vee) / \mathbb{C}\). We have shown that if these representations
have no $H$-invariants then $H$ contains $G^*$. Conversely, because $V$ is irreducible upon restriction to $G^*$, $V \otimes V^*/\mathbb{C}$ has no $G^*$-invariants, and thus neither does any irreducible subrepresentation, and the same argument works for \text{ad}G^*, so these have no invariants if $H$ contains $G^*$. This proves the statement in the case $c = 1$.

For greater $c$, we take our finite list to consist of, for each $1 \leq i \leq c$, the list in the $c = 1$ case composed with the projection onto the $i$th copy of $N(G^*)$, combined with, for each $1 \leq i < j \leq c$, the Lie algebra of the $i$th copy of $G^*$ tensored with the dual of the Lie algebra of the $j$th copy of $G^*$. These extra representations manifestly have no $(G^*)^c$-invariants. Conversely, if $H$ acts without invariants on these representations, then by the previous discussion we know that the projection of $H$ onto the $i$th copy of $N(G^*)$ contains $G^*$. Hence the projection of the commutator subgroup of the identity component of $H$ onto the $i$th copy of $G^*$ is $G^*$. By [34] Theorem on p. 1152, if the commutator subgroup of the identity component of $H$ is not $(G^*)^c$, then there is $i,j$ with $1 \leq i < j \leq c$ such that the projections of this subgroup to the $i$th and $j$th copy of $G^*$ are isomorphic, mod scalars. This isomorphism is unique and thus is defined over $H$. Then the Lie algebras of the $i$th and $j$th copy of $G^*$ must be isomorphic as representations of $H$, so the tensor product of one with the dual of the other has invariants, contradicting our assumption.

\begin{proof}
\end{proof}

\textbf{Remark 5.5.} Lists of representations satisfying the condition of Lemma 5.3 with smaller dimension here follow from Larsen's conjecture [39] Theorem 1.4] in the case that $G^*$ is a classical group, but these depend on the classification of finite simple groups. For some applications of results of this type, optimizing the dimension of the representations is relevant, but not here.

The following theorem is the analogue of Pink’s specialization theorem [32] Theorem 8.18.2], which shows, given any sheaf on the total space family of schemes, for “most” schemes in the family, the monodromy of the restricted sheaf is equal to a generic monodromy group. Our analogue shows that for “most” characters $\chi$, the monodromy of $\bigoplus_{c=1}^n R^{n-1}f_* (g^* L_{\chi_1})$ is (roughly) equal to a generic Tannakian monodromy group. This will connect our earlier investigations of the Tannakian group to our later arguments, which require control on monodromy groups.

\textbf{Theorem 5.6.} Assume that $Y_\pi$ is not translation-invariant by any nonzero element of $A$, that $G^*$ is a simple algebraic group acting by an irreducible representation, and that $Y$ is not equal to a constant family of hypersurfaces translated by a section of $A$.

Then for $\chi_1, \ldots, \chi_c$ characters of $\pi_1^G(A)$, with $(\chi_1, \ldots, \chi_c)$ avoiding some finite set of torsion translates of proper subtori of the dual torus to $\pi_1(A^c)$, the following conditions are satisfied:

- $R^k f_* (g^* L_{\chi_i}) = 0$ for $k \neq n - 1$, and
- the geometric monodromy group of $\bigoplus_{c=1}^n R^{n-1}f_* (g^* L_{\chi})$ contains as a normal subgroup $(G^*)^c$ acting on the sum of $c$ copies of the fixed irreducible representation.

\begin{proof}
First, we use the generic vanishing theorem to find a finite set of torsion translates of proper subtori of the dual torus to $\pi_1(A)$ such that, for $\chi$ avoiding them, $R^k f_* (g^* L_{\chi}) = 0$ for $k \neq n - 1$. We will then take the inverse images of these subtori under the duals of the $c$ inclusions $\pi_1(A) \to \pi_1(A^c)$ to be in our finite set of torsion translates of subtori of the dual torus to $\pi_1(A^c)$.

\end{proof}
Next, to calculate the monodromy, we will use the fact that the monodromy group is equal to the Zariski closure of the image of $\text{Gal}({\overline{\mathbb{Q}}}(\eta)/\mathbb{Q}(\eta))$ acting on the stalk at the geometric generic point.

Let us first check that the geometric monodromy group of $\bigoplus_{i=1}^c R^{n-1} f_*(g^* L_{X_i})$ is contained in $N(G^*)^c$. It suffices to show for each $i$ that the geometric monodromy group of $R^{n-1} f_*(g^* L_{X_i})$ is contained in $N(G^*)$. This follows from Lemma 3.8(2).

Now by Lemma 5.4 we can find an explicit list of representations of $N(G^*)^c$ such that any reductive subgroup of $N(G^*)^c$ contains $(G^*)^c$ if and only if its action on all these representations has no invariants. By Deligne’s theorem, the monodromy group of $\bigoplus_{i=1}^c R^{n-1} f_*(g^* L_{X_i})$ is reductive, and so we can apply this lemma.

By Lemma 5.7 each representation from the list of Lemma 5.4 corresponds to a perverse sheaf on $A^c_\eta$ in the arithmetic Tannakian category generated by $\bigoplus_{i=1}^c \xi_{i,s} Q_{\ell}[n-1]$. (We have to check that $\bigoplus_{i=1}^c \xi_{i,s} Q_{\ell}[n-1]$ is geometrically semisimple, but this is clear as the constant sheaf on any closed subvariety is semisimple. It follows that the action of $G_k$ on the representation associated to this complex factors through the normalizer of the action of $G_k$, and thus factors through $N(G^*)^c$, and so all representations of $N(G^*)^c$ correspond to geometrically semisimple perverse sheaves.)

Because $(G^*)^c$ acts nontrivially on all representations from Lemma 5.4 by Lemma 5.2 none of these perverse sheaves is a pullback from $A^c_\eta$, so by Lemma 5.3 outside some finite set of torsion translates of subtori of the dual torus of $\pi_1(A^c)$, the Galois group has no invariants for these representations. Thus, outside some finite set of torsion translates, the Galois group contains $(G^*)^c$.

\textbf{Lemma 5.7.} Suppose $A$ is defined over a number field $K$, and let $p$ be a prime at which $A$ has good reduction. For any natural number $c$, for any finite set $S$ of torsion translates of proper subtori of the dual torus $\Pi(A)^c$ to $\pi_1(A^c)$, there exist $\sigma_1, \ldots, \sigma_c \in \text{Gal}(\overline{\mathbb{Q}}_p|K_p)$ and some finite set $S'$ of torsion translates of proper subtori of the dual torus to $A$, such that for any $\chi$ outside the union of $S'$, the tuple $(\sigma_1(\chi), \ldots, \sigma_c(\chi))$ does not lie in the union of $S$.

\textbf{Proof.} By enlarging $K_p$ if necessary, we may assume that all the tori in $S$ are defined over $K_p$. Let $\sigma$ be a Frobenius element of $\text{Gal}(\overline{\mathbb{Q}}_p|K_p)$, acting on $\pi_1(A^c)$. We will prove this lemma as a consequence of the fact that this action of $\sigma$ is invertible, with no roots of unity as eigenvalues.

To illustrate the argument, let’s first consider the case where $S = \{T + \xi\}$ consists of only a single torsion translate of a torus. Let $Z$ be the quotient of $\text{Hom}(\pi_1^c(A^c), \overline{\mathbb{Q}}_\ell\) by $\text{Hom}(\pi_1^c(A^c), \overline{\mathbb{Q}}_\ell) \cap T$. Then $Z$ is an abelian group, and there is a natural homomorphism $f : \text{Hom}(\pi_1^c(A^c), \overline{\mathbb{Q}}_\ell) \to Z$. We can describe $T + \xi$ as the inverse image of $f(\xi)$. Because by assumption $T$ is defined over $K_p$, there is an action of $\text{Gal}(\overline{\mathbb{Q}}_p|K_p)$ on $Z$ making $f$ into $\text{Gal}(\overline{\mathbb{Q}}_p|K_p)$-equivariant homomorphism.

By the definition of subtori, there exists an abelian subvariety $B$ of $A$ such that $T$ consists of characters vanishing on $B$, and thus $Z = \text{Hom}(\pi_1^c(B), \overline{\mathbb{Q}}_\ell)$ and $f$ is dual to the inclusion $B \to A^c$.

We’ll take our Galois elements to be powers of Frobenius: $\sigma_i = \sigma^{e_i}$, for integers $e_1, \ldots, e_c$. Consider the linear map

$$g = g_{e_1, \ldots, e_c} : \text{Hom}(\pi_1^c(A), \overline{\mathbb{Q}}_\ell) \to Z$$
given by
\[ g_{e_1, \ldots, e_c}(x) = f(\sigma_1(x), \sigma_2(x), \ldots, \sigma_c(x)). \]

If \( g \) is not uniformly zero, then \( g^{-1}(f(\xi)) \) is a finite union of torsion translates of proper subtori, and we are done. (This is because \( g \) is dual to the homomorphism of abelian varieties \( B \to A^c \to A \) by composing the inclusion with \((x_1, \ldots, x_c) \mapsto \sigma_1(x_1) + \cdots + \sigma_c(x_c)\). If \( g \) is not uniformly zero then the image of this homomorphism includes some nontrivial abelian subvariety. Thus, characters pulled back to zero under this homomorphism vanish on some fixed finite étale cover of that nontrivial abelian subvariety. Hence, characters pulled back to zero under this homomorphism lie in a finite union of torsion translates of characters that vanish on that nontrivial abelian subvariety. The same is true for characters that pull back to \( f(\xi) \). Hence, we just need to find some \( e_1, \ldots, e_c \) for which \( g \) is not uniformly zero. We’ll say that the tuple \((e_1, \ldots, e_c)\) is bad for \( T \) if \( g \) is uniformly zero.

For \( 1 \leq r \leq c \), let \( \Pi_r(A^c) \) be the set of characters of \( \pi_1(A^c) = \pi_1(A)^c \) trivial on each direct summand \( \pi_1(A) \) except possibly the \( r \)-th, so we have the direct sum decomposition \( \Pi(A^c) = \bigoplus_{r=1}^c \Pi_r(A^c) \). Since \( T \) is a proper subtorus of the dual torus, there exists some index \( r \) such that \( T \) does not contain \( \Pi(A) \); in other words, \( f(0, \ldots, 0, x, 0, \ldots, 0)(\text{with } x \text{ in the } r\text{th place}) \) is not uniformly zero. We’ll call such an \( r \) nondegenerate for \( T \).

Now if \( e'_r \neq e_r \), the endomorphism
\[ x \mapsto \sigma^{e'_r}(x) - \sigma^{e_r}(x) \]
of \( \Pi(A) \) is surjective. If \((e_1, \ldots, e_r, \ldots, e_c)\) is bad for \( T \), then \((e_1, \ldots, e'_r, \ldots, e_c)\) is not bad for \( T \).

Now return to the general case, where \( S \) is a finite set of torsion translates of proper subtori \( T \) of \( \Pi(A) \). We need to show that there exists \((e_1, \ldots, e_r, \ldots, e_c)\) that is not bad for any of these finitely many \( T \). For each \( T \), let \( r(T) \) be the least index that is nondegenerate for \( T \). We’ll work by reverse induction on \( r \) (from \( c \) to \( 1 \)).

Suppose we have a tuple \((e_1, \ldots, e_r, \ldots, e_c)\) that is not bad for any \( T \) with \( r(T) > r \). Any such \( T \) contains \( \Pi_r(A^c) \), so changing \( e_r \) does not affect this property. Now consider all \( T \) such that \( r(T) = r \). For each such \( T \), only one value of \( e_r \) can give a bad tuple \((e_1, \ldots, e_r, \ldots, e_c)\). Thus, we can choose \( e'_r \) so that the tuple \((e_1, \ldots, e'_r, \ldots, e_c)\) is not bad for any \( T \) with \( r(T) \geq r \). Repeating this process for \( r(T) = c, c-1, \ldots, 1 \), we produce a tuple \((e_1, \ldots, e'_r, \ldots, e_c)\) for which no \( T \) is bad. \( \square \)

**Lemma 5.8.** Suppose \( A \) is defined over a number field \( K \). For any finite set of torsion translates of proper subtori of the dual torus to \( \pi_1(A) \), there exists a torsion character \( \chi \) of \( \pi_1(A) \) such that no \( \text{Gal}(\overline{K}/K) \)-conjugate of \( \chi \) lies in the union of that set. Furthermore, we can arrange that \( \chi \) is of order a power of \( \ell \), for any given prime \( \ell \).

**Proof.** For \( \chi \) to have no Galois conjugate in any of these translates of subtori, it suffices that \( \chi \) not lie in any Galois conjugate of these translates of subtori.

By definition, each proper subtorus of the dual torus corresponds to some abelian subvariety, which must be defined over a number field. Since every torsion point is defined over a number field, every torsion translate of a proper subtorus is defined
over a number field. Hence they have finitely many Galois conjugates, and the union of all their Galois conjugates is a finite union of torsion translates of proper subtori.

Each proper subtorus can contain at most \( \ell^{2k(n-1)} \ell^k \)-torsion characters, and so any translate of a proper subtorus can contain at most \( \ell^{2k(n-1)} \ell^k \)-torsion characters, while there are \( \ell^{2kn} \ell^{2k} \)-torsion characters in total, so as soon as \( \ell^{2k} \) is greater than this finite number of tori, there will be an \( \ell^k \)-torsion character not in any of them. \( \square \)

For the remainder of this section, we’ll suppose \( A \) is an abelian variety over a number field \( K \), \( X \) is a smooth scheme over \( \mathbb{Q} \), and \( Y \subseteq X \times \mathbb{Q} A = X_K \times_K A \) is a family of hypersurfaces in \( A \), smooth, proper and flat over \( X_K \). For every embedding \( \iota: K \to \mathbb{C} \), we can form schemes

\[ A_{\iota} = A \otimes_{\text{Spec} \mathbb{K}_{\iota}} \text{Spec} \mathbb{C} \]

and

\[ Y_{\iota} = Y \otimes_{\text{Spec} \mathbb{K}_{\iota}} \text{Spec} \mathbb{C}; \]

these are both schemes over \( \mathbb{C} \), and \( Y_{\iota} \) has a projection to \( X_{\mathbb{C}} = X \otimes_{\text{Spec} \mathbb{Q}} \text{Spec} \mathbb{C} \).

Let \( f: Y_{\iota} \to X_{\mathbb{C}} \) and \( g: Y_{\iota} \to A_{\iota} \) be the projections; for every torsion character \( \chi \) of \( \pi^\text{et}_1(A_{\iota}) \), let \( \mathcal{L}_{(\iota, \chi)} \) be the corresponding character sheaf on \( A_{\iota} \).

**Corollary 5.9.** Assume that \( Y_{\eta} \) is not translation-invariant by any nonzero element of \( A \), that \( G^* \) is a simple algebraic group acting by an irreducible representation, and that \( Y \) is not equal to a constant family of hypersurfaces translated by a section of \( A \).

Then for any prime \( \ell \), there exist an embedding \( \iota: K \to \mathbb{C} \) and a torsion character \( \chi \) of \( \pi^\text{et}_1(A_{\iota}) \), of order a power of \( \ell \), such that for every conjugate \( (\iota', \chi') \) of \( (\iota, \chi) \) by an element of \( \text{Gal}(\mathbb{Q}/K) \):

- for \( k \neq n - 1 \), we have \( R^k f_{\iota' *}(g_{\iota'}^* \mathcal{L}_{(\iota', \chi')}) = 0 \), and
- there exist \( \sigma_1, \ldots, \sigma_c \in \text{Gal}(\mathbb{Q}_p/K) \) such that the monodromy group of \( \bigoplus_{i=1}^c R^{n-1} f_{\iota' *}(g_{\iota'}^* \mathcal{L}_{\sigma_i(\iota', \chi')}) \) contains as a normal subgroup \( (G^*)^c \) acting on the sum of \( c \) copies of the fixed irreducible representation.

**Proof.** This follows from the previous 3 results, applying Theorem 5.6 to each of the finitely many pairs \( (Y_{\iota}, A_{\iota}) \). \( \square \)
6. Hodge–Deligne systems

The goal of the next few sections is to prove Theorem 8.21, which is analogous to Lemma 4.2, Prop. 5.3, and Thm. 10.1 in [40]. Roughly, the theorem says that, if a smooth variety \(X\) over \(\mathbb{Q}\) admits a Hodge–Deligne system that has big monodromy and satisfies two numerical conditions, then the integral points of \(X\) are not Zariski dense. We follow the same strategy as [40], but we’ll need to work in greater generality. First, [40] works only with the primitive cohomology of a family of varieties, but we’ll need to work with the cohomology with coefficients. Second, we’ll need to work with Galois representations valued in a disconnected reductive group. Finally, we are unable to precisely identify the Zariski closure of the image of monodromy; we only know that it is a \(c\)-balanced subgroup of \(G\).

We’ll begin by defining the notion of “Hodge–Deligne system”, which will figure in our statement of Theorem 8.21. Let \(X\) be a variety over a number field \(K\) (which will eventually be taken to be \(\mathbb{Q}\)). A smooth, projective family of varieties over \(X\) gives rise to various cohomology objects on \(X\). The argument of [40] relies on the interplay among several of these objects: a complex period map, a \(p\)-adic period map, and a family of \(p\)-adic global Galois representations on \(X\). Deligne has called the collection of these cohomology objects a “system of realizations” for a motive [14]; our notion of “Hodge–Deligne system” will be closely related to Deligne’s systems of realizations.

**Definition 6.1.** Let \(k\) be an integer, and \(q\) a prime power. A rational \(q\)-Weil number of weight \(k\) is an algebraic number, all of whose conjugates have complex absolute value \(q^{k/2}\).

An integral \(q\)-Weil number is a rational \(q\)-Weil number that is an algebraic integer.

When \(\ell\) is a prime of \(\mathcal{O}_K\), we write \(q_\ell\) for the cardinality of the residue field at \(\ell\).

**Definition 6.2.** Suppose given a number field \(K\) with a chosen embedding \(\iota_C: K \to \mathbb{C}\). Let \(X\) be a smooth variety over \(K\). Let \(S\) be a finite set of primes of \(K\), and let \(X\) be a smooth model of \(X\) over \(\mathcal{O}_K\). Let \(p\) be a prime of \(\mathbb{Q}\) not lying below any place of \(S\) and let \(v\) be a place of \(K\) lying over \(p\).

A Hodge–Deligne system on \(X\) at \(v\) consists of the following structures:

- A singular local system \(V_{\text{Sing}}\) of \(\mathbb{Q}\)-vector spaces on \(X_C\).
- An étale local system \(V_{\text{et}}\) of \(\mathbb{Q}_p\)-vector spaces on \(X_{\text{et}}\times_{\text{Spec} \mathbb{Z}} \text{Spec} \mathbb{Z}[1/p]\).
- A vector bundle \(V_{dR}\) on \(X\), an integrable connection \(\nabla\) on \(V_{dR}\), and a descending filtration \(\text{Fil}^i V_{dR}\) of \(V_{dR}\) by subbundles

\[
\text{Fil}^i V_{dR} \supseteq \text{Fil}^{i+1} V_{dR} \supseteq \cdots \supseteq \text{Fil}^M V_{dR} = 0
\]

(not necessarily \(\nabla\)-stable), each of which is locally a direct summand of \(V_{dR}\).

- A filtered \(F\)-isocrystal \(V_{\text{cris}}\) on \(X_{K_v}\) (see, for example, [57] end of §3.1]), with the following isomorphisms:

\[\text{Fil}^i V_{dR} = \text{Fil}^{-M} V_{dR} \supseteq \text{Fil}^{-M+1} V_{dR} \supseteq \cdots \supseteq \text{Fil}^M V_{dR} = 0\]

1It is important that we allow Weil numbers that are not algebraic integers, since we are going to work in a Tannakian setting.

2The name is meant to evoke variations of Hodge structure and Deligne’s systems of realizations.
(1) An isomorphism on $X_{C, \text{an}}$ between $V_{\text{Sing}} \otimes \mathbb{Q} \otimes \mathbb{Q}_p$ and the pullback of $V_{\text{et}}$ to $X_{C, \text{an}}$.  

(2) An isomorphism on $X_{C, \text{an}}$ between $V_{\text{Sing}} \otimes \mathbb{Q}_X \mathcal{O}_{X_{C, \text{an}}}$ and $V_{\text{et}} \otimes \mathcal{O}_X \mathcal{O}_{X_{C, \text{an}}}$.

(3) An isomorphism on an open neighborhood of the rigid analytic generic fiber of $X_{K_v}$ between the underlying vector bundle to $V_{\text{cris}}$ and the pullback of $V_{\text{dR}}$.

(4) An isomorphism on $X_{K_v, \text{proet}}$ between the $\mathcal{O}_B$-modules $V_{\text{cris}} \otimes \mathcal{O}_X$, $\mathcal{O}_{\text{cris}}$ and $V_{\text{et}} \otimes \mathbb{Q}_p \mathcal{O}_{\text{cris}}$.

and an increasing filtration $W_i$ of all four objects, compatible with all the isomorphisms, such that all this data satisfies the axioms:

- $\text{Fil}^i V_{\text{dR}}$ and $\nabla$ satisfy Griffiths transversality.
- The connection $\nabla$ is induced under the isomorphism (2) by the trivial connection on $\mathcal{O}_X$.
- For each point of $X_C$, the $i$th associated graded under $W_i$ of the stalk of $(V_{\text{Sing}}, V_{\text{dR}} \otimes K \mathbb{C}, \text{Fil}^i (2))$ at that point is a pure Hodge structure of weight $i$.
- The $i$th associated graded under $W_i$ of $V_{\text{et}}$ is pure of weight $i$, i.e. for each closed point $x$ of $X \times \text{Spec} \mathbb{Z}$ with residue field $\kappa_x$, the eigenvalues of Frobenius on the $i$th associated graded of $V_{\text{et}, x}$ are $|\kappa_x|$-Weil numbers of weight $i$.
- The $i$th associated graded of $V_{\text{cris}}$ under $W_i$ is pure of weight $i$, i.e. for each closed point $x$ of $X$ lying over $p$ with residue field $\kappa_x$, the eigenvalues of Frobenius on the $i$th associated graded of $V_{\text{cris}, x}$ are $|\kappa_x|$-Weil numbers of weight $i$.
- The connection $\nabla$ has regular singularities in a smooth simple normal crossings compactification of $X_K$.
- The isomorphism (3) is compatible with the connection.
- The isomorphism (4) is compatible with connection, filtration, and Frobenius.

We note that the isomorphism (2) and the first three axioms make up the definition of a variation of Hodge structures. The isomorphism (4) and the last axiom make up the definition of a crystalline local system [57, §1]. (Faltings calls these objects “dual-crystalline sheaves” [18, Theorem 2.6], at least in the situation where the Hodge–Tate weights are bounded between $0$ and $p - 2$.)

We say a Hodge–Deligne system is pure of weight $w$ if $W_{w-1}$ vanishes and $W_w$ is the whole system.

The rank of a Hodge–Deligne system $V$ is the rank of the local system $V_{\text{Sing}}$ of $\mathbb{Q}$-vector spaces. By the various isomorphisms, this is equal to the ranks of $V_{\text{et}}$, $V_{\text{dR}}$, and $V_{\text{cris}}$.

We will also need to work with polarized and integral variations of Hodge structure, and for that we need the following slight modifications of the notion of Hodge–Deligne system.

**Definition 6.3.** Let $K$, $X$, $S$, $\mathcal{X}$, $v$ be as above. An integral Hodge–Deligne system on $X$ consists of a Hodge–Deligne system on $X$ together with an integral structure on $V_{\text{Sing}}$ (i.e. a singular local system $V_{\text{int}}$ of free $\mathbb{Z}$-modules on $X_C$ together with an isomorphism $V_{\text{int}} \otimes \mathbb{Z} \mathbb{Q} \cong V_{\text{Sing}}$).³

³We do not use the étale-singular comparison; we could have left it out.
Definition 6.4. Let $K, X, S, X', v$ be as above. A polarized Hodge–Deligne system on $X$ consists of a Hodge–Deligne system on $X$, pure of some weight, together with a polarization of the variation of Hodge structures $(V_{\text{sing}}, V_{\text{dR}}, (2))$ (i.e. a morphism of local systems $V_{\text{sing}} \otimes V_{\text{sing}} \to \mathbb{Q}$ which restricted to the stalk at any point of $X(\mathbb{C})$ defines a polarization of the pure Hodge structure at that point.)

Definition 6.5. We say that a Hodge–Deligne system $V$ has integral Frobenius eigenvalues if the Weil numbers appearing as eigenvalues of Frobenius on $V_{\text{et}, x}$ and $V_{\text{cris}, x}$ are integral, for all closed points $x \in X \times \text{Spec } \mathbb{Z}$ and all closed points $x$ of $X'$ lying over $p$, respectively.

Definition 6.6. The differential Galois group of a Hodge–Deligne system $V$ is the differential Galois group of the underlying vector bundle with connection $V_{\text{dR}}$. (For the definition of differential Galois group, see [47, §§1.2 – 1.4]. A vector bundle with connection gives rise to a linear differential equation; by the differential Galois group of the vector bundle with connection, we mean the differential Galois group of a Picard–Vessiot ring of the corresponding differential equation.) The differential Galois group is the Zariski closure of the monodromy group of the variation of Hodge structure $V_H$; this follows from the Riemann–Hilbert correspondence, and the fact that the period map has regular singularities along a smooth normal crossings compactification.

Remark 6.7. Let $k_v$ be the residue field of $K$ at $v$. The “Frobenius automorphism” of $K_v$ is the element of $\text{Gal } K_v/\mathbb{Q}$ that acts as the $p$-th power map on $k_v$.

A filtered $F$-isocrystal $V_{\text{cris}}$ gives, for every $\pi \in X(k_v)$, a pair

$$(V_\pi, \phi_\pi),$$

where $V_\pi$ is a $K_v$-vector space, and $\phi_\pi$ is an endomorphism of $V_\pi$, semilinear over Frobenius. Furthermore, for every $x \in X(O_{K_v})$ belonging to the residue class of $\pi$, the object $V_{\text{cris}}$ defines a filtration on $V_\pi$, with an isomorphism to the filtered VS $V_{\text{dR}} \otimes K_v$. We’ll call the resulting data $V_{\text{cris}, x} = (V_{\text{cris}, x}, \phi_{\text{cris}, x}, F_{\text{cris}, x}).$

Example 6.8. (The trivial Hodge–Deligne system.) Let $K$ and $E$ be number fields. Take $X = \text{Spec } K$, and define the trivial Hodge–Deligne system $O_E$ on $\text{Spec } K$ by:

- $O_{E, \text{sing}} = E$.
- $O_{E, \text{et}} = E \otimes \mathbb{Q} \mathbb{Q}_p$, with the trivial Galois action.
- $O_{E, \text{dR}} = E \otimes \mathbb{Q}_p K_v$ with trivial connection and filtration (i.e. $\text{Fil}^1 O_{E, \text{dR}} = O_{E, \text{dR}},$ and $\text{Fil}^0 O_{E, \text{dR}} = 0$).
- $O_{E, \text{cris}}$ is determined by $V_{\text{dR}}$ and the requirement that Frobenius act on $E \otimes \mathbb{Q}_p K_v$ through the trivial action on $E$ and the Frobenius automorphism of $K_v$.
- The weight filtration is such that $O_E$ is concentrated in weight zero.

When $X$ is an arbitrary smooth $K$-variety, we define the system $O_E$ on $X$ by pullback from $\text{Spec } K$.

In general, Hodge–Deligne systems will come from families of varieties by taking cohomology.
Example 6.9. (Pushforward of Hodge–Deligne systems.)

Let $X$ be a variety over a number field $K$, and let $\pi: Y \to X$ be a smooth, projective family of relative dimension $n$. Let $V$ be a Hodge–Deligne system on $Y$, and choose some $k$ with $0 \leq k \leq n$. We define a Hodge–Deligne system $W = R^k\pi_* (V)$ on $X$, as follows.

- Spread $Y \to X$ out to a smooth projective family $\pi: \mathcal{Y} \to \mathcal{X}$ over $\mathcal{O}_{K,S}$, for some finite set $S$ of places of $K$. Call a prime $p$ of $\mathbb{Q}$ good if no element of $S$ lies over $p$.
- Fix a good prime $p$ and a place $v$ lying over $p$.
- Take $W_{\text{sing}} = R^k\pi_* V_{\text{sing}}$, with the pushforward taken in the analytic topology on $X$ and $Y$. (This is again a local system, by Ehresmann’s theorem.)
- Take $W_{\text{et}} = R^k\pi_* V_{\text{et}}$, with the pushforward taken in the étale topology. (See [28] for an introduction to the étale topology.)
- Take $W_{\text{dR}}$ to be the relative de Rham cohomology of $V_{\text{dR}}$ over $X$, i.e. the pushforward of $V_{\text{dR}}$ as a $D$-module, with its Hodge filtration $\text{Fil}^i$. This is a filtered vector bundle by Hodge theory.
- Take $W_{\text{cris}}$ to be the $v$-adic crystalline cohomology of $Y$. This is a filtered $F$-isocrystal by $p$-adic Hodge theory ([53, Thm. 8.8]). (See [9] or [10] for the construction of crystalline cohomology, and [44, §1] for its structure as filtered $F$-isocrystal.)
- The isomorphisms (1), (2), (3), (4) follow from Artin’s comparison theorem, Hodge theory, the de Rham-crystalline comparison, and relative $p$-adic Hodge theory ([53, Thm. 8.8], [57, Thm. 5.5]) respectively.
- The filtration $W_i$ is induced from the filtration of $\mathcal{V}$ after shifting by $k$. In particular, if $\mathcal{V}$ is pure of weight $w$, then $W$ is pure of weight $w + k$.

Definition 6.10. If $V$ and $W$ are two Hodge–Deligne systems on $X$, a morphism from $V$ to $W$ consists of:

- A map of analytic local systems $V_{\text{sing}} \to W_{\text{sing}}$.
- A map of étale local systems $V_{\text{et}} \to W_{\text{et}}$.
- A map of vector bundles $V_{\text{dR}} \to W_{\text{dR}}$, flat with respect to the connections on $V_{\text{dR}}$ and $W_{\text{dR}}$, and respecting the filtrations $\text{Fil}^i V_{\text{dR}}$ and $\text{Fil}^i W_{\text{dR}}$, and
- A map of filtered $F$-isocrystals $V_{\text{cris}} \to W_{\text{cris}}$.

compatible with all the comparison isomorphisms (1), (2), (3), (4).

Lemma 6.11. The systems of compatible periods on $X$ at $v$ form a Tannakian category with fiber functor given by $V_{\text{sing},x}$ for some $x \in X(\mathbb{C})$.

In this Tannakian category, the tensor product of two systems will be defined by separately tensoring the individual objects $V_{\text{sing}}, V_{\text{et}}, V_{\text{dR}},$ and $V_{\text{cris}},$ and similarly for the dual of a system.

Proof. Let us first check that they form an abelian category. Kernels and cokernels will be taken separately on the four components $f_{\text{sing}}, f_{\text{et}}, f_{\text{dR}},$ and $f_{\text{cris}}$. The verification of most of the axioms is routine; we’ll explain only two steps which are not immediate.

The first is existence of a cokernel for $f_{\text{dR}}$. Let $f: V \to W$ be a morphism of Hodge–Deligne systems. In general the cokernel of a morphism of vector bundles need not be a vector bundle. But the cokernel of $f_{\text{dR}}: V_{\text{dR}} \to W_{\text{dR}}$ must be a vector bundle because $f_{\text{dR}}$ is a flat map of vector bundles with connection.
The second is the equality of images and coinages. A priori, the image of \( f_{dR} \) has two possibly different filtrations, the image filtration and the coimage filtration. These filtrations agree because variations of Hodge structure form an abelian category.

So the category of Hodge–Deligne systems is abelian.

Furthermore, the category of Hodge-Deligne systems is manifestly \( \mathbb{Q} \)-linear.

To check it is a \( \mathbb{Q} \)-linear symmetric monoidal category, we use the symmetric monoidal structure on the categories of singular local systems, étale local systems, filtered vector bundles with flat connections, and \( F \)-isocrystals separately. To define the tensor product of two objects we individually tensor the \( V_{\text{sing}}, V_{\text{et}}, V_{dR}, V_{\text{cris}} \) and then tensor the isomorphisms (1), (2), (3), (4). This requires that the various pullback functors we use in defining these isomorphisms are compatible with tensor product, which is standard. We also need to check that the tensor product satisfies the axioms, which is clear in all cases. In addition to defining a product, the symmetric monoidal structure consists of isomorphisms and commutative diagrams. The fact that these isomorphisms are compatible with (1), (2), (3), (4) follows from their naturality and the fact that the pullback functors are symmetric monoidal.

The unit object is given by \( O_{\mathbb{Q}} \) from Example 6.8.

For rigidity, we use a similar argument. Each individual category has a dual object functor, unit and counit natural transformations, and commutative diagrams satisfied by these isomorphisms. Because the pullback functors are compatible with this data, we can define the dual of a Hodge-Deligne system, and the unit and counit are compatible with the isomorphisms (1), (2), (3), (4) and thus form a valid morphism of Hodge-Deligne systems. The commutative diagrams follow from the commutative diagrams on the four individual objects.

To check that \( V_{\text{sing},x} \) is an exact tensor functor to \( \mathbb{Q} \)-vector spaces, it suffices to check that \( V_{\text{sing}} \) is an exact tensor functor to \( \mathbb{Q} \)-local systems. The fact that it is a tensor functor follows from how we defined tensor product; the fact that it is exact follows from how we constructed kernels and cokernels. \( \square \)

If \( f: X' \to X \) is a map, then for any system \( V \) on \( X \), we can define the pullback \( f^*V \) on \( X' \) by pulling back the four components separately. When the map \( f \) is clear, we’ll sometimes write \( V|_{X'} \) instead of \( f^*V \).

**Definition 6.12.** Let \( X \) be a \( K \)-scheme. A constant Hodge–Deligne system on \( X \) is a system of the form \( f_{\ast} V \), where \( V \) is a Hodge–Deligne system on \( \text{Spec} \ K \).

Constant Hodge-Deligne systems will be much more general than most notions of motives. For instance nothing prevents us from combining the étale and crystalline cohomology of one variety with the Hodge structure of a different variety, as long as they have the same Hodge numbers. Despite this, the notion of Hodge-Deligne system is strong enough for the arguments that we will make.

6.1. \( H^0 \)-algebras. In order to make the arguments of [40] work, we need bounds on the centralizer of Frobenius.

The paper [40] works with Hodge–Deligne systems of the form \( R^k \pi_\ast (O_{\mathbb{Q}}) \), for \( \pi: Y \to X \) a family with geometrically disconnected fibers. In this context, the zeroth cohomology \( H^0(Y_x) \) has nontrivial Galois structure. The action of \( H^0(Y_x) \) on \( H^k(Y_x) \) gives rise to the bounds we need on the Frobenius centralizer, by means of the semilinearity of Frobenius.
For this work involving abelian varieties, we’ll have a group $G$ acting on the fibers of $Y \to X$; this $G$ will give a bound on the Frobenius centralizer in an analogous way.

The notion of module over an $H^0$-algebra generalizes both pictures. Loosely speaking, an $H^0$-algebra is a weight-zero algebra object in the category of Hodge–Deligne systems. Motives over extensions of $K$ (Example 6.20), motives with coefficients in a number field $E$ (Examples 6.14 and 6.22), and motives with an action of a finite algebraic group (Examples 6.19 and 6.22) will all turn out to be modules over various $H^0$-algebras.

**Definition 6.13.** A commutative $H^0$-algebra is a Hodge–Deligne system $E$ on a variety $X$ over a number field $K$, equipped with maps $e : \mathcal{O}_E \to E$ and $m : E \otimes E \to E$ satisfying the following properties.

- $E$ is pure of weight 0.
- The filtration on $E_{dR}$ is trivial: $\text{Fil}^0 E_{dR} = E_{dR}$ and $\text{Fil}^1 E_{dR} = 0$.
- The maps $e$ and $m$ make $E$ into a commutative algebra object.

When we say “$H^0$-algebra”, we will mean “commutative $H^0$-algebra”.

**Example 6.14.** (Trivial Hodge–Deligne system $\mathcal{O}_E$.)

Let $K$ and $E$ be number fields. The trivial Hodge–Deligne system $\mathcal{O}_E$ of Example 6.8 has an algebra structure coming functorially from the algebra structure on $E$.

**Definition 6.15.** If $X = \text{Spec} K$, we say that $E$ is étale if $E_{\text{sing}}$ is an étale $\mathbb{Q}$-algebra.

**Example 6.16.** (Étale $H^0$-algebras over a field.)

Let $X = \text{Spec} K$ be the spectrum of a number field. In this setting we can give a concrete description of étale $H^0$-algebras $E$ over $\text{Spec} K$.

The singular realization $E = E_{\text{sing}}$ has the structure of $\mathbb{Q}$-algebra, which we assume is étale; $E_{dR}$ is determined by $E_{dR} = E \otimes_{\mathbb{Q}} K$ with trivial filtration.

The étale realization $E_{\text{ét}}$ is the $\mathbb{Q}_p$-algebra $E \otimes_{\mathbb{Q}} \mathbb{Q}_p$, equipped with a continuous action of the Galois group $\text{Gal}_K$. By assumption, $E$ is an étale $\mathbb{Q}$-algebra, so $\text{Aut}(E \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ is a finite group. The action of $\text{Gal}_K$ descends to the maximal quotient $\text{Gal}_{K,S}$ of $\text{Gal}_K$ unramified outside some finite set $S$ of places of $K$, which (by $p$-adic Hodge theory) must include the place $v$ of $K$.

Finally we turn to $E_{\text{cris}}$. The structure of $K_v$-algebra is given by an isomorphism $E_{\text{cris}} \cong E \otimes_{\mathbb{Q}_p} K_v$.

The filtration is trivial, and the Frobenius (which we will notate $\sigma$) is the endomorphism $\sigma_1 \otimes \sigma_2$ of $E \otimes_{\mathbb{Q}_p} K_v$, where $\sigma_1$ gives the action of $\text{Frob}_v \in \text{Gal}_K$ on $E$, and $\sigma_2$ is the endomorphism of $K_v$ that acts as the $p$-th power map on residue fields. The
Frobenius \( \text{Frob}_v \in G_K \) is only well-defined up to conjugacy, but that is enough to determine \( E_{\text{cris}} \) up to isomorphism.

**Example 6.17. (H^0 of a family.)**

If \( \pi: Y \to X \) is a proper map, then the degree-zero cohomology of \( Y \), equipped with the cup product, gives an \( H^0 \)-algebra on \( X \).

The underlying Hodge-Deligne system is constructed as in Example 6.9 as \( \mathbf{R}^0 \pi_* (\mathcal{O}_L) \).

The Hodge filtration is trivial because the Hodge filtration on \( H^0_{dR} \) of any smooth scheme is trivial. The map \( \mathcal{O}_X \to E \) is the unit of the adjunction between \( \pi_* \) and \( \pi^* \), and the map \( E \otimes E \to E \) is given by cup product.

**Example 6.18. (Group algebra of a finite group.)**

For a finite group \( G \), define the \( H^0 \)-algebra \( \mathcal{O}[G] \) as follows.

- \( \mathcal{O}[G]_{\text{sing}} = \mathbb{Q}[G] \).
- \( \mathcal{O}[G]_{\text{et}} = \mathbb{Q}_p[G] \).
- \( \mathcal{O}[G]_{dR} = K[G] \) is the trivial vector bundle, with trivial connection.
- The filtration on \( \mathcal{O}[G]_{dR} \) is \( \mathcal{O}[G]_{dR} = \mathcal{O}_0 \supseteq \mathcal{O}_1 = 0 \).
- The filtered \( F \)-isocrystal is the constant vector bundle \( \mathcal{O}[G] \) with trivial connection. Its fiber at any point is the group algebra \( K_v[G] \), with Frobenius action coming from the Frobenius on \( K_v \).

Note that, for a group \( G \), it is natural to view the group algebra \( \mathbb{Q}[G] \) as a space of measures on \( G \), and thus dual to the space of functions on \( G \). Then the multiplication in the group algebra corresponds to convolution of measures. This suggests the right way to generalize the group algebra to group schemes, as the dual to their ring of functions. (Of course, the trace map makes their ring of functions self-dual.)

**Example 6.19. (Group algebra of a finite group scheme over \( K \).)**

Let \( G \) be a finite étale group scheme over a number field \( K \). The group operation \( G \times G \to G \) defines a Hopf algebra comultiplication \( \Gamma(G, \mathcal{O}_G) \to \Gamma(G, \mathcal{O}_G) \otimes \Gamma(G, \mathcal{O}_G) \). The dual map \( \Gamma(G, \mathcal{O}_G)^\vee \otimes \Gamma(G, \mathcal{O}_G)^\vee \to \Gamma(G, \mathcal{O}_G)^\vee \) gives \( \Gamma(G, \mathcal{O}_G)^\vee \) the structure of an algebra.

Denoting by \( \pi: G \to \text{Spec} \, K \) the structure map, we define \( E = (\pi_* \mathcal{O}_L|_G)^\vee \). A concrete description is as follows.

- \( E[G]_{\text{sing}} = \mathbb{Q}[G(\mathbb{C})] \) with the usual algebra structure.
- \( E[G]_{\text{et}} = \mathbb{Q}_p[G(\overline{K})] \) with its natural Galois action and structure of \( \mathbb{Q}_p \)-algebra.
- \( E[G]_{dR} = (\Gamma(G, \mathcal{O}_G))^\vee \) with is natural algebra structure.
- \( E[G]_{\text{cris}} = \left( \Gamma(G_{K_v}, \mathcal{O}_{G_{K_v}}) \right)^\vee \) with its natural algebra structure and Frobenius coming from the Galois action on \( G \).

After passing to an extension of \( K \) over which \( G \) splits, an element \( a \in G \) gives an element of each of \( E[G]_{\text{sing}}, E[G]_{\text{et}}, E[G]_{dR}, \) and \( E[G]_{\text{cris}}, \) and each of these four realizations is generated (as a vector space over the appropriate field) by \( G \).

**Example 6.20.** Let \( L/K \) be an extension of fields, and \( \pi: \text{Spec} \, L \to \text{Spec} \, K \) the corresponding map of schemes. If \( E \) is an \( H^0 \)-algebra on \( L \), then \( \pi_* E \) is an \( H^0 \)-algebra on \( K \).

6.2. Modules over an \( H^0 \)-algebra.
Definition 6.21. Let $E$ be an $H^0$-algebra. An $E$-module is a system of compatible periods $V$ with a map $m_V : E \otimes V \to V$, such that the composition

$$V \cong O_Q \otimes V \xrightarrow{e \otimes 1} E \otimes V \xrightarrow{m_V} V$$

is the identity map, and the diagram

$$(4) \quad \begin{array}{ccc}
E \otimes E \otimes V & \xrightarrow{1 \otimes m_V} & E \otimes V \\
\downarrow{m \otimes 1} & & \downarrow{m} \\
E \otimes V & \xrightarrow{m_V} & V
\end{array}$$

commutes.

Example 6.22. An $O_E$-module (Example 6.14) is analogous to a motive with coefficients in $E$. An $O[G]$-module (Example 6.19) is analogous to a motive with an action of the group scheme $G$.

If $V$ and $W$ are $E$-modules, then we define the tensor product

$$V \otimes_E W$$

as the coequalizer of two maps

$$E \otimes V \otimes W \rightrightarrows V \otimes W,$$

the first of which is induced from $E \otimes V \to V$, and the second from $E \otimes W \to W$. (See [12, §2.3].)

Definition 6.23. Let $E$ be a finite étale algebra over a field $E_0$. We say an $E$-module $V$ is equidimensional if it is free of finite rank.

Equivalently, writing $E$ as a product of fields $E_i$, we say that $V$ is equidimensional if

$$\dim_{E_i}(V \otimes_E E_i)$$

is independent of $i$.

In this case, we call that dimension the rank of $V$.

Definition 6.24. Suppose $E$ is a constant étale $H^0$-algebra, and $V$ is an $E$-module. We say $V$ is equidimensional if the stalks of $V_{\text{sing}}$ are equidimensional modules over the stalks of $E_{\text{sing}}$. In this case, the ranks of the stalks as $E_{\text{sing}}$-modules are constant; we call that rank the rank of $V$.

Lemma 6.25. If $V$ is an equidimensional $E$-module of rank $N$ on $X$, then the following statements hold.

- The stalks of $V_{\text{et}}$ are equidimensional modules of rank $N$ over the stalks of $E_{\text{et}}$.
- The stalks of $V_{\text{dR}}$ are equidimensional modules of rank $N$ over the stalks of $E_{\text{dR}}$.
- The stalks of $V_{\text{cris}}$ are equidimensional modules of rank $N$ over the stalks of $E_{\text{cris}}$.

Proof. Follows from the “comparison isomorphisms” in the definition of Hodge–Deligne system. □
**Definition 6.26.** Suppose $V$ is an equidimensional $E$-module of rank $N$. Then we say that $V$ is an $E$-module with $GL_N$-structure.

Suppose additionally that there exists a Hodge-Deligne system $L$ of rank 1, with a nondegenerate bilinear pairing

$$V \otimes_E V \to L.$$

If the pairing is alternating, we say that $V$ has $GSp_N$-structure; if symmetric, we say that $V$ has $GO_N$-structure.

Below (Definition 6.37) we will define a notion of “object with $G$-structure” in a Tannakian category. We warn the reader that Definition 6.26 is not consistent with Definition 6.37: Definition 6.26 applies to $E$-modules, for $E$ an $H^0$-algebra, while Definition 6.37 applies to objects of an $E$-linear tensor category, with $E$ a field.

In the next few sections, we’ll study the structure of $E$-modules with $GL$, $GSp$, or $GO$-structure, where $E$ is a constant, étale $H^0$-algebra.

### 6.3. Local systems on an abelian variety: construction of a Hodge–Deligne system.

Let $A$ be an abelian variety over a number field $K$, $X$ an arbitrary smooth variety over $\mathbb{Q}$, and $X_K$ its base change to $K$. Let

$$Y \subseteq X \times_{\mathbb{Q}} A = X_K \times_K A$$

be a subscheme, smooth, proper and flat over $X_K$. Let $f: Y \to X$ and $g: Y \to A$ be the projections. Fix a positive integer $r$. Let $L$ be a field, containing $K$, Galois over $\mathbb{Q}$, and over which $A[r]$ splits. Let $\chi$ be an order-$r$ character of $\pi_1^{\text{et}}(A)$, where $r$ is prime to $p$; let $\mathcal{L}_\chi$ be the corresponding character sheaf on $A$. (It is a $\mathbb{Q}_p[\mu_r]$-local system on the étale site of $A_L$.) Let $k = n - 1 = \dim Y$. We want to create a Hodge–Deligne system on $X$ whole base change to $L$ has $R^k f_* g^* \mathcal{L}_\chi$ as a direct summand.

The tensor product $K \otimes_{\mathbb{Q}} L$ splits as a direct sum

$$K \otimes_{\mathbb{Q}} L \cong \bigoplus_{\iota} L^{(\iota)},$$

indexed by the $[K: \mathbb{Q}]$ embeddings of $K$ into $L$. Here each $L^{(\iota)}$ is an isomorphic copy of $L$; the superscript $(\iota)$ is merely an index. We have the corresponding splitting

$$A \times_{\text{Spec } \mathbb{Q}} \text{Spec } L \cong \prod_{\iota} A_{\iota},$$

where for each $\iota$, we define

$$A_{\iota} = A \otimes_{\text{Spec } K, \iota} \text{Spec } L.$$

Similarly, define

$$Y_{\iota} = Y \otimes_{\text{Spec } K, \iota} \text{Spec } L,$$
the base change of $Y$ along $\iota$. Then we have the Cartesian diagram

\[
\begin{array}{ccc}
\prod \iota & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\prod \iota X_L \times_L A_i & \longrightarrow & X \times A \\
\downarrow & & \downarrow \\
X_L & \longrightarrow & X.
\end{array}
\]

Let $f_\iota : Y_\iota \to X_L$ and $g_\iota : Y_\iota \to A_i$ be the projections.

Let $r$ be prime to $p$, and let $\Pi^{K/Q}(A[r])$ be the set of all pairs $(\iota, \chi)$, where $\iota : K \to L$ is a $\mathbb{Q}$-linear embedding, and $\chi$ is a character of $\pi_1(A_\iota)$ of order dividing $r$. For fixed $\iota$, the set of characters $\chi$ is naturally identified with the set of maps $A_\iota[r] \to \mathbb{Q}[\mu_r]$:

\[
\Pi^{K/Q}(A[r]) \ni (\iota, \chi) \mapsto f_\iota \in A_\iota[r] \to \mathbb{Q}[\mu_r],
\]

thus, $\Pi^{K/Q}(A[r])$ has a natural action of

\[
\text{Gal}_Q \times \text{Gal}_{\mathbb{Q}[\mu_r]/Q},
\]

where $\text{Gal}_Q$ acts on the pairs $(\iota, \chi)$ via its action on $L$, and $\text{Gal}_{\mathbb{Q}[\mu_r]/Q}$ acts on $\mathbb{Q}[\mu_r]$.

**Lemma 6.27.** Let $\chi$ be character of $\pi_1^{et}(A)$ of some finite order $r$. Then there exists an $\mathcal{O}_{\mathbb{Q}[\mu_r]}$-module $L_\chi$ on $A|_L$ such that $(L_\chi)_{et}$ is the character sheaf associated with $\chi$, and $(L_\chi)_{sing}$ is the analytic $\mathbb{Q}[\mu_r]$-local system associated with $\chi$.

*Proof.* Descent. \hfill \Box

**Lemma 6.28. (Construction of $E_I$ and $V_I$.)** Let $I$ be an orbit of $\text{Gal}_Q \times \text{Gal}_{\mathbb{Q}[\mu_r]/Q}$ on $\Pi^{K/Q}(A)[r]$. There exist an $H^0$-algebra $E_I$ on $K$ and an $E_I$-module $V_I$ on $X$ with the following properties.

- After base change to $L$ and extension of coefficients to $\mathbb{Q}[\mu_r]$, we have the direct sum decomposition

\[
E_I|_L \otimes_{\mathbb{Q}} \mathcal{O}_{\mathbb{Q}[\mu_r]} \cong \bigoplus_{(\iota, \chi) \in I} \mathcal{O}_{\mathbb{Q}[\mu_r]}^{(\iota, \chi)},
\]

where each $\mathcal{O}_{\mathbb{Q}[\mu_r]}^{(\iota, \chi)}$ is a copy of $\mathcal{O}_{\mathbb{Q}[\mu_r]}$.

- The Galois representation $E_{I,et}$ is compatible with the isomorphism

\[
E_{I,et} \otimes_{\mathbb{Q}} \mathbb{Q}[\mu_r] \cong \bigoplus_{(\iota, \chi) \in I} \mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}[\mu_r],
\]

where $\text{Gal}_K$ acts on the right-hand side by permutation of the characters $\chi$.

Similarly, the Frobenius endomorphism of $E_{I,cris}$ is compatible with the isomorphism

\[
E_{I,cris} \otimes_{\mathbb{Q}} \mathbb{Q}[\mu_r] \cong \bigoplus_{(\iota, \chi) \in I} \mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}[\mu_r],
\]

where Frobenius acts on the right-hand side by permuting the pairs $(\iota, \chi)$, via the $\text{Gal}_Q$ action, with trivial action on $\mathbb{Q}[\mu_r]$. 

After base change to $L$ and extension of coefficients to $\mathbb{Q}[\mu_r]$, the module $V_I$ decomposes as the direct sum

$$V_I|_L \otimes_{\mathcal{O}_Q} \mathcal{O}_{\mathbb{Q}[\mu_r]} = \bigoplus_{(i,\chi) \in I} R^k f_* g^* L_X.$$

Furthermore, this decomposition is compatible with the decomposition of $E_I|_L \otimes_{\mathcal{O}_Q} \mathcal{O}_{\mathbb{Q}[\mu_r]}$ into fields.

$V_I$ can be made into a polarized, integral Hodge–Deligne system.

**Proof.** We have the map

$$id_X \times [r]: X \times \mathbb{Q} A \to X \times \mathbb{Q} A$$

where $[r]: A \to A$ is multiplication by $r$; let $Y_r = (id_X \times [r])^{-1}(Y)$.

We have an isomorphism of Hodge–Deligne systems

$$R^k h_* \mathcal{O}_Q \cong R^k f_* g^* [r]_* \mathcal{O}_Q.$$

Furthermore, on $A \times \text{Spec} \mathbb{Q} \text{Spec} L$, we have the direct sum decomposition of Hodge–Deligne systems

$$[r]_* \mathcal{O}_Q|_{A \times \mathbb{Q} L} \otimes_{\mathcal{O}_Q} \mathcal{O}_{\mathbb{Q}[\mu_r]} = \bigoplus_{(i,\chi) \in \Pi^{\mathcal{H}/\mathbb{Q}(A)|[r]}} L_{i,\chi},$$

where $L_{i,\chi}$ is the system $L_X$ on $A_i$, and the trivial system on all other components of $A \times \text{Spec} \mathbb{Q}$ Spec $L$. This gives a decomposition on $X_L$

$$(R^k h_* \mathcal{O}_Q)|_L \otimes \mathcal{O}_{\mathbb{Q}[\mu_r]} = \bigoplus_{(i,\chi) \in \Pi^{\mathcal{H}/\mathbb{Q}(A)|[r]}} R^k f_* g^* L_{i,\chi} = \bigoplus_{(i,\chi) \in \Pi^{\mathcal{H}/\mathbb{Q}(A)|[r]}} R^k f_* g^* L_{i,\chi}.$$

Let

$$V = R^k h_* \mathcal{O}_Q \cong R^k f_* g^* [r]_* \mathcal{O}_Q.$$

Let $E$ be the pullback to $X$ of the Hodge–Deligne system on $\mathbb{Q}$ coming from $E = (\pi_* \mathcal{O}_Q|_{A[r]})^\vee$,

where $\pi: A \to \mathbb{Q}$ is the projection. This $E$ has a structure of $H^0$-algebra, coming from the structure of group scheme on $A[r]$ (see Examples [6.19] and [6.20]). Furthermore, the group action $A[r] \times Y_r \to Y_r$ makes $V$ into a module over $E$.

After base change from $\mathbb{Q}$ to $L$ and extension of coefficients, we have a decomposition of $H^0$-algebras on $X$

$$E|_L \otimes_{\mathcal{O}_Q} \mathcal{O}_{\mathbb{Q}[\mu_r]} = \bigoplus_{(i,\chi) \in \Pi^{\mathcal{H}/\mathbb{Q}(A)|[r]}} \mathcal{O}_{\mathbb{Q}[\mu_r]}^{(i,\chi)}.$$
The set $\Pi_K/Q(A)[r]$ has commuting actions of $\text{Gal}_Q$ and $\text{Gal}_{Q^{cyc}}/Q$, the former coming from the base field $Q$, and the latter from the coefficient field $Q^{cyc}$. For each orbit $I$ of $\text{Gal}_Q \times \text{Gal}_{Q^{cyc}}/Q$, there exists an $H^0$-algebra $E_I$ over $Q$ such that

$$E_I|_L \otimes E_p = \bigoplus_{(\iota, \chi) \in I} O_{Q[\mu_r]}^{(\iota, \chi)}.$$ 

Over $Q$ and with $Q$-coefficients, $E$ splits as the direct sum of the algebras $E_I$.

The $E$-module $V$ also splits as the direct sum of the objects $V_I = V \otimes_E E_I$, and after base change and extension of coefficients we have

$$V_I|_L \otimes O_{Q[\mu_r]} = \bigoplus_{(\iota, \chi) \in I} R^k f_\iota^* g_\iota^* O_X.$$

Finally we have to explain the polarization and integral structure on $V_{I, \text{sing}}$. On $V_{\text{sing}}$ we have a standard polarization and integral structure: the polarization comes from Poincaré duality, and the integral structure is simply the integral structure on the singular cohomology of $Y_r$. These structures induce a polarization and integral structure on $V_{I, \text{sing}}$, by restricting the polarization and intersecting the integral lattice with $V_{I, \text{sing}}$.

Lemma 6.29. Fix notation as in Lemma 6.28. Then $V_I$ is an $E_I$-module with $GL_N$-structure, in the sense of Definition 6.26.

Furthermore, if $Y$ is equal to a translate of $[-1]^*Y$, then $V_I$ has $GSp_N$-structure if $n$ is even and $GO_N$-structure if $n$ is odd.

Proof. To prove that $V_I$ has $GL_N$-structure we only need to check that $V_I$ is equidimensional; this is a consequence of the transitive Galois action on the index set $I$.

Suppose $Y$ is equal to a translate of $[-1]^*Y$. This equality gives an involution $\iota: V \to V$.

The cup product pairing and the trace map compose to give a map

$$\langle -, - \rangle: V \otimes V = R^k h_* O_Q \otimes R^k h_* O_Q \to R^{2k} h_* O_Q \to O_Q(-k).$$

This pairing does not factor through $V \otimes_E V$, which would be equivalent to the identity

$$\langle ev, w \rangle = \langle v, ew \rangle$$

of maps $V \times E \times V \to O_Q(-k)$ (where $v, e, w$ are local sections of the sheaves underlying $V, E, \text{ and } V$.) Instead, for $a \in A[r]$, the pairing satisfies

$$\langle v, w \rangle = \langle av, aw \rangle.$$

However, when $Y$ is equal to a translate of $[-1]^*Y$, and $\iota$ is the involution of $V$ described above, we can form the pairing

$$\langle v \otimes w \rangle \mapsto \langle v, \iota w \rangle$$

as the composition

$$V \otimes V \overset{\text{1st}}{\to} V \otimes V \overset{\langle -, - \rangle}{\to} R^{2k} f_\iota^* g_\iota^* O_Q|_Y.$$

I claim that this pairing does satisfy (5), so that it descends to a pairing

$$V \otimes_E V \to R^{2k} f_\iota^* g_\iota^* O_Q|_Y.$$
It is enough to check (5) after extending both the base field and the coefficient field, so we can assume $E$ is the group algebra of the finite abelian group $A[r]$, and that $E$ splits as a direct sum over characters of that group. Now (5) follows from
\[
\langle a v, w \rangle = \langle v, a^{-1} w \rangle = \langle v, \iota(a w) \rangle,
\]
where $a \in A[r]$.

We have
\[
\langle w, w \rangle = \langle \iota w, \iota^2 v \rangle = \langle \iota w, v \rangle = (-1)^k \langle v, \iota w \rangle
\]
so this pairing is symplectic if $n$ is even (and thus $k$ is odd) and symmetric if $n$ is odd (and thus $k$ is even).

For a fixed $(\text{Gal}_{\mathbb{Q}} \times \text{Gal}_{\mathbb{Q}^{simpl}})$-orbit $I$ on $\Pi^{\mathbb{K}/\mathbb{Q}}(A)[r]$, define the reductive group $H$ as follows. Let $N$ be the rank of $V_I$. If $Y$ is equal to a translate of $[-1]^* Y$ and $n$ is even, let $H = GSp_N$. If $Y$ is equal to a translate of $[-1]^* Y$ and $n$ is odd, let $H = GO_N$. Otherwise, let $H = GL_N$. Then by Lemma 6.29, $V_I$ has $H$-structure.

In the following sections we will analyze this structure in some detail.

6.4. An algebraic group. Let $E_0$ be a field – we will mostly be interested in the case $E_0 = \mathbb{Q}_p$ – and let $E$ be a finite étale $E_0$-algebra. Let $H$ be a reductive group over $E_0$, and choose a representation $V_{\text{simp}}$ of $H$. We will assume that $H$ is one of $GL_N$, $GSp_N$, or $GO_N$, and $V_{\text{simp}}$ is the standard representation.

Let $G^0$ be the Weil restriction $G^0 = \text{Res}_{E_0}^E H$, and let $V = E \otimes_{E_0} V_{\text{simp}}$. By restriction of scalars, we will view $V$ as an $E_0$-vector space with actions of $E$ and $G^0$. Let $G$ be the normalizer of $G^0$ in the group of $E_0$-linear automorphisms of $V$.

Definition 6.30. Let $\sigma$ be an $E_0$-linear automorphism of $E$. We say that an $E_0$-linear endomorphism $\phi: V \to V$ is $\sigma$-semilinear (or semilinear over $\sigma$) if
\[
\phi(\lambda v) = \sigma(\lambda) \phi(v)
\]
for all $\lambda \in E$ and $v \in V$.

Lemma 6.31. If $H = GL_N$, then $G$ is the algebraic group whose $E_0$-points correspond to endomorphisms $\phi$ of $V$, semilinear over some $E_0$-linear automorphism $\sigma$ of $E$.

If $H$ is $GSp_N$ or $GO_N$, then $G^0$ preserves, up to scaling, an (alternating or symmetric) $E$-valued pairing $\langle -,- \rangle$ on $V$. In this setting $G$ is the group of endomorphisms $\phi$ of $V$, semilinear over some $E_0$-linear automorphism $\sigma$ of $E$, and satisfying
\[
\langle \phi(v_1), \phi(v_2) \rangle = C \sigma(\langle v_1, v_2 \rangle)
\]
for some $C \in E$, for all $v_1, v_2 \in V$.

In particular, we have an exact sequence of groups
\[
1 \to G^0 \to G \to \text{Aut}_{E_0} E \to 1.
\]

Proof. Any element of the normalizer of $G$ normalizes the center of $G^0$, which is $E$, acting by scalar multiplication. The $E_0$-linear automorphisms of $V$ that normalize the action of $E$ are exactly the semilinear automorphisms.

If $H$ is $GSp_N$ or $GO_N$, then for $\phi \in G$ which is $E$-semilinear over $\sigma$, the bilinear form $\langle \phi(v_1), \phi(v_2) \rangle$ is $E$-semilinear over $\sigma$ in both variables and is preserved up to scaling by $G^0$. Thus $\sigma^{-1}(\langle \phi(v_1), \phi(v_2) \rangle)$ is $E$-bilinear and preserved up to scaling.
by $G^0$. Hence it is a scalar multiple of $\langle v_1, v_2 \rangle$, which is the unique $E$-linear $G^0$-equivariant form. This gives \((6)\).

Conversely, any $E$-semilinear automorphism which, if $H$ is $GSp_N$ or $GO_N$, satisfies \((6)\), manifestly normalizes $G^0$.

We need a generalization of \([40\] Lemma 2.1].

**Lemma 6.32.** Suppose $\sigma \in \text{Aut}_{E_0} E$ is such that $E^\sigma = E_0$, and suppose $\phi \in G$ is semilinear over $\sigma$. Then the centralizer $Z_G(\phi)$ satisfies

$$\dim Z_G(\phi) \leq \dim H.$$

**Proof.** The proof goes through exactly as in \([40\]. By passing to the algebraic closure we may assume that $E = E_0^a$, so $G^0 = H^a$. The hypothesis implies that $\sigma$ acts transitively on the $d$ factors $H$ of $H^a$. A calculation shows that the projection $Z_{G^0}(\phi) \to H$ onto any single factor $H$ of $H^a$ is injective. The result follows because $Z_{G^0}(\phi)$ has finite index in $Z_G(\phi)$.

\(\square\)

### 6.5. Disconnected reductive groups.

We need to apply $p$-adic Hodge theory in a Tannakian setting: we’ll work with Galois representations valued in the disconnected algebraic group $G$, and the corresponding filtered $\phi$-modules. To this end, we need to study filtrations valued in a disconnected algebraic group.

We will use a notion of parabolic subgroup due to Richardson \([49\) (see also \([41\) and the forthcoming survey \([50\]). Let $G$ be an algebraic group over a field of characteristic zero whose identity component $G^0$ is reductive.

**Definition 6.33.** Let $f : \mathbb{G}_m \to G$ be a map of schemes. We say that $\lim_{t \to 0} f(t)$ exists if $f$ extends to a map $\hat{f} : \mathbb{A}^1 \to G$; in this case we write

$$\lim_{t \to 0} f(t) = \hat{f}(0).$$

Let $\mu : \mathbb{G}_m \to G$ be a cocharacter. Define the subgroups $P_\mu$, $L_\mu$, and $U_\mu$ of $G$ as follows.

- $P_\mu = \{g \in G \mid \lim_{t \to 0} \mu(t) g(t) \mu(t)^{-1} \text{ exists}\}$
- $U_\mu = \{g \in G \mid \lim_{t \to 0} \mu(t) g(t) \mu(t)^{-1} = 1\}$
- $L_\mu = \{\lim_{t \to 0} \mu(t) g(t) \mu(t)^{-1} \mid g \in P_\mu\}$

We say that a subgroup $P \subseteq G$ is parabolic if it is of the form $P_\mu$ for some $\mu$; in this case we say $L_\mu$ is an associated Levi subgroup, for any $\mu$ such that $P_\mu = P$.

In this setting, $U_\mu$ is unipotent, and $P_\mu$ is the semidirect product of $L_\mu$ by $U_\mu$. Furthermore, $L_\mu$ is the centralizer of $\mu$ in $G$ \([41\) Prop. 5.2].

**Example 6.34.** The purpose of this example is to show that a parabolic subgroup $P \subseteq G$ is not uniquely determined by the parabolic subgroup $P^0 = P \cap G^0$ of $G^0$.

Let $E = E_a^2$ and $H = GL_N$, and let $G = G$ be the group defined in Section 6.4. Then $G^0$ is the product of two copies of $GL_N$. Let $\mu_1$ be the cocharacter that acts as $t^2$ on one copy of $GL_N$ and $t^3$ on the other. Let $\mu_2$ be the cocharacter that acts as $t^2$ on both copies of $GL_N$. Then $P_{\mu_1} = G^0$ but $P_{\mu_2} = G$.

Now we turn to the notion of semisimplification of subgroups of $G$. See \([55\) for a discussion of this notion in the connected setting; it is applied in \([40\) §2.3]. For the general theory of complete reducibility for disconnected reductive groups, see \([5, 7, 4\), and \([6\]. We warn the reader that the theory is developed there over an arbitrary field, and many complications arise in positive characteristic that are irrelevant to us here.
Definition 6.35. We say that an algebraic subgroup $H \subseteq G$ is \textit{G-completely reducible} if, for every parabolic $P$ containing $H$, there is some associated Levi subgroup $L$ that also contains $H$.

If $H \subseteq G$ is an arbitrary algebraic subgroup, we define its \textit{semisimplification} with respect to $G$ as follows. Let $P$ be a parabolic subgroup of $G$, minimal containing $H$. Choose a Levi subgroup $L$ of $P$, and let $H^{ss}$ be the image of $H$ under the projection $P \to L$.

Lemma 6.36. Let $H \subseteq G$ be an algebraic subgroup. Then $H$ is $G$-completely reducible if and only if the identity component of $H$ is reductive. If we choose an embedding of $G$ into $GL_n$, then $H$ is $G$-completely reducible if and only if it is $GL_n$-completely reducible.

For a general algebraic subgroup $H \subseteq G$, the $G$-semisimplification $H^{ss}$ is well-defined up to $G$-conjugacy, and it is $G$-completely reducible.

Proof. (This is proven in the work in progress \cite{Bate} §4 by Bate, Martin, and Röhrle. Martin shared with us the argument, which we explain below.)

We begin with the claim that $H \subseteq G$ is $G$-completely reducible if and only if the identity component of $H$ is reductive. Over an algebraically closed field, this is \cite{Bate} §2.2]. If $G$ and $H$ are algebraic over a field $K$ that is not algebraically closed, then $H$ is $G$-completely reducible over $K$ if and only if it is so over $\overline{K}$; see \cite{Bate} Theorem 5.7(i)] and \cite{Bate} Theorem 9.3]. This proves the claim over arbitrary $K$ (of characteristic zero).

If $G \subseteq GL_n$, then by the above, $H$ is $GL_n$-completely reducible if and only if its identity component is reductive.

Now we turn to the final claim, that $H^{ss}$ is well-defined up to $G(K)$-conjugacy, when $K$ is not algebraically closed, follows from \cite{Bate} Theorem 1.3].

\hfill $\Box$

6.6. \textbf{Some Tannakian formalism.} We recall the notion of object with $G$-structure. See \cite{Deligne} Def. 1.3]; a general reference for the Tannakian formalism is \cite{Deligne}.

Definition 6.37. Let $G$ be an algebraic group over a field $E$ of characteristic zero, and let $C$ be an $E$-linear rigid abelian tensor category. An \textit{object in $C$ with $G$-structure} is an exact faithful tensor functor $\omega: \text{Rep}G \to C$, where $\text{Rep}G$ is the category of finite-dimensional $E$-linear representations of $G$.

If $G_1 \to G_2$ is a map of algebraic groups, then an object with $G_1$-structure gives rise to an object with $G_2$-structure by functoriality.

Remark 6.38. If $V$ is a faithful representation of $G$, then an object $\omega$ with $G$-structure is determined by $\omega(V)$. In practice, we will find it useful to specify $\omega$ by describing $\omega(V)$.

6.6.1. \textbf{Filtrations.} (Filtrations on $G$.)

We want to define a notion of “filtered vector space with $G$-structure” or “filtration on $G$.” Definition 6.37 does not apply, because the category of filtered vector spaces is not abelian. Instead, we will use the formalism of filtrations from \cite{Deligne} §IV.2]. Throughout this section, $G$ will be an algebraic group over a field $E$ of characteristic zero.
Definition 6.39. A filtration on $G$ (or $G$-filtration) is an exact tensor filtration of the forgetful functor from $\text{Rep}_G$ to $\text{Vec}_E$, in the sense of [50] IV.2.1.1].

Loosely speaking, a $G$-filtration is a choice of descending filtration on every finite-dimensional representation of $G$, compatible with tensor products, duals, and passage to subquotients.

If $G_1 \to G_2$ is a map of algebraic groups, then a $G_1$-filtration gives rise to a $G_2$-filtration by functoriality.

Since the base field $E$ is of characteristic zero, every $G$-filtration is “scindable” [50, Théorème IV.2.4] and hence also “admissible” [50] IV.2.2.1]. In particular, every filtration comes from a gradation (in general not unique) on $G$.

Let us explain this more carefully. A representation of $G_m$ on a finite-dimensional vector space $V$ gives a decomposition $V = \bigoplus V_j$, where $V_j$ is the $t^j$-eigenspace of $G_m$ on $V$. A cocharacter $\mu : G_m \to G$ defines a filtration on $G$ by defining, for all representations $V$ of $G$,

$$\text{Fil}^i V = \bigoplus_{j \geq i} V_j.$$ 

That every filtration on $G$ is “scindable” means that every filtration comes from some cocharacter $\mu$ (in general not unique).

Given a filtration on $G$, we can define two distinguished subgroups $P$ and $U$ of $G$ [50, §IV.2.1.3]. The subgroup $P \subseteq G$ is the group of elements that stabilize the filtration on every representation $V$ of $G$, and $U \subseteq P$ is the group of elements that stabilize the filtration and furthermore act as the identity on the associated graded. When $G$ is a group whose identity component is reductive, and the filtration is defined by a cocharacter $\mu$, the group $P$ coincides with the Richardson parabolic $P_\mu$ (Definition 6.33), and $U$ is its unipotent radical $U_\mu$.

Again supposing $G$ is a group whose identity component is reductive, define an equivalence relation on the set of cocharacters of $G$ as follows: we say that $\mu_1 \sim \mu_2$ if $P_{\mu_1} = P_{\mu_2}$ and $\mu_1$ is conjugate to $\mu_2$ in this parabolic. Then $\mu_1$ and $\mu_2$ give rise to the same filtration on $G$ if and only if $\mu_1 \sim \mu_2$.

If $G$ acts faithfully on $V$, then a $G$-filtration is determined by the corresponding filtration on $V$, as in Remark 6.38 we will use this without comment.

We mention in passing that, even if $G$ is reductive, the parabolic $P$ does not quite determine the filtration. If $G$ is reductive, then $P$ determines a filtration on every representation of $G$ only up to indexing.

6.6.2. $G$-filtrations vs $G^0$-filtrations. Let $G$ and $G^0$ be as in Section 6.4 (The discussion in this section applies to any group $G$ whose identity component is reductive, but we will apply it only to $G$ and $G^0$.)

By definition, a $G^0$-filtration $F^0$ gives rise to a $G$-filtration $F$, by composition

$$G_m \to G^0 \to G.$$ 

Since $G_m$ is connected, every $G$-filtration arises in this way. Conversely, given a $G$-filtration, we will see shortly (Lemma 6.40) that there is a unique $G^0$-filtration giving rise to it.

Lemma 6.40. Suppose $\mu_1, \mu_2 : G_m \to G$ are two characters defining the same filtration on $G$. Then $\mu_1$ and $\mu_2$ define the same filtration on $G^0$. 


Proof. We know there is some $g \in P_{\mu_1}$ such that
\[ g\mu_1 g^{-1} = \mu_2. \]
We need to show that we can in fact take $g \in P_{\mu_1}^0 \cap G^0$.

Let $g_L$ be the image of $g$ under the projection $P_{\mu_1} \to L_{\mu_1}$. Then $g_L$ centralizes $\mu_1$ by \[\text{[11 Prop. 5.2(a)]},\] and from the limit formula
\[ g_L = \lim_{t \to 0} \mu(t) g \mu(t)^{-1} \]
we see that $g g_L^{-1} \in G^0$. Thus $g g_L^{-1} \in G^0$ conjugates $\mu_1$ to $\mu_2$. □

Definition 6.41. We say that a $G^0$-filtration $F^0$ is associated to a $G$-filtration $F$ if there is a single cocharacter $\mu : G_m \to G^0 \subseteq G$ giving rise to both $F^0$ and $F$.

By the discussion above, if $F$ and $F^0$ are associated, then any $\mu$ giving rise to one necessarily gives rise to the other. Being associated defines a bijection between $G^0$-filtrations and $G$-filtrations.

6.6.3. $G$-objects in the category of weakly admissible filtered $\phi$-modules. Let $G$ be an algebraic group over $\mathbb{Q}_p$. (It is important that we work over $\mathbb{Q}_p$, and not an extension.) A $G$-object in the category of weakly admissible filtered $\phi$-modules gives rise to a $G$-filtration and an element (the Frobenius endomorphism) in $G$. In the spirit of Remark 6.38, we can describe such an object as a triple $(V, \phi, F)$, where $V$ is a vector space on which $G$ acts faithfully, $\phi \in G$ is an automorphism of $V$, and $F$ is a $G$-filtration on $V$, such that $(V, \phi, F)$ is weakly admissible. Somewhat imprecisely, we will call such objects “filtered $\phi$-modules with $G$-structure.”

6.6.4. $p$-adic Hodge theory. The next result is a Tannakian form of the crystalline comparison theorem, for representations valued in an arbitrary algebraic group $G$. In order to avoid problems with semilinearity in the target category, we restrict to representations of $\text{Gal}_{\mathbb{Q}_p}$.

Lemma 6.42. Let $G \subseteq GL_N$ be an algebraic group over $\mathbb{Q}_p$. We say that a local Galois representation
\[ \text{Gal}_{\mathbb{Q}_p} \to G \]
is crystalline if the representation
\[ \text{Gal}_{\mathbb{Q}_p} \to G_N \]
is crystalline in the usual sense. This property is independent of the choice of embedding $G \hookrightarrow GL_N$.

The functor $D_{\text{cris}}$ of $p$-adic Hodge theory \[\text{[21 Expose III]}\] associates to every crystalline representation of $\text{Gal}_{\mathbb{Q}_p}$ a $G$-object in the category of admissible filtered $\phi$-modules over $\mathbb{Q}_p$.

Proof. This follows from standard Tannakian properties of $D_{\text{cris}}$. □
6.7. Structure of $E$-modules. Let $E$ be an étale $H^0$-algebra, and $V$ an $E$-module with $H$-structure, where $H$ is $GL_N$, $GSp_N$, or $GO_N$. We described the structure of $E$ in Example 6.16. Now we are ready to describe $V$.

The local system $E_{et}$ is pulled back from a Galois representation on $\text{Spec } K$, which we’ll also call $E_{et}$. By definition this is a finite étale $\mathbb{Q}_p$-algebra $E$ with an action of $\text{Gal}_K$.

Let $E_0 = \mathbb{Q}_p$, and let $G$ and $G^0$ be as in Section 6.4. At any $x \in X(K)$, the fiber $V = V_{et,x}$ of the étale local system is a representation of the Galois group $\text{Gal}_K$. This $V$ has the structure of $E$-algebra, and if $H$ is $GSp$ or $GO$ then there is a pairing $V \otimes E V \to L$, where $L$ is a one-dimensional $\mathbb{Q}_p$-vector space with an action of $\text{Gal}_K$. The action of $\text{Gal}_K$ respects the pairing, and acts on $V$ semilinearly over its action on $E$. It follows (Lemma 6.31) that the representation of $\text{Gal}_K$ on $V$ is given by a map

$$\text{Gal}_K \to G,$$

and the quotient

$$\text{Gal}_K \to G/G^0 \to \text{Aut}_{E_0} E$$

is exactly the representation of $\text{Gal}_K$ on $E$ given by the structure of $E_{et}$.

Now we turn to de Rham cohomology and its cousins. For any point $x \in X(K)$, the group $G^0$ can be realized as the group of automorphisms of $V_{dR,x}$ respecting the $E_{dR}$-action and (where applicable) the bilinear pairing.

The differential Galois group $G_{mon}$ satisfies $G_{mon} \subseteq G^0$. The weaker inclusion $G_{mon} \subseteq G$ is immediate from the Tannakian formalism; the reason $G_{mon}$ is in fact contained in the finite-index subgroup $G^0$ is that the monodromy action on $E_{sing}$ is trivial.

Lastly, we describe the filtered $\phi$-module $V_{cris,x}$. For this, we assume that $K_v = \mathbb{Q}_p$. Recall the structure of $E_{cris}$ from Example 6.16: it is the $\mathbb{Q}_p$-algebra $E$, equipped with a Frobenius endomorphism $\sigma$. Then $V_{cris,x}$ is a vector space over $E$, and it is naturally a filtered $\phi$-module with $G$-structure. Its Frobenius endomorphism $\phi$ is semilinear over $\sigma \in \text{Aut}_E$.

6.8. Galois representations. The next result is a form of the Faltings finiteness lemma. Compare also [40, Lemma 2.4], which applies when $G$ is a connected reductive group.

**Lemma 6.43.** Let $E_0 = \mathbb{Q}_p$, and let $E$, $N$, $H$, $G$, $G^0$ be as in Section 6.4. The group $G$ has a standard embedding into $GL_N[E_{cris}(\mathbb{Q}_p)]$, coming from its action by endomorphisms on a free $E$-module of rank $n$.

Fix a number field $K$, a finite set $S$ of primes of $\mathcal{O}_K$, and an integer $w$. There are, up to $G$-conjugacy, only finitely many Galois representations

$$\rho: \text{Gal}_K \to G$$

satisfying the following conditions.

- The representation $\rho$ is semisimple (in the sense of Section 6.6).
- The representation $\rho$ is unramified outside $S$.
- The representation $\rho$ is pure of weight $w$ with integral Weil numbers: for every prime $\ell \not\in S$, the characteristic polynomial of Frobenius has all roots algebraic integers of complex absolute value $q^{w/2}$.
Proof. If \( \rho \) is semisimple in the sense of Section 5.5, then it is semisimple in the usual sense (as a representation into \( \text{GL}_n[E \otimes \mathbb{Q}_p](\mathbb{Q}_p) \)); see Definition 6.35 and Lemma 6.36.

Let \( \text{Gal}_L \subseteq \text{Gal}_K \) be the kernel of

\[ \text{Gal}_K \to G \to G/G^0. \]

The representation \( \rho \) restricts to a \( G^0 \)-valued representation

\[ \rho|_L : \text{Gal}_L \to G^0, \]

which is also semisimple (in the usual sense). Now \( G^0 \) is a connected reductive group, so we can use [40, Lemma 2.6] to conclude that there are only finitely many possibilities for \( \rho|_L \).

For each fixed choice of \( \rho|_L \), if \( \rho|_L \) extends to a map \( \rho \) compatible with the map \( \text{Gal}_K \to G/G^0 \), then the possible extensions form a torsor under \( H^1(\text{Gal}_L/K, \mathbb{Z}G^0(\text{Im}(\rho|_L))) \).

This cohomology group is a subgroup of \( H^1(\text{Gal}_K/K, \mathbb{Z}G^0(\text{Im}(\rho|_L))) \), which is finite by Serre’s \( H^1 \) lemma ([54, §III.4.3, Théorème 4]). \( \Box \)

Lemma 6.44. Let \( G \) be an algebraic group over a field \( E \) whose identity component is reductive.

Let \( H \subseteq G \) be an algebraic subgroup.

Then the set of Levi subgroups of \( G \) containing \( H \) and defined over \( E \) forms finitely many orbits under conjugation by the \( E \)-points of the centralizer \( Z_G(H) \).

Proof. A Levi subgroup \( L \) of \( G \) is the centralizer of a cocharacter \( \mu : \mathbb{G}_m \to G \); the subgroup \( L \) contains \( H \) if and only if \( \mu \) takes values in \( Z_G(H) \). Since all maximal \( E \)-split tori in \( Z_G(H) \) are conjugate, we can assume that \( \mu \) takes values in a fixed torus \( T \).

So we need to know that cocharacters \( \mu \) of \( G \), taking values in a given torus \( T \), define only finitely many different Levi subgroups \( L \) of \( G \). It is well-known that there are only finitely many possibilities for \( L^0 = L \cap G^0 \). But now we have

\[ L^0 \subseteq L \subseteq N_G(L^0), \]

and since \( N_G(L^0) \) contains \( L^0 \) with finite index, there are only finitely many possibilities for \( L \). \( \Box \)

Remark 6.45. Even if \( G^0 \) is a torus, and thus has only a single Levi subgroup, there can be multiple Levi subgroups of \( G \) if the component group of \( G \) acts nontrivially on the cocharacters of \( G^0 \), because a component will be in the Levi if and only if it fixes the cocharacter \( \mu \). However, there will only be finitely many.

Lemma 6.46. (only finitely many Levis to give the semisimplification)

Let

\[ \rho_0 : \text{Gal}_K \to G \]

be a semisimple Galois representation. Then there exists a finite collection of Levi subgroups \( L \subseteq G \) with the following property: for any Galois representation

\[ \rho : \text{Gal}_K \to G \]

whose semisimplification is \( G \)-conjugate to \( \rho_0 \), there exist \( g \in G \), an \( L \) from the finite collection, and a parabolic subgroup \( P \) containing \( L \), such that \( g \rho g^{-1} \) takes values in \( P \), and the composite map

\[ G \to g \rho g^{-1} \to P \to L \]
is $\rho_0$.

Proof. This is an immediate consequence of Lemma 6.44. $\Box$
7. Period maps and monodromy

7.1. Compatible period maps and Bakker–Tsimerman. The Bakker–Tsimerman theorem \[3\] is a strong result on the transcendence of complex period mappings. It implies a $p$-adic analogue, which we recall here. (See also \[40\], §9.)

The $p$-adic Bakker–Tsimerman theorem is an unlikely intersection statement for the $p$-adic period map attached to a Hodge–Deligne system. (For a detailed discussion of this period map, see \[40\], §3.3-3.4.) Suppose $V$ is a Hodge–Deligne system on $X$, and let $\Omega \subseteq X(K_v)$ be a $v$-adic residue disk. For all $x \in \Omega$,

$$V_{\text{cris},x}$$

is a filtered $\phi$-module $(V_x, \phi_x, \text{Fil}_x)$. The structure of $F$-isocrystal means that, for all $x \in \Omega$, the vector spaces $V_x$ are canonically identified, in such a way that $\phi_x$ is constant on $\Omega$. Using this identification, the filtration Fil$_x$ varies $p$-adically in $x$.

A priori, this defines a map

$$\Phi_p: \Omega \to H \cong GL_N/P$$

into some flag variety, where $N$ is the rank of $V$. We will show that the period map actually takes values in a smaller flag variety $G_{\text{mon}}/P_{\text{mon}}$, where $G_{\text{mon}}$ is the differential Galois group of $V$.

Lemma 7.1. If $G_{\text{mon}}$ is the differential Galois group of $V$, then in fact the image of $\Phi_p$ is contained in a single orbit of $G_{\text{mon}}$.

Proof. (This argument was suggested by Sergey Gorchinskiy.)

Fix a basepoint $x_0 \in X(K')$, for some field $K'$ with fixed embeddings into both $\mathbb{Q}_p$ and $\mathbb{C}$. \[4\] Let $H$ be the flag variety classifying filtrations with the same dimensional data as the filtration on $V_{\text{dR},x_0}$; we have an isomorphism $H \cong GL_N/P$, for $P$ some parabolic subgroup of $GL_N$.

Consider the complex period map $\Phi_C$ from the universal cover $\tilde{X}_{\text{an}}$ of $X_{\text{an}}$ to $H$. The theorem of the fixed part (\[51\], §7) implies that $\Phi_C$ lands in a single orbit of $G_{\text{mon}}$.

To transfer the result from $\Phi_C$ to $\Phi_p$, we use Picard–Vessiot theory. (For an introduction to Picard–Vessiot theory, see \[47\].)

To the vector bundle with connection underlying $V$, Picard–Vessiot theory attaches a $G_{\text{mon}}$-torsor $PV \to X_{K'}$, whose fiber over any $L$-point $x$ classifies vector space isomorphisms

$$V_{\text{dR},x} \cong V_{\text{dR},x_0}$$

respecting the $G_{\text{mon}}$-structure on both sides (but not, in general, the filtrations). Furthermore, we obtain a $G_{\text{mon}}$-equivariant map

$$\Phi_{PV}: PV \to GL_N/P$$

where a point of $PV$ over $x \in X$ gives an isomorphism $V_{\text{dR},x_0} \cong V_{\text{dR},x_0}$, and we use that isomorphism to identify the filtration on $V_{\text{dR},x}$ with a point of the flag variety $H$.

In the complex setting, integrating the connection gives a complex-analytic map

$$t_C: \tilde{X}_{\text{an}} \to PV_{\text{an}}.$$

\[4\]In our setting we are trying to prove finiteness of integral points, residue disk by residue disk, so there is no harm in taking $K' = K$: if the residue disk contains no rational points, there is nothing to prove.
in the \(p\)-adic setting, integration gives a rigid-analytic map
\[
t_p: \hat{X}^{rig} \to PV^{rig}.
\]
By definition, the complex and \(p\)-adic period maps are given by \(\Phi_{PV} \circ \iota_C\) and \(\Phi_{PV} \circ t_p\), respectively.

Since the image of \(\Phi_C\) is contained in a single \(G_{\text{mon}}\)-orbit on \(\mathcal{H}\), the same is true of the image of \(\Phi_{PV}\) by \(G_{\text{mon}}\)-equivariance. Hence, the image of \(\Phi_p\) is itself contained in a single \(G_{\text{mon}}\)-orbit. \(\square\)

**Lemma 7.2.** The construction above defines a \(p\)-adic period map
\[
\Phi_p: \Omega \to G_{\text{mon}}/P_{\text{mon}},
\]
where \(P_{\text{mon}}\) is a parabolic subgroup of \(G_{\text{mon}}\).

**Proof.** By Lemma 7.1, the \(p\)-adic period map takes values in \(G_{\text{mon}}/(P \cap G_{\text{mon}})\), where \(P \subseteq GL_N\) is the parabolic subgroup determined by the Hodge cocharacter \(\mu\). What remains is to show that \(P \cap G_{\text{mon}}\) is parabolic in \(G_{\text{mon}}\).

We know that \(\mu\) lies in the generic Mumford-Tate group, and hence normalizes \(G_{\text{mon}}\) [11 §5 Thm. 1]. Since the outer automorphism group of \(G_{\text{mon}}\) is finite, some power \(\mu^r\) acts on \(G_{\text{mon}}\) by inner automorphisms. Thus the adjoint action of \(\mu^r\) by conjugation on \(G_{\text{mon}}\) gives a map
\[
G_m \to \text{Aut } G_{\text{mon}}.
\]
Further increasing \(r\) if necessary, we can lift this to a cocharacter
\[
\nu: G_m \to G_{\text{mon}}.
\]
Then the parabolic defined by \(\nu\) in \(G_{\text{mon}}\) is exactly \(P \cap G_{\text{mon}}\). \(\square\)

From now on, the parabolic subgroup \(P_{\text{mon}} \subseteq G_{\text{mon}}\) will simply be called \(P\). We have defined a \(p\)-adic period map, valued in \(G_{\text{mon}}/P\).

**Theorem 7.3.** (\(p\)-adic Bakker–Tsimerman theorem). Let \(V\) be a polarized, integral Hodge–Deligne system on a smooth variety \(X\) over \(K\), and let \(G_{\text{mon}}\) be the differential Galois group of \(V\).

Choose a \(p\)-adic residue disk \(\Omega \subseteq X(C_{K_p})\), and a basepoint \(x_0 \in \Omega\). By Lemma 7.2 these give rise to a period map
\[
\Phi_p: \Omega \to G_{\text{mon}}/P,
\]
where \(P\) is a parabolic subgroup of \(G_{\text{mon}}\).

Suppose \(Z \subseteq G_{\text{mon}}/P\) is a closed subscheme such that
\[
\text{codim}_{G_{\text{mon}}/P} Z \geq \dim X.
\]
Then \(\Phi_p^{-1}(Z)\) is not Zariski-dense in \(X\).

**Proof.** This is [11 Lemma 9.3]. That lemma is stated only for \(G\) the full orthogonal or symplectic group, but the proof given there applies verbatim for all reductive \(G\), as does the underlying theorem of Bakker and Tsimerman [3] on which it is based. \(\square\)
7.2. **Complex monodromy.** Our next goal is Lemma 7.6, which shows that the differential Galois group of our \( V \) is strongly \( c \)-balanced. This is a “big monodromy” statement, analogous to \cite{40} Lemma 4.3 and \cite{40} Theorem 8.1.

**Lemma 7.4.** Let \( H \) be one of the algebraic groups \( SL_N, Sp_N, \) or \( SO_N \). Let \( G \) be a subgroup of \( H^d \) such that each of the \( d \) coordinate projections \( \pi_i: G \to H \) (for \( 1 \leq i \leq d \)) is surjective.

Define a relation \( \sim \) on the index set \( \{1, \ldots, d\} \) by declaring that \( i \sim j \) if and only if
\[
(\pi_i, \pi_j): G \to H^2
\]
is not surjective.

1. The relation \( \sim \) is an equivalence relation.
2. If \( i_1, \ldots, i_c \) are a complete set of representatives for \( \sim \), then the map
\[
(\pi_{i_1}, \ldots, \pi_{i_c}): G \to H^c
\]
is surjective with finite kernel.

**Proof.** This is an algebraic version of Goursat’s lemma. See \cite{40} Lemma 2.12 and \cite{48} Lemma 5.2.1.

The group \( H \) has a finite center; call it \( Z \). For any two indices \( i \) and \( j \) (possibly equal), there are two possibilities for the image of the projection
\[
G \xrightarrow{\pi_i, \pi_j} H^2 \to (H/Z)^2.
\]
Either the map is surjective, or its image is the graph of an automorphism of \( H/Z \).

In the former case, \((\pi_i, \pi_j)\) must surject (onto \( H^2 \)) as well.

A calculation shows that \( \sim \) is an equivalence relation. If \( i_1, \ldots, i_c \) is a complete system of representatives for \( \sim \), then repeated application of Goursat’s lemma shows that
\[
(\pi_{i_1}, \ldots, \pi_{i_c}): G \to H^c
\]
is surjective with finite kernel. 

**Definition 7.5.** Let \( H \) be one of the algebraic groups \( SL_N, Sp_N, \) or \( O_N \), and \( G \) a subgroup of \( H^d \). For \( 1 \leq i \leq d \), let \( \pi_i: G \to H \) be the coordinate projection, and suppose that each \( \pi_i \) is surjective. The **index classes of \( G \)** are the equivalence classes of the relation \( \sim \) of Lemma 7.4. We say that \( G \) is \( c \)-balanced (as a subgroup of \( H^d \)) if its index classes are all of equal size, and there are at least \( c \) of them.

Suppose now we are given a permutation \( \sigma \) of the index set \( \{1, \ldots, d\} \). We say \( G \) is strongly \( c \)-balanced (with respect to \( \sigma \)) if it is \( c \)-balanced, each orbit of \( \sigma \) on \( \{1, \ldots, d\} \) contains elements of at least \( c \) of the index classes of \( G \), and \( \sigma \) preserves the partition of \( \{1, \ldots, d\} \) into index classes.

Finally, let \( E_n, E, H, G, G^0 \) be as in Section 6.4 and let \( G^1 \) be:

- in the case \( H = GL_N \), the kernel of the determinant map \( G^0 \to \mathbb{G}_{m,E} \)
- in the case \( H = GSp_N \), the kernel of the similitude character \( G^0 \to \mathbb{G}_{m,E} \)
- in the case \( H = GO_N \), the intersection of the kernels of the determinant map \( G^0 \to \mathbb{G}_{m,E} \) and the similitude character \( G^0 \to \mathbb{G}_{m,E} \).

Then \( G^1 \) is a form of \( SL^d_N, Sp^d_N, \) or \( O^d_N \). We say that an algebraic subgroup \( G \subseteq G^0 \) is \( c \)-balanced (resp. strongly \( c \)-balanced with respect to \( \sigma \)) if \( (G \cap G^1)(\overline{K}) \) is \( c \)-balanced (resp. strongly \( c \)-balanced with respect to \( \sigma \)) as a subgroup of \( SL^d_N, Sp^d_N \) or \( O^d_N \).
Lemma 7.6. Suppose we have $K$, $p$, $v$, $A$, $X$, $Y$, $L$ as above. Fix some embedding $\iota_0: K \to L$. Suppose $\chi_0$ is as in Corollary 5.9, with $G^* = SL_N$, $Sp_N$, or $SO_N$, and let $E_0, E, H, G, G^0$ be as in Section 6.4. Let $I$ be the full $(\Gal_{Q} \times \Gal_{Q_{cyc}}/Q)$-orbit containing $(\iota_0, \chi_0)$, and let $V = V_I$ be the corresponding Hodge–Deligne system.

The Frobenius at $v$, an element of $\Gal_K$, acts on the set $I$; call this permutation $\sigma$.

Then the differential Galois group $G_{\text{mon}}$ of $V$ (base-changed to $\mathbb{Q}_p$) is a strongly $c$-balanced subgroup of $G^0$, with respect to $\sigma$.

Proof. The differential Galois group, after base change to $\mathbb{C}$, is the Zariski closure of the monodromy of the variation of Hodge structures. The variation of Hodge structure $V_H$ splits as the direct sum of $V_{H,(\iota,\chi)} = R^k f_\ast g_\ast L_{\chi}$. Clearly monodromy acts trivially on the set of pairs $(\iota, \chi)$, and when $H$ is $GSp$ or $GO$, there is a bilinear pairing on $R^k f_\ast g_\ast L_{\chi}$. Thus, $G_{\text{mon}} \subseteq G^0$.

Now Corollary 5.9 implies that the geometric monodromy group of $V_{H,(\iota_0,\chi_0)}$ is all of $H(\mathbb{C})$. By symmetry under the action of $(\Gal_{Q} \times \Gal_{Q_{cyc}}/Q)$, the same is true for all $(\iota, \chi) \in I$, so we can talk about the relation $\sim$ and the index classes of Definition 7.5.

The action of $(\Gal_{Q} \times \Gal_{Q_{cyc}}/Q)$ respects the relation $\sim$; in particular the index classes are all of the same size, and Frobenius respects the relation $\sim$. Corollary 5.9 shows that every $\sigma$-orbit in $I$ contains elements of at least $c$ index classes. $\square$
8. Hodge–Deligne systems and integral points

In this section we'll finish the proof of Theorem 8.21. Recall the setup from Section 6: we have a smooth variety $X$ over $\mathbb{Q}$ and a Hodge–Deligne system $V$ on $X$, satisfying certain conditions. Lemma 6.43 tells us that, as $x$ ranges over the integral points of $X$, there are only finitely many possibilities, up to semisimplification, for the global Galois representation $V_{x,ct}$. We will use this to bound the integral points of $X$.

A significant technical obstacle (both in [40] and here) is that Lemma 6.43 only applies to semisimple global Galois representations; there may be many different Galois representations arising as fibers of $V_{ct}$, all of which have the same semisimplification. If we assume the Grothendieck–Serre conjecture—that all Galois representations that arise are semisimple—this difficulty does not arise. Under this assumption, the argument is very simple; see Section 9.

To apply Lemma 6.43 without assuming semisimplicity, we need to recognize when two global Galois representations might have the same semisimplification. We only have access to the local representations, which are generally far from semisimple. The key idea is that any filtration on the global representation, when restricted to the local representation, must be of a special form.

Passing from local Galois representations via $p$-adic Hodge theory to filtered $\phi$-modules, we have the following situation. We're given a $\phi$-module $(V, \phi)$ with varying filtration $F = F^\bullet$; the variation of $F$ is described by a “monodromy group” $G_{mon}$ (the differential Galois group attached to our Hodge–Deligne system). We will consider the associated graded of $(V, \phi, F)$ with respect to a $\phi$-stable filtration $\mathfrak{f}$. By Lemma 8.8, if $\mathfrak{f}$ comes from a filtration on the global representation, then the associated graded of $F$ with respect to $\mathfrak{f}$ is a balanced filtration (Definition 8.6).

So we want to know, for how many choices of $F$ does there exist a $\phi$-stable $\mathfrak{f}$, such that the associated graded of $(V, \phi, F)$ with respect to $\mathfrak{f}$ lies in a given balanced isomorphism class? We will bound the dimension of such $F$ in the flag variety. This material is very similar to the combinatorial arguments in [40, §10-11].

To start with, we'll recall some results from [40] to limit the reducibility of global representations. Recall the following notion from [40, §2.2].

**Definition 8.1.** For a decreasing filtration $F^\bullet V$ on a vector space $V$ (with $F^0 V = V$) we define the weight of the filtration to be

$$\text{weight}_F(V) = \sum_{p \geq 0} p \dim \text{gr}^p(V) / \dim V,$$

where $\text{gr}^p(V) = F^p(V)/F^{p+1}(V)$ is the associated graded.

**Lemma 8.2.** Let $K$ be a number field having no CM subfield, and let $v$ be a place of $K$, unramified over $\mathbb{Q}$, and lying over the rational prime $p$. Let $V$ be a representation of $\text{Gal}_K$ on a $\mathbb{Q}_p$-vector space which is crystalline at all primes above $p$, and such that at all primes $\ell$ outside of a finite set $S$, the characteristic polynomial of Frobenius has algebraic coefficients and all roots rational $q\ell$-Weil numbers of weight $w$.

Let $V_{dR} = [V \otimes_{\mathbb{Q}_p} B_{\text{cris}}]^{\text{Gal}_K}$ be the filtered $K_v$-vector space that is associated to $\rho|_{\text{Gal}_K}$ by the $p$-adic Hodge theory functor $D_{\text{cris}}$ of [21, Expose III].

Then the weight of the Hodge filtration on $V_{dR}$ equals $w/2$. 
Proof. This is [10] Lemma 2.9; when \( K \) has no CM subfield, the condition that \( v \) be a “friendly place” simply means that \( K_v \) is unramified over \( \mathbb{Q}_p \).

Next we’ll rephrase this in terms of filtrations on reductive groups. We’ll work with filtrations and semisimplifications relative to a group \( G \) whose identity component is reductive. We will apply these results to the groups \( G \) and \( G^0 \) from Section 6.4 as well as similar groups. When \( G \) is disconnected, recall the notions of “parabolic subgroup,” “filtration” and so forth from Sections 6.4-6.6.

We’ll use the following notation (consistent with [10]): \( Q \) is the parabolic subgroup of \( G \) corresponding to the Hodge filtration \( F \), while \( P \) corresponds to the semisimplification filtration \( f \). The group \( M \) is the Levi quotient of \( P \), corresponding to the associated graded of \( f \).

Fix \( G, P, \) and \( M \). In the reductive case [10, Lemma 11.2] defines a map from filtrations \( F \) on \( G \) to filtrations \( F_M \) on \( M \); we’ll need to extend this to the group \( G \) as well. Recall (Section 6.6) that for any \( G \)-filtration \( F \), there is a cocharacter \( \mu : \mathbb{G}_m \to G \) defining \( F \). The substance of [10, Lemma 11.2] is that \( \mu \) can be chosen to take values in \( P \). Projecting from \( P \) to the Levi quotient \( M \) gives a filtration on \( M \), which is independent of the choice of \( \mu \).

Lemma 8.3. Suppose \( G \) is a group whose identity component is reductive. Let \( P \) be a parabolic subgroup of \( G \), and \( M \) its Levi quotient.

Fix a filtration \( F \) on \( G \). Then there exists a cocharacter \( \mu : \mathbb{G}_m \to P \) (with image in \( P \)) defining \( F \). Furthermore, if \( F_M \) is the filtration on \( M \) defined by the composite map \( \mathbb{G}_m \to P \to M \), then \( F_M \) is independent of the choice of \( \mu \).

Proof. For \( G \) connected reductive this is [10, Lemma 11.2].

In the general case, by [10, Lemma 11.2] applied to the identity component \( G^0 \), we know that \( \mu \) can be chosen with image in \( P \cap G^0 \), and the corresponding filtration on \( M \cap G^0 \) is independent of the choice of \( \mu \). But the filtration \( \mu \) defines on \( M \cap G^0 \) determines the filtration \( \mu \) defines on \( M \); see Section 6.6.2.

Definition 8.4. Given \( G, P, M, F \) as above, we call the filtration \( F_M \) on \( G \) the associated graded filtration, and write it as \( F_M = \text{Gr}_M F \).

This has a natural Tannakian meaning.

Lemma 8.5. Suppose \( F_P \) is a \( P \)-filtration, and \( F \) is the \( G \)-filtration obtained from \( F_P \) by functoriality with respect to the inclusion \( P \to G \). Then \( F_M \) is the \( M \)-filtration obtained from \( F_P \) by means of the quotient map \( P \to M \).

Proof. Follows immediately from the definitions.

We’ll need a generalization of the notion of “balanced filtration” from [10, §11.1, 11.4]. Given a group \( S \) whose identity component is reductive, we define
\[
a_S = X_*(Z_{S^0}) \otimes \mathbb{Q} = (X^*(S^0) \otimes \mathbb{Q})^\vee,
\]
where \( Z_{S^0} \) is the center of the identity component \( S^0 \) of \( S \). A cocharacter \( \mu \) of \( S \) defines a class in \( a_S \), which we call its weight \( w(\mu) = w_S(\mu) \).

Let \( G, P, M \) be as above. Then the inclusion \( Z(G^0) \to Z(M^0) \) gives a map \( \iota_{GM} : a_G \to a_M \). Furthermore, the parabolic \( P \) defines a preferred element of
(a_M)^\vee$, the modular character $\gamma_P$, defined as the inverse of the determinant of the adjoint representation of $M$ on the Lie algebra of $P$.

**Definition 8.6.** Suppose given $G$ an algebraic group whose identity component is reductive, $P$ a parabolic subgroup, and $M$ its Levi quotient. Let $F_M$ be a filtration on $M$, given by a cocharacter $\mu$.

- We say that $F_M$ is balanced with respect to $P$ if $w_M(\mu) = \iota_G M(w_G(\mu))$.
- We say that $F_M$ is weakly balanced if $\gamma_P(\mu) = 0$ for $\gamma_P$ the modular character of $P$.
- We say that $F_M$ is at least balanced if $\gamma_P(\mu) \leq 0$, for $\gamma_P$ the modular character of $P$.

We say a $G$-filtration $F$ is balanced (resp. weakly balanced, at least balanced) with respect to $P$ (or $M$, or $f$) if the associated graded $\text{Gr}_M F$ is so.

**Remark 8.7.** Balanced implies weakly balanced because $\gamma_P(\iota_G M(w_G(\mu))) = 0$; this identity boils down to the fact that the center of $G$ acts trivially through the adjoint representation on the Lie algebra of $P$.

Furthermore, weakly balanced implies at least balanced. (“At least balanced” should more accurately be called “at least weakly balanced”, but we’ll call it “at least balanced” to save space.)

**Lemma 8.8.** (Filtered $\phi$-modules coming from global representations are weakly balanced.)

Let $G$ be an algebraic group over $\mathbb{Q}_p$ whose identity component is reductive, and fix an embedding $G \hookrightarrow GL_N$. Let $\rho: \text{Gal}_{\mathbb{Q}} \to G_{\mathbb{Q}_p} \subseteq GL_N(\mathbb{Q}_p)$ be a representation satisfying the hypotheses of Lemma 8.2. Suppose $\rho$ in fact has image contained in some parabolic subgroup $P(\mathbb{Q}_p) \subseteq G(\mathbb{Q}_p)$, and let $M$ be a Levi subgroup of $P$.

Let $(V, \phi, F)$ be the filtered $\phi$-module over $\mathbb{Q}_p$ that is associated to the local representation $\rho|_{\mathbb{Q}_p}$. Then $F$ is in fact a $G$-filtration on $V$, and the associated graded $\text{Gr}_M(F)$ is a balanced filtration on $M$.

**Proof.** (Compare [40, Prop. 10.6(b), §11.4, §11.6].)

That $F$ is a $G$-filtration on $V$ is a consequence of the Tannakian formalism (see Lemma 6.42).

To see that $F_M$ is balanced, let $\mu: \mathbb{G}_m \to P$ be a cocharacter defining $F$. It is enough to show that every character $\chi: P \to \mathbb{G}_m$ annihilating $Z(G)$ also kills $\mu$.

For every such character $\chi$, a calculation shows that $\chi \circ \rho$ is pure of weight zero: the Frobenius eigenvalues at unramified primes are rational Weil numbers of weight zero.

Indeed, let $\rho'$ be the representation of $P$ obtained from summing $\rho$ with $\chi$, with $\chi$. Let $\text{Frob}_\ell^*$ be the semisimplification of a Frobenius element at $\ell$ acting on $\rho'$. Then $\text{Frob}_\ell^*$ lies in $P$. Fix an eigenbasis of $\text{Frob}_\ell^*$ and let $T$ be the subgroup of $P$ consisting of elements which have each element of this eigenbasis as eigenvectors. This is an algebraic subgroup of the torus with coordinates $\lambda_1, \ldots, \lambda_{N+1}$, hence
is defined by relations $\prod_i \lambda_i^{e_i} = 1$ for $e_i = \mathbb{Z}$. Letting $\lambda_{N+1}$ be the eigenvalue on $\chi$, because $Z(G)$ acts trivially on $\chi$, we know that every element of $T$ whose eigenvalues $\lambda_1, \ldots, \lambda_N$ are all equal acts by scalars on $\rho$, hence lies in $Z(G)$, and thus acts trivially on $\chi$. It follows by restricting all relations to the subtorus where $\lambda_1 = \cdots = \lambda_N$ that there must be some relation with $e_{N+1} \neq 0$ and $\sum_{i=1}^N e_i = 0$. Applying this relation to the eigenvalues of $\text{Frob}^s$, we see that

$$|\chi(\text{Frob}^s)|^{e_{N+1}} = |\chi(\text{Frob}^s)^{e_N}| = |\prod_{i=1}^N \lambda_i(\text{Frob}^s)^{e_{i}}| = \prod_{i=1}^N |\lambda_i(\text{Frob}^s)|^{-e_{i}} = N \prod_{i=1}^N (p^{w/2})^{-e_{i}}$$

$$= p^{-(w/2) \sum_{i=1}^N e_i} = p^0 = 1$$

and so $\chi$ is pure of weight 0.

By Lemma 8.2, the weight of the corresponding filtered $\phi$-module is also zero, which is what we needed to prove. \hfill \qed

**Definition 8.9.** Let $E_0$, $E$, $G^0$ and $G$ be as in Section 6.4 and fix $\sigma \in \text{Aut}_{E_0, E}$. A **$G$-bifiltered $\phi$-module** is a quadruple $(V, \phi, F, \mathfrak{f})$, with $V$ as in Section 6.4, $F$ and $\mathfrak{f}$ two $G$-filtrations on $V$, and $\phi \in G$ a $\sigma$-semilinear endomorphism of $V$ respecting $\mathfrak{f}$. In this setting, let $P$ and $Q$ denote the parabolic subgroups of $G$ corresponding to $\mathfrak{f}$ and $F$, respectively; to say that $\phi$ respects $\mathfrak{f}$ means that $\phi \in P$.

A **graded $G$-bifiltered $\phi$-module** is a quadruple $(V, \phi, \mathfrak{f}, F_M)$, where $V$ is as in Notation 6.4, $\mathfrak{f}$ is a $G$-filtration on $V$ with associated Levi quotient $M$, and $F_M$ is a filtration on $M$.

We say that two graded $G$-bifiltered $\phi$-modules $(V_1, \phi_1, F_{M,1})$ and $(V_2, \phi_2, F_{M,2})$ are equivalent if they agree up to $G$-conjugacy. More precisely, let $P_1$ and $P_2$ be the parabolic subgroups attached to $\mathfrak{f}_1$ and $\mathfrak{f}_2$, and let $M_1$ and $M_2$ be their Levi quotients. We require that there exist $g \in G$ satisfying the following conditions.

- $gP_1g^{-1} = P_2$
- The filtrations $gF_{M,1}g^{-1}$ and $F_{M,2}$ on $M_2$ agree.
- The two elements $g\phi_1 g^{-1}$ and $\phi_2$ of $P_2$ project to the same element of $M_2$.

There is an obvious functor

$$(V, \phi, F, \mathfrak{f}) \to (V, \phi, \mathfrak{f}, \text{Gr}_M F)$$

from $G$-bifiltered $\phi$-modules to graded $G$-bifiltered $\phi$-modules. We say that two $G$-bifiltered $\phi$-modules are semisimply equivalent if the corresponding graded $G$-bifiltered $\phi$-modules are equivalent.

We say that a $G$-filtered $\phi$-module $(V, \phi, F)$ is of the semisimplicity type $(V_0, \phi_0, F_0, \mathfrak{f}_0)$ if there exists a $G$-filtration $\mathfrak{f}$ on $V$ such that $(V, \phi, F, \mathfrak{f})$ and $(V_0, \phi_0, F_0, \mathfrak{f}_0)$ are semisimply equivalent.

**Remark 8.10.** Loosely speaking, the objects introduced should be understood as follows.

A $G$-bifiltered $\phi$-module comes (by $p$-adic Hodge theory) from a filtered Galois representation (i.e. a Galois representation on a vector space $V$, and a Galois-stable filtration on $V$).

A graded $G$-bifiltered $\phi$-module comes from the associated graded to a filtered Galois representation.
Two graded $G$-bifiltered $\phi$-modules are equivalent if the corresponding Galois representations are isomorphic; two $G$-filtered $\phi$-modules are of the same semisimplicity type if the associated gradeds of the corresponding filtered Galois representation agree up to isomorphism.

**Lemma 8.11.** Let $p$ be a prime. A representation $\rho_0: G_\mathbb{Q} \to G$

of the global Galois group $G_\mathbb{Q}$, crystalline at $p$, gives rise by $p$-adic Hodge theory to an admissible filtered $\phi$-module $(V_0, \phi_0, F_0)$ with $G$-structure. Suppose $\rho_0$ is semisimple. Suppose another crystalline global representation $\rho: G_\mathbb{Q} \to G$ has semisimplification $\rho_0$, and call the corresponding filtered $\phi$-module $(V, \phi, F)$. Then there exist $G$-filtrations $\mathfrak{f}_0$ on $V_0$ and $\mathfrak{f}$ on $V$ such that $(V, \phi, F, \mathfrak{f})$ and $(V_0, \phi_0, F_0, \mathfrak{f}_0)$ are semisimply equivalent.

Furthermore, we can take $\mathfrak{f}_0$ one of a list of finitely many candidates, depending only on $\rho_0$, and the filtration $F_0$ is balanced with respect to $\mathfrak{f}_0$.

**Proof.** By restriction to $G_K$, and the results of $p$-adic Hodge theory (Lemma 6.42), a crystalline representation $G_K \to G$ gives rise to a $G$-valued filtered $\phi$-module $(V, \phi, F)$, with $\phi$ semilinear over $\sigma$.

To say that $\rho_0$ is the semisimplification of $\rho$ means (see Lemma 6.36 and surrounding discussion) that there exist a parabolic subgroup $P \subseteq G$ with associated Levi $L$, and a Levi $L_0 \subseteq G$, such that $\rho_0$ takes values in $L_0$, and $\rho$ takes values in $P$, and the image of $\rho$ under the quotient map $P \to L$ is $G$-conjugate to $\rho_0$. Lemma 6.40 shows that, given $\rho_0$, we can take $L_0$ to be one of finitely many possible subgroups.

Fix $\rho_0$ and $P$, and let $M$ be the Levi quotient attached to $P$. By Lemma 6.42 we find that $\rho$ and $\rho_0$ must give rise to filtered $\phi$-modules with $P$-structure, such that the corresponding filtered $\phi$-modules with $M$-structure (obtained by functoriality via the quotient map $P \to M$) are isomorphic. It follows formally that $(V, \phi, F, \mathfrak{f})$ and $(V_0, \phi_0, F_0, \mathfrak{f}_0)$ are semisimply equivalent.

Finally, $F_0$ is weakly balanced with respect to $\mathfrak{f}_0$ by Lemma 8.8.

**Remark 8.12.** Recall (Definition 6.41 and surrounding discussion) that there is an equivalence between $G$-filtrations and $G^0$-filtrations; we will use this without comment, and work with $G^0$-filtrations on $V$.

**Definition 8.13.** (Adjoint Hodge numbers; see beginning of §10 of [40].)

Let $G$ be a reductive group, and suppose $F$ is a filtration on $G$.

In this setting, we define the **adjoint Hodge numbers** as follows: The filtration $F$ on $G$ gives, by definition, a filtration on every representation of $G$. We apply this to the adjoint representation of $G$ on $\text{Lie} G$, and call the adjoint Hodge numbers $h^p$ the dimensions of the associated graded of the resulting filtration:

$$h^p = \dim F^p V / F^{p-1} V.$$  

For any real $x \in [0, \sum_p h^p]$, we define the “sum of the topmost $x$ Hodge numbers” $T(x)$ by the continuous, piecewise-linear function $T: [0, \sum_p h^p] \to \mathbb{R}$ satisfying

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5The restriction to $\mathbb{Q}$, instead of an arbitrary number field $K$, is for two reasons. First, in the general setting, a filtered $\phi$-module would be *semilinear over $K_\mathbb{Q}$*, and we have not defined filtered $\phi$-modules with $G$-structure in the semilinear setting; this restriction is inessential. Second, we will need to apply Lemma 8.2 and for that we need $K$ to have no CM subfield.
\[ T(0) = 0 \] and
\[ T'(x) = k \]
for \( \sum_{p=k+1}^{\infty} h^p < x < \sum_{p=k}^{\infty} h^p \). (The sums are finite because only finitely many \( h^p \) are nonzero.)

When we want to emphasize the dependence on the group \( G \), we’ll write \( T_G(x) \) and \( h^p_G \) instead of \( T(x) \) and \( h^p \).

**Definition 8.14.** (Uniform Hodge numbers.)

Recall notation from Section 6.4. In particular, \( H \) is a reductive group over \( \mathbb{Q}_p \), and \( G^0 = \text{Res}_{\mathbb{Q}_p} H \) is a form of \( H^d \); the groups \( G^0 \) and \( G \) act on the vector space \( V \). Let \( F \) be a \( G^0 \)-filtration on \( V \).

After base change to \( \mathbb{Q}_p \), we obtain
\[ E \otimes \mathbb{Q}_p \cong \bigoplus \mathbb{Q}_p \]
the direct sum taken over all maps \( \iota : E \to \mathbb{Q}_p \). (The subscript on \( \mathbb{Q}_p \) is purely for notational convenience: each \( \mathbb{Q}_p \) is a copy of \( \mathbb{Q}_p \).) Similarly, the group \( G^0 \) splits as a direct sum of copies of \( H \), and \( (V,F) \) splits as a direct sum of filtered \( \mathbb{Q}_p \)-vector spaces \( (V_i,F_i) \), indexed by \( \iota \).

For each \( \iota \), let \( h^p_{\iota} \) be the adjoint Hodge numbers of the \( H \)-filtration \( F_{\iota} \). We say that \( F \) is uniform if the numbers \( h^p_{\iota} \) are independent of \( \iota \). In this situation, we write the Hodge numbers and associated \( T \)-function as
\[ h^p = h^p_H(V_{\iota}) \text{ and } T(x) = T_H(x). \]

**Lemma 8.15.** In the setting of Definition 8.14, let \( d = [E : \mathbb{Q}_p] \). We have
\[ h^p_{G^0} = d h^p_H \]
and
\[ T_{G^0}(dx) = dT_H(x). \]

**Proof.** The Hodge numbers can be computed after base change to \( \mathbb{Q}_p \). \( \square \)

**Lemma 8.16.** Take notation as in Section 6.4, and suppose \( E = E_{\mathbb{Q}_p} \).

Let \( P \) be a parabolic subgroup of \( G^0 = H^d \), corresponding to a \( G^0 \)-filtration \( \mathfrak{f} \), and let \( M \) be the corresponding Levi quotient.

Let \( Q_0 \) be another parabolic subgroup of \( G^0 \), corresponding to some filtration \( F_0 \), so that \( G^0/Q_0 \) parametrizes filtrations \( F \) that are \( G^0 \)-conjugate to \( F_0 \). Suppose \( F_0 \) is uniform in the sense of Definition 8.14, and let \( h^p = h^p_H \) be the adjoint Hodge numbers on \( H \). Let \( t \) be the dimension of a maximal torus in \( H \). Suppose \( e \) is a positive integer such that:

- (First numerical condition.)
\[ \sum_{p>0} h^p \geq \frac{e}{c} \]

- (Second numerical condition.)
\[ \sum_{p>0} ph^p > T\left(\frac{e}{c}\right) + T\left(\frac{1}{2}(h^0 - t) + \frac{e}{c}\right). \]
Then for any at-least-balanced filtration $F_{0,M}$ on $M$, the set of filtrations $F$ on $G^0$ that are $G^0$-conjugate to $F_0$ and satisfy $\text{Gr}_M F = F_{0,M}$ is of codimension at least $e$ in $G^0/Q_0$.

Proof. This is essentially [40, Prop. 11.3], applied to $G^0 = H^c$. Note that the Hodge numbers $h^p$ and the function $T$ used in [40] are $h_{H^c}$ and $T_{H^c}$. They are related to $h_H$ and $T_H$ by

$$h^p_{H^c} = ch^p_H$$

and

$$T_{H^c}(cx) = cT_H(x).$$

Similarly, the dimension of a maximal torus in $H^c$ is $ct$.

Two small modifications need to be made to the proof in [40]. First, we have replaced $\frac{1}{2}h^0$ with $\frac{1}{2}(h^0 - t)$. To get this stronger bound, replace the final inequality of [40, Equation 11.15] with

$$\dim(Q/B) + e \leq \frac{1}{2}(a_0 - t) + e.$$ (Here $Q$ is a parabolic subgroup, $a_0$ the dimension of its Levi quotient, and $B$ a Borel. There is no new idea here; this bound is stronger only because the bound in [40] was not sharp.)

Second, our hypothesis is weaker: in the above-referenced proposition, $F_{0,M}$ is assumed to be balanced, while here it is only assumed to be at least balanced. This is not a problem, since the inequalities work in our favor. In our context, [40, Equation 11.14] is replaced with the inequality

$$\sum_{\gamma \in \Sigma - \Sigma_P} \langle w\mu, \gamma \rangle \leq 0,$$

and [40, Equation 11.16] becomes

$$\sum_{\beta \in X} \langle \mu, w^{-1}\beta \rangle \leq \sum_{X'} -\langle \mu, w^{-1}\beta \rangle.$$ The rest of the proof goes through as in [40].

Our next goal (Lemma 8.19) is a slight generalization of [40, Equation 11.18]: the result in [40] only holds with $\phi$ contained in a connected reductive group (e.g. $G^0$), but here $\phi$ is semilinear, so it is contained in $G$, but not in $G^0$.

**Lemma 8.17.** Let $U$ be a unipotent algebraic group over a field of characteristic zero, and $\psi$ an automorphism of $U$, such that $\psi^r = \text{id}_U$. Suppose $u \in U$ is such that

$$u\psi(u)\psi^2(u) \cdots \psi^{r-1}(u) = 0.$$ Then there exists $v \in U$ such that

$$u = v^{-1}\psi(v).$$

Proof. By induction on $\dim U$; reduce to the case where $U = G^0_S$ is abelian.

**Lemma 8.18.** Let $G^0$ and $G$ be as in Section 6.4, $P$ a parabolic subgroup of $G^0$, $M$ the associated Levi quotient, and choose a section $M \rightarrow P$. Suppose $\phi \in G$ is semisimple and normalizes $P$.

Then $\phi$ is $P$-conjugate to an element that normalizes $M$. 

Proof. Suppose $\phi^r \in G^0$. Since $\phi^r$ is semisimple, it is contained in some maximal torus $T$ of $G^0$; we may conjugate by an element of $P$ to assure that $T \subseteq M$.

Since $\phi$ normalizes $P$, and all Levi subgroups in $P$ are $P$-conjugate, we can write $\phi M \phi^{-1} = u^{-1} M u$ for some $u \in P$. Since $u$ is well-defined up to left multiplication by $M$, we may take $u$ unipotent. Then $(u\phi)^r$ is an element of $U \phi^r$ that normalizes $M$, so

$$(u\phi)^r = \phi^r.$$ 

Now apply Lemma 8.17 with $\psi(x) = \phi x \phi^{-1}$, to conclude that $u = v^{-1} \psi(v) = v^{-1} \phi v \phi^{-1}$ for some $v \in P$. Then $v^{-1} \phi v$ normalizes $M$. \hfill \Box

Lemma 8.19. (Compare [40] Equation 11.18.)

Assume we are in the setting of Section 6.4. Fix a $G$-filtered module $(V_0, \phi_0, F_0, f_0)$, and another $\phi$-module $(V, \phi)$ with $G$-structure; suppose both $\phi$ and $\phi_0$ are semilinear over some $\sigma \in \text{Aut}_{E_0} E$.

Consider all pairs $(f, F_M)$, where $f$ is a filtration on $G$, $M$ is the Levi quotient, and $(V, \phi, f, F_M)$ is equivalent to $(V_0, \phi_0, f_0, (F_0)_{M_0})$, in the sense of Definition 8.9. The set of such pairs has dimension at most $\dim Z_{G^0}(\phi^{ss})$.

(Even though $\phi^{ss}$ is only defined up to conjugacy, the dimension $\dim Z_{G^0}(\phi^{ss})$ is well-defined, so the statement of the lemma makes sense.)

Proof. This is a question about the dimension of a variety, so we can pass to a finite extension of $K$. In particular, we may assume $E = K^d$ and $\sigma$ acts by permutation on the factors of $E = K^d$. We’ll index the factors $K_1$ through $K_d$, and we’ll regard $\sigma$ as a permutation of $\{1, \ldots, d\}$. Write $V_i = V \otimes_E K_i$ for the $i$-th factor of $V$, and write $H_i$ for the corresponding factor of $G^0 = H^d$ (it satisfies $H_i \cong H$). The semilinearity condition means that $\phi$ decomposes as a sum of maps

$$\phi_i : V_i \rightarrow V_{\sigma i}.$$ 

As in [40] third paragraph of section 11.6], we can assume $\phi$ is semisimple; if not, passing to the semisimplification will only increase the dimension of the set described. (If $\phi \in N_G P$ is semilinear over $\sigma$, then its semisimplification can be taken to have the same properties.)

A filtration $f$ on $G^0$ (which is the same as a filtration on $G$) decomposes as the product of filtrations $f_i$ on $H_i$. If $f$ is $\phi$-stable, then $f_i$ determines $f_{\sigma i}$. So to specify $f$ it is enough to specify $f_i$, for a single $i$ in each $\sigma$-orbit.

Since everything in sight splits as a direct product over $\sigma$-orbits, we may as well restrict attention to a single $\sigma$-orbit; call it $\{1, \ldots, r\}$. A $\phi$-stable filtration $f$ on $G^0$ is uniquely determined by a $\phi^0$-stable filtration $f_1$ on $H_1$. As in [40], the dimension of the set of such filtrations is exactly

$$\dim Z_{G^0}(\phi) - \dim Z_P(\phi),$$

where $Z_{G^0}(\phi) = Z_G(\phi) \cap G^0$, and $Z_P(\phi)$ is defined similarly.

To conclude, we need to show that, given $f$ (and its associated Levi quotient $M$), the set of filtrations $F_M$ such that $(V, \phi, f, F_M)$ is equivalent to $(V_0, \phi_0, f_0, (F_0)_{M_0})$ has dimension at most $\dim Z_P(\phi)$. The proof of this is the same as in [40], using Lemma 8.18 in place of [40] Equation 2.1]. \hfill \Box
Lemma 8.20. Assume we are in the setting of Section 6.4. Fix a $G^0$-bifiltered $\phi$-module $(V_0, \phi_0, F_0, f_0)$, and another $\phi$-module $(V, \phi)$ with $G$-structure; suppose both $\phi$ and $\phi_0$ are semilinear over some $\sigma \in \text{Aut}_{E_0} E$, and $F_0$ is balanced with respect to $f_0$.

Let $G_{\text{mon}}$ be a subgroup of $G^0$, strongly $c$-balanced with respect to $\sigma$.

Suppose $F_0$ is uniform in the sense of Definition 8.14, and let $h^p = h^p_{\text{simp}}$ be the adjoint Hodge numbers on $H$. Let $t$ be the dimension of a maximal torus in $H$.

Suppose $e$ is a positive integer satisfying the following numerical conditions.

- (First numerical condition.)
  \[
  \sum_{p>0} h^p \geq \frac{1}{c}(e + \dim H)
  \]

- (Second numerical condition.)
  \[
  \sum_{p>0} ph_p > T \left( \frac{1}{c}(e + \dim H) \right) + T \left( \frac{1}{2}(h^0 - t) + \frac{1}{c}(e + \dim H) \right).
  \]

Let $\mathcal{H} = G_{\text{mon}}/(Q^0 \cap G_{\text{mon}})$ be the flag variety parametrizing filtrations on $G^0$ that are conjugate to $F_0$ under the conjugation of $G_{\text{mon}}$. Then the filtrations $F$ such that $(V, \phi, F)$ is of semisimplicity type $(V_0, \phi_0, F_0, f_0)$ are of codimension at least $e$ in $\mathcal{H}$.

Proof. Let $P$ be the parabolic of $G^0$, and $M$ the Levi quotient, associated to a hypothetical semisimplification filtration $\mathfrak{f}$. There are only finitely many possibilities for the parabolic group $P$, up to $G^0$-conjugacy, so we may as well as fix a single $P$.

The dimension in question can be calculated after base change to an extension of $K$, so we can assume that $G^0 = H^d$. As in the proof of Lemma 8.19, we’ll call the $d$ factors $H_1, \ldots, H_d$. The whole setup factors over these $d$ factors, so we can write $P$ as the direct sum of parabolics $P_i \subseteq H_i$, and so forth. Again, $\sigma \in \text{Aut}_{E_0} E$ gives a permutation of the index set $\{1, \ldots, d\}$, which we also call $\sigma$. Semilinearity over $\sigma$ means that the map $\phi$ permutes the $d$ factors according to the permutation $\sigma$.

The strategy is as follows. We want to apply a result like Lemma 8.16, but we don’t have full monodromy group $G^0$. Instead, we know the group $G_{\text{mon}}$ is $c$-balanced; we’ll project onto $c$ of the $d$ factors, so that $G_{\text{mon}}$ projects onto the full group $(H/Z(H))^c$. We need the projection of $F_0$ to the $c$ factors to be at least balanced, and we can apply Lemmas 8.16 and 8.19 to the group $H^c$ to finish.

For any subset $I$ of the index set $\{1, \ldots, d\}$, let $H^I = \prod_{i \in I} H_i$; define $P^I$, $M^I$, and $F_0^I$ similarly. Any filtrations $F$ and $\mathfrak{f}$ on $G^0$ can be written as products of factors $F_i$ and $\mathfrak{f}_i$, so we can define $F^I$ and $\mathfrak{f}^I$.

I claim that, for any $\mathfrak{f}$ and any $F$ that is balanced with respect to $\mathfrak{f}$, we can find $I \subseteq \{1, \ldots, d\}$ satisfying the following properties.

- $I$ consists of exactly $c$ elements, from $c$ distinct index classes for $G_{\text{mon}}$.
- The elements of $I$ belong to a single orbit of $\sigma$ on $\{1, \ldots, d\}$.
- $F^I$ is at least balanced with respect to $\mathfrak{f}^I$.

To see this, let $\mu$ be a cocharacter defining $F$ as in Definition 8.6. We can write $\mu$ as a product over $\mu_i$ with $i \in \{1, \ldots, d\}$. Similarly, the character $\gamma_p$ splits over...
the factors, and we have:

\[ \gamma_P(\mu) = \sum_i \gamma_{P_i}(\mu_i). \]

Hence there exists an orbit \( J \) of \( \sigma \) on the \( d \) factors such that \( F^J \) is at least balanced. Since \( G_{\text{mon}} \) is strongly \( c \)-balanced, we can find a subset \( I \subset J \subset \{1, \ldots, d\} \) of the index set such that \( \# I = c \), the elements of \( I \) belong to \( c \) distinct index classes, and \( F^I \) is at least balanced. Since the elements of \( I \) belong to distinct index classes, the projection \( G_{\text{mon}} \to (H/Z(H))^J \) has image a union of connected components of the target, so it is smooth with equidimensional fibers, and the same is true of

\[ G_{\text{mon}}/(Q^0 \cap G_{\text{mon}}) \to H^I/(Q^0 \cap H^I). \]

We want to estimate the codimension in \( \mathcal{H} = G_{\text{mon}}/(Q^0 \cap G_{\text{mon}}) \) of the set of \( F \) such that \((V, \phi, F, f)\) is semisimply equivalent to \((V_0, \phi_0, F_0, f_0)\), for some choice of \( f \). Consider the projection

\[ (V, \phi, F, f) \to (f, F_M). \]

By Lemma 8.19 applied to \( H^J \), the set of pairs \((f^I, F_M^I)\) such that \((V^J, \phi^J, F_M^J, f^I)\) has dimension bounded by \( \dim Z_G((\phi^J)^{ss}) \); this dimension is at most \( \dim H \) by Lemma 6.32.

Now fix \((f^I, F_M^I)\). By Lemma 8.16 applied to \( H^I \), the set of \( F^I \) such that \( \text{Gr}_M F^I = F_M^I \) has codimension at least \( e + \dim H \) among all \( H^I \)-filtrations \( F^I \). The map \( F^J \to F^I \) from \( H^I \)-filtrations in a given \( G_{\text{mon}} \)-conjugacy class to \( H^I \)-filtrations is smooth with equidimensional fibers, so the set of \( F^J \) such that

\[ \text{Gr}_M(F^J) = F_M^J \]

again has codimension at least \( e + \dim H \).

It follows that the set of \( F \) satisfying the desired condition has codimension at least \( e \). \( \square \)

**Theorem 8.21.** (Basic theorem giving non-density of integral points.)

Let \( X \) be a variety over \( \mathbb{Q} \), let \( S \) be a finite set of primes of \( \mathbb{Z} \), and let \( \mathcal{X} \) be a smooth model of \( X \) over \( \mathbb{Z}[1/S] \).

Let \( \mathcal{E} \) be a constant \( H^0 \)-algebra on \( X \), and let \( H \) be one of \( GL_N, GSp_N, \) or \( GO_N \). Let \( V \) be a polarized, integral \( \mathcal{E} \)-module with \( H \)-structure, in the sense of Definition 6.26, having integral Frobenius eigenvalues (Def. 6.3). Suppose the Hodge numbers of \( V \) are uniform in the sense of Definition 8.14, and let \( h^p = h^p_{\text{simp}} \) be the adjoint Hodge numbers on \( H \). Let \( t \) be the dimension of a maximal torus in \( H \). Let \( p \) be as in Definition 6.2. Let \( G^0 \) and \( G \) be as in Section 6.4.

Let \( \Omega \subseteq \mathcal{X} \) be a residue disk modulo \( p \). Suppose there is a positive integer \( c \) such that \( \mathcal{V} \) satisfies the following conditions.

\[ \bullet \text{(Big monodromy.) If } G_{\text{mon}} \subseteq G^0 \text{ is the differential Galois group of } \mathcal{V}, \text{ then } G_{\text{mon}} \subseteq G^0 \text{ is strongly } c \text{-balanced with respect to Frobenius. (The Frobenius is determined from the structure of } \mathcal{E}; \text{ see Section 6.7.)} \]
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(First numerical condition.)
\[ \sum_{p>0} h_p \geq \frac{1}{c} \left( \dim X + \dim H \right) \]

(Second numerical condition.)
\[ \sum_{p>0} ph_p > T_G \left( \frac{1}{c} \left( \dim X + \dim H \right) \right) + T_G \left( \frac{1}{2} (h^0 - t) + \frac{1}{c} \left( \dim X + \dim H \right) \right) \]

Then the image of \( \mathcal{X}(\mathbb{Z}[1/S]) \cap \Omega \) in \( X \) is not Zariski dense.

In particular, if this holds for all (finitely many) mod-\( p \) residue disks \( \Omega \), then the image of \( \mathcal{X}(\mathbb{Z}[1/S]) \) is not Zariski dense in \( X \).

Proof. For every \( x \in \mathcal{X}(\mathbb{Z}[1/S]) \), the fiber \( \rho_x = \mathcal{V}_{et,x} \) of the \( p \)-adic étale local system is a global Galois representation valued in \( G \), having good reduction outside \( S \) and all Frobenius eigenvalues Weil numbers, by hypothesis. By Faltings’s finiteness lemma (in the form of Lemma 6.43), there are only finitely many possible isomorphism classes for the semisimplified representation \( \rho_x^{ss} \). So it is enough to show, for any fixed \( \rho_0 \), that the set \( \mathcal{X}(\mathbb{Z}[1/S], \rho_0) := \{ x \in \mathcal{X}(\mathcal{O}_{K,S}) | \rho_x^{ss} \cong \rho_0 \} \) is not Zariski dense in \( X \).

Lemma 8.11 gives a finite list of semisimplicity types such that, for every \( x \in \mathcal{X}(\mathbb{Z}[1/S], \rho_0) \), the filtered \( \phi \)-module \( V_{cris,x} \) belongs to one of them. Let \( \Omega \subseteq \mathcal{X}(\mathbb{Z}[1/S], \rho_0) \) be a mod-\( p \) residue disk, and fix a semisimplicity type \((V_0, \phi_0, F_0, f_0)\). By Lemma 8.11, \( F_0 \) is weakly balanced with respect to \( f_0 \). It is enough to show that the set \( \mathcal{X}(\mathcal{O}_{K,S}, \rho_0) := \{ x \in \mathcal{X}(\mathcal{O}_{K,S}) | V_{cris,x} \text{ is of semisimplicity type } (V_0, \phi_0, F_0, f_0) \} \) is not Zariski dense in \( X \).

We are now in the setting of Theorem 7.3. For \( x \in \Omega \), the filtered \( \phi \)-module \( V_{cris,x} \) is of the form \((V, \phi, F_x)\), where \((V, \phi)\) is independent of \( x \). (This is a property of \( F \)-isocrystals in general; it reflects the fact that if \((V, \phi, F)\) is the crystalline cohomology of a scheme over \( \mathbb{Z}_p \), then \((V, \phi)\) can be recovered from its reduction modulo \( p \). See [40, Section 3.3] for further discussion.) The variation of \( F_x \) with \( x \) is classified by a \( p \)-adic period map \( \Phi_p : \Omega \to G^0/Q \), where \( G^0/Q \) is the flag variety classifying \( G^0 \)-filtrations on \( V \).

In fact, the \( p \)-adic period map lands in a single \( G_{mon} \)-orbit on \( G^0/Q \); we can write the orbit as \( G_{mon}/(Q \cap G_{mon}) \). By Lemma 8.20 with \( e = \dim X \), there is a Zariski-closed set \( Z \subseteq G_{mon}/(Q \cap G_{mon}) \) of codimension at least \( \dim X \) such that, if \((V, \phi, F_x)\) is of semisimplicity type \((V_0, \phi_0, F_0, f_0)\), then \( x \in \Phi_p^{-1}(Z) \). We conclude by Theorem 7.3. \( \square \)
9. Hodge–Deligne systems and integral points, assuming global semisimplicity

In this section we prove Theorem 9.3, a variant of Theorem 8.21 that assumes the semisimplicity of certain global Galois representations. This material is not logically needed for the main argument; we include it to illustrate the main ideas of Section 8 without the complications coming from semisimplification.

Lemma 9.1. (Compare Lemma 8.11.)

Let $p$ be a prime. A semisimple representation $\rho_0 : G_\mathbb{Q} \to G$ of the global Galois group $G_\mathbb{Q}$, crystalline at $p$, gives rise by $p$-adic Hodge theory to an admissible filtered $\phi$-module $(V_0, \phi_0, F_0)$ with $G$-structure. Suppose another crystalline global representation $\rho : G_K \to G$ is isomorphic to $\rho_0$, and call the corresponding filtered $\phi$-module $(V, \phi, F)$. Then there is an isomorphism of filtered $\phi$-modules $(V_0, \phi_0, F_0) \cong (V, \phi, F)$.

In particular, if $(V_0, \phi_0) = (V, \phi)$, then there exists an automorphism $f : V \to V$ such that

- $f$ commutes with $\phi$, and
- $fF_0 = F$.

Proof. The functors of $p$-adic Hodge theory take isomorphic objects to isomorphic objects. $\square$

Lemma 9.2. (Compare Lemma 8.20.)

Assume we are in the setting of Section 6.4. Fix an admissible filtered $\phi$-module with $G$-structure $(V_0, \phi_0, F_0)$, and another $\phi$-module $(V, \phi)$ with $G$-structure; suppose both $\phi$ and $\phi_0$ are semilinear over some $\sigma \in \text{Aut}_{\mathbb{E}} E$.

Let $G_{\text{mon}}$ be a subgroup of $G^0$, strongly $c$-balanced with respect to $\sigma$.

Suppose $F_0$ is uniform in the sense of Definition 8.14, and let $h^p = h^p_{\text{simp}}$ be the adjoint Hodge numbers on $H$. Suppose $e$ is a positive integer satisfying the following numerical condition.

- (Numerical condition.)

$$\sum_{p > 0} h^p \geq \frac{1}{c} (e + \dim H)$$

Let $H = G_{\text{mon}}/(Q^0 \cap G_{\text{mon}})$ be the flag variety parametrizing filtrations on $G^0$ that are conjugate to $F_0$ under the conjugation of $G_{\text{mon}}$. Then the filtrations $F$ such that $(V, \phi, F)$ is isomorphic to $(V_0, \phi_0, F_0)$ are of codimension at least $e$ in $H$.

Proof. This is a question about the dimension of a variety over $\mathbb{Q}_p$; by passing to an extension, we may assume that $G^0 = H^d$ is split, and $\sigma$ acts by permuting the factors. Increasing $c$ if necessary, we obtain an isomorphism $G_{\text{mon}} \cong H^e$ by projecting onto some $c$ of the $d$ factors, and we may assume that $\sigma$ acts transitively on these $c$ factors.

By Lemma 9.1, the filtrations $F$ satisfying the condition described form at most one orbit under the action of $Z(\phi)$ on $H$. By Lemma 6.32 we have

$$\dim Z(\phi) \leq \dim H.$$
On the other hand,

\[ \dim \mathcal{H} = c \sum_{p > 0} h^p. \]

The result follows.

**Theorem 9.3.** (Compare Theorem 8.21)

Let \( X \) be a variety over \( \mathbb{Q} \), let \( S \) be a finite set of primes of \( \mathcal{O}_K \), and let \( \mathcal{X} \) be a smooth model of \( X \) over \( \mathbb{Z}[1/|S|] \).

Let \( \mathbb{E} \) be a constant \( H^0 \)-algebra on \( X \), and let \( \mathcal{H} \) be one of \( \text{GL}_N \), \( \text{GSp}_N \), or \( \text{GO}_N \).

Let \( \mathbb{V} \) be a polarized, integral, \( \mathbb{E} \)-module with \( \mathcal{H} \)-structure, in the sense of Definition 6.26, having integral Frobenius eigenvalues (Def. 6.3). Suppose the Hodge numbers of \( \mathbb{V} \) are uniform in the sense of Definition 8.14, and let \( h^p = h_{p}^{\text{simp}} \) be the adjoint Hodge numbers on \( \mathcal{H} \). Let \( p \) be as in Definition 6.2.

Let \( \Omega \subseteq X \) be a residue disk modulo \( p \). Suppose there is a positive integer \( c \) such that \( \mathbb{V} \) satisfies the following conditions.

- (Big monodromy.) If \( \mathcal{G}_{\text{mon}} \subseteq \mathcal{G}_0 \) is the differential Galois group of \( \mathbb{V} \), then \( \mathcal{G}_{\text{mon}} \subseteq \mathcal{G}_0 \) is strongly \( c \)-balanced with respect to Frobenius. (The Frobenius is determined from the structure of \( \mathbb{E} \); see Section 6.7.)

- (Numerical condition.)

\[ \sum_{p > 0} h^p \geq \frac{1}{c} (\dim X + \dim \mathcal{H}) \]

Let \( \mathcal{X}(\mathbb{Z}[1/|S|])^{**} \) be the subset of \( \mathcal{X}(\mathbb{Z}[1/|S|]) \) consisting of those \( x \) for which the global Galois representation \( \mathbb{V}_{\text{et},x} \) is semisimple. Then the image of \( \mathcal{X}(\mathbb{Z}[1/|S|])^{**} \cap \Omega \) in \( X \) is not Zariski dense.

In particular, if this holds for all (finitely many) mod-\( p \) residue disks \( \Omega \), then the image of \( \mathcal{X}(\mathbb{Z}[1/|S|]) \) is not Zariski dense in \( X \).

**Proof.** This follows from Lemmas 9.1 and 9.2 in the same way that Theorem 8.21 follows from Lemmas 8.11 and 8.11. We will sketch the proof here; the reader can find more details in the proof of Theorem 8.21.

For every \( x \in \mathcal{X}(\mathbb{Z}[1/|S|]) \), consider the semisimple global Galois representation \( \mathbb{V}_x = \mathbb{V}_{\text{et},x} \). By Lemma 6.43, there are only finitely many possible isomorphism classes for the semisimple representation \( \mathbb{V}_x \). So it is enough to show, for any fixed \( \rho_0 \), that the set

\[ \mathcal{X}(\Omega, (V_0, \phi_0, F_0)) : \{ x \in \mathcal{X}(\mathcal{O}_{K,S}) | \mathbb{V}_x \cong \rho_0 \} \]

is not Zariski dense in \( X \). By Lemma 9.1, it is enough to show that the set

\[ X(\Omega, (V_0, \phi_0, F_0)) : \{ x \in \Omega | (V, \phi, F) \cong (V_0, \phi_0, F_0) \} \]

is not Zariski dense in \( X \).

We are now in the setting of Theorem 7.3. For \( x \in \Omega \), the filtered \( \phi \)-module \( \mathbb{V}_{\text{cris},x} \) is of the form \( (V, \phi, F_x) \), where \( (V, \phi) \) is independent of \( x \). The variation of \( F_x \) with \( x \) is classified by a \( p \)-adic period map

\[ \Phi_p : \Omega \to \mathcal{G}_0^0 / Q, \]

where \( \mathcal{G}_0^0 / Q \) is the flag variety classifying \( \mathcal{G}_0^0 \)-filtrations on \( V \).

In fact, the \( p \)-adic period map lands in a single \( \mathcal{G}_{\text{mon}} \)-orbit on \( \mathcal{G}_0^0 / Q \); we can write the orbit as \( \mathcal{G}_{\text{mon}} / (Q \cap \mathcal{G}_{\text{mon}}) \). By Lemma 9.2 with \( e = \dim X \), there is a...
Zariski-closed set $Z \subseteq G_{mon}/(Q \cap G_{mon})$ of codimension at least $\dim X$ such that, if $(V, \phi, F_x) \cong (V_0, \phi_0, F_0)$, then $x \in \Phi_p^{-1}(Z)$. We conclude by Theorem 7.3. □
10. Verifying the numerical conditions

The goal of this section is to verify the two numerical conditions of Theorem 8.21. This is very similar to the estimates in [40, Section 10.2].

Our Hodge-Deligne systems arise from pushforwards of character sheaves of an abelian variety along families of hypersurfaces in that abelian variety. Thus, the Hodge structure on the stalk at a point arises from the cohomology of a hypersurface, twisted by a rank-one character sheaf.

For $A$ an abelian variety of dimension $n$, $Y$ a degree $d$ hypersurface in $A$, $\chi$ a finite-order character of $\pi_1(A)$, such that $H^i(Y, L_\chi)$ vanishes for $i \neq n - 1$, the Hodge structure $H^{n-1}(Y, L_\chi)$ has Hodge numbers $h^k = dA(n, k)$, where $A(n, k)$ is the Eulerian number, by Lemma 2.4. (The degree of a hypersurface in an abelian variety is the degree of the corresponding polarization.)

Recall from Section 6.3 that the various realizations of $H^{n-1}(Y, L_\chi)$ form a Hodge–Deligne system with $H$-structure, for $H$ one of GL, GSp, and GO. This system has uniform Hodge numbers (Definition 8.14), as computed in Lemma 2.4, so it makes sense to talk about the Hodge filtration as a filtration on $H$.

Lemma 10.1. Let $Y$ be a hypersurface in an $n$-dimensional abelian variety $A$, let $\chi$ be a finite-order character of $\pi_1(A)$ such that $H^i(Y, L_\chi)$ vanishes for $i \neq n - 1$, and let $V$ be the Hodge structure on $H^{n-1}(Y, L_\chi)$, regarded as a filtered vector space with $H$-structure, for $H$ one of GL, GSp, and GO. Let $h^p$ be the adjoint Hodge numbers (Definition 8.13), and let $t$ be the dimension of a maximal torus in $H$.

If $n \geq 2$, we have $\frac{1}{2}(h^0 - t) < \sum_{p>0} h^p$.

Proof. (See also [40, Lemma 10.3].)

Since the adjoint Hodge numbers satisfy $h^p = h^{-p}$, we have

$$\sum_{p>0} h^p = \frac{1}{2} \left( \sum_{p \in \mathbb{Z}} h^p \right) = \frac{1}{2} (\dim H - h^0).$$

Thus it is enough to show that

$$2h^0 < \dim H + t.$$

In each case, we will calculate $h^0$ and $\dim H$ in terms of Eulerian numbers $A(n, k)$, and then prove (8) by using the following inequality of Eulerian numbers, to be established later:

$$2 \sum_p d^2 A(n, p)^2 \leq d^2 \left( \sum_p A(n, p) \right)^2$$

We adopt the convention that $A(n, k) = 0$ when $k \notin \{0, \ldots, n - 1\}$.

- If $H = GL$ then

  $$h^0 = \sum_p d^2 A(n, p)^2$$

  $$\dim H = d^2 \left( \sum_p A(n, p) \right)^2$$

  $$t = d \sum_p A(n, p).$$
so
\[ 2h^0 = 2 \sum_p d^2 A(n, p)^2 \leq d^2 \left( \sum_p A(n, p) \right)^2 < d^2 \left( \sum_p A(n, p) \right)^2 + d \sum_p A(n, p) = \dim H + t. \]

- If \( H = \text{GSp} \) then
  \[ h^0 = \frac{1}{2} \left[ \left( \sum_p d^2 A(n, p)^2 \right) + dA(n, \frac{n-1}{2}) \right] + 1 \]
  \[ \dim H = \frac{1}{2} d \left( \sum_p A(n, p) \right) \left[ d \left( \sum_p A(n, p) \right) + 1 \right] + 1 \]
  \[ t = \frac{1}{2} d \sum_p A(n, p) + 1. \]

so
\[ 2h^0 = \left( \sum_p d^2 A(n, p)^2 \right) + dA(n, \frac{n-1}{2}) + 2 \leq d^2 \left( \sum_p A(n, p) \right)^2 + dA(n, \frac{n-1}{2}) + 2 \]
\[ < \frac{d^2}{2} \left( \sum_p A(n, p) \right)^2 + d \left( \sum_p A(n, p) \right) + 2 = \dim H + t. \]

- If \( H = \text{GO} \) then
  \[ h^0 = \frac{1}{2} \left[ \left( \sum_p d^2 A(n, p)^2 \right) - dA(n, \frac{n-1}{2}) \right] + 1 \]
  \[ \dim H = \frac{1}{2} d \left( \sum_p A(n, p) \right) \left[ d \left( \sum_p A(n, p) \right) - 1 \right] + 1 \]
  \[ t = \frac{1}{2} d \sum_p A(n, p) + 1. \]

(The formula for \( t \) in the GO case holds only for even-dimensional orthogonal groups, but \( d \sum_p A(n, p) = d \cdot n! \) is even.) Thus
\[ 2h^0 = \left( \sum_p d^2 A(n, p)^2 \right) - dA(n, \frac{n-1}{2}) + 2 < \left( \sum_p d^2 A(n, p)^2 \right) + 2 \leq \frac{d^2}{2} \left( \sum_p A(n, p) \right)^2 + 2 = \dim H + t. \]

We now prove (9). It is known that, as \( n \) grows large, the Eulerian numbers approximate a normal distribution with variance \( \sqrt{n/12} \). This purely qualitative result implies that (9) holds for sufficiently large \( n \).

To get precise bounds, we’ll use log concavity, together with a calculation of the second moment. The key idea is that a sequence of numbers that (i) is log-concave, and (ii) has large second moment, cannot be too concentrated at the middle term. Let
\[ a_i = \frac{\sum_p A(n, p)A(n, p - i)}{(\sum_p A(n, p))^2}; \]
this is normalized so that \( \sum a_i = 1 \).

Now we’ll prove that \( a_0 \leq 1/2 \) for all \( n \geq 2 \); this will be Lemma 10.6. This will be a consequence of log-concavity and a formula for the second moment.
Lemma 10.2. The sequence \((a_i)\) is log-concave and satisfies \(a_{-i} = a_i\).

Proof. This is proved in the first paragraph of the proof of [40, Lemma 10.3]. Symmetry is elementary; log-concavity follows from the classical fact that the Eulerian numbers are log-concave (see, for example, [45, Problems 4.6 and 4.8]) and the fact that log-concavity is preserved under convolution [30, Thms. 1.4, 3.3]. □

Lemma 10.3. The second moment of \((a_i)\) is

\[ \sum_i i^2 a_i = \frac{n+1}{6}. \]

Proof. This is [40, Eqn. 10.10]. □

Lemma 10.4. Suppose \(a_0 > 1/2\). Then for all \(k \geq 1\) we have

\[ \sum_{i=k}^{\infty} a_i < \sum_{i=k}^{\infty} \frac{1}{2 \cdot 3^i}. \]

Proof. Because \(a_0 > 1/2\), by symmetry, we have

\[ \sum_{i=1}^{\infty} a_i = \frac{1 - a_0}{2} < \frac{1}{4} = \sum_{i=1}^{\infty} \frac{1}{2 \cdot 3^i}. \]

so if

\[ \sum_{i=k}^{\infty} a_i \geq \sum_{i=k}^{\infty} \frac{1}{2 \cdot 3^i} \]

we must have \(a_j < \frac{1}{2 \cdot 3^j}\) for some \(1 \leq j < k\).

But then by log-concavity and the fact that \(a_0 > 0\), we have \(a_i < \frac{1}{2 \cdot 3^i}\) for all \(i > j\), and thus in particular for all \(i \geq k\), so

\[ \sum_{i=k}^{\infty} a_i < \sum_{i=k}^{\infty} \frac{1}{2 \cdot 3^i}. \]

□

Lemma 10.5. If \(a_0 > 1/2\) then \(\sum_k k^2 a_k < \frac{3}{2}\).

Proof. We have

\[ \sum_k k^2 a_k = 2 \sum_{k=1}^{\infty} k^2 a_k = 2 \sum_{k=1}^{\infty} (2k-1) a_i > 2 \sum_{k=1}^{\infty} (2k-1) \sum_{i=k}^{\infty} \frac{1}{2 \cdot 3^i} = 2 \sum_{k=1}^{\infty} k^2 \frac{1}{2 \cdot 3^i} = \frac{3}{2}. \]

Here we use symmetry, Lemma 10.4 and the identity

\[ \sum_{k>0} k^2 c^k = \frac{c(1+c)}{(1-c)^3}. \]

□

Lemma 10.6. If \(n \geq 2\) then \(a_0 \leq 1/2\).

Proof. If \(n \geq 9\), this is immediate from Lemmas 10.3 and 10.5.

For \(2 \leq n \leq 8\), we can prove this by computation. For even \(n\) this follows immediately from the symmetry \(A(n,k) = A(n,n-1-k)\), so it suffices to check \(n = 3, 5, 7\), which can be done by hand.

This proves Lemma 10.1.
11. Proof of main theorem

**Definition 11.1.** Let $A$ be an abelian variety over a number field $K$, and $H \subseteq A$ a hypersurface. We say that $H$ is \textit{primitive} if it is not invariant under translation by any $x \in A(\Q)$.

Recall the sequence $a(i)$ defined in Theorem 4.1.

**Theorem 11.2.** Let $A$ be an abelian variety of dimension $n$ over a number field $K$. Let $S$ be a finite set of primes of $\mathcal{O}_K$ including all the places of bad reduction for $A$.

Let $\phi$ be an ample class in the Neron-Severi group of $A$, and let $d = \frac{a(i) + a(i+1)}{a(i+1)/6}$ for any $i \geq 2$.

Then there are only finitely many smooth primitive hypersurfaces $H \subseteq A$ representing $\phi$, defined over $K$ with good reduction outside $S$, up to translation.

**Proof.** Choose a smooth model of $A$ over $\mathcal{O}_{K,S}$. Working over $\mathcal{O}_{K,S}$, let $\text{Hilb}$ be the Hilbert scheme of smooth hypersurfaces of class $\phi$, and let $H_{\text{univ}} \subseteq \text{Hilb} \times A$ be the universal family over $\text{Hilb}$. Let

$$\text{Hilb}(\mathcal{O}_{K,S})^{\text{prim}} \subseteq \text{Hilb}(\mathcal{O}_{K,S})$$

be the open subscheme of \textit{primitive} hypersurfaces in $A$ (Definition 11.1). (Note that, by definition, a hypersurface defined over $K$ with good reduction outside $S$ extends to an $\mathcal{O}_{K,S}$-point of $\text{Hilb}$.) The group $A$ acts on $\text{Hilb}$ by translation; we need to show that $\text{Hilb}(\mathcal{O}_{K,S})^{\text{prim}}$ is contained in the union of finitely many orbits of $A$.

Theorem 8.21 only works over $\Q$; we’ll pass from $K$ to $\Q$ by Weil restriction. (See [11, Theorem 7.6.4] for the notion of Weil restriction relative to a finite \'{e}tale extension of rings.) Enlarging $S$ if necessary, we can arrange that $\mathcal{O}_{K,S}$ is finite \'{e}tale over $\mathbb{Z}[1/S']$, where $S'$ is a finite set of primes of $\mathbb{Z}$. Over the Weil restriction

$$\text{Res}_{\mathbb{Z}[1/S']}^{\mathcal{O}_{K,S}} \text{Hilb}$$

we have the family

$$H_{\text{univ}, \Q} = H_{\text{univ}} \times_{\text{Hilb}} (\text{Res}_{\mathbb{Z}[1/S']}^{\mathcal{O}_{K,S}} \text{Hilb} \times_{\mathbb{Z}[1/S']} \mathcal{O}_{K,S});$$

and the fibers of this family are the same as the fibers of the original family $H_{\text{univ}}$. More precisely, the sets $\text{Hilb}(\mathcal{O}_{K,S})$ and

$$\text{Res}_{\mathbb{Z}[1/S']}^{\mathcal{O}_{K,S}} \text{Hilb}(\mathbb{Z}[1/S'])$$

are in canonical bijection. Let

$$\text{Res}_{\mathbb{Z}[1/S']}^{\mathcal{O}_{K,S}} \text{Hilb}(\mathbb{Z}[1/S'])^{\text{prim}} \subseteq \text{Res}_{\mathbb{Z}[1/S']}^{\mathcal{O}_{K,S}} \text{Hilb}(\mathbb{Z}[1/S'])$$

be the subset corresponding to $\text{Hilb}(\mathcal{O}_{K,S})^{\text{prim}}$ under this bijection. For any $x \in \text{Hilb}(\mathcal{O}_{K,S})$, call $\text{Res} x$ the corresponding point of $\text{Res}_{\mathbb{Z}[1/S']}^{\mathcal{O}_{K,S}} \text{Hilb}(\mathbb{Z}[1/S'])$. Then we have a canonical isomorphism of schemes

$$(H_{\text{univ}})_x \cong (H_{\text{univ}, \Q})_{\text{Res} x};$$
Lemma 6.29 gives from geometry; the polarization and integral structure are given in Lemma 6.28. Weil numbers is a consequence of the Weil numbers, since $V$ comes from geometry; the polarization and integral structure are given in Lemma 6.28.

Let $X_{\text{sing}}$ be an irreducible component of the Zariski closure of $\text{Res}^K_{\Q} \text{Hilb}(\Z[1/\S])_{\text{prim}}$ in $\text{Res}^K_{\Q} \text{Hilb}$, and choose a resolution of singularities $X \to X_{\text{sing}}$. Let $Y = H_{\text{univ}, \Q} \times_{\text{Res}^K_{\Q} \text{Hilb}} X$ be the pullback of our family of hypersurfaces to $X$. Note that $Y$ is a hypersurface in $X \times_{\Q} A = X_K \times_K A$.

We will apply Theorem 8.21 to show that $Y \subseteq X_K \times_K A$ is the translate of a constant hypersurface $H_0 \in A$ by a section $s \in A(X_K)$. In this case, all the points of $\text{Res}^K_{\Q} \text{Hilb}(\Z[1/\S])_{\text{prim}}$ lying in $X$ would correspond to points in the orbit $H_0$.

The map $X \to X_{\text{sing}}$ is proper, so it spreads out to a proper map

$$X \to \text{Res}_{\Z[1/\S]} \text{Hilb},$$

possibly after further enlarging the finite sets $S$ and $S'$. Possibly after further enlargement of $S$ and $S'$, the family $Y$ spreads out to a smooth proper family $Y \to \mathcal{X}$, which is a hypersurface in $A \times_{\Z[1/\S]} X$. Now, by properness, every $S$-integral point of Hilb lifts to an $S$-integral point of $\mathcal{X}$, so by construction the $S$-integral points are Zariski dense in $\mathcal{X}$.

Fix some $p \notin S'$. Assume that $Y \subseteq X_K \times_K A$ is not equal to the translate of a constant hypersurface $H_0 \in A$ by a section $s \in A(X_K)$. Let $\eta$ be the generic point of $X_K$. The fibers of $Y$ over points in a dense subset of $X$ are primitive hypersurfaces, so $Y_\eta$ is not translation-invariant by any nonzero element of $A$. Let $G$ be the Tannakian group of the constant sheaf on $Y_\eta$, and let $G^*$ be the commutator subgroup of the identity component of $A$. By Theorem 4.1 because of our assumptions on $n$ and $d$, we have $G^* = \text{SL}_N$, $\text{Sp}_N$, or $\text{SO}_N$, acting by its standard representation. Furthermore the $\text{Sp}_N$ case occurs exactly when $Y_\eta$ is equal to a translate of $[-1]^*Y_\eta$ and $n$ is even and the $\text{SO}_N$ case occurs exactly when $Y_\eta$ is equal to a translate of $[-1]^*Y_\eta$ and $n$ is even. In particular, $G^*$ is a simple algebraic group acting by an irreducible representation.

We can now apply Corollary 5.9 This gives us the existence of an embedding $\iota: K \to \mathbb{C}$ and a torsion character $\chi$ of $\pi_1(A_s)$, satisfying the big monodromy condition that is needed for Lemma 6.6. In Section 6.3 and Lemma 6.28 we have constructed a Hodge–Deligne system $V_I$ on $X_{\Q}$ attached to the orbit $I$ containing $(\iota, \chi)$ and the family $Y \subseteq X_K \times_K A$. By Lemma 7.6 the differential Galois group of $V_I$ is a strongly $c$-balanced subgroup of $G^0$. Corollary 5.9 gives vanishing of cohomology outside degree $n - 1$, so the Hodge numbers of $V_I$ are given by Lemma 2.7.

We apply Theorem 8.21 to $V = V_I$. That the eigenvalues of Frobenius on $V$ are integral Weil numbers is a consequence of the Weil numbers, since $V$ comes from geometry; the polarization and integral structure are given in Lemma 6.28. Lemma 6.29 gives $\mathcal{H}$-structure, with $\mathcal{H}$ chosen as in Lemma 6.29 (Note that this matches the group $G^*$ by our earlier calculation, because $Y_\eta$ is equal to a translate...
of \([-1]^*Y\) if and only if \(Y\) is equal to a translate of \([-1]^*Y\).) Lemma 7.6 gives that \(G_{\text{mon}}\) is strongly \(c\)-balanced with respect to Frobenius. The Hodge numbers of \(V\) are uniform because we have explicitly computed them, independently of \(\iota\), in Lemma 2.7.

The numerical conditions are satisfied because every term except \(c\) is independent of \(c\), so taking \(c\) arbitrarily large it suffices to have

\[
\sum_{p>0} h_p > 0
\]

\[
\sum_{p>0} ph_p > T_G \left( \frac{1}{2}(h^0 - t) \right).
\]

By Lemma 10.1 we have

\[
\sum_{p>0} h_p > \frac{1}{2}(h^0 - t).
\]

This implies the first condition because \(t \leq h_0\). It implies the second condition because, by the definition of \(T_G\), \(T_G(x)\) is strictly increasing for \(x \leq \sum_{p>0} h_p\) and \(T_G(\sum_{p>0} h_p) = \sum_{p>0} ph_p\).

Hence the hypotheses of Theorem 8.21 are satisfied, showing that the integral points are not Zariski dense, contradicting our earlier assumption.

\[\square\]

**Corollary 11.3.** Let \(K, A, S, \phi\) be as in Theorem 11.2. Suppose that either \(n \geq 4\) or \(n = 3\) and \(d\) is not a multiple of \((a(i)+a(i+1))/6\) for any \(i \geq 2\).

There are only finitely many smooth hypersurfaces \(H \subseteq A\) representing \(\phi\), with good reduction outside \(S\), up to translation.

**Proof.** Any hypersurface \(H \subseteq A\) is of the form \(\pi^{-1}H'\), where \(\pi: A \to A'\) is an isogeny defined over \(K\), and \(H' \subseteq A'\) is a primitive hypersurface defined over \(K\). In this case \(\phi\) is the pullback of an ample class \(\phi'\) along \(\pi\), and we have \(d = \phi^n/n! = (\phi^n/n!)\cdot\deg \pi\). Hence \(\deg \pi\) is bounded, so there are only finitely many possibilities for \(\pi\). For each one there is a unique \(\phi'\) with \(\phi = \pi^*\phi'\). Furthermore, \(d(\phi') \neq (a(i)+a(i+1))/6\) for any \(i \geq 2\). We conclude by applying Theorem 11.2 to each \((A', \phi')\).

\[\square\]

**Theorem 11.4.** Suppose \(\dim A \geq 4\). Fix an ample class \(\phi\) in the Neron-Severi group of \(A\). There are only finitely many smooth hypersurfaces \(H \subseteq A\) representing \(\phi\), with good reduction outside \(S\), up to translation.

**Proof.** This is one case of Corollary 11.3.

\[\square\]

**Theorem 11.5.** Suppose \(\dim A = 3\). Fix an ample class \(\phi\) in the Neron-Severi group of \(A\). Assume that the intersection number \(\phi \cdot \phi \cdot \phi\) is not divisible by \(d(i)\) for any \(i \geq 2\). There are only finitely many smooth hypersurfaces \(H \subseteq A\) representing \(\phi\), with good reduction outside \(S\), up to translation.

**Proof.** This is one case of Corollary 11.3 once we cancel the factor of \(3! = 6\) from the denominators.

\[\square\]
Appendix A. Combinatorics involving binomial coefficients and Eulerian numbers

This section is devoted to proving Proposition 4.11. Thus, throughout this section, we preserve the notation and assumptions of Proposition 4.11, which we review here.

We use $A(n, q)$ for the Eulerian numbers. We adopt the convention that $A(n, q) = 0$ unless $0 \leq q < n$; similarly, we take $\binom{n}{q}$ to vanish whenever $n$ is positive but $q < 0$ or $q \geq n$.

Recall from the introduction that $a(i)$ is the sequence satisfying

$$a(1) = 1, a(2) = 5, a(i + 2) = 4a(i + 1) + 1 - a(i).$$

**Proposition A.1** (Proposition 4.11). Let $n \geq 2$ and $d \geq 1$ be integers. Suppose that there exists a natural number $k$, function $m_H$ from the integers to the natural numbers and an integer $s$. Write $m = \sum m_H(i)$, and suppose that $1 < k < m - 1$.

Suppose the equation

$$(11) \quad \sum_{m_{n,q} \geq m_H(i)} \prod_{i \in n \mathbb{Z} + q} \binom{m_H(i)}{m_S(i)} = dA(n, q).$$

is satisfied for all $q \in \mathbb{Z}$. Then we have one of the cases

1. $m = 4$ and $k = 2$
2. $n = 2$ and $d = \binom{2k-1}{k}$ for some $k > 2$
3. $n = 3$ and $d = \binom{a(i)+a(i+1)}{a(i)}/6$ for some $i \geq 2$.

A tuple $(n, d, m, k, m_H, s)$ satisfying the conditions of the proposition will be called a solution.

We first handle the cases $n = 2, 3, 4$ directly, then give a general argument that handles cases $n \geq 5$. Thus Proposition A.1 will follow immediately once we have proven Lemmas A.1 (the $n = 2$ case), A.8 (the $n = 3$ case), A.9 (the $n = 4$ case), A.18, A.21, A.25, A.27 and A.31 (which collectively handle the $n \geq 5$ case).

We can assume without loss of generality that $k \leq m/2$.

Let $m_{\text{max}}$ and $m_{\text{min}}$ be the functions $m_S : \mathbb{Z} \to \mathbb{Z}$ satisfying $0 \leq m_S(i) \leq m_H(i)$, $\sum_i m_S(i) = k$, and maximizing, respectively, minimizing $\sum_i m_S(i)$.

**Lemma A.2.** There is a unique $w$ such that $m_{\text{min}}(i) = m_H(i)$ for all $i < w$, $m_{\text{min}}(i) = 0$ for all $i > w$, and $m_{\text{min}}(i) > 0$ for $i = w$.

Similarly, there is a unique $w'$ such that $m_{\text{max}}(i) = m_H(i)$ for all $i > w'$, $m_{\text{max}} = 0$ for all $i < w'$ and $m_{\text{max}} > 0$ for $i = w'$.

Furthermore $w \leq w'$.

**Proof.** We take $w$ to be the largest $i$ such that $m_{\text{min}}(i) > 0$. The last two conditions are then obvious and the fact that $m_{\text{min}}(i) = m_H(i)$ for $i < w$ follows by minimality - if it were not so, we could increase $m_{\text{min}}(i)$ by 1, reduce $m_{\text{min}}(w)$ by 1, and thereby reduce $\sum_i m_S(i)$ by $w - i$.

We take $w'$ to be the least $i$ such that $m_{\text{max}}(i) > 0$, and make a symmetrical argument.
Finally, for contradiction, assume \( w > w' \). Then
\[
\sum_i m_H(i) = m \geq 2k = \sum_i m_{\min}(i) + \sum_i m_{\max}(i)
\]
\[
> \sum_{i<w} m_{\min}(i) + \sum_{i>w} m_{\max}(i) = \sum_{i<w} m_H(i) + \sum_{i>w} m_H(i) \geq \sum_i m_H(i),
\]
a contradiction. \( \square \)

**Lemma A.3.** We have
\[
\sum_{i<w} (w - i)m_H(i) + \sum_{i>w} (i - w')m_H(i) + k(w' - w) = n - 1
\]

Note that all the terms on the left side of this equation are nonnegative.

**Proof.** Because the function \( q \mapsto dE(n, q) \) is supported on \( q \) ranging from 0 to \( n - 1 \), we must have
\[
n - 1 = (s + n - 1) - s = \sum_i im_{\max}(i) - \sum_i im_{\min}(i) = \sum_i (i - w')m_{\max}(i) - \sum_i (i - w)m_{\min}(i) + k(w' - w)
\]
\[
= \sum_{i<w} (w - i)m_H(i) + \sum_{i>w} (i - w')m_H(i) + k(w' - w)
\]
\( \square \)

**A.1. The case \( n \leq 4 \).**

**Lemma A.4.** Suppose \( n = 2 \). Then we must have \( d = \binom{2k - 1}{k} \). Furthermore if \( k = 2 \) then \( m = 4 \).

**Proof.** In (12), because the summands on the left side are nonnegative, one must be 1 and the others must vanish. The one that is 1 can only be the summand associated to \( i = w - 1 \) or \( i = w' + 1 \) as the other summands are integer multiples of something at least 2. By symmetry, we may assume the 1 comes from \( i = w + 1 \). Because the last summand vanishes, we must have \( w = w' \).

This gives \( m_H(i) = 0 \) unless \( i = w \) or \( w + 1 \) and \( m_H(w + 1) = 1 \).

Then the only possible solutions to \( 0 \leq m_S(i) \leq m_H(i) \) and \( \sum_i m_S(i) = k \) are \( m_S(w) = k, m_S(w + 1) = 0 \) and \( m_S(w + 1) = 1 \). These have \( \sum_i m_S(i) \) equal to \( kw \) and \( kw + 1 \) respectively, so we must have \( s = kw \). This implies
\[
\begin{pmatrix} m_S(w) \\ k \end{pmatrix} = \begin{pmatrix} m_S(w) \\ k \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = dE(2, 0) = d
\]
\[
\begin{pmatrix} m_S(w) \\ k - 1 \end{pmatrix} = \begin{pmatrix} m_S(w) \\ k - 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = dE(2, 1) = d
\]
and thus
\[
\begin{pmatrix} m_S(w) \\ k - 1 \end{pmatrix} = \begin{pmatrix} m_S(w) \\ k \end{pmatrix}
\]
which implies \( m_S(w) = 2k - 1 \) and thus \( d = \binom{2k - 1}{k} \).

In the \( k = 2 \) case, we have \( m = m_S(w) + m_S(w + 1) = 2k = 4 \), as desired. \( \square \)
Lemma A.5. Suppose \( n = 3 \). Then we must have \( m_H(i) = 0 \) for \( i \neq w-1, w, w+1 \) and \( m_H(w-1) = m_H(w+1) = 1 \) unless \( m = 4 \) and \( k = 2 \).

Proof. In (12), either two terms are 1 and the rest zero or one term is 2 and the rest are zero. In the first case, since the only terms that can be 1 are \( i = w-1 \) and \( i = w'+1 \), implying in particular that \( w = w' \) we have the stated conclusion. So it suffices to eliminate the case that one term is 2. The only possibilities are \( i = w-2, w-1, w'+1, w'+2 \), and the last term if \( k = 2 \). By symmetry, we are reduced to eliminating \( i = w'+1, i = w'+2 \), and the final term. In the first two cases we have \( w' = w \) and in the last case we have \( k = 2, w'-w = 1 \).

If \( m_H(i) = 0 \) for all \( i \) except \( w, w+1 \), and \( m_H(w+1) = 2 \), then there are three possibilities for \( m_S \): \( (m_S(w), m_S(w+1)) \) must equal \((k, 0), (k-1, 1)\), or \((k-2, 2)\). Using \( m_H(w) = m - 2 \), this gives

\[
\binom{m-2}{k} = \binom{m-2}{k} \binom{2}{0} = dE(3, 0) = d
\]

\[
2 \binom{m-2}{k-1} = \binom{m-2}{k-1} \binom{2}{1} = dE(3, 1) = 4d
\]

\[
\binom{m-2}{k-2} = \binom{m-2}{k-2} \binom{2}{2} = dE(3, 2) = d
\]

and this implies

\[
\frac{1}{2} = \binom{k-2}{k-1} = \frac{m-k-1}{k}
\]

\[
\frac{1}{2} = \binom{m-2}{k-2} = \frac{k-1}{m-k}
\]

so \( m-k = 2k-2 \) and \( k = 2m - 2k - 2 \) giving \( m = 4, k = 2 \).

If \( m_H(i) = 0 \) for all \( i \) except \( w, w+2 \), then \( \sum_i im_S(i) \equiv kw \mod 2 \) whenever \( 0 \leq m_S(i) \leq m_H(i) \) and \( \sum_i m_S(i) = k \). Hence the left side of (12) is nonzero only when \( s + q \equiv kw \mod 2 \). This contradicts the fact that \( E(2, q) \) is nonzero for \( q \) of both parities.

If \( m_H(i) = 0 \) for all \( i \) except \( w, w+1 \) and \( k = 2 \), then there are three possibilities for \( m_S \): \( (m_S(w), m_S(w+1)) \) must be \((2, 0), (1, 1)\), or \((0, 2)\). This gives

\[
\frac{m_H(w)(m_H(w) - 1)}{2} = \binom{m_H(w)}{2} \binom{m_H(w) + 1}{0} = dA(3, 0) = d
\]

\[
m_H(w)m_H(w+1) = \binom{m_H(w)}{1} \binom{m_H(w) + 1}{1} = dA(3, 1) = 4d
\]

\[
\frac{m_H(w+1)(m_H(w+1) - 1)}{2} = \binom{m_H(w)}{0} \binom{m_H(w) + 1}{2} = dA(3, 2) = d
\]

Because \( d > 0 \) this implies \( m_H(w) = m_H(w+1) \) and thus \( 2m_H(w)(m_H(w+1)) = m_H(w)^2 \) which implies \( m_H(w) = m_H(w+1) = 2 \) and thus \( m = 4, k = 2 \). \( \square \)

Lemma A.6. Suppose \( n = 3 \). Then unless \( m = 4 \) and \( k = 2 \), we have

\[
(m-k)^2 - 4(m-k)k + k^2 = m.
\]
Proof. By Lemma A.5, we have $m_H(i) = 0$ for $i \neq w-1, w, w+1$ and $m_H(w-1) = m_H(w+1) = 1$. This means there are four possibilities for $m_S$: $(m_S(w-1), m_S(w), m_S(w+1))$ must equal $(0, k, 0)$, $(0, k-1, 1)$, $(1, k-1, 0)$, or $(1, k-2, 1)$. Using $m_H(w) = m - 2$, and $\binom{m-2}{0} = \binom{m-2}{1} = 1$, we get
\[
\binom{m-2}{k-1} = dE(3, 0) = d
\]
\[
\binom{m-2}{k} + \binom{m-2}{k-2} = dE(3, 1) = 4d
\]
\[
\binom{m-2}{k-1} = dE(3, 2) = d
\]
so in other words we have
\[
\binom{m-2}{k} + \binom{m-2}{k-2} = 4\binom{m-2}{k-1}
\]
which dividing by $(m-2)!$ and multiplying by $k!(m-k)!$ is
\[
(m-k)(m-k-1) + k(k-1) = 4k(m-k)
\]
which is exactly the stated Diophantine equation. □

Lemma A.7. The positive integer solutions to $a^2 - 4ab + b^2 = a + b$ with $a \leq b$ have the form $(a, b) = (a(i), a(i+1))$ for some $i \geq 1$.

Proof. Let $a$ and $b$ be positive integers with $a^2 - 4ab + b^2$ and $b \geq a$. We will show that there exists $i \geq 1$ such that $a = a(i)$ and $b = a(i+1)$. By induction, it suffices to prove that either $a = 1, b = 5$ or that there exists an integer $b'$ with $0 < b' < a$, $b = 4a + 1 - b'$, and $(b', a)$ solving the same equation, as if $b' = a(i')$, $a = a(i' + 1)$ then $b = a(i' + 2)$.

To do this, let $b' = 4a + 1 - b$. Then because we can rewrite the equation as
\[
b(4a + 1 - b) = a^2 - a,
\]
as $b$ is a solution then $b'$ is a solution as well. Furthermore, if $a > 1$ then $a^2 - a > 0$ so $b' = \frac{a^2 - a}{b} > 0$, and we must have $b' < a$ because if $b' \geq a$ we have
\[
a^2 - a = b(4a + 1 - b) \geq a \cdot a > a^2,
\]
so if $a = 1$ we always have a solution. On the other hand, if $a = 1$ then $b(5-b) = 0$ so, because $b > 0$, we have $b = 5$, the base case. □

Lemma A.8. Suppose $n = 3$. Then either $m = 4$ and $k = 2$ or $m = a(i) + a(i+1)$ and $k = a(i)$ for some $i \geq 2$.

Proof. This follows from Lemma A.6 and A.7 once we observe that because $k \leq m/2$, we have $k \leq m - k$, and because $k \geq 2$, the case $k = a(1), m = a(2)$ cannot occur. □

Lemma A.9. There are no solutions for $n = 4$.

Proof. We first consider the contribution to (12) from $k(w' - w)$. Because $k \geq 2$ and this contribution is at most $(n-1) = 3$, we can only have $w' = w$ or $w' = w+1$, $k = 2$ or 3. Let us eliminate the $w' = w+1$ cases first.
In the $k = 3$ case we have $m_H(i) = 0$ unless $i = w$ or $w + 1$. Thus we have four possibilities for $m_S$ - we must have $(m_S(w), m_S(w + 1)) = (3, 0), (2, 1), (1, 2)$, or $(0, 3)$. This gives

$$\begin{pmatrix} m_H(w) \\ 3 \end{pmatrix} \begin{pmatrix} m_H(w + 1) \\ 0 \end{pmatrix} = dE(4, 0) = d$$

$$\begin{pmatrix} m_H(w) \\ 2 \end{pmatrix} \begin{pmatrix} m_H(w + 1) \\ 1 \end{pmatrix} = dE(4, 1) = 11d$$

$$\begin{pmatrix} m_H(w) \\ 1 \end{pmatrix} \begin{pmatrix} m_H(w + 1) \\ 2 \end{pmatrix} = dE(4, 2) = 11d$$

$$\begin{pmatrix} m_H(w) \\ 0 \end{pmatrix} \begin{pmatrix} m_H(w + 1) \\ 3 \end{pmatrix} = dE(4, 3) = d$$

Combining the first and last equations with $d > 0$, we see that $m_H(w) = m_H(w + 1)$. Dividing the second equation by the first, we get

$$\frac{3m_H(w)}{m_H(w) - 2} = 11$$

which implies $m_H(w) = \frac{11}{2}$, a contradiction.

In the $k = 2$ case, one more term must be 1, which is $i = w - 1$ or $i = w' + 1$. Without loss of generality, it is $i = w - 1$. Then we have $m_H(i) = 0$ unless $i = w - 1, w, w + 1$ and $m_H(w - 1) = 1$. Then the possible values of $(m_S(w - 1), m_S(w), m_S(w + 1))$ are $(1, 1, 0), (1, 0, 1), (0, 2, 0), (0, 1, 1)$, and $(0, 0, 2)$. This gives (ignoring factors of the form $(\binom{w}{0}$ or $(\binom{1}{1})$)

$$\begin{pmatrix} m_H(w) \\ 1 \end{pmatrix} = dE(4, 0) = d$$

$$\begin{pmatrix} m_H(w + 1) \\ 1 \end{pmatrix} + \begin{pmatrix} m_H(w) \\ 2 \end{pmatrix} = dE(4, 1) = 11d$$

$$\begin{pmatrix} m_H(w) \\ 1 \end{pmatrix} \begin{pmatrix} m_H(w + 1) \\ 1 \end{pmatrix} = dE(4, 2) = 11d$$

$$\begin{pmatrix} m_H(w + 1) \\ 2 \end{pmatrix} = d$$

Dividing the third equation by the first, we get $m_H(w + 1) = 11$. Thus by the fourth equation $d = \binom{11}{2} = 55$, and by the first equation $m_H(w) = 55$. Then the second equation gives $11 + \binom{55}{2} = 11 \cdot 55$, which is false, so there are no solutions.

We now handle the case $w = w'$. In this case, because the total sum of $4\binom{2}{1}$ is 3, and only the two terms $i = w + 1, i = w - 1$ can contribute a 1, we must have one term contributing 3 or one 2 and one 1. There are four terms that might contribute 3: $i = w + 3, i = w + 1, i = w - 1$, and $i = w - 3$, and four that might contribute 2: $i = w + 2, i = w + 1, i = w - 1$, and $i = w - 2$. This gives a total of 10 possibilities for $m_H$, or 4 up to symmetry: we may assume $(m_H(w - 1), m_H(w), m_H(w + 1), m_H(w + 2), m_H(w + 3)) = (0, m - 1, 0, 0, 1), (0, m - 3, 3, 0, 0), (0, m - 2, 1, 1, 0), (1, m - 2, 0, 1, 0), (1, m - 2, 2, 0, 0)$.

The case $(0, m - 1, 0, 0, 1)$ is easy to eliminate as it implies that the left side of (3) is nonvanishing only in a single residue class mod 3, but we know that the right side does not have that possibility.
The case \((0, m-3, 0, 0)\) gives us four possibilities for \(m_S\), implying
\[
\binom{m-3}{k} \binom{3}{0} = dE(4, 0) = d
\]
\[
\binom{m-3}{k-1} \binom{3}{1} = dE(4, 0) = 11d
\]
\[
\binom{m-3}{k-2} \binom{3}{2} = dE(4, 0) = 11d
\]
\[
\binom{m-3}{k-3} \binom{3}{3} = dE(4, 0) = d
\]
The first and fourth equations gives \(\binom{m-3}{k} = \binom{m-3}{k-3}\), which implies that \(m-3-k = k-3\) or \(m = 2k\). Dividing the third equation by the second, we then obtain
\[
11 = \frac{(2k-2)}{(k-3)} \binom{3}{3} = \frac{k}{k-2} \cdot 3
\]
whose unique solution is \(k = \frac{11}{4}\), a contradiction.

The case \((0, m-2, 1, 0)\) gives us four possibilities for \(m_S\), implying
\[
\binom{m-2}{k} \binom{1}{0} \binom{1}{0} = dE(4, 0) = d
\]
\[
\binom{m-2}{k-1} \binom{1}{1} \binom{0}{0} = dE(4, 1) = 11d
\]
\[
\binom{m-2}{k-1} \binom{1}{0} \binom{1}{1} = dE(4, 2) = 11d
\]
\[
\binom{m-2}{k-2} \binom{1}{1} \binom{1}{1} = dE(4, 3) = d
\]
By the second and third equations we have \(\binom{m-2}{k} = \binom{m-2}{k-2}\) which implies \(m-2-k = k-2\) or \(m = 2k\). Dividing the second equation by the first we get \(11 = \frac{k}{k-1}\) so \(k = \frac{11}{10}\), a contradiction.

The case \((1, m-2, 1, 0)\) gives us four possibilities for \(m_S\), implying
\[
\binom{1}{1} \binom{m-2}{k-1} \binom{1}{0} = dE(4, 0) = d
\]
\[
\binom{1}{0} \binom{m-2}{k} \binom{1}{0} = dE(4, 1) = 11d
\]
\[
\binom{1}{1} \binom{m-2}{k-2} \binom{1}{1} = dE(4, 2) = 11d
\]
\[
\binom{1}{0} \binom{m-2}{k-1} \binom{1}{1} = dE(4, 3) = d
\]
By the second and third equations, we have \(\binom{m-2}{k} = \binom{m-2}{k-2}\) which again implies \(m = 2k\). But then \(\binom{2k-2}{k-1} > \binom{2k-2}{k-2}\) which means \(d > 11d\), a contradiction.

The case \((1, m-2, 2, 0)\) has six possibilities for \(m_S\), implying
\[
\binom{1}{1} \binom{m-2}{k-1} \binom{2}{0} = dE(4, 0) = d
\]
\[
\begin{aligned}
\binom{1}{0} \binom{m - 2}{k} \binom{2}{0} + \binom{1}{1} \binom{m - 2}{k - 2} \binom{2}{1} &= dE(4, 1) = 11d \\
\binom{1}{1} \binom{m - 2}{k - 3} \binom{2}{2} + \binom{1}{0} \binom{m - 2}{k - 1} \binom{2}{1} &= dE(4, 2) = 11d \\
\binom{1}{0} \binom{m - 2}{k - 2} \binom{2}{2} &= dE(4, 3) = d
\end{aligned}
\]

By the first and fourth equations, we have
\[
\binom{m - 2}{k - 2} k - 1
\]
which implies
\[
m = 2k.
\]

But then since
\[
\binom{2k - 2}{k - 3} \leq \binom{2k - 2}{k - 2}
\]
we have by the third and fourth equations
\[
11d = \binom{2k - 2}{k - 3} + 2 \binom{2k - 2}{k - 1} \leq \binom{2k - 2}{k - 2} + 2 \binom{2k - 2}{k - 2} = 3d
\]
a contradiction. \(\square\)

A.2. **The case** \(n \geq 5\): **general setup.** We’ll write \(m_0 = m_{\min}\) for the rest of this section. The key equality for \(q = 0\) gives
\[
(13) \quad d = \binom{m_H(w)}{m_0(w)}.
\]

For any \(m_S : \mathbb{Z} \to \mathbb{Z}\) as in Equation (11) we’ll write
\[
N(m_S) = N(m_H, m_S) = \prod_i \binom{m_{H(i)}}{m_{S(i)}}.
\]

Now the key equality (11) becomes
\[
(14) \quad \sum_{m_S : \mathbb{Z} \to \mathbb{Z} \atop 0 \leq m_S(i) \leq m_H(i)} \frac{N(m_S)}{N(m_0)} = A(n, q).
\]

We’re going to get a contradiction from combinatorial considerations involving the terms associated to small \(q\) in Equation (3).

By abuse of notation, we’ll let \([i]\) denote the function \(m_{\{i\}}\) taking the value 1 on \(i\) and zero elsewhere; so any function \(\mathbb{Z} \to \mathbb{Z}\) is a linear combination of the elementary functions \([i]\).

There are at most two functions \(m_1\) that contribute to the \(q = 1\) case in Equation (14). These are
\[
m_0^q = m_0 + [w] - [w - 1]
\]
and
\[
m_1^q = m_0 + [w + 1] - [w].
\]

We compute
\[
\frac{N(m_0^q)}{N(m_0)} = \frac{m_H(w - 1)(m_H(w) - m_0(w))}{(m_0(w) + 1)}
\]
\[
\frac{N(m_1^q)}{N(m_0)} = \frac{m_H(w + 1)m_0(w)}{(m_H(w) - m_0(w) + 1)};
\]
the \(q = 1\) case of Equation (14) gives
\[
\frac{N(m_0^q)}{N(m_0)} + \frac{N(m_1^q)}{N(m_0)} = A(n, 1).
\]
Similarly, there are at most five nonzero terms in the $q = 2$ case Equation (14):
\[
\begin{align*}
m_a^2 &= m_0 + [w] - [w - 2] \\
m_b^2 &= m_0 + 2[w] - 2[w - 1] \\
m_c^2 &= m_0 + [w + 1] - [w - 1] \\
m_d^2 &= m_0 + 2[w + 1] - 2[w] \\
m_0^2 &= m_0 + [w + 2] - [w].
\end{align*}
\]

We have the following equalities.
\[
\begin{align*}
N(m_a^2) &= \frac{m_H(w - 2)(m_H(w) - m_0(w))}{(m_0(w) + 1)} \\
N(m_b^2) &= \frac{m_H(w - 1)(m_H(w - 1) - 1)(m_H(w) - m_0(w))(m_H(w) - m_0(w) - 1)}{2(m_0(w) + 1)(m_0(w) + 2)} \\
N(m_c^2) &= \frac{m_H(w) m_H(w + 2)}{m_H(w) - m_0(w) + 1}.
\end{align*}
\]

Equation (14) gives
\[
\sum_{* = a,b,c,d,e} \frac{N(m_*)}{N(m_0)} = A(n, 2).
\]

We conclude with a lemma that will be useful at several points in the argument.

**Lemma A.10.** We have
\[
\frac{N(m_a^2)}{N(m_0)} + \frac{N(m_b^2)}{N(m_0)} = A(n, 1)
\]
and
\[
\left( \frac{N(m_c^2)}{N(m_0)} \right) \left( \frac{N(m_d^2)}{N(m_0)} \right) < m_H(w - 1)m_H(w + 1).
\]

In particular, if
\[
m_H(w - 1)m_H(w + 1) < \frac{1}{4} A(n, 1)^2,
\]
let $\alpha < \beta$ be the real roots of
\[
X^2 - A(n, 1)X + m_H(w - 1)m_H(w + 1).
\]

Then we have
\[
\min \left( \frac{N(m_a^2)}{N(m_0)}, \frac{N(m_b^2)}{N(m_0)} \right) \leq \alpha
\]
and
\[
\max \left( \frac{N(m_d^2)}{N(m_0)}, \frac{N(m_e^2)}{N(m_0)} \right) \geq \beta
\]

**Proof.** The first equality is the $q = 1$ case of Equation (14). The second follows from the explicit formulas for the two quotients $\frac{N(m_a^2)}{N(m_0)}, \frac{N(m_d^2)}{N(m_0)}$.

The bounds in terms of $\alpha$ and $\beta$ follow from the first two inequalities. \qed
A.3. **The case \( n \geq 5 \), with \( N(m_1^q) \) big.** In the following sections we will use without proof a number of inequalities involving Eulerian numbers. The proofs of such inequalities are routine; we discuss the technique in Section B.

Recall notation from the beginning of Section A and the beginning of Section A.2.

**Lemma A.11.** If \( n \geq 5 \) and

\[
\frac{N(m_1^q)}{N(m_0)} \geq \frac{A(n, 1)}{2}
\]

then \( m_H(w - 1) = 1 \).

**Proof.** We’ll estimate \( N(m_2^b) \). The key point is that, combining (16) and (15), we have

\[
\frac{N(m_2^b)N(m_0)}{N(m_1^q)^2} \leq 4 \frac{A(n, 2)}{A(n, 1)^2}.
\]

We will focus on proving a lower bound for the left side, assuming \( m_H(w - 1) > 1 \), which which will give a contradiction for sufficiently large \( n \).

By hypothesis, we have

\[
\frac{A(n, 1)}{2} \leq \frac{m_H(w - 1)(m_H(w) - m_0(w))}{(m_0(w) + 1)}.
\]

By Lemma A.3 we have \( m_H(w - 1) \leq n - 1 \). Also, we know \( m_0(w) \geq 1 \), so

\[
m_H(w) - m_0(w) \geq \frac{A(n, 1)}{n - 1}.
\]

If \( n \geq 6 \), then \( \frac{A(n, 1)}{n - 1} \geq 9 \), which gives the bound

\[
m_H(w) - m_0(w) - 1 \geq \frac{8}{9}(m_H(w) - m_0(w))
\]

Assuming \( m_H(w - 1) \geq 2 \), we find

\[
\frac{N(m_2^b)N(m_0)}{N(m_1^q)^2} = \frac{m_H(w - 1)(m_H(w - 1) - 1)(m_H(w) - m_0(w))(m_H(w) - m_0(w) - 1)}{2(m_0(w) + 1)(m_0(w) + 2)} \cdot \frac{(m_0(w) + 1)^2}{m_H(w - 1)(m_H(w) - m_0(w))^2}
\]

\[
\geq \frac{1}{2} \cdot \frac{m_0(w) + 1}{m_0(w) + 2} \cdot \frac{m_H(w - 1) - 1}{m_H(w - 1)} \cdot \frac{m_H(w) - m_0(w) - 1}{m_H(w) - m_0(w)}
\]

\[
\geq \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{m_H(w) - m_0(w) - 1}{m_H(w) - m_0(w)} = \frac{1}{6} \cdot \frac{m_H(w) - m_0(w) - 1}{m_H(w) - m_0(w)}
\]

By combining this with (18), we conclude that

\[
\frac{N(m_2^b)N(m_0)}{N(m_1^q)^2} \geq 4/27
\]

which combines with (17) to give

\[
\frac{A(n, 2)}{A(n, 1)^2} \geq 1/27
\]
which is impossible for \( n \geq 11 \). (See the discussion in Appendix [B] and bound-A3a in the Python code.)

For smaller \( n \), we do a more precise version of the above analysis. For \( 5 \leq n \leq 10 \), we will improve on the bound (16) by using Lemma A.10. For \( n = 5 \), we will also need to replace the bound (18) by a slightly weaker one without the assumption \( n \geq 6 \). With these modifications, the argument will work for \( n \) from 5 to 10.

To apply Lemma A.10, we must give an upper bound for \( m_H(w - 1)m_H(w + 1) \), showing that the two roots \( \alpha, \beta \) are far apart. Consider

\[
m^2_a = m_0 + 2[w + 1] - 2[w - 1].
\]

We have

\[
A(n, 4) \geq \frac{N(m^4_a)}{N(m_0)} = \frac{[m_H(w + 1)(m_H(w + 1) - 1)] [m_H(w - 1)(m_H(w - 1) - 1)]}{4},
\]

so (still assuming \( m_H(w - 1) \geq 2 \)) we conclude that

\[
2A(n, 4) \geq m_H(w + 1)(m_H(w + 1) - 1).
\]

For \( 5 \leq n \leq 10 \) we have

\[
m_H(w + 1) \leq \sqrt{2A(n, 4) + 1} \leq \frac{1}{48} \frac{A(n, 1)^2}{n - 1}
\]

(see bound-A3b in the Python code). Thus we have

\[
m_H(w - 1)m_H(w + 1) \leq (n - 1) \cdot \frac{1}{48} \frac{A(n, 1)^2}{n - 1} = \frac{1}{48} A(n, 1)^2,
\]

so by Lemma A.10,

\[
\frac{N(m^4_a)}{N(m_0)} \geq \frac{45}{46} A(n, 1).
\]

When \( 6 \leq n \leq 10 \) we have from above (20) that

\[
\frac{N(m^b_2)N(m_0)}{N(m^4_1)^2} \geq 4/27,
\]

so

\[
\frac{A(n, 2)}{A(n, 1)^2} \geq \frac{4}{27} \left( \frac{45}{46} \right)^2,
\]

which does not hold for any \( 6 \leq n \leq 10 \).

Finally when \( n = 5 \) we use the estimate

\[
m_H(w) - m_0(w) \geq \frac{A(n, 1)}{n - 1} = \frac{13}{2}
\]

to deduce

\[
m_H(w) - m_0(w) - 1 \geq \frac{11}{13} (m_H(w) - m_0(w)).
\]

Now (19) implies

\[
\frac{N(m^b_2)N(m_0)}{N(m^4_1)^2} \geq \frac{1}{6} \cdot \frac{11}{13};
\]

combining this with (21), we get

\[
\frac{A(n, 2)}{A(n, 1)^2} \geq \frac{1}{6} \cdot \frac{11}{13} \left( \frac{45}{46} \right)^2,
\]

which is not true for \( n = 5 \).
Thus we arrive at a contradiction in every case. \qed

Lemma A.12. If
\[ \frac{N(m_0^2)}{N(m_0)} \geq \frac{A(n, 1)}{2} \]
then \( m_H(w - 2) \leq 2 \).

Proof. Assume \( m_H(w - 2) \geq 3 \), and consider
\[ m_0^2 = m_0 - 3[w - 2] - [w - 1] + 4[w] \]
We will show that \( N(m_0^2) \) is too big.

First, note that we must have \( n \geq 8 \) by Lemma A.3. By hypothesis, we have \( \frac{N(m_1^1)}{N(m_0)} > \frac{A(n, 1)}{2} \). From Lemma A.11, we know \( m_H(w - 1) = 1 \), so
\[ \frac{N(m_1^1)}{N(m_0)} = \frac{(m_H(w) - m_0(w))}{(m_0(w) + 1)} \]
and thus
\[ (m_H(w) - m_0(w)) \geq A(n, 1) \]
Since \( n \geq 8 \), this gives \( (m_H(w) - m_0(w)) \geq 247 \).

Now
\[ \frac{N(m_0^2)}{N(m_0)} = C \prod_{i=1}^{4} \frac{(m_H(w) - m_0(w) - i + 1)}{(m_0(w) + i)} \]
where
\[ C = \frac{m_H(w - 2)(m_H(w - 2) - 1)(m_H(w - 2) - 2)(m_H(w - 1))}{6} \geq 1 \]
Using the bounds \( (m_H(w) - m_0(w) - 3) > .98(m_H(w) - m_0(w)) \) and
\[ \frac{(m_0(w) + 1)^4}{(m_0(w) + 1)(m_0(w) + 2)(m_0(w) + 3)(m_0(w) + 4)} \geq \frac{2}{15} \]
we deduce that
\[ \frac{N(m_0^2)}{N(m_0)} \geq \frac{.92A(n, 1)^4}{120} \geq \frac{A(n, 1)^4}{140} \]
The inequality \( A(n, 7) < \frac{1}{140}A(n, 1)^4 \)
(see Appendix B and bound-A3c in the Python code) gives a contradiction. \qed

Lemma A.13. If \( n \geq 5 \) and
\[ \frac{N(m_0^2)}{N(m_0)} \geq \frac{A(n, 1)}{2} \]
then
\[ m_H(w + 1)(m_H(w + 1) - 1) \leq \frac{48}{11} A(n, 3) \frac{N(m_0^2)}{N(m_0)} \leq \frac{48}{11} A(n, 3) A(n, 1) \]
Furthermore, if \( m_0(w) \geq 2 \) then we have the stronger bound
\[ m_H(w + 1)(m_H(w + 1) - 1) \leq \frac{36}{11} A(n, 3) \frac{N(m_0^2)}{N(m_0)} \leq \frac{36}{11} A(n, 3) A(n, 1) \]
Proof. Let 
\[ m_3^w = m_0 + 2[w + 1] - [w] - [w - 1]. \]
The result will follow from 
\[ \frac{N(m_3^w)}{N(m_0)} \leq A(n, 3). \]

We have 
\[ A(n, 3) \geq \frac{N(m_3^w)}{N(m_0)} = \frac{m_H(w + 1)(m_H(w + 1) - 1)m_H(w - 1)m_0(w)}{2(m_H(w) - m_0(w) + 1)}. \]

On the other hand, 
\[ \frac{A(n, 1)}{2} \leq \frac{N(m_1^w)}{N(m_0)} = \frac{m_H(w - 1)(m_H(w) - m_0(w))}{(m_0(w) + 1)}. \]

We know \( m_{H}(w - 1) = 1 \) by Lemma A.11, so \( m_{H}(w) - m_0(w) \geq A(n, 1) \geq 11. \) Thus we can estimate 
\[ \frac{m_0(w)}{2(m_H(w) - m_0(w) + 1)} \cdot \frac{(m_H(w) - m_0(w))}{(m_0(w) + 1)} \geq \frac{1}{4} \cdot \frac{11}{12} = \frac{11}{48}. \]

The first bound follows.

To get the second bound, note that if \( m_0(w) \geq 2 \) then 
\[ \frac{m_0(w)}{m_0(w) + 1} \geq \frac{2}{3}, \]

so 
\[ \frac{m_0(w)}{2(m_H(w) - m_0(w) + 1)} \cdot \frac{(m_H(w) - m_0(w))}{(m_0(w) + 1)} \geq \frac{1}{3} \cdot \frac{11}{12} = \frac{11}{36}. \]

Lemma A.14. If \( n \geq 5 \) and 
\[ \frac{N(m_3^w)}{N(m_0)} \geq \frac{A(n, 1)}{2} \]
then 
\[ m_H(w + 2)(m_H(w + 2) - 1) \leq \frac{48}{11} A(n, 5) \frac{N(m_3^w)}{N(m_0)} \leq \frac{48}{11} A(n, 5) A(n, 1). \]

Furthermore, if \( m_0(w) \geq 2 \) then we have the stronger bound 
\[ m_H(w + 2)(m_H(w + 2) - 1) \leq \frac{36}{11} A(n, 5) \frac{N(m_3^w)}{N(m_0)} \leq \frac{36}{11} A(n, 5) A(n, 1). \]

Proof. Let 
\[ m_5^w = m_0 + 2[w + 2] - [w] - [w - 1]. \]
The result follows from 
\[ \frac{N(m_5^w)}{N(m_0)} \leq A(n, 5); \]
the proof is exactly analogous to the proof of Lemma A.13.

Lemma A.15. If 
\[ \frac{N(m_5^w)}{N(m_0)} \geq \frac{A(n, 1)}{2} \]
then \( n \leq 10. \)
Proof. Assume \( n \geq 5 \). We’ll estimate each \( \frac{N(m^2_2)}{N(m_0)} \), and show that for large \( n \) their sum is too small.

We have the following bounds. By Lemma A.12, we have \( m_H(w - 2) \leq 2 \), so

\[
\frac{N(m^2_2)}{N(m_0)} \leq 2 \frac{N(m^2_0)}{N(m_0)} \leq 2A(n, 1).
\]

Lemma A.11 tells us that \( m_H(w - 1) = 1 \) (if \( n \geq 5 \)), so

\[
\frac{N(m^2_0)}{N(m_0)} = 0.
\]

By Lemma A.13, we know that

\[
m_H(w + 1)(m_H(w + 1) - 1) \leq \frac{48}{11} A(n, 3) \frac{N(m^2_1)}{N(m_0)} \leq \frac{48}{11} A(n, 3) A(n, 1),
\]

so

\[
\frac{N(m^2_1)}{N(m_0)} = m_H(w - 1)m_H(w + 1) = m_H(w + 1) \leq \sqrt{\frac{48}{11} A(n, 3) A(n, 1) + 1}
\]

and

\[
\frac{N(m^2_1)}{N(m_0)} = \frac{(m_H(w + 1))(m_H(w + 1) - 1)(m_0(w))(m_0(w) - 1)}{2(m_H(w) - m_0(w) + 1)(m_H(w) - m_0(w) + 2)} \leq \frac{1}{2} \left( \frac{48}{11} A(n, 3) \frac{N(m^2_1)}{N(m_0)} \right) \left( \frac{N(m^2_0)}{N(m_0)} \right)^{-2} \leq \frac{48}{11} A(n, 3) A(n, 1).
\]

Finally, from Lemma A.14 we find

\[
m_H(w + 2) \leq \sqrt{\frac{48}{11} A(n, 5) \frac{N(m^2_1)}{N(m_0)} + 1}
\]

so

\[
\frac{N(m^2_1)}{N(m_0)} = \frac{m_0(w)m_H(w + 2)}{m_H(w) - m_0(w) + 1} \leq \left( \frac{N(m^2_1)}{N(m_0)} \right)^{-1} \left( \sqrt{\frac{48}{11} A(n, 5) \frac{N(m^2_1)}{N(m_0)} + 1} \right) \leq \sqrt{\frac{96}{11} A(n, 5) + 1}.
\]

These five bounds combine (see Appendix B, and bound-A3d in the Python code) to give

\[
\frac{N(m^2_2)}{N(m_0)} + \frac{N(m^2_1)}{N(m_0)} + \frac{N(m^2_0)}{N(m_0)} + \frac{N(m^2_1)}{N(m_0)} + \frac{N(m^2_2)}{N(m_0)} < A(n, 2)
\]

for \( n \geq 11 \), a contradiction.

\[\square\]

**Lemma A.16.** If \( 5 \leq n \leq 19 \) and

\[
\frac{N(m^2_1)}{N(m_0)} \geq \frac{A(n, 1)}{2}
\]
then \( m_0(w - 1) = 1 \) and \( m_0(w - s) = 0 \) for \( s \geq 2 \).

**Proof.** We have already seen in Lemma 11 that \( m_0(w - 1) = 1 \). Suppose for a contradiction that \( m_0(w - s) > 0 \) for some \( s \geq 2 \). Let

\[
m_{s+3} = m_0 + 2[w + 1] - |w - 1| - |w - s|.
\]

Then

\[
n! \geq A(n, s + 3) \geq \frac{N(m_{s+3})}{N(m_0)} \geq \frac{m_H(w + 1)(m_H(w + 1) - 1)}{2},
\]

so

\[
m_H(w + 1) \leq \sqrt{2n!} + 1.
\]

We will use this stronger bound to redo the estimates in Lemma 15. We have

\[
\frac{N(m_2^a)}{N(m_0)} = m_H(w + 1) \leq \sqrt{2n!} + 1
\]

and

\[
\frac{N(m_2^c)}{N(m_0)} = \frac{(m_H(w + 1))(m_H(w + 1) - 1)(m_H(w))}{2(m_H(w) - m_0(w) + 1)(m_H(w) - m_0(w) + 2)}
\]

\[
\leq \frac{1}{2} m_H(w + 1)^2 \left( \frac{N(m_1^a)}{N(m_0)} \right)^{-2}
\]

\[
\leq \frac{1}{2} \left( \sqrt{2n!} + 1 \right)^2 \left( \frac{A(n,1)}{2} \right)^{-2}.
\]

We use the same bounds on

\[
\frac{N(m_2^a)}{N(m_0)} \cdot \frac{N(m_2^b)}{N(m_0)} \cdot \frac{N(m_2^c)}{N(m_0)}
\]

as in Lemma 15, we conclude (see bound-A3e in the Python code) that

\[
\frac{N(m_2^a)}{N(m_0)} + \frac{N(m_2^b)}{N(m_0)} + \frac{N(m_2^c)}{N(m_0)} + \frac{N(m_2^d)}{N(m_0)} + \frac{N(m_2^e)}{N(m_0)} < A(n, 2)
\]

for \( 6 \leq n \leq 19 \).

Finally, suppose \( n = 5 \). Then since \( s + 3 \geq 4 \), we have the much stronger bound \( A(n, s + 3) \leq 1 \), which implies \( m_H(w + 1) \leq 2 \). We deduce as above

\[
\frac{N(m_2^a)}{N(m_0)} = m_H(w + 1) \leq 2
\]

\[
\frac{N(m_2^a)}{N(m_0)} \leq \left( \frac{N(m_1^a)}{N(m_0)} \right)^{-2} < 1
\]

and arrive at a contradiction as before. (See bound-A3f in the Python code.) \( \square \)

**Lemma A.17.** If \( n \geq 5 \) and

\[
\frac{N(m_1^a)}{N(m_0)} \geq \frac{A(n,1)}{2}
\]

then \( m_0(w) = 1 \).
\textbf{Proof.} Suppose $m_0(w) \geq 2$. Again, we’ll redo the estimates in Lemma \ref{A.15}. Since $n \geq 5$ and (by Lemma \ref{A.15}) $n \leq 10$, Lemma \ref{A.16} implies that

$$\frac{N(m_2^2)}{N(m_0)} = \frac{N(m_0^b)}{N(m_0)} = 0.$$ 

Lemmas \ref{A.13} and \ref{A.14} give stronger bounds in case $m_0(w) \geq 2$. As in Lemma \ref{A.15}, we deduce in this case that:

$$\frac{N(m_2^c)}{N(m_0)} \leq \sqrt{\frac{72}{11} A(n, 5) A(n, 1) + 1}.$$

Yet again, we conclude for $n$ in the given range that

$$\frac{N(m_2^a)}{N(m_0)} + \frac{N(m_2^b)}{N(m_0)} + \frac{N(m_2^c)}{N(m_0)} + \frac{N(m_2^d)}{N(m_0)} + \frac{N(m_2^e)}{N(m_0)} < A(n, 2),$$

a contradiction. (See Appendix \texttt{B} and bound-A3g in the Python code.) \hfill \Box

\textbf{Lemma A.18.} We cannot have $n \geq 5$ and

$$\frac{N(m_2^1)}{N(m_0)} \geq \frac{A(n, 1)}{2}.$$ 

\textbf{Proof.} Again, we’ll redo the estimates in Lemma \ref{A.15} in light of everything we now know. By Lemma \ref{A.15}, we may assume that $n \leq 11$.

Now Lemma \ref{A.16} implies that

$$\frac{N(m_2^a)}{N(m_0)} = \frac{N(m_0^b)}{N(m_0)} = 0,$$

while Lemma \ref{A.17} gives us

$$\frac{N(m_2^c)}{N(m_0)} = 0.$$

Yet again (see proof of Lemma \ref{A.15}), we have the bounds

$$\frac{N(m_2^e)}{N(m_0)} \leq \sqrt{\frac{48}{11} A(n, 3) A(n, 1) + 1}$$

and

$$\frac{N(m_2^e)}{N(m_0)} \leq \sqrt{\frac{96}{11} A(n, 5) A(n, 1) + 1}.$$

Yet again, we conclude that

$$\frac{N(m_2^a)}{N(m_0)} + \frac{N(m_2^b)}{N(m_0)} + \frac{N(m_2^c)}{N(m_0)} + \frac{N(m_2^d)}{N(m_0)} + \frac{N(m_2^e)}{N(m_0)} < A(n, 2),$$

a contradiction. (See Appendix \texttt{B} and bound-A3h in the Python code.) \hfill \Box
A.4. The case $n \geq 5$, with $k \geq n$. Recall notation from the beginning of Section A and the beginning of Section A.2.

We’ll treat the case where $k \geq n$ next. By Lemma A.3, if $k \geq n$, then $w = w'$.

**Lemma A.19.** If $k \geq n$ and $n \geq 5$ then one of the two ratios
\[
\frac{N(m^k_1)N(m^k_2)}{N(m_0)} \cdot \frac{N(m^k_1)}{N(m_0)}
\]
is less than 1, and the other is greater than $A(n,1) - 1$.

**Proof.** From Lemma A.3, we have
\[
m_H(w - 1) + m_H(w + 1) \leq n - 1
\]
so
\[
m_H(w - 1)m_H(w + 1) \leq \frac{(n - 1)^2}{2}.
\]
Then apply Lemma A.10 and the inequality
\[
\frac{(n - 1)^2}{2} < A(n,1) - 1
\]
(see Appendix B). □

**Lemma A.20.** If $n \geq 5$ and
\[
\frac{N(m^k_1)}{N(m_0)} > A(n,1) - 1
\]
then $m_H(w - 1) \leq 1$.

If $n \geq 5$ and
\[
\frac{N(m^k_1)}{N(m_0)} > A(n,1) - 1
\]
then $m_H(w + 1) \leq 1$.

**Proof.** The first case follows from Lemma A.11, we’ll prove the second. (As an alternative to Lemma A.11, the first case could be proven by an argument analogous to the argument below.)

So suppose $\frac{N(m^k_1)}{N(m_0)} > A(n,1) - 1$ and $m_H(w - 1) \geq 2$, and consider
\[
\frac{N(m^k_2)}{N(m^k_1)} = \frac{(m_H(w + 1) - 1)(m_0(w) - 1)}{2(m_H(w) - m_0(w) + 2)}.
\]
\[
\frac{N(m^k_2)N(m_0)}{N(m^k_1)^2} = \frac{1}{2} \cdot \frac{m_H(w + 1) - 1}{m_H(w + 1)} \cdot \frac{m_H(w) - m_0(w) + 1}{m_H(w) - m_0(w) + 2} \cdot \frac{m_0(w) - 1}{m_0(w)}.
\]
From
\[
\frac{m_0(w)}{(m_H(w) - m_0(w) + 1)} = \frac{1}{m_H(w + 1)} \cdot \frac{N(m^k_1)}{N(m_0)} \geq \frac{A(n,1) - 1}{n - 1} > 6
\]
we deduce that $m_0(w) > 6$, so by integrality $m_0(w) \geq 7$, so
\[
m_0(w) - 1 > \frac{6}{7} m_0(w).
\]
We also know that 
\[(m_H(w) - m_0(w) + 2) \leq 2(m_H(w) - m_0(w) + 1) \text{ and } (m_H(w + 1) - 1) \geq \frac{1}{2} m_H(w + 1),\]
so
\[\frac{A(n, 2)}{(A(n, 1) - 1)^2} \geq \frac{N(m_0^2)N(m_0)}{N(m_0^2)} \geq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{28}.\]
For \(n \geq 5\), this contradicts
\[A(n, 2) < \frac{3}{28}(A(n, 1) - 1)^2.\]
(See Appendix B and bound-A4a in the Python code.)

\[\square\]

**Lemma A.21.** For \(n \geq 5\) we cannot have \(k \geq n\).

**Proof.** Assume \(k \geq n\).

We'll consider the two cases given in Lemma A.19. The first case is ruled out by Lemma A.18; we'll prove the second. (Alternatively, the first case could be proven by an argument analogous to the argument below.)

So, suppose
\[N(m_0^2) > A(n, 1) - 1.\]

From
\[m_0(w) \geq \frac{m_0(w)}{m_H(w) - m_0(w) + 1} > A(n, 1) - 1\]
and the bounds
\[m_H(w - 2), m_H(w - 1) \leq n - 1 < A(n, 1) - 1\]
we deduce that
\[N(m_0^k) < m_H(w - 2) \left( \frac{m_0(w)}{m_H(w) - m_0(w) + 1} \right)^{-1} < 1\]
and
\[N(m_0^k) < \frac{1}{2} m_H(w - 1)^2 \left( \frac{m_0(w)}{m_H(w) - m_0(w) + 1} \right)^{-2} < 1.\]
By Lemma A.20 we have \(m_H(w + 1) \leq 1\), so
\[N(m_0^k) = m_H(w - 1)m_H(w + 1) \leq n - 1.\]
Also, \(m_H(w + 1) \leq 1\) implies that
\[N(m_0^k) = 0.\]
Finally, Lemma A.3 gives \(m_H(w + 2) \leq \frac{n - 1}{2}\), so
\[N(m_0^k) \leq \frac{n - 1}{2} A(n, 1).\]
Adding these bounds, we see that
\[A(n, 2) = \sum_{s=a,b,c,d,e} N(m_0^k) \leq n + 1 + \frac{(n - 1)}{2} A(n, 1).\]
But this contradicts the inequality
\[n + 1 + \frac{(n - 1)}{2} A(n, 1) < A(n, 2),\]
which holds for \( n \geq 5 \) (see Appendix [I] and bound-A4b in the Python code).

A.5. **The case** \( n \geq 5 \), **with** \( N(m_b^k) \) **big and** \( k < n \). Recall notation from the beginning of Section [A] and the beginning of Section [A.2]

**Lemma A.22.** Suppose \( k \leq n - 1 \) and

\[
\frac{N(m_b^k)}{N(m_0)} \geq A(n, 1)/2.
\]

Then

\[
m_H(w + 1) \geq \frac{A(n, 1)}{2(n - 1)}.
\]

In particular, if \( n \geq 5 \) then

\[
m_H(w + 1) \geq 4.
\]

**Proof.** We have

\[
m_H(w + 1) \geq \frac{N(m_b^k)}{N(m_0)} \cdot \frac{1}{m_0(w)} \geq \frac{A(n, 1)}{2m_0(w)} \geq \frac{A(n, 1)}{2(n - 1)}.
\]

The “in particular” follows from the fact that \( m_H(w + 1) \) is an integer.

**Lemma A.23.** Suppose \( n \geq 5 \), \( k \leq n - 1 \), and

\[
\frac{N(m_b^k)}{N(m_0)} \geq A(n, 1)/2.
\]

Then we cannot have simultaneously \( \frac{N(m_a^k)}{N(m_0)} > 0 \) and \( m_0(w) \geq 2 \).

**Proof.** Suppose \( \frac{N(m_a^k)}{N(m_0)} > 0 \) and \( m_0(w) \geq 2 \); the first condition implies \( m_H(w - 1) \geq 1 \) and \( m_H(w) - m_0(w) \geq 1 \). Consider

\[
m_4^b = m_0 - [w - 1] - 2[w] + 3[w + 1].
\]

We have

\[
\frac{N(m_a^k)}{N(m_0)} = \frac{N(m_a^k)/N(m_0)}{N(m_0)/N(m_0)} = \frac{N(m_b^k)}{N(m_0)} \cdot \frac{N(m_0)}{N(m_0)} = m_H(w - 1)m_H(w + 1).
\]

Thus we obtain

\[
\frac{N(m_b^k)}{N(m_0)} > \frac{1}{6} \left( \frac{N(m_a^k)}{N(m_0)} \right)^2 \frac{N(m_b^k)}{N(m_0)} \frac{N(m_0)}{N(m_0)} = \frac{N(m_b^k)}{N(m_0)} \frac{N(m_0)}{N(m_0)} = \frac{m_H(w - 1)(m_H(w) - m_0(w))}{m_0(w) + 1} \geq \frac{1}{n}.
\]

We conclude that

\[
\frac{N(m_b^k)}{N(m_1^k)} > \frac{1}{6} \left( \frac{N(m_a^k)}{N(m_0)} \right)^2 \frac{N(m_b^k)}{N(m_0)} \frac{N(m_0)}{N(m_0)} = \frac{1}{8} \left( \frac{N(m_a^k)}{N(m_0)} \right)^2 (A(n, 1) - \frac{N(m_a^k)}{N(m_0)})^2.
\]
so
\[
\frac{N(m_0^4)}{N(m_0)} > \frac{1}{48} \left( \frac{N(m_0^2)}{N(m_0)} \right) \left( A(n, 1) - \frac{N(m_0^4)}{N(m_0)} \right)^3
\]

As a function of \( \frac{N(m_0^4)}{N(m_0)} \), this right-hand side is minimized when \( \frac{N(m_0^4)}{N(m_0)} = \frac{1}{n} \), so we have
\[
A(n, 4) \geq \frac{N(m_0^4)}{N(m_0)} > \frac{1}{48n} \left( A(n, 1) - \frac{1}{n} \right)^3.
\]

This contradicts the inequality
\[
A(n, 4) < \frac{1}{78n} (A(n, 1) - 1)^3,
\]
which is valid for all \( n \). (See Appendix \[B\] and bound-A5a in the Python code.) \( \square \)

**Lemma A.24.** If \( n \geq 5 \), \( \frac{N(m_0^4)}{N(m_0)} = 0 \), \( k \leq n - 1 \), and
\[
\frac{N(m_0^4)}{N(m_0)} \geq A(n, 1)/2,
\]
then \( m_0(w) = 1 \) and \( N(m_0^4) = 0 \).

**Proof.** Since \( N(m_0^4) = 0 \), we have
\[
\frac{N(m_0^4)}{N(m_0)} = A(n, 1).
\]

Assuming \( m_0(w) \geq 2 \), let’s look at \( m_0^4 \). We have
\[
\frac{N(m_0^4)}{N(m_0)} = \frac{(m_H(w + 1))(m_H(w + 1) - 1)(m_0(w))(m_0(w) - 1)}{2(m_H(w) - m_0(w) + 1)(m_H(w) - m_0(w) + 2)}.
\]

We want to compare this with the inequality
\[
\frac{N(m_0^4)}{N(m_0)} \leq A(n, 2) < \frac{1}{10} A(n, 1)^2,
\]
which is valid for all \( n \). (See Appendix \[B\] and bound-A5b in the Python code.)

We have certainly
\[
\frac{N(m_0^4)}{N(m_0)} \geq \frac{1}{8} \frac{(m_H(w + 1) - 1)}{m_H(w + 1)} \left( \frac{N(m_0^4)}{N(m_0)} \right)^2 = \frac{1}{8} \frac{(m_H(w + 1) - 1)}{m_H(w + 1)} A(n, 1)^2.
\]

So we will be done if \( m_H(w + 1) \geq 5 \).

But as in the proof of Lemma A.22 we find (using now \( \frac{N(m_0^4)}{N(m_0)} = A(n, 1) \) in place of a weaker bound) that
\[
m_H(w + 1) \geq \frac{A(n, 1)}{(n - 1)},
\]
and so in particular \( m_H(w + 1) > 5 \) whenever \( n \geq 5 \). \( \square \)

**Lemma A.25.** If \( n \geq 5 \), \( k \leq n - 1 \), and
\[
\frac{N(m_0^4)}{N(m_0)} \geq A(n, 1)/2,
\]
then \( m_0(w) = 1 \).

In particular, there must be some \( r > 0 \) with \( m_0(w - r) > 0 \).
Proof. That \( m_0(w) = 1 \) follows from Lemmas A.23 and A.24, the second claim follows because \( k > 1 \). \( \square \)

Let \( w - r \) be the highest Hodge weight below \( w \). In other words, take \( r > 0 \) minimal such that \( m_H(w - r) > 0 \). (Such \( r \) exists by Lemma A.25.) For example, if \( N(m_1^a) \neq 0 \) then we must have \( m_H(w - 1) \neq 0 \), so \( r = 1 \).

A.5.1. Case: \( m_1^b \) big and \( r = 1 \). Recall the definitions of \( m_2^* \) from above (\( * = a, b, c, d, e \)).

Lemma A.26. If \( k \leq n - 1 \),
\[
\frac{N(m_1^b)}{N(m_0)} \geq A(n, 1)/2,
\]
and \( m_0(w) = 1 \), then \( m_0(w - 1) \leq 1 \).

Proof. Suppose for a contradiction \( m_0(w - 1) \geq 2 \), and consider
\[
m_5^b = m_0 + 3[w + 1] - [w] - 2[w - 1].
\]

We have
\[
\frac{N(m_5^b)}{N(m_1^b)} = \left( \frac{m_H(w - 1)(m_H(w - 1) - 1)}{2} \right) \left( \frac{(m_H(w + 1) - 1)(m_H(w + 1) - 2)}{6} \right)
\]
\[
\geq \frac{(m_H(w + 1) - 1)(m_H(w + 1) - 2)}{6}.
\]

From
\[
\frac{N(m_5^b)}{N(m_0)} = \frac{m_H(w + 1)m_0(w)}{m_H(w) - m_0(w) + 1}
\]
and \( m_0(w) = 1 \), we find that
\[
m_H(w + 1) \geq A(n, 1)/2.
\]

In particular, since \( n \geq 5 \), we have \( m_H(w + 1) \geq 13 \), so certainly
\[
(m_H(w + 1) - 1)(m_H(w + 1) - 2) \geq \frac{1}{2} m_H(w + 1)^2.
\]

We conclude that
\[
\frac{N(m_5^b)}{N(m_0)} = \left( \frac{N(m_5^b)}{N(m_1^b)} \right) \left( \frac{N(m_1^b)}{N(m_0)} \right) \geq \frac{1}{12} m_H(w + 1)^2 \frac{A(n, 1)}{2} \geq \frac{A(n, 1)^3}{96}.
\]

This contradicts the inequality
\[
A(n, 1)^3 > 341A(n, 5),
\]
which is valid for all \( n \). (See Appendix B and bound-A5c in the Python code.) \( \square \)

Lemma A.27. We cannot have \( n \geq 5 \), \( k \leq n - 1 \),
\[
\frac{N(m_1^b)}{N(m_0)} \geq A(n, 1)/2,
\]
and \( r = 1 \).
Proof. Assume the stated conditions hold. We will bound the five ratios \( \frac{N(m^2_n)}{N(m_0)} \), and show that their sum is too small. Lemmas A.25 and A.26 tell us that \( m_0(w - 1) = m_0(w) = 1 \). Hence, \( N(m^2_2) = N(m^4_2) = 0 \).

We have

\[
\frac{N(m^2_2)}{N(m_0)} = \frac{m_0(w - 2)(m_H(w) - 1)}{2}
\]

\[
\frac{N(m^2_n)}{N(m_0)} = \frac{m_H(w + 1)}{2}
\]

\[
\frac{N(m^2_n)}{N(m_0)} = \frac{m_H(w + 2)}{m_H(w)}.
\]

Since

\[
A(n, 1) = \frac{N(m^2_n)}{N(m_0)} = m_H(w) - \frac{1}{2}
\]

and \( m_0(w - 2) \leq n - 1 \), we have

\[
\frac{N(m^2_n)}{N(m_0)} \leq \frac{(n - 1)A(n, 1)}{2}.
\]

Let

\[
m^5_n = m_0 + 2[w + 1] - [w] - [w - 1].
\]

Dividing the bound

\[
A(n, 3) \geq \frac{N(m^5_n)}{N(m_0)} = \frac{(m_H(w + 1))(m_H(w + 1) - 1)}{2(m_H(w) - m_0(w) + 1)}
\]

by

\[
\frac{N(m^5_n)}{N(m_0)} = \frac{m_H(w + 1)m_0(w)}{(m_H(w) - m_0(w) + 1)} \geq \frac{A(n, 1)}{2},
\]

we obtain

\[
(m_H(w + 1) - 1) \leq \frac{4A(n, 3)}{A(n, 1)}\]

so

\[
\frac{N(m^5_n)}{N(m_0)} \leq \frac{4A(n, 3)}{A(n, 1)} + 1.
\]

Finally, let

\[
m^5_5 = m_0 + 2[w + 2] - [w] - [w - 1].
\]

We have

\[
A(n, 5) \geq \frac{N(m^5_5)}{N(m_0)} = \frac{m_H(w + 2)(m_H(w + 2) - 1)}{2m_H(w)} \geq \frac{1}{2} \left( \frac{N(m^5_5)}{N(m_0)} - 1 \right)^2
\]

so

\[
\frac{N(m^5_5)}{N(m_0)} \leq \sqrt{2A(n, 5)} + 1.
\]

We conclude that

\[
A(n, 2) = \sum_{*_{a,b,c}} \frac{N(m^5_n)}{N(m_0)} \leq \left( \frac{nA(n, 1)}{2} \right) + \left( \frac{4A(n, 3)}{A(n, 1)} + 1 \right) + \left( \sqrt{2A(n, 5)} + 1 \right),
\]

which contradicts the inequality

\[
A(n, 2) > \left( \frac{(n - 1)A(n, 1)}{2} \right) + \left( \frac{4A(n, 3)}{A(n, 1)} + 1 \right) + \left( \sqrt{2A(n, 5)} + 1 \right),
\]

valid for all \( n \geq 5 \). (See Appendix B and bound-A5d in the Python code.) □
A.5.2. Case: $N(m_b^k)$ big and $r \geq 2$. Suppose $n \geq 5$ and

$$\frac{N(m_b^k)}{N(m_0)} \geq A(n,1)/2.$$ 

From Lemma A.25 we have $m_0(w) = 1$, and because $r \geq 2$, we have $m_0(w - 1) = 0$.

**Lemma A.28.** Suppose $r \geq 2$, $m_0(w) = 1$, and

$$\frac{N(m_b^k)}{N(m_0)} \geq A(n,1)/2.$$ 

We have

$$m_0(w) = 1,$$

$$m_H(w + i) = dA(n,i)$$

for $0 \leq i \leq r - 1$, and

$$m_H(w + r) = dA(n,r) - \frac{d(d - 1)}{2} m_H(w - r).$$

**Proof.** We have already seen in Lemma A.25 that $m_0(w) = 1$.

For $w \leq q \leq w + r - 1$, there is only one nonzero term in Equation (3); for $q = w + r$, there are only two. \hfill \square

**Lemma A.29.** Suppose $n \geq 5$, $r \geq 2$, $m_0(w) = 1$, and

$$\frac{N(m_b^k)}{N(m_0)} \geq A(n,1)/2.$$ 

Then

$$A(n,3r - 2) \geq \frac{A(n,r - 1)(dA(n,r - 1) - 1)}{2}.$$ 

**Proof.** Let

$$m_{3r - 2} = m_0 + 2[w + r - 1] - [w] - [w - r].$$

We have

$$A(n,3r - 2) \geq \frac{N(m_{3r - 2})}{d} = \frac{(dA(n,r - 1))(dA(n,r - 1) - 1)m_H(w - r)}{2d(m_H(w) - m_0(w) + 1)} \geq \frac{A(n,r - 1)(dA(n,r - 1) - 1)}{2}.$$ 

\hfill \square

**Lemma A.30.** Suppose $n \geq 5$, $r \geq 2$, $m_0(w) = 1$, and

$$\frac{N(m_b^k)}{N(m_0)} \geq A(n,1)/2.$$ 

Then $r = 2$ and

$$d < \frac{3A(n,4)}{A(n,1)^2}.$$ 

**Proof.** By Lemma A.29, we have

$$A(n,3r - 2) \geq \frac{A(n,r - 1)(dA(n,r - 1) - 1)}{2} \geq \frac{A(n,r - 1)(A(n,r - 1) - 1)}{2}.$$ 

If $r \geq 3$, this is not possible by Lemma B.4.

If $r = 2$ the result follows from Lemma A.29 and

$$dA(n,r - 1) - 1 > \frac{2}{3}dA(n,r - 1).$$ 

\hfill \square
Lemma A.31. We cannot have \( n \geq 5, k \leq n - 1, r \geq 2, \) and \( m_0(w) = 1. \)

Proof. Assume we had a solution satisfying the given conditions. By Lemma A.30 we know that \( r = 2 \) and

\[
d < \frac{3A(n, 4)}{A(n, 1)^2}.
\]

This implies

\[
d < \frac{A(n, 2)}{2n},
\]

by the inequality

\[
6nA(n, 4) < A(n, 1)^2A(n, 2),
\]

which holds for all \( n \geq 3. \) (See Appendix [3] and bound-A5e in the Python code.)

Thus, since \( m_H(w - 2) < n, \) we have

\[
m_H(w + 2) = dA(n, 2) - \frac{d(d - 1)}{2}m_H(w - 2) \geq \frac{3}{4}dA(n, 2).
\]

Finally, taking

\[
m_a^6 = m_0 + 2[w + 2] - [w] - [w - 2],
\]

we find that

\[
A(n, 6) \geq \frac{N(m_a^6)}{N(m_0)} \geq \frac{m_H(w + 2)(m_H(w + 2) - 1)m_0(w)m_H(w - 2)}{2(m_H(w) - m_0(w) + 1)} = \frac{m_H(w + 2)(m_H(w + 2) - 1)m_H(w - 2)}{2d} \geq \frac{1}{2} \cdot \frac{3}{4}A(n, 2)(\frac{3}{4}dA(n, 2) - 1) \geq \frac{9}{32}(A(n, 2) - 1)^2.
\]

This contradicts the inequality

\[
(A(n, 2) - 1)^2 > 100A(n, 6),
\]

which holds for all \( n. \) (See Appendix [3] and bound-A5f in the Python code.) \( \square \)
APPENDIX B. COLLECTED INEQUALITIES INVOLVING EULERIAN NUMBERS

The argument above used several dozen inequalities involving Eulerian numbers. We will not give detailed proofs of them; aside from the inequality in Lemma B.4, each of the inequalities used can be proven using Lemma B.1 or Lemma B.3 for large \( n \), and then verifying by hand the finite number of remaining cases. One such proof is presented as Lemma B.5 to illustrate the method. The Python code used to verify the finite number of cases will be posted as an ancillary file alongside the arXiv submission.

The reader is encouraged to verify the plausibility of such inequalities for large \( n \), using the asymptotic approximation

\[
A(n,q) \sim (q+1)^n,
\]

which is valid for fixed \( q \) and large \( n \).

Recall our convention that \( A(n,q) = 0 \) if \( q \geq n \).

**Lemma B.1.** For all \( n \geq 1 \) and \( q \geq 0 \), we have

\[
(q+1)^n - (n+1)q^n \leq A(n,q) \leq (q+1)^n.
\]

**Proof.** Recall (§1.3) that the Eulerian number \( A(n,q) \) counts the number of permutations of \( \{1, 2, \ldots, n\} \) with exactly \( q \) ascents.

If we label the integers 1 to \( n \) with labels 1 through \( q+1 \), then we get a permutation with at most \( q \) ascents by giving all the numbers of label 1 in decreasing order, then all the numbers of label 2 in decreasing order, and so on. Every permutation with at most \( q \) ascents arises in this way; this proves the right-hand inequality.

If a permutation constructed this way has fewer than \( q \) ascents, then there must exist adjacent labels \( i \) and \( i+1 \) where all the numbers with label \( i \) are less than all the numbers with label \( i+1 \). If that happens, we can record a number \( j \) from 0 to \( n \) which is the number of elements in the sequence with label at most \( i \), then subtract one from the labeling of everything with label greater than \( i \). There are \( (n+1)q^n \) possibilities of this new data, and we can recover the original labeling by adding 1 to the label of everything that comes after the first \( j \) elements in the sequence. So the number of labelings giving permutations with fewer than \( q \) ascents is at most \( (n+1)q^n \). This proves the left-hand inequality. \( \square \)

**Lemma B.2.** For all \( n \geq 1 \) and \( q \geq 0 \), we have

\[
A(n,q) \leq n!.
\]

**Proof.** Follows from

\[
\sum_q A(n,q) = n!.
\]

The two bounds given patch well enough for our modest needs: if \( q \sim n/\log n \), then the bound in Lemma B.1 is close to \( n! \), at least in a power sense. Surely more precise asymptotics are known, but these weak bounds are enough for the proof of Lemma B.4.

**Lemma B.3.** If \( n \geq 2 \) and

\[
qu < \frac{n}{\log(n+1) + 1} - 1
\]

then

\[
(1 - 1/e)(q+1)^n \leq A(n,q) \leq (q+1)^n.
\]
Proof. We have
\[
1/(q + 1) < \log \left(\frac{q + 1}{q}\right) < 1/q
\]
\[
e^{n/(q+1)} < \left(\frac{q + 1}{q}\right)^n < e^{n/q}.
\]
If \(n/(q + 1) > \log(n + 1) + 1\) then
\[
(n + 1)^q < (q + 1)/e.
\]
Now use Lemma B.1. □

Lemma B.4. For \(n\) arbitrary and \(3 \leq r \leq n - 2\) we have
\[
A(n, r - 1)(A(n, r - 1) - 1) > 2A(n, 3r - 2).
\]
Proof. We’ll assume \(n \geq 21\); for smaller \(n\) there are only finitely many cases, which can be checked by hand. (See bound-B in the Python code.)

We can also assume \(r \leq n/2\); otherwise, the right-hand side is zero. We’ll split into two cases: either \(r < n \log(n + 1) + 1\) or \(r > \sqrt n\). Our hypothesis on \(n\) guarantees that there is some \(r_0\) with \(\sqrt n \leq r_0 < \frac{n}{\log(n + 1) + 1}\); a fortiori, one of the two cases always holds.

If \(r < \frac{n}{\log(n + 1) + 1}\), then Lemma B.3 gives
\[
A(n, r - 1) \geq (1 - 1/e)r^n,
\]
so
\[
A(n, r - 1)(A(n, r - 1) - 1) \geq (1 - 1/e)^2r^{2n} - (1 - 1/e)r^n.
\]
On the other hand, by Lemma B.1, we can bound the right-hand side:
\[
2A(n, 3r - 2) \leq 2(3r - 1)^n.
\]
The reader may verify that
\[
(1 - 1/e)^2r^{2n} > 2(3r - 1)^n + (1 - 1/e)r^n
\]
whenever \(r \geq 3\) and \(n \geq 14\); this proves the inequality we want.

Recall that we chose \(r_0\) such that \(\sqrt n \leq r_0 < \frac{n}{\log(n + 1) + 1}\). Lemma B.3 gives
\[
A(n, r_0 - 1)(A(n, r_0 - 1) - 1) > \frac{3}{10} r_0^{2n} \geq \frac{3}{10} n^n > 2 \ast n!.
\]
Now for any \(r\) with \(r_0 \leq r \leq n/2\), we have
\[
A(n, r) \geq A(n, r_0),
\]
so
\[
A(n, r - 1)(A(n, r - 1) - 1) > 2 \ast n! \geq 2A(n, 3r - 2)
\]
by Lemma B.2. □

We’ll conclude with Lemma B.5, whose proof is given merely to illustrate a routine technique. A number of inequalities were used without proof in Appendix A. They can all be proven by asymptotic estimates using Lemma B.3 for large \(n\), followed case-by-case verification for small \(n\). The following bound was used in the proof of Lemma A.11; we give a full proof here to illustrate the general method.

Lemma B.5. For \(n \geq 11\), we have
\[
\frac{A(n, 2)}{A(n, 1)^2} < 1/27.
\]
Proof. Lemma B.3 gives us

\[ A(n, 2) \leq 3^n \]

and

\[ A(n, 1) \geq (1 - 1/e)^2 n, \]

provided \( n \geq 6 \). Hence we will be done as soon as we can show that

\[ \frac{1}{(1 - 1/e)^2} \cdot \left(\frac{3}{4}\right)^n < 1/27, \]

which happens for \( n \geq 15 \). The cases \( 11 \leq n \leq 14 \) must be checked separately. \( \square \)
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