A YAU PROBLEM FOR VARIATIONAL CAPACITY

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Abstract. Through using the semidiameter (in connection to: the mean radius; the $p - 1$ integral mean curvature radius; the graphic ADM mass radius) of a closed convex hypersurface in $\mathbb{R}^n$ with $n \geq 2$ as an sharp upper bound of the variational $p$ capacity radius, this paper settles an extension of S.-T. Yau’s [71 Problem 59] from the surface area to the variational $(1, n) \ni p$ capacity whose limiting as $p \to 1$ actually induces the surface area.

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1. Theorem and Its Corollary

In his problem section of Seminar on Differential Geometry published by Princeton University Press 1982, S.-T. Yau raised the following problem (cf. [71 Page 683, Problem 59]):

Let $h$ be a real-valued function on $\mathbb{R}^3$. Find (reasonable) conditions on $h$ to insure that one can find a closed surface with prescribed genus in $\mathbb{R}^3$ whose mean curvature (or curvature) is given by $h$.

Since posed, this problem has received a lot of attention – see also: [63] [172,32,19] for the aspect of mean curvature; [52,53,6,16,65,66,64,9] 69 for the aspect of Gauss curvature; [30] [29] and their references for the aspect.
of curvature measure. In this paper, we study the above problem with genus zero from the perspective of the so-called variational \( p \) capacity. To be more precise, it is perhaps appropriate to review F. Almgren’s comments on the Yau’s problem (see the mid part of [71, Page 683, Problem 59]):

For “suitable” \( h \) one can obtain a compact smooth submanifold \( \partial A \) in \( \mathbb{R}^3 \) having mean curvature \( h \) by maximizing over bounded open sets \( A \subset \mathbb{R}^3 \) the quantity

\[
F(A) = \int_A h \, d\mathcal{L}^3 - \text{Area}(\partial A).
\]

A function \( h \) would be suitable, for example, in case it were continuous, bounded, and \( \mathcal{L}^3 \) summable, and \( \sup F > 0 \). However, the relation between \( h \) and the genus of the resulting extreme \( \partial A \) is not clear.

Note that \( \text{Area}(\partial A) \) is just the variational 1-capacity of \( \partial A \) whenever \( A \) is convex body, i.e., \( A \in \mathbb{K}^3 \), where \( \mathbb{K}^n \) comprises all compact and convex subsets of \( \mathbb{R}^n \) with nonempty interior (cf. [44], [23] and [47, Page 149]). So, as a variant of the Yau problem, it seems interesting to consider the maximizing problem below:

\[
\sup \left\{ F_{\text{pcap}}(A) = \int_A h \, d\mathcal{L}^n - \text{pcap}(A) : A \in C^n \right\}.
\]

In the above and below, \( C^n \) stands for the class of all compact and convex subsets of \( \mathbb{R}^n \) (the \( 2 \leq n \)-dimensional Euclidean space) and \( \text{pcap}(E) \) is the variational \( 1 \leq p < n \) capacity of an arbitrary set \( E \subset \mathbb{R}^n \):

\[
\text{pcap}(E) = \inf_{\text{open} U \supseteq E} \text{pcap}(U) = \inf_{\text{open} U \supseteq E} \left\{ \sup_{\text{compact } K \subseteq U} \text{pcap}(K) \right\},
\]

where for a compact subset \( K \subset \mathbb{R}^n \) one uses

\[
\text{pcap}(K) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla f|^p \, d\mathcal{L}^n : f \in C_0^\infty(\mathbb{R}^n) \text{ & } f \geq 1_K \right\},
\]

with \( d\mathcal{L}^n \) denoting the usual \( n \)-dimensional Lebesgue measure and \( 1_K \) being the characteristic function of \( K \).

According to [31] Page 32, we have \( \text{pcap}(A) = \text{pcap}(\partial A) \) provided that \( A \subset \mathbb{R}^n \) is compact. This yields

\[
\text{1cap}(A) = \text{Area}(\partial A) \quad \forall \quad A \in \mathbb{K}^n.
\]

Physically speaking, \( 2\text{cap}(A) \) of a compact set \( A \subset \mathbb{R}^3 \) expresses the total electric charge flowing into \( \mathbb{R}^3 \setminus A \) across the boundary \( \partial A \) of \( A \). Moreover, in accordance with Colesanti-Salani’s calculation in [14] we see that for \( p \in (1, n) \) the capacity \( \text{pcap}(A) \) of \( A \in \mathbb{K}^n \) can be determined via

\[
(1.1) \quad \text{pcap}(A) = \int_{\mathbb{R}^n \setminus A} |\nabla u_A|^p \, d\mathcal{L}^n = \int_{\partial A} |\nabla u_A|^{p-1} \, d\mathcal{H}^{n-1},
\]
where \(d'\mathcal{H}^{n-1}\) represents the \((n-1)\)-dimensional Hausdorff measure on \(\partial A\), \(u_A\) is the so-called \(p\)-equilibrium potential, i.e., the unique weak solution to the following boundary value problem:

\[
\begin{cases}
\text{div}(\vert \nabla u \vert^{p-2} \nabla u) = 0 & \text{in } \mathbb{R}^n \setminus A; \\
u = 1 & \text{on } \partial A \quad & \& u(x) \to 0 \text{ as } |x| \to \infty,
\end{cases}
\]

and the vector \(\nabla u_A\) exists almost everywhere as the non-tangential limit on \(\partial A\) with respect to \(d'\mathcal{H}^{n-1}\); see also Lewis-Nyström’s [42 Theorems 3-4].

Below is the main result of this paper.

**Theorem 1.1.** Given \(p \in (1, n), \alpha \in (0, 1)\) and a nonnegative integer \(k\), let 

\(h\) be a positive, continuous and \(L^1\)-integrable function on \(\mathbb{R}^n\).

(i) \(F_{pcap}(\cdot)\) attains its supremum over \(\mathcal{C}^n\) if and only if there exists \(A \in \mathcal{C}^n\) such that \(F_{pcap}(A) \geq 0\).

(ii) Suppose \(A \in \mathcal{K}^n\) is a maximizer of \(F_{pcap}(\cdot)\). Then such an \(A\) satisfies the variational Eikonal \(p\)-equation \((p - 1)\vert \nabla u_A \vert^p = h\) in the sense of

\[
\int_{S^{n-1}} \phi_s((p - 1)\vert \nabla u_A \vert^p \ d'\mathcal{H}^{n-1}) = \int_{S^{n-1}} \phi_s(h \ d'\mathcal{H}^{n-1}) \ \forall \ \phi \in C(S^{n-1}),
\]

where \(g_s(X \ d'\mathcal{H}^{n-1})\) is the push-forward measure of a given nonnegative measure \(X \ d'\mathcal{H}^{n-1}\) via the Gauss map \(g\) from \(\partial A\) to the unit sphere \(S^{n-1}\) of \(\mathbb{R}^n\):



\[
g_s(X \ d'\mathcal{H}^{n-1})(E) = \int_{g^{-1}(E)} X \ d'\mathcal{H}^{n-1} \ \forall \ \text{Borel set } E \subset S^{n-1},
\]

with \(g^{-1}\) being the inverse of the Gauss map \(g\). In particular, if \(\partial A\) is \(C^2\) strictly convex, then \((p - 1)\vert \nabla u_A \vert^p = h\) holds pointwisely on \(\partial A\).

(iii) If \(h\) is of \(C^{k, \alpha}\) and \(A\), with \(\partial A\) being \(C^2\) strictly convex, is a maximizer of \(F_{pcap}(\cdot)\), then \(\partial A\) is of \(C^{k+1, \alpha}\).

**Theorem 1.1** can actually give much more information than just a generalized solution to the Yau problem for \(pcap(\cdot)\) with \(1 < p < n\). To see this, recall two related facts. The first is:

\[
\text{div}(\vert \nabla u \vert^{p-2} \nabla u) = \vert u_v \vert^{p-2}((n - 1)Hu_v + (p - 1)u_{vv}),
\]

where \(\nu, u_v, u_{vv},\) and \(H\) denote the outer unit normal vector, the first-order derivative along \(\nu\), the second-order derivative along \(\nu\), and the mean curvature of the level surface of \(u\) respectively, and so,

\[
\text{div}(\vert \nabla u \vert^{-1} \nabla u) = ((n - 1)H)(\frac{u_v}{\vert u_v \vert})
\]
holds at least weakly. The second is Maz’ya’s isocapacitary inequality for $p \in [1, n]$ (cf. [46]):

\[
(1.5) \quad \left( \frac{L^n(E)}{\omega_n} \right)^\frac{1}{n} \leq \left( \frac{p - 1}{n - p} \right)^{p-1} \left( \frac{pcap(E)}{\sigma_{n-1}} \right)^{\frac{1}{p-1}} \quad \forall \ E \subset \mathbb{R}^n
\]

and Federer’s isoperimetric inequality (cf. [21, §3.2.43]):

\[
(1.6) \quad \left( \frac{L^n(E)}{\omega_n} \right)^\frac{1}{n} \leq \left( \frac{H^{n-1}(\partial E)}{\sigma_{n-1}} \right)^{\frac{1}{n-1}} \quad \forall \ E \in C^n.
\]

Here and henceforth, $\omega_n$ and $\sigma_{n-1} = n \omega_n$ stand for the volume and the surface area of the unit ball of $\mathbb{R}^n$ respectively. Of course, the equality in (1.5)/(1.6) holds as $A$ is a ball. Moreover, the left hand side of (1.5)/(1.6) is called the volume radius of $E$, and the right hand sides of (1.5) and (1.6) are called the variational $p$ capacity radius and the surface radius respectively.

Now, our issue is as follows - the treatment of Theorem 1.1 brings not only Corollary 1.2 - a generalized solution to a special case (i.e., genus $= 0$) of the original Yau problem, but also a new analytic approach to some related geometric problems.

**Corollary 1.2.** Let $h \in L^1(\mathbb{R}^n)$ be positive and continuous, $k$ be a nonnegative integer, $\alpha \in (0, 1)$, and

\[
F_{\mathcal{H}^{n-1}}(A) = \int_A h \, d\mathcal{L}^n - \mathcal{H}^{n-1}(A) \quad \forall \ A \in C^n.
\]

(i) $F_{\mathcal{H}^{n-1}}(\cdot)$ attains its supremum over $C^n$ if and only if there exists $A \in C^n$ such that $F_{\mathcal{H}^{n-1}}(A) \geq 0$.

(ii) Suppose $A \in \mathbb{R}^n$ is a maximizer of $F_{\mathcal{H}^{n-1}}(\cdot)$. Then there is a Borel measure $\mu_{\mathcal{H}^{n-1}}$ on $\mathbb{S}^{n-1}$ such that $d\mu_{\mathcal{H}^{n-1}} = g_*(h \, d\mathcal{H}^{n-1})$, namely,

\[
(1.7) \quad \int_{\mathbb{S}^{n-1}} \phi \, d\mu_{\mathcal{H}^{n-1}} = \int_{\mathbb{S}^{n-1}} \phi g_*(h \, d\mathcal{H}^{n-1}) \quad \forall \ \phi \in C(\mathbb{S}^{n-1}).
\]

In particular, if $\partial A$ is $C^2$ strictly convex, then such a maximizer $A$ satisfies the equation $h(\cdot) = H(\partial A, \cdot)$ - the mean curvature of $\partial A$.

(iii) If $h$ is of $C^{k,\alpha}$ and $A$, with $\partial A$ being $C^2$ strictly convex, is a maximizer of $F_{\mathcal{H}^{n-1}}(\cdot)$, then $\partial A$ is of $C^{k+2,\alpha}$.

2. **Five Lemmas and Their Proofs**

In order to prove Theorem 1.1 and Corollary 1.2 we will not only keep in mind the iso-capacitary/isoperimetric inequality (1.5)/(1.6) which shows that the volume radius serves as a sharp lower bound of the variational $p$ capacity radius and the surface radius, but also explore the optimal upper bounds of these two geometric quantities in terms of the semidiameter, the
mean radius, the $p - 1$ integral mean curvature radius, and the graphic ADM mass radius; see the coming-up next five lemmas. In short, under certain conditions on $A$ (and its boundary $\partial A$ as well as its interior $A^\circ$), $1 < p < n$, $H(\partial A, \cdot)$ and $f$, we will build the following decisive radius tree

$$
\left( \frac{L^n(A)}{\omega_n} \right)^{\frac{1}{p}} \leq \left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right)^{\frac{1}{p-1}} \leq \left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right)^{\frac{1}{p-1}} \left( \frac{H^{n-1}(\partial A)}{\sigma_{n-1}} \right)^{\frac{1}{p-1}} \leq \left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right)^{\frac{1}{p-1}} \left( \frac{H^{n-1}(\partial A)}{\sigma_{n-1}} \right)^{\frac{1}{p-1}} \leq \frac{\text{diam}(A)}{2},
$$

and surprisingly find that if all principal curvatures of a given $C^2$ boundary $\partial A$ are in the interval $[\alpha, \beta] \subset (0, \infty)$ then

$$
\left( \frac{p - 1}{n - p} \right)^{\beta} \left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right) \leq \left( \frac{H^{n-1}(\partial A)}{\sigma_{n-1}} \right) \leq \left( \frac{p - 1}{n - p} \right)^{\alpha} \left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right).
$$

2.1. Semidiameter. The isodiameter or Bieberbach’s inequality (cf. [20, Page 69] and [57, Page 318]) says that the diameter $\text{diam}(A)$ of $A \subset \mathbb{R}^n$ dominates the double of the volume radius of $A$:

$$
\left( \frac{L^n(A)}{\omega_n} \right)^{\frac{1}{p}} \leq \frac{\text{diam}(A)}{2}
$$

with equality if $A$ is a ball. Interestingly, (2.1) has been improved through the foregoing (1.5)/(1.6) and the following (2.2)/(2.3).

Lemma 2.1.
(i) If $p \in (1, n)$ and, $A \subset \mathbb{R}^n$ is a connected compact set, then

$$
\left( \frac{p - 1}{n - p} \right)^{\beta} \left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right) \leq \frac{\text{diam}(A)}{2}
$$

holds, where equality is valid as $A$ is a ball.

(ii) If $A \in C^\infty$, then

$$
\left( \frac{H^{n-1}(\partial A)}{\sigma_{n-1}} \right) \leq \frac{\text{diam}(A)}{2}
$$

holds, where equality is valid as $A$ is a ball.

Proof. Obviously, equalities in (2.2) and (2.3) occur when $A$ is a ball. Note that (2.3) is the well-known Kubota inequality (cf. [39, 45]). So, it suffices
to prove the remaining part of (2.2). To do so, suppose

\[
\begin{align*}
\text{dist}(x, A) &= \inf_{y \in A} |x - y|; \\
rB^n &= \{x \in \mathbb{R}^n : |x| < r\} \quad \forall \quad r > 0; \\
\mathbb{R}^n &= \mathbb{R}^n \cup \{\infty\}; \\
S(A, t) &= \mathcal{H}^{n-1}(\{x \in rB^n \setminus A : \text{dist}(x, A) = t\}) \quad \forall \quad t > 0.
\end{align*}
\]

The flat case of Gehring’s Theorem 2 in [25] implies that if

\[
A \subset rB^n \quad \& \quad \tau = \liminf_{x \to \mathbb{R}^n \setminus rB^n} \text{dist}(x, A),
\]

then

\[
(2.4) \quad \text{pcap}(rB^n, A) \leq \left( \int_0^\tau (S(A, t))^{\frac{1}{n-1}} \, dt \right)^{1-p},
\]

where

\[
\text{pcap}(rB^n, A) = \inf_u \int_{rB^n \setminus A} |\nabla u|^p \, d\mathcal{L}^n
\]

for which the infimum ranges over all functions \(u\) that are continuous in \(\mathbb{R}^n\) and absolutely continuous in the sense of Tonelli in \(\mathbb{R}^n\) with \(u = 0\) in \(A\) and \(u = 1\) in \(\mathbb{R}^n \setminus rB^n\).

Noting such an essential fact that if \(\hat{A}\) is the convex hull of \(A\) then

\[
\text{pcap}(A) \leq \text{pcap}(\hat{A}) \quad \& \quad \text{diam}(A) = \text{diam}(\hat{A}),
\]

without loss of generality we may assume that \(A\) is convex, and then restate Kubato’s inequality (cf. [39, 27]) for such an \(A:\)

\[
\frac{\mathcal{H}^{n-1}(\partial A)}{\sigma_{n-1}} \leq \left( \frac{\text{diam}(A)}{2} \right)^{n-1}.
\]

This in turn implies

\[
\frac{S(A, t)}{\sigma_{n-1}} \leq \left( \frac{\text{diam}(A) + 2t}{2} \right)^{n-1}.
\]
So, the last inequality, along with (2.4), gives
\[ \frac{pcap(rB^n, A)}{\sigma_{n-1}} \leq \left( \int_0^\tau \left( \frac{\text{diam}(A) + 2t}{2} \right)^{\frac{n-1}{p}} dt \right)^{1-p} \]
\[ = \left( \left( \frac{1 - p}{n - p} \right) \left( \frac{\text{diam}(A)}{2} + \tau \right)^{\frac{n-p}{1-p}} - \left( \frac{\text{diam}(A)}{2} \right)^{\frac{n-p}{1-p}} \right)^{1-p} \]
\[ \rightarrow \left( \frac{p - 1}{n - p} \left( \frac{\text{diam}(A)}{2} \right)^{\frac{n-p}{1-p}} \right)^{1-p} \text{ as } \tau \rightarrow \infty. \]

As a result, we get
\[
\frac{pcap(A)}{\sigma_{n-1}} = \lim_{r \rightarrow \infty} \frac{pcap(rB^n, A)}{\sigma_{n-1}} \leq \left( \frac{p - 1}{n - p} \right)^{1-p} \left( \frac{\text{diam}(A)}{2} \right)^{n-p}
\]
whence reaching the inequality of (2.2).

\[\square\]

2.2. **Mean radius.** For \( A \in C^n \), denote by (cf. [57, 1.7])
\[ h_A(x) = \sup_{y \in A} x \cdot y \quad \& \quad b(A) = \frac{2}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} h_A \, d\theta \]
the support function and the mean width of \( A \) (with \( d\theta \) being the standard area measure on \( \mathbb{S}^{n-1} \)) respectively, and then write \( b(A)/2 \) for the mean radius of \( A \) according to [54]. Clearly,
\[ \frac{b(A)}{2} \leq \frac{\text{diam}(A)}{2} \]
with equality if \( A \) is a ball. Interestingly, the Uryasohn inequality (cf. [57, (6.25)])
\[ \left( \frac{L^n(A)}{\omega_n} \right)^{\frac{1}{n}} \leq \frac{b(A)}{2} \]
holds with equality if \( A \) is a ball. Even more interestingly, the forthcoming lemma reveals that (2.5) can be further improved.

**Lemma 2.2.**
(i) If \( A \in C^n \) and \( p = n - 1 \), then
\[ \left( \frac{p - 1}{n - p} \left( \frac{pcap(A)}{\sigma_{n-1}} \right) \right)^{\frac{1}{p-1}} \leq \frac{b(A)}{2} \]
with equality if \( A \) is a ball.
(ii) If \( A \in C^n \), then

\[
(2.7) \quad \left( \frac{\mathcal{H}^{n-1}(\partial A)}{\sigma_{n-1}} \right)^{\frac{1}{n-1}} \leq \frac{b(A)}{2}
\]

with equality if \( A \) is a ball.

Proof. Since \((2.7)\) can be seen from Chakerian’s [7, (25)], it is enough to verify \((2.6)\). Note that

\[
(2.8) \quad \frac{|x|b(A)}{2} = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} h_A(|x|\theta) d\theta.
\]

is valid for any given \( x \in \mathbb{R}^n \), and importantly, an extension of [2, Example 7.4] to \( A \in C^n \) tells us that the right side of \((2.8)\) can be approximated by \( \sum_{j=1}^{m} h_A(|x|\theta_j) \lambda_j \) which is the support function of \( \sum_{j=1}^{m} \lambda_j R_j(A) \), where

\[
\begin{align*}
\lambda_j &\in (0, 1); \\
\sum_{j=1}^{m} \lambda_j &= 1;
\end{align*}
\]

and \( R_j(A) \) is an appropriate rotation of \( A \) associated to \( \theta_j \). Therefore, by employing Colesanti-Salani’s [14, Theorem 1] and by induction, we can readily obtain that if \( p = n - 1 \) then

\[
(2.9) \quad \text{pcap}(A) = \sum_{j=1}^{m} \lambda_j \text{pcap}(A) = \sum_{j=1}^{m} \lambda_j \text{pcap}(R_j(A)) \leq \text{pcap}\left( \sum_{j=1}^{m} \lambda_j R_j(A) \right).
\]

Here the rotation-invariance of \( \text{pcap}(\cdot) \) has been used; see e.g. [20, Page 151]. Note also that the left side of \((2.8)\) is the support function of a ball of radius \( b(A)/2 \). So, a combination of the above approximation, the correspondence between a support function and a convex set, \((2.9)\) and the well-known formula

\[
(2.10) \quad \text{pcap}(r \mathbb{B}^n) = \sigma_{n-1} \left( \frac{p - 1}{n - p} \right)^{1-p} r^{n-p},
\]

derives the left inequality of \((2.6)\). \(\square\)

2.3. \( p - 1 \) integral mean curvature radius. We should point out that if \( p = n - 1 = 2 \) then \((2.6)\) is just Pólya’s inequality [54, (5)] – here the fact that for a \( C^2 \) body \( A \in \mathbb{R}^3 \) the mean radius \( b(A)/2 \) is equal to \((4\pi)^{-1}\) times the surface integral of the mean curvature has been used. To see this more transparently, let us recall that for a convex set \( A \) with its boundary \( \partial A \) being \( C^2 \) hypersurface,

\[
m_j(A, x) = \begin{cases} 
1 & \text{for } j = 0; \\
\left( \frac{n - 1}{j} \right)^{-1} \sum_{1 \leq i_1 < \ldots < i_j \leq n-1} \kappa_{i_1}(x) \cdots \kappa_{i_j}(x) & \text{for } j = 1, \ldots, n - 1,
\end{cases}
\]
is the \( j \)-th mean curvature at \( x \in \partial A \), where \( \kappa_1(x), \ldots, \kappa_{n-1}(x) \) are the principal curvatures of \( \partial A \) at the point \( x \). Note that (see, e.g., [22])

\[
\begin{cases}
m_1(A, x) = H(\partial A, x) = \text{mean curvature of } \partial A \text{ at } x; \\
m_j(A, x) \leq (H(\partial A, x))^j \quad \text{for } j = 1, \ldots, n-1; \\
m_{n-1}(A, x) = G(\partial A, x) = \text{Gauss curvature of } \partial A \text{ at } x.
\end{cases}
\]

Such a higher order mean curvature \( m_j(A, \cdot) \) is used to produce the so-called \( j \)-th integral mean curvature of \( \partial A \):

\[
M_j(A) = \int_{\partial A} m_j(A, \cdot) \, dH^{n-1}(\cdot).
\]

Clearly, we have

\[
\begin{cases}
M_0 = H^{n-1}(\partial A); \\
M_1 = \int_{\partial A} H(\partial A, \cdot) \, dH^{n-1}(\cdot); \\
M_{n-2} = \sigma_{n-1} b(K)/2.
\end{cases}
\]

Moreover, if \( \nu(x) \) is the outer unit normal vector then (cf. [48])

\[
M_0 = \int_{\partial A} x \cdot \nu(x) H(\partial A, x) \, dH^{n-1}(x);
\]

if \( n = 2 \) then the Gauss-Bonnet formula gives \( M_1(A) = 2\pi \); and if \( p = n - 1 = 2 \) then (2.6) reduces to the above-mentioned Pólya’s inequality.

According to [56, (13.43)], the foregoing \( S(A, t) \) has the following decomposition

\[
S(A, t) = \sum_{j=0}^{n-1} \binom{n-1}{j} M_j(A) r^j.
\]

This formula is brought into (2.4) to deduce

\[
\int_0^\infty \left( \int_{\partial A} (1 + tH(\partial A, \cdot))^{n-1} \, dH^{n-1}(\cdot) \right)^{\frac{p}{n-p}} \, dt \right)^{1-p}
\]

with equality if \( A \) is a ball. The inequality (2.11) will be complemented through the forthcoming lemma which not only extends Freire-Schwartz’s [22, Theorem 2] (and Pólya’s inequalities [54, (3)-(4)] for \( n = 3 \)) from \( p = 2 \) to \( p \in [2, n) \), but also induces Willmore’s inequality (cf. [55] Page 87] or [8] for immersed hypersurfaces in \( \mathbb{R}^d \):

\[
\sigma_{n-1} \leq \int_{\partial A} (H(\partial A, \cdot))^{n-1} \, dH^{n-1}(\cdot)
\]

through letting \( p \to n \) in (2.13) whose special case \( p = 2 \) is essentially the Huisken’s result presented in [28, Theorem 6].
Lemma 2.3. Given $p \in [2, n)$, let $A \subset \mathbb{R}^n$ be a connected compact set with $C^2$ boundary $\partial A$.

(i) If $H(\partial A, \cdot) \geq 0$, then

\begin{equation}
\left( \frac{(p-1)^{p-1}(pcap(A))}{\sigma_{n-1}} \right)^\frac{1}{p} \leq \left( \frac{1}{\sigma_{n-1}} \int_{\partial A} (H(\partial A, \cdot))^{p-1} d\mathcal{H}^{n-1}(\cdot) \right)^\frac{1}{p}
\end{equation}

with equality if $A$ is a ball.

(ii) If $H(\partial A, \cdot) \geq 0$ and $\partial A$ is outer-minimizing, i.e., $K \supseteq A \Rightarrow H^{n-1}(\partial K) \geq H^{n-1}(\partial A)$, then

\begin{equation}
\left( \frac{H^{n-1}(\partial A)}{\sigma_{n-1}} \right)^\frac{1}{p} \leq \left( \frac{1}{\sigma_{n-1}} \int_{\partial A} (H(\partial A, \cdot))^{p-1} d\mathcal{H}^{n-1}(\cdot) \right)^\frac{1}{p}
\end{equation}

with equality if $A$ is a ball.

Proof. Geometrically speaking, the right hand side of (2.12)/(2.13) is said to be the $p-1$ integral mean curvature radius of $A$. Obviously, (2.12)/(2.13) and (1.5)/(1.6) are combined to deduce the following volume-integral-mean-curvature inequality

\begin{equation}
\left( \frac{L^n(A)}{\omega_n} \right)^\frac{1}{p} \leq \left( \frac{1}{\sigma_{n-1}} \int_{\partial A} (H(\partial A, \cdot))^{p-1} d\mathcal{H}^{n-1}(\cdot) \right)^\frac{1}{p}
\end{equation}

with equality if $A$ is a ball.

The equality cases of (2.12) and (2.13) are trivial. So, it remains to check their inequalities. To do so, we will write $(\Sigma_t)_{t \geq 0}$ for the level sets of the function induced by the weak solution $\phi$ to Huisken-Ilmanen’s Inverse Mean Curvature Flow (IMCF) (cf. [34, 35]) starting at $\partial A$: $\frac{\partial}{\partial t} = H^{-1} \nu$ – here $\nu$ is the outer unit normal vector and the level set formulation of this flow is decided by the Dirichlet problem:

\begin{equation}
\begin{cases}
\text{div}(|\nabla u|^{p-2} \nabla u) = |\nabla u| & \text{in } \mathbb{R}^n \setminus A; \\
u = 0 & \text{on } \partial A & u(x) \to \infty & \text{as } |x| \to \infty,
\end{cases}
\end{equation}

whose proper weak solution (cf. [49, 50]) can be obtained via letting $p \to 1$ in $\nu = \exp(u/(1-p))$ coupled with the boundary value problem (see also (1.22)):

\begin{equation}
\begin{cases}
\text{div}(|\nabla v|^{p-2} \nabla v) = 0 & \text{in } \mathbb{R}^n \setminus A; \\
u = 1 & \text{on } \partial A & v(x) \to 0 & \text{as } |x| \to \infty.
\end{cases}
\end{equation}

The following fundamental results (a)-(b)-(c)-(d) on IMCF are due to Huisken-Ilmanen (cf. [34, 35] and [4, 22]):
(a) There is a proper, locally Lipschitz function \( \phi \) such that: \( \phi \geq 0 \) in \( \mathbb{R}^n \setminus A^o \); \( \phi = 0 \) on \( \partial A \); for each \( t > 0 \),

\[
\Sigma_t = \partial \{ x \in \mathbb{R}^n \setminus A^o : \phi(x) \geq t \} \quad \& \quad \Sigma_t' = \partial \{ x \in \mathbb{R}^n \setminus A^o : \phi(x) > t \}
\]
define increasing families of \( C^{1,\alpha} \) hypersurfaces.

(b) \( \Sigma_t \) (resp. \( \Sigma_t' \)) minimize (resp. strictly minimize) area among surfaces homologous to \( \Sigma_t \) in \( \{ x \in \mathbb{R}^n \setminus A^o : \phi(x) \geq t \} \);

\[
\Sigma_t' = \partial \{ x \in \mathbb{R}^n \setminus A^o : \phi(x) > 0 \}
\]
strictly minimizes area among hypersurfaces homologous to \( \Sigma = \partial A \) in \( \mathbb{R}^n \setminus A^o \).

(c) For almost all \( t > 0 \), the weak mean curvature of \( \Sigma_t \) is defined and equal to \( |\nabla \phi|/(n - 1) \) that is positive almost everywhere on \( \Sigma_t \).

(d) For each \( t > 0 \), one has:

\[
\mathcal{H}^{n-1}(\Sigma_t) = e^t \mathcal{H}^{n-1}(\Sigma_t');
\]
and

\[
\mathcal{H}^{n-1}(\Sigma_t) = e^t \mathcal{H}^{n-1}(\partial A)
\]
if \( \partial A \) is outer-minimizing, i.e., \( \partial A \) minimizes area among all surfaces homologous to \( \partial A \) in \( \mathbb{R}^n \setminus A^o \).

According to (1) and the definition of pcap(\( \cdot \)), we have

\[
\text{pcap}(A) = \text{pcap}(\partial A) \leq \inf \int_{\mathbb{R}^n \setminus A^o} |\nabla f|^p \, d\mathcal{L}^n
\]
where the infimum is taken over all functions \( f = g \circ \psi \) that have the above-described level hypersurfaces \( (\Sigma_t)_{t \geq 0} \) and enjoy the property that \( g \) is a one-variable function with \( g(0) = 0 \) and \( g(\infty) = 1 \) and \( \psi \) is a nonnegative function on \( \mathbb{R}^n \setminus A^o \) with \( \psi|\partial A = 0 \) and \( \lim_{|x| \to \infty} \psi(x) = \infty \). Note that the classical co-area formula yields

\[
\int_{\mathbb{R}^n \setminus A^o} |\nabla f|^p \, d\mathcal{L}^n = \int_0^\infty |g'(t)|^p \left( \int_{\Sigma_t} |\nabla \psi|^{p-1} \, d\mathcal{H}^{n-1} \right) dt.
\]
So, upon choosing

\[
\begin{cases}
\psi = \phi; \\
g(t) = V_p(t) = \int_0^t \left( U_p(s)^{1-p} \right)^{1-p} ds; \\
U_p(t) = \frac{1}{\sigma_{n-1}} \int_{\Sigma_t} |\nabla \phi|^{p-1} \, d\mathcal{H}^{n-1},
\end{cases}
\]
we can achieve

\[
\frac{\text{pcap}(\partial A)}{\sigma_{n-1}} \leq \int_0^\infty U_p(t) \left| \frac{d}{dt} V_p(t) \right|^p dt,
\]
whence finding

\[ (2.14) \quad \frac{\text{pcap}(A)}{\sigma_{n-1}} \leq \left( \int_0^\infty (U_p(t))^{\frac{1}{1-p}} \, dt \right)^{1-p}. \]

Next, let us estimate the growth of \( U_p(\cdot) \). In fact, utilizing [35, Lemma 1.2, (ii)&(v)], an integration-by-part, the inequality

\[ H_t^2 - (n - 1)|\Pi_t|^2 \leq 0 \]

with \( 0 < H_t \) and \( \Pi_t \) being the mean curvature and the second fundamental form on \( \Sigma_t \) respectively, the assumption \( p \in [2, n) \), and the property (c) above, we get

\[
\frac{d}{dt} U_p(t) = \frac{d}{dt} \left( \frac{(n - 1)^{p-1}}{\sigma_{n-1}} \int_{\Sigma_t} H_t^{p-1} \, d\mathcal{H}^{n-1} \right) \\
= \frac{(n - 1)^{p-1}}{\sigma_{n-1}} \int_{\Sigma_t} \left( (p - 1)H_t^{p-2} \left( \frac{d}{dt} H_t \right) + H_t^{p-1} \right) \, d\mathcal{H}^{n-1} \\
= \frac{(n - 1)^{p-1}}{\sigma_{n-1}} \int_{\Sigma_t} \left( H_t - (p - 1)\frac{1}{H_t} \right)^2 \frac{d}{dt} H_t \, d\mathcal{H}^{n-1} \\
\leq \frac{n - p}{(n - 1)^{p-1}} \int_{\Sigma_t} \frac{d}{dt} \left( \frac{(n - 1)(p - 1)}{H_t} \right)^{p-1} \, d\mathcal{H}^{n-1} \\
= \left( \frac{n - p}{n - 1} \right) U_p(t). 
\]

Here, the author thanks Professor Guofang Wang for pointing out that \( \Delta H_{t-1} \) should appear in the above argument.

The last estimate in turn implies

\[ (2.15) \quad U_p(t) \leq U_p(0) \exp \left( t\frac{n - p}{n - 1} \right). \]

Using (2.14)-(2.15) we find

\[ \frac{\text{pcap}(A)}{\sigma_{n-1}} \leq U_p(0) \left( \frac{(n - 1)(p - 1)}{n - p} \right)^{1-p} \]

whence reaching (2.12) via (c).

Finally, in order to check (2.13), we apply (d) and the above-established differential inequality

\[ \frac{d}{dt} U_p(t) \leq \left( \frac{n - p}{n - 1} \right) U_p(t) \]
to discover that

\[ t \mapsto \Phi_p(t) = (\mathcal{H}^{n-1}(\Sigma_t))^{\frac{p}{n}} \int_{\Sigma_t} H_t^{p-1} d\mathcal{H}^{n-1} \]

is a decreasing function. But, since \( \Sigma_t \) tends to a round sphere as \( t \to \infty \), one concludes

\[ \Phi_p(\infty) = \lim_{t \to \infty} \Phi_p(t) = \sigma_{n-1}^{\frac{p-1}{n-1}}. \]

Therefore,

\[ (\mathcal{H}^{n-1}(\partial A))^{\frac{p}{n}} \int_{\partial A} (H(\partial A, \cdot))^{p-1} d\mathcal{H}^{n-1}(\cdot) = \Phi_p(0) \geq \Phi_p(\infty) = \sigma_{n-1}^{\frac{p-1}{n-1}}, \]

and consequently, (2.13) follows. \( \square \)

### 2.4. Variational \( p \) capacity radius vs surface radius.

In his paper [54], Pólya conjectured that of all members in \( C^3 \), with a given surface area, the round ball has the minimal electrostatic capacity \( 2\text{cap}(\cdot) \). While this conjecture has not yet been proved or disproved, the following Lemma 2.4 confirms partially the conjecture.

**Lemma 2.4.** Let \( p \in (1, n) \).

(i) If there is a constant \( \alpha > 0 \) such that \( A \subset \mathbb{R}^n \) is \( \alpha \)-convex, i.e., for any \( x \in \partial A \) there exists a closed ball \( B \) with radius \( \alpha^{-1} \) such that \( x \in \partial B \) and \( A \subseteq B \), then

\[ \left( \frac{p-1}{(n-p)\alpha} \right)^{p-1} \left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right) \geq \frac{\mathcal{H}^{n-1}(\partial A)}{\sigma_{n-1}} \]

with equality when and only when \( A \) is a ball of radius \( \alpha^{-1} \).

(ii) If \( A \subset \mathbb{R}^n \) is a connected compact set with \( C^2 \) boundary \( \partial A \) and there is a constant \( \beta > 0 \) such that \( 0 \leq H(\partial A, \cdot) \leq \beta \), then

\[ \left( \frac{p-1}{(n-p)\beta} \right)^{p-1} \left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right) \leq \frac{\mathcal{H}^{n-1}(\partial A)}{\sigma_{n-1}} \]

with equality when and only when \( A \) is a ball of radius \( \beta^{-1} \).

**Proof.** (i) To prove (2.16), let us keep in mind the fact that if \( \partial A \) is of \( C^2 \) then \( A \) is \( \alpha \)-convex if and only if each principal curvature \( \kappa_j \) of \( \partial A \) is not less than \( \alpha \), i.e., \( \kappa_j \geq \alpha \).

Following the argument for Hurtado-Palmer-Ritoré’s [36, Theorem 4.5] which is just the case \( p = 2 \) of (2.16) we set

\[ v(x) = \phi(d(x, A)) \quad \& \quad \phi(t) = (1 + \alpha t)^{\frac{p}{n}}. \]
Then $v$ is of $C^{1,1}$ in $\mathbb{R}^n \setminus A$. Given $t \in (0, \infty)$. If $x \in \mathbb{R}^n \setminus A$ is such a point that $d(x, A)$ is twice differentiable along the line minimizing $d(x, A)$ and if
\[ A_t = \{ y \in \mathbb{R}^n : \text{dist}(y, A) \leq t \}, \]
then on this line one utilizes (1.4) to derive
\[
\text{div}(|\nabla v|^{p-2} \nabla v) = |\phi'(d(x, A))|^{p-2} \left[ (n-1)H_t(x)\phi'(d(x, A)) + (p-1)\phi''(d(x, A)) \right]
\]
where $H_t$ stands for the mean curvature of the hypersurface $\partial A_t$ which is parallel to $\partial A$. Note that $A_t$ is $(t + \alpha^{-1})^{-1}$-convex. So, one has
\[ H_t \geq \alpha/(1 + \alpha t) \]
at the regular points in $\partial A_t$. Recall that $u = u_A$ is the $p$-equilibrium potential. A simple calculation gives
\[ \phi'(t) = \alpha \left( \frac{p-n}{p-1} \right) (1 + \alpha t)^{\frac{1-n}{p-1}} \leq 0. \]
This, along with (2.18) and a simple computation, shows that
\[
\text{div}(|\nabla v|^{p-2} \nabla v) \leq 0 = \text{div}(|\nabla u|^{p-2} \nabla u)
\]
holds whenever $x \mapsto d(x, A)$ is of $C^2$.

Next, we prove that $v \geq u$ holds in $\mathbb{R}^n \setminus A$. For the above given $t > 0$ let $u_t$ and $\phi_t$ be the $p$-equilibrium potentials of the rings
\[ (A_t, A) \quad \& \quad ((t + \alpha^{-1})B^n, \alpha^{-1}B^n) \]
respectively (cf. [41]), as well as, set $v_t = \phi_t(d(x, A))$. Then the last div-estimate, plus an integration-by-part argument, implies that
\[
\text{div}(|\nabla v_t|^{p-2} \nabla v_t) \leq \text{div}(|\nabla u_t|^{p-2} \nabla u_t) \quad \text{in} \quad A_t \setminus A
\]
is valid in the distributional sense. Now, from the weak comparison principle for $p$-Laplacian (see e.g. [61]) it follows that $v_t \geq u_t$ holds in $A_t \setminus A$, and so that $v \geq u$ is valid in $\mathbb{R}^n \setminus A$ via letting $t \to \infty$.

Note also that $\nabla u$ and $\nabla v$ have non-tangential limit $\mathcal{H}^{n-1}$-almost everywhere on $\partial A$. So, if $x \in \partial A$, then $\nabla u$ and $\nabla v$ can be defined at $x$. Upon extending $u$ and $v$ continuously to $x$ and $B$ being an exterior ball to $A$, and utilizing
\[
\begin{align*}
&\text{div}(|\nabla v|^{p-2} \nabla v) \leq \text{div}(|\nabla u|^{p-2} \nabla u) \quad \text{in} \quad B; \\
&u(x) = v(x) = 1 \quad \text{for} \quad x \in \partial A; \\
v(x) \geq u(x) \quad \text{for} \quad x \in \mathbb{R}^n \setminus A; \\
v - u \quad \text{continuous on} \quad B \cup \partial B,
\end{align*}
\]

(2.19)
as well as taking the Hopf maximum principle into account, we get
\[(2.20) \quad |\nabla v(x)| \leq |\nabla u(x)| \quad \forall \quad x \in \partial A.\]

An application of (1.1) gives that
\[
\text{pcap}(A) = \int_{\partial A} |\nabla u|^p \, dH^{n-1} \\
\geq \int_{\partial A} |\nabla v|^p \, dH^{n-1} \\
= (\phi'(0))^{p-1} \mathcal{H}^{n-1}(\partial A) \\
= \left( \frac{n-p}{p-1} \alpha \right)^{p-1} \mathcal{H}^{n-1}(\partial A),
\]

namely, (2.16) holds.

Of course, if \(A\) is a ball with radius \(\alpha^{-1}\), then equality of (2.16) trivially holds. Conversely, when equality of (2.16) is true, (2.21) is employed to derive that
\[|\nabla u(x)| = |\nabla v(x)| \quad \text{holds for } H^{n-1}\text{-almost every points } x \in \partial A.\]

Consequently,
\[u = v \quad \text{holds on any exterior ball to } A \quad \text{and therefore it still true in } \mathbb{R}^n \setminus A.\]

So, the level sets of \(u\) and \(v\) are the same. Thanks to \(u \in C^\infty(\mathbb{R}^n \setminus A)\) (cf. [14]), the level sets of \(u\) are \(C^\infty\) hypersurfaces. Since
\[|\nabla v(x)| = |\phi'(d(x,A))|/|\nabla d(x,A)| \neq 0 \quad \forall \quad x \in \mathbb{R}^n \setminus A,
\]
ones has that \(|\nabla u| = |\nabla v|\) does not vanish. Consequently,
\[
\begin{align*}
H_t &= \alpha/(1 + \alpha t); \\
\text{div}(|\nabla v|^{p-2} \nabla v) &= \text{div}(|\nabla u|^{p-2} \nabla u).
\end{align*}
\]

This in turn derives that the principal curvatures of \(\partial A_t\) equal \((t + \alpha^{-1})^{-1}\), and so that \((A_t)_{t>0}\) are concentric balls with radius \(\alpha^{-1} + t\). Therefore, \(A\) is a ball of radius \(\alpha^{-1}\).

(ii) Under the assumption that \(A \subset \mathbb{R}^n\) is a connected compact set with \(C^2\) boundary \(\partial A\) and \(0 \leq H(\partial A, \cdot) \leq \beta\) holds for a constant \(\beta > 0\), we may apply (2.12) to derive that under \(p \in [2, n)\) one has
\[
\left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right)^{1-p} \leq \left( \frac{1}{\sigma_{n-1}} \int_{\partial A} (H(\partial A, \cdot))^{p-1} \, dH^{n-1}(\cdot) \right)^{1-p} \leq \left( \frac{\mathcal{H}^{n-1}(\partial A)}{\beta^{1-p}} \right),
\]

and thus (2.17).

Nevertheless, the general inequality (2.17) can be also verified by slightly modifying the above argument for (i). The key is the selection of the function pair \((v, \phi)\) for (ii) - more precisely -
\[v(x) = \phi(d(x,A)) \quad \& \quad \phi(t) = (1 + \beta t)^{\frac{n}{n-1}}.\]
Under this choice, \( \alpha, (2.18), (2.20), \) and \( (2.21) \) will be replaced by

\[
\begin{aligned}
&\begin{cases}
\beta, \\
H_t \leq \beta/(1 + \beta t), \\
\text{div}(|\nabla v|^{p-2}\nabla v) \geq \text{div}(|\nabla u|^{p-2}\nabla u) \quad \text{in} \quad B;
\end{cases}
\quad u(x) = v(x) = 1 \quad \text{for} \quad x \in \partial A;
\quad v(x) \leq u(x) \quad \text{for} \quad x \in \mathbb{R}^n \setminus A;
\quad v - u \quad \text{continuous on} \quad B \cup \partial B,
\end{aligned}
\]

and

\[
\begin{aligned}
&\text{pcap}(A) \\
&= \int_{\partial A} |\nabla u|^{p-1} \, dH^{n-1} \\
&\leq \int_{\partial A} |\nabla v|^{p-1} \, dH^{n-1} \\
&= \left( \left( \frac{n-p}{p-1} \right)^{p-1} \right)^{p-1} \mathcal{H}^{n-1}(\partial A),
\end{aligned}
\]

as desired.

The argument for equality of \( (2.17) \) is similar to that for equality of \( (2.6) \) (but this time, just using the last estimation), and so left for the interested reader. \( \square \)

2.5. **Graphic ADM mass radius.** Following [40] we consider the so-called graphic ADM mass. For \( f(x) = f(x_1, \ldots, x_n) \) and \( i, j, k = 1, 2, \ldots, n \) we write

\[
\begin{aligned}
&f_i = \partial f/\partial x_i; \\
&f_{ij} = \partial^2 f/\partial x_i \partial x_j; \\
&f_{ijk} = \partial^3 f/\partial x_i \partial x_j \partial x_k; \\
&\delta_{ij} = 0 \text{ or } 1 \text{ as } i \neq j \text{ or } i = j.
\end{aligned}
\]

Suppose \( U \) is a bounded open set in \( \mathbb{R}^n \) with boundary \( \partial U \). We say that a smooth function \( f: \mathbb{R}^n \setminus U \hookrightarrow \mathbb{R} \) is asymptotically flat provided there is a constant \( \gamma > (n-2)/2 \) such that

\[
|f_i(x)| + |x||f_{ij}(x)| + |x|^2|f_{ijk}(x)| = O(|x|^{-\gamma/2}) \quad \text{as} \quad |x| \to \infty.
\]

Now, given such a smooth asymptotically flat function, the graph of \( f \), denoted by

\[
(\mathbb{R}^n \setminus U, \delta + df \otimes df) = (\mathbb{R}^n \setminus U, (\delta_{ij} + f_i f_j)),
\]

where \( \delta_{ij} \) is the Kronecker delta.
is a complete Riemannian manifold. And then, the ADM (named after three physicists: Arnowitt, Deser and Misner) mass of this graph is determined by

\[ m_{ADM}(\mathbb{R}^n \setminus U, \delta + df \otimes df) = \lim_{r \to \infty} \int_{S_r} \sum_{i,j=1}^{n} \frac{(f_{ij}f_j - f_if_j)f_i}{2(n - 1)\sigma_{n-1}(1 + |\nabla f|^2)} \, d\sigma, \]

where \( S_r \) is the coordinate sphere of radius \( r \) and \( d\sigma \) is the area element of \( S_r \). It is perhaps appropriate to point out that under \( n \geq 3 \) this definition of the ADM mass coincides the definition of the original ADM mass of an asymptotically flat manifold; see also [40] for a brief review on the Riemannian positive mass theorem (cf. Schoen-Yau ([58, 59]) and Witten [70]) and its strengthening - the Riemannian Penrose inequality for area outer minimizing horizon (cf. Huisken-Illmanen [34] and Bray [3]).

**Lemma 2.5.** Let \( A \subset \mathbb{R}^n \) be a connected compact set with \( C^2 \) boundary \( \partial A \) and \( H(\partial A, \cdot) \geq 0 \). Suppose \( f : \mathbb{R}^n \setminus A^c \mapsto \mathbb{R} \) is a smooth asymptotically flat function such that \( f(\partial A) \) is in a level set of \( f \), \( \lim_{x \to \partial A} |\nabla f(x)| = \infty \), and the scalar curvature of \((\mathbb{R}^n \setminus A^c, \delta + df \otimes df)\) is nonnegative.

(i) If \( p = 2 < n \), then

\[ \left( \frac{p}{n - p} \right)^{p-1} \frac{pcap(A)}{\sigma_{n-1}} \leq (2m_{ADM}((\mathbb{R}^n \setminus A^c, \delta + df \otimes df)))^{\frac{1}{n-2}}. \]

(ii) If \( n \geq 3 \) and \( \partial A \) is outer-minimizing, then

\[ \left( \frac{\mathcal{H}^{n-1}(\partial A)}{\sigma_{n-1}} \right)^{\frac{1}{n-1}} \leq (2m_{ADM}((\mathbb{R}^n \setminus A^c, \delta + df \otimes df)))^{\frac{1}{n-2}}. \]

**Proof.** Naturally, the right hand side of (2.22)/(2.23) is called the graphic ADMS mass radius. An application of both (2.22)/(2.23) and (1.5)/(1.6) gives the following volume-mass inequality

\[ \left( \frac{L^n(A)}{\omega_n} \right)^{\frac{1}{n}} \leq (2m_{ADM}((\mathbb{R}^n \setminus A^c, \delta + df \otimes df)))^{\frac{1}{n-2}}; \]

see also [60] for an analogous inequality for the conformally flat manifolds.

As the Penrose inequality for graphs with convex boundaries, (2.23) for \( A \in C^0 \) comes from Lam’s [40, Remark 8]. Since [40, Lemma 12] can be replaced by (2.13) under \( p = 2 \) and \( \partial A \) being outer-minimizing, (2.23) is valid for the case described in Lemma 2.5. Thus, it remains to verify (2.22).
Note that Lam’s [40, Theorem 6] actually says
\begin{equation}
2m_{ADM}(\mathbb{R}^n \setminus A^\circ, \delta + df \otimes df)
= \frac{1}{\sigma_{n-1}} \int_{\partial A} H(\partial A, \cdot) \, d\mathcal{H}^{n-1}(\cdot)
+ \frac{1}{(n-1)n\omega_n} \int_{\mathbb{R}^n \setminus A^\circ} R_f(\cdot) \, d\mathcal{L}^n(\cdot),
\end{equation}
where
\[ R_f = \sum_{j=1}^n \frac{\partial}{\partial x_j} \sum_{i=1}^n \left( \frac{f_i f_j - f_i j f_i}{1 + |\nabla f|^2} \right) \]
is the scalar curvature of the graph $(\mathbb{R}^n \setminus A^\circ, \delta + df \otimes df)$ of $f$; see also [40, Lemma 10] or [33, Proposition 5.4]. Thus, (2.24), which may be regarded as the Gauss-Bonnet like formula for the graphic ADM mass, along with $R_f \geq 0$, and (2.12), implies that if $p = 2 < n$ then
\begin{align*}
\left( \left( \frac{p-1}{n-p} \right)^{p-1} \frac{\text{pcap}(A)}{\sigma_{n-1}} \right)^{\frac{1}{n-p}} & \leq \left( \frac{1}{\sigma_{n-1}} \int_{\partial A} \left( H(\partial A, \cdot) \right)^{p-1} \, d\mathcal{H}^{n-1}(\cdot) \right)^{\frac{1}{n-p}} \\
& \leq \left( 2m_{ADM}(\mathbb{R}^n \setminus A^\circ, \delta + df \otimes df) \right)^{\frac{1}{n-p}}.
\end{align*}
In other words, (2.22) holds. \hfill \Box

3. Proofs of Theorem and Its Corollary

We are ready to prove Theorem 1.1 and its Corollary 1.2.

Proof of Theorem 1.1. (i) Owing to $h \in L^1(\mathbb{R}^n)$, we get
\[ F_{\text{pcap}}(A) \leq \|h\|_{L^1(\mathbb{R}^n)} - \text{pcap}(A) \quad \forall \quad A \in C^0. \]
Observe that if $(B_j)_{j \geq 1}$ is a sequence of closed balls converging to a point then $(F_{\text{pcap}}(B_j))_{j \geq 1}$ tends to 0. Thus, $\sup_{A \in C^0} F_{\text{pcap}}(A) \geq 0$. As a consequence, if $F_{\text{pcap}}(\cdot)$ attains its supremum at $A_0 \in C^0$ then there must be
\[ F_{\text{pcap}}(A_0) = \sup_{A \in C^0} F_{\text{pcap}}(A) \geq 0. \]
On the other hand, suppose there exists $A \in C^0$ so that $F_{\text{pcap}}(A) \geq 0$. Then $\sup_{K \in C^0} F_{\text{pcap}}(K) \geq 0$. If $(A_j)_{j \geq 1}$ is a sequence of maximizers for $F_{\text{pcap}}(\cdot)$ with $F_{\text{pcap}}(A_j) > 0$ and the inradius of $A_j$ having a uniform lower bound $r_0 > 0$ (if, otherwise, $A_j$ converges to a set of single point $\{a_0\}$, then
$F_{\text{pcap}}(A_j)$ tends to 0 and hence $\{a_0\} \in C^n$ is a maximizer. Using this, (2.10) and (2.2) of Lemma 2.1 we obtain

$$r_0 = \left(\frac{p-1}{n-p}\right)^{p-1}\left(\frac{\text{pcap}(r_0^n)}{\sigma_{n-1}}\right)^{\frac{1}{p}}$$

(3.1)

$$\leq \left(\frac{p-1}{n-p}\right)^{p-1}\left(\frac{\text{pcap}(A_j)}{\sigma_{n-1}}\right)^{\frac{1}{p}}$$

$$\leq 2^{-1}\text{diam}(A_j).$$

Because

$$F_{\text{pcap}}(A_j) \leq \|h\|_{L^1(\mathbb{R}^n)} - \sigma_{n-1}\left(\frac{p-1}{n-p}\right)^{1-p}\left(\frac{\mathcal{L}^n(A_j)}{\omega_n}\right)^{\frac{n}{n-p}},$$

(3.2)

one concludes that if $(\text{diam}(A_j))_{j \geq 1}$ is not bounded, then (3.1) and the definition of $r_0$ are employed to derive that $(\mathcal{L}^n(A_j))_{j \geq 1}$ is not bounded, and so $(F_{\text{pcap}}(A_j))_{j \geq 1}$ has a subsequence which goes to negative infinity. However, each $F_{\text{pcap}}(A_j)$ is assumed to be positive. Thus, $(\text{diam}(A_j))_{j \geq 1}$ has a uniform upper bound. Now, by the well-known Blaschke selection principle (see e.g. [57, Theorem 1.8.6]), we can choose a subsequence of $(A_j)_{j \geq 1}$ that converges to $A_0 \in \mathbb{K}^n$. Since $\text{pcap}(\cdot)$ is continuous (cf. [47, Pages 142-143]) and $h \in C(\mathbb{R}^n)$ (i.e., $h$ is continuous in $\mathbb{R}^n$), $F_{\text{pcap}}(\cdot)$ is continuous, and so, $A_0$ is a maximizer of $F_{\text{pcap}}(\cdot)$.

(ii) For $A, B \in \mathbb{K}^n$ and $t \in (0, 1)$ let $C_t = A + tB$. Then

$$C_t \in \mathbb{K}^n \quad \& \quad h_{C_t} = h_A + th_B.$$

Using Tso’s variational formula for $\int_A h d\mathcal{L}^n$ in [66, (4)] and the variational formula for $\text{pcap}(\cdot)$ in [13, Theorem 1.1] (see also [57, Corollary 3.16] or [38, Theorem 2.5] for $2\text{cap}(\cdot)$), we obtain

$$\frac{d}{dt}F_{\text{pcap}}(C_t)\big|_{t=0} = \int_{\partial A} h_B(g)h d\mathcal{H}^{n-1} - \int_{\partial A} h_B(g)(p-1)|\nabla u_A|^p d\mathcal{H}^{n-1}.$$  

(3.3)

Obviously, if $A$ is a maximizer of $F_{\text{pcap}}(\cdot)$, then it must be a critical point of $F_{\text{pcap}}(C_t)$ and thus

$$\frac{d}{dt}F_{\text{pcap}}(C_t)\big|_{t=0} = 0.$$

This and (3.3) derive

$$\int_{\partial A} h_B(g)(p-1)|\nabla u_A|^p d\mathcal{H}^{n-1} = \int_{\partial A} h_B(g)h d\mathcal{H}^{n-1}.$$  

(3.4)

A combined application of (3.4) and [57, Lemmas 1.7.9 & 1.8.10] gives that
\[
\int_{\mathbb{S}^{n-1}} \phi g_*(p-1)|\nabla u_A|^p \, d\mathcal{H}^{n-1} = \int_{\partial A} \phi(g)(p-1)|\nabla u_A|^p \, d\mathcal{H}^{n-1} = \int_{\partial A} \phi(g) h \, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} \phi g_* h \, d\mathcal{H}^{n-1}
\]

holds for any \( \phi \in C(\mathbb{S}^{n-1}) \), and thereby reaching (1.3). Moreover, if \( \partial A \) is \( C^2 \) strictly convex, then the Gauss map from \( \partial A \) to \( \mathbb{S}^{n-1} \) is a diffeomorphism, and hence (1.3) is equivalent to

\[ (p-1)|\nabla u_A(x)|^p = h(x) \quad \forall \quad x \in \partial A. \]

(iii) Suppose \( h \in C^{k,\alpha} \) with \( k \) being a nonnegative integer. Since \( \partial A \) is of \( C^2 \), an application of [43, Theorem 1] (cf. [24, 18, 62, 67, 26, 51]) yields that \( u_A \in C^{1,\hat{\alpha}}(A) \) is valid for some \( \hat{\alpha} \in (0,1) \). The last equation and \( h \in C^{k,\alpha}(\mathbb{R}^n) \) with \( \alpha \in (0,1) \) derive that

\[ |\nabla u_A| = \left( \frac{h}{p-1} \right)^{\frac{1}{p}} \]

is of \( C^{k,\alpha} \). Note that \( \partial A \) is \( C^2 \) strictly convex. So, if \( \partial A \) is represented locally as \( y_n = \psi(x_1, \ldots, x_{n-1}) \), then the map

\[ (x_1, \ldots, x_{n-1}) \mapsto \nabla u_A(x_1, \ldots, x_{n-1}, \psi(x_1, \ldots, x_{n-1})) \]

is of \( C^{k,\alpha} \). Thus, a combination of the chain rule (or the implicit function theorem) and the estimate \( 0 < \inf_{\partial A} h \leq \sup_{\partial A} h < \infty \) imply that \( \psi \) is of \( C^{1+k,\alpha} \). This in turn implies that \( \partial A \) is of \( C^{1+k,\alpha} \). \( \square \)

**Proof of Corollary 1.2.** The argument for Corollary (i) is very similar to that for Theorem 1.1(i) except that (3.1) and (3.2) are replaced respectively by their endpoint (\( p = 1 \)) cases:

\[ r_0 = \left( \frac{\mathcal{H}^{-1}(\partial A)}{\sigma_{n-1}} \right)^{\frac{1}{p-1}} \leq \left( \frac{\mathcal{H}^{-1}(\partial A_j)}{\sigma_{n-1}} \right)^{\frac{1}{p-1}} \leq 2^{-1} \text{diam}(A_j) \]

and

\[ F_{\mathcal{H}^{n-1}}(A_j) \leq \| h \|_{L^1(\mathbb{R}^n)} - \sigma_{n-1} \left( \frac{\mathcal{L}^n(A_j)}{\omega_n} \right)^{\frac{n-1}{n}}. \]

To reach Corollary (ii), recall that for \( C_t = A + tB \) with \( A, B \in \mathbb{R}^n \) and \( t \in (0,1) \) (in the proof of Theorem 1.1(ii)) there exists a curvature measure
\[ \int_{\partial A} h d\mu_{H^{n-1}} = (n-1) \int_{\partial A} h d\mu_{H^{n-1}}. \]

Since \( A \) is a maximizer of \( F_{H^{n-1}}(\cdot) \), it is a critical point of \( F_{H^{n-1}}(\cdot) \), and consequently,

\[ \frac{d}{dt} F_{H^{n-1}}(C_t) \bigg|_{t=0} = 0, \]

whence yielding (1.7) via

\[ d\mu_{H^{n-1}} = g^*(h dH^{n-1}). \]

Furthermore, if \( \partial A \) is \( C^2 \) strictly convex, then the Gauss map \( g : \partial A \mapsto \mathbb{S}^{n-1} \) is a diffeomorphic transformation, and hence (1.7) reduces to the mean curvature equation

\[ h(x) = H(\partial A, x) \quad \forall \quad x \in \partial A \]

through using the variational formula for \( H^{n-1} \) (see e.g. \[15, 12, 10, 11\])

\[ \frac{d}{dt} H^{n-1}(\partial C_t) \bigg|_{t=0} = \int_{\partial A} h_B(g) H(\partial A, \cdot) dH^{n-1}(\cdot). \]

To validate Corollary (iii), note once again that under \( \partial A \) being \( C^2 \) strictly convex one has that if \( A \in K_n \) is a maximizer of \( F_{H^{n-1}}(\cdot) \) then \( h(\cdot) = H(\partial A, \cdot) \) holds on \( \partial A \). Also, since (cf. \[17\], Page 197)

\[ (n-1)H(\partial A, x) = \Delta b_A(x) \quad \forall \quad x \in \partial A \]

where

\[ b_A = d_A - d_{\mathbb{R}^n \setminus A} \quad \& \quad d_E(x) = \text{dist}(x, E) = \min_{y \in E} |x - y| \quad \forall \quad E \in C^n, \]

one concludes that

\[ \Delta b_A(x) = (n-1)h(x) \quad \forall \quad x \in \partial A, \]

and so \( b_A \) is of \( C^{k+2,\alpha} \) provided \( h \) is of \( C^{k,\alpha} \), and consequently, \( \partial A \) is of \( C^{k+2,\alpha} \) due to Delfour-Zolésio’s \[17\] Theorem 5.5.

\[ \square \]

Remark. The previous arguments for Theorem 1.1 and its Corollary 1.2, (1.5)-(1.6), the classic variational formula for the volume, and regularities for the Monge-Ampère equations established in \[1, 5, 68\] can be used to produce a natural Minkowski type proposition – under the hypothesis that \( h \in L^1(\mathbb{R}^n) \) is positive and continuous, \( k \) is a nonnegative integer, \( \alpha \in (0, 1) \), and

\[ F_{L^p}(A) = \int_A h d\mathcal{L}^n - \mathcal{L}^n(A) \quad \forall \quad A \in C^n, \]

one has:
\( F_{L^p}(\cdot) \) attains its supremum over \( C^n \) if and only if there exists \( A \in C^n \) such that \( F_{L^p}(A) \geq 0 \).

- Suppose \( A \in \mathbb{R}^n \) is a maximizer of \( F_{L^p}(\cdot) \). Then there is a Borel measure \( \mu_{L^p,A} \) on \( S^{n-1} \) such that \( d\mu_{L^p,A} = g_*(h \, d\mathcal{H}^{n-1}) \), namely,

\[
\int_{S^{n-1}} \phi \, d\mu_{L^p,A} = \int_{S^{n-1}} \phi g_*(h \, d\mathcal{H}^{n-1}) \quad \forall \phi \in C(S^{n-1}).
\]

In particular, if \( \partial A \) is \( C^2 \) strictly convex, then such a maximizer \( A \) satisfies the inverse Gauss curvature equation \( h(\cdot) = (G(\partial A, \cdot))^{-1} \).

- If \( h \) is of \( C^{k,\alpha} \) and \( A \), with \( \partial A \) being \( C^2 \) strictly convex, is a maximizer of \( F_{L^p}(\cdot) \), then \( \partial A \) is of \( C^{k+2,\alpha} \).

\[ \square \]

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