Entanglement detection by Bragg scattering

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We show how to measure the structural witnesses proposed in [P. Krammer et al., Phys. Rev. Lett. 103, 100502 (2009)] for detecting entanglement in a spin chain using photon scattering. The procedure, moreover, allows one to measure the two-point correlation function of the spin array. This proposal could be performed in existing experimental platforms realizing ion chains in Paul traps or atomic arrays in optical lattices.

Multiparticle entanglement plays a crucial role in various tasks of quantum information, such as in multiparty quantum secret sharing \cite{ref1} and in the computational speed-up in quantum algorithms \cite{ref2, ref3}. This motivates the great interest in detecting entanglement in many-body systems. An experimentally feasible method to achieve this task for spin systems was recently proposed in Ref. \cite{ref4} and later tested in a quantum optical experiment \cite{ref5}. Such a method is based on the construction of entanglement witnesses related to structure factors operators and it is suitable to detect entanglement for various families of multiparticle entangled states, such as the Dicke states, without the need of any \textit{a priori} knowledge on the Hamiltonian describing the physical system under consideration. In this paper we discuss specifically how this method can be implemented in ion chains or atomic arrays in optical lattices. In addition, we propose how to measure all elements of the structure factor matrix, thereby allowing one to reconstruct the two-body density matrix of the spin system.

We first briefly review the structural witnesses method and define the structure factor matrix. Consider a chain of N spin 1/2 particles which are ordered in a one-dimensional array at the positions $r_i$ ($i = 1, \ldots, N$). The structural witness method is based on the measurement of the $3 \times 3$ matrix $S(q)$, which is a function of the transferred wave vector $q$ and whose elements are given by the static structure factors

$$S^{\alpha\beta}(q) = \sum_{i<j} e^{iq \cdot (r_i - r_j)} \langle S_i^\alpha S_j^\beta \rangle.$$  \hspace{1cm} (1)

In the above expression $S_i^\alpha$ is the $\alpha$ component of the spin operator of particle $i$ at position $r_i$ ($\alpha = x, y, z$) and the average is taken over the initial state of the $N$ spins, whose entanglement is to be detected. As shown in Ref. \cite{ref4}, for different values of $q$ and different linear combinations of $S^{\alpha\beta}(q)$ it is possible to detect different families of multiparticle entangled states. For example, with the choice $q = 0$ one can construct the witness operator $\hat{W}_D$

$$\hat{W}_D = 1 - \frac{2}{N(N-1)}(\hat{S}^{xx}(0) + \hat{S}^{yy}(0) - \hat{S}^{zz}(0)),$$  \hspace{1cm} (2)

where $1$ denotes the identity operator for the $N$-qubit system and $\hat{S}^{\alpha\beta}(q) = \sum_{i<j} e^{iq \cdot (r_i - r_j)} S_i^\alpha S_j^\beta$ are the structure factor operators whose expectation values are given in Eq. \cite{ref1}. Whenever operator $\hat{W}_D$ gives a negative expectation value on a density matrix of the spin array $\rho_0$, $\text{Tr}(\hat{W}_D \rho_0) < 0$, we are guaranteed that the $N$-spin state under examination is entangled. The witness in Eq. \cite{ref2} is suitable for detecting symmetric Dicke states with any number of excitations, namely its average value for symmetric Dicke states is always negative.

More generally, the structural witness operators take the form

$$\hat{W} = 1 - \sum_{\alpha=x,y,z} c_\alpha \hat{C}_\alpha(q^\alpha),$$  \hspace{1cm} (3)

where $c_\alpha$ are real coefficients with $|c_\alpha| \leq 1$, $\hat{C}_\alpha(q^\alpha) = \frac{1}{N(N-1)}(\hat{S}^{\alpha\alpha}(q^\alpha) + \hat{S}^{\alpha\alpha}(-q^\alpha))$ and $q^\alpha$ denote three possibly different values for the transferred wave vector $\vec{q}$.

The underlying detection method relies on the outcome of scattering experiments and is therefore particularly suited to detect entanglement in many-body systems where local addressing of the individual constituent particles is not available. The purpose of this paper is to provide a specific implementation scheme, suggesting a possible pump-probe experiment one can perform on an atomic array or on an ion chain. The setup is sketched in Fig. \cite{fig1}. We show that the measurement of the intensity of the probe light allows one to determine the entanglement witnesses constructed through linear superpositions of the quantities in Eq. \cite{eq1},

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{(Color online) An entanglement witness for a spin array can be constructed by means of a pump-probe experiment. The intensity of the scattered light allows one to determine the expectation value of the witness operator. The collection efficiency can be increased when the probe field is the mode of a high-finesse resonator \cite{fig2}.}
\end{figure}
as in the form (3). In addition, it also allows one to determine all elements of the structure form matrix, therefore measuring the two-point correlation function of the spin system and hence giving access to the two-body density matrix.

Let us first denote by $|0\rangle_j$ and $|1\rangle_j$ the two states of the spin $j = 1, \ldots, N$ at position $r_j = j d \hat{x}$, where $d$ is the interparticle distance and the array is along the $\hat{x}$-axis. The spins are prepared in the state $\rho_0$ and the elements of Eq. (1) are evaluated over $\rho_0$. These elements can be found by measuring the intensity of a weak probe after implementing a dynamics governed by Hamiltonian

$$H_{\text{eff}} = \hbar \sum_j \varrho(t) \left( \alpha_0 e^{i\varphi} S_j^\dagger + \alpha_1 e^{-i\varphi} S_j \right) e^{i\mathbf{q} \cdot \mathbf{r}_j} + \text{H.c.},$$

where $S_j^\dagger = |1\rangle_j \langle 0|$, $S_j = |0\rangle_j \langle 1|$, such that $S_j^2 = S_j^\dagger + S_j$ and $S_j^2 = -i(S_j^\dagger - S_j)$. The parameters $\alpha_0, \alpha_1$ are real-valued frequencies, $\phi$ is a phase, $\mathbf{q}$ is the difference between probe and pump wave vectors, and $\alpha$ the annihilation operator of the probe field. The dimensionless function $\varrho(t)$ gives the temporal form of the excitation, which warrants that the dynamics is in the linear response regime, so that the scattered light depends on the initial state. Here, $\varrho(t) = 0$ for $t < 0$ and $\max[\varrho(t)] = 1$. Moreover, the pulse temporal duration is finite and the variation is sufficiently smooth to warrant the effective dynamics described in Eq. (4). In the following we will characterize its characteristic temporal duration by $\Delta t$.

Since the number of scattered photons at each experimental run must be small, a high collection efficiency of the scattered light is required. This can be achieved when the probe is the mode of a high-finesse resonator [6] and could be realized in existing experimental platforms, where atomic arrays [7, 8] or ion crystals [9] have been confined and strongly coupled with a cavity field. In the following we shall consider that the probe field is one mode of a high-finesse ring cavity, which strongly couples with an optical dipole transition.

In order to give a detailed derivation of Hamiltonian (4), we first specify the atomic states composing the spin transition of the atoms composing the array. We assume that the spin of a single atom is composed by radiatively stable states of the hyperfine multiplet of state $S_{1/2}$. This example is taken for convenience, since the treatment in the following can be extended to other pairs of radiatively stable states. We denote the spin states by $|0\rangle \equiv |S_{1/2}, -1/2\rangle$ and $|1\rangle \equiv |S_{1/2}, +1/2\rangle$, while the excited states (which will be needed in order to obtain the effective Hamiltonian) are $|e_0\rangle \equiv |P_{1/2}, -1/2\rangle$ and $|e_1\rangle \equiv |P_{1/2}, +1/2\rangle$. For simplicity, we assume that the states $|0\rangle, |1\rangle$ are degenerate and their energy is set to zero, while states $|e_0\rangle, |e_1\rangle$ are both at frequency $\omega_0$, so that the Hamiltonian for the relevant electronic degrees of freedom reads $H_{\text{el}} = \hbar \omega_0 \sum_j \langle |e_0\rangle_j \langle e_0| + |e_1\rangle_j \langle e_1| \rangle$. The electronic levels are coupled by cavity field and lasers as shown in Fig. 2. The pump lasers, in particular, have wave vector $\mathbf{k}_L$, frequency $\omega_L$, and couple transitions $|1\rangle_j \rightarrow |e_0\rangle_j$ and $|0\rangle_j \rightarrow |e_1\rangle_j$ with maximum Rabi frequency $\Omega_0$ and $\Omega_1$, respectively. The cavity mode, which acts as a probe, is linearly polarized and at frequency $\omega_c$ and wave vector $\mathbf{k}$. We denote by $a_1$ and $a_1^\dagger$ the annihilation and creation operators of a cavity-mode photon, and by $a_2$ and $a_2^\dagger$ the corresponding operators for the second, degenerate cavity mode at wave vector $-\mathbf{k}$. Both modes drive transitions $|0\rangle_j \rightarrow |e_0\rangle_j$ and $|1\rangle_j \rightarrow |e_1\rangle_j$ with vacuum-Rabi frequency $g$. The Hamiltonian governing the coherent dynamics of spins and cavity modes reads

$$H_L = -\hbar \delta_c \left( a_1^\dagger a_1 + a_2^\dagger a_2 \right) - \hbar \Delta \sum_j \left( \langle e_0\rangle_j \langle e_0| + \langle e_1\rangle_j \langle e_1| \right) + H_{\text{el}} + H_{\text{int}} + H_{\text{phase}}$$

and is reported in the reference frame rotating at the carrier frequency $\omega_L$ of the laser pulses, with $\Delta = \omega_L - \omega_0$ and $\delta_c = \omega_L - \omega_c$, respectively. The phase $\phi$ determines the relative phase between the fields, since we assumed that $\Omega_0, \Omega_1, g$ are real-valued. The incoherent dynamics due to the radiative decay of the excited state at rate $\gamma$ can be suppressed by driving far-off resonant Raman transitions, which corresponds to choosing values of the detuning such that $|\Delta| \gg g, \Omega_j, |\delta_c|, \kappa, \gamma$, where $\kappa$ is the linewidth of the cavity modes. In this limit the atomic
state follows adiabatically the cavity field, namely
\[ |0\rangle_f e_0 = \frac{q_0}{\Delta} |0\rangle_f |0\rangle \left( a_1 e^{i k \cdot r_f} + a_2 e^{-i k \cdot r_f} \right) \] (6)
\[ + \frac{\Omega_0}{\Delta} \frac{g(t)}{|0\rangle_f} e^{-i \phi} e^{i k_{L_f} \cdot r_f} \]
\[ |1\rangle_f e_1 = \frac{q}{\Delta} |1\rangle_f |1\rangle \left( a_1 e^{i k \cdot r_f} + a_2 e^{-i k \cdot r_f} \right) \] (7)
\[ + \frac{\Omega_1}{\Delta} \frac{g(t)}{|1\rangle_f} e^{-i \phi} e^{i k_{L_f} \cdot r_f} . \]

These equations hold at the lowest non-vanishing order in the perturbative expansion in $1/|\Delta|$ and when the width of the pulse is such that $\Delta \tau \gg 1/|\Delta|$. These values are substituted in the Heisenberg-Langevin equation for the cavity mode, $\dot{a}_f = -\kappa a_f + \sqrt{2\kappa} a_{\ell,f} + |a_{\ell,f}, H_L(\ell)\rangle / (i\hbar)$, where $a_{\ell,f}(t)$ is the input noise field, satisfying the relations $\langle a_{\ell,f}(t) \rangle = 0$, $\langle a_{\ell,in}(t) a_{\ell,in}^\dagger(t') \rangle = \delta_{k\ell} \delta(t-t')$ ($j, k = 1, 2$ [10]). Setting $a \equiv a_1$, in the linear response regime the solution reads
\[ a \simeq e^{-\left(\kappa-i\delta'_c\right) t} a(0) + \sqrt{2\kappa} \int_0^t dt' e^{-\left(\kappa-i\delta'_c\right)(t-t')} a_{\ell}(t') \]
\[ -\frac{q^2}{\Delta} \sum_j e^{-i 2k \cdot r_j} \left( 1 - e^{-\left(\kappa-i\delta'_c\right) t} \right) a_2(0) \]
\[ - f(t) \sum_j \left( a_0 e^{i \phi} S_j(0) + a_1 e^{-i \phi} S_j^\dagger(0) \right) e^{i q \cdot r_j} \] (8)
where $q_1 = k_{L_f} - k$, $a_0 = g \Omega_0 / \Delta$, $a_1 = g \Omega_1 / \Delta$, and $f(t) = i \int_0^t dt' e^{-\left(\kappa-i\delta'_c\right)(t-t')} a_{\ell}(t') \cdot \delta_c = 0 + \sqrt{2\kappa} a_{\ell}(t) + a_{\ell,in}(t)$. It is evaluated using Eq. (9), the properties of the input operators, and assuming that the cavity modes are initially in the vacuum field and can be cast in the form $I_{\text{out}} = 2k |f(t)|^2 (I_0 + I_{\text{int}})$, where $I_0$ is the sum of the intensities scattered by each atom, while $I_{\text{int}}$ is the intensity due to the interference between the scattered waves at different sites. In detail,
\[ I_0 = N \sum_{k=1}^N \langle S_k^x \rangle / N , \] (9)
and
\[ I_{\text{int}} = \sum_{k \neq l} e^{-iq_1 \cdot (r_k-r_l)} \left( |a_x|^2 \langle S_k^x S_l^x \rangle + |a_y|^2 \langle S_k^y S_l^y \rangle + |a_z|^2 \langle S_k^z S_l^z \rangle \right) \]
\[ + \alpha_{a_0}^* \alpha_{a_0} \langle S_k^a S_l^a \rangle + \alpha_{a_1}^* \alpha_{a_1} \langle S_k^a S_l^a \rangle . \] (10)

The coefficients depend on the lasers amplitudes and relative phase as follows
\[ \alpha_x = \alpha_1 e^{i \phi} + \alpha_0 e^{-i \phi} \]
\[ \alpha_y = \frac{\alpha_1 e^{i \phi} - \alpha_0 e^{-i \phi}}{2i} . \] (11)

The possibility to vary these coefficients gives access to the general structural witness [3], as can be easily shown using the identity
\[ \sum_{k \neq l} e^{-i q \cdot (r_k-r_l)} \langle S_k^a S_l^a \rangle = S^{a3}(\mathbf{q}) + S^{a3}(\mathbf{-q}) . \] (13)

For instance, the element $C^{yx}(\mathbf{q})$ is found by setting $\alpha_y = 0$ and $\alpha_x = \alpha$, which corresponds to choosing $\Omega_0 = \Omega_1$ and $\phi = 0$.

This setup allows also to measure the off-diagonal components in the matrix [11]. For example, by performing the measurement for different values of $\Omega_0, \Omega_1, \phi$ and on the intensity of the second cavity mode allows one to determine the components $S^{yz}(\mathbf{q})$ and $S^{yz}(\mathbf{q})$. The other elements involving also the $z$ component, namely $S^{yz}(\mathbf{q})$ and $S^{yz}(\mathbf{q})$, are found by the same measurement procedure after the Hadamard transformation $U_{\text{H}}^a = \otimes_i (S_i^x + S_i^y) / \sqrt{2} (U_{\text{H}}^b = \otimes_j (S_j^y + S_j^z))$ has been made on the qubits of the initial state: the density matrix over which one performs the trace in Eqs. (9) and (10) is now $\rho' = U_{\text{H}}^a \rho U_{\text{H}}^b$, which corresponds to a Raman pulse composed by classical fields [12]. The net effect is hence to transform the operators $S_j^x$ into $S_j^y$ ($S_j^y$ into $S_j^z$) and vice versa, so that, for example, $\text{Tr}\{S_i^x S_i^y \rho'\} = \text{Tr}\{S_i^y S_i^z \rho\}$. In this way also the term $C^{yz}(\mathbf{q})$ of the general structural witness can be accessed, as well as all other elements needed in order to reconstruct the two-body density matrix.

These results show that in the case of Dicke states the entanglement witness [2] can be determined by controlling the amplitudes and phases of the driving laser. One possibility for extracting the value at $\mathbf{q} = \mathbf{0}$ is when the array is along the cavity axis and orthogonal to the laser direction, setting $d \mathbf{k} \cdot \mathbf{k} = 2\pi n$, where $n$ is an integer and we assumed that the chain is along the $x$ axis. We note that localization about the minima is required, so that there is no hopping from site to site. The effect of the vibrations about the minima will tend in general to lower the interferometric contrast, however in the Lamb-Dicke regime this detrimental effect is small [13].

In summary, we have presented a specific setup to implement a recently proposed multipartite entanglement detection method [4] in a chain of atoms or ions in an optical lattice. The example we provide here shows explicitly how the measurement could be performed with current-day technology [17,9]. These calculations can be extended to other setups based on the atomic microscopes in Ref. [14], or on quantum non-demolition measurement like the ones discussed in Ref. [15]. We mention
that recently an entanglement detection method based on
the proposal [4] and on a similar protocol [16] has been
experimentally realized for the motional degrees of free-

dom of cold atoms in optical lattices [17].

The present entanglement detection scheme is based on
the measurement of two-point correlation functions, and
provides, amongst others, a determination of the mag-
netic susceptibility, which at thermodynamic equilibrium
and for specific Hamiltonians constitute a macroscopic
entanglement witness [18]. We conclude by pointing out
that measurements of higher order correlation functions
of the scattered field in the proposed system allows one to
determine higher order correlation functions of the spin
operators [19], which are at the basis of witnesses, for
instance, of GHZ-like multipartite states [20].

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