Consistent Maximum Likelihood Estimation Using Subsets with Applications to Multivariate Mixed Models

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Abstract

We present new results for consistency of maximum likelihood estimators with a focus on multivariate mixed models. Our theory builds on the idea of using subsets of the full data to establish consistency of estimators based on the full data. It requires neither that the data consist of independent observations, nor that the observations can be modeled as a stationary stochastic process. Compared to existing asymptotic theory using the idea of subsets we substantially weaken the assumptions, bringing them closer to what suffices in classical settings. We apply our theory in two multivariate mixed models for which it was unknown whether maximum likelihood estimators are consistent. The models we consider have non-stochastic predictors and multivariate responses which are possibly mixed-type (some discrete and some continuous).

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1 Introduction

Mixed models are frequently used in applications and have been the subject of numerous articles and books [8, 16, 22]. Yet, it was unknown until recently whether MLEs are consistent even in some simple generalized linear mixed models (GLMMs) [18]. What complicates proving consistency in some mixed models is the dependence among response variables induced by certain random effect designs. Of course, not all types of dependence between responses are problematic – there is a vast literature on maximum likelihood estimation with dependent observations [1, 7, 14, 15, 28, 30, 33]. But, as we will discuss in more detail below, for some commonly used random effect designs such as those with crossed random effects, existing conditions for consistency of MLEs are hard to verify [18]. In a few GLMMs with crossed random effects, consistency has been proved using a novel argument that relates the likelihood for the full data to that of a subset consisting of independent and identically distributed (i.i.d.) random variables, “the subset argument” [17].

Fundamentally, however, the issue is not unique to GLMMs or even mixed models; any other parametric model appropriate for the same settings may present similar difficulties. Accordingly, it was recognized in the first work on consistency using subsets that the idea has the potential to be extended to more general models [17]. We address this by deriving weaker conditions, based in part on the use of subsets, that are sufficient for consistency of MLEs, without assuming a particular model. They help explain formally what makes the subset argument work, why it is useful in some settings where more classical ones are not, and when it can fail. We illustrate the usefulness of our conditions by proving consistency of MLEs in two multivariate GLMMs (MGLMMs) to which existing theory has not been applied successfully.

To fix ideas, let \( \Theta \) denote a parameter set, \( f_\theta \) a joint density for the random vector \( Y = (Y_1, \ldots, Y_n) \), and \( \theta^0 \) the “true” parameter. Let also \( L_n(\theta; Y) = f_\theta(Y) / f_{\theta^0}(Y) \) and \( \Lambda_n(\theta; Y) = \log L_n(\theta; Y) \). If \( \Theta \) is a finite set, then since \( L_n(\theta^0; Y) = 1 \), a necessary and sufficient condition for consistency of MLEs is that, as \( n \to \infty \),

\[
P(L_n(\theta; Y) \geq 1) \to 0 \quad \text{for all } \theta \neq \theta^0.
\]

When \( \Theta \) is not a finite set, (1) needs to be amended by a uniformity argument to be sufficient, but the main ideas are the same. There are many ways to establish (1). With i.i.d. observations and regularity conditions, (1) or stronger results follow from the law of large numbers applied to \( n^{-1}\Lambda_n(\theta; Y) \) [6, 9, 10, 31]. More generally, if \( Y \)
is a stochastic process, \( \Lambda_n(\theta; Y) \) may, suitably scaled, satisfy an ergodic theorem, leading again to (i) under regularity conditions. In the literature on maximum likelihood estimation with dependent observations, it is often assumed that some such limit law holds, either for \( \Lambda_n(\theta; Y) \) or its derivatives \([7, 14, 15]\), or that the moments of \( \Lambda_n(\theta; Y) \) converge in an appropriate way \([1, 28]\). Unfortunately, in many practically relevant settings, it is not clear that any such convergence holds and proving that it does is arguably the main obstacle to establishing consistency of MLEs. Let us illustrate using an MGLMM, commonly considered both in statistics and applied sciences \([4, 5, 12, 21, 32]\).

Let \( X = [x_1, \ldots, x_n]^T \in \mathbb{R}^{n \times p} \) be a matrix of non-stochastic predictors, \( Z = [z_1, \ldots, z_n]^T \in \mathbb{R}^{n \times r} \) a non-stochastic design matrix, and \( U \in \mathbb{R}^r \) a multivariate normal vector of random effects, with mean zero and covariance matrix \( \Sigma \). For the MGLMM, \( \Theta \subseteq \mathbb{R}^d \), for some \( d \geq 1 \), \( \beta = \beta(\theta) \), and \( \Sigma = \Sigma(\theta) \). The responses \( Y_1, \ldots, Y_n \) are conditionally independent given \( U \), with conditional densities in the form

\[
f_{\theta,i}(y_i \mid u) = k_i(y_i, \tau_i) \exp \left( \frac{y_i[x_i^T \beta + z_i^T u] - c_i(x_i^T \beta + z_i^T u)}{\tau_i} \right),
\]

where, for \( i = 1, \ldots, n \), \( c_i \) is the conditional cumulant function, \( \tau_i \) a dispersion parameter, and \( k_i(y_i, \tau_i) \) ensures \( f_{\theta,i}(y_i \mid u) \) integrates to one. Conditional independence implies \( f_{\theta}^n(y \mid u) = \prod_{i=1}^n f_{\theta,i}(y_i \mid u) \). Several of the responses could be from the same subject, hence the “multivariate”, and they can be of mixed type, some continuous and some discrete, for example.

The dependence among the linear predictors is easily characterized since \( X\beta + ZU \sim \mathcal{N}(X\beta, Z\Sigma Z^T) \). The relevant density for maximum likelihood estimation, however, is the marginal density,

\[
f_{\theta}^n(y) = \int_{\mathbb{R}^r} f_{\theta}^n(y \mid u) \phi_\theta(u) du,
\]

where \( \phi_\theta \) denotes the \( r \)-dimensional multivariate normal density with mean zero and covariance matrix \( \Sigma = \Sigma(\theta) \). The density \( f_{\theta}^n(y) \) typically does not admit a closed form expression. Moreover, the dependence among responses it implies is in general less transparent than that among the linear predictors. What we can say in general is that two responses are dependent only if their corresponding linear predictors are. That is, response component \( i \) and \( j \) are independent if \( z_i^T \Sigma z_j = 0 \).

It is convenient if \( Z\Sigma Z^T \) is, upon possible reordering of the responses, block diagonal since in that case the full vector of responses can be partitioned into independent sub-vectors. If these are of fixed length as \( n \) grows then one is back in

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the classical setting where the full data consists only of an increasing number of independent vectors. This setting is common to many articles on asymptotic theory in mixed models \[13, 24, 25, 29\]. Unfortunately, in applications the number of independent response vectors – the number of diagonal blocks in \(Z\Sigma Z^T\) – is often small. For example, Sung and Geyer \[29\] note that in the famous salamander data \[20\] there are 3 independent vectors, each of length 120. Thus, in their notation there are \(n = 3\) independent observations, but in our notation there are \(n = 3 \times 120 = 360\) possibly dependent observations. It seems more reasonable, then, to assume that \(n \to \infty\) without also assuming that the response vector \(Y\) consists only of an increasing number of independent sub-vectors. This type of limiting process has, in the context of mixed models, previously only been investigated carefully in special cases that do not allow for predictors or mixed-type responses \[17, 23\]. To be sure, Jiang’s \[17\] general theory does allow for predictors, but the specific applications do not.

The intuition behind the usefulness of the subset argument can be understood by considering the following simple LMM with crossed random effects. Suppose
\[
Y_{i,j} = \theta + U^{(1)}_i + U^{(2)}_j + E_{i,j},
\]
where \(U^{(1)}_i, U^{(2)}_j,\) and \(E_{i,j}\) are all i.i.d. standard normal, \(i = 1, \ldots, N, j = 1, \ldots, N\). It is easy to check that the \(Y_{i,j}\)'s cannot be partitioned into independent subsets. However, there are many subsets that, even though there is dependence among them, consist of independent random variables. For example, the two subsets \((Y_{1,1}, Y_{2,2}, \ldots, Y_{N,N})\) and \((Y_{1,2}, Y_{2,3}, \ldots, Y_{N-1,N})\) are dependent, but taken separately they both consist of i.i.d. random variables. The MLE of \(\theta\) based on either subset, i.e. a subset sample mean, is consistent as \(N \to \infty\). Intuitively, then, the MLE based on all of the \(N^2\) variables should be too. Of course, the subset argument is not needed to prove that in this simple example, but the intuition is the same for models where a direct proof is harder. How to formalize this intuition, without actually having to require the subset components to be either independent or identically distributed, is the topic of Section 2.

The rest of the paper is organized as follows. We develop our theory for consistency of MLEs using subsets in Section 2. These results do not assume any particular model. In Section 3 we apply the theory from Section 2 to two MGLMMs. Section 4 contains a brief discussion of our results and their implications. Many technical details are deferred to the appendices.
2 Consistency using subsets of the full data

Recall that \( Y = (Y_1, \ldots, Y_n) \) denotes a collection of random variables and let \( W = (W_1, \ldots, W_m) \) denote a collection of random variables that form a subset of those in \( Y \), i.e. \( \{W_1, \ldots, W_m\} \subseteq \{Y_1, \ldots, Y_n\} \). We will henceforth call \( W \) a subcollection of \( Y \) to avoid confusion with other subsets introduced later. The main results in this section give conditions for when subcollections can be used to prove consistency of maximizers of \( L_n(\theta; Y) \). Unless otherwise noted, all convergence statements are as \( n \) tends to infinity and the number of elements in a subcollection, \( m = m(n) \), tends to infinity as a function of \( n \).

All discussed random variables are defined on an underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\), with the elements of \( \Omega \) denoted \( \omega \). The parameter set \( \Theta \) is assumed to be a subset of a metric space \((T, d_T)\). We write, for any \( t \in T \) and \( \delta > 0 \), \( B_\delta(t) = \{t' \in T : d_T(t, t') < \delta\} \). For any \( A \subseteq T \), \( \bar{A} \) denotes its closure and \( \partial A \) its boundary. We assume the true parameter \( \theta_0 \) is the same for all \( n \) but the joint density \( f^n_{\theta_0}(y) \) of \( Y \), against a dominating, \( \sigma \)-finite product measure \( \nu = \nu_n \), can depend on \( n \) in an arbitrary manner. In particular, our setting allows for a triangular array of responses, \( Y_{n,1}, \ldots, Y_{n,n} \), though for convenience we do not make this explicit in the notation.

By \( \theta_0 \) being the true parameter we mean that \( \mathbb{P}(Y \in A) = \int_A f^n_{\theta_0}(y) \nu(dy) \) for any measurable \( A \) in the range space of \( Y \). That is, expectations and probabilities with respect to \( \mathbb{P} \) are the same as those taken with respect to distributions indexed by \( \theta_0 \). Densities for the subcollection and its components are denoted by \( g \) in place of \( f \); for example, \( L_m(\theta; W) = g^m_{\theta}(W)/g^m_{\theta_0}(W) \).

We will use subcollections to establish the following sufficient condition for consistency of maximizers of \( L_n(\theta; Y) \):

\[
\mathbb{P} \left( \sup_{\theta \in \Theta \cap B_\varepsilon(\theta_0)^c} L_n(\theta; Y) \geq 1 \right) \to 0, \quad \forall \varepsilon > 0.
\]

(2)

The appeal of using subcollections to prove (2), instead of directly working with the full data likelihood \( L_n(\theta; Y) \), can be explained using the following lemma.

**Lemma 2.1.** For every \( c \in (0, \infty) \), \( \theta \in \Theta \), and subcollection \( W \), \( \mathbb{P} \)-almost surely,

\[
\mathbb{P} \left( L_n(\theta; Y) \geq c \mid W \right) \leq c^{-1} L_m(\theta; W).
\]

Versions of Lemma 2.1 are well known [17, 18], but Appendix A contains a proof for completeness. From the lemma it follows that if \( L_m(\theta; W) \to 0 \), then
\[ \mathbb{E}[P(L_n(\theta; Y) \geq 1 \mid W)] = P(L_n(\theta; Y) \geq 1) \to 0 \] by dominated convergence. That is, up to a uniformity argument, (2) can be established by showing that the likelihood of the subcollection converges to zero in probability, outside of a neighborhood of \( \theta^0 \). Uniform versions of that convergence will play a crucial role in our results.

**Definition 2.1.** We say that a subset \( A \subseteq \Theta \) is identified by a subcollection \( W \) if

\[ \sup_{\theta \in A} L_m(\theta; W) \to 0. \]

If \( \sup_{\theta \in A} L_m(\theta; W) = O_p(a_n) \) for some sequence of constants \( \{a_n\}, n = 1, 2, \ldots, \) we call \( a_n \) an identification rate.

To understand this definition better, consider the case where the subcollection \( W \) consists of \( m \) i.i.d. random variables with common marginal density \( g_{\theta,1} \). Suppose also that there is no \( \theta \in A \) for which \( g_{\theta,1} = g_{\theta^0,1} \) \( \nu \)-almost everywhere. That is, \( \theta^0 \) is an identified parameter in the classical sense if we restrict attention to the parameter set \( A \cup \{\theta^0\} \). Then, under regularity conditions [10, Theorems 16 and 17], one has \( \sup_{\theta \in A} \mathbb{E}[\Lambda_m(\theta; W)] < 0 \) and, by a uniform strong law of large numbers,

\[ \lim_{m \to \infty} m^{-1} \sup_{\theta \in A} |\Lambda_m(\theta; W) - \mathbb{E}[\Lambda_m(\theta; W)]| = 0. \]

Using this, it is straightforward to show that \( A \) is identified by \( W \) with an identification rate that is exponentially fast in \( m \). That is, with i.i.d. components and regularity conditions, the classical definition of an identified parameter implies identification in the sense of Definition 2.1. However, we want to allow for subcollections that do not consist of i.i.d. components, and in that case the classical definition is not as useful. For example, we have independent but not identically distributed components in one of our MGLMMs. In this and more general cases, a parameter could be identified in the classical sense for all sample sizes \( n \), but, loosely speaking, the difference between the distributions for \( W \) indexed by some \( \theta \in A \) and that indexed by \( \theta^0 \) could vanish asymptotically, preventing \( W \) from identifying \( A \) in our sense. Finally, notice also that \( A \) being identified by \( W \) is essentially equivalent to MLEs based on \( W \) with the restricted parameter set \( A \cup \{\theta^0\} \) being consistent.

We can now be more precise about how to use subcollections to establish (2). The strategy is to first find a subcollection \( W \) that identifies \( B_\varepsilon(\theta^0)^c \cap \Theta \) for every \( \varepsilon > 0 \), and then use Lemma 2.1 to get the convergence for the full likelihood in (2). For this strategy to be useful, showing that \( W \) identifies \( B_\varepsilon(\theta^0)^c \cap \Theta \) has to be easier than showing that \( Y \) does since the latter would directly imply (2). That is, one has to be able to pick out a subcollection with more convenient properties than the full data. Our applications in Section 3 illustrate how this can be done.

It is useful to allow for several subcollections \( W^{(i)} \), consisting of \( m_i \) components, and subsets \( A_i, i = 1, \ldots, s \). By doing so, different subcollections can be used to
identify different subsets of the parameter set. For example, if the parameter set
is a product space, as is common in applications, then different subcollections
can be used to, loosely speaking, identify different elements of the parameter vector.
Assumption 1 makes precise what we need to identify $\Theta \cap B_\varepsilon(\theta^0)^c$
using several subcollections.

**Assumption 1.** For every small enough $\varepsilon > 0$, there are subsets $A_i = A_i(\varepsilon) \subseteq \Theta$
and corresponding subcollections $W(i), i = 1, \ldots, s$, such that $\cup_{i=1}^s A_i \supseteq \Theta \cap B_\varepsilon(\theta^0)^c$
and each $A_i$ is identified by $W(i)$ with some identification rate $a_{n,i}, n = 1, 2, \ldots, i = 1, \ldots, s$.

This assumption is somewhat similar to assumptions A2 and A3 made by Jiang [17], which are also assumptions about parameter identification using several subcollections. However, those assumptions are stated in terms of $E(\Lambda_{m_i}(\theta; W(i)))$
and $\text{Var}(\Lambda_{m_i}(\theta; W(i))), i = 1, \ldots, s$. The fact that we do not have to assume any-
thing about the variances of the log-likelihood ratios is an important improvement.
For example, if subcollection $i$ consists of i.i.d. components, the convergence of
$m_i^{-1}\Lambda_{m_i}(\theta; W(i))$ is immediate from the law of large numbers, but calculating its
variance may be difficult.

For finite parameter sets, Assumption 1 is enough to give consistency of MLEs
via Lemma 2.1. For more general cases we also need to control the regularity of the
log-likelihood for the full data. The following two assumptions are made to ensure
that the uniformity of the convergence detailed in Assumption 1 and Definition 2.1
carries over to $\Lambda_n(\theta; Y)$, in the sense of (2).

**Assumption 2.** For every $i \in \{1, \ldots, s\}$ and $n \in \{1, 2, \ldots\}$, $\Lambda_n(\theta; Y)$ is $P$-almost
surely Lipschitz continuous in $\theta$ on the $A_i$ defined in Assumption 1; that is, there
exists a random variable $K_{n,i}$ not depending on $\theta$ such that, $P$-almost surely and
for every $\theta, \theta' \in A_i$,

$$\left|\Lambda_n(\theta; Y) - \Lambda_n(\theta'; Y)\right| \leq K_{n,i} d_T(\theta, \theta').$$

**Assumption 3.** Each $A_i$ from Assumption 1 can be covered by $M_{n,i}$ balls of radius
$\delta_{n,i}$ such that

$$K_{n,i} \delta_{n,i} \xrightarrow{P} 0 \text{ and } M_{n,i} a_{n,i} \rightarrow 0,$$

where $a_{n,i}$ and $K_{n,i}, i = 1, \ldots, s, n = 1, 2, \ldots$, are the same as in Assumptions 1
and 2 respectively.

The assumptions give us the convergence in (2) and, consequently, the following lemma.
Lemma 2.2. If Assumptions 1 – 3 hold, then the probability that there exists a global maximizer of \( \Lambda_n(\theta; Y) \) in \( B_\varepsilon(\theta^0)^c \cap \Theta \) tends to zero as \( n \to \infty \), for every \( \varepsilon > 0 \).

Proof. We give an outline here and a detailed proof in Appendix A. Without loss of generality, we may assume \( s = 1 \), so there is one subcollection \( W \) that identifies \( A = \Theta \cap B_\varepsilon(\theta^0)^c \), for arbitrary, small \( \varepsilon > 0 \), with rate \( a_n \). It suffices to prove that \( P(\sup_{\theta \in A} L_n(\theta; Y) \geq 1) \to 0 \). For \( j = 1, \ldots, M_n \) let \( \theta^j \) be a point in the intersection of \( A \) and the \( j \)th ball in the cover of \( A \) given by Assumption 3. Some algebra and Assumption 2 gives

\[
P \left( \sup_{\theta \in A} L_n(\theta; Y) \geq 1 \right) \leq P \left( \max_{j \leq M_n} L_n(\theta^j; Y) \geq 1/2 \right) + P \left( e^{K_n \delta_n} \geq 2 \right).
\]

The second term is \( o(1) \) by Assumption 3. It remains to deal with the first. By conditioning on the subcollection and using Lemma 2.1 one gets

\[
P \left( \max_{j \leq M_n} L_n(\theta^j; Y) \geq 1/2 \mid W \right) \leq 2M_n \sup_{\theta \in A} L_{\max}(\theta; W).
\]

The right hand side is \( o(1) \) by Assumption 3 so the expectation of the left hand side is also \( o(1) \) by dominated convergence, which finishes the proof. \( \square \)

We will use Lemma 2.2 to establish both a Wald-type consistency, meaning consistency of sequences of global maximizers of \( L_n(\theta; Y) \), and a Cramér-type consistency, meaning consistency of a sequence of roots to the likelihood equations \( \nabla \Lambda_n(\theta; Y) = 0 \). It follows almost immediately from the lemma that if \( L_n(\theta; Y) \) has a global maximizer \( \hat{\theta}_n \), \( P \)-almost surely for every \( n \), then \( \hat{\theta}_n \xrightarrow{P} \theta^0 \). In particular, if \( \Theta \) is compact one gets Wald-type consistency with an additional continuity assumption. Since Assumption 2 implies \( L_n(\theta; Y) \) is continuous at every point except possibly \( \theta^0 \), assuming continuity also at the unknown \( \theta^0 \) should be insignificant in any application of interest.

Theorem 2.3. If \( \Theta \) is compact, \( L_n(\theta; Y) \) is \( P \)-almost surely continuous on \( \Theta \) for every \( n \), and Assumptions 1 – 3 hold, then a maximizer \( \hat{\theta}_n \) of \( L_n(\theta; Y) \) exists \( P \)-almost surely for every \( n \), and \( \hat{\theta}_n \xrightarrow{P} \theta^0 \) for any sequence of such maximizers.

Proof. Since continuous functions attain their suprema on compact sets, \( L_n(\theta; Y) \) has a maximizer on \( \Theta \), \( P \)-almost surely. By Lemma 2.2 all maximizers are in \( B_{\varepsilon}(\theta^0) \) with probability tending to one, for all small enough \( \varepsilon > 0 \). \( \square \)
Though compactness is a common assumption \[15, 34\], it is sometimes too restrictive or even unnecessary. If \(L_n(\theta; Y)\), or more commonly \(\Lambda_n(\theta; Y)\), is strictly concave in \(\theta\) on a convex \(\Theta\), then it is enough to verify the assumptions on a neighborhood of \(\theta^0\) (c.f. Theorem 2.4) to get consistency of the unique global maximizer. However, a global maximizer need not exist even as \(n \to \infty\), or perhaps the assumptions cannot be verified for other reasons. With a few additional assumptions, Lemma 2.2 can then be used to get the weaker Cramér-type consistency, which also only requires verifying assumptions for neighborhoods of \(\theta^0\).

**Theorem 2.4.** If \(\Theta \subseteq \mathbb{R}^d\) for some \(d \geq 1\), \(L_n(\theta; Y)\) is almost surely differentiable in \(\theta\) on a neighborhood of an interior \(\theta^0\) for every \(n\), and Assumptions 4 - 6 hold with \(\Theta\) replaced by \(\tilde{B}_\varepsilon(\theta^0)\) for all small enough \(\varepsilon > 0\), then, with probability tending to one as \(n \to \infty\), there exists a local maximizer of \(L_n(\theta; Y)\), and hence a root to the likelihood equation \(\nabla \Lambda_n(\theta; Y) = 0\), in \(B_\varepsilon(\theta^0)\), for all small enough \(\varepsilon > 0\).

**Proof.** Since \(\theta^0\) is interior we may assume \(\varepsilon > 0\) is small enough that all points of \(\tilde{B}_\varepsilon(\theta^0)\) are interior. Almost sure differentiability of \(L_n(\theta; Y)\) implies almost sure continuity. Thus, \(L_n(\theta; Y)\) attains a local maximum on the compact \(\tilde{B}_\varepsilon(\theta^0)\), \(P\)-almost surely. By Lemma 2.2 with probability tending to one, there are no such maximizers in \(\tilde{B}_\varepsilon(\theta^0) \setminus B_\varepsilon(\theta^0) = \partial B_\varepsilon(\theta^0)\). Thus, with probability tending to one, there exists a local maximizer in \(B_\varepsilon(\theta^0)\). Since \(L_n(\theta; Y)\) and hence \(\Lambda_n(\theta; Y)\) is \(P\)-almost surely differentiable, any such maximizer must be a root to the likelihood equation \(\nabla \Lambda_n(\theta; Y) = 0\).

In the next section we apply Theorem 2.4 to two special cases of the MGLMM described in Section 1. We also discuss in more detail how to think about the subcollections and subsets in specific models.

### 3 Application to multivariate mixed models

#### 3.1 Longitudinal linear mixed model

The first model we consider is an extension of the variance components model that has been studied previously [23]. In addition to dependence between subjects induced by crossed random effects the model incorporates autoregressive temporal dependence between measurements from the same subject. To make the discussion clearer we assume easy-to-specify fixed and random effect structures. This allows us to focus on the core issues, that is, on how to select subcollections and subsets that
can be used to verify the conditions of our theory. Our model includes a baseline mean and a treatment effect. A general fixed effect design matrix could be treated the same way as in our second example, discussed in Section 3.2. Before establishing consistency, we discuss the model definition and how to select appropriate subcollections.

Suppose for subjects \((i, j)\), \(i = 1, \ldots, N\), \(j = 1, \ldots, N\), and time points \(t = 1, \ldots, T\) we observe the response \(Y_{i,j,t}\), where for convenience we assume both \(N\) and \(T\) are even. Let the stacked vector of responses be

\[
Y = [Y_{1,1,1}, \ldots, Y_{1,1,T}, Y_{1,2,1}, \ldots, Y_{N,N,T}]^T \in \mathbb{R}^n, \quad n = TN^2.
\]

Recall from the introduction that the MGLMM is specified by the conditional distribution \(f_\theta(y \mid u)\) and the distribution of the random effects, \(\phi_\theta(u)\). For a linear mixed model we let \(f_\theta(y \mid u)\) be a multivariate normal distribution with mean \(X\beta + Zu\) and covariance matrix \(\theta_3I_N\), \(\theta_3 > 0\), where the two components of \(\beta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^T \in \mathbb{R}^2\) are a baseline mean and a treatment effect, respectively, and \(I_k\) denotes the \(k \times k\) identity matrix. Note, in the notation of the introduction, the dispersion parameter in the conditional distribution is \(\tau_i = \theta_3\), for all \(i\). We treat \(\theta_3\) as a parameter to be estimated and not as known, which is otherwise common in the literature.

Let \(h_n\) be a vector of zeros and ones where the \(i\)th element is one if it corresponds to an observation in time \(t \leq T/2\) and zero otherwise and let \(1_n\) denote an \(n\)-vector of ones. We take \(X = [1_n, h_n] \in \mathbb{R}^{TN^2 \times 2}\), which corresponds to a treatment being applied in the first half of the experiment. Notice that unless \(T\) is fixed, which we do not assume, this setup implies the predictors change with \(n\). Indeed, as \(T\) grows, a particular observation can go from being made in the latter half of the experiment to the earlier half. Thus, the responses form a triangular array.

Partition \(U\) into three independent sub-vectors, \(U^{(1)} \sim \mathcal{N}(0, \theta_4I_N)\), \(U^{(2)} \sim \mathcal{N}(0, \theta_5I_N)\), and \(U^{(3)} \sim \mathcal{N}(0, \theta_6I_{N^2} \otimes \Psi)\), where \(\Psi = (\Psi_{i,j}) = (\theta_i^{i-j})\) is a first order autoregressive correlation matrix, \(\theta_i > 0, i = 4, 5, 6\), and \(\theta_7 \in (-1, 1)\). We will use \(U^{(1)}\) and \(U^{(2)}\) as crossed random effects, inducing dependence between subjects, and \(U^{(3)}\) to get temporal dependence within subjects. To that end, let \(Z_1 = I_N \otimes 1_N \otimes 1_T, Z_2 = 1_N \otimes I_N \otimes 1_T,\) and \(Z = [Z_1, Z_2, I_{TN^2}]\). Then, with \(J_k = 1_k1_k^T\), the covariance matrix of the linear predictors \(X\beta + ZU\) is

\[
Z\Sigma Z^T = \theta_4I_N \otimes J_{NT} + \theta_5I_N \otimes I_N \otimes J_T + \theta_6I_{N^2} \otimes \Psi.
\]
More transparently, for the elements of $E(Y \mid U) = X\beta + ZU$, it holds that

$$\text{cov}[E(Y_{i,j,t} \mid U), E(Y_{i',j',t'} \mid U)] = \begin{cases} 
\theta_4 + \theta_5 + \theta_6\theta_7^{[t-t']} & i = i', j = j' \\
\theta_4 & i = i', j \neq j' \\
\theta_5 & i \neq i', j = j' \\
0 & \text{otherwise}
\end{cases}$$

The marginal density $f_{\theta}^n(y)$ admits a closed form expression in this example. Specifically, the marginal distribution for $Y$ is multivariate normal with mean $m(\theta) = X\beta(\theta)$ and covariance matrix $C(\theta) = \theta_3 I_{TN^2} + Z\Sigma(\theta)Z^T$. Note that the structure of $C(\theta)$ is similar to that of the covariance matrix of the linear predictors just discussed. In particular, there are many zeros in the covariance matrix $C(\theta)$, i.e. there are many independent observations, but $Y$ cannot be partitioned into independent vectors.

### 3.1.1 Subcollection selection

The model definitions imply that $\Theta = \mathbb{R} \times \mathbb{R} \times (0, \infty) \times (0, \infty) \times (0, \infty) \times (-1, 1)$, a subset of $(T, d_T) = (\mathbb{R}^7, \| \cdot \|)$, where $\| \cdot \|$ denotes the Euclidean norm when applied to vectors. We write $\theta = (\theta_1, \ldots, \theta_7)$.

Subcollections are selected for the purpose of verifying Assumption IX. The main idea guiding selection is suggested by the fact that identification follows, under regularity conditions, if the subcollection’s log-likelihood satisfies a law of large numbers. We will use $s = 2$ such subcollections and require that they together identify $\theta$ in the classical sense. By this we mean that, letting $\nu_\theta^i$ denote the distribution of subcollection $i$ implied by parameter $\theta$,

$$\{ \theta \in \Theta : \nu_\theta^1 = \nu_{\theta^0}^{\nu_1} \} \cap \{ \theta \in \Theta : \nu_\theta^2 = \nu_{\theta^0}^{\nu_2} \} = \{ \theta^0 \}.$$ 

With these properties in mind, we take $W^{(1)}$ to consist of the vectors $W_i^{(1)} = (Y_{2i-1,1}, Y_{2i,1}, Y_{2i,2}, Y_{2i,T}) \in \mathbb{R}^2, i = 1, \ldots, N/2$. Because these vectors do not share any random effects, they are independent. In fact, they are i.i.d. multivariate normal with common mean $m_1(\theta) = [\theta_1 + \theta_2, \theta_4]^T$ and common covariance matrix $C_1(\theta) = I_2(\theta_3 + \theta_4 + \theta_5 + \theta_6)$. Clearly, $\theta_1$ and $\theta_2$ are identified in the classical sense by this subcollection, but not $\theta_3, \ldots, \theta_7$. Note that even though the predictors, and hence the distributions, do not change with $N$ for this subcollection, it is strictly speaking a triangular array unless $T$ is fixed.
To identify the remaining parameters, take $W^{(2)}$ to consist of the vectors

$$W_i^{(2)} = (Y_{2i-1,2i-1,1}, Y_{2i-1,2i-1,2}, Y_{2i-1,2i-1,3}, Y_{2i-1,2i-1,1}, Y_{2i,2i-1,1}), \quad i = 1, \ldots, N/2.$$  

These are also i.i.d. multivariate normal, with common mean $m_2(\theta) = (\theta_1 + \theta_2)1_5$ and common covariance matrix

$$C_2(\theta) = \begin{bmatrix}
\sum_{i=3}^{6} \theta_i & \theta_4 + \theta_5 + \theta_6 \theta_7 & \theta_4 + \theta_5 + \theta_6 \theta_7^2 & \theta_4 & \theta_5 \\
\cdot & \sum_{i=3}^{6} \theta_i & \theta_4 + \theta_5 + \theta_6 \theta_7 & \theta_4 & \theta_5 \\
\cdot & \cdot & \sum_{i=3}^{6} \theta_i & \theta_4 & \theta_5 \\
\cdot & \cdot & \cdot & \sum_{i=3}^{6} \theta_i & 0 \\
\cdot & \cdot & \cdot & \cdot & \sum_{i=3}^{6} \theta_i
\end{bmatrix}.$$  

It is straightforward to check that $C_2(\theta) = C_2(\theta')$ implies $\theta_i = \theta'_i$, $i = 3, \ldots, 7$.

In summary, the two subcollections together identify $\theta$ in the classical sense. Moreover, since both subcollections consist of i.i.d. multivariate normal vectors, their log-likelihoods satisfy a law of large numbers as $N \to \infty$. With this we are equipped to verify that Assumptions 1–3 hold locally, leading to the main result of the section in Theorem 3.4.

### 3.1.2 Consistency

The purpose of this section is to verify the conditions of Theorem 2.4. The interesting part of that is to check that Assumptions 1–3 hold with $\Theta$ replaced by $\tilde{B}_\varepsilon(\theta^0)$, for all small enough $\varepsilon > 0$. For this purpose we will first prove two lemmas that roughly correspond to Assumptions 1 and 2. The limiting process we consider is that $N$ tends to infinity while $T$ can be fixed or tend to infinity with $N$, at rates discussed below. Thus, the statements $n \to \infty$ and $N \to \infty$ are equivalent. We will need the following result which is proved in Appendix B.

**Proposition 3.1.** If $\Theta$ is compact, $L_{m_i}(\theta; W^{(i)})$ is continuous in $\theta$ on $\Theta$ for every $w^{(i)}$ in the support of $W^{(i)}$, $i = 1, \ldots, s$, and $\cap_{i=1}^{s} \{\theta \in \Theta : \nu_{\theta}^i = \nu_{\theta^0}^i\} = \{\theta^0\}$, then for any $\varepsilon > 0$ there are compact sets $\tilde{A}_1, \ldots, \tilde{A}_s$ such that $\{\theta \in \Theta : \nu_{\theta}^i = \nu_{\theta^0}^i\} \cap \tilde{A}_i = \emptyset$, $i = 1, \ldots, s$, and $\cup_{i=1}^{s} \tilde{A}_i = \Theta \cap \tilde{B}_\varepsilon(\theta^0)^c$.

Note, when applying the proposition in the present application, $m_i = N$, $s = 2$, and $\Theta$ is replaced by $\tilde{B}_\varepsilon(\theta^0)$. The proposition is useful because the $\tilde{A}_i$s it gives are compact, as we will see in the proof of the following lemma. This lemma formalizes verification of Assumption 1.
**Lemma 3.2.** If \( \theta^0 \) is an interior point of \( \Theta \), then for all small enough \( \varepsilon > 0 \) there exist subsets \( A_1 \) and \( A_2 \) such that \( A_1 \cup A_2 = \partial B_\varepsilon(\theta^0) \),

1. \( N^{-1} \sup_{\theta \in A_i} E[\Lambda_{N/2}(\theta; W^{(i)})] = \sup_{\theta \in A_i} E[\Lambda_1(\theta; W^{(i)})]/2 < 0 \),
2. \( \mathbb{P} \)-almost surely, \( N^{-1} \sup_{\theta \in A_i} |\Lambda_{N/2}(\theta; W^{(i)})| - E[\Lambda_{N/2}(\theta; W^{(i)})]| \to 0 \), and, consequently;
3. \( A_i \) is identified by \( W^{(i)} \) with an identification rate \( a_{n,i} = o(\varepsilon^{e^{-\varepsilon N(n)}}) \) for some \( \epsilon > 0 \), \( i = 1,2 \).

**Proof.** We give an outline here and a detailed proof in Appendix B. It is easy to check that the requirements of Proposition 3.1 are satisfied with \( \Theta \) replaced by \( \bar{B}_\varepsilon(\theta^0) \). By taking the \( A_i \)s to be the \( \bar{A}_i \)s given by Proposition 3.1, proving points 1 – 2 is similar to proving that MLEs based on subcollection \( i \) are consistent if the parameter set is restricted to the compact set \( A_i \cup \{ \theta^0 \} \), \( i = 1,2 \). Since the subcollection components are i.i.d., this is straightforward using classical ideas [10, Theorems 16 and 17]. The only difference from the referenced work is that one subcollection is a triangular array and so we use a different strong law. Point 3 follows from points 1 and 2. \( \square \)

Note that, in this lemma and elsewhere, \( \epsilon \) is a small number that is defined in context whereas \( \varepsilon \) always denotes the radius of the neighborhood of \( \theta^0 \) we are considering. It remains to verify the assumptions concerned with the regularity of the log-likelihood of the full data. When the log-likelihood is differentiable, Lipschitz continuity follows from the mean value theorem if the gradient is bounded. The following lemma uses that to verify Assumption 2. The resulting Lipschitz constant, i.e. the bound of the gradient, is the same for both \( A_1 \) and \( A_2 \). The lemma also gives a probabilistic bound on the order of this Lipschitz constant as \( n \to \infty \) that will be useful when verifying Assumption 3.

**Lemma 3.3.** If \( \theta^0 \) is an interior point of \( \Theta \), then for every \( n \) and small enough \( \varepsilon > 0 \) there exists a random variable \( K_n \) such that, \( \mathbb{P} \)-almost surely,

\[
\sup_{\theta \in B_\varepsilon(\theta^0)} ||\nabla \Lambda_n(\theta; Y)|| \leq K_n = o_P(n^b),
\]

for some \( b > 0 \).

Proving Lemma 3.3 (see Appendix B) is largely an exercise in bounding the eigenvalues of the covariance matrix \( C(\theta) \) and its inverse on interior points of \( \Theta \). We are ready for the main result of the section.
Theorem 3.4. If $\theta^0$ is an interior point of $\Theta$ and $T = O(N^k)$ for some $k \geq 0$ as $N \to \infty$, then, $P$-almost surely, there exists a sequence $\hat{\theta}_n$ of roots to the likelihood equations $\nabla \Lambda_n(\theta; Y) = 0$ such that $\hat{\theta}_n \xrightarrow{P} \theta^0$.

Proof. We verify the conditions of Theorem 2.4. Fix an arbitrary $\epsilon > 0$. Since $\theta^0$ is interior we may assume $\epsilon$ is small enough that all points in $\bar{B}_\epsilon(\theta^0)$ are interior points of $\Theta$. By Lemma 3.3 $\ell_n(\theta; Y)$ is $P$-almost surely differentiable on $\bar{B}_\epsilon(\theta^0)$, so $\Lambda_n(\theta; Y) = \ell_n(\theta, Y) - \ell_n(\theta^0; Y)$ is, too. By Lemma 3.2 Assumption 1 holds with what is there denoted $\Theta$ replaced by $\bar{B}_\epsilon(\theta^0)$. The identification rate is exponentially fast in $\frac{N}{2}$, $a_n = o(e^{-\epsilon N})$ for some $\epsilon > 0$. Lemma 3.3 shows that $\Lambda_n(\theta; Y)$ is $K_n$-Lipschitz on both $A_1$ and $A_2$, and that $K_n = o_P(n^b)$ for some $b > 0$. This verifies Assumption 2. It remains only to verify that the rate conditions in Assumption 3 hold. The $\delta$-covering number of the sphere $\partial B_\epsilon(\theta^0)$ is $O((\epsilon/\delta)^{d-1})$ as $\delta \to 0$ [Lemma 1]. Thus, since $A_i \subseteq \partial B_\epsilon(\theta^0)$, by picking $\delta_{i,n} = n^{-b}$ we can have $M_{n,i} = O(n^{(d-1)b})$ as $n \to \infty$, $i = 1, 2$. Our choice of $\delta_{n,i}$ ensures $K_{n,i} \delta_{n,i} = K_n \delta_n = o_P(1)$, which is the first rate condition. Since the identification rate is exponential in $\frac{N}{2}$ for both subcollections, we have that $M_{n,i}a_{n,i} = O(N^{2b(d-1)Tb(d-1)}e^{-\epsilon N})$ for some $\epsilon > 0$, which is $o(1)$ as $N \to \infty$ since $T$ is of (at most) polynomial order in $N$. 

Two key components in the proof of Theorem 3.4 are that the subcollections consist of i.i.d. random vectors that identify the parameters and that the gradient of the log-likelihood is of polynomial order in $n$. We expect the same proof technique to work in many other cases. Essentially, all that is needed is that the subcollections grow faster than logarithmically in the total sample size $n$ and that the gradient is of polynomial order, no matter how large that order is.

It is possible that the assumption that $T = O(N^k)$, $k \geq 0$, could be relaxed by picking other subcollections that also make use of the variation in the time dimension. It is not trivial, however, since the dependence between any two responses sharing a random effect does not vanish as time between the observations increases. In the next section we examine how predictors and mixed-type responses affect the argument.

3.2 Logit-normal MGLMM

The model we consider in this section is an extension in several ways of the logistic GLMMs for which the technique based on subcollections was first developed [17]. The random effect structures are similar, i.e. crossed, but we have multivariate,
mixed-type responses, and predictors. The main ideas for verifying the assumptions of the theory from Section 2 are the same as in our LMM example. However, due to the inclusion of predictors, we use results from empirical process theory in place of the more classical strong laws used for the LMM. Showing existence of appropriate subsets of the parameter space that the subcollections identify also requires more work than with i.i.d. components. As before, we discuss the model definition and subcollection selection before establishing consistency.

Suppose for subjects \((i, j), i = 1, \ldots, N, j = 1, \ldots, N\), there are two responses, \(Y_{i,j,1}\) which is continuous and \(Y_{i,j,2}\) which is binary. The vector of all responses is

\[ Y = [Y_{1,1,1}, Y_{1,1,2}, Y_{1,2,1}, \ldots, Y_{N,N,2}]^T \in \mathbb{R}^n, \quad n = 2N^2. \]

For each subject we observe a vector of non-stochastic predictors \(x_{i,j} \in \mathbb{R}^p\), the same for both responses. Similarly, \(z_{i,j} \in \mathbb{R}^r\) is the same for both responses. Let 

\[ l_{i,j,k} = x_{i,j}^T \beta_k + z_{i,j}^T u \]

be the linear predictor, \(i = 1, \ldots, N, j = 1, \ldots, N, k = 1, 2\), where \(\beta_1 = [\theta_1, \ldots, \theta_p]^T, \beta_2 = [\theta_{p+1}, \ldots, \theta_{2p}]^T\). We assume that \(\|x_{i,j}\| \leq 1\) for all \(i, j\). In practice this only rules out the possibility that \(\|x_{i,j}\| = \infty\) since our setting allows for the standardization of predictors. The conditional density of the responses given the random effects that we consider is

\[
f_{\theta}^n(y \mid u) = (2\pi)^{-n/2} \exp \left[ -\sum_{i,j} (y_{i,j,1} - l_{i,j,1})^2/2 + y_{i,j,2} l_{i,j,2} - \log \left( 1 + e^{l_{i,j,2}} \right) \right].
\]

Given the random effects, \(Y_{i,j,1}\) is normal with mean \(l_{i,j,1}\) and variance 1, and \(Y_{i,j,2}\) is Bernoulli with success probability \(1/(1 + e^{-l_{i,j,2}})\) – a logistic GLMM. The choice of \(\tau_i = 1\) for all \(i\) is made for identifiability reasons for the Bernoulli responses, and for convenience for the normal responses. Setting the \(\tau_i\)s to some other known constants does not fundamentally change the results.

Consider two independent vectors, \(U^{(1)} \sim \mathcal{N}(0, \theta_d I_N)\) and \(U^{(2)} \sim \mathcal{N}(0, \theta_d I_N)\), and corresponding design matrices \(Z_1 = I_N \otimes 1_N \otimes 1_2\) and \(Z_2 = 1_N \otimes I_n \otimes 1_2\). With \(U = [U^{(1)}^T, U^{(2)}^T]^T\) and \(Z = [Z_1, Z_2]\) the linear predictors are 

\[ l_{i,j,k} = x_{i,j}^T \beta_k + u_{i}^{(1)} + u_{j}^{(2)}. \]

Thus, responses from the same subject share two random effects, responses from different subjects with one of the first two indexes in common share one random effect, and other responses share no random effects and are hence independent. The covariance matrix for the linear predictors is easily computed in the same way as in the LMM. The covariance matrix for responses, however, is less transparent. It is for simplicity that we assume in this section that all random effects have the same variance. It is not necessary for our theory to be operational but this simplification shortens proofs considerably and allows us to focus on the main ideas.
3.2.1 Subcollection selection

With \( p \) predictors the \((2p + 1)\)-dimensional parameter set is \( \Theta = \mathbb{R}^p \times \mathbb{R}^p \times (0, \infty) \), a subset of the metric space \((\mathbb{R}^d, \| \cdot \|)\). The intuition behind the selection of subcollections is that the normal responses should identify the coefficient \( \beta_1 \) and the variance parameter \( \theta_d \). Similarly, the Bernoulli responses should identify the coefficient vector \( \beta_2 \). With that in mind we take \( W^{(i)} = (Y_{1,1,i}, Y_{2,2,i}, \ldots, Y_{N,N,i}) \), \( i = 1, 2 \). Both of these subcollections consist of independent but not identically distributed random variables – independence follows from the fact that no components in the same subcollection share random effects. Notice that these subcollections are in practice often triangular arrays since the predictors may need to be scaled by \( 1/\max_{i \leq j \leq N} \| x_{i,j} \| \) to satisfy \( \| x_{i,j} \| \leq 1 \). All responses in the first subcollection have marginal normal distributions and all responses in the second have marginal Bernoulli distributions.

Identification is more complicated than in our previous example. One issue is that there can be many \( \theta_d \) and \( \beta_2 \) that give the same marginal success probability for the components in the second subcollection. A second issue is that, since the predictors can change with \( n \), classical identification for a fixed \( n \) does not necessarily lead to identification in the sense of Definition 2.1. Additionally, the approach used in the LMM to find appropriate subsets \( A_1 \) and \( A_2 \) by means of Proposition 3.1 only works in general when the subcollection components are i.i.d. Thus, we take a slightly different route to establishing consistency compared to the LMM.

3.2.2 Consistency

In this section we verify the conditions of Theorem 2.4. The limiting process is that \( N \to \infty \), which is equivalent to \( n \to \infty \) since \( n = 2N^2 \). We will first prove two lemmas that roughly correspond to Assumptions [1 and 2]

Let \( \lambda_{\min}(\cdot) \) denote the minimum eigenvalue of its matrix argument.

**Lemma 3.5.** If \( \theta^0 \) is an interior point of \( \Theta \) and

\[
\liminf_{N \to \infty} \lambda_{\min} \left( N^{-1} \sum_{i=1}^{N} x_{i,i} x_{i,i}^\top \right) > 0,
\]

then for all small enough \( \varepsilon > 0 \) there exist \( A_1 \) and \( A_2 \) such that \( A_1 \cup A_2 = \partial B_\varepsilon(\theta^0) \),

1. \( \limsup_{N \to \infty} N^{-1} \sup_{\theta \in A_1} \mathbb{E}[\Lambda_N(\theta; W^{(i)})] < 0 \),
2. \( \sup_{\theta \in A_1} N^{-1} \| \Lambda_N(\theta; W^{(i)}) - \mathbb{E}[\Lambda_N(\theta; W^{(i)})] \| \overset{P}{\to} 0 \), and, consequently;
3. $A_i$ is identified by $W^{(i)}$ with an identification rate $a_{n,i} = o(e^{-\epsilon N})$ for some $\epsilon > 0$, $i = 1, 2$.

**Proof.** A detailed proof is Appendix B, we here give the proof idea. Let $A_2 = \partial B_\varepsilon(\theta^0) \cap \{\theta : |\theta_d - \theta^0_d| \leq \eta\} \cap \{||\beta_2 - \beta^0_2|| \geq \varepsilon/2\}$, for some small $\eta > 0$. Let $A_1$ be the closure of $\partial B_\varepsilon(\theta^0) \cap A_2$. The idea is that if $\eta$ is small enough, so that $|\theta_d - \theta^0_d| \approx \eta$ and $||\beta_2 - \beta^0_2|| \geq \varepsilon/2$ on $A_2$, then the distributions of $W^{(2)}$ implied by $\theta \in A_2$ and $\theta^0$ are different if $X = [x_{1,1}, x_{2,2}, \ldots, x_{N,N}]^T$ has full column rank. That is, $W^{(2)}$ should be able to distinguish every $\theta \in A_2$ from $\theta^0$. Moreover, one can show that on $A_1$ it holds either that $|\theta_d - \theta^0_d| \geq \min(\eta, \varepsilon/4)$ or that $||\beta_1 - \beta^0_1|| \geq \varepsilon/4$. In either case, $W^{(1)}$ should be able to distinguish $\theta \in A_1$ from $\theta^0$. Formalizing this idea leads to point 1. Point 2 follows from checking the conditions of a uniform law of large numbers [26, Theorem 8.2] and point 3 from points 1 and 2. \qed

The explicit construction of the subsets $A_1$ and $A_2$, as opposed to using Proposition 3.1, warrants an additional comment. Recall, the proposition gives compact $\widetilde{A}_1$ and $\widetilde{A}_2$ such that $\widetilde{A}_1 \cup \widetilde{A}_2 = \partial B_\varepsilon(\theta^0)$ and $\nu_{\theta}^0 \neq \nu_{\theta_0}^0$, $\theta \in \widetilde{A}_i$, $i = 1, 2$. If one takes $A_i = \widetilde{A}_i$, then point 1 in Lemma 3.2 follows. Moreover, when the subcollection components are i.i.d., this in turn leads to point 1 in Lemma 3.5 which is what is really needed. However, when the distributions of the subcollection components are not identical, this last implication is not true in general.

Having selected appropriate subcollections and subsets it remains only to check that the log-likelihood for the full data satisfies the regularity conditions in Assumptions 2–3. The following lemma verifies Assumption 2 and establishes a rate needed for the verification of Assumption 3.

**Lemma 3.6.** If $\theta^0$ is an interior point of $\Theta$, then for every $n$ and small enough $\varepsilon > 0$ there exists a random variable $K_n$ such that, $\mathbb{P}$-almost surely,

$$\sup_{\theta \in B_\varepsilon(\theta^0)} ||\nabla \Lambda_n(\theta; Y)|| \leq K_n = o_P(n^b),$$

for some $b > 0$.

Upon inspecting the proof one sees that $b$ can be taken to be $1 + \epsilon$, for any $\epsilon > 0$. This is a better (slower) rate than that obtained in the linear mixed model (see the proof of Lemma 3.3). We are now ready to state the main result of the section.

**Theorem 3.7.** If $\theta^0$ is an interior point of $\Theta$ and

$$\liminf_{N \to \infty} \lambda_{\min} \left(N^{-1} \sum_{i=1}^N x_{i,i}x_{i,i}^T \right) > 0,$$
then, \( \mathbb{P} \)-almost surely, there exists a sequence \( \hat{\theta}_n \) of roots to the likelihood equations
\[
\nabla \Lambda_n(\theta; Y) = 0
\]
such that \( \hat{\theta}_n \xrightarrow{\mathbb{P}} \theta^0 \).

Proof. The proof is similar to that of Theorem 3.4 so we skip some details. We may assume all points in \( \bar{B}_\varepsilon(\theta^0) \) are interior points of \( \Theta \). By Lemma 3.5 \( \Lambda_n(\theta; Y) \) is differentiable on \( \bar{B}_\varepsilon(\theta^0) \). By Lemma 3.5 the identification rate is exponentially fast in \( N \) and Lemma 3.6 shows that \( \Lambda_n(\theta; Y) \) is \( K_n \)-Lipschitz on both \( A_1 \) and \( A_2 \), and that \( K_n = o_p(n^b) \) for some \( b > 0 \). This verifies Assumption 2. By picking \( \delta_{i,n} = n^{-b} \) we can have \( M_{n,i} = O(n^{(d-1)b}) \) as \( n \to \infty, i = 1, 2 \). Thus, \( K_n \delta_n = o_p(1) \) and \( M_{n,i}a_{n,i} = O(N^{2b[d-1]}e^{-\epsilon N}) \) for some \( \epsilon > 0 \), which is \( o(1) \) as \( N \to \infty \) since \( n = 2N^2 \).

4 Discussion

Our theory develops the current state-of-the-art asymptotic theory based on subcollections to cover more general cases. The assumptions we make highlight what makes the use of subcollections work. In particular, the interplay between the identification rates of subcollections and the regularity of the likelihood function for the full data is made precise. We note that when the subcollections consist of \( m \in \{1, 2, \ldots \} \) independent random variables, as in our examples, then if \( n = o(m^b) \) for some \( b > 0 \) and \( \nabla \Lambda_n(\theta; Y) = o(n^{b'}) \) for some \( b' > 0 \), uniformly on a compact \( \Theta \), the rate conditions are satisfied. This so because, under regularity conditions, the identification rate in a subcollection with \( m \) independent random variables is exponential in \( m = n^{1/b} \). Since this argument works for arbitrarily large \( b \) and \( b' \) our theory is operational in a wide range of models. Loosely speaking, if the score function is of less than exponential order in the sample size and there are subcollections of independent random variables that grow faster than logarithmically in the sample size, the MLE is consistent. The conditions should be verifiable in many models since they often require only standard asymptotic tools. For example, in the LMM example nothing more than a uniform law of large numbers and strict positivity of the K–L divergence between distributions corresponding to distinct, identified parameters is needed. Though not pursued here, by inspecting the assumptions of our theory one also sees that it has the potential to be extended to allow the dimension of the parameter set, \( d \), grow with \( n \). The rates required in our assumptions could be satisfied also if \( d \) grows, at least if at a slow enough rate.

Consistency of MLEs have not previously been established in either of the two cases.
models to which we apply the general theory. In particular, previous work on asymptotic theory for MLEs in mixed models often either assumes independent replications of a response vector, that there are no predictors, or no mixed-type responses. We have tried to keep the models here as simple as possible while still illustrating key ideas. Crossed random effects, temporal dependence, and predictors are included because they are challenging theoretically and are commonly used in practice. We have refrained from including things that do not require any new methods but make ideas less transparent. For example, it would be straightforward to include random effects that are not crossed, possibly at the expense of using more subcollections or subcollections consisting of independent vectors of larger dimension than what is now necessary. Similarly, adding several crossed random effects does not make things much harder, only less transparent.

Avenues for future research includes the rate of convergence of the MLEs as well as their asymptotic distribution. Intuitively, one expects MLEs based on the full data to converge at least as fast as the slowest of the subcollection MLEs, that is, the estimators one gets from using only a subset of the full data. There is some evidence of this, namely that, under regularity conditions, the Fisher information in the full data is always larger than that in any subcollection \[17\]. On the other hand, it is easy to show that, for the simple LMM example in the introduction, the full data MLE converges at the same rate as that based on a subcollection of \(N = \sqrt{n}\) i.i.d. observations; that is, at the rate \(n^{1/4}\). Given the similarities of the random effect structures, that convergence rate may in future work be a reasonable working hypothesis for MLEs in the MGLMM considered here.


Appendix A  Theory

A.1 Preliminary results

We first present some lemmas that will be useful when proving the main results.

Lemma A.1. For any positive random variables $X$, $Y$, $Z$, defined on the same probability space, and $c > 0$, $P(XY \geq Z) \leq P(X \geq c) + P(Y \geq Z/c)$.

Proof. If for positive constants $x, y, z, c$ it holds that $xy \geq z$, then either $x \geq c$ or $y \geq z/c$, since otherwise $xy < c(z/c) = z$. Thus, $\{\omega : X(\omega)Y(\omega) \geq Z(\omega)\} \subseteq \{\omega : X(\omega) \geq c\} \cup \{\omega : Y(\omega) \geq Z(\omega)/c\}$. By sub-additivity of measures, $P(XY \geq Z) \leq P(\{X \geq c\} \cup \{Y \geq Z/c\}) \leq P(X \geq c) + P(Y \geq Z/c)$.

Lemma A.2. Suppose $A_i, i = 1, \ldots, n$ are compact subsets of some metric space $(T, d_T)$ such that $\cap_{i=1}^n A_i = \emptyset$, then the open covers $C_i = \cup_{x \in A_i} B_{\delta}(x), i = 1, \ldots, s$, also have an empty intersection for all small enough $\delta > 0$.

Proof. Consider the covers $C_{k,i} = \cup_{x \in A_i} B_{1/k}(x), k = 1, 2, \ldots, i = 1, \ldots, n$. If $C_k = \cap_i C_{k,i} = \emptyset$ for some $k < \infty$, then we are done. Suppose for contradiction $C_k$ is non-empty for every $k < \infty$. By construction, every point $x_k \in C_k$ is within $1/k$ of at least one point in every $A_i$. That is, we can pick, for every $k \geq 1$ and $i = 1, \ldots, n$, an $x_k \in C_k$ and $y_{k,i} \in A_i$ such that $d(x_k, y_{k,i}) \leq 1/k$. Thus, by the triangle inequality, for every $k$, $d(y_{k,i}, y_{k,j}) \leq 2/k$. By compactness of $A_1$, say, $y_{k,1}$ has a convergent subsequence $y_{k_m,1} \rightarrow y_1$ as $m \rightarrow \infty$, for some $y_1 \in A_1$ by the fact that $A_1$ is closed as a compact subset of a metric space. But then, for every $i$, by the triangle inequality, $d(y_{k_m,i}, y_1) \leq d(y_{k_m,i}, y_{k_m,1}) + d(y_{k_m,1}, y_1) \leq 2/k_m + d(y_{k_m,1}, y_1) \rightarrow 0$ as $m \rightarrow \infty$. Thus, since every $A_i$ is closed, $y_1 \in A_i$ for every $i$, which is the desired contradiction.

Lemma A.3. Suppose $\Theta$ is a compact subset of some metric space and, for every $\theta \in \Theta$, $f_{\theta}$ is a probability density against some dominating measure $\nu$ which does not depend on $\theta$. Suppose also that $f_{\theta}(x)$ is continuous in $\theta$ for every $x$ and define the measures $\nu_\theta$ by $\nu_\theta(A) = \int_A f_{\theta}(x) d\nu(x)$ for any $\nu$-measurable $A$. Then for any $\theta^0 \in \Theta$, the set $\Theta^0 = \{\theta \in \Theta : \nu_\theta = \nu_{\theta^0}\}$ is compact.

Proof. Because $\Theta$ is a compact subset of a metric space, it suffices to show that $\Theta^0$ is closed. Note that $\Theta^0$ always includes the point $\theta^0$ and is thus non-empty. Pick an arbitrary converging sequence $\theta_n \in \Theta^0$, call the limit point $\theta^*$. By continuity of $\theta \mapsto f_{\theta}(x)$ for every $x$, $f_{\theta_n} \rightarrow f_{\theta^*}$ pointwise. Now for any $\nu$-measurable $A$,
Without loss of generality, consider $i$ where $\nu_0(A) - \nu_{\theta_n}(A)$, which vanishes as $n \to \infty$ by a generalized dominated convergence theorem \cite{27}. Theorem 19) – the dominating sequence of functions for which the integrals converge can be $f_{\theta_n}(x) \geq f_{\theta_n}(x)I_A(x)$ – so indeed $\theta^* \in \Theta^0$. 

\[ A.2 \text{ Main results} \]

For economical notation in the proofs we write $f_\theta(y) = f_\theta^n(y)$, $f_\theta(y_i) = f_{\theta,i}(y_i)$, $f_\theta(w) = g_\theta(w)$, $f_\theta(u) = \phi_\theta(u)$, and so on. That is, the letter $f$ is overloaded and the argument indicates which density we are referring to.

\textit{Proof Lemma 2.1.} Let $Y = (W, Z)$, where $Z$ consists of the components of $Y$ that are not in the subcollection $W$. Then $f_\theta(y) = f_\theta(w, z)$ and by (conditional) Markov’s inequality, for any $k > 0$,

\[ P(L_n(\theta; Y) \geq c \mid W) \leq c^{-1} E(L_n(\theta; Y) \mid W) = c^{-1} E\left(\frac{f_\theta(W, Z)}{f_{\theta_0}(W, Z)} \mid W\right). \]

Now the following calculation shows the random variable $L_n(\theta; W) = f_\theta(W)/f_{\theta_0}(W)$ is a version of $E(f_\theta(W, Z)/f_{\theta_0}(W, Z) \mid W)$:

\[ \int_Z \frac{f_\theta(w, z)}{f_{\theta_0}(w, z)} \frac{f_{\theta_0}(z \mid w)}{f_{\theta_0}(z)} \nu_Z(\theta d\nu_Z) = \int_Z \frac{f_\theta(w, z)}{f_{\theta_0}(w)} \frac{f_{\theta_0}(w, z)}{f_{\theta_0}(w)} \nu_Z(\theta d\nu_Z) = \int_Z \frac{f_\theta(w)}{f_{\theta_0}(w)} \nu_Z(\theta d\nu_Z), \]

where $\nu_Z$ is the measure against which the components in $Z$ have joint density $f_\theta(z)$ and $Z$ is the range space of $Z$. Since the conditional expectation is unique up to P-null sets, this finishes the proof. \hfill \Box

\textit{Proof of Lemma 2.2.} Fix some arbitrary $\varepsilon > 0$. If $\sup_{\theta \in A_i} L_n(\theta; Y) < 1$ for $i = 1, \ldots, s$, then, since $L_n(\theta^0; Y) = 1$, there are no global maximizers in $\bigcup_{i=1}^s A_i \supseteq \Theta \cap B_\varepsilon(\theta^0)^c$. Thus, it suffices to prove

\[ P\left(\bigcup_{i=1}^s \sup_{\theta \in A_i} L_n(\theta; Y) \geq 1\right) \leq \sum_{i=1}^s P\left(\sup_{\theta \in A_i} L_n(\theta; Y) \geq 1\right) \to 0. \]

Since $s$ is fixed it is enough that $P\left(\sup_{\theta \in A_i} L_n(\theta; Y) \geq 1\right) \to 0$ for every $i = 1, \ldots, s$. Without loss of generality, consider $i = 1$. Pick a cover of $A_1$ as given by Assumption 3 and, for every ball in the cover, pick a $\theta^i$ in the intersection of that ball with $A_1$. If there are some balls that do not intersect $A_1$, they may be discarded from the
cover, so we assume without loss of generality that all balls do intersect \( A_1 \). We then get \( M_{n,1} \) points such that every point in \( A_1 \) is within \( \delta_{n,1} \) of at least one of them. For any \( \theta \in A_1 \), let \( \theta^j(\theta) \) denote the \( \theta^j \) closest to it (pick an arbitrary one if there are many). Using the Lipschitz continuity given by Assumption 2 and that \( x \mapsto e^x \) is increasing we have,

\[
P\left( \sup_{\theta \in A_1} L_n(\theta; Y) \geq 1 \right) = P\left( \sup_{\theta \in A_1} \Lambda_n(\theta; Y) \geq 0 \right)
\]

\[
= P\left( \sup_{\theta \in A_1} \ell_n(\theta; Y) \geq \ell_n(\theta^0; Y) \right)
\]

\[
\leq P\left( \sup_{\theta \in A_1} \left[ \ell_n(\theta^j(\theta); Y) + K_{n,1}d_{\mathcal{T}}(\theta, \theta^j(\theta)) \right] \geq \ell_n(\theta^0; Y) \right).
\]

Because there are only \( M_{n,1} \) points \( \theta^j \), and \( d_{\mathcal{T}}(\theta^j(\theta), \theta) \leq \delta_{n,1} \) since \( \theta^j(\theta) \) is the one closest to \( \theta \), we get that the last line is upper bounded by

\[
P\left( \max_{j \leq M_{n,1}} \left[ \ell_n(\theta^j; Y) + K_{n,1}\delta_{n,1} \right] \geq \ell_n(\theta^0; Y) \right) = P\left( \max_{j \leq M_{n,1}} f_{\theta^j}(Y) e^{K_{n,1}\delta_{n,1}} \geq f_{\theta^0}(Y) \right).
\]

But by applying Lemma \[A.1] with \( c = 2 \),

\[
P\left( \max_{j \leq M_{n,1}} f_{\theta^j}(Y) e^{K_{n,1}\delta_{n,1}} \geq f_{\theta^0}(Y) \right) \leq P\left( 2 \max_{j \leq M_{n,1}} f_{\theta^j}(Y) \geq f_{\theta^0}(Y) \right) + P\left( e^{K_{n,1}\delta_{n,1}} \geq 2 \right)
\]

\[
= P\left( 2 \max_{j \leq M_{n,1}} f_{\theta^j}(Y) \geq f_{\theta^0}(Y) \right) + o(1)
\]

where the last line uses Assumption \[3\]. The choice of the constant 2 in the application of Lemma \[A.1\] is arbitrary – any number with positive logarithm works. The remaining term,

\[
P\left( 2 \max_{j \leq M_{n,1}} f_{\theta^j}(Y) \geq f_{\theta^0}(Y) \right) = P\left( \max_{j \leq M_{n,1}} L_n(\theta^j; Y) \geq 1/2 \right),
\]

we will deal with using Lemma \[2.1\] and dominated convergence. After conditioning on \( W^{(1)} \) we have

\[
P\left( \max_{j \leq M_{n,1}} L_n(\theta^j; Y) \geq 1/2 \mid W^{(1)} \right) \leq \sum_{j=1}^{M_{n,1}} 2L_{m_1}(\theta^j; W^{(1)})
\]

\[
\leq 2M_{n,1} \sup_{\theta \in A_1} L_{m_1}(\theta, W^{(1)}),
\]

\( P \)-almost surely, where the first inequality is by subadditivity and Lemma \[2.1\] and the second uses that \( L_n(\theta^j; W^{(1)}) \leq \sup_{\theta \in A_1} L_{m_1}(\theta; W^{(1)}) \) by definition. The expression in the last line vanishes as \( n \to \infty \) by Assumption 3. Thus,

\[
P\left( \max_{j \leq M_{n,1}} L_n(\theta^j; Y) \geq 1/2 \right) \to 0
\]

by dominated convergence. The dominating function can be the constant 1. This finishes the proof. \( \square \)
Appendix B  Applications

Let $\lambda_{\text{max}}(\cdot)$ and $\lambda_{\text{min}}(\cdot)$ denote the maximum and minimum eigenvalue of its matrix argument, respectively. For matrices, $\| \cdot \|$ denotes the spectral norm and $\| \cdot \|_F$ the Frobenius norm. Differentiation with respect to $\theta_i$ is denoted $\nabla_i$.

We will use the following well known fact repeatedly. It is stated as a lemma for easy reference.

Lemma B.1. If $h$ is a continuous function from some metric space $X$ to $\mathbb{R}$ and $A$ is a compact subset of $X$, then $\sup_{x \in A} h(x) = h(x^*)$ for some $x^* \in A$. In particular, $\inf_{x \in A} h(x) > c$ for some constant $c$ and every $x \in A$, then $\inf_{x \in A} h(x) > c$.

Of course, the same holds if the supremum is replaced by an infimum or if less than is replaced by greater than.

Lemma B.2. Let $X_{n,1}, \ldots, X_{n,n}$ be a triangular array with rows of i.i.d. multivariate normal $q$-vectors with mean $E(X_{n,i}) = \mu = \mu(\theta)$ and covariance matrix $\text{cov}(X_{n,i}) = \Sigma = \Sigma(\theta)$, $\theta \in \Theta$. Suppose that

$$0 < 1/c_1 \leq \inf_{\theta \in \Theta} \lambda_{\text{min}}(\Sigma(\theta)) \leq \sup_{\theta \in \Theta} \lambda_{\text{max}}(\Sigma(\theta)) \leq c_1 < \infty$$

and $\sup_{\theta \in \Theta} \| \mu(\theta) \| \leq c_2$ for some $c_1, c_2 \in (0, \infty)$, then

$$\sup_{\theta \in \Theta} \left| n^{-1} \sum_{i=1}^{n} \left\{ \Lambda_i(\theta; X_{n,i}) - E[\Lambda_1(\theta; X_{n,1})] \right\} \right| \to 0,$$

$\mathbb{P}$-almost surely, where $\Lambda_i(\theta; X_{n,i}) = \log f_\theta(X_{n,i}) / f_{\theta_0}(X_{n,i})$, and $f_\theta(X_{n,i})$ means the density for $X_{n,i}$ evaluated at $X_{n,i}$.

Proof. Theorem 16 in Ferguson [10] applies almost verbatim to triangular arrays in place of i.i.d. sequences. The only necessary modification to its proof is that the pointwise strong law of large numbers needs to be motivated. Write

$$\sup_{\theta \in \Theta} |\Lambda_1(\theta; x)| \leq \sup_{\theta \in \Theta} |\log f_\theta(x)| + |\log f_{\theta_0}(x)|$$

$$\leq \sup_{\theta \in \Theta} |\log \det \Sigma| + \sup_{\theta \in \Theta} \|x - \mu\|^2 \|\Sigma^{-1}\| + |\log f_{\theta_0}(x)|$$

$$\leq \sup_{\theta \in \Theta} |\log \det \Sigma| + \sup_{\theta \in \Theta} (\|x\| + \|\mu\|)^2 \sup_{\theta \in \Theta} \|\Sigma^{-1}\| + |\log f_{\theta_0}(x)|$$

$$\leq |\log (qc_1)| + (\|x\| + c_2)^2 c_1 + |\log f_{\theta_0}(x)|$$

$$=: K(x),$$
which is a quadratic function of $x$, not depending on $\theta$. Thus, since the $X_{n,i}$s are i.i.d. and normal random variables have all finite moments, $\Lambda_i(\theta; X_{n,i})$ has bounded fourth moment, uniformly in $i$, $n$, and $\theta$. Classical proofs for a strong law with finite fourth moment applies without change to triangular arrays. The other conditions of Ferguson’s Theorem 16 are easy to verify, using $K(x)$ as the dominating function.

**Proof Proposition 3.1.** Lemma A.3 gives that \( \{ \theta \in \Theta : \nu_i^j = \nu_{i0}^j \} \) is a closed set, $i = 1, \ldots, s$. Thus, the sets $D_i = \{ \theta \in \Theta : \nu_i^j = \nu_{i0}^j \} \cap B_\varepsilon(\theta^0)c$, $i = 1, \ldots, s$, are closed as intersections of closed sets and compact as a closed subset of a compact set, $\Theta$. By Lemma A.2 we can pick $\delta$ small enough that the open covers $B_i = \bigcup_{\theta \in D_i} B_\varepsilon(\theta) \supseteq D_i$ have an empty intersection, $\cap_{i=1}^s B_i = \emptyset$. Let $A_i = \Theta \cap B_\varepsilon(\theta^0)c \cap B_i^c$ and note $\cup_{i=1}^s A_i = \Theta \cap B_\varepsilon(\theta^0)c \cap (\cap_{i=1}^s B_i)^c = \Theta \cap B_\varepsilon(\theta^0)c$. Each $A_i$ closed as the intersection of closed sets, and compact as a closed subset of a compact set, $\Theta$. By construction, for any $\theta \in A_i$ it must be that $\theta \in B_\varepsilon(\theta^0)c$. Since $A_i$ is a subset of $B_\varepsilon(\theta^0)c$ by construction this implies $\theta \in \{ \theta \in \Theta : \nu_i^j = \nu_{i0}^j \}^c$, which finishes the proof. ∎

### B.1 Longitudinal linear mixed model

**Lemma B.3.** The log-likelihood $\ell_n(\theta; y)$ is differentiable in $\theta$ at any interior point of $\Theta$, for every $n \geq 1$ and every $y$ in the support of $Y$.

**Proof.** The multivariate normal log-likelihood $\ell_n(\theta; Y)$ is differentiable in its mean $m$ and covariance matrix $C$ everywhere $C = C(\theta)$ is positive definite. It is easy to see that $C$ is positive definite on all interior point since $\Psi$ is (c.f. Lemma B.4). Now $\ell_n(\theta; Y)$ is differentiable on all interior points by the chain rule since the elements of $m$ and $C$ are differentiable in $\theta$. ∎

**Proof Lemma 3.2.** Fix an $\varepsilon > 0$ small enough that all points of $\bar{B}_\varepsilon(\theta^0)$ are interior. By construction of the subcollections, the assumptions of Proposition 3.4 are satisfied with what is there denoted $\Theta$ replaced by $\bar{B}_\varepsilon(\theta^0)$. Take $A_1$ and $A_2$ to be the compact sets given by that proposition. The proof of point 1 is standard [10, p. 115] and hence omitted. Point 2 is proven by checking the conditions of Lemma B.2 with what is there denoted $\Theta$ replaced by the compact $A_i$, $i = 1, 2$. The following argument works for either subcollection. First note $\lambda_{\text{max}}(C_i) = \|C_i(\theta)\| \leq \|C_i(\theta)\|_F^2$. Since the Frobenius norm is the square root of the sum of squared entries and the entries are continuous functions of $\theta$, $\theta \mapsto \|C_i(\theta)\|_F$ is continuous and attains its

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supremum on the compact set \( A \), so \( \|C(\theta)\| \) is bounded above on \( \bar{B}_\varepsilon(\theta^0) \). By spectral decomposition of \( C_i \) it is immediate that \( \lambda_{\min}(C_i) = 1/\lambda_{\max}(C_i^{-1}) \). Thus, since \( C_i(\theta) \) is clearly positive definite on all interior points and the inverse is a continuous mapping at points where \( C_i \) is positive definite \[19\], we get by the same arguments that \( \lambda_{\max}(C_i^{-1}(\theta)) \) is bounded on \( A_i \). It is obvious that \( \theta \mapsto m_i(\theta) \) is continuous and hence attains its supremum on \( A_i \). This concludes the proof of point 2.

It remains to prove point 3. By point 1 we may pick an \( \varepsilon > 0 \) such that, for either subcollection,

\[
\sup_{\theta \in A_i} N^{-1}E[\Lambda_{N/2}(\theta; W^{(i)})] = \sup_{\theta \in A_i} E[\Lambda_1(\theta; W^{(i)})]/2 < -3\varepsilon.
\]

By point 2 we have, \( P \)-almost surely and for all large enough \( N \),

\[
\sup_{\theta \in A_i} N^{-1}|\Lambda_{N/2}(\theta; W^{(i)}) - E[\Lambda_{N/2}(\theta; W^{(i)})]| \leq \varepsilon.
\]

Thus we have that \( \sup_{\theta \in A_i} \Lambda_{N/2}(\theta; W^{(i)}) < -2N\varepsilon \), and hence that

\[
\sup_{\theta \in A_i} L_{N/2}(\theta; X^{(i)}) \leq e^{-2N\varepsilon},
\]

for all large enough \( N \), \( P \)-almost surely. The last right hand side is clearly \( o(e^{-\varepsilon N}) \) as \( N \to \infty \).

We will use the following results in the proof of Lemma \[3.6\].

**Lemma B.4.** The following hold when all points in \( \bar{B}_\varepsilon(\theta^0) \) are interior (the first inequality in 1 holds always):

1. \( \|\Psi\|_F \leq T \) and \( \sup_{\theta \in \bar{B}_\varepsilon(\theta^0)} \|\Psi^{-1}\|_F \leq c\sqrt{T} \) for some \( c > 0 \),
2. \( \sup_{\theta \in \bar{B}_\varepsilon(\theta^0)} \|C(\theta)\| \leq c_1 NT + c_2 T + c_3 \) for some \( c_1, c_2, c_3 > 0 \),
3. \( \sup_{\theta \in \bar{B}_\varepsilon(\theta^0)} \|C(\theta)^{-1}\| \leq c \) for some \( c > 0 \),
4. \( \sup_{\theta \in \bar{B}_\varepsilon(\theta^0)} \|\nabla_i \Sigma\| \leq NT + cT^2 \) for some \( c > 0 \) and every \( i \geq 3 \).
5. \( \sup_{\theta \in \bar{B}_\varepsilon(\theta^0)} \|Y - m(\theta)\| = o_P(n) \), and

**Proof.**

1. The Frobenius norm is the square root of the sum of squared elements, and all elements of \( \Psi \) are in the form \( \theta_k^7 \) for some integer \( k \) – this establishes the first inequality. The inverse of \( \Psi \) can be written as \((1 - \theta_k^2)^{-1}\) times a tri-diagonal matrix where the diagonal entries are 1 or 1 + \( \theta_k^2 \), and the leading off-diagonals have entries \(-\theta_7\). Thus, \( \|\Psi^{-1}\|_F \) is the square root of the sum of \( 3T \) possibly non-zero elements, each a continuous function of \( \theta \). The inequality now follows from Lemma \[B.1\].
2. Using that eigenvalues of the sum of two positive, semi-definite matrices must be at least as large as those of either summand and that the eigenvalues of Kronecker products are the products of the multiplicands’ eigenvalues \(^{10}\), we get

\[
\lambda_{\text{max}}(C) \leq \lambda_{\text{max}}(\theta_3 I_n) + \lambda_{\text{max}}(\theta_4 I_N \otimes J_{NT}) + \lambda_{\text{max}}(\theta_5 J_N \otimes I_N \otimes J_T) + \lambda_{\text{max}}(\theta_6 I_{N^2} \otimes \Psi) \\
\leq \theta_3 + \theta_4 NT + \theta_5 NT + \theta_6 T,
\]

where in the last step we also used \(\lambda_{\text{max}}(\Psi) \leq \|\Psi\|_F \leq T\) by 1. The existence of the constants \(c_1, c_2, c_3\) now follows from Lemma B.1.

3. Since \(Z\Sigma Z^T\) is positive definite, we get \(\lambda_{\text{min}}(C) = \lambda_{\text{min}}(\theta_3 I_n + Z\Sigma Z^T) \geq \theta_3\).

Since all points in \(\bar{B}_\varepsilon(\theta^0)\) are interior, \(\theta_3\) is lower bounded by some \(c^{-1} > 0\) on it (Lemma B.1). Thus, using that the eigenvalues of \(C^{-1}\) are the reciprocals of the eigenvalues of \(C\), we get \(\|C^{-1}\|_F \leq (nc^2)^{1/2} = N\sqrt{Tc}\).

4. Clearly, \(\nabla_2 C(\theta) = I_n\) which has eigenvalue 1 with multiplicity \(n\). If \(i = 4\) or \(i = 5\), then the derivative is either \(I_N \otimes J_N \otimes J_T\) or \(J_N \otimes I_N \otimes J_T\), which both have maximal eigenvalue \(NT\). If \(i = 6\), then the derivative is \(\Psi \otimes I_{N^2}\), which has maximal eigenvalue less than \(T\) by 1. If \(i = 7\), then the derivative is \(\theta_6 \nabla_7 \Psi\). We have \(\nabla_7 \Psi_{i,j} = |i - j| \theta_i^j |j|^{-1}\) if \(|i - j| \geq 1\) and \(\nabla_7 \Psi_{i,j} = 0\) otherwise. Thus, \(\nabla_7 \Psi_{i,j} \leq T\) and, consequently, \(\|\nabla_7 \Psi\|_F \leq T^2\). We conclude, by Lemma B.1 \(\nabla_7 C(\theta) \leq cT^2\) for some \(c > 0\).

5. Let \(UAU^T\) be the spectral decomposition of \(C\). Then \(\|Y - m(\theta)\| = \|U^T(Y - m(\theta))\|\). The vector \(U^T(Y - m(\theta))\) is multivariate normal with mean 0 and covariance matrix \(\Lambda\). Thus, since a Gaussian process is determined by its finite dimensional distributions, the stochastic process \(\|Y - m(\theta)\|^2\), \(\theta \in \bar{B}_\varepsilon(\theta^0)\), has the same distribution as the process \(\sum_{i=1}^n \Lambda_{i,i}(\theta)\xi_i^2\), where \(\xi_1, \ldots, \xi_n\) are i.i.d. standard normal. By point 2, the supremum of the latter process satisfies \(\sup_{\theta \in B_\varepsilon(\theta^0)} \sum_{i=1}^n \Lambda_{i,i}(\theta)\xi_i^2 \leq (c_1 NT + c_2 T + c_3) \sum_{i=1}^n \xi_i^2 = o_p(n^2)\), which follows from that the last sum is a positive random variable with mean \(n\), and hence it converges to zero in \(L_1\) when divided by anything of higher order than \(n\).

\[\square\]

Proof Proposition 3.3. Let \(e = e(\theta) = Y - m(\theta)\) and let \(\nabla_e\) and \(\nabla_C\) denote differentiation with respect to \(e\) and \(C\). Since \(e\) is linear in \(\theta_1\) and \(\theta_2\), and \(C(\theta)\) is differentiable in each \(\theta_i\), \(i \geq 3\), bounding the gradient for \(\theta\) is easily done after
establishing bounds for $\nabla C\ell_n(\theta; Y)$ and $\nabla e\ell_n(\theta)$. These derivatives exist for every $n$ because the covariance matrix $C(\theta)$ is positive-definite on $\bar{B}_\varepsilon(\theta^0)$ by Lemma B.4 and the multivariate normal log-likelihood is differentiable wherever the covariance matrix is non-singular [19]. We have

$$\nabla C\ell_n(\theta; Y) = -\frac{1}{2} \left[ C^{-1} + C^{-1}ee^T C^{-1} \right]$$

and $\nabla e\ell_n(\theta) = -C^{-1}e$.

Thus,

$$|\nabla_1\ell_n(\theta)| = |\nabla e\ell_n(\theta)^T \nabla_1 e(\theta)| = |e^T C^{-1} 1_n| \leq \|e\| \|C^{-1}\| N^2 T,$$

$$|\nabla_2\ell_n(\theta)| = |\nabla e\ell_n(\theta)^T \nabla_2 e(\theta)| = |e^T C^{-1} h_n| \leq \|e\| \|C^{-1}\| N^2 T/2,$$

and, for $i \geq 3$,

$$|\nabla_i\ell_n(\theta)| = |\text{vec}[\nabla C\ell_n(\theta)]^T \text{vec}[\nabla_i C]| = \frac{1}{2} \text{vec} \left[ C^{-1} + C^{-1}ee^T C^{-1} \right]^T \text{vec}[\nabla_i C]|$$

$$= \text{tr} \left[ (C^{-1} + C^{-1}ee^T C^{-1}) \nabla_i C \right]$$

$$\leq \|C^{-1}\|_F \|\nabla_i C\|_F + |e^T C^{-1} \nabla_i C^{-1} e|$$

$$\leq \|C^{-1}\|_F \|\nabla_i C\|_F + \|e\|^2 \|C^{-1}\|^2 \|\nabla_i C\|,$$

where vec($\cdot$) denotes the vectorization operator stacking the columns of its matrix argument. Thus, by Lemma B.4

$$\sup_{\theta \in \bar{B}_\varepsilon(\theta)} |\nabla_1\ell_n(\theta)| \leq \sup_{\theta \in \bar{B}_\varepsilon(\theta)} \|e\| \|C^{-1}\| N^2 T \leq o_p(n) O(NT + T)TN^2 = o_p(T^3 N^5),$$

$$\sup_{\theta \in \bar{B}_\varepsilon(\theta)} |\nabla_2\ell_n(\theta)| \leq \sup_{\theta \in \bar{B}_\varepsilon(\theta)} \|e\| \|C^{-1}\| N^2 T/2 \leq o_p(n) O(NT + T)TN^2 = o_p(T^3 N^5),$$

and, for $i \geq 3$,

$$\sup_{\theta \in \bar{B}_\varepsilon(\theta)} |\nabla_i\ell_n(\theta)| \leq \sup_{\theta \in \bar{B}_\varepsilon(\theta)} (\|C^{-1}\|_F \|\nabla_i C\|_F + \|e\|^2 \|C^{-1}\|^2 \|\nabla_i C\|).$$

By Lemma B.4, the supremum of each of the terms in the last line are of at most polynomial order, which finishes the proof.

**B.2 Logit-Normal MGLMM**

**Lemma B.5.** The log-likelihood $\ell_n(\theta; y)$ is differentiable in $\theta$ on $\bar{B}_\varepsilon(\theta^0)$, for every $n \geq 1$ and every $y$ in the support of $Y$.

**Proof.** To prove differentiability of $f_\theta(y)$ in $\theta$ on $\bar{B}_\varepsilon(\theta^0)$, checking the usual conditions for differentiation under the integral are sufficient [11, Theorem 2.27]. It’s
obvious that \( f_\theta(y \mid u)f_\theta(u) \) is differentiable in \( \theta \) on every interior point of \( \Theta \), so it suffices to find, for \( i = 1, \ldots, d \), functions \( K_i : \mathbb{R}^{2n} \times \mathbb{R}^{2N} \to [0, \infty) \), not depending on \( \theta \), such that \( |\nabla_i f_\theta(y \mid u)f_\theta(u)| \leq K_i(y, u) \) and \( \int K_i(y, u)du < \infty \). Clearly, \( |\nabla_i f_\theta(y \mid u)f_\theta(u)| \leq \| \nabla_\beta_i f_\theta(y \mid u)f_\theta(u) \| \), for any \( i \) such that \( \theta_i \) is a component of \( \beta_1 \), and similarly for the components of \( \beta_2 \). Thus, it suffices to find bounds for \( \| \nabla_\beta_i f_\theta(y \mid u)f_\theta(u) \|, i = 1, 2 \), and \( |\nabla_{\theta, i} f_\theta(y \mid u)f_\theta(u)| \). For the purposes of this integration, the responses \( y_{i,j,k} \) are constant and the sample size \( n \) is fixed. We prove the existence of integrable bounds in the following forms, where \( c_1, \ldots, c_4 > 0 \),

1. \( K_1(y, u) = c_1 \exp \left( -\frac{1}{2\sigma^2} u^T u \right) \sum_{i,j} \left( |y_{i,j,1}| + 1 + |u_{i}^{(1)}| + |u_{j}^{(2)}| \right) \geq \| \nabla_\beta_i f_\theta(y \mid u)f_\theta(u) \|, \)

2. \( K_2(y, u) = c_3 \exp \left( -\frac{1}{2\sigma^2} u^T u \right) \geq \| \nabla_\beta_2 f_\theta(y \mid u)f_\theta(u) \|, \)

3. \( K_3(y, u) = c_4 \exp \left( -\frac{1}{2\sigma^2} u^T u \right) (u^T u + 1) \geq |\nabla_{\theta_i} f_\theta(y \mid u)f_\theta(u)|. \)

It is clear that \( K_1, K_2, K_3 \) so defined are integrable because they are, up to scaling, moments of multivariate normal distributions. Thus, it remains only to prove the stated inequalities indeed hold.

By the triangle inequality, that \( f_\theta(y \mid u) \leq (2\pi)^{-n/2} \), and the fact that \( f_\theta(u) \) does not depend on \( \beta_1 \), we have

\[
\| \nabla_\beta_i f_\theta(y \mid u)f_\theta(u) \| = \left\| f_\theta(y \mid u)f_\theta(u) \sum_{i,j} (y_{i,j,1} - l_{i,j,1}) x_{i,j} \right\|
\leq (2\pi)^{-n/2} (2\pi \theta_d)^{-N} \exp \left( -\frac{1}{2\theta_d} u^T u \right) \sum_{i,j} \left( |y_{i,j,1}| + \| \beta_1 \| |x_{i,j}| + |u_{i}^{(1)}| + |u_{j}^{(2)}| \right) \|x_{i,j}\|
\]

The inequality in the definition of \( K_1 \) follows from Lemma [3.1] upon noting that \( \theta_d \) is bounded both away from zero and above on interior points, that \( \| \beta_1 \| \) is similarly upper bounded on such points, and that \( \|x_{i,j}\| \leq 1 \) by assumption.

For the inequality in the definition of \( K_2 \) we use that \( f_\theta(y \mid u) \leq (2\pi)^{-n/2} \) and that \( |y_{i,j,2} - 1/(1 + e^{-l_{i,j,2}})| \leq 1 \). The latter assertion follows from that \( y_{i,j,2} \in \{0, 1\} \) and that \( 1/(1 + e^t) \in (0, 1) \) for all \( t \in \mathbb{R} \). Thus, since \( f_\theta(u) \) does not depend on \( \beta_2 \),

\[
\| \nabla_\beta_2 f_\theta(y \mid u)f_\theta(u) \| = \left\| f_\theta(y \mid u)f_\theta(u) \sum_{i,j} (y_{i,j,2} - 1/(1 + e^{-l_{i,j,2}})) x_{i,j} \right\|
\leq n (2\pi)^{-n/2} (2\pi \theta_d)^{-N} \exp \left( -\frac{1}{2\theta_d} u^T u \right) \|x_{i,j}\|
\]

Now the desired inequality follows from again noting the bounds from below and above of \( \theta_d \) and that \( \|x_{i,j}\| \leq 1 \).
The inequality in the definition of $K_3$ follows similarly. First, $f_\theta(y \mid u)$ does not depend on $\theta_d$ so we get
\[
|\nabla_{\theta_d} f_\theta(y \mid u)f_\theta(u)| = |f_\theta(y \mid u)f_\theta(u)\left(\frac{N}{\theta_d} + \frac{u^T u}{2\theta_d^2}\right)| \\
\leq (2\pi)^{-n/2} (2\pi \theta_d)^{-N} \exp\left(-\frac{u^T u}{\theta_d} \left(\frac{N}{\theta_d} + \frac{u^T u}{2\theta_d^2}\right)\right).
\]
Now we are done upon again appealing to the lower and upper bounds of $\theta_d$ on $\bar{B}_\varepsilon(\theta^0)$. □

For the proof of Lemma 3.5 we will need the following lemmas.

**Lemma B.6.** Let $\mathcal{X}$ be a metric space and $f : \mathcal{X} \times \mathbb{R}^d \to \mathbb{R}$, for some $d > 0$, be continuous under the product metric. If $\mathcal{X}$ is compact, then $h(y) = \sup_{x \in \mathcal{X}} f(x, y)$ is continuous.

**Proof.** Fix some $y$ and consider the compact set $A = \mathcal{X} \times \bar{B}_1(y)$. Since $A$ is compact, $f$ is uniformly continuous on $A$. Thus, for any $\epsilon > 0$ we can pick $\delta$ such that for every $(x', y'), (x'', y'') \in A$, it holds that if $d((x', y'), (x'', y'')) < \delta$, then $|f(x', y') - f(x'', y'')| < \epsilon$. Thus, for any $y' \in B_\delta(y) \subseteq B_1(y)$, we have $|h(y) - h(y')| = \left|\sup_{x \in \mathcal{X}} f(x, y) - \sup_{x \in \mathcal{X}} f(x, y')\right| \leq \sup_{x \in \mathcal{X}} |f(x, y) - f(x, y')| = |f(x^*(y, y'), y) - f(x^*(y, y'), y')| < \epsilon$, where $x^*(y, y') = \arg\max_{x \in \mathcal{X}} \{f(x, y) - f(x, y')\}$. The arg max exists by Lemma B.1 since continuity of $f$ implies continuity in $x$ for every $y$. □

**Lemma B.7.** The K–L divergence from a Bernoulli distribution with parameter $p$ to one with parameter $q$ is lower bounded by $2(p - q)^2$.

**Proof.** By direct computation, the K–L divergence is $p \log(p/q) + (1 - p) \log((1 - p)/(1 - q))$. Now using that $t(1 - t) \leq 1/4$ for all $t \in \mathbb{R}$ and assuming $p > q$ we get
\[
p \log(p/q) + (1 - p) \log((1 - p)/(1 - q)) = \int_q^p \left(\frac{p}{t} - \frac{1 - p}{1 - t}\right) dt \\
= \int_q^p \left(\frac{p - t}{t(1 - t)}\right) dt \\
\geq 4 \int_q^p (p - t) dt \\
= 2(p - q)^2
\]
If instead $q > p$, then the same inequality results from letting $1 - p$ and $1 - q$ take the roles of $p$ and $q$. If $p = q$, then the inequality is an equality. □
Let $C(\delta, G, \| \cdot \|)$ denote the $\delta$-covering number of the set $G$ under the distance associated with the norm $\| \cdot \|$, that is, the least number of open balls of radius $\delta$ needed to cover $G$.

**Lemma B.8** (Theorem 8.2 [26]). Let $h_1(\omega, \theta), h_2(\omega, \theta), \ldots, \theta \in A \subseteq \Theta$, be independent processes with integrable envelopes $H_1(\omega), H_2(\omega), \ldots$, meaning $|h_i(\omega, \theta)| \leq H_i(\omega)$, for all $i$ and $\theta \in A$. Let $H = (H_1, \ldots, H_N)$ and $\mathcal{H}_{N, \omega} = \{[h_1(\omega, \theta), \ldots, h_N(\omega, \theta)] \in \mathbb{R}^N : \theta \in A\}$. If for every $\epsilon > 0$ there exists a $K > 0$ such that

1. $N^{-1} \sum_{i=1}^{N} E[H_i I(H_i > K)] < \epsilon$ for all $N$, and
2. $\log C(\epsilon \| H \|_1, \mathcal{H}_{N, \omega}, \| \cdot \|_1) = o_p(N)$ as $N \to \infty$,

then

$$\sup_{\theta \in A} N^{-1} \left| \sum_{i=1}^{N} h_i(\omega, \theta) - E(h_i(\omega, \theta)) \right| \xrightarrow{P} 0.$$

**Proof.** Pollard [26] proves this result with packing numbers replaced by covering numbers. Since [26, p. 10]

$$C(\epsilon, \mathcal{H}_{N, \omega}, \| \cdot \|_1) \leq D(\epsilon, \mathcal{H}_{N, \omega}, \| \cdot \|_1) \leq C(\epsilon/2, \mathcal{H}_{N, \omega}, \| \cdot \|_1),$$

where $D$ denotes packing numbers, there is nothing more to prove. \hfill \Box

**Proof Lemma 3.5.** Let us first prove that, given $\epsilon > 0$, there exists an $\eta > 0$, and hence $A_i = A_i(\epsilon, \eta), i = 1, 2$, such that point 1 in the lemma holds. The definition of $A_i(\epsilon, \eta)$ is as in the main text. Let $c(t) = \log(1 + e^t)$ denote the cumulant function in the conditional distribution of $Y_i,i,2$ given the random effects and define

$$p_i(\beta_2, \theta_d) = E\left[ c' \left( x_{i,i}^T \beta_2 + \sqrt{\theta_d/\theta_d^0} \left( U_i^{(1)} + U_i^{(2)} \right) \right) \right].$$

Recall, $E$ denotes expectation with respect to the distributions indexed by $\theta^0$, so $p_i(\beta_2, \theta_d)$ is the success probability of $Y_{i,i,2}$ when $\beta_2$ and $\theta_d$ are the true parameters.

Note that because the components in $W^{(2)}$ are independent, $E[\Lambda_N(\theta; W^{(2)})]$ is a sum of $N$ terms, each summand being the negative K-L divergence between two Bernoulli variables with parameters $p_i(\beta_2, \theta_d)$ and $p_i(\beta_2^0, \theta_d^0)$. Thus, by Lemma B.7, Jensen’s inequality, the reverse triangle inequality, and the triangle inequality,
respectively,
\[
N^{-1}E[\Lambda_N(\theta; W^{(2)})] \\
\leq -2N^{-1} \sum_{i=1}^{N} |p_i(\beta_2, \theta_d) - p_i(\beta_0^0, \theta_0^0)|^2 \\
\leq -2 \left[ N^{-1} \sum_{i=1}^{N} |p_i(\beta_2, \theta_d) - p_i(\beta_0^0, \theta_0^0)|^2 \right] \\
\leq -2 \left[ N^{-1} \sum_{i=1}^{N} \left| |p_i(\beta_2, \theta_d) - p_i(\beta_2, \theta_0^0)| - |p_i(\beta_2, \theta_0^0) - p_i(\beta_2, \theta_0^0)| \right| \right] \\
\leq -2 \left[ N^{-1} \sum_{i=1}^{N} |p_i(\beta_2, \theta_d) - p_i(\beta_2, \theta_0^0)| - N^{-1} \sum_{i=1}^{N} |p_i(\beta_2^0, \theta_0^0) - p_i(\beta_2, \theta_0^0)| \right]^2. \tag{3}
\]

Let us work separately with the averages in the last line. We will show that the second can be made arbitrarily small on \(A_2\) by selecting \(\eta\) small enough, and that the first is bounded away from zero on the same \(A_2\), leading to an asymptotic upper bound on \(\sup_{\theta \in A_2} N^{-1}E[\Lambda_N(\theta; W^{(2)})]\) away from zero. We start with the first average.

Let \(H\) be a compact subset of \(\mathbb{R}\) such that \(x^T_{i,i} \beta_2 \in H\) for all \(i\) and \(\theta \in \bar{B}_\varepsilon(\theta^0)\). Such \(H\) exists because the predictors are bounded and \(\beta_2\) is bounded on \(\bar{B}_\varepsilon(\theta^0)\).

Then, defining \(\tilde{p}_i(\gamma, \theta_d)\) as \(p_i(\beta_2, \theta_d)\) but with \(x^T_{i,i} \beta_2\) replaced by \(\gamma\), we get
\[
\sup_{\theta \in A_2} |p_i(\beta_2, \theta_d) - p_i(\beta_2, \theta_0^0)| \leq \sup_{\theta \in A_2} |\tilde{p}_i(\gamma, \theta_d) - \tilde{p}_i(\gamma, \theta_0^0)|.
\]

Since the random variable in the expectation defining \(\tilde{p}_i\) is bounded by 1 (it is the mean of a Bernoulli random variable), \(\tilde{p}_i\) is continuous by dominated convergence. Thus, since \(H\) is compact, \(\sup_{\gamma \in H} |\tilde{p}_i(\gamma, \theta_d) - \tilde{p}_i(\gamma, \theta_0^0)|\) is continuous in \(\theta_d\) by Lemma B.6. That is, we can make \(\sup_{\gamma \in H} |\tilde{p}_i(\gamma, \theta_d) - \tilde{p}_i(\gamma, \theta_0^0)|\) arbitrarily small on \(A_2 = A_2(\eta, \varepsilon)\) by picking \(\eta\) small enough, which is what we wanted to show. We next work with the second average in (3).

By the mean value theorem, for some \(\tilde{\beta}_{2,i}\) between \(\beta_2\) and \(\beta_0^0\),
\[
|p_i(\beta_2, \theta_0^0) - p_i(\beta_2, \theta_0^0)| = |E(c''(x^T_{i,i} \tilde{\beta}_{2,i} + U_i^{(2)})x^T_{i,i}(\beta_2 - \beta_0^0)|. 
\]

Here, differentiation under the expectation is permissible since \(c''\) is the variance of a Bernoulli random variable, hence bounded by 1/4, and
\[
|x^T_{i,i}(\beta_2 - \beta_0^0)| \leq \|x_{i,i}\|\|\beta_2 - \beta_0^0\|^2 \leq \varepsilon \text{ on } \bar{B}_\varepsilon(\theta^0). 
\]

By the same bound on \(c''\) we get that \(E(c''(\gamma + U_i^{(1)} + U_j^{(2)})\) is continuous in \(\gamma\). Thus, by Lemma B.14, \(\inf_{\gamma \in H} E(c''(\gamma + U_i^{(1)} + U_j^{(2)})\) \geq c_1 > 0. That \(c_1\) must be positive follows from that \(c''\) is strictly positive on all of \(\mathbb{R}\). We have thus proven that
\[
|p_i(\beta_2^0, \theta_0^0) - p_i(\beta_2, \theta_0^0)| \geq c_1|x^T_{i,i}(\beta_2 - \beta_0^0)|, \text{ uniformly on } \bar{B}_\varepsilon(\theta^0). 
\]

Using this and that
\[ |x_{i,i}^T(\beta_2 - \beta_2^0)| \leq \|x_{i,i}\| \|\beta_2 - \beta_2^0\| \leq \varepsilon \leq 1 \]

so that squaring it makes it smaller,

\[
N^{-1} \sum_{i=1}^{N} |p_i(\beta_2, \theta_2^0) - p_i(\beta_2, \theta_2^0)| \geq c_1 N^{-1} \sum_{i=1}^{N} |x_{i,i}^T(\beta_2 - \beta_2^0)|
\]

\[
\geq c_1 N^{-1} (\beta_2 - \beta_2^0)^T \left( \sum_{i=1}^{N} x_{i,i}x_{i,i}^T \right) (\beta_2 - \beta_2^0)
\]

\[
\geq c_1 \|\beta_2 - \beta_2^0\|^2 N^{-1} \lambda_{\min} \left( \sum_{i=1}^{N} x_{i,i}x_{i,i}^T \right)
\]

which lower limit as \( N \to \infty \) is bounded below by some strictly positive constant, say \( c_2 \), since \( \liminf_{N \to \infty} N^{-1} \lambda_{\min} \left( \sum_{i=1}^{N} x_{i,i}x_{i,i}^T \right) \geq c_3 > 0 \), for some \( c_3 \), and \( \|\beta_2 - \beta_2^0\| \geq \varepsilon/2 > 0 \) on \( A_2 \). To summarize, we may pick \( \eta \) so small that the second average in (3) is less than \( c_2/2 \), say, and hence get \( \sup_{\theta \in A_2} N^{-1} E[N(\theta; W(2))] \leq -2(c_2 - c_2/2)^2 < 0 \), for all but at most finitely many \( N \). This proves point 1 as it pertains to \( A_2 \).

Consider next

\[ A_1 = \partial B_\varepsilon(\theta^0) \cap \{ \{ \theta : |\theta_d - \theta_d^0| \geq \eta \} \cup \{ \theta : \|\beta_2 - \beta_2^0\| \leq \varepsilon/2 \} \}
\]

and \( W(1) \). Similarly to for \( W(2) \), \( E[N(\theta; W(1))] \) can due to independence be written as a sum of \( N \) terms in the form

\[
E\{\log[f_\theta(Y_{i,i,1})/f_{\theta^0}(Y_{i,i,1})]\} = -\frac{1}{2} \left[ \log \left( \frac{1 + 2\theta_d}{1 + 2\theta_d^0} \right) + \frac{1 + 2\theta_d^0 + [x_{i,i}^T(\beta_2 - \beta_2^0)]^2}{1 + 2\theta_d} - 1 \right].
\]

(4)

which is the negative K–L divergence between two univariate normal distributions.

Let us consider the possible values this can take for \( \theta \in A_1 \). If \( |\theta_d - \theta_d^0| \geq \eta \), then (4) is upper bounded by what is obtained when \( \beta_1 = \beta_1^0 \). This in turn is a continuous function in \( \theta_d \) and hence attains its supremum on the compact set \( \{ \theta_d : \eta \leq |\theta_d - \theta_d^0| \leq \varepsilon \} \), and hence on \( A_1 \). This supremum is strictly positive because the divergence can be zero only if \( \theta_d = \theta_d^0 \). If instead \( \|\beta_2 - \beta_2^0\| \leq \varepsilon/2 \), then either \( |\theta_d - \theta_d^0| \geq \varepsilon/4 \) or \( \|\beta_1 - \beta_1^0\| \geq \varepsilon/4 \), for otherwise it cannot be that \( \|\theta - \theta^0\| = \varepsilon \). If \( |\theta_d - \theta_d^0| \geq \varepsilon/4 \) the divergence in (4) has a lower bound away from zero by the same argument as for the cases \( |\theta_d - \theta_d^0| \geq \eta \). It remains to deal with the case \( \|\beta_1 - \beta_1^0\| \geq \varepsilon/4 \).

Write \( [x_{i,i}^T(\beta_1^0 - \beta_1)]^2 = (\beta_1^0 - \beta_1)^T x_{i,i} x_{i,i}^T (\beta_1^0 - \beta_1) \) to see that \( -2N^{-1} \Lambda_N(\theta; W(1)) \) is equal to

\[
\log \left( \frac{1 + 2\theta_d}{1 + 2\theta_d^0} \right) + \frac{1 + 2\theta_d^0 + N^{-1} \sum_{i=1}^{N} (\beta_1^0 - \beta_1)^T x_{i,i} x_{i,i}^T (\beta_1^0 - \beta_1)}{1 + 2\theta_d} - 1,
\]
which has a lower limit that is greater than
\[
\log \left( \frac{1 + 2 \theta_d}{1 + 2 \theta_d^0} \right) + \frac{1 + 2 \theta_d^0 + c_3(\varepsilon/4)^2}{1 + 2 \theta_d} - 1.
\]
This expression is in turn maximized in \( \theta_d \) at \( \theta_d = \theta_d^0 + c_3(\varepsilon/16)^2 \); this follows from a straightforward optimization in \( 1 + 2 \theta_d \). The corresponding maximum evaluates to \( \log(1 + 2 \theta_d^0 + c_3(\varepsilon/4)^2) - \log(1 + 2 \theta_d^0) > 0 \). This finishes the proof of point 1.

The proof of point 2 consists of checking the conditions of Lemma [B.8]. We first work with \( A_1 \) and \( W^{(1)} \). Let \( h_i(\omega, \theta) = \log[f_\theta(Y_{i,i,1}(\omega))/f_{\theta_0}(Y_{i,i,1}(\omega))] \) be the log-likelihood ratio for the \( i \)th observation in the first subcollection, \( i = 1, \ldots, N \). We equip \( \mathcal{H}_{N,\omega} \) with the \( L_1 \) norm \( \| \cdot \|_1 \), and \( \Theta \) is equipped with the \( L_2 \) norm as before. To facilitate checking the two conditions we will first derive envelopes with the following properties: \( \sup_{-\infty < i < \infty} \mathbb{E}H_i^k < \infty \) for every \( k \geq 0 \), \( \sup_{-\infty < i < \infty} \mathbb{P}(H_i \geq K) \to 0 \) as \( K \to 0 \), and each \( h_i(\omega, \theta) \) is \( H_i \)-Lipschitz in \( \theta \) on \( \bar{B}_\varepsilon(\theta_0) \), and hence on \( A_1 \), for every \( \omega \). We start with the Lipschitz property.

Let us use the slight abuse of notation that \( y_{i,i,1} = Y_{i,i,1}(\omega) \). Since the distribution of \( W^{(1)} \) does not depend on \( \beta_2 \) we have \( \nabla_{\beta_2} h_i(\omega, \theta) = 0 \), and for some \( c_1, c_2, c_3, c_4, c_5 > 0 \) (depending on \( \varepsilon \)), and every \( \theta \in \bar{B}_\varepsilon(\theta_0) \),
\[
\| \nabla_{\beta_2} h_i(\omega, \theta) \| = \| (y_{i,i,1} - x_{i,i}^T \beta_1) x_{i,i}/(1 + 2 \theta_d) \| \leq c_1 |y_{i,i,1}| + c_2
\]
\[
| \nabla_{\theta_d} h_i(\omega, \theta) | = \frac{1}{2} \left| \frac{1}{1 + 2 \theta_d} - (y_{i,i,1} - x_{i,i}^T \beta_1)^2/(1 + 2 \theta_d)^2 \right| \leq c_3 + c_4 (|y_{i,i,1}| + c_5)^2.
\]
The existence of these constants follow from Lemma [B.1]. Let \( H_i \) be the sum of the bounds, i.e.
\[
H_i(\omega) = c_1 |y_{i,i,1}| + c_2 + c_3 + c_4 (|y_{i,i,1}| + c_5)^2.
\]
By the mean value theorem, \(| h_i(\omega, \theta) - h_i(\theta', \omega) | = |(\theta - \theta')^T \nabla h_i(\omega, \tilde{\theta})| \leq \| \theta - \theta' \| H_i \) for some \( \tilde{\theta} \) between \( \theta \) and \( \theta' \). That is, \( h_i \) is \( H_i \)-Lipschitz on \( \bar{B}_\varepsilon(\theta_0) \). That \( H_i \) is an envelope for \( h_i \) follows from noting that \( h_i(\omega, \theta_0) = 0 \) so by taking \( \theta' = \theta_0 \) in the previous calculation, \( |h_i(\omega, \theta)| \leq H_i \| \theta - \theta_0 \| \leq H_i \) on \( \bar{B}_\varepsilon(\theta_0) \). That \( \sup_i \mathbb{E}(H_i^k) < \infty \) for every \( k > 0 \) and \( \sup_i \mathbb{P}(H_i > K) \to 0 \) as \( K \to \infty \) follow from that \( Y_{i,i,1} \) is normally distributed with variance \( 1 + 2 \theta_d^0 \), not depending on \( i \), and mean satisfying \(-\| \beta_1 \|^2 \leq x_{i,i}^T \beta_1 \leq \| \beta_1 \|^2 \). We are now ready to check the conditions of Lemma [B.8].

By the Cauchy–Schwartz inequality and the properties just derived, we have for every fixed \( N \) that
\[
N^{-1} \sum_{i=1}^N \mathbb{E}[H_i I(H_i > K)] \leq \sup_i \mathbb{E}[H_i^2] \sup_i \mathbb{P}(H_i \geq K) \to 0, \ K \to \infty,
\]
which verifies the first condition.

For the second condition, note that the derived Lipschitz property gives, for arbitrary \( h = (h_1(\omega, \theta), \ldots, h_N(\omega, \theta)) \) and \( h' = (h_1(\omega, \theta'), \ldots, h_N(\omega, \theta')) \) in \( \mathcal{H}_{N, \omega} \):

\[
\| h - h' \|_1 = \sum_{i=1}^{N} |h_i(\omega, \theta) - h_i(\omega, \theta')| 
\leq \sum_{i=1}^{N} \| \theta - \theta' \| H_i(\omega) 
= \| \theta - \theta' \| \| H \|_1.
\]

Thus, if we cover \( \partial B_\varepsilon(\theta^0) \) with \( \varepsilon \)-balls with centers \( \theta^j \), \( j = 1, \ldots, M \), then the corresponding \( L_1 \) balls in \( \mathbb{R}^N \) of radius \( \varepsilon \| H \|_1 \) with centers \( h^j = (h_1(\omega, \theta^j), \ldots, h_N(\omega, \theta^j)) \) cover \( \mathcal{H}_{N, \omega} \). This is so because for every \( \theta \in \partial B_\varepsilon(\theta^0) \) there is a \( \theta^j \) such that \( \| \theta - \theta^j \| \leq \varepsilon \), and hence by the Lipschitz property \( \| h(\omega, \theta) - h(\omega, \theta^j) \|_1 \leq \| H \|_1 \varepsilon \).

Thus, \( C(\varepsilon \| H \|_1, \mathcal{H}_{N, \omega}, \| \cdot \|_1) \leq C(\varepsilon, \partial B_\varepsilon(\theta^0), \| \cdot \|) \). Since \( C(\varepsilon, \partial B_\varepsilon(\theta^0), \| \cdot \|) \) is constant in \( N \), the second condition of Lemma 3.8 is verified for \( A_1 \) and \( W(1) \).

The arguments for \( A_2 \) and \( W(2) \) are similar, redefining \( h_i(\omega, \theta) \) with \( Y_{i,i,1} \) replaced by \( Y_{i,1,2} \), taking \( A_2 \) in place of \( A_1 \), and so on. We need only prove the existence of envelopes \( H_1, \ldots, H_N \) with the desired properties. Using that \( |y_{i,j,2} - c'(l_{i,2,1})| \leq 1 \) and that \( f_\theta(y_{i,i,2} | u)f_\theta(u)/f_\theta(y_{i,i,2}) = f_\theta(u | y_{i,i,2}) \) one gets,

\[
\| \nabla_{\beta_2} h_i(\omega, \theta) \| = \left\| \frac{1}{f_\theta(y_{i,i,2})} \int f_\theta(y_{i,i,2} | u)f_\theta(u)du \right\|
\leq \| x_{i,i} \| \leq 1.
\]

Using that \( U_1^{(1)} \) and \( U_2^{(2)} \) are the only random effects entering the linear predictor \( l_{i,j,2} \), and that \( f_\theta(y_{i,j,2} | u) \leq 1 \),

\[
\| \nabla_{\theta_d} h_i(\omega, \theta) \| = \left| \frac{1}{f_\theta(y_{i,i,2})} \int f_\theta(y_{i,i,2} | u)f_\theta(u_i^{(1)}, u_j^{(2)}) \left( \frac{(u_i^{(1)})^2 + (u_j^{(2)})^2}{2\theta_d^2} - \frac{1}{\theta_d} \right) du \right|
\leq \frac{1}{2\theta_df_\theta(y_{i,i,2})} \int f_\theta(u_i^{(1)}, u_j^{(2)}) \left( \frac{(u_i^{(1)})^2 + (u_j^{(2)})^2}{\theta_d} \right) du + \frac{1}{\theta_d}
= \frac{1}{\theta_d} + \frac{1}{\theta_d}.
\]

By Lemma 3.8, the quantity in the last line attains its supremum on \( \partial B_\varepsilon(\theta^0) \). This maximum is finite for both \( y_{i,i,2} = 1 \) and \( y_{i,i,2} = 0 \) since the marginal success probability cannot be one or zero on interior points of \( \Theta \). Thus, on \( \partial B_\varepsilon(\theta^0) \), \( \| \nabla h_i(\omega, \theta) \| \)
is bounded by a constant, say $H$, the largest needed for the two cases $y_{i,1,2} = 0$ and $y_{i,1,2} = 1$. By setting $H_i = H, i = 1, \ldots, N$, we have envelopes with the right properties and this completes the proof of point 2.

Finally, we prove point 3. Consider without loss of generality the first subset and subcollection. For economical notation we write $L_N(\theta) = L_N(\theta; W^{(1)})$ and $\Lambda_N(\theta) = \Lambda_N(\theta; W^{(1)})$. Point 1 gives that $\sup_{\theta \in A_1} E[\Lambda_N(\theta)] < -3\epsilon$ for some $\epsilon > 0$ and all large enough $N$. Assuming that $N$ is large enough that this holds, we get

$$
\mathbb{P}\left(e^{\epsilon N} \sup_{\theta \in A_1} L_N(\theta) > e^{-\epsilon N}\right) = \mathbb{P}\left(N^{-1} \sup_{\theta \in A_1} \Lambda_N(\theta) > -2\epsilon\right)
\leq \mathbb{P}\left(N^{-1} \sup_{\theta \in A_1} \Lambda_N(\theta) > \epsilon + \sup_{\theta \in A_1} E[\Lambda_N(\theta)]\right)
= \mathbb{P}\left(N^{-1} \sup_{\theta \in A_1} \Lambda_N(\theta) - \sup_{\theta \in A_1} E[\Lambda_N(\theta)] > \epsilon\right)
\leq \mathbb{P}\left(N^{-1} \sup_{\theta \in A_1} |\Lambda_N(\theta) - E[\Lambda_N(\theta)]| > \epsilon\right),
$$

which vanishes as $N \to \infty$ by point 2. Thus, since $e^{-\epsilon N} \to 0$, $e^{\epsilon N} \sup_{\theta \in A_1} L_n(\theta) \to 0$.$\Box$

**Proof Proposition 3.6.** We will find a Lipschitz constant (random variable) with the desired properties by bounding $\|\nabla \log f_\theta(y)\|$. We first consider derivatives with respect to $\theta_d$. Define

$$
J^n(\theta) = (2\pi \theta_d)^N f_\theta(y) = \int f_\theta(y \mid u) \exp\left(-\frac{u^T u}{2\theta_d}\right) du
$$

and

$$
K^n(\theta) = \int f_\theta(y \mid u) \exp\left(-\frac{u^T u}{2\theta_d}\right) \frac{u^T u}{2\theta_d^2} du.
$$

Then $\nabla_{\theta_d} J^n(\theta) = K^n(\theta)$, and hence

$$
\nabla_{\theta_d} \log f_\theta(y) = \nabla_{\theta_d} \log[(2\pi \theta_d)^{-N} J^n(\theta)] = -\frac{N}{\theta_d} + \frac{K^n(\theta)}{J^n(\theta)}.
$$

We focus on the second term first. Let $A_n = \{u \in \mathbb{R}^{2N} : u^T u \leq a_n\}$ for some constant $a_n$ (depending on the total sample size $n$). Let $K^n_1(\theta)$ be the integral defining $K^n(\theta)$ restricted to $A_n$, and let $K^n_2(\theta)$ be the same integral but instead restricted to $A_n^c$ so that $K^n(\theta) = K^n_1(\theta) + K^n_2(\theta)$. Then, since the integrands are non-negative,

$$
K^n_1(\theta)/J^n(\theta) = \frac{\int_{A_n} f_\theta(y \mid u) \exp\left(-\frac{u^T u}{2\theta_d}\right) \frac{u^T u}{2\theta_d^2} du}{\int f_\theta(y \mid u) \exp\left(-\frac{u^T u}{2\theta_d}\right) du} \leq \frac{a_n}{2\theta_d^2}.
$$

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and, hence,
\[ |\nabla_{\theta_d} \log f_\theta(y)| \leq \frac{N}{\theta_d^2} + \frac{a_n}{2\theta_d^2} + \frac{K^2_n(\theta)}{f_n(\theta)}. \]

On \( A_n^c \), we have by definition that \( u^T u \geq u^T u/2 + a_n/2 \). Thus, using that \( f_\theta(y | u) \leq (2\pi)^{-n/2} \),

\[ K^2_n(\theta) \leq \int_{A_n^c} f_\theta(y | u) \exp \left( -\frac{1}{2\theta_d^2}(u^T u/2 + a_n/2) \right) u^T u \frac{du}{2\theta_d^2} \]
\[ \leq \frac{1}{2\theta_d^2} e^{-\frac{a_n}{4\theta_d^2}} \int f_\theta(y | u) \exp \left( -\frac{u^T u}{4\theta_d} \right) u^T u du \]
\[ \leq \frac{1}{2\theta_d^2} e^{-\frac{a_n}{4\theta_d^2} (2\pi)^{-n/2}} \int \exp \left( -\frac{u^T u}{4\theta_d} \right) u^T u du \]
\[ = \frac{1}{2\theta_d^2} e^{-\frac{a_n}{4\theta_d^2} (2\pi)^{-n/2}} \int (4\pi\theta_d)^{-N} \exp \left( -\frac{u^T u}{4\theta_d} \right) u^T u du \]
\[ = 4N\theta_d e^{-\frac{a_n}{4\theta_d} (2\pi)^{-n/2}} (4\pi\theta_d)^N. \tag{5} \]

Using Lemma \[\text{Lemma B.1}\] can be upper bounded on \( \tilde{B}_c(\theta^0) \) by \( h_n^0 = \exp(c_1a_n + c_2n + c_3N + c_4\log N + c_5) \) for some constants \( c_1, \ldots, c_5 \). It will be important later to note that the constant \( c_1 \) is negative in this expression.

We next derive a lower bound on \( J^0(\theta) \). To that end, let \( B_n = \{ u \in \mathbb{R}^{2N} : |u_i| \leq 1, i = 1, \ldots, N \} \). Since the integrand in \( J^0(\theta) \) is positive, we may lower bound it by the same integral restricted to \( B_n \). We then get, using that \( \exp(-u^T u/(2\theta_d)) \geq \exp(-N/\theta_d) \) on \( B_n \) and that Lebesgue measure of \( B_n \) is \( 4N \),

\[ J^0(\theta) \geq \exp \left( -\frac{N}{\theta_d} \right) \int_{B_n} f_\theta(y | u) du \]
\[ \geq e^{-\frac{N}{\theta_d}} (2\pi)^{-n/2} \exp \left( -\sum_{i,j} y_{i,j,1}^2/2 + |y_{i,j,1}|(|x_{i,j,1}^T \beta_1| + 2) + |x_{i,j,1}^T \beta_1| + 2 \right)^2 \]
\[ \times \exp \left( -\sum_{i,j} |y_{i,j,2}|(|x_{i,j,2}^T \beta_2| + 2) + \log(1 + e^{|x_{i,j,2}^T \beta_2| + 2}) \right) 4N. \tag{6} \]

Here, the last inequality lower bounds all terms in the exponent by minus their absolute values. Again using Lemma \[\text{Lemma B.1}\] that the predictors are bounded, and that \( |y_{i,j,2}| \leq 1 \), we thus see that \( J^0(\theta) \) can be lower bounded on \( \tilde{B}_c(\theta^0) \) by \( h_2^0(y) = \exp(c_6N + c_7n + c_8 \sum_{i,j} y_{i,j,1}^2 + c_9 \sum_{i,j} |y_{i,j,1}| + c_{10}) \), for some constants \( c_6, \ldots, c_{10} \). Thus, by lower bounding \( \theta_d > c_{11}^{-1} \) on \( \tilde{B}_c(\theta^0) \) for some \( c_{11} > 0 \) we get

\[ \sup_{\theta \in \tilde{B}_c(\theta^0)} |\nabla_{\theta_d} \log f_\theta(y)| \leq c_{11}N + c_{11}^2 a_n/2 + \frac{h_1^n}{h_2^n(y)}. \]
Now, take \( a_n = n^{1+\epsilon/2} \) for some \( \epsilon > 0 \). Then the first two terms are \( O(a_n) \) as \( n \to \infty \). Moreover, since \( \sum_{i,j} E(Y_{i,j,1}^2) \leq n(1 + 2\theta_0^2) + n\|\theta_0\| = O(n) \) by boundedness of the predictors, both sums in the exponent of \( h_1^n/h_2^n(y) \) converges to zero in \( L_1 \) if divided by \( a_n \), and hence also in probability. It follows from the continuous mapping theorem that \( h_1/h_2^0(y) = O_P(1) \) since, as remarked above, \( c_1 < 0 \). We have thus proven that \( \sup_{\theta \in B_n^{(\theta')}} \| \nabla_{\theta_1} \log f_\theta(y) \| = O_P(a_n) = o_P(n^{1+\epsilon}) \), for every \( \epsilon > 0 \).

For \( \beta_1 \) we get by using the triangle inequality, boundedness of the predictors, \( t(1-t) \leq 1/4, t \in \mathbb{R} \), and \( f_\theta(y \mid u)f_\theta(u)/f_\theta(y) = f_\theta(u \mid y) \),

\[
\left\| \nabla_{\beta_1} \log f_\theta(y) \right\| = \frac{1}{f_\theta(y)} \left\| \int f_\theta(y \mid u)f_\theta(u) \sum_{i,j} \left[ y_{i,j,1} - l_{i,j,1} \right] x_{i,j} du \right\|
\]

\[
\leq \sum_{i,j} \left[ y_{i,j,1} - x_{i,j,1} - x_{i,j} \beta_1 \right] + \frac{1}{f_\theta(y)} \int f_\theta(y \mid u)f_\theta(u) \sum_{i,j} |u_{i}^{(1)} + u_{j}^{(2)}| du
\]

\[
\leq \sum_{i,j} \left[ y_{i,j,1} - x_{i,j} \beta_1 \right] + \frac{1}{f_\theta(y)} \int f_\theta(y \mid u)f_\theta(u) \sum_{i,j} \left[ 1/2 + (u_{i}^{(1)})^2 + (u_{j}^{(2)})^2 \right] du
\]

\[
= \sum_{i,j} \left[ y_{i,j,1} - x_{i,j} \beta_1 \right] + n/2 + \frac{1}{f_\theta(y)} \int f_\theta(y \mid u)f_\theta(u) u^T du
\]

\[
= \sum_{i,j} \left[ y_{i,j,1} - x_{i,j} \beta_1 \right] + n/2 + 2\theta_0^2 \frac{K_n(\theta)}{f_n(\theta)}
\]

Thus, by Lemma B.1 and the same arguments as for \( \nabla_{\theta_d} \log f_\theta(Y) \) we get that

\[
\sup_{\theta \in B_n^{(\theta')}} \| \nabla_{\beta_1} \log f_\theta(Y) \| = o_P(n^{1+\epsilon}) \) for any \( \epsilon > 0 \).

Finally, by the triangle inequality and that \( |y_{i,j,2} - c'(l_{i,j,2})| \leq 1 \) for all \( i \) and \( j \),

\[
\left\| \nabla_{\beta_2} \log f_\theta(y) \right\| = \left\| \frac{1}{f_\theta(y)} \int f_\theta(y \mid u) \sum_{i,j} \left[ y_{i,j,2} - c'(l_{i,j,2}) \right] x_{i,j} f_\theta(u) du \right\|
\]

\[
\leq n
\]

\[ \square \]

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