Abstract

We show that some finite $W$-superalgebras based on $gl(M|N)$ are truncation of the super-Yangian $Y(gl(M|N))$. In the same way, we prove that finite $W$-superalgebras based on $osp(M|2n)$ are truncation of the twisted super-Yangians $Y(gl(M|2n))^{\pm}$. Using this homomorphism, we present these $W$-superalgebras in an $R$-matrix formalism, and we classify their finite-dimensional irreducible representations.
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1 Introduction

\( \mathcal{W} \)-algebras have been introduced in the 2d-conformal models as a tool for the study of these theories. Then, these algebras and their finite-dimensional versions appeared to be relevant in several physical backgrounds. For more details on \( \mathcal{W} \)-algebras, see e.g. [1]. However, a full understanding of their algebraic structure (and of their geometrical interpretation) is lacking. The connection of some of these finite \( \mathcal{W} \)-algebras with Yangians appeared to be a solution at least for the algebraic structure. It could be surprising that Yangians [2], which play an important role in integrable systems, see e.g. [3], enter into the study of algebras originating from 2d-conformal models. Let us however note that such a connection have been already remarked in WZW models [4]. For more informations on the algebraic structure of Yangians, see e.g. [5] and ref. therein.

The existence of an algebra homomorphism between Yangian based on \( sl(N) \) and finite \( \mathcal{W}(sl(Np), N.sl(p)) \)-algebras was first proved in [6]. Such a connection plays a role in the study of physical models: for instance, in the case of the \( N \)-vectorial non-linear Schrödinger equation on the real line, the full symmetry is the Yangian \( Y(gl(N)) \equiv Y(N) \), but the space of states with particle number less than \( p \) is a representation of the \( \mathcal{W}(gl(Np), p.sl(N)) \) algebra [7].

Later, the connection between Yangians and finite \( \mathcal{W}(gl(Np), N.sl(p)) \)-algebras was proven in the FRT presentation [9] of the Yangian. It appears that in this framework the above \( \mathcal{W} \)-algebras are nothing but truncations of the Yangian \( Y(N) \), indicating the level where the truncation occurs. Thanks to this presentation, an (evaluated) \( R \)-matrix for these \( \mathcal{W} \)-algebras was given, and their finite-dimensional irreducible representations classified [8].

Then [10], this connection was extended to a class of \( \mathcal{W} \)-algebras, namely the algebras of type \( \mathcal{W}[so(2mp), m.sl(p)] \), \( \mathcal{W}[so((2m+1)p), m.sl(p) + so(p)] \) and \( \mathcal{W}[sp(2np), n.sl(p)] \), which where related to truncation of twisted Yangian \( Y^\pm(N) \). Note that although Yangians based on orthogonal and symplectic algebras exist [2], and admit an FRT presentation [11], it is the twisted Yangians introduced by Olshanski [12, 13] which enter into the game. These later are not Hopf algebras but only Hopf coideals in \( Y(N) \). Nevertheless, this relation allows to give an \( R \)-matrix presentation of the \( \mathcal{W} \)-algebras under consideration, with however the slight change that it is an "RSRS" relation which occurs, not an "RTT" one. The classification of finite-dimensional irreducible representations of the \( \mathcal{W} \)-algebras then follows [10].

The aim of the present article is to extend the above correspondence to the case of finite \( \mathcal{W} \)-superalgebras, based on Lie superalgebras \( gl(M|N) \) and \( osp(M|2n) \). As for \( gl(N) \) on the one hand, and \( so(m) \) and \( sp(2n) \) on the other hand, the treatment for \( gl(M|N) \) and for \( osp(M|2n) \) will be very different. Due to this difference, this article is divided in two main parts. In the first part, we show that \( \mathcal{W}(gl(Mp|Np), (M + N)gl(p)) \) superalgebras are truncation of the super-Yangian based on \( gl(M|N) \), leading to an "RTT" presentation of these \( \mathcal{W} \)-superalgebras. We use this property to classify the finite-dimensional irreducible representations of these \( \mathcal{W} \)-superalgebras. In the second part, we deal with \( \mathcal{W} \)-superalgebras based on \( osp(M|N) \) and twisted super-Yangians. We show that these \( \mathcal{W} \)-superalgebras are
truncations of twisted super-Yangians, leading to an "RSRS" presentation of the formers and a classification of their finite-dimensional irreducible representations.

2 Super-Yangian

The super-Yangian $Y(gl(M|N)) = Y(M|N)$ was first defined by Nazarov [14]. It can be obtained as the generalization of the construction for the Yangian $Y(M)$, based on the Lie algebra $gl(M)$, to the case of the Lie superalgebra $gl(M|N)$. Its representations have been studied by Zhang [15].

2.1 Introduction to $Y(M|N)$

The Lie superalgebra $gl(M|N)$ is a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$ spanned by the basis $\{E_{ab}|a, b = 1, 2, ..., M + N\}$. We introduce the gradation index $[\cdot]$:

$$[a] = \begin{cases} 0 & \text{if } a \leq M \\ 1 & \text{if } M < a \leq M + N \end{cases} \quad \text{and} \quad [E_{ab}] = [a] + [b]$$

The bilinear graded commutator associated to $gl(M|N)$ is defined as follows:

$$[\cdot, \cdot] : \begin{cases} gl(M|N) \otimes gl(M|N) \to gl(M|N) \\ \{E_{ab}, E_{cd}\} = \delta_{cb}E_{ad} - (-1)^{([a]+[b])([c]+[d])}\delta_{ad}E_{cb} \end{cases} \quad (2.1)$$

The super-Yangian $Y(M|N)$ is a $\mathbb{Z}_2$-graded Hopf algebra generated by an infinite set of elements $T_{(n)}^{ab}$, $a, b = 1, 2, ..., M + N$ and $n \in \mathbb{Z}_{>0}$. The $T_{(n)}^{ab}$ are even if $[a] + [b] \equiv 0 \pmod{2}$ and odd otherwise.

We introduce the generating function

$$T(u) = \sum_{a, b=1}^{M+N} T^{ab}(u) E_{ab} \quad \text{and} \quad T^{ab}(u) = \sum_{n=0}^{\infty} T_{(n)}^{ab} u^{-n} \quad (2.2)$$

with $T_{(0)}^{ab} = \delta^{ab}$, $u$ a spectral parameter and $E_{ab}$ the matrix with 1 at position $(a, b)$ and 0 elsewhere.

The following $R$ matrix

$$R(u) = 1 \otimes 1 - \frac{P}{u} \quad u \in \mathbb{C}$$

satisfies the graded Yang-Baxter equation. The permutation operator $P$ is defined by:

$$P_{12} = \sum_{i,j} (-1)^{[j]} E_{ij} \otimes E_{ji} \quad (2.3)$$
and the tensor product is chosen graded:
\[(E_{ij} \otimes E_{kl}) \cdot (E_{mn} \otimes E_{pq}) = (-1)^{(|k|)+(|l|)(|m|)+(|n|)}E_{ij}E_{mn} \otimes E_{kl}E_{pq}\] (2.4)

The defining relations in \(Y(M|N)\) can be written as follows:
\[R(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u - v)\] (2.5)

with
\[T_1(u) = \sum_{a,b=1}^{M+N} T^{ab}(u) \ u_{ab} \otimes 1 \ \text{and} \ T_2(v) = \sum_{a,b=1}^{M+N} T^{ab}(v) 1 \otimes u_{ab}\] (2.6)

We can rewrite equation (2.5) as follows:
\[\{T^{ab}(u), T^{cd}(v)\} = \frac{(-1)^{|c|(|a|)+|a|(|b|)+|b|(|d|)}}{u - v} \left( T^{cd}(u)T^{ad}(v) - T^{cb}(v)T^{ad}(u) \right)\] (2.7)

or equivalently:
\[\{T^{ab}_{(m)}, T^{cd}_{(n)}\} = \delta^{cb}T^{ad}_{(m+n-1)} - (-1)^{|c|(|a|)+|a|(|d|)+|b|(|d|)}}\delta^{ad}T^{cb}_{(m+n-1)} + \]
\[+ (-1)^{|c|(|a|)+|a|(|b|)+|b|(|d|)+|d|(|b|)+|b|(|d|)+|d|(|b|)+|b|(|d|)}) \sum_{r=1}^{min-1} \left\{ T^{cb}_{(r)}T^{ad}_{(m+n-1-r)} - T^{cb}_{(m+n-1-r)}T^{ad}_{(r)} \right\}\] (2.8)

where \(min\) stands for \(min(m, n)\).

The Hopf structure is given by:
\[\epsilon(T^{ab}(u)) = \delta^{ab} \ \text{and} \ S(T^{ab}(u)) = (T^{-1}(u))^{ab}\] (2.9)
\[\Delta(T^{ab}(u)) = \sum_{c=1}^{M+N} T^{ac}(u) \otimes T^{cb}(u)\] (2.10)

The super-Yangian \(Y(M|N)\) is a deformation of the enveloping algebra of a polynomial algebra (restricted to its positive modes) based on \(gl(M|N)\), noted \(U(gl(M|N)[x])\). The parameter \(\hbar\) can be recovered after rescaling of the generators by an appropriate power of \(\hbar^{-1} T^{ab}_{(n)} \rightarrow \hbar^{n-1} T^{ab}_{(n)}\).

### 2.2 Finite-dimensional irreducible representations of \(Y(M|N)\)

The finite-dimensional irreducible representations of \(Y(M|N)\) have been studied in [15]. We recall here the main results, using a different basis for the positive roots (see [17] for details).

We introduce \(\mathbb{N}_{M+N} = [1, M + N] \cap \mathbb{Z}_+\), \(\mathbb{N}_{M+N}^2 = \mathbb{N}_{M+N} \times \mathbb{N}_{M+N}\) and

\[\Phi^+ = \left\{ (a, b) \in \mathbb{N}_{M+N}^2, \ \text{with either} \begin{array}{l}
1 \leq a < b \leq M \\
M + 1 \leq a < b \leq M + 2n \\
1 \leq a \leq M \ \text{and} \ M + n + 1 \leq b \leq M + 2n \\
M + n + 1 \leq a \leq M + 2n \ \text{and} \ 1 \leq b \leq M 
\end{array} \right\}\] (2.11)
Definition 2.1: Let \( V \) be an irreducible \( Y(M|N) \)-module. A nonzero element \( v_+^\Lambda \in V \) is called highest weight vector if

\[
T_{(n)}^{ab} v_+^\Lambda = 0, \quad \forall (a, b) \in \Phi_+ \quad n > 0
\]

\[
T_{(n)}^{aa} v_+^\Lambda = \lambda_a^{(n)} v_+^\Lambda, \quad a = 1, \ldots, M + N, \quad n > 0, \quad \lambda_a^{(n)} \in \mathbb{C}.
\] (2.12)

An irreducible module is called a highest weight module if it admits a highest weight vector. We define

\[
\Lambda(u) \equiv (\lambda_1(u), \lambda_2(u), \ldots, \lambda_{M+N}(u)) \quad (2.13)
\]

with \( \lambda_a(u) = 1 + \sum_{n>0} \lambda_a^{(n)} u^{-n} \) and call \( \Lambda(u) \) a highest weight of \( V \).

Theorem 2.2: Every finite-dimensional irreducible \( Y(M|N) \)-module \( V \) contains a unique (up to scalar multiples) highest weight vector \( v_+^\Lambda \).

Corresponding to each \( \Lambda(u) \) of the form (2.13), there exists a unique irreducible highest weight \( Y(M|N) \)-module \( V(\Lambda) \) with highest weight \( \Lambda(u) \).

Theorem 2.3: The irreducible highest weight \( Y(M|N) \)-module \( V(\Lambda) \) is finite dimensional if only if its highest weight \( \Lambda(u) \) satisfies the following conditions:

\[
\frac{\lambda_a(u)}{\lambda_a+1(u)} = \frac{P_a(u+1)}{P_a(u)} \quad 1 \leq a < N + M, \quad a \neq M
\]

\[
\frac{\lambda_M(u)}{\lambda_M+1(u)} = \frac{\hat{P}_M(u)}{P_M(u)}
\] (2.14)

where, \( m_a \) being the degree of \( P_a \),

\[
P_a(u) = \prod_{i=1}^{m_a} (u - \gamma_a^{(i)}) \quad 1 \leq a < N + M \text{ and } a \neq M, \quad \gamma_a^{(i)} \in \mathbb{C}
\]

\[
\hat{P}_M(u) = \prod_{i=1}^{m_M} \left( 1 - \frac{r(i)}{u} \right) \quad \text{and} \quad P_M(u) = \prod_{i=1}^{m_M} \left( 1 - \frac{r(i)}{u} \right), \quad r(i), \hat{r}(i) \in \mathbb{C}
\] (2.15)

Among the finite-dimensional highest weight representations, there is a class of particular interest:

Definition 2.4 (Evaluation representations)

An evaluation representation \( \text{ev}_{\pi_{\mu}} \) is a morphism from the super-Yangian \( Y(M|N) \) to a highest weight irreducible representation \( \pi_{\mu} \) of \( \mathfrak{gl}(M|N) \). The morphism is given by:

\[
\text{ev}_{\pi_{\mu}}(T^{ab}(u)) = \delta^{ab} + \pi_{\mu}(\mathcal{E}^{ab}) u^{-1} \quad \forall a, b \in \{1, \ldots, M + N\}
\] (2.16)

that is

\[
\text{ev}_{\pi_{\mu}}(T^{ab}_{(0)}) = \delta^{ab} ; \quad \text{ev}_{\pi_{\mu}}(T^{ab}_{(1)}) = \pi_{\mu}(\mathcal{E}^{ab}) ; \quad \text{ev}_{\pi_{\mu}}(T^{ab}_{(r)}) = 0 \quad \text{for } r > 1
\] (2.17)
where $E^{ab}$ are the standard $gl(M|N)$ generators.

The highest weight $\mu(u) = (\mu_1(u), \ldots, \mu_{M+N}(u))$ of the representation $ev_{\pi_\mu}$ is given by:

$$\mu_a(u) = 1 + \mu_a u^{-1} \text{ for } a \in \{1, \ldots, M+N\}$$  \hspace{1cm} (2.18)

where $\mu = (\mu_1, \ldots, \mu_{M+N})$ is the highest weight of $\pi_\mu$.

Any finite-dimensional irreducible representation can be obtained through the tensor products of such evaluation representations |15|.

**Definition 2.5 (Tensor product of evaluation representations)**

Let be $\{ev_{\pi_i}\}_{i=1,\ldots,s}$ a set of evaluation representations. The tensor products of these representations $ev_{\hat{\pi}} = ev_{\pi_{i_1}} \otimes \ldots \otimes ev_{\pi_{i_s}}$ is a morphism from $Y(M|N)$ to the tensor product of $gl(M|N)$ representations $\hat{\pi} = \otimes_i \pi_i$ given by:

$$ev_{\hat{\pi}}(T^{ab}_r) = \bigoplus_{r_1+r_2+\ldots+r_n=r} \left( ev_{\pi_{i_1}}(T^{a1}_{r_1}) \otimes ev_{\pi_{i_2}}(T^{i_1i_2}_{r_2}) \otimes \ldots \otimes ev_{\pi_{i_s}}(T^{i_{s-1}b}_{r_s}) \right)$$  \hspace{1cm} (2.19)

where there is an implicit summation on the indices $i_1, i_2, \ldots, i_{s-1} = 1, \ldots, M+N$.

It satisfies:

$$ev_{\hat{\pi}}(T^{ab}_r) \neq 0 \text{ if and only if } r \leq s$$  \hspace{1cm} (2.20)

### 2.3 Truncated super-Yangians

We will proceed as in [8]: we introduce $T \equiv U(\{T^{ij}_n, n > p\})$ and the left ideal $\mathcal{I}_p \equiv Y(M|N) \cdot T$, generated by $T$. We then define the coset (truncation of the super-Yangian at order $p$):

$$Y_p(M|N) \equiv Y(M|N)/\mathcal{I}_p$$  \hspace{1cm} (2.21)

**Property 2.6** The truncated super-Yangian $Y_p(M|N)$ is a superalgebra ($\forall p \in \mathbb{Z}_{>0}$).

**Proof:** As in [8], the Lie superalgebra structure of $Y_p(M|N)$ can be proved by showing that $\mathcal{I}_p$ is a two-sided ideal. We first show that

$$[Y(M|N), T] \subset Y(M|N) \cdot T = \mathcal{I}_p$$  \hspace{1cm} (2.22)

The relation (2.8) shows that $[T^{ij}_m, T^{kl}_n]$ (for $n > p$) is the sum of two terms, the first being in $Y(M|N) \cdot T$, the second belonging to $T \cdot Y(M|N)$. Focusing on the latter, one rewrites it as

$$\sum_{r=0}^{\mu-1} \left( T^{ij}_{r} T^{kl}_{m+n-1-r} + (-1)^{|i|(|k|+|j|+|k|)} \sum_{s=0}^{r-1} (T^{ij} T^{kl}_{m+n-2-s} - T^{ij}_{m+n-2-s} T^{kl}_{s}) \right)$$  \hspace{1cm} (2.23)

$$= \sum_{r=0}^{\mu-1} T^{ij}_{r} T^{kl}_{m+n-1-r} + (-1)^{|i|(|k|+|j|)+|k||j|} \sum_{s=0}^{\mu-2} (\mu - s - 1) (T^{ij}_{s} T^{kl}_{m+n-2-s} - T^{ij}_{m+n-2-s} T^{kl}_{s})$$

*Note however that one has sometimes to make a quotient to get an irreducible representation from these tensor products.
where \( \mu \) stands for \( \min(m, n) \). In (2.23), the first sum belongs to \( T_p \), while the last sum belongs to \( T_p \cdot Y(M|N) \), with a summation which has one term less than the previous one: we can thus proceed recursively in a finite number of steps. The final result is an element of \( Y(M|N) \cdot T_p \). In the same way, one can show that

\[
[Y(M|N) \cdot T_p] \subset T_p \cdot Y(M|N)
\]

so that \( I_p = Y(M|N) \cdot T_p = T_p \cdot Y(M|N) \). 

Note that \( \Delta \) is not a morphism of this superalgebra (for the structure induced by \( Y(M|N) \)), i.e. \( Y_p(M|N) \) has no natural Hopf structure.

Finally, we remark that each \( Y_p(M|N) \) is a deformation of a truncated polynomial algebra based on \( gl(M|N) \). By truncated polynomial algebra, we mean the quotient of a usual \( gl(M|N) \) polynomial algebra (of generators \( T_{ij}(n) \)) by the relations \( T_{ij}(n) = 0 \) for \( n > p \). The construction is the same as for the full super-Yangian.

### 2.4 Poisson super-Yangians

In the following we will deal with classical super-Yangian, where the commutator is replaced by a \( \mathbb{Z}_2 \)-graded Poisson Bracket (PB). It corresponds to the usual classical limit of quantum groups. One sets:

\[
L(u) = \sum_{a,b=1}^{M+N} (-1)^{|b|} T_{ab}(u) \otimes E_{ba}
\]

\[
R_{12}(u) = 1 \otimes 1 + \hbar r_{12}(u) + o(\hbar) \quad \text{with} \quad r_{12}(u) = \frac{P_{12}}{u}
\]

\[
[\ , \ ] = \hbar \{\ , \ } + o(\hbar)
\]

The relation (2.5) is then expanded as a series in \( \hbar \). Since in a classical super-Yangian we have \( T_{ab} T_{cd} = (-1)^{(|a|+|b|)(|c|+|d|)} T_{cd} T_{ab} \), we obtain:

\[
\{T_{ab}(u), T_{cd}(v)\} = \frac{1}{u-v} (-1)^{|c|(|a|+|b|)+|a||b|} \left(T_{ch}(u)T_{ad}(v) - T_{ch}(v)T_{ad}(u)\right)
\]

which leads to:

\[
\{T_{ab}(m), T_{cd}(n)\} = \delta_{cb} T_{ad}(m+n-1) - (-1)^{(|a|+|b|)(|c|+|d|)} \delta_{ad} T_{cb}(m+n-1) + \sum_{\min(m,n)-1}^{m+n-1} \left(T_{ch}(r)T_{ad}(m+n-1-r) - T_{ch}(r)T_{ad}(m+n-1-r)\right)
\]

In classical super-Yangians, all the algebraic properties described above still apply.
For simplicity we note $\mathcal{W}_p(M|N) \equiv \mathcal{W}(gl(Mp|Np), (M + N)gl(p))$

### 3.1 Definition of $\mathcal{W}(\mathcal{G}, \mathcal{H})$ superalgebras and Dirac brackets

$\mathcal{W}(\mathcal{G}, \mathcal{H})$ (super)algebras can be constructed as Hamiltonian reduction on a Lie (super)algebra $\mathcal{G}$, with Poisson Brackets $\{.,.\}$. The construction is done as follows.

We start with an $sl(2)$ embedding in $\mathcal{G}$, this embedding being defined as the principal embedding in a regular sub(super)algebra $\mathcal{H} \subset \mathcal{G}$. We remind that the principal $sl(2)$-embedding of an algebra $\mathcal{H}$ is given by $e_+ = \sum_i e_i$, where $e_+$ is the positive root generator of $sl(2)$, and $e_i$ are the simple roots generators of $\mathcal{H}$. If $\mathcal{H}$ is a superalgebra, the principal $sl(2)$-embedding is defined as the principal embedding of its bosonic part.

Once the $sl(2)$ embedding in $\mathcal{G}$ is fixed (i.e. when $\mathcal{H} \subset \mathcal{G}$ is given), let $(e_\pm, h)$ be its generators, one decomposes $\mathcal{G}$ into $sl(2)$ representations. This amounts to take a $\mathcal{G}$-basis of the form $J^i_{jm}, -j \leq m \leq j$, and $i$ labeling the multiplicities, with

$$[e_\pm, J^i_{jm}] = \alpha_{jm} J^i_{jm \pm 1}, \quad [h, J^i_{jm}] = m J^i_{jm} \text{ with } \alpha_{jm} \in \mathbb{C} \quad (3.1)$$

We take $e_\pm = J^0_{1,\pm 1}$ and $h = J^0_{1,0}$. Then, one introduces a set of second class constrains (in Dirac terminology):

$$J^i_{jm} = \delta_i^0 \delta_{j,1} \delta_{m,-1} \text{ for } m < j, \forall j, \forall i \quad (3.2)$$

This remains to set to zero all the generators but the $sl(2)$-highest weights ones (which are left free), and $e_-$ which is set to 1.

The $\mathcal{W}(\mathcal{G}, \mathcal{H})$ (super)algebra is defined as the enveloping algebra generated by the $sl(2)$ highest weight generators, equipped with the Dirac brackets associated to the constrains $(3.2)$.

We remind that the Dirac brackets can be calculated as follows. If $\Phi = \{\phi_\alpha\}_{\alpha \in I}$ denotes the set of all the above constraints, we have

$$\Delta_{\alpha\beta} = \{\phi_\alpha, \phi_\beta\} \text{ is invertible: } \sum_{\gamma \in I} \Delta_{\alpha\gamma} \Delta_{\gamma\beta} = \delta_\alpha^\beta \text{ where } \Delta_{\alpha\beta} \equiv (\Delta^{-1})_{\alpha\beta} \quad (3.3)$$

The Dirac brackets are constructed as:

$$\{X,Y\}_* \sim \{X,Y\} - \sum_{\alpha,\beta \in I} \{X,\phi_\alpha\} \Delta_{\alpha\beta}\{\phi_\beta, Y\} \quad \forall X,Y \quad (3.4)$$

where the symbol $\sim$ means that one has to apply the constraints on the right hand side once the Poisson Brackets have been computed.

### 3.2 Soldering procedure

The soldering procedure is an alternative way to compute the PB of $\mathcal{W}(\mathcal{G}, \mathcal{H})$ algebras. We apply it to the superalgebra $gl(Mp|Np)$ with generators $\mathcal{E}^i_{jm}, 0 \leq j \leq p - 1, -j \leq m \leq j,$
Let $a, b = 1, \ldots, M+N$ (see appendix A). Let $M_{ab}^{jm}$ be the $(M+N)$ square matrices representing the generators $E_{ab}^{jm}$ in the fundamental representation of $gl(Mp|Np)$. Denoting $J_{jm}^{ab}$ the dual basis, we introduce the matrix

$$
\mathbb{J} \equiv \sum_{a,b=1}^{M+N} \sum_{j=0}^{p-1} \sum_{m=-j}^{j} J_{jm}^{ab} M_{ab}^{jm}
$$

(3.5)

Let us consider an infinitesimal transformation of parameters $\lambda_{jm}^{ab}$. For convenience we define the matrix $\lambda_{jm}^{ab} M_{ab}^{jm}$.

$$
\delta_\lambda \mathbb{J} \equiv (\delta_\lambda J_{jm}^{ab}) M_{ab}^{jm} = [\lambda, \mathbb{J}] = \{ str(\lambda \mathbb{J}), \mathbb{J} \}
$$

(3.6)

$$
= \lambda_{ef}^{rs} str (M_{ef}^{rs} M_{cd}^{tu}) \{ J_{cd}^{tu} J_{jm}^{ab} \} M_{ab}^{jm}
$$

(3.7)

where summation over repeated indices is assumed. $[,]$ denotes the commutator of $Z_2$-graded matrices, and $\{,,\}$ the PB.

We ask $\mathbb{J}$ to be of the form:

$$
\mathbb{J}_{|g.f.} = \epsilon_ - + \sum_{a,b=1}^{M+N} \sum_{j=0}^{p-1} W_{j}^{ab} M_{ab}^{jj}
$$

(3.8)

where $\epsilon_-$ is the $sl(2)$ negative root generator (see appendix A.1). This remains to constrain the generators $J_{jm}^{ab}$ to obey the following second class constraints:

$$
J_{jm}^{ab} = \delta_{j,1} \delta_{m+1,0} \delta_{ab}, \text{ for } -j \leq m < j, \forall j, \forall a,b
$$

(3.9)

We look for transformations leaving $\mathbb{J}_{|g.f.}$ with the same form:

$$
\delta_\lambda (\mathbb{J}_{|g.f.}) = [\lambda, \mathbb{J}_{|g.f.}] = (\delta_\lambda W_{j}^{ab}) M_{ab}^{jj}
$$

(3.10)

The parameters $\lambda_{jm}^{ab}$ are constrained and only $(M+N)^2 p$ of them are free. The equation (3.10) leads to:

$$
\lambda_{j,m+1}^{ab} = \sum_{k,r=0}^{p-1} \sum_{l=-k}^{k} \sum_{e=1}^{M+N} (\lambda_{kl}^{ae} W_{r}^{eb} < k,l ; r,r|jm > - W_{r}^{ae} \lambda_{kl}^{eb} < r,r ; k,l|jm >) (3.11)
$$

for $-j \leq m \leq j - 1$

$$
\delta_\lambda W_{j}^{ab} = \sum_{k,r=0}^{p-1} \sum_{l=-k}^{k} \sum_{e=1}^{M+N} (\lambda_{kl}^{ae} W_{r}^{eb} < k,l ; r,r|jj > - W_{r}^{ae} \lambda_{kl}^{eb} < r,r ; k,l|jj >) (3.12)
$$

where $< \cdot | \cdot >$ are real numbers defined in appendix A.1. All the coefficients $\lambda_{kl}$ can be expressed in terms of the parameters $\lambda_{k,-k}$ and the generators $W$, after a straightforward but tedious use of equations (3.11).
On the other hand we have:

\[ \delta \lambda W^a_b = \lambda^e_f \text{str}(M^r_s M^k{k}) \{ W^c_d, W^a_b \} \]  

(3.13)

With appendix A of [8] we obtain:

\[ \text{str}(M^r_s M^k{k}) = \delta^r_e \delta^s_k \delta^k_d \delta^d_e (-1)^{[d]} (-1)^k (2k)! (k!)^2 \left( \begin{array}{c} p + k \\ 2k + 1 \end{array} \right) \]  

(3.14)

We define

\[ \tilde{\lambda}_k^{ab} \equiv (-1)^k (2k)! (k!)^2 \left( \begin{array}{c} p + k \\ 2k + 1 \end{array} \right) \lambda^{ab}_{k,k} \]  

(3.15)

Equation (3.13) becomes:

\[ \delta \lambda W^a_b = \sum_{k=0}^{p-1} \sum_{c,d=1}^{M+N} (-1)^{[d]} \tilde{\lambda}^{ae}_{k,k} W^{eb}_{k} \{ W^{cd}_{k}, W^{a}_b \} \]  

(3.16)

If we now compare (3.12) and (3.16), the \( \tilde{\lambda}_k^{ab} \) being independent from one another, we get \( \{ W^{cd}_{k}, W^{a}_b \} \) as a polynomial in the \( W \)'s.

3.3 Calculation of Poisson Brackets

We now give two examples of PB calculations which will be needed in the following.

3.3.1 Calculation of \( \{ W^{ab}_{0}, W^{cd}_{j} \} \)

For \( j = 0 \) equation (3.12) becomes:

\[ \delta \lambda W^a_b = \sum_{k=0}^{p-1} \left( \lambda^{ae}_{k,k} W^{eb}_{k} < k, -k; k, k|0,0 > - W^{ae}_{k} \lambda^{eb}_{k,-k} < k, k; k, -k|0,0 > \right) \]

\[ = \frac{1}{p} \sum_{k=0}^{p-1} \left( \tilde{\lambda}^{ae}_{k,k} W^{eb}_{k} - (-1)^{[a]+[e]}(k) \tilde{\lambda}^{eb}_{k} W^{ae}_{k} \right) \]  

(3.17)

Equation (3.16) rewrites:

\[ \delta \lambda W^a_b = \sum_{k=0}^{p-1} \sum_{c,d=1}^{M+N} (-1)^{[d]} \tilde{\lambda}^{ae}_{k,k} \{ W^{cd}_{k}, W^{a}_b \} \]

\[ = \sum_{k=0}^{p-1} \sum_{c,d=1}^{M+N} (-1)^{[d]} (-1)^{[a]+[b]+[e]+[d]} \tilde{\lambda}^{dc}_{k} \{ W^{cd}_{k}, W^{a}_b \} \]  

(3.18)
Comparing the $\tilde{\lambda}^d_{k}$-components of both equations, we obtain:

$$
(-1)^{([a]+[b])([c]+[d])+[d]} \{W^a_{0}, W^c_{k}\} = \frac{1}{p} \left( \delta^{bc} (-1)^{([a]+[b])([c]+[d])} W^a_{k} - \delta^{ad} W^b_{k} \right)
$$  \hspace{1cm} (3.19)

If we define $\hat{W}^a_{k} \equiv (-1)^{[a]} W^a_{k}$, $\forall k$, equation (3.19) becomes:

$$
\{\hat{W}^a_{0}, \hat{W}^c_{k}\} = \frac{1}{p} \left( \delta^{cb} \hat{W}^a_{k} - \delta^{ad} (-1)^{([a]+[b])([c]+[d])} \hat{W}^b_{k} \right)
$$  \hspace{1cm} (3.20)

### 3.3.2 Calculation of $\{W^a_{1}, W^c_{j}\}$

Using the same procedure with $j = 1$ we get:

$$
\delta \lambda W^a_{1} = (-1)^{1+[d]+([a]+[b])([c]+[d])} \{W^a_{1}, W^c_{r}\}
$$

$$
= \frac{3}{p(p^2-1)} \sum_{k=1}^{p-1} k(p^2-k^2) \left[ \lambda_{k-1}, W^c_{k} \right]^{ab} + \frac{3}{p(p^2-1)} \sum_{k=1}^{p-1} \sum_{n \geq k} \left[ \lambda_{n}, W^c_{n-k} \right]^{ab} + \frac{3}{p(p^2-1)} \sum_{k=0}^{p-1} \frac{1}{2k+1} \left[ \lambda_{n-1}, W^c_{n-k} \right]^{ab} - \frac{3}{p(p^2-1)} \sum_{m \geq n \geq k \geq 0} \frac{1}{m(2k+1)} \left[ \lambda_{n}, W^c_{n-m} \right]^{ab}
$$

where $\left[ \lambda_{x}, W^c_{y} \right]^{ab} \equiv \sum_{e=1}^{M+N} \left( \lambda^a_{x} W^c_{y} \pm W^a_{y} \tilde{\lambda}^c_{x} \right)$

We use $\hat{W}^a_{k} \equiv (-1)^{[a]} W^a_{k}$ and identify the $\tilde{\lambda}^d_{k}$-components on both sides of the equation:

$$
\frac{p(p^2-1)}{3} \{\hat{W}^a_{r}, \hat{W}^c_{r}\} = \frac{(r+1)(p^2-(r+1)^2)}{2(r+1)+1} \left( \delta^{bc} \hat{W}^a_{r+1} - (-1)^{([a]+[b])([c]+[d])} \delta^{ad} \hat{W}^b_{r+1} \right) + \sum_{k=1}^{r} \left\{ \delta^{bc} (-1)^{[c]} \hat{W}^a_{k} \hat{W}^c_{r-k} - (-1)^{([a]+[b])([c]+[d])} \delta^{ad} (-1)^{[e]} \hat{W}^c_{r-k} \hat{W}^a_{k} \right\} + \sum_{k=0}^{r-1} \frac{r-k}{2k+1} \left\{ \delta^{bc} (-1)^{[c]} \hat{W}^a_{k} \hat{W}^c_{r-k} - (-1)^{([a]+[b])([c]+[d])} \delta^{ad} (-1)^{[e]} \hat{W}^c_{r-k} \hat{W}^a_{k} \right\}
$$  \hspace{1cm} (3.21)
The $\hat{W}$-basis is the one we will work on, we shall therefore omit the $^\circ$ on $W$ from now on.

3.4 $\mathcal{W}(sl(Mp|Np), (M + N)sl(p))$ superalgebras

The $sl(2)$ principal embedding in $(M + N)gl(p)$ is indeed an embedding in $(M + N)sl(p)$, i.e. it commutes with the $(M + N)gl(1)$ generators defined by $gl(p) = sl(p) \oplus gl(1)$. Moreover, considering these $(M + N)gl(1)$ subalgebras in $gl(Mp|Np)$ which commutes with $(M + N)sl(p)$, it is easy to see that none of its generators is affected by the constraints (3.9), since they are highest weights. Furthermore, these $gl(1)$ generators, while they do not commute with all the constraints, weakly commute with them. By weakly, we mean after use of the constraints (once the PB have been computed). Thus, their Dirac brackets coincide with their original PB. This implies that these $gl(1)$ generators still form $gl(1)$ subalgebras in the $\mathcal{W}$-superalgebra.

In addition, the diagonal $gl(1)$ of these $(M + N)gl(1)$ subalgebras, which corresponds to the decomposition $gl(Mp|Np) = sl(Mp|Np) \oplus gl(1)$, is central for the original PB. Therefore, this $gl(1)$ generator is still central for the Dirac brackets. In other words, one gets

$$\mathcal{W}_p(M|N) = \mathcal{W}(gl(Mp|Np), (M + N)gl(p)) = \mathcal{W}(gl(Mp|Np), (M + N)sl(p))$$

$$= \mathcal{W}[sl(Mp|Np) \oplus gl(1), (M + N)sl(p)] = U\left( \mathcal{W}[sl(Mp|Np), (M + N)sl(p)] \oplus gl(1) \right)$$

4 Truncated super-Yangians and $\mathcal{W}$-superalgebras

4.1 $\mathcal{W}_p(M|N)$ as a deformation of a truncated polynomial algebra

Property 4.1 The $\mathcal{W}_p(M|N)$ superalgebra is a deformation of the truncated polynomial superalgebra $gl(M|N)_p$. 

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Theorem: To see that the $\mathcal{W}_p(M|N)$ is a deformation of a truncated polynomial algebra based on $gl(M|N)$, we modify the constraints to

$$ \mathcal{J} = \frac{1}{\hbar} \epsilon_- + \sum_{a,b=1}^N \sum_{j=0}^{p-1} \sum_{0 \leq m \leq j} J_{jm}^a M_j^m \quad (4.1) $$

These constraints are equivalent to the previous ones as soon as $\hbar \neq 0$ (they correspond to a rescaling $J_{jm}^a \to \hbar^{-m} J_{jm}^a$). With these new constraints, the equations associated to the soldering procedure read:

$$ \lambda_{j,m+1}^{ab} = \hbar \sum_{k,r=0}^{p-1} \sum_{k,l=-k}^M \sum_{e=1}^{N+N} \left( \lambda_{kl}^{ae} W_{r}^{eb} < k \ , \ r \ , \ r \mid jm > - W_{r}^{ae} \lambda_{kl}^{eb} < r \ , \ r \mid k \ , \ l \mid jm > \right) $$

for $-j \leq m \leq j - 1 \quad (4.2)$

$$ \delta_j W_{j}^{ab} = \sum_{k,r=0}^{p-1} \sum_{k,l=-k}^M \sum_{e=1}^{N+N} \left( \lambda_{kl}^{ae} W_{r}^{eb} < k \ , \ l \ , \ r \mid jj > - W_{r}^{ae} \lambda_{kl}^{eb} < r \ , \ k \mid j \mid jj > \right) $$

This implies that the parameter $\lambda_{j,m}^{ab}$ behaves as $\hbar^{j+m}$. Then, the Poisson brackets of the $W$ generators take the form:

$$ \{ W_{j}^{ab} , W_{\ell}^{cd} \}_{\hbar} = \delta^{bc}_{\ell} W_{j+\ell}^{ad} - (-1)^{(a+[b])(c+[d])} \delta^{ad}_{\ell} W_{j+\ell}^{cb} - \hbar P_{h}^{abcd}(W) \quad (4.3) $$

where $P_{h}^{abcd}(W)$, polynomial in the $W$’s, has only positive (or null) powers of $\hbar$. This clearly shows that the $\mathcal{W}_p(M|N)$ superalgebra is a deformation of the superalgebra generated by $W_{j}^{ab} \equiv J_{jj}^{ab}$ and with defining (undeformed) Poisson brackets:

$$ \{ W_{j}^{ab} , W_{\ell}^{cd} \}_{0} = \delta^{bc}_{\ell} W_{j+\ell}^{ad} - (-1)^{(a+[b])(c+[d])} \delta^{ad}_{\ell} W_{j+\ell}^{cb} \quad \text{if } j + \ell < p \quad (4.4) $$

$$ = 0 \quad \text{if } j + \ell \geq p \quad (4.5) $$

One recognizes in this superalgebra a (enveloping) polynomial algebra based on $gl(M|N)$ quotiented by the relations $W_{j}^{ab} = 0$ if $j \geq p$. In other words, this algebra is nothing but a truncated polynomial algebra, and the $\mathcal{W}$-superalgebra is a deformation of it.

Property 4.2 There exist two sets of generators $\{ \pm W_{j}^{ab} \}_{j=0}^\infty$ in $\mathcal{W}_p(M|N)$ such that, \forall a, b, c, d = 1, ..., $M + N$:

$$ \forall j \geq 1 \quad \{ \pm W_{1}^{ab} , \pm W_{j}^{cd} \} = \delta^{cb}_{d} \pm W_{j+1}^{ad} - (-1)^{(a+[b])(c+[d])} \delta^{ad}_{d} \pm W_{j+1}^{cb} + \left( (-1)^{c(d+[b])+d[a]} \right) \left( \pm W_{0}^{cb} \pm W_{0}^{ad} - \pm W_{j}^{ad} \pm W_{j}^{cb} \right) $$

$$ \forall j \geq 0 \quad \{ \pm W_{0}^{ab} , \pm W_{j}^{cd} \} = \delta^{cb}_{d} \pm W_{j}^{ad} - (-1)^{(a+[b])(c+[d])} \delta^{ad}_{d} \pm W_{j}^{cb} $$

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The generators $\pm \bar{W}^{ab}_j$ are polynomials of degree $(j+1)$ in the original generators $W^{ab}_j$ and are recursively defined by:

\[
\bar{W}^{ab}_0 \equiv 
\pm W^{ab}_0 = -W^{ab}_0 = pW^{ab}_0
\]

\[
\pm W^{ab}_1 = \pm \frac{p(p^2 - 1)}{6} W^{ab}_1 + \frac{p(p \pm 1)}{2} \sum_{e=1}^{M+N} (-1)^{|e|} W^{ae}_0 W^{eb}_0
\]

and for $j > 1$:

\[
\pm W^{ab}_j = \sum_{n=1}^{j+1} \sum_{|\bar{s}|=j+1-n} \sum_{i_1, \ldots, i_{n-1}=1} (\pm)_{i_1+\ldots+|\bar{s}|} W^{ai_1}_{s_1} W^{i_1 i_2}_{s_2} \cdots W^{i_{n-1}b}_{s_n}
\]

for some numbers $\pm \alpha^{n,j}_{\bar{s}}$ determined by (4.6). The summation on $\bar{s}$ is understood as a summation on $n$ positive (or null) integers $(s_1 \ldots s_n) \equiv \bar{s}$ such that $|\bar{s}| = \sum_{i=1}^{n} s_i = j+1-n$.

The subsets $\{ \pm \bar{W}^{ab}_j \}_{j=0, \ldots, p-1}$ form two bases of $W_p(M|N)$, the other generators $\{ \pm \hat{W}^{ab}_j \}_{j \geq p}$ being polynomials in the basis elements.

Proof: As in [8] the relations (4.6) and (4.7) can be proven by recursion on $j$. Indeed, a direct calculation shows that (4.7) is obeyed by (4.10) for any numbers $\pm \alpha^{n,j}_{\bar{s}}$. Then, (4.6) uniquely determine these numbers, up to the choice made in (4.9).

Remark 1 The relations (4.6) allow to compute recursively all the PB of $W_p(M|N)$ but $\{ \pm \hat{W}^{0}_j, \pm \hat{W}^{0}_k \}$, where

\[
\pm \hat{W}^{0}_j = \sum_{a=1}^{M+N} \pm W^{a0}_j
\]

In the following, we will assume that

\[
\{ \pm \hat{W}^{0}_j, \pm \hat{W}^{0}_k \} = 0, ~\forall j, k
\]

Note that (4.6) and (4.7) prove that (4.12) is valid for $j = 0, 1$ and $\forall k$. Let us also remark that, since $W_p(M|N)$ is a deformation of $\mathfrak{gl}(M|N)$ (see below), the lemma B.1 ensures that $\{ \pm \hat{W}^{0}_j, \pm \hat{W}^{0}_k \}$ is central in $W_p(M|N)$.

The first and the last coefficients that appear in definition (4.10) can be computed by recursion ($\forall j \geq 0$):

\[
\pm \alpha^{1,j}_{j} = (-1)^j (j!)^2 \binom{p+j}{2j+1}
\]

\[
\alpha^{j,j+1}_{(0, \ldots, 0)} = \binom{p}{j+1}
\]

\[
\alpha^{j,j+1}_{(0, \ldots, 0)} = \binom{p+j}{j+1}
\]
The non-vanishing coefficients (4.13) show that the generators $\pm \bar{W}_{j}^{ab}$ for $j < p$ are indeed independent, since these generators write $\pm \bar{W}_{j}^{ab} = \pm \alpha_{j}^{i,j} W_{j}^{ab} + \text{lower}$, where lower is a polynomial in $W_{k}$'s with $k < j$.

**Corollary 4.3** The change of generators between $\{\pm \bar{W}_{j}^{ab}\}_{j=1,...}$ and $\{-\bar{W}_{j}^{ab}\}_{j=1,...}$ is given by:

$$
\pm \bar{W}_{j}^{ab} = \sum_{n=1}^{j+1} (-1)^{j+1+n} \sum_{[\bar{s}]=j+1-n}^{M+N} \sum_{i_{1},...,i_{n-1}=1}^{\bar{s}} \bar{W}_{s_{1}}^{ai_{1}} ... \bar{W}_{s_{n-1}}^{i_{n-1}b} (-1)^{[i_{1}]+...+[i_{n-1}]} (4.16)
$$

**Proof:** The procedure is the same as in [8]: a direct calculation shows that indeed the expression (4.16) satisfies (4.6-4.7), and that (4.16) is valid for $\pm \bar{W}_{1}^{ab}$.

**Corollary 4.4** The basis $\{-\bar{W}_{j}^{ab}\}_{j=1,...,p-1}$ is such that $-\bar{W}_{j}^{ab} = 0$ for $j \geq p$. In the basis $\{\pm \bar{W}_{j}^{ab}\}_{j=1,...,p-1}$ all the $\pm \bar{W}_{j}^{ab}$ generators ($j \geq p$) are not vanishing.

**Proof:** (4.13) shows that $\pm \bar{W}_{j}^{ab} \neq 0$ for $j \geq p$. Now, using (4.6) for $j = p$, with the form (4.10), one gets $\alpha_{j}^{n,p} = (-1)^{n} A$ with $A = 0$ or 1. Then, (4.14) shows that $A = 0$ for $-\bar{W}_{p}^{ab}$. Finally, (4.3) ensures that $-\bar{W}_{j}^{ab} = 0$, for $j > p$, as soon as $-\bar{W}_{p}^{ab} = 0$.

### 4.2 \(W_{p}(M|N)\) and \(Y_{p}(M|N)\)

We have shown that both $W_{p}(M|N)$ and $Y_{p}(M|N)$ are deformations of a truncated polynomial superalgebra based on $gl(M|N)$. It remains to show that these deformations coincide.

**Theorem 4.5** The $W_{p}(M|N)$ superalgebra is the truncated super-Yangian $Y_{p}(M|N)$

**Proof:** First, the map $-\bar{W}_{j}^{ab} \rightarrow T_{j-1}^{ab}, \forall 0 \leq j < p$, between basis vectors shows that $W_{p}(M|N)$ and $Y_{p}(M|N)$ are isomorphic as vector spaces (and indeed coincide with $gl(M|N)$). Since they are both deformations of $gl(M|N)_{p}$, we can introduce $\phi^{W}$ and $\phi^{T}$, the cochains associated to the deformation corresponding to $W_{p}(M|N)$ and $Y_{p}(M|N)$ respectively.

Now, remark that the two superalgebras have identical (in fact undeformed) PB on the couples $(-\bar{W}_{j}^{ab}, -T_{j}^{cd})$, which proves that the cochains $\phi^{W}$ and $\phi^{T}$ coincide (in fact vanish) on these points. It is also the case for the couples $(-\bar{W}_{0}^{ab}, -T_{0}^{cd})$, due to the formula (2.20) and assumption (4.12).

Moreover, the property (4.2) shows that the cochains $\phi^{W}$ and $\phi^{T}$ coincide on the couples $(-\bar{W}_{1}^{ab}, -T_{1}^{cd})$. Since $\phi^{W}$ and $\phi^{T}$ are cocycles, this is enough (using lemma 3.1) to prove that they are identical.
4.3 Representations of $\mathcal{W}_p(M|N)$

**Theorem 4.6** Any finite-dimensional irreducible representation of the $\mathcal{W}_p(M|N)$ superalgebra is highest weight. It has a unique (up to scalar multiplication) highest weight vector.

**Proof:** An irreducible representation $\pi$ of the $\mathcal{W}_p(M|N)$ superalgebra can be lifted to a representation of the whole super-Yangian by setting $\pi(T_{ij}^{(r)}) = 0$ for $r > n$. It is then obviously irreducible for the super-Yangian, and thus is highest weight by theorem 2.2.

**Theorem 4.7** Finite dimensional irreducible representations of $\mathcal{W}_p(M|N)$

Any finite-dimensional irreducible representation of the $\mathcal{W}_p(M|N)$ superalgebra is isomorphic to an evaluation representation or to the subquotient of tensor product of at most $p$ evaluation representations.

**Proof:** By evaluation representations for $\mathcal{W}_p(M|N)$ superalgebra, we mean the definitions 2.4 and 2.5 with the change $T_{ab}^r \rightarrow W_{ab}^{r-1}$ (i.e. the evaluation representations of the truncated super-Yangian). The property (2.20) clearly shows that the (subquotient of) tensor product of $n$ evaluation representations is a representation of the truncated super-Yangian as soon as $n \leq p$. It also shows that if it is irreducible for the super-Yangian, then it is also irreducible for the truncated super-Yangian and that they are finite dimensional.

Now conversely, an irreducible representation $\pi$ of the $\mathcal{W}_p(M|N)$ superalgebra can be lifted to a representation of the whole super-Yangian by setting $\pi(T_{ij}^{(r)}) = 0$ for $r > n$. It is then obviously irreducible for the super-Yangian, and thus is isomorphic to the (irreducible subquotient of) tensor product of evaluation representations.

5 Twisted super-Yangians

Twisted super-Yangian have been introduce in [17]. We remind here the main results.

We start with the super-Yangian $Y(M|2n)$, and introduce the transposition $t$ on matrices:

$$E_{ab}^t = (-1)^{[a][b]+1} \theta_a \theta_b E_{ba}$$

with

$$\begin{align*}
\bar{a} &= M + 1 - a \quad \text{for } 1 \leq a \leq M \\
\bar{a} &= 2M + 2n + 1 - a \quad \text{for } M + 1 \leq a \leq M + 2n
\end{align*}$$

(5.1)

where the $\theta_a$’s are given by

$$\begin{align*}
\theta_a &= 1 \quad \text{for } 1 \leq a \leq M \\
\theta_a &= \text{sg}(\frac{2M+2n+1}{2} - a) \quad \text{for } M + 1 \leq a \leq M + 2n
\end{align*}$$

(5.2)
Note that we have the relations
\[ (-1)^{[a]} \theta_a \theta_a = 1 \quad \text{and} \quad [a] = [\bar{a}] \quad \forall \ a \] (5.3)

Then, we define on \( Y(M|2n) \):
\[ \tau[T(u)] = \sum_{a,b} \tau[T^{ab}(u)] E_{ab} = \sum_{a,b} T^{ab}(-u) E_{ab}^t \] (5.4)

which reads for the super-Yangian generators:
\[ \tau(T^{ab}(u)) = (-1)^{[a][[b]+1]} \theta_a \theta_b T^{\bar{b}\bar{a}}(-u) \] (5.5)
\[ \tau \] is an algebra automorphism of \( Y(M|2n) \).

One defines in \( Y(M|2n) \):
\[ S(u) = T(u) \tau[T(u)] = \sum_{a,b=1}^{M+N} S^{ab}(u) E_{ab} = \mathbb{I} + \sum_{a,b=1}^{M+N} \sum_{n>0} u^{-n} S^{ab}_{(n)} E_{ab} \] (5.6)
\[ S^{ab}_{(n)} = \sum_{c=1}^{M+N} \sum_{p=0}^n (-1)^p (-1)^{[c][[b]+1]} \theta_c \theta_b T^{ac}_{(n-p)} T^{\bar{c}\bar{e}}_{(p)} \] (5.7)
\[ S^{ab}(u) = \sum_{c=1}^{M+N} (-1)^{[c][[b]+1]} \theta_c \theta_b T^{ac}(u) T^{\bar{b}\bar{e}}(-u) \] (5.8)

**Definition 5.1** \( S(u) \) defines a subalgebra of the super-Yangian, the twisted super-Yangian \( Y(M|2n)^+ \). It obeys the following relation:
\[ R_{12}(u-v) S_1(u) R'_{12}(u+v) S_2(v) = S_2(v) R'_{12}(u+v) S_1(u) R_{12}(u-v) \] (5.9)
where \( R(x) \) is the super-Yangian \( R \)-matrix,
\[ R'(x) = \mathbb{I} + \frac{1}{x} Q = R^{t_1}(-x) \quad \text{with} \quad Q = P^{t_1} \] (5.10)
and \( t_1 \) is the transposition \( [5,1] \) in the first auxiliary space.

Introducing \[ \tau(S(u)) = \sum_{a,b=1}^{M+N} S^{ab}(-u) E_{ab}^t \] (5.11)
one gets
\[ \tau(S^{ab}(u)) = (-1)^{[a][[b]+1]} \theta_a \theta_b S^{\bar{b}\bar{a}}(-u) \] (5.12)
Then, using the expression (5.8) and the commutation relations of the super-Yangian, one can show the symmetry relation:

$$\tau(S(u)) = S(u) + \frac{\theta_0}{2u}(S(u) - S(-u))$$ (5.13)

Note that the relation (5.9) is equivalent to the following commutator:

$$[S_1(u), S_2(v)] = \frac{1}{u-v} \left( P_{12} S_1(u) S_2(v) - S_2(v) S_1(u) P_{12} \right) +$$

$$- \frac{1}{u+v} \left( S_1(u) Q_{12} S_2(v) - S_2(v) Q_{12} S_1(u) \right) +$$

$$+ \frac{1}{u^2 - v^2} \left( P_{12} S_1(u) Q_{12} S_2(v) - S_2(v) Q_{12} S_1(u) P_{12} \right)$$ (5.14)

and also to

$$[S^{ab}(u), S^{cd}(v)] = \frac{(-1)^{[a]+[b][c]}}{u-v} \left( -1 \right)^{[a][b]} \left( S^{cb}(u) S^{ad}(v) - S^{cb}(v) S^{ad}(u) \right) +$$

$$- \frac{(-1)^{[a]+[b][c]}}{u+v} \left( (-1)^{[a][c]} \theta_b \theta_c S^{ae}(u) S^{bd}(v) - (-1)^{[b][d]} \theta_a \theta_d S^{ca}(v) S^{bd}(u) \right) +$$

$$+ \frac{(-1)^{[a]}	heta_a \theta_b}{u^2 - v^2} \left( -1 \right)^{[a]} \theta_a \theta_b \left( S^{ca}(u) S^{bd}(v) - S^{ca}(v) S^{bd}(u) \right)$$ (5.15)

As for $Y(M|N)$, one can show that $Y(M|2n)^+$ is a deformation of $\mathcal{U}(osp(M|2n)[x])$.

### 5.1 Finite-dimensional irreducible representations of twisted super-Yangians

The finite-dimensional irreducible representations of twisted super-Yangians have been studied in [17]. We recall here the main results. As for super-Yangian, they rely on the evaluation morphism:

**Property 5.2** The following map defines an algebra inclusion:

$$Y(M|2n)^+ \rightarrow \mathcal{U}[osp(M|2n)]$$

$$S(u) \rightarrow F(u) = I + \frac{1}{u + \frac{1}{2}} F$$ (5.16)

where the $osp(M|2n)$ generators $J^{ab}$ have been gathered in the matrix

$$F = \sum_{a,b=1}^{M+N} J^{ab} F_{ab} \quad \text{with} \quad F_{ab} = E_{ab} - (-1)^{[a][b]+1} \theta_a \theta_b E_{b\bar{a}}$$ (5.17)
Using the above inclusion, one constructs from any finite-dimensional irreducible representation of $osp(M|2n)$, a finite-dimensional irreducible representation of $Y(M|2n)^+$. 

**Theorem 5.3:** Every finite-dimensional irreducible $Y(M|2n)^+$-module contains a unique (up to scalar multiples) highest weight vector.

A sufficient condition for the existence of irreducible finite-dimensional representations have been given in [17]. It corresponds to an explicit construction of the representation as tensor product of $Y(M|N)$ evaluation representations and possibly one $osp(M|2n)$ representation (using the evaluation morphism). These sufficient conditions were conjectured to be necessary: we will assume this conjecture in the following.

### 5.2 Classical twisted super-Yangians

As for super-Yangians, one can introduce a classical (Poisson bracket) version of twisted super-Yangians. The calculation is the same as in section 2.4: one writes $R(u - v) = I + h r(u - v)$, $R'(u + v) = I + h r'(u + v)$, and consider the terms in $h$. One gets:

$$\{S_1(u), S_2(v)\} = r_{12}(u - v)S_1(u)S_2(v) - S_2(v)S_1(u)r_{12}(u - v) + S_2(v)r'_{12}(u + v)S_1(u) - S_1(u)r'_{12}(u + v)S_2(v)$$

(5.18)

In component, this reads:

$$\{S(q)_1, S(r)_2\} = \sum_{s=0}^{\mu-1} \left[ P_{12} S(s)_1 S_{(r+q-s-1)_2} - S_{(r+q-s-1)_2} S(s)_1 P_{12} \right]$$

$$+ (-1)^{q+s} \left[ Q_{12} S_{(r+q-s-1)_2} S(s)_1 Q_{12} - S_{(r+q-s-1)_2} Q_{12} S(s)_1 \right]$$

with $\mu = \min(q, p)$.

Let us remark that the symmetry relation (5.13), in its classical form, takes the form

$$\tau(S(u)) = S(-u)$$

(5.19)

because the $T^{ab}(u)$ generators are now $\mathbb{Z}_2$-commuting.

### 6 Folded $\mathcal{W}$-superalgebras revisited

It is well-known that the $gl(M|N)$ superalgebra can be folded (using an outer automorphism) into orthosymplectic ones (see e.g. [20]). In the same way, folded $\mathcal{W}$-superalgebras have been defined† in [18], and shown to be $\mathcal{W}$-superalgebras based on orthosymplectic superalgebras.

We present here a different proof of this property, adapted to our purpose, and generalized to the case of the automorphisms presented in section 5. For such a purpose, we use the Dirac bracket definition introduced in section 3.1.

†Strictly speaking, it is the folding of "affine" $\mathcal{W}$-superalgebras that has been defined in [18], but the folding of finite $\mathcal{W}$-superalgebras can be defined by the same procedure.
6.1 Automorphism of \( gl(Mp|2np) \) and \( \mathcal{W}_p(M|N) \)

As for the super-Yangian, one introduces an automorphism of \( gl(Mp|2np) \) defined by

\[
\tau(J_{jm}^{ab}) = (-1)^{j+1}(-1)^{[a][b]+1} \theta^a \theta^b J_{jm}^{\bar{a} \bar{b}} \tag{6.1}
\]

where \( \theta^a \) is defined in (5.2), and \( \bar{a} \) is given in (5.1).

To prove that \( \tau \) is an automorphism of \( gl(Mp|2np) \), we need the following property of the Clebsch-Gordan coefficient, which was proved in [8]. Note that we need this property only for the algebra \( gl(p) \), because of the decomposition \( gl(Mp|2np) \sim gl(M|2n) \otimes gl(p) \) used here (see appendix A).

**Property 6.1** The Clebsch-Gordan like coefficients obey the rule:

\[
<j, m; t, q | r, s> = (-1)^{j+t+r} <t, q; j, m | r, s> \tag{6.2}
\]

Note that in the above formula, it is not the \( \mathbb{Z}_2 \)-grades \( [j], [t] \) or \( [r] \) that are used, but really \( j, t \) and \( r \) themselves.

With this property, it is a simple matter of calculation to show that \( \tau \) defined in (6.1) is an automorphism of \( gl(Mp|2np) \).

6.2 Folding \( gl(Mp|2np) \) and \( \mathcal{W}_p(M|2n) \)

6.2.1 \( gl(Mp|2np) \)

One considers the subalgebra \( \text{Ker}(I - \tau) \) in \( gl(Mp|2np) \). It is generated by the combinations:

\[
K_{jm}^{ab} = J_{jm}^{ab} + \tau(J_{jm}^{ab}) = J_{jm}^{ab} - (-1)^j (-1)^{[a][b]+1} \theta^a \theta^b J_{jm}^{\bar{a} \bar{b}} \tag{6.3}
\]

which obey the symmetry relation

\[
\tau(K_{jm}^{ab}) = K_{jm}^{ab} \text{ i.e. } K_{jm}^{ab} = (-1)^{j+1} (-1)^{[a][b]+1} \theta^a \theta^b K_{jm}^{\bar{a} \bar{b}} \tag{6.4}
\]

Using the PB:

\[
\{J_{jm}^{ab}, J_{kr}^{cd}\} = \sum_{r=|j-k|}^{j+k} \sum_{s=-r}^{r} <j, m; k, \ell | r, s \> \left( \delta^{bc} J_{rs}^{ad} - (-1)^{([a]+[b])([c]+[d])} (-1)^{j+k+r} \delta^{ad} J_{rs}^{cb} \right)
\]

one can compute the commutation relations:

\[
\{K_{jm}^{ab}, K_{kl}^{cd}\} = \sum_{r=|j-k|}^{j+k} \sum_{s=-r}^{r} <j, m; k, \ell | r, s \> \left( \delta^{bc} K_{rs}^{ad} - (-1)^{j \theta^{a} \theta^{b}} (-1)^{[a][b]+1} \delta^{ac} K_{rs}^{bd} \right.

- (-1)^{j+k+r} (-1)^{([a]+[b])([c]+[d])} \left( \delta^{ad} K_{rs}^{cb} - (-1)^{j \theta^{a} \theta^{b}} (-1)^{[a][b]+1} \delta^{bd} K_{rs}^{ca} \right)
\]
After a rescaling of $K_{jm}^{ab}$, one recognizes the superalgebra $osp(M|2np)$.

Looking at the decomposition of the fundamental of $gl(M|2np)$ with respect to the principal embedding of $sl(2)$ in $(M + 2n).sl(p)$ (see [19] and [18] for the technic used here) one shows that the subalgebra $(M + 2n).sl(p)$, generated by the $J^{ab}_{jm}$'s, is folded into a $(m + n).sl(p)$ (resp. $(m + n).sl(p) \oplus so(p)$) when $M = 2m$ (resp. $M = 2m + 1$).

In the following, we will denote this subalgebra $[M.sl(p)]^{\tau} \oplus n.sl(p)$.

**6.2.2 $W_p(M|2n)$**

We are now dealing with the enveloping algebra of $gl(M|2np)$, that we denote $U[gl(M|2np)] \equiv U(M|2np)$. One introduces the coset:

$$U(M|2np)^+ \equiv U(M|2np)/K \quad \text{where} \quad K = U(M|2np) \cdot L$$

with $L$ spanned by $J^{ab}_{jm} - \tau(J^{ab}_{jm}), \forall a, b, j, m$

$$W_p(M|2n)^+ \equiv W_p(M|2n)/J \quad \text{where} \quad J = W_p(M|2n) \cdot I$$

with $I$ spanned by $W^{ab}_{j} - \tau(W^{ab}_{j}), \forall a, b, j$

We have the property

**Property 6.2** $\tau$ is an automorphism of $U(M|2np)$ provided with the Dirac brackets:

$$\tau \left( \{ J^{ab}_{jm}, J^{cd}_{kl} \}^* \right) = \{ \tau(J^{ab}_{jm}), \tau(J^{cd}_{kl}) \}^* \quad (6.5)$$

Hence, $\tau$ is also an automorphism of $W_p(M|2n)$.

**Proof:** It is obvious that $\tau$ is an automorphism of Poisson brackets on $U(M|2np)$. Moreover, due to the form of the constraints (3.9), $\tau$ acts as a relabeling (up to a sign) of the constraints:

$$\tau(\varphi_\alpha) = \epsilon_{\alpha'} \varphi_{\alpha'} \text{ where } \alpha' \equiv \tau(\alpha) \quad \text{and} \quad \epsilon_{\alpha'} = \epsilon_\alpha = \pm 1 \quad (6.6)$$

which shows that $\tau(\Phi) = \Phi$. We have also

$$\tau(\Delta_{\alpha\beta}) = \epsilon_{\alpha'} \epsilon_{\beta'} \Delta_{\alpha'\beta'} \quad (6.7)$$

This implies that

$$\tau \left( \{ A, \varphi_\alpha \} \Delta^{\alpha\beta} \{ \varphi_\beta, B \} \right) = \{ \tau(A), \varphi_{\alpha'} \} \Delta^{\alpha'\beta'} \{ \varphi_{\beta'}, \tau(B) \} = \{ \tau(A), \varphi_\alpha \} \Delta^{\alpha\beta} \{ \varphi_\beta, \tau(B) \} \quad (6.8)$$

This shows that this automorphism is compatible with the set of constraints $\Phi$ and thus $\tau$ is an automorphism of the Dirac brackets.

**Corollary 6.3** The Dirac brackets provide $W_p(M|2n)^+$ with an algebraic structure.
Proof: We define on $W_p(M|2n)^+$ a bracket which is just the previous Dirac bracket restricted to this coset. Since $W_p(M|2n)^+$ is generated by elements of the form $W + \tau(W)$, we have:

$$\{W + \tau(W), W' + \tau(W')\}_* = \{W, W'\} + \{\tau(W), \tau(W')\}_* + \{W, \tau(W')\}_*$$

$$\{W, W'\} + \{\tau(W), W'\}_* + \tau(\{W, W'\}_* + \{\tau(W), W'\}_*)$$

Indeed we have:

**Property 6.4** The $W_p(M|2n)^+$ superalgebra is the $W[osp(Mp|2np), [M.sl(p)]^\tau \oplus n.sl(p)]$ superalgebra.

Above, the $[M.sl(p)]^\tau$ (resp. $n.sl(p)$) subalgebra is understood as subalgebra of the orthogonal (resp. symplectic) algebra in $osp(Mp|2np)$.

**Proof:** On the coset, we have $J_{jm}^{ab} \equiv \tau(J_{jm}^{ab}) \equiv 2K_{jm}^{ab}$. We introduce on $U(Mp|2np)$

$$2D\varphi_a = \varphi_a - \tau(\varphi_a) ; 2S\varphi_a = \varphi_a + \tau(\varphi_a) \quad (6.9)$$

Since these generators satisfy $D\varphi_a = -\tau(D\varphi_a)$ and $S\varphi_a = \tau(S\varphi_a)$ and are in $gl(Mp|2np)$, we have

$$\{S\varphi_a, D\varphi_\beta\} \in \mathcal{I} \ i.e. \ \{S\varphi_a, D\varphi_\beta\} = 0 \ on \ W_p(M|2n)^+ \quad (6.10)$$

Similarly we define

$$D\Delta_{ab} = \{D\varphi_a, D\varphi_\beta\} ; S\Delta_{ab} = \{S\varphi_a, S\varphi_\beta\} ; \quad (6.11)$$

which obey the properties:

$$D\Delta_{ab} = \epsilon_{a'\beta'} D\Delta_{a'\beta'} = -\epsilon_{a'\beta'} D\Delta_{a'\beta'} = -\epsilon_{a'\beta'} D\Delta_{a'\beta'} \quad (6.12)$$

$$S\Delta_{ab} = \epsilon_{a'\beta'} S\Delta_{a'\beta'} = \epsilon_{a'\beta'} S\Delta_{a'\beta'} = \epsilon_{a'\beta'} S\Delta_{a'\beta'} \quad (6.13)$$

$$\Delta_{a\beta} = S\Delta_{a\beta} + D\Delta_{a\beta} \ on \ W_p(M|2n)^+ \quad (6.14)$$

We will say that a matrix is $\tau$-antisymmetric when it satisfies a relation like (6.12), and $\tau$-symmetric when it obeys (6.13). $\tau$-antisymmetric matrices are orthogonal to $\tau$-symmetric ones:

$$D\Delta \cdot S\Delta = 0 \ since (D\Delta \cdot S\Delta)_{a\beta} = \sum_{\gamma} D\Delta_{a\gamma} S\Delta_{\gamma\beta} = \sum_{\gamma'} D\Delta_{a\gamma'} S\Delta_{\gamma'\beta} = -\sum_{\gamma} D\Delta_{a\gamma} S\Delta_{\gamma\beta}$$

$S\Delta_{a\beta}$ is the matrix of constraints of $osp(Mp|2np)$ reduced with respect to $[M.sl(p)]^\tau \oplus n.sl(p)$. Thus, it is invertible and the associated Dirac brackets define the superalgebra $W(osp(Mp|2np), [M.sl(p)]^\tau \oplus n.sl(p))$. It remains to show that, on $W_p(M|2n)^+$, the previously defined Dirac brackets coincide with these latter Dirac brackets.
For that purpose, we use the form \( \Delta = \Delta_0 (\mathbb{I} + \hat{\Delta}) \), given in [8], where \( \Delta_0 \) is an invertible \( \tau \)-symmetric matrix and \( \hat{\Delta} \) is nilpotent (of finite order \( r \)). Introducing the \( \tau \)-symmetrized and antisymmetrized part of \( \hat{\Delta} \), one deduces

\[
\Delta^{-1} = \Delta_0^{-1} \sum_{n=0}^{r} (-1)^n (S\hat{\Delta} + D\hat{\Delta})^n = \Delta_0^{-1} \sum_{n=0}^{r} (-1)^n \left((S\hat{\Delta})^n + (D\hat{\Delta})^n\right) = S\Delta^{-1} + D\Delta^{-1}
\]

which shows that \( D\Delta \) is also invertible.

On \( \mathcal{W}_p(M|2n)^+ \), we have

\[
\{K^{ab}_{(m)}, K^{cd}_{(n)}\} = \{K^{ab}_{(m)}, K^{cd}_{(n)}\} - \{K^{ab}_{(m)}, D\varphi_\alpha + S\varphi_\alpha\} \Delta^{\alpha\beta}\{D\varphi_\beta + S\varphi_\beta, K^{cd}_{(n)}\} = \{K^{ab}_{(m)}, K^{cd}_{(n)}\} - \{K^{ab}_{(m)}, S\varphi_\alpha\} \Delta^{\alpha\beta}\{S\varphi_\beta, K^{cd}_{(n)}\}
\]

\[
\{K^{ab}_{(m)}, K^{cd}_{(n)}\} = \{K^{ab}_{(m)}, K^{cd}_{(n)}\} - \{K^{ab}_{(m)}, S\varphi_\alpha\} \Delta^{\alpha\beta}\{S\varphi_\beta, K^{cd}_{(n)}\} = \{K^{ab}_{(m)}, K^{cd}_{(n)}\} - \{K^{ab}_{(m)}, S\varphi_\alpha\} \Delta^{\alpha\beta}\{S\varphi_\beta, K^{cd}_{(n)}\}
\]

From the \( \tau \)-antisymmetry of \( D\Delta^{-1} \), we get

\[
\{., S\varphi_\alpha\} D\Delta^{\alpha\beta}\{S\varphi_\beta, .\} = \{., S\varphi_\alpha\} D\Delta^{\alpha\beta}\{S\varphi_\beta, .\} = -\{., S\varphi_\alpha\} D\Delta^{\alpha\beta}\{S\varphi_\beta, .\} = 0
\]

which leads to the Dirac brackets:

\[
\{K^{ab}_{(m)}, K^{cd}_{(n)}\} = \{K^{ab}_{(m)}, K^{cd}_{(n)}\} - \{K^{ab}_{(m)}, S\varphi_\alpha\} \Delta^{\alpha\beta}\{S\varphi_\beta, K^{cd}_{(n)}\}
\]

These Dirac brackets are just the ones of the \( \mathcal{W}(osp(Mp|2np), [M.sl(p)]\tau \oplus n.sl(p)) \) superalgebra, by definition of \( S\Delta \).

7 Folded \( \mathcal{W} \)-algebras as truncated twisted Yangians

7.1 Classical case

We start with the \( \mathcal{W}_p(M|2n) \) superalgebra in the Yangian basis. The Poisson brackets are

\[
\{T_{(m)1}, T_{(n)2}\} = \sum_{r=0}^{\min(m,n)-1} (P_{12} T_{(r)1} T_{(m+n-r)2} - T_{(r)2} T_{(m+n-r)1} P_{12})
\]

with the convention \( T_{(m)} = 0 \) for \( m > p \). The action of the automorphism \( \tau \), both for twisted super-Yangian and folded \( \mathcal{W}_p(M|2n) \) superalgebra, reads

\[
\tau(T_{(m)}) = (-1)^m T_{(m)}^t
\]

However, from the twisted super-Yangian point of view, one selects the generators

\[
S_{(m)} = \sum_{r+s=m} (-1)^s T_{(r)} T_{(s)}^t
\]

while in the folded \( \mathcal{W} \)-superalgebra case, one constrains the generators to \( T_{(m)} = (-1)^m T_{(m)}^t \). Although the procedures are different (and indeed lead to different generators), we have:
Theorem 7.1  As an algebra, the \( \mathcal{W} \)-superalgebra \( \mathcal{W}(\mathfrak{osp}(Mp|2np), [M, sl(p)]^\tau \oplus n, sl(p)) \) is isomorphic to the truncation (at level \( p \)) of the (classical) twisted super-Yangian \( \mathcal{Y}(M|2n)^+ \).

More precisely, we have the correspondences:

\[
Y_p(2m+1|2n)^+ \leftrightarrow \mathcal{W}[\mathfrak{osp}(mp+p|2np), (m+n), sl(p) \oplus \mathfrak{so}(p)]
\]

\[
Y_p(2m|2n)^+ \leftrightarrow \mathcal{W}[\mathfrak{osp}(2mp|2np), (m+n), sl(p)]
\]

Proof: We prove this theorem by showing that the Dirac brackets of the folded \( \mathcal{W} \)-superalgebra coincide with the Poisson brackets \( (5.18) \) with the truncation \( S(|m) = 0 \) for \( m > p \).

We start with the \( \mathcal{W}_p(M|2n) \) superalgebra in the truncated super-Yangian basis:

\[
\{T(q)_1, T(r)_2\} = \sum_{s=0}^{\mu-1} (P_{12}T_{(s)}1T_{(r+q-s-1)2} - T_{(r+q-s-1)2}T_{(s)}1P_{12})
\]

with \( T_{(s)} = 0 \) for \( s > p \) and \( \mu = \min(q, r, p) \) (7.4)

In this basis, we define

\[
2\varphi(s) = T_{(s)} - (-1)^s T^t_{(s)} \quad \text{and} \quad 2K(s) = T_{(s)} + (-1)^s T^t_{(s)}
\]

The folding (of the \( \mathcal{W} \)-superalgebra) corresponds to

\[
\varphi(s) = 0 \quad \text{i.e.} \quad K(s) = (-1)^s K^t_{(s)}
\]

(7.6)

It is a simple matter of calculation to get

\[
2\{K(q)_1, K(r)_2\} = \sum_{s=0}^{\mu-1} \left[ P_{12}K_{(s)}1K_{(r+q-s-1)2} - K_{(r+q-s-1)2}K_{(s)}1P_{12} + (-1)^q+s (K_{(s)}1Q_{12}K_{(r+q-s-1)2} - K_{(r+q-s-1)2}Q_{12}K_{(s)}1) \right]
\]

(7.7)

that is to say

\[
2\{K_1(u), K_2(v)\} = [r_{12}(u-v), K_1(u)K_2(v)] + K_2(v)r'_{12}(u+v)K_1(u) - K_1(u)r'_{12}(u+v)K_2(v)
\]

These PB are equivalent to the relation \( (5.18) \) for \( S(u) \equiv K(\frac{u}{2}) \). The constraint \( (7.6) \) is then rewritten as \( \tau(S(-u)) = S(u) \). Thus, the folded \( \mathcal{W} \)-superalgebra and the truncated twisted super-Yangian are defined by the same relations.

\[
\square
\]

7.2 Quantization and representations of \( \mathcal{W} \)-superalgebras

Now that folded \( \mathcal{W} \)-superalgebras have proved to be truncation of twisted super-Yangians at classical level, there quantization is very simple. It can be identified with the truncated
twisted super-Yangian at quantum level:

\[ R_{12}(u - v) S_1(u) R'_{12}(u + v) S_2(v) = S_2(v) R'_{12}(u + v) S_1(u) R_{12}(u - v) \quad (7.8) \]

with

\[
\begin{align*}
R_{12}(x) &= \mathbb{I} - \frac{1}{x} P_{12} ; \\
R'_{12}(x) &= \mathbb{I} - \frac{1}{x} Q_{12} \\
S(u) &= \sum_{m=0}^{p} u^{-m} S_m ; \quad S_0 = \mathbb{I}
\end{align*}
\quad (7.9)
\]

Using the representations classification of twisted super-Yangians given in [17], one can then deduce the classification of irreducible finite-dimensional representations for truncated twisted super-Yangians in the same way it has been done in [10] for ordinary twisted Yangians. For conciseness, we will just sketch the results. In particular, one gets the following theorems

**Theorem 7.2** Any finite-dimensional irreducible representation of the $W_p(M|2n)^+$ superalgebra is highest weight.

**Proof:** Same proof as for theorem 4.6.

**Theorem 7.3** Any finite-dimensional irreducible representation of the $W_p(M|2n)^+$ superalgebra is isomorphic to an evaluation representation, or to the (irreducible subquotient of) tensor product of at most $[p/2]$ evaluation representations of $Y(M|2n)$, and possibly an $osp(M|2n)$ representation.

**Proof:** Same proof as for twisted Yangians, see [10], using the results given in [17] for $Y(M|2n)^+$. Indeed, as for the $gl(M|N)$ case, one needs to have $S(r) = 0$ for $r \geq p$ to get a representation of the $W$-superalgebra. This constrains the number of evaluation representations allowed to be tensorised (to get a representation). The difference with the $Y(M|N)$ case lies in the quadratic form $S(u) = T(u) \tau(T(-u))$, which lowers the maximum number of terms in the tensor product. The occurrence of an $osp(M|2n)$ representation is due to the classification given in [17].

Reasoning as in [10], one can also get a condition on the weights of the representation. We omit it here, due to the lack of place.

**Remark 2** As for $W$-algebras based on $so(M)$ and $sp(2n)$, see [10] for more details, one could think that $W$-superalgebras based on $osp(M|2n)$ are related to super-Yangians based on $osp(M|2n)$ instead of twisted super-Yangians. However, a simple counting (using the method given in [19]) of the generators shows that it is the twisted super-Yangians that have to be considered.
A General settings on $gl(Mp|Np)$

A.1 Clebsch-Gordan like coefficients

We start with the $gl(Mp|Np)$ superalgebra in its fundamental representation, and consider the $sl(2)$ principal embedding in $(M + N)gl(p) \equiv gl(p) \oplus \ldots \oplus gl(p)$.

In the fundamental representation, one can view $gl(Mp|Np)$ as $gl(p) \otimes gl(M|N)$, so that the generators of this $sl(2)$ can be written as $\epsilon_{\pm,0} \equiv e_{\pm,0} \otimes 1_{M+N}$. The $e_{\pm,0}$ are the generators of the $sl(2)$ algebra principal in $gl(p)$ and verify $[e_0, e_{\pm}] = \pm e_{\pm}$ and $[e_+, e_-] = e_0$.

The generator $1_{M+N}$ is the identity generator in $gl(M|N)$.

Under the adjoint action of this $sl(2)$, $gl(p) \otimes gl(M|N)$ can be decomposed in $sl(2)$ multiplets: $M^{ab}_{jm} \equiv M_{jm} \otimes E^{ab}$ with $a, b = 1, \ldots, M+N ; -j \leq m \leq j ; 0 \leq j \leq p - 1$.

The $M^{ab}_{jm}$ are $p \times p$ matrices resulting from the decomposition of $gl(p)$ in $sl(2)$ multiplets.

Properties of the $M^{ab}_{jm}$ are gathered in appendix A of [8].

The $E^{ab}$ are $(M+N) \times (M+N)$ matrices with 1 at position $(a, b)$. They are the graded part of $M^{ab}_{jm}$ which is even if $a + b \equiv 0 \pmod{2}$ and odd otherwise.

Following appendix A of [8] we have

$$\text{(A.1)} [\epsilon_+, M^{ab}_{jm}] = \frac{j(j+1) - m(m+1)}{2} M^{ab}_{jm+1}$$
$$\text{(A.2)} [\epsilon_-, M^{ab}_{jm}] = M^{ab}_{jm-1}$$
$$\text{(A.3)} [\epsilon_0, M^{ab}_{jm}] = m M^{ab}_{jm}$$

The product law (in the fundamental representation) reads:

$$M^{ab}_{jm} \cdot M^{cd}_{ln} = \delta^{bc} \sum_{r=|j-l|}^{j+l} \sum_{s=-r}^{r} < j, m; l, n | r, s > M^{ad}_{rs}$$

which leads to the following commutation relations (valid in the abstract algebra):

$$[M^{ab}_{jm}, M^{cd}_{ln}] = \sum_{r=|j-l|}^{j+l} \sum_{s=-r}^{r} \left( \delta^{bc} < j, m; l, n | r, s > M^{ad}_{rs} - (-1)^{(|a|+|b|)(|c|+|d|)} \delta^{ad} < l, n; j, m | r, s > M^{cb}_{rs} \right)$$

The scalar product is:

$$\eta^{ab,cd}_{j,m,l,n} = strain(M^{ab}_{jm} \cdot M^{cd}_{ln}) = (-1)^{|a|} \delta^{ad} \delta^{cb} (-1)^{m} \delta_{j,l} \delta_{m+n,0} \eta_{j}$$

for some non-vanishing coefficient $\eta_{j}$, given in [8].

The ”Clebsch-Gordan like” coefficients are then given by

$$< jm, kl | r, s >= \frac{(-1)^{a}}{\eta_{r}} tr(M_{jm} M_{kl} M_{r,-s})$$

(A.6)
We remind that in (A.6), it is the usual trace operator which is involved, since we are in the $gl(p)$ Lie algebra.

**A.2 Structure constants**

We consider a Lie superalgebra $G$ in its fundamental representation, with homogeneous generators $t_a$. As usual, we can define a gradation index $[\cdot]$ such that:

$$
[a] = \begin{cases} 
0 & \text{if } t_a \text{ bosonic} \\
1 & \text{if } t_a \text{ fermionic}
\end{cases}
$$

The commutation relations are $[t_a, t_b] = f_{ab}^c t_c$ (summation over repeated indices). The structure constants have following property: $f_{ab}^c \neq 0 \Rightarrow [a] + [b] + [c] = 0$. They obey the graded Jacobi identity:

$$
f_{ab}^d f_{dc}^e = f_{bc}^d f_{da}^e + (-1)^{[b][c]} f_{ac}^d f_{db}^e
$$

(A.7)

We introduce its inverse $g_{ab}$ and use it to rise and lower the indices. For instance $t^a \equiv g^{ab} t_b$ and $f_{abc} \equiv g^{\alpha\beta\gamma} f_{\alpha\beta} g_{\gamma c}$. we therefore have $[t^a, t^b] = f_{ab}^c t^c$.

Defining the tensor $f_{abc} = f_{ab}^\gamma g_{\gamma c}$, one can take it totally (graded) antisymmetric:

$$
f_{abc} = -(-1)^{[a][b]} f_{bac} = -(-1)^{[b][c]} f_{abc}
$$

(A.10)

**B Deformations and cohomology**

Let us consider a Lie superalgebra $A$ with homogeneous generators $u_\alpha$ and Lie bracket:

$$
\{u_\alpha, u_\beta\} = f_{\alpha\beta}^{\gamma} u_{\gamma}
$$

(B.1)

The gradation index $[\cdot]$ is such that $[\alpha] = 0$ if $u_\alpha$ is bosonic and $[\alpha] = 1$ if $u_\alpha$ is fermionic.

We aim to construct a deformation of the Lie bracket (B.1), following e.g. [16].

We introduce $n$-cochains $(n \in \mathbb{Z}_{>0})$, i.e. linear maps $\chi^{(n)}$ from $A^n$ to $A$ with following property:

$$
\chi^{(n)}(u_{\alpha_1}, \ldots, u_{\alpha_i}, u_{\alpha_{i+1}}, \ldots, u_{\alpha_n}) = (-1)^{1+[\alpha_i][\alpha_{i+1}]} \chi^{(n)}(u_{\alpha_1}, \ldots, u_{\alpha_{i+1}}, u_{\alpha_i}, \ldots, u_{\alpha_n}).
$$

(B.2)
The Chevalley derivation $\delta$ maps $n$-cochains to $(n + 1)$-cochains:

$$(\delta \chi^{(n)})(u_{\alpha_0}, u_{\alpha_1}, \ldots, u_{\alpha_n}) = \sum_{i=0}^{n} (-1)^{i+\epsilon_i} \{u_{\alpha_i}, \chi^{(n)}(u_{\alpha_0}, \ldots, \hat{u}_{\alpha_i}, \ldots, u_{\alpha_n})\} \quad (B.3)$$

$$+ \sum_{0 \leq i < j \leq n} (-1)^{i+j+\epsilon_{ij}} \chi^{(n)}(\{u_{\alpha_i}, u_{\alpha_j}\}, u_{\alpha_0}, \ldots, \hat{u}_{\alpha_i}, \ldots, \hat{u}_{\alpha_j}, \ldots, u_{\alpha_n})$$

where $\epsilon_i = [\alpha_i](\sum_{k<i}[\alpha_k])$ and $\epsilon_{ij} = \epsilon_i + \epsilon_j + [\alpha_i][\alpha_j]$.

It obeys $\delta^2 = 0$, so that one can define $n$-cocycles, which are closed $n$-cochains ($\delta \chi^{(n)} = 0$), and coboundaries, which are exact $n$-cochains ($\chi^{(n)} = \delta \chi^{(n-1)}$). As usual, one considers closed cochains modulo exacts ones to study the cohomology associated to cochains:

$$\phi$$

We write the commutation relations of $\text{gl}(M|N)_p$:

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Here, we will be mainly concerned with the action of the Chevalley derivation on 2-cochains:

$$(\delta \chi)(u, v, w) = \{u, \chi(v, w)\} \cdot (-1)^{[u][v]} \{v, \chi(u, w)\} \cdot (-1)^{[u][w]} \{w, \chi(u, v)\} - \chi(\{u, v\}, w) - (-1)^{[v][w]} \chi(\{u, w\}, v) - (-1)^{[u][w]} \chi(\{v, w\}, u) \quad (B.4)$$

We now consider a deformation of the enveloping algebra $U(A)$:

$$\{u_\alpha, u_\beta\}_h = f_{\alpha\beta} u_\gamma + h \varphi_h(u_\alpha, u_\beta) \quad (B.5)$$

where $\varphi_h$ is a 2-cocycle which may depend on positive powers of $h$. Asking the bracket $\{\cdot, \cdot\}_h$ to obey the graded Jacoby identity is equivalent to say that $\varphi_h$ is a 2-cocycle:

$$(\delta \varphi_h)(u_\alpha, u_\beta, u_\gamma) = 0 \quad (B.6)$$

We now prove a result that is used in the present article.

**Lemma B.1** Let $\text{gl}(M|N)_p$ be the polynomial algebra based on $\text{gl}(M|N)$, truncated at order $p$, and $u_{ab}^j$ ($j < p$ and $a, b = 1, \ldots, M+N$) the corresponding generators. Let $\varphi$ be a 2-cocycle with values in $U(\text{gl}(M|N)_p)$. We introduce $u_{0a}^j = \sum_{n=1}^{M+N} u_{ja}^n$.

If $\varphi(u_{0b}^{ab}, u_{cd}^j)$ and $\varphi(u_{ab}^{cd}, u_{0d}^j)$, $\forall a, b, c, d = 1, \ldots, M + N$ and $\forall j = 0, \ldots, p - 1$ are known, then $\varphi$ is completely determined up to $\varphi(u_{0b}^{ab}, u_{0d}^0)$, $j, k > 1$, which is central in $U(\text{gl}(M|N)_p)$.

**Proof:** We write the cocycle condition for a triplet $(u_{cd}^j, u_{cd}^k, u_{cd}^e)$:

$$\varphi(\{u_{cd}^j, u_{cd}^k\}, u_{cd}^e) + (-1)^{[c]+[d]} \varphi(\{u_{cd}^j, u_{cd}^e\}, u_{cd}^k) = \varphi(\{u_{cd}^k, u_{cd}^e\}, u_{cd}^j)$$

$$- (-1)^{(j+k)} \varphi(\{u_{cd}^j, u_{cd}^k\}, u_{cd}^e) + (-1)^{[c]+[d]} \varphi(\{u_{cd}^j, u_{cd}^e\}, u_{cd}^k)$$

$$- (-1)^{(j+k)} \varphi(\{u_{cd}^j, u_{cd}^k\}, u_{cd}^e) - (-1)^{(j+k)} \varphi(\{u_{cd}^j, u_{cd}^k\}, u_{cd}^e)$$

We write the commutation relations of $\text{gl}(M|N)_p$ as:

$$\{u_{ab}^{cd}, u_{ab}^{cd}\} = \delta^{bc} u_{ab}^{cd} - (-1)^{(j+k)} [c]+[d] \delta^{cd} u_{ab}^{cb}$$

with $u_{ab}^{ab} = 0$, $n > p$, $\forall a, b$ (B.7)
Taking as special case $e = g = a \neq b$ and $\ell = 1$, one gets from (B.6):

\[
\varphi(u_{j+1}^{ab}, u_k^{cd}) = \left\{ u_j^{ab}, \varphi(u_k^{cd}, u_1^{aa}) \right\} - \varphi\left( \left\{ u_j^{ab}, u_k^{cd} \right\}, u_1^{aa} \right) + (-1)^{(|e|+|d|)(|a|+|b|)} \left( \delta^{\alpha a} - \delta^{\beta e} \right) \varphi(u_k^{cd}, u_j^{ab}) - \left\{ u_1^{aa}, \varphi(u_j^{ab}, u_k^{cd}) \right\} - \left\{ u_k^{cd}, \varphi(u_j^{ab}, u_1^{aa}) \right\}
\] (B.8)

Taking as special case $j = 1$, $k = 2$, this last equation shows that one can compute $\varphi(u_2^{ab}, u_2^{cd})$, for $a \neq b$, as soon as one knows $\varphi(u_1^{cd}, u_j^{ag})$, $\forall c, d, e, g, \forall j$. Then, in the same way, $j = 1$ allows to compute $\varphi(u_2^{ab}, u_{k+1}^{cd})$ as soon as one knows $\varphi(u_2^{ab}, u_k^{cd})$.

More generally, if one supposes by induction that $\varphi(u_j^{ab}, u_k^{cd})$, $\forall j'$, $\forall k$, and $\forall c, d, a \neq b$, are known, (B.8) shows that one can compute $\varphi(u_j^{ab}, u_k^{cd})$, for $a \neq b$ and $\forall k$.

Thus, by induction, we have shown that one can compute $\varphi(u_j^{ab}, u_k^{cd})$, for $a \neq b$, $\forall j$, $k$, $c, d$, from the knowledge of $\varphi(u_1^{cd}, u_k^{eg})$.

It remains to compute $\varphi(u_j^{aa}, u_k^{bb})$. For such a purpose, we start again with (B.6) with now $a = d \neq b = c$ and $e = g$:

\[
\varphi\left( u_{j+k}^{aa} - (-1)^{|a|+|b|} u_{j+k}^{bb}, u_{\ell}^{ee} \right) = (\delta^{ae} - \delta^{be}) \left( \varphi(u_{j+\ell}^{ab}, u_k^{ba}) + (-1)^{|a|+|b|} \varphi(u_k^{ba}, u_{j+\ell}^{ab}) \right)
\] + $\left\{ u_j^{ab}, \varphi(u_k^{ba}, u_{\ell}^{ee}) \right\} - (-1)^{|a|+|b|} \left\{ u_k^{ba}, \varphi(u_j^{ab}, u_{\ell}^{ee}) \right\}$
\[= \left\{ u_k^{cd}, \varphi(u_j^{ab}, u_1^{aa}) \right\}
\] (B.9)

All the terms in the r.h.s. of the above equation are known, so that one can compute $\varphi\left( (-1)^{|a|} u_j^{aa} - (-1)^{|b|} u_j^{bb}, u_k^{ee} \right)$, $\forall a, b, e, \forall j, k$.

Thus, only $\varphi(u_j^{00}, u_k^{00})$, where $u_j^{00} = \sum_{a=1}^{M+N} u_j^{aa}$, remains to be computed.

Once again, from (B.6), taking $a = b$ and $c = d$, and then summing over $a$ and $d$, one gets

\[
\left\{ u_k^{eg}, \varphi(u_j^{00}, u_k^{00}) \right\} = 0
\] (B.10)

which shows that $\varphi(u_j^{00}, u_k^{00})$ is central in $\mathcal{U}(gl(M|N)_p)$.

Thus, apart from the values $\varphi(u_2^{ab}, u_2^{cd})$ and the just mentioned central terms, we are able to compute all the expressions $\varphi(u_j^{ab}, u_k^{cd})$. This ends the proof.

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One should take $a \neq b$, but for $a = b$ one obviously gets 0.

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