Abstract—The growing size of modern datasets necessitates a massive computation into smaller computations and operate in a distributed manner for improving overall performance. However, the adversarial servers in the distributed computing system deliberately send erroneous data in order to affect the computation for their benefit. Computing Boolean functions is the key component of many applications of interest, e.g., the classification problem, verification functions in the blockchain and the design of cryptographic algorithm. In this paper, we consider the problem of computing the Boolean function in which the computation is carried out distributively across several workers with particular focus on security against Byzantine workers. We note that any Boolean function can be modeled as a multivariate polynomial which have high degree in general. Hence, the recent proposed Lagrange Coded Computing (LCC) can be used to simultaneously provide resiliency, security, and privacy. However, the security threshold (i.e., the maximum number of adversarial workers can be tolerated) provided by LCC can be extremely low if the degree of polynomial is high. Our goal is to design an efficient coding scheme which achieves the optimal security threshold with low decoding overhead. We propose three different schemes called coded Algebraic normal form (ANF), coded Disjunctive normal form (DNF) and coded polynomial threshold function (PTF). Instead of modeling the Boolean function as a general polynomial, the key idea of the proposed schemes is to model it as the concatenation of some low-degree polynomials and the threshold functions. In terms of the security threshold, we show that the proposed coded ANF and coded DNF are optimal. For the Boolean functions with the polynomial size of sparsity and weight, it is demonstrated that the proposed coded PTF outperforms LCC in terms of the security threshold and the decoding complexity.

I. INTRODUCTION

With the growing size of modern datasets for applications such as machine learning and data science, it is necessary to partition a massive computation into smaller computations and perform these smaller computations in a distributed manner for improving overall performance [1]. However, distributing the computations to some external entities, which are not necessarily trusted, i.e., adversarial servers make security a major concern [2]–[4]. Thus, it is important to provide security against adversarial workers that deliberately send erroneous data in order to affect the computation for their benefit.

Computing Boolean functions is the key component of many applications of interest. For instance, learning a Boolean function for the inference of classification in discrete attribute spaces from examples of its input/output behavior has been widely studied in the past few decades [5]. The examples in the classification problem are represented by binary (0 or 1) attributes, and the inference can be converted into a Boolean function which outputs the category of each example belongs to [6]. For hash functions based on bit mixing (e.g., SHA-2), the Boolean functions are used to represent the verification functions. Moreover, Boolean functions are also primarily used in in the design of cryptographic algorithm [7].

In this paper, we consider the problem of computing the Boolean function in which the computation is carried out distributively across several workers with particular focus on security against Byzantine workers. Specifically, using a master-worker distributed computing system with \(N\) workers, the goal is to compute the Boolean function \(f : \{0, 1\}^{m} \rightarrow \{0, 1\}\) over a large dataset \(X = (X_1, X_2, \ldots, X_K)\), i.e., \(f(X_1), \ldots, f(X_K)\), in which the (encoded) datasets are pre-stored in the workers such that the computations can be secure against adversarial workers in the system.

Any Boolean function can be modeled as an Algebraic normal form (i.e., multivariate polynomial). Thus, the recent proposed Lagrange Coded Computing (LCC) [8], an universal encoding technique for arbitrary multivariate polynomial computations, can be used to simultaneously alleviate the issues of resiliency, security, and privacy. The security threshold (maximum number of adversarial workers can be tolerated) provided by LCC is \(N - (K-1) \cdot \deg f - 1\) which can be extremely low if the degree of polynomial \(\deg f\) is high. Such degree problem can be further amplified in complex Boolean functions whose degree can grow exponentially in general. Thus, we aim at designing the efficient coding scheme achieves the optimal security threshold with low decoding overhead.

A. Main Contributions

As main contributions of the paper is that instead of modeling the Boolean function as a general polynomial, we propose the three proposed schemes modeling it as the concatenation of some low-degree polynomials and the threshold functions (see Figure [1]). To illustrate the main idea of the proposed schemes, consider an AND function of three input bits \(X[1], X[2], X[3]\) which is formally defined by \(f(X) = X[1] \land X[2] \land X[3]\). The function \(f\) can be modeled as a polynomial function (Algebraic normal form) \(X[1]X[2]X[3]\) which has a degree of 3. For this polynomial, LCC achieves the security threshold \(N - 3(K-1) - 1\). Instead of directly computing the degree-3 polynomial, our proposed approach is to model it as a linear threshold function \(\text{sgn}(X[1] + X[2] + [3] - \frac{1}{2})\) in which \(f(X) = 1\) if and only if \(\text{sgn}(X[1] + X[2] + [3] - \frac{1}{2}) > 0\). Then, a simple linear code (e.g., \((N, K)\) MDS code) can be used for computing the linear function \(X[1] + X[2] + [3] - \frac{5}{2}\), which provides the optimal security threshold \(\frac{N-K}{2}\).

We propose three different schemes called coded Algebraic normal form (ANF), coded Disjunctive normal form (DNF) and coded polynomial threshold function (PTF). The idea behind coded ANF (DNF) is to first decompose the Boolean function into some monomials (clauses) and then construct a linear threshold function for each monomial (clause). Then, an
Decoding Complexity

A tremendous success in various problems, such as straggler computing. In the past few years, coded computing has had alleviating the various issues that arise in large-scale distributed environments.

B. Related Prior Work

Coded computing broadly refers to a family of techniques that utilize coding to inject computation redundancy in order to alleviate the various issues that arise in large-scale distributed computing. In the past few years, coded computing has had a tremendous success in various problems, such as straggler mitigation and bandwidth reduction (e.g., [9]–[16]).

The Boolean function $f(X)$ can be represented by an Algebraic normal form (ANF) as follows:

$$ f(X) = \bigoplus_{S \subseteq [m]} \mu_f(S) \prod_{j \in S} X[j] \quad \text{(1)} $$

where $X[j]$ is the $j$-bit of data $X$ and $\mu_f(S) \in \{0, 1\}$ is the ANF coefficient of the corresponding monomial $\prod_{j \in S} X[j].$

We denote the degree of Boolean function $f$ by $\deg f$ and the sparsity (number of monomials) of $f$ by $w(f),$ i.e., $r(f) = \sum_{S \subseteq [m]} \mu_f(S).$

Furthermore, we denote the support of $f$ by $\text{Supp}(f),$ which is the set of vectors in $\{0, 1\}^m$ such that $f = 1,$ i.e., $\text{Supp}(f) = \{X \in \{0, 1\}^m : f(X) = 1\}.$ Let $w(f)$ be the weight of Boolean function $f$, defined by $w(f) = |\text{Supp}(f)|.$

Alternatively, each Boolean function $f$ can be represented by a Disjunctive normal form (DNF) as follows:

$$ f = T_1 \lor T_2 \lor \cdots \lor T_{w(f)} \quad \text{(2)} $$

where each clause $T_i$ has $m$ literals which corresponds to an input $Y_i$ such that $f(Y_i) = 1.$ For example, if $Y_i = 001,$ then the corresponding clause is $\sim Y_i[0] \land Y_i[1] \land Y_i[2].$

Prior to computation, each worker has already stored a fraction of the dataset in a possibly coded manner. Specifically, each worker $n$ stores $X_n = g_n(X_1, \ldots, X_K),$ where $g_n$ is the encoding function of worker $n.$ Each worker $n$ computes $h_n(X_n)$ and returns the result to the master, in which $h_n$ is the function decided by the master. Then, the master aggregates

\begin{table*}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
& Security Threshold & Decoding Complexity \\
\hline
LCC & $N - (K - 1)\log f - 1$ & $O(N \log^2 N \log \log N)$ \\
\hline
coded ANF & $\frac{N-K}{2}$ & $O(r(\mu)N \log^2 N \log \log N)$ \\
\hline
coded DNF & $\frac{N-K}{2}$ & $O(w(\mu)N \log^2 N \log \log N)$ \\
\hline
coded PTF & $N - (K - 1)(\log w(\mu) + 1) - 1$ & $O(N \log^2 N \log \log N)$ \\
\hline
Outer Bound & $\frac{N-K}{2}$ & - \\
\hline
\end{tabular}
\caption{Performance comparison of LCC and the proposed three schemes for the Boolean function $f(X)$ which has the sparsity $r(f)$ and weight $w(f)$.
}
\end{table*}
the results from the workers until it receives a decodable set of local computations. We say a set of computations is decodable if \( f(X_1), \ldots, f(X_K) \) can be obtained by computing decoding functions over the received results. More concretely, given any subset of workers that return the computing results (denoted by \( K \)), the master computes \( v_K(\{h_n(X_n)\}) \), where each \( v_K \) is a deterministic function. We refer to the \( v_K \)'s as decoding functions.

In particular, we focus on finding the coding scheme to be robust to as many adversarial workers as possible in the system. The following term defines the security which can be provided by a coding scheme.

**Definition 1** (Security Threshold). For an integer \( b \), we say a scheme \( S \) is \( b \)-secure if the master can be robust against \( b \) adversaries. The security threshold, denoted by \( \beta_S \), is the maximum value of \( b \) such that a scheme \( S \) is \( b \)-secure, i.e.,

\[
\beta_S \triangleq \sup \{ b : S \text{ is } b \text{-secure} \}.
\]  

(3)

Based on the above system model, the problem is now formulated as: *What is the coding scheme which achieves the optimal security threshold with low decoding complexity?*

**III. OVERVIEW OF LAGRANGE CODED COMPUTING**

In this section, we consider Lagrange Coded Computing (LCC) [8] and show how it works for our problem.

Since Lagrange coded computing requires the underlying field size to be at least the number of workers \( N \), we first extend the field size of \( \{0,1\}^m \) such that the size of extension field is at least the number of workers \( N \). More specifically, we embed each bit \( X_k[j] \in \{0,1\} \) of data \( X_k \) into a binary extension field \( \{0,1\}^{t} \) such that \( 2^t \geq N \). The embedding \( X_k[j] \) of the bit \( X_k[j] \) is generated such that

\[
\bar{X}_k[j] = \begin{cases} 
00 \cdots 0, & X_k[j] = 0, \\
00 \cdots 1, & X_k[j] = 1.
\end{cases}
\]  

(4)

Note that over extension field the output of Boolean function \( f \) is \( 00 \cdots 0 \) if the original result is \( 0 \); \( 00 \cdots 1 \) if the original result is \( 1 \).

For the data encoding by using LCC, we first select \( K \) distinct elements \( \beta_1, \beta_2, \ldots, \beta_K \) from extension field \( \{0,1\}^{t} \), and let \( u \) be the respective Lagrange interpolation polynomial:

\[
u(z) \triangleq \sum_{k=1}^{K} \bar{X}_k \prod_{l \in [K] \setminus \{k\}} \frac{z - \beta_l}{\bar{X}_k - \beta_l},
\]  

(5)

where \( u : \{0,1\}^{t} \to \{0,1\}^{mt} \) is a polynomial of degree \( K - 1 \) such that \( u(\beta_l) = \bar{X}_l \). Then we can select distinct elements \( \alpha_1, \alpha_2, \ldots, \alpha_N \) from extension field \( \{0,1\}^{t} \), and encode \( \bar{X}_1, \ldots, \bar{X}_K \) to \( \bar{X}_n = u(\alpha_n) \) for all \( n \in [N] \), i.e.,

\[
\bar{X}_n = u(\alpha_n) \triangleq \sum_{k=1}^{K} \bar{X}_k \prod_{l \in [K] \setminus \{k\}} \frac{\alpha_n - \beta_l}{\bar{X}_k - \beta_l}.
\]  

(6)

Each worker \( n \in [N] \) stores \( \bar{X}_n \) locally. Following the above data encoding, each worker \( n \) computes function \( f \) on \( \bar{X}_n \) and sends the result back to the master upon its completion.

In the following, we present the security threshold provided by LCC. By [8], to be robust to \( b \) adversarial workers (given \( N \) and \( K \)), LCC requires \( N \geq (K - 1)\deg f + 2b + 1 \); i.e., LCC achieves the security threshold

\[
\beta_{LCC} = \frac{N - (K - 1)\deg f - 1}{2}.
\]  

(7)

After receiving results from the workers, the master can obtain all coefficients of \( f(u(z)) \) by applying Reed-Solomon decoding [25], [26]. Having this polynomial, the master evaluates it at \( \beta_k \) for every \( k \in [K] \) to obtain \( f(u(\beta_k)) = f(\bar{X}_k) \).

The complexity of decoding a length-\( N \) Reed-Solomon code with dimension \( t \) is \( O(tN \log^2 N \log \log N) \). To have a sufficiently large field for LCC, we pick \( t = \lceil \log N \rceil \). Thus, the decoding process by the master requires complexity \( O(N \log^3 N \log \log N) \).

The security threshold achieved by LCC depends on the degree of function \( f \), i.e., the security guarantee is highly degraded if \( f \) has high degree. To mitigate such degree effect, we model the Boolean function as the concatenation of some low-degree polynomials and the threshold functions by proposing three schemes in the following sections.

**IV. SCHEME 1: CODED ALGEBRAIC NORMAL FORM**

In this section, we propose a coding scheme called coded Algebraic normal form (ANF) which computes the ANF representation of Boolean function by the linear threshold functions (LTF) and a simple linear code is used for the data encoding. We start with an example to illustrate the idea of coded ANF.

**Example 1.** We consider a function which has an ANF representation defined as follows:

\[
f(X) = X[1]X[2] \cdot X[m/2].
\]  

(8)

Then, we define a linear function over the real field:

\[
L(X) = \sum_{j=1}^{\frac{m}{2}} X[j] - \frac{m}{2} + \frac{1}{2}
\]  

(9)

where \( L(X) = \frac{1}{2} \) if and only if \( f(X) = 1 \). Otherwise, \( L(X) \leq -\frac{1}{2} \). Thus, we can compute \( f(X) \) by computing its corresponding linear threshold function \( \text{sgn}(L(X)) \), i.e., \( f(X) = 1 \) if \( \text{sgn}(L(X)) = 1 \); otherwise, \( f(X) = 0 \) if \( \text{sgn}(L(X)) = -1 \). Unlike computing the function \( f(X) \) with the degree \( \frac{m}{2} \) which results in low security threshold, computing the linear function \( L(X) \) allows us to apply a linear code on the computations.

**A. Formal Description of coded ANF**

Given the ANF representation defined in [1], we now present the proposed coded ANF as follows. For each monomial \( \prod_{j \in S} X[j] \) such that \( \mu_f(S) = 1 \), we define a linear function \( L_S(X) \) as follows:

\[
L_S(X) = \sum_{j \in S} X[j] - |S| + \frac{1}{2}
\]  

(10)

It is clear that \( L_S(X) = \frac{1}{2} \) if and only if \( \prod_{j \in S} X[j] = 1 \). Otherwise, \( L_S(X) \leq -\frac{1}{2} \). Thus, there are \( r(f) \) constructed
The master encodes $X_1, X_2, \ldots, X_K$ to $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_N$ over the real field using an $(N, K)$ MDS code. Each worker $n \in [N]$ stores $\tilde{X}_n$ locally. Each worker $n \in [N]$ computes the functions $\{L_S(\tilde{X}_n)\}_{\{S \subseteq [m], \mu_f(S) = 1\}}$ and then sends the results back to the master. After receiving the results from the workers, the master first recovers $L_S(\tilde{X}_k)$ for each $k \in [K]$ and each $S \in \{G : G \subseteq [m], \mu_f(G) = 1\}$. Then, the master has $\prod_{j \in S} X_k[j] = 1$ if $\text{sgn}(L_S(\tilde{X}_k)) = 1$; $\prod_{j \in S} X_k[j] = 0$ if $\text{sgn}(L_S(\tilde{X}_k)) = -1$. Lastly, the master recovers $f(X_1), \ldots, f(X_K)$ by summing the monomials.

**B. Security Threshold of Coded ANF**

To decode the $(N, K)$ MDS code, coded ANF applies Reed-Solomon decoding. Successful decoding requires the number of errors of computation results such that $N \geq K + 2b$. The following theorem shows the security achieved by coded ANF.

**Theorem 1.** Given a number of workers $N$ and a dataset $X = (X_1, \ldots, X_K)$, the proposed coded ANF can be robust to $b$ adversaries for computing $\{f(X_k)\}_{k=1}^K$ for any Boolean function $f$, as long as

$$N \geq K + 2b;$$

i.e., coded ANF achieves the security threshold

$$\beta_{\text{ANF}} = \frac{N - K}{2}.$$  

Whenever the master receives $N$ results from the workers, the master decodes the computation results using a length-$N$ Reed-Solomon code for each of $r(f)$ linear functions which incurs the total complexity $O(r(f)N \log^2 N \log \log N)$. Computing all the monomials via the signs of corresponding linear threshold functions incurs the complexity $O(Nr(f))$. Lastly, computing $f(X_1), \ldots, f(X_K)$ by summing the monomials incurs the complexity $O(Nr(f))$ since there are $r(f) - 1$ additions in function $f$. Thus, the total complexity of decoding step is $O(r(f)N \log^2 N \log \log N)$ which works well for small $r(f)$. Note that the operation of this scheme is over the real field whose size doesn’t scale with size of $m$.

**V. SCHEME 2: CODED DISJUNCTIVE NORMAL FORM**

In this section, we propose a coding scheme called **coded Disjunctive normal form (DNF)** which computes the DNF representation of Boolean function by LTFs and a simple linear code is used for the data encoding. We start with an example to illustrate the idea behind coded DNF.

**Example 2.** Consider a function which has an ANF representation defined as follows:

$$f(X) = (X[1] + \cdots + X[m]) \lor (X[1] \lor \cdots \lor X[m] \lor 1)$$

which has the degree $\deg f = m - 1$ and the number of monomials $r(f) = 2^m - 1$. Alternatively, this function has a DNF representation as follows:

$$f(X) = (X[1] \land \cdots \land X[m]) \lor (\sim X[1] \land \cdots \land \sim X[m])$$

which has the weight $w(f) = 2$.

For the clause $X[1] \land \cdots \land X[m]$, we define a linear function over the real field:

$$L_1(X) = X[1] + \cdots + X[m] - m + \frac{1}{2}$$

where $X[0] \land \cdots \land X[m] = 1$ if and only if $L_1(X) = \frac{1}{2}$. Otherwise, $L_1(X) \leq -\frac{1}{2}$. Similarly, for the clause $\sim X[0] \land \cdots \land \sim X[m]$, we define a linear function over the real field:

$$L_2(X) = -X[1] - \cdots - X[m] + \frac{1}{2}$$

where $\sim X[1] \land \cdots \land \sim X[m] = 1$ if and only if $L_2(X) = \frac{1}{2}$. Otherwise, $L_2(X) \leq -\frac{1}{2}$. Therefore, we can compute $f(X)$ by computing $\text{sgn}(L_1(X))$ and $\text{sgn}(L_2(X))$, i.e., $f(X) = 1$ if at least one of $\text{sgn}(L_1(X))$ and $\text{sgn}(L_2(X))$ is equal to 1. Otherwise, $f(X) = 0$. Unlike directly computing the function $f(X)$ with the degree of $m - 1$, computing the linear functions $L_1(X)$ and $L_2(X)$ allows us to apply a linear code on the computations.

**A. Formal Description of coded DNF**

Given the DNF representation defined in (2), we now present the proposed coded DNF as follows. For each clause $T_i$ with the corresponding input $Y_i$ such that $f(Y_i) = 1$, we define a linear function $L_i(X)$ over the real field:

$$L_i(X) = \sum_{j=1}^{m} Z[i,j]X[j] - \sum_{j=1}^{m} Y[i,j] + \frac{1}{2}$$

where

$$Z[i,j] = \begin{cases} 1, & \text{if } Y[i,j] = 1 \\ -1, & \text{if } Y[i,j] = 0. \end{cases}$$

It is clear that $L_i(Y_i) = \frac{1}{2}$ and $L_i(X) \leq -\frac{1}{2}$ for all other inputs $X \neq Y_i$. Thus, there are $w(f)$ constructed linear threshold functions, and each clause $T_i$ can be computed by its corresponding linear threshold function $\text{sgn}(L_i(X))$.

The master encodes $X_1, X_2, \ldots, X_K$ to $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_N$ over the real field using an $(N, K)$ MDS code. Each worker $n \in [N]$ stores $\tilde{X}_n$ locally. Each worker $n$ computes the functions $L_1(X_n), \ldots, L_{w(f)}(X_n)$ and then sends the results back to the master. After receiving the results from the workers, the master first recovers $L_i(X_k)$ for each $i \in [w(f)]$ and each $k \in [K]$ via MDS decoding. Then, the master has $T_i(X_k) = 1$ if $\text{sgn}(L_i(X_k)) = 1$; otherwise $T_i(X_k) = 0$. Lastly, the master has $f(X_k) = 1$ if at least one of $T_1(X_k), \ldots, T_{w(f)}(X_k)$ is equal to 1. Otherwise, $f(X_k) = 0$.

**B. Security Threshold of Coded DNF**

Similar to coded ANF, we present the following theorem shows the security threshold achieved by the coded DNF.

**Theorem 2.** Given a number of workers $N$ and a dataset $X = (X_1, \ldots, X_K)$, the proposed coded DNF can be robust to $b$ adversaries for computing $\{f(X_k)\}_{k=1}^K$ for any Boolean function $f$, as long as

$$N \geq K + 2b;$$
i.e., coded DNF achieves the security threshold
\[ \beta_{DNF} = \frac{N - K}{2}. \] (18)

Whenever the master receives \( N \) results from the workers, the master decodes the computation results using a length-\( N \) Reed-Solomon code for each of \( w(f) \) linear functions which incurs the total complexity \( O(w(f)N\log^2 N \log \log N) \). Computing all the clauses via the signs of corresponding linear threshold functions incurs the complexity \( O(Nw(f)) \). Lastly, computing \( f(X_i), \ldots, f(X_K) \) by checking all the clauses requires the complexity \( O(Nw(f)) \). Thus, the total complexity of decoding step is \( O(w(f)N\log^2 N \log \log N) \) which works well for small \( w(f) \).

VI. SCHEME 3: CODED POLYNOMIAL THRESHOLD FUNCTION

In this section, we propose a coding scheme called coded polynomial threshold function which computes the DNF representation of Boolean function by the polynomial threshold functions (PTF) and LCC is used for the data encoding.

A. Formal Description of coded PTF

Given the DNF representation defined in (2), we now construct the polynomial function \( P(X) \) with degree of at most \( \lfloor \log w(f) \rfloor + 1 \) as follows:

\[ P(X) = A_1C_1(X)L_1(X) + \cdots + A_{w(f)}C_{w(f)}(X)L_{w(f)}(X) \]

where \( A_1 \gg A_2 \gg A_3 \cdots \gg A_m > 0 \) are appropriately chosen positive values.

The master encodes \( X_1, X_2, \ldots, X_K \) to \( \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_N \) over the real field using LCC. Each worker \( n \in [N] \) stores \( \tilde{X}_n \) locally. Each worker \( n \) computes the function \( P(\tilde{X}_n) \) and then sends the result back to the master. After receiving the results from the workers, the master first recovers \( P(X_1), \ldots, P(X_K) \) via LCC decoding. Then, the master has \( f(X_k) = 1 \) if \( \text{sgn}(P(X_k)) = 1 \); otherwise \( f(X_k) = 0 \).

B. Security Threshold of Coded PTF

Since \( P(X) \) has degree of at most \( \lfloor \log_2 w(f) \rfloor + 1 \), to be robust to \( b \) adversaries, LCC requires the number of workers \( N \) such that \( N \geq (K-1)(\lfloor \log_2 w(f) \rfloor + 1) + 2b + 1 \). Thus, we have the following theorem.

Theorem 3. Given a number of workers \( N \) and a dataset \( X = (X_1, \ldots, X_K) \), the proposed coded polynomial threshold function can be robust to \( b \) adversaries for computing \( \{f(X_k)\}_{k=1}^K \) for any Boolean function \( f \), as long as

\[ N \geq (K-1)(\lfloor \log_2 w(f) \rfloor + 1) + 2b + 1; \] (19)

i.e., coded PTF achieves the security threshold

\[ \beta_{PTF} = \frac{N - (K-1)(\lfloor \log_2 w(f) \rfloor + 1) - 1}{2}. \] (20)

Whenever the master receives \( N \) results from the workers, the master decodes the computation results using a length-\( N \) Reed-Solomon code for the polynomial function which incurs the total complexity \( O(N\log^2 N \log \log N) \). Lastly, computing \( f(X_1), f(X_2), \ldots, f(X_K) \) by checking the signs requires the complexity \( O(N) \). Thus, the total complexity of decoding step is \( O(N\log^2 N \log \log N) \).

In the following example, we show that coded PTF outperforms LCC for the Boolean functions with the polynomial size of \( r(f) \) and \( w(f) \).

Example 3. Consider a function which has an ANF representation defined as follows:

\[ f(X) = (X[1] \oplus X[2]) \cdots (X[2m' - 1] \oplus X[2m']) \times X[2m' + 1] \cdots X[m] \] (21)

where \( m' = \lfloor \log_2 m^2 \rfloor \). Note that here we focus on the case that \( m \) is large enough such that \( m > m' = \lfloor \log_2 m^2 \rfloor \). The function \( f \) has the degree of \( m - \lfloor \log_2 m^2 \rfloor \), the sparsity of \( \approx m^2 \) and the weight of \( \approx m^2 \).

For the Boolean function considered in Example 3, coded PTF achieves the security threshold

\[ \frac{N - (K-1)(\lfloor \log_2 m^2 \rfloor + 1) - 1}{2} \]

which is greater than the security threshold

\[ \frac{N - (K-1)(m - \lfloor \log_2 m^2 \rfloor) - 1}{2} \]

provided by LCC. Although coded ANF and coded DNF achieve the optimal security threshold \( \frac{N-K}{2} \) but they require decoding complexity \( O(m^2 N \log^2 N \log \log N) \) which has the order of \( m^2 \), i.e., they only work for small \( m \). With the security slightly worse than coded ANF and coded DNF, coded PTF achieves the better decoding complexity which is independent of \( m \), i.e., coded PTF works for large \( m \).
