Microlocal Category

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1 Introduction

In this paper we associate a dg-category over Novikov ring with coefficients in \( \mathbb{Q} \) to a compact symplectic manifold \( M \) with its symplectic form having integral periods. We generalize the approach in [Tam2008], where we associate a certain category —denoted by \( D^>_{>0}(X \times \mathbb{R}) \)— to the cotangent bundle \( T^*X \). No pseudoholomorphic curves are used in the construction. The constructed category possesses properties similar to the celebrated Fukaya category [FOOO]. For example, every object has a support which is a closed subset of \( M \); the full sub-category of objects supported on a smooth Lagrangian manifold admits —modulo the maximal ideal of Novikov’s ring— the same description as in the Fukaya category, and likewise for the hom space between objects supported on transversal smooth Lagrangian manifolds. We also have an invariance under Hamiltonian flow property. These properties will be proven in a subsequent paper.

A different approach to associating a Fukaya-like category to a symplectic manifold based on deformation quantization is developed in [Tsyg2015].

Given a symplectic manifold \( M \) as specified above, we consider a family \( F \) of Its Darboux balls, \( I : F \times B_R \to M \) —given an \( f \in F \), we have a symplectic embedding \( I_f : B_R \to M \), where \( B_R \subset T^*\mathbb{R}^N \) is the symplectic ball of radius \( R \). We then consider a category \( D^>_{>0}(F \times \mathbb{R}^N \times \mathbb{R}) \) and build a version of a monad acting on it. More precisely, we consider a monoidal category \( D^>_{>0}((F \times \mathbb{R}^N)^2 \times \mathbb{R}) \) which acts on \( D^>_{>0}(F \times \mathbb{R}^N \times \mathbb{R}) \) by convolutions and define an \( A_\infty \) algebra \( A \) with curvature in \( D^>_{>0}((F \times \mathbb{R}^N)^2 \times \mathbb{R}) \). The desired microlocal category is then defined as that of \( A \)-modules in \( D^>_{>0}(F \times \mathbb{R}^N \times \mathbb{R}) \).

The algebra \( A \) is first constructed in the \( \varepsilon \)-quasi-classical limit. That is, in a modification of \( D^>_{>0}((F \times \mathbb{R}^N)^2 \times \mathbb{R}) \) having the same objects but the hom definition changes to

\[
\text{Hom}_\varepsilon(F, G) := \text{Cone}(\text{Hom}(F, T_{-\varepsilon}G) \to \text{Hom}(F, G)),
\]

where \( \varepsilon > 0 \) is a small number. The next part is the quantization, that is, lifting \( A \) to an \( A_\infty \)-algebra with curvature in \( D^>_{>0}((F \times \mathbb{R}^N)^2 \times \mathbb{R}) \). The quasi-classical part of the problem can be carried over \( \mathbb{Z} \) but the quantization requires passage to \( \mathbb{Q} \): our approach is based on obstruction theory which only vanish up-to torsion.

Let us now dwell on the quantization problem. We start with discussing:

1.1 Ground rings/categories for quantization

First of all, let us draw a parallel with the more traditional quantization setting, where we have a local complete ring \( \mathbb{Q}[[q]] \) and the problem is to lift a certain object over \( \mathbb{Q} \) to \( \mathbb{Q}[[q]] \). In our case it is more convenient to work in the category of \( \varepsilon \)-\( \mathbb{Z} \)-graded objects in the category of complexes of \( \mathbb{Q} \)-vector spaces. Let us denote this category by \( \mathcal{G} \) (this notation will only be used in this introduction). For such a \( V \in \mathcal{G} \), we denote by \( \text{gr}^{\varepsilon \mathbb{Z}}V \) its corresponding graded component.

Let \( \Lambda := \mathbb{Q}[q] \) be a \( \varepsilon \)-\( \mathbb{Z} \)-graded ring in \( \mathcal{G} \), where the variable \( q \) is of grading \(-\varepsilon \). This ring is a graded analogue of the celebrated Novikov’s ring.
Let us denote an appropriate dg-version of $D_{>0}(X \times \mathbb{R})$ by $\text{sh}(X \times \mathbb{R})$ so that the latter category is enriched over the category of complexes of $\mathbb{Q}$-vector spaces. The category $\text{sh}(X \times \mathbb{R})$ has an enrichment over $\mathcal{G}$. Let $F,G \in D_{>0}(X \times \mathbb{R})$. Set $\text{gr}^{-n\varepsilon}\text{Hom}(F,G) := \text{Hom}(F,T_{n\varepsilon}G)$, where $T_{n\varepsilon}$ denotes the shift along $\mathbb{R}$ by $n\varepsilon$. As was explained in [Tam2008], we have natural maps $T_{n\varepsilon}G \to T_{m\varepsilon}G$, $n \leq m$, whence induced maps $\tau_{n,m} : \text{gr}^{-n\varepsilon}\text{Hom}(F,G) \to \text{gr}^{-m\varepsilon}\text{Hom}(F,G)$, $n \leq m$. The action of $\mathbb{Q}[q]$ is as follows: the generator $q$ acts on $\text{gr}^{-n\varepsilon}\text{Hom}(F,G)$ by $\tau_{n,n+1}$. Observe that the properties of $D_{>0}(X \times \mathbb{R})$ do also imply the acyclicity of the following homotopy projective limit:

$$
\text{gr}^{-0}\text{Hom}(F,G) \leftarrow \text{gr}^{-1}\text{Hom}(F,G) \leftarrow \text{gr}^{-2}\text{Hom}(F,G) \leftarrow \cdots.
$$

This can be interpreted as the $(q)$-adic completeness of a $\mathbb{Q}[q]$-module $\text{Hom}(F,G)$.

Let us now consider 'the derived reduction mod $q$' which can be computed as follows

$$
\text{gr}^{-n\varepsilon}\text{Hom}_q(F,G) := \text{Cone}(q : \text{gr}^{-(n-1)\varepsilon}\text{Hom}(F,G) \to \text{gr}^{-n\varepsilon}(F,G))
$$

which coincides with the definition in [1].

The category of $\mathbb{Q}[q]$-modules causes problems, for example, when we want to tensor multiply over $\mathbb{Q}[q] —$ this requires derived tensor product. One can replace this category with a full sub-category of 'semi-free' objects. More precisely, we define a category $\text{Quant} \langle \varepsilon \rangle$ as follows.

We first define a dg-category $Q$ whose every object is a $\mathbb{Z},\varepsilon$-graded complex of $\mathbb{Q}$-vector spaces and we set

$$
\text{Hom}_Q(X,Y) := \prod_{n \geq m} \text{Hom}(\text{gr}^{n\varepsilon}X; \text{gr}^{n\varepsilon}Y).
$$

An object of $\text{Quant} \langle \varepsilon \rangle$ is a pair $(X,D_X)$, where $X \in Q$, and $D_X \in \text{Hom}^1(X,X)$ satisfies the Maurer-Cartan equation $dD_X + D_X^2 = 0$. Set

$$
\text{Hom}((X,D_X),(Y,D_Y)) := (\text{Hom}_Q(X,Y), D_{XY}),
$$

where we change the differential on the complex $\text{Hom}_Q(X,Y)$ as follows:

$$
D_{XY}f := df + D_Yf - (-1)^{\deg f}D_X f.
$$

Let $\mathcal{X} := (X,D_X) \in \text{Quant} \langle \varepsilon \rangle$. Let $D_{nm} \in \text{Hom}^1(\text{gr}^{n\varepsilon}X; \text{gr}^{m\varepsilon}X)$, $n \leq m$, be the components. Every object in $\text{Quant} \langle \varepsilon \rangle$ is isomorphic to that with $D_{nn} = 0$ for all $n$ which we will be assumed from now on.

Define a $\mathbb{Q}[q]$-module $\mathcal{X}[[q]] \in \mathcal{G}$ as follows:

$$
\text{gr}^{n\varepsilon} \mathcal{X}[[q]] := \prod_{p \geq n} \text{gr}^{p\varepsilon} X^{(n)}_\mathcal{X},
$$

where

$$
D^{(n)}_\mathcal{X} := \sum_{p,q|n \leq p \leq q} D_{pq}.
$$

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The functor $\mathcal{X} \mapsto \mathcal{X}[[q]]$ is fully faithful; from now on we replace the category of $Q[q]$-modules in $\mathcal{G}$ with $\textbf{Quant}(\varepsilon)$. As the $Q[q]$-module $\mathcal{X}[[q]]$ is semi-free, we can define its classical reduction as the reduction mod $(q)$ which is nothing else but $X$ viewed as a $\mathbb{Z}\varepsilon$-graded complex of $\mathbb{Q}$-vector spaces; in other words, we forget $D_X$ (since we assume that all the diagonal components $D_{nn}$ are equal to 0).

We thus denote by $\textbf{Classic}(\varepsilon)$ the category of $\mathbb{Z}\varepsilon$-graded complexes of $\mathbb{Q}$-vector spaces (with $\text{Hom}(X,Y) := \prod_n (\text{gr}^n X; \text{gr}^n Y)$) and define a functor $\text{red} : \textbf{Quant}(\varepsilon) \to \textbf{Classic}(\varepsilon)$ as explained above: $\text{red}(X,D) := X$.

Denote by $\text{sh}_q(X)$ the category $\text{sh}((\mathfrak{Fr} \times \mathbb{R}^N)^2)$ enriched over $\textbf{Quant}(\varepsilon)$ and by $\text{sh}_0(X)$ the triangulated hull of $\text{red}\text{sh}_q(X)$. Still denote by $\text{red} : \text{sh}_q(X) \to \text{sh}_0(X)$ the natural functor which has the meaning of the $\varepsilon$-quasi-classical reduction.

1.2 $\mathbb{Z} \times \mathbb{Z}$-equivariance

We have a $\mathbb{Z} \times \mathbb{Z}$-action on the category $\text{sh}_q((\mathfrak{Fr} \times \mathbb{R}^N)^2)$ where $(m,n)$ acts by $T_n[2m]$, where $T_n$ is the shift along $\mathbb{R}$ by $n$ units. We define $A_0$ as the $\mathbb{Z} \times \mathbb{Z}$-equivariant algebra. For simplicity, we neglect this aspect in this introduction.

1.3 Passage to operads

We have monoidal categories $\mathcal{M}_q := \text{sh}_q((\mathfrak{Fr} \times \mathbb{R}^N)^2)$, $\mathcal{M}_0 := \text{sh}_0((\mathfrak{Fr} \times \mathbb{R}^N)^2)$ and a monoidal functor $\text{red} : \mathcal{M}_q \to \mathcal{M}_0$. We also have an associative algebra $A_0$ in $\mathcal{M}_0$. Assume for simplicity that there exists an object $A \in \mathcal{M}_q$ where $\text{red}A \cong A_0$. Denote by $\mathcal{O}$ the full asymmetric operad of $A$ which is defined in the category $\textbf{Quant}(\varepsilon)$. The associative algebra structure on $A_0$ reflects in a map of asymmetric operads $\text{assoc} \to \text{red}\mathcal{O}$ (defined over $\textbf{Classic}(\varepsilon)$), where $\text{assoc}$ is the asymmetric operad of associative algebras, $\text{assoc}(n) = \mathbb{Q}$ for all $n$. The quantization problem is now to lift this map onto the level of $\textbf{Quant}(\varepsilon)$. This can be conveniently formulated as the problem of finding a Maurer-Cartan element in $\mathcal{O}$ whose classical reduction is equivalent to the Maurer-Cartan element determined by $A_0$. More precisely, we are to find elements $M_n^k \in (\text{gr}^{k\varepsilon} \mathcal{O}(n))^1$, where:

- $M_n^0 = 0$ for all $n$;
- $M_2^0$ equals the binary product of $A_0$; $M_n^0 = 0$ for all $n \neq 2$;
- $dM_n^k + \sum M_{m}^{l} \{M_{n-m+1}^{k-l} \} = 0$,

where $f\{g\}$ is the brace operation and we observe that the sum is essentially finite. Such a Maurer-Cartan element is equivalent to the structure of an $A_\infty$-algebra with curvature on $A$.

This type of deformation problems if well known to be controlled by the Hochschild complex of $A_0$ which does not vanish. In order to achieve an unobstructedness we enrich the structure.
1.3.1 Monoidal categories with a trace/Circular operads

The notion of a trace on a monoidal category is the categorification of the notion of a trace on an associative algebra. Let $\mathcal{M}$ be a monoidal category over enriched over a SMC $\mathcal{C}$. Informally speaking, a contravariant trace on $\mathcal{M}$ is a contravariant functor $\text{Tr} : \mathcal{M} \to \mathcal{C}$ such that the polyfunctors

$$(X_0, X_1, \ldots, X_n) \mapsto \text{Tr}(X_0 \otimes X_1 \otimes \cdots \otimes X_n), \quad X_i \in \mathcal{M}$$

are invariant under cyclic permutations of $(X_0, X_1, \ldots, X_n)$. See Sec 5.8 for a more precise definition. Given an object $A$ of a monoidal category with a trace, one associates to it the following collection of spaces:

$${\mathcal{O}}_A^{\text{noncyc}}(n) := \text{Hom}(A^\otimes n; A); \quad {\mathcal{O}}_A^{\text{cyc}}(n) := \text{Tr}(A_0 \otimes A_1 \otimes \cdots \otimes A_n).$$

The objects $${\mathcal{O}}_A(n)^{\text{cyc}}$$ carry a cyclic group of $(n+1)$-st order action, we also have insertion maps

$${\mathcal{O}}_A^{\text{cyc}}(n) \otimes {\mathcal{O}}_A^{\text{noncyc}}(k_0) \otimes {\mathcal{O}}_A^{\text{noncyc}}(k_1) \cdots \otimes {\mathcal{O}}_A^{\text{noncyc}}(k_n) \to {\mathcal{O}}_A^{\text{cyc}}(k_0 + k_1 + \cdots + k_n).$$

One also has an asymmetric operad structure on $${\mathcal{O}}_A^{\text{noncyc}}$$. These data satisfy certain associativity axioms. We call such a structure a circular operad abbreviated as CO. It is a restricted version of a modular operad defined in [GetzKap].

Given a monoidal category with a trace one defines a trace on an algebra $A$ in $\mathcal{M}$ as a map $t : \text{unit}_{\mathcal{M}} \to \text{Tr}(A)$ satisfying: let $m_n : A^\otimes n \to A$ be the $n$-fold product on $A$. Then each element $m_n t \in \text{Tr}(A^\otimes n)$ must be cyclically invariant.

Equivalently, define a circular operad $\text{assoc}$ whose every space is $\mathbb{Q}$, and all the composition and cyclic group action maps preserve $1 \in \mathbb{Q}$. The structure of an algebra with a trace on an object $A$ is now equivalent to a map of circular operads $\text{assoc} \to {\mathcal{O}}_A$.

Back to our setting, we endow the category $\mathcal{M}_Q$, hence $\mathcal{M}_0$, with a trace, we also endow the algebra $A_0$ with a trace resulting in a map of circular operads $\text{assoc} \to \text{red}{\mathcal{O}}_A$, where $\mathcal{O}_A$ is the full circular operad of $A$. One generalizes the definition of a MC-element to this setting which allows us to define the deformation problem in the circular operad setting. This problem is still obstructed — so we have to further enhance the structure which is done by means of the following tool. For future purposes we will now describe the obstruction complex.

1.3.2 Deformation complex of a map $a_0 : \text{assoc} \to \text{red}{\mathcal{O}}_A$

Given a map $a_0 : \text{assoc} \to \text{red}{\mathcal{O}}_A$, denote $\text{Hoch}(a_0)^n := {\mathcal{O}}_A^{\text{noncyc}}(n)$ and $\text{Hochcyc}(a_0)^n := {\mathcal{O}}_A^{\text{cyc}}(n)$. We have a standard co-simplicial structure on $\text{Hoch}(a_0)^\bullet$ and a co-cyclic structure on $\text{Hochcyc}(a_0)^\bullet$. Denote by $\text{Hoch}(a_0)$ the total cochain complex of $\text{Hoch}(a_0)^\bullet$. Likewise denote by $\text{Hochcyc}(a_0)$ the total cyclic cochain complex of $\text{Hochcyc}(a_0)^\bullet$ which computes $R\text{Hom}_{\mathcal{A}}(\mathbb{Q}; \text{Hochcyc}(a_0)^\bullet)$,

where $\mathbb{Q}$ is the constant cyclic object. The map $\text{red}a_0 : \text{assoc}^{\text{cyc}} \to \text{red}{\mathcal{O}}_A^{\text{cyc}}$ defines a cocycle $\Omega$ in $\text{Hochcyc}(a_0)$. The Lie derivative $X \mapsto L_X \Omega$ defines a map $\omega : \text{Hoch}(a_0) \to \text{Hochcyc}(a_0)[-1]$. One can show that the obstruction complex to lifting $a_0$ is as follows:

$$\mathcal{O}(a_0) = \text{Cone} \omega.$$
1.4 \(c_1\)-Localization

Let \(\Lambda\) be the cyclic category. Given a (co)-cyclic object \(X\), we have a natural map \(c_1 : X \to X[2]\), the first Chern class. One generalizes to the case of an arbitrary circular operad as follows. Let \(E\) be an asymmetric operad. Let us consider a category \(C\) whose every object is a COO with an identification \(O_{\text{noncyc}} = E\). One can construct a category \(Y(E)^{\text{cyc}}\) whose every object is of the form \((n), n \geq 0\), such that the structure of an object \(O \in C\) is equivalent to that of a functor \(F_O : Y(E)^{\text{cyc}} \to C\), where \(F((n)) = O^{\text{cyc}}(n)\). For example, \(Y(\text{assoc})^{\text{cyc}}\) is the \(\mathbb{Q}\)-span of the cyclic category. It turns out that the \(c_1\) map extends to the category of functors \(Y(E)^{\text{cyc}} \to C\). Let us now define the \(c_1\)-localization in the standard way: given \(X : Y(E)^{\text{cyc}} \to C\), we define an inductive sequence

\[X \xrightarrow{c_1} X[2] \xrightarrow{c_1} X[4] \xrightarrow{c_1} \cdots\]

and define the hocolim of this sequence as an ind-object in the category of functors \(Y(E)^{\text{cyc}} \to C\). Denote this ind-object by \(X_{\text{loc}}\). A similar procedure is used for the definition of the periodic cyclic homology.

Let us get back to our setting, where we have a circular operad \(O\) in \(\text{Quant}(\varepsilon)\) and a map of circular operads \(a_0 : \text{assoc} \to \text{red}O\). (2)

Consider the object \(O_{\text{loc}}^{\text{cyc}}\), which is an ind-functor \(Y^{\text{cyc}}(O) \to \text{Classic}(\varepsilon)\), whence an induced ind \(Y(\text{assoc})^{\text{cyc}}\)-structure on \(\text{red}O_{\text{loc}}^{\text{cyc}}\), which is the same as the structure of an ind-cocyclic object. It turns out that this structure is rigid — it admits a canonical lifting to the quantum level so that we get an ind-cyclic object structure on \(O_{\text{cyc}}^{\text{loc}}\). Denote by \(O_{\text{l}}\) the resulting circular operad \((\text{assoc}_{\text{noncyc}}, O_{\text{cyc}})\).

Let \(U := (\text{assoc}_{\text{noncyc}} \oplus O_{\text{noncyc}}^{\text{cyc}}, O_{\text{cyc}})\), where \(\text{assoc}_{\text{noncyc}}\) acts on \(O_{\text{cyc}}^{\text{cyc}}\) by zero. We then construct a homotopy map of CO \(U \to O_{\text{l}}\) which on the level of the underlying asymmetric operads reduces to the projection \(\text{assoc}_{\text{noncyc}} \oplus O_{\text{noncyc}}^{\text{cyc}} \to \text{assoc}_{\text{noncyc}}\).

1.4.1 Deformation complexes and VCO

Let us define a map \(i : \text{redassoc} \to \text{red}U\), where \( \text{redassoc}_{\text{noncyc}} \to \text{red}(\text{assoc}_{\text{noncyc}} \oplus O_{\text{noncyc}}^{\text{cyc}})\) is the diagonal map \(\text{Id} \oplus a_0^{\text{noncyc}}\) and the cyclic component coincides with \(a_0^{\text{cyc}}\), where \(a_0\) is as in (2). We thus get a diagram

\[
\begin{array}{ccc}
\text{assoc} & \longrightarrow & \text{red}U \\
\downarrow \text{Id} & & \downarrow \\
\text{assoc} & \longrightarrow & \text{red}O_{\text{l}} \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{assoc} & \longrightarrow & \text{red}U \\
\downarrow & & \downarrow \\
\text{assoc} & \longrightarrow & \text{red}O_{\text{l}} \\
\end{array}
\]

Whence an induced diagram of the deformation complexes

\[
\begin{array}{ccc}
D(\text{assoc} \to \text{red}U) & \longrightarrow & D(\text{assoc} \to \text{red}O_{\text{l}}) \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
D(\text{assoc} \to \text{assoc}) & \longrightarrow & D(\text{assoc} \to \text{ass}^{\text{noncyc}}) \\
\end{array}
\]

Denote by

\[
D'(\text{assoc} \to \text{red}U) := \text{Cone} \pi_1[-1]; \quad D'(\text{assoc} \to \text{red}O_{\text{l}}) := \text{Cone} \pi_2[-1]
\]

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We have an induced map

\[ D'(\text{assoc} \to \text{red} U) \to D'(\text{assoc} \to \text{red} O), \tag{4} \]

which we will now describe.

Let \( a \in \text{Classic\langle \varepsilon \rangle} \) be as follows

\[ \text{gr}_{\varepsilon} a = \bigoplus_{k \in \mathbb{Z}} \mathbb{Q}[2k], \]

if \( n \varepsilon \in \mathbb{Z} \), and \( \text{gr}_{\varepsilon} a = 0 \) otherwise. Let \( C_M \) be the singular chain complex on \( M \) with coefficients in \( \mathbb{Q} \) and let \( \Omega_M := C_M \otimes a \).

We have

\[ \text{Hoch(assoc \to redO)} \sim \Omega_M, \]

\[ \text{Hochcyc(assoc \to redO)} \sim \Omega_M[[u]], \]

where \( u \) is a variable of degree +2 — the \( c_1 \)-action is by the multiplication by \( u \). The freeness of the \( c_1 \)-action on \( \Omega_M[[u]] \) comes from the triviality of the circle action on the realization of the co-cyclic object \( \text{Hochcyc}^*(\text{assoc} \to \text{redO}) \).

We have

\[ D'(\text{assoc} \to \text{red} U) \sim u^{-1}\Omega_M[[u]][1]; \quad D'(\text{assoc} \to \text{red} O_l) \sim \Omega_M[u^{-1}, u][1]; \]

the map \([1]\) is then homotopy equivalent to the obvious embedding

\[ u^{-1}\Omega_M[[u]][1] \to \Omega_M[u^{-1}, u][1]. \]

It is crucial in our approach that this map admits a splitting. That is, there exists an object \( V_0 \in \text{Classic\langle \varepsilon \rangle} \) and a map \( V_0 \to D(\text{assoc} \to \text{red} O_l) \) such that the induced map

\[ D(\text{assoc} \to \text{red} U) \to \text{Cone}(V_0 \to D(\text{assoc} \to \text{red} O_l)). \]

Because of rigidity, the map \( V_0 \to D(\text{assoc} \to \text{red} O_l) \) lifts as follows. First we define \( V \in \text{Quant\langle \varepsilon \rangle} \) as \((V_0, 0)\). Next, we construct a homotopy map of circular operads

\[ (\text{assoc}^\text{noncyc}, V \otimes \text{assoc}^\text{cyc}) \to O_l. \]

We then have the following structure:

- circular operads \( U, O_l \) and an object \( V \), all the data in the category \( \text{Quant\langle \varepsilon \rangle} \);
- maps of circular operads \( U \to O \leftarrow (\text{assoc}^\text{noncyc}, V \otimes \text{assoc}^\text{cyc}). \)

We call such a structure \( \text{VCO (V from the object V)} \).

Denote the resulting VCO by \( W \). Let also \( \text{assoc} \) be the VCO with its both CO being \( \text{assoc} \) and \( V = 0 \), the structure maps

\[ \text{assoc} \to \text{assoc} \leftarrow (\text{assoc}^\text{noncyc}, 0) \]

are the obvious map. It follows that we have a map of VCO \( \text{assoc} \to \text{redW} \). The problem of lifting this map turns out to be unobstructed, which concludes the solution of the quantization problem.
1.5 Structure of the paper

First, we fix the notation and review basic operations with dg-categories such as formal direct sums, products, twists by a Maurer-Cartan element. Next, we define dg-versions of $D_{>0}(X \times \mathbb{R})$, referred to as $\text{sh}_q(X)$, and its quasi-classical reductions, $\text{sh}_\varepsilon(X)$, $\varepsilon > 0$. We then proceed to the quasi-classical and the quantization parts of the construction both having detailed introductions (see Sec 5 and Sec 17) to which we refer the reader.

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2 DG Categories

2.0.1 Fixing a ground category

Let $\mathbb{k}$ be either $\mathbb{Z}$ or $\mathbb{Q}$. Let $\mathbb{k}\text{-mod}$ be the symmetric monoidal category of complexes of $\mathbb{k}$-modules.

2.1 Category $\mathbf{D}\mathbf{C}$

Let $\mathcal{C}$ be a category enriched over $\mathbb{k}\text{-mod}$. Let $X \in \mathcal{C}$. A MC-element on $X$ is an element $D \in \text{Hom}^1(X, X)$ satisfying $dD + D^2 = 0$. Let $\mathbf{D}\mathbf{C}$ be a category enriched over $\mathbb{k}\text{-mod}$ whose every element is a pair $(X, D)$, where $X \in \mathcal{C}$ and $D$ is a MC-element on $X$. Let

$\text{Hom}_{\mathbf{D}\mathbf{C}}((X_1, D_1), (X_2, D_2)) := \langle \text{Hom}_{\mathcal{C}}(X_1, X_2), d' \rangle$,

where the modified differential $d'$ is defined by the formula

$d'f = df + D_2f - (-1)^{|f|}fD_1$.

A category $\mathcal{C}$ enriched over $\mathbb{k}\text{-mod}$ is called $D$-closed if the embedding $\mathcal{C} \rightarrow \mathbf{D}\mathbf{C}$, $X \mapsto (X, 0)$, is an equivalence of categories.

2.2 Ground category

A ground category is a SMC $\mathfrak{A}$ which is

— $D$-closed;
— closed under direct sums and products;
— the tensor product is compatible with direct sums and differentials, the latter means that the natural map

$$(X, D_X) \otimes (Y, D_Y) \rightarrow (X \otimes Y, D_X \otimes 1 + 1 \otimes D_Y)$$

is an isomorphism for all $(X, D_X), (Y, D_Y)$;
— has inner hom.
Let $\mathbb{K} -$freemod $\subset \mathbb{K}$-mod be the full sub-category of complexes of free $\mathbb{K}$-modules. We then have a fully faithful embedding $\mathbb{K} -$freemod $\rightarrow \mathfrak{A}$ which is a strict tensor functor preserving differentials and direct sums.

We use the term dg-category to mean a category enriched over $\mathfrak{A}$. Throughout the paper one can assume everywhere that $\mathfrak{A} = \mathbb{K}$-mod.

In the case one needs to use $\mathbb{Z}$ as a ground ring one can still use the category of complexes of abelian groups which satisfies all the properties of a ground category. However, one encounters problems coming from inexactness of the tensor product and hom. Alternatively, one can define a ground category swell($\mathbb{Z}$) which is free of these drawbacks. As this won’t be used in the paper, we will sketch the construction very briefly. We will use the notion of a co-filter on a set $S$, which, by definition, is a collection of subsets of $S$ satisfying:

— if $A \in \mathcal{F}; B \subset A$, then $B \in \mathcal{F}$;
— if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

Given a filter $\mathcal{F}$ on $S$ and $\mathcal{G}$ on $T$, let us define filters $\mathcal{F} \times \mathcal{G}, \text{Hom}(\mathcal{F}, \mathcal{G})$ on $S \times T$, where $\mathcal{F} \times \mathcal{G}$ is the smallest co-filter containing all the sets $A \times B, A \in \mathcal{F}, B \in \mathcal{G}$.

The co-filter $\text{Hom}(\mathcal{F}; \mathcal{G})$ consists of all subsets $U \subset S \times T$ satisfying:

— for every $t \in T$ and every $R \in \mathcal{F}$, the set $(U \cap R \times t)$ is finite.
— Let $R \in \mathcal{F}$. Define a sub-set $H_R \subset T$ consisting of all $t \in T$ such that the set $(U \cap (R \times t))$ is non-empty. Then $H_R \in \mathcal{G}$ for all $R \in \mathcal{F}$.

Given a set $S$, a filter $\mathcal{F}$ on $S$ and abelian groups $A_s, s \in S$, define the restricted product

$$\prod_{s \in S} \mathcal{F} A_s \subset \prod_{s \in S} A_s,$$

to consist of all families $(a_s)_{s \in S}, a_s \in A_s$, satisfying:

$$\{s \in S | a_s \neq 0\} \in \mathcal{F}.$$

One defines the restricted product of complexes of abelian groups by setting

$$(\prod_{s \in S} \mathcal{F} A_s)^k := \prod_{s \in S} \mathcal{F} A_s^k.$$

Let us now define a category swell$_0(\mathbb{Z})$ is the following collection of data: $X := (S_X, \mathcal{F}_X, \{X_s\}_{s \in S})$, where $\mathcal{F}_X$ is a filter on $S_X$ and each $X_s$ is a finite complex of free finitely generated abelian groups. Set

$$\text{Hom}(X, Y) := Z^0 \prod_{(s,t) \in S \times T} \text{Hom}(\mathcal{F}_X; \mathcal{F}_Y) \text{Hom}(X_s; Y_t),$$

where $Z^0$ denotes the group of 0-cycles. Define a tensor product on swell$_0(\mathbb{Z})$, where

$$X \otimes Y := (S \times T; \mathcal{F}_X \times \mathcal{F}_Y, \{X_s \otimes Y_t\}_{(s,t) \in S \times T}).$$

We now set swell($\mathbb{Z}$) := Dswell$_0(\mathbb{Z})$, one can check that swell($\mathbb{Z}$) satisfies all the properties of the ground category.
2.3 Weak \( t \)-structure on a ground category \( \mathfrak{A} \)

We will define the following weakening of \( t \)-structure on \( \mathfrak{A} \) which is given in terms of prescription of a full sub-category \( D_{\leq 0} \mathfrak{C} \) with the following properties:

1) \( D_{\leq 0} \mathfrak{A} \) contains the unit, is \( D \)-closed, closed under the tensor product, and is closed under direct sums. \( X \in D_{\leq 0} \mathfrak{A} \) implies \( X[n] \in D_{\leq 0} \mathfrak{A} \), \( n \geq 0 \).

Denote by \( D_{> 0} \mathfrak{A} \) the full sub-category of \( \mathfrak{A} \) consisting of all objects \( T \in D_{> 0} \mathfrak{A} \) satisfying \( H^0 \text{Hom}_\mathfrak{A}(X,T) = 0 \) for all \( X \in D_{\leq 0} \mathfrak{A} \).

2) Let us define the category \( \text{trunc}(\mathfrak{A}) \) consisting of all \( X \in \mathfrak{A} \) satisfying: There exists a free \( \mathbb{k} \)-module \( A \in \mathfrak{A} \) and a map \( A \to X \) in \( \mathfrak{A} \) such that \( \text{Cone}(A \to X) \in D_{> 0} \mathfrak{A} \). Then the functor \( T_X : D_{\leq 0} \mathfrak{A}^{\text{op}} \to \text{Sets} \), \( T_X(U) = Z^0 \text{Hom}_\mathfrak{A}(U,X) \) is representable (we view here \( D_{\leq 0} \mathfrak{A}^{\text{op}} \) as a category over \( \text{Sets} \)). Denote the represented object by \( \tau_{\leq 0} X \).

Furthermore, the induced map \( A \to \tau_{\leq 0} X \) is then a homotopy equivalence and admits a unique retraction.

Instead of writing \( X \in \text{trunc}(\mathfrak{A}) \) we will also say that \( X \) admits a truncation.

Denote \( A := H^0(X) \), we can define \( H^0(X) \) as the universal free abelian group endowed with a map \( \tau_{\leq 0} X \to H^0(X) \) so that \( H^0 \) is a functor from the category \( \text{trunc}(\mathfrak{A}) \) (enriched over \( \text{Sets} \)) to that of free abelian groups. We have a natural transformation \( \tau_{\leq 0} \to H^0 \).

It follows that \( \text{trunc}(\mathfrak{A}) \) is closed under differential, products, and positive cohomological shifts.

**Lemma 2.1** Let \( I \) be a category over \( \text{Sets} \) and let \( F : I \to \text{trunc}(\mathfrak{A}) \) be a functor and suppose that

\[
L^0 := \text{proj lim}_{i \in I} H^0(F(i))
\]

is a free abelian group. Then \( L := \text{holim}_{i \in I} F(i) \in \text{trunc}(\mathfrak{A}) \) and we have an isomorphism \( H^0(L) \cong L \).

Let us fix throughout the paper a ground category with a weak \( t \)-structure \( \mathfrak{A} \). For example, the category of complexes of \( \mathbb{Q} \)-vector spaces.

2.4 Tensor product

Let \( \mathfrak{C}, \mathfrak{D} \) be dg-categories. Let \( \mathfrak{C} \oplus \mathfrak{D} \) be a dg-category whose every object is a pair \( (X,Y) \in \text{Ob} \mathfrak{C} \times \text{Ob} \mathfrak{D} \) and

\[
\text{Hom}((X_1,Y_1);(X_2,Y_2)) := \text{Hom}_\mathfrak{C}(X_1,X_2) \otimes \text{Hom}_\mathfrak{D}(Y_1,Y_2).
\]

2.4.1 Categories \( \bigoplus \mathfrak{C}, \prod \mathfrak{C} \)

Let \( \bigoplus \mathfrak{C} \) be the category of formal direct sums in \( \mathfrak{C} \). An object of \( \bigoplus \mathfrak{C} \) is a collection of the following data:

- a set \( S \);
- a family of objects \( \{X_s\}_{s \in S} \).

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— a function $n : S \to \mathbb{Z}$

We refer to such an object as

$$\bigoplus_{s \in S} X[n_s]$$

Set

$$\text{Hom}(\bigoplus_{s \in S} X[n(s)]; \bigoplus_{t \in T} Y[m(t)]) := \prod_{s \in S} \bigoplus_{t \in T} \text{Hom}_C(X_s, Y_t)[m_t - n_s].$$

Similar to above, define the category $\prod C$, whose every object is defined by the same data as in $\bigoplus C$; an object of $\prod C$ is denoted as follows:

$$\prod_{s \in S} X[n(s)].$$

Set

$$\text{Hom}(\bigoplus_{s \in S} X[n(s)]; \bigoplus_{t \in T} Y[m(t)]) := \prod_{t \in T} \bigoplus_{s \in S} \text{Hom}_C(X_s, Y_t)[m(t) - n(s)].$$

We will often use the category $(D \bigoplus) C$ which is closed under direct sums and $D$-closed, or, briefly, $(D \bigoplus)$-closed. Likewise the category $D \prod C$ is $D \prod$-closed.

Let us list some properties of these operations.

a) Let $\mathcal{D}$ be a $(D \bigoplus)$-closed dg-category. Let $F : C \to \mathcal{D}$ be a dg-functor. The functor $F$ extends canonically to a functor

$$F : (D \bigoplus) C \to \mathcal{D}.$$

b) Every $(D \bigoplus)$-closed category $\mathcal{C}$ is tensored by the category of complexes free $\mathbb{k}$-modules.

c) We have a natural funtors

$$(D \bigoplus) C \otimes (D \bigoplus) \mathcal{D} \to (D \bigoplus) C \otimes \mathcal{D};$$

We have a natural funtors

$$D \prod C \otimes D \prod \mathcal{D} \to D \prod C \otimes \mathcal{D};$$

d) Suppose $\mathcal{C}$ is a SMC, then we have induced SMC structures on $(D \bigoplus) C$, $D \prod C$.

e) We have functors

$$\text{Hom} : ((D \bigoplus) C)^{\text{op}} \otimes (D \bigoplus) \mathcal{D} \to D \prod (C^{\text{op}} \otimes \mathcal{D});$$

$$\text{Hom} : ((D \bigoplus) C)^{\text{op}} \otimes D \prod \mathcal{D} \to D \prod (C^{\text{op}} \otimes \mathcal{D}).$$
2.5 Operation $\otimes^L$

Let $I$, $C$, $D$ be dg-categories.

Let $F : I \to C$; $G : I^{\text{op}} \to D$ be dg-functors. Define a functor

$$F \otimes^L I G \in (D \bigoplus) (C \otimes A \otimes D)$$

as follows.

Set

$$F \otimes^L I G := \bigoplus_{n=0}^{\infty} (F \otimes^L I G)_n, D$$

where

$$(F \otimes^L I G)_n = \bigoplus_{(X_0, X_1, \ldots, X_n) \in I} F(X_0) \otimes \text{Hom}_I(X_0, X_1) \otimes \text{Hom}_I(X_1, X_2) \otimes \cdots \otimes \text{Hom}_I(X_{n-1}, X_n) \otimes G(X_n)[n] \in (D \bigoplus) C \otimes D.$$

The MC-element $D$ is the sum of its components

$$D_n : (F \otimes^L I G)_n \to (F \otimes^L I G)_{n-1}$$

each of which is the standard bar differential.

2.6 $R\text{Hom}_I(F, G)$

Let now $F : I \to C$, $G : I \to cD$. Define

$$R\text{Hom}(F, G) \in D \prod (C^{\text{op}} \otimes A^{\text{op}} \otimes D),$$

where

$$R\text{Hom}(F, G) = R\text{Hom}^0(F, G) \to F\text{Hom}^1(F, G) \to \cdots \to R\text{Hom}^n(F, G) \to \cdots,$$

and

$$R^n\text{Hom}(F, G) = \prod_{X_0, X_1, \ldots, X_n} (F(X_0), \text{Hom}_I(X_0, X_1), \ldots, \text{Hom}_I(X_{n-1}, X_n), F(X_n)).$$

2.7 $\text{hocolim, holim}$

Let $\text{pt}$ be the category with one object whose endomorphism group is $\mathbb{K}$. Let $I$ be a category over $\text{Sets}$. Let $J := \mathbb{K}[I]$. Let $\mathbb{K} : J^{\text{op}} \to \text{pt}$ be the constant functor.

Let $F : J \to C$ be a dg-functor. Set

$$\text{hocolim}_I F := F \otimes^J \mathbb{K} \in (D \bigoplus) (C \otimes \text{pt}) \cong (D \bigoplus) C;$$

$$\text{holim}_I F = R\text{Hom}_I(\mathbb{K}, F) \in D \prod (C).$$

Suppose $C$ is $(D \bigoplus)$-closed and let $\tau : (D \bigoplus) C \to C$ be the canonical functor. We will write by abuse of notation $\text{hocolim}_I F \in C$ instead of $\tau(\text{hocolim}_I F)$. Likewise, whenever $C$ is $D \prod$ closed, we define $\text{holim}_I F \in C$. 

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3 Sheaves: a dg-model

3.1 Pre-sheaves

Let $X$ be a locally compact topological space. Let $\text{Open}_X$ be the category of open subsets of $X$, where we have a unique arrow $U \to V$ iff $U \subset V$. Denote by the same symbol the $\mathbb{k}$-span of $\text{Open}_X$. Let $\text{psh}(X) := (\mathbf{D} \bigoplus) \text{Open}_X^{\text{op}}$.

Let $U \in \text{Open}_X$. Denote by $h_U : \text{Open}_X^{op} \to (\mathbf{D} \bigoplus) \text{pt}$ the Yoneda functor $h_U(V) := \mathbb{k}[\text{Hom}(V, U)]$.

Still denote by $h_U$ the extension:

$h_U : (\mathbf{D} \bigoplus) \text{Open}_X^{op} \to (\mathbf{D} \bigoplus) \text{pt}$.

Let $F \in \text{psh}(X)$. Define a functor $H_F : \text{Open}_X \to (\mathbf{D} \bigoplus) \text{pt}$, where $H_F(U) := h_U(F)$.

3.2 Sheaves

We will define a full sub-category $\text{sh}(X) \subset \text{psh}(X)$ by the following conditions.

a) ‘stability’: let $\text{Id} : \text{Open}_X^{opp} \to \text{Open}_X^{opp}$ be the identity functor. Let

$$\rho : (\mathbf{D} \bigoplus) \text{pt} \otimes (\mathbf{D} \bigoplus) \text{Open}_X^{\text{op}} \to (\mathbf{D} \bigoplus) \text{Open}_X^{\text{op}}$$

be the natural functor.

$$G_F := \rho(H_F \otimes \text{Id}_{\text{Open}_X}) \in (\mathbf{D} \bigoplus) \text{Open}_X^{\text{op}} = \text{psh}(X).$$

We have a natural map $G_F \to F$ in $\text{psh}(X)$.

Call $F$ stable if this map is a homotopy equivalence.

b) Coverings.

Let $A$ be a set. Let $\{U_a\}_{a \in A}$ be a family of open sets in $X$. Let $B$ be the set of all finite intersections $U_{a_1} \cap U_{a_2} \cap \cdots \cap U_{a_n}$. For $b \in B$, denote by $U_b \subset X$ the corresponding open subset. Let $U$ be the union of all $U_b$, $b \in B$. View $B$ as a sub-poset of $\text{Open}_X$. We have a natural map

$$\text{hocolim}_B \; H_F|_B \to H_F(U).$$

We require this map to be a homotopy equivalence for every family $\{U_a\}_{a \in A}$.

3.3 The sheaf $\mathbb{k}_K$

3.3.1 Coverings

Let $K \subset X$ be a compact subset. A finite covering $U$ of $K$ is a collection $U_0, U_1, U_2, \ldots, U_N \in \text{Open}_X$ for some $N \in \mathbb{Z}_{\geq 0}$.

Denote by $\text{Cov}(K)$ the set of all finite coverings of $K$. 

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3.3.2 Chech complex

Denote by $\text{Chech}(\mathcal{U}) \in (\mathbf{D} \oplus \text{Open}^\text{op}_X)$ the following object: $\text{Chech}(\mathcal{U}) = (\bigoplus_{n=0}^{N} \text{Chech}_n^\mathcal{U}, D)$

$$\text{Chech}(\mathcal{U})^n := (\bigoplus_{0 \leq i_0 < i_1 < \ldots < i_n \leq N} \mathcal{U}_{i_0} \cap \mathcal{U}_{i_1} \cap \cdots \cap \mathcal{U}_{i_n})[-n].$$

The differential $D$ is the sum of its components $D^n : \text{Chech}(\mathcal{U})^n \to \text{Chech}(\mathcal{U})^{n+1}$ defined in the well-known way.

We have a natural map

$$X \to \text{Chech}(\mathcal{U}). \quad (5)$$

3.3.3 Cap-product

Let $\cap : \text{Open}_X \times \text{Open}_X \to \text{Open}_X$ be the intersection bi-functor which naturally extends to a bi-functor

$$\cap : \text{psh}(X) \otimes \text{psh}(X) \to \text{psh}(X).$$

We have

$$X \cap F \cong F \text{ for all } F \in \text{psh}(X). \quad (6)$$

Let $\mathcal{S}(K)$ be the poset of all finite subsets of $\text{Cov}(K)$. Set

$$\text{Chech}(\mathcal{S}) := \bigcap_{\mathcal{U} \in \mathcal{S}} \text{Chech}(\mathcal{U}).$$

Let $S_1 \subset S_2$. The maps (5), (6) induce a map $\text{Chech}(S_1) \to \text{Chech}(S_2)$. So that

$$\text{Chech} : \mathcal{S} \to \text{psh}(X).$$

Set

$$\mathcal{K}_K := \operatorname{hocolim}_{\mathcal{S}(K)} \text{Chech}.$$

**Proposition 3.1** We have $\mathcal{K}_K \in \text{sh}(X)$.

Let $K_1 \subset K_2$. Then $\text{Cov}_{K_2} \subset \text{Cov}_{K_1}$, whence an induced map

$$\mathcal{K}_{K_2} \to \mathcal{K}_{K_1}. \quad (7)$$

3.4 Sheaf $\mathcal{K}_U$

3.4.1 Compactification. Restriction functor

Let $\overline{X}$ be the one-point compactification of $X$. Let

$$r : \text{Open}^\text{op}_\overline{X} \to \text{psh}(X)$$
be given by \( r(U) = U \) if \( U \subset X \); \( r(U) = 0 \) otherwise. The functor \( r \) extends to a functor

\[
r : \text{psh}(\mathcal{X}) \to \text{psh}(X).
\]

### 3.4.2 Definition of \( \mathcal{K}_U \)

Let \( U \in \text{Open}_X \). Set

\[
\mathcal{K}_U := r \text{Cone}(\mathcal{K}_X \to \mathcal{K}_{X \setminus U})[-1].
\]

The maps \( \mathcal{K}_U \) give the rule \( U \mapsto \mathcal{K}_U \) the structure of a functor \( \text{Open}_X \to \text{sh}(X) \).

### 3.5 Derived category of sheaves on \( X \) versus \( \text{sh}(X) \)

Let \( \text{Com}(X) \) be the category of complexes of sheaves of \( \mathbb{K} \)-modules on \( X \) bounded from above, enriched over the category of complexes of \( \mathbb{K} \)-modules.

Define a functor \( \zeta : \text{Open}_X \to \text{Com}(X) \) by setting \( \zeta(U) := \mathcal{K}_U \in \text{Com}(X) \). We have an induced functor

\[
\text{ho}\zeta|_{\text{ho sh}(X)} : \text{ho sh}(X) \to \mathcal{D}(X).
\]

Let us construct the inverse functor. Let us construct a (non-additive) functorial free resolution functor from the category \( \text{ComAb}^- \) of complexes of abelian groups bounded from above to \( (\mathcal{D} \oplus)\text{pt} \).

For an abelian group \( A \) set \( F(A) := \mathbb{Z}[A]/\mathbb{Z}.0 \). We have a natural transformation \( F(A) \to A \). Next, \( F \) transfers every 0 arrow to a 0 arrow and preserves the 0 object. Therefore, \( F \) extends to a (non-additive) endofunctor on \( \text{ComAb}^- \).

Let \( X \in \text{ComAb}^- \). Define a complex

\[
\cdots \to R^{-n}(X) \xrightarrow{D_n} R^{-n+1}(X) \xrightarrow{D_{n-1}} \cdots \xrightarrow{D_2} R^{-1}(X) \xrightarrow{D_1} R^0(X) \xrightarrow{D_0} 0,
\]

by induction. Set \( R^0(X) := F(X), D_0 = 0 \). Set

\[
R^{-(n+1)}(X) = F(\text{Ker}D_n).
\]

Set

\[
D_{n+1} : R^{-(n+1)}(X) \to \text{Ker}D_n \to R^{-n}(X).
\]

Let also \( i_0 : R^0(X) = F(X) \to X \) be the canonical map. We have thus constructed a bicomplex \( R^\bullet(X) \); denote by \( R(X) \in (\mathcal{D} \oplus)\text{pt} \) the total of \( R^\bullet(X) \). The map \( i_0 \) induces a natural transformation \( R(X) \to X \) which is a quasi-isomorphism so that \( R \) is a functorial free resolution.

Let now \( F^\bullet \) be a complex of sheaves on \( X \) bounded from above. We then get a functor \( R(F) : \text{Open}_X^{\text{op}} \to (\mathcal{D} \oplus)\text{pt} \), where

\[
R(F)(U) := R(F(U)).
\]

Define an object \( \lambda(F) \in \text{sh}(X) \) by setting

\[
\lambda(F) := R(F)(U) \otimes_{U \in \text{Open}_X^{\text{op}}} \mathcal{K}_U.
\]
Let us extend $\lambda$ to the category of un-bounded complexes of sheaves on $X$. By setting

$$\lambda(F) := \text{hocolim}_n \lambda(\tau_{\leq n} F).$$

It follows that $\lambda$ converts quasi-isomorphisms into homotopy equivalence, whence an induced functor

$$\text{ho} \lambda : D(X, \mathbb{K}) \to \text{h} \text{sh}(X).$$

Consider the composition

$$\zeta \lambda(F) = \text{hocolim}_n \zeta \lambda(\tau_{\leq n} F) = \text{hocolim}_n R(\tau_{\leq n} F)(U) \otimes_{U \in \text{Open}_X^{\text{op}}} \zeta(\mathbb{K}_U).$$

We have a natural transformation $\zeta(\mathbb{K}_U) \to \mathbb{K}_U$ which is a quasi-isomorphism. We therefore get a zig-zag of natural transformations which are term-wise quasi-isomorphisms between $\zeta \lambda$ and $\text{Id}$.

Let us now consider

$$\lambda \zeta(F) \sim F(U) \otimes L \lambda(\mathbb{K}_U).$$

We have a natural transformation

$$\mathbb{K}_U \to \lambda \zeta \mathbb{K}_U$$

which is a homotopy equivalence. This proves the statement.

### 3.6 Base of topology

Let $B \subset \text{Open}_X$ be a base of topology, that is: $B$ is closed under intersections and every open subset of $X$ can be represented as a union of sets from $B$. View $B$ as a poset, hence as a dg-category. Let $\text{sh}(B) \subset (\mathbf{D} \bigoplus)(B^{\text{op}})$ be the full sub-category satisfying the stability condition and the covering condition with respect to all coverings of elements of $B$ whose all terms are in $B$.

Let us define a functor $\beta_B : \text{sh}(X) \to \text{sh}(B)$, where

$$\beta_B(F) = H_F|_B \otimes L \text{Id}_{B^{\text{op}}}. $$

It follows that $\beta_B$ is a weak equivalence of dg-categories.

### 3.7 Products

Set $\text{sh}(X|Y) := \text{sh}(\text{Open}_X \times \text{Open}_Y)$. We have a functor

$$\boxtimes : \text{sh}(X) \otimes \text{sh}(Y) \to (\mathbf{D} \bigoplus)(\text{Open}_X^{\text{op}} \times \text{Open}_Y^{\text{op}}).$$

It follows that

$$\boxtimes : \text{sh}(X) \otimes \text{sh}(Y) \to \text{sh}(X|Y).$$

3.7.1 Convolution

Define a functor

\[ g_X : \text{Open}_X^{\text{op}} \times \text{Open}_X^{\text{op}} \to (D \bigoplus)\text{pt} \]

via

\[ g_X(U, V) := k \]

if \( U \cap V \neq \emptyset \);

\[ g_X(U, V) := 0 \]

otherwise.

**Lemma 3.2** Let \( X \) be compact. We have a zig-zag homotopy equivalence

\[ g_X(U, V) \otimes_{(U, V) \in \text{Open}_X^{\text{op}} \times \text{Open}_X^{\text{op}}} \mathbb{K}_{U \times V} \to \mathbb{K}_{\Delta_{X \times X}}, \]

where \( X \) is the diagonal.

We have a bifunctor

\[ \circ : \text{sh}(X|Y) \otimes \text{sh}(Y|Z) \to \text{sh}(X|Z) \]

defined as follows:

\[ \text{sh}(X|Y) \otimes \text{sh}(Y|Z) \to \text{sh}(X|Y|Y|Z) \xrightarrow{g_Y} \text{sh}(X|Z). \]

We have an induced map on homotopy categories

\[ \text{ho}^* : D(X \times Y) \times D(Y \times Z) \to D(X \times Z). \]

**Proposition 3.3** The bifunctor \( \text{ho} \circ \) is isomorphic to the following one:

\[ (F, G) \mapsto F \ast_Y G, \]

where \( \ast_Y \) is the operator of composition of kernels.

4 Quantum and Classical sheaves

4.1 More on categories

4.1.1 The category of complexes

Let \( C \) be a category enriched over \( \mathcal{A} \). Let \( n \in \mathbb{Z}_{\geq 0} \). Let us define a category \( \text{Com}(C)(1/2^n)' \) enriched over \( \mathcal{A} \) whose every object is a \((1/2^n)\),\( \mathbb{Z} \)-graded object in \( C \). For such an \( X \) denote by \( \text{gr}^{k/2^n}X \) the corresponding graded component.

Set

\[ \text{Hom}(X, Y) := \prod_{m \leq n} \text{Hom}_C(\text{gr}^m X; \text{gr}^n Y) \in \mathcal{A}. \]
Set \( \text{Com}(C)(1/2^n) := D\text{Com}(C)(1/2^n)' \).

One can represent every object of \( \text{Com}(C)(1/2^n) \) as \((X, D)\), where \( X \in \text{Com}(C)(1/2^n)_0 \) and all the diagonal components \( D_{kk} : \text{gr}^k X \to \text{gr}^k X \) vanish.

Suppose the category is closed under direct products (resp. direct sums) then so is \( \text{Com}(C)(1/2^n) \), where
\[
\text{gr}^k(\prod_a X_a) = \prod_a \text{gr}^k X_a;
\]
\[
\text{gr}^k(\bigoplus_a X_a) = \bigoplus_a \text{gr}^k X_a.
\]

Suppose \( C \) is a SMC closed under direct sums and its tensor product is compatible with direct sums. This structure is inherited by \( \text{Com}(C)(1/2^n) \), where
\[
\text{gr}^k(X \otimes Y) := \bigoplus_{p \in \mathbb{Z}} \text{gr}^{k-p/2^n} X \otimes \text{gr}^{p/2^n} Y.
\]

Suppose \( C \) is, in addition to the hypotheses from the previous paragraph, closed under direct products and has an internal Hom. Then so is \( \text{Com}(C)(1/2^n) \).

We have
\[
\text{gr}^k \text{Hom}(X, Y) = \prod_{l \in \mathbb{Z}} \text{Hom}_C(\text{gr}^{l/2^n} X; \text{gr}^{(l/2^n)+k} Y).
\]

We have an obvious embedding functor
\[
i_{nm} : \text{Com}(C)(1/2^n) \to \text{Com}(C)(1/2^m),
\]
whenever \( 0 \leq n \leq m \). The embedding \( i_{nm} \) has a right adjoint, to be denoted by \( c_{mn} \). We have
\[
\text{gr}^{p/2^n}(X, D) = (\bigoplus_{2m-n \leq q < 2m-n(p+1)} \text{gr}^{q/2^n} X, D'),
\]

where the differential \( D' \) is induced by \( D \). In the case \( C \) is an SMC, \( i_{nm} \) has a strict tensor structure, and \( c_{mn} \) has a lax tensor structure, that is we have a natural transformation
\[
c_{mn}(X \otimes c_{mn}(Y) \to c_{mn}(X \otimes Y).
\]

Let also \( \text{Gr}(C)(1/2^n) \) be the category whose every object is a \( 1/2^n \mathbb{Z} \) graded object \( X \) in \( C \). Set
\[
\text{Hom}(X, Y) := \prod_n \text{Hom}(\text{gr}^n X; \text{gr}^n Y).
\]

The category \( \text{Gr}(C)(1/2^n) \) has similar properties to that of \( \text{Com}(C)(1/2^n) \).

We have a functor
\[
\text{Gr} : \text{Com}(C)(1/2^n) \to \text{Gr}(C)(1/2^n).
\]

Let \( (X, D) \in \text{Com}(C)(1/2^n) \), where \( X \in \text{Com}(C)(1/2^n)_0 \) and all the diagonal components of \( D \) vanish. Then \( \text{Gr}(X, D) := X \).
4.1.2 The categories \textbf{Classic}(C)\langle1/2^n\rangle, \textbf{Quant}(C)\langle1/2^n\rangle.

Let \( C \) be a \( D \)-closed category enriched over \( \mathfrak{A} \). Define a category \( \textbf{Quant}(C)\langle1/2^n\rangle' \) enriched over \( \text{Com}(\mathfrak{A})\langle1/2^n\rangle \) whose every object \( X \) is a \( \mathbb{R} \)-graded object in \( C \). Denote by \( \text{gr}^c X \) the corresponding graded component.

Set \( \text{Hom}(X,Y) = (\mathcal{H}, 0) \), where

\[
\text{gr}^k 2^n \mathcal{H} := \prod_{c \in \mathbb{R}} \oplus \text{Hom}(\text{gr}^c X; \text{gr}^{c+k/2^n+\delta} Y).
\]

Let

\[
\text{Quant}(C)\langle1/2^n\rangle := D\text{Quant}(C)\langle1/2^n\rangle'.
\]

Let us list some properties.

We have a functor

\[
T_{\text{Quant}} : \text{Quant}(C)\langle1/2^n\rangle \otimes \text{Quant}(D)\langle1/2^n\rangle \to \text{Quant}((D \bigoplus)C \otimes D), \tag{8}
\]

where

\[
\text{gr}^c T_{\text{Quant}}(X,Y) = \bigoplus_{a \in \mathbb{R}} (\text{gr}^a X, \text{gr}^{c-a} Y).
\]

Let \( X, Y \) be \( \mathbb{R} \)-graded objects in \( C \). We then have a natural isomorphism

\[
\text{Hom}_{\text{Quant}(C)\langle1/2^n\rangle}(X,Y) \cong c_{mn} \text{Hom}_{\text{Quant}(C)\langle1/2^m\rangle}(X,Y),
\]

whence functors

\[
i_{nm} : i_{nm} \text{Quant}(C)\langle1/2^n\rangle \to \text{Quant}(C)\langle1/2^m\rangle, \quad n \leq m.
\]

The functor \( i_{nm} \) is identical on objects. Let us define the action of \( i_{nm} \) on hom’s:

\[
i_{nm} : i_{nm} \text{Hom}(X,Y) = i_{nm} c_{mn} \text{Hom}(i_{nm}X; i_{nm}Y) \to \text{Hom}(i_{nm}X; i_{nm}Y),
\]

where the last arrow comes from the conjugacy.

Let \( \Gamma : \text{Com}(\mathfrak{A})\langle1/2^n\rangle \to \mathfrak{A} \) be given by \( \Gamma(X) = \text{Hom} \left( \text{unit}_{\text{Com}(\mathfrak{A})}; X \right) \). It now follows that \( i_{nm} \)

induces an equivalence of categories enriched over \( \mathfrak{A} \):

\[i_{nm} : \Gamma(\text{Quant}(C)\langle1/2^n\rangle) \to \Gamma(\text{Quant}(C)\langle1/2^m\rangle).
\]

Denote by \( \text{Quant}(C) \) any of \( \Gamma(\text{Quant}(C)\langle1/2^n\rangle) \).

Let us now define the category \( \textbf{Classic}(C)\langle1/2^n\rangle := D\text{GrQuant}(C)\langle1/2^n\rangle \) enriched over \( \text{Gr}(\mathfrak{A})\langle1/2^n\rangle \).

We have functors

\[
\text{Gr} : \text{GrQuant}(C)\langle1/2^n\rangle \to \text{Classic}(C)\langle1/2^n\rangle;
\]

\[
i_{nm} : i_{nm} \text{Classic}(C)\langle1/2^n\rangle \to \text{Classic}(C)\langle1/2^m\rangle.
\]

In the case \( C \) is an SMC, these functors have a tensor structure.
We have functors

\[
T_{\text{Quant}} : \text{Quant}(C) \otimes \text{Quant}(D) \to \text{Quant}(\bigoplus(D \otimes C)); \tag{9}
\]

\[
T_{\text{Classic}} : \text{Classic}(C)(1/2^n) \otimes \text{Classic}(D)(1/2^n) \to \text{Classic}(\bigoplus(D \otimes C))(1/2^n) \tag{10}
\]
defined similarly to (8).

4.1.3 Functors \( R_{\leq a}, R_{>a} \)

Let \( \text{Quant}(C)_{\leq a} \subset \text{Quant}(C) \) consist of all objects of the form \((X,D)\) with \(gr^bX = 0\) for all \(b > a\).

Suppose \( C = (D \bigoplus)V \) for some category \( V \) enriched over \( A \). We then have an equivalence

\[
\text{Quant}(C)_{\leq a} \cong (D \bigoplus)(V \otimes (-\infty,a]),
\]

where \((-\infty,a]\) is viewed as a poset, whence a category structure.

The embedding \( \text{Quant}(C)_{\leq a} \subset \text{Quant}(C) \) has a right adjoint, to be denoted by \( R_{\leq a} \), where \( gr^b(R_{\leq a}T) = 0, b > a; gr^b(R_{\leq a}T) = gr^bT, b \leq a \). Denote

\[
R_{>0}X := \text{Cone} R_{\leq a}X \to X.
\]

4.2 The category \( sh_q(X) \)

Let us define a full subcategory

\[
sh_q(X) \subset \text{Quant}(\bigoplus(\text{Open}_X^{\text{op}}))
\]
as follows.

Let us define a functor

\[
\eta : \text{Quant}(\bigoplus(\text{Open}_X^{\text{op}})) \to (\bigoplus(\text{Open}_X^{\text{op}}) \times \mathbb{R})
\]

Which is defined as follows:

- let us first define

\[
\eta_{\leq a} : \text{Quant}(\bigoplus(\text{Open}_X^{\text{op}})_{\leq a} \cong (D \bigoplus)(\text{Open}_X^{\text{op}} \times (-\infty,a]) \to (D \bigoplus)(\text{Open}_X^{\text{op}} \times \mathbb{R}),
\]

where the last arrow is induced by the map

\[
(U,b) \mapsto U \times (b,\infty).
\]

Set

\[
\eta(T) := \text{hocolim}_{a \to \infty} \eta_{\leq a} R_{\leq a}T.
\]

Let \( sh_q(X) \subset \text{Quant}(\bigoplus(\text{Open}_X^{\text{op}})) \) consist of all objects \( T \) with \( \eta(T) \in \text{sh}(X \times \mathbb{R}) \).

Let \( \text{sh}(X \times \mathbb{R})_{>0} \subset \text{sh}(X \times \mathbb{R}) \) be the full sub-category consisting of all objects \( F \) with \( F(U \times (-\infty,a)) \sim 0 \) for all \( U \in \text{Open}_X \) and all \( a \in \mathbb{R} \).
Proposition 4.1  The functor $\eta$ establishes a weak equivalence

$$\text{sh}_q(X) \to \text{sh}(X \times \mathbb{R})_{>0}.$$  

Let also

$$\text{sh}_{q,1/2^n}(X) \subset \text{Quant}((\text{Open}_{X}^\text{op})^{1/2^n})$$

be a full sub-category consisting of all the objects from $\text{sh}_q(X)$.

### 4.3 Full sub-category $\text{sh}_{1/2^n}(X) \subset \text{Classic}((D \bigoplus \text{Open}_{X}^\text{op})^{1/2^n})$

Let $\Gamma : \text{Classic}(\mathfrak{A})(1/2^n) \to \mathfrak{A}$ be given by $\Gamma(X) := \text{Hom(} \text{unit}; X \text{)}$.

We have functors

$$e_a : \Gamma((\text{Classic}(D \bigoplus \text{Open}_{X}^\text{op})^{1/2^n})) \to \text{Quant}((D \bigoplus \text{Open}_{X}^\text{op}), a \in \mathbb{R},$$

where $\text{gr}^b e_a(X) = \text{gr}^b X$, where $a \leq b < a + 1/2^n$ and $\text{gr}^b e_a(X) = 0$ otherwise.

Let

$$\text{sh}_{1/2^n}(X) \subset \text{Classic}((D \bigoplus \text{Open}_{X}^\text{op})^{1/2^n})$$

be the full sub-category consisting of all objects $T$ with $e_a(T) \in \text{sh}_q(X)$ for all $a \in \mathbb{R}$.

The functor

$$\text{red} : \text{Quant}((\text{Open}_{X}^\text{op})^{1/2^n}) \to \text{Classic}((\text{Open}_{X}^\text{op})^{1/2^n})$$

descends to a functor

$$\text{red} : \text{sh}_{q,1/2^n}(X) \to \text{sh}_{1/2^n}(X).$$

We also have functors

$$\text{red}_{1/2^n,1/2^m} : i_{1/2^n,1/2^m} \text{sh}_{1/2^n}(X) \to \text{sh}_{1/2^m}(X), \quad n \leq m.$$  

### 4.4 Product

The functor $T_{\text{Quant}}$ as in (9) gives rise to a functor

$$\text{Quant}((D \bigoplus \text{Open}_{X}^\text{op}) \otimes \text{Quant}((D \bigoplus \text{Open}_{Y}^\text{op}) \to \text{Quant}((D \bigoplus ((\text{Open}_{X}^\text{op}) \times \text{Open}_{Y}^\text{op})))$$

which descends onto the level of sheaves so that we obtain the following functors

$$\otimes : \text{sh}_q(X) \otimes \text{sh}_q(Y) \to \text{sh}_q(X[Y]);$$

$$\otimes : \text{sh}_{q,1/2^n}(X) \otimes \text{sh}_{q,1/2^n}(Y) \to \text{sh}_{q,1/2^n}(X[Y]).$$

Likewise, the functors $T_{\text{Classic}}$ as in (10) produce functors:

$$\otimes : \text{sh}_{1/2^n}(X) \otimes \text{sh}_{1/2^n}(Y) \to \text{sh}_{1/2^n}(X[Y]).$$
4.5 Convolution of kernels

Denote
\[ \circ : sh_q(X|Y) \otimes sh_q(Y|Z) \to sh_q(X|Y|Z) \overset{\partial_Y}{\to} sh_q(X|Z) \]

The induced bi-functor on the homotopy categories is isomorphic to
\[ (F,G) \mapsto Ra_!(F \ast_Y G), \]
where
\[ a : X \times R \times Z \times R \to X \times Y \times R, \]
\[ a(x,t_1,z,t_2) = (x,z,t_1 + t_2). \]

4.6 Functors \( f_!, f^{-1} \)

Let \( f : X \to Y \) be a map of locally compact topological spaces. Let \( \Gamma_f \subset X \times Y \) be the graph of \( f \). Set \( K_f \in sh_q(X|Y), K^t_f = \zeta_k^\Gamma_f \times [0,\infty) \). Let \( K^t_f \in sh_q(Y'|X) \) be the image of \( K_f \) under the equivalence \( sh_q(X|Y) \sim sh_q(Y'|X) \).

Set \( f_! : sh_q(X) \to sh_q(Y), f^{-1} : sh_q(Y) \to sh_q(X) \), be given by
\[ f_!(F) = F \ast_X K_f; \quad f^{-1} G := F \ast_Y K^t_f. \]

The functors \( f_!, f^{-1} \) lift the correspondent functors between \( D_{>0}(X \times R), D_{>0}(Y \times R) \).

4.7 Microsupport

Let \( F \in sh_q(X) \). We then have a sub-set
\[ SS \eta(F) \subset T^*(X \times R) = T^*X \times T^*R. \]

Let us refer to a point in \( T^*R \) as \((t,k)\), where \( t \in R \) and \( k \in T^*_kR \). Let \( T^*_k(X \times R) \) consist of all points whose \( k \)-coordinate is positive. The group \( R^*_k \) of positive dilations acts freely on \( T^*_k(X \times R) \) fiberwise. The quotient of this action is diffeomorphic to \( T^*X \times R \). The projection \( \pi : T^*_k(X \times R) \to T^*X \times R \) is as follows
\[ \pi(x,\omega,t,k) = (x,\omega/k,t), \]
where \( x \in X, \omega \in T^*_xX \).

Set
\[ \mu_S(F) := \pi(SS \eta(F) \cap T^*_k(X \times R)). \]

It follows that
\[ SS \eta F \cap T^*_k(X \times R) = \pi^{-1} \mu_S(F). \]

The homogeneous symplectic structure on \( T^*_k(X \times R) \) gives rise to a contact structure on \( T^*X \times R \), the corresponding contact form is \( \theta = dt + \alpha \), where \( \alpha \) is the Liouville form on \( T^*X \). We will always assume this contact structure on \( T^*X \times R \).
5 Classical theory: Introduction

5.1 A category associated to an open subset of $T^*X \times \mathbb{R}$

In the previous section, we have associated categories $\text{sh}_q(X)$, $\text{sh}_\varepsilon(X)$ to a smooth manifold $X$. Given an object $F \in \text{sh}_q(X)$, one associates a closed subset $\mu \mathcal{S}(F) \subset T^*X \times \mathbb{R}$. In [?] it was shown that Hamiltonian symplectomorphisms act by natural transformations on $D_{>0}(X \times \mathbb{R})$, an analogue of $\text{sh}_q(X)$. These observations suggest to associate the category $\text{sh}_q(X)$ to a contact manifold $T^*X \times \mathbb{R}$. Next, given an open subset $U \subset T^*X \times \mathbb{R}$, we can build categories $\text{sh}_q(X)[U]$, (resp. $\text{sh}_\varepsilon(X)[U]$), which are quotients by the full sub-category $C_q(U)$ (resp. $C_\varepsilon(U)$), of objects micro-supported away from $U$.

5.2 A category associated to a symplectic ball

The case of $U = B_R \times \mathbb{R} \subset T^*\mathbb{R}^N \times \mathbb{R}$, where $B_R \subset T^*\mathbb{R}^N$ is the standard open ball of radius $R$, was treated in [Chiu]. In particular, it was shown that the category $\text{sh}_q(\mathbb{R}^N)[B_R \times \mathbb{R}]$ is equivalent to the left orthogonal complement to $C_q(U)$. Same is true for $\text{sh}_\varepsilon(\mathbb{R}^N)[B_R \times \mathbb{R}]$. These results are reviewed in Sec[7].

5.3 Symplectic embeddings of a ball into $T^*\mathbb{R}^N \times \mathbb{R}$

Let now $i : B_R \to T^*\mathbb{R}^N$ be a symplectic embedding and consider the category $\text{sh}_q(\mathbb{R}^N)[i(B_R) \times \mathbb{R}]$. This category turns out to be weakly equivalent to $\text{sh}_q(\mathbb{R}^N)[B_R \times \mathbb{R}]$, however, there is no canonical way to choose a weak equivalence. Such a choice requires the following additional data

— the lifting of $i$ to a $\mathbb{R}$-equivariant contact embedding $i_c : B_R \times \mathbb{R} \to T^*\mathbb{R}$;

— let $d_{i_0} \in \text{Sp}(2N)$ be the differential of $i$ at 0, we then have to specify a lifting $D \in \overline{\text{Sp}}(2N)$ of $d_{i_0}$, where $\overline{\text{Sp}}(2N)$ is the universal cover.

Call these data the grading of $i$. Given such a graded $i$, one constructs an object

\[ \mathcal{P}_i \in \text{sh}_q(\mathbb{R}^N \times \mathbb{R}^N) \tag{11} \]

such that the convolution with $\mathcal{P}_i$ induces an equivalence from $\text{sh}_q(\mathbb{R}^N)[B_R \times \mathbb{R}]$ and $\text{sh}_q(\mathbb{R}^N)[i(B_R) \times \mathbb{R}]$.

5.4 Families of symplectic embeddings of a ball into $T^*\mathbb{R}^N$

One generalizes to families of grading embeddings as follows. Let $i : F \times B_R \to T^*\mathbb{R}^N$ be a family of symplectic embeddings of $B_R$ into $T^*\mathbb{R}^N$, let $d_F : F \to \text{Sp}(2N)$ be the map, where $d_F(f)$, $f \in F$, is the differential of $i|_{f \times B_R}$ at $(f, 0)$.

Define a grading of $i$ as:

— the lifting of $i$ to a family $i_c : F \times B_R \times \mathbb{R} \to T^*\mathbb{R}^N \times \mathbb{R}$ of $\mathbb{R}$-equivariant contact embeddings;

— the lifting of $d_F$ to a map $D_F : F \to \overline{\text{Sp}}(2N)$.

Given such a graded family, one constructs an object $\mathcal{P}_i \in \text{sh}_q(F \times \mathbb{R}^N \times \mathbb{R}^N)$, where the restriction $\mathcal{P}_i|_{f \times \mathbb{R}^N \times \mathbb{R}^N}$ is homotopy equivalent to $\mathcal{P}_{i|_{f \times B_R}}$ as in [11].
5.4.1 Example: parallel transitions

Consider the simplest example, where \( F = T^*\mathbb{R}^N \). Given an \( f \in F \), set \( i_f(v) = f + v \), where \( v \in B_R \) and we use the vector space structure on \( T^*\mathbb{R}^N \).

We will now pass to defining a grading on this family. Let us refer to a point of \( T^*\mathbb{R}^N \) as \((q,p)\), \( q \in \mathbb{R}^N \); \( p \in (\mathbb{R}^N)^* \). First, lift our family to a \( \mathbb{R} \)-equivariant family of contact embeddings \( i_c : T^*\mathbb{R}^N \times B_R \times \mathbb{R} \to T^*\mathbb{R}^N \times \mathbb{R} \). Set

\[
i_c((q_f,p_f),(q_b,p_b),t) = (q_f + q_b, p_f + p_b, t - pfqb).
\]

Next, the differential \( d_0 i : F \to \text{Sp}(2N) \) is identically the identity, thus admitting an obvious lifting to \( \overline{\text{Sp}}(2N) \).

The quantization can be shown to be as follows:

\[
\mathcal{P}_i := \mathcal{K}_{Z'}^* \mathcal{P}_R
\]

where \( Z \subset T^*\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \) consists of all points \((q_f,p_f,q,q_b,t)\) satisfying \( q = q'_b + q_f; t + pfqb \geq 0 \).

5.4.2 The adjoint functor

The convolution with \( \mathcal{P}_i \) defines a functor \( C : \text{sh}_q(F \times \mathbb{R}^N)[T^*F \times B_R \times \mathbb{R}] \to \text{sh}_q(\mathbb{R}^N) \). Explicitly: let \( A : F \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N \times \mathbb{R} \) be given by

\[
A((q_f,p_f),q_b,t) = (q_f + q_b, t + pfqb),
\]

then we have \( C(F) = A_! F \). Using the standard theorems from 6 functors formalism, one can show that \( C \) has a right adjoint, to be denoted by \( D \). One has \( D(F) \sim A!F \circ_{\mathbb{R}^N} \mathcal{P}_R \). Next, \( A \) is a smooth trivial fibration with its fiber being diffeomorphic to \( F \) so that \( A! \sim A^{-1}[2N] \) and \( DF \sim A^{-1}F \circ_{\mathbb{R}^N} \mathcal{P}_R[2N] \) is a convolution with a kernel, to be denoted by \( Q_i \), where

\[
Q_i := \mathcal{P}_R \circ_{\mathbb{R}^N} \mathcal{K}_{Z'}[2N],
\]

where \( Z' \subset \mathbb{R}^N \times T^*\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}, \)

\[
Z' := \{(q_f,p_f,q,t) | q = q'_b + q_f; t - pfqb \geq 0 \}
\]

The functor \( D \) has no right adjoint one, however, there is a map

\[
DC \to \text{Id}[2N],
\]

defined as follows:

\[
DCF = \mathcal{P}_R \circ_{\mathbb{R}^N} A^{-1}A_!F[2N] \to \mathcal{K}_{\Delta_{\mathbb{R}^N}} \circ_{\mathbb{R}^N} F[2N] \sim F[2N].
\]

The induced map

\[
\text{Hom}(F,C\mathcal{G}[-N]) \to \text{Hom}(DF; DC\mathcal{G}[-N]) \to \text{Hom}(DF; \mathcal{G})
\]

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is a homotopy equivalence whenever $G$ is right orthogonal to all objects microsupported away from $T^*F \times B_R \times \mathbb{R}$ and the support of $G$ is proper along the projection $F \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}^N \times \mathbb{R}$.

As $D$ is a right adjoint to $C$, we can form a corresponding monad: $A := DC$ acting on $sh_q(F \times \mathbb{R}^N)$. As the functor $C$ is essentially surjective, we get an equivalence between the category of $A$-modules in $sh_q(F \times \mathbb{R}^N)$ and $sh_q(\mathbb{R}^N)$.

This observation suggests an idea of producing a category associated to an arbitrary compact symplectic manifold $M^{2N}$: choose a family $F$ of symplectic embeddings $B_R \hookrightarrow M$; define an appropriate monad $A_M$ acting on $sh_q(F \times \mathbb{R}^N)$, and, finally, consider a category of $A_M$-modules in $sh_q(F \times \mathbb{R}^N)$. The problem thus reduces to finding a monad $A_M$, which, in turn, reduces to finding an algebra $A_M$ in the monoidal category $sh_q(F \times \mathbb{R}^N \times F \times \mathbb{R}^N)$, so that the $A_M$-action on $sh_q(F \times \mathbb{R}^N)$ is by the convolution with $A_M$.

Unfortunately, the way, the monad $A$ was constructed above is not generalizeable to an arbitrary $M$ — the construction uses the existent category $sh_q(\mathbb{R}^N)$. However, there is an alternative — more 'microlocal'— way to produce $A$. This alternative way works for more general symplectic manifolds. We will first work out a simplified version in the framework of the homotopy category $ho\, sh_q$. Next, we will generalize the construction to the case of a compact symplectic $M$ with its symplectic form having integer periods. We will finally carry this construction over to the dg-setting. We will start with formulating the key Lemma.

### 5.4.3 Averaging Lemma

Let $A : F \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}^N \times \mathbb{R}$ be as in (12). For $f \in F$, let $A_f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}^N \times \mathbb{R}$ be the restriction of $A$ onto $f \times \mathbb{R}^N \times \mathbb{R}$. The functor $A_f : ho\, sh_q(\mathbb{R}^N \times \mathbb{R}) \rightarrow ho\, sh_q(\mathbb{R}^N \times \mathbb{R})$ is then a quantization of the shift by $f$ in $T^*\mathbb{R}^N$.

Let $\gamma \in ho\, sh_q(\mathbb{R}^N \times \mathbb{R})$ be an object satisfying $\mu S (\gamma) \subset B_R \times B_R \times \mathbb{R} \subset T^*\mathbb{R}^N \times T^*\mathbb{R}^N \times \mathbb{R}$. Denote by $T_\gamma$ the following endofunctor on $ho\, sh_q(T^*\mathbb{R}^N \times \mathbb{R})$, $T_\gamma F := F *_{\mathbb{R}} \gamma$.

For $f \in F$, denote

$$T_\gamma^f := A_f T_\gamma A_{-f} : ho\, sh_q(\mathbb{R}^N) \rightarrow ho\, sh_q(\mathbb{R}^N).$$

This endofunctor is representable by a kernel $\gamma^f := ho\, sh_2(\mathbb{R}^N \times \mathbb{R}^N)$. By letting $f$ vary, we get an object

$$\gamma^F \in ho\, sh_q(F \times \mathbb{R}^N \times \mathbb{R}^N).$$

Define the **averaging** of $\gamma$ as

$$p_F \gamma^F \in ho\, sh_q(\mathbb{R}^N \times \mathbb{R}^N),$$

where $p_F : F \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N \times \mathbb{R}^N$ is the projection along $F$.

Denote by $F_\gamma : ho\, sh_q(\mathbb{R}^N) \rightarrow ho\, sh_q(\mathbb{R}^N)$ the endofunctor determined by the kernel $p_F \gamma^F$. We have

$$F_\gamma = CT_\gamma D.$$

As a result of averaging, $F_\gamma$ has the following simple structure. Let $\delta : \mathbb{R}^N \rightarrow \mathbb{R}^N \times \mathbb{R}^N$ be the diagonal embedding and let $p : \mathbb{R}^N \rightarrow pt$ be the projection. Set $f_\gamma := p_\delta^{-1} \gamma[N] \in ho\, sh_q(pt)$. By the
conjugacy, we have a natural map $\gamma \to f_\gamma \boxtimes \Delta_{pN}$. Let $G \in \text{ho.sh}(T^*\mathbb{R}^N \times \mathbb{R}^N)$. We have an induced map $T\gamma G \to f_\gamma \ast G$. Let $S_{f_\gamma}$ be the operation of convolution with $f_\gamma$. We have maps

$$CT_\gamma D \to CS_{f_\gamma}D \sim S_{f_\gamma}CD \to S_{f_\gamma},$$

where the rightmost arrow is induced by the map $CD \to \text{Id}$ coming from the conjugacy.

**Claim 5.1 (Averaging Lemma)** The through map $CT_\gamma D \to S_{f_\gamma}$ is an isomorphism of functors.

The above construction descends onto the level of $\text{ho.sh}_\varepsilon$.

### 5.4.4 The monoidal structure on the natural transformation (14)

Denote $A := \text{ho.sh}_q(\mathbb{R}^N \times \mathbb{R}^N)[B_R \times B_R]$ and $B := \text{ho.sh}_q(\mathbb{R}^N \times \mathbb{R}^N)$. We have convolution monoidal structures on both $A$ and $B$. Denote $I(\gamma) := CT_\gamma D[-2N]$; $J(\gamma) := S_{f_\gamma}[-2N]$. We replace functors with representing kernels, we interpret both $I$ and $J$ as functors $A \to B$. As it turns out, the functors $I, J$ can be endowed with lax tensor structure. The natural transformation $I \to J$ in (14) upgrades to an isomorphism of lax tensor functors.

The natural transformation $I(\gamma_1) \ast I(\gamma_2) \to I(\gamma_1 \ast \gamma_2)$ is as follows:

$$CT_{\gamma_1}DCT_{\gamma_2}D[-4N] \to CT_{\gamma_1}T_{\gamma_2}D[-2N] \sim CT_{\gamma_1 \ast \gamma_2}D[-2N],$$

where the first arrow is induced by the natural transformation $DC \to \text{Id}[2N]$ as in (13).

Let us define the natural transformation $J(\gamma_1) \ast J(\gamma_2) \to J(\gamma_1 \ast \gamma_2)$.

We first define a natural transformation $f_{\gamma_1} \ast f_{\gamma_2}[-4N] \to f_{\gamma_1 \ast \gamma_2}[-2N]$. Let $p : (\mathbb{R}^N)^4 \to \text{pt}$ be the projection. Denote the coordinates on $(\mathbb{R}^N)^4$ by $(q_1, q_2, q_3, q_4)$. We then have:

$$f_{\gamma_1} \ast f_{\gamma_2}[-4N] \sim p((\gamma_1 \boxtimes \gamma_2) \otimes \mathbb{K}_{q_1=q_2,q_3=q_4}[-2N]) \ast p((\gamma_1 \boxtimes \gamma_2) \otimes \mathbb{K}_{q_1=q_4,q_2=q_3}[N]) \sim f_{\gamma_1 \ast \gamma_2}[-2N],$$

where the arrow $(\ast)$ is induced by the canonical map

$$\mathbb{K}_{q_1=q_2,q_3=q_4} \rightarrow \mathbb{K}_{q_1=q_2,q_3=q_4} \rightarrow \mathbb{K}_{q_1=q_4,q_2=q_3}[N].$$

We now define the tensor structure on $J$:

$$J(\gamma_1) \ast J(\gamma_2) \sim S_{f_{\gamma_1}} \ast S_{f_{\gamma_2}}[-4N] \rightarrow S_{f_{\gamma_1 \ast \gamma_2}}[-2N] \sim J(\gamma_1 \ast \gamma_2).$$

### 5.4.5 Redefining a monad structure on $A$

It turns out that given $\varepsilon < \pi R^2$, one can choose $\gamma$ so that $f_\gamma$ is isomorphic to the unit of the monoidal category $\text{ho.sh}_\varepsilon(\text{pt})$ so that $S_{f_\gamma} \sim \text{Id}$. Unfortunately there is no analogous statement on the level of $\text{ho.sh}_q$, that’s why we have to switch to $\text{ho.sh}_\varepsilon$.

Thus, we have a map

$$\gamma \rightarrow \text{unit}$$

(16)
in \( \text{hosh}_\varepsilon(\mathbb{R}^N \times \mathbb{R}^N) \) which induces an isomorphism
\[
CT_\gamma D \to C \text{Id} D \to \text{Id}.
\]

Let us multiply by \( D \) from the left and by \( C \) from the right. This gives an isomorphism:
\[
AT_\gamma A \to A \text{Id} A \to A,
\]
where \( A = DC \), and the last map is induced by the product on \( A \). On the other hand, we have a natural transformation
\[
A A \to A = DC \to \text{Id}[2N].
\]
Denote \( A := A \), \( B := A[-2N] \), so that we have a natural transformation \( BA \to \text{Id} \). Let \( \zeta := \gamma[2N] \).

We can now rewrite (17) as an isomorphism
\[
AT_\zeta B \to A.
\]
Observe that an associative algebra structure on \( \zeta \) gives an induced associative algebra structure on \( AT_\zeta B \):
\[
AT_\zeta BAT_\zeta B \to AT_\zeta T_\zeta B \to AT_\zeta \zeta B = AT_\zeta B \to A
\]
The goal is to find an associative structure on \( \zeta \) so that (18) be an isomorphism of associative algebras. One gets the following sufficient condition for this. We first observe that the associative product \( \mu : \zeta \star \zeta \to \zeta \) gives, by the conjugacy, the following map
\[
\nu : \zeta \ast \zeta \to p_{14}^{-1} K_\Delta \otimes p_{23}^{-1} \zeta[N],
\]
where \( p_{ij} : (\mathbb{R}^N)^4 \to (\mathbb{R}^N)^2 \) are the projections.

We then have the following diagram
\[
\begin{array}{ccc}
\zeta \ast \zeta & \xrightarrow{\nu} & p_{14}^{-1} K_\Delta \otimes p_{23}^{-1} \zeta[N] \\
\downarrow & & \downarrow \\
K_{q_1=q_2,q_3=q_4}[4N] & \xrightarrow{\ast} & K_{q_1=q_4,q_2=q_3}[3N]
\end{array}
\]
where the vertical arrows are induced by the map \( \zeta \to K_\Delta[2N] \) induced by the map (16). The bottom horizontal arrow (\( \ast \)) is induced by the map (15).

**Claim 5.2** Suppose the diagram in (19) commutes. Then (18) is an isomorphism of algebras.

**Sketch of the proof** Follows from Sec 5.4.4.

An algebra \( \zeta \) satisfying the specified conditions is constructed (see Sec. 7.1). We, therefore, have an alternative construction of the monad \( A \). It only requires kernels \( A, B \in \text{hosh}_\varepsilon(\mathbb{F} \times \mathbb{R}^N \times \mathbb{F} \times \mathbb{R}^N) \) endowed with a natural transformation
\[
B \star_{\mathbb{F} \times \mathbb{R}^N} A \to K_{\Delta_{\mathbb{F} \times \mathbb{R}^N}} \times_{\mathbb{F} \times \mathbb{R}^N}.
\]
One then gets a monad by the formula \( \mathbb{A} := A \ast T_\zeta \ast B \). This approach applies to more general symplectic manifolds.
5.5 Generalization to a compact symplectic $M$ with its symplectic form $\omega$ having integer periods

Choose a pseudo-Kaehler metric $g$ on $M$. Let $P_g$ be the bundle of $g$-unitary frames on $M$. Given a point $m \in M$ and a $g$-unitary frame $f \in P_g|_m$, there is a standard way to choose Darboux coordinates near $m$. Therefore, for $R$ small enough, one has a family of symplectic embeddings of a ball $B_R$ into $M$ parameterized by $P_g$:

$$I : P_g \times B_R \to M.$$ 

Let now $p_L : L \to M$ be the pre-quantization bundle of $\omega$, that is a principal circle bundle with connection whose curvature is $\omega$ — the existence of such a bundle follows from the integrality of periods of $\omega$.

We have the connection 1-form $\theta$ on $L$ such that $p_L^* \omega = d\theta$ so that $(L, \theta)$ becomes an $S^1$-equivariant contact manifold. One can upgrade the family $I$ into a family of contact $S^1$-equivariant embeddings $J_R : P_g \times_M L \times (B_R \times S^1) \to L$. Denote $\text{Fr}_g := P_g \times_M L$.

Let $r > 0$ be a small enough number. Denote by $J_r$ the restriction of $J_R$ onto $\text{Fr}_g \times B_r \times S^1$. Consider a subset $U \subset \text{Fr}_g \times \text{Fr}_g$ consisting of all pairs $(f_1, f_2)$ such that $J_r(f_1 \times B_r) \subset J_R(f_2 \times B_R)$. We then get a family of $S^1$-equivariant contact embeddings

$$E : U \times B_r \times S^1 \to B_R \times S^1.$$  

Let $u \in U$. Let $d_E(u) \in \text{Sp}(2N)$ be the differential at $0 \in B_R$ of the composition

$$u \times B_R \times 0 \to U \times B_r \times S^1 \xrightarrow{E} B_R \times S^1 \xrightarrow{\text{pr}_{B_R}} B_R.$$ 

Let $s_E(u) \in S^1$ be the image of the following map:

$$u \times 0 \times 0 \to U \times B_r \times S^1 \to B_R \times S^1 \xrightarrow{\text{pr}_{S^1}} S^1.$$ 

Let $V \subset U$ consist of all points $v$ with $s_E(v) = e_{S^1}$ and $d_E(v)$ being a Hermitian matrix in $\text{Sp}(2N)$. Observe that every element $g \in \text{Sp}(2N)$ can be uniquely written as $g = hu$, where $h$ is hermitian symplectic and $u$ is unitary.

One can canonically lift $E|_{V \times B_r \times S^1}$ to a $\mathbb{R}$-equivariant family of graded symplectic embeddings

$$\varepsilon : V \times B_r \times \mathbb{R} \to B_R \times \mathbb{R}.$$ 

We now have a quantization $\mathcal{P}_\varepsilon \in \text{ho sh}_{\mathbb{q}}(V \times \mathbb{R}^N \times \mathbb{R}^N)$. Let

$$\mathcal{P} := j ! \mathcal{P}_\varepsilon \in \text{ho sh}_{\mathbb{q}}(\text{Fr}_g \times \mathbb{R}^N \times \text{Fr}_g \times \mathbb{R}^N),$$ 

where

$$j : V \times \mathbb{R}^N \times \mathbb{R}^N \subset \text{Fr}_g \times \mathbb{R}^N \times \text{Fr}_g \times \mathbb{R}^N$$ 

is the embedding. Let $\sigma$ be the permutation of the factors of $(\text{Fr}_g \times \mathbb{R}^N)^2$. Let $\mathcal{Q} := \sigma \mathcal{P}[2N]$. In particular we have a natural transformation $\mathcal{Q}_{\text{Fr}_g \times \mathbb{R}^N} \mathcal{P} \to \text{unit}$.

We then have the following replacements for the ingredients of the construction in the previous subsection.
— replace $T^*\mathbb{R}^N$ with $M$;
— replace $\text{ho sh}_\varepsilon(F \times \mathbb{R}^N \times F \times \mathbb{R}^N)$ with $\text{ho sh}_\varepsilon(\text{Fr}_g \times \mathbb{R}^N \times \text{Fr}_g \times \mathbb{R}^N)$, where $\varepsilon < \pi r^2$;
— set $A$ to be the endofunctor determined by $P$ and $B$ to the endofunctor determined by $Q$.

Let us now define a generalization for $\zeta$. We have an action of $U(N) \times S^1$ on $\text{Fr}_g$, where $U(N) \times S^1$ is the universal cover of $U(N) \times S^1$. Therefore the category $\text{ho sh}_\varepsilon(U(N) \times S^1 \times \mathbb{R}^N \times \mathbb{R}^N)$ acts on $\text{ho sh}_\varepsilon(\text{Fr}_g \times \mathbb{R}^N \times \text{Fr}_g \times \mathbb{R}^N)$ both from the right and the left. One then defines an $U(N) \times S^1$-equivariant version of $\zeta$, to be denoted by $\xi$, where $\xi$ is an algebra in $\text{ho sh}_\varepsilon(U(N) \times S^1 \times \mathbb{R}^N \times \mathbb{R}^N)$.

We now define the required monad $A_M \in \text{ho sh}_\varepsilon(\text{Fr}_g \times \mathbb{R}^N \times \text{Fr}_g \times \mathbb{R}^N)$ in the same way as above, namely,

$$A_M := A \ast \xi \ast B.$$

### 5.5.1 Reformulation in terms of a sequence of tensor functors

It is convenient to break this construction into a sequence of maps of monoidal categories.

1) Consider a monoidal category $\text{ho sh}_\varepsilon(\text{Fr}_g \times_M \text{Fr}_g \times \mathbb{R}^N \times \mathbb{R}^N)$ which is identified with a full subcategory of $\text{ho sh}_\varepsilon(\text{Fr}_g \times \text{Fr}_g \times \mathbb{R}^N \times \mathbb{R}^N)$ consisting of objects supported on $\text{Fr}_g \times_M \text{Fr}_g \times \mathbb{R}^N \times \mathbb{R}^N$.

This full subcategory is closed under the tensor product, whence an embedding tensor functor:

$$I : \text{ho sh}_\varepsilon(\text{Fr}_g \times_M \text{Fr}_g \times \mathbb{R}^N \times \mathbb{R}^N) \rightarrow \text{sh}_\varepsilon(\text{Fr}_g \times \text{Fr}_g \times \mathbb{R}^N \times \mathbb{R}^N).$$

Let also $JF := A \ast I(F) \ast B$, we have a tensor structure on $J : \text{ho sh}_\varepsilon(\text{Fr}_g \times_M \text{Fr}_g \times \mathbb{R}^N \times \mathbb{R}^N) \rightarrow \text{sh}_\varepsilon(\text{Fr}_g \times \text{Fr}_g \times \mathbb{R}^N \times \mathbb{R}^N)$.

Let now $\Phi$ be a pull-back of the following diagram

$$\text{Fr}_g \times_M \text{Fr}_g \rightarrow U(N) \times S^1 \leftarrow U(N) \times S^1.$$

We have a groupoid structure on $\Phi \Rightarrow \text{Fr}_g$ as well as a map of groupoids

$$\pi : (\Phi \Rightarrow \text{Fr}_g) \rightarrow (U(N) \times S^1 \Rightarrow \text{pt}). \quad (20)$$

The convolution product gives a monoidal structure to the category $\text{sh}_\varepsilon(\Phi)$.

Let us define a tensor functor

$$K : \text{sh}_\varepsilon(\Phi) \rightarrow \text{sh}_\varepsilon(\text{Fr}_g \times_M \text{Fr}_g \times \mathbb{R}^N \times \mathbb{R}^N)$$

as follows.

Let $q : \Phi \rightarrow \text{Fr}_g \times_M \text{Fr}_g$ be the covering map. We have maps

$$\begin{align*}
\Phi \times \mathbb{R}^N \times \mathbb{R}^N & \quad \xrightarrow{p_1} \quad \text{Fr}_g \times M \text{Fr}_g \times \mathbb{R}^N \times \mathbb{R}^N \\
\Phi \times \mathbb{R}^N \times \mathbb{R}^N & \quad \xrightarrow{p_2} \quad \text{Fr}_g \times M \text{Fr}_g \times \mathbb{R}^N \times \mathbb{R}^N \\
\text{Fr}_g \times M \text{Fr}_g \times \mathbb{R}^N \times \mathbb{R}^N & \quad \xrightarrow{q} \quad U(N) \times S^1 \times \mathbb{R}^N \times \mathbb{R}^N
\end{align*}$$
where $p_1$ is the projection onto $\Phi$, $p_2 = q \times \text{Id}_{R^N \times R^N}$, and $q : \Phi \rightarrow \overline{U(N) \times S^1}$ is induced by (20). Set $K(F) := p_2!(p_1^*F \otimes \pi^{-1}\xi)$.

### 5.5.2 The action of $\mathbb{Z} \times \mathbb{Z}$

For $a \in \mathbb{Z} \times \mathbb{Z}$, let $T_a^{\mathbb{Z} \times \mathbb{Z}} : \overline{U(N) \times S^1} \rightarrow \overline{U(N) \times S^1}$ be the action, where we consider $\mathbb{Z} \times \mathbb{Z} \subset \overline{U(N) \times S^1}$ as the kernel of the homomorphism $\overline{U(N) \times S^1} \rightarrow U(N) \times S^1$. Let $a = (a_1, a_2), a_1, a_2 \in \mathbb{Z}$. We have $T_a^{\mathbb{Z} \times \mathbb{Z}} \xi \cong T_{a_2}^\mathbb{R}\xi[2a_1]$, where, for every topological space $X$, $T_c^\mathbb{R}$ denotes the endofunctor on $\text{sh}_\xi(X)$ induced by the parallel transition $X \times \mathbb{R} \rightarrow X \times \mathbb{R}$, $(x, t) \mapsto (x, t + c)$. One naturally extends the $\mathbb{Z} \times \mathbb{Z}$-action onto $\Phi$—the action is by the deck transformations of the covering map $\Phi \rightarrow \mathbf{F}_g \times_M \mathbf{F}_g$.

It now follows that we have an isomorphism of functors

$$KT_a^{\mathbb{Z} \times \mathbb{Z}} \cong T_{a_2}^\mathbb{R}K[2a_1].$$

One can take this twisted equivariance into account as follows.

Let $\text{Class}ic^{\mathbb{Z} \times \mathbb{Z}}(\varepsilon)$ be the SMC of $\mathbb{Z} \times \mathbb{Z}$-graded objects in $\text{Class}ic(\varepsilon)$. For $X \in R_{\varepsilon}^{\mathbb{Z} \times \mathbb{Z}}$ and $c \in \mathbb{Z} \times \mathbb{Z}$, let $\text{gr}_{\varepsilon}^{\mathbb{Z} \times \mathbb{Z}}X$ be the $c$-th graded component of $X$.

Let us define the enrichment of $\text{sh}_\varepsilon(\Phi)$ over $\text{Class}ic^{\mathbb{Z} \times \mathbb{Z}}(\varepsilon)$, where we set

$$\text{gr}_{a}^{\mathbb{Z} \times \mathbb{Z}}\text{Hom}(F; G) := \text{Hom}(F, T_{a_2}T_a^{\mathbb{Z} \times \mathbb{Z}}G[2a_1]).$$

Denote the resulting category enriched over $\text{Class}ic^{\mathbb{Z} \times \mathbb{Z}}(\varepsilon)$ by $\text{sh}_{\varepsilon}^{\mathbb{Z} \times \mathbb{Z}}(\Phi)$.

We have a tensor functor $|| : \text{Class}ic^{\mathbb{Z} \times \mathbb{Z}}(\varepsilon) \rightarrow \text{Class}ic(\varepsilon)$;

$$|X| := \bigoplus_{a \in \mathbb{Z} \times \mathbb{Z}} \text{gr}_{a}^{\mathbb{Z} \times \mathbb{Z}}X.$$

Whence a category $|\text{sh}_{\varepsilon}^{\mathbb{Z} \times \mathbb{Z}}(\Phi)|$. We have a monoidal structure, enriched over $\text{ho}\text{Class}ic(\varepsilon)$, on $\text{ho}|\text{sh}_{\varepsilon}^{\mathbb{Z} \times \mathbb{Z}}(\Phi)|$.

The tensor functor $K$ now factors as the following sequence of tensor functors:

$$\text{ho}\text{sh}_\varepsilon(\Phi) \rightarrow \text{ho}|\text{sh}_{\varepsilon}^{\mathbb{Z} \times \mathbb{Z}}(\Phi)| \rightarrow \text{ho}\text{sh}_\varepsilon(\mathbf{F}_g \times_M \mathbf{F}_g \times \mathbb{R}^N \times \mathbb{R}^N).$$

The rightmost arrow admits the following reformulation. Let $\mathbb{Z}\mathbb{Z} := \mathbb{Z} \times \mathbb{Z} \cup \{\infty\}$ viewed as a partially ordered commutative monoid; where the elements of $\mathbb{Z} \times \mathbb{Z}$ are pairwise incomparable and $\infty$ is the greatest element. The addition on $\mathbb{Z} \times \mathbb{Z}$ is the group law and we set $\infty + \mathbb{Z}\mathbb{Z} = \infty$. This way, $\mathbb{Z}\mathbb{Z}$ becomes a SMC, where the category structure is determined by the partial order and the tensor product by the addition in the monoid.

Let $\text{Class}ic^{\mathbb{Z}\mathbb{Z}}(\varepsilon)$ be the SMC, enriched over $\text{Class}ic(\varepsilon)$, whose every object $X$ is a collection of objects $\text{gr}_{\varepsilon}^{\mathbb{Z}\mathbb{Z}}X \in \text{Class}ic(\varepsilon), c \in \mathbb{Z}\mathbb{Z}$. Set

$$\text{Hom}(X, Y) := \prod_{a, b \in \mathbb{Z}\mathbb{Z}|a \leq b} \text{Hom}(|\text{gr}_{a}^{\mathbb{Z}\mathbb{Z}}X; \text{gr}_{b}^{\mathbb{Z}\mathbb{Z}}Y|).$$

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The composition is well defined because for every $a \in \mathbb{Z}$ there are only finitely many elements $b$ such that $a \leq b$.

Next, we define the tensor product in $\text{Classic}^{\mathbb{Z}}(\varepsilon)$ by setting
\[
\text{gr}_{c}^{\mathbb{Z}}(X \otimes Y) := \bigoplus_{a,b|a+b=c} \text{gr}_{a}^{\mathbb{Z}}X \otimes \text{gr}_{b}^{\mathbb{Z}}Y.
\]

We have tensor functors $i : \text{Classic}^{\mathbb{Z} \times \mathbb{Z}}(\varepsilon) \to \text{Classic}^{\mathbb{Z}}(\varepsilon)$; $i_{\infty} : \text{Classic}(\varepsilon) \to \text{Classic}^{\infty}(\infty)$, where
\[
\text{gr}_{a}^{\mathbb{Z}}i(X) = \text{gr}_{a}^{\mathbb{Z} \times \mathbb{Z}}X, a \in \mathbb{Z} \times \mathbb{Z}, \quad \text{gr}_{\infty}^{\mathbb{Z}}i(X) = 0;
\]
\[
\text{gr}_{a}^{\mathbb{Z}}i_{\infty}(X) = 0, a \in \mathbb{Z} \times \mathbb{Z}, \quad \text{gr}_{\infty}^{\mathbb{Z}}(X) = X.
\]

We can now summarize our achievements in terms of a sequence of tensor functors of monoidal categories enriched over $\text{hoClassic}^{\mathbb{Z}}(\varepsilon)$:
\[
\text{hoi} \text{sh}_{c}^{\mathbb{Z} \times \mathbb{Z}}(\Phi) \to \text{hoi} \text{sh}_{c}(\text{Fr}_{\Phi} \times_{M} \text{Fr}_{\Phi} \times \mathbb{R}^{N} \times \mathbb{R}^{N}) \to \text{hoi} \text{sh}_{c}(\text{Fr}_{\Phi} \times \mathbb{R}^{N} \times \text{Fr}_{\Phi} \times \mathbb{R}^{N}). \quad (21)
\]

### 5.5.3 Building an associative algebra revisited

Instead of building an associative algebra in the rightmost monoidal category we can do it in the leftmost one. We have such a structure on the constant sheaf $A_{\Phi} := \mathbb{K}_{\Phi \times [0, \infty)}$. In fact we have a reacher structure on $A$: namely we have elements
\[
\tau_{a} : T_{a_{2}^{R}}\mathbb{K}[2a_{1}] \to \text{gr}_{a}^{\mathbb{Z}}\text{Hom}(A_{\Phi}, A_{\Phi})
\]

we have a commutative diagram
\[
T_{a_{2}^{R}}^{K}[2a_{1} + 2b_{1}] \longrightarrow \text{gr}_{a}^{\mathbb{Z}}\text{Hom}(A_{\Phi}, A_{\Phi}) \otimes \text{gr}_{b}^{\mathbb{Z}}\text{Hom}(A_{\Phi}, A_{\Phi}) \quad (22)
\]
\[
\text{gr}_{a+b}(A_{\Phi}; A_{\Phi}) \quad \text{gr}_{a+b}\text{Hom}(A_{\Phi} \otimes A_{\Phi}; A_{\Phi} \otimes A_{\Phi})
\]
\[
\text{gr}_{a+b}\text{Hom}(A_{\Phi} \otimes A_{\Phi}; A_{\Phi} \otimes A_{\Phi})
\]

### 5.6 Lifting to a dg-level

#### 5.6.1 As asymmetric operad as a lax version of monoidal category

As is usual when passing to dg structure, it is more convenient to use lax versions of various algebraic structures. Our lax version of a monoidal category will be that of a colored asymmetric operad $\mathcal{O}$ over a ground SMC $\mathcal{C}$. By such a structure we mean:

— a collection of objects $C_{\mathcal{O}}$;
— a collection of ‘operadic spaces’ \( \mathcal{O}(X_1, X_2, \ldots, X_n; X) \in \mathcal{C} \), \( X_1, X \in \mathcal{C} \), \( n \geq 0 \) endowed with the standard composition maps satisfying the associativity axioms.

Given a small monoidal category \( \mathcal{M} \) enriched over \( \mathcal{C} \), with its set of objects \( \text{Ob} \mathcal{M} \), we define an asymmetric operad \( \mathcal{O}_\mathcal{M} \) called the endomorphism operad of \( \mathcal{M} \), where \( \mathcal{O}_\mathcal{M}(X_1, \ldots, X_n; X) := \text{Hom}_\mathcal{M}(X_1 \otimes X_2 \otimes \cdots \otimes X_n, X) \). In the case \( n = 0 \), we set \( \mathcal{O}(X) := \text{Hom}_\mathcal{M}(\text{unit}_\mathcal{M}; X) \).

One has an inverse procedure of converting an asymmetric colored operad \( \mathcal{O} \) into a monoidal unital category \( \mathcal{M}^{\mathcal{O}} \). An object of \( \mathcal{M}^{\mathcal{O}} \) is an ordered collection \( (X_1, X_2, \ldots, X_n) \) of objects of \( \mathcal{C} \). Empty collection is also allowed — it is the unit of \( \mathcal{M}^{\mathcal{O}} \). We set

\[
(X_1, X_2, \ldots, X_n) \otimes (Y_1, Y_2, \ldots, Y_m) := (X_1, \ldots, X_n, Y_1, \ldots, Y_m).
\]

Let us now define

\[
\text{Hom}((X_1, X_2, \ldots, X_n); (Y_1, Y_2, \ldots, Y_m))
\]

Let \( S \subset \{1, 2, \ldots, n\} \) be a segment, that is \( S = [a, b] := \{x | a \leq x \leq b\} \) for some \( a, b \in S \), \( a \leq b \), or \( S = \emptyset \). Let \( i \in m \). Denote \( ([a, b]|i) := \text{Hom}(X_a, X_{a+1}, \ldots, X_b; Y_i) \), set \( (\emptyset|i) := \mathcal{O}(|Y_i) \).

Set

\[
\text{Hom}((X_1, X_2, \ldots, X_n); (Y_1, Y_2, \ldots, Y_m)) = \bigoplus_{f: \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}} (f^{-1}|i),
\]

if \( m > 0 \).

In the case \( m = 0 \), we set \( \text{Hom}((X_1, X_2, \ldots, X_n); \text{unit}) = 0 \) if \( n > 0 \) and \( \text{Hom}(\text{unit}|\text{unit}) = \text{unit}_\mathcal{C} \) if \( n = 0 \). Thus, \( \mathcal{M}^{\mathcal{O}} \) is the associated asymmetric PROP of \( \mathcal{O} \).

5.6.2 Further relaxing

We will need to further relax the definition of a colored asymmetric operad which can be done in the following standard way. First, the structure of an asymmetric operad is, in turn, can be formulated as that of an algebra over a certain operad (not asymmetric anymore!), denote it temporarily by \( T \). Next, one can choose a resolution \( T' \to T \) thereby producing the notion of a \( T' \)-algebra as a lax version of a \( T \)-algebra, i.e. an asymmetric operad.

Let us now discuss the operad \( T \). Of course, one can define it as a colored symmetric operad, but in fact one can do better by taking into account the specificity of the situation: the arguments of every meaningful operation in \( T \) correspond to vertices of a certain planar tree. This leads us to the definition of a tree operad \( \mathcal{T} \) as:

— a prescription of an object \( \mathcal{T}(t) \) for every isomorphism class of planar trees \( t \);
— Let \( t \) be a planar tree with its set of vertices \( V_t \). For each \( v \in V_t \) let \( E_v \) be the number of its inputs. Let also \( E_t \) be the number of inputs of \( t \). Given planar trees \( t_v \) for each \( v \in V_t \) such that \( E_v = E_{t_v} \), one can insert the \( t_v \)’s into \( t \) thus getting a new planar tree, to be denoted by \( t\{t_v\}_{v \in V_t} \). In each such an occasion one should have a composition map

\[
\mathcal{T}(t) \otimes \bigotimes_{v \in V} \mathcal{T}(t_v) \to \mathcal{T}(t\{t_v\}_{v \in V_t}).
\]
These composition maps must satisfy associativity and unitality axioms. Given a collection of objects $O(n) \in \mathcal{C}$, $n \geq 0$, one gets its endomorphism tree operad $E_0$, where

$$E_0(t) := \text{Hom}_{\mathcal{C}}(\bigotimes_{v \in V_t} O(E_v); O(E_t)).$$

Given a tree operad $\mathcal{T}$, a $\mathcal{T}$-algebra structure on $O$ is a map of tree operads $\mathcal{T} \to E_0$.

One develops a colored version of a tree operad, in which case we should first fix the set of colors $C$ and replace planar trees with $C$-colored planar trees (that is planar trees whose all edges, inputs, and the output, are $C$-colored). Given a tree operad $\mathcal{T}$ and a set of colors $C$, we define a $C$-colored tree operad $\mathcal{T}_C$ where $\mathcal{T}_C(t) = \mathcal{T}(\tilde{t})$, where $\tilde{t}$ is a planar tree with its coloring erased. One has a notion of an algebra over a colored tree operad. We call $\mathcal{T}_C$-algebras simply $\mathcal{T}$-algebras.

A $C$-colored asymmetric operad is then the same as an algebra over the trivial tree-operad $\text{triv}$ whose all operadic spaces are set to the unit of the ground category, and its every operadic composition sends the tensor product of a number of copies of the unit identically to the unit.

Define a contractible tree operad as a tree operad $\mathcal{T}$ endowed with a termwise homotopy equivalence $p_{\mathcal{T}} : \mathcal{T} \to \text{triv}$. One then defines a homotopy asymmetric operad as an algebra over a contractible $\mathcal{T}$.

If the ground category is $(\mathbb{D} \oplus)$-closed, then there is a straightening-out functor from the category of $\mathcal{T}$-algebras to that of $\text{triv}$-algebras so that a homotopy colored asymmetric operad can be converted to a colored asymmetric operad proper.

### 5.6.3 Homotopy asymmetric operad structures on categories of sheaves

Our goal is to lift monoidal categories as in [21] onto dg level. This can be done according the following scheme.

Let $X$ be a locally compact topological space and we have a monoidal structure on $\text{hosh}_c(X)$, to be denoted by $*$, enriched over $\text{Classic}_{\mathbb{ZZ}}(\varepsilon)$.

Given objects $\text{gr}^{\mathbb{ZZ}}_a K_n \in \text{sh}_c(X^n \times X)$, $n \geq 0$, $a \in \mathbb{ZZ}$, we have a $\text{sh}_c(X)$-colored collection, to be denoted by $\mathcal{B}^K$:

$$\text{gr}^{\mathbb{ZZ}}_a \mathcal{B}^K(F_1, F_2, \ldots, F_n; F) := \text{Hom}_{\text{sh}_c(X^n)}(F_1 \otimes F_2 \otimes \cdots \otimes F_n; \text{gr}^{\mathbb{ZZ}}_a K_n \circ_X F).$$  \hfill (23)

We assume that for $n \geq 0$ we have natural transformations in $\text{hosh}_c(X)$:

$$\text{Hom}_{\text{hosh}_c(X^n)}(F_1 \otimes F_2 \otimes \cdots \otimes F_n; \text{gr}^{\mathbb{ZZ}}_a K_n \circ_X F) \to \text{gr}^{\mathbb{ZZ}}_a \text{Hom}_{\text{hosh}_c(X)}(F_1 \ast F_2 \ast \cdots \ast F_n; F).$$  \hfill (24)

which are isomorphisms for all $n \geq 1$. As is seen from the examples, for $n = 0$, the situation is more delicate, the natural transformation is only an isomorphism for a certain full sub-category.

Let us consider for example the category $\text{sh}_c(F \times \mathbb{R}^N \times F \times \mathbb{R}^N)$. Denote $Y := F \times \mathbb{R}^N$ so that $X = Y \times Y$. Let us refer to a point of $X$ as $(y, z)$, $y, z \in Y$. Let $L_n \subset X^n \times X$ consist of all points

$$((y_1, z_1), (y_2, z_2), \ldots, (y_n, z_n), (y, z))$$

satisfying $z_1 = y_2$, $z_2 = y_3$, $\ldots$, $z_{n-1} = y_n$, $y_1 = y$, $z_n = z$ if $n > 0$. If $n = 0$, set $L_0 \subset Y$ to be the diagonal. Set $\text{gr}^{\mathbb{ZZ}}_a K_n := k_{L_n|[(n-1)N]}; \text{gr}^{\mathbb{ZZ}}_a K_n = 0, a \in \mathbb{Z} \times \mathbb{Z}$ for all $n$.

Let $i_n : Y^{n+1} \to (Y \times Y)^n$ be given by

$$i_n((y_0, y_1, \ldots, y_{n-1}, y_n)) = ((y_0, y_1), (y_1, y_2), \ldots, (y_n, y_{n+1})).$$
Let \( p : Y^{n+1} \to Y \times Y \) be given by
\[
i_n((y_0, y_1, \ldots, y_{n+1})) = (y_0, y_n).
\]

Let also \( q_n = p i_n \).

One now has the following maps in the homotopy category \((n > 0)\):
\[
\text{Hom}(F_1, F_2, \ldots, F_n, \mathbf{gr}_{\infty} \mathbf{Z} \mathbf{Z} K_n \circ_X F) \cong \text{Hom}(F_1 \boxtimes F_2 \boxtimes \cdots \boxtimes F_n; i_n! i_n^{-1} p^{-1} F[(n - 1)N])
\]
\[
\cong \text{Hom}(i_n^{-1}(F_1 \boxtimes \cdots \boxtimes F_n); q_i^{-1} F) \cong \text{Hom}(q_i^{-1}(F_1 \boxtimes \cdots \boxtimes F_n); F)
\]
\[
\cong \text{Hom}(F_1 \ast F_2 \ast \cdots \ast F_n; F).
\]

Consider the case \( n = 0 \). Let \( \Delta : Y \to Y \times Y \) be the diagonal map. We now have the following sequence of maps
\[
(\kappa_{\Delta(Y)} \circ F)[-N] \cong \Gamma_c(Y; \Delta^{-1} F[-N]) \to \Gamma(Y; \Delta^1 F) \cong \text{Hom}(\Delta! \Phi_Y; F) \cong \text{Hom}(\text{unit}, F)
\]
which is an isomorphism provided that \( F \) has a compact support and non-singular along the diagonal \( \Delta(Y) \).

Let us now define a tree operad \( \mathcal{E} \), defined over the ground category \( \mathfrak{A} \), which acts on a collection of objects \( \text{sh}_c(X) \) as follows. Given a planar tree \( t \), and elements \( a_v \in \mathbf{Z} \mathbf{Z} \) for each \( v \in V \), one associates the object \( K^{Z_0}_{E(v)} \) to every vertex \( v \in V \). Taking convolution with respect to every edge of \( t \), we get an object, to be denoted by \( K_t(\{a_v\}_{v \in V}) \in \text{sh}(X^{E_t} \times X) \). Set
\[
\mathcal{E}(t)(\{a_v\}_{v \in V}; a) := \text{Hom}(K_t(\{a_v\}_{v \in V}); K^{Z_0}_{E_t});
\]
\[
\mathcal{E}(t) := \prod_{a \geq \sum_v a_v} \mathcal{E}(t)(\{a_v\}_{v \in V}; a).
\]

The action of \( \mathcal{E} \) on \( \text{sh}_c(X) \) is straightforward. Suppose for simplicity that \( \mathfrak{A} \) is the category of complexes of \( \mathbb{Q} \)-vector spaces. One then can apply the functor \( \tau_{\leq 0} \) so as to get a tree operad \( \tau_{\leq 0} \mathcal{E}_K \) mapped into \( \mathcal{E}_K \), hence still acting on \( \mathcal{B} \). Suppose that
\[
\tau_{\leq 0} \mathcal{E}_K \text{ has only the cohomology in degree 0}.
\]

Therefore, we have a homotopy equivalence of tree operads \( \tau_{\leq 0} \mathcal{E}_K \to H^0(\mathcal{E}_K) \). On the other hand, the monoidal structure on \( \text{ho}(\text{sh}_c(X)) \) results in a map \( \text{triv} \to H^0(\mathcal{E}_K) \). Let \( C \) be a pull-back of the diagram \( \tau_{\leq 0} \mathcal{E}_K \to H^0(\mathcal{E}_K) \leftarrow \text{triv} \). As the left arrow is termwise surjective, the structure map \( C \to \text{triv} \) is a term-wise homotopy equivalence. On the other hand, \( C \) maps to \( \tau_{\leq 0} \mathcal{E}_K \), hence acts on \( C^K \), whence a homotopy colored asymmetric operad structure on \( \text{sh}_c(X) \). Call \( K \) a quasi-contractible collection of kernels, if the condition (27) is the case.

Let us denote \( C^F := \text{sh}_c(Fr_g \times \mathbb{R}^N \times Fr_g \times \mathbb{R}^N), C^\Psi := \text{sh}_c(Fr_g \times Fr_g \times \mathbb{R}^N \times \mathbb{R}^N) \), \( C^\Phi := \text{sh}^{\mathbb{Z} \times \mathbb{Z}}(\Phi) \).

These categories are exactly those used in Sec 5.5. The above outlined methods allow one to endow \( C^F, C^\Psi, C^\Phi \) with a homotopy colored asymmetric operad structure, as the corresponding collections of kernels happen to be quasi-contractible.
5.6.4 Homotopy maps of homotopy colored asymmetric operads

We follow the ideas from \cite{Fresse} to use various versions of operadic modules and bimodules. First of all, we switch to the language of Schur functors, appropriately modified. Fix a set of colors $C$ and a ground SMC $\mathcal{C}$. Let $\text{col}$ be the category, enriched over $\text{Sets}$, whose every object $O$ is a collection of objects $O(X_1, X_2, \ldots, X_n|X) \in \mathcal{C}$ for all $n \geq 0$ and all $X_1, X \in C$.

Let $t$ be a $C$-colored tree and $v$ a vertex of $T$. Let $X^v_1, X^v_2, \ldots, X^n_{E(v)}$ be the colors of the inputs of $v$, listed according to the order of the inputs, and let $X^v$ be the color of the output of $v$. Let $O \in \text{col}$. Set $O_v := O(X^v_1, X^v_2, \ldots, X^n_{E(v)})$. Set $O(t) := \bigotimes_{v \in V_t} O_v$.

Let also $\text{treecollec}$ be the category, enriched over $\text{Sets}$, whose every object $T$ is a collection of objects $T(t) \in \mathcal{C}$ for every isomorphism class of $C$-colored planar trees $t$.

Let us define a functor $S : \text{treecollec} \times \text{col} \rightarrow \text{col}$,

$$S_T O := \bigoplus_t T(t) \otimes O(t),$$

where the direct sum is taken over the set of all isomorphism classes of $C$-colored planar trees.

One has a monoidal structure on $\text{treecollec}$, denoted by $\circ$, such that we have a natural isomorphism

$$S_{T_1} S_{T_2} O \cong S_{T_1 \circ T_2} O.$$

One has

$$T_1 \circ T_2(t) = \bigoplus T_1(t_1) \otimes \bigotimes_{v \in V_{t_1}} T_2(t_v),$$

where the direct sum is taken over the set of all isomorphism classes of collections of $C$-colored planar trees $t_1, t_v$, where $t = t_1 \{t_v\}_{v \in V_{t_1}}$.

The structure of a tree operad on an object $T \in \text{treecollec}$ is equivalent to that of a unital monoid with respect to $\circ$. The structure of a $T$ algebra on an object $O \in \text{col}$ is equivalent to that of a left $T$-module with respect to the action $S$.

Let $T_1, T_2$ be $C$-colored tree operads, that is monoids in $\text{treecollec}$. Let $M \in \text{treecollec}$ be a $T_1 - T_2$-bi-module and let $O \in \text{col}$ be a left $T_2$-module. We have the following diagram

$$S_M S_{T_2} O_2 \Rightarrow S_M O_2 \quad (28)$$

where the arrows on the left are as follows. The upper arrow comes from the $T_2$-action on $M$:

$$S_M S_{T_2} O_2 \cong S_{M \circ T_2} O_2 \rightarrow S_M O_2;$$

the bottom arrow comes from the $T_2$-action on $O_2$:

$$S_M (S_{T_2} O_2) \rightarrow S_M O_2.$$

Suppose the diagram \eqref{28} admits a co-equalizer, to be denoted $S_M \circ_{T_2} O_2$. This co-equalizer then carries a left $T_1$-action.
One can show that if $M$ is good enough, namely if $M$ is a semi-free $T_2$-module, then $S_M \circ T_2 \mathcal{O}_2$ does exist for any $\mathcal{O}_2$.

Let $\mathcal{T}_1, \mathcal{T}_2$ be contractible tree operads. Define the notion of a contractible $\mathcal{T}_1 - \mathcal{T}_2$-bimodule as:

- a semi-free $\mathcal{T}_1 - \mathcal{T}_2$-bimodule $M$ and a map of triples 'a pair of tree operads and their bimodule':

$$(\mathcal{T}_1, \mathcal{M}, \mathcal{T}_2) \rightarrow (\text{triv}, \text{triv}, \text{triv}),$$

where the maps $\mathcal{T}_1, \mathcal{T}_2 \rightarrow \text{triv}$ are the ones coming from a contractible tree operad structure; we vies $\text{triv}$ as a bimodule over itself in the standard way, the map $\mathcal{M} \rightarrow \text{triv}$ is a term-wise homotopy equivalence.

Let $\mathcal{O}_k$ be a $\mathcal{T}_k$-algebra, $k = 1, 2$. Define a homotopy map $\mathcal{O}_2 \rightarrow \mathcal{O}_1$ as a prescription of a contractible $\mathcal{T}_1 - \mathcal{T}_2$-bimodule $\mathcal{M}$ and a map of $\mathcal{T}_1$-algebras:

$$f : \mathcal{M} \circ T_2 \mathcal{O}_2 \rightarrow \mathcal{O}_1.$$

As above, one has a straightening out procedure that converts such a generalized map into a pair $\mathcal{O}_1', \mathcal{O}_2'$ of strict colored asymmetric operads and their map $f' : \mathcal{O}_1' \rightarrow \mathcal{O}_2'$.

### 5.6.5 Producing contractible bimodules

Let $X^{(1)}, X^{(2)}$ be locally-compact topological spaces and suppose we have objects $\text{gr}^{ZZ} K_1^{(k)} \in \text{sh}_{\varepsilon}((X^{(k)})^n \times X^{(k)}), k = 1, 2, n \geq 0, a \in \mathbb{ZZ}$, similar to Sec. 5.6.3; in particular, we have monoidal structures on $\text{ho sh}_{\varepsilon}(X^{(k)}), k = 1, 2$ which agree with the kernels $\text{gr}^{ZZ} K_1^{(k)}$ in a way specified in (24).

We now have $X^{(k)}$-colored collections $B^{(k)}$ defined similar to (23). Next, we can define objects $K_t^{(k)}(a_v)_{v \in V_t} \in \text{sh}_{\varepsilon}((X^{(k)})^{E_t} \times X^{(k)}),$ the objects

$$\mathbb{E}^{(k)}(t)(\{a_v\}_{v \in V_t}; a) := \text{Hom}(K_t^{(k)}(a_v)_{v \in V_t}; \text{gr}^{ZZ} K^{(k)}_E),$$

and the tree operads $\mathbb{E}^{(k)}$ as in (25), (26). Suppose we also have a tensor functor $g : \text{ho sh}_{\varepsilon}(X^{(1)}) \rightarrow \text{ho sh}_{\varepsilon}(X^{(2)})$ which is induced by a kernel $H \in \text{sh}_{\varepsilon}(X^{(2)} \times X^{(1)})$ so that $g = ho h$, where $h : \text{sh}_{\varepsilon}(X^{(1)}) \rightarrow \text{sh}_{\varepsilon}(X^{(2)})$ is the convolution with $H$.

Set

$$M(t) := \text{Hom}_{\langle (X^{(2)})^{E_t} \times X \rangle}(H^{(2)}_{(X^{(1)})^{E_t}}K^{(1)}_t; K^{(2)}_E \circ X H).$$

One gets a $\mathbb{E}^{(2)} - \mathbb{E}^{(1)}$-bimodule structure on $M$. Let $N := \tau_{\leq 0} M$.

We have a map

$$S_N \circ O^{(1)} \rightarrow h^{-1} O^{(2)}$$

of $\mathbb{E}^{(2)}$-modules. If $\tau_{\leq 0} \mathbb{E}^{(k)}, k = 1, 2$, and $N$ only have cohomology in degree 0, we call a collection of kernels $(K^{(1)}, K^{(2)}, H)$ quasi-contractible.

Similar to Sec. 5.6.3 we now have a diagram of triples “a pair of operads and their bimodule”

$$(\tau_{\leq 0} \mathbb{E}^{(1)}, N, \tau_{\leq 0} \mathbb{E}^{(2)}) \rightarrow (H^{(0)} \mathbb{E}^{(1)}, H^{(0)} N, H^{(0)} \mathbb{E}^{(2)}) \leftarrow (\text{triv}, \text{triv}, \text{triv})$$

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whose pull-back, to be denoted \((C_1, N_{12}, C_2)\), is contractible. Let us choose a resolution \(N'_{12} \to N_{12}\) of a \(C_1 - C_2\) bimodule \(N_{12}\). The map in \((29)\) gives rise to a map
\[
\Sigma_{N'_{12}} \circ C_2 \to O^{(1)}.
\]
We get this way a homotopy map of homotopy colored asymmetric operads \(O^{(1)} \to O^{(2)}\).

The outlined construction applies to the monoidal categories in \((21)\) and their maps. We will get a sequence of homotopy colored asymmetric operads and their homotopy maps over \(\text{Classic}^{ZZ}(\varepsilon)\):
\[
iC^{\Phi} \to i_{\infty}O^{\psi} \to i_{\infty}O^{F}
\]
The straightening out procedure produces a corresponding sequence of strict colored asymmetric operads and their maps
\[
O^{\Phi} \to i_{\infty}O^{\psi} \to i_{\infty}O^{F}.
\]

5.7 Associative algebra structure

Let us now lift the structure from Sec \(22\) to the dg-level. Let \(a \in \text{Classic}^{ZZ}(\varepsilon)\) be given by
\[
\text{gr}^{ZZ} a = T_{a_2} \text{unit}[2a_1], \quad a \in \mathbb{Z} \times \mathbb{Z}, \quad \text{gr}^{ZZ}_{\infty} a = 0
\]
We have a commutative algebra structure on \(a\). Let \(\text{assoc}\) be an asymmetric operad with one color in \(\text{Classic}^{ZZ}(\varepsilon)\), where \(\text{assoc}(n) = \text{unit}\). Let, finally, \(O_A^{\Phi} := a \otimes \text{assoc}\), which is an asymmetric operad with one color in \(\text{Classic}^{ZZ}(\varepsilon)\).

Using the above method, one can get a homotopy map \(O_A^{\Phi} \to C^{\Phi}\) lifting the maps from Sec \(22\). The straightening out gives rise to a diagram of colored asymmetric operads and their maps
\[
O_A^{\Phi} \sim\sim\sim O_A^{\Phi} \to O^{\Phi}.
\]
Passing to the associated monoidal categories allows one to produce maps Let us pass to the associated PROPS, we then get a monoidal category \(\mathcal{M}^{\Phi}\), an object \(X\) in it, and a map from \(\mathcal{O}^A\) to the full operad of \(X\). Let us pass to \((D \boxplus)\mathcal{M}^{\Phi}_{\varepsilon}\). The straightening out procedure now produces an object \(X' \in (D \boxplus)\mathcal{M}^{\Phi}_{\varepsilon}\) homotopy equivalent to \(X\) and a map from \(\mathcal{O}^{A}_{\varepsilon}\) to the full operad of \(X'\). All in all, we get a sequence of monoidal categories and their maps
\[
\mathcal{M}^A \to \mathcal{M}^{\Phi} \to i_{\infty}\mathcal{M}^{\Psi} \to i_{\infty}\mathcal{M}^{F},
\]
where we denote \(\mathcal{M}^{F} := \mathcal{M}^{O^{F}}, \mathcal{M}^{\Psi} := \mathcal{M}^{O^{\Psi}}, \) etc.

We have constructed a \(\mathbb{Z} \times \mathbb{Z}\)-equivariant algebra in the monoidal category \(\mathcal{M}^{F}_{\varepsilon}\). In order to be able to quantize it we need to enrich a structure on it (via introduction of a trace) and to prove some properties of this structure. The rest of the Introduction will be devoted to reviewing the related results.
5.8 Traces

In this section we will enrich the structure on $\mathcal{M}^F$ and on the algebra by introducing a trace on both of them.

Let us define a (contravariant) trace on a monoidal category $\mathcal{M}$ over a ground category $\mathcal{C}$ as:
— a functor $\text{Tr} : \mathcal{M}^{\text{op}} \to \mathcal{C}$;
— natural isomorphisms $\sigma_{X,Y} : \text{Tr}(X \otimes Y) \xrightarrow{\sim} \text{Tr}(Y \otimes X)$.

These isomorphisms should satisfy:
— $\sigma_{X,Y} \sigma_{Y,X} = \text{Id}$;
— the following diagram should commute:

$$
\begin{array}{ccc}
\text{Tr}(X \otimes \text{unit}) & \xrightarrow{\sigma_{X,\text{unit}}} & \text{Tr}(\text{unit} \otimes X) \\
& \downarrow \quad \downarrow & \downarrow \\
& \text{Tr}(X) & \text{Tr}(X) \\
\end{array}
$$

where the diagonal arrows are induced by the structure maps in $\mathcal{M}$: $X \otimes \text{unit} \to X$; $\text{unit} \otimes X \to X$.

— the following diagram should commute:

$$
\begin{array}{ccc}
\text{Tr}(X \otimes Y \otimes Z) & \xrightarrow{\sigma_{X,Y \otimes Z}} & \text{Tr}(Y \otimes Z \otimes X) \\
& \downarrow \quad \downarrow & \downarrow \\
& \text{Tr}(Z \otimes X \otimes Y) & \text{Tr}(Z \otimes X \otimes Y) \\
\end{array}
$$

Let $(\mathcal{M}, \text{Tr}_\mathcal{M})$, $(\mathcal{N}, \text{Tr}_\mathcal{N})$ be monoidal categories with a trace. Define a functor $F : (\mathcal{M}, \text{Tr}_\mathcal{M}) \to (\mathcal{N}, \text{Tr}_\mathcal{N})$ of monoidal categories with trace as:

— a lax tensor functor $F : \mathcal{M} \to \mathcal{N}$ endowed with a natural transformation of functors $s : \text{Tr}_\mathcal{M} \to \text{Tr}_\mathcal{N} \circ F$ fitting into the following commutative diagram

$$
\begin{array}{ccc}
\text{Tr}_\mathcal{M}(X \otimes Y) & \xrightarrow{\sigma_{X,Y}} & \text{Tr}_\mathcal{M}(Y \otimes X) \\
& \downarrow \quad \downarrow & \downarrow \\
\text{Tr}_\mathcal{N}(F(X \otimes Y)) & \text{Tr}_\mathcal{N}(F(Y \otimes X)) \\
& \downarrow \quad \downarrow & \downarrow \\
\text{Tr}_\mathcal{N}(F(X) \otimes F(Y)) & \xrightarrow{\sigma_{F(X),F(Y)}} & \text{Tr}_\mathcal{N}(F(Y) \otimes F(X)) \\
\end{array}
$$

where the arrows $(\ast)$ are induced by the natural maps $F(X) \otimes F(Y) \to F(X \otimes Y)$ and $F(Y) \otimes F(X) \to F(Y \otimes X)$.
Let now $A$ be an associative algebra in $\mathcal{M}$. A trace on $A$ is an element $\text{tr}_A : \text{unit} \to \text{Tr}(A)$ such that the following diagram commutes

\[
\begin{array}{ccc}
\text{unit} & \xrightarrow{\text{tr}_A} & \text{Tr}(A) \\
\downarrow & & \downarrow \\
\text{Tr}(A) & \xrightarrow{\text{Tr}(m_A)} & \text{Tr}(A \otimes A)
\end{array}
\]

Define the trace on the category $\text{hoSh}_\mathbb{Z}(Y \times Y)$, where $Y = \mathbf{Fr}_g \times \mathbb{R}^N$, by setting $\text{Tr}(F) := \text{Hom}(F; \text{unit})$. One endows every category in (21) with a trace in a similar way, all the tensor functors in (21) can be defined as functors of monoidal categories with a trace.

5.9 Circular operads

In order to transfer the traces onto the dg level, we resort to operads. As was discussed above, every monoidal category gives rise to a colored asymmetric operad. In a similar fashion, every monoidal category with trace enriched over an SMC $\mathcal{C}$ produces a colored circular operad in $\mathcal{C}$ which, by definition, is

— a colored asymmetric operad $\mathcal{O}^{\text{noncyc}}$ in $\mathcal{C}$ with its set of colors $C$
— objects $\mathcal{O}^{\text{cyc}}(c_0, c_1, \ldots, c_n)$ for all $n \geq 0$ and all $c_k \in C$.
— let $f : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}$ be a cyclically monotone map. For each $k \in \{0, 1, \ldots, m\}$, $f^{-1}k$ is a sub-interval of $\{0, 1, \ldots, n\}$, including an empty set. Let $d_1, d_2, \ldots, d_n \in C$. For every sub-interval $I \subset \{0, 1, \ldots, n\}$, $I = [a, b]$, let $d_I$ be a sub-sequence $d_{i_0}, d_{i_{a+1}}, \ldots, d_{b}$. We then should have an insertion map

\[
\mathcal{O}^{\text{cyc}}(c_0, c_1, \ldots, c_m) \otimes_{k=0}^{m} \mathcal{O}^{\text{noncyc}}(d_{f^{-1}k} | c_k) \to \mathcal{O}(d_0, d_1, \ldots, d_n).
\]

The insertion maps should obey the obvious associativity and unitality axioms.

Every small monoidal category $\mathcal{M}$ with a trace enriched over $\mathcal{C}$ defines such a circular operad $\mathcal{O}$ whose set of colors is the set of objects of $\mathcal{M}$, where, as above,

\[
\mathcal{O}^{\text{noncyc}}(X_1, X_2, \ldots, X_n | X) := \text{Hom}_\mathcal{M}(X_1 \otimes X_2 \otimes \cdots \otimes X_n; X).
\]

Next, we set

\[
\mathcal{O}^{\text{cyc}}(X_0, X_1, \ldots, X_n) := \text{Tr}(X_0 \otimes X_1 \otimes \cdots \otimes X_n).
\]

The above constructions carry over onto the setting of circular operads straightforwardly so that we get a sequence of circular operads and their maps similar to (30). The straightening out procedure leads to a sequence of monoidal categories with a trace and their functors similar to (31).
5.10 Hochschild Complexes

Let $A$ be an associative algebra a monoidal category with trace $\mathcal{M}$ enriched over a SMC $\mathcal{C}$. One associates to $A$ two standard objects. One of them is a co-simplicial object $\text{Hoch}^\bullet(A)$, where $\text{Hoch}^0(A) := \text{Hom}(A^\otimes n; A)$. The totalization of this object is the standard Hochschild cochain complex of $A$. The trace on $\mathcal{M}$ allows one to define a co-cyclic object $\text{Hochcyc}^\bullet(A)$, where $\text{Hochcyc}^n(A) := \text{Tr}(A^\otimes n+1)$. Let us denote by $\text{Hoch}(A)$ and $\text{Hochcyc}(A)$ the totalizations.

Recall that we have constructed an associative algebra with trace in $\mathcal{M}^\Phi$, to be denoted by $A_\Phi$ (see (31)). Denote the images of $A_\Phi$ by $A_\Psi \in \mathcal{M}^\Psi$ and $A_F \in \mathcal{M}^F$. The tensor functors in (31) define maps of Hochschild complexes

$$\text{Hoch}(A_\Phi) \to \text{Hoch}(A_\Psi) \to \text{Hoch}(A_F); \quad \text{Hochcyc}(A_\Phi) \to \text{Hochcyc}(A_\Psi) \to \text{Hochcyc}(A_F).$$

One shows that each of these maps is a homotopy equivalence.

5.11 $Y(\mathcal{B})$-modules

Let us now take the $\mathbb{Z} \times \mathbb{Z}$-equivariant structure on our algebras into account. We will use the notion of an $\mathcal{O}$-bimodule, where $\mathcal{O}$ is an arbitrary circular operad over the set of colors $C$. It is convenient to define a set $Y(C)^{\text{noncyc}}$ whose every element is either a collection of the form $(c_1, c_2, \ldots, c_n|c)$, $n \geq 0$, $c_i, c \in C$, as well as a set $Y(C)^{\text{cyc}}$, whose every object is $(c_0, c_1, \ldots, c_n)$, $n \geq 0$, $c_i \in C$. Let finally set $Y(C) := Y(C)^{\text{noncyc}} \sqcup Y(C)^{\text{cyc}}$. $Y(C)$ can be interpreted as the set of ‘arities’.

Suppose we have a family of objects $\mathcal{M}(a) \in \mathcal{C}$ for all $a \in Y(C)$. Define an $\mathcal{O}$-bimodule structure on $\mathcal{M}$ as the structure of a circular operad on the direct sum $\mathcal{O} \oplus \mathcal{M}$ such that: — the natural inclusion/projection maps $\mathcal{O} \to \mathcal{O} \oplus \mathcal{M} \to \mathcal{O}$ are circular operad maps; — every composition map restricted onto $\mathcal{M}(a_1) \otimes \mathcal{M}(a_2)$, $a_1, a_2 \in Y(C)$, vanishes.

One can define a category $Y(\mathcal{O})$ over $\mathcal{C}$ whose set of objects is $Y(C)$ with the property that an $\mathcal{O}$-bimodule structure on $\mathcal{M}$ is equivalent to that of a functor $\mathcal{M} : Y(\mathcal{O}) \to \mathcal{C}$.

Given a map of circular operads $\mathcal{O} \to \mathcal{O}_1$, $\mathcal{O}_1$ has a natural structure of a $\mathcal{O}$-bimodule on $\mathcal{O}_1$, that is $\mathcal{O}_1$ is a functor $\mathcal{O}_1 : Y(\mathcal{O}) \to \mathcal{C}$.

In our case, we have $\mathbb{Z} \times \mathbb{Z}$-equivariant algebras $A_\Phi \in \mathcal{M}^\Phi$, $A_\Psi \in \mathcal{M}^\Psi$, $A_F \in \mathcal{M}^F$. Let $\mathcal{O}^{A_b}$ be a full circular operad of $A_\Phi$, and similar for $\mathcal{O}^{A_\Psi}$ and $\mathcal{O}^{A_F}$. We now have maps of circular operads

$$\mathcal{B} \to \mathcal{O}^{A_b} \to \mathcal{O}^{A_\Psi} \to \mathcal{O}^{A_F},$$

whence an induced sequence of $Y(\mathcal{B})$-modules $\mathcal{O}^{A_b} \to \mathcal{O}^{A_\Psi} \to \mathcal{O}^{A_F}$ in $\text{Classic}^{\mathbb{Z} \times \mathbb{Z}}(\mathcal{C})$. The monoidal functor

$$|| : \text{Classic}^{\mathbb{Z} \times \mathbb{Z}}(\mathcal{C}) \to \text{Classic}(\mathcal{C})$$

gives rise to an induced sequence of $Y(\mathcal{B}(\mathcal{C}))$-modules in $\text{Classic}(\mathcal{C})$:

$$|\mathcal{O}^{A_b}| \to |\mathcal{O}^{A_\Psi}| \to |\mathcal{O}^{A_F}|. \quad (32)$$

We will now review some properties of these modules.
5.11.1 Condensation

We have a map of circular operads \( \text{assoc} \to \mathcal{B} \) which gives a functor \( Y(\text{assoc}) \to Y(\mathcal{B}) \). Given a functor \( \mathcal{M} : Y(\mathcal{B}) \to \mathcal{C} \), we therefore have an induced structure \( \mathcal{M} \) on \( \mathcal{C} \). The latter structure give rise to cosimplicial structures on \( \mathcal{M}^{\text{noncyc}} \) and \( \mathcal{M}^{\text{cyc}} \). Call \( \mathcal{M} : Y(\mathcal{B}) \to \mathcal{C} \) quasi-constant if every arrow \( f \in \Delta \) acts on both \( \mathcal{M}^{\text{noncyc}} \) and \( \mathcal{M}^{\text{cyc}} \) by quasi-isomorphisms.

Call \( N : Y(\mathcal{B}) \to \mathcal{C} \) right orthogonal to quasi-constant objects if \( R\text{Hom}(\mathcal{M}; N) \sim 0 \) for every quasi-constant object \( \mathcal{M} \).

One can define an endofunctor \( \text{con} \) on the category of functors \( Y(\mathcal{B}) \to \mathcal{C} \) as well as a natural transformation \( \text{con} \to \text{Id} \) such that \( \text{con}(\mathcal{M}) \) is quasi-constant for every \( \mathcal{M} : Y(\mathcal{B}) \to \mathcal{C} \) and the cone of the induced map \( \text{con}(\mathcal{M}) \to \mathcal{M} \) is right orthogonal to quasi-constant objects. In a homotopical sense, \( \text{con}(\mathcal{M}) \) is therefore the universal quasi-constant object mapping into \( \mathcal{M} \).

**Claim 5.3** Apply \( \text{con} \) to (32):

\[
\text{con}|O^A| \to \text{con}|O^A| \to \text{con}|O^F|.
\]

All the arrows in this sequence are termwise equivalences.

This claims allows one to prove the properties of \( \text{con}|O^F| \) via passing to \( \text{con}|O^A| \).

5.11.2 The \( c_1 \)-action

In this section our basic category is that of complexes of \( \mathbb{Q} \)-vector spaces.

It is well known that every cyclic object \( X \) has a natural endomorphism \( c_1 : X \to X[2] \), ‘the first Chern class’. One has a similar property for every functor \( Y(\mathcal{B}) \to \mathcal{C} \). In particular, we have a \( c_1 \)-action on the object

\[
H := R\text{Hom}(\text{con}|O^A|, \text{con}|O^A|) \in \text{Classic}\langle \varepsilon \rangle.
\]

we show that such an action on every non-zero graded component \( \text{gr}^n H, n \neq 0 \) is homotopy nilpotent. That is there exists a number \( N(n) \geq 0 \) such that the map \( c_1^{N(n)} : \text{gr}^n H \to \text{gr}^n H[2N(n)] \) is homotopy equivalent to 0.

Finally, one studies an object

\[
T := R\text{Hom}_{Y(\mathcal{B})}(|\mathcal{B}|; \text{con}|O^A|) \sim R\text{Hom}_{Y(\mathcal{B})}(|\mathcal{B}|; |O^A|)
\]

which also carry a \( c_1 \)-action. We prove that there is a homotopy equivalence of objects with \( c_1 \)-action

\[
G[u] \to T,
\]

where \( u \) is a variable of degree +2, and \( c_1 \) acts on the LHS by the multiplication by \( u \).

5.12 Dependence on a pseudo-Kaehler metric

The above outlined approach depends on the choice of a pseudo-Kaehler metric \( g \). In order to make the construction invariant under symplectomorphisms of \( M \), one considers the set of all pseudo-Kaehler
metrics on \( M \) and carries over all the steps in this settings. One cannot use the category \( \text{Classic}(\varepsilon) \) anymore, as one cannot choose a \( \varepsilon \) uniformly for all the metrics. One however can build ground

categories \( R_q, R_0 \) as a replacement.

6 Action of \( \text{Sp}(2N) \)

Let \( G \) be the universal cover of \( \text{Sp}(2N) \). Let \( V = \mathbb{R}^{2N} \) be the standard symplectic vector space with the coordinates \((q, p)\) and let \( E = \mathbb{R}^N \) so that \( V = T^*E \). The group \( \text{Sp}(2N) \), hence \( G \), acts on \( V \).

6.1 Graph of the \( G \)-action on \( T^*E \)

Let \( a : T^*E \to T^*E \) be the antipode map \((q, p) \mapsto (q, -p)\). Let \( \Gamma \subset G \times V \times V \) consist of all points of the form \((g, v, gv)\) such that there exists a unique Legendrian sub-manifold \( L \subset T^*(G \times E \times E) \times \mathbb{R} \) which:

— diffeomorphically projects unto \( G \) under the projection \( T^*(G \times E \times E) \times \mathbb{R} \to G \times T^*(E \times E) \).
— contains all the points of the form \((e, v, v^a, 0)\), where \( e \) is the unit of \( G \) and \( v \in V \).

Let \( C \) be the full sub-category of \( \text{sh}_q(G \times E \times E) \) consisting of all objects \( F \) with \( \mu S (F) \subset \mathcal{L} \) such that there exists a homotopy equivalence in \( \text{sh}_q(E \times E) \):

\[
F|_{e \times E \times E} \sim \mathcal{K}_{\Delta_E \times [0, \infty]},
\]

where \( \Delta_E \subset E \times E \) is the diagonal.

Let \( \mathcal{A}_{\text{unit}} \subset \mathcal{A} \) be the full sub-category of all objects isomorphic to \( \text{unit}_{\mathcal{A}} \). We have a functor

\[
\Psi : \mathcal{C} \to \mathcal{A}_{\text{unit}},
\]

where

\[
\Psi(F) := \text{Hom}(\mathcal{K}_{\Delta_E \times [0, \infty]}; F|_{e \times E \times E}).
\]

Theorem 6.1 The functor \( \Psi \) is an equivalence of categories.

Sketch of the proof. Part 1: Let us construct at least one object \( \mathcal{S} \) of \( \mathcal{C} \)

1) For an open subset \( U \subset G \), let \( \mathcal{L}_U \subset T^*(U \times E \times E) \times \mathbb{R} \) be the restriction of \( \mathcal{L} \). Let \( \mathcal{C}_U \) be the full sub-category of \( \text{sh}_q(U \times E \times E) \) consisting of all objects \( F \) such that \( \mu S (F) \subset \mathcal{L}_U \) and there exists a homotopy equivalence \( F|_{e \times E \times E} \sim \mathcal{K}_{\Delta_E \times [0, \infty]} \).

2) Let \( \mathcal{U} \) be a small enough geodesically convex neighborhood of unit in \( \text{Sp}(2N) \) satisfying: for each \( g \in \mathcal{U} \) we have: \((q, p')\) is a non-degenerate system of coordinates, where \((q', p') = g(q, p)\). \( \mathcal{U} \) lifts uniquely to \( G \), to be denoted by the same letter.

3) Let \( F, F' \in \text{sh}_q(E \times E) \) be the Fourier kernels:

\[
F := \mathcal{K}_{\{\langle q_1, q_2, t \rangle | t + \langle q_1, q_2 \rangle \geq 0 \}}; \quad F' := \mathcal{K}_{\{\langle q_1, q_2, t \rangle | t - \langle q_1, q_2 \rangle \geq 0 \}} [2N].
\]
Let \( R : T^*E \times \mathbb{R} \to T^*E \times \mathbb{R} \) be the 'Fourier' contactomorphism given by
\[
R(q, p, t) = (-p, q, t + \langle p, q \rangle).
\]

Let
\[
R_1 : T^*E \times T^*E \times \mathbb{R} \to T^*E \times T^*E \times \mathbb{R},
\]
be defined by \( R_1(u_1, u_2, t) = (u_1, R^{-1}(u_2, t)) \). Let \( C'_U \subset \text{sh}_q(U \times E \times E) \) consist of all objects \( F \) such that

- there exists a homotopy equivalence \( F|_{e \times E \times E} \sim F' \).

- \( \mu S F \subset R_1(L_U) \).

It follows that the functor \( G \mapsto G \ast_E F \) induces a homotopy equivalence of categories \( C'_U \to C_U \).

4) The Legendrian manifold \( R_1L_U \subset T^*(G \times E \times E) \times \mathbb{R} \) projects uniquely onto the base \( G \times E \times E \), therefore, \( R_1L_U \) is of the form \( L_f \) for some smooth function \( f \) on \( G \times E \times E \).

Let \( B \subset \text{sh}_q(U \times E \times E) \) be the full sub-category of objects \( F \) satisfying:

- \( \mu S (F) \subset (T^*_U \times E \times E) \times 0 \);
- there exists a homotopy equivalence \( F|_{e \times E \times E} \sim \langle E \times E \rangle_{[0, \infty]} \).

It follows that \( B \) is the category consisting of all objects homotopy equivalent to \( \langle E \times E \rangle_{[0, \infty]} \). The convolution with \( \langle e, e, t, t + f(e) \rangle_{[0, \infty]} \) gives a homotopy equivalence of categories \( B \to C'_U \).

Fix an object \( S_U \in C_U \) along with a homotopy equivalence
\[
S_U|_{0 \times E \times E} \sim \langle E \times E \rangle_{[0, \infty]}.
\]

5) For \( h \in U \), set \( S_h := S_U|_{h \times E \times E} \). Every \( g = G \) can be written as \( g = g_1g_2 \cdots g_n \), where \( g_i, g_i^{-1} \in U \).

Set \( S_{g_1, \ldots, g_n} = S_{g_1} \ast_E S_{g_2} \ast_E \cdots \ast_E S_{g_n} \).

For each \( g \), choose an object \( S_{g,t} \) which is homotopy equivalent to one of \( S_{g_1, \ldots, g_n} \ast_E S_U \) for \( g_1 \cdots g_n = g \). Observe that the objects \( S_{g_1, \ldots, g_n} \) and \( S_{g_1', \ldots, g_m'} \), where \( g_1 \cdots g_n = g_1' \cdots g_m' = g \) are homotopy equivalent. It suffices to show that
\[
S_{g_1, \ldots, g_m, (g_n)^{-1}, \ldots, (g_1')^{-1}} \sim \langle E \rangle_{[0, \infty]}
\]
that is \( S_{g_1g_2 \cdots g_n} = \langle E \rangle_{[0, \infty]} \) whenever \( g_1g_2 \cdots g_n = e \). As \( U \) is geodesically closed, there is a unique shortest geodesic line joining \( g_1 \cdots g_k \) and \( g_1 \cdots g_{k+1} \). We will thus get a broken geodesic line starting and terminating at \( e \). As \( G \) is simply connected, this line can be contracted to a point. By possibly adding intermediate points, one can reduce the problem to the case when there exist smooth paths \( h_k : [0, 1] \to U \) such that \( h_1(t) \cdots h_n(t) = e, h_k(1) = e, h_k(0) = g_k \) for all \( k \). Let \( S_k \in \text{sh}(I \times E \times E) \), \( S_k := h_k^{-1}S_U \). Consider
\[
\Sigma := S_1 \ast_E S_2 \ast_E \cdots \ast_E S_n \in \text{sh}_q(I \times E \times E)|_{\Delta \times E \times E},
\]
where \( \Delta \subset I \) is the diagonal.

It follows that
\[
\Sigma_{1 \times E \times E} \sim \langle E \rangle_{[0, \infty]}; \quad \Sigma_{0 \times E \times E} \sim S_{g_1, g_2, \ldots, g_n}.
\]
Next, the singular support estimate shows that $\Sigma$ is locally constant along $\Delta_I$, which implies the statement.

6) Choose a covering $G = \bigcup_n g_nU$. Let $I$ be the poset consisting of all non-empty intersections $g_iU \cap \cdots g_kU$. Each element of $I$ is geodesically convex. It follows that all the restrictions $S_{g_iU|g_iU\cap \cdots g_kU}$ are homotopy equivalent. Indeed, choose a point $h \in g_iU \cap \cdots g_kU$; 4) implies that there is a homotopy equivalence of restrictions $S_{g_iU|h \times E \times E}$ with $S_h$. The statement now follows from 4).

For every $A \in I$, $A = g_{i_1}U \cap g_{i_2}U \cap \cdots g_{i_k}U$, choose an object $S_A \in C_A$ to be homotopy equivalent to each of the restrictions $S_{g_iU|A \times E \times E}$.

7) For each $V \in I$ let $j_V : V \to G$ be the embedding. Let $T_V := j_V ! S_V$.

8) Whenever $A \subset B$, $A, B \in I$, we have a homotopy equivalence $K \sim \text{Hom}(T_A, T_B)$. Let $r_{AB} : T_A \to T_B$ be the image of 1 $\in K$.

We have $r_{BC}r_{AB}$ is homotopy equivalent to $E_{ABC}r_{AC}$ for some $E_{ABC} \in K^\times$.

9) $E_{ABC}$ is a 2-cocycle on $I$. Since $H^2(G, K^\times) = 0$, $E_{ABC}$ is exact. Therefore, wlog we can assume that $E_{ABC} = 1$.

10) Denote $J(A, B) := \tau_{\leq 0} \text{Hom}(T_A, T_B)$, see Sec 2.2 for the definition of $\tau_{\leq 0}$.

Finally, we set $S := S_G := \mathbb{Z} \otimes_{J_{\mathrm{op}}} S$.

Part 2. Uniqueness The convolution with $S$ gives a pair of quasi-inverse maps between $C_G$ and the full subcategory of objects $S \in \text{sh}_q(G \times E \times E)$ with $\mu S \subset T_G^*G \times T_E^*(E \times E) \times \{0\}$, where there exists an isomorphism

$$S|_{E \times E} \sim \mathbb{K}_{\Delta_E \times [0, \infty)}.$$

Passing to $\tau_{\leq 0}$ yields the statement.

6.1.1 The object $S$

Fix an object $S \in C$ endowed with a homotopy equivalence

$$S|_{E \times E} \to \mathbb{K}_{\Delta_E \times [0, \infty)}.$$

7 Objects supported on a symplectic ball

7.1 Projector onto the ball

Let $i_0 : \mathbb{R}/2\pi \mathbb{Z} \to \text{Sp}(2N)$ be a one-parametric subgroup consisting of all transformations

$$q' = q \cos(2a) + p \sin(2a);$$

$$p' = -q \sin(2a) + p \cos(2a).$$
Let \( i : \mathbb{R} \hookrightarrow G \) be the lifting. Denote \( \mathbb{1} := i(\mathbb{R}) \). Let \( \mathcal{T} \in \text{sh}_q(\mathbb{1} \times E \times E) \) be the restriction of \( S \). The object \( \mathcal{T} \) is microsupported within the set

\[
\Sigma = \Sigma_0 \cup \{ (a, -(q^2+p^2), q, -p, q', p', -S(q, p, a)) | (q, p) \in V; a \in \mathbb{R}, \sin(2a) \neq 0 \} \subset T^*\mathbb{1} \times T^*E \times T^*E \times \mathbb{R}
\]

where

\[
\Sigma_0 = \{(\pi n, -(q^2+p^2), q, -p, q, p, 0) | (q, p) \in V, n \in \mathbb{Z} \} \cup \{(\pi(1/2+n); -(q^2+p^2), q, -p, -q, -p, 0) | (q, p) \in V, n \in \mathbb{Z} \};
\]

\[
S(q, p, a) = \frac{\cos(2a)(q^2 + (q')^2) + 2qq'}{2\sin(2a)}.
\]

Let \( B = \mathbb{R} \) with the coordinate \( b \). Let \( p_B : B \times E \times E \to E \times E \) be the projection. Set

\[
\mathcal{P}_R := p_B \mathcal{T} \ast_1 [k_{((a,b,t) \in B \times E | q < c \leq 2} \mathcal{P}_R[2N] \to \mathcal{P}_R.
\]

We have

\[
\mathcal{P}_R \sim p_B \mathcal{T} \ast_1 [k_{((a,b,t) | a \leq 0, t+aR^2 \geq 0} \mathcal{P}_R[2N] \to \mathcal{P}_R.
\]

where \( p_B : \mathbb{1} \times E \times E \to E \times E \) is the projection.

### 7.1.1 The map \( \alpha : T_{-\pi R^2} \mathcal{P}_R[2N] \to \mathcal{P}_R \)

We have a homotopy equivalence

\[
T^a_{-\pi} \mathcal{T}[-2N] \sim \mathcal{T},
\]

where \( T^a_{-\pi} \) is the translation along \( \mathbb{1} \) by \( -\pi \) units.

Thus, we have a map

\[
\mathcal{P}_R \sim p_B ! (T^a_{-\pi} \mathcal{T}) \ast_1 [k_{((a,b,t) | a \leq 0, t+aR^2 \geq 0} [-2N]) \sim p_B ! (\mathcal{T} \ast_1 [k_{((a_1,a_2,t) | a_2 \leq 0, a_1 = a_2 + \pi, t+aR^2 \geq 0} [-2N]) \sim
\]

\[
\sim p_B ! (\mathcal{T} \ast_1 [k_{((a_1,a_2,t) | a_1 \leq \pi, a_1 = a_2, t \geq R^2 - a_1 R^2} [-2N]) \sim
\]

\[
\sim T_{-\pi R^2} p_B ! (\mathcal{T} \ast_1 [k_{((a_1,a_2,t) | a_1 \leq \pi, a_1 = a_2, t+aR^2 \geq 0} [-2N]) \sim
\]

\[
\sim T_{-\pi R^2} p_B ! (\mathcal{T} \ast_1 [k_{((a_1,a_2,t) | a_1 \leq 0, a_1 = a_2, t+aR^2 \geq 0} [-2N]) \sim T_{-\pi R^2} \mathcal{P}_R[2N] \sim T_{-\pi R^2} \mathcal{P}_R[2N].
\]

This map can be rewritten as

\[
\alpha : T_{-\pi R^2} \mathcal{P}_R[2N] \to \mathcal{P}_R.
\]

### 7.1.2 \( \text{Hom}(T_c\mathcal{P}_R; \mathcal{P}_R) \)

Let \( (\nu - 1) \pi R^2 < c \leq \nu \pi R^2 \), where \( \nu \in \mathbb{Z} \). Let \( G_c := \text{Hom}(T_c \mathcal{P}_R; \mathcal{P}_R) \). Then

\[
G_c \sim \mathbb{Z}[-2N\nu] \text{ if } \nu \leq 0, \quad G_c = 0 \text{ if } \nu > 0.
\]

The natural map \( G_{\nu \pi R^2} \to G_c \) is a homotopy equivalence. The generator of \( G_{\nu \pi R^2} \), \( \nu < 0 \) is given by \( \alpha^{\nu} \).

The map \( \mathcal{P}_R \to k_{[\Delta E, 0]} \) induces a homotopy equivalence

\[
\text{Hom}(T_c \mathcal{P}_R; \mathcal{P}_R) \to \text{Hom}(T_c \mathcal{P}_R; k_{\Delta E \times [0, \infty)}).
\]

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7.1.3 \( P_R \) is a projector

We have a natural map
\[
\pr : P_R \to \mathcal{K}_{\Delta E \times [0, \infty)}.
\] (34)

Let \( \mathcal{C}_R \subset \text{sh}_q(E) \) be the full subcategory of objects supported away from \( \hat{B}_R \times \mathbb{R} \subset T^*E \times \mathbb{R} \). Let \( \text{sh}_q[\hat{B}_R] \subset \text{sh}_q(E) \) be the left orthogonal complement to \( \mathcal{C}_R \). We have \( P_R *_E F \in \text{sh}_q[\hat{B}_R] \); \( \text{Cone } P_R *_E F \to F \in \mathcal{C}_R \) so that \( P_R \) gives a semi-orthogonal decomposition.

7.1.4 Generalization

Denote by \( \text{sh}_{1/2}[T^*X \times \hat{B}_R \times \mathbb{R}] \subset \text{sh}_{1/2}(X \times E) \) be the left orthogonal complement to the full category of objects supported away from \( T^*X \times \hat{B}_R \times \mathbb{R} \). The convolution with \( P_R \) gives a semi-orthogonal decomposition.

7.1.5 The object \( \gamma = \text{Cone } \alpha \)

Let \( \gamma := \text{Cone } \alpha \). We have
\[
\gamma \sim T *_{1/2} \mathcal{K}_{\{(a_1, a_2) | a_1 = a_2; -\pi R^2 < a_1 \leq 0, t + aR^2 \geq 0\}}
\]
We have a homotopy equivalence
\[
E_c := \text{Hom}(T_c \gamma; P_R) \sim \text{Hom}(T_c \gamma; \mathcal{K}_{\Delta E \times [0, \infty)}) \sim \text{Hom}(\text{Cone}(T_c - \pi R^2 \gamma [2N] \to T_c \gamma); \mathcal{K}_{\Delta E \times [0, \infty)}) \\
\sim \text{Cone}(\text{Hom}(T_c \gamma; \mathcal{K}_{\Delta E \times [0, \infty)}) \to \text{Hom}(T_c - \pi R^2 \gamma; \mathcal{K}_{\Delta E \times [0, \infty)}[\{-2N\}][-1]) \\
\sim \text{Cone}(G_c \to G_{c - \pi R^2}[-2N][-1]),
\]
where the map is incuced by the multiplication by \( \alpha \).

Therefore,
\[
-E_c = \mathbb{K}[-2N - 1], \quad 0 < c \leq \pi R^2; \\
-E_c = 0 \text{ otherwise}.
\]

7.1.6 Singular support of \( \gamma \)

We have
\[
\mu S T *_{1/3} \mathcal{K}_{\{(a_1, a_2) | a_1 = a_2, -\pi R^2 < a_1 \leq 0, t + aR^2 \geq 0\}} \subset \{(a, R^2 + k, q, -p, q', p', t - aR^2) \in \Sigma | -\pi < a < 0 \} \cup S,
\]
where \( \Sigma \) is as in (33) and
\[
S = \{(-\pi, R^2 + k, q, -p, q, p, -\pi R^2)|k \leq -p^2 - q^2\} \cup \{(0, R^2 + k, q, -p, q, p, 0)|k \leq -p^2 - q^2\}.
\]
Therefore, we have
\[ \mu S \gamma \subset \{(q, -p, q', p', -aR^2 - S(a, q, q'))|p^2 + q^2 = R^2; -\pi < a < 0\} \cup \{(q, -p, q, -\pi R^2)|q^2 + p^2 \leq R^2\} \]
\[ \cup \{(q, -p, q, p, 0)|q^2 + p^2 \geq R^2\}. \]
It follows that \( 0 \leq -aR^2 - S(a, q, q') \leq \pi R^2 \) if \(-\pi < a < 0\).

7.1.7 Singular support of \( \mathcal{P} \)

Similarly, one can find
\[ \mu S \mathcal{P} \subset \{(q, -p, q', p', -aR^2 - S(a, q, q'))|p^2 + q^2 = R^2; a < 0\} \cup \{(q, -p, q, p, 0)|q^2 + p^2 \leq R^2\}. \]

7.1.8 Singular support of \( \text{Cone} \mathcal{P} \to \mathbb{K}_{\Delta E \times [0, \infty)} \)

We have
\[ \text{Cone}(\mathcal{P} \to \mathbb{K}_{\Delta E \times [0, \infty)}) \approx p_! \star \mathbb{K}_{\{(a_1, a_2)|a_1 = a_2, a_1 \leq 0, t + aR^2 \geq 0\}} \]
so that
\[ \mu S \mathcal{T} \star \mathbb{K}_{\{(a_1, a_2)|a_1 = a_2, a_1 \leq 0, t + aR^2 \geq 0\}} \subset \{(a, R^2 + k, q, -p, q', p', t - aR^2) \in \Sigma|a < 0\} \cup S', \]
where \( \Sigma \) is as in (33) and
\[ S' = \{(0, R^2 + k, q, -p, q, p, 0)|k \geq -p^2 - q^2\}. \]

Therefore,
\[ \mu S \text{Cone}(\mathcal{P} \to \mathbb{K}_{\Delta E \times [0, \infty)}) \subset \{(q, -p, q', p', -aR^2 - S(a, q, q'))|p^2 + q^2 = R^2; a < 0\} \cup \{(q, -p, q, p, 0)|q^2 + p^2 \geq R^2\}. \]

7.1.9 Corollaries

**Corollary 7.1** We have
\[ R_{\leq c}\text{Cone}(\mathcal{P} \to \mathbb{K}_{\Delta E \times [0, \infty)}) \approx 0; \]
\[ R_{\leq c}\text{Cone}(\mathcal{P} \otimes \mathcal{P} \to \mathbb{K}_{\Delta E \times \Delta E \times [0, \infty)}) \approx 0. \]
for all \( c \leq 0 \).

**Corollary 7.2** Let \( F \in \text{sh}(E \times E) \). Then the natural maps
\[ \text{Hom}(\mathbb{K}_{\Delta E \times \Delta E}; F) \xrightarrow{\sim} \text{Hom}(\mathcal{P} \otimes \mathcal{P}; F \otimes \mathbb{K}_{[0, \infty)}); \]
\[ \text{Hom}(\mathbb{K}_{\Delta E \times \Delta E}; F) \xrightarrow{\sim} \text{Hom}(T_{2\pi R^2} \gamma \otimes \gamma [-4N]; F \otimes \mathbb{K}_{[0, \infty)}) \]
are homotopy equivalences.
7.1.10 Convolution of $\gamma$ with itself

We have a homotopy equivalence

$$\gamma \star \gamma \sim \gamma \oplus T_{-\pi R^2}\gamma[2N].$$

Denote by $\mu : \gamma \star \gamma \to \gamma$ the projection.

We now have the following homotopy equivalence

$$\text{Hom}(T_c\gamma; \mathbb{K}_{\Delta E \times [0,\infty)}) \xrightarrow{\mu} \text{Hom}(T_c\gamma \star \gamma; \mathbb{K}_{\Delta E \times [0,\infty]}),$$

for all $c$ except those in $(\pi R^2, 2\pi R^2]$.

In particular, for $0 < c \leq \pi R^2$, we have:

$$\text{Hom}(T_c\gamma \star \gamma; \mathbb{K}_{\Delta E \times [0,\infty)}) \sim \mathbb{K}[-2N - 1];$$

For $c \leq 0$, the above expression is homotopy equivalent to 0.

Let $\Lambda \in \text{sh}_q(\text{pt}); \Lambda = \text{Cone}(\mathbb{K}_{[-\pi R^2, \infty}) \to \mathbb{K}_{[0,\infty]}).$

We have a chain of homotopy equivalences

$$\text{Hom}(\gamma; \mathbb{K}_{\Delta E \boxtimes \Lambda}) \xrightarrow{\mu} \text{Hom}(\gamma \star \gamma; \mathbb{K}_{\Delta E \boxtimes \Lambda}) \sim \mathbb{K}[-2N].$$

In particular, we have a homotopy equivalence

$$\text{Hom}(\gamma, \mathbb{K}_{\Delta E}[2N]) \sim \mathbb{K}.$$

Let

$$\nu : \gamma \to \Lambda \boxtimes \mathbb{K}_{\Delta E}[2N]$$

be the generator.

One also has a map $\varepsilon : \Lambda \boxtimes \mathbb{K}_{\Delta E} \to \gamma$ which has a homotopy unit property with respect to $\mu$, the through map

$$\gamma \sim \mathbb{K}_{\Delta E} \star \gamma \to \Lambda \boxtimes \mathbb{K}_{\Delta E} \star \gamma \to \gamma \star \gamma \to \gamma$$

is homotopy equivalent to the Identity.

The induced map

$$\text{Hom}(\gamma, \mathbb{K}_{\Delta E}[2N]) \xrightarrow{\varepsilon} \text{Hom}(\gamma, \gamma)$$

is a homotopy equivalence. The map $\nu$ on the LHS corresponds to $\text{Id}$ on the RHS.

7.1.11 Lemma on $\nu \boxtimes \nu$

Consider the following maps

$$\gamma \boxtimes \gamma \xrightarrow{\nu \boxtimes \nu} \Lambda \boxtimes \mathbb{K}_{\Delta E} \boxtimes \Lambda \boxtimes \mathbb{K}_{\Delta E}[4N] \to \Lambda \boxtimes \mathbb{K}_{\Delta E \times \Delta E}[4N];$$

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\[ \gamma \boxtimes \gamma \xrightarrow{\mu} p_{14}^{-1} \gamma \boxtimes p_{23}^{-1} K_{\Delta_E} \xrightarrow{\mu} \Lambda \boxtimes p_{14}^{-1} K_{\Delta_E} \boxtimes p_{23}^{-1} K_{\Delta_E} [3N] \to K_{\Delta_E \times \Delta_E} [4N]. \]  

(38)

Here the maps \( \overline{\mu} \) is obtained from \( \mu \) by conjugation. The last arrow is the generator of \( \text{Hom}(p_{23}^{-1} K_{\Delta_E} \otimes p_{14}^{-1} K_{\Delta_E}; K_{\Delta_E \times \Delta_E} [N]). \)

**Lemma 7.3** The maps \( \text{(37)} \) and \( \text{(38)} \) are homotopy equivalent.

**Sketch of the proof** One reformulates the statement as follows:

By the conjugacy, the map \( \nu \) corresponds to a homotopy equivalence

\[ \xi : \Lambda \to \gamma *_{E^2} K_{\Delta_E \times [0, \infty]}[n] \]

The problem reduces to showing that the map

\[ \Lambda \to \Lambda \approx (\gamma \boxtimes \gamma) *_{E^4} K_{\Delta \times \Delta}[2n] \to (\gamma \boxtimes \gamma) *_{E^4} K_{(v_1, v_2, v_3, v_4) \in E^4 | v_1 = v_4, v_2 = v_3}[n] \approx (\gamma *_{E^2} \gamma) *_{E^2} K_{\Delta}[n] \to \gamma *_{E^2} K_{\Delta}[n] \]

(39)

is homotopy equivalent to

\[ \Lambda \otimes \Lambda \to \Lambda \to \gamma *_{E^2} K_{\Delta}[n]. \]

(40)

We have a homotopy equivalence,

\[ \gamma *_{E^2} K_{\Delta}[n] \cong \text{Hom}(K_{\Delta}; \gamma). \]

The map \( \xi \) rewrites as \( \xi' : \Lambda \to \text{Hom}(K_{\Delta}; \gamma) \) which produces a map \( e : \Lambda \otimes K_{\Delta} \to \gamma. \)

The map \( \text{(39)} \) rewrites as

\[ \Lambda \otimes \Lambda \to \text{Hom}(K_{\Delta}; \gamma) \otimes \text{Hom}(K_{\Delta}; \gamma) \to \text{Hom}(K_{\Delta}; \gamma *_{E} \gamma) \to \text{Hom}(K_{\Delta}; \gamma). \]

The map \( \text{(40)} \) rewrites as

\[ \Lambda \otimes \Lambda \to \text{Hom}(K_{\Delta}; \gamma). \]

Homotopy equivalence of the two maps follows from the following maps being homotopy equivalent:

\[ \Lambda \otimes K_{\Delta} *_{E} \Lambda \otimes K_{\Delta} \xrightarrow{\text{ext}} \gamma *_{E} \gamma \to \gamma \]

and

\[ \Lambda \otimes \Lambda \to \Lambda \to \gamma. \]

The latter statement follows from Sec. 7.1.10.
7.1.12 \( \gamma \) as an object of \( \text{sh}_{\pi R^2}(E \times E) \)

We assume that \( \pi R^2 = 1/2^n \) for some \( n \in \mathbb{Z}_{\geq 0} \). It follows that \( \gamma \) is supported within the set \( E \times E \times [-\pi R^2; 0] \). Therefore, \( \gamma \) determines an object of \( \text{sh}_{\pi R^2}(E \times E) \), to be denoted by \( \Gamma \).

Using the bar-resolution for \( \Gamma \ast_E \Gamma \), we see that it is glued of \( \gamma \ast_E \Lambda^n \ast_E \gamma \). We therefore have the following homotopy equivalences (all the hom’s are in \( \text{sh}_{\pi R^2}(E \times E) \)):

\[
\text{Hom}(\Gamma; \mathbb{K}_{\Delta_E}) \xrightarrow{\xi} \text{Hom}(\Gamma \ast_E \Gamma; \mathbb{K}_{\Delta_E}) \sim \mathbb{K}[-2N].
\]

7.2 Study of the category \( \text{sh}_q(F \times E \times E)[T^*F \times \text{int}B_R \times \text{int}B_R \times \mathbb{R}] \)

7.2.1 The category \( \mathbb{B}_I \)

Let \( I \subset \mathbb{R} \) be an open subset. Denote by \( \mathbb{B}_I \) the full sub-category of \( \text{sh}_q(F \times E \times E)[T^*F \times \text{int}B_R \times \text{int}B_R \times \mathbb{R}] \) consisting of all objects \( X \), where

\[
\mu S(X) \cap T^*F \times \text{int}B_R \times \text{int}B_R \times I = \emptyset.
\]

7.2.2 Study of \( \mathbb{B}_{(a, \infty)} \)

Let \( F \in \mathbb{B}_{(a, \infty)} \).

We have a natural map

\[
F \ast (P_R \boxtimes P_R) \to (R_{\leq a}F) \ast (P_R \boxtimes P_R),
\]

where \( R_{\leq a} \) is as in Sec 4.1.3

**Lemma 7.4** The above map is a homotopy equivalence.

**Sketch of the proof** Equivalently, we are to show

\[
(R_{>a}F) \ast (P_R \boxtimes P_R) \sim 0.
\]

We have

\[
\text{hocolim}_{\leq a} R_{>c}F \xrightarrow{\sim} R_{>a}F,
\]

therefore, it suffices to show that

\[
(R_{>c}F) \ast (P_R \boxtimes P_R) \sim 0.
\] (41)

Let us study \( \mu S R_{>c}F \). Denote \( \Phi := F \times E \times E \). The functor \( R_{>c} \) descends onto \( D_{>0}(F \times E \times E \times \mathbb{R}) \), where it is isomorphic to the functor

\[
F \mapsto \text{Cone} (i_{\leq c} F|_{\Phi_{\times [c, \infty)}} \to F|_{\Phi_{\times c} \boxtimes \mathbb{K}_{[c, \infty)}})[-1]
\]
where

$$i_c : \Phi \times [c, \infty) \to \Phi \times \mathbb{R}$$

is the embedding. $F$ is therefore homotopy equivalent (as an object of $\text{sh}(\Phi \times \mathbb{R})$) to:

$$\text{Cone}(j_c^! F|_{\Phi \times (c, \infty)} \to F|_{\Phi \times c} \boxtimes K_{(c, \infty)})[-1],$$

where

$$j_c : \Phi \times (c, \infty) \to \Phi \times \mathbb{R}$$

is the embedding.

It follows that

$$\text{SS}(F|_{\Phi \times c} \boxtimes K_{(c, \infty)}) \cap T^*_>(\Phi \times \mathbb{R}) = \emptyset.$$  

The object on the left hand side is isomorphic in $D(\Phi \times \mathbb{R})$ to $F \otimes K_{t>c}$. Let us use the SS estimate from KS.

We have $\text{SS}(F)$ contains no points of the form $(f, \eta, v_1, \zeta_1, v_2, \zeta_2, t, k)$, where $k > 0$ and $|(v_i, \zeta_i/k)| < R$, $i = 1, 2$. Next, $\mu S (K_{t>c}) = \{(f,0, v_1, 0, v_2, 0, t, k)|k \leq 0, t \geq c, t > c \Rightarrow k = 0\}$. Therefore, $\Phi \boxtimes K_{t>c} \in D(\Phi \times \mathbb{R} \times \mathbb{R})$ is non-singular along the diagonal $\Phi \times \Delta \mathbb{R}$. Hence, $\text{SS}(F \otimes K_{t>c})$ is obtained via fiberwise adding of $\mu S (F)$ and $\mu S (K_{t>c})$. The resulting sum contains no points of the form $(f, \eta, v_1, \zeta_1, v_2, \zeta_2, t, k)$, where $k > 0$ and $|(v_i, \zeta_i/k)| < R$, $i = 1, 2$, which implies (41).

### 7.2.3 Study of $B_{(-\infty,a)}$

Let $\tau_{\geq a}, \tau_{< a} : \text{sh}(\Phi \times \mathbb{R}) \to \text{sh}(\Phi \times \mathbb{R})$ be given by

$$\tau_{\geq a} F = i_a! F|_{\Phi \times [a, \infty)}; \quad \tau_{< a} F = j_a! F|_{\Phi \times (-\infty, a)}.$$

**Lemma 7.5** Let $F \in B_{(-\infty,a)}$. Then $\tau_{< a} F \sim 0$.

**Sketch of the proof** It suffices to show that $R_{\leq c} F \sim 0$ for all $c < a$. Similar to the previous Lemma, we deduce that $R_{\leq c} F$ is non-singular on the set

$$T^* F \times \text{int}B_R \times \text{int}B_R \times \mathbb{R}.$$  

Next, we have homotopy equivalences

$$R_{\leq c}(F \ast (P_R \boxtimes P_R)) \sim R_{\leq c}(R_{\leq c} F \ast (P_R \boxtimes P_R)) \sim 0.$$  

This proves the statement.

### 7.2.4 Study of $B_{R\setminus a}$

Let $b_R \subset E$ be the open ball of radius $R$ centered at 0. We have functors

$$\alpha : \text{sh}(F \times b_R \times b_R) \to B_{R \setminus a},$$

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where
\[ \alpha(S) = (S \boxtimes K_{(a,\infty)}) \star (P_R \boxtimes P_R); \]
\[ \beta : \mathbb{B}_{R \setminus a} \to \text{sh}(F \times b_R \times b_R), \]
where
\[ \beta(T) = T|_{t=a}. \]

**Proposition 7.6** The functors \( \alpha, \beta \) establish homotopy inverse homotopy equivalences of categories.

**Sketch of the proof** Let \( S \in \mathbb{B}_{R \setminus a} \). According to the two previous subsections we have homotopy equivalences:
\[ S \sim R_{\leq a} \star (P_R \boxtimes P_R) \sim (\tau_{\geq a} R_{\leq a} S) \star (P_R \boxtimes P_R) \sim ((S \bullet K_{t \geq a}) \boxtimes K_{(a,\infty)}) \star (P_R \boxtimes P_R), \]
which implies the statement.

7.2.5 \( \mu_{(\alpha(F))} \)

**Proposition 7.7** Let \( C \) be a closed conic subset of \( T^* F \times T^* b_R \times T^* b_R \). \( F \in \mathbb{B}_{R \setminus a} \) and suppose \( \mu_{(\alpha(F))} \cap T^* F \times B_R \times B_R \times a \subset C \). Then \( \mu_{(\alpha(F) \boxtimes K_{(a,\infty)})} \subset C \times a \).

7.2.6 The category \( \mathbb{B}_{R \setminus a, \Delta} \)

Let \( \alpha : B_R \to B_R \) be the antipode map, \( \alpha(q,p) = (q,-p) \). Let
\[ \Delta^\alpha = \{(\alpha(v),v)|v \in \text{int} B_R\} \subset \text{int} B_R \times \text{int} B_R. \]

Let \( \mathbb{B}_{R \setminus a, \Delta} \subset \mathbb{B}_{R \setminus a} \) be the full sub-category of objects \( X \) where
\[ \mu_{(\alpha(F))} \cap T^* F \times \text{int} B_R \times \text{int} B_R \times \mathbb{R} \subset T^* F \times T^* \Delta^\alpha (\text{int} B_R \times \text{int} B_R) \times a. \]

Let \( A_F \subset \text{sh}(F \times b_R \times b_R) \) be the full sub-category of objects \( T \) where
\[ \mu_{(\alpha(F) \boxtimes K_{(a,\infty)})} \subset T^* F \times \Delta^\alpha (F \times b_R \times b_R) \times a. \]

According to the previous subsection, we have a homotopy equivalence
\[ \beta : A_F \to \mathbb{B}_{R \setminus a, \Delta}. \]

Furthermore, let \( \text{Loc}(F) \subset \text{sh}(F) \) be the full sub-category of objects \( T \) where
\[ \mu_{(\alpha(F) \boxtimes K_{(a,\infty)})} \subset T^* F \times a. \]

Let \( \gamma : \text{Loc}(F) \to A_F \) be given by
\[ \gamma(S) = S \boxtimes K_{\Delta^\alpha b_R}. \]

**Lemma 7.8** \( \gamma \) is a homotopy equivalence of categories.

Therefore,

**Proposition 7.9** the functor \( \zeta := \beta \gamma : \text{Loc}(F) \to \mathbb{B}_{R \setminus a, \Delta} \) is a homotopy equivalence of categories.
8 Families of symplectic embeddings of a ball into $T^*E$

Let
\[ \alpha := \sum p_i dq_i \]
be the Liouville form on $B_R$. Let $\theta = dt + \alpha$ be the contact form on $B_R \times \mathbb{R}$. Let $F$ be a smooth family and let
\[ I : F \times B_R \to T^*E \times \mathbb{R} \]
be a smooth map such that, for all $f \in F$, the restriction
\[ I_f := I|_{f \times B_R} : f \times B_R \to T^*E \times \mathbb{R} \]
satisfies
\[ I_f^* \theta = \alpha. \]
Let $\Gamma_I \subset F \times \text{int} B_R \times T^*E \times \mathbb{R}$ consist of all points of the form
\[ (f, (q, -p)), I(f, q, p)). \]
There is a unique Legendrian manifold $\mathcal{L} \subset T^*F \times \text{int} B_R \times T^*E \times \mathbb{R}$ which projects uniquely onto $\Gamma$ under the projection
\[ T^*F \times \text{int} B_R \times T^*E \times \mathbb{R} \to F \times \text{int} B_R \times T^*E \times \mathbb{R}. \]
Let $f \in F$. Let
\[ J_f : f \times B_R \to T^*E \times \mathbb{R} \to T^*E \]
be the through map, which is a symplectomorphic embedding.
Let $DI(f) \in \text{Sp}(2N)$ be the differential of $J_f$ in $0 \in B_R$. This way we get a map $DI : F \to \text{Sp}(2N)$.
Suppose we have a lifting $\overline{DI} : F \to \overline{\text{Sp}}(2N)$ of $DI$. Call such a collection of data $\mathcal{I} := (I, \overline{DI})$ a graded family of symplectic embeddings of a ball into $T^*E$.

8.1 The category $C_I$

Let $\text{sh}_q(F \times \mathbb{R}^n \times \mathbb{R}^n)|T^*F \times \text{int} B_R \times T^*\mathbb{R}^n \times \mathbb{R}|$ be the full sub-category of $\text{sh}_q(F \times \mathbb{R}^n \times \mathbb{R}^n)$ consisting of all objects $F$ which are left orthogonal to all objects non-singular on $T^*F \times \text{int} B_R \times T^*\mathbb{R}^n \times \mathbb{R}$, same as in Sec 7.1.4.
Below we will study the full sub-category
\[ C_I \subset \text{sh}_q(F \times \mathbb{R}^n \times \mathbb{R}^n)|T^*F \times \text{int} B_R \times T^*\mathbb{R}^n \times \mathbb{R}| \]
consisting of all objects $T$ satisfying $\mu S (T) \cap T^*F \times \text{int} B_R \times T^*\mathbb{R}^n \times \mathbb{R} \subset \mathcal{L}$.  

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8.2 Main Theorem

Let $A_F$ be a category as in (12).

**Theorem 8.1** We have a homotopy equivalence between the categories $C_T$ and $A_F$.

The proof of this theorem occupies the rest of the subsection.

1) Extend $I$ to $F \times [-1, 1] \times B_R$ as follows. For $t \in [-1, 1] \setminus 0$, set

$$J(f, t, x) = \frac{I(f, tx) - I(f, 0)}{t} + I(f, 0).$$

This map extends uniquely to a smooth map $J : F \times [-1, 1] \times B_R \to T^*\mathbb{R}^n$. The grading of $I$ extends uniquely to a grading $\mathcal{J}$ of $J$.

Let $K = J|_{F \times 0}$. It follows that $K$ is a family of linear symplectomorphisms of $T^*\mathbb{R}^n$ restricted to $B_R$. The grading $\mathcal{J}$ determines uniquely a map

$$\mu : F \to \mathbf{Sp}(2N) \times \mathbb{R}. \tag{43}$$

2) For every $(f, t) \in F \times B_R$ we have a Hamiltonian vector field on $B_R$, namely $\frac{dJ(f, t)}{dt}$. Let $H_{(f, t)}$ be a smooth function on $B_R$ corresponding to this vector field and satisfying $H_{(f, t)}(0) = 0$. It follows that $H : F \times I \times B_R \to \mathbb{R}$ is a smooth function. It extends to a smooth function on $F \times I \times T^*E$ whose support projects properly onto $F \times I$.

3) Let $\chi : \mathbb{R} \to [-1, 1]$ be a non-decreasing smooth function such that $\chi(t) = -1$ for all $t \leq -1$, $\chi(t) = 1$ for all $t \geq 1$, and $\chi(0) = 0$. Let $K(f, t) = J(f, \chi(t))$ and $h(f, t, v) = H(f, \chi(t), v)\chi'(t)$ so that $h(f, t, -)$ is the Hamiltonian function of the vector field $\frac{dK(f, t)}{dt}$. It follows that there exists a unique family of symplectomorphisms $M : F \times \mathbb{R} \times E \to E$ such that

a) $M|_{F \times 0}$ is the family of linear symplectomorphisms coinciding with $J|_{F \times 0} = K|_{F \times 0}$;

b) $\frac{dM(f, t)}{dt}$ is the Hamiltonian vector field of $h(f, t, -)$.

It also follows that $M|_{F \times I \times B_R} = K$.

4) The family $M$ defines a Legendrian submanifold $\mathcal{L}_M \subset T^*(F \times E \times E \times \mathbb{R})$ such that $\mathcal{L}_M \cap T^*F \times B_R \times T^*E \times \mathbb{R} = \mathcal{L}_K$.

5) According to the theorem of Guillermou-Kaschiwara-Schapira, there exists a quantization of $\mathcal{L}_M$: an object $Q \in \mathbf{sh}_q(F \times E \times E)$ such that $\mu S \circ Q \subset \mathcal{L}_M$ and $Q|_{t=0} = \mu^{-1}S$, where $\mu$ is as in (43).

6) Similarly, one defines a quantization $Q'$ of the family $M^{-1}$ of inverse symplectomorphisms.

7) Let $\Delta : F \times E \times E \to F \times I \times F \times I \times E \times E$ be the following embedding

$$\Delta(f, v_1, v_2) = (f, 1, f, 1, v_1, v_2).$$

We have endofunctors

$$S \mapsto S \ast_{F \times E} \Delta_!Q; \quad S \mapsto S \ast_{F \times E} \Delta_!Q'.$$
of \( \text{sh}_q(F \times E \times E) \) which descends to homotopy inverse homotopy equivalences between \( \mathcal{C}_O \) and \( \mathcal{C}_I \), where \( O : F \times B_R \to B_R \xrightarrow{\iota} E \) is the constant family, where \( \iota \) is the standard embedding.

By definition, \( \mathcal{C}_O = \mathcal{B}_{R, (0, \Delta)}. \) By Proposition \([7.9]\) we have a homotopy equivalence \( \zeta : \text{Loc}(F) \to \mathcal{C}_O. \) We thus have constructed a zig-zag homotopy equivalence between \( \text{Loc}(F) \) and \( \mathcal{C}_I. \) Denote by \( \mathcal{P}_I \in \mathcal{C}_I \) the object corresponding to \( \mathcal{K}_F \in \text{Loc}(F). \)

### 8.2.1 Inverse functor

We have \( \mathcal{P}_I \in \text{sh}_q(F \times b_R \times E). \)

Let \( I' : F \times B_R \to T^*E \) be given by \( I'(f, v) = \alpha I(f, \alpha(v)), \) where \( \alpha : T^*E \to T^*E, \alpha(q, p) = \alpha(q, -p). \)

Let \( Q_I := \sigma_I \mathcal{P}_I \in \text{sh}_q(F \times E \times b_R), \) where \( \sigma : b_R \times E \to E \times b_R \) is the permutation.

Let \( \Delta_F : F \to F \times F \) be the diagonal embedding.

**Proposition 8.2** We have

\[
\mathcal{Q}_I \ast_{F \times E} \Delta_F! \mathcal{P}_I \approx \mathcal{K}_F \boxtimes \mathcal{P}_R \in \text{sh}_q(F \times b_R \times b_R).
\]

### 8.2.2

Let \( \pi : T^*F \to F \) be the projection. Let \( G_I \subset T^*F \times T^*E \) be an open subset defined as follows

\[
G_I = \{ (\phi, v) | v \in I(\pi(f)) \times \text{int} B_R \}.
\]

Let us also define functors

\[
\mathcal{P} : \text{sh}_q(F \times b_R)[T^*F \times \text{int} B_R] \rightarrow \text{sh}_q(F \times E)[G_I];
\]

\[
\mathcal{Q} : \text{sh}_q(F \times E)[G_I] \rightarrow \text{sh}_q(F \times b_R)[T^*F \times \text{int} B_R],
\]

where

\[
\mathcal{P}_I(S) = S \ast_{F \times b_R} \Delta_F! \mathcal{P}_I; \quad \mathcal{Q}_I(T) = T \ast_{F \times E} \Delta_F! \mathcal{Q}_I.
\]

**Proposition 8.3** The functors \( \mathcal{P}_I, \mathcal{Q}_I \) establish homotopy mutually inverse homotopy equivalences between the categories \( \text{sh}_q(F \times b_R)[T^*F \times \text{int} B_R] \) and \( \text{sh}_q(F \times E)[G_I]. \)

### 8.3 Pair of consecutive families

Let \( u : F \times B_r \to B_R, v : F \times B_R \to E \) be graded families of symplectic embeddings. Let \( w : F \times B_r \to E \) be defined by \( w(f, b) = v(f, u(f, b)) \). The gradings define liftings \( g_u : F \times B_r \to \overline{\text{Sp}}(2N); \)

\( g_v : F \times B_R \to \overline{\text{Sp}}(2N) \) of the corresponding differential maps.

Let \( g_w : F \times B_r \to \overline{\text{Sp}}(2N) \) be given by \( g_w(f, b) = g_v(f, u(f, b))g_u(f, b). \) It follows that \( g_w \) lifts the differential map \( F \times B_r \to E \) determined by \( w. \) Therefore, \( g_w \) is a grading of \( w. \)

**Proposition 8.4** We have a homotopy equivalence \( \mathcal{P}_v \circ \mathcal{P}_u \sim \mathcal{P}_w. \)
**Sketch of the proof** As above, let us extend the family \( v \) to a family

\[
v_t : F \times [-1, 1] \times B_R \to E,
\]

where

\[
v_t(f, t, b) = \frac{v(f, tb) - v(f, 0)}{t} + v(f, 0).
\]

Let \( w_t : F \times [-1, 1] \times B_r \to E \), where \( w_t(f, t, b) = v_t(f, t, u(f, b)) \). The gradings from \( v \) and \( w \) extend to \( v_t, w_t \). We will show that there exists a homotopy equivalence

\[
P_{v_t} \circ P_u \sim \sim P_{w_t}. \tag{44}
\]

Restriction to \( t = 1 \) will then show the Proposition.

To show the existence of (44), it suffices to establish the homotopy equivalence of the restriction to \( t = 0 \). Observe that \( v_0 \) comes from a family of linear symplectomorphisms \( F \to \text{Sp}(2N) \) whose grading defines a lifting \( V_0 : F \to \text{Sp}(2N) \). Let \( V \in \text{sh}_q(F \times \times E \times E) \) be the corresponding object. We have a homotopy equivalence

\[
P_{v_0} \circ P_u \sim V \circ P_u
\]

so the problem reduces to establishing a homotopy equivalence \( V \circ P_u \sim \sim P_{v_0 u} \).

In a similar way (via considering the family \( u_t \)), one reduces the problem to the case when the family \( u \) is linear. The grading then defines an object \( U \in \text{sh}_\infty(F \times E \times E) \). Similarly, the linear family \( v_0 u \), along with its grading, defines an object \( W \in \text{sh}_\infty(F \times E \times E) \).

Next, we have homotopy equivalences \( U \circ P_{B_r} \sim \sim P_u ; W \circ P_{B_r} \sim \sim P_{v_0 u} \) so that the problem reduces to establishing a homotopy equivalence

\[
V \circ U \sim W,
\]

which follows from Sec 6.

### 8.4 Mobile families

#### 8.4.1 Definition

Let \( U \subset T^*E \) be an open subset let \( j : U \to T^*E \) be the corresponding open embedding. Let \( I : U \times B_R \to T^*E \) be a family of symplectic embeddings, where we assume \( I|_{U \times 0} = j \).

The family \( I \) defines a Lagrangian sub-manifold

\[
L_I \subset T^*U \times \text{int}B_R \times T^*E.
\]

Set \( F = E \oplus E^* \).

We have a natural identification \( T^*U = U \times F \). For each \( \xi \in U \) let \( L_{\xi} : T^*U \times \text{int}B_R \times T^*E \cap L_I \subset F \times \text{int}B_R \times T^*E \). Let \( P_{\xi} \subset F \times \text{int}B_R \) be the image of \( L_{\xi} \) under the projection along \( T^*E \). Call \( I \) mobile if for every \( \xi \), \( P_{\xi} \) is a graph of an embedding \( \text{int}B_R \to F \).
8.4.2 Main proposition

We have objects $\mathcal{P}_I, \mathcal{Q}_I \in \text{sh}_q(U \times E \times E)$. Let $p_1, p_2 : U \times E \times E \times E \times E \to U \times E \times E$ be the projections

$$p_1(u, e_1, f_1, e_2, f_2) = (u, e_1, f_1); \quad p_2(u, e_1, f_1, e_2, f_2) = (u, e_2, f_2).$$

Consider

$$R_I := p_1^{-1} \mathcal{P}_I \circ p_2^{-1} \mathcal{Q}_I.$$

Let $i : E^3 \to E^4; p : E^3 \to E^2$ be given by $i(a, b, c) = (a, b, b, c); p(a, b, c) = (a, c)$. According to the previous subsection, we have a map

$$p_i^{-1} R_I \to \mathcal{K}_{U \times \Delta_{E \times \infty}}$$

where $\Delta_E \subset E \times E$ is the diagonal.

By the conjugacy, we have a map

$$R_I \to \mathcal{K}_{U \times \Delta_{14} \times \Delta_{23} \times \infty}[[N]],$$

where $N = \dim E$ which, in turn, gives rise to a map

$$\alpha : \pi_U R_I \to \mathcal{K}_{\Delta_{14} \times \Delta_{23} \times \infty}[[N]],$$

where $\pi_U : U \times E^4 \to E^4$ is the projection along $U$.

Let $V \subset U$ be an open subset satisfying: for every $u \in U$, if $I(u \times B_R) \cap V \neq \emptyset$, then $I(u \times B_R) \subset U$.

Let $p_i : T^* E^4 \to T^* E$ be the projections $i = 1, 2, 3, 4$. Let $p_{ij} := p_i \times p_j : T^* E^4 \to T^* E^2$.

**Proposition 8.5** Let $A, B \in \text{sh}_q(E \times E)$ and assume that $\mu_S A \subset B_R \times B_R \times \mathbb{R}; \mu_S B \subset V$. Then $H := (\text{Cone } \alpha) \ast_{E^4} (p_{23}^{-1} A \circ p_{14}^{-1} B) \sim 0$.

Sketch of the proof. Let us define a family of symplectic embeddings

$$J : U \times (-1, 1) \times B_R \to T^* E$$

by means of dilations, same as above. One then defines an object $\pi_U^1 R_J \in \text{sh}_q((-1, 1) \times E^4)$, a map

$$\alpha_J : \pi_U^1 R_J \to \mathcal{K}_{(-1, 1) \times \Delta_{14} \times \Delta_{23} \times \infty}[[N]],$$

and an object

$$H_J := (\text{Cone } \alpha_J) \ast_{E^4} (p_{23}^{-1} A \circ p_{14}^{-1} B) \in \text{sh}_q((-1, 1)).$$

Singular support estimate (see below) shows that

$$\mu_S H_J \subset T^*_J I \times \mathbb{R}.$$

Therefore, it suffices to show that $H_J|_0 \sim 0$, in other words, the problem reduces to the case when $I$ is a family of linear symplectic embeddings. The latter case can be reduced to the case when every embedding is a parallel transfer which is straightforward.
Estimate of $\mu S H_J$. It suffices to show that

$$\mu S (\pi_U! R_J *_{E^4} (A \boxtimes B)) \subset T^*_\ast (-1, 1).$$

Let us identify

$$T^*(U \times \mathbb{R} \times E^4) \times \mathbb{R} = (U \times \mathbb{R}) \times (F \oplus \mathbb{R}) \times F^4 \times \mathbb{R}.$$ 

We have

$$\mu S (R_J) \subset \{ (\tau, \eta_J(\tau, v_1) - \eta_J(\tau, v_2), v_1^a, J(\tau, v_1), v_2^a, J(\tau, v_2)) | \tau \in U \times \mathbb{R}, v_i \in F, |v_i| < R \} \times \mathbb{R}$$

$$\cup \{ (\tau, \xi, v_1, v_2, w_2) | |v_1|, |v_2| \leq R; \max(|v_1|, |v_2|) = R \} \times \mathbb{R}.$$ 

Consider now $\mu S (R_J *_{E \times E} A)$. As $\mu S (A) \subset \{ (v_1, v_2) | |v_1|, |v_2| < R \}$, it follows that

$$\mu S (R_J *_{E \times E} A) \subset \{ (\tau, \eta_J(\tau, v_1) - \eta_J(\tau, v_2), J(\tau, v_1), J(\tau, v_2)) | |v_1|, |v_2| < R \} \times \mathbb{R}.$$ 

Let us estimate

$$\mu S ((R_J *_{E \times E} A) *_{E \times E} B).$$

It follows that there exists a compact subset $K \subset U$ such that

$$\mu S ((R_J *_{E \times E} A) *_{E \times E} B) \subset \{ (\tau, \eta_J(\tau, v_1) - \eta_J(\tau, v_2)) | \tau \in K \times (-1, 1), |v_1|, |v_2| < R \} \times \mathbb{R}.$$ 

Namely, one can choose $K = \{ u \in U | I(u, B_R) \cap V \neq \emptyset \}$.

Let now $\tau = (u, x) \in U \times (-1, 1)$. We have $\eta_J(\tau, v) \in F \oplus \mathbb{R}$. Let $f(\tau, v)$ be the $F$-component and $x(\tau, v)$ be the $\mathbb{R}$-component. Let us now estimate

$$\mu S (\pi_U!(R_J *_{E^4} (A \boxtimes B))).$$

As $\pi_U$ is proper on the support of $R_J *_{E^4} (A \boxtimes B)$, the singular support in question is determined by the condition $f(\tau, v_1) - f(\tau, v_2) = 0$. As the family $I$ is mobile, this condition implies $v_1 = v_2$, which implies $\eta_J(\tau, v_1) - \eta_J(\tau, v_2) = 0$ and

$$\mu S (\pi_U!(R_J *_{E^4} (A \boxtimes B))) \subset T^*_\ast (-1, 1) \times \mathbb{R}.$$ 

9 Tree operads and multi-categories

9.1 Planar/cyclic trees

Let us introduce a notation for a tree $t$. Denote by $\mathbf{inp}(t)$ the set of inputs of $t$, $V_t$ the set of inner vertices of $t$, for $v \in V_t$, denote by $E_v$ the set of inputs of $v$. Let $p_t$ be the principal vertex of $t$.

9.1.1 Planar trees

Define a planar tree as a tree with a total order on every set $E_v$; we then have an induced total order on $\mathbf{inp}(t)$.

We have a unique identification of ordered sets $E_v = \{1, 2, \ldots, n_v\}$, where $n_v = \# E_v$; $\mathbf{inp}_t = \{1, 2, \ldots, n_t\}$, where $n_t = \# \mathbf{inp}_t$. 

9.1.2 Cyclic trees

Define a cyclic tree as a tree with a total order on every set $E_v$, $v \neq p_t$, and a cyclic order on $p_t$. We then have an induced cyclic order on $\text{inp}_t$, in particular, we assume $\text{inp}_{p_t} \neq \emptyset$.

A rigid cyclic tree is a cyclic tree along with identifications $E_{p_t} = \{1, 2, \ldots, n_{p_t}\}$, $\text{inp}_{p_t} = \{1, 2, \ldots, n_t\}$ which agree with the cyclic order on both sets.

9.1.3 Inserting trees into a tree

Let $t$ be a planar tree. Let $t_v$, $v \in V_t$ be planar trees where $n_{t_v} = n_v$. One then can insert the trees $t_v$ into $t$. Denote the resulting tree by $t\{t_v\}_{v \in V_t}$.

Similarly, let $t$ be a rigid cyclic tree. Let $t_v$, $v \in V_t \setminus p_t$ be planar trees with $n_{t_v} = n_v$; let $t_{p_t}$ be a rigid cyclic tree with $n_{t_{p_t}} = n_{p_t}$. One then can define a similar insertion, to be denoted by $t\{t_v\}_{v \in V_t}$.

9.1.4 Isomorphism classes of trees

Let $\text{trees}$ be the set of isomorphism classes of planar trees and $\text{trees}_{\text{cyc}}$ be the set of isomorphism classes of rigid cyclic trees.

Let also $\text{trees}_n \subset \text{trees}$ be the subset consisting of all isomorphism classes of trees with $n_t = n$ and likewise for $\text{cyc trees}_n$. The above defined insertions are defined on the level of isomorphism classes.

9.1.5 Families parameterized by isomorphism classes of trees

Let $\mathfrak{A}$ be a $\bigoplus$-closed dg SMC. Let $\mathcal{T}(\mathfrak{A})$ be a category, enriched over sets, whose every object is a family of objects $X_t \in \mathfrak{A}$, $t \in \text{trees} \sqcup \text{cyc trees}$. Let $X, Y \in \mathcal{T}(\mathfrak{A})$. Let us define a new family $X \circ Y \in \mathcal{T}(\mathfrak{A})$ as follows:

$$X \circ Y(T) = \bigoplus_{T = t\{t_v\}_{v \in V_t}} X(t) \otimes \bigotimes_{v \in V_t} Y(t_v).$$

This way, $\mathcal{T}(\mathfrak{A})$ becomes a monoidal category. The unit object $\text{unit} \in \mathcal{T}(\mathfrak{A})$ is defined by setting $\text{unit}(t) = \text{unit}_{\mathfrak{A}}$ for all isomorphism classes of planar trees with one vertex (corollas) and all isomorphism classes $t$ of rigid cyclic trees with one vertex and matching numberings of $E_p$ and $\text{inp}_t$. Otherwise, $\text{unit}(t) = 0$.

9.1.6 Planar trees with marked right branch

Let $\text{trees}_m \subset \text{trees}$ be a subset consisting of all planar trees $t$ with $\text{inp}_t \neq \emptyset$. Let $r_t \in \text{inp}_t$ be the rightmost input. For $t \in \text{trees}_m$, let $V_t^R \subset V_t$ consist of all vertices for which $r_t$ is an output.

Let $t \in \text{trees}_m$; let $t_v \in \text{trees}$, $v \in V_t \setminus V_t^R$ and $t_w \in \text{trees}_m$, $w \in V_t^R$. Suppose $\#\text{inp}_v = \#E_v$ for all $v \in V_t$. We then have a well defined insertion $t\{t_v\}_{v \in V_t}$.
Let $T_M(\mathcal{A})$ be a category, enriched over sets, whose every object is a family of objects $X_t \in \mathcal{A}$, $t \in \text{trees} \sqcup \text{cycntrees} \sqcup \text{treesm}$. Let $X, Y \in T(\mathcal{A})$. We define a new family $X \circ Y \in T(\mathcal{A})$ by the same formula (45). This way, $T_M(\mathcal{A})$ becomes a monoidal category. The unit object $\text{unit} \in T(\mathcal{A})$ is defined by setting $\text{unit}(t) = \text{unit}_A$ for all isomorphism classes of planar trees or planar trees with marked right branch with one vertex (corollas) and all isomorphism classes $t$ of rigid cyclic trees with one vertex and matching numberings of $E_p$ and $\text{inp}_t$. Otherwise, $\text{unit}(t) = 0$.

### 9.2 Collections of functors

Let $\mathcal{C}, \mathcal{D}$ be categories enriched over $\mathcal{A}$ and tensored by $\mathcal{A}$. Suppose we are also given a functor $h : \mathcal{C} \otimes \mathcal{D} \to \mathcal{A}$.

Let us define a category over $\mathsf{Sets}$, $F(\mathcal{C}, \mathcal{D})$, as follows

$$F(\mathcal{C}, \mathcal{D}) := \prod_{n=0}^{\infty} (\mathcal{D} \bigoplus)(\mathcal{C}^n \otimes \mathcal{D}) \times \prod_{n=1}^{\infty} \mathcal{C}^n$$

so that an object $F \in F(\mathcal{C}, \mathcal{D})$ is a collection of objects $F^{[n]} \in (\mathcal{D} \bigoplus)(\mathcal{C}^n \otimes \mathcal{D})$, $n \geq 0$, and $F^{(n)} \in (\mathcal{D} \bigoplus)(\mathcal{C}^n)$, $n \geq 1$.

Let $t$ be a planar tree. Define an object $F(t) \in (\mathcal{D} \bigoplus)(\mathcal{C}^n_t \otimes \mathcal{D})$.

A) We have an equivalence of categories

$$\bigotimes_{v \in V_t} (\mathcal{C}^n_v \otimes \mathcal{D}) \cong \left( \bigotimes_{v \in V_t \setminus pt} \mathcal{C} \otimes \mathcal{D} \right) \otimes (\mathcal{C}^n_t \otimes \mathcal{D}),$$

coming from the bijection

$$\bigcup_{v \in V_t} E_v \cong V_t \sqcup \text{inp}_t \setminus pt$$

which associates to an edge its target.

As a result we have a through map (via applying the functor $h$):

$$\alpha_t : \bigotimes_{v \in V_t} (\mathcal{D} \bigoplus)(\mathcal{C}^n_v \otimes \mathcal{D}) \to (\mathcal{D} \bigoplus) \left( \bigotimes_{v \in V_t \setminus pt} \mathcal{C} \otimes \mathcal{D} \right) \otimes (\mathcal{C}^n_t \otimes \mathcal{D}) \to (\mathcal{D} \bigoplus)(\mathcal{C}^n_t \otimes \mathcal{D}).$$

C) Set $F(t) := \alpha_t \left( \bigotimes_{v \in V_t} F^{[n_v]} \right)$.

Let now $t$ be a rigid cyclic tree. Define a functor $F(t) \in (\mathcal{D} \bigoplus)(\mathcal{C}^n_t)$ in a similar way. Let

$$\alpha_t : \mathcal{C}^n_t \otimes \bigotimes_{v \in V_t \setminus pt} (\mathcal{C}^n_v \otimes \mathcal{D}) \to \mathcal{C}^n_t$$
be defined similar to above and set
\[ F(t) := \varpi_t \left( F^{(n_{pt})} \otimes \bigotimes_{v \in V_t \setminus pt} F^{[n_v]} \right). \]

### 9.2.1 Extended collection of functors

Let \( \mathcal{C}, \mathcal{D}, \mathcal{C}_R, \mathcal{D}_R \) be dg categories tensored by \( \mathfrak{A} \). Suppose we are also given functors \( h : \mathcal{C} \otimes \mathcal{D} \to \mathfrak{A} \), \( h_R : \mathcal{C}_R \otimes \mathcal{D}_R \to \mathfrak{B} \).

Let us define a category over \( \text{Sets} \), \( \mathcal{F}(\mathcal{C}, \mathcal{D}, \mathcal{C}_R, \mathcal{D}_R) \), as follows

\[ \mathcal{F}(\mathcal{C}, \mathcal{D}, \mathcal{C}_R, \mathcal{D}_R) := \prod_{n=0}^{\infty} (D \bigoplus) (C^n \otimes D) \times \prod_{n=1}^{\infty} C^n \times \prod_{n=0}^{\infty} (D \bigoplus) (C^n \otimes \mathcal{C}_R \otimes \mathcal{D}_R). \]

so that an object \( F \in \mathcal{F}(\mathcal{C}, \mathcal{D}) \) is a collection of objects \( F^{[n]} \in (D \bigoplus) (C^{\otimes n} \otimes D) \), \( n \geq 0 \), \( F^{(n)} \in (D \bigoplus) (C^{\otimes n}) \), \( n \geq 1 \), and \( F^{[n,1]} \in (D \bigoplus) (C^{\otimes n} \otimes \mathcal{C}_R \otimes \mathcal{D}_R) \).

### 9.2.2 Definition of \( F(t) \)

Let \( F \) be a collection of functors. Let \( t \) be a planar tree. Define an object
\[ F(t) \in (D \bigoplus) (C^{\otimes nt} \otimes D). \]

A) We have an equivalence of categories

\[ \bigotimes_{v \in V_t} (C^{\otimes n_v} \otimes D) \cong \left( \bigotimes_{v \in V_t \setminus pt} C \otimes D \right) \otimes (C^{\otimes nt} \otimes D), \]

coming from the bijection
\[ \bigsqcup_{v \in V_t} E_v \cong V_t \sqcup \text{inp} \setminus pt \]

which associates to an edge its target.

As a result we have a through map (via applying the functor \( h \)):

\[ \alpha_t : \bigotimes_{v \in V_t} (D \bigoplus) (C^{\otimes n_v} \otimes D) \to (D \bigoplus) \left( \bigotimes_{v \in V_t \setminus pt} C \otimes D \right) \otimes (C^{\otimes nt} \otimes D) \to (D \bigoplus) (C^{\otimes nt} \otimes D). \]

C) Set \( F(t) := \alpha_t \left( \bigotimes_{v \in V_t} F^{[n_v]} \right). \)

Let now \( t \) be a rigid cyclic tree. Define a functor \( F(t) \in (D \bigoplus) (C^{nt}) \) in a similar way. Let

\[ \alpha_t : C^{\otimes nt} \otimes \bigotimes_{v \in V_t \setminus pt} (C^{\otimes n_v} \otimes D) \to C^{\otimes nt} \]

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be defined similar to above and set
\[ F(t) := c_t \left( F^{(n_p t)} \otimes \bigotimes_{v \in V \setminus p_t} F^{[n_v]} \right). \]

Let now \( F \) be an extended collection of functors. We then define \( t(F) \) in a similar way.

### 9.3 Schur functors

Suppose \( C, D \) are enriched and tensored over \( \mathcal{A} \). Let \( X \in \mathcal{T}(\mathcal{A}) \) and \( F \in \mathcal{F}(C, D) \). Define an object \( S_X(F) \in \mathcal{F}(C, D) \) as follows
\[ S_X(F)[n] := \bigoplus_{t \in \text{trees}_n} t(F); \quad S_X(F)^{(n)} = \bigoplus_{t \in \text{cyctrees}_n} t(F). \]

We have natural isomorphisms
\[ S_X S_Y F \cong S_{X \circ Y} F; \quad S_{\text{unit}} F \cong F. \]

In fact, we have a \( \mathcal{T}(\mathcal{A}) \)-action on \( \mathcal{F}(C, D) \).

One defines a \( \mathcal{T}_M(\mathcal{A}) \)-action on \( \mathcal{F}(C, CC_R, D, DD_R) \) in a similar way.

### 9.4 Tree operads

A **tree operad in** \( \mathcal{T}(\mathcal{A}) \) is the same as a unital monoid in \( \mathcal{T}(\mathcal{A}) \). Respectively, an extended tree operad in \( \mathcal{T}_M(\mathcal{A}) \) is the same as a unital monoid in \( \mathcal{T}_M(\mathcal{A}) \).

#### 9.4.1 A tree operad \texttt{triv}

Let \( \texttt{triv} \in \mathcal{T}(\mathcal{A}) \) be given by \( \texttt{triv}(t) = \text{unit}_\mathcal{A} \) for all \( t \). Define \( \texttt{triv} \in \mathcal{T}_M(\mathcal{A}) \) in a similar way.

#### 9.4.2 Endomorphism tree operad

Let \( C, D \) be enriched and tensored over \( \mathcal{A} \). Let \( F, G \in \mathcal{F}(C, D) \). Consider a functor \( H_{F,G} : \mathcal{T}((D \oplus)\mathcal{A}) \to \text{Sets} \),
\[ H_{F,G}(X) = \text{Hom}(S_X F; G) \]

The functor \( H_{F,G} \) is representable. Denote the representing object by \( \mathcal{H}_{F,G} \). We have (\( t \) is planar):
\[ \mathcal{H}_{F,G}(t) = \text{Hom}_{(D \oplus)(C \otimes_{\text{nt}} \text{Cop})}(F(t); G^{[n_t]}); \]

if \( t \) is a rigid cyclic tree, we have:
\[ \mathcal{H}_{F,G}(t) = \text{Hom}_{(D \oplus)C^{\otimes nt}}(F(t); G^{(n_t)}). \]
Set $\text{End}_F := \mathcal{H}_{F,F}$. We have a natural tree operad structure on $\text{End}_F$. Furthermore, we have an $\text{End}_F - \text{End}_G$-bi-module structure on $\mathcal{H}_{F,G}$ (where we interpret tree operads $\text{End}_F, \text{End}_G$ as monoids in $\mathcal{T}((D \bigoplus)\mathfrak{A})$).

Let $F \in \mathcal{F}(\mathcal{C}, \mathcal{C} R, D, D R)$. We define an extended tree operad $\text{End}_F$ in a similar way.

### 9.5 Pull backs from $\mathcal{F}(\mathcal{C'}, \mathcal{D'})$ to $\mathcal{F}(\mathcal{C}, \mathcal{D})$

Let $\mathfrak{A}$ have internal hom. Let $\mathcal{C}, \mathcal{C}', \mathcal{D}, \mathcal{D}'$ be categories enriched over $\mathfrak{A}$; let $h : \mathcal{C} \otimes \mathcal{D} \to \mathfrak{A}$; $h' : \mathcal{C}' \otimes \mathcal{D}' \to \mathfrak{A}$ be functors.

Let $G \in \mathcal{F}(\mathcal{C}', \mathcal{D}')$. Let $L \in (D \bigoplus)(D \otimes \mathcal{C}')$.

Consider the following functor $H : \mathcal{F}(\mathcal{C}, \mathcal{D})^{\text{op}} \to \text{Sets}$ as follows.

1) We have functors

$$e_L : \mathcal{C}^n \otimes \mathcal{D} \otimes (D \otimes \mathcal{C}')^n \to (\mathcal{C}')^n \otimes \mathcal{D},$$

via using the hom-functor $h^\otimes : \mathcal{C}^n \otimes \mathcal{D} \to \mathfrak{A}$, as well as

$$f_L : (\mathcal{C}')^n \otimes \mathcal{D}' \otimes \mathcal{D} \otimes \mathcal{C}' \to (\mathcal{C}')^n \otimes \mathcal{D}.$$  

via the hom functor $h' : \mathcal{C}' \otimes \mathcal{D}' \to \mathfrak{A}$.

Similarly, one defines a cyclic version: set

$$e^{\text{cyc}}_L : \mathcal{C}^n \otimes \mathcal{D} \otimes (D \otimes \mathcal{C}')^n \to (\mathcal{C}')^n.$$  

2) set

$$H^{[n]}(F,G) := \text{Hom}(e_L(F^{[n]} \otimes \mathcal{L}^n); f_L(G^{[n]} \otimes \mathcal{L}));$$

$$H^{(n)}(F,G) := \text{Hom}(e^{\text{cyc}}_L(F^{(n)} \otimes \mathcal{L}^n); G^{(n)}).$$

Set

$$H(F,G) = \prod_{n \geq 0} H^{[n]}(F,G) \times \prod_{n > 0} H^{(n)}(F,G).$$

It follows that the functor $F \mapsto H(F,G)$ is representable. Denote the representing object by $L^{-1}G$. Let $X \in \mathcal{T}(\mathfrak{A})$. We have a natural map $S_X L^{-1}G \to L^{-1}S_X G$.

Therefore, $\mathcal{H}_{F,L^{-1}G}$ is a $\text{End}_F - \text{End}_G$-bimodule.

One generalizes this construction to the extended case straightforwardly.

### 9.5.1 Quasi-contracible tree operads

Let now $\mathfrak{A} = \text{pt}$ so that $(D \bigoplus)\mathfrak{A} = G \mathbb{Z}$. Call a tree operad $O \in \mathcal{T}(G \mathbb{Z})$ pseudo-contractible if

1) $O(t) \in G \mathbb{Z}$ admits a truncation for every $O(t)$. We therefore have an induced tree operad structure on $\tau_{\leq 0}O$ and a map of tree operads $\tau_{\leq 0}O \to O$.  

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2) Every object \(\tau_{\leq 0} O\) admits a truncation \(\tau_{\geq 0}\), to be denoted \(H^0 O(t)\) which is a finitely generated free \(K\)-module; we have an induced map of tree operads \(\tau_{\leq 0} O \to H^0 O\). We require this map to be a term-wise homotopy equivalence.

A quasi-contractible tree operad is a pseudo-contractible operad \(O\) endowed with a map of tree operads \(\text{triv} \to H^0 (O)\).

In this case there exists a splitting of the map \(\tau_{\leq 0} O(t) \to H^0 O(t)\), hence a pull-back of the diagram

\[
\text{triv} \to H^0 (O) \leftarrow \tau_{\leq 0} O,
\]

to be denoted by \(\text{triv}_{\!O} \) so that we have a diagram

\[
\text{triv} \xrightarrow{\sim} \text{triv}_{\!O} \to O.
\]

Let \(O_1, O_2\) be quasi-contractible operads and \(M\) a \(O_1 - O_2\)-bi-module. Call \(M\) pseudo-contractible if there exist truncations \(\tau_{\leq 0} M(t)\) and \(\tau_{\geq 0} \tau_{\leq 0} M(t) =: H^0 M(t)\), where each \(H^0 M(t)\) is a finitely generated free \(K\)-module.

A quasi-contractible \(O_1 - O_2\)-bi-module \(M\) is a pseudo-contractible \(O_1 - O_2\)-bi-module \(M\) endowed with a map

\[
(\text{triv}, \text{triv}, \text{triv}) \to (H^0 O_1, H^0 M, H^0 O_2)
\]

of triples: a pair of tree-operads and their bi-module.

Similar to above, we have a pull-back of the diagram

\[
(\text{triv}, \text{triv}, \text{triv}) \leftarrow (\tau_{\leq 0} O_1, \tau_{\leq 0} M, \tau_{\leq 0} O_2) \to (H^0 O_1, H^0 M, H^0 O_2),
\]

to be denoted by \((\text{triv}_{\!O_1}, \text{triv}_{\!M; \text{triv}_{\!O_2}})\) so that we have a diagram

\[
(\text{triv}, \text{triv}, \text{triv}) \xrightarrow{\sim} (\text{triv}_{\!O_1}, \text{triv}_{\!M, \text{triv}_{\!O_2}}) \to (O_1, M, O_2).
\]

10  Straightening out

Fix a ground category \(A\).

10.1  Pseudo-contractible sequences

Fix categories \(C_i, D_i, i = 1, n\); functors \(h_i : C_i \otimes D_i \to A\), objects \(F_i \in F(C_i, D_i)\) and \(L_{i+1,i} : (D \oplus (D_i+1 \otimes C_i)).\) Call such a collection of data a sequence. Call such a sequence pseudo-contractible if every triple

\[
(\text{End} F_i, \mathcal{H}_{F_{i+1}, L_{i+1,i} F_i, \text{End} F_{i+1}}),
\]

\(0 \leq i \leq n - 1\), is pseudo-contractible.

Denote by \((O_i, O_{i+1}, O_{i+1})\) the pull-back of the diagram

\[
\tau_{\leq 0}(\text{End} F_{i+1}, \mathcal{H}_{F_{i+1}, F_i, \text{End} F_i}) \longrightarrow H^0(\text{End} F_{i+1}, \mathcal{H}_{F_{i+1}, F_i, \text{End} F_i})
\]

\[
(\text{triv}, \text{triv}, \text{triv})
\]

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Denote by
\[ \pi_{i+1, i} : (O_i, O_{i, i+1}, O_{i+1}) \sim (\text{triv}, \text{triv}, \text{triv}) \]
the projection.

10.2 Straightening

Let us define new sequences of functors \( G_i \in \mathcal{F}(C_i, D_i) \) such that we have a \( \text{triv} \) -action on each \( G_i \) as well as maps of \( \text{triv} \)-modules \( G_{i+1} \rightarrow L_{i+1, i} G_i \). Namely, fix canonical quasi-free resolutions \( R_i \rightarrow O_i \) of \( O_i \) as of a bimodule over itself.

Set
\[
G_0 := \text{triv} \circ O_0 R_0 \circ O_0 F_0; \\
G_1 := \text{triv} \circ O_0 R_0 \circ O_0 O_{01} \circ O_1 R_i \circ O_1 F_0; \\
G_2 := \text{triv} \circ O_0 R_0 \circ O_0 O_{01} \circ O_1 O_{12} \circ O_2 R_2 \circ O_2 F_0; \\
\]
etc.

The maps of \( \text{triv} \)-modules \( G_{i+1} \rightarrow L_{i+1, i} G_i \) are induced by the maps
\[
O_{i, i+1} \circ O_{i+1} R_{i+1} \circ O_{i+1} F_{i+1} \rightarrow O_{i, i+1} \circ O_{i+1} F_{i+1} \rightarrow F_i.
\]

10.2.1 Extended case

Let \( \mathcal{F} \) be an extended collection of functors acted upon by a quasi-contractible extended tree operad. One then constructs an extended collection \( \mathcal{F}' \) with a \( \text{triv} \)-action in a similar way.

10.3 Definition of a monoidal category with trace

10.3.1 The category \( \tau(\mathcal{M}) \)

Let \( \mathcal{M} \) be a monoidal category over \( \mathfrak{A} \). Define a category \( \tau(\mathcal{M}) \) enriched over \( \mathfrak{A} \) as follows. An object of \( \tau(\mathcal{M}) \) is an object of \( \mathcal{M}^{\otimes n} \) for some \( n \geq 1 \).

Let \( \mathcal{X} := (X_1, X_2, \ldots, X_n) \), \( \mathcal{Y} := (Y_1, Y_2, \ldots, Y_m) \in \tau(\mathcal{M}) \), where \( X_i, Y_j \in \mathcal{M} \). Let \( f : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, m\} \) be a cyclically non-decreasing map. For \( p \in \{1, 2, \ldots, m\} \) we then have a total order on \( f^{-1} p \). Let
\[
X_p := \bigotimes_{i \in f^{-1} p} X_i \in \mathcal{M}.
\]

Let
\[
\text{Hom}_f(\mathcal{X}, \mathcal{Y}) := \bigotimes_{1 \leq p \leq m} \text{Hom}_\mathcal{M}(X_p; Y_p).
\]

Let
\[
\text{Hom}_{\tau(\mathcal{M})}(\mathcal{X}, \mathcal{Y}) := \bigoplus_{f} \text{Hom}_f(\mathcal{X}, \mathcal{Y}),
\]
where \( f \) runs through the set of all cyclically non-decreasing maps \( \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, m\} \).
10.3.2 Definition of a monoidal category with trace

A monoidal category with trace \((\mathcal{M}, \mathcal{M}^\text{cyc})\) is a pair consisting of a monoidal category \(\mathcal{M}\) and a category \(\mathcal{M}^\text{cyc}\) endowed with a functor \(\text{TR} : \tau(\mathcal{M}) \to \mathcal{M}^\text{cyc}\).

10.4 A monoidal category with trace \(U(F)\) from a collection \(F \in \mathcal{F}(\mathcal{C}, \mathcal{D})\) with a triv-action

As above, let \(F \in \mathcal{F}(\mathcal{C}, \mathcal{D})\). Suppose \(F\) carries a \text{triv}-action. Let us construct a monoidal category with trace \(U(F)\).

— An object of \(U(F)\) is an object of \((D \bigoplus)(C^{\boxplus n})\) for some \(n \geq 0\).
— Let \(X \in (D \bigoplus)(C^{\boxplus n})\) and \(Y \in (D \bigoplus)(C^{\boxplus m})\), \(m > 0\). Let \(f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}\) be a non-decreasing map. Let \(\phi_k := \# f^{-1}m\). Let

\[ F(f) := \bigotimes_{k=1}^m F(\phi_k) \in (D \bigoplus)(C^{\boxplus n} \boxtimes D^{\boxplus m}). \]

Set

\[ \text{Hom}_f(X, Y) := \text{Hom}(X; h^{\boxplus m}(F(f) \boxtimes Y)). \]

Set

\[ \text{Hom}(X, Y) := \bigoplus_f \text{Hom}_f(X, Y). \]

If \(n > 0\) and \(m = 0\), then we set \(\text{Hom}(X, Y) = 0\). Finally, if \(n = m = 0\), we set

\[ \text{Hom}(X_n, Y_m) = \text{unit}_A. \]

The \text{triv}-action on \(F\) induces maps

\[ \text{Hom}_f(X, Y) \otimes \text{Hom}_g(X, Y) \to \text{Hom}_{fg}(X, Y), \]

whence a \(\mathcal{A}\)-category structure on \(U(F)\).

Define the monoidal structure on \(U(F)\) by means of \(\boxtimes\) and natural maps

\[ \text{Hom}_f(X_1, Y_1) \otimes \text{Hom}_g(X_2, Y_2) \to \text{Hom}_{fg}(X_1 \boxtimes X_2; Y_1 \boxtimes Y_2). \]

The unit is a unique object in \(C^{\boxplus 0}\).

Define a trace on \(U(F)\) via prescribing restrictions

\[ \text{TR}_n : C^{\boxplus n} \to \mathcal{A}, \quad \text{TR}_n(X) = \text{Hom}(X; F^{(n)}). \]

Suppose we have a map of \text{triv}-modules \(\pi_{i+1,i} : G_{i+1} \to L_{i+1,i}^{-1}G_i\), as in the previous subsection. We then have an induced tensor functor \(\pi_{i+1,i} : U(G_{i+1}) \to U_G(i)\), where for \(X \in C^{\boxplus n}_{i+1}\) we set

\[ \pi_{i+1,i}(X) := h^{\boxplus m}(L_i^{\boxplus n} \boxtimes X) \in C^{\boxplus n}_i. \]
10.4.1 The category $U_1(F)$

Denote by $U_1(F) \subset U(F)$ the full sub-category of objects homotopy equivalent to an object from $\mathcal{C} \subset U(F)$.

10.4.2 The category $U^{\text{cyc}}(\mathcal{O})$

Let $U^{\text{cyc}}(\mathcal{O}) \subset \tau(U(F))$ be the full sub-category consisting of all objects of the form $(X_1, X_2, \ldots, X_n)$, where $n > 0$ and $X_i \in \mathcal{C}$.

We have a functor $I: U(F) \to U^{\text{cyc}}(F)$ which is identical on objects.

We have a functor $\mathbf{T} \mathbf{R}: U^{\text{cyc}}(F)^{\text{op}} \to \mathfrak{A}$.

Let $U^{\text{cyc}}_{\geq 0}(F)$ be a category which is obtained from $U^{\text{cyc}}(F)$ by adding an extra object, to be denoted by $(0)$, where $\text{Hom}((0), X) = 0$, $\text{Hom}(X, (0)) = \mathbf{T} \mathbf{R}(X)$, for all $X \in U^{\text{cyc}}(F)$, $\text{Hom}((0), (0)) = \text{unit}_\mathfrak{A}$.

10.4.3 Extended case: a monoidal category with a trace and its module

Let now $F$ be an extended collection of functors with $\text{triv}$-action. We then get a monoidal category with traces $U(F)$ and its module $U_M(F)$, that is, $U_M(F)$ is a category endowed with a functor $U(F) \otimes U_M(F) \to U_M(F)$ satisfying the associativity axiom.

10.5 Circular operads

A circular operad is a $\text{triv}$-module on an object $\mathcal{O} \in \mathcal{F}(p, p)$, where $p$ is a disjoint union of a finite number of copies of $\text{pt}$, the category with one object, to be denoted by $O$ and $\text{Hom}(O, O) = \text{unit}_\mathfrak{A}$. Set $\mathcal{O}(n) := \mathcal{O}^{[n]}(\text{pt}, \ldots, \text{pt})$; $\mathcal{O}^{\text{cyc}}(n) := \mathcal{O}^{(n)}(\text{pt}, \ldots, \text{pt})$.

A circular operad structure is equivalent to an asymmetric operad structure on a collection $\mathcal{O}$ and an $U^{\text{cyc}}(\mathcal{O})$-module structure on $\mathcal{O}^{\text{cyc}}$. In particular, we have a $\mathbb{Z}/n\mathbb{Z}$-action on $\mathcal{O}^{\text{cyc}}(n)$ for every $n \geq 1$. Call $\mathcal{O}$ cyclically semi-free if such is each $\mathbb{Z}/n\mathbb{Z}$-module $\mathcal{O}^{\text{cyc}}(n)$.

Let $\Lambda$ be the cyclic category. Denote its objects by $(n)$, $n > 0$ (observe the shift by 1 unit with respect to the traditional numbering!). Let $\mathcal{Z}^{\text{cyc}}(\mathcal{O})$ be the category with the same set of objects as in $U^{\text{cyc}}(\mathcal{O})$ and we set

$$\mathcal{Z}^{\text{cyc}}(\mathcal{O})((m), (n)) = U^{\text{cyc}}((m), (n)) \otimes \Lambda^{\text{op}}((m), (n)).$$

We have a functor $U^{\text{cyc}}(\mathcal{O}) \to \mathcal{Z}^{\text{cyc}}(\mathcal{O})$. Let $U : U^{\text{cyc}}(\mathcal{O})^{\text{op}} \to \mathfrak{A}$, $L : \Lambda \to \mathfrak{A}$ be functors. Set $U \circ L((n)) := U(n) \otimes L(n)$. We have

$$U \circ L : U^{\text{cyc}} \to \mathcal{Z}^{\text{cyc}} \to \mathfrak{A}.$$ 

Hence, we have an induced circular operad structure on $(\mathcal{O}, \mathcal{O}^{\text{cyc}} \circ L)$.

Fix a free resolution $\mathcal{R}_\Lambda \to \mathbb{K}$ of the constant $\Lambda$-module $\mathbb{K}$. It follows that $\mathcal{O}_\mathcal{R} := (\mathcal{O}, \mathcal{O}^{\text{cyc}} \circ \mathcal{R}_\Lambda)$ is a cyclically semi-free circular operad. We have a natural map of circular operads $\mathcal{O}_\mathcal{R} \to \mathcal{O}$. 

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10.6 Algebras over circular operads

Let $O$ be a circular operad. Let $(\mathcal{M}, \mathcal{M}^{cy})$ be a monoidal category with a trace. A $O$-algebra in $\mathcal{M}$ is a strict tensor functor of monoidal categories with a trace:

$$(U(O), U^{cy}(O)) \to (\mathcal{M}, \mathcal{M}^{cy}).$$

Suppose $\mathcal{M}$ is $\bigoplus$-closed and $O$ is cyclically semi-free. Then the structure of an $O$-algebra is equivalent to that of an algebra over a certain monad, to be denoted by $S_O$, acting on the category $K := \mathcal{M} \times \mathcal{M}^{cy}$ over Sets. By definition,

$$S_O(X, U) := (X', U'),$$

where

$$X' := \bigoplus_{n \geq 0} O(n) \otimes X^\otimes n;$$

$$U' := U \oplus \bigoplus_{n \geq 1} O^{cy}(n) \otimes_{\mathbb{Z}/n\mathbb{Z}} \text{TR}(X, \ldots, X).$$

There exists a monoidal structure on $F(\text{pt}, \text{pt})$, to be denoted by $\circ$, so that we have an isomorphism

$$S_X S_Y \cong S_{X \circ Y},$$

for all cyclically free $X, Y$. A structure of a circular operad on $O$ is equivalent to that of a unital monoid in $F(\text{pt}, \text{pt})$ whence a monad structure on $S_O$.

Let $O_2 \to O_1$ be a map of circular operads. We then have a $O_1 - O_2$-bimodule structure on $O_2 \in F(\text{pt}, \text{pt})$. Choose its semi-free resolution $R$.

Let $(X, Y)$ be an $O_2$-algebra in $\mathcal{M}$. One then has a $O_1$-algebra structure on

$$R \circ_{O_2} (X, Y),$$

which is well-defined as long as $(\mathcal{M}, \mathcal{M}^{cy})$ is $\bigoplus D$-closed.

**PART 3. MICROLOCAL CATEGORY: CLASSICAL LEVEL**

11 Geometric setting

11.1 Principal bundles

— Let $M$ be a compact symplectic $2N$-dimensional manifold whose symplectic form $\omega$ has integral periods.

— Let $L$ be a circle bundle whose first Chern class equals $\omega$.

— Let $P_0 \to M$ be the principal bundle of symplectic frames on $M$ with its structure group $\text{Sp}(2N)$. Let $P := P_0 \times_M L$. $P$ is a $\text{Sp}(2N) \times S^1$-principal bundle over $M$.

— Let $\mathcal{H} := P_0/U(N)$. We have a smooth fibration $\mathcal{H} \to M$ with contractible fiber. We have a principal $U(N) \times S^1$-bundle $P \to \mathcal{H}$.

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11.2 Pseudo-Kaehler metrics

—Let \( \text{Met} \) be the set of all pseudo-Kaehler metrics on \( M \) with their symplectic form being \( \omega \). One identifies \( \text{Met} \) with the set of sections of the bundle \( \mathfrak{H} \to M \). For \( g \in \text{Met} \), denote by \( i_g : M \to \mathfrak{H} \) the corresponding section and \( \text{Fr}_g := i_g^{-1} P \).

Fix retractions \( \pi_g : P \to \text{Fr}_g \) so that we have the following retraction of principal \( U(N) \times S^1 \)-bundles

\[
\begin{array}{c}
\text{Fr}_g \\
\downarrow \\
M
\end{array} \xrightarrow{\pi_g} \begin{array}{c} P \\
\downarrow \\
\mathfrak{H} \\
\downarrow \\
M
\end{array}
\]

The retraction \( \pi_g \) is constructed as follows. Fix a \( U(N) \times U(N) \)-equivariant retraction

\[
U(N) \to \text{Sp}(2N) \to U(N).
\]

We then set

\[
\pi_g : P = \text{Fr}_g \times_{U(N)} \text{Sp}(2N) \to \text{Fr}_g \times_{U(N)} U(N) = \text{Fr}_g.
\]

Denote by \( i_{\text{Met}} : \text{Met} \times M \to \mathfrak{H} \) the union of all \( i_g, g \in \text{Met} \). Let \( \text{Fr} := \bigsqcup_{g \in \text{Met}} \text{Fr}_g \). We have \( \text{Fr} = i_{\text{Met}}^{-1} P \). One has the following retraction of principal \( U(N) \)-bundles

\[
\begin{array}{c}
\text{Fr} \\
\downarrow \\
M \times \text{Met}
\end{array} \xrightarrow{\pi_{\text{Met}}} \begin{array}{c} P \times \text{Met} \\
\downarrow \\
\mathfrak{H} \times \text{Met} \\
\downarrow \\
M \times \text{Met}
\end{array}
\]

11.3 Groupoids

11.3.1 The groupoid \( P \times_M P \)

Let us start with a groupoid

\[
P \times_M P \rightrightarrows P.
\]

11.3.2 The groupoid \( Q \), a covering of \( P \times_M P \)

We have a natural map \( P \times_M P \to \text{Sp}(2N) \times S^1 \). Let \( \mathcal{H} \) be the universal cover of \( \text{Sp}(2N) \times S^1 \). Set

\[
Q := (P \times_M P) \times_{\text{Sp}(2N) \times S^1} \mathcal{H}.
\]

We have a groupoid structure on \( Q \rightrightarrows P \) and a map of groupoids

\[
(Q \rightrightarrows P) \to (P \times_M P \rightrightarrows P).
\]

The \( \mathbb{Z} \times \mathbb{Z} \)-action on \( \mathcal{H} \) carries over to \( Q \). Let us denote this action by \( T \). Let \( \sigma : Q \to Q \) be the inversion map.
One also has iterations

\[ C_n : Q \times_P Q \times_P \cdots \times_P Q \to Q, \]

where there are \( n \) occurrences of \( Q \) on the LHS.

Let \( I_n : Q \times_P Q \times_P \cdots \times_P Q \to Q^n \) be the embedding.

Let \( a \in \mathbb{Z} \times \mathbb{Z} \). Let \( K_n^a \) be the image of the map

\[ I_n \times a.C_n : Q \times_P Q \times_P \cdots \times_P Q \to Q^{n+1}. \]

Let also \( K_n^{\text{cyc}, a} \) be obtained from \( K_n^a \) by applying the permutation \( \sigma \) to the last factor of \( Q \).

### 11.3.3 The groupoid \( \Phi \)

The deformation retraction (48) induces a deformation retraction of groupoids

\[
\begin{array}{cccc}
\text{Fr} \times_M \text{Fr} & j \to & P \times_M P \times \text{Met} \times \text{Met} & \longrightarrow & \text{Fr} \times_M \text{Fr} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Fr} & \longrightarrow & P \times \text{Met} & \longrightarrow & \text{Fr} \\
\end{array}
\]

(49)

Let now \( \Phi \) be the \( j \)-pull-back of the covering

\[ Q \times \text{Met} \times \text{Met} \to P \times_M P \times \text{Met} \times \text{Met}. \]

We have an induced groupoid structure on \( \Phi \mathrel{\Rightarrow} \text{Fr} \); the map \( j \) then naturally extends to a map of groupoids

\[ (\Phi \mathrel{\Rightarrow} \text{Fr}) \to (Q \times \text{Met} \mathrel{\Rightarrow} P \times \text{Met}). \]

As \( j \) is a deformation retraction, we have a deformation retraction of groupoids:

\[
\begin{array}{cccc}
\Phi & j \to & Q \times \text{Met} \times \text{Met} & \longrightarrow & \Phi \\
\downarrow & & \downarrow & & \downarrow \\
\text{Fr} & \longrightarrow & P \times \text{Met} & \longrightarrow & \text{Fr} \\
\end{array}
\]

(50)

The arrows on the top commute with the \( \mathbb{Z} \times \mathbb{Z} \)-actions on all spaces.

Let \( \Phi_{g_1, g_2} \) be the pre-image of \((g_1, g_2) \in \text{Met} \times \text{Met}\) under the map

\[ \Phi \to \text{Met} \times \text{Met}. \]

We then have groupoid composition maps

\[ C(g_1 g_2 \cdots g_n) : \Phi_{g_1, g_2} \times_{\text{Fr}_{g_2}} \Phi_{g_2, g_3} \times_{\text{Fr}_{g_3}} \cdots \times_{\text{Fr}_{g_{n-1}}} \Phi_{g_{n-1}, g_n} \to \Phi_{g_1, g_n}. \]
Let us define subspaces
\[
K(g_1g_2 \cdots g_n)^a \subset \Phi_{g_1g_2} \times \Phi_{g_2g_3} \times \cdots \times \Phi_{g_{n-1}g_n} \times \Phi_{g_1g_n}
\]
and
\[
K(g_1g_2 \cdots g_n)^{\text{cyc},a} \subset \Phi_{g_1g_2} \times \Phi_{g_2g_3} \times \cdots \times \Phi_{g_{n-1}g_n} \times \Phi_{g_ng_1}
\]
similar to Sec. 11.3.2.

The retraction (50) induces deformation retractions:
\[
\begin{array}{c}
K(g_1g_2 \cdots g_n)^a \\
\Phi_{g_1g_2} \times \Phi_{g_2g_3} \times \cdots \times \Phi_{g_{n-1}g_n} \\
\end{array} \rightarrow 
\begin{array}{c}
K_n^a \\
Q^{n-1} \\
\Phi_{g_1g_2} \times \Phi_{g_2g_3} \times \cdots \times \Phi_{g_{n-1}g_n} \\
\end{array}
\]

11.3.4 The groupoid \( \mathcal{G} \)

Let \( S := P \times_{\mathfrak{g}} P \) we have a homotopy equivalence of groupoids
\[
(S \Rightarrow P) \rightarrow (P \times_M P \Rightarrow P),
\]

Let
\[
\mathcal{G} := S \times_{P \times_M P} Q.
\]
Equivalently, one can define \( \mathcal{G} \) as follows. Since \( P \rightarrow \mathfrak{g} \) is a principal \( U(N) \)-bundle, we have a map
\[
S = P \times_{\mathfrak{g}} P \rightarrow U(N).
\]
Let \( \overline{U}(N) \rightarrow U(N) \) be the universal cover. Then \( \mathcal{G} = S \times_{U(N)} \overline{U}(N) \).

We have an induced groupoid structure on \( \mathcal{G} \Rightarrow P \) as well as a \( \mathbb{Z} \times \mathbb{Z} \)-equivariant groupoid maps
\[
(\mathcal{G} \Rightarrow P) \rightarrow (Q \Rightarrow P) \xrightarrow{p_{g_1g_2}} (\Phi_{g_1g_2} \Rightarrow M).
\]

We define spaces \( K_n^a, K_n^{\mathcal{G}, \text{cyc}} \subset \mathfrak{g}^n \) similar to above.

11.3.5 The groupoid \( \Sigma \)

Let \( L := P/\text{SU}(N) \). We then get an \( S^1 \times S^1 \)-bundle \( L \rightarrow \mathfrak{g} \), hence, a map \( L \times_{\mathfrak{g}} L \rightarrow S^1 \). Let
\[
\Sigma := (L \times_{\mathfrak{g}} L) \times_{S^1 \times S^1} \mathbb{R} \times \mathbb{R}
\]
We have a groupoid structure on \( \Sigma \Rightarrow L \) as well as a \( \mathbb{Z} \times \mathbb{Z} \)-equivariant groupoid map
\[
(\mathcal{G} \Rightarrow P) \rightarrow (\Sigma \Rightarrow L).
\]

One defines the spaces \( K_n^\Sigma, K_n^{\Sigma, \text{cyc}} \subset \Sigma^n \) in the same way as above.
11.4 Symplectic balls

Let \( g \in \text{Met} \).

Let \( B_{g,R}M \subset TM \) be the sub-bundle of balls of \( g \)-radius \( R \). Let \( \pi_{g,R} : B_{g,R}M \to M \) be the projection.

One can define a function \( R'' : \text{Met} \to \mathbb{R}_{>0} \) satisfying:

1. for every \( g \in \text{Met} \), we have a map \( I^g : B_{R''(g)}^g M \to M \), where for each \( m \in M \), the induced map \( I_{g,m}^g : (\pi_{g,R''(g)})^{-1}m \to M \) is a symplectic embedding such that \( I_{g,m}^g(0) = m \). For \( R \leq R''(g) \), denote \( B'_{R,g}(m) := I_{g,m}^g((\pi_{g,R})^{-1}m) \).

2. for all \( R \leq R''(g) \) and all \( m \in M \), \( B_{g,R}(m) \subset M \) is \( g \)-geodesically convex.

3. Let \( m_0 \in M \) and fix an identification \( \psi_{m_0} : B_{R''(g)}^g \cong B_{g,R''(m_0)}. \) For each \( n \in B_{R''(g)}^g(m_0) \), let \( \phi_n : B_{R''(g)}^g \cong B_{g,R''(n)} \to M \) be the embedding such that the local coordinates near \( n \) coming from \( \psi_{m_0} \) and \( \phi_n \) have coincident differentials at \( n \). This way, we have a family of symplectic embeddings \( \phi : B_{R''(g)}^g \times B_{R''(g)} \to M \). We require this family to be mobile.

For any function \( \phi : \text{Met} \to \mathbb{R}_{>0} \) which is invariant under the action of symplectomorphisms, one can define another such a function \( \mu_\phi \) satisfying:

- let \( m_i, i = 1, 100 \) be points in \( M \) such that \( B_{g,\mu_\phi}(m_i) \cap B_{g,\mu_\phi}(m_i+1) \neq \emptyset \), \( 1 \leq i < 100 \). Then for all \( i, m_i \in B_{\phi(g)}^g(m_1) \).

Let \( R' = \mu_{R''}, R = \mu_R, \) and \( r = \mu_R \).

Let \( \text{Met}_R \) be the set of pairs \( (g,R) \), where \( g \in \text{Met} \) and we assume \( R < r(g) \). We have a partial order on \( \text{Met}_R \): \((g_1,R_1) \leq (g_2,R_2) \) if \( B_{g_1,R_1}(m) \subset B_{g_2,R_2}(m) \) for all \( m \in M \). Write \( (g_1,R_1) \ll (g_2,R_2) \) if for all \( m_1, m_2 \in M \), \( B_{R_1(g_1)}^g(m_1) \cap B_{R_2(g_2)}^g(m_2) \neq \emptyset \) implies that \( B_{R_2(g_2)}^g(m_2) \subset \text{int} B_{R_1(g_1)}^g(m_1) \). Note that \( \ll \) is not a partial order.

It follows immediately that for all \( i \in \text{Met}_R \) we have \( i \ll i \).

Let \( i_k = (g_k,R_k) \in \text{Met}_R \). We set \( \Phi_{i_1,i_2} := \Phi_{g_1,g_2} \).

12 More on categories

12.1 A poset \( \text{SMet}_R \)

Fix a subset \( \text{Met}_R' \subset \text{Met}_R \). We assume that \( \text{Met}_R' \) is filtered and that for every \( g \in \text{Met}_R' \) there exists an \( i \in \text{Met}_R \) such that \( g \ll i \).

Define a poset \( \text{SMet}_R \) as follows. An element of \( \text{SMet}_R \) is linearly ordered subset \( S \subset \text{Met}_R' \). Write \( S_1 \geq S_2 \) if \( S_1 \subset S_2 \) (sic!).

Let \( \text{SMet}_R' \subset \text{SMet}_R \) consist of all non-empty subsets \( S \).

Let \( \mu : \text{SMet}_R' \to \text{Met}_R' \) be the following monotone map \( \mu(S) = \min(S) \).

Suppose the minimal element of \( S \) is of the form \((g,R)\). We then write \( R_{\min}(S) = R \). Let also \( \text{val} : \text{SMet}_R' \to \mathbb{R}_{>0} \) be given by \( \text{val}(S) = \pi R_{\min}^2 \). Let us also set \( \text{val}() = \infty \).
12.2 Categories $R_q$, $R_0$

Let $\mathcal{C}$ be a category enriched over the ground category $\mathcal{A}$ and $(D \bigoplus)$-closed.

An object of $R_q(\mathcal{C})'$, enriched over $\mathfrak{A}$, is a family $\{X_S\}_{S \in \text{SMet}R}$, where

$$X_S \in \text{Com}(\mathcal{C})(\text{val}(S)).$$

Set

$$\text{Hom}(X, Y) := \prod_{T \subset S} \text{Hom}(i_{\text{val}T\text{val}S}X_T; Y_S) \in \mathfrak{A}.$$ 

The composition is well defined because, given $T, S \in \text{SMet}R$, there are only finitely many $R \in \text{SMet}R$, where $T \subset R \subset S$.

Set $R_q(\mathcal{C}) := DR_q(\mathcal{C})'$.

An object of $R_q(\mathcal{C})$ is a collection $\{X_S\}_{S \in \text{SMet}R}$, where $X_S \in \text{Gr}(\mathcal{C})(\text{val}S)$. Set

$$\text{Hom}(X, Y) := \prod_{T \subset S} \text{Hom}(i_{\text{val}T\text{val}S}X_T; Y_S) \in \mathfrak{A}.$$

Set $R_0(\mathcal{C}) := DR_0(\mathcal{C})'$.

Suppose $\mathcal{C}$ is a ground category, then so are $R_q(\mathcal{C})$, $R_0(\mathcal{C})$. Indeed, we have

$$\bigoplus_{\alpha} X^\alpha_S = \bigoplus_{\alpha} (X^\alpha)_S;$$

$$\prod_{\alpha} X^\alpha_S = \prod_{\alpha} (X^\alpha)_S.$$

Let $X, Y \in R_q(\mathcal{C})$ (resp $R_0(\mathcal{C})$). Define the tensor product by

$$(X \otimes Y)_S := \bigoplus_{S_1, S_2 | S_1 \cup S_2 = S} i_{\text{val}(S_1)\text{val}(S)}X_{S_1} \otimes i_{\text{val}(S_2)\text{val}(S)}X_{S_2}.$$ 

Define the inner hom

$$\text{Hom}(X, Y)_S := \prod_{T \subset S} \text{Hom}(i_{\text{val}T\text{val}S}X_T; Y_S).$$

In both cases above, the differential is determined by those on $X$ and $Y$.

12.3 Category $S_q(\mathcal{C})$ enriched over $R_q(\mathfrak{A})$

An object of $S_q(\mathcal{C})'$ is a collection $\{X^S\}_{S \in \text{SMet}R}$, where

$$X^S \in \text{Quant}(\mathcal{C})(\text{val}(S)).$$

Define $\text{Hom}(X, Y) \in R_q(\mathfrak{A})'$:

$$\text{Hom}(X, Y)_S := \prod_{T \subset S} \text{Hom}(X^S; i_{\text{val}T\text{val}S}Y_T) \in \text{Com}(\mathfrak{A})(\text{val}(S)).$$
The composition is well defined because, given \( T, S \in \text{SMetR} \), there are only finitely many \( R \in \text{SMetR} \), where \( T \subset R \subset S \).

Set \( S_q(C) := D S_q(C)' \).

An object of \( S'_0(C) \) is a collection \( \{X^S\}_{S \in \text{SMetR}} \), where \( X^S \in \text{Classic}(C)\langle \text{val}(S) \rangle \). Define \( \text{Hom}(X, Y) \in R_0(cA) \).

\[
\text{Hom}(X, Y)_S := \prod_{T \subset S} \text{Hom}(X^T, i_{\text{val}(T)}Y^T) \in \text{Gr}(\mathfrak{A})\langle \text{val}(S) \rangle.
\]

Set \( S_0(C) := DS_0(C)' \).

Suppose \( C \) is closed under direct products/sums, then so are \( S_q(C), S_0(C) \). Indeed, we have

\[
(\bigoplus_{\alpha} X^\alpha)^S = \bigoplus_{\alpha} (X^\alpha)^S;
\]
\[
(\prod_{\alpha} X^\alpha)^S = \prod_{\alpha} (X^\alpha)^S.
\]

Suppose \( C, D \) are categories enriched over \( \mathfrak{A} \). Define a functor

\[
\boxtimes S_q(C) \otimes S_q(D) \to S_q((D \bigoplus)C \otimes D).
\]

\[
(X \boxtimes Y)^S := \bigoplus_{S_1, S_2|S_1 \cup S_2 = S} (i_{\text{val}(S_1)\text{val}(S)}X^{S_1} \boxtimes i_{\text{val}(S_2)\text{val}(S)}Y^{S_2}),
\]

where \( \boxtimes \) on the RHS denotes the functor

\[
\boxtimes : \text{Classic}(C)\langle \text{val}(S) \rangle \otimes \text{Classic}(D)\langle \text{val}(S) \rangle \to \text{Classic}((D \bigoplus)C \otimes D)\langle \text{val}(S) \rangle.
\]

The differential is determined by those on \( X \) and \( Y \).

One defines a functor

\[
\boxtimes : S_0(C) \otimes S_0(D) \to S_0((D \bigoplus)C \otimes D),
\]

in a similar way.

It now follows that given an SMC \( C \) with direct sums, \( D \)-closed, such that the direct sums and differentials are compatible with tensor product, we have an induced SMC structure on \( S_q(C), S_0(C) \).

We also have a functor

\[
S_q(C) \otimes R_q(D) \to S_q((D \bigoplus)C \otimes D),
\]

where

\[
(X \otimes U)^S = (X^S \otimes \bigoplus_{T \subset S} U_T, D_S),
\]

where the differential \( D_S \) is induced by the differential on \( U \).
One defines a functor 

$$S_0(C) \otimes R_0(D) \to S_0((D \bigoplus)C \otimes D).$$

In particular, $S_q(C)$ is tensored over $R_q(D)$ if $C$ is tensored over $D$, and likewise for $S_0(C)$.

Let $X$ be a locally compact topological space. Let $D$ be a category. Denote

$$psh(X, D, S_q) := S_q(D \otimes \text{Open}_X), \quad psh(X, D, S_0) := S_0(D \otimes \text{Open}_X).$$

12.3.1 Lemma on truncation

The following Lemma follows from Lemma 2.1.

Lemma 12.1 Let $X \in R_0(\mathfrak{A})$ and suppose that

1) for each $S \in \text{SMet}_R$, we have $\text{gr}^0 X \leq S \in \text{trunc} \mathfrak{A};$

2) $L^0 := \text{proj lim}_{S \in \text{SMet}_R} H^0 \text{gr}^0 X \leq S$ is a free abelian group.

Then $L := \text{Hom}(\text{unit}_{R_0(\mathfrak{A})}; X) \in \text{trunc} \mathfrak{A}$ and $H^0(L) \cong L^0$.

12.4 The category $ZZ(C)$

12.4.1 A partially ordered monoid $ZZ$

Let $ZZ$ be a partially ordered commutative monoid, where $ZZ = \{\infty\} \sqcup \mathbb{Z} \times \mathbb{Z}$, where we endow $\mathbb{Z} \times \mathbb{Z}$ with the discrete poset structure and declare $\infty$ to be the greatest object. The addition on $ZZ$ is the group addition, when restricted on $\mathbb{Z} \times \mathbb{Z}$, and we set $\infty + a = \infty$ for all $a \in \mathbb{Z} \times \mathbb{Z}$.

12.4.2 A category $ZZ(C)$

Let $C$ be a SMC enriched over another SMC $A$ which is $D, \oplus, \prod$-closed.

Let $ZZ(C)'$ be a category, enriched over $\mathfrak{A}$, whose every object is a $ZZ$-graded object $X_\bullet$ in $C$ and we set

$$\text{Hom}(X,Y) = \prod_{s \in \mathbb{Z} \times \mathbb{Z}} (\text{Hom}(X_s,Y_t) \oplus \text{Hom}(X_s,Y_\infty)) \oplus \text{Hom}(X_\infty,Y_\infty).$$

That is,

$$\text{Hom}(X,Y) = \prod_{s \geq t} \text{Hom}(X_s,Y_t).$$

Set

$$(X \otimes Y)_s = \bigoplus_{t_1, t_2 \in ZZ | t_1 + t_2 = s} X_{t_1} \otimes Y_{t_2}.$$
for $a \in \mathbb{Z} \times \mathbb{Z}$, we have
\[
\text{Hom}(X, Y)_a = \prod_{b \in \mathbb{Z} \times \mathbb{Z}} (X_b, Y_{a+b});
\]
\[
\text{Hom}(X, Y)_{\infty} = \prod_{b \in \mathbb{Z}} \text{Hom}(X_b; Y_{\infty}).
\]
Set $\mathbb{Z}Z(C) := D\mathbb{Z}Z(C)'$. The category $\mathbb{Z}Z(C)$ has a tensor structure and is enriched over $\mathbb{Z}Z(A)$.

### 13 Constructing the categories

Throughout this section, the categories are assumed to be enriched over $R_0(\mathbb{Z}Z(A))$ unless otherwise specified.

#### 13.1 The categories $\mathbb{F}, \mathbb{F}_R$

Let $Fr := \bigsqcup_{i \in \text{Met}_R} Fr_i$. Let
\[
\mathbb{F} := \text{psh}(Fr \times E \times Fr \times E, S_q(\mathbb{Z}Z(A)));
\]
\[
\mathbb{F}_R := \text{psh}(Fr \times E, S_q(\mathbb{Z}Z(A))).
\]
Let $h : \mathbb{F} \boxtimes \mathbb{F} \to S_q(\mathbb{Z}Z(A))$ be given by $h(U, V) = \text{unit}$ if $U \cap V \neq \emptyset$ and $h_{\mathbb{F}}(U, V) = 0$ otherwise. Define the functor $h_R : \mathbb{F}_R \boxtimes \mathbb{F}_R \to S_q(\mathbb{Z}Z(A))$ in a similar way.

Denote $E := Fr \times E$. Let
\[
F_{\mathbb{F}}^{[n]} \in \text{psh}(E \times E)^n \times (E \times E), S_q(\mathbb{Z}Z(A))
\]
be given by
\[
F_{\mathbb{F}}^{[n]} := \mathbb{K}_{\Delta[n]} \otimes \text{unit},
\]
where $\Delta^{[n]} \subset (E \times E)^n \times (E \times E)$ consists of all points of the form
\[
((e_1, e_2), (e_2, e_3), \ldots, (e_{n-1}, e_n), (e_1, e_n))
\]
Define
\[
F_{\mathbb{F}_R}^{[n,1]} = F_{\mathbb{F}}^{[n]} \in \text{psh}((E \times E)^n \times E \times E), S_q(\mathbb{Z}Z(A))).
\]
Let $F_{\mathbb{F}_R}^{(n)} \in \text{psh}((E \times E)^{n+1}, S_q(\mathbb{Z}Z(A)))$ be given by
\[
F_{\mathbb{F}}^{(n)} := \mathbb{K}_{\Delta^{(n)}} \otimes \text{unit}[−2N],
\]
where $\Delta^{(n)} \subset (E \times E)^{n+1}$ consists of all points of the form
\[
((e_0, e_1), (e_1, e_2), \ldots, (e_{n-1}, e_n), (e_n, e_0)).
\]
Via the functor $\text{red} : S_q \to S_0$, the category $\mathbb{F}$ is also defined over $S_0$. 

13.2 The category $\mathcal{G}$

Set $\Phi := \bigsqcup_{i,j \in \text{MetR}} \Phi_{ij}$. Let $K_n^{[n]} \subset \Phi \times \Phi$ be the union

$$(K_n^{[n]})^{ab} := \bigsqcup_{i_0, i_1, \ldots, i_n \in \text{MetR}} K^{ab}(i_0, i_1, \ldots, i_n);$$

$$(K_n^{(n)})^{ab} := \bigsqcup_{i_0, i_1, \ldots, i_n \in \text{MetR}} (K^{\text{cyc}})^{ab}(i_0, i_1, \ldots, i_n);$$

Let $\Delta_n^E \subset (E \times E)^n \times (E \times E)$ consist of all points of the form $((e_0, e_1), (e_1, e_2), \ldots, (e_{n-1}, e_n); (e_0, e_n))$ and $\Delta_n^{(n)} \subset (E \times E)^{n+1}$ consist of all points of the form $((e_0, e_1), (e_1, e_2), \ldots, (e_{n-1}, e_n), (e_n, e_0))$.

Set

$$\mathcal{G} := \text{psh}(\Phi \times E \times E, S^0(\mathbb{Z}\mathbb{Z}(\mathfrak{A}))).$$

Set $h_G : \mathcal{G} \otimes \mathcal{G} \to S^0(\mathbb{Z}\mathbb{Z}(\mathfrak{A}))$ be defined by the same rule as $h_F$. Let

$$F_n^{[n]} \in \text{psh}((\Phi \times E \times E)^{n+1}, S^0(\mathbb{Z}\mathbb{Z}(\mathfrak{A})))$$

be given by

$$F_n^{[n]} := \bigoplus_{a,b} K^{[n] \times \Delta_n^E} \otimes (a, b) \otimes \text{unit}S_0;$$

let

$$F_n^{(n)} \in \text{psh}(\text{Open}_{(\Phi \times E \times E)}^{n+1}, S_0)$$

be given by

$$F_n^{(n)} := \bigoplus_{a,b} K^{(n) \times \Delta_n^{(n)}} \otimes (a, b) \otimes \text{unit}S_0;$$

13.2.1 Quasi-contractibility

The quasi-contractibility of both $F_F$, $F_G$ can be easily verified.

13.3 Connecting $\mathcal{G}$ and $\mathcal{F}$

13.3.1 Constructing an object $\Xi \in \text{psh}(\Phi \times E \times E \times \text{Fr} \times \text{Fr} \times E \times E, S^0(\mathbb{Z}\mathbb{Z}(\mathfrak{A})))$

Let $\pi : \Phi \to \text{Fr} \times \text{Fr} \times \text{Fr}$ be the projection. Let

$$\sigma : \Phi \times E \times E \times \text{Fr} \times \text{Fr} \times E \times E \to \text{Fr} \times \text{Fr} \times E \times E \times \text{Fr} \times \text{Fr} \times \text{Fr} \times E \times E$$

be the induced map. Let $pr : \text{Fr} \to \text{MetR}$ be the projection. Let

$$i : \text{Fr} \times \text{Fr} \times E \times E \times \text{Fr} \times \text{Fr} \times E \times E \to \text{Fr} \times \text{Fr} \times E \times E \times \text{Fr} \times \text{Fr} \times E \times E$$

be the closed embedding.
Let
\[ j : \mathbb{F} \times \text{MetR} \times \mathbb{F} \times \mathbb{E} \times \mathbb{F} \times \text{MetR} \times \mathbb{F} \times \mathbb{E} \times \mathbb{E} \times \mathbb{F} \times \mathbb{F} \times \mathbb{E} \times \mathbb{E} \to \mathbb{F} \times \mathbb{F} \times \mathbb{E} \times \mathbb{E} \times \mathbb{F} \times \mathbb{F} \times \mathbb{E} \times \mathbb{E} \]
be the closed embedding defined as follows:
\[ j(f_1, f_2, v_1, v_2, f'_1, f'_2, v'_1, v'_2) = (f_1, f'_1, v_1, v'_1, f_2, f'_2, v_2, v'_2). \]

We therefore have the following diagram
\[ \Phi \times \mathbb{E} \times \mathbb{E} \times \mathbb{F} \times \mathbb{F} \times \mathbb{E} \times \mathbb{E} \]
\[ \downarrow \sigma \]
\[ \mathbb{F} \times \mathbb{E} \times \mathbb{F} \times \mathbb{F} \times \mathbb{E} \times \mathbb{E} \]
\[ \downarrow i \]
\[ \mathbb{F} \times \text{MetR} \times \mathbb{F} \times \mathbb{E} \times \mathbb{E} \times \mathbb{F} \times \mathbb{F} \times \mathbb{E} \times \mathbb{E} \]

Let \( i = (g, r) \in \text{MetR} \) let \( R = \text{rad} g/2 \). Let \( V_{ii} \subset \mathbb{F} \times \mathbb{F} \) consist of all pairs \((f_1, f_2)\) where \( B(g, r)(f_1) \subset B(g, R)(f_2) \). We then have a family of symplectic embeddings \( V_{ii} \times B_r \to B_R \). Let \( D : V_{ii} \to \text{Sp}(2N) \) be the differential map at \( 0 \in B_r \). Let \( U_{ii} \subset V_{ii} \) be \( D^{-1} e_{\text{Sp}(2N)} \).

We now get a graded family of symplectic embeddings \( U_{ii} \times B_r \to B_R \)

Let \( P_i \in \text{sh}_q(\mathbb{F} \times \mathbb{R}^N \times \mathbb{R}^N) \) be the object determined by this family.

Let \( i : U_{ii} \to \mathbb{F} \times \mathbb{F} \); be the embedding.

Let
\[ \mathcal{P}_i := i^! \mathcal{P}_i \in \text{sh}_q(\mathbb{F} \times \mathbb{F} \times \mathbb{E} \times \mathbb{E}) \]
The objects \( \mathcal{P}_i \) give rise to an object Let \( \mathcal{P}_i \in \text{sh}_q(\mathbb{F} \times \text{MetR} \times \mathbb{F} \times \mathbb{E} \times \mathbb{E}) \).

Set
\[ \Xi' := \sigma^{-1} i^{-1} j^!(\mathcal{P}_i \boxtimes \mathcal{P}_i) \]

Set
\[ \Xi := \Xi' \otimes \text{units}_{S_0} \in \text{psh}(\Phi \times \mathbb{E} \times \mathbb{F} \times \mathbb{F} \times \mathbb{E} \times \mathbb{E}, S_0(\mathbb{Z}\mathcal{A})). \]

13.3.2 Quasi-Contractibility

Let \( t \) be a planar tree with \( n \) inputs. One sees that we have a homotopy equivalence \( F_G(t) \sim F_G^{[n]} \). Similarly, for a cyclic tree with \( n + 1 \) inputs we have a homotopy equivalence \( F_G(t) \sim F_G^{(n)} \).

The problem now reduces to showing the quasi-contraction of
\[ \text{Hom}(F_G^{[n]} \circ L^{S_0}; F_G^{[n]} \circ L); \quad (52) \]

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Let us first show the quasi-contractibility of the object in (52). Consider the following diagram

\[
\begin{array}{c}
(Fr \times Fr \times E \times E)^{2n} \\
\sigma \downarrow \\
(Fr \times Fr \times E \times E)^n \times (Fr \times Fr \times E \times E)^n \\
\downarrow i \\
K[n] \times E^{n+1} \times (Fr \times Fr \times E \times E)^n \rightarrow Fr_{\times M} \times Fr \times E \times E \times (Fr \times Fr \times E \times E) \\
\downarrow \tilde{p} \\
\Phi \times E \times (Fr \times Fr \times E \times E)^n \rightarrow Fr \times Fr \times E \times E \times (Fr \times Fr \times E \times E) \\
\downarrow j \downarrow p \\
\Phi \times E \times (Fr \times Fr \times E \times E)^n \rightarrow Fr \times Fr \times E \times E \\
\downarrow \tilde{p} \downarrow \rho \\
\Phi \times E \times Fr \times Fr \times E \times E \rightarrow Fr \times Fr \times E \times E \times Fr \times Fr \times E \times E
\end{array}
\]

Here:

- \(\sigma\) is given by
  \[
  \sigma((f_1, f_2, e_1, e_2), (f_3, f_4, e_3, e_4), \ldots, (f_{2n-1}, f_{2n}, e_{2n-1}, e_{2n})) \\
  \times ((f'_1, f'_2, e'_1, e'_2), (f'_3, f'_4, e'_3, e'_4), \ldots, (f'_{2n-1}, f'_{2n}, e'_{2n-1}, e'_{2n}))
  \]
  \[
  = ((f_1, f'_1, e_1, e'_1), (f_2, f'_2, e_2, e'_2), \ldots, (f_{2n}, f'_{2n}, e_{2n}, e'_{2n})).
  \]
  \(i = i_1 \times \text{Id}, \quad i_1 : Fr_{\times M} \times E^{n+1} \rightarrow (Fr \times Fr \times E \times E)^n\)

is given by

\[
i_1(f_0, f_1, f_2, \ldots, f_{n-1}, f_n, e_0, e_1, \ldots, e_n) = (f_0, f_1, f_2, f_3, f_4, \ldots, f_n, f_{n-1}, f_n, e_0, e_1, \ldots, e_{n-1}, e_n).
\]

- The map \(d\) is induced by the covering map \(K[n] \rightarrow Fr_{\times M}^{n+1}\);
- set \(\tilde{i} := i \circ d\);
- The map \(p\) is induced by the projection onto the marginal factors \(Fr_{\times M}^{n+1} \rightarrow Fr \times Fr; E^{n+1} \rightarrow E \times E\);
- The map \(\tilde{p}\) is defined in a similar way to \(p\);
- The map \(a\) is induced by the covering map \(\Phi \rightarrow Fr \times Fr\);
- The maps \(j, \tilde{j}\) are induced by the map
  \[
  j_1 : (Fr \times E)^{n+1} \rightarrow (Fr \times Fr \times E \times E)^n,
  \]

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where

\[ j_1(f_0, e_0, f_1, e_1, \ldots, f_n, e_n) = (f_0, e_0, f_1, e_1, \ldots, f_{n-1}, e_{n-1}, f_n, e_n) \].

— The map \( b \) is induced by the covering map \( b : \Phi \times \text{Fr} \times_M \text{Fr} \).

— The map \( \rho \) is induced by the closed embedding \( \text{Fr} \times_M \text{Fr} \to \text{Fr} \times \text{Fr} \) and by the projection onto the marginal factors

\[ (\text{Fr} \times E)^{n+1} \to \text{Fr} \times \text{Fr} \times E \times E. \]

The map \( \tilde{\rho} \) is induced by the above projection;

— the map \( c \) is a through map

\[ \Phi \to \text{Fr} \times_M \text{Fr} \to \text{Fr} \times \text{Fr}, \]

where the left arrow is the covering map and the left arrow is the closed embedding.

Let now

\[ \mathcal{P}_{2n} := \mathbb{P}^{[2n]} \in \text{sh}_q((\text{Fr} \times \text{Fr} \times E \times E)^{2n}). \]

Let

\[ \mathcal{P}_2 := \mathbb{P} \Box \mathbb{P} \in \text{sh}_q((\text{Fr} \times \text{Fr} \times E \times E)^2). \]

We then rewrite (52) as follows:

\[
\text{Hom}(\tilde{p}_i\tilde{i}^{-1}(\sigma_1^{-1}P_{2n}); (\tilde{j})^{-1}\rho^{-1}P_2) \sim \text{Hom}(\tilde{p}_i\tilde{i}^{-1}(\sigma_1^{-1}P_{2n}); (\tilde{j})^{-1}\rho^{-1}P_2)
\]

\[
\sim \text{Hom}(\tilde{p}_i\tilde{i}^{-1}(\sigma_1^{-1}P_{2n}); a^{-1}j_1\rho^{-1}P_2) \sim \text{Hom}(a_i\tilde{p}_i\tilde{i}^{-1}\sigma_1^{-1}P_{2n}; j_1\rho^{-1}P_2)
\]

\[
\sim \text{Hom}(p_i d_i d_i^{-1} \sigma_1^{-1}P_{2n}; a^{-1}j_1\rho^{-1}P_2) \sim \text{Hom}(p_i d_i d_i^{-1} \sigma_1^{-1}P_{2n}; (\tilde{j})^{-1}\rho^{-1}P_2)
\]

\[
\sim \text{Hom}(j_1^{-1}p_i d_i d_i^{-1} \sigma_1^{-1}P_{2n} \otimes \lambda; j_1\rho^{-1}P_2),
\]

where

\[ \lambda = d_i \psi_{K[n]} \times E^{n+1} \times (\text{Fr} \times \text{Fr} \times E)^n. \]

Let us now consider the following diagram:

\[
\begin{array}{c}
\text{(Fr x E x Fr x E)}^{2n} \\
\sigma \\
\text{(Fr x E)}^{2n} \times \text{(Fr x E)}^{2n} \\
i \\
\text{Fr}^{n+1} \times E^{n+1} \times \text{(Fr x E)}^{2n} \xrightarrow{k_1} \text{Fr}^{n+1} \times E^{n+1} \times \text{(Fr x E)}^{n+1} \xrightarrow{l_1} (\text{Fr} \times E)^{n+1} \times (\text{Fr} \times E)^{n+1} \\
\downarrow p_1 \\
\text{Fr}^{n+1} \times E^2 \times \text{(Fr x E)}^{2n} \xrightarrow{k_2} \text{Fr}^{n+1} \times E^2 \times (\text{Fr} \times E)^{n+1} \xrightarrow{l_2} \text{Fr}^{n+1} \times E^2 \times (\text{Fr} \times E)^{n+1} \\
\downarrow p_2 \\
\text{Fr} \times \text{Fr} \times E^2 \times \text{(Fr x E)}^{2n} \xrightarrow{j} \text{Fr} \times \text{Fr} \times E^2 \times (\text{Fr} \times E)^{n+1} \\
\downarrow \rho \\
\text{Fr} \times \text{Fr} \times E \times \text{Fr} \times \text{Fr} \times E \times E
\end{array}
\]
Here:

— $\sigma$ and $i$ are as above;

— $p_1, q_1, r$ are induced by the projection onto the marginal factors $E^{n+1} \to E^2$;

— $p_2, q_2$ are induced by the projection onto the marginal factors $Fr \times Fr \to Fr \times Fr$.

—the maps $k_1, k_2, j$ is induced by the following map

$$k' : (Fr \times E)^{n+1} \to (Fr \times E \times Fr \times E)^{2n} :$$

$$k'(f_0, e_0, f_1, e_1, \ldots, f_n, e_n) = (f_0, e_0, f_1, e_1, f_1, e_2, f_2, e_2, \ldots, f_{n-1}, e_{n-1}, f_n, e_n).$$

The map $j$ coincides with that on the previous diagram.

— The maps $l_1, l_2$ are induced by the closed embedding $Fr \times Fr \to Fr \times Fr$

— set $n = k' \times k'$, set $m = nl_1$.

We have $p = p_2p_1$. As all squares in the diagram are Cartesian, we have:

$$j^{-1}p_1(\lambda \otimes i^{-1}\sigma^{-1}P_{2n}) \sim j^{-1}p_2(\lambda \otimes p_1i^{-1}) \sim q_2!(\lambda \otimes k_2^{-1}p_1i^{-1}\sigma^{-1}P_{2n}) \sim q_2!(\lambda \otimes q_1^{-1}k_1^{-1}i^{-1}\sigma^{-1}P_{2n})$$

$$\sim q_2!(\lambda \otimes q_1^{-1}m^{-1}\sigma^{-1}P_{2n}) = q_2!(\lambda \otimes q_1^{-1}n^{-1}\sigma^{-1}P_{2n}) \sim q_2!(\lambda \otimes (l_2^{-1}r_1n^{-1}\sigma^{-1}P_{2n}))$$

Let

$$n_0 : Fr \times E \times Fr \times E \to (Fr \times E \times Fr \times E)^2$$

be the diagonal map

$$n_0(f_1, e_1, f_2, e_2) = ((f_1, e_1, f_2, e_2), (f_1, e_1, f_2, e_2)).$$

Let

$$r_0 : Fr \times E \times Fr \times E \to Fr \times Fr \times E$$

be the projection along the first factor of $E$.

Let

$$\tau : (Fr \times E)^{n+1} \times (Fr \times E)^{n+1} \to (Fr \times E \times Fr \times E)^{n+1};$$

be given by

$$\tau((f_0, e_0, f_1, e_1, \ldots, f_n, e_n), (f_0', e_0', f_1', e_1', \ldots, f_n', e_n'))$$

$$\to ((f_0, e_0, f_0', e_0'), (f_1, e_1, f_1', e_1'), \ldots, (f_n, e_n, f_n', e_n')).$$

We have

$$\sigma n = (Id_{Fr \times E \times Fr \times E} \times n_0^{x_{n-1}} \times Id_{Fr \times E \times Fr \times E}) \tau$$

Let

$$\tau_2 : Fr \times E \times Fr \times E \times (Fr \times Fr \times E)^{n-1} \times Fr \times E \times Fr \times E \to Fr^{n+1} \times E^2 \times (Fr \times E)^{n+1}$$

be given by

$$\tau_2(f_0, e_0, f_0', e_0'), (f_1, f_1', e_1'), (f_2, f_2', e_2'), \ldots, (f_{n-1}, f_{n-1}', e_{n-1}'), (f_n, e_n, f_n', e_n'))$$

$$= ((f_0, f_1, \ldots, f_n), (e_0, e_n), (f_0', e_0', f_1', e_1', \ldots, f_{n+1}', e_{n+1}')).$$
We now have
\[ r = \tau_2((\text{Id}_{Fr} \times E \times Fr \times E) \times \tau_0^{-1} \times \text{Id}_{Fr} \times E \times Fr \times E)\tau^{-1} \]
so that
\[ r_n^{-1} = \tau_2((\text{Id}_{Fr} \times E \times Fr \times E) \boxtimes (\tau_0^{-1} n_0^{-1} \times \text{Id}_{Fr} \times E \times Fr \times E)). \]
Therefore,
\[ r_n^{-1} P_{2n} \sim \tau_2(\mathbb{P} \boxtimes (\mathbb{P} \boxtimes \mathbb{P})) \boxtimes \mathbb{P}). \]
Let \( P_{\text{MetR}} \in \text{sh}_q(\text{Fr} \times E \times E) \) be defined as follows:
\[ P_{\text{MetR}}|_{Fr \times E} := k_{Fr} \boxtimes P_{r(i)}. \]

Let \( \delta : \text{Fr} \times E \rightarrow \text{Fr} \times E \times E \) be the diagonal embedding. Let \( V_{ii}, U_{ii} \subset \text{Fr}_i \times \text{Fr}_i \) be as in Sec. 13.3.
Let
\[ U := \bigcup_{i \in \text{MetR}} U_{ii} \subset \bigcup_{i \in \text{MetR}} \text{Fr}_i \times \text{Fr}_i \subset \text{Fr} \times \text{Fr}. \]
Let \( p_1 : \text{Fr} \times \text{Fr} \times E \rightarrow E \); \( p_2 : \text{Fr} \times \text{Fr} \times E \rightarrow \text{Fr} \times \text{Fr} \) be projections, let \( \delta : \text{Fr} \times E \rightarrow \text{Fr} \times E \times E \) be the diagonal embedding. We have
\[ r_0^{-1} P_{\text{2n}} \sim p_1^{-1} \delta^{-1} P_{\text{MetR}} \boxtimes p_2^{-1} k_U. \]
Let \( \kappa : (\text{Fr} \times \text{Fr})^{n+1} \rightarrow \text{Fr}^{n+1} \times \text{Fr}^{n+1} \) be the natural map.
Let \( A := l_2^{-1}(\kappa(U^{n+1}) \times E^2 \times E^{n+1}) \). It follows that \( q_2|_A : A \rightarrow \text{Fr} \times_M \text{Fr} \times E^2 \times (\text{Fr} \times E)^{n+1} ; \)
\( l_2|_A : A \rightarrow \text{Fr}^{n+1} \times E^2 \times (\text{Fr} \times E)^{n+1} \) are embeddings of a locally closed subset.
Denote \( i_A := q_2|_A \). Let us decompose
\[ \text{Fr} \times_M \text{Fr} \times E^2 \times (\text{Fr} \times E)^{n+1} = \text{Fr} \times_M \text{Fr} \times E^2 \times \text{Fr}^2 \times E^2 \times (\text{Fr} \times E)^{n-1} \]
Let
\[ i : \text{Fr} \times_M \text{Fr} \times E^2 \times \text{Fr}^2 \times E^2 \rightarrow \text{Fr} \times \text{Fr} \times E^2 \times \text{Fr} \times \text{Fr} \times E^2 \]
be the embedding. Let \( q : A \rightarrow \text{Fr} \times_M^{n+1} \) be the projection. We then have
\[ q_2((\lambda \otimes (l_2^{-1} r_n^{-1}) \mathbb{P}_{2n}) \sim i_A((\lambda^{-1} (\mathbb{P} \boxtimes \mathbb{P}) \boxtimes (\delta^{-1} P_{\text{MetR}})) \otimes q^{-1} \lambda). \]

Next, we have \( \rho^{-1} P_2 \sim (\mathbb{P} \boxtimes \mathbb{P}) \boxtimes k \).
Let \( B := \rho^{-1}(\mathcal{U} \times E^2 \times \mathcal{U} \times E^2) \). Observe that \( i_A(A) \subset B \) is an open subset. Therefore, we have
\[ \text{Hom}(i_A((\mathbb{P} \boxtimes \mathbb{P}) \boxtimes (\delta^{-1} P_{\text{MetR}}) \otimes q^{-1} \lambda); \rho^{-1} P_2) \sim \text{Hom}(i_A((\mathbb{P} \boxtimes \mathbb{P}) \boxtimes (\delta^{-1} P_{\text{MetR}})) \otimes q^{-1} \lambda; i_A^{-1} \rho^{-1}(\mathbb{P} \boxtimes \mathbb{P})) \]
The convolution with \( i_A^{-1} \rho^{-1}(\mathbb{Q} \boxtimes \mathbb{Q}) \) gives rise to a homotopy equivalence of the above hom with
\[ \text{Hom}(i_A^{-1} \Delta^{-1}(P_{\text{MetR}} \boxtimes P_{\text{MetR}}) \boxtimes (\delta^{-1} P_{\text{MetR}} \boxtimes (\delta^{-1} P_{\text{MetR}})) \boxtimes (8^{-1} \mathbb{P}_{\text{Fr} \times E}) \boxtimes (\mathbb{P}_{\text{Fr} \times E})^2(n-1))) \]
\[ \sim \text{Hom}(i_A^{-1} \Delta^{-1}(P_{\text{MetR}} \boxtimes P_{\text{MetR}}) \boxtimes (\delta^{-1} P_{\text{MetR}} \boxtimes (\delta^{-1} P_{\text{MetR}})) \boxtimes (\Delta^{-1}(\mathbb{K}_{\text{Fr} \times E} \times \Delta_{E,0}) \boxtimes (\mathbb{K}_{\text{Fr} \times E})^2(n-1))). \]
where
\[ \Delta : F \times_M F \times E \times E \times F \times F \times E \times E \rightarrow F \times F \times E \times E \times F \times F \times E \times E, \]
\[ \Delta_E \subset E^2 \text{ is the diagonal and the map } (*) \text{ is induced by the universal map} \]
\[ P_{\text{Met}} \rightarrow k_{F \times \Delta_E,0}. \]
The universality of this map implies that (*) is a homotopy equivalence. The latter hom is homotopy equivalent to
\[ \text{Hom}(\mathcal{cut}_{\leq 0} \Delta^{-1}(P_{\text{Met}} \boxtimes P_{\text{Met}}) \boxtimes (\delta^{-1} P_{\text{Met}} \boxtimes (\delta^{-1} P_{\text{Met}})^{n-1} ; i_A^{-1}(\Delta^{-1}(K_{F \times \Delta_E} \boxtimes (K_{F \times E})^\otimes(n-1)))) \]
The resulting space is pseudo-contractible, as both arguments of the hom are constant sheaves on locally closed constructible sub-sets of \( A \).
Consider now the case \([53]\).

\[ (F \times F \times E \times E)^{2n+2} \]
\[ K^{(n)} \times E^{n+1} \times (F \times F \times E \times E)^{n+1} \]
\[ (F \times F \times E \times E)^{n+1} \]
\[ (F \times E)^{n+1} \]
Let \( \delta : K^{(n)} \rightarrow F \times_M^{n+1} \) be the projection. Let \( \lambda^\text{cyc} := \delta[k_{K^{(n)}}]. \)
The space in \([53]\) can be rewritten as follows:
\[ \text{Hom}(\bar{p})^{-1}(i^{-1} P_{2n+2} ; j[k_{F \times E})^{n+1,0}(-2N)] \sim \text{Hom}(j^{-1} p d \delta^{-1} i^{-1}(P_{2n+2})[2N] ; k_{(F \times E)^{n+1,0}}) \]
\[ \sim \text{Hom}(j^{-1} p t([\lambda^\text{cyc} \boxtimes K^{(n)} \times (F \times F \times E \times E)^{n+1}) \otimes i^{-1} P_{2n+2})[2N] ; k_{((F \times E)^{n+1,0})}, \quad (54) \]
Let us now consider the following diagram
As every square in this diagram is Cartesian, and \( p = p_2p_1 \), we have

\[
j^{-1}p_1((\lambda^\text{cyc} \boxtimes K_{E^n+1 \times (Fr \times Fr \times E)^{n+1}}) \otimes i^{-1}\mathcal{P}_{2n+2})
\]

\[
\sim j^{-1}p_2p_1((\lambda^\text{cyc} \boxtimes K_{E^n+1 \times (Fr \times Fr \times E)^{n+1}}) \otimes i^{-1}\mathcal{P}_{2n+2})
\]

\[
\sim q_2k_2^{-1}p_1((\lambda^\text{cyc} \boxtimes K_{E^n+1 \times (Fr \times Fr \times E)^{n+1}}) \otimes i^{-1}\mathcal{P}_{2n+2})
\]

\[
\sim q_2q_1k_1^{-1}((\lambda^\text{cyc} \boxtimes K_{E^n+1 \times (Fr \times E)^{n+1}}) \otimes i^{-1}\mathcal{P}_{2n+2})
\]

\[
\sim q_2q_1((\lambda^\text{cyc} \boxtimes K_{E^n+1 \times (Fr \times E)^{n+1}}) \otimes l^{-1}n^{-1}\mathcal{P}_{2n+2})
\]

\[
\sim q_2((\lambda^\text{cyc} \boxtimes K_{(Fr \times E)^{n+1}}) \otimes l_2^{-1}r_1n^{-1}\mathcal{P}_{2n+2})
\]

We can now continue the chain of homotopy equivalences (54)

\[
\text{Hom}(j^{-1}p_1((\lambda^\text{cyc} \boxtimes K_{E^n+1 \times (Fr \times Fr \times E)^{n+1}}) \otimes i^{-1}\mathcal{P}_{2n+2})[2N]; K_{[[Fr \times E]^{n+1}]})
\]

\[
\sim \text{Hom}(q_2((\lambda^\text{cyc} \boxtimes K_{(Fr \times E)^{n+1}}) \otimes l_2^{-1}r_1n^{-1}\mathcal{P}_{2n+2})[2N]; K_{[[Fr \times E]^{n+1}]})
\]

\[
\sim \text{Hom}((\lambda^\text{cyc} \boxtimes K_{(Fr \times E)^{n+1}}) \otimes l_2^{-1}r_1n^{-1}\mathcal{P}_{2n+2}; q_2^1 K_{[[Fr \times E]^{n+1}]})[-2N])
\]

Similar to the previous proof, we have

\[
r_1n^{-1}\mathcal{P}_{2n+2} \sim \tau^{-1}(r_0n_0^{-1}\mathcal{P}_2)^{\otimes(n+1)}.
\]

where

\[
\tau : Fr^{n+1} \times (Fr \times E)^{n+1} \to (Fr \times Fr \times E)^{n+1}
\]

is a permutation.

Let \( \mathcal{W}_1 := \tau^{-1}(U \times E)^{n+1} \); \( \mathcal{W}_2 := l_2^{-1}\mathcal{W}_1 \); \( \mathcal{W}_3 := q_2(\mathcal{W}_2) \). Let \( l_2' : \mathcal{W}_2 \to \mathcal{W}_1 \); \( q_2' : \mathcal{W}_2 \to \mathcal{W}_1 \) be the induced maps. It follows that \( q_2' \) is a smooth fibration whose fiber is diffeomorphic to \( R^{2n} \) which implies \( (q_2')^1[-2N] \sim (q_2)^{-1} \). We have

\[
\text{Hom}((\lambda^\text{cyc} \boxtimes K_{(Fr \times E)^{n+1}}) \otimes l_2^{-1}r_1n^{-1}\mathcal{P}_{2n+2}; q_2^1 K_{[[Fr \times E]^{n+1}]})[-2N])
\]

\[
\sim \text{Hom}((\lambda^\text{cyc} \boxtimes K_{(Fr \times E)^{n+1}}) |_{\mathcal{W}_2} \otimes (l_2')^{-1}(\tau^{-1}(r_0n_0^{-1}\mathcal{P}_2)^{\otimes(n+1)}|_{\mathcal{W}_1}); (q_2')^1 K_{[\mathcal{W}_2,0]}[-2N])
\]

\[
\sim \text{Hom}((\lambda^\text{cyc} \boxtimes K_{(Fr \times E)^{n+1}}) |_{\mathcal{W}_2} \otimes (l_2')^{-1}(\tau^{-1}(r_0n_0^{-1}\mathcal{P}_2)^{\otimes(n+1)}|_{\mathcal{W}_1}); (q_2')^{-1} K_{[\mathcal{W}_2,0]})
\]

The statement now follows.

### 13.4 The category \( \mathbb{H} \)

Let

\[
\mathbb{H} := \text{psh}(\Sigma, S_0(\text{ZZ}(\mathfrak{A}))).
\]

Let

\[
P_{\mathbb{H}}^{[n]} := \bigoplus_{a,b} K_{[R^\Sigma, [a,b], a]} \otimes (a, b) \otimes \text{units}_{q} [2b]
\]

\[
P_{\mathbb{H}}^{(n)} := \bigoplus_{a,b} K_{[R^\Sigma, (a,b), a]} \otimes (a, b) \otimes \text{units}_{q} [2b]
\]

The quasi-contractibility is easy to check
13.5 Connecting $\mathbb{H}$ and $\mathbb{G}$

13.5.1 The objects $\mathcal{P}_{ki}$

Let $k, i \in \text{MetR}$, $k \leq i$. Let $k = (g_k; r_k); i = (g_i, r_i)$. We have a family of symplectic embeddings

$$\mathbf{F} r_k \times_M \mathbf{F} r_i \times B_{r_k} \to B_{r_i}. \quad (55)$$

The differential at $0 \in B_{r_k}$ gives rise to a map $\mathbf{F} r_k \times_M \mathbf{F} r_i \to \text{Sp}(2N)$. Let $\mathbf{F} r_{ki} \subset \mathbf{F} r_k \times_M \mathbf{F} r_i$ be the pre-image of the set of all symmetric matrices in $\text{Sp}(2N)$. The subset $\mathbf{F} r_{ki}$ projects diffeomorphically onto both $\mathbf{F} r_k$ and $\mathbf{F} r_i$. This way we get identifications

$$b_{ki} : \mathbf{F} r_k \rightarrow \mathbf{F} r_i; \quad c_{ki} : \mathbf{F} r_k \rightarrow \mathbf{F} r_{ki}$$

The restriction of $(55)$ onto $\mathbf{F} r_{ki} \cong \mathbf{F} r_k$ gives rise to a graded family of symplectic embeddings

$$\phi_{ki} : \mathbf{F} r_k \times B_{r_k} \rightarrow B_{r_i}.$$ 

Whence objects $\mathcal{P}_{ki} \in \text{sh}_q(\mathbf{F} r_k \times E \times E)$.

Let $s : \mathbf{F} r_k \times E \times E \to \mathbf{F} r_i \times E \times E$ be given by $s(f, e_1, e_2) = (b_{ki}(f), e_2, e_1)$.

Let $Q_{ik} := s_i \mathcal{P}_{ki} \in \text{sh}_q(\mathbf{F} r_i \times E \times E)$

Denote by $D_{ki}' : \mathbf{F} r_k \to \text{Sp}(2N)$ the differential of $\phi_{ki}$ at $0 \in B_{r_k}$. $D_{ki}'$ takes values in the space of symmetric matrices, therefore, we have a lifting $D_{ki} : \mathbf{F} r_k \rightarrow \text{Sp}(2N)$.

Let $k_1 \leq k_2 \leq i$. Let $b_{k_1 k_2}^i : \mathbf{F} r_{k_1} \rightarrow \mathbf{F} r_{k_2}$ be given by

$$b_{k_1 k_2}^i = b_{k_2}^{-1} b_{k_1}.$$ 

We have a unique graded family of symplectic embeddings

$$\phi_{k_1 k_2}^i : \mathbf{F} r_{k_1} \times B_{r_{k_1}} \rightarrow B_{r_{k_2}}$$

satisfying

$$\phi_{k_2 i}(b_{k_1 k_2}^i(f), \phi_{k_1 k_2}^i(f, b)) = \phi_{k_1 i}(f, b).$$

The family $\phi_{k_1 k_2}^i$ defines objects

$$\mathcal{P}_{k_1 k_2}^i \in \text{sh}_q(\mathbf{F} r_{k_1} \times E \times E); \quad Q_{k_2 k_1}^i \in \text{sh}_q(\mathbf{F} r_{k_2} \times E \times E).$$

Let

$$p_E : \mathbf{F} r_{k_1} \times E \times E \times E \to \mathbf{F} r_{k_1} \times E \times E$$

be the projection, where $p_E(f, e_1, e_2, e_3) = (f, e_1, e_3)$; let

$$p : \mathbf{F} r_{k_1} \times E \times E \times E \to \mathbf{F} r_{k_1} \times E \times E,$$

$p(f, e_1, e_2, e_3) = (f, e_1, e_2)$;

let

$$q : \mathbf{F} r_{k_1} \times E \times E \times E \to \mathbf{F} r_{k_1} \times E \times E,$$
$q(f, e_1, e_2, e_3) = (b_{k_1k_2}^i(f); e_2, e_3)$.

For $U, V \in \text{sh}_q(\text{Fr}_{k_1} \times E \times E)$, denote

$$U \circ_{E}^{\text{Fr}_{k_1}} V := p_{E!}(p^{-1}U \otimes q^{-1}V).$$

Let also $\beta_{k_1k_2}^i : \text{Fr}_{k_1} \times E \times E \to \text{Fr}_{k_2} \times E \times E$ be the map induced by $b_{k_1k_2}^i$. We now have

$$\mathcal{P}_{k_1i} \sim (\beta_{k_1k_2}^i)^{-1} \mathcal{P}_{k_2i} \circ_{E}^{\text{Fr}_{k_1}} \mathcal{P}_{k_1k_2}.$$  

**13.5.2 The objects $\Gamma_{ij}^k$**

Let $i, j, k \in \text{Met}_R$, $i, j \geq k$. Define an object

$$\Gamma_{ij}^k \in \text{sh}_{\pi_{k_1}}(\Phi_{ij} \times E \times E).$$

Let $\sigma : \Phi_{ij} \to \text{Sp}(2N)$ be the structure map. Let

$$p_i : \Phi_{ij} \to \text{Fr}_i, \quad p_j : \Phi_{ij} \to \text{Fr}_j;$$

$$\pi_i : \Phi_{ij} \times E \times E \to \text{Fr}_i \times E \times E; \quad \pi_j : \Phi_{ij} \times E \times E \to \text{Fr}_j \times E \times E$$

be the projections.

Define a map $\tau : \Phi_{ij} \to \text{Sp}(2N)$ by the condition

$$\sigma(f)D_{ki}(b_{ki}^{-1}p_i(f)) = D_{kj}(b_{kj}^{-1}p_j(f))\tau(f).$$

Let

$$t : \Phi_{ij} \times E \times E \to \text{Sp}(2N) \times E \times E$$

be the map induced by $\tau$.

Set

$$\Gamma_{ij}^k := \pi_i^{-1}\mathcal{P}_{ki} \circ_{E}^{\Phi_{ij}} \Gamma_{rk} \circ_{E}^{\Phi_{ij}} (\beta_{kj})!\mathcal{Q}_{jk} \in \text{sh}_{\pi_{k_1}}(\Phi_{ij} \times E \times E);$$

$$\gamma_{ij}^k := \pi_i^{-1}\mathcal{P}_{ki} \circ_{E}^{\Phi_{ij}} \gamma_{rk} \circ_{E}^{\Phi_{ij}} (\beta_{kj})!\mathcal{Q}_{jk} \in \text{sh}_{q}(\Phi_{ij} \times E \times E),$$

where $\beta_{kj} : \text{Fr}_k \times E \times E \to \text{Fr}_i \times E \times E$ is induced by $b_{kj}$.

Let $k_1 \leq k_2 \leq i, j$ and consider

$$G^i_{ij}(k_2, k_1) := \text{Hom}(\Gamma^k_{ij}; \Gamma^k_{ij}) \sim \text{Hom}(\pi_i^{-1}\mathcal{P}_{k_2i} \circ_{E}^{\Phi_{ij}} \gamma_{rk_2}; (\beta_{kj})!\mathcal{Q}_{jk}; \pi_i^{-1}\mathcal{P}_{k_1i} \circ_{E}^{\Phi_{ij}} \gamma_{rk_1}; (\beta_{kj})!\mathcal{Q}_{jk_1});$$

$$\sim \text{Hom}((\mathcal{P}_{k_2i} \boxplus \mathcal{P}_{k_2j}) \gamma_{rk_2}; (\mathcal{P}_{k_1i} \boxplus \mathcal{P}_{k_1j}) \gamma_{rk_1});$$

$$\sim \text{Hom}(\gamma_{rk_2}; (\mathcal{P}_{k_1i} \boxplus \mathcal{P}_{k_1j}) \gamma_{rk_1});$$

$$\sim \text{Hom}(\gamma_{rk_2}; \gamma_{rk_1});$$

$$\sim \text{Hom}((\mathcal{Q}_{k_2i} \boxplus \mathcal{Q}_{k_2j}) \gamma_{rk_2}; \gamma_{rk_1});$$

$$\sim \text{Hom}((\mathcal{Q}_{k_1i} \boxplus \mathcal{Q}_{k_1j}) \gamma_{rk_2}; \gamma_{rk_1});$$

$$\sim \text{Hom}(\gamma_{rk_2}; \gamma_{rk_1}).$$
It follows that $G'_{ij}(k_2, k_1)$ admits a truncation. Let

$$G_{ij}(k_2, k_1) := \tau_{\leq 0} G'_{ij}(k_2, k_1).$$

We have a homotopy equivalence $\pi : G_{ij}(k_2, k_1) \to \mathcal{K}$.

We can now define categories $G_{ij}, SMetR_{ij}$ whose objects consist of all $S \subset SMetR$, where

$$k \in S \Rightarrow k << i, j.$$

and

$$\text{Hom}_{G_{ij}}(S_2, S_1) = G_{ij}(\mu(S_2), \mu(S_1)); \quad \text{Hom}_{SMetR_{ij}}(S_2, S_1) = \mathcal{K}$$

if $S_2 \subset S_1$ and

$$\text{Hom}_{G_{ij}}(S_2, S_1) = \text{Hom}_{SMetR_{ij}}(S_2, S_1) = 0$$

otherwise.

We have a functor $\pi : G_{ij} \to SMetR_{ij}$ which is a weak equivalence of categories.

Let $\tau : (SMetR_{ij})^{op} \to S_0(\mathfrak{A})$ be given by:

$\tau(S)^T = 0$ if $S \neq T$ and $\tau(S)^S = \mathcal{K}$.

Let $\tau_{G_{ij}} := \pi^{-1} \tau$,

$$\tau_{G_{ij}} : (G_{ij})^{op} \to S_0(\mathfrak{A}).$$

Let $S \in SMetR$. We then have a functor $i_S : \text{Com}(\mathfrak{A})(\text{val}(S)) \to R_0(\mathfrak{A})$, where $i_S(X)^T = 0$ if $T \neq S$ and $i_S(X)^S = X$ otherwise. It induces a functor

$$i_S : \text{psh}_{\text{val}(S)}(X, \mathbb{Z} \mathbb{Z}(\mathfrak{A})) \to \text{psh}(X, R_0(\mathbb{Z} \mathbb{Z}(\mathfrak{A})))$$

in the obvious way. We now have a functor

$$\Gamma_{ij} : G_{ij} \to \text{psh}(\Phi_{ij} \times E \times E, R_0(\mathbb{Z} \mathbb{Z}(\mathfrak{A}))),$$

where

$$\Gamma_{ij}(S) = i_S(\Gamma_{ij}^\mu(S))$$

Set

$$\Gamma_{ij} := \Gamma_{ij} \otimes_{G_{ij}} \tau_{G_{ij}} \in \text{psh}(\Phi_{ij} \times E \times E, S_0(\mathbb{Z} \mathbb{Z}(\mathfrak{A}))),$$

where we have used the tensor product $R_0(\mathbb{Z} \mathbb{Z}(\mathfrak{A})) \otimes S_0(\mathbb{Z} \mathbb{Z}(\mathfrak{A})) \to S_0(\mathbb{Z} \mathbb{Z}(\mathfrak{A}))$.

### 13.6 The linking object $\Gamma$

Let us now construct an object

$$\Gamma \in \text{psh}((\Sigma \times \Phi \times E \times E, S_0(\mathbb{Z} \mathbb{Z}(\mathfrak{A}))))$$

by means of prescription its components

$$\Gamma_{ij} \in \text{psh}((\Sigma \times \Phi_{ij} \times E \times E, S_0(\mathbb{Z} \mathbb{Z}(\mathfrak{A})))).$$
As was explained above, we have a map

\[ a_{ij} : \Phi_{ij} \to \Sigma. \]

Let \( A_{ij} \subset \Phi_{ij} \times \Sigma \) be the graph of \( a_{ij} \). Let \( p_A : A_{ij} \times E \times E \to \Phi_{ij} \times E \times E \); \( i_A : A_{ij} \times E \times E \to \Phi_{ij} \times E \times E \) be the corresponding maps. Set

\[ \Gamma_{ij} := i_A p_A^{-1} \Gamma_{ij} [-N]. \]

13.6.1 Quasi-contractibility

Let us first consider

\[ H^{[n]} = \text{Hom}(F^1[n] \circ (\Gamma \boxtimes n); F^1[n] \circ \Gamma) \]

Consider the following diagram

\[
\begin{array}{ccc}
(\Sigma \times \Phi \times E \times E)^n & \xrightarrow{\alpha} & (\Phi \times E \times E)^n \\
\downarrow i & & \downarrow i \\
K^{\Sigma,[n],ab} \times \Phi^n \times E^{n+1} & \xrightarrow{\tilde{p}} & \Sigma \times (\Phi \times E \times E)^n \\
\downarrow j & & \downarrow j \\
K^{\Sigma,[n],ab} \times K^{[n],ab} \times E^{n+1} & \xrightarrow{p} & \Sigma \times K^{[n],ab} \times E^{n+1} \xrightarrow{q} \Sigma \times \Phi \times E \times E
\end{array}
\]

We have

\[ H^{[n]} = \text{Hom}(\tilde{p}^{-1} i^{-1} (\Gamma \boxtimes n); j^{-1} q^{-1} \Gamma) \sim \text{Hom}(j^{-1} \tilde{p}^{-1} i^{-1} (\Gamma \boxtimes n); q^{-1} \Gamma) \sim \text{Hom}(p_{n} j^{-1} i^{-1} (\Gamma \boxtimes n); q^{-1} \Gamma) \]

Let

\[ a := \bigsqcup_{ij} a_{ij} : \Phi \to \Sigma. \]

The map \( a \) induces maps

\[ a_n : K^{[n],ab} \to K^{\Sigma,[n],ab}. \]

We now have the following diagram:

\[
\begin{array}{ccc}
(\Phi \times E \times E)^n & \xrightarrow{\alpha} & (\Phi \times E \times E)^n \\
\downarrow i & & \downarrow i \\
K^{\Sigma,[n],ab} \times \Phi^n \times E^{n+1} & \xrightarrow{\tilde{\alpha}} & K^{\Sigma,[n],ab} \times \Phi^n \times E^{n+1} \\
\downarrow j & & \downarrow j \\
K^{\Sigma,[n],ab} \times K^{[n],ab} \times E^{n+1} & \xrightarrow{\tilde{b}} & K^{[n],ab} \times E^{n+1} \xrightarrow{\tilde{c}} \Phi \times E \times E \\
\downarrow e & & \downarrow \bar{c} \\
K^{\Sigma,[n],ab} \times K^{[n],ab} \times E^{n+1} & \xrightarrow{p} & \Sigma \times K^{[n],ab} \times E^{n+1} \xrightarrow{q} \Sigma \times \Phi \times E \times E
\end{array}
\]

Here:
\[ \alpha((\phi_1, e_1, e_2), (\phi_2, e_3, e_4), \ldots, (\phi_n, e_{2n-1}, e_{2n})) = ((a(\phi_1), \phi_1, e_1, e_2), (a(\phi_2), \phi_2, e_3, e_4), \ldots, (a(\phi_n), \phi_n, e_{2n-1}, e_{2n})) \]

\[ \tilde{\alpha} = a_n \times \text{Id}_{E^{n+1}}. \]

Let \( \pi_{k,k+1} : K^{\Sigma[n],ab} \to \Phi \) be the projection onto the corresponding factor, \( 0 \leq k \leq n \). Then

\[ i(\kappa, \phi_1, \phi_2, \ldots, \phi_n, e_0, e_1, e_2, \ldots, e_n) = ((\pi_{01}\kappa, \phi_1, e_1), (\pi_{12}\kappa, \phi_2, e_2), \ldots, (\pi_{n-1,n}\kappa, \phi_n, e_{n-1,n})). \]

\[ \tilde{i}((\phi_1, \phi_2, \ldots, \phi_n), (e_0, e_1, \ldots, e_n)) = ((\phi_1, e_0, e_1), (\phi_2, e_1, e_2), \ldots, (\phi_n, e_{n-1}, e_n)). \]

Let

\[ \pi := \pi_1 \times \pi_2 \times \cdots \times \pi_n : K^{[n],ab} \to \Phi^n. \]

Then

\[ j(\kappa, \kappa', e) = (\kappa, \pi(\kappa'), e); \quad \tilde{j} := \pi \times \text{Id}_{E^{n+1}}. \]

\[ b = a_n \times \text{Id}_{K^{[n],ab} \times E^{n+1}}; \]

Let \( \pi_{0n} : K^{\Sigma[n],ab} \to \Sigma \) be the projection onto the first and the last factors. Then

\[ p(\kappa_\sigma, \kappa, e) = (\pi_{0n}(\kappa_\sigma), \kappa, e). \]

Set \( c = pb; \)

Set \( \tilde{c}(\phi, e_1, e_2) = (a(\phi), \phi, e_1, e_2); \)

Set \( q(\sigma, \kappa, e_0, e_1, \ldots, e_n) = (\sigma, \pi_{0n}\kappa, e_0, e_n); \)

Set \( \tilde{q}(\kappa, e_0, e_1, \ldots, e_n) = (\pi_{0n}\kappa, e_0, e_n). \) All the squares in this diagram are Cartesian. We have

\[ (\Gamma)^{\otimes n} \sim (\alpha)!(\Gamma^{\otimes n}); \quad \Gamma \sim (\tilde{c})!\Gamma, \]

therefore,

\[ H^{[n]} \sim \text{Hom}(p_!j^{-1}i^{-1}(\alpha)!(\Gamma)^{\otimes n}; q^{-1}\tilde{c}_!\Gamma) \sim \text{Hom}(p_!b!(\tilde{i}j^{-1})^{-1}(\Gamma)^{\otimes n}; c!(\tilde{q})^{-1}\Gamma) \]

Observe that \( c = pb \) and that \( c \) is a closed embedding, therefore, we can continue:

\[ H^{[n]} \sim \text{Hom}(\tilde{(i)j}^{-1}(\Gamma[N+1])^{\otimes n}; (\tilde{q})^{-1}\Gamma[N+1]). \]
We have a decomposition

\[ K^{[n],ab} = \bigsqcup_{i_0i_1\cdots i_n \in \text{MetR}} K^{ab}_{i_0i_1\cdots i_n} \]

Accordingly \( H^{[n]} \) splits into a direct product of its components

\[ H^{[n]} = \prod_{i_0i_1\cdots i_n} H^{ab}_{i_0i_1\cdots i_n} \]

The problem reduces to considering each such a component.

Denote by \( \tau : K^{ab}_{i_0i_1\cdots i_n} \times E^{n+1} \to (\Phi \times E \times E)^n \)
the restriction of \( \tilde{\gamma} \). Let us denote by \( q' \) the restriction of \( \tilde{q} \) onto \( K^{ab}_{i_0\cdots i_n} \). Let us decompose \( q' = q_\Phi q_E \), where

\[
q_E : K^{ab}_{i_0\cdots i_n} \times E^{n+1} \to K^{ab}_{i_0\cdots i_n} \times E^2; \\
q_\Phi : K^{ab}_{i_0\cdots i_n} \times E^2 \to \Phi_{i_0i_n} \times E^2.
\]

Denote by \( K^{[n],ab}_{i_0i_1\cdots i_n} \in \text{psh} \left( \bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m} \times E \times E, S_0(\mathbb{Z}\mathbb{Z}(\mathfrak{A})) \right) \)
the restriction of \( \tilde{\gamma} \). Let us denote by \( q' \) the restriction of \( \tilde{q} \) onto \( K^{ab}_{i_0\cdots i_n} \). Let us decompose \( q' = q_\Phi q_E \), where

\[
q_E : K^{ab}_{i_0\cdots i_n} \times E^{n+1} \to K^{ab}_{i_0\cdots i_n} \times E^2; \\
q_\Phi : K^{ab}_{i_0\cdots i_n} \times E^2 \to \Phi_{i_0i_n} \times E^2.
\]

Let us consider

\[
\bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m} \times E \times E, S_0(\mathbb{Z}\mathbb{Z}(\mathfrak{A}))
\]

We have

\[
\bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m} \sim (\bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m}) \otimes^L (\bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m}),
\]

where \( \otimes^L \) is taken over the category \( \bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m} \).

We then have a homotopy equivalence

\[
H_{i_0\cdots i_n} \sim \text{Hom} \left( \iota^{-1}(\bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m} [-N]) \otimes^L (\bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m}), q^{-1}(\bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m} [-N]) \otimes^L (\bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m}) \right)
\]

\[
\sim \text{RHom} \left( \iota^{-1} \bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m} [-N]; \text{Hom}S_0(\bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m} [-N]) \otimes^L (\bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m}) \right). \quad (56)
\]

For \( S \in \text{SMetR} \), define an object \( f_S^0 \in S_0 \), \( (f_S^0)^T = 0 \) if \( T \neq S \), and \( (f_S^0)^S = \lambda \).

Observe that for every \( S \in \text{SMetR} \), the natural map

\[
q^{-1}(\bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m} [-N] \otimes^L \Phi_{i_0i_1\cdots i_m}) \to \text{Hom}S_0(\bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m} [-N] \otimes^L \Phi_{i_0i_1\cdots i_m})
\]

is a homotopy equivalence (because there are only finitely many \( T \in \mathcal{G}_{i_0i_n} \) satisfying \( T \subset S \). We can now continue (56):

\[
H(i_0i_1 \cdots i_n) \sim \text{RHom} \left( \iota^{-1} \bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m} [-N]; q^{-1}(\bigotimes_{m=0}^{n-1} \Phi_{i_0i_1\cdots i_m} [-N] \otimes^L \Phi_{i_0i_1\cdots i_m}) \right),
\]

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where 
\[ \lambda : \bigotimes_{m=0}^{n-1} \mathcal{G}(i_m i_{m+1}) \otimes \mathcal{G}(i_0 i_n)^* \to \mathfrak{A}; \]
\[ \lambda(S_1, S_2, \ldots, S_n; S) = \mathbb{K} \text{ if } S_1 \cup S_2 \cup \cdots \cup S_n \subset S \text{ and } \lambda(S_1, S_2, \ldots, S_n; S) = 0 \text{ otherwise.} \]
We have 
\[ (\tilde{q}^{-1}\Gamma_{i_0 i_n}[-N] \otimes \mathcal{G}_{i_0 i_n} \lambda)(S_1, S_2, \ldots, S_n) \sim q^{-1}\Gamma_{i_0 i_n}[-N] \]
if \( S_1 \cup S_2 \cup \cdots \cup S_n \) is linearly ordered and 
\[ (\tilde{q}^{-1}\Gamma_{i_0 i_n}[-N] \otimes \mathcal{G}_{i_0 i_n} \lambda)(S_1, S_2, \ldots, S_n) \sim 0 \]
otherwise.

The problem now reduces to showing quasi-contractibility of 
\[ \text{Hom}(\tau^{-1}(\mathcal{X}_m \Gamma_{i_m-1 i_m}^{k_m}[-N]); \tilde{q}^{-1}(\Gamma_{i_0 i_n}^{k}[-N])) \]
under assumptions that \( k_m \in G(i_{m-1}, i_m) \) and \( k \leq k_m \) for all \( m \).

Let 
\[ q_{E^1} : K_{i_0 i_1 \ldots i_n}^{ab} \times E^{n+1} \to K_{i_0 i_1 \ldots i_n}^{ab} \times E^2 \]
be the projection onto the first and the last factor of \( E^{n+1} \). The above space can be rewritten as 
\[ \text{Hom}(q_{E^1}^{-1}(\mathcal{X}_m \Gamma_{i_m-1 i_m}^{k_m}[-N])); \tilde{q}^{-1}(\Gamma_{i_0 i_n}^{k}[-N])) \]
\[ = \text{Hom}(q_{E^1}^{-1}(\mathcal{X}_m \Gamma_{i_m-1 i_m}^{k_m})); \tilde{q}^{-1}\Gamma_{i_0 i_n}^{k}) \]

Step 1. Let 
\[ \text{red} : \text{sh}_q(\Phi_{ij} \times E \times E) \to \text{shval}^k(\Phi_{ij} \times E \times E) \]
be the projection. The object \( \Gamma_{ij}^k \in \text{shval}^k(\Phi_{ij} \times E \times E) \) is glued of objects \( (\gamma_{ij}^k) \) of the form 
\[ (\gamma_{ij}^k)_l = \text{red} \text{T}_{-\tau \gamma_{ij}^k} \gamma_{ij}^k[2l]. \]
The object \( \gamma_{ij}^k \) is glued of \( \Phi_{ij}^k \) and \( \text{T}_{-\tau \gamma_{ij}^k} \Phi_{ij}^k[-2N - 1] \). The object \( (\gamma_{ij}^k)_l \) is therefore glued of the following objects 
\[ X_{ijk[l]}^1 := \text{T}_{-(l+1)\text{val}} \Phi_{ij}^k[2l + 2N + 1] \]
and 
\[ X_{ijk[l]}^0 := \text{T}_{-\text{val}} \Phi_{ij}^k[2l]. \]

Step 2. Let \( L = q_{E^1}^{-1}(\mathcal{X}_m \Phi_{i_m-1 i_m}^{k_m}) \) and \( M = q_{E^1}^{-1}(\mathcal{X}_m \Phi_{i_m-1 i_m}^{k_m}) \).
We then have a natural map \( M \to L \) because \( k \leq k_m \) for all \( m \). Singular support consideration shows that for any \( a \in \mathbb{R} \), 
\[ \text{Hom}(T_a L; \tilde{q}^{-1}\gamma_{i_0 i_n}^k[N]) \to R\text{Hom}(T_a M; \tilde{q}^{-1}\gamma_{i_0 i_n}^k[N]) \]
is a homotopy equivalence.
Step 3 We have a homotopy equivalence $M \sim \tilde{q}^{-1}P^{k}_{\delta_{\text{val}}}$.

Step 4 $\text{Hom}(P^{k}_{ij}; T_{a}T_{-\text{val}(k)}\gamma^{k}_{ij}) = \mathbb{K}$ if $0 \leq a < \text{val}k$ and 0 otherwise (use the representation of $\gamma^{k}_{ij}$ as the cone of $P^{k}_{ij}[2N + 1] \xrightarrow{1} T_{\pi^{2}k}P^{k}_{ij}$.

Step 5 $\tilde{q}_{u}q^{-1}F[\nu] = F \otimes \mathcal{L}$, where $\mathcal{L} \in \mathbf{sh}(\Phi_{i\delta_{\text{val}}} \times E \times E)$ is a local system obtained by gluing constant sheaves of the form $\mathbb{K}\Phi_{i\delta_{\text{val}}} \times E \times E[a]$, $a \geq 0$. Here $\nu$ is the relative dimension of $p$.

Step 6 Count. It suffices to show that all objects

$$\text{Hom}(q_{E}q^{-1} \otimes_{m} (X^{a}_{m}(i_{m-1}i_{m}k_{m}l_{m})); \tilde{q}^{-1}T_{-\text{val}(k)}\gamma^{k}_{i\delta_{\text{val}}}) \in \mathbf{GZtrunc}.$$ 

We have:

$$q_{E}q^{-1} \otimes_{m} (X^{a}_{m}(i_{m-1}i_{m}k_{m}l_{m})) = q_{E}q^{-1}(\otimes_{m} T_{-(l_{m} + a_{m})}\text{val}(k_{m})P^{k}_{i_{m-1}i_{m}}[2l_{m} + (2N + 1)a_{m}]) = T_{u}q_{E}q^{-1}(\otimes_{m} P^{k}_{i_{m-1}i_{m}})[v']$$

where

$$u' = \sum -(l_{m} + a_{m})\text{val}(k_{m})$$

and

$$v' = 2 \sum l_{m} + (2N + 1) \sum a_{m}.$$ 

So that we have

$$\text{Hom}(q_{E}q^{-1} \otimes_{m} (X^{a}_{m}(i_{m-1}i_{m}k_{m}l_{m})); \tilde{q}^{-1}T_{-\pi^{2}k}\gamma^{k}_{i\delta_{\text{val}}})$$

$$= \text{Hom}(T_{u}q_{E}q^{-1}P^{k}_{\delta_{\text{val}}} [v']; \tilde{q}^{-1}T_{-\pi^{2}k}\gamma^{k}_{i\delta_{\text{val}}})$$

$$= \text{Hom}(\tilde{q}^{-1}P^{k}_{\delta_{\text{val}}}; T_{-\pi^{2}k}^{-u}q^{-1}\gamma^{k}_{i\delta_{\text{val}}}[2N + 1 - v'])$$

$$= \text{Hom}(P^{k}_{\delta_{\text{val}}} \otimes \mathcal{L}; T_{-\pi^{2}k}^{-u}\gamma^{k}_{i\delta_{\text{val}}}[2N + 1 - v']).$$

As $\mathcal{L}$ is glued of $\mathbb{K}\Phi_{i\delta_{\text{val}}} \times E \times E[a]$, $a \geq 0$, it suffices to show that

$$\text{Hom}(P^{k}_{\delta_{\text{val}}}; T_{-\pi^{2}k}^{-u}\gamma^{k}_{i\delta_{\text{val}}}[2N + 1 - v']) \in D_{\geq 0}{\mathfrak{A}}.$$ 

This is true in each of the following cases

1) $-u' \notin [0, \pi^{2}k]$;

2) $-u' \in [0, \pi^{2}k]$ and $-v' \leq 0$.

However, $v' \geq 0$ which proves the statement.

13.7 The category $\mathcal{M}$

Let us first define a category $\mathcal{M}_{0}$ enriched over $\mathfrak{A}$.
Set \( M_0 := \text{Open}_M \). Let \( h : \mathcal{M}_0 \otimes \mathcal{M}_0 \to \mathfrak{A} \), \( h(U, V) = \mathbb{K} \) if \( U \cap V \neq \emptyset \); \( h(U, V) = 0 \) otherwise.

Let \( \Delta^n_M \subset M^n \times M \); \( \Delta^{(n)} \subset M^{n+1} \) be diagonals. Set
\[
F^{[n]}_{\mathcal{M}_0} := \mathbb{K}_{\Delta^n_M} \in \text{sh}(M^n \times M);
\]
\[
F^{(n)}_{\mathcal{M}_0} := \mathbb{K}_{\Delta^{(n)}_M} \in \text{sh}(M^{n+1}).
\]

Let \( \forall_0 \in \text{Classic}(\mathbb{Z}\mathbb{Z}(\mathfrak{A}))(1) \) be defined by
\[
\text{gr}^{a}\forall_0 := \bigoplus_b \langle a, b \rangle [2b].
\]

Set
\[
\forall := i_0(\forall_0) \in R_0(\mathbb{Z}\mathbb{Z}(\mathfrak{A})).
\]

We have an algebra structure on \( \forall \). Denote by \( \mathcal{B} \) the category, enriched over \( R_0(\mathbb{Z}\mathbb{Z}(\mathfrak{A})) \), with one object, whose endomorphism space is \( \forall \). Denote this object by \( e \). As \( \forall \) is a commutative algebra, we have a symmetric monoidal structure on \( \mathcal{B} \), where \( e \otimes e = e \).

Let now \( \mathcal{M} := \mathcal{M}_0 \otimes \mathcal{B} \). The functors \( F^{[n]}, F^{(n)} \) extend to \( \mathcal{M} \).

### 13.7.1 Connecting \( \mathcal{M} \) and \( \mathcal{H} \)

Let \( p_{\Sigma} : \Sigma \to M \) be the projection. Let \( G_{\Sigma} \subset \Sigma \times M \) be the graph of \( p_{\Sigma} \). Let
\[
\Delta := \mathbb{K}_{G_{\Sigma}} \otimes \text{units}_{S_0(\mathbb{Z}\mathbb{Z}(\mathfrak{A}))} \in (D \bigoplus)(\mathcal{M} \otimes \mathcal{H} \otimes S_0(\mathbb{Z}\mathbb{Z}(\mathfrak{A}))).
\]

### 13.8 The categories \( \mathcal{B}_\mathcal{T} \)

Fix a triangulation \( \mathcal{T} \) of \( M \). Let \( \text{Open}_\mathcal{T} \subset \text{Open}_M \) consist of \( M \) and all stars of \( \mathcal{T} \). Let \( \mathcal{B}_\mathcal{T} := \text{Open}_\mathcal{T} \) viewed as a discrete set. Let \( h_{\mathcal{B}} : \mathcal{B} \times \mathcal{B} \to \mathfrak{A} \) be given by \( h_{\mathcal{B}}(U, V) = \mathbb{K} \) if \( U = V \) and \( h_{\mathcal{B}}(U, V) = 0 \) otherwise.

Let
\[
F^{[n]}_{\mathcal{B}_{\mathcal{T}}} := \bigoplus_{U_1 \cap U_2 \cap \cdots \cap U_n \subset U} U_1 \otimes U_2 \otimes \cdots \otimes U_n \otimes \forall \otimes \text{units}_{S_0(\mathfrak{A})}
\]

Let
\[
F^{(n)}_{\mathcal{B}_{\mathcal{T}}} := \bigoplus_{U_0 \cap U_1 \cap \cdots \cap U_n \neq \emptyset} U_0 \otimes U_1 \otimes \cdots \otimes U_n \otimes \text{units}_{S_0(\mathfrak{A})}
\]

Let \( \mathcal{B} \subset \mathcal{B}_{\mathcal{T}} \) be the full sub-category consisting of a single object \( M \).
13.8.1 Connecting $\mathbb{B}_T$ and $\mathcal{M}$

Let $I : \text{SMetR}^{\text{op}} \to S_0$, $I(S) = f^S_0$. Let $I \in (D \bigoplus)S_0$,

$$I := \text{hocolim}_{S \in \text{SMetR}} I(S).$$

Set $\mathcal{E} \in \text{psh}(M, S_0(\text{ZZ}(\mathfrak{A})))$,

$$\mathcal{E}|_{U \times M} := K_U^{(0,0)} \otimes I.$$

Observe that the restriction of $\mathbb{B}_T$ and $\mathcal{E}$ onto $M \in \text{Open}_T$ is independent of the choice of $T$.

13.9 Straightening out

13.9.1 Changing the ground category

We have the constant tensor functor $\text{ZZ} \to \text{pt}$ which induces a tensor functor $R_0(\text{ZZ}(\mathfrak{A})) \to R_0(\mathfrak{A})$ by means of which the structure on the categories $\mathbb{B}, \mathcal{M}, \ldots$ carries over to $R_0(\mathfrak{A})$. All the categories below are thus enriched over $R_0$. The category $\mathcal{F}$ is enriched over $R_q$.

13.9.2 Straightening out

Applying the procedures from Sec 10 we produce a sequence of monoidal categories and their functors

$$U(\mathbb{B}) \to U(\mathbb{E}_U) \to (D \bigoplus) U(\mathcal{M}) \to (D \bigoplus) U(\mathcal{H}) \to (D \bigoplus) U(\mathcal{G}) \to (D \bigoplus) U(\mathcal{F}).$$

The category $U(\mathcal{F})$ is enriched over $R_q$. We also have a category $U(\mathcal{F}_R)$, where $(D \bigoplus)U(\mathcal{F}_R)$ enriched over $D \prod \bigoplus U(\mathcal{F})$.

We have a zig-zag map from $U(\mathcal{F}_0) \otimes U(\mathfrak{A})$ to $(D \bigoplus)U(\mathcal{M})$. We have sub-categories $U(*_1)$, where $* = \mathbb{B}, \mathcal{M}, \mathcal{H}, \mathcal{G}, \mathcal{F}$. These sub-categories are preserved by the above functors.

14 Hochschild complexes

14.1 Hochschild complexes of an algebra in a monoidal category with trace

Let $\mathcal{M}$ be a monoidal category with trace enriched over a ground category $\mathfrak{A}$. Let $A$ be an algebra in $\mathcal{M}$. We then can build the Hochschild cochain complex $\text{Hoch}^\bullet(A, A)$ in $\mathfrak{A}$, where $\text{Hoch}^n(A) = \text{Hom}_\mathcal{M}(A^\otimes n; A)$ as well as Hochschild chain complex $\text{Hochcyc}^{-\bullet}(A)$, where $\text{Hochcyc}^{-n}(A) = TR_\mathcal{M}(A^\otimes n+1)$, in the usual way.

Let $N$ be an $A$-bimodule in $\mathcal{M}$. One then defines Hochschild complexes $\text{Hoch}^\bullet(M, A)$ and $\text{Hochcyc}^\bullet(N)$, where

$$\text{Hoch}^n(M, A) = \bigoplus_{i+j=n, i,j \geq 0} \text{Hom}_\mathcal{M}(A^\otimes i \otimes N \otimes A^\otimes j; A);$$

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Hochcyc$^{-n}(M) = M \otimes A^\otimes n$.

Let $A$ be $A$ viewed as a bimodule over itself. we have a natural map

$$\text{Hoch}(A) \to \text{Hoch}(A, A).$$

We will now formulate a sufficient condition for this map to be a homotopy equivalence. Let $\text{unit} \in M$ be the unit. A map $\text{unit} \to A$ in $\mathcal{M}$ is called a homotopy unit of $A$ if the induced maps $A = A \otimes \text{unit} \to A \otimes A \to A$ and $A = \text{unit} \otimes A \to A \otimes A \to A$ are homotopy equivalent to $\text{Id}_A$.

**Proposition 14.1** Suppose $A$ admits a homotopy unit. Then the map (58) is a homotopy equivalence.

14.1.1 A map $I : \text{Hoch}(A) \to \text{Hochcyc}(A)$, where $A$ is an algebra with trace

Let $A$ be an algebra with trace $\text{Tr}$ in a monoidal category with trace $\mathcal{M}$, where $\text{Tr} : \text{unit} \to \text{TR}(A)$. Let us define maps

$$I^n : \text{Hoch}^n(A) \to \text{Hochcyc}^n(A).$$

To this end we first define maps $d_0 : \text{Hoch}^n(A) \to \text{Hoch}^{n+1}(A)$ as follows:

$$d_0 : \text{Hoch}^n(A) = \text{Hom}(A^\otimes n; A) \to \text{Hom}(A \otimes A^\otimes n; A \otimes A) \overset{m}{\cong} \text{Hom}(A^{n+1}; A).$$

We now set

$$I : \text{Hoch}^n(A) := \text{Hom}(A^\otimes n; A) \otimes \text{unit} \overset{d_0 \otimes \text{Tr}}\to \text{Hom}(A^\otimes n+1; A) \otimes \text{TR}(A)$$

$$\to \text{TR}(A^\otimes n+1) = \text{Hochcyc}^n(A).$$

It follows that $I$ gives rise to a map of complexes.

14.2 Hochschild complexes in $\mathcal{M}_F$ and $\mathcal{M}_H$

We have an algebra $M \in \mathcal{M}_A$ and its bimodules $M_U$ for every element $U \in \text{Open}_T$. We have a functor $U \mapsto M_U$ from the category $\text{Open}_T$ to the category of $M$-bimodules in $\mathcal{M}_B$.

Denote by $M^H, M^H_U, M^F, M^F_U$ their images in $\mathcal{M}_H$ and $\mathcal{M}_F$.

We have natural maps

$$\text{hocolim}_{U \in T} M^H_U \to M^H; \quad \text{hocolim}_{U \in T} M^F_U \to M^F.$$ \hspace{1cm} (59)

**Lemma 14.2** The map (59) is a homotopy equivalence.

We have induced maps

$$\text{Hoch}(M^H) \to \text{Hoch}(M^F); \quad \text{Hochcyc}(M^H) \to \text{Hoch}(M^F);$$

$$\text{Hoch}(M^H_U, M^H) \to \text{Hoch}(M^F_U, M^F); \quad \text{Hochcyc}(M^H_U, M^H) \to \text{Hochcyc}(M^F_U, M^F).$$ \hspace{1cm} (60)

(61)
Proposition 14.3 The maps in (60), (61) are homotopy equivalences.

The rest of the section is devoted to proving the Proposition.

14.2.1 Passage to the category Classic(valS)

By virtue of Lemma ??, it suffices to show that the induced maps

$$\tau_{\mathcal{S}\text{Hoch}}(M^{\mathbb{H}}) \rightarrow \tau_{\mathcal{S}\text{Hoch}}(M^{\mathbb{F}}); \quad \tau_{\mathcal{S}\text{Hochcyc}}(M^{\mathbb{H}}) \rightarrow \tau_{\mathcal{S}\text{Hochcyc}}(M^{\mathbb{F}});$$

(62)

$$\tau_{\mathcal{S}\text{Hoch}}(M^{\mathbb{H}}_{U}, M^{\mathbb{H}}) \rightarrow \tau_{\mathcal{S}\text{Hoch}}(M^{\mathbb{F}}_{U}, M^{\mathbb{F}}); \quad \tau_{\mathcal{S}\text{Hochcyc}}(M^{\mathbb{H}}_{U}, M^{\mathbb{H}}) \rightarrow \tau_{\mathcal{S}\text{Hochcyc}}(M^{\mathbb{F}}_{U}, M^{\mathbb{F}});$$

(63)

are homotopy equivalences in (D $\bigoplus$)Q-valS for every $S \in \text{SMetR}$. From now on we fix an element $S \in \text{SMetR}$.

Observe that it suffices to prove (63) for any refinement $U_S$ of the cover $U$, where $U_S$ may depend on $S$. We therefore make the following assumption without loss of generality:

Assumption 14.4 Let $\sigma$ be the minimal element of $S$, then every element $U$ from $U_S$ is contained in a ball $B_\sigma(m_0)$ for some point $m_0 \in M$.

14.2.2 Changing the set MetR$'$

Let us replace MetR$'$ with its subset $S$. It is clear that the induced map

$$\tau_{\mathcal{S}\text{Hoch}}^{\text{MetR}'}(M^{\mathbb{H}}_{U}, M^{\mathbb{H}}) \rightarrow \tau_{\mathcal{S}\text{Hoch}}^{S}(M^{\mathbb{H}}_{U}, M^{\mathbb{H}})$$

is an isomorphism and similar for all the remaining ingredients of (62), (63). We therefore will assume MetR$'$ = $S$ from now on.

14.2.3 Passage to homotopy categories

Let $K$ be a category enriched over the category of complexes of $k$-modules. Denote by hoK the category, enriched over the category of $k$-modules, where we set hoK(X, Y) := $H^0$Hom(X, Y). If $\mathcal{M}$ is a monoidal category with a trace enriched over a ground SMC $\mathfrak{A}$, then ho$\mathcal{M}$ is a monoidal category with a trace enriched over a SMC ho$\mathfrak{A}$.

14.2.4 When a homotopy equivalence in ho$\mathfrak{A}$ implies that in $\mathfrak{A}$?

Let $X^n \in \mathfrak{A}$ and let $d_{nm} \in \text{Hom}^0(X^n, X^m)$, $n < m$ be elements satisfying

$$dd_{nm} + \sum_{k|n<k<m} d_{km}d_{nk} = 0.$$ 

Call such data a complex in $\mathfrak{A}$.
Let us define an object $|X^*| \in \mathcal{B}$ as the representing object of

$$(\bigoplus_{n<0} X^n[-n] \oplus \prod_{n \geq 0} X^n[-n]) \cdot \sum_{nm} d_{nm})$$

A map of complexes $f: X^* \to Y^*$ is a collection of maps $f_{nm}: X_n \to X_m$, $m \geq n$ satisfying: let $f := \sum_{nm} f_{nm} \in \text{Hom}(|X^*|,|Y^*|)$, then $df = 0$.

One builds a complex Cone $f$ in $\mathfrak{A}$ in a standard way.

Let us now pass to $\text{hoA}$. Let $X^n, d_{nm}$ be as above. We have

$$dd_{n,n+1} = 0; \quad d_{n,n+1}d_{n+1}d_{n+2} + dd_{n,n+2} = 0,$$

which implies that we have the following complex in $\text{hoA}$, to be denoted by $\text{hoA}X^*$:

$$0 \to X^0 \xrightarrow{d_0} X^1 \xrightarrow{d_1} \cdots$$

It also follows that a map of complexes $f: X^* \to Y^*$ induces a map

$$\text{ho}_f: \text{hoA}X^* \to \text{hoA}Y^*$$

of complexes in $\text{hoB}$. One define a complex Cone $\text{ho}_f$ in $\text{hoB}$ in a standard way.

Call $\text{hoA}X^*$ acyclic if there exist elements $o_i \in \text{Hom}_{\text{hoB}}(X^i; X^{i-1})$ such that $d_{i-1,i}o_i + o_{i+1}d_{i,i+1} = \text{Id}_{X^i}$ in $\text{hoB}$.

Call a map $\text{ho}_f$ a homotopy equivalence if Cone $\text{ho}_f$ is acyclic.

**Proposition 14.5** 1) Suppose $\text{hoA}X^*$ is acyclic. Then so is $X^*$.

2) Let $f: X^* \to Y^*$ be a map of complexes in $\mathcal{B}$ and suppose $\text{ho}_f$ is a homotopy equivalence. Then so is $f$.

**Sketch of the proof** It is clear that 1) implies 2). Let us prove 1). Choose representatives of $o_i$ in $\text{Hom}_{\text{hoB}}(X^i; X^{i-1})$, to be denoted by $h_i$. We have $dh_i = 0$;

$$d_{i-1,i}h_i + h_{i+1}d_{i,i+1} = f_{ii} - du_{i}$$

for some $u_i \in \text{Hom}^{-1}_{\mathcal{B}}(X^i, X^i)$.

The sum of all $h_i$ and $d_i$ defines an element $H \in \text{Hom}^{-1}(\mathcal{X}, \mathcal{X})$ such that $dH = \text{Id} + K$, where $K: X^i \to \prod_{j \geq i+2} X^j$ so that $\text{Id} + K$ is an automorphism of $\mathcal{X}$. This finishes the proof.

### 14.2.5 Hochschild complexes in the homotopy category

Let $\mathcal{M}, \mathcal{N}$ are monoidal categories enriched over $\mathfrak{A}$, let $A$ be an algebra in $\mathcal{M}$ admitting a homotopy unit and let $K$ be an $A$-bimodule in $\mathcal{M}$. Let $f: \mathcal{M} \to \mathcal{N}$ be a tensor functor. We then have maps of Hochschild complexes

$$\text{Hoch}(K, A) \to \text{Hoch}(f(K), f(A)); \quad \text{Hochcyc}(K) \to \text{Hochcyc}(f(K)).$$ (64)
On the other hand we have an algebra \( \text{ho}A \) and its bi-module \( \text{ho}K \) in \( \text{ho}\mathcal{B} \) and Hochschild complexes in \( \text{ho}\mathcal{B} \):

\[
\text{Hoch}(\text{ho}K, \text{ho}A) \to \text{Hoch}(f(\text{ho}K), f(\text{ho}A)); \quad \text{Hochcyc}(\text{ho}K) \to \text{Hochcyc}(f(\text{ho}K)) \quad (65)
\]

It is clear that

\[
\text{ho}_{\text{Hoch}}(K, A) \cong \text{Hoch}(\text{ho}K, \text{ho}A); \quad \text{ho}_{\text{Hochcyc}}(K) \cong \text{Hochcyc}(\text{ho}K).
\]

Therefore,

**Claim 14.6** If maps in (65) are homotopy equivalences, then so are maps from (64).

In particular our problem reduces to showing that the following maps are homotopy equivalences

\[
\text{Hoch}(\text{ho}M_{U}^{F}, \text{ho}M_{V}^{F}) \to \text{Hoch}(\text{ho}M_{U}^{F}, \text{ho}M_{V}^{F}); \quad \text{Hochcyc}(\text{ho}M_{U}^{F}, \text{ho}M_{V}^{F}) \to \text{Hochcyc}(\text{ho}M_{U}^{F}, \text{ho}M_{V}^{F}) \quad (66)
\]

### 14.3 Describing \( \text{ho}(M_{i})_{1} \)

We have an equivalence of categories \( i : \text{ho} \mathcal{H} \to \text{ho}(M_{i})_{1} \). For each \( n \geq 0 \), consider the functor \( T_{n} : (\text{ho}\mathcal{H})^{\text{op}} \times (\text{ho}\mathcal{H}) \to \text{ho}\mathcal{R}_{0} \), where

\[
T_{n}(X_{1}, X_{2}, \ldots, X_{n}; X) := \text{Hom}_{\text{ho}\mathcal{H}}(i(X_{1}) \otimes i(X_{2}) \otimes \cdots \otimes i(X_{n}), i(X)).
\]

We have an isomorphism of functors

\[
T_{n}(X_{1}, X_{2}, \ldots, X_{n}; X) \cong \text{Hom}_{\text{ho}psh_{val}(\Sigma^{n})}(X_{1} \boxtimes X_{2} \boxtimes \cdots \boxtimes X_{n}; \bigoplus_{ab} T_{ab}T_{a}(K_{n})_{ab} \circ_{\Sigma} X[2b])
\]

Recall that we have a groupoid structure on \( \Sigma \Rightarrow \mathcal{L} \).

Let \( \mu_{n} : \Sigma^{n} \to \Sigma \) be the composition map of this groupoid.

We have \( (K_{n})^{ab} \cong \Sigma^{n} \) for all \( a, b \in \mathbb{Z} \). The projection \( p_{1n}^{ab} : (K_{n})^{ab} \to \Sigma \), under this identification reads as

\[
K_{n} \xrightarrow{\mu_{n}} \Sigma \Rightarrow \Sigma,
\]

Observe that \( \mu_{n} \) is a smooth fibration. Therefore, \( \mu_{n}^{-1} = \mu_{n}^{-1}[-(n-1)D] \), where

\[
(n-1)D = (n-1)(\dim \text{Sp}(2N) - \dim \text{SU}(N)),
\]

so that \( D = \dim \text{Sp}(2N) - \dim \text{SU}(N) \). Let also \( i_{n} : K_{n} \to \Sigma^{n} \) be the embedding, so that we can rewrite

\[
T_{n}(X_{1}, X_{2}, \ldots, X_{n}; X) = \text{Hom}(X_{1} \boxtimes \cdots \boxtimes X_{n}; \bigoplus_{ab} i_{n}^{-1}\mu_{n}^{-1}T_{ab}T_{a}X[2b])
\]

\[
= \text{Hom}(i_{n}^{-1}(X_{1} \boxtimes \cdots \boxtimes X_{n}); \bigoplus_{ab} \mu_{n}^{-1}[-(n-1)D]T_{ab}T_{a}X[2b])
\]

\[
\cong \text{Hom}(\mu_{n}^{-1}i_{n}^{-1}(X_{1}[D] \boxtimes \cdots \boxtimes X_{n}[D]); \bigoplus_{ab} T_{ab}T_{a}X[D][2b]).
\]

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The groupoid structure on $\Sigma \Rightarrow \mathcal{L}$ produces a convolution monoidal structure on $\text{ho sh}_q(\Sigma)$, where the tensor product, to be denoted by $\ast$ is given by

$$X_1 \ast X_2 \ast \cdots \ast X_n = \mu_{n}i_{n}^{-1}(X_1 \boxtimes \cdots \boxtimes X_n),$$

so that we finally have

$$T_n(X_1, X_2, \ldots, X_n; X) \cong \bigoplus_{ab} T_a\text{Hom}(X_1[D] \ast \cdots \ast X_n[D]; \bigoplus_{ab} T_{ab}T_a X[D][2b])$$

$$= \text{Hom}_{\text{ho}(\mathcal{M}_q)_1}(X_1[D] \ast \cdots \ast X_n[D]; X[D]). \quad (67)$$

This result has the following corollaries:

1) Let $\mathcal{H}'$ be the category enriched over $\text{hoR}_0$, whose objects are the same as in $\text{ho psh}(\Sigma, S_0)$ and we set

$$\text{Hom}_{\mathcal{H}'}(Y, X) = \text{Hom}_{\text{ho sh}_q(\Sigma)}(Y; \bigoplus_{ab} T_{ab}T_a X[2b])$$

We then have a monoidal structure on $\mathcal{H}'$ given by $\ast$. Next, we have a lax tensor functor

$$\Sigma_{\mathcal{H}'} : (\mathcal{H}', \ast) \to \text{hoM}_{\mathcal{H}1},$$

where $\Sigma_{\mathcal{H}'}(X) = i(X)[D]$. By lax tensor functor structure on $S$ we mean a natural transformation $\Sigma_{\mathcal{H}'}(X) \otimes \Sigma_{\mathcal{H}'}(Y) \to \Sigma_{\mathcal{H}'}(X \ast Y)$ which has an associativity property but need not be a homotopy equivalence. The induced map

$$\text{Hom}_{\mathcal{H}'}(X_1 \ast X_2 \ast \cdots \ast X_n; X) \to \text{Hom}_{\mathcal{M}_q}(\Sigma_{\mathcal{H}'}(X_1) \otimes \Sigma_{\mathcal{H}'}(X_2) \otimes \cdots \otimes \Sigma_{\mathcal{H}'}(X_n); \Sigma_{\mathcal{H}'}(X))$$

is an isomorphism for all $X_i, X \in \mathcal{H}'$.

14.4 The categories $\text{ho}(\mathcal{M}_G)_1, \text{ho}(\mathcal{M}_F)_1$

One gets similar results for $\mathcal{M}_G, \mathcal{M}_F$. Let

$$\mathcal{G}' := \text{ho psh}_{\text{val}S}(\Phi \times E \times E); \quad \mathcal{F}' := \text{ho psh}_{\text{val}S}(\Phi \times E^2).$$

Set

$$\text{Hom}_{\mathcal{G}'}(A, B) := \text{Hom}_{\text{ho psh}_{\text{val}S}}(\Phi \times E \times E)(A; \bigoplus_{ab} T_{ab}T_a B[2b]).$$

Set

$$\text{Hom}_{\mathcal{F}'}(A, B) = \text{Hom}_{\text{psh}_{\text{val}S}}(\Phi \times E^2)(A, B).$$

Let us denote by $\ast$ the convolution on $\mathcal{G}'$ coming from the groupoid

$$\Phi \times E \times E \Rightarrow \Phi \times E.$$

We thereby get a monoidal structure on $\mathcal{G}'$.  

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Let us also denote by $\ast$ the convolution on $\mathcal{F}'$ coming from the groupoid

$$(\mathcal{F} \times E)^2 \to \mathcal{F} \times E.$$  

This way we get a monoidal structure on $\mathcal{F}'$.

Let $D_G = \dim \mathcal{F} \times E - \dim M$; $D_F := \dim \mathcal{F} \times E$. Then we have lax monoidal functors $\Sigma_G : \mathcal{G}' \to \mathcal{M}_G$ and $\Sigma_F : \mathcal{F}' \to \mathcal{M}_F$, where $\Sigma_G(X) = i(X)[-D_G]$ and $\Sigma_F(X) = i(X)[-D_F]$.

The induced maps

$$\Hom_G(X_1 \ast X_2 \ast \cdots \ast X_n; X) \cong \Hom_{\mathcal{M}_G}(\Sigma_G(X_1) \otimes \Sigma_G(X_2) \otimes \cdots \otimes \Sigma_G(X_n); \Sigma_G(X));$$

$$\Hom_F(X_1 \ast X_2 \ast \cdots \ast X_n; X) \cong \Hom_{\mathcal{M}_F}(\Sigma_F(X_1) \otimes \Sigma_F(X_2) \otimes \cdots \otimes \Sigma_F(X_n); \Sigma_F(X)).$$

are isomorphism for all $X_i, X \in \mathcal{G}'$ (resp. all $X_i, X \in \mathcal{F}'$).

### 14.5 Reduction to $\mathcal{H}'$, $\mathcal{G}'$, and $\mathcal{F}'$

As the essential image of $\Sigma_\mathcal{H}$ is $\text{ho}(\mathcal{M}_\mathcal{H})_1$ and likewise for $\Sigma_\mathcal{G}$, $\Sigma_\mathcal{F}$, we have induced lax tensor functors

$$E_{\mathcal{H}'}: \mathcal{H}' \to \mathcal{G}', \quad E_{\mathcal{G}'}: \mathcal{G}' \to \mathcal{F}'.$$

Denote $E_{\mathcal{H}'}: = E_{\mathcal{G}'} : E_{\mathcal{F}'}$.

The algebra $M_\mathcal{H}' \in \mathcal{M}_\mathcal{H}$ as well as its bimodule $M_\mathcal{U}'^\mathcal{H}$ determines an algebra in $\mathcal{H}'$ and its bimodule, denoted by $A_{\mathcal{H}}', M_{\mathcal{H}}^\mathcal{F}'$. Likewise we have an algebra and its bimodule $A_{\mathcal{F}}', M_{\mathcal{F}}^\mathcal{F}'$ in $\mathcal{F}'$ which are isomorphic to $E_{\mathcal{H}'}(A_{\mathcal{H}}'), E_{\mathcal{F}'}(M_{\mathcal{F}}^\mathcal{F}')$. The Hochschild complexes for $M_{\mathcal{U}}^\mathcal{H}, M_{\mathcal{U}}^\mathcal{F}$ are isomorphic to those for $M_{\mathcal{U}}^\mathcal{H}', M_{\mathcal{U}}^\mathcal{F}'$, the map $E_{\mathcal{H}'}$ induces a map of Hochschild complexes isomorphic to that induced by the map $M_{\mathcal{H}} \to M_{\mathcal{F}}$. Thus, the problem is reduced to showing that the map $E_{\mathcal{H}'}$ induces an isomorphism of Hochschild complexes

$$\text{Hoch}(M_{\mathcal{U}}^\mathcal{H}', A_{\mathcal{H}}') \to \text{Hoch}(M_{\mathcal{U}}^\mathcal{F}; A_{\mathcal{F}}'); \quad \text{Hochcyc}(M_{\mathcal{U}}^\mathcal{H}') \to \text{Hochcyc}(M_{\mathcal{U}}^\mathcal{F}').$$

### 14.6 Description of the functor $E_{\mathcal{G}'}: \mathcal{G}' \to \mathcal{F}'$

Let $\pi : \Phi \times E \times E \to \mathcal{F} \times M \mathcal{F} \times E \times E \to (\mathcal{F} \times E)^2$.

We also have objects $P_{\text{MetR}}, Q_{\text{MetR}} \in \mathcal{F}'$. We now have

$$E_{\mathcal{G}'}(T) = Q_{\text{MetR}} \ast \pi T \ast P_{\text{MetR}}.$$

### 14.7 The functor $E_{\mathcal{H}'}$ and a natural transformation $P_{r_j} \ast E \pi T \ast E P_{r_1}[-2N] \to E_{\mathcal{H}'}(T)$.

Let $\Delta : \mathcal{F} \times E \times E \to \mathcal{F} \times \mathcal{F} \times E \times E$ be the embedding. Let us calculate $\Delta' P_{\text{MetR}}$

The object $P_{\text{MetR}}$ is supported on the subset $U_{ij} \times E \times E \subset \mathcal{F} \times \mathcal{F}$. The embedding $\Delta$ factors through $U_{ij} \times E \times E$ so that we have a closed embedding of smooth manifolds $\delta : \mathcal{F} \times E \times E \to U_{ij} \times E \times E$. The embedding $\delta$ is of codimension $2N$. 

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It follows that \( P_{\text{MetR}}|_{U_{ii} \times E \times E} \) is non-characteristic on \( T_{\text{Fr}_i \times E \times E}^{*} U_{ii} \times E \times E \), therefore we have
\[
\delta\P_{ii} \sim \Delta^{-1}\P_{ii}[-2N] \sim p_{\text{Fr}_i}^{-1} P_{\text{fr}}[-2N],
\]
where \( p_{\text{Fr}_i} : \text{Fr}_i \times E \times E \to E \times E \) is the projection.

Let
\[
\pi : \Phi_{ij} \times E \times E \to \text{Fr}_i \times M \text{Fr}_j \times E \times E \to \text{Fr}_i \times \text{Fr}_j \times E \times E
\]
be the through map. Let
\[
(E_{G^{G'}})_{ji}(T) := E_{G^{G'}}(T)|_{\text{Fr}_j \times \text{Fr}_i \times E \times E}
\]
be the corresponding component.

Let
\[
E'_{ji}(T) := P_{r_j} \circ E \pi \circ T \circ P_{r_i}[-2N].
\]
We have a natural transformation
\[
E'_{ji} \to (E_{G^{G'}})_{ji},
\]
where we have taken into account that \( Q_{\text{MetR}} = P_{\text{MetR}}[2N] \).

14.7.1 Splitting the groupoid map \( \Phi_{ff} \times_M U \to \Sigma_{ff} \times_M U \)

Let \( f \in S \) be the maximal element. In particular, we have \( f \gg f \).

Let \( \mathcal{L}_f := \text{Fr}_f / \text{SU}(N) ; \)
\[
\Sigma_{ff} = (\mathcal{L}_f \times_M \mathcal{L}_f) \times_{S^1 \times S^1} \mathbb{R} \times \mathbb{R},
\]
so that we have an embedding \( i_f : \Sigma_{ff} \to \Sigma \).

We have maps of groupoids
\[
(\Phi_{ff} \Rightarrow \text{Fr}_f) \to (\Sigma_{ff} \Rightarrow \mathcal{L}_f) \to (\Sigma \Rightarrow \mathcal{L}).
\]

Let us split the map of groupoids
\[
(\Phi_{ff} \Rightarrow \text{Fr}_f)|_U \to (\Sigma_{ff} \Rightarrow \mathcal{L}_f)|_U.
\]

First, fix a \( U(N) \times S^1 \)-equivariant trivialization of the bundle \( \text{Fr}_f|_U = U \times (U(N) \times S^1) \). Let
\[
s : (U(N) \times S^1) \times (U(N) \times S^1) \to S^1 \times S^1
\]
be given by \( s(g, h) = \det(g^{-1}h) \).

Let
\[
s_0 : (S^1 \times S^1) \times (S^1 \times S^1) \to S^1 \times S^1
\]
be given by \( s(g, h) = g^{-1}h \).

Let
\[
(U(N) \times S^1) \times (U(N) \times S^1) := ((U(N) \times S^1) \times (U(N) \times S^1)) \times_{s,S^1 \times S^1} \mathbb{R} \times \mathbb{R};
\]
\[
(S^1 \times S^1) \times (S^1 \times S^1) := ((S^1 \times S^1) \times (S^1 \times S^1)) \times_{s_0,S^1 \times S^1} \mathbb{R} \times \mathbb{R}
\]

The above map now can be rewritten as
\[
(U \times (U(N) \times S^1) \times (U(N) \times S^1) \Rightarrow U \times (U(N) \times S^1)) \rightarrow \left( U \times (S^1 \times S^1) \times (S^1 \times S^1) \Rightarrow U \times (S^1 \times S^1) \right).
\]

Let \( z : S^1 = U(1) \rightarrow U(N) \) be the embedding induced by the embedding \( C^1 \rightarrow C^N \). It defines a splitting \( (S^1 \times S^1) \times (S^1 \times S^1) \rightarrow (U(N) \times S^1) \times (U(N) \times S^1) \), whence the desired splitting
\[
z : \Sigma \Rightarrow \Phi.
\]

Denote by \( D_\zeta \) the codimension of a smooth embedding \( \zeta \).

We have the following natural transformation of functors
\[
E_{H \to F}(T) \cong E_{G \to F}(T) \Rightarrow E_{H \to F}(T) \Rightarrow E_{G \to F}(T) =: K(T).
\]

14.7.2 Applying the natural transformation to the Hochschild complexes

Consider the following commutative diagram in \( \text{ho}(D \oplus \text{CMetR}) \):

\[
\begin{array}{ccc}
\text{Hoch}(M_{U}^{bb}; A^{bb}) & \xrightarrow{\alpha} & \text{Hoch}(M_{U}^{bb}; A^{F}) \\
| & & |
\downarrow{\beta} & \downarrow{\lambda}
\text{Hom}_{BD}(M_{U}^{bb}; A^{bb}) & \xrightarrow{\mu} & \text{Hom}_{BD}(K(M_{U}^{bb}); A_{\epsilon}^{bb})
\end{array}
\]

Our goal is to show that \( \alpha \) is an isomorphism. We will deduce it from the following statements:

— \( \nu \mu \) is an isomorphism;
— \( \nu \lambda \) is an isomorphism;
— \( \beta \) is an isomorphism.

14.8 \( \nu \mu \) is an isomorphism

Let
\[
e : Fr \times_M Fr \times E \times E \rightarrow Fr \times Fr \times E \times E
\]
be the embedding.

We have a natural transformation \( e_{i} e^{-1} A_{\epsilon}^{bb}[2N] \rightarrow A_{\epsilon}^{F} \) which induces a homotopy equivalence
\[
\text{Hom}_{BD}(K(M_{U}^{bb}); e_{i} e^{-1} A_{\epsilon}^{bb}[2N]) \cong \text{Hom}_{BD}(K(M_{U}^{bb}); A_{\epsilon}^{F})
\]

The natural transformation \( K \rightarrow E_{BD} \) factors as
\[
K \rightarrow e_{i} e^{-1} E_{BD}[2N] \rightarrow E_{BD},
\]

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so that we have a map

\[ \text{Hom}_{F'}(M^b_U; A^{b'}) \to \text{Hom}_{F'}(K(M^b_U); e_1 e^{-1} A^{b'}_{f'f}[-2N]) \]

the problem reduces to showing this map to be a homotopy equivalence.

Let us rewrite the definition of the functor \( e_1^{-1} E_{f'f'} \).

Let \( f' = (g_f, R_f/2) \). Let \( W \subset M \times M \) consist of all pairs \((m_1, m_2)\), where \( B_f(m_2) \subset B_{f'}(m_1) \). Let \( \pi_1, \pi_2 : W \to M \) be the projections onto the factors. We have a projection

\[ U_{f'f} \to W. \]

Let

\[ u_0 := U_{f'f} \times_W U_{f'f}. \]

The projection \( U_{f'f} \to \Phi_{f'} \) induces an identification

\[ U_0 \to \Phi_{f'} \times_M \Phi_{f'} \times_M W. \]

We have projections

\[ U_0 \to \Phi_{f'} \times_M \Phi_{f'}; \quad U_0 \to \Phi_f \times_M \Phi_f. \]

Let also \( U := \Phi_{f'f'} \times_M W \). The above projections lift canonically to projections

\[ p_{f'} : U \to \Phi_{f'f'}; \quad p_f : U \to \Phi_{ff}. \]

Let \( u_1, u_2 : U \to U_{f'f} \) be the projections.

Let \( v_1, v_2 : U \times E \times E \times E \times E \to U_{f'f} \times E \times E \) be given by \( v_1(x, e_1, e_2, e_3, e_4) = (u_1(x), e_1, e_2) \); \( v_2(x, e_1, e_2, e_3, e_4) = (u_2(x), e_3, e_4) \); Let \( w : U \times E \times E \times E \times E \to U \times E \times E \) be the projection along the second and the third factors of \( E \):

\[ w(x, e_1, e_2, e_3, e_4) = (x, e_1, e_4). \]

Let \( \sigma : \Phi_{ff} \to \overline{U(N)} \times S^T \) be the projection.

Let

\[ \varepsilon : U \times E \times E \times E \times E \to \overline{U(N)} \times S^T \times E \times E \]

be the map induced by \( \sigma p_f : U \to \overline{U(N)} \times S^T \) and the projection onto the second and the third factors of \( E \).

Let us define an object \( Z_1 \in \text{psh}(U \times E \times E, S_0) \) as follows. Set

\[ Z_1 := w_*(v_1^{-1} \Phi_{f'f} \otimes v_2^{-1} \Phi_{ff} \otimes \varepsilon^{-1} \Xi). \]

Set

\[ Z := P_{r_f} \ast_E Z_1 \ast_E P_{r_f}. \]

Let \( q : U \to \Phi_{ff} \to \Sigma_{ff} \) be the through map. We have

\[ e^{-1} E_{f'f'}(T) \sim \rho p_{f'f} (Z \otimes q^{-1} T), \]

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where $\rho : \Phi_{f'} \to Fr_f \times_M Fr_{f'}$ is the covering map.

We have a natural transformation

$$p_{f'}!(Z \otimes q^{-1}T) \to p_{f'}! Z \otimes p_{f'}! q^{-1}T[2N],$$

which is a homotopy equivalence as long as $T$ is a locally constant object.

We have an identification $\Phi_{f'} = \Phi_f \times U(N) \times S^1$ which induces an identification

$$\mathcal{U} = U_f \times U(N) \times S^1.$$

Let is rewrite the definition of $Z_1$ under this identification.

Let $p_{12}, p_{34}, p_{14} : U_{f'} \times E \times E \times E \times E \to U_f \times E \times E$ be the projections along the last two factors of $E$ (resp. the first two factors of $E$, resp. along the 2-nd and the 3-rd factors).

Let $p_{23} : U_{f'} \times E \times E \times E \times E \to E \times E$ be the projection onto the 2-nd and the 3-rd factors of $E$.

Set

$$\mathcal{Y} := p_{14}((p_{12}^{-1}p_{f'} \otimes p_{23}^{-1}P \otimes p_{34}^{-1}Q_{f'}))$$

We have

$$Z \cong \mathcal{Y} \ast_E S.$$

Sec. 8.4 implies a homotopy equivalence $p_{f'!} \mathcal{Y} \sim t^{-1}P_{r_f}[2N]$, where $t : Fr_f \times E \times E \to E \times E$ is the projection.

Therefore, we have a natural transformation of functors

$$e^{-1}E_{f'}(T) \to P_{r_f} \ast_E \rho_t(p_{f!}q^{-1}T \ast_E S) \ast_E [2N] =: F(T)$$

which is a homotopy equivalence provided that $T$ is locally constant.

We have a natural transformation $\mathcal{K} \to E_{f'} \to F$, and the problem reduces to showing that the induced map

$$\text{Hom}_{f'}(M^*_{U_f}; A^{*f}) \to \text{Hom}_{f'}(\mathcal{K}(M^*_{U_f}); F(A^{*f}))$$

is a homotopy equivalence.

Let

$$\iota : \mathbb{R} \times \mathbb{R} = U(1) \times S^1 \to U(N) \times S^1$$

be the embedding.

Let $r : Fr_f \times U(N) \times S^1 \to \Sigma$ be the projection and let

$$\zeta : \Sigma = Fr_f \times U(1) \times S^1 \to Fr_f \times U(N) \times S^1$$

be the embedding induced by $\iota$. We have

$$\mathcal{K}(T) = \rho_t((\zeta_! T) \boxtimes \Xi) \ast_E S$$

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We now have the following chain of homotopy equivalences:

\[
\begin{align*}
\text{Hom}_{F}(\mathcal{K}(M_{U}^{fr} ; F(A^{fr})); & \sim \text{Hom}_{F}(\mathcal{K}(\zeta \mathcal{M}_{U}^{fr} ; \Xi) * E) * E[2N]; p^{1}P_{f^{*}} * E \rho_{1}(p_{f^{*}}q^{-1}A^{fr} * E) * E P_{f^{*}}[2N]) \\
& \sim \text{Hom}_{F}(\mathcal{K}(\zeta \mathcal{M}_{U}^{fr} ; \Xi) * E) * E[S_{U}] ; \bigoplus \left( T_{a}(p_{f^{*}}q^{-1}A^{fr} * E)[2N] \right) \\
& \sim \text{Hom}_{F}(\mathcal{K}(M_{U}^{fr} \otimes \Xi) ; \zeta^{-1} \bigoplus \left( T_{a}(p_{f^{*}}q^{-1}A^{fr} * E)[2N] \right) \otimes K_{\Delta_{E}})
\end{align*}
\]

We have used a homotopy equivalence

\[
p^{-1}p_{1}S \sim \bigoplus_{ab} T_{a} \pi^{-1}S[2a],
\]

where

\[
\pi : \Phi_{f^{*}} \times E \to \overline{U(N) \times S^{T} \times E \times E}
\]

is the projection. The map \( \Xi \to K_{\Delta_{E}} \) in shval\( E \times E \) induces a homotopy equivalence

\[
\text{Hom}_{\Sigma}(M_{U}^{fr} ; \bigoplus_{ab} T_{a} \zeta^{-1}(p_{f^{*}}q^{-1}A^{fr})[2N]) \sim \text{Hom}_{F}(M_{U}^{fr} \otimes \Xi ; \zeta^{-1} \bigoplus_{ab} T_{a}(p_{f^{*}}q^{-1}A^{fr}[2N]) \otimes K_{\Delta_{E}})
\]

Let the embedding

\[
\eta : \Phi_{f^{*}} \times \overline{U(N) \times S^{T}} \to U_{f^{*}} \times \overline{U(N) \times S^{T}}
\]

be induced by the diagonal embedding \( \Phi_{f^{*}} \to U_{f^{*}} \). We have an equivalence (provided that \( T \) is locally constant):

\[
T \sim \zeta^{-1} \eta^{-1} p_{f^{*}}q^{-1} T
\]

as well as a map

\[
\zeta^{-1} \eta^{-1} p_{f^{*}}q^{-1} T \to \zeta^{-1} p_{f^{*}}q^{-1} T[2N]
\]

which results in the following natural transformation

\[
T \to \zeta^{-1} p_{f^{*}}q^{-1} T[2N]. \quad (69)
\]

This transformation induces a map

\[
\text{Hom}_{\Sigma}(M_{U}^{fr} ; \bigoplus_{ab} T_{a} A^{fr}[2b]) \to \text{Hom}_{\Sigma}(M_{U}^{fr} ; \bigoplus_{ab} T_{a} \zeta^{-1}(p_{f^{*}}q^{-1}A^{fr})[2b][2N])
\]

It follows that this map is isomorphic to the map in (68). The problem therefore reduces to showing that the map

\[
\mathbb{K}_{\Sigma,0} \to \zeta^{-1} p_{f^{*}}q^{-1} \mathbb{K}_{\Sigma,0}[2N]
\]

induced by the natural transformation in (69) is a homotopy equivalence which is immediate.
14.9 \( \nu\lambda \) is an isomorphism

The second statement reduces to the following one. We have a map \( \mathcal{K}(M_{U}^{2f})_{ff} \rightarrow (M_{U}^{ff})_{ff} \). We are to show that the following composition is an isomorphism in \( \mathbb{F}^{'} \):

\[
A_{ij}^{ff} \cdot \mathcal{K}(M_{U}^{2f})_{ff} \cdot A_{ij}^{ff} \rightarrow A_{ij}^{ff} \cdot M_{U}^{ff} \cdot A_{ij}^{ff} \rightarrow (M_{U}^{ff})_{ij},
\]

(70)

A. Let \( T \subset S \), \( T \in \text{SMetR}^{'} \). Let

\[
h_{T} : S_{0}(\mathfrak{A}) \rightarrow \text{Classic}(\mathfrak{A})\langle \text{val}\mathcal{T} \rangle
\]

be the projection.

It suffices to show that

\[
h_{T}(A_{ij}^{ff}) \cdot h_{T}(\mathcal{K}(M_{U}^{2f})_{ff}) \cdot h_{T}(A_{ij}^{ff})
\]

\[
\rightarrow h_{T}(A_{ij}^{ff} \cdot \mathcal{K}(M_{U}^{2f})_{ff} \cdot A_{ij}^{ff}) \cong h_{T}(A_{ij}^{ff} \cdot M_{U}^{ff} \cdot A_{ij}^{ff})
\]

\[
\rightarrow h_{T}((M_{U}^{ff})_{ij}).
\]

(71)

is an isomorphism in \( \text{hoClassic}(\mathfrak{A})\langle \text{val}\mathcal{T} \rangle \).

Let \( \sigma \) be the minimum of \( T \). We have a natural map \( h_{T} \rightarrow h_{(\sigma)} \) which induces an isomorphism on both sides of the above through map. Therefore, we reduce the problem to the case when \( T = \{ \sigma \} \) is a one element set.

B. Let us reformulate [71]. Let \( p_{M} : \text{Fr}_{f} \rightarrow M \), \( p_{M} : \text{Fr}_{\sigma} \rightarrow M \) be the projections. Let

\[
\mathcal{J} \subset \text{Fr}_{i} \times \text{Fr}_{\sigma} \times \text{Fr}_{\sigma} \times \text{Fr}_{f} \times \text{Fr}_{\sigma} \times \text{Fr}_{f} \times \text{Fr}_{\sigma} \times \text{Fr}_{\sigma} \times \text{Fr}_{j}
\]

be the subset consisting of all points \((f_1, f_2, \ldots, f_{10})\), where \((f_2, f_1) \in U_{\sigma f_{R_i}}, (f_2, f_3) \in \text{Fr}_{\sigma} \times M \text{Fr}_{\sigma}, (f_3, f_4) \in U_{\sigma f_{R_j}}, (f_5, f_4) \in U_{\sigma f_{R_j}}, (f_5, f_6) \in \text{Fr}_{\sigma} \times M \text{Fr}_{\sigma}, (f_6, f_7) \in U_{\sigma f_{R_j}}, (f_8, f_7) \in U_{\sigma f_{R_j}}, (f_8, f_9) \in \text{Fr}_{\sigma} \times M \text{Fr}_{\sigma}, (f_9, f_{10}) \in U_{\sigma f_{R_j}}\).

Let \( p_{mn}, 1 \leq m < n \leq 10 \) be the projection from

\[
\text{Fr}_{i} \times \text{Fr}_{\sigma} \times \text{Fr}_{\sigma} \times \text{Fr}_{f} \times \text{Fr}_{\sigma} \times \text{Fr}_{f} \times \text{Fr}_{\sigma} \times \text{Fr}_{\sigma} \times \text{Fr}_{j}
\]

onto the \( m \)-th and the \( n \)-th factors. Still denote by \( p_{mn} \) the restriction of these projections onto \( \mathcal{J} \).

Let \( \pi : \mathcal{J} \rightarrow (U(N) \times S^{1})^{3} \) be the through map

\[
\mathcal{J}^{P_{23} \times P_{56} \times P_{89}} (\text{Fr}_{\sigma} \times M \text{Fr}_{\sigma})^{3} \rightarrow U(N) \times S^{1}^{3}
\]

be the composition.

Let

\[
\overline{\mathcal{J}} := \mathcal{J} \times \pi, (U(N) \times S^{1})^{3} \rightarrow \overline{U(N) \times S^{1}^{3}}
\]

Let \( q_{23}, q_{56}, q_{89} : \overline{\mathcal{J}} \rightarrow \overline{U(N) \times S^{1}} \) be the projections.
We then rewrite the LHS of (71) as
\[ p_{1,10}( p_{12}( P_{i,R}, \sigma ) \ast E q_{23}^{-1} \gamma_r \ast E q_{34}^{-1} Q_{\sigma_f} \ast E q_{45}^{-1} P_{\sigma_f} \ast E q_{56}^{-1} \gamma_r \ast E p_{67}^{-1} Q_{\sigma_f} \ast E p_{78}^{-1} P_{\sigma_f} \ast E q_{89}^{-1} \gamma_r \ast p_{9,10}^{-1} Q_{\sigma_{jR}} P_{r_j} ) \].

Let
\[ K \subset F_{r_j} \times ( F_{r_j} \times F_{r_j} )^3 \times F_{r_j} \]

consist of all points \((f_1, f_2, f_3, f_5, f_6, f_8, f_9, f_{10})\), where \((f_1, f_2) \in U_{i,R}; (f_2, f_3), (f_8, f_9) \in F_{r_j} \times_M F_{r_j}, (f_5, f_6) \in F_{r_j} \times_M F_{r_j} \mid U, (f_9, f_{10}) \in U_{jR} \).

Let \( q_{23}, q_{56}, q_{89} : K \to U(N) \) be the projections. Let \( \pi : q_{23} \times q_{56} \times q_{89}, \pi : K \to (U(N) \times S^1)^3 \).

Let \( p_K : J \to K \) be the projection. Let \( p_K^\sigma \) be the projection from \( K \) to the corresponding pair of factors.

The map \( P_{\sigma_f} \ast Q_{\sigma} \to K_{\Delta_{Fr \times E}} \) and the algebra structure on \( \gamma_r \) induce the following maps
\[
p_{1,10}( p_{12}( P_{i,R}, \sigma ) \ast E q_{23}^{-1} \gamma_r \ast E q_{34}^{-1} Q_{\sigma_f} \ast E q_{45}^{-1} P_{\sigma_f} \ast E q_{56}^{-1} \gamma_r \ast E p_{67}^{-1} Q_{\sigma_f} \ast E p_{78}^{-1} P_{\sigma_f} \ast E q_{89}^{-1} \gamma_r \ast p_{9,10}^{-1} Q_{\sigma_{jR}} P_{r_j} ) \\
\to p_{1,10}( p_{12}( P_{i,R}, \sigma ) \ast E q_{23}^{-1} \gamma_r \ast E p_{35}^{-1} K_{\Delta_{Fr \times E}} \ast E q_{56}^{-1} \gamma_r \ast E p_{68}^{-1} K_{\Delta_{Fr \times E}} \ast E q_{89}^{-1} \gamma_r \ast p_{9,10}^{-1} Q_{\sigma_{jR}} P_{r_j} ) \\
\to p_{1,10}( p_{12}( P_{i,R}, \sigma ) \ast E ( q_{235689}^{-1} \gamma_r |_{\Delta_{Fr \times E}} )^{-1} E p_{9,10}^{-1} Q_{\sigma_{jR}} P_{r_j} )
\]
The resulting map is isomorphic to the map \( (71) \).

C. Let \( \sigma = (g_\sigma, r_\sigma) \). For an \( r > 0 \) denote \( \sigma_r := (g_\sigma, r); \ i_r := (g_i, r) \), etc.

As \( \sigma < i, j \), there exists a number \( \rho > r_\sigma \) such that \( \sigma_\rho < i, j \). Fix such a \( \rho \).

In particular we have the following families of symplectic embeddings
\[ I_{\sigma, i} : U_{\sigma, i} \times B_\rho \to B_{R_i}; \quad I_{\sigma, j} : U_{\sigma, j} \times B_\rho \to B_{R_j} \]

2) There exists a number \( r > 0 \) satisfying:
\[ \forall n_1, n_2 \in M : B_\sigma(n_1) \cap B_\sigma(n_2) = \emptyset \Rightarrow B_\sigma(n_2) \subset \text{int} B_\sigma(n_1) \]

C. Define \( J_r \subset F_{r_j} \times J \times F_{r_j} \) to consist of all points \((f_0, f_1, \ldots, f_{10}, f_{11})\) satisfying \( B_{\sigma, f_0} \subset B_{i}(f_1) \), \( B_{\sigma, f_0} \cap B_{\sigma, f_2} = \emptyset \), \( B_{\sigma, f_0} \cap B_{\sigma, f_11} = \emptyset \), \( B_{j}(f_{10}) \supset B_{\sigma, f_{11}} \).

Let us define \( K_r \subset F_{r_j} \times K \times F_{r_j} \) in a similar way.

The inclusions \( B_{\sigma, f_0} \subset B_{i}(f_1), B_{j}(f_{10}) \supset B_{\sigma, f_{11}} \) define, via their differential at 0, the maps
\[ q_{01, 10, 11} : J_r \to K_r \to \text{Sp}(2N) \]

Let
\[ q_J : J_r \to K_r \to \text{Sp}(2N) \times \text{Sp}(2N) \]

be the product.
Let
\[ J_r := (J_r \times J_r) \times_{q_J, Sp(2N) \times Sp(2N)} Sp(2N) \times Sp(2N); \]
\[ K_r := (K_r \times K_r) \times_{q_K, Sp(2N) \times Sp(2N)} Sp(2N) \times Sp(2N). \]

We have a projection \( p_{012} : J_r \to K_r. \) Let \( J_r := J_r \times K_r, \) \( K_r^i := K_r \times_{K_r} K_r^i. \)

We have the following inclusions: \( B_{\sigma_r}(f_0) \subset B_{\sigma_r}(f_2) \subset B_i(f_1). \) Whence the following families of symplectic embeddings
\[ a : J_r \times B_{\sigma_r}(f_0) \to B_{\sigma_r}(f_2); \]
\[ b : J_r \times B_{\sigma_r}(f_2) \to B_i(f_1). \]

Let also
\[ b_r : J_r \times B_{\sigma_r}(f_2) \to B_i(f_1) \]
be the restriction of \( b. \) By the construction, \( b_r \) and \( ba \) are graded. Therefore, so is \( b \) and, hence \( a. \) Let \( c = ba. \) Let \( A^i, B^i \) be the quantizations of \( a, b. \) The quantization of \( b_r \) is then \( B^i * E P_{r_2}. \)

Let \( C^i := B^i * E A^i \) be the quantization of \( ba. \) Let \( A^i, B^i, C^i \) be the similar objects for \( j. \)

It suffices to show that the following map is an isomorphism
\[ p_{01,10,11!}((C^i)^t \ast E B^i \ast E P_{r_2} \ast E q_2^{23} \gamma_{r_\sigma} \ast E q_3^{34} Q_{r_\sigma f} \ast E q_4^{45} P_{r_\sigma}) \]
\[ \ast E q_5^{56} \epsilon_{i_\gamma} \gamma_{r_\sigma} \ast E P_{78} P_{r_\sigma} \ast E q_8^{89} \gamma_{r_\sigma} \ast E P_{r_\sigma} \ast E (B^i)^t \ast E (C^i)^t \]
\[ \to p_{01,10,11!}((C^i)^t \ast E (B^i)^t \ast E P_{r_\sigma} \ast E (q_2^{235689} \gamma_{r_\sigma} |_{\Delta_{r_\sigma}}) \ast E P_{r_\sigma} \ast E (B^i)^t \ast E (C^i)^t) \]  \( (72) \)

We have \( C^i \ast E B = A^t \ast E B^i \ast E B \ast E P_{r_\sigma} \equiv A^i \ast E P_{r_\sigma}. \) Accordingly, we rewrite:

\[ p_{01,10,11!}((A^i)^t \ast P_{r_\sigma} \ast E q_2^{23} \gamma_{r_\sigma} \ast E q_3^{34} Q_{r_\sigma f} \ast E q_4^{45} P_{r_\sigma}) \ast E q_5^{56} \epsilon_{i_\gamma} \gamma_{r_\sigma} \ast E P_{78} P_{r_\sigma} \ast E q_8^{89} \gamma_{r_\sigma} \ast E P_{r_\sigma} \ast E A^i \]
\[ \to p_{01,10,11!}((A^i)^t \ast E P_{r_\sigma} \ast E (q_2^{235689} \gamma_{r_\sigma} |_{\Delta_{r_\sigma}}) \ast E P_{r_\sigma} \ast E A^i) \]  \( (73) \)

Let \( \kappa : J_r \to Fr_{r_\sigma} \times Fr_{r_1} \times Fr_{r_2} \times Fr_{r_3} \times Fr_{r_4} \) be the projection. Denote by \( Q \) the image of \( \kappa. \) Let us number the factors from 1 to 10. denote by \( p_{ij} \) the projection from \( Q \) onto the corresponding pair of factors. As above, we have projections
\[ q_{34}, q_{56}, q_{78} : Q \to Fr_{r_\sigma} \times_M Fr_{r_\sigma} \to U(N) \times S^1. \]

Let \( q : Q \to (U(N) \times S^1)^3 \) be the product. Let
\[ \overline{Q} := Q \times_{q, (U(N) \times S^1)} U(N) \times S^1. \]

The projection \( \kappa \) lifts to a map
\[ k : J_r \to \overline{Q}. \]

Because of the geodesic convexity, \( k : J_r \to \overline{Q} \) is a fibration with contractible fibers.
Denote
\[
S := (A^i)^* E P_{r_s} * E q_{13}^{-1} \gamma_{r_s} * E q_{34}^{-1} Q_{r_s f} * E q_{45}^{-1} P_{f r_s} * E q_{56}^{-1} Q_{r_s f} * E q_{67}^{-1} P_{f r_s} * E q_{78}^{-1} \gamma_{r_s} * E P_{r_s} * E A^i)
\]
as in (73).

It follows that
(1) \(S\) is constant along the fibers of \(q\), in particular, the map \(k^{-1} k_s S \to S\) is a homotopy equivalence.

(2) The object \(k_s S\) is supported on the following subset \(\bar{Q}' \subset \bar{Q}\) consisting of all points \(p\) whose projection onto \(Q\), \((f_1, f_2, \ldots, f_{10})\), satisfies \(B_\sigma(f_4) \cap B_\sigma(f_5) \neq \emptyset\), \(B_\sigma(f_6) \cap B_\sigma(f_7) \neq \emptyset\).

Let \(\mathcal{N} \subset M^{10}\) consist of all points \((m_1, m_2, \ldots, m_{10})\) satisfying:
\[
B_\sigma(m_1) \subset B_2(m_2),
\]
\[
B_\sigma(m_1) \cap B_\sigma(m_3) \neq \emptyset, m_3 = m_4, B_\sigma(m_4) \cap B_\sigma(m_5) \neq \emptyset, m_5 = m_6 \in U, B_\sigma(m_6) \cap B_\sigma(m_7) \neq \emptyset, m_7 = m_8, B_\sigma(m_8) \cap B_\sigma(m_{10}) \neq \emptyset, B_\sigma(m_9) \supset B_\sigma(m_{10}).
\]

According to Assumption 14.4, \(U \subset B_\sigma(m_0)\) for some point \(m_0 \in M\). Let \(V := B_\sigma_{R_\sigma}(m_0)\) and \(W := B_\sigma_{R_{\sigma}'}(m_0)\), see Sec. 11.4 for definition of \(R_\sigma', R_{\sigma}'\). It follows that
(1) \(\mathcal{N} \subset V^{10}\);
(2) if \(n \in V\), then \(B_\sigma_{R_\sigma}(v) \subset W\).

Choose a section \(s\) of \(\text{Fr}_\sigma|_W\) (which exists because \(W\) is a ball).

Let \(\pi : \bar{Q}' \to \mathcal{N}\) be the projection. There exist maps \(\lambda_i : \mathcal{N} \to U(N), i = 1, 2, 3, 4\), such that for every point \(m = (m_1, \ldots, m_{10}) \in \mathcal{M}_M\), the set \(\pi^{-1} p\) consists of all points of the form
\[
(g_1 s(m_1), g_2 s_2(m_2), g_2 \lambda_1(p)s(m_3), c_1 \lambda_2(p)s(m_4), c_1 s(m_5), c_2 s(m_6), c_2 \lambda_3(p)s(m_7),
\]
\[
g_3 \lambda_4(p)s(m_8), g_3 s_j(m_9), g_4 s(m_{10})),
\]
where \(g_i \in \bar{U}(N) \times S^1, c_i \in R \times R = S^1 \times S^1\). We thus have an identification
\[
\bar{Q}' \cong \bar{U}(N) \times S^1 \times S^1 \times S^1 \times \mathcal{N}.
\]

Let us rewrite (73) under this identification. Denote by \(g_i : \bar{Q}' \to \bar{U}(N) \times S^1; c_i : \bar{Q}' \to S^1 \times S^1\) the corresponding projections.

Next, we have a graded family of symplectic embeddings
\[
\bar{U}(N) \times S^1 \times V \to U(N) \times S^1 \times V \cong \text{Fr}_\sigma|_V \times B_{R_\sigma} \to B_{R_\sigma}'.
\]

Denote \(\text{Fr}_\sigma := \bar{U}(N) \times S^1 \times V\). Denote by \(T \in \text{sh}_q(\text{Fr}_\sigma \times E \times E)\) the quantization of this family.

We have projections \(\pi_k : \bar{Q}' \to M \to \text{Fr}_\sigma, k = 0, 2, 3, \ldots, 8, 9, 11; g_i : \bar{Q}' \to \bar{U}(N) \times S^1\) be the projections.

We can now rewrite (73) as follows (the functions \(\lambda_i\) drop out from the expression)
Let reduces to showing that the following map is an isomorphism

\[ p_{1,2,9,10}(g_1^{-1}S^*E\pi_1^{-1}T^*E\pi_3^{-1}T|E*E\gamma_{r_\sigma}E*E\pi_4^{-1}T|U*E\gamma_{r_\sigma}E*E\pi_6^{-1}T|U*E\pi_7^{-1}T|E\gamma_{r_\sigma}E*E\pi_8^{-1}T|E\pi_{10}^{-1}T|Eg_4^{-1}S) \]

\[ \rightarrow p_{1,2,9,10}(g_1^{-1}S^*E\pi_1^{-1}T^*E\pi_3^{-1}T|U*E\pi_3^{-1}T|U*E\pi_4^{-1}T|U*E\pi_6^{-1}T|U*E\pi_7^{-1}T|U*E\pi_{10}^{-1}T|U*Eg_4^{-1}S) \]

Let \( p_i : W^5 \rightarrow W \) be projections. Let \( q_i : W^5 \rightarrow W \rightarrow M \rightarrow \mathfrak{fr}_\sigma \) be the projections. The statement reduces to showing that the following map is an isomorphism

\[ p_{1,5!}(q_1^{-1}T^*Eg_2^{-1}T^*E\gamma_{r_\sigma}Eg_3^{-1}T^*Eg_4^{-1}T^*Eg_5^{-1}T) \]

\[ \rightarrow p_{1,5!}(q_1^{-1}T^*Eg_2^{-1}T|U*E\pi_3^{-1}T|U*E\pi_4^{-1}T|U*E\pi_6^{-1}T|U*E\pi_7^{-1}T|U*E\pi_{10}^{-1}T|U*Eg_4^{-1}T) \]

The statement now follows from the mobility of the family \( T \), see Condition 3 in Sec [11.4]

### 15 \( Y(\mathbb{B}) \)-modules

#### 15.1 Infinitesimal operads

An infinitesimal circular operad (ICO) is a colored operad (CO) in the category \( \text{Com}_{[0,1]}(C) \).

Given an ICO \( \mathcal{O} \) let \( |\mathcal{O}| \) denote the CO over \( C \) obtained via applying the totalization functor \( \text{Com}_{[0,1]}(C) \rightarrow C \).

Let \( \text{Gr}_{[0,1]}(C) \) be a category whose every object is a pair \((X^0, X^1)\) of objects of \( C \). We set

\[ \text{Hom}((X^0, X^1); (Y^0, Y^1)) = \text{Hom}(X^0, Y^0) \oplus \text{Hom}(X^1, Y^1). \]

Define a SMC on \( \text{Gr}_{[0,1]}(C) \) by setting

\[ (X^0, X^1) \otimes (Y^0, Y^1) := X^0 \otimes Y^0; X^0 \otimes Y^1 \oplus X^1 \otimes Y^0. \]

We have functors \( \text{Gr}_{[0,1]}(C) \xrightarrow{i} \text{Com}_{[0,1]}(C) \xrightarrow{s} \text{Gr}_{[0,1]}(C) \).

A split ICO is a CO in the category \( \text{Gr}_{[0,1]}(C) \).

Given a split ICO \( \mathcal{O}' \), denote by \( X(\mathcal{O}') \) the set of all maps \( D : (\mathcal{O}')^0 \rightarrow (\mathcal{O}')^1[1] \) such that \((\mathcal{O}', D)\) is an ICO.

#### 15.2 Categories \( Y(\mathcal{O}), Y^\text{cyc}(\mathcal{O}) \)

Let \( \mathcal{O} \) be a CO in \( C \). Denote by \( S(\mathcal{O}) \) the category over \( \text{Sets} \) whose every object is a split ICO \( \mathcal{O}' \) along with an identification of VCO \((\mathcal{O}')^0 = \mathcal{O}\). One can construct a category \( \text{Y}(\mathcal{O}) \) enriched over \( C \), whose objects are of the form \((n)^{\text{noncyc}}, (n)^{\text{cyc}}\), such that \( S(\mathcal{O}) \) is equivalent to the category of functors \( \text{Y}(\mathcal{O})^{\text{op}} \rightarrow C \). Namely, given \( \mathcal{U} \in S(\mathcal{O}) \), the corresponding functor associates \( \mathcal{U}^1(n)^{\text{noncyc}} \) to \((n)^{\text{noncyc}}\) and \( \mathcal{U}^1(n)^{\text{cyc}} \) to \((n)^{\text{cyc}}\).

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Denote by \(Y^{\text{cyc}}(O) \subset Y(O)\) the full sub-category consisting of all objects \((n)^{\text{cyc}}, n > 0\). The category \(Y^{\text{cyc}}(O)\) only depends on the non-cyclic part of \(O\). We therefore can write \(Y^{\text{cyc}}(O^{\text{noncyc}})\) instead of \(Y^{\text{cyc}}(O)\).

It follows that the structure of a circular operad on a pair \((O^{\text{noncyc}}, O^{\text{cyc}})\) is equivalent to that of an asymmetric operad on \(O^{\text{noncyc}}\) and of a functor \(Y^{\text{cyc}}(O^{\text{noncyc}}) \to C\) on \(O^{\text{cyc}}\).

### 15.3 Studying \(Y(\text{grB})\)-modules

Let us get back to the sequence of monoidal categories with trace and their functors in \(57\). Denote by \(O^M\) the full circular operad of \(M^M\), and likewise for \(O^H, O^F\), etc. We therefore have a sequence of circular operads and their maps:

\[
gr\text{B} \to O^M \to O^H \to \text{gr}\text{B}^F.
\]

Hence we have an induced structure of \(Y(\text{grB})\)-modules and their maps

\[
O^M \to O^H \to \text{gr}\text{B}^F.
\]

In this section we will formulate and prove several statements about these modules. We start with introducing certain endofunctors on the category of \(Y(\text{grB})\)-modules and their maps.

### 15.4 Description of \(Y(\text{assoc})\), \(Y(B)\)

The full sub-category of \(Y(\text{assoc})\) consisting of objects \((n)^{\text{noncyc}}\) is isomorphic to \(\Delta\); the full sub-category consisting of all objects \((n)^{\text{cyc}}\) is isomorphic to the cyclic category \(\Lambda\). Denote by \(I : \Delta \to \Lambda\) the natural embedding \([n] \mapsto (n + 1)\). The category \(Y(\text{assoc})\) is then equivalent to the cone of \(I\).

Let us now describe the category \(Y(B)\). Let \(a \in R_q\),

\[
a := \bigoplus_{a, b \in \mathbb{Z}} T_a \text{unit}_{R_q}[2b].
\]

We have a unital commutative algebra structure on \(a\). We have a functor \(\text{Hochcyc}(a) : \Lambda^{\text{op}} \to R_q\). We have an induced commutative algebra structure on \(\text{Hochcyc}(a)\).

We have

\[
\text{Hom}_{Y(B)}((m)^{\text{noncyc}}, (n)^{\text{noncyc}}) = \text{Hochcyc}_m(a) \otimes \text{Hom}_{Y(\text{assoc})}((m)^{\text{noncyc}}, (n)^{\text{noncyc}});
\]

\[
\text{Hom}_{Y(B)}((m)^{\text{noncyc}}, (n)^{\text{cyc}}) = \text{Hochcyc}_m(a) \otimes \text{Hom}_{Y(\text{assoc})}((m)^{\text{noncyc}}, (n)^{\text{cyc}});
\]

\[
\text{Hom}_{Y(B)}((m)^{\text{cyc}}, (n)^{\text{cyc}}) = \text{Hochcyc}_{n-1}(a) \otimes \text{Hom}_{Y(\text{assoc})}((m)^{\text{cyc}}, (n)^{\text{cyc}}).
\]

Denote by \(Y(B)^{\text{noncyc}} \subset Y(B)\) the full sub-category consisting of all objects \((n)^{\text{noncyc}}\). Denote by \(Y(B)^{\text{cyc}} \subset Y(B)\) the full sub-category consisting of all objects \((m)^{\text{cyc}}\). We have a functor \(I : Y(B)^{\text{noncyc}} \to Y(B)^{\text{cyc}}\), where \(I(n)^{\text{noncyc}} = (n)^{\text{cyc}}\).
Let $F : \mathcal{Y}(\mathbb{B}) \to \mathbb{R}$ (or $F : \mathcal{Y}(\text{gr}\mathbb{B}) \to \mathbb{R}$) be a functor. Let $F^{\text{nocy}c} := F|_{\mathcal{Y}(\mathbb{B})^{\text{nocy}c}}; F^{\text{cyc}} := F|_{\mathcal{Y}(\mathbb{B})^{\text{cyc}}}$ be restrictions. The embedding $\text{assoc} \to \mathbb{B}$ gives rise to further restrictions $F^{\text{nocy}c}, \Delta, F^{\text{cyc}}, \Delta : \Delta \to \mathbb{R}$.

Denote $\Gamma^{\text{nocy}c}(F) := C^*(F^{\text{nocy}c}, \Delta); \Gamma^{\text{cyc}}(F) := C^*(F^{\text{cyc}}, \Delta)$.

15.5 Condensation

Let $\mathcal{E}$ be a colored operad in $\mathbb{A}$ with two colors, to be called $\text{a}$ and $\text{m}$. Let $c_1, c_2, \ldots, c_n, c$ be colors. Let $S_\text{a}, S_\text{m} \subset \{1, 2, \ldots, n\}$ be defined by $S_\text{a} = \{i|c_i = \text{a}\}; S_\text{m} = \{i|c_i = \text{m}\}$.

if $c = \text{m}$, set $\mathcal{E}^{\text{nocy}c}(c_1, c_2, \ldots, c_n|c) = \mathbb{K};$

if $c = \text{a}$ and $c_1 = c_2 = \cdots = c_n = \text{a}$, set $\mathcal{E}^{\text{nocy}c}(c_1, c_2, \ldots, c_n|c) = \mathbb{K}$. Otherwise, set $\mathcal{E}(c_1, c_2, \ldots, c_n|c) = 0$. Finally, set $\mathcal{E}^{\text{cyc}}(c_0, c_1, \ldots, c_n) = \mathbb{K}$ for all $n$.

Let $\pi : \{\text{a, m}\} \to \text{pt}$ be the projection. We have a natural map of circular operads $\mathcal{E} \to \pi^{-1}\mathbb{E}$. (76)

Define two functors $\iota, \epsilon : \mathcal{Y}(\mathcal{E}) \to \mathcal{Y}(\mathbb{E})$ as follows.

$-\iota$. On objects:

$\iota(c_1, c_2, \ldots, c_n)^{\text{nocy}c} = (n)^{\text{nocy}c}; \iota(c_1, c_2, \ldots, c_n)^{\text{cyc}} := (n)^{\text{cyc}}$.

On maps: induced by (76);

$-\epsilon$. On objects $\epsilon(c_1, c_2, \ldots, c_n)^{\text{cyc}} := (|S_\text{m}|)^{\text{cyc}}, \epsilon(c_1, c_2, \ldots, c_n)^{\text{nocy}c} := (|S_\text{m}|)^{\text{nocy}c}$.

On arrows, cyclic part: we have

$\mathcal{Y}(\mathcal{E})^{\text{cyc}}((c_1', c_2', \ldots, c_m')^{\text{cyc}}; (c_1, c_2, \ldots, c_n)^{\text{cyc}}) = \bigoplus_f \mathcal{E}(f)$,

where the direct sum is taken over all $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$, where $f$ is cyclic and $f(S_\text{m}) \subset S_\text{m}'$. We thus have an induced map $f' : S_\text{m} \to S_\text{m}'. Next, we have an isomorphism $\mathcal{E}(f) \to \mathbb{E}(f')$, whence an induced map

$\mathcal{Y}(\mathcal{E})^{\text{cyc}}((c_1, c_2, \ldots, c_n)^{\text{cyc}}; (c_1', c_2', \ldots, c_m')^{\text{cyc}}) \to \mathcal{Y}(\mathcal{E})^{\text{cyc}}((|S_\text{m}|)^{\text{cyc}}, (|S_\text{m}'|)^{\text{cyc}})$.

The action of $\epsilon$ on the non-cyclic arrows is defined in a similar way.
We now have an object
\[ \mathcal{K}_{\text{con}} := \varepsilon \otimes \lambda_{(E)}^L \in (D \bigoplus)(Y(\mathcal{B})^\text{op} \otimes Y(\mathcal{B})). \]

Let \( \mathcal{A}, \mathcal{B} \) be arbitrary categories enriched over \( \mathcal{C} \). Let \( K \in (D \bigoplus)(\mathcal{A}^\text{op} \otimes \mathcal{B}) \). Let \( F : \mathcal{B} \to \mathcal{C} \). Define the co-convolution with \( K \), \( \text{Hom}_\mathcal{B}(K; F) : \mathcal{A} \to \mathcal{C} \), where
\[
\text{Hom}_\mathcal{B}(K, F)(a) := \text{Hom}_\mathcal{B}(h_a \circ_A K; F), \quad a \in A.
\]

Let \( F : Y(\mathcal{B}) \to \mathfrak{A} \) be a functor. Let
\[
\text{con}(F) := \text{Hom}_{Y(\mathcal{B})^\text{op}}(K_{\text{con}}; F)
\]
so that \( \text{con} \) is an endofunctor on the category of functors \( Y(\mathcal{B}) \to \mathfrak{A} \).

We have a zig-zag natural transformation from \( \text{con} \) to \( \text{Id} \). It is defined as follows. Denote by \( \mu : Y(\mathcal{B}) \to Y(E)c, \mu((n)^{\text{cyc}}) = (m, m, \ldots, m)^{\text{cyc}}, ; \mu((n)^{\text{noncyc}}) = (m, m, \ldots, m)^{\text{noncyc}}, (n \text{ occurrences of } m) \). We have \( \iota \mu = \varepsilon \mu = \text{Id} \), whence the following sequence of maps:
\[
I_{Y(\mathcal{B})^\text{op}} \otimes \lambda_{(E)}^L I_{Y(\mathcal{B})} \cong \varepsilon \mu \otimes \lambda_{(E)}^L \iota \mu \to \varepsilon \otimes \lambda_{(E)}^L \iota = K_{\text{con}}.
\]
where
\[
I_{Y(\mathcal{B})^\text{op}} : Y(\mathcal{B})^\text{op} \to (D \bigoplus)(Y(\mathcal{B})^\text{op}), \quad I_{Y(\mathcal{B})} : Y(\mathcal{B}) \to (D \bigoplus)Y(\mathcal{B})
\]
are the canonical maps.

Denote \( K_I := I_{Y(\mathcal{B})^\text{op}} \otimes \lambda_{(E)}^L I_{Y(\mathcal{B})} \) so that we have a map
\[
K_I \to K_{\text{con}} \tag{77}
\]
We then have maps
\[
\text{con}F = \text{Hom}_{Y(\mathcal{B})}(K_{\text{con}}; F) \to \text{Hom}_{Y(\mathcal{B})}(K_I; F) \cong F
\]
which define the required zig-zag.

**Lemma 15.1** Let \( V : Y(\mathcal{B}) \to \mathfrak{A} \). Then the maps
\[
\Gamma^{\text{noncyc}}(\text{con}V) \to \Gamma^{\text{noncyc}}(\text{Hom}_{Y(\mathcal{B})}(K_I; V)), \tag{78}
\]
\[
\Gamma^{\text{cyc}}(\text{con}V) \to \Gamma^{\text{cyc}}(\text{Hom}_{Y(\mathcal{B})}(K_I; V)), \tag{79}
\]
induced by \( \tag{77} \), are homotopy equivalences.

**Sketch of the proof.** Consider the cyclic case \( \tag{79} \), the case \( \tag{78} \) is treated similarly. We have isomorphisms
\[
\Gamma^{\text{cyc}}(\text{con}(V)) \cong \text{Hom}_{Y(\mathcal{B})^{\text{cyc}}}(C_*(K_{\text{con}}|_{A^\text{op} \otimes Y(\mathcal{B})^{\text{cyc}}}); V);
\]
\[
\Gamma^{\text{cyc}}(\text{Hom}(K_I, V)) \cong \text{Hom}_{Y(\mathcal{B})^{\text{cyc}}}(C_*(K_I|_{A^\text{op} \otimes Y(\mathcal{B})^{\text{cyc}}}); V).
\]
The problem now reduces to showing that the following map
\[
C_*(K_{\text{con}}|_{A^\text{op} \otimes Y(\mathcal{B})^{\text{cyc}}}) \to C_*(K_I|_{A^\text{op} \otimes Y(\mathcal{B})^{\text{cyc}}})
\]
induced by \( \tag{77} \), is a homotopy equivalence.

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15.5.1 Semi-orthogonal decomposition

Let \( \mathcal{A} \) be a ground category. Call a functor \( F : \Delta \to \mathcal{A} \) quasi-constant if every arrow in \( \Delta \) acts by a homotopy equivalence. Equivalently, let \( R_\Delta \) be a semi-free resolution of the constant functor \( \mathbb{K} : \Delta \to \mathcal{A} \). Then \( F \) is quasi-constant iff the natural map

\[
\text{Hom}_\Delta(R_\Delta; F) \otimes R_\Delta \to F
\]

is a termwise homotopy equivalence.

Denote by \( \mathcal{C}_Y(\mathcal{B}) \) the category of functors \( Y(\mathcal{B})^{\text{op}} \to \mathcal{A} \), enriched over \( \text{Sets} \).

Let \( \mathcal{C}_Y(\mathcal{B}),_{\text{const}} \subset \mathcal{C}_Y(\mathcal{B}) \) be the full sub-category consisting of all objects \( X \), where \( X_{\text{cyc}}|_\Delta \) and \( X_{\text{noncyc}} \) are quasi-constant.

Let \( \mathcal{C}_Y(\mathcal{B}),_{\perp} \subset \mathcal{C}_Y(\mathcal{B}) \) be ‘the right orthogonal complement’ to \( \mathcal{C}_Y(\mathcal{B}),_{\text{const}} \) — it consists of all objects \( Y \) such that for every \( X \in \mathcal{C}_Y(\mathcal{B}),_{\text{const}} \), we have

\[
R\text{Hom}_Y(\mathcal{B})(X, Y) \sim 0.
\]

Lemma 15.2 We have \( X \in \mathcal{C}_Y(\mathcal{B}),_{\perp} \) iff \( \Gamma_{\text{noncyc}}(X) \sim 0 \) and \( \Gamma_{\text{cyc}}(X) \sim 0 \).

We have the following semi-orthogonal decomposition.

Proposition 15.3 Let \( X \in \mathcal{C}_Y(\mathcal{B}). \) Then

1) \( \text{con}X \in \mathcal{C}_Y(\mathcal{B}),_{\text{const}} \);

2) \( \text{Cone} (\text{con}X \to \text{Hom}_Y(\mathcal{B})(K_I; X)) \in \mathcal{C}_Y(\mathcal{B}),_{\perp} \).

15.5.2 Lemma on the map \( \mathcal{O}^H \to \text{gr}\mathcal{O}^F \)

Let us get back to the sequence (75).

Proposition 15.4 1) The cone of the map \( \mathcal{O}^H \to \text{gr}\mathcal{O}^F \) belongs to \( \mathcal{C}_Y(\mathcal{B}),_{\perp} \);

2) The induced map \( \text{con}\mathcal{O}^H \to \text{congr}\mathcal{O}^F \) is a term-wise homotopy equivalence.

Sketch of the proof. Follows from Prop. 14.3 and Lemma 15.2.

15.6 Studying the object \( \text{con}(\mathcal{O}^H) \)

15.7 The map \( I : (\mathcal{O}^H)^{\text{noncyc}}|_\Delta \to (\mathcal{O}^H)^{\text{cyc}}|_\Delta \) is a homotopy equivalence

We have a structure map

\[
i : (\mathcal{O}^H)^{\text{noncyc}}|_\Delta \to (\mathcal{O}^H)^{\text{cyc}}|_\Delta.
\]

The following Proposition is straightforward

Proposition 15.5 The map \( (80) \) is a term-wise homotopy equivalence.
15.8 The map \(c_1\)

We have a functor \(\delta : Y(\mathbb{B}) \rightarrow Y(\mathbb{B}) \otimes Y(\text{assoc});\) where

\[
\delta(a) = a \otimes a, \ a \in \text{Ob} \ Y(\mathbb{B}).
\]

For \(X : Y(\mathbb{B}) \rightarrow R_q; V : Y(\text{assoc}) \rightarrow R_q,\) set \(X \circ Y := \delta^{-1}(X \otimes Y)\).

Let \(\mathcal{R} : Y(\text{assoc}) \rightarrow \mathcal{A}\) be a semi-free resolution of the constant functor \(\mathbb{K}\) We have a termwise homotopy equivalence

\[
X \circ \mathcal{R} \rightarrow X \circ \mathbb{K} = X.
\]

Next, we have an element \(c_1 : \mathcal{R} \rightarrow \mathcal{R}[2].\)

We have an induced map

\[
c_1 := c_1^X : X \circ \mathcal{R} \rightarrow X \circ \mathcal{R}[2].
\]

where \(\otimes\) is the term-wise product of functors. We can iterate so as to get maps

\[
(c_1^X)^n : X \circ \mathcal{R} \rightarrow X \circ \mathcal{R}[2n].
\]

Call \(X\) \(c_1\)-nilpotent, if \((c_1^X)^n\) is homotopy equivalent to \(0\) for \(n\) large enough.

15.8.1 Formulating the Proposition

For each subset \(V \in \mathcal{R}\) and \(S \in \text{SMetR}\) we have a functor \(\tau_{V,S} : R_0 \rightarrow R_0,\) where \((\text{gr}^a(\tau_{V,S}X)_T = X_S^a\) if \(a \in V\) and \(T = S.\) Set \(\text{gr}^a(\tau_{V,S}X)_T = 0\) if either \(T \neq S,\) or \(a \notin V,\) or both.

**Proposition 15.6** Let \(a > 0\) and \(S \in \text{SMetR}.\) The object \(\tau_{(0,a),S}O^H\) is \(c_1\)-nilpotent.

The rest of the subsection will be devoted to proving this proposition.

15.8.2 Reduction to the case \(V\) has only one point

It follows that

\[
\tau_{(0,a),S} = \bigoplus_{b \in (0,a), b \in \text{val}_S Z} \tau_{\{b\},S}.
\]

Denote \(\tau_{b,S} := \tau_{\{b\},S}.\) Therefore, it suffices to show that for every \(a \in \text{val}_S Z_{>0}, \tau_{a,S}O^H\) is \(c_1\)-nilpotent.

15.8.3 Localizing with respect to a covering of \(M\)

Choose a triangulation \(\mathcal{T}\) of \(M\) as in Sec 13.8. Let \(U\) be the open covering of \(M\) consisting of all stars of \(\mathcal{T}.\) We now have a map \(O^{E_U} \rightarrow O^H.\) For each \(U \in \mathcal{U},\) we have an object \(M^H_U \in O^{E_U}.\) Its full operad is isomorphic to \(\mathbb{B}.\) We also have its image \(M^H_U \in O^H.\) Denote by \(O^H_U\) the full operad of \(M^H_U.\) We have a homotopy equivalence

\[
O^H \sim \text{holim}_{U \in \mathcal{U}} O^H_U.
\]

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As the covering $\mathcal{U}$ is finite, it suffices to show that
\[ \tau_{a,S}O_{U}^{\mathcal{H}} \]
is $c_1$-nilpotent for each $U \in \mathcal{U}$.

**15.8.4 Restriction to $\Delta$**

Denote
\[ X^{cyc} := (\tau_{a,S}O_{U}^{\mathcal{H}})^{cyc}|_{\Delta}. \]
It follows that $X^{cyc}|_{\Delta}$ is quasi-constant. Let $R_{\Delta} \to \mathcal{K}$ be a free resolution in the category of functors $\Delta \to \mathcal{A}$. It now follows that the natural map
\[ R_{\Delta} \otimes \text{Hom}_{\Delta}(R_{\Delta}; X^{cyc}|_{\Delta}) \to X^{cyc}|_{\Delta} \]
is a termwise homotopy equivalence, that is we have a homotopy equivalence
\[ H \otimes R_{\Delta} \sim X^{cyc}|_{\Delta} \]
for an appropriate $H \in \mathcal{A}$.

Let us now pass to $\Lambda$. Let $\mathcal{R}$ be a free resolution of the constant functor
\[ \mathcal{K}: \Lambda \to \mathcal{A}. \]
Consider an object
\[ \text{Hom}_{\Lambda}(\mathcal{R}; X^{cyc}|_{\Lambda}) \otimes \mathcal{R}. \]
The endomorphism $u: \mathcal{R} \to \mathcal{R}[2]$ acts on both tensor functors. Denote those actions by $u_1$ and $u_2$ respectively. We can now build a cone
\[ X_{\mathcal{R}} := \text{Cone(\text{Hom}_{\Lambda}(\mathcal{R}; X^{cyc}|_{\Lambda}) \otimes \mathcal{R}[−2] \xrightarrow{u_2−u_1} \text{Hom}_{\Lambda}(\mathcal{R}; X^{cyc}|_{\Lambda}) \otimes \mathcal{R})} \]
We have a natural map $X_{\mathcal{R}} \to X$ which is a homotopy equivalence. This implies:

**Lemma 15.7** Suppose $X|_{\Delta}$ is quasi-constant. Then $c_1^{\mathfrak{g}}$-action on $X$ is homotopy equivalent to 0 iff such is the induced $c_1^{\mathfrak{g}}$-action on $R\text{Hom}_{\Lambda}(\mathcal{R}; X)$.

**15.8.5 Passage to $\mathbb{Z}Z$**

Recall that the category $M^{\mathfrak{g}}$ is enriched over the SMC of $\mathbb{Z}Z$-graded objects in $R_0$, to be denoted by $R_0(\mathbb{Z}Z)$.

The tensor functor $\pi: R_0(\mathbb{Z}Z) \to R_0$ gives the induced structure over $R_0$. Denote by $g^{p}_{\mathbb{Z}Z}$ the corresponding graded component.
15.8.6 An algebra $\xi$

As the set $U$ is contractible, we can trivialize the groupoid

$$\Sigma|_U \Rightarrow \mathcal{L}_U$$

so that it becomes isomorphic to

$$((S^1 \times S^1)^2 \times U \Rightarrow S^1 \times S^1 \times U).$$

Choose a point $a \in S^1 \times S^1$. We then have an embedding of groupoids

$$i : (\mathbb{Z} \times \mathbb{Z} \times U \Rightarrow U) \rightarrow ((S^1 \times S^1)^2 \times U \Rightarrow S^1 \times S^1 \times U),$$

where the groupoid structure comes from the group law on $\mathbb{Z} \times \mathbb{Z}$.

Set $\xi := i^*M^H$. The full operad of the pair $(\xi, M^H)$ is pseudo-contractible (in the category $\mathbf{R}_0(\mathbb{Z}\mathbb{Z})$).

Therefore, via straightening out, we can construct a zig-zag map of functors $\Lambda \rightarrow \mathfrak{A}$:

$$\tau_a \mathcal{O}_\xi^H \rightarrow \tau_a, S \mathcal{O}_\xi^H,$$

where $\mathcal{O}_\xi$ is the full operad of $\xi$. It follows that the induced map

$$\text{Hom}(\mathcal{R}; \tau_a \mathcal{O}_\xi^H) \rightarrow \text{Hom}(\mathcal{R}; \tau_a, S \mathcal{O}_\xi^H))$$

is a homotopy equivalence.

Therefore, the problem reduces to showing that the $c^p_t$-action on the LHS is homotopy equivalent to 0.

Next, it follows, that the full operad of $\xi$ is homotopy equivalent to a full operad of an algebra $\eta$ in a monoidal category $G$ enriched over $\mathbf{R}_0(\mathbb{Z}\mathbb{Z})$, to be now defined.

An object of the category $G$ is a $\mathbb{Z} \times \mathbb{Z}$-graded object in $\mathbf{R}$. Set

$$\text{gr}_{\mathbb{Z}\mathbb{Z}}^{(a_1, a_2)} \text{Hom}_G(X, Y) := T_{a_1} \text{gr}_{\mathbb{Z}\mathbb{Z}}^{(a_1, a_2)} \text{Hom}(X; Y)[2a_2].$$

The tensor product on $G$ is defined as that of graded objects:

$$\text{gr}_G^c(X \otimes Y) = \bigoplus_{a \in \mathbb{Z} \times \mathbb{Z}} \text{gr}_G^a X \otimes \text{gr}_G^{c-a} Y, \quad a, c \in \mathbb{Z} \times \mathbb{Z}.$$ 

Let us now define the above mentioned algebra $\eta$ in $G$ by setting $\text{gr}_G^{a,b} \eta := \mathbb{K}$ for all $a, b \in \mathbb{Z}$. In other words $\eta = \mathbb{K}[u^{-1}, u, v^{-1}, v]$, where $u : \mathbb{K} \rightarrow T_{1} \text{gr}_{\mathbb{Z}\mathbb{Z}}^{1, 0} \eta; v : \mathbb{K} \rightarrow \text{gr}_{\mathbb{Z}\mathbb{Z}}^{0, 1} \eta[2]$ are generators. The statement now can be proven by direct computation based on HKR, provided that $\mathbb{K} = \mathbb{Q}$.

15.9 Studying the map $\mathcal{O}^M \rightarrow \mathcal{O}^H$

We have a zig-zag homotopy equivalence between $\mathcal{O}^M$ and $\mathcal{O}^B \circ \mathcal{O}^{M_0}$.
Let $X : \mathcal{Y}(\text{assoc}) \to R_0; N : \mathcal{Y}(\mathbb{B})^{\text{op}} \to R_0$ be functors. Denote

$$H(X; N) := \text{RHom}_{\mathcal{Y}(\text{assoc})}(O^\mathbb{B} \circ X; N).$$

Let $a \in \mathcal{Y}(\text{assoc})^{\text{op}}$. Let $h_a : \mathcal{Y}(\text{assoc}) \to R_0$ be the corresponding free object. Set

$$N^{Z \times Z}(a) := H(h_a; N).$$

We have a structure of a functor $\mathcal{Y}(\text{assoc}) \to R_0$ on $N^{Z \times Z}$.

Let $Z : \mathcal{Y}(\text{assoc}) \to R_0$ be a functor. We have maps

$$Z(a) \to \text{RHom}_{\mathcal{Y}(\text{assoc})}(h_a, Z) \to \text{RHom}_{\mathcal{Y}(\mathbb{B})}(O^\mathbb{B} \circ h_a; O^\mathbb{B} \circ Z),$$

whence a map

$$Z \to (\mathbb{B} \circ Z)^{Z \times Z}.$$

Therefore, we have a zig-zag map from $O^{\mathbb{B}_0}$ to $(O^{\mathbb{B}})^{Z \times Z}$. Denote by $C : \mathcal{Y}(\text{assoc}) \to R_0$ the cone of this map.

**Proposition 15.8** We have $C$ is right orthogonal to every quasi-constant functor $T : \mathcal{Y}(\text{assoc}) \to R_0$, that is $\text{RHom}(T, C) \sim 0$.

**Sketch of the proof** Let $C_U$ be the cone of the zig-zag map from $O^{\mathbb{B}_0}_U$ to $(O^{\mathbb{B}})^{Z \times Z}_U$. Denote by $C : \mathcal{Y}(\text{assoc}) \to R_0$ the cone of this map.

It suffices to show a similar statement for $C_U$, which can be checked directly.

**15.9.1 The structure of the object $O^{\mathbb{B}_0}$**

Call an object $X : \mathcal{Y}(\text{assoc}) \to R_0$ strictly constant if there is an object $A \in R_0$ and a zig-zag homotopy equivalence of functors $\Lambda \to R_0$ from $A \otimes K$ to $X$.

**Proposition 15.9** The object $O^{\mathbb{B}_0}$ is strictly constant

**Sketch of the proof** As above, fix a covering $U$ of $M$. Denote $T := O^{\mathbb{B}_0}_U; S_U := O^{\mathbb{B}_0}_U$. We have a functor $S : U \times \mathcal{Y}(\text{assoc}) \to R_0, S(U, F) := S_U(F)$. It follows that $S(U, n) \sim K$ for all $(U, n) \in U \times \mathcal{Y}(\text{assoc})$ which implies that we have a zig-zag termwise homotopy equivalence between $S$ and $K$.

We also have a homotopy equivalence of cyclic objects

$$T \Rightarrow \text{holim}_{U \in U} S(U, -).$$

The statement now follows.

**16 Reformulation**

**16.1 $c_1$-Localization**

**16.1.1 Category $\text{Fun}(\mathcal{D})$**

Let $\mathcal{D}, \mathcal{C}$ be categories enriched over $\mathfrak{A}$. Let $\mathcal{C}$ be tensored over $\mathfrak{A}$. Let $\text{Funct}(\mathcal{D})$ be the category of $\mathfrak{A}$-functors $\mathcal{D} \to \mathcal{C}$. Let $\text{Fun}(\mathcal{D}) := (\mathcal{D} \bigoplus (\mathcal{D} \otimes \mathcal{C})$. We have the resolution functor $R : \text{Funct}(\mathcal{D}) \to$
2) Let $$\mathsf{res} := \mathsf{Id}_D \otimes_{D} \mathsf{Id}_{D^{\text{op}}} \in \mathbf{Fun}(D \otimes D^{\text{op}})$$.

Let us define an endofunctor

$$\lambda : \mathbf{Fun}(D) \to \mathbf{Fun}(D),$$

$$\lambda(T) := \mathsf{res} \circ_{D} T.$$

### 16.1.2 $c_1$-Localization in $\mathbf{Fun}((\mathcal{O})^{\text{cyc}})^{\text{op}}$

Let now $$\mathcal{D} := \mathbf{Fun}((\mathcal{O})^{\text{cyc}})^{\text{op}}.$$ We have a natural transformation $$c_1 : \mathsf{res} \to \mathsf{res}[2]$$ which induces a natural transformation $$c_1 : \lambda 	o \lambda[2].$$ Let us define an endofunctor

$$\mathbf{loc} : (\mathbf{D} \bigoplus)\mathbf{Fun}((\mathcal{O})^{\text{cyc}})^{\text{op}} \to (\mathbf{D} \bigoplus)\mathbf{Fun}((\mathcal{O})^{\text{cyc}})^{\text{op}}.$$

$$\mathbf{loc}(T) := \text{hocolim}(\lambda(T) \xrightarrow{c_1} \lambda(T)[2] \xrightarrow{c_3} \lambda(T)[4] \xrightarrow{c_3} \cdots).$$

For $$F : \mathcal{O}^{\text{cyc}} \to \mathcal{C},$$ denote $$F_{\mathbf{loc}} := \mathbf{loc}(\mathcal{R}(F)).$$

It follows that $$F_{\mathbf{loc}} \sim 0$$ as long as the $$c_1$$-action on $$F$$ is nilpotent.

The functor $$\mathbf{loc}$$ extends to an endofunctor on

$$\mathbf{D} \bigoplus \prod \mathbf{Fun}((\mathcal{O})^{\text{cyc}})^{\text{op}}.$$
16.2.2 Co-convolution with a kernel

Let $\mathcal{D}_1, \mathcal{D}_2$ be categories enriched over $\mathcal{C}$. Let $K \in \text{Fun}(\mathcal{D}_1^{\text{op}} \otimes \mathcal{D}_2)$. and $F \in \text{Fun}(\mathcal{D}_2)$. We then have an object

$$\text{Hom}_{\mathcal{D}_2}(K, F) \in \mathcal{D} \prod \text{Fun}(\mathcal{D}_1).$$

We can now apply the functor $\mathcal{D}$ so as to get an object

$$\mathcal{D}\text{Hom}_{\mathcal{D}_2}(K, F) \in \mathcal{D} \prod \text{Fun}(\mathcal{D}_1).$$

Call $\mathcal{D}\text{Hom}$ the co-convolution with $K$.

One extends $\mathcal{D}\text{Hom}(K, -)$ to a functor

$$\mathcal{D} \prod \text{Fun}(\mathcal{D}_2) \to \mathcal{D} \prod \text{Fun}(\mathcal{D}_1).$$

16.2.3 Composition of co-convolutions

Let $K_{21} \in \text{Fun}(\mathcal{D}_2^{\text{op}} \otimes \mathcal{D}_1)$, $K_{32} \in \text{Fun}(\mathcal{D}_3^{\text{op}} \otimes \mathcal{D}_2)$.

Consider the composition

$$\mathcal{D}\text{Hom}(K_{32}, \mathcal{D}\text{Hom}(K_{21}; -)) : \mathcal{D} \prod \text{Fun}(\mathcal{D}_3) \to \mathcal{D} \prod \text{Fun}(\mathcal{D}_1).$$

and construct natural transformations which are termwise homotopy equivalences and whose composition is the identity:

$$\mathcal{D}\text{Hom}(K_{32} \circ_{\mathcal{D}_2} K_{21}; -) \to \mathcal{D}\text{Hom}(K_{32}, \mathcal{D}\text{Hom}(K_{21}; -)) \to \mathcal{D}\text{Hom}(K_{32} \circ_{\mathcal{D}_2} K_{21}; -). \quad (81)$$

Let us first do it for $K_{32} = \bigoplus_{c \in \mathcal{C}} \delta^c_3 \otimes d^c_2 \otimes U^c$, $K_{21} = \bigoplus_{b \in \mathcal{B}} \delta^b_2 \otimes d^b_1 \otimes U^b$,

where $U^c, U^b \in \mathcal{C}$, $\delta^c_3 \in \mathcal{D}_3^{\text{op}}$, $d^c_2 \in \mathcal{D}_2$, $\delta^b_2 \in \mathcal{D}_2^{\text{op}}$, $d^b_1 \in \mathcal{D}_1$. Let also

$$X = \prod_{z \in \mathcal{Z}} \bigoplus_{a \in \mathcal{A}} d^{za} \otimes U^{za}, \quad d^{za} \in \mathcal{D}_1, \ U^{za} \in \mathcal{C}.$$

We have

$$\mathcal{D}\text{Hom}(K_{21}; X) = \prod_{z \in \mathcal{Z}} \bigoplus_{b \in \mathcal{B}, a \in \mathcal{A}} L_{t \in \mathcal{D}_2} \text{Hom}_\mathcal{C} \left( U^b \otimes \text{Hom}_{\mathcal{D}_2}(\delta^b_2, t); U^{za} \otimes \text{Hom}_{\mathcal{D}_1}(d^b_1; d^{za}) \right).$$

Next,

$$\mathcal{D}\text{Hom}(K_{32}, \mathcal{D}\text{Hom}(K_{21}; X))$$

$$= \mathcal{D} \prod_{z \in \mathcal{Z}, b \in \mathcal{B}, c \in \mathcal{C}, a \in \mathcal{A}} \delta^c_3 \otimes \text{Hom}_{\mathcal{D}_2}(d^c_2, t) \otimes L_{t \in \mathcal{D}_2} \text{Hom}_\mathcal{C} \left( U^b \otimes U^c \otimes \text{Hom}_{\mathcal{D}_2}(\delta^b_2, t); U^{za} \otimes \text{Hom}_{\mathcal{D}_1}(d^b_1; d^{za}) \right).$$
In other words we can decompose

\[ \mathcal{D}\text{Hom}(K_{32}; \mathcal{D}\text{Hom}(K_{21}; X)) = GH(K_{32}, K_{21}, X), \]

where

\[
H : (D \bigoplus (\mathcal{D}_3^{\text{op}} \otimes \mathcal{D}_2 \otimes \mathcal{C}))^{\text{op}} \otimes (D \bigoplus (\mathcal{D}_2^{\text{op}} \otimes \mathcal{D}_1 \otimes \mathcal{C}))^{\text{op}} \otimes \mathcal{D}_1 \otimes \mathcal{C}
\rightarrow D \bigoplus (\mathcal{D}_3^{\text{op}} \otimes \mathcal{D}_2 \otimes \mathcal{C} \otimes \mathcal{D}_2^{\text{op}} \otimes \mathcal{D}_1 \otimes \mathcal{C})^{\text{op}} \otimes \mathcal{D}_1 \otimes \mathcal{C}
\]

\[
\xrightarrow{\text{Hom}} D \prod \bigoplus (\mathcal{D}_3 \otimes \mathcal{D}_2^{\text{op}} \otimes \mathcal{C}^{\text{op}} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1^{\text{op}} \otimes \mathcal{C}^{\text{op}} \otimes \mathcal{D}_1 \otimes \mathcal{C}).
\]

Next,

\[ G : D \prod \bigoplus (\mathcal{D}_3 \otimes \mathcal{D}_2^{\text{op}} \otimes \mathcal{C}^{\text{op}} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1^{\text{op}} \otimes \mathcal{C}^{\text{op}} \otimes \mathcal{D}_1 \otimes \mathcal{C}) \rightarrow D \prod \bigoplus (\mathcal{D}_3 \otimes \mathcal{C}), \]

is induced by the following functor

\[ G_0 : \mathcal{D}_3 \otimes \mathcal{D}_2^{\text{op}} \otimes \mathcal{C}^{\text{op}} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1^{\text{op}} \otimes \mathcal{C}^{\text{op}} \otimes \mathcal{D}_1 \otimes \mathcal{C} \rightarrow D \prod \bigoplus (\mathcal{D}_3 \otimes \mathcal{C}), \]

\[ G_0(\delta_3, d_2, U_c, \delta_2, d_1, U_b, d_1^2, U_a)
\]

\[
:= \mathcal{D} \left( \delta_3 \otimes \text{Hom}_{\mathcal{D}_2}(d_2; t) \otimes_{t \in \mathcal{D}_2} \text{Hom}_{\mathcal{C}}(U_b \otimes U_c \otimes \text{Hom}_{\mathcal{D}_2}(\delta_2, t); U_a \otimes \text{Hom}_{\mathcal{D}_1}(d_1; d_1^2)) \right).
\]

Likewise, we can decompose

\[ \text{Hom}_{\mathcal{D}_1}(K_{32} \circ \mathcal{D}_2 K_{21}; X) = EH(K_{32}, K_{21}, X), \]

where \( E \) is induced by the following functor

\[ E_0 : \mathcal{D}_3 \otimes \mathcal{D}_2^{\text{op}} \otimes \mathcal{C}^{\text{op}} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1^{\text{op}} \otimes \mathcal{C}^{\text{op}} \otimes \mathcal{D}_1 \otimes \mathcal{C} \rightarrow D \prod \bigoplus (\mathcal{D}_3 \otimes \mathcal{C}), \]

\[ E_0(\delta_3, d_2, U_c, \delta_2, d_1, U_b, d_1^2, U_a)
\]

\[
:= \mathcal{D} \left( \delta_3 \otimes \text{Hom}_{\mathcal{C}}(U_b \otimes U_c \otimes \text{Hom}_{\mathcal{D}_2}(\delta_2, d_2); U_a \otimes \text{Hom}_{\mathcal{D}_1}(d_1; d_1^2)) \right).
\]

The retraction \([81]\) is induced by the following retraction

\[ \text{Hom}_{\mathcal{C}}(U_b \otimes U_c \otimes \text{Hom}_{\mathcal{D}_2}(\delta_2, d_2); U_a \otimes \text{Hom}_{\mathcal{D}_1}(d_1; d_1^2)) \]

\[
\rightarrow \text{Hom}_{\mathcal{D}_2}(d_2, t) \otimes_{t \in \mathcal{D}_2} \text{Hom}_{\mathcal{C}}(U_b \otimes U_c \otimes \text{Hom}_{\mathcal{D}_2}(\delta_2, t); U_a \otimes \text{Hom}_{\mathcal{D}_1}(d_1; d_1^2))
\]

\[
\rightarrow \text{Hom}_{\mathcal{C}}(U_b \otimes U_c \otimes \text{Hom}_{\mathcal{D}_2}(\delta_2, d_2); U_a \otimes \text{Hom}_{\mathcal{D}_1}(d_1; d_1^2))
\]

where the leftmost arrow is induced by letting \( t = d_2 \) and the rightmost arrow is induced by the composition of hom's.
16.2.4 The identity kernel $K_I$

Let

$$K_I := \text{Id}_{\mathcal{D}^\text{op}} \otimes^L \text{Id}_{\mathcal{D}} \in \text{Fun}(\mathcal{D}^\text{op} \otimes \mathcal{D}).$$

For every $F \in \text{Fun}(\mathcal{D})$, we have a natural transformation

$$F \circ_{\mathcal{D}} K_I \sim F$$

which produces an element

$$i_F : K \to \text{Hom}(F \circ_{\mathcal{D}} K_I; F) = \mathbb{D}\text{Hom}(F \circ_{\mathcal{D}} K_I; F) \to \text{Hom}(F; \mathbb{D}\text{Hom}(K_I; F)),$$

whence a natural transformation

$$F \mapsto \mathbb{D}\text{Hom}(K_I; F). \quad (82)$$

This transformation extends onto all $F \in \mathcal{D}\prod\text{Fun}(\mathcal{D})$.

We will use the following Lemma

**Lemma 16.1** Let $L \in \text{Fun}(\mathcal{D}_1^\text{op} \otimes \mathcal{D})$ and $F \in \text{Fun}(\mathcal{D})$. The following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{D}\text{Hom}(L, F) & \xrightarrow{\iota} & \mathbb{D}\text{Hom}(L, \mathbb{D}\text{Hom}(K_I; F)) \\
\downarrow & & \downarrow \\
\mathbb{D}\text{Hom}(L \circ_{\mathcal{D}} K_I; F) & & \\
\end{array}
$$

where the map $\iota$ is induced by the natural map $L \circ_{\mathcal{D}} K_I \to L$.

16.2.5 The functor $\text{con}$

Let $\text{con} : \mathcal{D}\prod\text{Fun}((Y(\mathbb{B})^{\text{cyc}})^\text{op}) \to \mathcal{D}\prod\text{Fun}((Y(\mathbb{B})^{\text{cyc}})^\text{op})$ be defined by

$$\text{con}(F) := \mathbb{D}\text{Hom}(K_{\text{con}}; F).$$

The natural transformation $K_{\text{Id}} \to K_{\text{con}}$ induces natural transformations

$$\text{con}(F) \to \mathbb{D}\text{Hom}(K_{\text{Id}}; F) \leftarrow F. \quad (83)$$

**Lemma 16.2** The induced maps

$$\Gamma(\text{con}(F)) \to \Gamma(\mathbb{D}\text{Hom}(K_{\text{Id}}; F)) \leftarrow \Gamma(F).$$

are homotopy equivalences.
16.2.6 Reformulating the results

Claim 16.3 We have
\[ \tau > 0 \text{Hom}((\text{con}(\mathcal{O}_{\text{loc}}^F); \text{con}(\mathcal{O}_{\text{loc}}^F)) \sim 0 \]

This Claim follows from Prop 15.4 and 15.6.

Fix a semi-free resolution \( R_E \rightarrow \mathbb{B} \). Let \( Z := D \oplus \text{Fun}(Y(R_E)^{op}) \). Let
\[ R : \text{Funct}(Y(R_E); C) \rightarrow \text{Fun}(Y(R_E)^{op}) \]
be the canonical resolution functor.

Claim 16.4 The following map in \( R_0 \):
\[ \text{Hom}_Z(\text{gr} R_F; \text{gr} \mathcal{O}_F) \rightarrow \text{Hom}_Z(\text{gr} R_E; (\text{gr} \mathcal{O}_E)_{\text{loc}}) \]
admits a splitting up-to a homotopy.

This Claim follows from Prop 15.4 and 15.9.

Fix a homotopy equivalence in \( R_0 \): (from the above Claim):
\[ \text{Hom}_Z(\text{gr} R_E; (\text{gr} \mathcal{O}_F)^{\oplus} L_0 \rightarrow \text{Hom}_Z(\text{gr} R_E; (\text{gr} \mathcal{O}_E)_{\text{loc}}). \quad (84) \]

17 Quantization: Introduction

So far, we have constructed a CO \( \mathcal{O}_F \) in the category \( R_q \) as well as a map of CO \( \text{red} \mathbb{B} \rightarrow \text{red} \mathcal{O}_F \) (where \( \text{red} \) means that the map is defined over \( R_0 \)). In the rest of the paper we deal with the problem of lifting this map onto the level of \( R_q \), which we also call the quantization.

More precisely, we start with defining another CO, to be denoted by \( \mathcal{A} \), and a map of CO \( \mathcal{A} \rightarrow \mathbb{B} \) (over \( R_q \)), and we lift the through map \( \text{red} \mathcal{A} \rightarrow \text{red} \mathbb{B} \rightarrow \text{red} \mathcal{O}_F \) (over \( R_0 \)) to a map \( \mathcal{A} \rightarrow \mathcal{O}_F \) (over \( R_q \)). So far as \( \mathbb{B} \) can be interpreted as an operad of \( Z \times Z \)-equivariant algebras, \( \mathcal{A} \) does then control \( Z \times Z \)-equivariant \( A_{\infty} \)-algebras with 'curvature'. Putting aside the \( Z \times Z \)-equivariance, the structure of an '\( A_{\infty} \)-algebra with curvature' on an object \( X \) of a monoidal category enriched over \( R_q \) is a Maurer-Cartan element in the Hochschild complex of \( X \) (viewed as an associative algebra with zero).

The curvature version of the lifting problem happens to be more tractable, as it turns out to be controlled by the Hochschild complexes studied in Sec 14. On the other hand, this weaker version of the lifting problem is still sufficient for our main task — building the microlocal category. Indeed, the operad \( \mathcal{O}_F \) being a full operad of a certain object in a monoidal category \( D \oplus \text{U}(\mathbb{F}) \), the lifting provides for an \( A_{\infty} \)-algebra with curvature structure on that object over \( R_q \). Next, the monoidal category \( (D \oplus) \text{U}(\mathbb{F}) \) acts on the category \( (D \oplus) \text{U}(\mathbb{F}_R) \), and we define the microlocal category as that of \( A_{\infty} \) modules over this algebra in the category \( (D \oplus) \text{U}(\mathbb{F}_R) \).

Let us outline the main steps of the quantization.
First, we construct a tensor functor $\pi : R_q \to \text{Com}_{\geq 0} \frak{A}$ (see Sec. 18.1 below). The category $R_q$ is now enriched over $\text{Com}_{\geq 0} \frak{A}$ (it is crucial that the functor $\pi$ induces an isomorphism $$\text{Hom}(\text{unit}_{R_q}; X) \to \text{Hom}(\text{unit}_{\text{Com}_{\geq 0} \frak{A}}; \pi(X)).$$ Accordingly, we get a functor $\pi_0 : R_0 \to \text{Gr}_{\geq 0} \frak{A}$, where $\text{Gr}_{\geq 0} \frak{A}$ is the category of non-negatively graded objects in $\frak{A}$.

We can now solve the resulting quantization problem by induction. Namely, let $\text{Com}_{[0, k]} \frak{A}$ be a SMC whose every object is a complex in $\frak{A}$ concentrated in degrees from 0 to $k$. The tensor product is defined by setting

$$(X \otimes Y)^p := \bigoplus_{q \leq p \leq p} X^q \otimes Y^{p-q}, \quad 0 \leq p \leq k.$$  

We have tensor functors $\text{pr}_k : \text{Com}_{\geq 0} \frak{A} \to \text{Com}_{[0, k]} \frak{A}; \text{pr}_{lk} : \text{Com}_{[0, l]} \frak{A} \to \text{Com}_{[0, k]} \frak{A}$, $l \geq k$, where $\text{pr}_k(X)^s = X^s; \text{pr}_{lk}(X)^s = X^s, 0 \leq s \leq k$. One can now proceed similar to the case of the deformation quantization, exploiting the analogy between the category $\text{Com}_{\geq 0} \frak{A}$ and the category of (continuous) modules over the ring of formal series in one variable $h$. The category $\text{Com}_{[0, k]} \frak{A}$ is then analogous to the quotient ring $k[[h]]/(h^{k+1})$. Our strategy is thus as follows. First, we have categories $\text{pr}_k R_q$ enriched over $\text{Com}_{[0, k]} \frak{A}$. Observe that $\text{pr}_0 R_q$ is isomorphic to the category $R_0$ as a category enriched over $\frak{A}$. The starting data of the problem (i.e. a map of operads $\text{red}\frak{A} \to \text{red}\frak{B} \to \text{red}\text{Op}$) can, hence, be formulated within the category $\text{pr}_0 R_q$. This is the base of induction.

Let the $k$-th induction step consist of lifting a map $\text{pr}_k \frak{A} \to \text{pr}_k \text{Op}$ in $\text{pr}_k R_q$ to the level of $\text{pr}_{k+1} R_q$, that is to a map $\text{pr}_{k+1} \frak{A} \to \text{pr}_{k+1} \text{Op}$. One can develop an obstruction theory for this kind of problems (similar to the deformation quantization theory), the obstructions are controlled by the Hochschild complexes as in Sec. 14, unfortunately, they do not vanish. Let us describe the obstruction complex for the future sake. We start with defining the notion of an $A$-module, where $A$ is a CO, and of the object $D(M)$ classifying the derivations from $A$ to $M$.

Let $A$ be a CO in an SMC $C$ enriched over an SMC $B$. One first defines the notion of an $A$-module $M$ as a collection of objects $M(n)^{\text{nocy}}$, $M(n)^{\text{cy}}$, as well as a CO structure on the direct sum $A \oplus M$ such that the inclusion $A \to A \oplus M$ is a CO map; and $M$ is an ideal whose square is 0. Equivalently, $M$ is a $Y(A)$-module.

One has the notion of a derivation of $A$ with values in a $A$-module $M$, denote the set of such derivations by $\text{Der}(M)$. $A$ being quasi-free, the functor $M \mapsto \text{Der}(M)$ is representable and the representing object is denoted by $\Omega_A$ and called the module of Kaehler differential on $A$. If $A$ is freely generated by a collection $G$, then $\Omega_A$ is freely generated over $A$ by $G$. The following functor $F : B^\text{op} \to \text{Sets}$ is therefore representable $F(T) := \text{Hom}(\Omega_A \otimes T; M)$, where $\text{Hom}$ is in the category of $A$-modules. Denote the representing object by $D(M)$. Let $M^{\text{nocy}} := (M^{\text{nocy}}, 0)$, $M^{\text{cy}} := (0, M^{\text{cy}})$ with the obvious induced $A$-module structure. Let

$$D(M)^{\text{nocy}} := D(M^{\text{nocy}}), \quad D(M)^{\text{cy}} := D(M^{\text{cy}}).$$

We have a map $h : D(M)^{\text{cy}}[-1] \to D(M)^{\text{nocy}}$ so that $\text{Cone } h \cong D(M)$.

We will now define the obstruction complex. Let $A, B$ be circular operads in $R_q$ and let $f : \text{red} A \to \text{red} B$ be a map of CO. We then get a $\text{red} A$-module structure on $\text{red} B$, to be denote by $\text{red} B_f$. We
therefore have an object $\mathcal{D}(\text{red}B_f) \in \text{Gr}_{\geq 0}A$. This object is the above mentioned obstruction complex to lifting $f$.

As was mentioned above, the obstruction complex for the map $\text{red}A \to \text{red}\mathcal{O}^F$ as in (85), does not vanish. However, one can modify the statement of the problem in order to achieve the unobstructedness. As usual, the modification is via replacing the structure of CO with a richer structure that we call VCO. In order to motivate the definition of VCO we will first perform certain manipulations on $O$.

We start with the following categorical construction.

17.1 The category $IE$

Let $E$ be a (symmetric monoidal) category enriched over $\text{Com}_{\geq 0}A$. One then defines a (symmetric monoidal) category $IE$ enriched over $\text{Com}_{\geq 0}A$ as follows. One starts with defining a category $IE'$ whose every object $X$ is a collection of objects $\text{gr}^nX \in E$, $n \in \mathbb{Z}$. Let $X, Y \in IE'$. Denote $\mathcal{H}^k := \prod_{n \in \mathbb{Z}} \text{Hom}_E(\text{gr}^nX; \text{gr}^{n+k}Y) \in \text{Com}_{\geq 0}A$.

$$\text{Hom}_{IE'}(X, Y) := \bigoplus_{k > 0} \tau_{\geq k} \mathcal{H}^{-k} \bigoplus_{k \geq 0} \mathcal{H}^k,$$

where $\tau_{\geq k} : \text{Com}_{\geq 0}A \to \text{Com}_{\geq 0}A$ is the stupid truncation: $(\tau_{\geq k}U)^l = 0$ if $l < k$ and $(\tau_{\geq k}U)^l = U^l$ if $l \geq k$.

Observe that we have an isomorphism

$$\bigoplus_{k > 0} \tau_{\geq k} \mathcal{H}^{-k} \to \prod_{k \geq 0} \tau_{\geq k} \mathcal{H}^{-k}.$$

Let now set $IE := D(IE')$. We will also use full sub-categories $I_{\leq 0}E, I_{\geq 0}E \subset IE$ consisting of all objects $X$ with $\text{gr}^kX = 0$ if $k > 0$ (resp. if $k < 0$).

17.2 The operad $\mathcal{B} \oplus \mathcal{O}$ as an object of $/R_q$

It is convenient to switch from $\mathcal{O}$ to a direct sum

$$\mathcal{U} := \mathcal{B} \oplus \mathcal{O},$$

which is again a CO in $R_q$. Let

$$p : \mathcal{U} \to \mathcal{B}$$

be the projection. Let $i : \text{red}\mathcal{B} \to \text{red}\mathcal{U}$ be the diagonal map, that is $i = \text{Id}_B \oplus \iota$, where $\iota$ is as in (74). In particular $pi = \text{Id}_B$, which defines an isomorphism

$$\text{red}\mathcal{U}(a) \xrightarrow{\sim} \text{red}\mathcal{B}(a) \oplus \text{red}\mathcal{O}(a)$$

for each $a = (n)^{\text{noncyc}}$ or $a = (n)^{\text{cyc}}$, different from $\text{(86)}$. This isomorphism induces the following one in $R_q$: $\mathcal{U}(a) \cong (\mathcal{B}(a) \oplus \mathcal{O}(a), D_a)$ for an appropriate differential $D_a$ such that $\text{red}D_a = 0$. We can now
define objects $\mathcal{U}(a) \in IR_q$, where $gr^0\mathcal{U}(a) := \mathbb{B}(a)$; $gr^{-1}\mathcal{U}(a) = \mathcal{O}(a)$ and all other components of $\mathcal{U}(a)$ vanish. We finally set $\mathcal{U}(a) := (\mathcal{U}(a)', D_n)$, which is well defined since $redD_n = 0$. It also follows that $\mathcal{U}$ is a CO in $I_{\leq 0}R_q$. The projection $[87]$ induces a projection $p_{\mathcal{U}\mathcal{B}} : \mathcal{U} \to \mathbb{B}$, where we view $\mathbb{B}$ as an operad in $IR_q$ concentrated in the grading 0.

We are now ready to perform our first manipulation

### 17.3 Tensoring with $Y(\mathbb{B})^{\text{cyc}}$ over $Y(\mathcal{U})^{\text{cyc}}$

Here is our first manipulation. Let $(n)^{\text{cyc}} \in Y(\mathbb{B})^{\text{cyc}}$ be an object. Let $h_n : (Y(\mathbb{B})^{\text{cyc}})^{op} \to R_q$ be the Yoneda functor:

$$h_n((m)^{\text{cyc}}) := \text{Hom}_{Y(\mathbb{B})^{\text{cyc}}}((m)^{\text{cyc}}, (n)^{\text{cyc}}).$$

Via the projection $p_{\mathcal{U}\mathcal{B}}$, one can also view each $h_n$ as a functor $Y(\mathcal{U})^{\text{cyc}} \to IR_q$. Let us define a functor $V : Y(\mathbb{B}) \to IR_q$ as follows:

$$V((n)^{\text{cyc}}) := h_n \otimes_{Y(\mathcal{U})^{\text{cyc}}} U^{\text{cyc}}.$$

We have maps

$$V((n)^{\text{cyc}}) \xrightarrow{p_{\mathcal{U}\mathcal{B}}} h_n \otimes_{Y(\mathbb{B})^{\text{cyc}}} \mathbb{B}^{\text{cyc}} \to (n)^{\text{cyc}}.$$

We have a $Y(\mathbb{B})$-action on $V$ so that we can define a CO $IU$, where $IU^{\text{noncyc}} = \mathbb{B}^{\text{noncyc}}$ and $IU^{\text{cyc}} := V$. We have a map of CO $\mathcal{U} \to IU$ whose non-cyclic component is the above defined projection $p_{\mathcal{U}\mathcal{B}}^{\text{noncyc}}$, and the map $\mathcal{U}(n)^{\text{cyc}} \to V((n)^{\text{cyc}})$ is as follows:

$$\mathcal{U}(n)^{\text{cyc}} \to h_n(n) \otimes U^{\text{cyc}}(n) \to h_n \otimes_{Y(\mathcal{U})^{\text{cyc}}} U^{\text{cyc}} = V((n)^{\text{cyc}}),$$

where the first arrow is induced by the identity $\text{unit} \to h_n(n)$.

#### 17.3.1 Converting $V$ into an object of $(D \bigoplus (D \bigoplus (Y(\mathbb{B})^{\text{cyc}})^{op} \otimes R_q))$

Let us convert $V : Y(\mathbb{B})^{\text{cyc}} \to I_{\leq 0}R_q$ to a more convenient object. Let us start with a simpler situation, when we have a functor $W : Y(\mathbb{B}) \to I_{\leq 0}R_q$ so that we have graded components $gr^k W : Y(\mathbb{B})^{\text{cyc}} \to R_q$. We can now pass to resolutions

$$R_{gr^kW} \in (D \bigoplus (Y(\mathbb{B})^{\text{cyc}})^{op} \otimes R_q).$$

Denote

$$\text{Fun}(Y(\mathbb{B})^{\text{cyc}}) := (D \bigoplus (Y(\mathbb{B})^{\text{cyc}})^{op} \otimes R_q).$$

We can now produce an object

$$\Sigma W := \bigoplus_k R_{gr^kW} \in (D \bigoplus \text{Fun}(Y(\mathbb{B})^{\text{cyc}})).$$

In our case, when we have a functor $V : Y(\mathbb{B})^{\text{cyc}} \to IR_q$, we can still define an object $\Sigma V \in (D \bigoplus \text{Fun}(Y(\mathbb{B})^{\text{cyc}}))$ in a similar way. One can rewrite $V((n)^{\text{cyc}}) = (W((n)^{\text{cyc}}), D_n)$ for an appropriate $W$ as above. It turns out that the differentials $D_n$ induce a differential on $\Sigma W$, to be denoted by $D_\Sigma$, so that one can put $\Sigma V := (\Sigma W; D_\Sigma)$. 

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17.3.2 Generalizing the definition of CO

In order to be able to replace \( V \) with \( \Sigma V \) in the definition of the CO \( \mathcal{U} \), we need to generalize the notion of CO as follows.

Let us define a generalized CO (GCO) to be the following structure: an asymmetric operad \( \mathcal{O} \) and an object of \( (\mathcal{D} \bigoplus \mathcal{F}un(\mathcal{Y}(\mathcal{O}))) \) \footnote{We now have a GCO \( \mathcal{V} \), where \( \mathcal{V}^{\text{noncyc}} = \mathbb{B}^{\text{noncyc}} \) and \( \mathcal{V}^{\text{cyc}} = \Sigma V \).}

We now pass to our next manipulation.

17.4 \( c_1 \)-localization

The goal of this manipulation is to simplify the operad \( V \). The \( c_1 \)-localization was defined in Sec \[16.1.2\]. We repeat it here.

We generalize from the case of cyclic objects. Let \( K \) be the constant cyclic object. Let \( R \to K \) be a free resolution. One has an endomorphism \( c_1 : R \to R[2] \), ‘the first Chern class’. One can define a similar endomorphism for any cyclic object \( X \). Indeed, let \( X \odot R \) be a cyclic object, where \( X \odot R(n) := X(n) \otimes R(n) \) for every object \( n \) of the cyclic category. We therefore have a zig-zag

\[
X \xleftarrow{\sim} X \otimes R \xrightarrow{c_1} X \otimes R[2] \xrightarrow{\sim} X[2].
\]

Let now \( \mathcal{O} \) be a CO. One generalizes the \( c_1 \)-endomorphism to act on every object \( X : \mathcal{Y}(\mathcal{O})^{\text{cyc}} \to \mathfrak{A} \), where \( \mathfrak{A} \) is the ground category. Indeed, if \( X : \mathcal{Y}(\mathcal{O})^{\text{cyc}} \to \mathfrak{A} \) and \( T : \Lambda \to \mathfrak{A} \) (where \( \Lambda \) is the cyclic category), one defines a functor \( X \circ T : \mathcal{Y}(\mathcal{O})^{\text{cyc}} \to \mathfrak{A} \), where \( X \circ T(n) := X(n) \otimes T(n) \), whence a \( c_1 \)-map \( c_1 : X \circ \mathcal{R} \to X \circ \mathcal{R}[2] \).

Let us now try to invert \( c_1 \) (this is what we mean by the term ‘\( c_1 \)-localization’). The most straightforward approach, where one considers a functor \( X'_{\text{loc}} : \mathcal{Y}(\mathcal{O})^{\text{cyc}} \to \mathfrak{A} \),

\[
X'_{\text{loc}}(n) = \text{hocolim}(X \otimes \mathcal{R} \xrightarrow{c_1} X \otimes \mathcal{R}[2] \xrightarrow{c_1} X \otimes \mathcal{R}[4] \xrightarrow{c_1} \ldots)
\]

does fail, because such an \( F(n) \) is acyclic for each \( n \). However, one can define the localization as an ind-object. One now defines an object \( X_{\text{loc}} \in (\mathcal{D} \bigoplus \mathcal{F}un(\mathcal{Y}(\mathcal{O})^{\text{cyc}}) \) by the same formula as in \[89\]. As the hocolim is taken in the category of formal sums, \( X_{\text{loc}} \) is now not necessarily acyclic. Furthermore, one generalizes the operation \( X \mapsto X_{\text{loc}} \) to an endofunctor on \( (\mathcal{D} \bigoplus \mathcal{F}un(\mathcal{Y}(\mathcal{O})^{\text{cyc}}) \) in the obvious way.

Let us now apply the \( c_1 \)-localization to the GCO \( \mathcal{V} \) from Sec \[17.3.1\].

As was mentioned above, we have a map \( \mathcal{V}^{\text{cyc}} \to \mathcal{R}^{\text{cyc}}_{\text{loc}} \) in \( (\mathcal{D} \bigoplus \mathcal{F}un(\mathcal{Y}(\mathcal{B})^{\text{cyc}}) \). Consider the diagram

\[
\mathcal{V}^{\text{cyc}}_{\text{loc}} \to (\mathcal{R}^{\text{cyc}}_{\text{loc}})_{\text{loc}} \leftarrow \mathcal{R}_{\text{loc}}.
\]

This diagram has a colimit, to be denoted by \( \mathcal{V}^{\text{cyc}}_{\text{loc}} \). We thus has a GCO \( \mathcal{V}_l \), where \( \mathcal{V}^{\text{noncyc}}_l = \mathbb{B} \). We have a natural map of GCO \( \mathcal{V} \to \mathcal{V}_l \), whence a through map of GCO

\[
gU \to \mathcal{V}_l. \tag{90}
\]

The map \( (\mathcal{U})^{\text{noncyc}} \to \mathbb{B}^{\text{noncyc}} \) unduces a functor \( t : \mathcal{Y}(\mathcal{U})^{\text{cyc}} \to \mathcal{Y}(\mathbb{B})^{\text{cyc}} \) The map \( \mathcal{V} \to \mathcal{V}_l \) induces a map

\[
\text{red}t(\mathcal{U})^{\text{cyc}}_l \to \text{red}\mathcal{V}^{\text{cyc}}_l. \tag{91}
\]
in \( \text{red}(D \bigoplus \text{Fun}(Y(\mathbb{B}))^{\text{cyc}}) \). One can show (Lemma [21.1]) that this map is a homotopy equivalence, which is the main reason why we apply the localization.

The object on the LHS of (91) is still too complicated, whence our next manipulation.

### 17.5 Condensation

This operation was defined in Sec 15.5, we repeat it here.

Call a co-simplicial object in \( C \) quasi-constant if every arrow in the simplicial category acts by a homotopy equivalence. Recall that the category \( Y(\text{assoc})^{\text{cyc}} \) coincides with the cyclic category, therefore, given a functor \( X : Y(\mathbb{B})^{\text{cyc}} \to C \), one can restrict it to \( Y(\text{assoc})^{\text{cyc}} \) thus getting a co-cyclic object which we can further restrict to the simplicial category, thus getting a co-simplicial object, to be denoted by \( X|_\Delta \). Call \( X \) quasi-constant if such is \( X|_\Delta \). One defines an endofunctor \( \text{con} \) on the category of functors \( Y(\mathbb{B})^{\text{cyc}} \to C \) enriched over \( \text{Sets} \) and a natural transformation \( \text{con} \to \text{Id} \), where \( \text{con}X \) is quasi-constant and the induced map \( R\text{Hom}(T; \text{con}X) \to R\text{Hom}(T; X) \) is a homotopy equivalence for every quasi-constant \( T \). One extends \( \text{con} \) to an endofunctor on \( D \bigoplus \prod \text{Fun}(Y(\mathbb{B})^{\text{cyc}}) \) endowed with a natural transformation \( \text{con} \to \text{Id} \). We can now consider an object \( \text{con}\mathcal{V}_{\text{cyc}} \in D \bigoplus \prod \text{Fun}(Y(\mathbb{B})^{\text{cyc}}) \). The latter object can be identified up-to a homotopy equivalence as follows. First of all we have an induced homotopy equivalence \( \text{conred}(gU)^{\text{cyc}} \to \text{con}\mathcal{V}_{\text{cyc}} \).

As follows from Sec 14 the map of operads \( \text{red}\mathcal{O}_H \to \text{red}\mathcal{O}_F \) induces homotopy equivalences

\[
\mathbb{B} \oplus \text{red}(\mathcal{O}_H)^{\text{cyc}} \leftarrow \mathbb{B} \oplus \text{conred}(\mathcal{O}_H)^{\text{loc}} \to \text{conred}(gU)^{\text{cyc}} \to \text{redcon}\mathcal{V}_{\text{cyc}}.
\]

This identifies \( \text{redcon}\mathcal{V}_{\text{cyc}} \). Claim 16.3 easily implies that the object \( \text{red}(\mathcal{O}_H)^{\text{cyc}} \) is rigid: all its liftings to \( \text{Fun}(Y(\mathbb{B})^{\text{cyc}}) \) are homotopy equivalent to each other, which gives us the desired identification of \( \mathcal{V}_{\text{cyc}} \).

We now pass to the last manipulation.

### 17.6 Splitting the map of obstruction objects \( D(\mathcal{U}) \to D(\mathcal{U}_i) \)

This map admits a splitting according to Claim 16.4. We therefore can write \( D(\mathcal{U}_i^{\text{loc}}) \cong D(\mathcal{U}) \oplus \mathcal{V}_0 \) for an appropriate object \( \mathcal{V}_0 \). We thus have a map \( \mathcal{V}_0 \to D(\mathcal{U}_i^{\text{loc}}) \). It follows that this map factors as follows:

\[
\mathcal{V}_0 \to D(\mathcal{U}_i^{\text{loc}})^{\text{cyc}}
\]

As was mentioned above, we have a homotopy equivalence \( D(\text{redcon}\mathcal{V}_i^{\text{loc}})^{\text{cyc}} \sim \text{Hom}_Y(\text{red}\mathbb{B})^{\text{cyc}}(\text{red}\mathbb{B})^{\text{cyc}}; \text{redcon}\mathcal{V}_i^{\text{cyc}}) \). One deduces that there exists a homotopy equivalence \( \mathcal{V}_0 \sim \mathcal{V}_1 \) and a map \( \mathbb{B}^{\text{cyc}} \otimes \mathcal{V}_1 \to \text{redcon}\mathcal{U}_i^{\text{cyc}} \). According to the above established structure of \( \mathcal{U}_i^{\text{cyc}} \), this map lifts to a map

\[
\mathbb{B}^{\text{cyc}} \otimes \mathcal{V} \to \text{con}\mathcal{V}_i^{\text{cyc}} \to \mathcal{V}_i^{\text{cyc}}.
\]

(92)
17.7 Packing the data into a new (rigid) structure

Let us now define a structure that involves the above map as well as the constructed map $U \to V_l$. We call it GVCO (G=generalized; V stands for the term $V$ in (92)), see Sec 18.2.1 for the definition. We thus get a GVCO built from the above specified data. One redefines the notion of the obstruction space in the setting of GVCO, and it turns out to be acyclic in our case. This provides for the desired quantization.

We start with the general theory (Sec 18-Sec 20), where we give a definition of GVCO, its Kaehler differential module, and the obstruction space. We then show that vanishing of the obstruction space implies the lifting.

In the next Section 21 we perform the above outlined steps resulting in building a GVCO and producing an unobstructed quantization problem.

18 Quantization: general theory

18.1 The categories $B_q, B_0$

Fix the notation $B_q := \text{Com}_{\geq 0}$. Let also $B_0$ be the SMC of $\mathbb{Z}_{\geq 0}$-graded objects in $\mathcal{A}$. We have tensor functors $\text{gr} : B_q \to B_0$, $\text{q} : B_0 \to B_q$, where the functor $\text{gr}$ annihilates the differential and $\text{q}(X)$ is $X$ viewed as a complex with 0 differential.

18.1.1 Tensor functors $T_q : R_q \to B_q$, $T_0 : R_0 \to B_0$.

We have a ground SMC $R_q$. In order to do the quantization, we are going to build a lax tensor functor $T_q : R_q \to B_q$, where we set

$$(T_q X)^i = \prod_{S \in \text{SMet}\text{R}} \text{gr}^\text{valS} X_S,$$

$i \geq 0$. The differential on $T_q X$ is induced by that on $X$.

The functor $T_q$ gives $R_q$ the structure of a SMC enriched over $B_q$.

One defines a tensor functor $T_0 : R_0 \to B_0$ in a similar way so that $R_0$ is enriched over $B_0$.

18.2 VCO

A VCO in a SMC $\mathcal{C}$, is a collection $O_{\text{noncyc}}(n), O_{\text{cyc}}^1(n), O_{\text{cyc}}^2(n), O_{\text{cyc}}^3(n) \in \mathcal{C}$, $n \geq 0$, and $V \in \mathcal{C}$, endowed with the following structure:

— a circular operad structures on $(O_{\text{noncyc}}, O_{\text{cyc}}^1)$, $i = 1, 2, 3$ which induce the same asymmetric operad structure on $O_{\text{noncyc}}$;

— maps of circular operads with their restrictions onto $O_{\text{noncyc}}$ being the identity:

$$(O_{\text{noncyc}}, O_{\text{cyc}}^1) \to (O_{\text{noncyc}}, O_{\text{cyc}}^2) \leftarrow (O_{\text{noncyc}}, V \otimes O_{\text{cyc}}^3).$$
18.2.1 Definition of GVCO

The VCO structure can be reformulated as follows:
— an asymmetric operad \( O^{\text{noncyc}} \);
— \( Y(O^{\text{noncyc}})^{\text{cyc}} \)-modules \( O_i^{\text{cyc}} \); \( i = 1, 2, 3 \);
— an object \( V \in \mathcal{C} \);
— \( \text{maps of } Y(O^{\text{noncyc}})^{\text{cyc}} \)-modules \( O_1 \to O_2 \leftarrow V \otimes O_3 \).

One therefore can generalize: let GVCO be the following structure:
— an asymmetric operad \( O^{\text{noncyc}} \);
— objects \( O_i^{\text{cyc}} \) of \( \mathcal{D} \bigoplus \prod \text{Fun}(Y^{\text{cyc}}(O^{\text{noncyc}})^{\text{op}}) \); \( i = 1, 2, 3 \);
— an object \( V \in \mathcal{C} \);
— maps \( O_1 \to O_2 \leftarrow V \otimes O_3 \) in \( \mathcal{D} \bigoplus \prod \text{Fun}(Y^{\text{cyc}}(O^{\text{noncyc}})^{\text{op}}) \).

GVCO over \( \mathcal{C} \) form a category enriched over \( \text{Sets} \).

Given a GVCO \( O \) one gets a VCO operad \( [O] \), where \( [O]^{\text{noncyc}} = O^{\text{noncyc}} \);

\[
[O]_k^{\text{cyc}}((n)^{\text{cyc}}) := \text{Hom}_{\mathcal{D} \bigoplus \prod \text{Fun}(Y(O)^{\text{cyc}})^{\text{op}}}( (n)^{\text{cyc}} ; O_k^{\text{cyc}} )
\]

In fact, \([,] \) is a functor from the category of GVCO to that of VCO. We define a map \( f \) from a VCO \( O' \) to a GVCO \( O \) as a VCO map \( f : O' \to [O] \).

18.3 Semi-free operads

Let \( \text{Col} \) be a category (over \( \text{Sets} \)) whose every object is a collection of objects \( X(n)^{\text{noncyc}}, X(n)^{\text{cyc}}, n > 0, k = 1, 2, 3 \), and \( X^V \in \mathcal{C} \). We have a forgetful functor from the category of VCO to \( \text{Col} \) which has a left adjoint, to be denoted by \( \text{free} \). A VCO of the form \( \text{free}(X) \) is called \( \text{free} \).

A quasi-free VCO is that of the form \( [\text{free}(X), D] \), where \( D \) is an arbitrary differential.

A semi-free VCO is a quasi-free VCO of the form \( [\text{free}(X^*), D] \), where each \( X^*^{\text{noncyc}}(n), X^*^{\text{cyc}}(n)_k, X^V \in \mathcal{C}_{\leq 0}^{\text{cyc}} \) and \( \text{gr} D = 0 \).

Given a quasi-free VCO \( O \), we get a GVCO \( gO \), where

\[
gO_k \in \text{Fun}(Y^{\text{cyc}}(O^{\text{noncyc}})^{\text{op}}).
\]

A map \( gO_1 \to O \), where \( O_1 \) is a VCO and \( O \) a GVCO is the same as a map \( O_1 \to O \) as defined above.

18.4 Infinitesimal operads

An infinitesimal VCO (IVCO) is a VCO in the category \( \text{Com}_{[0,1]}(\mathcal{C}) \).

Given an IVCO \( O \) let \( |O| \) denote the VCO over \( \mathcal{C} \) obtained via applying the totalization functor \( \text{Com}_{[0,1]}(\mathcal{C}) \to \mathcal{C} \).
Let $\text{Gr}_{[0,1]}(\mathcal{C})$ be a category whose every object is a pair $(X^0, X^1)$ of objects of $\mathcal{C}$. We set
\[
\text{Hom}((X^0, X^1); (Y^0, Y^1)) = \text{Hom}(X^0, Y^0) \oplus \text{Hom}(X^1, Y^1).
\]

Define a SMC on $\text{Gr}_{[0,1]}(\mathcal{C})$ by setting
\[
(X^0, X^1) \otimes (Y^0, Y^1) := X^0 \otimes Y^0; X^0 \otimes Y^1 \oplus X^1 \otimes Y^0.
\]

We have functors
\[
\text{Gr}_{[0,1]}(\mathcal{C}) \xrightarrow{\iota} \text{Com}_{[0,1]}(\mathcal{C}) \xrightarrow{\pi} \text{Gr}_{[0,1]}(\mathcal{C}).
\]

A split IVCO is a circular operad in the category $\text{Gr}_{[0,1]}(\mathcal{C})$.

Given a split IVCO $\mathcal{O}'$, denote by $X(\mathcal{O}')$ the set of all elements $D \in \text{Hom}^1((\mathcal{O}')^0, (\mathcal{O}')^1)$ such that $(\mathcal{O}', D)$ is an IVCO.

18.5 The categories $S^G(\mathcal{O}), Y_V(\mathcal{O})$

Let $\mathcal{O}$ be a GVCO in $\mathcal{C}$. Denote by $S^G(\mathcal{O})$ be the category over Sets whose every object is a split IGVCO $\mathcal{O}'$ along with an identification of IVCO $(\mathcal{O}')^0 = \mathcal{O}$.

18.6 Kaehler differentials

Let $\mathcal{O}$ be a VCO in $\mathcal{C}$. We have a functor $\mathcal{O}' \mapsto X(\mathcal{O}')$ from $S(\mathcal{O})$ to Sets. Suppose $X$ is representable. Call the representing object $\omega_{\mathcal{O}}$.

**Proposition 18.1** Suppose $\mathcal{O}$ is quasi-free. Then $X$ is representable so that $\omega_{\mathcal{O}}$ is well-defined.

We have a canonical element in $X(\omega_{\mathcal{O}})$ which gives $\omega_{\mathcal{O}}$ an IVCO structure. It also follows that given an IVCO $\mathcal{O}'$ with $(\mathcal{O}')^0 = \mathcal{O}$ (as VCO), we have a canonical map of split IVCO
\[
\omega_{\mathcal{O}} \to \mathcal{O}'
\]
which is compatible with the infinitesimal circular operad structures on both sides.

Let us denote $\Omega_{\mathcal{O}} := (\omega_{\mathcal{O}})^1$. We have a $Y(\mathcal{O})$-module structure on $\Omega_{\mathcal{O}}$. Let $\mathcal{O}' \in S^G(\mathcal{O})$. We then have an identification
\[
X(\mathcal{O}') = \text{Hom}_{Y(\mathcal{O})}(\Omega_{\mathcal{O}}; (\mathcal{O}')^1).
\]

18.7 An IVCO $X(\mathcal{O}')$

Let $E$ be a category over Sets whose every object is a pair $(X \in \mathcal{C}, f : \text{unit}_{\mathcal{C}} \to X)$. An arrow $(X, f) \to (Y, g)$ is a map of complexes $\phi : \text{Cone} f \to \text{Cone} g$Such that
— the composition
\[
\text{Cone} f \xrightarrow{\phi} \text{Cone} g \to \text{unit}_{\mathcal{C}}[1]
\]

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equals the projection $\text{Cone } f \to \text{unit}_C[1]$.

Let $(X, f) \in E$ and let $O$ be a VCO in $\mathcal{C}$. Denote by $O^{X, f}$ the following IVCO in $\mathcal{C}$:

$$(O^{X, f})^0 = O; \quad (O^{X, f})^1 = X \otimes \Omega_O;$$

$$D : O \to X \otimes O$$

is defined as the composition

$$O \xrightarrow{D_\omega} \Omega_O \xrightarrow{f} X \otimes \Omega_O.$$

Let $u : (X, f) \to (Y, g)$ be an arrow in $E$ whose components are

$$e : \text{unit}_C[1] \to Y; \quad v : X \to Y; \quad \text{Id} : \text{unit}_C[1] \to \text{unit}_C[1].$$

In order for $u$ to be a map of complexes, one must have: $dv = 0; \quad de + v(f) + g = 0$.

Let us define a map $u_* : |O^{X, f}| \to |O^{Y, g}|$ via its components:

$$\text{Id} : O \to O; \quad D_\omega \otimes e : O \to \Omega_O \otimes Y; \quad \text{Id}_{\Omega_O} \otimes v : \Omega_O \otimes X \to \Omega_O \otimes Y.$$

This way, we get a functor $E \to S(\mathcal{O})$. Let $\mathcal{O}' \in S^G(\mathcal{O})$. We then get a functor $H_{\mathcal{O}'} : E \to \text{Sets}$, where

$$H_{\mathcal{O}'}((X, f)) = \text{Hom}_{S(\mathcal{O})}^G(O^{X, f} : \mathcal{O}').$$

Proposition 18.2 Suppose $\mathcal{O}$ is quasi-free. Then $H_{\mathcal{O}'}$ is representable.

Denote the representing object $(\mathcal{X}(\mathcal{O}'), D_{\mathcal{O}'})$.

19 Lifting

Let $\mathcal{A}$ be a ground category. Let $\mathcal{Q}_q$ be SMC enriched over $\text{Com}_{\leq 1}\mathcal{A}$. Let $\mathcal{Q}_0 := \pi_0\mathcal{Q}$. Let $\text{red} : \mathcal{Q}_q \to \mathcal{Q}_0$ be the reduction functor (over $\mathcal{A}$).

Let $\mathcal{A}^V, \mathcal{B}^V$ be quasi-free VCO in $\mathcal{Q}_q$. Let $\mathcal{O}$ be a GVCO in $\mathcal{Q}_q$. Suppose we are given a map $\mathcal{O} \to \mathcal{g}\mathcal{B}^V$ which quasi-splits as well as a map $\alpha : \mathcal{A}^V \to \mathcal{g}\mathcal{B}^V$ (which is the same as a VCO map $\alpha : \mathcal{A}^V \to \mathcal{B}^V$).

Suppose, finally, that we have a map $\beta : \text{red}\mathcal{A}^V \to \text{red}\mathcal{O}$ such that the through map $\text{red}\mathcal{A}^V \to \text{red}\mathcal{g}\mathcal{B}^V$ equals $\text{red}\alpha$.

Let us define the following functor $T : S(\mathcal{A}^V) \to \text{Sets}$. Set $T(\mathcal{A}^V')$ to consist of all maps $|\mathcal{A}^V'| \to \mathcal{O}$ such that the composition $|\mathcal{A}^V'| \to \mathcal{O} \to \mathcal{g}\mathcal{B}^V$ coincides with the composition

$$|\mathcal{A}^V'| \to \mathcal{A}^V \xrightarrow{\alpha} \mathcal{g}\mathcal{B}^V$$

and the induced map $\text{red}|\mathcal{A}^V'| \to \text{red}\mathcal{O}$ coincides with the composition

$$\text{red}|\mathcal{A}^V'| \to \text{red}\mathcal{A}^V \xrightarrow{\beta} \text{red}\mathcal{O}.$$
Proposition 19.1 The functor $T_E$ is representable.

Let $(\mathcal{Y}, g)$ be the representing object.

19.0.1 Explicit formula for $\mathcal{Y}$

Let $G_{\mathfrak{A}}$ be a category whose every object is a pair $(X^0, X^1)$ of objects from $\mathfrak{A}$. Set

$$\text{Hom}((X^0, X^1); (Y^0, Y^1)) = \text{Hom}(X^0, Y^0) \oplus \text{Hom}(X^1, Y^1).$$

The category $G_{\mathfrak{A}}$ has a symmetric monoidal structure via

$$(X^0, X^1) \otimes (Y^0, Y^1) := (X^0 \otimes Y^0, X^0 \otimes Y^1 \oplus X^1 \otimes Y^0).$$

We have a tensor functor $r : \text{Com}_{\leq 1}(\mathfrak{A}) \to G_{\mathfrak{A}}$, where $r(X^0 \to X^1) = (X^0, X^1)$.

Let us get back to a sequence of operads

$$\text{red}_{\mathcal{A}V} \to \text{red}\mathcal{O} \to \text{redgB}^V.$$

We can consider the above sequence as that of maps of $\mathcal{Y}(\text{redB}^V)^{\text{op}}$-modules. The arrow $\text{red}\mathcal{O} \to \text{redgB}^V$ admits a kernel, to be denoted by $K$, which is a $\mathcal{Y}(\text{redA}V)^{\text{op}}$-module.

We now have a well-defined object

$$K := \text{Hom}_{\mathcal{Y}(\mathcal{A}V)}(\Omega_{\mathcal{A}V}; K) \in G_{\mathfrak{A}}.$$

Proposition 19.2 We have $\mathcal{Y} \cong K^1$.

20 Lifting: Iterations

20.1 Preliminaries

Let $B_q := \text{Com}_{\geq 0}\mathfrak{A}$, as above. Recall that the category $R_q$ is enriched over $B_q$.

Let $F^{\leq i} : B_q \to B_q$ associate to a complex $(X^\bullet, d)$ the complex $(F^{\leq i}X^\bullet, d)$, where $(F^{\leq i}X)^k = X^k$, $k \leq i$, $(F^{\leq i}X)^k = 0$, $k > i$. We have a lax tensor structure on $F^{\leq i}$ so that we have categories $R_i := F^{\leq i}R_q$ enriched over $B_q$. We have tensor functors $\text{red}_{NM} : R_N \to R_M$, $N \geq M$. We also have quasi-splittings $i_{MN} : \text{Hom}_{R_M}^k(X, Y) \to \text{Hom}_{R_N}^k(X, Y)$ of $\text{red}_{NM}$ which are not compatible with the differential.

As above, let $\mathcal{A}V, B^V$, be quasi-free VCO in $R_q$ and $\mathcal{U}$ be a GVCO endowed with the following maps

$$\mathcal{U} \to gB^V; \quad \beta : \mathcal{A}V \to gB^V \quad \text{redA}V \to \text{red}\mathcal{U},$$

where the composition

$$\text{redA}V \to \text{red}\mathcal{U} \to \text{redgB}^V$$

coincides with $\text{red}\beta$. 141
20.2 Rigidity

We have the following diagram in $B_0$:

$$\text{Hom}_D \prod \oplus \text{Fun}_Y((\text{red}_A^V)^\text{op})(\Omega_{\text{red}_A^V}; \text{red} U) \rightarrow \text{Hom}_D \prod \oplus \text{Fun}_Y(\text{red}_A^V)^\text{op})(\Omega_{\text{red}_A^V}; \text{red} g B^V).$$

(93)

Call the data from the previous sub-section rigid if the cone of this diagram is acyclic.

In this sub-section we will construct a VCO $A^V$ in $R_q$, a homotopy equivalence $A^V' \rightarrow A^V$ and a lifting $A^V' \rightarrow U$ such that the induced map $\text{red} A^V' \rightarrow \text{red} U$ coincides with the composition $\text{red} A^V' \rightarrow \text{red} A^V \rightarrow \pi U$ and the through map

$$A^V' \rightarrow U \rightarrow g B^V$$

coincides with the composition $A^V' \rightarrow A^V \rightarrow g B^V$.

20.3 Categories $R_i$

Let $B_i := \text{Com}_{[0,i]} \mathfrak{A} \subset \text{Com}_{\geq 0} \mathfrak{A}$ be a full sub-category consisting of all complexes $(X^*,d)$, where $X^i = 0$ for all $i > 1$. Let also $\pi_{\leq i} : \text{Com}_{\geq 0} \mathfrak{A} \rightarrow \text{Com}_{[0,i]} \mathfrak{A}$ be given by

$$\pi_{\leq i} X^j = X^j, \ j \leq i; \quad \pi_{\leq i} X^j = 0, \ j > i.$$

Introduce a tensor structure on $\text{Com}_{[0,i]} \mathfrak{A}$ by setting

$$(X_1 \otimes X_2 \otimes \cdots \otimes X_n)_{\text{Com}_{[0,i]} \mathfrak{A}} := \pi_{\leq i} (X_1 \otimes X_2 \otimes \cdots \otimes X_n)_{\text{Com}_{\geq 0} \mathfrak{A}}.$$

The functor $\pi_{\leq i}$ has an obvious tensor structure. Let also $\iota_i : \text{Com}_{[0,i]} \rightarrow \text{Com}_{[0,\infty]}$ be the embedding. We have a lax tensor structure on $\iota_i$, i.e. the natural transformation

$$\iota_i (X) \otimes \iota_i (Y) \rightarrow \iota_i (X \otimes Y)$$

is not necessarily a homotopy equivalence.

Let $R_{\leq i} := \pi_{\leq i} R_q$ so that $R_{\leq i}$ is an SMC enriched over $\text{Com}_{[0,i]} (A^V)$, hence over $B$.

Let us define a tensor functor $C : \text{Com}_{[0,i+1]} (\mathfrak{A}) \rightarrow \text{Com}_{[0,1]} (\mathfrak{A})$, where

$$C(X) = (C(X)^0 \xrightarrow{D} C(X)^1)$$

with $C(X)^0 = \pi_{\leq i} X$ and $C(X^1) = X^{i+1}$. The differential $D$ is chosen so that we have an isomorphism $|C(X)| \equiv |X|$. This way $R_{i+1}$ is enriched over $\text{Com}_{\leq 1} (\mathfrak{A})$.

Let now $\rho (\text{Com}_{[0,1]} (\mathfrak{A})) \rightarrow \mathfrak{A}$ be a tensor functor defined as follows $\rho (X^0 \rightarrow X^1) := X^0$. We now have an equivalence of SMC enriched over $\mathfrak{A}$: $\rho R_{i+1} \simeq R_i$.

Let us construct:

— a projective sequence $\cdots \rightarrow A^V(2) \rightarrow A^V(1) \rightarrow A^V(0) = A^V$ of quasi-free VCO over $R_q$, where all the arrows are term-wise homotopy equivalences;

— maps $\alpha_i : A^V(i) \rightarrow U$ over $R_i$ compatible with the maps in the above projective sequence, where the map $\alpha_0$ coincides with the given one;

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— maps $\beta_i : \mathcal{A}^{(i)} \to g^B$ over $R_q$, compatible with the above projective sequence, where the map $\beta_0$ must coincide with the given one.

— the following diagram must commute in $R_i$;

\[
\begin{array}{ccc}
\mathcal{U} & \to & g^B \\
& \uparrow \mathcal{V}(i) & \downarrow \mathcal{A} \\
& \mathcal{A}^{(i)} & \\
\end{array}
\]

Let us do this inductively with respect to $i$. The base $i = 0$ is clear. The step (from $i$ to $i + 1$) is as follows. By the assumption, we have the following diagram

\[
\begin{array}{ccc}
\mathcal{U} & \to & g^B \\
& \uparrow \mathcal{V}(i) & \downarrow \mathcal{A} \\
& \mathcal{A}^{(i)} & \\
\end{array}
\]

where the through map is a term-wise homotopy equivalence. We can now use the tensor functor $C$ so as to apply Prop 19.1. We will then get an object $(\mathcal{Y}, g)$ and an operad $\mathcal{A}^{(i+1), \mathcal{Y}, g}$ defined over $Q_q$ fitting into the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{U} & \to & g^B \\
& \uparrow \mathcal{V}(i) & \downarrow \mathcal{A} \\
& \mathcal{A}^{(i+1), \mathcal{Y}, g} & \to & \mathcal{A}^{(i)} \\
\end{array}
\]

It follows that $\mathcal{Y}$ is acyclic, therefore, the natural map

$$\mathcal{A}^{(i+1), \mathcal{Y}, g} \to \mathcal{A}^{(i)}$$

is a homotopy equivalence.

We therefore can set $\mathcal{A}^{(i+1)}$ to be a semi-free resolution of $\mathcal{A}^{(i), \mathcal{Y}, g}$. This finishes the proof.

21 Quantization: our case

Let us construct a VCO $\mathcal{V}$ possessing all the features from the previous section.
21.1 More on categories: The category $\mathcal{I}E$, where $E$ is enriched over $B_q$

21.1.1 Truncation

Let $\tau_k : B_q \rightarrow B_q$ be the truncation ($\tau_k = \text{Id}$ for all $k \leq 0$).

21.1.2 Definition of $\mathcal{I}E$

Let $E$ be a category enriched over $B_q$. Define a new category $\mathcal{I}E$, enriched over $B_q$.

Let us first define a category $\mathcal{I}E'$, enriched over $B_q$, whose every object is a collection of objects $X^n \in E$, $n \in \mathbb{Z}$. Set

$$\text{Hom}(X,Y) := \prod_{n,m} \tau_{\geq n-m} \text{Hom}_E(X^n;Y^m) \in B_q.$$

Set $\mathcal{I}E := D(\mathcal{I}E')$. Suppose $E$ is an SMC, we the have an SMC structure on $\mathcal{I}E$, where

$$(X \otimes Y)^n = \bigoplus_m X^m \otimes Y^{(n-m)}.$$

Let $I_{\leq 0}E \subset \mathcal{I}E$ be the full sub-category of objects $(X^n,D)$, where $X^n = 0$ for all $n > 0$ and likewise for $I_{\geq 0}E \subset \mathcal{I}E$.

We have tensor functors

$$\text{red} : \text{red}(I_{\leq 0}E) \rightarrow \text{Com}_{\leq 0}E; \quad \text{red} : \text{red}(I_{\geq 0}E) \rightarrow \text{Com}_{\geq 0}E,$$

where $(X,D) \mapsto (\text{red}X,\text{red}D)$.

21.2 The operad $\mathbb{A}$

21.2.1 The operad MC

View $\mathfrak{A}$ as a category enriched over $B_q$, via the obvious embedding $i_B : \mathbb{A} \rightarrow B_q$, where $i_B(X)^0 = X$; $i_B(X)^p = 0$ if $p \neq 0$. For $T \in \mathfrak{A}$, denote by $T^{(k)} \in I\mathfrak{A}$, the following object: $(T^{(k)})^l = T$ if $l = k$; $(T^{(k)})^l = 0$ otherwise.

Let $MC$ be a CO in $I_{\geq 0}\mathfrak{A}$ freely generated by the elements:

$-m_n : k^{(0)}[2-n] \rightarrow \text{MC}(n)\text{noncyc}, n \geq 2$;

$-m_n : k^{(1)}[2-n] \rightarrow \text{MC}(n)\text{noncyc}, n \leq 2$;

$-\mu_n : k \rightarrow \text{MC}(n)\text{cyc}[1-n]$; it is assumed that each $\mu_n$ is $\mathbb{Z}/n\mathbb{Z}$-invariant.

The differential is defined as follows:

$$dm_n + \sum m_p\{m_{n+1-p}\} = 0; \quad d\mu_n + \mu_p\{m_{n+1-p}\} = 0.$$

We have a natural map $MC \rightarrow \text{assoc}$, which vanishes on all $m_n$, $n \neq 2$, and on all $\mu_n$, $n \neq 0$. We have an induced map

$$\mathfrak{a} \otimes MC \rightarrow \mathbb{B}.$$
Denote by \( \mathfrak{A} \) the canonical semi-free resolution of \( a \otimes MC \). We therefore have an induced map \( \mathfrak{A} \to \mathcal{R}_E \).

The embedding \( \mathfrak{A} \to R_q, X \mapsto X \otimes \text{unit} \), induces an embedding

\[
I_{\geq 0} \mathfrak{A} \to I_{\geq 0} R_q
\]

which is a tensor functor. This way, \( \mathfrak{A} \) is a CO in \( I_{\geq 0} R_q \).

### 21.3 Formulating the quantization problem

Based on the data from Sec. 16, we will construct a zig-zag map of operads in \( I_{\geq 0} R_q \):

\[
\mathfrak{A} \leftarrow \mathfrak{A}' \to \mathcal{O}
\]

which lifts the through map

\[
\mathfrak{A} \to \mathcal{R}_E \to \text{gr} \mathcal{O},
\]

where \( \mathcal{O} = \mathcal{O}^E \) and \( \iota \) are as in (74).

### 21.4 A CO \( \mathcal{U} \) in \( I_{\leq 0} R_q \)

Set \( \mathcal{U} := \mathcal{O} \oplus \mathcal{R}_E \). We have a splitting \( \sigma : \text{red} \mathcal{R}_E \to \text{red} \mathcal{U} \to \text{red} \mathcal{R}_E \), where \( \sigma|_{\text{red} \mathcal{R}_E} = \iota \) and \( \sigma|_{\mathcal{R}_E} = \text{Id} \). We therefore can represent

\[
\mathcal{U}(*) = (\mathcal{R}_E(*), K(*), D),
\]

where \( D : K(*) \to \mathcal{R}_E(*) \) is a differential and \( D \) vanishes in \( R_0 \), that is \( D \in \tau_{\geq 1} \text{Hom}^1(K(*), \mathcal{R}_E(*)) \).

Let us upgrade \( \mathcal{U} \) to a CO in \( I_{\leq 0} R_q \), where \( \mathcal{U}^0 = \mathcal{R}_E \) and \( \mathcal{U}^{-1}(*) = K(*) \), all other components of \( \mathcal{U}^\bullet \) vanish.

Let us also view \( \mathcal{R}_E \) as a circular operad in \( I_{\leq 0} R_q \) centered in degree 0. We still have a map of circular operads

\[
\mathcal{U}^\bullet \to \mathcal{R}_E.
\]

### 21.5 Tensoring with \( Y^{\text{cyc}}(\mathcal{R}_E) \)

Denote

\[
\mathcal{T}^{\text{cyc}} := Y^{\text{cyc}}(\mathcal{R}_E) \otimes \mathcal{T}^{\text{cyc}}(\mathcal{U}^\bullet) (\mathcal{U}^\bullet)^{\text{cyc}} : Y^{\text{cyc}}(\mathcal{R}_E) \to I_{\leq 0} R_q.
\]

We have a through map

\[
\mathcal{T}^{\text{cyc}} \to Y^{\text{cyc}}(\mathcal{R}_E) \otimes \mathcal{T}^{\text{cyc}}(\mathcal{R}_E) \mathcal{R}_E^{\text{cyc}} \to \mathcal{R}_E^{\text{cyc}}.
\]

We have a circular operad structure on the pair \( (\mathcal{R}_E^{\text{noncyc}}, \mathcal{T}^{\text{cyc}}) \) as well as maps of circular operads over \( I_{\leq 0} R_q \)

\[
\mathcal{U}^\bullet \to (\mathcal{R}_E^{\text{noncyc}}, \mathcal{T}^{\text{cyc}}) \to \mathcal{R}_E,
\]

where the composition coincides with the map [95].
The object \( \mathcal{I}^{\text{cyc}} \) is quasi-free, it therefore defines an object

\[
\mathcal{I}^{\text{cyc}} \in D \bigoplus ((Y(\mathcal{R}_B)^{\text{cyc}})^{\text{op}} \otimes I_{\leq 0} R_q).
\]  

(96)

Let us define functor

\[
\lambda: \bigoplus (Y(\mathcal{R}_B)^{\text{cyc}})^{\text{op}} \otimes I_{\leq 0} R_q' \rightarrow \bigoplus (Y(\mathcal{R}_B)^{\text{cyc}} \otimes R_q),
\]

where

\[
\lambda \bigoplus (k_a) \otimes X_a := \bigoplus \bigoplus (k_a) \otimes X^{-n}.
\]

This functor naturally gives rise to a tensor functor

\[
\lambda: D \bigoplus ((Y(\mathcal{R}_B)^{\text{cyc}})^{\text{op}} \otimes I_{\leq 0} R_q) \rightarrow D \bigoplus ((Y(\mathcal{R}_B)^{\text{cyc}})^{\text{op}} \otimes R_q).
\]

Hence, \((\mathcal{R}_B^{\text{noncyc}}, \lambda(\mathcal{I}^{\text{cyc}}))\) is a GCO, to be denoted below by \(\mathcal{I} \mathcal{U}\).

On the classical level, we have the following diagram

\[
\begin{array}{ccc}
\text{red} D \bigoplus ((Y(\mathcal{R}_B)^{\text{cyc}})^{\text{op}} \otimes I_{\leq 0} R_q) & \xrightarrow{\text{red} \lambda} & D \bigoplus ((Y(\mathcal{R}_B)^{\text{cyc}})^{\text{op}} \otimes R_0) \\
\downarrow \sim & & \downarrow \parallel \\
D \bigoplus ((Y(\mathcal{R}_B)^{\text{cyc}})^{\text{op}} \otimes \text{Com}_{\leq 0} R_0) & \longrightarrow & \text{Com}_{\leq 0} D \bigoplus ((Y(\mathcal{R}_B)^{\text{cyc}})^{\text{op}} \otimes R_0)
\end{array}
\]

which commutes up-to a natural isomorphism of the functors.

Below, we denote

\[
\text{Fun}_{\mathcal{R}_B} := D \bigoplus ((Y(\mathcal{R}_B)^{\text{cyc}})^{\text{op}} \otimes R_q).
\]

21.6 \( c_1 \) - Localization

Consider a diagram in \((D \bigoplus) \text{Fun}_{\mathcal{R}_B}\)

\[
\mathcal{I} \mathcal{U}^{\text{cyc}}_{\text{loc}} \xrightarrow{p_{\mathcal{R}_B^{\text{cyc}}}} \mathcal{R}^{\text{cyc}}_{\text{loc}} \leftarrow 0.
\]  

(98)

This right arrow of this diagram quasi-splits so that it admits a pull-back to be denoted by \(\mathcal{I} \mathcal{O}^{\text{cyc}}_l\).

Let \(\mathcal{I} \mathcal{O}_l := (\mathcal{R}_B^{\text{noncyc}}, \mathcal{I} \mathcal{O}^{\text{cyc}}_l)\).

As we have a map \(\text{red} \mathcal{R}_B \rightarrow \text{red} \mathcal{O}\), we have a \(Y(\text{red} \mathcal{R}_B)^{\text{cyc}}\)-module structure on \(\mathcal{O}^{\text{cyc}}\), hence an object \(\text{red} \mathcal{O}^{\text{cyc}}_{\text{loc}} \in (D \bigoplus) \text{Fun}_{\mathcal{R}_B}\).

We now have maps in \((D \bigoplus) \text{Fun}_{\text{red} \mathcal{R}_B}\):

\[
(\text{red} \mathcal{O}^{\text{cyc}})_{\text{loc}} \rightarrow \text{red} \mathcal{I} \mathcal{O}^{\text{cyc}}_l.
\]

Lemma 21.1 This map is a homotopy equivalence.
Sketch of the proof We have a functor

\[ F : \text{red}D \bigoplus ((Y(\mathcal{R}_\mathbb{E})^{\text{cyc}})^{\text{op}} \otimes I_{\leq 0}R_q) \to \text{Com}_{\leq 0}(D \bigoplus ((Y(\mathcal{R}_\mathbb{E})^{\text{cyc}})^{\text{op}} \otimes R_0) \]

which coincides with the composition of the left and the bottom arrows in (97). The same diagram implies that \( \text{red} \mathcal{F}^{\text{cyc}} \cong \mathcal{F}^{\text{cyc}} \), where \( \mathcal{F} \) is the totalization functor from \( \text{Com}_{\leq 0} \).

Denote by

\[ I_k := (\mathcal{F}^{\text{cyc}})^{-k} \in \text{Fun}((Y(\mathcal{R}_\mathbb{E})^{\text{cyc}})^{\text{op}} \otimes R_0) \]

the \(-k\)-th component of \( \mathcal{F}^{\text{cyc}} \) as a complex concentrated in non-positive degrees.

As the arrow \( p_\mathbb{E} \) in (98) induces an isomorphism on the 0-th component, and we have a homotopy equivalence

\[ \mathcal{O}^{\text{cyc}} \simeq (I_1)_{\text{loc}}, \]

it suffices to show that \((I_k)_{\text{loc}} \sim 0\) for all \( k > 1 \).

Next, we have \((k \geq 2)\):

\[ I_k \cong \text{red} \bigoplus_{r > 0, t_1 > 0, t_1 + t_2 + \cdots + t_r = k} Y(\mathbb{E})^{\text{cyc}} \otimes L_{\mathcal{Y}(\mathbb{E})^{\text{cyc}}} Y_{t_1}(\mathcal{U})^{\text{cyc}} \otimes L_{\mathcal{Y}(\mathbb{E})^{\text{cyc}}} Y_{t_2}(\mathcal{U})^{\text{cyc}} \otimes L_{\mathcal{Y}(\mathbb{E})^{\text{cyc}}} \mathcal{O}_{\text{cyc}} \]

\[ \sim \bigoplus_{r > 0, t_1 > 0, t_1 + t_2 + \cdots + t_r = k} Y_{t_1}(\mathcal{U})^{\text{cyc}} \otimes L_{\mathcal{Y}(\mathbb{E})^{\text{cyc}}} Y_{t_2}(\mathcal{U})^{\text{cyc}} \otimes L_{\mathcal{Y}(\mathbb{E})^{\text{cyc}}} \cdots \otimes L_{\mathcal{Y}(\mathbb{E})^{\text{cyc}}} Y_{t_r}(\mathcal{U})^{\text{cyc}} \otimes L_{\mathcal{Y}(\mathbb{E})^{\text{cyc}}} \mathcal{O}_{\text{cyc}}. \]

It therefore, suffices to show that given any \( M : Y(\mathbb{E})^{\text{cyc}} \to R_0 \), the \( c_1 \)-action on \( Y_{t}(\mathcal{U})^{\text{cyc}} \otimes L_{\mathcal{Y}(\mathbb{E})^{\text{cyc}}} M \) is 0.

Let us give an explicit description of

\[ \text{red}Y_{t}(\mathcal{U})^{\text{cyc}} : (Y(\mathbb{E})^{\text{cyc}})^{\text{op}} \otimes Y(\mathbb{E})^{\text{cyc}} \to R_0. \]

Let \((k) := \{0, 1, 2, \ldots, k\}\) viewed as a cyclically ordered set. Given a cyclically monotone map \( f : (m) \to (n) \), we have a total order on every pre-image \( f^{-1}i, i \in (n) \).

We have

\[ \text{red}Y_t(\mathcal{U})^{\text{cyc}}((n), (m)) = \text{red} \bigoplus_{f : (m) \to (n), S \subset (n), |S| = t} \bigotimes_{s \in S} \mathcal{O}_{\text{noncyc}}(f^{-1}s|s) \otimes \bigotimes_{t \notin S} \mathcal{E}_{\text{noncyc}}(f^{-1}t|t), \]

where the direct sum is taken over all cyclically monotone \( f : (m) \to (n) \) and all subsets \( S \subset (n) \) of cardinality \( t \). Denote

\[ \text{red}Y_t(\mathcal{U})^{\text{cyc}}((n), (m))^S := \bigotimes_{s \in S} \mathcal{O}_{\text{noncyc}}(f^{-1}s|s) \otimes \bigotimes_{t \notin S} \mathcal{E}_{\text{noncyc}}(f^{-1}t|t) \]

Let \( S_t(n) \) be the set of all \( t \)-element subsets of \( (n) \). Let us now define a functor \( \Sigma_t : Y^{\text{cyc}}(\text{assoc}) \to \mathcal{Q}\text{-mod} \), where we set \( \Sigma_t((n)) := \mathcal{Q}[S_t(n)] \) (the \( \mathcal{Q} \)-span of \( S_t(n) \)). Let \( f : (n) \to (m) \) be a cyclically monotone map. We then have an element \( f_\ast \in Y^{\text{cyc}}(\text{assoc})(\langle m \rangle, (n)) \). Let \( S \in S_t(n) \), let \( e_S \in \mathcal{Q}[S_t(n)] \)

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be the corresponding element. Set \( f, e_S := e_f(S) \) if \( f \) is injective on \( S \) and \( f, e_S := 0 \) otherwise. Denote by
\[
\Sigma'_t : Y^{\text{cyc}}(\text{assoc})^{\text{op}} \otimes Y^{\text{cyc}}(\text{assoc}) \to \mathbb{Q} \text{-mod}
\]
the functor \( \Sigma'_t((n), (m)) := \Sigma_t((m)) \). Let \( \circ \) be as in Sec 15.8. We now have a retraction of functors \( Y^{\text{cyc}}(\mathcal{B})^{\text{op}} \otimes Y^{\text{cyc}}(\mathcal{B}) \to R_0 \):
\[
Y_t(U)^{\text{cyc}} \to Y_t(U)^{\text{cyc}} \circ \Sigma'_t \to Y_t(U)^{\text{cyc}}.
\]
This retraction is defined as follows. Let \( \varepsilon : \mathbb{Q} \to \Sigma'_t((m), (n)) \) be given by \( \varepsilon(1) = e_S \). Set
\[
\varepsilon|_{Y_t(U)^S} := \text{Id} \otimes \varepsilon|_S.
\]
Set \( \pi|_{Y_t(U)^S} = \text{Id}; \pi|_{Y_t(U)^S} = 0 \) if \( T \neq S \).

The problem now reduces to showing the nilpotence of the \( c_1 \)-action on \( \Sigma_t \), which can be verified by the direct computation. Observe that this fact requires \( \mathbb{Q} \) as a base ring.

**Corollary 21.2** We have a homotopy equivalence in \((\mathcal{D} \oplus \prod \text{Fun}_{\mathcal{R}_B})^c:\)
\[
\text{redcon}(\mathcal{O}_l^{\text{cyc}}) \to \text{redcon}(\text{loc} \text{Fun}_{\mathcal{R}_B}(\text{con}(\mathcal{O}_l^{\text{cyc}})).
\]

**Corollary 21.3** Let
\[
A := \text{End}_{\mathcal{D} \oplus \prod \text{Fun}_{\mathcal{R}_B}}(\text{con}(\mathcal{O}_l^{\text{cyc}})).
\]
Then \( \text{gr}^{>0} A \) is acyclic.

### 21.6.1 Studying \( \text{Hom}_{\mathcal{D} \oplus \prod \text{Fun}_{\mathcal{R}_B}}((\mathcal{R}_B^{\text{cyc}}; \text{con}(\mathcal{O}_l^{\text{cyc}})) \)

Set
\[
H := \text{Hom}_{\mathcal{D} \oplus \prod \text{Fun}_{\mathcal{R}_B}}((\mathcal{R}_B^{\text{cyc}}; \text{con}(\mathcal{O}_l^{\text{cyc}})).
\]
\[
A := \text{End}_{\mathcal{D} \oplus \prod \text{Fun}_{\mathcal{R}_B}}(\text{con}(\mathcal{O}_l^{\text{cyc}})); \quad B := \text{End}_{\mathcal{D} \oplus \prod \text{Fun}_{\mathcal{R}_B}}((\mathcal{R}_B^{\text{cyc}}).
\]

\( H \) is therefore a \( B - A \)-bimodule. Let \( B = (B_0, d_B), A = (A_0, d_A) \). We have \( d_B = 0, d_A = d_A_0 + \text{Ad} x \)
As \( \text{gr}^{>0} A_0 \) is acyclic, we have a zig-zag homotopy equivalence between \( A \) and \( \text{gr}^{>0} A_0 \). Therefore, we have a homotopy equivalence in \( B_q \) between \( H \) and \( (H_0, 0) \), where \( H_0 \in B_0 \).

Using Corollary 21.2 the maps in (83) and Lemma 16.2 we get the following diagram of homotopy equivalences:

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{D} \oplus \prod \text{Fun}_{\mathcal{R}_B}}((\mathcal{R}_B^{\text{cyc}}; \text{redcon}(\mathcal{O}_l^{\text{cyc}})) \to & \text{Hom}_{\mathcal{D} \oplus \prod \text{Fun}_{\mathcal{R}_B}}(\text{con}(\mathcal{O}_l^{\text{cyc}})) := H_0 \\
\text{Hom}_{\mathcal{D} \oplus \prod \text{Fun}_{\mathcal{R}_B}}((\mathcal{R}_B^{\text{cyc}}; \text{redcon}(\mathcal{O}_l^{\text{cyc}})) \to & \text{Hom}_{\mathcal{D} \oplus \prod \text{Fun}_{\mathcal{R}_B}}(\text{redcon}(\mathcal{O}_l^{\text{cyc}})) \\
\end{array}
\]

(99)

Let
\[
G_0 := \text{Hom}_{\mathcal{D} \oplus \prod \text{Fun}_{\mathcal{R}_B}}((\mathcal{R}_B^{\text{cyc}}; \text{redcon}(\mathcal{O}_l^{\text{cyc}})).
\]

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We have the following zig-zag in $B_q$

$$(G_0, 0) \xleftarrow{\sim} T \rightarrow \text{Hom}_D \oplus \prod_{\text{Fun} \mathcal{B}} (R^cyc; O^cyc).$$

for an appropriate $T \in B_q$, where the left arrow quasi-splits.

Let $\mathcal{L}_0$ be as in (84) so that we get a map

$$\mathcal{L}_0 \rightarrow \text{Hom}_D \oplus \prod_{\text{Fun} \mathcal{B}} (R^cyc; \text{red} O^cyc) =: G_0.$$

Next, we have a map $\iota : R \rightarrow \text{red} O$ as in (94) (as well as in (74)) whence a commutative diagram

$$R \rightarrow \text{red} O \rightarrow \text{red} O^cyc.$$

Taking pull-back with respect to the arrow $T \xrightarrow{\sim} (G_0, 0)$, we get diagrams

for appropriate $V, I \in B_q$.

We therefore have an induced map

$$V \otimes R^cyc \rightarrow IO^cyc.$$

and a commutative diagram

$$\text{red} O^cyc \rightarrow \text{red} IO^cyc.$$ (100) (101)

21.7 Constructing VCO’s $\mathcal{V}, \mathcal{A}^V, \mathcal{B}^V$

21.7.1 $\mathcal{V}$

Let us now build a VCO in $IR_q$, where

$-\mathcal{V}^\text{noncyc} = \mathcal{U}^\text{noncyc},$
— $V_1^{\text{cyc}} = O^{\text{cyc}} \oplus I \otimes \mathcal{R}_B^{\text{cyc}}$;
— $V_2^{\text{cyc}} = I O^{\text{loc}}$;
— $V_3^{\text{cyc}} = \mathcal{R}_B^{\text{cyc}}$;
— $V = V \otimes \text{unit}_{R_q}$. The map $p_{12} : V_1^{\text{cyc}} \to V_2^{\text{cyc}}$ is as follows:

$$p_{12} : O^{\text{cyc}} \to I O_1^{\text{loc}}$$

is the canonical map and

$$p_{12} : I \otimes \mathcal{R}_B^{\text{cyc}} \to I O_1^{\text{loc}}$$

coincides with the map in (101). The map $V V \otimes V_3^{\text{cyc}} \to V_2^{\text{cyc}}$ is as in (100).

### 21.7.2 $\mathbb{B}^V$

Set $(\mathbb{B}^V)^{\text{noncyc}} = \mathcal{R}_B^{\text{noncyc}}$, $(\mathbb{B}^V)_1^{\text{cyc}} = \mathcal{R}_B^{\text{cyc}}$, $\mathbb{B}_2^{\text{cyc}} = 0$; $\mathbb{B}_3^{\text{cyc}} = \mathcal{R}_B^{\text{cyc}}$; $(\mathbb{B}^V) V = 0$.

### 21.7.3 $\mathcal{A}^V$

Set $(\mathcal{A}^V)^{\text{noncyc}} = \mathcal{A}^{\text{noncyc}}$, $(\mathcal{A}^V)_1^{\text{cyc}} = \mathcal{A}^{\text{cyc}} \otimes I$, $\mathcal{A}_2^{\text{cyc}} = \mathcal{A}^{\text{cyc}}$. Let $\mathcal{A}$ be a complex $0 \to A^{-1} \to A^0 \to 0$, where $A^0 = \text{unit} \oplus \text{unit}$ and $A^{-1} = \text{unit}$, the differential $A^{-1} \to A^0$ is $\text{Id} \oplus (-\text{Id})$. Let $i_0, i_1 : \text{unit} \to \mathcal{A}$ be the embeddings onto the first and the second summand of $A^0$. Set $\mathcal{A}_2^{\text{cyc}} = \mathcal{A}^{\text{cyc}} \otimes \mathcal{A}$. Set $(\mathcal{A}^V) V = \text{unit}$.

Set the map $\mathcal{A}_1^{\text{cyc}} \to \mathcal{A}_2^{\text{cyc}}$ to be the composition

$$\mathcal{A}_1^{\text{cyc}} \to \mathcal{A}^{\text{cyc}} \to \mathcal{A}_2^{\text{cyc}},$$

where the first arrow is induced by the map $I \to \text{unit}$ and the second arrow is $\text{Id}_{\mathcal{A}^{\text{cyc}}} \otimes i_0$; set the map $\mathcal{A}_3^{\text{cyc}} \to \mathcal{A}_2^{\text{cyc}}$ to be $\text{Id}_{\mathcal{A}^{\text{cyc}}} \otimes i_1$.

### 21.7.4 The map $p : V \to \mathbb{B}^V$

We have $V^{\text{noncyc}} = U^{\text{noncyc}} = (O^F)^{\text{noncyc}} \oplus \mathcal{R}_B$, (see (16)), whence the projection $p^{\text{noncyc}} : V^{\text{noncyc}} \to \mathcal{R}_B^{\text{noncyc}}$. Define the projection

$$p_1 : V_1^{\text{cyc}} \to \mathcal{R}_B^{\text{cyc}} \otimes I \to \mathcal{R}_B^{\text{cyc}}$$

in a similar way. Set all the remaining maps to 0.

### 21.7.5 The map $\beta : \mathcal{A}^V \to \mathbb{B}^V$

Set

$$\beta^{\text{noncyc}} : \mathcal{A}^{\text{noncyc}} \to \mathcal{R}_B^{\text{noncyc}} = \mathbb{B}^{\text{noncyc}}$$

to be induced by the map $\mathcal{A} \to \mathcal{R}_B$ as defined in Sec 21.2.1. Define

$$\beta_1 : \mathcal{A}_1^{\text{cyc}} \to \mathcal{R}_B^{\text{cyc}} = \mathbb{B}_1, \beta_2 : \mathcal{A}_2^{\text{cyc}} \to \mathcal{R}_B^{\text{cyc}} = \mathbb{B}_2, \beta_3 : \mathcal{A}_3^{\text{cyc}} \to \mathcal{R}_B^{\text{cyc}} = \mathbb{B}_3,$$
to be induced by the same map \( A \to R_B \) from Sec 21.2.1.

Define \( q^V : \text{unit} \to 0 \) to be the 0 map.

### 21.7.6 The map \( \alpha : \text{red} A^V \to \text{red} V \)

The map \( \alpha_1 : \text{red} I \otimes A^\text{cyc} \to \text{red} V \) is defined as the direct sum of:

\[
\alpha_{11} : \text{red} I \otimes A^\text{cyc} \to A^\text{cyc} \to O^\text{cyc}
\]

and

\[
\alpha_{12} = -\beta_{12} : \text{red} I \otimes A^\text{cyc} \to A^\text{cyc} \to R_B^\text{cyc},
\]

where \( \beta_{12} \) is the tensor product of the above defined arrows \( I \to \text{unit} \) and \( A^\text{cyc} \to R_B^\text{cyc} \). As a result, the composition

\[
\text{red} A^V_1 \to \text{red} B^V_1 \to \text{red} B^V_2
\]

is 0. We now can set \( \alpha_2 = 0; \alpha_3 : A \to R_B \) to be the above defined map, and, finally, \( \alpha^V = 0 \).

### 21.7.7 Checking the rigidity condition

The rigidity condition (93) follows from the homotopy equivalence in (84).

We therefore conclude the existence of a VCO \( A' \), a homotopy equivalence \( A' \to A^V \), and a map \( A' \to V \) such that \( \text{red} A' \to \text{red} V \) is homotopy equivalent to the above defined map \( \text{red} A^V \to \text{red} V \).

Restriction to the \( \text{noncyc} \)- and 1- components yields the following zig-zag of CO in \( R_q \):

\[
A \sim A' \to O,
\]

The reduction of this zig-zag to \( R_q \) is homotopy equivalent to the arrow

\[
\text{red} A \to \text{red} R_B \to \text{red} B \xrightarrow{\phi} \text{red} O,
\]

where \( \phi \) is as in (74). This solves the quantization problem.

### 21.8 Building the Microlocal category

Recall (Sec 13.9.2) that we have a category \( (D \coprod)U(F_R) \) enriched over the category \( D \coprod U(F) \).

Next, we have an object \( A \in U(F) \), where \( O \) is a quasi-free resolution of the full operad of \( A \). Therefore, \( A \) carries a \( A' \)-action. We have homotopy equivalences \( A' \to A \to a \otimes \text{MC} \), we can now construct a homotopy equivalent object \( B \to A \) with a \( a \otimes \text{MC} \)-action on it. The \( \text{red} A \otimes \text{MC} \)-action passes through the map \( \text{red} A \otimes \text{MC} \to \text{red} B \).

Let \( \text{MC}_R \) be a 2-colored operad in \( \mathfrak{A} \), such that a \( \text{MC}_R \)-structure on a pair \( (A, M) \) is the same as that of an \( A_\infty \)-algebra with a curvature on \( A \) and that of an \( A \)-module on \( M \). Let us define an \( A_\infty \)-category \( M_M \), enriched over \( R_q \), whose every object is a \( a \otimes \text{MC} \)-algebra structure on a pair \( (B, M) \), \( M \in (D \coprod)U(F_R) \), whose restriction onto \( B \) coincides with the existing one. The \( A_\infty \)-structure is defined in the standard way.

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