On the relation between the distributions of stopping time and stopped sum with applications

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Abstract

Let $T$ be a stopping time associated with a sequence of independent random variables $Z_1, Z_2, \ldots$. By applying a suitable change in the probability measure we present relations between the moment or probability generating functions of the stopping time $T$ and the stopped sum $S_T = Z_1 + Z_2 + \ldots + Z_T$. These relations imply that, when the distribution of $S_T$ is known, then the distribution of $T$ is also known and vice versa. Applications are offered in order to illustrate the applicability of the main results, which also have independent interest. In the first one we consider a random walk with exponentially distributed up and down steps and derive the distribution of its first exit time from an interval $(-a, b)$. In the second application we consider a series of samples from a manufacturing process and we let $Z_i, i \geq 1$, denoting the number of non-conforming products in the $i$-th sample. We derive the joint distribution of the random vector $(T, S_T)$, where $T$ is the waiting time until the sampling level of the inspection changes based on a $k$-run switching rule. Finally, we demonstrate how the joint distribution of $(T, S_T)$ can be used for the estimation of the probability $p$ of an item being defective, by employing an EM algorithm.

Key words and phrases: stopping time, stopped sum, exponentially tilted probability measure, random walk, first exit time, boundary crossing probabilities, acceptance sampling, $k$-run switching rule, sooner waiting time distribution, joint generating function, EM algorithm.

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1 Introduction

In several areas of applied science researchers are interested in studying the time $T$ to take a given action, based on sequentially observed random variables (rv’s) $Z_1, Z_2, \ldots$, as well as in the associated partial sums $S_n = Z_1 + Z_2 + \ldots + Z_n, n = 1, 2, \ldots$. The waiting time $T$ and the corresponding random sum $S_T$ are usually referred to as stopping time and stopped sum respectively. Stopping time problems arise in many diverse scientific areas such as sequential analysis, quality control, mathematical finance, operations research, biology, actuarial science, etc. For a gentle introduction to the theory of stopping times and stopped sums, the interested reader is referred to Karlin and Taylor (1975). For a more thorough investigation of the theory of stopped random walks we refer to Gut (2009).

When studying the distribution of $T$ in a sequence of independent and identically distributed (iid) trials, the stopped sum $S_T$ also provides useful information about the nature of the statistical experiment. The pioneering work of Abraham Wald (1945) in the area of sequential analysis established powerful identities that relate the distributional properties of $T$ and $S_T$. These identities are usually referred to as Wald’s (fundamental) Identity and Wald’s (first) equation and they are, respectively, given by

$$E((M_Z(w))^{-T} e^{wS_T}) = 1,$$

(1)

where $M_Z(w) = E(e^{wZ})$, and

$$E(S_T) = E(Z) E(T).$$

(2)

In a recent article Antzoulakos and Boutsikas (2007) established a particular relation between the distributions of $T$ and $S_T$. More specifically, they considered the waiting time $T_r$ until the $r$–th occurrence of a pattern $E$ in a sequence of binary trials $Z_1, Z_2, \ldots$ and the total number of successes $S_{T_r}$ observed until that time, and established a direct method to obtain the joint probability generating function (pgf) of $(T_r, S_{T_r})$ from the pgf of $T_r$ only. In this paper we extend the aforementioned result for any distribution of the $Z_i$’s and any stopping time $T$, determining the joint distribution of $(T, S_T)$ from the distribution of $T$ or $S_T$.

The organization of the paper is as follows: In Section 2 we state the main identities that connect the distributions of $T$ and $S_T$, along with the required theoretical backup. An important part of our work is comprised of the applications that are presented in Section 3. These applications, not only serve as an illustration of the applicability of the results of Section 2, but they also have an interest on their own. In the first one we consider the first exit time $T$ from an interval $(-a, b)$ ($a > 0$ or $a = \infty$) of a random walk $S_i$, $i = 1, 2, \ldots$, with exponentially distributed up and down steps. By identifying the distribution of $S_T$ we extract an exact formula for the pgf of the boundary crossing time $T$. In the second application we consider a sequence $Z_i$, $i = 1, 2, \ldots$, of measurements
taken from samples corresponding to lots of products from a manufacturing process (e.g. number of defective items in each sample). Denoting by $T$ the waiting time until the sampling level of the inspection changes using a $k$-run switching rule associated with $Z_i$’s, we obtain the joint pgf of $T$ and $S_T$ ($S_T$ denotes the total number of defective items observed until switching) by exploiting the fact that $T$ follows a geometric distribution of order $k$. Finally, we demonstrate how the joint distribution of $T$ and $S_T$ can be useful in the estimation of the probability $p$ of an item being defective, by employing an EM algorithm.

2 Identities connecting the distributions of stopped sum and stopping time.

Let $F_1, F_2, \ldots$ be a sequence of distributions on $R$ such that $\int_{\mathbb{R}} e^{wz} dF_i(z) < \infty$, $i = 1, 2, \ldots$, for every $w$ in an interval $W$ containing zero. We can always construct a sequence of independent rv’s $Z_1, Z_2, \ldots$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $Z_i \sim F_i, i = 1, 2, \ldots$. Moreover, if $F_i(\cdot|w)$ denotes the exponentially tilted $F_i$, i.e. $F_i(x|w) := \mathbb{E}(e^{wZ_i}I_{\{Z_i \leq x\}})/\mathbb{E}(e^{wZ_i}), w \in \mathbb{R}$, we can always change $\mathbb{P}$ to a new probability measure $\tilde{\mathbb{P}}_w$ on $(\Omega, \mathcal{F})$ under which $Z_1, Z_2, \ldots$ are still independent but now $Z_i \sim F_i(\cdot|w), i = 1, 2, \ldots$. A formal construction of the probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}}_w)$ is given in the Appendix.

We shall write $\tilde{\mathbb{E}}_w(\cdot)$ for the expected value with respect to the measure $\tilde{\mathbb{P}}_w$. We shall also use the notation $\mathbb{P} := \tilde{\mathbb{P}}_0, \mathbb{E} := \tilde{\mathbb{E}}_0$. It is easy to see that, in the special case when $Z_1, Z_2, \ldots$ possess the same density $f$ with respect to $\mathbb{P}$, their density $f_w$ with respect to $\tilde{\mathbb{P}}_w$ is given by

$$f_w(z) = \frac{e^{wz}f(z)}{\mathbb{E}(e^{wZ_i})}.$$ 

**Remark.** (The derivative $d\tilde{\mathbb{P}}_w/d\mathbb{P}$ on $\mathcal{F}_n$). Define $\mathcal{F}_n = \sigma(Z_1, Z_2, \ldots, Z_n) \subseteq \mathcal{R}^n$ to be the minimal $\sigma$-algebra generated by $Z_1, Z_2, \ldots, Z_n$. The sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ is a nondecreasing sequence of $\sigma$-algebras in $\mathcal{R}^n$. The Radon-Nikodym derivative of $\tilde{\mathbb{P}}_w$ with respect to $\mathbb{P}$ when both are restricted to $\mathcal{F}_n$ is $X_n = e^{w(Z_1 + Z_2 + \ldots + Z_n)}/\prod_{i=1}^{n} \mathbb{E}(e^{wZ_i})$ (that is, $\tilde{\mathbb{P}}_w(A) = \int_A X_n d\mathbb{P}, A \in \mathcal{F}_n$) and hence

$$\tilde{\mathbb{E}}_w(Y) = \frac{\mathbb{E}(Ye^{w(Z_1 + Z_2 + \ldots + Z_n)})}{\prod_{i=1}^{n} \mathbb{E}(e^{wZ_i})}$$

(3) for every $\mathcal{F}_n$-measurable random variable $Y$. It is worth mentioning that, even though $\mathbb{P}$ and $\tilde{\mathbb{P}}_w$ are equivalent on every $\mathcal{F}_n$, they are mutually singular on $\mathcal{F}_\infty = \mathcal{R}^n$ when $w \neq 0$ and $Z_1, Z_2, \ldots$ are identically distributed (that is, there exist disjoint sets $A, A'$ in $\mathcal{R}^n$ such that $\tilde{\mathbb{P}}_w(A) = 1$ and $\mathbb{P}(A') = 1$). This can be easily seen since there exists a set $B \in \mathcal{B}(\mathbb{R})$ such that $\tilde{\mathbb{P}}_w(Z_i \in B) \neq \mathbb{P}(Z_i \in B)$ while (invoking the strong law of large numbers) $\frac{1}{n} \sum_{i=1}^{n} I_{\{Z_i \in B\}}$ converges to $\tilde{\mathbb{P}}_w(Z_i \in B)$.
on some \( A \in \mathcal{R}^N \) with \( \hat{P}_w(A) = 1 \) and to \( \mathbb{P}(Z_i \in B) \) on some \( A' \in \mathcal{R}^N \) with \( \mathbb{P}(A') = 1 \). Since \( \hat{P}_w(Z_i \in B) \neq \mathbb{P}(Z_i \in B) \) we have that \( A \cap A' = \emptyset \). Thus \( \mathbb{P}(X_n \to 0) = 1 \) (see e.g. Theorem 35.8 in Billingsley (1986)) even though \( \mathbb{E}(X_n) = 1 \) for every \( n \). Therefore, in general, there does not exist a Radon-Nikodym derivative of \( \hat{P}_w \) with respect to \( \mathbb{P} \) on \( \mathcal{R}^N \) and hence \( \hat{P}_w \) cannot be constructed on \( \mathcal{R}^N \) from \( \mathbb{P} \) through a Radon-Nikodym derivative. This fact does not induce any problem since we have guaranteed the existence of \( \hat{P}_w \) via the Kolmogorov Existence Theorem (see Appendix).

Let now \( T \) be a stopping time associated with the sequence \( Z_1, Z_2, \ldots \), i.e. the set \( [T = n] = \{ \omega \in \Omega : T(\omega) = n \} \) belongs to \( \mathcal{F}_n = \sigma(Z_1, Z_2, \ldots, Z_n) \) for every \( n = 1, 2, \ldots \), and let \( S_T := Z_1 + Z_2 + \ldots + Z_T \). The next result relates the distributions of \( T \) and \( S_T \).

**Theorem 1** Let \( T \) be a stopping time associated with the sequence \( Z_1, Z_2, \ldots \), and let \( Y \) be a random variable such that \( Y \cdot I_{[T=n]} \) is \( \mathcal{F}_n \)-measurable. Then

\[
\mathbb{E}(Y e^{wS_T} I_{[T<\infty]}) = \hat{E}_w(Y \prod_{i=1}^T \mathbb{E}(e^{wZ_i}) I_{[T<\infty]})
\]

(4)

for all real \( w \) such that the above expectations exist.

**Proof.** If \( G_k := Y e^{wS_T} \sum_{n=1}^k I_{[T=n]} \) then \( |G_k| \leq |Y| e^{wS_T} I_{[T<\infty]} \) a.s. and \( \mathbb{E}(|Y| e^{wS_T} I_{[T<\infty]}) < \infty \), which, by the Dominated Convergence Theorem (DCT), implies that \( \mathbb{E} (\lim_k G_k) = \lim_k \mathbb{E}(G_k). \) Thus,

\[
\mathbb{E}(Y e^{wS_T} I_{[T<\infty]}) = \mathbb{E}(\lim_{k \to \infty} G_k) = \lim_{k \to \infty} \mathbb{E}(G_k) = \sum_{n=1}^\infty \mathbb{E}(Y I_{[T=n]} e^{wS_n}).
\]

By theorems’ assumptions, the r.v. \( Y I_{[T=n]} \) is \( \mathcal{F}_n \)-measurable and hence (see (3) above) \( \hat{E}_w(Y I_{[T=n]} = \mathbb{E}(Y I_{[T=n]} e^{wS_n})/\prod_{i=1}^n \mathbb{E}(e^{wZ_i}) \). Therefore,

\[
\mathbb{E}(Y e^{wS_T} I_{[T<\infty]}) = \sum_{n=1}^\infty \hat{E}_w(Y I_{[T=n]} \prod_{i=1}^n \mathbb{E}(e^{wZ_i})) = \sum_{n=1}^\infty \hat{E}_w(Y I_{[T=n]} \prod_{i=1}^T \mathbb{E}(e^{wZ_i}))
\]

which, invoking again the DCT, leads to (4) provided that \( \hat{E}_w((Y| \prod_{i=1}^T \mathbb{E}(e^{wZ_i}) I_{[T<\infty]}) < \infty. \)

The above result can be considered as a version of Wald’s Likelihood Ratio Identity (WLRI, see e.g. Siegmund (1985), or Lai (2004)).

In the sequel we focus on a special use of Equation (4). Our aim is to generalize the following result of Antzoulakos and Boutsikas (2007): If \( Z_1, Z_2, \ldots \) is a sequence of iid binary r.v’s (trials) with \( \mathbb{P}(Z_i = 1) = 1 - \mathbb{P}(Z_i = 0) = p \) and \( T \) denotes the waiting time (i.e. the number of trials) until a certain pattern \( \mathcal{E} \) occurs in \( Z_1, Z_2, \ldots \) then, the joint pgf of \( (T, S_T) \) follows from the pgf of \( T \) through the relation

\[
\mathbb{E}(u^T e^{wS_T}) = \hat{E}_w \left( (u(1 - p + pw))^T \right)
\]

(5)
for all \( w, u \) in a neighborhood of 0, where the expectation \( \tilde{E}_w \) is considered under \( \tilde{P}_w \) such that 
\[
\tilde{P}_w(Z_i = 1) = 1 - \tilde{P}_w(Z_i = 0) = \frac{pw}{pw + 1-p}.
\]
The above identity, reveals that, when the distribution of \( T \) is known then the joint distribution of \((T, S_T)\) is also known. In other words, the distribution of \( T \) uniquely determines the joint distribution of \((T, S_T)\) and consequently the distribution of \( S_T \).

A generalization of (5) could refer to any distribution for the \( Z_i \)'s and any stopping time \( T \). In addition, an inverse form of (5) could also be very useful implying that the distribution of \( S_T \) uniquely determines the joint distribution of \((T, S_T)\). As it is shown in the next two corollaries, generalizations of this form can be easily derived from Equation (4).

**Corollary 2** If \( \mathbb{P}(T < \infty) = \tilde{P}_w(T < \infty) = 1 \) then
\[
\mathbb{E}(u^T e^{wS_T}) = \tilde{E}_w((u\mathbb{E}(e^{wZ}))^T),
\]
for all real \( u, w \) such that the above expectations exist. In particular, \( \mathbb{E}(e^{wS_T}) = \tilde{E}_w(\mathbb{E}(e^{wZ})^T) \).

**Proof.** It follows from (4) by letting \( Z_1, Z_2, \ldots \) be a sequence of iid rv’s and by setting \( Y = u^T \) (note that \( u^T I_{[T=n]} = u^n I_{[T=n]} \) is \( \mathcal{F}_n \)-measurable).

**Corollary 3** If there exists a real function \( w_u \) such that \( \mathbb{E}(e^{w_u Z}) = u^{-1} \) and \( \tilde{P}_{w_u} \) is a probability measure with \( \mathbb{P}(T < \infty) = \tilde{P}_{w_u}(T < \infty) = 1 \), then
\[
\mathbb{E}(u^T e^{xS_T}) = \tilde{E}_{w_u}(e^{(x-w_u)S_T}),
\]
for all real \( u, x \) such that the above expectations exist. In particular, \( \mathbb{E}(u^T) = \tilde{E}_{w_u}(e^{-w_uS_T}) \).

**Proof.** By setting \( Y = u^T e^{(x-w_u)S_T} \) we have that the rv \( Y I_{[T=n]} = u^n e^{(x-w_u)(Z_1+\ldots+Z_n)} I_{[T=n]} \) is \( \mathcal{F}_n \)-measurable. Therefore by employing (4) with respect to the measures \( \mathbb{P} \) and \( \tilde{P}_{w_u} \) we get
\[
\mathbb{E}(u^T e^{(x-w_u)S_T} e^{w_u S_T}) = \tilde{E}_{w_u}(u^T e^{(x-w_u)S_T} \mathbb{E}(e^{w_u Z})^T)
\]
which readily leads to (7) since \( u\mathbb{E}(e^{w_u Z})^T = 1 \).

It is worth mentioning that, more generally, we can similarly get from (4) that,
\[
\mathbb{E}(Y u^T e^{wS_T} I_{[T<\infty]}) = \tilde{E}_w(Y (u\mathbb{E}(e^{wZ}))^T I_{[T<\infty]})
\]
and
\[
\mathbb{E}(Y u^T e^{xS_T} I_{[T<\infty]}) = \tilde{E}_{w_u}(Y e^{(x-w_u)S_T} I_{[T<\infty]})
\]
where \( Y \) is a rv such that \( Y I_{[T=n]} \) is \( \mathcal{F}_n \)-measurable. The above corollaries imply that, under appropriate conditions, the distribution of \( S_T \) uniquely determines the distribution of the stopping time \( T \) and vice versa. Two applications illustrating this fact are presented in the following section.
3 Applications

3.1 The distribution of the first exit time of a random walk

Let $Z_1, Z_2, \ldots$ be a sequence of (non-degenerate) iid rv’s representing the consecutive jumps of a random walk $S_n, n = 1, 2, \ldots$, that is, $S_n = Z_1 + Z_2 + \ldots + Z_n$. Define also the following stopping time

$$T = \inf\{n : S_n \geq b \text{ or } S_n \leq -a\}$$

for some $a, b > 0$. Obviously, $T$ expresses the steps of the random walk until it exits the set $(-a, b)$. It can be easily verified that $\mathbb{E}(T) < \infty$ (e.g. see Karlin and Taylor (1975), p.264) and thus $T$ is finite a.s.

Probabilities regarding the first passage, or boundary crossing times arise in a variety of contexts in applied probability and statistics, such as sequential analysis, ruin theory, queueing theory, stochastic finance etc. Usually, it is of interest to evaluate the probability $\mathbb{P}(S_T \geq b) = 1 - \mathbb{P}(S_T \leq -a)$, the distribution of $T$ and $\mathbb{E}(T), \mathbb{V}(T)$.

Initially, we find the probability $\mathbb{P}(S_T \geq b)$ via Wald’s Identity by using a standard technique (see e.g. Karlin and Taylor (1975), p.265). It can be verified that for $w^* = (1 - p)\theta_1 - p\theta_2$ we have that

$$\mathbb{E}(e^{wZ}) = p \int_{-\infty}^{\infty} e^{wx} f_1(x) dx + (1 - p) \int_{-\infty}^{\infty} e^{wx} f_2(-x) dx = \frac{p\theta_1}{\theta_1 - w} + \frac{(1 - p)\theta_2}{\theta_2 + w}.$$
\[ \mathbb{E}(e^{w^{*}Z}) = 1, \] and therefore from (6) (or from (1)) we get \[ \mathbb{E}(e^{w^{*}S_T}) = \hat{\mathbb{E}}_{w^{*}}(\mathbb{E}(e^{w^{*}Z})^T) = \hat{\mathbb{E}}_{w^{*}}(1^T) = 1. \] Hence, it follows that

\[ 1 = \mathbb{E}(e^{w^{*}S_T}) = \mathbb{E}\left(e^{w^{*}S_T}|S_T \geq b\right) \mathbb{P}(S_T \geq b) + \mathbb{E}\left(e^{w^{*}S_T}|S_T \leq -a\right) (1 - \mathbb{P}(S_T \geq b)) \]

and by solving with respect to \( \mathbb{P}(S_T \geq b) \) we get

\[ \mathbb{P}(S_T \geq b) = \frac{1 - \mathbb{E}\left(e^{w^{*}S_T}|S_T \leq -a\right)}{\mathbb{E}(e^{w^{*}S_T}|S_T \geq b) - \mathbb{E}(e^{w^{*}S_T}|S_T \leq -a)}. \] (10)

Invoking the memoryless property of the exponential distribution we have that

\[ \mathbb{E}\left(e^{wS_T}|S_T \geq b\right) = e^{wb}\mathbb{E}\left(e^{w(S_T-b)}|S_T - b \sim \mathcal{E}(\theta_1)\right) = \frac{\theta_1}{\theta_1-w} e^{wb}, \]

\[ \text{and combining the above we deduce that for } w^{*} \neq 0, \]

\[ \mathbb{P}(S_T \geq b) = \frac{1 - \frac{\theta_2 e^{-w^{*}a}}{\theta_1-w} - \frac{\theta_2 e^{-w^{*}a}}{w+\theta_2}}{\frac{\theta_1 e^{-w^{*}b}}{\theta_1-w} - \frac{\theta_2 e^{-((1-p)\theta_1-p\theta_2)a}}{(1-p)\theta_1 + \theta_2}}. \] (12)

For \( w^{*} = 0 \) (i.e. the case where \( (1-p)\theta_1 = p\theta_2 \)) we can take \( w^{*} \to 0 \) in the above formula and subsequently deduce that \( \mathbb{P}(S_T \geq b) = \frac{\theta_1 + p\theta_1 \theta_2}{\theta_1 + (1-p)(\theta_1 + \theta_2)} \).

Next, we derive the mgf of \( T \) by employing Corollary 3. A solution \( w_u \) of the equation \( \mathbb{E}(e^{wZ}) = w^{-1} \) with respect to \( w \) is

\[ w_u = \frac{\theta_1 - \theta_2 + u((1-p)\theta_2 - p\theta_1) + \sqrt{(\theta_1 - \theta_2 + u((1-p)\theta_2 - p\theta_1))^2 + 4(1-u)\theta_1\theta_2}}{2}. \] (13)

The function \( w_u \) is strictly decreasing for \( u \in [0,1] \) with \( w_0 = \theta_1, w_1 = \max\{0,(1-p)\theta_1 - p\theta_2\} \) and thus \( 0 < w_u < \theta_1 \) for \( u \in (0,1) \). Under the measure \( \hat{\mathbb{P}}_w \), the pdf \( f_{z_i} \) of each \( z_i \) takes on the form

\[ f_{z_i}(x) = \frac{e^{ux}f(x)}{\mathbb{E}(e^{wZ})} = \frac{e^{ux}(p_{\theta_1} e^{-(\theta_1 - u)x} I_{x \geq 0} + (1-p)\theta_2 e^{\theta_2 x} I_{x < 0})}{\theta_1 + (1-p)\theta_2}. \]

\[ = \begin{cases} c_w(\theta_1 - w) e^{-(\theta_1 - w)x}, & x \geq 0 \\ (1-c_w)(\theta_2 + w) e^{-(\theta_2 + w)(-x)}, & x < 0 \end{cases} \]

where \( c_w = \frac{p_{\theta_1}}{\theta_1-w}, (\frac{p_{\theta_1}}{\theta_1-w} + \frac{(1-p)\theta_2}{\theta_2+w})^{-1} \) \( (0 < c_w < 1 \text{ for } -\theta_2 < w < \theta_1) \). Hence, under \( \hat{\mathbb{P}}_{w_u}, u \in (0,1) \), we still have exponentially distributed up and down jumps, but now the parameters \( p, \theta_1 \) and \( \theta_2 \) are substituted by \( c_w = \frac{w u_{\theta_1}}{\theta_1 - u}, (\theta_1 - w_u) \), and \( (\theta_2 + w_u) \) respectively. Again, \( T \) is finite \( \hat{\mathbb{P}}_w \)-a.s.

Using Corollary 3 and (11) it follows that

\[ \mathbb{E}(u^T) = \hat{\mathbb{E}}_{w_u}(e^{-w_uS_T}) \]

\[ = \hat{\mathbb{E}}_{w_u}(e^{-w_uS_T}|S_T \geq b) \hat{\mathbb{P}}_{w_u}(S_T \geq b) + \hat{\mathbb{E}}_{w_u}(e^{-w_uS_T}|S_T \leq -a) (1 - \hat{\mathbb{P}}_{w_u}(S_T \geq b)) \]

\[ = (\theta_1 - w_u)e^{-w_u b} \hat{\mathbb{P}}_{w_u}(S_T \geq b) + (\theta_2 + w_u)e^{w_u a} \hat{\mathbb{P}}_{w_u}(S_T \geq b) (1 - \hat{\mathbb{P}}_{w_u}(S_T \geq b)). \] (14)
Also, using (12) under the probability measure \( \tilde{P}_{wu} \), we get
\[
\tilde{P}_{wu}(S_T \geq b) = \frac{1 - \frac{\theta_2 + w_u}{(1-c_{wu})(\theta_1+\theta_2)} e^{-\beta_u a}}{\frac{\theta_1-w_u}{c_{wu}(\theta_1+\theta_2)} e^{\beta_u b} - \frac{\theta_2 + w_u}{(1-c_{wu})(\theta_1+\theta_2)} e^{-\beta_u a}}
\]
(15)
where \( \beta_u = (1-c_{wu})(\theta_1-w_u) - c_{wu}(\theta_2 + w_u) \).

Combining (14) and (15) we deduce the following proposition.

**Proposition 4** Let \( S_n, n = 1, 2, \ldots \) be a random walk with step distribution \( F(x) = pF_1(x) + (1 - p)F_2(x) \), where \( F_i \sim \mathcal{E}(\theta_i), i = 1, 2, p \in (0, 1) \). If \( T \) denotes the time until the random walk exits \((-a,b), a,b > 0 \) then the probability generating function of \( T \) is given by
\[
\mathbb{E}(u^T) = \frac{(\theta_1-w_u) - (\theta_2+w_u)e^{\beta_u a}}{(\theta_1-w_u)e^{\beta_u b}} \left( 1 - \frac{(\theta_2+w_u)^2 e^{-\beta_u a}}{u(1-p)\theta_2(\theta_1+\theta_2)} \right) + \frac{(\theta_2 + w_u)e^{\beta_u a}}{\theta_2}, \quad u \in (0,1)
\]
where
\[
\beta_u = -\sqrt{(\theta_1 - \theta_2 + u((1-p)\theta_2 - p\theta_1))^2 + 4(1-u)\theta_1\theta_2}, \quad w_u = \frac{1}{2}(\theta_1 - \theta_2 + u((1-p)\theta_2 - p\theta_1) - \beta_u)
\]
(16)

Note that, for the special case \( p = \frac{\theta_1}{\theta_1+\theta_2} \), the above generating function can also be derived by employing results established by Khan (2008).

Apart from its theoretical interest, the above formula can also be used for the numerical determination of the distribution of \( T \) for given values of the parameters \( \theta_1, \theta_2, p, a \) and \( b \), since
\[
\mathbb{P}(T = m) = \left. \frac{\text{d}^m}{\text{d}u^m}(\mathbb{E}(u^T)) \right|_{u=0}.
\]
(17)
In practice, this can be easily accomplished by the use of appropriate mathematical software (e.g. using the function `SeriesCoefficient` of Wolfram Mathematica). In Figure 1 the distribution of \( T \) has been pictured for two sets of values of the parameters. The height of the bars represent the probabilities \( \mathbb{P}(T = m), m = 0, 1, \ldots, 50 \), while the small dots show the corresponding probabilities estimated by Monte Carlo simulation after \( 10^5 \) iterations.

An explicit formula for \( \mathbb{E}(T) \) can be easily derived by differentiating \( \mathbb{E}(u^T) \) given in Proposition 4 with respect to \( u \) and taking \( u \to 1 \). The details are left to the reader.

Employing Corollary 3 we can derive the joint gf of \( T \) and \( S_T \), which yields
\[
\mathbb{E}(u^T e^{xS_T}) = \mathbb{E}_{wu}(e^{xS_T} e^{-w_u S_T}) = \mathbb{E}_{wu}(e^{(x-w_u)S_T} | S_T \geq b) \tilde{P}_{wu}(S_T \geq b) + \mathbb{E}_{wu}(e^{(x-w_u)S_T} | S_T \leq -a) (1 - \tilde{P}_{wu}(S_T \geq b))
\]
\[
= \frac{(\theta_1 - w_u)e^{(x-w_u)b}}{(\theta_1 - w_u) - (x - w_u)} \tilde{P}_{wu}(S_T \geq b) + \frac{(\theta_2 + w_u)e^{(w-u)x}}{(x - w_u) + (\theta_2 + w_u)} (1 - \tilde{P}_{wu}(S_T \geq b))
\]
Figure 1: The probability mass function of $T$ ($p = 1/3$, $\theta_1 = 2$, $\theta_2 = 1$, $a = 8$, $b = 6$ and $p = 1/2$, $\theta_1 = 1$, $\theta_2 = 1$, $a = 4$, $b = 4$)

where $\tilde{P}_{wu}(S_T \geq b)$ and $w_u$ are given above.

Moreover, for the pgf of the conditional distribution of $T$, given that the random walk crossed the upper boundary, we observe that (9) with $Y = I_{[S_T \geq b]}$, $x = 0$, leads to

$$E(u^T | S_T \geq b) P(S_T \geq b) = E_{wu}(e^{-w_u S_T} I_{[S_T \geq b]}) = \tilde{E}_{wu}(e^{-w_u S_T} | S_T \geq b) \tilde{P}_{wu}(S_T \geq b).$$

Therefore we deduce the following result.

**Proposition 5** Let $S_n$, $n = 1, 2, \ldots$ be a random walk with step distribution $F(x) = pF_1(x) + (1 - p)F_2(x)$, where $F_i \sim E(\theta_i)$, $i = 1, 2$. If $T$ denotes the time until the random walk exits $(-a, b)$, $a, b > 0$ then the conditional pgf of $T$ given that $S_T \geq b$, is

$$E(u^T | S_T \geq b) = \frac{(\theta_1 - w_u)e^{-w_u b}}{\theta_1} \cdot \frac{\tilde{P}_{wu}(S_T \geq b)}{P(S_T \geq b)}, \quad u \in (0, 1)$$

where $w_u$, $P(S_T \geq b)$ and $\tilde{P}_{wu}(S_T \geq b)$, are as in (16), (12) and (15) respectively.

Proposition 5 along with (17) can be used for the calculation of the conditional probabilities $h(m) = P(T = m | S_T \geq b)$. In Figure 2, which was constructed similarly to Figure 1, we have plotted the conditional distribution of $T$ for two sets of values of the parameters.

Finally, it is worth mentioning that when the $Z_i$’s follow a Laplace distribution (i.e., $\theta_1 = \theta_2 = \theta$, $p = 1/2$) the pgf of $T$ takes on the simple form

$$E(u^T) = \frac{(e^{a\theta \tilde{u}} + e^{b\theta \tilde{u}})u}{1 - \tilde{u} + (1 + \tilde{u}) e^{(a+b)\theta \tilde{u}}}$$
where $\tilde{u} = \sqrt{1 - u}$. Also, $E(u^T e^{xS_T})$ now simplifies to
\[
E(u^T e^{xS_T}) = \frac{\theta(1-\tilde{u})e^{(x-\theta a)u}}{\theta-x} - \frac{\theta(1+\tilde{u})e^{(\theta a-x)u}}{x+\theta}(\theta u - (1 + \tilde{u})^2 e^{2\theta a u}) + \frac{\theta(1+\tilde{u})}{x+\theta} e^{(\theta x - u) 2\theta a u},
\]
while the conditional pgf of $T$ now reads
\[
E(u^T | S_T \geq b) = \frac{e^{\theta \tilde{u} b}(2 + a\theta + b\theta)(1 - \tilde{u})(-u + e^{2\theta a \tilde{u}}(2 - u + 2\tilde{u}))}{(1 + a\theta)((1 - e^{2(a+b)\theta \tilde{u}})(u - 2) + 2(1 + e^{2(a+b)\theta \tilde{u}})\tilde{u})}.
\]

Finally, by differentiating $E(u^T)$ and $E(u^T | S_T \geq b)$, with respect to $u$, taking $u \to 1$ and after some algebraic manipulations, we may also easily derive explicit formulae for $E(T)$, $V(T)$ and $E(u^T | S_T \geq b)$.

(b) We consider again the random walk $Z_1, Z_2, \ldots$ discussed in (a) with $a = \infty$ (i.e. now there exists only an upper barrier), that is $T$ denotes the waiting time (steps) until the random walk crosses $b > 0$. Exploiting the results of Section 2, we find the probability $P(T < \infty)$ and the conditional pgf of $T$ given that $T < \infty$. In this case, $P(T < \infty) = 1$ only when the mean step $E(Z) = \frac{p}{\theta_2} - \frac{1-p}{\theta_2}$ is positive. We conveniently observe that the mean step under the probability measure $P_{w_u}$ is always positive, that is,
\[
E_{w_u}(Z) = \frac{c_{w_u}}{\theta_1 - w_u} - \frac{1 - c_{w_u}}{\theta_2 + w_u} = \frac{p\theta_1 u}{(\theta_1 - w_u)^2} - \frac{(1 - p)\theta_2 u}{(\theta_2 + w_u)^2} > 0,
\]
for all $u \in (0, 1)$. This can be justified as follows: Note first that $w_u$ is strictly decreasing for $u \in [0, 1]$ with $w_0 = \theta_1$ and $w_1 = \max\{0, (1 - p)\theta_1 - p\theta_2\}$. It suffices to show that $g(w_u) > 0, u \in (0, 1)$, where $g(x) = (\theta_2 + x)^2p\theta_1 - (\theta_1 - x)^2(1 - p)\theta_2$. The function $g(x)$ is strictly increasing in $[0, \theta_1] (g'(x) > 0$ for $x \in [0, \theta_1])$. We examine the following three cases:
(i) If \( p\theta_2 - (1-p)\theta_1 > 0 \), then \( w_1 = 0 \) and hence \( g(w_u) > g(w_1) = g(0) = (p\theta_2 - (1-p)\theta_1)\theta_1\theta_2 > 0 \).

(ii) If \( p\theta_2 - (1-p)\theta_1 < 0 \), then \( w_1 = (1-p)\theta_1 - p\theta_2 > 0 \) and hence \( g(w_u) > g(w_1) = p(1-p)(\theta_2 + \theta_1)^2 ((1-p)\theta_1 - p\theta_2) > 0 \).

(iii) If \( p\theta_2 - (1-p)\theta_1 = 0 \), then directly, \( \hat{E}_{w_u}(Z) = \frac{uw_u}{(\theta_1-w_u)(\theta_2+w_u)} \left( \frac{\theta_1}{\theta_1-w_u} + \frac{\theta_2}{\theta_2+w_u} \right) > 0 \).

Therefore, \( \hat{P}_{wu}(T < \infty) = 1, u \in (0, 1) \), and from relation (9) we deduce that

\[
\mathbb{E}(u^T I_{T<\infty}) = \hat{E}_{w_u}(e^{-w_u S_T I_{T<\infty}}) = \hat{P}_{wu}(e^{-w_u S_T | T < \infty})\hat{P}_{wu}(T < \infty)
= \hat{E}_{w_u}(e^{-w_u S_T | T < \infty}) = \left( \frac{\theta_1 - w_u}{\theta_1} \right) e^{-w_u b}, u \in (0, 1).
\]

Letting \( u \to 1 \) we get that \( \mathbb{P}(T < \infty) = \frac{(\theta_1 - w_u)e^{-w_u b}}{\theta_1} \). Since \( \mathbb{E}(u^T I_{T<\infty}) = \mathbb{E}(u^T | T < \infty)\mathbb{P}(T < \infty) \) we readily deduce the following proposition.

**Proposition 6** Let \( S_n, n = 1, 2, \ldots \) be a random walk with step distribution \( F(x) = pF_1(x) + (1-p)F_2(x) \), where \( F_i \sim \mathcal{E}(\theta_i), i = 1, 2, p \in (0,1) \). If \( T \) denotes the time until the random walk crosses \( b > 0 \), then the conditional pgf of \( T \) given that \( T < \infty \) is

\[
\mathbb{E}(u^T | T < \infty) = \frac{\theta_1 - w_u}{\theta_1} e^{(w_1-w_u)b}, \quad u \in (0, 1)
\]

where \( w_u \) is as in (16). Moreover,

\[
\mathbb{P}(T < \infty) = \begin{cases} \frac{p(\theta_1 + \theta_2)}{\theta_1} e^{-(1-p)\theta_1 - p\theta_2 b}, & (1-p)\theta_1 - p\theta_2 \geq 0 \\ 1, & (1-p)\theta_1 - p\theta_2 < 0. \end{cases}
\]

By employing Proposition 6 we can easily compute the conditional probabilities \( s(m) = \mathbb{P}(T = m | T < \infty) \) through (17). In Figure 3 the conditional probabilities \( s(m) \) have been plotted for two sets of values of the parameters.

In the first case we have that \( \mathbb{P}(T < \infty) = \frac{14}{15} e^{-3/10} \approx 0.69143 \), while in the second case \( \mathbb{P}(T < \infty) = 1 \).

### 3.2 The distribution of the total number of defective items in a sampling system based on a \( k \)-run switching rule.

In the current paragraph we present an application in acceptance sampling which is a major component of the field of statistical process control. In acceptance sampling we frequently deal with sampling systems/plans that have at least two sampling levels controlled by switching rules that
are based on run and scan statistics. Two examples of such systems are the continuous sampling plans (see, for example, Schilling and Neubauer (2009)) and the Military Standard 105E (see, for example, Montgomery (2005)).

In acceptance sampling for attributes we take samples of fixed size corresponding to consecutive lots of items from a manufacturing process and we record the number $Z_i$, $i = 1, 2, ...$ of non-conforming (defective) items in the $i$-th sample. Let $c$ be the acceptance number of the “normal” sampling level, that is a lot is rejected if the corresponding sample contains more than $c$ non-conforming items. Assume that a switch in a more “tightened” (“reduced”) sampling level is instituted when each one of $k$-consecutive samples have more than (less than or equal) $c$ non-conforming items. We denote by $T$ the waiting time (i.e. number of lots) until the sampling level of the inspection changes. Our aim is to obtain the joint pgf of $T$ and $S_T$ by exploiting the fact that $T$ follows a known distribution. The study of the random variable $S_T$ is crucial, especially under a rectifying inspection program.

In the sequel we deal with a sampling system that begins under the normal sampling level and a switch is permitted only to the tightened one. More specifically, suppose that the size of the samples is fixed and equal to $n$ and that the probability of an item being defective is equal to $p \in (0, 1)$. Therefore, each $Z_i$, $i = 1, 2, ...$ follows a Binomial distribution with parameters $n, p$. The number $T$ of inspected lots until the tightened sampling level is instituted can be expressed as

$$T = \inf\{l \geq k : Z_{l-k+1} > c, ..., Z_l > c\}.$$ 

The stopped sum $S_T = \sum_{i=1}^{T} Z_i$ expresses the total number of defective items found until switching to the tightened sampling level.

Since $Z_i$’s are discrete rv’s we can conveniently set $t = e^w$ in Corollary 2 to get the following
relation for the joint pgf of \((T, S)\),
\[
\mathbb{E}(u^T t^S) = \mathbb{E}_t((u\mathbb{E}(t^Z)) T),
\]
(18)
where \(\mathbb{E}(t^Z) = (1 - p + pt)^n\). The distribution of the \(Z_i\)’s under the probability measure \(\tilde{P}_t\) is
\[
\tilde{P}_t(Z_i = x) = \frac{t^x\mathbb{P}(Z_i = x)}{\mathbb{E}(t^Z)} = \frac{n^x}{(1 - p + pt)^x(1 - p + pt)^{n-x}},\ x = 0, 1, \ldots, n.
\]
Therefore, under \(\tilde{P}_t\), \(Z_i\) follows a binomial distribution, with parameters \(n\) and \(p_t = \frac{pt}{1 - p + pt}, \ t > 0\).

The stopping time \(T\) can be considered as the first time a success run of length \(k\) occurs in a sequence of independent trials with success probability \(q = \mathbb{P}(Z_i > c)\). Hence, \(T < \infty\) and the distribution of \(T\) is known as the geometric distribution of order \(k\) (see, for example, Philippou et al. (1983) or Balakrishnan and Koutras (2002)) with pgf given by,
\[
\mathcal{M}(z, q) = \mathbb{E}(z^T) = \frac{(qz)^k(1 - qz)}{1 - z + (1 - q)q^kz^{k+1}},\ z \in [0, 1].
\]
(20)
Under the probability measure \(\tilde{P}_t\) we have
\[
q_t = \tilde{P}_t(Z_i > c) = 1 - \sum_{x=0}^{c} \binom{n}{x} p_t^x (1 - p_t)^{n-x},
\]
and thus, \(\tilde{E}_t(z^T)\), is given by (20), by replacing \(q\) with \(q_t\). Taking into account this observation, equality (18) leads to the following formula for the joint pgf of \((T, S_T)\),
\[
\mathbb{E}(u^T t^{S_T}) = \mathbb{E}_t((u(1 - p + pt)^n)^T) = \mathcal{M}(u(1 - p + pt)^n, q_t)
\]
(21)
for all \(u \in [0, 1]\) and \(t \in (0, 1]\) guaranteeing that \(u(1 - p + pt)^n \in [0, 1]\) and \(t > 0\), as required by (20) and (19).

The pgf \(\mathbb{E}(t^{S_T})\) follows readily from the above by setting \(u = 1\). The distribution of \(S_T\), which has support \(\{k(c+1), k(c+1)+1, \ldots\}\), can be numerically evaluated for specific values of the parameters \(n, p, c\) and \(k\) as described after formula (17). Using this procedure we calculate \(\mathbb{P}(S_T = m)\) for two sets of the parameters and the results are shown in Figure 4.

It should also be mentioned, that since \(S_T\) is a positive integer-valued rv, the generating function \(\mathcal{H}(t) = \sum_{m=0}^{\infty} \mathbb{P}(S_T > m)t^m, t \in (-1, 1)\) of the tail probabilities can be easily determined via the formula
\[
\mathbb{E}(t^{S_T}) = 1 - (1 - t)\mathcal{H}(t).
\]
Figure 4: The probability mass function of $S_T$ ($n = 20, p = 0.1, c = 1, k = 2$ and $n = 30, p = 0.2, c = 3, k = 3$)

The tail probabilities of the distribution of $S_T$ can be used in practice for the determination of the parameters of the above mentioned sampling plan. For various combinations of $c$ and $k$ it would be interesting to know the probability that the total number of defective items until switching exceeds a certain threshold. For example, consider the case where $n = 40, c = 1, k = 3$ and $p = 0.02$. For $u = 1$, Equation (21) provides the pgf of $S_T$ from which, by differentiation, we get that $\mathbb{E}(S_T) = 142.04$ (note that $\mathbb{E}(S_T)$ can also be evaluated via Wald’s first equation). Moreover, using $\mathcal{H}(t)$, we can compute the percentile points of the distribution of $S_T$, which provide complete knowledge about the performance of the sampling plan, in terms of the total number of defective items found until the switching. Since in that case the median of the distribution of $S_T$ is 100, we deduce that there is a probability lower than 50% that the total number of defective items will exceed 100 until switching.

It is worth mentioning that the above procedure could easily be expressed in a more general setting. For example, if the measurements $Z_i, i = 1, 2, ...$ from the inspected lots follow a general distribution with cdf $F$ (continuous, discrete or mixed) and a switching sampling level occurs at time $T$ according to some stopping rule (e.g. a $k/m$ scan rule), then following the methodology described above we can similarly determine the joint generating function of $(T, S_T)$ provided that the pgf of $T$ is known (e.g. is a geometric distribution of order $k/m$, see Balakrishnan and Koutras (2002)). In this respect we state without proof the following proposition.

**Proposition 7** Let $Z_i, i = 1, 2, ...$ be a sequence of iid measurements following a distribution $F$ and let $T$ be the waiting time (i.e. number of $Z_i$'s) until a switching sampling level occurs based on the $k/m$ scan switching rule: $k$ out of $m$ consecutive $Z_i$'s belong to a specific measurable set $A \subset \mathbb{R}$. If $\mathcal{M}_{k,m}(z,q) = \mathbb{E}(z^T), z \in \mathcal{W}$ denotes the pgf of the geometric distribution of order $k/m$
with success probability \( q \), then
\[
\mathbb{E}(u^T e^{wS_T}) = M_{k,m}(u \mathbb{E}(e^{wZ}), \frac{\mathbb{E}(e^{wZ} I(Z \in A))}{\mathbb{E}(e^{wZ})})
\]
for all \( u, w \) such that \( \mathbb{E}(e^{wZ}) < \infty \) and \( u \mathbb{E}(e^{wZ}) \in \mathcal{W} \).

The interested reader who wishes to study the general sampling system which permits a switch from the normal sampling level to the tightened or to the reduced sampling level may consult Ebnesahrashoob and Sobel (1990) for the pgf of the associated waiting time rv \( T \).

### 3.2.1 Estimating \( p \) via an EM algorithm.

In this last subsection we present an interesting application of the formula of \( \mathbb{E}(u^T e^{wS_T}) \) obtained above (cf. [21]), regarding the estimation of the probability \( p \) of an item being defective. Assume that \( \nu \) independent inspections are conducted according to the \( k \)-run switching rule described above and let \( T_i \) be the waiting time (i.e. number of lots) until the sampling level of the \( i \)-th inspection changes, \( i = 1, 2, ..., \nu \). Denote also by \( S_{T_i} \) the total number of defective items found until switching to the tightened sampling level has occurred in the \( i \)-th inspection, \( i = 1, 2, ..., \nu \). We are interested in estimating \( p \) when only the sample values \( \tau = (\tau_1, \tau_2, ..., \tau_\nu) \) of the \( \nu \) aforementioned waiting times are available.

Since the likelihood function 
\[
L(p; \tau) = \prod_{i=1}^{\nu} \mathbb{P}(T_i = \tau_i | p)
\]

does not have a convenient form in order to directly find the MLE of \( p \), we will show how we can alternatively employ an EM algorithm, considering \( S_{\tau} = (S_{T_1}, S_{T_2}, ..., S_{T_\nu}) \) as missing values (latent variables). The likelihood function 
\[
L(p; \tau, S_{\tau}) \propto \prod_{i=1}^{\nu} p^{S_{T_i}}(1-p)^{n_{\tau_i}-S_{T_i}} = (\frac{p}{1-p})^{\sum_{i=1}^{\nu} S_{T_i}}(1-p)^{n\sum_{i=1}^{\nu} \tau_i}.
\]

Since \( S_{\tau} \) is not available, we can find the MLE of \( p \) by iteratively applying the following two steps (EM algorithm; cf. Dempster et al. (1977)):

**E-step:** Given \( \tau \) and the estimate of \( p \) at the \( j \)-th step, say \( p^{(j)} \), compute the conditional expected value of the log likelihood function,
\[
Q(p | p^{(j)}) = \mathbb{E}_{S_{\tau} | \tau, p^{(j)}}(\log L(p; \tau, S_{\tau}))
\]
\[
= \sum_{i=1}^{\nu} \mathbb{E}(S_{T_i} | T_i = \tau_i, p^{(j)}) \log \frac{p}{1-p} + n \sum_{i=1}^{\nu} \tau_i \log(1-p).
\]

The expected value \( \mathbb{E}(S_{T_i} | T_i = \tau_i) \) can be calculated by
\[
\mathbb{E}(S_{T_i} | T_i = \tau_i) = \frac{1}{\mathbb{P}(T_i = \tau_i)} \sum_m m\mathbb{P}(S_{T_i} = m, T_i = \tau_i),
\]
which can be derived from Equation (21). More specifically, the sum \( \sum m \mathbb{P}(S_T = m, T = r) \) is the coefficient of the \( r \)-th order term in the power series expansion of \( \mathbb{E}(S_T u^T) \) (where \( \mathbb{E}(S_T u^T) = \frac{\partial}{\partial t} \mathbb{E}(u^T t^{S_T}) |_{t=1} \), and \( \mathbb{P}(T_i = \tau_i) \) can be derived from the series expansion of \( \mathbb{E}(u^T) \).

**M-step**: Find the parameter \( p^{(j+1)} \) that maximizes \( Q(p | p^{(j)}) \), i.e.,

\[
p^{(j+1)} = \arg \max_p Q(p | p^{(j)}) = \frac{\sum_{i=1}^{\nu} \mathbb{E}(S_T | T_i = \tau_i, p^{(j)})}{n \sum_{i=1}^{\nu} \tau_i}.
\]

The above two steps are repeated until we achieve the desired accuracy in the estimate \( \hat{p} \) of \( p \) (e.g. \( \hat{p} = p^{(j_0)} \) where \( j_0 = \min\{ j : |p^{(j)} - p^{(j-1)}| < \varepsilon \} \). From the above procedure we can also get an estimate, \( \mathbb{E}(S_T | T_i = \tau_i, \hat{p}) \), of the unobserved variable \( S_{\tau_i}, i = 1, 2, ..., \nu \). The observed Fisher information, which can be exploited for establishing approximate confidence intervals for \( p \), takes on the form

\[
I(\hat{p}) = \mathbb{E}_{S_T | \tau, \hat{p}} \left( -\frac{\partial^2 \log L(p; \tau, \mathbf{S}_T)}{\partial p^2} \bigg|_{p=\hat{p}} \right) = \frac{1}{(1 - \hat{p})^2 \hat{p}^2} \left( (1 - 2\hat{p}) \sum_{i=1}^{\nu} \mathbb{E}(S_T | T_i = \tau_i, \hat{p}) + n\hat{p}^2 \sum_{i=1}^{\nu} \tau_i \right).
\]

As an example of the above estimation procedure, suppose that a \( k \)-run switching rule is employed for \( \nu = 20 \) inspections with \( c = 4, k = 3, n = 50 \), and the resulted waiting times are:

\[
\tau = (10, 5, 17, 4, 19, 3, 25, 6, 16, 16, 5, 4, 4, 5, 6, 12, 7, 12, 12, 13)
\]

(actually, these are simulated values with \( p = 0.10 \)). By employing the EM algorithm we obtain \( \hat{p} = 0.0998513 \) \( (\varepsilon = 10^{-8}) \) while the estimates of \( S_{\tau_i}, i = 1, 2, ..., \nu \) are

\[
50, 27.4, 81.4, 22.4, 90.3, 19.4, 117.2, 32.4, 76.9, 76.9,
27.4, 22.4, 22.4, 27.4, 32.4, 59, 36.4, 59, 59, 63.5
\]

The estimated standard error of \( \hat{p} \) is \( I(\hat{p})^{-1/2} = 0.00299055 \) (cf. (22)) and the approximate 1 - \( a \) = 95% confidence interval for \( p \) is \( \hat{p} \pm I(\hat{p})^{-1/2} z_{a/2} = (0.0939898, 0.105713) \).

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4 Appendix

The formal construction of \( (\Omega, \mathcal{F}, \mathbb{P}_w) \) : Denote by \( \Omega = \mathbb{R}^N \) the collection of all maps from \( N = \{1, 2, \ldots\} \) to \( \mathbb{R} \). Each element \( x \) of the product space \( \mathbb{R}^N \) can be written as a sequence
\( x = (x_1, x_2, \ldots) \) with each \( x_i \) belonging to \( \mathbb{R} \). For each \( i \in \mathbb{N} \) consider the mapping \( Z_i : \mathbb{R}^N \rightarrow \mathbb{R} \) with \( Z_i(x) = x_i \) (that is, \( Z_i \) is a coordinate function or projection). Let \( \mathcal{F} = \mathcal{R}^N \) be the minimal \( \sigma \)-algebra such that \( Z_1, Z_2, \ldots \) are measurable, i.e. \( \mathcal{R}^N := \sigma(Z_1, Z_2, \ldots) = \sigma(\{x \in \mathbb{R}^N : x_i \in B\}, B \in \mathcal{B}(\mathbb{R}), i \in \mathbb{N}) \), where \( \mathcal{B}(\mathbb{R}) \) is the \( \sigma \)-algebra of the Borel sets of \( \mathbb{R} \). Next, denote by \( \mu_i \) the probability measure on \( \mathcal{B}(\mathbb{R}) \) that corresponds to \( F_i \), \( i = 1, 2, \ldots \). For every \( i = 1, 2, \ldots \) define the distribution \( F_i(\cdot | w) \) on \( \mathbb{R} \), such that

\[
F_i(x|w) := \frac{\int_{(-\infty,x]} e^{wz}dF_i(z)}{\int_{\mathbb{R}} e^{wz}dF_i(z)}, \quad x \in \mathbb{R}, \quad w \in \mathcal{W},
\]

which can be considered as the exponentially tilted \( F_i \). Obviously, \( F_i(x|0) = F_i(x) \). If \( \mu_i^w \) denotes the probability measure on \( \mathcal{B}(\mathbb{R}) \) corresponding to \( F_i(\cdot | w) \) then, equivalently, \( \mu_i^w(B) = \int_B e^{wx} \mu_i(dx) / \int_{\mathbb{R}} e^{wx} \mu_i(dx) \) for every \( B \in \mathcal{B}(\mathbb{R}) \). Therefore \( \mu_i^w \ll \mu_i \) and the Radon-Nikodym derivative for \( \mu_i^w \) with respect to \( \mu_i \) reads

\[
\frac{d\mu_i^w}{d\mu_i}(x) = \frac{e^{wx}}{\int_{\mathbb{R}} e^{wx} \mu_i(dx)}, \quad x \in \mathbb{R}, \quad w \in \mathcal{W}.
\]

Finally, invoking Kolmogorov’s Existence Theorem, there exists a probability measure \( \tilde{P}_w \) on \( \mathcal{R}^N \) such that the coordinate variable process \( Z_1, Z_2, \ldots \) on \( (\mathbb{R}^N, \mathcal{R}^N, \tilde{P}_w) \) consists of independent rv’s, with distributions \( \mu_1^w, \mu_2^w, \ldots \) respectively, and the construction is completed for all \( w \in \mathcal{W} \).
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