Staggered flux state of electron in two-dimensional t-J model

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The competition between the staggered flux state, or the d-density wave state, and the d-wave pairing state is analyzed in two-dimensional t-J model based on the U(1) slave boson mean-field theory. Not only staggered flux of spinon but also staggered flux of holon are considered. In this formalism, the hopping order parameter of physical electron is described by the product of hopping order parameters of spinon and holon. The staggered flux amplitude of electron is the difference of staggered flux amplitude of spinon and that of holon. In π-flux phase of spinon, staggered fluxes of spinon and holon cancel completely and staggered flux order of electron does not exist. However, in staggered flux phase of spinon whose staggered flux amplitude is not π, fluxes does not cancel completely and staggered flux amplitude of electron remains. Thus, the phase transition between these two phases, π-flux phase and staggered flux phase of spinon, becomes a second order transition in physical electron picture. The order parameter which characterizes this transition is staggered flux order parameter of electron. A mean-field phase diagram is shown. It is proved analytically that there is no coexistence of staggered flux and d-wave pairing. The temperature dependences of Fermi-surface and excitation gap at (0, π) are shown. These behaviors are consistent with angle-resolved photoemission spectroscopy (ARPES) experiments.

I. INTRODUCTION

Recently the staggered flux state [1,2] has been revived as a candidate of pseudo-gap phase of high $T_c$ superconductors. The staggered flux state is characterized by the staggered orbital-current. Time-reversal-symmetry is broken if staggered current is the current of physical electron. Weak magnetic signals were caught in recent experiments of underdoped YBa$_2$Cu$_3$O$_{6+x}$ (YBCO) by neutron scattering [3,4]. Static alternating magnetic signals were also caught by muon spin rotation experiment in the vortex cores of underdoped YBCO [5]. The staggered flux state contains a density wave ordering whose symmetry is $d_{x^2-y^2}$ [6]. Because of this, it is also called d-density wave. Thus, there is the gap whose symmetry is $d_{x^2-y^2}$. Although $d_{x^2-y^2}$-gap exists in both the staggered flux state and the $d_{x^2-y^2}$-wave pairing state, the structures of the fermion excitation in these states are different. The staggered flux state should have a segment-like Fermi-surface as observed in angle-resolved photoemission spectroscopy (ARPES) experiments [7-14]. On the contrary, $d_{x^2-y^2}$-wave pairing state has a point-like Fermi-surface. The absence of specific heat anomaly may be thought to deny the staggered-flux phase. However, we remark that there are arguments to justify the absence [15,16]. The lack of anomaly is either because of finite chemical potential [15], or because that this transition can be described by 6-vertex model [16].

The high-$T_c$ superconductors are doped Mott-insulators. Essential physics of these materials comes from strong Coulomb repulsion. This physics is described by t-J model [17,18]. A method that restrict double occupancy is the slave boson method. In both the slave boson methods, the U(1) method and the SU(2) method, the physical electron operator is described by a product of auxiliary fermions and bosons. In the U(1) slave boson theory, the physical electron operator $c_{i\sigma}$ is described by a product of an auxiliary spin-1/2 neutral fermion operator $f_{i\sigma}$ called spinon and a auxiliary spinless charged boson operator $b_i$ called holon; $c_{i\sigma} = b_i^\dagger f_{i\sigma}$ with a constraint $b_i^\dagger b_i + f_{i\sigma}^\dagger f_{i\sigma} = 1$. Here, the repeated spin index $\sigma$ is summed up over the two spin states. On the contrary, in the SU(2) slave boson theory [19,20], two auxiliary bosons are introduced. The physical electron operator $c_{i\sigma}$ is described by a product of an auxiliary isospin-1/2 neutral fermion operator $f_{i\sigma}$ and two auxiliary spinless charged boson operators, $b_{i1}, b_{i2}$; $c_{i\sigma} = h_i^\dagger \psi_i^\dagger / \sqrt{2} = (b_{i1}^\dagger f_{i\sigma} + b_{i2}^\dagger \epsilon_{\sigma\sigma'} f_{i\sigma'}^\dagger ) / \sqrt{2}$ with three constraints, $\frac{1}{2} \psi_i^\dagger \tau \psi_i^\dagger + h_i^\dagger h_i = 0$. Here, $h_i^\dagger = (b_{i1}, b_{i2})$, $\psi_i^T = (f_{i\sigma}, \epsilon_{\sigma\sigma'} f_{i\sigma'}^\dagger)$, and $\tau = (\tau_1, \tau_2, \tau_3)$ are Pauli matrices.

Wen and Lee proposed based on the SU(2) theory [20-22] that the pseudo-gap state is a mixture of staggered flux state and $d_{x^2-y^2}$ superconducting state. These orders are dynamical rather than static and that the time-reversal symmetry is not broken. They discussed this in the region where bosons are condensed, $(z_{i1}, z_{i2}) \neq 0$, based on the $O(4)$ sigma-model description. It is also proposed by Lee and Wen [21] that the static staggered-flux order can only exists in the vortex core. Short comings of this theory are that finite size Fermi-surface is not obtained naturally and that the condensation of bosons is assumed. The staggered current of physical electron and that of spinon are not equivalent in the staggered flux [1,2] phase at finite temperature because the stag-
gered flux phase of spinon exists above the temperature of Bose-condensation of holon \([19,20,23]\). Above the temperature of Bose-condensation, holon operators cannot be treated as classical numbers (c-number). \([24]\) The correspondence between the electron current and the spinon current is not obvious.

On the contrary, Chakravarty, Laughlin, Morr, and Nayak \([25]\) proposed that the staggered flux order exists as a static form in the pseudo-gap state. They proposed this at zero temperature based on Ginzburg-Landau theory. The staggered flux state they proposed is that of electron and time-reversal symmetry is broken. They also proposed that there exist three ground states. The ground state changes as doping increases from pure staggered flux phase to the coexisting phase of staggered flux state they proposed is that of electron current and the staggered current of spinon exists above the temperature corresponding to the electron current and the spinon current is not obvious.

\[
\text{Nayak} [25] \text{ proposed that the staggered flux order exists on a square lattice [18], results will be published elsewhere [26].}
\]

In this paper, we analyze the staggered-flux state of physical electron in two-dimensional \(t\)-\(J\) model by the \(U(1)\) slave boson method without assuming the Bose condensation of holon. The relation between the staggered current of electron and the staggered current of spinon is provided. In Sec. II, our formulation will be reviewed. Not only an auxiliary field \(\chi_{ij}\) that describes spinon-hopping but also an auxiliary field \(B_{ij}\) that describes holon-hopping will be introduced to decouple the hopping Hamiltonian. The advantages of this formulation will be stated in Sec. II A. For mean field solution, not only staggered flux of spinon but also staggered flux of holon will be considered. This solution provides the relation between the staggered flux state of spinon and that of electron. In Sec. III, The self-consistent equations will be derived. In Sec. IV, the phase diagram is shown. where the staggered-flux phase of electron exists. In Appendix, it is proved analytically that the staggered flux and the d-wave pairing do not coexist. The instability of these two states, the staggered flux state and the d-wave pairing state, are discussed. In Sec. V, the temperature dependences of Fermi surface and excitation gap at \((0, \pi)\) are shown. We show that the Fermi-surface in the staggered-flux state resembles segment-like Fermi-surface observed by ARPES experiments. Finally, conclusion will be given in Sec. VI. Part of the present results will be published elsewhere \([26]\).

II. FORMALISM

A. Slave boson \(t\)-\(J\) model and auxiliary fields

The starting Hamiltonian is two-dimensional \(t\)-\(J\) model on a square lattice \([18]\),

\[
H = -t \sum_{\langle i,j \rangle} P(c_{i\sigma}^\dagger c_{j\sigma} + \text{h.c.}) P + J \sum_{\langle i,j \rangle} S_i \cdot S_j. \tag{1}
\]

Here, \(<i,j>\) represents sum over the nearest-neighbor sites, the repeated spin index \(\sigma\) is summed up over the two spin states, and \(P\) is a projection operator to no doubly occupied state, \(S_i = \frac{1}{2} \sum_{\sigma \neq \sigma'} c_{i\sigma}^\dagger c_{i\sigma'}\), where \(\sigma = (\sigma_1, \sigma_2, \sigma_3)\) are Pauli matrices. One way to represent the projection to no double occupancy is the slave boson method. In the \(U(1)\) slave boson method, the electron operator \(c_{i\sigma}\) is described by a product of an auxiliary spin-1/2 neutral fermion operator \(f_{i\sigma}\) called spinon and an auxiliary spinless charged boson operator \(b_{i\sigma}\) called holon; \(c_{i\sigma} = b_{i\sigma}^\dagger f_{i\sigma}\) with a constraint \(b_{i\sigma}^\dagger b_{i\sigma} + f_{i\sigma}^\dagger f_{i\sigma} = 1\). Here, the repeated spin index \(\sigma\) is summed up over the two spin states. In the path integral formalism, the partition function \(Z(\beta) = \text{Tr} \exp(-\beta H)\) is described by following functional integral,

\[
Z = \int [db][df][d\lambda] \exp(-S),
\]

\[
S = \int_0^\beta d\tau \sum_i (b_{i\sigma}^\dagger \partial_{\tau} b_{i\sigma} + f_{i\sigma}^\dagger \partial_{\tau} f_{i\sigma}) + H,
\]

\[
H = -t \sum_{\langle i,j \rangle} (f_{i\sigma}^\dagger f_{j\sigma} b_{i\sigma}^\dagger b_{j\sigma} + \text{c.c.}) + J \sum_{\langle i,j \rangle} S_i \cdot S_j
\]

\[
+ i \sum_i \lambda_i (b_{i\uparrow}^\dagger b_{i\downarrow}^\dagger f_{i\sigma} - 1) - \mu_e \sum_i f_{i\sigma}^\dagger f_{i\sigma}, \tag{3}
\]

where \(\beta\) is the inverse temperature \(\beta = 1/T\), \(\tau\) is the imaginary time, \(f[db] = \int \prod_i db_i^\dagger(\tau) db_i(\tau)\) is a complex boson integral, \(f[df] = \int \prod_{\tau,\sigma} df_{i\sigma}^\dagger(\tau) df_{i\sigma}(\tau)\) is a complex Grassmann integral, \(f[d\lambda] = \int \prod_{\tau} d\lambda_i(\tau)\) is the Lagrange-multiplier integral that represents a constraint \(b_{i\sigma}^\dagger b_{i\sigma} + f_{i\sigma}^\dagger f_{i\sigma} = 1\), \(S_i = \frac{1}{2} f_{i\sigma}^\dagger(\sigma)_{\sigma'} f_{i\sigma'} + \mu_e\) is chemical potential. We introduce three complex auxiliary fields \(\chi_{ij}, \eta_{ij}\), and \(B_{ij}\) on links to decouple the Hamiltonian.

\[
H' = \sum_{\langle ij \rangle} \left[-t \left\{ B_{ij} f_{i\sigma}^\dagger f_{j\sigma} + \chi_{ij}^\dagger b_{ij} b_{ij}^\dagger \right\} + \text{c.c.} \right.
\]

\[
+ t \left\{ B_{ij} \chi_{ij}^\dagger b_{ij} + B_{ij} \chi_{ij} \right\} - \frac{3J}{8} \left\{ \chi_{ij} f_{i\sigma}^\dagger f_{j\sigma} + \eta_{ij} f_{i\sigma} f_{j\sigma}^\dagger \right\} + \text{c.c.} + \frac{3J}{8} \left\{ |\chi_{ij}|^2 + |\eta_{ij}|^2 \right\},
\]

\[
+ i \sum_i \lambda_i (b_{i\uparrow}^\dagger b_{i\downarrow}^\dagger f_{i\sigma} - 1) - \mu_e \sum_i f_{i\sigma}^\dagger f_{i\sigma}. \tag{4}
\]

Then partition function \(Z(\beta)\) is rewritten as:

\[
Z = \int [db][df][d\lambda][d\chi][d\eta][dB] \exp(-S'), \quad \text{where } S' = \int_0^\beta d\tau \sum_i (b_{i\sigma}^\dagger \partial_{\tau} b_{i\sigma} + f_{i\sigma}^\dagger \partial_{\tau} f_{i\sigma}) + H'.
\]

The following relations are obtained by differentiating the integrand in Eq. (4) by \(\chi_{ij}^\dagger, \eta_{ij}^\dagger, \text{ and } B_{ij}^\dagger [27],\)

\[
\tilde{\chi}_{ij} \equiv \langle \chi_{ij} \rangle = \langle f_{i\sigma}^\dagger f_{j\sigma} \rangle,
\]

\[
\tilde{\eta}_{ij} \equiv \langle \eta_{ij} \rangle = \langle j_{ij} f_{j\sigma}^\dagger - f_{ij} f_{ij}^\dagger \rangle,
\]

\[
B_{ij} \equiv \langle B_{ij} \rangle = \langle b_{ij}^\dagger b_{ij} \rangle. \tag{5}
\]

According to these relations, \(\chi_{ij}\) describes the hopping of spinon, \(\eta_{ij}\) describes the singlet pairing of spinon which is
called resonating-valence bond (RVB), and $B_{ij}$ describes the hopping of holon. The quantity $B_{ij} = \langle b_i^\dagger b_j \rangle$ can have a finite value even if there is no Bose-condensation of holon (see Sec. IV).

Present treatment is different from previous one by Ubbens and Lee [23]. In Ref. [23], $B_{ij}$ was not introduced. Then $\chi_{ij}$ cannot be interpreted as spinon hopping $\langle f_i^\dagger f_j \rangle$, but the correct relation becomes $\chi_{ij} = \langle f_i^\dagger f_j + \frac{\eta_i}{\eta_j} b_i^\dagger b_j \rangle$ as can be known from Eq. (4) without $B_{ij}$ term. In their method, the four-boson term $b_i^\dagger b_j b_i^\dagger b_j$ is created by the decoupling, but it is neglected based on an argument that the effect of this term is small at low doping. The situation is similar in the SU(2) method [19,20], if $B_{ij}$ is not introduced. Our treatment has a merit that the four-boson term does not appear in the decoupling.

### B. Saddle point solution

We approximate the integral by auxiliary fields $\chi_{ij}$, $\eta_{ij}$, and $B_{ij}$ with their saddle point values $\bar{\chi}_{ij}$, $\bar{\eta}_{ij}$, and $\bar{B}_{ij}$. For the saddle flux solution of $\chi_{ij}$ and $B_{ij}$, we considered not only the staggered flux order of the spinon $\phi_s$ but also the staggered flux order of the holon $\phi_h$ (Fig. 1), namely

\begin{align}
\bar{\chi}_{i+x,i} &= \chi e^{i(-1)^s x y_s / 4} = x_s + i(-1)^s y_s, \\
\bar{\chi}_{i+y,i} &= \chi e^{-i(-1)^s y_s / 4} = x_s - i(-1)^s y_s, \\
\bar{B}_{i+x,i} &= B e^{i(-1)^s x y_s / 4} = x_h + i(-1)^s y_h, \\
\bar{B}_{i+y,i} &= B e^{-i(-1)^s y_s / 4} = x_h - i(-1)^s y_h.
\end{align}

Here, $\hat{x}$ and $\hat{y}$ are unit vectors in the $x$ and $y$ direction, $x_s = \chi \cos(\phi_s / 4)$, $y_s = \chi \sin(\phi_s / 4)$, $x_h = B \cos(\phi_h / 4)$, $y_h = B \sin(\phi_h / 4)$ [28]. The staggered flux state contains a density wave (particle-hole pairing) ordering whose symmetry is $d_{\pm 2}\hat{y}$, which is called “d-density wave” [6]. The order parameters $y_s$ and $y_h$ correspond to the d-density wave order parameter of spinon and holon, respectively [29]. For the symmetry of spinon pairing, we considered $d_{\pm 2} y^2$ [30,31], namely

\begin{equation}
\bar{\eta}_{i+x,i} = - \bar{\eta}_{i+y,i} = \eta.
\end{equation}

![FIG. 1. Staggered flux of spinon $\phi_s$ and holon $\phi_h$.](image)

The integral by Lagrange multiplier field $\lambda_i$ is also approximated with its saddle point value $\bar{\lambda}_i = -i \lambda_0$. Then spinon and holon have chemical potentials, $\lambda = \mu_0 - \lambda_0$ and $\bar{\mu}_h = -\lambda_0$, respectively. These chemical potentials enforce the global constraints of spinon number and holon number, $\langle f_i^\dagger f_i \rangle = 1 - \delta$ and $\langle b_i^\dagger b_i \rangle = \delta$, respectively.

This method that treats the spinon and the holon as separated degrees of freedoms is justified because the low temperature phase of the $t$-$J$ model is in the deconfinement phase of spinon and holon even when full gauge fluctuations (fluctuations around this saddle point) are included. It is shown based on the compact U(1) treatments of the gauge fields [32,33]. As the gauge field couples with finite density fermion, the gauge coupling gets weaker. In the deconfinement phase of spinon and holon, the gauge field can be treated perturbatively.

Our formalism is an extension of the study by Ubbens and Lee [23]. The $f_i^\dagger f_j$ term couples not only to $(3J/8)\chi_{ij}$ but also to $t \bar{B}_{ij}$, namely the spinon feels the spin staggered-flux $(\phi_s)$ and the holon staggered-flux $(\phi_h)$. The expectation values of the holon, $B$ and $\phi_h$, have finite values for the solution of self-consistency equations. Two advantages exist in our formalism: 1) this is a new saddle point solution whose free energy is lower than the previous one, 2) this solution provides the relation between the staggered-flux of electron and that of spinon.

With holon order parameters $\bar{B}_{ij}$, the hopping order parameter of electron becomes a product of the hopping order parameters of spinon and holon,

\begin{equation}
\langle c_i^\dagger c_j \rangle = \langle f_i^\dagger f_j \rangle \langle b_i^\dagger b_i \rangle = \bar{\chi}_{ij} \bar{B}_{ij}^* = \chi B \bar{e}^{i(\phi_s - \phi_h) / 4}.
\end{equation}

The electron staggered-flux order parameter $\phi_e$ and the electron d-density wave order parameter $y_e$ are given by

\begin{equation}
\phi_e = \phi_s - \phi_h, \quad y_e = \chi B \sin(\phi_e / 4).
\end{equation}

According to eq. (12), the d-density wave order parameter of electron $y_e$ can also be written as, $y_e = -i \frac{1}{2N} \sum_k (\cos k_x - \cos k_y) \langle c_i^\dagger c_k \rangle \bar{e}^{i(\phi_e - \phi_h) / 4}$.

### III. SELF CONSISTENT EQUATIONS

This saddle point is described by the following partition function $Z^{MF} = \int \langle db \rangle |df| \exp(-S^{MF})$, where $S^{MF} = \int_0^\beta d\tau \left[ \sum_i (b_i^\dagger \partial_\tau b_i + f_i^\dagger \partial_\tau f_i) + H^{MF} \right]$, $H^{MF} = H^{MF}_{\text{matter}} - N (\mu - \mu_0)$

\begin{equation}
+ 2N \left[ \frac{3J}{8} (x_s^2 + y_s^2 + \eta^2) + 2t (x_s x_h + y_s y_h) \right] + \frac{1}{2N} \sum_k \left( \cos k_x - \cos k_y \right) \langle c_i^\dagger c_k \rangle \bar{e}^{i(\phi_e - \phi_h) / 4},
\end{equation}

the spinon chemical potential is $\lambda = \lambda_0$, and the holon chemical potential is $\mu_0 = -\lambda_0$. The saddle point value of $i\lambda_0$ is $i\lambda = \lambda_0$. Here, the matter part of this Hamiltonian is given by
\begin{align}
H_{\text{matter}}^{MF} &= \sum_k \langle \Psi_k^\dagger H_k \Psi_k + \bar{\Psi}_k^\dagger \bar{H}_k^\dagger \bar{\Psi}_k \rangle, \label{eq:hamiltonian}
\end{align}

where, \( \sum_k \) stands for a sum over the half Brillouin zone. \( \Psi_k \) and \( \bar{\Psi}_k \) are vector representations of spinon and holon, \( \Psi_k = (f_{k+1}^{\dagger}, f_{k+Q}^{\dagger}, f_{k-1}, f_{k-Q}) \), \( \bar{\Psi}_k = (b_k^{\dagger}, b_{k+Q}^{\dagger}) \). The matrices \( H_k \) and \( H_k^h \) are,

\begin{align}
H_k &= \begin{pmatrix}
(\epsilon_k - \mu) & iW_k & \Delta_k & 0 \\
-iW_k & -(\epsilon_k + \mu) & 0 & -\Delta_k \\
\Delta_k & 0 & -(\epsilon_k - \mu) & iW_k \\
0 & -\Delta_k & iW_k & (\epsilon_k + \mu)
\end{pmatrix}, \\
H_k^h &= \begin{pmatrix}
(\epsilon_k^h - \mu^h) & iW_k^h & 0 & 0 \\
-iW_k^h & -(\epsilon_k^h + \mu^h) & 0 & 0 \\
0 & 0 & iW_k^h & -(\epsilon_k^h + \mu^h)
\end{pmatrix},
\end{align}

where

\begin{align}
\epsilon_k &= -(2tx + \frac{3J}{4}x_y) \cos k_x + \cos k_y, \\
W_k &= (2ty + \frac{3J}{4}y_y) \cos k_x - \cos k_y, \\
\Delta_k &= \frac{3J}{4} \eta \cos k_x - \cos k_y, \\
\epsilon_k^h &= -2tx \cos k_x + \cos k_y, \\
W_k^h &= 2ty \cos k_x - \cos k_y.
\end{align}

The spinon couples not only to spinon order parameters, \( x \) and \( y \), but also to holon order parameters, \( x \) and \( y \). The holon couples only to spinon order parameters, \( x \) and \( y \). There exist two components, \( W_k \) and \( \Delta_k \), that generate the \( d_{x^2-y^2} \) wave gap.

After the Bogoliubov transformation, diagonalized Hamiltonian is obtained,

\begin{align}
H_{\text{matter}}^{MF} &= \sum_{k,s=\pm 1} \langle \gamma_{ks}^h \alpha_{ks}^h \beta_{ks}^\dagger + E_{ks}^h \alpha_{ks}^\dagger \alpha_{ks}^h \rangle, \label{eq:diagonalized_hamiltonian}
\end{align}

where

\begin{align}
E_{ks} &= \sqrt{(s \sqrt{\epsilon_k^2 + W_k^2} - \mu)^2 + \Delta_k^2}, \\
E_{ks}^h &= s \sqrt{\epsilon_k^2 + W_k^2} - \mu.
\end{align}

Both \( \alpha_{ks} \) and \( \beta_{ks} \) are fermionic fields, \( \alpha_{ks}^h \) is bosonic field, and \( E_{ks}^h \) is excitation spectrum of spinon (holon). Index \( s \) describes the band index, which takes the value \(+1\) or \(-1\).

After integrating out spinon field and holon field, the free energy has the following form,

\begin{align}
F &= -2T \sum_{k,s} \gamma_{ks} \ln \cosh(\beta E_{ks}/2) \\
&\quad + T \sum_{k,s} \gamma_{ks} \ln(1 - e^{-\beta E_{ks}^h}) \\
&\quad + 2N \left[ \frac{3J}{8} (x_s^2 + y_s^2 + \eta^2) + 2t(x_s x_h + y_s y_h) \right] \\
&\quad - N\delta(\mu - \mu^h).
\end{align}

By minimizing the free energy \( F \), we obtain the self-consistency equations,

\begin{align}
x_s &= (2tx + \frac{3J}{4}x_y) \frac{1}{N} \sum_{k,s} \gamma_{ks}^2, \\
y_s &= (2ty + \frac{3J}{4}y_y) \frac{1}{N} \sum_{k,s} \gamma_{ks}^2, \\
\eta &= \frac{3J}{4} \frac{1}{N} \sum_{k,s} \gamma_{ks}^2, \\
x_h &= (2tx \frac{1}{N} \sum_{k,s} \gamma_{ks}, \\
y_h &= (2ty \frac{1}{N} \sum_{k,s} \gamma_{ks}.
\end{align}

The chemical potential \( \mu \) and \( \mu^h \) are determined by

\begin{align}
\delta &= \frac{1}{N} \sum_{k,s} \gamma_{ks} \ln \cosh(\beta E_{ks}/2), \\
\delta &= \frac{1}{N} \sum_{k,s} \gamma_{ks} \ln e^{\beta E_{ks}^h} - 1,
\end{align}

where \( \gamma_{ks} = \cos k_x \pm \cos k_y \).

At half-filling, where spinon chemical potential \( \mu \) is zero and holon is absent (\( \bar{B}_{ij} = 0 \), i.e. \( x = y_h = 0 \)), \( E_{ks} = \sqrt{\epsilon_k^2 + (3J/4)^2(y_s^2 + \eta^2)(\cos k_x - \cos k_y)^2} \), and both \( y_s \) and \( \eta \) have the same self-consistent equations. Thus, staggered flux state and \( d \)-wave pairing state are degenerate. All states that have the same value of \( y_s^2 + \eta^2 \) are degenerate. However, when state has the lower energy at finite doping is known only after actual minimization of the free energy is done.
IV. PHASE DIAGRAM AND STAGGERED CURRENT

A. Phase diagram

FIG. 2. Mean field phase diagram for $t/J = 1$, where $\delta$ is hole concentration. The staggered-flux order of electron exists only in region 2. In this phase, staggered-fluxes of spinon and holon do not cancel completely. Thus the staggered flux of electron remains. In region 1, both spinon and holon are in the $\pi$-flux state. The fluxes cancel completely. The region 3 is the d-wave RVB phase where the $d_{x^2-y^2}$ pairing of spinon exists. The region 4 is the uniform RVB phase where hopping order parameters exist however it is real. In the region 5, all order parameters are zero. The phase diagram for $t/J = 2$ is quantitatively similar to the phase diagram for $t/J = 1$.

We solved the self-consistency equations numerically, and obtained the phase diagram (Fig. 2) where 1) a region of the staggered-flux phase of the electron exists, 2) staggered-flux and d-wave pairing do not coexist, and 3) the ground state is a purely superconducting state.

At half-filling, the holon order parameter $\bar{B}_{ij}$ is zero and the degeneracy of spinon states between the staggered-flux state and the d-wave pairing state exists due to local SU(2) symmetry [34]; $\chi \neq 0, \bar{B}_{ij} = 0, \eta_x^2 + \eta_y^2 = \text{const} \neq 0$. In region 1, the staggered-flux of electron does not exist although the staggered flux of spinon exists. The spinon and holon states are both $\pi$-flux state [35] respectively. In the electron picture, the staggered-flux is canceled completely; $\chi \neq 0, B \neq 0, \phi_s = \phi_h = \pi, \eta = 0$. The d-density wave order parameter of the electron is $y_e = \chi B \sin ((\pi - \pi)/4) = 0$. The staggered-flux order of the electron exists only in region 2. The staggered current of electron exists in this phase. The spinon staggered-flux $\phi_s$ and holon staggered-flux $\phi_h$ are not equal to $\pi$ nor 0, and the holon staggered-flux amplitude $\phi_h$ is not equal to the spinon staggered-flux amplitude $\phi_s$. $\phi_s \neq \phi_h; \chi \neq 0, B \neq 0, \phi_s \neq 0, \phi_h \neq 0, \eta = 0, and y_e = \chi B \sin ((\phi_h - \phi_s)/4) \neq 0$. In region 3, $d_{x^2-y^2}$-wave pairing exists; $\chi \neq 0 and B \neq 0, \phi_s = \phi_h = 0, \eta \neq 0. and y_e = 0$. In region 4, there exists only uniform hopping order; $\chi \neq 0, B \neq 0, \phi_s = \phi_h = \eta = 0, and y_e = 0$. In region 5, all order parameters are zero. Spinon and holon cannot hop; $\chi = B = \phi_s = \phi_h = \eta = 0$. and $y_e = 0$. The phase transitions, region 1 to region 3 and region 2 to region 3, are first order. Other phase transitions are second order. It is proved analytically that the staggered flux and the d-wave pairing do not coexist at finite doping. Details are given in Appendix.

With the boson order parameter $B_{ij}$, the $\pi$-flux phase and staggered-flux phase of spinon and holon extends to higher-doped region compared to the previous work [23], where $B_{ij}$ was not considered. The transition between region 1 and region 2 is a second order transition in our theory. (If one only focuses on spinon degree of freedom, this does not look like phase transition [19,20,23].) There exists an order parameter that characterizes this transition. It is the staggered-flux of the electron. This is shown in Sec. IV B.

Although the ground state is the d-wave superconducting phase at finite doping, the staggered-flux phase exists at finite temperature. The staggered-flux state has larger entropy than the d-wave pairing state, due to the larger number of the excitation around finite size Fermi-surface. On the contrary, the Fermi-surface in the d-wave pairing state is always point-like. Therefore the staggered-flux state arises when temperature increases. These are shown in Sec. V.

When the fluctuation around the saddle point solution (gauge field) is included, it is expected that the staggered flux phase extends to still higher doped region compared with the mean field phase diagram. There are three reasons, 1) the d-wave pairing of spinon will be destroyed, 2) the instability to the staggered-flux state exists, and 3) the symmetry breaking of the staggered-flux order is discrete. Let us elaborate these reasons. Firstly, the d-wave pairing of spinon is destroyed above the Bose-condensation temperature of holon when the fluctuation around the saddle point solution (gauge field) is included [36]. The d-wave RVB state without Bose-condensation does not occur, and the transition to the d-wave pairing state becomes the direct transition to the d-wave superconducting state. The second reason is that the instability to the staggered-flux state has been discovered also in the gauge theory that includes the full lattice structure [37,38]. It is to the staggered-flux state at low doped region and to the flux density-wave state (incommensurate staggered-flux state) at high doped region. The momentum that has anomaly is $Q = (\pi, \pi)$ to the staggered-flux state, and is $(\pi, \pi - \epsilon)$ to the flux density-wave. At finite temperature, these anomalies occur. The third reason is that staggered flux state is more robust against the fluctuation than the d-wave pairing (d-wave RVB) state. This comes from the difference in the symmetry breakings. The symmetry breaking in the staggered-flux state is discrete symmetry, $Z_2$. On the other hand, continuous symmetry, $U(1)$, is broken in the d-wave pairing state.
The spinon current $\delta$ is hole concentration.

At the transition point to d-wave pairing, the phase changes as doping increases from region 1 to region 2 and to finally region 3. In region 1 which is $\pi$-flux phase of spinon, staggered-fluxes of spinon and holon cancel completely and staggered-flux order of electron does not exist. However, in region 2, staggered flux amplitude is not $\pi$, fluxes does not cancel completely and staggered-flux amplitude of electron remains. The staggered-flux amplitude of holon decays rapidly as doping increases but it is not zero. At the transition point to d-wave pairing, staggered-flux order parameter becomes zero discontinuously.

\[
J_{ij}^s = i((\frac{3}{8}J_{ij}^s + tB_{ij}^s)f_{i\sigma}^\dagger f_{j\sigma} - h.c.), \tag{34}
\]

\[
J_{ij}^h = it(\bar{\chi}_{ij}^+b_j^\dagger - h.c.). \tag{35}
\]

The explicit form of the electron current in this saddle point is $J_{i+\delta j} = (-1)^{s_\delta}2t\chi\sin(\phi_e/4) = (-1)^{s_\delta}2ty_e$, $J_{i+\delta j} = (-1)^{s_\delta}2t\chi\sin(\phi_e/4) = (-1)^{s_\delta}2ty_e$. The current of the electron is small at lower doped region because it is in proportion to the $B = |\langle \hat{b}_i^\dagger b_j \rangle|$. At low doping, $B$ is proportional to $\delta$ and small. The magnetic field due to the orbital current of electron is same order with previous estimations of $10^{-3}T$ [2,25].

V. FERMI SURFACE AND EXCITATION GAP AT $(0, \pi)$

The temperature dependences of Fermi surface and excitation gap at $(0, \pi)$ are shown. These behaviors are consistent with angle-resolved photoemission spectroscopy (ARPES) experiments [7–14].

A. Fermi surface

Fermi-surface is the loci of gapless excitation in momentum space. It is defined by the zero-energy line of the electron excitation energy. The Fermi surface for $\delta = 0.02$ is shown in Fig. 4 at several temperatures. When temperature decreases, the Fermi-surface becomes smaller in the staggered-flux phase, and changes to a point-like Fermi-surface discontinuously at the transition temperature to the d-wave pairing phase. However, this discontinuity is small because Fermi surface in the staggered-flux phase has shrunk almost to a point at the transition point. The shape of Fermi-surface in the staggered-flux state of our theory is segment-like because the spectral weight has large enough value only along a part of the contour.
The contribution of holon Green function comes mainly from the bottom of the holon band. The momenta at the bottom of the boson band are (0,0) and (π,π) in the staggered flux state, and are (0,0), (π,0), (0,π), and (π,π) in the π-flux state. We can treat the bosons as if they are Bose condensed at these band minima. In this approximation, the spectral function of electron has the following form,

\[
A^e(\omega, k) = \frac{\delta}{2} \left\{ u_k^2 \delta(\omega - E_k^e) + |v_k|^2 \delta(\omega - E_k^s) \right\} + A^{in}(\omega, k),
\]

where \( u_k^2 = (1 + \epsilon_k/(\epsilon_k^2 + W_k^2)^{1/2})/2, |v_k|^2 = (1 - \epsilon_k/(\epsilon_k^2 + W_k^2)^{1/2})/2, A^{in}(\omega, k) \) is the incoherent part of the spectral function, \( E_k^e = \pm \sqrt{\epsilon_k^2 + W_k^2} - \mu \) are spectrums of spinon in the staggered-flux state. In reality, the boson excitations exist at finite temperature. They make the discrete δ-function peaks have a finite width. This effect, however, will not change the shape of the Fermi-surface. Thus, we show the zero-energy-line of spinon in the staggered-flux phase defined by \( E_k^- = -\sqrt{\epsilon_k^2 + W_k^2} - \mu = 0 \) in Fig.4. The upper band spectrum \( E_k^+ \) is always positive at finite doping where \(-\mu \geq 0\).

Although the zero-energy-line of fermion forms an ellipse, the Fermi-surface can be considered as segment-like. The reason is that the intensity of spectrum function \( \frac{\delta}{2} |v_k|^2 \) is not symmetric between in the inner region(\( |k_x| + |k_y| \leq \pi \)) and in the outer region(\( |k_x| + |k_y| \geq \pi \)). As shown in Fig.5 it is stronger in the inner region. For example, on the line \( k_x = k_y \), where \( d_{x^2-y^2} \)-gap due to staggered-flux order(\( W_k^2 \)) is always zero, the intensity is \( \frac{\delta}{2} \) at the inner side of the ellipse as \( |v_k|^2 = 1 \), but the intensity is zero at the outer side of the ellipse as \( |v_k|^2 = 0 \).

This result is quite different from the SU(2) mean-field theory, where the shape of Fermi-surface is always point-like because the particle hole symmetry(i.e.SU(2) symmetry) of fermion remains even at finite doping. As the filling of fermions are always half-filled, the staggered flux state and d-wave pairing state can be assumed to be gauge equivalent. Artificial introduction of the phenomenological interactions are needed for the fermion to have a segment-like Fermi-surface in the SU(2) theory.

Now we give the details. The electron Green function \( G^e \) is described by the product of spinon Green function \( G^s \) and holon Green function \( G^h \) [39];

\[
G^e(i\omega_n, k) = \frac{1}{N\beta} \sum_{ip}\ G^s(ip, p)G^h(ip - i\omega_n, p - k) \quad \text{(37)}
\]
The temperature dependence of the excitation gap at \((0, \pi)\) looks almost continuous at the transition point between the staggered-flux phase and the d-wave pairing phase (Fig.6), although it is a first order transition in the present mean field analysis. The energy at \((0, \pi)\) are 
\[ E_{\text{staggered}, \pm}(0, \pi) = \pm 4t y_0 + 3J y_0/2 \pm \mu \]
in the staggered flux state, and 
\[ E_{\text{d-pairing}, \pm}(0, \pi) = \pm \sqrt{\mu^2 + (3J \eta/2)^2} \]
in the d-wave RVB state, respectively.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig6.png}
\caption{The temperature dependence of the excitation gap at \((0, \pi)\) at \(\delta = 0.02\) for \(t/J = 2\). The phase changes as temperature decreases from the uniform phase to the staggered-flux phase and finally to the d-wave pairing phase.}
\end{figure}

C. Comparison with experiments

The ARPES experiments on Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ (BSCCO) [7–13] and recent one on La$_{2-x}$Sr$_x$CuO$_4$ (LSCO) [14] showed that a segment-like Fermi-surface exists near the \((\pi/2, \pi/2)\) in pseudo-gap region. [7–9]. There is the gap whose symmetry is \(d_{x^2-y^2}\). The gap is largest near \((\pi, 0)\) and does not exists on the \((0, 0)\)-\((\pi, \pi)\) line. The reason that these Fermi-surface often called segment-like, or arc, is due to the shape where the spectral weight is finite. It does not form a closed curve. When one scan on the \((0, 0)-\(\pi, \pi)\) line, the spectral weight only exists in inner region of Brillouin zone (where \(|k_x| + |k_y| \leq \pi\) [8]. Thus, this is called segment-like. When temperature decrease, the segment-like Fermi-surface become smaller in pseudo-gap phase and change to the Fermi-point in the \(d_{x^2-y^2}\)-wave superconducting state [10,13]. On the other hand, the temperature dependence of the excitation gap at \((0, \pi)\) almost looks continuous at the transition point between the pseudo-gap phase and the \(d_{x^2-y^2}\)-wave superconducting phase [13]. Our present results are consistent with these observations.

VI. SUMMARY

The competition between the staggered flux, or the d-wave, and the d-wave pairing is analyzed in two-dimensional \(t-J\) model based on U(1) slave boson mean-field theory. Not only staggered-flux of spinon but also staggered-flux of holon are considered, independently. In this formalism, the hopping order parameter of electron is described by a product of hopping order parameters of spinon and holon. The staggered-flux amplitude of the electron is a difference of staggered-flux amplitudes of spinon and holon. A phase diagram is obtained where 1) a region of the electron staggered-flux state exists, 2) the staggered flux and the d-wave pairing do not coexist, and 3) the ground state is a purely d-wave superconducting state. In \(\pi\)-phase flux of spinon, staggered-fluxes of spinon and holon cancel completely and the staggered-flux order of the electron does not exist. However, in the staggered-flux phase of spinon whose staggered-flux is not \(\pi\), fluxes do not cancel completely and staggered-flux order of electron exist. Thus, the phase transition between these two phases, \(\pi\)-flux phase and staggered-flux phase of spinon, becomes a second order transition in electron picture. The order parameter that characterizes this transition is the staggered-flux order parameter of the electron. The relation between the staggered current of electron and that of spinon is provided. The local cancellation of spinon current and holon current does not exclude the possibility of the electron staggered current. It is proved analytically that the staggered-flux and the d-wave pairing does not coexist except at half-filling. The condition for coexistence of staggered-flux and d-wave pairing as a saddle-point solution is provided. The instability of these two states, staggered flux and d-wave pairing, are discussed. The temperature dependence of following two quantities, Fermi surface and the excitation gap at \((0, \pi)\), are shown. The behaviors are consistent with ARPES experiments. When temperature decrease, the segments-like Fermi-surface in the staggered flux phase becomes smaller and changes to a point at the transition temperature to the d-wave pairing phase. The temperature dependence of the excitation gap at \((0, \pi)\) looks continuous at the transition point between the staggered-flux phase and the d-wave pairing phase although these two phases are qualitatively different.

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8
APPENDIX: COEXISTENCE CONDITION AND INSTABILITY

It is proved analytically that the staggered flux and the d-wave pairing coexist only at half-filling. The instability of these two states, staggered flux and d-wave pairing, are also discussed.

From self-consistent equations (27)/(28)/(30), following equation is derived. It must be satisfied for any states whose free energy is at saddle point.

\[ 0 = y_s \eta \left[ -(2t)^2 C(A + B) + \frac{3J}{4} B \right], \]  \hspace{1cm} (A1)

where the quantities, A, B, and C, are given by,

\[ A = \frac{1}{N} \sum_{k,s} \gamma_{k,s}^2 \tan \left( \frac{\beta E_{ks}/2}{E_{ks}} \right), \]  \hspace{1cm} (A2)

\[ B = -\mu \frac{1}{N} \sum_{k,s} \frac{\gamma_{k,s}^2}{2(2\epsilon_k + W_{k,s}^2)^{1/2}} \tan \left( \frac{\beta E_{ks}/2}{E_{ks}} \right), \]  \hspace{1cm} (A3)

\[ C = \frac{1}{N} \sum_{k,s} \frac{\gamma_{k,s}^2}{2(2\epsilon_k + W_{k,s}^2)^{1/2}} \left( -s \right), \]  \hspace{1cm} (A4)

where \( \gamma_{k,s} = \cos k_x \cos k_y \), \( A > 0 \), \( B \leq 0 \), \( C \leq 0 \). Here, the quantity B is different from \( \mu = 0 \) and is negative at finite doping. C is zero at half-filling or when all holon condense at \((0,0)\) or \((\pi,\pi)\), and is negative when excited holon exists or when holon condense at \((\pi,0)\) or \((0,\pi)\) which is the situation of \( \pi \)-flux state. If the staggered flux and the d-wave pairing coexist, \( y_s \eta \) is non-zero. Then in order for the coexistence, eq.(A1) requires that the quantity in the square bracket must vanish. At half-filling, \( B = C = 0 \), so the condition is satisfied. At finite doping \( B < 0 \), so if \( C = 0 \) there is no coexistence. When \( C \neq 0 \), we cannot expect the quantity in the square bracket vanish except at a special point in the phase space. However, from the continuity of the order parameters, it is impossible to have both \( y_s \) and \( \eta \) nonzero only at a single point.

The curvature of free energy at the saddle points of pure staggered-flux state and pure d-wave pairing state have the following forms respectively,

\[ \frac{\partial^2 F}{\partial y_s^2} \mid_{y_s \neq 0, y_s = 0} = 2N\left\{ \frac{3J}{4} B + (2t)^2 C \right\}, \]  \hspace{1cm} (A5)

\[ \frac{\partial^2 F}{\partial y^2} \mid_{y \neq 0, y = 0} = 2N\left\{ \frac{3J}{4} B + \frac{3J}{4} \left( 1 - \frac{1}{1 - \frac{4(2t)^2}{4J - C}} \right) \right\}. \]  \hspace{1cm} (A6)

The stability of the states depends on \( y_s \) and \( C \). There are three cases: (i) The case that \( B \) and \( C \) are both zero: This is realized at half-filling, and \( (\partial^2 F/\partial y_s^2) \mid_{y_s \neq 0, y_s = 0} = (\partial^2 F/\partial y^2) \mid_{y \neq 0, y = 0} = 0 \). We need not discuss this case. (ii) The case that \( B \) is negative and \( C \) is zero (this is the same situation that Zhang discussed [40]): In this case, pure d-wave pairing are always stable, and pure staggered-flux state is unstable against infinitesimal d-wave pairing. The curvature at each state are \( (\partial^2 F/\partial y_s^2) \mid_{y_s \neq 0, y_s = 0} = -2N(3J/4)^2B > 0 \) and \( (\partial^2 F/\partial y^2) \mid_{y \neq 0, y = 0} = 2N(3J/4)^2C < 0 \). (iii) The case that \( B \) and \( C \) are both negative: A region where pure d-density wave is stable and a region where d-wave pairing is stable can both exist.

The situation where \( B \) is zero and \( C \) is negative does not occur. \( B \) is zero only at half-filling where \( C \) is zero.

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To be more precise, these identities are not only identities between expectation values but also identities between operators, the electron and the spinon, which are fixed when a direction in the (\(\theta_{ij}\), \(\gamma_{ij}\)) space is chosen. However, above the Bose-condensation temperature of holon where \(z_i = z_{i1} = z_{i2} = 0\), above discussions cannot be used. Boson operators cannot be treated as c-number. Unfortunately, the staggered flux phase exists in this region.

Many choices of the \(\chi_{ij}\) and the \(B_{ij}\) are equivalent. These choices of the form of the \(\chi_{ij}\) and the \(B_{ij}\) corresponds to a gauge fixing in the staggered-flux state. The only gauge invariant property of \(\chi_{ij}\) and \(B_{ij}\) is its flux through a plaquette. If dimerized phase is excluded, these auxiliary fields, \(\chi_{ij}\) and \(B_{ij}\) can parameterized by their amplitude and phase, \(\chi_{ij} = e^{i\theta_{ij}}\) and \(B_{ij} = e^{i\phi_{ij}}\). The flux through a plaquette of spinon(holon) is \(\theta_{ij} + \theta_{jk} + \theta_{ki} + \theta_{li}\). In the U(1) slave boson method, electron operator \(c^{\dagger}_{\alpha} = f^{\dagger}_{\alpha}b^{\dagger}_{ij}\) is invariant under the internal local U(1) transformation, \(f^{\dagger}_{\alpha} \rightarrow f^{\dagger}_{\alpha}e^{i\phi_{\alpha}}\) and \(b_{ij} \rightarrow b_{ij}e^{i\phi_{ij}}\). The action is also invariant under this transformation. Then auxiliary fields also transform as following, \(\chi e^{i\theta_{ij}} \rightarrow \chi e^{i(\theta_{ij} + \theta_{ij} - \theta_{ij})}\), and \(B e^{i\phi_{ij}} \rightarrow B e^{i(\phi_{ij} + \phi_{ij} - \phi_{ij})}\). However, the flux through a plaquette of \(\chi\) and that of \(B\) are gauge invariant. Of course, the hopping of the electron, \(c_{\alpha}^{\dagger}c_{\alpha} = \chi_{ij}B_{ij}\), is also gauge invariant quantity, \(\chi Be^{i(\theta_{ij} - \theta_{ij})} \rightarrow \chi Be^{i(\theta_{ij} - \phi_{ij})}e^{i(\theta_{ij} - \theta_{ij}) - (\theta_{ij} - \theta_{ij})} = \chi Be^{i(\theta_{ij} - \phi_{ij})}\).

The order parameters \(y_{\alpha}\) and \(y_{\alpha}\) are also represented in the following forms, respectively, \(y_{\alpha} = -i\sum_{k}^{1/2} \langle \cos k_{\alpha} - \cos k_{\alpha} \rangle f_{\alpha}^{\dagger}f_{\alpha}Q_{\alpha}\) and \(y_{\alpha} = -i\sum_{k}^{1/2} \langle \cos k_{\alpha} - \cos k_{\alpha} \rangle (b_{11}b_{k+Q})\). These forms show that these order parameters represent a density wave whose symmetry is \(d_{x^2-y^2}\).

The staggered \(\pi\)-flux state is equivalent to the uniform \(\pi\)-flux state because these states transform each other by a shift of the 2\(\pi\)-flux. The 2\(\pi\)-flux does not change physics.