KINEMATIC SELF-SIMILARITY

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Abstract. Self-similarity in general relativity is briefly reviewed and the differences between self-similarity of the first kind (which can be obtained from dimensional considerations and is invariantly characterized by the existence of a homothetic vector in perfect fluid spacetimes) and generalized self-similarity are discussed. The covariant notion of a kinematic self-similarity in the context of relativistic fluid mechanics is defined. It has been argued that kinematic self-similarity is an appropriate generalization of homothety and is the natural relativistic counterpart of self-similarity of the more general second (and zeroth) kind. Various mathematical and physical properties of spacetimes admitting a kinematic self-similarity are discussed. The governing equations for perfect fluid cosmological models are introduced and a set of integrability conditions for the existence of a proper kinematic self-similarity in these models is derived. Exact solutions of the irrotational perfect fluid Einstein field equations admitting a kinematic self-similarity are then sought in a number of special cases, and it is found that: (1) in the geodesic case the 3-spaces orthogonal to the fluid velocity vector are necessarily Ricci-flat and (ii) in the further specialization to dust (i.e., zero pressure) the differential equation governing the expansion can be completely integrated and the asymptotic properties of these solutions can be determined, (iii) the solutions in the case of zero-expansion consist of a class of shear-free and static models and a class of stiff perfect fluid (and non-static) models, and (iv) solutions in which the kinematic self-similarity is parallel to the velocity vector are necessarily Friedmann-Robertson-Walker (FRW) models. Solutions in which the kinematic self-similarity is orthogonal to the velocity vector are also considered. In addition, the existence of kinematic self-similarities in FRW spacetimes is comprehensively studied. It is known that there are a variety of circumstances in general relativity in which self-similar models act as asymptotic states of more general models. Finally, the questions of under what conditions are models which admit a proper kinematic self-similarity asymptotic to an exact homothetic solution and under what conditions are the asymptotic states of cosmological models represented by exact solutions of Einstein’s field equations which admit a generalized self-similarity are addressed.

1. Introduction

Self-similar solutions were originally of interest since the governing equations of a given problem simplify and often systems of partial differential equations (PDEs) reduce to ordinary differential equations. Indeed, self-similarity in the broadest (Lie) sense refers to an invariance which simply allows the reduction of a system of PDEs. Self-similarity refers to the fact that the spatial distribution of the characteristics of motion remains similar to itself at all times during the motion and self-similar solutions represent solutions of degenerate problems in which all dimensional constant parameters entering the initial and boundary conditions vanish or become infinite (Barenblatt and Zeldovich, 1972). Indeed, such solutions describe the “intermediate-asymptotic” behaviour of solutions in the region in which a solution no longer depends on the details of the initial and/or boundary conditions. Cases in which the form of the self-similar asymptotes can be obtained from dimensional considerations are referred to as self-similar solutions of the first kind (Barenblatt and Zeldovich, 1972).

Similarity solutions are of importance within general relativity. For example, a strong explosion in a homogeneous background produces fluctuations which may be very complicated initially, but they...
tend to be described more and more closely by a spherically symmetric similarity solution as time evolves (Sedov, 1967), and this applies even if the explosion occurs in an expanding cosmological background (Schwartz et al., 1975; Ikeuchi et al., 1983). The evolution of voids is also described by similarity solutions at late times (Bertschinger, 1985). In addition, the expansion of the universe from the big bang and the collapse of a star to a singularity might (both) exhibit self-similarity in some form since it might be expected that the initial conditions ‘are forgotten’ in some sense.

In this paper we shall assume that the source of the gravitational field is that of a perfect fluid; i.e., the energy-momentum tensor is given by

$$T_{ab} = (\mu + p)u_a u_b + pg_{ab},$$  \hspace{1cm} (1.1)

where \( u^a \) is the normalized fluid 4-velocity and \( \mu \) and \( p \) are, respectively, the density and pressure. In natural units \( c = 8\pi G = 1 \), the Einstein field equations (EFEs) of general relativity then read

$$G_{ab} = T_{ab},$$  \hspace{1cm} (1.2)

where \( G_{ab} \) is the Einstein tensor.

Similarity solutions were first studied in general relativity by Cahill and Taub (1971), who did so in the cosmological context and under the assumption of a spherically symmetric distribution of a self-gravitating perfect fluid. They assumed that the solution was such that the dependent variables are essentially functions of a single independent variable constructed as a dimensionless combination of the independent variables and that the model contains no other dimensional constants. Cahill and Taub (1971) showed that the existence of a similarity of the first kind in this situation could be invariantly formulated in terms of the existence of a homothetic vector. A proper homothetic vector (HV) is a vector field \( \xi \) which satisfies (after a constant rescaling)

$$\mathcal{L}_\xi g_{ab} = 2 g_{ab},$$  \hspace{1cm} (1.3)

where \( g_{ab} \) is the metric and \( \mathcal{L} \) denotes Lie differentiation along \( \xi \).

It follows from equation (1.3) that

$$\mathcal{L}_\xi G_{ab} = 0.$$  \hspace{1cm} (1.4)

When the source of the gravitational field is a perfect fluid, in the case of self-similarity of the first kind it follows from dimensional considerations that the physical quantities transform according to

$$\mathcal{L}_\xi u^a = -u^a,$$  \hspace{1cm} (1.5)

and

$$\mathcal{L}_\xi \mu = -2\mu, \quad \mathcal{L}_\xi p = -2p.$$  \hspace{1cm} (1.6)

From these equations it follows that

$$\mathcal{L}_\xi T_{ab} = 0,$$  \hspace{1cm} (1.7)

which is therefore consistent with the EFEs (1.2). Indeed, in the case of a perfect fluid it follows that equations (1.5) and (1.6) result from equations (1.3) [through eqns. (1.2), (1.4) and (1.7)] so that the physical quantities transform appropriately (Cahill and Taub, 1971; Eardley, 1974). Hence in this case “geometric” self-similarity and “physical” self-similarity coincide. However, this need not be the case (see Coley, 1996, for details). The properties of the matter and those of the geometry are related through the EFEs, and in general there will be further constraints arising from the compatibility of the EFEs and the conditions of self-similarity (“integrability” conditions). We shall investigate these integrability conditions later.

A. Self-similarity of the second kind.

The existence of self-similar solutions of the first type is related to the conservation laws and to the invariance of the problem with respect to the group of similarity transformations of quantities with independent dimensions, in which case a certain regularity of the limiting process in passing from the original non-self-similar regime to the self-similar regime is implicitly assumed. However, in general such a passage to this limit need not be regular, whence the expressions for the self-similar variables are not determined from dimensional analysis of the problem alone. Solutions
are then called self-similar solutions of the second type. Characteristic of these solutions is that they contain dimensional constants that are not determined from the conservation laws (but can be found by matching the self-similar solutions with the non-self-similar solutions whose asymptotes they represent) (Barenblatt and Zeldovich, 1972).

Self-similarity in the broadest sense refers to the general situation in which a system is not restricted to be strictly invariant under the relevant group action, but merely to be appropriately rescaled. The basic condition characterizing a manifold vector field $\xi$ as a self-similar generator (Carter and Henriksen, 1991) is that there exist constants $d_i$ such that for each independent physical field $\Phi^i_A$,

$$L_\xi \Phi^i_A = d_i \Phi^i_A, \quad (1.8)$$

where the fields $\Phi^i_A$ can be scalar (e.g., $\mu$), vectorial (e.g., $u_a$) or tensorial (e.g., $g_{ab}$). In general relativity the gravitational field is represented by the metric tensor $g_{ab}$, and an appropriate definition of "geometrical" self-similarity is necessary. In the seminal work by Cahill and Taub (1971) the simplest generalization was effected whereby the metric itself satisfies an equation of the form (1.8), namely $\xi$ is a HV, this evidently corresponding to Zeldovich’s similarity of the first kind.

However, in relativity it is not the energy-momentum tensor itself that must satisfy (1.8), but each of the physical fields making up the energy-momentum tensor must separately satisfy an equation of the form (1.8). In the case of a fluid characterized by the timelike congruence $u_a$ the energy-momentum tensor can be uniquely decomposed with respect to $u_a$ (Ellis, 1971), and each of these uniquely defined components (each of which has a physical interpretation in terms of the energy, pressure, heat flow and anisotropic stress as measured by an observer comoving with the fluid) must separately satisfy an equation of the form (1.8). In the same way, if the metric can be uniquely, physically, and covariantly decomposed then the homothetic condition can be replaced by the conditions that each uniquely defined component must satisfy (1.8), maintaining self-similarity. For example, in the case of a fluid, the metric can be decomposed uniquely in terms of $u_a$, through the projection tensor

$$h_{ab} = g_{ab} + u_a u_b, \quad (1.9)$$

into parts $h_{ab}$ and (minus) $u_a u_b$. The projection tensor represents the projection of the metric into the 3-spaces orthogonal to $u^a$ (i.e., into the rest frame of the comoving observers), and if $u_a$ is irrotational these 3-spaces are surface forming, the decomposition is global, and $h_{ab}$ represents the intrinsic metric of these 3-spaces. $h_{ab}$ is the first fundamental form of the hypersurfaces orthogonal to $u^a$ and can be regarded as the relativistic counterpart of the Newtonian metric tensor, when the flow independent $u_a$ is defined as the relativistic counterpart of the preferred (irrotational) Newtonian time covector $-t^{,a}$ (Carter and Henriksen, 1989).

B. Kinematic self-similarity.

It is arguments similar to these, and, more importantly, a detailed comparison with self-similarity in a continuous Newtonian medium, that has led Carter and Henriksen (1989) to the covariant notion of kinematic self-similarity in the context of relativistic fluid mechanics. A kinematic self-similarity vector $\xi$ satisfies the condition

$$L_\xi u_a = \alpha u_a, \quad (1.10)$$

where $\alpha$ is a constant (i.e., $\xi$ is a continuous kinematic self-similar generator with respect to the flow $u_a$). Furthermore,

$$L_\xi h_{ab} = 2 h_{ab} \quad (1.11)$$

($\xi$ has been normalized so that the constant in (1.11) has been set to unity). Evidently, in the case $\alpha = 1$ it follows that $\xi$ is a HV (Cahill and Taub, 1971), corresponding to self-similarity of the first kind (Barrenblatt and Zeldovich, 1972). Carter and Henriksen (1989) then argue that the case $\alpha \neq 1$ ($\alpha = 0$) is the natural relativistic counterpart of self-similarity of the more general second kind (zeroth kind).

The parameter $\alpha$ represents the constant relative proportionality factor governing the rates of dilation of spatial length scale and amplification of time scale. Evidently, when $\alpha \neq 1$ (i.e., $\xi$ is not a HV), the relative rescaling of space and time (under $\xi$) are not the same (and in the zeroth case there is a space dilation without any time amplification).
C. Remarks.

1. All of the fields must satisfy equations of the form (1.8). However, not all of the \(d_i\) need be independent. In practice it is convenient to assume independence and then determine the constraints on the \(d_i\) (in addition to the constraints on the form of the solutions) that arise from the imposition of the equations of state and the EFEs. In the case of a perfect fluid we have that, in addition to the conditions of “kinematic” self-similarity and “geometric” self-similarity, as represented by equations (1.10) and (1.11), respectively, the further conditions of “physical” self-similarity, given by

\[
L_{ξμ} = aμ, \quad L_{ξν} = bp, \quad (1.12)
\]

must be satisfied, where \(a\) and \(b\) are constants. That is, in order for a perfect fluid spacetime to admit a proper kinematic self-similarity all of the conditions represented by equations (1.10)–(1.12) must be satisfied.

2. Additional constraints may arise from the imposition of an equation of state. The least restrictive case arises from the exceptional pressure-free case. Here \(p = 0\), whence

\[
L_{ξ} T_{ab} = (2α + a)T_{ab}, \quad (1.13)
\]

and no further restrictions apply. Usually a linear barotropic equation of state of the form

\[
p = (γ - 1)μ, \quad (1.14)
\]

is assumed, where the constant \(γ\) obeys \(1 \leq γ \leq 2\) for ordinary matter. [The case \(0 \leq γ < 2/3\) is of interest in studying models that undergo inflation.] If a barotropic equation of state \(p = p(μ)\) is assumed, it is well known that if a spacetime admits a non-trivial HV then equation (1.14) necessarily results (Cahill and Taub, 1971). If, on the other hand, a spacetime admits a non-trivial kinematic self-similarity then (except in the special case of dust) the polytropic equation of state \(p = po μ^{γ}\) follows from equations (1.12) (where \(γ\) is the polytropic index and \(γ = b/a\)).

We note from equation (1.13) that in the case of dust the total energy-momentum tensor \(T_{ab}\) itself trivially satisfies an equation of the form (1.8). When \(p \neq 0\), this is only possible in the special case that \(b = a + 2(α - 1)\).

3. The differential geometric properties of HV were studied by Yano (1955). The totality of HV on a spacetime form a Lie algebra \(H_n\) (of dimension \(n\)) which contains (if \(H_n\) is non-trivial) an \((n - 1)\) dimensional (sub)algebra of KV, \(G_{n-1}\). If a given spacetime is not conformal to an exceptional ‘plane wave spacetime’, it follows that if the orbits of \(H_n\) are \(r\)-dimensional, then the orbits of \(G_{n-1}\) are \((r - 1)\)-dimensional (Eardley, 1974), and if a spacetime is not conformally flat, the spacetime is conformally related to a spacetime for which the Lie algebra \(H_n\) is the Lie algebra of KV (Defrise-Carter, 1975).

The totality of kinematic self-similar vector fields on a spacetime also form a Lie algebra (Carter and Henriksen, 1991) which we shall denote here by \(K\). Now, \(H \subseteq K\), but \(K\) neither contains nor is contained within either the conformal algebra \(C\) in general or the inheriting algebra \(I\) (Castejon-Amenedo and Coley, 1992) in particular.

Let us suppose that \(ξ_1\) and \(ξ_2 \in K\), so that

\[
L_{ξ_1} h_{ab} = 2h_{ab}, \quad L_{ξ_1} u_a = α_1 u_a, \\
L_{ξ_2} h_{ab} = 2h_{ab}, \quad L_{ξ_2} u_a = α_2 u_a,
\]

where \(ξ_1\) and \(ξ_2\) are linearly independent and \(α_1 \neq 1\) and \(α_2 \neq 1\) (i.e., neither are HV). Then, by defining

\[
ξ = cξ_1 + dξ_2, \quad (1.15)
\]

where \(c\) and \(d\) are non-zero constants, we find that

\[
L_{ξ} h_{ab} = 2(c + d)h_{ab} \quad (1.16)
\]

and

\[
L_{ξ} u_a = (cα_1 + dα_2)u_a. \quad (1.17)
\]
[We note that equations of the form (1.12) are trivially satisfied for $\xi$ given by (1.15)]. Now, if $\alpha_1 \neq \alpha_2$ (where either $\alpha_1$ or $\alpha_2$ may be zero), then we can always choose $d = 1 - c$ and $c = (1 - \alpha_2)/(\alpha_1 - \alpha_2)$ so that $\mathcal{L}_\xi h_{ab} = 2h_{ab}$ and $\mathcal{L}_\xi u_a = u_a$; that is, $\xi$ is a HV. If, on the other hand, $\alpha_1 = \alpha_2$ (possibly zero), then we can choose $c + d = 0$, whence $\mathcal{L}_\xi h_{ab} = 0$ and $\mathcal{L}_\xi u_a = 0$ so that $\xi$ is, in fact, a KV. Therefore, we have shown that each non-trivial $K_n(n > 1)$ contains an $(n - 1)$-dimensional (sub)algebra of HV, $H_{n-1}$, where $H_{n-1}$ may be trivial (i.e., $H_{n-1}$ need not contain any proper HV). An illustration of this result can be found in section 4.C.

4. In the case of radiation (i.e., $p = \frac{\alpha}{3} \mu$, so that $T = 0$) the existence of a HV implies the existence of a conserved quantity. In general, if we define the current $P^a = T^{ab} \xi_b$, then from energy-momentum conservation, $P^a_{\alpha} = T^{ab} g_{ab} = T$. For radiation, $P^a = \frac{4}{3}(4u_a [u_b b] + \xi^a)$, and $P^a_{\alpha} = 0$. In the case of kinematic self-similarity it follows from equations (1.10) and (1.11) that $P^a_{\alpha} = 3\mu - \alpha \mu$, implying the existence of a conserved quantity for fluids with an equation of state $p = \frac{\alpha}{3} \mu$. In particular, in the case of kinematic self-similarity of the zeroth kind (i.e., $\alpha = 0$), there exists a conserved quantity in the special case of dust (i.e., $p = 0$).

5. Finally, there is another case of potential interest in which the ratio $\alpha/1$ (where $\alpha$ is defined in (1.10) and we recall that the constant in (1.11) has been normalized to unity) approaches infinity, and we could refer to this case as kinematic self-similarity of ‘infinite’ kind. This case could be covariantly defined by equation (1.10) (in which $\alpha$ could be normalized to unity) and equation (1.11) with zero right-hand side (i.e., $\mathcal{L}_\xi h_{ab} = 0$, and $\xi$ consequently represents a generalized “rigid motion”. This case will be investigated elsewhere.

In section 2 we shall describe the cosmological models under investigation and introduce their governing equations. We shall then obtain the set of integrability conditions for the existence of a kinematic self-similarity in the models. Finally, we shall deduce the equations in the relevant case of zero vorticity. In sections 3 and 4 we shall study the models in a number of special cases; namely, in the case of zero acceleration (3.A) and the further subcase of zero pressure (3.B), the case of zero expansion (which includes the special subcase of static models) (3.C), the cases in which the kinematic self-similarity is either parallel to or orthogonal to the fluid velocity vector (4.A and 4.B, respectively) and finally we shall investigate the existence of kinematic self-similar vectors in FRW spacetimes. In section 5 we shall summarize the main results and we shall discuss further avenues of research.

2. Analysis

A. The governing equations.

The covariant derivative of $u_a$ can be decomposed according to (Ellis, 1971)†

$$u_{a;b} = \sigma_{ab} + \frac{1}{3} \theta h_{ab} + \omega_{ab} - \dot{u}_a u_b,$$

(2.1)

where $\theta_{ab} \equiv h_{(a} \theta_{b)} u_{c;d}, \theta \equiv g^{ab} \theta_{ab}$ is the expansion, $\sigma_{ab} = \theta_{ab} - \frac{1}{3} \theta h_{ab}$ is the shear tensor and $\sigma^2 \equiv \frac{1}{2} \sigma_{ab} \sigma^{ab}$, $\omega_{ab} \equiv h_{[a} \omega_{b]} u_{c;d}$ is the vorticity tensor and $\omega^2 \equiv \frac{1}{2} \omega_{ab} \omega^{ab}$, and $\ddot{u}_a \equiv u_{a;b} u^b$ is the acceleration.† Using these definitions the governing equations can be written down (Ellis, 1971).

The conservation laws, in the case of a perfect fluid, become

$$\dot{\mu} + (\mu + p) \theta = 0,$$

(2.2)

$$\mu + p u_a = -p h_{a;b}.$$

(2.3)

where $\dot{\mu} \equiv \mu \dot{u}^a$ (for example). Equations (4.12), (4.15), (4.17), (4.18), (4.16) and (4.19) in Ellis

†We shall follow the notation and conventions in Ellis (1971); in particular, Roman indices range from 0 to 3 and Greek indices from 1 to 3.
(1971), become, respectively,

\[
\dot{\theta} + \frac{1}{3} \theta^2 - \dot{\omega}^a_{,a} + 2(\sigma^2 - \omega^2) + \frac{1}{2}(\mu + 3\rho) = 0,
\]

(2.4)

\[
h^a_b \left( \frac{(L^2 \omega)^a}{L^2} \right) = h^a_b \left( \omega^b + \frac{2}{3} \theta \omega^b \right) = \sigma^a_b \omega^b + \frac{1}{2} \eta^{abcd} u_b \dot{u}_{c,d},
\]

(2.5)

\[
h^c_b \left( \omega^c_{,c} - \sigma^c_{,c} + \frac{2}{3} \theta \right) + (\omega^c_b + \sigma^c_b) \dot{u}^b = 0,
\]

(2.6)

\[
\omega^a_{,a} = 2 \omega^b \dot{u}_b,
\]

(2.7)

\[
h^f_g \sigma_{fg} - h^g_f h^b_g \dot{u}_{(f:g)} - \dot{u}_a u_b + \omega_a \omega_b + \sigma_a \sigma_b + \eta^f_{ab} \mu + \sigma^f_{ab} + \frac{3}{2} \theta \sigma_{ab} + \frac{4}{3} \theta \omega^2 - \frac{2}{3} \omega^2 + \frac{1}{3} \theta^2 + 2 \mu = 0,
\]

(2.8)

where \( E_{ab} \) and \( H_{ab} \) are, respectively, the electric and magnetic parts of the Weyl tensor.\(^{\dagger}\)

The remaining equations are the Bianchi identities (cf. Ellis, 1971, eqns. (4.21a - d)) and the field equations (cf. Ellis, 1971, eqns (4.23) - (4.26)). In particular, in the case of zero vorticity (\( \omega = 0 \)) the Gauss-Codacci equations become

\[
3R_{ab} = (\dot{\omega}^f_{ab} - \dot{\sigma}^f_{ab} + \dot{u}^a_{(f:g)\cdot h}^g) h^f_g - \dot{u}_a \dot{u}_b + \frac{1}{3} \frac{\theta^2}{\sigma^2} + 2 \mu - \dot{\omega}^c_{,c},
\]

(2.10)

and

\[
3R = \frac{2}{3} \theta^2 + 2 \sigma^2 + 2 \mu,
\]

(2.11)

where \( 3R_{ab} \) is the Ricci tensor of the 3-spaces orthogonal to \( u^a \) and \( 3R \) is the corresponding Ricci scalar.

**B. Integrability Conditions.**

From equations (1.10) and (1.11), the existence of a kinematic self-similar vector implies that

\[
\mathcal{L}_\xi g_{ab} = 2g_{ab} + 2(1 - \alpha)u_a u_b,
\]

(2.12)

where \( \alpha \neq 1 \) is assumed hereafter. From Yano (1955) we have that

\[
\mathcal{L}_\xi R_{ab} = (\mathcal{L}_\xi \Gamma^c_{ab})_{,c} - (\mathcal{L}_\xi \Gamma^c_{ac})_{,b}
\]

(2.13)

where

\[
\mathcal{L}_\xi \Gamma^d_{bc} = \frac{1}{2} g^{dl} [(\mathcal{L}_\xi g_{dl})_{,c} + (\mathcal{L}_\xi g_{lc})_{,b} - (\mathcal{L}_\xi g_{bc})_{,l}].
\]

(2.14)

Hence we have that

\[
\mathcal{L}_\xi R_{ab} = (1 - \alpha) g^{cf} [(u_b,_{a},_{c} + (u_a)_{,c};_{b} - (u_b)_{,a};_{c}].
\]

(2.15)

Therefore, using (2.1), after a long calculation we obtain

\[
\frac{1}{1 - \alpha} \mathcal{L}_\xi R_{ab} = 2\sigma_{ab} + 2\theta \sigma_{ab} + 2\sigma_{bc} \omega^c_{,a} + 2\sigma_{ac} \omega^c_{,b}
\]

\[
+ 4 \omega_{ac} \omega^c_{,b} + \frac{2}{3} \theta g_{ab} [\dot{\theta} + \theta] + u_a [2 \omega_{cb} \omega^c_{,b} - 2 \sigma_{bc} \dot{u}^c - 2 \omega_{bc} \dot{u}^c]
\]

\[
+ u_b [2 \omega_{ca} \omega^c_{,b} - 2 \sigma_{ac} \dot{u}^c - 2 \omega_{ac} \dot{u}^c] + u_a u_b \left[ \frac{2}{3} \theta (\dot{\theta} + \theta) - 2 \dot{u}_c \omega^c_{,b} \right].
\]

(2.16)
Thus, decomposing equation (2.16) using $u^a$, we obtain

\[
\frac{1}{(1 - \alpha)} [\mathcal{L}_\xi R_{ab}] u^a u^b = -8\omega^2 - 2\dot{u}_c^{;c},
\]

(2.17)

\[
\frac{1}{(1 - \alpha)} [\mathcal{L}_\xi R_{ab}] h^{ab} = 2(\dot{\theta} + \theta^2 - 4\omega^2),
\]

(2.18)

\[
\frac{1}{(1 - \alpha)} [\mathcal{L}_\xi R_{ab}] u^a h^b = 2\omega_{da} u^a + 2\omega_{dc}^{;c} - 4\omega^2 u_d,
\]

(2.19)

\[
\frac{1}{(1 - \alpha)} [\mathcal{L}_\xi R_{ab}] h^a h^b = \frac{1}{3} h_{ef} h^{ab} = 2[\dot{\sigma}_{ef} - u_f \sigma_{eb} u^b - u_e \sigma_{fb} u^b
\]

\[+ \theta \sigma_{ef} + \sigma_{fe} e^e_e + \sigma_{ee} \omega^e_f + 2\omega_e^e \omega_c f + \frac{4}{3} h_{ef} \omega^2].
\]

(2.20)

In the case of a perfect fluid, from the EFEs we have that

\[
R_{ab} = \frac{1}{2} (\mu + 3p) u_a u_b + \frac{1}{2} (\mu - p) h_{ab},
\]

(2.21)

whence, using equations (1.10) and (1.11), and (1.12), we obtain

\[
\mathcal{L}_\xi R_{ab} = \frac{1}{2} \{(a + 2\alpha)\mu + 3(b + 2\alpha)p\} u_a u_b
\]

\[+ \frac{1}{2} \{(a + 2)\mu - (b + 2)p\} h_{ab}.
\]

(2.22)

Decomposing equation (2.22) yields

\[
\frac{1}{(1 - \alpha)} [\mathcal{L}_\xi R_{ab}] u^a u^b = \left[ \frac{a + 2\alpha}{2(1 - \alpha)} \right] \mu + \left[ \frac{3b + 2\alpha}{2(1 - \alpha)} \right] p,
\]

(2.23)

\[
\frac{1}{(1 - \alpha)} [\mathcal{L}_\xi R_{ab}] h^{ab} = \left[ \frac{3(a + 2)}{2(1 - \alpha)} \right] \mu + \left[ \frac{-3(b + 2)}{2(1 - \alpha)} \right] p,
\]

(2.24)

\[
[\mathcal{L}_\xi R_{ab}] u^a h^b = 0,
\]

(2.25)

\[
[\mathcal{L}_\xi R_{ab}] \left( h^a h^b - \frac{1}{3} h_{ef} h^{ab} \right) = 0.
\]

(2.26)

Hence, the integrability conditions for the existence of a proper kinematic self-similarity become

\[
-8\omega^2 - 2\dot{u}_c^{;c} = \left[ \frac{a + 2\alpha}{2(1 - \alpha)} \right] \mu + \left[ \frac{3b + 2\alpha}{2(1 - \alpha)} \right] p,
\]

(2.27)

\[
2(\dot{\theta} + \theta^2 - 4\omega^2) = \left[ \frac{3(a + 2)}{2(1 - \alpha)} \right] \mu + \left[ \frac{-3(b + 2)}{2(1 - \alpha)} \right] p,
\]

(2.28)

\[
2\omega_{da} \dot{u}^a + 2\omega_{dc}^{;c} - 4\omega^2 u_d = 0,
\]

(2.29)

and

\[
\dot{\sigma}_{ef} - u_f \sigma_{eb} u^b - u_e \sigma_{fb} u^b + \theta \sigma_{ef} + \sigma_{fe} e^e_e + \sigma_{ee} \omega^e_f + 2\omega_e^e \omega_c f + \frac{4}{3} h_{ef} \omega^2 = 0.
\]

(2.30)

Equations (2.27)-(2.30) must be satisfied in addition to equations (2.2)-(2.9).
C. Vorticity-free case.

In the Introduction we discussed the analogy with self-similarity in the Newtonian case where the irrotational case was singled out to be of special importance. Setting \( \omega = 0 \) in the above equations we obtain:

\[
\dot{\mu} + (\mu + p)\theta = 0, \tag{2.31}
\]

\[
(\mu + p)\dot{u}_a = -p_h \delta^b_a, \tag{2.32}
\]

\[
\dot{\theta} + \frac{1}{3}\theta^2 - \dot{u}^a_{\alpha} + 2\sigma^2 + \frac{1}{2}(\mu + 3p) = 0, \tag{2.33}
\]

and

\[
\left(-\sigma^{bc}_{\alpha c} + \frac{2}{3}\theta^b\right) h^e_b + \sigma^e_b \dot{u}^b = 0. \tag{2.34}
\]

The integrability conditions (2.27)-(2.30) become

\[
-2\dot{u}^c_{\alpha c} = \left[\frac{a + 2\alpha}{2(1 - \alpha)}\right]\mu + \left[\frac{3(b + 2\alpha)}{2(1 - \alpha)}\right] p, \tag{2.35}
\]

\[
2(\dot{\theta} + \theta^2) = \left[\frac{3(a + 2)}{2(1 - \alpha)}\right]\mu + \left[\frac{-3(b + 2)}{2(1 - \alpha)}\right] p, \tag{2.36}
\]

and

\[
\dot{\sigma}_{ef} - u_f \sigma_{eb} \dot{u}^b - u_e \sigma_{fb} \dot{u}^b + \theta \sigma_{ef} = 0. \tag{2.37}
\]

Finally, equations (2.10) and (2.11) must be satisfied. The remaining non-trivial equations are equations (2.8) and (2.9) which serve to define \( E_{ab} \) and \( H_{ab} \), respectively. We note that from equations (2.8), (2.10), (2.11) and (2.37) we have that

\[
3R_{ab} - \frac{1}{3}h_{ab} 3R = E_{ab} + \sigma_{af} \sigma_{fb} - \frac{1}{3}\theta \sigma_{ab} - \frac{2}{3}\theta^2 h_{ab}. \tag{2.38}
\]

Now, contracting equation (2.37), we obtain

\[
\sigma[\dot{\sigma} + \theta \sigma] = 0. \tag{2.39}
\]

Using (2.35), equation (2.33) becomes

\[
\dot{\theta} + \frac{1}{3}\theta^2 + 2\sigma^2 + \frac{(a + 2)}{4(1 - \alpha)} \mu + \frac{3(b + 2)}{4(1 - \alpha)} p = 0. \tag{2.40}
\]

Differentiating this expression, and using equations (2.39) and (2.40), yields

\[
\ddot{\theta} + \frac{8}{3}\theta \ddot{\theta} + \frac{2}{3}\theta^3 + \frac{(a + 2)}{4(1 - \alpha)} [\dot{\mu} + 2\theta \mu] + \frac{3(b + 2)}{4(1 - \alpha)} [\dot{p} + 2\theta p] = 0. \tag{2.41}
\]

Finally, using equations (2.31) and (2.36), we obtain (\( \theta \neq 0 \))

\[
\dot{\theta} + \theta^2 = \frac{3(a + 2)}{4(1 - \alpha)} (\mu - p). \tag{2.42}
\]

Hence from (2.36) we have that

\[
(a - b)p = 0. \tag{2.43}
\]

We shall study the special case of dust (\( p = 0 \)) in the next subsection. If \( p \neq 0 \), then necessarily \( a = b \).

In either case, equations (2.40) and (2.42) yield

\[
-\frac{2}{3}\theta^2 + 2\sigma^2 + \frac{(a + 2)(1 - \alpha)}{4(1 - \alpha)} \mu = 0. \tag{2.44}
\]

From equation (2.11) we then have that

\[
3R = -\frac{(a + 2)(1 - \alpha)}{4(1 - \alpha)} \mu. \tag{2.45}
\]

From the above equations we note a number of interesting special cases arising. When \( p \neq 0 \), we see that if \( p = \mu \) (stiff matter) or \( a = -2 \), equation (2.42) reduces to a simple first order DE for \( \theta \).

In addition, if \( a = -2 \), then from (2.44) we see that \( \theta^2 = 3\sigma^2 \). Finally, if \( a = -2\alpha \), then from (2.45) we see that the Ricci scalar curvature of the 3-spaces orthogonal to \( u^a \) vanishes.
3. Special Cases

A. Geodesic case.

Let us first consider the case in which the acceleration $\dot{u}_a$ is zero. First, we note that when $u_a$ is irrotational and geodesic there exists preferred coordinates in which [Coley and McManus, 1994]

$$ds^2 = -dt^2 + H_{\alpha\beta}(t, x^\gamma) dx^\alpha dx^\beta,$$

(3.1)

and

$$u_a = -\delta_a^0$$

(3.2)

(and in these coordinates we now have, for example, $\dot{\theta} = \theta, t$). We also note that if the shear is zero, $\sigma = 0$, then the spacetime is necessarily FRW (Ellis, 1971; Coley and McManus, 1994). Henceforward we shall assume that $\sigma \neq 0$ (FRW spacetimes will be considered later).

Immediately, from (2.32) we have that

$$p, b h_{ba} = 0$$

(3.3)

(or $p, a = 0$ in the preferred coordinates), and since

$$(a - b) p = 0,$$

(3.4)

equation (2.35) yields ($\mu + 3p \neq 0$)

$$a = -2\alpha,$$  

(3.5)

whence (in both cases $a = b$ and $p = 0$) it follows from equation (2.42) that

$$\dot{\theta} + \theta^2 - \frac{3}{2}\mu + \frac{3}{2}p = 0,$$

(3.6)

and consequently from (2.33) or (2.44) we obtain

$$-\frac{2}{3}\theta^2 + 2\sigma^2 + 2\mu = 0.$$  

(3.7)

Therefore, from (2.11) or (2.45), we find that

$$^3R = 0,$$

(3.8)

and since, from equation (2.37), we have that

$$\dot{\sigma}_{ef} + \theta \sigma_{ef} = 0,$$

(3.9)

and hence

$$\dot{\sigma} + \theta \sigma = 0,$$

(3.10)

from equation (2.10) we consequently have that

$$^3R_{ab} = 0;$$

(3.11)

i.e., the 3-spaces orthogonal to $u^a$ are, in fact, Ricci flat.

Spacetimes in which the 3-spaces orthogonal to $u$ are Ricci flat have been studied by a number of authors (e.g., Collins and Szafron, 1979; Stephani and Wolf, 1986; see also Kramer et al., 1980). In particular, examples in which $u$ is irrotational and geodesic include a subclass of orthogonal spatially homogeneous models (i.e., the Bianchi I spacetimes; see Kramer et al., 1980) and a subclass of the Szekeres cosmological models (Collins and Szafron, 1979; Goode and Wainwright, 1982). We shall return to this and attempt to exploit previous work elsewhere.
The remaining non-trivial equations are [(2.31) and (2.34)]

\[ \dot{\mu} + (\mu + p)\theta = 0, \]  

(3.12)

and

\[ (-\sigma^{bc}_{;c} + \frac{2}{3}\theta^{b}h^{c}_{b}) = 0, \]  

(3.13)

which implies that

\[ \sigma^{c}_{;c} = \frac{2}{3}\theta^{c} + \left[ 2\sigma^{2} + \frac{2}{3}\theta \right] u_{c}, \]  

(3.14)

and equations (2.8) and (2.9) yield simplified expressions for \( E_{ab} \) and \( H_{ab} \). Finally, differentiating (3.6), and using equations (3.6) and (3.12), we obtain

\[ \ddot{\theta} + 3\dot{\theta} + \theta^{3} = -\frac{3}{2}(\dot{p} + 2p\theta). \]  

(3.15)

In the preferred coordinates (3.1), equations (1.10) and (3.2) yield

\[ \xi^{a}_{o,a} = \alpha \delta^{a}_{o}; \quad \xi^{a} = \alpha t + \bar{c}, \]  

(3.16)

and equations (1.12) and (3.3) yield

\[ \frac{d}{dt}(p(t)) = \frac{b\dot{p}(t)}{\alpha t + c}, \]  

(3.17)

the solution of which can be written as (\( \alpha \neq 0 \))

\[ p = p_{0}t^{-2}, \]  

(3.18)

since from (3.5) either \( b = -2\alpha \) or \( p = 0 \), and we have set \( \bar{c} = 0 \) so that \( p \to \infty \) as \( t \to 0^{+} \). In addition\(^1\), equation (1.11) yields

\[ \xi^{a}_{o,t} = 0; \quad \xi^{a} = \xi^{a}(x^{\gamma}), \]  

(3.19)

and from equations (1.12) and (3.12) we obtain

\[ \mu_{,\alpha}\xi^{\alpha} = -2\alpha\mu + \alpha(p_{0} + \mu t^{2})\theta t^{-1}. \]  

(3.20)

Next, the differential equation (3.15) becomes

\[ \theta_{,tt} + 3\theta_{,t} + \theta^{3} = 3p_{0}t^{-3}(1 - \theta t). \]  

(3.21)

Defining the new time, \( \tau \), and the new variable, \( \psi \), by

\[ \frac{d\tau}{dt} = \theta, \quad \psi = \theta^{2}, \]  

(3.22)

equation (3.21) becomes

\[ \psi'' + 3\psi' + 2\psi = 3p_{0}t^{-3}(\psi^{-1/2} - t), \]  

(3.23)

where \( \psi' = \frac{d\psi}{d\tau} \) and

\[ t = \int \psi^{-1/2}d\tau. \]  

(3.24)

\(^1\)The special case \( \alpha = 0 \) will be dealt with separately at the end of this subsection.
In principle we can solve equations (3.23) and (3.24) for \( \psi = \psi(\tau, x^\gamma) \) and \( \tau = \tau(t, x^\gamma) \) in terms of arbitrary functions of \( x^\gamma \). We can then integrate equation (3.10) to obtain

\[
\sigma^2 = \Sigma^2(x^\gamma)e^{-2\tau}.
\] (3.25)

Equation (3.9) becomes

\[
\sigma'_{\alpha\beta} - \frac{2}{\theta}\sigma_{\alpha \gamma}\sigma_{\beta}^\gamma + \frac{1}{3}\sigma_{\alpha\beta} = 0.
\] (3.26)

In addition, equation (3.7) yields

\[
\mu = -\Sigma^2e^{-2\tau} + \frac{1}{3}\psi,
\] (3.27)

where (eqn. (3.12))

\[
\mu' + \mu = -p_o t^{-2}.
\] (3.28)

The remaining non-trivial equations serve to determine the metric functions \( H_{\alpha\beta} \) [e.g., eqns. (1.11) and (3.11)], or to constrain (through differential relationships) the various arbitrary functions of \( x^\gamma \) [e.g., eqns. (3.14), (3.20), (3.27) and (3.28)]. The special case \( p = 0 \) will be dealt with in the next subsection. If \( p \neq 0 \) and there exists an equation of state of the form \( p = p(\mu) \), then from equations (1.12) and (3.4) we obtain \( p = (\gamma - 1)\mu \) and hence

\[
\mu = \mu_o t^{-2}.
\] (3.29)

From (3.12) we then obtain \( (\mu_o + p_o \neq 0) \)

\[
\theta = \frac{2\mu_o}{\mu_o + p_o} t^{-1},
\] (3.30)

whence equation (3.6) yields

\[
\mu_o = p_o \text{ or } p_o = \frac{2}{\sqrt{3}}\mu_o^{1/2} - \mu_o.
\] (3.31)

In the latter case \( \theta^2 = 3\mu \) whence from (3.7) \( \sigma = 0 \) and the spacetime is necessarily FRW, and in the former case we have the special case of a stiff fluid with \( \theta = t^{-1} \) and \( \sigma^2 = \frac{1}{3}t^{-2}(1 - 3\mu_o) \) [and, of course, \( 3R_{ab} = 0 \); eqn. (3.20) is satisfied identically and \( \sigma_{ab} \) satisfies eqns. (3.9) and (3.25), where \( e^{-\tau} \equiv t^{-1} \), and \( \sigma_{ab} = 0 \)]. Finally, we note the special non-trivial solution of Benoit and Coley (1996) in the particular case of spherical symmetry.

**Special case** \( \alpha = 0 \).

In the special case of kinematic self-similarity of the zeroth kind (i.e., \( \alpha = 0 \)), the analysis up to equation (3.19) is similar resulting in

\[
a = b = 0,
\] (3.32)

and

\[
p = p_o; \ p_o \text{ constant },
\] (3.33)

where

\[
\xi^a = (\bar{\xi}, \xi^a(x^\gamma)).
\] (3.34)

The differential equation (3.23), obtained from (3.15), becomes

\[
\psi'' + 3\psi' + 2\psi = -6p_o.
\] (3.35)

This equation can be integrated to obtain

\[
\theta^2 = \psi = c(x^\alpha)e^{-2\tau} + d(x^\alpha)e^{-\gamma} - 3p_o,
\] (3.36)

and the analysis then essentially follows the same steps as in the zero-pressure case (see the details in the next subsection). Indeed, integrating equations (3.12) and (3.10) to obtain

\[
\mu = M(x^\alpha)e^{-\gamma} - p_o,
\] (3.37)

and

\[
\sigma^2 = \Sigma^2(x^\alpha)e^{-2\tau},
\] (3.38)

we can see that equations (3.6), (3.7) [(3.9) and (3.10)], and (1.12) [the analogue of (3.20)] are all invariant under \( \theta \rightarrow \bar{\theta}, \mu \rightarrow \bar{\mu}, \sigma \rightarrow \bar{\sigma} \) where \( \bar{\theta}^2 = \bar{\theta}^2 + 3p_o, \bar{\mu} = \mu + p_o, \bar{\sigma}^2 = \sigma^2 \). However, assuming that \( p_o > 0 \) (i.e., positive pressure), we see from (3.36) that the model is only valid for \( \tau \leq \tau_c \), for some critical value \( \tau_c \) (so that \( \bar{\theta}^2 \geq 0 \)), and for early times (\( \tau \to -\infty \) or \( t \to 0^+ \)) the model is indistinguishable from the zero-pressure model.
B. Pressure-free case.

The special case of dust is of particular importance since in this case the constraints from the imposition of self-similarity are the least restrictive and also because there are examples of dust models admitting a kinematic self-similarity (Carter and Henriksen, 1989). We also note that in the case of dust \( p = 0 \), it follows that \( \mathcal{E}T_{ab} = (a + 2\alpha)T_{ab} \); that is the total energy-momentum tensor satisfies an equation of the form (1.12).

When \( p = 0 \), the differential equation (3.23) reduces to

\[
\psi'' + 3\psi' + 2\psi = 0, \tag{3.39}
\]

with the solution

\[
\psi = \theta^2 = c(x^\alpha)e^{-2\tau} + d(x^\alpha)e^{-\tau}, \tag{3.40}
\]

\( \text{where} \ (cd \neq 0) \)

\[
t = \frac{2\sqrt{c}}{d} \left( 1 + \frac{c}{d} \right)^{1/2} - \frac{2\sqrt{c}}{d} \tag{3.41}
\]

(where the function of integration is chosen to ensure that \( t \to 0^+ \) as \( \tau \to -\infty \)). Alternatively, we can write

\[
\theta^2 = 4 \left( t + \frac{2\sqrt{c}}{d} \right)^2 \left[ 4c \frac{d}{d^2} - \left( t + \frac{2\sqrt{c}}{d} \right)^2 \right]^{-2}, \tag{3.42}
\]

where

\[
c^\tau = \frac{d}{4} \left( t + \frac{2\sqrt{c}}{d} \right)^2 - \frac{c}{d}. \tag{3.43}
\]

Integrating equation (3.28) yields

\[
\mu = M(x^\alpha)e^{-\tau}, \tag{3.44}
\]

and equation (3.25) gives

\[
\sigma^2 = \Sigma^2(x^\alpha)e^{-2\tau}, \tag{3.45}
\]

whence equations (3.6) and (3.7) (or (3.27)) are satisfied when

\[
M = \frac{1}{3}d, \tag{3.46}
\]

\[
\Sigma^2 = \frac{1}{3}c. \tag{3.47}
\]

In addition, since \( a = -2\alpha (\mu \neq 0) \), equation (1.12) yields

\[
2\alpha + \frac{M\alpha\xi^\alpha}{M} - \tau\alpha\xi^\alpha - \theta\xi^\alpha = 0, \tag{3.48}
\]

where

\[
\xi^\alpha = (\alpha t + \bar{c}, \xi^\alpha(x^\gamma)). \tag{3.49}
\]

Using equations (3.42), (3.43) and (3.46), equation (3.48) reduces to

\[
\left\{ \left( \frac{\sqrt{c}}{d} \right)_{,\alpha} \xi^\alpha - \frac{\alpha\sqrt{c}}{d} t + \left[ t + \frac{2\sqrt{c}}{d} \right] \frac{\bar{c}}{2} = 0, \tag{3.50}
\right.
\]

and hence \( \bar{c} = 0 \) (and \( \alpha \) is necessarily non-zero) and

\[
\left( \frac{\sqrt{c}}{d} \right)_{,\alpha} \xi^\alpha = \alpha \frac{\sqrt{c}}{d}. \tag{3.51}
\]
Equations (3.14), (1.11), (3.9) and (3.11), remain to be satisfied; these equations (through equation (3.1) and due to the definitions of the shear and expansion in terms of the metric functions, their time derivatives, and the inverse metric functions), constrain the metric functions $H_{\alpha\beta}$. We recall that the resulting spacetimes are 3-Ricci flat (i.e., $3R_{ab} = 0$).

Let us investigate the asymptotic behaviour of the solutions. As $t \to \infty$ ($\tau \to \infty$), we find that

$$\theta = \sqrt{d}e^{-\tau/2} = 2t^{-1}, \quad (3.52)$$

and from equations (3.46) and (3.47) we have that

$$\frac{\mu}{\theta^2} \to \frac{1}{3} \quad \text{and} \quad \frac{\sigma}{\theta} \to 0. \quad (3.53)$$

Therefore, the models are asymptotic (at late times) to an exact zero-pressure and flat FRW model with a power-law scale function; we note that this Einstein de Sitter model admits a homothetic vector.

On the other hand, as $t \to 0^+$ ($\tau \to -\infty$), we find that

$$\theta = \sqrt{d}e^{-\tau} = \frac{1}{t}, \quad (3.54)$$

and from equations (3.46) and (3.47) we have that

$$\frac{\mu}{\theta^2} \to 0 \quad \text{and} \quad \frac{\sigma^2}{\theta^2} \to \frac{1}{3}. \quad (3.55)$$

Hence the models are asymptotic (at early times) to an exact vacuum, 3-Ricci flat solution; in particular, this exact solution has

$$\mu = 0, \quad \theta = \frac{1}{t}; \quad \sigma^2 = \frac{1}{3} \theta^2 = \frac{1}{3} t^{-2}, \quad 3R_{ab} = 0.$$

From equation (3.1) and from the definitions of the shear and expansion, these equations yield the following equations for the metric functions $H_{\alpha\beta}(t,x^\gamma)$:

$$H^{\alpha\beta}H_{\alpha\beta,t} = \frac{2}{t^2}, \quad (3.56)$$

$$H^{\alpha\gamma}H^{\beta\delta}H_{\alpha\beta,t}H_{\gamma\delta,t} = \frac{4}{t^2}, \quad (3.57)$$

and (from equation (3.9))

$$H_{\alpha\beta,tt} - H^{\gamma\delta}H_{\alpha\gamma,t}H_{\beta\delta,t} + \frac{1}{t}H_{\alpha\beta,t} = 0. \quad (3.58)$$

In the special case of spatial homogeneity we necessarily obtain the Kasner model. In particular, in general in the case that $H_{\alpha\beta}(t,x^\gamma)$ is diagonal, i.e., $H_{\alpha\beta} = \text{diag} \{h_1(t,x^\gamma), h_2(t,x^\gamma), h_3(t,x^\gamma)\}$, these equations can be integrated to yield

$$h_1 = A_1(x^\gamma)t^{2a_1}, \quad h_2 = A_2(x^\gamma)t^{2a_2}, \quad h_3 = A_3(x^\gamma)t^{2a_3}, \quad (3.59)$$

where the constants in (3.59) obey

$$a_1 + a_2 + a_3 = a_1^2 + a_2^2 + a_3^2 = 1, \quad (3.60)$$

and hence we obtain the Kasner model. We note that this vacuum Bianchi I exact solution also admits a homothetic vector.

Finally, we note that the spherically symmetric dust solutions of Lynden-Bell and Lemos (1988) and Carter and Henriksen (1989), that are a particular case of the perfect fluid solutions of Benoit and Coley (1996), represent a special non-trivial solution of the above equations.
C. Case of Zero Expansion.
We shall also consider the special case in which $\theta = 0$, although this case is not of interest from a cosmological point of view. Let us choose comoving coordinates so that (with $\omega = 0$)

$$ds^2 = -(U^2)dt^2 + H_{\alpha\beta}(t,x)dx^\alpha dx^\beta,$$  \hspace{1cm} (3.61)

where

$$u^\alpha = \frac{1}{U}\delta^\alpha_0.$$  \hspace{1cm} (3.62)

Using these coordinates we have that

$$\dot{u}_a = (0, [ln U],_a),$$  \hspace{1cm} (3.63)

$\theta = 0$ implies that

$$H_{\alpha\beta}H_{\alpha\beta,t} = 0,$$  \hspace{1cm} (3.64)

and $\dot{\Phi} = 0$ implies that $\dot{\Phi} = 0$. In addition, from equations (1.10) and (1.11) we obtain

$$\xi^a = (\xi^0(t), \xi^a(x^\gamma)).$$  \hspace{1cm} (3.65)

From equations (2.31) and (2.39) we have that

$$\mu, t = 0 = \sigma, t,$$  \hspace{1cm} (3.66)

and from equation (3.41) we find that

$$(\mu + p)[ln U],_a = -p_a.$$  \hspace{1cm} (3.67)

The class of static solutions is contained within this special case presently under consideration.

Immediately, equation (2.36) yields

$$(a + 2)\mu - (b + 2)p = 0,$$  \hspace{1cm} (3.68)

whence equations (2.33) and (2.35) yield

$$8(1 - \alpha)\sigma^2 + (a + 2)\mu + 3(b + 2)p = 0.$$  \hspace{1cm} (3.69)

These two equations are best dealt with separately in two different subcases.

**Subcase (i):** Either $b = -2$ or $p = 0$. From equation (3.68) $a = -2$, and hence from equation (3.69), the shear is zero, $\sigma^2 = 0$. Neglecting the case $p = 0$ (since in this case we obtain $\dot{u}_a = 0$ from eqn. (2.32) and hence the resulting spacetime is a special static FRW spacetime), we consequently obtain

$$a = b = -2,$$  \hspace{1cm} (3.70)

and hence

$$\sigma_{ab} = 0;$$  \hspace{1cm} (3.71)

i.e.,

$$H_{\alpha\beta,t} = 0$$  \hspace{1cm} (3.72)

[and consequently $H_{\alpha\beta}$ can be diagonalized—$H_{\alpha\beta} = \text{diag} \{h_1(x^\gamma), h_2(x^\gamma), h_3(x^\gamma)\}$]. From equations (2.11) and (2.33) we then obtain

$$^3R = 2\mu(x^\gamma),$$  \hspace{1cm} (3.73)

and

$$\dot{u}^a_{\ ;a} = \frac{1}{2}(\mu + 3p).$$  \hspace{1cm} (3.74)
In addition, from equation (2.9), we have that
\[ H_{ab} = 0, \] (3.75)
i.e., the magnetic part of the Weyl tensor vanishes (note clumsy notation here), and equation (2.38) yields
\[ E_{ab} = 3R_{ab} - \frac{1}{3} 3R h_{ab}, \] (3.76)
From equation (1.10) we have that
\[ (\ln U)_t \xi^0 + (\ln U)_\alpha \xi^\alpha = \alpha - \xi^0_{,t}, \] (3.77)
and equations (1.12) yield
\[ \mu_{,\alpha} \xi^{\alpha} = -2\mu, \] (3.78)
\[ p_{,\alpha} \xi^0 + p_{,\alpha} \xi^\alpha = -2p. \] (3.79)
Finally, equation (3.67) gives
\[ (\mu + p)(\ln U)_\alpha \xi^\alpha = -p_{,\alpha} \xi^\alpha, \] (3.80)
and (the integrability conditions of (3.67) give)
\[ p_{,\alpha} \mu_{,\beta} = \mu_{,\alpha} p_{,\beta}. \] (3.81)
From this equation we obtain the solution
\[ \ln p = F(t, \ln \mu), \] (3.82)
whence on defining T by
\[ \frac{dT}{dt} = \frac{2}{\xi^0}, \] (3.83)
equation (3.79) becomes
\[ \frac{\partial F}{\partial (\ln \mu)} \frac{\partial F}{\partial T} = 1, \] (3.84)
with the solution
\[ F = \frac{1}{2}(\ln \mu + T) + \tilde{f}(\ln \mu - T), \]
which can be written as
\[ p = \mu f(\ln \mu - T), \] (3.85)
where f is an arbitrary function of a single variable. Equations (3.67), (3.77) and (3.80) then yield
\[ (\ln U)_T = \frac{1}{2}(\xi^0_{,t} - \alpha) + \left[ \frac{f + f'}{1 + f} \right], \]
and
\[ (\ln U)_\alpha = - \left[ \frac{f + f'}{1 + f} \right] (\ln \mu)_\alpha \]
which has the solution
\[ \ln U = \int \left[ \frac{\alpha - \xi^0_{,t}}{\xi^0} \right] dt - \int \frac{f(m) + f'(m)}{1 + f(m)} dm, \] (3.86)
where \( m \equiv (\ln \mu - T) \).
Now,
\[ \dot{u}^\alpha_{\cdot\alpha} = H^{\alpha\beta}(x^\gamma)[(lnU)_{\alpha\beta} + (lnU)_{\cdot\alpha}(lnU)_{\cdot\beta} - \Gamma^d_{\alpha\beta}(x^\delta)(lnU)_{\cdot\gamma}], \] (3.87)
whence, on defining
\[ F(m) = \int \frac{f(m) + f'(m)}{1 + f(m)} dm, \] (3.88)
equation (3.74) becomes
\[ [F']\{H^{\alpha\beta}(ln\mu)_{\cdot\alpha\beta} - H^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}(ln\mu)_{\cdot\gamma}\}
+ [F'' - (F')^2]\{H^{\alpha\beta}(ln\mu)_{\cdot\alpha}(ln\mu)_{\cdot\beta}\}
+ \left[ \frac{1}{2}(1 + 3f) \right] \{\mu\} = 0. \] (3.89)
This equation contains the metric functions \( H_{\alpha\beta}(x^\gamma) \) and \( \mu(x^\gamma) \) [and \( f(ln\mu - T(t)) (F) \) and \( \xi^0(t) \)].
The \( H_{\alpha\beta} \) [and \( \xi^\alpha(x^\gamma) \)] are further constrained by equations (3.73), (3.75) and (3.76) [and eqns. (3.78) and (1.11)], where \( \mu \) is given in terms of the metric functions through the EFEs.

We notice that equation (3.89) is of the form of the sum of (three) terms in which each term in square brackets depends on \( t \) [through eqns. (3.65), (3.83), (3.85) and (3.88)] and each term in curly brackets does not. A particular solution of equation (3.89) has \( f = \text{constant} \). Indeed, if \( f \) is not constant then in general (for \( \mu \neq 0 \)) we have that
\[ F'' - (F')^2 = c_1(1 + 3f); \quad F' = c_2(1 + 3f) [c_2 \neq 0], \] (3.90)
which on integration yields \( e^{-F} = d_2 - \frac{c_2}{c_1} d_1 e^{c_1 t} / c_2 (c_1 \neq 0) \) or \( e^{-F} = d_2 - d_1 m (c_1 = 0) \) and hence a particular functional form for \( f(m) \), whence for consistency equation (3.88) implies that \( d_1 = 0 \) (in either case), which contradicts the assertion that \( f \) is not constant.

Therefore, in general \( f \) is constant and hence from (3.85) we have that
\[ p = (\gamma - 1)\mu; \quad f \equiv \gamma - 1 \text{ (const.)}. \] (3.91)
In particular, we note that \( p = p(x^\gamma) \). Finally, equation (3.67) gives
\[ [lnU]_{\cdot\alpha} = \left( \frac{1 - \gamma}{\gamma} \right) (ln\mu)_{\cdot\alpha}, \] (3.92)
which integrates to yield
\[ U = u(t) \mu^{-1+1/\gamma}. \] (3.93)
However, since \( U \) is separable, a redefinition of the time coordinate in (3.61) can be employed to set \( u(t) = 1 \). Hence
\[ U = U(x^\gamma) = \mu^{-1+1/\gamma}, \] (3.94)
and the metric is independent of time and hence the model is completely static.

Equation (3.77) then yields
\[ \xi^0(t) = \left( \alpha + \frac{2(1 - \gamma)}{\gamma} \right) t + \tau, \] (3.95)
whence the \( \xi^\alpha(x^\gamma) \) are constrained by equations (3.78) and (1.11) while the \( H_{\alpha\beta}(x^\gamma) \) are themselves subject to equations (3.73), (3.75) and (3.76).

Finally, from (3.91) we have that
\[ F(m) = \int \frac{\gamma - 1}{\gamma} dm = \frac{\gamma - 1}{\gamma} (ln\mu - T), \] (3.96)
and equation (3.86) yields

\[
\ln U = -\frac{1}{2} \int (\alpha - \xi^0) dT - F(m)
= \int \frac{(1-\gamma)}{\gamma} dT - \frac{\gamma - 1}{\gamma} (\ln \mu - T) = \frac{1 - \gamma}{\gamma} \ln \mu + d,
\]

which is consistent with equation (3.93). In addition, in this case equation (3.89) reduces to

\[
\frac{(\gamma - 1)}{\gamma} H^{\alpha\beta} (\ln \mu)_{,\alpha} (\ln \mu)_{,\beta} - H^{\alpha\beta} (\ln \mu)_{,\alpha\beta}
+ H^{\alpha\beta} \Gamma^\gamma_{\alpha\beta} (\ln \mu)_{,\gamma} = \frac{(3\gamma - 2)}{2(\gamma - 1)} \mu.
\]

This equation must be satisfied in addition to equations (3.73), (3.75) and (3.76).

**Subcase (ii):** In this subcase we have that \( b \neq -2 \) and \( p \neq 0 \), so that equation (3.68) yields \( p = \frac{(a + 2)}{(a - 2)} \mu \), whence equations (1.12) imply that \( a = b (\neq -2) \) and hence \( p = \mu \) (note that \( p, t = 0 \)), and finally equation (3.69) yields \( 2(1 - \alpha) \sigma^2 + (a + 2) \mu = 0 \); that is, we have that

\[
(b + 2)p \neq 0, a = b, p = \mu, \sigma^2 = \frac{-(a + 2)}{2(1 - \alpha)} \mu.
\]

From equations (2.11) and (2.33) we also obtain

\[
3R = -\frac{(a + 2\alpha)}{(1 - \alpha)} \mu = \dot{u}^{\alpha : \alpha}.
\]

From equation (3.67) we obtain

\[
U = u(t) \mu^{-\frac{1}{2}}
\]

(whence we note is the special case of (3.93) with \( \gamma = 2 \) corresponding to \( p = \mu \)), whence by redefining the \( t \)-coordinates to set \( u(t) = 1 \) we have that

\[
U = \mu^{-\frac{1}{2}}.
\]

Therefore, although \( \mu, p \) and \( U \) are independent of \( t \), the metric functions \( H_{\alpha\beta} = H_{\alpha\beta}(t, x^\gamma) \) depend on \( t \) and the models are not static (since \( \sigma^2 \neq 0 \) for \( \mu \neq 0 \), \( \sigma_{ab} \neq 0 \) and hence \( H_{\alpha\beta, t} \) cannot vanish for all \( \alpha, \beta \)). However, the \( H_{\alpha\beta} \) are constrained by equations (3.64), (3.99) and (3.100), in addition to equations (2.34) and (2.37), etc. For example, using (3.63) and (3.64), equation (3.100) yields

\[
U[U_{,\alpha\beta} - \Gamma^\delta_{\alpha\beta} U_{,\delta}] H^{\alpha\beta} = -\frac{(a + 2\alpha)}{(1 - \alpha)} (\text{const.}).
\]

Using equations (3.99) and (3.102), equations (1.12) yield

\[
(lnU)_{,\alpha} \xi^\alpha = -\frac{a}{2} = -\frac{1}{2} (ln \mu)_{,\alpha} \xi^\alpha,
\]

whence equation (1.10) yields

\[
\xi^0_{,t} = \alpha + \frac{a}{2}.
\]

which integrates to give

\[
\xi^0(t) = \frac{1}{2}(a + 2\alpha) t + \tau.
\]

Equation (1.11) remains to be satisfied.
4. More Special Cases

There are two special cases of particular interest in which the kinematic self-similarity is either parallel or orthogonal to the velocity vector. Let us next consider these two cases separately.

A. $\xi$ parallel to $u$

In Coley (1991) it was shown that if a perfect fluid spacetime admits a proper HV parallel to the velocity vector then that spacetime is necessarily an FRW spacetime satisfying $\dot{\theta} = -\frac{1}{3}\theta^2, \theta \neq 0$ [with the very special and physically unreasonable equation of state $\mu + 3p = 0$; strictly speaking this result was only proven in the case $p = p(\mu)$ and $\mu + p \neq 0$].

Suppose that a spacetime admits a kinematic self-similar vector $\xi$ parallel to $u$, i.e.,

$$\xi^a = Au^a, \quad (4.1)$$

then, from equation (2.12),

$$A(u_{a;b} + u_{b;a}) + A_{\cdot b}u_{a} + A_{\cdot a}u_{b} = 2g_{ab} + 2(1 - \alpha)u_{a}u_{b}. \quad (4.2)$$

Contracting equation (4.2) with $g_{ab}$ and $u^{a}u^{b}$ in turn then yields

$$\dot{A} = \alpha, \quad A\theta = 3, \quad (4.3)$$

and hence

$$\dot{\theta} + \frac{\alpha}{3}\theta^2 = 0. \quad (4.4)$$

Contracting equation (4.2) with $h_{c}^{\cdot a}h_{d}^{b}$ then yields

$$\sigma_{ab} = 0, \quad (4.5)$$

i.e., the spacetime is shear-free, and finally contracting equation (4.2) with $u^{a}$ yields

$$\dot{u}_{a} = (\ln A^{-1})_{\cdot b}h_{b}^{a}. \quad (4.6)$$

Assuming that the vorticity is zero, equation (2.6) then yields

$$\theta_{\cdot b}h_{b}^{a} = 0. \quad (4.7)$$

Therefore, from equations (4.3) [$A = 3/\theta$] and (4.7) we see that

$$\dot{u}_{a} = 0, \quad (4.8)$$

i.e., the acceleration is zero.

Hence, since the shear, vorticity and acceleration are all zero, the spacetime is necessarily FRW (Ellis, 1971; Coley and McManus, 1994). Therefore, there exists coordinates in which the metric is of the form (4.48) and equation (4.4) yields

$$\theta_{t} + \frac{\alpha}{3}\theta^2 = 0; \quad \theta = \frac{3}{\alpha}t^{-1} \quad (4.9)$$

(definition the constant of integration so that $\theta \rightarrow \infty$ as $t \rightarrow 0^+$). The remaining equations to be solved reduce to

$$\frac{a + 2\alpha}{2(1 - \alpha)}\mu + \frac{3(b + 2\alpha)}{2(1 - \alpha)}p = 0, \quad (4.10)$$

$$2\theta^2 \left(1 - \frac{\alpha}{3}\right) = \frac{3(a + 2)}{2(1 - \alpha)}\mu - \frac{3(b + 2)}{2(1 - \alpha)}p, \quad (4.11)$$

$$2\theta^2(1 - \alpha) = -3\mu - 9p, \quad (4.12)$$
\[
\dot{\mu} + (\mu + p) \theta = 0. \tag{4.13}
\]
It can be easily shown that \(a \neq -2\alpha\) and \(p = 0\) lead to contradictions, so that (4.10) yields
\[
a = b = -2\alpha. \tag{4.14}
\]
Equations (4.11)-(4.13) then yield the consistent solution
\[
p = \left[\frac{2\alpha}{3} - 1\right] \mu \tag{4.15}
\]
and
\[
\mu = \frac{1}{3} \theta^2 = \frac{3}{\alpha^2} \xi^2. \tag{4.16}
\]
We note that the above solution reduces to the homothetic case when \(\alpha = 1\) (see, e.g., eqns. (4.4) and (4.15) and (4.16)). However, unlike the homothetic case there exist models with realistic equations of state; for example, \(0 < p < \mu\) for \(\frac{3}{2} < \alpha < 3\).

**B. \(\xi\) orthogonal to \(u\).**
McIntosh (1975) showed that a perfect fluid spacetime cannot admit a non-trivial homothetic vector which is orthogonal to the fluid 4-velocity unless \(p = \mu\).
We shall work in comoving coordinates in which the metric is given by (3.61) and the velocity vector by (3.62). In these coordinates the acceleration is then given by
\[
\dot{u}_0 = 0; \quad \dot{u}_\alpha = (\ln U)_{,\alpha}. \tag{4.17}
\]
Since \(\xi\) is orthogonal to \(u\), \(\xi^0 = 0\), and equation (1.11) implies that \(\xi^\alpha = \xi^\alpha (x^\gamma)\), and consequently equation (1.10) reduces to
\[
U_{,\alpha} \xi^\alpha = \alpha U. \tag{4.18}
\]
Equations (1.12) yield
\[
\mu_{,\alpha} \xi^\alpha = a \mu \tag{4.19}
\]
and
\[
p_{,\alpha} \xi^\alpha = b p. \tag{4.20}
\]
Also, we can further specify the coordinates so that
\[
\xi^\alpha = \xi (x^\gamma) \delta^\alpha_x. \tag{4.21}
\]
and in these coordinates (if \(a\)ab \(\neq 0\)) equations (4.18) - (4.20) yield
\[
(ln[U^{1/\alpha}])_{,x} = (ln[\mu^{1/\alpha}])_{,x} = (ln[p^{1/b}])_{,x} = \frac{1}{\xi}, \tag{4.22}
\]
which can be partially integrated to yield
\[
\mu = \overline{f}(t, y, z) U^{\alpha/a}; p = \overline{g}(t, y, z) U^{b/a}. \tag{4.23}
\]
Finally, the conservation equation (2.32) reduces to
\[
(\mu + p)(\ln U)_{,\alpha} = -p_{,\alpha}, \tag{4.24}
\]
whence, on contraction with \(\xi^\alpha\) and using equations (4.18) and (4.20), we obtain
\[
\alpha [\mu + p] = -bp. \tag{4.25}
\]
Apart from the special subcase $\alpha = b = 0$, which implies that

$$U_{,x} = 0 = p_{,x},$$

(4.26)
equation (4.25) implies that (i) if $b = 0$, then either $\alpha = 0$ (the special subcase above) or $\mu + p = 0$ (a case which we shall not consider here), (ii) if $\alpha = 0$, then either $b = 0$ (special subcase) or $p = 0$ (a case considered earlier), and (iii) if $b = -\alpha$, then either $\alpha = 0$ (and $b = 0$) or $\mu = 0$ (a case of no interest here; however, see the appendix), otherwise we have that in the general case

$$p = \frac{-\alpha}{(\alpha + b)}\mu; \quad \alpha b + \mu \neq 0.$$

(4.27)

Therefore, equations (4.27) and (1.12) immediately imply that

$$a = b.$$

(4.28)
The conservation equation (2.31) then yields

$$\mu_{,t} = -a(a + \alpha)^{-1}\mu U \theta,$$

(4.29)

which can be regarded as an equation for $\theta$ in terms of $\mu$ and $U$.

Using equations (4.27) and (4.28), equations (2.33) and (2.35) then yield

$$\dot{\theta} + \frac{1}{3} \theta^2 = -2\sigma^2 - \frac{(a + 2)(a - 2\alpha)}{4(a + \alpha)(1 - \alpha)}\mu,$$

(4.30)
equation (2.36) yields

$$\dot{\theta} + \theta^2 = \frac{3(a + 2)(a + 2\alpha)}{4(1 - \alpha)(a + \alpha)}\mu,$$

(4.31)
from which we deduce that

$$\frac{2}{3}\theta^2 = 2\sigma^2 + \frac{(a + 2)(1 - \alpha)}{(1 - \alpha)}\mu.$$

(4.32)
Differentiating this equation with respect to $t$, and using equations (4.29) and (4.31), we obtain ($\sigma \neq 0$)

$$\dot{\sigma} = -\sigma \theta,$$

(4.33)
which also follows from equation (2.37). Finally, from equations (4.32) and (2.45) we obtain

$$3R = \frac{(a + 2\alpha)}{(1 - \alpha)}\mu.$$

(4.34)
Now, defining $F(x, y, z)$ by $(\ln F)_{,x} = 1/\xi$, equations (4.22) then yield

$$\mu = f(t, y, z)F^a,$$

(4.35)
$$U = g(t, y, z)F^a.$$

(4.36)
Since $F_{,t} = 0$, equation (4.29) yields

$$\theta = \frac{(a + \alpha)}{a} \frac{\dot{f} F^{-a}}{f},$$

(4.37)
whence equation (4.31) then yields

$$- \left[ \frac{f_{,t}}{f g} \right]_{,t} + \frac{(a + \alpha)}{a} \frac{(f_{,t})^2}{f^2 g} = \frac{3a(a + 2)(a + 2\alpha)}{4(1 - \alpha)(a + \alpha)^2} \frac{f^a F^a + 2a}{f g}.$$
Because the only term in (4.38) that depends on $x$ is the term $F^{α+2α}$ (and since $F_x ≠ 0$ from $(lnF)_x = 1/ξ ≠ 0$), and assuming that $a ≠ 0$ (since $a = 0$ implies $b = 0$), the only way that this equation can then be satisfied is for either

$$ (i) \ a = -2, \ or \ (ii) \ a = -2α. \quad (4.39) $$

Hence, from equation (4.31) we have that

$$ \dot{θ} + θ^2 = 0. \quad (4.40) $$

From equations (4.38) and (4.40) we can then integrate to obtain

$$ g = J(y,z)f^{-(α+2α)/α}f_t. \quad (4.41) $$

Finally, let us consider the two cases in (4.39) separately.

(i) $a = -2$. In this case (4.25) implies that

$$ p = \frac{α}{2 - α} \mu, \quad (4.42) $$

and equations (4.32) and (4.34) imply that

$$ σ^2 = \frac{1}{3} θ^2, \quad (4.43) $$

and

$$ 3R = 2μ. \quad (4.44) $$

(ii) $a = -2α$. In this case (4.25) implies that

$$ p = μ \quad (4.45) $$

(i.e., the fluid is stiff), and from equations (4.32) and (4.34) we have that

$$ \mu + σ^2 = \frac{1}{3} θ^2, \quad (4.46) $$

and hence

$$ 3R = 0. \quad (4.47) $$

Further progress (e.g., further constraining the form of the functions $F(x,y,z), f(t,y,z)$ and $J(y,z)$ or determining the form of the metric functions $H_{αβ}(t,x,y,z)$) can be made in specific spacetimes. For example, spherically symmetric spacetimes have been studied [and it has been claimed that in such spacetimes if there exists a vector field $ξ$ satisfying (1.11) which is orthogonal to $u$, then the resulting spacetime metric is singular (Ponce de Leon, 1993)]. We shall investigate the existence of kinematic self-similar vectors in the special case of FRW spacetimes in the next subsection.

C. FRW models.

We recall from Maartens and Maharaj (1986) that (perfect fluid) FRW models, when written in the comoving form

$$ ds^2 = -dt^2 + R^2(t) \left\{ \frac{dv^2}{1 - kv^2} + r^2 dΩ^2 \right\}, \quad (4.48) $$

admit a homothetic vector $P_α(= θ_1)$ parallel to $u$ for all $k$ when $R(t) = dt, \ d$ constant (and hence $μ = -3p = 3(1 + kd^2)t^{-2}; \ μ + 3p = 0$), and admit a homothetic vector $H(= tθ_1 + rθ_2)$ when $R = dt^2$ but only in the case $k = 0$ (whence $μ = 3c^2t^{-2}, \ p = \gamma - 1)\mu$ with $γ = 2/3c$).
We cannot simply apply the results from section 3 here since it was explicitly assumed there that \( \sigma \neq 0 \). However, when \( \sigma = \omega = \dot{u}_a = 0 \), we immediately obtain from equation (2.35) that

\[
(a + 2\alpha)\mu + 3(b + 2\alpha)p = 0. \tag{4.49}
\]

This again suggests two subcases. First, when \( a = -2\alpha \), we have that either \( b = -2\alpha \) or \( p = 0 \), whence from equations (2.33) and (2.36) we obtain (in either case) \( \mu = \frac{1}{3}b^2 \) and hence from equation (2.11) we obtain \( 3\mu R = 0 \). Second, if \( a \neq -2\alpha \), then if \( b = a \), from (4.49) we obtain \( \mu + 3p = 0 \) [on the other hand, if we assume \( a \neq b \neq -2\alpha (p \neq 0) \), then from (4.49) we have that \( p = \frac{3}{2}(b + 2\alpha) \mu \), whence from equations (1.12) we deduce that \( a = b \), resulting in a contradiction]. Therefore, in order for an FRW spacetime to admit a kinematic self-similarity then necessarily either \( k = 0 \) (zero-curvature) or \( \mu + 3p = 0 \), in direct analogy with the homothetic case.

However, we shall not proceed with this type of analysis here since we wish to determine and display all the kinematical self-similarities in conventional forms; i.e., in the form corresponding to (4.48) and in a form that explicitly displays the self-similar form of the solutions.

In order to derive the spacetimes in manifestly self-similar form, we write the metric in comoving spherically symmetric coordinates adapted to \( \xi \), viz.,

\[
d^2s = -2\phi(t)dt^2 + e^{2\psi(t)}dr^2 + r^2S^2(\xi)d\Omega^2, \tag{4.50}
\]

where \( d\Omega^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 \), and

\[
u = e^{-\phi} \frac{\partial}{\partial t}, \tag{4.51}
\]

where, assuming \( \xi \) is neither parallel to nor orthogonal to \( \nu \), the kinematic self-similar vector \( \xi \) and the corresponding self-similar variable \( \xi \) can be written in one of the following forms (Carter and Henriksen, 1989):

- first kind (\( \alpha = 1 \); HV) : \( \xi = \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}; \quad \xi = r/t, \tag{4.52} \)
- second kind (\( \alpha \neq 0, 1 \)) : \( \xi = \alpha t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}; \quad \xi = r/(\alpha t)^{1/\alpha}, \tag{4.53} \)
- zeroth kind (\( \alpha = 0 \)) : \( \xi = \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}; \quad \xi = e^{-t}. \tag{4.54} \)

Now, if the acceleration is zero, we have that \( e^\psi = a_0 \), a constant, and if the shear is zero, we have that \( e^\psi = f_o S \), where \( f_o \) is a constant. Therefore, the FRW metric can be written in the form

\[
d^2s = -a_0^2dt^2 + S^2(\xi)[f_o^2 dr^2 + r^2 d\Omega^2]. \tag{4.55}
\]

(i) If \( \alpha \neq 0 \) and assuming \( S' \neq 0 \) (non-static case), the EFEs (for a perfect fluid source) then yield

\[
S = d\xi, \tag{4.56}
\]

and \( c \neq -1 \)

\[
f_o^2 = (1 + c)^2. \tag{4.57}
\]

The forms for \( \xi \) in (4.52) and (4.53) are invariant under the changes \( t \to at \) and \( r \to br \), and \( a \) and \( b \) can be chosen to set \( a_o = 1 \) and \( d = 1 \), and we can then write the metric as

\[
d^2s = -dt^2 + \xi^{2c} \left\{ \frac{dr^2}{(1 + c)^2} + r^2 d\Omega^2 \right\}. \tag{4.58}
\]

Recalling that \( \xi = r(\alpha t)^{-1/\alpha} (\alpha \neq 0) \), and defining a new radial coordinate \( \tilde{r} \) by \( \tilde{r} = \alpha^a r^{c+1} \), where \( a = -c/\alpha \), the FRW metric takes on its familiar form

\[
d^2s = -dt^2 + \tilde{r}^{2a} \{ d\tilde{r}^2 + \tilde{r}^2 d\Omega^2 \}, \tag{4.59}
\]
where $\xi$ is now given by

$$
\xi = \alpha t \frac{\partial}{\partial t} + (1 + c)r \frac{\partial}{\partial r},
$$

(4.60)

We note that each orthogonal 3-space is flat ($k = 0$) and that the simple power-law solutions give rise to an equation of state of the form $p = (\gamma - 1)\mu$, and equations (1.12) are automatically satisfied. The known homothetic case ($\xi = H$ for $k = 0$) is included here. In fact, each spacetime (4.59) (for each value of $a$) admits kinematical self-similar vectors for all values of $\alpha (\neq 0)$. [This is known to be true in the dust subcase (Carter and Henriksen, 1989)]. This, of course, arises due to the separability of $S(\xi)$ in (4.55) (e.g., see equations (4.53) and (4.56)).

(ii) If $\alpha = 0$, then the EFEs imply that

$$
S = \alpha \xi,
$$

(4.61)

and

$$
f_o^2 = 4.
$$

(4.62)

Since $\xi = re^{-t}$, under the transformation $t \to t + b$, $b$ can be chosen so that $c = 1$, and hence

$$
ds^2 = -a_o^2dt^2 + r^2e^{-2t}(4dr^2 + r^2d\Omega^2).
$$

(4.63)

Finally, defining the new variables $\bar{t} = a_o t, \bar{r} = r^2$, the metric becomes

$$
ds^2 = -d\bar{t}^2 + e^{-2t/a_o}(d\bar{r}^2 + \bar{r}^2d\Omega^2),
$$

(4.64)

and the kinematical self-similar vector is given by

$$
\xi = a_o \frac{\partial}{\partial \bar{t}} + 2\bar{r} \frac{\partial}{\partial \bar{r}}.
$$

(4.65)

Again, this FRW model is flat ($k = 0$). However, this case has no homothetic vector analogue. The equation of state for this FRW model is given by $\mu = -p = 12a_o^{-2}$, and the metric is, of course, the de Sitter metric. Hence, the de Sitter model admits a self-similarity of the zeroth kind. Equations (1.12) are trivially satisfied with $a = b = 0$.

Finally, we must deal with the two special cases not considered above in which $\xi$ is either parallel to or orthogonal to $u$. For illustrative purposes let us return to the more conventional coordinate system in which the metric is given by (4.48). Writing $\xi$ in the form

$$
\xi = \xi^o(r, t) \frac{\partial}{\partial t} + \xi^r(r, t) \frac{\partial}{\partial r},
$$

(4.66)

equations (1.10) and (1.11) yield

$$
\xi^o = \alpha t + \beta,
$$

(4.67)

$$
\xi^r = \frac{2 - c}{2}r,
$$

(4.68)

where $c$ is an arbitrary constant, and

$$
\frac{2\dot{R}}{R}(\alpha t + \beta) = c,
$$

(4.69)

and

$$
k(c - 2) = 0.
$$

(4.70)

From this last equation we see that either $k = 0$, and the FRW spacetime is flat, or $c = 2$, whence the kinematic self-similar vector is parallel to the fluid velocity vector (i.e., $\xi^r = 0$). Let us consider this latter case first.
(a) $\xi$ parallel to $u$. In this case $c = 2$ (i.e., $\xi^r = 0$) and $k$ is unrestricted by equations (4.67)–(4.70), whence equation (4.69) becomes $(\alpha^2 + \beta^2 \neq 0)$

$$\frac{\dot{R}}{R}(\alpha t + \beta) = 1. \quad (4.71)$$

If $\alpha \neq 0$, we can set $\beta = 0$ by a time translation, whence equation (4.71) yields

$$R = dt^{1/\alpha}, \quad (4.72)$$

so that when $\alpha = 1$ (homothetic vector case) we recover the usual FRW model with $\mu + 3p = 0$ (valid for all $k$), whereas for $\alpha \neq 1$ all power law solutions of the form (4.72) admit a vector field satisfying equations (1.10) and (1.11).

We note that if $k = 0$, then the spacetime (4.59) also admits additional kinematic self-similarities parallel to $u$ (in addition to those given by (4.60)). If, on the other hand, $k \neq 0$, we can see that the resulting FRW models with power law scale factors always admit a vector field parallel to the fluid velocity vector that satisfies equations (1.10) and (1.11). However, equations (1.12) need not be satisfied. Indeed, from equation (4.72) we find that

$$\mu = \frac{3}{\alpha^2} t^{-2} \mp \frac{3k}{d^2} t^{-2/\alpha}, \quad (4.73)$$

$$p = \frac{1}{\alpha} (2 - 3/\alpha) t^{-2} - \frac{k}{d^2} t^{-2/\alpha}, \quad (4.74)$$

so that equations (1.12) can only be satisfied if (either) $\alpha = 1$, whence $\xi$ is a homothetic vector and $\mu + 3p = 0$ as before (or $k = 0$).

It is curious to note, however, that if we consider the perfect fluid source to be due to two separate comoving perfect fluids (Coley and Tupper, 1986), so that

$$\mu = \mu_1 + \mu_2, \quad p = p_1 + p_2, \quad (4.75)$$

where

$$\mu_1 = \frac{3}{\alpha^2} t^{-2}; \quad p_1 = (2\alpha/3 - 1)\mu_1 \quad (4.76)$$

and

$$\mu_2 = \frac{3k}{d^2} t^{-2/\alpha}; \quad p_2 = -\frac{1}{3} \mu_2 \quad (4.77)$$

then

$$\mathcal{L}_{\xi^i} \mu_1 = -2\alpha \mu_1, \quad \mathcal{L}_{\xi^i} p_1 = -2\alpha p_1$$

$$\mathcal{L}_{\xi^i} \mathcal{L}_{\xi^i} \mu_2 = -2\mu_2, \quad \mathcal{L}_{\xi^i} \mathcal{L}_{\xi^i} p_2 = -2p_2 \quad (4.78)$$

where $\xi^i = \alpha \frac{\partial}{\partial t}$, so that each separate fluid satisfies equations of the form (1.12).

If $\alpha = 0$, then equation (4.71) yields

$$R = dt^{1/\beta}. \quad (4.79)$$

That is, FRW spacetimes with (4.79) admit a vector field $\xi^i = \alpha \frac{\partial}{\partial t} = \alpha t u$ satisfying equations (1.10) and (1.11). If $k = 0$, then spacetimes of the form (4.64) admit additional kinematic self-similarities parallel to $u$. There is no analogous result in the case of a homothetic vector. If $k \neq 0$, however, then equations (1.12) cannot be satisfied. Again, if we consider the two separate comoving perfect fluids interpretation (4.75) with

$$\mu_1 = \frac{3}{\beta^2} = -p_1, \quad (4.80)$$

$$\mu_2 = \frac{3k}{d^2} e^{-2t/\beta} = -3p_2, \quad (4.81)$$
then each separate fluid satisfies equations of the form (1.12).

If \( c \neq 2 \) (\( \xi^r 
eq 0 \)), then from equation (4.70) we must have \( k = 0 \); i.e., the FRW models are of zero curvature. If \( \alpha^2 + \beta^2 \neq 0 \), then solving equation (4.69) yields the spacetimes (4.59) and (4.64) obtained earlier. If \( \alpha = \beta = 0 \), then \( \xi^r = 0 \).

(b) \( \xi \) orthogonal to \( u \). If \( \alpha = \beta = 0 \), then \( \xi^r = 0 \) and the vector \( \xi \) is orthogonal to the fluid velocity vector. Equation (4.69) simply yields \( c = 0 \), so that

\[ \xi_\perp = r \frac{\partial}{\partial r}; \]  

that is, every flat FRW model admits a vector field of the form (4.82) which satisfies the conditions (1.10) and (1.11); in fact,

\[ \mathcal{L}_{\xi_\perp} u_a = 0. \]  

In addition, since \( \mu = \mu(t) \) and \( p = p(t) \), it follows immediately that

\[ \mathcal{L}_{\xi_\perp} \mu = 0 = \mathcal{L}_{\xi_\perp} p, \]  

so that equations (1.12) are trivially satisfied. Hence, every flat FRW model (for any \( R(t) \)) admits a vector field \( \xi_\perp \), given by (4.82), orthogonal to \( u \), that satisfies equations (1.10), (1.11), (1.12); in particular, \( \mathcal{L}_{\xi_\perp} h_{ab} = 2h_{ab} \) and equations (4.83) and (4.84) are satisfied.

5. Conclusion

After a brief review of self-similarity and its applications in general relativity, the covariant notion of a kinematic self-similarity in the context of relativistic fluid mechanics was introduced (Carter and Henriksen, 1989), which it was argued is the natural relativistic counterpart of self-similarity of the more general second (or zeroth) kind and hence is a generalization of a homothety which corresponds to self-similarity of the first kind (Cahill and Taub, 1971). Various mathematical and physical properties of spacetimes admitting a kinematic self-similarity were discussed. We note the relationship between a kinematic self-similarity and what Collins and Szafron (1979) term an intrinsic symmetry and what Tomita (1981) refers to as a partial homothety (however, see also Tomita and Jantzen, 1983).

The governing equations (adopted from Ellis, 1971) of the perfect fluid (cosmological) models under investigation were introduced, and a set of integrability conditions for the existence of a proper kinematic self-similarity in the spacetime models was derived. These important constraints, the integrability conditions, given by equations (2.16) in general and by equations (2.27) - (2.30) in the particular case of a perfect fluid, played a central role in the resulting analysis. All of the relevant equations were then given in the physically important case of zero vorticity.

A. Summary

Exact solutions of the irrotational perfect fluid Einstein field equations admitting a kinematic self-similarity were then sought in a number of special cases. Since the integrability conditions constitute very severe constraints, such solutions are necessarily of a particularly simple form.

First the geodesic (i.e., zero acceleration) case was considered in subsection 3.A. It was proven that (provided the shear is non-zero) the 3-spaces orthogonal to \( u \) are Ricci-flat, that is, \( ^3R_{ab} = 0 \). This case consequently merits further consideration since there have been various studies of spacetimes with vanishing 3-Ricci tensor (see, for example, Collins and Szafron, 1979, and Stephani and Wolf, 1986). It was further proven that \( a = -2\alpha \), and the form of the kinematic self-similar vector \( \xi \) and the pressure were given in the particular coordinates (3.1)/(3.2) by equations (3.16) and (3.18), respectively (\( a = b = 0 \) and \( p = \text{constant} \) in the special case \( \alpha = 0 \)). The expansion was shown to be governed by the differential equation (3.23)/(3.24), which was in fact integrated in the special case \( \alpha = 0 \) [see equation (3.36)].

The further specialization to dust (i.e., zero pressure) was then considered in subsection 3.B. In this case the governing differential equation (3.23) reduces to equation (3.32) which can be completely integrated to obtain the expansion (see eqns. (3.40) - (3.43)). The asymptotic properties of this class of solutions was studied, and it was found that the resulting models are asymptotic (at late times) to an exact flat FRW (Einstein-de Sitter) model and are asymptotic (at early times) to an exact vacuum (3-Ricci flat) model with \( 3\sigma^2 + \theta^2 = t^{-2} \) for a large class of models this was shown.
to be the Kasner (Bianchi I) solution. We note that these exact (asymptotic) solutions are known to admit a homothetic vector.

The case of zero expansion, studied in subsection 3.C, was shown to subdivide into two subcases. In the first subcase the models are necessarily shear-free with zero magnetic part of the Weyl tensor, and in general $p = (\gamma - 1)\mu$ (and equation (3.94) is satisfied) and the resulting models are completely static. In the second subcase the models are necessarily stiff ($p = \mu$) non-static perfect fluid models. In the coordinates (3.61)/(3.62) the form of the kinematic self-similarity was found to be given by equation (3.65) and either equation (3.95) or (3.106).

In subsection 4.A perfect fluid spacetimes admitting a kinematic self-similar vector $\xi$ parallel to the velocity vector $u$ were studied, and it was proven that such spacetimes are necessarily FRW spacetimes with $p = \frac{\alpha}{\mu} - 1$ and $\mu = \frac{\theta^2}{\alpha} - t^{-2}$. In the case that $\xi$ is orthogonal to $u$, it was shown in subsection 4.B that in general $\mu = 0$ or $p = (\gamma - 1)\mu$ with $\gamma = b/(\alpha + b)$ and $\theta$ satisfies the differential equation (4.40) (and $a = b$), and that either $a = -2$ whence equations (4.42) - (4.44) follow or $a = -2\alpha$ and equations (4.45) - (4.47) result.

Finally, in subsection 4.C, the existence of kinematic self-similarities were studied in FRW spacetimes. In the general case in which $\xi$ is neither parallel nor orthogonal to $u$, it was shown that if $\alpha \neq 0$ then the resulting FRW model is flat with $p = (\gamma - 1)\mu$ and the scale factor is of a simple power-law form and $\xi$ is given by (4.60) in conventional coordinates. Indeed, each such FRW model admits kinematic self-similar vectors for all values of $\alpha \neq 0$ in addition to a homothetic vector. On the other hand, if $\alpha = 0$ the FRW model is necessarily a flat de Sitter model with $\mu + p = 0$ and $\xi$ is given by (4.65). Hence the de Sitter spacetime admits a self-similarity of the zeroth kind (note that such a spacetime cannot admit a homothety). In addition, all FRW models with a scale factor of the power-law form (4.72) with $\alpha \neq 0$ [and with a scale factor of the exponential form (4.79) when $\alpha = 0$] were shown to admit a vector field $\xi$ parallel to $u$ which satisfies equations (1.10) and (1.11). Here the curvature need not be zero; however, in the flat case the FRW models admit these special vector fields (parallel to $u$) in addition to the kinematic self-similar vectors mentioned above. Finally, every flat FRW model (with a scale factor of any form) was shown to admit a vector field $\xi$ of the form (4.82), orthogonal to $u$, which satisfies equation (1.10) and equations (1.11) and (1.12) [see equations (4.83) and (4.84)].

**B. Discussion**

There are a variety of circumstances in general relativity theory, and particularly in cosmology, in which self-similar models act as asymptotic states of more general models. Indeed, in a number of classes of perfect fluid cosmological models with equation of state $p = (\gamma - 1)\mu$ and in which the governing equations reduce to a dynamical system, including, for example, spatially homogeneous models and silent universe models, and in some cases spherically symmetric models and $G_2$ models, it is known that exact solutions admitting a homothetic vector play an important role in describing the asymptotic properties of these models (see Coley, 1996, for a review and appropriate references).

For example, orthogonal spatially homogeneous models have attracted much attention since the governing equations reduce to a relativity simple finite dimensional system of autonomous ordinary differential equations (Wainwright and Ellis, 1996 - henceforward WE). Wainwright and collaborators (see Refs. in WE and Coley, 1996) have utilized an orthonormal frame approach and introduced an expansion-normalized (and hence dimensionless) set of variables to study these models. In particular, it was proven that all the singular points of the (corresponding “reduced”) system (of ordinary differential equations) correspond to exact solutions admitting a homothetic vector (Hsu and Wainwright, 1986). It is in this sense that self-similar models play an important role in describing the dynamics of spatially homogeneous models asymptotically. [The situation is complicated by the fact that in the more general classes of Bianchi models there exist more complicated attractors than simple singular points; for example, in models of Bianchi type IX (and VIII) there is oscillatory behaviour with chaotic-like characteristics as one follows the evolution into the past towards the initial singularity due to the existence of a 2-dimensional attractor in the 5-dimensional phase space (WE).]

In addition, in the class of inhomogeneous $G_2$ cosmological models (in which the spacetime admits two commuting spacelike Killing vectors acting orthogonally transitively) it has been shown by Hewitt and Wainwright (1990) that the governing Einstein field equations can be written as an (finite dimensional) autonomous system of first-order quasi-linear partial differential equations in terms of two independent dimensionless variables, and it was proven that the associated dynamical equilibrium states correspond to exact cosmological solutions that admit a homothetic vector (and
which are consequently self-similar). In a particular subclass of $G_2$ cosmologies, the separable diagonal $G_2$ models, it was shown that the models do indeed asymptote towards the dynamical equilibrium points, and Wainwright and Hewitt have conjectured that this may be the case for more general $G_2$ models. Hence self-similar models may play an important role in describing the asymptotic dynamical behaviour of these inhomogeneous cosmological models.

In this paper we have studied models which admit a kinematic self-similar vector. We note that in all cases in which we have either been able to integrate the equations to obtain exact solutions or we have been able to determine the asymptotic behaviour of a class of models, the asymptotic behaviour has been represented by an exact solution admitting a proper homothetic vector (and hence a model which is self-similar of the first kind). For example, in the pressure-free case studied in subsection 3.B, models were found to be asymptotic to the Einstein-de Sitter model (to the future) and the Kasner model (to the past), and both of these exact solutions admit a homothetic vector. In addition, the particular, exact, perfect fluid spherically symmetric solutions studied by Benoit and Coley (1996) [which include the dust solutions of Lynden-Bell and Lemos (1988) and Carter and Henriksen (1989)] were shown to be asymptotic (both to the past and to the future) to exact FRW models which admit a homothetic vector, and it was also shown in Benoit and Coley (1996) that the equilibrium points at finite values of the autonomous system of ordinary differential equations which govern the general class of perfect fluid spherically symmetric, kinematic self-similar models correspond to exact solutions which admit a homothetic vector. It appears that the same is true for the analogous solutions in the case of plane symmetry (work in progress).

It would be interesting to determine whether this is true in more generality. That is, it is an interesting and important question to determine the conditions under which models admitting a proper kinematic self-similarity are asymptotic to an exact homothetic solution. This might then shed light on when self-similar models of the second kind play an important role in determining the “intermediate asymptotic” behaviour of solutions (Barenblatt and Zeldovich, 1972), and the role of generalized self-similar models in describing the asymptotic properties of models.

The exception to the above is the de Sitter model. This flat FRW model with equation of state $\mu = -p = \text{constant}$ (or equivalently with a cosmological constant) does not admit a homothetic vector and is not asymptotic to a solution that does. Hence it cannot be true that kinematic self-similar models are asymptotic to exact homothetic solutions under all circumstances.

The de Sitter model does, however, admit a self-similarity of the zeroth kind (see subsection 4.C). It is curious to note that the de Sitter model acts as an asymptotic state of more general models (according to the so-called cosmic no-hair theorems). In particular, it was shown by Wald (1983) that spatially homogeneous Bianchi models with a cosmological constant (except those of Bianchi type IX which recollapse) are future asymptotic to the de Sitter model. The existence of the de Sitter model as a future asymptotic state is indicative of exponential inflation. Hence, it is of interest to determine under what conditions the asymptotic states of cosmological models are represented by solutions of Einstein’s field equations admitting a generalized self-similarity (i.e., not just a homothety). This question deserves to be studied further.

Recently Wainwright’s work has been generalized to the case of imperfect fluid Bianchi models satisfying the non-causal linear Eckart theory of irreversible thermodynamics (Coley and van den Hoogen, 1994) and the causal theory (both the truncated version and the full theory) of Israel and Stewart (see Coley et al., 1996, and references within). Dimensionless physical variables (similar to those used by Wainwright) were utilized and a set of “dimensionless equations of state” were assumed, whence it was again shown that in general the singular points of the resulting dynamical system are represented by exact homothetic solutions (Coley and van den Hoogen, 1994). In the exceptional cases the singular points correspond to models which violate the strong energy conditions and have constant expansion, and the models are analogues of the de Sitter solution (with the viscous terms mimicking a cosmological constant) and are consequently self-similar of the zeroth kind.

Moreover, in other work viscous fluid models have been studied (particularly in the case of simple FRW and Bianchi spacetimes) in which the governing equations reduce to a (simple) system of autonomous ordinary differential equations, but since particular equations of state were assumed that are not of a “dimensionless” form the associated singular points do not necessarily correspond to exact solutions admitting a homothetic vector. In this work the viscosity coefficients are modeled by both the non-causal Eckart and causal Israel-Stewart theories of irreversible thermodynamics; the reader is directed to the research papers of the Polish and Russian groups and the Russian and Spanish groups, respectively, which are fully referenced in the papers by Coley and collaborators.
cited here.] In a preliminary investigation of this work it appears that all singular points correspond either to an exact solution which is known to admit a homothetic vector, to an exact zero-curvature FRW model (not necessarily admitting a homothety), or to a de Sitter-like solution. Hence, all of these models appear to be asymptotic to models which admit a kinematic self-similarity of the zeroth, first or second kind. Clearly this needs to be studied further.

However, it is clear that kinematic self-similar models play an important role in describing the asymptotic properties of cosmological models. It is interesting to ask to what extent are cosmological models (or, rather, what is the class of solutions of Einstein’s field equations which are) asymptotic to self-similar solutions, when self-similarity is understood in its more general sense.

APPENDIX: THE VACUUM CASE.

Vacuum spacetimes admitting a homothetic vector were studied by McIntosh (1975), in which it was shown that a non-flat vacuum spacetime can only admit a non-trivial homothetic vector if that homothety is non-null and is not hypersurface orthogonal.

The study of kinematic self-similar vectors in vacuum spacetimes is not physically well-motivated, since in general there does not exist a physically or intrinsically defined timelike vector \( \mathbf{u} \) with respect to which the metric can be uniquely decomposed (unlike in the case of a perfect fluid spacetime, for example, in which there exists such a vector which has both an intrinsic physical and geometrical role - it is both tangent to the fluid flow and is the unique normalized timelike eigenvector of the Ricci tensor), and consequently there exists no such \( \mathbf{u} \) with respect to which the definition (1.10)/(1.11) can be applied. However, for curiosities sake, let us study the consequences of the existence of an intrinsically defined timelike vector \( \mathbf{u} \) which satisfies equations (1.10) and (1.11) in a vacuum spacetime.

From subsection 2B (for \( \omega = 0 \)), in the case \( \mu = p = 0 \) we obtain the equations

\[
\dot{c}^c = 0, \quad (A.1)
\]

and

\[
\dot{\theta} + \theta^2 = 0, \quad (A.2)
\]

and hence

\[
\sigma^2 = \frac{1}{3} \theta^2, \quad (A.3)
\]

and consequently

\[
3R = 0. \quad (A.4)
\]

Let us adopt coordinates so that the metric functions are defined through (3.61) and equations (3.62) and (3.63) are valid, whence on defining the new time parameter, \( \tau \), by

\[
\Phi' = \theta U \partial \Phi \partial t, \quad (A.5)
\]

we can then integrate equation (A.2) to obtain

\[
\theta = \Theta(x^\gamma)e^{-\tau}, \quad (A.6)
\]

whence equation (A.3) yields

\[
\sigma^2 = \frac{1}{3} \Theta^2 e^{-2\tau}, \quad (A.7)
\]

which also follows from (2.37) [when \( \sigma \neq 0 \); if \( \sigma = 0 \), then it follows that \( \theta = 0 \)]. By definition we have that

\[
H^{\alpha\beta} H'_{\alpha\beta} = 2, \quad (A.8)
\]

and

\[
\sigma_{\alpha\beta} = \frac{\theta}{2} H'_{\alpha\beta} - \frac{\theta}{3} H_{\alpha\beta}, \quad (A.9)
\]
and
\[ 3R_{\alpha\beta} = \frac{1}{U} U_{,\alpha\beta} - \frac{1}{U} \Gamma_{\alpha\beta}^\gamma U_{,\gamma} \quad (A.10) \]

(all other components of \( \sigma_{ab} \) and \( 3R_{ab} \) vanish). Finally, equation (2.37) yields
\[ \sigma'_{\alpha\beta} - H_{\gamma\delta} \sigma_{\gamma(\alpha} H'_{\beta)} + \sigma_{\alpha\beta} = 0, \quad (A.11) \]
and using (A.9) we consequently obtain
\[ H''_{\alpha\beta} - H_{\gamma\delta} H'_{\alpha\gamma} H'_{\beta\delta} = 0. \quad (A.12) \]

In the case that \( H_{\alpha\beta} \) is diagonal, i.e., \( H_{\alpha\beta}(\tau, x^\gamma) \equiv \text{diag}\{h_1, h_2, h_3\} \), we can integrate equations (A.12) to obtain
\[ h_\nu(\tau, x^\gamma) = G_\nu(x^\gamma) e^{2F_\nu(x^\gamma)} \tau \quad (\nu = 1, 2, 3), \quad (A.13) \]
whence equation (A.8) then implies that
\[ F_1 + F_2 + F_3 = 1. \quad (A.14) \]

We note that the spatially homogeneous vacuum Bianchi I (Kasner) model is a particular solution of equations (A.2) - (A.4), (A.8) and (A.12) in which \( U = 1 \) (and hence \( 3R_{ab} = 0 \)).

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REFERENCES

G. I. Barenblatt, 1952, Prikl. Mat. Mekh. 16, 67.
G. I. Barenblatt and Yu B. Zeldovich, 1972, Ann. Rev. Fluid Mech. 4, 285.
P. M. Benoit and A. A. Coley, 1996, Class. Q. Grav., submitted.
E. Bertshinger, 1985, Ap. J. 268, 17.
A. H. Cahill and M. E. Taub, 1971, Comm. Math. Phys. 21, 1.
B. Carter and R. N. Henriksen, 1989, Ann. Physique Supp. 14, 47.
B. Carter and R. N. Henriksen, 1991, J. Math. Phys. 32, 2580.
J. Castejon-Amendo and A. A. Coley, 1992, Class. Q. Grav. 9, 2203.
A. A. Coley, 1991, Class Q. Grav. 8, 955.
A. A. Coley, 1996, in the Proceedings of the Sixth Canadian Conference on General Relativity and Relativistic Astrophysics, the Fields Institute Communications Series (AMS), eds. S. P. Braham, J. D. Gegenberg and R. J. McKellar (Providence, RI).
A. A. Coley and R. J. van den Hoogen, 1994, J. Math Phys. 35, 4117.
A. A. Coley, R. J. van den Hoogen and R. Maartens, 1996, Phys. Rev. D 54, 1393.
A. A. Coley and D. J. McMamis, 1994, Class. Q. Grav. 11, 1261.
A. A. Coley and B. O. J. Tupper, 1986, J. Math. Phys. 27, 406.
C. B. Collins and D. A. Szafron, 1979, J. Math. Phys. 20, 2347.
L. Defrise-Carter, 1975, Comm. Math. Phys. 40, 273.
D. M. Eardley, 1974, Comm. Math. Phys. 37, 287.
G. F. R. Ellis, 1971, Relativistic Cosmology, in General Relativity and Cosmology, XLVII Corso, Varenna, Italy (1969), ed. R. Sachs (Academic, New York).
S. W. Goode and J. Wainwright, 1982, Phys. Rev. D 26, 3315.
R. N. Henriksen, 1989, MNRAS 240, 917.
R. N. Henriksen, A. G. Emslie and P. S. Wesson, 1983, Phys. Rev. D 27, 1219.
C. G. Hewitt and J. Wainwright, 1990, Class. Quantum Grav. 7, 2295.
L. Hsu and J. Wainwright, 1986, Class. Quantum Grav. 3, 1105.
S. Ikeuchi, K. Tomisaka and J. P. Ostriker, 1983, Ap. J. 265, 583.
D. Kramer, H. Stephani, M. A. H. MacCallum and E. Herlt, 1980, *Exact Solutions of Einstein’s Field Equations* (Cambridge University Press, Cambridge).

D. Lynden-Bell and J. P. S. Lemos, 1988, MNRAS 233, 197.

R. Maartens and S. D. Maharaj, 1986, Class. Q. Grav. 3, 1005.

C. B. G. McIntosh, 1975, Gen. Rel. Grav. 7, 199.

J. Ponce de Leon, 1993, Gen. Rel. Grav. 25, 865.

J. Schwartz, J. P. Ostriker and A. Yahil, 1975, Ap. J. 202, 1.

L. I. Sedov, 1967, *Similarity and Dimensional Methods in Mechanics* (New York, Academic).

H. Stephani and Th. Wolf, 1986, in *Galaxies, Axisymmetric Systems and Relativity*, ed. M. A. H. MacCallum, p.275 (Cambridge University Press, Cambridge).

K. Tomita, 1981, Prog. Theoret. Phys. 66, 2025.

K. Tomita and R. T. Jantzen, 1983, Prog. Theoret. Phys. 70, 886.

J. Wainwright and G. F. R. Ellis, 1996, *Dynamical Systems in Cosmology* (Cambridge University Press, Cambridge).

R. M. Wald, 1983, Phys. Rev. D. 28, 2118.

K. Yano, 1955, *The Theory of Lie Derivatives* (North Holland, Amsterdam).