ON THE EFFECTIVE MEMBERSHIP PROBLEM FOR POLYNOMIAL IDEALS

MATS ANDERSSON & ELIZABETH WULCAN

Abstract. We discuss the possibility of representing elements in polynomial ideals in $\mathbb{C}^N$ with optimal degree bounds. Classical theorems due to Macaulay and Max Noether say that such a representation is possible under certain conditions on the variety of the associated homogeneous ideal. We present some variants of these results, as well as generalizations to subvarieties of $\mathbb{C}^N$.

1. Introduction

Let $V$ be an algebraic subvariety of $\mathbb{C}^N$ of pure dimension $n$ and let $F_1, \ldots, F_m$ be polynomials in $\mathbb{C}^N$. We are interested in finding solutions to the polynomial division problem

$$F_1 Q_1 + \cdots + F_m Q_m = \Phi$$

on $V$ with degree estimates, provided $\Phi$ is in the ideal $(F_j)$ on $V$. By a result of Hermann, [15], if $\deg F_j \leq d$, there are polynomials $Q_j$ such that $\deg (F_j Q_j) \leq \deg \Phi + C(d, N)$, where $C(d, N)$ is like $2(2d)^{2N-1}$ for large $d$ and thus doubly exponential. It is shown in [24] (see also [10, Example 3.9]) that in general this estimate cannot be substantially improved.

If one imposes conditions on $V$ and $F_j$ one can, however, obtain much sharper estimates. The following two results in $\mathbb{C}^n$ are classical.

If $F_1, \ldots, F_m$ are polynomials in $\mathbb{C}^n$ of degrees $d_1 \geq \ldots \geq d_m$ with no common zeros even at infinity and $\Phi$ is any polynomial, then one can solve (1.1) with $\deg (F_j Q_j) \leq \max(\deg \Phi, d_1 + \ldots + d_n + 1 - n)$.

If $F_1, \ldots, F_n$ are polynomials in $\mathbb{C}^n$ such that their common zero set is discrete and does not intersect the hyperplane at infinity, and $\Phi$ belongs to the ideal $(F_j)$, then one can find polynomials $Q_j$ such that (1.1) holds and $\deg (F_j Q_j) \leq \deg \Phi$.

The first theorem is due to Macaulay, [23], and the second one is Max Noether’s AF+BG theorem, [25], originally stated for $n = 2$. Noether’s result is clearly optimal.

In this paper we present extensions of these results to the case of more general varieties $V \subset \mathbb{C}^N$, and also generalizations in which we relax the condition on (the zero set of) the $F_j$. It grew out of our paper [9], in which we extended to the singular setting a framework for solving polynomial ideal membership problems with residue techniques introduced in [34] and further developed in [5, 39, 31], see below. The proofs in this paper follow the same setup. However, at least some of the results also admit algebraic proofs, see Remark 6.2.

Date: May 5, 2014.

The first author was partially supported by the Swedish Research Council. The second author was partially supported by the Swedish Research Council and by the NSF.
Throughout we will let $X$ denote the closure of $V$ in $\mathbb{P}^N$, and $\text{reg } X$ the regularity of $X$, see Section 1 for the definition. For each $F_j$ we let $f_j$ denote the induced section of $O(deg F_j)|_X$.

We begin with an extension of Macaulay’ theorem to singular varieties; this can easily be proved by standard arguments, cf. Remark 0.2.

**Theorem 1.1.** Let $V$ be an algebraic subvariety of $\mathbb{C}^N$, with closure $X$ in $\mathbb{P}^N$, and let $F_1, \ldots, F_m$ be polynomials in $\mathbb{C}^N$ of degrees $d_1 \geq \ldots \geq d_m$. Assume that $f_j$ have no common zeros on $X$. Then for each polynomial $\Phi$ in $\mathbb{C}^N$ there are polynomials $Q_j$ such that (1.1) holds and

$$deg(F_jQ_j) \leq \max(deg \Phi, d_1 + \cdots + d_{n+1} - (n + 1) + \text{reg } X).$$

If $X$ is smooth, then reg $X \leq (n+1)(\text{deg } X - 1) + 1$; this is Mumford’s bound, see, e.g., [22, Example 1.8.48]. If $X$ is Cohen-Macaulay in $\mathbb{P}^N$ (and $N$ is minimal) then reg $X \leq \text{deg } X - (N - n)$, see, [17, Corollary 4.15]. In particular, if $V = \mathbb{C}^n$ so that $X = \mathbb{P}^n$, then reg $X = 1$; thus we get back Macaulay’s theorem. For a discussion of bounds on reg $X$ for a general $X$, see, e.g., [10, Section 3].

Let $Z^f$ denote the common zero set of $f_1, \ldots, f_m$ in $X$. Moreover, let $X_\infty := X \setminus V$. For smooth varieties we have the following version of Max Noether’s theorem.

**Theorem 1.2.** Let $V$ be an algebraic subvariety of $\mathbb{C}^N$ of dimension $n$ such that its closure $X$ in $\mathbb{P}^N$ is smooth, and let $F_1, \ldots, F_m$ be polynomials in $\mathbb{C}^N$ of degrees $d_1 \geq \ldots \geq d_m$. Assume that $m \leq n$, that

$$\text{codim } (Z^f \cap V) \geq m,$$

and that $Z^f$ has no irreducible component contained in $X_\infty$. If $\Phi$ is a polynomial that belongs to the ideal $(F_j)$ in $V$, then there is a representation (1.1) with

$$deg(F_jQ_j) \leq \max(deg \Phi, d_1 + \cdots + d_m - m + \text{reg } X).$$

If in addition $X$ is Cohen-Macaulay in $\mathbb{P}^N$ one can choose $Q_j$ so that

$$deg(F_jQ_j) \leq deg \Phi.$$

**Remark 1.3.** If $X$ is Cohen-Macaulay it suffices that $V$ is smooth to obtain (1.4). \hfill $\Box$

For $V = \mathbb{C}^n$ Theorem 1.2 appeared in [3, Theorem 1.2].

For a general $X$, in order to have a Max Noether theorem, we need the common zero set of the $f_j$ not to intersect the singular locus of $X$ too badly. To make this statement more precise we need to introduce what we call the intrinsic BEF-varieties

$$X^{n-1} \subset \cdots \subset X^1,$$

of $X \subset \mathbb{P}^N$. These are the sets where the mappings in a locally free resolution of $O_{\mathbb{P}^N}/J_X$ do not have optimal rank. They are intrinsically defined subvarieties of $X$ that are contained in $X^0 := X_{\text{sing}}$. The codimension of $X^\ell$ is at least $\ell + 1$, and if $X$ is locally Cohen-Macaulay $X^\ell$ is empty for $\ell \geq 1$, see Sections 2.3 and 2.4.

**Theorem 1.4.** Let $V$ be an algebraic subvariety of $\mathbb{C}^N$ of dimension $n$, with closure $X$ in $\mathbb{P}^N$, and let $F_j$ be as in Theorem 1.2. Assume that $Z^f$ satisfies (1.2), that $Z^f$ has no irreducible component contained in $X_\infty$, and moreover that

$$\text{codim } (Z^f \cap X^\ell) \geq m + \ell + 1, \quad \ell \geq 0.$$
If $\Phi$ is a polynomial that belongs to the ideal $(F_j)$ in $V$, then there is a representation (1.1) such that (1.3) holds. If in addition $X$ is Cohen-Macaulay in $\mathbb{P}^N$, and $m \leq n$, we can choose $Q_j$ such that (1.4) holds.

Notice that (1.5) forces that either $Z^f \cap X_\text{sing} = \emptyset$ or $m < n$. If $X$ is smooth, then (1.5) is vacuous, and thus Theorem 1.2 follows immediately from Theorem 1.4. If only $V$ is smooth but $X$ is Cohen-Macaulay, then by the assumption on $Z^f$ codim$(Z^f \cap X_\infty) \geq m + 1$ and since $X^0 \subset X_\infty$, (1.5) is satisfied. This proves the claim in Remark 1.3.

Next we will present some generalizations of Theorem 1.4 where we relax the hypotheses on the common zero set $Z^f$ of the $f_j$. First, we drop the size hypothesis (1.2) on $Z^f \cap V$. We then still get an estimate of the form (1.3) but the second entry on the right hand side is now replaced by a constant that depends on $F_j$ in a more involved manner. The condition that $Z^f$ has no irreducible component at infinity should now be understood as that the ideal sheaf $J_f$ over $X$ generated by the sections $f_1, \ldots, f_m$ has no associated variety, in the sense of [28], contained in $X_\infty$, see Section 3. This means that at each $x \in X_\infty$, $(J_f)_x$ has no (varieties of) associated prime ideals contained in $X_\infty$. Let $J_f$ be the homogeneous ideal in $\mathbb{C}[z_0, \ldots, z_N]$ associated with $J_f$, and let $\text{reg} J_f$ be the regularity of $J_f$, cf. Section 4.

**Theorem 1.5.** Let $V$ be an algebraic subvariety of $\mathbb{C}^N$, with closure $X$ in $\mathbb{P}^N$, and let $F_1, \ldots, F_m$ be polynomials in $\mathbb{C}^N$. Assume that $J_f$ has no associated variety contained in $X_\infty$. Then there is a constant $\beta = \beta(X,F_1,\ldots,F_m)$ such that if $\Phi \in (F_j)$, then there is a representation (1.1) on $V$ with

\[ \text{deg}(F_j Q_j) \leq \max(\text{deg} \Phi, \beta). \]

If $V = \mathbb{C}^N$, one can take $\beta = \text{reg} J_f$.

Conversely, if there is an associated prime of $J_f$ contained in $X_\infty$, then there is no $\beta$ such that one can solve (1.1) with (1.6) for all $\Phi$ in $(F_j)$.

In [27] Shiffman computed the regularity of a zero-dimensional homogeneous polynomial ideal $J_f$ to be $\leq d_1 + \ldots + d_{n+1} - n$. Using this he obtained (the first part of) Theorem 1.5 for $V = \mathbb{C}^N$ and $\dim Z^f = 0$ with $\beta = \text{reg} J_f = d_1 + \ldots + d_{n+1} - n$, i.e., the same bound as in Macaulay’s theorem, see [27] Theorem 2(iv). Theorem 1.5 can thus be seen as a generalization of Shiffman’s result.

The estimate (1.6) is clearly sharp if $\text{deg} \Phi \geq \beta$. If the ideal sheaf $J_f$ is locally Cohen-Macaulay, for instance locally a complete intersection, then there are no embedded primes of $J_f$, and so the hypothesis that $J_f$ has no associated variety at infinity just means that no irreducible component of $Z^f$ is contained in $X_\infty$. Thus we get back the hypothesis in Theorems 1.2 and 1.4.

Next, let us instead relax the condition that $Z^f$ has no irreducible components at infinity. If the degrees of $F_j$ are $\leq d$, we let $\tilde{f}_j$ denote the section of $O(d)|_X$ corresponding to $F_j$. We let $Z^f$ be the common zero set of $\tilde{f}_1, \ldots, \tilde{f}_m$ and $J_f$ the coherent analytic sheaf over $X$ generated by the $\tilde{f}_j$. Moreover, we let $c_\infty$ be the maximal codimension of the so-called (Fulton-MacPherson) distinguished varieties of $J_f$ that are contained in $X_\infty$, see Section 5.1. If there are no distinguished varieties contained in $X_\infty$, then we interpret $c_\infty$ as $-\infty$. Note that it is not sufficient that $Z^f \cap V = Z^f$, since there may be embedded distinguished varieties contained in $X_\infty$. 

ON THE EFFECTIVE MEMBERSHIP PROBLEM FOR POLYNOMIAL IDEALS 3
It is well-known that the codimension of a distinguished variety cannot exceed the number \( m \), see, e.g., Proposition 2.6 in [15], and thus \( c_\infty \leq \mu \), where \( \mu := \min(m, n) \).

**Theorem 1.6.** Let \( V \) be an algebraic subvariety of \( \mathbb{C}^N \), with closure \( X \) in \( \mathbb{P}^N \), and let \( F_1, \ldots, F_m \) be polynomials in \( \mathbb{C}^N \) of degree \( \leq d \). Assume that \( Z_{\tilde{f}} \) satisfies
\begin{align}
(1.7) \quad \text{codim } (Z_{\tilde{f}} \cap X) & \geq m \\
(1.8) \quad \text{codim } (Z_{\tilde{f}} \cap X^\ell) & \geq m + \ell + 1, \quad \ell \geq 0.
\end{align}
If \( \Phi \) is a polynomial that belongs to \((F_j)\) on \( V \), then there is a representation (1.1)
\begin{align}
(1.9) \quad \text{deg } (F_jQ_j) & \leq \text{max}(\text{deg } \Phi + \mu d c_\infty \text{deg } X, \text{deg } X).
\end{align}

Note that for most choices of \( F_j \) and \( \Phi \) the first entry in (1.9) is much larger than the second entry. For instance this is true for all \( \Phi \) if \( c_\infty \geq 2 \) and \( d \) is large enough. In particular, if \( X = \mathbb{P}^n \), so that \( \text{reg } X = 1 \), and \( c_\infty \geq 2 \), the first entry is the largest for all \( d \).

For \( X = \mathbb{P}^n \) Theorem 1.6 is due to the first author and Götmark, [5, Theorem 1.3]. In the case when \( \text{deg } F_j = d \), so that \( \tilde{f}_j = f_j \), Theorem 1.6 generalizes Theorems 1.1 – 1.4, see Remark 6.3.

**Example 1.7.** If the \( F_j \) have no common zeros on \( V \), then Theorem 1.6 gives a solution to
\[ F_1Q_1 + \cdots + F_mQ_m = 1 \]
with \( \text{deg } (F_jQ_j) \leq \mu d c_\infty \text{deg } X \) if \( d \) is large enough. Except for the annoying factor \( \mu \) we then get back is Jelonek’s optimal effective Nullstellensatz, [20]. \qed

Note that the estimates of \( \text{deg } (F_jQ_j) \) in the theorems above hold for representations of all \( \Phi \) in \((F_j)\). If one, instead of adding conditions on \( V \) and \( F_j \), imposes further conditions on \( \Phi \), then Hermann’s degree estimate for solutions to (1.1) can also be essentially improved. Theorem 1.1 in our recent paper [9] asserts that for any \( V \subset \mathbb{C}^N \) there is a number \( \mu_0 \) such that if \( F_1, \ldots, F_m \) are polynomials in \( \mathbb{C}^N \) of degree \( \leq d \) and \( \Phi \) is a polynomial such that \( |\Phi| \leq C|F|^{|\mu|+\mu_0} \) locally on \( V \), where \( |F|^2 = |F_1|^2 + \cdots + |F_m|^2 \), then one can solve (1.1) with
\begin{align}
(1.10) \quad \text{deg } (F_jQ_j) & \leq \text{max}(\text{deg } \Phi + (\mu + \mu_0) d c_\infty \text{deg } X, \text{deg } X).
\end{align}
The statement that \( |\Phi| \leq C|F|^{|\mu|+\mu_0} \) implies that there is a representation (1.1) is a direct consequence of Huneke’s uniform Briançon-Skoda theorem, [12] [19], and thus the degree estimate (1.10) can be seen as a global effective Briançon-Skoda-Huneke theorem.

**Acknowledgment.** We thank Richard Lärkäng for helpful discussions.
2. Residue currents

We will briefly recall some residue theory. For more details we refer to [9] and the references therein.

2.1. Currents on a singular variety. If nothing else is mentioned $X$ will be a reduced subvariety of $\mathbb{P}^N$ of pure dimension $n$. The sheaf $C_{\ell,k}$ of currents of bidegree $(\ell,k)$ on $X$ is by definition the dual of the sheaf $\mathcal{E}_{n-\ell,n-k}$ of smooth $(n-\ell,n-k)$-forms on $X$. If $i: X \to \mathbb{P}^N$ is an embedding of $X$, then $\mathcal{E}_{n-\ell,n-k}$ can be identified with the quotient sheaf $\mathcal{E}_{n-\ell,n-k}^{\mathbb{P}^N}/\text{Ker } i^*$, where $\text{Ker } i^*$ is the sheaf of forms $\xi$ on $\mathbb{P}^N$ such that $i^*\xi$ vanish on $X_{\text{reg}}$. It follows that the currents $\tau$ in $C_{\ell,k}$ can be identified with currents $\tau' = i_\ast \tau$ on $\mathbb{P}^N$ of bidegree $(N-n+\ell,N-n+k)$ that vanish on $\text{Ker } i^*$.

Given a holomorphic function $f$ on $X$, we write $1/f$ for the \textit{principal value distribution}, defined for instance as $\lim_{\epsilon \to 0} \chi(|f|^2/\epsilon)(1/f)$, where $\chi(t)$ is the characteristic function of the interval $[1, \infty)$ or a smooth approximand of it, or as the analytic continuation of $\lambda \to |f|^{2\lambda}(1/f)$ to $\lambda = 0$. It is readily checked that $f(1/f) = 1$ as distributions and that the \textit{residue current} $\partial(1/f)$ satisfies $f\partial(1/f) = 0$. We will need the fact that

$$v^\lambda |f|^{2\lambda} \left| \frac{1}{f} \right|_{\lambda=0} = \frac{1}{f}$$

if $v$ is a strictly positive smooth function; cf. [11] Lemma 2.1.

2.2. Pseudomeromorphic currents. The notion of pseudomeromorphic currents on manifolds was introduced in [8]. A slightly extended version appeared in [6]: A current on $X$ is \textit{pseudomeromorphic} if it is (the sum of terms that are) the push-forward under (a composition of) modifications, projections, and open inclusions of currents of the form

$$\frac{\xi}{s_1^{\alpha_1} \cdots s_{n-1}^{\alpha_{n-1}} \wedge \partial \frac{1}{s_n^{\alpha_n}}}$$

where $s$ is a local coordinate system and $\xi$ is a smooth form with compact support, see, e.g., [9] for details.

Pseudomeromorphic currents in many respects behave like positive closed currents. For example they satisfy the \textit{dimension principle}: If $\tau$ is a pseudomeromorphic current on $X$ of bidegree $(\ast,p)$ that has support on a variety of codimension $> p$, then $\tau = 0$.

Also, pseudomeromorphic currents allow for multiplication with characteristic functions of constructible sets so that ordinary computational rules hold. If $\tau$ is a pseudomeromorphic current on $X$ and $V$ is a subvariety of $X$, then the natural restriction of $\tau$ to the open set $X \setminus V$ has a canonical extension $1_{X \setminus V} \tau := |h|^{2\lambda}(1/f)|_{\lambda=0}$, where $h$ is any holomorphic tuple such that $\{v = 0\} = V$. It follows that $1_V \tau := \tau - 1_{X \setminus V} \tau$ is a pseudomeromorphic current with support on $V$. Note that if $\alpha$ is a smooth form, then $1_V \alpha \wedge \tau = \alpha \wedge 1_V \tau$ and if $W$ are $W'$ are constructible sets, then

$$1_W 1_{W'} \tau = 1_{W \cap W'} \tau.$$ (2.2)

Moreover, if $\pi: \tilde{X} \to X$ is a modification, $\tilde{\tau}$ is a pseudomeromorphic current on $\tilde{X}$, and $\tau = \pi_\ast \tilde{\tau}$, then

$$1_V \tau = \pi_\ast (1_{\pi^{-1} V} \tilde{\tau})$$ (2.3)
for any subvariety $V \subset X$. If $W$ is a subvariety of $X$ and $1_V \tau = 0$ for all subvarieties $V \subset W$ of positive codimension we say that $\tau$ has the *standard extension property*, SEP with respect to $W$, see \[11\].

Recall that a current is *semi-meromorphic* if it is the quotient of a smooth form and a holomorphic function. Following \[9\] we say that a current $\tau$ is *almost semi-meromorphic* in $X$ if there is a modification $\pi: \tilde{X} \to X$ and a semi-meromorphic current $\tilde{\tau}$ such that $\tau = \pi_* \tilde{\tau}$.

### 2.3. Residue currents associated with Hermitian complexes

Consider a complex of Hermitian holomorphic vector bundles over a complex manifold $Y$ of dimension $n$,

\[
(2.4) \quad 0 \to E_M \xrightarrow{f_M} \ldots \xrightarrow{f^3} E_2 \xrightarrow{f^2} E_1 \xrightarrow{f^1} E_0 \to 0,
\]

that is pointwise exact outside an analytic variety $Z \subset Y$ of positive codimension $p$. Suppose that the rank of $E_0$ is 1. In \[2, 7\] was associated to \((2.4)\) a $\bigoplus \text{Hom} (E_0, E_k)$-valued pseudomeromorphic current $R = R^I$; it has support on $Z$ and in a certain sense it measures the lack of exactness of the associated sheaf complex of holomorphic sections

\[
\begin{aligned}
(2.5) & \quad 0 \to \mathcal{O}(E_M) \xrightarrow{f_M} \ldots \xrightarrow{f^3} \mathcal{O}(E_2) \xrightarrow{f^2} \mathcal{O}(E_1) \xrightarrow{f^1} \mathcal{O}(E_0).
\end{aligned}
\]

**Proposition 2.1.** If $\phi$ is a holomorphic section of $E_0$ such that $R\phi = 0$, then $\phi \in \text{Im} f^1$. Moreover, if

\[
H^{k-1}(Y, \mathcal{O}(E_k)) = 0, \quad 1 \leq k \leq \min(M, n+1),
\]

then there is a global holomorphic section $q$ of $E_1$ such that $f^1 q = \phi$.

We also have the duality principle: If \((2.5)\) is exact, i.e., if it is a locally free resolution of the sheaf $\mathcal{O}(E_0)/\text{Im} f^1$, then $R\phi = 0$ if and only if $\phi \in \text{Im} f^1$.

As in \[9\] we will refer to a (locally) free resolution \((2.5)\) of $\mathcal{O}(E_0)/J$ together with Hermitian metrics on the corresponding vector bundles as a *Hermitian (locally) free resolution*.

Let us look at the construction of $R$ in a special case; see, e.g., \[9\] for more details and the general case. Let $R_k$ denote the component of $R$ that takes values in $\text{Hom} (E_0, E_k)$.

**Example 2.2** (The Koszul complex). Given Hermitian line bundles $S \to Y$ and $L_1, \ldots, L_m \to Y$ and a tuple $f$ of holomorphic sections $f_1, \ldots, f_m$ of $L_1, \ldots, L_m$, respectively, let \((2.4)\) be the (twisted) Koszul complex of $f$: Let $E^j$ be disjoint trivial line bundles with basis elements $e_j$, let $E = L_1^{-1} \otimes E^1 \oplus \cdots \oplus L_m^{-1} \otimes E^m$, and identify $f$ with a section $f = \sum f_j e_j^*$ of $E^*$, where $e_j^*$ are the dual basis elements. Moreover, let

\[
E_0 = S, \quad E_k = S \otimes \Lambda^k E,
\]

and let all $f^k$ in \((2.4)\) be interior multiplication $\delta_f$ by the section $f$.

The current associated with the Koszul complex was introduced in \[11\]; we will briefly recall the construction. Let $\sigma$ be the section of $E$ over $Y \setminus Z$ with pointwise minimal norm such that $f \cdot \sigma = \delta_f \sigma = 1$, i.e.,

\[
\sigma = \sum_j \frac{f^*_j e_j}{|f_j|^2}.
\]
where $f_j^*$ is the section of $L_j^{-1}$ of minimal norm such that $f_j f_j^* = |f_j|^2 g_j$ and $|f|^2 = |f_1|^2 g_1 + \cdots + |f_m|^2 g_m$. Then $R_k$ equals the analytic continuation to $\lambda = 0$ of
\begin{equation}
R^\lambda = R^\lambda e : = \bar{\partial} |f|^2 \lambda \wedge \sigma \wedge (\bar{\partial} \sigma)^{k-1}.
\end{equation}
Here the exterior product is with respect to the exterior algebra over $E \oplus T^* Y$ so that $d z_j / e_l = - e_l / d z_j$ etc; in particular, $\partial \sigma$ is a form of even degree.

If $m = 1$, then $\sigma$ is just $(1/f_1) e_1$ and $R = \bar{\partial} (1/f_1) \wedge e_1$. In general, the coefficients of $R$ are the Bochner-Martinelli residue currents introduced by Passare-Tsikh-Yger \cite{26}. The sheaf complex associated with the Koszul complex is exact if and only if $f$ is a complete intersection, i.e., codim $Z f = m$. In this case one can prove that (the coefficient of) $R = R_m$ coincides with the classical Coleff-Herrera residue current $\bar{\partial} (1/f_1) \wedge \cdots \wedge \bar{\partial} (1/f_m)$.

Since, in light of the above example, $R$ generalizes the classical Coleff-Herrera residue current (as well as the Bochner-Martinelli residue currents), we say that $R$ is the residue current associated with the Hermitian complex \eqref{2.4}.

The construction of $R$ in general involves the minimal inverse $\sigma_k$ of each $f^k$ in \eqref{2.4}; $R$ is defined as the analytic continuation to $\lambda = 0$ of a regularization $R^\lambda$ which generalizes \eqref{2.7}. The component $R_k$ is of the form $\bar{\partial} |f|^2 \lambda \wedge \sigma_k \bar{\partial} \sigma_k \cdots \bar{\partial} \sigma_1 |_{\lambda = 0}$; see, e.g., \cite{24} for a precise interpretation of this. It follows that outside the set $Z_k$ where $f^k$ does not have optimal rank,
\begin{equation}
R_k = \alpha_k R_{k-1},
\end{equation}
where $\alpha_k$ is a smooth $\text{Hom}(E_{k-1}, E_k)$-valued $(0, 1)$-form. If \eqref{2.5} is exact, these sets are independent of the resolution; we call them BEF varieties (which is an acronym for Buchsbaum-Eisenbud-Fitting, cf. \cite{9}) and denote them $Z^\text{bf}_k = Z^\text{bf}_k(\mathcal{J}_f)$. The Buchsbaum-Eisenbud theorem asserts that codim $Z^\text{bf}_k \geq k$; more precisely it says that the complex \eqref{2.5} is exact if and only if the codimension of the set where $f_k$ does not have optimal rank is $\geq k$, see, e.g., \cite{17} Theorem 3.3. If $\mathcal{J}_f$ has pure codimension $p$, then codim $Z^\text{bf}_k \geq k + 1$ for $k > p$, see \cite{16} Corollary 20.14. Also, note that if in addition $X$ is locally Cohen-Macaulay, then $Z_k = 0$ for $k > p$. The current $R_k$ has bidegree $(0, k)$, and thus, by the dimension principle, $R_k = 0$ for $k < p$, and for degree reasons, $R_k = 0$ for $k > n$.

If the complex \eqref{2.4} is twisted by a Hermitian line bundle, the residue current $R$ is not affected. This follows since the $\sigma_k$ are not affected by the twisting.

2.4. BEF-varieties on singular varieties. Let $i : X \to Y$ be a (local) embedding of $X$ of dimension $n$ into a smooth manifold $Y$ of dimension $N$. Note that if $\mathcal{J}_f$ is a coherent ideal sheaf on $X$, then $\mathcal{J}_f + \mathcal{J}_X$ is a well-defined sheaf on $Y$. Indeed, locally $\mathcal{J}_f$ is the pullback $i^* \tilde{\mathcal{J}}_f$ of an ideal sheaf on $Y$ and the sheaf $\mathcal{J}_f + \mathcal{J}_X$ is independent of the choice of $\tilde{\mathcal{J}}_f$. We define $k$th BEF-variety $Z^\text{bf}_k(\mathcal{J}_f)$ of $\mathcal{J}_f$ as $Z^\text{bf}_{k+N-n}(\mathcal{J}_f + \mathcal{J}_X)$, which clearly is a subvariety of $X$.

This definition is independent of the embedding $i$. To see this recall that (locally) $i$ can be factorized as $X \xrightarrow{\iota} \Omega \to \Omega \times \mathbb{C}^r = Y$, where $\iota$ is a minimal embedding. From a locally free resolution of $\mathcal{O}^\Omega / \mathcal{J}$, where $\mathcal{J}$ is a coherent ideal sheaf over $\Omega$, it is not hard to construct a locally free resolution of $\mathcal{O}^\Omega / (\mathcal{J} + \mathcal{J}_\Omega)$. By relating the sets where the mappings in these resolutions do not have optimal rank one can
show that the BEF-varieties of $J$ are independent of $i$, cf. \[1\] Remark 4.6 and \[9\] Section 3.

2.5. The structure form $\omega$ on a singular variety. Now assume that $X$ is as in Section 2.4 and let $R$ be the residue current associated with a Hermitian free resolution $O(E_*)$, $g^*$ of the sheaf $J_X$ of $X$, and let $\Omega$ be a global nonvanishing $(\dim \mathbb{P}^N, 0)$-form with values in $O(N + 1)$. It was shown in \[6\] Proposition 3.3 that there is a (unique) almost semi-meromorphic current $\omega = \omega_0 + \cdots + \omega_{n-1}$ on $X$, that is smooth on $X_{\text{reg}}$ and such that

$$i_* \omega = R \wedge \Omega.$$  

We say that $\omega$ is a structure form on $X$. Let $E^\ell$ denote the restriction of $E_{N-n+\ell}$ to $X$. Then the component $\omega^\ell$ is an $(n, \ell)$-form taking values in $\text{Hom}(E^0, E^\ell)$. Moreover, let $X^0 = X_{\text{sing}}$ and $X^\ell = X_{N-n+\ell}$, where $X_j$ are the BEF-varieties of $J_X$. In the language of the previous section $X^\ell$ is the $\ell$th BEF-variety of the zero sheaf. It follows from that section that the $X^\ell$ are independent of the embedding $i : X \to Y$ of $X$ into a smooth manifold $Y$; we therefore call them the intrinsic BEF-varieties of $X$. In light of (2.8) there are almost semi-meromorphic forms $\alpha^\ell$, smooth outside $X^\ell$, such that

$$\omega^\ell = \alpha^\ell \omega_{\ell-1}.$$ 

on $X$.

3. Gap sheaves and primary decomposition of sheaves

Recall that any ideal $a$ in a Noetherian ring $A$ admits a primary decomposition (or Noether-Lasker decomposition), i.e., it can be written as $a = \bigcap a_k$, where $a_k$ is $p_k$-primary ($ab \in a_k$ implies $a \in a_k$ or $b^s \in a_k$ for some $s$ and $\sqrt{a_k} = p_k$) for some prime ideal $p_k$. The primes in a minimal such decomposition are called the associated primes of $a$ and the set $\text{Ass}(a)$ of associated primes is independent of the primary decomposition.

Given a coherent subsheaf $J$ of $O_X$, Siu \[28\] gave a way of defining a “global” primary decomposition. Let us briefly recall his construction. First, for $p = 0, 1, \ldots, \dim X$, let $J_{[p]} \supset J$ be the $p$th gap sheaf (Lückergarbe), introduced by Thimm \[29\]. A germ $s \in O_x$ is in $(J_{[p]})_x$ if and only if there is a neighborhood $U$ of $x$ and a section $t \in J(U)$ such that $s_x = t_x$ and $t_y \in J_y$ for all $y \in U$ outside an analytic set of dimension at most $p$. It is not hard to see that $J_{[p]}$ is a coherent sheaf, see \[29\], and that the set $Y^p$ where $(J_{[p]})_x \neq J_x$ is an analytic variety of dimension at most $p$, see \[28\] Theorem 3. The irreducible components of $Y^p$, $p = 0, 1, \ldots, \dim X$, are called the associated (sub)varieties of $J$. A coherent sheaf $J$ is said to be primary if it has only one associated variety $Y$; we then say that $J$ is $Y$-primary. Theorem 6 in \[28\] asserts that each coherent $J \subset O_X$ admits a decomposition

$$J = \bigcap J_i,$$

where there is one $Y_i$-primary intersectand $J_i$ for each associated variety $Y_i$ of $J$. For a radical sheaf $J_X$, the decomposition (3.1) corresponds to decomposing $X$ into irreducible components.

By Theorem 4 in \[28\] if $Y$ is an associated prime variety of $J$, then at $x \in X$ the irreducible components $\text{Ass}(J_{Y_x})$ of $Y_x$ are germs of varieties of associated primes.
of \( J_x \). Furthermore, if \( Y_x \) is (the variety of) an associated prime of \( J_x \), then \( Y_x \) is contained in \( Y^p_x \) for \( p \geq \dim Y_x \). For fixed \( x \) we get that

\[
\bigcup_{Y \in \text{Ass}(J), Y \ni x} \text{Ass}(J_{Y_x})
\]

is a disjoint union of \( \text{Ass}(J_x) \). Thus we have

**Lemma 3.1.** The germ at \( x \) of \( J_{[p]} \) is precisely the intersection of the primary components of \( J_x \) that are of dimension \( > p \).

Given a subvariety \( Z \) of \( X \), the gap sheaf \( \mathcal{J}[Z] \supset J \) is defined as follows: A germ \( s \in \mathcal{O}_x \) in \( \mathcal{J}[Z]_x \) if and only if it extends to a section of \( \mathcal{J}(U) \) for some neighborhood \( U \) of \( x \), where \( s_y \in \mathcal{J}_y \) for all \( y \in U \setminus Z \). Note that \( \mathcal{J}[Z]_x \) is the intersection of all components in a primary decomposition of \( J_x \) for which the associated varieties are not contained in \( Z \). It is not hard to see that \( \mathcal{J}[Z] \) is coherent, see [29]. Observe that \( J_{[p]} = \mathcal{J}[Y^p] \).

**Remark 3.2.** We claim that in fact

\[
J_{[p]} = \mathcal{J}[Z_{n-p}^\text{bef}].
\]

To see this assume first that \( X \) is smooth. Then the (germs of) varieties of associated prime ideals of \( J \) of dimension \( \leq p \) are precisely the (germs of) varieties of associated prime ideals that are contained in \( Z_{n-p}^\text{bef} \), see, e.g., [16, Corollary 20.14]. Now (3.2) follows from Lemma 3.1.

For a general \( X \), let \( i : X \to Y \) be a local embedding of \( X \) into a manifold \( Y \) of dimension \( N \) and let \( \tilde{J} = J + J_X \), cf. Section 2.3. It is not hard to verify that if \( \mathfrak{a} \) is an ideal in \( \mathcal{O}_x^X \) and \( \mathfrak{a} := \mathfrak{a} + (J_X)_x \) is the corresponding ideal in \( \mathcal{O}_x^Y \), then \( \mathfrak{a} = \cap \mathfrak{a}_k \) is a primary decomposition of \( \mathfrak{a} \) if and only if \( \tilde{\mathfrak{a}} = \cap \tilde{\mathfrak{a}}_k \) is a primary decomposition of \( \tilde{\mathfrak{a}} \). Hence, in light of Lemma 3.1, \( i^* \tilde{J}[V] = \mathcal{J}[V \cap X] \) and \( i^* \tilde{J}_{[p]} = \mathcal{J}_{[p]} \). By the definition of BEF-varieties in Section 2.4, thus \( i^* \tilde{J}[Z_{N-p}^\text{bef}(\tilde{J})] = \mathcal{J}[Z_{N-p}^\text{bef}(J)] = \mathcal{J}[Z_{n-p}^\text{bef}(J)] \), which proves (3.2) since \( \tilde{J}_{[p]} = \tilde{J}[Z_{n-p}^\text{bef}(J)] \).

\[\square\]

Given a residue current \( R \) constructed from a Hermitian locally free resolution of \( \mathcal{O}^X/J \) on a smooth \( X \) as in Section 2.3 in [8] we showed that the germ \( R_x \) of the current \( R \) at \( x \in X \) can be written as \( R_x = \sum R^p \), where the sum is over the associated primes of \( J_x \), and \( R^p \) has support on the variety \( V(\mathfrak{p}) \) of \( \mathfrak{p} \) and has the SEP with respect to \( V(\mathfrak{p}) \).

4. **Resolutions of homogeneous ideals**

Let \( \mathcal{J} \) be a coherent ideal sheaf on \( \mathbb{P}^N \). Then there is a locally free resolution \( \mathcal{O}(E^\mathcal{J}_i), f^* \), where \( E_k \) is a direct sum of line bundles \( E_k = \bigoplus \mathcal{O}(-d_k^i) \) and \( f^k = (f^k_{ij}) \) are matrices of homogeneous forms with \( \deg f^k_{ij} = d_k^i - d_{k-1}^i \), see, e.g., [22, Ch.1, Example 1.2.21]. Let \( J \) denote the homogeneous ideal in the graded ring \( S = \mathbb{C}[z_0, \ldots, z_N] \) associated with \( \mathcal{J} \), and let \( S(\ell) \) denote the module \( S \) where all degrees are shifted by \( \ell \). Then \( \mathcal{O}(E^\mathcal{J}_i), f^* \) corresponds to a free resolution

\[
\ldots \to \bigoplus \mathcal{S}(-d_k^i) \to \ldots \to \bigoplus \mathcal{S}(-d_2^i) \to \bigoplus \mathcal{S}(-d_1^i) \to \mathcal{S}
\]

of the module \( S/J \). Conversely, any such free resolution corresponds to a locally free resolution \( \mathcal{O}(E^\mathcal{J}_i), f^* \).
Recall that the regularity of a homogeneous module with a minimal graded free resolution \([4, 1]\) is defined as \(\max_{k,i}(d_k^i - k)\), see, e.g., [17, Ch.4]. The regularity \(\text{reg}\ J\) of the ideal \(J\) equals \(\text{reg}\ (S/J) + 1\), cf. [17, Exercise 4.3].

If \(X\) is a subvariety of \(\mathbb{P}^N\), then the regularity of \(X\), \(\text{reg}\ X\), is defined as the regularity of \(J_X\). Notice that if \(X\) has pure dimension, then the ideal \(J_X\) has pure dimension in \(S\); in particular the ideal associated to the origin is not an associated prime ideal. Theorem 20.14 in [16] thus implies that \(Z_0^{\text{reg}}\) is empty. Therefore the depth of \(S/J_X\) is at least 1, and hence a minimal free resolution of \(S/J_X\) has length \(\leq N\). For such a resolution we thus get
\[
\text{reg } X = \max_{k \leq \min(M, N)} (d_k^i - k) + 1.
\]
A global section of \(O(s)|_X \to X\) extends to a global section of \(O(s) \to \mathbb{P}^N\) as soon as \(s \geq \text{reg } X - 1\), see, e.g., [17, Chapter 4].

5. Division problems on singular varieties

Let \(E^*_k, g^*\) be a complex that corresponds to a Hermitian free resolution of \(O^{\mathbb{P}^N}/J_X\) as above, and let \(E^f, f^*\) be an arbitrary Hermitian pointwise generically surjective complex over \(\mathbb{P}^N\). Then the product current
\[
R^f \wedge R^g := R^f \wedge R^g|_{\lambda = 0}
\]
is well-defined on \(\mathbb{P}^N\),
\[
R^f \wedge \omega := R^f \wedge \omega|_{\lambda = 0}
\]
is a well-defined current on \(X\), and \(i_*(R^f \wedge \omega) = R^f \wedge R^g\), see [9, Section 2]. In particular, \(R^f \wedge R^g\) and \(R^f \wedge \omega\) only depend on the restriction of \(f\) to \(X\), and thus these currents are well-defined even if \(f\) is only defined over \(X\). Moreover \(R^f \wedge R^g \phi = 0\) if and only if \(R^f \wedge \omega^s \phi = 0\). On \(X_{\text{reg}}\), \(R^f \wedge \omega\) is just the product of the current \(R^f\) and the smooth form \(\omega\).

The current \(R^f \wedge R^g\) is related to the tensor product complex \(E^*_h, h^*\), where
\[
E^*_h = \bigoplus_{i+j=k} E^f_i \otimes E^g_j,
\]
and \(h = f + g\), cf. [9, Section 2.5], in a similar way as is the current \(R^h\) associated with this complex, see [4]. In particular, if \(\phi\) is a section of \(E^*_h = E^*_0 \otimes E^*_0\) such that \(R^f \wedge R^g \phi = 0\), one can locally solve \(f^i q + g^i q' = \phi\). Moreover if \((2.6)\) is satisfied for the product complex there is a global such section \((q, q')\) of \(E^*_1 = E^*_1 \otimes E^*_0 \oplus E^*_1 \otimes E^*_1\). In general, however, \(R^f \wedge R^g\) does not coincide with \(R^h\).

In fact, the definition of \(R^f\) in Section [2.3] works also when \(Y\) is singular. However, Proposition [2.7] and the duality principle do not hold in general, see, e.g., [21], and therefore \(R^f\) itself is not so well suited for division problems.

Example 5.1. Assume that \(E^*_f, f^*\) is the Koszul complex generated by sections \(f_j\) of \(L_j = O(d_j)|_X\), where \(X \subset \mathbb{P}^N\), twisted by \(S = O(\rho)\), as in Example [2.2] and that \(E^*_g, g^*\) is a complex associated with a minimal Hermitian free resolution of \(S/J_X\) as in Section [4]. Note that then \(E^*_\ell\) is a direct sum of line bundles
\[
O(\rho - (d_{i_1} + \cdots + d_{i_\ell}) - d_k^i).
\]
Recall that
\[
H^k(\mathbb{P}^N, O(\ell)) = 0\quad \text{if}\quad \ell \geq -N\quad \text{or}\quad k < N,
\]
see, e.g., [13]. Thus (2.6) is satisfied if \( \rho \geq d_i + \cdots + d_{i\ell} + d_{N+1-\ell} - N \) for \( \ell = 1, 2, \ldots, \min(m, n+1) \) and all choices of \( i \) and \( i_j \). Notice that, cf., (2.4),
\[
d_{N+1-\ell} - N = (d_{N+1-\ell} - (N + 1 - \ell)) + 1 - \ell \leq \text{reg} X - \ell.
\]
Hence (2.6) is satisfied if
\[
\rho \geq d_1 + \cdots + d_{\min(m,n+1)} - \min(m,n+1) + \text{reg} X.
\]
Summing up we have:

If \( \rho \) satisfies (5.2) and \( \phi \) is a section of \( \mathcal{O}(\rho) \) on \( \mathbb{P}^N \) such that \( R^f \wedge R^g \phi = 0 \) (or equivalently \( R^f \wedge R^g i^* \phi = 0 \)) then there are global sections \( q_j \) of \( \mathcal{O}(\rho - d_j) \) such that \( f_1 q_1 + \cdots + f_m q_m = \phi \) on \( X \).

If \( X \) is Cohen-Macaulay we may assume that \( E^q_\phi, g^* \) ends at level \( N - n \). If moreover \( m \leq n \), then \( E^q_\phi, h^* \) ends at level \( \leq N \) and thus (2.6) is satisfied for any \( \rho \).

\[\square\]

Example 5.2. Let \( F_j \) be polynomials in \( \mathbb{C}^N \), let \( \hat{f}_j \) be the sections of \( \mathcal{O}(\deg F_j) \to \mathbb{P}^N \) corresponding to \( F_j \), and let \( J_f \) be the ideal sheaf on \( \mathbb{P}^N \) generated by the \( \hat{f}_j \).

Moreover, let \( E^0_\phi, f^* \) and \( E^0_\phi, g^* \) be complexes associated with minimal free resolutions of \( J_f \) and \( J_X \) as in Section 4, where \( X \) is a subvariety of \( \mathbb{P}^N \); say \( E^0_f = \bigoplus \mathcal{O}(\delta_k) \) and \( E^0_f = \bigoplus \mathcal{O}(d_k) \). Then \( E^0_k \) is a direct sum of line bundles \( \mathcal{O}(\delta_k - d_k - \ell) \), and thus (2.6) is satisfied if \( \rho \geq d_i + d_{N+1-\ell} - N \) for all \( i, j, \ell \), cf. Example 5.1. Notice that, in light of Section 4,
\[
d_i + d_{N+1-\ell} - N = (\delta_k - \ell) + (d_{N+1-\ell} - (N + 1 - \ell)) + 1 \leq \text{reg} J_f + \text{reg} X - 1,
\]
where \( J_f \) is the homogeneous ideal associated with \( J_f \). Thus (2.6) is satisfied if \( \rho \geq \text{reg} J_f + \text{reg} X - 1 \).

Let \( Z_k^f \) and \( Z_k^g \) be the BEF-varieties of \( J_f \) and \( J_X \), respectively. Theorem 4.2 in [4] asserts that if
\[
\text{codim} (Z_k^f \cap Z_k^g) \geq k + \ell,
\]
then \( R^f \wedge R^g \phi = 0 \) if and only if \( \phi \in J_f + J_X = J_f + J_X \), where \( J_f \) is the sheaf on \( X \) generated by the restrictions \( f_j \) of \( \hat{f}_j \), cf. Section 2.4. If moreover \( J_f \) and \( J_X \) are both Cohen-Macaulay and the resolutions \( \mathcal{O}(E^0_f), f^* \) and \( \mathcal{O}(E^0_g), g^* \) have minimal length, then \( R^f \wedge R^g = R^h \), see [4] Theorem 4.2.

\[\square\]

5.1. Distinguished varieties. Let \( X \) be a subvariety of \( \mathbb{P}^N \) and let \( \hat{f}_j \) be sections of \( L = \mathcal{O}(d)|_X \). Moreover, let \( \nu \colon X_+ \to X \) be the normalization of the blow-up of \( X \) along \( J_f \), and let \( W = \sum r_j W_j \) be the exceptional divisor; here \( W_j \) are irreducible Cartier divisors. The images \( Z_j := \nu(W_j) \) are called the (Fulton-MacPherson) distinguished varieties associated with \( J_f \), see, e.g., [22]. If we consider \( \hat{f} = (\hat{f}_1, \ldots, \hat{f}_m) \) as a section of \( E^* := \bigoplus \mathcal{O}(d) \), then \( \nu^* \hat{f} = \hat{f}^0 \hat{f}^0 \), where \( \hat{f}^0 \) is a section of the line bundle \( \mathcal{O}(-W) \) and \( \hat{f}^0 = (\hat{f}_1^0, \ldots, \hat{f}_m^0) \) is a nonvanishing section of \( \nu^* E^* \otimes \mathcal{O}(W) \), where \( \mathcal{O}(W) = \mathcal{O}(-W)^{-1} \). Furthermore, \( \omega_f := dd^c \log |f^0|^2 \) is a smooth first Chern form for \( \nu^* L \otimes \mathcal{O}(W) \). We will use the geometric estimate
\[
\sum r_j \deg L Z_j \leq \deg L X
\]
from [15] Proposition 3.1, see also [22] (5.20).

Let $R^\hat f$ be the residue current associated with the Koszul complex of the $\tilde f_j$ as in Example 2.2 and consider the regularization (2.7) of $R^\hat f$. Using the notation in Example 2.2 $\nu^*\sigma = (1/\tilde f_0)\sigma'$, where $1/\tilde f_0$ is a meromorphic section of $\mathcal{O}(W)$ and $\sigma'$ is a smooth section of $\nu^*E \otimes \mathcal{O}(-W)$. It follows that
$$\nu^*(\sigma \land (\bar{\partial}\sigma)^{k-1}) = \frac{1}{(\tilde f_0)^k}\sigma' \land (\bar{\partial}\sigma')^{k-1},$$
and hence
$$\nu^*R^\lambda_k = \partial[\tilde f_0] \tilde f_0^{2\lambda} \land \frac{1}{(\tilde f_0)^k}\sigma' \land (\bar{\partial}\sigma')^{k-1} \text{ for } \Re \lambda >> 0,$$
when $k \geq 1$. Since $\tilde f'$ is nonvanishing, by (2.1) the value at $\lambda = 0$ is precisely
$$\tag{5.5} R^+_k := \bar{\partial} \frac{1}{(\tilde f_0)^k} \sigma' \land (\bar{\partial}\sigma')^{k-1}. $$
Thus
$$\nu_*R^+_k = R^\hat f_k.$$

6. Proofs

Proof of Theorem 1.3. For $j = 1, \ldots, m$, let $\hat f_j$ be the deg $F_j$-homogenization of the polynomial $F_j$, considered as a section of $\mathcal{O}(\deg F_j) \rightarrow \mathbb{P}^N$. Moreover let $g_1, \ldots, g_r$ be global generators of the ideal sheaf $\mathcal{J}_X$; assume they are sections of $\mathcal{O}(d_1), \ldots, \mathcal{O}(d_r)$, respectively. Let $J = J_f + J_X = J_f + J_X$. Then there is a locally free resolution $\mathcal{O}(E^h_k)$, $h^\bullet$ of $\mathcal{O}/J$, where each $E^h_k$ is a direct sum of line bundles $E_k = \bigoplus \mathcal{O}(-d_k)$ and in particular $E^1 = \bigoplus_j \mathcal{O}(-\deg F_j) \oplus \bigoplus_j \mathcal{O}(-d_k)$ and $h^1 = (f_1, \ldots, f_m, g_1, \ldots, g_r) =: f + g$, cf. Section 4. Let $R = R^h$ be the residue current associated with $E^h_k$, $h^\bullet$.

Recall from Section 3 that for fixed $x \in X$, $R_x = \sum R^p$, where the sum is over $\text{Ass}(\mathcal{J}_x)$ and where $R^p$ has the SEP with respect to $V(p)$; in particular, $1_{H_{\infty}}R^p = R^p$ if $V(p) \subset H_{\infty}$ and $1_{H_{\infty}}R^p = 0$ otherwise. Thus
$$\tag{6.1} 1_{H_{\infty}}R_x = \sum_{p \in \text{Ass}(\mathcal{J}_x), V(p) \subset H_{\infty}} R^p. $$
In Remark 3.2 we saw that $\mathfrak{a} = \cap \mathfrak{a}_k$ is a primary decomposition of the ideal $\mathfrak{a}$ in $\mathcal{O}_x$ if and only if $\tilde{\mathfrak{a}} = \cap \tilde{\mathfrak{a}}_k$ is a primary decomposition of the ideal $\tilde{\mathfrak{a}} = \mathfrak{a} + (\mathcal{J}_x)_x$ in $\mathcal{O}_x$. Thus, that $\mathcal{J}_f$ has no associated varieties contained in $X_{\infty}$ implies that, for a fixed $x \in X$, $\mathcal{J}_x$ has no (variants of) associated primes contained in the hyperplane $H_{\infty}$ at infinity in $\mathbb{P}^N$. We conclude, in light of (6.1), that $1_{H_{\infty}}R = 0$. If $\phi$ is any homogenization of $\Phi$ then $1_{C_N}R\phi = 0$ because of the duality principle and hence $R\phi = 1_{H_{\infty}}R\phi + 1_{C_N}R\phi = 0$.

Assume that the complex $E^h_k$, $h^\bullet$ ends at level $M$ (by Hilbert’s syzygy theorem we may assume that $M \leq N + 1$) and let
$$\tag{6.2} \beta := \max_i d_{N+1}^i - N \text{ if } M = N + 1 \text{ and } \beta := 0 \text{ otherwise.}$$
If $\rho \geq \beta$ then (2.10) is satisfied for $E^h_k$, $h^\bullet$ twisted by $\mathcal{O}(\rho)$ in light of (5.1) and thus by Proposition 2.1 there are global holomorphic sections $q = (q_j)$ of $\bigoplus \mathcal{O}(\rho - \deg F_j)$ and $q' = (q'_j)$ of $\bigoplus \mathcal{O}(\rho - d_k)$ over $\mathbb{P}^N$ such that $\tilde{f}q + gq' = \phi$. Indeed, recall from
the end of Section 2.3 that $R$ is also the residue current associated with the twisted complex. Dehomogenizing gives polynomials $Q_j$, $Q_j'$, and $G_j$ in $\mathbb{C}^N$ such that
\[ \sum F_j Q_j + \sum G_j Q_j' = \Phi \]
and where $\deg (F_j Q_j) \leq \rho$. Since the $G_j$ vanish on $V$ we get the desired solution to (1.1) on $V$, and thus the first part of Theorem 1.5 follows with $\beta$ as in (6.2).

If $V = \mathbb{C}^N$, $\mathcal{O}_X$ should be interpreted as the zero sheaf. Then $E^*_h, h^*$ is a locally free resolution of $\mathcal{O}/\mathcal{J}_f$ and $\beta \leq \text{reg} \mathcal{J}_f$, cf. Section 3.

For the second part of Theorem 1.5 assume that $\mathcal{J}_f$ has an associated variety contained in $X_{\infty}$. We are to prove that for arbitrarily large $\ell$ there is a polynomial $\Phi = \Phi_\ell$ of degree $\geq \ell$ in $(\mathcal{F}_j)$ on $V$ for which one can not solve (1.1) with $\deg (F_j Q_j) \leq \deg \Phi_\ell$.

Let $L = \mathcal{O}(1)|_X$. The hypothesis on $\mathcal{J}_f$ then means that $\mathcal{J}_f|_{X_{\infty}}$ is strictly larger than $\mathcal{J}_f$. Therefore, since $L$ is ample, for some large enough $s_0$ there is a global section $\psi_0$ of $L^{\otimes s_0} \to X$ such that $\psi_0$ is in $\mathcal{J}_f|_{X_{\infty}}$ but not in $\mathcal{J}_f$. Moreover we can find a global section $\psi$ of $L^{\otimes s}$ for some $s \geq 1$ such that $\psi$ does not vanish identically on any of the associated varieties of $\mathcal{J}_f$ that are contained in $X_{\infty}$. We may assume that $s_0, s \geq \text{reg} X - 1$, so that $\psi_0$ and $\psi$ extend to global sections $\hat{\psi}_0$ and $\hat{\psi}$ of $\mathcal{O}(s_0)$ and $\mathcal{O}(s)$, respectively. Let $\Psi_0$ and $\Psi$ be the corresponding dehomogenized polynomials in $\mathbb{C}^N$. For $\ell \geq 0$, let $\phi_\ell = \psi_0 \psi^\ell$ and $\Phi_\ell = \Psi_0 \Psi^\ell$. Since $\mathcal{J}_f|_{X_{\infty}} x = (\mathcal{J}_f)_x$ for all $x \in V$, $\Phi_\ell$ is in the ideal $(\mathcal{F}_j)$ on $V$, and thus we can solve (1.1) for $\Phi = \Phi_\ell$ on $V$. Assume that there is a solution to (1.1) with $\deg (F_j Q_j) \leq \rho_\ell$. Then there are sections $q_j$ of $L^{\rho_\ell - \deg F_j}$ such that
\[ \sum f_j q_j = z_0^{\rho_\ell - (s_0 + s\ell)} \phi_\ell \]
on $X$. Since $\phi_\ell$ is not in $\mathcal{J}_f$ it follows that $\rho_\ell - (s_0 + s\ell) \geq 1$ and thus $\rho_\ell \geq 1 + (s_0 + s\ell) \geq 1 + \deg \Phi_\ell$. Since $\hat{\psi}$ does not vanish identically at $X_{\infty}$, $\deg \Psi \geq 1$ and hence $\deg \Phi_\ell \geq \ell$. Hence we have found $\Phi_\ell$ with the desired properties and the second part of Theorem 1.5 follows.

\[ \square \]

Remark 6.1. If $\mathcal{J}_j$ and $\mathcal{J}_X$ are Cohen-Macaulay and the BEF-varieties of $\mathcal{J}_j$ and $\mathcal{J}_X$ satisfy (5.3), then we can choose the complex $E^*_h, h^*$ in the above proof to be the tensor product of the complexes $E^*_g, f^*$ and $E^*_h, g^*$ corresponding to minimal resolutions of $\mathcal{J}_j$ and $\mathcal{J}_X$, see Example 5.2. In this case, by Example 5.2 we get Theorem 1.5 for $\beta = \text{reg} \mathcal{J}_j + \text{reg} X - 1$.

\[ \square \]

The residue current technique in the preceding proof is convenient and makes it possible to carry out the proof within our general framework, but it is not crucial.

Remark 6.2. (The algebraic approach). Let us first sketch an algebraic proof of the first part of Theorem 1.5. We use the notation from the proof above. To begin with we have to prove that $\phi$ is in $\mathcal{J}$, which of course precisely corresponds to proving that $R \phi = 0$. Since (the restriction to $V$ of) $\phi$ is in $\mathcal{J}_f$ on $V$ it follows that $\phi_{x'}$ is in $\mathcal{J}$ outside $H_{\infty}$. Since moreover $\mathcal{J} = \mathcal{O}^{\mathbb{C}^N}$ outside $X$, we have to prove that $\phi_{x'} \in \mathcal{J}_x$ for each $x \in X_{\infty}$. At such a point $x$ we have a minimal primary decomposition $\mathcal{J}_x = \cap \mathcal{J}_x^\ell$. Since $\mathcal{J}$ is coheren, $\mathcal{J} \subset \mathcal{J}_f^\ell$ in a neighborhood $U$ of $x$, where $\mathcal{J}_f^\ell$ is the
coherent sheaf defined by $\mathcal{J}_x^\ell$. Let $Z^\ell$ be the zero-set of $\mathcal{J}_x^\ell$. Since $\phi_{x'}$ is in $\mathcal{J}_{x'}$ for $x'$ outside $H_\infty$, it follows that $\phi_{x'}$ is in $\mathcal{J}_{x'}^\ell$ for $x' \in Z^\ell \setminus H_\infty$. Hence $F := (\mathcal{J}^\ell + (\phi))/\mathcal{J}^\ell$ is a coherent sheaf in $U$ with support on $Z^\ell \cap H_\infty$. Since by assumption $\mathcal{J}_x$ has no associated varieties contained in $X_\infty$ it follows that $Z^\ell \cap H_\infty$ has positive codimension in $Z^\ell$, cf. the proof of Theorem 1.5 above. Therefore, by the Nullstellensatz there is a holomorphic function $h$, not vanishing identically on $Z^\ell$ such that $hF = 0$. In particular, $h_x \phi_{x'} \in \mathcal{J}_{x'}^\ell$. Since $h_x$ is not in the radical of $\mathcal{J}_{x'}^\ell$ and $\mathcal{J}_{x'}^\ell$ is primary it follows that $\phi_{x'} \in \mathcal{J}_{x'}^\ell$. We conclude that $\phi_{x'} \in \mathcal{J}_{x'}$. Notice that the last arguments above can be thought of as an algebraic version of the SEP-argument in the proof of Theorem 1.5 above.

Next we would like to use that $\phi \in \mathcal{J}$ to conclude that there is a global holomorphic solution to $hq = \phi$. By a partition of unity, using that $E^{h \cdot}h^\bullet$ is exact, one can glue local such solutions together to obtain a global smooth solution to $(h - \partial)\psi = \phi$, cf. [9, Section 4]. By solving a certain sequence of $\partial$-equations in $\mathbb{P}^N$ we can modify $\psi$ to a global holomorphic solution $q$ to $hq = \phi$. These $\partial$-equations are solvable if $\rho \geq \beta$ defined by (6.2). Alternatively, one can directly refer to the well-known result that there is a solution to $hq = \phi$ if $\rho \geq \text{reg} J$, where $J$ is the homogeneous ideal corresponding to $\mathcal{J}$, see, e.g., [17, Proposition 4.16].

In the same way Theorem 1.1 and 1.2 follow without any reference to residues. Probably one can also find give an algebraic proof of Theorem 1.4.

In the next proof the residue technique plays a more decisive role.

**Proof of Theorem 1.2** Let

$$\rho = \max(\deg \Phi + \mu d^\infty \deg X, (d - 1) \min(m, n + 1) + \text{reg} X),$$

or if $X$ is Cohen-Macaulay and $m \leq n$ let $\rho = \deg \Phi + md^\infty \deg X$, and let $\phi$ be the $\rho$-homogenization of $\Phi$ considered as a section of $\mathcal{O}(\rho)|_X$. Note that then $\phi = z_0^{-\deg \Phi} \tilde{\phi}$, where $\tilde{\phi}$ is the deg $\Phi$-homogenization of $\Phi$. Moreover, let $R^\hat{f} \wedge \omega$ be the residue current associated with the (twisted) Koszul complex $E^{\hat{f}}, \hat{f}^\bullet$ of the sections $\hat{f}_j$ of $\mathcal{O}(d)|_X$ associated with $F_j$, and a complex $E_q^\bullet, q^\bullet$ associated with a minimal resolution of $\mathcal{O}/\mathcal{J}_X$ as in Example 5.1 (with $d_j = d$ for all $j$).

**Claim:** $R^{\hat{f}} \wedge \omega_0 \phi$ has support on $Z^{\hat{f}} \cap X^0$.

To prove the claim, since $\omega$ is smooth on $X_{\text{reg}}$, it is enough to show that $R^{\hat{f}} \phi = 0$ on $X_{\text{reg}}$. First, since codim $Z^{\hat{f}} \cap V \geq m$, the duality principle for a complete intersection, cf. Example 2.2 implies that $R^{\hat{f}} \phi = 0$ on $V_{\text{reg}}$.

Next, to prove that $1_{X_\infty \setminus X^0} R^{\hat{f}} \phi = 0$ we consider the normalization of the blow-up $\nu: X_+ \to X$, and let $R^+: = \sum R_k^+$ be as in Section 5.1. Let $W'$ be the union of the irreducible components of $W = \nu^{-1}Z^{\hat{f}}$ that are contained in $\nu^{-1}X_\infty$. We claim that

$$1_{X_\infty} R^{\hat{f}} = \nu_* (1_{W'} R^+).$$

In fact, by (2.3),

$$1_{X_\infty} R^{\hat{f}} = \nu_* (1_{\nu^{-1}X_\infty} R^+) = \nu_* (1_{\nu^{-1}X_\infty} (1_{W'} + 1_{W \setminus W'}) R^+).$$
By (2.2), \( 1_{\nu^{-1}X_{\infty}}W^\nu R^+ = 1_{W^\nu}R^+ \). Moreover,
\[
1_{\nu^{-1}X_{\infty}}W^\nu \partial (\frac{1}{(f^0)^k}) = 1_{\nu^{-1}X_{\infty} \cap (W \setminus W')} \partial (\frac{1}{(f^0)^k}) = 0
\]
by (2.2) and the dimension principle, since \( \nu^{-1}X_{\infty} \cap (W \setminus W') \) has codimension at least 2 in \( X_+ \). In view of (5.5) we conclude that \( 1_{\nu^{-1}X_{\infty}}W^\nu R^+ = 0 \), and thus (6.3) follows from (6.4).

It follows from (6.3) that \( 1_{X_{\infty} \setminus X_0} R_1^j \phi = 0 \) if \( 1_{W^\nu}R^+ \nu^* \phi = 0 \). To show that \( 1_{W^\nu}R^+ \nu^* \phi \) vanishes first note that it is sufficient to show that it vanishes in a neighborhood of each point \( x \) on \( W' \) where \( W' \) is smooth. Indeed, since \( W_{\text{sing}} \) has codimension at least 2 in \( W \), \( 1_{W_{\text{sing}}} \partial (1/(f^0)^k) = 0 \) by the dimension principle. Hence, using (5.5) and (2.2) we get that
\[
1_{W^\nu}R^+ = 1_{W^\nu}(1_{W_{\text{reg}}} + 1_{W_{\text{sing}}}) R^+ = 1_{W^\nu \cap W_{\text{reg}}} R^+.
\]
Consider now \( x \in 1_{W^\nu \cap W_{\text{reg}}} \); say \( x \) is contained in the irreducible component \( W_j \) of \( W' \). In a neighborhood of \( x \) we have that \( f^0 = s^j v \), where \( s \) is a local coordinate function and \( v \) is nonvanishing and \( r_j \) is as in Section 5.1. Since \( \phi = s_0^{\mu - \deg \Phi} \), by the choice of \( \rho, \nu^* \phi \) vanishes to order (at least) \( \mu d^\nu - \deg X \) on \( W' \).

If \( \Omega \) is a first Chern form for \( O(1)|_X \), e.g., \( \Omega = dd^c \log |z|^2 \), then \( d\Omega \) is a first Chern form for \( L = O(d)|_X \) on \( X \). (notice that \( d \) denotes the degree and not the differential).

By (5.4) we therefore have that
\[
\int_{Z_j} (d\Omega)^{\dim Z_j} \leq \int_X (d\Omega)^n,
\]
which implies that
\[
r_j \leq d^{\text{codim } Z_j} \deg X.
\]
It follows that \( \nu^* \phi \) vanishes (at least) to order \( \mu r_j \) on \( W_j \) and hence it has a factor \( s^{\mu r_j} \). In a neighborhood of \( x \),
\[
\partial (\frac{1}{(f^0)^k}) = \partial (\frac{1}{s^{kr_j}})^{\text{smooth}}
\]
and thus, in light of (5.5), \( R_k^j \nu^* \phi = 0 \) for \( k \leq \mu \) there. Hence \( 1_{W^\nu \cap W_{\text{reg}}} R_k^j \nu^* \phi = 0 \) for \( k \leq \mu \) and \( 1_{X_{\infty} \setminus X_0} R_1^j \phi = 0 \). We conclude that \( 1_{X_{\infty} \setminus X_0} R_1^j \phi = 1_{\text{sing}} \; R_1^j \phi + 1_{X_{\infty} \setminus X_0} R_1^j \phi = 0 \), which proves the claim that \( R_1^j \cap \omega_0 \phi \) has support on \( Z_1 \cap X_0 \).

By (1.8) and the dimension principle we conclude that \( R_1^j \cap \omega_0 \phi \) vanishes identically, since the bidegree of \( R_1^j \) is at most \((0, m)\) and \( \omega_0 \) has bidegree \((n, 0)\). Thus \( R_1^j \cap \omega_1 \phi = R_1^j \cap \omega_0 \phi \), see (2.9), vanishes outside \( X^1 \). By (1.8) and the dimension principle, it vanishes identically since the bidegree of \( R_1^j \cap \omega_1 \) is at most \((n, m + 1)\). By induction, it follows that \( R_1^j \cap \omega_\ell \phi = 0 \) for each \( \ell \). We conclude that \( R_1^j \cap \omega \phi = 0 \).

Since \( \rho \) satisfies (5.2) (with \( d_j = d \)) and \( R_1^j \cap \omega \phi = 0 \), by Example 5.4 there is a global section \( q = (q_j) \) of \( \sum_{\ell} O(\rho - d) \) such that \( f q = \phi \) on \( X \). Dehomogenizing gives polynomials \( Q_j \) such that (1.1) holds on \( V \) and \( \deg (F_j Q_j) \leq \rho \).

\[\square\]

\textbf{Proof of Theorems 1.1 and 1.4} Let
\[
\rho = \max(\deg \Phi, d_1 + \ldots + d_{\min(m, n + 1) - \min(m, n + 1) + \text{reg } X}),
\]
or if $X$ is Cohen-Macaulay and $m \leq n$ let $\rho = \deg \Phi$. Moreover let $\phi$ be the $\rho$-homogenization of $\Phi$ and let $R^f \land \omega$ be the residue current associated with the twisted Koszul complex $E^f_j, f^*$ of the deg $F_j$-homogenizations $f_j$ of $F_j$ and a minimal resolution of $\mathcal{O}/J_X$ as in Example 5.1.

We claim that under the hypotheses of both theorems $R^f \land \omega_0 \phi$ has support on $Z^f \cap X^0$. Since $\omega$ is smooth outside $X^0$ it is enough to show that $R^f \phi = 0$ there.

First in the case of Theorem 1.1, $R^f$ vanishes for trivial reasons, since $Z^f$ is empty.

In the case of Theorem 1.4, first $R^f \phi$ vanishes on $V_{\text{reg}}$ by the duality principle. Next, since by assumption (1.2) holds and $Z^f$ has no irreducible components in $X_\infty$, it holds that $\text{codim } (X_\infty \cap Z^f) > m$. Since the components of $R^f$ have bidegree at most $(0, m)$, we conclude that $\mathbf{1}_{X_\infty \setminus X_0} R^f = 0$ by the dimension principle. This proves that $R^f \land \omega \phi$ has support on $Z^f \cap X^0$.

Now arguing as in the end of the proof of Theorem 1.6, we get that $R^f \land \omega \phi = 0$, and the results follow from Example 5.1.

\[\square\]

Remark 6.3. If $\deg F_j = d$, then Theorems 1.1 and 1.4 follow directly from Theorem 1.6. First, notice that Theorem 1.1 follows if we apply Theorem 1.6 to $F_j$ with no common zeros on $X$. Indeed, since $Z^f$ is empty, $\text{codim } (Z^f \cap X) = \infty$ and thus (1.7) and (1.8) are satisfied, and moreover $c_\infty = -\infty$.

Next, assume that $F_j$ satisfy the hypothesis of Theorem 1.4. Since the codimension of a distinguished variety is at most $m$ the condition that $Z^f$ satisfies (1.2) and has no irreducible component contained in $X_\infty$ means that (1.7) is satisfied and no distinguished varieties can be contained in $X_\infty$. Thus $c_\infty = -\infty$ and $d^\infty = 0$ and Theorem 1.4 follows from Theorem 1.6.

\[\square\]

References

[1] M. Andersson: Residue currents and ideals of holomorphic functions, Bull. Sci. Math. 128, (2004), 481–512.
[2] M. Andersson: Integral representation with weights. II. Division and interpolation, Math. Z. 254 (2006), no. 2, 315–332.
[3] M. Andersson: The membership problem for polynomial ideals in terms of residue currents, Ann. Inst. Fourier 56 (2006), 101–119.
[4] M. Andersson: A residue criterion for strong holomorphicity, Arkiv Mat. 48 (2010), 1–15.
[5] M. Andersson & E. Götmark: Explicit representation of membership of polynomial ideals, Math. Ann. 349 (2011), 345–365.
[6] M. Andersson & H. Samuelsson: A Dolbeault-Grothendieck lemma on a complex space via Koppelman formulas, Invent. math. 190 (2012), 261–297.
[7] M. Andersson, E. Wulcan: Residue currents with prescribed annihilator ideals, Ann. Sci. École Norm. Sup. 40 (2007), 985–1007.
[8] M. Andersson & E. Wulcan: Decomposition of residue currents, J. Reine Angew. Math. 638 (2010), 103–118.
[9] M. Andersson & E. Wulcan: Global effective versions of the Briançon-Skoda-Huneke theorem, preprint, available at arXiv:1107.0388.
[10] D. Bayer & D. Mumford: What can be computed in algebraic geometry?, Computational algebraic geometry and commutative algebra (Cortona, 1991), 1–48, Sympos. Math., XXXIV, Cambridge Univ. Press, Cambridge, 1993.
[11] J.-B. Björk: Residues and $\mathcal{D}$-modules, The legacy of Niels Henrik Abel, 605651, Springer, Berlin, 2004.
[12] J. Briançon & H. Skoda: Sur la clôture intégrale d’un idéal de germes de fonctions holomorphes en un point de $\mathbb{C}^n$, C. R. Acad. Sci. Paris Sér. A 278 (1974), 949–951.
ON THE EFFECTIVE MEMBERSHIP PROBLEM FOR POLYNOMIAL IDEALS

[13] J-P Demailly: Complex Analytic and Differential Geometry, Monograph Grenoble (1997).
[14] A. Dickenstein & C. Sessa: Canonical representatives in moderate cohomology, Invent. Math. 80 (1985), 417–434.
[15] L. Ein & R. Lazarsfeld: A geometric effective Nullstellensatz, Invent. Math. 135 (1999), 427–448.
[16] D. Eisenbud: Commutative algebra. With a view toward algebraic geometry, Graduate Texts in Mathematics, 160. Springer-Verlag, New York, 1995.
[17] D. Eisenbud: The geometry of syzygies. A second course in commutative algebra and algebraic geometry, Graduate Texts in Mathematics, 229. Springer-Verlag, New York, 2005.
[18] G. Hermann: Die Frage der endlich vielen Schritte in der Theorie der Polynomideale, Math. Ann., 95 (1926), 736–788.
[19] C. Huneke: Uniform bounds in Noetherian rings, Invent. Math. 107 (1992) 203–223.
[20] Z. Jelonek: On the effective Nullstellensatz, Invent. Math. 162 1–17 (2005).
[21] R. Lärkäng: On the duality theorem on an analytic variety, Math. Ann. to appear, available at arXiv:1007.0139.
[22] R. Lazarsfeld: Positivity in algebraic geometry I and II, Springer-Verlag 2004.
[23] F.S. Macaulay: The algebraic theory of modular systems, Cambridge Univ. Press, Cambridge 1916.
[24] E. Mayr & A. Mayer: The complexity of the word problem for commutative semigroups and polynomial ideals, Adv. in math. 46 (1982), 305–329.
[25] M. Nöther: Über einen Satz aus der Theorie der algebraischen Functionen, Math. Ann. (1873), 351–359.
[26] M. Passare & A. Tsikh & A. Yger: Residue currents of the Bochner-Martinelli type, Publicacions Mat. 44 (2000), 85–117.
[27] B. Shiffman: Degree bounds for the division problem in polynomial ideals, Michigan Math. J. 36 (1989), no. 2, 163–171.
[28] Y.-T. Siu: Noether-Lasker decomposition of coherent analytic subsheaves, Trans. Amer. Math. Soc. 135 1969 375–385.
[29] W. Thimm: Lückengarben von kohärenten analytischen Modulgarben, Math. Ann. 148 1962 372–394.
[30] E. Wulcan: Sparse effective membership problems via residue currents, Math. Ann. 350 (2011), 661–682.
[31] E. Wulcan: Some variants of Macaulay’s and Max Noether’s theorems, J. Commut. Algebra 2 (2010), no. 4, 567–580.

Department of Mathematics, Chalmers University of Technology and the University of Gothenburg, S-412 96 GÖTEBORG, SWEDEN
E-mail address: matsa@chalmers.se & wulcan@chalmers.se