A row space method for solving a system of linear equations

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Abstract

A new algorithm is presented for computing a direct solution to a system of consistent linear equations. It produces a minimum norm particular solution, a generalized inverse (of type $\{124\}$), and a null space projection operator. In addition, the algorithm permits an online formulation so that computations may proceed as the data become available. The algorithm does not require the solution of a triangular system of equations, nor does it rely on block partitioned matrices.

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Key Words: row space; linear equations; system of linear equations; generalized inverse; minimum norm; null space projection; direct solution; online solution

1 Introduction

The problem of solving a system of linear equations is widespread across mathematics, science and engineering. When there are $m$ equations in $n$ unknowns, the equations are written as $Ax = b$, where $A = (a_{ij})$ is the coefficient matrix, $x = (x_j)$ is the vector of unknowns, and $b = (b_i)$. (The indices are $i = 1, \ldots, m$ and $j = 1, \ldots, n$.) The entries in $A$, $x$ and $b$ are in a field $\mathcal{F}$, which is assumed to be endowed with an inner product. The examples in this paper assume $\mathcal{F} = \mathbb{C}$. 
2 THE ALGORITHM

It is known that a solution to $Ax = b$ may be written as $x = x_p + x_h$, where the particular solution $x_p$ satisfies $Ax_p = b$ and the homogeneous solution $x_h$ satisfies $Ax_h = 0$. Note that $x_h$ is an element of the null space of $A$ (i.e., $\mathcal{N}(A)$). When the dimension of $\mathcal{N}(A)$ is greater than zero or when $A$ is rectangular, the inverse to $A$ does not exist, and hence it’s no longer possible to write the solution in the cogent form $x = A^{-1}b$. Nevertheless, when $A^{-1}$ doesn’t exist, one may still use the generalized inverse ($G$), which allows the two parts of the solution to be written as:

$$
x_p = Gb
\quad x_h = Py
\quad P = I_n - GA
$$

where $I_n$ is an $n$-by-$n$ identity matrix, $P$ is a null space projection operator and $y$ is an arbitrary $n$-dimensional vector. (By definition, $Py \in \mathcal{N}(A)$ for any $y \in \mathcal{F}^n$.) Different properties of the solution $Gb$ can be inferred by the type of the generalized inverse. (This is discussed in the literature \[8, 9, 4, 2, 1\] and will not be reviewed here.) The $G$ that will be computed here is of type \{124\}, which among other things yields a solution $Gb$ which has minimum Euclidean norm.

There are other interesting features, such as the fact that the algorithm may be formulated in an online mode, which is defined to mean that the solution $x$ can be formed while the input data ($A$ and $b$) are still being acquired. This is useful in cases when it takes a relatively long time to access/create the input data.

2 The algorithm

The solution will require $A$ to have its rows orthonormalized, perhaps using the usual Gram-Schmidt orthogonalization procedure. However, before discussing a modified orthonormalization procedure, it is helpful to define the following.

Definition: A quasi-orthonormal list of vectors consists of vectors with norm equal to 1 or 0. Those vectors of norm equal to 1 are mutually orthogonal.
**Definition:** The row orthonormalization procedure (ROP) of a matrix $Q$ is performed by left multiplying it by a series of matrices $M_s$ ($s = 1, 2, 3, ...$) which affect an orthonormalization procedure on the nonzero rows of $Q$. The result is that the rows of $Q$ form a quasi-orthonormal list of vectors.

Thus, after applying the ROP, the rows are either zero or part of an orthonormal set. (Pseudocode for the ROP may be found in Appendix A.) The ROP will here be applied to $A$ as it exists in the context of the equation $Ax = b$, causing the same operations to be applied to $b$ as well. It is implemented with the nonsingular, m-by-m matrices $M_s$ which results in

$$(\cdots M_2 M_1) A x = (\cdots M_2 M_1) b$$

(1)

The $M_i$ are applied to $A$, causing it to become $A' = MA$, where $M = (\cdots M_2 M_1)$. (Note that it is not required that the $M_s$ represent elementary row operations.) There are two versions for how to view the right-hand side of the previous equation: (1) apply the $M_s$ to $b$ so that it becomes $b'$ (where $b'$ equals $Mb$); (2) accumulate the $M_s$ as a prefactor so that it becomes $Mb$. To summarize, after applying the ROP to $Ax = b$, these two variations may be written as

$$A' x = b'$$

(2)

or

$$A' x = Mb$$

(3)

The significance of the difference has mainly to do with computer implementation issues, with respect to storage and reusability.

**Definition:** Let $W_Q$ be a set which consists of the indices of the non-zero rows of an m-by-m matrix $Q$. Note that $W_Q \subseteq \{1, 2, ..., m\}$. Define an index matrix by the following:

$$(I_Q)_{ij} = \begin{cases} 
1 & \text{if } i = j \text{ and } i \in W_Q, \\
0 & \text{otherwise.}
\end{cases}$$

This matrix is essentially just an identity matrix, except now the i-th row has a 1 only if $i \in W_Q$. This matrix is immediately applicable to $A'$, whose rows comprise a quasi-orthonormal list of vectors. The following identities
may be easily verified:

\[ A' (A')^* = I_{N'} \]  
\[ (A')^* = (A')^* I_{N'} \]  
\[ \mathcal{I}_{A'} b' = b' \]  

where the superscript * represents a conjugate transpose.

**Lemma:** An arbitrary vector \( x \in \mathcal{F}^n \) may be expressed as

\[ x = (A')^* w + x_h \]  

where \( A \) is an \( m \)-by-\( n \) matrix, \( w \in \mathcal{F}^m \) and \( x_h \) satisfies \( Ax_h = 0 \).

**Proof.** Beginning with \( A' x = b' \), observe that \( A' : \mathcal{F}^n \to \mathcal{F}^m \) is a linear transformation. It is then known (Thm. 18.3 [3])

\[ \mathcal{F}^n = \text{range}[(A')^*] \oplus \text{null}[A'] \]  

This direct sum decomposition may be used to rewrite an \( x \in \mathcal{F}^n \) as

\[ x = w_1 + w_2 \]  

where

\[ w_1 \in \text{range}[(A')^*] \]  
\[ w_2 \in \text{null}[A'] \]  

However, every vector that is in the range of some matrix \( Q \) may be expressed as \( Qv \), for an appropriate \( v \). Thus, for some \( w \in \mathcal{F}^m \), it follows that \( w_1 \) may be written as

\[ w_1 = (A')^* w \]  

Also, because \( M \) is nonsingular, \( \text{null}[A'] = \text{null}[A] \), so that \( w_2 \in \text{null}[A] \). Substituting (13) into (10) gives the desired result. \[ \square \]

What remains is to determine \( w \) and to find a means of computing \( x_h \).
Theorem: A solution to $Ax = b$ is

$$x = (A')^*b' + x_h$$  \hspace{1cm} (14)

where all variables are as defined earlier.

Proof. The proposed solution is verified by substitution into $Ax = b$

\[
Ax = A(A')^*b' + Ax_h = M^{-1}A'(A')^*b' = M^{-1}I_A'b' = M^{-1}b' = b
\]

One could also take a constructive approach toward obtaining the above solution. Upon substituting (8) into $A'x = b'$ one obtains

$$A'(A')^*w = b'.$$

which upon using (4) becomes $I_{A'} w = b'$. Equation (8) may now be re-expressed as

$$x = (A')^*w + x_h = (A')^*I_{A'}w + x_h = (A')^*b' + x_h$$

Discussion

One of the first things to notice is that $(A')^*b'$ is a particular solution to $Ax = b$, as well as a minimum norm solution. The reason for the latter is because it is an element of range$[(A')^*]$ which is the orthogonal complement of the null space of $A'$. In other words, the particular solution is orthogonal to the null space. The minimum norm nature of the particular solution will be revisited when the Penrose identities are checked.

The first variation (Eqn. (2)) transforms $Ax = b$ into $A'x = b'$. In terms of augmented matrices, one applies the ROP to $[A|b]$ to produce $[A'|b']$, where
the rows of $A'$ form a quasi-orthonormal list of vectors. The solution is formed as

$$x_p = (A')^* b'$$

$$x_h = Py$$

$$P = I_n - (A')^* A'$$

where $y$ is an arbitrary vector in $\mathbb{F}^n$. This approach may be preferred in digital computations, when creating storage for $M$ may be an issue. However, this variation would not be preferred if a solution is sought for additional $b$ vectors, since the ROP step would have to be repeated.

The second variation (Eqn. (3)) transforms $Ax = b$ into $A'x = Mb$. In terms of augmented matrices, one applies the ROP to the augmented matrix $[A|I_n]$ to produce $[A'|M]$, in which the rows of $A'$ again form a quasi-orthonormal list of vectors. In this case the solution is

$$x_p = Gb$$

$$x_h = Py$$

$$P = I_n - GA$$

$$G = (A')^* M$$

where $y \in \mathbb{F}^n$. Finally, note that in this variation it is still possible to compute $P$ as $I_n - (A')^* A'$, and to compute $x_p$ as $(A')^* Mb$. In other words, one doesn’t have to actually form $G$ to compute the solution. Finally, while this case requires storage for $M$, it also allows one to compute a solution for additional vectors $b$ without having to repeat the ROP step.

These differences, while trivial for small example problems, may become significant when the matrix dimensions are large, and computer storage space is limited. Otherwise, choosing one variation over the other is mainly a matter of convenience. Pseudocode for these variations is given in Appendix B.

**Penrose Conditions**

It is convenient to classify a generalized inverse ($G$) according to which Penrose identities it satisfies. In particular, different properties of the solution
$Gb$ follow if certain sets of Penrose identities are satisfied. The first Penrose identity is $AGA = A$, which is seen to always be true for our solution.

\[
AGA = A(A')^*MA \\
= M^{-1}MA(A')^*A' \\
= M^{-1}A'(A')^*A' \\
= M^{-1}I_{A'}A' \\
= M^{-1}A' \\
= A
\]

Likewise, the second identity $GAG = G$ is always true.

\[
GAG = [(A')^*M][[(A')^*M] \\
= (A')^*A'(A')^*M \\
= (A')^*I_{A'}M \\
= (A')^*M \\
= G
\]

The third Penrose identity $(AG = (AG)^*)$ is seen to be problematic

\[
AG = A(A')^*M \\
= M^{-1}MA(A')^*M \\
= M^{-1}A'(A')^*M \\
= M^{-1}I_{A'}M
\]

If $A$ is of full row rank, then $I_{A'}$ equals $I_m$, and $AG$ becomes $I_m$; the identity is satisfied, allowing the identity to be satisfied. However, if $A$ is not of full row rank, then this identity is not true in general. Finally, the fourth Penrose identity $GA = (GA)^*$ is seen to be true:

\[
GA = (A')^*MA \\
= (A')^*A' \\
= ((A')^*A')^* \\
= ((A')^*MA)^* \\
= (GA)^*
\]
In summary, the generalized inverse obtained by this algorithm is at least a \{124\}-inverse. If it’s additionally true that \(A\) is of full row rank, then it becomes a \{1234\}-inverse, a.k.a. a Moore-Penrose inverse. Recall that generalized inverses which are at least of type \{14\} (which is the case here) yield minimum norm solutions \(Gb\).

### Online Capabilities

To the author’s knowledge, algorithms for solving \(Ax = b\) require that all data (i.e., \(A, b\)) be available at the outset before the linear solver can begin. This algorithm is different: it can do the calculation as the data arrives (so long as it arrives in a certain manner). This style of computation is referred to as an online algorithm. Examples of when it is useful is in cases where it takes a large amount of time to compute (or acquire) all the entries in \(A\) and \(b\). This approach reduces the overall computation time. In addition, it will be shown that the updates to \(x\) are mutually orthogonal; this means that the estimation of \(\|x\|\) is monotonically non-decreasing throughout the computation.

Assume that the rows of \([A|b]\) become available one at a time, and for simplicity let the i-th row of \([A|b]\) be the i-th to arrive. Note that when the ROP is based on the classical Gram-Schmidt (CGS) \([\text{6, 11}]\) procedure, the 1st through i-th rows of \(A, M\) and \(b\) will no longer change following the i-th step of the algorithm. Since those rows are done changing at these points, they become available to be used in a computation of \(x_p\). The next step is to use the column-row expansion \([\text{5}]\) on the product \((A')^*b'\) to rewrite \(x_p\) as

\[
x_p = (A')^*b' = \sum_{i=1}^{m} x_p^{(i)}
\]

where

\[
x_p^{(i)} = \text{Col}_i[(A')^*] b_i'
\]

and \(\text{Col}_i\) signifies the i-th column. Following the i-th step in the algorithm, the i-th term on the right-hand side (i.e, \(x_p\)) may be computed. Thus the solution is accrued just by adding \(x_p^{(1)} + x_p^{(2)} + \cdots\). Furthermore, the updates to \(x_p\) are mutually orthogonal, i.e., \(< x_p^{(i)}, x_p^{(k)} >= 0\) for \(i \neq k\). This approach based on the first variation will be the basis of the following online
computation. An illustration of this technique on a numerical example is in Appendix D.

The same basic approach can also be taken for the second variation of the algorithm, except now the column-row expansion is used to rewrite the generalized inverse as

\[ G = (A')^* M = \sum_{i=1}^{m} G^{(i)} \]

where

\[ G^{(i)} = \text{Col}_i[(A')^*] \text{ Row}_i[M] \]

Note that \( G^{(i)} \) may be computed following the i-th step of the ROP. Following the computation of \( G \), the particular solution is easily computed from it.

**Final Remarks**

What is immediately noteworthy about the algorithm is that it doesn’t use an elimination or partitioning strategy; in particular, there was no solving of a triangular system of equations. The algorithm is similar to those based on matrix decompositions, in which \( A \) is written as a product of matrices, and then reinserted into \( Ax = b \). In the algorithm presented here, the factorization was done implicitly, by operating on \( A \) while it was in the context of the equation \( Ax = b \). Borrowing a term from metallurgy, this type of factorization might be called an ”in situ factorization”. Also, keep in mind that it is required that the equations represented by \( Ax = b \) be consistent. (Although, if they are inconsistent, a simple modification to the pseudocode makes it easy to discover that.)

The new algorithm is perhaps most similar to a version of the QR algorithm, in which the GS procedure operates on the rows of \( A \). However, in that approach, one still has to solve a triangular system of equations. This destroys the possibility of easily computing a generalized inverse or a null space projection operator, as well as formulating the solution in an online mode.

The solution found by the new method is always of minimum norm. When \( A \) additionally has full row rank, the solution is a least-square solution. These properties follow from the generalized inverse, which is type \{124\} generally, and type \{1234\} when \( A \) has full row rank. (Recall that if a generalized inverse \( G \) is at least type \{14\} the solution \( Gb \) has minimum norm, and if it
is at least type {13} the solution $Gb$ is a least squares solution.) Finally, it’s pointed out that it isn’t necessary to explicitly form $G$; the first variation side-steps that computation.

The orthonormalization procedure named “ROP” can be thought of as just the GS procedure, except that the resulting zero-norm row vectors are retained in the end result. Zero vectors are allowed to persist in $A$ only because it’s easier to leave them there. They could also be removed; in that case $A$ and $b$ would have to be re-sized. An extension of the ROP [12, 13] takes into account numerical precision and declares the norm of a vector to be zero if it is less than a small number $\epsilon$; the size of $\epsilon$ is related to the machine precision used to implement the algorithm [7].

Although the new algorithm was cast to solve $Ax = b$, it can easily solve [12] its matrix generalization: $AX = B$, where $X$ is $n$-by-$p$ and $B$ is $m$-by-$p$ (and $p \geq 1$). In that case the ROP proceeds as before, and the particular and homogeneous parts of the solution $X = X_p + X_h$ are

$$X_p = GB$$
$$X_h = PY$$

where $G$ and $P$ are the same as before, and $Y \in F^{n \times p}$. This also admits an online formulation.

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**APPENDIX A: pseudocode**

The following pseudocode, written in the style of the C programming language, illustrates the workings of the algorithm. It shows that while it was previously expedient to emphasize the role of the matrices $M_s$, it’s not necessary to explicitly form them. Separate pseudocode is presented for each of the variations, to aid exposition.

Two accommodations are made for machine precision, should this be implemented on a computer. The first is the variable 'eps', which might be
set to some multiple of the machine precision (cf. "ε-rank" \cite{7}). (For the examples herein, 'eps' is zero.) Also, it might be expedient to take further action on a row that has a norm less than 'eps'; this would be done in the code where the comment "zero-norm option" appears. (However, this option is not used in the examples herein.) Finally, the notation Row$_i[P]$ indicates the i-th row of a matrix $P$. The bars $||$ indicate a Euclidean norm, and the angle brackets $<,>$ indicate an inner product. Since exact arithmetic is assumed for the examples and the further discussion in this paper, take 'eps' to be zero.

In the first variation the algorithm begins with the input data $A$ and $b$. Row operations are done on the augmented matrix $[A|b]$, transforming it into $[A'|b']$. The rows of $A'$ subsequently form a quasi-orthonormal set. This version of the ROP is based on the modified Gram-Schmidt (MGS) \cite{6, 11} procedure.

//first variation
for (i = 1 to m){
  mag = $||$ Row$_i[A]$$||$
  if (mag > eps){
    //normalization
    Row$_i[A] = Row$_i[A] / mag$
    b$_i = b_i / mag$

    //orthogonalization
    for (k = i+1 to m){
      prod = $< Row_k[A], Row_i[A] >$
      Row$_k[A] = Row_k[A] - Row_i[A] * prod$
      b$_k = b_k - b_i * prod$
    }
  }else{
    //implement a "zero-norm option"
  }
}
Following this the various solution features (i.e., $x_p, G, P, ...$) are computed.

In the second variation, the ROP begins with the input of the coefficient matrix $A$ and an m-by-m matrix $M$ which is initialized as an identity matrix. In the pseudocode, the entries for $A$ and $M = I_m$ will be written over; the result at the end will be identified as $A'$ and $M = (\cdots M_2 M_1)$,
respectively. The ROP proceeds by doing row operations on the augmented matrix \([A|I_m]\), transforming it into \([A'|M]\), such that the rows of \(A'\) form a quasi-orthonormal set. This version of the ROP is also based on the MGS procedure.

```plaintext
//second variation
for (i = 1 to m){
    mag = || Row_i[A] ||
    if( mag > eps){
        //normalization
        Row_i[A] = Row_i[A] / mag
        Row_i[M] = Row_i[M] / mag
    }
    //orthogonalization
    for (k = i+1 to m){
        prod = < Row_k[A], Row_i[A] >
        Row_k[A] = Row_k[A] - Row_i[A] * prod
        Row_k[M] = Row_k[M] - Row_i[M] * prod
    }
} else{
    //implement a "zero-norm option"
}
```

Following this the various solution features are computed.

**APPENDIX B: example of 1st variation**

In this section the pseudocode for the first variation is used to compute the solution. The input data are

\[
A = \begin{bmatrix}
0 & -3i & 0 \\
2i & 1 & -1 \\
4i & 2 - 3i & -2
\end{bmatrix}, \quad b = \begin{pmatrix}
1 \\
2i \\
1 + 4i
\end{pmatrix}
\]

Note that \(A\) has rank 2.
Step 1

The \( i = 1 \) case in the loop in the pseudocode involves the normalization of the first row. The associated row operation is

\[
\text{Row}_1 \leftarrow \left( \frac{1}{3} \right) \text{Row}_1
\]

Following this operation, the intermediate values for \( A' \) and \( b' \) are

\[
[A|b] = \begin{bmatrix}
0 & -i & 0 & \frac{1}{3} \\
2i & 1 & -1 & 2i \\
4i & 2 - 3i & -2 & 1 + 4i
\end{bmatrix}
\]

Step 2

This step is the orthogonalization between the first and second rows of \( A \); it occurs when \( i = 1 \) and \( k = 2 \) in the pseudocode. The associated row operation is

\[
\text{Row}_2 \leftarrow \text{Row}_2 - (i) \text{Row}_1
\]

This leads to the following intermediate values for \( A' \) and \( b' \)

\[
[A|b] = \begin{bmatrix}
0 & -i & 0 & \frac{1}{3} \\
2i & 0 & -1 & \frac{2i}{3} \\
4i & 2 - 3i & -2 & 1 + 4i
\end{bmatrix}
\]

Step 3

This orthogonalization step is between the first and third rows, and occurs when \( i = 1 \) and \( k = 3 \). The associated row operation is

\[
\text{Row}_3 \leftarrow \text{Row}_3 - (3 + 2i) \text{Row}_1
\]

At this point the intermediate values for \( A' \) and \( b' \) are

\[
[A|b] = \begin{bmatrix}
0 & -i & 0 & \frac{1}{3} \\
2i & 0 & -1 & \frac{2i}{3} \\
4i & 0 & -2 & \frac{10i}{3}
\end{bmatrix}
\]
Step 4
The normalization step for \( i=2 \) is for the second row. The associated row operation is

\[
\text{Row}_2 \leftarrow \frac{1}{\sqrt{5}} \text{Row}_2
\]

Following this, \( A' \) and \( b' \) take on the intermediate values

\[
[A|b] = \begin{bmatrix}
0 & -i & 0 & \frac{1}{\sqrt{5}} \\
\frac{2}{\sqrt{5}}i & 0 & -\frac{1}{\sqrt{5}} & \frac{\sqrt{3}}{3}i \\
4i & 0 & -2 & \frac{10}{3}i
\end{bmatrix}
\]

Step 5
For \( i=2 \) and \( k=3 \), the third row is orthogonalized with respect to the second. The row operation associated with this is

\[
\text{Row}_3 \leftarrow \text{Row}_3 - (2\sqrt{5}) \text{Row}_2
\]

The intermediate values for \( A' \) and \( b' \) are now

\[
[A|b] = \begin{bmatrix}
0 & -i & 0 & \frac{1}{\sqrt{5}} \\
\frac{2}{\sqrt{5}}i & 0 & -\frac{1}{\sqrt{5}} & \frac{\sqrt{3}}{3}i \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Noteworthy is that the third row of the augmented matrix has become zero. Had the third entry in \( b' \) been nonzero the equations would be inconsistent.

Final steps
According to the pseudocode, the final step should be the normalization of the third row (for \( i=3 \)). However, since the third row has zero norm, this step is skipped (according to the if-statement in the pseudocode, since 'eps' is zero). It follows that what were identified as intermediate values for \( A' \) and \( b' \) in the fifth step are in fact final values. The particular solution may now be formed as

\[
x_p = (A')^*b' = \begin{bmatrix}
0 & -\frac{2}{\sqrt{5}}i & 0 \\
i & 0 & 0 \\
0 & -\frac{1}{\sqrt{5}} & 0
\end{bmatrix} \begin{pmatrix}
\frac{1}{\sqrt{5}}i \\
\frac{2}{\sqrt{5}}i \\
0
\end{pmatrix} = \frac{1}{3} \begin{pmatrix}
\frac{2}{i} \\
i
\end{pmatrix}
\]
Also, the null space projection operator is

\[ P = I_3 - (A')^* A' = \frac{1}{5} \begin{bmatrix} 1 & 0 & -2i \\ 0 & 0 & 0 \\ 2i & 0 & 4 \end{bmatrix} \]

As noted earlier, the homogeneous solution is formed as \( x_h = Py \), for arbitrary \( y \). Setting \( y = (y_1, y_2, y_3)^T \), where each entry is arbitrary, the result is

\[ x_h = \frac{1}{5} (y_1 - 2i y_3) \begin{pmatrix} 1 \\ 0 \\ 2i \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 2i \end{pmatrix} \]

where \( \alpha \) is an arbitrary element in \( \mathbb{C} \). Noteworthy is that the nullity of \( A \) is one, which corresponds to the above parametrization of \( x_h \) requiring only one vector. Also note that it was not necessary to explicitly construct the \( M_s \) that were used in the derivation of the algorithm.

**APPENDIX C: example of 2nd variation**

This is the same example as before, the same steps are done, except now \( M \) is changed instead of \( b \). The row operations have the same steps and the same row operation as in the example for the first variation. Hence, all that will be shown are the intermediate values for the augmented matrix \([A|M]\).

**Step 1**

\[
[A|M] = \begin{bmatrix} 0 & -i & 0 & \frac{1}{3} & 0 & 0 \\ 2i & 1 & -1 & 0 & 1 & 0 \\ 4i & 2 - 3i & -2 & 0 & 0 & 1 \end{bmatrix}
\]

**Step 2**

\[
[A|M] = \begin{bmatrix} 0 & -i & 0 & \frac{1}{3} & 0 & 0 \\ 2i & 0 & -1 & -\frac{2}{3}i & 1 & 0 \\ 4i & 2 - 3i & -2 & 0 & 0 & 1 \end{bmatrix}
\]

**Step 3**

\[
[A|M] = \begin{bmatrix} 0 & -i & 0 & \frac{1}{3} & 0 & 0 \\ 2i & 0 & -1 & -\frac{2}{3}i & 1 & 0 \\ 4i & 0 & -2 & -1 - \frac{2}{3}i & 0 & 1 \end{bmatrix}
\]
Step 4
\[
[A|M] = \begin{bmatrix}
0 & -i & 0 & | & \frac{1}{3} & 0 & 0 \\
\frac{2}{\sqrt{5}}i & 0 & -\frac{1}{\sqrt{5}} & | & -\frac{2\sqrt{5}}{15}i & \frac{\sqrt{5}}{5} & 0 \\
4i & 0 & -2 & | & -1 & -\frac{2}{3}i & 0 & 1
\end{bmatrix}
\]

Step 5
\[
[A|M] = \begin{bmatrix}
0 & -i & 0 & | & \frac{1}{3} & 0 & 0 \\
\frac{2}{\sqrt{5}}i & 0 & -\frac{1}{\sqrt{5}} & | & -\frac{2\sqrt{5}}{15}i & \frac{\sqrt{5}}{5} & 0 \\
0 & 0 & 0 & | & -1 & -2 & 1
\end{bmatrix}
\]

The final $A'$ and $M$ matrices are those found above, in step 5. The generalized inverse $G$ is found to be
\[
G = (A')^*M = \frac{1}{15} \begin{bmatrix} -2 & -6i & 0 \\ 5i & 0 & 0 \\ i & -3 & 0 \end{bmatrix}
\]

Besides permitting a computation of the particular solution via $x_p = Gb$, it also allows an easy computation of the null space projection operator ($P = I_n - GA$), which is the same as before. In short, the second variation can compute everything that the first variation did; the difference is that it can also compute $G$. Finally, note that it was not necessary to explicitly construct the $M_s$ that were used in the derivation of the algorithm.

**Additional remarks**

To complete the example, the matrices $M_s$ ($s = 1, 2, 3, 4, 5$) are shown for the row operations given in the above examples. ($M_s$ corresponds to the row operation in the $s$-th step.) While it was not necessary to actually construct and use these matrices, they may provide an aid to understanding for the reader.

\[
M_1 = \begin{bmatrix}
\frac{1}{3} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad M_2 = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad M_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-(3+2i) & 0 & 1
\end{bmatrix},
\]

\[
M_4 = \begin{bmatrix}
1 & 0 & 0 \\
0 & \frac{2\sqrt{5}}{5} & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad M_5 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -2\sqrt{5} & 1
\end{bmatrix}
\]

It is left as an exercise to verify that $M_5 \cdots M_1$ equals the $M$ that was computed in the second variation above.
APPENDIX D: online case

In the following example, the same data is used to illustrate the online case for the first variation. In the expressions below, the "input data" for the rows of $A$ and $b$ are from newly acquired data. (Double-hyphens in a matrix mean that no data has been entered there yet.) Also, the ”intermediate results” are how $A$, $b$, and $x_p$ appear following the $i$-th update. After $i = 3$, the updates are complete, and the last $A$ and $b$ may be identified with $A'$ and $b'$, respectively. Also, note how the norm of $x_p$ is non-decreasing with respect to $i$.

$i = 1$

input data:

Row$_1(A) = (0, -3i, 0)$  
Row$_1(b) = (1)$

intermediate results:

\[
A = \begin{bmatrix}
0 & -i & 0 \\
-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}
\end{bmatrix}, \quad b = \begin{bmatrix}
\frac{1}{3} \\
-\frac{1}{3}
\end{bmatrix}, \quad x_p^{(1)} = \begin{bmatrix}
0 \\
\frac{1}{3} i
\end{bmatrix}
\]

$i = 2$

input data:

Row$_2(A) = (2i, 1, -1)$  
Row$_2(b) = (2i)$

intermediate results:

\[
A = \begin{bmatrix}
\frac{2\sqrt{5}}{5}i & -i & 0 \\
0 & -\frac{\sqrt{5}}{5} & 0
\end{bmatrix}, \quad b = \begin{bmatrix}
\frac{1}{3} \sqrt{5} i \\
-\frac{1}{3} \sqrt{5}
\end{bmatrix}, \quad x_p^{(2)} = \begin{bmatrix}
\frac{2}{3} \\
-\frac{1}{3} i
\end{bmatrix}
\]
$i = 3$

input data:

Row\(_3\)(\(A\)) = \((4i, 2 - 3i, -2)\)
Row\(_3\)(\(b\)) = \((1 + 4i)\)

intermediate results:

\[ A = \begin{bmatrix} 0 & -i & 0 \\ \frac{2\sqrt{5}}{5}i & 0 & -\sqrt{5} \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} \frac{3}{5}i \\ \frac{4}{3} \end{bmatrix}, \quad x_p^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

The particular solution follows from adding all the updates, giving

\[ x_p = x_p^{(1)} + x_p^{(2)} + x_p^{(3)} = \frac{1}{3} \begin{bmatrix} 2 \\ i \\ -i \end{bmatrix} \]

which is the same as found earlier. As a final point, note that it is trivial to repeat the (CGS) orthonormalization step for each \(i\) during this online computation. Doing so would increase the accuracy of the solution\footnote{[12]}.

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[13] This algorithm, when it appears in certain applications, has patent protection. However, that has no bearing on its non-commercial use in education or research.

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