Open n-Qubit System as a Quantum Computer with Four-Valued Logic

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Abstract

In this paper we generalize the usual model of quantum computer to a model in which the state is an operator of density matrix and the gates are general superoperators (quantum operations), not necessarily unitary. A mixed state (operator of density matrix) of n two-level quantum system (open or closed n-qubit system) is considered as an element of $4^n$-dimensional operator Hilbert space (Liouville space). It allows to use quantum computer (circuit) model with 4-valued logic. The gates of this model are general superoperators which act on n-ququats state. Ququat is quantum state in a 4-dimensional (operator) Hilbert space. Unitary two-valued logic gates and quantum operations for n-qubit open system are considered as four-valued logic gates acting on n-ququats. We discuss properties of quantum 4-valued logic gates. In the paper we study universality for quantum four-valued logic gates.

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I Introduction

Usual models for quantum computer use closed n-qubit systems and deal only with unitary gates on pure states. In these models it is difficult or impossible to deal formally with measurements, dissipation, decoherence and noise. It turns out, that the restriction to pure states and unitary gates is unnecessary. One can describe an open system starting from a closed system if the open system is a part of the closed system. However, situations can arise where it is difficult or impossible to find a closed system comprising the given open system. This would render theory of dissipative and open systems a fundamental generalization of quantum mechanics. Understanding dynamics of open systems is important for studying quantum noise processes, quantum error correction, decoherence effects, in quantum computations and to perform simulations of open quantum systems.

In this paper we generalize the usual model of quantum computer to a model in which the state is a density matrix operator and the gates are general superoperators (quantum operations), not necessarily unitary. Pure state of n two-level closed quantum systems is an element of $2^n$-dimensional Hilbert space and it allows to realize quantum computer model with 2-valued logic. The gates of this computer model are unitary space and it allows to realize quantum computer model with operators (quantum operations) which act on general n-ququats (4-valued logic). The gates of this model are general superoperators, evolution equations for closed and open quantum systems (pure and mixed states, Liouville space and superoperators, acting on ququats.

In general, a quantum system is not in a pure state. Open quantum systems are not really isolated from the rest of the universe, so it does not have a well defined pure state. Landau and von Neumann introduced a mixed state and a density matrix into quantum theory. A density matrix is a Hermitian tensor product of Pauli matrices which is consists of the computational basis which has orthogonal unit vectors, i.e. $< k|l > = \delta_{kl}$, where $k,l \in \{0,1\}$. The Hilbert space of qubit is $\mathcal{H}_2 = \mathbb{C}^2$. The quantum system which corresponds to a quantum computer (quantum circuits) consists of n quantum two-state particles. The Hilbert space $\mathcal{H}^{(n)}$ of such a system is a tensor product of n Hilbert spaces $\mathcal{H}_2$ of two-state particle: $\mathcal{H}^{(n)} = \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_2$. The space $\mathcal{H}^{(n)}$ is a $2^n$ dimensional complex linear space. Let us choose a basis for $\mathcal{H}^{(n)}$ which is consists of the $N = 2^n$ orthogonal states $| k >$, where $k$ is in binary representation. A pure state $| \Psi(t) > \in \mathcal{H}^{(n)}$ is generally a superposition of the basis states

$$| \Psi(t) > = \sum_{k=0}^{N-1} a_k(t)|k >,$$

with $N = 2^n$ and $\sum_{k=0}^{N-1}| a_k(t) |^2 = 1$. The inner product between $| \Psi >$ and $| \Psi' >$ is denoted by $< \Psi | \Psi' >$ and

$$< \Psi | \Psi' > = \sum_{k=0}^{N-1} a^*_k a'_k.$$

II Quantum state and qubit

II.1 Pure states

A quantum system in a pure state is described by unit vector in a Hilbert space $\mathcal{H}$. In the Dirac notation a pure state is denoted by $| \Psi >$. The Hilbert space $\mathcal{H}$ is a linear space with an inner product. The inner product for $| \Psi_1 >, | \Psi_2 > \in \mathcal{H}$ is denoted by $< \Psi_1 | \Psi_2 >$. A quantum bit or qubit, the fundamental concept of quantum computations, is a two-state quantum system. The two basis states labeled $| 0 >$ and $| 1 >$, are orthogonal unit vectors, i.e.

$$< k|l > = \delta_{kl},$$

where $k,l \in \{0,1\}$. The Hilbert space of qubit is $\mathcal{H}_2 = \mathbb{C}^2$. The quantum system which corresponds to a quantum computer (quantum circuits) consists of n quantum two-state particles. The Hilbert space $\mathcal{H}^{(n)}$ of such a system is a tensor product of n Hilbert spaces $\mathcal{H}_2$ of two-state particle: $\mathcal{H}^{(n)} = \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_2$. The space $\mathcal{H}^{(n)}$ is a $2^n$ dimensional complex linear space. Let us choose a basis for $\mathcal{H}^{(n)}$ which is consists of the $N = 2^n$ orthonormal states $| k >$, where $k$ is in binary representation. A pure state $| \Psi(t) > \in \mathcal{H}^{(n)}$ is generally a superposition of the basis states

$$| \Psi(t) > = \sum_{k=0}^{N-1} a_k(t)|k >,$$

with $N = 2^n$ and $\sum_{k=0}^{N-1}| a_k(t) |^2 = 1$. The inner product between $| \Psi >$ and $| \Psi' >$ is denoted by $< \Psi | \Psi' >$ and

$$< \Psi | \Psi' > = \sum_{k=0}^{N-1} a^*_k a'_k.$$

II.2 Mixed states

In general, a quantum system is not in a pure state. Open quantum systems are not really isolated from the rest of the universe, so it does not have a well defined pure state. Landau and von Neumann introduced a mixed state and a density matrix into quantum theory. A density matrix is a Hermitian positive (\rho > 0) operator on $\mathcal{H}^{(n)}$ with unit trace ($Tr \rho = 1$). Pure states can be characterized by idempotent condition $\rho^2 = \rho$. A pure state $| \Psi > \in \mathcal{H}$ is represented by the operator $\rho = | \Psi < \Psi |$.

One can represent an arbitrary density matrix operator $\rho(t)$ for n-qubits in terms of tensor products of Pauli matrices $\sigma_{\mu_i}$:

$$\rho(t) = \frac{1}{2^n} \sum_{\mu_1,..,\mu_n} P_{\mu_1,..,\mu_n}(t) \sigma_{\mu_1} \otimes ... \otimes \sigma_{\mu_n}.$$

where each $\mu_i \in \{0,1,2,3\}$ and $i = 1,...,n$. Here $\sigma_{\mu_i}$ are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad$$
The real expansion coefficients \( P_{\mu_1...\mu_n}(t) \) are given by
\[
P_{\mu_1...\mu_n}(t) = Tr(\sigma_{\mu_1} \otimes ... \otimes \sigma_{\mu_n} \rho(t)).
\]
Normalization (\( Tr \rho = 1 \)) requires that \( P_0(t) = 1 \). Since the eigenvalues of the Pauli matrices are \( \pm 1 \), the expansion coefficients satisfy \( |P_{\mu_1...\mu_n}(t)| \leq 1 \). Let us rewrite (2) in the form:
\[
\rho(t) = \frac{1}{2^n} \sum_{\mu=0}^{N-1} \sigma_{\mu} P_{\mu}(t),
\]
where \( \sigma_{\mu} = \sigma_{\mu_1} \otimes ... \otimes \sigma_{\mu_n}, \mu = (\mu_1...\mu_n) \) and \( N = 4^n \).

An arbitrary general one-qubit state \( \rho(t) \) can be represented as
\[
\rho(t) = \frac{1}{2} \sum_{\mu=0}^{N-1} \sigma_{\mu} P_{\mu}(t),
\]
where \( P_{\mu}(t) = Tr(\sigma_{\mu} \rho(t)) \) and \( P_0(t) = 1 \). The pure state can be identified with Bloch sphere
\[
P_1^2(t) + P_2^2(t) + P_3^2(t) = 1.
\]
The mixed state can be identified with close ball
\[
P_1^2(t) + P_2^2(t) + P_3^2(t) \leq 1.
\]

Not all linear combinations of quantum states \( \rho_j(t) \) are states. The operator
\[
\rho(t) = \sum_j \lambda_j \rho_j(t)
\]
is a state iff \( \sum_j \lambda_j = 1 \).

### III Liouville space and superoperators

For the concept of Liouville space and superoperators see [28]-[54].

#### III.1 Liouville space

The space of linear operators acting on a \( N = 2^n \)-dimensional Hilbert space \( \mathcal{H}^{(n)} \) is a \( N^2 = 4^n \)-dimensional complex linear space \( \mathcal{H}^{(n)} \). We denote an element \( A \) of \( \mathcal{H}^{(n)} \) by a ket-vector \( |A \rangle \). The inner product of two elements \( |A \rangle \) and \( |B \rangle \) of \( \mathcal{H}^{(n)} \) is defined as
\[
(A|B) = Tr(A^\dagger B).
\]
The norm \( ||A|| = \sqrt{(A|A)} \) is the Hilbert-Schmidt norm of operator \( A \). A new Hilbert space \( \mathcal{H}^{(n)} \) with scalar product (3) is called Liouville space attached to \( \mathcal{H} \) or the associated Hilbert space, or Hilbert-Schmidt space [28]-[54].

Let \( \{|k \rangle \} \) be an orthonormal basis of \( \mathcal{H}^{(n)} \):
\[
<k|k'| = \delta_{kk'}, \sum_{k=0}^{N-1} |k > < k| = I.
\]

Then \( |k, l \rangle = ||k > < l| \) is an orthonormal basis of the Liouville space \( \mathcal{H}^{(n)} \):
\[
(k, l|k', l') = \delta_{kk'} \delta_{ll'}, \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} |k, l \rangle(k, l| = \hat{1},
\]
where \( N = 2^n \). This operator basis has \( 4^n \) elements. Note that \( |k, l \rangle \neq |k, l \rangle \geq |k > < l| > and
\[
|k, l \rangle = |k_1, l_1 \rangle \otimes |k_2, l_2 \rangle \otimes ... \otimes |k_n, l_n \rangle,
\]
where \( k_i, l_i \in \{0, 1\}, i = 1, ..., n \) and
\[
|k_i, l_i \rangle \otimes |k_j, l_j \rangle = |k_i > \otimes |k_j > , < l_i \otimes < l_j |).
\]

For an arbitrary element \( |A \rangle \) of \( \mathcal{H}^{(n)} \) we have
\[
|A \rangle = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} |k, l \rangle(k, l|A \rangle
\]
with
\[
(k, l|A \rangle = Tr(|l > < k|A \rangle) = < k|A|l >= A_{kl}.
\]

An operator \( \rho(t) \) of density matrix for \( n \)-qubits can be considered as an element \( |\rho(t)\rangle \) of the space \( \mathcal{H}^{(n)} \). From (3) we get
\[
|\rho(t)\rangle = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} |k, l \rangle(k, l|\rho(t)\rangle,
\]
where \( N = 2^n \) and
\[
\sum_{k=0}^{N-1} (k, k|\rho(t)\rangle = 1.
\]

#### III.2 Superoperators

Operators, which act on \( \mathcal{H}^{(n)} \), are called superoperators and we denote them in general by the hat.

For an arbitrary superoperator \( \hat{\Lambda} \) on \( \mathcal{H} \), which is defined by
\[
\hat{\Lambda}|A\rangle = |\Lambda(A)\rangle,
\]
we have
\[
(k, l|\hat{\Lambda}|A\rangle = \sum_{k'=0}^{N-1} \sum_{l'=0}^{N-1} (k, l|\hat{\Lambda}|k', l'|\rangle(k', l'|A \rangle = \sum_{k'=0}^{N-1} \sum_{l'=0}^{N-1} \Lambda_{kkl'} A_{k'l'},
\]
where \( N = 2^n \).

Let \( A \) be a linear operator in Hilbert space. Then the superoperators \( \hat{L}_A \) and \( \hat{R}_A \) will be defined by
\[
\hat{L}_A|B\rangle = |AB\rangle, \hat{R}_A|B\rangle = |BA\rangle.
\]
In the basis \(|k, l\rangle\) we have
\[
(k, l|\hat{L}_A|B) = \sum_{k'=0}^{N-1} \sum_{l'=0}^{N-1} (k, l|\hat{L}_A|k', l')(k', l'|B) = \sum_{k'=0}^{N-1} \sum_{l'=0}^{N-1} (\hat{L}_A)_{k'k'} < k'|B|l' >,
\]
Note that
\[
(k, l|AB) = < k|AB|l >= \sum_{k'=0}^{N-1} \sum_{l'=0}^{N-1} < k|A|k'> < k'|B|l' > < l'|l >.
\]
Finally, we obtain
\[
(\hat{L}_A)_{k'k'} = < k|A|k'> < l'|l > = A_{kk'} \delta_{ll'}.
\]
The superoperator \(\hat{P} = |A\rangle\langle B|\) is defined by
\[
\hat{P}|C) = |A\rangle\langle B|C| = |A\rangle Tr(B|C).
\]
A superoperator \(\hat{E}^\dagger\) is called the adjoint superoperator for \(\hat{E}\) if
\[
(\hat{E}^\dagger(A)|B) = (A|\hat{E}(B))
\]
for all \(|A\rangle\) and \(|B\rangle\) from \(\hat{H}\). For example, if \(\hat{E} = \hat{L}_A \hat{R}_B\), then \(\hat{E}^\dagger = \hat{L}_{A^\dagger} \hat{R}_{B^\dagger}\). If \(\hat{E} = \hat{L}_A\), then \(\hat{E}^\dagger = \hat{L}_{A^\dagger}\). If \(\hat{E} = \hat{L}_A \hat{R}_{A^\dagger}\), then \(\hat{E}^\dagger = \hat{L}_{A^\dagger} \hat{R}_A\).

A superoperator \(\hat{E}\) is called unital if \(\hat{E}(|I\rangle) = |I\rangle\).

IV Generalized computational basis and ququats

Let us introduce generalized computational basis and generalized computational states for \(4^n\)-dimensional operator Hilbert space (Liouville space).

IV.1 Pauli representation

Pauli matrices \(\{I, \sigma\}\) can be considered as a basis in operator space. Let us write the Pauli matrices \(\{I, \sigma\}\) in the form
\[
\sigma_1 = |0\rangle\langle 0| + |1\rangle\langle 1| = |0, 1\rangle + |1, 0\rangle,
\]
\[
\sigma_2 = -i|0\rangle\langle 1| + i|1\rangle\langle 0| = -i(|0, 1\rangle - |1, 0\rangle),
\]
\[
\sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1| = |0, 0\rangle - |1, 1\rangle,
\]
\[
\sigma_0 = I = |0\rangle\langle 0| + |1\rangle\langle 1| = |0, 0\rangle + |1, 1\rangle.
\]
Let us use the formulas
\[
|0, 0\rangle = \frac{1}{2}(|\sigma_0\rangle + |\sigma_3\rangle), \quad |1, 1\rangle = \frac{1}{2}(|\sigma_0\rangle - |\sigma_3\rangle),\quad |0, 1\rangle = \frac{1}{2}(|\sigma_1\rangle + i|\sigma_2\rangle), \quad |1, 0\rangle = \frac{1}{2}(|\sigma_1\rangle - i|\sigma_2\rangle).
\]

It allows to rewrite operator basis
\[
|k, l\rangle = |k_1, l_1\rangle \otimes |k_2, l_2\rangle \otimes \cdots \otimes |k_n, l_n\rangle
\]
by complete basis operators
\[
|\sigma_\mu\rangle = |\sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes \cdots \otimes \sigma_{\mu_n}\rangle,
\]
where \(\mu_i = 2k_i + l_i\), i.e. \(\mu_i \in \{0, 1, 2, 3\}\) and \(i = 1, ..., n\). The basis \(|\sigma_\mu\rangle\) is orthogonal
\[
(|\sigma_\mu\rangle|\sigma_{\mu'}\rangle) = 2^n \delta_{\mu\mu'},
\]
and complete operator basis
\[
\frac{1}{2^n} \sum_{\mu=0}^{N-1} |\sigma_\mu\rangle (\sigma_\mu | |A\rangle) = \hat{I}.
\]

For an arbitrary element \(|A\rangle\) of \(\hat{H}^{(n)}\) we have Pauli representation by
\[
|A\rangle = \frac{1}{2^n} \sum_{\mu=0}^{N-1} |\sigma_\mu\rangle (\sigma_\mu | |A\rangle)
\]
with the complex coefficients
\[
(|\sigma_\mu\rangle | A\rangle) = Tr(\sigma_\mu A).
\]

We can rewrite formulas (2) using the complete operator basis \(|\sigma_\mu\rangle\) in Liouville space \(\hat{H}^{(n)}\):
\[
|\rho(t)\rangle = \frac{1}{2^n} \sum_{\mu=0}^{N-1} |\sigma_\mu\rangle (\sigma_\mu | \rho(t)\rangle),
\]
where \(\sigma_\mu = \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n}\), \(\mu = (\mu_1 ... \mu_n)\) and \((\sigma_\mu | \rho(t)\rangle) = P_\mu(t)\).

The density matrix operator \(\rho(t)\) is a self-adjoint operator with unit trace. It follows that
\[
P_\mu^*(t) = P_\mu(t), \quad P_0(t) = (\sigma_0 | \rho(t)\rangle) = 1.
\]
In general case,
\[
\frac{1}{2^n} \sum_{\mu=0}^{N-1} P^2_\mu(t) = (\rho(t) | \rho(t)\rangle) = Tr(\rho^2(t)) \leq 1.
\]
Note that Schwarz inequality
\[
(|A\rangle | B\rangle)^2 \leq (A | A\rangle) (B | B\rangle)
\]
leads to
\[
(|I\rangle | \rho(t)\rangle)^2 \leq (I | I\rangle) (\rho(t) | \rho(t)\rangle),
\]
\[
1 = |Tr\rho(t)\rangle^2 \leq 2^n (\rho(t) | \rho(t)\rangle) = \sum_{\mu=0}^{N-1} P^2_\mu(t),
\]
i.e.
\[
\frac{1}{\sqrt{2^n}} \leq Tr(\rho^2(t)) \leq 1 \quad \text{or} \quad 1 \leq \sum_{\mu=0}^{N-1} P_{\mu}^2(t) \leq 2^n.
\]

An arbitrary general one-qubit state \( \rho(t) \) can be represented in Liouville space \( \mathcal{L}_2 \) as
\[
|\rho(t)\rangle = \frac{1}{2} \sum_{\mu=0}^{3} |\sigma_{\mu} \rangle P_{\mu}(t),
\]
where \( P_{\mu} = (\sigma_{\mu} | \rho \rangle \langle \rho | \sigma_{\mu} \rangle \), \( P_0 = 1 \) and \( P_{\mu}^2(t) + P_{\mu}^2(t) + P_{\mu}^2(t) \leq 1 \). Note that the basis \( |\sigma_{\mu} \rangle \) is orthogonal, but is not orthonormal.

**IV.2 Generalized computational basis**

Let us define the orthonormal basis \( |\mu\rangle \) of Liouville space \( \mathcal{L}^{(n)} \). In general case, the state \( \rho(t) \) of the n-qubits system is an element of Hilbert space \( \mathcal{H}^{(n)} \). The basis for \( \mathcal{H}^{(n)} \) consists of the \( N^2 = 2^{2n} \) orthonormal basis elements denoted by \( |\mu\rangle \).

**Definition** A basis of Liouville space \( \mathcal{L}^{(n)} \) is defined by
\[
|\mu\rangle = |\mu_1...\mu_n\rangle = \frac{1}{\sqrt{2^n}} |\sigma_{\mu_1} \otimes ... \otimes \sigma_{\mu_n} \rangle,
\]
where each \( \mu_i \in \{0, 1, 2, 3\} \) and
\[
(\mu | \mu' \rangle = \delta_{\mu\mu'}, \quad \sum_{\mu=0}^{N-1} |\mu\rangle |\mu\rangle = I,
\]
is called a generalized computational basis.

Here \( \mu \) is 4-valued representation of
\[
\mu = \mu_1 4^{n-1} + ... + \mu_{n-1} 4 + \mu_n.
\]

**Example.** In general case, one-qubit state \( \rho(t) \) of open quantum system is
\[
|\rho\rangle = |0\rangle \frac{1}{\sqrt{2}} + |1\rangle \rho_1 + |2\rangle \rho_2 + |3\rangle \rho_3,
\]
where four orthonormal basis elements are
\[
|0\rangle = \frac{1}{\sqrt{2}} |\sigma_0 \rangle = \frac{1}{\sqrt{2}} |I \rangle, \quad |1\rangle = \frac{1}{\sqrt{2}} |\sigma_1 \rangle,
\]
\[
|2\rangle = \frac{1}{\sqrt{2}} |\sigma_2 \rangle, \quad |3\rangle = \frac{1}{\sqrt{2}} |\sigma_3 \rangle.
\]

**Example.** Two-qubit state \( \rho(t) \) is an element of 16-dimensional Hilbert space with the orthonormal basis
\[
|00\rangle = \frac{1}{2} |I \otimes I \rangle, \quad |0k\rangle = \frac{1}{2} |I \otimes \sigma_k \rangle,
\]
\[
|k0\rangle = \frac{1}{2} |\sigma_k \otimes I \rangle, \quad |kl\rangle = \frac{1}{2} |\sigma_k \otimes \sigma_l \rangle,
\]
where \( k, l \in \{1, 2, 3\} \).

The usual computational basis \( \{|k\rangle\} \) is not a basis of general state \( \rho(t) \) which has a time dependence. In general case, a pure state evolves to mixed state.

Pure state of n two-level closed quantum systems is an element of \( 2^n \)-dimensional functional Hilbert space \( \mathcal{H}^{(n)} \). It leads to quantum computer model with 2-valued logic. In general case, the mixed state \( \rho(t) \) of n two-level (open or closed) quantum system is an element of \( 4^n \)-dimensional operator Hilbert space \( \mathcal{L}^{(n)} \) (Liouville space). It leads to 4-valued logic model for quantum computer.

The state of the quantum computation at any point time is a superposition of basis elements
\[
|\rho(t)\rangle = \sum_{\mu=0}^{N-1} |\mu\rangle \rho_{\mu}(t),
\]
where \( \rho_{\mu}(t) \) are real numbers (functions) satisfying normalized condition \( \rho_0(t) = 1/\sqrt{2^n} \), i.e.
\[
\sqrt{2^n} |\rho(t)\rangle = Tr(\rho(t)) = 1.
\]

Any state \( |\rho(t)\rangle \) for basis element \( |000...0\rangle \) has \( P_{00...0} = 1 \) in all cases.

**IV.3 Generalized computational states**

Generalized computational basis elements \( |\mu\rangle \) are not quantum states for \( \mu \neq 0 \). It follows from normalized condition \( (0 | \rho(t) \rangle = 1/\sqrt{2} \). The general quantum state in Pauli representation \( \{\}\) has the form
\[
|\rho(t)\rangle = \sum_{\mu=0}^{N-1} |\mu\rangle \rho_{\mu}(t),
\]
where \( P_{0}(t) = 1 \) in all cases. Let us define simple computational quantum states.

**Definition** A quantum states in Liouville space defined by
\[
|\mu\rangle = \frac{1}{2^n} \left( |\sigma_0 \rangle + |\sigma_\mu \rangle (1 - \delta_{\mu0}) \right)
\]
or
\[
|\mu\rangle = \frac{1}{\sqrt{2^n}} \left( |0 \rangle + |\mu\rangle (1 - \delta_{\mu0}) \right).
\]
is called generalized computational states.

Note that all states \(|\mu\rangle\), where \( \mu \neq 0 \), are pure states, since \(|\mu | |\mu\rangle = 1\). The state \(|0\rangle\) is maximally mixed state. The states \(|\mu\rangle\) are elements of Liouville space \( \mathcal{L}^{(n)} \).

Quantum state in a 4-dimensional Hilbert space is usually called ququat or qu-quart[55] or qudit [56, 57, 58, 59, 60] with \( D = 4 \). Usually ququat is considered as 4-level quantum states for \( \mu \neq 0 \). We consider ququat as general quantum state in a 4-dimensional operator Hilbert space.

**Definition** A quantum state in 4-dimensional operator Hilbert space (Liouville space) \( \mathcal{L}^{(4)} \) associated with single qubit of \( \mathcal{H}^{(1)} = \mathcal{H}_2 \) is called single ququat. A quantum state in \( 4^n \)-dimensional Liouville space \( \mathcal{L}^{(4^n)} \) associated with n-qubits system is called n-quqats.
Example. For the single ququat the states $|\mu\rangle$ are

$$|0\rangle = \frac{1}{2}|\sigma_0\rangle, \quad |k\rangle = \frac{1}{2}(|\sigma_0\rangle + |\sigma_k\rangle),$$

or

$$|0\rangle = \frac{1}{\sqrt{2}}|0\rangle, \quad |k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |k\rangle).$$

It is convenient to use matrices for quantum states. In matrix representation the single ququat computational basis $|\mu\rangle$ and computational states $|\mu\rangle$ can be represented by

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In this representation single qubit generalized computational states $|\mu\rangle$ is represented by

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$|2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

A general single ququat quantum state $|\rho\rangle = \sum_{\mu=0}^{N-1} |\mu\rangle\rho_\mu$ is represented

$$|\rho\rangle = \begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix},$$

where $\rho_0 = 1/\sqrt{2}$ and $\rho_1^2 + \rho_2^2 + \rho_3^2 \leq 1$.

We can use the other matrix representation for the states $|\rho\rangle$ which has no the coefficient $1/\sqrt{2^n}$. In this representation single qubit generalized computational states $|\mu\rangle$ is represented by

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

A general single ququat quantum state

$$|\rho\rangle = \begin{pmatrix} 1 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix},$$

where $P_1^2 + P_2^2 + P_3^2 \leq 1$, is a superposition of generalized computational states

$$|\rho\rangle = |0\rangle(1 - P_1 - P_2 - P_3) + |1\rangle P_1 + |2\rangle P_2 + |3\rangle P_3.$$

Note that density matrix operator $\rho$ as an element of Liouville space is represented by $|\rho\rangle$ and $|\rho\rangle$. We use different brackets only to emphasize the different matrix representations connected by coefficient $1/\sqrt{2^n}$. This coefficient can be neglected under the consideration of the quantum 4-valued logic gates.

V Evolution equations and quantum operations

In this section I review the description of open quantum systems dynamics in terms of evolution equations and quantum operations.

V.1 Evolution equation for pure state of closed systems

Let $H$ be the Hamilton operator, then in the Schroedinger picture the equation of motion for the pure state $|\Psi(t)\rangle >$ of closed system is given by the Schroedinger equation

$$\frac{d}{dt}|\Psi(t)\rangle > = -iH|\Psi(t)\rangle > . \quad (8)$$

The change in the state $|\Psi(t)\rangle >$ of a closed quantum system between two fixed times $t$ and $t_0$ is described by a unitary operator $U(t, t_0)$ which depends on those times

$$|\Psi(t)\rangle > = U(t, t_0)|\Psi(t_0)\rangle > .$$

If the Hamilton operator $H$ has no time dependence, then the unitary operator $U(t, t_0)$ has the form

$$U(t, t_0) = exp\{-i(t - t_0)H\}.$$

In general case, the unitary operator $U(t, t_0)$ is defined by

$$\frac{d}{dt}U(t, t_0) = -iHU(t, t_0), \quad U(t_0, t_0) = I.$$

A pure state $|\Psi\rangle \in \mathcal{H}^{(n)}$ of closed $n$-qubits system is generally a superposition of the orthonormal basis states $|k\rangle$

$$|\Psi(t)\rangle > = \sum_k a_k(t)|k\rangle > .$$

Let the Hamilton operator $H$ on the space $\mathcal{H}^{(n)}$ can be written in the form

$$H = \sum_{l,m} H_{lm}|l\rangle < m|.$$

Then equation (8) can be given in the form

$$\frac{d}{dt}a_k(t) = -i\sum_l H_{kl}a_l(t).$$

V.2 Evolution equation for mixed state of closed system

Let $H$ be the Hamiltonian, then in the Schroedinger picture the evolution equation for the mixed state $\rho(t)$ of closed system is given by the von Neumann equation

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)]. \quad (9)$$

This equation can be rewritten by

$$\frac{d}{dt}|\rho(t)\rangle = \hat{A}|\rho(t)\rangle,$$
where the Liouville superoperator $\hat{\Lambda}$ is given by

$$\hat{\Lambda} = -i(\hat{L}_H - \hat{R}_H).$$

A change of pure and mixed states of closed (Hamiltonian) quantum system is the unitary evolution. The final state $\rho(t) = \rho$ by unitary transformation $U = U(t, t_0)$:

$$\rho \rightarrow \rho(t) = U_t(\rho) = U\rho U^\dagger,$$

where $U t^\dagger = I$. The superoperator $\hat{U}$ is written in the form $\hat{U} = \hat{L}_U \hat{R}_U$: $|\rho(t)\rangle = \hat{U}|\rho\rangle$.

**V.3 Evolution equation for mixed state of open system**

A classification of norm continuous (or, equivalently, with bounded generators) dynamical semigroup [61] of the Bahach space of trace-class operators on $\mathcal{H}$, has been given by Lindblad [62]. The general form of the generator $\Lambda$ of such a semigroup is the following

$$\hat{\Lambda}\rho = -i[H, \rho] + \Phi(\rho) - \Phi(I) \circ \rho ,$$

where

$$\Phi(B) = \sum_j V_j BV_j^\dagger , \quad A \circ B = \frac{1}{2}(AB + BA).$$

Here $H$ is a bounded self-adjoint Hamilton operator, $\{V_j\}$ is a sequence of bounded operators, $\Phi(I)$ is a bounded operator.

The evolution equation has the form

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \sum_j \left( V_j \rho(t) V_j^\dagger - \rho(t) \circ (V_j V_j^\dagger) \right).$$

For the proofs of (11) and (12), we refer to [62, 66]. Using equation (12) the evolution equation for the mixed state $|\rho(t)\rangle$ can be written by

$$\frac{d}{dt}|\rho(t)\rangle = \hat{\Lambda}|\rho(t)\rangle,$$

where the Liouville superoperator $\hat{\Lambda}$ is given by

$$\hat{\Lambda} = -i(\hat{L}_H - \hat{R}_H) + \frac{1}{2} \sum_j \left( 2\hat{L}_{V_j} \hat{R}_{V_j}^\dagger - \hat{L}_{V_j} \hat{L}_{V_j}^\dagger - \hat{R}_{V_j} \hat{R}_{V_j}^\dagger \right).$$

In the case of a n-level system ($\text{dim}\mathcal{H} = n$), evolution equation (12) can be given in the form [61]:

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \sum_{k,l=1}^{n^2-1} C_{kl} \left( F_k \rho(t) F_l^\dagger - \rho(t) \circ (F_k F_l^\dagger) \right),$$

where

$$H^\dagger = H, \quad Tr(H) = 0, \quad Tr(F_k) = 0, \quad Tr(F_k^\dagger F_l) = \delta_{kl},$$

The matrix $\{C_{kl}\}$ is a positive matrix $(n^2 - 1) \times (n^2 - 1)$ and \{I, $F_k|k = 1, \ldots, n^2 - 1\}$ is an operator basis for the space of bounded operators on $\mathcal{H}$. The matrix $\{C_{kl}\}$ is called a positive matrix if all elements $C_{kl}$ are real ($C_{kl} = C_{lk}$) and positive $C_{kl} > 0$. For the proofs of (13), we refer to [61].

For a given $\Lambda$, operator $H$ is uniquely determined by the condition $Tr(H) = 0$, and the matrix $\{C_{kl}\}$ is uniquely determined by the choice of the $F_k$. The conditions $Tr(H) = 0$ and $Tr(F_k) = 0$ provide a canonical separation of the superoperator $\Lambda$ into a Hamiltonian plus dissipative part.

If the condition of completely positivity is replaced by the weaker requirement of simple positivity, the generator for a n-level system can be again written in the form (13), where the matrix $\{C_{kl}\}$ is a matrix of positive defined Hermitian form [57, 63], i.e.

$$\sum_{k,l=1}^{n^2 - 1} C_{kl} z_k z_l^* > 0,$$

for all $z_k \in \mathbb{C}$. The matrix $\{C_{kl}\}$ of Hermitian form is Hermitian matrix ($C_{kl} = C_{lk}$). It is known [53] that Hermitian form is positive if and only if

$$\det \begin{pmatrix} C_{11} & C_{12} & \ldots & C_{1k} \\ C_{21} & C_{22} & \ldots & C_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ C_{k1} & C_{k2} & \ldots & C_{kk} \end{pmatrix} > 0,$$

for all $k = 1, 2, \ldots, n^2 - 1$. This condition is equivalent to the condition of positivity for matrix eigenvalues.

Let us consider a two-level quantum system (qubit) [61], [63, 64, 65] for a positive trace-preserving semigroup. Let $\{F_\mu\}$, where $\mu \in \{0, 1, 2, 3\}$, be a complete orthonormal set of self-adjoint matrices:

$$F_0 = \frac{1}{\sqrt{2}} I, \quad F_1 = \frac{1}{\sqrt{2}} \sigma_1, \quad F_2 = \frac{1}{\sqrt{2}} \sigma_2, \quad F_3 = \frac{1}{\sqrt{2}} \sigma_3.$$

Let Hamilton operator $H$ and state $\rho(t)$ have the form

$$H = \sum_{k=1}^{3} H_k \sigma_k, \quad \rho(t) = \frac{1}{2} (P_0 I + P_k(t) \sigma_k),$$

where $P_0 = 1$ in all cases. Using the relations

$$\sigma_k \sigma_l = I \delta_{kl} + i \sum_{m=1}^{3} \varepsilon_{klm} \sigma_m, \quad [\sigma_k, \sigma_l] = 2i \sum_{m=1}^{3} \varepsilon_{klm} \sigma_m,$$

and $\varepsilon_{klm} \varepsilon_{ijm} = \delta_{kl} \varepsilon_{ij} - \delta_{kj} \varepsilon_{il}$, for (13) we obtain the equations:

$$\frac{d}{dt} P_k(t) = \sum_{l=1}^{3} \left( 2H_m \varepsilon_{klm} + \frac{1}{8} (C_{kl} + C_{lk}) - \frac{1}{4} C \delta_{kl} \right) P_l(t) - \frac{1}{4} \varepsilon_{ijl} (I_m C_{ij}) P_0,$$

where $C = \frac{1}{n} C_{mnm}$ and $k, l \in \{1, 2, 3\}$. We can rewrite this equation in the form

$$\frac{d}{dt} P_\mu(t) = \sum_{\nu=0}^{3} \mathcal{L}_{\mu \nu} P_\nu(t),$$

for all $\mu \in \{0, 1, 2, 3\}$. We can rewrite this equation in the form

$$\frac{d}{dt} P_\mu(t) = \sum_{\nu=0}^{3} \mathcal{L}_{\mu \nu} P_\nu(t),$$

for all $\mu \in \{0, 1, 2, 3\}$.
where $\mu, \nu \in \{0, 1, 2, 3\}$ and the matrix $L_{\mu\nu}$ is

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
B_1 & -C_{(22)} - C_{(33)} & C_{(12)} + 2H_3 & C_{(13)} + 2H_2 \\
B_2 & C_{(12)} + 2H_3 & -C_{(11)} - C_{(33)} & C_{(23)} - 2H_1 \\
B_3 & C_{(13)} + 2H_2 & C_{(23)} + 2H_1 & -C_{(11)} - C_{(22)}
\end{pmatrix}
\]

where

\[
B_k = -\frac{1}{4} \varepsilon_{ijk} (JmC_{ij}), \quad C_{(kl)} = \frac{1}{8} (C_{kl} + C_{lk}).
\]

If the matrix $C_{kl}$ and Hamilton operator $H$ are not time-dependent, then equation (14) has a solution

\[
P_\mu(t) = \sum_{\nu=0}^{3} E_{\mu\nu}(t, t_0) P_\nu(t_0),
\]

where the matrix $E_{\mu\nu}$ is

\[
E = \begin{pmatrix}
1 & 0 & 0 & 0 \\
T_1 & R_{11} & R_{12} & R_{13} \\
T_2 & R_{21} & R_{22} & R_{23} \\
T_3 & R_{31} & R_{32} & R_{33}
\end{pmatrix}.
\]

The matrices $T$ and $R$ of the matrix $E_{\mu\nu}$ are defined by

\[
T = (e^{rA} - I)(\tau A)^{-1}B = \sum_{n=0}^{\infty} \frac{\tau^{n-1}}{n!} A^{n-1},
\]

\[
R = e^{rA} = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} A^n,
\]

where $\tau = t - t_0$ and elements of the matrix $A$ are

\[
A_{kl} = 2H_{m} e_{kml} + \frac{1}{8} (C_{kl} + C_{lk}) - \frac{1}{4} C_\delta_{kl}.
\]

If $C_{kl}$ is a real matrix, then all $T_k = 0$, where $k = 1, 2, 3$.

### V.4 Quantum operation

Unitary evolution (10) is not the most general type of state change possible for quantum systems. A most general state change of a quantum system is a positive map $E$ which is called a quantum operation or superoperator. For the concept of quantum operations see [58, 69, 70, 4]. In the formalism of quantum operations the final (output) state $\rho'$ is related to the initial (output) state $\rho$ by a map

\[
\rho \rightarrow \rho' = \frac{E(\rho)}{Tr(E(\rho))}.
\]

By definition, $Tr(E(\rho))$ is the probability that the process represented by $E$ occurs, when $\rho$ is the initial state. The probability never exceed 1. The quantum operation $E$ is trace-decreasing, i.e. $Tr(E(\rho)) \leq 1$ for all density matrix operators $\rho$. This condition can be expressed as an operator inequality for $A_j$. The operators $A_j$ must satisfy

\[
\sum_{j=1}^{m} A_j^\dagger A_j \leq I.
\]

The normalized post-dynamics system state is defined by (15). The map (15) is nonlinear trace-preserving map. If the linear quantum operation $E$ is trace-preserving $Tr(E(\rho)) = 1$, then

\[
\sum_{j=1}^{m} A_j^\dagger A_j = I.
\]

Notice that a trace-preserving quantum operation $E(\rho) = A\rho A^\dagger$ must be a unitary transformation ($A^\dagger A = AA^\dagger = I$).

The example of nonunitary dynamics is associated with the measurement of quantum system. The system being measured is no longer a closed system, since it is interacting with the measuring device. The usual way [2] to describe a measurement (von Neumann measurement) is a set of projectors $P_k$ onto the pure state space of the system such that

\[
P_k P_l = \delta_{kl} P_k, \quad P_k^\dagger = P_k, \quad \sum_k P_k = I.
\]

The unnormalized state of the system after the measurement is given by

\[
E_k(\rho) = P_k \rho P_k.
\]

The probability of this measurement result is given by

\[
p(k) = Tr(E_k(\rho)).
\]

The normalization condition, $\sum_k p(k) = 1$ for all density matrix operators, is equivalent to the completeness condition $\sum_k P_k = I$. If the state of the system before the measurement was $\rho$, than the normalized state of the system after the measurement is

\[
\rho' = p^{-1}(k) E_k(\rho).
\]

### VI Quantum four-valued logic gates

In this section we consider some properties of four-valued logic gates. We connect quantum four-valued logic gates with unitary two-valued logic gates and quantum operations by the generalized computational basis.

#### VI.1 Generalized quantum gates

Quantum operations can be considered as generalized quantum gates that act on general (mixed) states. Let us define a quantum 4-valued logic gates.

**Definition** Quantum four-valued logic gate is a superoperator $E$ on Liouville space $\mathcal{H}^{(4)}$ which maps a density matrix
operator $|\rho\rangle$ of $n$-ququats to a density matrix operator $|\rho'\rangle$ of $n$-ququats.

A generalized quantum gate is a superoperator $\hat{E}$ which maps density matrix operator $|\rho\rangle$ to density matrix operator $|\rho'\rangle$. If $\rho$ is operator of density matrix, then $\hat{E}(\rho)$ should also be a density matrix operator. Any density matrix operator is self-adjoint (i.e., $\rho^\dagger(t) = \rho(t)$), positive (i.e., $\rho(t) > 0$) operator with unit trace ($Tr\rho(t) = 1$). Therefore we have some requirements for superoperator $\hat{E}$.

The requirements for a superoperator $\hat{E}$ to be a generalized quantum gate are as follows:

1. The superoperator $\hat{E}$ is real superoperator, i.e. $(\hat{E}(A^\dagger))^\dagger = \hat{E}(A^\dagger)$ for all $A$ or $(\hat{E}(\rho))^\dagger = \hat{E}(\rho)$. Real superoperator $\hat{E}$ maps self-adjoint operator $\rho$ into self-adjoint operator $\hat{E}(\rho)$: $(\hat{E}(\rho))^\dagger = \hat{E}(\rho)$.

2. The gate $\hat{E}$ is a positive superoperator, i.e. $\hat{E}$ maps positive operators to positive operators: $\hat{E}(A^2) > 0$ for all $A \neq 0$ or $\hat{E}(\rho) > 0$.

3. We have to assume the superoperator $\hat{E}$ to be not merely positive but completely positive. The superoperator $\hat{E}$ is completely positive map of Liouville space, i.e. the positivity is remained if we extend the Liouville space $\mathcal{L}^{(m)}$ by adding more qubits. That is, the superoperator $\hat{E} \otimes \hat{I}^{(m)}$ must be positive, where $\hat{I}^{(m)}$ is the identity superoperator on some Liouville space $\mathcal{L}^{(m)}$.

3. The superoperator $\hat{E}$ is trace-preserving map, i.e.

$$
(I|\hat{E}|\rho) = (\hat{E}(I)|\rho) = 1 \quad \text{or} \quad \hat{E}^\dagger(I) = I.
$$

In general case, the linear trace-decreasing superoperator is not a quantum four-valued logic gate, since it can be not trace-preserving. The generalized quantum gate can be defined as nonlinear trace-preserving gate $\hat{N}$ by

$$
\hat{N}|\rho\rangle = (\hat{E}|\rho\rangle I|\hat{E}|\rho\rangle)^{-1} \quad \text{or} \quad \hat{N}(\rho) = \frac{\hat{E}(\rho)}{Tr(\hat{E}(\rho))},
$$

where $\hat{E}$ is a linear completely positive trace-decreasing superoperator.

In the generalized computational basis the gate $\hat{E}$ can be represented by

$$
\hat{E} = \frac{1}{2^n} \sum_{\mu=0}^{N-1} \sum_{\nu=0}^{N-1} \epsilon_{\mu\nu}|\sigma_\mu\rangle\langle\sigma_\nu|,
$$

where $N = 4^n$, $\mu$ and $\nu$ are 4-valued representation of

$$
\mu = \mu_1 4^{N-1} + ... + \mu_{N-4} 4^{N-1} + \mu_N,
$$

$$
\nu = \nu_1 4^{N-1} + ... + \nu_{N-4} 4^{N-1} + \nu_N,
$$

$$
\sigma_\mu = \sigma_{\mu_1} \otimes ... \otimes \sigma_{\mu_m},
$$

$\mu_i, \nu_i \in \{0, 1, 2, 3\}$ and $\epsilon_{\mu\nu}$ are elements of some matrix.

VI.2 General quantum operation as four-valued logic gates

**Proposition 1** In the generalized computational basis $|\mu\rangle$ any linear two-valued logic quantum operation $\hat{E}$ can be represented as a quantum four-valued logic gate $\hat{E}$ defined by

$$
\hat{E} = \sum_{\mu=0}^{N-1} \sum_{\nu=0}^{N-1} \epsilon_{\mu\nu} |\mu\rangle \langle\nu|,
$$

where

$$
\epsilon_{\mu\nu} = \frac{1}{\sqrt{2^n}} Tr\left(\sigma_\mu \hat{E}(\sigma_\nu)\right),
$$

and $\sigma_\mu = \sigma_{\mu_1} \otimes ... \otimes \sigma_{\mu_m}$.

**Proof.** The state $\rho(t)$ in the generalized computational basis $|\mu\rangle$ has the form

$$
|\rho(t)\rangle = \sum_{\mu=0}^{N-1} |\mu\rangle \rho_\mu(t),
$$

where $N = 4^n$ and

$$
\rho_\mu(t) = (|\mu\rangle \langle\mu|) = \frac{1}{\sqrt{2^n}} Tr(\sigma_\mu \rho(t)).
$$

The quantum operation $\hat{E}$ define a four-valued logic gate by

$$
|\rho(t)\rangle = \hat{E}_I|\rho\rangle = |\hat{E}_I(\rho)\rangle = \sum_{\nu=0}^{N-1} |\hat{E}_I(\sigma_\nu)\rangle \frac{1}{\sqrt{2^n}} \rho_\nu(t_0).
$$
Then
\[
(\mu|\rho(t)) = \sum_{\nu=0}^{N-1} (\sigma_{\mu}|E_{\nu}(\sigma_{\nu})) \frac{1}{2^n} \rho_{\nu}(t_0).
\]
Finally, we obtain
\[
\rho_{\mu}(t) = \sum_{\nu=0}^{N-1} \mathcal{E}_{\nu\mu} \rho_{\nu}(t_0),
\]
where
\[
\mathcal{E}_{\nu\mu} = \frac{1}{2^n} (\sigma_{\mu}|E_{\nu}(\sigma_{\nu})) = \frac{1}{2^n} \text{Tr} \left( \sigma_{\mu} E_{\nu}(\sigma_{\nu}) \right).
\]
This formula defines a relation between general quantum operation \(\mathcal{E}\) and the real \(4^n \times 4^n\) matrix \(\mathcal{E}_{\nu\mu}\) of four-valued logic gate \(\hat{\mathcal{E}}\).

Four-valued logic gates \(\hat{\mathcal{E}}\) in the matrix representation are represented by \(4^n \times 4^n\) matrices \(\mathcal{E}_{\nu\mu}\). The matrix \(\mathcal{E}_{\nu\mu}\) of the gate \(\hat{\mathcal{E}}\) is
\[
\mathcal{E} = 
\begin{pmatrix}
\mathcal{E}_{00} & \mathcal{E}_{01} & \cdots & \mathcal{E}_{0a} \\
\mathcal{E}_{10} & \mathcal{E}_{11} & \cdots & \mathcal{E}_{1a} \\
\vdots & \vdots & \ddots & \vdots \\
\mathcal{E}_{a0} & \mathcal{E}_{a1} & \cdots & \mathcal{E}_{aa}
\end{pmatrix}.
\]
where \(a = N - 1 = 4^n - 1\).

In matrix representation the gate \(\hat{\mathcal{E}}\) maps the state \(|\rho\rangle\) to the state \(|\rho'\rangle = \sum_{\nu=0}^{N-1} |\mu\rangle \rho_{\mu}\) by
\[
|\rho'\rangle = \sum_{\nu=0}^{N-1} \mathcal{E}_{\nu\mu} \rho_{\nu}.
\]

**Proof.**

\[
\mathcal{E}_{\nu\mu} = \frac{1}{2^n} \sum_{j=1}^{m} \text{Tr} \left( \sigma_{\mu} A_j \sigma_{\nu} A_j^\dagger \right) = \frac{1}{2^n} \sum_{j=1}^{m} (A_j^\dagger \sigma_{\mu} \sigma_{\nu} A_j).
\]

**Proposition 3** Any real matrix \(\mathcal{E}_{\nu\mu}\) associated with linear (trace-preserving) quantum four-valued logic gates [18] has
\[
\mathcal{E}_{0\nu} = \delta_{0\nu}.
\]

**Proof.**

\[
\mathcal{E}_{0\nu} = \frac{1}{2^n} \text{Tr} \left( \sigma_0 \mathcal{E}(\sigma_{\nu}) \right) = \frac{1}{2^n} \text{Tr} \left( \mathcal{E}(\sigma_{\nu}) \right) = \\
= \frac{1}{2^n} \text{Tr} \left( \sum_{j=1}^{m} A_j \sigma_{\nu} A_j^\dagger \right) = \frac{1}{2^n} \text{Tr} \left( \left( \sum_{j=1}^{m} A_j A_j^\dagger \right) \sigma_{\nu} \right) = \\
= \frac{1}{2^n} \text{Tr} \sigma_{\nu} = \delta_{0\nu}.
\]

The general linear n-ququats quantum gate has the form:
\[
\mathcal{E} = 
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
T_1 & R_{11} & \cdots & R_{1N-1} \\
T_2 & R_{21} & \cdots & R_{2N-1} \\
\vdots & \vdots & \ddots & \vdots \\
T_{N-1} & R_{N-1,1} & \cdots & R_{N-1,N-1}
\end{pmatrix}.
\]

Completely positive condition leads to some inequalities [17], [88.59] for matrix elements \(\mathcal{E}_{\mu\nu}\).

In general case, linear quantum 4-value logic gate acts on \(|0\rangle\) by
\[
\hat{\mathcal{E}}|0\rangle = |0\rangle + \sum_{k=1}^{N-1} T_k|k\rangle.
\]

For example, single ququat quantum gate acts by
\[
\hat{\mathcal{E}}|0\rangle = |0\rangle + |1\rangle + |2\rangle + |3\rangle.
\]

If all \(T_k, \ k = 1, \ldots, N - 1\) is equal to zero, then \(\hat{\mathcal{E}}|0\rangle = |0\rangle\). The linear quantum gates with \(T = 0\) conserve the maximally mixed state \(|0\rangle\) invariant.

**Definition** A quantum four-valued logic gate \(\hat{\mathcal{E}}\) is called unitary gate or gate with \(T = 0\) if maximally mixed state \(|0\rangle\) is invariant under the action of this gate: \(\hat{\mathcal{E}}|0\rangle = |0\rangle\).

The output state of a linear quantum four-valued logic gate \(\hat{\mathcal{E}}\) is \(|00...0\rangle\) if and only if the input state is \(|00...0\rangle\). If \(\hat{\mathcal{E}}|00...0\rangle \neq |00...0\rangle\), then \(\hat{\mathcal{E}}\) is not unital gate.
**Proposition 4** The matrix $E_{\mu\nu}$ of linear trace-preserving $n$-ququats gate $\hat{E}$ is an element of group $TGL(4^n-1, \mathbb{R})$ which is a semidirect product of general linear group $GL(4^n-1, \mathbb{R})$ and translation group $T(4^n-1, \mathbb{R})$.

**Proof.** This proposition follows from proposition 3. Any element (gate matrix $E_{\mu\nu}$) of group $TGL(4^n-1, \mathbb{R})$ can be represented by

$$E(T, R) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ T & R \end{pmatrix},$$

where $T$ is a column with $4^n-1$ elements, 0 is a line with $4^n-1$ zero elements, and $R$ is a real $(4^n-1) \times (4^n-1)$ matrix $R \in GL(4^n-1, \mathbb{R})$. If $R$ is orthogonal $(4^n-1) \times (4^n-1)$ matrix ($R^T R = I$), then we have motion group $\{2, 95\}$. The group multiplication of elements $E(T, R)$ and $E(T', R')$ is defined by

$$E(T, R)E(T', R') = E(T + RT', RR').$$

In particular, we have

$$E(T, R) = E(T, I)E(0, R), \quad E(T, R) = E(0, R)E(R^{-1}T, I),$$

where $I$ is unit $(4^n-1) \times (4^n-1)$ matrix.

Therefore any linear quantum gate can be decompose on unital gate and translation gate. It allows to consider two types of linear trace-preserving gates:

1) Translation gates $E^{(T)}$ defined by matrices $E(T, I)$.
2) Unital quantum gates $E^{(T=0)}$ with the matrices $E(0, R)$.

Translation gate $E^{(T)}$ is

$$E^{(T)} = \sum_{\mu=0}^{N-1} |\mu\rangle \langle \mu| + \sum_{k=1}^{N-1} T_k |k\rangle \langle 0|,$$

has the matrix

$$E(T, I) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ T_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ T_{N-1} & 0 & \cdots & 1 \end{pmatrix}.$$

One-parameter subgroups $T(4^n-1, \mathbb{R})$ of $n$-ququats translation gates consist of one-parameters $4^n-1$ gates

$$E^{(T,k)}(t) = \sum_{\mu=0}^{N-1} |\mu\rangle \langle \mu| + t|k\rangle \langle 0|,$$

where $t$ is a real parameter and $k = 1, 2, ..., 4^n-1$. Generators of the gates are defined by

$$\hat{H}_k = \left( \frac{d}{dt} E^{(T,k)}(t) \right)_{t=0} = |k\rangle \langle 0|.$$ 

The quantum $n$-ququats unital gate can be represented by

$$\hat{E}^{(T=0)} = |0\rangle \langle 0| + \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} R_{kl} |k\rangle \langle l|,$$

where $N = 4^n$. The gate matrix $E(0, R)$ has the form

$$E(0, R) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & R_{11} & \cdots & R_{1N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & R_{N-11} & \cdots & R_{N-1N-1} \end{pmatrix}.$$

In matrix representation the linear trace-preserving gates with $T = 0$ can be described by group $GL(4^n-1, \mathbb{R})$ which define a set of all linear transformations of $\mathbb{R}^n$ by $[16, 17]$. The group $GL(4^n-1, \mathbb{R})$ has $(4^n-1)^2$ independent one-parameter subgroups $GL_{kl}(4^n-1, \mathbb{R})$ of one-parameter gates $E^{(kl)}(t)$ such that

$$\hat{E}^{(kl)}(t) = |0\rangle \langle 0| + t|k\rangle \langle l|.$$ 

Generators are defined by

$$\hat{H}_{kl} = \left( \frac{d}{dt} \hat{E}^{(kl)}(t) \right)_{t=0} = |k\rangle \langle l|.$$ 

where $k, l = 1, 2, ..., 4^n-1$. The generators $\hat{H}_{kl}$ of the one-parameter subgroup $GL_{kl}(4^n-1, \mathbb{R})$ are represented by $4^n \times 4^n$ matrix $H_{kl}$ with elements

$$(H_{kl})_{\mu\nu} = \delta_{k\mu}\delta_{l\nu}. $$

The set of superoperators $\{\hat{H}_{kl}\}$ is a basis of Lie algebra $gl(4^n-1, \mathbb{R})$ such that

$$[\hat{H}_{ij}, \hat{H}_{kl}] = \delta_{jk}\hat{H}_{il} - \delta_{il}\hat{H}_{jk}. $$

**VI.3 Decomposition for linear quantum gates**

Let us consider the n-ququats linear gate

$$\hat{E} = |0\rangle \langle 0| + \sum_{\mu=1}^{N-1} T_{\mu} |\mu\rangle \langle 0| + \sum_{\mu=1}^{N-1} \sum_{\nu=1}^{N-1} R_{\mu\nu} |\mu\rangle \langle \nu|,$$

where $N = 4^n$. The gate matrix $E(T, R)$ is an element of Lie group $TGL(4^n-1, \mathbb{R})$. The matrix $R$ is an element of Lie group $GL(4^n-1, \mathbb{R})$.

**Theorem 1.** (Singular Value Decomposition for Matrix)

Any real matrix $R$ can be written in the form $R = U_1 D U_2^T$, where $U_1$ and $U_2$ are real orthogonal $(N-1) \times (N-1)$ matrices and $D = diag(\lambda_1, ..., \lambda_{N-1})$ is diagonal $(N-1) \times (N-1)$ matrix such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} \geq 0$.

**Proof.** This theorem is proved in $[9, 12, 13, 85]$.

Let us consider the unital gates $E^{(T=0)}$ defined by $[19]$, where all $T_\mu = 0$.

**Theorem 2.** (Singular Value Decomposition for Gates)

Any unital linear gate $\hat{E}$ defined by $[13]$ with all $T_\mu = 0$ can be represented by

$$\hat{E} = \hat{U}_1 \hat{D} \hat{U}_2,$$

where $\hat{U}_1$ and $\hat{U}_2$ are unit orthogonal quantum gates

$$\hat{U}_i = |0\rangle \langle 0| + \sum_{\mu=1}^{N-1} \sum_{\nu=1}^{N-1} U^{(i)}_{\mu\nu} |\mu\rangle \langle \nu|,$$

for $i = 1, 2$.
\( \hat{D} \) is a unital diagonal quantum gate, such that
\[
\hat{D} = |0 \rangle \langle 0 | + \sum_{\mu=1}^{N-1} \lambda_{\mu} |\mu \rangle \langle \mu |,
\]
where \( \lambda_{\mu} \geq 0 \).

**Proof.** The proof of this theorem can be easily realized in matrix representation by using theorem 1.

In general case, we have the following theorem.

**Theorem 3.** (Singular Valued Decomposition for Gates)
Any linear quantum four-valued logic gate \([19]\) can be represented by
\[
\hat{E} = \hat{E}(T) \hat{U}_1 \hat{D} \hat{U}_2,
\]
where \( \hat{U}_1 \) and \( \hat{U}_2 \) are unital orthogonal quantum gates \([20]\), \( \hat{D} \) is a unital diagonal quantum gate \([21]\), \( \hat{E}(T) \) is a translation quantum gate, such that
\[
\hat{E}(T) = |0 \rangle \langle 0 | + \sum_{\mu=1}^{N-1} |\mu \rangle \langle \mu | + \sum_{\mu=1}^{N-1} T_{\mu} |\mu \rangle \langle 0 |.
\]

**Proof.** The proof of this theorem can be easily realised in matrix representation by using Proposition 4 and Theorem 1.

As a result we have that any trace-preserving gate can be realized by 3 types of gates: (1) unital orthogonal quantum gates \( \hat{U} \) with matrix \( U \in SO(4^n - 1, \mathbb{R}) \); (2) unital diagonal quantum gate \( \hat{D} \) with matrix \( D \in D(4^n - 1, \mathbb{R}) \); (3) nonunital translation gate \( \hat{E}(T) \) with matrix \( E(T) \in T(4^n - 1, \mathbb{R}) \).

**Proposition 5**  If the quantum operation \( \hat{E} \) has the form
\[
\hat{E}(\rho) = \sum_{j=1}^{m} A_j \rho A_j^\dagger,
\]
where \( A \) is a self-adjoint operator \( (A^\dagger = A) \), then quantum four-valued logic gate \( \hat{E} \) is described by symmetric matrix \( \hat{E}_{\mu \nu} = E_{\mu \nu} \).

**Proof.** If \( A_j^\dagger = A_j \), then
\[
\hat{E}_{\mu \nu} = \frac{1}{\sqrt{2^n}} \sum_{j=1}^{m} Tr(\sigma_{\mu} A_j \sigma_{\nu} A_j) = \frac{1}{\sqrt{2^n}} \sum_{j=1}^{m} Tr(\sigma_{\nu} A_j \sigma_{\mu} A_j) = \hat{E}_{\nu \mu}.
\]

This gate is trace-preserving if \( \hat{E}_{\mu 0} = \hat{E}_{0 \mu} = \delta_{\mu 0} \).

The symmetric \( n \)-quatum linear (trace-preserving) quantum gate has the form
\[
\hat{E}(S) = |0 \rangle \langle 0 | + \sum_{\mu=1}^{N-1} \sum_{\nu=1}^{N-1} S_{\mu \nu} |\mu \rangle \langle \nu |,
\]
where \( S_{\mu \nu} = S_{\nu \mu} \) and this gate is unital \((T_k = 0 \text{ for all } k)\).

**Theorem 4.** (Polar Decomposition for matrices)
Any real \((N - 1) \times (N - 1)\) matrix \( R \) can be written in the form \( R = UR U^\dagger \) or \( R = U^\dagger UR \), where \( U \) and \( U^\dagger \) are orthogonal \((N - 1) \times (N - 1)\) matrices and \( S \) and \( S^\dagger \) are symmetric \((N - 1) \times (N - 1)\) matrices such that \( S = \sqrt{R R^T}, S^\dagger = \sqrt{R^T R} \).

**Proof.** This theorem is proved in \([83]\).

**Theorem 5.** (Polar Decomposition for gates)
\( \hat{E} \) is a four-valued logic gate \([22]\) can be written in the form \( \hat{E} = \hat{U} \hat{E}(S) \hat{U}^\dagger \) or \( \hat{E} = \hat{E}(S^\dagger) \hat{U} \), where \( \hat{U} \) and \( \hat{U}^\dagger \) are orthogonal gates \([4]\).

\( \hat{E}(S) \) and \( \hat{E}(S^\dagger) \) are symmetric gates \([22]\).

**Proof.** The proof of this theorem can be easily realized in matrix representation by using Theorem 4.

**VI.4 Unitary two-valued logic gates as orthogonal four-valued logic gates**

Let us rewrite the representation \([3]\) for the mixed state \( |\rho(t)\rangle \) using generalized computational basis in the form
\[
|\rho(t)\rangle = \sum_{\mu=0}^{N-1} |\mu \rangle \rho_{\mu}(t),
\]
where
\[
\rho_{\mu}(t) = (\mu|\rho(t)|) = \frac{1}{\sqrt{2^n}} Tr(\sigma_{\mu} \rho(t)).
\]

Note that \( \rho_0(t) = (0|\rho(t)|) = 1/\sqrt{2^n} Tr \rho(t) = 1/\sqrt{2^n} \) for all cases.

**Proposition 6** In the generalized computational basis any unitary two-valued logic gate \( \hat{U} \) can be considered as a quantum four-valued logic gate:
\[
\hat{U} = \sum_{\mu=0}^{N-1} \sum_{\nu=0}^{N-1} U_{\mu \nu} |\mu \rangle \langle \nu |,
\]
where \( U_{\mu \nu} \) is a real matrix such that
\[
U_{\mu \nu} = \frac{1}{2^n} Tr(\sigma_{\nu} U \sigma_{\mu} U^\dagger).
\]

**Proof.** Let us consider unitary two-valued logic gate \( \hat{U} \). Using equation \([10]\), we get
\[
|\rho(t)\rangle = \hat{U}_1 |\rho(t_0)\rangle.
\]

Then
\[
(\mu|\rho(t)|) = (\mu|\hat{U}_1|\rho(t_0)|) = \sum_{\nu=0}^{N-1} (\mu|\hat{U}_1|\nu) (\nu|\rho(t_0)|).
\]

Finally, we obtain
\[
\rho_{\mu}(t) = \sum_{\nu=0}^{N-1} U_{\mu \nu} (t, t_0) \rho_{\nu}(t_0),
\]
where
\[
U_{\mu \nu} (t, t_0) = (\mu|\hat{U}_1|\nu) = \frac{1}{2^n} (\sigma_{\nu} |\hat{U}_1(\sigma_{\mu})| =
This formula defines a relation between unitary quantum two-valued logic gates $U$ and the real $4^n \times 4^n$ matrix $U$.

**Proposition 7** Any four-valued logic gate associated with unitary 2-valued logic gate by (24) is unit gate, i.e. gate matrix $U$ defined by (24) has $U_{\mu 0} = U_{0\mu} = \delta_{\mu 0}$.

**Proof.**

\[ U_{\mu 0} = \frac{1}{2^n} Tr \left( \sigma_\mu U \sigma_0 U^\dagger \right) = \frac{1}{2^n} Tr \left( \sigma_\mu U U^\dagger \right) = \frac{1}{2^n} Tr \sigma_\nu. \]

Using $Tr \sigma_\mu = \delta_{\mu 0}$ we get $U_{\mu 0} = \delta_{\mu 0}$.

Let us denote the gate $\hat{U}$ associated with unitary two-valued logic gate $U$ by $\hat{U}(U)$.

**Proposition 8** If $U$ is unitary two-valued logic gate, then in the generalized computational basis a quantum four-valued logic gate $\hat{U} = \hat{U}(U)$ associated with $U$ is represented by orthogonal matrix $\hat{U}(U)$:

\[ \hat{U}(U)(\hat{U}(U))^T = (\hat{U}(U))^T \hat{U}(U) = I. \quad (25) \]

**Proof.** Let $\hat{U}(U)$ is defined by

\[ \hat{U}(U)|\rho\rangle = |U \rho U^\dagger\rangle, \quad \hat{U}(U)|\rho\rangle = |U^\dagger \rho U\rangle. \]

If $UU^\dagger = U^\dagger U = I$, then

\[ \hat{U}(U)\hat{U}(U)^\dagger = \hat{U}(U)^\dagger \hat{U}(U) = \hat{I}. \]

In the matrix representation we have

\[ \hat{E}(U)^\dagger \hat{E}(U) = \hat{E}(U)^\dagger \hat{E}(U) = I. \]

Note that

\[ \hat{E}(U)^\dagger = (\hat{E}(U))^T. \]

Finally, we obtain (25).

In matrix representation orthogonal gates can be described by group $SO(4^n - 1, \mathbb{R})$ which is a set of all linear transformations of $\mathbb{R}^n$ such that $\sum_{\mu = 0}^{N-1} \rho_\mu^2 = const$ and $det[E_{\mu \nu}] = 1$. The group $SO(4^n - 1, \mathbb{R})$ has $(4^n - 1)(24^n - 1)$ independent one-parameter subgroups $SO_{kl}(4^n - 1, \mathbb{R})$ of one-parameter orthogonal gates which are

\[ \hat{E}(k,l)(\alpha) = \sum_{\mu \neq k,l} |\mu\rangle|\mu\rangle + \cos \alpha \left( |k\rangle(k) + |l\rangle(l) \right) + \sin \alpha \left( |l\rangle(k) - |k\rangle(l) \right). \]

This gate defines rotation in the flat $(k, l)$. Let us note that the generators of the one-parameter subgroup $SO_{kl}(4^n - 1, \mathbb{R})$ are represented by antisymmetric $(4^n - 1) \times (4^n - 1)$ matrix $X_{kl}$ with elements

\[ (X_{kl})_{\mu \nu} = \delta_{\mu k} \delta_{\nu l} - \delta_{\mu l} \delta_{\nu k}. \]

**Proposition 9** If $\hat{E}^\dagger$ is adjoint superoperator for linear trace-preserving gate $\hat{E}$, then matrices of the gates are connected by transposition $\hat{E}^\dagger = \hat{E}^T$:

\[ (\hat{E}^\dagger)_{\mu \nu} = \hat{E}_{\nu \mu}. \]

**Proof.** Using

\[ \hat{E} = \sum_{j=1}^m \hat{L}_{A_j} \hat{R}_{A_j}, \quad \hat{E}^\dagger = \sum_{j=1}^m \hat{L}_{A_j} \hat{R}_{A_j}, \]

we get

\[ E_{\mu \nu} = \frac{1}{2^n} \sum_{j=1}^m Tr(\sigma_\mu A_j \sigma_\nu A_j^\dagger), \]

\[ (E^\dagger)_{\mu \nu} = \frac{1}{2^n} \sum_{j=1}^m Tr(\sigma_\nu A_j \sigma_\mu A_j^\dagger) = \frac{1}{2^n} \sum_{j=1}^m Tr(\sigma_\nu A_j \sigma_\mu A_j). \]

Obviously, if

\[ \hat{E} = \sum_{\mu = 0}^{N-1} \sum_{\nu = 0}^{N-1} E_{\mu \nu} |\mu\rangle\langle\nu|, \]

then

\[ \hat{E}^\dagger = \sum_{\mu = 0}^{N-1} \sum_{\nu = 0}^{N-1} E_{\nu \mu} |\nu\rangle\langle\mu|. \]

**Proposition 10** If $\hat{E}^\dagger \hat{E} = \hat{E} \hat{E}^\dagger = \hat{I}$, then $\hat{E}$ is orthogonal gate, i.e. $\hat{E}^T \hat{E} = \hat{E} \hat{E}^T = I$.

**Proof.** If $\hat{E}^\dagger \hat{E} = \hat{I}$, then

\[ \sum_{\alpha = 0}^{N-1} (\mu |\hat{E}^\dagger|\alpha\rangle|\alpha\rangle|\nu\rangle) = (\mu |\hat{I}|\nu\rangle, \]

i.e.

\[ \sum_{\alpha = 0}^{N-1} (\hat{E}^\dagger)_{\mu \alpha} E_{\alpha \nu} = \delta_{\mu \nu}. \]

Using proposition 9 we have

\[ \sum_{\alpha = 0}^{N-1} (\hat{E})_{\mu \alpha} E_{\alpha \nu} = \delta_{\mu \nu}, \]

i.e. $\hat{E}^T \hat{E} = I$.

If $E_{\mu \nu}$ is real orthogonal matrix, then

\[ \sum_{\nu = 0}^{N-1} (E_{\mu \nu})^2 = 1. \]
Therefore all elements of orthogonal gate matrix never exceed 1, i.e. $|E_{\mu\nu}| \leq 1$.

Note that n-qubit unitary two-valued logic gate $U$ is an element of Lie group $SU(2^n)$. The dimension of this group is equal to $dim\ SU(2^n) = (2^n)^2 - 1 = 4^n - 1$. The matrix of n-ququant orthogonal linear gate $\hat{U} = \hat{E}(U)$ can be considered as an element of Lie group $SO(4n-1)$. The dimension of this group is equal to $dim\ SO(4^n - 1) = (4^n - 1)(2 \cdot 4^{n-1} - 1)$.

For example, if $n = 1$, then

$$dim\ SU(2) = 3, \quad dim\ SO(1^2 - 1) = 3.$$ 

If $n = 2$, then

$$dim\ SU(2^2) = 15, \quad dim\ SO(2^2 - 1) = 105.$$ 

Therefore not all orthogonal 4-valued logic gates for mixed and pure states are connected with unitary 2-valued logic gates for pure states.

### VI.5 Single ququat orthogonal gates

Let us consider single ququat 4-valued logic gate $\hat{U}$ associated with unitary single qubit 2-valued logic gate $U$.

**Proposition 11** Any single-qubit unitary quantum two-valued logic gate can be realized as the product of single ququat simple rotation gates $\hat{U}^{(1)}(\alpha), \hat{U}^{(2)}(\theta)$ and $\hat{U}^{(1)}(\beta)$ defined by

$$\hat{U}^{(1)}(\alpha) = |0\rangle\langle 0| + |3\rangle\langle 3| + \cos \alpha \left( |1\rangle\langle 1| + |2\rangle\langle 2| \right) + \sin \alpha \left( |2\rangle\langle 1| - |1\rangle\langle 2| \right),$$

$$\hat{U}^{(2)}(\theta) = |0\rangle\langle 0| + |2\rangle\langle 2| + \cos \theta \left( |1\rangle\langle 1| + |3\rangle\langle 3| \right) + \sin \theta \left( |1\rangle\langle 3| - |3\rangle\langle 1| \right),$$

where $\alpha, \theta$ and $\beta$ are Euler angles.

**Proof.** Let us consider a general single qubit unitary gate $U$. Every unitary one-qubit gate $U$ can be represented by 2 $\times$ 2-matrix

$$U(\alpha, \theta, \beta) = e^{-i\alpha \sigma_x/2} e^{-i\beta \sigma_y/2} e^{-i\beta \sigma_z/2} =$$

$$= \begin{pmatrix}
  e^{i(\alpha/2+\beta/2)} \cos \theta/2 & -e^{i(\alpha/2-\beta/2)} \sin \theta/2 \\
  e^{i(\alpha/2-\beta/2)} \sin \theta/2 & e^{i(\alpha/2+\beta/2)} \cos \theta/2
\end{pmatrix},$$

i.e.

$$U(\alpha, \theta, \beta) = U_1(\alpha)U_2(\theta)U_1(\beta),$$

where

$$U_1(\alpha) = \begin{pmatrix}
  e^{-i\alpha/2} & 0 \\
  0 & e^{i\alpha/2}
\end{pmatrix},$$

$$U_2(\theta) = \begin{pmatrix}
  \cos \theta/2 & -\sin \theta/2 \\
  \sin \theta/2 & \cos \theta/2
\end{pmatrix},$$

$$U_1(\beta) = \begin{pmatrix}
  e^{-i\beta/2} & 0 \\
  0 & e^{i\beta/2}
\end{pmatrix},$$

where $\alpha, \theta$ and $\beta$ are Euler angles. The correspondent $4 \times 4$-matrix $U(\alpha, \theta, \beta)$ of four-valued logic gate has the form

$$U(\alpha, \theta, \beta) = U^{(1)}(\alpha)U^{(2)}(\theta)U^{(1)}(\beta),$$

where

$$U^{(1)}(\alpha) = \frac{1}{2} Tr \left( \sigma_\mu U_1(\alpha)\sigma_\mu U_1^\dagger(\alpha) \right),$$

$$U^{(2)}(\theta) = \frac{1}{2} Tr \left( \sigma_\mu U_2(\theta)\sigma_\mu U_2^\dagger(\theta) \right),$$

Finally, we obtain

$$U^{(1)}(\alpha) = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & \cos \alpha & -\sin \alpha & 0 \\
  0 & \sin \alpha & \cos \alpha & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix},$$

$$U^{(2)}(\theta) = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & \cos \theta & 0 & \sin \theta \\
  0 & 0 & 1 & 0 \\
  0 & -\sin \theta & 0 & \cos \theta
\end{pmatrix},$$

where

$$0 \leq \alpha < 2\pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \beta \leq 2\pi.$$ 

Using $U(\alpha, \theta + 2\pi, \beta) = -U(\alpha, \theta, \beta)$, we get that 2-valued logic gates $U(\alpha, \theta, \beta)$ and $U(\alpha, \theta + 2\pi, \beta)$ map into single 4-valued logic gate $U(\alpha, \theta, \beta)$. The back rotation 4-valued logic gate is defined by the matrix

$$U^{-1}(\alpha, \theta, \beta) = U(2\pi - \alpha, \pi - \theta, 2\pi - \beta).$$

The simple rotation gates $\hat{U}^{(1)}(\alpha), \hat{U}^{(2)}(\theta), \hat{U}^{(1)}(\beta)$ are defined by matrices $\hat{U}^{(1)}(\alpha), \hat{U}^{(2)}(\theta)$ and $\hat{U}^{(1)}(\beta)$.

Let us introduce simple reflection gates by

$$\hat{R}^{(1)} = |0\rangle\langle 0| - |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|,$$

$$\hat{R}^{(2)} = |0\rangle\langle 0| + |1\rangle\langle 1| - |2\rangle\langle 2| + |3\rangle\langle 3|,$$

$$\hat{R}^{(3)} = |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| - |3\rangle\langle 3|.$$ 

**Proposition 12** Any single ququat linear gate $\hat{E}$ defined by orthogonal matrix $E : E E^T = I$ can be realized by

- simple rotation gates $\hat{U}^{(1)}$ and $\hat{U}^{(2)}$.
- inversion gate $\hat{I}$ defined by

$$\hat{I} = |0\rangle\langle 0| - |1\rangle\langle 1| - |2\rangle\langle 2| - |3\rangle\langle 3|.$$ 

**Proof.** Using Proposition 11 and

$$\hat{R}^{(3)} = \hat{U}^{(1)}\hat{I}, \quad \hat{R}^{(2)} = \hat{U}^{(2)}\hat{I}, \quad \hat{R}^{(1)} = \hat{U}^{(1)}\hat{U}^{(1)}\hat{I},$$

we get this proposition.
Example 1. In the generalized computational basis the Pauli matrices as two-valued logic gates are the four-valued logic gates with diagonal $4 \times 4$ matrix. The gate $I = \sigma_0$ is

$$\hat{U}^{(\sigma_0)} = \sum_{\mu=0}^{3} |\mu\rangle\langle\mu| = \hat{I},$$

i.e. $U^{(\sigma_0)}_{\mu\nu} = (1/2) Tr(\sigma_{\mu}\sigma_{\nu}) = \delta_{\mu\nu}$.

For the unitary two-valued logic gates are equal to the Pauli matrix $\sigma_k$, where $k \in \{1, 2, 3\}$, we have quantum four-valued logic gates

$$\hat{U}^{(\sigma_k)} = \sum_{\mu,\nu=0}^{3} U^{(\sigma_k)}_{\mu\nu} |\mu\rangle\langle\nu|,$$

with the matrix

$$U^{(\sigma_k)}_{\mu\nu} = 2\delta_{\mu 0}\delta_{\nu 0} + 2\delta_{\mu k}\delta_{\nu k} - \delta_{\mu\nu} .$$

Example 2. In the generalized computational basis the unitary logical gates ("negation") of two-valued logic

$$X = |0 \rangle \langle 0| + |1 \rangle \langle 1| = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

is represented by quantum four-valued logic gate

$$\hat{U}^{(X)} = |0\rangle\langle 0| + |1\rangle\langle 1| - |2\rangle\langle 2| - |3\rangle\langle 3|,$$

i.e. $4 \times 4$ matrix is

$$\hat{U}^{(X)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Example 3. The Hadamard two-valued logic gate

$$H = \frac{1}{\sqrt{2}} (\sigma_1 + \sigma_3)$$

can be represented as a four-valued logic gate by

$$\hat{E}^{(H)} = |0\rangle\langle 0| - |2\rangle\langle 2| + |3\rangle\langle 1| + |1\rangle\langle 3|,$$

with

$$E^{(H)}_{\mu\nu} = \delta_{\mu 0}\delta_{\nu 0} - \delta_{\mu 2}\delta_{\nu 2} + \delta_{\mu 3}\delta_{\nu 1} + \delta_{\mu 1}\delta_{\nu 3}.$$
The linear trace-decreasing superoperator for von Neumann measurement projector $|0><0|$ on pure state $|0>$ is
\[ \hat{E}^{(0)} = \frac{1}{2}(|0><0| + |3><3| + |0><3| + |3><0|). \]

**Example.** For the projection operator
\[ P_1 = |1><1| = \frac{1}{2}(\sigma_0 - \sigma_3) \]
Using formula (28) we derive
\[ E^{(1)}_\nu = \frac{1}{2}\left(\delta_{\nu 0}\delta_{\sigma 0} + \delta_{\nu 3}\delta_{\sigma 3} - \delta_{\nu 3}\delta_{\sigma 0} - \delta_{\nu 0}\delta_{\sigma 3}\right). \]
The linear superoperator $\hat{E}^{(1)}$ for von Neumann measurement projector onto pure state $|1>$ is
\[ \hat{E}^{(1)} = \frac{1}{2}\left(|0><0| + |3><3| - |0><3| - |3><0|\right), \]
i.e.
\[ E^{(1)} = \begin{pmatrix} 1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/2 & 0 & 0 & 1/2 \end{pmatrix}. \]

The superoperators $\hat{E}^{(0)}$ and $\hat{E}^{(1)}$ are not trace-preserving. The probabilities that processes represented by superoperators $\hat{E}^{(k)}$ occurs are
\[ p(0) = \frac{1}{\sqrt{2}}(\rho_0 + \rho_3), \quad p(1) = \frac{1}{\sqrt{2}}(\rho_0 + \rho_3). \]

**VI.7 Reversible quantum 4-valued logic gate**

In the paper [97], Mabuchi and Zoller have shown how a measurement on a quantum system can be reversed under appropriate conditions. In the papers [98, 99, 100, 12] was considered necessary and sufficient conditions for general quantum operations to be reversible.

Let us consider quantum operation $\mathcal{E}$ on a subspace $\mathcal{M}$ of the total state space.

**Theorem 6.** A quantum operation $\mathcal{E}$
\[ \mathcal{E}(\rho) = \sum_{j=1}^{m} A_j \rho A_j^\dagger \]
is reversible on subspace $\mathcal{M}$ if and only if there exists a positive matrix $M$ such that
\[ P_M A_j^\dagger A_j P_M = M_{jk} P_M. \] (29)
where $P_M$ is a projector onto subspace $\mathcal{M}$. The trace of $M$
\[ \sum_{j=1}^{m} M_{jj} = \mu^2 \]
is the constant value of $\text{Tr}(\mathcal{E}(\rho))$ on $\mathcal{M}$.

**Proof.** This result was proved in [12, 98, 99].

Let $\hat{E}^{(\mathcal{M})}$ is projection superoperator defined by
\[ \hat{E}^{\mathcal{M}}(\rho) = P_M \rho P_M. \]

**Note that**
\[ \hat{E}^{(\mathcal{M})} \hat{E}^{(\mathcal{M})} = \hat{E}^{(\mathcal{M})}, \quad (\hat{E}^{(\mathcal{M})})^\dagger = \hat{E}^{(\mathcal{M})}. \]

Let $\hat{E}_{\mathcal{M}}$ be the restriction of $\hat{E}$ to the subspace $\mathcal{M}$
\[ \hat{E}_{\mathcal{M}}(\rho) = \sum_{j=1}^{m} A_j P_M \rho P_M A_j^\dagger. \] (30)

Notice that $\hat{E}_{\mathcal{M}}(\rho) = \hat{E}(\rho)$ if $\rho$ lies wholly in $\mathcal{M}$. Note, that the adjoint superoperator for trace-decreasing quantum operation is generally not a quantum operation, since it can be trace-increasing, but it is always a completely positive map.

Equation (29) is equivalent to the requirement that superoperator $\hat{E}_{\mathcal{M}}^\dagger(\rho)$ be a positive multiple of identity operation on $\mathcal{M}$. This requirement can be formulated as theorem.

**Theorem 7.**
A necessary and sufficient condition for reversibility of linear superoperator $\hat{E}$ on the subspace $\mathcal{M}$ is
\[ \hat{E}^{\mathcal{M}} \hat{E} \hat{E}^{\mathcal{M}} = \gamma \hat{E}^{\mathcal{M}}. \]

**Proof.** For the proofs we refer to [99].

**VII Classical four-valued logic classical gates**

Let us consider some elements of classical four-valued logic.

**VII.1 Elementary classical gates**

A classical four-valued logic gate is called a function $g(x_1, ..., x_n)$ if following conditions hold:
- all $x_i \in \{0, 1, 2, 3\}$, where $i = 1, ..., n$.
- $g(x_1, ..., x_n) \in \{0, 1, 2, 3\}$.

It is known that the number of all classical logic gates with n-arguments $x_1, ..., x_n$ is equal to $4^n$. The number of classical logic gates $g(x)$ with single argument is equal to $4^1 = 256$.

| Single argument classical gates |
|-------------------------------|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

| Single argument classical gates |
|-------------------------------|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
The number of classical logic gates $g(x_1, x_2)$ with two-arguments is equal to 

$$4^{12} \times 4^{16} = 42949677296.$$ 

Let us write some of these gates.

| Two-arguments classical gates $(x_1, x_2)$ | $\land$ | $\lor$ | $V_4$ | $\sim V_4$ |
|-----------------|-------|-------|-------|--------|
| $(0;0)$ | 0 | 0 | 1 | 2 |
| $(0;1)$ | 0 | 1 | 0 | 1 |
| $(0;2)$ | 0 | 2 | 0 | 3 |
| $(0;3)$ | 0 | 3 | 0 | 3 |
| $(1;0)$ | 0 | 1 | 2 | 1 |
| $(1;1)$ | 1 | 1 | 2 | 1 |
| $(1;2)$ | 1 | 2 | 0 | 3 |
| $(1;3)$ | 1 | 3 | 0 | 3 |
| $(2;0)$ | 0 | 2 | 0 | 3 |
| $(2;1)$ | 1 | 2 | 0 | 3 |
| $(2;2)$ | 2 | 2 | 0 | 3 |
| $(2;3)$ | 2 | 3 | 0 | 3 |
| $(3;0)$ | 0 | 3 | 0 | 3 |
| $(3;1)$ | 1 | 3 | 0 | 3 |
| $(3;2)$ | 2 | 3 | 0 | 3 |
| $(3;3)$ | 3 | 3 | 0 | 3 |

Let us define some elementary classical 4-valued logic gates by formulas.

- **Luckasiewicz negation:** $\sim x = 3 - x$.
- **Cyclic shift:** $\varpi = x + 1 (\text{mod} 4)$.
- **Functions $I_i(x)$**, where $i = 0, \ldots, 3$, such that $I_i(x) = 3$ if $x = i$ and $I_i(x) = 0$ if $x \neq i$.
- **Generalized conjunction:** $x_1 \land x_2 = \min(x_1, x_2)$.
- **Generalized disjunction:** $x_1 \lor x_2 = \max(x_1, x_2)$.
- **Generalized Sheffer function:**

$$V_4(x_1, x_2) = \max(x_1, x_2) + 1 (\text{mod} 4).$$

Commutative law, associative law and distributive law for the generalized conjunction and disjunction are satisfied:

- **Commutative law**

$$x_1 \land x_2 = x_2 \land x_1, \quad x_1 \lor x_2 = x_2 \lor x_1$$

- **Associative law**

$$(x_1 \lor x_2) \lor x_3 = x_1 \lor (x_2 \lor x_3).$$

$$(x_1 \land x_2) \land x_3 = x_1 \land (x_2 \land x_3).$$

- **Distributive law**

$$x_1 \lor (x_2 \land x_3) = (x_1 \lor x_2) \land (x_1 \lor x_3).$$

$$x_1 \land (x_2 \lor x_3) = (x_1 \land x_2) \lor (x_1 \land x_3).$$

Note that the Luckasiewicz negation is satisfied

$$\sim (\sim x) = x, \quad \sim (x_1 \land x_2) = (\sim x_1) \lor (\sim x_2).$$

The shift $\varpi$ for $x$ is not satisfied usual negation rules:

$$\varpi \neq x, \quad \varpi_1 \land \varpi_2 \neq \varpi_1 \lor \varpi_2.$$ 

The analog of disjunction normal form of the n-arguments 4-valued logic gate is

$$g(x_1, \ldots, x_n) = \bigvee_{(k_1, \ldots, k_n)} I_{k_1}(x_1) \land \ldots \land I_{k_n}(x_n) \land g(k_1, \ldots, k_n).$$

**VIII.2 Universal classical gates**

Let us consider universal sets of universal classical gates of four-valued logic.

**Theorem 8.**

The set $\{0, 1, 2, 3, I_0, I_1, I_2, I_3, x_1 \land x_2, x_1 \lor x_2\}$ is universal.

The set $\{\varpi, x_1 \lor x_2\}$ is universal.

**Proof.** This theorem is proved in [75].

**Theorem 9.**

All logic single argument 4-valued gates $g(x)$ can be generated by functions:

- $g_1(x) = x - 1 (\text{mod} 4)$.
- $g_2(x): g_2(0) = 0, g_2(1) = 1, g_2(2) = 3, g_2(3) = 2$.
- $g_3(x) = 1$ if $x = 0$ and $g_3(x) = x$ if $x \neq 0$.

**Proof.** This theorem was proved by Picard in [78].

**VIII Quantum four-valued logic gates for classical gates**

**VIII.1 Quantum gates for single argument classical gates**

Let us consider linear trace-preserving quantum gates for classical gates $\sim, \varpi, I_0, I_1, I_2, I_3, 0, 1, 2, 3, g_1, g_2, g_3, \hat{\bullet}, \Box$.

**Proposition 14** Any single argument classical gate $g(\nu)$ can be realized as linear trace-preserving quantum four-valued logic gate by

$$\hat{\xi}(g) = |0\rangle\langle 0| + \sum_{k=1}^{3} |g(k))\langle k| + (1 - \delta_{0g(0)}) (|g(0))\langle 0| - \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} (1 - \delta_{\mu g(\nu)}) |\mu\rangle\langle \nu|).$$

**Proof.** The proof is by direct calculation in

$$\hat{\xi}(g)[\alpha] = |g(\alpha)|,$$

where

$$\hat{\xi}(g)[\alpha] = \frac{1}{\sqrt{2}} \left( \hat{\xi}(g)[0] + \hat{\xi}(g)[\alpha] \right).$$
Examples.
1. The generalized negation gate is
\[
\hat{\mathcal{E}}(\Lambda_N) = |0)(0| + |1)(2| + |2)(1| + |3)(0| - |3)(3|. \\
\mathcal{E}(\Lambda_N) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & -1
\end{pmatrix}.
\]
2. The four-valued logic gate \( I_0 \) can be realized by
\[
\hat{\mathcal{E}}(I_0) = |0)(0| + |3)(0| - \sum_{k=1}^{3} |3)(k|. \\
\mathcal{E}(I_0) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & -1 & -1 & -1
\end{pmatrix}.
\]
3. The gates \( I_k(x) \), where \( k = 1, 2, 3 \) is
\[
\hat{\mathcal{E}}(I_k) = |0)(0| + |3)(k|. \\
\mathcal{E}(I_k) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]
4. The gate \( \overline{x} \) can be realized by
\[
\hat{\mathcal{E}}(\overline{x}) = |0)(0| + |1)(0| + |2)(1| + |3)(2| - \sum_{k=1}^{3} |1)(k|. \\
\mathcal{E}(\overline{x}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & -1 & -1 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]
5. The constant gates \( 0 \) and \( k = 1, 2, 3 \) can be realized by
\[
\hat{\mathcal{E}}(0) = |0)(0|, \quad \hat{\mathcal{E}}(k) = |0)(0| + |k)(0|. \\
\mathcal{E}(0) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
6. The gate \( g_1(x) \) can be realized by
\[
\hat{\mathcal{E}}(g_1) = |0)(0| + |1)(2| + |2)(3| + |3)(0| - \sum_{k=1}^{3} |3)(k|. \\
\mathcal{E}(g_1) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & -1 & -1 & -1
\end{pmatrix}.
\]
7. The gate \( g_2(x) \) is
\[
\hat{\mathcal{E}}(g_2) = |0)(0| + |1)(1| + |3)(2| + |2)(3|. \\
\mathcal{E}(g_2) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]
8. The gate \( g_3(x) \) can be realized by
\[
\hat{\mathcal{E}}(g_3) = |0)(0| + |2)(2| + |3)(3) + |1)(0| - |1)(2| - |1)(3|. \\
\mathcal{E}(g_3) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & -1 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
9. The gate \( \triangleleft x \) is
\[
\hat{\mathcal{E}}(\triangleleft) = |0)(0| + \sum_{k=1}^{3} |3)(k|. \\
\mathcal{E}(\triangleleft) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
10. The gate \( \square x = \sim \triangleleft x \) is
\[
\hat{\mathcal{E}}(\sim \triangleleft) = |0)(0| + |3)(3|. \\
\mathcal{E}(\sim \triangleleft) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
Note that quantum gates \( \hat{\mathcal{E}}(\Lambda_N), \hat{\mathcal{E}}(I_0), \hat{\mathcal{E}}(k), \hat{\mathcal{E}}(g_1) \) are not unital gates.

VIII.2 Quantum gates for two-arguments classical gates

Let us consider quantum gates for two-arguments classical gates.

1. The generalized conjunction \( x_1 \land x_2 = min(x_1, x_2) \) and generalized disjunction \( x_1 \lor x_2 = max(x_1, x_2) \) can be realized by two-ququat gate with \( T = 0 \):

   \[
\begin{array}{c}
\text{x}_1 \\
\text{x}_2
\end{array}
\begin{array}{c}
\text{CD} \\
\text{x}_1 \lor \text{x}_2 \\
\text{x}_1 \land \text{x}_2
\end{array}
\]

   Let us write the quantum gate which realizes the CD gate in the generalized computational basis by
\[
\hat{E} = \sum_{\mu} \sum_{\nu} |\mu\nu)(\mu\nu| + \sum_{k=1}^{3} (|00) - |k0\rangle)(k0|).$

by linear unital quantum gates (It is interesting to consider a representation for classical gates
VIII.3 Unital quantum gates for single argument classical gates

It is interesting to consider a representation for classical gates by linear unital quantum gates ($\hat{E}[0] = |0\rangle$). There is a restriction for representation single argument classical (4-valued logic) gate by linear quantum four-valued logic gates with $T = 0$ (all $T_k = 0$). Any unital n-qupit quantum gate has the form

$$\hat{E}^{(T=0)} = |0\rangle\langle 0| + \sum_{\mu=1}^{N-1} \sum_{\nu=1}^{N-1} R_{\mu\nu}|\mu\rangle\langle \nu|,$$

i.e. $T_k = 0$ for all $k$ and the gate matrix is

$$\hat{E}^{(T=0)} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & R_{11} & R_{12} & \cdots & R_{1N-1} \\ 0 & R_{21} & R_{22} & \cdots & R_{2N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & R_{N-11} & R_{N-12} & \cdots & R_{N-1N-1} \end{pmatrix}.$$

**Proposition 15** If the single argument classical (4-valued logic) gate $g(x)$ such that $g(0) = k$, where $k \in \{1, 2, 3\}$ and exists $m \in \{1, 2, 3\}: g(m) = l$, where $l \neq k$, then there is no a representation of this gate by some unital quantum four-valued logic gates $\hat{E}$.

**Proof.** If $\hat{E}[0] = |k\rangle$ and $\hat{E}[m] = |l\rangle$, where $k \in \{1, 2, 3\}$, $l \neq k$, then

$$\hat{E}_{k0} = (k|\hat{E}|0) = (k|\hat{E}|0) = (k|k| =$$

$$= \frac{1}{\sqrt{2^n}}((k|0) + (k|k)) = \frac{1}{\sqrt{2^n}} \neq 0,$$

i.e. $T_k \neq 0$ in the matrix $E_{\mu\nu}$.

From this proposition we see that single argument classical gate $g(x)$ can be realized by single qupit quantum gate with $T = 0$ if and only if

1. $g(0) = 0$, or
2. $g(\mu) = const$, i.e. $g(0) = g(1) = g(2) = g(3) = k$ and $k \in \{1, 2, 3\}$.

For example, classical gates $\sim x, I_0, \overline{x}, g_1$ and $g_3$ can not be realized by single qupit unital quantum gates.

Single argument classical logic gates such that $g(0) \neq 0$ can not be realized by single qupit quantum gates $\hat{E}$ with $T = 0$. This classical gates can be realized by two-qubits unital quantum gates. Let us consider Luckasiewicz negation $\sim x = 3 - x$. If $x_2 \neq 0$ and $x_1, x_2 \in \{0, 1, 2, 3\}$ then we can define quantum Luckasiewicz negation gate by

This gate realizes Luckasiewicz negation for $x_1: LN_2|x_1 \otimes x_2 = |(\sim x_1) \otimes x_2$ iff $x_2 = 0$. If $x_2 = 0$, then the two-qupit gate must be following

Let us write the unital quantum four-valued logic gate which realizes Luckasiewicz negation in generalized computational basis by

$$\hat{E}^{(LN_2)} = |00\rangle(00) + \sum_{k=1,2,3} (|k3)(k0) + |k2)(k1| + $$

$$+ |k1)(k2) + |k0)(k3)|.$$
VIII.4 Unital quantum gates for two-arguments classical gates

By analogy with Proposition 15 we can prove the following.

Proposition 16 The classical n-arguments 4-valued logic gate $g(x_1, \ldots, x_n)$ can be realized as n-ququat unital quantum gate if and only if $g(0, \ldots, 0) = 0$, or $g(x_1, \ldots, x_n) = \text{const}$.

The two arguments nonconstant classical gate $g(x_1, x_2)$ can be realized by two-ququat linear unital quantum gate $\hat{E}$ if $g(0, 0) = 0$.

Two arguments nonconstant classical gates such that $g(0, 0) = 0$ can not be realized by two-ququat quantum gates $\hat{E}$ with $T = 0$. These classical gates can be realized by three-ququats unital quantum gates. Let us consider Sheffer function $V_4(x_1, x_2) = \max(x_1, x_2) + 1 \mod 4$. If $x_3 \neq 0$ and $x_1, x_2, x_3 \in \{0, 1, 2, 3\}$, then we can define unital quantum Sheffer gate by

Let us consider linear completely positive trace-decreasing superoperators on single ququat density matrices that consists of all one-ququat gates and the two-ququat gates exclusive-or (XOR) gate is universal in the sense that all unitary operations on arbitrary many qubits can be expressed as compositions of these gates. Recently in the paper [59] was considered universality for n-qudits quantum gates.

The same is not true for the general quantum operations (superoperators) corresponding to the dynamics of open quantum systems. In the paper [24] single ququat open quantum system with Markovian dynamics was considered and the resources needed for universality of general quantum operations was studied. An analysis of completely-positive trace-preserving superoperators on single ququat density matrices was realized in papers [84, 87, 88].

Let us study universality for general quantum four-valued logic gates. A set of quantum four-valued logic gates is universal if all quantum gates on arbitrary many ququats can be expressed as compositions of these gates. A set of quantum four-valued logic gates is universal if all unitary two-valued logic gates and general quantum operations can be represented by compositions of these gates. Single ququat gates cannot map two initially un-entangled ququats into an entangled state. Therefore the single ququat gates or set of single ququats gates are not universal gates. Quantum gates which are realization of classical gates cannot be universal by definition, since these gates evolve generalized computational states to generalized computational states and never to the superposition of them.

Let us consider linear completely positive trace-decreasing superoperator $\hat{E}$. This superoperator can be represented in the form

$$\hat{E} = \sum_{j=1}^{m} \hat{L}_A \hat{R}_A^j$$

where $\hat{L}_A$ and $\hat{R}_A$ are left and right multiplication superoperators on $\mathbb{F}^{(4)}$ defined by $\hat{L}_A|B\rangle = |AB\rangle$, $\hat{R}_A|B\rangle = |BA\rangle$.

The n-ququats linear gate $\hat{E}$ is completely positive trace-preserving superoperator such that the gate matrix is an element of Lie group $TGL(4^n - 1, \mathbb{R})$. In general case, the n-ququats nonlinear gate $\hat{N}$ is defined by completely positive trace-decreasing linear superoperator $\hat{E}$ such that the gate matrix is an element of Lie group $GL(4^n, \mathbb{R})$. The condition of completely positivity leads to difficult inequalities for gate matrix elements [89, 86, 87, 88]. In order to satisfy condition of completely positivity we use the representation [81]. To find the universal set of completely positive (linear or nonlinear) gates $\hat{E}$ we consider the universal set of the superoperators $\hat{L}_A$, and $\hat{R}_A^j$. The matrices of these superoperators are connected by complex conjugation. Obviously, the universal set of superoperators $\hat{L}_A$ defines a universal set of completely positive superoperators $\hat{E}$ of the quantum gates. The trace-preserving condition for linear superoperator [81] is equivalent to the requirement for gate matrix $\hat{E} \in TGL(4^n - 1, \mathbb{R})$, i.e. $\hat{E}_{00} = \delta_{00}$. The trace-decreasing condition can be satisfied by inequality of the following proposition.
Proposition 17 If the matrix elements $E_{\mu\nu}$ of a superoperator $\hat{E}$ is satisfied the inequality

$$\sum_{\mu=0}^{N-1} (E_{0\mu})^2 \leq 1,$$

then $\hat{E}$ is a trace-decreasing superoperator.

Proof. Using Schwarz inequality

$$\left(\sum_{\mu=0}^{N-1} E_{0\mu} \rho_\mu\right)^2 \leq \sum_{\mu=0}^{N-1} (E_{0\mu})^2 \sum_{\nu=0}^{N-1} (\rho_\nu)^2,$$

and the property of density matrix

$$Tr \rho^2 = \langle \rho | \rho \rangle = \sum_{\nu=0}^{N-1} (\rho_\nu)^2 \leq 1,$$

we have

$$|Tr \hat{E}(\rho)|^2 = |\langle 0 | \hat{E} | \rho \rangle |^2 = \left(\sum_{\mu=0}^{N-1} E_{0\mu} \rho_\mu\right)^2 \leq \sum_{\mu=0}^{N-1} (E_{0\mu})^2.$$

Using (23), we get $|Tr \hat{E}(\rho)| \leq 1$. Since $\hat{E}$ is completely positive (or positive) superoperator ($\hat{E}(\rho) \geq 0$), it follows that

$$0 \leq Tr \hat{E}(\rho) \leq 1,$$

i.e. $\hat{E}$ is trace-decreasing superoperator.

Let the superoperators $\hat{L}_A$ and $\hat{R}_A$ be called pseudo-gates. These superoperators can be represented by

$$\hat{L}_A = \sum_{\mu=0}^{N-1} \sum_{\nu=0}^{N-1} L^{(A)}_{\mu\nu} |\mu\rangle |\nu\rangle, \quad \hat{R}_A = \sum_{\mu=0}^{N-1} \sum_{\nu=0}^{N-1} R^{(A)}_{\mu\nu} |\mu\rangle |\nu\rangle.$$

Proposition 18 The matrix of the completely positive superoperator (37) can be represented by

$$E_{\mu\nu} = \sum_{j=1}^{m} \sum_{\alpha=0}^{N-1} L^{(jA)}_{\mu\alpha} R^{(jA\dagger)}_{\alpha\nu}.$$

Proof. Let us write the matrix $E_{\mu\nu}$ by matrices of superoperators $\hat{L}_A$ and $\hat{R}_A$.

$$E_{\mu\nu} = \langle \mu | \hat{E} | \nu \rangle = \sum_{j=1}^{m} \langle \mu | \hat{L}_A \hat{R}_A^{\dagger} | \nu \rangle = \sum_{j=1}^{m} \sum_{\alpha=0}^{N-1} \langle \mu | \hat{L}_A | \alpha \rangle \langle \alpha | \hat{R}_A^{\dagger} | \nu \rangle = \sum_{j=1}^{m} \sum_{\alpha=0}^{N-1} L^{(jA)}_{\mu\alpha} R^{(jA\dagger)}_{\alpha\nu}.$$
Proof. Using complex conjugation of the matrix elements \( (L_{\mu\nu}^{(jA)})^* = \frac{1}{2\pi} (\sigma_\mu \sigma_\nu |A|^*) = \frac{1}{2\pi} (A|\sigma_\mu \sigma_\nu) = R_{\mu\nu}^{(jA)} \),

we can write the gate matrix \( (L_{\mu\nu}^{(jA)})^* \) in the form

\[
\mathcal{E}_{\mu\nu} = \sum_{j=1}^{m} \sum_{\alpha=0}^{N-1} L_{\mu\nu}^{(jA)} (L_{\alpha\alpha}^{(jA)})^*.
\]

Proposition 20 The matrices \( L_{\mu\nu}^{(jA)} \) and \( R_{\mu\nu}^{(jA)} \) of the n-ququats quantum gate \( \hat{E} \) are the elements of Lie group \( GL(4^n, \mathbb{C}) \).

Proof. The proof is trivial.

A two-ququats gate \( \hat{E} \) is called primitive \(^{32} \) if \( \hat{E} \) maps tensor product of single ququats to tensor product of single ququats, i.e., if \( |\rho_1 \rangle \) and \( |\rho_2 \rangle \) are ququats, then we can find ququats \( |\rho_1' \rangle \) and \( |\rho_2' \rangle \) such that

\[
\hat{E} |\rho_1 \otimes \rho_2 \rangle = |\rho_1' \otimes \rho_2' \rangle.
\]

The superoperator \( \hat{E} \) is called imprimitive if \( \hat{E} \) is not primitive.

It can be shown that almost every pseudo-gate that operates on two or more ququats is universal pseudo-gate.

Proposition 21 The set of all single ququat pseudo-gates and any imprimitive two-ququats pseudo-gate are universal set of pseudo-gates.

Proof. This proposition can be proved by analogy with \(^{32} \). Let us consider some points of the proof. Expressed in group theory language, all n-ququats pseudo-gates are elements of the Lie group \( GL(4^n, \mathbb{C}) \). Two-ququats pseudogates \( \hat{L} \) are elements of Lie group \( GL(16, \mathbb{C}) \). The question of universality is the same as the question of what set of superoperators \( \hat{L} \) sufficient to generate \( GL(16, \mathbb{C}) \). The group \( GL(16, \mathbb{C}) \) has \((2^2)^{16} = 256\) independent one-parameter subgroups \( GL_{\mu\nu}(16, \mathbb{C}) \) of one-parameter pseudo-gates \( \hat{L}_{\mu\nu}(t) \) such that \( \hat{L}_{\mu\nu}(t) = \hat{L}_{\mu\nu}^0 \exp it \). Infinitesimal generators of Lie group \( GL(4^n, \mathbb{C}) \) are defined by

\[
\hat{H}_{\mu\nu} = \left( \frac{d}{dt} \hat{L}_{\mu\nu}(t) \right)_{t=0},
\]

where \( \mu, \nu = 0, 1, \ldots, 4^n - 1 \). The generators \( \hat{H}_{\mu\nu} \) of the one-parameter subgroup \( GL_{\mu\nu}(4^n, \mathbb{R}) \) are superoperators of the form \( \hat{H}_{\mu\nu} = |\mu\rangle \langle \nu| \) on \( \mathcal{H}_{(n)} \) which can be represented by \( 4^n \times 4^n \) matrix \( H_{\mu\nu} \) with elements

\[
(H_{\mu\nu})_{\alpha\beta} = \delta_{\mu\alpha} \delta_{\nu\beta}.
\]

The set of superoperators \( \hat{H}_{\mu\nu} \) is a basis (Weyl basis \(^{34} \)) of Lie algebra \( gl(16, \mathbb{R}) \) such that

\[
[\hat{H}_{\mu\nu}, \hat{H}_{\alpha\beta}] = \delta_{\nu\alpha} \hat{H}_{\mu\beta} - \delta_{\mu\beta} \hat{H}_{\nu\alpha},
\]

where \( \mu, \nu, \alpha, \beta = 0, 1, \ldots, 15 \). Any element \( \hat{H} \) of the algebra \( gl(16, \mathbb{C}) \) can be represented by

\[
\hat{H} = \sum_{\mu=0}^{15} \sum_{\nu=0}^{15} h_{\mu\nu} \hat{H}_{\mu\nu},
\]

where \( h_{\mu\nu} \) are complex coefficients.

As a basis of Lie algebra \( gl(16, \mathbb{C}) \) we can use 256 linearly independent self-adjoint superoperators

\[
H_{\alpha\alpha} = |\alpha\rangle \langle \alpha|, \quad H_{\alpha\beta} = |\alpha\rangle \langle \beta| + |\beta\rangle \langle \alpha|,
\]

\[
H_{\alpha\beta} = -i \left( |\alpha\rangle \langle \beta| - |\beta\rangle \langle \alpha| \right),
\]

where \( 0 \leq \alpha \leq \beta \leq 15 \). The matrices of these generators is Hermitian \( 16 \times 16 \) matrices. The matrix elements of 256 Hermitian \( 16 \times 16 \) matrices \( H_{\alpha\alpha}, H_{\alpha\beta} \) and \( H_{\alpha\beta} \) are defined by

\[
(H_{\alpha\alpha})_{\mu\nu} = \delta_{\mu\alpha} \delta_{\nu\alpha}, \quad (H_{\alpha\beta})_{\mu\nu} = \delta_{\mu\alpha} \delta_{\nu\beta} + \delta_{\mu\beta} \delta_{\nu\alpha},
\]

\[
(H_{\alpha\beta})_{\mu\nu} = -i(\delta_{\mu\alpha} \delta_{\nu\beta} - \delta_{\mu\beta} \delta_{\nu\alpha}).
\]

For any Hermitian generators \( \hat{H} \) exists one-parameter pseudo-gates \( \hat{L}(t) \) which can be represented in the form \( \hat{L}(t) = \exp i \hat{H} t \) such that \( \hat{L}(t) \hat{L}(t) = I \).

Let us write main operations which allow to derive new pseudo-gates \( \hat{L} \) from a set of pseudo-gates.

1) We introduce general SWAP (twist) pseudo-gate \( \hat{T}^{(SW)} \). A new pseudo-gate \( \hat{L}^{(SW)} \) defined by \( \hat{L}_{\mu\nu}^{(SW)} = \hat{T}^{(SW)} \hat{L} \hat{T}^{(SW)} \) is obtained directly from \( \hat{L} \) by exchanging two ququats.

2) Any superoperator \( \hat{L} \) on \( \mathcal{H}_{(2)} \) generated by the commutator \( i[H_{\mu\nu}, H_{\alpha\beta}] \) can be obtained from \( \hat{L}_{\mu\nu}(t) = \exp it \hat{H}_{\mu\nu} \) and \( \hat{L}_{\alpha\beta}(t) = \exp it \hat{H}_{\alpha\beta} \) because

\[
\exp t [\hat{H}_{\mu\nu}, \hat{H}_{\alpha\beta}] = \lim_{n \to \infty} \left( \hat{L}_{\alpha\beta}(-n) \hat{L}_{\mu\nu}(t_n) \hat{L}_{\alpha\beta}(t_n) \hat{L}_{\mu\nu}(-t_n) \right)^n,
\]

where \( t_n = 1/\sqrt{n} \). Thus we can use the commutator \( i[H_{\mu\nu}, H_{\alpha\beta}] \) to generate pseudo-gates.

3) Every transformation \( \hat{L}(a, b) = \exp \hat{H}(a, b) \) of \( GL(16, \mathbb{C}) \) generated by superoperator \( \hat{H}(a, b) = a \hat{H}_{\mu\nu} + b \hat{H}_{\alpha\beta} \), where \( a \) and \( b \) is complex, can be obtained from \( \hat{L}_{\mu\nu}(t) = \exp it \hat{H}_{\mu\nu} \) and \( \hat{L}_{\alpha\beta}(t) = \exp it \hat{H}_{\alpha\beta} \) by

\[
\exp i \hat{H}(a, b) = \lim_{n \to \infty} \left( \hat{L}_{\mu\nu}(\frac{a}{n}) \hat{L}_{\alpha\beta}(\frac{b}{n}) \right)^n.
\]

For other details of the proof, see \(^{32} \). X Quantum four-valued logic gates of order (n,m)

In general case, a quantum gate is defined to be the most general quantum operation \(^{3} \):

Definition Quantum gate \( \hat{G} \) of order (n,m) is a positive (completely positive) linear (nonlinear) trace preserving map from density matrix operator \( |\rho\rangle \) on n-ququats to density matrix operator \( |\rho'\rangle \) on m-ququats.
where
\[ \mu = \mu_1 4^{m-1} + ... + \mu_m 4^{m-14} + \mu_m , \]
\[ \nu = \nu_1 4^{n-1} + ... + \nu_n 4^{n-14} + \nu_n . \]

For the gate matrices we use \( N = 4^n - 1 \) and \( M = 4^m - 1 \).

The matrix \( \hat{G}_{\mu,\nu}^{(n,m)} \) of linear gate is a real \( 4^n \times 4^m \)-matrix with
\[
\hat{G}_{\mu,\nu}^{(n,m)} = \delta_{\mu,\nu}.
\]

In general case, linear gates \( \hat{G}_{\mu,\nu}^{(n,m)} \) of order \( (n, m) \) have \( \hat{G}_{\mu,0}^{(n,m)} \neq 0 \), i.e. this gate is not unital. i.e.
\[
\hat{G}^{(n,m)} = \begin{pmatrix}
1 & 0 & 0 & ... & 0 \\
T_1 & R_{11} & R_{12} & ... & R_{1N} \\
T_2 & R_{21} & R_{22} & ... & R_{2N} \\
... & ... & ... & ... & ... \\
T_M & R_{M1} & R_{M2} & ... & R_{MN}
\end{pmatrix}.
\]

**Theorem 10.** (Singular Valued Decomposition for Matrix) Any real \( N \times M \) matrix \( G \) can be written in the form
\[
G = U_M D_{NM} U_N^T,
\]
where
- \( U_M \) is an orthogonal \( M \times M \) matrix.
- \( U_N \) is an orthogonal \( N \times N \) matrix.
- \( D_{NM} \) is diagonal \( N \times M \) matrix such that
\[
D_{NM} = \text{diag}(\lambda_1, ..., \lambda_p), \quad p = \min\{N, M\},
\]
where \( \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p \geq 0 \).

**Proof.** This theorem is proved in [1, 2, 3, 4, 5].

Let us consider the unital gates with \( T = 0 \) defined by
\[
\hat{G}^{(n,m)} = \frac{1}{\sqrt{2^{2nM}(0)(0) + \sum_{\mu=1}^{N} \sum_{\nu=1}^{M} G_{\mu,\nu}^{(n,m)}|\sigma_\mu\rangle \langle \sigma_\nu|}}.
\]
**Theorem 12.** (Singular Valued Decomposition for Gates) Any unital linear gate \( \hat{G}^{(n,m)} \) of order \( (n, m) \) defined by \( \hat{G}^{(n,m)} \) can be represented by
\[
\hat{G}^{(n,m)} = \hat{U}^{(m,m)} \hat{D}^{(n,m)} \hat{U}^{(n,n)},
\]
where
- \( \hat{U}^{(m,m)} \) is an orthogonal quantum gate of order \( (m, m) \).
- \( \hat{U}^{(n,n)} \) is an orthogonal quantum gate of order \( (n, n) \).
- \( \hat{D}^{(n,m)} \) is a diagonal quantum gate of order \( (n, m) \), such that
\[
\hat{D}^{(n,m)} = \frac{1}{\sqrt{2^{2nM}}(0)(0) + \sum_{\mu=1}^{P} \lambda_\mu |\sigma_\mu\rangle \langle \sigma_\mu|},
\]
where \( p = \min\{N, M\} \) and \( \lambda_\mu \geq 0 \).

**Proof.** The proof of this theorem can be easy realized in matrix representation by using theorem 10.

In general case, we have the following theorem.

**Theorem 13.** (Singular Valued Decomposition for Gates) Any linear gate \( \hat{G}^{(n,m)} \) of order \( (n, m) \) can be represented by
\[
\hat{G}^{(n,m)} = \hat{T}^{(m,m)} \hat{U}^{(m,m)} \hat{D}^{(n,m)} \hat{U}^{(n,n)},
\]
where
- \( \hat{U}^{(m,m)} \) is an orthogonal quantum gate of order \( (m, m) \).
- \( \hat{U}^{(n,n)} \) is an orthogonal quantum gate of order \( (n, n) \).
- \( \hat{D}^{(n,m)} \) is a diagonal quantum gate of order \( (n, m) \).
- \( \hat{T}^{(m,m)} \) is a translation quantum gate of order \( (m, m) \):
\[
\hat{T}^{(m,m)} = \frac{1}{\sqrt{2^{2nM}}(0)(0) + \sum_{\mu=1}^{P} |\sigma_\mu\rangle \langle \sigma_\mu| + \sum_{\mu=0}^{M} T_\mu \sum_{\mu=0}^{M} \sum_{\mu=0}^{M} |\sigma_\mu\rangle \langle \sigma_\mu|},
\]
where \( p = \min\{N, M\} \) and \( \lambda_\mu \geq 0 \).

**Proof.** The proof of this theorem can be easy realized in matrix representation by using theorem 10.

Note that, any n-arguments classical gate \( g(\nu_1, ..., \nu_n) \) can be realized as linear trace-preserving quantum four-valued logic gate \( \hat{G}^{(n,1)} \) of order \( (n, 1) \) by
\[
\hat{G}^{(n,1)} = |0\rangle\langle 0| + \sum_{\nu_1, ..., \nu_n \neq 0, 0} |g(\nu_1, ..., \nu_n)\rangle \langle \nu_1, ..., \nu_n| + (1 - \delta_{0g(0,0,...,0)})|g(0, 0, ..., 0)\rangle\langle 0, 0, ..., 0| - (1 - \delta_{0g(0,0,...,0)}) \sum_{\mu=0}^{N-1} \sum_{\nu_1, ..., \nu_n} (1 - \delta_{g(0,0,...,0)}) |\mu\rangle \langle \nu_1, ..., \nu_n|.
\]

In general case, this quantum gate is not unital gate.

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