Randomized regularized extended Kaczmarz algorithms for tensor recovery
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Abstract
Randomized regularized Kaczmarz algorithms have recently been proposed to solve tensor recovery models with consistent linear measurements. In this work, we propose a novel algorithm based on the randomized extended Kaczmarz algorithm (which converges linearly in expectation to the unique minimum norm least squares solution of a linear system) for tensor recovery models with inconsistent linear measurements. We prove the linear convergence in expectation of our algorithm. Numerical experiments on a tensor least squares problem and a sparse tensor recovery problem are given to illustrate the theoretical results.

Keywords. Randomized regularized extended Kaczmarz, sparse tensor recovery, tensor least squares problem, linear convergence

AMS subject classifications: 65F10, 68W20, 90C25, 15A69

1 Introduction

Tensor recovery models have attracted much attention recently because of various applications, such as transportation, medical imaging, and remote sensing. With the assumption of consistent linear measurements, Chen and Qin [7] proposed a algorithmic framework based on the Kaczmarz-type algorithms [12, 29] for the following tensor recovery model:

\[ \hat{X} = \arg\min_{X \in \mathbb{R}^{N_2 \times N_3}} f(X), \quad \text{s.t.} \quad A \ast X = B, \]  

where the objective function \( f \) is strongly convex, the sensing tensor \( A \in \mathbb{R}^{N_1 \times N_2 \times N_3} \), the acquired measurement tensor \( B \in \mathbb{R}^{N_1 \times K \times N_3} \), and \( A \ast X \) is the tensor t-product [14]. In this paper, we assume that the linear measurement \( A \ast X = B \) is inconsistent and therefore consider the following constrained minimization problem:

\[ \hat{X} = \arg\min_{X \in \mathbb{R}^{N_2 \times N_3}} f(X), \quad \text{s.t.} \quad A^\top A \ast X = A^\top B, \]  

where \( A^\top \) denotes the transpose of \( A \) (see section 2.1). The solution \( \hat{X} \) of (2) is a least squares solution of the inconsistent system \( A \ast X = B \) with some desirable characteristics promoted by regularization terms of the objective function \( f \).

In recent years, randomized iterative algorithms for linear systems of equations with massive data sets have been greatly developed due to low memory footprints and good numerical performance, such as the randomized Kaczmarz (RK) algorithm [29], the randomized coordinate descent (RCD) algorithm [16], the randomized extended Kaczmarz (REK) algorithm [35], and their extensions, e.g., [24, 10, 20, 25, 1, 2, 3, 8, 22, 9, 1, 31, 32, 11, 33]. The RK algorithm converges linearly in expectation to a solution of consistent linear systems [29, 35] and to within a radius (convergence horizon) of a (least squares) solution of inconsistent linear systems [29].

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The REK algorithm converges linearly in expectation to a (least squares) solution of arbitrary linear systems \[35, 20, 8\]. In this paper, we replace the RK algorithm integrated in the randomized regularized Kaczmarz (RRK) algorithm of \[7, Algorithm 3.2\] with the REK algorithm and propose a randomized regularized extended Kaczmarz (RREK) algorithm for solving (2). The proposed RREK algorithm is called “regularized” since the objective function contains regularization terms for preserving some desirable characteristics of the underlying solution. We prove the linear convergence of the proposed algorithm. Special cases including tensor least squares problems and sparse tensor recovery problems are provided. Numerical experiments are given to illustrate our theoretical results.

The rest of this paper is organized as follows. In section 2, we provide clarification of notation, review basic concepts and results in tensor algebra and convex optimization, and also briefly introduce the RRK algorithm of \[7\]. In section 3, we describe the proposed RREK algorithm for solving (2) and establish its convergence theory. In section 4, we discuss two special cases including a tensor least squares problem and a sparse tensor recovery problem. In section 5, we report two numerical experiments to illustrate the theoretical results. Finally, we present brief concluding remarks in section 6.

2 Preliminaries

2.1 Basic notation

Throughout the paper, we use boldface uppercase letters such as \(A\) for matrices and calligraphic letters such as \(A\) for tensors. For an integer \(m \geq 1\), let \([m] := \{1, 2, 3, \ldots, m\}\). For any matrix \(A \in \mathbb{R}^{m \times n}\), we use \(A^\top\), \(A^\dagger\), \(\|A\|_2\), and \(\sigma_{\text{min}}(A)\) to denote the transpose, the Moore–Penrose pseudoinverse, the spectral norm, and the minimum nonzero singular values of \(A\), respectively.

For any random variables \(\xi\) and \(\zeta\), we use \(E[\xi]\) and \(E[\xi | \zeta]\) to denote the expectation of \(\xi\) and the conditional expectation of \(\xi\) given \(\zeta\), respectively.

2.2 Tensor basics

In this subsection, we provide a brief review of key definitions and facts in tensor algebra. We follow the notation used in \[14, 13, 21\].

For a third-order tensor \(A \in \mathbb{R}^{N_1 \times N_2 \times N_3}\), we denote its \((i, j, k)\) entry as \(A_{ijk}\) and use \(A_{i,:,:,}, A_{:,i,:}, A_{::,i}\) to denote respectively the \(i\)th horizontal, lateral and frontal slice. For notational convenience, the \(i\)th frontal slice \(A_{::,i}\) will be denoted as \(A_i\). We define the block circulant matrix \(\text{bcirc}(A)\) of \(A\) as,

\[
\text{bcirc}(A) := \begin{bmatrix} A_1 & A_{N_2} & \cdots & A_2 \\ A_2 & A_1 & \cdots & A_3 \\ \vdots & \vdots & \ddots & \vdots \\ A_{N_3} & A_{N_3-1} & \cdots & A_1 \end{bmatrix} \in \mathbb{R}^{N_1 N_3 \times N_2 N_3},
\]

We also define the operator \(\text{unfold}(\cdot)\) and its inversion \(\text{fold}(\cdot)\),

\[
\text{unfold}(A) := \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{N_3} \end{bmatrix} \in \mathbb{R}^{N_1 N_3 \times N_2}, \quad \text{fold} \left( \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{N_3} \end{bmatrix} \right) := A.
\]

**Definition 2.1 (t-product).** For \(A \in \mathbb{R}^{N_1 \times N_2 \times N_3}\) and \(B \in \mathbb{R}^{N_2 \times K \times N_3}\), the t-product \(A \ast B\) is defined to be a tensor of size \(N_1 \times K \times N_3\),

\[
A \ast B := \text{fold}(\text{bcirc}(A)\text{unfold}(B)).
\]
Definition 2.2 (transpose). The transpose of $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}$, denoted by $A^\top$, is the $N_2 \times N_1 \times N_3$ tensor obtained by transposing each of the frontal slices and then reversing the order of transposed frontal slices 2 through $N_3$.

For $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}$, we have

$$bcirc(A^\top) = (bcirc(A))^\top.$$  \hspace{1cm} (3)

Definition 2.3 (identity tensor). The identity tensor $I \in \mathbb{R}^{N_1 \times N_2 \times N_3}$ is the tensor whose first frontal slice is the $N \times N$ identity matrix, and whose other frontal slices are all zeros.

For $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}$ and $I \in \mathbb{R}^{N_2 \times N_2 \times N_3}$, it holds that

$$A_{i,:} = I_{i,:} * A.$$  \hspace{1cm} (4)

Definition 2.4 (inner product). The inner product between $A$ and $B$ in $\mathbb{R}^{N_1 \times N_2 \times N_3}$ is defined as

$$\langle A, B \rangle := \sum_{i,j,k} A_{ijk} B_{ijk}.$$  \hspace{1cm} (5)

For $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}$, $B \in \mathbb{R}^{N_2 \times K \times N_3}$, and $C \in \mathbb{R}^{N_1 \times K \times N_3}$, it holds that

$$\langle A * B, C \rangle = \langle B, A^\top * C \rangle.$$  \hspace{1cm} (6)

Definition 2.5 (1-norm, spectral norm, and Frobenius norm). The 1-norm, spectral norm, and Frobenius norm of $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}$ are defined as

$$\|A\|_1 := \sum_{i,j,k} |A_{ijk}|, \quad \|A\|_2 := \|bcirc(A)\|_2,$$

and

$$\|A\|_F := \sqrt{\langle A, A \rangle} = \sqrt{\sum_{i,j,k} (A_{ijk})^2},$$

respectively.

For $A$ and $B$ in $\mathbb{R}^{N_1 \times N_2 \times N_3}$, it holds that

$$|\langle A, B \rangle| \leq \|A\|_F \|B\|_F.$$  \hspace{1cm} (7)

For $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}$ and $B \in \mathbb{R}^{N_2 \times K \times N_3}$, it holds that

$$\|A * B\|_F \leq \|A\|_2 \|B\|_F.$$  \hspace{1cm} (8)

Definition 2.6 (K-range). The $K$-range of $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}$ is defined as

$$\text{range}_K(A) := \{A * \mathcal{Y} \mid \mathcal{Y} \in \mathbb{R}^{N_2 \times K \times N_3}\}.$$  

For $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}$, it holds that

$$\text{range}_K(A^\top * A) = \text{range}_K(A^\top).$$  \hspace{1cm} (9)

For $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}$ and all $\mathcal{X} \in \text{range}_K(A)$, it holds that

$$\|A^\top * \mathcal{X}\|_F^2 \geq \sigma_{\min}^2(bcirc(A)) \|\mathcal{X}\|_F^2.$$  \hspace{1cm} (10)
Definition 2.7 (pseudoinverse). The pseudoinverse of \( A \in \mathbb{R}^{N_1 \times N_2 \times N_3} \), denoted by \( A^\dagger \), is the \( N_2 \times N_1 \times N_3 \) tensor satisfying
\[
\text{bcirc}(A^\dagger) = \text{bcirc}(A)^\dagger.
\]
For \( A \in \mathbb{R}^{N_1 \times N_2 \times N_3} \) and \( B \in \mathbb{R}^{N_1 \times K \times N_3} \), it holds that
\[
A^\top * A * A^\dagger * B = A^\top * B.
\]
If \( X \in \mathbb{R}^{N_2 \times K \times N_3} \) satisfies \( A^\top * A * X = A^\top * B \), then it holds that
\[
A * X = A * A^\dagger * B.
\]

2.3 Convex optimization basics

To make the paper self-contained, we present basic definitions and properties about convex functions defined on tensor spaces in this subsection. We refer the reader to \([26, 5]\) for more definitions and properties.

Definition 2.8 (subdifferential). For a continuous function \( f : \mathbb{R}^{N_1 \times N_2 \times N_3} \to \mathbb{R} \), its subdifferential at \( \mathcal{X} \in \mathbb{R}^{N_1 \times N_2 \times N_3} \) is defined as
\[
\partial f(\mathcal{X}) := \{ Z : f(Y) \geq f(\mathcal{X}) + \langle Z, Y - \mathcal{X} \rangle \ \forall Y \in \mathbb{R}^{N_1 \times N_2 \times N_3} \}.
\]

Definition 2.9 (\( \gamma \)-strong convexity). A function \( f : \mathbb{R}^{N_1 \times N_2 \times N_3} \to \mathbb{R} \) is called \( \gamma \)-strongly convex for a given \( \gamma > 0 \) if the following inequality holds for all \( \mathcal{X}, \mathcal{Y} \in \mathbb{R}^{N_1 \times N_2 \times N_3} \) and \( Z \in \partial f(\mathcal{X}) \):
\[
f(\mathcal{Y}) \geq f(\mathcal{X}) + \langle Z, \mathcal{Y} - \mathcal{X} \rangle + \frac{\gamma}{2}\| \mathcal{Y} - \mathcal{X} \|^2.
\]

The function \( f(\mathcal{X}) = \frac{1}{2}\| \mathcal{X} \|^2 \) is differential and 1-strongly convex. Moreover, it is easy to show that the function \( h(\mathcal{X}) + \frac{1}{2}\| \mathcal{X} \|^2 \) is 1-strongly convex if \( h(\mathcal{X}) \) is convex.

Definition 2.10 (conjugate function). The conjugate function of \( f : \mathbb{R}^{N_1 \times N_2 \times N_3} \to \mathbb{R} \) at \( \mathcal{Y} \in \mathbb{R}^{N_1 \times N_2 \times N_3} \) is defined as
\[
f^*(\mathcal{Y}) := \sup_{\mathcal{X} \in \mathbb{R}^{N_1 \times N_2 \times N_3}} \{ \langle \mathcal{Y}, \mathcal{X} \rangle - f(\mathcal{X}) \}.
\]

If \( f(\mathcal{X}) \) is \( \gamma \)-strongly convex, then the conjugate function \( f^*(\mathcal{X}) \) is differentiable and for all \( \mathcal{X}, \mathcal{Y} \in \mathbb{R}^{N_1 \times N_2 \times N_3} \), the following inequality holds:
\[
f^*(\mathcal{Y}) \leq f^*(\mathcal{X}) + \langle \nabla f^*(\mathcal{X}), \mathcal{Y} - \mathcal{X} \rangle + \frac{1}{2\gamma}\| \mathcal{Y} - \mathcal{X} \|^2. \tag{10}
\]

For a strongly convex function \( f(\mathcal{X}) \), it can be shown that \([26, 5]\)
\[
\mathcal{Z} \in \partial f(\mathcal{X}) \iff \mathcal{X} = \nabla f^*(\mathcal{Z}). \tag{11}
\]

For a convex function \( h(\mathcal{X}) \), the conjugate function of \( \lambda h(\mathcal{X}) + \frac{1}{2}\| \mathcal{X} \|^2 \) is differentiable. Its gradient involves the proximal mapping of \( h(\mathcal{X}) \), and it holds that
\[
\nabla \left( \lambda h(\cdot) + \frac{1}{2}\| \cdot \|^2 \right)^* (\mathcal{X}) = \text{prox}_{\lambda h}(\mathcal{X}) := \arg\min_{\mathcal{Y} \in \mathbb{R}^{N_1 \times N_2 \times N_3}} \left\{ \lambda h(\mathcal{Y}) + \frac{1}{2}\| \mathcal{Y} - \mathcal{X} \|^2 \right\}.
\]

Definition 2.11 (Bregman distance). For a convex function \( f : \mathbb{R}^{N_1 \times N_2 \times N_3} \to \mathbb{R} \), the Bregman distance between \( \mathcal{X} \) and \( \mathcal{Y} \) with respect to \( f \) and \( \mathcal{Z} \in \partial f(\mathcal{X}) \) is defined as
\[
D_{f, \mathcal{Z}}(\mathcal{X}, \mathcal{Y}) := f(\mathcal{Y}) - f(\mathcal{X}) - \langle \mathcal{Z}, \mathcal{Y} - \mathcal{X} \rangle.
\]
It follows from \((Z, \mathcal{X}) = f(\mathcal{X}) + f^*(Z)\) if \(Z \in \partial f(\mathcal{X})\) that
\[
D_{f,Z}(\mathcal{X}, \mathcal{Y}) = f(\mathcal{Y}) + f^*(Z) - (Z, \mathcal{Y}).
\] (12)

If \(f\) is \(\gamma\)-strongly convex, then it holds that
\[
D_{f,Z}(\mathcal{X}, \mathcal{Y}) \geq \frac{\gamma}{2} \|\mathcal{X} - \mathcal{Y}\|^2_F.
\] (13)

**Definition 2.12** (restricted strong convexity \([15, 27]\)). Let \(f : \mathbb{R}^{N_1 \times N_2 \times N_3} \to \mathbb{R}\) be convex differentiable with a nonempty minimizer set \(X_f\). The function \(f\) is called restricted strongly convex on \(\mathbb{R}^{N_1 \times N_2 \times N_3}\) with a constant \(\mu > 0\) if it satisfies for all \(\mathcal{X} \in \mathbb{R}^{N_1 \times N_2 \times N_3}\) the inequality
\[
\langle \nabla f(P_{X_f}(\mathcal{X})) - \nabla f(\mathcal{X}), P_{X_f}(\mathcal{X}) - \mathcal{X} \rangle \geq \mu \|P_{X_f}(\mathcal{X}) - \mathcal{X}\|^2_F,
\]

where \(P_{X_f}(\mathcal{X})\) denotes the orthogonal projection of \(\mathcal{X}\) onto \(X_f\).

**Definition 2.13** (strong admissibility). Let \(A \in \mathbb{R}^{N_1 \times N_2 \times N_3}\) and \(B \in \mathbb{R}^{N_1 \times K \times N_3}\) be given. Let \(f : \mathbb{R}^{N_2 \times K \times N_3} \to \mathbb{R}\) be strongly convex. The function \(f\) is called strongly admissible if the function \(g(\mathcal{Y}) := f^*(A^\top * A * \mathcal{Y}) - \langle \mathcal{Y}, A^\top * B \rangle\) is restricted strongly convex on \(\mathbb{R}^{N_2 \times K \times N_3}\).

**Lemma 2.14.** Let \(\hat{X}\) be the solution of (4). If \(f\) is strongly admissible, then there exists a constant \(\nu > 0\) such that
\[
D_{f,Z}(\mathcal{X}, \hat{X}) \leq \frac{1}{\nu} \|A * (\mathcal{X} - \hat{X})\|^2_F,
\] (14)

for all \(\mathcal{X} \in \mathbb{R}^{N_2 \times K \times N_3}\) and \(Z \in \partial f(\mathcal{X}) \cap \text{range}_K(A^\top)\).

**Proof.** The solution \(\hat{X}\) of (4) satisfies the following optimality conditions:
\[
A^\top * A * \hat{X} = A^\top * B, \quad \partial f(\hat{X}) \cap \text{range}_K(A^\top A) \neq \emptyset.
\] (15)

The dual problem of (4) is the unconstrained problem
\[
\min_{\mathcal{Y} \in \mathbb{R}^{N_2 \times K \times N_3}} g(\mathcal{Y}),
\]

where
\[
g(\mathcal{Y}) = f^*(A^\top A * \mathcal{Y}) - \langle \mathcal{Y}, A^\top B \rangle.
\]

By the strong duality, we have
\[
f(\hat{X}) = -\min_{\mathcal{Y} \in \mathbb{R}^{N_2 \times K \times N_3}} g(\mathcal{Y}).
\]

Since \(Z \in \text{range}_K(A^\top) = \text{range}_K(A^\top A)\), we can write \(Z = A^\top A * \mathcal{Y}\) for some \(\mathcal{Y}\). Then
\[
D_{f,Z}(\mathcal{X}, \hat{X}) \overset{(12)}{=} f^*(Z) - (Z, \hat{X}) + f(\hat{X})
\]
\[
= f^*(A^\top A * \mathcal{Y}) - \langle A^\top A * \mathcal{Y}, \hat{X} \rangle + f(\hat{X})
\]
\[
\overset{(10)}{=} f^*(A^\top A * \mathcal{Y}) - \langle \mathcal{Y}, A^\top A * \hat{X} \rangle + f(\hat{X})
\]
\[
\overset{(19)}{=} f^*(A^\top A * \mathcal{Y}) - \langle \mathcal{Y}, A^\top B \rangle + f(\hat{X})
\]
\[
= g(\mathcal{Y}) - \min_{\mathcal{Y} \in \mathbb{R}^{N_2 \times K \times N_3}} g(\mathcal{Y}).
\]

Since \(g(\mathcal{Y})\) is restricted strongly convex on \(\mathbb{R}^{N_2 \times K \times N_3}\), there exists a constant \(\mu > 0\) such that
\[
\langle \nabla g(P_{X_f}(\mathcal{Y})) - \nabla g(\mathcal{Y}), P_{X_f}(\mathcal{Y}) - \mathcal{Y} \rangle \geq \mu \|P_{X_f}(\mathcal{Y}) - \mathcal{Y}\|^2_F.
\]
By $\nabla g(P_X g(Y)) = 0$ and the Cauchy–Schwarz inequality \cite{7}, we get
\[
\|\nabla g(Y)\|_F \geq \mu \|P_X g(Y) - Y\|_F.
\]
The convexity of $g(Y)$ implies
\[
g(Y) - g(P_X g(Y)) \leq \langle \nabla g(Y), Y - P_X g(Y) \rangle \leq \|\nabla g(Y)\|_F \|P_X g(Y) - Y\|_F \leq \frac{1}{\mu} \|\nabla g(Y)\|^2_F. \tag{16}
\]
The gradient of $g(Y)$ is
\[
\nabla g(Y) = A^\top A \ast \nabla f^\ast(A^\top \ast A \ast Y) - A^\top \ast B
\]
\[
= A^\top A \ast \nabla f^\ast(Z) - A^\top \ast B
\]
\[
= A^\top A \ast X - A^\top \ast A \ast \hat{X}. \tag{17}
\]
Therefore,
\[
D_{f,Z}(X, \hat{X}) = g(Y) - g(P_X g(Y)) \leq \frac{1}{\mu} \frac{\|A\|^2 \|A \ast \hat{X}\|_F^2}{\|A \ast (X - \hat{X})\|_F^2} \leq \frac{1}{\nu} \frac{\|A \ast (X - \hat{X})\|_F^2}{\|A \ast (X - \hat{X})\|_F^2}.
\]
This completes the proof. \hfill \qed

The constant $\nu$ depends on the tensor $A$ and the function $f$. In general, it is hard to quantify $\nu$. We refer the reader to \cite{27, 15, 7} for examples of strongly admissible functions.

### 2.4 The RRK algorithm

By combining the RK algorithm and the gradient of the conjugate function at the previous iterate, Chen and Qin \cite{7} proposed the RRK algorithm (see Algorithm 1) for solving the minimization problem \cite{1}. They proved a linear convergence rate if $A \ast X = B$ is consistent; see Theorem 3.9 of \cite{7}. Moreover, they also considered the noisy scenario (the perturbed constraint $A \ast X = \tilde{B}$ where $\tilde{B} = B + \epsilon$) and proved that the RRK algorithm linearly converges to with a radius of the solution of \cite{1}; see Theorem 3.10 of \cite{7}.

**Algorithm 1:** The RRK algorithm for solving \cite{1}

**Input:** $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}, B \in \mathbb{R}^{N_1 \times K \times N_3}$, stepsize $\alpha_r$, maximum number of iterations $M$, and tolerance $\tau$.

**Initialize:** $Z^{(0)} \in \text{range}_K(A^\top)$ and $\mathcal{X}^{(0)} = \nabla f^\ast(Z^{(0)})$.

for $k = 1, 2, \ldots, M$ do

Pick $i_k \in [N_1]$ with probability $\|A_{ik,:,:}\|_F^2 / \|A\|_F^2$

Set $Z^{(k)} = Z^{(k-1)} - \alpha_r (A_{ik,:,:})^\top \ast \frac{A_{ik,:,:} \ast \mathcal{X}^{(k-1)} - B_{ik,:,:}}{\|A_{ik,:,:}\|_F^2}$

Set $X^{(k)} = \nabla f^\ast(Z^{(k)})$

Stop if $\|X^{(k)} - X^{(k-1)}\|_F / \|X^{(k-1)}\|_F < \tau$

end

For the special case that
\[
f(\mathcal{X}) = \frac{1}{2} \|\mathcal{X}\|_F^2,
\]
we have

$$f^*(\mathcal{X}) = \frac{1}{2}||\mathcal{X}||^2_F, \quad \nabla f^*(\mathcal{X}) = \mathcal{X}.$$  

Algorithm 1 becomes a tensor randomized Kaczmarz (TRK) algorithm for solving the tensor system $\mathcal{A} \ast \mathcal{X} = \mathcal{B}$ with the iteration

$$\mathcal{X}^{(k)} = \mathcal{X}^{(k-1)} - \alpha_r (A_{i_{k},:,})^\top \ast \frac{A_{i_{k},:,} \ast \mathcal{X}^{(k-1)} - B_{i_{k},:,}}{||A_{i_{k},:,}||^2_F},$$  

which is different from the tensor randomized Kaczmarz algorithm proposed by Ma and Molitor [19].

3 The RREK algorithm

The REK algorithm [35] converges linearly in expectation to the minimum 2-norm least squares solution of a linear system. Motivated by this property, we consider replacing the RK algorithm integrated in the RRK algorithm with the REK algorithm and propose the following RREK algorithm (see Algorithm 2) for solving the constrained minimization problem (2). The RREK algorithm only uses one horizontal slice and one lateral slice of $A$ at each step and avoids forming $A^\top \ast A$ explicitly. We also note that $\mathcal{Z}^{(k)}$ in the RREK algorithm is the same as the $k$th iterate generated by the TRK iteration (18) applied to the consistent tensor system $A^\top \ast \mathcal{Z} = \mathbf{0}$.

![Algorithm 2: The RREK algorithm for solving (2)](algorithm_2.png)

Next we analyze the convergence of the RREK algorithm. Our analysis is similar to that of [9], but slightly more complicated. The convergence estimates depend on the positive numbers $\lambda_r$ and $\lambda_c$ defined as

$$\lambda_r := \max_{i \in [N_1]} ||A_{i,:,}:||^2_F / ||A||^2_F, \quad \lambda_c := \max_{j \in [N_2]} ||A_{:,j,:}||^2_F / ||A||^2_F.$$  

We give the convergence result of $\mathcal{Z}^{(k)}$ in the RREK algorithm in the following theorem.

Theorem 3.1. If $0 < \alpha_c < 2 / \lambda_c$, then the sequence $\{\mathcal{Z}^{(k)}\}$ generated by Algorithm 2 satisfies

$$\mathbb{E} [||\mathcal{Z}^{(k)} - (\mathcal{B} - \mathcal{A} \ast \mathcal{A}^\top \ast \mathcal{B})||^2_F] \leq \rho_c^k ||\mathcal{A} \ast \mathcal{A}^\top \ast \mathcal{B}||^2_F,$$

where

$$\rho_c = 1 - \frac{(2\alpha_c - \alpha_c^2 \lambda_c)\sigma^2_{\min}(bcirc(A))}{||A||^2_F}.$$  

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Proof. Introduce the auxiliary tensor sequence
\[ \mathcal{E}^{(k)} = \mathcal{E}^{(k-1)} - (B - A^* A^1 \ast B). \]
By \((A_{:,j_k,:})^\top \ast (B - A^* A^1 \ast B) = 0\) (because \(A^\top \ast (B - A^* A^1 \ast B) = 0\)), we have
\[ \mathcal{E}^{(k)} = \mathcal{E}^{(k-1)} - \alpha_c A_{:,j_k,:}^{\top} \ast \mathcal{E}^{(k-1)} \]

Then,
\[
\|\mathcal{E}^{(k)}\|_F^2 = \left\| \mathcal{E}^{(k-1)} - \alpha_c A_{:,j_k,:} \ast \mathcal{E}^{(k-1)} \right\|_F^2 \\
= \left\| \mathcal{E}^{(k-1)} \right\|_F^2 - \frac{2\alpha_c}{\|A_{:,j_k,:}\|_F^2} \langle \mathcal{E}^{(k-1)}, A_{:,j_k,:} \ast \mathcal{E}^{(k-1)} \rangle \\
+ \frac{\alpha_c^2}{\|A_{:,j_k,:}\|_F^2} \|A_{:,j_k,:} \ast \mathcal{E}^{(k-1)}\|_F^2 \\
\leq \left\| \mathcal{E}^{(k-1)} \right\|_F^2 - \frac{2\alpha_c}{\|A_{:,j_k,:}\|_F^2} \|A_{:,j_k,:}\|_F^2 \| \mathcal{E}^{(k-1)} \|_F^2 + \frac{\alpha_c^2}{\|A_{:,j_k,:}\|_F^2} \|A_{:,j_k,:} \ast \mathcal{E}^{(k-1)}\|_F^2 \\
\leq \left\| \mathcal{E}^{(k-1)} \right\|_F^2 - \frac{2\alpha_c}{\|A_{:,j_k,:}\|_F^2} \|A_{:,j_k,:}\|_F^2 \| \mathcal{E}^{(k-1)} \|_F^2 + \frac{\alpha_c^2 \lambda_c}{\|A_{:,j_k,:}\|_F^2} \|A_{:,j_k,:} \ast \mathcal{E}^{(k-1)}\|_F^2.
\]

Taking conditional expectation conditioned on \(\mathcal{E}^{(k-1)}\) gives
\[
\mathbb{E} \left[ \|\mathcal{E}^{(k)}\|_F^2 \mid \mathcal{E}^{(k-1)} \right] \leq \left\| \mathcal{E}^{(k-1)} \right\|_F^2 - \frac{2\alpha_c - \alpha_c^2 \lambda_c}{\|A\|_F^2} \|A\|_F^2 \| \mathcal{E}^{(k-1)} \|_F^2 \\
\leq \left( 1 - \frac{(2\alpha_c - \alpha_c^2 \lambda_c) \sigma_{\text{min}}^2 (\text{bcirc}(A))}{\|A\|_F^2} \right) \| \mathcal{E}^{(k-1)} \|_F^2.
\]

In the last inequality, we use the facts that \(2\alpha_c - \alpha_c^2 \lambda_c > 0\), \(\mathcal{E}^{(k)} \in \text{range}_R(A)\) (by induction), and \(\mathcal{E}^{(0)}\). Next, by the law of total expectation, we have
\[
\mathbb{E} \left[ \|\mathcal{E}^{(k)}\|_F^2 \right] \leq \left( 1 - \frac{(2\alpha_c - \alpha_c^2 \lambda_c) \sigma_{\text{min}}^2 (\text{bcirc}(A))}{\|A\|_F^2} \right) \mathbb{E} \left[ \|\mathcal{E}^{(k-1)}\|_F^2 \right].
\]

Unrolling the recurrence yields the result. \(\square\)

We give the main convergence result of the RREK algorithm in the following theorem.

**Theorem 3.2.** Let \( f \) be \( \gamma \)-strongly convex and strongly admissible, and \( \nu > 0 \) is the constant from (14). Let \( \hat{X} \) be the solution of (3). Assume that 0 < \( \alpha_c < 2/\lambda_c \) and 0 < \( \alpha_t < 2\gamma/\lambda_t \). For any \( \delta > 0 \), the sequences \( \{X^{(k)}\} \) and \( \{Y^{(k)}\} \) generated by Algorithm 2 satisfy
\[
\mathbb{E} \left[ D_f(X^{(k)}, \hat{X}) \right] \leq \frac{\delta + \gamma}{2\delta F_{\|A\|_F^2}} \|A \ast A^1 \ast B\|_F^2 \sum_{i=0}^{k-1} \rho_c^i \left( 1 + \frac{\delta}{\gamma} \right)^i \rho_t^i \\
+ \left( 1 + \frac{\delta}{\gamma} \right)^k \rho_t^k D_f(Y^{(0)}, \hat{X}),
\]
where
\[
\rho_c = 1 - \frac{(2\alpha_c - \alpha_c^2 \lambda_c) \sigma_{\text{min}}^2 (\text{bcirc}(A))}{\|A\|_F^2}, \quad \rho_t = 1 - \frac{(2\gamma \alpha_t - \alpha_t^2 \lambda_t) \nu}{2\gamma \|A\|_F^2}.
\]
Proof. Let
\[ \hat{\gamma}^{(k)} := \gamma^{(k-1)} - \alpha_t (A_{ik,:,:})^\top A_{ik,:,:}^\top A \gamma^{(k-1)} - A_{ik,:,:} A^\top B, \]
which is actually the one-step RRK update for the consistent constraint
\[ A \gamma = A A^\top B \]
from \( \gamma^{(k-1)} \) and \( \gamma^{(k-1)} \). We have
\[ \gamma^{(k)} - \hat{\gamma}^{(k)} = \alpha_t (A_{ik,:,:})^\top \frac{B_{ik,:,:} - Z_{ik,:,:} - A_{ik,:,:} A^\top B}{\|A_{ik,:,:}\|_F^2}. \]
Then,
\[
\| \gamma^{(k)} - \hat{\gamma}^{(k)} \|_F^2 = \frac{\alpha_t^2}{\|A_{ik,:,:}\|_F^2} \| (A_{ik,:,:})^\top (B_{ik,:,:} - Z_{ik,:,:} - A_{ik,:,:} A^\top B) \|_F^2 \\
\leq \frac{\alpha_t^2 \|A_{ik,:,:}\|_F^2}{\|A_{ik,:,:}\|_F^2} \| B_{ik,:,:} - Z_{ik,:,:} - A_{ik,:,:} A^\top B \|_F^2 \\
\leq \frac{\alpha_t^2 \lambda_t}{\|A_{ik,:,:}\|_F^2} \| B_{ik,:,:} - Z_{ik,:,:} - A_{ik,:,:} A^\top B \|_F^2. \quad (19)
\]
Let \( E_{k-1} [\cdot] \) denote the conditional expectation conditioned on \( Z^{(k-1)}, \gamma^{(k-1)}, \) and \( \gamma^{(k-1)} \). Let \( E_{k-1}^F [\cdot] \) denote the conditional expectation conditioned on \( Z(k), \gamma^{(k-1)}, \) and \( \gamma^{(k-1)} \). Then, by the law of total expectation, we have
\[ E_{k-1} [\cdot] = E_{k-1} \left[ E_{k-1}^F [\cdot] \right]. \]
Taking conditional expectation for (19) conditioned on \( Z^{(k-1)}, \gamma^{(k-1)}, \) and \( \gamma^{(k-1)} \), we obtain
\[ E_{k-1} \left[ \| \gamma^{(k)} - \hat{\gamma}^{(k)} \|_F^2 \right] = E_{k-1} \left[ E_{k-1}^F \left[ \| \gamma^{(k)} - \hat{\gamma}^{(k)} \|_F^2 \right] \right] \\
\leq \frac{\alpha_t^2 \lambda_t}{\|A\|_F^2} E_{k-1} \left[ \| Z^{(k)} - (B - A A^\top B) \|_F^2 \right]. \]
Then, by the law of total expectation and Theorem 3.1 we have
\[
E \left[ \| \gamma^{(k)} - \hat{\gamma}^{(k)} \|_F^2 \right] \leq \frac{\alpha_t^2 \lambda_t}{\|A\|_F^2} E \left[ \| Z^{(k)} - (B - A A^\top B) \|_F^2 \right] \\
\leq \frac{\alpha_t^2 \lambda_t \rho_{F}^k}{\|A\|_F^2} \|A A^\top B \|_F^2. \quad (20)
\]
Let
\[ W := \frac{A_{ik,:,:} * \gamma^{(k-1)} - A_{ik,:,:} A^\top B}{\|A_{ik,:,:}\|_F^2} \]
By \( A \hat{\gamma} = A A^\top B \), we have
\[ W = \frac{A_{ik,:,:} * (\gamma^{(k-1)} - \hat{\gamma})}{\|A_{ik,:,:}\|_F^2}. \quad (21) \]
Let
\[ \hat{\gamma}^{(k)} := \nabla f^* (\hat{\gamma}^{(k)}). \quad (22) \]
By (11), we have
\[ \hat{\gamma}^{(k)} \in \partial f(\hat{\gamma}^{(k)}). \quad (23) \]
Therefore, the Bregman distance between $\hat{X}^{(k)}$ and $\hat{X}$ with respect to $f$ and $\hat{Y}^{(k)}$ satisfies
\[
D_{f,Y}(\hat{X}^{(k)}, \hat{X}) \overset{12}{=} f^*(\hat{Y}^{(k)}) - (\hat{Y}^{(k)}, \hat{X}) + f(\hat{X})
= f^*(Y^{(k-1)} - \alpha_t (A_{i_k::i})^\top * W) - (Y^{(k-1)} - \alpha_t (A_{i_k::i})^\top * W, \hat{X}) + f(\hat{X})
\overset{13}{\leq} f^*(Y^{(k-1)}) - (\nabla f^*(Y^{(k-1)}), \alpha_t (A_{i_k::i})^\top * W) + \frac{\alpha_t^2}{2\gamma} \| (A_{i_k::i})^\top * W \|_F^2
\]
\[
- (Y^{(k-1)} - \alpha_t (A_{i_k::i})^\top * W, \hat{X}) + f(\hat{X})
\overset{12}{=} D_{f,Y}(\hat{X}^{(k-1)}, \hat{X}) - (X^{(k-1)}, \alpha_t (A_{i_k::i})^\top * W)
+ \frac{\alpha_t^2}{2\gamma} \| (A_{i_k::i})^\top * W \|_F^2
\]
\[
= D_{f,Y}(\hat{X}^{(k-1)}, \hat{X}) - (X^{(k-1)} - \hat{X}, \alpha_t (A_{i_k::i})^\top * W)
+ \frac{\alpha_t^2}{2\gamma} \| (A_{i_k::i})^\top * W \|_F^2
\]
\[
= D_{f,Y}(\hat{X}^{(k-1)}, \hat{X}) - \frac{\alpha_t}{\| A_{i_k::i} \|_F^2} \| A_{i_k::i} \|_F \| X^{(k-1)} - \hat{X} \|_F^2
\]
\[
+ \frac{\alpha_t^2}{2\gamma} \| A_{i_k::i} \|_F \| X^{(k-1)} - \hat{X} \|_F^2.
\]
Taking conditional expectation conditioned on $Z^{(k-1)}$, $Y^{(k-1)}$, and $X^{(k-1)}$, we have
\[
\mathbb{E}_{k-1} \left[ D_{f,Y}(\hat{X}^{(k)}, \hat{X}) \right] \overset{4}{\leq} D_{f,Y}(\hat{X}^{(k-1)}, \hat{X}) - \frac{2\gamma \alpha_t - \alpha_t^2 \lambda_t}{2\gamma \| A \|_F^2} \| A \|_F \| X^{(k-1)} - \hat{X} \|_F^2
\]
\[
\overset{11}{\leq} \left( 1 - \frac{(2\gamma \alpha_t - \alpha_t^2 \lambda_t)\nu}{2\gamma \| A \|_F^2} \right) D_{f,Y}(\hat{X}^{(k-1)}, \hat{X}).
\]
Thus, by the law of total expectation, we have
\[
\mathbb{E} \left[ D_{f,Y}(\hat{X}^{(k)}, \hat{X}) \right] \leq \rho_k \mathbb{E} \left[ D_{f,Y}(\hat{X}^{(k-1)}, \hat{X}) \right]. \quad (24)
\]
Now, we consider the Bregman distance $D_{f,Y}(X^{(k)}, \hat{X})$, which satisfies
\[
D_{f,Y}(X^{(k)}, \hat{X}) \overset{12}{=} D_{f,Y}(\hat{X}^{(k)}, \hat{X}) + f^*(Y^{(k)}) - f^*(\hat{Y}^{(k)}) - (Y^{(k)}, \hat{X}) + (\hat{Y}^{(k)}, \hat{X})
\overset{13}{\leq} D_{f,Y}(\hat{X}^{(k)}, \hat{X}) + (\nabla f^*(\hat{Y}^{(k)}), Y^{(k)} - \hat{Y}^{(k)}) + \frac{1}{2\gamma} \| Y^{(k)} - \hat{Y}^{(k)} \|_F^2
- (Y^{(k)} - \hat{Y}^{(k)}, \hat{X})
\overset{22}{\leq} D_{f,Y}(\hat{X}^{(k)}, \hat{X}) + \| \hat{X}^{(k)} - \hat{X}, Y^{(k)} - \hat{Y}^{(k)} \|_F^2 + \frac{1}{2\gamma} \| Y^{(k)} - \hat{Y}^{(k)} \|_F^2
\leq D_{f,Y}(\hat{X}^{(k)}, \hat{X}) + \delta \| \hat{X}^{(k)} - \hat{X} \|_F^2 + \frac{1}{2\gamma} \| Y^{(k)} - \hat{Y}^{(k)} \|_F^2
+ \frac{1}{2\gamma} \| Y^{(k)} - \hat{Y}^{(k)} \|_F^2 \quad \text{(by 7 and Young’s inequality)}
\overset{23}{\leq} \left( 1 + \frac{\delta}{\gamma} \right) D_{f,Y}(\hat{X}^{(k)}, \hat{X}) + \frac{\delta + \gamma}{2\gamma} \| Y^{(k)} - \hat{Y}^{(k)} \|_F^2.
\]
Taking expectation, we have

\[
\mathbb{E} \left[ D_{f,Y^{(k)}}(\mathcal{X}^{(k)}, \hat{\mathcal{X}}) \right] \leq \left( 1 + \frac{\delta}{\gamma} \right) \mathbb{E} \left[ D_{f,Y^{(k)}}(\hat{\mathcal{X}}^{(k)}, \hat{\mathcal{X}}) \right] + \frac{\delta + \gamma}{2\delta} \mathbb{E} \left[ \|Y^{(k)} - \hat{Y}^{(k)}\|^2_F \right]
\]

\[
\leq \frac{\delta + \gamma}{2\delta} \frac{\alpha^2 \lambda \tau}{\|A\|^2_F} \|A \ast A^\dagger \ast B\|^2_F + \left( 1 + \frac{\delta}{\gamma} \right) \rho \mathbb{E} \left[ D_{f,Y^{(k-1)}}(\mathcal{X}^{(k-1)}, \hat{\mathcal{X}}) \right]
\]

\[
\leq \delta + \gamma \frac{\alpha^2 \lambda \tau}{2\delta} \|A \ast A^\dagger \ast B\|^2_F \left( \rho^k + \rho^{k-1} \left( 1 + \frac{\delta}{\gamma} \right) \rho \right)
\]

\[
+ \left( 1 + \frac{\delta}{\gamma} \right)^2 \rho^k \mathbb{E} \left[ D_{f,Y^{(k-2)}}(\mathcal{X}^{(k-2)}, \hat{\mathcal{X}}) \right]
\]

\[
\leq \cdots
\]

\[
\leq \delta + \gamma \frac{\alpha^2 \lambda \tau}{2\delta} \|A \ast A^\dagger \ast B\|^2_F \sum_{i=0}^{k-1} \rho^{k-i} \left( 1 + \frac{\delta}{\gamma} \right)^i \rho^i
\]

\[
+ \left( 1 + \frac{\delta}{\gamma} \right)^k \rho^k D_{f,Y^{(0)}}(\mathcal{X}^{(0)}, \hat{\mathcal{X}}).
\]

This completes the proof. \(\square\)

**Remark 3.3.** Let \(\rho = \max\{\rho_c, \rho_r\}\). We have \(0 < \rho < 1\). Assume that \(\delta > 0\) satisfies \(0 < (1 + \delta/\gamma)\rho < 1\). We have

\[
\mathbb{E} \left[ D_{f,Y^{(k)}}(\mathcal{X}^{(k)}, \hat{\mathcal{X}}) \right] \leq \left( 1 + \frac{\delta}{\gamma} \right)^k \rho^k \left( D_{f,Y^{(0)}}(\mathcal{X}^{(0)}, \hat{\mathcal{X}}) + \frac{\delta + \gamma}{2\delta} \frac{\alpha^2 \lambda \tau}{\|A\|^2_F} \|A \ast A^\dagger \ast B\|^2_F \right),
\]

therefore,

\[
\mathbb{E} \left[ \|\mathcal{X}^{(k)} - \hat{\mathcal{X}}\|^2_F \right] \leq \left( 1 + \frac{\delta}{\gamma} \right)^k \rho^k \frac{2}{\gamma} \left( D_{f,Y^{(0)}}(\mathcal{X}^{(0)}, \hat{\mathcal{X}}) + \frac{\delta + \gamma}{2\delta} \frac{\alpha^2 \lambda \tau}{\|A\|^2_F} \|A \ast A^\dagger \ast B\|^2_F \right),
\]

which shows that the RREK algorithm converges linearly in expectation to \(\hat{\mathcal{X}}\) with the rate \((1 + \delta/\gamma)\rho\).

## 4 Special cases of the proposed algorithm

### 4.1 Tensor randomized extended Kaczmarz (TREK) for tensor least squares

In this subsection, we consider the following tensor least squares problem

\[
\hat{\mathcal{X}} = \arg\min_{\mathcal{X} \in \mathbb{R}^{N_2 \times N_3 \times N_3}} \frac{1}{2} \|\mathcal{X}\|^2_F, \quad \text{s.t.} \quad A^\dagger \ast A \ast \mathcal{X} = A^\dagger \ast \mathcal{B}.
\]

The function

\[
f(\mathcal{X}) = \frac{1}{2} \|\mathcal{X}\|^2_F
\]

is 1-strongly convex and strongly admissible. We have

\[
\hat{\mathcal{X}} = A^\dagger \ast \mathcal{B}, \quad \nabla f(\mathcal{X}) = \mathcal{X}, \quad f^*(\mathcal{X}) = \frac{1}{2} \|\mathcal{X}\|^2_F, \quad \nabla f^*(\mathcal{X}) = \mathcal{X},
\]

and

\[
D_{f,\mathcal{X}}(\mathcal{X}, \mathcal{Y}) = \frac{1}{2} \|\mathcal{X} - \mathcal{Y}\|^2_F.
\]

As a consequence, Algorithm 2 becomes Algorithm 3.
As a consequence, Algorithm 2 becomes Algorithm 4.

...shrinkage function have been widely considered; see, e.g., [6, 17, 28, 7] and references therein. Define the soft

...shrinkage function. We have (see, e.g., [5])

\[ \text{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases} \]

Algorithm 3: The TREK algorithm for solving (25)

**Input:** \( \mathbf{A} \in \mathbb{R}^{N_1 \times N_2 \times N_3}, \mathbf{B} \in \mathbb{R}^{N_1 \times K \times N_3} \), stepsizes \( \alpha_r \) and \( \alpha_c \), maximum number of iterations \( M \), and tolerance \( \tau \).

**Initialize:** \( Z^{(0)} = B, X^{(0)} \in \text{range}_K(A^\top) \)

for \( k = 1, 2, \ldots, M \) do

Pick \( j_k \in [N_2] \) with probability \( \|A_{:,j_k,:}\|^2_F / \|A\|^2_F \)

Set \( Z^{(k)} = Z^{(k-1)} - \alpha_c A_{:,j_k,:} \cdot (A_{:,j_k,:})^\top Z^{(k-1)} / \|A_{:,j_k,:}\|^2_F \)

Pick \( i_k \in [N_1] \) with probability \( \|A_{i_k,:,:}\|^2_F / \|A\|^2_F \)

Set \( X^{(k+1)} = X^{(k)} - \delta \alpha_r (A_{i_k,:,:})^\top * (A_{i_k,:,:})^\top \cdot (Z^{(k)} - B) / \|A_{i_k,:,:}\|^2_F \)

Stop if \( \|X^{(k)} - X^{(k-1)}\|_F / \|X^{(k-1)}\|_F < \tau \)

end

For any \( X \in \text{range}_K(A^\top) \), we have

\[ D_f,\lambda(X, \hat{X}) = \frac{1}{2} \|X - \hat{X}\|_F^2 \leq \frac{1}{2\sigma_{\min}(\text{bcirc}(A))} \|A \cdot (X - \hat{X})\|_F^2. \]

Then, the constant \( \nu \) in (14) is \( \nu = 2\alpha^2_{\min}(\text{bcirc}(A)) \). In this setting, Theorem 3.2 reduces to the following result, which implies that the TREK algorithm converges linearly in expectation to the minimum Frobenius norm least squares solution of the tensor system \( A \cdot X = B \).

**Corollary 4.1.** Assume that \( 0 < \alpha_c < 2/\lambda_c \) and \( 0 < \alpha_r < 2/\lambda_r \). For any \( \delta > 0 \), the sequence \( \{X^{(k)}\} \) generated by Algorithm 3 satisfies

\[ \mathbb{E} \left[ \|X^{(k)} - A^\top \cdot B\|_F^2 \right] \leq \frac{\delta + 1}{\delta} \frac{\alpha_r^2 \lambda_r}{\|A\|^2_F} \|A \cdot B\|_F^2 \sum_{i=0}^{k-1} \rho_c^{k-i} \left( 1 + \delta \right)^i \rho_r^i \]

\[ + (1 + \delta)^k \rho_c^k \|X^{(0)} - A^\top \cdot B\|_F^2, \]

where

\[ \rho_c = 1 - \left( 2\alpha_c - \alpha_c^2 \lambda_c \right) \sigma_{\min}(\text{bcirc}(A)) / \|A\|^2_F, \quad \rho_r = 1 - \left( 2\alpha_r - \alpha_r^2 \lambda_r \right) \sigma_{\min}(\text{bcirc}(A)) / \|A\|^2_F. \]

### 4.2 RREK for sparse tensor recovery

In this subsection, we consider the following constrained minimization problem

\[ \hat{X} = \arg\min_{X \in \mathbb{R}^{N_2 \times K \times N_3}} \frac{1}{2} \|X\|_F^2 + \lambda \|X\|_1, \quad \text{s.t.} \quad A^\top \cdot A \cdot X = A^\top \cdot B. \]

We mention that minimization problems with the objective function

\[ f(X) = \frac{1}{2} \|X\|_F^2 + \lambda \|X\|_1 \]

have been widely considered; see, e.g., [6, 17, 28, 7] and references therein. Define the soft shrinkage function \( S_\lambda(X) \) componentwise as

\[ (S_\lambda(X))_{i,j,k} = \max\{|X_{i,j,k}| - \lambda, 0\} \cdot \text{sgn}(X_{i,j,k}), \]

where \( \text{sgn}(\cdot) \) is the sign function. We have (see, e.g., [5])

\[ \nabla f^*(X) = S_\lambda(X). \]

As a consequence, Algorithm 2 becomes Algorithm 4.
where (see section 2.4) for solving the tensor least squares problem (25). We compare the performance of the TREK algorithm (Algorithm 2) and the TRK algorithm.

### 5.1 Tensor least squares

The tensor t-product toolbox [18] is used in our computations. In all experiments, the reported results are the average of 10 Core i7 processor, 16 GB memory, and Mac operating system. The tensor t-product toolbox [18] is used in our computations. In all experiments, the reported results are the average of 10 experiments are performed using MATLAB R2020b on a laptop with 2.7 GHz Quad-Core Intel Core i7 processor, 16 GB memory, and Mac operating system. The tensor t-product toolbox [18] is used in our computations. In all experiments, the reported results are the average of 10 independent trials.

#### 5.1 Tensor least squares

We compare the performance of the TREK algorithm (Algorithm 2) and the TRK algorithm (see section 2.4) for solving the tensor least squares problem [25].
In our experiment, we generate the sensing tensor $A$ and the acquired measurement tensor $B$ as follows:

$$A = \text{randn}(N_1, N_2, N_3), \quad B = A \ast \text{randn}(N_2, K, N_3) + \text{randn}(N_1, K, N_3)/10.$$  

We set $N_1 = 100$, $N_2 = 20$, $N_3 = 20$, and $K = 20$. The maximum number of iterations $M$ is 1000. We use the stepsizes $\alpha_r = 1.5/\lambda_r$ and $\alpha_c = 1.5/\lambda_c$. In Figure 1, we plot the relative error versus the number of iterations. We observe (i) that the TRK algorithm converges linearly to within a radius of $A^\dagger \ast B$ and (ii) that the TREK algorithm converges linearly to $A^\dagger \ast B$.

![Figure 1: The relative error versus the number of iterations of the TREK and TRK algorithms for a tensor least squares problem.](image)

5.2 Sparse tensor recovery

We compare the performance of the RREK algorithm (Algorithm 3) and the RRK algorithm [7, Algorithm 3.2] for solving the sparse tensor recovery problem (26).

In our experiment, we generate the sensing tensor $A$, the ground truth tensor $X_s$, and the acquired measurement tensor $B$ as follows:

$$A = \text{randn}(N_1, N_2, N_3), \quad A(N_1 - 9 : N_1 - 5, :, :) = A(N_1 - 4 : N_1, :, :),$$  

$$X_s = \text{randn}(N_2, K, N_3), \quad X_s = X_s \ast (X_s > 2.33),$$  

$$Z = \text{null}(\text{bcirc}(A^\top)), \quad q = \text{size}(Z, 2), \quad B = A \ast X_s \ast \text{fold}(Z \ast \text{randn}(q, K))/10.$$  

We set $N_1 = 100$, $N_2 = 200$, $N_3 = 10$, and $K = 20$. The maximum number of iterations $M$ is 20000. We use the stepsizes $\alpha_r = 1.5/\lambda_r$ and $\alpha_c = 1.5/\lambda_c$. The ground truth $X_s$ is a sparse nonnegative tensor with approximately density 0.01 and smallest nonzero entry 2.33. In Figure 2 we plot the relative error versus the number of iterations. We observe (i) that the RRK algorithm converges linearly to within a radius of the ground truth $X_s$ and (ii) that the RREK algorithm converges linearly to the ground truth $X_s$.

6 Concluding remarks

We have proposed a randomized regularized extended Kaczmarz algorithm for solving the tensor recovery problem [2]. At each step, only one horizontal slice and one lateral slice of the sensing tensor $A$ are used. Linear convergence of the proposed algorithm is proved under certain
assumptions. Numerical experiments on a tensor least squares problem and a sparse tensor recovery problem confirm the theoretical results. In the future, we will use the existing acceleration strategies such as those in [25, 9, 4, 31] to further improve the efficiency when applied to large scale problems. Moreover, we note that the tensor recovery model [2] can be used as a variable selection procedure and we are studying its performance compared with the elastic net [34] and the lasso [30].

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