An impulse to the ground to end rolling with slipping

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Abstract
Several scenarios used to teach mechanics feature a rolling motion with slipping that transitions to one without slipping through friction with the ground. We present a compact approach for determining the final velocity in these scenarios: we summarise the transition by introducing an impulse that is transferred to the ground and account for it in the conservation of angular and linear momentum. In contrast to common approaches, we do not require any assumptions on the friction. Our approach thus serves to illustrate how using conservation laws and collisions allows to simplify problems by summarising complex interactions. We exemplify our approach with three scenarios: a sliding ball starting to roll, a turning wheel being released onto the ground, and a braking monowheel.

Keywords: angular momentum, rotations, rolling, collisions, friction

(Some figures may appear in colour only in the online journal)

1. Introduction
In teaching, scenarios involving rolling with slipping (RWS) are commonly employed to illustrate and apply concepts from the topics of rotation and friction [1–10]. These scenarios feature a roller, i.e., a rigid solid with a shape allowing it to roll on a plane. The roller is oriented such that it can roll, but its linear velocity $v$ and angular velocity $\omega$ do not meet the rolling condition $v = \omega r$, with $r$ being the roller’s radius. Classical examples are a bowling ball that has just been thrown onto the ground or a rotating wheel that is dropped on the ground. In consequence, kinetic friction acts on the roller until the rolling condition is fulfilled and it rolls without slipping. Recently, RWS experiments have gained attention as digital video-tracking techniques have made them accessible for teaching [5–8, 10].
Figure 1. A pushed roller. The vectors for the frictional force $F_f$ and the impulse $p$ are vertically dislocated for readability.

We here present an alternative approach to solving RWS problems using a collision-like framework. We suggest to contrast it with the common approach to RWS problems that involves calculating the frictional force and resulting torque, in order to illustrate collisions as a technique. Moreover, we demonstrate that our approach facilitates solving more complex problems, thus illustrating the benefit of rotational conservation laws. In this paper, we apply our approach and alternatives to three RWS problems and discuss their practical and didactic merits. In the appendix, we provide exercises that illustrate and employ the proposed approach.

2. Pushed roller

This is arguably the most popular RWS scenario and has been discussed in textbooks [4, example 9-19] and didactic publications [1, 2, 5, 9]: a roller with mass $m$, moment of inertia $I$ and radius $r$ is brought onto a horizontal plane such that it moves with velocity $v$ but does not rotate at all, i.e., $\omega = 0$ (see figure 1). The kinetic friction between roller and plane depends only on the normal force (and thus is constant). Rolling friction and air resistance shall be ignored. The goal is to determine the roller’s velocity $v'$ when it is not slipping anymore. Typical examples for this scenario are a bowling ball that has just been thrown onto a lane or a billiard ball that has just been struck.

2.1. Using the frictional force

We first solve this problem with the typical approach to RWS problems [1–4, 7–10], employing the constant kinetic frictional force $F_f$ (see figure 1): during the duration $t$ of the slipping phase, the roller is subject to a constant negative acceleration $a$ and thus its final velocity is:

$$ v' = v - at = v - \frac{F_ft}{m}. \quad (1) $$

Simultaneously, it is subject to a constant angular acceleration $\alpha$ on account of the torque $\tau$ caused by $F_f$, resulting in the final angular velocity:

$$ \omega' = \alpha t = \frac{\tau t}{I} = \frac{r F_f t}{I} \implies t = \frac{I \omega'}{r F_f}. \quad (2) $$
To determine $v'$, we insert equation (2) into equation (1) and use that, after the slipping phase, the rolling condition $v' = \omega' r$ must be met:

$$
\bar{v}' = v - \frac{F_f t}{m} = v - \frac{F_f I \omega'}{m r^2} = v - \frac{I v'}{m r^2} \Rightarrow \bar{v}' = v - \frac{v}{1 + \frac{r}{mr^2}} = \frac{mr^2}{mr^2 + I} v. \quad (3)
$$

This approach hinges on determining the duration $t$ of the slipping phase, which in turn was facilitated by our assumption that the frictional force $F_f$ is constant for the entire phase. Without this assumption, equations (2) and (1) would have to include time integrals over a time-dependent $F_f(t)$. However, the final state does not depend on the friction force at all. This suggests that we do not need to make any assumptions on the friction (except that it exists) and that we can treat the transition in a more abstract manner, namely as a collision.

2.2. Using an impulse to the ground

We summarise the entirety of the frictional forces during the slipping phase into an impulse $p$ which the roller exerts on the ground in direction of motion at the point of contact (see figure 1). (We could just as well work with the opposite impulse that the ground exerts on the roller, but consider this to be slightly less intuitive.) We account for the momentum lost through this impulse in the conservation of (linear) momentum:

$$
m v = mv' + p. \quad (4)
$$

Also, this impulse is solely responsible for the angular momentum gained by the roller:

$$
pr = I \omega'. \quad (5)
$$

The slipping phase ends once the rolling condition is met: $v' = \omega' r$. It is only at this point that we exploit the ground’s special properties by neglecting its change of velocity, which vanishes due to its enormous mass. Combining the rolling condition with equations (5) and (4), we obtain:

$$
m v = mv' + p = mv' + \frac{I \omega'}{r} = mv' + \frac{I v'}{r^2} \Rightarrow \bar{v}' = \frac{mv}{m + \frac{r}{pr}} = \frac{mr^2}{mr^2 + I} v, \quad (6)
$$

which agrees with equation (3).

This confirms that the outcome does not depend on the details of the friction: it can as well be non-linear or velocity-dependent. We can even replace the friction with a different interaction that results in the rolling condition to be satisfied. For instance in the example of the bowling ball, suppose the inside of the grip hole collides with a rigid obstacle in the ground. Alternatively introduce a small strong demon who is firmly connected to the ground at the initial position of the roller and briefly holds it until the rolling condition is met.

As the last examples are easier to regard as instantaneous and thus like a collision, they may serve as an accessible, illustrative, and memorable introduction of our impulse $p$. From there it is only a small step to realising that the interaction can extend over a longer period of time without affecting the outcome. It is also illustrative that we lose the capability of determining the duration of the slipping phase—we regard it as an instantaneous collision after all.
To see how our approach is connected to the previous one, we can generalise equations (1) and (2) using an integral over a time-dependent friction force $F_f$:

$$v' = v - \frac{1}{m} \int_0^t F_f(\tilde{t}) \, d\tilde{t} = v - \frac{1}{m} \int_0^t \alpha(\tilde{t}) \, d\tilde{t},$$

$$\omega' = \int_0^t \alpha(\tilde{t}) \, d\tilde{t} = \frac{r}{I} \int_0^t F_f(\tilde{t}) \, d\tilde{t}.$$  \hfill (7)

As the same unknown integral appears in both equations, we can eliminate it, arriving at equation (3) again. This unknown integral is our unknown impulse $p = \int_0^t F_f(\tilde{t}) \, d\tilde{t}$.

2.3. Side note: using an instantaneous axis and Steiner’s theorem

We briefly want to discuss another way to solve this problem using conservation laws: for a rolling motion without slipping, the roller must perform a rotation around a moving axis through its point of contact with the ground, usually named instantaneous axis. According to Steiner’s theorem, the roller’s moment of inertia with respect to this axis is $I' = I + mr^2$. The conservation of angular momentum around this axis yields:

$$mvr = I\omega' = (I + mr^2) \frac{v'}{r} \quad \implies \quad v' = \frac{mr^2}{I + mr^2} \cdot v.$$  \hfill (8)

While this approach is even more compact than the previous one, we do not consider it as accessible as it relies on the difficult concept of instantaneous axes [1] and the interaction with the ground has become completely implicit. Also, we consider it to require greater care when applying it to more complex problems such as the scenario described in section 4. However this approach has its merit by illustrating Steiner’s theorem when compared with the previous approach: the rotation around an arbitrary axis can be decomposed into a rotation around the centre of mass and the linear motion of that centre of mass.

3. Hitting the ground rotating

This scenario can be regarded the opposite to the previous one: a roller is released slightly above the ground, while already rotating around its centre with angular velocity $\omega$ (see figure 2). However, the centre itself does not move, i.e., $v = 0$. Again, rolling friction and air resistance shall be ignored and the goal is to determine the roller’s final velocity $v'$. This scenario has also been featured in didactic publications [8, 11], in particular as a problem where physical intuition fails both, students and professors [11]. These references feature solutions using the frictional force as well as one similar to section 2.3.

We again introduce an impulse $p$ that the roller exerts on the ground to accelerate. This time it is opposed to the direction of motion. The conservation of linear momentum gives us $p = mv'$ and the conservation of angular momentum gives us:

$$I\omega = I\omega' + pr = \frac{Iv'}{r} + mv'r \quad \implies \quad v' = \frac{I\omega r}{I + mr^2}.$$  \hfill (9)

Note that this can be regarded as a superelastic collision (with a coefficient of restitution $e = \frac{\omega}{\omega'} = \infty$), where rotational energy is translated to (linear) kinetic energy.

Our approach may help to build up some of the lacking intuition for this problem [11]: to achieve the rolling condition, a given amount of momentum (our impulse) needs to be
transferred to the ground. The amount of momentum depends only on the initial velocity and properties of the roller (I and r). However, how this momentum is transferred is secondary, like for a perfectly inelastic collision. The fact that the detail of the friction do not matter can also be illustrated by the case of a bouncing roller: it is only subject to horizontal and rotational acceleration when it is in contact with the ground and thus bouncing phases can be ignored when determining the final state.

3.1. Replacing the ground with a cart

So far, we only considered interactions of a roller with the ground and exploited the latter’s special properties, namely its practically infinite mass. To illustrate the effects of this, we consider a variant of the previous scenario, where the roller is not dropped on the ground, but a cart (figure 3). The cart has a mass $m_c$, a plain and even surface, and can move frictionless on the ground (the mass of its wheels shall be negligible). It is sufficiently large that the roller does not drop off its edge during the slipping phase. In the beginning the cart has a velocity $v_c = 0$.

We only need to take one difference to the previous scenario (equation (9)) into account: our rolling condition is not $\omega' r = v'$ anymore, as we have to account for the cart’s movement as well. Instead it is:

$$\omega' r = v' + v'_c = v' + \frac{p}{m_c} = v' + \frac{mv'}{m_c} = \left(1 + \frac{m}{m_c}\right) v'. \quad (10)$$
Figure 4. A driving monowheel brakes. The impulse $p$ is vertically dislocated for readability.

With this we can proceed in analogy to equation (9):

$$I \omega = I \omega' + pr = \frac{I}{r} \left( 1 + \frac{m_{c}}{m} \right) v' + m v' r \implies v' = \frac{I \omega r}{I \left( 1 + \frac{m_{c}}{m} \right) + mr^2}.$$  

(11)

In the limit $m_{c} \to \infty$, i.e., for the ground, this turns into equation (9) as expected.

4. Braking with a monowheel

A monowheel [12] is an exotic vehicle, which consists of two parts (see figure 4): a giant wheel (w) and an interior (i) containing the driver, motor, etc. Interior and wheel can be turned against each other by motor or muscle power to make the vehicle roll. During regular operation, the wheel rolls, but the interior barely rotates ($\omega = 0$). For those unfamiliar with monowheels, we recommend to watch them on video [e.g., 13, 14].

We here consider the scenario of a monowheel driving with a velocity $v$. The vehicle has a mass $m$ and radius $r$. The wheel’s moment of inertia is $I_w$ and that of the interior is $I_i$, both with respect to the centre. Now, the monowheel makes a hard brake, i.e., wheel and interior are locked to each other afterwards. It briefly skids and then continues rolling. The goal is to determine the final velocity $v'$ after this. A possible unbalance of the interior leading to a torque from gravity shall be ignored; this is justified if the braking process happens sufficiently quickly or we only care about the maximum velocity if the monowheel continues rolling (a phenomenon known as gerbilling). Once more, rolling friction and air resistance shall be ignored.

This scenario features two points of friction: between interior and wheel as well as between wheel and ground. As a result, there are two simultaneous processes going on: hitting the brakes leads to a rotational collision [15, 16] of the two components of the monowheel, which causes a mismatch in the rolling condition and thus a rolling-with-slipping problem. Still, our approach allows to handle both at once, without requiring assumptions about either friction.

We first note that the rolling condition is met before and after the braking, i.e., $v = \omega r$ and $v' = \omega' r$. With an impulse $p$ exerted on the ground by the monowheel, we obtain for the conservation of momentum:

$$mv = mv' + p,$$  

(12)
and for the conservation of angular momentum:

\[ I_w \omega = (I_w + I_i) \omega' - pr \]

\[ \Rightarrow I_w v \frac{r}{r} = (I_w + I_i) \frac{v'}{r} - (mv - mv') r \]

\[ \Rightarrow I_w v = (I_w + I_i) \frac{v'}{r} - mvr - mv' r^2 \]

\[ \Rightarrow v' = \frac{mr^2 + I_w}{mr^2 + I_w + I_i} \cdot v. \]  

(13)

5. Conclusion

We demonstrated how to determine the final state in problems involving RWS by introducing an impulse that is transferred to the ground. In contrast to the usual approach [1–4, 7–10], we did not need to make any assumptions on the frictional force and thus could show that the results are more general. Moreover, we showed that our approach allows to solve the more complex problem of the braking monowheel, which features two frictional interactions.

We propose to use our approach to demonstrate how conservation laws and treating complex events as collisions may ease solving problems, in particular in the context of rotations. Notably, comparing our approach with the one using the frictional force poses a rare opportunity to unravel an inelastic collision without relying on more advanced concepts. We are only aware of such examples for elastic collisions (e.g., the slingshot manoeuvre [4, problem 107]), while inelastic collisions can usually only be unravelled into a complex rheological problem that goes far beyond the scope of an introduction to mechanics. Moreover, it may be illustrative to contrast our approach with ones using instantaneous axis and Steiner’s theorem (section 2.3) or thermodynamics [2].

We therefore consider our approach a valuable complement to mechanics courses to illustrate and apply collisions and rotational conservation laws.

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Appendix A. Exercises

We here provide a sequence of two exercises surrounding the presented approach using the formerly discussed scenarios. We also provide TikZ sources of figures 1–4 as a supplement for easy re-use and adaption.

In the first exercise, the students are first guided through the explicit approach of section 2.1 and are then introduced to the alternative presented in section 2.2. In the second exercise, the students are assumed to be familiar with the approach and are challenged to apply it to the problem of section 4, which is embedded in other tasks revolving around the monowheel. Both exercises are designed to be solved in groups. A small exercise on the problem discussed in section 3 can be inserted in between.
A.1. Exercise 1: bowling ball

You throw a bowling ball on a lane. Immediately afterwards, it slides with a velocity $v = 5 \text{ m s}^{-1}$ but it does not rotate. Then, kinetic friction causes it to rotate until it rolls losslessly with a velocity $v'$. The goal of this exercise is to compute the final velocity $v'$ in two different ways. Assume the bowling ball to be a perfect homogeneous sphere with radius $r = 10 \text{ cm}$ and weight $m = 5 \text{ kg}$. The coefficient of kinetic friction between bowling ball and ground is $\mu = 0.2$.

(a) Compute the frictional force $F_f$ exerted on the bowling ball and the resulting acceleration $A_1$ as well as the angular acceleration $\alpha$.

(b) After what time $T$ does the bowling ball meet the rolling condition?

(c) What is the ball’s final velocity $v'$?

(d) The results of the previous tasks showed that the transition happens quickly and that the result is independent of $\mu$. This suggests that we can treat the entire process as a collision. Introduce an unknown impulse $p$ that summarises the interaction of the ball with the ground. It acts on the point of contact and in direction of rolling. Then use this impulse in the conservation of momentum and angular momentum and solve the resulting equations for $v'$.

A.2. Exercise 2: braking with a monowheel

Watch the videos in references [13] or [14] to familiarise yourself with the monowheel. Now consider a monowheel which consists of two parts:

- The wheel, which is a homogeneous cylindrical shell, with inner radius $r_2 = 145 \text{ cm}$, outer radius $r_1 = 150 \text{ cm}$, width $b = 100 \text{ cm}$ and density $\rho_1 = 4 \text{ g cm}^{-3}$.

- An interior containing the driver, motor, etc, which is simplified as a homogeneous quarter cylinder, with radius $r_2 = 145 \text{ cm}$, width $b$ and density $\rho_1 = 0.5 \text{ g cm}^{-3}$.

Ignore air resistance and rolling friction.

(a) Determine the wheel’s moment of inertia $I_w$ and that of the interior $I_i$, both with respect to the wheel’s centre.

(b) When driving with a velocity $v = 30 \text{ km h}^{-1}$, the monowheel makes a ‘hard brake’, such that the interior and wheel move as one. Show that the velocity after braking is:

$$v' = \frac{mr_1^2 + I_w}{mr_1^2 + I_w + I_i} v,$$

where $m$ is the mass of the vehicle. Compute $v'$.

(c) You want to design a monowheel that optimises the coefficient of restitution $e := \frac{v'}{v}$ when making a hard brake. You can arbitrarily distribute the monowheel’s mass to wheel and interior and also arbitrarily position it with respect to the centre. (The interior must not be outside the wheel though.) What is the lowest $e$ you can achieve?

(d) Is there a better way to halt a monowheel? Why is this a technical and safety challenge? Compare with a car.

1 Capitalised to avoid confusion with $\alpha$. 

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