MONOTONICITY AND KÄHLER-RICCI FLOW

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§0 Introduction.

In this paper, we shall give a geometric account of the linear trace Li-Yau-Hamilton (which will be abbreviated as LYH) inequality for the Kähler-Ricci flow proved by Luen-Fai Tam and the author in [NT1]. To put the result, especially the Liouville theorem for the plurisubharmonic functions, into the right perspective we would also describe some dualities existed in both linear and nonlinear analysis first. The purpose is to show the reader that the linear trace LYH for the Kähler-Ricci flow can be thought as a ‘global’ parabolic version of the classical monotonicity formula for the analytic hypersurfaces in $\mathbb{C}^m$ (or more generally for the positive $(1,1)$ currents). We also include some new results. For example the Harnack inequality for the nondivergent elliptic operator on complete Kähler manifolds with nonnegative holomorphic bisectional curvature was proved in Theorem 1.5. Another is the LYH inequality for the Hermitian-Einstein flow coupled with Kähler-Ricci flow (cf. Theorem 3.5). We also prove that the sufficient and necessary condition for the equality in the linear trace LYH inequality is that the solution is a Kähler-Ricci soliton. (It has no restriction on the tensor satisfying the linear Lichnerowicz-Laplacian heat equation.) See, Theorem 4.1 and Theorem 4.2. A direct consequence of these result is that type II (III) limit solutions of Ricci (Kähler-Ricci) flow are gradient (expanding) solitons (Kähler-Ricci solitons). Theorem 4.1 and Theorem 4.2 generalizes and unifies the previous theorems (cf. [H3], [Ca2], [C-Z]) of Hamilton, Cao and more recently Chen-Zhu on the limit solutions to the Ricci flow considerably.

There are monotonicity formulae for the parabolic equations such as the harmonic map heat equation and mean curvature flow (cf. [Hu], [St] and [H2]). But in the author’s point of view they all are still ‘local’ in the sense that the precise monotonicity only holds for (or inside) locally symmetric manifolds. Another important distinction is that the ‘global monotonicity’ (which holds on complete Kähler manifolds with nonnegative bisectional curvature) derived from the LYH inequality here is a point-wise estimate instead of the monotonicity of an integral quantity as in the previous mentioned cases such as those in [St] and [Hu]. There have been some other geometric interpretations on LYH inequality proved by Hamilton, for example in [C-C1], as well as on the Chow-Hamilton’s linear trace

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LYH in [C-C2]. This observation here relates the minimal submanifold theory, in particular
the monotonicity of the volume, to the LYH inequality for Ricci flow, at least for complex
analytic case, for the first time. Hopefully this will introduce further work along this
direction and bring out more understanding and results to the study of Ricci flow. Let us
start with a duality for linear elliptic PDE, which the author learned from Professor Peter
Li during his graduate study at UCI back in 1997-1998 (cf. [L-W] for more works in this
direction). This duality is hinged on the Harnack inequality.

§1 Harnack inequality.

On \( \mathbb{R}^n \), the celebrated De Giorgi-Nash Moser theory proves the Harnack inequality for
the uniformly elliptic operator of divergence form.

**Theorem 1.1.** Let \( L = \sum_{ij} \partial_i (a_{ij}(x) \partial_j) \) be a uniformly elliptic operator on \( \mathbb{R}^n \) with
\( \lambda I \leq (a_{ij}(x)) \leq \Lambda I \). Here constants \( \Lambda \geq \lambda > 0 \). Let \( u \geq 0 \) be a (local) \( W^{1,2} \)-weak solution
to \( Lu = 0 \) in \( B_o(2R) \). Then there exists a constant \( C = C(n, \frac{\Lambda}{\lambda}) > 0 \) such that

\[
\sup_{B_o(R)} u \leq C \inf_{B_o(R)} u.
\]

The duality provided by the Harnack inequality is between the local \( C^\alpha \)-regularity of
any \( W^{1,2} \)-solution and the Liouville theorem for the positive solution of \( Lu = 0 \). As it is
well-known (cf. [G-T]) that applying (1.1) to balls with small radius \( R \) implies that any
\( W^{1,2} \) solution to \( Lu = 0 \) is \( C^\alpha \). It is also easy to see that if \( u \geq 0 \) is a solution of \( Lu = 0 \)
on \( \mathbb{R}^n \) then \( u \) must be a constant. This can be seen by applying (1.1) to \( u - \inf_{\mathbb{R}^n} u \) on
\( B_o(\bar{R}) \) and letting \( \bar{R} \to \infty \).

When study the analysis on general manifolds, since any manifold locally looks Eu-
clidean and the metric is quasi-isometric to the Euclidean one the \( C^\alpha \)-regularity still holds
for any weak solution. One can also have the Harnack. However, the Harnack estimating
constant \( C = C(n, \frac{\Lambda}{\lambda}) \) depends on the local geometry of \( M \), namely the metric tensor \( g_{ij} \),
more precisely the quotient of the maximum eigenvalue over the minimal eigenvalue, on
local coordinate charts. This is a serious drawbacks since in order to study the nonlinear
problems one also wish to obtain the Harnack inequality with the estimating constant de-
pending only on some global geometric-analytic information such as the lower bound of
the curvature or the the behavior of the volume functional, etc. In a certain sense, one
should think of the metric \( g_{ij} \) as being unknown in details, but satisfying certain geometric
(coordinate invariant) conditions. The corresponding theory on general manifolds starts
with the work of Bombieri and Giusti [B-G], where they carried out the geometric analysis
of Moser’s proofs to minimizing submanifolds and proved the Harnack inequality for the
uniformly elliptic operator of divergence form on such manifolds. Later in [Y], a gradi-
ent estimate method was initiated and a similar Harnack inequality as in Theorem 1.1
was derived out of a gradient estimate for positive harmonic functions on manifolds with
nonnegative Ricci curvature. In particular, in [C-Y], the following theorem on harmonic
functions was proved.

**Theorem 1.2.** Let \( M \) be a complete Riemannian manifold with nonnegative Ricci curva-
ture. Let \( u \) be a harmonic function on \( B_o(2\bar{R}) \). Then there exists a constant \( C = C(n) > 0 \)
such that
\[
\sup_{B_o(R)} |\nabla u|^2 \leq \frac{C(n)}{R^2} \sup_{B_o(2R)} u^2.
\]

In particular, any global harmonic function (defined on \(M\)) of the sublinear growth is a constant.

The result was later also established for general uniformly elliptic operator of divergence form in [Gr] and [Sa] independently. In fact, they proved the Harnack for the corresponding parabolic equations under some geometric assumptions which are than the nonnegativity of the Ricci curvature. For the sake of this paper we just state their result in the following weaker form for the elliptic operators.

**Theorem 1.3.** Let \(M\) be a complete Riemannian manifold with nonnegative Ricci curvature. Let \(L\) be a uniformly elliptic operator of divergence form. Let \(u \geq 0\) be a solution to \(Lu = 0\) on \(B_o(2R)\). Then there exists a constant \(C = C(n, L)\) such that
\[
(1.3) \quad \sup_{B_o(R)} u \leq C \inf_{B_o(R)} u.
\]

In particular, any positive global \(L\)-harmonic function is a constant.

There are also many developments in establishing the Harnack inequality for the uniform elliptic operator of nondivergence form. The theory for the Euclidean (local) case was developed by Krylov and Safonov based on the Alexanderoff maximum principle (cf. [G-T], Chapter 8 for the detailed theory). We just want to mention a corresponding global result for the manifolds with nonnegative Riemannian sectional curvature which was proved by Cabré [C] more recently.

**Theorem 1.4.** Let \(M\) be a complete Riemannian manifold with nonnegative sectional curvature. Let \(L\) be a uniformly elliptic operator of nondivergence form. Then (1.3) still holds for nonnegative solution \(u\) on \(B_o(2R)\).

Using a comparison theorem established in [C-N], we shall extend the above result to the complete Kähler manifolds with nonnegative holomorphic bisectional curvature. More precisely, let \(L\) be a elliptic operator which locally can be written as \(L = \sum a^{\alpha \bar{\beta}}(z) \partial_{z_\alpha \bar{z}_\beta}\) such that there exists \(\Lambda \geq \lambda > 0\) with \(\Lambda |\eta|^2 \geq a^{\alpha \bar{\beta}}(z) \eta_\alpha \bar{\eta}_\beta \geq \lambda |\eta|^2\) for any \(\eta = \eta_\alpha dz^\alpha \in T^*_z M\) over any \(z \in M\). The following result can be viewed as a generalization of Theorem 1.4.

**Theorem 1.5.** Let \(M\) be a complete Kähler manifold (of complex dimension \(m\)) with nonnegative holomorphic bisectional curvature. Let \(u\) be a smooth function in \(B_o(2R)\) satisfying \(u \geq 0\) in \(B_o(2R)\). Then
\[
(1.6) \quad \sup_{B_o(R)} u \leq C \left\{ \inf_{B_o(R)} u + \frac{R^2}{V_o(2R)^m} \|Lu\|_{L^m(B_o(2R))} \right\}
\]

where \(C = C(m, \lambda, \Lambda) > 0\). In particular, (1.3) holds for nonnegative solution to \(Lu = 0\) and any positive solution \(u\) of \(Lu = 0\) is a constant.

As in [C] the key is to prove the following proposition.
**Proposition 1.1.** Let $u$ be a smooth function in a ball $B_o(7R)$ satisfying $u \geq 0$ in $B_o(7R) \setminus B_o(5R)$ and $\inf_{B_o(2R)} u \leq 1$. Then

\begin{equation}
V_o(R) \leq \frac{(32)^m}{\lambda^{2m}} \int_{\{u \leq 6\} \cap B_o(5R)} \left\{ \left( \frac{1}{2m} R^2 Lu + \frac{A}{4} \right) + \right\}^{2m}.
\end{equation}

Once the above result holds one can use the argument of section 5-7 in [C], which only uses the nonnegativity of Ricci curvature, to obtain Theorem 1.5. To prove the above proposition we need the following two results.

**Proposition 1.2.** For any $y \in M$, let $d_y(x)$ be the distance function from $x$ to $y$. Then

\begin{equation}
(d_y^2)_{\alpha \beta} \leq g_{\alpha \beta}
\end{equation}

in the distribution sense. In particular, if $x$ does not lie inside the cut locus of $y$ the above holds point-wisely. Here $g_{\alpha \beta}$ is the Kähler metric tensor.

See [C-N] for the proof of this comparison result. The following lemma is from [C].

**Lemma 1.1.** Let $M$ be a Riemannian manifold. Let $v$ be a smooth function on $\Omega \subset M$. Consider the map $\phi : \Omega \to M$ defined by

$$
\phi(z) = \exp_z \nabla v(z).
$$

Let $x \in \Omega$ and suppose that $\nabla v \in U_x$. Here $U_x = \{ t\eta : \eta \in T_x M, |\eta| = 1, 0 \leq t \leq c(\eta) \}$, where $c(\eta) = \sup \{ t > 0 : \exp_x s\eta \text{ is minimizing geodesic on } [0, t] \} \leq \infty$. Set $y = \phi(x)$. Then

$$
\text{Jac}(\phi(x)) = \text{Jac}(\exp_x(\nabla v(x))) \cdot |\det(D^2(v + \frac{d_y^2}{2}))(x)|,
$$

where $\text{Jac}(\exp_x(\nabla v(x)))$ denotes the Jacobian of the exponential map at the point $\nabla v(x) \in T_x M$.

This is just the Lemma 3.2 from [C]. The rest will be devoted to the proof of Proposition 1.1. The argument is just an adaptation of the one given in [C] to the case with only control on the complex Hessian of the distance function. The key is the following linear algebra lemma

**Lemma 1.2.** Let $A$ be a $2m \times 2m$ positive semi-definite symmetric matrix with the form

$$
A = \begin{pmatrix}
A_{11}, & A_{12}, & \cdots, & A_{1n} \\
A_{21}, & A_{22}, & \cdots, & A_{2n} \\
\vdots, & \vdots, & \ddots, & \vdots \\
A_{n1}, & A_{n2}, & \cdots, & A_{nn}
\end{pmatrix}
$$

where $A_{ij}$ are $2 \times 2$ matrix. Then

$$
\det(A) \leq \prod_i \det(A_{ii}).
$$
In particular, if $f$ is a real valued function defined on $\Omega \subset \mathbb{C}^m$ and assumes its local minimum at a point $x \in \Omega$. Then

$$\det(D^2 f)(x) \leq 8^m |\det(f_{\alpha \bar{\beta}})|^2(x).$$

Here $D^2 f$ is the real Hessian and $f_{\alpha \bar{\beta}}$ is the complex Hessian.

**Proof.** By perturbation, one can assume that $A$ is positive definite. Then the statement can be proved by perform the elementary row and column operation and induction. We leave the details to the interested reader. To prove the second part, we can choose a coordinate such that $f_{\alpha \bar{\beta}}$ is diagonal. Then

$$|\det(f_{\alpha \bar{\beta}})|^2 = \left(\frac{1}{16}\right)^m \Pi_i (f_{x_i x_i} + f_{y_i y_i})^2.$$  

Since $D^2 f$ is positive semi-definite, we have that

$$(f_{x_i x_i} + f_{y_i y_i})^2 \geq (f_{x_i x_i}^2 + f_{y_i y_i}^2) \geq 2(f_{x_i x_i} f_{y_i y_i} - f_{x_i y_i}^2).$$

Now the result follows by applying the first part of the lemma.

**Proof of Proposition 1.1.** For the sake of the completeness we start with the argument from [C]. Let $w_y(x) = R^2 u(x) + \frac{1}{2} d_y^2(x)$, for $y \in B_o(R)$. Clearly we have that $\sup_{B_o(2R)} w_y \leq R^2 + \frac{(3R)^2}{2} = \frac{11}{2} R^2$. But in $B_o(7R) \setminus B_o(5R)$, we have $w_y \geq \frac{(4R)^2}{2} = 8R^2 \geq \frac{11}{2} R^2$. Therefore, the minimum of $w_y$ achieves inside $B_o(5R)$. Namely,

$$\inf_{B_o(7R)} w_y(z) = w_y(x)$$

for some $x \in B_o(5R)$. As in [C] we know that at such point $x$ we have that

$$y = \exp_x(\nabla (R^2 u)(x)).$$

Now define the map $\phi(z) = \exp_z(\nabla (R^2 u)(z))$ for all $z \in B_o(7R)$. We also define the measurable set $E = \{x \in B_o(5R) : \text{there exists } y \in B_o(R) \text{ such that } w_y(x) = \inf_{B_o(7R)} w_y\}$. What we have shown is that for any $y \in B_o(R)$ there exists such $x \in E$ such that $\phi(x) = y$. Therefore we have that

$$V_o(R) \leq \int_E \text{Jac}(\phi)dv.$$  

Also we know for $x \in E$, $w_y(x) \leq \frac{11}{2} R^2$. Therefore, $u \leq \frac{11}{2}$. This implies that $E \subset \{u \leq 6\} \cap B_o(5R)$. Therefore we further have

$$V_o(R) \leq \int_{\{u \leq 6\} \cap B_o(5R)} \text{Jac}(\phi)dv.$$
The rest is to estimate $\text{Jac}(\phi)$. The approximation argument in [C] shows that we can apply Lemma 1.1 without the loss of the generality even $\nabla (R^2u)(x)$ might not in $U_x$. Hence

\begin{equation}
\text{Jac}(\phi) = \text{Jac}(\exp_x)(R^2 \nabla u) | \det (D^2 (R^2u + \frac{1}{2}d_y^2))(x)
\end{equation}

\begin{equation}
\leq | \det (D^2 (R^2u + \frac{1}{2}d_y^2))(x).
\end{equation}

Here we have used the fact that $\exp$ is volume decreasing if $\text{Ricci}(M) \geq 0$. Now we applying Lemma 1.2 and Proposition 1.2 to estimate $| \det (D^2 (R^2u + \frac{1}{2}d_y^2))(x)$ for $x \in E$.

\begin{equation}
| \det (D^2 (R^2u + \frac{1}{2}d_y^2))(x) \leq 8^m | \det \left( \left( R^2u + \frac{1}{2}d_y^2 \right) \right) |^2
\end{equation}

\begin{equation}
\leq \frac{8^m}{\lambda^2m} | \det \left( \left( R^2u + \frac{1}{2}d_y^2 \right) \right) |^2 | \det (\alpha^\alpha \bar{\alpha})|^2
\end{equation}

\begin{equation}
\leq \frac{8^m}{\lambda^2m} \left\{ \frac{1}{m} \sum_{\alpha \beta} \left( R^2 \alpha \bar{\alpha} u_{\alpha \bar{\alpha}} + \frac{1}{2} \alpha \bar{\alpha} (d^2)_{\alpha \bar{\alpha}} \right) \right\}^{2m}
\end{equation}

\begin{equation}
\leq \frac{8^m}{\lambda^2m} \left\{ \frac{1}{m} R^2 Lu + \frac{1}{2} \Lambda \right\}^{2m}
\end{equation}

\begin{equation}
= \frac{(32)^m}{\lambda^2m} \left\{ \frac{1}{2m} R^2 Lu + \frac{1}{4} \right\}^{2m}
\end{equation}

Noticing that for $x \in E$, $\frac{1}{2m} R^2 Lu + \frac{1}{4} \geq 0$, therefore combining (1.9) and (1.10) we have that

\begin{equation}
\int_E \text{Jac}(\phi) dv \leq \frac{(32)^m}{\lambda^2m} \int_E \left\{ \frac{1}{2m} R^2 Lu + \frac{1}{4} \right\}^{2m} dv
\end{equation}

\begin{equation}
\leq \frac{(32)^m}{\lambda^2m} \int_{\{u \leq 6\} \cap B_\rho(5R)} \left\{ \left( \frac{1}{2m} R^2 Lu + \frac{1}{4} \right)^+ \right\}^{2m} dv.
\end{equation}

This completes the proof of Proposition 1.1.

We close the section by pointing out that establishing a similar result for the uniformly elliptic operator of nondivergent form on manifolds with nonnegative Ricci curvature is an interesting problem.

\section{Monotonicity formulae.}

Here we shall demonstrate a nonlinear analogy of the duality provided through the Harnack inequality. In this case, the hinge is the monotonicity formulae. There are various types of monotonicity formulae arising in different geometric-analytical problems. We mainly focus on the one on harmonic maps as well as the one on minimal submanifolds and its slightly metamorphose on the positive current, which is the content of the well-known Bishop-Lelong Lemma in several complex variables. Let us start with the harmonic maps. We refer [Sc] for the notations and proofs.
Proposition 2.1. Let $u$ be a $W^{1,2}$-stationary map defined on $B_o(R) \subset \mathbb{R}^n$ into a Riemannian manifold $N \subset \mathbb{R}^K$. Then for any point $x \in B_o(R)$ and $r > 0$ such that $B_x(r) \subset B_o(R)$ we have that

$$I(x, r) = \frac{1}{r^{n-2}} \int_{B_x(r)} |Du|^2 \, dx$$

is monotone increasing in $r$. Here $|Du|^2 = \sum_{i\alpha} \left( \frac{\partial u}{\partial x^i} \right)^2$, with $u = (u^1, \ldots, u^K) \in N \subset \mathbb{R}^K$. Moreover we have

$$\frac{\partial}{\partial \sigma} I(x, \sigma) = 2 \int_{\partial B_x(\sigma)} \sigma^{2-n} |\frac{\partial u}{\partial r}|^2 \, dA.$$

The above monotonicity plays an important role in the regularity theory of the harmonic maps (cf. [S-U]). However it is not an easy matter as the linear case indicated in the last section. Substantial work are required. One can refer [Si2] for an updated proof of the $\epsilon$-regularity theorem of Schoen-Uhlenbeck on the minimizing maps.

On the other hand, analogous to the duality in the last section, the monotonicity formula (2.2) also plays the crucial role in various Liouville type results on harmonic maps. For instance, there is a result due to Garber, Ruijsenaars, Siler and Burns [G-R-S-B] saying that any harmonic maps $u : \mathbb{R}^n \to S^m$ with finite energy is a constant map for $n > 3$, which is an easy consequence of the above proposition. In [J], the author proved that any harmonic map $u : \mathbb{R}^n \to N$ with $\lim_{x \to \infty} u(x) = p_0$, a fixed point in $N$, must be constant map. The proof again relies crucially on Proposition 2.1. The interested readers can refer the above mentioned papers for the detailed statement of the Liouville theorems and their proofs. We are not going to pursue further on the monotonicity of the harmonic maps here and just want to mention that it would be interesting to derive some sharp ‘global’ version of Proposition 2.1, namely to find certain monotonicity which holds for a class of domain manifolds such as the manifolds with nonnegative Ricci curvature. One expects that such a result would be very useful in establishing and clarifying the Liouville type results for the harmonic maps.

Now we want to turn our focus to the monotonicity formula for the minimal submanifolds in $\mathbb{R}^n$ and its slight variation, the monotonicity formula for positive currents. This will be shown to be directly related to the LYH inequality derived for the Kähler-Ricci flow in [N-T1] in the next section.

Let $M^n$ be a minimal submanifold in $\mathbb{R}^N$. The following monotonicity formula is well-known.

Proposition 2.2. Let $\Theta(x, r)$ be the density function defined as

$$\Theta(x, r) = \frac{1}{\omega_n r^n} \int_{B_x(r)} vol_M = \frac{Vol(M \cap B_x(r))}{\omega_n r^n}.$$

Then for $r \leq R$ such that $B_x(R) \cap \partial M = \emptyset$, $\Theta(x, r)$ is monotone increasing. Moreover, one has

$$\frac{\partial}{\partial \sigma} \Theta(x, \sigma) = \int_{\partial B_x(\sigma)} \frac{|D^\perp r|^2}{\omega_n r^n} \, dA.$$
Here $\omega$ is the area of the unit sphere of dimension $n - 1$, $D^\perp r$ is the normal part of $Dr$.

As in [Si1], the above result also holds for stationary varifolds and it is crucial in the regularity result such as the Allard regularity theorem (cf. [Si1]). Since complex subvarieties in $\mathbb{C}^m$ are area minimizing, the above result holds for the complex subvarieties. In fact, there are slightly more general

**Bishop-Lelong Lemma.** Let $T$ be a $(p, p)$ positive current in $\mathbb{C}^m$. Define

\begin{equation}
\Theta(x, r) = \frac{1}{r^{2m-2p}} \int_{B_x(r)} T \wedge \left( \frac{1}{\pi} \omega_{\mathbb{C}^m} \right)^{m-p}.
\end{equation}

Here $\omega_{\mathbb{C}^m}$ is the Kähler form of $\mathbb{C}^m$. Then

\begin{equation}
\frac{\partial}{\partial r} \Theta(x, r) \geq 0.
\end{equation}

One can refer [G-H] (pages 390-391) for a proof of this lemma. Analogous to the previous case we would like to relate the above monotonicity to a Liouville theorem on plurisubharmonic functions on $\mathbb{C}^m$.

**Proposition 2.3.** Let $u$ be a plurisubharmonic function defined on $\mathbb{C}^m$. If

\begin{equation}
\lim_{x \to \infty} \frac{u(x)}{\log r(x)} = 0
\end{equation}

$u$ must be a constant.

**Proof.** The easiest proof is to restrict $u$ to $\mathbb{C}$ and apply the 2-dimensional result for the subharmonic functions. But we would like to prove it using (2.6). The plurisubharmonicity of $u$ is equivalent to that $\sqrt{-1} \partial \overline{\partial} u$ is a positive $(1, 1)$ current. Define

\[ M(x, r) = \frac{1}{A(x, r)} \int_{B_x(r)} u \, dA. \]

Here $A(x, r)$ is the area of $\partial B_x(r)$. Direct calculation (cf. [Ho], Lemma 4.4.9) shows

\[ r \frac{\partial}{\partial r} M(x, r) = \Theta(x, r). \]

Namely

\[ \frac{\partial}{\partial \log r} M(x, r) = \Theta(x, r). \]

Then (2.6) just implies that

\[ \frac{\partial^2}{\partial t^2} M(x, r) \geq 0 \]

where $t = \log r$. Namely (2.6) is equivalent to the fact that $M(x, r)$ is a convex function of $\log r$. Hence, under the assumption (2.7), we know that $M(x, r)$ must be constant. It then implies that $\Delta u \equiv 0$ since

\[ \Theta(x, r) = \frac{1}{r^{2m-2}} \int_{B_x(r)} \left( \frac{1}{2} \Delta u \right) \left( \frac{1}{\pi} \omega_{\mathbb{C}^m} \right)^m \equiv 0. \]

The result then follows from Theorem 1.2.

To prove a similar result for the plurisubharmonic functions on general complete Kähler manifolds motivates the LYH inequality in the next section.
§3 Linear trace Li-Yau-Hamilton inequality. In this section we try extend the monotonicity formula (2.6) to the general complete Kähler manifolds with nonnegative holomorphic bisectional curvature. In fact, we shall show that a linear trace LYH inequality for Kähler-Ricci flow would be a parabolic version of (2.6). And this parabolic monotonicity formula is enough to prove Liouville theorem for the plurisubharmonic functions in some cases. Let us start with the trace version of Hamilton’s LYH inequality on the Ricci flow.

**Theorem 3.1.** Let \((M, g_{ij}(x, t))\) be a complete solution to the Ricci flow

\[
\frac{\partial}{\partial t} g_{ij}(x, t) = -2R_{ij}(x, t)
\]

with bounded sectional curvature and nonnegative curvature operator on \(M \times (0, T)\). Then

\[
\frac{\partial R}{\partial t} + 2\nabla R \cdot V + 2R_{ij}V_iV_j + \frac{R}{t} \geq 0
\]

for any vector field \(V\).

In Kähler category, Cao has the following

**Theorem 3.2.** Let \((M, g_{\alpha\overline{\beta}}(x, t))\) be a complete solution to the Kähler-Ricci flow

\[
\frac{\partial}{\partial t} g_{\alpha\overline{\beta}}(x, t) = -R_{\alpha\overline{\beta}}(x, t)
\]

with bounded nonnegative holomorphic bisectional curvature on \(M \times [0, T)\). Then

\[
\frac{\partial R}{\partial t} + \nabla_{\overline{\alpha}} RV_\alpha + \nabla_\alpha RV_{\overline{\alpha}} + R_{\alpha\overline{\beta}} V_\alpha V_\overline{\beta} + \frac{R}{t} \geq 0.
\]

Again we only state the trace version of Cao’s result. Later Chow-Hamilton extends Theorem 3.1 to the following more general case.

**Theorem 3.3.** Let \((M, g(t))\) be a complete solution to the Ricci flow (3.1) and let \(h\) be a symmetric 2-tensor satisfying

\[
\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} + 2R_{piqj}h_{pq} - R_{ip}h_{jp} - R_{jp}h_{ip}.
\]

Assume that the metric initially has bounded nonnegative curvature operator (which is preserved under the flow). If on addition, \(h_{ij} > 0\) initially \((h_{ij}(x, t)) > 0\) as long as the solution exists and for any vector field \(V\)

\[
Q(x, t) := \text{div}(\text{div}(h)) + R_{ij}h_{ij} + 2\text{div}(h) \cdot V + h_{ij}V_iV_j + \frac{H}{2t} > 0.
\]

Here \(H = g^{ij}h_{ij}\).
Remarks. i) We should point out in general, one does require some growth conditions on \( h_{ij}(x,t) \) to apply the maximum principle on tensor to get \( h_{ij}(x,t) > 0 \). This seems missing in the statement of Theorem 3.3. ii) Assume that the flow \((M, g(t))\) and the \( h_{ij} \) are only defined on \( M \times (0,T) \). Also assume that the curvature operator of \( g(t) \) and \( \| h_{ij}(x,t) \| \) are uniformly bounded on \( M \) for any \( t \). Then the strict inequality in (3.6) will be replaced with an inequality. The examples are the expanding solitons for the Ricci flow. iii) A perturbation on \( h_{ij} \) could allow the case \( h_{ij}(x,t) \geq 0 \).

In [NT1] the authors proved the Kähler version of Theorem 3.3.

**Theorem 3.4.** Let \((M, g(t))\) be a complete solution to the Kähler-Ricci flow (3.3) on \( M \times [0,T) \). Let \( h_{\alpha\overline{\beta}}(x,t) \) be a Hermitian symmetric 2-tensor satisfying

\[
(3.7) \quad \left( \frac{\partial}{\partial t} - \Delta \right) h_{\gamma\delta} = R_{\beta\overline{\gamma}\gamma\delta} h_{\alpha\overline{\beta}} - \frac{1}{2} (R_{\gamma\overline{p}\beta} h_{p\delta} + R_{p\delta\beta} h_{\gamma\overline{p}})
\]

also on \( M \times [0,T) \). Assume that \((M, g(0))\) has bounded nonnegative holomorphic bisectional curvature. Also assume that \((h_{\alpha\overline{\beta}}(x,0)) \geq 0 \) and \( \| h_{\alpha\overline{\beta}} \|(x,0) \) is uniformly bounded on \( M \). Then \((h_{\alpha\overline{\beta}}(x,t)) \geq 0 \) and \( Z(x,t) \geq 0 \). Here \( Z(x,t) \) is given by

\[
(3.8) \quad Z = g^{\alpha\overline{\beta}} g^{\gamma\delta} \left[ \frac{1}{2} (\nabla_{\beta} \nabla_\gamma + \nabla_{\overline{\gamma}} \nabla_{\overline{\beta}}) h_{\alpha\overline{\delta}} + R_{\alpha\overline{\delta} \gamma\beta} h_{\gamma\delta} + (\nabla_\gamma h_{\alpha\overline{\delta}} V_{\overline{\beta}} + \nabla_{\overline{\beta}} h_{\alpha\overline{\delta}} V_{\gamma}) + h_{\alpha\overline{\delta}} V_{\overline{\beta}} V_{\gamma} \right] + \frac{H}{t}.
\]

If the \((M, g(t))\) and \( h(x,t) \) are both ancient in the sense that they exists on \( M \times (-\infty, T) \). Then we have \( \dot{Z} \geq 0 \), where \( \dot{Z} = Z - \frac{H}{t} \).

**Remark.** The inequality \( Z(x,t) \geq 0 \) also holds if we only assume both the metrics \( g(t) \) and the symmetric tensor \( h_{\alpha\overline{\beta}} \) are only defined on \( M \times (0,T) \) provided that the curvature tensor of \((M, g(t))\) and \( \| h_{\alpha\overline{\beta}} \|(x,t) \) uniformly bounded on \( M \) for each \( t \).

The Theorem 3.4 is an extension of Theorem 3.2 since in the case \( h_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}}(x,t) \), the Ricci tensor, (3.8) simplifies to (3.4). The same for the real case. In [C-H], (3.6) was called linear trace Harnack inequality. Since we have adapted the notion Li-Yau-Hamilton inequality in [N-T1] due to the reasons explained therein we call (3.8) linear trace LYH inequality.

Before we explain how one can derive a parabolic monotonicity formula out of Theorem 3.4 we would like to indicated cases when \( h_{\alpha\overline{\beta}} \) satisfying (3.7) does arise except the above mentioned obvious case that \( h_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}} \). Let \((M, g(t))\) be as in Theorem 3.4. Let \((E,H)\) be a holomorphic vector bundle on \( M \) with Hermitian metric \( H(x,t) \) deformed by the Hermitian-Einstein flow:

\[
(3.9) \quad \frac{\partial H}{\partial t} H^{-1} = -\Lambda F_H + \lambda I.
\]

Here \( \Lambda \) means the contraction by the Kähler form \( \omega_t \), \( \lambda \) is a constant, which is a holomorphic invariant in the case \( M \) is compact, and \( F_H \) is the curvature of the metric \( H \), which locally
can be written as \( F^j_{i\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta e_i^* \otimes e_j \) with \( \{e_i\} \) a local frame for \( E \). The transition rule for \( H \) under the frame change is \( H^U_{ij} = f_i^k f_j^l H^V_{kl} \) with transition functions \( f_i^j \) satisfying \( e_i^U = f_i^j e_j^V \). Denote \( \Omega_{\alpha\beta} = \sum_i F^i_{i\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta \). One can associate \( E \) the determinant line bundle \((L, h)\). Since the transition functions for \( L \) is just \( \det(f_i^j) \) we have that \( \det(H^U) = |\det(f_i^j)|^2 \det(H^V) \). Namely \( \det(H) \) is the naturally induced Hermitian metric on \( L \). Using the formula
\[
F_H = \bar{\partial}(\partial HH^{-1})
\]
one can easily see that \( \Omega_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta \) is the first Chern form of \((L, \det(H))\). Namely
\[
\Omega_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta = -\bar{\partial}\partial \log(\det(H)).
\]
In this case, Theorem 3.4 can be applied to obtain the gradient estimates for \( \Omega = g^{\alpha\beta} \Omega_{\alpha\beta} \) if \((E, H)\) is a solution of (3.9).

**Theorem 3.5.** Assume that \( M \) is a compact Kähler manifold. Let \((M, g(t))\) be a solution to the Kähler-Ricci flow (3.3) with nonnegative holomorphic bisectional curvature, and let \((E, H)\) be a solution to (3.9). Then \( \Omega_{\alpha\beta}(x, t) \) satisfies (3.7) and if \( \Omega_{\alpha\beta}(x, 0) \geq 0 \) initially, then \( \Omega_{\alpha\beta}(x, t) \geq 0 \) for all \( t > 0 \). Moreover, for any vector field \( V \),
\[
(3.10) \quad \Omega_t + \nabla_\alpha \nabla V_\alpha + \nabla_{\bar{\alpha}} \nabla V_{\bar{\alpha}} + \Omega_{\alpha\beta} V_\alpha V_{\bar{\beta}} + \frac{\Omega}{t} \geq 0.
\]
Here \( \Omega(x, t) = \sum g^{\alpha\beta}(x, t) \Omega_{\alpha\bar{\beta}}(x, t) \).

**Proof.** Let us denote \( \det(H(x, t)) \) by \( \eta(x, t) \). As we pointed out that \( \eta(x, t) \) is the induced Hermitian metric on \( L \). Define
\[
u(x, t) = \log \frac{\eta(x, t)}{\eta(x, 0)}.
\]
Taking trace on (3.9) then implies that
\[
\left( \frac{\partial}{\partial t} H^k_i(x, t) \right) (H^{-1})_k^i = g^{\alpha\bar{\beta}} \left\{ (H^k_i)_{\alpha\bar{\beta}} (H^{-1})_k^i + (H^k_i)_\alpha [(H^{-1})_k^i]_{\bar{\beta}} \right\} + \lambda r.
\]
Here \( r \) is the rank of \( E \). Hence
\[
\nu_t = \frac{\partial}{\partial t} \eta(x, t)
= \left( \frac{\partial}{\partial t} H^k_i(x, t) \right) (H^{-1})_k^i
= g^{\alpha\bar{\beta}} \left\{ (H^k_i)_{\alpha\bar{\beta}} (H^{-1})_k^i + (H^k_i)_\alpha [(H^{-1})_k^i]_{\bar{\beta}} \right\} + \lambda r.
\]
Calculating the \( \Delta \nu \) writes
\[
\Delta \nu = g^{\alpha\bar{\beta}} \left\{ (H^k_i)_{\alpha\bar{\beta}} (H^{-1})_k^i + (H^k_i)_\alpha [(H^{-1})_k^i]_{\bar{\beta}} \right\} + g^{\alpha\bar{\beta}} \Omega_{\alpha\bar{\beta}}(x, 0).
\]
Combining them together we have

\begin{equation}
\frac{\partial u}{\partial t} = \Delta u - g^{\alpha\bar{\beta}}(x,t)\Omega_{\alpha\bar{\beta}}(x,0) + \lambda r
\end{equation}

Locally we can write $-\Omega_{\alpha\bar{\beta}}(x,0)$ as $\partial_\alpha \bar{\partial}_\beta \varphi(x)$ for some smooth function $\varphi$. Define locally $U(x,t) = u + \varphi - r\lambda t$. Then (3.11) implies that

$$U_t = \Delta U.$$  

Here $\Delta$ is the time-dependent Laplacian operator. We also have that locally $U_{\alpha\bar{\beta}}(x,t) = \Omega_{\alpha\bar{\beta}}(x,t)$. Now Lemma 2.1 of [N-T1] implies that $U_{\alpha\bar{\beta}}(x,t)$ satisfies (3.7). Therefore we have shown that $\Omega_{\alpha\bar{\beta}}$ satisfies (3.7). Now just apply Hamilton maximum principle for tensors. We then have $\Omega_{\alpha\bar{\beta}}(x,t) \geq 0$ if it is true initially. Due to the fact that one can locally express $\Omega_{\alpha\bar{\beta}}$ by the $\partial \bar{\partial} U$ the similar reduction as carried out in the proof of Theorem 2.1 still works. Namely if $h_{\alpha\bar{\beta}}$ in Theorem 3.4 is given by $\Omega_{\alpha\bar{\beta}} = U_{\alpha\bar{\beta}}$, then

\[ \text{div}(h)_\alpha = \nabla_\gamma U_{\alpha\gamma} = \nabla_\alpha (U_t) = \nabla_\alpha \Omega \quad \text{and} \quad \text{div}(h)_{\bar{\alpha}} = \nabla_{\bar{\alpha}} U_{\alpha\bar{\alpha}} = \nabla_{\bar{\alpha}} (U_t) = \nabla_{\bar{\alpha}} \Omega, \]

and

\begin{equation}
Z = \Delta(U_t) + R_{\alpha\bar{\beta}} U_{\bar{\alpha}\bar{\beta}} + \nabla_{\bar{\alpha}} \Omega V_{\alpha} + \nabla_\alpha \Omega V_{\bar{\alpha}} + U_{\alpha\bar{\beta}} V_{\alpha} V_{\beta} + \frac{\Omega}{t} \geq 0
\end{equation}

for any $(1,0)$ vector field $V$. Here we have used the fact that $\Omega = U_t = \Delta U$. Differentiate $\Delta U = U_t$ with respect to $t$ we have that

$$\Delta(U_t) + R_{\alpha\bar{\beta}} U_{\bar{\alpha}\bar{\beta}} = U_{tt} = \Omega_t.$$  

Then (3.12) implies (3.10).

Theorem 3.5 is just a LYH inequality for the Hermitian-Einstein flow for vector bundles coupled with the Kähler-Ricci flow. We should point out that on complete noncompact manifold one needs some growth assumptions on $\Omega_{\alpha\bar{\beta}}(x,0)$ in order to show that $\Omega_{\alpha\bar{\beta}}(x,t) \geq 0$ for later time $t > 0$. For example, it would be sufficient to assume that $\|\Omega_{\alpha\bar{\beta}}\|(x,0) \leq A$ for some constant $A > 0$. In [N-T1], we proved the nonnegativity for $\Omega_{\alpha\bar{\beta}}(x,t)$ under much weaker assumption. Please refer to the paper for the details. The interested reader can write the similar result for the noncompact case by modelling the assumptions in [N-T1]. The special case when $h_{\alpha\bar{\beta}}$ in Theorem 3.4 is given by the complex Hessian of a global defined function only make sense when $M$ is noncompact and it is clearly the special case of the noncompact version of Theorem 3.5. In the following we want to show that this special case implies a parabolic monotonicity formula for the plurisubharmonic functions.

Let us first collect some simple calculations from [N-T1]. Let $u(x)$ be a plurisubharmonic function on $M$. Consider the heat equation coupled with the Kähler-Ricci flow $(\frac{\partial}{\partial t} - \Delta) u(x,t) = 0$ with $u(x,0) = u(x)$. With the assumption that $\Delta u(x) \leq C \exp(ar^2(x))$ for some positive constants $a$ and $C$, one can show that $u(x,t)$ is still plurisubharmonic. Applying (3.10) we have the following

\[ w_t + \nabla_{\bar{\alpha}} w \nabla_\alpha + \nabla_\alpha w \nabla_{\bar{\alpha}} + u_{\alpha\beta} V_{\alpha} V_{\beta} + \frac{w}{t} \geq 0 \]
with \( w = u_t \), which implies that

\[
(3.13) \quad w_t + \frac{w}{t} \geq 0.
\]

Namely \((tw)_t \geq 0\). But

\[
tw = \frac{\partial (u(x,t))}{\partial (\log t)}.
\]

Therefore, (3.13) is equivalent to the fact \( u(x,t) \) is convex function of \( \log t \). We would like to call this fact or (3.13) the parabolic monotonicity formula for the plurisubharmonic functions. The reason we have pointwise monotonicity instead of integral one is that the heat equation has the averaging effect. This more or less says that the average quantity, \( M(x,r) \) in the last section, is equivalent to \( u(x,t) \). For the linear heat equation on complete Riemannian manifolds with nonnegative Ricci curvature, the author have proved a precise statement reflecting this principle for manifolds with nonnegative Ricci curvature (cf. [N], section 3, The moment type estimates). However due to the nonlinearity of the Kähler-Ricci flow this can only be proved for the time-dependent Laplacian-heat equation under the assumptions such as that there exists \( C > 0 \) with \( \int_{B_x(r)} R(x,0) \, dy \leq \frac{C}{r^2} \), for any \( x \in M \), which also ensures the long time existence of the Kähler-Ricci flow by the work of W.-X. Shi [Sh2]. We also need to assume that the bisectional curvature of \( M \) is uniformly bounded since this is needed for the short time existence of the Kähler-Ricci flow (cf. [Sh1]). Under all these assumptions the same argument as in the proof of Proposition 2.3 proves a Liouville theorem for the plurisubharmonic functions on complete Kähler manifolds with nonnegative holomorphic bisectional curvature. The interested reader can refer [N-T1] for more detailed statements and other cases. We should point out that using a different method in [N-T2], the authors improved the Liouville theorems to general complete Kähler manifolds with nonnegative bisectional curvature.

We would like to end this section by mentioning some questions arising from this consideration on LYH inequalities. First, one might wonder what is the ‘global’ elliptic monotonicity? (Noting that in [N-T2] the proof also uses parabolic equations, can we have an elliptic proof for the Liouville theorem on plurisubharmonic functions?) Namely what is the corresponding Bishop-Lelong Lemma on complete Kähler manifold with nonnegative bisectional curvature? Certainly, the Euclidean argument does not lead to anything new. One needs an essentially more general argument. The second question is what is the corresponding LYH inequality for the \((p,p)\) currents. Inequality (3.13) is the parabolic monotonicity for the plurisubharmonic functions, or more generally \((1,1)\) currents. What’s the parabolic monotonicity formula for \((p,p)\)-currents? The last question is how about the real case. Namely, is there anything analogous to what we described here for Theorem 3.3, the linear trace LYH of Chow-Hamilton? Will it relate the minimal submanifold theory to Ricci flow more explicitly?

§4 Equalities in the linear trace LYH inequalities. In this section we study the geometry when the equality in Theorem 3.5 holds for some point at space-time. We first prove the following result. For the simplicity of the proof we just consider the case that \( \|h_{\alpha\beta}\|(x,t) \) are uniformly bounded in space for any fixed \( t \).
Theorem 4.1. Let \((M,g(t))\) be a complete solution to the Kähler-Ricci flow (3.3) on \(M \times (0,T)\). Let \(h_{\alpha \beta}(x,t)\) be a Hermitian symmetric 2-tensor satisfying (3.7) on \(M \times (0,T)\) too. Assume that \((M,g(0))\) has bounded nonnegative holomorphic bisectional curvature, \((h_{\alpha \bar{\beta}}(x,t)) \geq 0\) and \(\|h_{\alpha \bar{\beta}}\|(x,t)\) is uniformly bounded on \(M\) for any \(t \in (0,T)\). We also assume that \(M\) is simply-connected. Then \(Z(x,t) = 0\) for some point \((x_0,t_0)\) if and only if \((M,g(t))\) is an expanding Kähler-Ricci soliton. In the case of ancient solution, namely both flows are defined on \(M \times (-\infty,T)\), \(\dot{Z}(x,t) = 0\) for some \((x_0,t_0)\) at space-time if and only if \((M,g(t))\) is a gradient Kähler-Ricci soliton.

This clearly implies Cao’s theorems on limits of solutions to the Kähler-Ricci flow in \([\text{Ca}2]\). The proof is simpler, in my opinion, and unifies the proof for type II and type III singularity models. The result is also more general. The similar argument works for the real case. Namely, by considering the equality in the LYH inequality of Chow-Hamilton’s singularity models, we can obtain a proof of Hamilton’s theorem on the eternal (Cf. \([\text{H}3]\)) as well as the more recent theorem on the type III singularity models proved by Chen-Zhu in \([\text{C-Z}]\). In fact we write the following more general result.

Theorem 4.2. Let \((M,g(t))\) be a complete solution to the Ricci flow (3.1) on \(M \times (0,T)\) and let \(h\) be a symmetric 2-tensor satisfying (3.5). Assume that the metric initially has bounded nonnegative curvature operator (which is preserved under the flow) and \(M\) is simply-connected. We also assume that \((h_{ij}(x,t)) > 0\) and \(\|h_{ij}\|(x,t)\) is uniformly bounded on \(M\) for any \(t \in (0,T)\). Then \(Q(x,t) = 0\) for some \((x_0,t_0)\) in the space-time if and only if \((M,g(t))\) is an expanding soliton. Similarly, in the case of ancient solutions, (3.6) can be improved to \(\dot{Q} \geq 0\) with \(\dot{Q} = Q - \frac{H}{t^2}\). Moreover, \(\dot{Q}(x_0,t_0) = 0\) for some point \((x_0,t_0)\) in the space-time if and only if \((M,g(t))\) is a gradient Ricci soliton.

Proof of Theorem 4.1. First, we claim that by the maximum principle that for \(t < t_0\) at every point \(x \in M\) there exists a vector \(V\) such that \(Z(z,t) = 0\). This can be shown using the similar argument as in Proposition 3.2 of \([\text{Ca}2]\), where the case for the Kähler-Ricci flow (not the linear trace with general \(h_{\alpha \bar{\beta}}\)) was proved. On the other hand \(Z(x,t) \geq 0\) for any arbitrary \(V\). Therefore the \(V\), which makes \(Z(x,t) = 0\), is the minimizing one. Note also here we have assumed \((h_{\alpha \bar{\beta}}(x,t)) > 0\), the calculation in \([\text{N-T}1]\), especially (1.40) shows that

\[
(\frac{\partial}{\partial t} - \Delta) Z = Y_1 + R_{\alpha \bar{\beta}} R_{\bar{\gamma} \bar{\alpha}} h_{\gamma \bar{\alpha}} \nabla_{\bar{\gamma}} V_{\bar{\alpha}} - R_{p \bar{\alpha}} h_{\alpha \gamma} \nabla_p V_{\gamma} \\
+ h_{\gamma \bar{\alpha}} \nabla_{\bar{s}} V_{\bar{\gamma}} \nabla_{\bar{s}} V_{\alpha} + h_{\gamma \bar{\alpha}} \nabla_{\bar{s}} V_{\bar{\gamma}} \nabla_{\bar{s}} V_{\alpha} - \frac{H}{t^2}.
\]

Here \(Y_1\) is given by

\[
Y_1 = \left[ \Delta R_{s \bar{t}} + R_{s \bar{t} \alpha \bar{\beta}} R_{\bar{\alpha} \beta} + \nabla_\alpha R_{s \bar{t}} V_{\alpha} + \nabla_{\bar{\alpha}} R_{s \bar{t}} V_{\alpha} + R_{s \bar{t} \alpha \beta} V_{\alpha} V_{\beta} + \frac{R_{s \bar{t}}}{t} \right] h_{s \bar{t}}.
\]

If we denote

\[
Y_2 = h_{\gamma \bar{\alpha}} \left[ \nabla_p V_{\bar{\gamma}} - R_{p \bar{\gamma}} - \frac{1}{t} g_{p \bar{\gamma}} \right] \left[ \nabla_{\bar{p}} V_{\alpha} - R_{\alpha \bar{p}} - \frac{1}{t} g_{\bar{p} \alpha} \right] + h_{\gamma \bar{\alpha}} \nabla_{\bar{p}} V_{\bar{\gamma}} \nabla_p V_{\alpha}
\]
we can write (4.1) as

\[
\left( \frac{\partial}{\partial t} - \Delta \right) Z = Y_1 + Y_2 - \frac{1}{t} \left[ -h_{\gamma\bar{\alpha}} \nabla_\alpha V_\gamma - h_{\gamma\bar{\alpha}} \nabla_\bar{\alpha} V_\gamma + 2R_{\alpha\bar{\gamma}} h_{\gamma\bar{\alpha}} + \frac{2H}{t} \right].
\]

On the other hand, since \( V \) now is the minimizing vector, the direct calculation, using the equalities from the first variation consideration, shows

\[
2Z = -h_{\gamma\bar{\alpha}} \nabla_\alpha V_\gamma - h_{\gamma\bar{\alpha}} \nabla_\bar{\alpha} V_\gamma + 2R_{\alpha\bar{\gamma}} h_{\gamma\bar{\alpha}} + \frac{2H}{t}.
\]

One can refer (1.36), (1.37) and (1.39) of [N-T1] for the detailed calculations. Combining (4.1)–(4.3), we have

\[
\left( \frac{\partial}{\partial t} - \Delta \right) Z = Y_1 + Y_2 - \frac{2Z}{t}.
\]

On the other hand, for the minimizing \( V \), from \( Z(x, t) = 0 \) we know that

\[
\left( \frac{\partial}{\partial t} - \Delta \right) Z(x, t) = 0.
\]

By the nonnegativity of \( Y_1 \) and \( Y_2 \) we have that

\[
Y_1 = Y_2 = 0.
\]

Noticing that \( h_{\alpha\bar{\beta}} \) is positive definite, \( Y_1 = 0 \) implies that

\[
\Delta R_{st} + R_{s\bar{\alpha}\beta} R_{\alpha\bar{\beta}} + \nabla_\alpha R_{s\bar{\alpha}} V_\alpha + \nabla_\bar{\alpha} R_{st} V_\alpha + R_{s\bar{\alpha}\beta} V_\alpha V_\beta + \frac{R_{st}}{t} = 0
\]

and \( Y_2 = 0 \) implies that

\[
\nabla_\bar{p} V_\bar{\gamma} - R_{p\bar{\gamma}} - \frac{1}{t} g_{p\bar{\gamma}} = \nabla_\bar{p} V_\alpha - R_{\alpha\bar{p}} - \frac{1}{t} g_{\alpha\bar{p}} = 0,
\]

as well as

\[
\nabla_\bar{p} V_\bar{\gamma} = \nabla_\bar{p} V_\alpha = 0.
\]

By the simply-connectness of \( M \), (4.5) and (4.6) imply that \( V \) is a holomorphic vector field and is the gradient of a holomorphic function. Namely we have proved that \((M, g(t))\) is an expanding Kähler-Ricci soliton.

Now suppose that \((M, g(t))\) is an expanding Kähler-Ricci soliton. Then we have (4.5) above. On the other hand, since for the minimizing vector \( V \), (4.3) holds. Plugging (4.5) into (4.3) we have \( Z(x, t) = 0 \) for the minimizing vector.

The case for the ancient solution follows similarly. In this case replace \( t \) by \( t - A \) and let \( A \to -\infty \) we can have that

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \tilde{Z} = \tilde{Y}_1 + R_{\alpha\bar{\beta}} R_{p\bar{s}} h_{s\alpha} - R_{\alpha\bar{\beta}} h_{\gamma\bar{\alpha}} \nabla_p V_\gamma - R_{p\alpha} h_{\alpha\bar{\gamma}} \nabla_\bar{p} V_\gamma
\]

\[
+ h_{\gamma\bar{\alpha}} \nabla_s V_\gamma \nabla_s V_\alpha + h_{\gamma\bar{\alpha}} \nabla_s V_\gamma \nabla_s V_\alpha
\]
where

\[(4.2') \quad \hat{Y}_1 = [\Delta R_{s\ell} + R_{s\ell\alpha\beta} R_{\alpha\beta} + \nabla_\alpha R_{s\ell} V_\alpha + \nabla_\alpha R_{s\ell} V_\alpha + R_{s\ell\alpha\beta} V_\alpha V_\beta] h_{s\ell}.\]

Then we have that

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \check{Z} = \hat{Y}_1 + \hat{Y}_2
\]

with

\[\hat{Y}_2 = h_{\gamma\bar{\alpha}} [\nabla_p V_\gamma - R_{p\gamma}] [\nabla_{\bar{p}} V_\alpha - R_{\alpha\bar{p}}] + h_{\gamma\bar{\alpha}} \nabla_p V_\gamma \nabla_{\bar{p}} V_\alpha.\]

Now the proof follows verbatim.

**Proof of Theorem 4.2.** Similarly, using the maximum principle we know that for \(t < t_0\), at every point \(x \in M\) there exists a (an unique) minimizing vector \(V\) such that \(Q(x, t) = 0\). One can refer Proposition 4.2 of [C-Z] for a proof. Then the same argument as in the above proof works. One only need to use (6.5) of Chow-Hamilton’s [C-H] to replace (4.1) above. Namely, the proof of Chow-Hamilton’s linear trace LYH (Theorem 3.3 in the last section) already implied both Hamilton’s Main Theorem in [H3] as well as the later Theorem 4.3 of Chen-Zhu on the type III singularity models.

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**References**

[B-G] E. Bombieri and E. Giusti, *Harnack's inequality for elliptic differential equations on minimal surfaces*, Invent. Math. **15** (1972), 24–46.

[C] X. Cabré, *Nondivergent elliptic equation on manifolds with nonnegative curvature*, Comm. Pure and Appl. Math. **L** (1997), 623–665.

[Ca1] H.-D. Cao, *On Harnack inequalities for the Kähler-Ricci flow*, Invent. Math. **109** (1992), 247–263.

[Ca2] H.-D. Cao, *Limits of solutions to the Kähler-Ricci flow*, J. Differential Geom. **45** (1997), 257–272.

[C-C1] B. Chow and S. Chu, *A geometric interpretation of Hamilton’s Harnack inequality for the Ricci flow*, Math. Res. Letter **2** (1995), 701–718.

[C-C2] B. Chow and S. Chu, *A geometric interpretation to the linear trace Harnack inequality for the Ricci flow*, Math. Res. Letter **3** (1996), 549–568.

[C-H] B. Chow and R. Hamilton, *Constrained and linear Harnack inequalities for parabolic equations*, Invent. Math. **129** (1997), no. 3, 213–238.

[C-N] H.-D. Cao and L. Ni, *Matrix Li-Yau-Hamilton estimates for heat equation on Kähler manifolds*, submitted.

[C-Y] S. Y. Cheng and S.-T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), 333–354.

[C-Z] B.-L Chen and X. Zhu, *Complete Riemannian manifolds with pointwise pinched curvature*, Invent. Math. **140** (2000), 423–452.
[G-H] P. Griffith and J. Harris, *Principles of Algebraic Geometry*, John Wiley and Sons, 1978.

[Gr] A. Grigor’yan, *The heat equation on noncompact Riemannian manifolds*, Math. USSR Sbornik 72 (1992), 47–77.

[G-R-S-B] W.-D. Garber, S. Ruijseenaars, E. Seiler and D. Burns, *On finite action solution of the nonlinear $\sigma$-model*, Ann. Phys. 119 (1979), 305–325.

[G-T] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd Edition., Springer-Verlag, Berlin-Heidelberg-New York, 1983.

[H1] R. Hamilton, *The Harnack estimate for the Ricci flow*, J. Differential Geom. 37 (1993), 225–243.

[H2] R. Hamilton, *A matrix Harnack estimate for the heat equation*, Comm. Anal. Geom. 1 (1993), 113–126.

[H3] R. Hamilton, *Eternal solutions to the Ricci flow*, J. Differential Geom. 38 (1993), 1–11.

[Ho] L. Hörmander, *Introduction to Complex Analysis in Several Variables*, 3rd Edition., North-Holland, 1990.

[Hu] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. 31 (1990), 285–299.

[J] Z. Jin, *Liouville theorems for harmonic maps*, Invent. Math. 108 (1992), 1–10.

[L-W] P. Li and J.-P. Wang, *Hölder estimates and regularity of holomorphic and harmonic functions*, J. Differential Geom. 58 (2001), 309–329.

[L-Y] P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. 156 (1986), 139–168.

[N] L. Ni, *Poisson equation and Hermitian-Einstein metrics on holomorphic vector bundles over complete noncompact Kähler manifolds*, Indiana Univ. Math. Jour. 51 (2002), 679–704.

[N-T1] L. Ni and L.-F.Tam, *Plurisubharmonic functions and Kähler Ricci flow*, submitted.

[N-T2] L. Ni and L.-F.Tam, *Lowvile properties of plurisubharmonic functions*, submitted.

[Sa] L. Saloff-Coste, *Uniformly elliptic operators on Riemannian manifolds*, J. Differential Geom. 36 (1992), 417–450.

[Sc] R. Schoen, *Analytic aspects of harmonic map problem*, Seminar on nonlinear PDEs, 321–358, 1984.

[S-U] R. Schoen and K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Differential Geom. 17 (1982), 307–336.

[Sh1] W.-X. Shi, *Deforming the metric on complete Riemannian manifolds*, J. Differential Geom. 30 (1989), 223–301.

[Sh2] W.-X. Shi, *Ricci deformation of metric on complete noncompact Kähler manifolds*, Ph. D. thesis Harvard University, 1990.

[Si1] L. Simon, *Lectures on Geometric Measure Theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, 3(1983).

[Si2] L. Simon, *Theorems on Regularity and Singularity of Energy Minimizing Maps*, ETH Lectures in Mathematics, Zürich, Birkhäuser, 1996.

[St] M. Struwe, *On the evolution of harmonic maps in higher dimension*, J. Differential Geom. 28 (1988), 485–502.

[Y] S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. 28 (1975), 201–228.