Exploration and Incentivizing Participation in Clinical Trials

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Abstract

Participation incentives are a well-known issue inhibiting randomized clinical trials (RCTs). We frame this issue as a non-standard exploration-exploitation tradeoff: an RCT would like to explore as uniformly as possible, whereas each patient prefers “exploitation”, i.e., treatments that seem best. We incentivize participation by leveraging information asymmetry between the trial and the patients. We measure statistical performance via worst-case estimation error under adversarially generated outcomes, a standard objective for RCTs. We obtain a near-optimal solution in terms of this objective: an incentive-compatible mechanism with a particular guarantee, and a nearly matching impossibility result for any incentive-compatible mechanism. We consider three model variants: homogeneous patients (of the same “type” comprising preferences and medical histories), heterogeneous agents, and an extension with estimated type frequencies.

1 Introduction

Randomized Clinical Trials (RCTs) are a standard way to evaluate safety and efficiency of medical treatments. A paradigmatic design considers \( n = 2 \) alternative treatments, a.k.a. arms, e.g., a new and an old drug for the same medical condition, or a new drug and a placebo. Each patient \( t \) is assigned an arm \( a_t \) independently and uniformly at random. An outcome \( \omega_t \) is observed and evaluated according to some predefined metric \( f(\cdot) \). The goal is to use the observations to estimate the counterfactual averages \( \frac{1}{T} \sum_{t \in [T]} f(\omega_{a_t}), a \in [n], \) where \( T \) is the number of patients. Independent per-patient randomization over arms is essential to remove selection bias and ensure statistical validity. Uniform randomization is usually preferred, as it minimizes the estimation error. In particular, one can obtain strong provable guarantees even if each arm’s outcomes are generated by an adversary, rather than sampled from some fixed (arm-dependent) distribution.

A well-known issue which inhibits RCTs is the difficulty of recruiting patients (e.g., see surveys [Ross et al., 1999; Mills et al., 2006; Rodríguez-Torres et al., 2021]). This issue may be particularly damaging for large-scale RCTs for widespread medical conditions with relatively inexpensive treatments. Then finding suitable patients and providing them with appropriate treatments would be fairly realistic, but incentivizing patients to participate in sufficient numbers may be challenging.

We focus on patients’ reluctance to participate that stems from their preferences among the available treatments (as opposed to other factors, e.g., logistical difficulties). We frame this issue as a non-standard variant of exploration-exploitation tradeoff: the RCT always prefers to explore as uniformly as possible, whereas each patient prefers to exploit (choose treatment that seems best), and might not participate otherwise.

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1We use “RCT” acronym in this sense throughout. Another common meaning, “randomized controlled trial”, is also consistent with our model, since one of the arms there is typically a “control”.
2This corresponds to (oblivious) adversary in adversarial bandits [Auer et al., 2002].
3The standard variant of this tradeoff has an algorithm balance exploration vs exploitation for its own sake, so as to converge on the best arm.
We initiate the study of incentivized participation in RCTs. Our objective is to incentivize participation while optimizing statistical performance. We leverage information asymmetry between the RCT and the patients. We use preliminary data from the ongoing trial to skew the random choice of arms in favor of a better treatment (i.e., towards exploitation), while retaining a sufficient amount of near-uniform exploration. This causes the patients to prefer participation to any alternative treatment options they might have (which also don’t have access to the trial’s data).

We make two assumptions that are common in economic theory: the RCT has the power to (i) prevent undesirable information disclosure to the patients, and (ii) commit to following a particular policy. While such assumptions can be problematic in other applications, they are realistic in RCTs. Indeed, it is a standard practice to not reveal the history of an ongoing RCT and the assigned treatment, to avoid biasing patients and doctors in favor of better-performing treatments (e.g., see Detry et al. (2012, pp. 26-30)). Moreover, the design of an RCT is typically required by regulation to be fixed in advance, spelling out all applicable “decision points”, and (essentially) it needs to be revealed to the patients to meet the standards of “informed consent” (Arango et al. 2012).

While RCTs with \( n = 2 \) arms is the paradigmatic case, our results seamlessly extend to \( n > 2 \). RCTs with \( \geq 3 \) arms are studied in biostatistics, e.g., (Hellmich, 2001; Freidlin et al., 2008), and are important in practice, e.g., (Parmar et al., 2014; Redig and Jäne, 2015).

Our modelling approach. We model “incentivized participation” as a mechanism design problem under a statistical objective and incentives constraints. An RCT (mechanism) sequentially interacts with self-interested patients (agents) like a multi-armed bandit algorithm: in each round, an agent arrives, the mechanism chooses an arm for this agent, and observes an outcome. The mechanism’s statistical objective is standard for RCTs: counterfactually estimate each arm’s performance on the whole population, even if the outcomes are generated by an adversary.\(^4\) If not for the incentives constraints, the mechanism would simply choose arms uniformly at random.

Participation incentives are modeled as Bayesian Individual Rationality (BIR), a standard notion in economic theory. The mechanism commits to a some randomized policy that assigns an arm to each agent, depending on the history of interactions with the previous agents. The agent does not observe the history or the chosen arm (as discussed above). BIR states that each agent must prefer participation to her “outside option”, in expectation over her Bayesian beliefs.

Crucially, the agents believe in much simpler world than the adversarial one allowed by our statistical model. Essentially, they believe one’s outcome distribution is determined by the assigned arm and the medical history. Such beliefs are fairly realistic, representing pre-existing medical knowledge, and appear necessary for tractability.

We consider three model variants. To isolate the core issues of incentivized participation, we consider on the fundamental case of homogeneous agents with the same preferences and medical histories. The general case of heterogeneous agents endows each agent with a type: a (public) medical history which determines the outcome distributions and (private) preferences over these outcomes. The third variant provides the mechanism with estimated type frequencies; this allows to mitigate the harmful influence of rare-but-difficult agent types.

Our results. We focus on mechanisms with simple two-stage designs: a relatively short warm-up stage of predetermined duration, followed by the main stage that needs to be counterfactually estimated. The data from the warm-up stage creates information asymmetry which drives participation incentives, whereas the data from the main stage determines the counterfactual estimates.\(^5\)

Our guarantees focus on the main stage, assuming that enough data is collected in the warm-up stage.

\(^4\)In contrast, the standard objective for multi-armed bandits is to optimize cumulative payoffs.

\(^5\)In particular, the warm-up data is discarded for the counterfactual estimates. This is a part the price of incentivizing participation in a (much larger) main stage.
stage. The aforementioned data can also be collected either endogenously (i.e., by a mechanism in our model), or exogenously, e.g., using paid volunteers. The key point is that a relatively small amount of data, determined by the prior and the utility structure, suffices to bootstrap the main stage for an arbitrarily large number of agents. Endogenous implementation of the warm-up stage can rely on prior work on incentivized exploration (more on this in “Related work” and Lemma 3.6).

We obtain a near-optimal solution for each model variants described above, from homogeneous agents to heterogeneous agents to estimated type frequencies. First, we put forward a benchmark: a hypothetical policy that observes the true “state of the world” according to agents’ beliefs and optimizes a certain objective. Then, we design a mechanism that attains a worst-case upper bound on the estimation error, stated in terms of this benchmark and scaling as $1/T$. Finally, we prove that no mechanism achieves better estimation error, up to constant factors. Conceptually, we connect the dynamic process and statistical estimators with a static information-design problem. Many ideas carry over from one variant to the next.

Our mechanisms are simple in that their “sampling distributions” (the distributions from which the arms are sampled) do not adapt to the data collected in the main stage. This is desirable in practice In contrast, our negative results apply to a wide class of mechanisms, including the “multi-stage” mechanisms from Footnote 7. In particular, the sampling distributions can change over time and in “almost all” rounds they can depend arbitrarily on all past observations.

**Discussion.** Our contribution is not the mechanisms per se, but the whole “journey”: from the new model to the benchmark to the matching upper/lower bounds, and from homogeneous agents to heterogeneous agents to estimated type frequencies. While our results are asymptotic, and hence most meaningful for RCTs with a large number of participants, participation incentives would make such trials more common and realistic. For concreteness, we also illustrate our results with simple numerical examples (Appendix A).

We emphasize that theoretical participation incentives come at a price: the warm-up samples are discarded for counterfactual estimates, and the main-stage sampling distributions are skewed. Whether this price is acceptable in practice depends on how much these incentives actually help in recruiting patients. Ultimately, the “price of incentives” should be included in the “price of experimentation” that the society pays for the sake of obtaining reliable medical knowledge.

This paper is only concerned with patients’ initial decision to participate in an RCT. Their incentives to remain in the trial, e.g., if the treatment appears ineffective, is outside of our scope.

**Map of the paper.** Section 2 spells out our model, focusing on the homogeneous agents. Section 3 works out the case of homogeneous agents, introducing key ideas with a more accessible exposition. Heterogeneous agents are treated in Section 4. Section 5 considers an extension with estimated type frequencies. The technical machinery in Sections 3 to 5 is introduced gradually, and heavily reused in the later sections.

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6 A mapping from arms (or type-arm pairs) to outcome distributions.
7 Because treating the patients and/or observing the outcomes may take a long time, and the doctors involved may find it difficult to coordinate with one another. Consequently, many/most patients may need to be treated in parallel. So, RCTs typically do not adapt to data, or have just a few pre-determined “review points” when one can make a data-adaptive decision, with non-data-adaptive “stages” in between.
8 “Almost all” rounds here means: all rounds outside of a contiguous time interval comprising a constant (but possibly very small) fraction of rounds.
9 This is indeed an issue in practice, as per [FDA 2001, Section 2.1.7.2]. However, it is unclear if any approach based on information asymmetry could possibly help here: once the patient learns about the actual outcome, the “principal” does not have much any informational advantage left.
1.1 Related work

Incentivized exploration (IE) is a closely related model, motivated by recommendation systems. The platform strives to incentivize users to explore when they would prefer to exploit. The platform issues recommendations (which the users can override), and makes them incentive-compatible via information asymmetry. Introduced in [Kremer et al., 2014; Che and Hörner, 2018], IE is well-studied, see [Slivkins 2019, Chapter 11] for a survey. Most technically related are Mansour et al. (2020); Immorlica et al. (2019).

The main difference concerns performance objectives. IE strives to maximize the total reward under time-invariant outcome distribution, or just explore each arm once. Neither serves our statistical objective, which requires independent near-uniform randomization in each round: reward-maximization pulls a mechanism in a very different direction, and merely exploring each arm \(N\) times does not ensure statistical validity under adversarially chosen outcomes. The economic constraints are different, too, because patients cannot observe or alter their treatments (but can choose whether to participate) \(10\) Mechanism’s power to control information and commit to a particular policy are substantial assumptions in IE (e.g., Bahar et al. (2016); Immorlica et al. (2020) are written to mitigate them), but they are very benign for RCTs, as discussed above. Hence, our mechanisms, analyses and technical guarantees are very different from those in IE. However, our mechanisms reuse IE for the “warm-up stage” (see Lemma 3.6).

A variant of IE creates incentives via payments (e.g., Frazier et al., 2014; Han et al., 2015). However, this work also assumes time-invariant reward distributions and optimizes total reward. Moreover, it lets the agents observe full history, which is not the case for RCTs.

Economic design. Using information asymmetry to incentivize agents to take particular action is a common goal in information design (Bergemann and Morris, 2019; Kamenica, 2019). However, this literature tends to have a very different technical flavor. The key distinction is that designer’s utility is (typically) directly derived from each agent’s actions. In contrast, our statistical objective depends on the sampling distributions and does not add up across agents.

Randomized Clinical Trials (RCTs). We only point out the key connections to the (huge) medical literature on RCTs, as a more detailed review is beyond our scope.

Difficulties in patient recruitment are widely acknowledged as a significant barrier (e.g., Adams et al., 1997; Grill and Karlawish, 2010). Its reasons are well-studied, e.g., see surveys (Ross et al., 1999; Mills et al., 2006; Rodríguez-Torres et al., 2021). Patients preferences among treatments are listed among the major factors, along with other factors such as insufficient trust, logistical difficulties, or aversion to randomization and loss of control. Prior work suggested increasing the probability of the “new” treatment as a possible remedy (Vozdolka et al., 2009; Jenkins and Fallowfield, 2000; Pocock, 1979; Karlawish et al., 2008), albeit without a formal model of participation incentives and without theoretical guarantees. These papers focus on the scenario when the new treatment appears ex-ante preferable compared to non-participation, and do not revise the treatment probabilities as the new data arrives. Other suggested remedies, focusing on eligibility criteria, patient outreach, trial logistics, etc., are not as relevant to this paper.

Data-adaptive designs for RCTs are not uncommon, e.g., see Chow and Chang (2008) for a research survey, and Detry et al. (2012); Pallmann et al. (2018) for practitioners’ guidance. However, while prior work used data-adaptivity to improve efficiency (the tradeoff between the trial’s size and statistical properties), we use it to create participation incentives. Accordingly, the novelty in our RCT design is in connecting it to the analysis of incentives (and the corresponding lower bounds on the statistical objective), not in the RCT design per se. As far as the design itself,

\(10\) This distinction is consequential only for \(n > 2\) arms. Then the economic constraint in IE is stronger than ours.
our RCT for the homogeneous-agent case follows a very standard “epsilon-greedy” approach from multi-armed bandits, whereas the design for the heterogeneous agents appears new.

Our mechanisms are superficially similar to patient preference trials (PPTs) (Torgerson and Sibbald, 1998), in that both allow patient preferences to impact treatment allocation. The key difference is that PPTs allow the patients to directly choose the chosen treatment (potentially at the expense of the internal validity), whereas in our setting a patient’s report only affects the sampling distribution (hence the internal validity is not compromised). Another distinction is that we elicit patient preferences over the treatment outcomes, not over the treatments themselves.

Statistics and machine learning. Our mechanisms use a standard counterfactual estimator called inverse propensity score (IPS). While more advanced estimators are known (e.g., Dudík et al., 2014; Swaminathan and Joachims, 2015; Wang et al., 2017), IPS is optimal in some worst-case sense (see Section 2.4 and Appendix D), which suffices for our purposes.

While our mechanism follows the protocol of multi-armed bandits (Slivkins, 2019; Lattimore and Szepesvári, 2020), it strives to choose arms uniformly at random, whereas a bandit algorithm strives to converge on the best arm. More related is “pure exploration” (starting from Mannor and Tsitsiklis, 2004; Even-Dar et al., 2006), where a bandit algorithm predicts the best arm under a stationary outcome distribution, after exploring “for free” (e.g., uniformly) for $T$ rounds. In contrast, our model allows adversarially chosen outcomes, and requires estimates for all arms and all outcomes. Note that bandit algorithms, whether regret-minimizing or “pure exploration”, are generally not geared to incentivize participation. Interestingly, both “stationary” and adversarially chosen outcome distributions – like, resp., agent beliefs and statistical objective in our model – are well-studied in bandits, as, resp., “stochastic” and “adversarial” bandits.

2 Our model: incentivized clinical trial

We first present our model for homogeneous agents (i.e., agents with the same “type”), then extend it to heterogeneous agents in Section 4. This is to disentangle the core model, which is quite complex even for a single agent type, from some subtlety and heavy notation needed for multiple types.

Thus, we consider a randomized clinical trial (RCT) with some patients, interpreted as a mechanism) and agents, respectively. The mechanism interacts with the agents as a multi-armed bandit algorithm, but optimizes a very different statistical objective, and operates under economic constraints to incentivize participation. Accordingly, we split the model description into three subsections: the bandit model for the interaction protocol the statistical model for its performance objective, and the economic model for the agents’ incentives.

2.1 Bandit model: the interaction protocol

We have $T$ agents and $n$ treatments, also called arms. The set of patients is denoted by $[T] := \{1, \ldots , T\}$, and the set of arms is denoted by $[n]$. The paradigmatic case is $n = 2$ arms, but our results meaningfully extend to an arbitrary $n$. Choosing a given arm for a given agent yields an observable, objective outcome $\omega \in \Omega$, e.g., whether and how fast the patient was cured, and with which side effects (if any). The set $\Omega$ of possible outcomes is finite and known.

The mechanism interacts with agents as follows. In each round $t \in [T]$,

1. a new agent arrives
2. The mechanism chooses the sampling distribution over arms, denoted by $p_t$, samples an arm $a_t \in [n]$ independently at random from this distribution, and assigns this arm to the agent.

\footnote{Under the standard objective of cumulative reward/loss optimization, as expressed by regret.}
3. The outcome $\omega_t \in \Omega$ is realized and observed by the mechanism and the agent.

Each outcome $\omega$ is assigned a score $f(\omega) \in [0, 1]$, which may e.g., represent this outcome’s value. The scoring function $f : \Omega \to [0, 1]$ is exogenously fixed and known (its role is explained in Remark 2.2). Note that the mechanism follows a standard protocol of multi-armed bandits: in each round it chooses an arm $a_t$ and observes a numerical score $f(\omega_t)$ for playing this arm.

The outcome table ($\omega_{a,t} \in \Omega : a \in [n]$, $t \in [T]$) is fixed in advance (where round $t$ indexes rows), so that $\omega_t = \omega_{a_t,t} \in \Omega$ for each round $t$. We posit that this entire table is drawn from some distribution, which we henceforth call an adversary. An important special case is stochastic adversary, whereby each row $t$ of the table is drawn independently from the same fixed distribution.

Remark 2.1. The agents cannot change the assigned arms. Indeed, Indeed, an RCT participant cannot change her assigned treatment and cannot even observe what this treatment is.

2.2 Statistical model: the performance objective

The mechanism should estimate the average score of each arm without any assumptions on the adversary, i.e., on how the outcomes are generated. Not relying on modeling assumptions is a standard desiderata in randomized clinical trials.

More precisely, the mechanism designates the first $T_0$ rounds as a warm-up stage, a small-scale pilot experiment to create incentives, for some $T_0 \leq T/2$ chosen in advance. We are interested in the average score over the remaining rounds, which constitute the main stage.

Fix an adversary $\text{adv}$. The average score of a given arm $a$ over the main stage is denoted by

$$f_{\text{adv}}(a) := \frac{1}{|S|} \sum_{t \in S} f(\omega_{a,t}), \quad \text{where } S = \{ T_0 + 1, \ldots, T \}. \tag{2.1}$$

After round $T$, the mechanism outputs an estimate $\hat{f}_{\text{mech}}(a) \in [0, 1]$ for $f_{\text{adv}}(a)$, for each arm $a$.

Remark 2.2. We focus on the following two scenarios. First, $f_{\text{adv}}(a)$ is the frequency of a given outcome $\omega^* \in \Omega$ in the main stage; then the scoring function is $f(\omega) = 1_{\{\omega = \omega^*\}}$. Second, if $f(\omega)$ is the outcome’s value (for the agent and/or the society), then $f_{\text{adv}}(a)$ is the average value of arm $a$. We keep $f$ abstract throughout to handle both scenarios in a uniform way.

The estimation error is defined as the largest mean squared error (MSE). Formally, for mechanism $\text{mech}$ and adversary $\text{adv}$ we define

$$\text{ERR}(\text{mech} | \text{adv}) = \max_{f: \Omega \to [0,1], a \in [n]} \mathbb{E} \left[ (\hat{f}_{\text{mech}}(a) - f_{\text{adv}}(a))^2 \right], \tag{2.2}$$

where the max is over all scoring functions $f$ and all arms $a$, and the expectation is over the randomness in the mechanism and the adversary.

The full specification of our mechanism is that it chooses $T_0$, then follows the interaction protocol from Section 2.1, and finally computes the estimates $\hat{f}_{\text{mech}}(a)$, $a \in [n]$. Thus, the performance objective is to minimize $\text{ERR}$ uniformly over all adversaries $\text{adv}$. Note that essentially requires the mechanism to handle all $f$ at once.

Remark 2.3. Without economic constraints, a trivial solution is to take $T_0 = 0$ and randomize uniformly over arms, independently in each round. Indeed, this is a solution commonly used in practice. A standard IPS estimator (see Section 2.4) achieves $\text{ERR}(\text{mech} | \text{adv}) \leq n/T$.

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12 This corresponds to “oblivious adversary” in multi-armed bandits, as if there is an adversary who chooses the outcome table upfront. An analogous game-theoretic terminology is a non-strategic player called “Nature”.

13 In particular, the average treatment effect can be expressed as $f_{\text{adv}}(\text{treatment}) - f_{\text{adv}}(\text{control})$. 

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2.3 Economic model: agents, beliefs and incentives

The agents’ information structure is as follows. (i) Each agent \( t \) does not observe anything about the previous rounds. Indeed, not revealing any information about an ongoing trial is strongly encouraged in practice (e.g., Arango et al., 2012). Consequently, agent \( t \) does not observe the chosen distribution \( p_t \) if it depends on the history. (ii) Each agent \( t \) does not observe the assigned arm \( a_t \). Indeed, RCTs typically present different treatments in an identical way, e.g., as a pill that looks and feels exactly the same; this is strongly encouraged whenever possible (e.g., Arango et al., 2012). (iii) Each agent knows the specification of the mechanism before (s)he arrives. In practical terms, the entity which runs the RCT (e.g., a hospital) commits to a particular design of the trial, and this design is revealed to all patients. This property is typically enforced via regulation.

The mechanism must incentivize participation, i.e., satisfy Individual Rationality (IR). We require Bayesian IR in expectation over the agents’ beliefs over the adversaries.\(^{14}\) Crucially, we posit that agents’ beliefs are over stochastic adversaries, i.e., the agents believe that a given treatment yields the same outcome distribution in all patients.\(^{15}\) All agents share the same belief, a common simplification in economic theory. Informally, the beliefs represents the current medical knowledge.

Let us specify the agents’ beliefs. The outcome of each arm \( a \) is an independent draw from some distribution \( \psi^*(a) \) over outcomes each time this arm is chosen. For better notation, we identify states as mappings \( \psi : [n] \rightarrow \Delta_\Omega \) from arms to distributions over outcomes, and \( \psi^* \) as the true state.\(^{16}\) Then, the agents believe that \( \psi^* \) is drawn from some Bayesian prior: distribution \( \mathcal{P} \) over states. We posit that \( \mathcal{P} \) is known to the mechanism, and assume that it has finite support.

An agent’s utility for a given outcome \( \omega \) is determined by the outcome, denote it \( u(\omega) \in [0,1] \). Agent’s expected utility for arm \( a \) and state \( \psi \) is given by \( U_\psi(a) := \mathbb{E}_{\omega \sim \psi(a)} [u(\omega)] \). The utility structure \( u : \Omega \rightarrow [0,1] \) is known to the mechanism and all agents.

We ensure that each agent weakly prefers participation to the “outside option”. To formalize the latter, we posit that an agent could refuse to participate in the mechanism and choose an arm that appears best based on the initial information. This achieves Bayesian-expected utility

\[
U_{\text{out}} := \max_{a \in [n]} \mathbb{E}_{\omega \sim \psi^*} [U_\psi(a)]. \tag{2.3}
\]

Finally, we can state the IR property that the mechanism is required to satisfy. The mechanism is Bayesian Individually Rational (BIR) if \( \mathbb{E} [U_{\psi^*}(a_t)] \geq U_{\text{out}} \) for each round \( t \). The mechanism is BIR on a given set \( S \) of rounds if this holds for each round \( t \in S \).

Remark 2.4. While \((2.3)\) is a natural “outside option” within our model, our analysis accommodates an arbitrary, exogenously given \( U_{\text{out}} \in [0,1] \), and all results carry over with minimal modifications. This allows us to model several effects. First, some medical treatments might not be available outside of a clinical trial. Then we could take the max in \((2.3)\) only over the treatments that are. Conversely, a standard treatment might not be available inside the trial, e.g., in a new treatment vs. placebo trials. Then the max in \((2.3)\) should also include the standard treatment. Finally, patients might assign positive utility to participation itself, e.g., because they value contributing

\(^{14}\)Satisfying the IR property for every adversary (ex-post IR) is clearly impossible, except for a trivial mechanism that always chooses the agent’s outside option.

\(^{15}\)Agents have misspecified beliefs, in the sense that their support does not necessarily include the actual adversary. In particular, the actual adversary need not be stochastic. Misspecified beliefs are studied in a considerable recent literature in economic theory (starting from Berk, 1966; Esponda and Pouzo, 2016). However, this literature tends to focus on convergence and equilibria (e.g., Fudenberg et al., 2021; Esponda and Pouzo, 2021; Esponda et al., 2021), whereas our paper takes the mechanism design perspective and is concerned with IPS-style statistical estimation.

\(^{16}\)We use a standard notation: \( \Delta_\Omega \) is the set of all distributions over \( \Omega \).
to science or they expect to receive higher-quality and/or less expensive medical care. This can be modeled (albeit in a rather idealized way) by decreasing $U_{out}$ in (2.3) by some known amount.

2.4 Preliminaries

The tuple $\text{hist}_t = (p_s, a_s, \omega_s : \text{rounds } s < t)$ denotes the history collected by the mechanism before a given round $t$. Let $P_{\text{Bayes}}[\cdot]$ denote the probability according to agents’ Bayesian beliefs.

Per-round recommendation policy. A single round $t$ of our mechanism can be interpreted as a stand-alone “recommendation policy” which inputs the $\text{hist}_t$, and outputs a recommended arm. Such policies are key objects in our analysis. We define them formally below, in a slightly generalized form. The history $\text{hist}_t$ can be interpreted as a signal that is jointly distributed with state $\psi^*$. Indeed, fixing the mechanism in the previous rounds and assuming that the agents in these rounds report truthfully, the prior $P$ on $\psi^*$ determines a joint distribution $Q^{(t)}$ for the pair $(\psi^*, \text{hist}_t)$. For the subsequent definitions, we replace the history with an abstract signal jointly distributed with $\psi^*$.

Definition 2.5. A signal $\text{sig}$ is a random variable which, under agents’ beliefs, is jointly distributed with state $\psi^*$: $(\psi^*, \text{sig}) \sim Q$, where the joint prior $Q$ is determined by $P$. A recommendation policy $\sigma$ with signal $\text{sig}$ is a distribution over arms, $\sigma(\text{sig})$, parameterized by the signal.

As a shorthand, the Bayesian-expected utility of policy $\sigma$ under truthful reporting is

$$U_P(\sigma) := \mathbb{E}_{(\psi, \text{sig}) \sim Q} \mathbb{E}_{a \sim \sigma(\text{sig})} U_\psi(a). \quad (2.4)$$

A policy is called BIR if any agent weakly prefers following the policy compared to the outside option: $U_P(\sigma) \geq U_{out}$. The policy is called strictly BIR if $U_P(\sigma) > U_{out}$.

The mechanism can be represented as a collection of recommendation policies $(\sigma_t : t \in [T])$, where policy $\sigma_t$ inputs $\text{hist}_t$ as a signal and outputs the sampling distribution as $p_t = \sigma_t(\text{hist}_t)$ for each round $t$. The mechanism is BIR in a given round $t$ if and only if policy $\sigma_t$ is BIR.

Inverse-Propensity Scoring (IPS) is a standard estimator we use in all positive results. For each arm $a$, letting $S = \{T_0 + 1, \ldots, T\}$ be the set of rounds in the main stage, the estimator is

$$\text{IPS}(a) := \frac{1}{|S|} \sum_{t \in S} 1_{\{a_t = a\}} \cdot f(\omega_t) / p_t(a). \quad (2.5)$$

Consider a mechanism $\text{mech}$ which chooses a fixed sequence of sampling distributions $(p_t : t \in S)$. If $\text{mech}$ uses the IPS estimator, $\hat{f}_{\text{mech}} = \text{IPS}$, then

$$\text{ERR}(\text{mech} \mid \text{adv}) \leq |S|^{-2} \max_{a \in [n]} \sum_{t \in S} 1/p_t(a) \quad \text{for any adversary } \text{adv}. \quad (2.6)$$

This guarantee is “folklore”, see Appendix E for a proof.

3 Homogeneous agents

This section treats the fundamental case of homogeneous agents. We obtain an optimal solution, with nearly matching upper/lower bounds, and introduce key ideas for the general case.
3.1 Benchmark

We express our results in terms of a benchmark which encapsulates the optimal dependence on the prior. To this end, we consider recommendation policies that input the true state (sig = ψ*); we call them state-aware. We optimize among all BIR, state-aware policies σ so as to minimize the error upper bound (2.6). More precisely, we’d like to minimize minₐ1/σₐ(ψ*) in the worst case over adversaries adv, i.e., in the worst case over all possible realizations of ψ*.

Formally, the state-aware benchmark is defined as follows:

\[
\text{bench}(\mathcal{P}) = \inf_{\text{BIR, state-aware policies } \sigma} \max_{\text{states } \psi \in \text{support}(\mathcal{P}), \text{arms } a \in [n]} \frac{1}{\sigma_a(\psi)}. \tag{3.1}
\]

An alternative characterization of this benchmark is based on epsilon-greedy, a well-known bandit algorithm captured (in our terms) by the next definition.

**Definition 3.1.** Fixing ϵ ≥ 0 and state ψ, (ϵ, ψ)-greedy is a recommendation policy that explores with probability ϵ ≥ 0, choosing an arm independently and uniformly at random, and exploits with the remaining probability, choosing the best arm a*(ψ) for a given state ψ. We use this policy with ψ = ψ* and the largest ϵ such that this policy is BIR.

**Lemma 3.2.** The supremum in Eq. (3.1) is attained by (ϵ, ψ*)-greedy, for some ϵ ≥ 0. Thus:

\[
\text{bench}(\mathcal{P}) = \inf_{\epsilon \geq 0: (\epsilon, \psi^*)\text{-greedy is BIR}} \frac{n}{\epsilon}. \tag{3.2}
\]

**Remark 3.3.** Both formulations, (3.1) and (3.2), are used in the analysis: the former for the lower bound and the latter for the upper bound. (ϵ, ψ)-greedy policy is also used in our mechanism. While epsilon-greedy is suboptimal as a regret-minimizing bandit algorithm, it suffices to achieve optimal performance for our purposes.

**Proof of Lemma 3.2.** The “≤” direction in (3.2) holds because whenever policy (ϵ, ψ*)-greedy is BIR, we have σₐ(ψ) ≥ ϵ/n for all arms a and all states ψ. Focus on the “≥” direction from here on. Fix δ > 0. Let σ be a BIR, state-aware policy which gets within δ of the supremum in (3.1), so that σₐ(ψ) ≥ \(\frac{1}{\text{bench}(\mathcal{P}) + \delta}\) for all arms a and all states ψ. Let σ' be the (ϵ, ψ*)-greedy policy with ϵ = \(\frac{n}{\text{bench}(\mathcal{P}) + \delta}\). Its expected utility is

\[
U_P(\sigma') = \mathbb{E}_{\psi \sim \mathcal{P}} \left[ (1 - \epsilon + \epsilon/n) \cdot U_\psi(a^*(\psi)) + \frac{\epsilon}{n} \cdot \sum_{a \neq a^*(\psi)} U_\psi(a) \right] \\
\geq \mathbb{E}_{\psi \sim \mathcal{P}} \left[ \sigma_{a^*(\psi)}(\psi) \cdot U_\psi(a^*(\psi)) + \sigma_a(\psi) \cdot \sum_{a \neq a^*(\psi)} U_\psi(a) \right] \tag{3.3}
\]

\[
= U_P(\sigma) \geq U_{\text{out}}.
\]

The inequality in (3.3) holds since σₐ(ψ) ≥ ϵ/n, so for each state the right-hand side shifts probabilities towards arms with weakly smaller expected utility. It follows that policy σ' is BIR. So, this policy gets within δ from the supremum in (3.2). This holds for any δ > 0.

The benchmark is finite as long as the best arm strictly improves over the outside option. To express this point, let a*(ψ) ∈ argmaxₐ∈[n] U_ψ(a) be the best arm: a utility-maximizing action for a given state ψ, ties broken arbitrarily. Its Bayesian-expected reward is \(U^*_P := \mathbb{E}_{\psi \sim \mathcal{P}} [U_\psi(a^*(\psi))].\)

\(^{17}\)Failing that, we have a degenerate case when the patients believe that the trial is completely useless.
Claim 3.4. If $U^*_P > U_{out}$ then $\text{bench}(P) < \infty$.

Proof. If $E_{\psi \sim P}[U_\psi(a^*(\psi))] > U_{out}$, there exists $\epsilon > 0$ such that policy $(\epsilon, \psi^*)$-greedy is BIR. Lemma 3.2 then implies that the benchmark is finite. \qed

3.2 Our mechanism: Main stage

We focus on the main stage, coming back to the warm-up stage in Section 3.3. We use the same recommendation policy in all rounds of the main stage. Namely, we use policy $(\epsilon, \psi)$-greedy, for some fixed $\epsilon, \psi$. Since the true state $\psi^*$ is not known, we instead use state $\overline{\psi}$ which summarizes the data from the warm-up stage. Specifically, let $\overline{\psi}(a)$ be the empirical distribution over outcomes observed in the rounds of the warm-up stage when a given arm $a$ is chosen. Note that the average state $\overline{\psi}(a)$ is not necessarily in support($P$), but the $(\epsilon, \overline{\psi})$-greedy policy is well-defined.

To recap, our mechanism uses $(\epsilon, \overline{\psi})$-greedy recommendation policy in all rounds, for some $\epsilon > 0$. Our guarantee is conditional on collecting enough samples in the warm-up stage.

Theorem 3.5. Assume $U^*_P > U_{out}$. Suppose mechanism mech uses $(\epsilon, \overline{\psi})$-greedy policy in each round of the main stage, where $\epsilon = \frac{n}{T - T_0} \cdot \text{bench}(P)$, and estimator $\hat{f}_{\text{mech}} = \text{IPS}$ as per Eq. (2.5). Then:

(a) $\text{ERR (mech | adv)} \leq \frac{2 \cdot \text{bench}(P)}{T - T_0}$ for any adversary adv.

(b) Suppose each arm appears in at least $N_P$ rounds of the warm-up stage, with probability 1 over the agents’ beliefs. Here $N_P < \infty$ is a parameter determined by the prior $P$, as per Eq. (3.5). Then the mechanism is BIR over the main stage.

The requisite number of warm-up samples, $N_P$, is expressed in terms of the prior gap,

$$\text{gap}(P) := E_{\psi \sim P}[U_\psi(a^*(\psi)) - \frac{1}{n} \cdot \sum_{a \in [n]} U_\psi(a)],$$

the Bayesian-expected difference in utility between the best arm and the uniformly-average arm. Note that $\text{gap}(P) > 0$ provided that $U^*_P > U_{out}$. Then

$$N_P = 32\alpha^{-2} \log(8n/\alpha), \quad \text{where } \alpha = n \cdot \text{gap}(P)/\text{bench}(P) > 0.$$

We illustrate $\text{bench}(P)$ and $N_P$ with a simple numerical example in Appendix A.

Proof of Theorem 3.5. Let’s apply concentration to the approximate state $\overline{\psi}$. By Azuma-Hoeffding inequality, for any state $\overline{\psi}$ and any $\delta > 0$ we have

$$\mathbb{P}_{\text{Bayes}}\left[ |U_{\overline{\psi}}(a) - U_\psi(a)| \geq \delta \mid \psi^* = \overline{\psi} \right] \leq p_{\text{err}} := 2n \cdot \exp\left(-2\delta^2 N_P\right) \quad \forall a \in [n].$$

So, one can compare the best arm $a^*(\psi)$ for the true state $\psi^* = \psi$ and the best arm $a^*(\overline{\psi})$ for $\overline{\psi}$:

$$\mathbb{P}_{\text{Bayes}}\left[ U_\psi(a^*(\overline{\psi})) \geq U_\psi(a^*(\psi)) - 2\delta \mid \psi^* = \psi \right] \geq 1 - p_{\text{err}}.$$  \hfill (3.6)
Next, we show that mech is BIR in the main stage. Henceforth, let \( \delta = \frac{n}{8} \cdot \text{gap}(\mathcal{P})/\text{bench}(\mathcal{P}) \).

The agent’s expected utility in a single round of the main stage is

\[
U_{\mathcal{P}}(\epsilon, \psi)_{\text{greedy}} = \mathbb{E}_{\psi \sim \mathcal{P}} \left[ (1 - \epsilon) \cdot U_{\psi}(a_{\psi}) + \frac{\epsilon}{n} \cdot \sum_{a \in [n]} U_{\psi}(a) \right]
\]

\[
\geq \mathbb{E}_{\psi \sim \mathcal{P}} \left[ (1 - \epsilon) \cdot U_{\psi}(a^*(\psi)) + \frac{\epsilon}{n} \cdot \sum_{a \in [n]} U_{\psi}(a) \right] - p_{\text{err}} - 2\delta
\]

\[
= \mathbb{E}_{\psi \sim \mathcal{P}} \left[ (1 - 2\epsilon) \cdot U_{\psi}(a^*(\psi)) + \frac{2\epsilon}{n} \cdot \sum_{a \in [n]} U_{\psi}(a) \right] + \epsilon \cdot \text{gap}(\mathcal{P}) - p_{\text{err}} - 2\delta
\]

\[
\geq U_{\mathcal{P}}(\sigma^*),
\]

where we denote policy \((2\epsilon, \psi^*)_{\text{greedy}} with \(\sigma^*\). The first inequality holds by (3.6) (and the fact that utilities are at most 1). The last inequality holds because \(\epsilon \cdot \text{gap}(\mathcal{P}) - p_{\text{err}} - 2\delta \geq 0\) by our choice of \(\epsilon, \delta\).

To complete the BIR proof, \(U_{\mathcal{P}}(\sigma^*) \geq U_{\text{out}}\) by Lemma 3.2 since \(2\epsilon = n/\text{bench}(\mathcal{P})\).

Finally, recall that in each round of the main stage, each arm is sampled with probability at least \(\epsilon/n = 1/(2 \cdot \text{bench}(\mathcal{P}))\). Thus, Theorem 3.5(a) follows from Eq. (2.6). \(\square\)

3.3 Our mechanism: Warm-up stage

We rely on prior work on incentivized exploration to guarantee warm-up samples. Restated in our notation, the relevant guarantee is as follows:

Lemma 3.6 [Mansour et al. 2020]. There exists a number \(M_\mathcal{P} \leq \infty\) which depends only on the prior \(\mathcal{P}\) (but not on the time horizon \(T\)) such that:

(a) if \(M_\mathcal{P} < \infty\) then there exists a BIR mechanism which samples all arms in \(M_\mathcal{P}\) rounds.

(b) if \(M_\mathcal{P} = \infty\) and \(n = 2\) arms then no strictly BIR mechanism can sample both arms.

Thus, sufficient warm-up stage for Theorem 3.5 lasts \(T_0 = M_\mathcal{P} \cdot N_\mathcal{P}\) rounds, and consists of \(N_\mathcal{P}\) consecutive runs of the mechanism from Lemma 3.6(a). The expression for \(M_\mathcal{P}\) is somewhat tedious, and not essential for this paper.\(^{18}\)

The warm-up stage can be replaced with a non-BIR mechanism that incentivizes agent participation by other means, e.g., by paying the volunteers, as commonly done in medical trials. Crucially, \(n \cdot N_\mathcal{P}\) volunteers suffice to bootstrap our mechanism to run for an arbitrarily large \(T\).

3.4 Lower bound

We prove that the guarantee in Theorem 3.5(a) is essentially optimal in the worst case.

\(^{18}\)However, let us present it for \(n = 2\) arms, for the sake of completeness. Consider \(\Delta_\psi := U_{\psi}(1) - U_{\psi}(2)\), the “gap” between the arms’ expected utilities. W.l.o.g., assume that \(E_\psi[\Delta] \geq 0\), i.e., arm 1 is (weakly) preferred initially. Let \(X_{(k)} := E_{\psi}[\Delta \mid S_{(k)}]\), where \(S_{(k)}\) is a tuple of \(k\) independent reward samples from arm 1. Then, suppose there exists some finite \(k = k_\mathcal{P}\) such that \(\Pr[X_{(k)} > 0] > 0\) (if not, set \(M_\mathcal{P} = \infty\)). Let \(Y = X_{(k_\mathcal{P})}\). Then

\[
M_\mathcal{P} = (n + 1) \max \left( k_\mathcal{P}, \frac{E_\psi[\Delta]}{E(Y \mid Y > 0) \Pr[Y > 0]} \right).
\]
Our result applies to all mechanisms that do not adapt to new observations over some constant (but possibly small) fraction of rounds, and are unrestricted otherwise. Formally, given a contiguous set $S$ of rounds, we say that a mechanism is non-data-adaptive over $S$ if for each round $t \in S$ the sampling distribution $p_t$ does not depend on the data collected during $S$.

**Theorem 3.7.** Fix prior $P$ and time horizon $T \geq \Omega \left( \text{bench}(P)^{-2} \right)$ and $T_0 \leq T/2$. Let $S$ be a contiguous subset of rounds in the main stage, with $|S| \geq c \cdot T$ for some absolute constant $c > 0$. Fix mechanism $\text{mech}$ that is non-data-adaptive over $S$. Then some adversary $\text{adv}$ has

$$\text{ERR} \left( \text{mech} \mid \text{adv} \right) = \Omega \left( \frac{\text{bench}(P)}{T} \right).$$

Thus, Theorem 3.7 allows per-round policies that change over time and can themselves be jointly chosen at random. In rounds $t \notin S$ the policies can depend on all past observations. Importantly, this includes “multi-stage” mechanisms that partition the time into a constant number of contiguous “stages” and are non-data-adaptive within each (possibly with a data-adaptive warm-up stage of duration, say, $< T/2$). Most medical trials in practice fit this model, see Footnote 7.

**Proof of Theorem 3.7.** Fixing the history of the first $t_0 = \min(S) - 1$ rounds, the mechanism draws the tuple of sampling distributions $(p_t : t \in S)$, perhaps jointly at random. We prove that

$$\text{ERR} \left( \text{mech} \mid \text{adv}^\dagger \right) \geq \Omega \left( \frac{1}{|S|^2} \max_{a \in [n]} \sum_{t \in S} \min \left\{ \frac{1}{|E[a(t)]}, \sqrt{|S|} \right\} \right),$$

for some adversary $\text{adv}^\dagger$ determined by the expected sampling probabilities $(E[p_t] : t \in S)$. This is a general statistical tool, see Appendix D for a standalone formulation, discussion, and proof.

Given state $\psi$, let $\text{adv}_\psi$ be the adversary restricted to the first $t_0$ rounds that samples outcome $\omega_{a,t}$ from distribution $\psi(a)$, independently for each arm $a$ and each round $t \leq t_0$. Recall that $\text{adv}_\psi$ and $\text{mech}$ jointly determine the sampling distributions $p_t$ for all rounds $t > t_0$. Let $N_\psi(a)$ be the number of times arm $a$ is selected during the rounds in $S$ under adversary $\text{adv}_\psi$.

We claim there exists a state $\psi_0 \in \text{supp}(P)$ and arm $a_0$ such that

$$E \left[ N_{\psi_0}(a_0) \right] \leq |S|/\text{bench}(P).$$

Suppose not. Then we can construct a state-aware policy $\sigma$ which contradicts (3.1). Specifically, policy $\sigma$ simulates a run of $\text{mech}$ under adversary $\text{adv}_{\psi^*}$, chooses a round uniformly at random from the main stage, and recommends the same arm as $\text{mech}$ in this round. Such policy is BIR, and therefore would make the right-hand side of (3.1) strictly larger than $\text{bench}(P)$, contradiction. Claim proved. From here on, we fix state $\psi_0$ and arm $a_0$ which satisfy (3.9).

The adversary $\text{adv}$ is constructed as follows. We use the adversary $\text{adv}_{\psi_0}$ for the first $t_0$ rounds, and adversary $\text{adv}^\dagger$ afterwards. It immediately follows from (3.8) that

$$\text{ERR} \left( \text{mech} \mid \text{adv} \right) \geq \Omega \left( \frac{1}{|S|^2} \sum_{t \in S} \max \left\{ \frac{1}{q_t}, \sqrt{|S|} \right\} \right),$$

where $q_t := E[p_t(a_0)]$

$$\geq \Omega \left( \frac{1}{|S|^2} \left( \sqrt{|S| \cdot |S_L| + \sum_{t \in S_R} \frac{1}{q_t}} \right) \right),$$

where $S_L = \left\{ t \in S : q_t \leq 1/\sqrt{|S|} \right\}$ and $S_R = \left\{ t \in S : q_t > 1/\sqrt{|S|} \right\}$. In words, $S_L$ (resp., $S_R$) comprises the rounds in the main stage with low (resp., high) expected sampling probability $q_t$. 

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In the remainder of the proof, we analyze the right-hand side of (3.10). This is an expression in terms of deterministic scalars \( q_t \in [0, 1], t \in S \). We leverage the fact that (3.9) holds, so that

\[
N_H := \sum_{t \in S_H} q_t \leq \sum_{t \in S} q_t = E \left[ N_{\psi_0}(a_0) \right] \leq |S|/\text{bench}(P).
\]

By harmonic-arithmetic mean inequality,

\[
\sum_{t \in S_H} \frac{1}{q_t} \geq |S_H|^2/N_H \geq |S_H|^2 \cdot \text{bench}(P)/|S|.
\]

Plugging this back into (3.10) and noting that \( |S_L| + |S_H| = |S| \geq T/2 \), we have

\[
\text{ERR} (\text{mech} \mid \text{adv}) \geq \Omega \left( \sqrt{|S|} \cdot |S_L| + |S_H|^2 \cdot \text{bench}(P)/|S| \right) \geq \Omega \left( \frac{\text{bench}(P)}{T} \right).
\]

4 **Heterogeneous agents**

We now consider the general case of heterogeneous agents. Agent heterogeneity is two-fold: different agent types may have different outcome distributions for the same arm, and different subjective (and not directly observable) utilities for the same outcome. We model this as follows: **public types determine outcome distributions** and the **private types determine subjective utilities**.

The intuition is that beliefs about objective outcomes are driven by the patient’s objective medical history, which is typically well-documented and available to the clinical trial. However, patients’ preferences over outcome distributions can be highly subjective, e.g., in evaluating the disutility of particular medical complications. These subjective preferences are not known to the mechanism. In our model, each agent reports its private type upon arrival, and the mechanism must incentivize truthful reporting (in addition to the BIR property).

4.1 **Overview**

We obtain matching upper and lower bounds like in Section 3, but with a new benchmark, mechanism, and analysis. This novelty is necessitated by the BIC condition which binds the types together.\(^{19}\) We define a new benchmark, denoted \( \text{bench}(P, \Theta) \), which generalizes (3.1) to a given set \( \Theta \) of agent types. Compared to the expression in (3.1), this benchmark requires all BIC and BIR per-round policies under the inf, and takes the max over \( \Theta \) (as well as over all states). Our mechanism attains the same error bound as in Theorem 3.5 relative to \( \text{bench}(P, \Theta) \), with a similar but much more technical sufficient condition for BIR. The lower bound carries over from Theorem 3.7, replacing \( \text{bench}(P) \) with \( \text{bench}(P, \Theta) \).

Our mechanism is more complex compared to Section 3. Exploration probabilities now depend on the arm and the agent type, and are coordinated across types due to the BIC constraint. We specify them indirectly, via a state-aware policy that optimizes the benchmark. The true state \( \psi^* \) is estimated via the maximum likelihood estimator (MLE) given the data from the warm-up stage.

In terms of proof techniques, our mechanism requires more subtlety compared to Section 3 so as to analyze the MLE estimator. The lower bound is proved using the same techniques; the novelty here lies in formulating the definitions so that the theorem and techniques indeed carry over.

\(^{19}\) Indeed, without private types one can use mechanism from Section 3 tailored to each particular type.
4.2 Model: heterogeneous agents

We denote agent’s type as \( \theta = (\theta_{\text{pub}}, \theta_{\text{pri}}) \), where \( \theta_{\text{pub}} \in \Theta_{\text{pub}} \) is the public type, and \( \theta_{\text{pri}} \in \Theta_{\text{pri}} \) is the private type. The set of all possible types, \( \Theta = \Theta_{\text{pub}} \times \Theta_{\text{pri}} \), is known to the mechanism. We assume that it is finite. The case of homogeneous agents is when \( |\Theta| = 1 \).

Below we spell out the necessary changes to the model, compared to Section 2.

**Bandit model.** In the interaction protocol in Section 2.1, step 1 is modified as follows:

1a. A new agent arrives, with type \( \theta_t = (\theta_{\text{pub}}^{(t)}, \theta_{\text{pri}}^{(t)}) \) and outcomes \( (\omega_{a,t} \in \Omega : a \in [n]) \).
1b. The mechanism observes the public type \( \theta_{\text{pub}}^{(t)} \), but not the private type \( \theta_{\text{pri}}^{(t)} \) or the outcomes.
1c. The agent reports its private type, not necessarily truthfully, as reported type \( \theta_{\text{rep}}^{(t)} \in \Theta_{\text{pri}} \).

The pair \( \theta_{\text{obs}}^{(t)} = \left( \theta_{\text{pub}}^{(t)}, \theta_{\text{rep}}^{(t)} \right) \) observed by the mechanism will be called the observed type.

The agents’ information structure is the same as in the homogeneous case, with one addition: agents know their own types. An additional economic constraint defined below (Definition 4.5) ensures truthful reporting of the private types: \( \theta_{\text{rep}}^{(t)} = \theta_{\text{pri}}^{(t)} \) for all rounds \( t \).

The adversary now specifies the agents’ types, in addition to the outcomes. Formally, the table \( (\theta_t, \omega_{a,t} \in \Omega : a \in [n])_{t \in [T]} \) is fixed in advance (where \( t \) indexes rows). The table drawn at random from some distribution called an adversary. A stochastic adversary is defined as before: it draws each row \( t \) from some fixed distribution.

The history now includes types: the tuple \( \text{hist}_t = (\theta_t, p_s, a_s, \omega_s) \) denotes the history collected by the mechanism before a given round \( t \), assuming truthful reporting of agents’ types in the previous rounds.

**Statistical model.** Given an adversary \( \text{adv} \), the average score \( f_{\text{adv}}(a) \) and the estimation error \( \text{ERR} (\text{mech} | \text{adv}) \) are defined exactly the same as in the homogeneous case, as per Eq. (2.1) and Eq. (2.2). The estimation error is defined assuming truthful reporting.

**Remark 4.1.** Under truthful reporting, the adversary controls the types observed by the mechanism (because reported types are just private types, which are selected by the adversary). An alternative interpretation for the statistical model, not contingent on truthful reporting, is that the adversary controls reported types directly (and then the private types are irrelevant).

**Economic model.** The definitions are extended to include the agents’ types, as per the semantics described above. In what follows, consider an agent with type \( \theta = (\theta_{\text{pub}}, \theta_{\text{pri}}) \in \Theta_{\text{pub}} \times \Theta_{\text{pri}} \).

Agents’ beliefs are extended as follows. When some arm \( a \) is chosen, the outcome is an independent draw from some distribution \( \psi^*(a, \theta_{\text{pub}}) \) over outcomes; note that the outcome distribution is determined by both the arm and the public type. Mappings \( \psi : [n] \times \Theta_{\text{pub}} \rightarrow \Delta_\Omega \) are identified as states, and \( \psi^* \) as the true state. As before, the agents believe that \( \psi^* \) is drawn from some Bayesian prior \( \mathcal{P} \) over finitely many possible states. In each round \( t \), type \( \theta_t \) is drawn independently from some fixed distribution \( D_{\text{type}} \). Both \( \mathcal{P} \) and \( D_{\text{type}} \) are known to the mechanism and the agents.

To simplify presentation, we assume that all outcome distributions \( \psi(a, \theta_{\text{pub}}) \) have full support over the set \( \Omega \) of possible outcomes, i.e., that this holds for all arms \( a \), public types \( \theta_{\text{pub}} \), and states \( \psi \in \text{support}(\mathcal{P}) \); we term this the full support assumption.

The agent’s utility for a given outcome \( \omega \) is \( u(\omega, \theta_{\text{pri}}) \in [0, 1] \); note that it is determined by both the outcome and the private type. The realized utility is subjective and not directly observable. The utility structure \( u : \Omega \times \Theta_{\text{pri}} \rightarrow [0, 1] \) is known to the mechanism and all agents.

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20 Prior \( \mathcal{P} \) represents common pre-existing medical knowledge, so it is reasonable for it to be same for all agents.

21 This assumption can be removed, at the cost of some technical complications, see Remark 4.13.
Thus, the agent’s expected utility and outside option are redefined as

\[ U_\psi(a, \theta) = \mathbb{E}_{\omega \sim \psi(a, \theta_{\text{pub}})} u(\omega, \theta_{\text{pri}}) \text{ for arm } a \text{ and state } \psi, \]

\[ U_{\text{out}}(\theta) = \max_{a \in \mathcal{A}} \mathbb{E}_{\psi \sim \mathcal{P}} [U_\psi(a, \theta)]. \]

**Remark 4.2.** Our analysis also accommodates an arbitrary known \( U_{\text{out}} : \Theta \to [0,1] \), like in the homogeneous case, see Remark 2.4. All results carry over with minimal modifications.

To extend the BIR property, we focus on Bayesian-expected utility from truthful reporting. Let \( \mathcal{E}_t^{\text{tru}} \) be the event that all agents \( s \leq t - 1 \) have reported their private type truthfully: \( \theta_{\text{rep}}^{(s)} = \theta_{\text{pri}}^{(s)} \).

Let \( \mathbb{E}_{\text{Bayes}}[\cdot | \mathcal{E}] \) denote expectation with respect to agents’ Bayesian beliefs given event \( \mathcal{E}^{22} \).

**Definition 4.3.** The mechanism is BIR if for each round \( t \),

\[ \mathbb{E}_{\text{Bayes}}[U_{\psi^*}(a_t, \theta) | \mathcal{E}_t^{\text{tru}} \text{ and } \theta_t = \theta] \geq U_{\text{out}}(\theta) \text{ for any type } \theta \in \Theta. \tag{4.1} \]

The mechanism is called BIR on a given set \( S \) of rounds if (4.1) holds for each round \( t \in S \).

**Remark 4.4.** We require the BIR property to hold for all agent types, even though it may be more difficult to satisfy for some types than for some others. Indeed, if some agent type chooses not to participate, the score estimates \( \hat{f}_{\text{mech}} \) computed by the mechanism would likely suffer from selection bias. Technically, the BIR property can be ensured w.l.o.g. for every agent (at the expense of estimation quality) simply by choosing this agent’s outside option deterministically.

We also require the mechanism to (weakly) incentivize truthful reporting of private types, as captured by the following definition.

**Definition 4.5.** The mechanism is called **Bayesian Incentive Compatible** (BIC) if for any round \( t \), type \( \theta \in \Theta \), and private type \( \theta_{\text{pri}}' \in \Theta_{\text{pri}} \) it holds that

\[ \mathbb{E}_{\text{Bayes}}[U_{\psi^*}(a_t, \theta) | \mathcal{E}_t^{\text{tru}}, \theta_t = \theta] \geq \mathbb{E}_{\text{Bayes}}[U_{\psi^*}(a_t, \theta) | \mathcal{E}_t^{\text{tru}}, \theta_{\text{pub}}^{(t)} = \theta_{\text{pub}}, \theta_{\text{rep}}^{(t)} = \theta_{\text{pri}}']. \tag{4.2} \]

The mechanism is called BIC on a given set \( S \) of rounds if (4.2) holds for each round \( t \in S \).

Given a BIC mechanism, it is a (weakly) best response for each agent to report the private type truthfully, assuming the previous agents did so. We assume then that all agents report truthfully.

**Recommendation policies** are redefined to also input an observed type. Formally, a recommendation policy \( \sigma \) with signal \( \text{sig} \) is a distribution over arms, parameterized by the signal and the observed type \( \theta \in \Theta \). We denote this distribution as \( \sigma(\text{sig}, \theta) \). (Recall that signal is defined as per Definition 2.5.) Accordingly, the Bayesian-expected utility of policy \( \sigma \) is redefined as

\[ U_P(\sigma, \theta) := \mathbb{E}_{(\psi, \text{sig}) \sim \mathcal{Q}} \mathbb{E}_{\omega \sim \sigma(\text{sig}, \theta)} U_\psi(a, \theta), \text{ where } \theta \in \Theta. \tag{4.3} \]

The notions of BIR and BIC carry over in a natural way. The policy is called **BIR** if any agent weakly prefers reporting truthfully and following the policy compared to the outside option:

\[ U_P(\sigma, \theta) \geq U_{\text{out}}(\theta) \quad \forall \theta \in \Theta. \tag{4.4} \]

\[ ^{22} \text{We use non-standard notation for the expectation to emphasize that it only needs to be well-defined under event } \mathcal{E}, \text{ even though it may be ill-defined unconditionally.} \]
The policy is called BIC if for any type \( \theta = (\theta_{\text{pub}}, \theta_{\text{pri}}) \in \Theta \) and any observed type \( \theta_{\text{obs}} = (\theta_{\text{pub}}, \theta_{\text{rep}}) \in \Theta \) it holds that

\[
U_P(\sigma, \theta) \geq \mathbb{E}_{(\psi, \mathbf{sig}) \sim Q} \mathbb{E}_{a \sim \sigma(\mathbf{sig}, \theta_{\text{obs}})} U_{\psi}(a, \theta).
\]  

(4.5)

The mechanism can be represented as a collection of recommendation policies (\( \sigma_t : t \in [T] \)), where policy \( \sigma_t \) inputs history \( \text{hist}_t \) as a signal and determines the sampling distribution as

\[
p_t = \sigma_t(\text{hist}_t, \theta_{\text{obs}}^{(t)}) \quad \text{for each round } t.
\]  

(4.6)

The mechanism is BIR (resp., BIC) in a given round \( t \) if and only if policy \( \sigma_t \) is BIR (resp., BIC).

### 4.3 Our results

We obtain matching upper and lower bounds like in \textsection 3, but with more technicalities and new benchmark, mechanism, and analysis. This novelty is necessitated by the presence of multiple private types and the BIC condition which binds the types together. Indeed, without private types (formally, \( |\Theta_{\text{pri}}| = 1 \)), the BIC condition vanishes, and the public types are not bound to one another by the BIR condition alone. Then one can treat each public type separately, and use the mechanism from \textsection 3 tailored to this type.

**Benchmark.** Our bounds are expressed in terms of the following benchmark:

\[
\text{bench}(P, \Theta) = \inf_{\text{BIC, BIR, state-aware } \Theta\text{-policies } \sigma} \max_{\psi \in \text{support}(P), a \in [n]} \max_{\text{types } \theta \in \Theta} \frac{1}{\sigma_a(\psi, \theta)}.
\]  

(4.7)

Here, \( \Theta\)-policies are recommendation policies with type set \( \Theta \), as defined above.

This benchmark generalizes (3.1), the benchmark for the homogeneous case, taking the max over all types inside the expression. In particular, (3.1) can be written as \( \text{bench}(P, \{\theta\}) \), where \( \theta \) is the unique type in the homogeneous case. In fact, the general benchmark admits a lucid alternative characterization via the worst type:

\[
\text{bench}(P, \Theta) = \max_{\theta' \in \Theta} \text{bench}(P, \{\theta'\}).
\]  

(4.8)

However, our analysis entirely relies on the original formulation (4.7).

**Proof of Eq. (4.8).** Denote the right-hand side of Eq. (4.8) with \( \text{bench}(P, \Theta) \). First, it is easy to observe that \( \text{bench}(P, \Theta) \leq \text{bench}(P, \{\theta\}) \), since \( \text{bench}(P, \Theta) \) only requires BIR without incentive constraints. Next, we show that for any BIR policy \( \sigma \) with worst-case benchmark value \( \text{bench}(P, \Theta) \), we can design a BIR and BIC policy that matches the same worst-case benchmark value.

In particular, consider another policy \( \tilde{\sigma} \) which for any type \( \theta \in \Theta \), offers the utility-maximizing option among the “per-type” policies \( \sigma(\cdot, \theta') \), \( \theta' \in \Theta \). That is, for all states \( \psi \) and types \( \theta \),

\[
\tilde{\sigma}(\psi, \theta) := \sigma(\psi, \theta^*(\theta)), \quad \text{where } \theta^*(\theta) \in \arg\max_{\theta' \in \Theta} U_P(\sigma(\cdot, \theta'), \theta),
\]

with ties broken in some fixed way. Policy \( \tilde{\sigma} \) is BIC by design, because \( U_P(\tilde{\sigma}, \theta) \geq U_P(\sigma(\cdot, \theta_{\text{obs}}), \theta) \) for any observed type \( \theta_{\text{obs}} \). Plugging in \( \theta_{\text{obs}} = \theta \), we see that

\[
U_P(\tilde{\sigma}, \theta) \geq U_P(\sigma, \theta) \geq U_{\text{out}}(\theta),
\]

where the last inequality holds by the BIR property of \( \sigma \). So, policy \( \tilde{\sigma} \) is BIR. Finally, the minimum sampling probability is the same for both policies:

\[
\min_{\psi \in \text{support}(P), a \in [n], \text{types } \theta \in \Theta} \sigma_a(\psi, \theta) = \min_{\psi \in \text{support}(P), a \in [n], \text{types } \theta \in \Theta} \tilde{\sigma}_a(\psi, \theta).
\]

\(\square\)
To state the suitable non-degeneracy conditions, let us extend the best-arm notation from Section 3. Fix type \( \theta = (\theta_{\text{pub}}, \theta_{\text{rep}}) \in \Theta \). Let \( a^*(\psi, \theta) \in \arg\max_{a \in \Theta} U^*_\psi(a, \theta) \) be the best arm for a given state \( \psi \). The Bayesian-expected reward of the best arm is then
\[
U^*_\psi(\theta) := \mathbb{E}_{\psi \sim \mathcal{P}} \left[ U^*_\psi \left( a^*(\psi, \theta) \right) \right] \quad (4.9)
\]
If the best arm is computed for the reported type \( \theta_{\text{rep}} \in \Theta_{\text{pri}} \) (which is not necessarily reported truthfully), its Bayesian-expected reward becomes
\[
U^*_\psi(\theta, \theta_{\text{rep}}) := \mathbb{E}_{\psi \sim \mathcal{P}} \left[ U^*_\psi \left( a^*(\psi, \theta') \right) \right], \quad \text{where} \ \theta' = (\theta_{\text{pub}}, \theta_{\text{rep}}). \quad (4.10)
\]
Thus, the non-degeneracy condition is:
\[
U^*_\psi(\theta) > U^*_\psi(\theta_{\text{rep}}) \quad \forall \theta \in \Theta, \ \theta_{\text{rep}} \in \Theta_{\text{pri}}. \quad (4.11)
\]
Remark 4.6. We can interpret the best arm \( a^* = a^*(\psi^*, \theta) \) as a state-aware recommendation policy, called the best-arm policy. Then condition (4.11) states that \( a^* \) is strictly BIR and strictly BIC.

Claim 4.7. Assume (4.11). Then \( \text{bench}(\mathcal{P}, \Theta) < \infty \).

Proof. By (4.11), the \((\epsilon, \psi^*)\)-greedy policy is BIC and BIR for some \( \epsilon > 0 \). This policy satisfies the conditions under the inf in (4.7), and has a finite “benchmark value”, namely \( \frac{1}{\epsilon} \). \( \square \)

Lower bound. The lower bound carries over from Theorem 3.7 and is proved using the same techniques. The novelty is in formulating the definitions so that the theorem and the techniques indeed carry over. We defer the proof to Appendix C.

Theorem 4.8. The statement in Theorem 3.7 holds for benchmark \( \text{bench}(\mathcal{P}, \Theta) \).

Positive results. Our mechanism for the main stage is more complex compared to the one in Section 3. Exploration probabilities now depend on the arm and the agent type, and are coordinated across types due to the BIC constraint. We specify them indirectly, via a state-aware policy that optimizes the benchmark. Specifically, let \( \sigma^\text{opt} \) be a state-aware policy which optimizes the inf in Eq. (4.7); such policy exists as an optimizer of a continuous function on a compact space. The true state \( \psi^* \) is estimated via the maximum likelihood estimator (MLE) given the data observed in the warm-up stage, denote it by \( \psi^* \sim \text{MLE} \in \text{supp}(\mathcal{P}) \).

Our mechanism is defined as follows. We choose uniformly between \( \sigma^\text{opt} \) and the best-arm policy \( a^* \). Formally: in each round \( t \) of the main stage, the sampling distribution \( p_t \) is the average between two distributions: \( \sigma^\text{opt}(\psi^*; \theta) \) and \( a^*(\psi^*; \theta) \), where \( \theta = (\theta_{\text{pub}}, \theta_{\text{rep}}) \) is the observed type. For estimator \( \hat{f}_{\text{mech}} \), we use the IPS estimator (2.5). This completes the description of the main stage. We denote our mechanism as \( \text{TwoStageMech}(T_0, \sigma^\text{opt}, a^*) \).

Theorem 4.9. Assume (4.11). Let mech be the mechanism \( \text{TwoStageMech}(T_0, \sigma^\text{opt}, a^*) \). Then:

(a) \[ \text{ERR} \left( \text{mech} \mid \text{adv} \right) \leq \frac{2 \cdot \text{bench}(\mathcal{P}, \Theta)}{T - T_0} \quad \text{for any adversary adv.} \]

(b) Suppose the following holds for the warm-up stage:
\[
\mathbb{P}_{\text{Bayes}} \left[ \text{each} \ (\text{arm, public type}) \ \text{pair appears in} \ \geq N_\mathcal{P} \ \text{rounds} \right] \geq 1 - \delta_\mathcal{P}, \quad (4.12)
\]
here \( N_\mathcal{P} < \infty \) and \( \delta_\mathcal{P} \geq 0 \) are some parameters determined by the prior \( \mathcal{P} \) and the utility structure. Then the mechanism is BIR and BIC over the main stage.

\footnote{As in Section 3, \((\epsilon, \psi^*)\)-greedy chooses the best-arm policy in the exploitation branch.}
4.4 More details for the positive result (Theorem 4.9)

The parameters in part (b) of Theorem 4.9 are driven by the degeneracy gap: the gap in the non-degeneracy condition (4.11), defined as follows:

\[ \eta^* = \min \left( \min_{\theta \in \Theta} U_P^*(\theta) - U_{\text{out}}(\theta), \min_{\theta \in \Theta, \theta_{\text{rep}} \in \Theta_{\text{pub}}} U_P^*(\theta, \theta_{\text{rep}}) \right). \]  

(4.13)

Another key parameter, \( R_{\text{min}}(P) > 0 \), measures the difficulty of learning the true state \( \psi^* \) via MLE. We defer its definition to Eq. (4.17). Then \( N_P \) is defined as follows:

\[ N_P(\eta^*) = 1 + \frac{1}{2 R_{\text{min}}(P)} \cdot \log \frac{4 |\text{supp}(P)|}{\eta^* \cdot R_{\text{min}}(P)}. \]  

(4.14)

Theorem 4.10. Theorem 4.9 holds with parameters \( \delta_P = \eta^*/8 \), where \( \eta^* \) is the non-degeneracy gap from (4.13), and \( N_P = N_P(\eta^*) \), as per Eq. (4.14).

The specification of \( R_{\text{min}}(P) \) is rather lengthy, as it requires some notation regarding log-likelihood ratios (LLR). Fix two distinct states \( \psi, \psi' \in \text{support}(P) \). For any arm \( a \), type \( \theta = (\theta_{\text{pub}}, \theta_{\text{pri}}) \in \Theta \) and outcome \( \omega \in \Omega \), let

\[ \text{LLR}_{a, \theta, \omega}(\psi, \psi') := \log \frac{\psi_\omega(a, \theta_{\text{pub}})}{\psi'_\omega(a, \theta_{\text{pub}})} \]

be the LLR between the two states. The corresponding KL-divergence is

\[ \text{KL}_{a, \theta}(\psi, \psi') := \mathbb{E}_{\omega \sim \psi(a, \theta_{\text{pub}})} \text{LLR}_{a, \theta, \omega}(\psi, \psi') = \text{KL}(\psi(a, \theta_{\text{pub}}), \psi'(a, \theta_{\text{pub}})). \]

Let is define the maximum change in LLR compared to its expectation,

\[ \ell_{\text{max}}(\psi, \psi') := \max_{a \in [n], \theta \in \Theta, \omega \in \Omega} |\text{LLR}_{a, \theta, \omega}(\psi, \psi') - \text{KL}_{a, \theta}(\psi, \psi')| < \infty. \]  

(4.15)

The minimum positive KL-divergence is defined as

\[ \text{KL}_{\text{min}}(\psi, \psi') := \min_{a, \theta} \text{KL}_{a, \theta}(\psi, \psi') > 0, \]  

(4.16)

where the \( \min \) is over all pairs \( (a, \theta) \in [n] \times \Theta \) such that \( \text{KL}_{a, \theta}(\psi, \psi') > 0 \). At least one such \((a, \theta)\) pair must exist because the two states are distinct. Finally:

\[ R_{\text{min}}(P) := \min_{\text{states } \psi, \psi' \in \text{supp}(P)} \left( \frac{\text{KL}_{\text{min}}(\psi, \psi')}{\ell_{\text{max}}(\psi, \psi')} \right)^2 > 0. \]  

(4.17)

Main stage: extensions. It may be desirable to use relaxed versions of the benchmark-optimizing policy \( \sigma_{\text{opt}}^* \) and the best-arm policy \( a^* \). Indeed, the exact versions may be too difficult to compute or too complex to implement in practice; also, the prior \( P \) or the utility structure may be not fully known to the mechanism. We relax the benchmark-optimizing policy by allowing additive slack \( \delta > 0 \), which shows up in the final regret bound. We relax the best-arm policy as an arbitrary state-aware policy \( \tilde{\sigma} \) which is BIR and BIC by some additive margin \( \eta \leq \eta^* \); this margin only affects the number of warm-up samples.

Definition 4.11. A recommendation policy is called \( \eta\text{-BIR} \) (resp., \( \eta\text{-BIC} \)) for some \( \eta \geq 0 \) if satisfies Eq. (4.4) (resp., Eq. (4.5)) with \( +\eta \) added to the right-hand side.
Theorem 4.12. Assume (4.11). Let \( \hat{\sigma}^{\text{opt}} \) be a state-aware policy which optimizes the inf in the benchmark (4.10) up to an additive factor \( \delta \geq 0 \). Let \( \hat{\sigma} \) be a state-aware policy which is \( \eta \)-BIR and \( \eta \)-BIC, for some \( \eta \in (0, \eta^* ] \). Then mechanism TwoStageMech(\( T_0, \hat{\sigma}^{\text{opt}}, \hat{\sigma} \)) satisfies part (b) in Theorem 4.9 with parameters \( \delta_P = \eta/8 \) and \( N_P = N_P(\eta) \), and satisfies

\[
\text{ERR} \left( \text{mech} \mid \text{adv} \right) \leq \frac{2 \text{bench}(\mathcal{P}, \Theta) + 2\delta}{T - T_0} \quad \text{for any adversary} \text{ adv.} \tag{4.18}
\]

We recover Theorem 4.9 since the best-arm policy is \( \eta^* \)-BIR and \( \eta^* \)-BIC by (4.13).

Remark 4.13. The “full support” assumption stated in Section 4.2 can be removed, let us outline how. First, our analysis extends to a relaxed version of this assumption, when for any given pair \( (a, \theta_{\text{pub}}) \in [n] \times \Theta_{\text{pub}} \), all outcome distributions \( \psi(a, \theta_{\text{pub}}), \psi \in \text{support}(\mathcal{P}) \) have the same support, but this support may depend on the \( (a, \theta_{\text{pub}}) \) pair. Second, if the relaxed assumption is violated, then outcome distributions \( \psi(a, \theta_{\text{pub}}) \) and \( \psi'(a, \theta_{\text{pub}}) \) have different supports, for some states \( \psi, \psi' \in \text{support}(\mathcal{P}) \), some arm \( a \), and some and public type \( \theta_{\text{pub}} \). More specifically, there is an outcome \( \omega \) that lies in the support of \( \psi(a, \theta_{\text{pub}}) \), but not in the support of \( \psi'(a, \theta_{\text{pub}}) \). Then with sufficiently many samples from the \( (a, \theta_{\text{pub}}) \) pair, one would observe outcome \( \omega \) with high probability as long as it is in the support of \( \psi^*(a, \theta_{\text{pub}}) \). This would rule out either the case \( \psi^* = \psi \) or the case \( \psi^* = \psi' \). We take \( N_P \) large enough for this to happen for every combination of \( \omega, a, \theta_{\text{pub}}, \psi, \psi' \). Consider the set \( \Psi_0 \) of all states in \( \text{support}(\mathcal{P}) \) that are not “ruled out” after the warm-up stage; with high probability, the relaxed assumption holds with respect to this set.

Warm-up stage. Warm-up data collection is treated separately, as in Section 3. It can be done either endogenously (via a BIR, BIC mechanism), or exogenously (e.g., by paying the volunteers). Either way, the sufficient amount of warm-up data depends only on the prior and the utility structure, whereas it bootstraps our mechanism to run for an arbitrarily large time horizon \( T \).

Endogenous data collection has been studied in prior work on incentivized exploration, for both public and private agent types. In particular, for public types one can collect samples for all type-arm pairs, under some assumptions (Mansour et al., 2020). For arbitrary types, one can, in some sense, explore all type-arm pairs that can possibly be explored (Immorlica et al., 2019). Recall that a sufficient amount of warm-up data needs to be guaranteed only w.r.t. agents’ beliefs, in particular, only for stochastic adversaries.

4.5 Proof of the positive result (Theorem 4.12)

First, we upper-bound the error probability for the MLE estimate, according to the agent’s beliefs.

Claim 4.14. \( \mathbb{P}_{\text{Bayes}} \left[ \psi_{\text{MLE}} \neq \psi^* \right] \leq \eta/4 \).

The proof is a lengthy argument about log-likelihood ratios, deferred to Appendix B.

Next we show that the mechanism is BIR and BIC throughout the main stage. Fix types \( \theta = (\theta_{\text{pub}}, \theta_{\text{pri}}), \theta' = (\theta_{\text{pub}}, \theta_{\text{rep}}) \in \Theta \). Using the notation from Definition 4.3 and Definition 4.5, denote

\[
U_{\text{mech}}(\psi, \theta, \theta') := \mathbb{E}_{\text{Bayes}} \left[ U_{\psi^*} \left( a_t, \theta \right) \mid s^t_{\text{tru}}, \theta_{\text{pub}}^{(t)} = \theta_{\text{pub}}, \theta_{\text{rep}}^{(t)} = \theta_{\text{rep}}, \psi_{\text{MLE}} = \psi \right] \tag{4.19}
\]

for each round \( t \) in the main stage and each state \( \psi \in \text{support}(\mathcal{P}) \). In words, this is the Bayesian-expected utility of the agent type \( \theta \) if the reported type is \( \theta_{\text{rep}} \) and the estimated state is \( \psi_{\text{MLE}} = \psi \), assuming truthful reporting in the previous rounds. By specification of the mechanism, this quantity is the same for all rounds in the main stage. Write \( U_{\text{mech}}(\psi, \theta) = U_{\text{mech}}(\psi, \theta, \theta) \) for brevity.
To verify the BIR constraints, note that the utility of agent type $\theta$ from the mechanism is
\[
U_{\text{mech}}(\psi_{\text{MLE}}, \theta) \geq (1 - \Pr[\psi_{\text{MLE}} \neq \psi^*]) \cdot U_{\text{mech}}(\psi^*, \theta)
\geq U_{\text{mech}}(\psi^*, \theta) - \Pr[\psi_{\text{MLE}} \neq \psi^*] \geq U_{\text{out}}(\theta).
\]
where the last inequality holds since $\Pr[\psi_{\text{MLE}} \neq \psi^*] \leq \frac{\eta}{4}$ and mechanism $\text{mech}$ chooses an $\eta$-BIR policy with probability $\frac{1}{2}$.
To verify the BIC constraints:
\[
U_{\text{mech}}(\psi_{\text{MLE}}, \theta) \geq U_{\text{mech}}(\psi^*, \theta) - \Pr[\psi_{\text{MLE}} \neq \psi^*]
\geq U_{\text{mech}}(\psi^*, \theta, \theta') + \frac{1}{2} \cdot \eta - 2 \Pr[\psi_{\text{MLE}} \neq \psi^*] \geq U_{\text{mech}}(\psi_{\text{MLE}}, \theta, \theta').
\]
The second inequality holds since mechanism $\text{mech}$ chooses an $\eta$-BIC policy with probability $\frac{1}{2}$.
Finally, we derive the error bound for statistical estimation. Invoking inequality (2.6), we have
\[
\text{ERR}(\text{mech} | \text{adv}) \leq \max_{a \in \{n\}} \frac{1}{(T - T_0)^2} \sum_{t \in [T_0, T]} \frac{1}{p_t(a)}
\leq \frac{2}{(T - T_0)^2} \max_{a \in \{n\}} \max_{\psi} \sum_{t \in [T_0, T]} \frac{1}{1} \cdot \frac{\sigma_{\text{opt}}(\psi, \theta)}{\sigma_a(\psi, \theta)}
\leq \frac{2 \text{bench}(P, \Theta) + 2\delta}{T - T_0}.
\]

5 Heterogeneous agents with estimated type frequencies
In this section, we consider heterogeneous agents from a different perspective, whereby we fix the distribution $F_{\text{type}}$ of type frequencies. The benchmark $\text{bench}(P, \Theta)$ in Section 4 takes a max over all types (see Eq. (4.8)), and may therefore be skewed towards “bad” types – ones with large benchmark value – that are difficult for the mechanism to handle. To mitigate this issue when the bad types are rare, we define an alternative benchmark parameterized by $F_{\text{type}}$, denoted $\text{bench}(P, F_{\text{type}})$, which replaces the max over all types with an expectation over $F_{\text{type}}$. Our mechanism attains guarantees with respect to this benchmark, when initialized with estimated type frequencies $F_{\text{type}}$ that are not too far from $F_{\text{type}}$. Such estimates can be constructed exogenously via surveys, medical records, or past clinical trials. When $F_{\text{type}} = F_{\text{type}}$, this guarantee is optimal, much like in Theorem 3.7, in the worst case over all adversaries with a given $F_{\text{type}}$.

With this approach, the influence of rare-but-difficult agent types can now be understood solely from analyzing the benchmark $\text{bench}(P, F_{\text{type}})$. The issue is quite intricate, however, because “difficulty” of a type for our benchmark is not well-defined in isolation from the other types. Instead, it is a property of the entire distribution over types. In Section 5.3, we consider a simple scenario with two distributions over types, “good” and “bad”, and mixtures thereof. We analyze the benchmark and show that the influence of the “bad” distribution over types is indeed mitigated.

5.1 Results
The realized type frequencies are defined over the main stage, in a natural way:
\[
F_{\text{type}}(\theta | \text{adv}) = \frac{1}{T - T_0} \sum_{t>T_0} 1(\theta_t = \theta) \quad \forall \theta \in \Theta.
\]
We posit that the mechanism is initialized with estimated type frequencies: a distribution over types \( \tilde{F}_{\text{type}} \) that estimates \( F_{\text{type}}(\cdot | \text{adv}) \) for the (actual) adversary \( \text{adv} \). Such estimates can be constructed exogenously via surveys, medical records, past clinical trials, etc.

**Remark 5.1.** We only need to approximate the empirical frequency of types. That is, the type sequence is still chosen by the adversary. Our guarantees do not require the agent types to be drawn i.i.d. in each round.

The new benchmark replaces the max over all types in (4.7) with an expectation:

\[
\text{bench}(\mathcal{P}, F_{\text{type}}) = \inf_{\text{BIC, BIR state-aware policies } \sigma} \sup_{\psi \in \text{support}(\mathcal{P}), a \in [n]} \mathbb{E}_{\theta \sim F_{\text{type}}} \left[ \frac{1}{\sigma_a(\theta, \psi)} \right]. \tag{5.2}
\]

Let \( \sigma^{\text{opt}} = \sigma^{\text{opt}}(\mathcal{P}, F_{\text{type}}) \) be the benchmark-optimizing policy, i.e., a state-aware policy that optimizes the inf in Eq. (5.2), ties broken arbitrarily. Such policy exists as an optimizer of a continuous function on a compact space.

**Remark 5.2.** The new benchmark (5.2) specializes to Eq. (3.1) for homogeneous agents, just like (4.7) is that the max over types is replaced with an expectation.

It is easy to see that \( \text{bench}(\mathcal{P}, F_{\text{type}}) = \mathbb{E}_{\theta \sim F_{\text{type}}} \text{bench}(\mathcal{P}, \{ \theta \}) \). (Indeed, the only difference between their resp. definitions, (5.2) and (4.7), is that the max over types is replaced with an expectation.)

We use mechanism \( \text{TwoStageMech}(T_0, \sigma^{\text{opt}}, a^*) \), where \( a^* \) is the best-arm policy and \( \sigma^{\text{opt}} \) is the benchmark-optimizing policy. The latter is defined relative to the estimated type distribution, \( \sigma^{\text{opt}} = \sigma^{\text{opt}}(\mathcal{P}, F_{\text{type}}) \). The number of warm-up samples is given by the same Eq. (4.14).

The performance guarantee becomes more complex, as it depends on \( F_{\text{type}} \) and \( F_{\text{type}} \).

**Theorem 5.4.** Assume (4.11). Let \( \tilde{F}_{\text{type}} \) be some distribution over types. Consider mechanism \( \text{TwoStageMech}(T_0, \sigma^{\text{opt}}, a^*) \), where policy \( \sigma^{\text{opt}} = \sigma^{\text{opt}}(\tilde{F}_{\text{type}}) \) optimizes the benchmark for distribution \( \tilde{F}_{\text{type}} \). This mechanism satisfies part (b) in Theorem 4.9 with parameters \( \delta = \eta^*/8 \) and \( N_P = N_P(\eta^*) \), as per Eqs. (4.13) and (4.14). The performance guarantee is as follows:

\[
\text{ERR (mech | adv)} \leq \frac{2 \left( \text{bench}(\mathcal{P}, F_{\text{type}}) + C_P(F_{\text{type}}, \tilde{F}_{\text{type}}) \cdot \| F_{\text{type}} - \tilde{F}_{\text{type}} \|_1 \right)}{T - T_0}, \tag{5.3}
\]

where \( F_{\text{type}}(\cdot) = F_{\text{type}}(\cdot | \text{adv}) \) and the normalization factor \( C_P(F_{\text{type}}, \tilde{F}_{\text{type}}) < \infty \) is determined by the prior \( \mathcal{P} \), the utility structure, and distributions \( F_{\text{type}}, \tilde{F}_{\text{type}} \).

In more detail, \( C_P(F_{\text{type}}, \tilde{F}_{\text{type}}) = C_P(F_{\text{type}}) + C_P(F_{\text{type}}) \), where

\[
C_P(F_{\text{type}}) := \left( \min_{\theta \in \Theta, \psi \in \text{support}(\mathcal{P}), a \in [n]} \sigma_a^{\text{opt}}(\psi, \theta) \right)^{-1}, \quad \sigma^{\text{opt}} = \sigma^{\text{opt}}(\mathcal{P}, F_{\text{type}}).
\]
Remark 5.5. We can reasonably hope that the difference term in Theorem 5.4
\[ \text{diff} := C_P(F_{\text{type}}, \bar{F}_{\text{type}}) \cdot \| \bar{F}_{\text{type}} - F_{\text{type}} \|_1, \]
vanishes as the time horizon $T \to \infty$. Indeed, suppose the realized type frequencies $F_{\text{type}}$ are fixed, and the estimates $\bar{F}_{\text{type}}$ are based on the observations from the warm-up stage. The estimation error $\| \bar{F}_{\text{type}} - F_{\text{type}} \|_1$ scales as $\tilde{O}\left(1/\sqrt{T_0}\right)$ with high probability under i.i.d. type arrivals, which is $\tilde{O}\left(T^{-1/4}\right)$ if the warm-up stage lasts for (say) $T_0 = \sqrt{T}$ rounds.\footnote{We use $\tilde{O}(\cdot)$ notation to hide $O\left(\sqrt{\log T}\right)$ factors.} Therefore, we can reasonably hope that the estimation error scales as some negative power of $T$ in practice: say, as at most $T^{-\gamma}$ for some $\gamma > 0$. Then, fixing an arbitrary $\delta \in (0, 1)$, for any $T > \delta^{-1/\gamma}$ we have
\[ \text{diff} \leq C_{P, \delta}(F_{\text{type}}) \cdot T^{-\gamma}, \text{ where } C_{P, \delta}(F_{\text{type}}) := \sup_{\bar{F}_{\text{type}}, \| \bar{F}_{\text{type}} - F_{\text{type}} \|_1 < \delta} C_P(F_{\text{type}}, \bar{F}_{\text{type}}). \]
The point here is that $C_{P, \delta}(F_{\text{type}}) < \infty$ does not depend on $\bar{F}_{\text{type}}$ or the time horizon $T$.

Theorem 5.4 can be extended to “relaxed” versions of $a^*$ and $\sigma^{\text{opt}}$, just like in Theorem 4.10. The exact formulation is omitted, as we believe it does not yield additional insights.

We prove that the guarantee in Eq. (5.3) (with $\bar{F}_{\text{type}} = F_{\text{type}}$) is essentially optimal in the worst case over all adversaries with given type frequencies $F_{\text{type}}$.

Theorem 5.6. Fix prior $P$ and time horizon $T \geq \Omega\left(\frac{\text{bench}(P, F_{\text{type}})^2}{T}\right)$ and $T_0 \leq T/2$. Let $F_{\text{type}}$ be some distribution over types such that $F_{\text{type}}(\cdot) \in \frac{Z}{T-T_0}$. Let $S$ be a contiguous subset of rounds in the main stage, with $|S| \geq c \cdot T$ for some absolute constant $c > 0$. Suppose mechanism $\text{mech}$ is non-data-adaptive over $S$. Then some adversary $\text{adv}$ satisfies $F_{\text{type}}(\cdot \mid \text{adv}) = F_{\text{type}}$ and
\[ \text{ERR}(\text{mech} \mid \text{adv}) = \Omega\left(\frac{\text{bench}(P, F_{\text{type}})}{T}\right). \]

Like in the previous section, the lower bound is proved using the techniques from the homogeneous case (Theorem 3.7). We defer the proof to Appendix C.

5.2 Proof outline for the positive result (Theorem 5.4)

We first bound the difference in the benchmark due to the estimated type frequency.

Lemma 5.7. Let $F_{\text{type}}, \bar{F}_{\text{type}}$ be distributions over types. Then
\[ \text{bench}(P, \bar{F}_{\text{type}}) - \text{bench}(P, F_{\text{type}}) \leq C_P(F_{\text{type}}) \cdot \| \bar{F}_{\text{type}} - F_{\text{type}} \|_1. \]

Proof. Letting $\sigma^{\text{opt}} = \sigma^{\text{opt}}(P, F_{\text{type}})$, we have
\[
\text{bench}(P, \bar{F}_{\text{type}}) \leq \sup_{\psi \in \text{support}(P), a \in [n]} \mathbb{E}_{\theta \sim F_{\text{type}}} \left[ \frac{1}{\sigma^{\text{opt}}(\theta, \psi)} \right] \\
\leq \sup_{\psi \in \text{support}(P), a \in [n]} \mathbb{E}_{\theta \sim F_{\text{type}}} \left[ \frac{1}{\sigma^{\text{opt}}(\theta, \psi)} \right] + C_P(F_{\text{type}}) \cdot \| \bar{F}_{\text{type}} - F_{\text{type}} \|_1 \\
= \text{bench}(P, F_{\text{type}}) + C_P(F_{\text{type}}) \cdot \| \bar{F}_{\text{type}} - F_{\text{type}} \|_1.
\]

Proof. Letting $\sigma^{\text{opt}} = \sigma^{\text{opt}}(P, F_{\text{type}})$, we have
\[
\text{bench}(P, \bar{F}_{\text{type}}) \leq \sup_{\psi \in \text{support}(P), a \in [n]} \mathbb{E}_{\theta \sim F_{\text{type}}} \left[ \frac{1}{\sigma^{\text{opt}}(\theta, \psi)} \right] \\
\leq \sup_{\psi \in \text{support}(P), a \in [n]} \mathbb{E}_{\theta \sim F_{\text{type}}} \left[ \frac{1}{\sigma^{\text{opt}}(\theta, \psi)} \right] + C_P(F_{\text{type}}) \cdot \| \bar{F}_{\text{type}} - F_{\text{type}} \|_1 \\
= \text{bench}(P, F_{\text{type}}) + C_P(F_{\text{type}}) \cdot \| \bar{F}_{\text{type}} - F_{\text{type}} \|_1.
\]
The proof that the mechanism satisfies BIC and BIR is identical to that for Theorem 4.10, essentially because these two properties do not depend on the type frequencies in the main stage. Hence the details are omitted here.

To derive the error bound, recall that by the specification of our mechanism, the sampling probabilities in the main stage can be lower-bounded as \( p_t(a) \geq \frac{1}{2} \sigma_a^\text{opt}(\psi_{\text{MLE}}, \theta_t) \) for each arm \( a \) and round \( t \), where the benchmark-optimizing policy is \( \sigma^\text{opt} = \sigma^\text{opt}(F_{\text{type}}) \). Invoking the generic guarantee (2.6) for IPS estimators, conditional on the data in the warm-up stage we have

\[
\text{ERR}(\text{mech} | \text{adv}) \leq \frac{2}{(T - T_0)^2} \max_a \max_{\psi \in \text{supp}(\mathcal{P})} \max_{\theta \in \Theta} \sum_{t \in [T_0, T]} \frac{1}{\sigma_a^\text{opt}(\psi_{\text{MLE}}, \theta_t)}.
\]

\[
\leq \frac{2}{T - T_0} \max_a \max_{\psi \in \text{supp}(\mathcal{P})} \sum_{\theta \in \Theta} \sum_{t \in [T_0, T]} \frac{1}{\sigma_a^\text{opt}(\psi, \theta)}
\]

\[
\leq \frac{2}{T - T_0} \max_a \max_{\psi \in \text{supp}(\mathcal{P})} \sum_{\theta \in \Theta} \bar{F}_\text{type}^\psi(\theta) + \frac{|\bar{F}_\text{type}^\psi(\theta) - F^\text{type}(\theta)|}{\sigma_a^\text{opt}(\psi, \theta)}
\]

\[
\leq \frac{2}{T - T_0} \left( \text{bench}(\mathcal{P}, \bar{F}_\text{type}) + C_{\mathcal{P}}(\bar{F}_\text{type}) \cdot \| \bar{F}_\text{type} - F^\text{type} \|_1 \right).
\]

The final guarantee, Eq. (5.3), follows by plugging in Lemma 5.7.

### 5.3 Vanishing impact of bad types

Let us analyze benchmark \( \text{bench}(\mathcal{P}, F^\text{type}) \) to clarify a key advantage thereof: the ability to limit the influence of agent types that are rare but difficult to incentivize. This point is somewhat intricate to formulate, because \( \text{bench}(\mathcal{P}, F^\text{type}) \) is not readily expressible in terms of the per-type benchmarks (see Remark 5.2).

We focus on a relatively simple scenario with two distributions over types, \( F^\text{good} \) and \( F^\text{bad} \). We posit that \( \text{bench}(\mathcal{P}, F^\text{bad}) \) is “large” and \( \text{bench}(\mathcal{P}, F^\text{good}) \) is “small”. The base case is that \( F^\text{good}, F^\text{bad} \) are each supported on a single type: resp., a “good type” and a “bad type”. More generally, \( F^\text{good} \) (resp., \( F^\text{bad} \)) can be supported on a subset of “good types” (resp., “bad types”); further, we allow their support sets to overlap on some “medium types”.

We consider mixtures of \( F^\text{good} \) and \( F^\text{bad} \): distributions

\[
F_\epsilon \triangleq (1 - \epsilon) \cdot F^\text{good} + \epsilon \cdot F^\text{bad}, \quad \epsilon \in [0, 1].
\]

Interestingly, the bad influence of \( F^\text{bad} \) persists even in these mixtures:

\[
\text{bench}(\mathcal{P}, F_\epsilon) \geq (1 - \epsilon) \cdot \text{bench}(\mathcal{P}, F^\text{good}) + \epsilon \cdot \text{bench}(\mathcal{P}, F^\text{bad}), \quad \forall \epsilon \in [0, 1].
\]

We prove that the influence of \( F^\text{bad} \) vanishes as \( \epsilon \to 0 \), in the sense that

\[
\lim_{\epsilon \to 0} \text{bench}(\mathcal{P}, F_\epsilon) = \text{bench}(\mathcal{P}, F^\text{good}).
\]

**Theorem 5.8.** Let \( F^\text{good}, F^\text{bad} \) be distributions over \( \Theta \) with \( \text{bench}(\mathcal{P}, F^\text{good}) < \text{bench}(\mathcal{P}, F^\text{bad}) < \infty \). Assume that restricting the set of types to \( \text{supp}(F^\text{good}) \), there exists a state-aware policy that is BIC and \( \eta \)-BIR, for some \( \eta > 0 \). Then Eq. (5.7) holds.
Proof. The “≥” direction in (5.7) holds by taking $\epsilon \to 0$ in (5.6). We focus on the “≤” direction.

Let $\sigma^{\text{good}}$ and $\sigma^{\text{bad}}$ be the optimal state-aware policy given type frequency $F_{\text{good}}$ and $F_{\text{bad}}$ respectively, modified to offer the outside option for any type $\theta \in \text{supp}(F_{\text{good}}) \cap \text{supp}(F_{\text{bad}})$. Formally, both $\sigma^{\text{good}}$ and $\sigma^{\text{bad}}$ return an arm $a \in A_{\text{out}}$ which maximizes $E_{\psi \sim \mathcal{P}} [U_\psi (a, \theta)]$. Let $\sigma^\eta$ be the BIC and $\eta$-BIR policy given type frequency $F_{\text{good}}$ for some fixed $\eta > 0$, modified similarly. Let

$$p_{\text{min}} = \min_{\theta \in \text{supp}(F_{\text{good}}), a \in [n], \psi \in \text{supp}(\mathcal{P})} \sigma^{\text{good}}_a (\psi, \theta).$$

(5.8)

Note that that $p_{\text{min}} > 0$ since $\text{bench}(\mathcal{P}, F_{\text{good}})$ is finite.

Consider the policy $\sigma^*$ that offers two options to the agent and the agent can choose the best option based on his type:

Option 1: follow policy $\sigma^{\text{good}}$ with probability $1 - \epsilon^{1/2}$, and follow policy $\sigma^\eta$ with probability $\epsilon^{1/2}$.

Option 2: follow policy $\sigma^{\text{bad}}$ with probability $\eta\epsilon^{1/2}$, and always recommends the outside option with probability $1 - \eta\epsilon^{1/2}$.

For any type $\theta \in \text{supp}(F_{\text{good}})$, we have

$$U_{\mathcal{P}} \left( (1 - \epsilon^{1/2}) \cdot \sigma^{\text{good}} + \epsilon^{1/2} \cdot \sigma^\eta \right) \geq (1 - \epsilon^{1/2})U_{\text{out}} + \epsilon^{1/2}(U_{\text{out}} + \eta)
\geq \eta\epsilon^{1/2} \cdot U_{\mathcal{P}} (\sigma^{\text{bad}}) + (1 - \eta\epsilon^{1/2}) \cdot U_{\text{out}}.$$

Thus any type $\theta \in \text{supp}(F_{\text{good}})$ will prefer option 1 to option 2. Moreover, given option 1, type $\theta$ will not have incentives to deviate his report since both $\sigma^{\text{good}}$ and $\sigma^\eta$ are BIC for the type space $\text{supp}(F_{\text{good}})$, and deviating the report to types in $\text{supp}(F_{\text{bad}}) \backslash \text{supp}(F_{\text{good}})$ will lead to outside option value, which is not beneficial for the patient.

For any type $\theta \in \text{supp}(F_{\text{bad}})$, either he will choose option 2, in which case in state $\psi$ each action $a$ is sampled with probability at least $\eta\epsilon^{1/2} \cdot \sigma^{\text{bad}}_a (\psi, \theta)$, or he will choose option 1 and deviate the report to another type in $\text{supp}(F_{\text{bad}})$, in which case each action $a$ is sampled with probability at least $p_{\text{min}}$ given any state.

Thus, by applying policy $\sigma^*$ to the benchmark problem with type frequency $F_\epsilon$, we have

$$\text{bench}(\mathcal{P}, F_\epsilon) \leq \frac{1 - \epsilon}{1 - \epsilon^{1/2}} \cdot \text{bench}(\mathcal{P}, F_{\text{good}}) + \epsilon \cdot \left( \frac{1}{(1 - \epsilon^{1/2})p_{\text{min}}} + \frac{1}{\eta\epsilon^{1/2}} \cdot \text{bench}(\mathcal{P}, F_{\text{bad}}) \right).$$

We complete the proof by taking the limit $\epsilon \to 0$. \qed

6 Conclusions and open questions

We introduce a model for incentivized participation in clinical trials. The model combines the statistical objective (which is standard for clinical trials and adversarial in nature) from the economic constraints (which are standard in economic theory and based on Bayesian-stochastic beliefs). The model emphasizes homogeneous agents as the paradigmatic case, and extends to heterogeneous agents. The latter are endowed with public types that affect objective outcomes, and private types that affect subjective preferences over those outcomes.

We consider three model variants: homogenous agents, heterogenous agents with no ancillary information, and heterogeneous agents with estimated type frequencies. The third variant mitigates the influence of rare-but-difficult agent types. For each model variant, we identify an abstract “comparator benchmark”, design a simple mechanism which attains this benchmark, and prove
that no mechanism can achieve better statistical performance. Our results focus on mechanisms that do not adapt to data over the main stage, as defined in Section 2.4 and motivated by practical considerations.

Our work sets the stage for further investigation. First, one could ask concrete questions that go slightly beyond our modelling approach. Can one match our guarantees for heterogeneous agents if the type space is not fully known, or the type frequencies are known very imprecisely? As the duration of the warm-up stage could be prohibitively large in practice, particularly in the presence of “difficult types”, can one reduce it and/or use some of its data for the counterfactual estimation? Second, one could ask a more “conceptual” question: what if the patients cannot fully understand or quantify their risk preferences, i.e., their own private types? What would be more realistic ways to elicit their preferences, and how to incorporate that into incentivized medical trials?

References

Adams, J., Silverman, M., Musa, D., and Peele, P. (1997). Recruiting older adults for clinical trials. *Controlled clinical trials*, 18(1):14–26.

Arango, J., Arts, K., Braunschweiger, P., an David G. Forster, G. L. C., and Hansen, K., editors (2012). *Good Clinical Practice Guide*. CITI Program at The University of Miami. Available at www.citiprogrampublications.org.

Auer, P., Cesa-Bianchi, N., Freund, Y., and Schapire, R. E. (2002). The nonstochastic multiarmed bandit problem. *SIAM J. Comput.*, 32(1):48–77. Preliminary version in 36th IEEE FOCS, 1995.

Bahar, G., Smorodinsky, R., and Tennenholtz, M. (2016). Economic recommendation systems. In 16th ACM Conf. on Electronic Commerce (ACM-EC).

Bergemann, D. and Morris, S. (2019). Information design: A unified perspective. *Journal of Economic Literature*, 57(1):44–95.

Berk, R. H. (1966). Limiting behavior of posterior distributions when the model is incorrect. *The Annals of Mathematical Statistics*, 37(1):51–58.

Che, Y.-K. and Hörner, J. (2018). Recommender systems as mechanisms for social learning. *Quarterly Journal of Economics*, 133(2):871–925. Working paper since 2013, titled ‘Optimal design for social learning’.

Chow, S.-C. and Chang, M. (2008). Adaptive design methods in clinical trials – a review. *Orphanet Journal of Rare Diseases*, 3(11):1750–1172.

Detry, M. A., Lewis, R. J., Broglio, K. R., Connor, J. T., Berry, S. M., and Berry, D. A. (2012). Standards for the design, conduct, and evaluation of adaptive randomized clinical trials. In *PCORI’s First Methodology Report*. Patient-Centered Outcomes Research Institute (PCORI), Washington, DC, USA. Available online at https://www.pcori.org/research-results/about-our-research/research-methodology/pcori-methodology-report.

Dudík, M., Erhan, D., Langford, J., and Li, L. (2014). Doubly robust policy evaluation and optimization. *Statistical Science*, 29(4):1097–1104.

Esponda, I. and Pouzo, D. (2016). Berk–nash equilibrium: A framework for modeling agents with misspecified models. *Econometrica*, 84(3):1093–1130.
Esponda, I. and Pouzo, D. (2021). Equilibrium in misspecified markov decision processes. *Theoretical Economics*, 16(2):717–757.

Esponda, I., Pouzo, D., and Yamamoto, Y. (2021). Asymptotic behavior of bayesian learners with misspecified models. *Journal of Economic Theory*, 195:105260.

Even-Dar, E., Mannor, S., and Mansour, Y. (2006). Action elimination and stopping conditions for the multi-armed bandit and reinforcement learning problems. *J. of Machine Learning Research (JMLR)*, 7:1079–1105.

FDA (2001). Guidance for industry: E10 choice of control groups and related issues in clinical trials. U.S. Food and Drug Administration (FDA).

Frazier, P., Kempe, D., Kleinberg, J. M., and Kleinberg, R. (2014). Incentivizing exploration. In *ACM Conf. on Economics and Computation (ACM-EC)*.

Freidlin, B., Korn, E. L., Gray, R., and Martin, A. (2008). Multi-arm clinical trials of new agents: Some design considerations. *Clinical Cancer Research*, 14(14):43–68.

Fudenberg, D., Lanzani, G., and Strack, P. (2021). Limit points of endogenous misspecified learning. *Econometrica*, 89(3):1065–1098.

Grill, J. D. and Karlawish, J. (2010). Addressing the challenges to successful recruitment and retention in alzheimer’s disease clinical trials. *Alzheimer’s research & therapy*, 2:1–11.

Han, L., Kempe, D., and Qiang, R. (2015). Incentivizing exploration with heterogeneous value of money. In *11th Intl. Conf. on Web and Internet Economics (WINE)*, pages 370–383.

Hellmich, M. (2001). Monitoring clinical trials with multiple arms. *Biometrics*, 57(3):892–898.

Immorlica, N., Mao, J., Slivkins, A., and Wu, S. (2019). Bayesian exploration with heterogeneous agents. In *The Web Conference (formerly known as WWW)*, pages 751–761.

Immorlica, N., Mao, J., Slivkins, A., and Wu, S. (2020). Incentivizing exploration with selective data disclosure. In *ACM Conf. on Economics and Computation (ACM-EC)*, pages 647–648. Working paper available at [https://arxiv.org/abs/1811.06026](https://arxiv.org/abs/1811.06026).

Jenkins, V. and Fallowfield, L. (2000). Reasons for accepting or declining to participate in randomized clinical trials for cancer therapy. *British journal of cancer*, 82(11):1783–1788.

Kamenica, E. (2019). Bayesian persuasion and information design. *Annual Review of Economics*, 11(1):249–272.

Karlawish, J., Cary, M. S., Rubright, J., and TenHave, T. (2008). How redesigning ad clinical trials might increase study partners’ willingness to participate. *Neurology*, 71(23):1883–1888.

Kremer, I., Mansour, Y., and Perry, M. (2014). Implementing the “wisdom of the crowd”. *J. of Political Economy*, 122(5):988–1012. Preliminary version in *ACM EC 2013*.

Lattimore, T. and Szepesvári, C. (2020). *Bandit Algorithms*. Cambridge University Press, Cambridge, UK.

Mannor, S. and Tsitsiklis, J. N. (2004). The sample complexity of exploration in the multi-armed bandit problem. *J. of Machine Learning Research (JMLR)*, 5:623–648.
Mansour, Y., Slivkins, A., and Syrgkanis, V. (2020). Bayesian incentive-compatible bandit exploration. *Operations Research*, 68(4):1132–1161. Preliminary version in *ACM EC 2015*.

Mills, E. J., Seely, D., Rachlis, B., Griffith, L., Wu, P., Wilson, K., Ellis, P., and Wright, J. R. (2006). Barriers to participation in clinical trials of cancer: a meta-analysis and systematic review of patient-reported factors. *The lancet oncology*, 7(2):141–148.

Pallmann, P., Bedding, A. W., Choodari-Oskooei, B., Dimairo, M., Flight, L., Hampson, L. V., Holmes, J., Mander, A. P., Odondi, L., Sydes, M. R., et al. (2018). Adaptive designs in clinical trials: why use them, and how to run and report them. *BMC medicine*, 16:1–15.

Parmar, M. K., Carpenter, J., and Sydes, M. R. (2014). More multiarm randomised trials of superiority are needed. *Lancet*, 384(9940):283–284.

Pinsker, M. S. (1964). *Information and information stability of random variables and processes*. Holden-Day.

Pocock, S. J. (1979). Allocation of patients to treatment in clinical trials. *Biometrics*, pages 183–197.

Redig, A. and Jänne, P. (2015). Basket trials and the evolution of clinical trial design in an era of genomic medicine. *J. of Clinical Oncology*, 33.

Rodríguez-Torres, E., González-Pérez, M. M., and Díaz-Pérez, C. (2021). Barriers and facilitators to the participation of subjects in clinical trials: an overview of reviews. *Contemporary clinical trials communications*, 23:100829.

Ross, S., Grant, A., Counsell, C., Gillespie, W., Russell, I., and Prescott, R. (1999). Barriers to participation in randomised controlled trials: a systematic review. *Journal of clinical epidemiology*, 52(12):1143–1156.

Slivkins, A. (2019). Introduction to multi-armed bandits. *Foundations and Trends® in Machine Learning*, 12(1-2):1–286. Published with *Now Publishers* (Boston, MA, USA). Also available at [https://arxiv.org/abs/1904.07272](https://arxiv.org/abs/1904.07272). Latest online revision: Jan 2022.

Swaminathan, A. and Joachims, T. (2015). The self-normalized estimator for counterfactual learning. In *28th Advances in Neural Information Processing Systems (NIPS)*, pages 3231–3239.

Torgerson, D. J. and Sibbald, B. (1998). Understanding controlled trials. what is a patient preference trial? *BMJ: British Medical Journal*, 316(7128):360.

Vozdolska, R., Sano, M., Aisen, P., and Edland, S. D. (2009). The net effect of alternative allocation ratios on recruitment time and trial cost. *Clinical trials*, 6(2):126–132.

Wang, Y., Agarwal, A., and Dudík, M. (2017). Optimal and adaptive off-policy evaluation in contextual bandits. In *34th Intl. Conf. on Machine Learning (ICML)*, volume 70, pages 3589–3597.
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A Simple numerical examples

Homogeneous agents. There are $n = 2$ treatments (henceforth, “arms”), the “old” and the “new”. As per our model, agents believe each arm gives the same utility distribution for all patients, and the state (which we identify with a mapping from arms to utility distributions) is initially drawn from some prior $P$ (shared by all agents). Specifically, the old arm gives mean utility 1 with probability $\frac{2}{3}$ and utility 0 with probability $\frac{1}{3}$. The new arm gives utility 2 and -1 with probability $\frac{1}{2}$ each. The prior is independent across arms. Given the mean utilities (and the fact that the realized utilities lie in the [0, 1] interval), the noise shape can be arbitrary for this example.\(^{25}\)

Agents’ alternative to participation is to use the old arm, whose prior-mean utility is $\frac{2}{3}$; so, the “outside option” is $U_{\text{out}} = \frac{2}{3}$. Note that it is ex-ante preferred to the new arm (whose prior-mean utility is only $\frac{1}{2}$), so the agents are incentivized against participation if the arm probabilities are data-independent.

Our participation-incentivizing RCT design from Section 3 works out as follows. Leveraging Lemma 3.2, consider $\epsilon^*$, the largest $\epsilon$ such that $(\epsilon, \psi^*)$-greedy policy is BIR. We claim $\epsilon^* = \frac{8}{9}$.

Indeed, note that $\epsilon^*$ is is the unique $\epsilon$ such that the Bayesian-expected utility of $(\epsilon, \psi^*)$-greedy, call it $U(\epsilon)$, equals the outside option $U_{\text{out}}$. And one can check that $U(\frac{8}{9}) = \frac{2}{3} = U_{\text{out}}$. (One way to do that is to sum over the four possible states, as per Footnote 25 and for each state calculate the expected utility of the policy; we omit the easy details.)

Thus, we obtain the benchmark value $\text{bench}(P) = \frac{n}{\epsilon^*} = \frac{9}{4}$.

Now let us calculate $N_P$, the number of samples required for the warm-up stage. The prior-expected utility of the best arm is $U_P(a^*) = \frac{4}{3}$ (the calculation is similar to that for $U(\epsilon^*)$.) So, the “prior gap” (see Eq. (3.4)) is $\text{gap}(P) = \frac{4}{3} - \frac{5}{12} = \frac{11}{12}$. Therefore, according to Eq. (3.5), we have $\alpha = \frac{22}{27}$ and $N_P < 144$. In other words, a warm-up stage with 144 independent samples of each arm suffices to guarantee incentive-compatibility for the main stage.

Heterogeneous agents. The worst-case benchmark coincides with the benchmark value for the worst type, by Eq. (3.8). The frequency-based benchmark, $\text{bench}(P, F_{\text{type}})$ from Section 5 can provide a significant improvement. We illustrate this point with a simple example below.

There are $n = 2$ treatments (arms: the ”old” and the ”new”), one public type, and two private types. There are 3 possible outcomes: “full recovery”, “side effect”, and “no effect”. Agents’ alternative to participation is to choose the old arm.

\(^{25}\)Thus, in the formalism of our model, we have four possible states, with mean utility pairs (1, 2), (1, -1), (0, 2) and (0, -1), and respective prior probabilities $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{6}$.
Since we have a unique public type, all agents believe that the outcome distribution for a given arm is the same for all agents, like in the homogeneous case. Further, they believe the state — mapping from arms to outcome distributions — is initially drawn from some prior \( \mathcal{P} \), shared by all agents. The prior is as follows. For each arm, there are two possible outcome distributions. The risky distribution has full recovery with probability \( \frac{1}{2} \), side effect with probability \( \frac{1}{3} \), while ineffective with probability \( \frac{1}{6} \). The conservative distribution has full recovery with probability \( \frac{1}{3} \), side effect with probability \( \frac{1}{6} \), while ineffective with probability \( \frac{1}{2} \). There are two possible states: the old arm leads to risky distribution and the new arm leads to conservative distribution (call it state \( \psi_0 \)), and vice versa (state \( \psi_1 \)). The prior \( \mathcal{P} \) assigns probability \( \frac{2}{3} \) to state \( \psi_0 \), and the remaining probability to state \( \psi_1 \). This completes the description of agents’ beliefs.

The two private types are \( \theta_0 \) and \( \theta_1 \), appearing with equal frequency. The associated subjective utilities are as follows. Type \( \theta_0 \) has utility 1 for full recovery and utility 0 otherwise. Type \( \theta_1 \) has utility 1 for full recovery, utility 0 for no effect, and utility -10 for side effects.

If the two types were considered separately, the benchmark values are 3 for type \( \theta_0 \) and 2 for type \( \theta_1 \) (with minimum sampling probabilities, resp., \( \frac{1}{3} \) and \( \frac{1}{2} \)). The computation is similar to that for the homogeneous example, we omit the details.

Let \( \sigma_i \) be the benchmark-optimizing BIR policy for type \( \theta_i \), \( i \in \{0, 1\} \). Let \( \sigma \) be the joint policy that maps type \( \theta_i \) to the resp. per-type policy \( \sigma_i \). This policy is BIR since \( \sigma_0 \) and \( \sigma_1 \) are BIR. To see that policy \( \sigma \) is BIC, note that for each state, the two types have different best arms, and so misreporting the type would only decrease the probability of choosing the optimal arm. Therefore, \( \sigma \) is a feasible policy for the definition (5.2) of \( \text{bench}(\mathcal{P}, F_{\text{type}}) \), and so the benchmark value is at most what one obtains by plugging in this policy, namely \( \text{bench}(\mathcal{P}, F_{\text{type}}) \leq \frac{5}{2} \). Thus, we obtain a significant improvement over the worst-case benchmark \( \text{bench}(\mathcal{P}, \Theta) = 3 \).

\section*{B Proof of Claim 4.14: a claim on MLE error probability}

Consider the log-likelihood ratio (LLR) between two states \( \psi, \psi' \in \text{support}(\mathcal{P}) \). Specifically, let \( \psi(t) = \psi(a_t, \theta_{\text{pub}}(t)) \) be the outcome distribution for state \( \psi \) in a given round \( t \), and let \( \psi'(t, \omega) \) be the probability of outcome \( \omega \) according to this distribution. The LLR is defined as

\[
\text{LLR}_t \left( \psi, \psi' \right) := \log \left( \frac{\psi'(t, \omega)}{\psi(t, \omega)} \right), \quad \omega = \omega_t, \\
\text{LLR}_S \left( \psi, \psi' \right) := \sum_{t \in S} \text{LLR}_t \left( \psi, \psi' \right) \quad \forall S \subset [T].
\]

The MLE estimator is correct, \( \psi_{\text{MLE}} = \psi^* \), as long as

\[
\text{LLR}_{\mathcal{P}} \left( \psi^*, \psi' \right) > 0 \quad \text{for any other state } \psi' \in \text{support}(\mathcal{P}) \setminus \{ \psi^* \}. \tag{B.1}
\]

In the rest of the proof, we show that

\[
P_{\text{Bayes}} \left[ \text{Eq. \eqref{B.1} holds} \right] \geq 1 - \eta/4. \tag{B.2}
\]

Fix two distinct states \( \psi, \psi' \in \text{support}(\mathcal{P}) \). For each round \( t \), define

\[
X_t := \text{LLR}_t \left( \psi, \psi' \right) - \text{KL} \left( \psi(t), \psi'(t) \right). \tag{B.3}
\]

\footnote{The \( \frac{5}{2} \) value is obtained by explicitly computing the per-type policies \( \sigma_0, \sigma_1 \) and then plugging \( \sigma \) into (5.2).}
Focus on the subset of the warm-up stage when the outcome distributions $\psi_{(t)}, \psi'_{(t)}$ are distinct:

$$S := \left\{ t \in [T_0] : \psi(t) \neq \psi'(t) \right\}.$$  

(B.4)

We apply concentration to $X_S := \sum_{t \in S} X_t$. We are interested in the low-probability deviation

$$X_S \leq -|S| \cdot KL_{\min}(\psi, \psi')$$  

(B.5)

when $\psi^* = \psi$ and $S$ is large enough. Specifically, we prove the following claim.

**Claim B.1.** Fix states $\psi, \psi' \in \text{support}(P)$ and an arbitrary $s_0 \in [T_0]$. Then

$$\mathbb{P}_{\text{Bayes}} \left[ |S| \geq s_0 \text{ implies } (B.5) \mid \psi^* = \psi \right] \leq \frac{1}{2R_{\min}(P)} \cdot e^{-2(s_0-1)R_{\min}(P)}.$$  

(B.6)

Let us defer the proof of Claim B.1 till later, and use it to complete the proof of Claim 4.14. To emphasize the dependence on the two states, write $S = S(\psi, \psi')$ and denote the event in (B.5) with $\mathcal{E}(\psi, \psi')$. Taking the union bound in (B.6) over all states $\psi'$ and setting $s_0 = N_P(\eta)$, and taking the Bayesian expansion over events $\{ \psi^* = \psi \}$, $\psi \in \text{support}(P)$, we derive

$$\mathbb{P}_{\text{Bayes}} \left[ |S(\psi^*, \psi')| \geq N_P(\eta) \text{ implies } \mathcal{E}(\psi^*, \psi') \text{ } \forall \psi' \in \text{support}(P) \setminus \{ \psi^* \} \right] \leq \frac{\eta}{8}. \quad (B.7)$$

By assumption, with probability at least $1 - \eta/8$, the mechanism collects at least $N_P(\eta)$ samples for each (action, public type) pair. Denote this event by $\mathcal{E}_0$. For any $\psi' \in \text{support}(P) \setminus \{ \psi^* \}$, there exists at least one (action, public type) pair $(a, \theta_{\text{pub}})$ such that $\psi^*(a, \theta_{\text{pub}}) \neq \psi'(a, \theta_{\text{pub}})$. Therefore, under event $\mathcal{E}_0$ we have $|S(\psi^*, \psi')| \geq N_P(\eta)$. It follows that with probability at least $1 - \eta/4$, event $\mathcal{E}(\psi^*, \psi')$ holds for all states $\psi' \in \text{support}(P) \setminus \{ \psi^* \}$. This implies Eq. (B.2), and therefore completes the proof of Claim 4.14.

**Proof of Claim B.1.** Consider the random subset $S$ of the warm-up stage defined by (B.4). Let $t(j)$ be the $j$-th element of $S$, for all $j \in [|S|]$. Define a new sequence of random variables: $X_0 = 0$ and

$$\hat{X}_j := 1_{\{j \leq |S|\}} \cdot X_{t(j)}, \quad \forall j \in [T_0].$$

Fix $\tau \in [T_0]$. Since $\mathbb{E}_{\text{Bayes}} \left[ \hat{X}_j \mid \hat{X}_{j-1} \right] = 0$ for all $j \in [T_0]$, the sum $\hat{X}_{[\tau]} := \sum_{j \leq \tau} \hat{X}_j$ is a martingale (in the probability space induced by the agents’ beliefs). This martingale’s increments are bounded as $\hat{X}_j \leq \ell_{\max}(\psi, \psi')$. Applying the standard Azuma-Hoeffding inequality with deviation term $-\tau \cdot KL_{\min}(\psi, \psi')$, we obtain

$$\mathbb{P}_{\text{Bayes}} \left[ \hat{X}_{[\tau]} \leq -\tau \cdot KL_{\min}(\psi, \psi') \mid \psi^* = \psi \right] \leq \exp \left( -\frac{2\tau \cdot KL^2_{\min}(\psi, \psi')}{\ell^2_{\max}(\psi, \psi')} \right) \leq \exp \left( -2\tau \cdot R_{\min}(P) \right).$$

Taking a union bound over all $\tau \in [s_0, T_0]$, we have

$$\mathbb{P}_{\text{Bayes}} \left[ \hat{X}_{[\tau]} \leq -\tau \cdot KL_{\min}(\psi, \psi'), \forall \tau \in [s_0, T_0] \mid \psi^* = \psi \right] \leq \sum_{\tau \in [s_0, T_0]} e^{-2\tau \cdot R_{\min}(P)} \leq \frac{1}{2R_{\min}(P)} \cdot e^{-2(s_0-1)R_{\min}(P)}.$$

We obtain (B.5) by taking $\tau = |S|$ in (B.8). \qed
C Lower bounds for heterogeneous agents

In this appendix, we prove the two lower bounds for heterogeneous agents: Theorem 4.8 (for worst-case type frequencies) and Theorem 4.8 (for estimated type frequencies). Both theorems are proved via the techniques from the homogeneous case (Theorem 3.7).

We start with some common notation. Fix some mechanism mech which is non-data-adaptive over the main stage, and an associated warm-up stage duration $T_0$. Fix some subset $S$ of rounds in the main stage such that $|S| \geq c \cdot T$ for some absolute constant $c > 0$. Given some state $\psi$, let $\text{adv}_\psi$ be the adversary restricted to the first $t_0 = \min(S) - 1$ rounds that samples outcome $\omega_{a,t}$ from distribution $\psi(a)$, independently for each arm $a$ and each round $t \leq t_0$. Note that $\text{adv}_\psi$ and mech jointly determine the sampling distributions $p_t$ for all rounds $t \in S$. Let $N_\psi(a, \theta)$ be the number of times arm $a$ is selected in the rounds $t \in S$ for type $\theta$ under adversary $\text{adv}_\psi$.

**Proof of Theorem 4.8.** There is a state $\psi_0 \in \text{supp}(P)$, type $\theta \in \Theta$ and action $a_0 \in [n]$ such that

$$\frac{1}{|S|} \cdot \mathbb{E} \left[ N_{\psi_0}(a_0, \theta) \right] \geq \text{bench}(P, \Theta).$$

The adversary $\text{adv}$ is constructed as follows. For the first $t_0$ rounds, types are drawn from the prior and we use the adversary $\text{adv}_{\psi_0}$. For rounds $t > t_0$, only type $\theta$ arrives, and we use adversary $\text{adv}^\dagger$ from (3.8), determined by the expected sampling probabilities ($\mathbb{E}[p_t] : t \in S$) (see Lemma D.1). This reduces to the single type estimation problem, and by applying the same argument as in the proof of Theorem 3.7, we have that

$$\text{ERR (mech | adv)} \geq \Omega \left( \frac{\text{bench}(P, \Theta)}{T} \right).$$

**Proof of Theorem 5.6.** There exists a state $\psi_0 \in \text{supp}(P)$ and action $a_0 \in [n]$ such that

$$\sum_{\text{types } \theta \in \Theta} \frac{F_{\text{type}}(\theta)}{|S| \cdot \mathbb{E} \left[ N_{\psi_0}(a_0, \theta) \right]} \geq \text{bench}(P, F_{\text{type}}).$$

The adversary $\text{adv}$ is constructed like in the homogeneous case: $\text{adv}_{\psi_0}$ for the first $t_0$ rounds, and $\text{adv}^\dagger$ from (D.1) afterwards. By the argument from the proof of Theorem 3.7, we have that

$$\text{ERR (mech | adv)} \geq \Omega \left( \frac{1}{|S|^2} \sum_{\text{types } \theta \in \Theta} \left( \sqrt{|S|^2} \cdot |S^0_\theta| + \frac{|S^0_\theta|^2}{\mathbb{E} \left[ N_{\psi_0}(a_0, \theta) \right]} \right) \right).$$

Here, letting $q_t := \mathbb{E}[p_t(a_0)]$ and $\theta \in \Theta$, we define

$$S^\theta_L = \left\{ t \in S : \theta_t = \theta \text{ and } q_t \leq 1/\sqrt{|S|} \right\},$$
$$S^\theta_H = \left\{ t \in S : \theta_t = \theta \text{ and } q_t > 1/\sqrt{|S|} \right\}.$$

If there exists some type $\theta \in \Theta$ such that $|S^\theta_L| \geq \text{bench}(P, F_{\text{type}}) \cdot \sqrt{|S|}$, then the desired guarantee (5.4) holds immediately. Otherwise, since $|S| \geq \Omega \left( \frac{\text{bench}(P)}{\min_{\theta} F_{\text{type}}(\theta)} \right)^2$, we have $|S^\theta_H| \geq \frac{1}{2} |S| \cdot F_{\text{type}}(\theta)$ for all type $\theta$ and (5.4) holds as well.
D  Statistical lower bound: Proof of Eq. (3.8)

We prove the lower bound from Eq. (3.8). Let’s provide a standalone formulation for this result. Formally, we interpret \( \hat{f}_{\text{mech}} \) as a mapping from arm \( a \) and history \( \text{hist}_{T+1} \) to \([0, 1]\), and call any such mapping an estimator.

**Lemma D.1.** Fix any subset \( S \subset [T] \). Consider a mechanism \( \text{mech} \) which draws the tuple of sampling distributions \((p_t : t \in S)\) from some fixed distribution, and does so before round \( \min(S) \). For any estimator \( \hat{f}_{\text{mech}} \) in the mechanism there is an adversary \( \text{adv} \) such that

\[
\text{ERR} ( \text{mech} | \text{adv} ) \geq \Omega \left( \frac{1}{T^2} \max_{a \in [n]} \sum_{t \in S} \min \left\{ \frac{1}{\mathbb{E}[p_t(a)]} ; \sqrt{|S|} \right\} \right). 
\]  

(D.1)

The adversary in (D.1) only depends on \( \hat{f}_{\text{mech}} \) and the expected sampling probabilities \( \mathbb{E}[p_t] \).

**Remark D.2.** To compare to the IPS upper bound in Eq. (2.6), consider the case when \( S \) is the entire main stage and the sampling distributions \((p_t : t \in S)\) are chosen deterministically. We see that IPS is worst-case optimal as long as the sampling probabilities are larger than \( 1/\sqrt{|S|} \).

Lemma D.1 was known for the special case when \( S = [T] \) and each distributions \( p_t \) is chosen as an independent random draw from some fixed distribution over \( \Delta_n \) [Dudik et al., 2011]. The general case, with an arbitrary subset \( S \) and a correlated choice of distributions \((p_t : t \in S)\) may be a tool of independent interest. However, we did not attempt to optimize the constants.

We prove Lemma D.1 in the remainder of this subsection. We use Pinsker’s Lemma to bound the difference in event probabilities.

**Lemma D.3 [Pinsker, 1964].** If \( P \) and \( Q \) are two probability distributions on a measurable space \((X, \Sigma)\), then for any measurable event \( A \in \Sigma \), it holds that

\[
|P(A) - Q(A)| \leq \sqrt{\frac{1}{2} \text{KL}(P\|Q)},
\]

where \( \text{KL}(P\|Q) = \int_X \ln \left( \frac{dP}{dQ} \right) dP \) is the Kullback–Leibler divergence.

We observe that the bound is monotone in \( p_t(a) \) for any arm \( a \) and any time \( t \in S \). This holds since the principal can always simulate the estimator with lower sampling probabilities by randomly ignore the observations. Thus it is sufficient to prove that if \( \mathbb{E}[p_t(a)] > 1/\sqrt{|S|} \) for all \( t \in S \) and all \( a \in [n] \), then there is an adversary \( \text{adv} \) such that

\[
\text{ERR} ( \text{mech} | \text{adv} ) \geq \Omega \left( \frac{1}{T^2} \max_{a \in [n]} \sum_{t \in S} \frac{1}{\mathbb{E}[p_t(a)]} \right) \quad \text{for any estimator } \hat{f}_{\text{mech}}.
\]

We divide the analysis into two cases.

**Case 1:** There exists an arm \( a^* \in [n] \) such that \( \sum_{t \in S} \frac{1}{\mathbb{E}[p_t(a^*)]} \geq 710T \). Consider the binary outcome space \( \{\omega_0, \omega_1\} \) and the estimator function \( f(\omega_t) = 1_{\{\omega_t = \omega_0\}} \). Consider two stochastic adversaries \( \text{adv} \) and \( \text{adv}' \). For arm \( a^* \), at each time \( t \in S \), outcome \( \omega_0 \) is sampled with probability \( \frac{1}{2} \) in \( \text{adv} \), and is sampled with probability \( \frac{1}{2} + \delta_t \) in \( \text{adv}' \) where \( \delta_t \in [0, \frac{1}{2}] \) to be specified later. Moreover, both \( \text{adv} \) and \( \text{adv}' \) have the same arbitrary sequence of outcomes for time periods not in \( S \). It is easy to verify that

\[
\mathbb{E} [ f_{\text{adv}}(a^*) ] - \mathbb{E} [ f_{\text{adv}'}(a^*) ] = \frac{1}{T} \sum_{t \in S} \delta_t.
\]
Moreover, since at any time $t$, arm $a^*$ is chosen with probability $p_t(a^*)$, the KL-divergence between $\text{adv}$ and $\text{adv}'$ is

$$ KL(\text{adv}||\text{adv}') = \mathbb{E} \left[ \sum_{t \in S} p_t(a^*) KL(\text{adv}_t||\text{adv}'_t) \right] $$

$$ = \mathbb{E} \left[ \sum_{t \in S} p_t(a^*) \left( \frac{1}{2} + \delta_t \right) \log(1 + 2\delta_t) + \left( \frac{1}{2} - \delta_t \right) \log(1 - 2\delta_t) \right] $$

$$ \leq 4 \sum_{t \in S} \mathbb{E} [p_t(a^*)] \delta_t^2. $$

Letting $P$ be the probability that the absolute difference between the estimator to $\mathbb{E} [f_{\text{adv}}(a^*)]$ is at most $\frac{1}{2T} \sum_{t \in S} \delta_t$ given adversary $\text{adv}$ and $P'$ be the same probability given adversary $\text{adv}'$. By Lemma D.3, we have that

$$ |P - P'| \leq \sqrt{\frac{1}{2} KL(\text{adv}||\text{adv}')} \leq \sqrt{2 \sum_{t \in S} \mathbb{E} [p_t(a^*)] \cdot \delta_t^2}. \quad (D.2) $$

Let us specify $\delta_t$'s to ensure that $|P - P'| \leq \frac{1}{2}$:

$$ \delta_t = \left( \frac{\mathbb{E} [p_t(a^*)]}{\sqrt{8 \sum_{t \in S} \frac{1}{\mathbb{E} [p_t(a^*)]}}} \right)^{-1}. \quad (D.3) $$

Note that $\mathbb{E} [p_t(a^*)] \geq 1/\sqrt{|S|}$ (since we’ve restricted attention to the case when all sampling probabilities are at least $1/\sqrt{|S|}$). Moreover, $\mathbb{E} [p_t(a^*)] \leq 1$ implies that

$$ \sum_{t \in S} \frac{1}{\mathbb{E} [p_t(a^*)]} \geq |S| $$

and therefore

$$ \delta_t \leq \frac{1}{\mathbb{E} [p_t(a^*)] \cdot \sqrt{8|S|}} \leq \frac{1}{2\sqrt{2}} $$

which implies that the choice of $\delta_t$ is feasible.

Thus, there exists an adversary, say $\text{adv}$, such that with probability at least $\frac{1}{4}$, the absolute difference between the estimator to $\mathbb{E} [f_{\text{adv}}(a^*)]$ is at least $\frac{1}{2T|S|} \sum_{t \in S} \delta_t$. Moreover, by Hoeffding’s inequality, we have

$$ \Pr \left[ |f_{\text{adv}}(a^*) - \mathbb{E} (f_{\text{adv}}(a^*))| \geq \frac{1}{4T} \sum_{t \in S} \delta_t \right] \leq 2 \exp \left( -\frac{2 \left( \frac{1}{4} \sum_{t \in S} \delta_t \right)^2}{T} \right) \leq \frac{1}{8}, $$

where the last inequality holds since $\sum_{t \in S} \frac{1}{\mathbb{E} [p_t(a^*)]} \geq 710T$. By union bound, with probability at least $\frac{1}{4}$, the distance between the estimator and the truth is at least $\frac{1}{4T} \sum_{t \in S} \delta_t$ and hence the expected mean square error is

$$ \mathbb{E} [\text{ERR (mech | adv)}] \geq \frac{1}{8} \left( \frac{1}{4T} \sum_{t \in S} \delta_t \right)^2 = \Omega \left( \frac{1}{T^2} \max_{a \in [n]} \sum_{t \in S} \frac{1}{\mathbb{E} [p_t(a)]} \right). $$

**Case 2:** $\sum_{t \in S} \frac{1}{\mathbb{E} [p_t(a)]} \leq 710T$ for all arms $a \in [n]$. Consider again the binary outcome space $\{\omega_0, \omega_1\}$ and the estimator function $f(\omega_t) = 1_{\{\omega_t = \omega_0\}}$. Consider the stochastic adversaries $\text{adv}$
such that for each arm $a$, at each time $t$, outcome $\omega_0$ is sampled with probability $\frac{1}{2}$ in $\text{adv}$. Let $S_a$ be the number of time periods such that arm $a$ is not chosen. Note that given realized sequence of observations, there exists an arm $a \in [n]$ such that $|S_a| \leq \frac{T}{2}$. Moreover, since the adversary is independent across different time periods, the observations is uninformative about the realization of the unobserved outcomes. Thus, for any estimator in the mechanism, the estimation error for arm $a^*$ is at least the variance of the outcomes in unobserved time periods and

$$
E \left[ \text{ERR (mech | adv)} \right] \geq E \left[ \max_{a \in [n]} \text{Var} \left[ f_{\text{adv}}^S(a) \right] \right] \geq \frac{1}{8T} = \Omega \left( \frac{1}{T^2} \max_{a \in [n]} \sum_{t \in S} \frac{1}{p_t(a)} \right).
$$

Combining the observations, there must exist a deterministic sequence of adversary such that the mean square error conditional on the adversary satisfies stated lower bound.

### E Upper bound for IPS: proof of Eq. (2.6)

We note that it is not essential to focus on a particular subset $S$ of rounds such as the main stage; instead, the result holds for an arbitrary subset $S \subset [T]$.

Fixing any oblivious adversary, since the IPS estimator is unbiased, the mean square error of the IPS estimator equals its variance. Moreover, in the IPS estimator, the random variable $\frac{1(\omega_{a,t}) \cdot f(\omega_{a,t})}{p_t(a)}$ is independent across different time periods $t$. Since the variance for the sum of independent random variables equals the sum of variance, we have

$$
\text{ERR (mech | adv)} = \max_{\omega \in \Omega, a \in [n]} \frac{1}{|S|^2} \sum_{t \in S} \text{Var} \left[ \frac{1(\omega_{a,t}) \cdot f(\omega_{a,t})}{p_t(a)} \right] = \max_{\omega \in \Omega, a \in [n]} \frac{1}{|S|^2} \sum_{t \in S} \left( \frac{1}{p_t(a)} - 1 \right) \cdot f(\omega_{a,t}) \leq \max_{a \in [n]} \frac{1}{|S|^2} \sum_{t \in S} \frac{1}{p_t(a)}.
$$

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