THE RICCI TENSOR OF AN ALMOST HOMOGENEOUS KÄHLER MANIFOLD

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ABSTRACT. We determine an explicit expression for the Ricci tensor of a K-manifold, that is of a compact Kähler manifold $M$ with vanishing first Betti number, on which a semisimple group $G$ of biholomorphic isometries acts with an orbit of codimension one. We also prove that the Kähler form $\omega$ and the Ricci form $\rho$ of $M$ are uniquely determined by two special curves with values in $\mathfrak{g} = \text{Lie}(G)$, say $Z_\omega, Z_\rho : \mathbb{R} \to \mathfrak{g} = \text{Lie}(g)$ and we show how the curve $Z_\rho$ is determined by the curve $Z_\omega$.

These results are used in another work with F. Podestà, where new examples of non-homogeneous compact Kähler-Einstein manifolds with positive first Chern class are constructed.

1. Introduction.

The objects of our study are the so-called $K$-manifolds, that is Kähler manifolds $(M, J, g)$ with $b_1(M) = 0$ and which are acted on by a group $G$ of biholomorphic isometries, with regular orbits of codimension one. Note that since $M$ is compact and $G$ has orbits of codimension one, the complexified group $G^\mathbb{C}$ acts naturally on $M$ as a group of biholomorphic transformations, with an open and dense orbit. According to a terminology introduced by A. Huckleberry and D. Snow in [HS], $M$ is almost-homogeneous with respect to the $G^\mathbb{C}$-action. By the results in [HS], the subset $S \subset M$ of singular points for the $G^\mathbb{C}$-action is either connected or with exactly two connected components. If the first case occurs, we will say that $M$ is a non-standard $K$-manifold; we will call it standard $K$-manifold in the other case.

The aim of this paper is furnish an explicit expression for the Ricci curvature tensor of a $K$-manifold, to be used for constructing (and possibly classify) new families of examples of non-homogeneous $K$-manifold with special curvature conditions. A successful application of our results is given in [PS1], where several new examples of non-homogeneous compact Kähler-Einstein manifolds with positive first Chern class are found.

Note that explicit expressions for the Ricci tensor of standard $K$-manifolds can be found also in [Sa], [KS], [PS] and [DW]. However our results can be applied to any kind of $K$-manifold and hence they turn out to be particularly useful for the

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non-standard cases (at this regard, see also [CG]). They can be resumed in the
following three facts.

Let \( g \) be the Lie algebra of the compact group \( G \) acting on the K-manifold
\((M, J, g)\) with at least one orbit of codimension one. By a result of [PS 1], we may
always assume that \( G \) is semisimple. Let also \( B \) be the Cartan-Killing form of
\( g \). Then for any \( x \) in the regular point set \( M_{\text{reg}} \), one can consider the following
\( B \)-orthogonal decomposition of \( g \):

\[
g = l + \mathbb{R}Z + m ,
\]

(1.1)

where \( l = g_x \) is the isotropy subalgebra, \( \mathbb{R}Z + m \) is naturally identified with the
tangent space \( T_x(G/L) \cong T_x(G \cdot x) \) of the \( G \)-orbit \( G/L = G \cdot x \), and \( m \) is naturally
identified with the holomorphic subspace \( m \cong D_x \).

\[
D_x = \{ v \in T_x(G \cdot x) : Jv \in T_x(G \cdot x) \} .
\]

(1.2)

Notice that for any point \( x \in M_{\text{reg}} \) the \( B \)-orthogonal decomposition (1.1) is uniquely
given; on the other hand, two distinct points \( x, x' \in M_{\text{reg}} \) may determine two
distinct decompositions of type (1.1).

Our first result consists in proving that any K-manifold has a family \( O \) of smooth
curves \( \eta : \mathbb{R} \to M \) of the form

\[
\eta_t = \exp(itZ) \cdot x_o ,
\]

where \( Z \in g \), \( x_o \in M \) is a regular point for the \( G^\mathbb{C} \)-action and the following
properties are satisfied:

1. \( \eta_t \) intersects any regular \( G \)-orbit;
2. for any point \( \eta_t \in M_{\text{reg}} \), the tangent vector \( \eta'_t \) is transversal to the regular
   orbit \( G \cdot \eta_t \);
3. any element \( g \in G \) which belongs to a stabilizer \( G_{\eta_t} \), with \( \eta_t \in M_{\text{reg}} \),
   fixes pointwise the whole curve \( \eta_t \); in particular, all regular orbits \( G \cdot \eta_t \) are
   equivalent to the same homogeneous space \( G/L \);
4. the decompositions (1.1) associated with the points \( \eta_t \in M_{\text{reg}} \) do not depend
   on \( t \);
5. there exists a basis \( \{ f_1, \ldots, f_n \} \) for \( m \) such that for any \( \eta_t \in M_{\text{reg}} \) the complex
   structure \( J_t : m \to m \), induced by the complex structure of \( T_{\eta_t}M \), is of the
   following form:

\[
J_t f_{2j} = \lambda_j(t)f_{2j+1} , \quad J_t f_{2j+1} = -\frac{1}{\lambda_j(t)} f_{2j} ;
\]

(1.3)

where the function \( \lambda_j(t) \) is either one of the functions \(-\tanh(t), -\tanh(2t),
-\coth(t) \) and \(-\coth(2t) \) or it is identically equal to 1.

We call any such curve an \emph{optimal transversal curve}; the basis for \( \mathbb{R}Z + m \subset g \) given
by \((Z, f_1, \ldots, f_{2n-1})\), where the \( f_i \)'s verify (1.3), is called \emph{optimal basis associated
with} \( \eta \). An explicit description of the optimal basis for any given semisimple Lie

*group \( G \) is given in §3.*
Acted on by the compact semisimple Lie group $G$, in particular it is totally independent on the choice of the $G$-invariant Kähler metric $g$. At the same time, the Killing fields, associated with the elements of an optimal basis, determine a 1-parameter family of holomorphic frames at the points $\eta_t \in M_{\text{reg}}$, which are orthogonal w.r.t. at least one $G$-invariant Kähler metric $g$. It is also proved that, for all K-manifold $M$ which do not belong to a special class of non-standard K-manifold, those holomorphic frames are orthogonal w.r.t. any $G$-invariant Kähler metric $g$ on $M$ (see Corollary 4.2 for details). From these remarks and the fact that $\eta'_t = J\dot{Z}_{\eta_t}$, where $Z$ is the first element of any optimal basis, it may be inferred that any curve $\eta \in \mathcal{O}$ is a reparameterization of a normal geodesics of some (in most cases, any) $G$-invariant Kähler metric on $M$.

Our second main result is the following. Let $\eta$ be an optimal transversal curve of a K-manifold, $\mathfrak{g} = \mathfrak{l} + \mathbb{R}\mathfrak{Z} + \mathfrak{m}$ the decomposition (1.1) associated with the regular points $\eta_t \in M_{\text{reg}}$ and let $\omega$ and $\rho$ be the Kähler form and the Ricci form, respectively, associated with a given $G$-invariant Kähler metric $g$ on $(M, J)$.

By a slight modification of arguments used in [PS], we show that there exist two smooth curves

$$Z_{\omega}, Z_{\rho} : \mathbb{R} \to C_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l}) + \mathfrak{a}, \quad \mathfrak{a} = C_{\mathfrak{g}}(\mathfrak{l}) \cap (\mathbb{R}\mathfrak{Z} + \mathfrak{m}),$$

(1.4)
satisfying the following properties (here $\mathfrak{z}(\mathfrak{l})$ denotes the center of $\mathfrak{l}$ and $C_{\mathfrak{g}}(\mathfrak{l})$ denotes the centralizer of $\mathfrak{l}$ in $\mathfrak{g}$): for any $\eta_t \in M_{\text{reg}}$ and any two element $X, Y \in \mathfrak{g}$, with associated Killing fields $\hat{X}$ and $\hat{Y},$

$$\omega_{\eta_t}(\hat{X}, \hat{Y}) = B(Z_{\omega}(t), [X, Y]), \quad \rho_{\eta_t}(\hat{X}, \hat{Y}) = B(Z_{\rho}(t), [X, Y]).$$

(1.5)

We call such curves $Z_{\omega}(t)$ and $Z_{\rho}(t)$ the algebraic representatives of $\omega$ and $\rho$ along $\eta$. It is clear that the algebraic representatives determine uniquely the restrictions of $\omega$ and $\rho$ to the tangent spaces of the regular orbits. But the following Proposition establishes a result which is somehow stronger.

Before stating the proposition, we recall that in [PS] the following fact was established: if $\mathfrak{g} = \mathfrak{l} + \mathbb{R}\mathfrak{Z} + \mathfrak{m}$ is a decomposition of the form (1.1), then the subalgebra $\mathfrak{a} = C_{\mathfrak{g}}(\mathfrak{l}) \cap (\mathbb{R}\mathfrak{Z} + \mathfrak{m})$ is either 1-dimensional or 3-dimensional and isomorphic with $\mathfrak{su}_2$. By virtue of this dichotomy, the two cases considered in the following proposition are all possible cases.

**Proposition 1.1.** Let $\eta_t$ be an optimal transversal curve of a K-manifold $(M, J, g)$ acted on by the compact semisimple Lie group $G$ and let $\mathfrak{g} = \mathfrak{l} + \mathbb{R}\mathfrak{Z} + \mathfrak{m}$ be the decomposition of the form (1.1) determined by the points $\eta_t \in M_{\text{reg}}$. Let also $Z : \mathbb{R} \to C_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l}) + \mathfrak{a}$ be the algebraic representative of the Kähler form $\omega$ or of the Ricci form $\rho$. Then:

1. if $\mathfrak{a}$ is 1-dimensional, then it is of the form $\mathfrak{a} = \mathbb{R}\mathfrak{Z}$ and there exists an element $I \in \mathfrak{z}(\mathfrak{l})$ and a smooth function $f : \mathbb{R} \to \mathbb{R}$ so that

$Z(t) = f(t)Z + I$;

(1.6)

2. if $\mathfrak{a}$ is 3-dimensional, then it is of the form $\mathfrak{a} = \mathfrak{su}_2 = \mathbb{R}\mathfrak{Z} + \mathbb{R}\mathfrak{X} + \mathbb{R}\mathfrak{Y}$, with $[Z, X] = Y$ and $[X, Y] = Z$ and there exists an element $I \in \mathfrak{z}(\mathfrak{l})$, a real
number $C$ and a smooth function $f : \mathbb{R} \to \mathbb{R}$ so that
\[
Z(t) = f(t)Z_D + \frac{C}{\cosh(t)}X + I. \tag{1.7}
\]

Conversely, if $Z : \mathbb{R} \to C_{\mathfrak{g}}(l)$ is a curve in $C_{\mathfrak{g}}(l)$ of the form (1.6) or (1.7), then there exists a unique closed $J$-invariant, $G$-invariant 2-form $\varpi$ on the set of regular points $M_{\text{reg}}$, having $Z(t)$ as algebraic representative.

In particular, the Kähler form $\omega$ and the Ricci form $\rho$ are uniquely determined by their algebraic representatives.

Using (1.5), Proposition 1.1 and some basic properties of the decomposition $\mathfrak{g} = \mathfrak{l} + \mathbb{R} \mathfrak{z} + \mathfrak{m}$ (see §5), it can be shown that the algebraic representatives $Z_\omega(t)$ and $Z_\rho(t)$ are uniquely determined by the values $\omega_{\eta_t}(\hat{X}, J\hat{X}) = \mathcal{B}(Z_\omega(t), [X, J_tX])$ and $\rho_{\eta_t}(\hat{X}, J\hat{X}) = \mathcal{B}(Z_\rho(t), [X, J_tX])$, where $X \in \mathfrak{m}$ and $J_t$ is the complex structure on $\mathfrak{m}$ induced by the complex structure of the tangent space $T_{\eta_t}M$.

Here comes our third main result. It consists in Theorem 5.1 and Proposition 5.2, where we give the explicit expression for the value $r_{\eta_t}(X, X) = \rho_{\eta_t}(X, JX)$ for any $X \in \mathfrak{m}$, only in terms of the algebraic representative $Z_\omega(t)$ and of the Lie brackets between $X$ and the elements of the optimal basis in $\mathfrak{g}$. By the previous discussion, this result furnishes a way to write down explicitly the Ricci tensor of the Kähler metric associated with $Z_\omega(t)$.

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Notation. Throughout the paper, if $G$ is a Lie group acting isometrically on a Riemannian manifold $M$ and if $X \in \mathfrak{g} = \text{Lie}(G)$, we will adopt the symbol $\hat{X}$ to denote the Killing vector field on $M$ corresponding to $X$.

The Lie algebra of a Lie group will be always denoted by the corresponding gothic letter. For a group $G$ and a Lie algebra $\mathfrak{g}$, $Z(G)$ and $\mathfrak{z}(\mathfrak{g})$ denote the center of $G$ and of $\mathfrak{g}$, respectively. For any subset $A$ of a group $G$ or of a Lie algebra $\mathfrak{g}$, $C_G(A)$ and $C_{\mathfrak{g}}(A)$ are the centralizer of $A$ in $G$ and $\mathfrak{g}$, respectively.

Finally, for any subspace $\mathfrak{n} \subset \mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$, the symbol $\mathfrak{n}^\perp$ denotes the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{g}$ w.r.t. the Cartan-Killing form $\mathcal{B}$.

2. Fundamentals of K-manifolds.

2.1 K-manifolds, KO-manifolds and KE-manifolds.

A K-manifold is a pair formed by a compact Kähler manifold $(M, J, g)$ and a compact semisimple Lie group $G$ acting almost effectively and isometrically (hence biholomorphically) on $M$, such that:

i) $b_1(M) = 0$;

ii) $G$ acts of cohomogeneity one with respect to the action of $G$, i.e. the regular $G$-orbits are of codimension one in $M$. 

In this paper, \((M, J, \gamma)\) will always denote a \(K\)-manifold of dimension \(2n\), acted on by the compact semisimple Lie group \(G\). We will denote by \(\omega(\cdot, \cdot) = g(\cdot, J\cdot)\) the Kähler fundamental form and by \(\rho = r(\cdot, J\cdot)\) the Ricci form of \(M\).

For the general properties of cohomogeneity one manifolds and of \(K\)-manifolds, see e.g. [AA], [AA1], [Br], [HS], [PS]. Here we only recall some properties, which will be used in the paper.

If \(p \in M\) is a regular point, let us denote by \(L = G_p\) the corresponding isotropy subgroup. Since \(M\) is orientable, every regular orbit \(G \cdot p\) is orientable. Hence we may consider a unit normal vector field \(\xi\), defined on the subset of regular points \(M_{\text{reg}}\), which is orthogonal to any regular orbit. It is known (see [AA1]) that any integral curve of \(\xi\) is a geodesic. Any such geodesic is usually called normal geodesic.

A normal geodesic \(\gamma\) through a point \(p\) verifies the following properties: it intersects any \(G\)-orbit orthogonally; the isotropy subalgebra \(G_{\gamma_t}\) at a regular point \(\gamma_t\) is always equal \(G_p = L\) (see e.g. [AA], [AA1]). We formalize these two facts in the following definition.

We call nice transversal curve through a point \(p \in M_{\text{reg}}\) any curve \(\eta : \mathbb{R} \to M\) with \(p \in \eta(\mathbb{R})\) and such that:

1. it intersects any regular orbit;
2. for any \(\eta_t \in M_{\text{reg}}\)
   \[\eta_t' \notin T_{\eta_t}(G \cdot \eta_t)\;\tag{2.1}\]
3. for any \(\eta_t \in M_{\text{reg}}, G_{\eta_t} = L = G_p\).

The following property of \(K\)-manifold has been proved in [PS].

**Proposition 2.1.** Let \((M, J, \gamma)\) be a \(K\)-manifold acted on by the compact semisimple Lie group \(G\). Let also \(p \in M_{\text{reg}}\) and \(L = G_p\) the isotropy subgroup at \(p\). Then:

1. there exists an element \(Z\) (determined up to scaling) so that
   \[\mathbb{R}Z \subseteq C_\gamma(\mathfrak{l}) \cap \mathfrak{t}^\perp, \quad C_\gamma(\mathfrak{l} + \mathbb{R}Z) = \mathfrak{z}(\mathfrak{l}) + \mathbb{R}Z;\]  
   in particular, the connected subgroup \(K \subset G\) with subalgebra \(\mathfrak{k} = \mathfrak{t} + \mathbb{R}Z\) is the isotropy subgroup of a flag manifold \(F = G/K\);
2. the dimension of \(\mathfrak{a} = C_\gamma(\mathfrak{l}) \cap \mathfrak{t}^\perp\) is either 1 or 3; in case \(\dim_\mathbb{R} \mathfrak{a} = 3\), then \(\mathfrak{a}\) is a subalgebra isomorphic to \(\mathfrak{su}_2\) and there exists a Cartan subalgebra \(\mathfrak{t}^\mathbb{C} \subset \mathfrak{l}^\mathbb{C} + \mathfrak{a}^\mathbb{C} \subset \mathfrak{g}^\mathbb{C}\) so that \(\mathfrak{a}^\mathbb{C} = \mathfrak{CH}_\alpha + \mathbb{C}E_\alpha + \mathbb{C}E_{-\alpha}\) for some root \(\alpha\) of the root system of \((\mathfrak{g}^\mathbb{C}, \mathfrak{t}^\mathbb{C})\).

Note that if for some regular point \(p\) we have that \(\dim_\mathbb{R} \mathfrak{a} = 1\) (resp. \(\dim_\mathbb{R} \mathfrak{a} = 3\)), then the same occurs at any other regular point. Therefore we may consider the following definition.

**Definition 2.2.** Let \((M, J, \gamma)\) be a \(K\)-manifold and \(L = G_p\) the isotropy subgroup of a regular point \(p\). We say that \(M\) is a \(K\)-manifold with ordinary action (or shortly, \(KO\)-manifold) if \(\dim_\mathbb{R} \mathfrak{a} = \dim_\mathbb{R}(C_\gamma(\mathfrak{l}) \cap \mathfrak{t}^\perp) = 1\).

In all other cases, we say that \(M\) is with extra-ordinary action (or, shortly, \(KE\)-manifold).
Another useful property of K-manifolds is the following. It can be proved that any K-manifold admits exactly two singular orbits, at least one of which is complex (see [PS1]). By the results in [HS], it also follows that if $M$ is a K-manifold whose singular orbits are both complex, then $M$ admits a $G$-equivariant blow-up $\tilde{M}$ along the complex singular orbits, which is still a K-manifold and admits a holomorphic fibration over a flag manifold $G/K = G^\mathbb{C}/P$, with standard fiber equal to $\mathbb{C}P^1$.

Several other important facts are related to the existence (or non-existence) of two singular complex orbits (see [PS1] for a review of these properties). For this reason, it is convenient to introduce the following definition.

**Definition 2.3.** We say that a K-manifold $M$, acted on by a compact semisimple group $G$ with cohomogeneity one, is **standard** if the action of $G$ has two singular complex orbits. We call it **non-standard** in all other cases.

**2.2 The CR structure of the regular orbits of a K-manifold.**

A *CR structure of codimension* $r$ on a manifold $N$ is a pair $(\mathcal{D}, J)$ formed by a distribution $\mathcal{D} \subset TN$ of codimension $r$ and a smooth family $J$ of complex structures $J_x : \mathcal{D}_x \to \mathcal{D}_x$ on the spaces of the distribution.

A CR structure $(\mathcal{D}, J)$ is called *integrable* if the distribution $\mathcal{D}^{10} \subset T\mathbb{C}N$, given by the $J$-eigenspaces $\mathcal{D}^{10}_x \subset \mathcal{D}_x^\mathbb{C}$ corresponding to the eigenvalue $+i$, verifies

$$[\mathcal{D}^{10}, \mathcal{D}^{10}] \subset \mathcal{D}^{10}.$$  

Note that a complex structure $J$ on manifold $N$ may be always considered as an integrable CR structure of codimension zero.

A smooth map $\phi : N \to N'$ between two CR manifolds $(N, \mathcal{D}, J)$ and $(N', \mathcal{D}', J')$ is called CR map (or holomorphic map) if:

a) $\phi_*(\mathcal{D}) \subset \mathcal{D}'$;

b) for any $x \in N$, $\phi_* \circ J_x = J'_{\phi(x)} \circ \phi_*|_{\mathcal{D}_x}$.

A CR transformation of $(N, \mathcal{D}, J)$ is a diffeomorphism $\phi : N \to N$ which is also a CR map.

Any codimension one submanifold $N \subset M$ of a complex manifold $(M, J)$ is naturally endowed with an integrable CR structure of codimension one $(\mathcal{D}, J)$, which is called induced CR structure; it is defined by

$$\mathcal{D}_x = \{ v \in T_x N : Jv \in T_x N \} \quad J_x = J|_{\mathcal{D}_x}.$$  

It is clear that any regular orbit $G/L = G \cdot x \in M$ of a K-manifold $(M, J, g)$ has an induced CR structure $(\mathcal{D}, J)$, which is invariant under the transitive action of $G$. For this reason, several facts on the global structure of the regular orbits of a K-manifolds can be detected using what is known on compact homogeneous CR manifolds (see e.g. [AHR] and [AS]).

Here, we recall some of those facts, which will turn out to be crucial in the next sections.
Let \((G/L, \mathcal{D}, J)\) be a homogeneous CR manifold of a compact semisimple Lie group \(G\), with an integrable CR structure \((\mathcal{D}, J)\) of codimension one. If we consider the \(\mathcal{B}\)-orthogonal decomposition \(\mathfrak{g} = \mathfrak{l} + \mathfrak{n}\), where \(\mathfrak{l} = \text{Lie}(L)\), then the orthogonal complement \(\mathfrak{n}\) is naturally identifiable with the tangent space \(T_o(G/L)\), \(o = eL\), by means of the map
\[
\phi : \mathfrak{n} \to T_o(G/L) , \quad \phi(X) = \hat{X}|_o .
\]
If we denote by \(\mathfrak{m}\) the subspace
\[
\mathfrak{m} = \phi^{-1}(\mathcal{D}_o) \subset \mathfrak{n} ,
\]
we get the following orthogonal decomposition of \(\mathfrak{g}\):
\[
\mathfrak{g} = \mathfrak{l} + \mathfrak{n} = \mathfrak{l} + \mathbb{R}Z_D + \mathfrak{m} .
\] \tag{2.3}
where \(Z_D \in (\mathfrak{l} + \mathfrak{m})^\perp\). Since the decomposition is \(\text{ad}_\mathfrak{g}\)-invariant, it follows that \(Z_D \in C_\mathfrak{g}(\mathfrak{l})\).

Using again the identification map \(\phi : \mathfrak{n} \to T_o(G/L)\), we may consider the complex structure
\[
J : \mathfrak{m} \to \mathfrak{m} , \quad J \overset{\text{def}}{=} \phi^*(J_o) .
\] \tag{2.4}
Note that \(J\) is uniquely determined by the direct sum decomposition
\[
\mathfrak{m}^C = \mathfrak{m}^{10} + \mathfrak{m}^{01} , \quad \mathfrak{m}^{01} = \overline{\mathfrak{m}^{10}} ,
\] \tag{2.5}
where \(\mathfrak{m}^{10}\) and \(\mathfrak{m}^{01}\) are the \(J\)-eigenspaces with eigenvalues \(+i\) and \(-i\), respectively.

In all the following, (2.3) will be called the structural decomposition of \(\mathfrak{g}\) associated with \(\mathcal{D}\); the subspace \(\mathfrak{m}^{10} \subset \mathfrak{m}^C\) (respectively, \(\mathfrak{m}^{01} = \overline{\mathfrak{m}^{10}}\)) given (2.5) will be called the holomorphic (resp. anti-holomorphic) subspace associated with \((\mathcal{D}, J)\).

We recall that a \(G\)-invariant CR structure \((\mathcal{D}, J)\) on \(G/L\) is integrable if and only if the associated holomorphic subspace \(\mathfrak{m}^{10} \subset \mathfrak{m}^C\) is so that
\[
\mathfrak{l}^C + \mathfrak{m}^{10} \text{ is a subalgebra of } \mathfrak{g}^C .
\] \tag{2.6}

We now need to introduce a few concepts which are quite helpful in describing the structure of a generic compact homogeneous CR manifold.

**Definition 2.4.** Let \(N = G/L\) be a homogeneous manifold of a compact semisimple Lie group \(G\) and \((\mathcal{D}, J)\) a \(G\)-invariant, integrable CR structure of codimension one on \(N\).

We say that a CR manifold \((N = G/L, \mathcal{D}, J)\) is a Morimoto-Nagano space if either \(G/L = S^{2n-1}, n > 1\), endowed with the standard CR structure of \(S^{2n-1} \subset \mathbb{C}P^n\), or there exists a subgroup \(H \subset G\) so that:

a) \(G/H\) is a compact rank one symmetric space (i.e. \(\mathbb{R}P^n = \text{SO}_{n+1}/\text{SO}_n \cdot \mathbb{Z}_2, \mathbb{S}^{n} = \text{SO}_{n+1}/\text{SO}_n, \mathbb{C}P^n = \text{SU}_{n+1}/\text{SU}_n, \mathbb{H}P^n = \text{Sp}_{n+1}/\text{Sp}_n \) or \(\mathbb{O}P^2 = \text{F}_4/\text{Spin}_9\));
b) \(G/L\) is a sphere bundle \(S(G/H) \subset T(G/H)\) in the tangent space of \(G/H\);
c) \((\mathcal{D}, J)\) is the CR structure induced on \(G/L = S(G/H)\) by the \(G\)-invariant complex structure of \(T(G/H) \cong \mathfrak{g}^C/H^C\).
If a Morimoto-Nagano space is $G$-equivalent to a sphere $S^{2n-1}$ we call it trivial; we call it non-trivial in all other cases.

A $G$-equivariant holomorphic fibering
\[ \pi : N = G/L \to F = G/Q \]
of $(N, D, J)$ onto a non-trivial flag manifold $(F = G/Q, J_F)$ with invariant complex structure $J_F$, is called CRF fibration. A CRF fibration $\pi : G/L \to G/Q$ is called nice if the standard fiber is a non-trivial Morimoto-Nagano space; it is called very nice if it is nice and there exists no other nice CRF fibration $\pi' : G/L \to G/Q$ with standard fibers of smaller dimension.

The following Proposition gives necessary and sufficient conditions for the existence of a CRF fibration. The proof can be found in [AS].

**Proposition 2.5.** Let $G/L$ be homogeneous CR manifold of a compact semisimple Lie group $G$, with an integrable, codimension one $G$-invariant CR structure $(D, J)$. Let also $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_D + \mathfrak{m}$ be the structural decomposition of $\mathfrak{g}$ and $\mathfrak{m}^{10}$ the holomorphic subspace, associated with $(D, J)$.

Then $G/L$ admits a non-trivial CRF fibration if and only if there exists a proper parabolic subalgebra $\mathfrak{p} = \mathfrak{r} + \mathfrak{n} \subset \mathfrak{g}^\mathbb{C}$ (here $\mathfrak{r}$ is a reductive part and $\mathfrak{n}$ the nilradical of $\mathfrak{p}$) such that:

\begin{enumerate}
  \item $\mathfrak{r} = (\mathfrak{p} \cap \mathfrak{g})^\mathbb{C}$
  \item $\mathfrak{l}^\mathbb{C} + \mathfrak{m}^{01} \subset \mathfrak{p}$
  \item $\mathfrak{l}^\mathbb{C} \subset \mathfrak{r}$
\end{enumerate}

In this case, $G/L$ admits a CRF fibration with basis $G/Q = G^\mathbb{C}/P$, where $Q$ is the connected subgroup generated by $\mathfrak{q} = \mathfrak{r} \cap \mathfrak{g}$ and $P$ is the parabolic subgroup of $G^\mathbb{C}$ with Lie algebra $\mathfrak{p}$.

Let us go back to the regular orbits of a K-manifold $(M, J, g)$ acted on by the compact semisimple group $G$. We already pointed out that each regular orbit $(G/L = G \cdot x, D, J)$, endowed with the induced CR structure $(D, J)$, is a compact homogeneous CR manifold. In the statement of the following Theorem we collect the main results on the one-parameter family of compact homogeneous CR manifolds given by the regular orbits of a K-manifold, which is a direct consequence Th. 3.1 in [PS1] (see also [HS] and [PS] Th.2.4).

**Theorem 2.6.** Let $(M, J, g)$ be a K-manifold acted on by the compact semisimple Lie group $G$.

\begin{enumerate}
  \item If $M$ is standard, then there exists a flag manifold $(G/K, J_o)$ with a $G$-invariant complex structure $J_o$, such that any regular orbit $(G \cdot x = G/L, D, J)$ of $M$ admits a CRF-fibration $\pi : (G/L, D, J) \to (G/K, J_o)$ onto $(G/K, J_o)$ with standard fiber $S^1$.
  \item If $M$ is non-standard, then there exists a flag manifold $(G/K, J_o)$ with a $G$-invariant complex structure $J_o$ such that any regular orbit $(G/L = G \cdot x, D, J)$ admits a very nice CRF fibration $\pi : (G/L, D, J) \to (G/K, J_o)$ where the standard fiber $K/L$ is a non-trivial Morimoto-Nagano space of dimension $\dim K/L \geq 3$.
\end{enumerate}
Furthermore, if the last case occurs, then the fiber $K/L$ of the CRF fibration $\pi : (G/L, D, J) \to (G/K, D)$ has dimension $\beta$ if and only if $M$ is a non-standard KE-manifold and $K/L$ is either $S(RP^2) \subset T(RP^2) = CP^2 \setminus \{ [z] : t \cdot z = 0 \}$ or $S(CP^1) \subset T(CP^1) = CP^1 \times CP^1 \setminus \{ [z] = [w] \}$.

3. The optimal transversal curves of a K-manifold.

3.1 Notation and preliminary facts.

If $G$ is a compact semisimple Lie group and $t^C \subset g^C$ is a given Cartan subalgebra, we will use the following notation:

- $B$ is the Cartan-Killing form of $g$ and for any subspace $A \subset g$, $A^\perp$ is the $B$-orthogonal complement to $A$;
- $R$ is the root system of $(g^C, t^C)$;
- $H_\alpha \in t^C$ is the $B$-dual element to the root $\alpha$;
- for any $\alpha, \beta \in R$, the scalar product $(\alpha, \beta)$ is set to be equal to $\langle H_\alpha, H_\beta \rangle$;
- $E_\alpha$ is the root vector with root $\alpha$ in the Chevalley normalization; in particular $B(E_\alpha, E_{-\beta}) = \delta_{\alpha\beta}$, $[E_\alpha, E_{-\alpha}] = H_\alpha$, $[H_\alpha, E_\beta] = (\beta, \alpha) E_\beta$ and $[H_\alpha, E_{-\beta}] = -\langle \beta, \alpha \rangle E_{-\beta}$;
- for any root $\alpha$,

$$F_\alpha = \frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha}) \quad G_\alpha = \frac{i}{\sqrt{2}}(E_\alpha + E_{-\alpha})$$

note that for $\alpha, \beta \in R$

$$B(F_\alpha, F_\beta) = -\delta_{\alpha\beta} = B(G_\alpha, G_\beta) \quad B(F_\alpha, G_\beta) = B(F_\alpha, H_\beta) = B(G_\alpha, H_\beta) = 0 ;$$

- the notation for the roots of a simple Lie algebra is the same of [GOV] and [AS].

Recall that for any two roots $\alpha, \beta$, with $\beta \neq -\alpha$, in case $[E_\alpha, E_\beta]$ is non trivial then it is equal to $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ where the coefficients $N_{\alpha,\beta}$ verify the following conditions:

$$N_{\alpha,\beta} = -N_{\beta,\alpha} \quad N_{\alpha,\beta} = -N_{-\alpha,-\beta} . \quad (3.1)$$

From (3.1) and the properties of root vectors in the Chevalley normalization, the following well known properties can be derived:

1. for any $\alpha, \beta \in R$ with $\alpha \neq \beta$

$$[F_\alpha, F_\beta], [G_\alpha, G_\beta] \in \text{span}\{F_\gamma \mid \gamma \in R\} \quad [F_\alpha, G_\beta] \in \text{span}\{G_\gamma \mid \gamma \in R\} ; \quad (3.2)$$

2. for any $H \in t^C$ and any $\alpha, \beta \in R$, $B(H, [F_\alpha, F_\beta]) = B(H, [G_\alpha, G_\beta]) = 0$ and

$$B(H, [F_\alpha, G_\beta]) = i\delta_{\alpha\beta} B(H, H_\alpha) = \delta_{\alpha\beta} \alpha(iH) ; \quad (3.3)$$
Finally, for what concerns the Lie algebra of flag manifolds and of CR manifolds, we adopt the following notation.

Assume that $G/K$ is a flag manifold with invariant complex structure $J$ (for definitions and basic facts, we refer to [Al], [AP], [BFR], [Ni]) and let $\pi: G/L \to G/K$ be a $G$-equivariant $S^1$-bundle over $G/K$. In particular, let us assume that $l$ is a codimension one subalgebra of $k$. Recall that $k = k_{ss} + z(k)$, with $k_{ss}$ semisimple part of $k$. Hence the semisimple part $l_{ss}$ of $l$ is equal to $k_{ss}$ and $k = l + \mathbb{R}Z = (l_{ss} + \mathfrak{z}(\mathfrak{k}) \cap \mathfrak{l}) + \mathbb{R}Z$ for some $Z \in \mathfrak{z}(\mathfrak{k})$.

Let $\mathfrak{t}^C \subset \mathfrak{k}^C$ be a Cartan subalgebra for $\mathfrak{g}^C$ contained in $\mathfrak{k}^C$ and $R$ the root system of $(\mathfrak{g}^C, \mathfrak{t}^C)$. Then we will use the following notation:

- $R_\alpha = \{ \alpha \in R, E_\alpha \in \mathfrak{k} \}$;
- $R_m = \{ \alpha \in R, E_\alpha \in \mathfrak{m} \}$;
- for any $\alpha \in R$, we denote by $g(\alpha)^C = \text{span}_C \{ E_{\pm \alpha}, H_\alpha \}$ and $g(\alpha) = g(\alpha)^C \cap g$;
- $m(\alpha)$ denotes the irreducible $\mathfrak{t}^C$-submoduli of $\mathfrak{m}^C$, with highest weight $\alpha \in R_m$;
- if $m(\alpha)$ and $m(\beta)$ are equivalent as $\mathfrak{t}^C$-moduli, we denote by $m(\alpha) + \lambda m(\beta)$ the irreducible $\mathfrak{t}^C$-module with highest weight vector $E_\alpha + \lambda E_\beta$, $\alpha, \beta \in R_m$, $\lambda \in \mathbb{C}$.

3.2 The structural decomposition $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_D + \mathfrak{m}$ determined by the CR structure of a regular orbit.

The main results of this subsection are given by the following two theorems on the structural decomposition of the regular orbits of a $K$-manifolds. The first one is a straightforward consequence of definitions, Theorem 2.6 and the results in [PS].

**Theorem 3.1.** Let $(M, J, g)$ be a standard $K$-manifold acted on by the compact semisimple group $G$ and let $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_D + \mathfrak{m}$ and $\mathfrak{m}^{10}$ be the structural decomposition and the holomorphic subspace, respectively, associated with the CR structure $(\mathcal{D}, J)$ of a regular orbit $G/L = G \cdot p$. Let also $J: \mathfrak{m} \to \mathfrak{m}$ be the unique complex structure on $\mathfrak{m}$, which determines the decomposition $\mathfrak{m}^C = \mathfrak{m}^{10} + \overline{\mathfrak{m}^{10}}$.

Then, $\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z_D$ is the isotropy subalgebra of a flag manifold $K$, and the complex structure $J: \mathfrak{m} \to \mathfrak{m}$ is $\text{ad}_\mathfrak{k}$-invariant and corresponds to a $G$-invariant complex structure $J$ on $G/K$.

In particular, there exists a Cartan subalgebra $\mathfrak{t}^C \subset \mathfrak{k}^C$ and an ordering of the associated root system $R$, so that $\mathfrak{m}^{10}$ is generated by the corresponding positive root vectors in $\mathfrak{m}^C = (\mathfrak{t}^+)^C$.

The following theorem describes the structural decomposition and the holomorphic subspace of a regular orbit of a non-standard $K$-manifold. Also this theorem can be considered as a consequence of Theorem 2.6, but the proof is a little bit more involved.
**Theorem 3.2.** Let \((M, J, g)\) be a non-standard K-manifold acted on by the compact semisimple group \(G\) and let \(\mathfrak{g} = \mathfrak{t} + \mathbb{R} \mathcal{Z}_D + \mathfrak{m}\) and \(\mathfrak{m}^{10}\) be the structural decomposition and the holomorphic subspace, respectively, associated with the CR structure \((\mathcal{D}, J)\) of a regular orbit \(G/L = G \cdot p\).

Then there exists a simple subalgebra \(\mathfrak{g}_F \subset \mathfrak{g}\) with the following properties:

a) denote by \(I_F = I \cap \mathfrak{g}_F\), \(I_0 = I \cap \mathfrak{g}_F^\perp\), \(\mathfrak{m}_F = \mathfrak{m} \cap \mathfrak{g}_F\) and \(\mathfrak{m}' = \mathfrak{m} \cap \mathfrak{g}_F^\perp\); then the pair \((\mathfrak{g}_F, I_F)\) is one of those listed in Table 1 and \(\mathfrak{g}\) and \(\mathfrak{g}_F\) admit the following \(\mathcal{B}\)-orthogonal decompositions:

\[
\mathfrak{g} = I_0 + (I_F + \mathbb{R} \mathcal{Z}_D) + (\mathfrak{m}_F + \mathfrak{m}'), \quad \mathfrak{g}_F = I_F + \mathbb{R} \mathcal{Z}_D + \mathfrak{m}_F;
\]

furthermore \([I_0, \mathfrak{g}_F] = \{0\}\) and the connected subgroup \(K \subset G\) with Lie algebra \(\mathfrak{k} = I_0 + \mathfrak{g}_F\) is the isotropy subalgebra of a flag manifold \(G/K\);

b) denote by \(\mathfrak{m}_F^{10} = \mathfrak{m}_F^C \cap \mathfrak{m}^{10}\); then there exists a Cartan subalgebra \(\mathfrak{t}_F^C \subset \mathfrak{t}_F^C + \mathbb{C} \mathcal{Z}_D\) and a complex number \(\lambda\) with \(0 < |\lambda| < 1\) so that the element \(\mathcal{Z}_D\), determined up to scaling, and the subspace \(\mathfrak{m}_F^{10}\), determined up to an element of the Weyl group and up to complex conjugation, are as listed in Table 1 (see §3.1 for notation):

| \(\mathfrak{g}_F\) | \(\mathfrak{t}_F\) | \(\mathcal{Z}_D\) | \(\mathfrak{m}_F^{10}\) |
|---|---|---|---|
| \(\mathfrak{su}_2\) | \(\{0\}\) | \(-\frac{i}{2}H_{\varepsilon_1 - \varepsilon_2}\) | \(\mathbb{C}(E_{\varepsilon_1 - \varepsilon_2} + \lambda E_{-\varepsilon_1 + \varepsilon_2})\) |
| \(\mathfrak{su}_{n+1}\) \(\oplus \mathbb{R}\) | \(\mathfrak{su}_{n-2}\) | \(-iH_{\varepsilon_1 - \varepsilon_2}\) | \((\mathbb{C}(E_{\varepsilon_1 - \varepsilon_2} + \lambda^2 E_{-\varepsilon_1 + \varepsilon_2}) \oplus (m(\varepsilon_1 - \varepsilon_2) + \lambda m(\varepsilon_2 - \varepsilon_3) + \lambda m(\varepsilon_3 - \varepsilon_1)))\) |
| \(\mathfrak{su}_2 \oplus \mathfrak{su}_2\) | \(\mathbb{R}\) | \(-\frac{i}{2}(H_{\varepsilon_1 - \varepsilon_2} + H_{\varepsilon_1 - \varepsilon_2}')\) | \(\mathbb{C}(E_{\varepsilon_1 - \varepsilon_2} + \lambda E_{-(\varepsilon_1' - \varepsilon_2')}) \oplus \mathbb{C}(E_{\varepsilon_1' - \varepsilon_2' + \lambda E_{-\varepsilon_1 - \varepsilon_2})\) |
| \(\mathfrak{so}_7\) | \(\mathfrak{su}_3\) | \(-\frac{2i}{3}(H_{\varepsilon_1 + \varepsilon_2} + H_{\varepsilon_3})\) | \((m(\varepsilon_1 + \varepsilon_2) + \lambda m(-\varepsilon_3)) \oplus m(-\varepsilon_3) + (\lambda m(\varepsilon_1 + \varepsilon_2))\) |
| \(\mathfrak{f}_4\) | \(\mathfrak{so}_7\) | \(-2H_{\varepsilon_1}\) | \((m(\varepsilon_1 + \varepsilon_2) + \lambda^2 m(-\varepsilon_1 + \varepsilon_2)) \oplus (m(1/2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) + \lambda m(1/2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)))\) |
| \(\mathfrak{so}_{2n+1}\) \(\oplus \mathfrak{so}_{2n-1}\) | \(\mathfrak{so}_2\) | \(-i2H_{\varepsilon_1}\) | \((m(\varepsilon_1 + \varepsilon_2) + \lambda m(-\varepsilon_1 + \varepsilon_2))\) |
| \(\mathfrak{so}_{2n}\) \(\oplus \mathfrak{so}_{2n-2}\) | \(\mathfrak{so}_2\) | \(-iH_{\varepsilon_1}\) | \((m(\varepsilon_1 + \varepsilon_2) + \lambda m(-\varepsilon_1 + \varepsilon_2))\) |
| \(\mathfrak{sp}_n\) \(\oplus \mathfrak{sp}_{n-2}\) | \(\mathfrak{sp}_1\) \(\oplus \mathfrak{sp}_{n-2}\) | \(-iH_{\varepsilon_1 + \varepsilon_2}\) | \((m(2\varepsilon_1) + \lambda^2 m(-2\varepsilon_2) \oplus (m(\varepsilon_1 + \varepsilon_2) + \lambda m(-\varepsilon_2 + \varepsilon_3))\) |

**Table 1**

c) the holomorphic subspace \(\mathfrak{m}^{10}\) admits the following orthogonal decomposition

\[
\mathfrak{m}^{10} = \mathfrak{m}_F^{10} + \mathfrak{m}'^{10}
\]

where \(\mathfrak{m}'^{10} = \mathfrak{m}_F^C \cap \mathfrak{m}^{10}\);

d) the complex structure \(J' : \mathfrak{m} \to \mathfrak{m}'\) associated with the eigenspace decomposition \(\mathfrak{m}_F^C = \mathfrak{m}^{10} + \mathfrak{m}'^{01}\), where \(\mathfrak{m}'^{01} = \mathfrak{m}_F^\perp 10\), is \(\text{Ad}_K\)-invariant and
determines a $G$-invariant complex structure on the flag manifold $G/K$; in particular the $J'$-eigenspaces are $\text{ad}_{\mathbb{R}Z_D}$-invariant:

$$[\mathbb{R}Z_D, m^{10}] \subset m^{10}, \quad [\mathbb{R}Z_D, m^{01}] \subset m^{01}. $$

The proof of Theorem 3.2 needs the following Lemma.

**Lemma 3.3.** Let $G/L = G \cdot p$ be a regular orbit of the non-standard $K$-manifold $(M, J, g)$. Let also $\pi : (G/L, D, J) \rightarrow (G/K, J_0)$ be the CRF fibration given in Theorem 2.6 and $(D^K, J^K)$ the CR structures of the standard fiber $K/L$. Then:

i) the 1-dimensional subspaces $\mathbb{R}Z_{D^K}$ and $\mathbb{R}Z_D$ of the structural decompositions of $\mathfrak{k}$ and $\mathfrak{g}$ at the point $p$ are the same, i.e. their structural decompositions are $\mathfrak{k} = 1 + \mathbb{R}Z_D + m_K$ and $\mathfrak{g} = 1 + \mathbb{R}Z_D + m = 1 + \mathbb{R}Z_D + (m_K + m)$;

ii) the holomorphic subspace $m^{10}$ of $(G/L, D, J)$ admits the $B$-orthogonal decomposition $m^{10} = m^{10}_K + m^{10}$ where $m^{10} = m^{10} \cap m^{10}$ and $m^{10}_K$ is the holomorphic subspace of $(K/L, D^K, J^K)$;

iii) $[\mathbb{R}Z_D, m^{10}] \subset m^{10}$ and $[\mathbb{R}Z_D, m^{01}] \subset m^{01}$.

**Proof.** Let $\mathfrak{k} = 1 + \mathbb{R}Z_{D^K} + m_K$ and $\mathfrak{g} = 1 + \mathbb{R}Z_D + m$ be the structural decompositions of $\mathfrak{k}$ and $\mathfrak{g}$ at the point $p$, associated with the CR structures $(D^K, J^K)$ and $(D, J)$, respectively. Denote also by $J^K$ and $J$ the induced complex structures on $m_K$ and $m$.

To prove i), we have to show that $\mathbb{R}Z_{D^K} = \mathbb{R}Z_D$. This is proved by the following observation. By definitions,

$$m_K = \{ X \in m : \pi_*(\hat{X}_{eL}) = 0 \} = m \cap \mathfrak{k}$$

and hence

$$\mathbb{R}Z_{D^K} = \mathfrak{k} \cap (1 + m_K)^{\perp} = \mathfrak{k} \cap (1 + (m \cap \mathfrak{k}))^{\perp} \subset \mathfrak{k} \cap (1 + m)^{\perp} = \mathfrak{k} \cap \mathbb{R}Z_D = \mathbb{R}Z_D.$$ 

ii) follows from the fact $J_K = J_{|m_K}$.

To prove iii), we recall that by Proposition 2.5, if $P$ is the parabolic subgroup such that $(G/K, J_F)$ is $G$-equivariantly biholomorphic to $G^C/P$, then the subalgebra $\mathfrak{p} = \text{Lie}(P) \subset \mathfrak{g}^C$ verifies

$$\mathfrak{t}^C + m^{01}_K + m^{01} \subset \mathfrak{p} = \mathfrak{t}^C + \mathfrak{n}$$

where $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$ and $\mathfrak{t}^C$ is a reductive complement to $\mathfrak{n}$. In particular, $m^{01} \subset \mathfrak{p} \cap (\mathfrak{t}^C)^\perp = \mathfrak{n}$. Moreover,

$$\dim_C m^{01} = \dim_C G/P = \dim_C \mathfrak{n}$$

and hence $m^{01} = \mathfrak{n}$. It follows that $[\mathbb{R}Z_D, m^{01}] \subset [\mathfrak{t}^C, \mathfrak{n}] \subset \mathfrak{n} = m^{01}$ and $[\mathbb{R}Z_D, m^{10}] = [\mathbb{R}Z_D, m^{01}] \subset m^{01} = m^{10}$. □
Proof of Theorem 3.2. Let $K \subset G$ be a subgroup so that any regular orbit $G/L$ admits a very nice CRF fibration $\pi : (G/L, D, J) \to (G/K, J_o)$ as prescribed by Theorem 2.6. Then, for any regular point $p$, the $K$-orbit $K/L = K \cdot p \subset G/L = G \cdot p$ (which is the fiber of the CRF fibration $\pi$) is a non-trivial Morimoto-Nagano space. In particular, $K/L$ is Levi non-degenerate, it is simply connected and the CR structure is non-standard (for the definition of non-standard CR structures and the properties of the CR structures of the Morimoto-Nagano spaces, see [AS]). Furthermore, by Lemma 3.3, the 1-dimensional subspace $\mathbb{R}Z_{DN}$ associated with the CR structure of $K/L$ coincides with the 1-dimensional subspace $\mathbb{R}Z_D$ associated with the CR structure of $G/L$.

Let $L_o \subset L$ be the normal subgroup of the elements which act trivially on $K/L$. Let also $G_F = K/L_o$ and $I_o = \text{Lie}(L_o)$. $g_F = \mathfrak{t} \cap (I_o)^\perp \cong \text{Lie}(G_F)$.

Note that Th. 1.3, 1.4 and 1.5 of [AS] apply immediately to the homogeneous CR manifold $G_F/L_F$, with $L_F = L \mod L_o$. In particular, since the CRF fibration $\pi : G/L \to G/K$ is very nice, $K/L = G_F/L_F$ is a primitive homogeneous CR manifold (for the definition of primitive CR manifolds, see [AS]) and $g_F$ is $\mathfrak{su}_n$, $\mathfrak{su}_2 + \mathfrak{su}_2$, $\mathfrak{so}_7$, $\mathfrak{so}_n$ ($n \geq 5$) or $\mathfrak{sp}_n$ ($n \geq 2$).

From Th.1.4, Prop. 6.3 and Prop. 6.4 in [AS] and from Lemma 3.3 i) and ii), it follows immediately that the subalgebra $g_F$ and the holomorphic subspace $m_{10}^F$, associated with the CR structure of the fiber $K/L = G_F/L_F$, verify a), b), c) and d). □

In the following, we will call the subalgebra $g_F$ the Morimoto-Nagano subalgebra of the non-standard $K$-manifold $M$. We will soon prove that the Morimoto-Nagano subalgebra is independent (up to conjugation) from the choice of the regular orbit $G \cdot p = G/L$.

We will also call $(g_F, I_F)$ and the subspace $m_{10}^F$ the Morimoto-Nagano pair and the Morimoto-Nagano holomorphic subspace, respectively, of the regular orbit $G/L = G \cdot p$.

3.3 Optimal transversal curves.

We prove now the existence of a special family of nice transversal curves called optimal transversal curves (see §1). We first show the existence of such curves for a non-standard $K$-manifold.

Theorem 3.4. Let $(M, J, g)$ be a non-standard $K$-manifold acted on by the compact semisimple group $G$. Then there exists a point $p_o$ in the non-complex singular orbit and an element $Z \in \mathfrak{g}$, such that the curve

$$\eta : \mathbb{R} \to M, \quad \eta_t = \exp(tZ) \cdot p_o$$

verifies the following properties:

1. it is a nice transversal curve; in particular the isotropy subalgebra $\mathfrak{g}_{\eta_t}$ for any $\eta_t \in M_{\text{reg}}$ is a fixed subalgebra $I$;
(2) there exists a subspace $\mathfrak{m}$ such that, for any $\eta_t \in \mathcal{M}_{\text{reg}}$, the structural decomposition $\mathfrak{g} = \mathfrak{l} + \mathbb{R} Z_D(t) + \mathfrak{m}(t)$ of the orbit $G/L = G \cdot \eta_t$ is given by $\mathfrak{m}(t) = \mathfrak{m}$ and $\mathbb{R} Z_D(t) = \mathbb{R} Z$;

(3) the Morimoto-Nagano pairs $(\mathfrak{g}_F(t), \mathfrak{l}_F(t))$ of the regular orbits $G \cdot \eta_t$ do not depend on $t$;

(4) for any $\eta_t \in \mathcal{M}_{\text{reg}}$, the holomorphic subspace $\mathfrak{m}^{10}(t)$ admits the orthogonal decomposition

$$\mathfrak{m}^{10}(t) = \mathfrak{m}_F^{10}(t) + \mathfrak{m}^{10}(t)$$

where $\mathfrak{m}^{10}(t) = \mathfrak{m}^{10} \subset \mathfrak{m}^C$ is independent on $t$ and $\mathfrak{m}_F^{10}(t)$ is a Morimoto-Nagano holomorphic subspace which is listed in Table 1, determined by the parameter $\lambda$ equal to $\lambda = \lambda(t) = e^{2t}$.

Moreover, if $\eta_t = \exp(\text{ti}Z) \cdot p_0$ is any of such curves and if $(\mathfrak{g}_F, \mathfrak{l}_F)$ is (up to conjugation) the Morimoto-Nagano pair of a regular orbits $G/L = G \cdot \eta_t$, then (up to conjugation) $Z$ is the element in the column "ZD" of Table 1, associated with the Lie algebra $\mathfrak{g}_F$.

For the proof of Theorem 3.4, we first need two Lemmata.

**Lemma 3.5.** Let $(\mathcal{M}, J, g)$ be a $K$-manifold acted on by the compact semisimple Lie group $G$. Let also $p$ be a regular point and $G/L = G \cdot p$ and $G^C/H = G^C \cdot p$ the $G$- and the $G^C$-orbit of $p$, respectively. Then:

(1) the isotropy subalgebra $\mathfrak{h} = \text{Lie}(G^C_p)$ is equal to

$$\mathfrak{h} = \mathfrak{l}^C + \mathfrak{m}^{01}$$

where $\mathfrak{m}^{01} = \overline{\mathfrak{m}^{10}}$ is the anti-holomorphic subspace associated with the CR structure of $G/L = G \cdot p$;

(2) for any $g \in G^C$, the isotropy subalgebra $\mathfrak{l}' = \mathfrak{g}_{p'}$ at $p' = g \cdot p$ is equal to

$$\mathfrak{l}' = \text{Ad}_g(\mathfrak{l}^C + \mathfrak{m}^{01}) \cap \mathfrak{g} ;$$

(3) let $g \in G^C$ and suppose that $p' = g \cdot p$ is a regular point; if we denote by $\mathfrak{g} = \mathfrak{l}' + \mathbb{R} Z_D' + \mathfrak{m}'$ and by $\mathfrak{m}^{10}$ the structural decomposition and the holomorphic subspace, respectively, given by the CR structure of $G \cdot p' = G/L'$, then

$$\mathfrak{m}'^{10} = \overline{\text{Ad}_g(\mathfrak{l}^C + \mathfrak{m}^{10})} ,$$

$$\mathfrak{m}' = \left( \text{Ad}_g(\mathfrak{l}^C + \mathfrak{m}^{10}) + \overline{\text{Ad}_g(\mathfrak{l}^C + \mathfrak{m}^{10})} \right) \cap \mathfrak{g} \cap \mathfrak{l}' \perp .$$

**Proof.** (1) Consider an element $V = X + iY \in \mathfrak{g}^C$, with $X, Y \in \mathfrak{g}$. Then $V$ belongs to $\mathfrak{h}$ if and only if

$$X + iY|_p = \hat{X}_p + J\hat{Y}_p = 0 .$$
This means that $J\dot{X}_p = -\dot{Y}_p$ is tangent to the orbit $G \cdot p$. In particular, $X, Y \in l + m$ and $V = X + iJX \in t^C + m^{01}$.

(2) Clearly, $L' = G \cap G_{p'}^C = G \cap (gHg^{-1})$ and $l' = g \cap \text{Ad}_g(h)$. The claim is then an immediate consequence of (1).

(3) From (1), it follows that

$$m^{10} = \overline{m^{01}} = \overline{h} \cap (t^C)^\perp = \overline{\text{Ad}_g(t^C + m^{01})} \cap (t^C)^\perp.$$  

From this, the conclusion follows. □

**Lemma 3.6.** Let $(M, J, g)$ be a K-manifold acted on by the compact semisimple Lie group $G$. Let also $p$ be a regular point and $g = l + \mathbb{R}Z_D + m$ the structural decomposition associated with the CR structure of $G/L = G \cdot p$. Then:

1. For any $g \in \exp(\mathbb{C}^*Z_D)$, the isotropy subalgebra $g_{p'}$ at the point $p' = g \cdot p$ is orthogonal to $\mathbb{R}Z_D$; moreover, $l \subseteq g_{p'}$ and, if $p'$ is regular, $l = g_{p'}$;

2. The curve

$$\eta : \mathbb{R} \rightarrow M, \quad \eta_t = \exp(itZ_D) \cdot p$$

is a nice transversal curve through $p$.

**Proof.** (1) From Lemma 3.5 (2), for any point $p' = \exp(\lambda Z_D) \cdot p$, with $\lambda \in \mathbb{C}^*$,

$$B(g_{p'}, \mathbb{R}Z_D) = B(\text{Ad}_{\exp(\lambda Z_D)}((t^C + m^{01}) \cap g, \mathbb{R}Z_D) =$$

$$= B((t^C + m^{01}) \cap g, \text{Ad}_{\exp(-\lambda Z_D)}(\mathbb{R}Z_D)) = B((t^C + m^{01}) \cap g, \mathbb{R}Z_D) = 0.$$  

Moreover, since $Z_D \in C_{g^C}(t^C)$, we get that

$$g_{p'} = (\text{Ad}_{\exp(\lambda Z_D)}((t^C + m^{01})) \cap g = l + \text{Ad}_{\exp(\lambda Z_D)}(m^{01}) \cap g \supset l.$$  

This implies that $l = g_{p'}$ if $p'$ is regular.

(2) From (1), we have that condition (2.1) and the equality $G \cdot \eta_t = G \cdot p = G/L$ are verified for any point $\eta_t \in M_{\text{reg}}$. It remains to show that $\eta$ intersects any regular orbit.

Let $\Omega = M \setminus G$ be the orbit space and $\pi : M \rightarrow \Omega = M \setminus G$ the natural projection map. It is known (see e.g. [Br]) that $\Omega$ is homeomorphic to $\Omega = [0, 1]$, with $M_{\text{reg}} = \pi^{-1}([0, 1])$. Hence $\eta$ intersects any regular orbit if and only if $(\pi \circ \eta)(\mathbb{R}) \ni [0, 1]$.

Let $x_1 = \inf(\pi \circ \eta)(\mathbb{R})$ and let $(t_n) \subseteq [0, 1]$ be a sequence such that $(\pi \circ \eta)_{t_n}$ tends to $x_1$. If we assume that $x_1 > 0$, we may select a subsequence $t_{n_k}$ so that $\lim_{n_k \rightarrow \infty} \eta_{t_{n_k}}$ exists and it is equal to a regular point $p_o$. From (1) and a continuity argument, we could conclude that $l$ is equal to the isotropy subalgebra $g_{p_o}$, that $\hat{Z}_D|_{p_o} \neq 0$ and that $J\hat{Z}_D|_{p_o}$ is not tangent to the orbit $G \cdot p_o$. In particular, it would follow that the curve $\exp(i\mathbb{R}Z_D) \cdot p_o$ has non-empty intersection with $\eta(\mathbb{R}) = \exp(i\mathbb{R}Z_D) \cdot p$ and that $p_o \in \eta(\mathbb{R})$; moreover we would have that $\eta$ is transversal to $G \cdot p_o$ and that $x_1 = \pi(p_o)$ is an inner point of $\pi \circ \eta(\mathbb{R})$, which is a contradiction.

A similar contradiction arises if we assume that $x_2 = \sup \pi \circ \eta(\mathbb{R}) < 1$. □
Proof of Theorem 3.4. Pick a regular point \( p \). Let \( \mathfrak{g} = \mathfrak{I} + \mathbb{R} Z_\mathcal{D} + \mathfrak{m} \) be the structural decomposition of the orbit \( G \cdot p \) and let \( \eta_t = \exp(it Z_\mathcal{D}) \cdot p \). From Lemmata 3.5 and 3.6 and Theorem 3.2, the structural decompositions \( \mathfrak{g} = \mathfrak{I} + \mathbb{R} Z_\mathcal{D}(t) + \mathfrak{m}(t) \) of all regular orbits \( G \cdot \eta_t \) are independent on \( t \). Moreover, from Lemma 3.5 and Theorem 3.2, it follows that the Morimoto-Nagano pair \((\mathfrak{g}_F, l_F)\) is the same for all regular orbits \( G \cdot \eta_t \) and the holomorphic subspace \( m^0_{10} \) of the orbit \( G \cdot \eta_t \) is of the form

\[
m^0_{10} = \text{Ad}_{\exp(it Z_\mathcal{D})}(m^0_{10}) = \text{Ad}_{\exp(-it Z_\mathcal{D})}(m^0_{10}(0)) + \text{Ad}_{\exp(-it Z_\mathcal{D})}(m^0_{10}(0)) \quad (3.4)
\]

where \( m^0_{10} = m^0_{10}(0) + m^0_{10}(0) \) is the decomposition of the holomorphic subspace of \( G \cdot \eta_0 \) given in Theorem 3.2 c). Since \( Z_\mathcal{D} \in \mathfrak{g}_F \), from (3.4) and Theorem 3.2 d), it follows that

\[
m^0_{10} = \text{Ad}_{\exp(-it Z_\mathcal{D})}(m^0_{10}(0)) + m^0_{10}(0) .
\]

This proves that the Morimoto-Nagano holomorphic subspace \( m^0_{10}(t) \) of the orbit \( G \cdot \eta_t \) is

\[
m^0_{10}(t) = \text{Ad}_{\exp(-it Z_\mathcal{D})}(m^0_{10}(0)) \quad (3.5)
\]

and that the \( B \)-orthogonal complement \( m^0_{10} = m^0_{10}(0) \) is independent on \( t \) and \( \text{ad}_{Z_\mathcal{D}} \)-invariant.

A simple computation shows that if \( \mathfrak{g}_F \) and \( m^0_{10}(t) = \text{Ad}_{\exp(-it Z_\mathcal{D})}(m^0_{10}(0)) \) appear in a row of Table 1 and if \( Z_\mathcal{D} \) is equal to \( Z_\mathcal{D} = AZ_o \), where \( Z_o \) is the corresponding element listed in the column "\( Z_\mathcal{D} \)", then \( m^0_{10}(t) \) is determined by a complex parameter \( \lambda = \lambda(t) \), which verifies the differential equation

\[
\frac{d\lambda}{dt} = 2A\lambda(t) .
\]

In particular, if we assume \( A = 1 \), then \( \lambda(t) = e^{2t + B_p} \) where \( B_p \) is a complex number which depends only on the regular point \( p \).

Let us replace \( p \) with the point \( p_o = \exp(-i \frac{B_p}{2} Z) \cdot p \): it is immediate to realize that the new function \( \lambda(t) \) is equal to

\[
\lambda(t) = e^{2t + B_p} .
\]

This proves that the curve \( \eta_t = e^{it Z_\mathcal{D}} \cdot p_o \) verifies (1), (2), (3) and (4).

It remains to prove that for any choice of the regular point \( p \), the point \( p_o = \exp(-i \frac{B_p}{2} Z) \cdot p \) is a point of the non-complex singular orbit of \( M \).

Observe that, since \( \eta(\mathbb{R}) \) is the orbit of a real 1-parameter subgroup of \( G^C \), the complex isotropy subalgebra \( \mathfrak{h}_t \subset \mathfrak{g}^C \) is (up to conjugation) independent on the point \( \eta_t \). Indeed, if \( \eta_{t_o} \) is a regular point with complex isotropy subalgebra \( \mathfrak{h}_{t_o} = \mathfrak{I}^C + m^0_{10} + m^{01} \), then for any other point \( \eta_t \), we have that

\[
\mathfrak{h}_t = \text{Ad}_{\exp(i(t-t_o) Z_\mathcal{D})}((I^C + m^0_{10} + m^{01}) .
\]

On the other hand, the real isotropy subalgebra \( \mathfrak{g}_{\eta_t} \subset \mathfrak{g} \) is equal to

\[
\mathfrak{g}_{\eta_t} = \mathfrak{h}_t \cap \mathfrak{g} = \text{Ad}_{\exp(i(t-t_o) Z_\mathcal{D})}((I^C + m^0_{10} + m^{01}) \cap \mathfrak{g} . \quad (3.6)
\]
From (3.6), Table 1 and (4), one can check that in all cases
\[ g_{\eta_0} \supseteq I + \mathbb{R}Z_D \]
and hence that \( \eta_0 = p_o \) is a singular point for the \( G \)-action. On the other hand \( p_o \) cannot be in the complex singular \( G \)-orbit, because otherwise this orbit would coincide with \( G^C \cdot p_o = G^C \cdot p \) and it would contradict the assumption that \( p \) is a regular point for the \( G \)-action. \( \square \)

The following is the analogous result for standard K-manifolds.

**Theorem 3.7.** Let \((M, J, g)\) be a standard K-manifold acted on by the compact semisimple group \( G \) and let \( p_o \) be any regular point for the \( G \)-action. Let also \( g = I + \mathbb{R}Z + m \) and \( m^{10} \) be the structural decomposition and the holomorphic subspace associated with the CR structure of the orbit \( G/L = G \cdot p_o \). Then the curve
\[ \eta : \mathbb{R} \to M, \quad \eta_t = \exp(tiZ) \cdot p_o \]
verifies the following properties:

1. It is a nice transversal curve; in particular the stabilizer in \( g \) of any regular point \( \eta_t \) is equal to the isotropy subalgebra \( I = g_{p_o} \).
2. For any regular point \( \eta_t \), the structural decomposition \( g = I + \mathbb{R}Z_D(t) + m(t) \) and the holomorphic subspace \( m^{10}(t) \) of the CR structure of \( G/L = G \cdot \eta_t \) is given by the subspaces \( m(t) = m, \mathbb{R}Z_D(t) = \mathbb{R}Z \) and \( m^{10}(t) = m^{10} \).

**Proof.** (1) is immediate from Lemma 3.6.

(2) It is sufficient to prove that \([Z, m^{10}] \subset m^{10}\). In fact, from this the claim follows as an immediate corollary of Lemmata 3.5 and 3.6.

Let \((G/K, J_F)\) be the flag manifold with invariant complex structure \( J_F \), given by Theorem 2.6, so that any regular orbit \( G \cdot x \) admits a CRF fibration onto \( G/K \), with fiber \( S^1 \). Let also \( P \) be the parabolic subalgebra of \( G^C \) such that \( G/K \) is biholomorphic to \( G^C/P \).

From Proposition 2.5, if we denote by \( p = \mathfrak{p}^C + \mathfrak{n} \) the decomposition of the parabolic subalgebra \( p \subset G^C \) into nilradical \( \mathfrak{n} \) plus reductive part \( \mathfrak{p}^C \), we have that
\[ \mathfrak{g} = \mathfrak{p} \cap \mathfrak{g}, \quad \mathfrak{f}^C \subset \mathfrak{p}^C, \quad \mathfrak{f}^C + m^{01} \subset \mathfrak{p}^C + \mathfrak{n}. \] (3.7)

Since the CRF fibration has fiber \( S^1 \), it follows that \( \mathfrak{f} = I + \mathbb{R}Z' \) for some \( Z' \in \mathfrak{g}(\mathfrak{f}) \subset \mathfrak{a} = C_{g}(I) \cap I^\perp \).

In case \( \mathfrak{a} = 1 \), we have that \( \mathfrak{a} = \mathbb{R}Z = \mathbb{R}Z' \) and hence \( m^{10} \subset (\mathfrak{f}^C + CZ)^\perp = (\mathfrak{f}^C)^\perp \). From (3.7) we get that \( m^{01} = \mathfrak{m} \) and that \([Z, m^{01}] \subset [\mathfrak{f}^C, \mathfrak{n}] \subset \mathfrak{n} = m^{01} \).

In case \( \mathfrak{a} \) is 3-dimensional, let us denote by \( \mathfrak{a}^\perp = \mathfrak{a} \cap \mathfrak{m} = \mathfrak{a} \cap (\mathbb{R}Z)^\perp \) and by \( \mathfrak{a}^{10} = \mathfrak{a}^C \cap m^{10}, \mathfrak{a}^{01} = \mathfrak{a}^C \cap m^{01} = \mathfrak{a}^{10} \) so that \((a^\perp)^C = a^{10} + a^{01} \). Consider also the orthogonal decompositions
\[ \mathfrak{g} = I + \mathbb{R}Z + m = I + \mathbb{R}Z + a^\perp + m', \quad m^{10} = a^{10} + m^{10}, \]
where $m^{t10} = m^{10} \cap m^C$. Let $l^{ss}$ be the semisimple part of $l$ and note that $l^{ss} = l^{ss}$.

By classical properties of flag manifolds (see e.g. [Al], [AP], [Ni]) the $ad_{l^{ss}}$-module $m'$ contains no trivial $ad_{l^{ss}}$-module and hence $m^{t10} = [l^{ss}, m^{t10}] = [l, m^{t10}]$. In particular, $m^{01} = m^{t10}$ is orthogonal to $l^C$ and hence it is included in $n$. So,

$$[Z, m^{01}] \subset [Z, n \cap (l^C + a^C)] \subset n \cap (l^C + a^C) = m^{01}.$$

From this, it follows that in order to prove that $[Z, m^{10}] \subset m^{10}$, one has only to show that $[Z, a^{10}] \subset a^{10} \subset m^{10}$.

By dimension counting, $a^{10} = CE$ for some element $E \in a^C \simeq sl_2(\mathbb{C})$. In case $E$ is a nilpotent element for the Lie algebra $a^C \simeq sl_2(\mathbb{C})$, we may choose a Cartan subalgebra $CH_\alpha$ for $a = sl_2(\mathbb{R})$, so that $E \in CE_\alpha$. In this case, we have that

$$Z \in (a^{10} + a^{01}) \perp = (CE_\alpha + CE_{-\alpha}) \perp = CH_\alpha$$

and hence $[Z, a^{10}] \subset [CH_\alpha, CE_\alpha] = CE_\alpha = a^{10}$ and we are done.

In case $E$ is a regular element for $a^C$, with no loss of generality, we may consider a Cartan subalgebra $CH_\alpha$ for $a^C$ so that $CE = CE_\alpha + tE_{-\alpha}$ for some $t \neq 0$. In this case, $a^{01} = a^{10} = CE_{-\alpha}$ and, since $a^{10} \cap a^{01} = \{0\}$, it follows that $t \neq 1/\bar{t}$. In particular, we get that $CZ = (a^{10} + a^{01}) \perp = CH_\alpha$. Now, by Lemma 3.5 (2), for any $\lambda \in C^*$, the isotropy subalgebra $l_{g_\lambda, p_o}$, with $g_\lambda = \exp(\lambda Z)$, is equal to

$$l_{g_\lambda, p_o} = Ad_{\exp(\lambda Z)}(l^C + a^{01} + m^{01}) \cap g = l^C + m^{01} + CE_\alpha + t e^{-2\lambda\alpha(Z)} E_{-\alpha} \cap g.$$

Therefore, if $\lambda$ is such that $te^{-2\lambda\alpha(Z)} = -1$, we have that $l_{g_\lambda, p_o} = l + R(E_\alpha - E_{-\alpha}) \supset l$ and hence that $p = g_\lambda \cdot p_o$ is a singular point for the $G$-action. On the other hand, $p$ is in the $G^C$-orbit of $p_o$ and hence the singular orbit $G \cdot p$ is not a complex orbit. But this is in contradiction with the hypothesis that $M$ is standard and hence that it has two singular $G$-orbits, which are both complex. \qed

Any curve $\eta_t = \exp(itZ) \cdot p_o$, which verifies the claim of Theorems 3.4 or 3.7, will be called optimal transversal curve.

### 3.4 The optimal bases along the optimal transversal curves.

In all the following, $\eta$ is an optimal transversal curve. In case $M$ is a non-standard K-manifold, we denote by $g = l + \mathbb{R}Z_D + m$, $(g_F, l_F)$, $m^{10}_F(t)$ and $m^{10} = m^{10}_F(t) + m^{10}$ the structural decomposition, the Morimoto-Nagano pair, the Morimoto-Nagano subspace and the holomorphic subspace, respectively, at the regular points $\eta_t \in M_{reg}$. The same notation will be adopted in case $M$ is a standard K-manifold, with the convention that, in this case, the Morimoto-nagano pair $(g_F, l_F)$ is the trivial pair ($\{0\}, \{0\}$) and that the Morimoto-Nagano holomorphic subspace is $m^{10}_F = \{0\}$.

We will also assume that $l = l_o + l_F$, where $l_o = l \cap l^t_F$. By $l^C = t^C_o + t^C_F \subset l^C \subset g^C$, with $t_o \subset l_o$ and $t_F \subset l_F$, we denote a Cartan subalgebra of $g^C$ with the property
that, the expressions of \( m_F^{10}(t) \) and \( Z_D \) in terms of the root vectors of \( (\omega^C, t^C) \) are exactly as those listed in Table 1, corresponding to the parameter \( \lambda = \ell^2 \).

Let \( R \) be the root system of \( (\omega^C, t^C) \). Then \( R \) is union of the following disjoint subsets of roots:

\[
R = R^o \cup R' = (R^o_1 \cup R^o_F) \cup (R'_F \cup R'_+ \cup R'_-),
\]

where

\[
R^o_1 = \{ \alpha, E_\alpha \in \omega^C \}, \quad R^o_F = \{ \alpha, E_\alpha \in t^C \},
\]

\[
R'_F = \{ \alpha, E_\alpha \in m_F^C \}, \quad R'_+ = \{ \alpha, E_\alpha \in m_1^{10} \}, \quad R'_- = \{ \alpha, E_\alpha \in m_0^{10} \}.
\]

Note that

\[
-R^o_1 = R^o_1, \quad -R^o_F = R^o_F, \quad -R'_F = R'_F, \quad -R'_+ = R'_-.
\]

Moreover, \( R^o_1 \) is orthogonal to \( R^o_F \) and \( R^o_1 \), \( R^o_F \) and \( R'_F \cup R'_F \) are closed subsystems.

Clearly, in case \( M \) is standard, we will assume that \( R^o_F = R'_F = \emptyset \).

We claim that for any \( \alpha \in R'_F \) there exists exactly one root \( \alpha^d \in R'_F \) and two integers \( \epsilon_\alpha = \pm 1 \) and \( \ell_\alpha = \pm 1, \pm 2 \) such that, for any \( t \in \mathbb{R} \),

\[
E_\alpha + \epsilon_\alpha e^{2\ell_\alpha t} \epsilon_\alpha E_{-\alpha^d} \in m_F^{10}(t), \quad (3.8)
\]

The proof of this claim is the following. By direct inspection of Table 1, the reader can check that any maximal \( \omega^C \)-isotopic subspace of \( m_F^C(t) \) (i.e. any maximal subspace which is sum of equivalent irreducible \( \omega^C \)-moduli) is direct sum of exactly two irreducible \( \omega^C \)-moduli (see also [AS]). Let us denote by \( (\alpha_i, -\alpha_i) \) (\( i = 1, 2, \ldots \)) all pairs of roots in \( R_F \) with the property that the associated root vectors \( E_{\alpha_i} \) and \( E_{-\alpha_i} \) are maximal weight vectors of equivalent \( \omega^C \)-moduli in \( m_F^C(t) \). Using Table 1, one can check that in all cases \( m_F^{10}(t) \) decomposes into non-equivalent irreducible \( \omega^C \)-moduli, with maximal weight vectors of the form

\[
E_{\alpha_i} + \lambda_t^{(i)} E_{-\alpha_i^d}
\]

where \( \lambda_t^{(i)} = (\lambda(t))^t \ell_i = e^t \ell_i \), where \( \ell_i \) is an integer which is either \( \pm 1 \) or \( \pm 2 \).

Hence \( m_F^{10}(t) \) is spanned by the vectors \( E_{\alpha_i} + \lambda_t E_{-\alpha_i^d} \) and by vectors of the form

\[
[E_{\beta}, E_{\alpha_i} + \lambda_t E_{-\alpha_i^d}] = N_{\beta, \alpha_i} E_{\alpha_i + \beta} + \lambda_t N_{\beta, -\alpha_i^d} E_{-\alpha_i^d + \beta}, \quad (3.9)
\]

for some \( E_{\beta} \in \omega^C \). Since the \( \omega^C \)-moduli containing \( E_{\alpha_i} \) and \( E_{-\alpha_i^d} \) are equivalent, the lengths of the sequences of roots \( \alpha_i + r \beta \) and \( -\alpha_i^d + r \beta \) are both equal to some given integer, say \( p \). This implies that for any root \( \beta \in R_F^o \)

\[
N_{\beta, \alpha_i}^2 = (p+1)^2 = N_{\beta, -\alpha_i^d}^2
\]

and hence that \( \frac{N_{\beta, \alpha_i}}{N_{\beta, -\alpha_i^d}} = \pm 1 \). From this remark and (3.9), we conclude that \( m_F^{10}(t) \) is generated by elements of the form

\[
E_\alpha + \epsilon_\alpha e^{t \ell_\alpha t} E_{-\alpha^d},
\]
where \( \beta \in R^\alpha_F, \alpha = \alpha_i + \beta, \alpha = \alpha_i + \beta, \alpha^d = \alpha_i^d + \beta \) and \( \epsilon_\alpha = \frac{N_{\beta, \alpha}}{N_{\beta, -\alpha^d}} \). This concludes the proof of the claim.

For any root \( \alpha \in R_F \), we call \textit{CR-dual root of \( \alpha \)} the root \( \alpha^d \) so that \( E_\alpha + \epsilon_\alpha e^{\ell_\alpha t} E_{-\alpha^d} \in m_{10}(t) \).

We fix a positive root subsystem \( R^+ \subset R \) so that \( R'_+ = R^+ \cap (R \setminus (R^0 \cup R^+_F \cup R^-_F)) \). Moreover, we decompose the set of roots \( R'_F \) into

\[ R'_F = R^{(+)}_F \cup R^{(-)}_F \]

where

\[ R^{(+)}_F = \{ \alpha \in R'_F : E_\alpha + \epsilon_\alpha e^{\ell_\alpha t} E_{-\alpha^d} \in m_{10}, \text{ with } \ell_\alpha = +1, +2 \} \]

\[ R^{(-)}_F = \{ \alpha \in R'_F : E_\alpha + \epsilon_\alpha e^{\ell_\alpha t} E_{-\alpha^d} \in m_{10}, \text{ with } \ell_\alpha = -1, -2 \} \]

Using Table 1, one can check that in all cases

\[ m_{10} = \text{span}_\mathbb{C} \{ E_\alpha + \epsilon_\alpha e^{\ell_\alpha t} E_{-\alpha^d}, \alpha \in R^{(+)}_F \} \]

and that if \( \alpha \in R^{(+)}_F \), then also the CR dual root \( \alpha^d \in R^{(+)}_F \). We will denote by \( \{\alpha_1, \alpha_2^d, \alpha_3^d, \ldots, \alpha_r, \alpha_r^d\} \) the set of roots in \( R^{(+)}_F \) and by \( \{\beta_1, \ldots, \beta_s\} \) the roots in \( R^+ = R^+ \cap R' \).

Observe that the number of roots in \( R^{(+)}_F \) is equal to \( \frac{1}{2} (\dim_{\mathbb{R}} G_F / L_F - 1) \), where \( G_F / L_F \) is the Morimoto-Nagano space associated with the pair \((g_F, l_F)\).

Finally, we consider the following basis for \( \mathbb{R} Z_D + m \cong T_{\eta_l} G \cdot \eta_l \). We set

\[ F_0 = Z_D \]

and, for any \( 1 \leq i \leq r \), we define the vectors \( F^+_i, F^-_i, G^+_i \) and \( G^-_i \), as follows: in case \( \{\alpha_i, \alpha_i^d\} \subset R^{(+)}_F \) is a pair of CR dual roots with \( \alpha_i \neq \alpha_i^d \), we set

\[ F^+_i = \frac{1}{\sqrt{2}} (F_{\alpha_i} + \epsilon_{\alpha_i} F_{\alpha_i^d}), \quad F^-_i = \frac{1}{\sqrt{2}} (F_{\alpha_i} - \epsilon_{\alpha_i} F_{\alpha_i^d}) \]

\[ G^+_i = \frac{1}{\sqrt{2}} (G_{\alpha_i} + \epsilon_{\alpha_i} G_{\alpha_i^d}), \quad G^-_i = \frac{1}{\sqrt{2}} (G_{\alpha_i} - \epsilon_{\alpha_i} G_{\alpha_i^d}) \]  

(3.10)

where \( \epsilon_{\alpha_i} = \pm 1 \) is the integer which is defined in (3.8); in case \( \{\alpha_i, \alpha_i^d\} \subset R^{(+)}_F \) is a pair of CR dual roots with \( \alpha_i = \alpha_i^d \), we set

\[ F^+_i = F_{\alpha_i} = \frac{E_{\alpha_i} - E_{-\alpha_i}}{\sqrt{2}}, \quad G^+_i = G_{\alpha_i} = \frac{E_{\alpha_i} + E_{-\alpha_i}}{\sqrt{2}} \]  

(3.10')

and we do not define the corresponding vectors \( F^-_i \) or \( G^-_i \). Finally, for any \( 1 \leq i \leq s = n - 1 - 2r \), we set

\[ F'_i = F_{\beta_i}, \quad G'_i = G_{\beta_i} \].  

(3.11)
Note that in case \( r \) is odd, there is only one root \( \alpha_i \in R_F^{(+)} \) such that \( \alpha_i = \alpha_i^d \).
When \( \mathfrak{g}_F = \mathfrak{su}_2 \), this root is also the unique root in \( R_F^{(+)} \).

In case \( \mathfrak{g}_F = \{0\} \), we set \( F_0 = Z_D \) and \( F_i' = F_{\beta_i} \), \( G_i' = G_{\beta_i} \) and we do not define the vector \( F_i(\pm) \) or \( G_i(\pm) \).

The basis \( (F_0, F_k^\pm, F_j', G_k^\pm, G_j') \) for \( \mathbb{R}Z_D + m \), which we just defined, will be called optimal basis associated with the optimal transversal curve \( \eta \). Notice that this basis is \( \mathcal{B} \)-orthonormal.

For simplicity of notation, we will often use the symbol \( F_k \) (resp. \( G_k \)) to denote any vector in the set \( \{F_0, F_j^\pm, F_j'\} \) (resp. in \( \{G_j^\pm, G_j'\} \)). We will also denote by \( N_F \) the number of elements of the form \( F_i^\pm \). Note that \( N_F \) is equal to half the real dimension of the holomorphic distribution of the Morimoto-Nagano space \( G_F/L_F \).

For any odd integer \( 1 \leq 2k - 1 \leq N_F \), we will assume that \( F_{2k-1} = F_k^+ \); for any even integer \( 2 \leq 2k \leq N_F \), we will assume \( F_{2k} = F_k^- \). If \( N_F \) is odd, we denote by \( F_{N_F} \) the unique vector defined by (3.10'). We will also assume that \( F_j = F_{j-N_F} \) for any \( N_F + 1 \leq j \leq n - 1 \).

In case \( M \) is a standard K-manifold, we assume that \( N_F = 0 \).

In the following lemma, we describe the action of the complex structure \( J_t \) in terms of an optimal basis.

**Lemma 3.8.** Assume that \( \eta_t \) is an optimal transversal curve and let

\[
(F_0, F_k^\pm, F_j', G_k^\pm, G_j')
\]

an associated optimal basis of \( \mathbb{R}Z_D + m \). Let also \( J_t \) be the complex structure of \( m \) corresponding to the CR structure of a regular orbit \( G \cdot \eta_t \).

Then \( J_t F_i' = G_i' \) for any \( 1 \leq i \leq s = n - 1 - N_F \). Furthermore, if \( M \) is non-standard (i.e., \( N_F > 0 \)) then:

1. if \( 1 \leq i \leq N_F \) and \( \{\alpha_i, \alpha_i^d\} \) is a pair of CR-dual roots in \( R_F^{(+)} \) with \( \alpha_i \neq \alpha_i^d \) then

\[
J_t F_i^+ = -\coth(\ell_i t)G_i^+ , \quad J_t F_i^- = -\tanh(\ell_i t)G_i^- ,
\]

where \( \ell_i \) is equal to 2 if \( F_i^\pm \in [m_F, m_F] \cap m_F^C \) and it is equal to 1 otherwise;

2. if \( 1 \leq i \leq N_F \) and \( \{\alpha_i, \alpha_i^d\} \) is a pair of CR-dual roots in \( R_F^{(+)} \) with \( \alpha_i = \alpha_i^d \), so that \( F_i^+ = F_{\alpha_i} \) then

\[
J_t F_i^+ = -\coth(\ell_i t)G_i^+ ,
\]

where \( \ell_i \) is equal to 2 if \( F_i^\pm \in [m_F, m_F] \cap m_F^C \) and it is equal to 1 otherwise.

Note that the case \( \ell_i = 2 \) may occur only if \( \mathfrak{g}_F = \mathfrak{f}_4 \) or \( \mathfrak{sp}_n \) - see Table 1.

**Proof.** The first claim is an immediate consequence of Theorem 3.2 d) and the property of invariant complex structures on flag manifolds.
In order to prove (3.12), let us consider a pair \( \{\alpha_i, \alpha_d^i\} \) of CR dual roots in \( R_F^+ \) with \( \alpha_i \neq \alpha_d^i \); by the previous remarks, there exist two integers \( \ell_i, \ell_d^i \), which are either +1 or +2, and two integers \( \epsilon_i, \epsilon_d^i = \pm 1 \), so that

\[
E_{\alpha_i} + \epsilon_i e^{2\ell_i t} E_{-\alpha_i}^d, \quad E_{\alpha_d^i} + \epsilon_d^i e^{2\ell_d^i t} E_{-\alpha_i} \in m_F^0(t)
\]

for any \( t \neq 0 \).

By direct inspection of Table 1, one can check that the integers \( \ell_d^i, \ell_d^j \) are always equal. We claim that also \( \epsilon_i = \epsilon_d^i \) for any CR dual pair \( \{\alpha_i, \alpha_d^i\} \subset R_F^+ \).

In fact, by conjugation, it follows that the following two vectors are in \( m_F^0(t) \) for any \( t \neq 0 \):

\[
E_{\alpha_i} + \frac{1}{\epsilon_i e^{2\ell_i t}} E_{-\alpha_i}^d, \quad E_{\alpha_d^i} + \frac{1}{\epsilon_d^i e^{2\ell_d^i t}} E_{-\alpha_i} \in m_F^0(t).
\]

(3.14)

At this point, we recall that \( \eta_0 \) is a singular point for the \( G \)-action and that, by the structure theorems in [HS] (see also [AS]), the isotropy subalgebra \( g_\eta_0 \) contains the isotropy subalgebra \( (g_F)_{\eta_0} \) of the non-complex singular \( G_F \)-orbit in \( M \), which is of c.r.o.s.s.s. In particular, one can check that \( \dim_{\mathbb{R}}(g_F)_{\eta_0} = \dim_{\mathbb{R}}(g_F) + \dim_{\mathbb{C}} m_F^0(0) \).

On the other hand, by Lemma 3.5 (2), we have that \( (g_F)_{\eta_0} = I_F + g \cap m_F^0(0) \) and hence that

\[
\dim_{\mathbb{R}}(g \cap m_F^0(0)) = \dim_{\mathbb{C}} m_F^0(0).
\]

(3.15)

Here, by \( m_F^0(0) \) we denote the subspace which is obtained from Table 1, by setting the value of the parameter \( \lambda \) equal to \( \lambda(0) = e^0 = 1 \). Note that this subspace is not a Morimoto-Nagano subspace.

From (3.14), one can check that (3.15) occurs if and only if

\[
\epsilon_d^i = \epsilon_i
\]

(3.16)

for any pair of CR dual roots \( \alpha_i, \alpha_d^i \). This proves the claim.

In all the following, we will use the notation \( \epsilon_i = \epsilon_d^i = \epsilon_{\alpha_d^i} \).

By some straightforward computation, it follows that, for any \( t \neq 0 \), the elements \( F_{\alpha_i}, F_{\alpha_d^i}, G_{\alpha_i}, \) and \( G_{\alpha_d^i} \) are equal to the following linear combinations of holomorphic and anti-holomorphic elements:

\[
F_{\alpha_i} = \frac{1}{\sqrt{2(1 - e^{4\ell_i t})}} \left\{ \left[ (E_{\alpha_i} + \epsilon_i e^{2\ell_i t} E_{-\alpha_i}^d) + \epsilon_i e^{2\ell_i t} (E_{\alpha_d^i} + \epsilon_d^i e^{2\ell_d^i t} E_{-\alpha_i}) \right] + \\
+ \left[ -e^{4\ell_i t} (E_{\alpha_d^i} + \epsilon_d^i e^{2\ell_d^i t} E_{-\alpha_i}^d) - \epsilon_i e^{2\ell_i t} (E_{\alpha_d^i} + \epsilon_d^i e^{2\ell_d^i t} E_{-\alpha_i}) \right] \right\},
\]

\[
F_{\alpha_d^i} = \frac{1}{\sqrt{2(1 - e^{4\ell_d^i t})}} \left\{ \left[ \epsilon_i e^{2\ell_i t} (E_{\alpha_i} + \epsilon_i e^{2\ell_i t} E_{-\alpha_i}) + (E_{\alpha_d^i} + \epsilon_d^i e^{2\ell_d^i t} E_{-\alpha_i}) \right] - \\
- \left[ e^{4\ell_i t} \epsilon_i (E_{\alpha_i} + \epsilon_i e^{2\ell_i t} E_{-\alpha_i^d}) + e^{4\ell_d^i t} (E_{\alpha_d^i} + \epsilon_d^i e^{2\ell_d^i t} E_{-\alpha_i}) \right] \right\},
\]
We then obtain that

\[
G_{\alpha_i} = \frac{i}{\sqrt{2(1 - \epsilon_4^\alpha)}} \left\{ \left( E_{\alpha_i} + \epsilon_i e^{2\ell_i t} E_{-\alpha_i} \right) - \epsilon_i e^{2\ell_i t} \left( E_{\alpha_i} + \epsilon_i e^{2\ell_i t} E_{-\alpha_i} \right) \right\} +
\]

\[
+ \left[ -\epsilon_i e^{2\ell_i t} \left( E_{\alpha_i} + \epsilon_i e^{2\ell_i t} E_{-\alpha_i} \right) + \epsilon_i e^{2\ell_i t} \left( E_{\alpha_i} + \frac{1}{\epsilon_i e^{2\ell_i t} E_{-\alpha_i}} \right) \right],
\]

\[
G_{\alpha_i}^d = \frac{i}{\sqrt{2(1 - \epsilon_4^\alpha)}} \left\{ \left[ -\epsilon_i e^{2\ell_i t} \left( E_{\alpha_i} + \epsilon_i e^{2\ell_i t} E_{-\alpha_i} \right) + \left( E_{\alpha_i} + \epsilon_i e^{2\ell_i t} E_{-\alpha_i} \right) \right] +
\]

\[
+ \left[ \epsilon_i e^{2\ell_i t} \left( E_{\alpha_i} + \epsilon_i e^{2\ell_i t} E_{-\alpha_i} \right) - \epsilon_i e^{2\ell_i t} \left( E_{\alpha_i} + \frac{1}{\epsilon_i e^{2\ell_i t} E_{-\alpha_i}} \right) \right].
\]

We then obtain that

\[
J_i F_{\alpha_i} = \frac{1}{1 - e^{4\ell_i t}} \left[ \left( E_{\alpha_i} + e^{2\ell_i t} E_{-\alpha_i} \right) -
\]

\[
- e^{2\ell_i t} \left( 1 + e^{2\ell_i t} \left( E_{\alpha_i} + e^{-2\ell_i t} E_{-\alpha_i} \right) \right) \right],
\]

\[
J_i F_{\alpha_i}^d = \frac{2e_i e^{2\ell_i t}}{1 - e^{4\ell_i t}} G_{\alpha_i} + \frac{1 + e^{4\ell_i t}}{1 - e^{4\ell_i t}} G_{\alpha_i}^d.
\] (3.17)

So, using the fact that \( \epsilon_i^2 = 1 \), we get \( J_i F_i^+ = \frac{1 + e^{2\ell_i t}}{1 + e^{2\ell_i t}} G_i^+ = -\coth(\ell_i t) G_i^+ \) and \( J_i F_i^- = \frac{1 - e^{2\ell_i t}}{1 + e^{2\ell_i t}} G_i^- = -\tanh(\ell_i t) G_i^- \). The proof of (3.13) is similar. It suffices to observe that for any \( t \neq 0 \)

\[
F_i^+ = \frac{1}{\sqrt{2(1 - e^{4\ell_i t})}} \left\{ (1 + e^{2\ell_i t}) (E_{\alpha_i} + e^{2\ell_i t} E_{-\alpha_i}) -
\]

\[
- e^{2\ell_i t} (1 + e^{2\ell_i t}) (E_{\alpha_i} + e^{-2\ell_i t} E_{-\alpha_i}) \right\},
\]

\[
G_i^+ = \frac{i}{\sqrt{2(1 - e^{4\ell_i t})}} \left\{ (1 - e^{2\ell_i t}) (E_{\alpha_i} + e^{2\ell_i t} E_{-\alpha_i}) +
\]

\[
+ e^{2\ell_i t} (1 - e^{2\ell_i t}) (E_{\alpha_i} + e^{-2\ell_i t} E_{-\alpha_i}) \right\},
\]

and hence that \( J_i F_i^+ = \frac{1 + e^{2\ell_i t}}{1 - e^{4\ell_i t}} G_i^+ = -\coth(\ell_i t) G_i^+ \). \( \square \)

4. The algebraic representatives of the Kähler and Ricci form of a K-manifold.

In this section we give a rigorous definition of the algebraic representatives of the Kähler form \( \omega \) and the Ricci form \( \rho \) of a K-manifold. We will also prove Proposition 1.1.

Indeed, we will give the concept of 'algebraic representative' for any bounded, closed 2-form \( \varpi \), which is defined on \( M_{\text{reg}} \) and which is \( G \)-invariant and \( J \)-invariant. Clearly, \( \omega|_{M_{\text{reg}}} \) and \( \rho|_{M_{\text{reg}}} \) belong to this class of 2-forms.

Let \( \eta : \mathbb{R} \to M \) be an optimal transversal curve. Since \( \mathfrak{g} \) is semisimple, for any \( G \)-invariant 2-form \( \varpi \) on \( M_{\text{reg}} \) there exists a unique \( \text{ad}_t \)-invariant element \( F_{\varpi,t} \in \text{Hom}(\mathfrak{g}, \mathfrak{g}) \) such that:

\[
\mathcal{B}(F_{\varpi,t}(X), Y) = \varpi_{\eta_t}([\hat{X}, \hat{Y}]), \quad X, Y \in \mathfrak{g}, \quad t \neq 0.
\] (4.1)
If $\varpi$ is also closed, we have that for any $X, Y, W \in \mathfrak{g}$

$$0 = 3d\varpi(\dot{X}, \dot{Y}, \dot{W}) = \varpi(\dot{X}, [\dot{Y}, \dot{W}]) + \varpi(\dot{Y}, [\dot{W}, \dot{X}]) + \varpi(\dot{W}, [\dot{X}, \dot{Y}]).$$

This implies that

$$F_{\varpi, t}([X, Y]), W) = [F_{\varpi, t}(X), Y] + [X, F_{\varpi, t}(Y)]$$

i.e. $F_{\varpi, t}$ is a derivation of $\mathfrak{g}$. Therefore, $F_{\varpi, t}$ is of the form

$$F_{\varpi, t} = \text{ad}(Z_{\varpi}(t))$$

for some $Z_{\varpi}(t) \in \mathfrak{g}$ and $\varpi_{\eta_t} = \varpi$. By definition, if the algebraic representative $Z_{\varpi}(t)$ is given, it is possible to reconstruct the values of $\varpi$ on any pair of vectors, which are tangent to the regular orbits $G \cdot \eta_t$. Actually, since for any point $\eta_t \in M_{\text{reg}}$ we have that $J(T_{\eta_t}G) = T_{\eta_t}M$, it follows that one can evaluate $\varpi$ on any pair of vectors in $T_{\eta_t}M$ if the value $\varpi_{\eta_t}(\dot{Z}_D, J\dot{Z}_D)$ is also given. However, in case $\varpi$ is a closed form, the following Proposition shows that this last value can be recovered from the first derivative of the function $Z_{\varpi}(t)$.

**Proposition 4.1.** Let $(M, J, g)$ be a $K$-manifold acted on by the compact semisimple Lie group $G$. Let also $\eta_t = \exp(tZ_D) \cdot p_o$ be an optimal transversal curve and $Z_{\varpi} : \mathbb{R} \to \mathfrak{g}(\mathfrak{l}) + \mathfrak{a}$ the algebraic representative of a bounded, $G$-invariant, $J$-invariant closed 2-form $\varpi$ along the optimal transversal curve $\eta_t$.

Then:

1. if $M$ is a standard $K$-manifold or a non-standard $KO$-manifold (i.e. if either $\mathfrak{a} = \mathbb{R}Z_D$ or $\mathfrak{a} = \mathfrak{su}_2$ and $M$ is standard), then there exists an element $I_{\varpi} \in \mathfrak{g}(\mathfrak{l})$ and a smooth function $f_{\varpi} : \mathbb{R} \to \mathbb{R}$ so that

$$Z_{\varpi}(t) = f_{\varpi}(t)Z_D + I_{\varpi};$$

2. if $M$ is a non-standard $KE$-manifold, then there exists a Cartan subalgebra $\mathfrak{t}^C \subset \mathfrak{g}^C + a^C$ and a root $\alpha$ of the corresponding root system, such that $Z_D \in \mathbb{R}(iH_\alpha)$ and $\mathfrak{a} = \mathbb{R}Z_D + \mathbb{R}F_\alpha + \mathbb{R}G_\alpha$; furthermore there exists an element $I_{\varpi} \in \mathfrak{g}(\mathfrak{l})$, a real number $C_{\varpi}$ and a smooth function $f_{\varpi} : \mathbb{R} \to \mathbb{R}$ so that

$$Z_{\varpi}(t) = f_{\varpi}(t)Z_D + \frac{C_{\varpi}}{\text{cosh}(t)}G_\alpha + I_{\varpi}.$$

Conversely, if $Z_{\varpi} : \mathbb{R} \to \mathfrak{g}(\mathfrak{l})$ is a curve in $\mathfrak{g}(\mathfrak{l})$ of the form (4.4) or (4.4'), then there exists a unique closed $J$-invariant, $G$-invariant 2-form $\varpi$ on $M_{\text{reg}}$, having $Z_{\varpi}(t)$ as algebraic representative; such 2-form is the unique $J$- and $G$-invariant form which verifies

$$\varpi_{\eta_t}(\dot{V}, \dot{W}) = B(Z_{\varpi}(t), [V, W]), \quad \varpi_{\eta_t}(J\dot{Z}_D, \dot{Z}_D) = -f_{\varpi}'(t)B(Z_D, Z_D).$$
for any \( V, W \in \mathfrak{m} \) and any \( \eta \in M_{reg} \).

**Proof.** Let \( \varpi \) be a closed 2-form which is \( G \)-invariant and \( J \)-invariant and let \( Z_\varpi(t) \) be the associated algebraic representative along \( \eta \). Recall that \( Z_\varpi(t) \in \mathfrak{g}(t) + \mathfrak{a} \). So, if the action is ordinary (i.e. \( \mathfrak{a} = \mathbb{R}Z_D \)), \( Z_\varpi(t) \) is of the form

\[
Z_\varpi(t) = f_\varpi(t)Z_D + I_\varpi(t),
\]

(4.6)

where the vector \( I_\varpi(t) \in \mathfrak{g}(t) \) may depend on \( t \).

In case the action of \( G \) is extraordinary (that is \( \mathfrak{a} = \mathfrak{su}_2 \)) by Lemma 2.2 in [PS], there exists a Cartan subalgebra \( \mathfrak{t} \subseteq \mathfrak{t}^C + \mathfrak{a}^C \), such that \( \mathfrak{a}^C = \mathbb{C}H_\alpha + \mathbb{C}E_\alpha + \mathbb{C}E_{-\alpha} \) for some root \( \alpha \) of the corresponding root system. By the arguments in the proof of Theorem 3.7, this Cartan subalgebra can be always chosen in such a way that \( Z_D \in \mathbb{R}(iH_\alpha) \) and hence that \( \mathfrak{a} = \mathbb{R}Z_D + \mathbb{R}F_\alpha + \mathbb{R}G_\alpha \).

Then the function \( Z_\varpi(t) \) can be written as

\[
Z_\varpi(t) = f_\varpi(t)Z_D + g_\varpi(t)F_\alpha + h_\varpi(t)G_\alpha + I_\varpi(t)
\]

(4.6’)

for some smooth real valued functions \( f_\varpi, g_\varpi \) and \( h_\varpi \) and some element \( I_\varpi(t) \in \mathfrak{g}(t) \).

We now want to show that, in case \( M \) is a non-standard KE-manifold, then \( g_\varpi(t) \equiv 0 \) and that \( h_\varpi(t) = \frac{C_\varpi}{\cosh(t)} \) for some constant \( C_\varpi \).

In fact, observe that if \( Z_\varpi(t) \) is of the form (4.6’) and if \( Z_D \) is as listed in Table 1 for \( \mathfrak{g}_F = \mathfrak{su}_2 \), then

\[
\varpi_\eta(\hat{Z}_D, \hat{G}_\alpha) = g_\varpi(t)B(F_\alpha, [Z_D, G_\alpha]) = -g_\varpi(t),
\]

\[
\varpi_\eta(\hat{Z}_D, \hat{F}_\alpha) = h_\varpi(t)B(G_\alpha, [Z_D, F_\alpha]) = h_\varpi(t).
\]

Consider now the facts that \( \varpi \) is closed, \( \hat{G}_\alpha \) and \( \hat{Z}_D \) are holomorphic vector fields and \( J\hat{Z}_D|_{\eta} = \eta’ \). It follows that \( g_\varpi \) verifies the following ordinary differential equation

\[
\frac{dg_\varpi}{dt}|_{\eta} = -\frac{d}{dt}(\varpi(\hat{Z}_D, \hat{G}_\alpha)|_{\eta}) = -J\hat{Z}_D\left(\varpi(\hat{Z}_D, \hat{G}_\alpha)|_{\eta}\right) = \hat{G}_\alpha(\varpi(J\hat{Z}_D, \hat{Z}_D)|_{\eta}) + \hat{Z}_D(\varpi(\hat{G}_\alpha, J\hat{Z}_D)|_{\eta}) - \varpi_\eta([J\hat{Z}_D, \hat{Z}_D], \hat{G}_\alpha) - \varpi_\eta(\hat{G}_\alpha, J\hat{Z}_D) - \varpi_\eta([\hat{Z}_D, \hat{G}_\alpha], J\hat{Z}_D) = -\varpi_\eta(\hat{Z}_D, J\hat{F}_\alpha) = \coth(t)\varpi_\eta(\hat{Z}_D, \hat{G}_\alpha) = -\coth(t)g_\varpi(t).
\]

(4.7)

We claim that this implies

\[
g_\varpi(t) \equiv 0.
\]

(4.8)

In fact, if we assume that \( g_\varpi(t) \) does not vanish identically, integrating the above equation, we have that \( g_\varpi(t) = \frac{C}{\sinh(t)} \) for some \( C \neq 0 \) and hence with a singularity at \( t = 0 \). But this contradicts the fact that \( \varpi \) is a bounded 2-form.
With a similar argument, we have that \( h_\varpi(t) \) verifies the differential equation

\[
\frac{dh_\varpi}{dt} \bigg|_{\eta_\varpi} = -\tanh(t)h_\varpi(t);
\]

by integration this gives

\[
h_\varpi(t) = \frac{C_\varpi}{\cosh(t)}
\]

for some constant \( C_\varpi \).

We show now that, in case \( M \) is a standard KE-manifold, then \( Z_\varpi(t) \) is of the form (4.4). In fact, even if a priori \( Z_\varpi(t) \) is of the form (4.6'), from Lemma 3.8 and the same arguments for proving (4.7), we obtain that

\[
\frac{dg_\varpi}{dt} \bigg|_{\eta_\varpi} = -\varpi_{\eta_\varpi}(\hat{Z}_D, J \hat{F}_\alpha) = -\varpi_{\eta_\varpi}(\hat{Z}_D, \hat{G}_\alpha) = g_\varpi(t).
\]

This implies that \( g_\varpi(t) = Ae^t \) for some constant \( A \). On the other hand, if \( A \neq 0 \), it would follow that \( \lim_{t \to -\infty} |\varpi_{\eta_\varpi}(\hat{Z}_D, \hat{G}_\alpha)| = \lim_{t \to -\infty} |g_\varpi(t)| = +\infty \), which is impossible since \( \varpi_{\eta_\varpi}(\hat{Z}_D, \hat{G}_\alpha) \) is bounded. Hence \( g_\varpi(t) \equiv 0 \).

A similar argument proves that \( h_\varpi(t) \equiv 0 \).

In order to conclude the proof, it remains to show that in all cases the element \( I_\varpi(t) \) is independent on \( t \) and that \( \varpi_{\eta_\varpi}(J \hat{Z}_D, Z_D) = -f_\varpi(t)B(Z_D, Z_D) \) for any \( t \).

We will prove these two facts only for the case \( a \simeq \mathfrak{sl}(\mathbb{R}) \) and \( M \) non-standard, since the proof in all other cases is similar.

Consider two elements \( V, W \in \mathfrak{g} \). Since \( \varpi \) is closed we have that

\[
0 = 3d\varpi_{\eta_\varpi}(J \hat{Z}_D, \hat{V}, \hat{W}) =
\]

\[
= J \hat{Z}_D(\varpi_{\eta_\varpi}(\hat{V}, \hat{W})) - \hat{V}(\varpi_{\eta_\varpi}(J \hat{Z}_D, \hat{W})) + W(\varpi_{\eta_\varpi}(J \hat{Z}_D, \hat{V})) -
\]

\[
-\varpi_{\eta_\varpi}(J \hat{Z}_D, \hat{V}) + \varpi_{\eta_\varpi}(J \hat{Z}_D, \hat{W}) - \varpi_{\eta_\varpi}(\hat{V}, \hat{W}) - J \hat{Z}_D |_{\eta_\varpi}(\varpi(\hat{V}, \hat{W})) - \varpi_{\eta_\varpi}(J \hat{Z}_D, [\hat{V}, \hat{W}]) =
\]

\[
= \frac{d}{dt} (B(Z_\varpi, [V, W]))_{\mid_t} + \varpi_{\eta_\varpi}(J \hat{Z}_D, [\hat{V}, \hat{W}]).
\]

On the other hand, we have the following orthogonal decomposition of the element \([V, W]_t\):

\[
[V, W] = \frac{B(Z_D, [V, W])}{B(Z_D, Z_D)} Z_D - B(F_\alpha, [V, W]) F_\alpha - B(G_\alpha, [V, W]) G_\alpha +
\]

\[
+ [V, W]_{(t+a)^\perp} + [V, W]_t ,
\]

where \([V, W]_t\) and \([V, W]_{(t+a)^\perp}\) are the orthogonal projections of \([V, W]\) into \( \mathfrak{l}\) and \((I + a)^\perp\), respectively. Then

\[
\varpi_{\eta_\varpi}(J \hat{Z}_D, [\hat{V}, \hat{W}]) = \frac{B(Z_D, [V, W])}{B(Z_D, Z_D)} \varpi_{\eta_\varpi}(J \hat{Z}_D, \hat{Z}_D) - B(F_\alpha, [V, W]) \varpi_{\eta_\varpi}(J \hat{Z}_D, \hat{F}_\alpha) -
\]
Corollary 4.2. Let \( \eta \) interpret the optimal bases (see also □ as we needed to prove. Then, \( \eta \) along an optimal transversal curve \( V, W \) since \( \eta \) the following holomorphic frame in \( T^C_{\eta} M \):

\[
e_0 = \hat{F}_0|_\eta - iJ\hat{F}_0|_\eta = \hat{Z}|_\eta - iJ\hat{Z}|_\eta, \quad e_i = \hat{F}_i|_\eta - iJ\hat{F}_i|_\eta \quad i \geq 1.
\]

Then,

1. if \( M \) is a KO-manifold or a standard KE-manifold, then the holomorphic frames \( F_i \) are orthogonal w.r.t. any \( G \)-invariant Kähler metric \( g \) on \( M \);
2. if \( M \) is a non-standard KE-manifold, then the holomorphic frames \( F_i \) are orthogonal w.r.t. any \( G \)-invariant Kähler metric \( g \) on \( M \), whose associated algebraic representative \( Z_\omega(t) \) has vanishing coefficient \( C_\omega = 0 \) (see Proposition 4.1 for the definition of \( C_\omega \)).

Proof. It is a direct consequence of definitions and Proposition 4.1. □
5. The Ricci tensor of a K-manifold.

From the results of §4, the Ricci form $\rho$ can be completely recovered from the algebraic representative $Z_\rho(t)$ along an optimal transversal curve $\eta$. On the other hand, using a few known properties of flag manifolds, the reader can check that the curve $Z_\rho(t) \in \mathfrak{z}(l) + \mathfrak{a}$ is uniquely determined by the 1-parameter family of quadratic forms $Q^\tau$ on $\mathfrak{m}$ given by

$$Q^\tau_\eta : \mathfrak{m} \to \mathbb{R} , \quad Q^\tau_\eta(E) = r_{\eta_i}(\hat{E}, \hat{E}) \quad ( = -\rho_{\eta_i}(\hat{E}, \hat{E}) = -\mathcal{B}(Z_\rho(t), [E, J_tE]) ) .$$

Since $\mathfrak{m}$ corresponds to the subspace $D_\eta \subset T_\eta G \cdot \eta$, this means that for any Kähler metric $\omega$, the corresponding the Ricci tensor $\tau$ is uniquely determined by its restrictions $\tau|_{D_i \times D_i}$ on the holomorphic tangent spaces $D_i$ of the regular orbits $G \cdot \eta$.

The expression for the restrictions $\tau|_{D_i \times D_i}$ in terms of the algebraic representative $Z_\omega(t)$ of the Kähler form $\omega$ is given in the following Theorem.

**Theorem 5.1.** Let $(M, J, g)$ be a K-manifold and $\eta = \exp(tZ_D) \cdot p_o$ be an optimal transversal curve. Using the same notation of §3, let also $(F_i, G_i) = (F_0, F_k^\pm, G_k^\pm, F_j', G_j')$ be an optimal basis for $\mathbb{R}Z_D + \mathfrak{m};$ finally, for any $1 \leq j \leq N_F$ let $\ell_j$ be the integer which appear in (3.12) for the expression of $J_i F_i$ and for any $N_F + 1 \leq k \leq n - 1$ let $\beta_k$ be the root so that $F_k = F_{\beta_k}$.

Then, for any $\eta \in \mathcal{M}_{\text{reg}}$ and for any element $E \in \mathfrak{m}$

$$\rho_{\eta_i}(\hat{E}, J_tE) = A_E(t) \left( \frac{1}{2} h'(t) - \sum_{i=1}^{N_F} \tanh(-1)^{i+1}(\ell_i t) \ell_i + \sum_{j=N_F+1}^{n-1} \beta_j(iZ_D) \right) + B_E(t)$$

where

$$h(t) = \log(\omega^n(F_0, J\hat{F}_0, \hat{F}_1, J\hat{F}_1, \ldots, JF_{n-1})|_{\eta_i}) , \quad A_E(t) = \frac{\mathcal{B}([E, J_tE], Z_D)}{\mathcal{B}(Z_D, Z_D)} , \quad B_E(t) = -\sum_{i=1}^{N_F} \tanh(-1)^{i+1}(\ell_i t) \mathcal{B}([E, J_tE]|_{\mathfrak{m}}, [[F_i, G_i]|_{\mathfrak{m}}) +$$

$$+ \sum_{j=N_F+1}^{n-1} \mathcal{B} (iH_{\beta_j}, [E, J_tE]|_{\mathfrak{g}(l)}) ,$$

and where, for any $X \in \mathfrak{g}$, we denote by $X_{l+m}$ (resp. $X_{\mathfrak{g}(l)}$) the projection parallel to $(l+m)^\perp = \mathbb{R}Z_D$ (resp. to $\mathfrak{g}(l)^\perp$) of $X$ into $l + \mathfrak{m}$ (resp. into $\mathfrak{g}(l)$).

**Proof.** Let $J_i$ be the complex structure on $\mathfrak{m}$ induced by the complex structure $J$ of $M$. For any $E \in \mathfrak{m}$ and any point $\eta$, we may clearly write that $\rho_{\eta_i}(\hat{E}, J_tE) = \rho_{\eta_i}(\hat{E}, J_i \hat{E})$ and hence, by Koszul's formula (see [Ko], [Be]),

$$\rho_{\eta_i}(\hat{E}, J_tE) = \frac{1}{2} \left( \mathcal{L}_{J[E, J_iE]} \omega^n \right)_{\eta_i} (F_0, J\hat{F}_0, \hat{F}_1, J\hat{F}_1, \ldots, JF_{n-1}) \left( \omega^n_{\eta_i}(F_0, J\hat{F}_0, \hat{F}_1, J\hat{F}_1, \ldots, JF_{n-1}) \right)$$

(5.5)
On the other hand,

\[ Y|_{\eta(t)} = \sum_{i \geq 0} \lambda_i \hat{F}_i|_{\eta(t)} + \sum_{i \geq 1} \mu_i J \hat{F}_i|_{\eta(t)} , \]

where

\[ \lambda_i = \frac{B(Y, F_i)}{B(F_i, F_i)} , \quad \mu_i = \frac{B(Y, J_t F_i)}{B(J_t F_i, J_t F_i)} . \]

Hence, for any \( i \)

\[ [J[E, J_t E], \hat{F}_i]|_{\eta(t)} = \sum_{j \geq 0} \frac{B([E, J_t E], F_j)}{B(F_j, F_j)} J \hat{F}_j|_{\eta(t)} + \sum_{j \geq 1} \frac{B([E, J_t E], J_t F_j)}{B(J_t F_j, J_t F_j)} J \hat{F}_j|_{\eta(t)} , \]

\[ [J[E, J_t E], J \hat{F}_i]|_{\eta(t)} = \sum_{j \geq 0} \frac{B([E, J_t E], F_j)}{B(F_j, F_j)} J \hat{F}_j|_{\eta(t)} + \sum_{j \geq 1} \frac{B([E, J_t E], J_t F_j)}{B(J_t F_j, J_t F_j)} J \hat{F}_j|_{\eta(t)} . \]

Therefore, if we denote \( h(t) = \log(\omega^n(F_0, J \hat{F}_0, J \hat{F}_1, \ldots, J \hat{F}_{n-1})|_{\eta(t)}) \), then, after some straightforward computations, (5.5) becomes

\[ \rho_{\eta(t)}(E, J_t E) = \frac{1}{2} J[E, J_t E](h)|_{\eta(t)} - \sum_{i \geq 1} \frac{B([E, J_t E], J_t F_i)}{B(J_t F_i, J_t F_i)} \cdot \]

We claim that

\[ J[E, J_t E](h)|_{\eta(t)} = A_E(t) h'_t . \]

In fact, for any \( X \in \mathfrak{g} \)

\[ \hat{X}(\omega(F_0, J \hat{F}_0, \ldots, J \hat{F}_{n-1})|_{\eta(t)} = \]

\[ \omega_{\eta(t)}([X, F_0], J \hat{F}_0, \ldots, J \hat{F}_{n-1}) - \omega(F_0, J[X, F_0], \ldots, J \hat{F}_{n-1}) - \cdots = 0 . \]

On the other hand,

\[ J[E, J_t E]|_{\eta(t)} = \frac{B([E, J_t E], Z_D)}{B(Z_D, Z_D)} J \hat{Z}_D|_{\eta(t)} + J \hat{X}|_{\eta(t)} = A_E(t) J \hat{Z}_D|_{\eta(t)} + J_t \hat{X}|_{\eta(t)} \]

for some \( X \in \mathfrak{m} \). From (5.11) and (5.10) and the fact that \( J \hat{Z}_D|_{\eta(t)} = \eta_t \), we immediately obtain (5.9).
Let us now prove that
\[
\sum_{i \geq 1}^{n-1} \frac{B([E, J_i E], [F_i, J_i F_i])}{B(J_i F_i, J_i F_i)} = A_E \left\{ \sum_{i=1}^{N_F} \tanh(-1)^{i+1}(\ell_i t)\ell_i - \sum_{j=N_F+1}^{n-1} \beta_j(iZ_D) \right\} - B_E
\]
(5.12)
First of all, observe that from definitions, for any \(1 \leq k \leq N_F\) we have that, for any case of Table 1, when \(\alpha_k \neq \alpha_k^d\),
\[
B(Z_D, [F_k, G_k]) = \frac{1}{2} B(Z_D, [F_{\alpha_k} + (-1)^{k+1}\epsilon_k F_{\alpha_k^d}, G_{\alpha_k} + (-1)^{k+1}\epsilon_k G_{\alpha_k^d}]) =
\]
\[
= \frac{i}{2} B(Z_D, H_{\alpha_k} + H_{\alpha_k^d}) = \ell_k,
\]
(5.13)
and, when \(\alpha_k = \alpha_k^d\),
\[
B(Z_D, [F_k, G_k]) = B(Z_D, [F_{\alpha_k}, G_{\alpha_k}]) = B(Z_D, iH_{\alpha_k}) = \ell_k.
\]
(5.13’)
Similarly, for any \(N_F + 1 \leq j \leq n - 1\)
\[
B(Z_D, [F_j, G_j]) = B(Z_D, iH_{\beta_j}) = \beta_j(iZ_D).
\]
(5.14)
So, using (5.13), (5.13’), (5.14) and the fact that \(B(F_i, F_i) = B(G_i, G_i) = -1\) for any \(1 \leq i \leq n - 1\), we obtain that for \(1 \leq k \leq N_F\),
\[
\frac{B([E, J_i E], [F_k, J_i F_k])}{B(J_i F_k, J_i F_k)} = \tanh(-1)^{k+1}(\ell_k t) \left( B(Z_D, [F_k, G_k]) \frac{B([E, J_i E], Z_D)}{B(Z_D, Z_D)} + \right.
\]
\[
+ B([E, J_i E]_{l+m}, [F_k, G_k]_{l+m}) \right) =
\]
\[
= \tanh(-1)^{k+1}(\ell_k t) [A_E(t)\ell_k + B([E, J_i E]_{l+m}, [F_k, G_k]_{l+m})],
\]
(5.15)
and for any \(N_F + 1 \leq j \leq N\)
\[
\frac{B([E, J_i E], [F_j, J_i F_j])}{B(J_i F_j, J_i F_j)} =
\]
\[
= - \frac{B([E, J_i E], Z_D)}{B(Z_D, Z_D)} B(Z_D, [F_j, G_j]) - B([E, J_i E]_{l+m}[F_j, G_j]_{l+m}) =
\]
\[
= - A_E\beta_j(iZ_D) - B(iH_{\beta_j}, [E, J_i E]_{l(1)})
\]
(5.16)
From (5.15) and (5.16), we immediately obtain (5.12) and from (5.8) this concludes the proof. □

The expressions for the functions \(A_E(t)\) and \(B_E(t)\) simplify considerably if one assumes that \(E\) is an element of the optimal basis. Such expressions are given in the following conclusive proposition.
Proposition 5.2. Let $(F_i, G_i)$ be an optimal basis along an optimal transversal curve $\eta$ of a $K$-manifold $M$. For any $1 \leq i \leq N_F$, let $\ell_i$ be as in Theorem 5.1 and denote by $\{\alpha_i, \alpha_i^t\} \subset R_F^+$ the pair of CR-dual roots, such that $F_i = \frac{1}{\sqrt{2}}(F_{\alpha_i} \pm \epsilon_i F_{\alpha_i^t})$ or $F_i = F_{\alpha_i}$, in case $\alpha_i = \alpha_i^t$; also, for any $N_F + 1 \leq j \leq n - 1$, denote by $\beta_j \in R_+$ the root such that $F_j = F_{\beta_j}$. Finally, let $A_F(t)$ and $B_F(t)$ be as defined in Theorem 5.1 and let us denote by

$$Z^\kappa = \sum_{k=N_F+1}^{n-1} iH_{\beta_k}. \quad (5.17)$$

(1) If $E = F_i$ for some $1 \leq i \leq N_F$, then

$$A_{F_i}(t) = -\ell_i \tanh^{-1}(\ell_i t) \frac{B(Z_D, Z_D)}{B(Z_D, Z_D)}; \quad (5.18)$$

and

$$B_{F_i}(t) = -\ell_i \tanh^{-1}(\ell_i t) \frac{B(Z_D, Z_D)}{B(Z_D, Z_D)} +$$

$$+ \tanh^{-1}(\ell_i t) \left( \sum_{j=1}^{N_F} \tanh^{-1}(\ell_j t) B([F_i, G_i]_{i+m}, [F_j, G_j]_{i+m}) \right), \quad (5.19)$$

(2) If $E = F_i$ for some $N_F + 1 \leq i \leq n - 1$, then

$$A_{F_i}(t) = \frac{B(Z_D, iH_{\beta_i})}{B(Z_D, Z_D)}, \quad B_{F_i}(t) = B(Z^\kappa, iH_{\beta_i}). \quad (5.20)$$

Proof. Formulae (5.18) and (5.19) are immediate consequences of definitions and of (5.13), (5.13') and (5.14). Formula (5.20) can be checked using the fact that $[F_{\beta_i}, J_i F_{\beta_i}] = [F_{\beta_i}, G_{\beta_i}] = iH_{\beta_i}$ for any $N_F + 1 \leq i \leq n - 1$, from properties of the Lie brackets $[F_i, G_i]$, with $1 \leq i \leq N_F$, which can be derived from Table 1, and from the fact that $R Z_D \subset [m', m']^\perp$. □

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