A vacuum-like configuration in General Relativity as a manifestation of a Lorentz-invariant mode of five-dimensional gravity

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Abstract

A Lorentz-invariant cosmological model is constructed within the framework of five-dimensional gravity. The five-dimensional theorem which is analogous to the generalized Birkhoff theorem is proved, that corresponds to the Kaluza’s “cylinder condition”. The five-dimensional vacuum Einstein equations have an integral of motion corresponding to this symmetry, the integral of motion is similar to the mass function in general relativity (GR). Space closure with respect to the extra dimensionality follows from the requirement of the absence of a conical singularity. Thus, the Kaluza-Klein (KK) model is realized dynamically as a Lorentz-invariant mode of five-dimensional general relativity. After the dimensional reduction and conformal mapping the model is reduced to the GR configuration. It contains a scalar field with a vanishing conformally invariant energy-momentum tensor on the flat space-time background. This zero mode can be interpreted as a vacuum configuration in GR. As a result the vacuum-like configuration in GR can be considered as a manifestation of the Lorentz-invariant empty five-dimensional space.

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I. INTRODUCTION

The data of the observation astronomy and cosmology indicate the existence of new forms of matter in the Universe such as “dark matter” and “dark energy” (for a review see [1]). One of the candidates for the dark energy role is vacuum. It is associated with a non-vanishing cosmological constant (see [2], [3], [4], and references therein). The vacuum is defined as a Lorentz invariant state of physical medium in space-time \( V^{(4)} \) [5], [6], [7]. This concept has become a basis for the popular model of inflation in the early Universe [8], [9], [10]. Here the vacuum configuration is a cosmological model with the metric admitting the Lorentz group as the isometry group. The gravitational field is generated by the Lorentz-invariant “vacuum-like” energy-momentum tensor

\[
T^\mu_\nu = \frac{\Lambda}{8\pi G} \delta^\mu_\nu \tag{1.1}
\]

where \( G \) is the Newtonian gravitational constant, \( \Lambda \) is the cosmological constant. The vacuum density is determined by this constant:

\[
\rho_v = \frac{\Lambda}{8\pi G} \left( \frac{c}{1} \right).
\]

Let the Lorentz group \( L = O(1,3) \) act on the coordinates \( x^\mu (\mu, \nu = 0, 1, 2, 3) \) of space-time \( V^{(4)} \) in the following way

\[
\tilde{x}^\mu = L^\mu_\nu x^\nu. \tag{1.2}
\]

Moreover, \( \eta_{\tilde{\mu}\tilde{\nu}} = L^\alpha_\mu L^\beta_\nu \eta_{\alpha\beta} \), \( \tilde{L}_\mu^\alpha L_\alpha^\beta = \delta^\alpha_\beta \), \( L^\mu_\nu L^\alpha_\mu = \delta^\alpha_\nu \) where \( \eta_{\mu\nu}, \eta_{\tilde{\mu}\tilde{\nu}} \) are Minkowski metrics. The invariance condition \( V^{(4)} \) with respect to the Lorentz group leads to the conformally flat space with the metric

\[
(4)\, ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \psi^2(\zeta) \eta_{\mu\nu} dx^\mu dx^\nu \tag{1.3}
\]

where

\[
\zeta = \eta_{\mu\nu} x^\mu x^\nu. \tag{1.4}
\]

The Weyl tensor vanishes for the metric (1.3). This leads to the curvature tensor

\[
(4)\, R_{\mu\nu\rho\sigma} = (4)\, R_{\rho[\mu \eta_{\nu\sigma]} - (4)\, R_{\sigma[\mu \eta_{\nu]\rho} - (1/3)(4)\, R g_{\rho[\mu \eta_{\nu]}\sigma} \text{ where the square brackets mean an alternation, i.e. } A_{\mu\nu\rho} = (A_{\mu\nu} - A_{\nu\mu})/2, (4)\, R_{\mu\nu} \text{ is the Ricci tensor, } (4)\, R \text{ is the scalar curvature. From the Einstein equations }

(4)\, G_{\mu\nu} = (4)\, R_{\mu\nu} - (1/2) (4)\, R g_{\mu\nu} = \Lambda g_{\mu\nu} \tag{1.5}
\]

it follows that \( (4)\, R_{\mu\nu} = -\Lambda g_{\mu\nu} \), \( (4)\, R = -4\Lambda \). Therefore \( (4)\, R_{\mu\nu\rho\sigma} = K (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \) where \( K = -1/a^2 = -\Lambda/3 \) and we come to the de Sitter space-time (see [11] for details).

In the generalized stereographic coordinates [12], [13] the metric of constant curvature spaces can be written down as follows [14]

\[
(4)\, ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (1 - \Lambda \zeta/12)^{-2} \eta_{\mu\nu} dx^\mu dx^\nu. \tag{1.6}
\]

The one-parameter family of vacuum-like configurations with the metric (1.6) is labeled by the parameter \( \Lambda \). The last defines the vacuum density \( \rho_v = \Lambda/8\pi G \). At present the vacuum density value \( \rho_v \) (or the constant \( \Lambda \)) is determined only from observations [3].
It turns out that it is possible to construct a nontrivial “vacuum” configuration without the cosmological constant. Indeed, we consider a scalar field on the flat space-time background. Let the scalar field be a Lorentz-invariant solution of the wave equation and has a vanishing conformally invariant energy-momentum tensor. This configuration appears as a zero mode of the GR equations. In the present paper the interpretation of classical vacuum in GR as a Lorentz-invariant zero mode of the GR equations for the coupled gravitational and scalar fields with conformal connection is proposed with using this vacuum-like configuration.

The model admits a geometrical formulation within the framework of five-dimensional gravity [15]. It is shown that the Lorentz-invariant Kaluza-Klein space can be considered as a Lorentz-invariant mode of the equations of five-dimensional gravity (without external sources) where the cylindricity and closedness conditions are realized dynamically. After the dimensional reduction and corresponding conformal transformation this model is reduced to the above four-dimensional vacuum configuration with a scalar field on the flat space-time background. Therefore the vacuum in GR can be interpreted as a manifestation of the Lorentz-invariant mode of five-dimensional gravity or that of the KK theory.

The plan of the paper is following. In Sec. II we introduce the necessary relations used in five-dimensional gravity for the vacuum Lorentz-invariant five-dimensional spaces. In Sec. III we establish the five-dimensional version of the generalized Birkhoff theorem and explicitly construct an extra Killing vector $\xi$.

In Sec. IV we find the “first integral” of the five-dimensional vacuum Einstein equations which leads to the conservation law $G = L^2(\nabla L)^2 - \epsilon L^2 = \text{const}$. In Sec. V we obtain the metric for conformally flat five-dimensional Lorentz-invariant cosmological model and ascertain the physical meaning of $G$. After the dimensional reduction and corresponding conformal transformation we reduce this model to the four-dimensional vacuum-like Lorentz-invariant configuration for the gravitational and conformal scalar fields with conformal connection. In Sec. VI we consider some geometrical properties of the model and the regularity condition. In Sec. VII we reduce the five-dimensional action for a geodesic line to the four-dimensional action for a particle with a variable mass in the flat space-time.

In Appendix A the generalized stereographic coordinates are introduced. In Appendix B the curvature tensor, the Ricci tensor, the scalar curvature, and also the quadratic curvature invariant for the obtained metric are calculated. In Appendix C the proof of the five-dimensional version of the Birkhoff theorem is completed.

Here $G$ is the gravitational constant, the velocity of light $c = 1$. The metric tensor $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) has a signature (+ - - -).

II. THE BASIC EQUATIONS OF LORENTZ-INvariant FIVE-DIMENSIONAL COSMOLOGICAL MODEL

In the framework of the five-dimensional general relativity [15] we consider the five-dimensional pseudo-Riemannian space $V^{(5)}$ with the metric $g_{AB}$ ($A, B = 0, 1, 2, 3, 4$) (the signature is (+ - - -)) which in general depends on all the coordinates $x^A$. This metric satisfies the five-dimensional Einstein equations in the vacuum

\[
(5)G^A_B = (5)R^A_B - \frac{1}{2} (5)R \delta^A_B = 0. \tag{2.1}
\]
These equations follow from the five-dimensional Hilbert variational principle for the action

\[ I^{(5)} = -\frac{1}{16\pi G} \int \sqrt{|(5)g|} \ (5)R \ d^5x \]  

(2.2)

where \( \hat{G} \) is the “five-dimensional gravitational constant”, \( (5)g = \text{det}((5)g_{AB}) \). We furthermore suppose that \( x^\mu (\mu, \nu = 0, 1, 2, 3) \) are space-time coordinates and \( x^4 \equiv Z \) is the extra fifth coordinate.

Now we consider the five-dimensional Lorentz-invariant cosmological model. The Lorentz group \( L \) acts on space \( V^{(5)} \) by the rule (1.2). The invariance condition of \( V^{(5)} \) with respect to the Lorentz group leads to the space admitting the families of conformally flat space-time local subspaces. Therefore, the five-dimensional metric can be written via generalizing the metric (1.6) of space-time \( V^{(4)} \). It is easy to see that the desired maximum general Lorentz-invariant metric \( V^{(5)} \) can be written in the following form

\[ (5)ds^2 = \psi^2(Z, \zeta)\eta_{\mu\nu}dx^\mu dx^\nu - \tilde{V}^2(Z, \zeta) (dZ + N(Z, \zeta)x_\mu dx^\mu)^2 \]  

(2.3)

where \( \psi, \tilde{V}, N \) are some functions, \( x_\mu = \eta_{\mu\nu}x^\nu \). According to the Frobenius theorem [16], we have \( dZ + N(Z, \zeta)x_\mu dx^\mu = d\tilde{Z} + \tilde{N}(Z, u_\zeta)du_\zeta = \beta dz \) where \( \tilde{N}(Z, u_\zeta) = u_\zeta N(Z, \epsilon u_\zeta^2), \beta \) and \( \beta \) are some functions of variables \( Z \) and \( \zeta \). Using a function \( z = z(Z, \zeta) \) as a new fifth coordinate, we can write the metric (2.3) in the for

\[ (5)ds^2 = (5)g_{AB}dx^Adx^B = \psi^2(z, \zeta)\eta_{\mu\nu}dx^\mu dx^\nu - V^2(z, \zeta)dz^2 \]  

(2.4)

where \( V^2 = \beta \tilde{V}^2 \). In A it will be shown that the relation

\[ (4)ds_0^2 = \eta_{\mu\nu}dx^\mu dx^\nu = \epsilon \ du_\zeta^2 - u_\zeta^2 \ (3)d\Omega^2_\epsilon \]  

(2.5)

takes place. Here

\[ u_\zeta = \sqrt{\epsilon \eta_{\mu\nu}x^\mu x^\nu} = \sqrt{\epsilon \zeta} \ (\epsilon = \zeta/|\zeta|) \]  

(2.6)

is the “radial” variable which labels the hyperboloids \( \epsilon \eta_{\mu\nu}x^\mu x^\nu = u_\zeta^2 \),

\[ (3)d\Omega^2_\epsilon = h_{ij}dy^idy^j = \left( 1 - \epsilon S^2_\epsilon /4 \right)^{-2} (\epsilon(dy^1)^2 + (dy^2)^2 + (dy^3)^2) \]  

(2.7)

is the “angular” part of metric (2.5) in the stereographic coordinates \( y^i \) [12], [13], \( S^2_\epsilon = \epsilon(y^1)^2 + (y^2)^2 + (y^3)^2 \), (i,j=1,2,3). The metric (2.4) can be written in the form

\[ (5)ds^2 = \psi^2(z, \zeta)(\epsilon \ du_\zeta^2 - u_\zeta^2 (3)d\Omega^2_\epsilon) - V^2(z, \zeta)dz^2 \]  

(2.8)

with taking into account (2.5).

One can find the symmetry of the model by solving the five-dimensional Killing equations

\[ \mathcal{L}_\xi \ g_{AB} = (5)g_{AB,C}\xi^C + (5)g_{AC}\xi^C_B + (5)g_{CB}\xi^C_A = \xi_A;B + \xi_B;A = 0 \]  

(2.9)

where \( \mathcal{L}_\xi \) is a Lie derivative with respect to the Killing vector \( \tilde{\xi} = \xi^4\partial_A = \xi^z\partial_z + \xi^4\partial_\mu \). Here “\( _A \)” \( \equiv \partial_A \equiv \partial/\partial x^A \) and “\( _z \)” \( \equiv \partial_z \equiv \partial/\partial z \) denote the partial derivatives with respect
to the coordinates \(x^A\) and \(z\), correspondingly. For the metric (2.4) these equations take the form

\[
\begin{align*}
(V\xi^z)_z + V^\mu \xi^\mu &= 0, \\
V^2\xi^z - \psi^2\xi_{\mu,z} &= 0, \\
2\eta_{\mu\nu}(\psi\xi^z + \psi\alpha^z) + \psi(\xi_{\mu,\nu} + \xi_{\nu,\mu}) &= 0
\end{align*}
\]  

(2.10), (2.11), (2.12)

where \(\xi^\mu = \eta_{\mu\nu}\xi^\nu\). The general nontrivial solution of these equations \(\tilde{\xi}_L = x_\mu\omega^{\mu\nu}\partial_\nu\) \((x_\mu = \eta_{\mu\nu}x^\nu)\) for arbitrary \(\psi(Z, \zeta)\), \(V(Z, \zeta)\) determines a general member of the Lie algebra of the Lorentz group. Here \(\omega^{\mu\nu} = -\omega^{\nu\mu}\) are parameters fixing a member of this algebra.

For the metric (2.4) the Einstein tensor have the form (see B)

\[
(5) G^\mu_\nu = (5) G^{\mu}_u u^\nu_\epsilon u^\nu_\epsilon + (5) G^{\mu}_h h^\mu_\nu, \\
(5) G^\mu_z = (5) R^\mu_z = (5) G^{\mu}_u u^\nu_\epsilon, \\
(5) G^z_{\nu} = 6\psi^{-3}(2\xi\psi\psi + 4\psi - \psi^2\psi) \\
\]

(2.13), (2.14), (2.15)

where

\[
(5) G^\mu_u = -\frac{\epsilon\psi^2}{V^2}(5) G^\mu_\nu = \epsilon(5) G^{x}_z \xi^\mu - \frac{6u_\nu}{V}\left(\left(\frac{\psi^2_\nu}{V^2} - \frac{(\psi^2)^2}{V^3} + \frac{\xi\psi}{\psi^2} + \frac{4\psi}{V} + 4\psi\xi\psi + 2\psi\xi\right)\right), \\
(5) G^\nu_u = \frac{3\epsilon}{V^\nu}(3\psi^2_\nu + \frac{3\psi^2}{V^2} - \frac{3\psi^2}{V^3} - \frac{4\psi V}{V} - \frac{6V}{\psi^2} - \frac{2\psi}{\psi^2} - \frac{2\psi}{\psi^2} - \frac{12\psi}{\psi^2} + \frac{4\psi^2}{\psi^2})
\]

(2.16), (2.17)

Here \(u^\mu = x^\mu/u_\epsilon\) \((u^\mu = \eta_{\mu\nu}u_\nu^\nu)\) and \(z_\nu = \partial/\partial z, \xi_\nu = \partial^2/\partial z^2, \psi_\nu = \partial/\partial \psi, \zeta_\nu = \partial^2/\partial \zeta^2\). Operations with indexes for quantities \(A^\mu\) and \(A_u\) are given by the formula (B17).

### III. FIVE-DIMENSIONAL VERSION OF THE GENERALIZED BIRKHOFF THEOREM AS A CYLINDER CONDITION

If the metric (2.4) satisfies the vacuum equations of five dimensional general relativity, the symmetry of the model turns out to be higher than the initial one. In this case the metric (2.4) admits extra one-parameter isometry group with the Killing vector \(\tilde{\xi} = \xi^A \partial_A = \xi^z \partial_z + \xi^\mu \partial_\mu\). This group should commute with the Lorentz group, therefore we have

\[
[\tilde{\xi}, \tilde{\xi}_L] = 0
\]

(3.1)

whence, using the formula \(\tilde{\xi}_L = x_\mu \omega^{\mu\nu} \partial_\nu\), we obtain the following defining equations

\[
x_\mu \omega^{\mu\nu} \xi^z = 0, \quad \xi_\mu \omega^{\nu\nu} - x_\mu \omega^{\nu\nu} \xi^z = 0.
\]

(3.2)
The solution of these equations has the form

\[ \xi^z = h(z, \zeta), \quad \xi^\nu = f(z, \zeta) x^\mu \]  \hspace{1cm} (3.3)

where \( h = h(z, \zeta), f = f(z, \zeta) \) are some functions. Therefore the Killing vector can be written as

\[ \vec{\xi} = h \partial_z + f x^\mu \partial_\mu . \]  \hspace{1cm} (3.4)

After substitution (3.3), the Killing equations (2.10) – (2.12) are converted as:

\[ (Vh)_z + 2f \zeta V_\zeta = 0 , \]  \hspace{1cm} (3.5)

\[ V^2 h - \psi^2 f_x x_\mu = 0 , \]  \hspace{1cm} (3.6)

\[ 2\eta_{\mu\nu} (h \psi_z + f \psi_\alpha x^\alpha + f \psi) + \psi f_\nu x_\mu + \psi f_x x_\nu = 0 . \]  \hspace{1cm} (3.7)

For arbitrary functions \( \Phi = \Phi(z, \zeta) \) the formulae take place

\[ \Phi_\mu = 2\Phi_\zeta x_\mu = \epsilon \Phi_{,\zeta} u_\mu \psi , \quad x^\mu \Phi_{,\mu} = 2\zeta \Phi_{,\zeta} = u_\epsilon \Phi_{,\zeta} (\zeta = \epsilon u_\epsilon^2 = \eta_{\mu\nu} x^\mu x^\nu) . \]  \hspace{1cm} (3.8)

Therefore the relations (3.5) – (3.7) can be rewritten as follows

\[ (Vh)_z + 2f \zeta V_\zeta = 0 , \]  \hspace{1cm} (3.9)

\[ 2V^2 h - \psi^2 f_x x_\mu = 0 , \]  \hspace{1cm} (3.10)

\[ \eta_{\mu\nu} (h \psi_z + f (\psi + 2\zeta \psi_\zeta)) + 2\psi f_\nu x_\mu = 0 . \]  \hspace{1cm} (3.11)

Using the formulae (A3) and (A4), we obtain from the last equation

\[ (h \psi_z + f (u_\epsilon \psi), u_\epsilon)) (\epsilon u_\epsilon^2 u_\nu - h_{\mu\nu}) + 2\psi u_\epsilon^2 f_\zeta u_\mu u_\nu = 0 . \]  \hspace{1cm} (3.12)

Hence it follows that

\[ h \psi_z + 2\epsilon f L_{,\zeta} \sqrt{\epsilon \zeta} + 2\zeta \psi f_\zeta = 0 , \]  \hspace{1cm} (3.13)

\[ h \psi_z + 2\epsilon f L_{,\zeta} \sqrt{\epsilon \zeta} = 0 \]  \hspace{1cm} (3.14)

where

\[ L = u_\epsilon \psi = \psi \sqrt{\epsilon \zeta} . \]  \hspace{1cm} (3.15)

The equations (3.13), (3.14) are equivalent to the relations

\[ h L_{,z} + 2f \zeta L_{,\zeta} = 0 , \quad f_{,\zeta} = 0 . \]  \hspace{1cm} (3.16)

Note that the continuity equation

\[ \left( \sqrt{(5)} g A_A \right) = 0 \]  \hspace{1cm} (3.17)

follows from the Killing equations (2.9). Taking into account the metric (2.4) and Killing vector (3.4), we obtain from here
\[(hV\zeta\psi^4)_z + 2(fV\zeta^2\psi^4)_\zeta = 0.\] (3.18)

This equation will be satisfied identically if we assume
\[h = \frac{2F_\zeta}{V\zeta\psi^4}, \quad f = -\frac{F_z}{V\zeta^2\psi^4}\] (3.19)

where \(F\) is some function of \(z\) and \(\zeta\). Substituting the expressions for \(h\) and \(f\) into Eq. (3.16), we find \(F, u \in L, z - F, z L, u = 0\). It follows that \(F = F(L)\). As a result, for the Killing vector (3.4) we have
\[\tilde{\xi} = V^{-1}L^{-4}F(L)(2\zeta L, \zeta \partial_z - L, z x^\mu \partial_\mu).\]

The remaining Killing equations (3.9), (3.10) and the last equation in (3.16) lead to the following relations:
\[\left(L^{-4} F, L, \zeta\right)_z - V^{-1}L^{-4}F, L, z V_\zeta = 0,\] (3.20)
\[4V^2 \left(V^{-1}L^{-4}\zeta F, L, \zeta\right)_\zeta + \psi^2 \left(V^{-1}L^{-4}F, L, z\right)_z = 0,\] (3.21)
\[\left(V^{-1}L^{-4}F, L, z\right)_\zeta = 0.\] (3.22)

The last equation can be rewritten as
\[\frac{\psi^2}{V^2} \left(\frac{F, L}{L^3}\right)_\zeta + \frac{F, L}{L^3} \left(\frac{\psi, z}{V^3}\right)_\zeta = 0.\] (3.23)

By virtue of the equations \((5)G^z_u = 0\) and (2.16), it results in \(\left(F, L/L^3\right)_L = 0\). This equation gives \(F, L = AL^3\) where \(A\) is a constant. In further we suppose that \(A = 1\).

Now the equation (3.20) takes the form
\[\left(L^{-1} L, \zeta\right)_z - L^{-1}V^{-1}L, z V_\zeta = 0.\] (3.24)

This equation is satisfied by virtue of the Einstein equation \((5)G^u_u = 0\) and formula (2.16). At last, the equation (3.21) can be rewritten as follows
\[4V^2 \left(\psi^{-1}V^{-1}\zeta \psi, \zeta\right)_\zeta + \psi^2 \left(\psi^{-1}V^{-1}\psi, z\right)_z - 2V_\zeta = 0.\] (3.25)

Taking into consideration (2.15) and (2.17), one can see that this equation is satisfied by virtue of the equations \((5)G^z_u = 0\) and \((5)G^u_u = 0\).

Finally, substituting the expressions \(F, L = L^3\) into Eq. (3.19) and (3.4), we obtain the desired Killing vector of an extra symmetry
\[\bar{\xi} = \psi^{-1}V^{-1}\left(L, u \partial_z - L, z u^\mu \partial_\mu\right)\] (3.26)
\[\left(u^\mu = x^\mu /u_e\right).\]

As a result, we come to the five-dimensional version of the generalized Birkhoff theorem for spherically symmetric gravity in four dimensions [16], [17]. According to the theorem, the metric for the Lorentz-invariant five-dimensional space \(V^{(5)}\) satisfying the vacuum five-dimensional Einstein equations admits an extra Killing vector \(\bar{\xi}\). The vector \(\bar{\xi}\) (3.26) is linearly independent of the Killing vectors \(\tilde{\xi}_L = x_\mu \omega^{\mu
u} \partial_\nu\) of the initial Lorentz group. It determines an extra symmetry \(V^{(5)}\) that corresponds to the Kaluza-Klein cylinder condition. Thus, the Kaluza-Klein cylinder condition for the vacuum Lorentz-invariant spaces of five-dimensional gravity are realized dynamically. The proof of this theorem in the framework of two-dimensional dilaton gravity and discussion of the related problems see in [18].
IV. FIVE-DIMENSIONAL VERSION OF THE MASS FUNCTION
AND ITS CONSERVATION LAW

In addition to the usual conservation law for the geodesic equations \( \xi_A dx^A/(5) ds = \text{const} \), the Killing vector (3.26) \( \xi \) allows to construct a nontrivial conservation law which will be satisfied by virtue of the five-dimensional vacuum Einstein equations. Indeed, using the contracted Bianchi identities \( (5) G^A_{B;A} = 0 \) and Killing equations (2.9), we get the continuity equation

\[
P^A = (5) G^A_{B} \xi^B = \psi^{-1} V^{-1} \left( (\psi^4 V P^z), z + (\psi^4 V P^\mu), \mu \right) = 0 \tag{4.1}
\]

for the vector

\[
P^A = (5) G^A_{B} \xi^B = \psi^{-1} V^{-1} \left( L_{,u_e} (5) G^A_z - L_{,z} (5) G^A_{u_e} \right) \tag{4.2}
\]

Hence, using the components of the Einstein tensor (2.13) – (2.16), we obtain

\[
P^z = \psi^{-1} V^{-1} \left( L_{,u_e} (5) G^z_z - \epsilon L_{,z} (5) G^z_{u_e} \right), \tag{4.3}
\]
\[
P^u = \psi^{-1} V^{-1} \left( L_{,u_e} (5) G^u_z - L_{,z} (5) G^u_{u_e} \right) \tag{4.4}
\]

where \( P^u = u^\epsilon_{i} P^\mu \).

The equation (4.1) can be rewritten in the form

\[
(\psi^3 \psi^4 V P^z), z + \epsilon (u^3 \psi^4 V P^u), u_e = 0 \tag{4.5}
\]

with taking into account the relations (3.8). We will search for the solutions of this equation as

\[
2u^3 \psi^4 V P^z = 3 G_{,u_e}, \quad 2\epsilon u^3 \psi^4 V P^u = -3 G_{,z} \tag{4.6}
\]

where \( G = G(\xi, z) \) is a certain function. Hence we find

\[
P^z = \frac{3 G_{,u_e}}{2u^3 \psi^4 V}, \quad P^u = -\frac{3\epsilon G_{,z}}{2u^3 \psi^4 V} \tag{4.7}
\]

Substituting the expressions for \( P^z \) and \( P^u \) from (4.3) and (4.4) into the relations (4.6), we obtain the equation which links the gradient of the function \( G \) with components of the Einstein tensor:

\[
3 G_{,u_e} = 2 L^3 \left( (5) G^z_z L_{,u_e} - \epsilon (5) G^z_{u_e} L_{,z} \right), \tag{4.8}
\]
\[
3 G_{,z} = 2 \epsilon L^3 \left( (5) G^u_z L_{,z} - (5) G^u_{u_e} L_{,z} \right) \tag{4.9}
\]

From here, with the help of Eqs. (2.15) – (2.17), we obtain the following expression \( G_{,a} = \left( 4\xi^2 \psi \psi\xi_\zeta + 4\xi^2 \psi^2 - \xi^2 \psi^2 \psi^2 / V^2 \right) \), whence we find (up to an additive constant)

\[
G = \zeta^2 \left( 4\psi \psi\xi + 4\xi^2 \psi^2 - \psi^2 \psi^2 / V^2 \right) \tag{4.10}
\]
Due to the vacuum Einstein equations (2.1), the conservation law $G = G(\zeta, z) = \text{const}$ follows from (4.8) – (4.9).

Let us now rewrite the metric (2.8) in the (2 + 3) decomposed form

$$
(5) \, ds^2 = (2) \, ds^2 - L^2 (3) \, d\Omega^2
$$

(4.11)

where $(2) \, ds^2$ is the metric of space $V^{(2)}$ which is a section $y^a = \text{const}$ of the space $V^{(5)}$. In the coordinates $\{u_\epsilon, z\}$ we have

$$
(2) \, ds^2 = \gamma_{ab} dx^a dx^b = \epsilon \psi^2(z, \zeta) \, du_\epsilon^2 - V^2(z, \zeta) \, dz^2.
$$

(4.12)

The metric describes subspace $\zeta = \eta_{\mu\nu}x^\mu x^\nu < 0$, when $\epsilon = 1$ or subspace $\zeta = \eta_{\mu\nu}x^\mu x^\nu > 0$, when $\epsilon = -1$. In the coordinates $\{\zeta = \epsilon u_\epsilon^2, z\}$ we have

$$
(2) \, ds^2 = 4^{-1} \zeta^{-1} \psi^2(z, \zeta) \, d\zeta^2 - V^2(z, \zeta) \, dz^2.
$$

(4.13)

This form is applicable in both subspaces $\zeta > 0$ and $\zeta < 0$. Let us write out also two-dimensional representation of the Killing vector (3.26) in the coordinates $\{\zeta = \epsilon u_\epsilon^2, z\}$

$$
\vec{\xi} = u_\epsilon V^{-1}L^{-1}(L^{-1}, \zeta \partial_z - L, \zeta \partial_u) = 2\zeta V^{-1}L^{-1}(L^{-1}, \zeta \partial_z - L, \zeta \partial_u).
$$

(4.14)

Now the integral of motion (4.10) can be written in the two-dimensional invariant form

$$
G = L^2(\nabla L)^2 - \epsilon L^2
$$

(4.15)

where $(\nabla L)^2 = \gamma^{ab}\nabla_a L \nabla_b L$, $\nabla_a \equiv a_\zeta$ is a covariant derivative with respect to the metric $\gamma_{ab}$. The function $G$ is analogical to the known mass function of GR [19], [20], [21].

V. LORENTZ-INVARIENT FIVE-DIMENSIONAL COSMOLOGICAL MODEL AND REDUCTION OF GRAVITATIONAL ACTION FOR $V^{(5)}$

From the five-dimensional generalized Birkhoff theorem it follows that for Lorentz-invariant five-dimensional vacuum spaces there is such a coordinate frame $\{z, u\}$ conserving the diagonal form of the metric (2.8) in which all metric functions do not depend on the fifth coordinate $z$ (see C). Then, for the metric (4.12) the conservation law (4.15) has the form

$$
\epsilon G = u_\epsilon^2 \left(\frac{dL}{du_\epsilon}\right)^2 - L^2.
$$

(5.1)

By setting $G > 0$, from here we obtain $L = \sqrt{G}(au_\epsilon - \epsilon/au_\epsilon)/2$ where $a > 0$ is constant. Using the scale transformation $u_\epsilon \rightarrow u_\epsilon = au_\epsilon \sqrt{G}/2$, we obtain

$$
\psi = L/au_\epsilon = 1 - G/4\zeta.
$$

(5.2)

By setting $G < 0$, we find the same solution.

From the equation $(5)G_u^u = 0$, taking into account (2.17), we have

$$
\frac{V_\zeta}{V} = -\frac{2\psi_\zeta}{\psi} \frac{\psi + \zeta \psi_\zeta}{\psi + 2\zeta \psi_\zeta}
$$

(5.3)
whence, with the help of (5.2), we obtain
\[ V^{-1}V' = -(G/2)(\zeta^2 - G^2/16)^{-1}. \]
Integration of this equation leads to the following result
\[ V = b \left| \frac{1 + G/4\zeta}{1 - G/4\zeta} \right| \]  
(5.4)
where \( b > 0 \) is a constant which can be converted into unity via the scale transformation of \( z \). The remaining vacuum equations of five-dimensional gravity are satisfied identically.

As a result, the metric (2.8) can be written as
\[ (5)ds^2 = -(1 + \frac{G}{4u^2})^2 dz^2 + (1 - \frac{G}{4u^2})^2 \epsilon du^2 - u^2(\frac{3}{d\Omega^2}). \]  
(5.5)
The metric describes the different subdomains of the space \( V^{(5)} \) for \( \epsilon = \pm 1 \). Taking into account (2.5), we can rewrite this metric in the form suitable for the whole space \( V^{(5)} \)
\[ (5)ds^2 = -(1 + \frac{G}{4\zeta})^2 dz^2 + (1 - \frac{G}{4\zeta})^2 \eta_{\mu\nu}dx^\mu dx^\nu. \]  
(5.6)
The physical sense of our model and the constant \( G \) are clarified by the dimensional reduction of action (2.2). For this purpose we shall rewrite the interval (5.6) in the more general (1+4)-form
\[ (5)ds^2 = (5)g_{AB}dx^Adx^B = \hat{g}_{\mu\nu}dx^\mu dx^\nu - V^2dz^2. \]  
(5.7)
For spaces under consideration the variables \( V, \hat{g}_{\mu\nu} \) do not depend on the coordinate \( z \). Therefore from action (2.2) we obtain through the (1+4)-decomposition of scalar curvature \( (5)R \) [22], integration on \( z \) and omitting a divergence
\[ I^{(4)} = -\frac{1}{16\pi \hat{G}} \int \sqrt{-\hat{g}} \ V^{(4)}\hat{R} \ d^4x \]  
(5.8)
where \( \hat{R} \) is a scalar curvature with respect to the metric \( \hat{g}_{\mu\nu}, \hat{g} = \det(\hat{g}_{\mu\nu}), \hat{G} = \hat{G}/L \) is a reduced gravitational constant, \( L \) is a constant with length dimensionality.

The metric (5.6) belongs to the following general class of metrics
\[ (5)ds^2 = -\left(\frac{1 - \varphi/\sqrt{6}}{1 + \varphi/\sqrt{6}}\right)^2 dz^2 + \left(1 + \frac{\varphi}{\sqrt{6}}\right)^2 (4)ds^2 \]  
(5.9)
where \( \varphi \) is an induced scalar field, \( (4)ds^2 = g_{\mu\nu}dx^\mu dx^\nu \) is the physical space-time metric of \( V^{(4)} \). In this parametrization we obtain [23] from action (5.8)
\[ (4)I = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left\{ \left(1 - \frac{\varphi^2}{6}\right)(4)\hat{R} - g_{\mu\nu}\varphi,_{\mu}\varphi,_{\nu} \right\} \]  
(5.10)
where \( (4)\hat{R} \) is a scalar curvature with respect to the metric \( g_{\mu\nu}, g = \det(g_{\mu\nu}). \)
The new action describes the interacting gravitational \( g_{\mu\nu} \) and scalar \( \varphi \) fields. The equations of motion of a new system have the form
\[(\Delta - \frac{1}{6}R)\varphi = 0,\] (5.11)

\[G_{\mu\nu} = 4\pi t_{\mu\nu}(\varphi) \equiv 4\pi T_{\mu\nu}(\varphi) + \frac{1}{6} (G_{\mu\nu} - \nabla_\mu \nabla_\nu + g_{\mu\nu} \Delta) \varphi^2,\] (5.12)

\[T_{\mu\nu}(\varphi) = \frac{1}{4\pi} \left( \varphi_\mu \varphi_\nu - \frac{1}{2}g_{\mu\nu}(\nabla\varphi)^2 \right),\] (5.13)

where \(t_{\mu\nu}\) is a conformally invariant energy-momentum tensor of the scalar field \(\varphi\), \(\Delta = \nabla_\mu \nabla_\mu\) and all quantities are calculated with respect to the metric \(g_{\mu\nu}\).

In our case of the Lorentz-invariant five-dimensional cosmological model with the metric (5.6) the scalar field and physical metric in (5.9) take the form

\[\varphi = -\frac{\sqrt{6}G}{4\zeta} = -\frac{\sqrt{6}G}{4\eta_{\mu\nu}x^\mu x^\nu}, \quad g_{\mu\nu} = \eta_{\mu\nu}.\] (5.14)

Hence it can be seen that the constant \(G\) can be interpreted as a charge of the scalar field \(\varphi\).

Note that the model contains the nontrivial scalar field \(\varphi\) (5.14) with vanishing conformally invariant energy-momentum tensor \(t_{\mu\nu}(\varphi) = 0\). If we consider the equation \(\hat{t}_{\mu\nu}(\varphi) \equiv t_{\mu\nu}(\varphi) = \lambda \delta_{\mu\nu}\varphi\) as the eigenvalue equations for the operator \(\hat{t}_{\mu\nu}\), then the scalar field \(\varphi\) is an eigenfunction of this operator corresponding to the zero eigenvalue. Therefore, the scalar field (5.14) can be interpreted as a classical zero mode of the conformally invariant energy-momentum tensor \(t_{\mu\nu}\). The flat space-time can be interpreted also as a zero mode of the Einstein equations since it is trivial solution of the vacuum Einstein equations. As a result, we can consider the Lorentz-invariant solution of the Einstein equations for the interacting gravitational and conformal scalar fields as a zero mode of GR, and the field system (5.14) itself as a vacuum-like configuration of GR.

If we take into account the above reasons, the indicated vacuum-like configuration of GR can be interpreted as a manifestation of a Lorentz-invariant mode of five-dimensional gravitation.

VI. GEOMETRY OF MODEL, REGULARITY CONDITIONS, AND COMPACTIFICATION OF THE FIFTH COORDINATE

The metric (5.6) of the space \(V^{(5)}\) has a singularity on a light cone \(\zeta \equiv \eta_{\mu\nu}x^\mu x^\nu = 0\). This singularity divides the hypersurface \(z = \text{const}\) into two regions: \(U\)-region when \(\zeta > 0\) (\(\epsilon = 1\)) and \(V\)-region when \(\zeta < 0\) (\(\epsilon = -1\)), where \(\epsilon = \zeta/|\zeta|\).

The singularity \(\zeta = 0\) has a coordinate character. It turns out that the invariant of curvature \(I = (5)R_{ABCD}^{(5)}R^{ABCD}\) is finite here (see B). Therefore, the metrics (5.5) with different \(\epsilon = \pm 1\) can be interpreted as the metrics of regions \(U \subset V^{(5)}\) and \(V \subset V^{(5)}\). Then we can consider the metric (5.6) as the analytical expansion of the metrics (5.5) on the whole space \(V^{(5)}\).

According to (2.6), we shall introduce coordinates in \(U\)- and \(V\)-regions

\[u = u_{+1} = \pm \sqrt{\eta_{\mu\nu}x^\mu x^\nu}, \quad v = u_{-1} = \sqrt{-\eta_{\mu\nu}x^\mu x^\nu}.\] (6.1)

We suppose that \(-\infty < u < \infty\) and \(0 < v < \infty\).
Note that the region $U$ is a union of subregions of the future $U^+$ ($u > 0$) and past $U^-$ ($u < 0$). The subregions $U^+$ and $U^-$ are divided into the subregions $U^+_+, U^+_-$ and $U^-_+$, $U^-_-$ by the singular space-like surfaces

$$u^2 = |G|/4,$$

so that $U^+ = U^+_+ \cup U^+_-$ and $U^- = U^-_+ \cup U^-_-$. While the region $V$ is divided into the subregions $V^-, V^+$ by the singular timelike surface

$$v^2 = |G|/4,$$

so that $V = V^- \cup V^+$. Thus, we have the following subregions of the hypersurface $z = \text{const}$

$$U^+_+ = \{ u : 0 < u < \sqrt{|G|/2} \}, \quad U^+_- = \{ u : \sqrt{|G|/2} < u < \infty \},$$

$$U^-_+ = \{ u : -\sqrt{|G|/2} < u < 0 \}, \quad U^-_- = \{ u : -\infty < u < -\sqrt{|G|/2} \},$$

$$V^- = \{ v : 0 < v < \sqrt{|G|/2} \}, \quad V^+ = \{ v : \sqrt{|G|/2} < v < \infty \}.$$

The metrics (5.5) and (5.6) are invariant with respect to the inversion

$$u = \epsilon_1 G/4 \tilde{u}, \quad x^\mu = \epsilon_1 G \tilde{x}^\mu/4 \tilde{\zeta}, \quad \tilde{\zeta} = \eta_{\mu\nu} \tilde{x}^\mu \tilde{x}^\nu > 0,$$

$$v = |G|/4 \tilde{v}, \quad x^\mu = |G| \tilde{x}^\mu/4 \tilde{\zeta}, \quad \tilde{\zeta} = \eta_{\mu\nu} \tilde{x}^\mu \tilde{x}^\nu < 0$$

where $\epsilon_1 = \pm 1$. These inversions induce maps of subregions (6.4)–(6.6):

$$U^+_+ \leftrightarrow U^-_-, \quad U^-_+ \leftrightarrow U^+_-, \quad (\epsilon_1 G > 0),$$

$$V^+ \leftrightarrow V^-.$$

According to (5.5), the $U$-region is described in the coordinates $\{u, z\}$ by the metric

$$(5) \, ds^2 = -\left(1 + G/4u^2\right)^2 dz^2 + \left(1 - G/4u^2\right)^2 (du^2 - u^2 d\Omega^2_+).$$

The point $u = 0$ is a regular point of the metric. The singularity character of surfaces $u^2 = G/4$ depends on a sign of $G$. In the case $G < 0$ we have a surface where extreme curvature is reached, $I_{\text{max}} = 72/G^2$, and in case $G > 0$ we have a surface of singular curvature: $\lim I \to \infty$, when $u^2 \to G/4$ (see B).

If one uses a coordinate

$$T = u - G/4u,$$

the metric (6.12) takes the form

$$(5) \, ds^2 = -\left(1 + G/T^2\right) dz^2 + \left(1 + G/T^2\right)^{-1} dT^2 - T^2 d\Omega^2_+.$$

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Hence it can be seen that coordinates \{z, T\} in the \(U\)-region are analogous to curvature coordinates in the \(T\)-region of the Schwarzschild solution in GR.

The coordinate \(T\) is a single-valued function of \(u\), except for \(u = 0\) where it has the point of discontinuity. The inverse function \(u = (T + \sqrt{T^2 + G})/2\) is a double-valued function of \(T\) and can have a branchpoint at \(T = \sqrt{-G}\) when \(G < 0\). It is easy to see that in the case \(G > 0\) each of the regions \(U^+\) and \(U^-\) in coordinates \(\{z, u\}\) corresponds to the same region \((-\infty < T < \infty)\) in coordinates \(\{z, T\}\). Thus, the regions \(U^+\) and \(U^-\) in coordinates \(\{z, T\}\) are identified. At the inverse \((6.7)\) the coordinate \(T\) (6.13) behaves as follows

\[
T = -\epsilon_1 \tilde{T}, \quad \tilde{T} = \ddot{u} - G/4\dot{u}.
\] (6.15)

Therefore, in the case \(\epsilon_1 > 0\) the corresponding map of regions \((6.9)\) and \((6.10)\) results in the change of a time coordinate (T) sign. In addition, the subregions \(U^+_+, U^+_+, U^-_+\) and \(U^-_+\) exchange their places in an appropriate way. The metric (6.14) contains a curvature singularity at the instant \(T = 0\). The singularity is an invariant with respect to the inversion (6.7) and corresponds to singular surfaces \(u = \pm \sqrt{G}/2\) for the metric (6.12).

If \(G < 0\), the ranges \(0 < u < \sqrt{G}/2\) and \(\sqrt{G}/2 < u < \infty\) of the variable \(u\) correspond to the ranges \(\infty < T < \sqrt{|G|}\) and \(|G| < T < \infty\) of the variable \(T\). Hence it can be seen that the regions \(U^+_+\) and \(U^{-}_-\) are identified in coordinates \(\{z, T\}\). Similarly, the ranges \(-\infty < u < -\sqrt{G}/2\) and \(-\sqrt{G}/2 < u < 0\) of the variable \(u\) correspond to the ranges \(-\infty < T < -\sqrt{|G|}\) and \(-\sqrt{|G|} < T < -\infty\) of the variable \(T\). Therefore, the regions \(U^+_+\) and \(U^-_+\) in coordinates \(\{z, T\}\) are identified as well.

Note that the regular surfaces \(u = \pm \sqrt{|G|}/2\) correspond to the extreme values of time \(T = \pm \sqrt{|G|}\) and the curvature invariant \(I = 72/G^2\) (see B). We also note that 3-spheres \(T = \pm \sqrt{|G|}, z = \text{const}\) (with respect to the metric (6.14)) are similar to the event horizon of the Schwarzschild metric in GR. However, a passage into the region \(|T| < \sqrt{|G|}\) which is similar to a passage into the \(T\)-region of the Schwarzschild solution of GR is forbidden in the classical theory. This is because of such a change of coordinate sense that we should perform the replacement \(z \rightarrow T, T \rightarrow z\). Then the metric will depend on the fifth coordinate \(z\), that does not correspond to the problem set up.

Hence it follows that at a negative scalar charge the Lorentz-invariant Universe in the KK theory exists only when coordinate time \(|T| > \sqrt{|G|}\), disappearing at \(T = -\sqrt{|G|}\) and arising at \(T = \sqrt{|G|}\) with spatial section as an isotropic expanding 3-sphere \(S^3\) with initial radius \(\sqrt{|G|}\).

According to (5.5), in coordinates \(\{v, z\}\) the \(V\)-region is described by the metric

\[
(5) ds^2 = -\left(\frac{1 - G/4v^2}{1 + G/4v^2}\right)^2 dz^2 - \left(1 + \frac{G}{4v^2}\right)^2 (dv^2 + v^2 d\Omega^2).
\] (6.16)

The point \(v = 0\) is a regular point of the metric. The character of a singularity of the surface \(v = \sqrt{|G|}/2\) depends on a sign of \(G\). In the case \(G > 0\) we have a surface where extreme curvature is reached, \(I_{\text{max}} = 72/G^2\) when \(v^2 = -G/4\), and in the case \(G < 0\) we have a surface of singular curvature \(I \rightarrow \infty\) when \(v^2 \rightarrow -G/4\) (see B).
If one uses a coordinate
\[ R = v + G/4v, \] (6.17)
the metric (6.16) takes the form
\[ (5) ds^2 = -\left(1 - G/R^2\right) dz^2 - \left(1 - G/R^2\right)^{-1} dR^2 - R^2 d\Omega^2_-. \] (6.18)

Hence it can be seen that coordinates \{z, R\} in the V-region are an analogous to curvature coordinates in the R-region of the Schwarzschild solution in GR. The coordinate R sense is a radius of the three-dimensional pseudosphere \( S(1, 2) \) with the induced metric \( R^2 d\Omega^2_- \). This metric is defined in the stereographic coordinates by the formula (2.7).

It is easy to see that in the case \( G > 0 \) the expression in the right-hand side of (6.17) is invariant with respect to the inversion (6.8), therefore the map of regions (6.11) occurs without changing the radial coordinate \( R \). Hence it follows that at the transformation (6.17) we have the regions \( V_- \) and \( V_+ \) overlapping, since these regions are mapped into the same region \( \sqrt{G} \leq R < \infty \). In addition, the minimum radius \( R = \sqrt{G} \) of the pseudosphere \( S(1, 2) \) corresponds to the regular surface \( v = \sqrt{G}/2 \) with the curvature invariant extremum \( I_{\max} = 72/G^2 \) on it (see B). The pseudosphere \( R = \sqrt{G} \) (with respect to the metric (6.18)) is similar to the event horizon of the Schwarzschild metric. The region \( R < \sqrt{G} \) is nonphysical because of the violation of the signature conditions. Therefore, the region \( R < \sqrt{G} \) falls out of the range of definition of the considered space that is characteristic for a so called topological texture [24]. In this case there are 3-spheres which are not contractible in a point. However, in the case under consideration we have an exotic case of the noncontractible 3-pseudospheres \( S(1, 2) \) instead of 3-spheres \( S^3 \).

If \( G < 0 \), the expression in the right-hand side (6.17) changes a sign at the inversion (6.8). Therefore, the map of regions (6.11) occurs by changing a coordinate \( R \) sign. Besides, it the subregions \( V_- \) and \( V_+ \) exchange their places. The range \( 0 < v < \infty \) of variable \( v \) corresponds to the range \( -\infty < R < \infty \) of the variable \( R \). The central singularity \( R = 0 \) of the metric (6.18) corresponds to the surface \( v = \sqrt{|G|}/2 \) for the metric (6.16) and is invariant with respect to the inversion (6.8).

Let us now consider the regularity conditions for \( V^{(5)} \) on the “horizon” \( R = \sqrt{G} \) for (6.18) that corresponds \( v = \sqrt{G}/2 \) (\( G > 0 \)) for (6.16). With this end in view we consider the induced metric
\[ dl^2 = \psi^2(v) dv^2 + V^2(v) dz^2 = \left(1 + \frac{G}{4v^2}\right)^2 dv^2 + \left(\frac{1 - G/4v^2}{1 + G/4v^2}\right)^2 dz^2 \] (6.19)
on the surface \( V^{(2)} \subset V^{(5)} \): \( y^i = \text{const} \) when \( d\Omega^2_- = 0 \) in the metric (6.16). The space \( V^{(2)} \) with the metric (6.19), as locally Euclidean, can be locally embedded into \( R^{(3)} \) as a surface of revolution (see, for example, [25]). Hence it follows that the space \( V^{(2)} \) is a closed space with respect to \( z \).

This surface radius
\[ r_v \sim V(v) = \frac{1 - G/4v^2}{1 + G/4v^2} \quad (\sqrt{|G|}/2 \leq v < \infty) \]
is a monotonically decreasing quantity as \( v \) decreases with the range \( \infty > v > \sqrt{|G|}/2 \) and \( r_v = 0 \) when \( v = \sqrt{G}/2 \) \((G > 0)\). Thus, we have a conical surface of revolution. With the help of transformation

\[
    r = r(v) = \int \frac{\psi(v)dv}{V(v)} = v + \frac{G}{4v} + \sqrt{G} \ln \left( \frac{2v - \sqrt{G}}{2v + \sqrt{G}} \right)
\]  

(6.20)

conserving a symmetry the metric (6.19) can be reduced to a conformally flat form \( V \) must be periodic with the range of changing \( 0 \leq r < \pi \). Hence this vacuum-like configuration of GR can be considered as a manifestation of the vacuum-like configuration of GR for the interacting gravitational and conformal scalar fields. Thus, we have a conical surface of revolution. With the help of transformation

\[
    r = r(v) = \int \frac{\psi(v)dv}{V(v)} = v + \frac{G}{4v} + \sqrt{G} \ln \left( \frac{2v - \sqrt{G}}{2v + \sqrt{G}} \right)
\]  

(6.20)

conserving a symmetry the metric (6.19) can be reduced to a conformally flat form \( dl^2 = V^2(r)(dr^2 + dz^2) \). The additional transformation \( r = \sqrt{G} \ln \rho \) leads to the following metric

\[
    dl^2 = W^2(\rho)(d\rho^2 + \rho^2 d\alpha^2) = Ge^{-2v/\sqrt{G}+\sqrt{G}/2v} \left( \frac{1 + \sqrt{G}/2v}{1 + G/4v^2} \right)^2 (d\rho^2 + \rho^2 d\alpha^2)
\]  

(6.21)

where \( \alpha = z/\sqrt{G} \) \((G > 0)\). The introduced here function \( \rho \) and conformal factor \( W(\rho) \) are already positive and analytical. In order to avoid a conical singularity at the “horizon” \( R = \sqrt{G} \) \((\rho = 0)\), \( \alpha \) must have the period \( 2\pi \), this implying that the extra coordinate \( z \) must be periodic with the range of changing \( 0 \leq z \leq 2\pi \sqrt{G} \).

From this analysis it is apparent that for the section of the space \( V^{(5)} \) at \( y^i = \text{const} \ (i = 0,1,2) \) the \( v-z \) surface has the shape of a semi-infinite conic surface of revolution when \( G > 0 \). This conical surface is smoothly rounded in its vertex and is open at infinity; thus, it is topologically equivalent to \( R^2 \). Therefore, the \( V \)-region has a structure \( R^2 \times S(1,2) \) with an isotropic throat of the radius \( R = \sqrt{G} \) \((G > 0)\) as an interior boundary. The above conic surface radius is \( r_R = \sqrt{G(1 - G/R^2)} \) with the range \( 0 < r_R < r_{\infty} = \sqrt{G} \) when \( \sqrt{G} < R < \infty \). The maximum \( r_R \) can be interpreted as a compactification radius \( r_\alpha = r_{\infty} = \sqrt{G} \) of the space \( V^{(5)} \). We see that what can be called a spontaneous compactification takes place here.

Note that another mechanism of the spontaneous compactification of \( V^{(5)} \) was considered in [26] where \( z \)-periodicity was an effect induced by a moving tachyon.

Since \( U \)- and \( V \)-regions are subregions of \( V^{(5)} \) with a common smooth boundary \( \zeta \equiv \eta_{\mu\nu}x^\mu x^\nu = 0 \), the condition \( z = \alpha \sqrt{G} \ (0 \leq \alpha < 2\pi) \) should be satisfied for the whole space \( V^{(5)} \). Therefore, when \( G > 0 \) and in accordance with (5.6), we can write

\[
    (5)ds^2 = -\left(1 + \frac{G/4\zeta}{1 - G/4\zeta}\right)^2 Gd\alpha^2 + \left(1 - \frac{G}{4\zeta}\right)^2 \eta_{\mu\nu}dy^\mu dy^\nu.
\]  

(6.22)

This metric is a Lorentz-invariant mode of the equations of five-dimensional gravity for which both cylindricity and compactification of the fifth dimension are fulfilled dynamically. Therefore the space under consideration is the KK space.

As already has been noted, after the dimensional reduction the model is reduced to the vacuum-like configuration of GR for the interacting gravitational and conformal scalar fields. Hence this vacuum-like configuration of GR can be considered as a manifestation of a Lorentz-invariant mode of five-dimensional gravity for which the Kaluza-Klein postulates are realized dynamically.

**VII. REDUCTION OF ACTION FOR A GEODESIC LINE ON \( V^{(5)} \)**

The geodesic equations in \( V^{(5)} \) follow from the action principle \( \delta^{(5)}S = 0 \) for an action
\( (5) S = - \int (5) ds = \int (5) Ld\lambda \)  

(7.1)

where \( \lambda \) is a parameter on the world line. A Lagrangian \( (5) L \) for the metric (2.4) has the form

\[ (5) L = - \sqrt{\psi^2 \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - V^2 \dot{z}^2} \]

(7.2)

where \( \dot{x}^\mu = dx^\mu / d\lambda \).

Let us note that the coordinate \( z \) is cyclical and can be eliminated from the Lagrangian \( (5) L \). For this purpose we introduce the Routh function \( R \) [27]:

\[ R = g \dot{z} - (5) L = - \psi^2 \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu / L \]

(7.3)

where

\[ g = \partial (5) L / \partial \dot{z} = - V^2 \dot{z} / L = \text{const} \]  

(7.4)

is an integral of motion. The Routh function \( R \) is a Lagrange function with respect to the coordinates \( x^\mu \) and is a Hamiltonian function with respect to the coordinate \( z \). Therefore, if only trajectories in the space-time \( V^4 \) are of interest for us, it is possible to consider the function \( R \) as a reduced Lagrangian \( (4) L \) containing \( g \) as a parameter.

Excluding the velocity \( \dot{z} \) from the Routh function (7.3) with the help of (7.4), we obtain

\[ (4) L = R = \psi \sqrt{(1 + g^2 V^{-2}) \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \]  

(7.5)

Taking into account (6.22), we find the Lagrangian of a test particle

\[ (4) L = H(\zeta) \sqrt{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \]  

(7.6)

where

\[ H(\zeta) = \left( 1 - \frac{G}{4\zeta} \right) \sqrt{1 + g^2 \left( \frac{1 - G/4\zeta}{1 + G/4\zeta} \right)^2} \]  

(7.7)

Using the Routh function (7.3), we can rewrite the five-dimensional action (7.3) as

\[ (5) S = \int (5) Ld\lambda = \int (g \dot{z} - R) d\lambda \].

(7.8)

Hence, omitting a total differential and taking into account (7.5) and (7.6), we come to the four-dimensional Lorentz-invariant action

\[ (5) S \to (4) S = - \int R d\lambda = - \int (4) Ld\lambda = - \int H(\zeta) ds_0 \].

(7.9)

The four-dimensional equations of motion

\[ \frac{d}{ds_0} \left( \eta_{\mu\nu} \frac{dx^\mu}{ds_0} \right) = \eta_{\mu\nu} \frac{\partial H}{\partial x^\nu} \]  

(7.10)

follow from the constructed action where \( ds_0^2 = \eta_{\mu\nu} dx^\mu dx^\nu \).

The obtained action can be interpreted as an action for a particle of a variable mass \( m = H(\zeta) \) in the flat space-time. In the case \( |\zeta| \gg 0 \) it is reduced to an action for a free test particle of mass \( m = \sqrt{1 + g^2} \) in special relativity. In the case \( \zeta \sim G/4 \) we have accelerated motion and in the case \( \zeta \to G/4 \) we deal with ultrarelativistic motion. Let us suppose that the compactification radius of \( V^5 \) has an order of Planck length \( L_{Pl} \): \( r_c = \sqrt{G} \sim L_{Pl} \). Then the effects of five-dimensional gravity of the considered type can show themselves only at the Planck scales.
VIII. CONCLUSION

The Lorentz-invariant configuration of a scalar field with conformal connection constructed in the paper has the vanishing conformally invariant energy-momentum tensor on the flat space-time background. Therefore, the obtained solution is interpreted as a vacuum-like configuration in GR without a cosmological constant.

On the other hand, it is shown that the Lorentz-invariant solution of equations of the KK theory can be considered as a Lorentz-invariant mode of equations of the five-dimensional gravity (without external sources) in which the cylindricity condition and closedness with respect to the extra coordinate $z$ are realized dynamically. After the dimensional reduction and appropriate conformal mapping the model is reduced to the above vacuum configuration of GR with the conformally invariant scalar field.

Therefore, the vacuum in GR can be interpreted as a manifestation of the Lorentz-invariant empty space of the KK theory which in turn is the Lorentz-invariant mode in the five-dimensional gravity. The approach discussed above, as well as the standard theory, leads to the one-parameter family of vacuum configurations. The parameter is a charge of the scalar field $G$ associated with the geometrical parameter of the model, that is the compactification radius of $r_c = \sqrt{G}$ of the Kaluza-Klein space $V^{(5)}$. In addition a five-dimensional action for a geodesic line in $V^{(5)}$ is reduced to a four-dimensional action by the dimensional reduction. We can interpret the obtained four-dimensional effective action as an action for a particle of the variable mass $m = H(\zeta)$ in the flat space-time.

The offered approach is based on the five-dimensional version of the generalized Birkhoff theorem proved in the paper. According to the theorem, the Lorentz-invariant metric $g_{AB}$ satisfying the vacuum five-dimensional Einstein equations admits the extra Killing vector $\xi$. This vector defines an extra symmetry of the five-dimensional Lorentz-invariant vacuum space $V^{(5)}$ corresponding to the Kaluza-Klein cylinder condition. The conservation law $G = L^2(\nabla L)^2 - \epsilon L^2 = \text{const}$ corresponding to this symmetry is similar to the mass function for the spherically-symmetric gravitational fields in GR. After the dimensional reduction of the five-dimensional Hilbert action and corresponding conformal mapping, $G$ is identified with a charge of the conformally invariant scalar field. Therefore, the function $G$ can be called a charge function, and the conservation law corresponds to the charge conservation for the conformally invariant scalar field.

The regularity condition of the space $V^{(5)}$ for the obtained metric implying the conical singularity absence leads to the closure condition of $V^{(5)}$ with respect to the fifth coordinate $z$ with the period $L = 2\pi\sqrt{G}$. The metric of the surface $V^{(2)} \subset V^{(5)}$ ($y^i = \text{const}, i = 0, 1, 2$) describes a semi-infinite conic surface of revolution with the smoothly rounded vertex. This conic surface radius is $r_R = \sqrt{G(1 - G/R^2)}$ with the range $0 < r_R < r_\infty = \sqrt{G}$ while $\sqrt{G} < R < \infty$. The maximum magnitude of $r_R$ which is reached at $R = \infty$ can be interpreted as a compactification radius $r_c = r_\infty = \sqrt{G}$ of the five-dimensional space $V^{(5)}$. Finally we note that the considered five-dimensional space for sufficiently large $\zeta = \eta_{\mu\nu} x^\mu x^\nu$ becomes a flat space and its manifestation, the physical space-time, becomes a usual empty space-time of special relativity. Thus, the constructed five-dimensional cosmological model describes the mechanism of a dynamic spontaneous compactification of the five-dimensional space and its effective manifestation as a vacuum-like configuration of GR containing the scalar field with a vanishing conformally invariant energy-momentum tensor on the Minkowski space-time.
In a sense, the Lorentz-invariant Kaluza-Klein spaces can be understood as the limiting case of spaces of the five-dimensional GR admitting maximally symmetric Lorentz-invariant space-time sections. We consider the obtained solution as the “lowest” nontrivial state of the five-dimensional geometry, which manifest themselves in the form of the vacuum-like configuration GR. Its nonsymmetric perturbations generated by classical or quantum excitations can be treated as induced matter [15]. But we must first study the problem of the stability of the space $V^{(5)}$ under small perturbations of the type $\delta g_{AB} e^{nz}$ by analogy with the Schwarzschild metric. Then these perturbations for stable modes can be interpreted as matter induced by the five-dimensional geometry.

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**APPENDIX A: GENERALIZED STEREOGRAPHIC COORDINATES**

The generalized stereographic coordinates for three-dimensional “angular” parts of the Minkowski metric can be introduced via using a generalized stereographic projection [12], [13]. Taking into account definition $u_\epsilon^2 = \epsilon \eta_{\mu\nu} x^\mu x^\nu$, we have

$$u_\epsilon^\mu = \epsilon \frac{\partial u_\epsilon}{\partial x^\mu} = \frac{\eta_{\mu\nu} x^\nu}{u_\epsilon} = \frac{x^\mu}{u_\epsilon}. \quad \text{(A1)}$$

This 4-vector is time-like ($\eta_{\mu\nu} u_\epsilon^\mu u_\epsilon^\nu = 1$) when $\epsilon = 1$ and space-like ($\eta_{\mu\nu} u_\epsilon^\mu u_\epsilon^\nu = -1$) when $\epsilon = -1$. The second derivative of $u_\epsilon$:

$$\epsilon \frac{\partial^2 u_\epsilon}{\partial x^\mu \partial x^\nu} = \frac{\partial u_\epsilon^\mu}{\partial x^\nu} = -\frac{1}{u_\epsilon} h_{\mu\nu} \quad \text{(A2)}$$

leads to the induced metric

$$h_{\mu\nu} = \epsilon u_\epsilon^\mu u_\epsilon^\nu - \eta_{\mu\nu}, \quad u_\epsilon^\mu h_{\mu\nu} = 0 \quad \text{(A3)}$$

defined on the surface $u_\epsilon^2 = \epsilon \eta_{\mu\nu} x^\mu x^\nu = \text{const}$. Hence it follows $\partial u_\epsilon^\mu / \partial x^\mu = u_\epsilon^{\mu,\mu} = 3/u_\epsilon$.

We introduce “degenerated” angular coordinates $u_\epsilon^\mu$

$$u_\epsilon^\mu = x^\mu / u_\epsilon = \eta^{\mu\nu} u_\epsilon^\nu. \quad \text{(A4)}$$

The coordinates $\{u_\epsilon^\mu\} = \{u_+^\mu, u_-^\mu\}$ satisfy to relations

$$\eta_{\mu\nu} u_+^\mu u_+^\nu = (u_+^0)^2 - (u_+^1)^2 - \delta_{ab} u_+^a u_+^b = \epsilon \quad (a, b = 2, 3), \quad \text{(A5)}$$

and $\eta_{\mu\nu} u_+^\mu dx^\nu = du_+^\epsilon, \eta_{\mu\nu} u_+^\mu du_+^\nu = 0$. Using these formulae and (A3), we obtain for the Minkowski metric
\[ (4) ds_0^2 = \epsilon du^2 - h_{\mu\nu} dx^\mu dx^\nu = \epsilon du^2 - u^2 (3) d\Omega^2_\epsilon \]

where
\[ (3) d\Omega^2_\epsilon = h_{\mu\nu} du^\mu du^\nu = -(\eta_{\mu\nu} du^\mu du^\nu)_{|u^2=\text{const}} \]

are induced metrics on the surfaces \( \eta_{\mu\nu} u^\mu u^\nu = \epsilon \). They represent “angular parts” of the Minkowski metrics inside and outside the light cone.

Now we proceed from the degenerated coordinates \( u^\mu \) to the nondegenerate generalized stereographic coordinates \( y^i \) by means of the formulae
\[ u^\mu = \frac{y^\mu}{1 - \epsilon S_\epsilon^2/4}, \quad u^1 = \frac{y^1}{1 + S_\epsilon^2/4}, \quad u^0 = \frac{y^1}{1 + S_\epsilon^2/4} \]

where \( S_\epsilon^2 = \epsilon (y^1)^2 + (y^2)^2 + (y^3)^2 \). Then, according to (A5), we have
\[ u^0_+ = \frac{1 + S_\epsilon^2/4}{1 - S_\epsilon^2/4}, \quad u^1_- = \frac{1 - S_\epsilon^2/4}{1 + S_\epsilon^2/4}. \]

Now the angular part of the Minkowski metrics (A7) takes the form (2.7). This metric describes the three-dimensional space of constant curvature \( (K_0 = -\epsilon) \).

**APPENDIX B: THE CURVATURE TENSOR, THE RICCI TENSOR AND CURVATURE SCALAR OF \( V^{(5)} \)**

In order to find the curvature tensor of space \( V^{(5)} \) with the metric (2.4) at first we shall write out derivatives of the variable \( \zeta = \eta_{\mu\nu} x^\mu x^\nu \):
\[ \zeta, \mu = \frac{\partial \zeta}{\partial x^\mu} = 2\eta_{\mu\nu} x^\nu = 2x^\mu, \quad \zeta, \mu\nu = \frac{\partial^2 \zeta}{\partial x^\mu \partial x^\nu} = 2\eta_{\mu\nu}. \]

Taking into account these formulae, we obtain the Christoffel symbols for the metric (2.4)
\[ \Gamma^5_{55} = \frac{V_c}{V}, \quad \Gamma^5_{5\mu} = \frac{2V_c}{\psi^2} x^\mu, \quad \Gamma^\mu_{55} = \frac{2V_c}{\psi^2} x^\mu, \quad \Gamma^5_{\mu\nu} = \frac{\psi^2}{V^2} \eta_{\mu\nu}, \]

\[ \Gamma^\nu_5 = \frac{\psi x^\nu}{\psi}, \quad \Gamma^\mu_{\nu\rho} = \frac{\psi}{\psi^2} \left( \delta^\mu_\rho x_\nu - \delta^\mu_\nu x_\rho - \eta_{\nu\rho} x^\mu \right) \]

where the indexes “\( \zeta \)” and “\( z \)” denote the partial derivatives \( \partial/\partial \zeta \) and \( \partial/\partial z \), accordingly.

The components of the curvature tensor for the metric (2.4) have the form
\[ (5) R^z_{\mu\nu\rho} = \left( \frac{\psi^2}{\psi} - \frac{\psi^2}{\psi^2} \right) \eta_{\mu\nu} + 4 \left( -\frac{V_c}{V} + \frac{2V_c}{\psi^2} \right) x_\mu x_\nu, \]
\[ (5) R^\mu_{\nu\rho\sigma} = 2 \left( \frac{\psi^2}{\psi} - \frac{\psi^2}{\psi^2} \right) \left( x^\mu \eta_{\nu\rho} - x_\nu \delta^\mu_\rho \right), \]
\[ (5) R^\mu_{\nu\rho\sigma} = \frac{4V_c^2}{\psi^2} + \frac{4\psi}{\psi^2} \left( \frac{\psi}{V} \right) \left( \delta^\mu_\sigma \eta_{\nu\rho} - \delta^\mu_\rho \eta_{\nu\sigma} \right) + \]
\[ + 4 \left( \frac{\psi^2}{\psi} - \frac{2\psi^2}{\psi^2} \right) \left( x^\mu (x_\sigma \eta_{\nu\rho} - x_\rho \eta_{\nu\sigma}) - x_\nu (x_\sigma \delta^\mu_\rho - x_\rho \delta^\mu_\sigma) \right). \]
Hence we find the components of the Ricci tensor and curvature scalar

\[ \begin{align*}
(5) R_{zz} & = -\frac{4V^2}{\psi^2} \left( \frac{\psi \psi_{zz}}{V^2} - \frac{\psi \psi_z V_z}{V^3} - \frac{2\zeta \psi \zeta V_z}{\psi V} - \frac{2V_{z}}{V} - \frac{\zeta V_{z\zeta}}{V} \right), \\
(5) R_{\mu z} & = -6 \left( \frac{\psi \zeta}{\psi} - \frac{\psi \psi_z \zeta}{\psi^2} - \frac{\psi \zeta V_z}{\psi V} \right) x_\mu, \\
(5) R_{\mu \nu} & = 4 \left( -\frac{2\psi \zeta V_z}{\psi^2} + \frac{4\psi^2}{\psi^2} - \frac{V_{z\zeta} V_z}{V^2} + \frac{2\psi \zeta V_z}{\psi V} \right) x_\mu x_\nu + \\
& + \left( \frac{\psi \psi_{zz}}{V^2} - \frac{\psi \psi_z V_z}{V^3} + \frac{3\psi^2}{V^2} - \frac{4\zeta \psi \zeta}{\psi^2} - \frac{4\psi^2}{\psi^2} - \frac{12\psi \zeta}{\psi} - \frac{4\psi \zeta V_z}{\psi V} - \frac{2V}{V} \right) \eta_{\mu \nu},
\end{align*} \]  

At last we find components of the Einstein tensor

\[ \begin{align*}
(5) G_{zz} & = -6V^2 \left( \frac{2\zeta \psi \zeta}{\psi} + \frac{4\psi^2}{\psi^2} - \frac{\psi^2}{V^2} \right), \\
(5) G_{\mu z} & = (5) R_{\mu z} = -6 \left( \frac{\psi \zeta}{\psi} - \frac{\psi \psi_z \zeta}{\psi^2} - \frac{\psi \zeta V_z}{\psi V} \right) x_\mu, \\
(5) G_{\mu \nu} & = 4 \left( -\frac{2\psi \zeta V_z}{\psi^2} + \frac{4\psi^2}{\psi^2} - \frac{V_{z\zeta} V_z}{V^2} + \frac{2\psi \zeta V_z}{\psi V} \right) x_\mu x_\nu - \\
& - \left( \frac{3\psi \psi_{zz}}{V^2} + \frac{3\psi^2}{V^2} - \frac{3\psi \psi_z V_z}{V^3} - \frac{4\zeta \psi \zeta}{V^2} - \frac{4\psi^2}{\psi^2} - \frac{12\psi \zeta}{\psi} - \frac{4\psi \zeta V_z}{\psi V} \right) \eta_{\mu \nu}.
\end{align*} \]  

Further we introduce the induced metric (A3) and projection operator $P^\mu_\nu = -h^\mu_\nu = -\epsilon u^\mu_* u^\nu_* + \delta^\mu_\nu$ where $u^\mu_* P^\nu_\mu = 0$, $P^\mu_\nu P^\nu_\rho = P^\mu_\rho$. Here $\epsilon = \zeta / |\zeta|$, $\zeta = \eta_{\mu \nu} x^\mu x^\nu = \epsilon u^2_* u^\nu_*$, and $u^\mu_* u^\nu_* = \epsilon$. Using these formulae, we can rewrite components of the curvature tensor (B4), (B5), and (B6) in $(1+1+3)$-decomposed form [22]

\[ \begin{align*}
(5) R^z_{\mu z \nu} & = \epsilon \left( \frac{\psi \psi_{zz}}{V^2} - \frac{\psi \psi_z V_z}{V^3} - \frac{4\zeta \psi \zeta}{V^2} + \frac{4\zeta \psi \zeta}{\psi V} - \frac{2V_{z}}{V} \right) u^\nu_* u^\epsilon_* + \\
& + \left( -\frac{\psi \psi_{zz}}{V^2} + \frac{\psi \psi_z V_z}{V^3} + \frac{4\zeta \psi \zeta}{\psi V} + \frac{2V_{z}}{V} \right) h_{\mu \nu}, \\
(5) R^\mu_{\nu \rho z} & = 2\psi^{-1} \sqrt{\epsilon \zeta} \left( \psi z_\zeta - \psi^{-1} \psi \zeta \psi z - \psi z V^{-1} \zeta \right) \left( u^\nu_* h^\mu_\rho - u^\rho_* h_{\nu \rho} \right), \\
(5) R^\mu_{\nu \rho \sigma} & = \left( \frac{4\zeta \psi \zeta}{\psi^2} + \frac{4\psi \zeta}{\psi^2} - \frac{\psi^2}{V^2} \right) \left( h^\mu_\sigma h_{\nu \rho} - h^\mu_\rho h_{\nu \sigma} \right) + \\
& + 4\epsilon \left( \frac{\zeta \psi \zeta}{\psi} - \frac{\zeta \psi \zeta}{\psi^2} + \frac{\psi \zeta}{\psi} - \frac{\psi^2}{4V^2} \right) \left( u^\mu_* (u^\nu_* h_{\rho \sigma} - u^\rho_* h_{\nu \sigma}) - u^\nu_* (u^\mu_* h^\rho_\sigma - u^\rho_* h^\mu_\sigma) \right).
\end{align*} \]
Therefore, the components of the curvature tensor (B24), (B26) can be rewritten as

\[ R_{\mu\nu\rho\sigma} = G^2 \left( h_{\sigma\rho} h_{\nu\mu} - h_{\rho\sigma} h_{\nu\mu} \right) + 4 \epsilon \left( \frac{\partial^2 \psi}{\partial \psi^2} - \frac{\partial^2 \phi}{\partial \psi^2} \right) \left( u^\mu_i (u^\nu_i h_{\rho\sigma} - u^\nu_i h_{\sigma\rho}) - u^\nu_i (u^\nu_i h_{\rho\sigma} - u^\nu_i h_{\sigma\rho}) \right). \]

In our case the five-dimensional Einstein equations (2.1), with taking into account (2.15)–(2.18), can be written as

\[ u^\mu_i (u^\nu_i h_{\rho\sigma} - u^\nu_i h_{\sigma\rho}) - u^\nu_i (u^\nu_i h_{\rho\sigma} - u^\nu_i h_{\sigma\rho}) = 0. \]

Further, for components of the curvature tensor (B14)–(B16) we have

\[ R_{\mu\nu\rho\sigma} = 0 \] and

\[ R_{\mu\nu\rho\sigma} = 4 \epsilon \frac{\partial^2 \psi}{\partial \psi^2} (u^\mu_i (u^\nu_i h_{\rho\sigma} - u^\nu_i h_{\sigma\rho}) - u^\nu_i (u^\nu_i h_{\rho\sigma} - u^\nu_i h_{\sigma\rho})). \]

Using the conservation law \( G = \text{const} \) and equations (B19) and (B21), we have

\[ \psi \left( 4 \epsilon \frac{\partial^2 \psi}{\partial \psi^2} \right) = \frac{\partial \psi}{\partial \psi^2}. \]

Therefore, the components of the curvature tensor (24), (26) can be rewritten as

\[ R_{\mu\nu\rho\sigma} = G^2 \frac{\partial^2 \psi}{\partial \psi^2} (3 \epsilon u^\mu_i u^\nu_i + h_{\mu\nu}) \]

or, using the explicit form of the metric (5.6), we obtain

\[ I = 72G^2 \frac{\partial^2 \psi}{\partial \psi^2} \left( 1 - G \frac{4G}{4\epsilon} \right)^{-8} = 72G^2 \frac{\partial \psi}{\partial \psi^2} \left( u^\mu_i - \epsilon G/4u^\epsilon_i \right)^{-8}. \]
APPENDIX C: FIVE-DIMENSIONAL VERSION OF THE BIRKHOFF THEOREM

In terms of differential geometry the metric tensor of manifold \( V^{(2)} \subset V^{(5)} \): \( y^i = \text{const} \), according to (4.13), has the form

\[
g = \frac{\psi^2(z, \zeta)}{4\zeta} d\zeta \otimes d\zeta - V^2(z, \zeta) dz \otimes dz .
\]  
(C1)

The metric tensor determines the scalar product of vectors \( \vec{A}, \vec{B} \) and one-forms \( \alpha, \beta \):

\[
(\vec{A}, \vec{B}) = \langle \alpha, \beta \rangle = g^{-1}(\alpha, \beta).
\]  
(C2)

Let us introduce a new basis of the differential forms \( \{\omega_L, \omega_\xi\} \) as follows

\[
\omega_L = i_\xi \Omega = -\Omega(. , \bar{\zeta}), \quad \omega_\xi = g(. , \bar{\zeta})
\]  
(C3)

where \( \Omega = (1/2)\psi V(|\zeta|)^{-1/2} d\zeta \wedge dz \) is a two-form of volume on \( V^{(2)} \) and \( \bar{\zeta} \) is the Killing vector (4.14). We have

\[
\omega_L = -\epsilon(L, \zeta d\zeta + L, dz) = -\epsilon dL ,
\]  
(C4)

\[
\omega_\xi = -\frac{\psi^2 L, \zeta d\zeta + 4\zeta V^2 L, dz}{2\psi V \sqrt{\epsilon \zeta}}.
\]  
(C5)

According to the Frobenius theorem [16], [17], we have \( \omega_\xi = -\beta d\bar{\zeta} \) where \( \bar{\zeta} \) and \( \beta \) are some functions \( \{z, \zeta\} \). Using the variables \( \bar{\zeta} \) and \( L \) as new coordinates, we obtain \( \beta \gamma^{\bar{\zeta} \bar{\zeta}} = \beta \langle dL, d\bar{\zeta} \rangle = \epsilon \langle \omega_L, \omega_\xi \rangle = 0 \). Then we find

\[
(\bar{\zeta})^2 = g(\bar{\zeta}, \bar{\zeta}) = -\epsilon (\nabla L)^2 = -\epsilon \gamma^{LL} .
\]  
(C6)

On the other hand,

\[
(\bar{\zeta})^2 = g(\bar{\zeta}, \bar{\zeta}) = \langle \omega_L, \omega_\xi \rangle = \beta^2 \gamma^{\bar{\zeta} \bar{\zeta}} .
\]  
(C7)

Hence we obtain \( \gamma^{LL} = -\epsilon \beta^2 \gamma^{\bar{\zeta} \bar{\zeta}} \). Therefore, we have \( \gamma_{\bar{\zeta} \bar{\zeta}} = -\epsilon \beta^2 \gamma^{LL} \). In coordinates \( \{\bar{\zeta}, L\} \) the metric tensor (C1) of space \( V^{(2)} \) becomes

\[
g = \gamma^{LL}(dL \otimes dL - \epsilon \beta^2 d\bar{\zeta} \otimes d\bar{\zeta}) , \quad g^{-1} = \gamma^{-1}_{LL}(\partial_L \otimes \partial_L - \frac{\epsilon}{\beta^2} \partial_{\bar{\zeta}} \otimes \partial_{\bar{\zeta}}) .
\]  
(C8)

Then for the Killing vector (4.14) we find \( \bar{\zeta} = g^{-1}(., \omega_\xi) = \epsilon \beta^{-1} \gamma^{-1}_{LL} \partial_{\bar{\zeta}} \). From \( (L, L) \)-component of the Killing equations for the metric (C8) it follows \( \gamma^{LL} = \gamma^{LL}(L) \). Note that \( (z, z) \)-component of the Killing equations are satisfied identically. From \( (z, L) \)-component of the Killing equations the relation \( \partial (\beta \gamma^{LL}) / \partial L = 0 \) is obtained. Hence \( \beta = f(\bar{\zeta}) / \gamma^{LL} \) where \( f(\bar{\zeta}) \) is a certain function of \( \bar{\zeta} \). In addition, we can suppose that \( f(\bar{\zeta}) = 1 \). Thus, in the coordinates \( \{\bar{\zeta}, L\} \) the metric (4.11) takes the form
The unknown function $\gamma_{LL}$ can be found from the conservation law (4.15):

$$\gamma^{LL} = \gamma^{ab} L_a L_b = (\nabla L)^2 = \epsilon + G L^{-2}. \quad (C10)$$

Finally the metric (C9) can be written as [12]

$$(5) \, ds^2 = -\epsilon \frac{1}{\gamma_{LL}} d\tilde{z}^2 + \gamma_{LL} \left( \frac{dL}{du} \right)^2 du^2 - L^2 (3) d\Omega^2_\epsilon. \quad (C11)$$

Let us now return from the coordinates $\{\tilde{z}, L\}$ to the conformal coordinates $\{z, u\}$ in which the space-time part of the metric takes the conformally flat form (2.8). As a result of substituting $L = L(u)$ into (C9), we get

$$(5) \, ds^2 = -\epsilon \left( \epsilon + G/L^2 \right) d\tilde{z}^2 + \left( \epsilon + G/L^2 \right)^{-1} dL^2 - L^2 (3) d\Omega^2_\epsilon. \quad (C12)$$

Comparing this metric with the metric (2.8), we conclude that

$$\gamma_{LL} L_{u\epsilon}^2 = \epsilon \psi^2, \quad L = u\epsilon \psi, \quad V^2 = \epsilon/\gamma_{LL}. \quad (C13)$$

From here it follows that $\pm u\epsilon \sqrt{\epsilon \gamma_{LL}} dL = L du\epsilon$. In the case $\epsilon \gamma_{LL} > 0$ it has the a solution in the implicit form

$$u\epsilon = C \exp \int dL \left( \pm \sqrt{\epsilon \gamma_{LL}} / L \right). \quad (C14)$$

Thus, from the metric (C9) we can return to the metric (2.8) wherein all functions do not depend on a fifth coordinate $z$ this proves the statement.
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