ARITHMETIC GENUS OF INTEGRAL SPACE CURVES

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Abstract. We give an estimation for the arithmetic genus of an integral space curve, which are not contained in a surface of degree $k - 1$. Our main technique is the Bogomolov-Gieseker type inequality for $\mathbb{P}^3$ proved by Macrì.

1. Introduction

A classical problem, which goes back to Halphen [6], is to determine, for given integers $d$ and $k$, the maximal genus $G(d, k)$ of a smooth projective space curve of degree $d$ not contained in a surface of degree $< k$. This problem is actually very natural, and has been investigated by many people (see [5, 7, 8, 9]).

In this paper, we consider the same problem for an integral space curve. Our main result is:

Theorem 1.1. Let $C$ be an integral complex projective curve in $\mathbb{P}^3$ of degree $d$. Let $p_a(C)$ be its arithmetic genus. If $C$ is not contained in a surface of degree $< k$. Then

$$p_a(C) \leq \begin{cases} \frac{2d^2}{3} + \frac{1}{3}d(k - 6) + 1, & \text{if } k^2 < d, \\ d(\sqrt{d} - 2) + 1, & \text{if } k^2 \geq d. \end{cases}$$

The idea of the proof of Theorem 1.1 is to establish the tilt-stability of $\mathcal{I}_C$ via computing its walls, then the Bogomolov-Gieseker type inequality for $\mathbb{P}^3$ proved by Macrì [12] implies Theorem 1.1. This Bogomolov-Gieseker type inequality naturally appears in the construction of Bridgeland stability conditions on threefolds (cf. [4, 3, 2]). There are also some other interesting applications of the Bogomolov-Gieseker type inequality in [1] and [13].

Our tilt-stability of $\mathcal{I}_C$ can also gives a version of the Halphen Speciality Theorem:

Theorem 1.2. Let $C \subset \mathbb{P}^3$ be an integral complex projective degree $d$ curve not contained in any surface of degree $< k$. Then $h^2(\mathcal{I}_C(l)) = h^1(\mathcal{O}_C(l)) = 0$, if $l > \frac{2d}{k} - 4$ when $k^2 < d$, or $l > 2\sqrt{d} - 4$ when $k^2 \geq d$.

Our paper is organized as follows. In Section 2 we review basic properties of tilt-stability, the conjectural inequality proposed in [3, 2] and variants of the classical Bogomolov-Gieseker inequality satisfies by tilt-stable objects. Then in Section 3 the tilt-stability of $\mathcal{I}_C$ has been established via computing its walls. Finally, we show the proof of Theorem 1.1 and 1.2 in Section 4.
Notation. In this paper, we will always denote by $C$ an integral projective curve in the three dimensional complex projective space $\mathbb{P}^3$ and by $I_C$ its ideal sheaf in $\mathbb{P}^3$. We let $p_a(C) := h^1(C, \mathcal{O}_C)$ be the arithmetic genus of $C$. By $X$ we denote a complex smooth projective threefold and by $D^b(X)$ its bounded derived category of coherent sheaves.

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2. Preliminaries

In this section, we review the notion of tilt-stability for threefolds introduced in [3, 2]. Then we recall the Bogomolov-Gieseker type inequality for tilt-stable complexes proposed there.

Let $X$ be a smooth projective threefold over $\mathbb{C}$, and let $H$ be an ample divisor on $X$. Let $\alpha > 0$ and $\beta$ be two real numbers. We write $\chi^\beta(E) = e^{-\beta H} \chi(E)$ denotes the Chern character twisted by $\beta H$. More explicitly, we have

- $\chi_0^\beta(E) = \chi_0 = \text{rank}$
- $\chi_2^\beta = \chi_2 - \beta H \chi_1 + \frac{\alpha^2}{2} H^2 \chi_0$
- $\chi_3^\beta = \chi_3 - \beta H \chi_2 + \frac{\alpha^2}{2} H^2 \chi_1 - \frac{\alpha^3}{6} H^3 \chi_0$.

Slope-stability. We define the slope $\mu_\beta$ of a coherent sheaf $E \in \text{Coh}(X)$ by

$$\mu_\beta(E) = \begin{cases} +\infty, & \text{if } \chi_0^\beta(E) = 0, \\ \frac{\chi^\beta_2(E)}{\chi^\beta_0(E)}, & \text{otherwise}, \end{cases}$$

Definition 2.1. A coherent sheaf $E$ on $X$ is slope-(semi)stable (or $\mu_\beta$-(semi)stable) if, for all non-zero subsheaves $F \hookrightarrow E$, we have

$$\mu_\beta(F) < (\leq) \mu_\beta(E/F).$$

Harder-Narasimhan filtrations (HN-filtrations, for short) with respect to slope-stability exist in $\text{Coh}(X)$: given a non-zero sheaf $E \in \text{Coh}(X)$, there is a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that: $G_i := E_i/E_{i-1}$ is slope-semistable, and $\mu_\beta(G_1) > \cdots > \mu_\beta(G_n)$. We set $\mu_\beta^+(E) := \mu_\beta(G_1)$ and $\mu_\beta^-(E) := \mu_\beta(G_n)$.

Tilt-stability. There exists a torsion pair $(\mathcal{T}_\beta, \mathcal{F}_\beta)$ in $\text{Coh}(X)$ defined as follows:

$$\mathcal{T}_\beta = \{ E \in \text{Coh}(X) : \mu_\beta^+(E) > 0 \}$$
$$\mathcal{F}_\beta = \{ E \in \text{Coh}(X) : \mu_\beta^-(E) \leq 0 \}$$

Equivalently, $\mathcal{T}_\beta$ and $\mathcal{F}_\beta$ are the extension-closed subcategories of $\text{Coh}(X)$ generated by slope-stable sheaves of positive and non-positive slope, respectively.

Definition 2.2. We let $\text{Coh}^\beta(X) \subset D^b(X)$ be the extension-closure

$$\text{Coh}^\beta(X) = \langle \mathcal{T}_\beta, \mathcal{F}_\beta[1] \rangle.$$
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By the general theory of torsion pairs and tilting [10], \( \text{Coh}^\beta(X) \) is the heart of a bounded t-structure on \( D^b(X) \); in particular, it is an abelian category.

Now we can define the following slope function on \( \text{Coh}^\beta(X) \): for an object \( E \in \text{Coh}^\beta(X) \), we set

\[
\nu_{\alpha,\beta}(E) = \begin{cases} 
+\infty, & \text{if } H^2 \text{ch}_1^\beta(E) = 0, \\
\frac{H^2 \text{ch}_1^\beta(E) - \frac{1}{2} \alpha^2 H^3 \text{ch}_0^\beta(E)}{H^2 \text{ch}_0^\beta(E)}, & \text{otherwise}.
\end{cases}
\]

**Definition 2.3.** An object \( E \in \text{Coh}^\beta(X) \) is tilt-(semi)stable (or \( \nu_{\alpha,\beta} \)-(semi)stable) if, for all non-trivial subobjects \( F \hookrightarrow E \), we have \( \nu_{\alpha,\beta}(F) < (\leq) \nu_{\alpha,\beta}(E/F) \).

Lemma 3.2.4 in [3] shows that the Harder-Narasimhan property holds with respect to \( \nu_{\alpha,\beta} \)-stability, i.e., for any \( E \in \text{Coh}^\beta(X) \) there is a filtration in \( \text{Coh}^\beta(X) \)

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_n = E
\]

such that: \( F_i := E_i/E_{i-1} \) is \( \nu_{\alpha,\beta} \)-semistable with \( \nu_{\alpha,\beta}(F_1) > \cdots > \nu_{\alpha,\beta}(F_n) \).

**Definition 2.4.** In the above filtration, we call \( E_1 \) the \( \nu_{\alpha,\beta} \)-maximal subobject of \( E \in \text{Coh}^\beta(X) \). If \( E \) is \( \nu_{\alpha,\beta} \)-semistable, we say \( E \) itself to be its \( \nu_{\alpha,\beta} \)-maximal subobject.

**Bogomolov-Gieseker type inequality.** We now recall the Bogomolov-Gieseker type inequality for tilt-stable complexes proposed in [3, 2].

**Definition 2.5.** We define the generalized discriminant

\[
\Delta^\beta_H := (H^2 \text{ch}_1^\beta)^2 - 2H^3 \text{ch}_0^\beta \cdot (H \text{ch}_2^\beta).
\]

A short calculation shows \( \Delta^\beta_H = (H^2 \text{ch}_1)^2 - 2H^3 \text{ch}_0 \cdot (H \text{ch}_2) \). Hence the generalized discriminant is independent of \( \beta \).

**Theorem 2.6 ([3, Theorem 7.3.1]).** Assume \( E \in \text{Coh}^\beta(X) \) is \( \nu_{\alpha,\beta} \)-semistable. Then

\[
(2.1) \quad \Delta^\beta_H(E) \geq 0.
\]

**Conjecture 2.7 ([2, Conjecture 4.1]).** Assume \( E \in \text{Coh}^\beta(X) \) is \( \nu_{\alpha,\beta} \)-semistable. Then

\[
(2.2) \quad \alpha^2 \Delta^\beta_H(E) + 4 \left( H \text{ch}_2^\beta(E) \right)^2 - 6H^2 \text{ch}_1^\beta(E) \text{ch}_3^\beta(E) \geq 0.
\]

Such inequality was proved by Macrì [12] in the case of the projective space \( \mathbb{P}^3 \):

**Theorem 2.8.** The inequality (2.2) holds for \( \nu_{\alpha,\beta} \)-semistable objects in \( D^b(\mathbb{P}^3) \).

3. TILT-STABILITY OF IDEAL SHEAVES OF SPACE CURVES

In this section, we establish the tilt-stability of ideal sheaves of spaces curves via computing their walls. Then from Theorem (2.8), we can deduce a Castelnuovo type inequality for integral curves in \( \mathbb{P}^3 \).

Throughout this section, let \( C \) be an integral projective curve in \( \mathbb{P}^3 \) of degree \( d \) not contained in a surface of degree \( < k \), and let \( \mathcal{I}_C \) be the ideal sheaf of \( C \) in
$\mathbb{P}^3$. We keep the same notation as that in the previous section for $X = \mathbb{P}^3$ and $H = \text{a plane of } \mathbb{P}^3$. To simplify, we directly identify $H^{3-i} ch_0^2(E) = ch_0^2(E)$ for $E \in D^b(\mathbb{P}^3)$. The tilted slope becomes:

$$
\nu_{\alpha,\beta} = \frac{ch_2^\beta - \frac{1}{2} \alpha^2 ch_0^\beta}{ch_1^\beta} = \frac{ch_2 - \beta ch_1 + \frac{1}{2}(\beta^2 - \alpha^2) ch_0}{ch_1 - \beta ch_0}.
$$

The following lemma is a key observation for us to establish the tilt-stability of $\mathcal{I}_C$.

**Lemma 3.1.** Let $E$ be the $\nu_{\alpha,\beta}$-maximal subobject of $\mathcal{I}_C \in \text{Coh}^\beta(\mathbb{P}^3)$ for some $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$. If $2\alpha^2 + \beta^2 \geq 4d$, then $ch_0(E) = 1$.

**Proof.** By the long exact sequence of cohomology sheaves induced by the short exact sequence

$$0 \to E \to \mathcal{I}_C \to Q \to 0$$

in $\text{Coh}^\beta(\mathbb{P}^3)$, one sees that $E$ is a torsion free sheaf with $ch_0(E) \geq 1$. If $\mathcal{I}_C$ is $\nu_{\alpha,\beta}$-semistable, then $E = \mathcal{I}_C$ by our definition. Hence $ch_0(E) = 1$.

Now we assume that $\mathcal{I}_C$ is not $\nu_{\alpha,\beta}$-semistable. One deduces

$$
\nu_{\alpha,\beta}(E) = \frac{ch_2^\beta(E) - \frac{1}{2} \alpha^2 ch_0(E)}{ch_1^\beta(E)} > \nu_{\alpha,\beta}(\mathcal{I}_C) = \frac{\frac{1}{2}(\beta^2 - \alpha^2) - d}{-\beta},
$$

i.e.,

$$
ch_2^\beta(E) > \frac{\frac{1}{2}(\beta^2 - \alpha^2) - d}{-\beta} ch_1^\beta(E) + \frac{1}{2} \alpha^2 ch_0(E).
$$

By Theorem 2.6 we obtain

$$
\left(\frac{ch_1^\beta(E)}{2 ch_0(E)}\right)^2 \geq ch_2^\beta(E).
$$

Combining (3.1) and (3.2), one sees that

$$
\alpha^2 (ch_0(E))^2 + \frac{\beta^2 - \alpha^2 - 2d}{-\beta} ch_1^\beta(E) ch_0(E) - \left(ch_1^\beta(E)\right)^2 < 0.
$$

This implies

$$
ch_0(E) < \left(\frac{\beta^2 - \alpha^2 - 2d}{\beta} + \sqrt{\left(\frac{\beta^2 - \alpha^2 - 2d}{\beta}\right)^2 + 4\alpha^2} \right) \frac{ch_1^\beta(E)}{2\alpha^2}.
$$

Since $E$ is a subobject of $\mathcal{I}_C$ in $\text{Coh}^\beta(\mathbb{P}^3)$, by the definition of $\text{Coh}^\beta(\mathbb{P}^3)$, we deduce that

$$
0 < ch_1^\beta(E) \leq ch_1^\beta(\mathcal{I}_C) = -\beta.
$$

From (3.3), it follows that

$$
ch_0(E) < \frac{\alpha^2 - \beta^2 + 2d}{2\alpha^2} + \sqrt{\frac{(\beta^2 - \alpha^2 - 2d)^2 + 4\alpha^2 \beta^2}{2\alpha^2}}.
$$

On the other hand, since $2\alpha^2 + \beta^2 \geq 4d$, a direct computation shows

$$
\frac{\alpha^2 - \beta^2 + 2d}{2\alpha^2} + \sqrt{\frac{(\beta^2 - \alpha^2 - 2d)^2 + 4\alpha^2 \beta^2}{2\alpha^2}} < 2.
$$

Therefore, by (3.4), we conclude that $ch_0(E) < 2$, i.e., $ch_0(E) = 1$. $\square$
Lemma 3.2. Let \( r \beta < \theta \) i.e.,
\[
2 - \frac{\alpha^2 + \beta^2}{\theta - r \beta} \leq (\theta - r \beta) \leq (\theta - 0) < 0.
\]
Then \( \nu_{\alpha, \beta}(E) \leq (\nu_{\alpha, \beta}(I_C)) \) if and only if
\[
\frac{\theta}{2}(\alpha^2 + \beta^2) - (c + rd)\beta + \theta d \leq < 0.
\]

Proof. Since \( E \) is a subobject of \( I_C \) in \( \text{Coh}^{\beta}(\mathbb{P}^3) \), one has
\[
0 < \text{ch}_1^{\beta}(E) = \theta - r \beta \leq \text{ch}_1^{\beta}(I_C) = -\beta,
\]
i.e., \( r \beta < \theta \leq (r - 1)\beta \leq 0 \).

Hence
\[
\nu_{\alpha, \beta}(E) = \frac{\frac{\theta}{2}(\beta^2 - \alpha^2) - \beta \theta + c}{\theta - r \beta} \leq (\nu_{\alpha, \beta}(I_C) = \frac{\frac{\theta}{2}(\beta^2 - \alpha^2) - d}{-\beta}
\]
is equivalent to
\[
-\beta \left( \frac{\theta}{2}(\beta^2 - \alpha^2) - \beta \theta + c \right) \leq (\theta - r \beta) \left( \frac{1}{2}(\beta^2 - \alpha^2) - d \right),
\]
i.e.,
\[
\frac{\theta}{2}(\alpha^2 + \beta^2) - (c + rd)\beta + \theta d \leq (\theta - 0).
\]
\( \square \)

Proposition 3.3. If \( k^2 < d \), then \( I_C \) is \( \nu_{\alpha, \beta} \)-semistable for any \( \alpha > 0 \) and \( \beta = -\frac{2d}{k} \).

Proof. We let \( \alpha_0 \) be an arbitrary positive real number, \( \beta_0 = -\frac{2d}{k} \), and let \( E \) be the \( \nu_{\alpha_0, \beta_0} \)-maximal subobject of \( I_C \in \text{Coh}^{\beta_0}(\mathbb{P}^3) \).

Since \( k^2 < d \), one sees that \( 2\alpha_0^2 + \beta_0^2 > \beta_0^2 > 4d \). Hence, by Lemma 3.1 one has \( \text{ch}_0(E) = 1 \), and \( E \) is subsheaf of \( I_C \). We can write \( E = W(-l) \), where \( W \subset \mathbb{P}^3 \) is a scheme of dimension \( 1 \) and \( l \geq 0 \). The Chern characters of \( W(-l) \) are
\[
\text{ch}_0(I_W(-l)), \text{ch}_1(I_W(-l)), \text{ch}_2(I_W(-l))) = (1, -l, -\frac{1}{2}l^2 + \text{ch}_2(I_W)).
\]
Since \( I_W(-l) \) is a subobject of \( I_C \) in \( \text{Coh}^{\beta_0}(\mathbb{P}^3) \), one deduce
\[
0 < \text{ch}_1^{\beta_0}(I_W(-l)) = -l + \beta_0 \leq \text{ch}_1^{\beta_0}(I_C) = -\beta_0,
\]
i.e.,
\[
(3.5) \quad 0 \leq l < -\beta_0.
\]
If \( C \subseteq W \), then \( \text{ch}_2(I_W) \leq \text{ch}_2(I_C) = -d \). Thus one sees that
\[
-\frac{l}{2}(\alpha_0^2 + \beta_0^2) - (\frac{1}{2}l^2 + \text{ch}_2(I_W) + d)\beta_0 - ld \leq -\frac{l}{2}\beta_0^2 + (\frac{1}{2}l^2 - d)\beta_0
\]
\[
= \frac{-\beta_0}{2}(l + \beta_0)
\]
\[
\leq 0.
\]
By Lemma 3.2 we conclude that $\nu_{\alpha,\beta_0}(\mathcal{I}_W(-l)) \leq \nu_{\alpha,\beta_0}(\mathcal{I}_C)$. Therefore the $\nu_{\alpha,\beta_0}$-maximal subobject of $\mathcal{I}_C$ in Coh$^{\beta_0}(\mathbb{P}^3)$ is $\mathcal{I}_C$ itself. Namely, $\mathcal{I}_C$ is $\nu_{\alpha,\beta_0}$-semistable.

If $C \not\subseteq W$, then $\mathcal{I}_W(-l) \subseteq \mathcal{I}_C$ implies $\mathcal{O}_{ps}(-l) \subseteq \mathcal{I}_C$. Thus $l \geq k$. One deduces by (3.5) that

$$\frac{-l}{2}(\alpha^2_0 + \beta^2_0) - \left(\frac{1}{2}l^2 + \text{ch}_2(I_W) + d\right)\beta_0 - ld < \frac{-l}{2} \beta^2_0 - \left(\frac{1}{2}l^2 + d\right)\beta_0 - ld$$

$$= \frac{-l}{2} \left(\beta^2_0 + \left(l + \frac{2d}{l}\right)\beta_0 + 2d\right)$$

$$= \frac{-l}{2} (\beta_0 + l)(\beta_0 + \frac{2d}{l})$$

$$= \frac{-l}{2} (\beta_0 + l)(\frac{2d}{l} - \frac{2d}{k})$$

$$\leq 0.$$  

(3.6)

From Lemma 3.2 it follows that $\mathcal{I}_C$ is also $\nu_{\alpha,\beta_0}$-semistable in this case. □

**Proposition 3.4.** If $k^2 \geq d$, then $\mathcal{I}_C$ is $\nu_{\alpha,\beta}$-semistable for any $\alpha > 0$ and $\beta = -2\sqrt{d}$.

**Proof.** The proof is almost the same as that of Proposition 3.3. We let $\alpha_0$ be an arbitrary positive real number, $\beta_0 = -2\sqrt{d}$, and let $E$ be the $\nu_{\alpha_0,\beta_0}$-maximal subobject of $\mathcal{I}_C \in$ Coh$^{\beta_0}(\mathbb{P}^3)$.

By Lemma 3.1 the assumption $\beta_0 = -2\sqrt{d}$ makes sure that $\text{ch}_0(E) = 1$. We can still write $E = \mathcal{I}_W(-l)$ as in the proof of Proposition 3.3. When $C \subseteq W$, the same proof of Proposition 3.3 shows that $\mathcal{I}_C$ is $\nu_{\alpha,\beta_0}$-semistable.

In the case of $C \not\subseteq W$, one sees that $l \geq k$. Thus it follows from (3.6) and (3.5) that

$$\frac{-l}{2}(\alpha^2_0 + \beta^2_0) - \left(\frac{1}{2}l^2 + \text{ch}_2(I_W) + d\right)\beta_0 - ld < \frac{-l}{2} (\beta_0 + l)(\beta_0 + \frac{2d}{l})$$

$$\leq \frac{-l}{2} (\beta_0 + l)(\frac{2d}{l} - 2\sqrt{d}).$$

The assumption $k^2 \geq d$ guarantees that the left hand side of the above inequality is negative. Therefore we are done by Lemma 3.2. □

4. **THE PROOF OF THE MAIN THEOREMS**

Now we can prove Theorem 1.1 and 1.2 easily.

**Proof of Theorem 1.1.** Since $C$ is an integral curve, one sees that

$$\text{ch}_3^\beta(\mathcal{I}_C) = -\frac{1}{6} \beta^3 + d\beta + 2d - \chi(\mathcal{O}_C).$$

If $\mathcal{I}_C$ is $\nu_{\alpha,\beta}$-semistable, then from Theorem 2.8 it follows that

$$\alpha^2 \Sigma^\beta_H(\mathcal{I}_C) + 4 \left(\text{ch}_2^\beta(\mathcal{I}_C)\right)^2 - 6H^2 \text{ch}_1^\beta(\mathcal{I}_C) \text{ch}_3^\beta(\mathcal{I}_C)$$

$$= 2\alpha^2 d + 4d^2 + \beta^3 - 4\beta^2 d - 6(-\beta) \left(-\frac{1}{6} \beta^3 + d\beta + 2d - \chi(\mathcal{O}_C)\right)$$

$$= 2\alpha^2 d + 4d^2 + 2\beta^2 d + 6\beta(2d - \chi(\mathcal{O}_C))$$

$$\geq 0,$$
Thus $O$ is equivalent to $h$ (4.1) i.e.,

(4.1) $h^1(O_C) - 1 = -\chi(O_C) \leq \frac{2d^2 + (\alpha^2 + \beta^2)d}{3(-\beta)} - 2d$.

By Proposition 3.3 and 3.4 one can substitute $(\alpha, \beta) = (0, -\frac{2d}{k})$ and $(\alpha, \beta) = (0, -2\sqrt{d})$ into (4.1) respectively to obtain our desired conclusion. \hfill \Box

Proof of Theorem 1.2. The short exact sequence

$$0 \rightarrow I_C(m) \rightarrow O_{\mathbb{P}^3}(m) \rightarrow O_C(m) \rightarrow 0$$

induces a long exact sequence

$$H^1(O_{\mathbb{P}^3}(m)) \rightarrow H^1(O_C(m)) \rightarrow H^2(I_C(m)) \rightarrow H^2(O_{\mathbb{P}^3}(m)).$$

Since $H^1(O_{\mathbb{P}^3}(m)) = H^2(O_{\mathbb{P}^3}(m)) = 0$, we deduce $h^2(I_C(m)) = h^1(O_C(m))$.

Now we assume that

Assumption 4.1. $m > \frac{2d}{k}$, $k^2 < d$ and $\beta_0 = -\frac{2d}{k}$.

One sees that

$$\text{ch}_1^\beta(O_{\mathbb{P}^3}(-m)) = -m + \frac{2d}{k} < 0.$$ 

Thus $O_{\mathbb{P}^3}(-m)[1] \in \text{Col}^\beta(\mathbb{P}^3)$. It turns out that

$$\nu_{\alpha_0, \beta_0}(O_{\mathbb{P}^3}(-m)[1]) = \frac{-\frac{1}{2}(m + \beta_0)^2 + \frac{1}{2}\alpha_0^2}{m + \beta_0} < \nu_{\alpha_0, \beta_0}(I_C) = \frac{\frac{1}{2}(\beta_0^2 - \alpha_0^2) - d}{-\beta_0}$$

is equivalent to

$$-\beta_0 \left( \frac{1}{2}(m + \beta_0)^2 + \frac{1}{2}\alpha_0^2 \right) < (m + \beta_0)(\frac{1}{2}(\beta_0^2 - \alpha_0^2) - d),$$

i.e.,

$$\alpha_0^2 + \beta_0^2 + (m + \frac{2d}{m})\beta_0 + 2d < 0.$$ 

Assumption 4.1 implies

$$\beta_0^2 + (m + \frac{2d}{m})\beta_0 + 2d = (\beta_0 + m)(\beta_0 + \frac{2d}{m}) = (\beta_0 + m)(\frac{2d}{m} - \frac{2d}{k})$$

$$< (\beta_0 + m)(k - \frac{2d}{k})$$

$$< 0.$$ 

Thus we can find an $\alpha_0 > 0$ such that $\nu_{\alpha_0, \beta_0}(O_{\mathbb{P}^3}(-m)[1]) < \nu_{\alpha_0, \beta_0}(I_C)$. On the other hand, by [3, Proposition 7.4.1] and Proposition 4.1, one deduces that $O_{\mathbb{P}^3}(-m)[1]$ and $I_C$ are both $\nu_{\alpha_0, \beta_0}$-semistable. We conclude that

$$\text{Hom}_{\text{D}(\mathbb{P}^3)}(I_C, O_{\mathbb{P}^3}(-m)[1]) = 0.$$ 

By the Serre duality theorem, one obtains $h^2(I_C(m - 4)) = 0$. Therefore we conclude that $h^2(I_C(l)) = h^1(O_C(l)) = 0$ if $l > \frac{2d}{k} - 4$ and $k^2 < d$.

Similarly, one can show $h^2(I_C(l)) = h^1(O_C(l)) = 0$ if $l > 2\sqrt{d} - 4$ and $k^2 \geq d$. \hfill \Box
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