Quantum mechanical analogue
of the zeroth law of thermodynamics

(On the problem of incorporating Thermodynamics into Quantum Theory)

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Abstract

This approach to the incorporation of stochastic thermodynamics into quantum theory is based on the conception of consistent inclusion of the holistic stochastic environmental influence described by wave functions of the arbitrary vacuum, which was proposed in our paper previously.

In this study, we implement the possibility of explicitly incorporating the zeroth law of stochastic thermodynamics in the form of the saturated Schrödinger uncertainty relation into quantum theory. This allows comparatively analyzing the sets of states of arbitrary vacuums, namely, squeezed coherent states (SCSs) and correlated coherent states (CCSs). In addition, we compare the results of the construction of stochastic thermodynamics using SCSs and TCCSs with the versions involving the incorporation of thermodynamics into quantum theory developed previously and based on thermodfield dynamics and quantum statistical mechanics.

Key words: uncertainty relation, thermal equilibrium, zeroth law, invariance, squeezed coherent states, correlated coherent states.

1. Introduction

The concept of thermal equilibrium in thermodynamics is traditionally associated with the zeroth law, i.e., with the equality (if only on the average) of Kelvin temperatures of the object $T$ and the environment $T_0$, with which the model of classical thermostat is brought in correspondence. Of course, the concept of thermal equilibrium in the case of the zero temperature is not introduced because the thermostat in the traditional understanding (as a thermal environment) is not meant. The tacitly taken thermal stochastic environmental influence is not taken into account under these conditions, which are typical of quantum mechanics.

At the same time, in the case of sufficiently low temperatures where the thermal influence must be already taken into account, the quantum stochastic influence continues to be a significant factor. Thus, quantum and thermal fluctuations are produced simultaneously but they turn out to be nonadditive. We introduced [1] the concept of quantum thermostat (arbitrary vacuum in the quantum language) as a model of environment to describe such a case at low temperatures. To extend the concept of thermal equilibrium to the case of the contact with the quantum thermostat, proceeding from empirical considerations, we previously introduced the effective temperature [2]

$$T = \frac{\hbar \omega}{2k_B \coth \frac{\hbar \omega}{2k_B T}},$$

1The work was supported by RFBR (project no.12-01-90423)
where \( k_B \) is the Boltzmann constant and \( T \) is the Kelvin temperature, and proposed the generalized zeroth law in the form

\[
T = T_0 \pm \Delta T,
\]

(2)

where \( T_0 \) is the effective temperature of the quantum thermostat and \( \Delta T \) is the standard deviation of the effective object temperature. In the limit when the Kelvin temperature tends to zero \( T \to 0 \), the minimum effective temperature for the normal vacuum mode (with the frequency \( \omega \)) \( T_0^{\text{min}} \equiv \frac{\hbar \omega}{2k_B} \neq 0 \). In this case the expression (2) takes the form

\[
T^{\text{min}} = T_0^{\text{min}} \pm \Delta T^{\text{min}},
\]

(3)

and retains the meaning of the condition of equilibrium with the cold vacuum.

We now consider whether expression (2) can be treated fundamentally on the microlevel. To do this, we turn to the Hamiltonian of the arbitrary vacuum

\[
\hat{H}_{\tau,\varphi} = \hat{H}_{\tau,\varphi} - \hat{H}_{\text{inf}}_{\tau,\varphi},
\]

(4)

which we introduced in [3]. In this expression, \( \hat{H}_{\tau,\varphi} \) is the Hamiltonian of the system that is modeled by a quantum oscillator with the frequency \( \omega_0 \), and \( \hat{H}_{\text{inf}}_{\tau,\varphi} \) is the energy operator of the holistic (i.e. quantum and thermal) stochastic environmental influence at an arbitrary Kelvin temperature. We recall that the subscripts \( \tau \) and \( \varphi \) are the parameters \( (u,v) \) of the Bogolyubov transformation allowing us to pass from the initial (cold) vacuum to an arbitrary one (including the thermal one):

\[
u = \text{ch} \tau \cdot e^{i\varphi}; \quad v = \text{sh} \tau \cdot e^{-i\varphi}.
\]

(5)

The system Hamiltonian \( \hat{H}_{\tau,\varphi} \) contained in (4) is expressed through these parameters in the following way:

\[
\hat{H}_{\tau,\varphi} \equiv (\text{ch} 2\tau - \text{sh} 2\tau \cdot \cos 2\varphi)\hat{K}_{\tau,\varphi} + (\text{ch} 2\tau + \text{sh} 2\tau \cdot \cos 2\varphi)\hat{\Pi}_{\tau,\varphi},
\]

(6)

where \( \hat{K}_{\tau,\varphi} \) and \( \hat{\Pi}_{\tau,\varphi} \) are the respective kinetic and potential energy operators.

The second operator \( \hat{H}_{\text{inf}}_{\tau,\varphi} \) in (4) consists also of two terms reflecting the contributions of both the purely quantum influence \( \hat{j}_0 = \frac{\hbar}{2}\hat{f} \) of the cold vacuum and the additional influence \( \hat{\sigma}_{\tau,\varphi} \), which appears during the passage from the initial cold vacuum (i.e., at the environmental temperature \( T = 0 \)) to the arbitrary one admitting cases with nonzero environmental temperatures. In this case,

\[
\hat{H}_{\text{inf}}_{\tau,\varphi} = \omega \hat{j}_0 + \omega \hat{\sigma}_{\tau,\varphi}.
\]

(7)

Here,

\[
\hat{\sigma}_{\tau,\varphi} \equiv (\text{sh} 2\tau \cdot \sin 2\varphi)\frac{1}{2} (\hat{p} \hat{q} + \hat{q} \hat{p}).
\]

(8)

For \( \tau = 0 \) and \( \varphi = 0 \), operator (7), of course, transforms into the energy operator of the initial cold vacuum.

To calculate the average value of operator (4), below we use the complex wave function \( \psi_{\tau,\varphi} \) of the arbitrary vacuum. In the coordinate representation, it has the form

\[
\psi_{\tau,\varphi}(q,\omega) = [2\pi(\Delta q_{\tau,\varphi})^2]^{-1/4} \exp \left\{ -\frac{q^2}{4(\Delta q_{\tau,\varphi})^2} (1 - i\beta_{\tau,\varphi}) \right\},
\]

(9)
where \((\Delta q_{\tau,\varphi})^2\) is the coordinate variance for the arbitrary vacuum and \(\beta_{\tau,\varphi}\) is a significant parameter determining the phase of the wave function. Function (9) is the eigenfunction of Hamiltonian \(\hat{H}_{\tau,\varphi}\) (4) of the arbitrary vacuum with the zero eigenvalue. \[3\]
\[
\hat{H}_{\tau,\varphi}\psi_{\tau,\varphi}(q,\omega) = 0 \cdot \psi_{\tau,\varphi}(q,\omega) = 0.
\]
The quantities \((\Delta q_{\tau,\varphi})^2\) and \(\beta_{\tau,\varphi}\) in (9) are expressed in terms of the parameters \(\tau\) and \(\varphi\) as follows:
\[
(\Delta q_{\tau,\varphi})^2 = (\Delta q_0)^2 (\text{ch}^2 \tau - \text{sh}^2 \tau \cdot \cos^2 \varphi);\quad (\Delta q_0)^2 = \frac{\hbar}{2\omega_0};
\]
\[
\beta_{\tau,\varphi} = \text{sh} 2\tau \cdot \sin 2\varphi.
\]

The momentum variance for the state \(\ket{\psi_{\tau,\varphi}(q,\omega)}\) has the form
\[
(\Delta p_{\tau,\varphi})^2 = (\Delta p_0)^2 (\text{ch} 2\tau + \text{sh} 2\tau \cdot \cos 2\varphi),\quad (\Delta p_0)^2 = \frac{\hbar \omega_0}{2}.
\]

In this paper, we return to the study of criteria that allow separating states corresponding to the thermal equilibrium with the arbitrary vacuum from the set of states \(\psi_{\tau,\varphi}(q,\omega)\) generated by the Bogolyubov \(u, v\) transformation. Squeezed coherent states (SCSs) and correlated coherent states (CCSs) are candidates for this role because the expressions for the internal energy of the quantum oscillator in these states can be reduced to the form coinciding with the Planck formula \(U_T = \frac{\hbar \omega_0}{2} \coth \left( \frac{\hbar \omega}{2k_B T} \right)\), where \(k_B\) is the Boltzmann constant and \(T\) is the Kelvin temperature. To do this, it is possible to bring the parameter \(\tau\) formally in correspondence with the temperature in accordance with the relations
\[
\text{ch} 2\tau \equiv \coth \left( \frac{\hbar \omega}{2k_B T} \right);\quad \text{sh} 2\tau \equiv \left( \text{sh} \frac{\hbar \omega}{2k_B T} \right)^{-1}.
\]
and to convince ourselves that the obtained results and the Planck ones are similar in appearance. However, the fundamental concept of temperature has no pre-image in the microworld and is related to the concept of thermal equilibrium. Thus, the procedure for introducing another temperature-dependent parameter instead of \(\tau\) does not yet allow drawing the final conclusion on the equilibrium character of the given states and on the possibility of interpreting them as analogues of the thermal ones. Thus, it is required to formulate, in the quantum language, the condition that is analogous to that of thermal equilibrium with the thermostat at the temperature \(T\) adopted in thermodynamics and to verify the fulfillment of this condition in the cases of the SCSs and CCSs.

2. Analysis of squeezed coherent states of the arbitrary vacuum

In the paper \[4\], we preliminary indicated that SCSs can be regarded as thermal ones only conditionally, keeping in mind that in thermodynamics, only cases of the contact with the cold vacuum (i.e., at the Kelvin temperature \(T = 0\)) can be brought in correspondence to them. Based on the study of the form of the Schrödinger uncertainty relations for these states, we now adduce additional arguments in favor of this statement. In the case of the SCS where \(\varphi = 0\), wave function (9) becomes real because \(\beta_{\tau,\varphi}\) vanishes, and we let \(\psi_{\tau,0} \equiv \psi_0\) denote it. Substituting relations (12) in formula (6), we formally change the parametrization of the system Hamiltonian. However, we show below that this procedure
turns out to be meaningless in spite of the fact that we introduce another notation $\hat{H}_T \equiv \hat{H}_{\tau,\varphi}$ for it instead of $\hat{H}_{\tau,\varphi}$. Indeed, assuming that the oscillator mass is $m = 1$, we obtain

$$
\hat{H}_T = [\coth \frac{\hbar \omega}{2k_B T} - (\sinh \frac{\hbar \omega}{2k_B T})^{-1}]\frac{1}{2} \hat{p}^2 + [\coth \frac{\hbar \omega}{2k_B T} + (\sinh \frac{\hbar \omega}{2k_B T})^{-1}]\frac{1}{2} \omega^2 \hat{q}^2
$$

(13)
in this case. Now the operator $\hat{\sigma}_{\tau,0} \equiv (\sinh 2\varphi \cdot \sin 2\varphi)\frac{1}{2}(\hat{p} \hat{q} + \hat{q} \hat{p})_{\varphi=0}$ corresponding to the correlation between coordinate and momentum fluctuations in formula (7) vanishes because $\sin 2\varphi = 0$. Thus, the stochastic influence is characterized only by the operator $\hat{j}_0$, which is typical of the initial cold vacuum so that expression (7) becomes simplified: $\hat{H}_{\tau,\varphi} \equiv \omega \hat{j}_0$.

If the fact that the average value of the vacuum Hamiltonian $\hat{H}_{\tau,\varphi}$ vanishes is taken into account, the equality

$$
\langle \psi_T | \hat{H}_T | \psi_T \rangle = \langle \psi_T | \omega \hat{j}_0 | \psi_T \rangle.
$$

(14)
can be obtained from formula (4). We attract attention to the fact that the left- and right-hand sides of (14) contain physically different quantities: $\hat{H}_T$ characterizes the system modeled by the quantum oscillator, and $\hat{j}_0$, its environment. Thus, equality (14) means that the average energy of the system (oscillator) coincides with that of the quantum environmental influence. This assertion formulated in the quantum language is, in fact, equivalent to the zeroth law of thermodynamics, but for the limiting case of the contact of the system with the cold vacuum.

It follows from formulas (14) and (13) that

$$
\left[ \coth^2 \left( \frac{\hbar \omega}{2k_B T} \right) - \left( \sinh^2 \left( \frac{\hbar \omega}{2k_B T} \right) \right)^{-1} \right] U_0 = \frac{\hbar \omega}{2}.
$$

(15)

However, we note that the expression in brackets in (15) is identically equal to unity. In this case, $U_0 = \langle \psi_0 | \hat{H}_0 | \psi_0 \rangle = \frac{\hbar \omega}{2}$ is the average energy of the oscillator in the cold vacuum so that this expression is independent of the parameter $T$. In view of this argument, the parameter $T$ cannot be treated as temperature. In other words, the appropriateness of the change in the parametrization in form (12) turns out to be very doubtful, as was expected. At the same time, this procedure allows obtaining certain useful information from (15). So, taking comments to formulas (10) and (11) into account, we can state that

$$
\frac{U_0}{\omega} = \Delta q_0 \cdot \Delta p_0 = (UP)_0,
$$

where the notation $(UP)_0$ is introduced for the product of uncertainties $\Delta p_0 \cdot \Delta q_0$. Then expression (15) is, in fact, the partial realization of the Schrödinger UR, i.e., the saturated Heisenberg UR

$$(UP)_0 = \frac{\hbar}{2} \quad ; \quad (*)$$

which is not related to thermal effects in any way.

Thus, the set of SCSs describes the specific case of the system equilibrium only with the cold vacuum, i.e., at $T = 0$, which corresponds to a "maximally" isolated (from the nonclassical point of view) system, i.e., the system experiencing only stochastic quantum influence. In other words, for the SCS at any nonzero Kelvin temperature having the
standard physical meaning, it is improper to put a question of the equilibrium in the sense that is meant by the zeroth law of thermodynamics. In this case, there is no "warm" thermostat providing the thermal contact. This confirms the conclusion that probably, it is no mere chance that Umezawa called squeezed states not truly thermal ones but only thermal-like states.

Thus, we draw the conclusion in accordance to which the saturation of the UR in the Heisenberg form (*) in the quantum language is not a sufficient condition for the description of the thermal equilibrium at \( T \neq 0 \). Below, we show that the saturated Schrödinger UR plays this role.

3. Analysis of correlated coherent states of the arbitrary vacuum based on the Schrödinger UR

For the CCSs, another case occurs because the parameter \( \varphi = \pi/4 \). In this case, for the zero average value of vacuum Hamiltonian (4), we obtain the relation

\[
\langle \psi_{\tau,\varphi} | \mathcal{H} | \psi_{\tau,\varphi} \rangle = \langle \psi_{\tau,\varphi} | \hat{S}_{\tau}^{inf} | \psi_{\tau,\varphi} \rangle.
\] (16)

In accordance with (7) and (8), the operator \( \hat{S}_{\tau}^{inf} \) in it contains two nonzero terms. If the parametrization is changed now and the temperature is introduced in accordance with formula (12), then expressions (10) and (11) for variances become

\[
(\Delta q_T)^2 = (\Delta q_0)^2 \coth \left( \frac{\hbar \omega}{2k_B T} \right); \quad (\Delta p_T)^2 = (\Delta p_0)^2 \coth \left( \frac{\hbar \omega}{2k_B T} \right).
\] (17)

Accordingly, after such an operation, the operator of additional influence \( \hat{\sigma}_{\tau,\varphi} \) (8) corresponding to the correlation between coordinate and momentum fluctuations transforms (for \( \varphi = \pi/4 \)) to the form

\[
\hat{\sigma}_{\tau} \equiv (\text{sh} \, 2\tau \cdot \sin 2\varphi)^{-1/2} \left( \hat{p} \hat{q} + \hat{q} \hat{p} \right) = \frac{\hbar}{2 \, \text{sh} \, \frac{\hbar \omega}{2k_B T}} \frac{1}{2} \left( \hat{p} \hat{q} + \hat{q} \hat{p} \right)
\] (18)

Then taking into account (6) and (7) and also (17) and (18), from (16), we obtain

\[
\coth \left( \frac{\hbar \omega}{2k_B T} \right) \left[ \frac{\hbar \omega}{2} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \right] = \frac{1}{\text{sh} \, \left( \frac{\hbar \omega}{2k_B T} \right)} \left[ \frac{\hbar \omega}{2} \right] \frac{1}{\text{sh} \, \left( \frac{\hbar \omega}{2k_B T} \right)} + \frac{\hbar \omega}{2}.
\] (19)

If formulas (18) are now taken into account and the notation \( (\Delta p_T)^2 (\Delta q_T)^2 \equiv (UP_T)^2 \) for the product of variances is used again, the left-hand side of expression (16) can be written in the form of the following equality after the multiplication by \( \hbar/2\omega \):

\[
\left[ \frac{\hbar}{\omega} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \right]^2 = \left( \frac{\mathbb{U}_T}{\omega} \right)^2 = (UP_T)^2.
\] (20)

where \( \mathbb{U}_T \) is the Planck oscillator energy.

Accordingly, the right-hand side of (16) after the multiplication by \( \hbar/2\omega \) can be rewritten in the form that is more convenient for the interpretation from the physical point of view:

\[
\left[ \frac{\hbar}{2} \text{sh} \, \left( \frac{\hbar \omega}{2k_B T} \right) \right]^2 + \left( \frac{\hbar}{2} \right)^2 = \left( \frac{\hbar}{2} \cdot \coth \left( \frac{\hbar \omega}{2k_B T} \right) \right)^2 = \left| \langle \psi_T | \hat{p} \cdot \hat{q} | \psi_T \rangle \right|^2 \equiv \mathbb{J}_T^2.
\] (21)
where the quantity $J_T = \frac{\hbar}{2} \cdot \coth \frac{\hbar \omega}{2k_B T}$ has the meaning of the holistic effective stochastic environmental influence, to which both Hermitian operators $\hat{\sigma}_T$ and $\hat{j}_0$ contribute. Thus, equality (17) typical of the arbitrary vacuum with the wave function $\psi_T$ is nothing but the saturated Schrödinger uncertainty relation (SUR)

$$\langle UP \rangle_T^2 = \sigma_T^2 + \frac{\hbar^2}{4}$$

(22)

At the same time, the direct comparison of the right-hand sides of formulas (20) and (21) allows representing formula (17) in the form

$$\langle UP \rangle_T = J_T.$$  

(23)

In this case, we stress that the left- and right-hand sides of formula (23) determined by expressions (20) and (21) have different physical meanings in spite of the coincidence between their dimensionalities. The expression $\langle UP \rangle_T$ can be interpreted as a system response $J_{\text{syst}}$ to the stochastic environmental influence $J_T$, so that the equality between the influence on the system and its response

$$J_{\text{syst}} = J_T,$$

(24)

holds. We call the expression (24) the zeroth law of stochastic thermodynamics. Applying the general formula $J = \frac{\hbar \omega}{2} \cdot T$ for the effective stochastic influence to (24) on the left (for the system) and on the right (for the quantum thermostat), we obtain the relation

$$T_{\text{syst}} = T_T,$$

(25)

which is the zeroth law of stochastic thermodynamics written in more traditional form for effective temperatures of the environment and the object. Thus, saturated Schrödinger UR (SUR) (22) for thermal CCSs closely related to the fundamental description on the microlevel acquires the status of the quantum analogue of the zeroth law of stochastic thermodynamics. Therefore, equality (24) is the essential requirement of the theory.

Concerning the zeroth law in form (24), we want to point out the fact emphasizing the significant role of the phase of the wave function. Indeed, the right-hand side of (26) is mainly determined by precisely the phase of the wave function. Its role is especially significant at high temperatures when the corresponding term becomes

$$\langle J_T \rangle = \sqrt{\left(\frac{\hbar}{2 \cdot \text{sh} \frac{\hbar \omega}{2k_B T}}\right)^2 + \left(\frac{\hbar}{2}\right)^2}.$$

(26)

The first term in it is determined by the average value of the operator $\hat{\sigma}_T$, which is dependent on the phase of the wave function. It is not difficult to see that, except for the case $T \to 0$, the right-hand side of (26) is mainly determined by precisely the phase of the wave function. Its role is especially significant at high temperatures when the corresponding term becomes

$$\left(\frac{\hbar}{2 \cdot \text{sh} \frac{\hbar \omega}{2k_B T}}\right)^2 \to \left(\frac{k_B T}{\omega}\right)^2 \gg \left(\frac{\hbar}{2}\right)^2.$$

In this case, zeroth law (24) itself reduces to the equality condition for Kelvin temperatures $T = T_0$, which is taken as a definition of the thermal equilibrium in the classical thermodynamics and in quantum statistical mechanics (QSM) (in this case, the symbol $T_0$
denotes the thermostat temperature). However, it is important to stress that even in the cases where temperature fluctuations of the system (consisting of many particles, for example) can be neglected in the zeroth approximation, its definition, which is adequate to this concept at the asymptotic, has the form

\[ T \rightarrow \frac{\omega}{k_B}(U\mathcal{P})_T \]  

(27)

Thus, this quantity depends on the momentum and coordinate uncertainties, which indicate its initially random character.

In the cases where there is no phase, for example, when real wave functions of the SCS type are used, formula (24) becomes

\[ \mathcal{J}_{syst} = \mathcal{J}_{qu}, \]  

(28)

where \( \mathcal{J}_{qu} = \frac{\hbar}{2} \) is the quantum influence. In this case, the zeroth law is independent of the parameter \( T \), which allows fixing the ”thermal” equilibrium only with the cold vacuum, as was mentioned above, i.e., treating it in the ”Pickwick” sense.

Thus, to describe the thermal equilibrium with the arbitrary vacuum, it is insufficient to have only two requirements, namely, the definition of vacuum and the coincidence between the expression for the internal energy of the quantum oscillator formally expressed in terms of the parameter \( T \) and the Planck formula. Both sets of states (SCSs and CCSs) satisfy these requirements. However, only the saturated Schrödinger UR corresponds to the inclusion of the thermal influence on the microlevel. Therefore, it is possible to endow the quantity \( T \) with the physical meaning that is adequate to the concept of temperature only in the case of CCSs of the \( |\psi_{\tau,\omega}\rangle = |\psi_{\tau,\frac{\omega}{2}}\rangle \equiv |\psi_{\tau}\rangle \) type, which can be regarded as ”thermal” ones. In other words, to separate the states characterizing the thermal equilibrium with the arbitrary vacuum (for \( T \neq 0 \)) from all states generated by the Bogolyubov \((u,v)\) transformations, one necessary requirement must be introduced. It consists in that the equivalent of the zeroth law of thermodynamics in the form of the saturated Schrödinger UR

\[ \Delta p \cdot \Delta q = \left| \langle \psi | \hat{p} \cdot \hat{q} | \psi \rangle \right| \]

is included in the apparatus of quantum theory. In the state of the ”thermal” vacuum \( |\psi_T(q)\rangle \), it reduces to the condition

\[ (U\mathcal{P})_T = \frac{\hbar}{2} \coth \left( \frac{\hbar \omega}{2k_B T_0} \right), \]

where we use \( T_0 \), which is the Kelvin thermostat temperature. We stress that in this relation, the left-hand side is expressed in terms of the object characteristics, and the right-hand side, in terms of the environmental characteristics.

4. Invariance of the zeroth law of stochastic thermodynamics

We now analyze the above obtained fundamental formulation of the zeroth law from the point of view of its invariance under the Bogolyubov \((u,v)\) transformations. We attract attention again to the fact that qualitatively different quantities are contained in the right- and left-hand sides of the corresponding equalities; however, their dimensionalities coincide.
In particular, formula (28) means that at the zero temperature,
\[ J_{qu} = \hbar^2 = \Delta p_0 \cdot \Delta q_0 \equiv (UP)_0, \]  
(29)
where \((UP)_0\) is regarded as a holistic quantity characterizing the object under study. We recall that \((u, v)\) transformations retaining canonical permutation relations have the meaning of transformations for two symmetry groups — the group \(SU(1, 1)\) and its locally isomorphic Lorentz group \(O(2, 1)\) — simultaneously. Their common invariant is \(\hbar^2/4\).

We now discuss how the presence of the corresponding invariance affects the quantities contained in the left- and right-hand sides of formula (29). As follows from the preceding, under transformations with the real parameters \(u\) and \(v\) (i.e., for \(\varphi = 0\) in formula (5)), the expression in the right-hand side of (29) remains the same because, in this case, the average value of the operator of quantum thermal stochastic influence \(\hat{\sigma}\) remains equal to zero for any values of the parameter \(\tau\). At the same time, the expression in the left-hand side of (29) remains invariant; however, it can have different forms, depending on the choice of the parameter \(\tau \neq 0\). In other words, the expression \((UP)^2\) in the general case can have the form
\[ (UP)^2 = (UP)^2_0 \cosh^2 2\tau - \sinh^2 2\tau = \left(\frac{\hbar}{2}\right)^2, \]  
(30)
which corresponds to different ways of realizing SCSs.

If formulas (12) and also (10) and (11) are taken into account, expression (30) can be written as a determinant of the form
\[
(UP)^2 = (UP)^2_0 \begin{vmatrix} \cosh \frac{\hbar \omega}{2kT} + \frac{1}{\sinh \frac{\hbar \omega}{2kT}} & 0 \\ 0 & \cosh \frac{\hbar \omega}{2kT} - \frac{1}{\sinh \frac{\hbar \omega}{2kT}} \end{vmatrix} = \\
\begin{vmatrix} (\Delta p_T)^2 & 0 \\ 0 & (\Delta q_T)^2 \end{vmatrix} = \left(\frac{\hbar}{2}\right)^2.
\]  
(31)

If, in this case, the momentum and coordinate variances are calculated by averaging with the real wave function that follows from formula (9) for \(\beta_{\tau,\varphi} = 0\), which is said to be thermal-like, then (in accordance with the Umezawa terminology [5]) we deal with single-mode squeezed states. Trying to generalize formula (31), Umezawa introduced two independent sets of quanta, namely, ordinary quanta related to the standard operators of annihilation \(\hat{a}\) and creation \(\hat{a}^+\) and so-called thermal quanta related to the operators \(\hat{\tilde{a}}\) and \(\hat{\tilde{a}}^+\) with tildes. Because of this, he doubled the Hilbert space, which allows him to introduce two-mode squeezed states. They are obtained using the creation operators formed by means of the Bogolyubov \((u, v)\) transformations entangling operators from two sets, which he introduced, one with another. As a result, expression (30) becomes
\[ (UP)^2 = \begin{vmatrix} (\Delta \hat{p})_T^2 & (\Delta p \Delta \hat{p})_T \\ (\Delta q \Delta \hat{q})_T & (\Delta \hat{q})_T^2 \end{vmatrix} = \left(\frac{\hbar}{2}\right)^2 \]  
(32)
where the averaging is carried out over the thermal-like states, as in (31).

We stress that the off-main-diagonal terms in determinant (31) have the forms of correlators of homogeneous quantities \((\hat{p}, \hat{p})\) or \((\hat{q}, \hat{q})\); however, in this case, correlators of heterogeneous quantities of the \((\hat{p}, \hat{q})\) type, and so on, remain equal to zero. Thus, the right-hand-side of equality (29) remains the same, which corresponds to the property of the introduced states to be squeezed.
We note that the quantities contained in formulas (31) and (32) are averaged using real wave functions. At the same time, in the Umezawa’s opinion, analogous results can also be obtained using the real density matrix. Recently, Park made a successful effort of this sort for the quantum oscillator in the thermal equilibrium [6].

He proposed a method for calculating the elements of the determinant contained in (32) using the averaging with the Gibbs–von Neumann density operator

$$\hat{\rho}_T = Z^{-1} \exp[-\frac{\hbar \omega}{2k_B T}(\hat{N}_a + \frac{1}{2}\hat{I})]$$

where $Z = \text{Sp}[\exp[-\frac{\hbar \omega}{2k_B T}(\hat{N}_a + \frac{1}{2}\hat{I})]]$, $\hat{I}$ is the unity operator.

The result obtained by Park allows writing determinant (32=83) in the form

$$\langle \mathcal{U} \mathcal{P} \rangle^2 = \left| \begin{array}{cc} \text{Sp}[\Delta \hat{p}^2 \hat{\rho}_T] & \text{Sp}[\Delta \hat{p} \Delta \hat{p}^\dagger \Delta \hat{\rho}_T] \\ \text{Sp}[\Delta \hat{q} \Delta \hat{q}^\dagger \Delta \hat{\rho}_T] & \text{Sp}[\Delta \hat{q}^2 \hat{\rho}_T] \end{array} \right|$$

(34)

We attract attention to the fact that, to provide the invariance of $\mathcal{U} \mathcal{P}$, the off-main-diagonal terms that are absent in traditional approaches to the calculation of $\mathcal{U} \mathcal{P}$ are added by Park. From the mathematical point of view, the necessity of such a generalization is due to the fact that, in this case, the operators $(\Delta \hat{p})^2$ and $\hat{\rho}_T$ or $(\Delta \hat{q})^2$ and $\hat{\rho}_T$ under the trace sign do not commute one with the other. Therefore, the method for calculating variances adopted in the classical probability theory or in quantum mechanics in the case where pure states are used requires generalization, which was emphasized in the paper of Wigner and Yanase [7].

In accordance with Park, the off-diagonal terms in determinant (34) can be transformed using formula (33) and the commutation rules

$$\hat{\rho}_T^\dagger \hat{a} = \exp\left(\frac{\hbar \omega}{2k_B T}\right) \hat{a} \hat{\rho}_T^\dagger; \quad \hat{\rho}_T^\dagger \hat{a}^+ = \exp\left(\frac{\hbar \omega}{2k_B T}\right) \hat{a}^+ \hat{\rho}_T^\dagger$$

(35)

To do this, it is necessary to take into account that, in this case, $(\Delta \hat{p})^2 = (\hat{p})^2$ and $(\Delta \hat{q})^2 = (\hat{q})^2$ and to express $\hat{p}$ and $\hat{q}$ in terms of $\hat{a}$ and $\hat{a}^+$. Therefore, the relation $(\mathcal{U} \mathcal{P})^2$ in the left-hand side of (34) expressed in terms of them can be written as the determinant

$$\langle \mathcal{U} \mathcal{P} \rangle^2 = \left| \begin{array}{cc} (\Delta \hat{p})^2 & (\Delta \hat{p}) \hat{\rho}_T \hat{p} \\ (\Delta \hat{p}) \hat{\rho}_T \hat{p} & (\Delta \hat{q})^2 \hat{\rho}_T \end{array} \right|$$

(40)
in which the calculation of all average quantities with the density matrix $\hat{\rho}_T$ leads to the left-hand side of (34). The comparison of (40) with the left-hand side of expression (32) shows that the result of calculating $(\mathcal{U}^{\mathcal{P}})^2$ by Park using the real density matrix $\hat{\rho}_T$ does not differ significantly from the similar result obtained by Umedzawa using real two-mode squeezed states. However, it is important to stress that both results (32) and (41) are related to the case of the equilibrium of the system with the cold vacuum because the right-hand side of zeroth law (29) remains the same during the averaging over real pure and mixed states. Thus, the formal use of real states or real elements of the density matrix does not allow describing the equilibrium of the system with the thermal vacuum not only in the case of squeezed states, but also in the cases of the Umedzawa thermofield dynamics and the QSM-based thermodynamics even after the generalization of the latter in the Wigner and Yanase spirit.

We now discuss the invariance of the zeroth law with respect to the transitions between equilibrium thermal CCSs at nonzero temperatures. To do this, we use the invariance of the zeroth law with respect to the Lorentz group $O(2,1)$. It can be demonstrated in the simplest way using the right-hand side of condition (29) by replacing $(\mathcal{I}_0^{\mathcal{P}})^2$ in accordance with formula (21). Then we obtain

$$
(\mathcal{J}_{qu})^2 = (\mathcal{J}_T)^2 - \sigma_T^2 = \left(\frac{\hbar}{2}\right)^2
$$

Hence it follows that for thermal CCSs, the set of quantities $\mathcal{J}_T, \sigma_T$ can be regarded as a two-dimensional time-like vector in the pseudo-Euclidean space of states, and the quantity $\hbar^2/4$, as a length of this vector or the invariant of the Lorentz group $O(1,1)$ that is analogous to the eigenvalue in the standard special theory of relativity in the world of events (the group dimensionality decreases because the average value of the Lagrangian is zero for thermal CCSs). The roles of the traditional Lorentz factors $\beta_L$ and $\gamma_L$ in this case are played by the quantities

$$
\beta_{\text{term}} = [\coth \frac{\hbar \omega}{2 k_B T}]^{-1}; \quad \gamma_{\text{term}} = [\coth \frac{\hbar \omega}{2 k_B T}]
$$

In this case, the limits of the parameter $\beta_{\text{term}}$ as $T \to \infty$ and $T = 0$ in the space of states correspond to the limiting values $\beta_L = 0$ and $\beta_L \to 1$ in the space of events. Thus, the state of the cold vacuum in the group of Bogolyubov transformations corresponds to the proper inertial reference system in the special theory of relativity for the group $O(1,1)$ in the space of states. Other inertial reference systems that are equivalent to the proper inertial reference system correspond to different states of the arbitrary vacuum.

Thus, we are convinced once more that, among so far existing theoretical approaches, only $(\hbar, k)$ dynamics [1] with temperature-dependent complex wave functions provides the fulfillment of the zeroth law at nonzero temperatures. Of course, the possibility of constructing an adequate theory using the complex density matrix remains open. At the same time, we note that the authors of approaches to the description of thermal phenomena based on real wave functions or real elements of the density matrix understood intuitively the importance of the thermal equilibrium condition in form (24) as $T \neq 0$. However, in the corresponding theories, this condition always has the same trivial form (31), (32) or (34)

$$
\left(\frac{\hbar}{4}\right)^2 \left[ \coth^2 \left(\frac{1}{2} \beta \hbar \omega\right) - \left(\sinh^2 \left(\frac{1}{2} \beta \hbar \omega\right)\right)^{-1}\right] \mathcal{E}_0 = \frac{\hbar \omega}{2} = \left(\frac{\hbar}{4}\right)^2,
$$

$\text{(43)}$
where the parameter \( \beta \) in both hyperbolic functions has the same value \( \frac{1}{kT} \). We see the principal difference between formula (44) and the following one

\[
\left( \frac{\hbar}{4} \right)^2 \coth^2 \left( \frac{\hbar \omega}{2k_B T} \right) = \left( \frac{\hbar}{4} \right)^2 + \left( \frac{\hbar}{4} \right)^2 \left( \sinh^2 \left( \frac{\hbar \omega}{2k_B T_0} \right) \right)^{-1},
\]

(44)
in spite of their external similarity.

To obtain the desired result that allows treating the parameter \( T \) in formula (44) as a temperature, the authors of these papers simply rewrote equality (44=95) identically, carrying one of the terms from the left to the right. From the mathematical point of view, this action is, of course, admissible. However, it is improper from the physical point of view. It is very important to note that the left- and right-hand sides of equality (45=96) are related to different physical entities: the characteristics of the system with the temperature \( T \), which is able to fluctuate, is in the left-hand side, and the characteristic of the stochastic environment with the fixed temperature \( T_0 \), which does not fluctuate because of an infinitely large number of degrees of freedom of the quantum thermostat, is in the right-hand side. They must be calculated independently using the averaging over thermal CCSs.

That the above-mentioned formal procedure is inadmissible becomes especially obvious in the limit of high temperatures when formula (44=95) transforms into the identity

\[
\left( \frac{\hbar}{4} \right)^2 \coth^2 \left( \frac{\hbar \omega}{2k_B T} \right) \rightarrow \left( \frac{\hbar}{4} \right)^2 \left( \sinh^2 \left( \frac{\hbar \omega}{2k_B T_0} \right) \right)^{-1}.
\]

(45)

Summary

1. We have demonstrated that the physically significant subgroups of the group of Bogolyubov \((u,v)\) transformations generate two types of states: real squeezed coherent states providing the saturation of the Heisenberg "coordinate–momentum" UR at the zero temperature and complex correlated coherent states providing the saturation of the Schrödinger "coordinate–momentum" UR at finite temperatures. 2. On the microlevel, we have formulated a quantum analogue of the zeroth law of stochastic thermodynamics in the form of the saturated Schrödinger UR. 3. We have substantiated the invariance of the saturated Schrödinger UR under the Bogolyubov \(u,v\) transformations. 4. We have proposed to regard the saturated Schrödinger UR as an initial concept of quantum theory at finite temperatures. 5. We have shown that in theories using real wave functions or real elements of the density matrix, there is no correlation between the coordinate and momentum fluctuations. Therefore, in this case, the concept of thermal equilibrium occurring as a result of the correlation between coordinate and momentum fluctuations can be introduced only conditionally for the states at the zero temperature, i.e., if there is no thermal environmental influence. 6. We have established that in the theory involving complex wave functions of the vacuum (thermal CCSs), thermodynamics can be incorporated into quantum theory at any temperatures (unlike the Umezawa thermofield dynamics).

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