On intrinsic time measure in the modeling of cyclic behavior of a Nitinol cubic block

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Abstract

In this paper, the cyclic behavior of a superelastic-plastic nitinol cubic block is described by using the Bouc–Wen model coupled to an intrinsic time measure other than clock time, which governs the behavior of the materials. As a consequence, the thermodynamic admissibility of the Bouc–Wen model is provided by the endochronic theory of plasticity. The role of the intrinsic time measure is described by capturing the stiffness and strength degradation and the opposite phenomena. Such behavior is due to the permanent-strain addition of residual martensite and alterations in the properties of the texture during phase transformation.

Keywords: Bouc–Wen model, endochronic theory of plasticity, intrinsic time measure, nitinol polycrystals

(Some figures may appear in colour only in the online journal)

1. Introduction

Enumeration of the cyclic hysteretic features of a superelastic-plastic nitinol cubic block subjected to cyclic biaxial deformations in the [111] direction is the aim of this paper.

It is known that the energy variation during hysteretic cycles is accompanied by significant changes in stiffness and strength under different external stimuli [1–3]. This is due to the superelasticity and plasticity features observed in the behavior of nitinol [4–8]. The constitutive laws of the nitinol alloy subjected to multiaxial loading are primarily based on experimental results and are generally not available in the literature. Most tests refer to 1D wires, and as a consequence, most of the phenomenological features of the 3D constitutive models remain unexplained.

The tension–torsion tests conducted on thin-walled tubes of nitinol alloy using various loading/unloading paths under isothermal conditions were performed by McNaney, Imbeni, and Jung [9]. Their experimental results show significant variations in the mechanical response along the loading axes.

In this paper, a virtual cyclic biaxial test for a nitinol cubic block is proposed based on the Bouc–Wen model coupled to the intrinsic time measure from the endochronic theory. The goal is to obtain information about the effect of martensitic variants as well as the microstructural texture on stiffness and strength degradation and the opposite phenomena.

The endochronic theory is capable of capturing some features of the elastoplastic deformation of materials under loading and unloading, such as linear and nonlinear hardening; retardation of the vector and scalar properties of materials when a break in the strain path takes place; hysteresis, and stabilization of hysteresis under cyclic loading; and the effects of cyclic creep.

Other hysteretic features such as loss of lateral stiffness in loading cycles (stiffness degradation) and the degradation of strength when cyclical loading is done to the same displacement level (strength degradation) are strongly followed by a significant change in response.

This approach belongs to the class of theories of functional materials with internal variables [10–12].
Starting from the relationship between the Bouc model and the endochronic theory, the thermodynamic admissibility of the Bouc–Wen model was proved in [13] by adopting an intrinsic time measure.

We are interested in analyzing the behavior of a cubic block of nitinol fixed along the normal direction at one end and subjected to biaxial deformation in the [111] direction under isothermal conditions. The analysis is made by including the intrinsic time measure in the Bouc–Wen model.

This paper is organized as follows: section 2 is devoted to the relationship between the endochronic theory and the Bouc–Wen hysteresis model. The Bouc–Wen model was proposed without a thermodynamical analysis, but the thermodynamic admissibility of this model is achieved by adopting the intrinsic time measure. Section 3 summarizes the constitutive equations and the Bouc–Wen model coupled with the intrinsic time measure are presented. The cyclic biaxial loadings of the cubic block and the change in the properties of the cubic block during cyclic loading are discussed in section 4. The theory of small, infinitesimal elastic deformations superimposed on a finite elastic deformation is used to analyze the effect of deformation on the properties of the material. The state of austenite and martensite before deformation is cubic, characterized by two elastic Lamé constants. During biaxial loading, the material becomes orthorhombic, characterized by nine constants. The capturing of the incipient stiffness and strength degradation and the opposite phenomena after several cycles is discussed in section 4. The last section is devoted to conclusions.

2. Time-intrinsic measure

The intrinsic time measure $\theta$ was introduced by Valanis in 1971 as a non-decreasing function which depends on the strain tensor $\varepsilon$. It can be expressed as [13–15]

$$d\theta = (d\varepsilon : d\varepsilon)^{1/2},$$

where the double-dot product of two tensors is denoted by $\ldots$, i.e., $A : B = \delta_{ij} \delta_{kl} A_{ij} B_{kl}$, with $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. For $p = I$, the relation (2.1) is reduced to $d\theta = ||d\varepsilon||$.

We suppose that the Helmholtz free energy density $\Psi$ depends on a single internal variable tensor $\chi$:

$$\Psi = (C_0/2) \operatorname{tr} (\varepsilon)^2 + (C_2/2) \varepsilon_d : \varepsilon_d + B_0 \operatorname{tr} (\varepsilon) \operatorname{tr} (\chi) + B_2 \varepsilon_d : \chi_d + (D_0/2) \operatorname{tr} (\chi)^2 + (D_2/2) \chi_d : \chi_d,$$

where $\varepsilon_d = \varepsilon - 1/3 \operatorname{tr} (\varepsilon) I$ and $\chi_d = \chi - 1/3 \operatorname{tr} (\chi) I$ are the deviatoric parts of the strain tensor $\varepsilon$ and of the internal variable tensor $\chi$, respectively; $I$ is the unit tensor; and the constants verify the conditions $C_0 = C_1 + C_2/3$, $C_2 > 0$, $D_0 = D_1 + D_2/3$, $D_1, D_2 > 0$, and $B_1^2 \leq C_1 D_1$, $B_2^2 \leq C_2 D_2$, deduced from thermodynamic considerations [10, 13].

The state equation $\sigma = \frac{\partial \Psi}{\partial \varepsilon}$, where $\sigma$ is the stress tensor, leads to

$$\sigma = (1/3) \operatorname{tr} (\varepsilon) I + \sigma_d = (C_0 \operatorname{tr} (\varepsilon) + B_0 \operatorname{tr} (\chi)) I + C_2 \varepsilon_d + B_2 \chi_d,$$

where $\sigma_d$ is the deviatoric part of the stress tensor. The thermodynamic force $\tau = \frac{\partial \Psi}{\partial \chi}$ associated with the internal variable tensor $\chi$ becomes

$$\tau = (1/3) \operatorname{tr} (\varepsilon) I + \tau_d = (B_0 \operatorname{tr} (\varepsilon) + D_0 \operatorname{tr} (\chi)) I + B_2 \varepsilon_d + D_2 \chi_d,$$

where $\tau_d$ is the deviatoric part of the thermodynamic force and $\tau = 3B_0 \operatorname{tr} (\varepsilon) I$ is the elastic hydrostatic response. The second principle for small isothermal transformations states that the intrinsic mechanical dissipation $\Phi_i$ must be non-negative:

$$\Phi_i = \sigma : \dot{\varepsilon} - \Psi \geq 0,$$

or

$$\Phi_i = -\frac{\partial \Psi}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial \theta} = -\tau : \dot{\chi} \geq 0,$$

The second principle inequality (2.6) is rewritten by introducing $\theta$ [13]:

$$\Phi_i = -\tau : \frac{d\theta}{dt} \geq 0,$$

and it is satisfied if there exists a positive convex dissipation potential $\varphi \left( \frac{d\chi}{d\theta} \right)$ such that

$$\varphi \left( \frac{d\chi}{d\theta} \right) = I_0 \left( \operatorname{tr} \left( \frac{d\chi}{d\theta} \right) \right) + \left( b_2/2 \right) \left\| \frac{d\chi}{d\theta} \right\|^2 \geq 0, \varphi(0) = 0, \frac{d\theta}{dt} \geq 0,$$

where $b_2 > 0$. The term $I_0 \left( \operatorname{tr} \left( \frac{d\chi}{d\theta} \right) \right)$ is zero if $\operatorname{tr} \left( \frac{d\chi}{d\theta} \right) = 0$, and $\infty$ elsewhere.

In addition, it is easy to show that the hydrostatic part of the internal variable remains constant and that it is vanishing if $\operatorname{tr} (\chi (0)) = 0$, i.e.,

$$\operatorname{tr} \left( \varphi \left( \frac{d\chi}{d\theta} \right) \right) = 0, \frac{d\theta}{dt} = -\frac{\tau_d}{b_2}.$$

From the deviatoric part of (2.4) we get

$$\tau_d = B_2 \varepsilon_d + D_2 \chi_d.$$
The solution to (2.10) is given by [13]:

\[ \tau_d = B_2 \int_0^\beta \exp \left( -\frac{D_2}{b_2} (\theta - \theta') \frac{\partial e_d(\theta')}{\partial \theta'} \right) d\theta ', \]  

(2.11)

for \( \tau_d(0) = 0 \). By substituting (2.10) and (2.11) into (2.3) we have

\[ \sigma_d = \left( C_2 - \frac{B_2^2}{D_2} \right) \varepsilon_d + \frac{B_2}{D_2} \tau_d = \left( C_2 - \frac{B_2^2}{D_2} \right) \varepsilon_d + \frac{B_2}{D_2} \int_0^\beta \exp \left( -\frac{D_2}{b_2} (\theta - \theta') \frac{\partial e_d(\theta')}{\partial \theta'} \right) d\theta ', \]  

(2.12)

where \( C_2 - \frac{B_2^2}{D_2} \geq 0, \frac{B_2}{D_2} > 0 \). The term \( \left( C_2 - \frac{B_2^2}{D_2} \right) \varepsilon_d \) describes the hardening effect, whereas the integral corresponds to a smooth hysteretic behavior.

By taking \( A_0 = C_2 - \frac{B_2^2}{D_2}, A = \frac{B_2}{D_2}, \beta = \frac{D_2}{b_2}, \) and \( \mu(\theta) = A \exp(-\beta \theta), \) equation (2.12) becomes

\[ \sigma_d = A_0 \varepsilon_d + \frac{B_2}{D_2} \tau_d = A_0 \varepsilon_d + A \int_0^\beta \mu(\theta - \theta') \frac{\partial e_d(\theta')}{\partial \theta'} d\theta '. \]  

(2.13)

If we are taking

\[ z(t) = \int_0^\beta \mu(\theta - \theta') \frac{\partial e_d(\theta')}{\partial \theta'} d\theta ', \]  

(2.14)

Equations (2.13) and (2.14) become

\[ \sigma_d = A_0 \varepsilon_d + z, \quad dz = Ad\varepsilon_d - \beta z d\theta. \]  

(2.15)

We recognize in (2.15) the hysteresis model proposed by Bouc [16] in the differential form

\[ w(t) = A_0 u(t) + z(t), \quad dz = Adu - \beta z d\theta, \]  

(2.16)

where \( u(t) \) and \( w(t) \) are the input and output time-dependent functions and \( A_0 \geq 0 \). The function \( z(t) \) represents the hysteretic auxiliary variable, which describes the time history of the input variable \( u \).

The function \( \mu(\theta - \theta') \geq 0 \) is continuous, bounded, and positive and non-decreasing in its interval. This function is known in the literature as the hysteretic kernel. In particular, the hysteretic kernel has an exponential form

\[ \mu(\theta) = A \exp(-\beta \theta), \quad A, \beta > 0. \]  

(2.17)

The time function \( \theta \) is positive and non-decreasing and, according to Bouc, may represent the total variation of \( u \):

\[ \theta(t) = \int_0^t \frac{du}{d\tau} d\tau, \]  

(2.18)

or

\[ d\theta = |du|, \quad \text{with } \theta(0) = \theta_0. \]  

(2.19)

More general formulation for (2.16) was proposed in the literature. For example, Bouc suggested the form [17]

\[ dz = Adu - \beta \varepsilon |du| - \gamma |z| du, \quad \gamma < \beta. \]  

(2.20)

Wen [18] has proposed another model, with \( n > 0 \):

\[ dz = Adu - (\beta \text{ sign } (z du) + \gamma) |z| \int |du|. \]  

(2.21)

Baber and Wen [19] have advanced the stiffness and strength degradation model

\[ \eta dz = Adu - \nu (\beta \text{ sign } (z du) + \gamma) |z| \int |du|, \]  

(2.22)

where \( A \) is a parameter that controls the tangent stiffness and ultimate hysteretic strength; \( \beta, \gamma, n \) are the hysteretic shape parameters; and \( \nu, \eta \) are the strength and stiffness degradation parameters (for \( \nu = \eta = 1 \) the model does not degrade). These parameters depend on the dissipated hysteretic energy.

All parameters that appear in (2.20)–(2.22) for the hysteretic restoring force control the scale and general shape of the hysteretic loop, whereas \( n \) controls the smoothness of the loop. \( \beta \) and \( \gamma \) describe softening or hardening: if \( \beta + \gamma \) is positive the system exhibits softening, whereas if \( \beta + \gamma \) is negative, the system exhibits hardening. If \( \beta \) decreases, the width of the loop becomes large because the dissipation energy due to the hysteresis becomes larger.

These models were proposed without a thermodynamical analysis. By adopting the intrinsic time measures from the endochronic theory of plasticity, the thermodynamic admissibility of the Bouc–Wen model is proved. The conditions \( A > 0 \) and \( -\beta \leq \gamma \leq \beta \) are necessary and sufficient for the thermodynamic admissibility of the Bouc–Wen model [25].

3. Polycrystalline model

Let us consider a cubic block of nitinol with side \( L \) fixed along the normal direction at one end and subjected to cyclic biaxial deformations in the [111] direction.

Starting with (2.22), we describe the behavior of the cubic block by using the Bouc–Wen model, expressed by a set of differential equations

\[ \ddot{u} + aku + (1 - \alpha)kz = F(x, t), \]  

(3.1)

and

\[ \eta \dot{z} = A\dot{u} - \nu (\beta \text{ sign } (z \dot{u}) + \gamma) |z| \int \dot{u}. \]  

(3.2)

where \( u \) is displacement, \( k \) is the linear stiffness coefficient, and \( F(x, t) \) is the cyclic biaxial loading force. This force depends on the temperature which governs the phase transformations from austenite into martensite, \( A \rightarrow M \), and from martensite into austenite, \( M \rightarrow A \).

This force is expressed as \( F(x, t) = \frac{\tilde{\beta} \delta_T}{\tilde{\beta} M} \), where \( x \) is the loading direction, which is, in the case of a nitinol sample, the [111] direction; \( T(x, t) \) is the relative temperature related to the absolute temperature \( \Theta \) through the relation \( T = \Theta - \Theta_{ref} \), with \( \Theta_{ref} \) being a reference temperature; and \( \beta \) is the density of the nitinol, \( \beta = (3\alpha_a + 2\mu_a) \alpha_a \) with \( \alpha_a \) being the coefficient of linear thermal expansions.

The primary Lamé elastic constants for cubic symmetry are \( \lambda_a \) and \( \mu_a \). The Lamé constants and \( \alpha_a \) obey the rule of
mixture
\[ \lambda_\alpha = (1 - \xi M) \lambda_A + \xi M \lambda_{AM}, \quad \mu_\alpha \]
\[ = (1 - \xi M) \mu_A + \xi M \mu_{AM}, \quad \alpha_\alpha \]
\[ = (1 - \xi M) \alpha_A + \xi M \alpha_{AM}, \]  \hspace{1cm} (3.3)

where the superscripts \( A \) and \( M \) are for austenite and martensite, respectively.

The model (equations (3.1) and (3.2)) can be generalized for a continuum system such as the deformation of a nitinol cubic block. For this we attach to the theory a model which is able to reproduce the superelastic effect and also plasticity for states of stress. This model is based on the works [1, 6–8].

Hooke’s law for nitinol is given by
\[ \sigma = \lambda_\alpha \left( \varepsilon - \varepsilon'' \right) \delta + 2 \mu_\alpha \left( \varepsilon - \varepsilon'' \right) \]
\[ - \beta_\delta \delta \left( T - T_0 \right), \]  \hspace{1cm} (3.4)

where \( T_0 \) is the initial temperature and \( \delta \) is the delta symbol.

The hysteretic restoring force \( z \) is in the form of (3.1). The non-damping restoring force is composed of the linear restoring force \( \alpha dz \) and the hysteretic restoring force \( \left( 1 - \alpha \right) z \), where \( 0 < \alpha < 1 \) is the rigidity ratio representing the relative participations of the linear and nonlinear terms. The quantity \( z \) is known as the hysteretic restoring force.

For the time function measure, \( \theta \) is the total variation in the strain \( \varepsilon = \frac{\Delta \varepsilon}{\varepsilon}; \), i.e.,
\[ d\theta = |d\varepsilon|, \quad \left( t \right) = \int_0^t \frac{d\varepsilon}{dr} \, dr, \quad \theta (0) = \theta_0. \]  \hspace{1cm} (3.5)

The hysteretic operator (3.2) possesses non-symmetrical characteristics due to the term \( \Delta \varepsilon \) \( \text{sgn} \) \( \theta \), with \( \delta \) being a non-symmetrical factor.

The control variables are the uniaxial stress \( \sigma \) and the relative temperature \( T \). We suppose that any strain increment is decomposed into three possible components: elastic, recoverable transformation, and plastic:
\[ \Delta \varepsilon = \Delta \varepsilon'^{el} + \Delta \varepsilon'' + \Delta \varepsilon'^{pl}. \]  \hspace{1cm} (3.6)

The residual transformation component \( \varepsilon'^{res} \) is created instantaneously at the point of the switch to cyclic behavior, as a fraction of the recoverable transformation. This assumes the existence of a permanent (residual) fraction of martensite \( \xi^{res} \).

Thus, the total Lagrangian strain has four possible components: an elastic component \( \varepsilon'^{el} \), a plastic component \( \varepsilon'^{pl} \), a recoverable transformation component \( \varepsilon'' \), and a residual transformation component \( \varepsilon'^{res} \):
\[ \varepsilon = \varepsilon'^{el} + \varepsilon'' + \varepsilon'^{pl} + \varepsilon'^{res}. \]  \hspace{1cm} (3.7)

As internal variables, we choose the single-variant martensite fraction \( \xi_M \), for which the following relation must be verified:
\[ \xi_M + \xi_A + \xi^{res} = 1, \quad \xi_M + \xi_A = 0. \]  \hspace{1cm} (3.8)

The constitutive law for the transformation strain rate \( \dot{\varepsilon''} \) is given by
\[ \dot{\varepsilon''} = \beta_1 \xi_M \left( \frac{s}{2 \sigma_{eq}} + \alpha_1 I \right), \]  \hspace{1cm} (3.9)

where \( \alpha_1, \beta_1 \) are material constants, \( s = \sigma - \sigma_{eq}I \) is the deviatoric stress tensor, \( \sigma_{eq} = \frac{1}{3} tr \sigma \) is the hydrostatic stress, and \( I \) is the second-order unit tensor.

The parameter \( \alpha_1 \) is related to the transformation volume strain. The influence of hydrostatic stress on transformation is obtained by \( \alpha_1 \sigma_{eq} \). The maximum transformation strain in the loading direction is obtained after integrating the volumetric term in (3.9), which is \( \varepsilon_{vol}^{tr} = 3 \alpha_1 \beta_1 \). Also, the maximum transformation strain in the tensile direction is obtained by \( \varepsilon_{11}^{tr} = \beta_1 (1 + \alpha_1) \). The parameters \( \alpha_1 \) and \( \beta_1 \) can be determined after the values of \( \varepsilon_{vol}^{tr} \) and \( \varepsilon_{11}^{tr} \) are known. The following results are obtained: \( \alpha_1 = -3.15\% \) and \( \beta_1 = 4.13\% \) [9].

The constitutive law for plastic strain rate is given by
\[ \dot{\varepsilon'} = \frac{3}{4} \frac{s}{\sigma_{eq} Y_{pl}} \frac{s}{\mu_{eq} H_{pl}} \delta, \]  \hspace{1cm} (3.10)

where \( \sigma_{eq} = \sqrt{\frac{s}{2}} \) is the equivalent Cauchy stress based on the von Mises yield criteria, \( H_{pl} = \frac{d Y_{pl}}{d \sigma_{eq}} \) is the plastic hardness, \( Y_{pl} \) is the plastic hardening function of the material, and \( \dot{\varepsilon'} = \int_{\text{history}} \frac{2}{\sigma_{eq} Y_{pl}} \, d\varepsilon' ; \, d\varepsilon' \) is the equivalent plastic strain.

4. Biaxial loading of the cubic block

Let us assume a nitinol cubic block biaxially deformed in the [111] direction. The loading path consists in a simultaneous tension–torsion in tensile and torsional axes under isothermal conditions. Rates for each axis are independently controlled so that the end strain is reached simultaneously. The virtual test mimics a biaxial torsion and tension test with cycling up and down of 25 mm (at 5 cm min\(^{-1}\)) in the [111] direction, while twisting the cube back and forth 45 degrees.

The theory of small, infinitesimal elastic deformations superimposed on a finite elastic deformation is used to analyze the effect of loading on the behavior of the material. For this, we consider, simultaneously, three distinct configurations of the material points relative to the same origin \( O \) [23].

1. The first configuration, \( C_0 \). The natural or stress-free configuration \( C_0 \) is represented by the system \( (\xi_0) = (\xi_1, \xi_2, \xi_3) \) (figure 1). This system is associated with the preferred directions of the elastic material in order to take the maximum advantage of the material symmetry for simplifying the form of the constitutive equations. In this system we also have a simple form of the definition of elastic constants. The primary elastic constants are denoted by \( C_{ij}^{0} \) in Voigt’s notation. The initial undeformed system is a cubic one characterized by two constants. Due to a biaxial deformation, the points of the body are carried from the configuration \( C_0 \) to the
where \( \theta = \arccos \frac{1}{\sqrt{3}} \), \( \varphi = 45^\circ \). The geometrical elastic constants \( C'_{ij} \) in the configuration \((x')\) are related to the primary elastic constants \( C^0_{pq} \) in \((\xi)\) by

\[
C'_{ij} = R_{ij} R_{pq} C^0_{pq},
\]

The average elastic constants in the [111] direction in the cubic materials are defined as [23]

\[
a_{ij} = \frac{1}{2\pi} \int_0^{2\pi} C'_{ij} \, d\psi,
\]

where the angle \( \psi \) is defined in figure 1.

By denoting the constants \( a_{ijkl} \) by \( C'_{ijkl} \), we obtain from (4.5) and (4.6)

\[
\begin{pmatrix}
C'_{11} & C'_{12} & C'_{13} & 0 & 0 & 0 \\
C'_{11} & C'_{13} & 0 & 0 & 0 & 0 \\
C'_{12} & C'_{13} & C'_{33} & 0 & 0 & 0 \\
C'_{44} & 0 & 0 & C'_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2 \left( C'_{11} - C'_{12} \right)
\end{pmatrix}
\]

with

\[
\begin{align*}
C'_{11} &= C'_{22} = \alpha_{11} = \alpha_{22} = \left( C^0_{11} + C^0_{12} + 2 C^0_{14} \right)/2, \\
C'_{33} &= \alpha_{33} = \left( C^0_{11} + 2 C^0_{12} + 4 C^0_{14} \right)/3, \\
C'_{12} &= \alpha_{12} = \alpha_{21} = \left( C^0_{11} + 5 C^0_{12} - 2 C^0_{14} \right)/6, \\
C'_{13} &= C'_{23} = \alpha_{32} = \alpha_{31} = \alpha_{31} = \left( C^0_{11} + 2 C^0_{12} - 2 C^0_{14} \right)/3, \\
C'_{44} &= \alpha_{44} = \alpha_{55} = \alpha_{44} = \left( C^0_{11} - C^0_{12} + C^0_{14} \right)/3, \\
C'_{66} &= \alpha_{66} = \left( C^0_{11} - C^0_{12} + 4 C^0_{14} \right)/6 \\
&= 1/2 \left( C'_{11} - C'_{12} \right).
\end{align*}
\]

The relation of \( C'_{ij} \) to the primary Lamé constants is given by

\[
\begin{align*}
C'_{11} &= C'_{22} = \alpha_{11} = \alpha_{22} = \left( \lambda_a + 2 \mu_a \right), \\
C'_{33} &= \alpha_{33} = 4 \left( \lambda_a + \mu_a \right)/3, \\
C'_{12} &= \alpha_{12} = \lambda_a, \\
C'_{13} &= C'_{23} = \alpha_{23} = \alpha_{32} = \alpha_{31} = \lambda_a, \\
C'_{44} &= \alpha_{44} = \alpha_{55} = \mu_a, \\
C'_{66} &= \alpha_{66} = 1/2 \left( C'_{11} - C'_{12} \right) = \mu_a.
\end{align*}
\]

The tensor \( C' = a \) has five independent elements, expressed as a combination of the three primary elastic constants \( C^0_{11}, C^0_{12}, C^0_{44} \). All directions in \((x', x_2')\) are equivalent.

Thus, in the geometrical system, the material has a transversely isotropic symmetry.
3. The third configuration, $\tilde{C}$. Due to biaxial deformation the points of the body are carried from $C_0$ to the initially deformed equilibrium configuration $\tilde{C}$. In this configuration the material has a well-textured structure. Interaction between grains is sensitive to the generally non-uniform distribution of individual crystal orientations in the nitinol material. So biaxial loading may affect the texture properties during the phase transformation of the material [19–21].

The initially deformed equilibrium configuration $\tilde{C}$ has the coordinates $(X) = (X_1, X_2, X_3)$. This configuration is obtained from $(x') = O x_1 x_2 x_3'$ with a rotation around $x'_2 \equiv X_3$ of angle $\theta = \arcsin(1/\sqrt{3})$. The transformation from $(\xi) = (\xi_1, \xi_2, \xi_3)$ to $(X) = (X_1, X_2, X_3)$ is defined as [23]

$$X_i = (\xi_1 + \xi_2 + \xi_3)/\sqrt{3}, \quad X_2 = (-\xi_1 + \xi_2)/\sqrt{2}, \quad X_3 = (-\xi_1 - \xi_2 + 2\xi_3)/\sqrt{6}. \quad (4.9)$$

The systems $(x')$ and $(X)$ are related by the transformation

$$X_i = x'_i (1 + \varepsilon_i), \quad i = 1, 2, 3, \quad (4.10)$$

where $\varepsilon_i$ are the biaxial deformations in Voigt’s notation:

$$\varepsilon_{11} = \varepsilon_1, \quad \varepsilon_{22} = \varepsilon_2, \quad \varepsilon_{33} = \varepsilon_3, \quad 2\varepsilon_{12} = \varepsilon_4, \quad 2\varepsilon_{13} = \varepsilon_5, \quad 2\varepsilon_{23} = \varepsilon_6, \quad \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0, \quad (4.11)$$

The biaxial stresses are

$$\sigma_1 = \sigma_2 = \sigma, \quad \sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = 0. \quad (4.12)$$

The relation between $\varepsilon_1$ and $\varepsilon$ in (4.11) is obtained from the condition $\sigma_3 = 0$.

By substituting (4.3) into (4.10) we have the relation between $(\xi)$ and $(X)$:

$$X_i = R_{ik} \xi_k (1 + \varepsilon_i), \quad i = 1, 2, 3, \quad (4.13)$$

The relation between $C_{ij}$ and $\tilde{C}_{ij}$ is given by

$$\tilde{C}_{ij} = \left(\frac{X}{\xi}\right)^{-1} \partial X_i / \partial \xi_j C_{kl} \partial X_k / \partial \xi_l. \quad (4.14)$$

So the elastic constants in the deformed configuration are given by

$$\tilde{A}_{ij} = \tilde{C}_{ij} + \sigma \delta_{ij}. \quad (4.15)$$

The system loses transverse symmetry and becomes orthorhombic with nine constants $\tilde{A}_{ijkl}$:

$$\tilde{A}_{11} = \tilde{C}_{11} + \sigma, \quad \tilde{A}_{22} = \tilde{C}_{22} + \sigma, \quad \tilde{A}_{33} = \tilde{C}_{33}, \quad \tilde{A}_{12} = \tilde{C}_{12}, \quad \tilde{A}_{13} = \tilde{C}_{13}, \quad \tilde{A}_{23} = \tilde{C}_{23}, \quad \tilde{A}_{44} = \tilde{C}_{44} + \sigma, \quad \tilde{A}_{55} = \tilde{C}_{55} + \sigma, \quad \tilde{A}_{66} = \tilde{C}_{66} + \sigma. \quad (4.16)$$

where

$$\tilde{C}_{ij} = \frac{(1 + \varepsilon)^2}{36(1 + \varepsilon^3)} C_{11}, \quad \tilde{C}_{22} = \frac{(1 + \varepsilon)^2}{12(1 + \varepsilon^3)} C_{11}, \quad \tilde{C}_{12} = \frac{(1 + \varepsilon)^2}{12(1 + \varepsilon^3)} C_{12}, \quad \tilde{C}_{13} = \frac{1}{18} (1 + \varepsilon^3) C_{13}, \quad \tilde{C}_{33} = \frac{(1 + \varepsilon)^3}{9(1 + \varepsilon^2)} C_{33}, \quad \tilde{C}_{23} = \frac{1}{6} (1 + \varepsilon^3) C_{13}, \quad \tilde{C}_{44} = \frac{1}{6} (1 + \varepsilon^3) C_{44}, \quad \tilde{C}_{55} = \frac{1}{6} (1 + \varepsilon^3) C_{44}, \quad \tilde{C}_{66} = \frac{(1 + \varepsilon)^2}{12(1 + \varepsilon^3)} C_{66}. \quad (4.17)$$

Thus, we can write the constants as (4.16)

$$\begin{pmatrix}
\tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & 0 & 0 & 0 \\
\tilde{A}_{22} & \tilde{A}_{23} & 0 & 0 & 0 & 0 \\
\tilde{A}_{33} & 0 & 0 & 0 & 0 & 0 \\
\tilde{A}_{44} & 0 & 0 & 0 & 0 & 0 \\
\tilde{A}_{55} & 0 & 0 & 0 & 0 & 0 \\
\tilde{A}_{66} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \quad (4.18)$$

The system loses transverse symmetry and becomes orthorhombic with nine constants. These constants depend on the initial state of strain and stress.

5. Deformation of the nitinol cubic block

A cubic block of nitinol with side $L = 10\text{ mm}$ is fixed along the normal direction at one end and is subjected to cyclic biaxial deformations in the [111] direction. The loading path consists in simultaneous tension–torsion in the tensile and torsional axes. The rates for each axis are independently controlled so that the end strain is reached simultaneously. The unloading path unloads tensile and torsional axes simultaneously to zero strain.

The tensile strains range from 0% to 6.0%. The maximum shear strain is unlimited because the numerical tests show no predisposition to buckling. The strain rates are not fixed for the entire test and range between $10^{-5} \text{ s}^{-1}$ and $10^{-2} \text{ s}^{-1}$. For pure tension and pure torsion, the strain rate during loading/unloading is $10^{-4} \text{ s}^{-1}$. During mixed-mode proportional loading/unloading tests, the tensile strain rate is fixed at $10^{-4} \text{ s}^{-1}$, whereas the shearing strain rate is adjusted to maintain the prescribed loading path.

The state of austenite and martensite before deformation is cubic, characterized by two elastic Lamé constants, $\lambda_a$ and $\mu_a$. During biaxial loading, the material becomes orthorhombic, characterized by nine constants. The material parameters in the configuration $C_0$ are shown in table 1. The material constants in the deformed configuration are calculated from (4.15)–(4.18).

Interaction between grains is sensitive to the generally non-uniform distribution of individual crystal orientations in
The nitinol material. So cyclic biaxial loading may affect the texture properties during the phase transformation of the material [19–21].

To capture the stiffness and strength degradation and the opposite phenomena, the equivalent Cauchy stress \( \sigma_{eq} \) versus the equivalent Lagrangian strain response \( \varepsilon_{eq} \) under biaxial loading/unloading cycles is computed for different values of the parameters \( \nu \) and \( \eta \).

The ratio \( r \) between the maximum equivalent Cauchy stress and the maximum equivalent Lagrangian strain is defined as [9]

\[
\begin{align*}
    r &= \frac{\max(\sigma_{eq})}{\max(\varepsilon_{eq})}, \\
    \sigma_{eq} &= \sqrt{\sigma_T^2 + 3\sigma_I^2}, \\
    \varepsilon_{eq} &= \frac{\varepsilon_T^2}{\varepsilon_I^2 + (4/3)\varepsilon_T^2},
\end{align*}
\]

(5.1)

where \( \sigma_I(\sigma_T) \) and \( \varepsilon_I(\varepsilon_T) \) are the tensile (shearing) stress and strain.

The other parameters are fixed to be \( A = 0.75, \beta = -0.25, \gamma = 1 \).

The variation in \( r \) with respect to the number of cycles is presented in figure 2 for \( \nu = 1.25 \) and \( \eta = 1.25 \). The ratio degradation in the first 6 cycles (blue lines) can be explained by alterations in the texture properties during the cyclic loadings. The opposite phenomenon of stopping the ratio degradation appears at about 7 cycles and is characterized by progressive enhancement of the ratio (red line). The numerical experiments for an increased number of biaxial loading/unloading cycles (blue line) show long lives and high fatigue limits at about \( 10^7 \) cycles. This implies that there is an accumulation of plasticity with cycling.

Figure 3 displays the variation in ratio \( r \) versus \( \eta \) (for \( \nu = 1 \) and \( \nu = 1.1 \)) and \( \nu \) (for \( \eta = 1 \) and \( \eta = 1.1 \)) in the first six cycles.

Figure 4. Stiffness degradation (\( \nu = 1, \eta = 1.25 \)).

Figure 5. Strength degradation (\( \nu = 1.25, \eta = 1 \)).

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Table 1. Material parameters of a nitinol cubic block in the configuration \( C_0 \) [26].

| Parameter | Symbol | Value | Unit |
|-----------|--------|-------|------|
| Density   | \( \rho_0 \) | \( 6.45 \times 10^3 \) | Kg m\(^{-3}\) |
|           | \( \lambda, \lambda_M \) | \( 28.26, 6.3 \) | GPa |
| Elastic Lamé moduli | \( \mu_A, \mu_M \) | \( 18.85, 4.19 \) | GPa |
| Poisson’s ratio | \( \nu_A = \nu_M \) | 0.3 |
| Coefficient of linear thermal expansions | \( \alpha_A, \alpha_M \) | \( 12.5 \times 10^{-6} \) | 1/°C |
To understand stiffness and strength degradation and their opposite phenomena, the same computations are made separately for $\nu = 1$ and $\eta = 1.25$, and $\nu = 1.25$ and $\eta = 1$, respectively.

Figure 4 displays the incipient stiffness degradation associated with the asymmetry of the material response because of the texture, as expected. The stiffness degradation occurs when the elastic stiffness degrades with increasing ductility. The sequence of occurrence of the cycles is introduced in all figures.

Figure 5 displays the incipient strength degradation, also associated with the asymmetry of the material response. The strength degradation is described by reducing the capacity in the backbone curve. The incipient degradation of strength is done to the same displacement level.

Figure 6 shows the initial stiffness degradation for $\nu = 1$ and $\eta = 1.1$. The progressive loss of stiffness in each loading cycle is shown in figures 5(a)–(c) after three, five, and six cycles, respectively.

Figures 7(a)–(c) show the initial strength degradation for $\nu = 1.1$ and $\eta = 1$ after three, four, and five cycles, respectively.

These results are consistent with the experimental measurement of a nitinol cubic block subjected to tension and compression loadings [22].

The stopping of the stiffness and strength degradations is captured next, after several loading/unloading paths. Figure 8 shows the stopping of the stiffness degradation and the progressive enhancement of the stiffness (red lines) for $\nu = 1$ and $\eta = 1.25$. A possible explanation can be the interaction between grains, the redistribution of individual crystal orientations in the texture of the polycrystalline specimen, and the phase transformation. In support of this idea, there is the experimental observation according to which the grain boundaries make dislocation movement difficult, and thus they are important in achieving optimum superelasticity [9]. Hysteresis is the result of the complex interactions between grains and competing energies among the material’s grains and phases and their interaction with the external stimulus [20, 27].

We point out that the intrinsic time measure is used to model the entire behavior of the material with a self-consistent method.
Similarly, figure 9 shows the change in response by the strength degradation stopping and significant strengthening (red lines) for $\nu = 1.25$ and $\eta = 1$.

Such behavior is due to the permanent-strain addition of residual martensite and alterations of the texture properties. In support of this explanation there is the experimental observation of the rearrangement of the microstructure in order to prohibit the mobility of small grains [21, 27]. The residual martensite is accompanied by a grain-size reduction, and this mechanism hinders small-
grain motion and renders the material stronger than previously. Determination of residual martensite is not sufficient to describe the small-grain motion. A better approximation is done by using the intrinsic time measure as an internal variable. With this variable the evolution of the material can be fully described.

We observe from figures 8 and 9 that the material does not remember the early degradation in both stiffness and strength and still behaves in a unique manner, with significant growth in both stiffness and strength.

At this stage we can only assume that this growth will decrease progressively as the number of cycles of loadings increases, preparing the material for breaking, as shown in figure 2.

Finally, figure 10 shows the comparison of the plots \( \sigma_{eq} - \varepsilon_{eq} \) obtained experimentally [9] and computed according to our theory and the unmodified Bouc–Wen model [28], respectively. These plots refer to the equivalent stress–strain in simultaneous tension 6% and torsion 2% with simultaneous unloading. The results show that the paths calculated with our theory are closer to those obtained experimentally than those obtained by the unmodified Bouc–Wen model. The loading paths are similar to those under pure tension or torsion. Smooth plateaus are obtained at approximately constant stresses. The experimentally sharp changes in the stresses during loading and unloading are effected by the generation of martensitic variants, whereas existing variants lead to higher stresses due to their energetic incompatibility with the current loading state [9].

6. Conclusions

This paper addresses a modified Bouc–Wen model that is able to reproduce the cyclic behavior of a superelastic-plastic nitinol cubic block subjected to cyclic biaxial deformations in the [111] direction.

The modified Bouc–Wen model consists of the classical Bouc–Wen model coupled to an intrinsic time measure other than clock time, which governs the behavior of the materials. As a consequence, the thermodynamic admissibility of the Bouc–Wen model is provided by the endochronic theory of plasticity. The role of the intrinsic time measure is described by capturing the stiffness and strength degradation and the opposite phenomena. Such behavior is due to the permanent-strain addition of residual martensite and alterations in the properties of the texture during the phase transformation.

To capture the stiffness and strength degradation and the opposite phenomena, the equivalent Cauchy stress versus equivalent Lagrangian strain response under loading/unloading cycles are computed. The superelastic effect ensures not only the stopping of the incipient degradation in stiffness and strength but also the recovery of large deformations in loading/unloading cycles. A peculiar aspect is observed in the behavior of the material. The nitinol cubic block does not remember the early degradation in stiffness and strength and still behaves in a unique manner, with significant growth in both stiffness and strength.

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