Dressing the Quark with QCD Condensates

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Abstract: A condensate of $p = 0$ partons in the perturbative vacuum gives rise to a term $\propto \delta^4(p)$ in the free PQCD propagators. The leading condensate contribution to the quark propagator can be exactly summed since there is a factor $\delta^4(p)$ associated to each loop. We calculate the dressed quark propagator in the presence of either a gluon or a quark condensate, for a number of colors $N \to \infty$. The dressed quark propagator satisfies a Dyson-Schwinger type equation which can be exactly solved within our framework.

In the case of a gluon condensate the dressed quark propagator has no pole, hence quarks cannot appear in asymptotic states, and moreover the DS equation has a solution which spontaneously breaks chiral symmetry. We also calculate the dressed quark-photon vertex and verify that the corresponding Ward-Takahashi identity is satisfied, and that the dressed self-energy correction to the photon propagator does not shift the physical photon pole.

Keywords: Perturbative QCD, Quark and Gluon Condensates, $1/N$ Expansion.

*Research supported in part by the European Commission under contract HPRN-CT-2000-00130.
1. Introduction

In this paper we consider the contributions to the quark propagator from a $p = 0$ condensate term in the free gluon and quark propagators,

$$iD_{ab}^{\mu\nu}(p) = -\delta_{ab} g^{\mu\nu} \left[ \frac{i}{p^2 + i\epsilon} + \lambda_g^2 (2\pi)^4 \delta^4(p) \right]$$

$$iS_{AB}(p) = \delta_{AB} \left[ \frac{i\hat{p}}{p^2 + i\epsilon} + \lambda_q^3 (2\pi)^4 \delta^4(p) \right]$$

Due to the $\delta^4(p)$ factors which constrain loop integrals, the leading order contributions in the condensate parameters $\lambda_g$ and $\lambda_q$ arising from Feynman diagrams of arbitrary complexity may be resummed. (We simplify the topology of the diagrams by considering only contributions that are of leading order also in the number of colors $N$). We show that the quark propagator thus dressed satisfies a Dyson-Schwinger (DS) type equation (Figs. 2 and 7) which can be solved exactly. For $\lambda_g \neq 0$ the resulting propagator has a branch cut instead of a pole at $p^2 = 0$. Hence the quark does not propagate to the in and out states at $t = \pm\infty$. The DS equation also has a non-perturbative, chiral symmetry breaking solution which does not reduce to the free propagator in the $\lambda_g^2 \to 0$ limit.

We calculate the dressed quark-photon vertex and verify that the Ward-Takahashi identity is satisfied. This ensures that the quark loop correction to the photon propagator maintains the masslessness of the photon.

The condensate terms in Eqs. (1.1) and (1.2) arise when one starts the perturbative expansion from a free state which already contains gluon or quark pairs. Consider for simplicity the free Hamiltonian of a scalar field $\phi(x)$,

$$H_0(t) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left[ |\pi(t,p)|^2 + (p^2 + m^2)|\phi(t,p)|^2 \right]$$

where $\pi$ is the canonical momentum. $H_0$ is a sum (over 3-momentum $p$) of uncoupled harmonic oscillator Hamiltonians. The ground state wave function is thus

$$\Psi_0(t) \propto \exp \left[ -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} E_p |\phi(t,p)|^2 \right]$$

where $E_p = \sqrt{p^2 + m^2}$. Correspondingly, in a path integral formulation of perturbation theory it is straightforward to verify that imposing a wave function of the form (1.4) at $t = \pm T$ and then letting $T \to \infty$ one recovers the standard Feynman propagator of the scalar field. On the other hand, if one uses a gaussian wave function with a more general coefficient $C(p)$,

$$\Psi_C(t) \propto \exp \left[ -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} C(p) |\phi(t,p)|^2 \right]$$

then the on-shell part of the free propagator of momentum $p$ is modified by a term proportional to

$$i [C(p) - E_p] \delta(p^0 - E_p).$$
The usual Feynman rules for calculating higher order contributions apply provided the ‘condensate’ term (1.4) is always included in the free propagator.

The $\lambda_g$, $\lambda_q$ terms in the free gluon and quark propagators similarly arise from gaussian wave functions of the form (1.5) at $t = \pm \infty$, with $m = 0$ and $C(p = 0) \neq 0$. Explicit Lorentz invariance is maintained since the wave function is modified only at $p = 0$.

The modified gaussian wave function (1.5) may be interpreted as describing a coherent superposition (or condensate) of particle pairs by expanding the exponential,

$$\exp \left[ -\frac{1}{2}C(p)|\phi|^2 \right] = \sum_n \frac{(-1)^n}{2^n n!} [C(p) - E_p]^n |\phi(t, p)|^{2n} \exp \left[ -\frac{1}{2}E_p |\phi(t, p)|^2 \right] \quad (1.7)$$

Analogously, the condensate terms in Eqs. (1.1) and (1.2) may be viewed as arising from the presence of $p = 0$ gluon and quark pairs in the asymptotic in and out states.

Formally, a perturbative expansion around any state that has a non-vanishing overlap with the true ground state is equally justified. Thus to our knowledge there is no theoretical reason to exclude the condensate terms in the free gluon and quark propagators in a perturbative expansion of QCD. As we shall see, these terms strongly modify long distance physics but give only higher-twist corrections at short distance. The summation of leading condensate contributions that we consider here provides a kind of ‘tree’ approximation, to which corrections corresponding to higher powers of $g^2$ may be systematically added. This tree approximation allows studies of analyticity and unitarity in a novel setting, which may be closer to the confinement domain than standard PQCD.

Our approach may be compared to that of the QCD sum rules [3], where external $p = 0$ lines connect to vacuum matrix elements of quark and gluon operators. Assuming that the strong coupling freezes in the long distance regime at a sufficiently small value to make the perturbative expansion relevant [4], we need only a single parameter $\lambda_g$ ($\lambda_q$) in the gluon (quark) propagator to calculate any Green function. This avoids the use of sum rules, which require assumptions concerning their saturation and parametrizations of vacuum expectation values.

In section 2 we calculate the effects of a quark condensate ($\lambda_g = 0$, $\lambda_q \neq 0$) on the dressed quark propagator, quark-photon vertex and photon self-energy. We simplify the topology of the contributing diagrams by taking the limit of a large number of colors, $N \to \infty$ with $g^2N$ fixed [3]. The quark condensate introduces a parameter $\mu_q$ with the dimension of mass,

$$\mu_q^3 = g^2N \lambda_q^3 \quad (1.8)$$

The full expansion of any Green function $G$ is then a double sum of the form

$$G_q = \sum_{\ell=0}^{\infty} (g^2N)^\ell \sum_{n=0}^{\infty} C_q^{\ell,n} \mu_q^{3n} \quad (1.9)$$

where the index $q$ on $G_q$ indicates the presence of a quark condensate ($\lambda_q \neq 0$). We calculate the complete sum over $n$ for $\ell = 0$. This is possible since for $\ell = 0$ there is a $\delta^4(p)$ factor from the free quark propagator (1.2) in each loop integral. The remaining sum over
ℓ is then of usual perturbative form, involving higher powers of the coupling \( g^2 N \) and an increasing number of non-trivial loop integrals.

We neglect contributions where the condensate \( p = 0 \) lines are dressed, giving factors of \( \delta^4(0) \) proportional to the volume of space-time. Such terms should factorize from physical quantities. Similarly contributions where condensate lines interact with themselves are dropped, being singular and of a non-local nature.

We find that the quark acquires an effective, ‘constituent’ mass, as expected from the fact that the inclusion of quark pairs in the asymptotic states (\( \lambda_q \neq 0 \)) breaks chiral symmetry \[2\]. Gauge invariance is preserved and the photon remains massless.

In section 3 we study in a similar way the effects of a gluon condensate (\( \lambda_g \neq 0, \lambda_q = 0 \)). The gluon condensate mass parameter is

\[
\mu_g^2 = g^2 N \lambda_g^2
\]

and the full QCD expansion of any Green function is of the form

\[
G_g = \sum_{\ell=0}^{\infty} (g^2 N)^\ell \sum_{n=0}^{\infty} C_{\ell,n}^g \mu_g^{2n}
\]

We evaluate the full \( \ell = 0 \) ‘tree’ contribution in the \( N \rightarrow \infty \) limit. The DS equation (Fig. 7) for the dressed quark propagator is an ordinary second order algebraic equation. The solution has a cut instead of a pole at \( p^2 = 0 \), implying that the quark is removed from the spectrum of on-shell states. Surprisingly, there is also a non-perturbative solution which breaks chiral invariance. We verify the Ward-Takahashi identity, which ensures the masslessness of the photon within dimensional regularization.

We do not evaluate here the dressed gluon propagator, nor do we consider the case of a simultaneous gluon and quark condensate (\( \lambda_g, \lambda_q \neq 0 \)). The topological structure of the contributing Feynman diagrams is more complicated in these cases, principally due to contributions from the four-gluon vertex. We have verified that the gluon condensate contributions of order \( \mu_g^2 \) and \( \mu_g^4 \) (for \( \ell = 0 \)) to the gluon self-energy have the required transverse structure. We postpone to a future work the calculation of the ‘tree’ (\( \ell = 0 \)) dressed gluon propagator to all orders in the gluon condensate parameter \( \mu_g^2 \). Section 4 contains an outlook.

2. Quark condensate: \( \lambda_g = 0, \lambda_q \neq 0 \).

2.1 The dressed quark propagator

The first correction to the quark propagator (\( \ell = 0, n = 1 \) in \[1.9\]) is given by the diagram of Fig. 1a, where only the \( \lambda_q^3 \) term of the free propagator \[1.2\] is kept. This condensate term is proportional to \( \delta^4(k) \), where \( k \) is the internal quark momentum, and is indicated pictorially by a cut line. The quark self-energy, defined by amputating the external quark lines, is \[2\]:

\[
\Sigma_q(p) = \frac{2\mu_q^3}{p^2 + i\varepsilon}
\]
Figure 1: (a) The quark self-energy $\Sigma_q$ at $\mathcal{O}(\mu_q^3)$. The cut internal quark line indicates that only the term $\propto \delta^4(k)$ is kept for the internal propagator. (b) Diagrams which might a priori contribute to order $\mu_q^6$. (c) A contribution proportional to $\delta^4(0)$. (d) Another singular contribution.

From now on the color indices and trivial color factors such as $\delta_{AB}$ will be implicit. In \cite{2.1} we approximated $C_F = (N^2 - 1)/2N$ by its large $N$ limit $C_F \simeq N/2$.

At the next order ($n = 2$) the five Feynman diagrams of Fig. 1b, which have at least two internal quark lines, may contribute. Of these, the last two diagrams are suppressed in the large $N$ limit. The second diagram can contribute only via the two possible cuts shown in Figs. 1c and 1d. Fig. 1c is a self-interaction of a condensate line and thus proportional to $\delta^4(0)$, i.e., to the volume $\int d^4x$ of space-time. We expect such contributions to factorize from connected Green functions and do not consider them further. Fig. 1d involves scattering of condensate lines and is similarly non-local. The third diagram of Fig. 1b can be dropped for the same reason. Hence only the first diagram of Fig. 1b contributes at order $\ell = 0$, $n = 2$.

The above analysis generalizes in a straightforward way to higher powers of $\mu_q^3$, i.e., to terms in (1.3) with $\ell = 0$ and arbitrary $n$. As a result, only diagrams corresponding to
the geometric series dress the quark propagator $S_q(p)$:

$$iS_q(p) = \frac{i}{p} + \frac{i}{p} (i\Sigma_q(p)) \frac{i}{p} + \frac{i}{p} (i\Sigma_q(p)) \frac{i}{p} + \frac{i}{p} (i\Sigma_q(p)) \frac{i}{p} + \ldots$$  \hspace{1cm} (2.2)

implying the self-consistency equation shown in Fig. 2,

$$S_q(p) = \frac{1}{p} - \frac{1}{p} \Sigma_q(p)S_q(p)$$  \hspace{1cm} (2.3)

with solution

$$S_q(p) = \frac{1}{p} \frac{1}{\Sigma_q(p)} = \frac{1}{p} \frac{1}{\Sigma_q(p)} = \frac{p^4 p - 2\mu_q^3 p^2}{p^6 - (2\mu_q^3)^2}$$  \hspace{1cm} (2.4)

The poles of $S_q(p)$ at $(p^2)^3 = (2\mu_q^3)^2$ correspond to a ‘constituent’ quark mass

$$M_q = \mu_q \sqrt{2}$$  \hspace{1cm} (2.5)

and two complex poles at $p^2 = M_q^2 \exp(\pm 2i\pi/3)$. The appearance of a constituent quark mass $M_q \neq 0$ could be anticipated from the fact that the quark condensate gives a chirally non-invariant free propagator (1.2).

We stress that the quark condensate contributions are power suppressed at short distance,

$$p^2 \to \infty \Rightarrow S_q(p) = \frac{1}{p} + O\left(\frac{\mu_q^3}{p^6}\right)$$  \hspace{1cm} (2.6)

### 2.2 Dressed quark-photon vertex

In the $N \to \infty$ limit, the leading behaviour in $\mu_q$ of the quark-photon vertex $\Gamma_\mu^q(k, \bar{k})$ is given by the series shown in Fig. 3, where $\bar{k} = k - p$. This specific structure arises because cutting both the quark and antiquark lines appearing in between two successive gluon exchanges would prevent the momentum $p$ from flowing through the diagram from left to right. One can easily check that

$$\Gamma_\mu^q = V_\mu^q + W_\mu^q - \gamma^\mu$$  \hspace{1cm} (2.7)

where the effective vertices $V_\mu^q$ and $W_\mu^q$ satisfy the coupled equations shown in Fig. 4:

$$V_\mu(k, \bar{k}) = \gamma^\mu - \frac{\mu_q^3}{2k^2} \gamma^\nu S_q(p)W_\mu(p, 0)\gamma^\nu$$

$$W_\mu(k, \bar{k}) = \gamma^\mu - \frac{\mu_q^3}{2k^2} \gamma^\nu V_\mu(0, -p)S_q(-p)\gamma^\nu$$  \hspace{1cm} (2.8)
\[
\Gamma_{q}^{\mu}(k,\bar{k}) = \gamma_{\mu} + 2 \frac{\mu_{q}^2}{\not{p}} \left( \frac{1}{k^2} - \frac{1}{\bar{k}^2} \right) p^{\mu} - \frac{2 \mu_{q}^6}{2 \mu_{q}^6 - p^2} \left( \frac{1}{k^2} + \frac{1}{\bar{k}^2} \right) (\not{p} p^{\mu} - p^2 \gamma_{\mu})
\] (2.9)

Using the results of Appendix A for \(V^{\mu}\) and \(W^{\mu}\) one has:

\[
\Pi_{q}^{\mu\nu}(p) = e^2 N \lambda_3^3 \left\{ \text{Tr} \left[ \gamma^{\nu} V^{\mu}(0,-p) S(-p) \right] + \text{Tr} \left[ \gamma^{\nu} S(p) W^{\mu}(p,0) \right] \right\}
\] (2.11)

Using the results of Appendix A for \(V^{\mu}\) and \(W^{\mu}\) one has:
Figure 5: Photon self-energy $\Pi_{\mu\nu}^q$ in a quark condensate.

\[
V^\mu(0,-p) = \gamma^\mu - \frac{\mu_g^2}{2p^2} \left[ A\gamma^\mu + Bp^\mu + C\gamma p^\mu \right]
\]

\[
W^\mu(p,0) = \gamma^\mu - \frac{\mu_g^2}{2p^2} \left[ A\gamma^\mu - Bp^\mu + C\gamma p^\mu \right]
\]

(2.12)

where $A, B, C$ are given in (A.5). A short calculation yields:

\[
\Pi_{\mu\nu}^q(p) = 16e^2 \frac{N\lambda^3_q\mu_g^2}{2\mu_g^2 - p^2} \left( p^2 g_{\mu\nu} - p^\mu p^\nu \right) \equiv \Pi_{\mu\nu}^q(p^2) \left( p^2 g_{\mu\nu} - p^\mu p^\nu \right)
\]

(2.13)

The fact that $\Pi_{\mu\nu}^q(p^2 = 0)$ is finite implies that the photon remains massless after the self-energy correction.

3. Gluon condensate: $\lambda_g \neq 0, \lambda_q = 0$.

3.1 Dressed quark propagator

At first order in $\mu_g^2$ ($\ell = 0, n = 1$ in (1.11)), the quark propagator is modified by the correction shown in Fig. 6a. The internal gluon line is cut, indicating that only the condensate term proportional to $\lambda_g^2 \delta^4(k)$ is kept in the free gluon propagator (1.1). At second order, namely $\mu_g^4$ ($\ell = 0, n = 2$), only the two diagrams shown in Fig. 6b contribute. Other diagrams, such as the three last ones in Fig. 1b (with two internal cut gluon instead of quark lines) can be neglected for reasons similar to the case $\lambda_q \neq 0, \lambda_g = 0$ studied in the previous section. They are either suppressed by powers of $1/N$ in the large $N$ limit, or represent non-local contributions which should factorize from physical quantities.

At higher orders ($\ell = 0, n > 2$ in (1.11)) the relevant diagrams contributing to the quark propagator in the large $N$ limit are of the type shown in Fig. 6c, where all internal gluon lines are cut. All loop integrals are trivial due to the $\delta^4(k)$ factors from the condensate terms, showing that we are in effect dealing with a tree approximation. Note that we take into account one-particle irreducible as well as reducible diagrams, thus including all $\ell = 0$ contributions.

One may readily check that the complete set of relevant diagrams for the quark propagator $S_g(p)$ is generated by iterating the implicit equation shown in Fig. 7, which reads

\[
p S_g(p) = 1 - \frac{1}{2} \mu_g^2 \gamma^\mu S_g(p) \gamma_{\mu} S_g(p)
\]

(3.1)

where we used $C_F = N/2$ at leading order in $N$. 

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Figure 6: Quark propagator $S_g(p)$ in a gluon condensate, (a) at order $\mu_g^2$ and (b) at order $\mu_g^4$. (c) A generic diagram of order $\mu_g^{2n}$.

$S_g(p) = \frac{p}{p^2 + \sqrt{p^2(p^2 - 4\mu_g^2)}}$

Figure 7: Implicit equation for the dressed quark propagator $S_g(p)$. The full blob denotes the gluon condensate dressing.

Lorentz invariance constrains the quark propagator to be of the form

$$S_g(p) = a(p^2) \not{p} + b(p^2)$$

(3.2)

Substituting this in (3.1) one finds the equations

$$b(1 + \mu_g^2 a) = 0$$

$$a p^2 (1 - \mu_g^2 a) = 1 - 2 \mu_g^2 b^2$$

(3.3)

A chiral symmetry conserving quark propagator must have $b = 0$. With this constraint the second order equation for $a$ gives

$$S_{g1}(p) = \frac{2\not{p}}{p^2 + \sqrt{p^2(p^2 - 4\mu_g^2)}}$$

(3.4)

where we chose the sign of the square root so as to ensure that the propagator approaches the free one in the $p^2 \rightarrow \infty$ limit:

$$p^2 \rightarrow \infty \Rightarrow S_{g1}(p) = \frac{1}{p^2} \left[ 1 + \mathcal{O}\left(\frac{\mu_g^2}{p^2}\right) \right]$$

(3.5)

It is important to note that the $p^2 = 0$ quark pole of the ($\lambda_q = 0$) free propagator (1.2) has been removed by the dressing. The dressed propagator $S_{g1}(p)$ instead has branch
point singularities at $p^2 = 0$ and $p^2 = 4\mu_g^2$. In Appendix B (cf. Eq. (3.9)) we show that this removes the quark from the set of in and out states, in the sense that the Fourier transformed propagator vanishes at large times,

$$|S_{g1}(t, \vec{p})| \sim O\left(1/\sqrt{|t|}\right)$$

This raises interesting questions about analyticity and unitarity, which we hope to return to in a future paper.

If one does not impose chiral symmetry and thus allows $b \neq 0$ in (3.3) one finds

$$S_{g2}(p) = -\frac{1}{\mu_g^2} \left( \gamma \pm \sqrt{p^2 + \frac{1}{2}\mu_g^2} \right)$$

This solution is singular for $\mu_g^2 \to 0$ and hence does not have a power expansion in $\mu_g^2$ of the form (1.11). It emerges as a ‘non-perturbative’ solution of the implicit equation (3.1). Like the chiral symmetry conserving solution $S_{g1}$, the propagator $S_{g2}$ has no quark pole, only a branch point at $p^2 = -\mu_g^2/2$.

The solution $S_{g2}$ does not approach the perturbative propagator $1/\gamma$ at large $p^2$, and must therefore be discarded on physical grounds at short distance. However, we note that $S_{g2} = S_{g1}$ at $p^2 = -\mu_g^2/2$. In a loop integral over $p^0$ it is then possible (for $p^2 > \mu_g^2/2$) to choose the $S_{g2}$ solution in the interval $-\sqrt{p^2 - \mu_g^2/2} \leq p^0 \leq \sqrt{p^2 - \mu_g^2/2}$. This will give Green functions that break chiral symmetry. It is interesting to find chiral symmetry breaking without having introduced a quark condensate in the asymptotic states ($\lambda_q = 0$).

### 3.2 Dressed quark-photon vertex

The $q\bar{q}\gamma$ vertex $\Gamma^\mu_g(k, \bar{k})$ is given by the series shown in Fig. 8 for large $N$ and $\lambda_g \neq 0$. Again, the loop integrals are trivially performed in this $\ell = 0$ approximation of (1.11) since all internal gluon lines are cut. The infinite sum satisfies the implicit equation of Fig. 9,

$$\Gamma^\mu_g(k, \bar{k}) = \gamma^\mu - \frac{1}{2}\mu_g^2 \gamma^\nu S_g(k)\Gamma^\mu_g(k, \bar{k})S_g(\bar{k})\gamma^\nu$$

When the vertex is expanded on its independent Dirac components this equation reduces to a set of linear equations for the components. We do this explicitly in Appendix C for the case $S_g = S_{g1}$ (cf. Eqs. (C.4) and (C.7)). Thus the solution is unique when $S_g$ is given.

Multiplying (3.8) by $p_\mu = (k - \bar{k})_\mu$ and noting the propagator identity (3.1),

$$\frac{1}{2}\mu_g^2 \gamma^\mu S_g(k)\gamma_\mu = S_g^{-1}(k) - \frac{i}{\gamma}$$

$$\Gamma^\mu_g(k, \bar{k}) = \ldots$$

**Figure 8:** Quark-photon vertex $\Gamma^\mu_g(k, \bar{k})$ in a gluon condensate.
one easily verifies that $p_\mu \Gamma^\mu_g = S^{-1}_g(k) - S^{-1}_g(\bar{k})$ is a solution. Hence we conclude that the unique solution of (3.8) respects the Ward-Takahashi identity (2.10).

For highly virtual momenta $k^2, \bar{k}^2 \to \infty$, we have $S_g(k) \to 1/k, S_g(\bar{k}) \to 1/\bar{k}$ and (3.8) thus implies

$$k^2, \bar{k}^2 \to \infty \Rightarrow \Gamma^\mu_g(k, \bar{k}) = \gamma^\mu + \mathcal{O} \left( \frac{\mu_2^2}{k^2}, \frac{\mu_2^2}{\bar{k}^2} \right)$$

(3.10)

This may also be verified from the explicit expression (C.9) of the dressed vertex.

3.3 Photon self-energy $\Pi^\mu_\nu(p)$

Given the novel analytic structure of the dressed ($\lambda_g \neq 0$) quark propagator (cf. (3.4) and (3.7)) it is of interest to verify that the physical $p^2 = 0$ pole remains in the photon propagator. At large $N$ a single (dressed) quark loop dominates the photon self-energy $\Pi^\mu_\nu_g(p)$. Thus (cf. Fig. 10),

$$\Pi^\mu_\nu_g(p) = ie^2N \int \frac{d^Dk}{(2\pi)^D} \text{Tr} \left[ \gamma^\nu S_g(k)\Gamma^\mu_g(k, \bar{k})S_g(\bar{k}) \right]$$

(3.11)

At highly virtual loop momenta the condensate dressing is power suppressed, cf. Eqs. (3.3) and (3.10). We dimensionally regularize in $D$ dimensions the ultraviolet quadratic and logarithmic divergences, which are independent of $\mu_2^2$.

Multiplying the self-energy (3.11) by $p_\mu$ and using the Ward-Takahashi identity (2.10) one finds $p_\mu \Pi^\mu_\nu_g(p) = 0$. Hence the self-energy is transverse,

$$\Pi^\mu_\nu_g(p) = \Pi_g(p^2) (p^2 g^{\mu\nu} - p^\mu p^\nu)$$

(3.12)
In order to prove that the photon remains massless, we show that $\Pi_\mu^g(p) = 3p^2 \Pi_g(p^2)$ vanishes in the $p^2 \to 0$ limit. Since this quantity only depends on $p^2$ we may equivalently consider the 4-vector $p = k - \bar{k} \to 0$ limit.

Following Fradkin [6], we note that the $k \to \bar{k}$ limit of the Ward-Takahashi identity (2.10) reads

$$p^\mu \Gamma_g^\mu(k, k) = p^\mu \partial S - \frac{1}{g} \frac{\partial S_g(k)}{\partial k^\mu} \partial k^\mu$$

which vanishes within dimensional regularization since the integrand is a full derivative.

Hence the resummed photon propagator

$$\frac{1}{p^2[1 - \Pi_g(p^2)]} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2}\right)$$

has the physical photon pole at $p^2 = 0$. We give an explicit integral expression for $\Pi_g(p^2)$ in Eq. (C.13), for the case of the quark propagator $S_g$ given in (3.4).

4. Outlook

Our work raises several issues which merit further study. First of all, we did not consider the condensate corrections to the gluon propagator. The four-gluon coupling causes a richer set of diagrams to contribute in this case. However, this appears to be only a technical complication. The lowest order ($\ell = 0, n = 1$ in (1.11)) gluon condensate correction to the gluon propagator was evaluated in [7] and found to have the required transverse structure. We have checked that the $n = 2$ correction is also transverse. The expansion of the gluon self-energy dressed with a gluon condensate starts as

$$\Pi^{\mu\nu}(p) = \Pi_0^{\mu\nu}(p) + \mu_g^2 \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) \left(-2 + \frac{21}{2} \frac{\mu_g^2}{p^2} + \ldots\right)$$

where $\Pi_0^{\mu\nu}(p)$ is the standard perturbative ($\mu_g^2 = 0$) result.

It will also be worthwhile to study the case where both $\lambda_g$ and $\lambda_q$ are non-zero, which we have not considered here. The richer set of diagrams then appears not only for the gluon, but also for the quark propagator.

We took the $N \to \infty$ limit to simplify the set of contributing diagrams. The algebraic nature of the resummation holds for any $N$. It would be interesting to derive the generalizations to finite $N$ of DS equations like (2.3) and (3.1).
We considered only the case of a vanishing bare quark mass, cf. (1.2). For \( m_q \neq 0 \) the on-shell condition (1.6) of a quark condensate with \( p = 0 \) would have the form \( \delta(p^0 - m_q) \) and thus would break explicitly the Lorentz symmetry. On the other hand, it is straightforward to generalize our analysis to \( m_q \neq 0 \) in the presence of a gluon condensate \( \lambda_g \neq 0, \lambda_q = 0 \), which preserves Lorentz invariance. One finds that the modified implicit equation (3.1) for the quark propagator \( S_q(p) \) is then of fourth order in the parameters \( a, b \) of Eq. (3.2).

In this paper we did not sufficiently address the treatment of contributions proportional to the volume of space-time \( \delta^4(0) \). Such terms correspond to interactions between the \( p = 0 \) condensate particles themselves and we expect that they factorize from measurable quantities. A more systematic treatment of these effects would be desirable.

The restriction to \( p = 0 \) condensates required for explicit Lorentz invariance is very constraining. \( C(p = 0) \) in the asymptotic wave function (1.5) is then the only parameter, which translates into \( \lambda_g \) and \( \lambda_q \) in the free gluon and quark propagators. We believe that systematic studies of the properties of perturbative expansions with a \( p = 0 \) condensate term are warranted.

Our result that the quark propagator dressed with a gluon condensate (3.4) has a cut and hence vanishes at large times (cf. (3.6)) addresses the question of the analytic structure of confined partonic Green functions [8, 9]. This obviously also has a profound effect on unitarity relations – as it should to describe a physical situation where completeness sums involve hadrons rather than partons.

**Acknowledgments**

We thank T. Binoth, S. Brodsky, A. Cabo and E. Pilon for valuable discussions. S. P. would also like to thank the people of the SPhT in Saclay for their interesting comments on our work.

**APPENDIX**

**A. Quark-photon vertex** \( \Gamma^\mu_q(k, \bar{k}) \): \( \lambda_g = 0, \lambda_q \neq 0 \).

The quark-photon vertex \( \Gamma^\mu_q(k, \bar{k}) \) is given in terms of the quantities \( V^\mu \) and \( W^\mu \), see (2.7). Here we solve the equations (2.8) for \( V^\mu \) and \( W^\mu \). By iterating these equations one can infer that \( V^\mu \) and \( W^\mu \) have the general form:

\[
V^\mu(k, \bar{k}) = \gamma^\mu - \frac{\mu^3_q}{2k^2} \left[ A\gamma^\mu + Bp^\mu + C\phi p^\mu \right]
\]

\[
W^\mu(k, \bar{k}) = \gamma^\mu - \frac{\mu^3_q}{2k^2} \left[ A'\gamma^\mu + B'p^\mu + C'\phi p^\mu \right]
\]

where \( A, B, C, A', B', C' \) are functions of \( p^2 \) only. Writing the quark propagator (2.4) as

\[
S_q(p) = a\phi + b
\]
one then inserts \((A.1)\) and \((A.2)\) into \((2.8)\). A straightforward calculation yields six equations for the unknown quantities \(A, B, C, A', B', C'\):

\[
A = -2b (1 - \frac{\mu_3^3}{2p_2^3} A') \quad B = 4a - \frac{2\mu_3^3}{p_2^2} (aA' + bB' + ap^2C') \quad C = \frac{\mu_3^3}{p_2^2} (aB' + bC')
\]

\[
A' = -2b (1 - \frac{\mu_3^3}{2p_2^3} A) \quad B' = 4a - \frac{2\mu_3^3}{p_2^2} (aA - bB + ap^2C) \quad C' = \frac{\mu_3^3}{p_2^2} (-aB + bC)
\]

The solutions are found to be:

\[
A = A' = \frac{4\mu_3^3L^2}{p^6 - 2\mu_3^6} \quad B = -B' = \frac{4}{p^2} \quad C = C' = \frac{4\mu_3^3}{2\mu_0^2 - p^6}
\]

Using these expressions in \((A.1)\) and \((2.7)\) gives the result \((2.9)\) for the dressed quark-photon vertex \(\Gamma^\mu_q(k, \bar{k})\).

**B. Asymptotic behaviour of the dressed quark propagator \(S_q(t, \vec{p})\)**

We derive in this appendix the asymptotic time behaviour of the dressed \((\lambda_g \neq 0, \lambda_q = 0)\), chirally symmetric quark propagator \(S_{q1}(p)\) given in Eq. \((3.4)\). This propagator may be written as

\[
S_{q1}(p) = \frac{2p^2}{\mu^2} \left[1 - \frac{p^2 - \mu^2}{\sqrt{p^2 + i\varepsilon}\sqrt{p^2 - \mu^2 + i\varepsilon}}\right]
\]

where we define \(\mu = 2\mu_g\) in the present appendix, and the \(i\varepsilon\) prescription arises from the usual Feynman prescription of the free \((p \neq 0)\) quark propagator \(\hat{p}/(p^2 + i\varepsilon)\). The Fourier transformed propagator

\[
S_q(t, \vec{p}) = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} S_q(p) \exp(-itp_0)
\]

is thus

\[
S_q(t, \vec{p}) = \frac{2(i\gamma^0 \partial_t - \frac{\vec{p} \cdot \vec{\gamma}}{\mu^2})}{p^2 - \mu^2} \left[\delta(t) + (p^2 + \mu^2 + \partial_t^2)J(t, p^2, \mu^2)\right]
\]

\[
J(t, p^2, \mu^2) = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{\exp(-itp_0)}{\sqrt{p_0^2 - p^2 + i\varepsilon}\sqrt{p_0^2 - \mu^2 + i\varepsilon}}
\]

The function \(J(t, p^2, \mu^2)\) can be evaluated using Feynman parametrization,

\[
\frac{1}{\sqrt{A + i\varepsilon\sqrt{B + i\varepsilon}}} = \frac{1}{\pi} \int_{0}^{1} \frac{dx}{\sqrt{x(1 - x)}} \frac{1}{(1 - x)A + xB + i\varepsilon}
\]

and doing the \(p_0\)-integral using Cauchy’s theorem. The result is:

\[
J(t, p^2, \mu^2) = \frac{e^{-ipt}}{2i\pi} \int_{0}^{1} \frac{dx}{\sqrt{x(1 - x)}} \frac{1}{\sqrt{p^2 + x\mu^2}} \exp\left[\frac{-it|x|\mu^2}{\sqrt{p^2 + x\mu^2 + |\vec{p}|}}\right]
\]
The behaviour of \( J(t, p^2, \mu^2) \) for \(|t| \to \infty\) can be inferred by noticing that the integrand in (B.6) is peaked at \( x \to 0 \) in this limit. With the change of variable \( y = |t| \mu^2 x/(2|p|) \) one obtains

\[
J(t, p^2, \mu^2) \xrightarrow{|t| \to \infty} \frac{e^{-i|t|p}}{t \pi \sqrt{2|t|p} \mu^2} \int_0^\infty \frac{dy}{\sqrt{y}} e^{-iy} = -1 + i \frac{e^{-i|t|p}}{2\sqrt{\pi} \sqrt{|t|p} \mu^2}
\]

(B.7)

where we used

\[
\int_0^\infty \frac{dy}{\sqrt{y}} \cos(y) = \int_0^\infty \frac{dy}{\sqrt{y}} \sin(y) = \sqrt{\pi} / 2
\]

(B.8)

Using (B.7) in (B.3) gives the asymptotic time behaviour

\[
S_g(t, p) \xrightarrow{|t| \to \infty} -\frac{1 + i}{2\sqrt{\pi}} \frac{|p| \gamma^0 - p \cdot \gamma^5}{\sqrt{|t|p} \mu_g^2} \exp(-i|t|p)
\]

(B.9)

C. Quark-photon vertex \( \Gamma_\mu^\nu(k, \bar{k}) \): \( \lambda_g \neq 0, \lambda_q = 0 \).

In this appendix we solve the implicit equation (3.8) for the \( q\bar{q}\gamma \) vertex \( \Gamma_\mu^\nu \) in the case of the chirally invariant quark propagator (3.4),

\[
S_{g1}(p) = a_p \gamma^\mu \ ; \ a_p = \frac{1}{2\mu_g^2} \left( 1 - \sqrt{1 - 4\mu_g^2/p^2} \right)
\]

(C.1)

Using this expression for \( S_g \) Eq. (3.8) becomes

\[
\Gamma_\mu^\nu(k, \bar{k}) = \gamma_\mu - \frac{1}{2} f\gamma^\nu \Gamma_\mu^\nu(k, \bar{k}) \bar{k}\gamma_\nu
\]

where we introduced another dimensionful parameter \( f \),

\[
f = \mu_g^2 a_k a_{\bar{k}}
\]

(C.2)

Chiral and parity invariance restricts \( \Gamma_\mu^\nu(k, \bar{k}) \) to the form

\[
\Gamma_\mu^\nu(k, \bar{k}) = A_0 \gamma_\mu + A_1 k^\mu \bar{k} + A_2 k^\nu \bar{k} + A_3 k^\mu \bar{k} + A_4 \bar{k}^\mu \bar{k} + iA_5 \gamma_5 \epsilon^\mu(\gamma, k, \bar{k})
\]

(C.4)

where we defined

\[
\epsilon^\mu(\gamma, k, \bar{k}) = \epsilon^{\mu\rho\sigma\delta} \gamma_\rho \gamma_\sigma k_\delta \bar{k}_\sigma
\]

(C.5)

We find the coefficients \( A_i \) by inserting (C.4) into (C.2) and using

\[
\bar{k}^\mu \gamma_\mu = k^\mu \bar{k} + \bar{k}^\mu k - k \cdot \bar{k} \gamma_\mu - i\gamma_5 \epsilon^\mu(\gamma, k, \bar{k})
\]

\[
i\gamma_5 \epsilon^\mu(\gamma, k, \bar{k}) = -i\gamma_5 \epsilon^\mu(\gamma, k, \bar{k}) k \cdot \bar{k} + \gamma^\mu \left[ k^2 \bar{k} - (k \cdot \bar{k})^2 \right] - k^\mu \left[ k^2 \bar{k} - k \cdot \bar{k} \bar{k} \right]
\]

(C.6)

This gives the conditions

\[
A_0 = 1 - f k \cdot \bar{k} (A_0 + k \cdot \bar{k} A_5) + f k^2 \bar{k}^2 A_5
\]

\[
A_1 = f k^2 (A_2 - A_5)
\]

\[
A_2 = f (A_0 + k \cdot \bar{k} A_5 + k^2 A_1)
\]

\[
A_3 = f (A_0 + k \cdot \bar{k} A_5 + \bar{k}^2 A_4)
\]

\[
A_4 = f k^2 (A_3 - A_5)
\]

\[
A_5 = -f (A_0 + k \cdot \bar{k} A_5)
\]

(C.7)
with solutions
\[
A_0 = \frac{1 + f k \cdot \bar{k}}{1 + 2 f k \cdot k + f^2 k^2 \bar{k}^2} ; \quad A_5 = \frac{-f}{1 + 2 f k \cdot k + f^2 k^2 \bar{k}^2} \\
\]
\[
k^2 A_1 = \bar{k}^2 A_4 = -\frac{2 f k^2 \bar{k}^2}{1 - f^2 k^2 \bar{k}^2} A_5 ; \quad A_2 = A_3 = -\frac{1 + f^2 k^2 \bar{k}^2}{1 - f^2 k^2 \bar{k}^2} A_5
\] 

(C.8)

The result for \( \Gamma^\mu_g(k, \bar{k}) \) then follows from (C.4):
\[
\Gamma^\mu_g(k, \bar{k}) = \frac{1}{1 + 2 f k \cdot \bar{k} + f^2 k^2 \bar{k}^2} \left\{ (1 + f k \cdot \bar{k}) \gamma^\mu - f i \gamma_5 \epsilon^{\mu
u\rho\sigma} \gamma_\nu \gamma_\rho \gamma_\sigma \right\}
\]
\[
+ \frac{2 f^2}{1 - f^2 k^2 \bar{k}^2} (k^\mu \bar{k}^2 \bar{k} + \bar{k}^\mu \bar{k}^2 \bar{k}) + \frac{f (1 + f^2 k^2 \bar{k}^2)}{1 - f^2 k^2 \bar{k}^2} (k^\mu \bar{k} + \bar{k}^\mu \bar{k}) \}
\] 

(C.9)

Given this expression for the vertex one can directly show that it satisfies the Ward-Takahashi relation (2.10). Straightforward algebra yields:
\[
p_\mu \Gamma^\mu_g(k, \bar{k}) = \bar{k} \frac{1 - f \bar{k}^2}{1 - f^2 k^2 \bar{k}^2} - \bar{k} \frac{1 - f k^2}{1 - f^2 k^2 \bar{k}^2}
\] 

(C.10)

Using (C.3) and Eq. (3.3) (for \( b = 0 \)),
\[
a_p - \frac{1}{p^2} = \mu^2 a_p
\] 

(C.11)

one gets
\[
\frac{1 - f \bar{k}^2}{1 - f^2 k^2 \bar{k}^2} = \frac{1}{k^2 a_k} ; \quad \frac{1 - f k^2}{1 - f^2 k^2 \bar{k}^2} = \frac{1}{k^2 a_k}
\] 

(C.12)

Substituting these expressions in (C.10) gives the Ward-Takahashi relation (2.10).

The explicit expression in (C.9) for the vertex function allows us to write the photon self-energy \( \Pi_g(p^2) \) defined by (3.11) and (3.12) as
\[
\Pi_g(p^2) = \frac{8 i e^2 N}{3 p^2} \int \frac{d^D k}{(2\pi)^D} \left\{ \frac{\mu^2 g^2 k^2 a_k^2}{1 - \mu^2 g^2 k^2 a_k^2} \left[ \frac{k \cdot \bar{k} a_k a_{\bar{k}}}{} + \frac{\mu^2 g^2 k^2 a_k^2 a_{\bar{k}}^2}{1 + 2 \mu^2 g^2 k \cdot k a_k a_{\bar{k}}} + \frac{\mu^2 g^2 k^2 a_k^2 a_{\bar{k}}^2}{1 + 2 \mu^2 g^2 k \cdot k a_k a_{\bar{k}}} + \right] + \frac{2 \epsilon}{\bar{k} \cdot a_k a_{\bar{k}}} \left[ \right] \right\}
\] 

(C.13)

The factor \( \epsilon \) in (C.13) arises from the identity \( \gamma_\mu \gamma^\mu = D = 4 - \epsilon \). In the \( p = k - \bar{k} \to 0 \) limit the integrand reduces to
\[
\lim_{p \to 0} p^2 \Pi_g(p^2) = \frac{2 i e^2 N}{3 \mu^2 g^2} \int \frac{d^D k}{(2\pi)^D} \left\{ \frac{\left(1 - \sqrt{1 - 4 \mu^2 g^2 / k^2}\right)^3}{\sqrt{1 - 4 \mu^2 g^2 / k^2}} - \epsilon \sqrt{1 - 4 \mu^2 g^2 / k^2} \right\} = 0
\] 

(C.14)

The integral is found to vanish using Feynman parametrization as in (B.3) and performing a standard Wick rotation. This confirms the general result (3.15).
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