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with Linear-like Closed-loop Behavior

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Model Reference Adaptive Control with Linear-like Closed-loop Behavior

Mohamad T. Shahab and Daniel E. Miller

Abstract—It is typically proven in adaptive control that asymptotic stabilization and tracking holds, and that at best a bounded-noise bounded-state property is proven. Recently, it has been shown in both the pole-placement control and the $d$-step ahead control settings that if, as part of the adaptive controller, a parameter estimator based on the original projection algorithm is used and the parameter estimates are restricted to a convex set, then the closed-loop system experiences linear-like behavior: exponential stability, a bounded gain on the noise in every $p$-norm, and a convolution bound on the exogenous inputs; this can be leveraged to provide tolerance to unmodelled dynamics and plant parameter time-variation. In this paper, we extend the approach to the more general Model Reference Adaptive Control (MRAC) problem and demonstrate that we achieve the same desirable linear-like closed-loop properties.

I. INTRODUCTION

Adaptive control is an approach used to deal with systems with uncertain and/or time-varying parameters. In the classical approach to adaptive control, one combines a linear time-invariant (LTI) compensator together with a tuning mechanism to adjust the compensator parameters to match the plant. The first general proofs came around 1980, e.g. see [2], [3], [17], [19] and [20]. However, the original controllers are typically not robust to unmodelled dynamics, do not tolerate time-variations well, have poor transient behavior and do not handle noise/disturbances well, e.g. see [21]. During the following two decades, a good deal of research was carried out to alleviate these shortcomings; a number of small controller design changes were proposed, such as the use of signal normalization, deadzones and $\sigma$-modification, e.g. see [5], [6], [10], [9], and [25]; also, simply using projection onto a convex set of admissible parameters turned out to be powerful, e.g. see [8], [18], [26], [27] and [28]. However, in general these redesigned controllers provide asymptotic stability and not exponential stability, with no bounded gain on the noise\(^{1}\); that being said, some of them, especially those using projection, provide a bounded-noise bounded-state property, as well as tolerance of some degree of unmodelled dynamics and/or time-variations.

Recently, for discrete-time LTI plants, in both the $d$-step ahead control setting [12], [15], [16], and the pole-placement control setting [13], [14], a new approach has been proposed which not only provides exponential stability and a bounded gain on the noise, but also a convolution bound on the exogenous inputs; the resulting convolution bound is leveraged to prove tolerance to a degree of time-variations and to a degree of unmodelled dynamics [23]. As far as the authors are aware, such linear-like convolution bounds have never before been proven in the adaptive setting.

The key idea is to use the original projection algorithm in conjunction with a restriction of the parameter estimates to a convex set, although this convexity requirement was relaxed in [22], [14] and [24]. The goal of the present paper is to extend these linear-like results in the $d$-step ahead control setting to the more general Model Reference Adaptive Control (MRAC) problem.

Model reference adaptive control is an important approach to adaptive control where a pre-designed stable reference model is used to model the desired closed-loop behavior. Here we build on the results proven for the $d$-step-ahead adaptive control problem in [15] and [16], which is a special case of the more general MRAC problem considered here; because we are seeking stronger closed-loop properties than what is normally proven in the literature, more detailed analysis is needed in dealing with the MRAC setup, since the introduction of the reference model into the analysis brings extra complexity. We prove that the desirable linear-like closed-loop properties of exponential stability, a bounded gain on the noise in every $p$-norm and a convolution bound on the exogenous inputs, are achieved using a model reference adaptive controller; we also prove a stronger tracking result than what is usually found in the literature.

Notation. We use standard notation throughout the paper. We denote $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{C}$ as the set of real numbers, integers and complex numbers, respectively. We will denote the Euclidean-norm of a vector and the induced norm of a matrix by the subscript-less default notation $\| \cdot \|$. Let $S(\mathbb{R}^{p \times q})$ denote the set of $\mathbb{R}^{p \times q}$-valued sequences. Also, $\ell_\infty$ denotes the set of bounded sequences. For a signal $f \in \ell_\infty$, define the $\infty$-norm by $\| f \|_\infty := \sup_{t \in \mathbb{Z}} | f(t) |$. For a closed and convex set $\Omega \subseteq \mathbb{R}^p$, let the function $\text{Proj}_\Omega \{ \cdot \} : \mathbb{R}^p \mapsto \Omega$ denote the projection onto the set $\Omega$; because the set $\Omega$ is closed and convex, the function $\text{Proj}_\Omega$ is well-defined. If $\Omega \subseteq \mathbb{R}^p$ is a compact (closed and bounded) set, we define $\| \Omega \| := \max_{\| x \|_\infty \leq 1} \| x \|$. Let $I_p$ denote the identity matrix of size $p$. Define the normal vector $\mathbf{e}_j \in \mathbb{R}^p$ of appropriate length $p$ as $\mathbf{e}_j := \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^\top$. Last of all, for a signal $f \in S(\mathbb{R})$ which is sufficiently well-behaved to have a $z$-transform, we let $F(z)$ denote this quantity.

II. THE SETUP

In this paper we consider the following linear time-invariant (LTI) discrete-time plant:

$$\sum_{i=0}^{n} a_i y(t-i) = \sum_{i=0}^{m} b_i u(t-d-i) + w(t), \quad t \in \mathbb{Z}, \quad (1)$$

with $y(t) \in \mathbb{R}$ as the measured output, $u(t) \in \mathbb{R}$ as the control input, and $w(t) \in \mathbb{R}$ as the noise/disturbance. The plant parameters are regularized such that $a_0 = 1$, and the system delay is exactly $d$, i.e. $b_0 \neq 0$. Associated with this plant are the polynomials $A(z^{-1}) := \sum_{i=0}^{n} a_i z^{-i}$ and

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\(^{1}\)An exception is the work of Ydstie [28] where a bounded gain is proven.
$\mathbf{B}(z^{-1}) := \sum_{i=0}^{m} b_i z^{-i}$, the transfer function $z^{-d} \frac{\mathbf{B}(z^{-1})}{\mathbf{A}(z^{-1})}$, and the plant parameter vector:

$$\theta := [a_1, a_2, \ldots, a_n, b_0, b_1, \ldots, b_m];$$

we assume that $\theta$ belongs to a known set $S_{ab} \subset \mathbb{R}^{n+m+1}$. Observe that such a plant can be expressed in the $z$-transform domain as

$$\mathbf{A}(z^{-1})Y(z) = z^{-d}\mathbf{B}(z^{-1})U(z) + W(z). \quad (2)$$

The control objective is closed-loop stability and asymptotic tracking of a given reference signal $y^*(t) \in \mathbb{R}$ generated as the output of a stable reference model; more specifically, given pre-designed polynomials $\mathbf{H}(z^{-1}) := \sum_{i=0}^{n'} h_i z^{-i}$ and $\mathbf{L}(z^{-1}) := 1 + \sum_{i=1}^{n'} l_i z^{-i}$ (with $n' \leq n$), and given a bounded exogenous signal $r(t) \in \mathbb{R}$, we utilize the following reference model expressed in the $z$-transform form:

$$\mathbf{L}(z^{-1})Y^*(z) = z^{-d}\mathbf{H}(z^{-1})R(z). \quad (3)$$

We assume that the roots of $\mathbf{L}(z^{-1})$ belongs to the open unit disk, i.e. the reference model is stable. If we define the tracking error $\varepsilon$ by

$$\varepsilon(t) := y(t) - y^*(t), \quad (4)$$

then the goal is to drive $\varepsilon$ to zero asymptotically.

**Remark 1.** Notice that for the $d$-step-ahead control problem the reference model is simply $Y^*(z) = z^{-d}R(z)$.

We impose the following assumptions on the set of admissible parameters.

**Assumption 1.** $S_{ab}$ is closed and bounded (compact), and for each $\theta \in S_{ab}$, the corresponding $\mathbf{B}(z^{-1})$ has roots in the open unit disk and the sign of $b_0$ is always the same.

The boundedness requirement on $S_{ab}$ is reasonable in practical situations; it is used here to prove uniform bounds and decay rates on the closed-loop behavior. The constraint on the roots of $\mathbf{B}(z^{-1})$ is a requirement that the plant be minimum phase; this is necessary to ensure tracking of bounded reference signals [11]. Knowledge of the sign of the high-frequency gain $b_0$ is common in adaptive control [4].

**Remark 2.** It is implicit in the assumptions that we know the system delay $d$ as well as the upper bounds on the orders of $\mathbf{A}(z^{-1})$ and $\mathbf{B}(z^{-1})$.

To proceed, we use a parameter estimator together with an adaptive control law based on the Certainty Equivalence Principle. It is convenient to put the plant into the so-called *predictor form*. To this end, by long division we can find $\mathbf{F}(z^{-1}) = \sum_{i=0}^{d-1} f_i z^{-i}$ and $\alpha(z^{-1}) = \sum_{i=0}^{n-1} \alpha_i z^{-i}$ that satisfy the following:

$$\mathbf{L}(z^{-1}) \mathbf{A}(z^{-1}) = \mathbf{F}(z^{-1}) + z^{-d} \frac{\alpha(z^{-1})}{\mathbf{A}(z^{-1})}.$$ 

if we now define $\beta(z^{-1}) = \sum_{i=0}^{m+n+d-1} \beta_i z^{-i} := \mathbf{F}(z^{-1})\mathbf{B}(z^{-1})$, then it is easy to verify that the following is true:

$$z^{-d} \frac{\mathbf{B}(z^{-1})}{\mathbf{A}(z^{-1})} = \frac{\beta(z^{-1})}{z^{-d} \mathbf{L}(z^{-1}) - \alpha(z^{-1})}. \quad (5)$$

So comparing (5) with the plant equation in (2), we are able to re-write the plant equation as

$$\mathbf{L}(z^{-1})[z^dY(z)] = \alpha(z^{-1})Y(z) + \beta(z^{-1})U(z) + \mathbf{W}(z), \quad (6)$$

with $\mathbf{W}(z) := z^d\mathbf{F}(z^{-1})W(z)$. Now define a weighted sum of the system output $\overline{y}$ by

$$\overline{y}(t) := y(t) + \sum_{j=1}^{n'} l_j y(t-j); \quad (7)$$

clearly the $z$-transform of $\overline{y}(t)$ is $\mathbf{L}(z^{-1})Y(z)$, so the time-domain counterpart of (6) in predictor form is

$$\overline{y}(t + d) = \phi(t)^\top \theta^* + \overline{w}(t), \quad (8)$$

with

$$\phi(t) := \begin{bmatrix} y(t) \\ y(t-1) \\ \vdots \\ y(t-n+1) \\ u(t) \\ u(t-1) \\ \vdots \\ u(t-m-d+1) \end{bmatrix}, \quad \theta^* := \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{m+d-1} \end{bmatrix}. \quad (9)$$

Let $S_{ab} \subset \mathbb{R}^{n+m+d}$ denote the set of admissible $\theta^*$ that arise from the original plant parameters which lie in $S_{ab}$; it is clear that the associated mapping from $S_{ab}$ to $S_{a\beta}$ is analytic, so the compactness of $S_{ab}$ means that $S_{a\beta}$ is compact as well. Furthermore, it is easy to see that $\beta_0 = b_0$. It is convenient that the set of admissible parameters in the new parameter space be convex and closed; so at this point let $S \subset \mathbb{R}^{n+m+d}$ be any compact and convex set containing $S_{a\beta}$ for which the $(n+1)$th element is never zero; the convex hull of $S_{a\beta}$ will do, although it may be more convenient to use a hyperrectangle (for projection purposes). We will show an example on obtaining such a set in the simulation section.

Now define $\overline{Y}(z) := \mathbf{L}(z^{-1})Y^*(z)$; then the model reference control law is given by

$$\overline{y}(t + d) = \phi(t)^\top \theta^*.$$

In the absence of noise, and assuming the controller is applied for all $t \in \mathbb{Z}$, we can show that we have $y(t) = y^*(t)$ for all $t \in \mathbb{Z}$. In our case of unknown parameters, we seek an adaptive version of the control law which is applied after some initial time, i.e. $t \geq t_0$.

**A. Initialization**

In most adaptive control results, the goal is to prove asymptotic behavior, so the details of the initial condition are unimportant. On the other hand, here we wish to obtain a bound on the transient behavior so we must proceed carefully. Here we adopt the approach used in the $d$-step ahead control setting of [15] and [16]. With the definition (7) in mind and $n' \leq n$, observe that if we wish to solve (8) for $y(\cdot)$ starting at time $t_0$, then it is clear that we need an initial condition of

$$x_0 := [y(t_0) \ y(t_0-1) \ \ldots \ y(t_0-n-d+2) \ u(t_0) \ u(t_0-1) \ \ldots \ u(t_0-m-2d+2)]^\top;$$

observe that this is sufficient information to obtain $\phi(t_0), \phi(t_0-1), \ldots, \phi(t_0-d+1)$. 
B. Parameter Estimation

We can re-write the plant equation (8) as
\[ \bar{y}(t + 1) = \phi(t - d + 1)\bar{\theta}^* + \bar{w}(t - d + 1), \quad t \geq t_0. \] (9)

Given an estimate \( \hat{\theta}(t) \) of \( \theta^* \) at time \( t \), we define the prediction error by
\[ e(t + 1) := \bar{y}(t + 1) - \phi(t - d + 1)\bar{\theta}(t); \] (10)

this is a measure of the error in \( \hat{\theta}(t) \). A common way to obtain a new estimate is from the solution of the optimization problem
\[ \arg\min_{\bar{\theta}} \left\{ \|\theta - \hat{\theta}(t)\| : \bar{y}(t + 1) = \phi(t - d + 1)\bar{\theta}(t) \right\}, \]

yielding the (ideal) Projection Algorithm:
\[ \hat{\theta}(t + 1) = \begin{cases} \hat{\theta}(t) & \text{if } e(t + 1) = 0 \\ \hat{\theta}(t) + \frac{\phi(t - d + 1)}{\|\phi(t - d + 1)\|} e(t + 1) & \text{otherwise} \end{cases} \] (11)

at this point, we can also constrain it to \( S \) by projection. Of course, if \( \|\phi(t - d + 1)\| \) is close to zero, numerical problems may occur, so it is the norm in the literature (e.g. [4] and [3]) to add a constant to the denominator; however as pointed out in our earlier work [13], [14] and [16], this can lead to the loss of exponential stability and a loss of a bounded gain on the noise. As proposed in [13], [14] and [16], we turn off the estimation if it is clear that the noise is swamping the estimation error. To this end, with \( \delta \in (0, \infty) \), we turn off the estimator if the update is larger than \( 2\|S\| + \delta \) in magnitude; so define

\[ \rho(t) := \begin{cases} 1 & \text{if } |e(t + 1)| < (2\|S\| + \delta) \|\phi(t - d + 1)\| \\ 0 & \text{otherwise} \end{cases} \]

given the initial condition of \( \hat{\theta}(t_0) = \theta_0 \in \mathbb{R}^{m+n+d} \), for \( t \geq t_0 \) we define\(^2\)
\[ \hat{\theta}(t + 1) = \hat{\theta}(t) + \rho(t)\frac{\phi(t - d + 1)}{\|\phi(t - d + 1)\|} e(t + 1) \] (12a)
\[ \hat{\theta}(t + 1) = \text{Proj}_S \{ \hat{\theta}(t + 1) \}. \] (12b)

Analyzing the closed-loop system requires a careful examination of the estimation algorithm. First define the parameter error by \( \tilde{\theta}(t) := \hat{\theta}(t) - \theta^* \). The following result lists properties which are equivalent to those of Proposition 1 in [16] for the \( d \)-step ahead adaptive control setup.

Proposition 1. For every \( t_0 \in \mathbb{Z}, x_0 \in \mathbb{R}^{n+m+3d-2}, \theta_0 \in S, \theta \in S_{\theta_0}, w \in \mathcal{L}_\infty, \) and \( \delta \in (0, \infty) \), when the estimator (12) is applied to the plant (1), the following holds:
\[ \|\dot{\hat{\theta}}(t + 1) - \dot{\hat{\theta}}(t)\| \leq \rho(t) \frac{|e(t + 1)|}{\|\phi(t - d + 1)\|}, \quad t \geq t_0, \]
\[ \|\dot{\hat{\theta}}(t)\|^2 \leq \|\dot{\hat{\theta}}(\tau)\|^2 + \sum_{j=\tau}^{t-1} \rho(j) \left[ \frac{1}{2} \frac{|e(j + 1)|^2}{\|\phi(j - d + 1)\|^2} + \frac{2|\bar{w}(j - d + 1)^2}{\|\phi(j - d + 1)\|^2} \right], \quad t > \tau \geq t_0. \]

C. The Control Law

With the natural partitioning
\[ \hat{\theta}(t) := [\hat{\alpha}_0(t) \cdots \hat{\alpha}_{n-1}(t) \hat{\beta}_0(t) \cdots \hat{\beta}_{m+d-1}(t)]^\top, \]
the model reference adaptive control law (based on the Certainty Equivalence principle) is
\[ \bar{y}(t + d) = \phi(t)^\top \hat{\theta}(t); \]
solving this for \( u(t) \) and using the reference model (3), we have
\[ u(t) = \frac{1}{\beta_0(t)} \left[ - \sum_{i=0}^{n-1} \hat{\alpha}_i(t) y(t - i) - \sum_{i=1}^{m+d-1} \hat{\beta}_i(t) u(t - i) + \sum_{i=0}^{n'-d} h_i r(t - i) \right], \quad t \geq t_0. \] (13)

It is convenient for analysis to define an auxiliary tracking error:
\[ \bar{z}(t) := \bar{y}(t) - \bar{y}^*(t); \] (14)

it is easy to show that
\[ \bar{z}(t) = -\phi(t - d)^\top \hat{\theta}(t - d) + \bar{w}(t - d), \quad t \geq t_0 + d, \] (15)
\[ e(t) = -\phi(t - d)^\top \hat{\theta}(t) - 1 + \bar{w}(t - d), \quad t \geq t_0 + 1, \] (16)
as well as
\[ \bar{z}(t) = e(t) + \phi(t - d)^\top \left[ \bar{\theta}(t - 1) - \hat{\theta}(t - d) \right], \quad t \geq t_0 + d. \] (17)

Observe that we can compute \( \bar{z}(t), t \in \{t_0, t_0 + 1, \ldots, t_0 + d - 1\} \), from \( x_0, w \) and \( y^* \).

In the next section we develop several models used in the analysis, after which we state and prove our result. The approach borrows ideas from our previous work on the \( d \)-step ahead setup [16] and extends them to the Model Reference Adaptive Control (MRAC) case.

III. The Analysis

In the pole-placement adaptive control setup of our earlier work [14], a key closed-loop model consists of a update equation for \( \phi(t) \), with the state matrix consisting of controller and plant estimates; this was effective because the characteristic polynomial of this matrix is time-invariant and has all roots in the open unit disk. If we were to apply the same idea in our case here, then the characteristic polynomial would have roots which are time-varying, with some at zero and the rest at the roots of the corresponding naturally defined polynomial \( \beta(t, z^{-1}) \), which is time varying, and it may not have roots in the open unit disk. On the other hand, in the \( d \)-step ahead adaptive control setup of our earlier work [15] and [16], these difficulties were dealt with by constructing three different models for use in the analysis: a model that does not use parameter estimates but is driven by the tracking error, a crude model to bound the size of growth of \( \phi(t) \), and a crucial model which is driven by perturbed versions of the present and past values of \( \phi(t) \). Here in this paper, which deals with the more general MRAC problem, we construct similar, though not identical, models, but they need more careful analysis than ones in the \( d \)-step ahead control case.

A. A Good Model

Here we first obtain an equation which avoids using parameter estimates, though it is driven by the weighted

\[ \text{An exception is [1] where the ideal algorithm (11) is used and Lyapunov stability is proven, but a convolution bound on the exogenous inputs is not proven, and the high-frequency gain is assumed to be known.} \]

\[ \text{If } \delta = \infty, \text{ then we adopt the understanding that } \infty \times 0 = 0, \text{ in which case this formula in (12a) collapses into the original version (11).} \]
sum of the tracking error \( \tau(t) \). By extending the idea from [15] and [16], using the definition of \( \varepsilon \) we obtain a formula for \( y(t+1) \), and using the plant equation (1) we obtain a formula for \( u(t+1) \); then, it is easy to see that there exists a matrix \( A_y \in \mathbb{R}^{(n+m+d) \times (n+m+d)} \) (which depends implicitly on \( \theta \in S_{ab} \)) so that the following holds:

\[
\phi(t+1) = A_y \phi(t) + e_1 \varepsilon(t+1) + \frac{1}{b_0} e_{n+1} \sum_{i=0}^{d} a_{d-i} \varepsilon(t+1+i) + e_1 y^\tau(t+1) + \frac{1}{b_0} e_{n+1} \sum_{i=0}^{d} a_{d-i} y^\tau(t+1+i) - w(t+d+1). \tag{18}
\]

The characteristic polynomial of \( A_y \) is \( \frac{1}{b_0} z^{n+m+d} B(z^{-1}), \) so all of its roots are in the open unit disk.

The model in (18) is similar to the good model obtained in the analysis in the d-step ahead control case in [15] and [16] where it is driven by the tracking error \( \varepsilon \). However in the case considered here we would like to obtain a model which is, instead, driven by \( \tau \); this will turn out to be crucial in analyzing the closed-loop behavior. To this end, from (14) and the definitions of \( \tau \) and \( \tau^\tau \), it is easy to see that

\[
E(z) = \frac{1}{L(z^{-1})} E(z); \tag{19}
\]

so we can represent \( \varepsilon(t) \) as the output of an \( n' \)th-order system driven by \( \tau \) as follows: with \( \zeta(t) := [\varepsilon(t) \ \varepsilon(t-1) \ \cdots \ \varepsilon(t-n' + 1)]^\top \), and \( A_t \in \mathbb{R}^{n' \times n'} \) defined by

\[
A_t := \begin{bmatrix}
-l_{t1} & -l_{t2} & \cdots & -l_{tn'-1} & -l_{tn'} \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & 0 \\
\end{bmatrix},
\]

we have

\[
\zeta(t+1) = A_t \zeta(t) + e_1 \tau^\top \zeta(t). \tag{20a}
\]

\[
\varepsilon(t) = e_1 \zeta(t). \tag{20b}
\]

Note that (18) is driven on the RHS by \( d+1 \) terms of \( \varepsilon(\cdot) \); but from (20b) we have

\[
\varepsilon(t+1+j) = e_1 \zeta(t+1+j), \quad j = 0, 1, \ldots, d. \tag{21}
\]

With this in mind, we construct the following \((n'(d+1))\)-th-order system driven by \( \tau^\tau(\cdot) \):

\[
\begin{bmatrix}
\zeta(t+d+2) \\
\zeta(t+d+1) \\
\vdots \\
\zeta(t+2) \\
\zeta(t+1) \\
\end{bmatrix} = A_t
\begin{bmatrix}
\zeta(t+d+1) \\
\zeta(t+d) \\
\vdots \\
\zeta(t+1) \\
\zeta(t) \\
\end{bmatrix} + e_1 \tau(t+d+2) + B_1(t) \varepsilon(t+d+1) + B_2(t) w(t+d+1) + B_3(t) \tau(t+d+2) + B_4(t) y^\tau(t+d+2) + e_{n+m+d+1} \tau^\tau(t+d+2), \tag{22}
\]

At this point we can combine the models (18) and (22) together with the linking equation (21) to obtain a model driven by the exogenous inputs and \( \tau^\tau(\cdot) \) (rather than \( \varepsilon(\cdot) \); with

\[
\eta(t) := \frac{1}{b_0} e_{n+1} \sum_{i=0}^{d} a_{d-i} y^\tau(t+1+i) - w(t+d+1) + e_1 y^\tau(t+1), \tag{23}
\]

which depends continuously on \( \theta \in S_{ab} \), to obtain the following \((n+m+d+n'(d+1))\)-th-order system:

\[
\begin{bmatrix}
\phi(t+1) \\
\zeta(t+1) \\
\end{bmatrix} = \begin{bmatrix} A_y & B \end{bmatrix} \begin{bmatrix} \phi(t) \\
\zeta(t) \\
\end{bmatrix} + \eta(t) + e_{n+m+d+1} \tau(t+d+2). \tag{24}
\]

Before presenting an even better model suitable for analysis we need to analyze a couple of crude models of the closed-loop behavior.

B. Crude Models

At times, we will need to use crude models to bound the size of the growth of \( \phi(t) \) and the size of the growth of \( \tau(t) \) in terms of the exogenous inputs. Following a similar analysis of the crude model in [15] and [16], we now use (1) to describe \( y(t+1) \), and use the control law (13) together with the obtained equation for \( y(t+1) \) to describe \( w(t+1) \); so, we can appropriately define matrices \( A_1(t) \), \( B_1(t) \) and \( B_2(t) \) in terms of \( \theta \in S_{aa} \) and \( \theta(t+1) \in S \) so that we have the following crude model of the behavior of \( \phi(t) \):

\[
\phi(t+1) = A_1(t) \phi(t) + B_1(t) \tau^\tau(t+d+1) + B_2(t) w(t+1), \quad t \geq t_0. \tag{25}
\]

Furthermore, we combine (25) and (22) to obtain an equation for \( \phi(t) \); we can appropriately define matrices \( A_2(t) \), \( B_3(t) \), \( B_4(t) \), \( B_5(t) \) so that the following crude model of the behavior of \( \phi(t) \) is obtained:

\[
\phi(t+1) = A_2(t) \phi(t) + B_3(t) \tau^\tau(t+d+1) + B_4(t) w(t+1) + B_5(t) y^\tau(t+d+2) + e_{n+m+d+1} \tau^\tau(t+d+2), \quad t \geq t_0. \tag{26}
\]

Now we want to find a representation for \( \tau(t+d+2) \) on the RHS above in terms of \( \phi(t) \); from (15) we have \( \tau(t+d+2) = -\theta(t+2)^\top \phi(t+2) + \tau^\tau(t+2) \), so we use (25) to find a representation of \( \phi(t+2) \) in terms of \( \phi(t) \) and substitute into (26); then we can appropriately define matrices \( A_2(t) \), \( B_3(t) \), \( B_4(t) \), \( B_5(t) \), \( B_6(t) \) so that the following crude model of the behavior of \( \phi(t) \) is obtained:

\[
\phi(t+1) = A_2(t) \phi(t) + B_3(t) \tau^\tau(t+d+1) + B_4(t) w(t+1) + B_5(t) y^\tau(t+d+2) + B_6(t) \tau^\tau(t+d+2) + e_{n+m+d+1} \tau^\tau(t+d+2), \quad t \geq t_0. \tag{27}
\]

Due to the compactness of \( S_{ab} \), \( S_{aa} \) and \( S \), we can obtain the following immediately.

Proposition 2. There exists a constant \( c_1 \geq 1 \) such that for every \( t_0 \in \mathbb{Z}, x_0 \in \mathbb{R}^{n+m+3d-2}, c_0 \in S_{ab}, r,w \in \ell_\infty, \) and \( \delta \in (0,\infty) \), when the adaptive controller (12) and (13) is applied to the plant (1), the following holds:

\[
\| A_1(t) \| \leq c_1, \quad \| A_2(t) \| \leq c_1, \quad \| B_1(t) \| \leq c_1, \quad \| B_2(t) \| \leq c_1, \quad \| B_3(t) \| \leq c_1, \quad \| B_4(t) \| \leq c_1, \quad \| B_5(t) \| \leq c_1, \quad \| B_6(t) \| \leq c_1, \quad \| B_7(t) \| \leq c_1, \quad \| B_8(t) \| \leq t_0.
\]

C. A Better Model

The good closed-loop model (24) is driven by a future value of \( \tau^\tau(\cdot) \). We now combine it with the crude model (27) to obtain a new model which is driven by perturbed version of \( \tau^\tau(\cdot) \), with weights associated with the parameter estimation updates. Before proceeding, motivated by the
form of the term in the parameter estimator, we define
\[ \nu(t) := \rho(t) \frac{\phi(t - d + 1)}{||\phi(t - d + 1)||^2} e(t + 1). \]

**Proposition 3.** There exists a constant \(c_2\) so that for every \(t_0 \in \mathbb{Z}\), \(x_0 \in \mathbb{R}^{n+m+3d-2}\), \(\theta_0 \in \mathcal{S}\), \(\theta \in \mathcal{S}_{ab}\), \(r, w \in \ell_{\infty}\), \(\delta \in (0, \infty)\), when the adaptive controller (12) and (13) is applied to the plant (1), the following holds:
\[ \bar{\phi}(t + 1) = [\bar{A}_g + \Delta(t)]\bar{\phi}(t) + \bar{\eta}(t), \quad t \geq t_0, \quad (28) \]
with
\[ ||\Delta(t)|| \leq c_2 \sum_{j=2}^{d+1} ||\nu(t + j)||, \quad (29) \]
and
\[ ||\bar{\eta}(t)|| \leq c_2 \left( 1 + \sum_{j=2}^{d+1} ||\nu(t + j)|| \right) \left[ \sum_{j=1}^{\text{max}(3, d+1)} \left( ||y^*(t + j)|| + \left| \sum_{j=1}^{t} \gamma_j \right| + \left| \sum_{j=1}^{t} \delta_j \right| + ||\bar{W}(t + j)|| \right) \right]. \quad (30) \]

**Proof.** See the Appendix. \[ \Box \]

The result in Proposition 3 looks very similar, though not identical, to the analysis leading up to the main result of [15] and [16] on the \(d\)-step ahead adaptive control problem. Notice that the matrix \(\bar{A}_g\) is a function of \(\theta \in \mathcal{S}_{ab}\), and the coefficients of \(L(z^{-1})\); it lies in a corresponding compact set \(A \subseteq \mathbb{R}^{(n+m+d+3d'(d+1))\times(n+m+d+3d'(d+1))}\). Furthermore, the eigenvalues of \(\bar{A}_g\) are at the origin, the roots of \(L(z^{-1})\), and the roots of \(B(z^{-1})\). For all \(t > 0\), there is an open unit disk; so we can use classical arguments to prove that for the desired reference model there exist constants \(\gamma\) and \(\sigma \in (0, 1)\) so that for all \(\theta \in \mathcal{S}_{ab}\), we have
\[ ||\bar{A}_g^k|| \leq \gamma \sigma^k, \quad k \geq 0. \quad (31) \]
Indeed, we can choose any \(\sigma\) larger than
\[ \lambda := \max_{\theta \in \mathcal{S}_{ab}} \{ \lambda : \lambda \in \mathbb{C}, B(\lambda^{-1}) = 0 \text{ and } L(\lambda^{-1}) = 0 \}. \]

Equations of the form given in (28) appear in classical adaptive control approaches. While we can view (28) as a linear time-varying system, we have to keep in mind that \(\Delta(t)\) and \(\bar{\eta}(t)\) are implicit nonlinear functions of \(\theta, \theta_0, x_0, r, w\). However, this linear time-varying interpretation is very convenient for analysis; to this end, let \(\Phi_A\) denote the state transition matrix of a general time-varying square matrix \(A\). The following result is useful in analyzing our closed-loop system.

**Proposition 4.** (7). With \(\sigma \in (\lambda_1, 1)\), suppose that \(\gamma \geq 1\) and \(\sigma \geq 1\) are such that (31) is satisfied for every \(\bar{A}_g \in A\). For every \(\mu \in (\sigma, 1)\), \(g_0 \geq 0, g_1 \geq 0, g_2 \geq 0\), \(\bar{A}_g \in \mathcal{S}(\mathbb{R}^{(n+m+n'(d+1))\times(n+m+n'(d+1))})\) satisfying
\[ \sum_{j=1}^{t-1} ||\Delta(j)|| \leq g_0 + g_1(t - \tau)^{\frac{1}{2}} + g_2(t - \tau), \quad \bar{\tau} \geq t > \tau \geq 1, \]
we have
\[ \|\Phi_{\bar{A}_g + \Delta}(t, \tau)\| \leq \bar{\gamma}\mu^{t - \tau}, \quad \bar{\tau} \geq t > \tau \geq 1. \]

Next, we present the main result proving that the closed-loop system enjoys very desirable linear-like behavior.

**IV. THE MAIN RESULT**

**Theorem 1.** For every \(\delta \in (0, \infty)\) and \(\lambda \in (\lambda_1, 1)\), there exists a constant \(c > 0\) so that for every \(t_0 \in \mathbb{Z}\), \(\theta \in \mathcal{S}_{ab}\), \(r, w \in \ell_{\infty}\), \(\theta_0 \in \mathcal{S}\), and \(\delta \in (0, \infty)\), when the adaptive controller (12) and (13) is applied to the plant (1), the following bound holds:
\[ ||\phi(t)|| \leq c \lambda^{-t-t_0} ||x_0|| + \sum_{j=t_0}^{t} c \lambda^{t-j} (\|r(j)\| + \|w(j)\|), \quad t \geq t_0. \quad (32) \]

Furthermore, if \(w = 0\), then
\[ \sum_{k=t_0+d}^{\infty} e(k)^2 \leq c(||x_0||^2 + ||r\|_{\infty}^2). \quad (33) \]

**Remark 3.** The above result shows that the closed-loop system experiences linear-like behavior. There is a uniform exponential decay bound on the effect of the initial condition, and a convolution bound on the effect of the exogenous inputs. This implies that the system has a bounded gain (from \(w\) and \(r\) to \(y\)) in every 2-norm. For example, for \(p = \infty\), we see from the above bound that
\[ ||\phi(t)|| \leq \frac{1}{\lambda^t} (\lambda^{-t-t_0} ||x_0|| + ||r\|_{\infty} + ||r\|_{\infty}), \quad t \geq t_0. \]

**Remark 4.** In the absence of noise, most adaptive controllers provide that the tracking error is square summable, e.g. see [4]. Here we prove a stronger result, namely, an upper bound on the 2-norm in terms of the size of \(x_0\) and \(r\).

**Proof of Theorem 1.** Fix \(\delta \in (0, \infty)\) and \(\lambda \in (\lambda_1, 1)\). Let \(t_0 \in \mathbb{Z}\), \(\theta \in \mathcal{S}_{ab}\), \(r, w \in \ell_{\infty}\), \(\theta_0 \in \mathcal{S}\), and \(x_0 \in \mathbb{R}^{n+m+3d-2}\) be arbitrary. Now choose \(\sigma \in (\lambda_1, \lambda)\). We will analyze (28) to obtain a bound on \(\phi(t)\).

Before proceeding, we see that there exists \(\gamma_1\) so that for every \(\bar{A}_g \in A\), \(||\bar{A}_g^k|| \leq \gamma_1 \sigma^k, k \geq 0\). Also, we need to compute a bound on the sum of \(||\Delta(t)||\); since there are \(d\) terms on the RHS of (29), by the Cauchy-Schwarz inequality we obtain
\[ \sum_{j=t}^{t-1} ||\Delta(j)|| \leq d c_2 \sum_{j=t+1}^{t+\tau} ||\nu(j + 1)|| \]
\[ \leq d^2 c_2 \left[ \sum_{j=t+1}^{t+\tau} ||\nu(j + 1)||^2 \right]^{\frac{1}{2}} (t - \tau + d - 1)^{\frac{1}{2}}, \quad t > \tau \geq t_0; \quad (34) \]
but \((t_2 - t_1 + d - 1)^{\frac{1}{2}} \leq d(t_2 - t_1)^{\frac{1}{2}}, t_2 > t_1\), so incorporating this and the definition of \(\nu(t)\) we have
\[ \sum_{j=t}^{t-1} ||\Delta(j)|| \leq d^3 c_2 \left[ \sum_{j=t+1}^{t+\tau+2} ||\nu(j + 1)||^2 \right]^{\frac{1}{2}} (t - \tau)^{\frac{1}{2}}, \quad \tau \geq t \geq t_0; \quad (35) \]

Also for ease of notation, let us define
\[ \hat{w}(t) := \sum_{j=1}^{\text{max}(3, d+1)} (||y^*(t+j)|| + ||\nu(t+d+j)|| + ||w(t+j)|| + ||\eta(t+j)||). \]
Now we consider the closed-loop system behavior. To proceed, we partition the timeline into two parts: one in which the noise $\overline{\varphi}(\cdot)$ is small versus $\phi(\cdot)$ and one where it is not. To this end, with $\nu > 0$ to be chosen shortly, define

$$S_{\text{good}} := \left\{ j \geq t_0 : \phi(j) \neq 0 \text{ and } \frac{\varphi(j)^2}{||\varphi||^2} < \nu \right\},$$
$$S_{\text{bad}} := \left\{ j \geq t_0 : \phi(j) = 0 \text{ or } \frac{\varphi(j)^2}{||\varphi||^2} \geq \nu \right\}.$$

Clearly $\{ j \in \mathbb{Z} : j \geq t_0 \} = S_{\text{good}} \cup S_{\text{bad}}$. Observe that this partition implicitly depends on $\theta \in S_{\text{bad}}$, as well as the initial conditions. We will easily obtain bounds on the closed-loop system behavior on $S_{\text{bad}}$; we will apply Proposition 4 to analyze the behavior on $S_{\text{good}}$. Before proceeding, we partition the timeline into intervals which oscillate between $S_{\text{good}}$ and $S_{\text{bad}}$. To this end, it is easy to see that we can define a (possibly infinite) sequence of intervals of the form $[k_i, k_{i+1})$ satisfying: (i) $k_0 = t_0$; (ii) $[k_i, k_{i+1})$ either belongs to $S_{\text{good}}$ or $S_{\text{bad}}$; and (iii) if $k_{i+1} \neq \infty$ and $[k_i, k_{i+1})$ belongs to $S_{\text{good}}$ (respectively, $S_{\text{bad}}$), then the interval $[k_i, k_{i+2})$ must belong to $S_{\text{bad}}$ (respectively, $S_{\text{good}}$).

Now we analyze the closed-loop behavior on each interval.

**Case 1:** The behavior on $S_{\text{bad}}.$

Let $j \in [k_i, k_{i+1}) \subset S_{\text{bad}}$ be arbitrary. In this case, we have either $\phi(j) = 0$ or $\frac{\varphi(j)^2}{||\varphi||^2} \geq \nu$. In either case, we have

$$\|\phi(j)\| \leq \frac{1}{\sqrt{\nu}} \varphi(j), \quad j \in [k_i, k_{i+1});$$

then from the crude model (25) and Proposition 2, we have

$$\|\phi(j+1)\| \leq \frac{1}{\sqrt{\nu}} \varphi(j) + c_1 (j + d + 1) + c_1 |w(j+1)|, \quad j \in [k_i, k_{i+1});$$

combining this with (36) yields:

$$\|\phi(j)\| \leq \left\{ \begin{array}{ll}
\frac{1}{\sqrt{\nu}} \varphi(j), & j = k_i \\
ck1 (\frac{1}{\sqrt{\nu}} + 1) + \frac{1}{\varphi(j) + d} + |w(j)|, & j = k_i + 1, \ldots, k_{i+1}.
\end{array} \right.$$ (36)

**Case 2:** The behavior on $S_{\text{good}}.$

Suppose that $[k_i, k_{i+1})$ lies in $S_{\text{good}}$; notice that the bound on $\|\phi(t)\|$ in (35) extends outside $S_{\text{good}}$; so we handle the first $d + 1$ and last $d + 1$ times steps separately.

To this end, first suppose that $k_{i+1} - k_i \leq 2(d+1)$; then using the crude model on $\phi$ in (25) and Proposition 2, it is easy to show that if we define $\gamma_2 := \left(\frac{4}{\kappa}\right)^{d+1}$, then we have

$$\|\phi(t)\| \leq \gamma_2 \lambda^{t-k_i} \|\phi(k_i)\| + \sum_{j=k_i}^{t-1} \gamma_2 \lambda^{t-j-1} (|\phi(j+1)| + |w(j+1)|), \quad t \in [k_i, k_{i+1}).$$ (37)

Now suppose that $k_{i+1} - k_i > 2(d+1)$. Define $k_i := k_i + d + 1$ and $k_{i+1} := k_{i+1} - d - 1$. By the second part of Proposition 1 and using the facts that $||\theta(t)|| \leq 2 ||S||$ and that $\frac{\varphi(j)^2}{||\varphi||^2} \leq \nu$ for $j \in [k_i, k_{i+1})$, we obtain from (35):

$$\|\Delta(j)\| \leq 4 \nu (t - t + d - 1) + 4 \nu (t - t + d - 1) + 2(t - t + d - 1) \|\theta(t)\|^2,$$

$$k_{i+1} \geq t > t \geq k_i.$$ (38)

If we restrict $\nu \leq 1$, and define $\gamma_3 := d^3 c_2 \left[8 ||S||^2 + 4 \nu (t - t + d - 1) + 2(t - t + d - 1) \right], \quad k_{i+1} \geq t > t \geq k_i,$

then we obtain

$$\sum_{j=t}^{t-1} \|\Delta(j)\| \leq \gamma_3 (t - t)^2 + \gamma_3 v^2 (t - t), \quad k_{i+1} \geq t > t \geq k_i.$$ (40)

We now apply Proposition 4; set $y_0 = 0, g_1 = \gamma_3 g_2 = \gamma_3 v^2, \mu = \lambda, \gamma = \gamma_4$; we need $\gamma_3 v^2 < \frac{\lambda - \nu}{\gamma_4}$, so if we set $v := \min \left\{ 1, \frac{1}{\sqrt{\nu}} \right\}$, then from Proposition 4 we see that there exists a constant $\gamma_4$ so that the state transition matrix $\Phi_{\hat{A} + \Delta}(t, \tau)$ satisfies

$$\|\Phi_{\hat{A} + \Delta}(t, \tau)\| \leq \gamma_4 \lambda^t, \quad k_{i+1} \geq t > t \geq k_i.$$ (41)

Next we obtain a bound in terms of $\phi$. It is obvious that $\|\phi(t)\| \leq \|\phi(t)\|, \quad t \in [k_i, k_{i+1})$. Then from the definitions of $\phi(\cdot), \zeta(\cdot)$ and $\zeta(\cdot)$, it is easy to see that there exists a constant $c_3$ such that $||\phi(k_i)|| \leq c_3 \sum_{j=1}^{k_{i+1}} \|\phi(k_i + j)\| + c_3 \sum_{j=k_{i+1}}^{d+1} \|y^*(k_i + j)\|$. We use the crude model on $\phi(\cdot)$ in (25) and Proposition 2 to obtain bounds on $||\phi(k_i + j)\|, j = 1, 2, \ldots, d+1$, in terms of $\|\phi(k_i)\|$. Incorporating all of the above into (40) and after simplification, we see that there exists a constant $\gamma_8$ so that

$$\|\phi(t)\| \leq \gamma_8 \lambda^{t-k_i} \|\phi(k_i)\| + \sum_{j=k_i}^{t-1} \gamma_8 \lambda^{t-j-1} w(j) + \gamma_8 \sum_{q=1}^{n-2} |y^*(k_i - q)|, \quad t \in [k_i, k_{i+1}],$$ (42)

which we combine with (37) to conclude Case 2.

We now glue together the bounds obtained on $S_{\text{good}}$ and $S_{\text{bad}}$ to obtain a bound which holds on all of $[t_0, \infty)$ using the identical argument used in gluing together similar bounds in the proof of Theorem 1 of [16]. Last of all, we simplify the resulting quantity by using causality arguments to remove extraneous terms, and end up with the bound in (32).

Finally we prove asymptotic tracking. Suppose that $w = 0$; then we see from the estimation algorithm that in this case, $\rho(t) = 1 \Leftrightarrow ||\phi(t - d)\| \neq 0$. So from (17), the first part of

If the noise is zero, the $S_{\text{good}}$ may be the whole timeline $[t_0, \infty)$.
Proposition 1, and the Cauchy-Schwarz inequality, we have
\[ \rho(t-1) \frac{\tau(t)^2}{\| \phi(t-d) \|^2} \leq d \sum_{j=0}^{d-1} e(t-j)^2 \]
so by the second part of Proposition 1 we can see that
\[ \sum_{t=t_0+d, \| \phi(t-d) \| \neq 0}^{\infty} \frac{\tau(t)^2}{\| \phi(t-d) \|^2} \leq 8d^2 \| S \|^2. \]
Then it is easy to see by the boundedness of \( \phi \) proven in (32) and by the fact that \( \tau(t) = 0 \) when \( \phi(t-d) = 0 \), that
\[ \sum_{t=t_0+d}^{\infty} \frac{\tau(t)^2}{\| \phi(t-d) \|^2} \leq 8d^2 \| S \|^2, \]
and unmodelled dynamics. We have shown that the corresponding model reference adaptive controller provides a convolution bound with gain \( c \) and decay rate \( \lambda \) when applied to the time-invariant nominal plant; we can apply Theorems 1 and 2 of [23] to show that, in the presence of a degree of time-variation (slow enough parameter time-variations and/or occasional jumps) and small enough unmodelled dynamics, the controller still provides linear-like properties. Furthermore, we can also obtain bounds on the average tracking error both in the case of no noise under slow time-variations, as well as in the noisy case, by adapting arguments used in the proofs of Theorems 4 and 5 of [16].

V. ROBUSTNESS
It turns out that the convolution bounds proven in Theorem 1 will guarantee robustness to a degree of time-variations and unmodelled dynamics. We have shown that the corresponding model reference adaptive controller provides a convolution bound with gain \( c \) and decay rate \( \lambda \) when applied to the time-invariant nominal plant; we can apply Theorems 1 and 2 of [23] to show that, in the presence of a degree of time-variation (slow enough parameter time-variations and/or occasional jumps) and small enough unmodelled dynamics, the controller still provides linear-like properties. Furthermore, we can also obtain bounds on the average tracking error both in the case of no noise under slow time-variations, as well as in the noisy case, by adapting arguments used in the proofs of Theorems 4 and 5 of [16].

VI. A SIMULATION EXAMPLE
We now provide a simulation example to illustrate the results of this paper. Consider the time-varying plant
\[ y(t + 1) = -a_1(t)y(t) - a_2(t)y(t - 1) + b_0(t)u(t) + b_1(t), \]
with \( a_1(t) \in [-2, 2], a_2(t) \in [-2, 2], b_0(t) \in \left[ \frac{2}{5}, 5 \right] \) and \( b_1(t) \in [-1, 1] \). Note here that the delay \( d = 1 \). We want to apply an adaptive controller such that the closed-loop system follows the behavior of a reference model (3) with \( n = 2 \); following the discussion at the beginning of Section II, we transform the plant into the predictor form by way of long division: we see that \( a_0(t) = a_1(t) - 1, a_1(t) = a_2(t) - 2, b_0(t) = b_0(t), \) \( b_1(t) = b_1(t) \). We choose a reference model represented by \( L(z^{-1}) := 1 - \frac{2}{5} z^{-2} \), and \( H(z^{-1}) := \frac{1}{z^2} \), which has poles in the open unit disk as required; then we can set
\[ S := \mathcal{S}_{\alpha, \beta} = \left\{ \begin{array}{c} \alpha_0 \\ \alpha_1 \\ \beta_1 \end{array} \right\} \in \mathbb{R}^4 : \alpha_0 \in [-2, 2], \alpha_1 \in [-\frac{2}{5}, \frac{2}{3}], \\
\beta_1 \in [-1, 1] \right\}. \]

We apply the adaptive controller (12) and (13) (with \( \delta = \infty \)) to this plant with the plant parameters given by: \( a_1(t) = 2 \cos(\frac{1}{50} t), a_2(t) = -2 \sin(\frac{1}{50} t), b_0(t) = \frac{2}{5} - \frac{2}{3} \cos(\frac{1}{50} t), \) and \( b_1(t) = -\cos(\frac{1}{50} t) \), and the disturbance given by:
\[ w(t) = \begin{cases} \frac{1}{10} \cos(10 t), & 200 < t < 500 \\ 0, & \text{otherwise}. \end{cases} \]

Fig. 1. The first plot shows both \( y(t) \) (solid) and \( y^*(t) \) (dashed); the second plot shows the control input \( u(t) \).

Fig. 2. The plots show the parameter estimates \( \hat{\theta}(t) \) (solid) as well as the actual parameters \( \theta^* \) (dashed).

We set \( r(t) \) to be a unit square wave with period of 200 steps. We set \( y(-1) = y(0) = -1, u(-1) = 0 \), and the initial parameter estimates to the midpoint of the respective intervals. Figures 1 and 2 show the results. The controller does a good job of tracking when there is no disturbance; the tracking degrades when the disturbance enters the system but tracking performance improves when the disturbance returns to zero. You can also see that the estimator tracks the time-varying parameters fairly well.

VII. CONCLUSION
In this paper, we show that a model reference adaptive controller using a parameter estimator based on the original projection algorithm provides desirable linear-like closed-loop properties: exponential stability, a bounded gain on the noise in every p-norm, and a convolution bound on the square sum of the tracking error.

We would like to extend these linear-like results to the case when the sign of the high-frequency gain is unknown by using multiple estimators along the lines of [22] and [24]. At present, our proof does not extend to that case since the key transition matrix is not deadbeat, but we are working on an alternative proof approach.

APPENDIX

Proof of Proposition 3. First, define a square matrix of size \( (n+m+d+n'(d+1)) \), \( P := \begin{bmatrix} I_n+m+d & 0 \end{bmatrix} \); then we have
\[ \overline{\phi}(t)^T P = [\phi(t)^T, 0], \]
(41)
so $\overline{\phi}(t)^T P \overline{\phi}(t) = \phi(t)^T \phi(t) = ||\phi(t)||^2$. Now define

$$\Delta(t) := \rho(t-1) \frac{|\bar{e}(t)|}{||\phi(t-d)||^2} e_{n+m+d+1}(t-d)^T P, \ t \geq t_0 + 1,$$

and $\eta(t) := \begin{cases} 1 - \rho(t-1) |\bar{e}(t)|, & t \geq t_0 + 1; \end{cases}$ then by using these definitions, we can represent the term containing $\bar{e}(t+d+2)$ in the RHS of (24) as:

$$e_{n+m+d+1}(t + d + 2) = \Delta(t + d + 2)\phi(t + 2) + e_{n+m+d+1}(t + d + 2).$$

We now use (27) (which is valid for $t \geq t_0$) to represent $\overline{\phi}(t+2)$ in the RHS of (43) in terms $\overline{\phi}(t)$; so if we do this and incorporate the result into (43), then we are now able to rewrite the model (24) in the desired form of (28) by defining

$$\Delta(t) := \Delta(t + d + 2)A_2(t + 1)A_2(t), \ t \geq t_0,$$

and by grouping the remaining terms (containing exogenous signals) and defining

$$\eta(t) := \eta(t) + e_{n+m+d+1}(t + d + 2) + \Delta(t + d + 2)A_2(t + 1)B_3(t)\bar{\gamma}(t + d + 1) + A_2(t + 1)B_6(t)w(t + 1) + (A_2(t + 1)B_7(t) + B_5(t + 1))\bar{\gamma}(t + d + 2) + (A_2(t + 1)B_8(t) + B_1(t + 1)w(t + 2) + A_2(t + 1)e_{n+m+d+1}(t + 2) + B_7(t + 1)\bar{\gamma}(t + d + 3) + B_8(t + 1)w(t + 3) + e_{n+m+d+1}(t + 3),$$

concluding the first part of the proof.

We now prove the desired bound on $\Delta(t)$. By (41) we obtain from (42): $||\Delta(t)|| \leq \rho(t-1) ||\overline{\phi}(t)||, \ t \geq t_0 + 1$; from (17) and the first proposition of Proposition 1, we can see that

$$||\Delta(t)|| \leq \rho(t-1) \frac{|\bar{e}(t)|}{||\phi(t-d)||} \leq \rho(t-1) \frac{|e(t)|}{||\phi(t-d)||} + ||\bar{\theta}(t-1) - \theta(t-d)|| \leq \sum_{j=0}^{d-1} \rho(t - 1 - j) \frac{|e(t-j)|}{||\phi(t-d-j)||} = \sum_{j=1}^{d} ||\nu(t-j)||, \ t \geq t_0 + d.$$

Then from (44), using the bound in (46) and Proposition 2, we can easily show that there exist a constant so that (29) is proven.

Finally, we prove the bound on $\eta$. From (15) and the definition of $\rho()$, it is easy to show that $|\eta(t)| \leq (1 + \frac{4 \|P\|}{L_2}) \|\overline{\phi}(t-d)||, \ t \geq t_0 + d$; then by incorporating this bound and using the definition of $\eta(t)$ in (23) along with the bound in (29) into (45), it is easy to see that there exists a constant so that we obtain the desired bound (30).

REFERENCES

[1] S. Akhtar and D. S. Bernstein, “Lyapunov-stable discrete-time model reference adaptive control,” in Proc. Amer. Control Conf., Jun. 2005, pp. 3174–3179.

[2] A. Feuer and A. S. Morse, “Adaptive control of single-input, single-output linear systems,” IEEE Trans. Autom. Control, vol. 23, no. 4, pp. 557–569, Aug. 1978.

[3] G. C. Goodwin, P. Ramadge, and P. Caines, “Discrete-time multivariable adaptive control,” IEEE Trans. Autom. Control, vol. 25, no. 3, pp. 449–456, Jun. 1980.

[4] G. C. Goodwin and K. S. Sin, Adaptive Filtering Prediction and Control. New York, NY, USA: Dover Publications, Inc., 1984.

[5] P. A. Ioannou and K. S. Tsakalis, “A robust direct adaptive controller,” IEEE Trans. Autom. Control, vol. 31, no. 11, pp. 1033–1043, Nov. 1986.

[6] G. Kreisselmeier and B. D. O. Anderson, “Robust model reference adaptive control,” IEEE Trans. Autom. Control, vol. 31, no. 2, pp. 177–179, Feb. 1986.

[7] G. Kreisselmeier, “Adaptive control of a class of slowly time-varying plants,” Syst. Control Lett., vol. 8, no. 2, pp. 97–103, Dec. 1986.

[8] Y. Li and H.-F. Chen, “Robust adaptive pole placement for linear time-varying systems,” IEEE Trans. Autom. Control, vol. 41, no. 5, pp. 714–719, May 1996.

[9] R. H. Middleton and G. C. Goodwin, “Adaptive control of time-varying linear systems,” IEEE Trans. Autom. Control, vol. 33, no. 2, pp. 150–155, 1988.

[10] R. H. Middleton, G. C. Goodwin, D. J. Hill, and D. Q. Mcaye, “Design issues in adaptive control,” IEEE Trans. Autom. Control, vol. 33, no. 1, pp. 50–58, Jan. 1988.

[11] D. E. Miller, “On necessary assumptions in discrete-time model reference adaptive control,” Int. J. Adapt. Control Signal Process., vol. 10, no. 6, pp. 589–602, 1996.

[12] D. E. Miller, “A parameter adaptive controller which provides exponential stability: The first order case,” Syst. Control Lett., vol. 103, pp. 23–31, May 2017.

[13] D. E. Miller, “Classical discrete-time adaptive control revisited: Exponential stabilization,” in Proc. IEEE Conf. Control Technol. Appl. IEEE, Aug. 2017, pp. 1975–1980.

[14] D. E. Miller and M. T. Shahab, “Classical pole placement adaptive control revisited: Linear-like convolution bounds and exponential stability,” Math. Control Signals Syst., vol. 30, no. 4, p. 19, Nov. 2018.

[15] D. E. Miller and M. T. Shahab, “Classical d-step-ahead adaptive control revisited: Linear-like convolution bounds and exponential stability,” in Proc. Amer. Control Conf., Jul. 2019, pp. 417–422.

[16] D. E. Miller and M. T. Shahab, “Adaptive tracking with exponential stability and convolution bounds using vigilant estimation,” Math. Control Signals Syst., vol. 32, pp. 241–291, 2020.

[17] A. S. Morse, “Global stability of parameter-adaptive control systems,” IEEE Trans. Autom. Control, vol. 25, no. 3, pp. 433–439, Jun. 1980.

[18] S. M. Naik, P. R. Kumar, and B. E. Ydstie, “Robust continuous-time adaptive control by parameter projection,” IEEE Trans. Autom. Control, vol. 37, no. 2, pp. 182–197, 1992.

[19] K. S. Narendra and Y.-H. Lin, “Stable discrete adaptive control,” IEEE Trans. Autom. Control, vol. 25, no. 3, pp. 456–461, Jun. 1980.

[20] K. S. Narendra, Y.-H. Lin, and L. Valavani, “Stable adaptive controller design, part II: Proof of stability,” IEEE Trans. Autom. Control, vol. 25, no. 3, pp. 440–448, Jun. 1980.

[21] C. Rohrs, L. Valavani, M. Athans, and G. Stein, “Robustness of continuous-time adaptive control algorithms in the presence of unmodeled dynamics,” IEEE Trans. Autom. Control, vol. 30, no. 9, pp. 881–889, Sep. 1985.

[22] M. T. Shahab and D. E. Miller, “Multi-estimator based adaptive control which provides exponential stability: The first-order case,” in Proc. IEEE Conf. Decis. Control, Dec. 2018, pp. 2223–2228.

[23] M. T. Shahab and D. E. Miller, “The Inherent Robustness of a New Approach to Adaptive Control,” in Proc. IEEE Conf. Control Technol., Apr. 2020, pp. 510–515.

[24] M. T. Shahab and D. E. Miller, “Asymptotic Tracking and Linear-like Behavior Using Multi-Model Adaptive Control,” IEEE Trans. Autom. Control, 2021, to appear.

[25] K. S. Tsakalis and P. A. Ioannou, “Adaptive control of linear time-varying plants: a new model reference controller structure,” IEEE Trans. Autom. Control, vol. 34, no. 10, pp. 1038–1046, 1989.

[26] C. Wen, “A robust adaptive controller with minimal modifications for discrete time-varying systems,” IEEE Trans. Autom. Control, vol. 39, no. 5, pp. 987–991, May 1994.

[27] C. Wen and D. J. Hill, “Global boundedness of discrete-time adaptive control just using estimator projection,” Automatica, vol. 28, no. 6, pp. 1143–1157, Nov. 1992.

[28] B. E. Ydstie, “Transient performance and robustness of direct adaptive control,” IEEE Trans. Autom. Control, vol. 37, no. 8, pp. 1091–1105, 1992.