BLACK HOLE GROWTH IN DARK MATTER AND THE $M_{BH}$-$\sigma$ RELATION

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ABSTRACT

In this paper, we consider the growth of seed black holes immersed in dark matter halos. We investigate first the adiabatic growth in various initial distribution functions (isothermal; power law; and Navarro, Frenk, and White) and find the resulting density, radial velocity, and anisotropy profiles. In addition, we estimate the growth rate for a given black hole mass in the corresponding adiabatically modified dark matter distribution function. Only in the isothermal case is there a convincing black hole mass-age relation. By calculating the line-of-sight velocity dispersion for the various cases as a function of the black hole mass, we find the predicted adiabatic $M_{BH}$-$\sigma$ relation, which never approaches the recently observed power law. We conclude by abandoning adiabaticity, suggesting that the black hole grows proportionally to the dark matter halo itself on a dynamic timescale. This allows us to relate the observed $M_{BH}$-$\sigma$ relation to the cosmological power spectrum on galactic scales by using dimensional scaling arguments.

Subject headings: black hole physics — dark matter — galaxies: evolution — galaxies: halos — galaxies: nuclei

1. INTRODUCTION

Recent results by Gebhardt et al. (2000) and Ferrarese & Merritt (2000) that establish a strong correlation between central black hole (BH) mass $M_{BH}$ and the velocity dispersion $\sigma$, measured at $r_e/8$, indicate that the central black hole is intimately related to the dynamic structure of the galaxy. An earlier result by Magorrian et al. (1998) relates $M_{BH}$ linearly to the mass of the bulge, which suggests a similar conclusion, while Merrifield, Forbes, & Terlevich (2000) establish a link between $M_{BH}$ and the age of the stellar system: a massive central black hole seems to grow over a period of 1 Gyr or so.

These results have already stimulated the emission of various theories that create a feedback mechanism between bulge star formation and accretion of gas onto the black hole (Burkert & Silk 2001). These theories generally tend to establish the desired correlations but at the cost of some rather complicated physics simply, perhaps oversimply, described.

The object of the present paper is first to revisit the correlations established by the adiabatic growth of a black hole in a galaxy, since this process is relatively free of physical assumptions once the initial distribution function (DF) is chosen. This part of the work is very much in the spirit of van der Marel (1999), who examined the effect of the black hole on the fundamental plane relations but did not consider explicitly the $M_{BH}$-$\sigma$ relation. We choose DFs, moreover, that are appropriate for collisionless dark matter halos, the philosophy being that these are the dynamically dominant components of massive galaxies and should therefore dictate the observed velocities. These are chosen to be an isothermal or Gaussian DF (to test our code against previous work and because this may be the maximum entropy state according to Nakamura 2000), an isotropic steady state power-law DF found by Henriksen & Widrow (1995) for comparison purposes, and the DF that corresponds to the Navarro, Frenk, and White (NFW) density profile, as approximated by Widrow (2000), as the best measured approximation to a dark matter halo.

Such an approach does not, of course, explain the origin of the black hole, but rather yields only the perturbed DF that is created by its adiabatically established presence. An earlier attempt to grow the black hole during the formation of the galaxy (Ostriker 2000) relied on dissipative dark matter, which is fraught unfortunately with badly known parameters. In the present approach the black hole growth is limited to the particle flux across the event horizon of a seed black hole that is peculiar to the initial or adiabatically modified DF.

The adiabatic approach fails to yield either the correct form or extent of the observed correlations between the black hole mass and the modified galaxy. We therefore suggest a possible explanation of the correlations based on a nonadiabatic process of black hole growth on the formation timescale of the dark matter halo. The argument is essentially dimensional at this stage and must be examined numerically in greater detail.

In § 2, we review the adiabatic growth approximation and verify our code with the isothermal DF. In § 3, we give the results in the power law and NFW DFs. In § 4, we discuss the resulting $M_{BH}$ versus $\sigma$ in the various DFs and show the extent of the black hole influence in the galaxy. In § 5, we give our dimensional derivation of the observed relations based on a self-similar (but not necessarily spherically symmetric) growth of black hole and dark halo. Finally, in § 6 we give our conclusions.

2. ADIABATIC GROWTH

2.1. History

We consider the slow growth of an initially small seed BH located at the center of a collisionless, spherical system of stars and dark matter. Using an algorithm that makes use of the fact that, under these circumstances, the radial and transverse actions are conserved, the final state of the system containing a supermassive BH can be calculated.
This technique was first suggested by Peebles (1972), who used it to show that an adiabatic cusp with $\rho \sim R^{-3/2}$ would form in an isothermal sphere. This work was confirmed numerically by Young (1980). His algorithm is used here essentially unchanged. Lee & Goodman (1989) used Young’s algorithm to explore the more general case of a stellar system with a net rotation, although they treat the potential as spherically symmetric rather than axisymmetric. They found that while the rotation-to-dispersion metric. They found that while the rotation-to-dispersion

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surface density is about one-half of its central value in a fit-
core radius of the galaxy, similar to the usual definition of

It can then be calculated from Poisson’s equation:

The integral form of Poisson’s equation, however, is often
more convenient. We take the reference potential to be zero
at the center of the system and suppose the halo mass to
tend to zero there, which is true for a density profile less

3.3. The adiabatic growth framework is then as follows. A
particle of the initial system with energy $\epsilon$ and angular
momentum $j$ will have a different energy $\epsilon^*$ (but identical
angular momentum $j$) after the BH has grown. Its radial
action $i_R$, however, will be the same before and after growth:
thus, it is possible to invert $i_R(\epsilon^*, j)$ to get $\epsilon$ as a function
of $(\epsilon^*, j)$. The final DF is then found from
$F(\epsilon, j) = F(\epsilon^*(\epsilon, j), j)$.
This is implemented in an algorithm similar to that used
first by Young (1980). See the Appendix for details.

2.4. Isothermal Sphere

We apply the adiabatic growth framework first to the iso-
thermal sphere, characterized by the Gaussian DF:

This is a well-studied distribution and was Young’s
(1980) initial system. We present the adiabatic results for
the isothermal sphere here mainly as a check that the code
developed gave the same results that Young (1980) found.
There are in fact reasons to believe (Nakamura 2000; Henriksen & Le Delliou 2001) that this is ultimately the DF
of interest, especially near the center of the system. Of

Results are shown in Figure 1 for black hole masses from
zero to 0.001, 0.01, 0.1, and 1.0, as labeled by increasing
effect. These dimensionless masses correspond to physical
masses in the range of about $10^7$–$10^{10} M_\odot$ if we use the values
of $\rho_0$ and $r_0$ given in § 3.3.

$F(\epsilon) = (2\pi)^{-3/2} e^{-\epsilon}.$
We find the same characteristic density cusp of $R^{-3/2}$ as obtained by Young (1980), and the radial velocity profiles are also similar. Although Young (1980) did not show results for the anisotropy parameter, defined as

$$\beta = 1 - \langle V_T^2 \rangle / \langle V_R^2 \rangle,$$

we observe that there is at most about a 10% anisotropy in favor of tangential motion due to the increasing binding energy of a particle and the resultant decrease in eccentricity at constant angular momentum. Moreover, the perturbation extends as far as the core radius only when the mass is comparable to that of the core, as is to be expected. These results are the main test of our program.

3. CUSPY DARK MATTER HALOS

We explore here the adiabatic growth of a central black hole in a collisionless dark matter halo that possesses a central cusp. For initial systems, we choose two very different starting points. The first is an isotropic self-similar system, meant to represent the final state of a halo undergoing self-similar relaxation (Henriksen & Widrow 1995). The other system is the NFW system of Navarro, Frenk, & White (1996), which fits a wide range of dark matter halo sizes.

3.1. Self-similar Distribution

The self-similar system is described by the DF

$$F(\epsilon) = F_0 \epsilon^{-(3\delta-1)/(2\delta-2)},$$

where $\delta$ is a free parameter ($\frac{2}{3} < \delta < 1$, $\delta > 1$) that essentially controls the logarithmic slope of the initial density and potential, given by

$$\varrho = R^{-2/\delta},$$

$$\psi = R^{2-2/\delta}.$$  

The constant $F_0$ depends only on the value of $\delta$ and can be solved for in terms of the core radius $r_0$ and the density $\rho_0$ by using equations (2) and (4).

Using this system as a starting point for the adiabatic algorithm discussed above leads to rather different results than are found for the isothermal sphere. The results for two different choices of $\delta$ are given: Figure 2 is for the parameter $\delta = 2$, while Figure 3 shows results with $\delta = \frac{3}{2}$.

As can be seen in Figure 2, the initial density profile of the self-similar system with $\delta = 2$ is a power law with a logarithmic slope of $-1$. Adding an adiabatically grown BH to the system induces a steeper cusp region at small radii, and the outer radius at which this region begins depends, of course, on the final BH mass. The new cusp slope is $-7/3$, greater than the density cusp induced in the isothermal sphere, but the difference between the before and after slopes is not as...
great here as in the isothermal case (a gain of ~1.3 here vs. 1.5 for the isothermal sphere). The major difference with the isothermal sphere, however, is best seen in the anisotropy parameter. The system becomes quite tangentially anisotropic at small radii.

That the cusp slope will not increase as greatly for an initial system that already has a steep density cusp is shown best in Figure 3, where the grown BH seems to have left no visible mark on the density slope (to within graphical resolution: the initial density slope is \( \frac{\rho}{\rho_0} = 3 \) while the measured final slope is \( \frac{\rho}{\rho_0} = 4 \)). This result is for \( \delta = \frac{3}{4} \). Notice, however, that the BH growth has disturbed the radial velocity, which takes on the same \( \frac{1}{R} \) shape (although only for the largest masses) seen previously in both the \( \delta = 2 \) self-similar system and the isothermal sphere. That these systems all have similar velocities at small radii is simply indicative of their similar Keplerian potentials there.

### 3.2. NFW Profile

The NFW system, given by the universal density profile

\[
\rho = \frac{1}{R(1+R)^2},
\]

is more complicated to describe than the other two systems discussed above since it does not have an analytic DF. We use here an analytic fit of a numerical calculation of the DF (Widrow 2000), which in our units is

\[
F(\epsilon) = F_1(1 - \epsilon)^{3/2}e^{-3/2}\left[ -\frac{\ln(1 - \epsilon)}{\epsilon} \right] e^P,
\]

where \( P = \sum p_i e_i \) and the parameters are

\[
F_1 = 9.1967 \times 10^{-2}, \quad q = -2.7419, \quad p_1 = 0.3625, \quad p_2 = -0.5669, \quad p_3 = -0.0802, \quad p_4 = -0.4945.
\]

The results for this system are given in Figure 4. As expected, the BH growth induces cusps in both the density and the velocity. The logarithmic slope for the density is \( \frac{\rho}{\rho_0} = 3 \), which is the same as for the self-similar system with \( \delta = 2 \); they also both began with the same slope \((\rho \propto 1/R)\) in the inner region where the BH disturbance is greatest. The velocity cusp is the usual \( \frac{1}{R} \). We notice that the velocity tangential anisotropy is also comparable to the system with \( \delta = 2 \) at small radii for all black hole masses and that the velocity perturbation can extend nearly to the core radius for \( m_{\text{BH}} \geq 0.1 \).

### 3.3. Growth Timescales

It is of some interest to explore the rate at which a black hole would grow in the above dark matter distributions in view of the black hole mass/galactic age relation reported by Merrifield et al. (2000). Such growth represents an alter-
native to advection-dominated accretion flow (ADAF; see Narayan, Igumenshchev, & Abramowicz 2000 and references therein) as a means to grow a black hole invisibly. We consider as an illustration the time to grow the black hole by a factor of 10 in the various DFs.

In order to calculate the timescale for the BH to grow by a factor of 10, we use the following expression for the growth of the BH as it accretes matter:

\[
\frac{dM_{BH}}{dt} = 4\pi r_s^2 \rho(r_s)\langle v_r \rangle |_{r_s},
\]

(15)

where \(r_s\) is the radius of the last stable orbit, \(r_s = 3R_s\), and \(R_s = 2GM_{BH}/c^2\) is the Schwarzschild radius of the BH. In our units, and with \(\langle v_r \rangle\) given as an integral over the DF, this expression becomes

\[
\frac{dM_{BH}}{dt} = 8\pi^2 (4\pi G)^{1/2} \rho_0^{3/2} r_0^3 \int_{\psi_i}^{\psi(\infty)} d\psi \int_0^{\psi_{max}} j \, df(\epsilon, j),
\]

(16)

where \(\psi_x = \psi(r_x)\) is the potential evaluated at the radius of the last stable orbit. Note that this equation requires the adiabatic growth of a central BH, the timescale becomes then results in a timescale of about 9 billion yr. This is close to the BH to grow 10\(^{10}\)\(M_\odot\). This may be accounted for by the more pronounced anisotropy of the NFW DF at larger radii.

It should be noted that changing the initial constants \(r_0, \rho_0\), and the initial BH mass \(M_0\) can vary these numbers by a few orders of magnitude in both directions. From the work of van der Marel (1999), which fits an isothermal sphere to a variety of galaxies, it seems that the core radius can be as small as a few tens of parsecs for "power-law" galaxies and as large as a few kiloparsecs for "core" galaxies. Assuming the same scaling laws van der Marel (1999) uses, the density \(\rho_0\) will depend on the value we take for the core radius. According to these scaling laws, a radius of \(r_0 = 20\) pc, for example, corresponds to a density of \(\rho_0 = 8360 \, M_\odot\) pc\(^{-3}\), while at \(r_0 = 2000\) pc the density will be \(\rho_0 = 3 \, M_\odot\) pc\(^{-3}\).

Using these two sets of values and assuming that the initial seed BH mass can vary between \(10^4\) and \(10^9\)\(M_\odot\), we see that the isothermal sphere can grow by a factor of 10 anywhere between 1,000,000 and 10\(^{14}\) yr. For the self-similar system with \(\delta < 1\), changing the constants has essentially no effect, since the timescale is too short; for \(\delta \approx 1\), however, the timescale can approach a few billion years. The timescale for the NFW system will not go beyond a few hundred million years.

It seems, then, that only the isothermal or Gaussian DF can yield a BH mass-age relation of the type detected by Merrifield et al. (2000) by adiabatic growth from a collisionless DF. The other systems studied here grow more quickly than the isothermal sphere, mainly because of their stronger central density cusp.

4. ADIABATIC \(M_{BH-\sigma}\) RELATIONSHIP

Recent observations (Ferrarese & Merritt 2000; Gebhardt et al. 2000) have found a strong correlation between the mass of the central BH and the line-of-sight velocity dispersion in the bulge of its host galaxy. Although various theories have been suggested to explain this relationship (e.g., Haehnelt & Kauffmann 2000; Adams, Graff, & Richstone 2001), none have yet been proven conclusively. This relationship has been shown to follow

\[
M_{BH} \propto \sigma^\alpha,
\]

(19)

where \(\alpha\) is somewhere between 3.5 and 5.

The adiabatic growth model has been used by van der Marel (1999) to explain various observational properties of black holes, such as the central density cusp and its correlation with the luminosity of the galactic bulge. His analysis included properties both intrinsic to the adiabatic growth as well as scaling relations based on fundamental planelike observations. We repeat his analysis here to explore the \(M_{BH-\sigma}\) relation.

First, we calculate any intrinsic relation between the BH mass and the velocity dispersion of the bulge that may arise naturally from the adiabatic growth of the central BH. Keep in mind, however, that the calculations that follow cannot
be compared directly with observations since they are noise-
free and have an infinite resolution. Regardless, they should
give a sense of whether or not the BH growth can give a
relation like equation (19).

The line-of-sight velocity dispersion is found by projec-
ting the radial and transverse velocity moments on the plane
of the sky. The velocity moments are calculated from

$$\langle V_R^m V_T^n \rangle = \frac{4\pi}{\theta} \int \langle V_R^m V_T^n \rangle \, d\theta \int_0^{\theta_{\max}} \frac{dd}{r^4} F(\epsilon, j) V_R^m V_T^n \, ;$$

(20)

projecting them on the sky gives

$$\langle V_R^2 \rangle (R_p) = \frac{2}{\Sigma(R_p)} \int_{R_p}^{\infty} \frac{dR R\rho}{\sqrt{R^2 - R_p^2}}$$

where $R_p$ is the projected radius and $\Sigma$ is the projected
density given by

$$\Sigma(R_p) = 2 \int_{R_p}^{\infty} \frac{dR R\rho}{\sqrt{R^2 - R_p^2}} .$$

(22)

It is a simple matter to predict the dispersion near the
center of the system. As stated above, the velocity moments
simply reflect the Keplerian potential near the BH; thus, they
take the form

$$\langle V^2 \rangle \propto \frac{m_{\text{BH}}}{R} ,$$

(23)

and this form is identical regardless of the initial system.
Writing the dispersion as $\sigma = [\langle V^2 \rangle^{1/2}]$, this relation is simply

$$m_{\text{BH}} = \sigma^2 .$$

(24)

Of course, this relation is applicable only in the innermost
regions. Observations are usually done much farther from the
center—typically near the effective, or half-light, radius of the
bulge. This radius corresponds to the core radius $r_0$ for the unperturbed isothermal sphere. To calculate any
relationship between the BH mass and the velocity disper-
sion away from the center of the system, we must use the DF
calculated from the adiabatic growth framework.

Figure 5 shows the results for the three systems studied,
calculated at three different radii: $R = 10^{-3}$, near the center; $R = 1$, at the core radius; and $R = 100$, well outside of the
central region. The simple predictions made above (eq. [24])
are confirmed to exist in this model asymptotically as the
mass of the hole becomes large (Fig. 5, solid lines). As the
calculations are done farther out, however, the relation
exists only at large and larger masses (Fig. 5, dashed and
dotted lines).

Similar calculations for the self-similar and NFW distri-
butions yield almost identical results. The NFW DF is nota-
ble for attaining the asymptotic relation at smaller black
hole masses than for the isothermal case (the self-similar DF
is intermediate), but in all cases no linear relation steeper
than that of equation (24) is found to exist. There is, as can
be seen in the figures, a steep shoulder during the approach
to the asymptotic limit, but this is nonlinear and of insignifi-
cant extent.
The essential idea is that under this assumption, one can write the mass inside any surface (the surface in space would be defined by holding $M$ constant at a fixed time) in the halo as

$$M = \mathcal{M}(X)e^{(3\delta - 2\alpha)T}. \quad (25)$$

Here the vector $X \equiv r e^{(\alpha T)}$ is a scaled position vector and $e^{(\alpha T)} \equiv \alpha t$. The quantities $\delta$ and $\alpha$ are scales in space and time that allow dimensional information to be included in the expressions for the various quantities. Thus, the mass scale $\mu$ is determined in terms of $\delta$ and $\alpha$ by the condition that $G$ is a constant of the problem. This yields $\mu = 3\delta - 2\alpha$, as used above.

The velocity of any particle in the halo may be written consistently as

$$v = Ye^{(\delta - \alpha)T}. \quad (26)$$

The quantities $X$ and $Y$ are independent of $T$ during the self-similar collapse and so define a steady state phase space. Consequently, on a fixed spatial surface and averaged over the line-of-sight rms velocity, we can eliminate $T$ between equation (25) and the averaged equation (26) to obtain

$$\log M \propto \left(\frac{3\delta / \alpha - 2}{\delta / \alpha - 1}\right) \log \sigma, \quad (27)$$

where $\sigma$ is the velocity dispersion along the line of sight. It does not matter which mass surface is chosen in the system if it is truly self-similar, of course. Thus, the preceding relation applies to the “bulge” mass at the bulge scale and to the black hole mass on the black hole scale. However, the black hole mass will be simply proportional to the bulge mass during self-similar growth so that both masses will obey equation (27). This, of course, also requires that the black hole mass and the bulge mass are also proportional, but given that we are talking about total masses and recalling the vagaries of star formation, this would not necessarily imply a tight bulge luminosity/black hole mass correlation.

We may proceed to require that the constant of proportionality in equation (27) is equal to the observed (Gebhardt et al. 2000; Ferrarese & Merritt 2000) constant, say, $a$. Then we find that

$$\frac{\delta}{\alpha} = \frac{a - 2}{a - 3}, \quad (28)$$

so that, for example, if $a = 4$, then $\delta / \alpha = 2$; and if $a = 4.5$, then $\delta / \alpha = 5/3$. Should $a < 4$ (but greater than 3, which appears here as a kind of lower permissible limit), say, 15/4, then $\delta / \alpha = 7/3$. The principal numerical fact to note is that $\delta / \alpha$ is greater than or less than 2 depending on whether $a$ is less than or greater than 4.

The reason for the numerical discussion of the preceding paragraph is that the value $\delta / \alpha = 2$ is highly significant in spherical models of dark matter halo growth (see, e.g., Henriksen & Widrow 1999 for a summary). In these models, this ratio is given in terms of the power-law index $-\epsilon$ of the initial cosmological density perturbation by

$$\frac{\delta}{\alpha} = \frac{2}{3} \left(1 + \frac{1}{\epsilon}\right). \quad (29)$$

Consequently, we can infer from equations (28) and (29) that the initial cosmological overdensity had the power $-\epsilon$,

$$\epsilon = 2 \left(1 - \frac{3}{a}\right). \quad (30)$$

If we finally relate $\epsilon$ to the power spectrum index $n$ of the primordial density through the rms profile of such perturbations (other choices are possible, e.g., Hoffman & Shaham 1985, but similar results are found), then

$$n = 2\epsilon - 3. \quad (31)$$

Consequently, we arrive at a direct link between $a$ and the primordial power spectrum index on the scale of galaxy halos as

$$n = 1 - \frac{12}{a}. \quad (32)$$

Thus, under our interpretation of the black hole mass/velocity dispersion correlation, we are led to conclude that the primordial power spectrum on the scale of galaxies has the index $n$ of equation (32). For $a = 4$, this yields $n = -2$, while $a = 15/4$ and $a = 9/2$ yield $n = -11/5$ and $n = -5/3$, respectively. These values for $n$ on the scale of galaxies are all in good agreement with observation, which favors a value near $n = -2$. We conclude that our interpretation of the mass/velocity dispersion relation as originating in the primordial density profile is consistent with cosmological evidence.

6. CONCLUSIONS

In this paper, we have explored the implications of growing a black hole in various dark matter halo DFs. Our principal approach was to assume that the black hole grows adiabatically on a timescale long compared with the dynamic time of the halo. The method gives definite predictions for the modified density, radial velocity, and anisotropy profiles in the isothermal, self-similar power law and NFW dark matter distributions. The isothermal calculations reproduced and extended slightly previous work, but the calculations for the self-similar and NFW dark matter halos are new. Depending on the mass of the black hole, the disturbances can be noticeable out to nearly the core radius of the galaxy. Moreover, estimates of black hole growth timescales in the adiabatically modified DFs are given. Only in the isothermal DF is there found a reasonable black hole mass/galactic age relation. In no case, however, does the adiabatic argument give an $M_{BH-\sigma}$ relation that is close to that observed.

Thus, in the concluding section we explored by dimensional argument based on the concept of multidimensional self-similarity (Carter & Henriksen 1991; Henriksen 1997) the possibility that the central black hole grew on a dynamic timescale with the dark matter halo. A simple argument predicts $a = 3/(1 - \epsilon/2)$, where $a$ is the observed power in the $M_{BH-\sigma}$ relation and $-\epsilon$ is the power in $n$ of the initial cosmological density perturbation that produced the galactic halo. Under certain assumptions, this power can, in turn, be related to the power $n$ of the primordial cosmological power spectrum on the scale of galactic halos, so that, reasonably, $a = 12/(1 - n)$. This gives $n = -2$ for $a = 4$, which is close to that observed in both cases. We conclude that this sugges-
tion is promising and more work should be done to confirm or infirm the idea that the black hole can grow self-similarly with the dark matter core.

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APPENDIX

ALGORITHM

This appendix describes the framework that was used to calculate the adiabatic growth models presented above. We implement this framework using an algorithm similar to that of Young (1980), which solves directly for the final state of the system containing the supermassive black hole. Given a DF that describes the initial system, we do the following:

1. Compute the self-consistent potential $\psi$ and density $\rho$ for the initial system (eqs. [2] and [4]).
2. Compute the radial action $i_R(\epsilon, j)$ (eq. [6]) for the initial potential $\psi$.
3. Approximate the final potential $\psi^*$ for a given black hole mass by
   \[
   \psi^* = \psi - \frac{m_{BH}}{R}. 
   \] (A1)
4. Calculate the radial action $i_R^*(\epsilon, j)$ for this new potential $\psi^*$.
5. Equate the initial radial action $i_R(\epsilon, j)$ and the final action $i_R^*(\epsilon, j)$ to find the energy $\epsilon$ that has become $\epsilon^*$. Then, the new DF will be
   \[
   F^*(\epsilon^*, j) = F(\epsilon^{*}, j), j. \] (A2)
6. Compute a self-consistent density $\rho^*$ and new potential $\psi^*$ using $F^*$, continue back to step 4, and iterate until the density has converged.

The convergence criterion for the algorithm was taken to be when $\rho^*$ changed by less than $10^{-4}$ at all radii.

The computer program developed to implement this algorithm, written in C, uses a radial grid $R_i$, with points spaced logarithmically between an inner radius of $R = 10^{-5}$ and an outer radius of $R = 10^5$. The various properties of the system are described on this grid: $\rho_i = \rho(R_i)$, $\psi_i = \psi(R_i)$, and so on. The DF is described on a grid of energy and angular momentum points, where the energy points have for convenience the values of the potential on the radial grid, $\epsilon_i = \psi_i$, and the angular momentum is evaluated as $x = j/j_c$, where $j_c$ is the circular angular momentum and $x$ goes from 0 to 1.

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