The Orbit Method in the Finite Zone Integration Theory

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Abstract

A construction of integrable hamiltonian systems associated with different graded realizations of untwisted loop algebras is proposed. Such systems have the form of Euler - Arnold equations on orbits of loop algebras. The proof of completeness of the integrals of motion is carried out independently of the realization of the loop algebra. The hamiltonian systems obtained are shown to coincide with hierarchies of higher stationary equations for some nonlinear PDE’s integrable by inverse scattering method.

We apply the general scheme for the principal and homogeneous realizations of the loop algebra $sl_3(\mathbb{R}) \otimes \mathcal{P}(\lambda, \lambda^{-1})$. The corresponding equations on the degenerated orbit are interpreted as the Boussinesq’s and two-component modified KDV equations respectively. The scalar Lax representation for the Boussinesq’s equation is found in terms of coordinates on the orbit applying the Drinfeld - Sokolov reduction procedure.

1 Introduction

It is known that non-linear equations integrable by the inverse scattering method admit the zero-curvature representation

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0,$$

where $U(x, t)$ and $V(x, t)$ belong to some loop algebra $\tilde{\mathfrak{g}}$ (algebra of Laurent polynomials with the values in a finite-dimensional semisimple Lie algebra $\mathfrak{g}$). This representation is invariant under the gauge transformation by the corresponding loop group. Investigations stimulated by [1] resulted in a fact that the gauge non-equivalent equations (1) correspond to the different constructions of the basic representation of the loop algebra. Each the construction is related with a choice of the Heisenberg subalgebra, that determines a loop algebra realization. The explicit construction of all inequivalent graded Heisenberg subalgebras was given in [4]. Any of those is determined by the finite-order automorphism of the associated finite-dimensional Lie algebra. That automorphisms themselves are intimately related with the conjugacy classes of the Weyl group of the algebra. For example, in the $sl_n(\mathbb{R})$ – case the Weyl group is isomorphic to the symmetric group $S_n$. Its irreducible representations and hence conjugacy classes are classified by partitions of $n$:

$$n = n_1 + n_2 + \ldots + n_r,$$

where $n_1 \geq n_2 \geq \ldots \geq n_r \geq 1$.

The partition $n = 1 + 1 + \ldots + 1$ corresponds to the homogeneous construction, and the partition $n = n$ is related with the principal one. When applied to the loop algebra $sl_2(\mathbb{R})$, that cases lead
to the hierarchies of higher modified Korteweg - de Vries (mKDV) equations and higher Korteweg - de Vries (KDV) equations respectively (cf. [11, 12]). The investigation of the hierarchies of equations related with the other constructions is an actual problem (cf. [4, 5, 6]).

Let us remind that the functional phase space of a hamiltonian system that represents a non-linear integrable PDE contains the finite-dimensional subspaces being invariant under the actions of hamiltonian flows generated by all the integrals of motion of that system. That finite-dimensional configurations arise as solutions of the higher stationary (Novikov’s) equations [7]. The solutions provide a finite number of instability zones in the specter of the associated linear differential operator ($L$ – operator). It was shown [11] that Novikov’s equations are equivalent to the Euler - Arnold equations [13] on orbits of the coadjoint representation of the appropriate loop group. The time evolution is realized in that scheme naturally also. That facts were stated considering the homogeneous realization of the loop algebra $\tilde{sl}_2(\mathbb{R})$. The corresponding equations were interpreted as the higher stationary mKDV and sine- (sh-) Gordon equations.

This article develops the scheme of constructing the higher stationary equations on orbits of loop algebras of rank $\geq 1$ (section 2). An accent is made on different realizations of the loop algebra. The "intermediate" hierarchies that are related with realizations differing from the two mentioned above are of special interest. But in fact, the paper deals mostly with the principal realization because of its fundamental place among the others: every intermediately constructed affine algebra is a "modification" of the principally realized one. Third section concerns the examples of the principal and homogeneous realizations of the loop algebra $sl_3(\mathbb{R})$. As a result, the equations are obtained to be interpreted as the stationary Boussinesq’s equation and the two-component mKDV – type equation respectively. We will also obtain the Lax representation for the Boussinesq’s equation in terms of the coordinates on the orbit applying the Drinfeld - Sokolov reduction procedure [16].

To conclude the introduction, the following should be stressed. The orbit interpretation of the finite-zone integration theory (presented in [11] and in this paper) allows to construct all the theory of non-linear completely integrable PDE’s in a non-traditional way. Considering integrable hamiltonian equations on orbits as the ground of the theory and interpreting them as higher stationary equations for some (unknown yet) evolutionary equations, the problem of enumeration of integrable PDE’s is reduced to an algebraic-geometrical problem to classify the loop algebras and their orbits. It is also possible to construct the integrals of motion for the evolutionary equations starting from those for the stationary equations on the orbit. The latters are got easily as expansion coefficients (relative to the complex loop parameter) of the Casimir functions in the enveloping algebra of the loop algebra stated into the base of the theory.

2 Constructing higher stationary equations: the orbit scheme

1. The general case. Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra of rank $R$ and $\mathcal{P}(\lambda, \lambda^{-1})$ the associative algebra of Laurent polynomials with respect to the complex parameter $\lambda$ belonging to the unit circle. Let us consider the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathcal{P}(\lambda, \lambda^{-1})$ with the commutator:

$$[\sum A_i \lambda^i, \sum B_j \lambda^j] = \sum [A_i, B_j] \lambda^{i+j}.$$  

Below we will operate with the homogeneous and principal realizations of $\tilde{\mathfrak{g}}$. A manifestation of the difference between them is their different gradations with the gradation operators $d_h$ and $d_p$.  

respectively (see, for example, [3]). From now and later on the expressions "the realization of a loop algebra" and "the gradation in a loop algebra" are used as equivalent. Define the family of $Ad$--invariant non-degenerate forms on $\tilde{g}$:

$$\langle A, B \rangle_k = \sum_{i+j=k} (A_i, B_j), k \in \mathbb{Z},$$

where $(,)$ denotes the Killing form in $g$. Decompose $\tilde{g}$ in the direct sum of two subalgebras:

$$\tilde{g} = \tilde{g}_- \oplus \tilde{g}_+,$$

where

$$\tilde{g}_+ = \left\{ \sum_{i \geq 0} A_i \lambda^i \right\}, \quad \tilde{g}_- = \left\{ \sum_{i < 0} A_i \lambda^i \right\}.$$

Then $\tilde{g}_+$ and $\tilde{g}_-$ is the dual pair relative to $\langle , \rangle_-$, with the coadjoint action

$$ad^*_{A} \mu = P_+ [\mu, A], \quad A \in \tilde{g}_-, \, \mu \in \tilde{g}_+^* \simeq \tilde{g}_+,$$

(2)

where $P_+$ denotes the projector onto $\tilde{g}_+$. Let $\{Q_i\}_{i=0}^{\dim g}$ be a basis in $g$. Keeping in mind our further purposes, it is more convenient to fix a dual basis $\{Q^*_i\}_{i=0}^{\dim g}$ determined by $(Q^*_i, Q_j) = \delta_{ij}$. The finite-dimensional subspaces

$$M^{N+1} = \left\{ \mu \in \tilde{g}^*_- : \mu = \sum_{l=0}^{N+1} \sum_{i=1}^{\dim g} \mu_i^l Q_i^* \lambda^i \right\} \subset \tilde{g}_+^*, \quad N = 0, 1, 2, \ldots < \infty,$$

where $\mu_i^l = (\mu, Q_i^{-l-1})_{-1}$ are the coordinates on $M^{N+1}$, stay invariant under the action of $\tilde{g}_-$. The coadjoint action of $\tilde{g}_+$ is defined on $M^{N+1}$ also. Then $\mu(A) = \langle \mu, A \rangle_{N+1}, A \in \tilde{g}_+$, and we identify $\tilde{g}_+^*$ with the subspace $\tilde{g}_-^* \oplus M^{N+1}$ in such a case. The coordinates on $M^{N+1}$ can be written down as $\mu_i^l = (\mu, Q_i^{-l-N+1})_{N+1}$, and $M^{N+1}$ stay invariant under the action of $\tilde{g}_+$. The coadjoint actions induce the family of Lie - Poisson structures on $M^{N+1}$:

$$\{f_1, f_2\}_\sigma = \sum_{l=0}^{N+1} \sum_{i=1}^{\dim g} W_{ij}^{l,s}(\sigma) \frac{\partial f_1}{\partial \mu_i^l} \frac{\partial f_2}{\partial \mu_j^s}, \quad \forall f_1, f_2 \in \mathcal{C}^{\infty}(M^{N+1}),$$

(3)

where

$$W_{ij}^{l,s}(\sigma) = \left\langle ad^*_\sigma Q_i^{-l-s} \mu, Q_j^{-s} \right\rangle_{\sigma}, \quad \sigma \in \mathbb{Z} .$$

(4)

**Definition 1** Symplectic leaves of the Poisson structures $W(\sigma)$ will be called generic orbits of the corresponding loop subalgebras acting on $M^{N+1}$.

The two following cases are important. Let $\mathcal{O}^{\text{gen}}_+$ denote the generic orbit of the finite-dimensional quotient algebra $\tilde{g}_+ / \lambda^{N-1} \tilde{g}_-$ that acts effectively on $M^{N+1}$. The generic orbit of $\tilde{g}_+ / \lambda^{N+2} \tilde{g}_+$ will be denoted $\mathcal{O}^{\text{gen}}_+$. Let $H^\nu, \nu = 2, 3, \ldots, R+1$, be the Casimir functions in the enveloping algebra of $g$. They are polynomials of the variables $\mu_k = (\mu, Q_k)$ on the dual $g^*$ of $g$. The substitution $\mu_k \mapsto \mu_k(\lambda) = \sum_{l=0}^{N+1} \mu_k^l \lambda^l$ provides the continuation of $H^\nu$ to $\mathcal{C}^{\infty}(M^{N+1})$:

$$H^\nu = \sum_{\alpha=0}^{\nu(N+1)} h^\nu_{\alpha} \lambda^\alpha, \quad h^\nu_{\alpha} \in \mathcal{C}^{\infty}(M^{N+1}).$$

(5)
Theorem 1  

1. The functions \( \{ h^\nu_\alpha \} \) constitute an involutive collection in \( C^\infty(M^{N+1}) \), relative to the Poisson structures \( W(-1) \) and \( W(N+1) \).

2. The functions \( \{ h^\nu_\alpha \} , \alpha \geq (\nu - 1)(N + 1) \), annihilate the Poisson structure \( W(-1) \).

3. The functions \( \{ h^\nu_\alpha \} , \alpha = 0, 1, \ldots, N + 1 \), annihilate the Poisson structure \( W(N+1) \).

Proof. Let \( \tilde{Q}_i^{l+\sigma}(\sigma) \) be the tangent vector field corresponding to the basis element \( Q_i^{-l+\sigma} \) and the coadjoint action (2) of \( \tilde{g}_- \) (resp. \( \tilde{g}_+ \)) for \( \sigma = -1 \) (resp. \( \sigma = N + 1 \)). Then, \( \forall f \in C^\infty(M^{N+1}) \),

\[
\tilde{Q}_i^{l+\sigma}(\sigma) f(\mu) = \frac{d}{d\tau} f(Ad_{\exp \tau Q_i^{-l+\sigma}}^\nu \mu) \big|_{\tau=0} = \sum_{k,l} \frac{\partial f}{\partial \mu_{ik}^l} \frac{d\mu_{ik}^l(\tau)}{d\tau} \big|_{\tau=0},
\]

where

\[
\mu_{ik}^l(\tau) = \left\langle Ad_{\exp \tau Q_i^{-l+\sigma}}^\nu \mu, Q_k^{-r+\sigma} \right\rangle_{\sigma}.
\]

Next,

\[
\left. \frac{d\mu_{ik}^l(\tau)}{d\tau} \right|_{\tau=0} = \left\langle ad_{Q_i^{-l+\sigma}}^\nu, Q_k^{-r+\sigma} \right\rangle_{\sigma} = W_{ik}^l(\sigma).
\]

Then

\[
\tilde{Q}_i^{-l+\sigma}(\sigma) = \sum_{k=1}^{\dim g} \sum_{r=0}^{N+1} W_{ik}^l(\sigma) \frac{\partial}{\partial \mu_k^r}.
\]

Note that

\[
\frac{\partial}{\partial \mu_k^r} = \frac{\partial \mu_k(\lambda)}{\partial \mu_k^r} \frac{\partial}{\partial \mu_k(\lambda)} = \lambda^\nu \frac{\partial}{\partial \mu_k(\lambda)}.
\]

By (3) and (4),

\[
\sum_l \left( \lambda^{l+1} \tilde{Q}_i^{-l-1}(-1) + \lambda^{l-N-1} \tilde{Q}_i^{-l+N+1}(N+1) \right) = \sum_{j,k} C^j_{ik} \mu_j(\lambda) \frac{\partial}{\partial \mu_k(\lambda)}
\]

where \( C^j_{ik} \) denotes the structure constants of \( g \). The \( ad^\nu \) - invariance of \( H^\nu \) means

\[
\sum_{j,k} C^j_{ik} \mu_j(\lambda) \frac{\partial}{\partial \mu_k(\lambda)} H^\nu = 0,
\]

or, by previous formula,

\[
\sum_l \left( \lambda^{l+1} \tilde{Q}_i^{-l-1}(-1) + \lambda^{l-N-1} \tilde{Q}_i^{-l+N+1}(N+1) \right) H^\nu = 0.
\]

Substitute (3) herein and equate the coefficients at the same degrees of \( \lambda \). Then the following consequences arise:

\[
\tilde{Q}_i^{-l-1}(-1) h^\nu_\alpha = 0, \quad \alpha \geq (\nu - 1)(N + 1); \tag{7}
\]

\[
\tilde{Q}_i^{-l+N+1}(N+1) h^\nu_\alpha = 0, \quad 0 \leq \alpha \leq N + 1; \tag{8}
\]

\[
\tilde{Q}_i^{-l-1}(-1) h^\nu_\alpha + \tilde{Q}_i^{-l+N+1}(N+1) h^\nu_\alpha = 0, \quad N + 1 < \alpha < (\nu - 1)(N + 1). \tag{9}
\]

Second and third assertions of the theorem follow immediately from (3) and (4) respectively.
The first assertion is clear if \( \nu \neq \mu \). Let \( \nu = \mu \); (3) leads to the sequence of equations:

\[
\{ h_\alpha^\nu, h_\beta^\nu \}_{-1} = \{ h_\alpha^\nu - N - 2, h_\beta^\nu + N + 2 \}_{-1} = \cdots = \{ h_\alpha^\nu - m(N + 2), h_\beta^\nu + m(N + 2) \}_{-1},
\]

where \( m \) is a natural number. For every pair of non-negative integer numbers \( \alpha \) and \( \beta \) that are less then \((\nu - 1)(N + 1)\) there exists a number \( m \) such that one of the following inequalities holds: \( \alpha - m(N + 2) < 0 \), or \( \beta + m(N + 2) \geq (\nu - 1)(N + 1) \). The first one implies that \( h_\alpha^\nu - m(N + 2) \equiv 0 \), and the second that \( h_\beta^\nu + m(N + 2) \) annihilates the Poisson structure \( W(-1) \). In both of the cases the vanishing of the bracket \( \{ h_\alpha^\nu, h_\beta^\nu \}_{-1} \) is provided. The involutivity of the functions \( h_\alpha^\nu \) and \( h_\nu^\nu \) with respect to \( \{ , \}_{N + 1} \) can be proved in the same way.

\( \square \)

**Remark 1.** This proof is a realization of the "Adler scheme" \( \{8,9\} \) and carried out in a gradation invariant way. A.G. Reyman and M.A. Semenov-Tian-Shansky proved an analogue of Theorem 1 for homogeneous gradation case applying the \( R \)-matrix technique \( \{10\} \).

By Theorem 1, the generic orbit \( \mathcal{O}_{-}^{gen} \) is the real algebraic manifold embedded into \( M^{N + 1} \) by the constraints \( h_\alpha^\nu = C_\alpha^\nu, \alpha \geq (\nu - 1)(N + 1) \). Next, fixation of the functions \( h_\alpha^\nu, \alpha = 0, 1, \ldots, N + 1 \), determines the real algebraic manifold being the generic orbit \( \mathcal{O}_{+}^{gen} \). Set the hamiltonian flows of the form

\[
\frac{d\mu}{d\tau_\alpha} = \{ \mu, h_\alpha^\nu \}_\sigma = ad_{\mu_\alpha}^\nu \mu = [\mu, dh_\alpha^\nu] ,
\]

on \( \mathcal{O}_{-}^{gen} \) (resp. \( \mathcal{O}_{+}^{gen} \)), where \( \alpha < (\nu - 1)(N + 1) \) (resp. \( \alpha > N + 1 \)) and \( dh_{\alpha}^\nu \) is the differential of the hamiltonian \( h_{\alpha}^\nu \) (\( \tau_{\alpha} \) is the corresponding trajectory parameter).

**Remark 2.** Consider the Poisson structure \( W(-1) \). By (10), where \( \sigma = -1 \), the coordinates \( \mu_\alpha^{N + 1} \) do not change under the action of any hamiltonian \( h_\alpha^\nu \). The constraints \( \mu_k^{N + 1} = \text{const} \) determine, first of all, the embedding \( M^N \hookrightarrow M^{N + 1} \) and, second, the initial point of the generic orbit \( \mathcal{O}_{-}^{gen} \). Thus, the symplectic structure \( W(-1) \) is meant as restricted to \( M^N \). Considering the coadjoint action of \( \mathfrak{g}_-^\nu / \lambda^{-N - 1} \mathfrak{g}_-^\nu \), the functions \( h_{\nu(N + 1)}^\nu \) are fixed constants on \( M^N \) and hence must be neglected. Similarly, \( h_0^\nu \) are constants on \( M^{N + 1} \setminus \text{span}_{\mathcal{R}}{\mu_k^0} \) (with \( \setminus \) being set minus) if the action of \( \mathfrak{g}_+^\nu / \lambda^{N + 2} \mathfrak{g}_+^\nu \) is considered with respect to \( W(N + 1) \).

**Lemma 1** Let \( \mathfrak{g} \simeq sl_n(\mathcal{R}) \).

1. The dimensions of the generic orbits \( \mathcal{O}_{-}^{gen} \) and \( \mathcal{O}_{+}^{gen} \) are equal to

\[
\dim \mathcal{O}_{-}^{gen} = \dim \mathcal{O}_{+}^{gen} = (N + 1)(n - 1)n.
\]

2. The number of the non-annihilators on the generic orbits is equal to

\[
\#(\text{Ham}) = \frac{(N + 1)(n - 1)n}{2}.
\]

**Proof.** By Theorem 1 and keeping in mind Remark 2, the assertions follow from the straightforward calculations. The only point to check is the functional independence of the \( W(-1) \)–annihilators \( h_\alpha^\nu, (\nu - 1)(N + 1) \leq \alpha < \nu(N + 1), \) in the space \( M^N \), and the \( W(N + 1) \)–annihilators \( h_\alpha^\nu, 0 < \alpha \leq N + 1, \) in the space \( M^{N + 1} \). These facts are proved in the next proposition. \( \square \)
**Proposition 1** 1. Let not all $\mu_{i}^{N+1}$ be equal to zero.

(a) The functions \( h_{\nu}^{\alpha}, \alpha < (\nu-1)(N+1) \), are functionally independent almost everywhere on $M^{N}$.

(b) The functions \( h_{\nu}^{\alpha}, 0 \leq \alpha < (\nu-1)(N+1) \), are functionally independent almost everywhere on $O_{\text{gen}}^{-}$.

2. Let not all $\mu_{i}^{0}$ be equal to zero.

(a) The functions \( h_{\nu}^{\alpha}, 0 \leq \alpha < N+1 \), are functionally independent on $M_{N}^{-}$.

(b) The functions \( h_{\nu}^{\alpha}, \alpha > N+1 \), are functionally independent on $O_{\text{gen}}^{+}$.

**Proof.** We prove only two first statements, the second two are proved quite similarly. If $\nu \neq \mu$, the statements are obvious. Consider the matrix with the rows formed by the partial derivatives of the functions \( h_{\nu}^{\alpha}, (\nu-1)(N+1) \leq \alpha < \nu(N+1) \), with respect to the coordinates on $M^{N}$. It is quasitriangular. One can construct the minor of order $N+1$ from the entries of the matrix. That minor is not equivalent to zero in all the points of $M_{N}^{S}$ except \( \mu_{i}^{N} = 0 \). So the statement 1.a is proved. Next, consider the similar matrix of the derivatives but for $0 \leq \alpha < \nu(N+1)$ and where the derivatives are meant to be with respect to the coordinates on the orbit $O_{\text{gen}}^{-}$. It is a $\nu(N+1) \times (N+1)(n-1)n$ - matrix. There exist not less than $(\nu-1)(N+2)$ different minors of order $\nu(N+1)$ constituted by the entries of that matrix. Construct such minors for every $\nu$. Equating the minors obtained to zero one gets the system of the algebraic equations determining the set of singular points $M_{S}^{N} \subset M^{N}$. Its dimension is less than or equal to $\sum_{\nu=2}(\nu-1)(N+2) = (n-1)(N^{2}+2) + 1$. So $\dim M_{S}^{N} < \dim O_{\text{gen}}^{-}$, which proves the functional independence of the functions \( h_{\nu}^{\alpha}, 0 \leq \alpha < \nu(N+1) \) almost everywhere on $O_{\text{gen}}^{-}$. Now the statement 1.b becomes obvious. □

**Corollary 1** Hamiltonian flows (10) are integrable in the Liouville sense.

**Proof.** By Liouville theorem on the complete integrability [13], the assertion follows from Theorem 1, Lemma 1 and Proposition 1. □

Given hamiltonian $h_{\alpha}^{\nu}$, a Legendre-type transformation \( h_{\alpha}^{\nu} \mapsto L(h_{\alpha}^{\nu}) \) can be defined, and the corresponding hamiltonian system

\[
\frac{d\mu}{d\tau_{\alpha}} = \{\mu, h_{\alpha}^{\nu}\}_{\sigma}
\]

on the orbit admits the form of the Euler - Lagrange equation:

\[
\frac{\delta L(h_{\alpha}^{\nu})}{\delta \mu} = 0,
\]

where $L(h_{\alpha}^{\nu})$ is the Lagrange function associated with $h_{\alpha}^{\nu}$. The crucial point is that the hamiltonian flows generated by non-annihilators $h_{\alpha}^{\nu}$ are invariantly embedded one into another. That means, for example, that the hamiltonian flow generated by $h_{\alpha}^{\nu}$ on the orbit $O_{\text{gen}}^{+}$ can be written as (11), where

\[
L(h_{\alpha}^{\nu}) = \int \left\{ c_{0}L(h_{0}^{\nu}) + c_{1}L(h_{1}^{\nu}) + \ldots + c_{\alpha-1}L(h_{\alpha-1}^{\nu}) + c_{\alpha}L(h_{\alpha}^{\nu}) \right\} d\tau_{\alpha}^{\nu},
\]
with the constants $c_i$ being linear combinations of the annihilators determining the orbit. This fact enables us to identify the Euler - Arnold equation of the form (10) with the higher stationary equation for some evolutionary integrable system. The hierarchy of the higher stationary equations arise since the number $N$ can be chosen as large as necessary but finite. The lagrangian densities $\mathcal{L}(\cdot)$ appear as the densities of integrals of motion for the hierarchy of evolutionary equations, and are constructed purely algebraically without using the associated linear problem.

An additional ("time") evolution is realized as the action of any hamiltonian flow generated by non-annihilators $h^\mu_\beta \neq h^\nu_\alpha$ on stationary trajectory points of the hamiltonian system (10). As the result the system of equations on the orbit arises:

$$\frac{\partial \mu}{\partial \tau^\nu_\alpha} = \{\mu, h^\nu_\alpha\}_\sigma, \quad \frac{\partial \mu}{\partial \tau^\mu_\beta} = \{\mu, h^\mu_\beta\}_\sigma. \quad (12)$$

Its compatibility condition has the zero-curvature representation form for the restriction of an evolutionary equation onto the orbit:

$$\frac{\partial h^\nu_\alpha}{\partial \tau^\beta_\mu} - \frac{\partial h^\beta_\mu}{\partial \tau^\alpha_\nu} + [dh^\mu_\beta, dh^\nu_\alpha] = 0.$$

2. The $sl_n(\mathbb{R})$-case. Let $\mathfrak{g} \simeq sl_n(\mathbb{R})$. Then $R = n - 1$. Let $E_{ij}$ denote $n \times n$-matrix with the unit at the $(ij)$-entry and zeros elsewhere. Fix the dual basis of $sl_n(\mathbb{R})$:

$$\{Q^*_i\}_{1}^{\dim \mathfrak{g}} \equiv \left\{ H^*_1, \ldots, H^*_R, X^*_1, \ldots, X^*_R, Y^*_1, \ldots, Y^*_R \right\},$$

where

$$H^*_i = E_{ii} - E_{i+1,i+1};$$
$$X^*_1 = E_{12}, X^*_2 = E_{23}, \ldots, X^*_R = E_{R,R+1}, X^*_R = E_{13}, X^*_2 = E_{24}, \ldots, X^*_2R = E_{R-1,R+1}, \quad \ldots \ldots .$$
$$X^*_R = E_{1,R+1}; \quad Y^*_i = X^*_iT.$$

The homogeneous and principal gradation operators are given by

$$d_h = \lambda \frac{d}{d\lambda};$$
$$d_p = n\lambda \frac{d}{d\lambda} + ad_{\text{diag}}(\frac{1}{n+1}, \ldots, \frac{-1}{n+1});$$

respectively. The elements of the dual basis of $\widetilde{sl}_3(\mathbb{R})$ have the following grades with respect to $d_h$ and $d_p$:

$$d_h \left( \lambda^k \cdot Q^*_i \right) = k \cdot \left( \lambda^k \cdot Q^*_i \right), \quad i = 1, 2, \ldots, \dim \mathfrak{g};$$
$$d_p \left( \lambda^k \cdot H^*_i \right) = nk \cdot \left( \lambda^k \cdot H^*_i \right), \quad i = 1, 2, \ldots, R;$$
$$d_p \left( \lambda^k \cdot X^*_i \right) = (nk + 1) \cdot \left( \lambda^k \cdot X^*_i \right), \quad i = 1, 2, \ldots, R;$$
\[ d_p \left( \lambda^k \cdot X_i^p \right) = (nk + 2) \cdot \left( \lambda^k \cdot X_i^p \right), \quad i = R + 1, R + 2, \ldots, 2R - 1; \]
\[ d_p \left( \lambda^k \cdot X_{R(R+1)}^p \right) = (nk + R) \cdot \left( \lambda^k \cdot X_{R(R+1)}^p \right); \]
\[ d_p \left( \lambda^k \cdot Y_i^p \right) = (nk - 1) \cdot \left( \lambda^k \cdot Y_i^p \right), \quad i = 1, 2, \ldots, R; \]
\[ d_p \left( \lambda^k \cdot Y_{R(R+1)}^p \right) = (nk - R) \cdot \left( \lambda^k \cdot Y_{R(R+1)}^p \right). \]

The Casimir functions in the enveloping algebra of \( sl_n(\Re)e \) are \( H^\nu = \frac{1}{\nu} \text{tr} A^\nu, \nu = 2, 3, \ldots, n, \) where \( A \) belongs to the dual of \( sl_n(\Re)e \).

3 The higher stationary equations on orbits of the principally and homogeneously realized loop algebra \( sl_3(\Re)e \)

1. The principal realization. An element \( \mu \in M^{N+1} \) is given by
\[
\mu = \sum_{l=0}^{N+1} \left\{ \frac{2}{3} \left( \alpha_1^{3l} H_i^* \lambda^l + \beta_1^{3l+1} X_i^* \lambda^l + \gamma_1^{3l-1} Y_i^* \lambda^l \right) + \beta_3^{3l+2} X_3^* \lambda^l + \gamma_3^{3l-2} Y_3^* \lambda^l \right\},
\]
where the lower coordinate indices relate to the dual basis elements of \( sl_n(\Re)e \), and the upper indices denote the grades of the corresponding dual basis elements of \( sl_3(\Re)e \). Consider the Poisson manifold \( (M^{N+1}, W(-1)) \).

It follows from the theory of finite-dimensional Lie algebras that an initial point of a coad- joint representation orbit is determined uniquely by the point in the Weyl chamber in the span of the Cartan subalgebra (cf. [14]). By the analogy with that, we determine an initial point of the generic orbits fixing a point in the Weyl chamber for \( sl_3(\Re)e \) that is the coefficient at \( \lambda^{N+1} \). In the case of principally realized algebra \( sl_3(\Re)e \), the Weyl chamber of \( sl_3(\Re)e \) is (cf. [2]) \( \text{span}_\Re \{ \delta_1 \mathcal{W}_1 + \delta_2 \mathcal{W}_2 \} \), \( \delta_1, \delta_2 \in \Re^+, \) where \( \mathcal{W}_1 = E_{12} + E_{23} + E_{31}, \mathcal{W}_2 = E_{13} + E_{21} + E_{32} \). Let us choose the initial point on the boundary of the chamber: \( \delta_1 = 0, \delta_2 = 1 \). That means in coordinate terms: \( \beta_1^{3(N+1)+1} = \beta_2^{3(N+1)+1} = \beta_3^{3(N+1)+1} = 0, \gamma_1^{3(N+1)-1} = \gamma_2^{3(N+1)-1} = \gamma_3^{3(N+1)+2} = 1 \).

The embedding \( M^N \hookrightarrow M^{N+1} \) is completed putting \( \alpha_1^{3(N+1)} = \alpha_2^{3(N+1)} = 0 \).

Put \( N = 0 \). By Theorem 1, the generic orbits are fixed by
\[
\begin{align*}
    h_1^2 &= \beta_1^2 \gamma_1^2 + \beta_2^2 \gamma_2^2 + \beta_3^2 \gamma_3^2 = C_1^2, \\
    h_2^3 &= \beta_3^2 \gamma_1^2 + \beta_2^2 \gamma_2^2 + \beta_1^2 \gamma_3^2 = C_2^3, \\
\end{align*}
\]
where \( C_1^2 \) and \( C_2^3 \) are constants. The explicit forms of the \( W(-1) \)-annihilators \( h_1^2 \) and \( h_2^3 \) provide the trivial topological structure of the real algebraic manifold \( O_{\text{gen}} \simeq \Re^6 \). Consider the additional degeneration of the orbit determined by \( \gamma_3^2 = 0 \). The functions
\[
\begin{align*}
    h_0^2 &= (\alpha_1^0)^2 + (\alpha_2^0)^2 - \alpha_1^0 \alpha_2^0 + \beta_1^0 \gamma_1^{-1} + \beta_2^0 \gamma_2^{-1}, \\
    h_0^3 &= \alpha_2^0 \beta_1 \gamma_1^{-1} - \alpha_1^0 \beta_2 \gamma_1^{-1} - \alpha_1^0 (\alpha_2^0)^2 + \alpha_2^0 (\alpha_1^0)^2 + \beta_2^0 \gamma_1^{-1} \gamma_2^{-1}, \\
    h_1^3 &= \alpha_0^0 \beta_1 \gamma_1^{-1} - \alpha_0^0 \beta_2 \gamma_2^{-1} + \beta_2^0 \gamma_1^{-1} \gamma_2^{-1} + \beta_3^0 \gamma_1^{-1} \gamma_2^{-1}, \\
\end{align*}
\]
create integrable hamiltonian flows on the degenerated orbit. Fix the $W(1)$—annihilators: $C_1 = 0, C_2 = 1$, and consider the hamiltonian system
\[
\frac{d\mu}{d\tau_0} = \{\mu, h_0^3\}_{-1} = ad_{h_0^3}\mu
\] (13)

**Proposition 2** The equation (13) admits the restriction onto the immovable points of the following involution of the dual of $sl_3(\mathbb{R})$: $H_1^* \mapsto H_2^*, H_2^* \mapsto H_1^*, X_1^* \mapsto -X_2^*, X_2^* \mapsto -X_1^*, Y_1^* \mapsto -Y_2^*, Y_2^* \mapsto -Y_1^*, X_3^* \mapsto X_3^*, Y_3^* \mapsto Y_3^*$.

**Proof.** The set of immovable points is determined by the constraints $\alpha_0^1 = -\alpha_2^1, \gamma_1^1 = -\gamma_2^1, \beta_1^1 = -\beta_2^1$. This and the explicit form of (13) make the assertion obvious. □

Denote $\alpha_0^1 = -\alpha_2^1 \equiv \alpha, \gamma_1^1 = -\gamma_2^1 \equiv \gamma, \beta_1^1 = -\beta_2^1 \equiv \beta = const$ (the latter is by (13)). Then (13) reduces to the system of two ordinary differential equations:
\[
\begin{align*}
\alpha' + \gamma + \beta \alpha &= 0 \\
-\gamma' + 3\alpha^2 + \beta \gamma &= 0
\end{align*}
\] (14)

where $(\cdot)' \equiv \frac{d(\cdot)}{d\tau_0}$.

**Proposition 3** The hamiltonian system (14) admits the Euler - Lagrange form:
\[
\begin{align*}
\frac{\delta L}{\delta \gamma} &= 0 \\
\frac{\delta L}{\delta \alpha} &= 0 \\
L &= \int (L_2 + \beta L_1)d\tau_0^3
\end{align*}
\]

where the lagrangian densities
\[
L_1 = \alpha \gamma, \quad L_2 = \gamma \alpha' + \alpha^3 + \frac{1}{2} \gamma^2.
\]

**Proof.** Straightforward verification. □

The lagrangian densities $L_1$ and $L_2$ coincide (up to total derivatives) with the corresponding densities of integrals of motion for the Boussinesq’s equation (cf. [15]). Hence, the Euler - Arnold equation (13) on the degenerated orbit is interpreted as the stationary Boussinesq’s equation. Furthermore, the lagrangian density $L_2$ has the form of the Legendre - type transformation
\[
h_0^3 \mapsto \mathcal{L}(h_0^3) \equiv \mathcal{L}_2 = \frac{1}{2} \left(\gamma \frac{\partial h_0^3}{\partial \gamma} - h_0^3\right).
\]

The hierarchy of higher stationary Boussinesq’s equations appears when the subspaces $M^{N+1}, N = 0, 1, 2, \ldots < \infty,$ are involved. The densities of higher integrals of motion are calculated in the same way as for $N = 0$.

Remind the scalar Lax representation for the Boussinesq’s equation (cf., for example, [16]):
\[
\frac{dL}{dt} = \left[ L, (L^2)_+\right], \quad L = \frac{\partial^3}{\partial x^3} + u \frac{\partial}{\partial x} + v,
\]

9
with the explicit form
\[
\begin{align*}
\frac{\partial u}{\partial t} &= -\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial v}{\partial x}, \\
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} - 2 \frac{\partial^2 u}{\partial x^3} - 2 \frac{\partial u}{\partial x}.
\end{align*}
\]

We are going to connect the standard unknown functions \(u\) and \(v\) with the variables on the orbit. To do that consider the coadjoint action of the subalgebra \(\tilde{g}_+\) on the Poisson manifold \((M^1, W(0))\). One can easily check that the Hamiltonian flow
\[
\frac{d\mu}{d\tau_1^3} = \{\mu, h_1^3\}_0 = ad_{dh_1^3} \mu = \left[\mu, dh_1^3\right],
\]
coincides with (14) on the degenerate orbit. Since both of the \(\mu\) and the \(dh_1^3\) belong to \(\tilde{g}_+\) (i.e. contain only non-negative degrees of the parameter \(\lambda\)), the equation (15) is the stationary zero-curvature representation in a usual form, where \(\tau_1^3 = x\).

**Proposition 4** There exists the unique matrix
\[
L^{\text{can}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda - v & -u & 0 \end{pmatrix},
\]
and the lower triangular matrix \(S = S(\tau_1^3)\) with units at the diagonal entries, such that the following gauge transformation takes place:
\[
S \left( \frac{d\mu}{d\tau_1^3} + L^{\text{can}} \right) S^{-1} = \frac{d\mu}{d\tau_1^3} + dh_1^3.
\]

**Proof.** The statement follows immediately from Theorem 3.1 in [16] once the explicit form of the matrix \(dh_1^3\) is written down. \(\square\)

By this proposition, the quite simple expressions arise:
\[
u = 2\alpha, \quad v = \gamma,
\]
describing the restriction of \(u\) and \(v\) (up to a possible rescaling) from the infinite-dimensional phase space onto the trajectories of the equation (15) on the degenerate orbit.

Let us construct the equation to be interpreted as the restriction of the evolutionary Boussinesq’s equation onto the orbit. Consider the action of the subalgebra \(\tilde{g}_+\) on the Poisson manifold \((M^2, W(1))\). The Hamiltonian flow generated by the coefficient function \(h_2^3\) describes the stationary Boussinesq’s equation. Set the action of the Hamiltonian flow generated by \(h_2^3\) on the trajectories of the Hamiltonian \(h_2^3\). Then the system of partial differential equations
\[
\begin{align*}
\frac{\partial \mu}{\partial \tau_2^3} &= ad_{dh_2^3} \mu, \\
\frac{\partial \mu}{\partial \tau_3^3} &= ad_{dh_3^3} \mu,
\end{align*}
\]
holds. Its compatibility condition has the zero-curvature representation form for the Boussinesq’s equation:
\[ \frac{\partial dh_2^3}{\partial \tau_2^3} - \frac{\partial dh_3^3}{\partial \tau_2^3} + \left[ dh_3^3, dh_2^3 \right] = 0. \]

2. The homogeneous realization. In such a case an element \( \mu \in M^{N+1} \) reads
\[ \mu = \sum_{l=0}^{N+1} \left\{ \sum_{i=1}^{2} \alpha_i^l H_i^* \lambda^l + \sum_{i=1}^{3} \left( \beta_i^l X_i^* \lambda^l + \gamma_i^l Y_i^* \lambda^l \right) \right\}. \]

The homogeneous realization of \( sl_3(\mathbb{R}) \) corresponds to the Weyl chamber for \( sl_3(\mathbb{R}) \) defined by (cf. [3])
\[ \text{span}_{\mathbb{R}} \left\{ \kappa_1 \text{diag} \left( \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right) + \kappa_2 \text{diag} \left( \frac{1}{3}, \frac{1}{3}, -\frac{2}{3} \right) \right\}, \quad \kappa_1, \kappa_2 \in \mathbb{R}_+. \]

Set the initial point on the boundary of the chamber putting \( \kappa_1 = 0, \kappa_2 = 1 \) or, in coordinate terms, \( 2\alpha_1^{N+1} - \alpha_2^{N+1} = 0, 2\alpha_2^{N+1} - \alpha_1^{N+1} = 1 \). The constraints \( \beta_1^{N+1} = \beta_2^{N+1} = \beta_3^{N+1} = \gamma_1^{N+1} = \gamma_2^{N+1} = \gamma_3^{N+1} = 0 \), determine the embedding \( M^N \hookrightarrow M^{N+1} \). Consider the Poisson manifold \( (M^{N+1}, \mathcal{W}(-1)) \). The generic orbits are fixed by Theorem 1.

Put \( N = 2 \). The generic orbit \( \mathcal{O}_{\text{gen}} \) is the real algebraic manifold determined by the following system of conics and cubics in \( \mathbb{R}^{24} \):
\[ h_3^2 = C_3^2, \quad h_2^2 = C_4^2, \quad h_5^2 = C_5^2, \quad h_6^3 = C_6^3, \quad h_7^3 = C_7^3, \quad h_8^3 = C_8^3. \]

These algebraic equations can be solved with respect to the variables \( \alpha_1^l, \alpha_2^l, \ l = 0, 1, 2 \). Furthermore, for every fixed \( l \), the equations for finding \( \alpha_1^l \) and \( \alpha_2^l \) are linear if the variables \( \alpha_1^{l+k} \) and \( \alpha_2^{l+k} \), \( k = 1, 2 \) are found before. That implies the diffeomorphism \( \mathcal{O}_{\text{gen}} \cong \mathbb{R}^{18} \), and the variables \( \beta_1^{l}, \gamma_1^{l}, \ i = 1, 2, 3, \ l = 0, 1, 2 \), are the global coordinates on the orbit.

Consider the integrable hamiltonian system on the orbit:
\[ \frac{d\mu}{d\tau_1} = \{ \mu, h_2^2 \}_{-1}. \]

Let \( \theta : A \mapsto (-A)^T, \ \forall A \in sl_3(\mathbb{R}) \), be the Cartan involution of \( sl_3(\mathbb{R}) \). Define the ”extended” Cartan involution of the loop algebra: \( \hat{\theta} : \lambda^k A_k \mapsto (-1)^{k+1} \lambda^k A_k^T \).

Proposition 5 The system (16) admits the restriction onto the set of immovable points of the involution \( \hat{\theta} \).

Proof. The assertion is proved as that of Proposition 2. \( \square \)

The Euler - Arnold equation (16) on such the degenerate orbit takes the form:
\[ \left\{ \begin{aligned}
\beta_2'' + 6(\beta_2')^2 \beta_2' + 4\beta_2' \beta_3^2 \beta_3' + 2(\beta_3')^2 \beta_3' &= 0 \\
\beta_3'' + 6(\beta_3')^2 \beta_3' + 4\beta_3' \beta_2^2 \beta_2' + 2(\beta_2')^2 \beta_2' &= 0,
\end{aligned} \right. \]

where \( \cdot' \equiv \frac{d}{d\tau_1} \). This system can be interpreted as a stationary two-component mKDV - type equation.
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