DIFFERENT TYPES OF TOPOLOGICAL COMPLEXITY ON HIGHER HOMOTOPIC DISTANCE

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Abstract. We first study the higher version of the relative topological complexity by using the homotopic distance. We also introduced the generalized version of the relative topological complexity of a topological pair on both the Schwarz genus and the homotopic distance. With these concepts, we give some inequalities including the topological complexity and the Lusternik-Schnirelmann category, the most important parts of the study of robot motion planning in topology. Finally, by defining the parametrised topological complexity via the homotopic distance, we present some estimates on the higher setting of this concept.

1. Introduction

Studies on determining the topological complexity TC first start with M. Farber [11]. As Farber [13] states, one of the basic methods followed in topological complexity studies is to study the different versions of the number TC and obtain a relationship between these versions and TC. This generally leads to some natural bounds for TC. For instance, if $X$ is a path-connected topological space and $Y$ is a subset of $X \times X$, then the relative topological complexity $TC_X(Y)$ satisfies the following inequality [13]:

$$TC_X(Y) \leq TC(X).$$

Along with the definition of higher topological complexity $TC_n$ by Y. Rudyak [24], introducing the improved version of relative topological complexity also presents a natural lower bound for $TC_n$. In Section 3 we first focus on these higher settings of the relative topological complexity $TC_{n,X}(Y)$ via relative homotopic distance, where $Y$ is a subset of $X^n$.

Let $(A, B)$ be a pair of topological spaces with the condition $B \subseteq A$. Then the relative topological complexity of a pair $TC(A, B)$ is defined by R. Short [26] on the notion of Schwarz genus and admits the following fact:

$$TC(A, B) = TC_A(A \times B).$$

In other words, if one rewrites $TC(A, B)$ as $TC_{n,A}(B)$ (for $n = 2$), then one can define the relative higher topological complexity of a pair $TC_n(A, B)$ by using the relative homotopic distance such as the relative higher topological complexity of a space. Before such a definition on the homotopic distance, in Section 4 we first introduce $TC_n(A, B)$ in terms of the Schwarz genus.

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In recent years, topological robotics studies have focused on computing the new fiber homotopy equivalent invariant parametrised topological complexity [6, 7, 14, 27]. For the Farber’s topological complexity $TC$, one considers the motion planning problem as follows [11].

$$\pi : PX \to X \times X, \quad \pi(\gamma) = (\gamma(0), \gamma(1)),$$

is a fibration, where $PX$ contains all continuous paths on the path-connected topological space $X$. $PX$ has a compact-open topology. $TC(X)$ is the Schwarz genus of $\pi$ and $TC(X) = n$ means that the motion planning algorithm $s_j : A_j \subset X \times X \to PX$ must be continuous with the condition $\pi \circ s_j = 1_{A_j}$ for each $j \in \{1, \ldots, n\}$.

In the parametrised version, there are some basic changes, but the main idea is familiar. This is to take the endpoints of the space and form a continuous motion between these endpoints. One has some extra conditions. The first is that the initial and final states are located in the same fiber of the fibration $q : E \to B$. The other one is to restrict the motion planning algorithm to the same fiber. We shall explain this idea mathematically. Let $q : E \to B$ be a fibration with a nonempty, path-connected fibre $X = q^{-1}(b)$ for any $b \in B$. Let $E_B^I$ denote the subspace of $E$ that consists of all paths $\alpha$ in $E$ such that $q \circ \alpha$ is the constant map. $E \times_B E$ is a subset of $E \times E$ that includes the point $(e_1, e_2)$ for which $q(e_1)$ equals $q(e_2)$, i.e., the pair $(e_1, e_2)$ is located in the same fiber. Then we consider the fibration $\pi' : E_B^I \to E \times_B E$ with $\pi'(\alpha) = (\alpha(0), \alpha(1))$. Note that the path-connected fiber of this fibration is $\Omega X$. The parametrised topological complexity $TC[q : E \to B]$ is the Schwarz genus of $\pi'$ [7].

In the sense of robot motion planning problems, the parametrised topological complexity gives an extra meaning to the base space $B$ of the fibration $q : E \to B$. The external conditions in the system can be parameterized by means of the space $B$. With the algebraic topology (especially homotopy) tools, the parametrised topological complexity determines the degree of navigational complexity as a positive number for the system when the initial and final states have the same external conditions. We first rewrite the definition of the parametrised topological complexity with respect to the homotopic distance and then we update this definition for $n > 1$ to exhibit the parametrised higher topological complexity $TC_n[q : E \to B]$ in Section 5.

Topological complexity number varies such as relative, symmetric, monoidal, parametric etc., and accordingly, describing the higher version of all these numbers is a requirement for understanding the general concept of robot motion planning algorithms. For instance, see [15] and [4] for the simplicial complexity. Davis investigates the symmetric complexity of a circle [9] and the geodesic complexity of Klein bottles with the dimension $n$ [10], respectively. In this study, we investigate the general setting of certain topological complexities. First, we recall some definitions and facts about $TC$ and the related invariants such as cat (Lusternik-Schnirelmann category, see for a detail information [8]), Schwarz genus, and the homotopic distance. Then we give the homotopic distance definitions of relative higher topological complexity of a space, relative higher topological complexity of a pair, and the parametrised higher topological complexity of a fibration in the respective sections. We also mention on some equalities and inequalities including
TC_n, cat, and TC of a fibration with considering particular homotopy facts. In addition, we obtain lower and upper bounds for Hopf fibration and examine the Stiefel and Grasmann manifolds in the sense of parametrised topological complexity.

2. Preliminaries

This section is dedicated to providing brief information about the different types of topological complexities and their related invariants. Note that we frequently use these facts in the following sections.

2.1. Schwarz Genus and Homotopic Distance.

Definition 2.1. [25] Let \( q : E \to B \) be a fibration. Assume that \( B \) has an open cover \( \{B_1, B_2, \cdots, B_r\} \) for the minimum possible positive integer \( r \) such that there is a continuous map \( s_j : B_j \to E \) satisfying that \( q \circ s_j = 1_{B_j} \) for each \( j \in \{1, 2, \cdots, r\} \). Then the Schwarz genus of \( q \) is \( r \).

The Schwarz genus of \( q \) is generally denoted by \( \text{genus}(q) \) or \( \text{secat}(q) \).

Definition 2.2. [5, 21] Let \( f_j : A \to B \) be a continuous map for each \( j \in \{1, 2, \cdots, m\} \). Assume that \( A \) has an open cover \( \{A_1, A_2, \cdots, A_r\} \) for the minimum possible positive integer \( r \) such that the condition \( f_{i_1}|_{A_1} \simeq f_{i_2}|_{A_1} \simeq \cdots \simeq f_{i_m}|_{A_1} \) holds for all \( i \in \{1, \cdots, r\} \). Then the higher homotopic distance of degree \( m \) (or the \( m \)-th homotopic distance) is \( r \).

The higher homotopic distance is denoted by \( D(f_1, f_2, \cdots, f_m) \) and is simply called the higher homotopic distance. For \( m = 2 \), the notion is particularly called the homotopic distance [21].

Theorem 2.3. [21] Let \( f_1, f_2 : A \to B \) be two maps and \( \pi : PB \to B \times B \). Assume that \( p : P \to A \) is the pullback of \( \pi \) by \((f_1, f_2) : A \to B \times B \), where

\[
P = \{(a, \beta) \in A \times PB : \beta(0) = f_1(a) \text{ and } \beta(1) = f_2(a)\},
\]

i.e., the following diagram is commutative:

\[
\begin{array}{ccc}
P & \xrightarrow{pr_2} & PB \\
\downarrow{p} & & \downarrow{\pi} \\
A & \xrightarrow{(f_1, f_2)} & B \times B.
\end{array}
\]

Then \( D(f_1, f_2) = \text{genus}(p) \).

We have some useful notes on the (higher) homotopic distance for the forthcoming sections as follows [5, 21]:

Proposition 2.4. [5, 21] Each of the following satisfies:

a) If \( f_j : A \to B \) is homotopic to \( g_j : A \to B \) for each \( j \in \{1, \cdots, n\} \), then \( D(f_1, \cdots, f_n) = D(g_1, \cdots, g_n) \).
b) Let $1 < k < l$ and $f_1, \cdots, f_k, \cdots, f_l : A \to B$ be maps. Then we have that $D(f_1, \cdots, f_k) \leq D(f_1, \cdots, f_l)$.

c) Let $f_j : A \to B$ and $p_j : B \to C$ be maps with $p_{j-1} \simeq p_j$ for each $j \in \{2, \cdots, n\}$. Then $D(p_1 \circ f_1, \cdots, p_n \circ f_n) \leq D(f_1, \cdots, f_n)$.

d) Let $f_j : A \to B$ and $p_j : C \to A$ be maps with $p_{j-1} \simeq p_j$ for each $j \in \{2, \cdots, n\}$. Then $D(f_1 \circ p_1, \cdots, f_n \circ p_n) \leq D(f_1, \cdots, f_n)$.

e) The higher homotopic distance is a homotopy invariant.

f) Let $g, g' : C \to A$ and $f_1, f_2 : A \to B$ be maps. If $f_1 \circ g \simeq f_2 \circ g'$, then $D(f_1 \circ g, f_2 \circ g) \leq D(g, g')$.

2.2. Topological Complexity and LS-Category. Topological complexity, the main part of the studies of topological robotics, can be expressed in two ways. For simplicity, we use SG and HD as abbreviations for any definition in the sense of Schwarz genus and homotopic distance, respectively.

Definition 2.5. Let $X$ be a path-connected topological space.

- (SG) Let $\pi : PX \to X \times X$, $\pi(\alpha) = (\alpha(0), \alpha(1))$, be a path fibration. Then the topological complexity of $X$ is the genus($\pi$).
- (HD) Let $p_i : X^2 \to X$ be a projection map for each $i \in \{1, 2\}$. Then the topological complexity of $X$ is $D(p_1, p_2)$.

TC of a contractible space is 1 and the converse is also true, that is, if $TC(X) = 1$, then $X$ is a contractible space $[11]$.

Proposition 2.6. Let $X$ be a paracompact and locally contractible space with $\dim X = n$. Then $TC(X) \leq 2n + 1$.

An important result of Proposition 2.6 is that $2\text{cat}(X) - 1$ is an upper bound for $TC(X)$ when $X$ is paracompact. cat is also a natural lower bound $TC$. Thus, we conclude that $\text{cat}(X) \leq TC \leq 2\text{cat}(X) - 1$.

Theorem 2.7. The topological complexity of the complex Stiefel manifold $V_r(\mathbb{C}^k)$, denoted by $TC(V_r(\mathbb{R}^k))$, is less than or equal to $2r(k-r) + 1$.

The topological complexity is denoted by $TC$ and generalized for $n > 1$ as follows.

Definition 2.8. Let $X$ be a path-connected space.

- (SG) Let $e_n : P_n(X) \to X^n$, $e_n(\gamma) = (\gamma_1(1), \gamma_2(1), \cdots, \gamma_n(1))$, be a fibration, where $P_n(X)$ is the space of all multipaths in $X$ such that the initial point of all paths in each multipath is the same. Then the $n$-th topological complexity (simply called the higher topological complexity) of $X$ is the genus($e_n$).
- (HD) Let $p_i : X^n \to X$ be a projection map for each $i \in \{1, 2, \cdots, n\}$. Then the higher topological complexity of $X$ is $D(p_1, p_2, \cdots, p_n)$.
The higher topological complexity is denoted by $TC_n$ and it is useful for $n > 1$ because $TC_1(X) = 1$ for any space $X$. One of the important observations on $TC_n$ is the equality $TC_2(X) = TC(X)$. Another is the inequality $TC_n(X) \leq TC_{n+1}(X)$.

The Lusternik Schnirelmann category (simply called LS-category or denoted by cat) is another important homotopy invariant that inspired TC. It is also a natural bound for the topological complexity number.

**Definition 2.9.** [8][21] Let $X$ be a path-connected topological space

- (SG) Let $\eta : P_0X \to X$, $\eta(\gamma) = (\gamma(1))$, be a Serre Fibration, where $P_0(X)$ is the special case of $P_n(X)$, i.e., the space of all paths in $X$ for which the first point of these paths is a constant point $x'$ in $X$. Then the LS-category of $X$ is genus($\eta$).

- (HD) Let $x'$ be any point of $X$. For the continuous maps $i_1 : X \to X \times X$, $i_1(x) = (x, x')$ and $i_2 : X \to X \times X$, $i_2(x) = (x', x)$, the LS-category of $X$ is $D(i_1, i_2)$.

**Proposition 2.10.** [15] Let $A$ and $B$ be paracompact and path-connected spaces. Then $cat(A \times B) \leq cat(A) + cat(B)$.

Assume that $q : E \to B$ is a fibration with the fiber $X$. Then we have that $TC(E) \leq TC(X) + cat(B \times B) + 1$. [22] Another important observation is that the numbers TC and cat coincide when the topological space is a connected Lie group [12].

**Theorem 2.11.** [3] The LS-category of the complex Grassmannian $G_r(\mathbb{C}^k)$, denoted by $cat(G_r(\mathbb{C}^k))$, equals $rk$.

**Theorem 2.12.** [19] The LS-category of the quaternionic Grassmannian $G_r(\mathbb{H}^k)$, denoted by $cat(G_r(\mathbb{H}^k))$, equals $k(r - k)$.

It is possible to compute TC or $TC_n$ of a fibration as well as TC of a path-connected topological space [17][20][23].

**Definition 2.13.** [17][20][23] Let $q : E \to B$ be a surjective fibration.

- (SG) For the fibration $\pi'' : PE \to E \times B$ defined as $\pi''(\gamma) = (\gamma(0), q \circ \gamma(1))$, the topological complexity of $q$, denoted by $TC(q)$, is genus($\pi''$).

- (HD) Let $\pi_1 : E \times B \to E$ be the first projection map, and $\pi_2 : E \times B \to B$ the second projection map. Then the topological complexity of $q$ is $D(\pi_1, \pi_2)$.

**Proposition 2.14.** [23] Let $q_1 : E_1 \to B_1$ and $q_2 : E_2 \to B_2$ be two fibrations. Then

$$\max\{TC(q_1), TC(q_2)\} \leq TC(q_1 \times q_2) \leq TC(q_1) + TC(q_2) - 1.$$

**Proposition 2.15.** [23] Let $q : E \to B$ be a fibration. Then

$$cat(B) \leq TC(q) \leq \min\{TC(B), cat(E \times B)\}.$$
The general case of TC(q) is given in \[17\] with the following definition:

**Definition 2.16.** [17] Let \( q : (q_1, q_2, \cdots, q_n) : E \to B^n \) be a surjective fibration.

- (SG) For a fibration \( e^n_i : P_i(E) \to Y^n \), the \( n \)-th (higher) topological complexity of \( q \) is genus\( e^n_i \).

- (HD) For each projection map \( p_i : E^n \to E \) with \( i \in \{1, 2, \cdots, n\} \), the \( n \)-th (higher) topological complexity of \( q \) is \( D(q \circ p_1, q \circ p_2, \cdots, q \circ p_n) \).

**Theorem 2.17.** [17] If \( q_1 \simeq q_2 : E \to B^n \) are homotopic fibrations, then

\[
TC_n(q_1) = TC_n(q_2).
\]

**Theorem 2.18.** [2] Let \( X \) be a path-connected space. Then

\[
cat(X^{n-1}) \leq TC_n(X) \leq cat(X^n).
\]

2.3. Different Types of Topological Complexity. First, we recall the notions the relative (subspace) homotopic distance and the relative topological complexity. Later, we remind the definition of the parametrised topological complexity.

**Definition 2.19.** [20] Let \( f, g : A \to B \) be two continuous maps with the subset \( Y \subset A \). Then the relative homotopic distance (or subspace distance) on \( Y \) is the homotopic distance of two maps \( f|_Y \) and \( g|_Y \), informal saying,

\[
D_A(Y; f, g) = D(f|_Y, g|_Y).
\]

**Definition 2.20.** [13] [20] Let \( X \) be a path-connected topological space and \( Y \subset X \times X \).

- (SG) Let \( \pi^Y : P_Y X \to Y \), \( \pi^Y(\alpha) = (\alpha(0), \alpha(1)) \), be a path fibration, where \( P_Y X \) is a subset of \( PX \) and contains all the paths in \( X \) with the property that \( (\alpha(0), \alpha(1)) \) lies in \( Y \). Then the relative topological complexity of \( X \) with respect to the subspace \( Y \) is genus\( \pi^Y \).

- (HD) Let \( p_i : X^2 \to X \) be the projection map for each \( i \in \{1, 2\} \) and \( i_Y : Y \to X \times X \) be the inclusion map. Then the relative topological complexity of \( X \) with respect to the subspace \( Y \) is \( D_{X \times X}(Y; p_1, p_2) \).

The relative topological complexity of \( X \) with respect to the subspace \( Y \) is denoted by \( TC_Y(X) \).

**Definition 2.21.** [20] (SG) Let \( A \) be a path-connected topological space and \( B \subset A \). Let \( \pi^{A \times B} : P_{A \times B} \to A \times B \), \( \pi^{A \times B}(\gamma) = (\gamma(0), \gamma(1)) \), be a fibration, where \( P_{A \times B} \) is the space of all paths in \( A \) with the property that \( \gamma(0) \in A \) and \( \gamma(1) \in B \). Then the relative topological complexity of a pair \( (A, B) \) is genus\( \pi^{A \times B} \).

The relative topological complexity of a pair \( (A, B) \) is denoted by \( TC(A, B) \). We mention a similar result to the ordinary TC number:

**Proposition 2.22.** [20] Let \( (A, B) \) be the pair. Then

\[
TC(A, B) = 1 \iff A \text{ is contractible.}
\]
Proof. If \(q:E \to B\) be a fibration with nonempty, path-connected topological space \(X = q^{-1}(b)\) for any \(b \in B\). Let

\[
E_B^I = \{ \beta \in PE : q \circ \beta \text{ is constant} \} \subseteq PE
\]

and

\[
E \times_B E = \{ (a, b) \in E \times E : q(a) = q(b) \} \subseteq E^2.
\]

Then for the fibration \(\pi_B: E_B^I \to E \times_B E\) defined by \(\pi_B(\beta) = \beta(0), \beta(1)\), the parametrised topological complexity of \(q\), denoted by \(TC[q:E \to B]\), is genus(\(\pi_B\)).

**Proposition 3.2.** Let \(q_1: E_1 \to B_1\) and \(q_2: E_2 \to B_2\) be two fibrations with fibers \(X_1\) and \(X_2\), respectively, for metrisable spaces \(E_1, E_2, B_1,\) and \(B_2\). For the fibration \(q = q_1 \times q_2: E_1 \times E_2 \to B_1 \times B_2\) with the fiber \(X_1 \times X_2\), we have

\[
TC[q: E_1 \times E_2 \to B_1 \times B_2] \leq TC[q_1: E_1 \to B_1] + TC[q_2: E_2 \to B_2].
\]

3. Relative Higher Topological Complexity of a Space

**Definition 3.1.** (HD) Let \(X\) be a path-connected topological space and \(Y \subseteq X^n\) be a subspace. Then the relative higher topological complexity is defined as

\[
TC_{n,X}(Y) = D_{X^n}(Y;p_1,\cdots, p_n),
\]

where \(p_i : X^n \to X\) is a projection onto the \(i\)-th factor for each \(i = 1,\cdots, n\).

In particular, if \(Y\) is chosen as \(X^n\), then we find \(TC_{n,X}(X^n) = TC_n(X)\). Indeed,

\[
TC_{n,X}(X^n) = D_{X^n}(X^n;p_1,\cdots, p_n) = D(p_1,\cdots, p_n) = TC_n(X).
\]

**Proposition 3.2.** Let \(Y\) be a subspace of \(X^n\) and \(p_i : X^n \to X\) be the corresponding projection map for each \(i = 1,\cdots, n\). Then \(TC_{n,X}(Y) = 1\) if and only if the projections \((p_i)|_Y : Y \to X\) with \(i = 1,\cdots, n\) are homotopic to each other.

**Proof.** If \(Y\) is a subspace of \(X^n\), then we get

\[
TC_{n,X}(Y) = 1 \iff D_{X^n}(Y; p_1, \cdots, p_n) = 1 \iff (p_i)|_Y \simeq \cdots \simeq (p_n)|_Y.
\]

\(\Box\)

**Proposition 3.3.** Let \(Y\) be a subspace of \(X^n\). Then \(TC_{n,X}(Y) \leq TC_n(X)\).

**Proof.** If \(Y\) is a subspace of \(X^n\), then we have an inclusion map \(i_Y : Y \to X^n\). Therefore, by (d) part of Proposition 2.4 we observe that

\[
TC_{n,X}(Y) = D_{X^n}(Y;p_1,\cdots, p_n) = D(p_1 \circ i_Y,\cdots, p_n \circ i_Y) \leq D(p_1,\cdots, p_n) = TC_n(X).
\]

\(\Box\)

**Proposition 3.4.** Let \(Y \subset Z \subset X^n\). Then \(TC_{n,X}(Y) \leq TC_{n,X}(Z)\).
Proof. Let \( i_Y : Y \to X^n, i_Z : Z \to X^n \), and \( i : Y \to Z \) be three inclusion maps. Then \( i_Y \) can be rewritten as the composition of \( i \) and \( i_Z \) i.e., \( i_Y = i_Z \circ i \). From Proposition 2.4 (d), we have

\[
TC_{n,X}(Y) = D_{X^n}(Y; p_1, \ldots, p_n) = D(p_1 \circ i_Y, \ldots, p_n \circ i_Y) = D(p_1 \circ i_Z \circ i, \ldots, p_n \circ i_Z \circ i) \leq D(p_1 \circ i_Z, \ldots, p_n \circ i_Z) = TC_{n,X}(Z).
\]

Proposition 3.5. If \( Y \) is a retract of \( X \), then \( TC_{n,X}(Y^n) \geq TC_n(Y) \).

Proof. Let \( r : X \to Y \) be a retraction map. Let \( p_j : X^n \to X \) and \( q_j : Y^n \to Y \) be the projection maps for each \( j = 1, \ldots, n \). By (c) part of Proposition 2.4, we find

\[
TC_{n,X}(Y^n) = D_{X^n}(Y^n; p_1, \ldots, p_n) = D(p_1 \circ i_{Y^n}, \ldots, p_n \circ i_{Y^n}) \geq D(r \circ p_1 \circ i_{Y^n}, \ldots, r \circ p_n \circ i_{Y^n}) = D(q_1, \ldots, q_n) = TC_n(Y)
\]

with considering the fact that \( q_j = r \circ p_j \circ i_{Y^n} \) for each \( j \).

Corollary 3.6. If \( Y \) is a retract of \( X \) then, \( TC_n(Y) \leq TC_n(X) \).

Proof. Since \( Y \subset X \) is a retract, \( Y^n \subset X^n \). Then Proposition 3.5 says that \( TC_{n,X}(Y^n) \leq TC_{n,X}(X^n) = TC_n(X) \). Using Proposition 3.5, we conclude that \( TC_n(Y) \leq TC_n(X) \).

Proposition 3.7. Let \( \{Y_1, \cdots, Y_m\} \) be an open covering of \( X^n \). Then

\[
TC_{n,X}(Y_1) + \cdots + TC_{n,X}(Y_m) \leq m \cdot TC_n(X).
\]

Proof. Let \( Y_1, \cdots, Y_m \) be an open subsets of \( X^n \) with the fact \( X^n = Y_1 \cup \cdots \cup Y_m \). Then we have that

\[
TC_{n,X}(Y_1) + \cdots + TC_{n,X}(Y_m) = D_{X^n}(Y_1; p_1, \ldots, p_n) + \cdots + D_{X^n}(Y_m; p_1, \ldots, p_n) = D(p_1 \circ i_{Y_1}, \cdots, p_n \circ i_{Y_1}) + \cdots + D(p_1 \circ i_{Y_m}, \cdots, p_n \circ i_{Y_m}) \leq D(p_1, \cdots, p_n) + \cdots + D(p_1, \cdots, p_n) = m \cdot TC_n(X)
\]

from (d) part of Proposition 2.4.

Theorem 3.8. Let \( Y \subseteq X^2 \). Then \( TC_{2,X}(Y) \leq \text{cat}_{X \times X}(Y) \).

Proof. Let \( p_1, p_2 : X \times X \to X \) be projections and \( *=Y \to X \times X \) a constant map. The fact \( p_1 \circ * \simeq p_2 \circ * \) implies \( D(p_1 \circ i_Y, p_2 \circ i_Y) \leq D(i_Y, *) \) from (f) of Proposition 2.4. Thus, we conclude that \( TC_{2,X}(Y) \leq \text{cat}_{X \times X}(Y) \).
Corollary 3.9. Let $Y \subseteq X^n$. Then $TC_{n,X}(Y) \leq cat_{X^n}(Y)$.

Theorem 3.10. Let $Y, Z \subseteq X^n$ such that $Y$ and $Z$ have the same homotopy type. Then $TC_{n,X}(Y) = TC_{n,X}(Z)$.

Proof. Let $\beta : Z \to Y$ be the homotopy equivalence map. Assume that $i \in \{1, \cdots, n\}$. Consider the following commutative diagram for the projection $p_i : X^n \to X$ with the inclusions $i_Y : Y \to X^n$ and $i_Z : Z \to X^n$:

$$
\begin{array}{ccc}
Y & \xrightarrow{p_i \circ i_Y} & X \\
\beta \downarrow & & \downarrow 1_X \\
Z & \xrightarrow{p_i \circ i_Z} & X.
\end{array}
$$

Hence, by Theorem 3.10, we get

$$D(p_1 \circ i_Y, \cdots, p_n \circ i_Y) = D(p_1 \circ i_Z, \cdots, p_n \circ i_Z)$$

which concludes that $TC_{n,X}(Y) = TC_{n,X}(Z)$. \qed

4. RELATIVE HIGHER TOPOLOGICAL COMPLEXITY OF A PAIR

Definition 4.1. (SG) Let $A$ be a path-connected space and $B \subseteq A$. Set

$$P'_{A \times B} = \{\alpha \in A^{I_n} : \alpha(0) \in A, \alpha(1) = (\alpha_1(1), \cdots, \alpha_n(1)) \in B\} \subseteq A^{I_n}.$$

Then for a fibration $e'_n : P'_{A \times B} \to B^n$ defined with $e'(\alpha) = (\alpha_1(1), \cdots, \alpha_n(1))$, the relative higher topological complexity of the pair $(A, B)$ is defined as

$$TC_n(A, B) = genus(e'_n).$$

Note that $e'_n : P'_{A \times B} \to B^n$ is indeed a fibration because the restriction of a fibration $e_n : A^{I_n} \to A^n$, $e_n(\alpha) = (\alpha_1(1), \cdots, \alpha_n(1))$, to a subset $B^n \subset A^n$ is $e'_n$.

Proposition 4.2. a) $TC_1(A, B) = 1$. This means that the notation $TC_n(A, B)$ is significative for $n > 1$.

b) $TC_n'(A, B) = genus(d'_n)$, where $d'_n : B \to B^n$ is a diagonal map since $e'_n$ is a fibration substitute of $d'_n$.

c) For $n = 2$, $TC_2(A, B) = TC(A, B)$.

Proof. a) Let $n = 1$. Then for a fibration $e'_1 : P'_{A \times B} \to B$ with $e'_1(\alpha) = \alpha_1(1)$, we construct a map $s : B \to P'_{A \times B}$ such that $s$ takes any point $y$ of $B$ to the constant path $\epsilon_y$ at this point. Therefore, we get

$$e'_1 \circ s(y) = e'_1(\epsilon_y) = y = 1_B(y).$$

Thus, genus($e'_1$) equals 1.
b) Take a homotopy equivalence \( h : B \to P'_{A \times B} \) defined as \( h(y) = \epsilon_y \), where \( \epsilon_y \) is a constant path at \( y \). Then we get
\[
e' \circ h(y) = e'(\epsilon_y) = (y, \cdots, y) = d'(y).
\]

c) Let \( n = 2 \). Define \( e''_2 : P_{A \times B} \to A \times B \) as \( e''_2(\alpha) = (\alpha(0), \alpha(1)) \). Then \( e''_2 \) is a fibrational substitute of the diagonal map \( d'_2 : B \to B^2 \) because \( h' : B \to P_{A \times B}, h(y) = \epsilon_y \), is a homotopy equivalence and the condition \( e''_2 \circ h' = d'_2 \) holds with considering that \( B \subseteq A \). Thus, we find \( \text{TC}_2(A, B) = \text{genus}(e''_2) = \text{TC}(A, B) \). \( \Box \)

One of the well-known results of Schwarz [25] leads to us having an important relationship between the relative higher topological complexity and the Lusternik-Schnirelmann category:

**Corollary 4.3.** For a path-connected space \( A \) with its subset \( B \), we have \( \text{TC}_n(A, B) \leq \text{cat}(B^n) \). In addition, if \( A \) is contractible, then we conclude that \( \text{TC}_n(A, B) = \text{cat}(B^n) \).

By Corollary 4.3, we immediately have that if \( B \) is contractible, then we get
\[
\text{TC}_n(A, B) \leq \text{cat}(B^n) = 1,
\]
i.e., \( \text{TC}_n(A, B) = 1 \).

**Example 4.4.** For any point \( x_0 \) in a path-connected space \( A \), we obtain that \( \text{TC}_n(A, \{x_0\}) = 1 \).

It is possible to improve Corollary 4.3 with using Proposition 2.10.

**Corollary 4.5.** Let \( A \) be path-connected and \( B \) be a path-connected and paracompact subset of \( A \). Then we have \( \text{TC}_n(A, B) \leq n \cdot \text{cat}(B) \).

**Proof.** By Corollary 4.3, we obtain \( \text{TC}_n(A, B) \leq \text{cat}(B^n) \). Since \( B \) is path-connected and paracompact, we observe that \( \text{cat}(B^n) \leq n \cdot \text{cat}(B) \). \( \Box \)

Besides Schwarz genus, the relative higher topological complexity \( \text{TC}_n(A, B) \) of a pair \((A, B)\) can also be defined by higher homotopic distance:

**Definition 4.6.** (HD) Let \( A \) be a path-connected space and \( B \subseteq A \). Then
\[
\text{TC}_n(A, B) = D_{A^n}(A \times B \times \cdots \times B; p_1, p_2, \cdots, p_n)
\]
for \( n > 1 \), where each \( p_i \) is a projection from \( A \times A \times \cdots \times A \) to \( A \) onto the \( i \)-th factor for \( i = 1, \cdots, n \).

We assume that \( \text{TC}_1(A, B) \) always equals 1. If \( B = A \), then we conclude that \( \text{TC}_n(A, A) = \text{TC}_n(A) \). We observe that \( \text{TC}_n(A, B) \leq \text{TC}_{n+1}(A, B) \) as well as \( \text{TC}_n \) of a space or a fibration.

**Proposition 4.7.** Let \( B_1 \subset B_2 \subset A \). Then \( \text{TC}_n(A, B_1) \leq \text{TC}_n(A, B_2) \).
Proof. When we consider three inclusion maps $i_{B_1} : B_1 \to A$, $i_{B_2} : B_2 \to A$, $i : B_1 \to B_2$ such that $i_{B_1} = i_{B_2} \circ i$, the remaining part of the proof goes similar to the proof of Proposition 3.4. □

Proposition 4.8. If $A$ is path-connected with a subset $B \subset A$, then we have $TC_n(A, B) \leq TC_n(A)$.

Proof. Let $i : A \times B \times B \times \cdots \times B \to A^n$ be an inclusion map. Then Proposition 2.4 (d) gives us that

\[
TC_n(A, B) = D(A \times B \times B \times \cdots \times B; p_1, p_2, \cdots, p_n)
\]

\[
= D(p_1 \circ i, p_2 \circ i, \cdots, p_n \circ i)
\]

\[
\leq D(p_1, p_2, \cdots, p_n)
\]

\[
= TC_n(A).
\]

□

5. PARAMETRISED (HIGHER) TOPOLOGICAL COMPLEXITY USING HOMOTOPI

Distance

The task in this section is to mention the homotopic distance definition of the parametrised topological complexity. Let $q : E \to B$ be a fibration and $X \neq \emptyset$ is a path-connected fiber for the fibration $q$. Let $E_B^n$ be a set consisting of all continuous paths $\gamma$ in $E$ such that $q \circ \gamma$ is a constant path. Moreover, $E \times_B E$ is a subset of $E \times E$ and it contains all points $q(e, e')$ with the condition $q(e)$ equals $q(e')$. Recall that the map $\pi : E_B^n \to E \times_B E$ with $\pi(\gamma) = (\gamma(0), \gamma(1))$ is a fibration with fibre $\Omega X$. Then consider the following diagram (see also Theorem 2.3):

\[
P \xrightarrow{\pi_2} E_B^n \xrightarrow{\pi} E \times_B E
\]

Here $p^B_i : E \times_B E \to E$ is a projection map onto the $i$th factor for $i = 1, 2$. Similarly, $\pi_i$ is another projection map for each $i = 1, 2$. Also, $P$ is given by the set $\{(e, e', \gamma) : \gamma(0) = p_1(e, e') = e, \gamma(1) = p_2(e, e') = e'\}$, and it is clearly a subset of $E \times_B E \times E_B^n$. This yields that $D(p^B_1, p^B_2) = genus(\pi_1)$. Since $(p^B_1, p^B_2) = E \times_B E$, we observe that $genus(\pi_1) = genus(\pi)$. Combining this result with the Definition 2.2, we have a new statement of the parametrised topological complexity on homotopic distance:

Definition 5.1. (HD) The parametrised topological complexity is defined as

\[
TC[q : E \to B] = D(p^B_1, p^B_2)
\]

for the projection map $p^B_i : E \times_B E \to E$ with each $i = 1, 2$.

For the improved version of Definition 5.1, we note that the parametrised higher topological complexity is defined as

\[
TC_n[q : E \to B] = D(p^B_1, \cdots, p^B_n)
\]
for $n > 1$ and the map $p_i^B : E \times_B E \cdots \times_B E \to E$ with each $i = 1, \ldots, n$.

If $n = 1$, then $TC_1[q : E \to B]$ is always 1. The second observation states that the inequality $TC_n[q : E \to B] \leq TC_{n+1}[q : E \to B]$ holds. Furthermore, one can easily observe the equality $TC_2[q : E \to B] = TC[q : E \to B]$ by using the higher homotopic distance.

**Proposition 5.2.** Let $q : E \to B$ be a fibration. Let $q|_{B'} = q' : E' \to B'$ be another fibration with $B' \subset B$ and $E' = q^{-1}(B')$. Then

$$TC[q' : E' \to B'] \leq TC[q : E \to B].$$

**Remark 5.3.** Proposition [5.2] is first expressed in [2]. We now give explicit proof by using the homotopic distance.

**Proof.** By Definition [5.1], $TC[q : E \to B]$ and $TC[q' : E' \to B']$ are respectively equal to $D(p_i^B, p_2^B)$ and $D(p_i^B, p_2^B)$ for the projection maps $p_i^B : E \times_B E \to E$ and $p_i^B : E' \times_B E' \to E'$ for each $i = 1, 2$. The projection map $p_i^B$ can be thought as $p_i^B \circ j$, where $j : E' \times_B E' \to E' \times_B E$ is an inclusion. Therefore, we have

$$D(p_i^B, p_2^B) = D(p_1^B \circ j, p_2^B \circ j)$$

via (d) of Proposition [2.4]. Finally, the inequality $D(p_1^B \circ j, p_2^B \circ j) \leq D(p_1^B, p_2^B)$ concludes that $TC[q' : E' \to B'] \leq TC[q : E \to B]$. \hfill \square

It is possible that the parametrised topological complexity can be defined by the relative topological complexity. Indeed,

$$TC[q : E \to B] = D(p_1^B, p_2^B)$$

$$= D(p_1 \circ i_{E \times_B E}, p_2 \circ i_{E \times_B E})$$

$$= D_{E^2}(E \times_B E; p_1, p_2)$$

$$= TC_E(E \times_B E).$$

This fact is also improved with the following equality:

$$TC_n[q : E \to B] = TC_{n,E}(E \times_B E \cdots \times_B E).$$

**Proposition 5.4.** $TC_n[q : E \to B] \leq TC_n(E)$.

**Proof.** Let $j : E \times_B E \cdots \times_B E \to E \times E \cdots \times E$ be the inclusion map. Then, by Proposition [2.4](d), we have

$$TC_n[q : E \to B] = D(p_1^B, \ldots, p_n^B)$$

$$= D(p_1 \circ j, \ldots, p_n \circ j)$$

$$\leq D(p_1, \ldots, p_n) = TC_n(E).$$

\hfill \square

**Corollary 5.5.** $TC[q : E \to B] \leq TC(E)$.

**Theorem 5.6.** $TC_n(q) \leq TC_n[q : E \to B]$. 
Proof. Let \( j : E \times_B E \cdots \times_B E \rightarrow E \times E \cdots \times E \) be an inclusion map and \( p_i : E^n \rightarrow E \) be the projection map for each \( i = 1, \ldots, n \). Then we find
\[
TC_n(q) = D(q \circ p_1, \ldots, q \circ p_n) \\
\leq D(p_1, \ldots, p_n) \\
\leq D(p_1 \circ j, \ldots, p_n \circ j) \\
= D(p_1^n, \ldots, p_n^n) = TC_n[q : E \rightarrow B]
\]
by using (c) and (d) parts of Proposition 2.4, respectively. \( \square \)

Corollary 5.7. \( TC(q) \leq TC[q : E \rightarrow B] \).

Corollary 5.8. a) Let \( q_1 : E_1 \rightarrow B_1 \) be a fibration with a path-connected fiber \( X_1 \) and \( q_2 : E_2 \rightarrow B_2 \) be another fibration with a path-connected fiber \( X_2 \). If \( E_1, B_1, E_2 \) and \( B_2 \) are metrisable, then
\[
\max\{TC(q_1), TC(q_2)\} \leq TC[q_1 : E_1 \rightarrow B_1] + TC[q_2 : E_2 \rightarrow B_2].
\]
b) \( cat(B) \leq TC[q : E \rightarrow B] \).

Proof. a) Let \( q_1 \times q_2 : E_1 \times E_2 \rightarrow B_1 \times B_2 \) be a fibration with a path-connected fiber \( X_1 \times X_2 \). Then by Proposition 2.11 Corollary 5.7 and Proposition 2.24 respectively, we have
\[
\max\{TC(q_1), TC(q_2)\} \leq TC(q_1 \times q_2) \\
\leq TC[q_1 \times q_2 : E_1 \times E_2 \rightarrow B_1 \times B_2] \\
\leq TC[q_1 : E_1 \rightarrow B_1] + TC[q_2 : E_2 \rightarrow B_2].
\]
b) This is obvious from Proposition 2.15 and Corollary 5.7. \( \square \)

Example 5.9. Hopf Fibration: Consider the Hopf fibration \( q : S^3 \rightarrow S^2 \) with the path-connected fiber \( S^1 \). We shall show that \( 2 \leq TC_n[q : S^3 \rightarrow S^2] \leq n \) for \( n \geq 2 \). Since \( TC_n(S^1) = n \) [24], by Corollary 5.6 \( TC_n[q : S^3 \rightarrow S^2] \leq n \). On the other hand, \( cat(S^2) = 2 \) yields that \( TC_n[q : S^3 \rightarrow S^2] \geq 2 \) via (b) part of Corollary 5.8.

Example 5.10. Stiefel and Grassmann Manifolds: Consider the complex case, i.e.,
\[
U(r) \rightarrow V_r(C^k) \xrightarrow{q_1} G_r(C^k).
\]
Since \( cat(G_r(C^k)) = rk \) by Theorem 2.11 Corollary 5.8 (b) states that
\[
rk \leq TC[q_1 : V_r(C^k) \rightarrow G_r(C^k)].
\]
On the other hand, we have that \( TC(V_r(C^k)) \leq 2r(k - r) + 1 \) from Theorem 2.7
By Corollary 5.4 we get \( TC[q_1 : V_r(C^k) \rightarrow G_r(C^k)] \leq 2r(k - r) + 1 \). Finally, we conclude that
\[
rk \leq TC[q_1 : V_r(C^k) \rightarrow G_r(C^k)] \leq 2r(k - r) + 1
\]
for \( n = 2 \) in the sense of \( TC_n \).
If we assume that the quaternionic case, namely that,
\[
Sp(r) \rightarrow V_r(H^k) \xrightarrow{q_2} G_r(H^k),
\]

\[
\]
then the inequality
\[ k(r - k) \leq \text{TC}[q_2 : V_r(\mathbb{R}^k) \to G_r(\mathbb{R}^k)] \]
holds from the fact that \( k(r - k) = \text{cat}(G_r(\mathbb{R}^k)) \) by Theorem 2.6. Now consider the real case:
\[ O(n) \to V_r(\mathbb{R}^k) \xrightarrow{q_2} G_r(\mathbb{R}^k). \]
First, assume that \( r = 2 \) and \( k = 2p + 1 \) with any integer \( p > 0 \). Then, by Theorem 2.8 of [1], we get \( \text{TC}[q_3 : V_2(\mathbb{R}^{2p+1}) \to G_2(\mathbb{R}^{2p+1})] \geq 2^{p+1} - 2 \). For the upper bound, we shall use Corollary 5.5. Since \( \dim(V_2(\mathbb{R}^{2p+1})) = 2^{p+1} - 1 \), Proposition 2.6 says that \( \text{TC}(V_2(\mathbb{R}^{2p+1})) \leq 2^{p+2} - 1. \) As a consequence,
\[ 2^{p+1} - 2 \leq \text{TC}[q_3 : V_2(\mathbb{R}^{2p+1}) \to G_2(\mathbb{R}^{2p+1})] \leq 2^{p+2} - 1. \]
Now, assume that \( r = 2 \) and \( k = 2p + 2 \) with any integer \( p \). Similar to the previous case, by using Theorem 2.11 in [1], we have the following inequalities:
\[ 2^{p+1} - 1 \leq \text{TC}[q_3 : V_2(\mathbb{R}^{2p+2}) \to G_2(\mathbb{R}^{2p+2})] \leq 2^{p+2} + 3 \]
with considering the fact that \( \dim(V_2(\mathbb{R}^{2p+2})) = 2^{p+1} + 1. \)

**Theorem 5.11.** If \( q_1 : E_1 \to B_1 \) and \( q_2 : E_2 \to B_1 \) are fiber homotopy equivalent with the same nonempty path-connected fiber \( X_1 \) for both two fibrations \( q_1 \) and \( q_2 \), then
\[ \text{TC}_n[q_1 : E_1 \to B_1] = \text{TC}_n[q_2 : E_2 \to B_1], \]
that is, the parametrised higher topological complexity is a fiber homotopy equivalent invariant.

**Proof.** Let \( q_1 : E_1 \to B_1 \) and \( q_2 : E_2 \to B_1 \) be fiber homotopy equivalent with the fiber \( X_1 \). Then we have two maps \( h : E_1 \to E_2 \) and \( k : E_2 \to E_1 \) satisfying two conditions \( h \circ k \simeq 1_{E_2} \) and \( k \circ h \simeq 1_{E_1} \). For simplicity, we rewrite \( E_1 \) and \( E_2 \) as \( E_1 \times B_1, E_1 \times B_1, \ldots, E_1 \) and \( E_2 \times B_1, E_2 \times B_1, \ldots, E_2 \), respectively. Consider the homotopy equivalence map \( \beta : E_2 \to E_1 \). Assume that \( p_i^B : E_i \to B_1 \) and \( q_i^B : E_i \to E_2 \) are projections for each \( i \in \{1, \ldots, n\} \). Then the following commutative diagram
\[
\begin{array}{ccc}
E_1 & \xrightarrow{p_1^B} & E_1 \\
\downarrow{\beta} & & \downarrow{h} \\
E_2 & \xrightarrow{q_1^B} & E_2
\end{array}
\]
states that \( D(p_1^B, \ldots, p_n^B) = D(q_1^B, \ldots, q_n^B) \). As a consequence, the equality \( \text{TC}_n[q_1 : E_1 \to B_1] = \text{TC}_n[q_2 : E_2 \to B_1] \) holds. \( \square \)

6. Conclusion

The relative topological complexity is one of the first examples of different types of TC. With this approach, a lower bound for TC is obtained. A similar approach to \( \text{TC}_n \) is done by the relative higher topological complexity on the notion homotopic distance in this study. Interestingly, it also gives a new way to introduce
the parametrised topological complexity, which has been one of the most popular numbers among several TC versions in the last few years due to the different constructions of its motion planning algorithm. The higher homotopic distance again leads to define the parametrised higher topological complexity. These higher settings of TC always provide a different and strong perspective on the subject of robot motion planning problems in daily life.

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