Some Approaches to the Estimation of the Stopping Time of the Cross-boundary Event for the Process with the Change-point

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Abstract. In the paper two approaches of estimation of the first boundary crossing time moment are considered. As the first estimate (for the process with the change-point on the interval of observation [0,t]) the conditional mean value expectation of the stopping-time is considered. The second considered estimate is the measurable and observable stopping-time moment for which the conditional mean value of the appropriate process is equal to the value of the boundary.

1. Introduction

In the study of problems of applied significance (in Engineering, in Biology), processes that change at random times, reflecting events, namely, the «change-point» time, are often encountered.

A synonym for the word «change-point» is a violation, derangement, according to the directory of a technical translator «change-point» is a violation of normal work.

The «change-point» is used in the statistical control, the theory of detection. In papers (see, for example, [1-9]), methods for detecting instants of changes in the probability characteristics of random processes, i.e., the change-point time.

Detection is faster (with a certain level of false alarms) and more precisely allows you to continue managing observations in stochastic systems and control moving objects under incomplete information.

Various methods of change-point detecting help determine whether a process has reached a statistically controlled state at a properly specified level or remains in that state, and then maintain control and a high degree of homogeneity of the most important characteristics of a product or service by continuously recording information about product quality during production.

The use of various detection methods and their careful analysis lead to a better understanding and improvement of the processes.

The classical problem with the change-point was formulated as early as the 60s by A. Wald [4] and A.N. Shiryayev [10]. Further this direction was developed in the works of A.L. Presman [11], G. Robbins [12], and today the development of this task is reflected in the works of M.L. Nikolaev [13], V.V. Mazalova [14], G.I. Salova [15] and others. Technical areas of science are traditionally
considered as an applied application. Also in the last decade, a number of articles have appeared on the application of problems of disorder in Biology and Medicine.

It is customary to distinguish the following two classes of theoretical problems:
1. the analysis of the main and alternative hypotheses about the onset of the moment of disorder;
2. the numerical determination of the probability of occurrence of the moment of disorder to a specific time.

The reviews of various methods for detecting moments of changes in probabilistic properties can be seen, for example, in [16,17 and others].

In this paper, we consider two problems of estimating the time of the first cross-boundary event based on the process with the change-point (see paragraph 4 of chapter IV in [3]).

In this case, it is possible to construct the conditional mean for every time of observation \( t \geq 0 \), the condition for which is the observation of the process until time.

As an alternative method of estimation, a formalization of the approach is proposed, which consists of the measurable times consistent construction (subject for the observations up to the time of estimation), in such a way that the process with the change-point at this time has a conditional expectation equal to the value of the boundary.

2. The problem of the estimation based on the observations of the process with the change-point up to the time moment values

Let \( R = (\Omega, F, \mathbb{F} = (F_t)_{t \geq 0}, P) \) be the stochastic basis with usual conditions of completeness and regularity, \( \mathbb{F} = (F_t)_{t \geq 0} \) - non-decreasing filtration on probability space \( (\Omega, F, P) \) (basic definitions and properties, see, e.g., paragraph 1 ch.1 in [18]).

Let us consider the problem of estimating the moment of boundary crossing by the process with the change-point in the following a little simplified formulation.

Let the independent standard Wiener process \( W = (W_t)_{t \geq 0} \) and an exponentially distributed random variable \( \theta \) with the parameter \( \lambda > 0 \) be given on the stochastic basis \( R \):

\[
P\{\theta \leq 0\} = 0 \quad \text{with} \quad x \leq 0; \\
P\{\theta \leq x\} = 1 - \exp\{- \lambda x\} \quad \text{for} \quad x > 0.
\]

For the parameter \( \gamma > 0 \), we consider the process with the change-point \( X = (X_t)_{t \geq 0} \) with \( x_0 = 0 \) and

\[
X_t = \gamma \cdot \int_0^t I\{\theta \leq s\} \, ds + W_t,
\]

where \( I\{\cdot\} \) is the indicator function (having the value 1 for all elements of \( \{\cdot\} \) and the value 0 for all elements not in \( \{\cdot\} \)). Thus, the process is a simplification (compared with considered in [3]) in the sense that the diffusion coefficient is equal to unity and the probability of the change-point at the initial time is zero. These suggestions do not limit the generality, but will help to simplify the presentation. We denote \( A \) as the level of the boundary.

The first time of crossing of \( A \) by the process \( X \) in the present paper is considered as \( t = \tau(A) \) for \( t = \tau(A) \):

\[
\tau(A) = \inf\{t : t > 0, X_t \geq A\}.
\]

We define the non-decreasing family of \( \sigma \) - algebra \( F^X = (F^X_t)_{t \geq 0} \) (with \( F^X_0 = \emptyset, \Omega \)) and \( F^X_t \subseteq F \) for every \( t \geq 0 \) and \( F^X_t = \sigma(X_s ; s \leq t) \).
The problem is to estimate the time \( \tau = \tau(A) \) given the observations \( X \) up to the time \( t \), i.e. \( F_t^X \).

We consider the classical approach to the construction of the estimate, consisting in finding of the conditional mean:

\[
\hat{\tau} = (\hat{\tau}_t)_{t \geq 0} \quad \text{with} \quad \hat{\tau}_t = E(\tau \mid F_{\tau -}^X).
\]

The following is true.

**Theorem.**

If \( \gamma > 0 \), \( A > 0 \) and the conditions probability is \( \pi_t = P\{\theta \leq t \mid F_{\tau -}^X\} \) for every \( t \geq 0 \), the estimate \( \hat{\tau}_t = E(\tau \mid F_{\tau -}^X) \) is equal to

\[
\hat{\tau}_t = \tau \cdot I\{\tau \leq t\} + \left\{t + \frac{A - X_t}{\gamma} \cdot \pi_t + f(A - X_t)(1 - \pi_t)\right\} \cdot I\{\tau > t\}, \tag{3}
\]

where \( f(A) = E(\tau \mid A) \). In this case (2) (see paragraph 4, Chapter IV in [3]) for the process \( \pi_t = (\pi_t)_{t > 0} \),

where \( \pi_t = P\{\theta \leq t \mid F_{\tau -}^X\} \), the following is valid:

\[
\pi_t = \lambda \cdot \left[(1 - \pi_s)ds + \int_0^t \pi_s \cdot (1 - \pi_s) \cdot (dX_s - \gamma \cdot \pi_s \cdot ds)\right], \tag{4}
\]

allowing to fulfill out the consistent finding estimation of the probability \( \pi_t \) based on the observations \( (X_s)_{0 \leq s \leq t} \).

**Proof.**

For \( t = 0 \), the statement of the theorem is obvious from the form of the function \( f(A) \) (as \( F_0^X = \emptyset \)). If \( t > 0 \), for the time of the change-point \( \theta \) we define auxiliary notations for events

\( B = \{\theta \leq t\} \quad \text{and} \quad \overline{B} = \{\theta > t\} . \)

For the sake of brevity we denote \( P_t^X(\cdot) = P(\cdot \mid F_{\tau -}^X), \ E_t^X(\cdot) = E(\cdot \mid F_{\tau -}^X) \),

and for any event \( D \in F_{\tau -}^X \) and \( \rho_t^X(\cdot \mid D) = P(\cdot \mid F_{\tau -}^X \cap D) \) and \( E_t^X(\cdot \mid D) = E(\cdot \mid F_{\tau -}^X \cap D) \).

For the time \( \tau \) the following is true:

\[
\tau = \tau \cdot I\{\tau \leq t\} + \tau \cdot I\{\tau > t\} = \tau \cdot I\{\tau \leq t\} + \tau \cdot I(\overline{B}) \cdot I\{\tau > t\},
\]

which implies:

\[
\hat{\tau}_t = E_t^X(\tau) = \tau \cdot I\{\tau \leq t\} + E_t^X(\tau \cdot I(\overline{B}) \mid B) \cdot I\{\tau > t\} \cdot P_t^X(B) + E_t^X(\tau \cdot I(\overline{B}) \mid B) \cdot I\{\tau > t\} \cdot P_t^X(B),
\]

where random values \( I\{\tau > t\} \) and \( \tau \cdot I\{\tau \leq t\} \) are \( F_{\tau -}^X \) - measurable.

The value \( E_t^X(\tau \cdot I(\overline{B}) \mid X) \) coincides on the set \( \tau > t \) with the conditional expectation of the time of cross-boundary \( A \) by the Wiener process \( (W_u)_{u \geq 0} \) with the constant trend \( \gamma \cdot (u - t) \) and with the defined initial value of the process \( X_t \).

However, for this process \( \Psi_t = (\Psi_{\tau -}^X)_{u \geq t} \) with \( \Psi_t = x \) and

\[
\Psi_t = x + \gamma(u - t) + (W_u - W_{\tau -}^X).
\]

The expected value \( g(\beta) = E\tilde{\gamma}(A) \) of the time \( \tilde{\gamma} = \tilde{\gamma}(A) \) of cross-boundary of \( A \)

\[
\tilde{\gamma}(A) = \inf(s : s > 0, \Psi_s \geq A),
\]

is defined from the equality...
\[ \Psi \gamma(A) = A \]

and from the equality
\[ x + \gamma \cdot (\gamma(A) - t) + W \gamma(A) = A \, . \]

According to the martingale stopping theorem (see, for example, [3]), the following equality is fulfilled
\[ EW \gamma(A) = 0 \, , \]
and therefore
\[ g(A) = t + \frac{A - x}{\gamma} \]  (5)

As a result, we receive the following expression:
\[ E_{\gamma}^t (\tau : I(B) \mid B) \cdot I(\tau > t) = \frac{A - X}{\gamma} + t \cdot I(\tau > t) \, . \]  (6)

We note that the function \( f(x) = E_\tau(x) \) for every \( x \geq 0 \) is defined by the following expression:
\[ f(x) = \int_0^\infty \lambda \cdot e^{-\lambda u} \left[ u \cdot \frac{x}{\sqrt{2\pi s}} \cdot e^{-x^2/2s} ds + \int_u^\infty (u + \frac{x - y}{\alpha}) \cdot \frac{1}{\sqrt{2\pi u}} (e^{-y^2/(2u)} - e^{-(y-x)^2/(2u)}) dy \right] \, du \, . \]  (8)

The values of the function \( f(x) \) defined in the formula (8), can be obtained only with the numerical integration methods, that are obviously approximate. An alternative could be a simulation of the process \( \xi \) according to the of empirical means for \( E_\tau(x) \), as an approximation.

We remark, that for every value \( u \geq 0 \) the identity is obvious:
\[ E\tau = E\tau \cdot I\{\tau \leq u\} + E\tau \cdot I\{\tau > u\} \]  (9)

and
\[ E(\tau \mid \theta = u) = E(\tau \cdot I\{\tau \leq u\} \mid \theta = u) + E(\tau \cdot I\{\tau > u\} \mid \theta = u) \]  (10)

We denote
\[ r_\theta(u) = E(\tau \cdot I\{\tau \leq u\} \mid \theta = u) \]

and
\[ r_\tau(u) = E(\tau \cdot I\{\tau > u\} \mid \theta = u) \, . \]

Since the time \( \theta \) has the distribution density \( \lambda \cdot e^{-\lambda u} \) for \( u \geq 0 \), from (9) and (10) it follows that
\[ E\tau = \int_0^\infty \lambda \cdot e^{-\lambda u} (r_\tau(u) + r_\theta(u)) du \, . \]  (11)

To find \( r_\tau(u) \), we note that under the condition \( \{\theta = u\} \) that the process reaches \( X \) the level \( x > 0 \) (for the value \( x = 0 \) is obvious \( f(0) = 0 \) because \( X = 0 \) ) at time moments \( \tau \leq u \) it is carried out with the density of the distribution of the passage time of the level \( x \) by the Wiener process, since time of the drift \( \theta \) is zero.
Since the density (see, for example, p. 425 of reference book [19]) is equal to 
\[ \frac{x}{\sqrt{2\pi s^3}} e^{-x^2/(2s)} , \]
then
\[ r_1(u) = \int_{0}^{u} \frac{x}{\sqrt{2\pi s^3}} e^{-x^2/(2s)} ds . \]  

(12)

The joint distribution of the Wiener process and maximum of its values it has the density (on variables $y$ process $W_u$ with $y \leq x$ and maximum which is less $x$, see also [19]):
\[ \rho(y; u, x) = \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{\infty} e^{-s^2/(2u)} e^{-x^2/(2u)} ds = \frac{1}{\sqrt{2\pi u}} (e^{-y^2/(2u)} - e^{-(y-x)^2/(2u)}) . \]  

(13)

The expected value cross-boundary time moment by of the value of the boundary $x > 0$ process, which is the value $y < x$ under the condition $\tau > u$, is similar to (8) and equal to
\[ r_2(u) = \int_{-\infty}^{x} \frac{1}{a} (u + \frac{x-y}{a}) \cdot \rho(y; u, x) dx . \]  

(14)

From the formulas (11) - (14) it follows that (8) is true.

3. An empirical approach to estimating of the cross boundary time moment of the process with the change-point

Studying the cross-boundary moments $A > 0$ for every time $t \geq 0$ we consider $F_t^X$ - measurable and the random variable $\beta(t)$, at that the process $X$ has the conditional mean value equal to the boundary level:
\[ E_t^X(X; \beta(t)) = E(X; \beta(t) | F_t^X) = A \]  

(15)

At the same time, due to the relation $\{t \leq \tau\} \in F_t^X$, it is obvious, that
\[ \beta(t) \cdot 1\{t \leq \tau\} = \tau \cdot 1\{t \leq \tau\} \]  

(16)

The estimate $\hat{\beta}(t)$ of the time $\tau$ is called empirical (sure, conditionally), since it is calculated sequentially from observations $(X_s)_{0 \leq s \leq t}$.

Also the estimate of $\beta(t)$ is not unbiased, the convergence for $t \to \infty$ holds due to $\tau$ is finite:
\[ \beta(t) \to \tau \quad P_{-n.m} , \quad t \to \infty \]  

(17)

We note that in applied areas, such estimates are implemented and used in problems of approximate estimation and forecasting (see, for example, [20]).

For a deterministic case $X_t = EX_t$ the finding of $\beta$ is in search of the value $\beta(0)$, such that
\[ \hat{X}_t = \beta(0) = a . \]  

(18)

Obviously, the fulfillment of sequential estimation in the form of finding of $\beta = (\beta(t))_{t \geq 0}$ with - $F_t^X$ measurable non-decreasing time $\beta(t) \geq \beta(s)$ (for $t \geq s$) is a consistent $F_t^X$ localization of the predictable stopping time $\tau$.

For the case (18) from the equality
\[ \hat{X}_t = \int_{0}^{t} (1-e^{-\lambda s}) ds = \gamma(t - \frac{1}{\lambda}(1-e^{-\lambda t})) . \]  

(19)

the values of $\beta(0)$ are defined by solving the transcendental equation:
\[ \beta(0) = \frac{A}{\gamma} (1 - e^{-\beta(0)}) = \frac{A}{\gamma} \]  

(20)
Therefore, it is quite easy to generalize deterministic case (18) if the \( F^X_t \) is an adapted case (15):

**Statement.**

The value of \( \beta = (\beta(t))_{t \geq 0} \) is determined from relations (21) and (22) for every \( t \geq 0 \):

\[
\begin{align*}
\beta(t) \cdot I[\tau \leq t] &= \tau \cdot I[\tau \leq t] \\
\{(\beta_t - t) - \frac{1 - \pi^s_t}{\lambda}(1 - e^{-\lambda(t-t)})\} I[\tau > t] &= \frac{(A - X_t)}{\lambda} I[\tau > t]
\end{align*}
\]

(21)\hspace{1cm}(22)

where \( \pi^s_t = P^X_t \{ \theta \leq t \} \).

In the considered case (2) (see paragraph 4 of chapter IV in [3]) the relation for the process \( \pi = (\pi_t)_{t \geq 0} \) holds:

\[
\pi_t = \lambda \int_0^t (1 - \pi^s_s) ds + \gamma \int_0^t \pi^s_s (1 - \pi^s_s) (dX_s - \gamma \cdot \pi^s_s ds),
\]

(23)

**Proof.**

We consider the auxiliary process \( (v_t(u))_{u \geq t} \) for every \( t \geq 0 \) with

\[
v_t(u) = A \cdot I[\tau \leq t] + X^s_t \cdot I[\tau > t]
\]

or

\[
v_t(u) = \beta \cdot I[\tau \leq t] + (X_t + \alpha \cdot \int_0^t I[\theta \leq s] ds + \int_0^t dW_s)
\]

(24)

 desn, \( v_t(t) = X^s_t \). The equality \( (v_t(t))_{t \geq 0} \) under the condition (15) for every \( t \geq 0 \) is true:

\[
E^X_t v_t(\beta(t)) = A.
\]

(25)

Due to expression the \( F^X_t \)-measurability of \( \beta(t) \) from (25) follows the relation (26):

\[
A \cdot I[\tau \leq t] + (X_t + \gamma \cdot \int_0^t E^X_s \pi^s_s ds) \cdot I[\tau > t] = A.
\]

(26)

From (26) is follows that

\[
(\int_0^t E^X_s \pi^s_s ds) \cdot I[\tau > t] = \frac{A - X_t}{\gamma} \cdot I[\tau > t]
\]

(27)

It is known that he process \( (\int_0^t (1 - \pi^s_s)(dX_s - \gamma \cdot \pi^s_s ds))_{t \geq 0} \) is a Wiener process, with respect to \( (F^X_t, \mathcal{P}) \), [3].

Therefore, we obtain a simple relation for \( E^X_t \pi^s_s \) for \( s \geq t \):

\[
E^X_t \pi^s_s = \pi^s_t + \lambda \int_t^s (1 - E^X_t \pi^s_u) du
\]

The solution of which defines the left side of the equation (27). The statement is proved.

4. Conclusion

The results obtained in this paper are applicable in the problems of the change-point of measurable processes. Processes with the change-point are used in various fields: Biology, Medicine, Economics, Technology, etc.

Finding of the time of the change-point (a significant change in the values of processes) is an important problem of estimation, since the time of change-point is the time of the beginning of damages or destructions of system.
Described in this article the methods for detecting change-point, determining the stopping time the system (the moment of stopping for the process) show the need to use several methods in the simulation of real objects (operation of airplanes, ships, spacecraft, nuclear power plants, computer networks), and when setting system parameters based on real data, choose the most optimal method under the given constraints.

Detection of such a moment allows us to perform corrective measures to prevent further destruction of the observed systems.

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