Minimizers of $L^2$-Subcritical Inhomogeneous Variational Problems with A Spatially Decaying Nonlinearity

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Abstract

We study the minimizers of $L^2$-subcritical inhomogeneous variational problems with spatially decaying nonlinear terms, which contain $x = 0$ as a singular point. The limit concentration behavior of minimizers is proved as $M \to \infty$ by establishing the refined analysis of the spatially decaying nonlinear term.

Keywords: $L^2$-subcritical variational problems; Spatially decaying nonlinearity; Minimizers; Mass concentration

1 Introduction

In this paper, we consider the minimizers of the following $L^2$-subcritical constraint inhomogeneous variational problem

$$I(M) := \inf_{\{u \in H, \|u\|_2^2 = 1\}} E_M(u), M > 0,$$

where the Gross-Pitaevskii (GP) energy functional $E_M(u)$ contains a spatially decaying nonlinearity and is defined by

$$E_M(u) := \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)\,dx - \frac{2M^{p-1}}{p+1} \int_{\mathbb{R}^N} \frac{|u|^{p+1}}{|x|^b}\,dx, \quad N \geq 1,$$

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and the space $\mathcal{H}$ is defined as

$$ \mathcal{H} := \left\{ u(x) \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u(x)|^2 \, dx < \infty \right\} $$

with the associated norm $\|u\|_{\mathcal{H}} = \left\{ \int_{\mathbb{R}^N} \left( |\nabla u(x)|^2 + |u(x)|^2 + V(x)|u(x)|^2 \right) \, dx \right\}^{\frac{1}{2}}$. Here positive constants $b > 0$ and $p > 0$ of (1.1) satisfy

$$ 0 < b < \min\{2, N\}, \quad 1 < p < 1 + \frac{4 - 2b}{N}, \quad \text{where} \quad N \geq 1, \quad (1.3) $$

so that $E_M(u)$ admits $x = 0$ as a singular point in its nonlinear term. We always assume that the trapping potential $V(x) \geq 0$ satisfies

(V). $V(x) \in L^\infty(\mathbb{R}^N) \cap C^\alpha_{loc}(\mathbb{R}^N)$ with $\alpha \in (0, 1)$, $\{x \in \mathbb{R}^N : V(x) = 0\} = \{0\}$ and

$$ \lim_{|x| \to \infty} V(x) = \infty. $$

The variational problem (1.1) arises in various physical contexts, including the propagation of a laser beam in the optical fiber, Bose-Einstein condensates (BECs), and nonlinear optics (cf. [1, 3, 28]), where the constant $M > 0$ often represents the attractive interaction strength, and $V(x) \geq 0$ denotes an external potential. The variational problem (1.1) and its associated elliptic equation have attracted a lot of attentions over the past few years, due to the appearance of the singular point $x = 0$ in the nonlinear term, see [2, 8, 9, 12, 13, 15, 30] and the references therein.

When $b = 0$, (1.1) is a homogeneous constraint variational problem, for which there are many existing results (1.1) (cf. [6, 14, 18, 20, 22, 26, 27, 29, 34]), including the existence and nonexistence of minimizers, and their quantitative properties of all kinds. More precisely, when $p > 1 + \frac{4}{N}$, one can use the energy estimates to obtain the nonexistence of minimizers for (1.1) with $b = 0$ as soon as $M > 0$ (cf. [6, 7]), which is essentially in the $L^2$-supercritical case. However, if $p = 1 + \frac{4}{N}$, then (1.1) with $b = 0$ reduces to the $L^2$-critical case, which was addressed widely by the second author and his collaborators, see [18, 20, 22] and the references therein. As for the case where $1 < p < 1 + \frac{4}{N}$, (1.1) with $b = 0$ is in the $L^2$-subcritical case and admits generally minimizers for all $M \in (0, \infty)$. In this case, the uniqueness, symmetry breaking and concentration behavior of minimizers were investigated recently as $M \to \infty$, see [21, 26, 29] and the references therein.

When $b \neq 0$, the variational problem (1.1) contains the inhomogeneous nonlinear term $m(x)|u|^{p+1}$, where $m(x) = \frac{1}{|x|^b}$ admits $x = 0$ as a singular point. We remark that the inhomogeneous $L^2$-constraint variational problems were analyzed recently in [10, 11, 29] and the references therein. However, as far as we know, the above mentioned works handle mainly with the inhomogeneous nonlinear term $m(x)|u|^{p+1}$ where $m(x)$ satisfies $m(x) \in L^\infty(\mathbb{R}^N)$ without any singular point. On the other hand,
Ardila and Dinh obtained recently in \cite{2} the existence of minimizers and the stability of the standing waves, for which they studied the associated constraint variational problem (1.1), in the $L^2$-subcritical case where the harmonic potential satisfies $V(x) = \gamma^2|x|^2(\gamma > 0)$, $b > 0$ and $p > 0$ satisfy (1.3).

Under the assumptions \ref{V} and (1.3), we comment that it is standard to obtain the existence of minimizers for $I(M)$ for all $M > 0$, see \cite[Theorem 1.8]{2} and the related argument. Motivated by above mentioned works, in this paper we mainly study the limit behavior of minimizers $u_M$ for $I(M)$ as $M \to \infty$, and the main purpose of this paper is to investigate the impact of the singular point $x = 0$ on the behavior of $u_M$ as $M \to \infty$.

We now assume that $u_M$ is a minimizer of $I(M)$ for any $M > 0$. It then follows from the variational theory that $u_M$ satisfies the following Euler-Lagrange equation

$$ - \Delta u_M + V(x)u_M - M^{\frac{p-1}{2}} \frac{u_M^p}{|x|^b} = \mu_M u_M \text{ in } \mathbb{R}^N, $$

where $\mu_M \in \mathbb{R}^N$ is a suitable Lagrange multiplier associated to $u_M$. By the form of the energy functional $E_M(\cdot)$, one can obtain from \cite[Theorem 6.17]{25} that $E_M(u) = E_M(|u|)$ holds for any $u \in \mathcal{H}$, which implies that $|u_M|$ is also a minimizer of $I(M)$. By the strong maximum principle, one can further derive from (1.4) that $|u_M| > 0$ holds in $\mathbb{R}^N$. Therefore, $u_M$ must be either positive or negative. Without loss of generality, in the following we only consider positive minimizers $u_M > 0$ of $I(M)$.

Under the assumption (1.3), we next recall the following sharp Gagliardo-Nirenberg (GN) inequality (cf. \cite[Theorem 1.2]{13}): 

$$ \int_{\mathbb{R}^N} \frac{|u|^{p+1}}{|x|^b} dx \leq C_{GN}^{-1} \|
abla u\|^{\frac{N(p-1)}{2}} \|u\|^{p+1-\frac{N(p-1)}{2}} - b, \quad u \in H^1(\mathbb{R}^N), $$

where $C_{GN} > 0$ satisfies 

$$ C_{GN} = \left( \frac{N(p-1) + 2b}{2(p+1) - N(p-1) - 2b} \right)^{\frac{N(p-1)+2b}{2(p+1) - N(p-1) - 2b}} \frac{2(p+1) - N(p-1) - 2b}{2(p+1)} \|w\|^{p-1}, $$

and $w$ is the unique positive radially symmetric solution (cf. \cite{4,15,16,24,33}) of 

$$ - \Delta w + w - \frac{w^p}{|x|^b} = 0 \text{ in } \mathbb{R}^N, \quad w \in H^1(\mathbb{R}^N). $$

The equality in (1.5) is achieved at $u = w$. Moreover, $w$ satisfies the following Pohozaev identity 

$$ \|
abla w\|^2_2 = \frac{N(p-1) + 2b}{2(p+1)} \int_{\mathbb{R}^N} \frac{|w|^{p+1}}{|x|^b} dx = \frac{N(p-1) + 2b}{2(p+1) - N(p-1) - 2b} \|w\|^2_2. $$

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Recall also from [13, Theorem 2.2] that there exist positive constants $\delta > 0$ and $C > 0$ such that $w(x)$ satisfies

$$w(x), \ |\nabla w(x)| \leq Ce^{-\delta|x|} \text{ as } |x| \to \infty.$$  

(1.9)

All above properties of $w$ are often used in the refined analysis of minimizers for $I(M)$ as $M \to \infty$.

Using above notations, the main result of the present paper can be stated as the following theorem.

**Theorem 1.1.** Under the assumptions $(V)$ and \((1.3)\), let $u_k$ be a positive minimizer of $I(M_k)$, where $M_k \to \infty$ as $k \to \infty$. Then there exists a subsequence, still denoted by $\{u_k\}$, of $\{u_k\}$ such that $u_k$ satisfies

$$w_k(x) := \epsilon_k^N u_k(\epsilon_k x) \to \frac{w(x)}{\sqrt{a^*}} \text{ uniformly in } L^\infty(\mathbb{R}^N) \text{ as } k \to \infty,$$

(1.10)

where $\epsilon_k := \left(\frac{M_k}{a^*}\right)^{-\frac{4-N(p-1)-2\theta}{4-N(p-1)}} > 0$, $a^* := \|w\|_2^2 > 0$, and $w > 0$ is the unique positive solution of \((1.7)\). Moreover, $u_k$ decays exponentially in the sense that for sufficiently large $k > 0$,

$$w_k(x) \leq Ce^{-\sqrt{\theta}|x|} \text{ and } |\nabla w_k(x)| \leq Ce^{-\theta|x|} \text{ as } |x| \to \infty,$$

(1.11)

where $0 < \theta < 1$ and $C > 0$ are independent of $k > 0$.

The proof of Theorem 1.1 shows essentially that as $M_k \to \infty$, $u_k$ prefers to concentrate near the singular point $x = 0$ of $I(M_k)$, instead of a minimum point for $V(x)$. The proof of Theorem 1.1 depends on the refined estimate of $\int_{\mathbb{R}^N} V(x)|u_k|^2dx$ as $k \to \infty$, for which we shall consider the following constraint variational problem without the trap:

$$\bar{I}(M) := \inf_{\{u \in H^1(\mathbb{R}^N), \|u\|_2^2 = 1\}} \bar{E}_M(u), \quad N \geq 1,$$

(1.12)

where $\bar{E}_M(u)$ is defined by

$$\bar{E}_M(u) := \int_{\mathbb{R}^N} |\nabla u|^2dx - \frac{2M^{\frac{p+1}{2}}}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}\left|\frac{x}{b}\right|dx.$$  

(1.13)

By deriving the energy estimates between $\bar{I}(M_k)$ and $I(M_k)$ as $M_k \to \infty$, we shall verify that $I(M_k) - \bar{I}(M_k) \to 0$ as $M_k \to \infty$, which further implies that $\int_{\mathbb{R}^N} V(x)|u_k|^2dx \to 0$ as $M_k \to \infty$. Furthermore, the $L^\infty$-uniform convergence of \((1.10)\), which is established by analyzing delicately the singular nonlinear term of $I(M_k)$, seems crucial in the further refined investigations on the minimizers of $I(M_k)$ as $M_k \to \infty$.

This paper is organized as follows. Section 2 is devoted to the refined energy estimates of $I(M)$ as $M \to \infty$, based on which we shall complete in Section 3 the proof of Theorem 1.1 on the limit behavior of minimizers for $I(M)$ as $M \to \infty$. 

4
2 Energy estimates of $I(M)$

This section is devoted to establishing the energy estimates of $I(M)$ as $M \to \infty$ by analyzing the energy of $\tilde{I}(M)$ defined in (1.12). Employing the concentration-compactness principle, one can deduce that $\tilde{I}(M)$ admits minimizers for any $M \in (0, \infty)$, see, e.g., [6] [26] [27]. Moreover, without loss of generality, we may consider positive minimizers of $\tilde{I}(M)$ defined in (1.12). We start with the following energy estimates of $\tilde{I}(M)$.

Lemma 2.1. Under the assumption (1.3), assume that $V(x)$ satisfies (V), and let $\tilde{u}_M$ be a positive minimizer of $\tilde{I}(M)$. Then for any $M > 0$,

$$\tilde{I}(M) = -\lambda_0 \left( \frac{M}{a^*} \right)^{\frac{2(p-1)}{2(p-1)-2p}},$$

and

$$\tilde{u}_M(x) = \frac{1}{\sqrt{a^*}} \tilde{\alpha}_M^\frac{N}{2} w(\tilde{\alpha}_M x),$$

where $\tilde{\alpha}_M := \left( \frac{M}{a^*} \right)^{\frac{p-1}{2(p-1)-2p}} > 0$, $\lambda_0 := -\frac{N(p+1)+2b-1}{2(p+1)-N(p-1)-20b} > 0$ and $a^* := \|w\|^2_2$. Here $w > 0$ is the unique positive solution of the equation (1.7).

Proof. Assume that $\tilde{u}_M$ is a positive minimizer of $\tilde{I}(M)$ and $\tilde{u}_1$ is a positive minimizer of $\tilde{I}(1)$. We claim that for any $M > 0$,

$$\tilde{I}(M) = M^{\frac{2(p-1)}{4-N(p-1)-2b}} \tilde{I}(1)$$

and

$$\tilde{u}_M(x) = \alpha_M^{-\frac{N}{2}} \tilde{u}_1(\alpha_M x),$$

where $\alpha_M := M^{\frac{p-1}{4-N(p-1)-2b}} > 0$. Indeed, setting $\tilde{w}_1(x) := \frac{N}{2} \tilde{u}_M(\alpha_M^{-1} x)$, one can deduce from (1.12) that

$$\tilde{I}(M) = \hat{E}_M(\tilde{u}_M) = \int_{\mathbb{R}^N} |\nabla \tilde{u}_M|^2 dx - \frac{2M^{\frac{p+1}{2}}}{p+1} \int_{\mathbb{R}^N} \frac{|\tilde{u}_M|^{p+1}}{|x|^b} dx$$

$$= \alpha_M^2 \int_{\mathbb{R}^N} |\nabla \tilde{w}_1|^2 dx - \frac{2M^{\frac{p+1}{2}}}{p+1} \cdot \alpha_M^{-\frac{N(p+1)}{2}} \cdot \alpha_M^{-\frac{N}{2}} \cdot \alpha_M^b \int_{\mathbb{R}^N} \frac{|\tilde{w}_1|^{p+1}}{|x|^b} dx$$

$$= M^{\frac{2(p-1)}{4-N(p-1)-2b}} \left[ \int_{\mathbb{R}^N} |\nabla \tilde{w}_1|^2 dx - \frac{2}{p+1} \int_{\mathbb{R}^N} \frac{|\tilde{w}_1|^{p+1}}{|x|^b} dx \right]$$

$$\geq M^{\frac{2(p-1)}{4-N(p-1)-2b}} \tilde{I}(1).$$

(2.4)

Similarly, setting $\tilde{w}_M(x) := \alpha_M^{-\frac{N}{2}} \tilde{u}_1(\alpha_M x)$ as a test function of $\tilde{I}(M)$, one can get that

$$\tilde{I}(M) \leq \hat{E}_M(\tilde{w}_M) = M^{\frac{2(p-1)}{4-N(p-1)-2b}} \tilde{I}(1).$$

(2.5)

Following (2.4) and (2.5), we conclude that the first equality of (2.3) holds. Furthermore, one can check that $\tilde{w}_1$ is a minimizer of $\tilde{I}(1)$ and $\tilde{w}_M$ is a minimizer of $\tilde{I}(M)$. This proves the second equality of (2.3). Therefore, the claim (2.3) holds true.
We next prove that for any $M > 0$,

$$\tilde{I}(1) = -\lambda_0(a^*)^{-\frac{2(b-1)}{4-N(p-1)-2b}}, \text{ where } \lambda_0 := -\frac{N(p-1)+2b-4}{2(p+1)-N(p-1)-2b} > 0, \quad (2.6)$$

and

$$\tilde{u}_1(x) = (a^*)^{-\frac{2-b}{4-N(p-1)-2b}} w((a^*)^{-\frac{p-1}{4-N(p-1)-2b}} x). \quad (2.7)$$

Consider a test function $0 < \tilde{v}_0 \in H^1(\mathbb{R}^N)$ satisfying $||\tilde{v}_0||^2_2 = 1$. Set $\tilde{v}_\epsilon(x) := \epsilon^\frac{1}{2} \tilde{v}_0(\epsilon x)$, where $\epsilon > 0$ is small enough. One can get that for sufficiently small $\epsilon > 0$,

$$\tilde{I}(1) \leq \tilde{E}_1(\tilde{v}_\epsilon) = \int_{\mathbb{R}^N} |\nabla \tilde{v}_\epsilon|^2 \, dx - \frac{2}{p+1} \int_{\mathbb{R}^N} \frac{(|\tilde{v}_\epsilon|^{p+1})_x}{|x|^b} \, dx
\leq \epsilon^2 \int_{\mathbb{R}^N} |\nabla \tilde{v}_0|^2 \, dx - \frac{2\epsilon^{N(p-1)+b}}{p+1} \int_{\mathbb{R}^N} \frac{(|\tilde{v}_0|^{p+1})_x}{|x|^b} \, dx < 0, \quad (2.8)$$

due to the assumption $\text{(1.3)}$. Let $\tilde{u}_1 > 0$ be a positive minimizer of $\tilde{I}(1)$. Then $\tilde{u}_1$ satisfies the following Euler-Lagrange equation

$$-\Delta \tilde{u}_1(x) = \tilde{\mu}_1 \tilde{u}_1(x) + \frac{\tilde{u}_1^p(x)}{|x|^b} \text{ in } \mathbb{R}^N, \quad (2.9)$$

where $\tilde{\mu}_1 \in \mathbb{R}$ is the Lagrangian multiplier associated to $\tilde{u}_1$. Applying $(2.8)$ and $(2.9)$, we get that

$$\tilde{\mu}_1 = \int_{\mathbb{R}^N} |\nabla \tilde{u}_1|^2 \, dx - \int_{\mathbb{R}^N} \frac{|\tilde{u}_1|^{p+1}}{|x|^b} \, dx
= \tilde{I}(1) - \frac{p-1}{p+1} \int_{\mathbb{R}} \frac{|\tilde{u}_1|^{p+1}}{|x|^b} \, dx < 0. \quad (2.10)$$

Since $w > 0$ is the unique positive solution of $(1.7)$, one can conclude from $(1.7)$ and $(2.9)$ that

$$\tilde{u}_1(x) = (-\tilde{\mu}_1)^{\frac{2-b}{N(p-1)}} w((-\tilde{\mu}_1)^{\frac{1}{2}} x),$$

where $-\tilde{\mu}_1 < 0$ holds by $(2.10)$. Moreover, since

$$1 = ||\tilde{u}_1||^2_2 = (-\tilde{\mu}_1)^{\frac{4-2b-2N(p-1)}{2(p+1)}} ||w||^2_2 = (-\tilde{\mu}_1)^{\frac{4-2b-N(p-1)}{2(p+1)}} a^*,$$

one can derive that

$$\tilde{\mu}_1 = -(a^*)^{2-2b-2N(p-1)} < 0 \text{ and } \tilde{u}_1(x) = (a^*)^{\frac{2-b}{N(p-1)}} w((a^*)^{\frac{1}{2}} x) > 0.$$

Hence, $(2.7)$ is proved. On the other hand, substituting $(2.7)$ and $(2.8)$ into $(1.13)$, we
get that
\[
\tilde{I}(1) = \tilde{E}_1(\tilde{u}_1) = \int_{\mathbb{R}^N} |\nabla \tilde{u}_1|^2 \, dx - \frac{2}{p+1} \int_{\mathbb{R}^N} |\tilde{u}_1|^{p+1} \, dx
\]
\[
= (a^*)^{\frac{2(p-2)}{p-2-N(p-1)}} \cdot (a^*)^{\frac{2(1-p)}{p-2-N(p-1)}} \cdot (a^*)^{\frac{N(p-1)}{p-2-N(p-1)}} \int_{\mathbb{R}^N} |\nabla w|^2 \, dx
\]
\[
= (a^*)^{\frac{2(p-1)}{p-2-N(p-1)}} \cdot (a^*)^{\frac{2(p-1)}{2p+1-N(p-1)}} \cdot \frac{N(p-1) - 2b}{2(p+1) - N(p-1) - 2b} \cdot a^*
\]
\[
= -\lambda_0(a^*)^{-\frac{2(p-1)}{4-N(p-1)-2b}},
\]
which thus implies that (2.6) holds.

We finally conclude from (2.3)–(2.7) that for any \( M > 0 \),
\[
\tilde{I}(M) = M^{\frac{2(p-1)}{4-N(p-1)-2b}} \cdot (a^*)^{-\frac{2(p-1)}{4-N(p-1)-2b}} \cdot (-\lambda_0) = -\lambda_0 \left( \frac{M}{a^*} \right)^{\frac{2(p-1)}{4-N(p-1)-2b}}
\]
and
\[
\tilde{u}_M(x) = \tilde{a}_M \tilde{u}_1(\alpha_M x) = \alpha_M^{\frac{N}{4-N(p-1)-2b}} w \left( a^* \right)^{\frac{2b}{4-N(p-1)-2b}} \cdot \alpha_M x
\]
\[
= M^{\frac{4-N(p-1)-2b}{4-N(p-1)-2b}} \cdot \left( a^* \right)^{\frac{2b}{4-N(p-1)-2b}} \cdot w \left( \frac{M}{a^*} \right)^{\frac{4-N(p-1)-2b}{4-N(p-1)-2b}}
\]
\[
= \frac{1}{\sqrt{a^*} \tilde{a}_M} \tilde{a}_M w(\tilde{a}_M x).
\]
Therefore, the proof of Lemma 2.1 is completed.

Applying Lemma 2.1 we now establish the energy estimates of \( I(M) \).

**Lemma 2.2.** Under the assumption \( (I.3) \), assume that \( V(x) \) satisfies \( (V) \). Then we have
\[
\lim_{M \to \infty} \left( \frac{I(M)}{M^{\frac{2(p-1)}{4-N(p-1)-2b}}} \right) = -\lambda_0,
\]
where \( a^* := \|w\|_2^2 \), \( \lambda_0 := -\frac{N(p-1) + 2b - 4}{2(p+1) - N(p-1) - 2b} > 0 \), and \( w > 0 \) is the unique positive solution of the equation (1.7).

**Proof.** We first establish the lower bound of \( I(M) \) as \( M \to \infty \). Let \( u_M > 0 \) be a
Positive minimizer of $I(M)$. Under the assumption $(V)$, we get from (1.12) that

$$I(M) = \int_{\mathbb{R}^N} |\nabla u_M|^2 dx + \int_{\mathbb{R}^N} V(x) u_M^2(x) dx - \frac{2M^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} |u_M|^{p+1} |x|^b dx$$

$$\geq \int_{\mathbb{R}^N} |\nabla u_M|^2 dx - \frac{2M^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} |u_M|^{p+1} |x|^b dx$$

(2.12)

$$\geq \overline{I}(M) = -\lambda_0 \left( \frac{M}{a^*} \right)^{\frac{2(p-1)}{(p-1)^2-2b}}$$ as $M \to \infty$.

where $\lambda_0 > 0$ is as in Lemma 2.1. This thus implies the lower bound of $I(M)$ as $M \to \infty$.

We next estimate the upper bound of $I(M)$ as $M \to \infty$. Define

$$u_\tau(x) := A_\tau \frac{w(\tau x)}{\|w\|_2} \varphi(x), \quad \tau > 0,$$

(2.13)

where $0 \leq \varphi(x) \in C^\infty(\mathbb{R}^N)$ is a cut-off function satisfying

$$\varphi(x) = \begin{cases} 1, \quad |x| \leq 1; \\ 0, \quad |x| \geq 2, \end{cases}$$

$w > 0$ is the unique positive solution of (1.7), and $A_\tau > 0$ is a suitable constant such that $\|u_\tau(x)\|_2^2 = 1$. Applying the exponential decay of $w$ in (1.9), one can check that as $\tau \to \infty$,

$$1 \leq A_\tau^2 = \frac{\|w\|_2^2}{\int_{\mathbb{R}^N} w^2(x)\varphi^2(\frac{x}{\tau}) dx} \leq 1 + \frac{\int_{B_2} w^2(x) dx}{\int_{B_2} w^2(x) dx} \leq 1 + Ce^{-2b\tau},$$

(2.14)

where $C > 0$ is independent of $\tau > 0$. Substituting (2.13) into (1.2) and applying the exponential decay of $w$ in (1.9) and the identity (1.8), direct calculations yield that as $\tau \to \infty$,

$$\int_{\mathbb{R}^N} |\nabla u_\tau|^2 dx = \frac{A_\tau^2}{\|w\|_2} \int_{\mathbb{R}^N} \left[ \varphi(\frac{x}{\tau}) \tau \nabla w(x) + w(x) \nabla \varphi(\frac{x}{\tau}) \right]^2 dx$$

$$= \frac{A_\tau^2}{\|w\|_2} \int_{\mathbb{R}^N} \left[ \tau^2 \varphi^2(\frac{x}{\tau}) \nabla w(x) \right]^2 + w^2(x) \nabla \varphi(\frac{x}{\tau})^2$$

$$+ 2\tau \nabla w(x) \varphi(\frac{x}{\tau}) \nabla \varphi(\frac{x}{\tau}) w(x) \right] dx$$

$$\leq (1 + Ce^{-2b\tau}) \frac{N(p-1) + 2b}{2(p+1) - N(p-1) - 2b} \tau^2,$$
The key of proving (3.1) is to verify that

\[ I(M) - \tilde{I}(M) \to 0 \quad \text{as} \quad M \to \infty. \]  

**Proof.** The key of proving (3.1) is to verify that

\[ I(M) \leq E_M(u_\tau) \leq \frac{N(p-1)+2b}{2(p+1)-N(p-1)-2b} \tau^2 - \tau^{2(p-1)+b} \frac{4(M/a^*)^{p-1}}{2(p+1)-N(p-1)-2b} + V(0) + C e^{-\delta \sqrt{\tau}}. \]

It then follows from (2.15)–(2.17) that as \( \tau \to \infty \),

\[ \int_{\mathbb{R}^N} |u_\tau|^{p+1} dx = \frac{A^{p+1}_2}{\|w\|_2^{p+1}} \int_{\mathbb{R}^N} |w(x)\varphi(x)|^{p+1} dx = A^{p+1}_2 \tau^{N(p-1)+b} \int_{\mathbb{R}^N} |w(x)\varphi(x)|^{p+1} dx \geq A^{p+1}_2 \tau^{N(p-1)+b} \frac{2(p+1)}{2(p+1)-N(p-1)-2b} (a^*)^{\frac{1}{p+1}} - Ce^{-\delta \sqrt{\tau}}. \]

Setting \( \tau = \left( \frac{M}{a^*} \right)^{\frac{p-1}{1-N(p-1)-2b}} \) into the above estimate, it then gives that as \( M \to \infty \),

\[ \int_{\mathbb{R}^N} V(x)u_\tau^2(x)dx = \frac{A^2}{\|w\|_2^2} \int_{\mathbb{R}^N} V(\frac{x}{\tau})w^2(\frac{x}{\tau})dx \]

and

\[ \int_{\mathbb{R}^N} \frac{|u_\tau|^{p+1}}{|x|^b} dx = \frac{A^{p+1}_2}{\|w\|_2^{p+1}} \int_{\mathbb{R}^N} \frac{|w(\tau x)\varphi(\frac{x}{\tau})|^{p+1}}{|x|^b} dx \geq A^{p+1}_2 \tau^{N(p-1)+b} \frac{2(p+1)}{2(p+1)-N(p-1)-2b} (a^*)^{\frac{1}{p+1}} - Ce^{-\delta \sqrt{\tau}}. \]

Setting \( \tau = \left( \frac{M}{a^*} \right)^{\frac{p-1}{1-N(p-1)-2b}} \) into the above estimate, it then gives that as \( M \to \infty \),

\[ I(M) \leq \tilde{I}(M) \leq \frac{N(p-1)+2b}{2(p+1)-N(p-1)-2b} \tau^2 - \tau^{2(p-1)+b} \frac{4(M/a^*)^{p-1}}{2(p+1)-N(p-1)-2b} + V(0) + C e^{-\delta \sqrt{\tau}}. \]

where \( \lambda_0 := -\frac{N(p-1)+2b-a^*}{2(p+1)-N(p-1)-2b} > 0 \). Thus, (3.11) follows from (2.12) and (2.18), which completes the proof of Lemma 3.2

#### 3 Proof of Theorem 1.1

In this section, we shall complete the proof of Theorem 1.1 on the limit behavior of minimizers for (1.1) by the blow-up analysis. We first establish the following lemma.

**Lemma 3.1.** Under the assumption (1.3), assume that \( V(x) \) satisfies (V), and let \( u_M \) be a positive minimizer of \( I(M) \). Then we have

\[ \int_{\mathbb{R}^N} V(x)u_M^2(x)dx \to 0 \quad \text{as} \quad M \to \infty. \]  

**Proof.**
Indeed, if (3.2) holds, then one derive from (1.12) and (1.13) that
\[
\int_{\mathbb{R}^N} V(x) u_M^2(x) dx = I(M) - \tilde{E}_M(u_M) \leq I(M) - \tilde{I}(M) \to 0 \text{ as } M \to \infty, \tag{3.3}
\]
which thus implies that (3.1) holds.

We now prove (3.2). By Lemma 2.1, we deduce from (2.18) that
\[
I(M) \leq \tilde{I}(M) + o(1) \text{ as } M \to \infty. \tag{3.4}
\]
On the other hand, it follows from (1.12) and (1.13) that
\[
I(M) - \tilde{I}(M) \geq E_M(u_M) - \tilde{E}_M(u_M) = \int_{\mathbb{R}^N} V(x) u_M^2(x) dx \geq 0 \text{ as } M \to \infty. \tag{3.5}
\]
Therefore, the estimate (3.2) now follows from (3.4) and (3.5), and we are done. □

Motivated by [20,22,32], we next establish the following lemma.

**Lemma 3.2.** Under the assumption (1.3), assume that
\[
V(x) \text{ satisfies } (V),\]
and let \( u_k \) be a positive minimizer of \( I(M_k) \), where \( M_k \to \infty \) as \( k \to \infty \).
Define
\[
w_k(x) := \epsilon_N^{\frac{N}{2}} u_k(\epsilon_k x), \tag{3.6}
\]
where \( \epsilon_k := \left( \frac{M_k}{a^*} \right)^{-\frac{p-1}{2-N(p-1)-2b}} \to 0 \text{ as } k \to \infty. \)
Then there exists a subsequence, still denoted by \{w_k\}, of \{w_k\} such that
\[
w_k(x) \to \frac{w(x)}{\sqrt{a^*}} \text{ strongly in } H^1(\mathbb{R}^N) \text{ as } k \to \infty, \tag{3.7}
\]
where \( a^* := \|w\|_2^2 \) and \( w > 0 \) is the unique positive solution of (1.7).

**Proof.** We first prove that there exist some positive constants \( C_1, C_2, C_1', C_2' \), which are independent of \( k \), such that as \( k \to \infty \),
\[
0 < C_1 \leq \| \nabla w_k \|_2^2 \leq C_2 \text{ and } 0 < C_1' \leq \int_{\mathbb{R}^N} \frac{|w_k|^{p+1}}{|x|^b} dx \leq C_2'. \tag{3.8}
\]
Indeed, using Lemmas 2.2 and 3.1, we deduce from (1.2) and (3.6) that as \( k \to \infty \),
\[
\epsilon_k^2 I(M_k) = \epsilon_k^2 \left( \int_{\mathbb{R}^N} |\nabla u_k|^2 dx + \int_{\mathbb{R}^N} V(x) u_k^2 dx - \frac{2M_k^{p-1}}{p+1} \int_{\mathbb{R}^N} \frac{|u_k|^{p+1}}{|x|^b} dx \right)
\]
\[
= \left( \int_{\mathbb{R}^N} |\nabla w_k|^2 dx - \frac{2(a^*)^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} \frac{|w_k|^{p+1}}{|x|^b} dx + o(\epsilon_k^2) \right) \to -\lambda_0 < 0, \tag{3.9}
\]
which implies that
\[
\frac{2(a^*)^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} \frac{|w_k|^{p+1}}{|x|^b} dx \to 1 \text{ as } k \to \infty. \tag{3.10}
\]
By contradiction, assume that \( \|\nabla w_k\|_2^2 \to \infty \) as \( k \to \infty \). Define \( \gamma_k^2 := \|\nabla w_k\|_2^2 > 0 \) and \( v_k(x) = \gamma_k^{-2} w_k(\gamma_k^{-1} x) \), so that \( \gamma_k^2 \to \infty \) as \( k \to \infty \). It then follows that \( \|v_k\|_2^2 = 1 \) and \( \|\nabla v_k\|_2^2 = 1 \) for all \( k \geq 1 \). Further, we deduce from the GN inequality \([1.5]\) that

\[
\int_{\mathbb{R}^N} \frac{|v_k|^{p+1}}{|x|^b} dx \leq C_{GN}^{-1} \|\nabla v_k\|_{2}^{(N(p-1)+b)/2-\frac{N(p-1)-b}{2}} = C_{GN}^{-1}, \tag{3.11}
\]

where \( C_{GN} > 0 \) is given in \([1.6]\). Under the assumption \([1.3]\), since \( \|v_k\|_2^2 = \|\nabla v_k\|_2^2 = 1 \) for all \( k \geq 1 \), it follows from \((3.11)\) that

\[
\frac{2(a^*)^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} \frac{|v_k|^{p+1}}{|x|^b} dx = \frac{2(a^*)^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} \frac{|v_k|^{p+1}}{|x|^b} dx \int_{\mathbb{R}^N} \frac{|\nabla v_k|_2^2}{2} dx \gamma_k^{-1} \rightarrow 0 \quad \text{as} \quad k \to \infty,
\]

which however contradicts to \((3.10)\). Hence, we conclude that \( \|\nabla w_k\|_2^2 \leq C_2 \) holds uniformly as \( k \to \infty \). Applying the GN inequality \([1.5]\) and the fact that \( \|w_k\|_2^2 = 1 \), we deduce from above that \( \int_{\mathbb{R}^N} \frac{|w_k|^{p+1}}{|x|^b} dx \leq C_2 \) holds uniformly as \( k \to \infty \). On the other hand, one can obtain from \((3.9)\) that \( \int_{\mathbb{R}^N} \frac{|w_k|^{p+1}}{|x|^b} dx \geq C_1' \) holds uniformly as \( k \to \infty \), together with \([1.5]\), which then imply that \( \|\nabla w_k\|_2^2 \geq C_1 \) holds uniformly as \( k \to \infty \). We therefore conclude that \((3.8)\) holds.

From \((3.6)\) and \((3.8)\), we deduce that \( w_k \) is bounded uniformly in \( H^1(\mathbb{R}^N) \), which implies that there exist a subsequence, still denoted by \( \{w_k\} \), of \( \{w_k\} \) and \( 0 \leq w_0 \in H^1(\mathbb{R}^N) \) such that

\[
w_k \rightharpoonup w_0 \geq 0 \quad \text{weakly in} \quad H^1(\mathbb{R}^N) \quad \text{as} \quad k \to \infty. \tag{3.13}
\]

We now prove that \( w_0 \neq 0 \). Motivated by \([2,15]\), we first claim that

\[
\int_{\mathbb{R}^N} |x|^{-b} w_k^{p+1} dx \to \int_{\mathbb{R}^N} |x|^{-b} w_0^{p+1} dx \quad \text{as} \quad k \to \infty. \tag{3.14}
\]

Actually, we have

\[
\begin{align*}
|\int_{\mathbb{R}^N} |x|^{-b} w_k^{p+1} dx - \int_{\mathbb{R}^N} |x|^{-b} w_0^{p+1} dx| & \leq \int_{\mathbb{R}^N} |x|^{-b} |w_k^{p+1} - w_0^{p+1}| dx \\
& = \int_{B_R} |x|^{-b} |w_k^{p+1} - w_0^{p+1}| dx + \int_{\mathbb{R}^N \setminus B_R} |x|^{-b} |w_k^{p+1} - w_0^{p+1}| dx := A_k + B_k,
\end{align*}
\]

where \( R > 0 \) is arbitrary. Under the assumption \([1.3]\), we have \( 1 < p < 1 + \frac{4-2b}{N} \) and

\[
1 + \frac{4-2b}{N} < 1 + \frac{4-2b}{N-2} \quad \text{if} \quad N \geq 3.
\]
By Hölder inequality, we then have

\[ A_k = \int_{B_R} |x|^{-b} |w_k^{p+1} - w_0^{p+1}| \, dx \]

\[ \leq \left( \int_{B_R} |x|^{-br} \, dx \right)^{\frac{1}{r}} \left( \int_{B_R} |w_k^{p+1} - w_0^{p+1}|^t \, dx \right)^{\frac{1}{t}} \]

\[ \leq C \left( \int_{B_R} |w_k^{p+1} - w_0^{p+1}|^t \, dx \right)^{\frac{1}{t}}, \tag{3.15} \]

where \( \frac{1}{r} + \frac{1}{t} = 1 \), \( t > 1 \) and \( r > 1 \) satisfies \( \frac{b}{N} < \frac{1}{r} \). Note that \( \frac{1}{t} = 1 - \frac{1}{r} < \frac{N-b}{N} \).

Consider \( p_1 > 0 \) and \( q_1 > 0 \) satisfying

\[ \frac{p}{p_1} + \frac{1}{q_1} = 1 \quad \text{at} \quad \frac{N-b}{N}. \tag{3.16} \]

which then yields from (3.15) that

\[ A_k \leq C \left( \int_{B_R} |w_k^{p+1} - w_0^{p+1}|^t \, dx \right)^{\frac{1}{t}} \]

\[ \leq C \left( \|w_k\|_{L^p(B_R)}^p + \|w_0\|_{L^p(B_R)}^p \right) \|w_k - w_0\|_{L^q(B_R)}. \tag{3.17} \]

Similar to [2, Theorem 1.5], choose suitable constants \( p_1 > 0 \) and \( q_1 > 0 \) satisfying (3.16), so that

\[ \|w_k\|_{L^p(B_R)}^p + \|w_0\|_{L^p(B_R)}^p \leq C \quad \text{and} \quad \|w_k - w_0\|_{L^q(B_R)} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty, \]

where \( C > 0 \) is independent of \( k > 0 \). This further implies from (3.17) that

\[ A_k \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty, \tag{3.18} \]

On the other hand, for any \( \epsilon > 0 \), there exists \( R \geq \epsilon^{\frac{1}{N}} \) such that

\[ B_k = \int_{\mathbb{R}^N \setminus B_R} |x|^{-b} |w_k^{p+1} - w_0^{p+1}| \, dx \]

\[ \leq \epsilon \int_{\mathbb{R}^N \setminus B_R} (w_k^{p+1} + w_0^{p+1}) \, dx \leq C \epsilon \quad \text{as} \quad k \rightarrow \infty, \]

due to Sobolev’s embedding theorem and the uniform boundedness of \( w_k \) in \( H^1(\mathbb{R}^N) \).

Since \( \epsilon > 0 \) is arbitrary, we conclude from above that the claim (3.14) holds true.

Following (3.13) and (3.14), one can deduce that \( w_0 \neq 0 \).

Next, we prove that \( \|w_0\|_2^2 = 1 \). By contradiction, we assume that \( \|w_0\|_2^2 = l \), where \( l \in (0, 1) \). Set \( w_l := \sqrt{l} w_0 \). By (3.13), we may assume that \( w_k \rightarrow w_0 \) a.e. in \( \mathbb{R}^N \) as \( k \rightarrow \infty \). Using the Brézis-Lieb lemma, we obtain that

\[ \|\nabla w_k\|_2^2 = \|\nabla w_l\|_2^2 + \|\nabla (w_k - w_l)\|_2^2 + o(1) \quad \text{as} \quad k \rightarrow \infty. \tag{3.19} \]
From \((2.11), (3.1)\), \((3.3), (3.14)\) and \((3.19)\), we derive that as \(k \to \infty\),
\[-\lambda_0 = \lim_{k \to \infty} \epsilon_k^2 I(M_k)\]
\[= \lim_{k \to \infty} \epsilon_k^2 \left( \int_{\mathbb{R}^N} |\nabla u_k|^2 dx + \int_{\mathbb{R}^N} V(x) u_k^2 dx - \frac{2M_k}{p+1} \int_{\mathbb{R}^N} |u_k|^{p+1} dx \right)\]
\[\geq \lim_{k \to \infty} \left( \int_{\mathbb{R}^N} |\nabla w_k|^2 dx - \frac{2(a^*)^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} |w_k|^{p+1} dx \right)\]
\[= \int_{\mathbb{R}^N} |\nabla w_0|^2 dx - \lim_{k \to \infty} \int_{\mathbb{R}^N} |\nabla (w_k - w_0)|^2 dx - \frac{2(a^*)^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} |w_0|^{p+1} dx \]
\[\geq \int_{\mathbb{R}^N} |\nabla w_0|^2 dx - \frac{2(a^*)^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} |w_0|^{p+1} dx\]
\[> l\bar{I}(a^*) = -l\lambda_0 < 0,
\]
which is a contradiction. Hence, \(\|w_0\|_2^2 = 1\) holds true.

Since \(\|w_k\|_2^2 = \|w_0\|_2^2 = 1\), we have
\[w_k(x) \to w_0(x) \quad \text{strongly in } L^2(\mathbb{R}^N) \quad \text{as } k \to \infty. \tag{3.21}\]

By the weak lower semicontinuity, \((2.1)\) and \((3.1)\), we then derive from \((3.9)\) that
\[\nabla w_k(x) \to \nabla w_0(x) \quad \text{strongly in } L^2(\mathbb{R}^N) \quad \text{as } k \to \infty. \tag{3.22}\]

Note from \((3.9)\) that \(\{w_k\}\) is a minimizing sequence of \(\bar{I}(a^*)\). One then deduces from \((3.14)\) and \((3.22)\) that \(w_0\) is a minimizer of \(\bar{I}(a^*)\). By \((2.2)\), we obtain that \(w_0(x) = \frac{w(x)}{\sqrt{a^*}}\).

Combining \((3.21)\) and \((3.22)\), we obtain that
\[w_k(x) \to w_0(x) = \frac{w(x)}{\sqrt{a^*}} \quad \text{strongly in } H^1(\mathbb{R}^N) \quad \text{as } k \to \infty, \tag{3.23}\]
which gives \((3.7)\). The lemma is thus proved. \(\square\)

Applying above lemmas, we are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1**: 1. We first prove the exponential decay \((1.11)\). Let \(u_k > 0\) be a minimizer of \(I(M_k)\), and consider the sequence \(\{w_k\}\) defined in Lemma 3.2 where \(M_k \to \infty\) as \(k \to \infty\). We claim that there exists a subsequence, still denoted by \(\{w_k\}\), of \(\{w_k\}\) such that
\[w_k(x) \to 0 \quad \text{as } |x| \to \infty \quad \text{uniformly for sufficiently large } k > 0. \tag{3.24}\]

Indeed, one can derive from \((3.7)\) that for any \(2 \leq \alpha < 2^*\),
\[\int_{|x| \geq \gamma} |w_k|^\alpha dx \to 0 \quad \text{as } \gamma \to \infty \quad \text{uniformly for sufficiently large } k > 0. \tag{3.25}\]
On the other hand, it follows from (1.4) and (3.6) that \( w_k \) satisfies the following equation

\[
- \Delta w_k + \epsilon^2_k V(\epsilon_k x) w_k - (a^*) \frac{p-1}{p} w^{p-1}_k \frac{w_k}{|x|^b} = \mu_k \epsilon^2_k w_k \quad \text{in} \quad \mathbb{R}^N, \tag{3.26}
\]

where \( \mu_k \in \mathbb{R} \) is the Lagrange multiplier. Applying (1.8), (3.6) and (3.14), we deduce from Lemma 2.2 that

\[
\epsilon^2_k \mu_k = \epsilon^2_k I(M_k) - \frac{p-1}{p+1} M_k^{\frac{p+1}{2}} \int_{\mathbb{R}^N} \frac{|u_k|^{p+1}}{|x|^b} dx \leq \epsilon^2_k I(M_k) - \frac{p-1}{p+1} (a^*)^{\frac{p+1}{2}} \int_{\mathbb{R}^N} \frac{|w_k|^{p+1}}{|x|^b} dx \tag{3.27}
\]

\[-\Delta w_k - c(x) w_k \leq 0 \quad \text{in} \quad \mathbb{R}^N, \quad \text{where} \quad c(x) = (a^*)^{\frac{p+1}{2}} \frac{w^{p-1}_k(x)}{|x|^b}. \tag{3.28}\]

Furthermore, one can check from Hölder inequality that

\[c(x) \in L^t(\mathbb{R}^N), \quad \text{where} \quad t \in \left( \frac{2N}{N(p-1) + 2b}, \frac{2N}{(N-2)(p-1) + 2b} \right).\]

Applying De Giorgi-Nash-Moser theory (cf. [23, Theorem 4.1]) to (3.28), we deduce that

\[
\max_{B_1(\xi)} w_k(x) \leq C \left( \int_{B_2(\xi)} |w_k(x)|^\alpha dx \right)^{\frac{1}{\alpha}} \quad \text{for sufficiently large} \quad k > 0, \tag{3.29}
\]

where \( \xi \in \mathbb{R}^N \) is arbitrary, and \( C > 0 \) depends only on the bound of \( \|c(x)\|_{L^t(B_2(\xi))} \). Thus, (3.24) follows from (3.25) and (3.29).

Due to the smallness of \(|x|^{-b}\) for large \(|x| > 0\), we now derive from (3.24), (3.26) and (3.27) that there exists a sufficiently large constant \( R > 0 \), which is independent of \( k \), such that as \( k \to \infty \),

\[-\Delta w_k(x) + \theta w_k(x) \leq 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_R(0), \tag{3.30}\]

where \( 0 < \theta < 1 \) is independent of \( k \). By the comparison principle [5, Theorem 6.4.2], we obtain from (3.30) that as \( k \to \infty \),

\[w_k(x) \leq Ce^{-\sqrt{\theta}|x|} \quad \text{in} \quad \mathbb{R}^N \setminus B_R(0), \tag{3.31}\]

which implies the exponential decay (1.11) for \( w_k \) as \( k \to \infty \). Moreover, under the assumption \((V)\), since the term \(|x|^{-b}\) is small for large \(|x|\), applying the local elliptic estimate (cf. (3.15) in [17]) yields from (3.31) that as \( k \to \infty \),

\[|\nabla w_k(x)| \leq Ce^{-\theta|x|} \quad \text{for} \quad |x| > R,\]
which thus gives the exponential decay (1.11) for $\nabla w_k$ as $k \to \infty$. This proves (1.11).

2. We next prove that (1.10) holds true. Recall from Lemma 3.2 that 
\[
w_k(x) \to \frac{w(x)}{\sqrt{a^*}} \text{ in } H^1(\mathbb{R}^N) \text{ as } k \to \infty, \tag{3.32}
\]
where the convergence holds for the whole sequence $\{w_k(x)\}$, due to the uniqueness of $w(x) > 0$. Following (3.32), the $L^\infty$-uniform convergence (1.10) for the case $N = 1$ can be directly obtained by applying Sobolev’s embedding theorem $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.

We now prove the $L^\infty$-uniform convergence (1.10) for the case $N \geq 2$. Rewrite (3.26) as
\[
-\Delta w_k(x) = G_k(x) \text{ in } H^1(\mathbb{R}^N), \tag{3.33}
\]
where
\[
G_k(x) := \mu_k \epsilon_k^2 w_k - \epsilon_k^2 V(\epsilon_k x) w_k + (a^*)^{\frac{p-1}{2}} \frac{w_k^p}{|x|^b}.
\]
Since $w_k$ is bounded uniformly in $H^1(\mathbb{R}^N)$ as $k \to \infty$, we derive from (3.29) that $w_k$ is bounded uniformly in $L^\infty(\mathbb{R}^N)$.

Using Hölder inequality, we deduce that for any $R > 0$,
\[
\int_{B_R(0)} \frac{w_k^p(x)^r}{|x|^b} \, dx \leq \left( \int_{B_R(0)} \frac{1}{|x|^{br}} \, dx \right)^{\frac{1}{r'}} \left( \int_{B_R(0)} |w_k^p|^{rt'} \, dx \right)^{\frac{1}{t'}} < \infty, \tag{3.35}
\]
where $r \in (1, \frac{N}{b})$, $\frac{1}{r} + \frac{1}{r'} = 1$, and $t' = 1 + \max \left\{ \frac{N}{N-br'}, \frac{b}{pr} \right\}$. We then obtain from (3.35) that
\[
w_k^p(x)|x|^{-b} \in L^r_{\text{loc}}(\mathbb{R}^N) \text{ for any } r \in (1, \frac{N}{b}), \tag{3.36}
\]
which implies that $G_k(x)$ is bounded uniformly in $L^r_{\text{loc}}(\mathbb{R}^N)$. For any large $R > 0$, it thus follows from [17, Theorem 9.11] that
\[
\|w_k(x)\|_{W^{2,r}(B_R)} \leq C \left( \|w_k(x)\|_{L^r(B_{R+1})} + \|G_k(x)\|_{L^r(B_{R+1})} \right), \tag{3.37}
\]
where $C > 0$ is independent of $k > 0$ and $R > 0$. By the compactness of the embedding $W^{2,r}(B_R) \hookrightarrow L^\infty(B_R)$ for $2r > N$, cf. [17, Theorem 7.26], we conclude that there exists a subsequence, still denoted by $\{w_k\}$, of $\{w_k\}$ such that
\[
w_k(x) \to \tilde{w}_0(x) \text{ uniformly in } L^\infty(B_R) \text{ as } k \to \infty. \tag{3.38}
\]
Since $R > 0$ is arbitrary, we obtain from (3.32) that
\[
w_k(x) \to \frac{w(x)}{\sqrt{a^*}} \text{ uniformly in } L^\infty_{\text{loc}}(\mathbb{R}^N) \text{ as } k \to \infty. \tag{3.39}
\]
On the other hand, we deduce from (1.9) and (1.11) that for any \( \epsilon > 0 \), there exists a constant \( R_\epsilon > 0 \), independent of \( k > 0 \), such that

\[
|w_k(x)|, \quad \frac{|w(x)|}{\sqrt{a^*}} < \frac{\epsilon}{4} \quad \text{for any } |x| > R_\epsilon,
\]

which implies that

\[
\sup_{|x| > R_\epsilon} \left| w_k(x) - \frac{w(x)}{\sqrt{a^*}} \right| \leq \sup_{|x| > R_\epsilon} \left( |w_k(x)| + \frac{|w(x)|}{\sqrt{a^*}} \right) \leq \frac{\epsilon}{2}.
\]

Recall from (3.39) that for sufficiently large \( k > 0 \),

\[
\sup_{|x| \leq R_\epsilon} \left| w_k(x) - \frac{w(x)}{\sqrt{a^*}} \right| \leq \frac{\epsilon}{2}.
\]

We now conclude from above that the \( L^\infty \)-uniform convergence (1.10) holds true for all \( N \geq 2 \). The proof of Theorem 1.1 is therefore complete.

\[ \square \]

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