Cavity-QED interactions of two correlated atoms

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Abstract

We consider the resonant van der Waals (vdW) interaction between two correlated identical two-level atoms (at least one of which being excited) within the framework of macroscopic cavity quantum electrodynamics in linear, dispersing and absorbing media. The interaction of both atoms with the body-assisted electromagnetic field of the cavity is assumed to be strong. Our time-independent evaluation is based on an extended Jaynes–Cummings model. For a system prepared in a superposition of its dressed states, we derive the general form of the vdW forces, using a Lorentzian single-mode approximation. We demonstrate the applicability of this approach by considering the case of a planar cavity and showing the position dependence of Rabi oscillations. We also show that in the limiting case of weak coupling, our results reproduce the perturbative ones for the case where the field is initially in vacuum state while the atomic state is in a superposition of two correlated states sharing one excitation.

Keywords: correlated identical atoms, van der Waals interaction, cavity-QED, strong atom–field coupling

(Some figures may appear in colour only in the online journal)

1. Introduction

The van der Waals (vdW) force has been extensively studied for a long time. It has traditionally been derived from the interaction of two atoms with the continuous and unbounded spectrum of the electromagnetic field. With this assumption, the atom–field interaction can be treated as perturbation. Being evaluated as the position-dependent energy shift of two fluctuating dipoles interacting instantaneously, the vdW force in the weak coupling limit can be derived using time-independent perturbation theory [1].

London first found this force between two ground-state atoms for small interatomic separations which are much smaller than the atomic transition wavelengths [2]. Casimir and Polder formulated it taking into account retardation due to the finite speed of light [3]. Later the ideas have been extended to magnetic atoms [4] and also the effect of material environment was taken into account [5]. The theory had been generalised to identical atoms [6, 7] and also to atoms in excited states, where conflicting oscillatory [8–10] and non-oscillatory [11, 12] results, are predicted (for a review, see [1, 13]).

On the other side, the properties of atoms coupled to a discrete mode spectrum in a resonator-like structure are the subject of cavity quantum electrodynamics (cavity-QED). Historically, this field started with evaluating the effect of this coupling on the spontaneous emission probability of the atom, which was predicted to increase considerably due to the coupling with the cavity field [14].

For strong interaction of one excited atom with an initially empty, un-damped discrete mode field, a model proposed by Jaynes and Cummings (JC model) predicts an oscillatory exchange of energy between the atomic and field systems that...
occurs at a rate $\Omega$, known as the Rabi frequency [15]. The model involves calculating position-dependent eigenenergies for the energy eigenstates of the coupled system (known as dressed states). Generalising the static approach based on perturbation theory, the position-dependent part of these eigenvalues can be related to dispersion forces [16, 17]. If $\Omega$ is much larger than the rate at which photons escape from the cavity, then the interaction is strong and Rabi oscillations can be observed, otherwise the radiated photon may escape through the cavity walls before interacting with the atom [18].

In this article, using a extension of JC model to absorbing cavities, we study the vdW force between two identical two-level atoms with only two independent states respectively. Considering the simple case of two identical atoms the states that carry the excitation of two atoms can be assumed as a superposition of two states with only one atom being excited in each of them. Following [19], the frequency dependence of the atomic responses to the field is taken into account in our calculations where the spectral structure of the field is assumed to be Lorentzian.

The article is organised as follows: first, we introduce the appropriate Hamiltonian for the system under consideration using a suitable definition for the interaction Hamiltonian (section 2); then in section 3 we evaluate the vdW force by combining the dressed state method, used for dealing with the strong atom–field coupling regime, with the primary idea of Casimir and Polder. We also evaluate the force in the weak coupling regime using leading order perturbation theory and show that it is in agreement with the appropriate limits of our results. An example of a planar cavity is provided in section 4 to demonstrate our model and finally a summary is given in section 5.

2. Atom–field interaction

In order to derive the general form for the strong vdW force, it is required to introduce the system Hamiltonian in the framework of macroscopic QED.

Consider two neutral atoms (collections of charged particles) A and B positioned at $\mathbf{r}_A$ and $\mathbf{r}_B$, within an arbitrary arrangement of dispersing and absorbing magneto-electric bodies. The atoms are considered to interact strongly with the body-assisted electromagnetic field. The total Hamiltonian describing such a system in the multipolar coupling scheme reads

$$
\hat{H} = \hat{H}_F + \sum_{A'=A,B} (\hat{H}_{A'} + \hat{H}_{A'}^F),
$$

where

$$
\hat{H}_{A'} = \sum_n E_{A'}^n |n_A\rangle \langle n_A|,
$$

is the internal Hamiltonian of atom $A'$, with $E_{A'}^n$ and $|n_A\rangle$ indicating the unperturbed eigenvalues and eigenstates, respectively. Considering the simple case of two identical two-level atoms with only two independent states $|0\rangle$ and $|1\rangle$, and discarding a state-independent constant, $\hat{H}_{A'}$ reduces to

$$
\hat{H}_{A'} = \frac{h}{2} \omega_{A'}^{10} \hat{\sigma}_{A'},
$$

with $\omega_{A'}^{10} = (E_{A'}^{01} - E_{A'}^{00})/h$ and $\hat{\sigma}_{A'} = |1_A\rangle \langle 1_A| - |0_A\rangle \langle 0_A|$. In order to introduce $\hat{H}_F$, we use bosonic operators $\hat{r}_A(\mathbf{r}, \omega)$ and $\hat{r}_B(\mathbf{r}, \omega)$ as the creation and annihilation operators of the body-assisted field excitation obeying the commutation relations

$$
[\hat{r}_A(\mathbf{r}, \omega), \hat{r}_B^\dagger(\mathbf{r}', \omega')] = \delta(\mathbf{r} - \mathbf{r}')\delta(\omega - \omega')\delta_{\lambda\lambda'}^c,
$$

and

$$
[\hat{r}_A(\mathbf{r}, \omega), \hat{r}_B(\mathbf{r}', \omega')] = [\hat{r}_A^\dagger(\mathbf{r}, \omega), \hat{r}_B^\dagger(\mathbf{r}', \omega')] = 0.
$$

The field Hamiltonian then reads

$$
\hat{H}_F = \hbar \sum_{\lambda=e,m} \int d^3r \int_0^\infty d\omega \, \hat{r}_\lambda(\mathbf{r}, \omega) \cdot \hat{r}_\lambda^\dagger(\mathbf{r}, \omega).
$$

Now, having defined the atom and field Hamiltonians, we need to specify their interaction. In the electric dipole approximation, the multipolar Hamiltonian describing the interaction of an atom with the field can be written as

$$
\hat{H}_{AF} = -\hat{d}_{A'} \cdot \mathbf{E}(\mathbf{r}_A),
$$

where for two-level atoms, the electric dipole moment is defined as

$$
\hat{d}_{A'} = d_{A'}^{\mu \nu} \hat{\sigma}_{A'}^\mu + \text{H.c.},
$$

with $d_{A'}^{\mu \nu} = \langle m_A | \hat{d}_{A'} | n_A\rangle$, $\hat{\sigma}_{A'}^\mu = |0_A\rangle \langle 1_A| \hat{\sigma}_{A'} |1_A\rangle$, and $\mathbf{E}$ being the electric field operator.

In order to facilitate giving the electric field as a linear combination of fundamental operators $\hat{r}_A$ and $\hat{r}_A^\dagger$, we employ the auxiliary tensors $G_\varepsilon$ and $G_\mu$ defined in terms of the Green’s tensor $G$ as [20]

$$
G_{\varepsilon}(\mathbf{r}, \mathbf{r}', \omega) = \frac{i}{\varepsilon} \frac{\hbar}{\pi \varepsilon_0} \text{Im} \varepsilon(\mathbf{r}', \omega) G(\mathbf{r}, \mathbf{r}', \omega),
$$

$$
G_{\mu}(\mathbf{r}, \mathbf{r}', \omega) = \frac{i}{\varepsilon} \frac{\hbar}{\pi \varepsilon_0} \text{Im} \mu^{-1}(\mathbf{r}', \omega) [\nabla' \times G(\mathbf{r}', \mathbf{r}, \omega)]',
$$

The Green’s tensor obeys the differential equation

$$
\nabla \times \mu^{-1}(\mathbf{r}, \omega) \nabla \times G(\mathbf{r}, \mathbf{r}', \omega) - \frac{\hbar^2}{\varepsilon_0^2} \varepsilon(\mathbf{r}, \omega) G(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}'),
$$

subject to the boundary conditions imposed by the particular arrangement of magneto-dielectric bodies at hand. It also satisfies a useful integral relation

$$
\sum_{\lambda=e,m} \int d^3s \, G(\mathbf{r}, \mathbf{s}, \omega) G_{\lambda}^T(\mathbf{r}', \mathbf{s}, \omega) = \frac{\hbar \mu_0}{\pi} \omega^2 \text{Im} G(\mathbf{r}, \mathbf{r}', \omega).
$$

In equations (9)–(12), $\varepsilon$ and $\mu$ are the position- and frequency-dependent relative electric permittivity and magnetic...
permeability of the surrounding media. With these definitions, the electric field can be written as follows

\[
\hat{E}(r) = \sum_{\lambda=r,m} \int_0^\infty d\omega \int d^3r' \mathbf{G}_\lambda(r, r', \omega) \cdot \hat{f}_\lambda(r', \omega) + \text{H.c.}
\]

(13)

Using the field expansion (13) together with equation (8), the interaction Hamiltonian (7) reads

\[
\hat{H}_{\text{IT}} = -\sum_{\lambda=r,m} \int_0^\infty d\omega \int d^3r (\mathbf{d}_\lambda^0 \hat{\sigma}_{\lambda}^+ + \mathbf{d}_\lambda^0 \hat{\sigma}_\lambda^-) \times \mathbf{G}_\lambda(r, r', \omega) \cdot \hat{f}_\lambda(r', \omega) + \text{H.c.}
\]

(14)

Following the suggestion proposed in [19], we introduce an additional set of position-dependent creation and annihilation operators as

\[
\hat{a}_\lambda'(\omega) = -\frac{1}{\hbar} \frac{\mathbf{g}_\lambda(\omega)}{\mathbf{g}_\lambda'(\omega)} \sum_{\lambda', r} \int d^3r \mathbf{d}_\lambda^0 \cdot \mathbf{G}_\lambda(r, r', \omega) \cdot \hat{f}_\lambda'(r, \omega),
\]

(15)

with

\[
g_{\lambda,\lambda'}(\omega) = \frac{\mu_0^2}{\hbar^2} \mathbf{d}_\lambda^0 \cdot \text{Im} \mathbf{G}(r, r', \omega) \cdot \mathbf{d}_\lambda', \quad \lambda', \lambda'' \in \{A, B\}
\]

(16)

defined as the atom–field coupling strength. Substitution of these into \(\hat{H}_{\text{IT}}\) and applying the rotating wave approximation yields

\[
\hat{H}_{\text{IT}} = \hbar \int_0^\infty d\omega \mathbf{g}_\lambda(\omega) \cdot \hat{a}_\lambda(\omega) \hat{\sigma}_\lambda^+ + \hat{\sigma}_\lambda^-] + \hat{a}_\lambda'(\omega, \omega')] = \frac{\mathbf{g}_{\lambda,\lambda'}^2(\omega)}{\mathbf{g}_{\lambda,\lambda'}(\omega) \mathbf{g}_{\lambda,\lambda'}(\omega)} \delta(\omega - \omega').
\]

(18)

We define single quantum excitation states

\[
|\mathbf{r}_\lambda, \omega\rangle = \hat{a}_\lambda(\omega)|0\rangle
\]

(19)

which are eigenstates of \(\hat{H}_\text{IT}\),

\[
\hat{H}_\text{IT}|\mathbf{r}_\lambda, \omega\rangle = \hbar \omega |\mathbf{r}_\lambda, \omega\rangle,
\]

(20)

and have the orthogonality relation

\[
\langle \mathbf{r}_\lambda', \omega |\mathbf{r}_\lambda, \omega'\rangle = \frac{\mathbf{g}_{\lambda,\lambda'}^2(\omega)}{\mathbf{g}_{\lambda,\lambda'}(\omega) \mathbf{g}_{\lambda,\lambda'}(\omega)} \delta(\omega - \omega'),
\]

(21)

as is evident from equation (18).

These states are orthogonal with respect to frequency \(\omega\) but not position \(r\). This reflects the fact that there is a non-zero probability for the photons emitted by atom positioned at \(\mathbf{r}_A\) to be reabsorbed by an atom at a different position \(\mathbf{r}_B\).

3. Van der Waals forces

In agreement with the pioneering idea of Casimir and Polder [3], two atoms that are weakly coupled to the body-assisted electromagnetic field experience a vdW force deduced from an associated vdW potential. This in turn can be interpreted as the position-dependent energy shift due to a perturbation of the energy of the initial atomic state, arising from the atom–field coupling, so that the force is given by

\[
F_{\text{vdW}}(\mathbf{r}_A, \mathbf{r}_B) = -\nabla_{\text{vdW}} \mathcal{U}(\mathbf{r}_A, \mathbf{r}_B),
\]

(22)

where \(\mathcal{U}(\mathbf{r}_A, \mathbf{r}_B)\) is the vdW potential between two atoms A and B. However, in the case of excited atoms that are strongly coupled to the electromagnetic field, the interaction energy may not be sufficiently small and hence the perturbative approach may no longer be valid. This is the scenario we consider here.

For the following, we restrict our attention to the strong-coupling vdW force between two identical two-level atoms. The paradigmatic model of an atom strongly interacting with an electromagnetic field is the JC model [15]. This consists of a single two-level atom (or molecule) interacting with a single near-resonant quantised mode of the electromagnetic field of an ideal cavity, and predicts an oscillatory atom–field excitation exchange.

Here we extend this model to a system comprising two electrically polarisable atoms interacting with one cavity mode of the body-assisted electromagnetic field, generalising the simplified picture of standing-wave modes in an ideal cavity.

3.1. Static approximation

The considered system consists of two identical atoms, which is to say they have the same energy spacing \(\omega_A^{10} = \omega_B^{10} = \omega_{10}\). One of the atoms is assumed to be excited, and hence, in a strong interaction with a single cavity mode. The model is based on the assumption that we may approximate the single cavity mode by a narrow \((\gamma_v \ll \omega_v)\) Lorentzian-type spectrum;

\[
g_{\lambda,\lambda'}^2(\omega) = g_{\lambda,\lambda'}^2(\omega_v) \frac{\gamma_v^2/4}{(\omega - \omega_v)^2 + \gamma_v^2/4}
\]

(23)

with the parameter \(\omega_v\), being the central frequency of the peak and \(\gamma_v\) describing the spectral width.

In the case of two identical atoms, the Hilbert space can be spanned by three coupled states \(|u_1\rangle \equiv \frac{1}{\sqrt{2}} (|1_A\rangle|0_B\rangle + |0_A\rangle|1_B\rangle)|0_v\rangle, |u_2\rangle \equiv |0_A\rangle|0_B\rangle|1_v\rangle\) and \(|u_3\rangle \equiv \frac{1}{\sqrt{2}} (|1_A\rangle|0_B\rangle - |0_A\rangle|1_B\rangle)|0_v\rangle\) with the excitation being shared between atoms \((|1_A\rangle)\) and the field \((|1_v\rangle)\). It is worth noting that the bare states are chosen according to the fact that the total number of excitations remains constant in the rotating wave approximation.
The state \( |l_\nu\rangle \) represents an excitation of a single-mode \( \nu \) and is defined as
\[
|l_\nu\rangle = \frac{\sqrt{\gamma_\nu}}{2\pi N} \sum_{A,B} \int_{-\infty}^{\infty} d\omega \frac{g_A^2(\omega_i)}{\sqrt{\omega - \omega_i}^2 + \gamma_\nu^2/4} |r_A; \omega\rangle,
\]
(24)

Here, \( \delta_\omega \) is regarded as the distance between two neighbouring modes, which in our single-mode assumption must always be much larger than the width \( \gamma_\nu \) of that mode. The quantity \( N \) is the normalisation factor. Note that according to the definition of Lorentzian functions, as \( \gamma_\nu \) tends to zero, \( g_A^2(\omega) \) tends to infinity in such a way that the product of this with a Lorentzian takes the following form:
\[
\lim_{\gamma_\nu \to 0} g_A^2(\omega) = \frac{1}{2\pi} g_A^2(\omega) \delta_\omega (\omega - \omega_i).
\]
(25)

This fact allows the states \( |l_\nu\rangle \) to become normalised to unity. The explicit normalisation factor \( N \) can be calculated using equations (21) and (23) together with (25), and is found to be
\[
N = g_A^2(\omega_i) + g_B^2(\omega_i) + 2g_{AB}^2(\omega_i).
\]
(26)

For the following it is appropriate to construct a dressed basis. Using equations (1) together with equation (3), (6) and (17) and the definitions (24) and (26), one can obtain the matrix representation of \( \hat{H} \) on the subspace spanned by three states \( |u_1\rangle, |u_2\rangle \) and \( |u_3\rangle \) as
\[
\hat{H} = \begin{pmatrix}
0 & \frac{\hbar}{2} \sqrt{\gamma_\nu \pi N} & 0 \\
\frac{\hbar}{2} \sqrt{\gamma_\nu \pi N} & \hbar \Delta & \frac{\hbar}{2} \sqrt{\gamma_\nu \pi N} \\
0 & \frac{\hbar}{2} \sqrt{\gamma_\nu \pi N} & [g_A^2(\omega_i) - g_B^2(\omega_i)]
\end{pmatrix}.
\]
(27)

Diagonalising the Hamiltonian (27) and introducing the definitions
\[
\Omega_{AB}(r_A, r_B) = -\sqrt{2\gamma_\nu \pi} \sqrt{\frac{[g_A^2(\omega_i) + g_{AB}^2(\omega_i)]^2 + [g_B^2(\omega_i) + g_{AB}^2(\omega_i)]^2}{g_A^2(\omega_i) + g_B^2(\omega_i) + 2g_{AB}^2(\omega_i)}} \]
(28)
\[
\Delta = \omega_i - \omega_{10}
\]
(29)
of the vacuum Rabi frequency and detuning, respectively, yields the three eigenvalues
\[
E_{1,2} = \frac{\hbar}{2} \Delta \pm \frac{\hbar}{2} \Omega(r_A, r_B), \quad E_3 = 0
\]
(30)

Finally, the respective dressed states of the combined atom–field system can be written in the form
\[
|\psi_1\rangle = \sin \theta_1 \sin \varphi_1 |u_1\rangle + \cos \theta_1 |u_2\rangle + \sin \theta_1 \cos \varphi_1 |u_3\rangle,
\]
(31)
\[
|\psi_2\rangle = \cos \theta_1 \sin \varphi_1 |u_1\rangle - \sin \theta_1 |u_2\rangle + \cos \theta_1 \cos \varphi_1 |u_3\rangle,
\]
(32)
\[
|\psi_3\rangle = \cos \varphi_1 |u_1\rangle - \sin \varphi_1 |u_3\rangle,
\]
(33)

where \( \theta_1 \) and \( \varphi_1 \) are the coupling angles defined by
\[
\tan \theta_1 = \frac{\Omega(r_A, r_B) + \Delta}{\Omega(r_A, r_B) - \Delta}, \quad \tan \varphi_1 = \frac{g_A^2(\omega_i) + g_{AB}^2(\omega_i) + 2g_{AB}^2(\omega_i)}{g_A^2(\omega_i) - g_B^2(\omega_i)}.
\]
(34)

The eigenstate \( |\psi_3\rangle \) which corresponds to the zero eigenenergy is called the dark state. The position-dependent parts of the other two eigenenergies can be regarded as dispersion potentials
\[
U_{1,2} = \pm \frac{\hbar}{2} \Omega(r_A, r_B).
\]
(36)

The total dispersion force on atom A consists of a single-atom CP force and the cavity-enhanced vdW force from atom B. The latter is defined as
\[
F_{AB}^3(r_A, r_B) = -\nabla_A U_{1,2}(r_A, r_B).
\]
(37)

In order to obtain a more general expression for the force, let us consider the system to be prepared in a superposition state \( \psi \) which has projections on all the eigenstates
\[
|\psi\rangle = \sin \theta \sin \varphi |u_1\rangle + \cos \theta |u_2\rangle + \sin \theta \cos \varphi |u_3\rangle
\]
(38)
\[
= [\sin \theta \sin \varphi |u_1\rangle \cos \varphi - \varphi_1 + \cos \theta \cos \varphi |u_3\rangle |\psi_1\rangle \]
\[
+ [\sin \theta \cos \varphi |u_1\rangle \cos \varphi + \sin \theta \sin \varphi |u_3\rangle |\psi_2\rangle
\]
\[
+ [\sin \theta \sin \varphi |u_3\rangle |\psi_3\rangle],
\]
(39)
where the last equality is written according to the definitions (31)–(33). Making use of equations (36) and (38), the potential of the system prepared in the state \( \psi \) can be calculated as
\[
U_\psi = [\langle \psi | \psi \rangle]^2 U_1(r_A, r_B) + [\langle \psi | \psi_2 \rangle]^2 U_2(r_A, r_B)
\]
(39)
\[
+ [\langle \psi | \psi_3 \rangle]^2 U_3(r_A, r_B)
\]
where
\[
U_1 = \Omega(\cos 2\theta_1 \cos^2 \varphi - \sin^2 \theta \cos^2 \varphi - \varphi_1)
\]
(39)
\[ \Omega = \Omega[\{r_A, r_B\}], \] where the position dependence arises from \( \theta_c \) and \( \varphi_c \), as well as \( \Omega \), so the general form for the vdW force on atom A can be evaluated as

\[
F_V^\lambda(r_A, r_B) = -\frac{\hbar}{2} \left[ \cos 2\theta_c \cos^2 \theta - \sin^2 \theta \cos^2 (\varphi - \varphi_c) \right] + \cos (\varphi - \varphi_c) \sin 2\theta_c \sin 2\theta \nabla_A \Omega \\
- \frac{\hbar}{2} \Omega \nabla_A \left[ \cos 2\theta_c \cos^2 \theta - \sin^2 \theta \cos^2 (\varphi - \varphi_c) \right] + \cos (\varphi - \varphi_c) \sin 2\theta_c \sin 2\theta \right].
\]

(40)
The differentiations in the right-hand side can be performed using equations (34), (35) and the identity \((1 + \tan^2 \beta)^{-1/2} = \cos \beta\), so we find

\[
\nabla_A \theta_c = -\frac{\cos^2(2\theta_c)}{2\Delta} \nabla_A \Omega_R, \\
\nabla_A \varphi_c = \frac{\cos^2(2\varphi_c)}{2\Delta} \nabla_A \left\{ N \left[ g^a_0(\omega_c) - g^b_0(\omega_c) \right] \left[ g^b_0(\omega_c) - g^b_0(\omega_c) \right] - N^2 \right\}
\]

(41)
\[
[\Omega_R = \Omega(r_A, r_B)], \text{ using which, together with equation (34), we obtain } \sin(2\theta_c) = \Omega_R / \Omega. \text{ Finally, using the definition of } \Omega(r_A, r_B), \text{ we have}
\]

\[
\nabla_A \Omega = \sin(2\theta_c) \nabla_A \Omega_R. \quad (42)
\]

Let us study the state dependence of the force by considering special cases. It can be seen if that \( \theta = \theta_c, \varphi = \varphi_c \) or \( \theta = \theta_c + \pi/2, \varphi = \varphi_c \) (which is equivalent to \( |\psi\rangle = |\psi_1\rangle \) or \( |\psi\rangle = |\psi_2\rangle \), respectively), the previous results of equation (37) can be recovered.

It is worth noting that the rotating wave approximation used in derivation of our strong coupling results is only applicable in near-resonant interactions, and if the coupling strength satisfies \( g \ll \omega, \omega_{10} \) [21, 22]. We have defined the strong coupling regime by \( \Delta \ll \Omega_R \). As long as this condition is valid, the rotating wave approximation holds even for moderate detuning. As shown explicitly in [19] for the case of a single atom coupled to the cavity field, for weakly coupled interactions with large detuning, the neglect of off-resonant contributions is no longer justified. We will study the weak coupling limits of the force in the next section.

### 3.2. Weak coupling

Let us return to the primary idea of Casimir and Polder and consider a continuous and unbounded spectrum for the field. According to the perturbation theory for two identical atoms, the lowest-order energy shift depending on the positions of both atoms is second order and is defined as

\[
\Delta E = -\sum_i \sum_{A=A,B} \frac{(\langle \hat{H}_{A,B}^\dagger \vert I \rangle \langle I \vert \hat{H}_{A,B}^\dagger \rangle)}{E_i - E_i}
\]

(43)
in which \( \vert I \rangle \) and \( \vert I \rangle \) are the initial and intermediate states, respectively, and \( E_i \) and \( E_i \) are their respective unperturbed eigenenergies. We consider \( \vert I \rangle = \frac{1}{\sqrt{2}} (\vert 1A \rangle \vert 0B \rangle + \vert 0A \rangle \vert 1B \rangle) \{01\} \) and \( \vert I \rangle = \{0A \rangle \{0B \rangle \vert I_A (r, \omega) \rangle, \) with \( \vert I_A (r, \omega) \rangle = \hat{J}^\dagger (r, \omega) \{01\} \).

Note that the particular choice of \( \vert I \rangle \) ensures that we only deal with a real photon exchange process, so the potential will be resonant and the sum over intermediate state in equation (43) is in fact the sum over polarisations \( \lambda \) and integrals over \( r \) and \( \omega \).

Using equations (7), (12) and (13) together with Cauhcy’s theorem, the resonant potential for two identical atoms (one of them being excited) that are weakly coupled to the body-assisted field reads

\[
U = -\frac{\mu_0}{2} \omega_{10}^2 \hat{d}_A^{(0)} \cdot \text{Re} \hat{G}(r_A, r_B, \omega_{10}) \cdot \hat{d}_B^{(0)} \\
- \frac{\mu_0}{2} \omega_{10}^2 \hat{d}_B^{(0)} \cdot \text{Re} \hat{G}(r_B, r_A, \omega_{10}) \cdot \hat{d}_A^{(0)} \\
- \mu_0 \omega_{10}^2 \hat{d}_A^{(0)} \cdot \text{Re} \hat{G}(r_B, r_B, \omega_{10}) \cdot \hat{d}_B^{(0)}.
\]

(44)
The first and second terms represent the single-atom potentials for atom A and B, respectively, while the third is the interaction potential of the two atoms. To make contact with the perturbative results obtained by McLone and Power [6] for two identical atoms in free space, let us consider the Green function in free space [23]

\[
G_{mn}(k, r) = \frac{e^{ikr}}{4\pi r} \left[ \left( 1 + \frac{i k r - 1}{k^2 r^2} \right) \delta_{mn} + \left( -1 + \frac{3 - 3 i k r}{k^2 r^2} \right) \frac{r_m r_n}{r^2} \right].
\]

(45)
where \( k = \omega / c, r = r_2 - r_1, r = |r|, r_m = e_m \cdot r (\alpha = m, n). \) Substituting the real part of this tensor in the third part of equation (44), leads to

\[
U(r) = -d_{nn} d_{nn} \left[ \delta_{mn} - \frac{r_m r_n}{r^2} \right] k^2 \cos(kr) \\
\]

\[
\frac{k^2}{r^2} \cos(kr) \right] \] (46)
(P: principal value) in agreement with previous results of [6].

To make contact with our method, we consider the limit case of weak coupling in equation (36). In this limit \( \Delta \gg \Omega_R(r_A, r_B), \) the first coupling angle approaches \( \theta_c = \pi/2. \) For instance, as \( g_A = g_B \) is true in free space, the other coupling angle approaches \( \varphi_c = \pi/2 \) and also \( |\psi_1\rangle = -|u_1\rangle, |\psi_2\rangle = |u_2\rangle \) and \( |\psi_3\rangle = -|u_3\rangle. \) The potential associated with states \( |u_1\rangle \) and \( |u_2\rangle \) respectively can then be written as

\[
U_{u_1} = \frac{\hbar \gamma_c \pi N}{4 \Delta},
\]

(47)

\[
U_{u_2} = \frac{\hbar \gamma_c \pi N}{4 \Delta}.
\]

(48)
Employing the Kramers–Kronig relation

\[
\omega^2 \text{Re} \hat{G}(r_A, r_B, \omega) = \frac{1}{\pi} \int_0^\infty \frac{d\omega}{\omega'} \omega^2 \text{Im} \hat{G}(r_A, r_B, \omega'),
\]

(49)
for a sufficiently narrow mode (recalling equations (23) and (25)) we find:
\[
\begin{align*}
\mu_0 \omega^2 d_{\text{0}}^0 \cdot \text{Re} \mathbf{G}^\omega (\mathbf{r}_A, \mathbf{r}_A', \omega) \cdot d_{\text{0}}^0 \\
= \frac{\mu_0 \omega^2}{\pi \omega_\text{0}^2} d_{\text{0}}^0 \cdot \text{Im} \mathbf{G}^\omega (\mathbf{r}_A, \mathbf{r}_A', \omega_\text{0}) \cdot d_{\text{0}}^0 \\
\times \mathbf{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega'} \omega' \omega_\text{0}^2 + \gamma_\text{c}^2/4 \\
= \frac{\mu_0 \omega_\text{0}^2}{2(\omega_\text{0}^2 - \omega)} \omega^2 d_{\text{0}}^0 \cdot \text{Im} \mathbf{G}^\omega (\mathbf{r}_A, \mathbf{r}_A', \omega_\text{0}) \cdot d_{\text{0}}^0,
\end{align*}
\]

Note that the sign \( L \) has been added to indicate the assumption of dealing with single-resonance, Lorentzian cavity field. Using this result, equation (47) and (48), respectively, can be given as

\[
U_{w1} = -\frac{1}{2} \mu_0 \omega_\text{0}^2 d_{\text{0}}^0 \cdot \text{Re} \mathbf{G}^\omega (\mathbf{r}_A, \mathbf{r}_A, \omega_\text{0}) \cdot d_{\text{0}}^0 \\
- \frac{1}{2} \mu_0 \omega_\text{0}^2 d_{\text{0}}^0 \cdot \text{Re} \mathbf{G}^\omega (\mathbf{r}_B, \mathbf{r}_B, \omega_\text{0}) \cdot d_{\text{0}}^0 \\
- \mu_0 \omega_\text{0}^2 d_{\text{0}}^0 \cdot \text{Re} \mathbf{G}^\omega (\mathbf{r}_A, \mathbf{r}_B, \omega_\text{0}) \cdot d_{\text{0}}^0
\]

and

\[
U_{w2} = -U_{w1},
\]

Equation (51) is in complete agreement with the time-independent perturbative results for the resonant part of the dispersion potential of two identical two-level atoms (one of them being prepared in the upper state) that are weakly coupled to a body-assisted electromagnetic field, while equation (52) is the weak coupling potential for the state with excited field and ground-state atoms, and has not been predicted by perturbation theory so far.

In complete analogy to the derivation of equations (51) and (52), one can show that in weak coupling limit (large detuning) the potential for a system prepared in state \(|\psi\rangle\), can be written as

\[
U_\omega (\mathbf{r}_A, \mathbf{r}_B) = -\frac{\hbar}{4\Delta} [\cos^2(\theta) - \sin^2(\theta)\cos^2(\varphi)] \mathbf{G}_\omega (\mathbf{r}_A, \mathbf{r}_B)
\]

\[
= \frac{1}{2} [\cos^2(\theta) - \sin^2(\theta)\cos^2(\varphi)] \mu_0 \omega_\text{0}^2 \times d_{\text{0}}^0 \cdot \text{Re} \mathbf{G}^\omega (\mathbf{r}_A, \mathbf{r}_A, \omega_\text{0}) \cdot d_{\text{0}}^0
\]

\[
- \frac{1}{2} [\cos^2(\theta) - \sin^2(\theta)\cos^2(\varphi)] \mu_0 \omega_\text{0}^2 \times d_{\text{0}}^0 \cdot \text{Re} \mathbf{G}^\omega (\mathbf{r}_B, \mathbf{r}_B, \omega_\text{0}) \cdot d_{\text{0}}^0
\]

\[
- [\cos^2(\theta) - \sin^2(\theta)\cos^2(\varphi)] \mu_0 \omega_\text{0}^2 \times d_{\text{0}}^0 \cdot \text{Re} \mathbf{G}^\omega (\mathbf{r}_A, \mathbf{r}_B, \omega_\text{0}) \cdot d_{\text{0}}^0
\]

\[
(53)
\]

\section*{4. Planar cavity}

To apply our model, we assume two atoms, A and B, to be placed in a planar cavity of width \( d \). The cavity plates are assumed to be identical and -almost- perfectly reflecting and the atoms are placed along an axis perpendicular to the cavity plates (\( z \) direction), see figure 1.

We use the Green’s tensor as written in [13]:

\[
\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) = \frac{i}{8\pi} \int \frac{d^2 k}{k^\bot} e^{ik(\mathbf{r} - \mathbf{r}')} \\
\times \sum_{\sigma = \pm} \left\{ \frac{\rho_{\sigma} e^{2ikd}}{D_\sigma} [\mathbf{e}_\sigma \cdot \mathbf{e}_{\sigma'} e^{i(k - k')z'} + \mathbf{e}_\sigma \cdot \mathbf{e}_{\sigma'} e^{-i(k - k')z'}] \\
+ \frac{1}{D_\sigma} [\mathbf{e}_\sigma - \mathbf{e}_{\sigma'} e^{i(k - k')z'} + \mathbf{e}_\sigma - \mathbf{e}_{\sigma'} e^{-i(k - k')z'}] \right\},
\]

with

\[
D_\sigma = 1 - r^2 \rho^2 e^{2ikd},
\]

\[
k^\bot = \sqrt{k^2 - k^2_f}.
\]

The vectors \( e_{s\pm} \) and \( e_{p\pm} \) are the unit vectors for \( s/p \)-polarised waves. The reflection coefficients are \( r_{s\pm} = -r_{p\pm} = 1 - b \), in which \( b \), regarded as a small deviation of the plates from being perfectly reflecting, can be related to the transmission coefficient of the plates through \( b = P^2/2 \).

On cavity resonances (\( \omega_\nu = \nu \pi c/d \)), \( k^\bot = 0 \) carries the main contribution to the integral. Taking one parallel component of Green’s tensor and carrying out the angular integral, we get to a simplified version of respective Green’s tensor, by which we find:

\[
\omega_\nu^2 \text{Im} \mathbf{G}_{\omega\nu}(z_A, z_B, \omega_\nu) = -\frac{\omega_\nu^2}{16\pi c^2} \left\{ \cos \left[ \frac{2d - z_A - z_B}{c} \right] \omega_\nu \frac{1}{c} \\
- \cos \left[ \frac{2d + z_A - z_B}{c} \right] \omega_\nu \frac{1}{c} \\
- \cos \left[ \frac{z_A + z_B}{c} \right] \omega_\nu \frac{1}{c} \right\}.
\]

It can be shown that equation (57), shows Lorentzian behaviour equation (23), in the vicinity of each cavity resonance with \( \gamma_\nu = 2c\delta/d \).
According to equation (28), in the case of exact resonance ($\Delta = 0$), the cavity-induced Rabi frequencies can be written as

$$\Omega^2 = \Omega_b^2 = 2\gamma_0 \pi \times \left\{ \left[ \gamma_A^2(\omega_0) + \gamma_B^2(\omega_0) \right]^2 + \left[ \gamma_A^2(\omega_i) + \gamma_B^2(\omega_i) \right]^2 \right\} / \left[ \gamma_A^2(\omega_0) + \gamma_B^2(\omega_0) + 2\gamma_{AB}^2(\omega_i) \right].$$

(58)

where using the definition for $\gamma_0$, together with equations (16) and (57), $\gamma_0 \pi \gamma_{AB}^2(\omega_i)$ is calculated to be

$$\gamma_0 \pi \gamma_{AB}^2(\omega_i) = \frac{3\gamma_0}{8d} \left\{ \cos \left[ \frac{(2d - z_A - z_B)\omega_i}{c} \right] - \cos \left[ \frac{(2d - z_A + z_B)\omega_i}{c} \right] - \cos \left[ \frac{(z_A + z_B)\omega_i}{c} \right] \right\}.$$

(59)

with

$$\Gamma_0 = \frac{\omega_{d0}^2 \Gamma_0^2}{3\pi z_0/\hbar c^3}$$

(60)

being the free space decay rate. Note that $\gamma_0 \pi \gamma_{AB}^2(\omega_0)$ and $\gamma_0 \pi \gamma_{AB}^2(\omega_i)$ can be calculated from equation (59) as the limiting case of $z_A = z_B$.

The position dependence of the dimensionless squared Rabi frequencies is shown in figure 2 and Rabi oscillations are observed, where we separately display the cases of only atom A present ($g_A^2 = g^{2}_{AB} = 0$), only atom B present ($g_B^2 = g^{2}_{AB} = 0$), both atoms with no interaction ($g_{AB}^2 = 0$) and the fully interacting case. It can be seen that, as the two atoms are identical, their contribution to total squared Rabi frequency has the same position dependence and also the same intensity, while the peaks of $\Omega^2$ are placed in the same position as its interaction contribution but with higher...
frequency dependence on the position of one atom, it has been demonstrated that the second atom would be completely invisible to the first one, if it is positioned at each cavity node. Conversely, it may have the equal contribution as the first one, if it is positioned at one of the anti-nodes.

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