Label-Guided Graph Exploration with Adjustable Ratio of Labels

Meng Zhang\textsuperscript{a}, Yi Zhang\textsuperscript{b}, Jijun Tang\textsuperscript{c}

\textsuperscript{a}College of Computer Science and Technology, Jilin University, Changchun, China
\textsuperscript{b}Department of Computer Science, Jilin Business and Technology College, Changchun, China
\textsuperscript{c}Department of Computer Science & Engineering, Univ. of South Carolina, USA

Abstract

The graph exploration problem is to visit all the nodes of a connected graph by a mobile entity, e.g., a robot. The robot has no a priori knowledge of the topology of the graph or of its size. Cohen et al. \cite{3} introduced label guided graph exploration which allows the system designer to add short labels to the graph nodes in a preprocessing stage; these labels can guide the robot in the exploration of the graph. In this paper, we address the problem of adjustable 1-bit label guided graph exploration. We focus on the labeling schemes that not only enable a robot to explore the graph but also allow the system designer to adjust the ratio of the number of different labels. This flexibility is necessary when maintaining different labels may have different costs or when the ratio is pre-specified. We present 1-bit labeling (two colors, namely black and white) schemes for this problem along with a labeling algorithm for generating the required labels. Given an \(n\)-node graph and a rational number \(\rho\), we can design a 1-bit labeling scheme such that \(n/b \geq \rho\) where \(b\) is the number of nodes labeled black. The robot uses \(O(\rho \log \Delta)\) bits of memory for exploring all graphs of maximum degree \(\Delta\). The exploration is completed in time \(O(n \Delta^{16/\rho} + 7^{3/\rho} + \Delta^{40/\rho} + 10^{3/\rho})\). Moreover, our labeling scheme can work on graphs containing loops and multiple edges, while that of Cohen et al. focuses on simple graphs.

1 Introduction

This paper concerns the task of graph exploration by a finite automaton guided by a graph labeling scheme. A finite automaton \(\mathcal{R}\), called a robot, must be able to visit all the nodes of any unknown anonymous undirected graph \(G = (V, E)\). The robot has no a priori information about the topology of \(G\) and its size. While visiting a node the robot can distinguish between the edges that are incident on this node. At each node \(v\) the edges incident on it are ordered and labeled by consecutive integers \(0, \ldots, d - 1\) called port numbers, where \(d = \deg(v)\) is the degree of \(v\). We will refer to port ordering as a local orientation. We use Mealy Automata to model the robot. The robot has a transition function \(f\) and a finite number of states. If the automaton in state \(s\) knows the port \(i\) through which it enters a node of degree \(d\), it switches to state \(s'\) and exits the node through port \(i'\), that is, \(f(s, i, d) = (s', i')\).

The graph exploration by mobile agents (robots) recently received much attention, and different graph exploration scenarios have been investigated. In the case of tree exploration, it is shown by Diks et al. \cite{5} that the exploration of \(n\)-node trees such that the robot can stop once exploration is completed, requires a robot with memory size \(\Omega(\log \log \log n)\) bits, and \(\Omega(\log n)\) bits are necessary for exploration with return. Moreover, they constructed an algorithm of exploration with return for all trees of size at most \(n\), using \(O(\log^2 n)\) bits of memory. In the work of Ambühl et al. \cite{11}, the memory is lowered to \(O(\log n)\) bits for exploration with return. Flocchini et al. \cite{11} later showed that a team of \(\Omega(n)\) asynchronous oblivious robots are necessary for most \(n\)-node trees, and that it is possible to explore the tree by \(O(\log n / \log \log n)\) robots only if the maximum degree of the tree is 3.

The memory size of the robot is widely adopted as the measurement of the efficiency \cite{18, 7, 10, 8, 17}. Fraigniaud et al. \cite{8} proved that a robot needs \(\Theta(D \log \Delta)\) bits of memory to explore all graphs of diameter
$D$ and maximum degree $\Delta$. By the result of Reingold [17], a robot equipped with $O(\log n)$ bits of memory is able to explore all $n$-node graphs in the perpetual exploration model, where the return to the starting node is not required. The lower bound of memory bits $\Omega(\log n)$ is proved by Rollik [18].

In the scenario adopted in [2, 7, 10], the robot is provided with a pebble that can be dropped on a node and used to identify the node later. The authors in [2] showed that a robot can explore the graph with only one pebble if it knows an upper bound on the number of nodes, otherwise $\Theta(\log \log n)$ pebbles are necessary and sufficient. Flocchini et al. [12] studied a dynamic scenario where the exploration is on a class of highly dynamic graphs.

Recently, much research is focused on the exploration of anonymous graphs guided by labeling the graph nodes [15, 9, 14, 3, 13, 16, 4]. The periodic graph exploration requires that the automaton has to visit every node in an undirected graph infinitely many times in a periodic manner. Ilicinkas [14] considered minimizing the length of the exploration period by appropriate assignment of local port numbers. Gasieniec et al. [13] improved the upper bound of the exploration period $\pi$ from $4n - 2$ to $3.75n - 2$ in an $n$-node graph, providing the agent with a constant memory. For an oblivious agent, [6] achieved a period of $10n$. Recently, Cyzyowicz et al. [4] showed a period of length at most $4\frac{1}{4} n$ for oblivious agents and a period of length at most $3.5n$ for agents with constant memory. Kosowski et al. [16] provided a new pebble labeling which leads to shorter exploration cycles, improving the bound to $\pi \leq 4n - 2$ for oblivious agents.

Cohen et al. [3] introduced the exploration labeling schemes. The schemes consist of an algorithm $L$ and a robot $R$ such that given any simple graph $G$ with any port numbering, the algorithm $L$ labels the nodes of $G$, and $R$ explores $G$ with the help of the labeling produced by $L$. It is shown that using only 2-bit (actually, 3-valued) labels a robot with a constant memory is able to explore all graphs, and the exploration is completed in time $O(m)$ in any $m$-edge simple graph. The authors also presented a 1-bit labeling scheme (two kinds of labels, namely black and white) on bounded degree graphs and an exploration algorithm for the colored graph. The robot uses a memory of at least $O(\log \Delta)$ bits to explore all simple graphs of maximum degree $\Delta$. The robot stops once the exploration is completed. The completion time of the exploration is $O(\Delta^O(1)m)$.

1.1 Our Results

We consider the problem of adjustable label guided graph exploration. Since maintaining different labels may have different costs, it is necessary to limit the number of some labels. For example, in a 1-bit labeling scheme, if we use a lighting lamp to represent ‘1’ and a turned off lamp to represent ‘0’, the number of lighted lamps (label ‘1’) may be limited to reduce the cost. For a 1-bit labeling scheme on an $n$-node graph $G$ where the number of nodes labeled black is $b$, we define $N$-ratio as the ratio of the number of nodes to the number of nodes colored black, that is, $n/b$. Given a rational number $\rho$, we can design a 1-bit labeling scheme on $G$ such that the $N$-ratio is not less than $\rho$.

The 1-bit labeling scheme in [3] does not guarantee an arbitrary $N$-ratio and works specifically on simple graphs, i.e., undirected graphs without loops or multiple edges. This scheme employs the function of counting the number of neighbors for a node, which is impossible in a non-simple graph with multi-edges and loops. Using only the port numbering will not allow a robot to know whether two neighbors of a node are the same.

We present 1-bit labeling schemes that can adjust the $N$-ratio and can work on non-simple graphs. We first investigate a family of $N$-ratio tunable labeling schemes where the $N$-ratio can be changed but not in a precise way. We classify the nodes in $G$ by the distances between each node, and a specific node $r$ is assigned as the root. Each class of nodes in the classification is called a layer. In this family of labeling schemes, all nodes in the same layer are labeled similarly. We call $\rho = bl/l$ the $L$-ratio of the labeling scheme where $l$ is the number of layers, and $bl$ is the number of black layers. We introduce the $L$-ratio tunable labeling schemes, enabling a robot to explore all graphs of maximum degree $\Delta$. Starting from any node, the robot returns to the root once the exploration is completed. We also design a procedure for a robot to label the graph. But we need an extra label to indicate that a node is not labeled yet.

Based on the $L$-ratio tunable labeling schemes, we introduce the $N$-ratio adjustable labeling schemes. Precisely, given an expected $N$-ratio $2 \leq \rho \leq (D + 1)/4$, we derive a series of labelings from an $L$-ratio tunable labeling. Throughout the paper, we use $\rho'$ to denote the $L$-ratio and $\rho$ to denote the expected $N$-ratio.
Table 1: Comparison of the labeling schemes in [3] and ours. The first two rows are from [3].

| Label size (#bits) | Robot’s memory (#bits) | Time (#edge-traversals) | Ratio (#black nodes) | Works on |
|-------------------|------------------------|-------------------------|----------------------|----------|
| 2                 | $O(1)$                 | $O(n)$                  | –                    | simple graphs |
| 1                 | $O(\log \Delta)$      | $O(\Delta^{O(1)} n)$   | no guarantee\(^*\)   | simple graphs |
| 1                 | $O(\rho \log \Delta)$ | $O(n \Delta^{16 \rho/3}/\rho + \Delta^{40 \rho/3+1})$ | $\leq n/\rho$      | non-simple graphs |

We prove that a labeling scheme with $N$-ratio not less than $\rho$ can be found in these labeling schemes. The exploration is completed in time $O(n \Delta^{16 \rho/3}/\rho + \Delta^{40 \rho/3+1})$; the robot need $O(\rho \log \Delta)$ bits of memory.

Table 1 compares our approach with the work of Cohen et al. [3]. In the case of $\rho = 2$, our approach extends the 1-bit labeling scheme in [3] from simple graphs to non-simple graphs. The exploration algorithms are different, but their space and time complexities are similar for simple graphs. When working on a simple graph labeled by the 1-bit labeling scheme in [3], our exploration algorithm runs in time $O(\Delta^{10} n)$ as in [3]. Both approaches derive a spanning tree from the graph by the labeling. In [3], the tree contains all nodes; in our approach the tree contains only black nodes, and the edges are paths of the graph. To find a path of length $l$, the robot performs at most $\Delta^{2i+2}$ traversals. Moreover, we use a new method to identify the root and its neighbors for non-simple graphs.

When $\rho$ comes close to the diameter, the amount of memory used by the robot is not far from that of the situation where all nodes are white (that is, there is no labeling). It is known that $\Omega(D \log \Delta)$ \[^\#\; bits\ of\ memory\] bits of memory are necessary without pre-labeling of the graph, which is the same bound as ours when $\rho$ comes close to the diameter.

## 2 $L$-ratio Tunable 1-Bit Labeling Schemes for Bounded Degree Graphs

In this section, we describe an $L$-ratio tunable exploration labeling scheme using 1-bit labels. Let $G$ be an $n$-node graph of degree bounded by $\Delta$. It is possible to color the nodes of $G$ with two colors namely black and white, while the $L$-ratio of the labeling is tunable. There exists a robot that can explore the graph $G$ by the aid of the labeling, starting from any node and terminating after identifying that the entire graph has been traversed.

### 2.1 Notions

Let $v$ and $u$ be nodes connected by edge $e$. Denote by $\text{port}(e, u)$ the port number of the port of $u$ which $e$ is incident on. A path $P$ in a non-simple graph is defined as a series of edges $e_0, e_1, \ldots, e_k$ such that for a series of nodes $n_0, n_1, \ldots, n_k$, edge $e_i$ connects $n_i$ and $n_{i+1}$ ($0 \leq i \leq k$). The string $p_0 p_1 \ldots p_{2k+1}$, where $p_i = \text{port}(e_{\lceil i/2 \rceil}, n_{\lceil i/2 \rceil})$ ($0 \leq i \leq 2k+1$), is called the label of $P$. We denote by $P^{-1}$ the reversal path of $P$. We say that a path $P$ is greater than path $P'$, if the label of $P$ is lexicographically greater than the label of $P'$. The distance between two nodes $u, v$ is the number of edges in the shortest path from $u$ to $v$, denoted by $d(u, v)$. Let $L_i$ denote the set of nodes that are at distance $i$ from $r$, and $L_0 = \{r\}$. For layers $L$ and $L'$, we let $d(r, L)$ denote the distance between any node in layer $L$ and $r$, $d(L, L')$ denote $|d(r, L) - d(r, L')|$.

### 2.2 Labeling Schemes

The following is a class of $L$-ratio tunable 1-bit labeling schemes.

**Labeling $\mathcal{AL}$**. Pick an arbitrary node $r \in V$ and assign it the root of $\mathcal{AL}$. Label $r$ black. Select two different non-negative integers $d_1, d_2$ satisfying $d_1 \geq 2$ and $\lfloor d_2/2 \rfloor \geq d_1$. Define four classes of nodes $A, B, C,$ and $D$ as follows:

\[^*\]The number of black nodes in [3] can vary (without any control) from $\Theta(1)$ to $\Theta(n)$, depending on the cases.
$C = \{ v \in V \mid d(r, v) \mod (d_1 + d_2 + 2) = 0 \}$,
$D = \{ v \in V \mid d(r, v) \mod (d_1 + d_2 + 2) = 1 \}$,
$A = \{ v \in V \mid d(r, v) \mod (d_1 + d_2 + 2) = d_2 + 1 \}$,
$B = \{ v \in V \mid d(r, v) \mod (d_1 + d_2 + 2) = d_1 + d_2 + 1 \}$.

Label all the nodes in class $A, B, C$, and $D$ black and label all the nodes left white. The $\mathcal{AL}$ labeling is denoted by $(r, d_1, d_2)$.

An example of $\mathcal{AL}$ labeling schemes is shown in Figure 1. A layer is called a white (black) layer if all nodes in this layer are white (black). Denote black layers by $BL_0, BL_1, \dots, BL_{D_B}$, where $BL_0 = L_0, D_B + 1$ is the number of black layers, and $d(r, BL_i) < d(r, BL_j)$ if $i < j$. For $X \in \{A, B, C, D\}$, layer $BL_i$ is said to be an $X$-layer if $BL_i \subset X$. Two black layers are said to be adjacent if one is $BL_i$ and another is $BL_{i+1}$. The black nodes whose neighbors are all black are called B-nodes.

[[Figure 1: An $\mathcal{AL}$ labeling scheme. Each line represents a layer. Black lines represent black layers, and white lines represent white layers.]]

The $L$-ratio of the labeling can be altered by adjusting $d_1$ and $d_2$, but it cannot be adjusted precisely to guarantee that the $L$-ratio is not less than a given rational value. We assume that $D \geq d_1 + d_2 + 1$, that is, there are at least four black layers. Then the upper bound on the $L$-ratio is $(D + 1)/4$. The minimal $L$-ratio is of an $\mathcal{AL}$ labeling where $d_1 = 2, d_2 = 4$, and $D = 9$, and there are six black layers in the labeled graph. We have the $L$-ratio $\rho' \geq 5/3$.

For $\mathcal{AL}$ labeling schemes, we will prove the following in the remaining of Section 2.

**Theorem 1.** Let $G$ be an $n$-node graph of degree bounded by an integer $\Delta$, and let $G$ be labeled by an $\mathcal{AL}$ labeling scheme. There exists a robot that can explore the graph $G$, starting from any given node and terminating at $r$ after identifying that the entire graph has been traversed. The robot has $O(\rho' \log \Delta)$ bits of memory, and the total number of edge traversals by the robot is $O(\Delta^{12\rho' - 3} n) + o(\rho' \Delta n)$, where $\rho'$ is the $L$-ratio of the labeling.

For a black node $u$, we identify two subsets of nodes that can be reached by a path from $u$. For $u \in BL_i$ ($0 < i \leq D_B$), $\text{pred}(u)$ is the set of nodes in $BL_{i-1}$ such that for any $x \in \text{pred}(u)$, $d(u, x) = d(BL_i, BL_{i-1})$. For $u \in BL_i$ ($0 < i < D_B$), $\text{succ}(u)$ is the set of nodes in $BL_{i+1}$ such that for any $x \in \text{succ}(u)$, $d(u, x) = d(BL_i, BL_{i+1})$. For root $r$, we set $\text{pred}(r) = \emptyset$, and we have $\text{succ}(r) = BL_1$. For $u \in BL_{D_B}$, $\text{succ}(u) = \emptyset$.

In the following, we derive an implicit spanning tree of black nodes rooted at $r$ from an $\mathcal{AL}$ labeling scheme. For $u \in BL_i$ ($0 \leq i < D_B$), denote by $\text{succ}_\text{path}(u)$ the set of paths of length $d(BL_i, BL_{i+1})$ whose starting node is $u$ and ending node is in $BL_{i+1}$. For $u \in BL_{D_B}$, $\text{succ}_\text{path}(u) = \emptyset$. For $u \in BL_i$ ($0 < i \leq D_B$), denote by $\text{pred}_\text{path}(u)$ the set of paths of length $d(BL_i, BL_{i-1})$ whose starting node is $u$ and ending node is in $BL_{i-1}$. The path in $\text{pred}_\text{path}(u)$ with the lexicographically smallest label is called the parent path of $u$, denoted by $\text{parent}_\text{path}(u)$. We set $\text{pred}_\text{path}(r) = \emptyset$. The ending node of $\text{parent}_\text{path}(u)$ is called the parent of $u$, denoted by $\text{parent}(u)$. The set of nodes whose parent is $u$ is denoted by $\text{child}(u)$. We have $\text{child}(u) \subseteq \text{succ}(u)$ and $\text{parent}(u) \in \text{pred}(u)$. The reversal paths of the parent paths of the nodes in $\text{child}(u)$ are called child paths of $u$. All black nodes, their parent paths, and their child paths form an implicit spanning tree.

[[Figure 2: $R_W(v) = 1$, $R_W(v') = 2$. By property 3, node $u$ and $u'$ can be distinguished by $R_W(v)$ and $R_W(v')$.]]
2.3 Properties of \( \mathcal{AL} \) Labeling Schemes

In this section we describe three properties on \( \mathcal{AL} \) labeling schemes. These properties are the basis of the exploration algorithm. Since for any node \( u \) there is a shortest path from \( u \) to \( r \), we have the following property.

**Property 1.** Let \( u \neq r \) be a node, and let \( L_i \) be a black layer such that \( i < d(r, u) \). There exists at least a node \( x \in L_i \) such that \( d(x, u) = d(r, u) - i \).

A useful corollary of Property 1 is that any class \( D \) node has a \( B \)-node neighbor.

Assume that the nearest black nodes to some node \( v \) are at distance \( \ell \). Then the white-radius of \( v \) is \( \ell - 1 \), denoted by \( R_W(v) \). Property 2 gives the upper bound on the white radius of white nodes between two adjacent black layers. Figure 2 gives an example.

**Property 2.** Let \( u \) be a white node, and let \( d(r, BL_i) < d(r, u) < d(r, BL_{i+1}) \). We have \( R_W(u) \leq d(BL_i, BL_{i+1}) - 2 \).

Let \( P \) be a path from \( u \) to \( v \) of length \( \ell \) where only \( u \) and \( v \) are allowed to be black. Path \( P \) is called a white-path from \( u \), or precisely, an \( \ell \)-white-path. Let \( u \in BL_i \) \((i \neq 0)\), and let \( \ell = d(BL_i, BL_{i-1}) \). According to Property 1, there is at least one \( \ell \)-white-path from \( u \) to a node in \( BL_{i-1} \). The maximal white radius of nodes in this path is \( \lceil \ell/2 \rceil - 1 \), which leads to the following property.

**Property 3.** Let \( u \in BL_i \) \((i \neq 0)\), and let \( \ell = d(BL_i, BL_{i-1}) \). There exists a white path from \( u \) that reaches a white node whose white radius is not less than \( \lfloor \ell/2 \rfloor - 1 \).

These properties are used in our exploration algorithm. For example, we can distinguish between a class \( A \) node and a class \( B \) node by applying these properties. For \( u \in A \), there exists a white node \( x \) that can be reached by a white path from \( u \) such that \( R_W(x) = \lfloor d_2/2 \rfloor - 1 \). But for a class \( B \) node \( u \), the maximal white radius of white nodes that can be reached by a white path from \( u \) is not greater than \( d_1 - 2 \). Since \( d_1 \leq \lfloor d_2/2 \rfloor \) (see the definition of the \( \mathcal{AL} \) labeling), \( d_1 - 2 \) is less than \( \lfloor d_2/2 \rfloor - 1 \). Figure 2 gives an illustration.

2.4 The Local Search Procedure

The following local search procedure can be used to visit all nodes at distance not greater than a given radius from a node.

**Procedure** LocalSearch\((u, \ell, \text{inport})\)

**Input:** \( u \) is the starting node, \( \ell \) is the radius, and \( \text{inport} \) is the port from which \( \mathcal{R} \) enters \( u \).

1. if \( \ell = 0 \) then report(\( u \))\(^3\)
2. else
3. for \( \text{outport} \) from 0 to \( \deg(u) - 1 \) and \( \text{outport} \neq \text{inport} \) do
4. \( v \leftarrow \) the neighbor of \( u \) which \( \text{outport} \) leads to
5. \( \mathcal{R} \) moves to \( v \)
6. \( \text{inport}' \leftarrow \) the port from which \( \mathcal{R} \) enters \( v \)
7. LocalSearch\((v, \ell - 1, \text{inport}')\)
8. \( \mathcal{R} \) moves back to \( u \)
9. return

By the call LocalSearch\((u, \ell, -1)\), the robot explores all neighbors of \( u \) up to distance \( \ell \). In the local search from \( u \) within radius \( \ell \), there are at most \( LS(\ell) = 2\Delta \sum_{\ell=1}^{\ell-1}(\Delta - 1)^\ell = O(\Delta^\ell) \) edge traversals, and at most \( \Delta(\Delta - 1)^{\ell-1} \) nodes are reported. Note that an edge may be visited more than once, and a node could be reported more than once. The robot is in node \( u \) when the procedure terminates. We summarize the results on the LocalSearch procedure in the following lemma.

\(^3\)When the robot reports a node, it does not exit from the procedure nor makes any movement.
**Lemma 1.** In the local search from node $u$ within radius $\ell$, a robot with $O(\ell \log \Delta)$ bits of memory visits all nodes at distance not greater than $\ell$ from $u$ without visiting any other node. There are at most $O(\Delta^\ell)$ edge traversals and at most $\Delta(\Delta - 1)^{\ell-1}$ nodes being reported. The robot is in node $u$ when the local search terminates.

We can revise the procedure to explore only the paths that are greater than a given path $P$ from $u$ as follows. The robot first moves to the end of $P$ via $P$ and restores the context of the procedure for $P$ in its memory and then starts the procedure.

### 2.5 Exploration Guided by Labeling

The overall exploration performed by the robot is a depth first search (DFS) of the implicit spanning tree. All nodes will be visited in the DFS. The robot maintains a state $s \in \{\text{up, down}\}$. Initially, $\mathcal{R}$ is at the root $r$ of an $\mathcal{AL}$ labeling and leaves $r$ by the port numbered 0 in state down. Assume that $\mathcal{R}$ enters a black node $u$ via a path $P$ that belongs to the implicit spanning tree. If $\mathcal{R}$ is in state down, it searches for the minimal child path of $u$. If $\mathcal{R}$ is in state up, it moves down to the starting node of $P$ and searches for the minimal child path of $u$ that is greater than $P^{-1}$. In both cases, if $\mathcal{R}$ does not find the desired path, $\mathcal{R}$ moves to $\text{parent}(u)$ via the parent path of $u$ and transits the state to up; otherwise $\mathcal{R}$ moves to the end node of the path found and transits the state to down. The correctness of these procedures will be proved later.

To know whether a path belongs to the spanning tree, we use the following procedures.

1. $\text{Get\_Par\_Path}(u)$ identifies the parent path of $u \not\in \{r\} \cup BL_1$ and $\text{parent}(u)$. If $v = \text{parent}(u)$ is found, the procedure returns $v$, and $\mathcal{R}$ has moved to $v$ and recorded the parent path of $u$ in its memory; otherwise the procedure returns “false”.

2. $\text{Next\_Child\_Path}(u, P)$ identifies the minimal child path from $u \not= r$ that is greater than $P$ where $P$ is a child path of $u$ or $\varnothing$. When such a child path, say $P'$, is found, the procedure returns the end of $P'$, and $\mathcal{R}$ has moved to the end of $P'$. If no path is found, the robot goes back to $u$, and the procedure returns “false”.

All these procedures use a revised local search procedure, namely white local search. Given a radius $d$, a node $u$, and a path $P$ from $u$, the white local search procedure enumerates all the $d$-white-paths from $u$ that are greater than $P$. It returns “true” if such path exists and “false” otherwise. In both cases, the robot is in $u$ when the procedure terminates. This procedure is derived from $\text{LocalSearch}$, and the following line should be inserted into $\text{LocalSearch}$ between line 2 and line 3.

**if** $u$ is black and $\ell \not= \text{the initial radius of the local search} \text{ then return}

This procedure has the same property as Lemma[1] The term “local search” refers to the white local search procedure in the remainder of the paper.

#### 2.5.1 Procedure $\text{Get\_Par\_Path and Next\_Child\_Path}$

We first present procedures that will be used many times in the exploration procedures.

**Procedure $\text{Is\_B}$**

The $\text{Is\_B}$ procedure takes as input a black node $x$ that belongs to class $B$, $C$, or $D$ and returns “$B$” iff $x$ is in class $B$. The robot first checks whether $x$ is a $B$-node. If it is, $\text{Is\_B}(x)$ returns “$B$-node”. If not, the robot performs a local search from $x$ within radius $d_1$ (denote the local search by $LS_1$). Once a black node $y$ that has no $B$-node neighbor is reported, $\text{Is\_B}(x)$ returns “$B$”. If no such black node $y$ is reported or no node is reported, $\text{Is\_B}(x)$ returns “$D$”. In any case, $\mathcal{R}$ is in node $x$ when the procedure returns.

**Procedure $\text{C\_or\_D}$**

The $\text{C\_or\_D}$ procedure takes as input a black node $x$ ($x \not= r$) that belongs to class $C$ or $D$ and returns the class in which $x$ is. If $x$ is not a $B$-node $\text{C\_or\_D}(x)$ returns “$D$”. Otherwise the robot performs a local search from $x$ within radius 1 (denoted by $LS_1$). For each black neighbor $y$ of $x$ reported, perform $\text{Is\_B}(y)$. Once

---

* $P$ can be replaced by the label of $P$ as the initial node of $P$ is also input.
Is_B(y) returns “B”, C_or_D(x) returns “C”. If for every y, Is_B(y) does not return “B”, then C_or_D(x) returns “D”. In any case, R is in node x when the procedure returns.

Procedure A_or_B

The A_or_B procedure takes as input a black node x. If x belongs to class A or B, A_or_B(x) returns the class to which x belongs. The robot performs a local search (denoted by LS_1) within radius d_1 from x. A_or_B(x) returns “A” if in this local search a white node is reported whose white radius is d_1 - 1 and returns “B” otherwise. In any case, R is in node x when the procedure returns.

Now we present procedure Get_Par_Path and Next_Child_Path.

Procedure Get_Par_Path(u)

Assume that R starts from a node u ∈ BL_i (i ≥ 2). R aims at identifying the parent path of u and moving to parent(u). According to the class that u belongs to, we consider four cases. In the following, “X → Y” means that R is in an X-layer node u and tries to move to parent(u) in the adjacent Y-layer. In each case, the robot first calls procedure PathEnumeration (PE for short) and then calls procedure NodeChecking (NC for short) for each path enumerated by PE. The functions of these two procedures are: (1) PE: Enumerating (reporting) a set of white paths comprising pred_path(u) and their ends. (2) NC: Checking whether a path enumerated by PE is in pred_path(u). Since the local search enumerates paths in lexicographic order, in the following cases, the node in pred(u) firstly found by NC is parent(u), and the path recorded by the robot is the parent path of u. Figure 3 gives an illustration.

Figure 3: Four cases in Get_Par_Path. The automaton starts from node u. It can reach node x and x’ by PE. Node x is in pred set of node u while x’ is not.

Case(1) C → B
PE: Perform a local search from u within radius 1 (LS_1).
NC: For each black node x reported by PE, call Is_B(x). Once Is_B(x) returns “B”, we return x.

Case(2) B → A
PE: Perform a local search from u within radius d_1 (LS_1).
NC: For each black node x reported by PE, call A_or_B(x). Once A_or_B(x) returns “A”, we return x.

Case(3) A → D
PE: (i) Perform a local search from u within radius d_1 (LS_1). (ii) From each white node v reported, perform a local search within radius d_1 - 1 (LS_2). (iii) If all nodes visited in LS_2 are not black, perform local search within radius d_2 - d_1 from v (LS_3).
NC: For each black node x reported by PE, if x has a B-node neighbor, we return x.

Case(4) D → C
PE: (i) Perform a local search from u within radius d_1 (LS_1). (ii) From each white node v reported, perform a local search within radius d_1 - 1 (LS_2). (iii) If all nodes visited in LS_2 are not black, perform local search within radius d_2 - d_1 + 1 from v (LS_3).
NC: For each black node x reported by PE, if x has a B-node neighbor, we return x.

Figure 4: Four cases in Next_Child_Path. The automaton starts from node u. It can reach node x and x’ by PE. Node x is in succe set of node u while x’ is not.
Case(4) $D \to C$

PE: Perform a local search from $u$ within radius 1 ($LS_1$).
NC: For each black node $x$ reported by PE, call $C_{or\_D}(x)$. Once $C_{or\_D}(x)$ returns “C”, we return $x$.

In the above cases, if $x$ is returned by NC, then $x$ is parent($u$), the path recorded in $R$ is the parent path of $u$, and the robot has moved to parent($u$); otherwise we go back to PE to enumerate another $x$ for NC.

Identification of the Root and $BL_1$ Nodes. In $D \to C$, we distinguish a $C$-layer node from a $D$-layer node by checking whether the node has a neighbor in a $B$-layer. Since the nodes in $BL_1 \cup \{r\}$ have no ancestor in any $B$-layer, for $u \in BL_1 \cup \{r\}$, $D \to C$ in Get_Par_Path($u$) will fail to find the parent of $u$. For any other nodes, Get_Par_Path will succeed in finding their parents. Thus if Get_Par_Path($u$) fails, then $u$ is in $BL_1 \cup \{r\}$.

The next problem is how the robot identifies the root. The solution is that when leaving the root, the robot memorizes the ports in the arrived nodes by which it should return to the root. We revise Get_Par_Path($u$) as follows. If $D \to C$ fails to find parent($u$), we have $u \in BL_1 \cup \{r\}$; $R$ goes to $r$ through the port it memorized, and the procedure returns $r$.

Procedure Next_Child_Path($u$, $P$)

For a node $u \in BL_i$ ($i \geq 0$), and a path $P$ from $u$, the procedure identifies the minimal child path of $u$ greater than $P$. The robot calls the Enumerating procedure to enumerate some paths from $u$ greater than $P$ and calls the Identifying procedure to check whether an enumerated path is a child path. If such a path is found, we return its end node; otherwise we return “false”, and the robot backtracks to $u$, that is, $P$ is the maximal child path of $u$.

Enumerating. The procedure contains two parts: (i) PE: Use a local search to enumerate all $d$-white-paths $P'$ from $u$ that are greater than $P$, where $d = d(BL_i, BL_{i+1})$. If $P'$ does not exist, return “false”. (ii) NC: Check whether the end node of $P'$ is in succ($u$), if so, return this node. We consider the following cases. Figure 4 gives an illustration.

Case(1). $C \to D$
PE: Perform a local search from $u$ within radius 1 starting from $P$ ($LS_1$).
NC: For each black node $x$ reported by PE, call Is_B($x$). If “D” is returned, we return $x$. If “B-node” is returned, we call $C_{or\_D}(x)$; if “D” is returned, we return $x$.

Case(2). $D \to A$
PE: Perform a local search from $u$ within radius $d_2$ starting from $P$ ($LS_1$).
NC: For each black node $x$ reported by PE, if $x$ has no B-node neighbor, we return $x$.

Case(3). $A \to B$
PE: Perform a local search from $u$ within radius $d_1$ starting from $P$ ($LS_1$).
NC: For each black node $x$ reported by PE, call A_or_B($x$). Once “B” is returned, we return $x$.

Case(4). $B \to C$
PE: Perform local search from $u$ within radius 1 starting from $P$ ($LS_1$).
NC: The black nodes without any white neighbor reported by PE are in succ($u$) ($LS_2$). We return the first such node.

In the above cases, if $x$ is returned by NC, then $x$ is in succ($u$); otherwise we check another $x$ reported by PE.

Identifying. When Enumerating has found a shortest path $P'$ from $u$ to a node $x$ in succ($u$), $R$ has moved to $x$ and recorded $P'$ in its memory. $R$ then checks whether this path is a child path of $u$. If it is, the parent path of $x$ should be $P'$. We use Check_Par_Path($x$, $P'$) to verify it. The Check_Par_Path procedure is similar to the Get_Par_Path procedure except that the former’s PE part is performed in decreasing lexicographic order. If Check_Par_Path($x$, $P'$) finds a node in pred($x$), then $P'$ is not the parent path of $x$, and Check_Par_Path returns “false”; otherwise $P'$ is the parent path of $x$, and Check_Par_Path returns “true”. In both cases, $R$ is in node $x$ when Check_Par_Path terminates. If $P'$ is not a child path of $u$, we go back to Enumerating to enumerate another path. If a child path of $u$ is found, then Next_Child_Path($u$, $P$) returns the end node of $P$; otherwise returns “false”.

Exploration from an Arbitrary Node
When starting from an arbitrary node \( x \), the robot should first find the root. If \( x \) is a white node, the robot performs a normal local search within radius \( d_2 - 1 \) from \( x \) and stops when reaching a black node \( u \) (\( u \) is not a B-node). If \( x \) is a B-node, the robot performs a normal local search from \( x \) within radius 2 and stops when reaching a black node that is not a B-node. A B-node is either in class C or in class D. For a B-node in class C, a non-B-node black node will be reached by a local search within radius 1; for a B-node in class D, such node will be reached by a local search with radius 2. Therefore, in all cases the robot can reach a black node \( u \) that is not a B-node. The robot then identifies in which class \( u \) is. If \( u \) has a B-node neighbor, the robot performs \( \text{Is}_B(u) \). If “B” is returned, then \( u \in B \). If “D” is returned, then \( u \in D \). (Note that \( \text{Is}_B(u) \) cannot answer “B-node”.) If \( u \) does not have a B-node neighbor, the robot calls \( \text{A}_or_B(u) \). We have \( u \in A \) if “A” is returned and \( u \in B \) if “B” is returned.

After knowing the class of the starting node, the robot calls procedure \( \text{Get}_Par\_Path \) all the way to find the root. But our exploration cannot identify \( r \) without memorizing the port returning \( r \). Fortunately, the robot knows whether it is in a \( BL_1 \) node from the previous section. Let \( r' \) be the first \( BL_1 \) node found by the robot. Then \( r' \) is one of the B-node neighbors of \( r' \). We use every B-node neighbor of \( r' \) as a root and perform explorations from them. The robot should memorize the port by which it will return to \( r' \). At least, the exploration rooted at \( r \) will be performed that visits all the nodes in \( G \). The number of edge traversals in this case is at most \( \Delta \) times as large as that in exploration from \( r \).

### 2.6 Correctness of the Exploration

**Lemma 2.** For a black node \( x \) that belongs to class B, C, or D, \( \text{Is}_B(x) \) returns “B” iff \( x \) is in class B and returns “D” iff \( x \) is in D and not a B-node. The robot is in node \( x \) when \( \text{Is}_B(x) \) exits. In the call \( \text{Is}_B(x) \), the robot needs \( O(d_1 \log \Delta) \) bits of memory, and the total number of edge traversals of the robot is \( O(\Delta^{d_1+2}) \).

**Proof.** If \( x \in B \), \( LS_1 \) will report at least one node in class \( A \) according to Property 1. Since any node in class \( A \) has no B-node neighbor and has a white neighbor, \( LS_1 \) will report at least one such node. Therefore, if \( x \) is a node in class \( B \), \( \text{Is}_B(x) \) returns “B”.

If \( x \) is a B-node, \( \text{Is}_B(x) \) returns “B-node”. It can be easily verified that \( x \) is in class C or in class D. Let \( x \) be a class D node and not a B-node. We have that either \( LS_1 \) does not report any node or any node reported by \( LS_1 \) belongs to class \( D \). Since \( d_2 > d_1 \), \( LS_1 \) will not reach any class \( A \) node. By Property 1 any node in class \( D \) has at least a neighbor in class \( C \) (that is a B-node), then any black node reported by \( LS_1 \) has a B-node neighbor, and thus \( \text{Is}_B(x) \) returns “D”. Therefore, for a node \( x \) that belongs to class B, C, or D, \( \text{Is}_B(x) \) returns “B” iff \( x \) is in class B and “D” iff \( x \) is in D and not a B-node.

By definition, \( R \) is in node \( x \) when \( \text{Is}_B(x) \) exits. The number of edge traversals of \( \text{Is}_B(x) \) is not greater than that of a local search from \( x \) within radius \( d_1 + 2 \). By Lemma 1 there are at most \( O(\Delta^{d_1+2}) \) edge traversals in the call \( \text{Is}_B(x) \), and the memory space of \( R \) is \( O(d_1 \log \Delta) \) bits.

**Lemma 3.** For a black node \( x \neq r \) that belongs to class C or D, \( \text{C}_or_D(x) \) returns “C” iff \( x \in C \) and “D” iff \( x \in D \). The robot is in node \( x \) when \( \text{C}_or_D(x) \) exits. In the call \( \text{C}_or_D(x) \), the robot needs \( O(d_1 \log \Delta) \) bits of memory, and the total number of edge traversals of the robot is \( O(\Delta^{d_1+3}) \).

**Proof.** If \( x \in C \setminus \{r\} \), then \( x \) has a neighbor \( y \) in class \( B \), and thus \( \text{Is}_B(y) \) returns “B” by Lemma 2. Therefore, \( \text{C}_or_D(x) \) returns “C”. If \( x \in D \), then all neighbors of \( x \) belong to class \( C \) or \( D \). Thus, for any neighbor \( y \) of \( x \), \( \text{Is}_B(y) \) does not return “B” by Lemma 2. Therefore, \( \text{C}_or_D(x) \) returns “D”.

By definition, \( R \) is in node \( x \) when \( \text{C}_or_D(x) \) exits. The number of edge traversals of \( \text{C}_or_D(x) \) is not greater than that of a local search from \( x \) within radius \( d_1 + 3 \). By Lemma 1 there are at most \( O(\Delta^{d_1+3}) \) edge traversals in the call \( \text{C}_or_D(x) \), and the memory space of \( R \) is \( O(d_1 \log \Delta) \) bits.

**Lemma 4.** For a black node \( x \) that belongs to class A or B, \( \text{A}_or_B(x) \) returns “B” if \( x \) is in class B, and returns “A” otherwise. The robot is in \( x \) when \( \text{A}_or_B(x) \) exits. In the call \( \text{A}_or_B(x) \), the robot needs \( O(d_1 \log \Delta) \) bits of memory, and the total number of edge traversals of the robot is \( O(\Delta^{2d_1-1}) \).


Proof. According to Property 3, for $x \in A$, there exists a white node whose white radius is not less than $\lceil d_2/2 \rceil - 1$ that can be reached by a white path from $x$. According to Property 2, for $x \in B$, the white radius of nodes that have a white path to $x$ are not greater than $d_1 - 2$. By $A_C$, we have $\lceil d_2/2 \rceil \geq d_1$. Therefore, we know whether $x$ is in class $A$ or in class $B$ by checking the maximal white radius of nodes that have a white path to $x$. Thus, if $x \in A$, $LS_1$ will find a node with white radius $d_1 - 1$, and $A_{or\_B}(x)$ returns "A"; if $x \in B$, no such node will be found, and $A_{or\_B}(x)$ returns "B".

By definition, $R$ is in node $x$ when $A_{or\_B}(x)$ exists. The number of edge traversals in the call $A_{or\_B}(x)$ is not greater than that of the local search from $x$ within radius $2d_1 - 1$. By Lemma 1, there are at most $O(\Delta^{2d_1-1})$ edge traversals in the call $A_{or\_B}(x)$, and the memory space of $R$ is $O(d_1 \log \Delta)$ bits.

Lemma 5. For a black node $u$, let $R$ know to which class $u$ belongs. For $u \notin BL_1 \cup \{r\}$, when Get_Par_Path($u$) exits, parent($u$) is returned, and $R$ is in parent($u$) and recorded the parent path of $u$ in its memory. For $u \in BL_1$, Get_Par_Path($u$) can identify whether $u$ is in $BL_1$ and makes $R$ return to $r$. In a call to Get_Par_Path, there are at most $O(\Delta^{d_2+2})$ edge traversals, and the robot needs $O(d_2 \log \Delta)$ bits of memory space.

Proof. Let $R$ initiate at a node $u \in BL_1$, $i \geq 2$, when Get_Par_Path($u$) is called. We check separately the four cases in the procedure. In each case, two parts are to be proved: (1) PE can enumerate all paths in pred_path($u$) and their ends; (2) NC can identify whether the end nodes reported by PE are in pred($u$).

Case(1). $u \in C (C \rightarrow B)$
In this case, pred($u$) is a subset of the neighbors of $u$, since $d(BL_i, BL_{i-1}) = 1$. Therefore, all paths in pred_path($u$) can be enumerated by PE, so do the nodes in pred($u$).

Any neighbor $x$ of $u$ belong to class $B (BL_{i-1})$, class $C (BL_i)$, or class $D (BL_{i+1})$. By Lemma 2, if and only if $Is_B(x)$ returns "B", $u$ is in class $B$, i.e., pred($u$).

Case(2). $u \in B (B \rightarrow A)$
In this case, $d(BL_i, BL_{i-1}) = d_1$. By the local search from $u$ within radius $d_1$ (PE), all paths in pred_path($u$) and all nodes in pred($u$) can be reported.

The reported black nodes belong to class $A (BL_{i-1})$ or $B (BL_i)$; among them only the nodes in class $A$ are in pred($u$). According to Lemma 3, if and only if $A_{or\_B}(x)$ returns "A" then $u$ is in class $A$. Thus by calling $A_{or\_B}$ the nodes in pred($u$) can be identified.

Case(3). $u \in A (A \rightarrow D)$
In this case, $d(BL_i, BL_{i-1}) = d_2$. By $LS_1$ and $LS_2$, all white nodes at distance $d_1$ from both $u$ and $BL_i$ can be reported. As $d(BL_i, BL_{i+1}) = d_1$, these white nodes are between $BL_i$ and $BL_{i-1}$. The paths in pred_path($u$) containing such a white node can be enumerated by $LS_3$. Since every path in pred_path($u$) contains such a white node, PE can enumerate all paths in pred_path($u$) and their ends.

A black node $x$ reported by PE belongs to class $D$ or $A$. Through the observations on $A_C$, any node in class $D$ has at least one B-node neighbor, and any node in class $A$ has no B-node neighbor. So if $x$ has a B-node neighbor then $x$ belongs to class $D$. Therefore nodes in pred($u$) can be identified.

Case(4). $u \in D (D \rightarrow C)$
In this case, pred($u$) is a subset of the neighbors of $u$. PE can enumerate all paths in pred_path($u$) and all nodes in pred($u$).

For any neighbor $x$ of $u$, $x \in C$ or $x \in D$. By Lemma 3, $C_{or\_D}(x)$ can determine whether $x$ is in class $C$ which means $x \in pred(u)$.

All the local searches in this procedure are performed in increasing lexicographic order. According to $A_C$, in the above cases, the node in pred($u$) first be found is parent($u$), and the path stored in the memory of $R$ is the parent path of $u$. Since parent($u$) exists, Get_Par_Path($u$) returns parent($u$).

For $u \in BL_1 \cup \{r\}$, let $R$ take $u$ as a class $D$ node. We can verify that $Is_B(u)$ returns "D", and $D \rightarrow C$ will fail to find the parent path of $u$. By the above discussion, for nodes $x \notin BL_1 \cup \{r\}$, Get_Par_Path($x$) returns parent($x$). Thus Get_Par_Path($u$) identifies that $u$ is in $BL_1 \cup \{r\}$. $R$ then moves to $r$ from $u$ by the memorized port.

The worst-case number of edge traversals occurs in Case $A \rightarrow D$. By Lemma 1, this number is not greater than $LS(d_1) + O(\Delta^{d_1})(LS(d_1 - 1) + LS(d_2 - d_1 + 2)) = O(\Delta^{d_2+2})$. For the memory of $R$, in the worst case
Lemma 6. Let $u \notin BL_1 \cup \{r\}$ be a black node, and let $P$ be a white path from $u$. Let the robot know to which class $u$ belongs. $check_{\text{Par}Path}(u, P)$ returns “true” if $P$ is the parent path of $u$ and returns “false” otherwise. For $u \in BL_1 \cup \{r\}$, let $R$ take $u$ as a class $D$ node. $check_{\text{Par}Path}(u, P)$ returns “true” for any path $P$ from $u$ to $r$ containing one edge. When the procedure exits, $R$ is in node $u$. There are at most $O(d_2 \log \Delta + 2)$ edge traversals in a call to $check_{\text{Par}Path}$, and the memory space of $R$ is $O(d_2 \log \Delta)$ bits.

Proof. Procedure $check_{\text{Par}Path}$ is similar to $get_{\text{Par}Path}$ except that the PE part of $check_{\text{Par}Path}$ is performed in decreasing lexicographic order. By Lemma 5, for $u \notin BL_1 \cup \{r\}$, providing the robot knows in which class $u$ is, $check_{\text{Par}Path}(u, P)$ will find a path in $pred_{\text{path}}(u)$ that is lexicographically smaller than $P$ if this path exists. Whenever a path in $pred_{\text{path}}(u)$ is found by $check_{\text{Par}Path}(u, P)$, $P$ is not the parent path of $u$ according to the definition of parent path. $R$ returns to $u$ via the recorded path. If no such path is found, $P$ is the minimal path in $pred_{\text{path}}(u)$, i.e., the parent path of $u$. $R$ returns to $u$ by Lemma 1. Therefore $check_{\text{Par}Path}(u, P)$ can tell whether $P$ is the parent path of $u$. For any $x \in BL_1 \cup \{r\}$, $is_B(x)$ returns “$D$”, and thus $C_{or}D(x)$ returns “$D$”. Therefore, $check_{\text{Par}Path}(u, P)$ returns “true” for any path $P$ from $u$ to $r$ of length 1. The time and space complexity is similar as $get_{\text{Par}Path}$.

Lemma 7. Let $u \neq r$ be a black node, and let $P$ be a white path from $u$. Let $P'$ be the minimal child path of $u$ greater than $P$ if this path exists, and let $R$ know to which class $u$ belongs. Procedure $next_{\text{Child}Path}(u, P)$ returns the end of $P'$ if $P'$ exists, and $R$ is in the end node of $P'$ when the procedure exits. If $P'$ does not exist, then $next_{\text{Child}Path}$ returns “false” and $R$ moves to $u$. There are at most $O(d_2 \log \Delta + 2)$ edge traversals in $next_{\text{Child}Path}$, and the memory space of $R$ is $O(d_2 \log \Delta)$ bits.

Proof. Let $R$ start from node $u \in BL_i$ ($i \geq 1$). We first discuss four cases in the $Enumerating$ procedure. In each case, two parts are to be proved: (1) PE can enumerate all paths in $succ_{\text{path}}(u)$ that are greater than $P$; (2) NC can identify whether the end nodes of the paths reported by PE are in $succ(u)$.

Case(1). $u \in C (C \rightarrow D)$

In this case, $succ(u)$ is a subset of the neighbors of $u$, since $d(BL_i, BL_{i+1}) = 1$. Therefore, all paths in $succ_{\text{path}}(u)$ that are greater than $P$ can be enumerated by PE.

The neighbors of $u$ belong to class $B (BL_{i-1})$, class $C (BL_i)$, or class $D (BL_{i+1})$. Let $x$ be in $succ(u)$. By Lemma 2 if and only if $x$ is not a $B$-node, $is_B(x)$ returns “$D$”. By Lemma 3 if and only if $x$ is a $B$-node, $C_{or}D(x)$ returns “$D$”. Thus NC can identify whether $x$ is in $succ(u)$.

Case(2). $u \in D (D \rightarrow A)$

In this case, $d(BL_i, BL_{i+1}) = d_2$, the local search from $u$ within radius $d_2$ can report all paths in $succ_{\text{path}}(u)$ that are greater than $P$ and their end nodes.

Any black node reported by PE belongs to either class $A$ or class $D$. Only the nodes in $A$ belong to $succ(u)$. From the observations of $AL$ (i.e., any node in class $D$ has at least one $B$-node neighbor, but any node in class $A$ has none), $D \rightarrow A$ identifies whether a reported node is in $succ(u)$.

Case(3). $u \in A (A \rightarrow B)$

For $d(BL_i, BL_{i+1}) = d_1$, the local search from $u$ within radius $d_1$ can report all paths in $succ_{\text{path}}(u)$ that are greater than $P$ and their end nodes.

All the nodes in $succ(u)$ can be reported by $LS_1$. For the reported black nodes, only the nodes in class $B$ are in $succ(u)$. According to Lemma 4, $u$ is in class $B$ iff $A_{or}B(x)$ returns “$B$”. Thus by calling $A_{or}B$ the nodes in $succ(u)$ can be identified.

Case(4). $u \in B (B \rightarrow C)$

For $succ(u)$ are the neighbors of $u$, $LS_1$ can report all paths in $succ_{\text{path}}(u)$ greater than $P$ and their end nodes. Any layer $C$ node is a black node without white neighbors. Thus NC can identify whether the end nodes of the paths reported by PE are in $succ(u)$.

In all above cases, if a node reported by PE is identified as a node in $succ(u)$ by NC, then the path $P'$ reported by PE is in $succ_{\text{path}}(u)$. 

$(A \rightarrow D)$, the robot records a path of length $d_2 + 2$ and maintains a constant number of variables, therefore the space is $O(d_2 \log \Delta)$ bits.
Now we consider the Identifying procedure. Let $x$ be the node returned by Enumerating. According to Lemma 6 Check_Par_Path($x$, $P^{t-1}$) can tell whether $P^{t-1}$ is the parent path of $x$, and if so, $R$ returns to $x$. Therefore, the minimal child path of $u$ that is greater than $P$ will be identified if it exists. If it does not exist, all paths reported by PE do not pass Identifying. $R$ returns to $u$ in the end, and the procedure returns “false”.

Denote by $T_{x ightarrow y}$ the number of edge traversals of each case in Next_Child_Par and Get_Par. By Lemma 7 Case $D ightarrow A$ has the maximal number of edge traversals that is $T_{D ightarrow A} \leq LS(d_2 + 2) + O(\Delta d_2)T_{A ightarrow D} = O(\Delta^{2d_2+2})$. For the memory of $R$, in the worst case ($D \rightarrow A$), the robot records two paths of length $d_2$ and $d_2 + 2$ and maintains a constant number of variables, thus the space is $O(d_2 \log \Delta)$ bits. ☐

We consider the cases that $r$ is an input of Next_Child_Par.

Lemma 8. Let $P$ be a path from $u = r$, containing only one edge $e$ (e maybe a self loop). Let $port(e, r) \neq \deg(r) - 1$, and let $R$ know that $u$ is a class C node. Next_Child_Par($u$, $P$) identifies the path $P'$, containing only one edge $e'$ from $r$ such that $port(e', r) = port(e, r) + 1$, as the minimal child path of $r$ that is greater than $P$.

Proof. We can verify that for any $x \in BL_1 \cup \{r\}$, Is_B(x) returns “D”. Thus the following two statements hold. (1) In the Enumerating part of Next_Child_Par($r$, $P$), the end node $u$ of $P''$ will be returned. (2) For any path $P''$ from $u$ to $r$ that contains one edge, Check_Par_Path($u$, $P''$) returns “true”. Thus the lemma is proved. ☐

The overall exploration performed by our algorithm is the DFSs of subtrees rooted at each $BL_1$ node of the implicit spanning tree along with an exploration of $BL_1 \cup \{r\}$. The robot starts from $r$ and explores each node in $BL_1$ and then explores the subtree rooted at each node. By Lemma 6 starting from any $x \in BL_1$, the robot can conduct a DFS of the subtree rooted at $x$. Since the robot can identify nodes in $BL_1$, it can identify whether the DFS of a subtree is finished. If there are multi-edges between $r$ and $x$, the subtree of $x$ will be explored more than once. If $r$ has self loops, $r$ will be identified as a $D$ layer node but without any child in the spanning tree. In DFSs all the black nodes will be visited. For any white node $y$, let $BL_1$ be the black layer such that $d(r, BL_1) < d(r, y)$, and $\ell = d(y, BL_1)$ is the minimal. By Property 1 there exists $x \in BL_1$ such that there is an $\ell$-white path from $y$ to $x$. Thus, the PE procedure of Next_Child_Par($u$) in a DFS will visit $y$. Therefore, all white nodes will also be visited by DFSs, and thus all the nodes in $G$ will be visited. The robot stops once the exploration is completed, i.e., the robot returns to $r$ via the largest port at $r$.

2.7 Bound on the Number of Edge Traversals

By Lemma 7 the maximal number of edge traversals of one call to an exploration procedure is $O(\Delta^{2d_2+2})$. In the DFS, when the robot moves from node $u$ to parent($u$) through the parent path $P$ of $u$ in state up, the robot has to move back to $u$ to search for the minimal child path greater than $P^{-1}$. The total number of edge traversals of these moving backs is not greater than $bd_2$ where $b = o(n)$ is the number of black nodes. By Lemma 8 the edges from $r$ are all identified as child paths in the DFS. If there are $q$ edges between $r$ and a $BL_1$ node $x$, the subtree rooted at $x$ will be traversed $q$ times. Denote by $T_{all}$ the total number of edge traversals by the robot. We have $T_{all} \leq \Delta(\Delta^{2d_2+2} + d_2)o(n) = O(\Delta^{2d_2+5n} + o(d_2)\Delta n)$.

For simple graphs, the repetitive traversals can be avoided. When using our algorithm to explore a simple graph labeled by the 1-bit labeling scheme of $(r, 2, 4)$, the total number of edge traversals by the robot is $O(\Delta^{10n})$ which is similar to that in 3.

Given an $\mathcal{AL}$ labeling $\langle r, d_1, d_2 \rangle$ on $G$ with $L$-ratio $\rho'$. If there are six black layers and $D = d_1 + d_2 + 3$, the labeling has the minimal $L$-ratio $\frac{d_1 + d_2 + 4}{6}$, i.e., $\rho' \geq \frac{d_1 + d_2 + 4}{6}$, so $d_1 + d_2 \leq 6\rho' - 4$. For $d_1 \geq 2$, we have $2d_2 + 3 \leq 2(d_1 + d_2) - 1 \leq 12\rho' - 9$. Thus our exploration algorithm completes in time $O(\Delta^{12\rho' - 9n} + o(d_2)\Delta n)$. Since no more than a constant number of paths need to be stored at the same time and the length of such a path is not greater than $d_2$, $O(d_2 \log \Delta) = O(\rho' \log \Delta)$ bits of memory is necessary for the robot to explore the graph.
3 Exploration While Labeling

We present an algorithm allowing the robot to label the graph according to an $\mathcal{AC}$ labeling. As in [3], we assume that before labeling, the graph nodes are labeled by an initial color named “blank” that the robot can identify. The labeling algorithm takes as input an $\mathcal{AC}$ labeling $L = \langle r, d_1, d_2 \rangle$ and labels the black layers in order. Denote by $G_i$ the subgraph of graph $G$ induced by all nodes at distance at most $d(r, BL_i)$ from the root. In phase $i$ ($i \geq 2$) of the algorithm, the robot starts from the root and traverses all nodes in $G_i$ and colors the nodes in $BL_i$ black and colors the nodes in layers between $BL_{i-1}$ and $BL_i$ white. At the end of phase $i$, the robot has colored $G_i$ according to $L$ and returned to the root. During the labeling, the labeling algorithm labels each node only once.

In phase $i$, we call $BL_{i-1}$ the border layer, nodes in the border layer border nodes, and the set of nodes that are in the layers between $BL_{i-1}$ and $BL_i$ the working interval. In this section, we always use $BL_{i-1}$ to denote the border layer.

Initially, the robot labels the root (phase 0) and its neighbors black (phase 1). It then returns to the root. In phase $i$ ($i \geq 2$), if the border layer belongs to class $A$ or $D$, the labeling procedure has two stages:

(1) The robot colors all nodes in $BL_i$ black and returns to $r$.
(2) The robot colors all nodes in the working interval white and returns to $r$.

If the border layer belongs to class $B$ or $C$, there is only stage 1. We use $X.x$ to denote the stage of the labeling algorithm in which the border layer belongs to class $X$ and the stage is $x$. A 3-bit variable $\text{stage}$ is used to store the stage, initialized to $D.1$ in phase 2.

The labeling algorithm includes two procedures: (1) the exploration procedure; (2) the labeling procedure. The exploration procedure is a revision of the exploration procedure in Section 2.5. In a stage of phase $i$, the robot identifies some border nodes by the exploration procedure and calls the labeling procedure from each of these nodes to label the blank nodes. After calling the labeling procedure from a node, the robot sets state to up and moves up to the parent of the node. When the robot returns to $r$ from the largest port, variable $\text{stage}$ transforms according to the following diagram.

$$D.1 \rightarrow D.2 \rightarrow A.1 \rightarrow A.2 \rightarrow B.1 \rightarrow C.1$$

3.1 Labeling the Nodes

The robot uses the $\text{Label}_\text{Succ}$ procedure to color nodes. In stage $\ast.1$, for a node $u$ in the border layer, $\text{Label}_\text{Succ}$ colors all nodes in $\text{succ}(u)$ black. In stage $A.2$ and $D.2$, procedure $\text{Label}_\text{Succ}$ colors all nodes in the working interval white.

$\text{Label}_\text{Succ}$ accepts a parameter: $u$, a node in the border layer. The detail of $\text{Label}_\text{Succ}$ is given in the following where we consider six cases.

(1) stage = $B.1$ or $C.1$. The robot labels all blank neighbors of $u$ black.

(2) stage = $D.1$. The robot performs a local search from $u$ within radius $d_2$. For each reported blank node $x$, it performs a local search from $x$ within radius $d_2 - 1$. If all black nodes visited in the local search have no $B$-node neighbor, then the robot colors $x$ black.

(3) stage = $D.2$. The robot performs a local search from $u$ within radius $d_2$. For every visited blank node, the robot colors the node white.

(4) stage = $A.1$. The robot performs a local search from $u$ within radius $d_1$. For every blank node $x$ reported, it performs a local search from $x$ within radius $d_1 - 1$. If all black nodes visited in the local search do not have any white neighbor, then the robot colors $x$ black.

(5) stage = $A.2$. The robot performs a local search from $x$ within radius $d_1$. For every visited blank node, the robot colors the node white.
Table 2: In each case, for each stage the robot performs an operation when visiting a blank node. “-” means that the robot will not visit a blank node in a combination of a case and a stage. “◊” denotes the operation to return to $u$ and call $\text{Label}_{\text{Succ}}(u)$. “♦” denotes the operation to ignore the blank node.

| Case \ stage | D.1 | D.2 | A.1 | A.2 | B.1 | C.1 |
|--------------|-----|-----|-----|-----|-----|-----|
| $D \rightarrow A$ | ◊ | ◊ | ◊ | ◊ | - | - |
| $A \rightarrow B$ | - | ◊ | ◊ | ◊ | ◊ | - |
| $B \rightarrow C$ | ◊ | ◊ | - | ◊ | ◊ | ◊ |
| $C \rightarrow D$ | ◊ | ◊ | - | - | ◊ | ◊ |

3.2 Revising the Exploration Procedure

We revise the exploration procedure in Section 2.5 to explore the colored subgraph and color the uncolored subgraph. In a local search, when we say that the robot ignores a node, we mean that as soon as the robot moves in the node, it leaves this node by the port from which it moves in, not visiting any neighbor of the node, and continues the local search. The revisions are given as follows.

The revised $\text{Get}_{\text{Par}} \cdot \text{Path}$ procedure ignores all blank nodes it visited. The revised $\text{Next}_{\text{Child}} \cdot \text{Path}$ procedure ignores all blank nodes it visited except in the case where $\text{Next}_{\text{Child}} \cdot \text{Path}(u,P)$ visits a blank node in the case $X \rightarrow Y$ and stage $= X, \ast$. In this case, the robot returns to $u$ and calls $\text{Label}_{\text{Succ}}(u)$.

Table 2 gives the operation that the robot performs for each case of the $\text{Next}_{\text{Child}} \cdot \text{Path}$ procedure when visiting a blank node in different stages. When $\text{Label}_{\text{Succ}}(u)$ terminates, the robot is in node $u$, and it then backtracks to $\text{parent}(u)$ with state up and continues the exploration.

3.3 Correctness

For a black node $u$, let $u \in BL_k$, $rd = d(BL_k, BL_{k+1})$, denote by $\text{wdisc}(u)$ the set of nodes that the robot visits in a white local search within radius $rd$ from $u$. It is easy to verify that the whole graph $G$ is colored according to a labeling scheme $L$, if all the nodes in $\text{wdisc}(u)$ are colored according to $L$ for any black node $u$. A blank node $u \in L_k$ ($k \geq 0$) that has no neighbor in $L_{k+1}$ is called a leaf node.

We prove the correctness of the $\text{Label}_{\text{Succ}}$ procedure in the following.

Lemma 9. Let $u \in BL_{i-1}$, and let $G_{i-1}$ ($i \geq 2$) be colored according to an $AC$ labeling $L$. If stage $= B.1$ or $C.1$ or $A.1$ or $D.1$ and some nodes in $BL_i$ are colored black, $\text{Label}_{\text{Succ}}(u)$ colors all blank nodes in $\text{succ}(u)$ black, not coloring any other nodes. If stage $= A.2$ or $D.2$ and $BL_i$ is colored according to $L$ and some nodes in the working interval are colored white, $\text{Label}_{\text{Succ}}(u)$ colors all blank nodes in $\text{wdisc}(u)$ white, not coloring any other nodes.

Proof. Let the border layer belong to class $C$ or $B$, and stage $= C.1$ or $B.1$ accordingly. Since $G_{i-1}$ and part of $BL_i$ are colored according to $L$, all blank neighbors of $u$ are in $\text{succ}(u)$. $\text{Label}_{\text{Succ}}(u)$ only labels all blank neighbors of $u$ black. Therefore, $\text{Label}_{\text{Succ}}(u)$ colors all blank nodes in $\text{succ}(u)$ black, not coloring any other nodes.

Let $u \in D$, and stage $= D.1$. Let $x$ be a blank node reported by the local search from $u$ within radius $d_2$. In this case, $G_{i-1}$ and part of $BL_i$ are colored according to $L$, and all nodes in the working interval are blank. If $x \in \text{succ}(u)$, nodes at distance not greater than $d_2 - 1$ from $x$ are either blank nodes or black nodes in $BL_i$; otherwise there is at least one node in $BL_{i-1}$ at distance less than $d_2 - 1$ from $x$ by Property 1. Layer $BL_{i-1}$ is a $D$-layer in which every node has a $B$-node neighbor, while each node in $BL_i$ has no $B$-node neighbor. Therefore, $\text{Label}_{\text{Succ}}(u)$ can determine whether $x$ is in $\text{succ}(u)$. So $\text{Label}_{\text{Succ}}(u)$ colors all blank nodes in $\text{succ}(u)$ black, not coloring any other nodes.

Let $u \in A$, and stage $= A.1$. Let $x$ be a blank node visited by the local search from $u$ within radius $d_1$. If $x \in \text{succ}(u)$, all nodes at distance not greater than $d_1 - 1$ from $x$ are either blank nodes or black nodes in $BL_i$; otherwise some of these nodes may belong to $BL_{i-1}$. Since $G_{i-1}$ has been colored according to $L$ and $BL_{i-1}$
is an $A$-layer, every node in $BL_{i-1}$ has a white neighbor. For nodes in the working interval are blank in stage $A.1$, any node in $BL_i$ has no white neighbor. By this observation, $Label_{\text{Succ}}(u)$ can determine whether $x$ is in $\text{succ}(u)$. So $Label_{\text{Succ}}(u)$ colors all blank nodes in $\text{succ}(u)$ black, not coloring any other nodes.

If $G_{i-1}$ and $BL_i$ are colored according to $L$ and stage = $D.2$ or $A.2$, by definition, $Label_{\text{Succ}}(u)$ colors all blank nodes in $wdisc(u)$, not coloring any other nodes.

Now we prove the correctness of the labeling algorithm.

**Theorem 2.** By the end of the execution of the labeling algorithm taking as input an $AL$ labeling $L = \langle r, d_1, d_2 \rangle$, the graph is fully colored according to $L$, and the robot has explored the entire graph, terminating at the root.

**Proof.** For each $i \geq 0$, we say that $Property(i)$ holds at the end of phase $i$, if

1. The robot colors all nodes of $G_i$ according to $L$ and returns to the root.
2. Only nodes of $G_i$ are colored.

We now prove that, at the end of phase $i$, $Property(i)$ holds. Initially, $Property(1)$ holds at the end of phase 1. For $i \geq 1$, assume that at the end of phase $i-1$, $Property(i-1)$ holds. We prove that $Property(i)$ holds at the end of phase $i$.

By the induction hypothesis, during phase $i$, all nodes of $G_{i-1}$ are colored according to the labeling $L$, and all other nodes are blank.

We first prove that in $X \rightarrow Y$ of $\text{NextChildPath}$ from $u$, if a blank node is visited and $\text{stage} = X.*$ then $u$ is a border node. By definition, for $v \in BL_i$, $\text{NextChildPath}$ from $v$ will not visit any node in $L_i$ such that $t > d(r, BL_{i+2})$. For a class $X$ node $v \in BL_i$ ($s < i - 1$), we have $s \leq (i - 1) - 4$, since all blank nodes are in layers after $BL_{i-1}$, $\text{NextChildPath}$ from $v$ will not visit any blank node. Therefore, according to Table 2, $u$ is a border node if $Label_{\text{Succ}}(u)$ is called. The robot returns to $parent(u)$ with state up when $Label_{\text{Succ}}(u)$ terminates.

Suppose that for $u \in BL_{i-1}$, in a call to $\text{NextChildPath}$ from $u$, the robot does not visit any blank node and arrives at a child of $u$ say $v$. By definition, all neighbors of $v$ will be visited in $\text{NextChildPath}$ from $u$. Therefore, node $v$ has no neighbor in layer after $BL_i$, and $v$ is a leaf node. In the followed exploration from $v$, blank nodes will be ignored (see Table 2). $\text{NextChildPath}(v, \varnothing)$ returns “false”, and $\mathcal{R}$ will return from $v$ to $u$ in state up. If all calls to $\text{NextChildPath}$ from $u$ do not find a blank node then all children of $u$ are leaf nodes, which implies that $wdisc(u)$ has been colored according to $L$, and $\mathcal{R}$ returns to $parent(u)$ in state up not visiting any node beyond $G_i$.

By the above argument, for $u \in BL_{i-1}$, no matter whether $Label_{\text{Succ}}(u)$ is called, the robot will return to $parent(u)$ in state up. For $u \notin BL_{i-1}$, the blank nodes will be ignored in explorations from $u$ (see Table 2). For $G_{i-1}$ is colored correctly, by Theorem 1 in phase $i$, all border nodes are visited, and finally the robot returns to the root.

In the end of phase $i$, for $u \in BL_{i-1}$, either $Label_{\text{Succ}}(u)$ is called or $wdisc(u)$ has been colored according to $L$. By Lemma 9 for every border node $u$, nodes in $wdisc(u)$ are colored correctly. Therefore all the nodes in $G_i$ is colored correctly. Since $\bigcup_{u \in BL_{i-1}} wdisc(u) \subseteq G_i$, only nodes of $G_i$ are colored.

In summary, for each $i \geq 0$, $Property(i)$ holds at the end of phase $i$. It follows that after $\lceil (D + 1)/(d_1 + d_2 + 2) \rceil$ phases, the robot has fully colored and explored the entire graph. In the end, the last phase is performed, in which the robot finds that the exploration and the coloring are completed.

\[ \square \]

## 4 Labeling Schemes Enabling Adjusting the Ratio of Black Nodes

Based on $AL$ labeling schemes, we introduce the labeling schemes that allow the adjustment of the $N$-ratio. We will prove the following in the remaining of Section 4.
Theorem 3. There exists a robot with the property that for any $n$-node graph $G$ of degree bounded by integer $\Delta$, it is possible to color the nodes of $G$ with two colors (black and white), while the $N$-ratio is not less than a given rational number $\rho \in (2, (D + 1)/4]$. Using the labeling, the robot can explore the graph $G$, starting from a node $v$ and terminating at $v$ after identifying that the entire graph has been traversed. The robot has $O(\rho \log \Delta)$ bits of memory, and the total number of edge traversals by the robot is $O((n\Delta^{10\rho+7}/\rho + \Delta^{40\rho+10})).$

In the remainder of the paper, word “ratio” refers to “$N$-ratio” if not mentioned.

4.1 From $L$-ratio Tunable to $N$-ratio Adjustable

We generalize the $AC$ labeling to the periodic layer oriented labeling (PL in short). A PL labeling of a graph is composed of a root node and the sets of layers are colored black and white. A PL labeling colors the graph in a periodic manner, that is, $L_i$ and $L_{i+p}$ are colored with the same color where $p$ is the period. We can represent a PL labeling by a triple $(r, p, BL)$, where $r$ is the root, $0 < p \leq D + 1$ is an integer denoting the period, and $BL$ is an integer set on $[0, p - 1]$ denoting the black layers within a period. The set of black layers of the labeling $(r, p, BL)$ is $\{L_i | (i \mod p) \in BL, 0 \leq i \leq D\}$. We call the interval $[ip, (i + 1)p - 1], i \geq 0$, the $i$th unit of the labeling. For example, the labeling in Figure 1 can be denoted by $(r, 11, \{0, 1, 7, 10\})$. Let $S_1 = (r, p, BL), S_2 = (r, p, BL')$ be two PL labeling schemes with the same root and period. The union of $S_1$ and $S_2$ is denoted by $S_1 \cup S_2 = (r, p, BL \cup BL')$. Denote by $N(L_i)$ the number of nodes in layer $L_i$ of a labeling scheme $P$. Denote by $BN(P)$ the number of black nodes in $P$. Denote by $\rho(P) = n/BN(P)$ the $N$-ratio of the labeling scheme $P$.

We relax some restrictions of the $AC$ labeling and define the following:

Labeling $MP$. $MP = (r, p, BL)$ is a PL labeling, where $BL = \{PB, PC, PD, PA\}$ satisfies the following properties.

- $(PC - PB) \mod p = 1$,
- $(PD - PC) \mod p = 1$,
- $(PB - PA) \mod p = d_{AB}, d_{AB} \geq 2$,
- $(PA - PD) \mod p = d_{DA}$,
- $d_{DA}/2 \geq d_{AB}$.

We call $MP$ labelings the elementary labelings, and any $AC$ labeling is an elementary labeling. In an elementary labeling, $p = d_{AB} + d_{DA} + 2$. For convenience, we use quadruple $(r, PA, d_{AB}, d_{DA})$ to denote an elementary labeling, e.g., the $AC$ labeling can be denoted by $(r, d_{DA} + 1, d_{AB}, d_{DA})$. The labeling in Figure 2 can be denoted by $(r, 7, 3, 6)$. In this section, all labelings are elementary labelings or combinations of elementary labelings.

As for $AC$ labelings, we partition the black nodes in an elementary labeling to the following four sets:

$C = \{v \in V | d(r, v) \mod p = PC\}$,
$D = \{v \in V | d(r, v) \mod p = PD\}$,
$A = \{v \in V | d(r, v) \mod p = PA\}$,
$B = \{v \in V | d(r, v) \mod p = PB\}$.

An $AC$ labeling scheme cannot guarantee that its $N$-ratio is not less than its $L$-ratio. The following lemma implies a method to close the gap.

Lemma 10. Given a rational number $1 \leq \rho \leq D + 1$, let $\rho = m/\ell$, where $m > 0$ and $\ell > 0$ be integers. Let $PS$ be a set of labeling schemes of $G$ that have the same root, and $|PS| = m$. If $\sum_{P \in PS} BN(P) = \ell n$, then there exists $P \in PS$ such that $\rho(P) \geq \rho$. 
Proof. From $\sum_{P \in PS} BN(P) = tn$, we have
\[
\sum_{P \in PS} \frac{BN(P)}{n} = \sum_{P \in PS} \frac{BN(P)}{n} = \sum_{P \in PS} \frac{1}{\rho(P)} = t.
\]
(1)

By the pigeonhole principle, for $|PS| = m$, there exists $P \in PS$ such that $1/\rho(P) \leq t/m$. Therefore, there exists $P \in PS$ such that $\rho(P) \geq m/t = \rho$. \qed

If we find a set of labelings that satisfies Lemma[10], then we can find a labeling where the $N$-ratio is not less than a given rational number. To generate such labelings, we introduce the circular shifts of a labeling. For a labeling $P = \langle r, p, BL \rangle$ and an integer $0 \leq l \leq D$, denote $P^l = \langle r, p, BL^l \rangle$, where $BL^l = \{ i \mid (i - l) \mod p \} \in BL, 0 \leq i < p \}$, called a circular shift of $P$. We give some $N$-ratio adjustable labeling schemes as follows.

Let $\rho \in [4, (D + 1)/4]$ be an integer, and let $S = \langle r, 4\rho, BL \rangle$, where $BL = \{0, 5, 6, 7\}$. We have that $\bigcup_{i=0}^{p-1} BL^{4i} = [0, 4\rho - 1]$, and for any $0 \leq i, j \leq \rho - 1$, $BL^{4i} \cap BL^{4j} = \emptyset$. Let $H = \{ S^0, S^4, \ldots, S^{4(\rho - 1)} \}$. We have $\sum_{i=0}^{p-1} BN(S^{4i}) = n$. By Lemma[10] there exists $S^{4i} \in H$ such that $\rho(S^{4i}) \geq \rho$. This method does not work for $\rho = 3$. We give a solution in Figure 5 where we use six different labeling schemes $T_1, \ldots, T_6$ with period 12, and each layer is colored black by exactly two labeling schemes. Thus $\sum_{i=1}^{6} BN(T_i) = 2n$. By Lemma[10] there is $T_i \in \{ T_1, \ldots, T_6 \}$ such that $\rho(T_i) \geq 3$.

4.2 N-ratio Adjustable Labeling Schemes

In this subsection, we introduce a general method to construct the $N$-ratio adjustable labelings based on Lemma[10] We only discuss the cases where $\rho \geq 2$. For $1 < \rho < 2$, we can compute $\rho'' = \rho/(\rho - 1) \geq 2$. A labeling with $N$-ratio $\rho$ can be derived from a labeling with $N$-ratio $\rho''$ by reversing the color of each node. Given a rational number $\rho \geq 2$, let $\rho = m/t$ where $m$ and $t$ are relatively prime. The idea is to find a labeling scheme $P = \langle r, 4m, BL \rangle$ where $|BL| = 4t$, $\rho(P) = \rho$. Let $D + 1 \geq 4m$, we demonstrate that such $P$ exists. If $m$ is so huge that $D + 1 < 4m$, we have to find $m'$ and $t'$ that are relatively primes such that $\rho < m'/t'$ and $D + 1 \geq 4m'$. Then we try to find a labeling scheme $P = \langle r, 4m', BL \rangle$ where $|BL| = 4t'$, $\rho(P) = m'/t' > \rho$. Let the length of the unit be $p = 4m, x = 4m \mod t$. Partition each unit into $t$ disjoint intervals. The first $x$ intervals are of length $|p/t|$, the others are of length $|p/t|$. Let $d_{AB} = |(\lfloor p/t \rfloor - 2)/3|$, $d'_{DA} = |p/t| - 2 - d_{AB}$, and $d''_{DA} = |p/t| - 2 - d_{AB}$. We have $d_{DA}/2 \geq d_{AB}$ and $d''_{DA}/2 \geq d_{AB}$. The following $t$ elementary labelings with the same root and period can be derived.

![Figure 5: Examples of adjustable labeling schemes.](image1)

![Figure 6: Above is a unit of a labeling scheme with a rational $\rho$ where $m = 7$, $t = 3$. Below is the root unit of $P$. Intervals in a unit are in dotted boxes.](image2)
$$S_i = \begin{cases} 
(r, \lfloor 4m/t \rfloor i, d_{AB}, 4m - d_{AB} - 2), & 0 \leq i \leq x - 1, \\
(r, \lfloor 4m/t \rfloor (i - x), d_{AB}, 4m - d_{AB} - 2), & x \leq i \leq t - 1.
\end{cases}$$

Let $P = \bigcup_{i=1}^{t} S_i = \langle r, 4m, BL \rangle$. We have $|BL| = 4t$. In $P$, we classify the nodes into four classes $A, B, C,$ and $D$. Class $X \in \{A, B, C, D\}$ of $P$ is the union of class $X$ of all $S_i (0 \leq i \leq t - 1).$ There are totally $4m$ circular shift labelings of $P$. For every layer $L_k$, there are $4t$ circular shift labelings of $P$ where $L_k$ is labeled black. Therefore, $\sum_{k=1}^{m} BN(P^k) = 4tn$. By Lemma 10 there exists a circular shift of $P$, say $P^*$, such that $\rho(P^*) \geq m/t$. An example of $P$ is shown in Figure 6.

### 4.2.1 Transformation of $P^*$

We call the labelings where $r$ is in class $C$ the $R^C$ labelings. All $AL$ labelings are $R^C$ labelings. Let $P^*$ be the labeling such that $\rho(P^*) \geq \rho$, we can see that $P^*$ is not necessarily a $R^C$ labeling. In this section, we give a method to transform $P^*$ which is not a $R^C$ labeling to a $R^C$ labeling; the exploration algorithm for $AL$ labelings can be used after minor revisions. The transformation of $P^*$ is as follows.

Label $r$ and its neighbors black. Let the first $A$-layer in $P^*$ be $L_i$. If $\lfloor (l - 1)/2 \rfloor \geq d_{AB}$, we label the layers between $L_1$ and $L_1$ white. Otherwise, we label the layers between $L_1$ and the second $A$-layer white if this layer exists. If there is only one $A$-layer, then we label the layers after $L_1$ white. Denote the resulted labeling by $\hat{P}^*$. We redefine the units of $\hat{P}^*$ as follows: the interval $[0, d_r]$ is the 0th unit called root unit, where $d_r$ is either the distance between the root and the first $A$-layer if $A$-layers exist or the diameter of $G$ if no $A$-layers exist; the interval $[(i - 1)p + d_r, i p + d_r - 1]$ is the $i$th unit. We have $d_r \geq d_{DA}$.

It is possible that $BN(\hat{P}^*) > BN(P^*)$, and therefore $\rho(\hat{P}^*) < \rho$. To make sure $\rho(\hat{P}^*) \geq \rho$, we modify the transformation as follows. The root is chosen as a node with the minimal number of neighbors, say $\Delta'$. Label $L_0$ and $L_1$ black. If there exists an $A$-layer in $P^*$, say $L_k$, such that there is only one $C$-layer before $L_k$, we label the layers between $L_1$ and $L_k$ white. If no such $A$-layer exists, the diameter $D$ of $G$ is so short that we label the layers after $L_1$ white.

We prove that $\rho(P^*) \geq \rho$ as follows. Suppose that $L_k$ exists. Let $nb_1$ be the total number of black nodes in layers before $L_k$ in $P^*$, and let $nb_2 = N(L_1) + 1 = \Delta' + 1$ be that in $P^*$ which is the number of neighbors of $r$ plus 1. We have $BN(P^*) - BN(\hat{P}^*) = nb_1 - nb_2$. In $P^*$, before layer $L_k$, there are a $C$-layer and a $D$-layer, thus if the root is in a $C$-layer in $P^*$, then $P^*$ and $\hat{P}^*$ are similar; otherwise there are three adjacent $B, C,$ and $D$ layers before $L_k$. Because $\Delta'$ is the minimal number of neighbors of a node in the graph and all the neighbors of nodes in the middle $C$-layer are involved in the three adjacent black layers, the number of black nodes in these three layers is not less than $\Delta' + 1 = nb_2$. Therefore, $nb_1 - nb_2 \geq 0$. Thus $BN(P^*) - BN(\hat{P}^*) \geq 0$. We have $\rho(\hat{P}^*) \geq \rho$.

Suppose that $L_k$ does not exist. Since $\rho \leq (D + 1)/4$, there are at least four black layers in the first unit of $P^*$. In this case, we have $D \leq d_r + d_2 - 1$, and in $P^*$ there is only one $A$-layer, and there are only one $B$-layer and one $C$-layer after this $A$-layer. When there are three adjacent $B, C,$ and $D$ layers after the $A$-layer, based on the above discussions, we have $\rho(\hat{P}^*) \geq \rho$. When there are no three adjacent $B, C$, and $D$ layers after the $A$-layer, the last two layers are a $B$-layer followed by a $C$-layer. Since all the neighbors of the nodes in the last $C$-layer are involved in the last two black layers, the number of black nodes in the last two black layers is not less than $\Delta' + 1 = nb_2$. Therefore, $nb_1 - nb_2 \geq 0$, and $BN(P^*) - BN(\hat{P}^*) \geq 0$. As a result, we have $\rho(\hat{P}^*) \geq \rho$.

### 4.2.2 Exploration Algorithm

We revise the graph exploration algorithm in Section 2 to explore the graph labeled by $\hat{P}^*$ as follows. First, the memory of $\mathcal{R}$ increases to $O(d_r \log \Delta)$ bits. Second, add a 1-bit flag $fr$. If $\mathcal{R}$ is in the root unit, $fr = 1$, otherwise $fr = 0$. Third, in the following cases, $\mathcal{R}$ first determines the distance between a $D$-layer and the adjacent $A$-layer (we call this distance “$d_2$” of the current interval) is $d_{DA}$ or $d_{DA} + 1$ or $d_r$ as follows.

1. $D \rightarrow A$ of Next Child Path.

Assume that $\mathcal{R}$ is currently in a $D$-layer node $u$. If $fr = 1$, we set $d_2 = d_r$ and execute the procedure. If $D \rightarrow A$ succeeds we set $fr = 0$. 
Let \(fr = 0\). We first determine whether \(u\) is a leaf node; if not, we determine the distance between a \(D\)-layer and the adjacent \(A\)-layer. Then we backtrack from \(u\) or call \(D \rightarrow A\) with the correct \(d_r\). The distinguishing procedure is as follows. Perform a local search from \(u\) within radius \(d_{DA} (LS_1)\). If a black node in \(A\) is visited then \(d_2 = d_{DA}\). If no class \(A\) node is visited then perform a local search from \(u\) within radius \(d_{DA} + 1 (LS_2)\). If a black node \(v\) in \(A\) is visited then perform a local search from \(v\) within radius \(\lfloor d_{DA}/2 \rfloor\). For each white node \(x\) reported, check whether \(R_{W}(x) = \lfloor d_{DA}/2 \rfloor - 1\), and if so, perform a local search within radius \(\lfloor d_{DA}/2 \rfloor\) from \(x\). If a node with a \(B\)-node neighbor is reported, we have \(d_2 = d_{DA}\) and \(u\) is a leaf node; otherwise \(d_2 = d_{DA} + 1\). If no class \(A\) node is found in \(LS_1\) and \(LS_2\) then \(u\) is a leaf node.

(2) \(A \rightarrow D\) of Get_Par_\(Path\). Assume that \(R\) is currently in an \(A\)-layer node \(u\). We first set \(d_2 = d_{DA}\) and call the procedure. If \(A \rightarrow D\) fails to find the parent of \(u\), we set \(d_2 = d_{DA} + 1\) and redo \(A \rightarrow D\). If it fails again, we set \(d_2 = d_r\) and \(fr = 1\) and redo the procedure.

Now we consider the space and the time complexity of the exploration algorithm. For \(d_{DA} = \lfloor p/t \rfloor - 2 - d_{AB}\), we have \(d_{DA} = \lfloor 4p \rfloor - 2 - \lfloor \lfloor 4p \rfloor - 2 \rfloor/3 \leq 8p/3\). If \(L_0\) is a \(D\)-layer in \(P^*\), then \(\hat{P}^*\) has the maximal \(d_r\). In this case, \(d_r = d_{DA} + 1 + \lfloor p/t \rfloor \leq 29p/3\). Thus, the memory of \(R\) is still \(O(\rho \log \Delta)\). Since \(d_r \geq d_{PA}\), the number of edge traversals in exploring the root unit is increased comparing with \(AC\) labelings. The increased number of traversals is \(O(\Delta^2 2d_2 + 2) = O(\Delta 20\rho + 10)\). The total number of edge traversals is \(O(n\Delta^{2d_{DA}+1} + \Delta^{2d_r}) = O(n\Delta^{16\rho + 5}/\rho + \Delta^{40\rho + 10})\).

4.3 Labeling Algorithm

We use the algorithm in Section 4 with minor revisions to label a graph according to \(\hat{P}^*\). The parameters of \(\hat{P}^*\) are determined by system designers, including: \(r, d_{AB}, d_{DA},\) and \(d_r\). The robot takes as input these parameters and labels the graph. The revisions of the exploration procedures are as follows. When \(R\) explores from a \(D\)-layer node or an \(A\)-layer node, the robot has to know whether the distance from the \(D\)-layer to the adjacent \(A\)-layer (denoted by \(d_2\)) is \(d_{DA}\) or \(d_{DA} + 1\). We define a variable \(c\) of \(\lfloor \log t \rfloor\) bits to indicate that \(R\) is in the \(c\)th interval in a unit. Let there be \(j\) intervals before the first \(A\)-layer of \(\hat{P}^*\) in the first unit of \(P^*\). According to the definition of \(\hat{P}^*\), \(d_2 = d_{DA} + 1\) if \((c + j + t) \mod 4 < 4m \mod t\), and \(d_2 = d_{DA}\) otherwise. In this description, all arithmetic operations are modulo \(t\). Initially, variable \(c\) is set to \(t - 1\). \(c\) increases by 1 after \(R\) traversed from a \(D\)-layer node down to an \(A\)-layer node and decreases by 1 after \(R\) traversed from an \(A\)-layer node up to a \(D\)-layer node.

When starting from a class-\(A\) node or a class-\(D\) node, by \(c\) and \(fr\), the robot knows exactly \(d_2\) of the current interval. So the original exploration procedures in Subsection 2.5 can be used to explore the graph when \(c\) and \(fr\) is introduced.

Procedure Label_Succ does not need revision, since the robot knows \(d_2\) of the current interval. Then using the revised exploration algorithm in this section, the labeling algorithm in Section 5 can label the graph according to \(P^*\).

5 Future Work

Further interesting questions include whether there exist labeling schemes that are not spanning tree based, and whether there exists a labeling algorithm for an \(AC\) labeling that only uses two colors. The parameters of the \(N\)-ratio adjustable labeling scheme, i.e., the root, are determined by system designers. A question is whether there exists a finite state automaton that takes as input a valid \(N\)-ratio and labels the graph accordingly.

Acknowledgement

The authors thanks Leszek Gąsieniec and the anonymous referees for their constructive suggestions.

\[\text{\textsuperscript{1}}\text{If } d_{DA} \text{ is even, } \lfloor d_{DA}/2 \rfloor = d_{DA}/2. \text{ If } d_{DA} \text{ is odd, } \lfloor d_{DA}/2 \rfloor = \lfloor d_{DA}/2 \rfloor + 1\]
References

[1] C. Ambühl, L. Gąsieniec, A. Pelc, T. Radzik, X. Zhang, Tree exploration with logarithmic memory. *ACM Transactions on Algorithms* 7(2): 17, 2011.

[2] M. A. Bender, A. Fernandez, D. Ron, A. Sahai, S. Vadhan, The power of a pebble, Exploring and mapping directed graphs, *Inform. and Comput.* 176 (1): 1–21, 2002.

[3] R. Cohen, P. Fraigniaud, D. Ilcinkas, A. Korman, D. Peleg, Label-guided graph exploration by a finite automaton, *ACM Transactions on Algorithms* 4(4): 2008.

[4] J. Czyzowicz, S. Dobrev, L. Gąsieniec, D. Ilcinkas, J. Jansson, R. Klasing, I. Lignos, R. Martin, K. Sadakane, and W. K. Sung, More Efficient Periodic Traversal in Anonymous Undirected Graphs, In: *16th Colloquium on Structural Information and Communication Complexity (SIROCCO)*, LNCS 5869, 167–181, 2009.

[5] K. Diks, P Fraigniaud, E. Kranakis, A. Pelc, Tree Exploration with Little Memory, *Journal of Algorithms* 51 (1), 38–63, 2004.

[6] S. Dobrev, J. Jansson, K. Sadakane, W. K. Sung: Finding short right-hand-on-the-wall walks in graphs, In: *Proc. 12th Colloquium on Structural Information and Communication Complexity (SIROCCO)*. LNCS 3499, 127–139, 2005.

[7] P. Fraigniaud, D. Ilcinkas, Digraphs exploration with little memory, In: *21st Ann. Symp. on Theor. Aspects of Comput. Sci. (STACS)*, LNCS 2996, 246–257, 2004.

[8] P. Fraigniaud, D. Ilcinkas, G. Peer, A. Pelc, D. Peleg, Graph exploration by a finite automaton, *Theor. Comput. Sci.* 345(2–3): 331–344, 2005.

[9] P. Fraigniaud, D. Ilcinkas, A. Pelc: Tree exploration with advice. *Inf. Comput.* 206(11): 1276–1287, 2008.

[10] P. Fraigniaud, D. Ilcinkas, S. Rajsbaum, S. Tixeuil, Space lower bounds for graph exploration via reduced automata, In: *Proc. 12th Colloquium on Structural Information and Communication Complexity (SIROCCO)*, LNCS 3499, 140–154, 2005.

[11] P. Flocchini, D. Ilcinkas, A. Pelc, N. Santoro, Remembering without memory: tree exploration by asynchronous oblivious robots, *Theor. Comput. Sci.* 411, 1583–1598, 2010.

[12] P. Flocchini, B. Mans, N. Santoro, Exploration of Periodically Varying Graphs. In: *20th Int. Symp. on Algorithms and Computation (ISAAC)*, LNCS 5878, 534–543, 2009.

[13] L. Gąsieniec, R. Klasing, R. Martin, A. Navarra, X. Zhang, Fast periodic graph exploration with constant memory, *J. Comput. Syst. Sci.* 74(5): 808–822, 2008.

[14] D. Ilcinkas, Setting port numbers for fast graph exploration, *Theor. Comput. Sci.* 401(1–3): 236–242, 2008.

[15] A. Korman, S. Kutten, D. Peleg, Proof labeling schemes. In: proc. *24th Annual ACM Symposium on Principles of Distributed Computing (PODC 2005)*, 9–18, 2005.

[16] A. Kosowski, A. Navarra, Graph Decomposition for Improving Memoryless Periodic Exploration, In: *34th International Symposium on Mathematical Foundations of Computer Science (MFCS)*, LNCS 5734, 501–512, 2009.

[17] O. Reingold, Undirected connectivity in log-space. *J. ACM* 55(4): 2008.

[18] H. Rollik, Automaten in planaren Graphen, *Acta Inform.* 13: 287–298, 1980.