ABSTRACT

Adaptive beamformers (ABFs) suppress interferers by placing a notch in the beampattern at the interferer direction. This suppression improves detection of a weaker signal in the presence of strong interferers. Hence the notch depth plays a crucial role in determining the adaptive gain obtained from using ABF over conventional beamforming. This research derives models for the mean notch depth of a diagonally loaded MVDR ABF for a single interferer case. The model describes the mean notch depth as a function of number of snapshots, the number of sensors in the array, the interferer to noise ratio (INR) level, the interferer direction and the diagonal loading level. The derivation uses random matrix theory results on the behavior of the eigenvectors of sample covariance matrix. The notch depth predicted by the model is shown to be in close agreement with simulation results over a range of INRs and snapshots.

Index Terms— adaptive beamforming, MVDR, notch depth, random matrix theory, sample covariance matrix, diagonal loading

1. INTRODUCTION

A common array processing problem is to detect a low power source in presence of high power interferers. A conventional beamformer (CBF) produces a static beampattern which attenuates interferers by a fixed amount at each bearing. At the output, the weaker signal of interest will be masked by the higher power interferer and undermines detection. Alternatively, adaptive beamformers (ABF) can suppress interferers by placing deep notches in the beampattern in the interferer direction. ABFs rely on the knowledge of the data covariance matrix to compute the beamformer weights. In reality, the ensemble covariance matrix (ECM) for data is unknown a priori. The traditional approach is to replace the ECM by the sample covariance matrix (SCM) to compute the beamformer weights.

A class of sample matrix inversion (SMI) ABFs involve inverting the SCM to compute the beamformer weights [1, Sec. 7.3]. If the number of snapshots ($L$) is less than or approximately equal to the number of array sensors ($N$), the SCM is unstable or ill-conditioned for inversion. A common approach is diagonally loading the SCM to make it invertible for computing the ABF weights. The minimum variance distortionless response (MVDR) beamformer is one of the most extensively used SMI ABFs [2]. The main focus of this paper is to characterize the mean notch depth of a diagonally loaded (DL) MVDR ABF.

Prior work by Richmond [3] derived expressions for the mean and the variance of the SCM based MVDR beampattern. However the derivation in [3] only considers the snapshot sufficient case ($N < L$) and does not include diagonal loading. More recently, Buck and Wage [4] used Random Matrix Theory (RMT) results to develop a model for the mean notch depth of the dominant mode rejection (DMR) ABF. DMR is a variant of the MVDR ABF that uses a constrained SCM instead of diagonal loading [5]. Mestre and Lagunas [6] have used RMT results to derive a deterministic expression for asymptotic output signal-to-interferer-plus-noise ratio (SINR) of a DL-MVDR ABF. Their analysis is focused on deriving an estimator for the optimum loading factor ($\delta$). Similarly, Pajovic et al. [7] used RMT results to derive an analytic expression for the output power of a DL minimum power distortionless response (MPDR) beamformer. MPDR assumes source signal is present in the training data [1, Sec. 6.2.4].

The results presented in this paper are similar in spirit to the work in [4], but for the DL-MVDR ABF also considered in [6]. Recent results from RMT are used to derive an approximate model for the mean notch depth of a the DL-MVDR. The model will describe the notch depth as a function of the diagonal loading level ($\delta$) in addition to the number of snapshots ($L$), number of sensors ($N$), the interferer to noise ratio (INR), and the interferer location ($\theta_1$).

The rest of the paper is organized as follows: The next section describes the MVDR beamformer and defines related terminologies. Sec. 3 summarizes the notch depth model derivation. The simulation results are discussed in Sec. 4 followed by a brief conclusion in Sec. 5.

2. THE MVDR BEAMFORMER

The MVDR beamformer is one of the most extensively used ABFs [1,2]. The weight vector for the MVDR ABF steered to bearing direction $\theta_0$ is,

$$w = \Sigma^{-1} v_0 / \left( v_0^H \Sigma^{-1} v_0 \right)$$

where $\Sigma$ is the $N \times N$ ECM and $v_0 = v(\theta_0)$ is the array steering vector corresponding to the look direction $\theta_0$. Assuming a stationary narrowband interferer with power $\sigma_1^2$ at bearing $\theta_1$ and unit power white background noise, the ECM is

$$\Sigma = \sigma_1^2 v_1 v_1^H + I = \sum_{i=1}^{N} \gamma_i \xi_i \xi_i^H,$$

where $\gamma_1 > \gamma_2 = \ldots = \gamma_N = 1$ are the eigenvalues and $\xi_i$ are the corresponding eigenvectors. In the single interferer case $\xi_1 = v_1 / \sqrt{N}$,
i.e., the principal eigenvector is a scaled version of the interferer steering vector \((v_1)\). The MVDR ABF places a notch in the direction corresponding to \(\xi_1\).

In practice the ECM is estimated by computing the SCM \((S)\) from \(L\) data snapshot vectors \(x_l\),

\[
S = \frac{1}{L} \sum_{l=1}^L x_l x_l^H = \sum_{i=1}^N g_i e_i e_i^H .
\]

where \(x_l = a_l v_1 + n_l\) such that \(a_l \sim C \mathcal{N}(0, \sigma_1^2)\) and \(n_l \sim C \mathcal{N}(0, I)\). Since the noise power is unity, the INR is equal to \(\sigma_1^2\). Similarly \(g_1 > g_2 > \ldots > g_N\) are the eigenvalues and \(e_i\) are the eigenvectors of the SCM. The DL SCM is computed as \(S_\delta = S + \delta I\) where \(\delta > 0\). The eigenvectors are invariant to DL. Hence the sample principal eigenvector \(e_1\) estimates the interferer direction. The DL-MVDR ABF weights \((\hat{w}_\delta)\) are computed by replacing \(S\) with \(S_\delta\) in (1).

2.1. Notch Depth

The notch depth is defined as the magnitude of the beampattern at true interferer direction, i.e., \(ND = |w_0 v_1|^2\). The ensemble notch depth for the DL-MVDR is

\[
ND_{ens} = \frac{\cos^2(v_0, v_1)(1 + \delta)^2}{|1 + \delta + N \sigma_1^2 \sin^2(v_0, v_1)|},
\]

where \(\cos^2(v_0, v_1)\) is the generalized cosine between \(v_0\) and \(v_1\) as defined in (3). \(ND_{ens}\) is the ideally achievable notch depth assuming the ECM is known. Computing with the weights and the SCM results in a notch depth \(ND\) which is shallower than the ensemble \(ND_{ens}\). The mismatch between the sample and ensemble principal eigenvectors is the main cause of this loss in notch depth (3). RMT has results on bias of eigenvectors \((e_i)\) of the SCM, which will be used to derive the mean notch depth model in the next section.

3. MODEL

This section derives two models for the DL-MVDR ABF notch depth. The models characterize notch depth as a function of the number of sensors \(N\), the number of snapshots \(L\), the INR \((\sigma_1^2)\), the interferer direction \((\theta_i)\) and the diagonal loading level \((\delta)\). The first model treats snapshots \((L)\) as the independent variable and INR \((\sigma_1^2)\) as a parameter. The second model characterizes the ND as a function of the INR \((\sigma_1^2)\) while treating the snapshots \((L)\) as a parameter. The derivation uses the RMT results on the fidelity of the sample principal eigenvectors.

The first part of the RMT result on the eigenvectors of SCM gives an expression for the magnitude of the projection between the sample and the ensemble principal eigenvector, (10) (11) (12),

\[
|e_i^H \xi_1^\perp|^2 \xrightarrow{a.s.} \begin{cases} 
1 - c/(N \sigma_1^2) & \sigma_1^2 > \sqrt{c}/N \\
1 + c/(N \sigma_1^2) & \sigma_1^2 \leq \sqrt{c}/N 
\end{cases}
\]

where \(c = N/L\). This result holds in the RMT asymptotic sense, i.e., \(N, L \rightarrow \infty, N/L \rightarrow c\). It implies that for a sufficiently strong interferer \((N \sigma_1^2 > \sqrt{c})\) the sample principal eigenvector \(e_1\) is a biased estimate of its ensemble counterpart.

The second part of the result states that the noise eigenvectors are uniformly distributed over a unit sphere (10) Thm. 6. This implies the magnitude of projection of sample principal eigenvector on the orthogonal vector \(\xi_\perp\) is

\[
|e_1^H \xi_\perp|^2 = (1 - |e_1^H \xi_1|^2)/(N - 1).
\]

Further, the look direction steering vector \(v_0\) can be decomposed into two orthogonal unit vectors \(\xi_1\) and \(\xi_\perp\),

\[
v_0 = \alpha \xi_1 + \beta \xi_\perp
\]

where \(\alpha = \sqrt{N} \cos(v_0, v_1)\) and \(\beta = \sqrt{N} \sin(v_0, v_1)\).

3.1. Notch Depth vs Snapshots

The derivation of notch depth vs snapshots model begins from the expression for \(ND_\delta\) by substituting for \(v_0\) from (3) and setting \(v_1 = \sqrt{N} \xi_1\). This substitution results in expressions containing quadratic terms \(|e_1^H \xi_1|^2\) and \(|e_1^H \xi_\perp|^2\). The two terms are then replaced using RMT results in Eq. (4) and (5). Collecting common terms in \(L\) and factoring appropriately simplifies the notch depth expression to

\[
ND_\delta \approx \frac{\cos^2(v_0, v_1)(1 + \delta)^2}{|1 + \delta + N \sigma_1^2 \sin^2(v_0, v_1)|} \left|f_1(L)f_2(L)\right|^2/\left|f_3(L)\right|^2,
\]

where

\[
\begin{align*}
f_1(L) &= N + L(1 + \delta + (N \sigma_1^2) \sin^2(v_0, v_1)) \\
f_2(L) &= \sqrt{L} - \sqrt{N} \sigma_1^2 \cot(v_0, v_1) \\
f_3(L) &= \sqrt{L} - \sqrt{N} \sigma_1^2 \tan(v_0, v_1).
\end{align*}
\]

This derivation assumes that the array is sufficiently long \((N \gg 1)\), the interferer power is strong enough \((N \sigma_1^2 \gg 1)\) and the interferer lies outside the main lobe of CBF \((\sigma_1^2 \tan^2(v_0, v_1) \gg 1)\). For the snapshot sufficient case of \(c \leq 1\) DL is constrained to \(\delta > (1 - \sqrt{c})\).

The model in Eq. (7) can be visualized as a linear piecewise function of \(L\) in a log-log scale. This interpretation of the model is similar to Bode plot approach to interpret system transfer functions (13). The same approach was used to interpret DMR notch depth model in (4) Sec. 3.1. As \(L\) increases each factor in (8) becomes significant over a different range of values of \(L\). The increase in magnitude of each factor dictates the slope of the linear piecewise function. The values of \(L\) for which the summation in each factor become equal predict the breakpoints,

\[
\begin{align*}
L_1 &= N/\left(\delta + \sigma_1^2 N \sin^2(v_0, v_1)\right) \quad \text{(2nd order)} \\
L_2 &= N \cot^2(v_0, v_1) / \sigma_1^2 \quad \text{(1st order)} \\
L_3 &= N \sigma_1^2 \tan^2(v_0, v_1)/(1 + \delta)^2 \quad \text{(1st order)}.
\end{align*}
\]
The notch depth vs INR model is developed following similar steps used in the simulations. The solid line represents the ensemble behavior over the range of INRs between INR, and INR. Again, the model predicted notch depth matches the averaged notch depth observed in the simulations.

For most practical array sizes and strong interferers, the breakpoints predicted in Eq. (9) are such that $L_1 < 1$, $L_2 \approx 1$ and $L_3$ is practically unattainable. Hence, the first two breakpoints are limited to a theoretical interpretation of the model and are not observed in practice. In practice the region of operation lies between $L_2$ and $L_3$ where notch depth grows by 10 dB for every decade increase in $L$.

Fig. 5 compares the notch depth as a function of INR, predicted by the RMT model in Eq. (9) and the notch depth estimated from simulations for different INR levels. The dashed lines represent notch depth predicted by the model. The discrete markers represent the average notch depth obtained from 500 trial Monte Carlo experiments. The solid line represents the ensemble behavior over the range of INR. Again, the model predicted notch depth matches the averaged notch depth observed in the simulations.

The range of INRs between INR, and INR, is where the DL-MVDR ABF is adapting to the change in interferer power. The notch depth grows by 20 dB for every 10 dB rise in interferer power once the interferer power exceeds a certain threshold. However, increasing the loading level makes the ensemble notch depth ND ens shallower.
rise. Consequently the interferer power in the beamformer output remains unchanged. Hence there is no additional adaptive gain in using DL-MVDR ABF in this range of INR levels.

Fig. 5 compares the notch depth as function of INR for snapshots $L = 2N$ among different DL levels. The figure shows that the DL-MVDR ABF with higher DL can suppress interferers with higher power $\sigma_i^2$. On the other hand, increasing DL means that the INR must grow larger before the DL-MVDR ABF actually begins adapting. INR$_1$ is effectively the minimum interferer power required for the DL-MVDR ABF to depart from CBF performance and begin adapting.

5. CONCLUSION

This paper presents RMT based models for the mean notch depth of a DL-MVDR ABF in a single interferer case. The simulation results verify the accuracy of the notch depth predicted by the two models. The derived models indicate that increasing the diagonal loading reduces snapshots required to converge to the ensemble notch depth. Similarly, the ability to suppress an interferer with higher power increases with higher diagonal loading. The improved suppression comes at a cost of shallower DL ensemble notch depth.

6. REFERENCES

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