Data-compatible solutions of constrained convex optimization

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Abstract

The data-compatibility approach to constrained convex optimization, proposed here, strives to a point that is “close enough” to the solution set and whose target function value is “close enough” to the constrained minimum value. These notions can replace analysis of asymptotic convergence to a solution point of infinite sequences generated by specific algorithms. We define and study data-compatibility
with the data of a constrained minimization problem in a Hilbert space and demonstrate it on a problem of minimizing a convex function over the intersection of the fixed point sets of nonexpansive mappings. An iterative algorithm, which we call the Hybrid Subgradient Method (HSM), is proposed and investigated with regard to its ability to generate data-compatible points for the problem at hand. A string-averaging HSM is obtained as a by-product.

**Keywords:** Data-compatibility, constrained convex minimization, fixed point sets, hybrid method, subgradient, string-averaging, common fixed points, proximity function, nonexpansive operators.

## 1 Introduction

The data of a constrained minimization problem \( \min \{ f(x) \mid x \in C \} \) consists of a target function \( f \) and a constraints set \( C \). For this problem to be meaningful, \( C \) needs to be nonempty, and for asymptotic convergence analysis of an algorithm for solving the problem one commonly needs that the solution set of the problem be nonempty, i.e., that there exists at least one point, say \( x^* \), in \( C \) with the property that \( f(x^*) \leq f(x) \) for all \( x \in C \).

In real-world practical situations these nonemptiness assumptions cannot always be guaranteed or verified. To cope with this we define the notion of data-compatibility with the data of a constrained minimization problem in a Hilbert space. Such data-compatibility is a finite, not an asymptotic, notion. Even when the sets \( C \) and the solution set of the constrained minimization problem are nonempty, striving for data-compatibility is a worthwhile aim because it can be “reached”, contrary to asymptotic limit points.

To demonstrate the notion of data-compatibility with the data of a constrained minimization problem we consider the problem of minimizing a convex function over the intersection of the fixed point sets of nonexpansive mappings. This problem has been treated extensively in the literature, of which we reference a few works below. But in all these earlier works the asymptotic convergence of algorithms is the central theme, not data-compatibility.

We propose an iterative algorithm, which we call the Hybrid Subgradient Method (HSM), and we investigate the algorithm’s ability to generate data-compatible points for the problem at hand. The term HSM is in analogy with the established term, coined by Yamada [12], of the Hybrid Steepest Descent Method (HSDM). The structural similarity of the HSM with the HSDM is
that the former uses subgradient steps instead of the steepest descent steps used by the latter.

As an important special case of the general algorithmic formulation we discuss a string-averaging algorithmic scheme. The string-averaging algorithmic notion has a quite general structure in itself. Invented in [3] and spurred many extensions and applications since then, e.g., [1][10] and the book [13], it works in general as follows. From a current iteration point, it performs consecutively specified iterative algorithmic steps “along” different “strings” of individual constraints sets and then takes a combination, convex or other, of the strings’ end-points as the next iterate. The string-averaging algorithmic scheme gives rise to a variety of specific algorithms by judiciously choosing the number of strings, their assignments and the nature of the combination of the strings’ end-points. Details are given in the sequel.

Earlier works on minimizing convex functions over the intersection of the fixed point sets of nonexpansive mappings are all based on asymptotic convergence of the algorithms and investigate the problem and prove convergence of algorithms under various conditions. These include, to name but a few, the papers of Iiduka [7], [8], [9], the work of Maingé [11], and publications by Hayashi and Iiduka [6] and Deutsch and Yamada [5].

The paper is organized as follows. In Section 2 we define the notion of data-compatibility of a point with the data of a constrained minimization problem. The problem of minimizing a convex function over the intersection of the fixed point sets of nonexpansive mappings is defined in Section 3 along with the proposed Hybrid Subgradient Method (HSM) for its solution. Inexact iterates are discussed in Section 4 followed in Section 5 by work on the main result that proves the ability of the HSM to generate a data-compatible point for the problem. We present the string-averaging variant of the HSM in Section 6. In Section 7 we conclude with a specific situation wherein the data of a constrained minimization problem does not necessarily obey feasibility of the constraints, i.e., does not demand that $C = \bigcap_{i=1}^{m} C_i$ is nonempty.

2 Data-compatibility

In this section we define the notion of data-compatibility of a point with the data of a constrained minimization problem. Let $\Omega \subseteq H$ be a given nonempty set in the Hilbert space $H$ and let there be given, for $i = 1, 2, \ldots, m$,
nonempty sets $C_i \subseteq \Omega$. We denote by $\Gamma := \{C_i\}_{i=1}^m$ the family of sets and refer to it as the “constraints data $\Gamma$”, or, in short, the data $\Gamma$.

We introduce a set $\Delta$ such that $\Omega \subseteq \Delta \subseteq H$ and assume that we are given a function $f : \Delta \to R$ which is referred to as “the target function $f$” or, in short, the data $f$. A pair $(\Gamma, f)$ is referred to as the “data pair $(\Gamma, f)$”.

### 2.1 Data-compatibility with constraints

First we look at compatibility with the constraints data alone. For this we need a proximity function.

**Definition 1** Proximity function. Given constraints data $\Gamma = \{C_i\}_{i=1}^m$, a proximity function $\text{Prox}_\Gamma : \Omega \to R_+$ (the nonnegative orthant) has the property that $\text{Prox}_\Gamma(x) = 0$ if and only if $x \in C := \cap_{i=1}^m C_i \neq \emptyset$. It measures how incompatible an $x \in \Omega$ is with the constraints of $\Gamma$. The lower the value of $\text{Prox}_\Gamma(x)$ is – the less incompatible $x$ is with the constraints.

A proximity function does not require that $C \neq \emptyset$ and it is a useful tool, particularly for situations when $C \neq \emptyset$ does not hold, or cannot be verified. An enlightening discussion of proximity functions and their relations with the convex feasibility problem can be found in Cegielski’s book [2, Subsection 1.3.4]. An important and often used proximity function is

$$\text{Prox}_\Gamma(x) := \frac{1}{2} \sum_{i=1}^m w_i \|P_{C_i}(x) - x\|^2,$$

where $P_{C_i}(x)$ is the orthogonal (metric) projection onto $C_i$ and $\{w_i\}_{i=1}^m$ is a set of weights such that $w_i \geq 0$ and $\sum_{i=1}^m w_i = 1$.

**Definition 2** $\gamma$-compatibility with constraints data $\Gamma$. Given constraints data $\Gamma$, a proximity function $\text{Prox}_\Gamma$, and a $\gamma \geq 0$, we say that a point $x \in \Omega$ is “$\gamma$-compatible with $\Gamma$” if $\text{Prox}_\Gamma(x) \leq \gamma$. We define the set of all points that are $\gamma$-compatible with $\Gamma$ by $\Pi(\Gamma, \gamma) := \{x \in \Omega | \text{Prox}_\Gamma(x) \leq \gamma\}$.

The set $\Pi(\Gamma, \gamma)$ need not be nonempty for all $\gamma$. If, however, $\Pi(\Gamma, 0) \neq \emptyset$ then $\Pi(\Gamma, 0) = C$. We have used the notion of $\gamma$-compatibility with constraints earlier in our work on the superiorization method, see, e.g., [4].
2.2 Data-compatibility with constrained minimization

We propose the next definition of compatibility with a data pair \((\Gamma, f)\). For \(\gamma\), for which \(\Pi(\Gamma, \gamma) \neq \emptyset\), we define

\[
S(f, \Pi(\Gamma, \gamma)) := \{x \in \Pi(\Gamma, \gamma) \mid f(x) \leq f(y), \text{ for all } y \in \Pi(\Gamma, \gamma)\}.
\]

If \(f\) is the zero function or if \(f = \text{constant}\) then \(S(f, \Pi(\Gamma, \gamma)) = \Pi(\Gamma, \gamma)\). We use the distance function between a point \(x\) and a set \(S\) defined as

\[
d(x, S) := \inf\{d(x, y) \mid y \in S\}
\]

where \(d(x, y)\) is the distance between points \(x\) and \(y\).

Definition 3 \((\tau, \bar{L})\)-compatibility with a data pair \((\Gamma, f)\). Given constraints data \(\Gamma\), a proximity function \(\text{Prox}_\Gamma\), a target function \(f\), a \(\gamma \geq 0\) such that \(\Pi(\Gamma, \gamma) \neq \emptyset\), a \(\tau \geq 0\), and a real number \(\bar{L} > 1\), we say that a point \(x \in \Omega\) is \(\text{“}(\tau, \bar{L})\text{-compatible with the data pair } (\Gamma, f)\text{”}\) if \(S(f, \Pi(\Gamma, \gamma)) \neq \emptyset\) and the following two conditions hold

\[
d(x, S(f, \Pi(\Gamma, \gamma))) \leq \tau
\]

and

\[
f(x) \leq f(z) + \tau \bar{L}, \text{ for all } z \in S(f, \Pi(\Gamma, \gamma)).
\]

This definition does not require nonemptiness of the intersection of the constraints \(C = \bigcap_{i=1}^{n} C_i\) neither does it require that the constrained minimization problem \(\min\{f(x) \mid x \in C\}\), has a nonempty solution set \(\text{SOL}(f, C)\) which is defined by

\[
\text{SOL}(f, C) := S(f, C) = \{x \in C \mid f(x) \leq f(y), \text{ for all } y \in C\}.
\]

It relies on the weaker assumptions that \(\Pi(\Gamma, \gamma) \neq \emptyset\) and \(S(f, \Pi(\Gamma, \gamma)) \neq \emptyset\). Therefore, these notions make it possible to deviate from the nonemptiness assumptions which usually lie at the heart of asymptotic analyses in optimization theory.

Inspired by [4, Definition 2.1], we suggest the following definition.

Definition 4 The \((\tau, \bar{L})\)-output of a sequence. Given constraints data \(\Gamma\), a proximity function \(\text{Prox}_\Gamma\), a target function \(f\), a \(\gamma \geq 0\), a \(\tau \geq 0\), and a real number \(\bar{L} > 1\), we consider a sequence \(X = \{x^k\}_{k=0}^{\infty}\) of points in \(\Omega\).
Let \( \text{OUT}((\Gamma, f), \gamma, (\tau, \bar{L}), \mathcal{X}) \) denote the point \( x \in \Omega \) that has the following properties: (i) \( x \) fulfills (4)–(5), and (ii) there exists a nonnegative integer \( K \) such that \( x^K = x \) and for all nonnegative integers \( k < K \) at least one of the two conditions (4)–(5) is violated. If there is such an \( x \), then it is unique. If there is no such \( x \) then we say that \( \text{OUT}((\Gamma, f), \gamma, (\tau, \bar{L}), \mathcal{X}) \) is undefined, otherwise it is defined.

If \( \mathcal{X} \) is a sequence generated by an iterative process, then \( \text{OUT}((\Gamma, f), \gamma, (\tau, \bar{L}), \mathcal{X}) \) is the output produced by that process when we add to it instructions that make it terminate as soon as it reaches a point that is \((\tau, \bar{L})\)-compatible with a data pair \((\Gamma, f)\).

In the special case that \( \gamma = 0 \) and \( \Pi(\Gamma, 0) \neq \emptyset \) we obtain from the above the following definition.

**Definition 5 \((\tau, \bar{L})\)-compatibility with a data pair \((\Gamma, f)\) in the consistent case.** Given consistent constraints data \( \Gamma \) via \( C := \bigcap_{i=1}^{m} C_i \neq \emptyset \), a target function \( f \), a \( \tau \geq 0 \), and a real number \( \bar{L} > 1 \), we say that a point \( x \in \Omega \) is “\((\tau, \bar{L})\)-compatible with the consistent data pair \((\Gamma, f)\)” if \( \text{SOL}(f, C) \neq \emptyset \) and the following two conditions hold

\[
d(x, \text{SOL}(f, C)) \leq \tau \tag{7}
\]

and

\[
f(x) \leq f(z) + \tau \bar{L}, \text{ for all } z \in \text{SOL}(f, C). \tag{8}
\]

In what follows we work in the framework of Definition 5 and study the behavior of an iterative algorithm for convex minimization over fixed point sets of nonexpansive operators. Rather than generating infinite sequences that asymptotically converge to a point in \( \text{SOL}(f, C) \), we specify conditions under which the algorithm generates solutions that are \((\tau, \bar{L})\)-compatible with the data pair \((\Gamma, f)\). The advantage of these data-compatibility notions is that they can cater better to practical situations regardless of the nonemptiness of the sets \( C \) or \( \text{SOL}(f, C) \).

In the sequel (Section 7) we discuss a specific situation wherein the data pair \((\Gamma, f)\), with \( \Gamma := \{C_i\} \) a family of closed and convex subsets of \( H \), not necessarily obeying that \( C = \bigcap_{i=1}^{m} C_i \) is nonempty.
3 The problem and the algorithm

Let \((X, \rho)\) be a metric space and let \(T : X \rightarrow X\) be an operator. The fixed point set of \(T\) is define by

\[
\text{Fix}(T) := \{x \in X \mid T(x) = x\}.
\] (9)

An operator \(T\) is nonexpansive if it satisfies

\[
\rho(T(x), T(y)) \leq \rho(x, y), \text{ for all } x, y \in X.
\] (10)

Given a nonempty set \(E \subseteq X\) define the distance of a point \(x \in X\) from it by

\[
d(x, E) := \inf \{\rho(x, y) \mid y \in E\}.
\] (11)

We denote the ball with center at a given \(x \in X\) and radius \(r > 0\) by \(B(x, r)\). The execution of the operator \(T\) for \(n\) times consecutively on an initial given point \(x\) is denoted by \(T^n x\), and \(T^0 x := x\).

We look at a constrained minimization problem of the form

\[
\min \{f(x) \mid x \in \text{Fix}(T)\}
\] (12)

where \(f\) is a convex target function from \(X\) into the reals, \(T\) is a given nonexpansive operator. Solving this problem means to find a point \(x\) in \(\text{SOL}(f, \text{Fix}(T))\),

\[
\text{SOL}(f, \text{Fix}(T)) := \{x \in \text{Fix}(T) \mid f(x) \leq f(y) \text{ for all } y \in \text{Fix}(T)\}.\] (14)

For this task we construct and employ an iterative Hybrid Subgradient Method (HSM) that uses the powers of the operator \(T\) combined with subgradient steps. We denote by \(\partial f(x_k)\) the subgradient set of \(f\) at \(x_k\).

\textbf{Algorithm 6 Hybrid Subgradient Method (HSM).}

\textbf{Initialization:} Let \(\{\alpha_k\}_{k=0}^\infty \subset (0, 1]\) be a scalar sequence and let \(x^0\), \(s^0 \in X\) be arbitrary initialization vectors.

\textbf{Iterative step:} Given a current iteration vector \(x^k\) and a current vector \(s^k\), calculate the next vectors as follows:
If $0 \in \partial f(x^k)$ then set $s^k = 0$ and calculate
\[ x^{k+1} = T(x^k). \] (15)

If $0 \notin \partial f(x^k)$ then choose and set $s^k \in \partial f(x^k)$ and calculate
\[ x^{k+1} = T \left( x^k - \alpha_k \frac{s^k}{\|s^k\|} \right). \] (16)

As mentioned above, our data-compatibility result, presented in Theorem 16 below, will not be about asymptotic convergence but rather specify conditions that guarantee the existence of a solution that is $(\tau, \bar{L})$-compatible with the data pair $(\Gamma = \text{Fix}(T), f)$. I.e., that for every $\tau \in (0, 1)$, and any sequence $\{x^k\}_{k=0}^\infty$, generated by Algorithm 6, there exists an integer $K$ so that, for all $k \geq K$:
\begin{align*}
  d(x^k, \text{SOL}(f, \text{Fix}(T))) &\leq \tau \quad (17) \\
  f(x^k) &\leq f(z) + \tau \bar{L} \text{ for all } z \in \text{SOL}(f, \text{Fix}(T)) \tag{18}
\end{align*}

where $\bar{L}$ is some well-defined constant.

4 Inexact iterates

In this sections we establish some properties of sequences of the form $\{T_j y^0\}_{j=0}^\infty$, for any $y^0 \in X$, with “computational errors”. These will serve as tools in proving the main result. In our work we need to focus on operators that have the property formulated in the next condition.

\textbf{Condition 7} Let $X$ be a metric space, assume that $T : X \to X$ is a nonexpansive operator such that $\lim_{j \to \infty} T_j y^0$ exists for any $y^0 \in X$.

This condition is met in a variety of cases, for example, when $T$ is nonexpansive and the interior of $\text{Fix}(T)$ is nonempty [2, Theorem 3.8.1].

\textbf{Proposition 8} Let $X$ be a metric space, assume that $T : X \to X$ is a nonexpansive operator and $\lim_{j \to \infty} T_j y^0$ exists for some $y^0 \in X$. Then $\lim_{j \to \infty} T_j y^0$ is a fixed point of $T$ and, consequently, $\text{Fix}(T) \neq \emptyset$. 

8
Proof. The proof is obvious. ■

Proposition 9  Let $X$ be a compact metric space, assume that $T : X \to X$ is a nonexpansive operator and let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that for each $x \in X$ satisfying $\rho(x, Tx) \leq \delta$ we have

$$d(x, \text{Fix}(T)) \leq \varepsilon. \quad (19)$$

Proof. Assume to the contrary that for each integer $k \geq 1$ there exists a point $x^k \in X$ such that

$$\rho(x^k, T x^k) \leq k^{-1} \text{ and } d(x^k, \text{Fix}(T)) > \varepsilon. \quad (20)$$

Since $X$ is compact, extracting a subsequence and re-indexing, if necessary, we may assume without loss of generality that, the sequence $\{x^k\}_{k=1}^\infty$ so generated by the repeated use of the above negation, converges and denote

$$z := \lim_{k \to \infty} x^k. \quad (21)$$

Since $T$ is nonexpansive, inequality (20) and the limit (21) yield, for all integers $k \geq 1$,

$$\rho(z, Tz) \leq \rho(z, x^k) + \rho(x^k, T x^k) + \rho(T x^k, Tz) \leq 2 \rho(z, x^k) + k^{-1} \to 0, \text{ as } k \to \infty, \quad (22)$$

thus,

$$z \in \text{Fix}(T). \quad (23)$$

In view of (21), for all sufficiently large integers $k$,

$$d(x^k, \text{Fix}(T)) \leq d(x^k, z) < \varepsilon. \quad (24)$$

This contradicts (20) and completes the proof. ■

Lemma 10  Let $X$ be a compact metric space, assume that $T : X \to X$ is a nonexpansive operator for which Condition 7 holds, and let $\mu > 0$. Then there exists an integer $k_1$ such that for each $x \in X$ there exists $j \in \{0, 1, \ldots, k_1\}$ such that

$$d(T^j x, \text{Fix}(T)) \leq \mu. \quad (25)$$
Proof. Assume to the contrary that for each integer \( k \geq 1 \) there exists a point \( x^k \in X \) such that
\[
d(T^j x^k, \text{Fix}(T)) > \mu, \text{ for all } j = 0, 1, \ldots, k. \tag{26}
\]
Since \( X \) is compact, extracting a subsequence and re-indexing, if necessary, we may assume without loss of generality that, the sequence \( \{x^k\}_{k=1}^\infty \) so generated by the repeated use of the above negation, converges and let
\[
z = \lim_{k \to \infty} x^k. \tag{27}
\]
By Proposition 8 and Condition 7 we conclude that
\[
d(T^j z, \text{Fix}(T)) < \mu. \tag{28}
\]
By (27) and since \( T \) is nonexpansive, for all sufficiently large integers \( k \),
\[
d(T^j x^k, \text{Fix}(T)) < \mu, \tag{29}
\]
contradicting (26), thus, concluding the proof. \( \blacksquare \)

Theorem 11 Let \( X \) be a compact metric space, assume that \( T : X \to X \) is a nonexpansive operator for which Condition 7 holds, and let \( \varepsilon > 0 \). Then there exists a natural number \( k_0 \) such that for each \( x \in X \) and each integer \( k \geq k_0 \),
\[
\rho(T^k x, \lim_{i \to \infty} T^i x) \leq \varepsilon. \tag{30}
\]

Proof. By Lemma 10 there exists an integer \( k_0 \) such that for each \( x \in X \) there exists \( j \in \{0, 1, \ldots, k_0\} \) so that
\[
d(T^j x, \text{Fix}(T)) < \varepsilon / 2. \tag{31}
\]
This implies that there exist a
\[
j \in \{0, 1, \ldots, k_0\} \tag{32}
\]
and a
\[
z \in \text{Fix}(T) \tag{33}
\]
such that
\[
\rho(T^j x, z) < \varepsilon / 2. \tag{34}
\]
Since $T$ is nonexpansive, we get, by (33) and (34), that for all integers $k \geq j$,
\[ \rho(T^k x, z) \leq \rho(T^j x, z) < \varepsilon/2, \] \tag{35}
yielding,
\[ \rho(\lim_{i \to \infty} T^i x, z) \leq \varepsilon/2. \] \tag{36}
Together with (32) and (35) this implies that for all integers $k \geq k_0$,
\[ \rho(T^k x, \lim_{i \to \infty} T^i x) \leq \rho(T^k x, z) + \rho(z, \lim_{i \to \infty} T^i x) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon, \] \tag{37}
completing the proof. \IEEEQEDclosed

**Proposition 12** Under the assumptions of Theorem 11, there exist an integer $k_0$ and a $\delta > 0$ such that for each finite sequence $\{x^i\}_{i=0}^{k_0-1} \subset X$ satisfying
\[ \rho(x^{i+1}, T x^i) \leq \delta, \text{ for all } i = 0, 1, \ldots, k_0 - 1, \] \tag{38}
the inequality
\[ d(x^{k_0}, \text{Fix}(T)) \leq \varepsilon \] \tag{39}
holds.

**Proof.** Theorem 11 implies that there exists an integer $k_0$ such that for each $x \in X$,
\[ d(T^{k_0} x, \text{Fix}(T)) \leq \varepsilon/4. \] \tag{40}
Define
\[ \delta := 4^{-1} \varepsilon(k_0)^{-1}, \] \tag{41}
assume that $\{x^i\}_{i=0}^{k_0} \subset X$ satisfies (38) and set
\[ y^0 := x^0, \ y^{i+1} := T y^i, \text{ for all } i = 0, 1, \ldots, k_0 - 1. \] \tag{42}
In view of (40) and (42),
\[ d(y^{k_0}, \text{Fix}(T)) \leq \varepsilon/4. \] \tag{43}
Next we show, by induction, that
\[ \rho(y^i, x^i) \leq i\delta, \text{ for all } i = 0, 1, \ldots, k_0. \] \tag{44}
Equation (42) implies that (44) holds for $i = 0$. Let $i \in \{0, 1, \ldots, k_0 - 1\}$ for which (44) holds. By the nonexpansiveness of $T$, (38), (43) and (44),

\[
\rho(y^{i+1}, x^{i+1}) = \rho(T y^i, x^{i+1}) \\
\leq \rho(T y^i, T x^i) + \rho(T x^i, x^{i+1}) \leq \rho(y^i, x^i) + \delta \leq (i + 1)\delta,
\]

in particular,

\[
\rho(y^{k_0}, x^{k_0}) \leq k_0\delta. \tag{46}
\]

It follows now from (41), (43) and (46) that

\[
d(x^{k_0}, \text{Fix}(T)) \leq d(x^{k_0}, y^{k_0}) + d(y^{k_0}, \text{Fix}(T)) \leq \varepsilon/4 + \varepsilon/4, \tag{47}
\]

which concludes the proof.

**Theorem 13** Under the assumptions of Theorem 11, there exist an integer $k_0$ and a $\delta > 0$ such that for each sequence $\{x^i\}_{i=0}^{\infty} \subset X$ satisfying $\rho(x^{i+1}, T x^i) \leq \delta$, for all $i = 0, 1, \ldots$, the inequality

\[
d(x^i, \text{Fix}(T)) \leq \varepsilon \tag{48}
\]

holds for all integers $i \geq k_0$.

**Proof.** The proof follows from Proposition 12. □

Theorem 13 implies the next result.

**Theorem 14** Under the assumptions of Theorem 11, if we take a sequence

\[
\{\mu_k\}_{k=1}^{\infty} \subset (0, \infty), \quad \lim_{k \to \infty} \mu_k = 0, \tag{49}
\]

then there exists an integer $k_1 > 0$ such that for each sequence $\{x^i\}_{i=0}^{\infty} \subset X$ satisfying

\[
\rho(x^{i+1}, T x^i) \leq \mu_{i+1}, \quad i = 0, 1, \ldots, \tag{50}
\]

the inequality $d(x^k, \text{Fix}(T)) \leq \varepsilon$ holds for all integers $k \geq k_1$. 
5 Reaching data-compatibility by the hybrid subgradient method

Our main data-compatibility result in Theorem 16 below is obtained under the following assumptions: \( Y \) is a convex and compact subspace of the Hilbert space \( H \), \( T : Y \to Y \) is a nonexpansive operator for which Condition 7 holds, and \( f : Y \to \mathbb{R} \) is a convex function.

Since \( Y \) is compact, there exists a ball \( B(0, M) \), with \( M > 0 \), such that
\[
\text{Fix}(T) \subset X \subset B(0, M),
\]
which means that the set \( \text{Fix}(T) \) is bounded. Moreover, the function \( f \) is continuous due to its convexity. By Condition 7 and Proposition 8 we get that \( \text{Fix}(T) \) is nonempty, this with the continuity of \( f \), the boundedness of \( \text{Fix}(T) \) and the fact that \( \text{Fix}(T) \) is closed, implies that there exists a point \( x \in \text{SOL}(f, \text{Fix}(T)), \) i.e., \( \text{SOL}(f, \text{Fix}(T)) \neq \emptyset \).

By the continuity of \( f \), it is Lipschitz on the compact subspace \( Y \), therefore, there exists a number \( \bar{L} > 1 \) such that
\[
|f(z^1) - f(z^2)| \leq \bar{L}||z^1 - z^2||, \text{ for all } z^1, z^2 \in Y \cap B(0, 3M + 2). \tag{52}
\]

We need the following lemma to prove the main result.

**Lemma 15** Assume that \( Y \) is a convex and compact subspace of \( H \) and that \( f : Y \to \mathbb{R} \) is a convex. Assume that \( T : Y \to Y \) is a nonexpansive operator for which Condition 7 holds. Let \( \bar{x} \in \text{SOL}(f, \text{Fix}(T)) \) and let \( \Delta \in (0, 1], \alpha > 0 \) and \( x \in Y \) satisfy
\[
\|x\| \leq 3M + 2, \ f(x) > f(\bar{x}) + \Delta. \tag{53}
\]

Further, let \( v \in \partial f(x) \). Then \( v \neq 0 \) and
\[
y := T \left( x - \alpha||v||^{-1}v \right) \tag{54}
\]
satisfies
\[
\|y - \bar{x}\|^2 \leq \|x - \bar{x}\|^2 - 2\alpha(4\bar{L})^{-1}\Delta + \alpha^2, \tag{55}
\]
where \( \bar{L} \) is as in (52). Moreover,
\[
d(y, \text{SOL}(f, \text{Fix}(T)))^2 \leq d(x, \text{SOL}(f, \text{Fix}(T)))^2 - 2\alpha(4\bar{L})^{-1}\Delta + \alpha^2. \tag{56}
\]
Proof. From (53) \( v \neq 0 \). For \( \bar{x} \in \text{SOL}(f, \text{Fix}(T)) \), we have, by (52) and (51), that for each \( z \in B(\bar{x}, 4^{-1}\Delta \bar{L}^{-1}) \),

\[
f(z) \leq f(\bar{x}) + \bar{L}\|z - \bar{x}\| \leq f(\bar{x}) + 4^{-1}\Delta. \tag{57}
\]

Therefore, (53) and \( v \in \partial f(x) \), imply that

\[
\langle v, z - x \rangle \leq f(z) - f(x) \leq -(3/4)\Delta, \text{ for all } z \in B(\bar{x}, 4^{-1}\Delta \bar{L}^{-1}). \tag{58}
\]

From this inequality we deduce that

\[
\langle \|v\|^{-1}v, z - x \rangle < 0, \text{ for all } z \in B(\bar{x}, 4^{-1}\Delta \bar{L}^{-1}), \tag{59}
\]

or, setting \( \bar{z} := \bar{x} + 4^{-1}\bar{L}^{-1}\Delta \|v\|^{-1}v \), that

\[
0 > \langle \|v\|^{-1}v, \bar{z} - x \rangle = \langle \|v\|^{-1}v, \bar{x} + 4^{-1}\bar{L}^{-1}\Delta \|v\|^{-1}v - x \rangle. \tag{60}
\]

This leads to

\[
\langle \|v\|^{-1}v, \bar{x} - x \rangle < -4^{-1}\bar{L}^{-1}\Delta. \tag{61}
\]

Putting \( \bar{y} := x - \alpha\|v\|^{-1}v \), we arrive at

\[
\|\bar{y} - \bar{x}\|^2 = \|x - \alpha\|v\|^{-1}v - \bar{x}\|^2 = \|x - \bar{x}\|^2 - 2\langle x - \bar{x}, \alpha\|v\|^{-1}v \rangle + \alpha^2 \\
\leq \|x - \bar{x}\|^2 - 2\alpha(4\bar{L})^{-1}\Delta + \alpha^2. \tag{62}
\]

From all the above we obtain

\[
\|y - \bar{x}\|^2 = \|T\bar{y} - \bar{x}\|^2 \leq \|\bar{y} - \bar{x}\|^2 \\
\leq \|x - \bar{x}\|^2 - 2\alpha(4\bar{L})^{-1}\Delta + \alpha^2, \tag{63}
\]

which completes the proof. \( \blacksquare \)

Now we present the main theorem showing that sequences generated by the hybrid subgradient method (HSM) have a \((\tau, \bar{L})\)-output, i.e., contain an iterate that is data-compatible.

**Theorem 16** Assume that \( Y \) is a convex and compact subspace of \( H \) and that \( T : Y \to Y \) is a nonexpansive operator for which Condition 7 holds. Let \( f : Y \to \mathbb{R} \) be a convex function, let

\[
\{\alpha_k\}_{k=0}^\infty \subset (0, 1], \text{ be a sequence such that } \lim_{k \to \infty} \alpha_k = 0 \text{ and } \sum_{k=0}^\infty \alpha_k = \infty,
\]

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and let $\tau \in (0, 1)$. Then there exist an integer $K$ and a real number $\bar{L}$ such that for any sequence $\{x^k\}_{k=0}^\infty \subset Y$, generated by Algorithm 6, the inequalities

\begin{equation}
  d(x^k, \text{SOL}(f, \text{Fix}(T))) \leq \tau \tag{65}
\end{equation}

and

\begin{equation}
  f(x^k) \leq f(z) + \tau \bar{L} \text{ for all } z \in \text{SOL}(f, \text{Fix}(T)) \tag{66}
\end{equation}

hold for all integers $k \geq K$.

**Proof.** Fix an $\bar{x} \in \text{SOL}(f, \text{Fix}(T))$. It is not difficult to see that there exists a number $\tau_0 \in (0, \tau/4)$ such that for each $x \in Y$ satisfying $d(x, \text{Fix}(T)) \leq \tau_0$ and $f(x) \leq f(\bar{x}) + \tau_0$ we have

\begin{equation}
  d(x, \text{SOL}(f, \text{Fix}(T))) \leq \tau/4. \tag{67}
\end{equation}

Since $\{x^k\}_{k=0}^\infty$ is generated by Algorithm 6 we know, from (15), (16) and (10), that

\begin{equation}
  \|x^k - Tx^{k-1}\| \leq \alpha_{k-1}, \text{ for all } k \geq 1. \tag{68}
\end{equation}

Thus, by Theorem 14 and (64), there exists an integer $n_1$ such that

\begin{equation}
  d(x^k, \text{Fix}(T)) \leq \tau_0, \text{ for all } k \geq n_1. \tag{69}
\end{equation}

This, along with (51), guarantees that

\begin{equation}
  \|x^k\| \leq M + 1, \text{ for all } k \geq n_1. \tag{70}
\end{equation}

Choose a positive $\tau_1$ for which $\tau_1 < (8\bar{L})^{-1}\tau_0$, by (64) there is an integer $n_2 > n_1$ such that

\begin{equation}
  \alpha_k \leq \tau_1 (32)^{-1}, \text{ for all } k > n_2, \tag{71}
\end{equation}

and so, there is an integer $n_0 > n_2 + 4$ such that

\begin{equation}
  \sum_{k=n_2}^{n_0-1} \alpha_k > 8(2M + 1)^2\bar{L}\tau_0^{-1}. \tag{72}
\end{equation}

We show now that there exists an integer $p \in [n_2 + 1, n_0]$ such that $f(x^p) \leq f(\bar{x}) + \tau_0$. Assuming the contrary means that for all $k \in [n_2 + 1, n_0]$, $f(x^k) > f(\bar{x}) + \tau_0$. \hfill (73)
By (73), (64), (70) and using Lemma 15, with \( \Delta = \tau_0, \alpha = \alpha_k, x = x^k, y = x^{k+1}, v = s^k \), we get, for all \( k \in [n_2 + 1, n_0] \),

\[
d(x^{k+1}, \text{SOL}(f, \text{Fix}(T)))^2 \\
\leq d(x^k, \text{SOL}(f, \text{Fix}(T)))^2 - 2\alpha_k(4\bar{L})^{-1}\tau_0 + \alpha_k^2.
\] (74)

According to the choice of \( \tau_1 \) and by (71) this implies that for all \( k \in [n_2 + 1, n_0] \),

\[
d(x^k, \text{SOL}(f, \text{Fix}(T)))^2 - d(x^{k+1}, \text{SOL}(f, \text{Fix}(T)))^2 \\
\geq \alpha_k[(2\bar{L})^{-1}\tau_0 - \alpha_k] \\
\geq \alpha_k(4\bar{L})^{-1}\tau_0,
\] (75)

which, together with (70) and (51), gives

\[
(2M + 1)^2 \\
\geq d(x^{n_2+1}, \text{SOL}(f, \text{Fix}(T)))^2 \\
\geq \sum_{k=n_2+1}^{n_0} (d(x^k, \text{SOL}(f, \text{Fix}(T)))^2 - d(x^{k+1}, \text{SOL}(f, \text{Fix}(T)))^2) \\
\geq (4\bar{L})^{-1}\tau_0 \sum_{k=n_2+1}^{n_0} \alpha_k,
\] (76)

and

\[
\sum_{k=n_2+1}^{n_0} \alpha_k \leq (2M + 1)^24\bar{L}\tau_0^{-1}.
\] (77)

This contradicts (72), proving that there is an integer \( p \in [n_2 + 1, n_0] \) such that \( f(x^p) \leq f(\bar{x}) + \tau_0 \). Thus, by (63) and (67),

\[
d(x^p, \text{SOL}(f, \text{Fix}(T))) \leq \tau/4.
\] (78)

We show that for all \( k \geq p, d(x^k, \text{SOL}(f, \text{Fix}(T))) \leq \tau \). Assuming the contrary,

\[
\text{there exists a } q > p \text{ such that } d(x^q, \text{SOL}(f, \text{Fix}(T))) > \tau.
\] (79)

We may assume, without loss of generality, that

\[
d(x^k, \text{SOL}(f, \text{Fix}(T))) \leq \tau, \text{ for all } p \leq k < q.
\] (80)
One of the following two cases must hold: (i) \( f(x^{q-1}) \leq f(\bar{x}) + \tau_0 \), or (ii) \( f(x^{q-1}) > f(\bar{x}) + \tau_0 \). In case (i), since \( p \in [n_2 + 1, n_0] \), (69), (70) and (67) show that
\[
d(x^{q-1}, \text{SOL}(f, \text{Fix}(T))) \leq \tau / 4.
\] (81)

Thus, there is a point \( z \in \text{SOL}(f, \text{Fix}(T)) \) such that \( \| x^{q-1} - z \| < \tau / 3 \). Using this fact and (68), (10), (9) and (71), yields
\[
\| x^{q-1} - z \| \leq \alpha_{q-1} + \| x^{q-1} - \bar{z} \| \leq \tau / 4 + \tau / 3,
\] (82)

proving that \( d(x^q, \text{SOL}(f, \text{Fix}(T))) \leq \tau \). This contradicts (79) and implies that case (ii) must hold, namely that \( f(x^{q-1}) > f(\bar{x}) + \tau_0 \). This, along with (70), (71), the choice of \( \tau_1 \), (80) and Lemma 15 with \( \Delta = \tau_0 \), \( \alpha = \alpha_{q-1} \), \( x = x^{q-1} \), \( y = x^q \), shows that
\[
d(x^q, \text{SOL}(f, \text{Fix}(T)))^2 \\
\leq d(x^{q-1}, \text{SOL}(f, \text{Fix}(T)))^2 - 2\alpha_{q-1}(4L)^{-1}\tau_0 + \alpha_{q-1}^2 \\
\leq d(x^{q-1}, \text{SOL}(f, \text{Fix}(T)))^2 - \alpha_{q-1}((2\bar{L})^{-1}\tau_0 - \alpha_{q-1}) \\
\leq d(x^{q-1}, \text{SOL}(f, \text{Fix}(T)))^2 \leq \tau^2,
\] (83)

namely, that \( d(x^q, \text{SOL}(f, \text{Fix}(T))) \leq \tau \). This contradicts (79), proving that, for all \( k \geq p \), \( d(x^k, \text{SOL}(f, \text{Fix}(T))) \leq \tau \). Together with (51) and (52) this implies that, for all \( k \geq n_0 \),
\[
f(x^k) \leq f(z) + \tau \bar{L} \text{ for all } z \in \text{SOL}(f, \text{Fix}(T)),
\] (84)
and the proof is complete. ■

6 Using the string-averaging algorithmic scheme

Assume that \( S \) is a convex and compact subspace of \( H \) and that \( O_1, O_2, \ldots, O_m \) are nonexpansive operators mapping \( S \) into \( S \), for which
\[
\mathcal{F} := \bigcap_{i=1}^{m} \text{Fix}(O_i) \neq \emptyset
\] (85)
Let $f : S \to R$ be a convex function. We are interested in solving the following problem by using a string-averaging algorithmic scheme.

$$\min \{ f(x) \mid x \in \mathcal{F} \}$$

(86)

whose solution means to

find a point $x$ in $\text{SOL}(f, \mathcal{F})$,

(87)

where

$$\text{SOL}(f, \mathcal{F}) := \{ x \in \mathcal{F} \mid f(x) \leq f(y), \text{ for all } y \in \mathcal{F} \}. $$

(88)

For $t = 1, 2, \ldots, \Theta$, let the string $I_t$ be an ordered subset of $\{1, 2, \ldots, m\}$ of the form

$$I_t = (i^t_1, i^t_2, \ldots, i^t_{m(t)}),$$

(89)

with $m(t)$ the number of elements in $I_t$. For any $x \in S$, the product of operators along a string $I_t$, $t = 1, 2, \ldots, \Theta$, is

$$F_t(x) := O^t_{i^t_{m(t)}} \cdots O^t_{i^t_2} O^t_{i^t_1} (x),$$

(90)

and is called a “string operator”.

We deal with string-averaging of fixed strings and fixed weights. To this end we assume that

$$\{1, 2, \ldots, m\} \subset \bigcup_{t=1}^{\Theta} I_t$$

(91)

and that a system of nonnegative weights $w_1, w_2, \ldots, w_{\Theta}$ such that $\sum_{t=1}^{\Theta} w(t) = 1$ is fixed and given. We define the operator

$$O(x) := \sum_{t=1}^{\Theta} w_t F_t(x).$$

(92)

This operator will be called “fit” if the strings that define it obey (91). We will need the following condition.

**Condition 17** For all $i = 1, 2, \ldots, m$, the following holds: For any $y \in S \setminus \text{Fix}(O_i)$ there exist $x \in \mathcal{F} = \bigcap_{i=1}^{m} \text{Fix}(O_i)$ such that $\|O_i(y) - x\| < \|y - x\|$. 

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Proposition 18  Let $O_1, O_2, \ldots, O_m$ be nonexpansive operators $O_i : S \to S$, and let $O = \sum_{t=1}^{\Theta} w_t F_t(x)$, be as in (92). If (91) and condition 17 hold, then $\text{Fix } (O) = \mathcal{F}$.

Proof. Clearly, $\mathcal{F} \subset \text{Fix } (O)$, therefore, it is sufficient to prove that $\text{Fix } (O) \subset \mathcal{F}$. Assume by negation that $\hat{\gamma} \in \text{Fix } x (O)$ such that $\hat{\gamma} \notin \mathcal{F}$. This means that there is an $1 \leq \hat{i} \leq m$ such that $\hat{\gamma} \notin \text{Fix } (O_{\hat{i}})$. Condition 17 implies that there exist an $\overline{\tau} \in \mathcal{F}$ that satisfies $\|O_{\hat{i}} (\hat{\gamma}) - \overline{\tau}\| < \|\hat{\gamma} - \overline{\tau}\|$. From this inequality, since $O_1, O_2, \ldots, O_m$ are nonexpansive operators, it is easy to see that

$$
\|O (\hat{\gamma}) - \overline{\tau}\| = \left\| \sum_{t=1}^{\Theta} w_t F_t (\hat{\gamma}) - \overline{\tau} \right\| \leq \sum_{t=1}^{\Theta} w_t \|F_t (\hat{\gamma}) - \overline{\tau}\| < \|\hat{\gamma} - \overline{\tau}\|,
$$

and, consequently, that $\hat{\gamma} \notin \text{Fix } (O)$. This contradicts the negation assumption made above and completes the proof. ■

We propose the following hybrid string-averaging subgradient method (HSASM) for solving the problem (86).

Algorithm 19 Hybrid String-Averaging Subgradient Method (HSASM).

Initialization: Let $\{\alpha_k\}_{k=0}^{\infty} \subset (0, 1]$ be a scalar sequence and let $x^0$, $s^0 \in S$ be arbitrary initialization vectors.

Iterative step: Given a current iteration vector $x^k$ and a current vector $s^k$, calculate the next vectors as follows:

If $0 \in \partial f(x^k)$ then set $s^k = 0$ and calculate

$$
x^{k+1} = O(x^k).
$$

If $0 \notin \partial f(x^k)$ then choose and set $s^k \in \partial f(x^k)$ and calculate

$$
x^{k+1} = O \left( x^k - \alpha_k \frac{s^k}{\|s^k\|} \right).
$$

The next theorem shows that sequences generated by the hybrid string-averaging subgradient method (HSASM) have a $(\tau, L)$-output, i.e., contain an iterate that is data-compatible.

Theorem 20 Let $S$ be a convex and compact subspace of $H$ and let $O_1, O_2, \ldots, O_m$ be nonexpansive operators mapping $S$ into $S$, such that $\mathcal{F} = \bigcap_{i=1}^{m} \text{Fix } (O_i) \neq \emptyset$. 19
Let \( O = \sum_{t=1}^\Theta w_t F_t(x) \) be as in [92] and assume that \( \lim_{j \to \infty} O^j y^0 \) exists for any \( y^0 \in S \). Let \( f : S \to R \) be a convex function, let \( \{\alpha_k\}_{k=0}^\infty \subset (0, 1) \) be a sequence such that
\[
\lim_{k \to \infty} \alpha_k = 0 \quad \text{and} \quad \sum_{k=0}^\infty \alpha_k = \infty,
\]
and let \( \tau \in (0, 1) \). If (97) and condition [17] hold then there exist an integer \( K \) and a real number \( \bar{L} \) such that for any sequence \( \{x_k\}_{k=0}^\infty \subset S \), generated by Algorithm [19], the inequalities
\[
d(x_k, \text{SOL}(f, F)) \leq \tau \quad \text{(97)}
\]
and
\[
f(x_k) \leq f(z) + \tau \bar{L} \quad \text{for all } z \in \text{SOL}(f, F) \quad \text{(98)}
\]
hold for all integers \( k \geq K \).

**Proof.** Since \( O_1, O_2, \ldots, O_m \) are nonexpansive and \( O \) is a fit operator, it follows that that \( O \) is nonexpansive. Moreover, condition [17] and proposition [18] ensure that \( \text{Fix}(O) = F \). This, along with the other assumptions of the theorem, enable the use of Theorem [16] to complete the proof. ■

7 Data-compatibility with constrained minimization when the constraints are inconsistent

In this section we consider a data pair \((\Gamma, f)\), assuming that \( \Gamma := \{C_i\}_{i=1}^m \) is a family of closed and convex subsets of \( H \), not necessarily obeying that \( C := \bigcap_{i=1}^m C_i \neq \emptyset \). Let \( \{w_i\}_{i=1}^m \) be a set of weights such that \( w_i \geq 0 \) and \( \sum_{i=1}^m w_i = 1 \). It is well-known that the operator \( P_w := \sum_{i=1}^m w_i P_{C_i}(x) \) is nonexpansive and satisfies
\[
\text{Fix}(P_w) = \text{Arg min}\{\text{Prox}_\Gamma(x) \mid x \in H\},
\]
where \( \text{Prox}_\Gamma(x) := \frac{1}{2} \sum_{i=1}^m w_i \|P_{C_i}(x) - x\|^2 \), see the succinct [2] Subsection 5.4 on the simultaneous projection method. If \( C \neq \emptyset \) then \( \text{Fix}(P_w) = C \). If, however, \( C = \emptyset \) then \( \text{Fix}(P_w) = \Pi(\Gamma, \gamma) \) for \( \gamma = \min\{\text{Prox}_\Gamma(x) \mid x \in H\} \) and is nonempty. Moreover, for any \( y^0 \in H \) the limit \( \lim_{k \to \infty} (P_w)^ky^0 \) exists and belong to \( \text{Fix}(P_w) \).
Assume that $S$ is a convex and compact subspace of $H$, such that $C_i \subseteq S$ for all $i = 1, 2, \ldots, m$, and consider the following algorithm.

**Algorithm 21 Hybrid Simultaneous Projection Subgradient Method (HSPSM).**

**Initialization:** Let $\{\alpha_k\}_{k=0}^{\infty} \subset (0, 1]$ be a scalar sequence and let $x^0$, $s^0 \in S$ be arbitrary initialization vectors.

**Iterative step:** Given a current iteration vector $x^k$ and a current vector $s^k$, calculate the next vectors as follows:

If $0 \in \partial f(x^k)$ then set $s^k = 0$ and calculate

$$x^{k+1} = P_w(x^k).$$

(100)

If $0 \notin \partial f(x^k)$ then choose and set $s^k \in \partial f(x^k)$ and calculate

$$x^{k+1} = P_w\left(x^k - \alpha_k \frac{s^k}{\|s^k\|}\right).$$

(101)

From the above assumptions and discussion we obtain the following theorem as a consequence of Theorem 16. It does not assume consistency of the underlying constraints $\Gamma = \{C_i\}_{i=1}^{m}$, and shows that sequences generated by the hybrid simultaneous projection subgradient method (HSPSM) have a $(\tau, \bar{L})$-output, i.e., contain an iterate that is data-compatible.

**Theorem 22** Assume that $S$ is a convex and compact subspace of $H$, and that $P_w(x) \in S$ for all $x \in S$. Let $f : S \to \mathbb{R}$ be a convex function, let

$$\{\alpha_k\}_{k=0}^{\infty} \subset (0, 1], \text{ be a sequence such that } \lim_{k \to \infty} \alpha_k = 0 \text{ and } \sum_{k=0}^{\infty} \alpha_k = \infty,$$

and let $\tau \in (0, 1)$. Then there exist an integer $K$ and a real number $\bar{L}$ such that, for any sequence $\{x^k\}_{k=0}^{\infty} \subset S$ generated by Algorithm 21, the inequalities

$$d(x^k, \text{SOL}(f, \text{Fix}(P_w))) \leq \tau$$

and

$$f(x^k) \leq f(z) + \tau \bar{L} \text{ for all } z \in \text{SOL}(f, \text{Fix}(P_w))$$

hold for all integers $k \geq K$.

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