On Morin Configurations of Higher Length

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This paper studies finite Morin configurations $F$ of planes in $\mathbb{P}^5$ having higher length—a question naturally related to the theory of Gushel–Mukai varieties. The uniqueness of the configuration of maximal cardinality 20 is proven. This is related to the canonical genus 6 curve $C_\ell$ union of the 10 lines in a smooth quintic Del Pezzo surface $Y$ in $\mathbb{P}^5$ and to the Petersen graph. More in general an irreducible family of special configurations of length $\geq 11$, we name as Morin–Del Pezzo configurations, is considered and studied. This includes the configuration of maximal cardinality and families of configurations of length $\geq 16$, previously unknown. It depends on 9 moduli and is defined via the family of nodal and rational canonical curves of $Y$. The special relations between Morin–Del Pezzo configurations and the geometry of special threefolds, like the Igusa quartic or its dual Segre primal, are focused.

1 Introduction

Let $\mathbb{G}$ be the Grassmannian of $n$-spaces of $\mathbb{P}^{2n+c}$, $c \geq 1$. For any $u \in \mathbb{G}$ we denote by $P_u$ its corresponding $n$-space and by $\sigma_u$ the codimension $c$ Schubert variety

$$\sigma_u := \{ e \in \mathbb{G} \mid P_u \cap P_e \neq \emptyset \}. \quad (1.1)$$

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A scheme of incident $n$-spaces is a closed scheme $F \subseteq \mathbb{G}$ satisfying the condition

$$F \subseteq \bigcap_{u \in F} \sigma_u.$$  \hspace{1cm} (1.2)

This implies $P_u \cap P_v \neq \emptyset$, $\forall u, v \in F$. We say that $F$ is complete if the equality holds.

**Definition 1.1.** A Morin configuration is a complete scheme $F$ of incident $n$-spaces.

Integral Morin configurations $F$ of planes in $\mathbb{P}^5$ were classified in 1930 by Morin himself if $\dim F > 0$, see [17]. In the same paper the following problem is posed:

**Problem 1.2.** Classify finite Morin configurations of planes in $\mathbb{P}^5$.

Notice that, as Zak [24] points out, the analogous classification in $\mathbb{P}^{2n+c}$ is elementary in the case $2n+c+1 \neq \binom{n+2}{2}$. Morin problem in $\mathbb{P}^5$, which is specially related to hyperkähler geometry, was readdressed in [8] by Dolgachev and Markushevich. They construct and study configurations of minimal cardinality 10 (resp. 13) and their families. In [20] O’Grady proved the existence of configurations of cardinality $k$ for any $10 \leq k \leq 16$. Next he showed that a finite Morin configuration of planes in $\mathbb{P}^5$ has length $k \leq 20$ and asked about the missing cases. The main result of [9] is the construction of a finite Morin configuration of planes in $\mathbb{P}^5$ of cardinality 20. In this paper we study the geometry of the configurations in several ways. Our constructions are building up on the theory of Gushel–Mukai manifolds [4, 5, 16] and EPW sextics [15, 18, 19]. We work over the complex field, let us summarize our results as follows.

Along the paper we construct in $\mathbb{P}^5$ an irreducible family of Morin configurations $F$ of any length $k$ between 11 and 20. This family depends on 9 moduli and defines a divisor in the moduli space of finite Morin configurations. A general configuration has instead length 10. For reasons soon to be evident, the members of our family will be called Morin–Del Pezzo configurations. Relying on the geometry of singular genus 6 canonical curves, we describe these configurations of length $k \in [11, 20]$. We prove that any smooth configuration $F$ of length $k \geq 16$ is Morin–Del Pezzo and, moreover, that:

**Theorem 1.3.** Up to $\text{Aut}\mathbb{P}^5$ a unique Morin configuration of planes in $\mathbb{P}^5$ exists having maximal cardinality 20.

See Sections 6 to 9. The central core of the paper is dedicated to show several relations connecting Morin configurations of planes to the beautiful geometry of some
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classical projective varieties. Our methods rely indeed on these relations, which seem to be of independent interest. This include the following:

1. The geometry related to a quintic Del Pezzo surface and the Segre primal.
2. The family of threefolds $V \in |O_{\mathbb{P}^2 \times \mathbb{P}^2}(2, 2)|$ with isolated singularities.
3. Highly singular canonical curves of genus 6 and possibly higher. Let us discuss each of these points separately.

1. To reasonably summarize these relations let us consider a smooth quintic Del Pezzo surface $Y \subset \mathbb{P}^5$. The linear system $\mathbb{P}_4^3 Y$, of the quadrics through $Y$, is a 4-space. Along the way we prove the following result, see 5.6.

**Theorem 1.4.** The discriminant hypersurface in $\mathbb{P}_4^3 Y$ is twice the Segre cubic $\Delta_Y$.

Then we consider the union of the ten lines of $Y$. This is a stable canonical curve

$$C_\ell \subset Y \subset \mathbb{P}^5$$

of genus 6. The linear system $\mathbb{P}_5^5$ of the quadrics through $C_\ell$ is a 5-space and $\mathbb{P}_4^3 Y$ is a hyperplane in it. It is known that the locus in $\mathbb{P}_4^3 Y$ of all quadrics of rank $\leq 4$ is union of five planes $P_1 \ldots P_5$ of $\Delta_Y$. Moreover, $\text{Sing } C_\ell$ is a set of 15 nodes and, for each $z \in \text{Sing } C_\ell$, the linear system $P_z := \{Q \in \mathbb{P}_5^5 \mid z \in \text{Sing } Q\}$ is a plane. $P_z$ is not in $\mathbb{P}_4^3 Y$. Let $z_1, z_2 \in \text{Sing } C_\ell$, one can show that $P_{z_1} \cap P_{z_2} \neq \emptyset$. Then it is possible to deduce as in Section 8 that the mentioned planes define a Morin configuration

$$F_\ell := \{P_1 \ldots P_5, P_z \mid z \in \text{Sing } C_\ell\}. \quad (1.4)$$

In particular, Theorem 1.3 can be also stated as follows.

**Theorem 1.5.** $F_\ell$ is the unique Morin configuration of cardinality 20 up to $\text{Aut } \mathbb{P}^5$.

2. The Lagrangian Grassmannian $LG(10, 20)$ and Morin configurations are strictly related. To be more precise let us fix some conventions, to be used throughout all the paper. We assume $\mathbb{P}^5 = \mathbb{P}(W)$ and denote the natural symplectic pairing of $\wedge^3 W$ as

$$w : \wedge^3 W \times \wedge^3 W \to \wedge^6 W.$$
Let $G \subset \mathbb{P}(\wedge^3 W)$ be the Grassmannian of planes of $\mathbb{P}^5$. As is well known a closed scheme $F \subset G$ is a Morin configuration iff its linear span is $\mathbb{P}(A)$, where $A$ belongs to the Lagrangian Grassmannian $LG(10, \wedge^3 W)$ and

$$F = \mathbb{P}(A) \cdot G.$$ \hfill (1.6)

We fix a point $u \in G$: for any finite $F$ considered $u$ will be a smooth point of $F$. Let $P_u^\perp$ be the net of hyperplanes through the plane $P_u$, for clarity we also fix the notation:

$$\mathbb{P}^2 \times \mathbb{P}^2 := P_u \times P_u^\perp.$$ \hfill (1.7)

This paper also relies on the construction, given in Section 2, where we associate to a Morin configuration $F$ a hypersurface of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. Indeed, $F$ spans $\mathbb{P}(A)$ as above and $F$ is pointed by $u$. We show that the pair $(A, u)$ uniquely defines a hypersurface $V_A \subset \mathbb{P}^2 \times \mathbb{P}^2$ and prove the following theorem.

**Theorem 1.6.** There exists a natural biregular map between $F - \{u\}$ and $\text{Sing } V_A$.

This relates the study of Morin configurations $F$ of higher length to the study of hypersurfaces $V$ of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ such that $\text{Sing } V$ is finite. In particular, let $V_\ell \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the threefold associated to the maximal configuration $F_\ell$. In the paper we describe its very interesting geometry as follows.

- $V_\ell$ contains a configuration of eight planes $a_i \times \mathbb{P}^2$ and $\mathbb{P}^2 \times b_j$, $1 \leq i, j \leq 4$, such that the sets $\alpha = \{a_1 \ldots a_4\}$ and $\beta = \{b_1 \ldots b_4\}$ are in general position in $\mathbb{P}^2$.
- Let $p: V_\ell \rightarrow \mathbb{P}^2$ be the 1st, (2nd), projection and $\Gamma_p \subset \mathbb{P}^2$ its branch sextic. Then $\Gamma_p$ is the union of the singular conics of the pencil whose base locus is $\alpha$, $\beta$.

Let $\mathcal{I}$ be the ideal sheaf of the set $\{(a_1, b_1) \ldots (a_4, b_4)\} \subset \mathbb{P}^2 \times \mathbb{P}^2$. Then $|\mathcal{I}(1, 1)|$ defines a degree 2 rational map $\pi: \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^4$, recently considered in [3, 7, 12]. Its branch divisor is the *Igusa quartic threefold*, that is, the dual of the Segre cubic. As a consequence of the mentioned results and of our description, it follows:

**Theorem 1.7.** $V_\ell$ is the ramification divisor of $\pi$ and $\pi(V_\ell)$ is the Igusa quartic.

(3) Going back to the quintic Del Pezzo $Y$, let $C \in |C_\ell|$ be a reduced singular curve and $\mathbb{P}^5_C$ the 5-space of the quadrics through $C$. Note that $C$ is a 1-dimensional Gushel–Mukai manifolds and analogous systems of quadrics were analyzed in [4, 16]. As in the
case of \( C \) we can reconstruct from \( \text{Sing} \ C \), in the Grassmannian of planes of \( \mathbb{P}^5_C \), the family of planes

\[
F_C := \{ P_1 \ldots P_5, P_z \in \text{Sing} \ C \}, \tag{1.8}
\]

where \( P_1 \ldots P_5 \) are the nets of rank 4 quadrics through \( Y \) and \( P_z \subset \mathbb{P}^5_C \) is the net of quadrics, which are singular at \( z \). The next theorem is proven in Section 8.

**Theorem 1.8.** Suppose that \( \text{Sing} \ C \) be not in a hyperplane. Then \( F_C \) is a Morin configuration.

The special feature of \( F_C \) is that \( \{ P_1, \ldots, P_5 \} \) is a smooth linear section of the Grassmannian \( \mathbb{G}_Y \) of planes of \( \mathbb{P}^4_Y \), see [6, 8.5.3]. Then the corresponding points \( p_1 \ldots p_5 \) only span a 3-space. Since a finite Morin configuration spans a 9-space, it follows that \( F_C \) has length \( k \geq 11 \) and, moreover, \( \text{Sing} \ C \) necessarily spans \( \mathbb{P}^5_C \).

**Definition 1.9.** A Morin–Del Pezzo configuration is a finite Morin configuration that contains with multiplicity one a 5-tuple projectively equivalent to \( \{ P_1 \ldots P_5 \} \).

In Sections 6, 7, and 8 we construct an integral family whose members are the Morin–Del Pezzo configurations and describe their properties. Let \( F = \mathbb{P}(A) \cdot \mathbb{G} \) be one of these and \( V_A \subset \mathbb{P}^2 \times \mathbb{P}^2 \) the bidegree \( (2, 2) \) hypersurface defined by \((A, u)\). We prove that \( F = \mathbb{P}(C) \) for some \( C \in |C| \) and that \( V_A \) contains a plane. Then \( V_A \) is rational and is reconstructed from \( C \) as follows. In the ambient space of \( C \) the base locus of the net \( P_5 \) is a Segre product \( \mathbb{P}^1 \times \mathbb{P}^2 \). Let \( J \) be the ideal sheaf of \( C \) in it, then \( |J(2, 2)| \) defines a rational map \( q : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2 \). Let \( p : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2 \) be the projection map, then:

**Theorem 1.10.** \( q \times p : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2 \times \mathbb{P}^2 \) is a birational embedding with image \( V_A \).

These results are used to deduce Theorem 1.3 and enumerate configurations. This is quickly done in Section 9. Then some concluding remarks follow: we note that a stable canonical \( C \) of genus \( g \geq 7 \) defines an analogous scheme of incident \( (g-4) \)-spaces in the dual of the space \( \mathbb{I}_C \) of quadrics through \( C \). That is, \( F'_C := \{ P_z, z \in \text{Sing} \ C \} \), where \( P_z \) is the orthogonal of \( \mathbb{I}_z := \{ Q \in \mathbb{I}_C \mid z \in \text{Sing} \ Q \} \). The involved dimensions satisfy the mentioned Zak’s equality. This makes interesting the following question:

\[
\text{When is } F'_C \text{ a Morin configuration and has maximal cardinality?} \tag{1.9}
\]
Here canonical graph curves, like $C_\ell$, could come into play. These are union of $2g - 2$ lines and have $3g - 3$ nodes. Each is uniquely defined by its dual associated graph. For $C_\ell$ this is the well-known Petersen graph (cf. [9]).

We discuss some example generalizing $C_\ell$ and chances that $3g - 3$ be the maximal cardinality. In this paper we also revisit O'Grady's bound for $g = 6$ and discuss realizations of singular plane sextics as $3 \times 3$ determinant of quadratic forms, see Section 3, Remark 7.12, and [20].

Further notations: $(X)$ linear span of $X$. $[X]$ vector space generated by $X$.

2 Morin Configurations of Planes in $\mathbb{P}^5$ and $V$-Threefolds

We start studying finite Morin configurations of planes in $\mathbb{P}^5 = \mathbb{P}(W)$. Our aim in this section is to associate to such a configuration with a distinguished plane $\mathbb{P}(U)$ a nodal $(2, 2)$ divisor in $\mathbb{P}^2 \times \mathbb{P}^2$. This construction was described in [15, Rem. 3.7] but here we aim to perform it in an intrinsic way, proving its independence from the choice of a direct sum decomposition $W = U \oplus E$. In Theorem 2.13 we show that the number of singular points of the associated $V$ threefold is one less than the cardinality of the configuration. Moreover, the singular points on $V$ are the images through a central projection of the points from the configuration. In such a way the Morin problem is reduced to the classification of singular $V$ threefolds.

We begin from the point $u$ we have already fixed in the Grassmannian $G$. Let $U \subset W$ be its corresponding space and let $1 \leq i \leq 3$. As in [19] we consider the pairing

$$ p_i : \wedge^i U \times \wedge^{3 - i} W \rightarrow \wedge^3 W, $$

(2.1)
defined by the wedge product, and its image \( W^i_u \subset \wedge^3 W \). We have the filtration
\[
\wedge^3 W := W^0_u \supset W^1_u \supset W^2_u \supset W^3_u = \wedge^3 U.
\] (2.2)

On the other hand we start dealing with maximal isotropic spaces \( A \) of \( w \).

**Proposition 2.1.** Let \( A \) be such a space then: \( A \subset W^1_u \Leftrightarrow \wedge^3 U \subset A \).

**Proof.** \( W^1_u \) is the orthogonal space of \( \wedge^3 U \), hence \( A \subset W^1_u \) iff \( A + \wedge^3 U \) is isotropic. Since \( A \) is maximal isotropic then \( A + \wedge^3 U \) is isotropic iff \( \wedge^3 U \subset A \). \( \blacksquare \)

Next we fix the following assumption on the maximal isotropic space \( A \):
\[
A \subset W^1_u \quad \text{and} \quad A \cap W^2_u = \wedge^3 U.
\] (2.3)

This is equivalent to say that \( u \in \mathbb{P}(A) \) and that the scheme \( \mathbb{P}(A) \cdot G \) is smooth of dimension 0 at \( u \). To see this just notice that \( W^2_u/W^3_u \) is naturally isomorphic to the Zariski tangent space to \( G \) at \( u \). Hence, its projective completion is embedded as
\[
T_u := \mathbb{P}(W^2_u) \subset \mathbb{P}(\wedge^3 W).
\] (2.4)

Therefore, the assumption is satisfied iff \( T_u \cap \mathbb{P}(A) = \{ u \} \). We will be mainly interested in the following loci in \( LG(10, 20) \), to be repeatedly considered.

**Definition 2.2.**
\[
\begin{align*}
\mathcal{A} & := \{ A \in LG(10, 20) \mid \text{the scheme } \mathbb{P}(A) \cdot G \text{ is finite} \}. \\
\mathcal{A}^c & := \{ A \in \mathcal{A} \mid \mathbb{P}(A) \cdot G \text{ is a Morin configuration} \}.
\end{align*}
\]

Now we consider the planes \( P_u = \mathbb{P}(U) \) and \( P^\perp_u := \mathbb{P}(\wedge^2 W/U) \) and the embedding
\[
P_u \times P^\perp_u \subset \mathbb{P}^8 := \mathbb{P}(U \otimes \wedge^2 (W/U))
\] (2.5)
by the Segre map. Moreover, let us also assume the natural identification
\[
U \otimes \wedge^2 (W/U) = W^1_u/W^2_u.
\] (2.6)
which is provided by the isomorphism sending \(a \otimes (b + U)\) to \((a \wedge b) + W^2_u\). Our aim is to reconstruct, from a maximal isotropic space \(A\) as above, a hypersurface

\[
V_A \subset \mathbb{P}_u \times \mathbb{P}^\perp_u \subset \mathbb{P}^8
\]  

of bidegree \((2, 2)\), which is naturally associated to \(A\). To construct \(V_A\) we begin with an auxiliary space \(E \subset W\) such that \(W = U \oplus E\). This provides the decomposition

\[
W^1_u = U \wedge (\wedge^2 E) \oplus (\wedge^2 U) \wedge E \oplus \wedge^3 U,
\]

and the new identification

\[
U \otimes (\wedge^2 W/U) = U \otimes (\wedge^2 E) = U \wedge (\wedge^2 E).
\]

Moreover, the decomposition \(W = U \oplus E\) induces a symmetric bilinear form

\[
w_{A,E} : A \times A \to \mathbb{C},
\]

which is defined as follows. Let \(a, a' \in A\) then, as for any vector of \(W^1_u\), we can uniquely write \(a = a_1 + a_2 + a_3\) and \(a' = a_1' + a_2' + a_3'\), where \(a_i, a_i' \in (\wedge^i U) \wedge (\wedge^{3-i} E), i = 1, 2, 3\). Since \(A\) is isotropic and \(a_1 \wedge a_1' = 0\), it follows that

\[
a \wedge a' = a_1 \wedge a_2' + a_2 \wedge a_1' = 0.
\]

**Definition 2.3.** \(w_{A,E}\) is the bilinear map such that \(w_{A,E}(a, a') = a_1 \wedge a_2'\).

Clearly, \(w_{A,E}\) is symmetric by the above equality. It is easy to see that \(a \in \text{Ker} w_{A,E}\) iff \(a_1 = a_2 = 0\) iff \(a \in \wedge^3 U\). Hence, \(w_{A,E}\) defines a quadric \(\hat{Q}_{A,E} \subset \mathbb{P}(A)\) such that

\[
\text{Sing} \hat{Q}_{A,E} = \{u\}.
\]

Now let us consider the projection map \(p_1 : W^1_u \to U \wedge (\wedge^2 E)\) and its restriction to \(A\)

\[
p_1|A : A \to U \wedge (\wedge^2 E).
\]
Under our assumption on $A$ this map is surjective and $\wedge^3 U$ is its Kernel. In particular, $p_1|A$ induces a linear projection of center the point $u$

$$p_{A,E} : \mathbb{P}(A) \to \mathbb{P}^8 = \mathbb{P}(W_u^1/W_u^2).$$

(2.13)

Since $u \in \text{Sing} \hat{Q}_{A,E}$ it is clear that the image of $\hat{Q}_{A,E}$ by $p_{A,E}$ is a quadric

$$Q_{A,E} \subset \mathbb{P}^8.$$

(2.14)

**Definition 2.4.** $Q_{A,E}$ is the quadric of $\mathbb{P}^8$ associated to the pair $(A, E)$.

**Remark 2.5.** So far some useful characterizations of the ambient space $\mathbb{P}^8$ have been introduced. In the same way, a useful more abstract construction of its quadric $Q_{A,E}$ is worth to be considered. Let us consider the quotient space

$$\overline{W}_u^1 := W_u^1/W_u^3$$

(2.15)

and its subspaces $\overline{A} := A/W_u^3$, $\overline{W}_u^2 = W_u^2/W_u^3$. On $\overline{W}_u^1$ the wedge product $w$ descends to a symplectic form $\overline{w}$ so that $\overline{A}$ and $\overline{W}_u^2$ are isotropic. Since $A \cap W_u^2 = W_u^3$ we have

$$\overline{W}_u^1 = \overline{A} \oplus \overline{W}_u^2.$$ 

(2.16)

Let $\overline{W}_E := (U \wedge (\wedge^2 E) + W_u^3)/W_u^3$ then $\overline{W}_E^1$ is isotropic for $\overline{w}$. Moreover, the previous subspaces $\overline{A}, \overline{W}_u^2, \overline{W}_E$ are two by two complementary in $\overline{W}_u^1$. In particular, $\overline{w}$ defines a duality $\overline{w} : \overline{W}_E \times \overline{W}_u^2 \to \mathbb{C}$. Putting under this duality $\overline{W}_u^2 = \overline{W}_E^\vee$, one finally obtains

$$\overline{A} \subset \overline{W}_E \oplus \overline{W}_E^\vee = \overline{W}_u^1.$$ 

(2.17)

Since $\overline{A} \cap \overline{W}_u^2 = 0$ then $\overline{A}$ is the graph of an isomorphism $\alpha : \overline{W}_E \to \overline{W}_E^\vee$. This is symmetric, see [15, Rem. 3.7]. It is straightforward to check that $\alpha$ defines $Q_{A,E}$.

Now recall our identification $U \otimes W/U = U \wedge (\wedge^2 E)$ and notice that the set of decomposable vectors $u \otimes (w+U)$ always corresponds to the set of decomposable vectors $u \wedge (e_1 \wedge e_2)$, independently from $E$. This means that, in the identity

$$\mathbb{P}^8 = \mathbb{P}(U \otimes W/U) = \mathbb{P}(U \wedge (\wedge^2 E)),$$
the Segre product $P_u \times P_u^\perp$ stays unchanged if we replace $E$ by any $F$ such that $U \oplus F = W$.

Let us consider the intersection scheme

$$V_{A,E} = Q_{A,E} \cdot (P_u \times P_u^\perp) \subset \mathbb{P}^g,$$

we are now in position to prove the following result.

**Theorem 2.6.** $V_{A,E}$ does not depend on the choice of $E$.

**Proof.** Let $\overline{P_u \times P_u^\perp} \subset \mathbb{P}(A)$ be the cone of vertex $u$ over $P_u \times P_u^\perp$. It suffices to show that the form $w_{A,E} : A \times A \to C$, defining the quadric cone $Q_{A,E}$, is proportional to the symmetric bilinear form $w_{A,F} : A \times A \to C$, defined in the same way of $w_{A,E}$ from any space $F \subset W$ such that $U \oplus F = W = E \oplus F$. To prove this recall that $w_{A,E}(a, a) = a_1 \wedge a_2$, where $a = a_1 + a_2 + a_3$ and $a_i \in \wedge^i U \wedge (\wedge^{3-i} E), i = 1, 2, 3$. Then consider any vector $a \in P_u \times P_u^\perp$. This means that $a_1$ is decomposable, that is, $a_1 = u \wedge e' \wedge e''$, with $u \in U$ and $e', e'' \in E$. Let $a_2 = \sum_k u_k' \wedge u_k'' \wedge e_k$ then we have

$$w_{A,E}(a, a) = \sum_k u \wedge u_k' \wedge u_k'' \wedge e' \wedge e'' \wedge e_k.$$ 

Now consider any $F$ satisfying $U \oplus F = W = E \oplus F$ and the planes $P_e, P_f$ defined by the points $e = [\wedge^3 E], f = [\wedge^3 F]$. Then we can define a natural embedding

$$\iota_f : P_e \to P_u \times P_f$$

as follows. Let $z \in P_e$ then $\iota_f(z) := (x, y)$, where $\{x\} = l \cap P_u$, $\{y\} = l \cap P_f$ and $l$ is the unique line passing through $z$ and incident to $P_u$ and $P_f$. The image of $\iota_f$ is the graph of a projective isomorphism $\alpha_f : P_u \to P_f$. This is defined, up to a nonzero constant factor, by an isomorphism $\alpha_f : U \to F$. Moreover, $E$, as a subspace of $U \oplus F = W$, is the graph of $\alpha_f$, that is, $E = \{v \oplus \alpha_f(v), v \in U\}$. Replacing, in $a_1, a_2$ and $w_{A,E}$, the vectors of $E$ by their images via $\alpha_f$, we obtain $w_{A,F}(a, a)$. Let $\alpha_{e,f} : E \to F$ be the isomorphism sending $v \oplus \alpha_f(v) \in E$ to $\alpha_f(v)$. Then, for every $a \in A$, $w_{A,E}(a, a)$ and $w_{A,F}(a, a)$ are proportional by the determinant of $\alpha_{e,f}$. This implies the statement. \hfill \blacksquare

From now on, since it does not depend on $E$, we will denote $V_{A,E}$ as $V_A$ and as $V_A$ a global section of $\mathcal{O}_{P_u \times P_u^\perp}(2, 2)$, which defines it. As we will see $V_A$ is proper in $P_u \times P_u^\perp$. 

Following some use we will say that a bidegree $(2, 2)$ hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ is a Verra threefold, for short a $V$-threefold. Let us introduce the following definitions.

**Definition 2.7.** $V_A := \text{div}(v_A)$ is the $V$-threefold associated to $A$.

**Definition 2.8.** $F_A := \mathbb{P}(A) \cdot G$ is the scheme of incident planes of $A$.

To conclude the section we characterize the scheme $F_A$ when it is finite. Before, we add further notation and geometric information on the previous constructions. Let $G_u := \{ e \in G | P_e \cap P_u \neq \emptyset \} = \mathbb{P}(W^1_u) \cap G$ be the codimension one Schubert cycle defined by $P_u$. In $\mathbb{P}(W^1_u)$ we also have the projective tangent space to $G$ at $u$, namely the space $T_u = \mathbb{P}(W^2_u) \subset \mathbb{P}(W^1_u)$. Let $\tau_u : G_u \rightarrow \mathbb{P}^8$ be the restriction to $G_u$ of the linear projection of center $T_u$, say

$$\tau : \mathbb{P}(W^1_u) \rightarrow \mathbb{P}^8.$$  

Describing $\tau_u$ is an exercise in Grassmannians we sketch in the next remark.

**Remark 2.9.** Assume $e \in G_u$ and $e$ not in $T_u$. Then for the point $e$ we have $e = [u_1 \wedge e_2 \wedge e_3]$ so that $u_1 \wedge e_2 \wedge e_3$ is a decomposable vector and $[u_1] = E \cap U$, where $E$ is the vector space $[u_1, e_1, e_2]$ and $u_1 \in U$. In particular, $x := [u_1]$ is the intersection of the planes $P_u$ and $P_e$ and the construction of $x$ defines a rational map

$$\gamma : G_u \rightarrow P_u$$

such that $\gamma(e) := x$. On the other hand let $\mathcal{I}_{P_u}$ be the ideal sheaf of $P_u$ in $\mathbb{P}^5$, then $P_u^\perp = \mathbb{P}(\wedge^2 W / U)$ can be obviously identified to the net $|\mathcal{I}_{P_u}(1)|$ of hyperplanes through $P_u$. It is also clear that $e$ uniquely defines an element $y \in P_u^\perp$, namely the hyperplane in $\mathbb{P}^5$ containing $P_u$ and the points $[e_2]$ and $[e_3]$. This defines a 2nd rational map

$$\gamma^\perp : G_u \rightarrow P_u^\perp.$$
such that $\gamma^\perp(e) := y$. Finally, since $\mathbb{P}^8 = \mathbb{P}(W_1^1/W_2^2)$, we have also a natural map

$$s : P_u \times P_u^\perp \to \mathbb{P}^8$$

such that $s(x, y) := [u_1 \land e_2 \land e_3 \mod W_2^2]$. This is the Segre embedding of $P_u \times P_u^\perp$ we have already considered. In particular, the next property follows.

**Proposition 2.10.** $\tau : G_u \to \mathbb{P}^8$ factors as in the next diagram:

$$
\begin{array}{ccc}
G_u & \xrightarrow{\gamma \times \gamma^\perp} & P_u \times P_u^\perp \\
& & \xrightarrow{s} \mathbb{P}^8
\end{array}
$$

Let $W_{x,y}$ be the 4-dimensional vector space $(U + E)/[u_1]$. Notice that the fibre $\tau_u^*(x, y)$ is naturally embedded in $G_u$ as the Grassmannian of lines of $\mathbb{P}(W_{x,y})$.

Now we will describe $F_A$ in terms of the singular locus of $V_A$. We fix the notation

$$\tau_A : \mathbb{P}(A) \to \mathbb{P}^8$$

for the restriction to $\mathbb{P}(A)$ of the projection $\tau : \mathbb{P}(W_1^1) \to \mathbb{P}^8$. Let $(x, y) \in P_u \times P_u^\perp$ we also fix the notation $A_x$ and $A_y$ for the 4-dimensional subspaces of $A$ such that

$$\mathbb{P}(A_x) = \tau_A^{-1}([x] \times P_u^\perp) \text{ and } \mathbb{P}(A_y) = \tau^{-1}(P_u \times \{y\}).$$

On $A_x$ and $A_y$ we have the symmetric bilinear forms induced by $w_{A,E}$, say

$$<,>_{y} : A_x \times A_x \to \mathbb{C} \text{ and } <,>_{x} : A_y \times A_y \to \mathbb{C}.$$ \hspace{1cm} (2.26)

These define two quadric cones of vertex $u$ in $\mathbb{P}(A_x)$ and $\mathbb{P}(A_y)$, cutting on the planes $[x] \times P_u^\perp$ and $P_u \times \{y\}$ the conics $V_A \cdot [x] \times P_u^\perp$ and $V_A \cdot P_y \times \{y\}$. Let

$$F'_A := F_A - \{u\},$$

from now on we assume that $F_A$ is finite, up to different advice.

**Lemma 2.11.** The map $\tau_{|F'_A} : F'_A \to P_u \times P_u^\perp \subset \mathbb{P}^8$ is biregular to its image.
Proof. We have $\tau(F'_A) \subset \tau(G_u) \subset P_u \times P_u^\perp$. To prove that $\tau|_{F'_A}$ is biregular to $\tau(F'_A)$ consider any scheme $\zeta \subset F'_A$ of length 2. We have $\zeta \subset P(A)$. Moreover, the restriction of $\tau$ to $P(A)$ is the projection from $\zeta$ to $u$. Hence, $\tau|_{F'_A}$ is not biregular to its image iff the line $\langle \zeta \rangle$ contains $u$. Since $u \not\subset \zeta$, this is equivalent to say that the scheme $\langle \zeta \rangle \cdot G$ has length $\geq 3$. Then it follows $\langle \zeta \rangle \subset G$, because $G$ is intersection of quadrics, and hence $F_A$ is not finite: against our assumption. This implies the statement. □

Now let us consider the cone of vertex $u$ over $F'_A$, that is, 

$$C(F'_A) := \tau|_{P(A)}^* \tau(F'_A). \quad (2.28)$$

Let $p = [a] \in P(A) - \{u\}$, keeping our constructions and conventions we can assume 

$$W^1_u = U \wedge (^2E) \oplus (^2U) \wedge E \oplus ^3U. \quad (2.29)$$

Then $a$ uniquely decomposes as $a = a_1 + a_2 + a_3$ with $a_i \in \wedge^i U \wedge (^3-i)E$, $i = 1, 2, 3$.

Lemma 2.12. $C(F_A) = \{[a_1 + a_2 + a_3] \in P(A) \mid a_2 = 0\}$.

Proof. It is clear that $p \in F'_A$ iff the line $\langle u, p \rangle$ is in the cone $C(P_u \times P_u^\perp)$ of vertex $u$ over $P_u \times P_u^\perp$, that is, iff $\langle u, p \rangle \subset P(A_y)$ for some $y \in P_u^\perp$. Equivalently, one has $a = u_1 \wedge u_2 \wedge u_3 + u_1 \wedge e_2 \wedge e_3$ for some $e_2 \wedge e_3 \in \wedge^2 E$, that is, $a_2 = 0$. □

Of course the condition defines as well the embedding $\tau(F'_A) \subset P_u \times P_u^\perp$. Now we study $\tau(F'_A)$. To this purpose let $p = [a_1 + a_2 + a_3] \in P(A)$ as usual. If $p \in C(F_A)$ then we have $\tau(p) = (x, y) \in P_u \times P_u^\perp$. At first we remark that the condition $a_2 = 0$ is precisely equivalent to the property that the polar forms 

$$< \cdot, a_2 >_y : A_x \to \mathbb{C} \quad \text{and} \quad < \cdot, a_2 >_x : A_y \to \mathbb{C} \quad (2.29)$$

of the vector $a_1 + a_2 + a_3$ are identically zero. This immediately translates in the following simple condition on a point $o := (x, y) \in P_u \times P_u^\perp$:

Both the planes $P_u \times \{y\}$ and $\{x\} \times P_u^\perp$ are tangent to $V_A$ at $o$. \quad (2.30)

Since these planes generate the embedded tangent space in the Segre embedding of $P_u \times P_u^\perp$, it follows that $o \in \tau(F'_A)$ iff $o$ is singular for $V_A$. To take in account multiplicities
let us write explicitly the equations of $\tau(F'_A)$. Under our notation we have

$$P_u \times P_u^\perp = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8.$$ 

On $\mathbb{P}^2 \times \mathbb{P}^2$ we fix projective coordinates $(x_1 : x_2 : x_3) \times (y_1 : y_2 : y_3)$ defining the point $(x, y)$ and then we consider the equation $f$ of $V$. Therefore, we have

$$f = \sum a_{ij} x_i x_j = 0, \quad (2.31)$$

where the $a_{ij}$'s are quadratic forms in $y$. By the condition 2.30 the partials

$$f_{x,i} := \frac{\partial f}{\partial x_i}, \quad f_{y,i} = \frac{\partial f}{\partial y_i} \quad i = 1, 2, 3,$$

define $\tau(F'_A)$, so the next theorem follows.

**Theorem 2.13.** $F'_A$ is biregular to the scheme defined by the above derivatives, that is, to the singular locus of $V_A$.

We are grateful to M. Kapustka for discussions around this result. We remark that $\tau(F'_A)$ fits in the standard exact sequence

$$0 \to T_{V_A} \to T_{\mathbb{P}^2 \times \mathbb{P}^2|_{V_A}} \to O_{V_A}(2, 2) \to O_{F'_A} \to 0 \quad (2.32)$$

of tangent and normal sheaves realizing the singular locus of $V_A$. This complete the proof of Theorem 1.6. black

**3 V-Threefolds With Isolated Singularities**

In the previous section we associated to a Morin configuration a singular $V$-threefold. The aim of this section is to study the geometry of such singular threefolds. In Theorem 3.9 we show that highly singular $V$-threefolds must contain planes.

We consider a $V$-threefold $V = V(f) \subset \mathbb{P}^2 \times \mathbb{P}^2$ such that Sing $V$ is finite. We want to discuss more on Sing $V$. Let us consider the projections

$$\mathbb{P}^2 \xleftarrow{\pi_x} V \xrightarrow{\pi_y} \mathbb{P}^2 \quad (3.1)$$
and the schemes $R_x \subset V$ and $R_y \subset V$ respectively defined by the ideals

$$J_{V,x} := (f_{x,1}, f_{x,2}, f_{x,3}), \quad J_{V,y} := (f_{y,1}, f_{y,2}, f_{y,3}).$$

(3.2)

It is clear that $R_x$ and $R_y$ are the ramification schemes of $\pi_y$ and $\pi_x$, respectively. Their supports are the loci where the tangent maps $d\pi_y$ and $d\pi_x$ have rank $\leq 1$. Now, in the Chow ring $\text{CH}^*(\mathbb{P}^2 \times \mathbb{P}^2)$, let $h_x$ and $h_y$ be the classes of the pull-back of a line by $\pi_x$ and $\pi_y$, respectively. Then it is very easy to see that $f_{x,i}$ and $f_{y,i}$ define divisors

$$D_{x,i} := \text{div}(f_{x,i}) \in |h_x + 2h_y|, \quad D_{y,i} := \text{div}(f_{y,i}) \in |2h_x + h_y|.$$

The next properties we show for $R_x$ are true for $R_y$ with the same arguments.

**Lemma 3.1.** Let $o \in \text{Sing} V$. If the plane $\mathbb{P}^2 \times \pi_y(o)$ is not in $V$ then $o \in \text{Sing} R_x$.

**Proof.** Cf. [12] 1.2. Let $(x_1 : x_2 : x_3) \times (t_1, t_2)$ be coordinates on $\mathbb{P}^2 \times [y_3 \neq 0]$ so that $t_i := \frac{y_i}{y_3}$ for $i = 1, 2$ and $o$ is $(0 : 0 : 1) \times (0, 0)$. On it the equation of $V$ is

$$qx_3^2 + (at_1x_1 + bt_2x_1 + ct_1x_2 + dt_2x_2)x_3 + p = 0,$$

where $q \in \mathbb{C}[t_1, t_2], p \in \mathbb{C}[x_1, x_2]$ are quadratic forms. The partials $\frac{\partial}{\partial x_i}, i = 1, 2, 3$, give local equations of $R_x$. In affine coordinates $u_1 := \frac{x_1}{x_3}, u_2 := \frac{x_2}{x_3}$, these equations are

$$2q + at_1u_1 + bt_2u_1 + ct_1u_2 + dt_2u_2 =$$

$$at_1 + bt_2 + 2p_{11}u_1 + 2p_{12}u_2 = ct_1 + dt_2 + 2p_{22}u_2 + 2p_{12}u_2 = 0,$$

where $p = \sum p_{ij}u_iu_j$. Clearly, the tangent space $T_{R_x,o}$ is defined by the latter two equations so that $\dim T_{R_x,o} \geq 2$. In Theorem 3.4 we prove that the only irreducible surfaces in $R_x$ are planes $\mathbb{P}^2 \times y$. Since the only one through $o$ is not, it follows that $o \in \text{Sing} R_x$. \hfill ■

**Theorem 3.2.** Assume that $\text{Sing} V$ is finite and the intersection scheme $R_x = D_{x,1} \cdot D_{x,2} \cdot D_{x,3}$ is proper, then the length of the singular locus $\text{Sing} V$ is $\leq 15$.

**Proof.** Since it is proper, $R_x$ is complete intersection of the divisors $D_{x,1}, D_{x,2}, D_{x,3}$ of class $h_x + 2h_y$. Hence, one computes that $R_x$ has arithmetic genus 10 and class $6h_x^2h_y + 12h_xh_y^2$ in $\text{CH}^*(\mathbb{P}^2 \times \mathbb{P}^2)$. Since $\text{Sing} V$ is finite, no component of $R_x$ is a fixed component of
the net of divisors generated by $D_{y,1}, D_{y,2}, D_{y,3}$. Hence, an element $D$ of this net intersects $R_x$ properly and $\text{Sing} \ V$ is embedded in the finite scheme $R_x \cdot D$. By Lemma 3.1 each point $o \in \text{Sing} \ V$ is singular for $R_x$. Then, since $R_x \cdot D$ has length 30 and its multiplicity is $\geq 2$ at each $o \in \text{Sing} \ V$, the length of $\text{Sing} \ V$ is $\leq 15$. ■

**Lemma 3.3.** If $\text{Sing} \ V$ is finite the discriminant of $\pi_y|V : V \to \mathbb{P}^2$ is a reduced curve.

**Proof.** Let $f = \pi_y|V$. From the finiteness of $\text{Sing} \ V$ and generic smoothness it follows that the discriminant of $f$ is a curve. Assume $B$ is a nonreduced, irreducible component of it. Let $y \in B$ be a general point then $\text{Sing} \ V \cap f^*(y) = \emptyset$. Moreover, it follows from [2, Lem. 6.6] that $f^*(y)$ is a conic of rank 1. Let $S \subset V$ be the closure of the union of the lines $f^{-1}(y)$, where $y \in B$ is general. Then $f : S \to B_{\text{red}}$ is a $\mathbb{P}^1$-bundle and $f^*B_{\text{red}}$ has multiplicity 2 along $S$. In particular, there exists an affine open set $U = \text{Spec} \ R \subset \mathbb{P}^2$ so that $U \cap B \neq \emptyset$ and the equation of $V$ in $U \times \mathbb{P}^2$ is $da^2 - bc$, where $a, b, c, d \in R[x_1, x_2, x_3]$. Moreover, $b \in R$ is the equation of $B_{\text{red}}$ in $U$, $d \in R$ and $a, c \in R[x_1, x_2, x_3]$ are forms of degrees 1 and 2, respectively, in $(x_1, x_2, x_3)$. Since $d \notin (b)$ and $V$ is irreducible, we can assume $d = 1$ up to shrinking $U$. Now consider in $U \times \mathbb{P}^2$ the set $Z = \{a^2 = b = c = 0\}$. It is clear that $Z$ is nonempty and hence of dimension 1. Moreover, $Z$ is contained in $\text{Sing} \ V$: this contradicts the finiteness of $\text{Sing} \ V$. ■

It easily follows that the only surfaces possibly contained in $R_x \cup R_y$ are planes.

**Theorem 3.4.** Let $S \subset R_x$ be an irreducible surface then $S = \mathbb{P}^2 \times o$, for some $o \in \mathbb{P}^2$.

**Proof.** $\pi_y(S)$ is an irreducible component of the discriminant curve $\Gamma$ of $\pi_y|V$. By the lemma $\Gamma$ is reduced Assume $\pi_y(V)$ is a curve, then the previous lemma and its proof imply that the general fibre of $V$ over $\pi_y(S)$ is a conic of rank 2. This is impossible because implies dim $S = 1$. Hence, $\pi_y(S)$ is a point and $S$ is a plane, fibre of $\pi_y|V$. ■

Now we assume that $R_x$ contains a plane $P := \mathbb{P}^2 \times o$.

**Proposition 3.5.** Let $b := P \cdot \text{Sing} \ V$ then $b$ is the base locus of a pencil of conics.

**Proof.** We can assume that $P = \{y_1 = y_2 = 0\}$. Then the equation of $V$ is:

$$f = q_{11}y_1^2 + q_{22}y_2^2 + q_{12}y_1y_2 + q_{13}y_1y_3 + q_{23}y_2y_3.$$
Restricting the derivatives \( f_{y,1}, f_{y,2}, f_{y,3} \) to \( P_u \) we conclude that

\[
\text{Sing } V \cdot P = \{ y_1 = y_2 = q_{13} = q_{23} = 0 \}.
\]

Hence, \( b \) is the base locus of the pencil of conics \( \lambda q_{13} + \mu q_{23} = 0 \).

**Remark 3.6.** The locus \( b \) is the complete intersection \( \{ q_{13} = q_{23} = 0 \} \). Let \( \sigma : Y \to \mathbb{P}^2 \) be the blowing up of \( o = \pi_y(P) \) and \( E = \sigma^{-1}(o) \), one has the Cartesian square

\[
\begin{array}{ccc}
\tilde{V} & \xrightarrow{\phi} & V \\
\tilde{\pi}_y \downarrow & & \downarrow \pi_y \\
Y & \xrightarrow{\sigma} & \mathbb{P}^2.
\end{array}
\]

Let \( \tilde{P} \) be the strict transform of \( P \) then \( \tilde{\pi}_y(\tilde{P}) = E \) and the morphism \( \tilde{\pi}_y : \tilde{P} \to E \) is \( \phi \) composed with the rational map defined by the pencil of conics \( \lambda q_{13} + \mu q_{23} = 0 \).

It is now useful to define the sets

\[
\mathcal{P}_x := \{ x \in \mathbb{P}^2 \mid x \times \mathbb{P}^2 \subset V \} \quad \mathcal{P}_y := \{ y \in \mathbb{P}^2 \mid \mathbb{P}^2 \times y \subset V \}.
\]

Let \( P \in \mathcal{P}_x \cup \mathcal{P}_y \) then \( V \) has multiplicity one at \( P \). Since \( \text{Sing } V \) is finite, this just follows writing the local equation of \( V \) at a general point of \( P \).

**Definition 3.7.** \( t_x(V) \) and \( t_y(V) \) are the cardinalities of \( \mathcal{P}_x \) and \( \mathcal{P}_y \).

**Lemma 3.8.** Both \( \mathcal{P}_x \) and \( \mathcal{P}_y \) have at most four points and no three are collinear.

**Proof.** Assume \( \mathcal{P}_x \) contains five points \( o_1 \ldots o_5 \) and let \( C \) a conic through these. It is easy to see that then \( V \) properly contains \( \mathbb{P}^2 \times C \) against the irreducibility of \( V \). The same proof applies to \( V \) and \( \mathbb{P}^2 \times L \), where \( L \) is a line through three points of \( \mathcal{P}_x \).

It follows from the previous results that \( R_x \) and \( R_y \) admit the decompositions

\[
R_x = R_x^1 \cup R_x^2 \quad \text{and} \quad R_y = R_y^1 + R_y^2,
\]

where \( R_x^1, R_x^2 \) are curves and \( R_y^2, R_y^2 \) disjoint unions of planes of class \( h_x^2, h_y^2 \), respectively. Relying on this, let us give now an alternative proof of O’Grady’s bound |\text{Sing } V| \leq
This will be useful for our further purposes. In particular, the next theorem suggests that a $V$-threefold $V$ such that $|\text{Sing } V| = 19$ contains four disjoint planes.

**Theorem 3.9.** If $\text{Sing } V$ is finite then it has length $\leq 15 + t$ with $t \leq 4$ and $t := \min \{ t_x(V), t_y(V) \}$.

**Proof.** Let $t = 1$ then $R_{x}^2$ is a plane $\mathbb{P}^2 \times o$. Consider a general member $D$ of the net generated by $D_{y,1}, D_{y,2}, D_{y,3}$. As in the proof of Theorem 3.2, we can assume that $D$ intersects the curve $R_{x}^1$ properly and that $B := D \cdot R_{x}^2$ is a general conic in the pencil with base scheme $b := \text{Sing } V \cdot R_{x}^2$. Then either $B$ is smooth and disjoint from $R_{x}^2 \cdot R_{x}^1$ or $p = \text{Supp } b$ is a single point and $B$ is singular exactly at $p$. Now we consider the scheme $D \cdot R_{x}$. This is defined by 3 divisors of class $h_x + 2h_y$ and one of class $2h_x + h_y$. In 3.4 this intersection was proper and of length 30. Here it is not proper and $B$ is its excess intersection curve. Applying excess intersection formula to $B$, [14, 6.3], one computes $D \cdot R_{x} = B \cup Z$, where $Z$ has length 24. Arguing as in the proof of 3.4, each $o \in Z$ has multiplicity $\geq 2$ and hence the length of $\text{Sing } V$ is bound by $12 + \deg b = 16$. This proves the statement for $t = 1$. The argument easily extends to $t \leq 4$. ■

**Remark 3.10.** Consider a general $\overline{D}$ of the net generated by $D_{x,1}, D_{x,2}, D_{x,3}$. Then $S = D \cdot \overline{D}$ is a complete intersection of class $2h_x^2 + 5h_x h_y + 2h_y^2$. Let $S$ be integral with finitely many nodes, which is the general case. Then $S$ is a K3 surface through $R_{x}^1 \cup B$. Let $\sigma : S' \to S$ be its minimal desingularization and $B'$ and $H'$ the pull-back of $B$ and $H \in |O_S(1,2)|$. It is easily seen that $Z$ has length $(H' - B')^2 = 30 + B^2 - 2H'B' = 24$. This recovers the above excess intersection formula for $B$, cfr. [10, 13.3.6].

## 4 Highly Singular V-Threefolds and the Igusa Quartic

Before introducing the main family of Morin configurations to be considered, (and explicitly reconstruct in it the unique one of maximal cardinality), we already use the previous results. We describe the associated V-threefold with the maximal number of singular points and its relation to a well-known threefold in $\mathbb{P}^4$, namely the Igusa quartic.

**Definition 4.1.** Let $o \in \text{Sing } V$, we say that $o$ is a tangential singularity if

$$(\pi_x(o) \times \mathbb{P}^2) \cup (\mathbb{P}^2 \times \pi_y(o)) \subset V.$$
Moreover, we denote by $t(V)$ the number of these singularities on $V$.

Assume $V$ is defined by $A$ so that $F_A := \mathbb{P}(A) \cdot G$ is reduced. By the theorem $F_A$ has length $\leq 16 + t(V)$ with $t(V) \leq 4$. This gives a constructive way to produce families of finite Morin configurations of higher length $\ell$ in the range $16 \leq \ell \leq 20$. Indeed, let $t := t(V) \geq 1$ then $\text{Sing } V$ is necessarily endowed with a set of tangential singularities $O_t := \{o_1 \ldots o_t\} \subset \text{Sing } V$

such that the projection maps $\pi_x : O_t \to \mathbb{P}^2$ and $\pi_y : O_t \to \mathbb{P}^2$ are injective. In particular, $V$ contains $2t$ distinct planes, say $\{u_i\} \times \mathbb{P}^2 \setminus \mathbb{P}^2 \times \{v_j\}$ with $1 \leq i,j \leq t$. Then a set $O_t$ as above is $\{o_i := (u_i, v_i), \ i = 1 \ldots t\}$. By Lemma 3.8 the sets

$$\{u_1 \ldots u_t\} \ \{v_1 \ldots v_t\}$$

are sets of distinct points so that no three are collinear. To construct configurations of length $\ell \geq 16$ we consider the union of planes

$$U_t := \bigcup_{1 \leq i,j \leq t} ([u_i] \times \mathbb{P}^2) \cup ([v_j] \times \mathbb{P}^2)$$

and its ideal sheaf $I_t$ in $\mathbb{P}^2 \times \mathbb{P}^2$. This defines the linear system of $V$-threefolds

$$|I_t(2, 2)|.$$

The case $t = 4$ leads to Morin configurations of length $\ell \geq 16$, in particular to the maximal one with 20 planes. In what follows we assume $t = 4$. Since the points $u_1 \ldots u_4$ and $v_1 \ldots v_4$ are in general position, we can fix coordinates $(x, y)$ on $\mathbb{P}^2 \times \mathbb{P}^2$ so that $(u_i, v_i)$ is in the diagonal $\{x - y = 0\}$. We can also assume that

$$u_1 = (1 : 0 : 0), \ u_2 = (0 : 1 : 0), \ u_3 = (0 : 0 : 1), \ u_4 = (1 : 1 : 1). \quad (4.1)$$

Let $q_1(x)$ and $q_2(x)$ be quadratic forms in $x$ generating the ideal of $\{u_1 \ldots u_4\}$. Then $q_1(y)$ and $q_2(y)$ generate the ideal of $\{v_1 \ldots v_4\}$ and the next theorem easily follows.

**Theorem 4.2.** $|I_t(2, 2)|$ is the 3-dimensional linear system

$$\lambda q_1(x)q_1(y) + \mu q_2(x)q_2(y) + \nu q_1(x)q_2(y) + \rho q_2(x)q_1(y) = 0.$$
Let $V \in |I_4(2, 2)|$ be general then $\text{Sing } V$ is the set of 16 tangential singularities \[ \{o_{ij} = (u_i, v_j), \ 1 \leq i, j \leq 4\}. \]

Later in this paper we will see that the branch sextic $\Gamma : V \to \mathbb{P}^2$ is the union of three conics of the pencil $\lambda q_1(y) + \mu q_2(y) = 0$. The most interesting case of $V$ arises when $\Gamma$ is the union of the three singular conics of the pencil, that is, \[ \Gamma = \{y_1y_2y_3(y_1 - y_2)(y_1 - y_3)(y_2 - y_3) = 0\}. \]

Then it turns out that $V$ has 19 ordinary double points: the 16 tangential singularities and 3 other points, one over each double point of $\Gamma$. We will also show that a unique $V$ satisfies $|\text{Sing } V| = 19$ and that it is defined by a complete Morin configuration of 20 planes in $\mathbb{P}^6$, which is unique as well. For reasons to be made clear in the end of this section, we fix for such a $V$ the notation $V_{\text{ram}}$. Its equation is \[ (x_1 - x_2)x_3(y_1 - y_2)y_3 + x_1(x_2 - x_3)y_1(y_2 - y_3) + x_2(x_1 - x_3)y_2(y_1 - y_3) = 0. \] (4.2)

We continue this section by some constructions useful to put $V_{\text{ram}}$ in its due geometric perspective. Let $O_4 \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the set of four points as above and let $B_4$ be the linear system of $V$-threefolds singular at $O_4$. We consider the linear projection of center $O_4$ \[ \phi : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^4. \] (4.3)

of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. The map $\phi$ is defined by the linear system $|I(1, 1)|$, where $I$ is the ideal sheaf of $O_4$ in $\mathbb{P}^2 \times \mathbb{P}^2$. The base scheme of $|I(1, 1)|$ is precisely $O_4$. Then, since the Segre product $\mathbb{P}^2 \times \mathbb{P}^2$ has degree six, it follows $\deg \phi = 2$.

The ramification divisor of $\phi$ is strictly related to the subject of this paper and to a very well-known threefold and its dual. We recall that the Segre primal is the unique, up to projective equivalence, cubic threefold $\Delta$ whose singular locus consists of ten double points, which is the maximum for a cubic hypersurface with isolated singularities in a 4-space. Of equivalent interest is its dual hypersurface \[ \Delta^* \subset \mathbb{P}^4. \] (4.4)

This is in turn a quartic threefold which is very well known. It is the Igusa quartic, see, for example, [3] and [6]. In particular, in the recent paper [3], it is shown that:
Theorem 4.3. \( \Delta^* \) is the branch divisor of \( \phi \).

Now let us consider as in 4.2 the threefold \( V_{ram} \), which defines the unique complete Morin configuration of 20 planes in \( \mathbb{P}^5 \). Relying on its equation, and on the equations of \( \phi \), it is not difficult to compute the image of \( V_{ram} \) by \( \phi \) and conclude as follows.

Theorem 4.4. \( V_{ram} \) is the ramification of \( \phi \) and \( \phi(V_{ram}) \) is the Igusa quartic.

5 Del Pezzo 5-Tuples of Planes and the Segre Primal

In what follows \( G_{\mathbb{P}^4} \) is the Grassmannian of planes of \( \mathbb{P}^4 \) embedded by its Plücker map, then \( \deg G_{\mathbb{P}^4} = 5 \). Let us consider any transversal 0-dimensional linear section

\[
h := \{h_1 \ldots h_5\} \subset G_{\mathbb{P}^4}, \tag{5.1}\]

then \( h \) spans a 3-space. It is known that its points are in general position in \( \langle h \rangle \).

Definition 5.1. We say that \( h \) is a Del Pezzo 5-tuple of planes of \( \mathbb{P}^4 \).

All Del Pezzo 5-tuples are projectively equivalent. So it is not restrictive fixing a Del Pezzo 5-tuple of special geometric interest as follows. Let \( Y \subset \mathbb{P}^5 \) be a smooth quintic Del Pezzo surface and \( \mathcal{I}_Y \) its ideal sheaf. Then we shall see in Proposition 5.4 that \( |\mathcal{I}_Y(2)| \) is a 4-space endowed with a natural Del Pezzo 5-tuple cf. [4, 16]. We restart from \( Y \) assuming that \( \mathbb{P}^4 \) is

\[
\mathbb{H} := |\mathcal{I}_Y(2)| \tag{5.2}\]

and \( h \) is the Del Pezzo 5-tuple considered in Lemma 5.4. More precisely, \( \mathbb{H} \) is a 4-space of quadrics of \( \mathbb{P}^5 \) and the locus of its quadrics of rank \( \leq 4 \) is the union of five nets of quadrics. These planes of \( \mathbb{H} \) are the elements of \( h \). From now on we fix the notation

\[
G_{\mathbb{H}} \text{ and } G_{\mathbb{H}^*} \tag{5.3}\]

for the Grassmannians of planes and of lines of \( \mathbb{H} \) in their Plücker spaces, respectively. At first we want to describe the discriminant sextic hypersurface of \( \mathbb{H} \), that is, the
scheme of the singular quadrics $Q \in \mathbb{H}$. Omitting the most standard steps, let us summarize this description as follows. Consider the correspondence

$$\tilde{\Delta} := \{(z, Q) \in Y \times \mathbb{H} \mid z \in \text{Sing } Q\},$$

together with its natural projection maps

$$Y \leftarrow \tilde{\Delta} \xrightarrow{q_2} \mathbb{H}$$

and notice that $q_1 : \tilde{\Delta} \to Y$ is a $\mathbb{P}^1$-bundle. Indeed, any projection $\pi_z : Y \to \mathbb{P}^4$ from a point $z \in Y$ defines an integral complete intersection of two quadric hypersurfaces

$$Y_z := \pi_z(Y) \subset \mathbb{P}^4. \quad (5.4)$$

The pencil of quadrics through $Y_z$ pulls back to a pencil of quadrics

$$L_z \subset \mathbb{H} \quad (5.5)$$

singular at $z$. It turns out that $q_1$ is a $\mathbb{P}^1$-bundle such that $q_1^*(z) = \{z\} \times L_z$.

**Lemma 5.2.** Assume $Q \in \mathbb{H}$ is singular, then $\text{Sing } Q \cap Y \neq \emptyset$.

**Proof.** The projection from $\text{Sing } Q$ defines a rational map $f : Y \to \mathbb{P}^r$ so that $f(Y) \subset \overline{Q}$, where $\overline{Q}$ is a smooth quadric and $r \leq 4$. Let $\text{Sing } Q \cap Y = \emptyset$ then $f$ is a morphism. But then $f(Y)$ is a quintic surface in $\overline{Q}$, which is impossible. Hence, $\text{Sing } Q \cap Y \neq \emptyset$. ■

The irreducibility of $Y$ implies the irreducibility of the closed set

$$\Delta := q_2(\tilde{\Delta}). \quad (5.6)$$

Since a general $Q \in \mathbb{H}$ is smooth then, by the previous lemma, $\Delta$ is a hypersurface and the support of the sextic discriminant of $\mathbb{H}$. The name of $\Delta$ is well known, see [6, 8.5]. Before of coming to it we recall more on its geometry, which is determined by $Y$. The surface $Y$ has exactly five pencils of conics. Each of these defines a distinct Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$, let us say

$$\Sigma_i \subset \mathbb{P}^5, \ i = 1 \ldots 5. \quad (5.7)$$
\( \Sigma_i \) is union of the supporting planes of the conics of a pencil. As is well known,

\[
Y = \Sigma_1 \cap \cdots \cap \Sigma_5 \subset \mathbb{P}^5.
\] (5.8)

Let \( \mathcal{I}_i \) be the ideal sheaf of \( \Sigma_i \). Notice also that \( |\mathcal{I}_i(2)| \) is the net of quadrics

\[
P_i := \{ Q \in \mathbb{H} \mid \text{Sing } Q \text{ is a line } \mathbb{P}^1 \times \{ t \} \subset \mathbb{P}^1 \times \mathbb{P}^2 = \Sigma_i \}.
\] (5.9)

Therefore, \( P_1 \ldots P_5 \) are planes in \( \Delta \). Now consider in \( \mathbb{G}_\mathbb{H} \) the corresponding set

\[
h = \{ h_1 \ldots h_5 \} \subset \mathbb{G}_\mathbb{H}
\] (5.10)

of five points and, in the Plücker space of \( \mathbb{G}_\mathbb{H} \), the hyperplane \( H_i \) such that

\[
H_i \cap \mathbb{G}_{\mathbb{H}^*} = \{ L \in \mathbb{G}_{\mathbb{H}^*} \mid L \cap P_i \neq \emptyset \}.
\]

We observe that the previous \( \mathbb{P}^1 \)-bundle \( q_1 : \tilde{\Delta} \to Y \) defines a morphism

\[
\iota : Y \to \mathbb{G}_{\mathbb{H}^*}
\] (5.11)

sending \( z \) to the parameter point of the pencil \( L_z \). We point out the following:

**Lemma 5.3.** \( P_i \cap L_z \) is not empty, so that \( \iota(Y) \subset H_1 \cap \cdots \cap H_5 \).

**Proof.** Let \( \pi_z : \Sigma_i \to \mathbb{P}^4 \) be the projection from \( z \). Since \( z \in \Sigma_i \) and \( \Sigma_i \) is smooth of degree 3 then \( \overline{Q}_i := \pi_z(\Sigma_i) \) is a quadric in \( \mathbb{P}^4 \). Let \( Q_i \) be its pull-back by \( \pi_z \), then \( Q_i \) is singular at \( z \) and contains \( \Sigma_i \). Hence, \( Q_i \in P_i \cap L_z \) and \( \iota(z) \in H_1 \cap \cdots \cap H_5 \).

Let \( \langle h \rangle^\perp \) be the orthogonal of the linear span \( \langle h \rangle \) in the Plücker space of \( \mathbb{G}_{\mathbb{H}^*} \), then

\[
\langle h \rangle^\perp = H_1 \cap \cdots \cap H_5.
\] (5.12)

Then we can consider \( \iota \) as a morphism \( \iota : Y \to \langle h \rangle^\perp \) with image in \( \langle h \rangle^\perp \cdot \mathbb{G}_{\mathbb{H}^*} \). The next statement, essentially well known, will be also useful in the next sections.

**Proposition 5.4.** \( \iota : Y \to \langle h \rangle^\perp \) is the anticanonical embedding of \( Y \) and \( h \) is a Del Pezzo 5-tuple of \( \mathbb{H} \). Moreover, \( \iota(Y) \) is a linear section of the Grassmannian \( \mathbb{G}_{\mathbb{H}^*} \).
Proof. Consider the Euler sequence of $\mathbb{P}^5$ restricted to $Y$

$$0 \to \mathcal{O}_Y(-1) \to H^0(\mathcal{O}_{\mathbb{P}^5}(1))^* \otimes \mathcal{O}_Y \to T_{|\mathbb{P}^5|}(\mathbb{P}^5)_{|\mathbb{P}^5|}(1) \to 0.$$ 

Then its dual defines a monomorphism $\nu : S^*_Y \to H^0(\mathcal{O}_{\mathbb{P}^5}(2)) \otimes \mathcal{O}_Y$, where we have put $S := \text{Sym}^2 T_{|\mathbb{P}^5|}(1)$. Let $z \in Y$, then $S^*_{Y,z}$ is the vector space of quadratic forms singular at $z$ and $\nu_z : S^*_{Y,z} \to H^0(\mathcal{O}_{\mathbb{P}^5}(2))$ is the inclusion map. Now let $U$ be the pull-back by $\iota$ of the universal bundle of $\mathbb{G}_{\mathbb{H}^*}$, observe that $P(U) = \tilde{\Delta}$ and that

$$
P(U) \xrightarrow{q_2} |T_Y(2)| \times Y \\
\downarrow \quad \downarrow \\
\mathbb{P}(S^*_Y) \xrightarrow{\nu} |\mathcal{O}_{\mathbb{P}^5}(2)| \times Y$$

is a Cartesian square where the vertical maps are inclusions. The properties of the rank two vector bundle $U$ are well known: $\mathcal{O}_{\tilde{\Delta}}(1)$ is the anticanonical sheaf. Moreover, the map $\iota : Y \to \mathbb{G}_{\mathbb{H}^*}$, defined by $U$, is the anticanonical embedding of $Y$, followed by the inclusion $\iota(Y) \subset \mathbb{G}_{\mathbb{H}^*}$ as a linear section. This implies the statement. ■

Definition 5.5. The Del Pezzo surface defined by $h$ is $Y_h := \langle h \rangle^\perp \cdot \mathbb{G}_{\mathbb{H}^*}$.

Finally, we go back to the hypersurface $\Delta$.

Theorem 5.6. $\Delta$ is the Segre primal and $2\Delta$ is the sextic discriminant of $\mathbb{H}$.

Proof. In the Chow ring of the Grassmannian of lines of a 4-space a 2-dimensional linear section has class $(2, 3)$. Since $q_2 : \tilde{\Delta} \to \Delta$ is a birational morphism it follows $\deg \Delta = 3$. Since $Y_h$ is smooth, it is well known that $\Delta$ is the Segre cubic primal. ■

Remark 5.7. As remarked the planes $P_1 \ldots P_5$ are in $\Delta$. It is easy to see that a unique quadric $Q_{ij}$ satisfies $P_i \cap P_j = \{Q_{ij}\}$, $i < j$. This implies that $Q_{ij} \in \text{Sing} \Delta$ and describes the ten singular points of $\Delta$. Notice also that $\text{Sing} Q_{ij}$ is one of the ten lines in $Y$ and that the obvious map $\{Q_{ij}, \ i < j\} \to \{\text{lines of } Y\}$ is bijective.

Remark 5.8. The previous statement has somehow a classical flavor, however, we are not aware of any reference for it. We thank Igor Dolgachev for his useful comments.
6 Morin–Del Pezzo Configurations

Now we use $Y$ and the natural Del Pezzo 5-tuple of planes $P_1 \ldots P_5 \subset \mathbb{H}$ to describe an interesting family of special Morin configurations. We fix a linear embedding

$$\mathbb{H} \subset \mathbb{P}(W),$$

(6.1)

the choice of it is irrelevant up to $\text{Aut}\mathbb{P}(W)$. We fix the notation $W_Y := H^0(I_Y(2))$ so that it follows $\mathbb{P}(\wedge^3 W_Y) \subset \mathbb{P}(\wedge^3 W)$ and $G_{\mathbb{H}} \subset G$. We will also assume that

$$u \in \{h_1 \ldots h_5\} = h.$$

(6.2)

**Definition 6.1.** A subspace $A \subset \wedge^3 W$ is Del Pezzo marked if $\langle h \rangle \subset P(A)$.

The space $\wedge^3 W_Y$ is obviously isotropic. We recall that a Morin configuration $F \subset G$ is, by definition, a configuration of incident planes, which is finite and complete. As we know, this is equivalent to say that $F$ is finite and, moreover, that there exists a maximal isotropic space $A \in LG(10, \wedge^3 W)$ such that $F = \mathbb{P}(A) \cdot G$ and $\langle F \rangle = \mathbb{P}(A)$.

**Definition 6.2.** Let $F$ be a Morin configuration: we say that $F$ is a Morin–Del Pezzo configuration if $F$ contains $h$ and $\langle h \rangle \cdot G = h$.

Let us point out that $h \subset F$ implies $A \cap \wedge^3 W_Y = [h]$, that is,

$$\langle h \rangle = \mathbb{P}(A) \cap \mathbb{P}(\wedge^3 W_Y).$$

(6.3)

This follows because, counting dimensions, the intersection $L \cap G_{\mathbb{H}}$ is not finite for any space $L \subset \mathbb{P}(\wedge^3 W_Y)$ that contains $\langle h \rangle$ properly. Let

$$F' := \mathbb{P}(A) \cdot (\mathbb{P}(\wedge^3 W) - \mathbb{P}(\wedge^3 W_Y)),$$

(6.4)

then the condition $\langle h \rangle \cdot G = h$ is equivalent to say that

$$F = F' \cup h.$$

(6.5)

We fix the notation $F'$ for the subscheme of $F$ occurring in this decomposition.
Remark 6.3. In this part of the paper we describe Morin–Del Pezzo configurations and give a method for their explicit construction in any possible length. As we will see, these configurations are strictly related to the family of $V$-threefolds containing a plane and to the Severi variety of quadratic sections $C$ of $Y$ such that $\text{Sing } C$ has length $\geq 6$.

We stress, however, that our construction only gives Morin configurations of special type. The reason is that $h$ spans a 3-space. Since any Morin configuration spans a 9-space, otherwise it is not complete, it follows that the length of $F \cup h$ is at least 11, while a general configuration has length 10. Nevertheless, this construction recovers most families of Morin configurations for any length $k \in [11, 20]$. As we will see, the family of Morin–Del Pezzo configurations is irreducible and depend on 9 moduli.

To begin, let us fix since now a vector $f \notin W_Y$ and the decomposition

$$W = F \oplus W_Y,$$  \hspace{1cm} (6.6)

where $F$ is generated by $f$. Moreover, we fix the identification

$$\wedge^2 W_Y = \{f \wedge b, \ b \in \wedge^2 W\}$$

and the decomposition $\wedge^3 W = \wedge^3 W_Y \oplus \wedge^2 W_Y$. So far we then have

$$\wedge^3 (W_Y \oplus F) = \wedge^3 W = \wedge^3 W_Y \oplus \wedge^2 W_Y.$$  \hspace{1cm} (6.7)

In particular any two vectors $v, v' \in \wedge^3 W$ are uniquely decomposed as $v = a + f \wedge b$ and $v' = a' + f \wedge b'$, where $a, a' \in \wedge^3 W_Y$ and $b, b' \in \wedge^2 W_Y$. Therefore, we have

$$w(v, v') = v \wedge v' = -f \wedge (a \wedge b' + a' \wedge b).$$  \hspace{1cm} (6.8)

Notice that $w$ is induced by the natural pairing $\wedge^3 W_Y \times \wedge^2 W_Y \to \wedge^5 W_Y$, up to a nonzero factor the choice of $f$ is irrelevant. The proof of the next lemma is immediate.

Lemma 6.4. The subspaces $\wedge^3 W_Y$ and $\wedge^2 W_Y$ are isotropic spaces of $w$.

Let $r : \wedge^3 W \to \wedge^2 W_Y$ be the map sending $a + f \wedge b$ to $b$, then $r$ has a geometric meaning. Indeed, $r$ defines the projection of center $\mathbb{P}(\wedge^2 W_Y)$

$$\bar{r} : \mathbb{P}(\wedge^3 W) \to \mathbb{P}(\wedge^2 W_Y).$$  \hspace{1cm} (6.9)
Now let $G_{H^*} \subset \mathbb{P}(\wedge^2 W_Y)$ be the Grassmannian of lines of $H$, then we have:

**Lemma 6.5.** Let $o \in G$, the assignement $o \rightarrow P_o \cap H$ defines the rational map

$$\overline{\nu}|G : G \rightarrow G_{H^*} \subset \mathbb{P}(\wedge^2 W_Y).$$

**Proof.** Let $v \in \wedge^3 W - \wedge^3 W_Y$ be decomposable and defining $o$. Then we have $v = b \wedge f'$ with $b$ decomposable in $\wedge^2 W_Y$ and $f' \in W - W_Y$. We can write $f'$ as $f' = kf + c$ with $c \in W_Y$. Then $v = a + kf \wedge b$ with $a = b \wedge c$. Hence, $P_o \cap H$ is the line defined by the vector $-kb = r(v)$ and the statement follows. □

**Remark 6.6.** In particular, the fibre of $\overline{\nu}|G$ at $\overline{\nu}(u)$ is the $\mathbb{P}^3$ of planes of $I$ containing the line $P_o \cap H$ and the next commutative diagram solves the indeterminacy of $\overline{\nu}|G$:

$$
\begin{array}{ccc}
\tilde{G} & \xrightarrow{\gamma} & \tilde{r} \\
\downarrow & & \downarrow \\
G & \xrightarrow{\overline{\nu}|G} & G_{H^*} \\
\end{array}
$$

(6.10)

In it $\tilde{G}$ is the correspondence defined below and $\gamma, \tilde{r}$ are its projections. $\tilde{r}$ is a $\mathbb{P}^3$-bundle.

$$\tilde{G} := \{(L, P) \in G_{H^*} \times G \mid L \subset P\}. \quad (6.11)$$

Now we consider the family of all isotropic spaces $A$ in $\wedge^3 W$, which are marked by the Del Pezzo 5-tuple $h$, that is, such that $h = \{h_1 \ldots h_5\} \subset \mathbb{P}(A)$. Let $i = 1 \ldots 5$ and let $s_i \in \wedge^3 W$ be a vector defining the point $h_i$, then we have the orthogonal space

$$H^{ort} := \{s_1 \ldots s_5\}^\perp \subset \wedge^3 W. \quad (6.12)$$

Since $s_1 \ldots s_5$ generate a subspace of dimension 4 it follows $\dim H^{ort} = 16$. Let

$$H_Y := r(H^{ort}) \subset \wedge^2 W_Y, \quad (6.13)$$

since $\text{Kerr} = \wedge^3 W_Y$ we have the exact sequence of vector spaces

$$0 \rightarrow \wedge^3 W_Y \rightarrow H^{ort} \xrightarrow{r} H_Y \rightarrow 0. \quad (6.14)$$
Let $\wedge^3 W_Y \times \wedge^2 W_Y \to \wedge^5 W_Y$ be the natural pairing. It follows from the geometric description of $r$ in 6.9 that $H_Y$ is the orthogonal of $\{s_1 \ldots s_5\}$ under such a pairing. Let $h \subset A$, where $A \subset \wedge^3 W$ is an isotropic subspace, then we have $A \subset H^{ort}$ and

$$r(A) \subseteq H_Y.$$ 

Furthermore, under the previous pairing, we have the equality:

$$r(A) = (A \cap \wedge^3 W_Y)^\perp. \quad (6.15)$$

Then, since $H_Y$ is 6-dimensional, the next lemma follows.

**Lemma 6.7.** Let $A$ be maximal isotropic then $r(A) = H_Y$ iff $A \cap \wedge^3 W_Y = [h]$.

Let $H_1 \ldots H_5 \subset \mathbb{P}(\wedge^2 W_Y)$ be the hyperplanes defined by $h_1 \ldots h_5$, respectively. As in 5.5, $\mathbb{P}(H_Y)$ is the 5-space spanned by the smooth Del Pezzo quintic surface

$$Y_h = H_1 \cap \cdots \cap H_5 \cap G_{\mathbb{H}^+}. \quad (6.16)$$

Now assume that $F' := (\mathbb{P}(A) - \langle h \rangle) \cdot G$ is finite, then we have:

**Lemma 6.8.** $\overline{r}$ restricted to $F'$ is an embedding.

**Proof.** Let $\zeta \subset F'$ be a scheme of length 2 such that $\overline{r}|\zeta$ is not an embedding. Then the line $\langle \zeta \rangle$ intersects $\langle h \rangle$ and $\zeta$ is contained in a fibre of $\overline{r}$. This, by remark 6.6, is a 3-space linearly embedded in $G$. It is the family of planes containing a fixed line of $\mathbb{H}$. But then $\langle \zeta \rangle$ is a pencil of planes contained in $F'$ and $F'$ is not finite: a contradiction. \hfill $\blacksquare$

Now we concentrate on Morin–Del Pezzo configurations. We start more in general from a maximal isotropic $A$. Keeping our notation we assume

$$F = \mathbb{P}(A) \cdot G = h \cup F',$$

where $\langle h \rangle \cdot G = h$ and $F'$ is finite. Let $V_A$ be the $V$-threefold defined by $A$, we want to reconstruct it explicitly and see that it is rational. Notice that $u \in h$. We put $u = h_5$ and consider the projection map from which $V_A$ is constructed. We know that this is the
restriction to \( \mathbb{P}(A) \) of the tangential projection of \( \mathbb{P}(\wedge^3 W) \) from the embedded tangent space to \( G \) at \( u \). This is just the projection from \( u \), we denote since now as

\[
p : \mathbb{P}(A) \to \mathbb{P}^8,
\]

see 2.4. \( \mathbb{P}^8 \) is the space of the Segre embedding \( \mathbb{P}^2 \times \mathbb{P}^2 \) of \( P_u \times P_u^\perp \) and we know that

\[
V_A \subset \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8.
\]

Since \( h \) spans a 3-space containing \( u \) we can add to our play the plane

\[
P_h := p(\langle h \rangle).
\]

Moreover, we will also consider the set of four points

\[
h_u := \{ p_u(h_1) \ldots p_u(h_4) \} \subset P_h.
\]

These are in general position in \( P_h \), since the same is true in \( \langle h \rangle \) for \( h \).

**Theorem 6.9.** \( V_A \) contains the plane \( P_h \), in particular \( V_A \) is rational.

**Proof.** We know that \( V_A \) has bidegree \((2, 2)\) and isolated singularities. Now assume that \( P_h \subset \mathbb{P}^2 \times \mathbb{P}^2 \). Then, since \( V_A \) is singular at the four points of \( h_u \), it is clear that \( P_h \cdot V_A \) cannot be a conic. This implies that \( P_h \subset V_A \). Hence, it suffices to show that

\[
P_h \subset \mathbb{P}^2 \times \mathbb{P}^2.
\]

Assume \( P_h \) is not in \( \mathbb{P}^2 \times \mathbb{P}^2 \) and consider the scheme \( D := P_h \cdot (\mathbb{P}^2 \times \mathbb{P}^2) \). Then \( D \) contains the set \( h_u \) of four points in general position but \( D \neq P_h \). We claim that then \( D \) is a conic. Let us prove this fact: the variety \( \Sigma \) of bisecant lines to \( \mathbb{P}^2 \times \mathbb{P}^2 \) is a well-known cubic hypersurface and a Severi variety. In particular, \( \Sigma \) contains the six lines joining two by two the points of \( h_u \). Hence, \( P_h \) is in \( \Sigma \), though not in \( \mathbb{P}^2 \times \mathbb{P}^2 \). It is known that every such a plane cuts exactly a conic of bidegree \((1, 1)\) on \( \mathbb{P}^2 \times \mathbb{P}^2 \), cfr. [21, Chapter 5]. Then \( D \) is a conic and its projections in the factors are lines \( L_1 \) and \( L_2 \). We have \( D \subset L_1 \times L_2 \) and \( L_1 \times L_2 \) is embedded in \( \mathbb{P}^2 \times \mathbb{P}^2 \) as a quadric. Assume \( L_1 \times L_2 \) is not in \( V_A \) then \( V_A \cdot (L_1 \times L_2) \) is a quadratic section of \( L_1 \times L_2 \), singular at the set of coplanar points \( h_u \). This implies \( V_A \cdot (L_1 \times L_2) = 2D \). Notice also that \( D \) spans \( P_h \).
Now we can fix coordinates \((x_1 : x_2 : x_3) \times (y_1 : y_2 : y_3)\) on \(\mathbb{P}^2 \times \mathbb{P}^2\) so that
\[
L_1 \times L_2 = \{ x_3 = y_3 = 0 \} \quad \text{and} \quad D = \{ x_3 = y_3 = d = 0 \},
\] (6.20)

\(d\) being a form of bidegree \((1,1)\) in \((x_1 : x_2) \times (y_1 : y_2)\). Then \(2D\) is the complete intersection \(\{ x_3 = y_3 = d^2 = 0 \}\) and the equation of \(V_A\) is \(ax_3 + by_3 + kd^2 = 0\), where \(a\) and \(b\) are forms of bidegrees \((1,2)\) and \((2,1)\) and \(k \neq 0\). If \(L_1 \times L_2\) is in \(V_A\) we have \(k = 0\). One computes that \(\text{Sing} \ V_A \cap D\) is defined by the equations \(a = b = d = x_3 = y_3 = 0\). Moreover, \(a, b, d\) define in \(L_1 \times L_2\) curves \(C_a, C_b, D\) of bidegrees \((1,2), (2,1), (1,1)\) and \(\text{Sing} \ V_A\) is finite. Since \(C_aD = C_bD = 3\), it follows that \(\text{Sing} \ V_A \cap D\) contains at most three singular points of \(V_A\). Since \(h_u\) has cardinality 4 this is a contradiction. Hence, we can conclude that \(P_h \subset V_A\). Finally, the rationality of \(V_A\) follows from the explicit birational map \(V_A \to \mathbb{P}^1 \times \mathbb{P}^2\) we construct in the next section. ■

**Remark 6.10.** Let \(F = \mathbb{P}(A) \cdot G\) be any Morin configuration, smooth at \(u\) as we assume in this paper, and \(V_A\) its associated \(V\)-threefold. If \(F\) has length \(\geq 16\) then Theorem 3.2 implies that \(V_A\) contains a plane. Up to \(\text{Aut} \mathbb{P}^2 \times \mathbb{P}^2\) we can assume that this is \(P_h\). Thus, Morin configurations of length \(\geq 16\) are basically Morin–Del Pezzo configurations.

### 7 The \(V\)-Threefold of a Morin–Del Pezzo Configuration

In this section we construct \(V\)-threefolds associated to Morin–Del Pezzo configurations. In particular, we prove that \(\pi : V \to \mathbb{P}^2\) admits a plane transversal to \(\pi\).

Let \(F = \mathbb{P}(A) \cdot G\) be a Morin–Del Pezzo configuration and let
\[
P_h := \{ o \} \times \mathbb{P}^2
\] (7.1)

be the plane contained in the threefold \(V_A\). Now we consider the projection
\[
P_h : \mathbb{P}^8 \to \mathbb{P}^5
\] (7.2)

of \(\mathbb{P}^8\) from \(P_h\) and study \(p_h|V_A\). Let us point out that \(p_h\) factors as in the diagram
\[
\begin{array}{c}
\mathbb{P}(A) \\
\downarrow p \quad \downarrow \tilde{\pi}|_{\mathbb{P}(A)} \\
\mathbb{P}^8 \\
\downarrow p_h \\
\mathbb{P}^5
\end{array}
\] (7.3)
where \( \overline{r} \) is as in Section 6. Indeed, \( \overline{r}|_{\mathbb{P}(A)} \) is the projection from \( \langle h \rangle \), while \( p \) and \( p_h \) are the projections from \( u \) and \( \langle h_u \rangle \). Then, since \( h_u = p(h) \), it follows \( \overline{r}|_{\mathbb{P}(A)} = p_h \circ p \).

Remark 7.1. Notice that \( \mathbb{P}^5 = \mathbb{P}(H_Y) \subset \mathbb{P}(\wedge^2 W_Y) \) and that \( \mathbb{P}(H_Y) \cdot G_H \) is the quintic Del Pezzo surface defined by \( < h > \perp \). This, by the definition of \( \overline{r} \), is the locus

\[
Y_h = \{ \ell_o : o \in Y \}, \tag{7.4}
\]

where \( \ell_o \) is the pencil of quadrics of \( \mathbb{H} \) singular at \( o \). See 6.14 and also Lemma 6.8.

Let \( \sigma : \mathbb{P} \to \mathbb{P}^2 \times \mathbb{P}^2 \) be the blowing of \( P_h \) then we have the commutative diagram

\[
\begin{array}{ccc}
\mathbb{P} & \xrightarrow{\sigma} & \mathbb{P}^2 \times \mathbb{P}^2 \\
\downarrow \tilde{p}_h & & \downarrow p_{h|\mathbb{P}^2 \times \mathbb{P}^2} \\
\mathbb{P}^1 \times \mathbb{P}^2 & \longrightarrow & \mathbb{P}^5,
\end{array} \tag{7.5}
\]

where \( \tilde{p}_h \) is a \( \mathbb{P}^1 \)-bundle and the bottom arrow is the Segre embedding. Let us consider the projection \( p_o : \mathbb{P}^2 \to \mathbb{P}^1 \) from the point \( o \), then we have

\[
\tilde{p}_h \circ \sigma^{-1} = p_o \times id_{\mathbb{P}^2} : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^2. \tag{7.6}
\]

Moreover, let \( E \subset \mathbb{P} \) be the exceptional divisor of \( \sigma \). Since \( P_h \) has trivial normal bundle the morphism \( \tilde{p}_h : E \to \mathbb{P}^1 \times \mathbb{P}^2 \) is biregular and its inverse defines a regular section

\[
s : \mathbb{P}^1 \times \mathbb{P}^2 \longrightarrow E \subset \mathbb{P}. \tag{7.7}
\]

We want to study the diagram more in detail with respect to \( V_A \). Denoting by \( \tilde{V}_A \) the strict transform of \( V_A \) via \( \sigma \), and by \( p_A \) the restriction of \( p_h \) to \( V_A \), we have

\[
\begin{array}{ccc}
\tilde{V}_A & \xrightarrow{\sigma} & V_A \\
\downarrow \tilde{p}_A & & \downarrow p_h \\
\mathbb{P}^1 \times \mathbb{P}^2 & \longrightarrow & \mathbb{P}^5,
\end{array} \tag{7.8}
\]

with \( \tilde{p}_A = \tilde{p}_h|\tilde{V}_A \). It is clear that \( \tilde{V}_A \) is rational, because it is an integral member of

\[
|O_\mathbb{P}(2\tilde{H} - E)|, \tag{7.9}
\]
where $O_p(\tilde{H}) \cong \sigma^* O_{\mathbb{P}^2 \times \mathbb{P}^2}(1,1)$. Since $2\tilde{H} - E$ has degree one on the fibers of $p_h$ then

$$\tilde{p}_A : \tilde{V}_A \longrightarrow \mathbb{P}^1 \times \mathbb{P}^2$$

is birational. Let $E_h = E \cdot \tilde{V}_A$ be the strict transform of $P_h$ by $\sigma_{|\tilde{V}_A}$, then

$$\sigma_{|V_A} : \tilde{V}_A - E_h \longrightarrow V_A - P_h \quad (7.10)$$

is a biregular map.

**Lemma 7.2.** $\sigma_{|E_h} : E_h \rightarrow P_h$ is the blowing up of $h_u$ and $E_h$ is a smooth quintic Del Pezzo surface.

**Proof.** We have $E = \mathbb{P}^1 \times P_h$ so that $\sigma_{|E} : E \rightarrow P_h$ is the natural projection. Let us compute the bidegree $(m, n)$ of $E_h$ in $\mathbb{P}^1 \times E_h$. Since $\tilde{V}_A$ has degree one on the fibers of $\tilde{p}_A$, it follows $m = 1$. Now notice that $\text{Sing} V_A : P_h = h_u$, because $(h) \cdot G = h$. Then, writing a local equation for a $V$-threefold containing a plane like $P_h$, it is easy to deduce that the pencil of conics through $h_u$ lifts, by $\sigma_{|E_h}$, to a pencil of conics. This is cut by the ruling of planes of $E$. Hence, $n = 2$ and $E_h$ is the blowing up of $P_h$. Since $h_u$ is a set of four points in general position, then $E_h$ is a smooth quintic Del Pezzo surface. $\blacksquare$

Notice also that $O_{E_h}(1,1) \cong \omega_{E_h}^{-1}$, therefore the Segre embedding of $E$ restricts to the anticanonical map of $E_h$. Moreover, the next theorem follows.

**Theorem 7.3.** $\sigma : \tilde{V}_A \rightarrow V_A$ is the small contraction of four disjoint copies of $\mathbb{P}^1$.

Let us fix the notation $Y_h := \tilde{p}_A(E_h)$. This is a smooth quintic Del Pezzo surface

$$Y_h \subset \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5. \quad (7.11)$$

Now we describe the birational morphism $\tilde{p}_A : \tilde{V}_A \rightarrow \mathbb{P}^1 \times \mathbb{P}^2$ in order to invert it. To this purpose it is useful to consider the conic bundle $\pi : V_A \rightarrow \mathbb{P}^2$ defined by the projection of $\mathbb{P}^2 \times \mathbb{P}^2$ onto the 2nd factor. We have the commutative diagram

$$\begin{array}{ccc}
E_h & \longrightarrow & \tilde{V}_A & \longrightarrow & V_A \\
\downarrow {\tilde{p}_A}_{|E_h} & & \downarrow \sigma & & \downarrow \pi \\
Y_h & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2
\end{array} \quad (7.12)$$
where $\tilde{\pi}$ is the projection map. Indeed, let $t \in \mathbb{P}^2$ then $\pi^*(t)$ is $V_A \cdot (\mathbb{P}^2 \times \{t\})$ and $P_h \cdot \pi^*(t) = (o, t)$. Moreover, $\tilde{p}_A \circ \sigma^{-1} | \pi^*(t)$ is precisely the projection from $(o, t)$

$$p_{o,t} : \pi^*(t) \to \mathbb{P}^1 \times \{t\}.$$ 

Notice that $(o, t) \in \pi^*(t) \subset P_h = \{o\} \times \mathbb{P}^2$. It is clear that the tangent space to $V_A$ at $(o, t)$ has dimension 4 if $(o, t) \in \text{Sing} \pi^*(t)$. This implies the next lemma.

**Lemma 7.4.** Assume $(o, t) \in P_h - h_u$ then $\pi^*(t)$ is smooth at $(o, t)$.

Let $\Gamma \subset \mathbb{P}^2$ be the discriminant sextic of $\pi$ and $t \in \mathbb{P}^2 - \Gamma$, then $\pi^*(t)$ is a smooth conic. Let $\pi^*(t)'$ be its strict transform by $\sigma$, then $\pi^*(t)' = (\tilde{p}_A \circ \tilde{\pi})^*(t)$ and

$$\tilde{p}_A | \pi^*(t)' : \pi^*(t)' \to \mathbb{P}^1 \times \{t\}$$

is biregular and induced by $p_{o,t}$. Moreover, $\tilde{p}_A$ is regular on $\pi^*(t)'$. In $E$ we define

$$C_h = E \cdot (\pi \circ \sigma)^* \Gamma.$$  \hspace{1cm} (7.13)

$C_h$ is a curve embedded in the Del Pezzo surface $E_h = E \cdot \tilde{V}_A$. Let

$$s_o : \mathbb{P}^2 \to V_A$$ \hspace{1cm} (7.14)

be the linear isomorphism such that $s_o(t) = (o, t)$ and let $\Gamma_h := s_o(\Gamma)$. Then the next lemma is standard, we omit further details.

**Lemma 7.5.** $C_h$ is the strict transform of $\Gamma_h$ by the blowing up $\sigma_{| E_h} : E_h \to P_h$. In particular $C_h$ is a quadratic section of the anticanonical embedding of $E_h$.

Finally, let us define and consider the following varieties:

**Definition 7.6.** $C := p_{h*} C_h$ and $F := p_h^* C$.

$F$ is a $\mathbb{P}^1$-bundle over $C$ and $C$ is the biregular to $C_h$ via $p_h$. We have

$$C \subset Y_h \subset \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5.$$ 

We recall that $C$ is complete intersection in $\mathbb{P}^1 \times \mathbb{P}^2$ of $Y_h$ and a quadratic section.
Theorem 7.7. \( \tilde{p}_A : \tilde{V}_A \to \mathbb{P}^1 \times \mathbb{P}^2 \) is the contraction of \( F \).

Proof. Let \( \zeta \subset \tilde{V}_A \) be a scheme of length 2. Assume that the morphism \( \tilde{p}_A \) is not an embedding on \( \zeta \). Then \( \zeta \subset f \) for a fibre \( f \) of \( p_h : \mathbb{P} \to \mathbb{P}^1 \times \mathbb{P}^2 \). Since \( \tilde{V}_A \) has intersection index 1 with \( f \), it follows \( f \subset \tilde{V}_A \). Notice also that, as every fibre of \( p_h \), \( \sigma_* f \) is a line in a plane \( \mathbb{P}^2 \times \{ t \} \). Hence, the fibre \( \pi^*(t) \) cannot be a smooth conic, since it contains the line \( f \). Then we have \( \sigma_* f \subset \pi^* \Gamma \) and \( f \subset F \). This implies the statement. \( \blacksquare \)

Remark 7.8. In a more descriptive way let \( t \) be a point of \( \Gamma \) such that \( t \notin h_u \). Then \( \pi^*(t) \) is a rank 2 conic and it is not singular at \( (o, t) \), as remarked. Let \( f + \bar{f} \subset \tilde{V}_A \) be its strict transform by \( \sigma \). Then a summand, say \( f \), is a fibre of \( p_h \) and intersects \( F_h \). For the other summand the map \( \tilde{p}_A : f \to \mathbb{P}^1 \times \{ t \} \) is a linear isomorphism.

Remark 7.9. Let \( g : \tilde{\Gamma} \to \Gamma \) be the degree 2 cover defined by \( \pi : V_A \to \mathbb{P}^2 \). Then \( \tilde{\Gamma} \) parametrizes the lines in \( \pi^*(t) \), \( t \in \Gamma \). At a general \( t \) we have \( g^*(t) = \{ f, \bar{f} \} \). Since \( f \) and \( \bar{f} \) are distinguished by the intersection with \( P_h \), then \( \tilde{\Gamma} \) is split over \( \Gamma \). If \( \Gamma \) is nodal one can see that \( g \) is a Wirtinger cover of \( \Gamma \), in the sense of [1, Section 5].

We can now reconstruct \( V_A \), describing explicitly the inverse map

\[
\sigma \circ \tilde{p}^{-1}_A : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^8 \tag{7.15}
\]

and its image \( V_A \subset \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8 \). Let \( \mathcal{J}_{P_h} \) be the ideal sheaf of \( P_h \) in \( \mathbb{P}^2 \times \mathbb{P}^2 \), then the rational map \( p_h \) is defined by \( |\mathcal{J}_{P_h}(1, 1)| \). Since we have

\[
F_h = (\sigma|\tilde{V}_A)^* P_h \quad \text{and} \quad Y_h = \tilde{p}_A^* F_h, \tag{7.16}
\]

it follows

\[
\tilde{p}_A^*(\sigma|\tilde{V}_A)^*|\mathcal{J}_{P_h}(1, 1)| = Y_h + |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1, 1)| \subset |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2, 3)|, \tag{7.17}
\]

where \( Y_h + |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1, 1)| \) denotes the linear system of divisors of bidegree \( (2, 3) \) having \( Y_h \) as a fixed component. This is contained in the linear system \( \mathcal{J} \), defining the rational map \( \sigma \circ \tilde{p}^{-1}_A : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^8 \). Let \( \mathcal{J}_C \) be the ideal sheaf of \( C \). Then we have

\[
Y_h + |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1, 1)| \subset \mathcal{J} \subseteq |\mathcal{J}_C(2, 3)|, \tag{7.18}
\]
just because $C$ is in the indeterminacy of $\sigma \circ \tilde{p}_A^{-1}$. Now the target space of this rational map is $\mathbb{P}^8$, since $V_A$ is not contained in a hyperplane. This implies $\dim \mathcal{J} = 8$ and makes our reconstruction much simpler.

**Theorem 7.10.** $\sigma \circ \tilde{p}_A^{-1} : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^8$ is defined by the linear system $|\mathcal{I}_C(2,3)|$.

**Proof.** It suffices to show that $\dim |\mathcal{J}_C(2,3)| = 8$. This follows, with the usual notation, from the standard exact sequence of ideal sheaves

$$0 \to \mathcal{J}_{Yh}(2,3) \to \mathcal{J}_C(2,3) \to \mathcal{J}_{C|Yh}(2,3) \to 0.$$  

It is easy to see that this is actually the sequence

$$0 \to \mathcal{O}_{Yh}(1,1) \to \mathcal{J}_C(2,3) \to \mathcal{O}_{Yh}(0,1) \to 0.$$  

Passing to the associated long exact sequence it follows $h^0(\mathcal{J}_C(2,3)) = 9$. 

Finally, let us remark that $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1,1)) = h^0(\mathcal{J}_C(2,2)) = 3$ and let

$$\mu : H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(0,1)) \otimes H^0(\mathcal{J}_C(2,2)) \to H^0(\mathcal{I}_C(2,3)) \quad (7.19)$$

be the multiplication map. Consider the rational maps

$$\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2 \times \mathbb{P}^2 \quad (7.20)$$

respectively defined by the net of surfaces $|\mathcal{J}_C(2,2)|$ and $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(0,1)|$. We claim that $\mu$ is an isomorphism. Then $\sigma \circ \tilde{p}_A^{-1} : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^8$ clearly factors through the product map $\pi_1 \times \pi_2$ and the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$. This makes clear how to effectively reconstruct $V_A$ from $\pi_1 \times \pi_2$. Let us prove our claim.

**Theorem 7.11.** $\mu$ is an isomorphism.

**Proof.** Consider the standard exact sequence of ideal sheaves of $\mathbb{P}^1 \times \mathbb{P}^2$

$$0 \to \mathcal{J}_{Yh}(2,2) \to \mathcal{J}_C(2,2) \to \mathcal{J}_{C|Yh}(2,2) \to 0.$$
Since $Y_h$ has bidegree $(1, 2)$ this is just
\[ 0 \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1, 0) \to \mathcal{J}(2, 2) \to \mathcal{O}_{Y_h} \to 0. \]

Tensor it by $L \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}$ with $L := H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(0, 1))$. Passing to the corresponding long exact sequences, one obtains the exact commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & L \otimes H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1, 0)) & \longrightarrow & L \otimes H^0(\mathcal{I}(2, 2)) & \longrightarrow & L \otimes H^0(\mathcal{O}_{Y_h}) & \longrightarrow & 0 \\
& & \downarrow \mu_1 & & \downarrow \mu & & \downarrow \mu_2 & & \\
0 & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1, 1)) & \longrightarrow & H^0(\mathcal{I}(2, 2)) & \longrightarrow & L & \longrightarrow & 0
\end{array}
\]

where $\mu_1, \mu_2$ are multiplication maps and isomorphisms. Then $\mu$ is an isomorphism. ■

Finally, we conclude this section by the following remark.

**Remark 7.12.** As above let $\Gamma \subset \mathbb{P}^2$ be the discriminant sextic of $\pi : V_A \to \mathbb{P}^2$. The set $\text{Sing}\Gamma$ contains the set of four points $\pi(h_u)$. Let $\mathcal{I}$ be its ideal sheaf in $\mathbb{P}^2$, then the product map $H^0(\mathcal{I}(2)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathcal{I}_{\mathbb{P}^2}(3))$ is an isomorphism. Moreover, the pencil of conics $|\mathcal{I}(2)|$ defines a rational map $q : \mathbb{P}^2 \to \mathbb{P}^1$ and hence the birational embedding

\[ q \times \text{id}_{\mathbb{P}^2} : \mathbb{P}^2 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5, \]

whose image in $\mathbb{P}^1 \times \mathbb{P}^2$ is $Y_h$. We know that $C$ is the strict transform of $\Gamma$ by $q \times \text{id}_{\mathbb{P}^2}$. Composing $q \times \text{id}_{\mathbb{P}^2}$ with the product map $\pi_1 \times \pi_2$ we obtain the plane $P_h$. Moreover, the image of $\pi_1 \times \pi_2$ in $\mathbb{P}^2 \times \mathbb{P}^2$ is the $V$-threefold $V_A$ and we retrieve $\Gamma$ as the discriminant curve of its projection $\pi : V_A \to \mathbb{P}^2$. Clearly this construction always works: under the only assumption that the sextic $\Gamma$ contains four singular points in general position. This shows the next property.

**Theorem 7.13.** Let $\Gamma \subset \mathbb{P}^2$ be any sextic with four singular points in general position. Then $\Gamma$ is the discriminant curve of a conic bundle $\pi : V_\Gamma \to \mathbb{P}^2$ such that:

1. $V_\Gamma$ is a bidegree $(2, 2)$ threefold in $\mathbb{P}^2 \times \mathbb{P}^2$,
2. $\pi : V_\Gamma \to \mathbb{P}^2$ is one of the two projections,
3. $V_\Gamma$ contains a plane $P$ transversal to $\pi$. 

As in remark 7.9, \( \pi \) defines a double cover \( g : \tilde{\Gamma} \to \Gamma \) which is split over \( \Gamma \). If \( \Gamma \) is nodal \( \pi \) is a Wirtinger cover. Hence, \( \tilde{\Gamma} \) is the gluing, according to the prescriptions, of two copies of the partial normalization of \( \Gamma \) at the above mentioned four nodes.

8 Geometry of Morin–Del Pezzo Configurations

Now we describe the truly geometric construction of a Morin–Del Pezzo configuration like \( F \). We construct the configuration considering the discriminant locus of quadrics containing a singular curve \( C \subset \mathbb{P}^5 \). We infer that such configurations form an irreducible family.

It turns out that \( F \) is determined by the curve \( C \subset \mathbb{P}^5 \) considered in (7.6) and \( \text{Sing } C \) as follows. Let \( \nu : C^\mathbb{Q} \to C \) be the normalization map, then \( \text{Sing } C \) is defined by the exact sequence

\[
0 \to \mathcal{O}_C \xrightarrow{\nu^*} \mathcal{O}_{C^\mathbb{Q}} \to \mathcal{O}_{\text{Sing } C} \to 0
\]

as usual. Restricting to \( F' \) the commutative diagram of linear maps 7.3, we obtain

\[
\begin{array}{ccc}
\mathbb{P}^8 & \xrightarrow{p_h} & \mathbb{P}^5 \\
\downarrow{\tilde{\varphi}} & & \downarrow{\bar{\varphi}} \\
F' & \xrightarrow{p_{F'}} & \bar{F}'
\end{array}
\]

Here \( \tilde{\varphi}_{F'} \) is an embedding by Lemma 6.8 and \( p_{F'} \) embeds \( F' \) in \( U := V_A - P_h \). Hence, \( F' \) is biregular to \( p(F') \) and \( p_h \) embeds \( p(F') \) in \( \mathbb{P}^5 \). On the other hand let \( R \subset V_A \) be the ramification scheme of \( \pi : V_A \to \mathbb{P}^2 \). Then \( \sigma^* R \) is contained in the fundamental divisor \( Z \) of \( \tilde{\varphi}_A : \tilde{V}_A \to \mathbb{P}^1 \times \mathbb{P}^2 \). More precisely, we have \( \tilde{\varphi}_A(Z) = C \) so that \( \tilde{\varphi}_A : Z \to C \) is a \( \mathbb{P}^1 \)-bundle. Then \( \sigma^* R \) is a birational section of it cutting on \( F \cdot U \) the locus of the singular points of the singular fibers of \( \pi \). Then Theorem 2.13 implies that

\[
p(F') = U \cdot \text{Sing } R.
\]

Since \( \sigma^{-1} : U \to \tilde{V}_A \) is an open embedding and \( p_h|U = \tilde{\varphi}_A \circ \sigma^{-1}|U \), it follows:

Lemma 8.1. The rational map \( \bar{\varphi} \) embeds \( F' \) in \( \text{Sing } C \).

Let \( F_h := \bar{\varphi}(F') \subset \mathbb{P}^5 \), the next lemma will be useful.

Lemma 8.2. \( h^0(I_{F_h}(1)) = 0 \) that is \( F_h \) spans \( \mathbb{P}^5 \).
Proof. We have $\langle F \rangle = \mathbb{P}(A)$ since $F = F' \cup h$ is complete. Moreover, $\overline{r} |_{F'} : F' \to \mathbb{P}^5$ is an embedding. Assume $h^0(I_{Fh}(1)) \geq 1$, then $F_h$ is contained in a hyperplane $L$. But then the pull-back of $L$ by $\overline{r} |_{\mathbb{P}(A)}$ contains $\langle F \rangle$: a contradiction. ■

Now we assume $W = H^0(I_C(2))$ for our usual vector space $W$ and that the inclusion of $\mathbb{H} = |I_Y(2)|$ in $\mathbb{P}(W)$ is induced by the standard exact sequence of global sections

$$0 \to H^0(I_Y(2)) \to H^0(I_C(2)) \to H^0(O_{Y_h}) \to 0.$$

As already remarked this is not restrictive up to projective equivalence. As in the proof of Lemma 5.4 let $S = \text{Sym}^2 T_{\mathbb{P}^5}(-1)$. Then $S^*_o \subset H^0(O_{\mathbb{P}^5}(2))$ is the space of quadratic forms singular at $o \in \mathbb{P}^5$ and this inclusion defines a monomorphism

$$\nu : S^* \to H^0(O_{\mathbb{P}^5}(2)) \otimes O_{\mathbb{P}^5}.$$ 

Restricting $\nu$ to $\text{Sing} C$ we then construct the Cartesian square

$$\begin{array}{ccc}
N & \to & W \otimes O_{\text{Sing} C} \\
\downarrow & & \downarrow \\
S^*_o \to H^0(O_{\mathbb{P}^5}(2)) \otimes O_{\text{Sing} C}.
\end{array} \quad (8.3)$$

$N$ is a rank 3 vector bundle over the finite scheme $\text{Sing} C$. Indeed, we have

$$N_o = H^0(I_C(2)) \cap H^0(I^2_o(2)) \quad (8.4)$$

and we know that $L_o := H^0(I_Y(2)) \cap H^0(I^2_o(2))$ has dimension 2. Since $C$ is a quadratic section of $Y_h$, singular at $o$, the above exact sequence implies $\dim N_o = 3$. Let $N_o := \mathbb{P}(N_o)$, then $N_o$ is the net of quadrics through $C$ singular at $o$. In particular, it is clear that the map associated to $N$ is the embedding sending $o$ to $N_o$, say

$$f_N : \text{Sing} C \to \mathbb{G}. \quad (8.5)$$

Now $\overline{r} : \mathbb{G} \to \mathbb{G}^*_o$ associates to $N_o$ the pencil $\mathbb{P}(L_o)$. Moreover, $Y_h$ is the locus in $\mathbb{G}^*_o$ of the pencils of quadrics $\mathbb{P}(L_o)$, singular at $o \in Y_h$ and containing $Y_h$, see 6.14 and Remark 7.1. Hence, it follows $\overline{r} \circ f_N = \text{id}_{\text{Sing} C}$ and therefore we have

$$f_N(\text{Sing} C) \subseteq F'. \quad (8.6)$$
Since \( F = h \cup F' \) is by assumption the Morin configuration defined by \( A \), we have \( \langle F \rangle = \mathbb{P}(A) \) and \( F = \mathbb{P}(A) \cdot G \). This implies that \( F' = f_N(\text{Sing } C) \).

After these remarks we can describe explicitly Morin–Del Pezzo configurations and construct an irreducible family which includes all these configurations. To this purpose we invert now the previous construction and start from a reduced and construct an irreducible family which includes all these configurations. To this purpose we invert now the previous construction and start from a reduced and construct an irreducible family which includes all these configurations. To this purpose we invert now the previous construction and start from a reduced and construct an irreducible family which includes all these configurations. To this purpose we invert now the previous construction and start from a reduced and construct an irreducible family which includes all these configurations. To this purpose we invert now the previous construction and start from a reduced and construct an irreducible family which includes all these configurations. To this purpose we invert now the previous construction and start from a reduced and construct an irreducible family which includes all these configurations.

\[
F' := f_N(\text{Sing } C) \quad \text{and} \quad \mathbb{P}(A) := \langle h \cup F' \rangle.
\]  

**Theorem 8.3.** \( A \) is maximal isotropic.

**Proof.** Let \( \ell \) be the length of \( F' \). We show by induction on \( 0 \leq k \leq \ell \) that, for any subscheme \( S' \subset F' \) of length \( k \), \( \langle h \cup S' \rangle = \mathbb{P}(A_{S'}) \) where \( A_{S'} \) is isotropic. For \( k = 0 \) \( h \cup S' = h \) and the statement is clearly true. Let \( S' \) be of length \( k + 1 \) then we observe that \( S' \) is the biregular image of \( f_N(S) \), where \( S \) is a subscheme of length \( k \) of \( \text{Sing } C \). Let \( o \in S \), then \( f_N(o) \) is the parameter point of the net of quadrics \( N_o \), containing \( C \) and singular at \( o \). It is also clear that \( \langle S' \rangle \) is spanned by the lines \( \langle \zeta'_1 \rangle, \ldots, \langle \zeta'_k \rangle \), where \( \zeta'_i = f_N(\zeta_i) \) and \( \zeta_i \) denotes a subscheme of length 2 of \( S \) containing the point \( o \). If \( o_i \in \zeta_i \) and \( o_i \neq o \) we denote by \( n_i \) a vector defining the point \( f_N(o_i) \). If \( o = o_i \) then \( n_i \) denotes a nonzero tangent vector to \( G \) at \( f_N(o) \) defining the tangent line \( \langle \zeta'_i \rangle \). Finally, let \( n \) be a vector defining \( f_N(o) \) and let \( s_1, s_5, n, n_1, \ldots, n_k \) be vectors defining \( h_1 \ldots h_5 \), respectively. Then \( A_{S'} \) is generated by \( s_1, s_5, n, n_1, \ldots, n_k \). By induction \( s_1, s_5, n, n_1, \ldots, n_k \) generate an isotropic space. Moreover \( n \) is isotropic. Hence, \( A_{S'} \) is isotropic if

\[
w(n, s_i) = w(n, n_j) = 0
\]

for \( i = 1 \ldots k \) and \( j = 1 \ldots 5 \). Since the tangent space to \( G \) at any point is isotropic, we have \( w(n, n_j) = 0 \) for every \( n_j \) such that \( \langle \zeta'_j \rangle \) is tangent to \( G \) at \( o \). Otherwise, we have \( o \neq o_j \) and we are left to show that \( w(n, n_j) = 0 \). To prove this we argue as follows, leaving some details to the reader. Let \( N_o \) and \( N_j \) be the net of singular quadrics defined by \( o \) and \( o_j \), respectively, as above. To prove \( w(n, n_j) = 0 \) it suffices to show that \( N_o \cap N_j \) is nonempty. Let \( \beta : Y_h \to \mathbb{P}^3 \) be the projection from the line \( b := \langle oo_j \rangle \). If \( b \) is not in \( Y_h \) then \( \beta(Y_h) \) is an integral cubic surface. Moreover, \( \beta(C) \) is a 4-nodal canonical curve. In particular, it follows that \( \beta(C) = \overline{Q} \cdot \beta(Y) \), where \( \overline{Q} \) is a quadric surface. Let \( Q = \beta^* \overline{Q} \), then \( Q \) is a quadric of rank 4, singular along the line \( \langle oo_j \rangle \) and contains \( C \). Hence, we have \( Q \in N_o \cap N_i \). Finally, it is clear from Section 5 that \( w(n, s_i) = 0 \). This shows by
induction that \( \langle h \cup F' \rangle \) is the projectivization of an isotropic space. Since it is isotropic, \( A \) has dimension \( \leq 10 \). On the other hand the assumption that \( \text{Sing} \ C \) spans \( \mathbb{P}^5 \) implies that \( r(A) \) is 6-dimensional. Since \( [h] \subseteq \text{Ker} \ r \), it follows \( \dim A = 10 \). ■

In what follows we will denote by \( C \) the family of curves like \( C \), that is,

\[
C := \{ C \in |\mathcal{O}_{Y_h}(2)| \mid h^0(\mathcal{I}_{\text{Sing} \ C(1)}) = 0 \text{ and } C \text{ is reduced} \}.
\] (8.8)

Notice that then \( \text{Sing} \ C \) has length \( \geq 6 \). Now let \( \mathcal{V} \) be the family of all reduced curves \( D \in |\mathcal{O}_{Y_h}(2)| \) such that \( \text{Sing} \ D \) has length \( \geq 6 \), it is known that \( \mathcal{V} \) is integral [22]. Moreover, it is easy to see that a general \( D \) in the family is an integral nodal curve such that \( \text{Sing} \ D \) consists of six nodes in general position in \( Y_h \subseteq \mathbb{P}^5 \). In particular, the conditions defining \( C \) are open and not empty on \( \mathcal{V} \), so that \( C \) is integral. In a similar way we can define and use the universal singular point over \( C \), that is, the family

\[
S := \{ (C, o) \in C \times Y_h \mid o \in \text{Sing} \ C \}.
\] (8.9)

Fixing \( o \in Y_h \), let \( S_o \) the fibre the projection \( S \to Y_h \) and let \( \sigma : Y_o \to Y_h \) be the blowing up of \( o \). Then \( Y_o \) is a quartic Del Pezzo surface. Moreover, the strict transform by \( \sigma \) of the family of curves \( S_o \) is just an open set in the variety \( \mathcal{V}' \) of all antibicanonical curves \( C' \subseteq Y_o \), which are reduced and such that \( \text{Sing} \ C' \) has length \( \geq 5 \). Again \( \mathcal{V}' \) is known to be integral of constant dimension 7 [22]. Hence, the next lemma follows.

**Lemma 8.4.** \( S \) and \( C \) are integral.

Now, to globalize slightly, we fix our notation as follows. Let \( C \in \mathcal{C} \), then we set

\[
W_C := H^0(\mathcal{I}_C(2))
\]

and consider the rank 6 vector bundle \( \pi : \mathcal{W} \to C \), whose fibre at \( C \) is \( W_C \). Passing to wedge product, we have the Grassmann bundle

\[
\mathcal{G} \subset \mathbb{P}(\wedge^3 \mathcal{W}) \xrightarrow{\wedge^3 \pi} C,
\] (8.10)

whose fibre \( \mathcal{G}_C \) is the Grassmannian of planes of \( \mathbb{P}(H^0(\mathcal{I}_C(2))) \), and the \( \mathbb{P}^9 \)-bundle

\[
\mathcal{P} \subset \mathbb{P}(\wedge^3 \mathcal{W}) \xrightarrow{\wedge^3 \pi} C,
\] (8.11)

whose fibre \( \mathcal{P}_C \) is \( \mathbb{P}(A_C) \) and \( A_C \subset \wedge^3 W_C \) is the isotropic space \( A \) as above. Let
The universal Morin–Del Pezzo configuration over $\mathbb{C}$ is the closed set

$$Z := G \cap P \subset \mathbb{P}(\wedge^3 W). \quad (8.12)$$

Some comments now are due. Let $f : S \to P$ be the morphism defined by the assignment $(C, o) \to N_o$, where $N_o$ is a net of quadrics as above. It is clear that

$$Z' := f(S) \quad (8.13)$$

is an irreducible component of $Z$. $Z$ contains as well the five irreducible components

$$H_i := s_i(C), \ i = 1 \ldots 5, \quad (8.14)$$

where $s_i : C \to P$ is the section such that $s_i(C) := h_i \in H = |I_{Yh}(2)| \subset P_C = \mathbb{P}(A_C)$. Let $H := \cup H_i$, so far we have $Z' \cup H \subseteq G \cap P$. In the next theorem we show that the latter is an equality. Of course this implies that each fibre of the family

$$\wedge^3 \pi : G \cap P \longrightarrow C \quad (8.15)$$

is a finite and complete configuration of incident planes, in particular a Morin–Del Pezzo configuration. This completes our description of these configurations.

**Theorem 8.6.** $G \cap P = H_1 \cup \ldots H_5 \cup Z'$.

To prove the theorem we proceed as follows. Let $z \in G \cap P$ be a point in the fibre over $C \in C$. Then $z$ is the parameter point of a net of quadrics $N \subset |I_C(2)|$ and we have to show that $z \in Z' \cup H$. If $z$ is in $h$ then there is nothing to show. Hence, we can assume that $z$ is not in $h$, in other words that $N$ is not in the hyperplane $\mathbb{H}$ of quadrics through $Y_h$. Then $L := N \cap \mathbb{H}$ is a pencil of quadrics singular at some point $v \in Y_h$. Its base scheme is a cone $B_v$ of vertex $v$ over an integral complete intersection of two quadrics in $\mathbb{P}^4$, see 5.4. Hence, the base scheme of $N$ is

$$X = Q \cdot B_v, \quad (8.16)$$

where $Q \in N - L$. In particular, it is clear that $X \cdot Y = Q \cdot Y = C$.

**Lemma 8.7.** $\text{Sing } C \subset \text{Sing } X$. 
Proof. Let \( o \in \text{Sing} \, C \), we can assume \( o \neq v \). Since \( N \) defines a point of \( \mathbb{P}(A) \) and \( A \) is isotropic we have \( N \cap N_o \neq \emptyset \). Hence, there exists a quadric \( Q_o \in N \), which is singular at \( o \). If \( Q_o \) is not in \( \mathbb{H} \) then \( X = Q_o \cdot B_v \) and \( o \in \text{Sing} X \). If \( Q_o \) is in \( \mathbb{H} \) then \( Q_o \in B_v \). In this case \( \text{Sing} \, Q_o \) contains \( z, v \) and the line \( E := \langle ov \rangle \). Let \( \pi_E : Y \to \mathbb{P}^3 \) be the projection from \( E \) then \( \pi_E(Y) \) is a quadric. Moreover, it is easy to deduce that then \( E \subset Y \) and that the cone \( B_v \) is singular along \( E \). Hence, we have \( o \in X \cap \text{Sing} B_v \subset \text{Sing} X \). ■

Lemma 8.8. Let \( v \) be as above then \( v \in \text{Sing} \, C \).

Proof. Let \( X = Q \cdot B_v \) and let \( q_v \in H^0(\mathcal{O}_{\mathbb{P}^5}(1)) \) be the polar form of \( v \) with respect to \( Q \). If \( q_v = 0 \) then \( Q \) is singular at \( v \) and the statement follows. If \( q_v(v) \neq 0 \) then \( v \) is not in \( X \) nor in \( Q \). In this case consider the projection \( \pi_v : X \to \mathbb{P}^4 \) from the vertex \( v \) of \( B_v \). Then \( \pi_v \) is a finite double covering of \( \pi_v(X) \), which is an integral complete intersection of two quadrics. Since \( X = Q \cdot B_v \) the ramification divisor of \( \pi_v \) is the hyperplane section of \( X \) by \( \{ q_v = 0 \} \). In particular, \( q_v \) vanishes on \( \text{Sing} \, X \). But then, by the previous lemma, \( q_v \) vanishes on \( \text{Sing} \, C \). Since we are assuming \( h^0(\mathcal{I}_{\text{Sing} \, C}(1)) = 0 \), we have a contradiction. If \( q_v(v) = 0 \) and \( q_v \neq 0 \) assume \( v \notin \text{Sing} \, C \) and observe that the line \( \langle vp \rangle \) is in \( Q \) for each \( p \in \text{Sing} \, C \). Indeed, \( \langle vp \rangle \) is tangent to \( Q \) at \( v \) and contains \( p \). Then \( q_v \) vanishes in \( \text{Sing} \, C \) and the same contradiction follows. Hence, \( v \in \text{Sing} \, C \). ■

The lemma implies that the net \( N \), corresponding to \( z \in G \cap P \), is the net \( N_v \) of all quadrics through \( C \) singular at \( v \). Hence, \( z \in \mathcal{Z}' \) and the proof of the theorem follows.

Remark 8.9. We point out that, as a consequence of our description, a general Morin–Del Pezzo configuration is obtained from a nodal, integral canonical curve \( C \subset Y_h \) with exactly 6 nodes. Notice also that \( C \subset \mathbb{P}^1 \times \mathbb{P}^2 \) so that its projection in \( \mathbb{P}^2 \) is a nodal sextic with 10 nodes.

9 Morin Configurations of Higher Length via Canonical Curves

Finally, we apply the previous results and constructions to deduce the uniqueness, up to projective equivalence in \( \mathbb{P}^5 \), of the finite Morin configuration having maximal cardinality 20. We also outline the simple description of those families of configurations of length \( k \geq 16 \) having the one of maximal cardinality as a limit. We rely as previously on stable, highly singular canonical curves of genus 6.

Let \( F = \mathbb{P}(A) \cdot G \) be a finite Morin configuration of planes in \( \mathbb{P}^5 \) of length \( k \geq 16 \). By Theorem 3.2 the \( V \)-threefold \( V_A \) of \( A \) contains a plane \( P_h = \{ o \} \times \mathbb{P}^2 \) as in 7.1. By
Proposition 3.5 \( b := P_h \cdot \text{Sing } V_A \) is the finite base scheme of a pencil of conics. Assume \( F \) has maximal cardinality 20. Then, by Theorem 3.9 and [20, Section 4], \( F \) is smooth because its length is \( \leq 20 \). Since \( F - \{ u \} \) is biregular to \( \text{Sing } V_A \), it follows that \( b \) is a smooth complete intersection of two conics. Now let us recall from Section 7 that \( V_A \) is the birational image of the product map considered in 7.20, namely

\[ \pi_1 \times \pi_2 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2 \times \mathbb{P}^2. \]

More precisely we have \( C \subset Y \subset \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5 \), where \( Y \) is a smooth quintic Del Pezzo surface and \( C \in |\mathcal{O}_Y(2)| \) is a canonical curve, that is, a Gushel–Mukai curve [16]. Then the map \( \pi_1 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2 \) is defined by the net of divisors \( |\mathcal{J}(2,2)| \), where \( \mathcal{J} \) is the ideal sheaf of \( C \), and \( \pi_2 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2 \) is the projection. Moreover, \( F \) is a Morin–Del Pezzo configuration and we have shown so far that \( \text{Sing } C \) is biregular to \( \text{Sing}(V_A - P_h) \).

Therefore, a smooth \( F \) of cardinality 20 is defined, up to \( \text{Aut}\mathbb{P}^5 \), by a nodal curve \( C \subset Y \) such that \( |\text{Sing } C| = 15 \). Finally, it is well known, and easy to see, that the unique curve \( C \) such that \( \text{Sing } C \) is smooth of cardinality 15 is the curve \( C_\ell \) union of the 10 lines of \( Y \). This proves the next uniqueness theorem.

**Theorem 9.1.** Up to \( \text{Aut}\mathbb{P}^5 \) a unique finite Morin configuration of 20 planes exists and it is the Morin–Del Pezzo configuration defined by the curve \( C_\ell \).

Notice that \( C_\ell \) is invariant under the action of \( \text{Aut}Y \), which is the symmetric group \( S_5 \). Actually, \( C_\ell \) is a stable graph curve that is uniquely defined by its associated graph \( \Gamma \). This has 10 vertices corresponding to the 10 lines of \( C_\ell \). Each edge of \( \Gamma \) corresponds to a node \( o \in \text{Sing } C_\ell \) and joins the vertices corresponding to the two lines through \( o \). In our situation \( \Gamma \) is the famous Petersen graph \( \Gamma \).

We do not address a systematic study of the stratification by their length of Morin–Del Pezzo configurations. We simply outline here some simple ways of smoothing partially \( C_\ell \) so to obtain some of the missed families of length \( k \in [16,19] \). To this purpose just consider suitable connected subgraphs \( \lambda \) of arithmetic genus zero and consider the family of graph curves defined by the graph \( \Gamma_\lambda \), obtained from \( \Gamma \) after contracting \( \lambda \) to a point. Let \( L \subset C_\ell \) be the curve defined by \( \lambda \), that is, \( L = L_1 + \cdots + L_n \) where the summands correspond to the vertices of \( \lambda \). Then the linear system \( |L| \) is very ample. We have \( L^2 = n - 2 \) and \( C_\ell D = 2n \). Let \( D \in |L| \) be general and

\[ C = C_\ell - L + D. \]
It is easy to see that $C$ is nodal and that $|\text{Sing } C| = 15 - n$. Moreover, for $1 \leq n \leq 4$ the construction provides a curve $C$ such that $\text{Sing } C$ spans $\mathbb{P}^5$. Let $\mathbb{P}^5_C$ be 5-space of quadrics through $C$, then $\text{Sing } C$ defines in it, as usual, a Morin–Del Pezzo configuration $20 - n$ planes. Iterating the contraction to a point of a genus 0 subgraph, one can describe all the irreducible families of Morin configurations of length $k \geq 16$ and their quotients by $\text{Aut} Y$. Such families are naturally related to appropriated moduli spaces of Gushel–Mukai fourfolds [5] and EPW sextics [18]. Hopefully this matter will be reconsidered elsewhere.

**Concluding remarks**

Some constructions in this paper, involving singular canonical curves of genus 6, admit natural extensions to higher genus. Indeed let $W_g$ be a vector space whose dimension is the triangular number $(g-2)/2$. We can assume that $W_g$ is the dual of the space of quadratic forms vanishing on a nodal canonical curve

$$C \subset \mathbb{P}^{g-1}.$$  

(9.2)

Then the equality considered by Zak in [24] has, as a special case, the following one

$$(g - 3) + \binom{g - 3}{2} = \binom{g - 2}{2}$$

(9.3)

and this makes interesting to consider Morin configurations of $(g - 4)$-spaces in the projective space $\mathbb{P}(W_g)$. Let $F$ be a finite Morin configuration in the Grassmannian $G$ of $(g - 4)$-spaces of $\mathbb{P}(W_g)$. Among many other questions it is natural to ask:

**What one can say about the maximal length of $F$?**

Stable canonical curves $C$ of genus $g$ with many nodes provide interesting examples of finite families of incident $(g - 4)$-spaces. Indeed let $I_C$ be the linear system of quadrics through a stable $C$ and let $I_z := \{Q \in I \mid z \in \text{Sing } Q, z \in \text{Sing } C\}$. It turns out that the orthogonal $P_z \subset \mathbb{P}(W_g)$ is a subspace of dimension $g - 4$. Then the family

$$F_C := \{P_z, z \in \text{Sing } C\}$$

is an example of family of incident $g - 4$-spaces. Indeed, let $z_1, z_2 \in \text{Sing } C$ be distinct points and let $P_{z_1}, P_{z_2} \subset \mathbb{P}(W_g)$ be the orthogonal $(g - 4)$-spaces of $I_{z_1}, I_{z_2}$, respectively. Then, with the same argument used in genus 6, the codimension of the space spanned
by \( \mathbb{I}_{z_1} \cup \mathbb{I}_{z_2} \) turns out to be \((g-4)\) = \( \dim W_{g-2} \). Equivalently \( P_{z_1} \cap P_{z_2} \) is a point. Hence, \( F_C \) is a finite family of incident \((g-4)\)-spaces of \( \mathbb{P}^{(g-2)} \).

Now stable canonical curves \( C \) which are union of lines are \( 3g-3 \)-nodal and provide smooth families \( F_C \) of cardinality \( 3g-3 \). Each curve \( C \) of this type is uniquely defined by a suitable graph as in the case of the Petersen graph. For instance a generalized Petersen graph \( G(2k-1,1) \), see [13], uniquely defines a stable canonical curve

\[
C_{g,\ell} \subset \mathbb{P}^1 \times \mathbb{P}^{k-1} \subset \mathbb{P}^{g-1}
\]

of even genus \( g = 2k \). Omitting the discussion of the odd genus case and several details, this is readily constructed in \( \mathbb{P}^1 \times \mathbb{P}^{k-1} \) as follows. In the Segre embedding \( \mathbb{P}^1 \times \mathbb{P}^{k+1} \) consider the lines \( L_i := \mathbb{P}^1 \times t_i, \ i = 1 \ldots k+1 \), where \( t := \{t_1 \ldots t_{k+1}\} \) is a set of points in general position in \( \mathbb{P}^{k-1} \). On the other hand one can construct in \( \mathbb{P}^{k-1} \) three nodal rational normal curves \( R'_1, R'_2, R'_3 \) that are union of lines, have no common component, and contain \( t \) as a subset of smooth points. Then we can define the curve

\[
C_{g,\ell} = R_1 \cup R_2 \cup R_3 \cup L_1 \cup \cdots \cup L_{k+1},
\]

where \( R_j := u_j \times R'_j, j = 1, 2, 3 \) and \( u_j \in \mathbb{P}^1 \). Let \( F_{g,\ell} := \{P_z, z \in \text{Sing } C_{g,\ell}\} \) be the set of incident \((g-4)\)-spaces defined by \( \text{Sing } C_{g,\ell} \). For \( g \geq 8 \) it is natural to ask whether \( F_{g,\ell} \) is a Morin configuration and has maximal cardinality. In any case the study of graph curves like \( C_{g,\ell} \) seems to be interesting in order to study Morin configurations and their relations to the geometry of canonical curves.

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