On a conjecture of Szemerédi and Petruska

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April 11, 2019

Abstract

Consider a 3-uniform hypergraph of order $n$ with clique number $k$ such that the intersection of all its $k$-cliques is empty. Szemerédi and Petruska [7] proved $n \leq 8m^2 + 3m$, for fixed $m = n - k$, and they conjectured the sharp bound $n \leq \binom{m+2}{2}$. Gyárfás, Lehel, and Tuza [4] improved the bound, proving $n \leq 2m^2 + m$. Here we combine a decomposition process introduced by Szemerédi and Petruska with the skew version of Bollobás’s theorem to prove $n \leq m^2 + 6m + 2$. This also improves an upper bound for the maximum order of a $\tau$-critical 3-uniform hypergraph with transversal number $m$.

1 Introduction

Let $\mathcal{N} = \{N_1, \ldots, N_\ell\}$ be a collection of $k$-subsets of $[n] = \{1, \ldots, n\}$. Set $V = \bigcup_{i=1}^{\ell} N_i$. Assume that $n = |V|$, $\ell \geq 2$, and $k \geq 3$. Set $m = n - k$; that is, $|N_i| = |V - N_i| = m$. We further assume that $\mathcal{N}$ satisfies the following two properties:

(i) $\bigcap_{i=1}^{\ell} N_i = \emptyset$, but $\bigcap_{j \neq i} N_j \neq \emptyset$ for all $i = 1, \ldots, \ell$.

(ii) For any $X \subseteq V$ such that $|X| = k + 1$, there exists a subset $T \subseteq X$ such that $|T| = 3$ and $T \not\subseteq N_i$, for all $i = 1, \ldots, \ell$.

We shall refer to a system $\mathcal{N}$ satisfying these constraints as an $(n, m)$-structure. Szemerédi and Petruska [7] conjectured the following:

**Conjecture 1.** Any $(n, m)$-structure satisfies $n \leq \binom{m+2}{2}$.

Szemerédi and Petruska give a construction to show that this conjecture, if true, would be sharp. Indeed it has been conjectured (by us and others) that this construction is the unique extremal structure for $m \geq 4$. Gyárfás, Lehel, and Tuza [4] proved $n \leq 2m^2 + m$. Here we prove $n \leq m^2 + 6m + 2$ (Theorem 10). Our proof adapts an iterative decomposition method.
process introduced by Szemerédi and Petruska and applies the skew version of Bollobás’s theorem [2] on the size of intersecting set pair systems. The skew version of Bollobás’s theorem was first proven by Frankl [3] and also Kalai [6] (see also Theorem 5.6 of Babai and Frankl’s book [1]).

As noted by Gyárfás, Lehel, and Tuza [4], the Szemerédi and Petruska problem is equivalent to determining the maximum order of a $\tau$-critical 3-uniform hypergraph with transversal number $m$. They also determined that $O(m^{r-1})$ is the correct order of magnitude for the maximum order of a $\tau$-critical $r$-uniform hypergraph with transversal number $m$. Theorem 10 improves, by a factor of two, the constant on the leading term of the upper bound in the case $r = 3$. We suspect that the methods presented here, a proof-of-concept for improved linear algebra bounds, may yield improved upper bounds for the order of any $\tau$-critical $r$-uniform hypergraph, for $r > 3$, with transversal number $m$. This is the focus of future research which also will explain a connection to a conjecture of Lehel and Tuza (Problem 18 of [8]) and a theorem of Hajnal [5].

Section 2 introduces notation and recalls the process, introduced by Szemerédi and Petruska, to decompose $(n, m)$-structures. Section 3 introduces a recursive procedure, based on this decomposition process, to select special private pairs. Section 4 defines a large subset of free private pairs chosen from this selection of special private pairs. A skew $(2, m)$-system ultimately arises from this subset of free private pairs in Section 5, where Theorem 10 is finally presented. The last section includes remarks on possible improvements.

2 The Decomposition Process

We begin by giving definitions and recalling the process, introduced by Szemerédi and Petruska, to decompose $(n, m)$-structures. Much of the presentation in this section is lifted verbatim from their paper [7]. We assume $\ell \geq 4$ (Szemerédi and Petruska resolve the $\ell = 2, 3$ cases). Let $\mathcal{N} = \{N_1, \ldots, N_\ell\}$, be an $(n, m)$-structure. Define a collection of objects iteratively in stages, which are also called times, starting with stage 0. Set $\ell_0 = \ell$, $\mathcal{N}^{(0)} = \mathcal{N}$ and $N_i^{(0)} = N_i$. For $i = 1, \ldots, \ell_0$, fix a choice of vertex $x_i^{(0)} \in \bigcap_{j \neq i} N_j$. By definition, $x_i^{(0)} \neq x_j^{(0)}$, for $i \neq j$. The set $A^{(0)} = \{x_1^{(0)}, \ldots, x_{\ell_0}^{(0)}\}$ is called the kernel at stage 0; $x_1^{(0)}, \ldots, x_{\ell_0}^{(0)}$ are the kernel vertices at stage 0.

Assume at stage $j$ ($j \geq 0$) that $\ell_j \geq 4$, $\mathcal{N}^{(j)} = \{N_1^{(j)}, \ldots, N_{\ell_j}^{(j)}\}$, and $A^{(j)} = \{x_1^{(j)}, \ldots, x_{\ell_j}^{(j)}\}$ are defined. Also assume that the minimal substructures of the “remainder” structure

$$R^{(j)} = \left\{N_1^{(j)} - \bigcup_{i=0}^{j} A^{(i)}, \ldots, N_{\ell_j}^{(j)} - \bigcup_{i=0}^{j} A^{(i)}\right\}$$

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1We have endeavored to use the same notation introduced by Szemerédi and Petruska. Important exceptions include that they use $n$ and $k$ to mean the quantities we refer to as $k$ and $\ell$, respectively.
satisfy (i). We now explain the definition of $\ell_{j+1}$, $\mathcal{N}^{(j+1)}$, and $A^{(j+1)}$. Consider substructures in $R^{(j)}$ that are minimal structures with respect to property (i). Stop if there are no such substructures with more than three sets. Otherwise, let

$$
\mathcal{N}^{(j+1)} = \left\{ N_1^{(j+1)}, \ldots, N_{\ell_{j+1}}^{(j+1)} \right\} \subset \mathcal{N}^{(j)}
$$

be chosen so that $\ell_{j+1} \geq 4$ and the corresponding remainders

$$
\left\{ N_1^{(j+1)} - \bigcup_{i=0}^{j} A^{(i)}, \ldots, N_{\ell_{j+1}}^{(j+1)} - \bigcup_{i=0}^{j} A^{(i)} \right\}
$$

form a substructure in $R^{(j)}$ that satisfies (i). For $i = 1, \ldots, \ell_{j+1}$, fix a choice of vertex

$$
x_i^{(j+1)} \in \bigcap_{r \neq i} \left( N_r^{(j+1)} - \bigcup_{i=0}^{j} A^{(i)} \right).
$$

By definition, $x_r^{(j+1)} \neq x_s^{(j+1)}$, for $1 \leq r < s \leq \ell_{j+1}$. The kernel at stage $j + 1$ is

$$
A^{(j+1)} = \left\{ x_1^{(j+1)}, \ldots, x_{\ell_{j+1}}^{(j+1)} \right\}.
$$

This process defines $\ell_j$, $\mathcal{N}^{(j)}$, and $A^{(j)}$ for $0 \leq j \leq t$, for some $t$. Note that $t$ has been defined here as the length of the iterative process.

Because $\mathcal{N} = \{ N_1, \ldots, N_0 \}$ is an arbitrary enumeration of $\mathcal{N}$, we may assume that

$$
\mathcal{N}^{(j)} = \left\{ N_1, \ldots, N_t \right\}, \text{ for } j = 0, \ldots, t.
$$

Define, for $i = 1, \ldots, \ell_0$, the last time (or stage) that the truncation of $N_i$ appears in a substructure of this decomposition process, denoted $t_i$, as

$$
t_i = \max \left\{ j : N_i \in \mathcal{N}^{(j)} \right\}.
$$

By definition, $t = t_1 \geq \cdots \geq t_{\ell_0} \geq 0$.

The next lemma gathers several properties of this iterative process.

**Lemma 1.** Some observations:

(a) $A^{(j)}$ are pairwise disjoint, for $j = 0, \ldots, t$.

(b) $\ell_j = |A^{(j)}| \geq 4$, for $j = 0, \ldots, t$.

(c) $|N_i \cap A^{(r)}| = \ell_r - 1$, for $1 \leq i \leq \ell, 0 \leq r \leq t_i$.

(d) $t < m$.

(e) $\ell_0 + \cdots + \ell_t \geq k - 2m = n - 3m$.

**Proof.** Properties (a) – (c) are immediate from the definition. Properties (d) and (e) are respectively Lemma 5 and Lemma 6 of [7].

Define $A = \bigcup_{i=0}^{t} A^{(i)}$; it is the set of kernel vertices. Let $G = V - A$ denote the garbage vertices; that is, the vertices remaining after the aforementioned kernel-defining decomposition process terminates.
3 Selection of private pairs

In this section we define a process to select private pairs. Much of the beginning of this is a review of results from the paper by Szemerédi and Petruska [7].

A pair of elements $p \subset N_i$ is single-covered with respect to $N^{(j)}$, for some $j$ satisfying $0 \leq j \leq t_i$, if $N_i$ is the only set in $N^{(j)}$ that contains $p$ as a subset. If $p \subset N_i$ is single-covered with respect to $N^{(j)}$, then it is a private pair for $N_i$ at time $j$, or simply a private pair. Observe that if a pair is private for $N_i$ at time $j$, then it remains a private pair for $N_i$ until (and including) time $t_i$. A pair that is contained in at least two sets in $N^{(j)}$ is called a double-covered pair (at time $j$).

The following lemma is a reformulation of Lemma 7 from [7].

**Lemma 2.** For all $j = 0, \ldots, t$,

(a) Every pair from $A$ is double-covered by $N^{(j)}$.

(b) For all $N_i \in N^{(j)}$ and any subset $Y \subseteq N_i$ such that $|Y| = j$, there exists a pair in $N_i \setminus Y$ that is single-covered with respect to $N^{(j)}$.

**Proof.** Properties (a) and (b) are, respectively, the proof of Lemma 7 part (b) and the proof of Lemma 7 part (a) of [7].

An important consequence of Lemma 2 (a) is the following.

**Corollary 3.** Any private pair must contain at least one vertex from $G$.

Next we describe a process to select a collection of private pairs for each $N_i$. For each $i \in \{1, \ldots, \ell\}$, we define, by induction on time, a set $P_i = \{p_i^{(j)} : 0 \leq j \leq t_i\}$ of $t_i + 1$ private pairs for $N_i$. The pair $p_i^{(j)}$ will be chosen from among the private pairs for $N_i$ at time $j$. By Corollary 3 the pair $p_i^{(j)}$ contains at least one vertex from $G$. Choose $g_i^{(j)} \in p_i^{(j)} \cap G$; it is the anchor of $p_i^{(j)}$. The other element of $p_i^{(j)}$ is the non-anchor of $p_i^{(j)}$; it is denoted $u_i^{(j)}$. Naturally it is possible that the non-anchor is also an element of $G$, but we shall distinguish $g_i^{(j)}$ as the anchor. Auxiliary sets $P_i^{(j)}$ and $G_i^{(j)}$ will also be defined; they are the initial segments of the private pairs and anchors for the private pairs selected for $N_i$.

Initially, for $i \in \{1, \ldots, \ell\}$, let $p_i^{(0)} = \{u_i^{(0)}, g_i^{(0)}\}$ be a private pair for $N_i$ at time zero. Such a private pair exists by an application of Lemma 2 part (b) in which $Y = \emptyset$. Set $G_i^{(0)} = \{g_i^{(0)}\}$ and $P_i^{(0)} = \{p_i^{(0)}\}$.

For $j > 0$ and $i \in \{1, \ldots, \ell\}$, assume that the sets $P_i^{(j-1)}$ and $G_i^{(j-1)}$ have already been defined. Also assume that a private pair $p_i^{(j)} = \{u_i^{(j)}, g_i^{(j)}\}$ has already been chosen for each $N_i$ with $j \leq t_i$. Now define

$$G_i^{(j)} = \begin{cases} G_i^{(j-1)} \cup \{g_i^{(j)}\} & \text{if } j \leq t_i, \\ G_i^{(j-1)} & \text{if } j > t_i, \end{cases}$$

$$P_i^{(j)} = \begin{cases} P_i^{(j-1)} \cup \{p_i^{(j)}\} & \text{if } j \leq t_i, \\ P_i^{(j-1)} & \text{if } j > t_i, \end{cases}$$
and similarly define,

\[ P_i^{(j)} = \begin{cases} P_i^{(j-1)} \cup \{ p_i^{(j)} \} & \text{if } j \leq t_i \\ P_i^{(j-1)} & \text{if } j > t_i. \end{cases} \]

This definition yields \( P_i^{(j)} = \{ p_i^{(0)}, \ldots, p_i^{(j)} \} \) and \( G_i^{(j)} = \{ g_i^{(0)}, \ldots, g_i^{(j)} \} \), for all \( 0 \leq j \leq t_i \).

In particular, note that \( |P_i^{(j)}| = |G_i^{(j)}| = j + 1 \), for all \( 0 \leq j \leq t_i \). Also \( \bigcup_{h=0}^{t} P_h^{(j)} \) represents the set of private pairs defined up through time \( j \).

To complete the iterative process, it remains to describe how to select a private pair \( p_i^{(j)} \), for all \( i \in \{1, \ldots, \ell_j\} \). Apply Lemma 2 part (b) with \( Y = G_i^{(j-1)} \) to produce a pair \( p_i^{(j)} = \{ u_i^{(j)}, g_i^{(j)} \} \) private for \( N_i^{(j)} \) satisfying \( p_i^{(j)} \cap Y = \emptyset \). In particular, \( g_i^{(j)} \in G \) and \( g_i^{(j)} \notin Y = G_i^{(j-1)} = \{ g_i^{(0)}, \ldots, g_i^{(j-1)} \} \).

If the non-anchor of \( p_i^{(j)} \) is in \( \bigcup_{s=0}^{j-1} A^{(s)} \), say \( u_i^{(j)} = x_a^{(b)} \), for some \( b < j \), then replace \( u_i^{(j)} \) with either \( x_a^{(j)} \), if \( j \leq t_a \), or \( x_a^{(t_a)} \) otherwise. Observe that after this replacement the new pair is still private to \( N_i \) at time \( j \). In other words, if \( u_i^{(j)} \in \bigcup_{s=0}^{j-1} A^{(s)} \), then \( u_i^{(j)} = x_a^{(t_a)} \), for some \( a \in \{1, \ldots, \ell\} \).

Finally, suppose that \( u_i^{(j)} \in A^{(j)} \), say \( u_i^{(j)} = x_a^{(j)} \), for some \( a \in \{1, \ldots, \ell\} \). If \( j < t_a \) and \( j < t_i \) then set \( u_i^{(j)} = x_a^{(j+1)} \). After this replacement the new pair is still private to \( N_i \) at time \( j \). In other words, if \( u_i^{(j)} = x_a^{(j)} \), for some \( a \in \{1, \ldots, \ell\} \), then \( j \in \{t_a, t_i\} \).

Let \( P^{(j)} = \{ p_i^{(j)} : 1 \leq i \leq \ell_j \} \) denote the private pairs defined by this process at time \( j \) and let \( P_i = \{ p_i^{(j)} : 0 \leq j \leq t_i \} \) be the set of \( t_i + 1 \) private pairs defined for \( N_i \) by this process. The collection of all selected pairs is defined as

\[ P = \bigcup_{j=0}^{t} P^{(j)} = \bigcup_{i=0}^{\ell} P_i. \]

The following lemma gathers observations about private pairs in \( P \).

**Lemma 4.** The pairs in \( P \) satisfy:

(a) \( P^{(j_1)} \cap P^{(j_2)} = \emptyset \), for \( 0 \leq j_1 < j_2 \leq t \).

(b) Any pair in \( \bigcup_{s=0}^{j} P^{(s)} \) is at most single-covered by \( N^{(j)} \), for \( j = 0, \ldots, t \).

(c) \( |P^{(j)}| = \ell_j \) and every \( |P_i \cap P^{(j)}| = 1 \), for all \( 0 \leq j \leq t \) and \( 1 \leq i \leq \ell_j \).

(d) \( |P| = \bigcup_{j=0}^{t} P^{(j)} = \sum_{j=0}^{t} \ell_j \geq n - 3m \).

(e) If \( u_i^{(j)} \in \bigcup_{s=0}^{j-1} A^{(s)} \), then \( u_i^{(j)} = x_a^{(t_a)} \), for some \( a \in \{1, \ldots, \ell\} \).
(f) If \( u_i^{(j)} = x_a^{(j)} \), for some \( a \in \{1, \ldots, \ell \} \), then \( j \in \{ t_a, t_i \} \).

Proof. Parts (a), (b), and (c) respectively, follow from the same arguments given to prove parts (a*), (**), and (***) of Lemma 8 of [7]. Part (d) is essentially a consequence of the arguments given to prove Lemma 1(e). Part (e) and (f) reiterate the observations in the paragraphs defining the selection of the private pairs in \( P \). □

4 Free pairs

In this section we define a special subset of private pairs in \( P \) that is used in Section 5 to define a large skew \( (2, m) \)-system. Recall that \( A \) is the set of kernel vertices. Every element of \( A \) has the form \( x_i^{(j)} \), where \( 1 \leq i \leq \ell \) and \( 0 \leq j \leq t_i \).

Create a digraph \( D \) on the vertex set \( A \) in which there is an arc from \( x_i^{(s)} \) to \( x_i^{(j)} \) if \( u_i^{(s)} = x_i^{(j)} \) and \( j \neq t_i \).

Lemma 5. If there is an arc in \( D \) from \( x_i^{(s)} \) to \( x_i^{(j)} \) then

(a) \( s \leq j \), and

(b) if \( s = j \), then \( s = t_r \).

Proof. Because \( j \neq t_i \), Lemma 4(e) implies that \( s \leq j \). If \( s = j \), then Lemma 4(f) guarantees that \( j \in \{ t_r, t_i \} \). Consequently \( t_r = j = s \). □

Lemma 6. The digraph \( D \) is acyclic and has out-degree at most one.

Proof. Lemma 4 part (c) guarantees that the out-degree of vertex \( x_i^{(s)} \) in \( D \) is at most one. Suppose, to the contrary, that there are arcs forming a directed cycle:

\[ x_i^{(s_1)} \rightarrow x_i^{(s_2)} \rightarrow \cdots \rightarrow x_i^{(s_h)} \rightarrow x_i^{(s_1)} . \]

Lemma 5(a) yields \( s_1 \leq s_2 \leq \cdots \leq s_h \leq s_1 \), so \( s_1 = s_i \), for all \( 1 \leq i \leq h \). Lemma 5(b) then implies that \( t_{r_i} = s_1 \), for all \( 1 \leq i \leq h \). But the arc from \( x_i^{(s_h)} \) to \( x_i^{(s_1)} \) requires \( s_1 \neq t_{r_1} \). □

The graph obtained from \( D \) by removing direction on the arcs is a forest that contains a maximum independent set of vertices; call it \( F \). Because forests are 2-colorable, it follows that \( |F| \geq |A|/2 \). Define free pairs in \( P \) this way: a pair \( p_i^{(s)} \in P \) is free if \( x_i^{(s)} \in F \).

The most important consequence of Lemma 4 is a lower bound on the number of free pairs.

Corollary 7. The number of free pairs in \( P \) is at least \( \frac{n-3m}{2} \).

Proof. By part (d) of Lemma 4, \( |A| \geq n-3m \). Because \( |F| \geq |A|/2 \), the result follows. □
5 A skew system

In this section we apply the following theorem, first proven by Frankl [3]; it is the skew version of a theorem due to Bollobás [2]. This theorem is also presented in the book by Babai and Frankl ([1], pages 94–95).

Theorem 8. (Bollobás’s Theorem - Skew Version) If \( A_1, \ldots, A_h \) are \( r \)-element sets and \( B_1, \ldots, B_h \) are \( s \)-element sets such that

(a) \( A_i \) and \( B_i \) are disjoint for \( i = 1, \ldots, h \),

(b) \( A_i \) and \( B_j \) intersect whenever \( 1 \leq i < j \leq h \)

then \( h \leq \binom{r + s}{r} \).

A system of sets, \( \{(A_i, B_i)\}_{i=1}^h \), satisfying the hypotheses of Theorem 8 is called a skew intersecting set pair \((r, s)\)-system; abbreviate this to skew \((r, s)\)-system.

The goal in this section is to apply Theorem 8 to a skew \((2, m)\)-system derived from the free pairs in \( P \). First use all of the pairs in \( P \) to define, iteratively, a collection of \( m \)-sets this way. To each \( N_i \) associate \( t_i + 1 \) \( m \)-sets denoted \( M_i^{(0)}, \ldots, M_i^{(t_i)} \). At stage 0, set \( M_i^{(0)} = N_i \), for all \( i = 1, \ldots, \ell \). For \( i = 1, \ldots, \ell \) and \( j = 1, \ldots, t_i \), recursively define

\[
M_i^{(j)} = \begin{cases} 
M_i^{(j-1)} - \{x_i^{(j-1)}\} + \{g_i^{(j-1)}\} & \text{if } p_i^{(j-1)} \text{ is free} \\
M_i^{(j-1)} & \text{if } p_i^{(j-1)} \text{ is not free}
\end{cases}
\]

Note that, because \( |M_i^{(0)}| = m \), it follows that \( |M_i^{(j)}| = m \), for all \( i = 1, \ldots, \ell \) and \( j = 1, \ldots, t_i \). Also observe that this recursive process will never remove \( x_i^{(t_i)} \) from \( M_i^{(0)} \) because the process halts at stage \( t_i \).

Now define the system

\[
\mathcal{F} = \left\{ (p_i^{(j)}, M_i^{(j)}) : p_i^{(j)} \in P \text{ is free} \right\},
\]

where \( \mathcal{F} \) is ordered linearly and chronologically via lexicographical order:

\[
(p_r^{(s)}, M_r^{(s)}) < (p_i^{(j)}, M_i^{(j)}) \iff (s < j) \text{ or } (s = j \text{ and } r < i).
\]

Theorem 9. \( \mathcal{F} \) is a skew \((2, m)\)-system.

Proof. Clearly \( |p_i^{(j)}| = 2 \) and \( |M_i^{(j)}| = m \), for all \( (p_i^{(j)}, M_i^{(j)}) \in \mathcal{F} \). Because \( p_i^{(j)} \) is private to \( N_i \) at time \( j \), it follows that \( p_i^{(j)} \subset N_i \); so, \( p_i^{(j)} \cap M_i^{(0)} = \emptyset \). Observe that the anchor, \( g_i^{(j)} \), for \( p_i^{(j)} \) can not be added to \( M_i^{(j)} \) by the recursive process generating the \( m \)-sets because \( g_i^{(j)} \) is added at time \( j + 1 \). Moreover, by the iterative choice of private pairs, the
pair \( p_i^{(j)} \) was chosen to be disjoint from \( \{ g_i^{(0)}, \ldots, g_i^{(j-1)} \} \), implying that the non-anchor of \( p_i^{(j)} \) is never added to \( M_i^{(0)} \) in the process to produce \( M_i^{(j)} \). Therefore, \( p_i^{(j)} \cap M_i^{(j)} = \emptyset \), showing that hypothesis (a) is satisfied in Theorem 8.

Now we prove the system satisfies hypothesis (b). Suppose \( (p_r^{(s)}, M_r^{(s)}) , (p_i^{(j)}, M_i^{(j)}) \in \mathcal{F} \) and \( (p_r^{(s)}, M_r^{(s)}) < (p_i^{(j)}, M_i^{(j)}) \). We must prove \( p_i^{(s)} \cap M_i^{(j)} \neq \emptyset \). If \( r = i \), then \( s < j \) so \( g_r^{(s)} \) is in \( M_i^{(s+1)} \) (and therefore \( M_i^{(j)} \)) because \( p_r^{(s)} \) is free. Consequently we may assume \( r \neq i \).

If \( g_r^{(s)} \in M_i^{(j)} \), then \( p_r^{(s)} \cap M_i^{(j)} \neq \emptyset \). So we may assume \( g_r^{(s)} \notin M_i^{(j)} \). Elements from \( G \) are only added to \( M_i^{(0)} \) to get to \( M_i^{(j)} \), so \( g_r^{(s)} \notin M_i^{(0)} = \overline{N}_i \) which implies \( g_r^{(s)} \in N_i \). Since \( s \leq j \) and \( p_r^{(s)} \) is private to \( N_r \) at time \( s \), we conclude that \( u_r^{(s)} \notin N_i \). So \( u_r^{(s)} \in \overline{N}_i = M_i^{(0)} \).

If \( u_r^{(s)} \neq x_i^{(a)} \), for some \( 0 \leq \alpha \leq j - 1 \), then \( u_r^{(s)} \in M_i^{(j)} \) meaning \( p_r^{(s)} \cap M_i^{(j)} \neq \emptyset \).

So we may assume that \( u_r^{(s)} = x_i^{(a)} \), for some \( 0 \leq \alpha \leq j - 1 \). Because \( \alpha < j \leq t_i \), note that \( \alpha \neq t_i \). By definition there is arc in the digraph \( D \) from \( x_r^{(s)} \) to \( x_i^{(a)} \). But \( p_r^{(s)} \) is free, so \( x_i^{(a)} \in F \). Since \( F \) is an independent set in \( D \), it follows that \( x_i^{(a)} \notin F \). Accordingly \( x_i^{(a)} \) is never removed from \( M_i^{(0)} \) in the production of \( M_i^{(j)} \). Therefore \( x_i^{(a)} = u_r^{(s)} \in p_r^{(s)} \cap M_i^{(j)} \). \( \square \)

Now we are ready to state the main theorem of the paper.

**Theorem 10.** Any \((n,m)\)-structure satisfies \( n \leq m^2 + 6m + 2 \).

**Proof.** Corollary 7 proves \( \frac{n-3m}{2} \leq |\mathcal{F}| \). Combining Theorem 8 with Theorem 9 yields \( |\mathcal{F}| \leq \left( \begin{array}{c} m+2 \\ 2 \end{array} \right) \); therefore \( n \leq m^2 + 6m + 2 \). \( \square \)

It is possible to reduce the upper bound in Theorem 10. Natural reductions can be achieved in two ways: increasing the bound on \(|P|\) that appears in Lemma 4(d) and enlarging the set of pairs used to define \( \mathcal{F} \). However the improvements that we have found do not reduce the leading term of the bound and so for simplicity’s sake we have opted to omit them.

More tantalizingly hopeful is a linear algebra approach (using a dimension argument similar to Lovász’s proof of Bollobás’s theorem as presented in the book by Babai and Frankl [1]) to prove Conjecture 1. Small computations confirm the linear independence of appropriately chosen homogeneous polynomials of degree two in \( m+1 \) variables associated with carefully selected private pairs for each \( N_i \) and each vertex from \( G \). Unfortunately a general proof of the linear independence of these polynomials has eluded us. The argument presented here essentially uses the skew version of Bollobás’s theorem to verify the linear independence of a large number of these polynomials. Numerous small extremal structures and the unwieldy form of the general conjecture (Problem 18 of [8]) also pose serious obstacles for larger uniformity (\( r > 3 \)).

**Acknowledgments**

We thank Jenő Lehel for helpful remarks and discussions.
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