THE WAVELET GALERKIN OPERATOR

DORIN E. DUTKAY

ABSTRACT. We consider the eigenvalue problem
\[ R_{m_0,m_0}h = \lambda h, \quad h \in C(\mathbb{T}), \quad |\lambda| = 1 \]
where \( R_{m_0,m_0} \) is the wavelet Galerkin operator associated to a wavelet filter \( m_0 \). The solution involves the construction of representations of the algebra \( \mathfrak{A}_N \) - the \( C^\ast \)-algebra generated by two unitaries \( U, V \) satisfying \( UVU^{-1} = V^N \) introduced in [Jor98].

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1. INTRODUCTION

The wavelet Galerkin operator appears in several different contexts such as wavelets (see for example [Law91], [Dau95], [CoDa96], [LLS96], [CoRa90]), ergodic theory and \( g \)-measures ([Kea82]) or quantum statistical mechanics ([Rue68]). For some of the applications of the Ruelle operator we refer the reader to the book by V. Baladi [Bal00]. It also bears many different names in the literature: the Ruelle operator, the Perron-Frobenius-Ruelle operator, the Ruelle-Araki operator, the Sinai-Bowen-Ruelle operator, the transfer operator and several others. We used the name wavelet Galerkin operator as suggested in [Law91], because of its close connection to wavelets that we will be using in the sequel. We will also use the name Ruelle operator and transfer operator.

The Ruelle operator considered in this paper is defined by
\[ R_{m_0,m_0'}f(z) = \frac{1}{N} \sum_{w^N=z} m_0(w)m_0'(w)f(w), \quad (z \in \mathbb{T}) \]
where \( m_0, m_0' \in L^\infty(\mathbb{T}) \) are nonsingular (i.e. they do not vanish on a set of positive measure ), \( \mathbb{T} \) is the unit circle \( \{z \in \mathbb{C} \mid |z| = 1\} \), \( N \geq 2 \) is an integer. A large amount of information about this operator is contained in [Bra]. One of the main objectives of this paper is to do a peripheral spectral analysis for the Ruelle operator, that is to solve the equation
\[ R_{m_0,m_0}h = \lambda h, \quad |\lambda| = 1, \quad h \in C(\mathbb{T}). \]

The restrictions that we will impose on \( m_0 \) are:
\[ m_0 \in \text{Lip}_1(\mathbb{T}) \]

(1.1)
where

\[ \text{Lip}_1(T) = \{ f : T \to \mathbb{C} \mid f \text{ is Lipschitz} \}. \]

(1.2) \( m_0 \) has a finite number of zeros.

(1.3) \( R_{m_0,m_0} 1 = 1 \)

(1.4) \( m_0(1) = \sqrt{N} \)

In ergodic theory the Ruelle operators are used in the derivation of correlation inequalities (see [Sto01] and [Dol98]) and in understanding the Gibbs measures. The role played by the Ruelle operator in wavelet theory is somewhat similar. It can be used to make a direct connection to the cascade approximation and orthogonality relations.

In the applications to wavelets, the function \( m_0 \) is a wavelet filter, i.e., its Fourier expansion

(1.5) \[ m_0(z) = \sum_{k \in \mathbb{Z}} a_k z^k \]

yields the masking coefficients of the scaling function \( \varphi \) on \( \mathbb{R} \), i.e. the function which results from the the fixed-point problem

(1.6) \[ \varphi(x) = \sqrt{N} \sum_{k \in \mathbb{Z}} a_k \varphi(Nx - k) \]

Then the solution \( \varphi \) is used in building a multiresolution for the wavelet analysis. If, for example, conditions can be placed on (1.5) which yield \( L^2(\mathbb{R}) \)-solutions to (1.6), then the closed subspace \( V_0 \) spanned by the translates \( \{ \varphi(x - k) \mid k \in \mathbb{Z} \} \) is invariant under the scaling operator

(1.7) \[ Uf(x) = \frac{1}{\sqrt{N}} f \left( \frac{x}{N} \right), \quad (x \in \mathbb{R}) \]

i.e. \( U(V_0) \subset V_0 \). Setting \( V_j := U^j(V_0) \), \( j \in \mathbb{Z} \) we get the resolution

\[ \ldots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \ldots \]

from which wavelets can be constructed as in [Dau92].

The cascade operator is defined on \( L^2(\mathbb{R}) \) from the masking coefficients by:

\[ M_a \psi = \sqrt{N} \sum_{n \in \mathbb{Z}} a_n \psi(N \cdot -n). \]

The scaling function \( \varphi \) is then a fixed point for the cascade operator, it satisfies the scaling equation

\[ M_a \varphi = \varphi. \]

Now set

\[ p(\psi_1, \psi_2) (e^{it}) = \sum_{n \in \mathbb{Z}} e^{int} \int_{\mathbb{R}} \overline{\psi_1(x)} \psi(x - n) dx. \]

The relation between the Ruelle operator \( R_{m_0,m_0} \) and the cascade operator \( M_a \) is

\[ R_{m_0,m_0} (p(\psi_1, \psi_2)) = p(M_a \psi_1, M_a \psi_2), \]
and this makes the transfer operator an adequate tool in the analysis of the orthogonality relations.

One of the fundamental problems in wavelet theory is to give necessary and sufficient conditions on \(m_0\) such that the translates of the scaling function \(\{\varphi(\cdot - n) \mid n \in \mathbb{Z}\}\) form an orthonormal set. There are two well known results that answer this question: one due to Lawton ([Law91a]), which says that such a condition is that \(R_{m_0, m_0}\), as an operator on continuous functions, has 1 as a simple eigenvalue, the other, due to A.Cohen ([Co90]), which says that the orthogonality is equivalent to the fact that \(m_0\) has no nontrivial cycles (a cycle is a set \(\{z_1, \ldots, z_p\}\) with \(z_1^N = z_2, \ldots, z_{p-1}^N = z_p, z_p^N = z_1^N = \sqrt{N}\) for all \(i \in \{1, \ldots, p\}\); the trivial cycle is \(\{1\}\)).

The peripheral spectral analysis in this paper will elucidate, among other things, why these two conditions are equivalent.

The wavelet theory gives a representation of the algebra \(\mathfrak{A}_N\) (i.e. the \(C^*\)-algebra generated by two unitary operators \(U\) and \(V\) subject to the relation \(UVU^{-1} = V^N\)) on \(L^2(\mathbb{R})\). \(U\) is the scaling operator in (1.7) and \(V\) is the translation by 1 \(V: \psi \mapsto \psi(\cdot - 1)\). In fact we also have a representation of \(L^\infty(T)\) on \(L^2(R)\) given by

\[
\pi(f) \psi = \sum_{n \in \mathbb{Z}} c_n \psi(\cdot - n),
\]

for \(f = \sum_{n \in \mathbb{Z}} c_n z^n \in L^\infty(T)\).

The scaling equation (1.6) can be rewritten as

\[
U \varphi = \pi(m_0) \varphi.
\]

This representation of \(\mathfrak{A}_N\) together with the scaling function \(\varphi\) is called the wavelet representation.

In [Jor98] it is proved that there is a 1-1 correspondence between positive solutions to \(R_{m_0, m_0} h = h\) and representations of \(\mathfrak{A}_N\). These representations are in fact given by the unitary \(U: \mathcal{H} \to \mathcal{H}\), a representation \(\pi: L^\infty(T) \to \mathcal{B}(\mathcal{H})\) satisfying

\[
U \pi(f) = \pi(\{f(z^N)\}) U, \quad (f \in L^\infty(T))
\]

and \(\varphi \in \mathcal{H}\) with \(U \varphi = \pi(m_0) \varphi\).

We reproduce here the theorem:

**Theorem 1.1.**

(i) Let \(m_0 \in L^\infty(T)\), and suppose \(m_0\) does not vanish on a subset of \(T\) of positive measure. Let

\[
(1.8) \quad (Rf)(z) = \frac{1}{N} \sum_{w^N = z} |m_0(w)|^2 f(w), \quad f \in L^1(T).
\]

Then there is a one-to-one correspondence between the data \(a\) and \(b\) below, where \(b\) is understood as equivalence classes under unitary equivalence:

(a) \(h \in L^1(T), h \geq 0, \) and \(R(h) = h.\)

(b) \(\tilde{\pi} \in \text{Rep}(\mathfrak{A}_N, \mathcal{H}), \varphi \in \mathcal{H}, \) and the unitary \(U\) from \(\tilde{\pi}\) satisfying

\[
(1.10) \quad U \varphi = \pi(m_0) \varphi.
\]
Moreover, a unique fixed point of the Ruelle operator is in fact the cyclic representation corresponding to Definition 1.2.

Given $S$, there exists a unique operator $h$ corresponding to $\pi, \varphi$. Then there exists a unique $h$ given to hold for some $s$, and $SU^{(a)}$, the correspondence is given by

\begin{equation}
(h(\varphi)) = h \varphi \in L^1(\mathbb{T}), h, h' \geq 0, R_{m_0, m_0}(h) = h, R_{m_0', h'}(h') = h'. \end{equation}

Let $\pi, \varphi$, $(\pi', \varphi')$ be the cyclic representations corresponding to $h$ and $h'$ respectively. If $h_0 \in L^1(\mathbb{T})$, $R_{m_0, m_0}'(h_0) = h_0$ and $|h_0|^2 \leq chh'$ for some $c > 0$ then there exists a unique operator $S : \mathcal{H}' \to \mathcal{H}$ such that

\begin{equation}
SU' = US, \quad S\pi'(f) = \pi(f)S, \quad (f \in L^\infty(\mathbb{T}))
\end{equation}

\begin{equation}
\langle \varphi, \pi(f)S\varphi' \rangle = \int_T fh_0 \, d\mu, \quad (f \in L^\infty(\mathbb{T}))
\end{equation}

Also $\|S\| \leq \sqrt{c}$.

Theorem 1.4. Let $m_0, m_0', h, h', (\pi, \varphi), (\pi', \varphi')$ be as in theorem 1.3. Suppose $S : \mathcal{H}' \to \mathcal{H}$ is a bounded operator that satisfies

\begin{equation}
SU' = US, \quad S\pi'(f) = \pi(f)S, \quad (f \in L^\infty(\mathbb{T}))
\end{equation}

Then there exists a unique $h_0 \in L^1(\mathbb{T})$ such that

\begin{equation}
R_{m_0, m_0}'h_0 = h_0
\end{equation}

and

\begin{equation}
\langle \varphi, S\pi'(f)\varphi' \rangle = \int_T fh_0 \, d\mu, \quad (f \in L^\infty(\mathbb{T}))
\end{equation}
Moreover \[ |h_0|^2 \leq \|S\|^2 hh' \text{ almost everywhere on } \mathbb{T}. \]

We will see how each cycle of \( m_0 \) gives rise to a representation of \( \mathfrak{A}_N \) (hence to a positive solution for \( R_{m_0,m_0} h = h \)).

We will also give the concrete form for the cyclic representation corresponding to the constant function 1 when \( m_0 \) satisfies \([1],[2],[3]\). When the wavelets are not orthogonal (the case of tight frames), the representations become more complicated.

2. Peripheral spectral analysis

We begin this section by analyzing the intertwining operators a little bit further. We will see that the commutant of the cyclic representation associated to a positive \( h \) with \( R_{m_0,m_0} h = h \) is abelian and we will find the eigenfunction \( h \) that corresponds to the composition of two intertwining operators that correspond to \( h_1 \) and \( h_2 \) respectively.

In \([4],[5]\), corollary 3.9 it is proved that the cyclic representation \( (\mathcal{H}_h, \pi_h, \varphi_h) \) corresponding to some \( h \geq 0 \) with \( R_{m_0,m_0} h = h \) is given by:

\[
\mathcal{H}_h := \left\{ (\xi_0, ..., \xi_n, ...) \mid \int_{\mathbb{T}} R_{m_0,m_0}^n \left( |\xi_n|^2 h \right) d\mu < \infty, R_{m_0,m_0} (\xi_{n+1} h) = \xi_n h \right\}
\]

\[
\pi_h (f) (\xi_0, ..., \xi_n, ...) = (f(x) \xi_0, ..., f (z^N) \xi_n, ...) , \quad (f \in L^\infty (\mathbb{T}))
\]

\[
U_h (\xi_0, ..., \xi_n, ...) = \left( m_0 (z) \xi_1, ..., m_0 (z^N) \xi_{n+1}, ... \right)
\]

\[
\langle (\xi_0, ..., \xi_n, ...) | (\eta_0, ..., \eta_n, ...) \rangle = \lim_{n \to \infty} \int_{\mathbb{T}} R_{m_0,m_0}^n (\overline{\xi}_n \eta_n h) d\mu
\]

and

\[ \varphi_h = (1,1,1,1, ...). \]

Also, we have the subspaces \( H_0^h \subset H_1^h \subset ... \subset H_n^h \subset \subset \mathcal{H}_h \) whose union is dense in \( \mathcal{H}_h \) where

\[
H_n^h := \left\{ (\xi_0, ..., \xi_n, ...) \in \mathcal{H}_h | \xi_{n+k} (z) = \xi_n (z^{N^k}) \right\}, \text{ for } k \geq 0\]

The set

\[
\gamma_n^h := \left\{ U_{m_0}^{-n} \pi_h (f) \varphi_h | f \in L^\infty (\mathbb{T}) \right\}
\]

is dense in \( H_n^h \) for all \( n \geq 0 \) and \( U_{m_0}^n H_0^h = H_n^h \).

Some notations: if \( m_0 \) and \( h \) are as in theorem \([1],[2],[3]\) then, we denote by \( (\mathcal{H}_h, \pi_h, \varphi_h) \) the cyclic representation associated to \( h \).

If \( m_0, m_0', h, h', \text{ and } h_0 \) are as in theorem \([1],[2],[3]\) then denote by \( S_{h,h',h_0} \) the intertwining operator from \( \mathcal{H}_{h'} \) to \( \mathcal{H}_h \) given by the aforementioned theorem.

Sometime we will omit the subscripts.

**Lemma 2.1.** Let \( P_{H_0^h} \) be the projection onto the subspace \( H_0^h \).

Then \( P_{H_0^h} S_{h,h',h_0} P_{H_0^{h'}} \) is multiplication by \( \frac{h_0}{h} \) on \( H_0^{h'} \) i.e.

\[
P_{H_0^h} S_{h,h',h_0} P_{H_0^{h'}} \left( \xi (z), \xi (z^N), ..., \xi (z^{N^k}), ... \right) =
\]

\[
= \left( \frac{h_0 (z)}{h (z)} \xi (z), \frac{h_0 (z^N)}{h (z^N)} \xi (z^N), ..., \frac{h_0 (z^{N^k})}{h (z^{N^k})} \xi (z^{N^k}), ... \right)
\]
This calculation shows that
\[ \phi_0^S = \frac{\mu_0}{\mu}. \]
Consider again an \( f \in L^\infty(\mathbb{T}) \) arbitrary.
\[ P^n_{H_0^h} SP^n_{H_0^h} \pi_h(f) \phi_h = P^n_{H_0^h} \pi_h(f) \phi_h = P^n_{H_0^h} \pi_h(f) \phi_h = P^n_{H_0^h} \pi_h(f) \phi_h = \]
\[ = P^n_{H_0^h} \left( f(z) \phi_0^S, \ldots, f \left( z^{N^n} \right) \phi_n^S, \ldots \right) = \]
\[ = \left( f(z) \phi_0^S, \ldots, f \left( z^{N^n} \right) \phi_n^S \left( z^{N^n} \right), \ldots \right) \]
This calculation shows that \( P^n_{H_0^h} SP^n_{H_0^h} \) is multiplication by \( \frac{\mu_0}{\mu} \) on \( \mathcal{H}_0^h \), so, by density, on \( H_0^h \).

Lemma 2.2. \( P^n_{H_0^h} \mathcal{H}_h, \mathcal{H}_h \) converges to \( S \mathcal{H}_h, \mathcal{H}_h \) in the strong operator topology.

Proof. Let \( \xi \in \mathcal{H}_h \).
\[ \left\| P^n_{H_0^h} \mathcal{H}_h \xi - \xi \right\| \leq \left\| P^n_{H_0^h} \mathcal{H}_h \xi - P^n_{H_0^h} S \xi \right\| + \left\| P^n_{H_0^h} S \xi - \xi \right\| \]
\[ \leq \left\| P^n_{H_0^h} \right\| \left\| S \right\| \left\| P^n_{H_0^h} \xi - \xi \right\| + \left\| P^n_{H_0^h} S \xi - \xi \right\| \to 0, \text{ as } n \to \infty \]
because the subspaces \( H_0^h \) form an increasing sequence whose union in dense in \( \mathcal{H}_h \)
(and similarly for \( H_0^h \)).

Theorem 2.3. The commutant \( \pi_h(\mathfrak{A}_n)' \) is abelian.

Proof. Consider \( S_1, S_2 \in \pi_h(\mathfrak{A}_n)' \). Then, according to theorem 1.2.1 \( S_1 = S_{h_1}, S_2 = S_{h_2}, \) for some \( h_1, h_2 \) with \( R_{h_0, h} h_1 = h_i, |h_i| \leq c h, i \in \{1, 2\} \). Let \( \xi \in \mathcal{H}_h \).

It has a decomposition \( \xi = \xi_0 + \eta \) with \( \xi_0 \in H_0^h \) and \( \eta \in H_0^h \). Using lemma 2.2.
\[ \left( P^n_{H_0^h} S_1 P^n_{H_0^h} \right) \left( P^n_{H_0^h} S_2 P^n_{H_0^h} \right) \xi = P^n_{H_0^h} S_1 P^n_{H_0^h} S_2 \xi_0 = \]
\[ = P^n_{H_0^h} S_2 P^n_{H_0^h} S_1 \xi_0 = \left( P^n_{H_0^h} S_2 P^n_{H_0^h} \right) \left( P^n_{H_0^h} S_1 P^n_{H_0^h} \right) \xi \]
Since \( P^n_{H_0^h} = U^{-n} P^n_{H_0^h} U^n \) it follows that \( P^n_{H_0^h} S_1 P^n_{H_0^h} \) and \( P^n_{H_0^h} S_2 P^n_{H_0^h} \) also commute. Lemma 2.2 can be used to get \( S_1 S_2 \) as the strong limit of \( \left( P^n_{H_0^h} S_1 P^n_{H_0^h} \right) \left( P^n_{H_0^h} S_2 P^n_{H_0^h} \right) \).

Similarly for \( S_2 S_1 \). And as the limit is unique we must have \( S_1 S_2 = S_2 S_1 \). 

Next, suppose we have two intertwining operators \( S_1 : \mathcal{H}_h \to \mathcal{H}_h', S_2 : \mathcal{H}_h' \to \mathcal{H}_h \) which come from \( h_1 \) and \( h_2 \) respectively. Then \( S_2 S_1 \) is also an intertwining operator so it must come from some \( h_3 \). We want to find the relation between \( h_1, h_2 \) and \( h_3 \).

Theorem 2.4. If \( S_{h_1} : \mathcal{H}_h \to \mathcal{H}_h' \) and \( S_{h_2} : \mathcal{H}_h' \to \mathcal{H}_h'' \) are intertwining operators
then, if \( S_{h_3} = S_{h_2} S_{h_1} \). We have for all \( f \in L^\infty(\mathbb{T}) \):
\[ \int_{\mathbb{T}} |f(z)|^2 R^n_{h_0, h_0} \left( \frac{|h_1| h_2}{h_0 h'} - \frac{|h_3|}{h_0 h''} \right)^2 \mu \to 0 \]
Proof. We begin with a calculation. For $f \in L^\infty(\mathbb{T})$:

$$P_{H_n} S_1 P_{H_n}^{T} (U_{h}^{-n} \pi_h(f) \varphi_h) = U_{h}^{-n} P_{H_n}^{T} U_{h}^{-n} S_1 P_{H_n}^{T} U_{h}^{-n} \pi_h(f) \varphi_h =$$

$$= U_{h}^{-n} P_{H_n}^{T} S_1 P_{H_n} \pi_h(f) \varphi_h$$

$$= U_{h}^{-n} \pi_{h'} \left( f \frac{h_1}{h'} \right) \varphi_{h'}$$

For the second equality we used the fact that $S_1$ is intertwining and for the last one, lemma 2.1

$$\left( P_{H_n}^{T} S_2 P_{H_n}^{T} \right) \left( P_{H_n}^{T} S_1 P_{H_n}^{T} \right) (U_{h}^{-n} \pi_h(f) \varphi_h) =$$

$$= \left( P_{H_n}^{T} S_2 P_{H_n}^{T} \right) U_{h}^{-n} \pi_{h'} \left( f \frac{h_1}{h'} \right) \varphi_{h'}$$

(2.1)

Similarly

$$\left( P_{H_n}^{T} S_2 S_1 P_{H_n}^{T} \right) (U_{h}^{-n} \pi_h(f) \varphi_h) = U_{h}^{-n} \pi_h \left( f \frac{h_3}{h''} \right) \varphi_h$$

Using 2.1, 2.2 and the notation $m_0^{(n)}(z) := m_0(z) m_0(z^N) ... m_0(z^{N-1})$, we have

$$\left\| \left( P_{H_n}^{T} S_2 P_{H_n}^{T} \right) \left( P_{H_n}^{T} S_1 P_{H_n}^{T} \right) (\pi_h(f) \varphi_h) - \left( P_{H_n}^{T} S_2 S_1 P_{H_n}^{T} \right) (\pi_h(f) \varphi_h) \right\|_{H_{h',h''}} =$$

$$= \left\| \left( P_{H_n}^{T} S_2 P_{H_n}^{T} \right) \left( P_{H_n}^{T} S_1 P_{H_n}^{T} \right) U_{h}^{-n} \pi_h \left( f \left( z^{N'} \right) m_0^{(n)} \right) \varphi_h -$$

$$- \left( P_{H_n}^{T} S_2 S_1 P_{H_n}^{T} \right) U_{h}^{-n} \pi_h \left( f \left( z^{N'} \right) m_0^{(n)} \right) \varphi_h \right\|_{H_{h',h''}} =$$

$$= \left\| U_{h}^{-n} \left( \pi_{h'} \left( f \left( z^{N'} \right) m_0^{(n)} \right) \frac{h_1}{h'} \frac{h_2}{h''} \right) \varphi_{h'} -$$

$$- U_{h}^{-n} \left( \pi_{h'} \left( f \left( z^{N'} \right) m_0^{(n)} \right) \frac{h_3}{h''} \right) \varphi_{h'} \right\|_{H_{h',h''}} =$$

$$= \int_T \left| f \right|^2 \left| \frac{m_0^{(n)}(z)}{h_1 h_2 h'} \right|^2 \frac{h_1}{h'} \frac{h_2}{h''} \frac{h_3}{h''} d\mu =$$

$$= \int_T \left| f \right|^2 R_{mn,mo}^{h} \frac{h_1 h_2 h'}{h''} d\mu$$

But, by lemma 2.2 the first term in this chain of equalities converges to 0 for all $f \in L^\infty(\mathbb{T})$ so we obtain the desired conclusion.

**Corollary 2.5.** If $S_{h_1}, S_{h_2} \in \pi_h(\mathbb{A}_N)'$, $S_{h_3} = S_{h_1} S_{h_2}$ and $h \in L^\infty(\mathbb{T})$ then

$$\int_T |g| \left| R_{mn,mo}^{h} \left( \frac{h_1 h_2}{h'} \right) - h_3 \right| d\mu \to 0, \quad (g \in L^\infty(\mathbb{T}))$$
Proof. We will need the following inequality

\( |R_{m_0,m_0}^n(\xi h)|^2 \leq R_{m_0,m_0}^n(|\xi|^2 h) h \)  

This can be proved using Schwartz’s inequality:

\[
|R_{m_0,m_0}^n(\xi h)|^2 = \left| \frac{1}{N^n} \sum_{w/N^n = z} |m_0^{(n)}(w)|^2 \xi(w)h(w) \right|^2 
\leq \left( \frac{1}{N^n} \sum_{w/N^n = z} |m_0^{(n)}(w)|^2 |\xi(w)|^2 h(w) \right) \left( \frac{1}{N^n} \sum_{w/N^n = z} |m_0^{(n)}(w)|^2 h(w) \right) = R_{m_0,m_0}^n(|\xi|^2 h) R_{m_0,m_0}^n h = R_{m_0,m_0}^n(|\xi|^2 h) h.
\]

Now take \( g \in L^\infty(\mathbb{T}) \) and \( f = gh^{1/2} \) in theorem 2.4 (\( h = h' = h'' \)). We have:

\[
\left( \int_T |g| R_{m_0,m_0}^n \left( \frac{h_1 h_2}{h} \right) - h_3 \right) d\mu \right) \leq \int_T |g|^2 \left| R_{m_0,m_0}^n \left( \frac{h_1 h_2}{h} \right) - h_3 \right|^2 d\mu = 
\]

\[
= \int_T |g|^2 R_{m_0,m_0}^n \left( \left( \frac{h_1 h_2}{h} - \frac{h_3}{h} \right) h \right)^2 d\mu 
\leq \int_T |g|^2 h R_{m_0,m_0}^n \left( \frac{h_1 h_2}{h} - \frac{h_3}{h} \right)^2 h d\mu 
= \int_T |f|^2 R_{m_0,m_0}^n \left( \frac{h_1 h_2}{h} - \frac{h_3}{h} \right)^2 h d\mu \to 0
\]

\[\square\]

In the sequel, we consider intertwining operators that correspond to continuous eigenfunctions \( h \). We will prove that if \( h_1 \) and \( h_2 \) are continuous and \( S_{h_3} = S_{h_1} S_{h_2} \) then \( h_3 \) must be also continuous. The fundamental result needed here is from [BraJo]:

**Theorem 2.6.** Let \( m_0 \) be a function on \( \mathbb{T} \) satisfying \( m_0 \in \text{Lip}_1(\mathbb{T}) \), \( R_{m_0,m_0}1 \leq 1 \) and consider the restriction of \( R_{m_0,m_0} \) to \( \text{Lip}_1(\mathbb{T}) \) going into \( \text{Lip}_1(\mathbb{T}) \). It follows that \( R_{m_0,m_0} \) has at most a finite number \( \lambda_1, \ldots, \lambda_p \) of eigenvalues of modulus 1, \( |\lambda_i| = 1 \), and \( R \) has a decomposition

\[ R_{m_0,m_0} = \sum_{i=1}^p \lambda_i T_{\lambda_i} + S, \]

where \( T_{\lambda} \) and \( S \) are bounded operators from \( \text{Lip}_1(\mathbb{T}) \) to \( \text{Lip}_1(\mathbb{T}) \), \( T_{\lambda} \) have finite-dimensional range, and

\[ T_{\lambda}^2 = T_{\lambda}, \quad T_{\lambda_i}T_{\lambda_j} = 0 \text{ for } i \neq j, \quad T_{\lambda_i}S = ST_{\lambda_i} = 0. \]

and there exist positive constants \( M, h \) such that

\[ \|S^n\|_{\text{Lip}_1(\mathbb{T}) \to \text{Lip}_1(\mathbb{T})} \leq M/(1 + h)^n \]
for \( n = 1, 2, \ldots \). Furthermore \( \| R_{m_0, m_0} \|_{\infty \to \infty} \leq 1 \), and there is a constant \( M \) such that
\[
(2.7) \quad \| S^n \|_{\infty \to \infty} \leq M
\]
for \( n = 1, 2, \ldots \).

Finally, the operators \( T_{\lambda_i} \) and \( S \) extend to bounded operators \( C(\mathbb{T}) \to C(\mathbb{T}) \), and the properties (2.4) and (2.5) still hold for this extension. Moreover
\[
\lim_{n \to \infty} S^n f = 0, \quad f \in C(\mathbb{T}),
\]
for \( n = 1, 2, \ldots \).

**Proof.** Everything is contained in [BraJo], theorem 3.4.4, proposition 4.4.4 and its proof. \qed

**Theorem 2.7.** Assume \( m_0 \) is Lipschitz, \( R_{m_0, m_0} \leq 1 \), \( h \geq 0 \) is continuous, \( R_{m_0, m_0} h = h \). If \( S_{h_1}, S_{h_2} \in \pi_h (\mathfrak{A}_N)' \), with \( h_1, h_2 \) continuous and \( S_{h_3} = S_{h_1} S_{h_2} \) then \( h_3 \) is also continuous and
\[
h_3 = T_1 \left( \frac{h_1 h_2}{h} \right) = \lim_{n \to \infty} R_{m_0, m_0} \left( \frac{h_1 h_2}{h} \right), \quad \text{uniformly.}
\]

**Proof.** By corollary (2.6) we have:
\[
(2.8) \quad \int_T g R^n \left( \frac{h_1 h_2}{h} \right) d\mu \to \int_T g h_3 d\mu \quad (g \in L^\infty (\mathbb{T}))
\]

Also, observe that \( \frac{h_1 h_2}{h} \) is continuous because \( |h_1| \leq c_1 h, \ |h_2| \leq c_2 h \) for some positive constants \( c_1, c_2 \), and if \( x_0 \in \mathbb{T} \) with \( h(x_0) = 0 \) then \( h_1(x_0) = 0, h_2(x_0) = 0 \) and \( |\frac{h_1 h_2}{h}| \leq c_2 h_1 \) Relation (2.8) implies that for all \( g \in L^\infty (\mathbb{T}) \)
\[
\int_T \frac{1}{m} \sum_{n=0}^{m-1} R^n \left( \frac{h_1 h_2}{h} \right) d\mu \to \int_T g h_3 d\mu
\]
However, by theorem (2.6) we have
\[
\frac{1}{m} \sum_{n=0}^{m-1} R^n \left( \frac{h_1 h_2}{h} \right) \to T_1 \left( \frac{h_1 h_2}{h} \right), \quad \text{uniformly.}
\]
Therefore \( h_3 = T_1 \left( \frac{h_1 h_2}{h} \right) \).

Next we want to prove that \( R^n \left( \frac{h_1 h_2}{h} \right) \to h_3 \) uniformly. By [BraJo], proposition 4.4.4, this is equivalent to \( T_{\lambda_i} \left( \frac{h_1 h_2}{h} \right) = 0 \) for \( \lambda_i \neq 1 \).

From (2.8) it follows, using theorem (2.6) that
\[
(2.9) \quad \sum_{\lambda_i \neq 1} \lambda_i^n \int_T g T_{\lambda_i} \left( \frac{h_1 h_2}{h} \right) d\mu \to 0
\]
for all \( g \in L^\infty (\mathbb{T}) \).

But \( T_{\lambda_i} \left( \frac{h_1 h_2}{h} \right) \) are eigenvectors corresponding to different eigenvalues so, some are 0 and the rest are linearly independent. For all \( i \) with \( T_{\lambda_i} \left( \frac{h_1 h_2}{h} \right) \neq 0 \) we can find a \( g_i \in L^\infty (\mathbb{T}) \) such that \( \int_T g_i T_{\lambda_i} \left( \frac{h_1 h_2}{h} \right) d\mu = 1 \) and \( \int_T g_i T_{\lambda_j} \left( \frac{h_1 h_2}{h} \right) d\mu = 0 \) for \( \lambda_j \neq \lambda_i \) (this can be obtain from the fact that \( L^\infty (\mathbb{T}) \) is the dual of \( L^1 (\mathbb{T}) \) which contains the vectors \( T_{\lambda_i} \left( \frac{h_1 h_2}{h} \right) \)). Then, if we use (2.8) for \( g_i \), we get that \( \lambda_i^n \to 0 \).
whenever $T_{\lambda_i} \left( \frac{x \lambda_i}{\mu} \right) \neq 0$, $\lambda_i \neq 1$, which is clearly absurd unless all $T_{\lambda_i} \left( \frac{x \lambda_i}{\mu} \right)$ are 0, for $\lambda_i \neq 1$. Thus, as we have mentioned before, this implies that $R^n \left( \frac{x \lambda_i}{\mu} \right) \rightarrow h_i$.

**Corollary 2.8.** If $h \in C(\mathbb{T})$, $h \geq 0$, $R_{m_0,m_0}h = h$ then the space

$$\{ h_0 \in C(\mathbb{T}) \mid R_{m_0,m_0}h_0 = h_0, |h_0| \leq ch \}$$

is a finite dimensional abelian $C^*$-algebra under the pointwise addition and multiplication by scalars, complex conjugation and the product given by $h_1 * h_2$ defined by $S_{h_1}S_{h_2} = S_{h_1}S_{h_2}$.

**Proof.** Everything follows from theorem 2.7 and theorem 2.3. For the finite dimensionality see [BraJo] or [CoRa90]. □

**Remark 2.9.** When $h = 1$ the $C^*$-algebra structure given in corollary 2.8 is the same as the one introduced in [BraJo], theorem 5.5.1.

Now we will show how each $m_0$-cycle (see definition 2.10 below) gives rise to a continuous solution $h \geq 0$, $R_{m_0,m_0}h = h$. In the end we will see that any eigenfunction $R_{m_0,m_0}h = h$ is a linear combination of eigenfunctions coming from such cycles.

**Definition 2.10.** Let $m_0 \in C(\mathbb{T})$. An $m_0$-cycle is a set $\{ z_1, ..., z_p \}$ contained in $\mathbb{T}$ such that $z_i^N = z_i$ for $i \in \{ 1, ..., p - 1 \}$, $z_p^N = z_1$ and $|m_0(z_i)| = \sqrt{N}$ for $i \in \{ 1, ..., p \}$.

First, we consider the eigenfunction that corresponds to the cycle $\{ 1 \}$. This appears in many instances and it is the one that defines the scaling function in the theory of multiresolution approximations (see [Dau92], [BraJo]).

**Proposition 2.11.** Let $m_0 \in Lip_1(\mathbb{T})$ with $m_0(1) = \sqrt{N}$, $R_{m_0,m_0}1 = 1$. Define

$$\varphi_{m_0,1}(x) = \prod_{k=1}^{\infty} \frac{m_0 \left( \frac{x}{2^{k-1}} \right)}{\sqrt{N}}, \quad (x \in \mathbb{R})$$

(i) $\varphi_{m_0,1}$ is a well defined, continuous function and it belongs to $L^2(\mathbb{R})$.

(ii) If $h_{m_0,1} = \text{Per} |\varphi_{m_0,1}|^2$ is Lipschitz (trigonometric polynomial if $m_0$ is one ), where

$$\text{Per}(f)(x) := \sum_{k \in \mathbb{Z}} f(x + 2k\pi), \quad (x \in [0, 2\pi], f : \mathbb{R} \rightarrow \mathbb{C})$$

Also $R_{m_0,m_0}h_{m_0,1} = h_{m_0,1}$, $h_{m_0,1}(1) = 1$, $h_{m_0,1}$ is 0 on every $m_0$-cycle disjoint of $\{ 1 \}$.

(iii) If $U_1 : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, $(U_1 \xi)(x) = \sqrt{N} \xi(Nx)$ and $\pi_1(f) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, $\pi_1(f)(\xi) = f\xi$ for all $f \in L^\infty(\mathbb{T})$, then $(U_1, \pi_1, \varphi_{m_0,1})$ define the cyclic representation corresponding to $h_{m_0,1}$.

(iv) The commutant of the representation from (iii) is

$$\{ M_f \mid f \in L^\infty(\mathbb{R}), f(Nx) = f(x) \text{ a.e. } \}$$

where $M_f$ is the operator of multiplication by $f$.

(v) $h_{m_0,1}$ is minimal, in the sense that if $0 \leq h' \leq ch_{m_0,1}$, $c > 0$, $h'$ continuous and $R_{m_0,m_0}h' = h'$ then there exists a $\lambda \geq 0$ such that $h' = \lambda h_{m_0,1}$.

(vi) If $h \geq 0$ is continuous, $R_{m_0,m_0}h = h$ and $h(1) = 1$ then $h \geq h_{m_0,1}$.
Proof. (i) See [Dau92] or [BraJo].

(ii) See [BraJo], theorem 5.1.1 and lemma 5.5.6.

For (iii) see [Dut]. Also, in (Dut) it is proved that we are dealing with a representation of \( \mathfrak{a}_N \) (it is the Fourier transform of the wavelet representation mentioned in the introduction). We only need to check that \( \varphi_{m_0,1} \) is cyclic for this representation.

Consider \( P \) the projection onto the subspace generated by \( \pi_1 (\mathfrak{a}_N) \varphi_{m_0,1} \). We prove first that \( P \) commutes with the representation. Take \( A \in \pi_1 (\mathfrak{a}_N) \), \( A \) self-adjoint. If \( B \in \pi_1 (\mathfrak{a}_N) \) then \( A(B\varphi_{m_0,1}) \in \pi_1 (\mathfrak{a}_N) \varphi_{m_0,1} \) so \( PA(B\varphi_{m_0,1}) = A(B\varphi_{m_0,1}) \). So \( PAP = AP \). Then

\[
AP = PAP = (PAP)^* = (AP)^* = PA
\]

so \( P \) commutes with \( A \), and since any member of \( \pi_1 (\mathfrak{a}_N) \) is a linear combination of selfadjoint operators from this set, it follows that \( P \) lies in the commutant of the representation. Then, by (iv), \( P = M_f \) for some \( f \in L^\infty (\mathbb{R}) \) with \( f(Nx) = f(x) \) a.e. As \( P \) is a projection \( f^2 = f = \overline{f} \) so \( f = \chi_A \) for some subset \( A \) of the real line. But \( P\varphi_{m_0,1} = \varphi_{m_0,1} \) so \( \varphi_{m_0,1} \chi_A = \varphi_{m_0,1} \) a.e. Since \( \varphi_{m_0,1}(0) = 1 \) and \( \varphi_{m_0,1} \) is continuous, it follows that \( A \) contains a neighbourhood of 0. This, coupled with the fact that \( \chi_A (Nx) = \chi_A (x) \) a.e., imply that \( \chi_A = 1 \) a.e. so \( P \) is the identity and thus \( \pi_1 (\mathfrak{a}_N) \varphi_{m_0,1} \) is dense, which means exactly that \( \varphi_{m_0,1} \) cyclic.

Consider \( h' \) as mentioned in the hypothesis. Then \( h' \) induces a member of the commutant \( S_{h'} \). By (iv), \( S_{h'} = M_{f_{h'}} \) for some \( f_{h'} \in L^\infty (\mathbb{R}) \) with \( f_{h'}(Nx) = f_{h'}(x) \) a.e. We have

\[
\langle \varphi_{m_0,1} | S_{h'} \pi_1 (f) \varphi_{m_0,1} \rangle = \int_\mathbb{T} f_{h'} \ dt, \quad (f \in L^\infty (\mathbb{T}))
\]

which implies that

\[
h' = \text{Per} (\varphi_{m_0,1} S_{h'} \varphi_{m_0,1}) = \text{Per} \left( f_{h'} | \varphi_{m_0,1} |^2 \right)
\]

We prove that \( f_{h'} \) is continuous at 0.

(2.10) \[
h'(x) = f_{h'}(x) |\varphi_{m_0,1}|^2 (x) + \sum_{k \neq 0} f_{h'}(x + 2k\pi) |\varphi_{m_0,1}|^2 (x + 2k\pi)
\]

As

\[
h_{m_0,1}(x) = |\varphi_{m_0,1}|^2 (x) + \sum_{k \neq 0} |\varphi_{m_0,1}|^2 (x + 2k\pi)
\]

and \( h_{m_0,1}(0) = |\varphi_{m_0,1}|^2 (0) = 1 \) and \( h_{m_0,1}, \varphi_{m_0,1} \) are continuous, it follows that

\[
\sum_{k \neq 0} |\varphi_{m_0,1}|^2 (x + 2k\pi) \rightarrow 0 \quad \text{as} \quad x \rightarrow 0
\]

Then, as \( x \rightarrow 0 \),

\[
\left| \sum_{k \neq 0} f_{h'}(x + 2k\pi) |\varphi_{m_0,1}|^2 (x + 2k\pi) \right| \leq \| f_{h'} \|_\infty \sum_{k \neq 0} |\varphi_{m_0,1}|^2 (x + 2k\pi) \rightarrow 0.
\]

Using this in (2.10) we obtain that \( \lim_{x \rightarrow 0} f_{h'}(x) = h'(0) \). But \( f_{h'}(Nx) = f_{h'}(x) \) a.e. so \( f_{h'} = h'(0) \) a.e. which implies that \( h' = h'(0) h_{m_0,1} \).
where \(\alpha\) and after periodization

Proposition 2.12. Let \(m_0 \in \text{Lip}_1(\mathbb{T})\), \(z_0 \in \mathbb{T}\) with \(z_0^N = z_0\), \(m_0(z_0) = \sqrt{N}e^{i\theta_0}\), \(R_{m_0,m_0}1 = 1\). Define

\[
\varphi_{m_0,z_0}(x) = \prod_{k=1}^{\infty} \frac{e^{-i\theta_0} \alpha_{z_0}(m_0)(\frac{x}{N^k})}{\sqrt{N}}, \quad (x \in \mathbb{R})
\]

where \(\alpha_{\rho}(f)(z) = f(\rho z)\) for \(z, \rho \in \mathbb{T}\) and \(f \in L^\infty(\mathbb{T})\).

(i) \(\varphi_{m_0,z_0}\) is a well defined continuous function that belongs to \(L^2(\mathbb{R})\).

(ii) \(h_{m_0,m_0} := \alpha_{z_0}^{-1}\left(\text{Per } |\varphi_{m_0,z_0}|^2\right)\) is Lipschitz (trigonometric polynomial if \(m_0\) is one ), \(R_{m_0,m_0}h_{m_0,m_0} = h_{m_0,z_0}, h_{m_0,z_0} = 1\), \(h_{m_0,z_0} = 0\) on every \(m_0\)-cycle disjoint of \(\{z_0\}\).

(iii) If \(U_{z_0} : L^2(\mathbb{R}) \to L^2(\mathbb{R})\), \(U_{z_0} \xi = e^{i\theta_0}U_1 \xi\) and \(\pi_{z_0}(f)(\xi) = \pi_1(\alpha_{z_0}(f)(\xi)\) for \(f \in L^\infty(\mathbb{T})\), then \((U_{z_0}, \pi_{z_0}, \varphi_{m_0,z_0})\) define the cyclic representation corresponding to \(h_{m_0,z_0}\).
(iv) The commutant of this representation is
\[ \{ M_f \mid f \in L^\infty(\mathbb{R}) , f(Nx) = f(x) \text{ a.e.} \} . \]

(v) \( h_{m_0,z_0} \) is minimal (see proposition 2.11 (iv)).

(vi) If \( h \geq 0 \) is continuous, \( R_{m_0,m_0} h = h \) and \( h(z_0) = 1 \) then \( h \geq h_{m_0,z_0} \).

Proof. Consider \( m_0' := e^{-ih_0} \alpha_{z_0}(m_0) \). We check that \( m_0' \) satisfies the hypotheses of proposition 2.11. Clearly \( m_0' \) is Lipschitz, \( m_0'(1) = \sqrt{N} \),
\[ R_{m_0',m_0'}(z) = \frac{1}{N} \sum_{w \in z} |m_0|^2 (w) \alpha_{z_0}^{-1} (h_{m_0',1}) (w) \]
\[ = \frac{1}{N} \sum_{w \in z} |m_0|^2 (w) h_{m_0',1} (w z_0^{-1}) \]
\[ = \frac{1}{N} \sum_{y \in z_0^{-1}} |m_0|^2 (y z_0) h_{m_0',1} (y) \]
\[ = R_{m_0',m_0'} (z z_0^{-1}) = h_{m_0,z_0}(z) \]

Thus we can apply proposition 2.11 to \( m_0' \).

\( \varphi_{m_0,z_0} = \varphi_{m_0',1} \) and everything follows.

\( h_{m_0,z_0} = \alpha_{z_0}^{-1} (h_{m_0',1}) \)

Also \( h_{m_0,z_0}(z_0) = h_{m_0',1}(z_0 z_0^{-1}) = 1 \) and, if \( C \) is an \( m_0 \)-cycle disjoint of \( \{ z_0 \} \) then \( z_0^{-1} C \) is an \( m_0' \)-cycle disjoint of \( \{ 1 \} \) and again proposition 2.11 applies.

\( \varphi_{m_0,z_0} = \varphi_{m_0',1} \) can also be deduced from proposition 2.11. The relation
\[ U_{z_0} \pi_{z_0} (f) = \pi_{z_0} (f (z N)) U_{z_0} \]
follows from the identity \( \alpha_{z_0} (f (z N)) = \alpha_{z_0} (f) (z N) \).

\( \varphi_{m_0,z_0} \) is given, then \( \alpha_{z_0} (h') \) satisfies: \( 0 \leq \alpha_{z_0} (h') \leq c \alpha_{z_0} (h_{m_0,z_0}) = c h_{m_0',1} \) and \( R_{m_0,m_0} \alpha_{z_0}(h') = \alpha_{z_0} (R_{m_0,m_0} h') = \alpha_{z_0} (h') \). Then, by proposition 2.11
\[ \alpha_{z_0} (h') = \lambda h_{m_0',1} \] for some \( \lambda \geq 0 \) so \( h' = \lambda h_{m_0,z_0} \).

The argument is similar to the one used in \( \varphi_{m_0,z_0} \).

Using proposition 2.12, we are now able to prove that each \( m_0 \)-cycle gives rise to a continuous solution for \( R_{m_0,m_0} h = h \).

Proposition 2.13. Let \( m_0 \in Lip_1(\mathbb{T}) \), \( R_{m_0,m_0} \) = 1 and let \( C = \{ z_1, z_2 = z_1^N, \ldots, z_p = z_{p-1}^N \} \), \( z_N = z_1 \), be an \( m_0 \)-cycle, \( m_0(z_k) = \sqrt{N} e^{i\theta_k} \) for \( k \in \{ 1, \ldots, p \} \). Denote by \( \theta_C = \theta_1 + \ldots + \theta_p \). Define
\[ \varphi_{k,m_0,C}(x) = \prod_{k=1}^{\infty} e^{-i\theta_k} \alpha_{z_k} \left( m_0^{(p)} \left( \frac{x}{N^p} \right) \right) \frac{N}{N^p}, \ (k \in \{ 1, \ldots, p \}) \]

(i) \( \varphi_{k,m_0,C} \) is a well defined continuous function that belongs to \( L^2(\mathbb{R}) \).
(ii) Define $g_{k,m_0,C} = \alpha_{k-1} \left( \operatorname{Per} |\varphi_{k,m_0,C}|^2 \right)$ for all $k \in \{1, \ldots, p\}$. $g_{k,m_0,C}$ is Lipschitz (trigonometric polynomial if $m_0$ is one). Also

$$R_{m_0,m_0}^p g_{k,m_0,C} = g_{k,m_0,C}$$

$$R_{m_0,m_0} g_{k,m_0,C} = g_{k+1,m_0,C}$$

(we will use the notation mod $p$ that is $z_{p+1} = z_1$, $g_{p+2,m_0,C} = g_{2,m_0,C}$ etc.)

$g_{k,m_0,C}(z_j) = \delta_{kj}$, $g_{k,m_0,C}$ is 0 on every $m_0$-cycle disjoint of $C$.

(iii) Define $h_{m_0,C} = \sum_{k=1}^{p} g_{k,m_0,C}$. Then $h_{m_0,C}$ is Lipschitz (trigonometric polynomial if $m_0$ is one). $R_{m_0,m_0} h_{m_0,C} = h_{m_0,C}$, $h_{m_0,C}(z_k) = 1$ for all $k \in \{1, \ldots, p\}$ and $h_{m_0,C}$ is 0 on every $m_0$-cycle disjoint of $C$.

(iv) $h_{m_0,C}$ is minimal.

(v) If $h \geq 0$ is continuous, $R_{m_0,m_0} h = h$ and $h = 1$ on $C$ then $h \geq h_{m_0,C}$.

(vi) If $U_C : L^2(\mathbb{R})^p \to L^2(\mathbb{R})^p$,

$$U_C(\xi_1, \ldots, \xi_p) = (e^{i\theta_1} U_1 \xi_2, \ldots, e^{i\theta_{p-1}} U_1 \xi_p, e^{i\theta_p} U_1 \xi_1)$$

and for $f \in L^\infty(\mathbb{T})$, $\pi_C(f) : L^2(\mathbb{R})^p \to L^2(\mathbb{R})^p$,

$$\pi_C(f)(\xi_1, \ldots, \xi_p) = (\pi_1(\alpha_{z_1}(f))\xi_1, \ldots, \pi_1(\alpha_{z_p}(f))\xi_p)$$

then $(U_C, \pi_C, (\varphi_{1,m_0,C}, \ldots, \varphi_{p,m_0,C}))$ is the cyclic representation corresponding to $h_{m_0,C}$.

(vii) The commutant of this representation is

$$\{ M_{f_1} \oplus \cdots \oplus M_{f_p} \mid f_k \in L^\infty(\mathbb{R}), f_{k+1}(Nx) = f_k(x) \ a.e \, \text{ for } k \in \{1, \ldots, p\} \}$$

Proof. Let $m_0' := m_0^{(p)}$. Observe that

$$m_0'(z_i) = m_0^{(p)}(z_i) = m_0(z_i) m_0(z_{N}^{i}) \cdots m_0(\bar{z}_{i}^{N-1}) =$$

$$m_0(z_1) m_0(z_2) \cdots m_0(z_p) = \sqrt{NP} e^{i\theta C}$$

Note that $R_{m_0',m_0} = R_{m_0,m_0}^p$ so $R_{m_0',m_0}^{(p)} 1 = 1$. Thus follows from proposition [2.4] (replace $N$ by $NP$ when working with $m_0^{(p)}$).

If $y_1, y_2 = y_1^N, \ldots, y_q = y_q^N$ is an $m_0$-cycle, then $\{y_i\}$ is an $m_0^{(p)}$-cycle. Therefore, all assertions in (iii), except the one that relates $g_{k,m_0,C}$ and $g_{k+1,m_0,C}$, follow from proposition [2.4] (iii).

We check now (v). $U_C$ is unitary as a composition of unitary operators. For $f \in L^\infty(\mathbb{T})$ we have:

$$U_C \pi_C(f)(\xi_1, \ldots, \xi_p) = (e^{i\theta_1} \pi_1(\alpha_{z_1}(f)(z_{N}^{i})) U_1 \xi_2, \ldots$$

$$\ldots e^{i\theta_{p-1}} \pi_1(\alpha_{z_p}(f)(z_{N}^{i})) U_1 \xi_p, e^{i\theta_p} \pi_1(\alpha_{z_1}(f)(z_{N}^{i})) U_1 \xi_1)$$

$$= (e^{i\theta_1} \pi_1(\alpha_{z_1}(f)(z_{N}^{i})) U_1 \xi_2, \ldots, e^{i\theta_{p-1}} \pi_1(\alpha_{z_{p-1}}(f)(z_{N}^{i})) U_1 \xi_p, e^{i\theta_p} \pi_1(\alpha_{z_1}(f)(z_{N}^{i})) U_1 \xi_1)$$

$$= \pi_C(f(z_{N}^{i})) U_C(\xi_1, \ldots, \xi_p)$$

Here we used that $\alpha_{z_{i+1}}(f)(z_{N}^{i}) = \alpha_{z_i}(f(z_{N}^{i})).$

We must check also that

$$U_C(\varphi_{1,m_0,C}, \ldots, \varphi_{p,m_0,C}) = \pi_C(m_0)(\varphi_{1,m_0,C}, \ldots, \varphi_{p,m_0,C})$$
To do this observe that
\[\alpha z_1 \left( m_0^{(p)} \right) (z) = \alpha z_1 (m_0(z)) \alpha z_1 (m_0(z^N)) \alpha z_1 \left( m_0 \left( z^{N^{p-1}} \right) \right) = \alpha z_1 (m_0(z)) \alpha z_2 (m_0(z)) \ldots \alpha z_p (m_0(z^{N^{p-1}})) \]

Thus
\[
\varphi_{1,m_0,C}(x) = \frac{e^{-i\theta_1 \alpha z_1 (m_0)(\frac{x}{N})}}{\sqrt{N}} \frac{e^{-i\theta_2 \alpha z_2 (m_0)(\frac{x}{N})}}{\sqrt{N}} \frac{e^{-i\theta_3 \alpha z_3 (m_0)(\frac{x}{N})}}{\sqrt{N}} \ldots
\]

so
\[
\varphi_{1,m_0,C}(x) = \prod_{k=1}^{\infty} \frac{e^{-i\theta_1 \alpha z_1 (m_0)(\frac{x}{N})}}{\sqrt{N}}
\]

Similarly
\[
\varphi_{i,m_0,C}(x) = \prod_{k=1}^{\infty} \frac{e^{-i\theta_1 \alpha z_1 (m_0)(\frac{x}{N})}}{\sqrt{N}} \quad \text{for } i \in \{1, \ldots, p\}
\]

Using this formula we obtain:
\[
U_1 \varphi_{i+1,m_0,C} = \sqrt{N} \varphi_{i+1,m_0,C} (N x)
\]

\[
e^{-i\theta_1 \alpha z_1 (m_0)} \prod_{k=2}^{\infty} e^{-i\theta_{i+1+k} \alpha z_1 \ldots \alpha z_k (m_0)(\frac{x}{N})} \frac{1}{\sqrt{N}}
\]

\[
e^{-i\theta_1 \alpha z_1 (m_0)} \varphi_{1,m_0,C}
\]

which shows that
\[
U C (\varphi_{1,m_0,C}, \ldots, \varphi_{p,m_0,C}) = \pi_C (m_0)(\varphi_{1,m_0,C}, \ldots, \varphi_{p,m_0,C})
\]

Next we compute the commutant. Consider \( A : L^2(\mathbb{R})^p \to L^2(\mathbb{R})^p \) commuting with the representation. Let \( P_i \) be the projection onto the \( i \)-th component, and let \( A_{ij} = P_i A P_j \). Note that
\[
U_C^p (\xi_1, \ldots, \xi_p) = (e^{-i\theta_C U^p} \xi_1, \ldots, e^{-i\theta_C U^p} \xi_p)
\]

Also, since \( z_i^{N^p} = z_i \), \( z_i = \frac{2\pi i k_i}{N^p - 1} \) for some integer \( k_i \). Take any \( \frac{2\pi i}{N^p - 1} \)-periodic essentially bounded function, \( g \). Then \( \alpha_z (g) = g \) so
\[
\pi_C (g) (\xi_1, \ldots, \xi_p) = (\pi_1 (g) \xi_1, \ldots, \pi_p (g) \xi_p)
\]

Then \( P_i \) commute with \( U_C^p \) and \( \pi_C (g) \) so \( A_{ij} \) commute with \( U_C^p \) and \( \pi_1 (g) \) and, using the argument in [Dut] (proof of theorem 4.1), (see also the proof of lemma 2.13 below), it follows that \( A_{ij} = M_{f_{ij}} \) for some \( f_{ij} \in L^\infty(\mathbb{T}) \).

Since \( A \) and \( \pi_C (g) \) commute for all \( f \in L^\infty(\mathbb{T}) \), we have for \( i \in \{1, \ldots, p\} \)
\[
\sum_{j=1}^{p} f_{ij} \pi_1 (\alpha z_j (f)) \xi_j = \pi_1 (\alpha z_i (f)) \sum_{j=1}^{p} f_{ij} \xi_j
\]

Fix \( k \) and take \( \xi_j = 0 \) for all \( j \neq k \), then
\[
f_{ik} \pi_1 (\alpha z_k (f)) \xi_k = \pi_1 (\alpha z_i (f)) f_{ik} \xi_k
\]
so $f_{ik} = 0$ for $i \neq k$. Then, since $A$ commutes with $U$ we have
\[ (e^{i\theta_1} \sqrt{N} f_{22}(Nx) \xi_2(Nx), \ldots, e^{i\theta_{p-1}} \sqrt{N} f_{pp}(Nx) \xi_p(Nx), e^{i\theta_p} \sqrt{N} f_{11}(Nx)) = \]
\[ = (e^{i\theta_1} f_{11}(x) \sqrt{N} \xi_2(Nx), \ldots, e^{i\theta_{p-1}} \sqrt{N} f_{p-1p-1}(x) \xi_p(Nx), e^{i\theta_p} \sqrt{N} f_{pp}(Nx)) \]
Therefore
\[ f_{22}(Nx) = f_{11}(x) \text{ a.e.} \]
\[ f_{33}(Nx) = f_{22}(x) \text{ a.e.} \]
\[ \vdots \]
\[ f_{11}(N(x) = f_{pp}(x) \text{ a.e.} \]
and (vii) follows.

The cyclicity of $(\varphi_{1,m_0,C}, \ldots, \varphi_{p,m_0,C})$ follows as in the proof of proposition 2.11 (iii).

We check that $R_{m_0,m_0} \varphi_{i,m_0,C} = \varphi_{i+1,m_0,C}$. Take $f \in L^\infty(\mathbb{T})$. We have:
\[ \int \mathbb{T} f \varphi_{i+1,m_0,C} \, d\mu = \langle \varphi_{i+1,m_0,C} | \pi_1 (\alpha_{z_{i+1}}(f) \varphi_{i+1,m_0}) \rangle \]
\[ = \langle U_1 \varphi_{i+1,m_0,C} | U_1 \pi_1 (\alpha_{z_{i+1}}(f) \varphi_{i+1,m_0,C}) \rangle = \]
\[ = \langle e^{-i\theta_1} \pi_1 (\alpha_{z_{i}}(m_0)) \varphi_{i,m_0,C} | e^{-i\theta_1} \pi_1 (\alpha_{z_{i+1}}(f) (z^N)) \pi_1 (\alpha_{z_{i}}(m_0)) \varphi_{i,m_0,C} \rangle \]
\[ = \langle \varphi_{i,m_0,C} | \pi_1 (\alpha_{z_{i}} (f (z^N)) \alpha_{z_{i}} (|m_0|^2) \varphi_{i,m_0,C} \rangle \]
\[ = \int \mathbb{T} f (z^N) |m_0|^2 \varphi_{i,m_0,C} \, d\mu = \int \mathbb{T} f(z) R_{m_0,m_0} \varphi_{i,m_0,C} \, d\mu. \]

Hence
\[ R_{m_0,m_0} \varphi_{i,m_0,C} = \varphi_{i+1,m_0,C}. \]

(iii) follows from (ii).

Next we prove that $h_{m_0,C}$ is minimal. Take a continuous $h'$ with $0 \leq h' \leq h_{m_0,C}$.

$R_{m_0,m_0} h' = h'$. Then $R_{m_0,m_0} (p) h' = R_{m_0,m_0} (p) h' = h'$ and
\[ 0 \leq h' \leq c (g_{1,m_0,C} + \ldots + g_{p,m_0,C}) \]
Now we use the fact that the space
\[ \{ g \in C(\mathbb{T}) \mid R_{m_0,p} (p) g = g \} \]
is a $C^*$-algebra isomorphic to $C\{ 1, \ldots, d \}$ for some $d$ (see corollary 2.8), and
by proposition 2.12 (iv) $g_{i,m_0,C}$ are minimal. It follows that $h'$ can be written uniquely as
\[ h' = \alpha_1 g_{1,m_0,C} + \ldots + \alpha_p g_{p,m_0,C} \]
with $\alpha_1, \ldots, \alpha_p \in \mathbb{C}$ (the uniqueness comes from the fact that $g_{i,m_0,C}$ are linearly independent, which, in turn, is implied by (iii)). Then
\[ R_{m_0,m_0} h' = \alpha_1 g_{2,m_0,C} + \ldots + \alpha_{p-1} g_{p,m_0,C} + \alpha_p g_{1,m_0,C} \]
so, by uniqueness $\alpha_1 = \alpha_2 = \ldots = \alpha_p = \alpha_1$ and
\[ h' = \alpha_1 (g_{1,m_0,C} + \ldots + g_{p,m_0,C}) = \alpha_1 h_{m_0,C}. \]

For (viii) we use a similar argument: take $h'$ as given in the hypothesis. Then
\[ R_{m_0,p} (p) h' = R_{m_0,m_0} (p) h' = h', h'(z_i) = 1 \]
for all $i$. Using proposition 2.12 (vii) we get $h' \geq g_{i,m_0,C}$ for all $i$. 

Now we use again the fact that \( \{g \in C(T) \mid R_{m_0^\epsilon(m_0^\epsilon)}g = g\} \) is a \( C^* \)-algebra isomorphic to \( C(\{1, \ldots, d\}) \) and \( g_{i,m_0} \) are minimal, so
\[
h' \geq (g_{i,m_0,C} + \ldots + g_{p,m_0,C}) = h_{m_0,C}
\]
\( \square \)

**Lemma 2.14.** Consider \( m_0, m_0' \) satisfying (1.1)–(1.4). Let \( C: z_1^N = z_2^N = \ldots = z_k^N = z_k^1 \) be an \( m_0 \)-cycle and \( C': z_1^N = z_2^N = \ldots = z_k^N = z_k^1 \) be an \( m_0' \)-cycle, \( m_0(z_k) = \sqrt{N} e^{i\theta_k} \), \( m_0'(z_k') = \sqrt{N} e^{i\theta_k} \) for all \( k \). Consider the cyclic representations associated to this cycles as in proposition 2.13, \((U_C, \pi_C, \gamma_C),(U_{C'}, \pi_{C'}, \gamma_{C'})\) and let \( S: L^2(\mathbb{R})^p \to L^2(\mathbb{R})^p \) be an intertwining operator. Then \( S = 0 \) if \( C \neq C' \). If \( C = C' \) and, after relabeling, \( z_k = z_k' \) for all \( k, p = p' \) then, there exist \( f_1, \ldots, f_p \in L^\infty(\mathbb{R}) \) such that
\[
S(\xi_1, \ldots, \xi_p) = (f_1\xi_1, \ldots, f_p\xi_p)
\]
with
\[
f_1(x) = e^{i(\theta_1 - \theta_1')} f_2(Nx), \text{ a.e.,}
\]
\[
\vdots
\]
\[
f_{p-1}(x) = e^{i(\theta_{p-1} - \theta_{p-1}')} f_p(Nx), \text{ a.e.,}
\]
\[
f_p(x) = e^{i(\theta_p - \theta_p')} f_1(Nx), \text{ a.e.}
\]

**Proof.** Note that
\[
U_C^p = e^{i\theta_C U_1^P} \oplus \ldots \oplus e^{i\theta_C U_1^P}
\]
where \( \theta_C = \theta_1 + \ldots + \theta_p \). Similarly for \( U_{C'}^p \). This shows that \( U_{C}^p \) commutes with the projections \( P_i \) onto the \( i \)-th component.

We have \( S U_C^p = U_{C'}^p S \) so \( (P_i S P_j) U_C^p = U_{C'}^p (P_i S P_j) \), therefore
\[
S_{ij} e^{i\alpha_0 C U_1^p} = e^{i\alpha_{ij} U_1^p} S_{ij}
\]
where \( S_{ij} = P_i S P_j \).

Also, since \( z_k^N = z_k, z_k \) has the form \( e^{i2\pi m} \) for all \( k \) and similarly for \( z_k' \). If we take \( f \in L^\infty(\mathbb{T}) \) to be \( \frac{2\pi}{mm'} \)-periodic, then \( \alpha_{z_k} f = f, \alpha_{z_k'} f = f \) for all \( k \) so
\[
\pi_C(\xi_1, \ldots, \xi_p) = (\pi_1(f)\xi_1, \ldots, \pi_1(f)\xi_p)
\]
\[
\pi_{C'}(\xi_1, \ldots, \xi_p) = (\pi_1(f)\xi_1, \ldots, \pi_1(f)\xi_p)
\]
and again
\[
S_{ij} \pi_1(f) = \pi_1(f) S_{ij}
\]
Hence \( S_{ij} \) commutes with \( \pi_1(f) = M_f \) whenever \( f \in L^\infty(\mathbb{R}) \) is \( \frac{2\pi}{mm'} \)-periodic.

But then also
\[
\left( U_{-pp'} \pi_1(f) U_{pp'}^p \right) S_{ij} = S_{ij} \left( U_{-pp'} \pi_1(f) U_{pp'}^p \right)
\]
and \( U_{-pp'} \pi_1(f) U_{pp'}^p = M_g \) where \( g (N_{pp'}^p x) = f(x) \) for \( x \in \mathbb{R} \) and \( g \) is \( \frac{2\pi}{mm'}N_{pp'}^p \)-periodic. By induction, it follows that \( S_{ij} \) commutes with \( M_f \) whenever \( f \in L^\infty(\mathbb{R}) \) is \( \frac{2\pi}{mm'}N_{pp'}^p \)-periodic, \( l \in \mathbb{N} \).

Now take \( f \in L^\infty(\mathbb{R}) \). Define \( f_1(x) = f(x) \) on \([-\frac{2\pi}{mm'}, \frac{2\pi}{mm'} N_{pp'}^p]\) and extend it to \( \mathbb{R} \) such that \( f_1 \) is \( \frac{2\pi}{mm'}N_{pp'}^p \)-periodic.
We prove that \( M_f \) converges to \( M_f \) in the strong operator topology. Take \( \psi \in L^2(\mathbb{R}) \).

\[
\| M_f \psi - M_f \psi \|_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |f_i - f|^2 |\psi|^2 \, dx
\]

\[
= \int_{|x| \geq \frac{1}{m} N_{N'} |1_{\mathbb{N}_{|N|'}} \cdot |\psi|^2 \, dx
\]

\[
\leq \left( 2 \|f\|_2 \right) \int_{\mathbb{R}} 1_{\{ |x| \geq \frac{1}{m} N_{N'} \}} |\psi|^2 \, dx
\to 0, \text{ as } l \to \infty
\]

Consequently the limit holds and \( M_f \) will commute also with \( S_{ij} \). As \( f \) was arbitrary in \( L^\infty(\mathbb{R}) \), using theorem IX.6.6 in [Con90], we obtain that \( S_{ij} = M_{f_{ij}} \) for some \( f_{ij} \in L^\infty(\mathbb{R}) \).

Having this, we rewrite the intertwining properties. First, we have for all \( f \in L^\infty(\mathbb{T}) \):

\[
\sum_{j=1}^{p'} f_{ij} \alpha_{j'}(f) \xi_j = \alpha_{z_i}(f) \sum_{j=1}^{p'} f_{ij} \xi_j, \quad (i \in \{1,...,p\})
\]

Fix \( k \in \{1,...,p'\} \) and take \( \xi_j = 0 \) for \( j \neq k \). Then

\[
f_{ik} \alpha_{z_i}(f) \xi_k = \alpha_{z_i}(f) f_{ik} \xi_k
\]

Since \( f \in L^\infty(\mathbb{T}) \) is arbitrary, it follows that \( f_{ik} = 0 \) unless \( z'_k = z_i \).

If \( z'_k = z_i \) then we get \( C = C' \). If \( C \neq C' \) then \( C \cap C' = \emptyset \) so \( f_{ij} = 0 \) for all \( i, j \) and \( S = 0 \).

It remains to consider the case \( C = C' \) and, relabeling \( z_k = z'_k \) for all \( k, p = p' \).

Equation (2.12) implies that \( f_{ij} = 0 \) for \( i \neq j \) so

\[
S(\xi_1,...,\xi_p) = (f_{i1}\xi_1,...,f_{ip}\xi_p)
\]

(we used the notation \( f_i = f_{ii} \)).

The fact that \( SU_{C'} = UC'S \) can be rewritten:

\[
f_1(x)e^{i\theta_1} \sqrt{N} \xi_2(Nx) = e^{i\theta_1} \sqrt{N} f_2(Nx) \xi_2(Nx)
\]

\[\vdots\]

\[
f_{p-1}(x)e^{i\theta_{p-1}} \sqrt{N} \xi_p(Nx) = e^{i\theta_{p-1}} \sqrt{N} f_p(Nx) \xi_p(Nx)
\]

\[
f_{p}(x)e^{i\theta_{p'}} \sqrt{N} \xi_1(Nx) = e^{i\theta_{p'}} \sqrt{N} f_1(Nx) \xi_1(Nx)
\]

so

\[
f_1(x) = e^{i(\theta_1-\theta'_1)} f_2(Nx), \text{ a.e.,}
\]

\[\vdots\]

\[
f_{p-1}(x) = e^{i(\theta_{p-1}-\theta'_{p-1})} f_p(Nx), \text{ a.e.,}
\]

\[
f_p(x) = e^{i(\theta_{p}-\theta'_p)} f_1(Nx), \text{ a.e.}
\]
Theorem 2.15. Let $m_0$ satisfy (1.1)-(1.4). Let $C_1, ..., C_n$ be the $m_0$-cycles. Then, each $h \in C(T)$ with $R_{m_0,m_0}h = h$ can be written uniquely as

$$h = \sum_{i=1}^{n} \alpha_i h_{m_0,C_i}$$

with $\alpha_i \in \mathbb{C}$. Moreover $\alpha_i = h|_{C_i}$. In particular

$$1 = \sum_{i=1}^{n} h_{m_0,C_i}$$

Proof. Proposition 2.13 (iii) shows that $h_{m_0,C_i}$ are linearly independent. Since the dimension of \{ $h \in C(T) \mid R_{m_0,m_0}h = h$ \} is $n$ (see [BraJo]), it follows that $h_{m_0,C_i}$ form a basis for this space. so

$$h = \sum_{i=1}^{n} \alpha_i h_{m_0,C_i}$$

for some $\alpha_i \in \mathbb{C}$. An application of proposition 2.13 (iii) shows that $\alpha_i = h|_{C_i}$. □

Theorem 2.16. Suppose $m_0$ satisfies the conditions (1.1)-(1.4). Let $C_1, ..., C_n$ be the $m_0$-cycles. For each $i$ consider $(U_{C_i}, \pi_{C_i}, \varphi_{C_i})$ which give the cyclic representation corresponding to $h_{m_0,C_i}$ (see proposition 2.13). Define

$$U = U_{C_1} \oplus ... \oplus U_{C_n},$$

$$\pi = \pi_{C_1} \oplus ... \oplus \pi_{C_n},$$

$$\varphi = \varphi_{C_1} \oplus ... \oplus \varphi_{C_n}.$$  

Then $(U, \pi, \varphi)$ give the cyclic representation corresponding to the constant function 1. Each element $S$ in the commutant of this representation has the form

$$S = S_{C_1} \oplus ... \oplus S_{C_n}$$

where $S_{C_i}$ is in the commutant of $(U_{C_i}, \pi_{C_i}, \varphi_{C_i})$.

Proof. Since

$$1 = \sum_{i=1}^{n} h_{m_0,C_i}$$

, for the first statement it is enough to check that $\varphi$ is cyclic. For this we will need the commutant and then the reasoning is the same as the one in the proofs of proposition 2.11 (iii) or proposition 2.13 (vi). But lemma 2.14 makes it clear that the elements of the commutant have the form mentioned in the hypothesis (see also the proof of theorem 2.17). We also need to prove that if $S$ is in the commutant, $S = S^2 = S^*$ and $S\varphi = \varphi$ then $S$ is the identity. But,

$$S = S_{C_1} \oplus ... \oplus S_{C_n}$$

so $S_{C_i} = S^2_{C_i} = S^*_{C_i}$ and $S_{C_i}\varphi_{C_i} = \varphi_{C_i}$ and, as $\varphi_{C_i}$ is cyclic in the corresponding representation, it follows that $S_{C_i}$ is the identity so $S = I$. □
Theorem 2.17. Suppose $m_0$ satisfies (1.4). Let $C_1, ..., C_n$ be the $m_0$-cycles, $C_i : z_{1i}, z_{2i} = z_{1i}^N, ..., z_{pi,i} = z_{pi,i}^N$, for $i \in \{1, ..., n\}$. Let $g_{k, m_0, C_i}$ be as in proposition 2.13, $k \in \{1, ..., p_i\}$, $i \in \{1, ..., n\}$.

If $h \in C(T)$, $h \neq 0$ and $R_{m_0, m_0} h = \lambda h$ for some $\lambda \in \mathbb{T}$, then there exists an $i \in \{1, ..., n\}$ such that $\lambda^{p_i} = 1$ and there exist $\alpha_i \in \mathbb{C}$, $i \in \{1, ..., n\}$ such that

$$h = \sum_{i=1}^{n} \alpha_i \left( \sum_{k=1}^{p_i} \lambda^{-k+1} g_{k, m_0, C_i} \right)$$

and $\alpha_i = 0$ if $\lambda^{p_i} \neq 1$.

Proof. First note that instead of $m_0$ we can take $\lvert m_0 \rvert$ and the problem remains the same. We have

$$\frac{1}{N} \sum_{w \in \mathbb{T}} \overline{\lambda m_0(w)} m_0(w) h(w) = h(z), \quad (z \in \mathbb{T})$$

so $R_{\lambda m_0, m_0} h = h$. Using theorem 1.3, it follows that $h$ induces an intertwining operator $S : \mathcal{H}_{m_0} \rightarrow \mathcal{H}_{\lambda m_0}$, where $(\mathcal{H}_{m_0}, \pi_{m_0}, \varphi_{m_0})$ is the cyclic representation corresponding to the constant function $1$ and $m_0$, and $(\mathcal{H}_{\lambda m_0}, \pi_{\lambda m_0}, \varphi_{\lambda m_0})$ is the cyclic representation corresponding to $1$ and $\lambda m_0$.

Using theorem 2.13 and proposition 2.13, we see that $\mathcal{H}_{m_0} = \mathcal{H}_{\lambda m_0}$, $\pi_{m_0}(f) = \pi_{\lambda m_0}(f)$, for $f \in L^\infty(\mathbb{T})$, $\varphi_{m_0} = \varphi_{\lambda m_0}$ and $U_{\lambda m_0} = \lambda U_{m_0}$.

The intertwining property of $S$ implies that

$$SU_{m_0} = \lambda U_{m_0} S$$

$$S\pi_{m_0}(f) = \pi_{m_0}(f) S, \quad (f \in L^\infty(\mathbb{T}))$$

If $P_i$ is the projection onto the components corresponding to the cycle $C_i$ then we see that $P_i$ commutes with both $U_{m_0}$ and $\pi_{m_0}(f)$, for $f \in L^\infty(\mathbb{T})$. Therefore

$$(P_i, S P_i) U_{C_i} = \lambda U_{C_i}, \quad (P_i, S P_i), \quad (P_i, S P_i) \pi_{C_i}(f) = \pi_{C_i}(f) (P_i, S P_i), \quad (f \in L^\infty(\mathbb{T}))$$

Using lemma 2.14 we obtain, $(P_i, S P_i) = 0$ if $i \neq j$ and for each $i \in \{1, ..., n\}$ there exist $f_{1i}, ..., f_{pi,i} \in L^\infty(\mathbb{R})$ such that

$$(P_i, S P_i) (\xi_1, ..., \xi_{pi}) = (f_{1i} \xi_1, ..., f_{pi,i} \xi_{pi}),$$

$$f_{1i}(x) = \lambda f_{2i}(N x) \text{ a.e.,}$$

$$\vdots$$

$$f_{pi-1,i}(x) = \lambda f_{p_i,i}(N x) \text{ a.e.,}$$

$$f_{p_i,i}(x) = \lambda f_{1i}(N x) \text{ a.e..}$$

Also, as

$$\int \varphi_{m_0} \varphi_{m_0}(f) \varphi_{m_0} \varphi_{m_0}, \quad (f \in L^\infty(\mathbb{T}))$$

after periodization we get

$$h = \sum_{i=1}^{n} \sum_{k=1}^{p_i} \alpha_{z_{ki}^k} \left( \operatorname{Per} \left( f_{ki} | \varphi_{k, m_0, C_i} |^2 \right) \right)$$
We want to prove that each \( f_{ki} \) is continuous at \( 0 \). Take \( i \in \{1,\ldots,n\}, k \in \{1,\ldots,p_i\} \).

We know from proposition 2.13 that \( g_{k,m_0,C_i} \) is 1 at \( z_{ki} \) and 0 at every other \( z_{lj} \).

Then
\[
|\alpha_{z_{lj}} \left( \text{Per} \left( f_{lj} \left| \varphi_{l,m_0,C_j} \right|^2 \right) \right) | \leq \| f_{lj} \|_{\infty} g_{l,m_0,C_j}
\]
so this function has limit 0 at \( z_{ki} \) for \((l,j) \neq (i,k)\). The argument used in the proof of proposition 2.11 (v) can be repeated here to obtain that \( \lim_{x \to 0} f_{ki}(x) = h(z_{ki}) \).

On the other hand we have
\[(2.13) \quad f_{ki}(N^{p_i}x) = \lambda^{-p_i} f_{ki}(x) \]
so if we let \( x \to 0 \), we obtain \( h(z_{ki}) = \lambda^{-p_i} h(z_{ki}) \). Consequently, \( h(z_{ki}) = f_{ki} = 0 \) or \( \lambda^{p_i} = 1 \). Since \( h \neq 0 \), there exists an \( i \in \{1,\ldots,n\} \) with \( \lambda^{p_i} = 1 \).

For an \( i \) with \( \lambda^{p_i} \neq 1 \) we have \( f_{ki} = 0 \) for all \( k \in \{1,\ldots,p_i\} \). Now take an \( i \) with \( \lambda^{p_i} = 1 \). From (2.13) and the fact that \( f_{ki} \) is continuous at \( 0 \), it follows that \( f_{ki} \) is constant. Let \( \alpha_i = f_{1i} \). Then \( f_{2i} = \lambda^{-1} \alpha_i, \ldots, f_{p_i} = \lambda^{-p_i+1} \alpha_i \) and the last assertion of the theorem is proved.

**Corollary 2.18.** Let \( m_0 \) as in theorem 2.17. For an eigenvalue \( \lambda \in \mathbb{T} \) and \( i \) with \( \lambda^{p_i} = 1 \), define
\[
h_{m_0,C_i}^\lambda = \sum_{k=1}^{p_i} \lambda^{-k+1} g_{k,m_0,C_i}
\]
Then for each eigenvalue \( \lambda \in \mathbb{T} \), the eigenfunctions \( h_{m_0,C_i}^\lambda \) with \( \lambda^{p_i} = 1 \) are linearly independent. Moreover if we define the measures
\[
\nu_i^\lambda = \frac{1}{p_i} \sum_{k=1}^{p_i} \lambda^{k-1} \delta_{z_{ki}}, \quad i \in \{1,\ldots,n\}, \lambda \in \mathbb{T}, \lambda^{p_i} = 1,
\]
where \( \delta_z \) is the Dirac measure at \( z \), then
\[
T_\lambda(f) = \sum_{i=1, \lambda^{p_i} = 1}^{n} \nu_i^\lambda(f) h_{m_0,C_i}^\lambda.
\]

**Proof.** First, we see that theorem 2.17 implies that \( h_{m_0,C_i}^\lambda \) with \( \lambda^{p_i} = 1 \) span the eigenspace corresponding to the eigenvalue \( \lambda \). Then we also note that, using proposition 2.13 (v) we have:
\[(2.14) \quad \nu_i^\lambda \left( h_{m_0,C_i}^\lambda \right) = \delta_{ij}.
\]
This shows that \( h_{m_0,C_i}^\lambda \) are linearly independent.
On the other hand we have for all \( f \in C(\mathbb{T}) \), using the fact that \( C_i \) is an \( m_0 \)-cycle:

\[
\nu^\lambda_i (R_{m_0,m_0}(f)) = \frac{1}{p_i} \sum_{k=1}^{p_i} \lambda^{k-1} \delta_{z_k} (R_{m_0,m_0}(f)) = \\
= \frac{1}{p_i} \sum_{k=1}^{p_i} \lambda^{k-1} \frac{1}{N} \sum_{w^N = z_k} |m_0(w)|^2 f(w) \\
= \frac{1}{p_i} \sum_{k=1}^{p_i} \lambda^{k-1} \left( \frac{1}{N} \sum_{w^N = z_k} |m_0(z_{k-1,i})|^2 f(z_{k-1,i}) \right) \\
+ \sum_{w^N = z_k, w \neq z_{k-1,i}} |m_0(w)|^2 f(w) \\
= \frac{1}{p_i} \sum_{k=1}^{p_i} \lambda^{k-1} (z_{k-1,i}) \\
= \lambda \nu^\lambda_i (f)
\]

Then, according to theorem 2.6,

\[
\nu^\lambda_i (T_\lambda(f)) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \lambda^{-k} \nu^\lambda_i (R_{m_0,m_0}^k(f)) = \\
= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \lambda^{-k} \lambda^k \nu^\lambda_i (f) \\
= \nu^\lambda_i (f)
\]

This, together with (2.14) and the fact that \( h^\lambda_{m_0,C_i} \) form a basis for the eigenspace, imply the last equality of the corollary. \( \square \)

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E-mail address: ddutkay@math.uiowa.edu