CHARACTERIZATION OF BOUNDEDNESS ON WEIGHTED MODULATION SPACES OF $\tau$-WIGNER DISTRIBUTIONS

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Abstract. This paper is devoted to give several characterizations on a more general level for the boundedness of $\tau$-Wigner distributions acting from weighted modulation spaces to weighted modulation and Wiener amalgam spaces. As applications, sharp exponents are obtained for the boundedness of $\tau$-Wigner distributions on modulation spaces with power weights. We also recapture the main theorems of Wigner distribution obtained in [8, 4]. As consequences, the characterizations of the boundedness on weighted modulation spaces of several types of pseudodifferential operators are established. In particular, we give the sharp exponents for the boundedness of pseudodifferential operators with symbols in Sjöstrand’s class and the corresponding Wiener amalgam spaces.

1. INTRODUCTION

The study of cross-Wigner distribution has a long history. It was first introduced in 1932 in E.Wigner’s ground-breaking paper [27], and then introduced in 1948 by J.Ville [25] in the field of signal analysis. Let us recall the definition as follows.

Given two functions $f_1, f_2 \in L^2(\mathbb{R}^d)$, the cross-Wigner distribution $W(f_1, f_2)$ is defined by

$$W(f_1, f_2)(x, \xi) := \int_{\mathbb{R}^d} f_1(x + \frac{t}{2}) f_2(x - \frac{t}{2}) e^{-2\pi i t \cdot \xi} dt.$$  

Let $T_s$ be the symmetric coordinate change defined by

$$T_s F(x, t) = F(x + \frac{t}{2}, x - \frac{t}{2}),$$

and let $\mathcal{F}_2$ be the partial Fourier transform in the second variable defined by

$$\mathcal{F}_2 F(x, \xi) = \int_{\mathbb{R}^d} F(x, t) e^{-2\pi i t \cdot \xi} dt.$$  

The cross-Wigner distribution can be written as

$$W(f_1, f_2) = \mathcal{F}_2 T_s (f_1 \otimes \bar{f}_2).$$

For $f = f_1 = f_2$, $W f = W(f, f)$ is simply called the Wigner distribution of $f$. For simplicity, in the remaining part of this paper, we omit the word “cross” no matter whether $f_1 = f_2$ or not.

As an important time-frequency representation, the Wigner distribution is closed related to the short-time Fourier transform (STFT) defined by

$$V_g f(x, \xi) := \int_{\mathbb{R}^d} f(t) g(t-x) e^{-2\pi i t \cdot \xi} dt, \quad f, g \in L^2(\mathbb{R}^d).$$

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In fact, a direct calculation shows that
\[ W(f, g)(x, \xi) = 2^d e^{4\pi i x \cdot \xi} V_{Ig} f(2x, 2\xi), \quad f, g \in L^2(\mathbb{R}^d), \]
where \( I g(t) = g(-t) \).

On the other hand, the pseudodifferential operator in the Weyl form, i.e., the Weyl operator \( L_\sigma \) with symbol \( \sigma \in S'(\mathbb{R}^{2d}) \) can be defined by means of duality pairing between the symbol and the Wigner distribution:
\[ \langle L_\sigma f, g \rangle = \langle \sigma, W(g, f) \rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d). \]

The localization operator \( A^{\varphi_1, \varphi_2}_a \) with symbol \( a \in \mathcal{S}'(\mathbb{R}^d) \), analysis window \( \varphi_1 \in \mathcal{S}(\mathbb{R}^d) \), and synthesis window function \( \varphi_2 \in \mathcal{S}(\mathbb{R}^d) \) can be regarded as the Weyl operator whose symbol is the convolution of \( a \) with the Wigner distribution of the windows \( \varphi_1 \) and \( \varphi_2 \):
\[ A^{\varphi_1, \varphi_2}_a = L_{a* W(\varphi_2, \varphi_1)}. \]

For \( \tau \in [0, 1] \), a more general time-frequency representation, namely, the cross-\( \tau \)-Wigner distribution of \( f_1, f_2 \in L^2(\mathbb{R}^d) \) is defined by
\[ W_\tau(f_1, f_2)(x, \xi) := \int_{\mathbb{R}^d} f_1(x + \tau t) f_2(x - (1 - \tau) t) e^{-2\pi i \xi \cdot t} dt. \]
For \( f = f_1 = f_2 \), \( W_\tau f := W_\tau(f, f) \) is simply called the \( \tau \)-Wigner distribution of \( f \). For simplicity, we omit the word “cross” in the remaining part of this paper.

Note that \( \tau \)-Wigner distribution is a generalization of the Wigner distribution. Varying the parameter \( \tau \), \( W_\tau(f_1, f_2) \) is a family of time-frequency representations:
- For \( \tau = 1/2 \), \( W_{1/2}(f_1, f_2) \) becomes the Wigner distribution \( W(f_1, f_2) \).
- For \( \tau = 0 \), \( W_0(f_1, f_2) \) coincides with the Rihaczek distribution \( R(f_1, f_2) \):
\[ W_0(f_1, f_2)(x, \xi) = R(f_1, f_2)(x, \xi) = e^{-2\pi i \xi \cdot f_1(x) \overline{f_2(\xi)}}. \]
- For \( \tau = 1 \), \( W_1(f_1, f_2) \) coincides with the conjugate Rihaczek distribution \( R^*(f_1, f_2) \):
\[ W_1(f_1, f_2)(x, \xi) = R^*(f_1, f_2)(x, \xi) = \overline{R(f_2, f_1)(x, \xi)} = e^{2\pi i \xi \cdot \overline{f_2(x)} \hat{f}_1(\xi)}. \]

By the Rihaczek distribution, the famous Kohn-Nirenberg operator \( K_\sigma \) with symbol \( \sigma \) can be defined weakly as:
\[ \langle K_\sigma f, g \rangle = \langle \sigma, R(g, f) \rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d), \ \sigma \in \mathcal{S}'(\mathbb{R}^{2d}). \]
In general, for \( \tau \in [0, 1] \), the so-called \( \tau \)-operators or Shubin operators [21] can be defined as
\[ \langle \text{OP}_\tau(\sigma) f, g \rangle = \langle \sigma, W_\tau(g, f) \rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d). \]
Note that \( \text{OP}_0(\sigma) \) coincides with the Kohn-Nirenberg operator \( K_\sigma \), \( \text{OP}_{1/2}(\sigma) \) is just the Weyl operator \( L_\sigma \). As the adjoint operator of \( \text{OP}_0(\sigma) \), \( \text{OP}_1(\sigma) \) is also called anti-Kohn-Nirenberg operator.

According to the above relations, the boundedness of several important operators has direct connections with the corresponding boundedness of \( \tau \)-Wigner distributions. Hence, it is important to establish the boundedness results of \( \tau \)-Wigner distribution on function spaces. Among them, the boundedness acting on modulation spaces has its important position, since it has close relationship with time-frequency analysis.

Modulation spaces were invented by H. Feichtinger [11] in 1983. Nowadays, they have been fully recognized as the “right” function spaces for time-frequency analysis. More precisely,
modulation spaces are defined by measuring the decay and integrability of the STFT as following:

\[ M^{p,q}_m(\mathbb{R}^d) = \{ f \in S'(\mathbb{R}^d) : V_g f \in L^{p,q}_m(\mathbb{R}^{2d}) \} \]

endowed with the obvious (quasi-)norm, where \( L^{p,q}_m(\mathbb{R}^{2d}) \) are weighted mixed-norm Lebesgue spaces with the weight \( m \), see Section 2 for more details. We use \( \mathcal{M}^{p,q}_m(\mathbb{R}^d) \) to denote the \( \mathcal{S}(\mathbb{R}^d) \) closure in \( M^{p,q}_m(\mathbb{R}^d) \).

For the power weights

\[ v_s(z) = (z_1)^s(z_2)^t, \quad v_s(z) = (1 + |z|^2)^{s/2} \quad s, t \in \mathbb{R}, \quad z = (z_1, z_2) \in \mathbb{R}^{2d}, \]

the problem for the boundedness of \( \tau \)-Wigner distribution acting from weighted modulation spaces to weighted modulation spaces (BMM) is to find the full range of exponents of \( p_i, q_i, p, q \in (0, \infty), \ s_i, t_i \in \mathbb{R}, \ i = 1, 2 \) for the boundedness:

\[ W_\tau : \mathcal{M}^{p_1,q_1}_{v_{s_1},t_1}(\mathbb{R}^d) \times \mathcal{M}^{p_2,q_2}_{v_{s_2},t_2}(\mathbb{R}^d) \to M^{p,q}_1(\mathbb{R}^{2d}), \]

that is,

\[ \| W_\tau(f_1, f_2) \|_{M^{p,q}_1} \lesssim \| f_1 \|_{\mathcal{M}^{p_1,q_1}_{v_{s_1},t_1}} \cdot \| f_2 \|_{\mathcal{M}^{p_2,q_2}_{v_{s_2},t_2}}, \quad f, g \in \mathcal{S}(\mathbb{R}^d), \]

where we write \( (1 \otimes v_s)(z, \zeta) := v_s(\zeta) \) for \( (z, \zeta) \in \mathbb{R}^{4d} \).

Note that, to avoid the fact that \( \mathcal{S}(\mathbb{R}^d) \) is not dense in some endpoint spaces, such as \( M^{p,q}_m \) with \( p = \infty \) or \( q = \infty \), we only consider the action of \( \tau \)-Wigner distribution on Schwartz function spaces. Similarly, we only consider the action of \( \tau \)-operator \( OP_\tau(\sigma) \) on Schwartz function spaces.

This problem restricted to \( s = s_i = t_i = 0 \), namely,

\[ W : \mathcal{M}^{p_1,q_1}(\mathbb{R}^d) \times \mathcal{M}^{p_2,q_2}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^{2d}) \tag{1.1} \]

was studied by Toft [24, Theorem 4.2], and then refined very recently by Cordero-Nicola [8, Theorem 1.1] and Cordero [4, Theorem 3.2]. In [8, 4], the authors find the sharp conditions for \((1.1)\) of exponents \( p_i, q_i, p, q \in (0, \infty), \ i = 1, 2 \). Under the same conditions, they also obtain the following estimate:

\[ W : \mathcal{M}^{p_1,q_1}_{\| \cdot \|_{1/2}}(\mathbb{R}^d) \times \mathcal{M}^{p_2,q_2}_{\| \cdot \|_{1/2}}(\mathbb{R}^d) \to M^{p,q}_{1/2}(\mathbb{R}^{2d}). \]

In the present paper, our first major goal is to consider BMM problem on a more general level. For suitable weight function \( m, m_1, m_2 \) on \( \mathbb{R}^{2d} \) (see Section 2 for more precise definitions of weights), our first main theorem shows that BMM can be characterized by the corresponding convolution inequalities of discrete mixed-norm spaces.

**Theorem 1.1.** (First characterization of BMM) Assume \( p_i, q_i, p, q \in (0, \infty], \ i = 1, 2, \tau \in [0, 1] \). Suppose that \( m, m_i \in \mathcal{P}(\mathbb{R}^{2d}), \ i = 1, 2 \). We have

\[ W_\tau : \mathcal{M}^{p_1,q_1}_{m_1}(\mathbb{R}^d) \times \mathcal{M}^{p_2,q_2}_{m_2}(\mathbb{R}^d) \to M^{p,q}_{1/2}(\mathbb{R}^{2d}) \]

if and only if for all \( \bar{a}, \bar{b} \),

\[ \|(a_{k_1,k_2}b_{n_1-k_1,n_2-k_2})\|_{\| \cdot \|_{1/2}} \lesssim \| \bar{a} \|_{\mathcal{M}^{p_1,q_1}_{m_1}(\mathbb{Z}^{2d})} \| \bar{b} \|_{\mathcal{M}^{p_2,q_2}_{m_2}(\mathbb{Z}^{2d})}. \tag{1.2} \]

In particular, for \( p < \infty \), this is equivalent to

\[ \mathcal{M}^{p_1/q_1}_{m_1/p}(\mathbb{Z}^{2d}) \ast \mathcal{M}^{p_2/q_2}_{m_2/p}(\mathbb{Z}^{2d}) \subset \mathcal{M}^{p,q}_{m_1/m_2}(\mathbb{Z}^{2d}). \tag{1.3} \]

Here we write \( \mathcal{M}_{m_2}(z) = m_2(-z), \ m_f(z) = m(Jz) = m(z_2, -z_1) \) for \( z \in \mathbb{Z}^{2d} \), where \( J \) is the symplectic matrix (see Section 2).
Furthermore, for submultiplicative weight $m$ and variable-recipearable weights $m_1$ and $m_2$, namely,
\[ m(z_1 + n_1, z_2 + n_2) \lesssim m(z_1, z_2)m(n_1, n_2), \quad m_1 = \omega_1 \otimes \mu_1, \quad m_2 = \omega_2 \otimes \mu_2, \]
our second main theorem shows that BMM can be further characterized by some convolution or embedding inequalities of discrete norm spaces.

**Theorem 1.2. (Second characterisation of BMM)** Assume $p_i, q_i, p, q \in (0, \infty)$, $i = 1, 2$, $\tau \in [0, 1]$. Suppose that $m \in \mathcal{P}(\mathbb{R}^{2d})$ is submultiplicative, $\omega_i, \mu_i \in \mathcal{P}(\mathbb{R}^d)$, $i = 1, 2$. We have
\[ W_\tau : \mathcal{M}^{p_1,q_1}_{\omega_1,\mu_1}(\mathbb{R}^d) \times \mathcal{M}^{p_2,q_2}_{\omega_2,\mu_2}(\mathbb{R}^d) \rightarrow \mathcal{M}^{p,q}_{1\otimes m}(\mathbb{R}^{2d}) \]
if and only if
\begin{align*}
&p_1^{1/p}(Z^d) \ast p_2^{1/p}(Z^d) \subset l_1^{q/p}(Z^d), \quad l_1^{p_1}(Z^d) \ast l_1^{p_2}(Z^d) \subset l_1^{q/p}(Z^d), \quad p < \infty, \quad (1.4) \\
&l_1^{p_1}(Z^d), \quad l_1^{p_2}(Z^d) \subset l_1^{q/p}(Z^d), \quad l_1^{q_1}(Z^d), \quad l_1^{q_2}(Z^d) \subset l_1^{q_2}(Z^d), \quad p \geq q. \quad (1.5)
\end{align*}

Here, we write $m_\alpha(z_1) = m(z_1, 0)$ and $m_\beta(z_2) = m(0, z_2)$ for $z_1, z_2 \in \mathbb{R}^d$.

As an application, we return to the case of power weights. Our third characterization shows that in this case BMM can be characterized by some convolution or embedding inequalities of discrete norm spaces with power weights. Some further characterizations of exponents will be shown in Section 5.

**Theorem 1.3. (Third characterisation of BMM)** Assume $p, q, p_i, q_i \in (0, \infty)$, $s_i, t_i \in \mathbb{R}$, $i = 1, 2$, $\tau \in [0, 1]$. We have
\[ W_\tau : \mathcal{M}^{p_1,q_1}_{v_i,\tau_i}(\mathbb{R}^d) \times \mathcal{M}^{p_2,q_2}_{v_2,\tau_2}(\mathbb{R}^d) \rightarrow \mathcal{M}^{p,q}_{1\otimes v}(\mathbb{R}^{2d}) \]
if and only if
\begin{align*}
&l_1^{p_1}(Z^d) \ast l_1^{p_2}(Z^d) \subset l_1^{q/p}(Z^d), \quad l_1^{q_1}(Z^d), \quad l_1^{q_2}(Z^d) \subset l_1^{q_2}(Z^d), \quad p < \infty, \quad (1.6) \\
&l_1^{p_1}(Z^d), \quad l_1^{p_2}(Z^d) \subset l_1^{q/p}(Z^d), \quad l_1^{q_1}(Z^d), \quad l_1^{q_2}(Z^d) \subset l_1^{q_2}(Z^d), \quad p \geq q. \quad (1.7)
\end{align*}

It is well known that the boundedness property of $\tau$-pseudodifferential operators with symbols in modulation spaces are independent with $\tau \in [0, 1]$, see [23, Remark 1.5]. As expected, the characterizations of BMM is also independent of $\tau$ as shown in Theorems 1.1 to 1.3. However, situation changes in the problem of the boundedness of $\tau$-Wigner distribution acting from weighted modulation spaces to weighted Wiener amalgam spaces (BMW).

In this paper, we consider the Wiener amalgam spaces $W(\mathcal{F}L^p, L^q_m)(\mathbb{R}^{2d})$, which are the image of modulation spaces $M^{p,q}_{1\otimes m}(\mathbb{R}^{2d})$ under the Fourier transform, see the next section for its precise definition. In contrast with the fruitful works on BMM, there are only few results of BMW. In [9, 6], some sufficient conditions of BMW (with weight on the first component of Wiener amalgam space, namely, $W(\mathcal{F}L^p_m, L^q)(\mathbb{R}^{2d})$) was established for $\tau \in (0, 1)$, a negative result of a special form of BMW (see [9, Proposition 4.4]) was shown for $\tau = 0, 1$.

Our second major goal is to give some characterizations for BMW on a general level. For suitable weight function $m, m_1, m_2$ on $\mathbb{R}^{2d}$ (see Section 2 for more precise definitions of weights), our first main theorem for BMW is as follows.

**Theorem 1.4. (First characterization of BMW)** Assume $p_i, q_i, p, q \in (0, \infty)$, $i = 1, 2$, $\tau \in [0, 1]$. Suppose that $m, m_i \in \mathcal{P}(\mathbb{R}^{2d})$, $i = 1, 2$. Denote $m(\zeta_1, \zeta_2) = m((1 - \tau)\zeta_1, \tau\zeta_2)$, $m_2(z_1, z_2) = m_2(1 - \tau z_1, \tau z_2)$. We have
\[ W_\tau : \mathcal{M}^{p_1,q_1}_{m_1}(\mathbb{R}^d) \times \mathcal{M}^{p_2,q_2}_{m_2}(\mathbb{R}^d) \rightarrow W(\mathcal{F}L^p, L^q_m)(\mathbb{R}^{2d}) \]
if and only if for all $\tilde{a}, \tilde{b}$,
\[
\|(a_{k_1,k_2}b_{n_1-k_1,n_2-k_2})\|_{1,0,\tau}^{p,q} \lesssim \|\tilde{a}\|_{l_1^{p_1,q_1}(\mathbb{Z}^{2d})}^{p_1,q_1} \|\tilde{b}\|_{l_2^{p_2,q_2}(\mathbb{Z}^{2d})}^{p_2,q_2} \quad \tau \in (0,1),
\]
\[
\|(a_{n_1,k_2}b_{k_2,n_2})\|_{1,0,\tau}^{p,q} \lesssim \|\tilde{a}\|_{l_1^{p_1,q_1}(\mathbb{Z}^{2d})}^{p_1,q_1} \|\tilde{b}\|_{l_2^{p_2,q_2}(\mathbb{Z}^{2d})}^{p_2,q_2} \quad \tau = 0,
\]
\[
\|(a_{n_1,k_2}b_{k_2,n_2})\|_{1,0,\tau}^{p,q} \lesssim \|\tilde{a}\|_{l_1^{p_1,q_1}(\mathbb{Z}^{2d})}^{p_1,q_1} \|\tilde{b}\|_{l_2^{p_2,q_2}(\mathbb{Z}^{2d})}^{p_2,q_2} \quad \tau = 1.
\]

In particular, for $p < \infty$, the condition (1.8) is equivalent to
\[
\|p_1/p,q_1/p(\mathbb{Z}^{2d}) \ast p_2/p,q_2/p(\mathbb{Z}^{2d}) \subseteq \|p,q/p(\mathbb{Z}^{2d}).
\]

As in the case of BMM, if $m$ is submultiplicative, $m_1$ and $m_2$ are variable-separable, we obtain a further characterization of BMW. See the definition of $l_{m}^{(p,q)}$ in Definition 2.13.

**Theorem 1.5. (Second characterization of BMW)** Assume $p_i,q_i,p,q \in (0,\infty]$, $i = 1,2$, $\tau \in [0,1]$. Suppose that $m \in \mathcal{P}(\mathbb{R}^d)$ is submultiplicative, $\omega_i,\mu_i \in \mathcal{P}(\mathbb{R}^d)$, $i = 1,2$. Denote
\[
\tilde{m}(\zeta_1,\zeta_2) = m((1-\tau)\zeta_1,\sigma_2), \tilde{\omega}_2(z_1) = \omega_2(1/\tau, z_1), \tilde{\mu}_2(z_2) = \mu_2(1/\tau, z_2).
\]
We have
\[
W_{\tau} : M_{\omega_1^{q_1}\mu_1}^{p_1,q_1}(\mathbb{R}^d) \times M_{\omega_2^{q_2}\mu_2}^{p_2,q_2}(\mathbb{R}^d) \longrightarrow W(\mathcal{F}L^p, L^q_m)(\mathbb{R}^{2d})
\]
if and only if
\[
\|p_1/p_2,q_1/p_2(\mathbb{Z}^{2d}) \ast p_2/p_1,q_2/p_1(\mathbb{Z}^{2d}) \subseteq \|p,q/p(\mathbb{Z}^{2d}), \quad p < \infty,
\]
\[
\|p_1,p_2,q_1,q_2(\mathbb{Z}^{2d}) \subseteq \|p,q(\mathbb{Z}^{2d}), \quad p \geq q,
\]
for $\tau \in (0,1)$, and
\[
\|p_1,q_1(\mathbb{Z}^{2d}) \subseteq \|q_1(\mathbb{Z}^{2d}), \|p_2,q_2(\mathbb{Z}^{2d}) \subseteq \|p_2(\mathbb{Z}^{2d}), \quad \tau = 0,
\]
\[
\|p_2,q_2(\mathbb{Z}^{2d}) \subseteq \|p_1,q_1(\mathbb{Z}^{2d}), \|p_2,q_2(\mathbb{Z}^{2d}) \subseteq \|p_2,q_2(\mathbb{Z}^{2d}), \quad \tau = 1.
\]
Here, we write $m(z_1) = \tilde{m}(z_1,0)$ and $m(z_2) = \tilde{m}(0, z_2)$ for $z_1, z_2 \in \mathbb{R}^d$.

For the case of power weight, we have following further characterization. See the characterizations of exponents in Section 5.

**Theorem 1.6. (Third characterization of BMW)** Assume $p_i,q_i,p,q \in (0,\infty]$, $s_i,t_i \in \mathbb{R}$, $i = 1,2$, $\tau \in [0,1]$. We have
\[
W_{\tau} : M_{s_1,t_1}^{p_1,q_1}(\mathbb{R}^d) \times M_{s_2,t_2}^{p_2,q_2}(\mathbb{R}^d) \longrightarrow W(\mathcal{F}L^p, L^q_m)(\mathbb{R}^{2d})
\]
if and only if
\[
\|p_1/q_1,p_2/q_2(\mathbb{Z}^{2d}) \subseteq \|q_1/p_1,q_2/p_2(\mathbb{Z}^{2d}), \quad p < \infty,
\]
\[
\|p_1/p_2,q_1/q_2(\mathbb{Z}^{2d}) \subseteq \|p_2/p_1,q_2/p_1(\mathbb{Z}^{2d}), \quad p \geq q,
\]
for $\tau \in (0,1)$, and
\[
\|p_1,q_1(\mathbb{Z}^{2d}) \subseteq \|p(\mathbb{Z}^{2d}), \|p_2,q_2(\mathbb{Z}^{2d}) \subseteq \|p_2(\mathbb{Z}^{2d}), \quad \tau = 0,
\]
\[
\|p_2,q_2(\mathbb{Z}^{2d}) \subseteq \|p_1,q_1(\mathbb{Z}^{2d}), \|p_2,q_2(\mathbb{Z}^{2d}) \subseteq \|p_2,q_2(\mathbb{Z}^{2d}), \quad \tau = 1.
\]
As mentioned in the beginning of this paper, the boundedness property of Wigner distribution has closed connections with some important operators, for which we can deduce fruitful new boundedness results from our main Theorems 1.1 to 1.6. Here, we focus on the boundedness of pseudodifferential operators with symbols in modulation and Wiener amalgam spaces.

Let us mention that the study of pseudodifferential operators has a long history in the field of classical harmonic analysis, we refer the reader to the pioneering works of Kohn–Nirenberg [19] and Hörmander [18]. See also the famous Hörmander class in [18]. The classical Calderon-Vaillancourt theorem [3] gives the $L^2$-boundedness of Kohn-Nirenberg operator with symbols belonging to the Hörmander’s class $S^{0,0}_0$, in which all the derivatives of symbols are required to be bounded.

In the field of time-frequency analysis, the earliest work of pseudodifferential operators is due to Sjöstrand [22], where the boundedness on $L^2$ of pseudodifferential operators with symbols in $M^{\infty,1}$ (Sjöstrand’s class) was obtained. Since $S^{0,0}_0 \subsetneq M^{\infty,1}$, Sjöstrand’s result essentially extended the Calderon-Vaillancourt theorem. Then, Gröchenig–Heil [15] and Gröchenig [13] extended Sjöstrand’s result to the boundedness on all modulation spaces $M^{p,q}$ with $1 \leq p, q \leq \infty$.

In this paper, we consider the problems for the boundedness on modulation spaces of pseudodifferential operators with symbols in modulation spaces (BPM) and the boundedness on modulation spaces of pseudodifferential operators with symbols in Wiener amalgam spaces (BPW). By an equivalent characterization between BMM (or BMW) and BPM (or BPW), we give several characterizations for BPM and BPW. See Section 6 for more details.

We also point out that our methods and theorems for BPM and BPW can be extended to the bilinear and even multilinear cases. See [2, 1] for the boundedness on modulation spaces of multilinear pseudodifferential operators with symbols in modulation spaces, and see a recent contribution in [20] for symbols in some modified modulation spaces. We may revisit this topic of multilinear cases in the future.

The rest of this paper is organized as follows. In Section 2, we recall some definitions of function spaces we shall use. We also list some basic time-frequency representations associated with Wigner distribution, and recall the Gabor expansion of modulation spaces, which are the key tools for our first characterizations in Theorems 1.1 and 1.4.

Section 3 is devoted to the first characterizations of BMM and BMW. First, Theorem 1.1 is proved by the help of the time-frequency tools mentioned in Section 2. Then, we establish the relations between BMM and BMW in Proposition 3.3. Combining this with Theorem 1.1, we give the proof for the non-endpoint case of Theorem 1.4. Like Theorem 1.1, the endpoint case of 1.4 will be proved directly by the time-frequency tools.

In Section 4, under some reasonable assumptions of weights, we give further characterizations of BMM and BMW. The separation of convolution inequality, i.e. Proposition 4.1, yields the proof for Theorem 1.2 and the non-endpoint case of Theorem 1.5. The separation of mixed-norm embedding inequality, i.e. Proposition 4.2, yields the proof for the endpoint case of Theorem 1.5.

The power weight case will be handled in Section 5. The proof for Theorem 1.3 and the non-endpoint case of Theorem 1.6 follows directly by Theorems 1.2 and 1.5. The proof of endpoint case of Theorem 1.6 follows by the further separation of mixed-norm embedding, namely, Proposition 5.10. We also list Lemmas 5.1, 5.2 and 5.7 for further exponent characterizations. Then, several characterizations of exponents are established for BMM and BMW, see Theorems
5.3, 5.4, 5.5, 5.6, 5.8 and 5.9 for the sharp exponents of BMM and non-endpoint cases of BMW. See Theorem 5.11 for the sharp exponents of endpoint cases of BMW.

In Section 6, by establishing some equivalent relations in Propositions 6.1 and 6.2, we give several useful characterizations of BPM and BPW. In particular, the sharp exponents of unweighted version of BPM and BPW will be given in Theorems 6.3 and 6.10. The sharp exponents for BPM with Sjöstrand’s class and for BPW with symbols in weighted version of BPM and BPW will be given in Theorems 6.6, 6.7 and 6.11, and Remarks 6.8 and 6.12 will be prepared for the comparisons between them. At the end of this section, we give the sharp exponents for the boundedness on Sobolev spaces $H^s$ of pseudodifferential operators with symbols in Wiener amalgam spaces, showing that the boundedness on Sobolev spaces can not happen with $W(\mathscr{F}L^1, L^1)(\mathbb{R}^{2d})$ symbols.

**Notations:** Throughout this paper, we will adopt the following notations. Let $C$ be a positive constant that may depend on $d, p, q, p_i, q_i, s, t, m, m_i, \omega, \mu_i, (i = 1, 2)$. The notation $X \lesssim Y$ denotes the statement that $X \leq CY$, and The notation $X \sim Y$ means the statement $X \lesssim Y \lesssim X$. The Schwartz function space is denoted by $S(\mathbb{R}^d)$, and the space of tempered distributions by $S'(\mathbb{R}^d)$. We use the brackets $\langle f, g \rangle$ to denote the extension to $S'(\mathbb{R}^d) \times S(\mathbb{R}^d)$ of the inner product $(f, g) = \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx$ for $f, g \in L^2(\mathbb{R}^d)$. We set $Tf(x) = f(-x)$ and $D\lambda f(x) = f(\lambda x)$ for $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^d$.

### 2. PRELIMINARIES

**2.1. Time-frequency representations.** The translation operator $T_x$ and modulation operator $M_\xi$ are defined as

$$T_x f(t) = f(t - x), \quad M_\xi f(t) = e^{2\pi i \xi t} f(t).$$

We recall that, as a bilinear map on $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, the STFT $V_g f$ can be extended to be a map from $S'(\mathbb{R}^d) \times S(\mathbb{R}^d)$ into $S'(\mathbb{R}^{2d})$ by

$$V_g f(x, \xi) = \langle f, M_\xi T_x g \rangle.$$

In fact, for $f \in S'(\mathbb{R}^d)$ and $g \in S(\mathbb{R}^d)$, $V_g f$ is a continuous function on $\mathbb{R}^{2d}$ with polynomial growth, see [14, Theorem 11.2.3]. The so-called fundamental identity of time-frequency analysis is as follows:

$$V_g f(x, \xi) = e^{-2\pi i x \cdot \xi} V_g f(x, -\xi), \quad (x, \xi) \in \mathbb{R}^{2d}.$$

Next, we calculate the linear transform of STFT.

**Lemma 2.1** (Linear transform of STFT). Assume $f, g \in L^2(\mathbb{R}^d)$. Let $L$ be a invertible linear transform on $\mathbb{R}^d$. For a function $f$, denote $f_L(x) := f(Lx)$. We have

$$V_{\phi_L} f_L(x, \xi) = |\det(L)|^{-1} V_{\phi} f(Lx, (L^{-1})^T \xi).$$

**Proof.** By a direct calculation, we have

$$V_{\phi_L} f_L(x, \xi) = \int_{\mathbb{R}^d} f(Lt) \overline{(Lt - Lx)e^{-2\pi i Lt 
}(L^{-1})^T \xi}} dt$$

$$= |\det(L)|^{-1} \int_{\mathbb{R}^d} f(t) \overline{(t - Lx)e^{-2\pi i t 
}(L^{-1})^T \xi}} dy = |\det(L)|^{-1} V_{\phi} f(Lx, (L^{-1})^T \xi).$$

$\square$
In the next lemma, we calculate the STFTs of τ-Wigner distributions, which are the key tools for the estimates of τ-Wigner distributions on modulation spaces. We refer the readers to [6] for the process of calculations.

**Lemma 2.2** (STFT of τ-Wigner distribution). Consider τ ∈ [0, 1]. Let \( \Phi_\tau = W_\tau(\phi_1, \phi_2) \) for nonzero functions \( \phi_1, \phi_2 \in S(\mathbb{R}^d) \). Then the STFT of \( W_\tau(f_1, f_2) \) with respect to the window \( \Phi_\tau \) is given by

\[
V_{\Phi_\tau}(W_\tau(f_1, f_2))(z, \zeta) = e^{-2\pi i z_1 \zeta_2} V_{\phi_1} f_1(z_1 - \tau \zeta_2, z_2 + (1 - \tau) \zeta_1) V_{\phi_2} f_2(z_1 + (1 - \tau) \zeta_2, z_2 - \tau \zeta_1).
\]

In particular, for \( \tau = 0 \),

\[
V_{\Phi_0}(W_0(f_1, f_2))(z, \zeta) = e^{-2\pi i z_1 \zeta_2} V_{\phi_1} f_1(z_1, z_2 + \zeta_1) V_{\phi_2} f_2(z_1 + \zeta_2, z_2).
\]

For \( \tau = 1 \), we have

\[
V_{\Phi_1}(W_1(f_1, f_2))(z, \zeta) = e^{-2\pi i z_1 \zeta_2} V_{\phi_1} f_1(z_1 - \zeta_2, z_2) V_{\phi_2} f_2(z_1, z_2 - \zeta_1).
\]

For \( \tau = \frac{1}{2} \), we have

\[
V_{\Phi_2}(W_2(f_1, f_2))(z, \zeta) = e^{-2\pi i z_1 \zeta_2} V_{\phi_1} f_1(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2}) V_{\phi_2} f_2(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2}),
\]

where \( J \) is the canonical symplectic matrix in \( \mathbb{R}^{2d} \) defined by

\[
J = \begin{pmatrix}
0_{d \times d} & I_{d \times d} \\
-I_{d \times d} & 0_{d \times d}
\end{pmatrix}.
\]

**Lemma 2.3** (Connection between STFT and τ-Wigner distribution I). For \( \tau \in (0, 1) \), \( f_1, f_2 \in L^2(\mathbb{R}^d) \), we have

\[
W_\tau(f_1, f_2)(x, \xi) = \tau^{-d} e^{2\pi i \tau^{-1} x \cdot \xi} V_{D_1^{1-\tau} f_2} f_1(\frac{1}{1 - \tau} x, \frac{1}{\tau} \xi).
\]

**Proof.**

\[
W_\tau(f_1, f_2)(x, \xi) = \int_{\mathbb{R}^d} f_1(x + \tau t) f_2(x - (1 - \tau) t) e^{-2\pi i t \cdot \xi} dt
\]

\[
= \int_{\mathbb{R}^d} f_1(x + \tau t) (D_1^{1-\tau} I f_2)(\tau t - \frac{\tau}{1 - \tau} x) e^{-2\pi i t \cdot \xi} dt
\]

\[
= \tau^{-d} \int_{\mathbb{R}^d} f_1(x + t) (D_1^{1-\tau} I f_2)(t - \frac{\tau}{1 - \tau} x) e^{-2\pi i t \cdot \xi} dt
\]

\[
= \tau^{-d} e^{2\pi i \tau^{-1} x \cdot \xi} \int_{\mathbb{R}^d} f_1(t) (D_1^{1-\tau} I f_2)(t - \frac{1}{1 - \tau} x) e^{-2\pi i \tau^{-1} t \cdot \xi} dt
\]

\[
= \tau^{-d} e^{2\pi i \tau^{-1} x \cdot \xi} V_{D_1^{1-\tau} I f_2} f_1(\frac{1}{1 - \tau} x, \frac{1}{\tau} \xi).
\]

**Lemma 2.4** (Connection between STFT and τ-Wigner distribution II).

\[
\mathcal{F} W_\tau(f_1, f_2)(z) = e^{-2\pi i z_1 \cdot z_2} V_{f_2} f_1(-J z)
\]
Proof. Write
\[ W_\tau(f_1, f_2)(x, \xi) = \mathcal{F}_2^{-1}(f \cdot (x - \tau \cdot) f_2(x + (1 - \tau) \cdot))(\xi). \]
Then,
\[ \mathcal{F}W_\tau(f_1, f_2)(z) = \mathcal{F}_1(f_1(\cdot - \tau z_2) f_2(\cdot + (1 - \tau) z_2))(z_1) \]
\[ = \int_{\mathbb{R}^d} f_1(x - \tau z_2) f_2(x + (1 - \tau) z_2) e^{-2\pi i x z_1} dx \]
\[ = e^{-2\pi i z_1 \cdot z_2} \int_{\mathbb{R}^d} f_1(x) f_2(x + z_2) e^{-2\pi i x z_1} dx \]
\[ = e^{-2\pi i z_1 \cdot z_2} V_2 f_1(-z_2, z_1) = e^{-2\pi i z_1 \cdot z_2} V_2 f_1(-J z). \]

2.2. Function spaces. As we mentioned above, modulation spaces are defined as a measure of the STFT of \( f \in S' \). In order to draw a more accurate portrait of the decay and summability properties of STFT, modulation space is usually be measured by the weighted norm. Let us recall the definitions of weights we shall use.

A weight function \( v \) on \( \mathbb{R}^d \) is called submultiplicative if \( v(z_1 + z_2) \leq v(z_1)v(z_2) \) for all \( z_1, z_2 \in \mathbb{R}^d \), a weight function \( m \) on \( \mathbb{R}^d \) is called \( v \)-moderate if
\[ m(z_1 + z_2) \leq C v(z_1)m(z_2), \quad z_1, z_2 \in \mathbb{R}^d. \]
In this paper, we consider the \( v \)-moderate weights where \( v \) is submultiplicative with polynomial growth. We use the notation \( \mathcal{P}(\mathbb{R}^d) \) to denote the cone of all non-negative functions which are \( v \)-moderate. Similarly, we can define \( \mathcal{P}(\mathbb{R}^{2d}) \).

The weighted mixed-norm spaces used to measure the STFT is defined as following.

**Definition 2.5 (Weighted mixed-norm spaces,).** Let \( m \in \mathcal{P}((\mathbb{R}^d)^d) \), \( p, q \in (0, \infty] \). Then the weighted mixed-norm space \( L_{m^p,q}^p(\mathbb{R}^{2d}) \) consists of all Lebesgue measurable functions on \( \mathbb{R}^{2d} \) such that the (quasi-)norm
\[ \|F\|_{L_{m^p,q}^p(\mathbb{R}^{2d})} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |F(x, \xi)|^p m(x, \xi)^p \, dx \right)^{q/p} d\xi \right)^{1/q} \]
is finite, with usual modification when \( p = \infty \) or \( q = \infty \).

Now, we recall the definition of modulation space.

**Definition 2.6.** Let \( 0 < p, q \leq \infty, m \in \mathcal{P}(\mathbb{R}^{2d}) \). Given a non-zero window function \( \phi \in S(\mathbb{R}^d) \), the (weighted) modulation space \( M_{m^p,q}^p(\mathbb{R}^d) \) consists of all \( f \in S'(\mathbb{R}^d) \) such that the norm
\[ \|f\|_{M_{m^p,q}^p(\mathbb{R}^d)} := \|V_\phi f(x, \xi)\|_{L_{m^p,q}^p(\mathbb{R}^{2d})} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_\phi f(x, \xi) m(x, \xi)|^p \, dx \right)^{q/p} d\xi \right)^{1/q} \]
is finite. We write \( M_{m^p,q}^p \) for modulation space with \( m \equiv 1 \).

Recall that the above definition of \( M_{m^p,q}^p \) is independent of the choice of window function \( \phi \). The readers may see this fact in [14] for the case \((p, q) \in [1, \infty]^2 \), and in [12] for full range \((p, q) \in (0, \infty]^2 \). In particular, we point out that in [12], the author find a admissible windows, denoted by \( \mathfrak{M}_{m^p,q} \), depending on \( p, q \), for the modulation space \( M_{m^p,q}^p \).
Denote by $\mathcal{M}^{p,q}_m(\mathbb{R}^d)$ the $\mathcal{S}(\mathbb{R}^d)$ closure in $M^{p,q}_m(\mathbb{R}^d)$. Note that $\mathcal{M}^{p,q}_m(\mathbb{R}^d) = M^{p,q}_m(\mathbb{R}^d)$ for $p, q \neq \infty$.

Next, we turn to the definition of Wiener amalgam space. As we mentioned in Section 1, in this paper we consider the Wiener amalgam space of the type of the image of modulation space under Fourier transform. Write $\mathcal{F}$ as

$$\| f \|_{\mathcal{F} M^{p,q}_m(\mathbb{R}^d)} = \| \mathcal{F}^{-1} f \|_{M^{p,q}_m(\mathbb{R}^d)} = \| V_{\phi} f(x, \xi) \|_{L^{p,q}_m(\mathbb{R}^{2d})} = \| V_{\phi} f(J(x, \xi)) \|_{L^{p,q}_m(\mathbb{R}^{2d})}.$$ 

The Wiener amalgam space can be also defined by the weighted mixed-norm of STFT.

**Definition 2.7.** Let $0 < p, q \leq \infty$, $m \in \mathcal{P}(\mathbb{R}^{2d})$. Given a non-zero window function $\phi \in \mathcal{S}(\mathbb{R}^d)$, the (weighted) Wiener amalgam space $\mathcal{F}(M^{p,q}_m)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the norm

$$\| f \|_{\mathcal{F} M^{p,q}_m(\mathbb{R}^d)} = \| V_{\phi} f(x, -x) \|_{L^{p,q}_m(\mathbb{R}^{2d})} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_{\phi} f(\xi, -x) m(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q}$$

is finite.

In particular, for $f \in \mathcal{S}'(\mathbb{R}^d)$, $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\| f \|_{\mathcal{F} M^{p,q}_{1 \otimes m}(\mathbb{R}^{2d})} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_{\phi} f(\xi, -x)|^p dx \right)^{q/p} m(x)^q d\xi \right)^{1/q}.$$ Using the notation of Wiener amalgam space in [10], we have

$$\mathcal{F} M^{p,q}_m(\mathbb{R}^{2d}) = W(\mathcal{F} L^p, L^q_m)(\mathbb{R}^{2d}).$$

This representation makes us more clear that $f$ belongs to $\mathcal{F} M^{p,q}_{1 \otimes m}(\mathbb{R}^{2d})$ means it lies locally in $\mathcal{F} L^p(\mathbb{R}^{2d})$ and globally in $L^q_m(\mathbb{R}^{2d})$.

Next, we collect following calculations for the linear transform of modulation and Wiener amalgam spaces.

**Lemma 2.8.** Let $0 < p, q \leq \infty$, $L$ be a invertible linear transform on $\mathbb{R}^d$, $m \in \mathcal{P}(\mathbb{R}^{2d})$. For a function $f$ on $\mathbb{R}^d$, denote $f_L(x) := f(Lx)$. We have

$$\| f_L \|_{M^{p,q}_m(\mathbb{R}^d)} \sim \| f \|_{M^{p,q}_m(\mathbb{R}^d)},$$

where $\widetilde{m}(x, \xi) = m(L^{-1}x, L^T \xi)$, $x, \xi \in \mathbb{R}^d$, $i = 1, 2$.

**Proof.** Using Lemma 2.1, we write

$$\| f_L \|_{M^{p,q}_m} = \| V_{\phi_L} f_L(x, \xi) \|_{L^{p,q}_m} \sim \| V_{\phi} f(Lx, (L^{-1})^T \xi) \|_{L^{p,q}_m} \sim \| V_{\phi} f(x, \xi) \|_{L^{p,q}_m} \sim \| f \|_{M^{p,q}_m}. \qed$$

**Lemma 2.9.** Let $0 < p, q \leq \infty$, $L$ be a invertible linear transform on $\mathbb{R}^{2d}$, $m \in \mathcal{P}(\mathbb{R}^{2d})$. For a function $f$ on $\mathbb{R}^{2d}$, denote $f_L(x) := f(Lx)$. We have

$$\| f_L \|_{W(\mathcal{F} L^p, L^q_m)(\mathbb{R}^{2d})} \sim \| f \|_{W(\mathcal{F} L^p, L^q_m)(\mathbb{R}^{2d})},$$

where $\widetilde{m}(z) = m(L^{-1}z)$.
Proof. By Lemma 2.1 and Definition 2.7, we have
\[ \|f_l\|_{W(\mathcal{F}L^p, L^q_m)(\mathbb{R}^{2d})} = \|\phi_0 f_l(\zeta, -z)\|_{L^p_{1\otimes m}} \]
\[ \sim \|\phi_0 f(L\zeta, -(L^{-1})^T z)\|_{L^p_{1\otimes m}} \]
\[ \sim \|\phi_0 f(\zeta, -z)\|_{L^p_{1\otimes m}} \sim \|f\|_{W(\mathcal{F}L^p, L^q_m)(\mathbb{R}^{2d})}. \]
\[ \square \]

Next, we recall a multiplication property of Wiener amalgam space.

Lemma 2.10. Let \(0 < p, q \leq \infty, \hat{p} = \min\{p, 1\}\). We have
\[ W(\mathcal{F}L^p, L^q_m) \cdot W(\mathcal{F}L^\hat{p}, L^{\infty}) \subset W(\mathcal{F}L^p, L^q_m). \]

Proof. Using the relation between modulation and Wiener amalgam space, the desired conclusion is equivalent to
\[ M^{p,q}_{1\otimes m}(\mathbb{R}^{2d}) \ast M^{\hat{p}, \infty} \subset M^{p,q}_{1\otimes m}(\mathbb{R}^{2d}). \]
This is a direct conclusion of \([16, \text{Theorem } 1.3]\) with the fact
\[ l^p \ast l^\hat{p} \subset l^p, \quad l^q_m \ast l^{\infty} \subset l^q_m. \]
\[ \square \]

Lemma 2.11 (Chirp function). For any \(\lambda \neq 0\), we have
\[ G_\lambda(x, \xi) = e^{2\pi i \lambda x \cdot \xi} \in W(\mathcal{F}L^\hat{p}, L^{\infty})(\mathbb{R}^{2d}). \]
Moreover, for any function \(F\) on \(\mathbb{R}^{2d}\), we have
\[ \|FG_\lambda\|_{W(\mathcal{F}L^p, L^q_m)(\mathbb{R}^{2d})} \sim \|F\|_{W(\mathcal{F}L^p, L^q_m)(\mathbb{R}^{2d})}. \]

Proof. Using Lemma 2.9, we only need to consider the case \(\lambda = 1\), i.e., to verify that \(G_1 = e^{2\pi i x \cdot \xi} \in W(\mathcal{F}L^\hat{p}, L^q_m)\). Denote by \(g_0(x, \xi) = e^{-\pi(|x|^2 + |\xi|^2)}\) the Gaussian function. By the calculation in \([5, \text{Proposition } 3.2]\), we have
\[ |V_{g_0} G_1(z, \xi)| = 2^{-d/2}e^{-\frac{\pi}{2}|z_1 - \xi_2|^2} e^{-\frac{\pi}{2}|z_2 - \xi_1|^2}. \]
By Definition 2.7, we conclude that
\[ \|G_1\|_{W(\mathcal{F}L^p, L^{\infty})(\mathbb{R}^{2d})} = \|V_{g_0} G_1(\zeta, -z)\|_{L^{\hat{p}, \infty}(\mathbb{R}^{4d})} \]
\[ \sim \| e^{-\frac{\pi}{2}|z_1 + 2z_2|^2} e^{-\frac{\pi}{2}|z_2 + z_1|^2} \|_{L^{\hat{p}, \infty}(\mathbb{R}^{4d})} \sim 1. \]
This completes the proof of \(G_\lambda \in W(\mathcal{F}L^\hat{p}, L^{\infty}). \)
Moreover, by the multiplication property (Lemma 2.10), we deduce that
\[ \|FG_\lambda\|_{W(\mathcal{F}L^p, L^q_m)} \lesssim \|F\|_{W(\mathcal{F}L^p, L^q_m)} \|G_\lambda\|_{W(\mathcal{F}L^\hat{p}, L^{\infty})} \]
\[ \lesssim \|F\|_{W(\mathcal{F}L^p, L^q_m)} = \|FG_\lambda G - \lambda\|_{W(\mathcal{F}L^p, L^q_m)} \lesssim \|FG_\lambda\|_{W(\mathcal{F}L^p, L^q_m)}. \]
From this, we obtain \(\|FG_\lambda\|_{W(\mathcal{F}L^p, L^q_m)} \sim \|F\|_{W(\mathcal{F}L^p, L^q_m)}. \)
\[ \square \]

In order to measure the summability and decay properties of Gabor coefficients, we recall the discrete weighted mixed-norm space.
Definition 2.12 (Discrete mixed-norm spaces I). Let $0 < p, q \leq \infty$, $m \in \mathcal{P}(\mathbb{R}^{2d})$. The space $l^{p,q}_m(\mathbb{Z}^{2d})$ consists of all sequences $\vec{a} = \{a_{k,n}\}_{k,n \in \mathbb{Z}^d}$ for which the (quasi-)norm

$$
\|\vec{a}\|_{l^{p,q}_m(\mathbb{Z}^{2d})} = \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} |a_{k,n}|^p m(k,n)^{p/q} \right)^{q/p} \right)^{1/q}
$$

is finite.

In Theorem 1.5, we use another type of discrete weighted mixed-norm space as following.

Definition 2.13 (Discrete mixed-norm spaces II). Let $0 < p, q \leq \infty$, $m \in \mathcal{P}(\mathbb{R}^{2d})$. The space $l^{(p,q)}_m(\mathbb{Z}^{2d})$ consists of all sequences $\vec{a} = \{a_{k,n}\}_{k,n \in \mathbb{Z}^d}$ for which the (quasi-)norm

$$
\|\vec{a}\|_{l^{(p,q)}_m(\mathbb{Z}^{2d})} = \left( \sum_{k \in \mathbb{Z}^d} \left( \sum_{n \in \mathbb{Z}^d} |a_{k,n}|^q m(k,n)^{q/p} \right)^{p/q} \right)^{1/p}
$$

is finite.

As usual for $\omega \in \mathcal{P}(\mathbb{R}^d)$, the space $l^p_\omega(\mathbb{Z}^d)$ consists of all $\vec{b} = \{b_k\}_{k \in \mathbb{Z}^d}$ for which the (quasi-)norm

$$
\|\vec{b}\|_{l^p_\omega(\mathbb{Z}^d)} = \left( \sum_{k \in \mathbb{Z}^d} |b_k|^p \omega(k)^{p} \right)^{1/p}
$$

is finite. For $\omega = \nu_s$, we write $l^p_{\nu_s} := l^p_s$ for simplicity.

2.3. Gabor analysis of modulation spaces. Comparing with the classical definition of modulation space in Definition 2.6, or the semi-discrete definition such as in [16, Proposition 2.1] in the same way as Besov spaces, the modulation spaces can be also characterized by the summability and decay properties of their Gabor coefficients, this is an important reason why the modulation spaces play the central role in the field of time-frequency analysis.

We recall some important operators which are the key tools for the discretization of modulation spaces.

Definition 2.14. Assume that $g, \gamma \in L^2(\mathbb{R}^d)$ and $\alpha, \beta > 0$. The coefficient operator or analysis operator $C_{g}^{\alpha,\beta}$ is defined by

$$
C_{g}^{\alpha,\beta} f = \langle f, T_{\alpha k} M_{\beta n} g \rangle_{k,n \in \mathbb{Z}^d}.
$$

The synthesis operator or reconstruction operator $D_{g}^{\alpha,\beta}$ is defined by

$$
D_{g}^{\alpha,\beta} \vec{c} = \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} c_{k,n} T_{\alpha k} M_{\beta n} \gamma.
$$

The Gabor frame operator $S_{g}^{\alpha,\beta}$ is defined by

$$
S_{g}^{\alpha,\beta} f = D_{g}^{\alpha,\beta} C_{g}^{\alpha,\beta} f = \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \langle f, T_{\alpha k} M_{\beta n} g \rangle T_{\alpha k} M_{\beta n} \gamma.
$$

In order to extend the boundedness result of analysis operator and synthesis operator to the modulation spaces of full range, following admissible window class was introduced in [12].
Define the space of admissible windows $M_v^{p,q}$ for the modulation space $M_m^{p,q}$ to be

$$M_v^{p,q} = \bigcup_{r_1 > d/p, s_1 > d/s, 1 \leq p_1 < \infty} M_{\omega_1, s_1}^{p_1, s_1}.$$ 

Based on the window class mentioned above, we recall the boundedness of $C_g^{\alpha,\beta}$ and $D_g^{\alpha,\beta}$, which works on the full range $p, q \in (0, \infty]$.

**Lemma 2.16.** Assume that $m$ is $v$-moderate, $p, q \in (0, \infty]$, and $g$ belongs to the subclass $M_{\omega_1, s_1}^{p_1, s_1}$ of $M_v^{p,q}$. Then the analysis operator $C_g^{\alpha,\beta}$ is bounded from $M_m^{p,q}$ into $l_m^{p,q}$, and the synthesis operator $D_g^{\alpha,\beta}$ is boundedness form $l_m^{p,q}$ into $M_m^{p,q}$ for all $\alpha, \beta > 0$, where $\tilde{m}(k, n) = \overline{m}(ak, \beta n)$.

Now, we recall the main theorem in [12], which extends the Gabor expansion of modulation spaces to the full range $0 < p, q \leq \infty$.

**Theorem 2.17.** *(see [12])* Assume that $m$ is $v$-moderate, $p, q \in (0, \infty]$, $g, \gamma \in M_v^{p,q}$, and that the Gabor frame operator $S_g^{\alpha,\gamma} = D_g^{\alpha,\beta}C_g^{\alpha,\beta} = I$ on $L^2(\mathbb{R}^d)$. Then

$$f = \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \langle f, T_{\alpha k} M_{\beta n} g \rangle T_{\alpha k} M_{\beta n} \gamma = \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \langle f, T_{\alpha k} M_{\beta n} \gamma \rangle T_{\alpha k} M_{\beta n} g$$

with unconditional convergence in $M_m^{p,q}$ if $p, q < \infty$, and with weak-star convergence in $M_m^{1,\infty}$ otherwise. Furthermore there are constants $A, B > 0$ such that for all $f \in M_m^{p,q}$

$$A \|f\|_{M_m^{p,q}} \leq \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} |\langle f, T_{\alpha k} M_{\beta n} g \rangle|^p \overline{m}(ak, \beta n)^p \right)^{q/p} \right)^{1/q} \leq B \|f\|_{M_m^{p,q}}$$

with obvious modification for $p = \infty$ or $q = \infty$. Likewise, the quasi-norm equivalence

$$A' \|f\|_{M_m^{p,q}} \leq \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} |\langle f, T_{\alpha k} M_{\beta n} \gamma \rangle|^p \overline{m}(k, n\beta)^p \right)^{q/p} \right)^{1/q} \leq B' \|f\|_{M_m^{p,q}}$$

holds on $M_m^{p,q}$.

The following well known theorem provide a way to find the Gabor frame of $L^2(\mathbb{R}^d)$. Recall that $\|g\|_{W(L^\infty, L^1(\mathbb{R}^d))} = \sum_{n \in \mathbb{Z}^d} \|g\chi_{Q_0+n}\|_{L^\infty}$ with $Q_0 = [0, 1]^d$.

**Theorem 2.18.** *(Walnut [26])* Suppose that $g \in W(L^\infty, L^1(\mathbb{R}^d))$ satisfying

$$A \leq \sum_{k \in \mathbb{Z}^d} |g(x - ak)|^2 \leq B \quad a.e.$$ 

for constants $A, B \in (0, \infty)$). Then there exists a constant $\beta_0$ depending on $\alpha$ such that $G(g, \alpha, \beta)$ is a Gabor frame of $L^2(\mathbb{R}^d)$ for all $\beta \leq \beta_0$. 


Theorem 2.19. Assume that \( g \in M^1_v(\mathbb{R}^d) \) and that \( \{T_{\alpha k}M_{\beta n}g\}_{k,n} \) is a Gabor frame for \( L^2(\mathbb{R}^d) \). Then the Gabor frame operator \( S_{g,g}^{\alpha,\beta} \) is invertible on \( M^1_v(\mathbb{R}^d) \). As a consequence, \( S_{g,g}^{\alpha,\beta} \) is invertible on all modulation spaces \( M^p_m^{\alpha q}(\mathbb{R}^d) \) for \( 1 \leq p, q \leq \infty \) and \( m \in \mathcal{P}(\mathbb{R}^d) \).

3. First characterizations: discretizations by time-frequency tools

3.1. Discretization by Gabor coefficients for BMM.

Proposition 3.1. Assume \( p_i, q_i, p, q \in (0, \infty) \), \( i = 1, 2 \). For any \( \alpha > 0 \), we have
\[
W_0 : M^{p_1,q_1}_{m_1}(\mathbb{R}^d) \times M^{p_2,q_2}_{m_2}(\mathbb{R}^d) \to M^{p,q}_{m_1 \otimes m_2}(\mathbb{R}^d)
\]
if and only if
\[
\|(a(k)b(k + Jn))\|_{L^{p,q}_{m_1 \otimes m_2}(\mathbb{Z}^d \times \mathbb{Z}^d)} \lesssim \|(a(k))\|_{L^{p_1,q_1}_{m_1}(\mathbb{Z}^d)} \cdot \|(b(k))\|_{L^{p_2,q_2}_{m_2}(\mathbb{Z}^d)}.
\]
Here, we denote \( \widetilde{m}(z) := m(\alpha z) \) and \( \widetilde{m}_i(z) := m_i(\alpha z) \) for \( z \in \mathbb{Z}^d \), \( i = 1, 2 \).

Proof. We divide the proof into two parts.

"Only if" part. Let \( \varphi \) be a smooth function supported in \( (-\alpha/2, \alpha/2)^d \), satisfying \( \hat{\varphi}(x) \geq 0 \) and \( \hat{\varphi}(0) = 1 \). For any \( \alpha > 0 \) and two nonnegative truncated (only finite nonzero items) sequences \( \bar{a} = (a_j, j, l)_{j,l} \in \mathbb{Z}^d \) and \( \bar{b} = (b_j, j, l)_{j,l} \in \mathbb{Z}^d \), we set
\[
f_1 = D^{\alpha,\alpha}_{\varphi,\varphi} \bar{a} = \sum_{j,l} a_{j,l} T_{\alpha k} M_{\alpha \varphi}, \quad f_2 = D^{\alpha,\alpha}_{\varphi,\varphi} \bar{b} = \sum_{j,l} b_{j,l} T_{\alpha k} M_{\alpha \varphi}.
\]
Take the window \( \Phi = W_0(\hat{\varphi}, \varphi) \) with \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) supported in \( (-\alpha/2, \alpha/2)^d \), satisfying \( \hat{\varphi}(x) \geq 0 \) and \( \hat{\varphi}(0) = 1 \). By the sampling property of STFT (see Lemma 2.16),
\[
\|W_0(f_1, f_2)\|_{L^{p,q}_{m_1 \otimes m_2}} \lesssim \|W_0(f_1, f_2)\|_{M^{p,q}_{m_1 \otimes m_2}}.
\]
Let us turn to the lower bound estimates of the first term in (3.2). Using Lemma 2.2, we write
\[
\|W_0(f_1, f_2)\|_{L^{p,q}_{m_1 \otimes m_2}} = \|W_0 f_1(z_1, z_2 + \zeta) W_0 f_2(z_1 + \zeta, z_2)\|_{\alpha \mathbb{Z}^d \times \alpha \mathbb{Z}^d} \lesssim \lim_{\zeta \to 0} \|W_0 f_1(z_1, z_2 + \zeta) W_0 f_2(z_1 + \zeta, z_2)\|_{\alpha \mathbb{Z}^d \times \alpha \mathbb{Z}^d} \lesssim \|W_0 f_1(z_1, z_2) W_0 f_2(z_1 + n_2, \alpha(k_2 - n_1))\|_{\alpha \mathbb{Z}^d \times \alpha \mathbb{Z}^d}.
\]
By the support of \( \varphi \) and \( \tilde{\varphi} \), we estimate \( |W_0 f_1(\alpha k_1, \alpha k_2)| \) by
\[
|W_0 f_1(\alpha k_1, \alpha k_2)| = |\langle f_1, T_{\alpha k_1} M_{\alpha \varphi} \rangle| = \sum_{j,l} |\sum_{j,l} a_{j,l} T_{\alpha k_1} M_{\alpha \varphi} f_1| = \sum_{l} |\sum_{j} a_{k_1,l} T_{\alpha k_1} M_{\alpha \varphi} f_1| = \sum_{l} |\sum_{j} a_{k_1,l} \langle M_{\alpha \varphi}, M_{\alpha k_2} \rangle| = \sum_{l} |\sum_{j} a_{k_1,l} \langle \varphi, M_{\alpha (k_2 - l)} \rangle|,
\]
where the interchange of the order of summation and integration is valid since the number of terms is finite. Using the nonnegativity of \( \bar{a}, \bar{b}, \hat{\varphi} \) and \( \hat{\tilde{\varphi}} \), we obtain
\[
\sum_{l} a_{k_1,l} \langle \varphi, M_{\alpha (k_2 - l)} \rangle = \sum_{l} a_{k_1,l} \langle \hat{\tilde{\varphi}}, T_{\alpha (k_2 - l)} \hat{\varphi} \rangle \geq a_{k_1,k_2} \langle \hat{\tilde{\varphi}}, \hat{\varphi} \rangle \geq a_{k_1,k_2}.
\]
From the above two estimates, we get the lower bound of $|V_0 f_1(\alpha k_1, \alpha k_2)|$:

$$|V_0 f_1(\alpha k_1, \alpha k_2)| \gtrsim a_{k_1, k_2}.$$ 

A similar calculation yields the lower bound of $|V_0 f_2(\alpha (k_1 + n_2), \alpha (k_2 - n_1))|$:

$$|V_0 f_2(\alpha (k_1 + n_2), \alpha (k_2 - n_1))| \gtrsim b_{k_1+n_2, k_2-n_1}.$$ 

By the lower bounds of $|V_0 f_1(\alpha k_1, \alpha k_2)|$ and $|V_0 f_2(\alpha (k_1 + n_2), \alpha (k_2 - n_1))|$, we obtain

$$\|V_0 (W_0 f_1, f_2)\|_{\mathcal{Z}^{d^2} \times \mathbb{Z}^d}^{p,q,m} = \|V_0 f_1(\alpha k_1, \alpha k_2) V_0 f_2(\alpha (k_1 + n_2), \alpha (k_2 - n_1))\|_{\mathcal{Z}^{d^2} \times \mathbb{Z}^d}^{p,q,m} \gtrsim \|(a_{k_1, k_2} b_{k_1+n_2, k_2-n_1})\|_{\mathcal{Z}^{d^2} \times \mathbb{Z}^d}^{p,q,m} = \|a(k)(b(k+Jn))\|_{\mathcal{Z}^{d^2} \times \mathbb{Z}^d}^{p,q,m}.$$ 

Using this and (3.2), we obtain the lower estimate of $\|W_0 (f_1, f_2)\|_{\mathcal{M}^{p,q}}$:

$$\|W_0 (f_1, f_2)\|_{\mathcal{M}^{p,q}} \gtrsim \|a(k)b(k+Jn)\|_{\mathcal{Z}^{d^2} \times \mathbb{Z}^d}^{p,q,m}.$$ (3.3) 

On the other hand, using the boundedness of synthesis operator in Lemma 2.16, we get the upper bound estimates of $\|f_1\|_{\mathcal{M}^{p,q,m}}$:

$$\|f_1\|_{\mathcal{M}^{p,q,m}} = \|D^{\alpha, \alpha}_\phi \tilde{a}\|_{\mathcal{M}^{p,q,m}} \lesssim \|\tilde{a}\|_{\mathcal{M}^{p,q,m}} \lesssim \|a(k)\|_{\mathcal{Z}^{d^2} \times \mathbb{Z}^d}^{p,q,m}.$$ (3.4)

Combining the estimates (3.3) and (3.4), we conclude the desired inequality by

$$\|a(k)b(k+Jn)\|_{\mathcal{Z}^{d^2} \times \mathbb{Z}^d}^{p,q,m} \lesssim \|W_0 (f_1, f_2)\|_{\mathcal{M}^{p,q,m}} \lesssim \|f_1\|_{\mathcal{M}^{p,q,m}} \|f_2\|_{\mathcal{M}^{p,q,m}} \lesssim \|\tilde{a}\|_{\mathcal{M}^{p,q,m}} \|\tilde{b}\|_{\mathcal{M}^{p,q,m}}.$$ 

By a standard limiting argument, the above inequality is valid for all sequences $\tilde{a}$ and $\tilde{b}$.

**“If” part.** For any fixed $\alpha > 0$, if (3.1) is valid, we claim that it also holds for $\tilde{\Phi}$ with any positive integer $N$: that is, (3.1) implies

$$\|(a(k)b(k+Jn))\|_{\mathcal{Z}^{d^2} \times \mathbb{Z}^d}^{p,q,m} \lesssim \|a(k)\|_{\mathcal{M}^{p,q,m}} \|b(k)\|_{\mathcal{M}^{p,q,m}}.$$ 

where $\tilde{m}(z) = \tilde{m}(\tilde{\Phi}) = m(\tilde{\Phi})$, $\tilde{m}_i(z) = \tilde{m}_i(\tilde{\Phi}) = m_i(\tilde{\Phi})$. Denote

$$\Lambda = \{0, N\}^d \cap \mathbb{Z}^d, \quad \Gamma_{j,l} = (j,l) + N^d, (j,l) \in \Lambda.$$ 

We have $\mathbb{Z}^d = \bigcup_{(j,l) \in \Lambda} \Gamma_{j,l}$ and

$$\|(a(k)b(k+Jn))\|_{\mathcal{Z}^{d^2} \times \mathbb{Z}^d}^{p,q,m} \sim \sum_{(j,l) \in \Lambda} \|(a(k)(b(k+Jn))\|_{\mathcal{Z}^{d^2} \times \mathbb{Z}^d}^{p,q,m}(\Gamma_{j,l})$$

$$\sim \sum_{(j,l) \in \Lambda} \|a(Nk+j)(b(Nk+Jn) + (j+l)m(Nn+l))\|_{\mathcal{Z}^{d^2} \times \mathbb{Z}^d}^{p,q,m}$$

$$\sim \sum_{(j,l) \in \Lambda} \|a(Nk+j)\|_{\mathcal{Z}^{d^2} \times \mathbb{Z}^d}^{p,q,m} \|b(Nk+Jn) + (j+l)m(n)\|_{\mathcal{Z}^{d^2} \times \mathbb{Z}^d}^{p,q,m}$$

$$= \sum_{(j,l) \in \Lambda} \|a(Nk+j)(b(Nk+Jn) + (j+l))\|_{\mathcal{Z}^{d^2} \times \mathbb{Z}^d}^{p,q,m}.$$
Then, we continue this estimate by applying (3.1):
\[
\sum_{(j,l) \in \Lambda} \|a(Nk + j)b(Nk + Jn + (j + Jl))\|^p_{\tilde{m}_1, \tilde{m}_2} \leq \sum_{(j,l) \in \Lambda} \|a(Nk + j)\|^p_{\tilde{m}_1, \tilde{m}_2} \|b(Nk + j + Jl)\|^p_{\tilde{m}_1, \tilde{m}_2}
\]
\[
= \sum_{(j,l) \in \Lambda} \|a(Nk + j)\|_{\tilde{m}_1} \|b(Nk + j + Jl)\|_{\tilde{m}_2}
\]
\[
\sim \sum_{(j,l) \in \Lambda} \|a(Nk + j)\|_{\tilde{m}_1} \|b(Nk + j + Jl)\|_{\tilde{m}_2}
\]
where in the last inequality we use the fact $|\Lambda| < \infty$. The claim follows by the above two estimates. From this claim, we find that if (3.1) holds for some $\alpha > 0$, then $\alpha$ can be taken to be sufficiently small. Now, we turn to verify the boundedness of $W_0$ under the assumption that (3.1) holds for some $\alpha > 0$.

Note that $\Phi = W_0(\phi, \phi) \in \mathcal{S}(\mathbb{R}^d)$ with $\phi \in \mathcal{S}(\mathbb{R}^d)$. There exists a sufficiently large integer $N_1$ such that for suitable constants $A, B \in (0, \infty)$
\[
A \leq \sum_{k \in \mathbb{Z}^d} |\Phi(x - \frac{\alpha}{N_1}k)|^2 \leq B.
\]
Denote $\tilde{\alpha} = \frac{\alpha}{N_1}$. Using Theorem 2.18, there exists a constant $\beta = \tilde{\alpha}/N_2 = \frac{\alpha}{N_1N_2}$ with sufficiently large integer $N_2$ such that $G(\Phi, \tilde{\alpha}, \beta)$ is a Gabor frame of $L^2(\mathbb{Z}^d)$. Let $\Psi = (S_{\Phi, \tilde{\alpha}, \beta})^{-1}\Phi$ be the canonical dual widow of $\Phi$. Note that $\Phi \in \mathcal{S} \subset \mathcal{M}_p^\infty$, then Definition 2.15 and Theorem 2.19 implies that $\Psi \in \mathcal{M}_p^\infty$. By the definitions of $\Phi$ and $\Psi$, we have $S_{\Phi, \tilde{\alpha}, \beta} = D_{\Psi, \tilde{\alpha}, \beta} C_{\Phi, \tilde{\alpha}, \beta} = I$ on $L^2(\mathbb{R}^d)$. Then Theorem 2.17 yields that $f = S_{\Phi, \tilde{\alpha}, \beta}f$ for all $f \in M_{1,0}^{p,q}$. Recalling $\beta = \tilde{\alpha}/N_2$ and using Lemma 2.2, we find that
\[
\|C_{\Phi, \tilde{\alpha}, \beta}W_0(f_1, f_2)\|_{\tilde{m}_1, \tilde{m}_2} = \|V_\Phi(W_0(f_1, f_2))(z, \zeta)\|_{\tilde{m}_1, \tilde{m}_2} \leq \|V_\Phi(W_0(f_1, f_2))(z, \zeta)\|_{\tilde{m}_1, \tilde{m}_2}
\]
\[
= \|V_\phi f_1(z_1, z_2 + \zeta_1) V_\phi f_2(z_1 + \zeta_2, z_2)\|_{\tilde{m}_1, \tilde{m}_2} \|V_\phi f_2(z + J \zeta)\|_{\tilde{m}_1, \tilde{m}_2}
\]
\[
= \|V_\phi f_1(z) V_\phi f_2(z + J \zeta)\|_{\tilde{m}_1, \tilde{m}_2} \|f_2\|_{M_{1,0}^p}\|f_1\|_{M_{1,0}^q}
\]
Using the inequality of discrete mixed-norm spaces (3.1) with $\frac{\alpha}{N_1N_2}$ and Lemma 2.16, we continue the above estimate by
\[
\|V_\phi f_1(\beta k) V_\phi f_2(\beta (k + Jn))\|_{\tilde{m}_1, \tilde{m}_2} \leq \|V_\phi f_1(\beta k)\|_{\tilde{m}_1, \tilde{m}_2} \|V_\phi f_2(\beta k)\|_{\tilde{m}_1, \tilde{m}_2}
\]
\[
= \|V_\phi f_1(z)\|_{\tilde{m}_1, \tilde{m}_2} \|V_\phi f_2(z)\|_{\tilde{m}_1, \tilde{m}_2} \|f_2\|_{M_{1,0}^p} \|f_1\|_{M_{1,0}^q} \leq \|f_1\|_{M_{1,0}^p} \|f_2\|_{M_{1,0}^q}.
\]
The above two estimates imply that
\[
\|C_{\Phi, \tilde{\alpha}, \beta}W_0(f_1, f_2)\|_{\tilde{m}_1, \tilde{m}_2} \leq \|f_1\|_{M_{1,0}^p} \|f_2\|_{M_{1,0}^q}.
\]
Hence,
\[
\| W_0(f_1, f_2) \|_{M_{1\otimes m}^{p,q}} = \| D_\Phi^{\alpha,\beta} C_\Phi^{\alpha,\beta} W_0(f_1, f_2) \|_{M_{1\otimes m}^{p,q}} \\
\lesssim \| C_\Phi^{\alpha,\beta} W_0(f_1, f_2) \|_{M_{1\otimes m}^{p,q}}^{\frac{p}{p-q}} \lesssim \| f_1 \|_{M_{m_1}^{p_1,q_1}} \cdot \| f_2 \|_{M_{m_2}^{p_2,q_2}}.
\]

We have now completed the proof. \(\square\)

We are now in a position to give the proof of Theorem 1.1. As we will see, it follows by Proposition 3.1 with some changes of variables.

Proof of Theorem 1.1. Let \(\Phi_\tau = W_\tau(\phi, \phi)\) with nonzero function \(\phi \in \mathcal{S}(\mathbb{R}^d)\). Using Lemma 2.2, we obtain that
\[
\| W_\tau(f_1, f_2) \|_{M_{1\otimes m}^{p,q}} = \| V_\Phi f_1, V_\Phi f_2 \|_{L_{1\otimes m}^{p,q}} \\
= \| V_\Phi (z_1 - \tau \zeta_2, z_2 + (1 - \tau) \zeta_1) V_\Phi f_1, V_\Phi f_2 (z_1 + (1 - \tau) \zeta_2, z_2 - \tau \zeta_1) \|_{L_{1\otimes m}^{p,q}} \\
= \| V_\Phi (z_1, z_2 + \zeta_1) V_\Phi f_1, V_\Phi f_2 (z_1 + \zeta_2, z_2) \|_{L_{1\otimes m}^{p,q}} \\
= \| \Phi_\tau W_0(f_1, f_2) \|_{L_{1\otimes m}^{p,q}} = \| W_0(f_1, f_2) \|_{M_{1\otimes m}^{p,q}}.
\]

Hence, the boundedness property:
\[
W_\tau: M_{m_1}^{p_1,q_1}(\mathbb{R}^d) \times M_{m_2}^{p_2,q_2}(\mathbb{R}^d) \rightarrow M_{1\otimes m}^{p,q}(\mathbb{R}^{2d})
\]
is independent with \(\tau \in [0, 1]\). We only need to consider the case \(\tau = 0\).

Write (3.1) with \(\alpha = 1\) by
\[
\left( \sum_{n_1, n_2 \in \mathbb{Z}^d} \left( \sum_{k_1, k_2 \in \mathbb{Z}^d} |a_{k_1, k_2} b_{k_1 + n_2, k_2 - n_1}|^p \right)^{1/p} m(n_1, n_2)^q \right)^{1/q}
\leq \left( \sum_{n_1 \in \mathbb{Z}^d} \left( \sum_{k_1 \in \mathbb{Z}^d} |a_{k_1, n_1} m_1(k_1, n_1)|^{p_1} \right)^{1/q_1} \right)^{1/q_1} \left( \sum_{n_2 \in \mathbb{Z}^d} \left( \sum_{k_2 \in \mathbb{Z}^d} |b_{k_2, n_2} m_2(k_2, n_2)|^{p_2} \right)^{1/q_2} \right)^{1/q_2},
\]
which is equivalent to
\[
\left( \sum_{n_1, n_2 \in \mathbb{Z}^d} \left( \sum_{k_1, k_2 \in \mathbb{Z}^d} |a_{k_1, k_2} b_{-n_2 - k_1, n_1 - k_2}|^p \right)^{1/p} m(n_1, n_2)^q \right)^{1/q}
\leq \left( \sum_{n_1 \in \mathbb{Z}^d} \left( \sum_{k_1 \in \mathbb{Z}^d} |a_{k_1, n_1} m_1(k_1, n_1)|^{p_1} \right)^{1/q_1} \right)^{1/q_1} \left( \sum_{n_2 \in \mathbb{Z}^d} \left( \sum_{k_2 \in \mathbb{Z}^d} |b_{-k_2, -n_2} m_2(k_2, n_2)|^{p_2} \right)^{1/q_2} \right)^{1/q_2},
\]
After some change of variables, one can find the following equivalent form:
\[
\left( \sum_{n_1, n_2 \in \mathbb{Z}^d} \left( \sum_{k_1, k_2 \in \mathbb{Z}^d} |a_{k_1, k_2} b_{n_1 - k_1, n_2 - k_2}|^p \right)^{1/p} m(n_2, -n_1)^q \right)^{1/q}
\leq \left( \sum_{n_1 \in \mathbb{Z}^d} \left( \sum_{k_1 \in \mathbb{Z}^d} |a_{k_1, n_1} m_1(k_1, n_1)|^{p_1} \right)^{1/q_1} \right)^{1/q_1} \left( \sum_{n_2 \in \mathbb{Z}^d} \left( \sum_{k_2 \in \mathbb{Z}^d} |b_{k_2, n_2} m_2(-k_2, -n_2)|^{p_2} \right)^{1/q_2} \right)^{1/q_2}.
\]
This is equivalent to the desired inequality
\[
\| (a_{k_1, k_2} b_{n_1 - k_1, n_2 - k_2}) \|_{L_{1\otimes m}^{p,q}} \lesssim \| a \|_{L_{m_1}^{p_1,q_1}(\mathbb{Z}^{2d})} \| b \|_{L_{m_2}^{p_2,q_2}(\mathbb{Z}^{2d})}.
\]
In particular, when $p < \infty$, this is equivalent to the convolution inequality:

$$
\frac{p_1}{m_1} \cdot \frac{p_1}{p_2} (\mathbb{Z}^{2d})^{\ast} \cdot \frac{q_1/q_2}{m_2} (\mathbb{Z}^{2d}) \subset \frac{q_1/q_2}{m_2} (\mathbb{Z}^{2d}).
$$

Hence, the conclusion in Theorem 1.1 (with $\tau = 0$) follows directly by Proposition 3.1.

\[\square\]

**Remark 3.2.** The reader may observe that, in the proof of Theorem 1.1, we only use Proposition 3.1 with $\alpha = 1$. However, in order to verify Proposition 3.1 with $\alpha = 1$, we actually need (3.1) for sufficiently small $\alpha$. Thus, we would like to keep a stronger version with any $\alpha > 0$.

### 3.2. Relations between BMM and BMW for $\tau \in (0, 1)$

**Proposition 3.3.** Assume $p_i, q_i, p, q \in (0, \infty], i = 1, 2$. For $\tau \in (0, 1)$, we have

$$
\|W_\tau(f_1, f_2)\|_{M^{p,q}_{1\otimes \tilde{m}_{j-1}}} \lesssim \|f_1\|_{M^{p_1,q_1}_{m_1}} \cdot \|f_2\|_{M^{p_2,q_2}_{m_2}}
$$

if and only if

$$
\|W_\tau(f_1, f_2)\|_{W(\mathcal{F}L^p, L^q_{\tilde{m}_{j-1}})} \lesssim \|f_1\|_{M^{p_1,q_1}_{m_1}} \cdot \|f_2\|_{M^{p_2,q_2}_{m_2}},
$$

where $\tilde{m}(\zeta_1, \zeta_2) = m((1-\tau)\zeta_1, \tau \zeta_2)$, $\tilde{m}_2(z_1, z_2) = m_2(\frac{1-\tau}{\tau}z_1, \frac{1-\tau}{\tau}z_2)$.

**Proof.** Using Lemmas 2.3, 2.9 and 2.11, we obtain

$$
\|W_\tau(f_1, f_2)\|_{W(\mathcal{F}L^p, L^q_{\tilde{m}_{j-1}})} \sim \|e^{-2\pi i\tau u_1 u_2} V_{2f_2} f_1(-J u)\|_{W(\mathcal{F}L^p, L^q_{\tilde{m}_{j-1}})}
$$

where $\tilde{m}(\zeta_1, \zeta_2) = m((1-\tau)\zeta_1, \tau \zeta_2)$. On the other hand, recalling $W(\mathcal{F}L^p, L^q_{m_{j-1}}) = \mathcal{F}M^{p,q}_{1\otimes m_{j-1}}$, and using Lemma 2.4 and Lemma 2.1, we obtain

$$
\|W_\tau(f_1, f_2)\|_{M^{p,q}_{1\otimes \tilde{m}_{j-1}}} = \|\mathcal{F}(W_\tau(f_1, f_2))\|_{W(\mathcal{F}L^p, L^q_{\tilde{m}_{j-1}})}
$$

Observing that $J^{-1} = -J$, and applying Lemma 2.9, we continue the above estimate by

$$
\|W_\tau(f_1, f_2)\|_{M^{p,q}_{1\otimes \tilde{m}_{j-1}}} = \|V_{f_2} f_1(-J u)\|_{W(\mathcal{F}L^p, L^q_{\tilde{m}_{j-1}})} \sim \|V_{f_2} f_1\|_{W(\mathcal{F}L^p, L^q_{m_{j-1}})}.
$$

Using (3.7) and (3.8), we obtain

$$
\|W_\tau(f_1, f_2)\|_{W(\mathcal{F}L^p, L^q_{\tilde{m}_{j-1}})} \sim \|W_\tau(f_1, D_{1-\tau}^{-1} f_2)\|_{M^{p,q}_{1\otimes \tilde{m}_{j-1}}}.
$$

Moreover, by Lemma 2.8, we have

$$
\|f_2\|_{M^{p_2,q_2}_{m_2}} = \|D_{1-\tau}^{-1} f_2\|_{M^{p_2,q_2}_{m_2}} \sim \|D_{1-\tau}^{-1} f_2\|_{M^{p_2,q_2}_{m_2}},
$$

where $\tilde{m}_2(z_1, z_2) = m(\frac{1-\tau}{\tau}z_1, \frac{1-\tau}{\tau}z_2)$.

If (3.5) holds, (3.6) follows by

$$
\|W_\tau(f_1, f_2)\|_{W(\mathcal{F}L^p, L^q_{\tilde{m}_{j-1}})} \sim \|W_\tau(f_1, D_{1-\tau}^{-1} f_2)\|_{M^{p,q}_{1\otimes \tilde{m}_{j-1}}} \lesssim \|f_1\|_{M^{p_1,q_1}_{m_1}} \cdot \|D_{1-\tau}^{-1} f_2\|_{M^{p_2,q_2}_{m_2}} \sim \|f_1\|_{M^{p_1,q_1}_{m_1}} \cdot \|f_2\|_{M^{p_2,q_2}_{m_2}}.
$$
Vice versa, if (3.6) holds, then (3.5) follows by
\[
\|W_\tau(f_1, f_2)\|_{M^{p,q}_{1\otimes m_{j-1}}} = \|W_\tau(f_1, D \frac{1}{\tau} T D \frac{1}{\tau} I f_2)\|_{M^{p,q}_{1\otimes m_{j-1}}} \\
\sim \|W_\tau(f_1, D \frac{1}{\tau} T f_2)\|_{W(\mathcal{F}L^p, L^q_m)} \\
\lesssim \|f_1\|_{M^{p_1,q_1}_{m_1}} \cdot \|D \frac{1}{\tau} T f_2\|_{M^{p_2,q_2}_{m_2}} \sim \|f_1\|_{M^{p_1,q_1}_{m_1}} \|f_2\|_{M^{p_2,q_2}_{m_2}}.
\]

By the above proposition, we can prove Theorem 1.4 for \( \tau \in (0, 1) \).

**Proof of Theorem 1.4 for \( \tau \in (0, 1) \).** Observe that \( (\tilde{m}_{j-1})_I = \tilde{m} \) and \( I(\tilde{m}_2) = \tilde{m}_2 \). The desired conclusion follows by Theorem 1.1 and Proposition 3.3.

\]

3.3. Discretization by Gabor coefficients for BMW with endpoints \( \tau = 0, 1 \).

**Proof of Theorem 1.4 for \( \tau = 0 \).** We divide the proof into two parts.

**“Only if” part.** Let \( f_1, f_2, \Phi \) be the same functions in the proof of Theorem 3.1. Write
\[
\|W_0(f_1, f_2)\|_{W(\mathcal{F}L^p, L^q_m)} = \|\mathcal{F}^{-1}(W_0(f_1, f_2))\|_{L^{p,q}_m} = \|V_\Phi(\mathcal{F}^{-1}(W_0(f_1, f_2)))(z, \zeta)\|_{L^{p,q}_m} \\
= \|V_\Phi(\mathcal{F}^{-1}(W_0(f_1, f_2))((\zeta, -z))\|_{L^{p,q}_m} = \|V_\Phi f_1(\zeta_1, \zeta_2 - z_1) V_\Phi f_2(\zeta_1 - z_2, \zeta_2)\|_{L^{p,q}_m} \\
= \|V_\Phi f_1(\zeta_1, z_1) V_\Phi f_2(z_2, \zeta_2)\|_{L^{p,q}_m}.
\]

By the same argument in the proof of Proposition 3.1, we have the lower estimate
\[
\|W_0(f_1, f_2)\|_{W(\mathcal{F}L^p, L^q_m)} \gtrsim \|V_\Phi f_1(n_1, k_1) V_\Phi f_2(k_2, n_2)\|_{\mathbb{Z}^d \times \mathbb{Z}^d} \gtrsim \|(a_{n_1, k_1} b_{k_2, n_2})\|_{L^{p,q}_m},
\]
and the upper bound estimates \( \|f_1\|_{M^{p_1,q_1}_{m_1}} \lesssim \|(a_{n_1, k_1} b_{k_2, n_2})\|_{L^{p,q}_m} \), \( \|f_2\|_{M^{p_2,q_2}_{m_2}} \lesssim \|(a_{n_1, k_1} b_{k_2, n_2})\|_{L^{p,q}_m} \), respectively. Here \( \tilde{m}(z) := m(z) \), and \( \tilde{m}_i(z) := m_i(z) \), \( z \in \mathbb{R}^{2d}, N \in \mathbb{N}, i = 1, 2 \).

**“If” part.** As in the proof of Theorem 3.1, for \( \Phi \in S(\mathbb{R}^{2d}) \) mentioned above, there exists \( \tilde{\Psi} \in \mathcal{M}^{p,q}_{2d} \), such that \( \tilde{\Psi} \mathcal{F} \Phi = D^{\alpha, \beta} \mathcal{F} \Phi = I \) on \( M^{p,q}_{1\otimes m_1} \), where \( \alpha = 1/N_1, \beta = \alpha/N_2 \), with some large integers \( N_1, N_2 \).

Applying Lemma 2.2, we find that
\[
\|C^{\alpha, \beta}_\Phi \mathcal{F}^{-1} W_0(f_1, f_2)\|_{L^{p,q}_m} = \|V_\Phi(\mathcal{F}^{-1} W_0(f_1, f_2))(z, \zeta)\|_{\alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d} \|_{L^{p,q}_m} \\
\leq \|V_\Phi(\mathcal{F}^{-1} W_0(f_1, f_2))(z, \zeta)\|_{\alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d} \|_{L^{p,q}_m} \\
= \|V_\Phi f_1(\zeta_1, z_1) V_\Phi f_2(z_2, \zeta_2)\|_{\alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d} \|_{L^{p,q}_m} \\
\lesssim \|V_\Phi f_1(\beta(n_1, k_1)) V_\Phi f_2(\beta(k_2, n_2))\|_{L^{p,q}_m}.
\]
Using the inequality of discrete mixed-norm spaces and the sampling property of STFT, we continue the above estimate by
\[
\|V_{\phi} f_1(\beta(n_1,k_1))V_{\phi} f_2(\beta(n_2,k_2))\|_{1 \otimes m}^{p,q}(\mathbb{Z}^{2d} \times \mathbb{Z}^{2d}) \lesssim \|V_{\phi} f_1(\beta k)\|_{p_1 \cdot q_1}^{p_1} \cdot \|V_{\phi} f_2(\beta k)\|_{p_2 \cdot q_2}^{p_2}.
\]

The above two estimates imply that
\[
\|C_{\Phi,\beta}^\alpha \mathcal{F}^{-1} W_0(f_1, f_2)\|_{p,q} \lesssim \|f_1\|_{M_{m_1}^{p_1 \cdot q_1}} \cdot \|f_2\|_{M_{m_2}^{p_2 \cdot q_2}}.
\]

Hence,
\[
\|W_0(f_1, f_2)\|_{W(\mathcal{F} L^p, L^q_m)} = \|\mathcal{F}^{-1} W_0(f_1, f_2)\|_{M_{1 \otimes m}^{p,q}} \lesssim \|C_{\Phi,\beta}^\alpha \mathcal{F}^{-1} W_0(f_1, f_2)\|_{M_{1 \otimes m}^{p,q}} \lesssim \|f_1\|_{M_{m_1}^{p_1 \cdot q_1}} \cdot \|f_2\|_{M_{m_2}^{p_2 \cdot q_2}}.
\]

We have now completed the proof.

\[\square\]

**Proof of Theorem 1.4 for \(\tau = 1\).** A direct calculation yields that
\[
\|W_1(f_1, f_2)\|_{W(\mathcal{F} L^p, L^q_m)} = \|W_0(f_1, f_2)\|_{W(\mathcal{F} L^p, L^q_m)} = \|W_0(f_2, f_1)\|_{W(\mathcal{F} L^p, L^q_m)} \lesssim \|f_1\|_{M_{m_1}^{p_1 \cdot q_1}} \cdot \|f_2\|_{M_{m_2}^{p_2 \cdot q_2}}.
\]

The desired result follows by this and the case of \(\tau = 0\). \[\square\]

## 4. Second characterizations: separations of mixed-norm inequalities

### 4.1. Separation of mixed-norm convolution inequality.

**Proposition 4.1** (Separation of convolution). Assume \(p_i, q_i, p, q \in (0, \infty), i = 1, 2\). Suppose that \(m\) is submultiplicative, \(m_i = \omega_i \otimes \mu_i, i = 1, 2\). We have
\[
\|\left(ak_{1,k_2} b_{n_1-k_1,n_2-k_2}\right)\|_{1 \otimes m}^{p,q}(\mathbb{Z}^{2d}) \lesssim \|\tilde{a}\|_{1 \otimes m}^{p_1,q_1}(\mathbb{Z}^{2d}) \|\tilde{b}\|_{1 \otimes m}^{p_2,q_2}(\mathbb{Z}^{2d}) \quad (4.1)
\]

if and only if
\[
\begin{align*}
\ell^{p_1/p}_{\omega_1}(\mathbb{Z}^d) \ast \ell^{p_2/p}_{\omega_2}(\mathbb{Z}^d) \subset \ell^{q_1/p}_{\mu_1}(\mathbb{Z}^d), & \quad \ell^{q_1/p}_{\mu_1}(\mathbb{Z}^d) \subset \ell^{q_2/p}_{\mu_2}(\mathbb{Z}^d), \quad p < \infty, \quad (4.2) \\
\ell^{p_1/q}_{\mu_1}(\mathbb{Z}^d), & \quad \ell^{p_2/q}_{\mu_2}(\mathbb{Z}^d) \subset \ell^{q_2/q}_{\mu_2}(\mathbb{Z}^d), \quad p \geq q. \quad (4.3)
\end{align*}
\]

Here, we denote \(m_\alpha(z_1) = m(z_1,0)\) and \(m_\beta(z_2) = m(0,z_2)\) for \(z_1, z_2 \in \mathbb{R}^d\).

**Proof.** We divide the proof into two parts.

**“Only if” part.** In this part, we separate (4.1) by means of testing it by several constructed discrete sequences.
**Test 1.** Set \( a_{k_1,n_1} = 0 \) if \( n_1 \neq 0 \mathbb{Z}^d \), and \( b_{k_2,n_2} = 0 \) if \( n_2 \neq 0 \mathbb{Z}^d \). The inequality (4.1) says that

\[
\left( \sum_{n_1 \in \mathbb{Z}^d} \left( \sum_{k_1 \in \mathbb{Z}^d} |a_{k_1,0}b_{n_1-k_1,0}|^p \right)^{q/p} m(n_1,0)^{q} \right)^{1/q} \lesssim \left( \sum_{k_1 \in \mathbb{Z}^d} |a_{k_1,0}m_1(k_1,0)|^{p_1} \right)^{1/p_1} \left( \sum_{k_2 \in \mathbb{Z}^d} |b_{k_2,0}m_2(k_2,0)|^{p_2} \right)^{1/p_2}. \tag{4.4}
\]

Note that for \( p < \infty \) the above inequality is equivalent to \( l^{p_1/p}(\mathbb{Z}^d) \ast l^{p_2/p}(\mathbb{Z}^d) \subset l^{q/p}(\mathbb{Z}^d) \). Moreover, if we further assume \( b_{k_2,n_2} = 0 \) for all \( n_2, k_2 \neq 0 \mathbb{Z}^d \), (4.4) implies the embedding relation \( l^{q_1}(\mathbb{Z}^d) \subset l^{q_2}(\mathbb{Z}^d) \). On the other hand, by taking \( a_{k_1,n_1} = 0 \) for all \( k_1, n_1 \neq 0 \mathbb{Z}^d \), we get \( l^{q_2}(\mathbb{Z}^d) \subset l^{q_1}(\mathbb{Z}^d) \).

**Test 2.** Set \( a_{k_1,n_1} = 0 \) if \( k_1 \neq 0 \mathbb{Z}^d \), and \( b_{k_2,n_2} = 0 \) if \( k_2 \neq 0 \mathbb{Z}^d \). The inequality (4.1) says that

\[
\left( \sum_{n_2 \in \mathbb{Z}^d} \left( \sum_{k_2 \in \mathbb{Z}^d} |a_{0,k_2}b_{n_2-k_2,0}|^p \right)^{q/p} m(0,n_2)^q \right)^{1/q} \lesssim \left( \sum_{n_1 \in \mathbb{Z}^d} |a_{0,n_1}m_1(0,n_1)|^{q_1} \right)^{1/q_1} \left( \sum_{n_2 \in \mathbb{Z}^d} |b_{n_2,0}m_2(0,n_2)|^{q_2} \right)^{1/q_2}. \tag{4.5}
\]

If \( p < \infty \), the above inequality is equivalent to \( l^{q_1/p}(\mathbb{Z}^d) \ast l^{q_2/p}(\mathbb{Z}^d) \subset l^{q/p}(\mathbb{Z}^d) \). By taking \( b_{k_2,n_2} = 0 \) for all \( n_2, k_2 \neq 0 \mathbb{Z}^d \), (4.4) becomes the embedding relation \( l^{q_1}(\mathbb{Z}^d) \subset l^{q_2}(\mathbb{Z}^d) \). Similarly, by taking \( a_{k_1,n_1} = 0 \) for all \( k_1, n_1 \neq 0 \mathbb{Z}^d \), we get \( l^{q_2}(\mathbb{Z}^d) \subset l^{q_1}(\mathbb{Z}^d) \).

From Tests 1 and 2, we get (4.2) and (4.3).

**“If” part.** In this part, we consider following cases.

**Case 1: \( p < q \).** In this case, we will verify (4.2) implies (4.1). Using the Minkowski inequality and \( m(n_1,n_2) \lesssim m_\alpha(n_1)m_\beta(n_2) \), we conclude that

\[
\left( \sum_{n_1,n_2 \in \mathbb{Z}^d} \left( \sum_{k_1,k_2 \in \mathbb{Z}^d} |a_{k_1,k_2}b_{n_1-k_1-k_2,0}|^p \right)^{q/p} m(n_1,n_2)^q \right)^{1/q} \lesssim \left( \sum_{n_1 \in \mathbb{Z}^d} \left( \sum_{k_1 \in \mathbb{Z}^d} |a_{k_1}b_{n_1-k_1,0}|^p \right)^{q/p} m_\alpha(n_1)^q m_\beta(n_2)^q \right)^{1/q} \lesssim \left( \sum_{k_1 \in \mathbb{Z}^d} \left( \sum_{n_1 \in \mathbb{Z}^d} \left( \sum_{n_2 \in \mathbb{Z}^d} |a_{k_1,k_2}b_{n_1-k_1-k_2,0}|^p \right)^{q/p} m_\alpha(n_1)^q \right)^{p/q} m_\beta(n_2)^q \right)^{1/q}.
\]

Using the convolution inequality \( l^{p_1/p}(\mathbb{Z}^d) \ast l^{p_2/p}(\mathbb{Z}^d) \subset l^{q/p}(\mathbb{Z}^d) \), we continue the above estimate by

\[
\left( \sum_{n_1,n_2 \in \mathbb{Z}^d} \left( \sum_{k_1,k_2 \in \mathbb{Z}^d} |a_{k_1,k_2}b_{n_1-k_1-k_2,0}|^p \right)^{q/p} m_\alpha(n_1)^q \right)^{1/q} \lesssim \left( \sum_{n_1,n_2 \in \mathbb{Z}^d} \left( \sum_{k_1 \in \mathbb{Z}^d} |a_{k_1,k_2}b_{n_1-k_1-k_2,0}|^p \right)^{q/p} m_\alpha(n_1)^q \right)^{1/q} \lesssim \left( \sum_{k_1 \in \mathbb{Z}^d} \left( \sum_{n_1 \in \mathbb{Z}^d} \left( \sum_{n_2 \in \mathbb{Z}^d} |a_{k_1,k_2}b_{n_1-k_1-k_2,0}|^p \right)^{q/p} m_\alpha(n_1)^q \right)^{p/q} m_\beta(n_2)^q \right)^{1/q}.
\]


Then, the convolution inequality \( l_{\mu^1}^p(\mathbb{Z}^d) \ast l_{\mu^2}^q(\mathbb{Z}^d) \subset l_{m}^q(\mathbb{Z}^d) \) further implies that
\[
\left( \sum_{n_1, n_2 \in \mathbb{Z}^d} \left( \sum_{k_1, k_2 \in \mathbb{Z}^d} \| (a_{k_1, k_2})_{k_1} \|_{\mu^1}^p \| (b_{n_1, n_2-k_2})_{n_1} \|_{\mu^2}^q \right)^{q/p} m_{\beta}(n_2)^{q/p} \right)^{1/q} \leq \| (a_{k_1, k_2})_{k_1} \|_{\mu^1}^p \| (b_{n_1, n_2-k_2})_{n_1} \|_{\mu^2}^q m_{\beta}(n_2)^{q/p}.
\]

**Case 2:** \( p \geq q \). In this case, we will verify (4.3) implies (4.1), then the conclusion (4.2) \( \Rightarrow \) (4.1) follows by the fact that (4.2) implies (4.3) for \( p < \infty \). By the well known embedding relation \( l^q \subset l^p \) for \( p \geq q \), and the submultiplicative property of \( m \), we have
\[
\left( \sum_{n_1, n_2 \in \mathbb{Z}^d} \left( \sum_{k_1, k_2 \in \mathbb{Z}^d} |a_{k_1, k_2}|^q |b_{n_1, n_2-k_2}|^q m(k_1, k_2)^q m(n_1 - k_1, n_2 - k_2)^q \right)^{1/q} \right) \leq \| a \|_{p, q, m_{\alpha} \otimes m_{\beta}} \| b \|_{p, q, m_{\alpha} \otimes m_{\beta}}.
\]

Using (4.3), i.e., the embedding relations \( l_{\omega^1}{\mu^1}(\mathbb{Z}^d) \), \( l_{\omega^2}{\mu^2}(\mathbb{Z}^d) \subset l_{m_{\alpha}}^q(\mathbb{Z}^d) \) and \( l_{\mu^1}{\omega^1}(\mathbb{Z}^d) \), \( l_{\mu^2}{\omega^2}(\mathbb{Z}^d) \subset l_{m_{\beta}}^q(\mathbb{Z}^d) \), we obtain the following embedding relations for discrete mixed-norm spaces:
\[
l_{\omega_{\mu^1}}^q(\mathbb{Z}^d) \subset l_{m_{\alpha}}^q(\mathbb{Z}^d), \quad l_{\omega_{\mu^2}}^q(\mathbb{Z}^d) \subset l_{m_{\beta}}^q(\mathbb{Z}^d).
\]

Now, we continue our main estimate by
\[
\left( \sum_{n_1, n_2 \in \mathbb{Z}^d} \left( \sum_{k_1, k_2 \in \mathbb{Z}^d} |a_{k_1, k_2}|^q |b_{n_1, n_2-k_2}|^q m(k_1, k_2)^q m(n_1 - k_1, n_2 - k_2)^q \right)^{1/q} \right) \leq \| a \|_{p, q, m_{\alpha} \otimes m_{\beta}} \| b \|_{p, q, m_{\alpha} \otimes m_{\beta}} \leq \| a \|_{p, q, \omega_{\mu^1} \otimes \omega_{\mu^2}} \| b \|_{p, q, \omega_{\mu^2} \otimes \omega_{\mu^2}}.
\]
This concludes the proof.

**Proof of Theorem 1.2.** This proof follows directly by Theorem 1.1 and Proposition 4.1 with the fact that \( (m_{\beta})_{\alpha} = I m_{\beta} \) and \( (m_{\beta})_{\beta} = m_{\alpha} \).

### 4.2. Separation of mixed-norm embedding inequality.

**Proposition 4.2** (Separation of embedding). Assume \( p_i, q_i, p, q \in (0, \infty), i = 1, 2 \). Suppose that \( m \) is submultiplicative, \( m_i = \omega_i \otimes \mu_i, i = 1, 2 \). We have
\[
\| (a_{n_1, k_1} b_{n_2, k_2}) \|_{l_{\mu_i}^p(\mathbb{Z}^d)} \leq \| a \|_{l_{\mu^1}^p(\mathbb{Z}^d)} \| b \|_{l_{\mu^2}^q(\mathbb{Z}^d)} \quad (4.6)
\]
if and only if the following two convolution relations:
\[
l_{\omega_i}^p \subset l^p, \quad l_{\mu_i}^q \subset l_{m_i}^q,
\]
hold. Here, we denote \( m_{\alpha}(z_1) = m(z_1, 0) \) and \( m_{\beta}(z_2) = m(0, z_2) \) for \( z_1, z_2 \in \mathbb{R}^d \).
Proof. We divide the proof into two parts.

"Only if" part. Write (4.6) by

\[
\left( \sum_{n_1,n_2 \in \mathbb{Z}^d} \left( \sum_{k_1,k_2 \in \mathbb{Z}^d} |a_{n_1,k_1} b_{k_2,n_2}|^p \right)^{q/p} m(n_1,n_2)^q \right)^{1/q} \\
\lesssim \left( \sum_{n_1 \in \mathbb{Z}^d} \left( \sum_{k_1 \in \mathbb{Z}^d} |a_{n_1,k_1} m_1(k_1,n_1)|^{p_1} \right)^{q_1/p_1} \right)^{1/q_1} \left( \sum_{n_2 \in \mathbb{Z}^d} \left( \sum_{k_2 \in \mathbb{Z}^d} |b_{k_2,n_2} m_2(k_2,n_2)|^{p_2} \right)^{q_2/p_2} \right)^{1/q_2}.
\]

(4.9)

The above inequality will be tested by several constructed discrete sequences.

**Test 1.** Set \(a_{k_1,n_1} = 0\) if \((k_1,n_1) \neq 0_{\mathbb{Z}^d \times \mathbb{Z}^d}\), and \(b_{k_2,n_2} = 0\) if \(n_2 \neq 0_{\mathbb{Z}^d}\). The inequality (4.9) says that

\[
\left( \sum_{k_2 \in \mathbb{Z}^d} |b_{k_2,0}|^p \right)^{1/p} \lesssim \left( \sum_{k_2 \in \mathbb{Z}^d} |b_{k_2,0} m_2(k_2,0)|^{p_2} \right)^{1/p_2},
\]

which implies the embedding relation \(l^r_{p_2} \subset l^q_{p_2}\) in (4.7).

**Test 2.** Set \(a_{k_1,n_1} = 0\) if \((k_1,n_1) \neq 0_{\mathbb{Z}^d \times \mathbb{Z}^d}\), and \(b_{k_2,n_2} = 0\) if \(k_2 \neq 0_{\mathbb{Z}^d}\). The inequality (4.9) says that

\[
\left( \sum_{n_1 \in \mathbb{Z}^d} |b_{0,n_2}|^q m(0,n_2)^q \right)^{1/q} \lesssim \left( \sum_{n_2 \in \mathbb{Z}^d} |b_{0,n_2} m_2(0,n_2)|^q \right)^{1/q_2}.
\]

This is just the embedding relation \(l^q_{p_2} \subset l^q_{b_2}\) in (4.7).

**Test 3.** Set \(b_{k_2,n_2} = 0\) if \((k_2,n_2) \neq 0_{\mathbb{Z}^d \times \mathbb{Z}^d}\). The inequality (4.9) says that

\[
\left( \sum_{n_1 \in \mathbb{Z}^d} \left( \sum_{k_1 \in \mathbb{Z}^d} |a_{n_1,k_1}|^p m(n_1,0)^q \right)^{q/p} \right)^{1/q} \lesssim \left( \sum_{k_1 \in \mathbb{Z}^d} \left( \sum_{n_1 \in \mathbb{Z}^d} |a_{k_1,n_1} m_1(k_1,n_1)|^{q_1/p_1} \right)^{q_1} \right)^{1/q_1}.
\]

This is just the embedding relation \(l^{p_1,q_1}_{\omega_1} \subset l^{q_1}_{\omega_1} \otimes \mu_1\) in (4.8).

"If" part. Recall \(m(n_1,n_2) \lesssim m_\alpha(n_1)m_\beta(n_2)\). Write

\[
\left( \sum_{n_1,n_2 \in \mathbb{Z}^d} \left( \sum_{k_1,k_2 \in \mathbb{Z}^d} |a_{n_1,k_1} b_{k_2,n_2}|^p \right)^{q/p} m(n_1,n_2)^q \right)^{1/q}
\]

\[
\lesssim \left( \sum_{n_1 \in \mathbb{Z}^d} \left( \sum_{k_1 \in \mathbb{Z}^d} |a_{n_1,k_1}|^p m_\alpha(n_1)^q \right)^{q/p} \right)^{1/q} \left( \sum_{n_2 \in \mathbb{Z}^d} \left( \sum_{k_2 \in \mathbb{Z}^d} |b_{k_2,n_2}|^p m_\beta(n_2)^q \right)^{q/p} \right)^{1/q} \\
\lesssim \left( \sum_{n_1 \in \mathbb{Z}^d} \left( \sum_{k_1 \in \mathbb{Z}^d} |a_{n_1,k_1}|^p m_\alpha(n_1)^q \right)^{q_1/p_1} \right)^{1/q_1} \left( \sum_{n_2 \in \mathbb{Z}^d} \left( \sum_{k_2 \in \mathbb{Z}^d} |b_{k_2,n_2}|^p m_\beta(n_2)^q \right)^{q_2/p_2} \right)^{1/q_2},
\]

where in the last inequality we use (4.7) and (4.8). \(\square\)
Proof of Theorem 1.5. Note that \( \tilde{m} \) is submultiplicative if \( m \) is submultiplicative. Observe that \( \tilde{m}_2 = \tilde{\omega}_2 \otimes \mu_2 \). The case \( \tau \in (0, 1) \) follows directly by Theorem 1.4 and Proposition 4.1. The endpoint cases \( \tau = 0, 1 \) follows by Theorem 1.4 and Proposition 4.2. \( \square \)

5. Third characterizations: applications for power weights

5.1. Sharp exponents for convolution inequalities. Note that Theorem 1.3 is a direct conclusion of Theorem 1.2. Observe that in Theorem 1.5, \( \tilde{m} \sim m, \tilde{\omega}_2 \sim \omega_2 \) and \( \tilde{\mu}_2 \sim \mu_2 \), for \( m = v_s, \omega_2 = v_{s_2}, \mu_2 = v_{s_2} \). Then Theorem 1.6 with \( \tau \in (0, 1) \) follows by Theorem 1.5. See the proof of Theorem 1.6 for \( \tau = 0, 1 \) in the next subsection.

If we want to get the sharp exponents for the convolution inequalities mentioned in Theorem 1.3 and 1.6, the following two lemmas is needed.

**Lemma 5.1.** *(See [17, Theorem 1.1])* Suppose \( 1 \leq q, q_1, q_2 \leq \infty, s, s_1, s_2 \in \mathbb{R} \). Then

\[
{I}^q_{s_1} (\mathbb{R}^d) + {I}^q_{s_2} (\mathbb{R}^d) \subset {I}^q_s (\mathbb{R}^d)
\]

if and only if \( (q, s) = (q, q_1, q_2, s, s_1, s_2) \) satisfies one of the following conditions \( A_i, i = 1, 2, 3, 4 \).

\[
(A_1) \quad \begin{cases}
    s \leq s_1, s \leq s_2, & 0 \leq s_1 + s_2,
    
    1 + \left( \frac{1}{q} + \frac{s}{d} \right) \lor 0 < \left( \frac{1}{q_1} + \frac{s_1}{d} \right) \lor 0 + \left( \frac{1}{q_2} + \frac{s_2}{d} \right) \lor 0,

    \frac{1}{q} + \frac{s}{d} \leq \frac{1}{q_1} + \frac{s_1}{d}, \quad \frac{1}{q} + \frac{s}{d} \leq \frac{1}{q_2} + \frac{s_2}{d}, \quad 1 \leq \frac{1}{q_1} + \frac{s_1}{d} + \frac{1}{q_2} + \frac{s_2}{d},

    (q, s) = (q_1, s_1) \text{ if } \frac{1}{q} + \frac{s}{d} = \frac{1}{q_1} + \frac{s_1}{d},

    (q, s) = (q_2, s_2) \text{ if } \frac{1}{q} + \frac{s}{d} = \frac{1}{q_2} + \frac{s_2}{d},

    (q_1, -s_1) = (q_2, s_2) \text{ if } 1 = \frac{1}{q_1} + \frac{s_1}{d} + \frac{1}{q_2} + \frac{s_2}{d};
\]

\[
(A_2) \quad \begin{cases}
    s = s_1 = s_2 = 0,

    \frac{1}{q} + \frac{s}{d} = 1 \text{ or } q = q_2, q_1 = 1 \text{ or } q = \infty, \frac{1}{q_1} + \frac{1}{q_2} = 1;
\]

\[
(A_3) \quad \begin{cases}
    s \leq s_1, s \leq s_2,

    \frac{1}{q_1} + \frac{1}{q_2} = 1, \quad s_1 + s_2 = 0,

    \frac{1}{q} + \frac{s}{d} < 0 \leq \frac{1}{q_1} + \frac{s_1}{d} + \frac{1}{q_2} + \frac{s_2}{d};
\]

\[
(A_4) \quad \begin{cases}
    s \leq s_1, s \leq s_2, \quad 0 \leq s_1 + s_2,

    1 + \frac{1}{q} + \frac{s}{d} = \frac{1}{q_1} + \frac{s_1}{d} + \frac{1}{q_2} + \frac{s_2}{d}, \quad \frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2},

    \frac{1}{q} + \frac{s}{d} < \frac{1}{q_1} + \frac{s_1}{d}, \quad \frac{1}{q} + \frac{s}{d} < \frac{1}{q_2} + \frac{s_2}{d}, \quad \frac{1}{q} + \frac{s}{d} > 0,

    \frac{1}{q} \neq \infty, \quad q_1, q_2 \neq 1, \text{ if } q = s_1 \text{ or } s = s_2.
\]

Here, we use the notation

\[
a \lor b = \max\{a, b\}.
\]

**Lemma 5.2.** *(see [17, Proposition 2.5])* Suppose \( 0 < q, q_1, q_2 \leq \infty \). Then

\[
{I}^q_{s_1} (\mathbb{R}^d) + {I}^q_{s_2} (\mathbb{R}^d) \subset {I}^q_s (\mathbb{R}^d)
\]

holds if and only if

\[
1 + \frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2}, \quad \frac{1}{q_1} \leq \frac{1}{q}, \quad \frac{1}{q} \leq \frac{1}{q_2}.
\]

Moreover, if (5.7) holds, we have

\[
{I}^q_{s_1} (\mathbb{R}^d) + {I}^q_{s_2} (\mathbb{R}^d) \subset {I}^q_s (\mathbb{R}^d).
\]
Proof. We only point out that (5.8) is a direct conclusion of (5.6) and $(j)^s \lesssim |j - l|^s |l|^s$ for all $s \in \mathbb{R}$. □

By the above two lemmas, we obtain the following results.

**Theorem 5.3.** Suppose that $p < \infty$ and $p \leq p_i, q_i, q$ for $i = 1, 2$, $\tau \in [0, 1]$. We have

$$W_\tau : \mathcal{M}^{p_1, q_1}_{v_1, t_1}(\mathbb{R}^d) \times \mathcal{M}^{p_2, q_2}_{v_2, t_2}(\mathbb{R}^d) \rightarrow M^{p, q}_{1 \circ v_3}(\mathbb{R}^{2d})$$

if and only if

$$(q/p, p_1/p, p_2/p, ps, ps_1, ps_2), (q/p, q_1/p, q_2/p, ps, pt_1, pt_2) \in \mathcal{A}.$$  

**Proof.** By Theorem 1.3, we have

$$W_\tau : \mathcal{M}^{p_1, q_1}_{v_1, t_1}(\mathbb{R}^d) \times \mathcal{M}^{p_2, q_2}_{v_2, t_2}(\mathbb{R}^d) \rightarrow M^{p, q}_{1 \circ v_3}(\mathbb{R}^{2d})$$

if and only if

$$\mathcal{L}_{ps}^{p_1/p}(\mathbb{Z}^d) \ast \mathcal{L}_{ps}^{p_2/p}(\mathbb{Z}^d) \subset \mathcal{L}_{ps}^{q/p}, \quad \mathcal{L}_{pt_1}^{p_1/p} \ast \mathcal{L}_{pt_2}^{q_2/p} \subset \mathcal{L}_{ps}^{q/p}.$$  

By Lemma 5.1, we find that the above two convolution inequalities are equivalent to

$$(q/p, p_1/p, p_2/p, ps, ps_1, ps_2), (q/p, q_1/p, q_2/p, ps, pt_1, pt_2) \in \mathcal{A}.$$  

□

**Theorem 5.4.** Suppose that $p < \infty$ and $p \leq p_i, q_i, q$ for $i = 1, 2$, $\tau \in (0, 1)$. We have

$$W_\tau : \mathcal{M}^{p_1, q_1}_{v_1, t_1} \times \mathcal{M}^{p_2, q_2}_{v_2, t_2} \rightarrow W(\mathcal{F} L^p, L^q_{L^m})$$

if and only if

$$(q/p, p_1/p, p_2/p, ps, ps_1, ps_2), (q/p, q_1/p, q_2/p, ps, pt_1, pt_2) \in \mathcal{A}.$$  

**Proof.** As the proof of Theorem 5.3, this is a direct conclusion of Theorem 1.6 and Lemma 5.1. □

Next, using Lemma 5.2, we recapture the main results of BMM in [8, 4].

**Theorem 5.5** (see also [8, 4]). Let $0 < p, q, p_i, q_i \leq \infty$ for $i = 1, 2$, $\tau \in [0, 1]$. We have

$$W_\tau : \mathcal{M}^{p_1, q_1}_{|s|} \times \mathcal{M}^{p_2, q_2}_{s} \rightarrow M^{p, q}_{s}$$

if and only if

$$p_i, q_i \leq q, \quad i = 1, 2$$  

and

$$\frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{p} + \frac{1}{q}. \quad (5.9)$$

Moreover, if (5.9) and (5.10) hold, we have

$$W_\tau : \mathcal{M}^{p_1, q_1}_{|s|} \times \mathcal{M}^{p_2, q_2}_{s} \rightarrow M^{p, q}_{s}.$$  

**Proof.** The case $p < \infty$ follows by Theorem 1.3 and Lemma 5.2. For $p = \infty$, the corresponding results can be verified by Theorem 1.3 and Lemma 5.7. □

Similarly, we also have following result for BMW.
Theorem 5.6. Let $0 < p, q, p_i, q_i \leq \infty$ for $i = 1, 2$, $\tau \in (0, 1)$. We have

$$W_{\tau} : \mathcal{M}^{p_1,q_1} \times \mathcal{M}^{p_2,q_2} \rightarrow W(\mathcal{F} L^p, L^q)$$

if and only if

$$p_i, q_i \leq q, \quad i = 1, 2$$

and

$$\frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{p} + \frac{1}{q}.$$  \hspace{1cm} (5.11)

Moreover, if (5.11) and (5.12) hold, we have

$$W_{\tau} : \mathcal{M}^{p_1,q_1}_{|s|} \times \mathcal{M}^{p_2,q_2}_{|s|} \rightarrow W(\mathcal{F} L^p, L^q_s).$$

5.2. Sharp exponents for embedding relations. In order to get the sharp exponents of embedding relations mentioned in Theorems 1.3 and 1.6, we recall the following lemma.

Lemma 5.7 (Sharpness of embedding, discrete form). Suppose $0 < q, q_1, q_2 \leq \infty$, $s, s_1, s_2 \in \mathbb{R}$. Then

$$l^{q_1}_{v_1} (\mathbb{R}^d) \subset l^{q_2}_{v_2} (\mathbb{R}^d)$$

holds if and only if

$$s_2 \leq s_1, \quad \frac{1}{q_2} + \frac{s_2}{d} < \frac{1}{q_1} + \frac{s_1}{d} \quad \text{or} \quad \begin{cases} s_2 = s_1 \\ q_2 = q_1. \end{cases}$$

Using this lemma, we conclude the sharp exponent characterizations for BMM and BMW.

Theorem 5.8. Suppose $p \geq q$, $\tau \in [0, 1]$. We have

$$W_{\tau} : \mathcal{M}^{p_1,q_1}_{v_1,v_1} (\mathbb{R}^d) \times \mathcal{M}^{p_2,q_2}_{v_2,v_2} (\mathbb{R}^d) \rightarrow M^{p,q}_{1\otimes v_2} (\mathbb{R}^{2d})$$

if and only if

$$s \leq s_1, s_2, t_1, t_2, \quad \frac{1}{q} + \frac{s}{d} < \frac{1}{p_1} + \frac{s_1}{d} \quad \text{or} \quad (q, s) = (p_1, s_1),$$

$$\frac{1}{q} + \frac{s}{d} < \frac{1}{p_2} + \frac{s_2}{d} \quad \text{or} \quad (q, s) = (p_2, s_2),$$

$$\frac{1}{q} + \frac{s}{d} < \frac{1}{q_1} + \frac{t_1}{d} \quad \text{or} \quad (q, s) = (q_1, t_1),$$

$$\frac{1}{q} + \frac{s}{d} < \frac{1}{q_2} + \frac{t_2}{d} \quad \text{or} \quad (q, s) = (q_2, t_2).$$

Proof. By Theorem 1.3, we have

$$W_{\tau} : \mathcal{M}^{p_1,q_1}_{v_1,v_1} (\mathbb{R}^d) \times \mathcal{M}^{p_2,q_2}_{v_2,v_2} (\mathbb{R}^d) \rightarrow M^{p,q}_{1\otimes v_2} (\mathbb{R}^{2d})$$

if and only if

$$l^{q_1}_{s_1} (\mathbb{Z}^d), l^{q_2}_{s_2} (\mathbb{Z}^d) \subset l^q_{t_1} (\mathbb{Z}^d), \quad l^{q_1}_{t_2} (\mathbb{Z}^d), l^{q_2}_{t_2} (\mathbb{Z}^d) \subset l^q_{t_2} (\mathbb{Z}^d).$$

Then, the desired conclusion follows by Lemma 5.7. \hfill \Box

Theorem 5.9. Suppose $p \geq q$, $\tau \in (0, 1)$. We have

$$W_{\tau} : \mathcal{M}^{p_1,q_1}_{v_1,v_1} \times \mathcal{M}^{p_2,q_2}_{v_2,v_2} \rightarrow W(\mathcal{F} L^p, L^q_m)$$

if and only if

$$s \leq s_1, s_2, t_1, t_2, \quad \frac{1}{q} + \frac{s}{d} < \frac{1}{p_1} + \frac{s_1}{d} \quad \text{or} \quad (q, s) = (p_1, s_1),$$

$$\frac{1}{q} + \frac{s}{d} < \frac{1}{p_2} + \frac{s_2}{d} \quad \text{or} \quad (q, s) = (p_2, s_2),$$

$$\frac{1}{q} + \frac{s}{d} < \frac{1}{q_1} + \frac{t_1}{d} \quad \text{or} \quad (q, s) = (q_1, t_1),$$

$$\frac{1}{q} + \frac{s}{d} < \frac{1}{q_2} + \frac{t_2}{d} \quad \text{or} \quad (q, s) = (q_2, t_2).$$
**Proof.** As the proof of Theorem 5.3, this is a direct conclusion of Theorem 1.6 and Lemma 5.7.

Following proposition is prepared for further separation of the endpoint cases of BMW with power weights, i.e., for the proof of Theorem 1.6 with $\tau = 0, 1$.

**Proposition 5.10.** Assume $p_1, q_1, p, q \in (0, \infty)$, $s, t \in \mathbb{R}$. Then

$$l^{p_1, q_1} \subset l^{(q, p)}_{v_{s,t}}$$

holds if and only if the following three embedding relations:

$$l^{p_1} \subset l^q_{v_s}, \quad l^q \subset l^p_{v_t}, \quad l^q \subset l^q_{v_{s+t}}.$$  

(5.14)

**Proof.** We divide this proof into two parts.

"Only if" part. Write (5.13) by

$$\left( \sum_{k \in \mathbb{Z}^d} \left( \sum_{n \in \mathbb{Z}^d} |a_{k,n}|^{p \langle n \rangle} \langle k \rangle^q \right)^{q/p} \langle k \rangle^s \right)^{1/q} \lesssim \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} |a_{k,n}|^{p_1} \right)^{q_1/p_1} \right)^{1/q_1}.$$  

(5.15)

Then, we text the above inequality by several constructed sequences.

**Test 1.** Set $a_{k,n} = 0$ if $n \neq 0 \mathbb{Z}^d$. The inequality (5.15) becomes

$$\left( \sum_{k \in \mathbb{Z}^d} |a_{k,0}|^q \langle k \rangle^q \right)^{1/q} \lesssim \left( \sum_{k \in \mathbb{Z}^d} |a_{k,0}|^{p_1} \right)^{1/p_1}.$$  

This implies the embedding relations $l^{p_1} \subset l^q_{v_s}$.

**Test 2.** Set $a_{k,n} = 0$ if $k \neq 0 \mathbb{Z}^d$. The inequality (5.15) becomes

$$\left( \sum_{n \in \mathbb{Z}^d} |a_{0,n}|^p \langle n \rangle^{tp} \right)^{1/p} \lesssim \left( \sum_{n \in \mathbb{Z}^d} |a_{0,n}|^{q_1} \right)^{1/q_1}.$$  

We obtain the embedding relation $l^{q_1} \subset l^p_{v_t}$.

**Test 3.** Set $a_{k,n} = 0$ if $k \neq n$. From the inequality (5.15) we have

$$\left( \sum_{k \in \mathbb{Z}^d} |a_{k,k}|^q \langle k \rangle^{(s+t)q} \right)^{1/q} \lesssim \left( \sum_{k \in \mathbb{Z}^d} |a_{k,k}|^{q_1} \right)^{1/q_1}.$$  

This is just the embedding relation $l^{q_1} \subset l^q_{v_{s+t}}$.

"Only if" part. Applying Lemma 5.7 to the embedding relations $l^{p_1} \subset l^q_{v_s}$ and $l^{q_1} \subset l^p_{v_t}$, we obtain $s \leq 0$ and $t \leq 0$. From this, we consider following three cases.

**Case 1:** $s = 0$. In this case, the embedding relations (5.14) can be written as

$$l^{p_1} \subset l^q, \quad l^{q_1} \subset l^p_{v_t}, \quad l^{q_1} \subset l^q_{v_{s+t}}.$$  

Correspondingly, our target is to verify

$$l^{p_1, q_1} \subset l^{(q, p)}_{v_{0, t}}.$$  

If $q/p \geq 1$, the Minkowski inequality implies that

$$\left( \sum_{k \in \mathbb{Z}^d} \left( \sum_{n \in \mathbb{Z}^d} |a_{k,n}|^{p \langle n \rangle} \langle k \rangle^p \right)^{q/p} \langle n \rangle^{tp} \right)^{1/p} \lesssim \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} |a_{k,n}|^q \langle n \rangle^{tp} \right)^{p/q} \langle k \rangle^{ tp} \right)^{1/q}.$$  

(5.16)
Then, we use the embedding relations $l^p \subset l^q$, $l^{q_1} \subset l^{p_1}$ to deduce that
\[
\left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} |a_{k,n}|^q \langle n \rangle^p \right)^{p/q} \right)^{1/p} \lesssim \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} |a_{k,n}|^{p_1} \right)^{q_1/p_1} \right)^{1/q_1}.
\]
The desired conclusion follows by the above two estimates.

If $q/p < 1$, we use the embedding relations $l^q \subset l^p$ to deduce that
\[
\left( \sum_{k \in \mathbb{Z}^d} \left( \sum_{n \in \mathbb{Z}^d} |a_{k,n}|^p \langle n \rangle^{q/p} \right)^{q/p} \right)^{1/q} \lesssim \left( \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} |a_{k,n}|^q \langle n \rangle^q \right)^{1/q}.
\]
Then, the embedding relations $l^{p_1} \subset l^q$, $l^{q_1} \subset l^{p_1}$ further implies that
\[
\left( \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} |a_{k,n}|^q \langle n \rangle^q \right)^{1/q} \lesssim \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} |a_{k,n}|^{p_1} \right)^{q_1/\nu_1} \right)^{1/q_1}.
\]
The desired conclusion follows by the above two estimates.

**Case 2:** $t = 0$. In this case, the embedding relations (5.14) can be written as
\[
l^{p_1} \subset l^q_{v_1}, \quad l^{q_1} \subset l^p, \quad l^{q_1} \subset l^q_{v_2}.
\]
Correspondingly, our target is to verify
\[
l^{p_1}, q_1 \subset l^{(q,p)}_{v_{1,2}}.
\]

If $p_1/q_1 \geq 1$, we use the embedding relations $l^{p_1} \subset l^q_{v_1}, l^{q_1} \subset l^p$ to deduce that
\[
\left( \sum_{k \in \mathbb{Z}^d} \left( \sum_{n \in \mathbb{Z}^d} |a_{k,n}|^p \langle k \rangle^{q/p} \right)^{q/p} \right)^{1/q} \lesssim \left( \sum_{k \in \mathbb{Z}^d} \left( \sum_{n \in \mathbb{Z}^d} |a_{k,n}|^{q_1} \right)^{p_1/q_1} \right)^{1/q_1}.
\]
Then, we use the Minkowski inequality to continue this estimate by
\[
\left( \sum_{k \in \mathbb{Z}^d} \left( \sum_{n \in \mathbb{Z}^d} \left| a_{k,n} \right|^{q_1} \right)^{p_1/q_1} \right)^{1/q_1} \lesssim \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} \left| a_{k,n} \right|^{p_1} \right)^{q_1/p_1} \right)^{1/q_1}.
\]
The desired conclusion follows by the above two estimates.

If $p_1/q_1 < 1$, we use the embedding relations $l^{q_1} \subset l^q_{v_1}, l^{q_1} \subset l^p$ to deduce that
\[
\left( \sum_{k \in \mathbb{Z}^d} \left( \sum_{n \in \mathbb{Z}^d} |a_{k,n}|^p \langle k \rangle^{q/p} \right)^{q/p} \right)^{1/q} \lesssim \left( \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} |a_{k,n}|^{q_1} \right)^{1/q_1}.
\]
Then, the well known embedding $l^{p_1} \subset l^{q_1}$ yields that
\[
\left( \sum_{n \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} |a_{k,n}|^{q_1} \right)^{1/q_1} \lesssim \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} \left| a_{k,n} \right|^{p_1} \right)^{q_1/p_1} \right)^{1/q_1}.
\]
The desired conclusion follows by the above two estimates.

**Case 3:** $s, t < 0$. In this case, by Lemma 5.7 and the embedding relations (5.14), we obtain that
\[
\frac{1}{q} + \frac{s}{d} < \frac{1}{p_1}, \quad \frac{1}{q} + \frac{1}{d} - \frac{t}{d}, \quad \frac{1}{q} + \frac{s}{d} < \frac{1}{q_1} - \frac{t}{d}.
\]
From this, there exists a sufficiently small constant $\epsilon > 0$ such that
\[
\frac{1}{q} + \frac{s}{d} + \epsilon < \frac{1}{p_1}, \quad \frac{1}{q} + \frac{1}{d} - \frac{t}{d} - \epsilon, \quad \frac{1}{q} + \frac{s}{d} + \epsilon < \frac{1}{q_1} - \frac{t}{d} - \epsilon.
\]
Set
\[
\frac{1}{\rho} := \max\{\frac{1}{q} + \frac{s}{d} + \epsilon, 0\}, \quad \frac{1}{r} := \frac{1}{q_1} - \frac{t}{d} - \epsilon.
\]

We have \(\frac{1}{\rho} < \frac{1}{r}\) and the following relations:
\[
\frac{1}{\rho} + \frac{s}{d} < \frac{1}{p_1}, \quad \frac{1}{r} + \frac{t}{d} < \frac{1}{q_1}.
\]

From this and Lemma 5.7, we obtain the following embedding relations
\[
l^{p_1} \subset l^\rho \subset l^{p_1}, \quad l^{q_1} \subset l^r \subset l^{q_1}.
\]

Using these embedding relations and the Minkowski inequality with \(\frac{1}{\rho} < \frac{1}{r}\), we deduce that
\[
\left(\sum_{k \in \mathbb{Z}^d} \left(\sum_{n \in \mathbb{Z}^d} |a_{k,n}|^{p/(n)} \langle n \rangle^{q/p} \langle k \rangle^q \right)^{1/q} \right)^{1/r} \lesssim \left(\sum_{n \in \mathbb{Z}^d} \left(\sum_{k \in \mathbb{Z}^d} |a_{k,n}|^{r/p} \langle n \rangle^r \langle k \rangle^q \right)^{1/r} \right)^{1/q}.
\]

This is the desired conclusion. \(\square\)

**Proof of Theorem 1.6.** The case \(\tau \in (0, 1)\) follows directly by Theorem 1.5. For \(\tau = 0\), observe that
\[
l^{p_1, q_1, p_1} \subset l^{(q, p)} \iff l^{p_1, q_1} \subset l^{(q, p)}.
\]

Then, Proposition 5.10 tells us
\[
l^{p_1, q_1} \subset l^{(q, p)} \iff l^{p_1} \subset l^{q_1}, \quad l^{q_1} \subset l^{p_1}, \quad l^{q_1} \subset l^{q_1 - t_1},
\]

which is equivalent to
\[
l^{p_1} \subset l^{q_1}, \quad l^{q_1} \subset l^{t_1}, \quad l^{q_1} \subset l^{q_1 + t_1} \subset l^q.
\]

Hence, we have
\[
l^{p_1, q_1} \subset l^{(q, p)} \iff l^{p_1} \subset l^{q_1}, \quad l^{q_1} \subset l^{p_1}, \quad l^{q_1} \subset l^{q_1 + t_1} \subset l^q.
\]

Combining this with Theorem 1.5, we obtain the desired conclusion. The case \(\tau = 1\) follows by the same argument of \(\tau = 0\), we omit the detail. \(\square\)

Using Lemma 5.7 and Theorem 1.6, we obtain the following exponents characterization for BMW with endpoints.

**Theorem 5.11.** Assume \(p_i, q_i, p, q \in (0, \infty], \, i = 1, 2, \, \tau = 0, 1\). We have
\[
W_\tau : \mathcal{M}^{p_1, q_1}_{v_{1,1}}(\mathbb{R}^d) \times \mathcal{M}^{p_2, q_2}_{v_{2,2}}(\mathbb{R}^d) \rightarrow W(\mathcal{F}L^p, L^q_\tau)(\mathbb{R}^{2d})
\]
if and only if
\[
(\tau = 0) \begin{cases}
s \leq s_1, t_2, \quad 0 \leq t_1, s_2, \\
\frac{1}{p} \leq \frac{1}{q_1} + \frac{t_1}{d} & \text{or} \quad (q_1, t_1) = (p, 0), \\
\frac{1}{p} \leq \frac{1}{q_2} + \frac{s_2}{d} & \text{or} \quad (p_2, s_2) = (p, 0), \\
\frac{1}{q} + \frac{t_1}{d} \leq \frac{1}{q_1} + \frac{s_2}{d} & \text{or} \quad (q, s) = (p_1, s_1), \\
\frac{1}{q} + \frac{t_1}{d} \leq \frac{1}{q_1} + \frac{s_1 + t_1}{d} & \text{or} \quad (q, s) = (q_1, s_1 + t_1), \\
\frac{1}{q} + \frac{t_1}{d} \leq \frac{1}{q_2} + \frac{t_2}{d} & \text{or} \quad (q, s) = (q_2, t_2),
\end{cases}
\]

and
Then, the duality of modulation spaces implies (iii).

\[ (\tau = 1) \]
\[ s \leq s_2, t_1, \quad 0 \leq t_2, s_1, \]
\[ \frac{1}{p} \leq \frac{1}{q_2} + \frac{t_2}{d} \quad \text{or} \quad (q_2, t_2) = (p, 0), \]
\[ \frac{1}{p} \leq \frac{1}{p_1} + \frac{s_1}{d} \quad \text{or} \quad (p_1, s_1) = (p, 0), \]
\[ \frac{1}{q} + \frac{s}{d} \leq \frac{1}{p_2} + \frac{s_2}{d} \quad \text{or} \quad (q, s) = (p_2, s_2), \]
\[ \frac{1}{q} + \frac{t}{d} \leq \frac{1}{q_2} + \frac{s_2 + t_2}{d} \quad \text{or} \quad (q, s) = (q_2, s_2 + t_2), \]
\[ \frac{1}{q} + \frac{t}{d} \leq \frac{1}{q_1} + \frac{t_1}{d} \quad \text{or} \quad (q, s) = (q_1, t_1). \]

6. Complements: pseudodifferential operators on modulation spaces

6.1. Relations between BMM (BMW) and BPM (BPW). In order to give characterizations of BPM and BPW, we would like to first establish some equivalent relations associated with BMM and BMW. We point out that these equivalent relations follows by the dual arguments of function spaces, which have been wildly used before, for instance, one can see the proof of [8, Theorem 5.1] for this direction.

**Proposition 6.1.** Assume \(1 \leq p, q, p_i, q_i \leq \infty, i = 1, 2, \tau \in [0, 1].\) Then the following statements are equivalent:

(i) \(\forall \sigma \in M^{p,q}_{1 \otimes m}(\mathbb{R}^d) \implies O_{\tau}(\sigma) : M^{p_i,q_i}_{m_i}(\mathbb{R}^d) \rightarrow M^{p_2,q_2}_{m_2}(\mathbb{R}^d),\)

(ii) \(\|O_{\tau}(\sigma)f\|_{M^{p_2,q_2}_{m_2}(\mathbb{R}^d)} \lesssim \|\sigma\|_{M^{p,q}_{1 \otimes m}(\mathbb{R}^d)} \|f\|_{M^{p_1,q_1}_{m_1}(\mathbb{R}^d)}, \quad f \in \mathcal{S}(\mathbb{R}^d),\)

(iii) \(W_{\tau} : M^{p_2,q_2}_{m_2^{-1}}(\mathbb{R}^d) \times M^{p_1,q_1}_{m_1}(\mathbb{R}^d) \rightarrow M^{p,q}_{1 \otimes m^{-1}}(\mathbb{R}^d).\)

**Proof.** (ii) \(\implies (i)\) is clear. In order to verify \((i) \implies (ii)\), we apply the Closed Graph Theorem as in [7, Proposition 4.7]. The map acting from \(M^{p,q}_{1 \otimes m}\) into \(B(M^{p_1,q_1}_{m_1}, M^{p_2,q_2}_{m_2})\) is defined by \(P : \sigma \rightarrow O_{\tau}(\sigma).\)

Given any sequence of pairs that \((\sigma_n, O_{\tau}(\sigma_n))\) tends to \((\sigma, T)\) in the topology of graph of \(P\), for any \(f, g \in \mathcal{S}(\mathbb{R}^d)\) we have

\[
\langle O_{\tau}(\sigma)f, g \rangle = \langle \sigma, W_{\tau}(g, f) \rangle = \lim_{n \to \infty} \langle \sigma_n, W_{\tau}(g, f) \rangle = \lim_{n \to \infty} \langle O_{\tau}(\sigma_n)f, g \rangle = \langle Tf, g \rangle.
\]

From this, we obtain \(T = O_{\tau}(\sigma)\). Then the graph of \(P\) is closed. Hence, \(P\) is bounded, i.e., (ii) is valid.

Next, we turn to the proof of \( (i) \iff (iii)\). This follows by a standard dual argument. If (ii) holds, for any \(f, g \in \mathcal{S}(\mathbb{R}^d)\) and \(\sigma \in M^{p,q}_{1 \otimes m}\), we have

\[
|\langle W_{\tau}(f, g), \sigma \rangle| = |\langle f, O_{\tau}(\sigma)g \rangle| \lesssim \|f\|_{M^{p_2,q_2}_{m_2^{-1}}} \|O_{\tau}(\sigma)g\|_{M^{p_2,q_2}_{m_2}} \lesssim \|\sigma\|_{M^{p,q}_{1 \otimes m}} \|g\|_{M^{p_1,q_1}_{m_1}}.
\]

Then, the duality of modulation spaces implies (iii).

Viceversa, if (iii) holds, for any \(f, g \in \mathcal{S}(\mathbb{R}^d)\) and \(\sigma \in M^{p,q}_{1 \otimes m}\), we have

\[
|\langle O_{\tau}(\sigma)f, g \rangle| = |\langle \sigma, W_{\tau}(g, f) \rangle| \lesssim \|\sigma\|_{M^{p,q}_{1 \otimes m}} \|W_{\tau}(g, f)\|_{M^{p,q}_{1 \otimes m^{-1}}} \lesssim \|\sigma\|_{M^{p,q}_{1 \otimes m}} \|g\|_{M^{p_2,q_2}_{m_2^{-1}}} \|f\|_{M^{p_1,q_1}_{m_1}}.
\]

Then, (iii) follows by the duality of modulation spaces. \(\square\)
By a similar argument, we give following equivalent relations between BMW and BPW.

**Proposition 6.2.** Assume $1 \leq p, q, p_i, q_i \leq \infty$, $i = 1, 2$, $\tau \in [0, 1]$. Then the following statements are equivalent:

(i) $\forall \sigma \in \mathcal{F} M_{1\otimes m}^{p,q}(\mathbb{R}^{2d}) \implies O P_{\tau}(\sigma) : \mathcal{M}_{m_1}^{p_1,q_1}(\mathbb{R}^d) \rightarrow M_{m_2}^{p_2,q_2}(\mathbb{R}^d)$,

(ii) $\|O P_{\tau}(\sigma)f\|_{M_{m_2}^{p_2,q_2}(\mathbb{R}^d)} \leq \|\sigma\|_{\mathcal{F} M_{1\otimes m}^{p,q}(\mathbb{R}^{2d})} \|f\|_{M_{m_1}^{p_1,q_1}(\mathbb{R}^d)}$, $f \in \mathcal{S}(\mathbb{R}^d)$,

(iii) $W_{\tau} : \mathcal{M}_{m_2}^{p_2,q_2}(\mathbb{R}^d) \times \mathcal{M}_{m_1}^{p_1,q_1}(\mathbb{R}^d) \rightarrow \mathcal{F} M_{1\otimes m-1}^{p',q'}(\mathbb{R}^{2d})$.

### 6.2. Sharp exponents of BPM and BPW

By propositions 6.1 and 6.2, the estimates of the $\tau$-Wigner distribution can be translated in ones into the corresponding results for $\tau$-operators. Following, we collect some important cases, the interested reader can deduce the results they need.

Following result is a direct conclusion by Proposition 6.1 and Theorem 5.5.

**Theorem 6.3.** Assume $1 \leq p, q, p_i, q_i \leq \infty$, $i = 1, 2$, $\tau \in [0, 1]$. We have

$$\forall \sigma \in M_{1\otimes v}^{p,q}(\mathbb{R}^{2d}) \implies O P_{\tau}(\sigma) : \mathcal{M}_{v_{s_1}}^{p_1,q_1}(\mathbb{R}^d) \rightarrow M_{v_{s_2}}^{p_2,q_2}(\mathbb{R}^d)$$

if and only if

$$p_1, q_1, p_2', q_2' \leq q', \; \; i = 1, 2$$

and

$$\frac{1}{p_1} + \frac{1}{p_2'} \geq \frac{1}{p'} + \frac{1}{q'}, \; \; \frac{1}{q_1} + \frac{1}{q_2'} \geq \frac{1}{p'} + \frac{1}{q'}.$$

Next, we want to establish the sharp exponents for the boundedness on modulation spaces with power weights of pseudodifferential operators with symbols in Sjöstrand’s class. Before this, we first give following characterization of BPM, which can be directly deduced by Proposition 6.1 and Theorem 1.3.

**Theorem 6.4.** Assume $1 \leq p, q, p_i, q_i \leq \infty$, $i = 1, 2$, $\tau \in [0, 1]$. We have

$$\forall \sigma \in M_{1\otimes v_i}^{p,q}(\mathbb{R}^{2d}) \implies O P_{\tau}(\sigma) : \mathcal{M}_{v_{s_1}}^{p_1,q_1}(\mathbb{R}^d) \rightarrow M_{v_{s_2}}^{p_2,q_2}(\mathbb{R}^d)$$

if and only if

$$l_{p_2/s_2}^{q_2}(\mathbb{Z}^d) \ast l_{p_1/s_1}^{q_1}(\mathbb{Z}^d) \subset l_{p_1/s_1}^{q_1}(\mathbb{Z}^d) \cap l_{p_2/s_2}^{q_2}(\mathbb{Z}^d) \subset l_{p_1/s_1}^{q_1}(\mathbb{Z}^d) \cup l_{p_2/s_2}^{q_2}(\mathbb{Z}^d), \; \; p > 1, \; \; (6.1)$$

$$l_{p_2}^{q_2}(\mathbb{Z}^d), \; l_{p_1}^{q_1}(\mathbb{Z}^d) \subset l_{p_1}^{q_1}(\mathbb{Z}^d), \; \; l_{p_2}^{q_2}(\mathbb{Z}^d), \; l_{p_1}^{q_1}(\mathbb{Z}^d) \subset l_{p_1}^{q_1}(\mathbb{Z}^d), \; \; p \leq q. \; \; (6.2)$$

We recall a special case of Lemma 5.1 as follows.

**Lemma 6.5.** Suppose $1 \leq q_1, q_2 \leq \infty$, $s_1, s_2 \in \mathbb{R}$. Then

$$l_{s_1}^{q_1}(\mathbb{R}^d) \ast l_{s_2}^{q_2}(\mathbb{R}^d) \subset l^{\infty}(\mathbb{R}^d) \; \; (6.3)$$

if and only if

$$s_1 = s_2 = 0, \; \; \frac{1}{q_1} + \frac{1}{q_2} = 1 \; \; \text{or} \; \; \begin{cases} 0 \leq s_1, s_2, & \\
1 \leq \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_1} + \frac{s_1}{q_2}, & \\
(q_1, s_1) = (\infty, 0) \; \text{if} \; \frac{1}{q_1} + \frac{s_1}{q_2} = 0, & \\
(q_2, s_2) = (\infty, 0) \; \text{if} \; \frac{1}{q_2} + \frac{s_2}{q_2} = 0. & \\
\end{cases} \; \; (6.4)$$

Now, we are in a position to give the sharp exponents of BPM with Sjöstrand’s class.
Theorem 6.6. Assume $1 \leq p, q, p_i, q_i \leq \infty$, $i = 1, 2$, $\tau \in [0, 1]$. We have
\begin{equation}
\forall \sigma \in M^{\infty, 1}(\mathbb{R}^{2d}) \implies OP_{\tau}(\sigma) : M^{p_1,q_1}_{\nu_1,\tau_1}(\mathbb{R}^d) \rightarrow M^{p_2,q_2}_{\nu_2,\tau_2}(\mathbb{R}^d)
\end{equation}
if and only if
\begin{equation}
s_1 = s_2 = 0, \quad p_1 = p_2 \quad \text{or} \quad \begin{cases} s_2 \leq 0 \leq s_1, \\
\frac{1}{p_2} + \frac{s_2}{d} < \frac{1}{p_1} + \frac{s_1}{d}, \quad (p_1, s_1) = (\infty, 0) \text{ if } \frac{1}{p_1} + \frac{s_1}{d} = 0, \\
(p_2, s_2) = (1, 0) \text{ if } \frac{1}{p_2} + \frac{s_2}{d} = 1,
\end{cases}
\end{equation}
and
\begin{equation}
t_1 = t_2 = 0, \quad q_1 = q_2 \quad \text{or} \quad \begin{cases} t_2 \leq 0 \leq t_1, \\
\frac{1}{q_2} + \frac{t_2}{d} < \frac{1}{q_1} + \frac{t_1}{d}, \quad (q_1, t_1) = (\infty, 0) \text{ if } \frac{1}{q_1} + \frac{t_1}{d} = 0, \\
(q_2, t_2) = (1, 0) \text{ if } \frac{1}{q_2} + \frac{t_2}{d} = 1.
\end{cases}
\end{equation}
In particular, when $s_i = t_i = 0$, $i = 1, 2$, we have (6.5) holds if and only if
\begin{equation}
p_1 \leq p_2, \quad q_1 \leq q_2.
\end{equation}

Proof. Using Theorem 6.4, we obtain that (6.5) holds if and only if
\begin{equation}
l^{p_2}_{s_2}(\mathbb{Z}^d) * l^{p_1}_{s_1}(\mathbb{Z}^d) \subset l^\infty(\mathbb{Z}^d), \quad l^{q_2}_{t_2}(\mathbb{Z}^d) * l^{q_1}_{t_1}(\mathbb{Z}^d) \subset l^\infty(\mathbb{Z}^d).
\end{equation}
Then, Lemma 6.5 tells us that the above two convolution inequalities are equivalent to (6.6) and (6.7). \hfill \Box

Observe that Wiener amalgam space $W(\mathcal{F}L^q, L^p)$ has the same local regularity with $M^{p,q}$. Moreover, for $p > q$ they have the following inclusion relations:
\begin{equation}
M^{p,q} \subset W(\mathcal{F}L^q, L^p), \quad M^{p,q} \supset W(\mathcal{F}L^p, L^q).
\end{equation}
So, there is a natural question that for $p > q$ whether the boundedness of pseudodifferential operator with symbols in $M^{p,q}$ or $W(\mathcal{F}L^p, L^q)$ can be preserved with symbols in $W(\mathcal{F}L^q, L^p)$ or $M^{p,q}$ respectively. With the help of our full characterizations of BMM and BMW, one can find that the answer is negative unless the trivial case happen, i.e., $p = q$. Here, we only give a detailed comparison for the Sjöstrand’s class $M^{\infty, 1}$ and the corresponding larger space $W(\mathcal{F}L^1, L^\infty)$. Let us being with the sharp exponents for the non-endpoint case of BPW.

Theorem 6.7. Assume $1 \leq p, q, p_i, q_i \leq \infty$, $i = 1, 2$, $\tau \in (0, 1)$. We have
\begin{equation}
\forall \sigma \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d}) \implies OP_{\tau}(\sigma) : M^{p_1,q_1}_{\nu_1,\tau_1}(\mathbb{R}^d) \rightarrow M^{p_2,q_2}_{\nu_2,\tau_2}(\mathbb{R}^d)
\end{equation}
if and only if
\begin{equation}
s_2 \leq 0 \leq s_1, \\
\frac{1}{p_2} + \frac{s_2}{d} < \frac{1}{p_1} + \frac{s_1}{d}, \quad (p_1, s_1) = (1, 0), \\
\frac{1}{p_2} + \frac{s_2}{d} < 0 \quad \text{or} \quad (p_2, s_2) = (\infty, 0),
\end{equation}
and
\begin{equation}
t_2 \leq 0 \leq t_1, \\
\frac{1}{q_2} + \frac{t_2}{d} < \frac{1}{q_1} + \frac{t_1}{d}, \quad (q_1, t_1) = (1, 0), \\
\frac{1}{q_2} + \frac{t_2}{d} < 0 \quad \text{or} \quad (q_2, t_2) = (\infty, 0).
\end{equation}
In particular, when \(s_i = t_i = 0, i = 1, 2\), we have (6.9) holds if and only if
\[
p_1 = q_1 = 1, \quad p_2 = q_2 = \infty.
\] (6.10)

**Proof.** By Proposition 6.2 and Theorem 1.6, we conclude that (6.9) holds if and only if
\[
l^p s_2 (Z^d), l^p s_1 (Z^d) \subset l^1 (Z^d), \quad l^p t_2 (Z^d), l^p t_1 (Z^d) \subset l^1 (Z^d).
\]

Then, the desired conclusion follows by Lemma 5.7. \(\square\)

**Remark 6.8.** Comparing (6.10) with (6.8), we find that the range of exponents for BPW \((\tau \in (0, 1))\) with symbols in \(W(FL^1, L^\infty)(R^{2d})\) is strictly small than that for BPM with Sjöstrand’s class.

Next, we handle that endpoint case of BPW. We first give following characterization for the endpoint cases of BPW, which can be directly deduced by Proposition 1.6 and Proposition 6.2.

**Theorem 6.9.** Assume \(1 \leq p, q, p_i, q_i \leq \infty, i = 1, 2, \tau = 0, 1\). We have
\[
\forall \sigma \in W(FL^p, L^q)(R^{2d}) \implies OP_{\tau} (\sigma): M^p q_1 (R^d) \to M^p q_2 (R^d)
\]
if and only if
\[
l_{p_2} (Z^d), l_{q_2} (Z^d) \subset l^{p_1} (Z^d), \quad l_{q_1} (Z^d) \subset l^{p_2} (Z^d), \quad \tau = 0, \tag{6.11}
\]
\[
l_{p_1} (Z^d), l_{q_1} (Z^d) \subset l^{p_2} (Z^d), \quad l_{q_2} (Z^d) \subset l^{p_1} (Z^d), \quad \tau = 1. \tag{6.12}
\]

Then, corresponding to Theorem 5.6, we establish the sharp exponents for the endpoint cases of BPW with constant weights.

**Theorem 6.10.** Assume \(1 \leq p, q, p_i, q_i \leq \infty, i = 1, 2, \tau = 0, 1\). We have
\[
\forall \sigma \in W(FL^p, L^q)(R^{2d}) \implies OP_{\tau} (\sigma): M^p q_1 (R^d) \to M^p q_2 (R^d)
\]
if and only if
\[
q_2, p_1 \leq p', \quad p_2, q_2, q_1 \leq q', \quad \tau = 0, \quad (6.14)
\]
\[
q_1, p_2 \leq p', \quad p_1, q_1, q_2 \leq q', \quad \tau = 1. \quad (6.15)
\]

**Proof.** It follows by Theorem 6.9 that (6.13) is equivalent to
\[
l^{p_2} (Z^d), l^{p_1} (Z^d) \subset l^{q_1} (Z^d), \quad l^{p_1} (Z^d), l^{p_2} (Z^d) \subset l^{q_2} (Z^d), \quad \tau = 0, \tag{6.11}
\]
\[
l^{q_2} (Z^d), l^{q_1} (Z^d) \subset l^{p_1} (Z^d), \quad l^{q_1} (Z^d), l^{q_2} (Z^d) \subset l^{p_2} (Z^d), \quad \tau = 1. \quad (6.12)
\]

Then, the desired conditions can be deduced by Lemma 5.7. \(\square\)

Now, we return to BPW with symbols in \(W(FL^1, L^\infty)(R^{2d})\).

**Theorem 6.11.** Assume \(1 \leq p, q, p_i, q_i \leq \infty, i = 1, 2, \tau = 0, 1\). We have
\[
\forall \sigma \in W(FL^1, L^\infty)(R^{2d}) \implies OP_{\tau} (\sigma): M^p q_1 (R^d) \to M^p q_2 (R^d)
\]
if and only if
\[
(\tau = 0) \begin{cases} 
    s_2, t_2 \leq 0 \leq s_1, t_1, & \text{or } (p_1, s_1) = (\infty, 0), \\
    \frac{1}{p_2} + \frac{t_2}{d} < 0 & \text{or } (p_2, s_2) = (\infty, 0), \\
    \frac{1}{q_1} + \frac{s_1}{d} < 0 & \text{or } (q_1, t_1) = (1, 0), \\
    \frac{1}{q_2} + \frac{t_2}{d} < 1 & \text{or } (q_2, t_2) = (1, 0), \\
    \frac{1}{q_2} + \frac{s_2 + t_2}{d} < 0 & \text{or } (q_2, s_2, t_2) = (\infty, 0, 0)
\end{cases}
\]
and

\[
(\tau = 1) \left\{
\begin{aligned}
& s_2, t_2 \leq 0 \leq s_1, t_1, \\
& 1 < \frac{1}{p_1} + \frac{\tau}{d} \quad \text{or} \quad (p_1, s_1) = (1, 0), \\
& \frac{1}{p_2} + \frac{\tau}{d} < 1 \quad \text{or} \quad (p_2, s_2) = (1, 0), \\
& 0 < \frac{1}{q_1} + \frac{\tau}{d} \quad \text{or} \quad (q_1, t_1) = (\infty, 0), \\
& \frac{1}{q_2} + \tau < 0 \quad \text{or} \quad (q_2, t_2) = (\infty, 0), \\
& 1 < \frac{1}{q_1} + \frac{\tau + t_1}{d} \quad \text{or} \quad (q_1, s_1, t_1) = (1, 0, 0)
\end{aligned}
\right.
\]

In particular, when \( s_i = t_i = 0, \ i = 1, 2 \), we have (6.16) holds if and only if

\[
\begin{align*}
q_1 &= 1, \ p_2 = q_2 = \infty, \quad \tau = 0, \\
q_2 &= \infty, \ p_1 = q_1 = 1, \quad \tau = 1.
\end{align*}
\] (6.17) (6.18)

Proof. It follows from Theorem 6.9 that (6.16) is equivalent to

\[
\begin{align*}
& l^p_{\tau_{t_2}}(\mathbb{Z}^d), l^p_{\tau_{s_2}}(\mathbb{Z}^d) \subset l^\infty(\mathbb{Z}^d), \quad l^p_{\tau_{s_2}}(\mathbb{Z}^d), l^p_{\tau_{s_2+t_2}}(\mathbb{Z}^d), l^p_{\tau_{s_2}}(\mathbb{Z}^d) \subset l^1(\mathbb{Z}^d), \quad \tau = 0, (6.19) \\
& l^{q_1}(\mathbb{Z}^d), l^{q_2}(\mathbb{Z}^d) \subset l^\infty(\mathbb{Z}^d), \quad l^{q_1}(\mathbb{Z}^d), l^{q_1+t_1}(\mathbb{Z}^d), l^{q_2}(\mathbb{Z}^d) \subset l^1(\mathbb{Z}^d). \quad \tau = 1. (6.20)
\end{align*}
\]

Then, the desired conclusion follows from Lemma 5.7. \( \square \)

Remark 6.12. Comparing (6.17), (6.18) with (6.8), one can find that the range of exponents for BPW \((\tau = 0, 1)\) with symbols in \(W(\mathcal{F} L^1, L^\infty)(\mathbb{R}^d)\) is strictly small than that for BPM with Sjöstrand’s class.

Finally, we consider the boundedness on Sobolev spaces \(H^s\) of pseudodifferential operators with symbols in \(W(\mathcal{F} L^p, L^q)(\mathbb{R}^d)\). Note that \(M_{\tau_{t_2}}^{2,2} = H^s\).

Theorem 6.13. Assume \(1 \leq p, q, p_i, q_i \leq \infty, \ i = 1, 2, \ \tau = 0, 1\). We have

\[
\forall \sigma \in W(\mathcal{F} L^p, L^q)(\mathbb{R}^d) \implies OP_{\tau_{t_1}}(\sigma) : M_{\tau_{t_1}}^{2,2}(\mathbb{R}^d) \rightarrow M_{\tau_{t_2}}^{2,2}(\mathbb{R}^d)
\] (6.21)

if and only if

\[
t_2 \leq 0 \leq t_1, \quad p, q \leq 2, \quad \tau = 0, 1.
\]

Proof. Recall that \(M_{\tau_{t_2}}^{2,2} = H^s\), by Theorem 6.9, we conclude that (6.21) is equivalent to

\[
\begin{align*}
& l^p_{\tau_{t_2}}(\mathbb{Z}^d), l^p(\mathbb{Z}^d) \subset l^p(\mathbb{Z}^d), \quad l^p(\mathbb{Z}^d), l^p_{\tau_{t_2}}(\mathbb{Z}^d), l^p_{\tau_{t_2}}(\mathbb{Z}^d) \subset l^q(\mathbb{Z}^d), \quad \tau = 0, \\
& l^{q_1}(\mathbb{Z}^d), l^{q_2}(\mathbb{Z}^d) \subset l^q(\mathbb{Z}^d), \quad l^q(\mathbb{Z}^d), l^{q_1}(\mathbb{Z}^d), l^{q_2}(\mathbb{Z}^d) \subset l^q(\mathbb{Z}^d), \quad \tau = 1,
\end{align*}
\]

which implies the desired conclusion by Lemma 5.7. \( \square \)

Remark 6.14. From Theorem 6.13, for \(\tau = 0, 1\), we observe that for any \(s \geq 0\), there exists a symbol \(\sigma \in W(\mathcal{F} L^\infty, L^1)(\mathbb{R}^d)\) such that the corresponding pseudodifferential operators \(OP_{\tau_{t_1}}(\sigma)\) are not bounded from \(L^2(\mathbb{R}^d)\) to \(H^s(\mathbb{R}^d)\).

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