Analytical criteria for magnetization reversal in $\varphi_0$ Josephson junction

A. A. Mazanik, 1, 2, 3, 4, 5 I. R. Rahmonov, 3, 4, 6 and Yu. M. Shukrinov 1, 5, 7

1) BLTP, JINR, Dubna, 141980, Moscow Region, Russia
2) MIPT, Dolgoprudny, 141700, Moscow Region, Russia
3) BLTP, JINR, Dubna, 141980, Moscow Region, Russia
4) Umarov Physical Technical Institute, TAS, Dushanbe, 734063, Tajikistan
5) Dubna State University, Dubna, Moscow Region, 141980, Russia

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The $\varphi_0$ Josephson junctions formed by ordinary superconductors and magnetic, non-centrosymmetric interlayer are studied. We derive an analytical solution for the magnetization dynamics induced by arbitrary current pulse and formulate the criteria for the magnetization reversal. Using the obtained results, the form and duration of the current pulse are optimized. The agreement between analytical and numerical investigations is reached in a case of large product of a ratio of the Josephson energy to the magnetic energy, strength of spin-orbit interaction and minimum value of flowing current. The obtained results allow to predict magnetization reversal at chosen system parameters.

Keywords: Josephson junction, magnetisation reversal, Phi-0 junction

It is well-known that Ohmic heat generation is a limiting factor for semiconductor technology, whereas one of the main features for the superconducting state is the absence of the resistivity, so the superconducting electronics stands out by ultra-low energy dissipation. One of the milestones for such electronics is a creation of cryogenic memory.

Different realizations for such devices were proposed including devices based on $\varphi_0$ Josephson junctions. Goldobin et al. demonstrated the use of such $\varphi_0$ Josephson junction as a memory cell (classical bit), where $\varphi_0$ junction was constructed as a combination of 0 and $\pi$ SFS junctions.

Guarcello and Bergeret proposed another approach based on $\varphi_0$ SFS junction formed by ordinary superconductors and magnetic interlayer without inversion symmetry. This setup was considered by Buzdin in Ref. 8. In such systems anomalous phase shift occurs due to interplay between Rashba spin-orbit interaction (SOI), due to a lack of inversion symmetry, and the exchange field. Current-phase relation reads as $I_s(\varphi) = I_s \sin(\varphi - \varphi_0)$, where $\varphi_0$ is proportional to a strength of SOI and the magnetic moment perpendicular to the gradient of the asymmetric spin-orbit potential, so direct coupling between the magnetic moment of a ferromagnetic layer and superconducting current is realized. Recently anomalous phase shift was experimentally observed in a quantum dot geometry, in the system fabricated with the topological insulator $Bi_2Se_3$ submitted to an in-plane magnetic field and in $InAs/Al$ heterostructures. In this scheme bit of information is associated with direction of the magnetic moment along or opposite direction of an easy axis of the ferromagnetic layer. The writing is carried out as a reversal of the magnetic moment by a pulse of current and readout is performed by detection of the magnetic flux by SQUID inductively coupled to the $\varphi_0$ junction.

In Ref. 15 it was found that one can realize the full magnetization reversal by applying an electric current pulse. A detailed pictures representing the intervals of the damping parameter $\alpha$, Josephson to magnetic energy relation $G$ and spin-orbit coupling parameter $r$ were obtained with full magnetization reversal. It was demonstrated that appearance of the reversal is sensitive to changing of system parameters and shows some periodic structure. Guarcello and Bergeret in Ref. 7 developed method of Ref. 17 by taking into account a term preserving a gauge-invariance. They also explored the robustness of the current-induced magnetization reversal against thermal fluctuations, suggested a way of decoupling the Josephson phase and the magnetization dynamics by tuning the Rashba SOI strength via a gate voltage. A suitable non-destructive readout scheme based on a dc-SQUID inductively coupled to the $\varphi_0$ junction are discussed as well. In all mentioned above works the magnetization reversal was studied numerically only.

In this work we derive an analytical solution for the magnetization dynamics induced by arbitrary current pulse and formulate the criteria for the magnetization reversal, which explains all the periodicity structure obtained in Ref. 17. Our theory works in a case of large product of the ratio of the Josephson to the magnetic energy, SOI strength and minimal value of flowing current.

The dynamics of the magnetic moment in $\varphi_0$ JJ is described by Landau-Lifshitz-Gilbert (LLG) equation
\begin{equation}
\frac{dM}{dt} = \gamma_m H_{eff} \times M + \frac{\alpha}{M_0} \left( M \times \frac{dM}{dt} \right),
\end{equation}
with effective magnetic field $H_{eff}$, $\gamma_m$ is gyromagnetic ratio, $\alpha$ is dimensionless parameter of Gilbert damping, $M_0 = ||M||$, $M_i$ are components of $M$. As it was calcu-
lated in Refs. [12] and [13] the effective field takes a form
\[ H_{\text{eff}} = \frac{K}{M_0} \left[ G r \sin \left( \varphi - \frac{M_y}{M_0} \right) e_y + \frac{M_z}{M_0} e_z \right]. \]  

(2)

Here \( \varphi \) is phase difference between two superconducting condensates, \( \varphi_0 = \gamma \frac{N_y}{M_0} \), \( r = \nu \nu F /\nu F \), \( l = 4 h L / h V \), \( L \)-length of \( F \) layer, \( h \)-exchange field of the \( F \) layer \( G = E_j / (K V) \), \( E_j = \Phi_0 I_c / 2 \pi \) is the Josephson energy, \( \Phi_0 \) is the flux quantum, \( I_c \) is the critical current, \( V \) is Fermi velocity, the parameter \( \nu \) characterizes a relative strength of \( S O I \), \( K \) is the anisotropic constant, and \( V \) is the volume of the \( F \) layer, so \( E_M = -K V m_2^2 /2 \). We will use time normalized to the inverse ferromagnetic resonance frequency \( \omega_x = \gamma K / M_0 \); \(( t \to t \omega_x) \) and magnetization components normalized to \( M_0 \) (\( m_i = \frac{M_i}{M_0} \)).

Let us consider dynamics of the magnetic moment in \( \varphi_0 \) Josephson junction induced by arbitrary current pulse \( I_p(t) \). According to extended RSJ model[14] the current flowing through the system is
\[ I_p(t) / I_c = w \left[ \frac{d\varphi}{dt} - r \frac{dm_y}{dt} \right] + \sin (\varphi - rm_y) \cdot w \frac{d\Phi}{dt} + \sin \Phi. \]

Here \( \Phi = \varphi - rm_y \), \( w = \frac{le_F}{\nu_F} \), \( \omega_r = \frac{2 \sqrt{FR}}{R} \).

In comparison with Refs. [12] [14] and [17] we include here term which comes from \( \frac{d\varphi_0}{dt} = wr \cdot dm_y / dt \) to preserve gauge-invariance.

We see that in terms of function \( \Phi = \varphi - rm_y \) the equation (3) can be solved analytically. According to function \( \sin \Phi \) takes a role of a field applied to the magnetic moment.

Our theory is based on few key observations. The first observation is that if conditions \( w = \omega_F / \omega_R \ll 1 \) and \( I_p(t) / I_c < 1 \) during the pulse are fulfilled then in equation (3) we can neglect the term \( w \cdot d\Phi / dt \) what implies useful relation
\[ I_p(t) / I_c = \sin \Phi(t). \]  

(4)

The second observation is \( w \ll 1 \) that means \( w = \omega_F / \omega_R \sim \text{const} \cdot 1 / G \ll 1 \), so we can vary \( G \) parameter in \( G \gg 1 \) region and use LLG equation in the limit \( G \gg 1 \).

As it was estimated in[12], it is plausible for \( G \) to vary in a wide range from \( G \ll 1 \) to \( G \sim 100 \gg 1 \).

The third observation is that the Gilbert damping can be relatively small \( \alpha \approx 10^{-2 \div 2} \), so if the duration of the current pulse is not very long, the damping has not enough time to influence the magnetization significantly and the system may be considered as in the situation \( \alpha = 0 \). Estimations for such damping will be given below.

According to the previous remarks, using [4], we can write LLG equation during the pulse as
\[ \begin{cases} \dot{m}_x = Gr m_z \sin \Phi(t) = Gr \frac{I(t)}{I_c} m_z, \\ \dot{m}_y = m_x m_z, \\ \dot{m}_z = -Gr m_x \sin \Phi(t) = -Gr \frac{I(t)}{I_c} m_x. \end{cases} \]  

(5)

As it was noticed in Ref. [12], here we may apply \( m_y(t) \approx \text{const} \cdot 0 \) and for applicability such method we also need \( Gr I_p(t) / I_c \gg 1 \) during the pulse. In opposite case zeroes of \( I_p(t) \) destroys the predominance of used terms and more careful consideration should be done. For \( M(t = t_0) = M_0(0, 0, 1) \) we directly find
\[ \begin{align*}
    m_x(t) &= \sin \phi(t), \\
    m_z(t) &= \cos \phi(t), \\
    \phi(t) &= Gr \int_{t_0}^{t} dt_1 \frac{I_p(t_1)}{I_c}. 
\end{align*} \]  

(6)

After the pulse has ended, the field \( \sin \Phi \) has the fast drop to 0 due to \( w \ll 1 \) and the dynamics of the magnetic moment is determined by \( \alpha \) which destroys the deviation from the easy axis[16]. So, the reversal occurs when
\[ \cos \left( Gr \int_{t_0}^{t_0+\delta t} dt_1 \frac{I_p(t_1)}{I_c} \right) < 0, \]

(7)

where \( \delta t \) is the pulse duration.

When the damping factor \( \alpha \) is not small, the amplitude of oscillations of \( m_y \) during the pulse starts to decrease due to the essential deviation of \( m_y(t) \) from 0. To estimate it we write for the first non-neglecting term in \( \alpha \)
\[ m_y = m_z m_z + \alpha Gr(1 - m_y^2) \sin \Phi(t). \]  

(8)

In the beginning of the pulse \( m_y(t) \approx 0 \), \( m_x \) makes fast oscillations due to \( Gr \gg 1 \), so rising of \( m_y \) is determined by \( \alpha Gr \sin \Phi(t) \). For applicability of (8) we need to save \( m_y(t) \approx 0 \) which imposes condition for the small damping regime
\[ \int_{t_0}^{t_0+\delta t} dt_1 \alpha Gr \sin \Phi(t_1) = \alpha Gr \int_{t_0}^{t_0+\delta t} dt_1 \frac{I_p(t_1)}{I_c} \ll 1. \]

(9)

We demonstrate this idea in Fig (a) and (b) for rectangular pulse \( I_p(t) = I_0 \left[ \theta(t - t_0) - \theta(t - t_0 - \delta t) \right] \) with \( I_0 = 0.5 I_c \) for two pulse durations \( \delta t = 1 \) and \( \delta t = 3 \). Parameters \( G = 100, r = 0.1, \alpha = 0.005, w = 0.01 \) were used. Our criteria (10) gives here cos (\( Gr I_0 \delta t/I_c \)) = 0.284 < 0, so the reversal is absent, whereas for \( \delta t = 3 \) we get cos (\( Gr I_0 \delta t/I_c \)) = −0.760 < 0 and the reversal occurs. We see how the solution (10) presented by blue dashed curve coincide with numerical one presented by green solid curve for complete equations (3) and (1) using (4) during the pulse. When the pulse has been off, the damping destroys any deviations from the easy axis \( m_z = \pm 1 \). It was demonstrated in the insets.

It should be noted that the magnetization reversal is not affected by a form of the current pulse, but only by its integral over the pulse duration. This is demonstrated in Fig (c) for the pulse \( I_p(t)/I_c = 0.75 - |t - t_0 - \delta t/2|/3, \delta t = 3 \). The integral \( \int_0^{\delta t} dt_1 I_p(t_1) \) for such pulse is the same as for the pulse in Fig (b) and the reversal appearance is also the same as in Fig (b). One may expect that the reversal independence on the pulse form gives also independence on the current white-noise fluctuations because \( \langle I_{th}(t) \rangle = 0 \) and therefore \( \int_{t_0}^{t_0+\delta t} dt_1 I_{th}(t_1) = 0 \).
the magnetic moment becomes aligned in the driving force in LLG equation (1). In this situation such curves are the curves of a constant amplitude for pulse duration in order to make the fastest reversal. We see from (11) that the lowest one is realized for low. According to (7), the reversal under \(I_r = 0\) occurs in hyperbolic areas in \(G - r\) space where

\[
\frac{\pi}{2} + 2\pi n \leq G_n r I_0 \delta t / I_c \leq \frac{3\pi}{2} + 2\pi n
\]

for \(n = 0, \pm 1, \ldots\), whereas the most efficient reversal appears when the condition

\[
\cos (Gr I_0 \delta t / I_c) = -1, \quad Gr I_0 \delta t / I_c = \pi + 2\pi n
\]

is fulfilled. It gives also hyperbolic curves with \(G_n r \delta t = \pi + 2\pi n\) in \(G - r\) space. From physical point of view such curves are the curves of a constant amplitude for the driving force in LLG equation (11). In this situation the magnetic moment becomes aligned in the \(m_z = -1\) direction exactly after the pulse has been off and the relevant time scale is determined only by the pulse duration, not by the Gilbert damping. It helps us to optimize the pulse duration in order to make the fastest reversal. We see from (11) that the lowest one is realized for \(n = 0\)

\[
\delta t_{eff} = \frac{\pi I_c}{Gr I_0}.
\]

This situation is demonstrated in Fig.1(d) for \(G = 100, r = 0.1, \alpha = 0.005, w = 0.01, I_0 / I_c = 0.5\) and \(\delta t_{eff} = 0.628\). It leads to \(\delta t_{reversal} \approx 0.6 \cdot 10^{-10}\) s for typical \(\omega_p \sim 10\) GHz. This time is two orders of magnitude smaller than in work 13.

For simplicity we consider rectangular pulses below. According to (7), the reversal under \(I_p(t) = I_0[\theta(t - t_0) - \theta(t - t_0 - \delta t)]\) occurs in hyperbolic areas in \(G - r\) space

\[
\frac{\pi}{2} + 2\pi n \leq G_n r I_0 \delta t / I_c \leq \frac{3\pi}{2} + 2\pi n
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Similar hyperbolic profiles of \(1/\delta t_{reversal}\) on \(I_p(t)\) were obtained theoretically 15 and experimentally in Refs. 13 and 14 for spin-transfer-induced magnetization reversal setup in current-perpendicular spin-valve nanomagnetic junctions. In contrast to our situation, such setup needs some critical spin-polarized current for magnetization reversal.

In order to test the obtained results we calculate numerically areas where the reversal appears in \(G - r\) diagram using complete equations (3) and (1) with (2). Then we compare them with analytical areas (10) and curves (11). This comparison is demonstrated in Fig.2. We see a perfect agreement between numerical and analytical calculations. It should be noticed that such areas were first observed in work 17 for non gauge-invariant scheme. The term \(\varphi_0\) only slightly shifts these areas and gives a possibility to the analytical solution of (3).

For simplicity we consider rectangular pulses below. According to (7), the reversal under \(I_p(t) = I_0[\theta(t - t_0) - \theta(t - t_0 - \delta t)]\) occurs in hyperbolic areas in \(G - r\) space where

\[
\frac{\pi}{2} + 2\pi n \leq G_n r I_0 \delta t / I_c \leq \frac{3\pi}{2} + 2\pi n
\]

for \(n = 0, \pm 1, \ldots\), whereas the most efficient reversal appears when the condition

\[
\cos (Gr I_0 \delta t / I_c) = -1, \quad Gr I_0 \delta t / I_c = \pi + 2\pi n
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is fulfilled. It gives also hyperbolic curves with \(G_n r \delta t = \pi + 2\pi n\) in \(G - r\) space. From physical point of view such curves are the curves of a constant amplitude for the driving force in LLG equation (11). In this situation the magnetic moment becomes aligned in the \(m_z = -1\) direction exactly after the pulse has been off and the relevant time scale is determined only by the pulse duration, not by the Gilbert damping. It helps us to optimize the pulse duration in order to make the fastest reversal. We see from (11) that the lowest one is realized for \(n = 0\)

\[
\delta t_{eff} = \frac{\pi I_c}{Gr I_0}.
\]

This situation is demonstrated in Fig.1(d) for \(G = 100, r = 0.1, \alpha = 0.005, w = 0.01, I_0 / I_c = 0.5\) and \(\delta t_{eff} = 0.628\). It leads to \(\delta t_{reversal} \approx 0.6 \cdot 10^{-10}\) s for typical \(\omega_p \sim 10\) GHz. This time is two orders of magnitude smaller than in work 13.
FIG. 3. Demonstration of periodicity in $G - \alpha$ diagram. Magnetization reversal is shown by orange points. Dashed blue lines correspond to the areas $|10|$. The calculation is performed for $G$ interval with the step $\Delta G = 0.5$, and $\alpha$ interval with the step $\Delta \alpha = 0.0001$. Other parameters are $r = 0.1$, $w = 0.01$, $I_0/I_c = 0.5$, $\delta t = 3$.

realization of superconducting memory elements.

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Supplement material on Analytical criteria for magnetization reversal in
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A. A. Mazanik\textsuperscript{1,2}, I. R. Rahmonov\textsuperscript{1,3}, and Yu. M. Shukrinov\textsuperscript{1,4}

\textsuperscript{1}BLTP, JINR, Dubna, Moscow Region, 141980, Russia
\textsuperscript{2}Moscow Institute of Physics and Technology Dolgoprudny, Moscow region, 141700, Russia
\textsuperscript{3}Umarov Physical Technical Institute, TAS, Dushanbe, 734063, Tajikistan
\textsuperscript{4}Dubna State University, Dubna, 141980, Russia

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1 Relation between \( I_p(t) \) and \( \sin \Phi(t) \)

To investigate the relation between the rectangular pulse \( I_p(t) = I_0 [\theta(t-t_0) - \theta(t-t_0-\delta t)] \) and \( \sin \Phi(t) \), where \( \Phi(t) = \varphi(t) - rm_p(t) \), we have to solve the following equation

\[
\frac{I_p(t)}{I_c} = w [\frac{d\varphi(t)}{dt} - r \frac{dm_p(t)}{dt}] + \sin (\varphi(t) - rm_p(t)) = w \frac{d\Phi(t)}{dt} + \sin \Phi(t),
\]

(1)

when the pulse has been started with \( \Phi(t = t_0) = 0 \) condition. After straightforward integration it gives us

\[
\tan \Phi(t)/2 = \frac{I_0}{I_c} \cdot \frac{\sinh [\frac{t-t_0}{\tau_0}]}{\sinh [\frac{t-t_0}{\tau_0}] + \sqrt{1 - \left(\frac{I_0}{I_c}\right)^2} \cosh [\frac{t-t_0}{\tau_0}]},
\]

(2)

Here \( \tau_0^{-1} = \sqrt{1 - \left(\frac{I_0}{I_c}\right)^2} \). When the pulse has been switched off,

\[
\sin \Phi(t) = \frac{2 \tan (\Phi(t_0 + \delta t)/2) \exp [-\frac{t-t_0-\delta t}{w}]}{1 + \tan^2 (\Phi(t_0 + \delta t)/2) \exp [-\frac{t-t_0-\delta t}{w}]}
\]

(3)

exponentially drops to 0 with time scale \( \tau_1 \sim w \).

So, when conditions \( w \ll \delta t \), and more accurate \( \sqrt{\frac{w}{\sqrt{1 - \left(\frac{I_0}{I_c}\right)^2}}} \ll \delta t \), are fulfilled, \( \sin \Phi(t) \) approaches \( I_0/I_c = \sin \Phi^* \) for \( \Delta t \sim \tau_0 \sim \sqrt{\frac{w}{\sqrt{1 - \left(\frac{I_0}{I_c}\right)^2}}} \) and is approximately constant. It means that \( \sin \Phi(t) \) shows nearly rectangular form with height \( I_0/I_c \) and duration \( \delta t \). So, the magnetic moment feels approximately constant field \( \sin \Phi(t) \) during such pulse. This limit was used in the article and demonstrated in Fig.(a).

When the conditions \( w \ll 1 \) and \( I_p(t)/I_c < 1 \) are violated, the profile of \( \sin \Phi(t) \) becomes more complicated. If we change \( w \), \( \sin \Phi(t) \) will not coincide with \( I_p(t)/I_c \). Firstly, some pumping process of \( \sin \Phi(t) \) to \( I_0/I_c \) occurs. Secondly, new region emerges where \( I_p(t) = 0 \), but \( \sin \Phi(t) \neq 0 \), which influences the magnetization dynamics. This situation is shown in Fig.(b) in comparison with Fig.(a). Here \( m_z(t_0 + \delta t) > 0 \), but the reversal still happens due to the region after the pulse, where \( \sin \Phi(t) > 0 \). If we make \( I_p(t) > I_c \), resistive state will occur resulting in oscillations of \( \sin \Phi(t) \). This situation is demonstrated in Fig.(c). A combination of \( I_p(t) > I_0 \) and \( w \sim 1 \) is shown in Fig.(d) resulting in superposition of effects \( I_p(t) > I_0 \) and \( w \sim 1 \) simultaneously.
Figure 1: The zoomed part of dynamics of $m_z$ based on numerical solution of LLG equation and extended RSJ model (red curve). The current pulse is shown by blue curve. The profile $\sin \Phi(t)$ is shown by green curve. Parameters of calculations are $G = 100$, $r = 0.1$, $\alpha = 0.005$, $\delta t = 3$, (a) $w = 0.01$, $I_0 = 0.5I_c$; (b) $w = 0.5$, $I_0 = 0.5I_c$; (c) $w = 0.3$, $I_0 = 1.1I_c$; (d) $w = 1$, $I_0 = 1.1I_c$. 