Scaling and Universality in City Space Syntax: between Zipf and Matthew

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Abstract

We report about universality of rank-integration distributions of open spaces in city space syntax similar to the famous rank-size distributions of cities (Zipf’s law). We also demonstrate that the degree of choice an open space represents for other spaces directly linked to it in a city follows a power law statistic. Universal statistical behavior of space syntax measures uncovers the universality of the city creation mechanism. We suggest that the observed universality may help to establish the international definition of a city as a specific land use pattern.

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1 Graphs and space syntax of urban environments

Urban space is of rather large scale to be seen from a single viewpoint; maps provide us with its representations by means of abstract symbols facilitating our perceiving and understanding of a city. The
middle scale and small scale maps are usually based on Euclidean geometry providing spatial objects with precise coordinates along their edges and outlines.

The widespread use of graph theoretic analysis in geographic science had been reviewed in [1] establishing it as central to spatial analysis of urban environments. In [2], the basic graph theory methods had been applied to the measurements of transportation networks.

Network analysis has long been a basic function of geographic information systems (GIS) for a variety of applications, in which computational modelling of an urban network is based on a graph view in which the intersections of linear features are regarded as nodes, and connections between pairs of nodes are represented as edges [3]. Similarly, urban forms are usually represented as the patterns of identifiable urban elements such as locations or areas (forming nodes in a graph) whose relationships to one another are often associated with linear transport routes such as streets within cities [4]. Such planar graph representations define locations or points in Euclidean plane as nodes or vertices \( \{i\}, i = 1, \ldots, N \), and the edges linking them together as \( i \sim j \), in which \( \{i, j\} = 1, 2, \ldots, N \). The value of a link can either be binary, with the value 1 as \( i \sim j \), and 0 otherwise, or be equal to actual physical distance between nodes, \( \text{dist}(i, j) \), or to some weight \( w_{ij} > 0 \) quantifying a certain characteristic property of the link. We shall call a planar graph representing the Euclidean space embedding of an urban network as its primary graph. Once a spatial system has been identified and represented by a graph in this way, it can be subjected to the graph theoretic analysis.

A spatial network of a city is a network of the spatial elements of urban environments. They are derived from maps of open spaces (streets, places, and roundabouts). Open spaces may be broken down into components; most simply, these might be street segments, which can be linked into a network via their intersections and analyzed as a networks of movement choices. The study of spatial configuration is instrumental in predicting human behavior, for instance, pedestrian movements in urban environments [6]. A set of theories and techniques for the analysis of spatial configurations is called space syntax [7]. Space syntax is established on a quite sophisticated speculation that the evolution of built form can be explained
in analogy to the way biological forms unravel [5]. It has been developed as a method for analyzing space in an urban environment capturing its quality as being comprehensible and easily navigable [6]. Although, in its initial form, space syntax was focused mainly on patterns of pedestrian movement in cities, later the various space syntax measures of urban configuration had been found to be correlated with the different aspects of social life, [8].

Decomposition of a space map into a complete set of intersecting axial lines, the fewest and longest lines of sight that pass through every open space comprising any system, produces an axial map or an overlapping convex map respectively. Axial lines and convex spaces may be treated as the spatial elements (nodes of a morphological graph), while either the junctions of axial lines or the overlaps of convex spaces may be considered as the edges linking spatial elements into a single graph unveiling the topological relationships between all open elements of the urban space. In what follows, we shall call this morphological representation of urban network as a dual graph.

The encoding of cities into non-planar dual graphs reveals their complex structure. The transition to a dual graph is a topologically non-trivial transformation of a planar primary graph into a non-planar one which encapsulates the hierarchy and structure of the urban area and also corresponds to perception of space that people experience when travelling along routes through the environment. The dual transformation replaces the 1D open segments (streets) by the zero-dimensional nodes. The sprawl like developments consisting of a number of blind passes branching off a main route are changed to the star subgraphs having a hub and a number of client nodes. Junctions and crossroads are replaced with edges connecting the corresponding nodes of the dual graph. Places and roundabouts are considered as the independent topological objects and acquire the individual IDs being nodes in the dual graph. Cycles are converted into cycles of the same lengths. The dual graph representation of a regular grid pattern is a complete bipartite graph, where the set of vertices can be divided into two disjoint subsets such that no edge has both end-points in the same subset, and every line joining the two subsets is present, [9]. These sets can be naturally interpreted as those of the vertical and horizontal edges in the primary graphs (streets and avenues). It is the dual graph transformation which al-
allows to separate the effects of order and of structure while analyzing a transport network on the morphological ground. It converts the repeating geometrical elements expressing the order in the urban developments into the twins nodes, the pairs of nodes such that any other is adjacent either to them both or to neither of them.

Integration is a centrality measure used in space syntax theory in order to express the degree to which a node is integrated or segregated from the whole urban texture. In the present paper, we report on the Rank-Integration distributions observed for the several compact urban patterns similar to the famous Rank-Size distribution ("Zipf’s law") (see Sec. 3). Furthermore, we show that the control value parameter used in space syntax theory for the evaluation of the degree of choice each node represents for nodes directly linked to it follows a power law statistic in the intermediate range of scales (see Sec. 3). Universal statistical behavior of space syntax measures uncovers the universality of the creation mechanism responsible for the appearance of nodes of high centrality which acts over all cities independently of their backgrounds. In Sec. 3.1 and 3.2, we discuss the possible mechanisms of city formation which may be responsible for the appearance of universality in all studied compact urban patterns.

We have studied five different compact urban patterns. Two of them are situated on islands: Manhattan (with an almost regular grid-like city plan) and the network of Venice canals (imprinting the joined effect of natural, political, and economical factors acting on the network during many centuries). In the old city center of Venice that stretches across 122 small islands in the marshy Venetian Lagoon along the Adriatic Sea in northeast Italy, the canals serve the function of roads.

We have also considered two organic cities founded shortly after the Crusades and developed within the medieval fortresses: Rothenburg ob der Tauber, the medieval Bavarian city preserving its original structure from the 13th century, and the downtown of Bielefeld (Altstadt Bielefeld), an economic and cultural center of Eastern Westphalia.

To supplement the study of urban canal networks, we have investigated that one in the city of Amsterdam. Although it is not actually isolated from the national canal network, it is binding to the delta of the Amstel river, forming a dense canal web exhibiting
a high degree of radial symmetry.

The scarcity of physical space is among the most important factors determining the structure of compact urban patterns. Some characteristics of studied dual city graphs are given in Tab. 1. There, \( N \) is the number of open spaces (streets/canals and places) in the urban pattern (the number of nodes in the dual graphs), \( M \) is the number of junctions (the number of edges in the dual graphs); the graph diameter, \( \text{diam}(\mathcal{G}) \) is the maximal depth (i.e., the graph-theoretical distance) between two vertices in a dual graph.

2 Space syntax measures

In space syntax theory, graph-based models of space are used in order to investigate the influence of the shape and configuration of environments on human spatial behavior and experience. A number of configurational measures have been introduced in so far in quantitative representations of relationships between spaces of urban areas and buildings. Below we give a brief introduction into the measures commonly accepted in space syntax theory.

Although similar parameters quantifying connectivity and centrality of nodes in a graph have been independently invented and extensively studied during the last century in a varied range of disciplines including computer science, biology, economics, and sociology, the syntactic measures are by no means just the new names for the well known quantities. In space syntax, the spaces are understood as voids between buildings restraining traffic that dramatically changes their meanings and the interpretation of results.

The main focus of the space syntax study is on the relative proximity (or accessibility) between different locations which involves calculating graph-theoretical distances between nodes of the dual graphs and associating these distances with densities and intensities of human activity which occur at different open spaces and along the links which connect them [10, 11, 4].

Space adjacency is a basic rule to form axial maps: two axial lines intersected are regarded as adjacency. Two spaces, \( i \) and \( j \), are held to be adjacent in the dual graph \( \mathcal{G} \) when it is possible to move freely from one space to another, without passing through any intervening.
The adjacency matrix \( A_G \) of the dual graph \( G \) is defined as follows:

\[
(A_G)_{ij} = \begin{cases} 
1, & i \sim j, \\
0, & \text{otherwise.}
\end{cases}
\] (1)

Let us note that rows and columns of \( A_G \) corresponding to the twins nodes are identical. \textit{Depth} is a topological distance between nodes in the dual graph \( G \). Two open spaces, \( i \) and \( j \), are said to be at depth \( d_{ij} \) if the least number of syntactic steps needed to reach one node from the other is \( d_{ij} \), [12]. The concept of depth can be extended to \textit{total depth}, the sum of all depths from a given origin,

\[
D_i = \sum_{j=1}^{N} d_{ij},
\] (2)

in which \( N \) is the total number of nodes in \( G \). The average number of syntactic steps from a given node \( i \) to any other node in the dual graph \( G \) is called the mean depth,

\[
\ell_i = \frac{D_i}{N-1}.
\] (3)

The mean depth (3) is used for quantifying the level of integration/segregation of the given node, [7].

\textit{Connectivity} is defined in space syntax theory as the number of nodes that connect directly to a given node in the dual graph \( G \), [12]. In graph theory, the space syntax connectivity\footnote{In graph theory, connectivity of a node is defined as the number of edges connected to a vertex. Note that it is not necessarily equal to the degree of node, \( \deg(i) \), since there may be more than one edge between any two vertices in the graph.} of a node is called the node \textit{degree}:

\[
\text{Connectivity}(i) = \deg(i) = \sum_{j=1}^{N} (A_G)_{ij}.
\] (4)

The \textit{accessibility} of a space is considered in space syntax as a key determinant of its spatial interaction and its analysis is based on an implicit graph-theoretic view of the dual graph.

\textit{Integration} of a node is by definition expressed by a value that indicates the degree to which a node is integrated or segregated from a system as a whole (\textit{global integration}), or from a partial system.
consisting of nodes a few steps away (local integration), \cite{13}. It is measured by the Real Relative Asymmetry (RRA) \cite{9},

$$\text{RRA}(i) = 2 \frac{\ell_i - 1}{D_N (N - 2)},$$ \hspace{1cm} (5)

in which the normalization parameter allowing to compare nodes belonging to the dual graphs of different sizes is

$$D_N = 2 \frac{N \left( \log_2 \left( \frac{N+2}{3} \right) - 1 \right) + 1}{(N - 1)(N - 2)}. \hspace{1cm} (6)$$

Another local measure used in space syntax theory is the control value (CV). It evaluates the degree to which a space controls access to its immediate neighbors taking into account the number of alternative connections that each of these neighbors has. The control value is determined according to the following formula, \cite{7}:

$$\text{CV}(i) = \sum_{i \sim j} \frac{1}{\text{deg}(j)} = \sum_{j=1}^{N} \left( A G D^{-1} \right)_{ij},$$ \hspace{1cm} (7)

where the diagonal matrix is $D = \text{diag} \left( \text{deg}(1), \text{deg}(2), \ldots, \text{deg}(N) \right)$. A dynamic global measure of the flow through a space $i \in G$ commonly accepted in space syntax theory is the global choice, \cite{14}. It captures how often a node may be used in journeys from all spaces to all others spaces in the city. Vertices that occur on many shortest paths between other vertices have higher betweenness than those that do not. Global choice can be estimated as the ratio between the number of shortest paths through the node $i$ and the total number of all shortest paths in $G$,

$$\text{Choice}(i) = \frac{\text{#shortest paths through } i}{\text{#all shortest paths}}. \hspace{1cm} (8)$$

A space $i$ has a strong choice value when many of the shortest paths, connecting all spaces to all spaces of a system, passes through it. The Dijkstra’s classical algorithm which visits all nodes that are closer to the source than the target before reaching the target can be implemented in order to compute the value Choice($i$).

The integration and the global choice index are the centrality measures which capture the relative structural importance of a node in a dual graph.
3 Scaling and universality in city space syntax

In urban studies, scaling and universality have been found with a remarkable regularity.

The famous rank-size distribution of city sizes all over the world is known as Zipf’s Law. If we calculate the natural logarithm of the city rank in some countries and of the city size (measured in terms of its population) and then plot the resulting data in a diagram, we obtain a remarkable linear pattern with the slope of the line equals $-1$ (or $+1$, if cities have been ranked in the ascending order), [15].

A possible explanation for the rank-size distributions of the human settlements based on simple stochastic models of settlement formation and growth has been recently proposed in [16].

![Figure 1: The log-log plot of the Rank-Integration distribution calculated the nodes of five dual graphs: the street grid of Manhattan, Rothenburg, Bielefeld, the canal networks in Venice and Amsterdam. The linear pattern is given by a straight line.](image)

Here we report on the similar rank distributions for the values of

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2 The empirical validity of Zipf’s Law for cities have been checked out recently using new data on the city populations over 73 countries by two different estimation methods in [15]. The use of various estimators justifies the validity of Zipf’s Law for from 20 to 43 of 73 investigated countries. It has been suggested in [15] that variations in the value of Zipf’s exponent are better explained by political economy variables than by economic geography variables.
the space syntax measures quantifying the centrality of nodes in the dual graphs calculated for the compact urban patterns.

We have ranked all open spaces in the dual graphs of the investigated five compact urban patterns in accordance to their centrality indices. The nodes with the worse centrality have been assigned to the first (lowest) rank, while to those of the best centrality the highest rank has been given. The diagrams for Rank-Centrality distributions have been calculated for all dual graphs representing two German medieval cities (Rothenburg of der Tauber and Bielefeld), the street grid in Manhattan, and two city canal networks, in Amsterdam and in Venice, within the same frame. It is worth to mention that no matter how the centrality level was estimated, either by the integration values or by the global choice values, the data from all centrality indicators demonstrate a surprising universality being fitted perfectly with the linear pattern, with the slope of the line equals 1.

Figure 2: The log-log plot of the Rank-Global Choice (Betweenness) distribution calculated the nodes of five dual graphs: the street grid of Manhattan, Rothenburg, Bielefeld, the canal networks in Venice and Amsterdam. The linear pattern is given by a straight line.

The matching of power law behaviors (the same scaling exponent) can have a deeper origin in the background dynamical process re-
sponsible for such a power-law relation. Being diverse in their sizes, forms, economical and political history, cities display nevertheless the identical *scaling behavior* and probably share the similar fundamental mechanisms of the open spaces creation. Formally, such a common dynamics can be referred to as *universality*, so that those cities with precisely the same critical exponents of rank-integration statistics are said to belong to the same *universality class*.

The ubiquity of power-law relations in complex systems are often thought to be signatures of hierarchy and robustness. The area distribution of satellite cities around large urban centers has been reported to obey a power-law with exponent \( \simeq 2 \), [17]. The fractal dimension of urban aggregates as a global measure of areal coverage have been studied extensively for many cities around the world during the last decades (see [18],[19] for a review). The scaling property has also been observed recently in concern with the space syntax studies, in the distribution of the length of open space linear segments (axial lines) [20].

In the present subsection, we discuss a scaling property of control values distribution calculated for the dual graphs of compact urban patterns.

The \( CV(i) \)-parameter quantifies the degree of choice the node \( i \in \mathcal{G} \) represents for other nodes directly connected to it.

Provided random walks, in which a walker moves in one step to another node randomly chosen among all its nearest neighbors are defined on the graph \( \mathcal{G} \), the parameter \( CV(i) \) acquires a probabilistic interpretation. Namely, it specifies the expected number of walkers which is found in \( i \in \mathcal{G} \) after one step if the random walks starts from a uniform configuration, in which all nodes in the graph have been uniformly populated by precisely one walker.

Then, a graph \( \mathcal{G} \) can be characterized by the probability

\[
P(m) = \Pr [i \in \mathcal{G} | CV(i) = m] \tag{9}
\]

of that the control value of a node chosen uniformly at random among all nodes of the graph \( \mathcal{G} \) equals to \( m > 0 \).

The log-log plot of (9) is shown in Fig. 3. It is important to mention that the profile of the probability decay exhibits the approximate scaling well fitted by the cubic hyperbola, \( P(m) \simeq m^{-3} \), universally for all five compact urban patterns we have studied.

Universal statistical behavior of the control values for the nodes
Figure 3: The log-log plot of the probability distribution that a node randomly selected among all nodes of the dual graph $G$ will be populated with precisely $m$ random walkers in one step starting from the uniform distribution (one random walker at each node). The dashed line indicates the cubic hyperbola decay, $P(m) = m^{-3}$.

representing a relatively strong choice for their nearest neighbors,

$$\Pr [i \in G|CV(i) = m] \simeq \frac{1}{m^3}, \quad (10)$$

uncovers the universality of the creation mechanism responsible for the appearance of the "strong choice" nodes which acts over all cities independently of their backgrounds. It is a common suggestion in space syntax theory that open spaces of strong choice are responsible for the public space processes driven largely by the universal social activities like trades and exchange which are common across different cultures and historical epochs and give cities a similar global structure of the "deformed wheel" [22].

It has been shown long time ago by H. Simon [26] that the power law distributions always arise when the "the rich get richer" principle works, i.e. when a quantity increases with its amount already
present. In sociology this principle is known as the Matthew effect \cite{27} (this reference appears in \cite{28}) following the well-known biblical edict. In the next section, we discuss the graph evolution algorithms related to ”the rich get richer” principle.

3.1 Cameo principle of scale-free urban developments

It is interesting to consider possible city development algorithms that could lead to the emergence of scaling invariant degree structure in urban environments.

Among the classical models in which the degree distribution of the arising graph satisfies a power-law is the graph generating algorithms based on the \textit{preferential attachment} approach. Within preferential attachment algorithms, the growth of a network starts with an initial graph of \(n_0 \geq 2\) nodes such that the degree of each node in the initial network is at least 1. The celebrated Barabási-Albert model \cite{23} have been proposed in order to model the emergency and growth of scale-free complex networks. New nodes in the model \cite{23} are added to the network one at a time. Each new node is connected to \(n\) of the existing with a probability that is biased being proportional to the number of links that the existing node already has,

\[
p_i = \frac{\text{deg}(i)}{\sum_{j=1}^{N} \text{deg}(j)}.
\]

It is clear that the nodes of high degrees tend to quickly accumulate even more links representing a strong \textit{preference choice} for the emerging nodes, while nodes with only a few links are unlikely to be chosen as the destination for a new link. The preferential attachment forms a positive feedback loop in which an initial random degree variation is magnified with time, \cite{24}. It is fascinating that the expected degree distribution in the graph generated in accordance to the algorithm proposed in \cite{23} asymptotically approaches the cubic hyperbola,

\[
\Pr [i \in \mathcal{G} | \text{deg}(i) = k] \simeq \frac{1}{k^3}.
\]

It is however obvious that the mechanisms governing the city creation and development certainly do not follow such a simple preferential attachment principle as that discussed in \cite{23}. Indeed, when
new streets (public or private) are created as a result of site subdivision or site redevelopment, they can hardly be planed in such a way as to meet the streets that already have the ever maximal number of junctions with other streets in a city. The challenge of city modelling calls for the more realistic heuristic principles that could catch the main features of city creation and development.

It is clear that a prominent model describing the urban developments should take into account the structure of embedding physical space: the size and shape of landscape, and the local land use patterns. A suitable algorithm describing the development of complex networks which takes into account has been recently proposed in [25]. It is called the Cameo-principle having in mind the attractiveness, rareness and beauty of the small medallion with a profiled head in relief called Cameo. It is exactly their rareness and beauty which gives them their high value.

In the Cameo model [25], the local attractiveness of a site determining the creation of new spaces of motion in that is specified by a real positive parameter \( \omega > 0 \). Indeed, it is rather difficult if ever be possible to estimate exactly the actual value \( \omega(i) \) for any site \( i \in \mathcal{G} \) in the urban pattern, since such an estimation can be referred to both the local believes of city inhabitants and may be to the cultural context of the site that may vary over the different nations, historical epochs, and even over the certain groups of population.

Therefore, in the framework of the probabilistic approach, it seems natural to consider the value \( \omega \) as a real positive independent random variable distributed over the vertex set of the graph representation of the site uniformly in accordance to a smooth monotone decreasing probability density function \( f(\omega) \).

Let us suggest that there is just a few distinguished sites which are much more attractive then an average one in the city, so that the density function \( f \) has a right tail for large \( \omega \gg \bar{\omega} \) such that \( f(\omega) \ll f(\bar{\omega}) \).

Each newly created space of motion \( i \) (represented by a node in the dual city graph \( \mathcal{G}(N) \) containing \( N \) nodes) may be arranged in such a way to connect to the already existed space \( j \in \mathcal{G}(N) \) depending only on its attractiveness \( \omega(j) \) and is of the form

\[
\Pr[i \sim j \mid \omega(j)] \simeq \frac{1}{N \cdot f^\alpha(\omega(j))} \quad (13)
\]
with some $\alpha \in (0, 1)$. The assumption (13) implies that the probability to create the new space adjacent to a space $j$ scales with the rarity of sites characterized with the same attractivity $\omega$ as $j$.

The striking observation under the above assumptions is the emergence of a scale-free degree distribution independent of the choice of distribution $f(\omega)$. Furthermore, the exponent in the asymptotic degree distribution becomes independent of the distribution $f(\omega)$ provided its tail, $f(\omega) \ll f(\bar{\omega})$, decays faster than any power law.

In the model of growing networks proposed in [25], the initial graph $G_0$ has $N_0$ vertices, and a new vertex of attractiveness $\omega$ taken independently uniformly distributed in accordance to the given density $f(\omega)$ is added to the already existed network at each time click $t \in \mathbb{Z}_+$. Being associated to the graph, the vertex establishes $k_0 > 0$ connections with other vertices already present in that. All edges are formed accordingly to the Cameo principle (13).

The main result of [25] is on the probability distribution that a randomly chosen vertex $i$ which had joined the Cameo graph $G$ at time $\tau > 0$ with attractiveness $\omega(i)$ amasses precisely $k$ links from other vertices which emerge by time $t > \tau$. It is important to note that in the Cameo model the order in which the edges are created plays a role for the fine structure of the graphs. The resulting degree distribution for $t - \tau > k/k_0$ is irrelevant to the concrete form of $f(\omega)$ and reads as following

\[ P(k) \simeq \frac{1}{t} \sum_{0 < \tau < t} \frac{k_0^{1/\alpha}}{k^{1+1/\alpha+\alpha(1)}} \ln^{1/\alpha} \left( \frac{t}{\tau} \right) \approx \frac{1}{k^{1+1/\alpha+\alpha(1)}}. \]

In order to obtain the asymptotic probability degree distribution for an arbitrary node as $t \to \infty$, it is necessary to sum (14) over all $\tau < t$ that gives

\[ P(k) \simeq \frac{1}{t} \sum_{0 < \tau < t} \frac{k_0^{1/\alpha}}{k^{1+1/\alpha+\alpha(1)}} \ln^{1/\alpha} \left( \frac{t}{\tau} \right) = \frac{1}{k^{1+1/\alpha+\alpha(1)}}. \]

The emergence of the power law (15) demonstrates that graphs with a scale-free degree distribution may appear naturally as the result of a simple edge formation rule based on choices where the probability to chose a vertex with affinity parameter $\omega$ is proportional to the
frequency of appearance of that parameter. If the affinity parameter \( \omega \) is itself power law like distributed one could also use a direct proportionality to the value \( \omega \) to get still a scale free graph.

3.2 Trade-offs models of urban sprawl creation

The Vermont Forum on Sprawl defines sprawl as ”dispersed development outside of compact urban and village centers along highways and in rural countryside” (the quote appears in [32]). The term urban sprawl generally has negative connotations due to the health and environmental issues that sprawl creates [29]. Residents of sprawling neighborhoods tend to emit more pollution per person and suffer more traffic fatalities. Urban sprawl is a growing problem in many countries and in the so called ”smart growth” policies, which became popular in US during the 1990s, trumping such mainstays as education and crime in many polls and contributing to the election of anti-sprawl politicians [30].

There are currently many indicators of sprawl but the majority of them are economic or land-use related rather than intrinsically spatial or morphological.

One of the early study devoted to the significant reduction of the number of dead-ends in the urban network, caused by the merging of the cart path and road networks, suggested that the amount of ”ringiness” [31] in the system might serve as a good morphological indicator of sprawl. In [32], it has been suggested that the differences between suburban and urban developments and even sprawl are clearly discernable by the proportion and distribution of cycle lengths in the dual graph of the network. Typical suburban developments usually contain a high ratio of cul-de-sacs, the dead-end streets with only one inlet, together with just a few entrances accessed from central roads.

However, while observing the highly interlaced patterns of modern housing subdivisions sprawled out over rural lands at the fringe of many urban areas in USA and Canada, one can see that such

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3For instance, low population density is an indicator of sprawl.

4In accordance to the US Bureau of Census data on Urbanized Areas, Over a 20-year period, the 100 largest Urbanized Areas in US sprawled out over an additional 14,545 square miles. That was more than 9 million acres of natural habitats, farmland and other rural space that were covered over by the asphalt, buildings and sub-divisions of suburbia. See http://www.sprawlcity.org/libis/index.html for details.
subdivisions may contain no cycles at all being an arborescence offering only a few places to enter and exit the development, causing traffic to use high volume collector roads.

Probably, the most fascinatingly reparative morphological element of urban sprawl is an individual access from a private household to the central path. Designating the individual parking places as the client nodes and the high volume road as a hub in the dual graph, we obtain a star graph as the typical syntactic motive pertinent to urban sprawl.

Figure 4: A segment of Atlanta’s suburban sprawl (USA). The spatial morphology of sprawl is represented by the star graph.

The dual graph corresponding to the segment of suburban sprawl shown in Fig. 4 forms a star graph, in which 32 individual spaces representing the private parking places are connected to a hub, the only sinuous central road.

Star graphs have been observed in many technical and informational networks. In particular, the algorithms generating the tree like graphs combined from a number of star subgraphs have been extensively studied and modelled in concern with the Internet topology (see [33]-[35] and many others).

Extensive experiments suggest that the hierarchical trees containing the numerous star subgraphs could arise as a result of trade-off process minimizing the weighted sums of two or more objectives.

In the simple model of Internet growth [35], a tree is built as nodes arrive uniformly at random in the unit square (the shape is,
as usual, inconsequential). When the $i$-th node arrives, it attaches itself on one of the previous nodes $j$ that minimizes the weighted sum of the two objectives:

$$\text{cost}(i, j) = \alpha \cdot d_{ij} + c_j, \quad \alpha \geq 0,$$

where $d_{ij}$ is the geometrical (Euclidean) distance between two nodes $i$ and $j$, and $c_j$ is the centrality value of $j$.

It is newsworthy that the behavior of the trade-off model (16) depends crucially on the value of the tuning parameter $\alpha$ which can be naturally interpreted as the "last mile" cost reckoning the construction and maintenance expenditures.

If $\alpha$ is taken less than a particular constant depending upon the landscape shapes and certain economical conditions, then Euclidean distance is of no importance, and the network produced by the trade-off algorithm is easily seen to form a star. A star graph consists of a central node (hub) characterized by the uttermost connectivity and a number of terminal vertices linked to the hub.

It seems natural to apply the simple trade-off models to dual graphs in order to predict the appearance of urban sprawl with the local land-use scheme. The decisive factor for the emergence of star graphs is the supremacy of the centrality (integration) objective, while the physical (Euclidean) distance between graph vertices is of no importance. It is well known that the humbleness of physical distances is among main factors shaping the sprawl land use patterns. In return, being highly dependent on automobiles for transportation, the low density sprawl development consumes much more land than traditional urban developments.

The other way around, if we suggest that the last mile costs $\alpha$ grow up with the network size $N$ at least as fast as $\sim \sqrt{N}$, then the Euclidean distance between nodes becomes an important factor shaping the form of the network. A graph that arises in the trade-off process in such a case constitutes a dynamic version of the Euclidean minimum spanning tree, in which high degrees would occur, but with exponentially vanishing probability.

Eventually, if $\alpha$ exceeds a certain constant, but grows slower than $\sqrt{N}$, then, almost certainly, the degrees obey a power law, and the resulting dual graph forms a fractal.

We suggest that the emergence of a complicated highly inhomogeneous structure which we observe in urban developments can
generally involve trade-offs, the optimization problems between the multiple, complicated and probably conflicting objectives. In order to support this proposition, we have simulated on a trade-off model minimizing two different objectives simultaneously. The trading between the geometrical distance and centrality of nodes has been accounted by the objective (16). We have also used another cost function,

\[ \text{cost}(i, j) = \omega \cdot d_{ij} + d_{ij}, \quad \omega \geq 0, \tag{17} \]

in which \( d_{ij} \) is the Euclidean distance and \( d_{ij} \) is the graph theoretical distance between nodes \( i \) and \( j \) (the number of hopes), as the second objective. In general, the structure of graph that would arise in such a complex trade-off process crucially depends upon the magnitudes of \( \omega \) and \( \alpha \), their relative values, and the way they depend on the network size \( N \). In general, it may also depend upon the local curvature of the simulation domain that describes a landscape shape and its distribution.

In the simplest case of plain domains, for the relatively small values of tuning parameters, \( \alpha \approx \omega \approx 0 \), the geometrical distances
between nodes are of no importance, and the structure of the resulting graph (shown in [5]) is shaped by the simultaneous optimization between the graph distance and centrality objectives. It is worth mentioning that the resulting graph sketched in [5] contains a valuable fraction of twins nodes (while $\alpha = \omega = 10^{-3}$, 84% of graphs nodes are twins) forming almost a complete bipartite dual graph.

4 Discussion and Conclusion

Although, nowadays the majority of people live in cities [36] there is no one standard international definition of a city: the term may be used either for a town possessing city status; for an urban locality exceeding an arbitrary population size; for a town dominating other towns with particular regional economic or administrative significance. In most parts of the world, cities are generally substantial and nearly always have an urban core, but in the US many incorporated areas which have a very modest population, or a suburban or even mostly rural character, are designated as cities.

A universal social dynamic underlying the scaling phenomena observed in cities implies that an increase in productive social opportunities, both in number and quality, leads to quantifiable changes in individual behavior of humans in a city integrating them into a complex dynamical network [37]. The impact of urban landscapes on the construction of social relations draws attention in the fields of ethnography, sociology, and anthropology. In particular, it has been suggested that the urban space combining social, economic, ideological and technological factors is responsible for the technological, socioeconomic, and cultural development, [38]. It is worth to mention that the processes relating urbanization to economic development and knowledge production are very general, being shared by all cities belonging to the same urban system and sustained across different nations and times.

We suggest that the observed universality of the integration and control value statistics we report about in the present paper may help to establish the international definition of a city as a specific land use pattern in which streets, places, street junctions and crossroads form an inhomogeneous complex network.
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Table 1: Some features of studied dual city graphs

| Urban pattern                  | $N$ | $M$ | diam($G$) |
|-------------------------------|-----|-----|-----------|
| Rothenburg ob d.T.            | 50  | 115 | 5         |
| Bielefeld (downtown)          | 50  | 142 | 6         |
| Amsterdam (canals)            | 57  | 200 | 7         |
| Venice (canals)               | 96  | 196 | 5         |
| Manhattan                     | 355 | 3543| 5         |