The **coinvariant ring** is the quotient $R_n = \mathbb{C}[x_1, \ldots, x_n]/(e_1, \ldots, e_n)$ of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ by the ideal generated by the $n$ elementary symmetric polynomials $e_1, \ldots, e_n$. Since the polynomials $e_1, \ldots, e_n$ are symmetric, the quotient $R_n$ carries the structure of a graded $S_n$-module and is one of the most important representations in algebraic combinatorics. In geometric terms, Borel [10] proved that the coinvariant ring presents the cohomology of the flag variety: we have $H^\bullet(\text{Fl}(n)) = R_n$.

In the early 1990s, Garsia and Haiman [19, 36] initiated the study of the diagonal coinvariant ring $DR_n$ obtained by modding out the polynomial ring $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by the ideal generated by positive-degree invariants of the diagonal action of $S_n$. Setting the $y$-variables in $DR_n$ to zero recovers $R_n$, so that $DR_n$ is a bigraded extension of $R_n$. A decade later, Haiman [35] used deep algebraic geometry to express the bigraded $S_n$-isomorphism type of $DR_n$ as the symmetric function $\nabla e_n$, where $\nabla$ is a remarkable symmetric function operator which has the Macdonald polynomials as its eigenbasis.

Thanks to Haiman’s result, finding the bigraded $S_n$-isomorphism type of $DR_n$ amounts to finding the expansion of $\nabla e_n$ in the Schur basis $\{s_\lambda : \lambda \vdash n\}$ of symmetric functions. Although some coefficients in this expansion are known [28], this remains an open problem. Haglund, Haiman, Remmel, Loehr, and Ulyanov [31] conjectured, and Carlsson and Mellit [12] proved, the *Shuffle Theorem*: a formula for the monomial expansion of $\nabla e_n$.

The Shuffle Theorem is a purely combinatorial statement about the symmetric function $\nabla e_n$. In the decade between its conjecture and its proof, various extensions and refinements were proposed [7, 33] in an attempt make $\nabla e_n$ vulnerable to inductive attack. The last of these to appear before the Shuffle Theorem was proved was the Delta Conjecture of Haglund, Remmel, and Wilson [34]. This conjecture depends on two parameters $k \leq n$ and predicts the monomial expansion $\Delta_{e_{k-1}} e_n$ where the delta operator $\Delta_{e_{k-1}}$ is another Macdonald eigenoperator. The Delta Conjecture reduces to the Shuffle Theorem when $k = n$. The ‘Rise Version’ of the Delta Conjecture was proven recently and independently by D’Adderio-Mellit [16] and Blasiak-Haiman-Morse-Pun-Seelinger [9], but the full conjecture remains open.

After the formulation of the Delta Conjecture, Haglund, Rhoades, and Shimozono [35] introduced a family $R_{n,k}$ of singly-graded quotients of $\mathbb{C}[x_1, \ldots, x_n]$ whose isomorphism type is (essentially) the $t = 0$ specialization of the symmetric function $\Delta'_{e_{k-1}} e_n$. The ring $R_{n,k}$ reduces to the classical coinvariant ring $R_n$ when $k = n$. Whereas the algebra of $R_n$ is governed by the combinatorics of permutations in $S_n$, the algebra of $R_{n,k}$ is governed by the combinatorics of ordered set partitions of $[n] := \{1, \ldots, n\}$ into $k$ blocks. Many algebraic properties of $R_n$ generalize naturally to $R_{n,k}$; these quotient rings constitute a coinvariant algebra for the Delta Conjecture.

Returning to geometry, Pawlowski and Rhoades [45] introduced a variety $X_{n,k}$ of spanning line configurations which is homotopy equivalent to the flag variety $\text{Fl}(n)$ when $k = n$. A point in $X_{n,k}$ is a tuple $(\ell_1, \ldots, \ell_n)$ of 1-dimensional subspaces $\ell_i \subseteq \mathbb{C}^k$ such that we have the vector space sum $\ell_1 + \cdots + \ell_n = \mathbb{C}^k$. The ring $R_{n,k}$ presents the cohomology $H^\bullet(X_{n,k})$, and geometric properties of $X_{n,k}$ are governed by combinatorial properties of ordered set partitions (recast as Fubini words). The variety $X_{n,k}$ is a flag variety for the Delta Conjecture. Griffin, Levinson, and Woo [27] gave a
different geometric model for the Delta Conjecture which simultaneously generalizes the theory of Springer fibers.

In Section 1 we provide the necessary background on symmetric function theory, as well as an introduction to the combinatorics of the Delta Conjecture. In Section 2 we introduce the generalized coinvariant rings $R_{n,k}$ and use orbit harmonics to study them. Section 3 is devoted to the variety $X_{n,k}$ of spanning line configurations; we show that $R_{n,k}$ presents its cohomology. In Section 4 we cover remarkable work of Griffin [23] and Griffin-Levinson-Woo [27] which unifies the delta coinvariant theory with the theory of Tanisaki quotients and Springer fibers. We close in Section 5 with future directions and conjectures related largely to the structure of the full symmetric function $\Delta'_{k,e}$ (rather than its $t = 0$ specialization). These are largely algebraic, with ties to rings of differential forms. Given the work of Haiman on Hilbert schemes and diagonal coinvariants, it is our hope that geometry will appear in the more general setting of $\Delta'_{k,e}$, as well.

0.1. A note to the geometrically inclined reader. If one wants to learn about the varieties $X_{n,k}$ right away, it is advised to skip to Section 3 (accepting certain combinatorial results in Section 1 and algebraic results in Section 2 as black boxes). On the other hand, the symmetric function theory of Section 1 was the primary motivation for defining the quotient rings $R_{n,k}$ of Section 2 which was in turn the primary motivation for finding a variety $X_{n,k}$ with $H^\bullet(X_{n,k}) = R_{n,k}$. So the combinatorially or algebraically inclined reader should not miss out on these!

0.2. A word on coefficients. We work over the complex field $\mathbb{C}$ for convenience, but the geometric results presented here can be shown to hold over the ring of integers $\mathbb{Z}$ with slightly more work. We describe the form of the arguments used in passing from $\mathbb{C}$ to $\mathbb{Z}$ after presenting the cohomology of $X_{n,k}$ (see Remark 3.9).

1. Symmetric functions and delta operators

Much of the algebraic and geometric material in this chapter involves the machinery of symmetric functions. We quickly introduce the necessary background and refer the reader to [29] for a comprehensive treatment.

1.1. Symmetric functions and $S_n$-modules. Let $x = (x_1, x_2, \ldots)$ be an infinite list of variables. For a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$, the monomial symmetric function $m_{\lambda} = m_{\lambda}(x)$ is the formal power series

$$m_{\lambda} := \sum_{1 \leq i_1 < \cdots < i_k} \sum_{a_1, \ldots, a_k} x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$$

where the internal sum is over all rearrangements $(a_1, \ldots, a_k)$ of the parts of $\lambda$. The ring of symmetric functions over a commutative ring $R$ is the $R$-submodule $\Lambda$ of the formal power series ring $R[[x]] = R[[x_1, x_2, \ldots]]$ given by $\Lambda := \text{span}_R\{m_{\lambda} : \lambda \text{ a partition}\}$. It is easily seen that $\Lambda$ is closed under multiplication, and so forms a subring of $R[[x]]$. We take our ground ring to be $R = \mathbb{C}(q,t)$ in this chapter, where $q$ and $t$ are variables.

The ring $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ is graded, with the degree $n$ piece $\Lambda_n$ having basis $\{m_{\lambda} : \lambda \vdash n\}$. There are many other interesting bases of $\Lambda_n$ indexed naturally by partitions $\lambda \vdash n$; we introduce those which will be of interest to us. For $n \geq 0$, the elementary, homogeneous, and power sum symmetric functions are given by

$$e_n := \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n} \quad h_n := \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n} \quad p_n := \sum_{i \geq 1} x_i^n$$

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, we extend this definition by setting

$$e_{\lambda} := e_{\lambda_1} e_{\lambda_2} \cdots \quad h_{\lambda} := h_{\lambda_1} h_{\lambda_2} \cdots \quad p_{\lambda} := p_{\lambda_1} p_{\lambda_2} \cdots$$
Each of the sets \( \{e_\lambda : \lambda \vdash n\}, \{h_\lambda : \lambda \vdash n\}, \) and \( \{p_\lambda : \lambda \vdash n\}\) are bases of \( \Lambda_n\).

The power sum basis gives rise to an operation on formal power series called plethysm. Let \( k \geq 1\) and let \( E = E(t_1, t_2, \ldots)\) be a rational expression which depends on variables \( t_1, t_2, \ldots\). The plethystic substitution \( p_k[E] \) of \( E \) into \( p_k \) is the rational function

\[
p_k[E] := E(t_1^k, t_2^k, \ldots)
\]

obtained by replacing each \( t_i \) in \( E \) with \( t_i^k \). More generally, if \( F \) is any symmetric function, we define \( F[E] \) by the rules

\[
(F_1 + F_2)[E] := F_1[E] + F_2[E] \quad (F_1 \cdot F_2)[E] := F_1[E] \cdot F_2[E] \quad c[E] := c
\]

for any symmetric functions \( F_1, F_2 \in \Lambda \) and all constants \( c \). Since \( \{p_1, p_2, \ldots\} \) is an algebraically independent generating set of \( \Lambda \), the expression \( F[E] \) is well-defined.

For a partition \( \lambda \vdash n \), a semistandard Young tableau of shape \( \lambda \) is a filling \( T \) of the Young diagram of \( \lambda \) with positive integers which increase weakly across rows and strictly down columns. For example, the tableau

\[
\begin{array}{ccc}
1 & 1 & 3 \\
2 & 4 & 4 \\
4 & & \\
\end{array}
\]

is semistandard of shape \((3, 3, 1)\). The Schur function \( s_\lambda \) is given by

\[
s_\lambda := \sum_T x^T
\]

where the sum is over all semistandard Young tableaux \( T \) of shape \( \lambda \) and \( x^T \) denotes the monomial \( x_1^{c_1}x_2^{c_2} \cdots \) where \( c_i \) is the number of \( i \)'s in \( T \); the sequence \( \mu = (c_1, c_2, \ldots) \) is the content of \( T \). Our example tableau has content \((2, 1, 1, 3)\) and contributes \( x_1^2x_2x_3x_4^3 \) to the Schur function \( s_{331} \).

The set \( \{s_\lambda : \lambda \vdash n\} \) forms the Schur basis of \( \Lambda_n \). The Hall inner product \( \langle - , - \rangle \) on \( \Lambda \) is obtained by declaring the Schur basis to be orthonormal, i.e.

\[
\langle s_\lambda, s_\mu \rangle := \delta_{\lambda, \mu}.
\]

The omega involution is the isometry \( \omega \) of this inner product given by \( \omega(s_\lambda) = s_{\lambda'} \), where \( \lambda' \) is the partition conjugate to \( \lambda \). By construction, we have \( \omega \circ \omega = \text{id} \); it can also be shown that \( \omega \) is a ring map \( \Lambda \to \Lambda \).

The combinatorics of symmetric functions is closely related to the representation theory of the symmetric group. Irreducible representations of \( S_n \) over \( \mathbb{C} \) are in one-to-one correspondence with partitions of \( n \). If \( \lambda \vdash n \) is a partition, write \( S^\lambda \) for the corresponding \( S_n \)-irreducible. For any finite-dimensional representation \( V \) of \( S_n \), there are unique multiplicities \( c_\lambda \) such that \( V = \bigoplus_{\lambda \vdash n} c_\lambda S^\lambda \).

The Frobenius image of \( V \) is the symmetric function

\[
\text{Frob}(V) := \sum_{\lambda \vdash n} c_\lambda \cdot s_\lambda \in \Lambda_n
\]

obtained by replacing each irreducible \( S^\lambda \) with the corresponding Schur function \( s_\lambda \).

We will consider representations of \( S_n \) equipped with both single and multiple gradings. If \( V = \bigoplus_{i \geq 0} V_i \) is a graded \( S_n \)-module, the graded Frobenius image is

\[
\text{grFrob}(V; q) := \sum_{i \geq 0} \text{Frob}(V_i) \cdot q^i.
\]

Generalizing further, if \( V = \bigoplus_{i,j \geq 0} V_{i,j} \) is a bigraded \( S_n \)-module, the bigraded Frobenius image is

\[
\text{grFrob}(V; q, t) := \sum_{i,j \geq 0} \text{Frob}(V_{i,j}) \cdot q^i t^j.
\]
The Frobenius map provides a dictionary between the representation theory of $S_n$ and the combinatorics of $\Lambda$. For example, if $V$ is an $S_n$-module and $W$ is an $S_m$-module, we have

$$\text{Frob}(V \circ W) = \text{Frob}(V) \cdot \text{Frob}(W)$$

where $V \circ W := \text{Ind}_{S_n \times S_m}^{S_{n+m}}(V \otimes W)$ is the induction product. The omega involution on symmetric functions corresponds to a sign twist, viz.

$$\text{Frob}((\text{sign} \otimes V)) = \omega(\text{Frob}(V))$$

where sign is the 1-dimensional sign representation of $S_n$. A highly desirable way to show that a given symmetric function $F \in \Lambda_n$ is Schur-positive (i.e., has Schur expansion with nonnegative integer coefficients) is to find an $S_n$-module $V$ with $\text{Frob}(V) = F$; this will be a recurring theme in this chapter.

1.2. Hall-Littlewood and Macdonald polynomials. The symmetric functions considered in the previous subsection are rather specialized members of a rich hierarchy of symmetric functions involving the auxiliary parameters $q$ and $t$. Two families in this hierarchy will be important for us: the Hall-Littlewood and Macdonald polynomials.

The Hall-Littlewood basis $\tilde{H}_\mu(x; q)$ of $\Lambda$ depends on the parameter $q$. We define these polynomials combinatorially using the cocharge statistic of Lascoux and Schützenberger [44]. We first describe cocharge on the level of words, and then extend to the case of tableaux.

Given any entry $w_j$ of a word $w$ and a positive integer $k$ which appears in $w$, write $\text{cprev}(k, w_j)$ for the cyclically previous $k$ before $w_j$ in $w$. More precisely, $\text{cprev}(k, w_j)$ is the rightmost $k$ appearing to the left of $w_j$ (if such a $k$ exists), and the rightmost $k$ in $w$ otherwise. If $k$ does not appear in $w$, we leave $\text{cprev}(k, w_j)$ undefined.

Let $\mu \vdash n$ be a partition with $k$ parts and let $w = w_1 \ldots w_n$ be a word with content $\mu$. That is, the word $w$ has $\mu_j$ copies of the letter $j$ for each $j$. We decompose $w$ as a disjoint union of permutations $w^{(1)}, w^{(2)}, \ldots$ as follows. Let $w_{i_1} = 1$ be the rightmost 1 in $w$ and recursively define $i_2, i_3, \ldots, i_k$ by

$$w_{i_{j+1}} = \text{cprev}(j + 1, w_{i_j})$$

for $j = 1, 2, \ldots, k - 1$. The first standard subword $w^{(1)}$ of $w$ is the subword of $w$ consisting of the entries $w_{i_1}, w_{i_2}, \ldots, w_{i_k}$. The word $w^{(1)}$ is well-defined because $w$ has partition content. Extending this idea, the standard subword decomposition $w^{(1)}, w^{(2)}, w^{(3)}, \ldots$ of $w$ is defined by letting $w^{(i)}$ be the first standard subword of the word obtained from $w$ by erasing the letters in $w^{(i)}, w^{(2)}, \ldots, w^{(i-1)}$.

An example should clarify these notions. Let $w = 424511133$, a word of partition content. We bold the rightmost 1 in $w$, then bold the cyclically previous 2, then bold the cyclically previous 3, and so on resulting in 4224511133. The permutation $w^{(1)} = [2, 4, 5, 1, 3]$ formed by the bolded letters is the first standard subword of $w$. Erasing these bolded letters yields the smaller word 42113. Repeating this procedure yields 42113 so that the second standard subword is $w^{(2)} = [4, 2, 1, 3]$. Erasing the bolded letters once more yields the one-letter word 1, so that $w^{(3)} = [1]$.

Let $w$ be a word with partition content. The cocharge $\text{cocharge}(w)$ is defined as follows. If $v \in S_n$ is a permutation of size $n$, the cocharge sequence $(cc_1, \ldots, cc_n)$ of $v$ is defined by the initial condition $cc_1 := 0$ and the recursion

$$cc_{i+1} := \begin{cases} cc_i & \text{if } v^{-1}(i + 1) > v^{-1}(i) \\ cc_i + 1 & \text{if } v^{-1}(i + 1) < v^{-1}(i) \end{cases}$$

and the cocharge of $v$ is the sum $\text{cocharge}(v) := cc_1 + \cdots + cc_n$ of its cocharge sequence. We extend the definition of cocharge to the partition-content word $w$ by setting

$$\text{cocharge}(w) := \text{cocharge}(w^{(1)}) + \text{cocharge}(w^{(2)}) + \cdots$$
where \(w^{(1)}, w^{(2)}, \ldots\) is the standard subword decomposition of \(w\). For the word \(w\) in the previous paragraph, we have

\[
\text{cocharge}(4224511133) = \text{cocharge}[2, 4, 5, 1, 3] + \text{cocharge}[4, 2, 1, 3] + \text{cocharge}[1] = 6 + 4 + 0 = 10.
\]

We are finally ready to define the polynomials \(\tH_\mu(x; q)\). The reading word \(\text{read}(T)\) of a semistandard tableau \(T\) is obtained by reading its entries from left to right within rows, proceeding from bottom to top. If \(\mu \vdash n\) is a partition, the \((\text{modified})\) Hall-Littlewood polynomial \(\tH_\mu(x; q)\) has Schur expansion

\[
(1.15) \quad \tH_\mu(x; q) := \sum_T q^{\text{cocharge(\text{read}(T))}} \cdot s_{\text{shape}(T)}
\]

where the sum is over all semistandard tableaux \(T\) of content \(\mu\). For example, if \(\mu = (3, 2, 2, 2, 1)\), the semistandard tableau

\[
\begin{array}{cccc}
1 & 1 & 1 & 3 \\
2 & 2 & 4 & 5 \\
\end{array}
\]

has content \(\mu\) and reading word 4224511133. Our earlier computations show that this tableau contributes \(q^{10} \cdot s_{541}\) to \(\tH_\mu(x; q)\).

Our final and most involved basis is that of \((\text{modified})\) Macdonald polynomials, depending on two parameters \(q\) and \(t\). These polynomials may be characterized plethystically as follows.

Let \(\geq\) be the dominance order on partitions given by

\[
\lambda \geq \mu \text{ if and only if } \lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \text{ for all } i.
\]

Macdonald [45] showed that there is a unique family \(\{\tH_\mu(x; q, t) : \mu \vdash n\}\) of symmetric functions whose Schur expansions satisfy the *normalization axiom*

\[
(1.17) \quad \langle \tH_\mu(x; q, t), s_{(n)} \rangle = 1.
\]

Here we use the standard shorthand

\[
(1.18) \quad x(1 - q) := x_1 + x_2 + \cdots - qx_1 - qx_2 - \cdots \quad \text{and} \quad x(1 - t) := x_1 + x_2 + \cdots - tx_1 - tx_2 - \cdots
\]

for expressions inside plethystic brackets. Haglund, Haiman, and Loehr [30] gave a combinatorial formula for the polynomials \(\tH_\mu(x; q, t)\) involving tableaux. Haiman [37] proved that the \(\tilde{H}_\mu(x; q, t)\) are the Frobenius images \(\text{grFrob}(V_\mu; q, t)\) of a certain bigraded \(S_n\)-module \(V_\mu\), and hence Schur-positive.

### 1.3. The diagonal coinvariant ring \(DR_n\)

Let \(x_n = (x_1, \ldots, x_n)\) and \(y_n = (y_1, \ldots, y_n)\) be two lists of \(n\) variables and let \(\mathbb{C}[x_n, y_n]\) be the polynomial ring over these variables. We consider the *diagonal action* of \(S_n\) on \(\mathbb{C}[x_n, y_n]\) given by

\[
(1.19) \quad w \cdot x_i := x_{w(i)} \quad w \cdot y_i := y_{w(i)} \quad w \in S_n, \quad 1 \leq i \leq n
\]

and let \(\mathbb{C}[x_n, y_n]^{S_n} \subseteq \mathbb{C}[x_n, y_n]\) be the space of \(S_n\)-invariants with vanishing constant term.

In the early 1990s, Garsia and Haiman [19, 36] initiated the study of the *diagonal coinvariant ring*. This is the quotient

\[
(1.20) \quad DR_n := \mathbb{C}[x_n, y_n]/(\mathbb{C}[x_n, y_n]^{S_n})
\]

obtained by modding out \(\mathbb{C}[x_n, y_n]\) by the ideal generated by the \(S_n\)-invariants with vanishing constant term. By considering \(x\)-degree and \(y\)-degree separately, the ring \(DR_n = \bigoplus_{i,j \geq 0} (DR_n)_{i,j}\) is a doubly graded \(S_n\)-module.
Setting the \( y \)-variables to zero in \( DR_n \), we recover the classical \( S_n \)-coinjective ring

\[
R_n := \mathbb{C}[x_1, \ldots, x_n]/(\mathbb{C}[x_1, \ldots, x_n]^{S_n}) = \mathbb{C}[x_1, \ldots, x_n]/(e_1, e_2, \ldots, e_n)
\]

obtained from \( \mathbb{C}[x_1, \ldots, x_n] \) by modding out by the elementary symmetric polynomials. The singly graded \( S_n \)-module \( R_n \) is among the most well-studied representations in algebraic combinatorics. As an ungraded module, we have \( R_n \cong S_n \mathbb{C}[x_n] \), so that \( R_n \) is a graded refinement of the regular representation of \( S_n \). The ring \( R_n \) has many interesting bases indexed by permutations \( w \in S_n \).

The quotient ring \( DR_n \) has remarkable properties. Using the algebraic geometry of the Hilbert scheme of \( n \) points in \( \mathbb{C}^2 \), Haiman proved \([38]\) that \( DR_n \) has vector space dimension \((n + 1)^{n-1}\). No elementary proof of this dimension formula is known.

The ungraded \( S_n \)-structure of \( DR_n \) is governed by size \( n \) parking functions; these are sequences \((a_1, \ldots, a_n)\) of positive integers whose nondecreasing rearrangement \((b_1 \leq \cdots \leq b_n)\) satisfies \( b_i \leq i \) for all \( i \). The symmetric group \( S_n \) acts on the set \( \text{Park}_n \) of length \( n \) parking functions by subscript permutation. A beautiful combinatorial argument called the Cycle Lemma gives the enumeration \( |\text{Park}_n| = (n + 1)^{n-1} \). Haiman proved that

\[
DR_n \cong \mathbb{C}[\text{Park}_n] \otimes \text{sign}
\]

as ungraded \( S_n \)-modules. Carlsson and Oblomkov \([13]\) used the theory of affine Springer fibers to find a monomial basis of \( DR_n \) naturally indexed by parking functions. Their work gives an alternative (still geometric) proof of the dimension formula \( \dim DR_n = (n + 1)^{n-1} \).

Haiman refined the isomorphism \([1, 22]\) by giving a description of the bigraded Frobenius image \( \text{grFrob}(DR_n; q, t) \) of the diagonal coinvariants, where \( q \) tracks \( x \)-degree and \( t \) tracks \( y \)-degree. In order to state Haiman’s result, we will need the nabla operator on symmetric functions introduced by F. Bergeron, Garsia, Haiman, and Tesler \([6]\). This is the Macdonald eigenoperator \( \nabla : \Lambda \to \Lambda \) characterized by

\[
\nabla : \bar{H}_{\mu}(x; q, t) \mapsto q^{\kappa(\mu)} t^{\kappa(\mu')} \cdot \bar{H}_{\mu}(x; q, t)
\]

where \( \kappa \) is the partition statistic \( \kappa(\lambda) := \sum_{i \geq 1} (i - 1) \cdot \lambda_i \). For example, suppose \( \mu = (3, 2) \) so that \( \mu' = (2, 2, 1) \). We compute \( \kappa(\mu) = 2 \) and \( \kappa(\mu') = 4 \) so that

\[
\nabla : H_{32}(x; q, t) \mapsto q^{2} t^{4} \cdot H_{32}(x; q, t).
\]

Haiman used algebraic geometry \([38]\) to prove

\[
\text{grFrob}(DR_n; q, t) = \nabla e_n,
\]

cementing a connection between the representation theory of \( DR_n \) and the combinatorics of \( \Lambda \).

Thanks to Haiman’s theorem \([1, 24]\), finding the bigraded \( S_n \)-isomorphism type of \( DR_n \) amounts to finding the coefficients of the Schur expansion \( \nabla e_n = \sum_{\lambda \vdash n} c_{\lambda}(q, t) \cdot s_{\lambda} \). The polynomial \( c_{\lambda}(q, t) \) is the bigraded multiplicity of \( S^\lambda \) \( DR_n \), so has nonnegative coefficients and satisfies the \( q, t \)-symmetry \( c_{\lambda}(q, t) = c_{\lambda}(t, q) \). Finding explicit manifestly positive formulas for the \( c_{\lambda}(q, t) \) has proven daunting. When \( \lambda = (a, 1^b) \) is a hook, Haglund’s \( q, t \)-Schröder Theorem \([28]\) describes \( c_{\lambda}(q, t) \) as the bivariate generating function \( \text{Sch}_{\lambda}(q, t) \) for a certain pair of statistics on Motzkin paths. Although \( \text{Sch}_{\lambda}(q, t) \) is manifestly positive, it is not manifestly symmetric; no combinatorial proof of \( q, t \)-symmetry \( \text{Sch}_{\lambda}(q, t) = \text{Sch}_{\lambda}(t, q) \) is known. For general partitions \( \lambda \vdash n \), there is not even a conjectural combinatorial expression for \( c_{\lambda}(q, t) \).

When finding the Schur expansion of a symmetric function \( F \in \Lambda \) is too difficult, it can be easier to find its monomial expansion. Using a dazzling array of symmetric function machinery, in 2015 Carlsson and Mellit \([12]\) proved the Shuffle Theorem

\[
\nabla e_n = \sum_{\lambda} q^{\text{area}(\lambda)} t^{\text{dinv}(\lambda)} x^\lambda
\]
where the sum is over size \( n \) word parking functions \( P \), thus giving a combinatorial monomial expansion of \( \nabla e_n \). The ingredients in the right hand side of Equation (1.25) may be described as follows.

A Dyck path of size \( n \) is a lattice path from \((0,0)\) to \((n,n)\) which never sinks below the diagonal \( y = x \). A word parking function \( P \) is obtained from a Dyck path by labeling its vertical steps with nonnegative integers which increase going up vertical runs. To any word parking function \( P \), we associate a monomial weight \( x^P = x_1^{c_1}x_2^{c_2}\cdots \) where \( c_i \) is the number of copies of \( i \) in \( P \). In the example below we have \( n = 5 \) and \( x^P = x_1^3x_2^1x_3^6 \).

If \( P \) is a size \( n \) word parking function, the area of \( P \) is the number \( \text{area}(P) \) of full boxes between \( P \) and the diagonal. In particular, the area of \( P \) depends only on the Dyck path underlying \( P \), not on its labeling with positive integers. In the above example we have \( \text{area}(P) = 2 \).

The other statistic \( \text{dinv}(P) \) appearing in (1.25) depends on the labeling of \( P \) (not just the Dyck path) and is more involved. A pair of labels \( a < b \) in \( P \) form a diagonal inversion if
- \( a \) and \( b \) appear in the same diagonal of \( P \), and \( a \) appears in a lower row than \( b \) (this is a primary diagonal inversion), or
- \( a \) appears one diagonal below \( b \) in \( P \), and \( a \) appears in a higher row than \( b \) (this is a secondary diagonal inversion).

In the above example, the single primary diagonal inversion is \((1,2)\) on the main diagonal. We have the secondary diagonal inversions \((1,6)\), \((2,6)\), and \((1,2)\) involving the main diagonal and the ‘superdiagonal’. We let \( \text{dinv}(P) \) be the total number of primary and secondary diagonal inversions, so that \( \text{dinv}(P) = 4 \) in our case.

The identity (1.25) was conjectured by Haglund, Haiman, Loehr, Remmel, and Ulyanov \[31\] in 2005. In the decade between its conjecture and its proof, many more general symmetric function identities were conjectured \[7, 33, 34\] in an attempt to place (1.25) in a position for inductive attack. Like (1.25) itself, these identities have two sides: a ‘symmetric function side’ involving the action of operators such as \( \nabla \) on the ring \( \Lambda \) and a ‘combinatorial side’ involving statistics on combinatorial objects. The Carlsson-Mellit proof \[12\] of (1.25) went through one such refinement: the Compositional Shuffle Conjecture of Haglund, Morse, and Zabrocki \[33\]. Shortly before the work of \[12\], Haglund, Remmel and Wilson \[34\] formulated another such refinement called the Delta Conjecture. This refinement proved more difficult than the Shuffle Theorem itself; its ‘Rise Version’ resisted proof for five years and its ‘Valley Version’ remains open. The Delta Conjecture involves a family of symmetric function operators generalizing \( \nabla \).

### 1.4. Delta operators.
Given a partition \( \mu \), define a polynomial \( B_\mu = \sum_{(i,j)} q^{i-1}t^{j-1} \), where the sum is over all matrix coordinates \((i,j)\) of cells in \( \mu \). For example, if \( \mu = (3,2) \) the cells of \( \mu \) contribute the monomials

\[
\begin{array}{cccc}
1 & q & q^2 \\
\hline
1 & q & qt \\
\end{array}
\]

so that \( B_\mu = 1 + q + q^2 + t + qt \). If \( F \in \Lambda \) is any symmetric function, the delta operator \( \Delta_F : \Lambda \to \Lambda \) indexed by \( F \) is the Macdonald eigenoperator

\[
(1.26) \quad \Delta_F : \tilde{H}_\mu(x; q, t) \mapsto F[B_\mu] \cdot \tilde{H}_\mu(x; q, t)
\]
or, in non-plethystic language,

\[ \Delta_F : \tilde{H}_\mu(x; q, t) \mapsto F(\ldots, q^{i-1}t^{j-1}, \ldots) \cdot \tilde{H}_\mu(x; q, t) \]

where \((i, j)\) range over all cells of \(\mu\). For example, we have

\[ \Delta_F : \tilde{H}_{32}(x; q, t) \mapsto F(1, q, q^2, t, qt) \cdot \tilde{H}_{32}(x; q, t) \]

where the notation \(F(1, q, q^2, t, qt)\) means that the variables \(x_1, x_2, x_3, x_4, x_5\) in \(F(x_1, x_2, \ldots)\) are set to \(1, q, q^2, t, qt\) and all other variables \(x_6, x_7, \ldots\) are set to zero. We will use a primed version of the delta operators \(\Delta'_F : \Lambda \to \Lambda\) characterized by

\[ \Delta'_F : \tilde{H}_\mu(x; q, t) \mapsto F[\beta_{i\mu} - 1] \cdot \tilde{H}_\mu(x; q, t) \]

so that the arguments \(F(\ldots, q^{i-1}t^{j-1}, \ldots)\) of the eigenvalue of \(\tilde{H}_\mu(x; q, t)\) do not contain the north-west cell of \(\mu\) with coordinates \((1, 1)\). In our example, this corresponds to the filling

\[
\begin{array}{cccc}
q & q^2 \\
t & qt
\end{array}
\]

and we have

\[ \Delta'_F : \tilde{H}_{32}(x; q, t) \mapsto F(q, q^2, t, qt) \cdot \tilde{H}_{32}(x; q, t). \]

The delta operators were defined by F. Bergeron and Garsia. A quick check on the Macdonald basis shows that

\[ \Delta_{e_n} = \Delta'_{e_{n-1}} = \nabla \]

as operators on the space \(\Lambda_n\) of degree \(n\) symmetric functions, so the Shuffle Theorem gives the monomial expansion of \(\Delta'_{e_{n-1}} e_n\). For \(k \leq n\), the Delta Conjecture predicts monomial expansions of the more general symmetric functions \(\Delta'_{e_{k-1}} e_n\). Roughly speaking, these expansions are obtained by refining the parking function statistics \(\text{area}\) and \(\text{dinv}\).

The statistic \(\text{area}(P)\) admits a natural refinement. If \(P\) is a size \(n\) word parking function and \(1 \leq i \leq n\), we set

\[ a_i(P) := \text{the number of full boxes in row } i \text{ between } P \text{ and the diagonal.} \]

The numbers \(a_1(P), \ldots, a_n(P)\) depend only on the Dyck path underlying \(P\), not on the choice of labeling. We have \(\text{area}(P) = a_1(P) + \cdots + a_n(P)\).

Refining \(\text{dinv}(P)\) is more complicated. For \(1 \leq i \leq n\), define a number \(d_i(P)\) by

\[ d_i(P) := |\{i < j \leq n : a_i(P) = a_j(P), \ell_i(P) < \ell_j(P)\}| + |\{i < j \leq n : a_i(P) = a_j(P) + 1, \ell_i(P) > \ell_j(P)\}| \]

where \(\ell_i(P)\) is the label in row \(i\) of \(P\). Roughly speaking, \(d_i(P)\) counts diagonal inversions in \(P\) which ‘originate in’ row \(i\). We have \(\text{dinv}(P) = d_1(P) + \cdots + d_n(P)\). For our example parking function, the values of \(a_i(P)\) and \(d_i(P)\) are as follows.

\[
\begin{array}{cccc}
i & a_i & d_i \\
5 & 0 & 0 \\
4 & 0 & 1 \\
3 & 1 & 1 \\
2 & 1 & 2 \\
1 & 0 & 0
\end{array}
\]
The Shuffle Theorem gives an expression for \( \nabla e_n \) in terms of the statistics \( \text{area} = a_1 + \cdots + a_n \) and \( \text{dinv} = d_1 + \cdots + d_n \). To get an expression for \( \Delta'_{e_{k-1}} e_n \), we cancel out the contributions of some of the \( a_1, \ldots, a_n, d_1, \ldots, d_n \) to these statistics. If \( P \) is a labeled Dyck path of size \( n \), the set \( \text{Val}(P) \) of contractible valleys of \( P \) is

\[
\text{Val}(P) := \{ 2 \leq i \leq n : a_i(P) < a_{i-1}(P) \} \cup \{ 2 \leq i \leq n : a_i(P) = a_{i-1}(P), \ell_i(P) > \ell_{i-1}(P) \}. 
\]

In other words, a valley (i.e. an east step followed by a north step) in row \( i \) is contractible if moving this valley one box east gives a valid labeled Dyck path. In the example above we have \( \text{Val}(P) = \{4, 5\} \).

**Conjecture 1.1.** (Haglund-Remmel-Wilson [34]) (The Delta Conjecture) For positive integers \( k \leq n \),

\[
(1.32) \quad \Delta'_{e_{k-1}} e_n = \left\{ z^{n-k} \right\} \sum_{P \in \text{LD}_n} q^{\text{dinv}(P)} \prod_{i : a_i(P) > a_{i-1}(P)} \left( 1 + z/q^{a_i(P)} \right) x^P \]

\[
= \left\{ z^{n-k} \right\} \sum_{P \in \text{LD}_n} q^{\text{dinv}(P)} \prod_{i \in \text{Val}(P)} \left( 1 + z/q^{d_i(P)+1} \right) x^P .
\]

Here the operator \( \left\{ z^{n-k} \right\} \) extracts the coefficient of \( z^{n-k} \).

We denote by \( \text{Rise}_{n,k}(x; q, t) \) and \( \text{Val}_{n,k}(x; q, t) \) the two combinatorial expressions on the right-hand side, so the Delta Conjecture may be written more succinctly as

\[
(1.35) \quad \Delta'_{e_{k-1}} e_n = \text{Rise}_{n,k}(x; q, t) = \text{Val}_{n,k}(x; q, t).
\]

The expression \( \text{Rise}_{n,k} \) is obtained from the decomposition \( \text{area} = a_1 + \cdots + a_n \) whereas \( \text{Val}_{n,k} \) is obtained from \( \text{dinv} = d_1 + \cdots + d_n \). The Delta Conjecture specializes to the Shuffle Theorem when \( k = n \).

There has been substantial progress on the Delta Conjecture since its introduction in [34]. The ‘Rise Version’, i.e. the equality \( \Delta'_{e_{k-1}} e_n = \text{Rise}_{n,k}(x; q, t) \) was proven independently by D’Adderio-Mellit [16] and Blasiak-Haiman-Morse-Pun-Seelinger [9] using different methods. D’Adderio and Mellit proved a refined ‘compositional’ version of the equality \( \Delta'_{e_{k-1}} e_n = \text{Rise}_{n,k}(x; q, t) \) conjectured by D’Adderio-Iraci-Wyngaard [15] involving ‘\( \Theta \)-operators’ on symmetric functions. In [16] these \( \Theta \)-operators are used to promote the Carlsson-Mellit proof of the Shuffle Theorem to a proof of the Rise Version of the Delta Conjecture. Blasiak et. al. proved an ‘extended’ version of \( \Delta'_{e_{k-1}} e_n = \text{Rise}_{n,k}(x; q, t) \) which gives a monomial expansion of \( \Delta_h \Delta'_{e_{k-1}} e_n \). The proof in [9] uses an action of Schiffmann algebra \( \mathcal{E} \) on the ring of symmetric functions. Neither result in [15] or [9] contains the other.

Far less is known about the ‘Valley Version’ \( \Delta'_{e_{k-1}} e_n = \text{Val}_{n,k}(x; q, t) \) of the Delta Conjecture. It is not even known whether \( \text{Val}_{n,k}(x; q, t) \) is symmetric in the \( x \)-variables. However, the Valley Version is proven when one of the variables \( q, t \) is set to zero. Combining results in [18, 34, 53], we have

\[
(1.36) \quad \Delta'_{e_{k-1}} e_n \big|_{t=0} = \text{Val}_{n,k}(x; q, 0) = \text{Val}_{n,k}(x; 0, q).
\]

We focus on \( t = 0 \) specialization \( \Delta'_{e_{k-1}} e_n \big|_{t=0} \) for most of this chapter; in the final section we look at \( \Delta'_{e_{k-1}} e_n \) itself.
2. Quotient rings

2.1. The ring \( R_{n,k} \).

The motivation for the Shuffle Theorem was to develop a combinatorial understanding of \( \nabla e_n \), the bigraded Frobenius image of the diagonal coinvariants \( DR_n \). In turn, the ring \( DR_n \) was initially motivated by the classical coinvariant ring \( R_n \) with its ties to the flag variety \( F1_n \). As a significant member of a family of (sometimes conjectural) identities developed to prove the Shuffle Theorem, it is natural to ask whether there is an analogous algebraic and geometric theory tied to \( \Delta_{e_{k-1}} e_n \). To this end, Haglund, Rhoades, and Shimozono [35] introduced the following quotient rings.

**Definition 2.1.** Let \( k \leq n \) be positive integers. We define an ideal \( I_{n,k} \subset \mathbb{C}[x_1, \ldots, x_n] \) by

\[
I_{n,k} := \langle e_n, e_{n-1}, \ldots, e_{n-k+1}, x_1^k, x_2^k, \ldots, x_n^k \rangle
\]

and denote the corresponding quotient ring by

\[
R_{n,k} := \mathbb{C}[x_1, \ldots, x_n]/I_{n,k}
\]

The ideal \( I_{n,k} \) is homogeneous, so \( R_{n,k} \) is a graded ring. Since the generating set of \( I_{n,k} \) is stable under the action of \( S_n \), the quotient \( R_{n,k} \) is a graded \( S_n \)-module. When \( k = 1 \), we recover the ground field \( R_{n,1} = \mathbb{C} \) since \( x_1, \ldots, x_n \in I_{n,1} \). When \( k = n \), it is not hard to see that the variable powers \( x_i^n \) are in the classical coinvariant ideal \( I_n = \langle e_1, \ldots, e_n \rangle \), so that \( I_{n,n} = I_n \).

The algebra \( R_n \) and geometry \( Fl(n) \) are governed by the combinatorics of \( S_n \). The following combinatorial objects play an analogous role for \( R_{n,k} \) and \( X_{n,k} \).

**Definition 2.2.** A word \( w = [w(1), \ldots, w(n)] \) over the positive integers is a Fubini word if for all \( i > 1 \) such that the letter \( i \) appears in \( w \), the letter \( i - 1 \) also appears in \( w \). We let \( W_{n,k} \) denote the family of Fubini words of length \( n \) with maximum letter \( k \).

As an example, we have \( W_{3,2} = \{ [1,1,2], [1,2,1], [2,1,1], [1,2,2], [2,1,2], [2,2,1] \} \). Words in \( W_{n,k} \) may be thought of as surjective maps \( w : [n] \rightarrow [k] \), giving a natural identification \( W_{n,n} = S_n \).

There is another way to think of Fubini words that will be useful for us. An ordered set partition of \( [n] \) is a sequence \( \sigma = (B_1 \mid \cdots \mid B_k) \) of nonempty subsets of \( [n] \) such that we have a disjoint union \( [n] = B_1 \sqcup \cdots \sqcup B_k \). We write \( OP_{n,k} \) for the family of all ordered set partitions of \( [n] \) into \( k \) blocks. There is a natural bijection between \( W_{n,k} \) and \( OP_{n,k} \), viz.

\[
[3,2,4,1,3,2] \mapsto (5 \mid 237 \mid 16 \mid 4).
\]

This bijection \( W_{n,k} \overset{\sim}{\rightarrow} OP_{n,k} \) is the inverse map \( w \mapsto w^{-1} \) on the symmetric group \( S_n \) when \( k = n \), so that Fubini words and ordered set partitions are ‘inverse objects’. We have

\[
|W_{n,k}| = |OP_{n,k}| = k! \cdot \text{Stir}(n,k)
\]

where \( \text{Stir}(n,k) \) is the (signless) Stirling number of the second kind counting \( k \)-block set partitions of \( [n] \).

We will see that Fubini words govern the structure of \( R_{n,k} \). The group \( S_n \) acts on the set \( W_{n,k} \) by

\[
v \cdot [w(1), \ldots, w(n)] := [w(v^{-1}(1)), \ldots, w(v^{-1}(n))] \quad v \in S_n, \ [w(1), \ldots, w(n)] \in W_{n,k}.
\]

The permutation module \( \mathbb{C}[W_{n,k}] \) is the regular representation \( \mathbb{C}[S_n] \) when \( k = n \). For general \( k \leq n \), we will prove that \( R_{n,k} \cong \mathbb{C}[W_{n,k}] \) as ungraded \( S_n \)-modules, so \( R_{n,k} \) is a graded refinement of \( \mathbb{C}[W_{n,k}] \).
2.2. Coinversion codes. Our first step towards proving \( R_{n,k} \cong \mathbb{C} \left[ \mathcal{W}_{n,k} \right] \) is to attach monomials in \( \mathbb{C}[x_1, \ldots, x_n] \) to ordered set partitions in \( \mathcal{OP}_{n,k} \) (or, equivalently, Fubini words in \( \mathcal{W}_{n,k} \)). The gadget we use for this is the *coinversion code* of an ordered set partition \[55\]. We review the classical case of permutations in \( S_n \), and then explain how to generalize to \( \mathcal{OP}_{n,k} \).

The *inversion number* \( \text{inv}(w) \) and the *coinversion number* \( \text{coinv}(w) \) of a permutation \( w \in S_n \) are
\[
\text{inv}(w) := \left| \{ 1 \leq i < j \leq n \mid w(i) > w(j) \} \right| \\
\text{coinv}(w) := \left| \{ 1 \leq i < j \leq n \mid w(i) < w(j) \} \right|.
\]
These statistics are complementary: we have \( \text{coinv}(w) + \text{inv}(w) = \binom{n}{2} \) for any \( w \in S_n \). Their common generating function is the *Mahonian distribution*
\[
\sum_{w \in S_n} q^{\text{inv}(w)} = \sum_{w \in S_n} q^{\text{coinv}(w)} = [n]_q!.
\]
where
\[
[n]_q! := (1 + q)(1 + q + q^2) \cdots (1 + q + q^2 + \cdots + q^{n-1})
\]
is the standard \( q \)-analogue of \( n! \). We work primarily with coinversions in this subsection.

The statistic coinv can be broken into pieces. The *coinversion code* associated to a permutation \( w \in S_n \) is the sequence \( \text{code}(w) = (c_1, \ldots, c_n) \) where
\[
c_i := \left| \{ j > i \mid w(j) > w(i) \} \right|
\]
For example, we have \( \text{code}([3,1,5,2,4]) = (2,3,0,1,0) \). The coinversion code refines \( \text{coinv} \) in the sense that \( \text{coinv}(w) = c_1 + \cdots + c_n \).

Let \( E_n := \{(a_1, \ldots, a_n) : 0 \leq a_i \leq n-i \} \) be the set of words over the positive integers which are componentwise \( \leq \) the staircase \( (n-1,n-2,\ldots,1,0) \). It is not hard to see that \( \text{code}(w) \in E_n \) for any \( w \in S_n \). In fact, the map \( \text{code} : w \mapsto \text{code}(w) \) is a bijection \( S_n \xrightarrow{\sim} E_n \); see \[62\] Sec. 1.3] for details.

Our aim is to extend the story above from \( S_n \) to \( \mathcal{OP}_{n,k} \), beginning with the notion of inversion and coinversion pairs. Let \( \sigma = (B_1 \mid \cdots \mid B_k) \in \mathcal{OP}_{n,k} \) be an ordered set partition and let \( 1 \leq a, b \leq n \) be a pair of letters belonging to distinct blocks of \( \sigma \) such that \( b \) is minimal in its block. The pair \( \{a, b\} \) is an *inversion pair* of \( \sigma \) if the block of \( a \) is to the right of the block of \( b \) and \( a < b \). Otherwise, the pair \( \{a, b\} \) is a *coinversion pair*. We have statistics
\[
\text{inv}(\sigma) := \# \text{ of inversion pairs in } \sigma \\
\text{coinv}(\sigma) := \# \text{ of coinversion pairs in } \sigma
\]
As an example of these concepts, let \( \sigma = (6 \mid 14 \mid 237 \mid 5) \in \mathcal{OP}_{7,4} \). The inversion pairs of \( \sigma \) are \{16, 26, 56, 24, 75\} so that \( \text{inv}(\sigma) = 5 \) and the coinversion pairs are \{12, 15, 25, 36, 13, 35, 46, 45, 67, 17\} so that \( \text{coinv}(\sigma) = 10 \). Observe that (for example) 34 is not an inversion pair 4 is not minimal in its block. The statistics \( \text{inv} \) and \( \text{coinv} \) are complementary on \( \mathcal{OP}_{n,k} \). More precisely, we have
\[
\text{inv}(\sigma) + \text{coinv}(\sigma) = \binom{k}{2} + (n-k)(k-1)
\]
for all \( \sigma \in \mathcal{OP}_{n,k} \).

The statistic \( \text{inv} \) on \( \mathcal{OP}_{n,k} \) was introduced by Steingrímsson \[63\] in a general study of ordered set partition statistics and rediscovered by Remmel-Wilson \[52\] in the context of the symmetric function \( \text{Rise}_{n,k}(x; q, t) \). Steingrímsson proved that the generating function of \( \text{inv} \) is
\[
\sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\text{inv}(\sigma)} = [k]_q! \cdot \text{Stir}_q(n, k)
\]
where the *\( q \)-Stirling number* is defined recursively by
\[
\text{Stir}_q(n, k) := \begin{cases} \\
\delta_{n,k} & n = 1 \\
\text{Stir}_q(n-1, k-1) + [k]_q \cdot \text{Stir}_q(n-1, k) & n > 1
\end{cases}
\]
Unlike in the permutation case, this distribution is not palindromic. For example, we have
\[ \sum_{\sigma \in OP_{n,k}} q^{\text{inv}(\sigma)} = q^2 + 3q + 2. \] The distribution of coinv is the reversal
\[ (2.11) \quad \sum_{\sigma \in OP_{n,k}} q^{\text{coinv}(\sigma)} = \text{rev}_q([k]_q! \cdot \text{Stir}_q(n, k)) = q^{\binom{k}{2} + (n-k)(k-1)} \cdot [k]_q! \cdot \text{Stir}_{q-1}(n, k) \]
so that (for example) \( \sum_{\sigma \in OP_{n,k}} q^{\text{coinv}(\sigma)} = 2q^2 + 3q + 1 \). Here \( \text{rev}_q \) acts on polynomials in \( q \) by reversing their coefficient sequences.

The statistic inv is tied to the Delta Conjecture. Haglund, Remmel, and Wilson \[34\] showed that
\[ (2.12) \quad \langle \text{Rise}_{n,k}(x; q, 0), p_1^n \rangle = \sum_{\sigma \in OP_{n,k}} q^{\text{inv}(\sigma)}, \]
giving a connection between the statistic inv and delta operators. The same authors defined three other statistics (dinv, maj, and minimaj) on \( OP_{n,k} \) which satisfy analogous identities
\[ (2.13) \quad \begin{cases} 
\langle \text{Rise}_{n,k}(x; 0, q), p_1^n \rangle = \sum_{\sigma \in OP_{n,k}} q^{\text{maj}(\sigma)} \\
\langle \text{Val}_{n,k}(x; q, 0), p_1^n \rangle = \sum_{\sigma \in OP_{n,k}} q^{\text{dinv}(\sigma)} \\
\langle \text{Val}_{n,k}(x; 0, q), p_1^n \rangle = \sum_{\sigma \in OP_{n,k}} q^{\text{minmaj}(\sigma)}, 
\end{cases} \]
so that ordered set partitions model the Delta Conjecture at \( t = 0 \). Wilson \[67\] and Rhoades \[53\] used a generalization of these statistics to ‘ordered multiset partitions’ to prove
\[ (2.14) \quad \text{Rise}_{n,k}(x; q, 0) = \text{Rise}_{n,k}(x; 0, q) = \text{Val}_{n,k}(x; q, 0) = \text{Val}_{n,k}(x; 0, q), \]
giving combinatorial evidence for the Delta Conjecture.

Just as in the permutation case, the coinversion statistic on \( OP_{n,k} \) can be broken into pieces. Let \( \sigma = (B_1 \mid \cdots \mid B_k) \in OP_{n,k} \) be an ordered set partition. The coinversion code of \( \sigma \) is the sequence \( \text{code}(\sigma) = (c_1, \ldots, c_n) \) where
\[ (2.15) \quad c_i := \begin{cases} 
\{ j > \ell : \min(B_j) > i \} & \text{if } i = \min(B_\ell) \text{ is minimal in its block,} \\
(\ell - 1) + \{ j > \ell : \min(B_j) > i \} & \text{if } i \in B_\ell \text{ is not minimal in its block.} 
\end{cases} \]
In our example \( \sigma = (6 \mid 14 \mid 237 \mid 5) \in OP_{7,4} \) we have \( \text{code}(\sigma) = (c_1, \ldots, c_7) = (2, 1, 3, 2, 0, 0, 2) \). As in the case of \( S_n \), we have \( \text{coinv}(\sigma) := c_1 + \cdots + c_n \) for any \( \sigma \in OP_{n,k} \).

The map \( \sigma \mapsto \text{code}(\sigma) \) is injective on \( OP_{n,k} \); to describe its image we need some terminology. Recall that a shuffle of two sequences \((a_1, \ldots, a_r)\) and \((b_1, \ldots, b_s)\) is an interleaving \((c_1, \ldots, c_{r+s})\) of these sequences which preserves the relative order of the \( a \)'s and the \( b \)'s. An \((n, k)\)-staircase a shuffle of the length \( k \) staircase \((k-1, k-2, \ldots, 1, 0)\) with the sequence \((k-1, \ldots, k-1)\) consisting of \( n - k \) copies of \( k - 1 \). For example, when \( n = 5 \) and \( k = 3 \), the \((n, k)\)-staircases are the shuffles of \((2, 1, 0)\) and \((2, 2)\) shown here
\[
\begin{align*}
(2, 1, 0, 2, 2) & \quad (2, 1, 2, 0, 2) & \quad (2, 2, 1, 0, 2) & \quad (2, 1, 2, 0, 2) & \quad (2, 2, 1, 2, 0) & \quad (2, 2, 2, 1, 0)
\end{align*}
\]
The number of \((n, k)\)-staircases is the binomial coefficient \( \binom{n-1}{k-1} \).

**Definition 2.3.** For \( k \leq n \), a word \((c_1, \ldots, c_n)\) over the positive integers is \((n, k)\)-substaircase if it is componentwise \( \leq \) at least one \((n, k)\)-staircase. Let \( E_{n,k} \) be the family of \((n, k)\)-substaircase sequences.

As an example, we have \((1, 1, 0, 2, 0) \in E_{5,3} \) because \((1, 1, 0, 2, 0) \leq (2, 1, 0, 2, 2) \) componentwise. We also have \((1, 1, 0, 2, 0) \leq (2, 1, 2, 2, 0) \) componentwise: sequences in \( E_{n,k} \) can fit under more than one \((n, k)\)-staircase.
An alternative characterization of the set $E_{n,k}$ can be formulated in terms of “skip sequence avoidance”. For any subset $S = \{s_1 < s_2 < \cdots < s_r\} \subseteq [n]$, define the skip sequence $\gamma(S) = (\gamma(S)_1, \ldots, \gamma(S)_n)$ by

$$\gamma(S)_i = \begin{cases} i - j + 1 & \text{if } i = s_j \in S \\ 0 & \text{if } i \notin S \end{cases}$$

For example, when $n = 8$ we have $\gamma(2458) = (0, 2, 0, 3, 3, 0, 0, 5)$. Thus, the first nonzero entry in $\gamma(S)$ is $\min(S)$ and every time the set $S$ ‘skips’ an element of $[n]$, the corresponding entry of $\gamma(S)$ is incremented.

**Definition 2.4.** A length $n$ sequence $(a_1, \ldots, a_n)$ of nonnegative integers is $(n,k)$-nonskip if

1. we have $a_i < k$ for all $i$ and
2. whenever $|S| = n - k + 1$, the componentwise inequality $\gamma(S) \leq (a_1, \ldots, a_1)$ does not hold.

Note the reversal $(a_n, \ldots, a_1)$ in the second condition.

For example, if $(n,k) = (5, 3)$ the sequence $(1, 1, 0, 2, 0)$ is $(n,k)$-nonskip because all of its entries are $< 3$ and it is not componentwise $\leq$ any of the reverse skip sequences

$$(0, 0, 1, 1, 1) \quad (0, 2, 0, 1, 1) \quad (0, 2, 2, 0, 1) \quad (0, 2, 2, 2, 0).$$

For arbitrary $k \leq n$ it turns out that

$$E_{n,k} = \{\text{all } (n,k)\text{-nonskip sequences } (a_1, \ldots, a_n)\}$$

so that substaircase and nonskip are equivalent. Coinversion codes give yet another characterization of $E_{n,k}$: we have $\text{coinv}(\sigma) \in E_{n,k}$ for all $\sigma \in OP_{n,k}$, and the function

$$\text{code} : OP_{n,k} \xrightarrow{\sim} E_{n,k}$$

is a bijection. The inverse $i : E_{n,k} \rightarrow OP_{n,k}$ of this map is the following insertion algorithm.

Given a sequence $(B_1 \mid \cdots \mid B_k)$ of possibly empty sets of positive integers, the coinversion labeling is obtained by first labeling the empty sets with $0, 1, 2, \ldots, j - 1$ from left to right (where there are $j$ empty sets), and then labeling the nonempty sets with $j, j + 1, \ldots, k - 1$ from right to left. If $(c_1, \ldots, c_n) \in E_{n,k}$, we define $i(c_1, \ldots, c_n) = (B_1 \mid \cdots \mid B_k)$ by starting with a sequence of $k$ empty sets and iteratively inserting $i$ into the set with coinversion label $c_i$, updating coinversion labels as we go. For example, if $(c_1, \ldots, c_n) = (2, 1, 3, 2, 0, 0, 2)$ this insertion is as follows

$$\begin{array}{c|c|c} i & c_i & (B_1 \mid \cdots \mid B_k) \\ \hline 1 & 2 & (\varnothing^3 \mid \varnothing^2 \mid \varnothing^1 \mid \varnothing^0) \\ 2 & 1 & (\varnothing^2 \mid 1^3 \mid \varnothing^1 \mid \varnothing^0) \\ 3 & 3 & (\varnothing^1 \mid 1^2 \mid 2^3 \mid \varnothing^0) \\ 4 & 2 & (\varnothing^1 \mid 1^2 \mid 2^3 \mid \varnothing^0) \\ 5 & 0 & (\varnothing^0 \mid 1^4 \mid 2^3 \mid \varnothing^0) \\ 6 & 0 & (\varnothing^0 \mid 1^4 \mid 2^3 \mid \varnothing^0) \\ 7 & 2 & (\varnothing^0 \mid 1^4 \mid 2^3 \mid \varnothing^0) \end{array}$$

and we conclude $i(2, 1, 3, 2, 0, 0, 2) = (6 \mid 1^4 \mid 2^3 \mid 5^3)$. Our results on codes and ordered set partitions may be summarized as follows (see [34, 55]).

**Theorem 2.5.** Let $k \leq n$ and let $(a_1, \ldots, a_n)$ be a length $n$ sequence of nonnegative integers. The following are equivalent.

1. The sequence $(a_1, \ldots, a_n)$ is componentwise $\leq$ some $(n,k)$-staircase.
2. The sequence $(a_1, \ldots, a_n)$ is $(n,k)$-nonskip.
3. We have $\text{code}(\sigma) = (a_1, \ldots, a_n)$ for some ordered set partition $\sigma \in OP_{n,k}$.
2.3. Demazure characters and Gröbner theory. Theorem 2.5 shows that the collection of monomials
\begin{equation}
\mathcal{A}_{n,k} := \{ x_{a_1} \cdots x_{a_n} : (a_1, \ldots, a_n) \in E_{n,k} \}
\end{equation}
is in bijection with \( \mathcal{OP}_{n,k} \) (or \( \mathcal{W}_{n,k} \)). When \( k = n \), E. Artin proved \( \mathbb{A} \) that the family \( \mathcal{A}_n := \mathcal{A}_{n,n} \) of ‘sub-staircase monomials’ descends to a basis of the classical coinvariant ring \( \mathcal{R}_{n} \). In this subsection and the next, we will see that \( \mathcal{A}_{n,k} \) descends to a basis of \( \mathcal{R}_{n,k} \). We show that \( \mathcal{A}_{n,k} \) descends to a spanning set of \( \mathcal{R}_{n,k} \) using Gröbner theory; we start by reviewing the “Gröbner basics”.

A total order \(<\) on the monomials in \( \mathbb{C}[x_1, \ldots, x_n] \) is a term order if
- for any monomial \( m \) we have \( 1 \leq m \), and
- \( m < m' \) implies \( m''m < m''m' \) for any monomials \( m, m', m'' \).

Given a term order \(<\), for any nonzero polynomial \( f \in \mathbb{C}[x_1, \ldots, x_n] \) denote by \( \text{in}_<(f) \) the largest monomial under \(<\) appearing in \( f \).

In this chapter, we shall exclusively use the neglex term order, which is the lexicographical term order with respect to the variable ordering \( x_n > \cdots > x_2 > x_1 \). Explicitly, we have \( x_1^{a_1} \cdots x_n^{a_n} < x_1^{b_1} \cdots x_n^{b_n} \) if there exists \( 1 \leq i \leq n \) so that \( a_i < b_i \) and \( a_{i+1} = b_{i+1}, \ldots, a_n = b_n \).

If \( I \subseteq \mathbb{C}[x_1, \ldots, x_n] \) is an ideal, the corresponding initial ideal is the monomial ideal \( \text{in}_<(I) := \langle \text{in}_<(f) : 0 \neq f \in I \rangle \). A finite subset \( G = \{ g_1, \ldots, g_r \} \subseteq I \) of nonzero polynomials in \( I \) is a Gröbner basis if \( \text{in}_<(I) = \langle \text{in}_<(g_1), \ldots, \text{in}_<(g_r) \rangle \). This condition implies that \( I = \langle G \rangle \). Furthermore, the set
\begin{equation}
\{ m : m \text{ a monomial in } \mathbb{C}[x_1, \ldots, x_n] \text{ not contained in } \text{in}_<(I) \} = \{ m : m \text{ a monomial in } \mathbb{C}[x_1, \ldots, x_n] \text{ such that } \text{in}_<(g) \nmid m \text{ for all } g \in G \}
\end{equation}
of monomials not contained in \( \text{in}_<(I) \) descends a basis for \( \mathbb{C}[x_1, \ldots, x_n]/I \) as a \( \mathbb{C} \)-vector space. This is the standard monomial basis of \( \mathbb{C}[x_1, \ldots, x_n]/I \); it is determined by the term order \(<\) and the ideal \( I \).

Although Gröbner bases are not unique, there is a canonical choice of Gröbner basis for an ideal \( I \) and a term order \(<\). A Gröbner basis \( G \) is called minimal if for any distinct elements \( g, g' \in G \) we have \( \text{in}_<(g) \nmid \text{in}_<(g') \). The Gröbner basis \( G \) is reduced if it is minimal and, for any distinct \( g, g' \in G \), no monomial appearing in \( g \) is divisible by \( \text{in}_<(g') \). Every ideal \( I \subseteq \mathbb{C}[x_1, \ldots, x_n] \) has a unique reduced Gröbner basis for a given term order.

Gröbner bases are of central importance in many computational algorithms in commutative algebra. However, even for ideals \( I \subseteq \mathbb{C}[x_1, \ldots, x_n] \) with a nice generating set and of algebraic or geometric significance, the polynomials appearing in a Gröbner basis can have unpredictable monomials with ugly coefficients. Miraculously, the Gröbner bases of our ideals \( I_{n,k} \) are very structured. We present the classical case of \( R_n \), and then outline the generalization to \( R_{n,k} \).

The coinvariant ideal \( I_n = \langle e_1, e_2, \ldots, e_n \rangle \) has the alternate presentation \( I_n = \langle h_1, h_2, \ldots, h_n \rangle \) in terms of the complete homogeneous symmetric polynomials in \( x_1, x_2, \ldots, x_n \). From this one can show that we have
\begin{equation}
h_{n-i}(x_1, x_2, \ldots, x_i) \in I_n
\end{equation}
for \( 1 \leq i \leq n \). With respect to the neglex order \(<\), we have \( \text{in}_<(h_{n-i}(x_1, x_2, \ldots, x_i)) = x_{i}^{n-i} \) so that the standard monomial basis of \( R_n \) is contained in
\begin{equation}
\{ \text{monomials } m \text{ in } x_1, \ldots, x_n : x_i^{n-i} \nmid m \text{ for } 1 \leq i \leq n \} = \mathcal{A}_n,
\end{equation}
the family of substaircase monomials. Since \( \dim R_n = n! = |\mathcal{A}_n| \), this implies that \( \mathcal{A}_n \) is the standard monomial basis of \( R_n \) and \( \{ h_{n-i}(x_1, x_2, \ldots, x_i) : 1 \leq i \leq n \} \) is a Gröbner basis of \( I_n \). It is easily seen that this Gröbner basis is reduced.

In order to generalize this story to \( R_{n,k} \), we need an analogue of the \( h_{n-i}(x_1, x_2, \ldots, x_i) \). These will be the Demazure characters (or key polynomials) \( \kappa_{\gamma} \) indexed by weak compositions (i.e. sequences of nonnegative integers) \( \gamma = (\gamma_1, \ldots, \gamma_n) \) of length \( n \). Given \( \gamma \), the polynomial \( \kappa_{\gamma} \) is defined
We outline the general approach, and then explain how it applies in our setting.

\[ \text{Orbit harmonics.} \]

By Lemma 2.6, Equation (2.25), and the fact that the variable powers $x_1^k, x_2^k, \ldots, x_n^k$ lie in $I_{n,k}$, we see that $A_{n,k}$ contains the standard monomial basis for $R_{n,k}$ with respect to the neglex term order.

2.4. Orbit harmonics. Proposition 2.7 bounds the dimension of $R_{n,k}$ from above. We bound this dimension from below using orbit harmonics, a general technique for turning ungraded group actions on finite sets into graded actions on quotient rings. Orbit harmonics dates back to at least the work of Kostant [43] for the classical coinvariant ring and was later applied by Garsia-Procesi [21] to Tanisaki quotients and by Garsia-Haiman [20] in the context of Macdonald polynomials. We outline the general approach, and then explain how it applies in our setting.

For any finite subset $Z \subseteq \mathbb{C}^n$, let $I(Z) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be the ideal

\[ I(Z) := \{ f \in \mathbb{C}[x_1, \ldots, x_n] : f(z) = 0 \text{ for all } z \in Z \} \]

of polynomials which vanish on $Z$. The coordinate ring $\mathbb{C}[x_1, \ldots, x_n]/I(Z)$ is the ring of polynomial maps $Z \rightarrow \mathbb{C}$. Since $Z$ is finite, Lagrange interpolation guarantees that any function $Z \rightarrow \mathbb{C}$ is the restriction of a polynomial map $f \in \mathbb{C}[x_1, \ldots, x_n]$ and have

\[ \mathbb{C}[Z] \cong \mathbb{C}[x_1, \ldots, x_n]/I(Z) \]

where $\mathbb{C}[Z]$ is the vector space with basis $Z$. The ideal $I(Z) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is usually not homogeneous, so the ring $\mathbb{C}[x_1, \ldots, x_n]/I(Z)$ is usually not graded. However, there is a standard way to produce a graded ideal from $I(Z)$ as follows.
Given a nonzero polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, let $\tau(f)$ denote the highest degree component of $f$. That is, if $f = f_d + \cdots + f_1 + f_0$ with $f_i$ homogeneous of degree $i$ and $f_d \neq 0$, we have $\tau(f) = f_d$.

For any ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$, the associated graded ideal is

$$
\text{gr } I := \langle \tau(f) : f \in I - \{0\} \rangle \subseteq \mathbb{C}[x_1, \ldots, x_n]
$$

By construction, the ideal gr$I$ is homogeneous so that the quotient $\mathbb{C}[x_1, \ldots, x_n]/\text{gr} I$ is a graded ring. The rings $\mathbb{C}[x_1, \ldots, x_n]/I$ and $\mathbb{C}[x_1, \ldots, x_n]/\text{gr} I$ share a number of features; an important one for us is as follows.

**Lemma 2.8.** Let $B \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be a set of homogeneous polynomials and let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be an ideal. If $B$ descends to a basis of $\mathbb{C}[x_1, \ldots, x_n]/\text{gr} I$ then $B$ descends to a basis of $\mathbb{C}[x_1, \ldots, x_n]/I$.

The isomorphism (2.27) and Lemma 2.8 imply that for any finite point locus $Z \subseteq \mathbb{C}^n$, we have isomorphisms

$$
\mathbb{C}[Z] \cong \mathbb{C}[x_1, \ldots, x_n]/I(Z) \cong \mathbb{C}[x_1, \ldots, x_n]/\text{gr} I(Z)
$$

of vector spaces where the quotient $\mathbb{C}[x_1, \ldots, x_n]/\text{gr} I(Z)$ is graded. We upgrade these vector space isomorphisms by incorporating group actions.

Let $G \subseteq GL_n(\mathbb{C})$ be a finite matrix group. In addition to the natural action of $G$ on $\mathbb{C}^n$, we have an action of $G$ on $\mathbb{C}[x_1, \ldots, x_n]$ by linear substitutions. If $Z \subseteq GL_n(\mathbb{C})$ is $G$-stable, the ideals $I(Z)$ and $\text{gr} I(Z)$ are $G$-stable, as well. If $Z \subseteq \mathbb{C}^n$ is finite, we have isomorphisms of ungraded $G$-modules

$$
\mathbb{C}[Z] \cong_G \mathbb{C}[x_1, \ldots, x_n]/I(Z) \cong_G \mathbb{C}[x_1, \ldots, x_n]/\text{gr} I(Z)
$$

where the quotient $\mathbb{C}[x_1, \ldots, x_n]/\text{gr} I(Z)$ has the additional structure of a graded $G$-module. The first isomorphism $\mathbb{C}[Z] \cong_G \mathbb{C}[x_1, \ldots, x_n]/I(Z)$ in (2.30) is standard; the second isomorphism may be justified as follows.

Let $B \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be a set of homogeneous polynomials which descends to a basis of $\mathbb{C}[x_1, \ldots, x_n]/I(Z)$. By Lemma 2.8 the set $B$ also descends to a basis of $\mathbb{C}[x_1, \ldots, x_n]/\text{gr} I(Z)$. The set $B$ decomposes as a disjoint union

$$
B = B_0 \sqcup B_1 \sqcup \cdots \sqcup B_r
$$

where $B_d$ consists of polynomials $f \in B$ of degree $d$.

Fix a group element $g \in G$. Then $g$ acts on both $\mathbb{C}[x_1, \ldots, x_n]/I(Z)$ and $\mathbb{C}[x_1, \ldots, x_n]/\text{gr} I(Z)$. If we express these actions with respect to the basis $B = B_0 \sqcup B_1 \sqcup \cdots \sqcup B_r$ we obtain matrices of block form

$$
\begin{pmatrix}
A_0 & * & \cdots & * \\
A_1 & * & \cdots & * \\
\vdots & & \ddots & \\
A_r & & & 
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
A_0 \\
A_1 \\
\vdots \\
A_r 
\end{pmatrix}
$$

where the diagonal block $A_d$ has size $|B_d|$ and is the same in either matrix. In particular, these matrices have the same trace. Since two representations of a finite group with the same character are isomorphic, we obtain the second isomorphism $\mathbb{C}[x_1, \ldots, x_n]/I(Z) \cong_G \mathbb{C}[x_1, \ldots, x_n]/\text{gr} I(Z)$ in (2.30).

By varying the point locus $Z$ (and the group $G$, but we mostly consider $G = S_n$), a wide family of graded quotient modules $\mathbb{C}[x_1, \ldots, x_n]/\text{gr} I(Z)$ can be obtained. The Hilbert series of $\mathbb{C}[x_1, \ldots, x_n]/\text{gr} I(Z)$ is typically an interesting statistic on the set $Z$. It will turn out that $R_{n,k}$ may be realized as such a quotient. Before going further, we give two simple examples of orbit harmonics which demonstrate an algebraic subtlety of this technique.

For our first example, let $G = S_2$ act on $\mathbb{C}^2$ by reflection across the line $x_1 = x_2$. The point locus $Z = \{(1,0), (1,1), (0,1)\}$ is $S_2$-stable and $x_1(x_1 - 1), x_2(x_2 - 1), (x_1 - 1)(x_2 - 1) \in \mathbb{C}[x_1, x_2]$ vanish on $Z$. Consequently, we have $\langle x_1(x_1 - 1), x_2(x_2 - 1), x_1x_2 \rangle \subseteq I(Z)$ and, taking highest degrees, we
have \( \langle x_1^2, x_2^2, x_1 x_2 \rangle \subseteq \text{gr} I(Z) \). Since \( \dim \mathbb{C}[x_1, x_2]/\text{gr} I(Z) = |Z| = 3 = \dim \mathbb{C}[x_1, x_2]/\langle x_1^2, x_2^2, x_1 x_2 \rangle \), we see that
\[
\mathbb{C}[x_1, x_2]/\text{gr} I(Z) = \mathbb{C}[x_1, x_2]/\langle x_1^2, x_2^2, x_1 x_2 \rangle.
\]
Geometrically, the transformation \( \mathbb{C}[x_1, x_2]/I(Z) \rightarrow \mathbb{C}[x_1, x_2]/\text{gr} I(Z) \) deforms the reduced scheme \( Z \) to a ‘triple point’ in the origin as shown.

The Hilbert series and graded Frobenius image of \( \mathbb{C}[x_1, x_2]/\text{gr} I(Z) \) are easily computed to be
\[
\text{Hilb}(\mathbb{C}[x_1, x_2]/\text{gr} I(Z); q) = 1 + 2q \quad \text{grFrob}(\mathbb{C}[x_1, x_2]/\text{gr} I(Z); q) = s_2 + (s_2 + s_{11}) \cdot q
\]
yielding a graded refinement of the action of \( S_2 \) on \( Z \). We remark that \( \text{grFrob}(\mathbb{C}[x_1, x_2]/\text{gr} I(Z); q) \) may be also expressed in the Hall-Littlewood basis as \( \tilde{H}_{11}(x; q) + \tilde{H}_2(x; q) \cdot q \).

For our second example, we again consider the action of \( G = S_2 \) on \( \mathbb{C}^2 \), but take the locus \( Z' = \{ (2, 0), (1, 1), (0, 2) \} \) as shown below. This locus is combinatorially equivalent to the locus \( Z \) of the previous paragraph, but the graded ring obtained from applying orbit harmonics is different.

Indeed, the polynomials \( x_1(x_1 - 1)(x_1 - 2), x_2(x_2 - 1)(x_2 - 2), x_1 + x_2 - 2 \in \mathbb{C}[x_1, x_2] \) vanish on \( Z' \) and it is not hard to check that \( \text{gr} I(Z') = \langle x_1^3, x_2^3, x_1 + x_2 \rangle \) is generated by the top degree components of these polynomials. From this we have the Hilbert and Frobenius series
\[
\text{Hilb}(\mathbb{C}[x_1, x_2]/\text{gr} I(Z'); q) = 1 + q + q^2 \quad \text{grFrob}(\mathbb{C}[x_1, x_2]/\text{gr} I(Z'); q) = s_2 + s_{11} \cdot q + s_2 \cdot q^2
\]
which are different than those for the ring \( \mathbb{C}[x_1, x_2]/\text{gr} I(Z) \), so we obtain a different graded refinement of the same combinatorial action of \( S_2 \). As they must, the graded Frobenius images \( \text{grFrob}(\mathbb{C}[x_1, x_2]/\text{gr} I(Z); q) \) and \( \text{grFrob}(\mathbb{C}[x_1, x_2]/\text{gr} I(Z'); q) \) have the same \( q \rightarrow 1 \) specialization, \( 2 \cdot s_2 + s_{11} \).

As with the case of \( Z \), the Frobenius series for \( Z' \) is positive in the \( \tilde{H} \)-basis: we have \( \text{grFrob}(\mathbb{C}[x_1, x_2]/\text{gr} I(Z'); q) = \tilde{H}_{11}(x; q) + q^2 \cdot \tilde{H}_2(x; q) \). We will see various other instances of Hall-Littlewood positivity of orbit harmonics modules throughout the paper.

**Question 2.9.** For which finite \( S_n \)-stable subsets \( Z \subseteq \mathbb{C}^n \) does the expansion of the symmetric function
\[
(2.32) \quad \text{grFrob}(\mathbb{C}[x_1, \ldots, x_n]/\text{gr} I(Z); q)
\]
in the Hall-Littlewood basis \( \{ \tilde{H}_\lambda(x; q) \} \) have coefficients in \( \mathbb{Z}_{\geq 0}[q] \)?

When \( Z \) is a single \( S_n \)-orbit, the answer to Question 2.9 is ‘yes’ by the work of Garsia-Procesi [21]; more generally, the answer to Question 2.9 is ‘yes’ for all loci \( Z \) considered in this paper. Griffin [25] analyzed the case where \( Z \) is union of two \( S_n \)-orbits; when the coordinate sums of these orbits are different, he proved that \( \text{grFrob}(\mathbb{C}[x_1, \ldots, x_n]/\text{gr} I(Z); q) \) expands positively in the Hall-Littlewood basis. However, he showed [25, Ex. 1] that if \( n = 3 \) and \( Z = S_3 \cdot \{(4, 0, 0), (2, 1, 1)\} \) then Hall-Littlewood positivity fails. Reineke, Rhoades, and Tewari [51] give a locus \( Z \subseteq \mathbb{C}^n \) of size \( |Z| = n^{n-2} \) such that \( \text{Res}^{S_n}_{S_{n-1}} \mathbb{C}[x_1, \ldots, x_n]/\text{gr} I(Z) \) is the coarea-graded parking representation of \( S_{n-1} \), but \( \text{grFrob}(\mathbb{C}[x_1, \ldots, x_n]/\text{gr} I(Z); q) \) does not expand positively in the basis \( \{ \tilde{H}_\lambda(x; q) \} \). Since the locus \( Z \) in [51] consists of the lattice points in a trimmed permutohedron, it seems that
the answer to Question 2.9 is more likely to be ‘no’ when the coordinate sums of the $S_n$-orbits in $Z$ are ‘too often equal’.

For our next example, we explain how the classical coinvariant ring $R_n$ may be regarded as an orbit harmonics quotient.

**Example 2.10.** The regular representation of $S_n$ on $\mathbb{C}[S_n]$ may be regarded as an action on a point locus. More specifically, if $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are distinct complex numbers we have $\mathbb{C}[S_n] \cong \mathbb{C}[Z]$ where

$$Z = \{(\alpha_{w(1)}, \ldots, \alpha_{w(n)}) : w \in S_n\}$$

is the (regular) orbit of $(\alpha_1, \ldots, \alpha_n)$ under coordinate permutation. Said differently, $Z$ is the vertex set of the permutahedron.

For $1 \leq d \leq n$, the difference $e_d(x_1, \ldots, x_n) - e_d(\alpha_1, \ldots, \alpha_n)$ vanishes on $Z$ so that $\langle e_1, e_2, \ldots, e_n \rangle \subseteq \text{gr} \ I(Z)$. To prove that $\langle e_1, e_2, \ldots, e_n \rangle = \text{gr} \ I(Z)$ we argue as follows.

It is well-known that the polynomials $e_1, e_2, \ldots, e_n$ form a homogeneous regular sequence in the rank $n$ polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. It follows that

$$\dim \mathbb{C}[x_1, \ldots, x_n]/\langle e_1, e_2, \ldots, e_n \rangle = (\deg e_1) \cdot (\deg e_2) \cdots (\deg e_n) = n! = |X|$$

which forces $\text{gr} \ I(Z) = \langle e_1, e_2, \ldots, e_n \rangle$. We derive the ungraded $S_n$-structure of the coinvariant ring

$$R_n = \dim \mathbb{C}[x_1, \ldots, x_n]/\langle e_1, e_2, \ldots, e_n \rangle = \mathbb{C}[x_1, \ldots, x_n]/\text{gr} \ I(Z) \cong \mathbb{C}[Z] \cong \mathbb{C}[S_n],$$

recovering a result of Chevalley [14].

This argument extends to any complex reflection group $G \subseteq GL_n(\mathbb{C})$. The set $Z$ may be taken to be any regular $G$-orbit and the elementary symmetric polynomials $e_1, e_2, \ldots, e_n$ are replaced with a fundamental system of invariants $f_1, f_2, \ldots, f_n$ satisfying $\mathbb{C}[x_1, \ldots, x_n]^G = \mathbb{C}[f_1, f_2, \ldots, f_n]$. Example 2.10 generalizes to express $R_{n,k}$ as an orbit harmonics quotient. Let $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ be distinct complex numbers and consider the locus

$$Z_{n,k} := \{(\alpha_{w(1)}, \ldots, \alpha_{w(n)}) : [w(1), \ldots, w(n)] \in W_{n,k} \text{ is a Fubini word}\}$$

of points in $\mathbb{C}^n$. For example, when $n = 3$ and $k = 2$ we have

$$Z_{3,2} = \{(\alpha_1, \alpha_1, \alpha_2), (\alpha_1, \alpha_2, \alpha_1), (\alpha_2, \alpha_1, \alpha_1), (\alpha_1, \alpha_1, \alpha_2), (\alpha_2, \alpha_1, \alpha_2), (\alpha_2, \alpha_2, \alpha_1)\}.$$

The correspondence $(\alpha_{w(1)}, \ldots, \alpha_{w(n)}) \leftrightarrow [w(1), \ldots, w(n)]$ is an $S_n$-equivariant bijection between $Z_{n,k}$ and $W_{n,k}$.

**Theorem 2.11.** We have $\text{gr} \ I(Z_{n,k}) = I_{n,k}$ so that $R_{n,k} = \mathbb{C}[x_1, \ldots, x_n]/\text{gr} \ I(Z_{n,k}) \cong \mathbb{C}[W_{n,k}]$ as ungraded $S_n$-modules. Furthermore, the set

$$\{\kappa_{\text{rev}(\gamma(S))}(x_1, \ldots, x_n) : S \subseteq [n], |S| = n - k + 1\} \cup \{x_1^k, x_2^k, \ldots, x_n^k\}$$

forms a Gröbner basis of $I_{n,k}$. The set $A_{n,k}$ of substaircase monomials is the standard monomial basis for $R_{n,k}$ in the neglex term order.

In the range $1 < k < n$, the Gröbner basis of Theorem 2.11 becomes reduced when one removes the $\kappa_{\text{rev}(\gamma(S))}$'s for sets $S$ containing $n$. The proof of Theorem 2.11 is a more elaborate version of Example 2.10.

**Proof.** We begin by listing some polynomials which vanish on $Z_{n,k}$. For $1 \leq i \leq n$, the $i^{th}$ coordinate of any point $(x_1, \ldots, x_n) \in Z_{n,k}$ lies in $\{\alpha_1, \ldots, \alpha_k\}$. Said differently, we have $(x_i - \alpha_1) \cdots (x_i - \alpha_k) \in \text{I}(Z_{n,k})$ so that $x_i^k \in \text{gr} \ I(Z_{n,k})$ for $1 \leq i \leq n$. To show that the remaining generators $e_r(x_1, \ldots, x_n)$ of $I_{n,k}$ lie in $\text{gr} \ I(Z_{n,k})$, we have a trick at our disposal.

Introduce a new variable $t$ and consider the rational expression

$$\frac{(1 + x_1 t)(1 + x_2 t) \cdots (1 + x_n t)}{(1 + \alpha_1 t)(1 + \alpha_2 t) \cdots (1 + \alpha_k t)}.$$
When \((x_1, \ldots, x_n) \in Z_{n,k}\) the \(k\) factors in the denominator cancel with \(k\) factors in the numerator, yielding a polynomial in \(t\) of degree \(n - k\). For \(r > n - k\), we see that

\[
0 = \sum_{a+b=r} (-1)^b e_a(x_1, x_2, \ldots, x_n) h_b(\alpha_1, \alpha_2, \ldots, \alpha_k)
\]

so that \(\sum_{a+b=r} (-1)^b e_a(x_1, x_2, \ldots, x_n) h_b(\alpha_1, \alpha_2, \ldots, \alpha_k) \in I(Z_{n,k})\) and taking the highest degree component gives \(e_r(x_1, x_2, \ldots, x_n) \in \text{gr} I(Z_{n,k})\). We conclude that

\[
I_{n,k} \subseteq \text{gr} I(Z_{n,k})
\]

so we have a canonical projection

\[
\mathbb{C}[x_1, \ldots, x_n]/\text{gr} I(Z_{n,k}) \twoheadrightarrow R_{n,k}.
\]

Lemma 2.6 implies that \(\text{dim } R_{n,k} \leq k! \cdot \text{Stir}(n, k)\) which forces (2.41) to be an isomorphism and (2.40) to be an equality. Proposition 2.7 (and its proof) imply that the given set of polynomials is indeed a Gröbner basis of \(R_{n,k}\) and that \(A_{n,k}\) is the standard monomial basis of \(R_{n,k}\). \(\square\)

2.5. Short exact sequences and graded module structure. Theorem 2.11 determines the ungraded \(S_n\)-module structure of \(R_{n,k}\). In this subsection we describe its graded \(S_n\)-structure. This was originally achieved in [35] using skewing operators; we outline a simpler approach due to Gillespie and Rhoades [23] using short exact sequences. The modules appearing in these sequences are a three-parameter version of \(R_{n,k}\).

**Definition 2.12.** For nonnegative integers \(k \leq s \leq n\), let \(I_{n,k,s} \subset \mathbb{C}[x_1, \ldots, x_n]\) be the ideal

\[
I_{n,k,s} := \langle x_1^{s+1}, \ldots, x_n^{s+1}, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle
\]

and let

\[
R_{n,k,s} := \mathbb{C}[x_1, \ldots, x_n]/I_{n,k,s}
\]

be the corresponding quotient ring. \(\square\)

We have \(R_{n,k,k} = R_{n,k}\) so that the rings in Definition 2.12 reduce to \(R_{n,k}\) when \(k = s\). For \(k = 0\), there are no elementary symmetric polynomials among the generators of \(I_{n,0,s}\) and the quotient has the simple form \(R_{n,0,s} = \mathbb{C}[x_1, \ldots, x_n]/\langle x_1^s, \ldots, x_n^s \rangle\).

Like the rings \(R_{n,k}\), the rings \(R_{n,k,s}\) may be analyzed using orbit harmonics. In particular, let \(\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{C}\) be \(s\) distinct complex numbers and consider the locus \(Z_{n,k,s} \subset \mathbb{C}^n\) given by

\[
Z_{n,k,s} := \{(x_1, \ldots, x_n) : \{\alpha_1, \ldots, \alpha_k\} \subseteq \{x_1, \ldots, x_n\} \subseteq \{\alpha_1, \ldots, \alpha_s\}\}.
\]

In other words, the locus \(Z_{n,k,s}\) consists of points \((x_1, x_2, \ldots, x_n) \in \mathbb{C}^n\) whose coordinates are drawn from the list \((\alpha_1, \alpha_2, \ldots, \alpha_s)\) in which the first \(k\) entries \(\alpha_1, \ldots, \alpha_k\) must occur. Using methods similar to those for \(X_{n,k}\), orbit harmonics applies to give

\[
I_{n,k,s} = \text{gr} I(Z_{n,k,s}) \text{ so that } R_{n,k,s} = \mathbb{C}[x_n]/I_{n,k,s} \cong \mathbb{C}[Z_{n,k,s}]
\]

where the isomorphism is of ungraded \(S_n\)-modules.

Definition 2.12 gives the appropriate interpolation between the rings \(R_{n,k}\) of study and the simpler rings \(R_{n,0,s}\). This interpolation takes the form of a short exact sequence. Aside from an easy boundary case, the following is [35, Lem 6.9].

**Lemma 2.13.** Let \(k < s \leq n\) be nonnegative integers. There exists a short exact sequence of \(S_n\)-modules

\[
0 \to R_{n,k,s-1} \xrightarrow{\varphi} R_{n,k,s} \xrightarrow{\pi} R_{n,k+1,s} \to 0
\]

where the first map is homogeneous of degree \(n - k\) and the second is homogeneous of degree 0.

---

1The roles of \(k\) and \(s\) in the ring \(R_{n,k,s}\) agree with those in [24] and elsewhere, but are switched relative to their original appearance in [35].
The first map $\varphi$ in Lemma 2.13 is multiplication by $e_{n-k}$ while the second map $\pi$ is the canonical projection coming from the ideal containment $I_{n,k,s} \subseteq I_{n,k+1,s}$. In the language of symmetric functions, Lemma 2.13 implies

\begin{equation}
grFrob(R_{n,k,s}; q) = q^{n-k} \cdot grFrob(R_{n,k,s-1}; q) + grFrob(R_{n,k+1,s}; q)
\end{equation}

for all nonnegative $k < s \leq n$. The graded $S_n$-structure of $R_{n,0,s} = \mathbb{C}[x_1, \ldots, x_n]/(x_1^s, \ldots, x_n^s)$ is fairly simple, so the recursion (2.45) can be solved for $grFrob(R_{n,k,s}; q)$. In the case $k = s$, this gives the graded $S_n$-structure of $R_{n,k}$.

In order to describe the symmetric function $grFrob(R_{n,k}; q)$, we need some terminology. Recall that for a partition $\lambda \vdash n$, a standard Young tableau $T$ of shape $\lambda$ is a filling of the boxes of $\lambda$ with $1, 2, \ldots, n$ which increases across rows and down columns. An example standard tableau of shape $(4, 3, 1) \vdash 8$ is shown below.

```
1 3 4 7
2 5 8
3 6
```

We let $SYT(\lambda)$ denote the family of standard tableaux of shape $\lambda$. If $T$ is a standard tableau with $n$ boxes, and index $1 \leq i \leq n - 1$ is a descent of $T$ if $i$ appears in a strictly higher row than $i + 1$; the above tableau has descents 1, 4, 5, and 7. The descent number $des(T)$ is the number of descents in $T$ and the major index $maj(T)$ is the sum of the descents; in our example we have $des(T) = 4$ and $maj(T) = 1 + 4 + 5 + 7 = 17$. We also use the $q$-binomial and $q$-multinomial coefficients

\begin{equation}
[n]_q = \left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]!}{[k]!_q [n-k]!_q} \\
\quad q^{-r} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \лонон

Theorem 2.14. Let $k \leq n$ be positive integers. The graded Frobenius image $grFrob(R_{n,k}; q)$ is given by

$$
grFrob(R_{n,k}; q) = \sum_{\lambda \vdash n} \sum_{T \in SYT(\lambda)} q^{maj(T)} \cdot \left[ \begin{array}{c} n - des(T) - 1 \\ n - k \end{array} \right]_q \cdot s_{\lambda}(x)$$

$$= q^{(n-k)(k-1)} \cdot \sum_{\ell(\lambda) = k} q^{\sum_{i}(\ell(\lambda) - \sum_{i}(\ell(\lambda) - 1))} \cdot \left[ \begin{array}{c} k \\ m_1(\lambda), \ldots, m_n(\lambda) \end{array} \right]_q \cdot \tilde{H}_\lambda(x; q)$$

where $m_i(\lambda)$ is the multiplicity of $i$ as a part of $\lambda$.

The three expressions for $grFrob(R_{n,k}; q)$ in Theorem 2.14 are interesting for different reasons. The first expression is a tableau formula for the Schur expansion of $grFrob(R_{n,k}; q)$ and directly gives the decomposition of $R_{n,k}$ into irreducibles. The second expression shows that $grFrob(R_{n,k}; q)$ expands positively in the Hall-Littlewood basis (as in Question 2.9). The final expression ties $R_{n,k}$ to delta operators.

3. Spanning line configurations

Theorem 2.14 states that the quotient ring $R_{n,k}$ plays the same role for the delta operator expression $\Delta'_{e_{k-1}e_n}$ that the classical coinvariant algebra $R_n$ plays for $\nabla e_n$. That is, the ring $R_{n,k}$ is a coinvariant algebra for the Delta Conjecture. In this section, we describe a variety $X_{n,k}$ of spanning line configurations defined by Pawlowski and Rhoades [48] whose cohomology is presented by $R_{n,k}$. The variety $X_{n,k}$ is a flag variety for the Delta Conjecture.
Presenting the cohomology of $X_{n,k}$ uses two main geometric tools: affine pavings and Chern classes of vector bundles. The ring $H^\bullet(X_{n,k})$ is generated by the Chern classes $x_1, \ldots, x_n$ of certain obvious line bundles $L_1, \ldots, L_n$, subject only to relations coming from the spanning condition and the Whitney Sum Formula. This ‘Chern calculus’ also serves to present the cohomology of several related varieties of subspace configurations satisfying rank conditions. We give a quick overview of these concepts with an emphasis on their combinatorial aspects; for a more thorough resource, see e.g. [17].

3.1. The variety $X_{n,k}$. A line in a vector space $V$ is a 1-dimensional linear subspace $\ell \subseteq V$. Let $\mathbb{P}^{k-1}$ be the $(k-1)$-dimensional projective space of lines in $\mathbb{C}^k$ and let $(\mathbb{P}^{k-1})^n := \mathbb{P}^{k-1} \times \cdots \times \mathbb{P}^{k-1}$ be its $n$-fold self product. Our variety of study is as follows.

**Definition 3.1.** Given $k \leq n$, we let $X_{n,k}$ be the subset of $(\mathbb{P}^{k-1})^n$ given by

$$X_{n,k} := \{(\ell_1, \ldots, \ell_n) : \ell_i \subseteq \mathbb{C}^k \text{ a line and } \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}.$$

It is not hard to see that $X_{n,k}$ is a Zariski open subvariety of $(\mathbb{P}^{k-1})^n$ and, as such, is a smooth complex manifold. We will see that $X_{n,k}$ has cohomology presentation $H^\bullet(X_{n,k}) \cong R_{n,k}$. When $k = 1$, we have the one-point space $X_{1,1} = \{\ast\}$ and $R_{1,1} = \mathbb{C}$, so the result holds in this case. At the other extreme, when $k = n$ we claim that $X_{n,n}$ is homotopy equivalent to the flag variety $\text{Fl}(n)$. To see this, we argue as follows.

When $k = n$, for any point $(\ell_1, \ldots, \ell_n) \in X_{n,n}$ we have a direct sum $\mathbb{C}^n = \ell_1 \oplus \cdots \oplus \ell_n$. There is a natural projection

$$\pi : X_{n,n} \to \text{Fl}(n)$$

which sends the tuple $(\ell_1, \ldots, \ell_n)$ of lines to the flag $(V_1, \ldots, V_n)$ whose $i^{th}$ piece is $V_i = \ell_1 \oplus \cdots \oplus \ell_i$. The projection $X_{n,n} \xrightarrow{\pi} \text{Fl}(n)$ is a fiber bundle with fiber isomorphic to

$$U_n := \{\text{all lower triangular } n \times n \text{ complex matrices with 1's on the diagonal}\},$$

the lower triangular unipotent subgroup of $GL_n(\mathbb{C})$. Since $U_n$ is affine, we conclude that $\pi$ is a homotopy equivalence.

The variety $X_{n,k}$ is always smooth, but unlike the flag variety $\text{Fl}(n)$ it is almost never compact. Indeed, it could not be both. The Hilbert series of $R_{n,k}$ is rarely palindromic: for example, $\text{Hilb}(R_{3,2}; q) = 1 + 3q + 2q^2$. Therefore, no space with cohomology $R_{n,k}$ can be both smooth and compact (otherwise it would satisfy Poincaré Duality). In the next section we describe an alternative geometric model of $R_{n,k}$ (and, in fact, a more general family of rings) which is compact but not smooth.

The defect of $X_{n,k}$’s failure to be compact is ameliorated from the perspective of $S_n$-actions. The isomorphism $H^\bullet(\text{Fl}(n)) \cong R_n$ is known to hold not only as graded rings, but also in the category of graded $S_n$-modules. While the $S_n$-structure of $R_n$ is simply variable permutation, the flag variety $\text{Fl}(n)$ admits no natural action of $S_n$. On the other hand, the group $S_n$ acts directly on the variety $X_{n,k}$ by

$$w \cdot (\ell_1, \ldots, \ell_n) := (\ell_{w^{-1}(1)}, \ldots, \ell_{w^{-1}(n)})$$

(giving an induced action on cohomology. When $k = n$, the homotopy equivalence $\text{Fl}(n) \simeq X_{n,n}$ induces an action of $S_n$ on $H^\bullet(\text{Fl}(n))$. For general $k$, this action will be compatible with our presentation $H^\bullet(X_{n,k}) \cong R_{n,k}$.

Our basic strategy for studying $X_{n,k}$ is to make use of compactification, a general technique in geometry where one embeds a space of study $X$ into a better-understood compact space $Y$ in such a way that $\overline{X} = Y$. If this embedding is sufficiently nice, geometric properties of $Y$ may be transferred to $X$. 

The $n$-fold self product $(\mathbb{P}^{k-1})^n$ may be thought of as the family of all $n$-tuples of lines $(\ell_1, \ldots, \ell_n)$ in $\mathbb{C}^k$, spanning or otherwise. The space $(\mathbb{P}^{k-1})^n$ is compact and has well-understood cohomology. Considering spanning tuples only gives a natural inclusion
\begin{equation}
\iota : X_{n,k} \hookrightarrow (\mathbb{P}^{k-1})^n
\end{equation}
which (since a generic collection of $n \geq k$ lines will span $\mathbb{C}^k$) satisfies $X_{n,k} = (\mathbb{P}^{k-1})^n$. We use this compactification to transfer properties of the well-understood product $(\mathbb{P}^{k-1})^n$ to $X_{n,k}$. To do this, we need to show that the inclusion $\iota$ is sufficiently nice. This will be made precise in the next subsection using affine pavings.

3.2. Affine pavings and Fubini words. An important method for understanding a space $X$ is to decompose it as a disjoint union $X = \bigsqcup_i C_i$ of cells $C_i$ which fit together to form a finite CW complex. Since any finite CW complex is compact and $X_{n,k}$ is not compact, we need a different kind of decomposition in our context.

Definition 3.2. Let $X$ be a complex variety. A filtration of $X$ by Zariski closed subvarieties
\[ \emptyset = Z_0 \subset Z_1 \subset \cdots \subset Z_m = X \]
is an affine paving of $X$ if each difference $Z_i - Z_{i-1}$ is isomorphic to a disjoint union $\bigsqcup_j A_{ij}$ of affine spaces $A_{ij}$. The spaces $A_{ij}$ are the cells of the affine paving. Given a disjoint union decomposition $X = \bigsqcup_{ij} A_{ij}$ of $X$ into affine spaces, we say that the $A_{ij}$ induce an affine paving if they arise as the cells of an affine paving of $X$.

As an example, projective space $\mathbb{P}^{k-1}$ admits an affine paving
\begin{equation}
\emptyset \subset [*:0: \cdots :0] \subset [*:*: \cdots :0] \subset \cdots \subset [*:*:\cdots:*] = \mathbb{P}^{k-1}
\end{equation}
with $k$ cells, one of each complex dimension $0, 1, \ldots, k-1$. We refer to this as the standard affine paving of $\mathbb{P}^{k-1}$. It is also a CW decomposition of $\mathbb{P}^{k-1}$. More generally, the CW decomposition of any flag variety $G/P$ into Schubert cells may be regarded as an affine paving.

Varieties with affine pavings enjoy the following two useful properties. Suppose $X$ is a smooth variety which admits an affine paving $\emptyset = Z_0 \subset Z_1 \subset \cdots \subset Z_m = X$ with cells $A_{ij}$. Recall that any closed subvariety $Z \subseteq X$ has a corresponding class $[Z]$ in the cohomology ring $H^\bullet(X)$.

1. The classes $[\overline{A_{ij}}]$ of the Zariski closures of these cells form a basis of $H^\bullet(X)$.
2. If $0 \leq r \leq m$ and $U := X - X_r$, the inclusion map $\iota : U \hookrightarrow X$ induces a surjection $\iota^* : H^\bullet(X) \twoheadrightarrow H^\bullet(U)$ on cohomology.

More explicitly, the map $\iota^*$ sends the classes $[\overline{A_{ij}}]$ for which $A_{ij} \cap U = \emptyset$ to zero while mapping the cells $[\overline{A_{ij}}]$ for which $A_{ij} \subseteq U$ onto a linear basis of $H^\bullet(U)$.

Our goal is to prove that $((\mathbb{P}^{k-1})^n, X_{n,k})$ forms a pair $(X, U)$ as in (2) above. More precisely, we want to find an affine paving $\emptyset = Z_0 \subset Z_1 \subset \cdots \subset Z_m = (\mathbb{P}^{k-1})^n$ for which $X_{n-k} = (\mathbb{P}^{k-1})^n - X_r$ is a terminal part.

If $X$ and $Y$ are varieties which admit affine pavings with cells $\{A_{ij}\}$ and $\{B_{rs}\}$, the products $A_{ij} \times B_{rs}$ of these cells induce a product paving of $X \times Y$. In particular, the standard paving (3.5) of $\mathbb{P}^{k-1}$ induces a product paving of $(\mathbb{P}^{k-1})^n$. Unfortunately, this paving is not suitable for our purposes: cells $C$ of this product paving typically satisfy neither $C \subseteq X_{n,k}$ nor $C \cap X_{n,k} = \emptyset$. For example, if $1 := \text{span}(1, \ldots, 1)$ is the line of constant vectors in $\mathbb{C}^k$, then $(1, \ldots, 1)$ will always be in the largest cell $C_0$ of the product paving and yet $(1, \ldots, 1) \notin X_{n,k}$ for $1 < k < n$. To get around this hurdle, we introduce a nonstandard paving of $(\mathbb{P}^{k-1})^n$. In order to describe this paving, we coordinatize as follows.

Write $\text{Mat}_{k \times n}$ for the affine space $\mathbb{A}^{k \times n}$ of $k \times n$ complex matrices equipped with an action of $GL_k(\mathbb{C}) \times T_n$ where $GL_k(\mathbb{C})$ acts on rows and the torus $T_n := (\mathbb{C}^\times)^n$ acts on columns. We introduce
nested subsets $U_{n,k} \subseteq V_{n,k}$ of $\text{Mat}_{k \times n}$ by
\begin{equation}
V_{n,k} := \{ A \in \text{Mat}_{k \times n} : A \text{ has no zero columns} \}
\end{equation}
and
\begin{equation}
U_{n,k} := \{ A \in \text{Mat}_{k \times n} : A \text{ has no zero columns and is of full rank} \}
\end{equation}
so that $U_{n,n} = GL_n(\mathbb{C})$. We have the natural identifications
\begin{equation}
(\mathbb{P}^{k-1})^n = V_{n,k}/T_n \quad \text{and} \quad X_{n,k} = U_{n,k}/T_n
\end{equation}
of quotient spaces. The sets $V_{n,k}$ and $U_{n,k}$ are both closed under the action of the product group $GL_k(\mathbb{C}) \times T_n$. If $U_k \subseteq GL_k(\mathbb{C})$ denotes the unipotent subgroup of lower triangular matrices with 1’s on the diagonal, restriction gives an action of $U_k \times T_n$ on these sets.

A variant of row reduction gives convenient representatives of the $U_k \times T_n$-orbits inside $V_{n,k}$. These will be certain pattern matrices corresponding to words $w = [w(1), w(2), \ldots, w(n)] \in [k]^n$ defined as follows. A position $1 \leq j \leq n$ in such a word is initial if $w(j)$ is the first of its letter in $w$. For example, in $[2, 5, 2, 5, 1]$ the initial positions are 1, 2, and 5.

**Definition 3.3.** Let $w = [w(1), w(2), \ldots, w(n)] \in [k]^n$ be a word. The pattern matrix $\text{PM}(w)$ is the $k \times n$ matrix with entries in $\{0, 1, *\}$ whose entries are determined by the following procedure.

1. Set the entries $\text{PM}(w)_{w(j),j} = 1$ for $j = 1, 2, \ldots, n$.
2. If $1 \leq j \leq n$ and $j$ is initial, set $\text{PM}(w)_{i,j} = *$ for all positions $(i, j)$ which are north and east of a 1.
3. If $1 \leq j \leq n$ and $j$ is not initial, set $\text{PM}(w)_{i,j} = *$ for all positions $(i, j)$ which are east of an initial 1.
4. Set all other entries of $\text{PM}(w)$ equal to 0.

A $k \times n$ matrix $A$ fits the pattern of $w$ if it can be obtained by replacing the *’s in $\text{PM}(w)$ with complex numbers.

For example, if $n = 8$, $k = 5$, and $w = [2, 4, 1, 1, 4, 5, 5, 4]$ we have

$$
\text{PM}[2, 4, 1, 1, 4, 5, 5, 4] = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & * & 0 \\
1 & * & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & * & 1 & 0 & * & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
$$

Given a word $w \in [k]^n$, let $U_k(w) \subseteq U_k$ be the subgroup of lower triangular unipotent $k \times k$ matrices $A$ for which $A_{ij} = 0$ whenever $j$ does not appear as a letter of $w$. In the above example, we have

$$
\text{U}_5[2, 4, 1, 1, 4, 5, 5, 4] = \begin{pmatrix}
1 & 0 & 0 & 0 \\
* & 1 & 0 & 0 \\
* & * & 1 & 0 \\
* & * & 0 & 1 \\
* & * & 0 & * & 1
\end{pmatrix}
$$

where the subdiagonal entries of the third column are zero since 3 does not appear in $w$. In particular, if every letter in $[k]$ appears in the word $w$, then $U_k(w) = U_k$ is the full lower triangular unipotent subgroup.

The following theorem of linear algebra is proven in [8].

**Theorem 3.4.** (Echelon form) Let $A \in V_{n,k}$ be a $k \times n$ matrix with no zero columns. The matrix $A$ may be written uniquely as

$$
A = uBt
$$
w ∈ [k]^n is a word, u ∈ U_k(w), B fits the pattern of w, and t ∈ T_n is a member of the diagonal torus. The matrix A represents a spanning line configuration (ℓ_1, . . . , ℓ_n) ∈ X_{n,k} if and only if each of the letters 1, 2, . . . , k appear in the word w.

Theorem 3.4 is best understood by example. Suppose k = 3, n = 6, and A is the matrix

$$A = \begin{pmatrix} 0 & -1 & 2 & 0 & 0 & -1 \\ 2 & 1 & 0 & 0 & 3 & -1 \\ 2 & 3 & -4 & 3 & 3 & 2 \end{pmatrix}.$$ 

In order to obtain B from A, we process columns from left to right. The subgroup U_3 ⊆ GL_3(ℂ) allows for downward row operations yielding

$$\begin{pmatrix} 0 & -1 & 2 & 0 & 0 & -1 \\ 2 & 1 & 0 & 0 & 3 & -1 \\ 2 & 3 & -4 & 3 & 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & 2 & 0 & 0 & -1 \\ 2 & 1 & 0 & 0 & 3 & -1 \\ 0 & 2 & -4 & 3 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & 2 & 0 & 0 & -1 \\ 2 & 0 & 2 & 0 & 3 & -2 \\ 0 & 0 & 0 & 3 & 0 & 1 \end{pmatrix}$$

so that u ∈ U_3 is given by

$$u = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}.$$ 

At this stage, we know that B fits the pattern of w = [2,1,1,3,2,3]. The fact that each letter 1, 2, 3 appears in w corresponds to the fact that the columns of A have the full span ℂ^3. Finally, we use the action of T_6 on columns to scale the pivot positions to ones, yielding

$$B = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

and

$$t = \text{diag} (2, -1, 2, 3, 3, 1) ∈ T_6.$$ 

Theorem 3.4 gives rise to a disjoint union decomposition of the projective space product (ℙ_{k−1})^n. Given a word w ∈ [k]^n, define $\hat{X}_w := U_k(w) \cdot \{ B : B \text{ fits the pattern of w } \} \cdot T_n/T_n$.

As sets, we have $(ℙ_{k−1})^n = \bigsqcup_{w ∈ [k]^n} \hat{X}_w$. When k = 2 and n = 3, the eight strata of this decomposition of (ℙ^1)^3 look as follows

$$X_{[2,2,1]} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad X_{[1,1,2]} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \quad X_{[1,2,1]} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & * & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$X_{[2,1,1]} = \begin{pmatrix} 1 & 0 \\ 1 & * \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & * \end{pmatrix} \quad X_{[1,2,2]} = \begin{pmatrix} 1 & 0 \\ 1 & * \end{pmatrix} \cdot \begin{pmatrix} 1 & * & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad X_{[2,1,2]} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$X_{[1,1,1]} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad X_{[2,2,2]} = \begin{pmatrix} 1 & 0 \\ 1 & * \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$ 

Recall that the Fubini words $W_{n,k}$ are words w ∈ [k]^n of length n in which each letter 1, 2, . . . , k appears. The cells $X_{[1,1,1]}$ and $X_{[2,2,2]}$ on the third line are indexed by words [1, 1, 1] and [2, 2, 2] which are not Fubini. Correspondingly, these cells have empty intersection with the subvariety $X_{3,2}$. Their union $X_{[1,1,1]} \cup X_{[2,2,2]}$ is the diagonal copy $\{(ℓ, ℓ, ℓ) : ℓ ∈ ℙ^1\}$ of ℙ^1 inside (ℙ^1)^3. The remaining six cells indexed by the Fubini words in $W_{3,2}$ form a partition of $X_{3,2} ⊂ (ℙ^1)^3$.

Theorem 3.4 is valid over any field. Before focusing on the complex geometry of $X_{n,k}$, we give an application over finite fields. The dimension $\dim(w)$ of a Fubini word w is the number of *'s in
its pattern matrix PM(w). Equivalently, dim(w) is the complex dimension of ˚Xw, less the quantity \( k \choose 2 \) coming from the unitary group U_k. One can use the q-Stirling recursion (2.10) to verify that

\[
\sum_{w \in \mathcal{W}_{n,k}} q^{\dim(w)} = [k]_q \cdot \text{Stir}_q(n, k)
\]

so that dimension is a Mahonian statistic on \( \mathcal{W}_{n,k} \). Theorem 3.4 has the following enumerative consequence; see [15].

**Corollary 3.5.** The number of \( n \)-tuples \( (\ell_1, \ldots, \ell_n) \) of lines in \( \mathbb{F}_q^n \) for which \( \ell_1 + \cdots + \ell_n = \mathbb{F}_q^n \) is \( q^\binom{k}{2} \cdot \left[ \frac{n}{k} \right]_q \cdot \text{Stir}_q(n, k) \).

We point out that the q-Stirling numbers have arisen in geometry before. Billey and Cozkun [8] introduced rank varieties which are certain closed subvarieties \( X(M) \) of the Grassmannian \( \text{Gr}(k, n) \) of \( k \)-planes in \( \mathbb{C}^n \). The generating function \( \sum_M q^{\dim X(M)} \) for the dimensions of these varieties for fixed \( n, k \) equals \( \text{Stir}_q(n+1, n-k+1) \) [8 Cor. 4.25]. The author is unaware of a direct connection between rank varieties and \( X_{n,k} \).

In the classical setting, the Schubert cells \( \{ X_w : w \in S_n \} \) assemble to give a CW-decomposition of the flag variety \( \text{Fl}(n) \). This cannot be true for the decomposition \( X_{n,k} = \bigsqcup_{w \in \mathcal{W}_{n,k}} \hat{X}_w \) since any finite CW complex is compact but \( X_{n,k} \) is not. The next result says that we have the next best thing: the collection \( \{ \hat{X}_w : w \in \mathcal{W}_{n,k} \} \) form the cells of an affine paving of \( X_{n,k} \).

**Theorem 3.6.** Let \( k \leq n \) be positive integers.

1. For any word \( w \in [k]^n \), the subset \( \hat{X}_w \subseteq (\mathbb{P}^{k-1})^n \) is affine with complex dimension equal to the number of star's in the pattern matrix \( \text{PM}(w) \) plus the dimension of the affine space \( U_k(w) \).
2. We have set-theoretic disjoint unions

\[
(\mathbb{P}^{k-1})^n = \bigsqcup_{w \in [k]^n} \hat{X}_w \quad \text{and} \quad X_{n,k} = \bigsqcup_{w \in \mathcal{W}_{n,k}} \hat{X}_w
\]

3. The cells \( \hat{X}_w \) induce an affine paving \( \emptyset = X_0 \subset X_1 \subset \cdots \subset X_m = (\mathbb{P}^{k-1})^n \) of \( (\mathbb{P}^{k-1})^n \) for which \( X_{n,k} = (\mathbb{P}^{k-1})^n - X_i \) for some \( i \).

Items (1) and (2) of Theorem 3.6 are direct consequences of Theorem 3.4. Item (3) requires a bit more work; the basic idea is as follows.

For any position \( (i, j) \) in a \( k \times n \) matrix \( A \), we define

\[
\text{rk}(i, j, A) := \text{rank of the northwest } i \times j \text{ submatrix of } A
\]

Since these northwest ranks are invariant under column scaling, we have a function \( \text{rk}(i, j, -) \) on \( (\mathbb{P}^{k-1})^n \). This gives a disjoint union decomposition

\[
(\mathbb{P}^{k-1})^n = \bigsqcup_{\rho} \Omega_{\rho}
\]

where the disjoint union ranges over all functions \( \rho : [k] \times [n] \to \mathbb{Z}_{\geq 0} \) and

\[
\Omega_{\rho} := \{ A : T_n : \text{rk}(i, j, A) = \rho(i, j) \text{ for all positions } (i, j) \}
\]

One shows that the nonempty \( \Omega_{\rho} \) decompose as a product \( \mathbb{A}^d \times \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_r} \) of an affine space with projective spaces of various dimension depending on \( \rho \). The disjoint union decomposition in Theorem 3.6 (2) refines the decomposition (3.11) and the cells \( \hat{X}_w \) contained in a fixed \( \Omega_{\rho} \) trace out the standard product paving of its decomposition \( \mathbb{A}^d \times \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_r} \).
3.3. Chern classes and cohomology presentation. Our next task is to leverage the linear algebra of the last subsection to compute the cohomology ring \( H^\bullet(X_{n,k}) \). To perform this computation, we need some basic facts about vector bundles.

Let \( X \) be a complex algebraic variety. For \( r \geq 0 \), a complex vector bundle of rank \( r \) over \( X \) is an algebraic variety \( E \) equipped with a surjective morphism \( \pi : E \rightarrow X \) such that, for each \( x \in X \), the fiber \( E_x := \pi^{-1}(x) \) has the structure of a complex vector space. The morphism \( E \rightarrow X \) is required to be locally trivial in the sense that, for all \( x \in X \), there is a neighborhood \( U \) of \( x \) and an isomorphism \( \pi^{-1}(U) \cong \mathbb{C}^r \times U \) making the following diagram commute

\[
\begin{array}{ccc}
\pi^{-1}(U) & \cong & \mathbb{C}^r \times U \\
\downarrow & & \downarrow \\
U & \rightarrow & U
\end{array}
\]

where the map \( \mathbb{C}^r \times U \rightarrow U \) is the canonical projection. A vector bundle \( E \) of rank 1 is called a line bundle.

Vector space operations on fibers can be used to construct new bundles from old. The direct sum (or Whitney sum) of two vector bundles \( E \) and \( F \) over \( X \) is defined by

\[
(E \oplus F)_x := E_x \oplus F_x \quad x \in X
\]

so that \( \text{rank}(E \oplus F) = \text{rank}(E) + \text{rank}(F) \). The dual of a vector bundle \( E \) over \( X \) has fibers

\[
E^*_x := (E_x)^* = \text{Hom}(E_x, \mathbb{C}) \quad x \in X
\]

so that \( \text{rank}(E^*) = \text{rank}(E) \).

A vector bundle \( E \) over \( X \) can give geometric information about \( X \). For \( 1 \leq i \leq r \), the Chern class

\[
c_i(E) \in H^{2i}(X)
\]

lives in the cohomology ring of \( X \). The total Chern class is the generating function of these individual Chern classes

\[
c(E) := 1 + c_1(E) + c_2(E) + \cdots + c_d(E) \in H^\bullet(X)
\]

which is a typically inhomogeneous element of cohomology. In the special case where \( E = \mathcal{L} \) is a line bundle, the total Chern class has the simple form \( c(\mathcal{L}) = 1 + c_1(\mathcal{L}) \).

For \( r \geq 0 \), the trivial vector bundle of rank \( r \) over \( X \) is \( E = \mathbb{C}^r \times X \) equipped with the canonical projection \( \mathbb{C}^r \times X \rightarrow X \). By standard abuse, we denote this trivial bundle by \( \mathbb{C}^r \). The definition of this bundle has nothing to do with \( X \), and thus carries no cohomological data: we have \( c(\mathbb{C}^r) = 1 \).

When \( X \) is a moduli space of complex vector spaces, natural choices of vector bundles \( E \) can give more information.

**Example 3.7.** Let \( \mathbb{P}^{k-1} \) be the variety of lines through the origin in \( \mathbb{C}^k \). The tautological line bundle \( \mathcal{L} \) over \( \mathbb{P}^{k-1} \) has fiber \( \ell^* = \text{Hom}(\ell, \mathbb{C}) \) over a point \( \ell \in \mathbb{P}^{k-1} \). If \( x = c_1(\mathcal{L}) \), we have the cohomology presentation

\[
H^\bullet(\mathbb{P}^{k-1}) = \mathbb{C}[x]/(x^k)
\]

as a truncated polynomial ring.

Recall that the Künneth Theorem gives an isomorphism

\[
H^\bullet(X \times Y) \cong H^\bullet(X) \otimes H^\bullet(Y)
\]

for any spaces \( X \) any \( Y \). Therefore, the \( n \)-fold self-product \( (\mathbb{P}^{k-1})^n \) has cohomology presentation

\[
H^\bullet((\mathbb{P}^{k-1})^n) = \mathbb{C}[x_1, \ldots, x_n]/(x_1^k, \ldots, x_n^k)
\]

where \( x_i = c_1(\mathcal{L}_i) \), the Chern class of the tautological line bundle over the \( i^{th} \) copy of \( \mathbb{P}^{k-1} \).
We shall bootstrap Example 3.7 to present the cohomology of $X_{n,k}$. In order to do this, we will need a piece of Chern calculus. Given a variety $X$, a short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0 \tag{3.19}$$

of vector bundles over $X$ consists of morphisms $\mathcal{E}' \to \mathcal{E} \to \mathcal{E}''$ of varieties commuting with the maps to $X$ such that for each point $x \in X$, the induced maps $\mathcal{E}'_x \to \mathcal{E}_x \to \mathcal{E}''_x$ form a short exact sequence of vector spaces. The direct sum gives a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E}' \oplus \mathcal{E}'' \to \mathcal{E}'' \to 0$ as expected, but more complicated examples are possible.

Given a short exact sequence as above, the Chern classes of $\mathcal{E}, \mathcal{E}'$, and $\mathcal{E}''$ are related in the ring $H^\bullet(X)$. This relationship is most cleanly stated in terms of total Chern classes:

$$c(\mathcal{E}) = c(\mathcal{E}') \cdot c(\mathcal{E}'') \tag{3.20}$$

When $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$, the corresponding relation

$$c(\mathcal{E} \oplus \mathcal{F}) = c(\mathcal{E}) \cdot c(\mathcal{F}) \tag{3.21}$$

inside $H^\bullet(X)$ is called the Whitney sum formula. The Whitney sum formula is a key ingredient for presenting the cohomology of $X_{n,k}$.

**Theorem 3.8.** For $1 \leq i \leq n$, let $\mathcal{L}_i$ be the vector bundle over $X_{n,k}$ whose fiber over a point $(\ell_1, \ldots, \ell_n)$ is the dual space $\ell_i^\perp$ of $\ell_i$. The cohomology ring $H^\bullet(X_{n,k})$ has presentation

$$H^\bullet(X_{n,k}) = R_{n,k} = \mathbb{C}[x_1, \ldots, x_n]/\langle x_1^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle$$

where $x_i \leftrightarrow c_1(\mathcal{L}_i)$.

**Proof.** Let us make the abbreviation $y_i := c_1(\mathcal{L}_i) \in H^\bullet(X_{n,k})$. By the definition of $X_{n,k}$, for any point $(\ell_1, \ldots, \ell_n) \in X_{n,k}$ we have a linear surjection

$$\ell_1 \oplus \cdots \oplus \ell_n \to \mathbb{C}^k \tag{3.22}$$

given by $(v_1, \ldots, v_n) \mapsto v_1 + \cdots + v_n$. Dualizing this surjection yields an injection of vector bundles $(\mathbb{C}^k)^* \to \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$ over $X_{n,k}$ which may be completed to a short exact sequence

$$0 \to (\mathbb{C}^k)^* \to \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n \to \mathcal{E} \to 0 \tag{3.23}$$

where $(\mathbb{C}^k)^*$ is a trivial rank $k$ bundle and $\mathcal{E}$ has rank $n-k$. Applying Equations (3.20) and (3.21), we have

$$(1 + y_1) \cdots (1 + y_n) = c((\mathbb{C}^k)^*) \cdot c(\mathcal{E}) = 1 \cdot c(\mathcal{E}) = c(\mathcal{E}) \tag{3.24}$$

where $c((\mathbb{C}^k)^*) = 1$ since $(\mathbb{C}^k)^*$ is a trivial bundle. As a member of the graded ring $H^\bullet(X)$, the element $c(\mathcal{E})$ has degree $\leq 2(n-k)$ so that

$$c_d(y_1, \ldots, y_n) = 0 \text{ whenever } d > n-k \tag{3.25}$$

since each $y_i = c_1(\mathcal{L}_i)$ has degree 2. Furthermore, since $\mathcal{L}_i$ is the pullback to $X_{n,k}$ of the tautological line bundle over the $i^{th}$ factor in $(\mathbb{P}^{k-1})^n$, we have

$$y_i^k = 0 \text{ for } i = 1, 2, \ldots, n. \tag{3.26}$$

Consequently, we have a well-defined homomorphism

$$\varphi : R_{n,k} \to H^\bullet(X_{n,k}) \tag{3.27}$$

given by $\varphi : x_i \mapsto y_i$.

We claim that $\varphi$ is surjective. If $\iota : X_{n,k} \to (\mathbb{P}^{k-1})^n$ is the inclusion, Theorem 3.6 implies that the induced map

$$\iota^* : H^\bullet((\mathbb{P}^{k-1})^n) \to H^\bullet(X_{n,k}) \tag{3.28}$$

on cohomology is a surjection. Example 3.7 and the subsequent discussion show that the cohomology classes of the tautological line bundles over the $n$ factors in $(\mathbb{P}^{k-1})^n$ generate the source ring
$H^\bullet(\mathbb{P}^{k-1})^n$. Consequently, the Chern classes $y_1, \ldots, y_n$ generate the target ring $H^\bullet(X_{n,k})$ and $\varphi$ is a surjection.

Finally, we show that $\varphi$ is an isomorphism. Theorem 3.6 implies that $\{\tilde{X}_w : w \in \mathcal{W}_{n,k}\}$ induces an affine paving of $X_{n,k}$, so that $H^\bullet(X_{n,k})$ has vector space dimension $|\mathcal{W}_{n,k}|$, the number of length $n$ Fubini words with maximum letter $k$. Theorem 2.11 shows that the domain $R_{n,k}$ of $\varphi$ has the same dimension, forcing the epimorphism $\varphi$ to be an isomorphism. 

We can consider Theorem 3.8 for varying $n$ and $k$. The cohomology rings $H^\bullet(\text{Fl}(n))$ of the flag varieties $\text{Fl}(n)$ exhibit favorable stability properties as $n \to \infty$ (see [17]). Anlogously, we can consider ind-varieties formed by the glueing $X_{n,k}$ along either of the sequences $(n,k) \leadsto (n+1,k)$ or $(n,k) \leadsto (n+1,k+1)$ which preserve the condition $n \geq k$ necessary for $X_{n,k}$ to be nonempty. We refer the reader to [48, 49] for details.

**Remark 3.9.** (From $\mathbb{C}$ to $\mathbb{Z}$.) Theorem 3.8 as well as the other geometric results presented in this chapter, hold over $\mathbb{Z}$ as well as $\mathbb{C}$. We outline the basic argument used in this setting for deducing an integral cohomology presentation from a complex cohomology presentation.

Let $I_{n,k} \subseteq \mathbb{Z}[x_1, \ldots, x_n]$ be the ideal in $\mathbb{Z}[x_1, \ldots, x_n]$ with the same generators as $I_{n,k}$:

$$I_{n,k} := \langle e_n, e_{n-1}, \ldots, e_{n-k+1}, x_1, x_2, \ldots, x_n \rangle \subseteq \mathbb{Z}[x_1, \ldots, x_n]$$

and denote the corresponding quotient ring by

$$R_{n,k} := \mathbb{Z}[x_1, \ldots, x_n]/I_{n,k}.$$ 

We claim that the family $\mathcal{A}_{n,k}$ of substaircase monomials in $x_1, \ldots, x_n$ descends to a $\mathbb{Z}$-basis of $R_{n,k}$. Linear independence over $\mathbb{Z}$ follows from linear independence over $\mathbb{C}$. To show spanning, one can use the Gröbner-theoretic part of Theorem 2.11 together with explicit witness relations [35, Lem. 3.4] which realize the relevant Demazure characters $\alpha_{\text{rev}(\gamma(S))} \in I_{n,k}$, or an explicit straightening algorithm appearing in Tanisaki [64] and Garisa-Procesi [21] which was adapted by Griffin [24] to this setting. In particular, the ring $R_{n,k}$ is a free $\mathbb{Z}$-module of rank $k! \cdot \text{Stir}(n,k)$.

The affine paving result Theorem 3.6 implies that the integral cohomology ring $H^\bullet(X_{n,k}; \mathbb{Z})$ is also a free $\mathbb{Z}$-module of rank $k! \cdot \text{Stir}(n,k)$, with basis given by the classes $[X_w]$ of the closures $X_w$ of cells $\tilde{X}_w$ indexed by Fubini words $w \in \mathcal{W}_{n,k}$. Theorem 3.6 also guarantees that the map $\iota^\bullet : H^\bullet(X_{n,k}; \mathbb{Z}) \to H^\bullet(\mathbb{P}^{k-1})^n; \mathbb{Z})$ on integral cohomology induced by $\iota : X_{n,k} \hookrightarrow (\mathbb{P}^{k-1})^n$ is a surjection. The same Whitney sum reasoning as in the proof of Theorem 3.8 yields a surjective map

$$\varphi : R_{n,k} \to H^\bullet(X_{n,k}; \mathbb{Z})$$

of $\mathbb{Z}$-algebras. To see that $\varphi$ is an isomorphism, we use the result that any epimorphism between two $\mathbb{Z}$-modules of the same finite rank is automatically an isomorphism (a fact easily proven using Smith Normal Form).

### 3.4 Fubini word Schubert polynomials

Given a Fubini word $w$, let $X_w \subseteq X_{n,k}$ be the closure of $\tilde{X}_w$. Theorem 3.6 guarantees that the classes $[X_w]$ of these cells forms a basis of $H^\bullet(X_{n,k})$ as $w$ varies over $\mathcal{W}_{n,k}$. In this subsection we describe explicit polynomial representatives $\mathcal{S}_w$ for the $[X_w]$ under the cohomology presentation $H^\bullet(X_{n,k}) = R_{n,k}$. The $\mathcal{S}_w$ will specialize to the usual Schubert polynomials when $k = n$ and $w \in S_n$ is a permutation.

In order to define our generalized Schubert polynomials, we will need some combinatorial definitions related to words. A word $v = [v(1), \ldots, v(n)]$ is convex if it contains no subword of the form $\cdots i \cdots j \cdots$ where $i \neq j$. The convexification of an arbitrary word $w = [w(1), \ldots, w(n)]$ is the unique convex word $\text{conv}(w) = [v(1), \ldots, v(n)]$ obtained by rearranging the letters of $w$ in which the initial letters appear in the same order. The sorting permutation $\text{sort}(w) \in S_n$ of $w$ is the unique
Bruhat minimal \( u \in S_n \) for which \( \{ w(u(1)), \ldots, w(u(n)) \} = \text{conv}(w) \). The \textit{standardization} of a convex Fubini word \( v = [v(1), \ldots, v(n)] \in W_{n,k} \) is the permutation \( \text{st}(v) \in S_n \) whose one-line notation is obtained by replacing the non-initial letters in \( v \), from left to right, with \( k+1, k+2, \ldots, n \).

**Definition 3.10.** Let \( w \in W_{n,k} \) be a Fubini word. Define a polynomial \( \mathcal{G}_w \in \mathbb{Z}[x_1, \ldots, x_n] \) by
\[
\mathcal{G}_w := \text{sort}(w)^{-1} \cdot \mathcal{G}_{\text{st}(\text{conv}(w))}
\]
where \( \mathcal{G}_{\text{st}(\text{conv}(w))} \) is the usual Schubert polynomial attached to the permutation \( \text{st}(\text{conv}(w)) \in S_n \).

As an example, consider the Fubini word \( w = [2, 1, 2, 1, 3, 3, 2, 3] \in W_{8,3} \). The convexification of \( w \) is \( \text{conv}(w) = [2, 2, 1, 1, 3, 3, 2, 3] \) and is \( \text{sort}(w) = [1, 3, 7, 2, 4, 5, 6, 8] \in S_8 \) the Bruhat-minimal permutation which sorts \( \text{conv}(w) \) into \( w \). The standardization of \( \text{conv}(w) \) is the permutation \( \text{st}(\text{conv}(w)) = [2, 4, 5, 1, 6, 3, 7, 8] \in S_8 \) whose Schubert polynomial is
\[
\mathcal{G}_{\text{st}(\text{conv}(w))} = x_1^2 x_2^2 x_3^3 + x_1^2 x_2^2 x_3 x_4 + x_1^2 x_2 x_3^2 x_4 + x_1^2 x_2 x_3 x_4^2 + x_1^2 x_2 x_3 x_4 + x_1^2 x_2 x_3 x_4.
\]
In order to obtain \( \mathcal{G}_w \) itself, we apply the permutation \( \text{sort}(w)^{-1} = [4, 1, 6, 2, 3, 5, 7, 8] \) to the subscripts in the above polynomial yielding
\[
\mathcal{G}_w = x_1^2 x_2^2 x_3^2 + x_2^2 x_3^2 x_4^2 + x_1^2 x_2 x_3^2 x_4 + x_1^2 x_2 x_3 x_4^2 + x_1^2 x_2 x_3 x_4 + x_1^2 x_2 x_3 x_4.
\]

The usual permutation-indexed Schubert polynomials represent closures of Schubert cells in the flag variety \( \text{Fl}_n \). This extends to the setting of \( X_{n,k} \); the following is a result of Pawlowski and Rhoades [48].

**Theorem 3.11.** For any \( w \in W_{n,k} \), the cohomology class \([X_w]\) is represented by \( \mathcal{G}_w \) under the cohomology presentation \( H^*(X_{n,k}) = R_{n,k} \) in Theorem 3.8. In particular, the set \( \{ \mathcal{G}_w : w \in W_{n,k} \} \) descends to a basis of \( R_{n,k} \).

Theorem 3.11 is proven using Fulton’s theory of degeneracy loci and matrix Schubert varieties. When \( w \) is convex, one observes that the closure of \( X_w \) inside \( \mathbb{P}^{(k-1)n} \) is an appropriate rank variety \( \Omega_\mu \) as in (3.12). Degeneracy locus theory applies to show that the representative of \([X_w]\) is as claimed. If \( w \) is not convex, consider the Bruhat-minimal permutation \( \text{sort}(w) \in S_n \) which carries \( w \) to its convexification \( \text{conv}(w) \). The permutation \( \text{sort}(w) \) has a reduced factorization \( \text{sort}(w) = s_{i_1} \cdots s_{i_r} \) where each \( s_{i_j} = (i_j, i_j + 1) \) interchanges at most one initial letter. It follows from the definitions that \( \text{sort}(w) : X_{\text{conv}(w)} = X_w \) as sets, and Theorem 3.8 allows us to reduce to the convex setting.

Unlike in the case of the flag variety \( \text{Fl}(n) \), the Schubert basis of Theorem 3.11 does not have positive structure constants in general. For example, if \( n = 4 \) and \( k = 3 \) we have
\[
\mathcal{G}_{[1,1,2,3]} \cdot \mathcal{G}_{[1,2,3,2]} = -\mathcal{G}_{[1,1,3,2]} + 2\mathcal{G}_{[2,2,1,3]}
\]
inside \( R_{4,3} \).

### 3.5. Variations

The ideas used to present the cohomology of \( X_{n,k} \) apply to more general moduli spaces of spanning configurations. In each case, the cohomology of the space in question is generated by the Chern classes of a natural family of vector bundles, subject only to relations coming from the Whitney Sum Formula. We give three examples of this paradigm.

Our first example, due to Pawlowski and Rhoades [48], gives a geometric interpretation of the rings \( R_{n,k,s} \) of Definition 2.12. Fix three integers \( k \leq s \leq n \) and let \( \pi : \mathbb{C}^s \to \mathbb{C}^k \) be the projection which forgets the last \( s - k \) coordinates. Let \( X_{n,k,s} \) be the open subvariety of \( \mathbb{P}^{s-1} \) given by
\[
X_{n,k,s} := \left\{ (\ell_1, \ldots, \ell_n) : \begin{array}{l}
\text{each } \ell_i \text{ is a line in } \mathbb{C}^s \\
\pi(\ell_1 + \cdots + \ell_n) = \mathbb{C}^k
\end{array} \right\}.
\]
This space reduces to \( X_{n,k} \) when \( k = s \). Let \( \mathcal{L}_i \) be the line bundle over \( X_{n,k,s} \) whose fiber over \((\ell_1, \ldots, \ell_n)\) is the dual space \( \ell_i^* \).
Just as was the case for \(X_{n,k}\), we have an inclusion \(\iota : X_{n,k,s} \hookrightarrow (\mathbb{P}^{s-1})^n\) and a nonstandard affine paving of \((\mathbb{P}^{s-1})^n\) in which \((\mathbb{P}^{s-1})^n - X_{n,k,s}\) forms an initial part. Thus, the cohomology map \(\iota^* : H^\bullet((\mathbb{P}^{s-1})^n) \to H^\bullet(X_{n,k,s})\) is a surjection and the ring \(H^\bullet(X_{n,k,s})\) is generated by the Chern classes \(x_i := c_1(L_i) \in H^2(X_{n,k,s})\) for \(i = 1, 2, \ldots, n\). The number of cells contained in \(X_{n,k,s}\) is the number of \(s\)-block ordered set partitions \((B_1 | \cdots | B_s)\) of \([n]\) in which the last \(s - k\) blocks \(B_{k+1}, B_{k+2}, \ldots, B_s\) are allowed to be empty; these objects index a basis of \(H^\bullet(X_{n,k,s})\).

The Chern class relations \(x_i^k = 0\) already hold over the projective space product \((\mathbb{P}^{s-1})^n\). For any point \((\ell_1, \ldots, \ell_n) \in X_{n,k,s}\) we have a surjection

\[
\ell_1 \oplus \cdots \oplus \ell_n \to \mathbb{C}^k
\]

given by \((v_1, \ldots, v_n) \mapsto \pi(v_1 + \cdots + v_n)\). Dualizing gives a short exact sequence

\[
0 \to (\mathbb{C}^k)^* \to \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n \to \mathcal{E} \to 0
\]

where \((\mathbb{C}^k)^*\) is trivial of rank \(k\) and \(\mathcal{E}\) has rank \(n - k\). The Whitney Sum Formula implies that the terms of degree \(> n - k\) in the product \((1 + x_1) \cdots (1 + x_n)\) vanish. Consequently, we have a surjection

\[
R_{n,k,s} \to H^\bullet(X_{n,k,s}).
\]

As before, we see that the domain \(R_{n,k,s}\) and codomain \(H^\bullet(X_{n,k,s})\) are vector spaces of the same dimension, so we have an isomorphism \(H^\bullet(X_{n,k,s}) 
\cong R_{n,k,s}\).

For our second example, due to Rhoades and Wilson [55], we consider spanning configurations of lines in which an initial set is required to be linearly independent. For parameters \(r \leq k \leq n\), we consider the open subvariety

\[
X_{n,k}^{(r)} := \left\{ (\ell_1, \ldots, \ell_n) : \ell_i \text{ a line in } \mathbb{C}^k, \ell_1 + \cdots + \ell_n = \mathbb{C}^k, \text{ and } \text{the sum } \ell_1 \oplus \cdots \oplus \ell_r \text{ is direct} \right\}
\]

of \((\mathbb{P}^{k-1})^n\). When \(r = 1\) this space reduces to \(X_{n,k}\).

We define the line bundles \(\mathcal{L}_i = \ell_i^*\) over \(X_{n,k}^{(r)}\) and their Chern classes \(x_i = c_1(\mathcal{L}_i) \in H^\bullet(X_{n,k}^{(r)})\) as before. The same paving used to study the embedding \(X_{n,k} \subseteq (\mathbb{P}^{k-1})^n\) shows that we have a cohomology surjection \(\iota^* : H^\bullet((\mathbb{P}^{k-1})^n) \to H^\bullet(X_{n,k}^{(r)})\) induced by \(\iota : X_{n,k}^{(r)} \hookrightarrow (\mathbb{P}^{k-1})^n\) so that the \(x_i\) generate \(H^\bullet(X_{n,k}^{(r)})\).

We examine relations among the Chern classes \(x_i \in H^\bullet(X_{n,k}^{(r)})\). The identities \(x_i^k = 0\) already hold over \((\mathbb{P}^{k-1})^n\). The same exact sequence of vector bundles used to study \(H^\bullet(X_{n,k})\) applies to show \(e_d(x_1, \ldots, x_n) = 0\) for \(d > n - k\). Since the first \(r\) members of a point \((\ell_1, \ldots, \ell_n) \in X_{n,k}^{(r)}\) have full \(r\)-dimensional span, vector addition gives an injection

\[
\ell_1 \oplus \cdots \oplus \ell_r \hookrightarrow \mathbb{C}^k
\]

which leads to a short exact sequence

\[
0 \to \mathcal{F} \to (\mathbb{C}^k)^* \to \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r \to 0
\]

of vector bundles over \(X_{n,k}^{(r)}\) where the middle bundle is trivial and \(\mathcal{F}\) has rank \(k - r\). The Whitney Sum Formula implies

\[
c(\mathcal{F}) \cdot (1 + x_1) \cdots (1 + x_r) = c((\mathbb{C}^k)^*) = 1,
\]

so that terms of degree \(> k - r\) in the expression

\[
\frac{1}{(1 + x_1) \cdots (1 + x_r)} = \sum_{d \geq 0} (-1)^d \cdot h_d(x_1, \ldots, x_r)
\]
vanish in $H^\bullet(X_{n,k}^{(r)})$. In summary, if we introduce the ideal
\begin{equation}
I_{n,k}^{(r)} := \langle x_1^k, x_2^k, \ldots, x_n^k \rangle + \langle e_d(x_1, x_2, \ldots, x_n) \rangle + \langle h_d(x_1, x_2, \ldots, x_r) \rangle : d > k - r \rangle
\end{equation}
in $\mathbb{C}[x_1, \ldots, x_n]$ generated by these relations, we have a surjection
\begin{equation}
\varphi : \mathbb{C}[x_1, \ldots, x_n]/I_{n,k}^{(r)} \to H^\bullet(X_{n,k}^{(r)}).
\end{equation}

Orbit harmonics and paving theory show that the domain and codomain of (3.42) are vector spaces of the same dimension, so (3.42) is an isomorphism.

For our final example (see [54]) we consider configurations of subspaces of arbitrary dimension $d \geq 1$. Recall that $\text{Gr}(d, k)$ denotes the Grassmannian of $d$-dimensional subspaces $W \subseteq \mathbb{C}^k$.

Extending this notation, if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is any list of $n$ integers $1 \leq \alpha_i \leq k$, we write $\text{Gr}(\alpha, k)$ for the product
\begin{equation}
\text{Gr}(\alpha, k) := \text{Gr}(\alpha_1, k) \times \cdots \times \text{Gr}(\alpha_n, k) = \{(W_1, \ldots, W_n) : W_i \subseteq \mathbb{C}^k, \dim W_i = \alpha_i\}.
\end{equation}

The spanning subspace configurations
\begin{equation}
X_{\alpha,k} := \{(W_1, \ldots, W_n) : W_1 + \cdots + W_n = \mathbb{C}^k\}
\end{equation}
in $\text{Gr}(\alpha, k)$ form an open subvariety which is nonempty if and only if $\alpha_1 + \cdots + \alpha_n \geq k$.

The varieties $X_{\alpha,k}$ generalize classical objects in type $A$ Schubert calculus.
- If $\alpha_i = k$ for any $i$, the space $X_{\alpha,k} = \text{Gr}(\alpha, k)$ is a product of Grassmannians.
- If $\alpha_1 + \cdots + \alpha_n = k$, the assignment $(W_1, W_2, \ldots, W_n) \mapsto (W_1, W_1 + W_2, \ldots, W_1 + \cdots + W_n)$ gives a homotopy equivalence $X_{\alpha,k} \simeq \text{Fl}(\alpha)$ between $X_{\alpha,k}$ and the partial flag variety $\text{Fl}(\alpha)$ indexed by $\alpha$. At the level of algebraic groups, this equivalence is the canonical surjection $GL_k(\mathbb{C})/L_\alpha \to GL_k(\mathbb{C})/P_\alpha$, where $L_\alpha$ and $P_\alpha$ are the Levi and parabolic subgroups of $GL_k(\mathbb{C})$ indexed by $\alpha$, respectively.

When $\alpha = (1, \ldots, 1)$ consists of $n$ copies of 1, the space $X_{\alpha,k}$ specializes to $X_{n,k}$.

For $\alpha = (\alpha_1, \ldots, \alpha_n)$ arbitrary, we let $E_i$ be the vector bundle over $X_{\alpha,k}$ whose fiber over $(W_1, \ldots, W_n)$ is the dual space $W_i^*$. The bundle $E_i$ has rank $\alpha_i$. The total Chern class $c(E_i)$ factors
\begin{equation}
c(E_i) = 1 + c_1(E_i) + c_2(E_i) + \cdots + c_{\alpha_i}(E_i) = (1 + x_{i,1})(1 + x_{i,2}) \cdots (1 + x_{i,\alpha_i})
\end{equation}
inside $H^\bullet(X_{\alpha,k})$ where $X_{\alpha,k}'$ is a flag extension of $X_{\alpha,k}$ and $x_{i,1}, x_{i,2}, \ldots, x_{i,\alpha_i} \in H^2(X_{\alpha,k}')$ are the Chern roots of $E_i$. Writing $N := \alpha_1 + \cdots + \alpha_n$, the sum of the $\alpha_i$, we refer to the total list of these Chern roots as
\begin{equation}
(x_1, \ldots, x_N) := (x_{1,1}, x_{1,1}, x_{2,1}, x_{2,2}, \ldots, x_{n,1}, \ldots, x_{n,\alpha_n}).
\end{equation}

Any polynomial in the variables $(x_1, \ldots, x_N)$ which is invariant under $S_n = S_{\alpha_1} \times \cdots \times S_{\alpha_n}$ is an element of the subring $H^\bullet(X_{\alpha,k}) \subseteq H^\bullet(X_{\alpha,k}')$.

Write $\iota : X_{n,k} \hookrightarrow \text{Gr}(\alpha, k)$ for the embedding of $X_{n,k}$ into the Grassmannian product. The Schubert cell decomposition of the Grassmannian induces a product paving of $\text{Gr}(\alpha, k)$, but this is not adapted to the map $\iota$. However, the echelon form of Theorem 3.4 generalizes to yield a nonstandard affine paving of $\mathbb{C} = Z_0 \subset Z_1 \subset \cdots \subset Z_m = \text{Gr}(\alpha, k)$ where $\text{Gr}(\alpha, k) - X_{\alpha,k} = Z_j$ for some $j$. Consequently, we have a surjection
\begin{equation}
\iota^* : H^\bullet(\text{Gr}(\alpha, k)) \to H^\bullet(X_{\alpha,k}).
\end{equation}

The vector bundle $E_i$ is the pullback to $X_{n,k}$ of the tautological bundle over the $i^{th}$ factor $\text{Gr}(\alpha_i, k)$ of $\text{Gr}(\alpha, k)$. Since $\iota^*$ is a surjection, we conclude that $H^\bullet(X_{n,k})$ is generated by the Chern classes of these bundles.
Given any configuration \((W_1, \ldots, W_n) \in X_{\alpha,k}\), for each \(i\) we have a short exact sequence of vector spaces
\[
0 \to W_i \to \mathbb{C}^k \to \mathbb{C}^k/W_i \to 0
\]
which dualizes to a short exact sequence
\[
0 \to \mathcal{F}_i \to (\mathbb{C}^k)^* \to \mathcal{E}_i \to 0
\]
of vector bundles where \(\mathcal{F}_i\) has rank \(k - \text{rank}(\mathcal{E}_i) = k - \alpha_i\). Since \(c(\mathcal{F}_i)c(\mathcal{E}_i) = c((\mathbb{C}^k)^*) = 1\), the rational expression
\[
\frac{1}{(1 + x_{i,1})(1 + x_{i,2})\cdots(1 + x_{i,\alpha_i})}
\]
is a polynomial of degree \(\leq k - \alpha_i\) so that \(h_d(x_{i,1}, x_{i,2}, \ldots, x_{i,\alpha_i}) = 0\) inside \(H^*(X_{\alpha,k})\) whenever \(d > k - \alpha_i\). These relations also hold over the Grassmannian \(\text{Gr}(\alpha, k)\). Over \(X_{\alpha,k}\) the spanning condition yields an additional short exact sequence
\[
0 \to (\mathbb{C}^k)^* \to \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n \to \mathcal{F} \to 0
\]
where \(\mathcal{F}\) has rank \(N - k\). Reasoning as in the case of \(X_{n,k}\), we have \(e_d(x_1, \ldots, x_N) = 0\) inside \(H^*(X_{\alpha,k})\) whenever \(d > N - k\). These relations suffice to present the cohomology of \(X_{\alpha,k}\). An orbit harmonics argument leads to a presentation
\[
H^*(X_{\alpha,k}) = (\mathbb{C}[x_1, \ldots, x_N]/I_{\alpha,k})^{S_\alpha}
\]
as a \(S_\alpha\)-invariant subring where \(I_{\alpha,k} \subseteq \mathbb{C}[x_1, \ldots, x_N]\) is the ideal
\[
I_{\alpha,k} := \langle e_d(x_1, x_2, \ldots, x_n) : d > N - k \rangle + \sum_{i=1}^{n} \langle h_d(x_{i,1}, x_{i,2}, \ldots, x_{i,\alpha_i}) : d_i > k - \alpha_i \rangle.
\]
When \(\alpha = (1, \ldots, 1)\) is a sequence of \(n\) copies of 1, the presentation (3.51) is the assertion \(H^*(X_{n,k}) = R_{n,k}\) of Theorem 3.8.

4. Generalized Springer fibers

The Garsia-Procesi-Tanisaki rings are quotients \(R_\lambda\) of \(\mathbb{C}[x_1, \ldots, x_n]\) which depend on a partition \(\lambda \vdash n\). Like the rings \(R_{n,k}\), they are instances of orbit harmonics quotients. Geometrically, the \(R_\lambda\) present [39] the cohomology of the Springer fibers \(B_\lambda\).

In his Ph.D. thesis, Griffin [24] introduced a common generalization \(R_{n,\lambda,s}\) of the two families of rings \(R_{n,k}\) and \(R_\lambda\). Many algebraic properties of \(R_{n,k}\) have been generalized to \(R_{n,\lambda,s}\). Griffin, Levinson, and Woo [27] defined a \(\Delta\)-Springer fiber \(Y_{n,\lambda,s}\) whose cohomology is presented by \(R_{n,\lambda,s}\). Unlike the variety \(X_{n,k}\) of spanning line configurations (which is smooth but not compact), the varieties \(Y_{n,\lambda,s}\) are compact but not smooth. As such, they give a different geometric model for the symmetric function \(\Delta_{s_{k-1}}^e e_n\) at \(t = 0\). We recall the classical story of Springer fibers, and then discuss their recent generalization.

4.1. Springer fibers. Given a partition \(\lambda \vdash n\), let \(B_\lambda\) be the associated Springer fiber [61] defined as follows. There is a natural action of \(GL_n(\mathbb{C})\) on the flag variety \(\text{Fl}(n)\) given by
\[
X \cdot (V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n) = (XV_0 \subset XV_1 \subset \cdots \subset XV_{n-1} \subset XV_n).
\]
For any invertible matrix \(X\), the set \(\{V_\bullet \in \text{Fl}(n) : X \cdot V_\bullet = V_\bullet\}\) of flags fixed by \(X\) forms a closed subvariety of \(\text{Fl}(n)\). If \(X = U_\lambda\) is a unipotent \(n \times n\) matrix of Jordan type \(\lambda\), the Springer fiber
\[
B_\lambda := \{V_\bullet \in \text{Fl}(n) : U_\lambda \cdot V_\bullet = V_\bullet\}
\]
is the associated fixed space. For example, when \(\lambda = (1^n)\) we have \(U_\lambda = I\) so that \(B_{(1^n)} = \text{Fl}(n)\). The only fixed point of \(U_{(n)}\) is the standard flag \(0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_n \rangle\) so that \(B_{(n)} = \{\ast\}\) is a single point. The space \(B_\lambda\) is compact, but is typically singular.
The cohomology ring of $\mathcal{B}_\lambda$ has a nice presentation. Given a subset $S \subseteq [n]$, let $e_d(S)$ be the elementary symmetric polynomial of degree $d$ in the restricted variable set $\{x_i : i \in S\}$. In particular, we have $e_d(S) = 0$ whenever $d > |S|$. We write $\lambda$ and

$$m_{i,n}(\lambda) := \lambda'_n + \lambda'_{n-1} + \cdots + \lambda'_{n-i+1}$$

for the number of boxes in the Young diagram of $\lambda$ which do not lie in the first $n-i$ columns. The Tanisaki ideal is

$$I_\lambda := \langle e_d(S) : S \subseteq [n], \ d > n - m_{|S|,n}(\lambda) \rangle \subseteq \mathbb{C}[x_1, \ldots, x_n].$$

We write $R_\lambda := \mathbb{C}[x_1, \ldots, x_n]/I_\lambda$ for the corresponding quotient ring. Hotta and Springer [39] presented the cohomology of $\mathcal{B}_\lambda$ as

$$H^*(\mathcal{B}_\lambda) = R_\lambda.$$  

As with the flag manifold $Fl(n)$, the symmetric group does not act on the variety $\mathcal{B}_\lambda$ in any natural way. Despite this, the Springer construction gives a symmetric group action on the cohomology of $\mathcal{B}_\lambda$. With respect to the presentation (4.5), this $S_n$-action is subscript permutation. Hotta and Springer [39] proved that the top-degree piece $H^{\operatorname{top}}(\mathcal{B}_\lambda)$ carries the irreducible representation $S^\lambda$ of $S_n$, giving a geometric model for irreducible $S_n$-modules.

Garsia-Procesi and Tanisaki [21, 66] studied the $S_n$-action on $R_\lambda$ from a combinatorial point of view. For distinct parameters $\alpha_1, \alpha_2, \ldots \in \mathbb{C}$ and a partition $\lambda \vdash n$, let $Z_\lambda \subset \mathbb{C}^n$ be the locus of points $(x_1, \ldots, x_n)$ in which the number $\alpha_i$ appears with multiplicity $\lambda_i$ among the coordinates. For example, we have $Z_{(2,1)} = \{(\alpha_1, \alpha_1, \alpha_2), (\alpha_1, \alpha_2, \alpha_1), (\alpha_2, \alpha_1, \alpha_1)\}$. It is evident that $Z_\lambda$ is closed under the coordinate permuting action of $S_n$ and carries a copy of the parabolic coset representation $\mathbb{C}[S_n/S_\lambda]$. The graded ideal $\operatorname{gr} I(Z_\lambda) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ arising from orbit harmonics was computed [21, 66] to be

$$\operatorname{gr} I(Z_\lambda) = I_\lambda$$

so that as an ungraded $S_n$-module, the ring $R_\lambda$ is an orbit harmonics quotient:

$$R_\lambda = \mathbb{C}[x_1, \ldots, x_n]/\operatorname{gr} I(Z_\lambda) \cong \mathbb{C}[S_n/S_\lambda].$$

At the level of Frobenius images, we have

$$\operatorname{Frob}(R_\lambda) = h_\lambda.$$  

Garsia and Procesi [21] refined (4.8) by showing

$$\operatorname{gr} \operatorname{Frob}(R_\lambda; q) = \widetilde{H}_\lambda(x; q)$$

so that the graded character of $R_\lambda$ is a Hall-Littlewood polynomial.

4.2. The rings $R_{n,\lambda,s}$. We are ready to give Griffin’s common generalization of the rings $R_{n,k}$ and $R_\lambda$. These rings depend on a partition $\lambda$ with at most $n$ boxes and an integer $s \geq \ell(\lambda)$.

**Definition 4.1.** (Griffin [24]) Let $n$ and $s$ be positive integers and let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s \geq 0)$ be a partition with $\leq s$ positive parts with $|\lambda| = \lambda_1 + \cdots + \lambda_s \leq n$. Let $I_{n,\lambda,s} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be the ideal

$$I_{n,\lambda,s} = \langle e_d(S) : S \subseteq [n], \ d > n - m_{|S|,n}(\lambda) \rangle + \langle x_1^s, x_2^s, \ldots, x_n^s \rangle$$

where $m_{i,n}(\lambda) = \lambda'_n + \lambda'_{n-1} + \cdots + \lambda'_{n-i+1}$. Write

$$R_{n,\lambda,s} := \mathbb{C}[x_1, \ldots, x_n]/I_{n,\lambda,s}$$

for the corresponding quotient ring.

The rings $R_{n,\lambda,s}$ have the following interesting specializations.

- When $\lambda = (1^n)$ is a single column of length $n$, for any $s \geq n$ the ring $R_{n,(1^n),s}$ is the classical coinvariant ring $R_n$. 

• When $\lambda = (1^k)$ is a single column of length $k \leq n$ and $s = k$, we have $R_{n,(1^k),k} = R_{n,k}$.

More generally, for $s \geq k$, we have $R_{n,(1^k),s} = R_{n,k,s}$.

• When $|\lambda| = n$, we recover the Tanisaki quotient $R_{n,\lambda,s} = R_\lambda$ for any $s \geq \ell(\lambda)$.

The algebra of $R_{n,\lambda,s}$ is described by the combinatorics of ordered set partition-like objects. More precisely, given $(n,\lambda,s)$ as in Definition 4.1, let $\mathcal{OP}_{n,\lambda,s}$ be the family of length $s$ sequences $(B_1 \mid \cdots \mid B_s)$ of pairwise disjoint subsets of $[n]$ such that $B_1 \sqcup \cdots \sqcup B_s = [n]$ and the set $B_i$ has at least $\lambda_i$ elements. Thus $\mathcal{OP}_{n,(1^k),k} = \mathcal{OP}_{n,k}$ are the ordered set partitions we met before. Observe that a block $B_i$ of $(B_1 \mid \cdots \mid B_s) \in \mathcal{OP}_{n,\lambda,s}$ is allowed to be empty whenever $\lambda_i = 0$.

The symmetric group $S_n$ acts on the set $\mathcal{OP}_{n,\lambda,s}$ of generalized ordered set partitions. Griffin extended the orbit harmonics techniques of [21] and [35] to prove the following result.

**Theorem 4.2.** We have the isomorphism $R_{n,\lambda,s} \cong \mathbb{C}[\mathcal{OP}_{n,\lambda,s}]$ of ungraded $S_n$-modules.

Griffin refined [26] Theorem 4.2 by giving a positive expansion of $\text{grFrob}(R_{n,\lambda,s};q)$ into the basis $\tilde{H}_\mu(x;q)$ of Hall-Littlewood polynomials, giving another positive instance of Question 2.9. Finding the graded $S_n$-structure of $R_{n,\lambda,s}$ uses substantially different (and more involved) methods than those used for $R_{n,k}$.

### 4.3. The $\Delta$-Springer fibers.

Like the quotient rings $R_\lambda$, the rings $R_{n,\lambda,s}$ also present cohomology rings of varieties $Y_{n,\lambda,s}$. When $\lambda \vdash n$ and $s = \ell(\lambda)$, these are precisely the Springer fibers $B_\lambda$. In general, the variety $Y_{n,\lambda,s}$ consists of certain ‘initial’ partial flags $(V_0 \subset V_1 \subset \cdots \subset V_n)$ where $\dim V_i = i$ sitting inside a high-dimensional vector space $\mathbb{C}^K$.

Let $\lambda \vdash k$ be a partition with at most $s$ parts. We form a larger partition

$$\tilde{\lambda} := (n-k + \lambda_1, n-k + \lambda_2, \ldots, n-k + \lambda_s)$$

by adding $n-k$ to every part of $\lambda = (\lambda_1, \ldots, \lambda_s)$, including any trailing zeros. The partition $\tilde{\lambda}$ is a partition of $K := s(n-k) + k$. We let $X_{\tilde{\lambda}} \in M_K(\mathbb{C})$ be a nilpotent endomorphism of $\mathbb{C}^K$ of Jordan type $\tilde{\lambda}$. Griffin, Levinson, and Woo used $X_{\tilde{\lambda}}$ to define the following delta-extension of the Springer fiber.

**Definition 4.3.** (Griffin-Levinson-Woo [27]) For a positive integer $n$ and a partition $\lambda$ of $k \leq n$ having $\leq s$ parts, the $\Delta$-Springer fiber is the variety $Y_{n,\lambda,s}$ given by

$$Y_{n,\lambda,s} := \{ V_\bullet = (V_0 \subset V_1 \subset \cdots \subset V_n) : V_i \subseteq \mathbb{C}^K, \dim V_i = i, X_{\tilde{\lambda}} V_i \subseteq V_i, \text{ and } X_{\tilde{\lambda}}^{n-k} \mathbb{C}^K \subseteq V_n \}.$$  

The $\Delta$-Springer fiber $Y_{n,\lambda,s}$ is a closed subvariety of the partial flag variety $\text{Fl}(1^n, K-n)$. As such, the variety $Y_{n,\lambda,s}$ is compact, but rarely smooth. Intuitively, the condition $X_{\tilde{\lambda}}^{n-k} \mathbb{C}^K \subseteq V_n$ on the largest subspace in flags $V_\bullet \in \text{Fl}(1^n, K-n)$ means that the space $Y_{n,\lambda,s}$ is constructed out of Springer fibers. This is made precise in [27] where it is shown that $Y_{n,\lambda,s}$ can be filtered with Springer fibers crossed with affine spaces.

**Theorem 4.4.** (Griffin-Levinson-Woo [27]) We have the presentation $H^\bullet(Y_{n,\lambda,s}) = R_{n,\lambda,s}$. In this presentation, the variable $x_i$ represents the Chern class $c_1(L_i)$ of the line bundle $L_i$ whose fiber over $V_\bullet = (V_0 \subset \cdots \subset V_n)$ is the dual space $(V_i/V_{i-1})^*$.  

The proof of Theorem 4.4 uses some of the same ideas as the spanning lines case of Theorem 3.8 but is significantly more difficult. Giving a full treatment is beyond our scope, but we sketch the main ideas.

Coordinate permutation yields an embedding $i : Y_{n,\lambda,s} \hookrightarrow Y_{n,\varnothing,s}$ of $Y_{n,\lambda,s}$ into the space $Y_{n,\varnothing,s}$ labelled by the ‘empty partition’ $\varnothing = (0, \ldots, 0)$ consisting of $s$ zeros. A combinatorial affine paving of $Z_0 \subset Z_1 \subset \cdots \subset Z_m$ of $Y_{n,\varnothing,s}$ is constructed for which $i(Y_{n,\lambda,s}) = Z_i$ for some $i$. By virtue of this paving, the induced map $i^* : H^\bullet(Y_{n,\varnothing,s}) \rightarrow H^\bullet(Y_{n,\lambda,s})$ is a surjection.
An argument using iterated projective bundles shows that $Y_{n,\varnothing,s}$ has the same cohomology as $(\mathbb{P}^s)^n$, i.e.

$$H^\bullet(Y_{n,\varnothing,s}) = \mathbb{C}[x_1, \ldots, x_n]/\langle x_1^s, \ldots, x_n^s \rangle$$

(4.12)

where $x_i = c_1(\mathcal{L}_i)$; by naturality $x_i^s$ also vanishes in $H^\bullet(Y_{n,\lambda,s})$. Checking that the relevant elementary symmetric polynomials $e_d(S)$ vanish in $H^\bullet(Y_{n,\lambda,s})$ is more involved and uses work of Brundan and Ostrik [11] on Spaltenstein varieties. This given, the generators of $I_{n,\lambda,s}$ vanish in $H^\bullet(Y_{n,\lambda,s})$ and we have a map of rings $\varphi : R_{n,\lambda,s} \to H^\bullet(Y_{n,\lambda,s})$. The surjectivity of $\iota^* : H^\bullet(Y_{n,\varnothing,s}) \to H^\bullet(Y_{n,\lambda,s})$ implies that $\varphi$ is an epimorphism. Since the domain and target of $\varphi$ are vector spaces of the same dimension, $\varphi$ is an isomorphism.

5. Future Directions

In this final section, we indicate directions for future research on the algebraic combinatorics of delta operators. In particular, we use anticommuting variables to give models for the full symmetric function $\Delta_{\lambda,\mu}$ rather than just its $t = 0$ specialization. The material in this section is more algebraic than geometric, but the rich connections between $DR_n$, Hilbert schemes, and affine Springer fibers [13, 38] suggest that geometry will ultimately play a large role in this story.

5.1. Superspace, the Fields Conjecture, and the Zabrocki Conjecture. The diagonal coinvariant ring $DR_n$ is obtained from the classical coinvariant ring $R_n$ by adding in a new set of commuting variables. In the past few years, various authors have considered coinvariant quotients involving anticommuting variables.

For $n \geq 0$, the superspace ring of rank $n$ is the unital $\mathbb{C}$-algebra $\Omega_n$ generated by the $2n$ symbols $x_1, \ldots, x_n, \theta_1, \ldots, \theta_n$ subject to the relations

$$x_ix_j = x_jx_i \quad x_i\theta_j = \theta_jx_i \quad \theta_i\theta_j = -\theta_j\theta_i$$

(5.1)

for $1 \leq i, j \leq n$. This algebra admits a decomposition

$$\Omega_n = \mathbb{C}[x_1, \ldots, x_n] \otimes \{\theta_1, \ldots, \theta_n\}$$

(5.2)

into a symmetric algebra tensor an exterior algebra.

The term “superspace” comes from supersymmetry in physics. In this context the $x_i$ correspond to bosons (so that $x_i^2$ represents two indistinguishable bosons in state $i$) and the $\theta_i$ correspond to fermions (so that $\theta_i^2 = 0$ reflects the Pauli Exclusion Principle: two fermions cannot occupy state $i$ at the same time). We may also view $\Omega_n$ as the ring of holomorphic differential forms on affine $n$-space.

The symmetric group $S_n$ acts diagonally on $\Omega_n$

$$w \cdot x_i := x_{w(i)} \quad w \cdot \theta_i := \theta_{w(i)} \quad (w \in S_n, \ 1 \leq i \leq n).$$

(5.3)

and members of the invariant ring $\Omega_n^{S_n}$ are superspace symmetric functions. Bases of $\Omega_n^{S_n}$ correspond to pairs $(\lambda, \mu)$ of partitions where $\mu$ has distinct parts and $\ell(\lambda) + \ell(\mu) \leq n$. Blondeau-Fournier, Desrosiers, Lapointe, and Mathieu [5] extended Macdonald polynomials $\tilde{H}_\mu(x; q, t)$ to the setting $\Omega_n^{S_n}$ of superspace symmetric functions.

Let $(\Omega_n^{S_n})_+$ be the superspace symmetric functions with vanishing constant term and let $I \subseteq \Omega_n$ be the (two-sided) ideal in $\Omega_n$ generated by $(\Omega_n^{S_n})_+$. The superspace coinvariant algebra is the quotient

$$SR_n := \Omega_n/I.$$ 

(5.4)

Like the diagonal coinvariants $DR_n$, the ring $SR_n$ is a bigraded $S_n$-module. The Fields Institute Combinatorics Group (see [68]) posed a conjecture relating $SR_n$ to delta operators.
Conjecture 5.1. (Fields Conjecture) The bigraded Frobenius image of $SR_n$ is

\begin{equation}
\text{grFrob}(SR_n; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot \Delta_{e_{k-1}}' e_n \mid_{t=0}
\end{equation}

where $q$ tracks commuting degree and $z$ tracks anticommuting degree.

Equivalently, the Fields Conjecture predicts that $SR_n$ is related to the quotient rings $R_{n,k}$ and the spaces $X_{n,k}$ of spanning line configurations via

\begin{equation}
\text{grFrob}(SR_n; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot (\text{rev}_q \circ \omega)\text{grFrob}(R_{n,k}; q) = \sum_{k=1}^{n} z^{n-k} \cdot (\text{rev}_q \circ \omega)\text{grFrob}(H^*(X_{n,k}); q).
\end{equation}

In terms of bigraded Hilbert series, this specializes to

\begin{equation}
\text{Hilb}(SR_n; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot [k]_q \cdot \text{Stir}_q(n, k)
\end{equation}

so that $\dim SR_n$ is the total number of ordered set partitions of $[n]$.

Despite the prominence of the ring $\Omega_n$ throughout mathematics and the natural definition of the quotient $SR_n$, the Fields Conjecture has proven quite challenging. Rhoades and Wilson proved [59] the predicted formula (5.7) for the bigraded Hilbert series of $SR_n$. Swanson and Wallach proved [65] that the alternating subspace $SR_n^{\text{sign}} \subset SR_n$ has bigraded Hilbert series as predicted by Conjecture 5.1, i.e.

\begin{equation}
\text{Hilb}(SR_n^{\text{sign}}; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot q \cdot [n-1]_{[k-1]/q}.
\end{equation}

Ideally, the quotient $SR_n$ will have some direct connection to the geometry of $X_{n,k}$, but such a link is yet to be discovered.

The ring $R_{n,k}$, the variety $X_{n,k}$ and (conjecturally) the superspace coinvariants $SR_n$ give algebraic and geometric models for the $t = 0$ specialization of $\Delta_{e_{k-1}}' e_n$. Zabrocki defined [68] an extension of $SR_n$ which (conjecturally) gives an algebraic model for $\Delta_{e_{k-1}}' e_n$ itself. Let $\Omega_n[y_1, \ldots, y_n]$ be the superspace ring with $n$ commuting variables $y_1, \ldots, y_n$ adjoined. Formally, we have

\begin{equation}
\Omega_n[y_1, \ldots, y_n] = \mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}[y_1, \ldots, y_n] \otimes \wedge\{\theta_1, \ldots, \theta_n\}.
\end{equation}

The group $S_n$ acts ‘triply diagonally’ on this space. Modding out by the $S_n$-invariants with vanishing constant term gives the superspace diagonal coinvariants

\begin{equation}
SDR_n := \Omega_n[y_1, \ldots, y_n]/\langle\Omega_n[y_1, \ldots, y_n]S_n\rangle
\end{equation}

which carry a triply graded action of $S_n$.

Conjecture 5.2. [68] (Zabrocki Conjecture) The triply graded Frobenius image of $SDR_n$ is given by

\begin{equation}
\text{grFrob}(SDR_n; q, t, z) = \sum_{k=1}^{n} z^{n-k} \cdot \Delta_{e_{k-1}}' e_n
\end{equation}

where $q$ tracks $x$-degree, $t$ tracks $y$-degree, and $z$ tracks $\theta$-degree.

Setting the $y$-variables to zero in the Zabrocki Conjecture recovers the Fields Conjecture 5.1. Setting the $\theta$-variables to zero recovers Haiman’s expression $\text{grFrob}(DR_n; q, t) = \nabla e_n$ for the bigraded Frobenius image of the diagonal coinvariant ring [38]. As the only known proof of $\text{grFrob}(DR_n; q, t) = \nabla e_n$ involves isospectral Hilbert schemes, a proof of the Zabrocki Conjecture is likely to involve substantial geometric innovation.
5.2. Superspace Vandermondes and Harmonics. While the coinvariant quotients $SR_n$ and $SDR_n$ have so far resisted analysis, there is a related family of modules defined using superspace which is better understood. To motivate these modules, we recall an alternative perspective on quotients of $\mathbb{C}[x_1, \ldots, x_n]$.

Given $f = f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$, let $\partial f$ be the differential operator obtained by replacing each variable $x_i$ with the corresponding partial derivative $\partial/\partial x_i$:

$$\partial f := f(\partial/\partial x_1, \ldots, \partial/\partial x_n).$$

The operators $\partial f$ give rise to a bilinear pairing $\langle - , - \rangle$ on $\mathbb{C}[x_1, \ldots, x_n]$ defined on monomials by

$$\langle m, m' \rangle := \text{constant term of } (\partial m) m'.$$

If $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is a homogeneous ideal, we have a direct sum decomposition $\mathbb{C}[x_1, \ldots, x_n] = I \oplus I^\perp$ where

$$I^\perp = \{ f \in \mathbb{C}[x_1, \ldots, x_n] : \langle f, g \rangle = 0 \text{ for all } g \in I \}$$

and the composite map

$$I^\perp \hookrightarrow \mathbb{C}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{C}[x_1, \ldots, x_n]/I$$

is an isomorphism of graded vector spaces. The space $I^\perp$ is the harmonic space (or Macaulay inverse system) of the quotient $\mathbb{C}[x_1, \ldots, x_n]/I$. Although $I^\perp$ is almost never closed under multiplication, it allows for the study of $\mathbb{C}[x_1, \ldots, x_n]/I$ without the use of cosets.

The harmonic space of the classical coinvariant ring $R_n$ has a nice description. The Vandermonde determinant $\delta_n \in \mathbb{C}[x_1, \ldots, x_n]$ is given by

$$\delta_n := \varepsilon_n \cdot (x_1^n x_2^{n-2} \cdots x_{n-1}^1 x_n^0)$$

where $\varepsilon_n := \sum_{w \in S_n} \text{sign}(w) \cdot w \in \mathbb{C}[S_n]$ is the antisymmetrizing element of the symmetric group algebra. The harmonic space $V_n := I_n^\perp$ of the coinvariant quotient $R_n = \mathbb{C}[x_1, \ldots, x_n]/I_n$ is the smallest linear subspace of $\mathbb{C}[x_1, \ldots, x_n]$ containing $\delta_n$ which is closed under the operators $\partial/\partial x_i$ for all $1 \leq i \leq n$. The harmonic spaces of the generalized coinvariant rings $R_{n,k}, R_\lambda$, and $R_{n,\lambda,s}$ also admit descriptions in terms of Vandermondes [58].

Harmonic theory extends to the superspace ring $\Omega_n$. For $1 \leq i \leq n$, let $\partial/\partial \theta_i : \Omega_n \to \Omega_n$ by the $\mathbb{C}[x_1, \ldots, x_n]$-linear operator which acts on $\theta$-monomials by

$$\partial/\partial \theta_i : \theta_{j_1} \cdots \theta_{j_r} = \begin{cases} (-1)^{s-1} \theta_{j_1} \cdots \tilde{\theta}_{j_{s}} \cdots \theta_{j_r} & \text{if } i = j_s \\ 0 & \text{if } i \neq j_1, \ldots, j_r \end{cases}$$

for all distinct indices $1 \leq j_1, \ldots, j_r \leq n$ where $\tilde{\theta}_j$ denotes omission. The map $\partial/\partial \theta_i$ is called a contraction operator. The $\partial/\partial x_i$ and $\partial/\partial \theta_i$ satisfy the same relations as the superspace generators: we have

$$\partial/\partial x_i \partial/\partial x_j = \partial/\partial x_j \partial/\partial x_i \quad \partial/\partial x_i \partial/\partial \theta_j = \partial/\partial \theta_j \partial/\partial x_i \quad \partial/\partial \theta_i \partial/\partial \theta_j = -\partial/\partial \theta_j \partial/\partial \theta_i$$

for $1 \leq i, j \leq n$. Consequently, for any $f = f(x_1, \ldots, x_n, \theta_1, \ldots, \theta_n) \in \Omega_n$ we have an endomorphism $\partial f : \Omega_n \to \Omega_n$ given by

$$\partial f = f(\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial \theta_1, \ldots, \partial/\partial \theta_n).$$

The operators $\partial f$ give rise to an inner product on $\Omega_n$ characterized by

$$\langle m, m' \rangle = \text{constant term of } (\partial m) m'$$

whenever $m, m' \in \Omega_n$ are superspace monomials (expressions of the form $x_1^{a_1} \cdots x_n^{a_n} \theta_{j_1} \cdots \theta_{j_r}$). If $I \subset \Omega_n$ is a bihomogeneous ideal, the composite

$$I^\perp \hookrightarrow \Omega_n \twoheadrightarrow \Omega_n/I$$
is an isomorphism where \( I^\perp = \{ f \in \Omega_n : \langle f, g \rangle = 0 \text{ for all } g \in I \} \). As in the commutative setting, the space \( I^\perp \) is called the harmonic space of \( \Omega_n/I \).

The harmonic space of the superspace coinvariant ring \( SR_n \) may be described as follows. For \( j \geq 1 \), let \( d_j : \Omega_n \rightarrow \Omega_n \) be the linear operator

\[
(5.22) \quad d_j(f) := \sum_{i=1}^n \theta_i \cdot \frac{\partial^j f}{\partial x_i^j}.
\]

When \( j = 1 \), the operator \( d_1 \) is the usual total derivative on differential forms. These ‘higher’ total derivatives conjecturally generate the superspace harmonics. The following result was conjectured by the Fields Group (personal communication) and proven by Rhoades and Wilson [59].

**Theorem 5.3. (Rhoades-Wilson [59]) (Operator Theorem)** The harmonic space to the superspace coinvariant ring \( SR_n \) is the smallest linear subspace of \( \Omega_n \) which is closed under the operators \( (\partial/\partial x_1), \ldots, (\partial/\partial x_n) \) and contains the \( 2^{n-1} \) superspace elements

\[
d_1^{\epsilon_1} d_2^{\epsilon_2} \cdots d_{n-1}^{\epsilon_{n-1}} (\delta_n) \in \Omega_n
\]

where \( \delta_n \in \mathbb{C}[x_1, \ldots, x_n] \) is the Vandermonde determinant and \( \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1} \in \{0, 1\} \).

Haiman proved [38] an analogous result to Theorem 5.3 which describes the harmonic space of the diagonal coinvariant ring \( DR_n \) in terms of polarization operators. Theorem 5.3 gives a model for \( SR_n \) involving honest superspace elements rather than cosets, thus potentially giving a method for finding the bigraded \( S_n \)-structure of \( SR_n \) and proving the Fields Conjecture. Josh Swanson has made substantial progress in uncovering this module structure of this harmonic space by relating the superspace elements \( d_1^{\epsilon_1} d_2^{\epsilon_2} \cdots d_{n-1}^{\epsilon_{n-1}} (\delta_n) \) to Tanisaki quotients [64], but much remains to be done.

Rhoades and Wilson [57] exploited the anticommuting \( \theta \)-variables to generalize the Vandermonde \( \delta_n \) to \( \Omega_n \) and give an alternative to the harmonic space of Conjecture 5.3. For \( k \leq n \), the superspace Vandermonde \( \delta_{n,k} \in \Omega_n \) is given by

\[
(5.23) \quad \delta_{n,k} := \varepsilon_n \cdot \left( x_1^{k-1} \cdots x_{n-k}^{k-1} x_{n-k+1}^{k-2} \cdots x_{n-k}^0 \cdot \theta_1 \cdots \theta_{n-k} \right)
\]

so that, for example, we have

\[
\delta_{3,2} = \varepsilon_3 \cdot (x_1 x_2 \theta_1) = x_1 x_2 \theta_1 - x_1 x_2 \theta_2 - x_1 x_3 \theta_1 + x_2 x_3 \theta_2 + x_1 x_3 \theta_3 - x_2 x_3 \theta_3.
\]

Thanks to the \( \theta \)-variables, the operator \( \varepsilon_n \) does not annihilate the generating monomial of \( \delta_{n,k} \) and we obtain nonzero elements of \( \Omega_n \).

The monomials in \( \delta_{n,k} \) appear among those in the elements \( d_1 d_2 \cdots d_{n-k}(\delta_n) \) of Conjecture 5.3 but \( \delta_{n,k} \neq d_1 d_2 \cdots d_{n-k}(\delta_n) \) as elements of \( \Omega_n \). In keeping with the harmonic paradigm, we use the \( \delta_{n,k} \) as seeds to grow modules.

**Definition 5.4.** Let \( V_{n,k} \) be the smallest linear subspace of \( \Omega_n \) which contains \( \delta_{n,k} \) and is closed under the operators \( \partial/\partial x_i \) for \( 1 \leq i \leq n \).

The spaces \( V_{n,k} \) are concentrated in \( \theta \)-degree \( n - k \). Considering \( x \)-degree alone gives \( V_{n,k} \) the structure of a singly graded \( S_n \)-module. Rhoades and Wilson proved [57] that

\[
(5.24) \quad \text{grProb}(V_{n,k}; q) = \Delta_{k-1}^t \varepsilon_n |_{t=0}
\]

and conjectured that the composite

\[
(5.25) \quad \bigoplus_{k=1}^n V_{n,k} \hookrightarrow \Omega_n \twoheadrightarrow SR_n
\]

is an isomorphism. If (5.25) is bijective, the Fields Conjecture would follow. Superspace Vandermonde can also be used to define modules inside \( \Omega_n[y_1, \ldots, y_n] \).
Definition 5.5. Let \( \mathbb{V}_{n,k} \) be the smallest linear subspace of \( \Omega_n[y_1, \ldots, y_n] \) which

- contains the superspace Vandermonde \( \delta_{n,k} \) (in the \( x \)-variables and \( \theta \)-variables alone),
- is closed under the differentiation operators \( \partial/\partial x_i \) and \( \partial/\partial y_i \) for \( 1 \leq i \leq n \), and
- if closed under the polarization operator \( y_1 \partial^j / \partial x_1^j + \cdots + y_n \partial^j / \partial x_n^j \) for each \( j \geq 1 \).

The polarization operators in Definition 5.5 lower \( x \)-degree by \( j \) and raise \( y \)-degree by 1. The space \( \mathbb{V}_{n,k} \) is closed under the action of \( S_n \) and concentrated in \( \theta \)-degree \( n - k \). By considering \( x \)-degree and \( y \)-degree, the space \( \mathbb{V}_{n,k} \) is a doubly graded \( S_n \)-module. Its bigraded \( S_n \)-structure is conjecturally governed by delta operators.

Conjecture 5.6. (Rhoades-Wilson [57]) We have \( \text{grFrob}(\mathbb{V}_{n,k}; q, t) = \Delta'_e \cdot e_n \). Furthermore, the composite map

\[
\bigoplus_{k=1}^n \mathbb{V}_{n,k} \hookrightarrow \Omega_n[y_1, \ldots, y_n] \twoheadrightarrow SDR_n
\]

is bijective.

Conjecture 5.6 implies Zabrocki’s Conjecture 5.2. Since the first assertion of Conjecture 5.6 is known at \( t = 0 \) but the corresponding coinvariant statement (the Fields Conjecture 5.1) remains open, it appears that Vandermondes could give an easier road to delta operator modules than coinvariants.

5.3. The Theta and Boson-Fermion coinvariant conjectures. The modules in the Fields and Zabrocki Conjectures suggest a natural generalization. Given integers \( n, r, \ell \geq 0 \), consider an \( r \times n \) matrix \( X_{r \times n} = (x_{i,j}) \) of commuting variables and an \( \ell \times n \) matrix \( \Theta_{\ell \times n} = (\theta_{i,j}) \) of anticommuting variables and let

\[
\mathcal{S}(n; r, \ell) := \mathbb{C}[X_{r \times n}] \otimes \wedge\{\Theta_{\ell \times n}\}
\]

be the \( \mathbb{C} \)-algebra generated by these variables. This is a symmetric algebra of rank \( r \times n \) tensor an exterior algebra of rank \( \ell \times n \). The symmetric group \( S_n \) acts on the columns of the matrices \( X_{r \times n} \) and \( \Theta_{\ell \times n} \). We let \( I \subset \mathcal{S}(n; r, \ell) \) be the ideal generated by \( S_n \)-invariants with vanishing constant terms and write

\[
\mathcal{R}(n; r, \ell) := \mathcal{S}(n; r, \ell)/I
\]

for the corresponding quotient ring. The ring \( \mathcal{R}(n; r, \ell) \) is multigraded, with \( r \) flavors of commuting graded and \( \ell \) flavors of anticommuting grading. We have met a number of special cases of \( \mathcal{R}(n; r, \ell) \).

- When \( r = 1 \) and \( \ell = 0 \) we have the classical coinvariant ring \( \mathcal{R}(n; 1, 0) = R_n \).
- When \( r = 2 \) and \( \ell = 0 \) we have the diagonal coinvariants \( \mathcal{R}(n; 2, 0) = DR_n \).
- When \( r = \ell = 1 \) we have the superspace coinvariants \( \mathcal{R}(n; 1, 1) = SR_n \).
- When \( r = 2 \) and \( \ell = 1 \) we have the module \( \mathcal{R}(n; 2, 1) = SDR_n \).

The ring \( \mathcal{R}(n; r, \ell) \) has the structure of a multigraded \( S_n \)-module. D’Adderio, Iraci, and Vanden Wyngaerd [15] introduced the \textit{theta operators} on the ring of symmetric functions and used them to conjecturally describe (part of) the modules \( \mathcal{R}(n; 2, \ell) \) in the case of \( r = 2 \) sets of commuting variables. We define their operators and state their conjecture.

Let \( \Pi : \Lambda \rightarrow \Lambda \) be the Macdonald eigenoperator defined by

\[
\Pi : \tilde{H}_\mu(x; q, t) \mapsto \prod_{(i,j)} (1 - q^{-1} t^{-1}) \cdot \tilde{H}_\mu(x; q, t)
\]

where the product ranges over the coordinates \( (i,j) \neq (1,1) \) of cells in the Young diagram of \( \mu \). For example, if \( \mu = (3, 2) \) we fill the Young diagram as in

\[
\begin{array}{ccc}
\cdot & q^2 & t^2 \\
\cdot & t & qt
\end{array}
\]
so that
\[ \Pi : \tilde{H}_{(3,2)}(x; q, t) \mapsto (1 - q)(1 - q^2)(1 - t)(1 - qt) \cdot \tilde{H}_{(3,2)}(x; q, t). \]

Omitting the cell \((1, 1)\) in the eigenvalue of \(\tilde{H}_\mu(x; q, t)\) ensures that the operator \(\Pi : \Lambda \to \Lambda\) is invertible.

Given a symmetric function \(F \in \Lambda\), the theta operator\(^2\) \(\Theta_F : \Lambda \to \Lambda\) is defined by
\[
(5.29) \quad \Theta_F : G \mapsto \Pi \cdot F^* \cdot \Pi^{-1} \cdot G
\]
for any symmetric function \(G\). Here \(F^*\) denotes multiplication by the plethystically transformed version \(F \left[ \frac{x}{(1-q)(1-t)} \right]\) of the symmetric function \(F\) labeling \(\Theta_F\).

The theta operators were introduced in \([15]\) to give a compositional refinement of the symmetric function side \(\Delta_{\ell+1} e_n\) of the Delta Conjecture involving the \(\nabla\) operator. This refinement was the basis of the D’Adderio-Mellit proof \([16]\) of the Rise Version of the Delta Conjecture. Theta operators are also expected to have ties to coinvariant theory.

**Conjecture 5.7.** (D’Adderio-Iraci-Vanden Wyngaerd [15]) Let \(j = (j_1, j_2)\) be a pair of nonnegative integers with sum \(j_1 + j_2 < n\). The piece \(R(n; 2, 2)_j\) of \(\theta\)-bidegree \(j\) is, by considering commuting degrees, a bigraded \(S_n\)-module. We have
\[
grFrob(R(n; 2, 2)_j; q, t) = \Theta_{e_{j_1}} \Theta_{e_{j_2}} \nabla e_{n - (j_1 + j_2)}. \]

We have (see [40, 41]) that \(R(n; 2, 2)_j = 0\) whenever \(j_1 + j_2 \geq n\) so that Conjecture 5.7 gives a complete description of the multigraded \(S_n\)-structure of \(R(n; 2, 2)\). However, in the case of \(\ell > 2\) fermionic sets of variables there are nonzero pieces \(R(n; 2, \ell)_j\) for which \(j_1 + \cdots + j_\ell \geq n\). D’Adderio et. al. show that Conjecture 5.7 implies the Zabrocki Conjecture when \(\ell = 1\).

The \(\ell = 2\) case of Conjecture 5.7 is especially intriguing due to the symmetry of the ring \(R(n; 2, 2)\). The quotient of \(R(n; 2, 2)\) obtained by setting the anticommuting variables to zero is the diagonal coinvariant ring \(DR_n\); the corresponding specialization of Conjecture 5.7 is equivalent to Haiman’s [38] formula \(grFrob(DR_n; q, t) = \nabla e_n\). The ‘fermionic diagonal coinvariant ring’ \(FDR_n\) obtained from \(R(n; 2, 2)\) obtained by setting the commuting variables to zero was introduced and studied by Kim and Rhoades [41]. In terms of combinatorics, the ring \(FDR_n\) encodes a kind of crossing resolution in set partitions of \(\{n\}\); see [42]. Iraci, Rhoades, and Romero [40] proved the case of Conjecture 5.7 involving \(FDR_n\).

Returning to the general setting of \(R(n; r, \ell)\), F. Bergeron has a remarkable conjecture relating these modules when \(r \to \infty\) or \(\ell \to \infty\). In order to state his conjecture, we need to consider additional structure on \(R(n; r, \ell)\).

The action of the general linear groups \(GL_r(\mathbb{C})\) and \(GL_\ell(\mathbb{C})\) on the rows of \(X_{r \times n}\) and \(\Theta_{\ell \times n}\) gives \(R(n; r, \ell)\) the structure of a \(GL_r(\mathbb{C}) \times GL_\ell(\mathbb{C})\)-module. Since the actions of \(S_n\) and \(GL_r(\mathbb{C}) \times GL_\ell(\mathbb{C})\) commute, we have an action of the product group
\[
(5.30) \quad G(n; r, \ell) := S_n \times GL_r(\mathbb{C}) \times GL_\ell(\mathbb{C})
\]
on \(R(n; r, \ell)\).

We encode \(G(n; r, \ell)\)-structure of \(R(n; r, \ell)\) as a formal power series. More precisely, we define the character of \(R(n; r, \ell)\) by
\[
(5.31) \quad \text{ch} \ R(n; r, \ell) := \sum_{\mathbf{i}, \mathbf{j}} \text{Frob}(R(n; r, \ell)_{\mathbf{i}, \mathbf{j}}) \cdot q_1^{i_1} \cdots q_r^{i_r} \cdot z_1^{j_1} \cdots z_\ell^{j_\ell}
\]
where the sum is over all \(r\)-tuples \(\mathbf{i} = (i_1, \ldots, i_r)\) and \(\ell\)-tuples \(\mathbf{j} = (j_1, \ldots, j_\ell)\) of nonnegative integers and \(R(n; r, \ell)_{\mathbf{i}, \mathbf{j}}\) is the piece of \(R(n; r, \ell)\) with commuting multidegree \(\mathbf{i}\) and anticommuting \(\mathbf{j}!\) not to be confused with the anticommuting \(\theta\)-variables!
multidegree \( j \). By the structure of polynomial representations of general linear groups, we have an expansion
\[
(5.32) \quad \text{ch} \mathcal{R}(n; r, \ell) = \sum_{\lambda, \mu, \nu} d_{\lambda, \mu, \nu} \cdot s_{\lambda}(x) \cdot s_{\mu}(q_1, \ldots, q_r) \cdot s_{\nu}(z_1, \ldots, z_{\ell})
\]
where \( \lambda \) ranges over partitions of \( n \), \( \mu \) and \( \nu \) range over arbitrary partitions, and the \( d_{\lambda, \mu, \nu} \) are nonnegative integers.

Temporarily consider the rings \( \mathcal{R}(n; r, 0) \) involving commuting variables alone. F. Bergeron proved \([2]\) that the limit
\[
(5.33) \quad \mathcal{E}_n(x, q) := \lim_{r \to \infty} \text{ch} \mathcal{R}(n; r, 0)
\]
converges in the ring of formal power series where \( q = (q_1, q_2, \ldots) \) is an infinite alphabet tracking commuting multidegrees. We may write
\[
(5.34) \quad \mathcal{E}_n(x, q) = \sum_{\lambda, \mu} b_{\lambda, \mu} \cdot s_{\lambda}(x) \cdot s_{\mu}(q)
\]
for nonnegative integers \( b_{\lambda, \mu} \).

Although \( \mathcal{E}_n \) does not explicitly involve anticommuting information, F. Bergeron conjectured \([4]\) that all of the characters \( \text{ch} \mathcal{R}(n; r, \ell) \) are determined by \( \mathcal{E}_n \) alone. Let \( z = (z_1, z_2, \ldots) \) be an infinite alphabet of anticommutative tracking variables.

**Conjecture 5.8.** (Combinatorial Supersymmetry \([4]\)) For any \( r, \ell \geq 0 \), the character \( \text{ch} \mathcal{R}(n; r, \ell) \) may be obtained from the ‘universal’ symmetric function
\[
(5.35) \quad \sum_{\lambda, \mu, \nu, \rho} b_{\lambda, \mu} \cdot c_{\nu, \rho}^{\mu} \cdot s_{\lambda}(x) \cdot s_{\nu}(q) \cdot s_{\rho}(z)
\]
by evaluating \( q_i = 0 \) for \( i > r \) and \( z_j = 0 \) for \( j > \ell \). Here the \( b_{\lambda, \mu} \) are as in the expression \((5.34)\) for \( \mathcal{E}_n(x, q) \) and \( c_{\nu, \rho}^{\mu} \) is a Littlewood-Richardson coefficient.

Notice that the partition \( \rho \) is unconjugated in the Littlewood-Richardson coefficient \( c_{\nu, \rho}^{\mu} \) in \((5.35)\) but is conjugated in the Schur function \( s_{\rho}(z) \). The polynomial \((5.35)\) may be expressed more succinctly in plethystic notation as \( \mathcal{E}_n(x, q - \varepsilon z) = \sum_{\lambda, \mu} b_{\lambda, \mu} \cdot s_{\lambda}(x) \cdot s_{\mu}(q - \varepsilon \cdot z) \).

**Remark 5.9.** As stated, Conjecture 5.8 implies that the characters \( \text{ch} \mathcal{R}(n; r, \ell) \) of the general modules \( \mathcal{R}(n; r, \ell) \) are determined by those of the ‘purely commuting’ modules \( \mathcal{R}(n; r, 0) \). However, there is an equivalent formulation involving the ‘purely anticommuting’ modules \( \mathcal{R}(n; 0, \ell) \). More precisely, let
\[
(5.36) \quad \mathcal{F}_n(x, z) := \lim_{\ell \to \infty} \text{ch} \mathcal{R}(n; 0, \ell)
\]
be the limit of the ‘purely anticommuting characters’. As in the commuting case, this limit exists and there are nonnegative integers \( d_{\lambda, \mu} \) such that
\[
(5.37) \quad \mathcal{F}_n(x, z) = \sum_{\lambda, \mu} d_{\lambda, \mu} \cdot s_{\lambda}(x) \cdot s_{\mu}(z).
\]
Motivated by the relationship between \((5.35)\) and \( \mathcal{E}_n(x, q) \), we consider the symmetric function
\[
(5.38) \quad \sum_{\lambda, \mu, \nu, \rho} d_{\lambda, \mu} \cdot c_{\nu, \rho}^{\mu} \cdot s_{\lambda}(x) \cdot s_{\nu}(q) \cdot s_{\rho}(z)
\]
where the partition \( \nu \) is unconjugated in \( c_{\nu, \rho}^{\mu} \) but conjugated in \( s_{\nu}(q) \). Conjecture 5.8 is equivalent to the assertion that, for any \( r, \ell \geq 0 \), the symmetric function \( \text{ch} \mathcal{R}(n; r, \ell) \) is obtained from \((5.38)\) by evaluating \( q_i = 0 \) for \( i > r \) and \( z_j = 0 \) for \( j > \ell \).
At a high level, Remark 5.9 says that the anticommuting modules $\mathcal{R}(n; 0, \ell)$ should ‘contain the same information as’ the commuting modules $\mathcal{R}(n; r, 0)$ (and, in particular, the diagonal coinvariant ring $DR_n = \mathcal{R}(n; 2, 0)$). Thus, the modules $\mathcal{R}(n; 0, \ell)$ will likely prove very challenging to study as $\ell \to \infty$.

Another remarkable conjecture of F. Bergeron’s states that the symmetric function $E_n(x, q)$ “contains the data of” symmetric functions appearing in the Delta Conjecture. For any symmetric function $F$, let $F^\perp : \Lambda \to \Lambda$ be the adjoint operation to multiplication by $F$ under the (nondegenerate) Hall inner product. The map $F^\perp$ is characterized by

\begin{equation}
\langle F^\perp G, H \rangle = \langle G, FH \rangle \quad \text{for all } G, H \in \Lambda.
\end{equation}

**Conjecture 5.10.** (Skewing Conjecture [3]) For any $n = a + b + 1$, we have

\begin{equation}
\Delta'_{e_b} e_n = \left[ e_b(q)^\perp E_n(x, q) \right]_{q \to (q, t, 0, 0, \ldots)}
\end{equation}

where the skewing operator $e_b(q)^\perp$ acts on the grading $q$-variables alone.

At a combinatorial level, the action of $e_b(q)^\perp$ on a Schur function $s_\lambda(q)$ in the $q$-variables is a sum over Schur functions $s_\mu(q)$ such that $\lambda/\mu$ is a skew diagram of size $b$ in which every row has at most one box. Algebraically, the action of $e_b(q)^\perp$ on a $GL_\infty(\mathbb{C})$-module corresponds to restriction to the ‘parabolic subgroup’ $GL_b(\mathbb{C}) \times GL_\infty(\mathbb{C})$, and then considering the ($\{1d_1\} \times GL_\infty(\mathbb{C})$)-action on the $(1^b)$-weight space of $GL_b(\mathbb{C})$. Even though the formula in Conjecture 5.10 relates symmetric functions in the two parameters $q, t$ alone, more parameters $q_1, q_2, \ldots$ are necessary before specialization for the formula to hold.

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