On decoding algorithms for polar codes
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Abstract
We survey the known list decoding algorithms for polar codes and compare their complexity.

Index terms: Polar codes; Reed-Muller codes; successive cancellation decoding.

1 A brief survey of recursive decoding algorithms
Successive cancellation (SC) decoding was considered in [1] for general Reed-Muller codes \( RM(r, m) \) of order \( r \) and dimension \( m \). It was also proposed in [1] to set to zeros those information bits that are the least protected in SC decoding. Simulation results of [1] show that the resulting subcodes with frozen bits significantly outperform the original codes \( RM(r, m) \). Subsequent papers [2, 3] extend SC technique to the list decoding of RM codes and their bit-frozen subcodes. Simulation results of these papers show that the optimal selection of the frozen bits in RM codes brings SC list decoding close to the maximum likelihood decoding for the code lengths \( n \).

A breakthrough in this area was achieved by E. Arikan [6], who proved that the bit-frozen subcodes of the full code \( RM(m, m) \) - now well known as polar codes - achieve the channel capacity of a symmetric memoryless channel as \( m \to \infty \). Paper [6] also employs a novel analytical technique and reveals new properties of probabilistic recursive processing, such as bit polarization. We also note that the specific choice of the maximal order \( r = m \) is immaterial in this case since the results of [6] hold for the optimized bit-frozen subcodes of any code \( RM(r, m) \) of rate \( R \to 1 \), in particular if \( r/m > 1/2 \) for \( m \to \infty \).

SC list decoding was later considered in [7]. This paper cites a similar algorithm of [3] but relates the algorithm of [5] to RM codes only. This is incorrect. All papers [2]-[5] address list decoding of the optimized bit-frozen subcodes, and all emphasize large improvements that these subcodes achieve over the original RM codes. Paper [7] also incorrectly asserts that the recursive processing of \( L \) codewords in [5] may require \( n^2 \log n \) operations per one codeword, as opposed to only \( n \log n \) operations for \( L = 1 \). In fact, SC list decoding of [2]-[5] yields complexity of \( n \log n \) per one codeword and the overall complexity has the order of \( L n \log n \) for both RM codes and their subcodes.

In summary, papers [2, 5] and [7] use a similar decoding algorithm and apply it to the same class of the bit-frozen subcodes of RM codes. On the other hand, we also note that the design of polar codes in [7] complements the earlier constructions of polar codes by using the precoded information blocks and the fast analytical technique of [8], which gives the output bit error rates without any simulation.

Below, we discuss recursive design and decoding of polar codes in more detail.

2 Recursive design of RM and polar codes
We first describe some recursive properties of RM or polar codes similarly to papers [2, 5]. Consider boolean polynomials \( f(x) \) of degree \( r \) or less in \( m \) binary variables \( x_1, \ldots, x_m \), where \( r \leq m \). Vectors \( x = (x_1, \ldots, x_m) \) will mark the positions of our code. We also use short notation \( x_{i \mid j} = (x_i, \ldots, x_j) \) for a punctured vector \( x \), where \( i \leq j \). Each map \( f(x) : \mathbb{F}_2^m \to \mathbb{F}_2 \) generates a codeword \( c = c(f) \) of a code \( RM(r, m) \). Below, we take any sequence \( (i_1, \ldots, i_m) \in \mathbb{F}_2^m \) and describe the recursive decomposition

\[
f(x) = f_0(x_{2 \mid m}) + x_1 f_1(x_{2 \mid m}) = \ldots = \sum_{i_1, \ldots, i_m} x^{i_1}_{1} \cdots x^{i_m}_{m} f_{i_1,\ldots,i_m}(x_{i+1 \mid m}) = \ldots = \sum_{i_1, \ldots, i_m} f_{i_1,\ldots,i_m} x^{i_1}_{1} \cdots x^{i_m}_{m}
\]

\[\tag{1}
\]

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Two polynomials $f_0(x_2|m)$ and $x_1f_1(x_2|m)$ derived in the first step generate codewords $(c_0, c_0)$ and $(0, c_1)$ in $F_2^n$. For each $i_1 = 0, 1$, the codeword $c_{i_1}$ belongs to the code $RM(r - i_1, m - 1)$. This yields the Plotkin construction $c = c_0, c_0 + c_1$ of code $RM(r, m)$. Similarly, any step $\ell = 2, ..., m$ decomposes the polynomial $f(x)$ with respect to various monomials $x_1^\ell \cdot \cdots \cdot x_m^\ell$. We define these monomials using the binary strings $\xi_{1|\ell} = i_1, ..., i_\ell$, which we call binary paths of length $\ell$. For each path $\xi_{1|\ell}$, the remaining part $f_{i_1, ..., i_\ell}(x_{\ell+1}|m)$ of decomposition $\mathbf{I}$ defines some codeword $c_{i_1, ..., i_\ell}$ of length $2^{m-\ell}$. Finally, any full path $\xi = (i_1, ..., i_m)$ of step $m$ defines the single monomial

$$x_\xi \equiv x_1^{i_1} \cdot \cdots \cdot x_m^{i_m}$$

which has some coefficient $f(\xi) = f_{i_1, ..., i_m} = 0, 1$. Note that any path $\xi$ that ends with an information bit $f(\xi) = 1$ gives some vector $c(\xi) \equiv c(x_\xi)$ of length $n$ and weight $2^{m-w(\xi)}$, where $w(\xi)$ is the Hamming weight of the string $\xi$. If $f(\xi) = 0$, then $c(\xi) = 0$. RM codes $RM(r, m)$ include only $k(r,m)$ paths of weight $w(\xi) \leq r$, where $k(r,m)$ is the dimension of the code $RM(r, m)$.

Decomposition $\mathbf{I}$ is also shown in Fig. 1 and 2. Here the full code $RM(4,4)$ is depicted in Fig. 1. Each decomposition step $\ell = 1, ..., 4$ is marked by the splitting monomial $x_\xi^\ell$. For example, path $\xi = 0110$ gives the coefficient $f_{0110}$ associated with the monomial $x_\xi \equiv x_2x_3$.

Fig. 2 depicts code $RM(2,5)$. Here we only include all paths $\xi$ of weight $w(\xi) \leq 2$. Note that any two paths $\xi_{1|\ell}$ entering some node have the same weight $w$ and generate the same code $RM(r - w, m - \ell)$ on their extensions $\xi_{1|\ell}, i_{\ell+1}, ..., i_m$. For example, path $\xi = 01100$ proceeds from $RM(2,5)$ to the single bit $RM(0,0)$ via nodes $RM(2,4), RM(1,3), RM(0,2)$, and $RM(0,1)$.

Now consider some subset of $N$ paths $T$. Then we encode $N$ information bits via their paths and obtain codewords $c(T) = \sum_{\xi \in T} c(\xi)$. These codewords form a linear code $C(m, T)$. Note also that at any level $\ell$ and at any node $\xi_{1|\ell}$, encoding only needs to add two codewords of level $\ell + 1$ entering this node. Thus, encoding performs the order of $2^{m-\ell}$ operations on each of $2^\ell$ nodes $\xi_{1|\ell}$ and has the overall complexity of $n\log_2 n$ summed up over all levels $\ell = 1, ..., m$.

## 3 Recursive decoding algorithms

**Recursive decoding of RM codes.** Consider a discrete memoryless channel (DMC) $W$ with inputs $\pm 1$ defined by the map $x \rightarrow (-1)^x$ for $x = 0, 1$. Then we define the codewords $c = (u, uv)$ of a code $RM(r, m)$, where vector $uv$ is the component-wise product of vectors $u$ and $v$ with symbols $\pm 1$. For any codeword $c$, let $y_{0j}, y_{1j} \in \mathbb{R}^n/2$ be the two output halves corrupted by noise. We use double index $i, j$ for any position $j = 1, ..., n/2$ in a half $i = 0, 1$. Define the posterior probability (PP) $q_{i,j} = \Pr\{c_{i,j} = 1 \mid y_{i,j}\}$ that 1 is sent in a position $i, j$. We will also use two related quantities, which we call “the offsets” $g_{i,j}$ and the
likelihoods \( h_{i,j} \):

\[
g_{i,j} = 2q_{i,j} - 1, \quad h_{i,j} = q_{i,j}/(1 - q_{i,j})
\]

(2)

Thus, we form vectors \( \mathbf{q} = (q_{i,j}) \), \( \mathbf{g} = (g_{i,j}) \) and \( \mathbf{h} = (h_{i,j}) \). The following recursive algorithm \( \Psi^m(\mathbf{q}) \) of [1, 2] performs SCD of information bits in codes \( RM(r, m) \) or their bit-frozen subcodes \( C(m, T) \). We first derive PP of symbols \( v_j \) in the \((\mathbf{u}, \mathbf{uv})\) construction:

\[
g^{(1)}_j \equiv \Pr\{v_j = 1 \mid q_{0,j}, q_{1,j}\}
\]

Namely, it is easy to verify that the offsets \( g^{(1)}_j \) of symbols \( q^{(1)}_j \) satisfy simple recalculation

\[
g^{(1)}_j = g_{0,j}g_{1,j}
\]

(3)

We may then apply some decoding algorithm \( \Psi^{m-1}_{r-1} \) to the vector \( \mathbf{q}^{(1)} \equiv (q^{(1)}_j) \) and obtain a vector \( \mathbf{v} \in RM(r - 1, m - 1) \) of length \( n/2 \). Then each half of the vector \( (\mathbf{y}_0, \mathbf{y}_1\mathbf{v}) \) forms a corrupted version of vector \( \mathbf{u} \) in the \((\mathbf{u}, \mathbf{uv})\) construction. As a result, every symbol \( u_j \) of vector \( \mathbf{u} \) has the likelihoods \( h_{0,j} \) and \( h_{1,j} \) on these halves. This gives the overall likelihood of every symbol \( u_j \):

\[
h^{(0)}_j = h_{0,j}(h_{1,j})^{\mathbf{v}_j}
\]

(4)

Given the vectors \( \mathbf{h}^{(0)} = (h^{(0)}_j) \) and \( \mathbf{q}^{(0)} \), we can now apply some algorithm \( \Psi^{m-1}_{r-1} \) and decode \( \mathbf{q}^{(0)} \) into a vector \( \mathbf{u} \in RM(r, m - 1) \). With respect to polar codes, observe that recalculation \( \mathbf{h}^{(0)} \) degrade the original channel, whereas recalculation \( \mathbf{q}^{(0)} \) upgrade it.

Recalculation \( \mathbf{h}^{(0)} \) and \( \mathbf{q}^{(0)} \) form the level \( \ell = 1 \) of SC decoding. We can also use recalculation \( \mathbf{h}^{(1)} \) and \( \mathbf{q}^{(1)} \) for vectors \( \mathbf{q}^{(1)} \) instead of decoding them. Then levels \( \ell = 2, \ldots, m \) are processed similarly, moving decoding back and forth along the paths of Fig. 1 or Fig. 2. For any current path \( \eta = \xi_{1|\ell} \), decoder has an input vector \( \mathbf{q}(\eta) \) that consists of \( 2^{m-\ell} \) PP. In essence, this vector represents the output channel of this path \( \eta \). Then we process the \( v \)-extension \( (\eta, 1) \) using recalculation \( \mathbf{h}^{(1)} \). After processing, the node \( (\eta, 1) \) returns its current output \( \mathbf{v}(\eta) \) to the node \( \eta \). Similarly, we then continue with recalculation \( \mathbf{h}^{(1)} \) for the \( u \)-extension \( (\eta, 0) \). Thus, \( v \)-extensions (marked with 1 on Fig. 1) precede the \( u \)-extensions, and all paths \( \xi_{1|\ell} \) are ordered lexicographically in each step.

Next, consider all full paths \( \xi(S), S = 1, \ldots, k(r, m) \). Every path \( \xi(S) \) ends with one information bit \( f(S) \equiv f(\xi(S)) \) and gives its likelihood \( q(S) = \Pr\{f(S) = 0 \mid \mathbf{q}\} \). We then choose the more reliable value for \( f(S) \). The result is the current sequence \( F(S) = f(1), \ldots, f(S) \) of the first \( S \) information bits. The decoding ends if \( S = k(r, m) \).

It is easy to verify that \( m \) decomposition steps give complexity \( 2n \log_2 n \). Indeed, any level \( \ell = 1, \ldots, m \) includes at most \( 2^\ell \) paths \( \eta = \xi_{1|\ell} \). Each path \( \eta \) recalculate vectors \( \mathbf{g}(\eta) \) and \( \mathbf{h}(\eta) \) of length \( 2^{m-\ell} \). Recalculations \( \mathbf{g}^{(0)}, \mathbf{h}^{(0)} \) on these vectors have complexity order of \( 2^{m-\ell} \). Thus, each level \( \ell \) of recursion requires the order of \( 2^m \) operations.

**Recursive decoding of polar codes.** Any subcode \( C(m, T) \) with \( N \) ordered paths \( \xi(1), \ldots, \xi(N) \) is decoded similarly. Here we simply drop all frozen paths \( \xi \notin T \) that give information bits \( f(\xi) \equiv 0 \). This gives the following algorithm:

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Algorithm \( \Psi(m, T) \) for code \( C(m, T) \).

Given: a vector \( \mathbf{q} = (q_{i,j}) \) of PP.
Take \( S = 1, \ldots, N \) and \( \ell = 1, \ldots, m \).
For a path \( \xi(S) = i_1(S), \ldots, i_m(S) \) in step \( \ell \) do:
  Apply recalculation \( \mathbf{h}^{(0)} \) if \( i_\ell(S) = 1 \)
  Apply recalculation \( \mathbf{h}^{(1)} \) if \( i_\ell(S) = 0 \).
Output the bit \( f(S) \) for \( \ell = m \).
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**Recursive list decoding.** For the bit-frozen subcodes \( C(m, T) \) of RM codes, list decoding version \( \Psi(m, T, L) \) of this algorithm was employed in [2, 5]. Consider processing of any path \( \xi(S) \). Then the algorithm already has the list of \( L \) most probable code candidates \( t = 1, \ldots, L \) obtained on the previous paths. Each candidate is defined by the sequence \( F_t(S - 1) = [f_t(1), \ldots, f_t(S - 1)] \) of \( S - 1 \) information
bits and by the current vector \(q_\ell\) of posterior probabilities derived by processing these \(S - 1\) paths. For each candidate \(t\), we then recalculate the vector \(q_\ell\) on the path \(\xi(S)\). This is similar to the case \(L = 1\). Namely, any intermediate node \(\eta = \xi_{1, m-1}(S)\) of the path \(\xi(S)\) is given \(L\) most probable vectors \(q_\ell(\eta)\) of length \(2^{m-\ell}\). If the path \(\xi(S)\) has the new bit \(i_\ell(S) = 1\), then we follow v-extension \((\eta, 1)\) and perform recalculation \(3\) for each vector \(q_\ell(\eta)\). Otherwise, path \(\eta\) receives \(L\) vectors \(v_\ell(\eta)\) from the v-extension \((\eta, 0)\) and proceeds with recalculation \(4\) on its u-extension \((\eta, 0)\).

Our recalculation are slightly different in the final step \(\ell = m\). Given the prefix \(\eta = \xi_{1, m-1}(S)\) of the path \(\xi(S)\), we continue with the same calculations \(3\) or \(4\) depending on the new symbol \(i_m(S)\). However, now we consider both values \(f_\ell(S) = 0, 1\) of a new information bit \(i_m(S)\). As a result, we obtain two posterior probabilities \(q_\ell(\eta)\) and \(q_\ell'(\eta)\) for each candidate \(t = 1, \ldots, L\) on the full path \(\xi(S)\). Thus, \(L\) candidates yield two presorted lists \(\{q_\ell(\eta)\}\) and \(\{q_\ell'(\eta)\}\). To proceed further, we select \(L\) most probable codewords in the combined list, which requires the order of \(O(L)\) operations.

The result is the new list of information bits \(F_\ell(S)\). Note that this list can exclude some candidates \(t\) but keep both values \(f_\ell(S) = 0, 1\) for some other \(t\) until we select the single most probable codeword in the end. In processing, we also keep the current posterior probabilities \(q_\ell(\xi(S))\), which will be used in the next steps for the path \(\xi(S+1)\). A more detailed description of this decoding algorithm is also given in \(5\).

Note that each level \(\ell\) includes at most \(2^\ell\) nodes \(\xi_{1, \ell}\) each of which processes \(2L\) vectors of length \(2^{m-\ell}\). Given some constant number \(c\) operations per code symbol, we only perform \(2cLn\) operations in step \(\ell\), and then relegate decoding to step \(\ell + 1\). Thus, we can bound complexity \(|\Psi_\ell(m, T, L)|\) of level \(\ell\) and the overall complexity \(|\Psi(m, T, L)|\) of the list decoding by the order of \(Ln \log_2 n\):

\[
|\Psi_\ell(m, T, L)| \leq |\Psi_{\ell+1}(m, T, L)| + 2cLn
\]

\[
|\Psi(m, T, L)| \leq \sum_{\ell=1}^{m} 2cLn = 2cLn \log_2 n
\]

This concludes our description of the list decoding algorithm.

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