Light polarization oscillations induced by photon-photon scattering

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We consider the Heisenberg-Euler action for an electromagnetic field in vacuum, which includes quantum corrections to the Maxwell equations induced by photon-photon scattering. We show that, in some configurations, the plane monochromatic waves become unstable, due to the appearance of secularities in the dynamical equations. These secularities can be treated using a multiscale approach, introducing a slow time variable. The amplitudes of the plane electromagnetic waves satisfy a system of ordinary differential nonlinear equations in the slow time. The analysis of this system shows that, due to the effect of photon-photon scattering, in the unstable configurations the electromagnetic waves oscillate periodically between left-hand-sided and right-hand-sided polarizations. Finally, we discuss the physical implications of this finding, and the possibility of disclosing traces of this effect in optical experiments.

INTRODUCTION

Despite the fact that the equations of the classical electromagnetic field in vacuum are linear, quantum corrections due to photon-photon scattering introduce nonlinear effects. Photon-photon scattering consists in the interaction of two photons $\gamma_a$ and $\gamma_b$ of wave vectors $\vec{k}$ and $\vec{b}$ that are scattered elastically, so that after the interaction the scattered photons $\gamma'_a$ and $\gamma'_b$ will have wave vectors $\vec{k}'$ and $\vec{b}'$. Of course, in order to scatter, the two colliding photons can not travel in the same direction, and this trivial fact is reflected in the properties of the nonlinear terms in the equations that we will study below.

The search for signatures of photon-photon scattering in optics is still an open issue \[1-25\], while indirect evidence for this process has been found in particle accelerations \[26-33\].

The quantum corrections due to photon-photon scattering were calculated a long time ago by Heisenberg and Euler \[34\], and extensively studied by other authors \[35-38\]. The effective Lagrangian of the electromagnetic field, obtained retaining only one electron loop corrections, is

$$L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \epsilon^2 \left[ (F_{\mu\nu} F^{\mu\nu})^2 - \frac{7}{16} \left( F_{\mu\nu} \tilde{F}^{\mu\nu} \right)^2 \right], \quad (1)$$

where $F_{\mu\nu} = A^{\mu,\nu} - A^{\nu,\mu}$ is the electromagnetic field \[51\], $A^\mu$ is the electromagnetic four-potential, $\tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$, and

$$\epsilon^2 = \frac{\alpha^2 (\hbar/m_e c)^3}{90 m_e c^2}, \quad (2)$$

where $\alpha = e^2/4\pi\epsilon_0 c \approx 1/137$ is the fine structure constant, $\epsilon_0$ the dielectric permeability of vacuum, and $m_e$ the electron mass. Such a Lagrangian is fit for this purpose when it is possible to neglect other quantum effects. For instance, for low energetic photons of energies $E_\gamma \ll m_e c^2$ particles creation is inhibited, and the photon-photon scattering is the only process inducing quantum correction to the Maxwell equations.

The terms $\propto \epsilon^2$ in the Lagrangian \[1\] take into account photon-photon scattering, and induce cubic corrections in the equations for the four-potential $A^\mu$. Since $\epsilon^2 \approx 4 \times 10^{-41} m^2 / J$, so that $\epsilon^2 F_{\mu\nu} F^{\mu\nu}$ is extremely small, such corrections are usually negligible with high accuracy. However, in many physical situations, tiny perturbations produce huge effects on a system, due to the action of hidden resonances.

A paradigmatic example of this fact is given by rogue waves, which are transient waves triggered by noise, appearing in several physical contexts, in particular in hydrodynamics \[39\], nonlinear optics \[40\], and Bose-Einstein condensates \[41\]. The simplest nonlinear model for the description of such a phenomenon is the integrable focusing Nonlinear Schrödinger Equation (NLS) $i \partial_t \psi + \partial_x^2 \psi + 2 |\psi|^2 \psi = 0 \[42\]$, with the nonlinear mechanism of the modulation instability \[43\] at the basis of the rogue waves generation (see also Refs. \[44\] for recent theoretical developments).

In this paper we show that the dynamics described by the Lagrangian \[1\] is unstable for some configurations of the electromagnetic field. However, this instability is quite different from the modulation instability \[43\] responsible for rogue waves generation, but it is not less interesting. In fact, we find that, in the unstable configurations, the electromagnetic waves oscillate periodically between left-hand-sided and right-hand-sided polarizations, due to the effect of photon-photon scattering.

We consider two plane counterpropagating \[52\] electromagnetic waves in vacuum, and we show that the nonlinear terms in \[1\] introduce resonant (or secular) corrections in the equations of the electromagnetic field. Thus, in a standard perturbative treatment, any small ($\sim \epsilon^2$) correction to the zeroth order solution of the modified Maxwell equations will explode at a finite time $T_i$, which is estimated as $T_i \sim 1/(\epsilon^2 c k < \rho >)$, where $k$ is the wavenumber of the electromagnetic waves, and $< \rho >$ their energy-density.
Furthermore, we show that, introducing a slow time variable, the secular terms can be treated in a multi-scale scheme. In fact, the amplitudes of the two counter-propagating waves satisfy a system of nonlinear coupled ordinary differential equations in the slow time. The analysis of this system shows that, for some initial conditions, nonlinear corrections have negligible effects, implying only small $\sim \epsilon^2$ corrections to the energy-momentum dispersion relation of the photons; while for other initial conditions, the effect of photon-photon scattering is unexpectedly relevant, consisting of a continuous oscillation in the polarization of the two beams between left-hand-sided and right-hand-sided components.

Finally, we discuss the physical implications of this finding, and we speculate on the possibility of detecting signatures of the polarization oscillations in cosmological observations and in optical experiments.

Without loss of generality, hereafter we will use the Lorentz gauge

$$\partial_\alpha A^\alpha = 0. \quad (3)$$

Starting from the Lagrangian (1), it is easy to show that the modified Maxwell equations for the electromagnetic four-potential $A^\alpha$ in the Lorentz gauge are

$$\square A^\alpha \left(1 + 8 \epsilon^2 F_{\mu\nu} F^{\mu\nu}\right) +$$

$$+ 8 \epsilon^2 \left[F^{\alpha\beta} \partial_\beta (F_{\mu\nu} F^{\mu\nu}) - \frac{7}{16} \tilde{F}^{\alpha\beta} \partial_\beta \left(F_{\mu\nu} \tilde{F}^{\mu\nu}\right)\right] = 0, \quad (4)$$

where $\square$ is the d’Alembertian operator.

In principle, the smallness of the parameter $\epsilon$ justifies a perturbative treatment of (4). However, as we will see below, a naive perturbative approach is doomed to failure, due to the presence of resonant terms.

At zeroth order in $\epsilon$, Eq. (4) reduces to the Maxwell equations in vacuum $\square A^\alpha = 0$, which can be solved exactly. Let us consider a zeroth order solution $A^{(0)\alpha}$ corresponding to a system of two plane electromagnetic waves propagating in the $x^3$ direction. Let us express the four-potential $A^{(0)\alpha}$ in the form

$$\left\{\begin{array}{l}
A^{(0)\alpha} = a^\alpha + b^\alpha + \text{c.c.}, \\
a^\alpha = \xi^\alpha e^{ikx}, \\
b^\alpha = \zeta^\alpha e^{ihx}
\end{array}\right. \quad (5)$$

where c.c. stands for complex conjugate and the four-dimensional polarization and wave vectors are constant and are given by

$$\left\{\begin{array}{l}
k = (k_0, 0, 0, k_3), \\
h = (h_0, 0, 0, h_3),
\end{array}\right. \quad \xi = (0, \xi^1, \xi^2, 0), \quad \zeta = (0, \zeta^1, \zeta^2, 0), \quad \xi^3 \equiv (0, 0, 1), \quad \zeta^3 \equiv \zeta^3, \quad \text{where } \zeta^\dagger \text{ and } \zeta^\ast \text{ are the vector and scalar products respectively and } \zeta^\ast \text{ is the complex conjugate of } \zeta. \quad (6)$$

with $|k_0/k_3| = |h_0/h_3| = 1$.

We can study the evolution of the small perturbations of the zeroth order solution (5). Let us consider a perturbative expansion of the four-potential in powers of $\epsilon$

$$A^\alpha = A^{(0)\alpha} + \epsilon^2 \delta A^{(2)\alpha}. \quad (7)$$

Inserting this expression in (4), one has the equations for the perturbations in the form

$$\square \delta A^{(2)\alpha} + B^\alpha = 0, \quad (8)$$

where we have defined

$$B^\alpha \equiv 8 \left[F^{\alpha\beta} \partial_\beta (F_{\mu\nu} F^{\mu\nu}) - \frac{7}{16} \tilde{F}^{\alpha\beta} \partial_\beta \left(F_{\mu\nu} \tilde{F}^{\mu\nu}\right)\right]. \quad (9)$$

The four-vector $B^\alpha$ in (9) must be evaluated on the solution (5). Direct calculation gives

$$F_{\mu\nu} F^{\mu\nu} = 2 \left[(A_{1,3})^2 - (A_{1,0})^2 + (A_{2,3})^2 + (A_{2,0})^2\right] =$$

$$= 4 (k_0 h_0 - k_3 h_3) \left[(\overline{\xi} \cdot \overline{\zeta}) e^{i(k+h)x} - (\overline{\zeta} \cdot \overline{\xi}) e^{i(k-h)x}\right] +$$

$$+ \text{c.c.}, \quad (10)$$

and

$$F_{\mu\nu} \tilde{F}^{\mu\nu} = 8 \left[A_{2,0} A_{1,3} - A_{1,0} A_{2,3}\right] =$$

$$= 8 (k_0 h_3 - k_3 h_0) \xi^3 \times \left[(\overline{\xi} \wedge \overline{\zeta}) e^{i(k+h)x} - (\overline{\zeta} \wedge \overline{\xi}) e^{i(k-h)x}\right] +$$

$$+ \text{c.c.}, \quad (11)$$

where $\overline{\xi} = (\xi^1, \xi^2, 0)$, $\overline{\zeta} = (\zeta^1, \zeta^2, 0)$, $\xi^3 \equiv (0, 0, 1)$, $\wedge$ and $\times$ are the vector and scalar products respectively and $\overline{\zeta}$ is the complex conjugate of $\zeta$. After some algebra one has

$$F^{03} \partial_\beta (F_{\mu\nu} F^{\mu\nu}) = F^{3\beta} \partial_\beta (F_{\mu\nu} F^{\mu\nu}) = 0, \quad (12a)$$

$$F^{3\beta} \partial_\beta (F_{\mu\nu} F^{\mu\nu}) = 4 (k_0 h_0 - k_3 h_3)^2 \times$$

$$\left\{\left[(\overline{\xi} \cdot \overline{\zeta}) \overline{\xi}^\dagger + (\overline{\zeta} \cdot \overline{\xi}) \overline{\zeta}^\dagger\right] e^{i kx} + \left[(\overline{\zeta} \cdot \overline{\xi}) \overline{\xi}^\dagger + (\overline{\xi} \cdot \overline{\zeta}) \overline{\zeta}^\dagger\right] e^{i hx}\right\} +$$

$$+ \text{c.c.} + \text{N.R.T.} \quad (12b)$$

and

$$\tilde{F}^{03} \partial_\beta \left(F_{\mu\nu} \tilde{F}^{\mu\nu}\right) = \tilde{F}^{3\beta} \partial_\beta \left(F_{\mu\nu} \tilde{F}^{\mu\nu}\right) = 0, \quad (13a)$$
\[ \tilde{F}^{j \beta} \partial_{\beta} \left( F_{\mu \nu} \tilde{F}^{\mu \nu} \right) = 16 \left( k_0 h_0 - k_3 h_3 \right)^2 e^{ir} \times \]
\[ \left\{ \left[ \epsilon^3 \cdot (\hat{\xi} \wedge \hat{\zeta}) \right] e^{i\xi x} + \epsilon^3 \cdot (\hat{\xi} \wedge \hat{\zeta}) \right\} e^{i\tilde{\xi} x} + \]
\[ + \left[ \epsilon^3 \cdot (\hat{\xi} \wedge \hat{\zeta}) \right] e^{i\xi x} + \epsilon^3 \cdot (\hat{\xi} \wedge \hat{\zeta}) \right\} e^{i\tilde{\xi} x} + \]
\[ + c.c. + N.R.T. \]

where hereafter \( j = 1, 2 \) and N.R.T stands for non resonant terms.

First, we note that from Eqs. (12a) and (13a) one has \( B^0 = B^3 = 0 \), and therefore from (8) it follows that the components \( \delta A^{(2)\beta} \) and \( \delta A^{(2)3} \) are stable.

Furthermore, from Eqs. (12b) and (13b) it follows that, if \( k_0/k_3 = -h_0/h_3 \), i.e. the two plane waves \( a^\alpha \) and \( b^\alpha \) in (6) are counterpropagating, the components \( B^3 \) contain terms \( \sim e^{ikx} \) and \( \sim e^{i\tilde{k}x} \) that are resonant, since they are a solution of the wave equation. Due to such resonant terms, the components \( \delta A^j \) grow linearly with time,

\[ \delta A^j \sim \epsilon^2 k^3 (A^{(0)})^3 x^0 e^{ikx} \sim \epsilon^2 t c k^3 (A^{(0)})^3 e^{ikx}, \quad (14) \]

where we have assumed for simplicity that \( k_0 \sim h_0 \sim k \), and \( |\hat{\xi}| \sim |\hat{\zeta}| \sim A^{(0)}. \) From (14) it is evident that the perturbative expansion in Eq. (7) fails when \( A^{(0)} \sim \epsilon^2 \delta A \), which gives the time scale of the secularities as

\[ T_i \sim \frac{1}{c k^3 \epsilon^2 (A^{(0)})^2} \sim \frac{1}{c k \epsilon^2 < \rho >}, \quad (15) \]

where we \( \rho > \sim (k A^{(0)})^2 \) is the energy density of the electromagnetic field.

We note that, if \( k_0/k_3 = h_0/h_3 \), i.e. the two plane waves \( a^\alpha \) and \( b^\alpha \) in (6) propagate in the same direction, the nonlinear terms in Eq. (4) disappear. In fact, in this case the four-vector \( A^\mu \) is a wave propagating in the positive \( x^3 \) direction, i.e., \( A^\mu = A^\mu(x^0 - x^3) \), or in the negative \( x^3 \) direction, i.e., \( A^\mu = A^\mu(x^0 + x^3) \). From (10) and (11) we can immediately see that in such a case one has \( F_{\mu \nu} \tilde{F}^{\mu \nu} = 0 \) and \( \tilde{F}_{\mu \nu} \tilde{F}^{\mu \nu} = 0 \). Thus, from (9) it comes that all the components \( B^\mu \) are null, and the zeroth order solution (6) is an exact solution of Eq. (4).

Therefore, when the two waves \( a^\mu \) and \( b^\mu \) propagate in the same direction, the effect of quantum corrections due to photon-photon scattering disappear. This reflects the fact that, in order to scatter, two photons cannot propagate in the same direction.

At that point, we focus on the case of two counterpropagating plane waves, in which photon-photon scattering plays an important role. Hereafter, we consider the case \( k_0 = k_3 > 0 \) and \( h_0 = -h_3 > 0 \). In this case the occurrence of resonant terms in (8) implies the linear divergence (14) of the perturbation \( \delta A^\mu \). In what follows, we show that the failure of the perturbative approach described above is amenable of a multiscale treatment. In fact, in perturbation theory, the presence of secularities is often due to a wrong perturbative approach, in problems in which the solutions depend simultaneously on widely different scales. In such cases, the divergences can be eliminated introducing suitable slow variables, i.e., dealing with a multiscale expansion. This analysis will finally clarify the physical meaning of the secularities described above.

We introduce suitable slow variables as

\[ y^0 = \epsilon^2 x^0, \quad y^1 = \epsilon (x^0 + x^3), \quad y^2 = \epsilon (x^0 - x^3), \quad (16) \]

As we will see, the variables \( y^1 \) and \( y^2 \) will play no role in the multiscale equations, so that the slow-scale evolution of the system will depend only on the slow time \( y^0 \). Using (16) one has

\[ \partial_{x^0} \to \partial_{y^0} + \epsilon (\partial_{y^1} + \partial_{y^2}) + c^2 \partial_{y^0}, \]

\[ \partial_{x^3} \to \partial_{x^0} + \epsilon (\partial_{y^1} - \partial_{y^2}), \quad (17) \]

which finally gives the d’Alembertian in terms of the derivatives with respect to slow and fast variables as

\[ \square \to \square + 2\epsilon \left[ (\partial_{y^1} + \partial_{y^2}) \partial_{x^0} - (\partial_{y^1} - \partial_{y^2}) \partial_{x^3} \right] + + 2\epsilon^2 \partial_{x^0} \partial_{y^0} + 2\partial_{y^1} \partial_{y^2} + o(\epsilon^3), \quad (18) \]

where \( \square \) is the d’Alembertian with respect to the fast variables \( x^0 \) and \( x^3 \).

The multiscale approach is useful when the dynamics evolves on widely different scales. In this case, the dependence of the solutions is split into fast and slow variables. Therefore, to find meaningful (on long times) approximated solutions of Eq. (4), we assume that the polarization vectors \( \xi \) and \( \zeta \) in (6) are no longer constant, but depend on the slow variables \( y^0, y^1, y^2 \). Moreover, we express the polarization vectors in terms of the left and right polarizations \( \hat{e}_L = (1, i, 0)/\sqrt{2} \) and \( \hat{e}_R = (1, -i, 0)/\sqrt{2} \) as

\[ \xi^\mu = a_L \hat{e}_L + a_R \hat{e}_R \quad (19) \]

\[ \zeta^\mu = b_L \hat{e}_L + b_R \hat{e}_R \]

where the coefficients \( a_L, a_R, b_L \) and \( b_R \) are the complex amplitudes of the different polarizations of the counterpropagating plane waves \( a^\mu \) and \( b^\mu \). Such amplitudes depend on the slow variables only as

\[ a_L = a_L(y^0, y^1, y^2), \quad a_R = a_R(y^0, y^1, y^2), \]

\[ b_L = b_L(y^0, y^1, y^2), \quad b_R = b_R(y^0, y^1, y^2) \quad (20) \]
Therefore, we search the solutions of Eqs (4) in the form (9) with the conditions (19)-20. Inserting (7) in (4) and using (18), we have

\[
\begin{align*}
\Box + 2 \epsilon \left[ (\partial_{y^1} + \partial_{x^2}) \partial_{x^0} - (\partial_{y^1} - \partial_{x^2}) \partial_{y^0} \right] \\
+ 2 \epsilon^2 \left[ \partial_{y^0} \partial_{y^0} + 2 \partial_{y^1} \partial_{y^2} \right] A^0 + \epsilon^2 \Box A^0 + \epsilon^2 B^0 = 0
\end{align*}
\]

Equation (21) is automatically satisfied at zeroth order in \( \epsilon \), while at the first order \( \sim \epsilon \) it implies that the functions \( a_L, a_R, b_L, \) and \( b_R \) depend on the slow variables as

\[
a_L = a_L(y^0, y^1), \quad a_R = a_R(y^0, y^1) \\
b_L = b_L(y^0, y^2), \quad b_R = b_R(y^0, y^2).
\]

To write the second order equations in a compact form, we define the vectors

\[
\begin{align*}
\vec{A}^{(0)} &= (0, A^{0,1}, A^{0,2}, 0) \\
\delta \vec{A}^{(0)} &= (0, \delta A^1, \delta A^2, 0) \\
\vec{B} &= (0, B^1, B^2, 0).
\end{align*}
\]

Using the fact that \( A^{0,0} = A^{0,3} = \delta A^0 = \delta A^3 = B^0 = B^3 = 0 \), so that the 0 and 3 components of (21) are automatically satisfied, at second order \( \sim \epsilon^2 \), Eq. (21) gives

\[
2 \left[ \partial_{x^0} \partial_{y^0} + 2 \partial_{y^1} \partial_{y^2} \right] A^{(0)} + R.T. (\vec{B}) = 0
\]

and

\[
\Box \delta \vec{A} + N.R.T. (\vec{B}) = 0,
\]

where \( N.R.T. (\vec{B}) \) and \( R.T. (\vec{B}) \) are, respectively, the non resonant and resonant terms in \( \vec{B} \).

Indeed, the terms \( \sim \epsilon^2 A^{(0)} \) are used to cancel the secularities contained in (21). On the other hand, \( \delta A^0 \) depends only on the fast variables \( x^0 \) and \( x^3 \), and it is used to cancel the non resonant terms in \( B^0 \), so it is a stable and small perturbation of \( A^{(0)} \), and we neglect it in the following discussion.

Let us write (24) in explicit form. Using (3), (19), (22) and (23) we have

\[
\left[ \partial_{x^0} \partial_{y^0} + 2 \partial_{y^1} \partial_{y^2} \right] A^{(0)} =
\]

\[
\left[ (ik_0 \partial_{y^0} a_L) \partial_{x^0} + (ik_0 \partial_{y^0} a_R) \partial_{x^0} \right] e^{ikx} +
\]

\[
\left[ (ih_0 \partial_{y^0} b_L) \partial_{x^0} + (ih_0 \partial_{y^0} b_R) \partial_{x^0} \right] e^{ikx} + c.c.
\]

and

\[
\vec{B} = 2 \epsilon^2 \hat{k}_0^2 h_0^2 \times
\]

\[
\left\{ \left[ -3a_L \left( |a_L|^2 + |b_R|^2 \right) + 22a_R b_L e^{ikx} \right] e^{ikx} +
\]

\[
\left[ -3a_R \left( |a_R|^2 + |b_L|^2 \right) + 22a_L b_R e^{ikx} \right] e^{ikx} \}
\]

Substituting these expressions in (24), we obtain the dynamical equations for the complex amplitudes as

\[
i \partial_x a_L + 16k_0 e^{ikx} = 0
\]

\[
i \partial_x a_R + 16k_0 e^{ikx} = 0
\]

\[
i \partial_x b_L + 16k_0 e^{ikx} = 0
\]

\[
i \partial_x b_R + 16k_0 e^{ikx} = 0.
\]

Since the amplitudes \( a_L \) and \( a_R \) do not depend on \( y^2 \), while \( b_L \) and \( b_R \) do not depend on \( y^1 \), the only possible dependence on the variables \( y^1 \) and \( y^2 \) is

\[
a_L = \alpha_L(y^0) e^{iqy^1}, \quad a_R = \alpha_R(y^0) e^{iqy^1},
\]

\[
b_L = \beta_L(y^0) e^{ipy^2}, \quad b_R = \beta_R(y^0) e^{ipy^2},
\]

with \( q \) and \( p \) real arbitrary numbers. However, this exponential dependence on \( y^1 \) and \( y^2 \) corresponds simply to an \( \epsilon^2 \) correction to the wave vectors \( k \) and \( h \); moreover \( q \) and \( p \) do not appear in (25), so we set \( q = s = 0 \).

Therefore, hereafter the complex amplitudes will depend only on the slow time \( y^0 \), and (28) becomes an ordinary differential system.

Let us study (28) in detail. It is quite immediate to recognize that the energy densities \( \rho_a \) and \( \rho_b \) are constant. Therefore, the intensities of the two plane waves \( a^\mu \) and \( b^\mu \) are conserved separately. Furthermore, the spin conservation implies that the quantity

\[
S = k_0 \left( |a_L|^2 + |a_R|^2 \right) + h_0 \left( |b_L|^2 + |b_R|^2 \right) \]

is also constant. Exploiting these relations, the system (28) can be simplified and then integrated (see the Appendix). However, to have a better understanding of the dynamics under study, we continue our discussion of (25).

First, we study stable configurations. Let us consider the choice \( a_R = s_1 a_L \) and \( b_R = s_2 b_L \), where \( s_1^2 = s_2^2 = 1 \) for two counterpropagating linearly polarized beams, and \( s_1 = s_2 = 0 \) for two left-hand circularly polarized waves.
by linearity is a correction to the frequency of light given
only the phase of the waves. The only effect of the non-
constant modulus, and the quantum corrections affect
where the frequencies and its solution is
\[
ia'_L + 16k_0h_0^3\left[-3(1+s_2^2)+22s_1s_2\right]a_L|b_L|^2 = 0
\]
\[
ib'_L + 16k_0^2h_0\left[-3(1+s_1^2)+22s_1s_2\right]b_L|a_L|^2 = 0
\]
(30)
and its solution is
\[
a_R = s_1a_L = s_1a_0^0e^{i\omega_0} \quad b_R = s_2b_L = s_2b_0^0e^{i\gamma_0},
\]
where the frequencies \(\omega\) and \(\gamma\) are given by
\[
\omega = 16\left[-3(1+s_2^2)+22s_1s_2\right]k_0h_0^3|b_L|^2
\]
\[
\gamma = 16\left[-3(1+s_1^2)+22s_1s_2\right]k_0^2h_0|a_L|^2
\]
(32)
In this class of solutions the complex amplitudes have
can constant modulus, and the quantum corrections affect
only the phase of the waves. The only effect of the non-
linearity is a correction to the frequency of light given by
\[
k'_0 = k_0 + e^2\omega \quad h'_0 = h_0 + e^2\gamma,
\]
(33)
and the relative frequency shifts are
\[
\Delta\frac{k_0}{k_0} \sim e^2h_0^2|b_L|^2 \sim e^2 < \rho_b >
\]
\[
\Delta\frac{h_0}{h_0} \sim e^2k_0^2|a_L|^2 \sim e^2 < \rho_a >.
\]
(34)
We mention that many quantum gravity models predict
a violation of the Lorentz symmetry \(45\), implying a
deformation of the photon energy-dispersion relation,
that might be observed in photons of astrophysical origin,
e.g., in gamma ray bursts \(46\). However, one should keep
in mind that an apparent breaking of the Lorentz symmetry
in photons might be due to photon-photon scattering
rather than quantum gravitational effects.

At that point, we study most interesting configurations,
in which the initial polarizations of the light beams
change dramatically during the evolution of the system.
We choose initial conditions in such a way that at least
one of the products \(a_La_R\) or \(b_Lb_R\) is initially nonzero.
In facts, the last terms in Eq. (28) are responsible of the
oscillatory behavior that we describe below. Solving (28)
umerically it is possible to see that the polarization of the
two counterpropagating waves oscillate periodically
between left-handed and right-handed polarizations.
For instance, in Figs. 1 and 2 we plot the square modulus of
the amplitudes for \(k_0 = h_0 = 1\) and initial values \(a_R^0 = 0,\)
\(a_L^0 = 1, b_L^0 = 1, b_R^0 = i\). From (28) it is easy to recog-
nize that the time scale of variation of the amplitudes is
given by \(15\), thus we plot the solutions in a time interval
\(\Delta t \sim 30T_i\), corresponding to an interval \(\Delta y^0 \sim e^2\Delta t_i\)
in which the oscillatory behavior of the dynamics is shown
completely.

From Fig. 1 we see that \(|a_R|\) is initially zero, but it
grows rapidly to \(|a_R| = |a_L^0|\), while \(|a_L|\) goes from \(|a_L^0|\)
to zero. Thus, the \(a^\mu\) beam initially in the left-handed
polarization switches to the right-handed polarization.
It remains in this state most of the time, until it jumps
back to its initial left-handed configuration after the first
period \(\tau \sim 13T_i\). After that, this behavior is repeated
periodically.

The behavior of the beam \(b^\mu\) is similar. In facts, \(|b_R|\)
goes to zero rapidly, while \(|b_L|\) goes to \(\sqrt{|b_L^0|^2 + |b_R^0|^2}\),
and it remains constant most of the time until, after a
time \(\tau \sim 13T_i\), \(|b_L|\) and \(|b_R|\) go back to their initial
values. The difference with respect the beam \(a^\mu\) is that
\(|b_L|\) does never reach the zero.

We emphasize that the reason why the system spends
most of the time in the configuration \(|a_L| \approx |b_R| \approx 0\), \(|a_L| \approx |a_R^0|\), and \(|b_L| \approx \sqrt{|b_L^0|^2 + |b_R^0|^2}\) is that this configuration corresponds approximately to two circularly polarized counterpropagating beams, which is analogous to the solution (31), which is stable.

Numerical investigation of (28) suggests that this oscillatory behavior is not affected (qualitatively) by the choice of the parameters in (28), while the recurrence time of the polarization oscillations depends on the wavelength and energy density of the electromagnetic field as in (15).

Finally, we note that the equivalence between the space and time coordinates \(x^0\) and \(x^3\), which is evident in the covariant formalism, implies that the oscillatory behavior may occur also in the space variable \(x^3\). However, this issue goes beyond the purpose of this paper and will be discussed elsewhere [47].

To finish, we estimate the recurrence times for realistic physical situations in cosmology and optics.

Due to the extreme smallness of \(\varepsilon^2\), one expects a huge recurrence time \(T_i\). For that reason, it is natural expect a \(T_i\) of the order of cosmological times \(1/H_0 \sim 10^{11}\) s [48]. Indeed, we can ask whether this instability might be important in cosmology, so we can estimate \(T_i\) in the case of the cosmic microwave background (CMB) radiation [45], which is an almost perfect blackbody radiation with a temperature of 2.7 K. The energy density of the CMB radiation is measured as \(< \rho_a > \sim 10^{-14} J/m^3\), while its wavelength is in the microwave range \(\lambda \sim 1\) mm, which gives a time \(T_i \sim 10^{22}\) s much greater than the age of the universe. It might be argued that it could be possible to have \(T_i \sim 1/H_0\) in the early universe. However, at high redshifts photons are highly energetic, and therefore they can produce other particles (e.g., electron-positron pairs). Indeed, other quantum effects are not negligible, so that the Lagrangian (1) is no longer fit for our purposes. Therefore, one concludes that the polarization oscillations cannot be observed in the CMB radiation.

Finally, we consider the possibility of observing the polarization oscillations in optical experiments. The search for signatures of the photon-photon scattering in optics is in progress [1–25]. For instance, Ref. [7] investigates the possibility of observing vacuum birefringence and dichroism induced by photon-photon interactions in ultra-strong laser fields.

We can estimate the time of recurrence of the polarization oscillations for light beams produced in petawatt class lasers, which will be available in the near future. The intensities attainable in these lasers reach \(I \sim 10^{22} W/cm^2\) [49–50], giving a recurrence time \(T_i \sim 4 \times 10^2 (\lambda/m)\) s, where \(\lambda/m\) is the laser wavelength in meters (we used \(k \sim h \sim 2\pi/\lambda\) and \(k^2 a^2 \sim k^2 b^2 \sim < \rho > 1/c\). Therefore, for realistic lasers with \(\lambda \sim 1\) mm, observation times can be of the order of \(10^{-3}\) s (to be compared with those estimated in Ref. [7]). This lets us hope to be able to observe polarization oscillations in two counterpropagating petawatt laser beams.

In conclusion, in this paper we have shown that the extremely weak photon-photon interaction might be responsible for surprisingly strong deviations from the free dynamics of electromagnetic waves. We have shown that, in the case of two counterpropagating laser beams, one of which has circular polarization and the other is not circularly polarized, the evolution of the electromagnetic waves consists in slow oscillations in the polarizations of the beams. We have estimated the recurrence of the polarization oscillations, and we have shown that, while unobservable in the cosmological context, this oscillatory behavior might be revealed in realistic optical experiments.

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Appendix

In this appendix we show that it is possible to solve the system (28). First, it is immediate to recognize that (28) implies the conservations of the energy densities \(< \rho_a > = k_0^2 (|a_L|^2 + |a_R|^2)\) and \(< \rho_b > = k_0^2 (|b_L|^2 + |b_R|^2)\). Therefore, it is natural to express the amplitudes in the form

\[
\begin{align*}
a_L(y^0) &= a^0 \sin(\phi(y^0)) e^{i(\omega y^0 + \gamma y^0)} \\
b_L(y^0) &= b^0 \sin(\varphi(y^0)) e^{i(\gamma y^0 + \psi_L y^0)} \\
a_R(y^0) &= a^0 \cos(\phi(y^0)) e^{i(\omega y^0 + \gamma y^0)} \\
b_R(y^0) &= b^0 \cos(\varphi(y^0)) e^{i(\gamma y^0 + \psi_L y^0)},
\end{align*}
\]

(35)

where \(a^0, b^0, \theta_L^0, \varphi_L^0, \psi_L^0, \psi_R^0\) and \(\psi_L^0\) are arbitrary constants such that \(\theta_R^0 - \theta_L^0 + \psi_L^0 - \psi_R^0 = \pi/2\) and

\[
\omega = -48k_0 |b_L^0|^2, \quad \gamma = -48k_0 b_0 |a_0|^2.
\]

(36)

Substituting (35) in (28), one obtains the following reduced system for the two variables \(\phi\) and \(\varphi\):

\[
\begin{align*}
\phi' + 176k_0 h_0^2 |b_0|^2 \sin(2\varphi) &= 0 \\
\varphi' - 176k_0^2 b_0 |a_0|^2 \cos(2\phi) &= 0,
\end{align*}
\]

(37)

that can be solved using the spin conservation \(k_0 |a_0|^2 \cos(2\phi) + h_0 |b_0|^2 \cos(2\varphi) + S = 0\). Therefore (28) is integrable and its solutions are periodic. However, we preferred to discuss (28) instead of (37), since the separate analysis of the behavior of the four amplitudes \(a_L, a_R, b_L, b_R\) makes the oscillatory dynamics of the polarizations more evident.
Phys. D: Nonlinear Phenomena 238 (5) (2009) 540-548.

[44] P. G. Grinevich and P. M. Santini, The exact rogue wave recurrence in the NLS periodic setting via matched asymptotic expansions, for 1 and 2 unstable modes, arXiv:1708.04535 [nlin.SI]; P. G. Grinevich and P. M. Santini, Numerical instability of the Akhmediev breather and a finite-gap model of it, arXiv:1708.00762 [nlin.PS]; P. G. Grinevich and P. M. Santini, The finite gap method and the analytic description of the exact rogue wave recurrence in the periodic NLS Cauchy problem, arXiv:1707.05669 [nlin.SI].

[45] G. Amelino-Camelia, Living Rev.Rel. 16 (2013) 5, arXiv:0806.0339 [gr-qc].

[46] G. Amelino-Camelia et al., Nature 393 (1998) 763-765.

[47] F. Briscese, Collective behavior of light in vacuum, arXiv:1710.07703 [physics.optics].

[48] Planck 2015 results, XIII, Cosmological parameters, Planck Collaboration: P. A. R. Ade et al., A&A 594, A13 (2016), arXiv:1502.01589 Planck 2015, XX, Constraints on inflation, Planck Collaboration: P. A. R. Ade et al., A&A 594, A20 (2016), arXiv:1502.02114.

[49] C. Danson, D. Hillier, N. Hopps, and D. Neely, Petawatt class lasers worldwide, High Power Laser Science and Engineering 3, e3 (2015).

[50] T. M. Jeong and J. Lee, Femtosecond petawatt laser, Ann. Phys. 526, 157172 (2014).

[51] In this paper we use the covariant formalism, so that the zeroth coordinate is defined as $x^0 = ct$.

[52] As we will see, the condition that the two waves are counterpropagating is necessary in order to have non-vanishing nonlinear corrections. This reflects the fact that such corrections come from photon-photon scattering, which require that the two photons have opposite velocities to collide.

[53] $H_0$ is the hubble constant today, and its inverse represents a typical cosmological timescale.