Diffusive Propagation of High Energy Cosmic Rays in Galaxy: Effect of Hall Drift

Hideyoshi Arakida
School of Education, Waseda University
arakida@edu.waseda.ac.jp

Shuichi Kuramata
Graduate School of Science and Technology, Hirosaki University

Received Day Month Year
Revised Day Month Year

We phenomenologically developed a propagation model of high energy galactic cosmic rays. We derived the analytical solutions by adopting the semi-empirical diffusion equation, proposed by Berezinskii et al. (1990) and the diffusion tensor proposed by Ptuskin et al. (1993). This model takes into account both the symmetric diffusion and the antisymmetric diffusion due to the particle Hall drift. Our solutions are an extension of the model developed by Ptuskin et al. (1993) to a two-dimensional two-layer (galactic disk and halo) model, and they coincide completely with the solution derived by Berezinskii et al. (1990) in the absence of antisymmetric diffusion due to Hall drift. We showed that this relatively simple toy model can be used to explain the variation in the exponent of the cosmic ray energy spectrum, $\gamma$, around the knee $E \approx 10^{15}$ eV.

Keywords: Cosmic Rays; Propagation; Diffusion Equation; Galactic Magnetic Field; Hall Drift

PACS numbers: 96.50.S-, 98.70.Sa, 96.50.sb

1. Introduction

Cosmic ray propagation is one of the most important and interesting subjects in astrophysics and high energy particle physics. It is believed that the observed cosmic ray data such as the cosmic ray energy spectrum includes information about the space through which the cosmic rays pass. In fact, it is possible to evaluate the thickness of matter, and it is also thought that details about the galactic magnetic field can be extracted from the observed data because the cosmic rays experience frequent collision and scattering with both the interstellar gas and the galactic magnetic field during their propagation in the galaxy. Thus, a reliable cosmic ray propagation model may enable us to obtain further knowledge about the galactic structure. Thus far, several cosmic ray propagation models have been proposed and
discussed (see [12] and references therein). Further, some researchers have successfully derived analytical solutions using the diffusion equation [3,4,5,6].

The spectrum of cosmic rays shows one of the most distinctive features, known as “knee”, around energy \( E \approx 10^{15} \) eV at which the exponent of the energy spectrum, \( \gamma \), changes from 2.6 – 2.7 for \( 10^{10} \leq E \leq 10^{15} \) eV to 3 – 3.1 for \( 10^{15} \leq E \leq 10^{18} \) eV. Currently, it is not clear why the exponent changes around \( E \approx 10^{15} \) eV. Thus far, several models have been proposed to explain this spectral property: a shock wave acceleration model based on the acceleration of cosmic ray particles by the shock wave front [7,8,9,10,11,12,13], a diffusive propagation model based on the leakage and diffusive propagation of cosmic rays in the galaxy [14,15,16,17,18,19,20], an interaction model based on the interaction of cosmic rays with the background particles in the galaxy [21,22,23,24], a reaction model based on the reaction of cosmic rays with the atmosphere of Earth [25,26], etc. Among these models, the shock wave acceleration model seems to be widely accepted as the explanation for the knee. However, the diffusive propagation model can provide the exposition for the first knee and the second knee and for the observed compositions and anisotropies [6,18,19,20]. The diffusive propagation model is characterized by introducing the particle Hall drift effect; thus, it is a theoretically simple model.

In this study, we derived solutions for the diffusive propagation model of cosmic rays and confirmed the validity of this model. We adopted the semi-empirical diffusion equation introduced in [3] and the diffusion tensor described in [6]. Then, we extended the propagation model in [6] to a two-dimensional two-layer (comprising the galactic disk and the halo) model in cylindrical coordinates.

This paper is organized as follows: In Section 2 we derive the analytical solutions for the diffusion equation of cosmic rays. In Section 3, we qualitatively show that our model can explain the spectral feature of the observed cosmic rays, namely, the exponential variation around the knee. In Section 4, we present the conclusions of our study.

2. Solutions for Diffusion Equation

2.1. Galactic Structure

Fig. 1 shows a schematic diagram of our galactic model. We assume that our galaxy has a cylindrical structure and it is divided into two parts: the galactic disk (\( D \)) and the halo (\( H \)). The signs + and − indicate the direction along the \( z \)-axis. In Fig. 1, \( R \) is the radius of the galaxy and \( 2h_g \) and \( 2h \) are the height of the disk region and the galaxy, respectively. We suppose that the source of cosmic rays is distributed uniformly within the galactic disk area only. The source region of the \( i \)-th particle is given by radius \( R_i \) (\( 0 < R_i \leq R \)) in the shaded area in Fig. 1.
2.2. Basic Equation, Diffusion Tensor, and Hall Drift

We consider the following transport equation

\[- \nabla_\alpha (D_{\alpha\beta}(z) \nabla_\beta) N_i(E, r, z) + n(z) \sigma_i N_i(E, r, z) = Q_i(E, r, z), \tag{1}\]

where \(N_i(E, r, z)\) is the number density of the \(i\)-th cosmic ray component per unit time, unit volume, and unit energy; \(D_{\alpha\beta}\) is the diffusion tensor (indexes \(\alpha\) and \(\beta\) denote the coordinates); \(v\) is the velocity of the cosmic ray; \(\sigma_i\) is the inelastic scattering cross-section between the \(i\)-th cosmic ray particle and the interstellar gas; \(n(z)\) is the density of the interstellar gas; and \(Q_i(E, r, z)\) represents the distribution of the source.

We adopt the diffusion tensor given in \(6\)

\[D_{\alpha\beta} = (D_\parallel - D_\perp) b_\alpha b_\beta + D_\perp \delta_{\alpha\beta} + D_A \epsilon_{\alpha\beta\gamma} b_\gamma,\tag{2}\]

where \(b_\alpha\) is the unit vector of the magnetic field, \(D_\parallel\) denotes the diffusion along the magnetic field line, \(D_\perp\) denotes the diffusion perpendicular to the magnetic field line, \(D_A\) is diffusion due to the antisymmetric drift of the particle, \(\delta_{\alpha\beta}\) is the Kronecker’s delta symbol, and \(\epsilon_{\alpha\beta\gamma}\) is the complete antisymmetric Levi-Civita tensor.

The observed data show that the magnetic field in the galaxy has a spiral structure, and it is regarded as an approximate toroidal structure \(27\). Thus, we assume the unit vector of the magnetic field as

\[b_r = b_z = 0, \quad b_\phi = 1.\tag{3}\]

It should be noted that the magnetic field line in the galactic disk changes the sign almost every 3 kpc along the radial direction \(28\). Nevertheless, for the sake of simplicity, we assume that the global magnetic field points in the same direction anywhere in the galactic disk. On the other hand, the structure of the magnetic field in the galactic halo is not well known; so we suppose that the magnetic field in the halo also has a toroidal structure given by

\[b^{(D)}_r = b^{(H)}_r = b^{(D)}_z = b^{(H)}_z = 0, \quad b^{(D)}_\phi = b^{(H)}_\phi = 1,\tag{4}\]
where \( D \) and \( H \) denote the disk and the halo, respectively. From Eq. (4), the diffusion terms in (1) becomes
\[
- \nabla_\alpha (D_{\alpha \beta} \nabla_\beta) = - \frac{1}{r} \frac{\partial}{\partial r} (r D_\perp) \frac{\partial}{\partial r} - \frac{\partial}{\partial z} D_\perp \frac{\partial}{\partial z} - \frac{\partial}{\partial z} D_A \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (r D_A) \frac{\partial}{\partial z}.
\] (5)

These coefficients are determined in [6], as
\[
D_\parallel = \frac{l v}{3}, \quad D_\perp = g A^4 \frac{l v}{3}, \quad D_A = \frac{r R}{3}.
\] (6)

Here \( l \) is the mean free path that is characterized by the turbulent exponent \( m \), \( A \) denotes the relative value of a random magnetic field within the characteristic scale \( L \), and \( g \) is a not well-determined parameter (see 6). From the gyro-radius
\[
\gamma = \frac{c}{Ze B} \frac{B \times p}{B^2},
\] (7)
we obtain the drift velocity of the particles
\[
V_{D\gamma} = - \nabla_\gamma (D_A \epsilon_{\alpha \beta \gamma} b_\alpha) = \frac{pcv}{3Ze} \epsilon_{\alpha \beta \gamma} \frac{\partial}{\partial x_\alpha} \frac{B_\beta}{B^2}.
\] (8)

Then, we substitute the coefficients of the diffusion tensor given in 6
\[
D_\perp (r, z) = \text{constant}, \quad D_A (r, z) = D_A (z) \frac{r}{R}, \quad D_A (z) = \text{constant}.
\] (9)

Then the diffusion term reduces to
\[
- \nabla_\alpha (D_{\alpha \beta} \nabla_\beta) = \frac{D_\perp}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} - \frac{\partial}{\partial z} \frac{\partial}{\partial z} + \frac{2D_A}{R} \frac{\partial}{\partial z}.
\] (10)

From Eq. (9), the drift velocity \( V_D \) is written as
\[
V_{Dr} = - \frac{\partial}{\partial z} D_A = 0, \quad V_{Dz} = \frac{1}{r} \frac{\partial}{\partial r} (r D_A) = \frac{2D_A}{R} = \text{constant}.
\] (11)

Thus, the drift effect appears only along the \( z \)-direction in this case.

2.3. Boundary and Continuity Conditions

Let us summarize the boundary and continuity conditions. First, at the edge of the galaxy, we assume that the cosmic rays leak from the galaxy
\[
N_i (r = R, z) = N_i (r, z = \pm h) = 0.
\] (12)

\footnote{It is suggested that the turbulent exponent in the magnetic field in the galactic disk is practically characterized by the power law with exponent \( m = 0.3 \pm 0.3 \) [20,30]. For instance, \( m = 1/3 \) is related to the Kolmogorov turbulence spectrum; \( m = 1/2 \), turbulence spectrum of Kraichnan hydromagnetic one [31]; and \( m = 0 \), Bykov-Toptygin spectrum [32]. In the case of the halo, we do not have sufficient knowledge about the magnetic turbulence; therefore, we assume that the same power law holds}
Diffusive Propagation of High Energy Cosmic Rays

This condition seems to be appropriate in terms of the cosmic ray $L/M$ ratio. Second, at the boundaries of the disk and the halo, $z = \pm h_g$, both $N_i^{(D)}$ and $N_i^{(H)}$ must be continuously connected. Then,

$$N_i^{(H\pm)}(\pm h_g, r) = N_i^{(D\pm)}(\pm h_g, r), \quad \frac{dN_i^{(H\pm)}(\pm h_g, r)}{dz} = \frac{dN_i^{(D\pm)}(\pm h_g, r)}{dz}, \quad (13)$$

and

$$N_i^{(D+)}(0, r) = N_i^{(D-)}(0, r), \quad \frac{dN_i^{(D+)}(0, r)}{dz} = \frac{dN_i^{(D-)}(0, r)}{dz}. \quad (14)$$

### 2.4. Solution for Transport Equation

Because Eq. (11) has the same form as Eqs. (3.10) or (3.11) in[8], we follow the same approach. To obtain the solution for Eq. (11), we first obtain the Green’s function $\Phi(r, z; r_0)$, which satisfies

$$- \nabla_\alpha (D_{\alpha \beta}(z)\nabla_\beta)\Phi(r, z; r_0) + n(z)v\sigma_i\Phi(r, z; r_0) = \theta(h_g - |z|)\frac{\delta(r - r_0)}{4\pi h_g r_0}, \quad (15)$$

where $r_0$ represents the position of the cosmic ray source and $\theta(h_g - |z|)$ is the step function. $\Phi(r, z; r_0)$ is related to $N_i$ in[8],

$$N_i = 4\pi h_g g_i E^{-\gamma_o} \int_0^{R_i} \chi(r_0)\Phi(r, z; r_0) r_0 dr_0, \quad (16)$$

where $g_i$ is the constant characterizing the $i$-th particle, $\chi(r_0)$ represents the radial distribution of the source of cosmic rays, and we presume the energy spectrum at the source obeys the power law $E^{-\gamma_o}$ for the $i$-th particle. We also suppose the radial component of $\Phi(r, z; r_0)$ is expanded by using the zeroth-order Bessel function

$$\Phi(r, z; r_0) = \sum_{s=1}^{\infty} a_s(z; r_0) J_0 \left( \frac{\nu_s r}{R} \right). \quad (17)$$

In this expression, $\nu_s$ is the $s$-th root of the equation $J_0(\nu_s) = 0$. Hence, Eq. (15) becomes

$$\sum_{s=1}^{\infty} \left\{ \left[ \frac{-D_\perp(z)}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} - D_\perp(z) \frac{\partial^2}{\partial z^2} + \frac{2D_{\|}(z)}{R} \frac{\partial}{\partial z} \right] + n(z)v\sigma_i \right\} a_s(z; r_0) J_0 \left( \frac{\nu_s r}{R} \right)$$

$$= \theta(h_g - |z|)\frac{\delta(r - r_0)}{4\pi h_g r_0}. \quad (18)$$

From the orthogonality of the Bessel function, the delta function is expressed as

$$\delta(r - r_0) = \sum_{s=1}^{\infty} \frac{2}{|J_1(\nu_s)|^2} \frac{r_0}{R^2} J_0 \left( \frac{\nu_s r_0}{R} \right) J_0 \left( \frac{\nu_s r}{R} \right), \quad (19)$$

where $J_1$ is the first-order Bessel function. Noting that

$$\frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} J_0 \left( \frac{\nu_s r}{R} \right) = - \left( \frac{\nu_s r}{R} \right)^2 J_0 \left( \frac{\nu_s r}{R} \right), \quad (20)$$
In the galactic halo, where

From the boundary condition in Eq. (12), Eqs. (26) and (27) are re written as

and the general solution has the form,

Then the following two solutions are obtained:

Thus, the problem reduces to solving the ordinary differential equation

where

Thus, the problem reduces to solving the ordinary differential equation

with

D, D, and n(z) are chosen as

In the galactic halo, C = 0 because of the step function. Therefore,

and the general solution has the form,

where A, A', B and B' are the integration constants and \( \alpha \) and \( \beta \) are the solutions of the following equation that is associated with Eq. (25),

Then the following two solutions are obtained:

From the boundary condition in Eq. (12), Eqs. (26) and (27) are rewritten as

\[ a_s^{(H+)}(z ; r_0) = B\left[ -\exp\left( (\beta_s - \alpha_s) h \right) e^{\alpha_s z} + e^{\beta_s z} \right], \quad z > 0 \]
\[ a_s^{(H-)}(z ; r_0) = B'\left[ -\exp\left( -(\beta_s - \alpha_s) h \right) e^{\alpha_s z} + e^{\beta_s z} \right], \quad z < 0. \]
In the galactic disk, the general solution takes the form

\[ a_{s}^{(D+)}(z; r_0) = P e^{\gamma_i z} + Q e^{\delta_i z} + C', \quad z \geq 0, \]  
\[ a_{s}^{(D-)}(z; r_0) = P' e^{\gamma_i z} + Q' e^{\delta_i z} + C', \quad z < 0, \]  
where \( P, P', Q \) and \( Q' \) are also integration constants and

\[ C' = \frac{C}{(\frac{\nu_s}{R})^2 + \frac{n_g v \sigma_i}{D}}, \]  
\( \gamma_i \) and \( \delta_i \) are the solutions to an equation similar to Eq. 25.

\[ \gamma_i = \frac{D_{A0}}{R D_{\perp}} - \sqrt{\left(\frac{D_{A0}}{R D_{\perp}}\right)^2 + \left(\frac{\nu_s}{R}\right)^2 + \frac{n_g v \sigma_i}{D}}, \]  
\[ \delta_i = \frac{D_{A0}}{R D_{\perp}} + \sqrt{\left(\frac{D_{A0}}{R D_{\perp}}\right)^2 + \left(\frac{\nu_s}{R}\right)^2 + \frac{n_g v \sigma_i}{D}}. \]  

We suppose the cosmic ray source distributes uniformly; then

\[ \chi(r_0) = \theta(R_i - r_0), \quad 0 < R_i \leq R. \]  
Furthermore,

\[ \int_{0}^{R_i} \theta(R_i - r_0) r_0 J_0 \left( \frac{\nu_s r_0}{R} \right) dr_0 = \frac{R R_i}{\nu_s} J_1 \left( \frac{R_i \nu_s}{R} \right), \]  
and after straightforward but bit tedious calculations, we obtain the number density of the \( i \)-th cosmic ray particles

\[ N_{ij}^{(H+)}(E, r, z) = \sum_{s=1}^{\infty} \frac{2 g_i e^{-\gamma_i E} R_i J_1 \left( \frac{R_i \nu_s}{R} \right)}{|J_1(\nu_s)|^2 \nu_s D_{\perp} R \left( \frac{\nu_s}{R} \right)^2 + \frac{n_g v \sigma_i}{D}} \left[ (\gamma_i - \alpha_i) e^{\gamma_i h_s} - (\gamma_i - \beta_i) e^{\beta_i h_s} \right] \]
\[ \times \left( (\delta_i - \gamma_i) \left[ (\alpha_i + \beta_i) (\gamma_i + \delta_i - 2 \gamma_i \delta_i - (\alpha_i^2 + \beta_i^2)) \right] e^{(\beta_i - \gamma_i) h_s} - e^{-(\delta_i - \gamma_i) h_s} \right) \]
\[ - \left[ (\alpha_i - \delta_i) (\gamma_i - \beta_i) e^{(\delta_i - \gamma_i) h_s} - (\beta_i - \delta_i) (\gamma_i - \alpha_i) e^{-(\delta_i - \gamma_i) h_s} \right] e^{(\beta_i - \alpha_i) (h_i - h_s)} \]
\[ - \left[ (\beta_i - \delta_i) (\gamma_i - \alpha_i) e^{(\delta_i - \gamma_i) h_s} - (\alpha_i - \beta_i) (\gamma_i - \beta_i) e^{-(\delta_i - \gamma_i) h_s} \right] e^{-(\beta_i - \alpha_i) (h_i - h_s)} \]
\[ \times \left[ (\alpha_i + \beta_i) (\gamma_i - \alpha_i) e^{-\gamma_i h_s} - (\alpha_i + \beta_i) (\gamma_i - \beta_i) e^{\gamma_i h_s} \right] e^{-(\beta_i - \alpha_i) (h_i)} \]
\[ - \left[ (\alpha_i - \beta_i) e^{-\gamma_i h_s} - (\beta_i - \alpha_i) e^{\gamma_i h_s} \right] e^{(\beta_i - \alpha_i) (h_i - h_s)} \]
\[ - \left[ (\alpha_i - \beta_i) e^{-\gamma_i h_s} - (\beta_i - \alpha_i) e^{\gamma_i h_s} \right] e^{(\beta_i - \alpha_i) (h_i - h_s)} \]
\[ \times [- \exp{(\beta_i - \alpha_i) h_i} e^{\alpha_i z} + e^{\beta_i z}], \]
\[
N_i^{(D+)}(E, r, z) = \sum_{s=1}^{\infty} \frac{2g_i E^{-\gamma \omega} R_i J_1 \left( \frac{R_i}{\nu_s D} \right) J_0 \left( \frac{\nu_s D}{R_i} \right)}{|J_1(\nu_s)|^2 \nu_s D \left( \frac{\nu_s^2 D}{R_i^2} + \frac{\nu_s^2 D}{D^2} \right) \left[ (\gamma_i - \alpha_i) \exp\{ (\beta_i - \alpha_i) h \} e^{\alpha_i h_y} - (\gamma_i - \beta_i) e^{\beta_i h_y} \right]}
\times \left\{ \left[ (\alpha_i + \beta_i) (\gamma_i + \delta_i) - 2\gamma_i \delta_i - (\alpha_i^2 + \beta_i^2) \right] e^{(\delta_i - \gamma_i) h_y} - e^{-(\delta_i - \gamma_i) h_y} \right\}^{-1}
\times \left\{ \left[ (\alpha_i + \beta_i) (\gamma_i - \alpha_i) - (\alpha_i^2 + \beta_i^2) \right] (e^{\gamma_i h_y} - e^{-\gamma_i h_y}) \right\}^{-1}
\times \left[ (\gamma_i - \alpha_i) e^{\gamma_i h_y} - \alpha_i (\gamma_i - \beta_i) e^{-\gamma_i h_y} \right] e^{(\beta_i - \alpha_i) (h - h_y)}
\times \left[ (\alpha_i + \beta_i) (\gamma_i - \alpha_i) - (\alpha_i^2 + \beta_i^2) \right] (e^{\gamma_i h_y} - e^{-\gamma_i h_y})
\times \left[ \beta_i (\gamma_i - \alpha_i) e^{\gamma_i h_y} - \alpha_i (\gamma_i - \beta_i) e^{-\gamma_i h_y} \right] e^{(\beta_i - \alpha_i) (h - h_y)}
\times \left[ (\alpha_i + \beta_i) (\gamma_i - \alpha_i) - (\alpha_i^2 + \beta_i^2) \right] (e^{\gamma_i h_y} - e^{-\gamma_i h_y})
\times \left[ \beta_i (\gamma_i - \alpha_i) e^{\gamma_i h_y} - \alpha_i (\gamma_i - \beta_i) e^{-\gamma_i h_y} \right] e^{(\beta_i - \alpha_i) (h - h_y)}
\times \left[ (\alpha_i + \beta_i) (\gamma_i - \alpha_i) - (\alpha_i^2 + \beta_i^2) \right] (e^{\gamma_i h_y} - e^{-\gamma_i h_y})
\times \left[ \beta_i (\gamma_i - \alpha_i) e^{\gamma_i h_y} - \alpha_i (\gamma_i - \beta_i) e^{-\gamma_i h_y} \right] e^{(\beta_i - \alpha_i) (h - h_y)}
\times \left[ (\alpha_i + \beta_i) (\gamma_i - \alpha_i) - (\alpha_i^2 + \beta_i^2) \right] (e^{\gamma_i h_y} - e^{-\gamma_i h_y})
\times \left[ \beta_i (\gamma_i - \alpha_i) e^{\gamma_i h_y} - \alpha_i (\gamma_i - \beta_i) e^{-\gamma_i h_y} \right] e^{(\beta_i - \alpha_i) (h - h_y)}
+ \alpha_i \exp\{ (\beta_i - \alpha_i) h \} e^{\alpha_i h_y} - \beta_i e^{\beta_i h_y} \right\} e^{\gamma_i z} + 1 \right],
\]
\[\begin{align*}
N_i^{(H^-)}(E, r, z) &= \sum_{s=1}^{\infty} \frac{2g_i E^{-\gamma_i} R_i J_1 \left( \frac{R_i}{R} \right) J_0 \left( \frac{\nu}{R} \right)}{|J_1(\nu_r)^2 \nu_s D_\perp R \left( \frac{\nu}{R} \right)^2 + \frac{\nu_s}{\nu_D}} \\
& \times \left[ \left[ (\gamma_i - \alpha_i) \exp\left\{ -\left( \beta_i - \alpha_i \right) h \right\} e^{-\alpha_i h_y} - (\gamma_i - \beta_i) e^{-\beta_i h_y} \right] \\
& \times \left[ (\delta_i - \gamma_i) \left\{ (\alpha_i + \beta_i)(\gamma_i + \delta_i) - 2\gamma_i \delta_i - (\alpha_i^2 + \beta_i^2) \right\} \left[ e^{(\delta_i - \gamma_i) h_y} - e^{-(\delta_i - \gamma_i) h_y} \right] \\
& - \left[ (\alpha_i - \delta_i)(\gamma_i - \beta_i)e^{(\delta_i - \gamma_i) h_y} - (\beta_i - \delta_i)(\gamma_i - \alpha_i)e^{-(\delta_i - \gamma_i) h_y} \right] e^{(\beta_i - \alpha_i) (h - h_y)} \\
& - \left[ (\beta_i - \delta_i)(\gamma_i - \alpha_i) e^{(\delta_i - \gamma_i) h_y} - (\alpha_i - \beta_i)(\gamma_i - \beta_i)e^{-(\delta_i - \gamma_i) h_y} \right] e^{-(\beta_i - \alpha_i) (h - h_y)} \right]^{-1} \\
& \times \left\{ \left[ (\alpha_i + \beta_i)\gamma_i - (\alpha_i^2 + \beta_i^2) \right] e^{\gamma_i h_y} - e^{-\gamma_i h_y} \right\} \\
& - \left[ \beta_i(\gamma_i - \alpha_i) e^{\gamma_i h_y} - \alpha_i(\gamma_i - \beta_i)e^{-\gamma_i h_y} \right] e^{(\beta_i - \alpha_i) (h - h_y)} \\
& - \left[ \alpha_i(\gamma_i - \beta_i)e^{\gamma_i h_y} - \beta_i(\gamma_i - \alpha_i)e^{-\gamma_i h_y} \right] e^{-(\beta_i - \alpha_i) (h - h_y)} \right] e^{-\delta_i h_y - \gamma_i} e^{-\gamma_i h_y} \\
& \times \left[ - \exp\left\{ -(\beta_i - \alpha_i) h \right\} e^{\alpha_i z} + e^{\beta_i z} \right]. \tag{42}
\end{align*}\]
In this section, we discuss the features of the derived solutions, especially focusing on the question whether the diffusion model can reproduce the exponential variation of cosmic ray spectrum due to particle drift.

We note that these solutions reduce to those in \[3\] in the absence of antisymmetric diffusion.

\[\begin{align}
N_i^{(D-)}(E, r, z) &= \sum_{s=1}^{\infty} \frac{2g_i E^{-\gamma_i} R_s J_1 \left( \frac{R_s}{R} \right) J_0 \left( \frac{R_s}{R} \right)}{|J_1(\nu_s)|^2 \nu_s D \| R |^2 + \frac{\nu_s \nu_i}{D}} \\
&= \left[ \frac{1}{[(\gamma_i - \alpha_i) \exp{-((\beta_i - \alpha_i)h) e^{-\alpha_i h_y} - (\gamma_i - \beta_i) e^{-\beta_i h_y}}} \\
&\times \left\{ [(\alpha_i - \delta_i) \exp{-((\beta_i - \alpha_i)h) e^{-\alpha_i h_y} - (\gamma_i - \beta_i) e^{-\beta_i h_y}}] \\
&\times \left\{ [(\alpha_i + \beta_i)(\gamma_i - \delta_i) - 2\gamma_i \delta_i - (\alpha_i^2 + \beta_i^2)] \right\} e^{(\delta_i - \gamma_i)h_y} - e^{(\delta_i - \gamma_i)h_y}] \\
&- [(\alpha_i - \delta_i)(\gamma_i - \beta_i) e^{(\delta_i - \gamma_i)h_y} - (\beta_i - \delta_i)(\gamma_i - \alpha_i) e^{(\delta_i - \gamma_i)h_y}] e^{(\beta_i - \alpha_i)(h-h_y)} \\
&- [(\beta_i - \delta_i)(\gamma_i - \alpha_i) e^{(\delta_i - \gamma_i)h_y} - (\alpha_i - \delta_i)(\gamma_i - \beta_i) e^{(\delta_i - \gamma_i)h_y}] e^{(\beta_i - \alpha_i)(h-h_y)} \\
&- \left\{ [(\alpha_i + \beta_i)(\gamma_i - \delta_i) - 2\gamma_i \delta_i - (\alpha_i^2 + \beta_i^2)] \right\} e^{(\gamma_i h_y - \gamma_i h_y)} \\
&- [(\alpha_i - \delta_i)(\gamma_i - \beta_i) e^{(\gamma_i h_y - \gamma_i h_y)} - (\beta_i - \delta_i)(\gamma_i - \alpha_i) e^{(\gamma_i h_y - \gamma_i h_y})}\right\}^{-1} \\
&+ \alpha_i \exp{-((\beta_i - \alpha_i)h) e^{-\alpha_i h_y} - (\beta_i e^{-\beta_i h_y}) e^{\gamma_i h_y} - (\beta_i e^{-\beta_i h_y}) e^{\gamma_i h_y}} e^{\gamma_i z} \\
&+ \left\{ [(\alpha_i + \beta_i)(\gamma_i + \delta_i) - 2\gamma_i \delta_i - (\alpha_i^2 + \beta_i^2)] \right\} e^{(\delta_i - \gamma_i)h_y} - e^{(\delta_i - \gamma_i)h_y}] \\
&- [(\alpha_i - \delta_i)(\gamma_i + \beta_i) e^{(\delta_i - \gamma_i)h_y} - (\beta_i - \delta_i)(\gamma_i - \alpha_i) e^{(\delta_i - \gamma_i)h_y}] e^{(\beta_i - \alpha_i)(h-h_y)} \\
&- [(\beta_i - \delta_i)(\gamma_i + \alpha_i) e^{(\delta_i - \gamma_i)h_y} - (\alpha_i - \delta_i)(\gamma_i + \beta_i) e^{(\delta_i - \gamma_i)h_y}] e^{(\beta_i - \alpha_i)(h-h_y)} \\
&- \left\{ [(\alpha_i + \beta_i)(\gamma_i - \delta_i) - 2\gamma_i \delta_i - (\alpha_i^2 + \beta_i^2)] \right\} e^{(\gamma_i h_y - \gamma_i h_y)} \\
&- [(\alpha_i - \delta_i)(\gamma_i + \beta_i) e^{(\gamma_i h_y - \gamma_i h_y)} - (\beta_i - \delta_i)(\gamma_i - \alpha_i) e^{(\gamma_i h_y - \gamma_i h_y})}\right\}^{-1} \\
&+ \alpha_i \exp{-((\beta_i + \alpha_i)h) e^{-\alpha_i h_y} - (\beta_i e^{-\beta_i h_y}) e^{\gamma_i h_y} - (\beta_i e^{-\beta_i h_y}) e^{\gamma_i h_y}} e^{\gamma_i z} + 1 \right]. \tag{43}
\end{align}\]

We note that these solution reduce to those in \[3\] in the absence of antisymmetric diffusion due to particle drift.

### 3. Exponential Variation of Cosmic Ray Spectrum

In this section, we discuss the features of the derived solutions, especially focusing on the question whether the diffusion model can reproduce the exponential variation of the cosmic ray spectrum. Here, we present qualitative examples and then omit the scales in the following figures.

In accordance with \[6\], we assume the diffusion coefficients have the following energy dependence

\[D_\perp = D_{D\perp} E^{m_D}, \quad d_\perp = d_{H\perp} E^{m_H}, \quad D_{A0} = D_{D\perp} E, \quad d_{A0} = d_{H\perp} E, \quad \tag{44}\]
log $I_i(E)$

$E$ Dependence of $I_i$

where the turbulence exponents in the disk and halo are denoted by $m_D$ and $m_H$, respectively, and for the energy spectrum at the source, we choose $\gamma_{i0} = 2.2$.

Fig. 2 shows the energy ($E$) dependence of the intensity $I_i$ of the model by Ptuskin and out three solutions: (1) $D_{D\perp} = d_{H\perp}$ and $D_{DA0} = d_{HA0}$ (shown as $D_D = d_H$); (2) the diffusion coefficient of the halo is larger than that of the disk, $D_{D\perp} < d_{H\perp}$ and $D_{D\perp} < d_{H\perp}$ (shown as $D_D < d_H$); and (3) the diffusion coefficient of the disk is larger than that of the halo, $D_{D\perp} > d_{H\perp}$ and $D_{D\perp} > d_{H\perp}$ (shown as $D_D > d_H$). In the case of our solution, the exponents of magnetic turbulence in the disk and the halo are equivalent, $m_D = m_H$. In Fig. 3, we multiplied $E^{2.7}$ with $I_i$ to emphasize the exponent variation. The three cases — $D_D = d_H$, $D_D < d_H$ and $D_D > d_H$ — can reproduce the variation in the cosmic ray spectrum in the same way, as the model developed by Ptuskin. Fig. 3 shows the $m$ dependence of our solutions: (1) the exponents of the disk and the halo are equivalent, $m_D = m_H$; (2) the exponent of the halo is larger than that of the disk, $m_D < m_H$; and (3) the exponent of the disk is larger than that of the halo, $m_D > m_H$. In all cases, we fixed $D_{D\perp} = d_{H\perp}$ and $D_{DA0} = d_{HA0}$. We found that all the solutions exhibit a similar trend, as shown in Fig. 2. Further, we may say that the difference between the two exponents around the knee, $\Delta \gamma = \gamma_{E < E_{\text{knee}}} - \gamma_{E_{\text{knee}} < E}$, is larger value for $m_D < m_H$ than that for $m_D > m_H$.

4. Conclusions

We phenomenologically proposed a propagation model of galactic cosmic rays based on the semi-empirical diffusion equation developed by Berezinskii et al. (1990) and the diffusion tensor introduced by Ptuskin et al. (1993). This model takes into account both the symmetric diffusion and the antisymmetric diffusion due to the particle Hall drift. The derived solutions are an extension of the model developed by Ptuskin et al. (1993) to a two-dimensional two-layer (galactic disk and halo) model,
and they coincide completely with the solutions derived by Berezinskii et al. (1990) in the absence of antisymmetric diffusion due to particle drift. We shown that this relatively simple model can be used to explain the variation in the exponent of the cosmic ray energy spectrum, $\gamma$, around the knee $E \approx 10^{15}$ eV.

In this paper, we showed that although the diffusive cosmic ray propagation model can be used to explain the observed cosmic ray spectrum, especially the exponential variation around the knee, our model is actually a more simple toy model based on assumptions such as the cylindrical structure of the galaxy and simplification of magnetic field. To further test the validity of the diffusion model, we must conduct numerical simulations under more realistic situations. This is a difficult task; nonetheless, it may help us to gain a deeper understanding of astroparticle physics and the galactic structure through which the cosmic rays pass.

References

1. C. J. Cesarsky, *Ann. Rev. Astron. Astrophys.*, 18, 289 (1980).
2. V. S. Ptsuskin, *Space Sci. Rev.*, 99, 281 (2001).
3. V. S. Berezinskii, S. V. Bulanov, V. A. Dogiel, V. L. Ginzburg and V. S. Ptsuskin, *Astrophysics of Cosmic Rays*, (North-Holland, 1990).
4. J. A. D. F. Pacheco, *A&A*, 13, 58 (1971).
5. P. L. Guet and J. A. D. F. Pacheco, *A&A*, 23, 337 (1973).
6. V. S. Ptsuskin, S. I. Rogovaya, V. N. Zirakashvili, L. G. Chuvilgin, G. B. Kristiansen, E. G. Klepach and G. V. Kulikov, *A&A*, 268, 726 (1993).
7. E. G. Berezhko and L. T. Ksenofontov, *JETP*, 89, 391 (1999).
8. T. Stanev, P. L. Biermann and T. K. Gaisser, *A&A*, 274, 902 (1993).
9. K. Kobayakawa, Y. S. Honda and T. Samura, *Phys. Rev. D*, 66, 083004 (2002).
10. L. G. Sveshnikova, *A&A*, 409, 799 (2003).
11. A. D. Erlykin and A. W. Wolfendale, *J. Phys. G: Nucl. Part. Phys.*, 27, 1005 (2001).
12. H. J. Völk and V. N. Zirakashvili, *Proc. 28th International Cosmic Ray Conference, 4*, 2031 (2003).
13. R. Plaga, *New Astron.*, **7**, 317 (2002).
14. S. P. Swordy, *Proc. 24th International Cosmic Ray Conference*, **2**, 697 (1995).
15. A. A. Lagutin, Yu. A. Nikulin and V. V. Uchaikin, *Nucl. Phys. B Proc. Suppl.*, **97**, 267 (2001).
16. S. Ogio and F. Kakimoto, *Proc. 28th International Cosmic Ray Conference*, **1**, 315 (2003).
17. E. Roulet, *Int. J. Mod. Phys. A*, **19**, 1133 (2004).
18. J. Candia, S. Mollerach and E. Roulet, *JHEP*, **0212**, 032 (2002).
19. J. Candia, E. Roulet and L. N. Epele, *JHEP*, **0212**, 033 (2002).
20. J. Candia, S. Mollerach and E. Roulet, *J. Cosm. Astropart. Phys.*, **0305**, 003 (2003).
21. W. Tkaczyk, *Proc. 27th International Cosmic Ray Conference*, **5**, 197309 (2001).
22. S. Karakula and W. Tkaczyk, *Astropart. Phys.*, **1**, 229 (1993).
23. M. T. Dova, L. N. Epele and J. D. Swain, astro-ph/0112191 (2001).
24. J. Candia, L. N. Epele and E. Roulet, *Astropart. Phys.*, **17**, 23 (2002).
25. D. Kazanas and A. Nicolaidis, *Proc. 27th International Cosmic Ray Conference*, 1760 (2001).
26. D. Kazanas and A. Nicolaidis, *Gen. Rel. Grav.*, **35**, 1117 (2003).
27. C. Heiles, *Interstellar Processes*, 171 (1987).
28. R. J. Rand and S. R. Kulkarni, *ApJ*, **343**, 760 (1989).
29. J. W. Armstrong and B. J. Rickett, *Mon. Not. R. Astro. Soc.*, **194**, 623 (1981).
30. A. A. Ruzmaikin, A. M. Shukurov and D. D. Sokoloff, *Magnetic Fields of Galaxies*, (Kluwer, Dordrecht, 1988).
31. R. H. Kraichnan, *Phys. Fluids*, **8**, 1385 (1965).
32. A. M. Bykov and I. N. Toptygin, *Astrophys. Space. Sci.*, **138**, 341 (1987).