THE COLORED HOMFLY-PT POLYNOMIALS OF THE TREFOIL KNOT, THE FIGURE-EIGHT KNOT AND TWIST KNOTS

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ABSTRACT. We give a rigorous proof of the colored HOMFLY-PT polynomials of the trefoil knot, the figure-eight knot and twist knots. For the trefoil knot and the figure-eight knot, it is expressed by a single sum, and for a twist knot, it is expressed by a double sum.

1. Introduction

After the articles [7, 8], the author feels that an explicit formula of the colored HOMFLY-PT polynomial of the figure-eight knot is desirable [1]. Actually, in [7], we devote to calculate the colored HOMFLY-PT polynomial of the figure-eight knot at a root of unity, which is expressed by a single sum. In [8], we treat twist knots, which is expressed by more than double sum depending on the number of full twists. Other expressions are also conjectured for the figure-eight knot [11] and the twist knot [12], which is expressed by as a single sum and a double sum, respectively. Therefore, in this article, we give a rigorous proof of the colored HOMFLY-PT polynomial of twist knots. This article is a generalization of the colored Jones polynomials [2, 3, 11].

The proof is similar as [11]. That is, for a positive integer \( n \), we construct an element \( \omega = \omega_n^+ \) of the HOMFLY-PT skein module satisfying Figure 1, where the horizontal arc consists of the two \( i \)-th \( q \)-symmetrizers with opposite orientations \( (i = 0, \ldots, n) \):

\[
\begin{array}{c}
\omega
\end{array}
\]

\[ = \]

\[
\begin{array}{c}
\omega
\end{array}
\]

Figure 1. \( \omega \) and a positive full twist

This article is organized as follows. In Section 2, we recall some notations and notions concerning the skein module derived from the HOMFLY-PT skein relations. In Section 3, we develop some eigenvalues of an element of the skein module. Using the eigenvalues, we define the element \( \omega \). In Section 4, for twist knots, we discuss a formula of \( p \) full twists. In Section 5, we calculate the colored HOMFLY-PT polynomial of twist knots. As a corollary, for the trefoil and the figure-eight knot, we can express it as a single sum.

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Although we only treat twist knots in this article, we hope that the twisting formula of Section 4 leads an application to the colored HOMFLY-PT polynomial of knots which contain twisting arcs.

2. Preliminaries

In this section, we review some notation including $q$-integers and $q$-symmetrizers. Let $a$ and $q$ be non-zero variables in $\mathbb{C}$. For an integer $n$, we define the symbols by

\[
[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad \{n\} = q^n - q^{-n}, \quad \{n; a\} = aq^n - a^{-1}q^{-n}.
\]

For integers $n > 0$, $i \geq 0$, we define the products of $i$ terms of these symbols by

\[
[n]_i = [n][n-1] \cdots [n-i+1], \quad \{n\}_i = \{n\}\{n-1\} \cdots \{n-i+1\},
\]

\[
\{n; a\}_i = \{n; a\}\{n-1; a\} \cdots \{n-i+1; a\}, \quad \{-n; a\}_i = \{-n; a\}\{-n+1; a\} \cdots \{-n+i-1; a\},
\]

where they are defined to be 1 if $i = 0$. Furthermore, we define

\[
[n]! = [n]_n, \quad \{n\}_i = \{n\}_n, \quad \binom{n}{i} = \frac{[n]!}{[i]![n-i]!}.
\]

We recall the HOMFLY-PT skein module, $S(M)$, of an oriented 3-manifold $M$. $S(M)$ is the free $\mathbb{C}$-module generated by isotopy classes of framed links in $M$ modulo the submodule generated by the HOMFLY-PT skein relations:

(i) $L \cup (\text{a trivial knot with 0 framing}) = \{n; a\}_1 L$, and $\emptyset = 1$,

(ii) $\tikzfig{diagram1} = (q - q^{-1}) \tikzfig{diagram2}$,

(iii) $\tikzfig{diagram3} = a \tikzfig{diagram4}$, \quad $\tikzfig{diagram5} = a^{-1} \tikzfig{diagram6}$.

We call a crossing positive or negative if it is the same crossing as the first or second terms in (ii), respectively.

Following [5][6][8], we review the definitions of the $q$-symmetrizer, the $q$-antisymmetrizer, and their properties. For an integer $n \geq 1$, we recursively define the $n$-th $q$-symmetrizer and the $n$-th $q$-antisymmetrizer by Figure 2 where the $q$-symmetrizer is denoted by a white rectangle and the $q$-antisymmetrizer is denoted by a black rectangle.

\[
\begin{align*}
\begin{tikzpicture}
\node (a) at (1,0) {
\begin{tikzpicture}
\fill[white] (-0.5,0) rectangle (0.5,0);
\end{tikzpicture}
};
\node (b) at (2,0) {
\begin{tikzpicture}
\fill[white] (-0.5,0) rectangle (0.5,0);
\end{tikzpicture}
};
\node (c) at (1.5,0) {
\begin{tikzpicture}
\fill[white] (-0.5,0) rectangle (0.5,0);
\end{tikzpicture}
};
\draw (a) -- (b); \draw (b) -- (c);
\end{tikzpicture}
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}
\node (a) at (1,0) {
\begin{tikzpicture}
\fill[white] (-0.5,0) rectangle (0.5,0);
\end{tikzpicture}
};
\node (b) at (2,0) {
\begin{tikzpicture}
\fill[white] (-0.5,0) rectangle (0.5,0);
\end{tikzpicture}
};
\node (c) at (1.5,0) {
\begin{tikzpicture}
\fill[white] (-0.5,0) rectangle (0.5,0);
\end{tikzpicture}
};
\draw (a) -- (b) -- (c);
\end{tikzpicture}
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}
\node (a) at (1,0) {
\begin{tikzpicture}
\fill[white] (-0.5,0) rectangle (0.5,0);
\end{tikzpicture}
};
\node (b) at (2,0) {
\begin{tikzpicture}
\fill[white] (-0.5,0) rectangle (0.5,0);
\end{tikzpicture}
};
\node (c) at (1.5,0) {
\begin{tikzpicture}
\fill[white] (-0.5,0) rectangle (0.5,0);
\end{tikzpicture}
};
\draw (a) -- (b) -- (c);
\end{tikzpicture}
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}
\node (a) at (1,0) {
\begin{tikzpicture}
\fill[white] (-0.5,0) rectangle (0.5,0);
\end{tikzpicture}
};
\node (b) at (2,0) {
\begin{tikzpicture}
\fill[white] (-0.5,0) rectangle (0.5,0);
\end{tikzpicture}
};
\node (c) at (1.5,0) {
\begin{tikzpicture}
\fill[white] (-0.5,0) rectangle (0.5,0);
\end{tikzpicture}
};
\draw (a) -- (b) -- (c);
\end{tikzpicture}
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}
\node (a) at (1,0) {
\begin{tikzpicture}
\fill[white] (-0.5,0) rectangle (0.5,0);
\end{tikzpicture}
};
\node (b) at (2,0) {
\begin{tikzpicture}
\fill[white] (-0.5,0) rectangle (0.5,0);
\end{tikzpicture}
};
\node (c) at (1.5,0) {
\begin{tikzpicture}
\fill[white] (-0.5,0) rectangle (0.5,0);
\end{tikzpicture}
};
\draw (a) -- (b) -- (c);
\end{tikzpicture}
\end{align*}
\]

Figure 2. The definitions of symmetrizers
The integer \( n \) beside an arc means \( n \) copies of the arc. When we can easily find the number of copies, we omit it.

The \( q \)-symmetrizer and the \( q \)-antisymmetrizer have the following properties, where the positive crossing in Figure 3 means only one crossing:

\[
\begin{align*}
\begin{array}{c}
\text{Figure 3. Properties of symmetrizers}
\end{array}
\end{align*}
\]

Using the \( m \)-th and \( n \)-th \( q \)-symmetrizers, we define the \((m,n)\)-th \( q \)-symmetrizer by

\[
\begin{align*}
\begin{array}{c}
\text{Figure 4. The definition of the \((m,n)\)-th \( q \)-symmetrizer}
\end{array}
\end{align*}
\]

where \( x_{m,n}^i \) is given by

\[
x_{m,n}^i = (-1)^i \binom{m}{i} \binom{n}{i} \frac{i!}{(m+n-2; a)_i}.
\]

These three symmetrizers have idempotent properties and vanishing properties as in Figure 5:

\[
\begin{align*}
\begin{array}{c}
\text{Figure 5. Idempotent and vanishing properties}
\end{array}
\end{align*}
\]

The next three lemmas are useful for calculations of the HOMFLY-PT skein module in Section 3.

**Lemma 2.1** (§). *The following hold.*
\[ \sum_{i=0}^{\min\{m,n\}} \alpha_{m,n}^i \]

Figure 6.

where \( \alpha_{m,n}^i = \alpha_{m,n}^i(a,q) \) is given by

\[ (-a)^{-i} q^{-i(m+n) + i(i+3)} \frac{\{m\}_i \{n\}_i}{\{i\}!} \]

In the same way as Lemma 2.1 by applying the vanishing property of \( H_m \) and \( E_n \), we have the next lemma.

**Lemma 2.2.** The following hold.

\[ m_n = -a^{-1} q^{m-n} \frac{\{m\}\{n\}}{\{1\}} \]

Figure 7.

**Remark 2.3.** In Figure 6 and 7 when all the negative crossings change into positive crossings, the coefficients also change so that \( a,q \) are replaced by \( a^{-1}, q^{-1} \), respectively. For our convenience, we set \( \bar{\alpha}_{m,n}^i = \alpha_{m,n}^i(a^{-1}, q^{-1}) \).

**Lemma 2.4.** For integers \( 0 \leq i \leq j \leq \min\{m,n\} \), we have

\[ \sum_{k=0}^{i} \beta_{k,i,j;m,n} \]

Figure 8.

where \( \beta_{k,i,j;m,n} = \beta_{k,i,j;m,n}(a,q) \) is given by

\[ \frac{\{m-j\}_k \{n-j\}_k \{j\}_{i-k}}{\{m\}_i \{n\}_i \{i\}_k} (m+n-j-k-1; a)_{i-k} \]

**Remark 2.5.** In Figure 6 and Figure 8 when all the \( q \)-symmetrizers change into \( q \)-antisymmetrizers, the coefficients also change so that \( q \) is replaced by \( -q^{-1} \).

The skein module of the solid torus \( S^1 \times D^2 \) is denoted by \( \mathcal{S} \). Here, we consider submodules of \( \mathcal{S} \). For a circle along \( S^1 \) of the solid torus, we define \( H_n \in \mathcal{S} \) by \( n \) copies of the circles inserted by the \( n \)-th \( q \)-symmetrizer. Similarly, we define \( E_n \) by the \( n \)-th \( q \)-antisymmetrizer. Each symmetrizer is compatible with the orientation of the circle. We define \( H_{m,n} \) by two copies of \( H_n \), one is anticlockwise and the other is clockwise. For two circles along \( S^1 \), one is anticlockwise and the other is clockwise, we define \( D_{m,n} \) by \( m \) copies of the anticlockwise circle and \( n \) copies of the clockwise inserted by the \( (m,n) \)-th \( q \)-symmetrizer.
Let $\mathcal{H}_{n,n}$ and $\mathcal{D}_{n,n}$ be the submodules spanned by $H_{i,i}$ and $D_{i,i}$ for $i = 0, \ldots, n$, respectively. Then, Lemma 2.4 implies that $\mathcal{D}_{n,n}$ is spanned by $H_{i,i}$ for $i = 0, \ldots, n$, and vice versa. Hence, we have $\mathcal{D}_{n,n} = \mathcal{H}_{n,n}$.

Let $\langle \cdot \rangle$ be the linear map on $\mathcal{S}$ to $\mathbb{C}$ defined by evaluating it in $S^3$. Let $t: \mathcal{S} \rightarrow \mathcal{S}$ be the twist map induced by one right-handed twist on the solid torus as shown in the right-hand side of Figure [1] where the twist is induced at the bottom of the solid torus. Similarly, let $t^{-1}$ be the twist map induced by one left-hand twist. For $x \in \mathcal{S}$, let $e_x: \mathcal{S} \rightarrow \mathcal{S}$ be the map encircling an element of $\mathcal{S}$ by $x$ as shown in the left-hand side of Figure [1] where $x$ slides into the bottom of the solid torus and encircles the element. In Section 3, we find two elements $\omega_n^+$ and $\omega_n^-$ so that $e_{\omega_n^\pm}(D_{n,n}) = t^{\pm 1}(D_{n,n})$. We give some examples of these maps:

$$
\langle H_n \rangle = \frac{\{n-1; a\}_n}{\{n\}!},
$$
$$
\langle E_n \rangle = \frac{\{-n+1; a\}_n}{\{n\}!},
$$
$$
t(H_n) = a^n q^{n(n-1)} H_n,
$$
$$
t(E_n) = a^n (-q)^{-n(n-1)} E_n,
$$
$$
t(D_{m,n}) = a^m n^q^{m(m-1)+n(n-1)} D_{m,n}.
$$

**Lemma 2.6.** For positive integers $m \geq n$, we have the following:

$$
\langle D_{m,n} \rangle = \frac{\{m+n-1; a\} \{m-2; a\}_{m-1} \{n-2; a\}_{n-1} (-1; a)}{\{m\}! \{n\}!}.
$$

**Proof.** Since $\langle D_{m,n} \rangle$ is described as follows,

\begin{align*}
\sum_{i=0}^{n} x^i_{m,n} &= x^0_{m,n} \quad = \sum_{i=0}^{n} x^i_{m,n} \\
\sum_{i=0}^{n} x^i_{m,n} &= x^0_{m,n} \quad = \sum_{i=0}^{n} x^i_{m,n}
\end{align*}

**Figure 9.** The calculation of $\langle D_{m,n} \rangle$

$\langle D_{m,n} \rangle$ is given by

$$
\sum_{i=0}^{n} x^i_{m,n} \frac{\{m-1; a\}_{m-i} \{n-1; a\}_{n-i} \{i-1; a\}_i}{\{m\}! \{n\}! \{i\}!} = \sum_{i=0}^{n} (-1)^i \frac{\{m+n-1; a\}_i \{m-1; a\}_{m-i} \{n-1; a\}_{n-i}}{\{m\}! \{n\}! \{i\}!} = \frac{\{m-1; a\}_m}{\{m+n-2; a\}_n \{m\}! \{n\}!} \times \sum_{i=0}^{n} (-1)^i \{m\}_i \{n\}_i \{m+n-i-2; a\}_{n-i} \{n-1; a\}_{n-i}.
$$

Here, we use the following transformation:

$$
\{m-1; a\} \{n-1; a\} = \{m+n-i-1; a\} \{i-1; a\} + \{m-i\} \{n-i\}.$$
Then, each term in the sum is expressed by

\[ \{m\}_i\{n\}_i\{m + n - i - 2; a\}_{n-i}\{n - 1; a\}_{n-i} = \{m\}_i\{n\}_i\{m + n - i - 2; a\}_{n-i-1}\{m - 1; a\}_{n-i}\{n - 1; a\}_{n-i-1} = \{m\}_i\{n\}_i\{m + n - i - 1; a\}_{n-i}\{n - 2; a\}_{n-i} + \{m\}_{i+1}\{n\}_{i+1}\{m + n - i - 2; a\}_{n-i-1}\{n - 2; a\}_{n-i-1}. \]

Hence, we have

\[ \sum_{i=0}^{n} (-1)^i\{m\}_i\{n\}_i\{m + n - i - 2; a\}_{n-i}\{n - 1; a\}_{n-i} = \sum_{i=0}^{n-1} (-1)^i\{m\}_i\{n\}_i\{m + n - i - 1; a\}_{n-i}\{n - 2; a\}_{n-i} + \{m\}_{i+1}\{n\}_{i+1}\{m + n - i - 2; a\}_{n-i-1}\{n - 2; a\}_{n-i-1} \]

\[ + \{m\}_n\{n\}_n = \{m + n - 1; a\}_n\{n - 2; a\}_n. \]

Therefore, we obtain

\[ \langle D_{m,n} \rangle = \frac{\{m - 1; a\}_m\{m + n - 2; a\}_n\{m + n - 1; a\}_n\{n - 2; a\}_n}{\{m\}_m\{n\}_n!} = \frac{\{m + n - 1; a\}_m\{m - 2; a\}_n\{n - 2; a\}_n\{n - 1; a\}}{\{m\}_m\{n\}_n!}. \]

\[ \square \]

3. Eigenvalues of $D_{n,n}$

In this section, we find several eigenvalues of $D_{n,n}$ concerning the encircling map $e$ and the twist map $t$.

3.1. The eigenvalue for the encircling map of $H_i$.

**Lemma 3.1.** For an integer $i \geq 0$, we have $e_{H_i}(D_{n,n}) = \sigma_{n,i}D_{n,n}$, where

\[ \sigma_{n,i} = \sum_{0 \leq j, k \leq i, j < k} \left( (-1)^j a^{-j+k}q^{j,k} + (-1)^k a^{-k+j}q^{k,j} \right) \frac{\{n\}_j\{n\}_k}{\{i - j - k\}!} \{i - 1; a\}_{i-j-k} \]

\[ + \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^j \frac{\{n\}_j\{n\}_j}{\{i - 2j\}!} \{i - 1; a\}_{i-2j}, \]

\[ \epsilon_{j,k} = (-j + k)(i + n) + \frac{j(j + 3)}{2} - \frac{k(k + 3)}{2}. \]

**Proof.** $e_{H_i}(D_{n,n})$ is described by
where $k'_i = k - j + l, j'_i = j - k + l$. From the vanishing property of $D_{n,n}$, the first double sum does not vanish if $l = j$, and the second double sum does not vanish if $l = k$. Hence, we obtain

$$\sigma_{n,i} = \sum_{0 \leq j \leq k \leq i} \alpha_{i,n}^j \bar{\alpha}_{i,n}^k \beta_{i-k,i-j,i,i}^j + \sum_{0 \leq k < j \leq i} \alpha_{i,n}^j \bar{\alpha}_{i,n}^k \beta_{i-j,i-k,i,i}^k.$$  

Furthermore, since $\beta$ has the following symmetry:

$$\beta_{i-k,i-j,i,i}^j = \frac{j!}{(i)_i (i-k)_k} \binom{i-k}{j} [i-1; a]_{i-j-k},$$

$\beta$ has the following symmetry:

$$\beta_{i-j,i-k,i,i}^k = \beta_{i-j,i-k,i,i}^k.$$  

$\sigma_{n,i}$ is given by

$$\sigma_{n,i} = \sum_{0 \leq j+k \leq i} (\alpha_{i,n}^j \bar{\alpha}_{i,n}^k + \alpha_{i,n}^k \bar{\alpha}_{i,n}^j) \beta_{i-k,i-j,i,i}^j + \sum_{j=0}^{\left\lfloor \frac{i}{2} \right\rfloor} \alpha_{i,n}^j \bar{\alpha}_{i,n}^k \beta_{i-j,i-j,i,i}^j$$

$$= \sum_{0 \leq j+k \leq i} \left( (-1)^j a^{-j+k} q^{s,i,k} + (-1)^k a^{-k+j} q^{s,j}q^{s,i} \right) \{n\}_j \{n\}_k \{i-1; a\}_{i-j-k}$$

$$+ \sum_{j=0}^{\left\lfloor \frac{i}{2} \right\rfloor} (-1)^j \{n\}_j \{n\}_j \{i-1; a\}_{i-2j}.$$  

□
3.2. The eigenvalue for the encircling map of $E_i$.

**Lemma 3.2.** For an integer $i \geq 0$, we have $e_{E_i}(D_{n,n}) = \tau_{n,i}D_{n,n}$, where

$$\tau_{n,i} = \frac{-i + 1}{i!} + \frac{(-i;a)_{i-1}}{(i-1)!} \{n\} \{n-1; a\}.$$

**Proof.** $e_{E_i}(D_{n,n})$ is described by

$$\begin{array}{c}
\includegraphics[width=0.8\textwidth]{figure11.png}
\end{array}$$

Then, the eigenvalue is given by

$$\begin{align*}
\frac{-i + 1}{i!} + (aq^{n-i} - a^{-1}q^{-n+i}) \frac{\{i\}\{n\}}{(1)} \\
- \frac{(i)^2}{1^2} \frac{n^2}{\{i-1\}^2} \beta_{i-1,i-1,i}^1 (a, -q^{-1}) \\
= \frac{-i + 1}{i!} + \frac{(-i;a)_{i-1}}{(i-1)!} \{n\} \{n-1; a\}.
\end{align*}$$

\[ \Box \]

**Remark 3.3.** For $n = 0$, we have the following:

$$\sigma_{0,i}(H_i) = \frac{(i-1; a)_{i}}{\{i\}!}, \quad \tau_{0,i}(E_i) = \frac{-i + 1}{i!}.$$

3.3. The eigenvalue for the encircling map of $R_i$. In this subsection, for $i = 0, \ldots, n$, we define elements $R_i \in S$, which are an important role to calculate the colored HOMFLY-PY polynomial. Actually, because the behavior of $H_n$ concerning the encircling map is very complicated, we set $R_n$ as a linear combination of $H_i$ ($i = 0, \ldots, n$) so that the behavior of $R_n$ is simple.
Definition. For \( i = 0, \ldots, n \), we recursively define elements \( R_i \) as follows:

\[
R_0 = H_0 = 1,
R_1 = H_1 - \frac{\{0; a\}}{\{1\}},
R_2 = H_2 - \frac{\{2; a\}}{\{1\}} R_1 - \frac{\{1; a\}_2}{\{2\}!} R_0,
\]

\vdots

\[
R_n = H_n - \sum_{i=0}^{n-1} \frac{\{n - 1 + i; a\}_{n-i}}{(n-i)!} R_i.
\]

Proposition 3.4. For an integer \( i \geq 0 \), we have \( e_{R_i}(D_{n,n}) = \theta_{n,i} D_{n,n} \), where \( \theta_{n,i} = \{ n \} \{ n + i - 2; a \} \).

Proof. By the definition of \( R_n \), each \( H_i \) is expressed by

\[
H_i = \sum_{j=0}^{i} \frac{\{i - 1 + j; a\}_{i-j}}{(i-j)!} R_j.
\]

To show the statement, it is enough to prove that the map \( e_{H_i}(D_{n,n}) \) has the eigenvalue

\[
\sum_{j=0}^{i} \frac{\{i - 1 + j; a\}_{i-j}}{(i-j)!} \theta_{n,j}.
\]

We will show it by induction. The statement holds for \( i = 0, 1, 2 \) by the definition of \( H_0 \) and Lemma 3.1. To use the hypothesis of induction, we express \( H_i \) by \( H_j \) for \( j = 0, \ldots, i - 1 \). By the determinant formula of Proposition 4 in \( [5] \), we have

\[
H_i = \sum_{j=1}^{i} (-1)^{i-j-1} H_{i-j} E_j
= H_{i-1} E_1 - H_{i-2} E_2 + H_{i-2} E_2 - \cdots + (-1)^{i-1} E_i.
\]

We apply the hypothesis of induction to \( H_j(j = 0, \ldots, i - 1) \) and Lemma 3.2 to \( E_j(j = 1, \ldots, i) \). To make the proof easy to read and avoid long equations, we set \( \eta_{n,k} \) by

\[
\eta_{n,k} = \{ n - k \} \{ n + k - 1; a \}.
\]
Then, the eigenvalue induced by $e_H$ is given by

\[
\begin{align*}
&\left( \sum_{k=0}^{i-1} \frac{\{i-2+k; a\}_i}{\{i-1-k\}!} \theta_{n,k} \right) \left( \eta_{n,0} + \frac{\{0; a\}}{\{1\}} \right) \\
&- \left( \sum_{k=0}^{i-2} \frac{\{i-3+k; a\}_i}{\{i-2-k\}!} \theta_{n,k} \right) \left( \eta_{n,0} + \frac{\{-2; a\}_2}{\{2\}!} \right) \\
&+ \left( \sum_{k=0}^{i-3} \frac{\{i-4+k; a\}_i}{\{i-3-k\}!} \theta_{n,k} \right) \left( \eta_{n,0} + \frac{\{-2; a\}_3}{\{3\}!} \right) \\
&\vdots \\
&+ (-1)^{i-1} \left( \frac{\{-i; a\}_i}{\{i-1\}!} \eta_{n,0} + \frac{\{-i+1; a\}_i}{\{i\}!} \right).
\end{align*}
\]

To explain more simply, we first demonstrate how $\theta_{n,i}, \theta_{n,i-1}, \theta_{n,i-2}$ and their coefficients are obtained. We remark that $\eta_{n,0}$ satisfies the following identities:

\[
\eta_{n,0} = \eta_{n,i-1} + \eta_{i-1,0} = \eta_{n,i-2} + \eta_{i-2,0} = \eta_{n,i-3} + \eta_{i-3,0}.
\]

We apply these identities to the first three terms $[1], [2], [3]$. Then, the eigenvalue is described by

\[
\begin{align*}
&\left( \theta_{n,i-1} \left( \eta_{n,i-1} + \eta_{i-1,0} + \frac{\{0; a\}}{\{1\}} \right) \\
&+ \frac{\{2i-4; a\}_2}{\{1\}!} \theta_{n,i-2} \left( \eta_{n,i-2} + \eta_{i-2,0} + \frac{\{0; a\}}{\{1\}} \right) \\
&+ \frac{\{2i-5; a\}_3}{\{2\}!} \theta_{n,i-3} \left( \eta_{n,i-3} + \eta_{i-3,0} + \frac{\{0; a\}}{\{1\}} \right) \\
&+ \left( \sum_{k=0}^{i-4} \frac{\{i-2+k; a\}_i}{\{i-1-k\}!} \theta_{n,k} \right) \left( \eta_{n,0} + \frac{\{0; a\}}{\{1\}} \right) \right) \\
&- \left( \theta_{n,i-2} \left( \frac{\{-2; a\}_2}{\{1\}!} \eta_{n,i-2} + \eta_{i-2,0} + \frac{\{-1; a\}_2}{\{2\}!} \right) \\
&+ \frac{\{2i-6; a\}_4}{\{1\}!} \theta_{n,i-3} \left( \frac{\{-2; a\}_2}{\{1\}} \eta_{n,i-3} + \eta_{i-3,0} + \frac{\{-1; a\}_2}{\{2\}!} \right) \\
&+ \left( \sum_{k=0}^{i-4} \frac{\{i-3+k; a\}_i}{\{i-2-k\}!} \theta_{n,k} \right) \left( \frac{\{-2; a\}_2}{\{1\}!} \eta_{n,0} + \frac{\{-1; a\}_2}{\{2\}!} \right) \right) \\
&+ \left( \theta_{n,i-3} \left( \frac{\{-3; a\}_3}{\{2\}!} \eta_{n,i-3} + \eta_{i-3,0} + \frac{\{-2; a\}_3}{\{3\}!} \right) \\
&+ \left( \sum_{k=0}^{i-4} \frac{\{i-4+k; a\}_i}{\{i-3-k\}!} \theta_{n,k} \right) \left( \frac{\{-3; a\}_3}{\{2\}!} \eta_{n,0} + \frac{\{-2; a\}_3}{\{3\}!} \right) \right) + \cdots.
\end{align*}
\]

Using the following identity:

\[
\theta_{n,k-1} \eta_{n,k-1} = \theta_{n,k} \quad (k = 1, \ldots),
\]

where $\theta_{n,k}$ are the coefficients of the eigenvalue.
Furthermore, we rearrange the following equation:

\[
\theta_{n,i} + \left( n_{-1,0} + \frac{\{0; a\}}{\{1\}} + \frac{\{2i - 4; a\}}{\{1\}} - \frac{\{-2; a\}}{\{1\}} \right) \theta_{n,i-1} \\
+ \left( \frac{\{2i - 4; a\}}{\{1\}} \left( n_{-2,0} + \frac{\{0; a\}}{\{1\}} + \frac{\{2i - 5; a\}_2}{\{2\}} - \frac{\{-2; a\}}{\{1\}} \right) n_{-2,0} \right) \eta_{n,i-2} + \ldots
\]

Next, we consider a general situation. The first term \([1]\) generates \(\theta_{n,i}, \theta_{n,i-1}, \ldots, \theta_{n,0}\). The second term \([2]\) generates \(\theta_{n,i-1}, \theta_{n,i-2}, \ldots, \theta_{n,0}\), and the third term \([3]\) generates \(\theta_{n,i-2}, \theta_{n,i-3}, \ldots, \theta_{n,0}\). Furthermore, this demonstration implies that the \(j\)-th term generates \(\theta_{n,i+j+1}, \theta_{n,i+j}, \ldots, \theta_{n,0}\). Therefore, \(\theta_{n,i+j+1}\) comes from the first, the second, ..., the \(j\)-th terms. By applying the identity

\[
\eta_{n,0} = \eta_{n,i} + \eta_{i,0} \quad (i = 1, \ldots),
\]

we express the \(l\)-th term as follows:

\[
(-1)^{l-1} \sum_{k=0}^{i-l} \frac{(i - l - 1 + k; a)_{i-l-k} \theta_{n,k}}{(i - l - k)!} \left( \frac{\{-l; a\}_{i-l} \eta_{n,0} + \{-l + 1; a\}_l}{\{l\}!} \right)
\]

\[
= (-1)^{l-1} \sum_{k=0}^{i-l} \frac{(i - l - 1 + k; a)_{i-l-k} \theta_{n,k}}{(i - l - k)!} \left( \frac{\{-l; a\}_{i-l} \eta_{n,k} + \eta_{n,0} + \{-l + 1; a\}_l}{\{l\}!} \right)
\]

\[
= (-1)^{l-1} \sum_{k=0}^{i-l} \frac{(i - l - 1 + k; a)_{i-l-k} \theta_{n,k+1}}{(i - l - k)!} \left( \frac{\{-l; a\}_{i-l} \eta_{n,k} + \{-l + 1; a\}_l}{\{l\}!} \right) \theta_{n,k}
\]

We consider the contribution of the coefficient of \(\theta_{n,i-j+1}\) for \(l = 1, \ldots, j\). In the upper term including \(\theta_{n,k+1}\), the contribution yields when \(k = i - j\). In the lower term including \(\theta_{n,k}\), the contribution yields when \(k = i - j + 1\). Hence, the coefficient of \(\theta_{n,i-j+1}\) is described by

\[
\sum_{l=1}^{j} (-1)^{l-1} \frac{(2i - j - l - 1; a)_{j-l} \{-l; a\}_{l-1}}{(j - l)!} \frac{\{-l; a\}_{l-1}}{(l - 1)!}
\]

\[
+ \sum_{l=1}^{j-1} (-1)^{l-1} \frac{(2i - j - l; a)_{j-l-1} \{-l; a\}_{l-1}}{(j - l - 1)!} \frac{\{-l; a\}_{l-1}}{(l - 1)!} \eta_{n,j+1,0} + \frac{\{-l + 1; a\}_l}{\{l\}!}
\]

Furthermore, we rearrange the following equation:

\[
\frac{(2i - j - l - 1; a)_{j-l} \{-l; a\}_{l-1}}{(j - l)!} \frac{\{-l; a\}_{l-1}}{(l - 1)!}
\]

\[
+ \frac{(2i - j - l; a)_{j-l-1} \{-l; a\}_{l-1}}{(j - l - 1)!} \frac{\{-l; a\}_{l-1}}{(l - 1)!} \eta_{n,j+1,0} + \frac{\{-l + 1; a\}_l}{\{l\}!}
\]
Using the identity
\[ \eta_{n,a} = \{n\} \{n-1; a\} = \frac{\{2n; a\}}{\{1\}} - \frac{\{2(n-1); a\}}{\{1\}} + \frac{\{-2; a\}}{\{1\}} - \frac{\{0; a\}}{\{1\}}, \]
it is expressed by
\[
\begin{aligned}
\frac{\{2i - j - l; a\}_{j-l-1}}{\{j - l - 1\}! \{l-1\}!} & \cdot \left( \frac{\{2(i - j + 1); a\}}{\{1\}} - \frac{\{2(i - j); a\}}{\{1\}} \right) \\
&+ \frac{\{2i - j - l; a\}_{j-l-1}}{\{j - l - 1\}! \{l-1\}!} \cdot \left( \frac{\{-2; a\}}{\{1\}} - \frac{\{0; a\}}{\{1\}} \right) \\
&+ \frac{\{2i - j - l; a\}_{j-l-1}}{\{j - l - 1\}! \{l-1\}!} \cdot \left( \frac{\{-l + 1; a\}}{\{1\}} \right)
\end{aligned}
\]
\[
\begin{aligned}
&= \left( \frac{\{2i - j - l; a\}_{j-l-1}}{\{j - l - 1\}! \{l-1\}!} \cdot \left( \frac{\{2(i - j + 1); a\}}{\{1\}} - \frac{\{2(i - j); a\}}{\{1\}} \right) \\
&+ \frac{\{2i - j - l; a\}_{j-l-1}}{\{j - l - 1\}! \{l-1\}!} \cdot \left( \frac{\{-2; a\}}{\{1\}} - \frac{\{0; a\}}{\{1\}} \right) \\
&+ \frac{\{2i - j - l; a\}_{j-l-1}}{\{j - l - 1\}! \{l-1\}!} \cdot \left( \frac{\{-l + 1; a\}}{\{1\}} \right)
\end{aligned}
\]
\[
\begin{aligned}
&= \frac{\{2i - j - l - 1; a\}_{j-l-2}}{\{j - l\}! \{l-1\}!} \cdot \left( \{2i - j - l; a\} \left( \frac{\{2(i - j + 1); a\}}{\{1\}} - \frac{\{2(i - j); a\}}{\{1\}} \right) \\
&+ \{2(i - j) + 1; a\} \{2(i - j); a\} \right) \\
&+ \frac{\{2i - j - l; a\}_{j-l-1}}{\{j - l - 1\}! \{l-1\}!} \cdot \left( \frac{\{-l + 1; a\}}{\{1\}} \right) \\
&\times \left( \{l\} \{-l; a\} \left( \frac{\{-2; a\}}{\{1\}} - \frac{\{0; a\}}{\{1\}} \right) + \{1; a\} \{0; a\} \right)
\end{aligned}
\]
\[
\begin{aligned}
&= \frac{\{2i - j - l - 1; a\}_{j-l-2}}{\{j - l\}! \{l-1\}!} \cdot \left( \{2i - j - l + 1; a\} \{2i - j - l; a\} + \{j - l\} \{j - l - 1\} \right) \\
&+ \frac{\{2i - j - l; a\}_{j-l-1}}{\{j - l - 1\}! \{l-1\}!} \cdot \left( \frac{\{-l + 1; a\}}{\{1\}} \right) \\
&\times \left( \{l\} \{-l; a\} \left( \frac{\{-2; a\}}{\{1\}} - \frac{\{0; a\}}{\{1\}} \right) + \{1; a\} \{0; a\} \right)
\end{aligned}
\]
Finally, it is given by
\[
\begin{aligned}
\frac{\{2i - j - l + 1; a\}_{j-l-1}}{\{j - l\}! \{l-1\}!} & + \frac{\{2i - j - l - 1; a\}_{j-l-2}}{\{j - l - 2\}! \{l-1\}!} \\
&+ \frac{\{2i - j - l; a\}_{j-l-1}}{\{j - l - 1\}! \{l-1\}!} + \frac{\{2i - j - l; a\}_{j-l-2}}{\{j - l - 2\}! \{l-2\}!}
\end{aligned}
\]
Here, we consider the term \( \{ l : a \}_i \) as 0 when \( i \) is negative. Then, the above expression is valid for \( l = 1, \ldots, j \). We sum up these terms from \( l = 1 \) to \( l = j \) with multiplication by \((-1)^{l-1}\). It is equal to

\[
\frac{\{2i - j; a\}_{j-1}}{(j - 1)!}.
\]

This is the coefficient of \( \theta_{n,i-j+1} \). This completes the proof. \( \square \)

Since \( \{n\}_i = 0 \) for \( i > n \), we have the following corollary.

**Corollary 3.5.** For an integer \( i > n \), we have \( e_R(D_{n,n}) = 0 \).

### 3.4. The eigenvalue for the encircling map of \( \omega_n \)

In the last subsection, we discuss a relation between the encircling map \( e \) and the twist maps \( t \). That is, for any \( x \in \mathcal{H}_{n,n} \), we would like to find an element \( \omega_n = \omega_n^x \in S \) such that \( e_{\omega_n}(x) = t(x) \) for the positive twist map \( t \). In fact, \( \omega_n \) is given by the following definition.

**Definition.** For an integer \( n \geq 0 \), we define \( \omega_n \in S \) by

\[
\omega_n = \sum_{i=0}^{n} t_i R_i, \quad \text{where} \quad t_i = a^i q^{\frac{(i-1)}{2}}.
\]

Since \( \{D_{i,i} : i = 0, \ldots, n\} \) is a basis of \( \mathcal{H}_{n,n} \) and \( t(D_{i,i}) = a^{2i} q^{2(i-1)} D_{i,i} \), if we would like to show \( e_{\omega_n}(x) = t(x) \), it is enough to show that \( e_{\omega_n}(D_{i,i}) = a^{2i} q^{2(i-1)} D_{i,i}, \quad (i = 0, \ldots, n) \).

However, by Corollary 3.5, it is sufficient to show that

\[
e_{\omega_n}(D_{i,i}) = a^{2i} q^{2(i-1)} D_{i,i}, \quad (i = 0, 1, \ldots).
\]

Since the map \( e_{\omega_n} \) induces the eigenvalue

\[
\sum_{i=0}^{n} a^i q^{\frac{(i-1)}{2}} \{ \theta_{n,i} \} = \sum_{i=0}^{n} a^i q^{\frac{(i-1)}{2}} \left[ \sum_{i=0}^{n} \binom{n}{i} \{ n + i - 2; a \}_i \right],
\]

we will prove the following statement.

**Proposition 3.6.** The following identity holds:

\[
a^{2n} q^{2n(n-1)} = \sum_{i=0}^{n} a^i q^{\frac{(i-1)}{2}} \left[ \sum_{i=0}^{n} \binom{n}{i} \{ n + i - 2; a \}_i \right].
\]

**Proof.** We consider a triangular array like Pascal’s triangle. The \( (0,0) \) entry is \( a^{2n} q^{2n(n-1)} \). First, we split \( a^{2n} q^{2n(n-1)} \) into

\[
a^{2n} q^{2n(n-1)} = a^{2n} q^{2n(n-1)} - a^{2n-2} q^{2(n-1)(n-1)} + a^{2n-2} q^{2(n-1)(n-1)}
\]

\[
= a^{2n-1} q^{2(n-1)(n-1)} \{ n - 1; a \} + a^{2n-2} q^{2(n-1)(n-1)}.
\]

The first term of the last equation is the \( (1,0) \) entry and the second term of it is the \( (1,1) \) entry in the triangle. Next, we split the \( (1,0) \) and the \( (0,1) \) entries. The
(1, 0) entry splits into
\[
a^{2n-1}q^{(2n-1)(n-1)} \{n-1; a\} = (a^{2n-1}q^{(2n-1)(n-1)} - a^{2n-3}q^{2n(n-1)-3n+1}) \{n-1; a\} + a^{2n-3}q^{2n(n-1)-3n+1} \{n-1; a\} = a^{2n-2}q^{2n(n-1)-2n+1} \{n-1; a\} \{n; a\} + a^{2n-3}q^{2n(2n-2)} \{n-1; a\},
\]

and the (1, 1) entry splits into
\[
a^{2n-2}q^{2n(n-1)(n-1)} = a^{2n-2}q^{2n(n-1)-2n+1} \{n; a\} + a^{2n-3}q^{2n(n-1)-3n+2} \left[\frac{2}{1}\right] \{n-1; a\} + a^{2n-4}q^{2n(n-1)(n-2)}.
\]

Then, we have the following split as the second row:
\[
a^{2n}q^{2n(n-1)} = a^{2n-2}q^{2n(n-1)-2n+1} \{n; a\} + a^{2n-3}q^{2n(n-1)-3n+2} \left[\frac{2}{1}\right] \{n-1; a\} + a^{2n-4}q^{2n(n-1)(n-2)}.
\]

The first, the second, the third term of the right-hand equation are the (2, 0), (2, 1), (2, 2) entries in the triangle. Let \(\lambda_{i,j}\) be
\[
\lambda_{i,j} = a^{2n-i-j}q^{\kappa_{i,j}} \left[\begin{array}{c} i \\ j \end{array}\right] \{n+i-j-2; a\}_{i,j},
\]
where \(\kappa_{i,j} = 2n(n-1) - n(i+j) - \frac{i(i-3)}{2} + \frac{j(j+1)}{2}\).

We show that the entry in the triangle is given by \(\lambda_{i,j}\) if we demonstrate a suitable split. We assume that the entry in the triangle is expressed by \(\lambda_{i,j}\). Then, we split \(\lambda_{i,j}\) into
\[
\lambda_{i,j} = \lambda_{i,j} - a^{-2}q^{-2(n+i-j-1)}\lambda_{i,j} + a^{-2}q^{-2(n+i-j-1)}\lambda_{i,j} = a^{-1}q^{(n+i-j-1)}\lambda_{i,j} \{n+i-j-1; a\} + a^{-2}q^{-2(n+i-j-1)}\lambda_{i,j}.
\]

We remark that this split agrees with the first row \((\lambda_{1,0}, \lambda_{1,1})\), and the second row \((\lambda_{2,0}, \lambda_{2,1}, \lambda_{2,2})\). This split means that the first term moves into the \((i+1, j)\) entry and the second term moves into \((i+1, j+1)\) entry. Here, we calculate the \((i+1, j+1)\) entry derived from the \((i, j)\) and the \((i, j+1)\) entries. It is expressed
by
\[a^{-2}q^{-(n+i-j-1)}\lambda_{i,j} + a^{-1}q^{-(n+i-(j+1)-1)}\lambda_{i,j+1}\{n+i-(j+1)-1;a\}\]
\[=a^{2n-i-j-2}q^{\kappa_{i,j}}{^{i+j-1}} \left\{n+i-j-2; a\right\}_{i-j}
+ a^{2n-i-j-2}q^{\kappa_{i,j+1}}{^{i-j-2}} \left\{n+i-j-3; a\right\}_{i-j-1}\{n+i-j-2; a\}\]
\[=a^{2n-i-j-2}\frac{[i]!}{[j+1]!(i-j)!}\{n+i-j-2; a\}_{i-j}
\times \left( q^{\kappa_{i,j}}{^{i+j-1}}(j+1) + q^{\kappa_{i,j+1}}{^{i-j-2}}[i-j] \right)\]
\[=a^{2n-i-j-2}\frac{[i]!}{[j+1]!(i-j)!}\{n+i-j-2; a\}_{i-j} \times q^{\kappa_{i,j+1}}{^{i+j+1}}[i+1].\]

This agrees with \(\lambda_{i+1,j+1}\). Therefore, when we take the sum of the \(i\)-th row, and we obtain
\[a^{2n}q^{2n(n-1)} = \sum_{k=0}^{i} \lambda_{i,k}.\]

Finally, put \(i = n\). The assertion is obtained. \(\square\)

For the negative twist map \(t^{-1}\), we consider the mirror image of elements of the skeins module. That is, we define \(\omega_n^- \in S\) by
\[\omega_n^- = \sum_{i=0}^{n} t_i R_i, \quad \text{where} \quad \bar{t}_i = (-1)^i a^{-i} q^{-i(i-1)} \frac{[i]!}{[i]!.}\]
Then, we have \(e_{\omega_n^-}(D_{i,i}) = t^{-1}(D_{i,i})\) for \(i = 0, \ldots, n\). Summarizing the above, we have the following proposition.

**Proposition 3.7.** For any \(x \in H_{m,n}\), we have \(e_{\omega_n^+}(x) = t(x)\) and \(e_{\omega_n^-}(x) = t^{-1}(x)\).

4. A TWISTING FORMULA

In this section, we would like to find a twisting formula of \(p\) full twists. As a preparation, we need the following lemma.

**Lemma 4.1.** The following holds:

\[m \left| n \right| \rightarrow \sum_{i=0}^{\min\{m,n\}} y_{m,n}^i \left| n - i \right| \rightarrow \begin{array}{c}
\vdots \\
\end{array}
\]

where \(y_{m,n}^i\) is defined by
\[y_{m,n}^i = \frac{\{m\}_i\{n\}_i}{\{i\}!\{m+n-i-1;a\}_i}.\]

**Proof.** The definition of the \((m, n)\)-th \(q\)-symmetrizer implies that the left-hand side is described by a linear combination of the \((m-i, n-i)\)-th \(q\)-symmetrizers with some coefficients \(y_i\) for \(i = 0, \ldots, \min\{m, n\}\). We select the \(j\)-th term in the sum, which includes the \((m-j, n-j)\)-th \(q\)-symmetrizer. Then, we connect the \(j\)-th
term to both sides. From Lemma 2.4 and the idempotent property, we obtain the coefficient of the $j$-th term, $y_j$, which is equal to $y_{m,n}^j$ above. □

We set the element $\omega_n^p \in S$ by

$$\omega_n^p = \sum_{i=0}^{n} t_{i,p} R_i,$$

for some $t_{i,p} \in \mathbb{C}$, and we would like to determine coefficients $t_{i,p}$ so that $e_{\omega_n^p}(x) = t^p(x)$ for $x \in D_{n,n}$.

By the idempotent and vanishing properties of $D_{n,n}$, for some $T_{n,p} \in \mathbb{C}$, the left-hand side of Figure 12 is transformed into $D_{n,n}$ with the multiplication of $T_{n,p}$.

![Figure 12](image)

To determine $t_{n,p}$, we calculate $T_{n,p}$ in two ways. One is to make use of the $(n - i, n - i)$-th $q$-symmetrizer ($i = 0, \ldots, n$), and the other is to encircle $p$ full twists by $\omega_n^p$. These two ways are due to [11].

First, using Lemma 4.1, we replace $2n$ arcs by the $(n - i, n - i)$-th $q$-symmetrizer ($i = 0, \ldots, n$). Next, we undo the $i$ arcs. This procedure is shown in Figure 13.

![Figure 13](image)

Because of the HOMFLY-PT skein relation and the vanishing property of the $(n - i, n - i)$-th $q$-symmetrizer, all crossings are canceled shown in Figure 14.
From the vanishing property of the $(n,n)$-th $q$-symmetrizer, Figure 14 does not vanish if $j = n - i$. Hence we have

$$T_{n,p} = \sum_{i=0}^{n} y_{n,n}(a^{-i}q^{-i(i-1)-2i(n-i)})^{2p} x_{n-i,n-i}^{n-i}$$

$$= \sum_{i=0}^{n} (a^{-i}q^{-i(i-1)-2i(n-i)})^{2p} \frac{\{n\}_i\{n\}_i}{\{i\}!\{2n-i-1; a\}_i} (-1)^{n-i} \frac{\{n-i\}!}{\{2n-2i-2; a\}_{n-i}}$$

$$= (a^{-n}q^{-n(n-1)})^{2p}\{n\}_i \sum_{i=0}^{n} (-1)^{i} (a^i q^{i(i-1)})^{2p} \frac{\{n\}_i\{n-i\}_i}{\{n-i\}_i!\{n+i-1; a\}_{n+i+1}}$$

Second, we encircle $2p$ full twists arcs by $\omega_{n}^p$. Here, we consider the number of arcs of $\omega_{n}^p$ at the horizontal line. From the vanishing property of the $(n,n)$-th $q$-symmetrizer, The non-vanishing term has $2n$ arcs, which is $R_n$. Therefore, we have the transformation shown in Figure 15.

Furthermore, we apply Lemma 4.1 to two $n$ arcs which go through $R_n$. By Corollary 3.5, they vanish except the $(n,n)$-th $q$-symmetrizer. Therefore, $T_{n,p}$ is given by

$$T_{n,p} = (a^n q^{n(n-1)})^{-2p} t_{n,p} \theta_{n,n} x_{n,n}^n$$

$$= (a^n q^{n(n-1)})^{-2p} \{n\}_i! \{2n-2; a\}_n (-1)^n \frac{\{n\}_i!}{\{2n-2; a\}_n} t_{n,p}$$

$$= (-1)^n (a^n q^{n(n-1)})^{-2p} (\{n\}_i!)^2 t_{n,p}$$

Finally, by comparing both $T_{n,p}$’s, we have the following proposition.

**Proposition 4.2.** $t_{n,p}$ is given by

$$t_{n,p} = (-1)^n \sum_{i=0}^{n} (-1)^{i} (a^i q^{i(i-1)})^{2p} \frac{\{2i-1; a\}}{\{i\}_i!\{n-i\}_i!\{n+i-1; a\}_{n+i}}$$
For $|p| \geq 2$, we need the $(n, n)$-th $q$-symmetrizer to calculate the twisting formula. But for $p = \pm 1$, we can use Lemma 2.1. Hence, by combining Proposition 3.7 we have the following corollary.

**Corollary 4.3.** For $p = \pm 1$, $t_{n,p}$ is given by

\[
t_{n,1} = a^n q^{\frac{n(n-1)}{2}} \binom{n}{n}! = t_n,
\]

\[
t_{n,-1} = (-1)^n a^{-n} q^{\frac{n(n-1)}{2}} \binom{n}{n}! = \bar{t}_n.
\]

5. **Calculations of the colored HOMFLY-PT polynomial**

Now, we are ready to calculate the colored HOMFLY-PT polynomial of twist knots. The $n$-th colored HOMFLY-PT polynomial for a knot $K$ is given by

\[
\mathcal{H}_n(K) = \frac{\langle K(H_n) \rangle}{\langle \text{Unknot}(H_n) \rangle} = \frac{\{n\}!}{\{n-1; a\}_n} \langle K(H_n) \rangle,
\]

where $K(H_n)$ stands for $K$ cabled by $H_n$ with compatible orientations. $\mathcal{H}_n(K)$ is normalized so that it is to be 1 for the unknot. We assume that the framing of the knot $K$ is to be 0.

For an integer $p$, the twist knot $K_p$ is described in Figure 16.

![Figure 16](16.png)

**Figure 16.** The twist knot $K_p$

Especially, $K_1$ is a left-handed trefoil, and $K_{-1}$ is a figure-eight knot.

Let us calculate $\langle K_p(H_n) \rangle$. Some transformations are also due to [11]. Since $H_n$ has the following presentation:

\[
H_n = \sum_{i=0}^{n} \frac{(n + i - 1; a)_{n-i}}{(n-i)!} R_i,
\]

we insert this presentation along $K_p$ instead of $H_n$, and we unknot the $p$ full twists with the encircling map by $\omega^p_n$. Here, we remark a pair $R_i \in K_p(H_n)$ and $R_j \in \omega^p_n$. The pair is a cabling of the Whitehead link, of which one component pierces the other twice in the opposite direction each other. Moreover, $R_i$ and $R_j$ are a linear combination of $D_{k,k}$ with $k \leq i$ and $k \leq j$, respectively. Therefore, by Corollary 3.5 the pair vanishes if $i \neq j$. Hence, we have $\langle K_p(H_n) \rangle$ as follows:

\[
\langle K_p(H_n) \rangle = \sum_{i=0}^{n} \frac{(n + i - 1; a)_{n-i}}{(n-i)!} t_{i,p} R_i.
\]
Next, we calculate the part of Figure above. The first equality comes from the fact that $R_i - H_i$ is a linear combination of $R_j$ with $j < i$. The second equality comes from Lemma 4.1 and Corollary 3.5.

\[
R_i - H_i = (a^i q^{i(i-1)})^2 \left( \begin{array}{c} i \\ R_i \end{array} \right) - (a^i q^{i(i-1)})^2 \alpha_{i,i} \left( \begin{array}{c} i \\ R_i \end{array} \right)
\]

Finally, the $i$-th term in the sum of $\langle K_p(H_n) \rangle$ normalized by $\langle \text{Unknot}(H_n) \rangle$ is expressed by

\[
\frac{\{n\}!}{\{n-1; a\}_n} \times \frac{(n+i-1; a}_{n-i} t_{i,p} \times (-a)^i q^{i(i-1)} \left( \begin{array}{c} 2i-1; a \end{array} \right) \left( \begin{array}{c} i \end{array} \right) \{2i-1; a\}_{i-1}^2 \{1; a\}_{i+1} \}
\]

Summarizing the above, we obtain the following theorem.

**Theorem 5.1.** For a twist knot $K_p$, the $n$-th colored HOMFLY-PT polynomial of $K_p$ is given by

\[
\mathcal{H}_n(K_p) = \sum_{i=0}^{n} (-a)^i q^{i(i-1)} \left( \begin{array}{c} i \\ i, p \end{array} \right) \left[ \frac{n}{i} \right] \left\{ n+i-1; a \right\}_{i} \left\{ i-2; a \right\}_{i}
\]

where

\[
s_{i,p} = \sum_{k=0}^{i} (-1)^k \binom{a^k q^{k(k-1)}}{k} \frac{1}{k!} \frac{2k-1; a}{\{i-k\}! \{i+k-1; a\}_{i+1}}.
\]
Corollary 5.2. For the left-handed trefoil $3_1$ and the figure-eight knot $4_1$, the $n$-th colored HOMFLY-PT polynomial is given by

$$
\mathcal{H}_n(3_1) = \mathcal{H}_n(K_1) = \sum_{i=0}^{n} (-1)^i a^{2i} q^{i(i-1)} \binom{n}{i} \{n+i-1; a\}_i \{i-2; a\}_i,
$$

$$
\mathcal{H}_n(4_1) = \mathcal{H}_n(K_{-1}) = \sum_{i=0}^{n} \binom{n}{i} \{n+i-1; a\}_i \{i-2; a\}_i.
$$

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References

[1] M. Aganagic and C. Vafa. Large N Duality, Mirror Symmetry, and a Q-deformed A-polynomial for Knots. arXiv:1204.4709.
[2] K. Habiro. On the colored Jones polynomial of some simple links. In: Recent Progress Towards the Volume Conjecture, Research Institute for Mathematical Sciences (RIMS) Kokyuroku 1172, September 2000.
[3] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu. A new polynomial invariant of knots and links. Bull. Amer. Math. Soc. (N.S.) 12 (1985), no. 2, 239–246.
[4] H. Itoyama, A. Mironov, A. Morozov, A. And. Morozov. HOMFLY and superpolynomials for figure eight knot in all symmetric and antisymmetric representations. J. High Energy Phys. 2012, no. 7, 131, front matter + 20 pp. arXiv:1203.5978.
[5] K. Kawagoe. On the skeins in the annulus and applications to invariants of 3-manifolds. J. Knot Theory Ramifications 7 (1998), no. 2, 187–203.
[6] K. Kawagoe. Skeins associated with Homfly and Kauffman polynomials and invariants of graphs. Arch. Math. (Basel) 77 (2001), no. 2, 200–208.
[7] K. Kawagoe. Limits of the HOMFLY polynomials of the figure-eight knot. Intelligence of low dimensional topology 2006, 143–150, Ser. Knots Everything, 40, World Sci. Publ., Hackensack, NJ, 2007.
[8] K. Kawagoe. On the formulae for the colored HOMFLY polynomials. J. Geom. Phys. 106 (2016), 143–154.
[9] T. T. Q. Le. Quantum invariants of 3-manifolds: integrality, splitting, and perturbative expansion. Proceedings of the Pacific Institute for the Mathematical Sciences Workshop "Invariants of Three-Manifolds" (Calgary, AB, 1999). Topology Appl. 127 (2003), no. 1-2, 125–152.
[10] W. B. R. Lickorish. An introduction to knot theory, Graduate Texts in Mathematics, 175. Springer-Verlag, New York, 1997.
[11] Gregor Masbaum. Skein-theoretical derivation of some formulas of Habiro. Algebraic & Geometric Topology, 3 (2003), 537–556.
[12] S. Nawata, P. Ramadevi, Zodinmawia, X. Sun. Super-A-polynomials for Twist Knots. J. High Energy Phys. 2012, no. 11, 157, front matter + 38 pp. arXiv:1209.1409.