Pathwise mild solutions for quasilinear stochastic partial differential equations

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October 5, 2018

Abstract

Stochastic partial differential equations (SPDEs) have become a key modelling tool in applications. Yet, there are many classes of SPDEs, where the existence and regularity theory for solutions is not completely developed. Here we contribute to this aspect and prove the existence of mild solutions for a broad class of quasilinear Cauchy problems, including - among others - cross-diffusion systems as a key application. Our solutions are local-in-time and are derived via a fixed point argument in suitable function spaces. The key idea is to combine the theory of deterministic quasilinear parabolic partial differential equations (PDEs) with recent theory of evolution semigroups. We also show, how to apply our theory to the Shigesada-Kawasaki-Teramoto (SKT) model. Furthermore, we provide examples of blow-up and ill-posed operators, which can occur after finite-time.

Keywords: quasilinear stochastic partial differential equations, maximal local pathwise mild solution, stochastic Shigesada-Kawasaki-Teramoto model.

1 Introduction

In this work, we study SPDEs as abstract quasilinear Cauchy problems

\[
\begin{aligned}
\frac{du(t)}{dt} &= [A(u(t)) u(t) + F(t, u(t))] \, dt + \sigma(t, u(t)) \, dW(t), \quad t \in (0, \infty), \ u(t) \in \mathbb{R}^d, \\
u(0) &= u_0,
\end{aligned}
\]

(1.1)

where the precise assumptions on the coefficients are stated in Section 3. One particular motivation is the SKT cross-diffusion model [58] with \( d = 2, \ A(u) = (\Delta(p_1(u)), \Delta(p_2(u)))^T \) for quadratic polynomials \( p_{1,2} \) in \( u = (u_1, u_2)^T \), \( \Delta \) denoting the Laplacian, and \( F \) also being a quadratic polynomial. The deterministic SKT system (i.e., \( \sigma \equiv 0 \)) and its variants have been studied very intensively [5, 15, 44, 45, 48]. Furthermore, there are many other motivations as the form of the deterministic part (or drift terms) \( A \) and \( F \) encompasses a much wider class of PDEs [6, 63]. For many applications, it is very important to consider noise terms (\( \sigma \neq 0 \)) due to intrinsic finite system-size noise or external fluctuations acting on the system.

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Here we aim to develop a theory for quasilinear stochastic evolution equations (1.1) using a semigroup approach. The main theme is to extend the very general deterministic theory of quasilinear Cauchy problems [3, 61, 63]. The key idea is to employ a modified definition of mild solutions [57] for the quasilinear case in comparison to the more classical parabolic SPDE setting [54]. Before we describe our approach in more detail, we briefly review some other techniques and solution concepts used for (certain subclasses of) the SPDE (1.1). Instead of mild solutions, one may instead use weak, or martingale, solutions [13, 20, 24, 35] of (1.1); here weak solution is interpreted in the classical PDE sense while these solutions are also sometimes referred to as strong solutions from a probabilistic perspective [55]. There are also several works exploiting the additional assumption of monotone coefficients [43] particularly in the case of the stochastic porous medium equation [9, 10], where \( A(u) = \Delta(a(u)) \) for a maximal monotone map \( a \) and \( F \equiv 0 \). Other approaches to quasilinear SPDEs are based upon a gradient structure [29], approximation methods [38, 39], kinetic solutions [20, 30], or directly looking at strong (in the PDE sense) solutions [36].

One may ask, why one might want to prove the existence of pathwise mild solutions obtained by a suitable variations-of-constants/Duhamel formula [34] instead of working with weak solutions obtained in a formulation via test functions [26]? One reason is that a mild formulation is often more natural to work with in the context of (random) dynamical systems for SPDEs [18, 34]. In fact, many classical results regarding dynamics and long-time behavior of semilinear SPDEs are often crucially based upon the mild formulation and semigroups [54]. We expect this theory to generalize a lot easier also in the quasilinear case if one does not have to work with weak(er) solutions. If we take the SKT system again as a motivation, then there are deterministic results regarding the existence of attractors using weak [53] as well as mild [62] solutions concepts.

We intend to investigate the existence of random attractors for the stochastic SKT equation using the mild formulation in a future work, since it perfectly fits into the framework of random dynamical systems. A second reason to consider mild solutions is that it should be easier to derive space-time regularity [21] of the solution for (1.1). Estimates for the nonlinear terms also tend to simplify in a mild solution setting already for SODEs [11]. A third reason to consider mild solutions is that they are more natural in the setting of regularity structures [32, 33, 28], where convolution with the heat kernel is a key tool; in this context generalizations of regularity structures to quasilinear SPDEs turn out to be very subtle [12, 28]. So a better understanding for more regular noises should be helpful. To deal with rough noises an alternative to regularity structures is to stay closer to a paracontrolled approach [31], which has recently been proposed for certain quasilinear SPDEs [8, 27, 50, 51].

Since the linear operator \( A \) depends on the solution itself, which will be in our case a stochastic process, we cannot apply the standard fixed-point argument as in [3, 63]. Namely, if we denote with \( U^u \) the random evolution operator generated by \( A(u) \), one naturally expects that the mild solution of (1.1) should be given by the variation-of-constants formula

\[
 u(t) = U^u(t, 0)u_0 + \int_0^t U^u(t, s)F(s, u(s)) \, ds + \int_0^t U^u(t, s)\sigma(s, u(s)) \, dW(s).
\]

(1.2)

As already observed in [57], and justified in Sections 2 and 3, the random evolution operator
$U^u(t, s, \omega)$ does not satisfy the necessary adaptedness properties required in order to define the Itô-integral. More precisely, it turns out to be only $\mathcal{F}_t$-adapted and not $\mathcal{F}_s$-adapted. Consequently, the stochastic convolution given in (1.2) is not well-defined in the Itô-sense. This situation is commonly met for instance in the theory of stochastic evolution equations with time-dependent random generators [57]. A way out of this situation is to introduce a new concept of mild solution for (1.1), which is based on the integration-by-parts formula for stochastic convolutions. This is motivated in [57, Sec. 4] as well as in Appendix A here for convenience. Using this approach, we prove by means of fixed-point arguments that the mild solution of (1.1) is given by

$$u(t) = U^u(t, 0)u_0 + U^u(t, 0) \int_0^t \sigma(r, u(r)) \, dW(r) + \int_0^t U^u(t, s)F(s, u(s)) \, ds$$

$$- \int_0^t U^u(t, s)A(u(s)) \int_s^t \sigma(r, u(r)) \, dW(r) \, ds. \tag{1.3}$$

One can show that the stochastic convolutions appearing in the formula above can be defined pathwise, therefore we will call this a pathwise mild solution of (1.1); see Section 3.

Additionally to the adaptedness issue stated above, note that there are several technical difficulties required in order to obtain sufficient space-time regularity results for (1.3) which are necessary to set-up the fixed-point argument from the deterministic theory of [61]. Therefore, it is by no means straightforward to see why (1.3) is the right solution concept for our original problem and how these theories fit together. We also emphasize that up to now there is no theory available for mild solutions for quasilinear SPDEs in contrast to the deterministic case. The semigroup approach has turned out to be a very powerful tool for the analysis of quasilinear PDEs, see [52], [5], [61] and the references specified therein. This work represents an important step in exploiting the semigroup methods from the deterministic setting in the stochastic one.

Another important feature of this approach is that it can be applied to more general stochastic processes $(S(t))_{t \in [0, T]}$ not just the Brownian motion, since one needs to establish an integration theory only for

$$\int_0^T \sigma(t) \, dS(t)$$

under suitable assumptions on $\sigma$. In this case $S$ does not even have to be a semimartingale, so one can consider (1.1) perturbed by an additive-fractional noise thereby generalizing results in [41, 42]; we intend to explore this in a future work.

We deal here with local in time existence of solutions for (1.1). To obtain global-in-time solutions for (1.1), one has to solve two further issues: (a) finite-time blow-up and (b) degeneration of the operator $A(u)$. The latter is often related in practice to preserving certain positivity assumptions present in the initial data and naturally relates to stochastic maximum principles [9, 22, 23, 49]. The issue (a) of blow-up certainly also occurs already for many classical SPDEs (see e.g. [19].
but also plays a key role for quasilinear SPDE problems \cite{36}. For completeness, we provide two very simple quasilinear counter-examples involving (a) and (b) to demonstrate that we cannot expect global-in-time existence for (1.1) in general; see Section 5. Yet, we conjecture that for many quasilinear SPDEs, where global existence is known for the PDE ($\sigma \equiv 0$), there is a natural choice of noise term $\sigma \not\equiv 0$ such that also the SPDE has global-in-time existence. The choice of noise term is definitely case-dependent but it should be possible to deal with many cases arising directly from modelling considerations, which is another interesting direction for future research.

2 Preliminaries

Throughout this work we fix a time horizon $T > 0$ and a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with a complete, right-continuous filtration. We also use the standard notation $\omega$ for elements in $\Omega$.

We first collect basic properties and results concerning parabolic evolution families generated by random nonautonomous operators $\{A(t, \omega)\}_{t\in[0,T]}$ on a separable Banach space $X$. We make the following assumptions, which are going to ensure that $A(t, \omega)$ generates a parabolic evolution system, which is a family of linear operators depending on two-time parameters and having similar properties to analytic $C_0$-semigroups \cite[Section II.2]{6}:

(A1) The spectrum of $A(t, \omega)$ is contained in an open sectorial domain, more precisely

$$\sigma(A(t, \omega)) \subset \Sigma_{\varphi} := \{ \lambda \in \mathbb{C} : |\arg \lambda| < \varphi \}, \text{ for } (t, \omega) \in [0, T] \times \Omega$$

with a fixed angle $0 < \varphi < \frac{\pi}{2}$.

(A2) There exists a constant $M \geq 1$ such that the resolvent estimate

$$||| (\lambda \text{Id} - A(t, \omega))^{-1} |||_{\mathcal{L}(X)} \leq \frac{M}{|\lambda| + 1}$$

holds true for $\lambda \not\in \Sigma_{\varphi}$ and $(t, \omega) \in [0, T] \times \Omega$, where $\mathcal{L}(X)$ is the space of linear operators on $X$ with norm $|| \cdot ||_{\mathcal{L}(X)}$ and $\text{Id}$ is the identity map.

(A3) The Acquistapace-Terreni condition \cite{1} is fulfilled, namely there exist two exponents $\nu, \delta \in (0, 1]$ with $\nu + \delta > 1$ such that for every $\omega \in \Omega$ there exists a constant $L(\omega) \geq 0$ such that for all $s, t \in [0, T]$ we have

$$||| A'(t, \omega)(A(t, \omega)^{-1} - A(s, \omega)^{-1}) |||_{\mathcal{L}(X)} \leq L(\omega)|t - s|^{\delta}. \quad (2.1)$$

Let $\Gamma := \{(s, t) \in [0, T]^2 : s \leq t\}$. In the following, for notational simplicity we drop the $\omega$-dependence of $A(t)$ and only indicate it explicitly for certain statements, where its crucial role is clarified.

Convention: Since certain expressions and constants depend on several parameters, i.e. $A(t, \omega)$ we always omit the last symbol whenever this dependence is clear.

By applying the results in \cite{1} pointwise in $\Omega$, see also \cite{57} Theorem 2.2, we obtain:
Theorem 2.1 There exists a unique map \( U : \Gamma \times \Omega \rightarrow \mathcal{L}(X) \) such that

(T1) for all \( t \in [0, T] \), \( U(t, t) = \text{Id} \);

(T2) for all \( r \leq s \leq t \), \( U(t, s)U(s, r) = U(t, r) \);

(T3) for every \( \omega \in \Omega \), the map \( U(\cdot, \cdot, \omega) \) is strongly continuous;

(T4) there exists a mapping \( C : \Omega \rightarrow \mathbb{R}_+ \), such that for all \( s \leq t \), one has

\[ ||U(t, s)||_{\mathcal{L}(X)} \leq C. \]

(T5) for every \( s < t \) it holds \( \frac{\partial}{\partial t} U(t, s) = A(t)U(t, s) \) and \( \frac{\partial}{\partial s} U(t, s) = -U(t, s)A(s) \) pointwise in \( \Omega \). Moreover, there exists a mapping \( C : \Omega \rightarrow \mathbb{R}_+ \) such that

\[ ||A(t)U(t, s)|| \leq C(t - s)^{-1}. \]

Consequently, if (A1)-(A3) hold true \( \{A(t, \cdot)\}_{t \in [0, T]} \) generates the evolution system/family \( \{U(t, s, \omega)\}_{0 \leq s \leq t \leq T} \). In our case, see Section 3 the evolution system \( U^u(t, s, \omega) \) will additionally depend on the solution \( u \) of a quasilinear SPDE.

In contrast to the deterministic setting all constants specified above depend on \( \omega \) which causes several technical difficulties, more precisely \( C \) depends in general on \( L(\omega) \) and on \( \delta \). For instance applying Theorem 4.4.1 in [6] one has

\[ ||U(t, s, \omega)||_{\mathcal{L}(X)} \leq \tilde{C} e^{\mu(\omega)(t-s)}, \text{ for } (t, s) \in \Gamma, \quad (2.2) \]

where \( \mu(\omega, \delta) = \mu(\omega) = \tilde{C}(\delta)L(\omega)^{1/\delta} \). Here \( \tilde{C} \) stands for an arbitrary constant and \( \tilde{C}(\delta) \) indicates the dependence of \( \tilde{C} \) on the Hölder exponent from (2.1). We point out following fact regarding this issue which is crucial for the computation in Section 3.

Remark 2.2 1) One can assume that the mapping \( L : \Omega \rightarrow \mathbb{R}_+ \) introduced in (2.1) is bounded in \( \Omega \), analogously to [57, Section 5.2]. The general case can be treated by a localization argument [57, Section 5.3], namely one introduces appropriate stopping times \( (\tau_n)_{n \in \mathbb{N}} \) and considers \( A_n(t, \omega) := A(t \wedge \tau_n(\omega), \omega) \).

2) In our case the generators will depend on the solution itself, so it is meaningful to control the solution process \( u \) in order to make sure that the corresponding evolution operator \( U^u \) is indeed well-defined. Consequently we deal with \( A_n(u(t, \omega)) := A(u(t \wedge \tau_n(\omega), \omega)) \) as stated in (A1’)-(A3’) in Section 3. Therefore, all constants arising from the estimates involving \( A(u) \) and \( U^u \) will depend on \( \tau_n(\omega) \), as precisely specified in Section 3 below.

The following estimates for analytic \( C_0 \)-semigroups and parabolic evolution operators are essential for the computation in Section 3. These can be looked up in [52, Section 2.6, p. 69], [63, Section 8.1, p. 154] and the references specified therein; note that they hold pointwise in \( \omega \in \Omega \) similar to Theorem 2.1.
Theorem 2.3 There exists a mapping \( C : \Omega \rightarrow \mathbb{R}_+ \) such that for \( 0 \leq s < t \leq T \) and \( \tilde{\alpha}, \tilde{\beta} \in (0,1] \) we have
\[
\| A^{\tilde{\beta}}(t)U(t,s) \|_{L(X)} \leq C(t-s)^{-\tilde{\beta}}, \quad \| U(t,s)A^{\tilde{\beta}}(t) \| \leq C(t-s)^{-\tilde{\beta}}, \quad (2.3)
\]
as well as
\[
\| A^{\tilde{\beta}}(t)U(t,s)A^{-\tilde{\alpha}}(s) \|_{L(X)} \leq C(t-s)^{\tilde{\alpha}-\tilde{\beta}}. \quad (2.4)
\]
Again, \( C \) depends in general on \( L(\omega) \) and on \( \delta \) introduced in (2.1). For more details and properties of fractional powers of sectorial operators \( A^{\tilde{\beta}} \) for \( \tilde{\beta} > 0 \) and the usual fractional space \( X(\cdot) \) we refer the reader to [52, Theorem 6.13, p. 74].

Remark 2.4 Note that the first two assumptions imply that \( A(t) \) generates an analytic \( C_0 \)-semigroup which is denoted by \( e^{-rA(t)} \). In this case there exists a mapping \( C : \Omega \rightarrow \mathbb{R}_+ \) such that the estimate
\[
\| A^{\tilde{\theta}}(t)e^{-rA(t)} \|_{L(X)} \leq Cr^{-\tilde{\theta}}, \quad \text{for } \tilde{\theta} > 0 \text{ and } r > 0
\]
holds true.

Furthermore, we recall the next result [57, Proposition 2.4], which deals with the measurability of \( U \):

Proposition 2.5 The evolution system \( U : \Gamma \times \Omega \rightarrow \mathcal{L}(X) \) is strongly measurable in the uniform operator topology. Moreover, for each \( t \geq s \), the mapping \( \omega \mapsto U(t,s,\omega) \in \mathcal{L}(X) \) is strongly \( \mathcal{F}_t \)-measurable in the uniform operator topology.

For the sake of completeness, we provide now some known results concerning stochastic calculus, which will be required further on. We let \( H \) and \( Z \) stand for two separable Hilbert spaces and \((W(t))_{t \in [0,T]}\) for an \( H \)-cylindrical Brownian motion, meaning that
\[
W(t) = \sum_{n=1}^{\infty} w_n(t)e_n,
\]
with \((w_n(\cdot))_{n \geq 1}\) being mutually independent real-valued standard Wiener processes relative to \((\mathcal{F}_t)_{t \geq 0}\) and \((e_n)_{n \geq 1}\) an orthonormal basis in the separable Hilbert space \( H \). With \( \mathcal{L}_2(H,Z) \) we denote the space of Hilbert-Schmidt operators from \( H \) to \( Z \). As justified in Section 3, for our aims it will be enough to analyze the stochastic integral respectively the stochastic process
\[
\left( \int_0^t \sigma(r) \ dW(r) \right)_{t \in [0,T]} \quad (2.5)
\]
only for strongly-measurable, adapted stochastic processes \( \sigma \in L^0(\Omega; L^2(0,T;\mathcal{L}_2(H,Z))) \). Here \( L^0 \) indicates measurability and \( L^p \) is going to denote the usual Lebesgue spaces.
We recall (consult \cite{57} Section 4.1] and the references specified therein) that the process exists and is pathwise continuous for $\sigma \in L^0(\Omega; L^2(0, T; L_2(H, Z)))$. Moreover, one has the one-sided estimate

$$||J(\sigma)||_{L^p(\Omega; C([0,T]; Z))} \leq C||\sigma||_{L^p(\Omega; L^2(0, T; L_2(H, Z)))}, \quad (2.6)$$

where for $t \in [0, T]$ we set $J(\sigma)(t) := \int_0^t \sigma(r) \, dW(r)$.

Furthermore, for $J(\sigma)$ the following regularity results \cite{56} Proposition 4.4] are available and will be employed in Section 3.

**Proposition 2.6** Let $0 < \hat{\alpha} < 1/2$, $p \in [2, \infty)$ and $\sigma$ be a strongly measurable adapted process, belonging to $L^0(\Omega; L^p(0, T; L_2(H, Z)))$. Then $J(\sigma) \in W^{\hat{\alpha}, p}(0, T; Z)$ almost surely (a.s.), where $W^{\hat{\alpha}, p}$ is the notation for the usual Sobolev spaces.

Due to the embedding

$$W^{\hat{\alpha}, p}(0, T; Z) \hookrightarrow C^{\hat{\alpha} - \frac{1}{p}}(0, T; Z) \quad \text{for } \frac{1}{p} < \hat{\alpha} < \frac{1}{2},$$

one obtains Hölder regularity of the integral process, namely $J(\sigma) \in C^{\hat{\alpha} - \frac{1}{p}}(0, T; Z)$ a.s.. The next crucial result will be used throughout the next subsection, see \cite{57} Proposition 4.1] for the full generality of the statement.

**Proposition 2.7** Let $0 < \hat{\alpha} < 1/2$, $p \in [2, \infty)$ and $\sigma$ be a strongly measurable adapted process, belonging to $L^0(\Omega; L^p(0, T; L_2(H, Z)))$. For $\frac{1}{p} < \hat{\alpha} < \frac{1}{2}$ there exists a $\sigma$-independent positive constant $C_T$ which tends to 0 as $T \to 0$, such that

$$||J(\sigma)||_{L^p(\Omega; C^{\hat{\alpha} - \frac{1}{p}}(0, T; Z))} \leq C_T ||\sigma||_{L^p(\Omega; L^p(0, T; L_2(H, Z)))}. \quad (2.7)$$

**Remark 2.8** The assertions above remain valid for type-2 Banach spaces (e.g. $L^p$-spaces for $p \geq 2$, Sobolev-spaces $W^{k,p}$ for $p \geq 2$), consult \cite{57} and \cite{56}. In this case one has to replace the Hilbert-Schmidt operators by $\gamma$-radonifying ones. Given a separable Hilbert space $H$ and a separable Banach space $Z$, we call an operator $R : H \to Z$ a $\gamma$-radonifying operator if

$$E \left| \sum_{n=1}^{\infty} \gamma_n R e_n \right|_Z^2 < \infty,$$

where $(\gamma_n)_{n \geq 1}$ is a sequence of independent standard Gaussian random variables on $(\Omega, F, P)$ and $(e_n)_{n \geq 1}$ is an orthonormal basis in $H$. The space of $\gamma$-radonifying operators $\gamma(H, X)$ is then endowed with the norm

$$\left( E \left| \sum_{n=1}^{\infty} \gamma_n R e_n \right|_Z^2 \right)^{1/2},$$

which does not depend on the choice of $(\gamma_n)_{n \geq 1}$ and $(e_n)_{n \geq 1}$. If $Z$ is isomorphic with a Hilbert space then $\gamma(H, Z)$ isometrically coincides with $L_2(H, Z)$. In summary, the computations in Section 3 carry over to the Banach space-valued setting, although we present them here only in a Hilbert space setting.
3 Quasilinear SPDEs

In this section we analyze stochastic quasilinear SPDEs using fixed-point arguments. In the deterministic case, this technique is known and can be found in [5], [63, Chapter 5] or [61]. As already emphasized, since the linear part also depends on the solution itself, the corresponding parabolic evolution operators will no longer have the necessary measurability properties required to define the Itô-integral, recall Proposition 2.5. Therefore, our ansatz is similar to the one used in [57] to deal with parabolic SPDEs with time-dependent random generators. Combining this approach with the fixed-point arguments of [63], [61] and [5], we are able to prove short-time existence for quasilinear SPDEs. In contrast to the non-autonomous random case we have to deal here with several technical difficulties, such as finding the appropriate function spaces for the fixed-point argument of [60] and using the right localization techniques. Note that even in the deterministic case, quasilinear PDEs may not possess global-in-time solutions without further assumptions, see for instance [5], [37] and Section 5.

In the following, let $X, Y$ and $Z$ denote three separable Hilbert spaces such that $Z \hookrightarrow Y \hookrightarrow X$ and let $K$ stand for an arbitrary open ball in $Z$. More precisely $K := \{V \in Z : ||V||_Z < R\}$ for a deterministic fixed $R > 0$.

The first step is to consider the quasilinear Cauchy problem

$$\begin{cases}
  du(t) = [(Au(t))(u(t)) + f(t)] \, dt + \sigma(t) \, dW(t), & t \in [0, T] \\
  u(0) = u_0 \in K \text{ a.s.}
\end{cases} \tag{3.1}$$

The results obtained for the inhomogeneous problem (3.1) will be further extended to the nonlinear case, more precisely we will include nonlinearities of semilinear type.

**Definition 3.1 (Local solution)** A local pathwise mild solution for (3.1) is a pair $(u, \tau)$, where $\tau$ is a strictly positive stopping time and the stochastic process $\{u(t) : t \geq 0\}$ is $(\mathcal{F}_t)_{t \in [0, \tau)}$-adapted and satisfies almost surely for every $t \geq 0$

$$u(t) = U^n(t, 0)u_0 + U^n(t, 0) \int_0^t \sigma(s) \, dW(s) + \int_0^t U^n(t, s)f(s) \, ds - \int_0^t U^n(t, s)A(u(s)) \int_s^t \sigma(\tau) \, dW(\tau) \, d\tau. \tag{3.2}$$

**Remark 3.2** The concept pathwise mild solution introduced in [57] is justified by the integration by parts formula for stochastic convolutions as motivated in [57, Section 4.2] as well as Appendix A. In this way, one overcomes the difficulty that the Itô-integral

$$\int_0^t U^n(t, s, \omega)\sigma(s) \, dW(s)$$

is not a true integral. Instead, it is evaluated using a stochastic Fubini theorem, see Appendix A.
cannot be defined since the mapping \( \omega \mapsto U^u(t, s, \omega) \) introduced in Section 2 is according to Proposition 2.5 only \( \mathcal{F}_t \)-measurable and not \( \mathcal{F}_s \)-measurable. Furthermore, as shown in [57, Theorem 3.4-3.5] the convolution-type integrals above can be defined in a pointwise sense. This fact, together with certain pathwise regularity results, will be exploited in the construction of solutions below.

**Definition 3.3** A local pathwise mild solution \( \{ u(t) : t \in [0, \tau) \} \) for (3.1) is unique if for any other local pathwise mild solution \( \{ \tilde{u}(t) : t \in [0, \tilde{\tau}) \} \) of (3.1), the processes \( u \) and \( \tilde{u} \) are equivalent on \([0, \tau \wedge \tilde{\tau})\).

**Definition 3.4** (Maximal and global solution) We call \( \{ u(t) : t \in [0, \tau) \} \) a maximal local pathwise mild solution of (3.1) if for any other local pathwise mild solution \( \{ \tilde{u}(t) : t \in [0, \tilde{\tau}) \} \) satisfying \( \tilde{\tau}(t) \geq \tau \) a.s. and \( \tilde{u}|_{[0,\tau)} \) is equivalent to \( u \), one has \( \tilde{\tau} = \tau \) a.s. If \( \{ u(t) : t \in [0, \tau) \} \) is a maximal local pathwise mild solution for (3.1), then the stopping time \( \tau \) is called its lifetime. If \( \mathbb{P}(\tau = \infty) = 1 \) then \( \{ u(t) : t \in [0, \tau) \} \) is a global pathwise mild solution for (3.1).

**Remark 3.5** We emphasize that by a maximal local pathwise mild solution of (3.24) we understand a triple \( (u, (\tau_n)_{n \geq 1}, \tau_\infty) \) such that each pair \( (u, \tau_n) \) is a local pathwise mild solution, \( (\tau_n)_{n \geq 1} \) is an increasing sequence of stopping times such that \( \tau_\infty := \lim_{n \uparrow \infty} \tau_n \) a.s. and

\[
\lim_{t \uparrow \tau_\infty} \sup_{s \in [0, t]} ||u(s)||_Z = \infty, \text{ a.s. on the set } \{ \omega : \tau_\infty(\omega) < \infty \},
\]

see also [14, Proposition 3.11]. For a global solution it holds that \( \tau_\infty = \infty \) a.s., which means that for every \( T > 0 \) the quantity

\[
\sup_{t \in [0, T]} ||u(t)||_Z
\]

is almost surely finite on the set \( \{ \omega : \tau_\infty(\omega) = \infty \} \). Furthermore, note that if uniqueness of local solutions holds true, then the same remains valid for maximal local solutions [14, Section 3].

For more details regarding local and maximal local mild solutions for SPDEs consult [65, 14, Section 3], [38] and the references specified therein.

In order to ensure the well-posedness of (3.11) we further state suitable assumptions for the generators.

**Remark 3.6** Note that since the operator \( A \) depends on the solution itself it can degenerate at some point already in the deterministic setting. A simple example is the change from a forward to a backward heat equation. Of course, the same issue arises for general multi-component degenerate quasilinear parabolic problems [4]. The conditions we consider below will exclude this situation in a certain time-horizon. For more information on quasilinear degenerate SPDEs consult [20].

**Definition 3.7** Denote by \( \mathcal{U}_T \) the set of all stochastic processes \( (u(t))_{t \in [0, T]} \) satisfying the following conditions:
• \( u(t) \) is \((\mathcal{F}_t)_{t \in [0,T]}\)-adapted;
• \( u(t) \) has a.s. continuous trajectories in \( X \);
• \( u(t) \) has a.s. bounded trajectories in \( Z \) and satisfies \( ||u(t) - u_0||_Z < R \) a.s. for \( t \in [0,T] \).

Recall that \( R > 0 \) (deterministic) was introduced at the beginning of this section. In the following sequel, \( B([0,T];Z) \) stands for the Banach space of all bounded \( Z \)-valued functions on \([0,T]\).

Similar to \[63\] and \[60\] one can make certain structural assumptions which ensure that \( A(u) \) exists for \( u \in \mathcal{U}_T \) and generates an evolution system \( U^u \). However, since all these assertions will depend on \( u \) we localize them as follows.

We introduce the sequence of stopping times \((\tau_n)_{n \geq 1}\) as

\[
\tau_n := \inf\{t \geq 0 : ||u(t)||_Z \geq n\} \tag{3.4}
\]

and impose several main assumptions \((A1')-(A3')\) that ensure the existence of a parabolic evolution operator \( U^u(\cdot,\cdot) \) together with a local Lipschitz continuity of the generators \( A(\cdot) \):

\((A1')\) For \( u \in \mathcal{U}_{T \wedge \tau_n}, A(u) \) is a sectorial operator of angle \( 0 < \varphi < \frac{\pi}{2} \), namely

\[
\sigma(A(u)) \subset \Sigma_\varphi := \{ \lambda \in \mathbb{C} : |\arg \lambda| < \varphi \}, \text{ for } u \in \mathcal{U}_{T \wedge \tau_n}.
\]

\((A2')\) For \( u \in \mathcal{U}_{T \wedge \tau_n} \) the resolvent operator \((\lambda\text{Id} - A(u))^{-1}\) satisfies the Hille-Yosida estimate, i.e., there exists \( M \geq 1 \) such that

\[
||(\lambda\text{Id} - A(u))^{-1}||_{\mathcal{L}(X)} \leq \frac{M}{|\lambda| + 1}, \text{ for } \lambda \notin \Sigma_\varphi, \text{ for } u \in \mathcal{U}_{T \wedge \tau_n}.
\]

\((A3')\) Let \( 0 < \nu \leq 1 \) be fixed. Then

\[
||A'(u)(A(u)^{-1} - A(v)^{-1})||_{\mathcal{L}(X)} \leq n||u - v||_Y, \text{ for } u, v \in \mathcal{U}_{T \wedge \tau_n}. \tag{3.5}
\]

Furthermore, recall that \( Y \) is a third Hilbert space such that \( Z \rightarrow Y \rightarrow X \).

Assumptions \((A1')\) and \((A2')\) imply according to Theorem \[2.1\] that \( A(u) \) generates an evolution system \( U^u \) for all \( u \in \mathcal{U}_{T \wedge \tau_n} \).

Moreover, regarding \[2.2\] we infer that

\[
||U^u(t,s)||_{\mathcal{L}(X)} \leq \tilde{C}(\delta)e^{n^{1/\delta}(t-s)}, \text{ for } u \in \mathcal{U}_{T \wedge \tau_n},
\]

so

\[
\sup_{0 \leq s \leq t \leq T \wedge \tau_n} ||U^u(t,s)||_{\mathcal{L}(X)} \leq C(\delta)e^{Tn^{1/\delta}}, \text{ for } u \in \mathcal{U}_{T \wedge \tau_n}. \tag{3.6}
\]
(A4') We require that there exist $\alpha, \beta$ with $0 \leq \alpha < \beta < \nu \leq 1$, $\beta + \nu > 1 + \alpha$ and $\beta \geq 1/2$, such that $D(A(u)^\alpha) =: Y$ and $D(A(u)^\beta) =: Z$; see [33] Chapter 5, Section 1.1.

**Remark 3.8**
- We emphasize that the domains $D(A(u))$ are in general allowed to depend on $u$ as discussed in [31], [33] Chapter 5] and [60] Section 3. We assume here for the case of brevity only constant domains (i.e. $\nu = 1$ in (A3')), but the techniques applied in this framework can be extended to non-constant domains assuming for instance $D(A(u)) \subset D(A(u)^\nu)$ for $u, v \in U_{\tilde{T} \wedge \tau_n}$ and $D(A(u)^\beta) \mapsto Z$, $D(A(u)^\alpha) \mapsto Y$.

- Note that one can take $\alpha = 0$ which means that $Y = X$, see [61] Chapter 5).
- Instead of (3.5) one can impose a global Lipschitz continuity of the generators, see [36] and thereafter introduce a cut-off function.

The assumption (A4') is meaningful from the point of view of the applications we want to consider as justified in Section 4. Since we are in the parabolic setting, we specify that we can identify the domains of the fractional powers of these generators with Sobolev spaces, namely

$$X_{\tilde{\mu}} := D(A(u)^{\tilde{\mu}}) = H^{2\tilde{\mu}}(G) \quad \text{or} \quad D(A(u)^{\tilde{\mu}}) = H^{2\tilde{\mu}}_N(G) \quad \text{or} \quad D(A(u)^{\tilde{\mu}}) = H^{2\tilde{\mu}}_D(G),$$

depending on the range of $\tilde{\mu} \geq 0$ and on the boundary conditions of $A$, see Section 4. Here $D$ and $N$ stand for Dirichlet respectively Neumann boundary conditions and $G \subset \mathbb{R}^n$ is an open bounded $C^2$-domain; Section 4 provides concrete examples for this setting. In the following we let $\| \cdot \|_{\tilde{\mu}}$ denote the norm in $X_{\tilde{\mu}}$.

We introduce a time-horizon $\tilde{T}$ such that $0 < \tilde{T} \leq T$ which will be chosen small enough as required in the fixed-point argument presented below and specify the set of processes we consider. Note that the following definition is meaningful since all terms involved in our computation exist pathwise.

**Definition 3.9** We define for $1 - \nu < \delta < \beta - \alpha \wedge \gamma$ and an arbitrary positive deterministic constant $k$ the set $\mathcal{K}$ of all $(\mathcal{F}_t)_{t \in [0, \tilde{T} \wedge \tau_n]}$-adapted stochastic processes $u : [0, \tilde{T} \wedge \tau_n] \times \Omega \rightarrow Y$ such that

1) $u(0) = u_0$ a.s.

2) $u \in \mathcal{B}([0, \tilde{T} \wedge \tau_n]; Z)$ with $\sup_{0 \leq t \leq \tilde{T} \wedge \tau_n} \|u(t) - u_0\|_Z \leq r$ a.s.

3) $u \in \mathcal{C}^\delta([0, \tilde{T} \wedge \tau_n]; Y)$ with $\sup_{0 \leq t \leq \tilde{T} \wedge \tau_n} \frac{\|u(t) - u(s)\|_Y}{(t-s)^\gamma} \leq k$ a.s.

Here we let $0 < r < R$, where the deterministic constant $R$ introduced above describes the radius of an arbitrary open ball in $Z$ and $\gamma$ denotes the Hölder exponent of $J(\sigma)$, recall Proposition 2.7.

**Remark 3.10**
- For computational simplicity, it is enough to consider $\mathcal{C}([0, \tilde{T} \wedge \tau_n]; Y)$ instead of $\mathcal{C}^\delta([0, \tilde{T} \wedge \tau_n]; Y)$, i.e. one shows that the trajectories of $u$ are a.s. bounded in $Z$ and a.s. continuous in $Y$. For optimal space-time regularity results we work with $\mathcal{K}$ as defined above.
Note that $\mathcal{K}$ is a closed subset of $\mathcal{C}([0, \bar{T} \land \tau_n]; Y)$. The first key result we prove is:

**Theorem 3.11** Let $f \in \mathcal{C}^\delta([0, T]; X)$, $\sigma \in L^0(\Omega; L_2(\mathcal{H}, X_2))$ and $(A1')-(A4')$ be satisfied. Then the stochastic evolution equation \ref{eq:3.7} has a unique local pathwise mild solution $u \in L^0(\Omega; B([0, \bar{T} \land \tau_n]; Z)) \cap L^0(\Omega; C^\delta([0, \bar{T} \land \tau_n]; Y))$.

**Remark 3.12**
1) The assumption $\sigma(\cdot) \in L_2(\mathcal{H}, X_2)$ will be required in order to prove the contraction property and is justified in Lemma 3.21. For suitable estimates of the generalized stochastic convolution given in (3.1), one only needs $\sigma(\cdot) \in L_2(\mathcal{H}, Z)$ as indicated in the following computations. For similar regularity conditions and further applications we refer the reader to [17, Section 2].

2) Note that the representation formula \ref{eq:3.8} holds true under this additional space-regularity assumption on $\sigma$.

We proceed to the proof of Theorem 3.11. To this aim, let $v \in \mathcal{K}$ a.s. and let $A_v(t)$ denote the family of sectorial operators $A_v(t) := A(v(t, \omega))$. For notational simplicity, the $\omega$-dependence will be dropped but it has to be kept in mind. We first consider the evolution equation

$$\begin{cases}
\mathrm{d}u(t) = [A_v(t)u(t) + f(t)] \, \mathrm{d}t + \sigma(t) \, \mathrm{d}W(t), \quad t \in [0, \bar{T}] \\
u(0) = u_0 \in K \text{ a.s.}
\end{cases} \tag{3.7}$$

Note that \ref{eq:3.7} represents a linear parabolic stochastic Cauchy problem, with time-dependent, random drift. Using [57, Theorem 5.3] we infer that \ref{eq:3.7} has a pathwise mild solution given by

$$u(t) = U^v(t, 0)u_0 + \int_0^t U^v(t, s)\sigma(s) \, \mathrm{d}W(s) + \int_0^t U^v(t, s)f(s) \, \mathrm{d}s - \int_0^t \int_s^t U^v(t, r)A_v(r) \sigma(r) \, \mathrm{d}W(r) \, \mathrm{d}s \text{ a.s.,} \tag{3.8}$$

where $U^v(t, s)$ is the random parabolic evolution operator generated by $A_v$.

**Remark 3.13** Note that using [57] Theorems 3.4-3.5, the previous convolutions can be defined in a pointwise sense and are pathwise continuous, which means that the solution formula \ref{eq:3.8} holds for almost all $\omega \in \Omega$ and for all $t \in [0, \bar{T}]$; we also refer to the proof of Lemma 3.15 below.

We define the mapping

$$\Phi(v) := u, \quad \text{for } v \in \mathcal{K}$$

and are going to prove that $\Phi$ maps $\mathcal{K}$ into itself and that it is a contraction with respect to the norm in $L^2(\Omega; \mathcal{C}([0, \bar{T} \land \tau_n]; Y))$ if one chooses $\bar{T}$ small enough. We split the proof into several steps.
Throughout this section we frequently use the estimates for analytic $C_0$-semigroups and parabolic evolution operators as stated in Theorem 2.3. Namely, according to (2.3), (2.4) there exists a constant $C : \Omega \rightarrow \mathbb{R}_+$ such that for $0 \leq s < t \leq \tilde{T} \wedge \tau_n$ and $\tilde{\alpha}, \tilde{\beta} \in (0, 1)$ we have

$$\|A^\beta(t)U^\nu(t, s)\|_{\mathcal{L}(X)} \leq C(t - s)^{-\tilde{\beta}}, \quad \|A^\beta(t)U^\nu(t, s)A^{-\tilde{\alpha}}(s)\|_{\mathcal{L}(X)} \leq C(t - s)^{-\tilde{\alpha} - \tilde{\beta}}. \quad (3.9)$$

Furthermore, one can estimate the difference between $U^\nu(t, s)$ and $e^{-(t-s)A_v(t)}$ or $e^{-(t-s)A_v(s)}$ for $0 \leq s \leq t \leq \tilde{T} \wedge \tau_n$ as follows from [31, Section 3]:

$$\|A^\beta(t)U^\nu(t, s)A^{-\tilde{\alpha}}(s) - e^{-(t-s)A_v(s)}\|_{\mathcal{L}(X)} \leq C(t - s)^{\tilde{\beta} + \nu - 1}, \quad 0 \leq \tilde{\beta} \leq 1 \quad (3.10)$$

$$\|A^\beta(t)(U^\nu(t, s) - e^{-(t-s)A_v(t)})A^{-\tilde{\alpha}}(s)\|_{\mathcal{L}(X)} \leq C(t - s)^{\tilde{\beta} - \tilde{\alpha} + \delta + \nu - 1}, \quad 0 \leq \tilde{\beta}, \tilde{\alpha} \leq 1. \quad (3.11)$$

Recall that $\delta$ stands for the Hölder exponent in the Acquistapace-Terreni condition (2.1).

**Remark 3.14** Note that $C$ depends on the stopping times, more precisely, according to (3.6) one has $C \leq \tilde{C}(\delta)e^{\tilde{T}n^{1/\delta}}$. For notational simplicity we drop this dependence, but it should be kept in mind.

Regarding all these we proceed towards our fixed-point argument.

**Lemma 3.15** If $\tilde{T}$ is sufficiently small, then $\Phi$ maps $\mathcal{K}$ into itself.

**Proof.** The regularity results stated in Proposition 2.6 are the key ingredients, which are required in order to estimate pathwise the terms containing stochastic integrals. Keeping in mind that $Z = D(A^\beta_v(t))$ for $t \in [0, \tilde{T} \wedge \tau_n]$ and $v \in \mathcal{K}$ a.s., the first of these estimates entails

$$\sup_{0 \leq t \leq \tilde{T} \wedge \tau_n} \|U^\nu(t, 0) \int_0^t \sigma(s) \, dW(s)\|_Z = \sup_{0 \leq t \leq \tilde{T} \wedge \tau_n} \|A^\beta_v(t)U^\nu(t, 0) \int_0^t \sigma(s) \, dW(s)\|_X$$

$$= \sup_{0 \leq t \leq \tilde{T} \wedge \tau_n} \|A^\beta_v(t)U^\nu(t, 0)A^{-\tilde{\alpha}}(0) \int_0^t \sigma(s) \, dW(s)\|_X$$

$$\leq C \sup_{0 \leq t \leq \tilde{T} \wedge \tau_n} \|A^\beta_v(t)U^\nu(t, 0)A^{-\tilde{\alpha}}(0)\|_{\mathcal{L}(X)} \|A^\beta(0) \int_0^t \sigma(s) \, dW(s)\|_X$$

$$\leq C \sup_{0 \leq t \leq \tilde{T} \wedge \tau_n} \| \int_0^t \sigma(s) \, dW(s)\|_Z \leq C ||J(\sigma)||_{\mathcal{L}([0, \tilde{T} \wedge \tau_n]; \mathbb{R})}.$$
Furthermore, due to Proposition 2.6 \( J(\sigma) \in C^\gamma([0, \bar{T} \land \tau_n]; Z) \) a.s. for \( \gamma < 1/2 \). We obtain

\[
\sup_{0 \leq t \leq \bar{T} \land \tau_n} \| \int_0^t U^v(t, s) A_v(s) \int_0^t \sigma(r) \, dW(r) \, ds \|_Z
\]

\[
= \sup_{0 \leq t \leq \bar{T} \land \tau_n} \| \int_0^t A_v^\beta(t) U^v(t, s) A_v^{-\beta}(s) A_v^\beta(s) \int_0^t \sigma(r) \, dW(r) \, ds \|_X
\]

\[
\leq C \sup_{0 \leq t \leq \bar{T} \land \tau_n} \int_0^t (t-s)^{-1/2} \left\| \int_0^t \sigma(r) \, dW(r) \right\|_Z \, ds
\]

\[
\leq C \sup_{0 \leq t \leq \bar{T} \land \tau_n} \int_0^t (t-s)^{-1} (t-s)^\gamma \| J(\sigma) \|_{C^\gamma([0, \bar{T} \land \tau_n]; Z)} \, ds \leq C \bar{T}^{\gamma} \| J(\sigma) \|_{C^\gamma([0, \bar{T} \land \tau_n]; Z)}.
\]

We can also estimate the term involving the initial condition

\[
\sup_{0 \leq t \leq \bar{T} \land \tau_n} \| U^v(t, 0) u_0 \|_Z = \sup_{0 \leq t \leq \bar{T} \land \tau_n} \| A_v^\beta(t) U^v(t, 0) A_v^{-\beta}(0) A_v^\beta(0) u_0 \|_X
\]

\[
\leq C \sup_{0 \leq t \leq \bar{T} \land \tau_n} \| A_v^\beta(t) U^v(t, 0) A_v^{-\beta}(0) \|_{L(X)} \| A_v^\beta(0) u_0 \|_X \leq C \| u_0 \|_Z.
\]

We immediately obtain that

\[
\sup_{0 \leq t \leq \bar{T} \land \tau_n} \| \int_0^t U^v(t, s) f(s) \, ds \|_Z \leq C \bar{T}^{1-\beta} \| f \|_{C^\beta([0, \bar{T} \land \tau_n]; X)}.
\]

Consequently, we may conclude based upon the previous estimates that

\[
\sup_{0 \leq t \leq \bar{T} \land \tau_n} \| u(t) \|_Z \leq C \left( \| u_0 \|_Z + \bar{T}^{\gamma} \| J(\sigma) \|_{C^\gamma([0, \bar{T} \land \tau_n]; Z)} + \bar{T}^{1-\beta} \| f \|_{C^\beta([0, \bar{T} \land \tau_n]; X)} \right).
\]

Furthermore, one can derive regularity results for \( u \) in appropriate function spaces. This can be shown using arguments from [63] Proposition 5.1 (deterministic estimates) and [67] Theorem 4.4 (pathwise Sobolev/Hölder regularity of the generalized stochastic convolution). For the convenience of the reader, we shortly indicate the main computation which justifies the Hölder-continuity of \( u \) in \( Y \). Let \( 0 \leq s < t \leq \bar{T} \land \tau_n \). Then building the difference of the solution at
these two time points yields

\[ u(t) - u(s) = (U^v(t, s) - \text{Id})u(s) + U^v(t, 0) \int_s^t \sigma(r) \, dW(r) + \int_s^t U^v(t, \tau) f(\tau) \, d\tau \]

\[ + \int_s^t U^v(t, \tau) A_v(\tau) \int_s^\tau \sigma(r) \, dW(r) + \int_s^t U^v(t, \tau) A_v(\tau) \int_s^\tau \sigma(r) \, dW(r) \, d\tau. \]

We write the first term as

\[ (U^v(t, s) - \text{Id})u(s) = \left[ (U^v(t, s) - e^{-(t-s)A_v(t)}) A_v^{-\beta}(s) + (e^{-(t-s)A_v(t)} - \text{Id})(A_v^{-\beta}(s) - A_v^{-\beta}(t)) \right. \]

\[ + \left. (e^{-(t-s)A_v(t)} - \text{Id})A_v^{-\beta}(t) \right] A_v^\beta(s)u(s). \quad (3.12) \]

Recalling that \( || \cdot ||_Y = ||A_v^\beta(t) \cdot ||_X \) we start estimating all the terms above in the appropriate norm. Using (3.11) we have for the first term

\[ ||A_v^\beta(t)(U^v(t, s) - e^{-(t-s)A_v(t)}) A_v^{-\beta}(s)||_{L(X)} \leq C(t-s)^{-\alpha+\delta+\nu-1}. \]

Recall that for \( \delta \) introduced in (2.1) we imposed that \( 1 - \nu < \delta < \beta - \alpha \wedge \gamma \).

Similarly,

\[ ||A_v^\beta(t)(e^{-(t-s)A_v(t)} - \text{Id})(A_v^{-\beta}(s) - A_v^{-\beta}(t))||_{L(X)} \]

\[ \leq ||(e^{-(t-s)A_v(t)} - \text{Id})A_v^{\alpha-\alpha'}(t)||_{L(X)} ||A_v^{\alpha'}(t)(A_v^{-\beta}(s) - A_v^{-\beta}(t))||_{L(X)} \leq C(t-s)^{\alpha'-\alpha+\delta} \]

and

\[ ||A_v^\beta(t)(e^{-(t-s)A_v(t)} - \text{Id})A_v^{-\beta}(t)||_{L(X)} \leq C(t-s)^{\beta-\alpha}. \]

Here \( \alpha' \) is an exponent satisfying \( \alpha < \alpha' < \beta + \nu - 1 \).

Obviously,

\[ \left\| A_v^\beta(t) \int_s^t U^v(t, \tau) f(\tau) \, d\tau \right\|_X \leq C(t-s)^{1-\alpha} \| f \|_{C^\delta([0, T \wedge \tau_0]; X)}. \]

Now, we turn to the terms of (3.12) that contain stochastic integrals and verify their regularity.

First of all

\[ \left\| A_v^\beta(t) U^v(t, 0) A_v^{-\beta}(0) A_v^0(0) \int_s^t \sigma(r) \, dW(r) \right\|_X \leq C t^{\beta-\alpha} ||J(\sigma)(t) - J(\sigma)(s)||_Z \]

\[ \leq C t^{\beta-\alpha} (t-s)^\gamma ||J(\sigma)||_{C^\gamma([0, T \wedge \tau_0]; Z)}. \]
Furthermore
\[
\left\| A^n_u(t) \int_s^t U^v(t, \tau) A_v(\tau) \int_r^s \sigma(r) \, dW(r) \, d\tau \right\|_X
\]
\[
= \left\| A^n_u(t) \int_s^t U^v(t, \tau) A_v(\tau) A_v^{-\beta}(\tau) A_v^\beta(\tau) \int_r^s \sigma(r) \, dW(r) \, d\tau \right\|_X
\]
\[
\leq C \int_s^t (t - \tau)^{3-1-\alpha} \| J(\sigma)(t) - J(\sigma)(\tau) \|_Z \, d\tau
\]
\[
\leq C(t - s)^{3-\alpha+\gamma} \| J(\sigma) \|_{C^\gamma([0, \widetilde{T} \wedge \tau_n]; Z)}.
\]

Finally
\[
\left\| A^n_u(t) \int_0^s U^v(t, \tau) A_v(\tau) \int_s^t \sigma(r) \, dW(r) \, d\tau \right\|_X
\]
\[
= \left\| A^n_u(t) \int_0^s U^v(t, \tau) A_v(\tau) A_v^{-\beta}(\tau) A_v^\beta(\tau) \int_0^s \sigma(r) \, dW(r) \, d\tau \right\|_X
\]
\[
\leq (t - s)^{3-\alpha+\gamma} \| J(\sigma) \|_{C^\gamma([0, \widetilde{T} \wedge \tau_n]; Z)}.
\]

In conclusion, choosing \( \widetilde{T} \) sufficiently small, we have that \( \| u(t) - u_0 \|_Z \leq R \) and \( \| u \|_{C^k([0, \widetilde{T} \wedge \tau_n]; Y)} \leq k \) a.s. This means that \( \Phi \) maps \( K \) into itself as claimed. \( \square \)

We also provide mean-square estimates for \( u \). The arguments employed in this computation will be required later on when we prove the contraction property of \( \Phi \) with respect to the norm in \( L^2(\Omega; C([0, \widetilde{T} \wedge \tau_n]; Y)) \).

**Lemma 3.16** We have
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_n} \| u(t) - u_0 \|^2_Z \right] \leq CR^2 + C \left( \widetilde{T}^{2(1-\beta)} \| f \|_{C^1([0, \widetilde{T} \wedge \tau_n]; \mathcal{X})}^2 + \widetilde{T} \| \sigma \|_{C^1([0, \widetilde{T} \wedge \tau_n]; \mathcal{L}_2(H, Z))}^2 \right)
\]
as well as
\[
\mathbb{E} \left[ \| u \|_{C^k([0, \widetilde{T} \wedge \tau_n]; Y)}^2 \right] \leq k^2.
\]

**Proof.** Let \( 0 < t \leq \widetilde{T} \wedge \tau_n \). We start to prove the first assertion considering each term of the solution separately. By similar deliberations combined with \( 3.10 \) and \( 3.11 \) we have
\[
\| (U^v(t, 0) - \text{Id}) u_0 \|_Z = \| A^n_v(t)(U^v(t, 0) - \text{Id}) u_0 \|_X
\]
\[
\leq C \| A^n_v(t)(U^v(t, 0) - \text{Id}) A_v^{-\beta}(0) \|_{C^k(\mathcal{X})} \| A_v^\beta(0) u_0 \|_X \leq CR,
\]
since \( u_0 \in K \) a.s. Consequently, we get
\[
E \left[ \sup_{0 \leq t \leq \bar{T} \land \tau_n} ||U^v(t, 0)u_0 - u_0||^2_Z \right] \leq CR^2.
\]

As argued above
\[
E \left[ \sup_{0 \leq t \leq \bar{T} \land \tau_n} \| \int_0^t U^v(t, s)f(s) \, ds \|_Z^2 \right] \leq C\bar{T}^{2(1-\beta)}||f||^2_{C([0, \bar{T} \land \tau_n], X)}.
\]

We now estimate the terms containing the stochastic integrals in \( L^2(\Omega, Z) \) following the strategy in [57] Lemma 5.2. The first stochastic integral yields
\[
||U^v(t, 0) \int_0^t \sigma(s) \, dW(s)||_{L^2(\Omega, Z)} = ||A^\beta_v(t)U^v(t, 0) \int_0^t \sigma(s) \, dW(s)||_{L^2(\Omega, X)}
\]
\[
\leq C||A^\beta_v(t)U^v(t, 0)||_{L(Z, X)} \int_0^t ||\sigma(s)|| \, dW(s)||_{L^2(\Omega, Z)}
\]
\[
\leq C\sqrt{t}||\sigma||_{C([0, t]; L_2(H, Z))}.
\]

Applying the Burkholder-Davis-Gundy inequality gives
\[
E \left[ \sup_{0 \leq t \leq \bar{T} \land \tau_n} \| \int_0^t \sigma(s) \, dW(s) \|_Z^2 \right] \leq C\bar{T}||\sigma||^2_{C([0, \bar{T} \land \tau_n]; L_2(H, Z))}.
\]

The next step is to analyze the generalized stochastic convolution. This yields
\[
\left\| \int_0^t U^v(t, s)A_v(s) \int_s^t \sigma(r) \, dW(r) \, ds \right\|_{L^2(\Omega, Z)} \leq \left\| \int_0^t A^\beta_v(t)U^v(t, s)A_v(s) \int_s^t \sigma(r) \, dW(r) \, ds \right\|_{L^2(\Omega, X)}
\]
\[
\leq C \int_0^t ||A^\beta_v(t)U^v(t, s)A_v(s)||_{L(Z, X)} \int_s^t ||\sigma(r)|| \, dW(r)||_{L^2(\Omega, Z)} \, ds
\]
\[
\leq C \int_0^t (t-s)^{-1/2} ||\sigma||_{L_2(H, Z)} \, ds \leq C\sqrt{t}||\sigma||_{C([0, \bar{T} \land \tau_n]; L_2(H, Z))}.
\]

Furthermore, one easily obtains that
\[
E \left[ \sup_{0 \leq t \leq \bar{T} \land \tau_n} \left\| \int_0^t U^v(t, s)A_v(s) \int_s^t \sigma(r) \, dW(r) \, ds \right\|_Z^2 \right]
\]
\[
\leq CE \int_0^t ||U^v(t, s)A_v(s)||_{L_2(H, Z)} \int_s^t ||\sigma(r)||^2 \, ds \leq C\bar{T}||\sigma||^2_{C([0, \bar{T} \land \tau_n]; L_2(H, Z))}.
\]
In summary, we have obtained
\[
E \left[ \sup_{0 \leq t \leq T \wedge \tau_n} \| u(t) - u_0 \| Z^2 \right] \leq CR^2 + C \left( \tilde{T}^{2(1-\beta)} \| f \|_{C^\beta([0,\tilde{T} \wedge \tau_n];X)} + \tilde{T} \| \sigma \|_{C([0,\tilde{T} \wedge \tau_n];L_2(\mathbb{H},Z))} \right),
\]
which proves the first statement. In order to prove the second one we only analyze the terms containing stochastic integrals, since the remaining ones are obvious due to Lemma 3.15. We set
\[
u(t) := U^v(t,0)(J(\sigma)(t)) - \int_0^t U^v(t,s)A_v(s)(J(\sigma)(t) - J(\sigma(s)) \ ds.
\]
From the estimates employed in Lemma 3.15 we infer that
\[
\| \nu \|_{C^\beta([0,\tilde{T} \wedge \tau_n];Y)} \leq C_\tilde{T} \| J(\sigma) \|_{C^\gamma([0,\tilde{T} \wedge \tau_n];Z)}
\]
where \( C_\tilde{T} \to 0 \) as \( \tilde{T} \to 0 \). Now, we can apply Proposition 2.7 which gives estimates of the second moment of the Hölder-norm of \( J(\sigma) \) and obtain
\[
\| \nu \|_{L^2(\Omega;C^\beta([0,\tilde{T} \wedge \tau_n];\mathbb{Y}))} \leq C \| J(\sigma) \|_{L^2(\Omega;C^\gamma([0,\tilde{T} \wedge \tau_n];\mathbb{Z}))} \leq CC_\tilde{T} \tilde{T}^{1/2} \| \sigma \|_{C^\gamma([0,\tilde{T} \wedge \tau_n];L_2(\mathbb{H},Z))}.
\]
Consequently, we find
\[
\| u \|_{L^2(\Omega;C^\beta([0,\tilde{T} \wedge \tau_n];\mathbb{Y}))} \leq CC_\tilde{T} \tilde{T}^{1/2} \| \sigma \|_{C^\gamma([0,\tilde{T} \wedge \tau_n];L_2(\mathbb{H},Z))} + \| u_0 \|_\mathbb{Z} + \| f \|_{C^\beta([0,\tilde{T}]\times \mathbb{X})}.
\]

The next step is to establish that \( \Phi : \mathcal{K} \to \mathcal{K} \) is a contraction with respect to the norm in \( L^2(\Omega;C([0,\tilde{T} \wedge \tau_n];\mathbb{Y})) \). To this aim consider two solutions having the same initial condition
\[
u_j(t) = U^{v_j}(t,0)u_0 + U^{v_j}(t,0) \int_0^t \sigma(s) \ dW(s) + \int_0^t U^{v_j}(t,s)f(s) \ ds - \int_0^t U^{v_j}(t,s)A_{v_j}(s) \int_s^t \sigma(r) \ dW(r) \ ds, \quad j \in \{1,2\}. \tag{3.13}
\]
Taking the difference between these two solutions entails
\[
u_1(t) - \nu_2(t) = (U^{v_1}(t,0) - U^{v_2}(t,0))u_0 + (U^{v_1}(t,0) - U^{v_2}(t,0)) \int_0^t \sigma(s) \ dW(s)
\]
\[
+ \int_0^t (U^{v_1}(t,s) - U^{v_2}(t,s))f(s) \ ds - \int_0^t (U^{v_1}(t,s)A_{v_1}(s) - U^{v_2}(t,s)A_{v_2}(s)) \int_s^t \sigma(r) \ dW(r) \ ds
\]
\[
=: (a^{v_1}(t) - a^{v_2}(t)) + (b^{v_1}(t) - b^{v_2}(t)) + (c^{v_1}(t) - c^{v_2}(t)) - (d^{v_1}(t) - d^{v_2}(t)).
\]
In order to verify the contraction property we will separately analyze the terms containing initial data, the deterministic drift term, and the stochastic/noise term(s).

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Lemma 3.17 Let $u_0 \in K$ a.s. and consider $0 \leq t \leq \hat{T} \wedge \tau_n$. Then the following estimate holds:

$$E \left[ \|a_{v_1} - a_{v_2}\|^2 \right] \leq C(Rn)^{2(\nu + \beta - \alpha - 1)}E\|v_1 - v_2\|^2_{C([0, \hat{T} \wedge \tau_n]; Y)}.$$ 

Proof. The proof follows the deterministic quasilinear setting discussed in [60, Lemma 3.1]. Knowing that for $0 \leq s \leq \tau \leq t \leq \hat{T} \wedge \tau_n$ and $u_0 \in K$ a.s.

$$\frac{\partial}{\partial \tau} U^{v_1}(t, \tau)U^{v_2}(\tau, s)u_0 = U^{v_1}(t, \tau)(A_{v_2}(\tau) - A_{v_1}(\tau))U^{v_2}(\tau, s)u_0,$$

we have

$$U^{v_2}(t, s)u_0 - U^{v_1}(t, s)u_0 = \int_s^t U^{v_1}(t, \tau)(A_{v_2}(\tau) - A_{v_1}(\tau))U^{v_2}(\tau, s)u_0 \, d\tau. \quad (3.14)$$

Therefore, from (3.14) using Yosida approximations $A_{v_j, m}$ for $j \in \{1, 2\}$ of the generators, we obtain for $0 \leq \theta < \nu$ that

$$A_{v_1, m}^\theta(t)(U^{v_1, m}(t, 0) - U^{v_2, m}(t, 0))A_{v_2, m}^{-\beta}(0)
= \int_0^t A_{v_1, m}^\theta(t)U^{v_1, m}(t, s)A_{v_1, m}(s)(A_{v_1, m}(s)^{-1} - A_{v_2, m}(s)^{-1})A_{v_2, m}(s)U^{v_2, m}(s, 0)A_{v_2, m}^{-\beta}(0) \, ds.$$ 

Letting $m \to \infty$ in the previous identity we conclude

$$A_{v_1}^\theta(t)(U^{v_1}(t, 0) - U^{v_2}(t, 0))A_{v_2}^{-\beta}(0)
= \int_0^t A_{v_1}^\theta(t)U^{v_1}(t, s)A_{v_1}^{1-\nu}(s)A_{v_1}(s)(A_{v_1}(s)^{-1} - A_{v_2}(s)^{-1})A_{v_2}(s)U^{v_2}(s, 0)A_{v_2}^{-\beta}(0) \, ds.$$ 

Taking into account (2.3), (5.5) and (3.11) we get

$$\|A_{v_1}^\theta(t)(U^{v_1}(t, 0) - U^{v_2}(t, 0))A_{v_2}^{-\beta}(0)\|_{C(X)} \leq Cn \int_0^t (t - s)^{\nu - \theta - 1}s^{-\beta - 1}\|v_1(s) - v_2(s)\|_Y \, ds
\leq Cnt^{\nu - \theta - \beta - 1}\|v_1 - v_2\|_{C([0, \hat{T} \wedge \tau_n]; Y)}, \quad (3.15)$$

from which we infer that

$$\|A_{v_1}^\theta(t)(U^{v_1}(t, 0) - U^{v_2}(t, 0))u_0\|_X \leq Cnt^{\nu - \theta - \beta - 1}\|v_1 - v_2\|_{C([0, \hat{T} \wedge \tau_n]; Y)}\|u_0\|_{\beta}. \quad (3.16)$$

Setting $\theta := \alpha$ in (3.16) entails

$$\|A_{v_1}^\alpha(t)(U^{v_1}(t, 0) - U^{v_2}(t, 0))u_0\|_X \leq Cnt^{\nu + \beta - \alpha - 1}\|v_1 - v_2\|_{C([0, \hat{T} \wedge \tau_n]; Y)}\|u_0\|_{\beta}.$$
which means that
\[
\| (U^{v_1}(t, 0) - U^{v_2}(t, 0))u_0 \|_Y \leq C n^\tilde{T}^{\nu+\beta-\alpha-1} \| v_1 - v_2 \|_{C([0, \tilde{T} \wedge \tau_n]; Y)} \| u_0 \|_\beta
\]
\[
\leq C R n^\tilde{T}^{\nu+\beta-\alpha-1} \| v_1 - v_2 \|_{C([0, \tilde{T} \wedge \tau_n]; Y)},
\]
since \( u_0 \in K \) a.s. Taking expectation in the previous inequality leads to the desired estimate. Recall that we assumed \( 1 - \nu < \beta - \alpha \) in (A4'). □

Lemma 3.18 The estimate
\[
E \left[ \left\| b^{v_1} - b^{v_2} \right\|^2_{C([0, \tilde{T} \wedge \tau_n]; Y)} \right] \leq C n^2 \tilde{T}^{2(\nu+\beta-\alpha)-1} \| \sigma \|^2_{L^2([0, \tilde{T} \wedge \tau_n]; L^2(\Omega, Z))} E \left[ \| v_1 - v_2 \|^2_{C([0, \tilde{T} \wedge \tau_n]; Y)} \right]
\] (3.17)
is valid.

Proof. Let \( 0 < t \leq \tilde{T} \wedge \tau_n \). We have
\[
\| (U^{v_1}(t, 0) - U^{v_2}(t, 0)) \int_0^t \sigma(s) \, dW(s) \|_{L^2(\Omega, Y)}
\]
\[
\leq \| A^{\alpha}_{v_1}(t)(U^{v_1}(t, 0) - U^{v_2}(t, 0))A^{-\beta}_{v_2}(0) \|_{C(X)} \int_0^t \| \sigma(s) \, dW(s) \|_{L^2(\Omega, Z)}
\]
\[
\leq C \sqrt{t} \| A^{\alpha}_{v_1}(t)(U^{v_1}(t, 0) - U^{v_2}(t, 0))A^{-\beta}_{v_2}(0) \|_{C(X)} \| \sigma \|_{C([0, t]; L^2(\Omega, Z))}.
\]
As already seen, (3.16) entails
\[
\| A^{\alpha}_{v_1}(t)(U^{v_1}(t, 0) - U^{v_2}(t, 0)) \|_{C(Z, X)} \leq C n^\tilde{T}^{\nu-\alpha+\beta-1} \| v_1 - v_2 \|_{C([0, \tilde{T} \wedge \tau_n]; Y)}.
\]
Putting all this together proves (3.17). □

Lemma 3.19 For \( 0 \leq t \leq \tilde{T} \wedge \tau_n \) we have
\[
E \left[ \left\| c^{v_1} - c^{v_2} \right\|^2_{C([0, \tilde{T} \wedge \tau_n]; Y)} \right] \leq C n^2 \tilde{T}^{(\nu-\alpha)} \| f \|^2_{L^2([0, \tilde{T} \wedge \tau_n]; X)} E \left[ \| v_1 - v_2 \|^2_{C([0, \tilde{T} \wedge \tau_n]; Y)} \right].
\]

Proof. Consider \( 0 \leq \theta < \nu \leq 1 \). Using again Yosida approximations we infer that
\[
A^{\theta}_{v_1}(t) \int_0^t (U^{v_1}(s, t) - U^{v_2}(s, t)) f(s) \, ds
\]
\[
= \int_0^t A^{\theta}_{v_1}(t) U^{v_1}(t, r) A_{v_1}(r) (A^{-1}_{v_1}(r) - A^{-1}_{v_2}(r)) A_{v_2}(r) U^{v_2}(r, s) f(s) \, dr \, ds
\]
\[
= \int_0^t A^{\theta}_{v_1}(t) U^{v_1}(t, r) A_{v_1}(r) (A^{-1}_{v_1}(r) - A^{-1}_{v_2}(r)) A_{v_2}(r) \int_0^r U^{v_2}(r, s) f(s) \, ds \, dr
\]
\[
= \int_0^t A^{\theta}_{v_1}(t) U^{v_1}(t, r) A^{1-\nu}_{v_1}(r) A_{v_1}(r) (A^{-1}_{v_1}(r) - A^{-1}_{v_2}(r)) A_{v_2}(r) \int_0^r U^{v_2}(r, s) f(s) \, ds \, dr.
\]
Regarding that

\[ \left\| A_{v_2}(r) \int_0^r U^{v_2}(r, s) f(s) \, ds \right\|_X \leq C_\delta \| f \|_{C^\gamma([0,T \land \tau_n]; X)} \]

we conclude similar to the proof of Lemma 3.15

\[ \left\| A_{v_2}^\beta(t) \int_0^t (U^{v_1}(t, s) - U^{v_2}(t, s)) f(s) \, ds \right\|_X \]

\[ \leq Cn \int_0^t (t-r)^{\nu-\theta} ||v_1(r) - v_2(r)||_Y \, dr \]

\[ \leq Cn \int_0^t (t-r)^{\nu-\theta} \, dr ||v_1 - v_2||_{C([0,T \land \tau_n]; Y)} ||f||_{C^\delta([0,T \land \tau_n]; X)} \]

\[ \leq CnT^{\nu-\theta} ||v_1 - v_2||_{C([0,T \land \tau_n]; Y)} ||f||_{C^\delta([0,T \land \tau_n]; X)}. \] (3.18)

Obviously, one also obtains

\[ \left\| A_{v_1}^\beta(t) \int_0^t (U^{v_1}(t, s) - U^{v_2}(t, s)) f(s) \, ds \right\|_X \leq Cn^{\nu-\theta} ||v_1 - v_2||_{C([0,T \land \tau_n]; Z)} ||f||_{C^\delta([0,T \land \tau_n]; X)}. \]

Again setting \( \theta := \alpha \) in (3.18) leads to

\[ \left\| \int_0^t (U^{v_1}(t, s) - U^{v_2}(t, s)) f(s) \, ds \right\|_Y \leq Cn^{\nu-\alpha} ||f||_{C^\gamma([0,T \land \tau_n]; X)} ||v_1 - v_2||_{C([0,T \land \tau_n]; Y)}. \]

Taking expectation yields the claimed result. \( \square \)

**Remark 3.20** Instead of taking \( f \) Hölder-continuous with values in \( X \) one can let \( f \) be just continuous with values in \( X_{\hat{\rho}} \), for a suitable chosen \( \hat{\rho} > 0 \), consult [63] and the references specified therein.

We now analyze the generalized stochastic convolution. To this aim the higher space-regularity of \( \sigma \) is required. Such a condition is natural for this technique, since one needs additional regularity assumptions when building the difference of two evolution systems, compare for instance Lemma 3.17

**Lemma 3.21** We have

\[ \mathbb{E} \left[ ||d^{v_1} - d^{v_2}||_2^2_{C([0,T \land \tau_n]; Y)} \right] \leq Cn^2 T^{2(\nu-\alpha+2\beta)-1} ||\sigma||_2^2_{C([0,T \land \tau_n]; L_2(H, X_{2\beta}))} \mathbb{E} \left[ ||v_1 - v_2||^2_{C([0,T \land \tau_n]; Y)} \right]. \]
Proof. Let $0 \leq t \leq \tilde{T} \wedge \tau_n$. Then

\[
\int_0^t \left( U^{v_1}(t, s)A_{v_1}(s) - U^{v_2}(t, s)A_{v_2}(s) \right) \int_s^t \sigma(r) \, dW(r) \, ds = \int_0^t \left( U^{v_1}(t, s) - U^{v_2}(t, s) \right)A_{v_1}(s) \int_s^t \sigma(r) \, dW(r) \, ds \\
+ \int_0^t U^{v_2}(t, s)(A_{v_1}(s) - A_{v_2}(s)) \int_s^t \sigma(r) \, dW(r) \, ds \\
=: (e^{v_1}(t) - e^{v_2}(t)) + (f^{v_1}(t) - f^{v_2}(t)).
\]  

(3.19)

Recalling Lemma 3.17 we rewrite the first term as

\[
\int_0^t \left( U^{v_1}(t, s) - U^{v_2}(t, s) \right)A_{v_1}(s) \int_s^t \sigma(r) \, dW(r) \, ds
\]

\[
= \int_0^t \int_s^t U^{v_2}(t, q)A_{v_2}(q)(A_{v_1}^{-1}(q) - A_{v_2}^{-1}(q))A_{v_1}(q)U^{v_1}(q, s)A_{v_1}^{1-2\beta}(s) \, dq \, A_{v_1}^{2\beta}(s) \int_s^t \sigma(r) \, dW(r) \, ds.
\]

Let $0 \leq \theta < \nu$. Regarding that

\[
\left\| \int_0^t \left( U^{v_1}(t, s) - U^{v_2}(t, s) \right)A_{v_1}(s) \int_s^t \sigma(r) \, dW(r) \, ds \right\|
\]

\[
= \left\| \int_0^t \int_s^t U^{v_2}(t, \tau)A_{v_2}(\tau)(A_{v_1}^{-1}(\tau) - A_{v_2}^{-1}(\tau))A_{v_1}(\tau)U^{v_1}(\tau, s)A_{v_1}(s)A_{v_1}^{-2\beta}(s) \, d\tau \, A_{v_1}^{2\beta}(s) \int_s^t \sigma(r) \, dW(r) \, ds \right\|
\]

we get

\[
\left\| \int_0^t A_{v_1}^{\theta}(t)(U^{v_1}(t, s) - U^{v_2}(t, s))A_{v_1}(s) \int_s^t \sigma(r) \, dW(r) \, ds \right\|_{L^2(\Omega; X)}
\]

\[
\leq Cn \int_0^t (t - s)^{\nu/2 + 2\beta - 1/2} \, ds \left\| \sigma \right\|_{C([0, t]; L^2(H, X_{2\beta}))} \left\| v_1 - v_2 \right\|_{C([0, t]; Y)}
\]

\[
\leq Cnt^{\nu/2 + 2\beta - 1/2} \left\| \sigma \right\|_{C([0, t]; L^2(H, X_{2\beta}))} \left\| v_1 - v_2 \right\|_{C([0, t]; Y)}.
\]
Therefore, choosing \( \tilde{T} \) small enough we obtain that 
\[
\Phi \in \mathcal{L}(X),
\]
and we estimate the second term in (3.19) as 
\[
\left\lVert \int_0^t A_{v_2}(t) U^{v_2}(t, s) (A_{v_1}(s) - A_{v_2}(s)) \int_s^t \sigma(r) \, dW(r) \, ds \right\rVert_{L^2(\Omega; X)} \leq C n^{\theta/2} |||\sigma|||_{C([0,\tilde{T} \wedge \tau_n]; \mathcal{L}(H, X_{\beta}))} \left[ |||v_1 - v_2|||_{C([0,\tilde{T} \wedge \tau_n]; Y)}^2 \right].
\]
Summarizing the previous calculations, we obtain 
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq \tilde{T} \wedge \tau_n} \left( \left\lVert \int_0^t A_{v_2}(t) U^{v_2}(t, s) (A_{v_1}(s) - A_{v_2}(s)) \int_s^t \sigma(r) \, dW(r) \, ds \right\rVert_{X}^2 \right) \right] \leq C n^{2 \tilde{T}^2(\nu - \theta + 2\beta - 1)} |||\sigma|||_{C([0,\tilde{T} \wedge \tau_n]; \mathcal{L}(H, X_{\beta}))} \left[ |||v_1 - v_2|||_{C([0,\tilde{T} \wedge \tau_n]; Y)}^2 \right].
\] (3.21)

Setting \( \theta := \alpha \) in (3.20) and (3.21), the triangle-inequality in \( L^2(\Omega; Y) \) proves the statement. \( \square \)

Collecting all these results finally yields.

**Lemma 3.22** The mapping \( \Phi : \mathcal{K} \to \mathcal{K} \) is a contraction for a sufficiently small \( \tilde{T} \).

**Proof.** We obtained that 
\[
\mathbb{E} \left[ ||u_1 - u_2||_{C([0,\tilde{T} \wedge \tau_n]; Y)}^2 \right] \leq C(Rn)^2 \tilde{T}^{2(\nu + \beta - 1)} \mathbb{E} \left[ ||v_1 - v_2||_{C([0,\tilde{T} \wedge \tau_n]; Y)}^2 \right] + C n^{2 \tilde{T}^2(\nu - \alpha + \beta - 1)} |||\sigma|||_{C([0,\tilde{T} \wedge \tau_n]; \mathcal{L}(H, Z))} \mathbb{E} \left[ ||v_1 - v_2||_{C([0,\tilde{T} \wedge \tau_n]; Y)}^2 \right] + C n^{2 \tilde{T}^2(\nu - 2\beta - 1)} \mathbb{E} \left[ ||v_1 - v_2||_{C([0,\tilde{T} \wedge \tau_n]; X)}^2 \right] + C n^{2 \tilde{T}^2(\nu - \alpha + 2\beta - 1)} |||\sigma|||_{C([0,\tilde{T} \wedge \tau_n]; \mathcal{L}(H, X_{\beta}))} \mathbb{E} \left[ ||v_1 - v_2||_{C([0,\tilde{T} \wedge \tau_n]; Y)}^2 \right].
\]
Therefore, choosing \( \tilde{T} \) small enough we obtain that \( \Phi \) is a contraction. \( \square \)
In conclusion, due to Lemma 3.22, Banach’s fixed-point theorem proves Theorem 3.11.

We now derive analogously to the proof of Theorem 1.3 in [38] assertions regarding the corresponding stopping times.

Lemma 3.23 (Positivity of the stopping times) Let $0 < \varepsilon < 1$ be fixed. Then under the assumptions of Theorem 3.11 we have

$$
\mathbb{P}(\tau_n > \varepsilon) > 1 - \varepsilon^2 \left( \tilde{C} \mathbb{E}[|u_0|^2_Z] + \tilde{C} \varepsilon^2 |\sigma|^2_{L_2(H,Z)} + \tilde{C} \varepsilon^{2(1-\beta)} |f|^2_{L_2([0,\varepsilon];X)} \right),
$$

(3.22)

where the positive constant $\tilde{C} = \tilde{C}(\delta)$ is independent of $\varepsilon$ and $u_0$.

Proof. For $0 < \varepsilon < 1$ there exists a positive number $n$ such that

$$
\frac{1}{n+1} \leq \varepsilon < \frac{1}{n}.
$$

Applying Theorem 3.11 we obtain that $(u, \tilde{T} \wedge \tau_n)$ is the pathwise mild solution of (3.1). Our aim is to derive estimates for $\mathbb{E} \sup_{0 \leq t \leq \varepsilon} ||u(t \wedge \tau_n)||^2_Z$.

Using the definition of $\tau_n$ we know that

$$
\left\{ \omega : \sup_{0 \leq t \leq \varepsilon} ||u(t \wedge \tau_n)||_Z < n \right\} \subset \{ \omega : \tau_n(\omega) > \varepsilon \}.
$$

(3.24)

Therefore, if we show (3.23), Chebyshev’s inequality proves the statement.

By the same computation as performed in Lemma 3.16 we have

$$
\mathbb{E} \sup_{0 \leq t \leq \tilde{T} \wedge \tau_n} ||U^u(t,0)u_0||^2_Z \leq \tilde{C} e^{n^2/\tilde{T}} \mathbb{E}[|u_0|^2_Z].
$$

Furthermore,

$$
\mathbb{E} \sup_{0 \leq t \leq \tilde{T} \wedge \tau_n} ||U^u(t,0) \int_0^t \sigma(s) \, dW(s)||^2_Z \leq \tilde{C} e^{n^2/\tilde{T}} \mathbb{E}[||\sigma||^2_{L_2([0,\tilde{T});L_2(H,Z))}]
$$

and

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq \tilde{T} \wedge \tau_n} \left\| \int_0^t U^u(t,s)A_v(s) \int_s^t \sigma(r) \, dW(r) \, ds \right\|^2_Z \right]
$$

$$
\leq \mathbb{C} \mathbb{E} \int_0^{\tilde{T} \wedge \tau_n} ||U^v(t,s)A_v(s) \int_s^t \sigma(r) \, dW(r)||^2_Z \, ds \leq \tilde{C} e^{n^2/\tilde{T}} \mathbb{E}[||\sigma||^2_{L_2([0,\tilde{T});L_2(H,Z))}],
$$

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This means that
\[
\mathbb{E} \sup_{0 \leq t \leq \tau_n} ||u(t)||_Z^2 \leq \tilde{C}e^{n^{2/3} \varepsilon} \left( \mathbb{E}||u_0||_Z^2 + \tilde{T}||\sigma||_{\mathcal{C}(\Omega, f; C^2(\mathbb{R}))}^2 + \tilde{T}^{2(1-\beta)}||f||_{C^2(\Omega, f; \mathbb{R})}^2 \right),
\]
which implies
\[
\mathbb{E} \sup_{0 \leq t \leq \varepsilon} ||u(t \wedge \tau_n)||_Z^2 \leq \tilde{C}e^{n^{2/3} \varepsilon} \left( \mathbb{E}||u_0||_Z^2 + \varepsilon||\sigma||_{C(\Omega, f; C^2(\mathbb{R}))}^2 + \varepsilon^{2(1-\beta)}||f||_{C^2(\Omega, f; \mathbb{R})}^2 \right). \quad (3.25)
\]
From (3.21) and Chebyshev’s inequality we have that
\[
\mathbb{P}(\tau_n > \varepsilon) > \mathbb{P}(\sup_{0 \leq t \leq \varepsilon} ||u(t \wedge \tau_n)||_Z < n) \geq 1 - \frac{1}{n^2} \mathbb{E} \sup_{0 \leq t \leq \varepsilon} ||u(t \wedge \tau_n)||_Z^2
\]
\[
\geq 1 - \tilde{C}e^{n^{2/3} \varepsilon} \left( \mathbb{E}||u_0||_Z^2 + \varepsilon||\sigma||_{C(\Omega, f; C^2(\mathbb{R}))}^2 + \varepsilon^{2(1-\beta)}||f||_{C^2(\Omega, f; \mathbb{R})}^2 \right)
\]
\[
\geq 1 - \varepsilon^2 \left( \tilde{C}\mathbb{E}||u_0||_Z^2 + \tilde{C}\varepsilon||\sigma||_{C(\Omega, f; C^2(\mathbb{R}))}^2 + \tilde{C}\varepsilon^{2(1-\beta)}||f||_{C^2(\Omega, f; \mathbb{R})}^2 \right). \quad \square
\]

As already discussed, the next step is to extend the results established in Theorem 3.11 and include nonlinearities of semilinear type. More precisely, we consider
\[
\begin{cases}
\frac{du(t)}{dt} = [(Au(t))(u(t)) + F(t,u(t)) + \sigma(t,u(t))] \ dt + \sigma(t,u(t)) \ dW(t), \ t \in [0,T] \\
u(0) = u_0 \in K \ a.s.
\end{cases}
\quad (3.26)
\]
and make standard local Lipschitz and growth assumptions on $F : \Omega \times [0,T] \times X \to X$ and $\sigma : \Omega \times [0,T] \times X \to \mathcal{L}_2(H,X_{2\beta})$. In particular, we assume that there exist constants $L_F = L_F(n), l_F = l_F(n), L_\sigma = L_\sigma(n), l_\sigma = l_\sigma(n) > 0$ such that
\[
||F(u) - F(v)||_X \leq L_F||u - v||_X, \ ||u||_Z \leq n, ||v||_Z \leq n, \quad (3.27)
\]
\[
||F(u)||_X \leq l_F(1 + ||u||_X), \ ||u||_Z \leq n, \quad (3.28)
\]
and respectively
\[
||\sigma(u) - \sigma(v)||_{\mathcal{L}_2(H,X_{2\beta})} \leq L_\sigma||u - v||_X, \ ||u||_Z \leq n, ||v||_Z \leq n, \quad (3.29)
\]
\[
||\sigma(u)||_{\mathcal{L}_2(H,X_{2\beta})} \leq l_\sigma(1 + ||u||_X), \ ||u||_Z \leq n. \quad (3.30)
\]
We remark that since $\mathcal{L}_2(H,X_{2\beta}) \hookrightarrow \mathcal{L}_2(H,X_\beta)$ and $Y \hookrightarrow X$ we also get
\[
||\sigma(u) - \sigma(v)||_{\mathcal{L}_2(H,X_\beta)} \leq CL_\sigma||\sigma(u) - \sigma(v)||_{\mathcal{L}_2(H,X_{2\beta})} \leq CL_\sigma||u - v||_X \leq CL_\sigma||u - v||_Y. \quad (3.31)
\]
We can now state our main result of this work.
Theorem 3.24 The quasilinear SPDE (3.26) possesses a unique local pathwise mild solution $u \in L^0(\Omega; \mathcal{B}([0, \tilde{T} \wedge \tau_n]; Z)) \cap L^0(\Omega; \mathcal{C}^\delta([0, T \wedge \tau_n]; Y))$ given by

$$u(t) = U^u(t, 0)u_0 + U^u(t, 0) \int_0^t \sigma(r, u(r)) \, dW(r) + \int_0^t U^u(t, s)f(s, u(s)) \, ds$$

$$- \int_0^t U^u(t, s)A(u(s)) \int_s^t \sigma(r, u(r)) \, dW(r) \, ds.$$  (3.32)

We note that all the results from the linear case concerning the definition and regularity properties of the generalized stochastic convolution can be extended to the nonlinear setting as discussed in [57, Section 5.1].

In order to prove Theorem 3.24 analogously to the linear case, we first let $v \in K$ a.s., set $f_v(t) := F(t, v(t)), \sigma_v(t) := \sigma(t, v(t))$ and consider the Cauchy problem

$$\left\{ \begin{array}{l}
\frac{du}{dt} = [(Au(t))(u(t)) + f_v(t)] \, dt + \sigma_v(t) \, dW(t), \ t \in [0, \tilde{T}] \\
u(0) = u_0 \in K \text{ a.s.}
\end{array} \right.$$  (3.33)

Note that all the assumptions of Theorem 3.11 are satisfied. This means that the quasilinear inhomogenous equation (3.33) possesses a unique pathwise mild solution $u \in L^0(\Omega; \mathcal{B}([0, \tilde{T} \wedge \tau_n]; Z)) \cap L^0(\Omega; \mathcal{C}^\delta([0, T \wedge \tau_n]; Y))$ such that

$$u(t) = U^u(t, 0)u_0 + U^u(t, 0) \int_0^t \sigma_v(r) \, dW(r) + \int_0^t U^u(t, s)f_v(s) \, ds$$

$$- \int_0^t U^u(t, s)A(u(s)) \int_s^t \sigma_v(r) \, dW(r) \, ds.$$  (3.34)

In order to obtain a solution for (3.26) by a fixed-point argument, we define just as before the mapping

$$\Phi(v) := u, \text{ for } v \in K.$$  (3.35)

One can show analogously to the proof of Lemma 3.15 that this maps $K$ into itself if one chooses $\tilde{T}$ small enough.

We now verify the contraction property with respect to the norm in $L^2(\Omega; \mathcal{C}([0, \tilde{T} \wedge \tau_n]; Y))$. The computation relies on similar estimates as for (3.7) combined with the local Lipschitz continuity and growth boundedness of $F$ and $\sigma$.

Lemma 3.25 The mapping $\Phi$ is a contraction if $\tilde{T}$ is sufficiently small.
Proof. Let \( 0 < t \leq \tilde{T} \land \tau_n \). Considering the difference between two solutions yields

\[
\begin{align*}
    u_1(t) - u_2(t) &= (U^{u_1}(t,0) - U^{u_2}(t,0))u_0 + (U^{u_1}(t,0) - U^{u_2}(t,0)) \int_0^t \sigma_{v_1}(r) \ dW(r) \\
    &+ U^{u_2}(t,0) \int_0^t (\sigma_{v_2}(r) - \sigma_{v_2}(r)) \ dW(r) + \int_0^t (U^{u_1}(t,s) - U^{u_2}(t,s))f_{v_1}(s) \ ds \\
    &+ \int_0^t U^{u_2}(t,s)(f_{v_1}(s) - f_{v_2}(s)) \ ds - \int_0^t U^{u_2}(t,s)A_{u_2}(s) \int_0^t (\sigma_{v_1}(r) - \sigma_{v_2}(r)) \ dW(r) \ ds \\
    &- \int_0^t (U^{u_1}(t,s) - U^{u_2}(t,s))A_{u_1}(s) \int_0^t \sigma_{v_1}(r) \ dW(r) \ ds \\
    &- \int_0^t U^{u_2}(t,s)(A_{u_1}(s) - A_{u_2}(s)) \int_0^t \sigma_{v_1}(r) \ dW(r) \ ds \\
    &=: (\tilde{a}^{u_1}(t) - \tilde{a}^{u_2}(t)) + (\tilde{b}^{u_1}(t) - \tilde{b}^{u_2}(t)) + (\tilde{c}^{u_1}(t) - \tilde{c}^{u_2}(t)) + (\tilde{d}^{u_1}(t) - \tilde{d}^{u_2}(t)) \\
    &+ (\tilde{e}^{u_1}(t) - \tilde{e}^{u_2}(t)) + (\tilde{f}^{u_1}(t) - \tilde{f}^{u_2}(t)) + (\tilde{g}^{u_1}(t) - \tilde{g}^{u_2}(t)) + (\tilde{h}^{u_1}(t) - \tilde{h}^{u_2}(t)).
\end{align*}
\]

We now provide suitable estimates for each of the terms above in appropriate function spaces. For the first one, as discussed in Lemma \([3.17]\) we have

\[
    \mathbb{E} \left[ ||\tilde{a}^{u_1} - \tilde{a}^{u_2}||^2_{C([0,\tilde{T} \land \tau_n];Y)} \right] \leq C(Rn)^{2\tilde{T}^2(\nu+\beta-\alpha-1)} \mathbb{E} \left[ ||u_1 - u_2||^2_{C([0,\tilde{T} \land \tau_n];Y)} \right].
\]

From Lemma \([3.18]\) applying the Burkholder-Davis-Gundy inequality and \([3.30]\), the second term yields

\[
\begin{align*}
    \mathbb{E} \left[ ||\tilde{b}^{u_1} - \tilde{b}^{u_2}||^2_{C([0,\tilde{T} \land \tau_n];Y)} \right] &= \mathbb{E} \left[ \sup_{0 \leq t \leq \tilde{T} \land \tau_n} ||A_{v_1}^0(t)(U^{u_1}(t,0) - U^{u_2}(t,0)) \int_0^t \sigma_{v_1}(r) \ dW(r)||^2_{Y} \right] \\
    &\leq CE \left[ \sup_{0 \leq t \leq \tilde{T} \land \tau_n} ||A_{v_1}^0(t)(U^{u_1}(t,0) - U^{u_2}(t,0))A_{v_2}^{-\beta}(0)||^2_{L(\mathcal{X})} || \int_0^t \sigma_{v_1}(s) \ dW(s)||^2_{Z} \right] \\
    &\leq Cn^{2\tilde{T}^2(\nu-\alpha+\beta)-2}||u_1 - u_2||^2_{C([0,\tilde{T} \land \tau_n];Y)} \mathbb{E} \left[ \int_0^{\tilde{T} \land \tau_n} ||\sigma_{v_1}(t)||^2_{L^2(\mathcal{H},Z)} \ dt \right] \\
    &\leq C(l_n)^{2\tilde{T}^2(\nu-\alpha+\beta)-1}||u_1 - u_2||^2_{C([0,\tilde{T} \land \tau_n];Y)} \mathbb{E} \left[ \sup_{0 \leq t \leq \tilde{T} \land \tau_n} (1 + ||v_1(t)||^2_Z) \right] \\
    &\leq C(l_n)^{2\tilde{T}^2(\nu-\alpha+\beta)-1}||u_1 - u_2||^2_{C([0,\tilde{T} \land \tau_n];Y)} \\
    &\leq C(l_n)^{2\tilde{T}^2(\nu-\alpha+\beta)-1}||u_1 - u_2||^2_{C([0,\tilde{T} \land \tau_n];Y)} (1 + R^2 + \mathbb{E} [||u_0||^2_Z]) \\
    &\leq C(CRl_n)^{2\tilde{T}^2(\nu-\alpha+\beta)-1} \mathbb{E} [||u_1 - u_2||^2_{C([0,\tilde{T} \land \tau_n];Y)}].
\end{align*}
\]
According to (3.29) we get

$$\mathbb{E}\left[\left\|\hat{v}^1 - \hat{v}^2\right\|^2_{\mathcal{C}([0,\hat{T}\wedge \tau_n];Y)}\right] = \mathbb{E}\left[\sup_{0 \leq t \leq \hat{T}\wedge \tau_n} \left\|U^{u_1}(t,0) \int_{0}^{t} (\sigma_{v_1}(r) - \sigma_{v_2}(r)) \, dW(r)\right\|^2_Y\right]$$

$$\leq CL_\sigma^2 \hat{T}^{2(\beta - \alpha)+1} \mathbb{E}\left[||v_1 - v_2||^2_{\mathcal{C}([0,\hat{T}\wedge \tau_n];Y)}\right].$$

Keeping Lemma 3.19 in mind, together with the fact that $v_1 \in \mathcal{K}$ a.s., yields due to (3.28)

$$\left\|\int_{0}^{t} (U^{u_1}(t,s) - U^{u_2}(t,s))f_{v_1}(s) \, ds\right\|_Y \leq \left\|\int_{0}^{t} A_{u_1}(t)(U^{u_1}(t,s) - U^{u_2}(t,s))f_{v_1}(s) \, ds\right\|_X$$

$$\leq Cn \int_{0}^{t} (t - \tau)^{\nu - \alpha - 1} ||u_1(\tau) - u_2(\tau)||_Y ||f_{v_1}||_{\mathcal{C}([0,t];X)} \, d\tau$$

$$\leq CCRL_F \hat{T}^{\nu - \alpha} ||u_1 - u_2||_{\mathcal{C}([0,\hat{T}\wedge \tau_n];Y)}.$$

Consequently, we find taking the expectation that

$$\mathbb{E}\left[\left\|\hat{v}^1 - \hat{v}^2\right\|^2_{\mathcal{C}([0,\hat{T}\wedge \tau_n];Y)}\right] \leq C(CRnL_F)^2 \hat{T}^{2(\nu - \alpha)} \mathbb{E}\left[||u_1 - u_2||^2_{\mathcal{C}([0,\hat{T}\wedge \tau_n];Y)}\right].$$

From (3.24) and using the Lipschitz assumption for $f$ we can directly infer, via similar calculations as above that

$$\mathbb{E}\left[\left\|\hat{v}^1 - \hat{v}^2\right\|^2_{\mathcal{C}([0,\hat{T}\wedge \tau_n];Y)}\right] \leq CL_\sigma^2 \hat{T}^{2(\nu - \alpha)} \mathbb{E}\left[||u_1 - u_2||^2_{\mathcal{C}([0,\hat{T}\wedge \tau_n];Y)}\right].$$

Similar computations as in the proof of Lemma 3.10 together with (3.29) imply

$$\mathbb{E}\left[\left\|\hat{f}^{u_1} - \hat{f}^{u_2}\right\|^2_{\mathcal{C}([0,\hat{T}\wedge \tau_n];Y)}\right] \leq CL_\sigma^2 \hat{T}^{2(\beta - \alpha)+1} \mathbb{E}\left[||u_1 - u_2||^2_{\mathcal{C}([0,\hat{T}\wedge \tau_n];Y)}\right].$$

The last two terms can be estimated as in Lemma 3.21 applying (3.30). The result of the computations is

$$\mathbb{E}\left[\left\|\hat{g}^{u_1} - \hat{g}^{u_2}\right\|^2_{\mathcal{C}([0,\hat{T}\wedge \tau_n];Y)}\right] \leq C(CRnL_\sigma)^2 \hat{T}^{2(\nu - \alpha + 2\beta)} \mathbb{E}\left[||u_1 - u_2||^2_{\mathcal{C}([0,\hat{T}\wedge \tau_n];Y)}\right].$$

Finally, another computation very similar to the previous ones gives us the following estimates

$$\mathbb{E}\left[\left\|\hat{h}^{u_1} - \hat{h}^{u_2}\right\|^2_{\mathcal{C}([0,\hat{T}\wedge \tau_n];Y)}\right] \leq C(CRnL_\sigma)^2 \hat{T}^{2(\nu - \alpha)+1} \mathbb{E}\left[||u_1 - u_2||^2_{\mathcal{C}([0,\hat{T}\wedge \tau_n];Y)}\right].$$

Collecting all these estimates for the terms defined in (3.31) and choosing $\hat{T}$ small enough proves the statement. \hfill \square

The following result is the analogue of Lemma 3.23 in the semilinear case.
Lemma 3.26 Let $0 < \varepsilon < 1$. Under the assumptions of Theorem 3.24 it holds
\[ \mathbb{P}(\tau_n > \varepsilon) > 1 - \varepsilon^2 (C_1 E||u_0||_Z^2 + C_2 \varepsilon), \]
where the positive constants $C_1$ and $C_2$ are independent of $u_0$.

Proof. The statement can be shown analogously to Lemma 3.23, compare the proof of Theorem 1.3 in [38]. One can find a positive number $n$ such that $\frac{1}{n+1} \leq \varepsilon < \frac{1}{n}$ and can use (3.32) to derive estimates for
\[ \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n} ||u(t)||_Z^2, \]
which provide bounds for
\[ \mathbb{E} \sup_{0 \leq t \leq \varepsilon} ||u(t \wedge \tau_n)||_Z^2. \]
Using these as in (3.30) and regarding the local growth boundedness of $f$ and $\sigma$ specified in (3.28) and (3.30), one infers that
\[ \mathbb{E} \sup_{0 \leq t \leq \varepsilon} ||u(t \wedge \tau_n)||_Z^2 \leq C \mathbb{E}||u_0||_Z^2 + \varepsilon^{2(1-\beta)} C(l_F(n) + 1) \varepsilon \int_0^\varepsilon \mathbb{E}||u(s \wedge \tau_n)||_Z^2 ds \\
+ C \varepsilon (l_\sigma(n) + 1) \int_0^\varepsilon \mathbb{E}||u(s \wedge \tau_n)||_Z^2 ds. \]
Gronwall’s Lemma and Chebyshev’s inequality prove the statement as argued in Lemma 3.23.

□

Remark 3.27 Of course, one could also make global Lipschitz assumptions on $f$ and $\sigma$ and thereafter use suitable cut-offs as in the semilinear case or as in [36]. Namely one can consider the standard cut-off function $h_n : Z \to Z$ defined as
\[ h_n u := \begin{cases} 
    u, & \text{if } ||u||_Z \leq n \\
    \frac{n}{||u||_Z}, & \text{if } ||u||_Z > n
\end{cases} \]
and show that $f_n := h_n f$ and $\sigma_n := h_n \sigma$ are globally Lipschitz continuous. Here we have directly localized the assumption.

From all these deliberations we finally conclude

Theorem 3.28 There exists a unique maximal local pathwise mild solution of (3.24) $u \in L^0(\Omega; B([0, \tau_\infty); Z)) \cap L^0(\Omega; C^b([0, \tau_\infty); Y))$, where $\tau_\infty := \lim_{n \uparrow \infty} \tau_n$ a.s.
Proof. The proof in \[64\] Section 4 and \[14\] Section 3 adapts to our setting. We denote by $\mathcal{S}$ the set of all stopping times such that $\tau \in \mathcal{S}$ if and only if there exists a process $(u(t))_{t \in [0,\tau)}$ such that $(u, \tau)$ is the unique local pathwise mild solution of (3.24). For each $n \in \mathbb{N}$ we take $\tau_n$ such that $(u_n, \tau_n)$ is the unique local pathwise mild solution of (3.24). This means that for each $n \in \mathbb{N}$ the pair $(u_n, \tau_n)$ is the local mild solution of (3.24) where $\tau_n := \inf \{t \geq 0 : \|u_n(t)\|_Z \geq n\} \wedge T$ for some $T > 0$. We now show that $\{\tau_n, n \in \mathbb{N}\}$ is an increasing sequence of stopping times and possesses therefore a limit. This will give us the lifetime of $u$. To this aim for $n < m$ let $\tau_{n,m} := \inf \{t \geq 0 : \|u_n(t)\|_Z \geq n\} \wedge T$. One can show arguing by contradiction that $\tau_n < \tau_m$ a.s. if $n < m$. Since $\tau_{n,m} \leq \tau_m$ a.s. for $n < m$ we obtain that $(u_m, \tau_{n,m})$ is a local solution of (3.24) as well as $(u_n, \tau_n)$. If $\tau_n > \tau_{n,m}$ a.s. then due to the uniqueness of the pathwise mild solution of (3.24) we infer that $u_n(t) = u_m(t)$ a.s. for all $t \in [0, \tau_{n,m}) = [0, \tau_{n,m}]$. This means that $\tau_{n,m}$ is the first exit time for $u_n$ with $\tau_{n,m} < \tau_n$ a.s., which is obviously a contradiction. Therefore, we conclude that $(\tau_n)_{n \in \mathbb{N}}$ is an increasing sequence of stopping times and possesses the limit $\tau_\infty := \lim_{n \uparrow \infty} \tau_n$ a.s. Let $\{u(t) : t \in [0, \tau_\infty)\}$ be the stochastic process defined by

$$u(t) := u_n(t), \text{ for } t \in [\tau_{n-1}, \tau_n), \ n \geq 1,$$

where $\tau_0 := 0$. Again, due to uniqueness we have that $u(t \wedge \tau_n) = u_n(t \wedge \tau_n)$ for $t > 0$. All in all we obtained a local pathwise mild solution $(u, \tau_\infty)$ of (3.24). The last step is to show that this is indeed a maximal local pathwise mild solution. To this aim, we infer that a.s. on the set $\{\omega : \tau_\infty(\omega) < T\}$

$$\lim_{t \uparrow \tau_\infty} \sup_{0 \leq s \leq t} \|u(s)\|_Z \geq \lim_{n \uparrow \infty} \sup_{0 \leq s \leq \tau_n} \|u_n(s)\|_Z = \lim_{n \uparrow \infty} \sup_{0 \leq s \leq \tau_n} \|u_n(s)\|_Z = \infty.$$

Consequently, $(u, \tau_\infty)$ is a maximal local pathwise mild solution of (3.24). \[\square\]

**Remark 3.29** Since

$$\left\{ \omega : \sup_{0 \leq t \leq \varepsilon} \|u(t \wedge \tau_n)\|_Z < n \right\} \subset \{\omega : \tau_n(\omega) > \varepsilon\} \subset \{\omega : \tau_\infty(\omega) > \varepsilon\},$$

obviously

$$\mathbb{P}(\tau_\infty > \varepsilon) > \mathbb{P}(\tau_n > \varepsilon) > 0.$$  

**Remark 3.30** In order to show that a solution is global-in-time one, it would remain to prove that $\tau_\infty = \infty$ a.s. As we would expect and as we shall see in Section 5 local-in-time existence can obviously fail to hold. However, in many applications, additional structure of the quasilinear PDE may be enough to also obtain global results. For example, if the deterministic PDE part is a cross-diffusion system with an entropy structure \[37\] and if the noise is multiplicative, we expect that the maximal local pathwise mild solution obtained in Theorem 3.28 is indeed a global one. We plan to investigate this in a future work using for instance using Khoshminksi’s test for non-explosion; see for example \[64\] Lemma 4.1, \[16\] Theorem 3.2 or \[40\] Section 5.

30
4 Applications: The Shigesada-Kawasaki-Teramoto Model

Let $G \subset \mathbb{R}^2$ be an open bounded $C^2$-domain. We fix parameters $k_1, k_2, \delta_{11}, \delta_{21} > 0$. We want to study a cross-diffusion SPDE, which has been originally introduced by Shigesada, Kawasaki and Teramoto [58] in the deterministic setting in order to analyze population segregation by induced cross-diffusion. Note that the nonlinear term correspond to those arising in the classical Lotka-Volterra competition model. The stochastic SKT system is given by

$$
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta (k_1 u + au v + cu^2) + \delta_{11} u - \gamma_{11} u^2 - \gamma_{12} uv) \ dt + \sigma_1(u, v) \ dW_1(t), \quad t > 0, \ x \in G \\
\frac{\partial v}{\partial t} &= \Delta (k_2 v + bw v + dv^2) + \delta_{21} v - \gamma_{21} uv - \gamma_{22} v^2) \ dt + \sigma_2(u, v) \ dW_2(t), \quad t > 0, \ x \in G \\
\frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0, \\
u(x, 0) &= c_0(x), \quad v(x, 0) = v_0(x) \geq 0
\end{align*}
$$

(4.1)

where $W_1, W_2$ are stochastic processes as defined in (Y4) below. Here $u = u(x, t)$ and $v = v(x, t)$ denote the densities of two competing species $S_1$ and $S_2$ in a certain position $x \in G$ at time $t$. The coefficients $\gamma_{11}, \gamma_{22} > 0$ denote the intraspecies competition rates in $S_1$, respectively in $S_2$ and $\gamma_{12}, \gamma_{21} > 0$ stand for the interspecies competition rates between $S_1$ and $S_2$. Furthermore, the terms $\Delta(cu^2)$ and $\Delta(dv^2)$ represent the self-diffusions of $S_1$ and $S_2$ with rates $c, d \geq 0$, and $\Delta(auv), \Delta(buv)$ represent the cross-diffusions of $S_1$ and $S_2$ with rates $a, b \geq 0$. We study the SKT model (4.1) in its divergence form with linear part $\text{div}(A(U)\nabla U)$, where

$$A(U) = \begin{pmatrix}
k_1 + 2cu + av & au \\
bv & k_2 + 2dv + bu
\end{pmatrix}$$

where $U := (u, v)^T$. We denote the nonlinear term by

$$F(U) = \begin{pmatrix}
\delta_{11} u - \gamma_{11} u^2 - \gamma_{12} uv \\
\delta_{21} v - \gamma_{21} uv - \gamma_{22} v^2
\end{pmatrix}.$$ 

We assume here that the parameters are chosen so that $A(U)$ is positive definite. For applications, the most interesting case occurs under the restriction that $u, v \geq 0$ should be preserved. If we would know this, then it suffices to impose

$$a^2 < 8cb \quad \text{and} \quad b^2 < 8da,$$

which is a necessary and sufficient condition for positive definiteness of $A(u)$. One can replace this by the even weaker condition $ab < 64cd$ [63, Chapter 15, Section 3].

Our aim is to formulate equation (4.1) as an abstract quasilinear SPDE, as investigated in Section 8 on $X := L^2(G) = L^2(G) \times L^2(G)$. Throughout this section we use the same notations as in Section 8. We set $Z := H^{1+\varepsilon}(G) = H^{1+\varepsilon}(G) \times H^{1+\varepsilon}(G)$, for a fixed $0 < \varepsilon < 1/2$ and rewrite (4.1) as

$$
\begin{align*}
dU(t) &= (A(U(t))U(t) + F(t, U(t))) \ dt + \sigma(t, U(t)) \ dW(t), \quad t \in [0, T], \\
U(0) &= U_0 \in K \ a.s.,
\end{align*}
$$

(4.2)
where \( \mathcal{W} := (W_1, W_2) \). According to \[63\] Proposition 15.1, there is a sectorial operator \( A(U) \), defined via the matrix \( A(u) \) in a standard way \[62\], of angle \( 0 < \varphi < \frac{
}{2} \) for \( U \in \mathcal{U}_T \), so we are justified to introduce \( X_{\bar{\mu}} := D(A(U)^{\bar{\mu}}) \), for \( \bar{\mu} \geq 0 \), see below.

The following assertions regarding the deterministic part of \( (4.1) \) are stated and proved in \[63\] Chapter 5 and \[61\] Section 3. Therefore all the assumptions made in the previous section are satisfied for this example.

**Remark 4.1** For more general deterministic quasilinear problems and assumptions on the coefficients for which the next statements hold true, see \[3\] Section 10.

(Y1) For \( U \in \mathcal{U}_T \) due to \[63\] Proposition 15.2 \( D(A(U)) = H^2_{\mathcal{W}}(G) \times H^2_{\mathcal{W}}(G) := H^2_{\mathcal{W}}(G) \) and due to \[63\] Proposition 15.3

\[
\begin{align*}
X_{\hat{\alpha}} &= H^{2\hat{\alpha}}(G), \quad \text{for } 0 \leq \hat{\mu} < \frac{3}{4} \\
X_{\hat{\alpha}} &= H^{2\hat{\alpha}}(G), \quad \text{for } \frac{3}{4} < \hat{\mu} \leq 1.
\end{align*}
\]

In this context we infer that \( Z = D(A(U)^{\beta}) \) for \( \beta = 1/2 + \varepsilon/2 \).

(Y2) According to \[63\] (15.10), p. 492, the following local Lipschitz continuity of the generators holds true: there exist a constant \( \bar{L} = \bar{L}(U, V) > 0 \) such that

\[
||A(U) - A(V)||_{L(H^2_{\mathcal{W}}(G), X)} \leq \bar{L}(U, V)||U - V||_Y, \quad \text{for } U, V \in \mathcal{U}_T.
\]  

(4.5)

Here \( Y := D(A(U)^{\alpha}) \) for \( 0 \leq \alpha \leq \frac{1+\varepsilon_0}{2} \), where \( 0 < \varepsilon_0 < \varepsilon \), see also \[61\].

(Y3) Let \( F : \Omega \times \mathcal{U}_T \to X \). There exist constants \( L_F = L_F(u), l_F = l_F(u) > 0 \) such that

\[
||F(U) - F(V)||_X \leq L_F||U - V||_X, \quad \text{for } U, V \in \mathcal{U}_T
\]

and

\[
||F(U)||_X \leq l_F(1 + ||U||_X), \quad \text{for } U \in \mathcal{U}_T.
\]

The local Lipschitz continuity is satisfied since \( F \) is a square function of \( u \) and \( v \), see \[61\] and \[63\].

(Y4) \( \mathcal{W} = (W_1, W_2) \) is an \( H \)-cylindrical Brownian motion, \( \sigma : \Omega \times \mathcal{U}_T \to \mathcal{L}_2(H, X_{2\beta}) \), where \( H \) stands for a separable Hilbert space. Furthermore, there exist constants \( L_\sigma = L_\sigma(u, v), l_\sigma = l_\sigma(u) > 0 \) such that

\[
||\sigma(U) - \sigma(V)||_{\mathcal{L}_2(H, X_{2\beta})} \leq L_\sigma||U - V||_X, \quad \text{for } U, V \in \mathcal{U}_T;
\]

respectively

\[
||\sigma(U)||_{\mathcal{L}_2(H, X_{2\beta})} \leq l_\sigma(1 + ||U||_X), \quad \text{for } U \in \mathcal{U}_T.
\]  

(4.6)

Keeping this in mind, we conclude that all assumptions made in Section \[3\] are fulfilled and the restrictions on the exponents \( \alpha, \beta \) and \( \nu \) imposed in (A4′) hold.
Remark 4.2 Note that non-negativity of local solutions for (4.1) is not ensured by (4.6). There is actually a trade-off: if we allow for additive noise as in (4.6), then we need a very strong assumption of uniform positive definiteness for $A(u)$ but if we allow for more general matrices $A(u)$, then we need more assumptions on the noise, e.g., we conjecture that the assumption

$$||\sigma(U)||_{L^2(H,X^2)} \leq L_0||U||_{X}, \text{ for } U \in U_T,$$

(4.7)

together with $u_0 > 0, v_0 > 0$ uniformly in space, will imply short-time existence up to a stopping time and preserve positivity, see [7] or Theorem 1.3 in [38].

Regarding the assumptions (Y1)-(Y4), we infer that in the context of Section 8 we have $Z = X_\beta$ for $\beta = 1 + \varepsilon$ and $Y = X_\alpha$ for $\alpha$ specified in (4.6). Note that (4.5) implies that (4.1) and (5.5) are fulfilled with $\nu = 1$. Therefore, we apply for (4.1) the abstract results proved in Section 3 and infer:

Theorem 4.3 Under the assumptions stated in this section the stochastic SKT equation (4.1) possesses a unique maximal local pathwise mild solution $U \in L^0(\Omega; B([0, \tau_\infty); Z)) \cap L^0(\Omega; C^4([0, \tau_\infty); Y))$.

5 Examples

As already known and well-established in the deterministic case, quasilinear PDEs do not possess global solutions without further assumptions. The aim of this subsection is to present simple examples of stochastic PDEs with cross-diffusion so that their solution cannot exist globally.

5.1 A Cross-Diffusion SPDE with Finite-Time Blow-Up

In this setting we give an example of a cross-diffusion equation that exhibits finite-time blow up in the deterministic case and prove that this holds true also in the stochastic one. This fact is not surprising, since we consider here only white noise [40] Theorems 4.1-4.3 but it seems useful to have for completeness. To this aim, we denote by $\phi$ the normalized eigenfunction associated to the first eigenvalue $\lambda_1$ of the Dirichlet-Laplacian in $G$, where $G \subset \mathbb{R}^n$ is an open-bounded $C^2$-domain. More precisely, we have that

$$\begin{cases}
\Delta \phi = -\lambda_1 \phi, \text{ in } G; \\
\phi = 0, \text{ } x \in \partial G; \\
\int_\overline{G} \phi(x) \, dx = 1.
\end{cases}
$$

(5.1)

Note that $\phi(x) > 0$ for $x \in G$. Keeping this in mind, we consider the following SPDE:

Example 5.1

$$\begin{align*}
\frac{du}{dt} &= (\Delta u + \frac{1}{2} \Delta v + u^2 + \frac{1}{2} v) \, dt + \sigma(u) \, dW(t), & t > 0, \text{ } x \in G \\
\frac{dv}{dt} &= (\Delta v + (\lambda_1 + k) v) \, dt, & t > 0, \text{ } x \in G, \\
u|_{\partial G} &= v|_{\partial G} = 0, & t > 0, \\
[u(x, 0) = u_0(x)] \geq 0, \text{ } v(x, 0) = \phi(x), & x \in G.
\end{align*}
$$

(5.2)
In this case we set \( U := (u, v)^T \) and have
\[
A(U) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Delta \quad \text{and} \quad F(U) = \begin{pmatrix} u^2 + \frac{\lambda_1}{2} v \\ (\lambda_1 + k)v \end{pmatrix}.
\]

The only requirement for \( \sigma \) is local-Lipschitz continuity in order to ensure the existence of a local solution for (5.2). As we will see in the next computation the stochastic term will not play a role due to the fact that the expectation of the Itô-integral is 0. In the deterministic case we know that the first component of the solution of (5.2) blows up in finite-time. We now show that this remains valid in the stochastic setting.

**Lemma 5.2** Consider the SPDE (5.2). There exists \( u_0 \geq 0 \) and a finite time \( T^* \) such that
\[
\lim_{t \uparrow T^*} \mathbb{E} \left[ \sup_{x \in G} u(x, t) \right] = +\infty. \quad (5.3)
\]

**Proof.** Since \( v(x, t) = e^{kt} \phi(x) \) is the solution of the second equation, the first one results in
\[
\begin{aligned}
\left\{ 
\begin{array}{l}
du = (\Delta u + u^2) \, dt + \sigma(u) \, dW(t) \\
u(0) = u_0(x) \geq 0.
\end{array}
\right.
\end{aligned} \quad (5.4)
\]

Note that (5.4) is a parabolic SPDE. Under the above assumptions, it is known that this possesses a local positive solution which exhibits finite-time blow-up [46, Theorem 4.1]. For the convenience of the reader we indicate the proof of this statement. Assuming that there exists a global solution \( u \) of (5.4) such that for any \( T > 0 \)
\[
\sup_{0 \leq t \leq T} \mathbb{E} \left[ \sup_{x \in G} u(x, t) \right] < \infty. \quad (5.5)
\]
we immediately also get that
\[
\sup_{0 \leq t \leq T} \mathbb{E} \left[ \int_G u(x, t) \phi(x) \, dx \right] \leq \sup_{0 \leq t \leq T} \mathbb{E} \left[ \sup_{x \in G} u(x, t) \right] < \infty.
\]

We set
\[
y(t) := \int_G u(x, t) \phi(x) \, dx \quad \text{for} \ t \geq 0,
\]
multiply (5.4) by \( \phi \), take the expectation and obtain via a direct application of integration-by-parts and Fubini’s Theorem that
\[
\mathbb{E} \left[ y(t) \right] = \int_G u_0(x) \phi(x) \, dx - \lambda_1 \int_0^t \mathbb{E} \left[ y(s) \right] \, ds + \int_0^t \mathbb{E} \left[ \int_G u^2(x, s) \phi(x) \, dx \right] \, ds.
\]

\[=: (u_0, \phi) \]
Setting \( \tilde{y}(t) := \mathbb{E}[y(t)] \) for \( t > 0 \) and differentiating with respect to \( t \) we obtain

\[
\begin{aligned}
\frac{d\tilde{y}(t)}{dt} &= -\lambda_1 \tilde{y}(t) + \mathbb{E} \left[ \int_G u^2(x,t) \phi(x) \right] dx \\
\tilde{y}(0) &= (u_0, \phi).
\end{aligned}
\]

From Jensen’s and Cauchy-Schwarz inequality we have

\[
\tilde{y}^2(t) = \mathbb{E} \left[ \int_G u(x,t) \phi(x) dx \right]^2 \leq \mathbb{E} \left[ \int_G u(x,t) \phi(x) dx \right] \mathbb{E} \left[ \int_G \phi(x) dx \right],
\]

Consequently, using (5.1)

\[
\begin{aligned}
\frac{d\tilde{y}(t)}{dt} &\geq -\lambda_1 \tilde{y}(t) + \tilde{y}^2(t) \\
\tilde{y}(0) &= (u_0, \phi).
\end{aligned}
\] (5.6)

So \( \tilde{y} \) must blow up in a finite time for a suitable \( u_0 \), which is a contradiction to (5.5). This proves the assertion. \( \square \)

### 5.2 A Cross-Diffusion SPDE with Degenerating Quasilinear Operator

We first construct an example, in which the solution blows up in finite time and cannot remain positive starting with a positive initial condition. Therefore, in this case the maximum principle is not valid. To this aim, letting \( k > \lambda_1 \) as in Lemma 5.2, we consider

Example 5.3

\[
\begin{aligned}
du &= (\Delta u + \frac{1}{2} \Delta v + u(2\lambda_1 - u)) \ dt + \sigma(u) \ dW(t), \quad t > 0, \ x \in G, \\
dv &= (\Delta v + (\lambda_1 + k)v) \ dt, \quad t > 0, \ x \in G, \\
u|_{\partial G} &= v|_{\partial G} = 0, \quad t > 0, \\
u(x,0) &= u_0(x) \geq 0, \ v(x,0) = \phi(x), \quad x \in G.
\end{aligned}
\] (5.7)

Here we have for \( U = (u, v)^T \)

\[
A(U) = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \Delta \quad \text{and} \quad F(U) = \begin{pmatrix} u(2\lambda_1 - u) \\ (\lambda_1 + k)v \end{pmatrix}.
\]

In the deterministic case it is known that \( u \) blows up in finite time and cannot remain positive [40, Theorem 1.6]. We investigate now this situation by similar methods as in Lemma 5.2 in the stochastic framework.
Lemma 5.4 Consider the SPDE \([5.7]\). There exists a finite time \(T^*\) such that

\[
\lim_{t \to T^* -} \mathbb{E} \left[ \sup_{x \in G} u(x, t) \right] = -\infty. \tag{5.8}
\]

Proof. Since \(v(x, t) = e^{kt} \phi(x)\) is the solution of the second equation, the first one results in

\[
du = (\Delta u - \frac{\lambda_1}{2} e^{kt} \phi + 2\lambda_1 u - u^2) \, dt + \sigma(u) \, dW(t). \tag{5.9}
\]

We prove the assertion by similar methods to those presented in Example 5.2 by setting

\[
y(t) := \int_G u(x, t) \phi(x) \, dx
\]

and by also defining \(\tilde{y}(t) := \mathbb{E}[y(t)]\). A direct calculation yields

\[
\begin{cases}
\frac{d\tilde{y}(t)}{dt} = \lambda_1 \tilde{y}(t) - \frac{\lambda_1}{2} e^{kt} \int_G \phi^2(x) \, dx - \mathbb{E} \int_G u^2(s, x) \phi(x) \, dx \\
\tilde{y}(0) = (u_0, \phi).
\end{cases}
\]

Again, we infer due to Jensen’s inequality that

\[
\begin{cases}
\frac{d\tilde{y}(t)}{dt} \leq \lambda_1 \tilde{y}(t) - \frac{\lambda_1}{2} e^{kt} \int_G \phi^2(x) \, dx - \tilde{y}^2(t) \\
\tilde{y}(0) = (u_0, \phi).
\end{cases}
\]

In order to reach a contradiction, we combine the inequalities

\[
\frac{d\tilde{y}(t)}{dt} \leq \lambda_1 \tilde{y}(t) - \frac{\lambda_1}{2} e^{kt} \int_G \phi^2(x) \, dx \quad \text{and} \quad \frac{d\tilde{y}(t)}{dt} \leq \lambda_1 \tilde{y}(t) - \tilde{y}^2(t).
\]

The first one entails using Gronwall’s Lemma

\[
\tilde{y}(t) \leq e^{\lambda_1 t} \left( (u_0, \phi) - \frac{\lambda_1 (e^{(k-\lambda_1)t} - 1)}{2(k - \lambda_1)} \int_G \phi^2(x) \, dx \right), \quad \text{for} \ t > 0. \tag{5.10}
\]

Since \(k > \lambda_1\), the estimate \((5.10)\) implies that there exists \(t_1 > 0\) such that \(\tilde{y}(t) < 0\) for \(t \geq t_1\), also

\[
\mathbb{E} \left[ \int_G u(t, x) \phi(x) \, dx \right] < 0, \quad \text{for} \ t \geq t_1.
\]

Consequently, for \(t \geq t_1\), the inequality \(\frac{d\tilde{y}(t)}{dt} \leq \lambda_1 \tilde{y}(t) - \tilde{y}(t)^2\) implies that

\[
\frac{d\tilde{y}(t)}{dt} \frac{1}{\tilde{y}^2(t)} - \lambda_1 \frac{1}{\tilde{y}(t)} \leq -1.
\]
Setting $w := \frac{1}{y}$, we obtain for $t \geq t_1$ that $w(t) > 0$ and
\[
\frac{dw(t)}{dt} + \lambda_1 w(t) \leq -1.
\]
It is now elementary to check that $w(t) \to 0$ as $t \to T^*$ for a finite time $T^*$. This implies $\tilde{y}(t) \to -\infty$ as $t \to T^*$—proving the claim. \hfill \Box

To conclude, regarding Example 5.7 one can now easily construct a cross-diffusion quasilinear SPDE, which becomes ill-posed. Indeed, we just have in a third component an equation degenerating into the backward heat-equation.

Example 5.5
\[
\begin{aligned}
du &= (\Delta u + \frac{1}{2} \Delta v + u(2\lambda_1 - u)) \, dt + \sigma(u) \, dW(t), \quad t > 0, \; x \in G, \\
dv &= (\Delta v + (\lambda_1 + k)v) \, dt, \quad t > 0, \; x \in G, \\
dw &= u \Delta w \, dt, \quad t > 0, \; x \in G, \\
u|_{\partial G} = v|_{\partial G} = 0 = w|_{\partial G} = 0, \\
u(x,0) = u_0(x) \geq 0, \; v(x,0) = \phi(x), \\
\end{aligned}
\]
(5.11)

A An integration by parts formula

We shortly indicate the computation for the pathwise mild solution of the linear SPDE
\[
dU(t) = A(t)U(t) \, dt + G(t) \, dW(t), \quad (A.1)
\]
obtained using an integration by parts formula.

The strong solution of (A.1)
\[
U(t) = \int_0^t A(s)U(s) \, ds + \int_0^t G(s) \, dW(s) \quad (A.2)
\]
can be written as
\[
U(t) = S(t,0) \int_0^t G(s) \, dW(s) - \int_0^t S(t,s)A(s) \int_0^t G(r) \, dW(r) \, ds, \quad (A.3)
\]
where $S(\cdot, \cdot)$ is the evolution operator generated by $A(\cdot)$. If $A(\cdot)$ is bounded, a straightforward computation \cite[Section 4.2, p. 18]{57} immediately proves the claim. Namely, from (A.3) we have
\[
U(t) = S(t,0) \int_0^t G(s) \, dW(s) - \int_0^t S(t,s)A(s) \int_0^t G(r) \, dW(r) \, ds \\
+ \int_0^t S(t,s)A(s) \int_0^r G(r) \, dW(r) \, ds.
\]
Since
\[ \frac{\partial}{\partial s} S(t, s) = -S(t, s) A(s) \]
we have
\[ \int_0^t S(t, s) A(s) \, ds = -S(t, t) + S(t, 0), \]
so
\[ U(t) = S(t, 0) \int_0^t G(s) \, dW(s) + S(t, t) \int_0^t G(s) \, dW(s) - S(t, 0) \int_0^t G(s) \, dW(s) \]
\[ + \int_0^t S(t, s) A(s) \int_0^s G(r) \, dW(r) \, ds \]
\[ = \int_0^t G(s) \, dW(s) + \int_0^t S(t, s) A(s) \int_0^s G(r) \, dW(r) \, ds. \]
Furthermore, using Fubini’s theorem
\[ \int_0^t A(r) U(r) \, dr = \int_0^t A(r) \int_0^r G(s) \, dW(s) \, dr + \int_0^t A(r) S(t, s) A(s) \int_0^s G(\tau) \, dW(\tau) \, ds \, dr \]
\[ = \int_0^t A(r) \int_0^r G(s) \, dW(s) \, dr + \int_0^t \int_0^r A(r) S(r, s) A(s) \int_0^s G(\tau) \, dW(\tau) \, dr \, ds \]
\[ = \int_0^t A(s) \int_0^s G(r) \, dW(r) \, ds + \int_0^t S(t, s) A(s) \int_0^s G(r) \, dW(r) \, ds \]
\[ - \int_0^t A(s) \int_0^s G(r) \, dW(r) \, ds \]
\[ = \int_0^t S(t, s) A(s) \int_0^s G(r) \, dW(r) \, ds. \]
For the last part we used that
\[ \frac{\partial}{\partial t} S(t, s) = A(t) S(t, s), \]
so
\[ \int_s^t A(r) S(r, s) \, dr = S(t, s) - S(s, s). \]
If \( A(\cdot) \) is unbounded, one can repeat the previous computation under suitable assumptions (as in Section [3]) which ensure the existence of the integrals above.
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