OSCILLATION CRITERIA FOR DIFFERENCE EQUATIONS WITH SEVERAL RETARDED ARGUMENTS

КРИТЕРІЇ КОЛИВАНЬ ДЛЯ РІЗНИЦЕВИХ РІВНЯНЬ З ДЕЯКИМИ ЗАГАЮВАЛЬНИМИ АРГУМЕНТАМИ

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The difference equation

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k) u(\tau_i(k)) = 0,$$

is considered, where $m \in \mathbb{N}$, the functions, $p_i: \mathbb{N} \to \mathbb{R}_+$, $\tau_i: \mathbb{N} \to \mathbb{N}$, $\lim_{k \to +\infty} \tau_i(k) = +\infty$, $\tau_i(k) \leq k - 1$, $i = 1, \ldots, m$, are defined on a set of natural numbers, and $\Delta u(k) = u(k+1) - u(k)$ is the difference operator. New oscillation criteria for all solutions of the equation are established.

1. Introduction. Consider the difference equation

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k) u(\tau_i(k)) = 0,$$

where $m \geq 1$ is a natural number, $p_i: \mathbb{N} \to \mathbb{R}_+$, $\tau_i: \mathbb{N} \to \mathbb{N}$ are functions defined on the set $\mathbb{N} = \{1, 2, \ldots\}$ and $\Delta u(k) = u(k+1) - u(k)$. Everywhere below it is assumed that

$$\lim_{k \to +\infty} \tau_i(k) = +\infty \quad \text{and} \quad \tau_i(k) \leq k - 1, \quad i = 1, \ldots, m.$$  (1.2)

For each $n \in \mathbb{N}$ denote $N_n = \{n, n+1, \ldots\}$.

Definition 1.1. Let $n \in \mathbb{N}$. We will call a function $u: \mathbb{N} \to \mathbb{R}$ a proper solution of equation (1.1) on the set $N_n$, if it satisfies (1.1) on $N_n$ and $\sup\{|u(i)|: i \geq k\} > 0$ for any $k \in N_n$.

Definition 1.2. We say that a proper solution $u: N_n \to \mathbb{R}$ of equation (1.1) is oscillatory, if for any $k \in N_n$ there exist $n_1, n_2 \in N_k$ such that $u(n_1)u(n_2) \leq 0$. Otherwise the solution is called nonoscillatory.
**Definition 1.3.** Equation (1.1) is said to be oscillatory, if any of its the proper solution is oscillatory.

The problem of oscillation of solution of linear difference equation (1.1) for \( m = 1 \) has been studied by several authors, see [1 – 3] and reference therein. As to investigation of the analogous problem for equation (1.1) and for differential equations see [4 – 10].

Some results of this paper without proof is given in [11].

**Remark 1.1.** Below we assume that

\[
\lim_{k \to +\infty} p_i(k) < +\infty, \quad i = 1, \ldots, m,
\]

otherwise equation (1.1) is oscillatory.

**2. Main results.** Throughout the paper we assume that

\[
\tau_s(k) = \min \{ \tau_i(k): i = 1, \ldots, m \},
\]

\[
\eta^{\tau_s}(k) = \max \{ s: s \in N, \tau_s(s) \leq k \},
\]

\[
\eta^{\tau_s}(k) = \eta^{\tau_s}(k), \quad \eta^{\tau_s}(k) = \eta^{\tau_s} \circ \eta^{\tau_s}_{k-1} \quad \text{for} \quad k \in N, \quad i = 2, 3, \ldots.
\]

By (1.2) it is obvious that \( \eta^{\tau_s}(k) < +\infty \) for any \( k \in N \) and \( \eta^{\tau_s}(k) \uparrow +\infty \) for \( k \uparrow +\infty \).

Let \( k \in N \). Denote

\[
\psi_1(k) = \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left(1 + mp^{1/m}(j)\right) \quad \text{for} \quad k \geq k_0,
\]

\[
\psi_s(k) = \left( \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left(1 + mp^{1/m}(s)\psi_{s-1}(j)\right) \right)^{1/m} \quad \text{for} \quad k \geq \eta^{\tau_s}(k_0), \quad s = 2, 3, \ldots,
\]

\[
p(j) = \prod_{i=1}^{m} p_i(j).
\]

**Theorem 2.1.** Let there exist \( s_0 \in N \) and non-decreasing functions \( \sigma_i: N \to N, \quad i = 1, \ldots, m, \) such that

\[
\tau_i(k) + 1 \leq \sigma_i(k) \leq k \quad \text{for} \quad k \in N, \quad i = 1, \ldots, m,
\]

and

\[
\lim_{k \to +\infty} \prod_{\ell=1}^{m} \left( \prod_{i=1}^{m} \sum_{s=\sigma_i(k)}^{k} p_i(s) \prod_{j=\tau_i(s)}^{\sigma_i(k)-1} \left(1 + mp^{1/m}(j)\psi_{s_0}^{1/m}(j)\right) \right)^{1/m} > \frac{1}{m^m}.
\]

Then equation (1.1) is oscillatory, where \( \psi_{s_0} \) is given by (2.2), (2.3) and (2.4), when \( s = s_0 \).

**Corollary 2.1.** Let there exist non-decreasing functions \( \sigma_i: N \to N, \quad i = 1, \ldots, m, \) such that

\[
\lim_{k \to +\infty} \prod_{\ell=1}^{m} \left( \prod_{i=1}^{m} \sum_{s=\sigma_i(k)}^{k} p_i(s) \prod_{j=\tau_i(s)}^{\sigma_i(k)-1} \left(1 + mp^{1/m}(j)\right) \right)^{1/m} > \frac{1}{m^m}.
\]

Then equation (1.1) is oscillatory, where \( p \) is given by (2.4).
Theorem 2.2. Let (2.5) be fulfilled,
\[ \liminf_{j \to +\infty} p(j) > 0 \]  
and
\[ \min \left\{ \liminf_{k \to +\infty} \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left( 1 + m^{1/m} p^{1/m}(j) \right)^{1+\frac{1}{m} \sum_{i=1}^{m} n_i} : \alpha \geq 1 \right\} > 1. \]

Then equation (1.1) is oscillatory, where \( p \) is given by (2.4).

Corollary 2.2. Let for large \( j \in \mathbb{N} \)

\[ p_i(j) \geq p_i > 0, \quad i = 1, \ldots, m, \]  

and

\[ \overline{p} \liminf_{k \to +\infty} \frac{(m + \sum_{i=1}^{m} (k - \tau_i(k)))^{m+\sum_{i=1}^{m} (k-\tau_i(k))}}{(\sum_{i=1}^{m} (k - \tau_i(k)))^{\sum_{i=1}^{m} (k-\tau_i(k))}} > 1. \]

Then equation (1.1) is oscillatory, where \( \overline{p} = \prod_{i=1}^{m} p_i \).

Corollary 2.3. Let \( n_i \in \mathbb{N}, \ i = 1, \ldots, m, \) for large \( k \in \mathbb{N} \)

\[ \tau_i(k) \leq k - n_i, \quad n_i \in \mathbb{N}, \quad p_i(k) \geq p_i > 0, \quad i = 1, 2, \ldots, m, \]  

and

\[ \overline{p} \left( m + \sum_{i=1}^{m} n_i \right)^{m+\sum_{i=1}^{m} n_i} \left( \sum_{i=1}^{m} n_i \right)^{\sum_{i=1}^{m} n_i} > 1. \]

Then equation (1.1) is oscillatory, where \( \overline{p} = \prod_{i=1}^{m} p_i \).

Corollary 2.4. Let \( n_i \in \mathbb{N}, \ i = 1, \ldots, m, \) for large \( k \in \mathbb{N}, (2.12) \) holds and

\[ \overline{p} \sum_{i=1}^{m} n_i > \left( \frac{\sum_{i=1}^{m} n_i}{m + \sum_{i=1}^{m} n_i} \right)^{1+\frac{1}{m} \sum_{i=1}^{m} n_i}. \]

Then equation (1.1) is oscillatory, where \( \overline{p} = \prod_{i=1}^{m} p_i \).

Remark 2.1. It is obvious that

\[ \left( \frac{\sum_{i=1}^{m} n_i}{m + \sum_{i=1}^{m} n_i} \right)^{1+\frac{1}{m} \sum_{i=1}^{m} n_i} \to \frac{1}{e} \]

when \( \sum_{i=1}^{m} n_i \to +\infty \).
3. Some auxiliary statements. In this section we establish the estimate of the quotient

\[ \frac{u(\tau_i(k))}{u(k+1)}, \quad i = 1, \ldots, m, \quad \text{and} \quad \left( \prod_{i=1}^{m} \frac{u(\tau_i(k))}{u(k+1)} \right)^{1/m} , \]

where \( u : N_{k_0} \to (0, +\infty) \) is nonoscillatory solution of equation (1.1).

**Lemma 3.1.** Let \( k_0 \in N \) and \( u : N_{k_0} \to (0, +\infty) \) is a positive solution of equation (1.1). Then for any \( s \in \{1, 2, \ldots\} \) we have

\[ \left( \prod_{i=1}^{m} u(\tau_i(k)) \right)^{1/m} \geq \left[ \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left( 1 + m \left( \prod_{\ell=1}^{m} p_{\ell(j)} \right)^{1/m} \psi_s(j) \right) \right]^{1/m} \]

for \( k \geq \eta_s^*(k_0), \quad s = 1, 2, \ldots, \)

where functions \( \psi_s \) and \( \eta_s^* \) are defined by (2.1) – (2.4).

**Proof.** From (1.1) we have

\[ \frac{u(k)}{u(k+1)} = 1 + \sum_{i=1}^{m} p_i(k) \frac{u(\tau_i(k))}{u(k+1)} \quad \text{for} \quad k \geq \eta_1(k_0) \]

and

\[ \frac{u(\tau_i(k))}{u(k+1)} = \prod_{j=\tau_i(k)}^{k} \left( 1 + \sum_{\ell=1}^{m} p_{\ell(j)} \frac{u(\tau_\ell(j))}{u(j+1)} \right) \quad \text{for} \quad k \geq \eta_2(k_0), \quad i = 1, \ldots, m. \]

Using the arithmetic mean-geometric mean inequality, from the last equality we get

\[ \frac{u(\tau_i(k))}{u(k+1)} \geq \prod_{j=\tau_i(k)}^{k} \left( 1 + m \left( \prod_{\ell=1}^{m} p_{\ell(j)} \right)^{1/m} \left( \prod_{\ell=1}^{m} \frac{u(\tau_\ell(j))}{u(j+1)} \right)^{1/m} \right) \]

for \( k \geq \eta_2(k_0), \quad i = 1, \ldots, m. \)

Therefore,

\[ \left( \prod_{i=1}^{m} \frac{u(\tau_i(k))}{u(k+1)} \right)^{1/m} \geq \left( \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left( 1 + m \left( \prod_{\ell=1}^{m} p_{\ell(j)} \right)^{1/m} \left( \prod_{\ell=1}^{m} \frac{u(\tau_\ell(j))}{u(j+1)} \right)^{1/m} \right) \right)^{1/m} \]

for \( k \geq \eta_2(k_0). \)

Since

\[ \left( \prod_{\ell=1}^{m} \frac{u(\tau_\ell(k))}{u(k+1)} \right)^{1/m} \geq 1, \]

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the inequality (3.1) is obviously fulfilled for \( s = 1 \). Assume its validity for some \( s \in \{1, 2, \ldots\} \). Then by (3.3) we obtain

\[
\left( \prod_{i=1}^{m} u(\tau_i(k)) \right)^{1/m} \geq \left( \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left( 1 + m \left( \prod_{i=1}^{m} p_j(i) \right)^{1/m} \psi_s(j) \right) \right)^{1/m} \quad \text{for} \quad k \geq \eta_{s+1}(k_0).
\]

where function \( \psi_s, \ s = 1, 2, \ldots \) are given by (2.2) and (2.3). The proof of the lemma is complete.

**Lemma 3.2.** Let

\[
\lim \inf_{k \to +\infty} p(k) > 0 \quad (3.4)
\]

and

\[
\min \left\{ \lim \inf_{k \to +\infty} \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left( 1 + m p_j^{1/m}(j) \right)^{\alpha} : \alpha \geq 1 \right\} > 1. \quad (3.5)
\]

Then

\[
\lim_{s \to +\infty} \left( \lim \inf_{k \to +\infty} \psi_s(k) \right) = +\infty, \quad (3.6)
\]

where functions \( p \) and \( \psi_s \) are given by (2.2) – (2.4).

**Proof.** Suppose contrary. Let

\[
\lim_{s \to +\infty} \left( \lim \inf_{k \to +\infty} \psi_s(k) \right) = \beta < +\infty. \quad (3.7)
\]

Then by (3.4) for any \( \varepsilon \in (0, \beta - 1) \) there exist \( k_0, s_0 \in \mathbb{N} \) such that

\[
\psi_s(k) \geq \beta - \varepsilon \geq 1 \quad \text{for} \quad k \in \mathbb{N}_{k_0} \quad \text{and} \quad s \in \mathbb{N}_{s_0}. \quad (3.8)
\]

According to (3.8) from (2.3) we get

\[
\psi_s(k) \geq \left( \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left( 1 + m p_j^{1/m}(j) (\beta - \varepsilon) \right) \right)^{1/m} \quad \text{for} \quad s \geq \eta_{s_0},
\]

where \( p \) is given by (2.4).

That is

\[
\lim \inf_{k \to +\infty} \psi_s(k) \geq \lim \inf_{k \to +\infty} \left( \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left( 1 + m p_j^{1/m}(j) (\beta - \varepsilon) \right) \right)^{1/m}.
\]

By (3.8) of the last inequality, we have

\[
\beta \geq \lim \inf_{k \to +\infty} \left( \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left( 1 + m p_j^{1/m}(j) (\beta - \varepsilon) \right) \right)^{1/m}.
\]
Therefore,
\[
\liminf_{k \to +\infty} \left( \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left( 1 + m(\beta - \varepsilon) p^{1/m(j)} \right) \right)^{1/m} \leq 1.
\] (3.9)

According to (3.9) for any \( \varepsilon \in (0, \beta) \) we get
\[
\min \left\{ \liminf_{k \to +\infty} \left( \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left( 1 + m p^{1/m(j)} \alpha \right) \right)^{1/m} : \alpha \geq 1 \right\} \leq \frac{\beta}{\beta - \varepsilon},
\]

with implies
\[
\min \left\{ \liminf_{k \to +\infty} \left( \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left( 1 + m p^{1/m(j)} \beta (\beta - \varepsilon) \right) \right)^{1/m} : \alpha \geq 1 \right\} \leq 1,
\]

where \( p \) is given by (2.4). This however contradicts (3.5). The obtained contradiction proves that (3.6) holds.

4. Proofs of the theorems. Proof of Theorem 2.1. Suppose to the contrary, that equation (1.1) has a non oscillatory proper solution \( u: N_{k_0} \to R \). Since \(-u(k)\) is also a solution (1.1), we confine ourselves only to the case that \( u \) is eventually positive solution of equation (1.1). Then there exists \( k_1 \geq k_0 \) such that \( u(\tau_i(k)) > 0 \) for \( k \in N_i(k), i = 1, \ldots, m \). As seen while proving Lemma 3.1
\[
\left( \prod_{i=1}^{m} \frac{u(\tau_i(k))}{u(k+1)} \right)^{1/m} \geq \psi_s(k) \quad \text{for} \quad k \geq \eta_s^{r_}\eta_s^{r_}(k_1), \quad s = 1, 2, \ldots,
\] (4.1)

where the functions \( \eta_s^{r_}\) and \( \psi_s \) are defined by (2.1)–(2.4).

From (1.1) by (2.1), (2.5) and (3.2) we have
\[
u(\sigma_i(k)) \geq \sum_{s=\sigma_i(k)}^{k} \sum_{j=1}^{m} p_j(s) u(\tau_j(s)) \quad \text{for} \quad k \geq \eta_s^{r_}(k_1), \quad s = 1, 2, \ldots,
\] (4.2)

and
\[
u(\tau_j(s)) \geq \nu(\sigma_j(k)) \prod_{t=\tau_j(s)}^{\sigma_j(k)-1} \left( 1 + \sum_{t=1}^{m} p_{s}(t) \frac{\nu(\sigma(t))}{\nu(t+1)} \right)
\] (4.3)
where functions \( \eta^M \) > 0. By (2.5) and (2.8), choose 

\[
\eta(k_1) \leq s \leq k, \quad j = 1, \ldots, m.
\]

Therefore, by (4.2) and (4.3) we get 

\[
u(\sigma_j(k)) = \sum_{j=1}^{m} \left( \sum_{s=\sigma_i(k)}^{k} p_j(s) u(\sigma_j(k)) \prod_{t=\tau_j(s)}^{\sigma_j(k)-1} \left( 1 + m \sum_{\ell=1}^{m} p_\ell(t) \frac{u(\sigma_j(t))}{u(t+1)} \right) \right) = 
\]

\[
= \sum_{j=1}^{m} u(\sigma_j(k)) \left( \sum_{s=\sigma_i(k)}^{k} p_j(s) \prod_{t=\tau_j(s)}^{\sigma_j(k)-1} \left( 1 + m \sum_{\ell=1}^{m} p_\ell(t) \frac{u(\sigma_j(t))}{u(t+1)} \right) \right), \quad i = 1, \ldots, m.
\]

Using the arithmetic mean-geometric mean inequality, from the last inequality we get 

\[
u(\sigma_i(k)) \geq m \left( \prod_{j=1}^{m} u(\sigma_j(k)) \right)^{1/m} \left( \prod_{j=1}^{m} s_{\sigma_i(k)} p_j(s) \times \prod_{t=\tau_j(s)}^{\sigma_j(k)-1} \left( 1 + m \left( \prod_{\ell=1}^{m} p_\ell(t) \frac{u(\sigma_j(t))}{u(t+1)} \right) \right) \right)^{1/m}
\]

for \( k \geq \eta^*(k_1), \quad i = 1, \ldots, m. \)

On the other hand by Lemma 3.1 we have

\[
\prod_{i=1}^{m} u(\sigma_i(k)) \geq m^m \prod_{i=1}^{m} u(\sigma_i(k)) \prod_{i=1}^{k} \left( \sum_{s=\sigma_i(k)}^{k} p_i(s) \prod_{t=\tau_j(s)}^{\sigma_j(k)-1} \left( 1 + m \left( \prod_{\ell=1}^{m} p_\ell(t) \frac{u(\sigma_j(t))}{u(t+1)} \right) \psi_s(t) \right) \right)^{1/m}
\]

for \( k \geq \eta^*(k_1), \quad s = 1, 2, \ldots, \)

where functions \( \eta^*_s \) and \( \psi_s \) are given by (2.1) – (2.3). Therefore, for \( s = k_0 \)

\[
\limsup_{k \to +\infty} \prod_{i=1}^{m} \left( \sum_{j=1}^{k} p_j(s) \prod_{t=\tau_j(s)}^{\sigma_j(k)-1} \left( 1 + m \left( \prod_{\ell=1}^{m} p_\ell(t) \frac{1}{\psi_k(t)} \right) \psi_s(t) \right) \right)^{1/m} \leq \frac{1}{m^m},
\]

which contradicts (2.6). The proof of the theorem is complete.

**Proof of Corollary 2.1.** Since \( \psi_*(k) \geq 1 \) for large \( s, k \in \mathbb{N} \), this corollary immediately follows from Theorem 2.1.

**Proof of Theorem 2.2.** Suppose to the contrary, that equation (1.1) has a nonoscillatory solution \( u: \mathbb{N}_{k_0} \to (0, +\infty) \). Then according to (2.8), (2.9) and Lemma 3.2 condition (3.6) holds. By (2.5) and (2.8), choose \( M > 0 \) such that for large \( k \) we have

\[
\prod_{\ell=1}^{m} \left( \prod_{i=1}^{k} p_i(s) \prod_{j=\tau_j(s)}^{\sigma_j(k)-1} \left( 1 + m \frac{1}{m_\ell} M_{\ell} \right) \right)^{1/m} > \frac{2}{m^m}. \quad (4.4)
\]
On the other hand according to (3.6) there exist \( s_0 \in \mathbb{N} \) and \( k_0 \in \mathbb{N} \) such that \( \psi_{s_0}(k) \geq M \) for \( k \geq k_0 \). Therefore, by (4.4) the condition (2.6) be fulfilled for \( s = s_0 \), which proved the validity of the theorem.

**Proof of Corollary 2.2.** It is suffices to show that condition (2.9) holds. Indeed by (2.10) for large \( k \) we have

\[
\liminf_{k \to +\infty} \prod_{i=1}^{m} \prod_{j=1}^{k} \left( 1 + m \alpha^{1/m} p^{1/m} \right) \geq \\
\geq \liminf_{k \to +\infty} \prod_{i=1}^{m} \left( 1 + m \alpha^{1/m} p^{1/m} \right)^{(k - \tau_i(k) + 1)} = \\
= \liminf_{k \to +\infty} \frac{(1 + m \alpha^{1/m} p^{1/m})^{\sum_{i=1}^{m} (k - \tau_i(k)) + m}}{\alpha} \geq \\
\geq \liminf_{k \to +\infty} \left\{ \min_{\alpha \geq 1} \frac{(1 + m \alpha^{1/m} p^{1/m})^{m + \sum_{i=1}^{m} (k - \tau_i(k))}}{\alpha} : \alpha \geq 1 \right\}, \tag{4.5}
\]

where \( p = \prod_{i=1}^{m} p_i \). Since

\[
\min_{\alpha \geq 1} \frac{(1 + m \alpha^{1/m} p^{1/m})^{m + \sum_{i=1}^{m} (k - \tau_i(k))}}{\alpha} = \\
= \frac{p \left( m + \sum_{i=1}^{m} (k - \tau_i(k)) \right)^{m + \sum_{i=1}^{m} (k - \tau_i(k))}}{\left( \sum_{i=1}^{m} (k - \tau_i(k)) \right)^{m + \sum_{i=1}^{m} (k - \tau_i(k))}}, \tag{4.6}
\]

according to (4.5) and (4.6) condition (2.9) holds, which proved the validity of the corollary.

**Proof of Corollary 2.3.** Since function \( (m + x)^{m+x} / x^x \) \( (1 \leq x < +\infty) \) is non-decreasing, according to (2.12) and (2.13) condition (2.9) be fulfilled, which proved the validity of the corollary.

**Remark 4.1.** It is obvious that if (2.12) be fulfilled and \( m = 1 \), then Corollary 2.4 is Theorem 7.5.1, [2].

**Remark 4.2.** The following example illustrate the significance of our results.

Consider difference equation

\[
\Delta u(k) + p(k) u(k - 1) = 0
\]

then condition (7.5.2) [2] is reduced to the condition

\[
\liminf_{k \to +\infty} p(k) > \frac{1}{4} \tag{4.7}
\]

and condition (2.9) is reduced by the condition

\[
\liminf_{k \to +\infty} \left( \sqrt{p(k - 1)} + \sqrt{p(k)} \right)^2 > 1. \tag{4.8}
\]
It is obvious that condition (4.7) implies condition (4.8).

On the other hand if \( \varepsilon \in \left(0, \frac{1}{4}\right) \), \( p(2k) = \varepsilon \) and \( p(2k + 1) = 1 - \varepsilon \), \( k = 1, 2, \ldots \) then it is obvious that condition (4.8) be fulfilled but condition (4.7) is not fulfilled.

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