Yangian realisations from finite $\mathcal{W}$-algebras

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Abstract

We construct an algebra homomorphism between the Yangian $Y(sl(n))$ and the finite $\mathcal{W}$-algebras $\mathcal{W}(sl(np), n.sl(p))$ for any $p$. We show how this result can be applied to determine properties of the finite dimensional representations of such $\mathcal{W}$-algebras.
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1 Introduction

In the year 1985, the mathematical physics literature was enriched with two new types of symmetries: $\mathcal{W}$ algebras [1] and Yangians [2]. $\mathcal{W}$ algebras showed up in the context of two-dimensional conformal field theories. They benefited from development owing in particular to their property to be algebras of constant of motion for Toda field theories, themselves defined as constrained WZNW models [3]. Yangians were first considered and defined in connection with some rational solutions of the quantum Yang-Baxter equation. Later, their relevance in integrable models with non Abelian symmetry was remarked [4]. Yangian symmetry has been proved for the Haldane-Shastry $SU(n)$ quantum spin chains with inverse square exchange, as well as for the embedding of this model in the $SU(2)_1$ WZNW one; this last approach leads to a new classification of the states of a conformal field theory in which the fundamental quasi-particles are the spinons [3] (see also [4]). Let us also emphasize the Yangian symmetry determined in the Calogero-Sutherland-Moser models [3, 4]. Coming back to $\mathcal{W}$ algebras, it can be shown that their zero modes provide algebras with a finite number of generators and which close polynomially. Such algebras can also be constructed by symplectic reduction of finite dimensional Lie algebras in the same way usual –or affine– $\mathcal{W}$ algebras arise as reduction of affine Lie algebras: they are called finite $\mathcal{W}$ algebras [5] (FWA). This definition extends to any algebra which satisfies the above properties of finiteness and polynomiality [6]. Some properties of such FWA’s have been developed [5, 6] and in particular a large class of them can be seen as the commutant, in a generalization of the enveloping algebra $U(G)$, of a subalgebra $\hat{G}$ of a simple Lie algebra $G$ [7]. This feature for FWA’s has been exploited in order to get new realizations of a simple Lie algebra $G$ once knowing a $G$ differential operator realization. In such a framework, representations of a FWA are used for the determination of $\hat{G}$ representations. This method has been applied to reformulate the construction of the unitary, irreducible representations of the conformal algebra $so(4, 2)$ and of its Poincaré subalgebra, and compared it to the usual induced representation technics [8]. It has also been used for building representations of observable algebras for systems of two identical particles in $d = 1$ and $d = 2$ dimensions, the $\hat{G}$ algebra under consideration being then symplectic ones; in each case, it has then been possible to relate the anyonic parameter to the eigenvalues of a $\mathcal{W}$-generator [9].

In this paper, we show that the defining relations of a Yangian are satisfied for a family of FWA’s. In other words, such $\mathcal{W}$ algebras provide Yangian realizations. This remarkable connection between two a priori different types of symmetry deserves in our opinion to be considered more closely. Meanwhile, we will use results on the representation theory of Yangians and start to adapt them to this class of FWA’s. In particular, we will show on special examples –the algebra $\mathcal{W}(sl(4), n.sl(2))–$ how to get the classification of all their irreducible finite dimensional representations.

It has seemed to us necessary to introduce in some detail the two main and a priori different algebraic objects needed for the purpose of this work. Hence, we propose in section 2 a brief reminder on $\mathcal{W}$-algebras with definitions and properties which will become useful to establish our main result. In particular, a short paragraph presents the Miura transformation. The
structure of $\mathcal{W}(sl(np), n.sl(p))$ algebras is also analysed. Then, the notion of Yangian $Y(\mathcal{G})$ is introduced in section 3, with some basic properties on its representation theory.

Such preliminaries allow us to arrive well-equipped for showing the main result of our paper, namely that there is an algebra homomorphism between the Yangian $Y(sl(n))$ and the finite $\mathcal{W}(sl(np), n.sl(p))$ algebra (for any $p$). This property is proven in section 4 for the classical case (i.e. $\mathcal{W}$-algebras with Poisson brackets) and generalized to the quantum case (i.e. $\mathcal{W}$-algebras with usual commutators) in the section 5. The proof necessitates the explicit knowledge of commutation relations among $\mathcal{W}$ generators. Such a result is obtained in the classical case via the soldering procedure [13]. Its extension to the quantum case leads to determine $sl(n)$ invariant tensors with well determined symmetries. In order not to overload the paper, all these necessary intermediate results are gathered in the appendices. Finally, as an application, the representation theory of $\mathcal{W}(sl(2n), n.sl(2))$ algebras is considered in section 6. General remarks and a discussion about some further possible developments conclude our study.

2 Finite $\mathcal{W}$ algebras: notation and classification.

As mentioned above, the $\mathcal{W}$ algebras that we will be interested in can be systematically obtained by the Hamiltonian reduction technique in a way analogous to the one used for the construction and classification of affine $\mathcal{W}$ algebras [3, 14, 15]. Actually, given a simple Lie algebra $\mathcal{G}$, there is a one-to-one correspondence between the finite $\mathcal{W}$ algebras one can construct in $\mathcal{U}(\mathcal{G})$ and the $sl(2)$ subalgebras in $\mathcal{G}$. We note that any $sl(2)$ subalgebra is principal in a subalgebra $\mathcal{S}$ of $\mathcal{G}$. The step generator $E_+$ in the $sl(2)$ subalgebra which is principal in $\mathcal{S}$ is then written as a linear combination of the simple root generators of $\mathcal{S}$: $E_+ = \sum_{i=1}^{s} E_{\beta_i}$ where $\beta_i$, $i = 1, \ldots, s = \text{rank}(\mathcal{S})$ are the simple roots of $\mathcal{S}$. It can be shown that one can complete uniquely $E_+$ with two generators $E_-$ and $H$ such that $(E_+, H)$ is an $sl(2)$ algebra. It is rather usual to denote the corresponding $\mathcal{W}$ algebra as $\mathcal{W}(\mathcal{G}, \mathcal{S})$. It is an algebra freely generated by a finite number of generators and which has a second antisymmetric product. Depending on the nature of this second product, we will speak of classical (the product is a Poisson bracket) or a quantum (the product is a commutator) $\mathcal{W}$-algebra.

2.1 Classical $\mathcal{W}(\mathcal{G}, \mathcal{S})$ algebras

To specify the Poisson structure of the $\mathcal{W}$-algebra, we start with the Poisson-Kirillov structure on $\mathcal{G}^*$. It mimicks the Lie algebra structure on $\mathcal{G}$, and we will still denote by $H_i$, $E_{\pm\alpha}$ the generators in $\mathcal{G}^*$. $\mathcal{U}(\mathcal{G}^*)$ is then a Poisson-Lie enveloping algebra. We construct the classical $\mathcal{W}(\mathcal{G}, \mathcal{S})$ algebra from an Hamiltonian reduction on $\mathcal{U}(\mathcal{G}^*)$, the constraints being given by the $sl(2)$ embedding as follows.

The Cartan generator $H$ of the $sl(2)$ subalgebra under consideration provides a gradation of $\mathcal{G}$:

$$\mathcal{G} = \bigoplus_{i=-N}^{N} \mathcal{G}_i \text{ with } [H, X] = iX, \forall X \in \mathcal{G}_i$$

(2.1)
The root system of $\mathcal{G}$ is also graded: $\Delta = \oplus \Delta_i$. We have

$$H \in \mathcal{G}_0, \; E_\pm \in \mathcal{G}_{\pm 1} \text{ and } E_- = \sum_{\alpha \in \Delta_{-1}} \chi_\alpha E_\alpha, \; \chi_\alpha \in \mathbb{C}. \quad (2.2)$$

Then, the first class constraints are

$$\Phi_\alpha = \Phi(E_\alpha) = E_\alpha = 0 \text{ if } E_\alpha \in \mathcal{G}_{<-1} \text{ and } \Phi_\alpha = \Phi(E_\alpha) = E_\alpha - \chi_\alpha = 0 \text{ if } E_\alpha \in \mathcal{G}_{-1} \quad (2.3)$$

the second class constraints being given by

$$\Phi(X) = X = 0 \text{ if } X \in \mathcal{G}_{\geq 0} \text{ and } \{E_+, X\} \neq 0 \quad (2.4)$$

Note that considering $\mathcal{G}^*$ as a module of the above mentionned $sl(2)$, the generators of $\mathcal{W}(\mathcal{G}, S)$ are in one-to-one correspondence with the highest weights of this $sl(2)$.

The Poisson bracket structure of the $\mathcal{W}(\mathcal{G}, S)$ algebra is then given by the Dirac brackets associated to the constraints. More explicitely, one labels the constraints $\Phi_i, \; i = 1, \ldots, I_0$ and denotes them by $C_{ij} \sim \{\Phi_i, \Phi_j\}$ where the symbols $\sim$ means that one has to apply the constraints after calculation of the Poisson-Kirillov brackets. Then, the Dirac brackets are given by

$$\{A, B\} \sim \{A, B\} - \sum_{i,j=1}^{I_0} \{A, \Phi_i\} C^{ij} \{\Phi_j, B\} \text{ with } C^{ij} = (C^{-1})_{ij} \quad (2.5)$$

They have the remarquable property that any generator in $\mathcal{U}(\mathcal{G}^*)$ has vanishing Dirac brackets with any constraint, so that the $\mathcal{W}(\mathcal{G}, S)$ algebra can be seen as the quotient of $\mathcal{U}(\mathcal{G}^*)$ provided with the Dirac bracket by the ideal generated by the constraints.

### 2.2 Quantum $\mathcal{W}(\mathcal{G}, S)$ algebras

These algebras are a quantization of the classical $\mathcal{W}(\mathcal{G}, S)$ algebras, where the Poisson structure has been replaced by a commutator. As we have an algebra on which one has imposed some constraints, there are two ways to quantise it: either, one first quantize $\mathcal{U}(\mathcal{G}^*)$ and then impose the constraints at the quantum level; or one quantize directly the $\mathcal{W}$-algebra. In the first case, one can just look at $\mathcal{U}(\mathcal{G})$ as the quantization of $\mathcal{U}(\mathcal{G}^*)$ and then use a BRS formalism. This has been developped in [16], where the BRS operator is constructed, and its cohomology computed. The quantum version of the $\mathcal{W}(\mathcal{G}, S)$ algebra is then the zeroth cohomological space, which inherits an algebraic structure from $\mathcal{U}(\mathcal{G})$. It is possible to explicitely construct a representative of each cohomology classes, using the highest weights in $\mathcal{G}$ of the $sl(2)$ under consideration. Note that for super $\mathcal{W}$-algebras, the same treatment can be applied, the factorisation of spin $\frac{1}{2}$ fields (and other gauging properties) being replaced by a filtration of the cohomological spaces in that formalism [17].

Here, we will look directly at the quantization of the $\mathcal{W}$-algebra. In that approach, we start with the classical $\mathcal{W}(\mathcal{G}, S)$ algebra and ask for an algebra which has a non commutative
product law, the associated commutator admitting as a limit the Poisson bracket (PB). This more pedestrian and less powerful approach will be sufficient for our purpose. In fact, we will only need to know that the quantum $\mathcal{W}(G, S)$ algebra has the same number of generators as the classical one, as well as the same leading term in the commutator. By leading term, we mean that in each commutator, the term of highest “conformal spin” is the same as the right hand side of the corresponding PB. In the following, we will choose this term to be a symmetrized product (see below).

2.3 Miura representations

Associated to the gradation of $G$, there exists a representation of the $\mathcal{W}(G, S)$-algebra, called the Miura map. It is an algebra morphism from $\mathcal{U}(G_0^*)$ to $\mathcal{W}(G, S)$. Note that one can show that $\mathcal{U}(G_0^*)$ has the same dimension as $\mathcal{W}(G, S)$, but the Miura map is neither an algebra isomorphism, nor a vector space isomorphism. It is based on a restriction of the Hamiltonian reduction that leads from $G$ to $\mathcal{W}(G, S)$. More explicitly, for classical finite $\mathcal{W}$-algebras, we start with the $(G^*\text{-valued})$ matrix

$$J_0 = e_- + j^k t_k \text{ with } [h, t_k] = 0$$

(2.6)

where $e_-$ and $t_k$ are in the fundamental representation of $G$ ($\{t_k\}$ is a basis of $G_0$) and $j^k \in G_0^*$. Then, we consider the transformations

$$J_0 \to J^g = g J_0 g^{-1} \text{ with } g \in G_+, \text{ Lie}(G_+) = G_+$$

(2.7)

It can be shown that there exists a unique element $g$ such that

$$J_0 \to J^g = J_{hw} = e_- + W^k m_k \text{ with } [e_+, m_k] = 0$$

(2.8)

where $m_k$ (in the fundamental representation of $G$) are the highest weights of $(h, e_\pm)$, the $sl(2)$ algebra under consideration. $\{W^k\}$ generate the classical $\mathcal{W}$-algebra and are determined by the Miura map (2.3). They are expressed in terms of the $j^k$ generators, and thus this map allows to construct the $\mathcal{W}(G, S)$-algebra generators as polynomials in $G_0$ ones (see e.g. [3] for details).

At the quantum level, one can still proceed in two ways: either work on cohomological space (the Miura map in that case corresponds to a restriction to the zero grade of the general cohomological construction), or directly quantize the classical Miura construction.

Using the Miura construction then leads to (finite dimensional) representations of $\mathcal{W}(G, S)$ associated to (finite dimensional) representations of $G_0$. Note however that the irreducibility of these $\mathcal{W}(G, S)$-representations is a priori not known, even if one starts from an irreducible representation of $G_0$ (see [18] for a counter example).

2.4 Example: $\mathcal{W}(sl(np), n.sl(p))$ algebras

As an example, let us first consider the $\mathcal{W}(sl(4), sl(2) \oplus sl(2))$ algebra.
It is made of seven generators \( J_i, \) \( S_i \) \((i = 1, 2, 3)\) and a central element \( C_2 \) such that:

\[
\begin{align*}
[J_i, J_j] &= i\epsilon_{ijk} J_k \\
[J_i, S_j] &= i\epsilon_{ijk} S_k \\
[S_i, S_j] &= i\epsilon_{ijk} J_k \left( C_2 - 2 \bar{J}^2 \right) \\
[C_2, J_i] &= [C_2, S_i] = 0 \\
& \quad \text{with } \bar{J}^2 = J_1^2 + J_2^2 + J_3^2
\end{align*}
\]  

(2.9)

We recognize the \( sl(2) \) subalgebra generated by the \( J_i \)'s as well as a vector representation (i.e. \( S_i \) generators) of this \( sl(2) \) algebra. We note that the \( S_i \)'s close polynomially on the other generators.

The same type of structure can be remarked, at a higher level, for the class of algebras \( \mathcal{W}(sl(np), n.sl(p)) \) where \( n.sl(p) \) stands for the regular embedding \( sl(p) \oplus \cdots \oplus sl(p) \) \((n \text{ times})\). We recall that an algebra \( \mathcal{S} \) is said to be regular in \( \mathcal{G} \), itself generated by the Cartan generators \( H_i, \) \( i = 1, \ldots, \text{rank } (\mathcal{G}) \) and the root generators \( E_\alpha, \) \( \alpha \in \Delta \), if \( \mathcal{S} \) is, up to a conjugation, generated by a subset of the above Cartan part, as well as a subset \( \{E_\alpha\}, \) \( \alpha \in \delta \subset \Delta \) of the \( \mathcal{G} \)-root generator set. In order to determine the number and the “conformal spin” of the \( \mathcal{W} \) generators, one has first to decompose the adjoint representation of \( \mathcal{G} \) under the \( sl(2) \) which is principal in \( \mathcal{S} = n.sl(p) \): \( \mathcal{G} = \oplus_j \mathcal{D}_j \) with \( \mathcal{D}_j \) the \((2j + 1)\)-dimensional irreducible representation of \( sl(2) \). Then to each \( \mathcal{D}_j \) will correspond one \( \mathcal{W} \) generator of “conformal spin” \((j + 1)\) if we keep in mind that such a generator can be seen as the zero mode of a primary field in a (Toda) conformal field theory. Actually, it is known that the \( \mathcal{G} \) adjoint representation can be seen as arising from the direct product of the \( \mathcal{G} \) fundamental representation by itself. Since this later representation reduces with respect to the \( \mathcal{S} \) principal \( sl(2) \) as \( n\mathcal{D}_{\frac{n-1}{2}} \), it leads to:

\[
n\mathcal{D}_{\frac{n-1}{2}} \times n\mathcal{D}_{\frac{n-1}{2}} \longrightarrow n^2(\mathcal{D}_{p-1} + \mathcal{D}_{p-2} + \cdots + \mathcal{D}_1) + (n^2 - 1)\mathcal{D}_0
\]

(2.10)

A more careful study will allow to recognize in the \((n^2 - 1)\mathcal{D}_0\) the generators of an \( sl(n) \) algebra, and to associate to each set of \((n^2 - 1)\mathcal{D}_k, \) \( k = 1, 2, \ldots, p - 1, \) an irreducible (adjoint) representation under the above determined \( sl(n) \) algebra, the corresponding elements being of “conformal spin” \((k + 1)\). To each remnant \( \mathcal{D}_k \) will finally be associated a spin \((k + 1)\) element which commutes with all the \( \mathcal{W} \) generators: these central elements will be denoted \( C_2, C_3, \ldots, C_p \). They can be identified with the first \( p \) Casimir operators of the \( sl(np) \) algebra.

Using a notation which will become clear in the next sections, we call \( W^a_0, \) \( a = 1, \ldots, n^2 - 1 \) the \( sl(n) \) generators, and \( W^a_k, \) \( k = 1, 2, \ldots, p - 1, \) the \( \mathcal{W} \) generators of respective spin \((k + 1)\).

We can gather the above assertions in the following commutation relations:

\[
\begin{align*}
[W^a_0, W^b_0] &= f^{abc} W^c_0 & a &= 1, 2, \ldots, n^2 - 1 \\
[W^a_0, W^b_k] &= f^{abc} W^c_k & k &= 1, 2, \ldots, p - 1 \\
[C_i, W^a_0] &= [C_i, W^a_k] = 0 & i &= 2, 3, \ldots, p
\end{align*}
\]

(2.11)

\(^1\)For details about the conformal spin contents of \( \mathcal{W} \)-algebras computed using \( sl(2) \) representations see[13].
The remaining commutator takes the form

\[ [W_k^a, W_\ell^b] = P_{k\ell}^{ab}(W) \]  \hspace{1cm} (2.12)

where \( P_{k\ell}^{ab} \) is a polynomial in the \( \mathcal{W} \)-generators, which is of “conformal spin” \((k + \ell - 1)\). It is determined using the technics described above.

The \( \mathcal{G}_0 \) subalgebra associated to \( \mathcal{W}(sl(np), n, sl(p)) \) is \( p.sl(n) \oplus (p - 1).gl(1) \) which one can denote \( s(p, gl(n)) \), i.e. traceless matrices of \( gl(n) \oplus gl(n) \oplus \cdots gl(n) \) \((p \text{ times})\). Thus, the Miura map for this kind of \( \mathcal{W} \)-algebras leads to a realisation in term of generators of the enveloping algebra of \( s(p, gl(n)) \).

3 Yangians \( \mathcal{Y}(\mathcal{G}) \)

3.1 Definition

We briefly recall some definitions about Yangians, most of them being gathered in [19]. Yangians are one of the two well-known families of infinite dimensional quantum groups (the other one being quantum affine algebras) that correspond to deformation of the universal enveloping algebra of some finite-dimensional Lie algebra, called \( \mathcal{G} \). As such, it is a Hopf algebra, topologically generated by elements \( Q_0^a \) and \( Q_1^a \), \( a = 1,\ldots, \dim \mathcal{G} \) which satisfy the following defining relations:

\[ Q_0^a \text{ generate } \mathcal{G} : [Q_0^a, Q_0^b] = f^{ab}_c Q_0^c \]  \hspace{1cm} (3.1)

\[ Q_1^a \text{ form an adjoint rep. of } \mathcal{G} : [Q_0^a, Q_1^b] = f^{ab}_c Q_1^c \]  \hspace{1cm} (3.2)

\[ [Q_1^a, [Q_0^b, Q_1^c]] + [Q_1^b, [Q_0^c, Q_1^a]] + [Q_1^c, [Q_0^a, Q_1^b]] = f^{ab}_p f^{bc}_q f^{cd}_r f^{de}_s s_3(Q_0^p, Q_0^q, Q_0^r) \]  \hspace{1cm} (3.3)

\[ [[Q_1^a, Q_1^b], [Q_0^c, Q_1^d]] + [[Q_1^b, Q_1^c], [Q_0^d, Q_1^a]] = \left( f^{ab}_p f^{bc}_q f^{cd}_g f^{de}_y f^{yz}_x g + f^{ab}_p f^{bc}_q f^{cd}_y f^{yz}_g f^{xz}_x \right) s_3(Q_0^p, Q_0^q, Q_1^d) \]  \hspace{1cm} (3.4)

where \( f^{ab}_c \) are the totally antisymmetric structure constants of \( \mathcal{G} \), \( \eta^{ab} \) is the Killing form, and \( s_n(\ldots, \ldots) \) is the totally symmetrized product of \( n \) terms. The generators \( Q_n^a \) for \( n > 1 \) are defined recursively through

\[ f^{ab}_c [Q_n^b, Q_{n-1}^c] = c_v Q_n^a \quad \text{with} \quad c_v \eta^{ab} = f^{ab}_c f^{bcd} \]  \hspace{1cm} (3.5)

It can be shown that for \( \mathcal{G} = sl(2) \), (3.3) is a consequence of the other relations, while for \( \mathcal{G} \neq sl(2) \), (3.4) follows from (3.3). The coproduct on \( \mathcal{Y}(\mathcal{G}) \) is given by

\[ \Delta(Q_0^a) = 1 \otimes Q_0^a + Q_0^a \otimes 1 \]

\[ \Delta(Q_1^a) = 1 \otimes Q_1^a + Q_1^a \otimes 1 + \frac{1}{2} f^{ab}_c Q_0^b \otimes Q_0^c \]  \hspace{1cm} (3.6)

In the following, we will focus on the Yangians \( \mathcal{Y}(sl(n)) \).
3.2 Evaluation representations of $Y(sl(n))$

When $G = sl(n)$, there is a special class of finite dimensional irreducible representations called evaluation representations. They are defined from the algebra homomorphisms

$$
ev_A^\pm \left\{ \begin{array}{l}
Y(sl(n)) \to \mathcal{U}(sl(n)) \\
Q_0^a \to t^a \\
Q_1^a \to A t^a + d_{bc}^a t^b t^c
\end{array} \right. \quad \text{with } A \in \mathbb{C}$$

(3.7)

where the $t^a$’s form a $sl(n)$ basis, and $d_{bc}^a$ is the totally symmetric invariant tensor of $sl(n)$ (we set $d_{bc}^a = 0$ when $n = 2$). It can be shown that $\ev_A^+$ and $\ev_A^-$ are isomorphic (and indeed $\ev_A^+ = \ev_A^- = \ev_A$ when $G = sl(2)$).

An evaluation representation of $Y(G)$ is defined by the pull-back of a $G$-representation (with the help of the evaluation homomorphism $\ev_A^\pm$). The corresponding representation space will be denoted generically by $V_A^\pm(\pi)$ where $\pi$ is a representation of $sl(n)$. We select hereafter two properties [19] which will be used in section 6.

Theorem 1: Any finite-dimensional irreducible $Y(sl(n))$ module is isomorphic to a sub-quotient of a tensor product of evaluation representations.

Theorem 2: When $G = sl(2)$, let $V_A(j)$ be the $(2j+1)$-dimensional irreducible representation space of $ev_A$ ($j \in \frac{1}{2}\mathbb{Z}$). Then, $V_A(j) \otimes V_B(k)$ is reducible if and only if $A-B = \pm(j+k-m+1)$ for some $0 < m \leq \min(2j, 2k)$.

In that case, $V_A(j) \otimes V_B(k)$ is not completely reducible, and not isomorphic to $V_B(k) \otimes V_A(j)$; otherwise, $V_A(j) \otimes V_B(k)$ is irreducible and isomorphic to $V_B(k) \otimes V_A(j)$.

4 Yangians and classical $\mathcal{W}$-algebra

In this section, we want to show that there is an algebra morphism between the Yangian $Y(sl(n))$ and the classical $\mathcal{W}(sl(np), n.sl(p))$ algebras ($\forall p$). For such a purpose, we need to compute some of the PB of the $\mathcal{W}$-algebra. It is done using the soldering procedure [13], the calculation being quite tricky (see appendices). For the generic case of $\mathcal{W}(sl(np), n.sl(p))$ algebras, the result is

$$\{W_1^a, W_1^b\} = \frac{1}{5} f_{ab}^c W_2^c - \frac{p^3}{16} (d_{bc}^a v f^b_{cu} - d_{bc}^a v f^a_{cu}) d^e_{de} W_0^c W_0^d W_0^e$$

(4.1)

where the indices run from 0 to $n^2 - 1$, with the notation $W_0^0 = 0$ and $W_0^0 = C_2$ and the normalisation:

$$\{W_0^a, W_k^b\} = \frac{1}{p} f_{ab}^c W_k^c \quad k = 0, 1, 2, ...$$

(4.2)

When $p = 2$, we have the constraint $W_2^a = 5 d_{bc}^a W_0^b W_1^c$. Let us stress that the tensors $f_{ab}^c$ and $d_{bc}^a$ are $gl(n)$ tensors, not $sl(n)$ ones: see appendix [3] for clarification.
For the case of $\mathcal{W}(sl(2p), 2.sl(p))$, the relations simplify to:

\[
\begin{align*}
\{W_1^a, W_1^b\} &= f^{ab}_{\ c} \left[ \frac{1}{5} W_2^c - \frac{p^3}{2} W_0^c \vec{W}_0^2 \right] \\
\{W_1^a, W_2^b\} &= f^{ab}_{\ c} \left[ \frac{3}{14} W_3^c + \frac{6(3p^2 - 7)}{p(p^2 - 1)} W_1^c W_1^0 + \frac{(p^2 - 9)(p^2 - 4)}{2(p^2 - 1)} W_1^c \vec{W}_0^2 + \\
&\quad + 3 W_0^c W_2^c - 30 W_0^c (\vec{W}_1 \cdot \vec{W}_0) \right]
\end{align*}
\] (4.3)

(4.4)

together with the constraints $W_2^a = 10 \left[ W_0^a W_1^0 + \delta_0^a (\vec{W}_1 \cdot \vec{W}_0) \right]$ for $p = 2$, and $W_3^a = 0$ for $p = 2$ or 3.

In this basis, the map is

\[
\rho_p \left\{ \begin{array}{c}
Y(sl(n)) \rightarrow \mathcal{W}(sl(np), n.sl(p)) \\
Q_k^a \rightarrow \beta_k W_k^a & \text{for } k = 0, 1, \ldots, p \\
Q_{p+l}^a \rightarrow P_{p+l}^a (W_0, W_1, \ldots, W_p) & \text{for } l > 0
\end{array} \right.
\] (4.5)

where $P_l^a$ are some homogeneous polynomials which preserve the “conformal spin” of $W_k^a$.

A careful computation shows (see appendix B), using the PBs (4.1–4.4), that the generators $W_k^a$ obey the relations (3.1–3.4), the commutators being replaced by PBs. As $Y(sl(n))$ is topologically generated by $Q_0^a$ and $Q_1^a$, it is sufficient to give $\beta_0$ and $\beta_1$. Indeed, once (4.3) is satisfied for $k = 0$ and 1, the relation (3.3) together with the PB of the $\mathcal{W}$-algebra ensure that (4.3) can be iteratively constructed for all $k$. We show in the appendix B that in our basis this relation is indeed satisfied for

\[
\beta_0 = p \quad \text{and} \quad \beta_1 = 2
\] (4.6)

We can thus conclude:

**Proposition 1:**

The classical algebra $\mathcal{W}(sl(np), n.sl(p))$ provides a representation of the Yangian $Y(sl(n))$, the map being given by $\rho_p$ defined in (4.3) and (4.4).

## 5 Yangians and quantum $\mathcal{W}(sl(np), n.sl(p))$ algebras

We can use the above study to deduce the same result for quantum $\mathcal{W}(sl(np), n.sl(p))$ algebras. In fact, as these algebras are a quantisation of the classical ones, we can deduce that the most general form of the commutator is

\[
[W_0^a, W_k^b] = \frac{1}{p} f^{ab}_{\ c} W_k^c
\] (5.1)
Now, it is easy to show that the following tensors indeed belong to these spaces

\[ [W_1^a, W_1^b] = f_{ab}^c \frac{1}{5} W_c^c - \frac{p^3}{16} \left( d^{au} v f_{cu}^b - d^{bu} v f_{cu}^a \right) d^{de} s_3(W_0^c, W_0^d, W_0^e) + + t_{cd}^e W_c^c + t_{cd}^{ab} s_2(W_0^c, W_0^d) + t_{cd}^{ab} W_e^c \]

for \( sl(n) \), and in the special case of \( sl(2) \)

\[ [W_1^a, W_2^b] = f_{ab}^c \left[ \frac{3}{14} W_3^c + \frac{6(3p^2 - 7)}{p(p^2 - 1)} s_2(W_1^c, W_1^0) + 3 s_2(W_2^0, W_0^c) + + \left( \frac{(p^2 - 9)(p^2 - 4)}{2(p^2 - 1)} \eta_{dg}\eta_e^c - 30 \eta_{de}\eta_g^c \right) s_3(W_0^g, W_0^d, W_1^e) \right] + + g_{cd}^e W_2^c + g_{cd}^{ab} W_c^a W_0^d + g_{cd}^{ab} s_3(W_0^c, W_0^d, W_0^e) + g_{cd}^{ab} W_1^c + + g_{cd}^{ab} s_2(W_0^c, W_0^d) + g_{cd}^{ab} W_0^c \]

for some tensors \( t_{ab}^{a_1 a_2 \ldots a_k} \) and \( g_{ab}^{a_1 a_2 \ldots a_k} \). By construction, the \( t \)-tensors are symmetric in the lower indices and antisymmetric in the upper ones. The \( g \)-tensors are only symmetric in the lower indices, except for \( g_{cd}^{ab} \) which has no symmetry property. Moreover, the Jacobi identities with \( W_0^a \) show that they are invariant tensors. Hence, we are looking for objects which belong to the \( \Lambda_2(G) \otimes \Lambda_2(G) \) for \( G = sl(n) \):

\[ M_0[\Lambda_2(G) \otimes G] = 1 \]

\[ M_0[S_2(G) \otimes G] = \begin{cases} 1 & \text{if } n \neq 2 \\ 0 & \text{for } sl(2) \end{cases} \]

\[ M_0[\Lambda_2(G) \otimes \Lambda_2(G)] = \begin{cases} 3 & \text{if } n \neq 2 \\ 1 & \text{for } sl(2) \end{cases} \]

Now, it is easy to show that the following tensors indeed belong to these spaces:\(^2\)

\[ \Lambda_2(G) \otimes G : \ t_{cd}^{ab} = f_{ab}^c \quad S_2(G) \otimes G : \ t_{cd}^{ab} = d_{cd}^{ab} \]

\[ \Lambda_2(G) \otimes S_2(G) : \ t_{cd}^{ab} = f_{ab}^c e_{cd}^{de} \]

As they are evidently independent and give the correct multiplicities (with the convention that the \( d \)-tensor is null for \( sl(2) \)), we deduce that the most general form one gets is:

\[ [W_1^a, W_1^b] = \frac{p^3}{16} \left( d^{au} v f_{zu}^b - d^{au} v f_{zu}^a \right) d^{de} s_3(W_0^c, W_0^d, W_0^e) + + \]

\(^2\)More general formulae are given in the appendix [D].
\[
W_{\mathfrak{sl}(n \cdot p), n \cdot \mathfrak{sl}(p))} \text{ for the algebra } \mathcal{W}(sl(np), n\cdot sl(p)). \text{ This commutator is the only one needed to prove that the algebra satisfies the defining relations of the Yangian when } n \neq 2.
\]

For the algebra \(\mathcal{W}(sl(2p), 2\cdot sl(p))\), we need also the relation:

\[
[W^a_1, W^b_2] = f^{ab}_{\ c} \left[ \frac{3}{14} W^c_3 + \frac{6(3p^2 - 7)}{p(p^2 - 1)} s_2(W^c_1, W^0_1) + 3 s_2(W^0_2, W^c_0) + \frac{(p^2 - 9)(p^2 - 4)}{2(p^2 - 1)} \eta_{dg} \eta_{e} - 30 \eta_{de} \eta_{g} \right] s_3(W^g_0, W^d_0, W^e_1) + \nu_1 W^c_2 + \nu'_1 W^c_1 + \nu''_1 W^c_0 + \nu_2 f^c_{\ de} W^d_1 W^e_0 + \nu_3 s_3(W^c_0, W^d_0, W^e_0)
\]

Then, one can show that the commutators (5.6–5.7) obeys to the defining relations of the Yangians \(Y(sl(n))\) for the same normalisations as in the classical case. It is done in the appendices C and D.

**Proposition 2:**

The quantum algebra \(\mathcal{W}(sl(np), n\cdot sl(p))\) provide a representation of the Yangian \(Y(sl(n))\), the map being given by \(\rho_p\) defined in (4.5) and (4.6).

At this point, let us note that the Yangian structure of the algebra \(\mathcal{W}(sl(4), 2\cdot sl(2))\) has been already remarked \([4, 20]\) and used for quantum mechanics applications \([20]\).

### 6 Representations of \(\mathcal{W}(sl(2n), n\cdot sl(2))\) algebras

Owing to the above identification, it is possible to adapt some known properties on Yangian representation theory to finite \(\mathcal{W}\) representations. We first illustrate this assertion in the case of \(\mathcal{W}(sl(2n), n\cdot sl(2))\).

**Proposition 3:**

Any finite dimensional irreducible representation of the algebra \(\mathcal{W}(sl(4), 2\cdot sl(2))\) is either an evaluation module \(\mathcal{V}_A(j)\) or the tensor product of two evaluation modules \(\mathcal{V}_A(j) \otimes \mathcal{V}_{(-A)}(k)\).

Conversely, \(\mathcal{V}_A(j)\) for any \(A\), and \(\mathcal{V}_A(j) \otimes \mathcal{V}_{(-A)}(k)\) \((A \neq 0)\) with \(2A \neq \pm(j + k - m + 1)\) for any \(m\) such that \(0 < m \leq \min(2j, 2k)\), are finite dimensional irreducible representations of the algebra \(\mathcal{W}(sl(4), 2\cdot sl(2))\).

The tensor product is calculated via the Yangian coproduct defined in (3.6).
The proof is done by direct calculation, using the theorem 2 of section 3.2. As a (irreducible) representation of the \( W(sl(4), 2.sl(2)) \) algebra must be a (irreducible) representation of the Yangian \( Y(sl(2)) \), we deduce that the (finite dimensional) irreducible representations of \( W(sl(4), 2.sl(2)) \) are in the set of evaluation modules \( V_A(j) \) or \( V_A(j) \otimes V_B(k) \otimes \cdots \otimes V_C(\ell) \).

For \( V_A(j) \), it is obvious that we have an irreducible representation, where the value of the \( W \) Casimir operator \( C_2 \) is related to \( A \):

\[
C_2(A, j) = (2j(j+1) + A^2) I
\]

For a product \( V_A(j) \otimes V_B(k) \), calculations show that we must have

\[
A + B = 0 \quad \text{and} \quad C_2(A, j; B, k) = \left( 2j(j+1) + 2k(k+1) + \frac{1}{2}(A^2 + B^2) \right) I \otimes I
\]

in order to get a representation of the \( W \)-algebra. The irreducibility is fixed by the first part of theorem 2, section 3.2, i.e. when there is no \( m \in \left[ 0, \min(2j, 2k) \right] \) such that \( 2A = \pm(j+k-m+1) \).

It is the second part of the theorem which ensures that the above construction exhausts the set of irreducible finite dimensional representations of \( W(sl(4), 2.sl(2)) \). Indeed, in the product \( V_A(j) \otimes V_B(k) \otimes V_C(\ell) \), we already know that we must have \( B = -A \) and \( C = -B \) for \( V_A(j) \otimes V_B(k) \) and \( V_B(k) \otimes V_C(\ell) \) to be representations of the \( W \)-algebra. Then, the irreducibility of \( V_{(-A)}(k) \otimes V_A(\ell) \) implies that this last representation is isomorphic to \( V_A(\ell) \otimes V(-A)(k) \). Therefore, \( V_A(j) \otimes V(-A)(k) \otimes V_A(\ell) \) is isomorphic to \( V_A(j) \otimes V_A(\ell) \otimes V(-A)(k) \). But the product \( V_A(j) \otimes V_A(\ell) \) is not a representation of the \( W \)-algebra, so that the triple product is not either.

Note that we get the surprising result that the tensor product of two representations of the \( W(sl(4), 2.sl(2)) \) algebra (\( V_A(j) \) and \( V_B(k) \)) is not always a representation of this algebra. In some sense, this result can be interpreted as a no-go theorem for the existence of a coproduct for \( W \)-algebras.

Let us also remark that the above representations are those obtained through the Miura map (see sections 2.3 and 2.4), so that we have proved that the Miura map gives all the irreducible finite dimensional representations of this \( W \)-algebra. Moreover, as the \( g_0 \) algebra we have to consider is just \( sl(2) \oplus sl(2) \oplus gl(1) = s(2, gl(2)) \), the condition \( A + B = 0 \) in the tensor product \( V_A(j) \otimes V_B(k) \) can just be interpreted as the traceless condition on \( s(2, gl(2)) \). Indeed, a representation of \( 2gl(2) \) is given by a representation space \( D_j \otimes D_k \) of \( 2sl(2) \), together with the values \( A \) and \( B \) of the two \( gl(1) \) generators, while for \( s(2, gl(2)) \), one has to impose \( A + B = 0 \).

In fact, we are able to prove a more general result:

\footnote{In fact, for \( A = 0 \), the tensor product indeed provides a representation of the \( W \)-algebra (it is just a representation of \( sl(2) \)). However, in that case, the tensor product is not irreducible.}
Proposition 4:

Any finite dimensional irreducible representation of the algebra \( \mathcal{W}(sl(2n), n.sl(2)) \) must be either an evaluation module \( \mathcal{V}_A^\pm(\pi) \) or the tensor product of two evaluation modules \( \mathcal{V}_A^\pm(\pi) \otimes \mathcal{V}_{(-A)}^{\mp}(\pi') \), where \( \pi \) and \( \pi' \) are irreducible finite dimensional representations of \( sl(n) \), the tensor product being calculated via the Yangian coproduct defined in (3.4).

All these representations can be obtained from the Miura map:

\[
\text{s}(2.gl(n)) \equiv 2.sl(n) \oplus gl(1) \rightarrow \mathcal{W}(sl(2n), n.sl(2))
\]

Note that these algebras are just the ones used in [11] to construct the finite \( \mathcal{W} \)-algebras as commutants in \( \mathcal{U}(\mathcal{G}) \).

It seems rather natural to conjecture that this situation will remain valid in the general case of \( \mathcal{W}(sl(np), n.sl(p)) \) algebras [21].

7 Conclusion

A rather surprising connection between Yangians and finite \( \mathcal{W} \)-algebras has been developed in this paper. We have proved directly that finite \( \mathcal{W} \)-algebras of the type \( \mathcal{W}(sl(np), n.sl(p)) \) satisfy the defining relations of the Yangian \( Y(sl(n)) \). In particular, we have been led to explicitly compute rather non trivial commutators of \( \mathcal{W} \) generators, namely spin 2 - spin 2 and spin 2 - spin 3 ones, a result which is interesting in itself. The question is now to understand more deeply this relationship between Yangian and finite \( \mathcal{W} \)-algebras.

Of course, the structure of the \( \mathcal{W}(sl(np), n.sl(p)) \) algebra (see section 2.4) reveals the special role played by its (spin one) Lie subalgebra \( sl(n) \). The \( \mathcal{W} \) generators of equal spin gather into adjoint representations of this \( sl(n) \) algebra, inducing some resemblance with the \( Y(sl(n)) \) yangian structure.

At this point, let us remark another common point between \( Y(sl(n)) \) and \( \mathcal{W}(sl(np), n.sl(p)) \), namely the construction of their finite dimensional representations with the help of \( sl(n) \) ones. Indeed, the evaluation homomorphism (in the case of Yangians) and the Miura map (for \( \mathcal{W} \)-algebras) play identical roles for such a construction: the former allows to represent \( Y(sl(n)) \) on the tensor product of \( sl(n) \) representations (with the use of additional constant numbers), while the later uses a representation of the \( \mathcal{G}_0 \) algebra \( p.sl(n) \oplus (p-1).gl(1) \). This clearly shows a one-to-one correspondence.

Let us also stress another feature of the \( \mathcal{W}(sl(np), n.sl(p)) \) algebras: for \( p = 2 \), they are the commutant in (a localisation of) \( \mathcal{U}(sl(n)) \) of an Abelian subalgebra \( \tilde{\mathcal{G}} \) of \( sl(n) \) [11, 18], the case \( p > 2 \) being with no doubt generalisable.

Finally, in the seek of understanding our results, one could think to a R-matrix approach. This point of view looks natural, since a R-matrix definition of the Yangians is available, while our \( \mathcal{W} \)-algebras are symmetry algebras of (integrable) non-Abelian lattice Toda models.
Due to the wide class of \( \mathcal{W}(\mathcal{G}, \mathcal{S}) \)-algebras, it seems natural to think of generalisations of our work. First of all, one could imagine to study Yangians \( Y(\mathcal{G}) \) (with \( \mathcal{G} \neq sl(n) \)) from the \( \mathcal{W} \)-algebras point of view. However, a rapid survey of \( \mathcal{W}(\mathcal{G}, \mathcal{S}) \)-algebras shows that \( \mathcal{W}(sl(np), n.sl(p)) \) algebras are the only \( \mathcal{W}(\mathcal{G}, \mathcal{S}) \)-algebras where the generators are all gathered in adjoint representations of the Lie \( \mathcal{W} \)-subalgebra. Inversely, \( \mathcal{W}(\mathcal{G}, \mathcal{S}) \)-algebras might be a way to generalize the notion of Yangians \( Y(\mathcal{G}) \) to cases where the generators are in any representation of \( \mathcal{G} \). In that case, the Hopf structure remains to be determined. Finally, it would be of some interest to look for an extension to affine \( \mathcal{W} \)-algebras.

Let us end with two comments concerning applications. The first one concerns the representation theory of finite \( \mathcal{W} \)-algebras. Preliminary results have been given in section 6 and deals with the classification of finite dimensional representations of \( \mathcal{W}(sl(2n), n.sl(2)) \). More complete results will be available soon [21]. Secondly, the possibility of carrying out the tensor product of \( \mathcal{W} \) representations, although only in special cases, allows to imagine the construction of spin chain models based on a finite \( \mathcal{W}(sl(np), n.sl(p)) \) algebra.

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Appendices

A The soldering procedure

The soldering procedure [13] allows to compute the Poisson brackets of the \( \mathcal{W} \)-algebras. The basic idea is to implement the \( \mathcal{W}(\mathcal{G}, \mathcal{S}) \)-transformations from \( \mathcal{G} \) ones. Indeed, as the \( \mathcal{W}(\mathcal{G}, \mathcal{S}) \)-algebra can be realised from an Hamiltonian reduction on \( \mathcal{G} \), one can see the \( \mathcal{W} \) transformations as a particular class of (field dependent) \( \mathcal{G} \) conjugations that preserve the constraints we have imposed. Thus, the soldering procedure just says that the PBs of the \( \mathcal{W}(\mathcal{G}, \mathcal{S}) \) algebra can be deduced from the commutators in \( \mathcal{G} \). It applies to any \( \mathcal{W}(\mathcal{G}, \mathcal{S}) \) algebra, but we will focus on the \( \mathcal{W}(sl(np), n.sl(p)) \) ones.

For such a purpose, we define

\[
J = \sum_{a=1}^{(np)^2-1} J^a t_a \text{ where } t_a \text{ are } (np) \times (np) \text{ matrices and } J^a \in \mathcal{G}^* \quad (A.1)
\]
Then, we introduce the highest weight basis for the $sl(2)$ under consideration ($E_{\pm}, H$):

$$J = E_- + \sum_{i=1}^{p(n^2)-1} W^i M_i \text{ with } [E_+, M_i] = 0$$ (A.2)

where $M_i$ are $(np) \times (np)$ matrices and $E_+$ is considered here in the fundamental representation $E_+ = \sum_{i=1}^{np-1} E_{i,i+1}$, with $E_{ij}$ the matrix whose elements are $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

To compute the PB of the generator $W^i$ of the $\mathcal{W}$-algebra, one writes the variation of $J$ under the infinitesimal action of one of the $\mathcal{W}$-generators in two ways, namely:

$$\delta_\varepsilon J = \{ \text{tr}(\varepsilon J), J \}_{PB} = [\varepsilon J, J]$$ (A.3)

where $\{ \text{tr}(\varepsilon J), J \}_{PB}$ is the matrix of PB:

$$\{ \text{tr}(\varepsilon J), J \}_{PB} = \{ \text{tr}(\varepsilon J), W^i \}_{PB} M_i$$ (A.4)

and $[\varepsilon J, J]$ is a commutator of $(np) \times (np)$ matrices:

$$[\varepsilon J, J] = f_{abc}^d \varepsilon^b J^c t_a$$ (A.5)

$\varepsilon$ is an $np \times np$ matrix such that $\delta_\varepsilon J = [\varepsilon J, J]$ keeps the form (A.2) with of course $\delta_\varepsilon E_- = 0$. This matrix $\varepsilon$ has $n(p^2) - 1$ free entries, which is the right number of parameters needed to describe a gauge transformation by a general element in the $\mathcal{W}$-algebra. Identifying the matrix of PB with the commutator of matrices leads to the PB of the $\mathcal{W}$-algebra.

We now use the property $gl(np) \sim gl(n) \otimes gl(p)$ to explicitly compute some of the PBs. In $gl(n) \otimes gl(p)$, a general element can be written as

$$J = \sum_{\alpha=0}^{n^2-1-p^2-1} \sum_{s=0}^{p^2-1} J^{\alpha s} t_\alpha \otimes \tau_s$$ (A.6)

with $t_\alpha, n \times n$ matrices and $\tau_s, p \times p$ matrices. The principal $sl(2)$ in $n.sl(p)$ takes the form

$$H = I_n \otimes h \text{ and } E_{\pm} = I_n \otimes e_{\pm}$$ (A.7)

where $(h, e_{\pm})$ form the principal $sl(2)$ in $sl(p)$, and $I_n$ is the identity in $sl(n)$. Then,

$$J = I_n \otimes e_- - \sum_{k=0}^{p-1} W_k \otimes m_k$$ (A.8)

where $e_-$ is viewed as a $p \times p$ matrix, $e_- = \sum_{i=1}^{p-1} E_{i,i+1}$ and $m_k$ are $p \times p$ matrices representing the highest weights of the principal $sl(2)$ in $sl(p)$. They have been computed in [22]:

$$m_k = \sum_{i=1}^{p-k} a^{i-k}_k E_{i,i+k} \text{ with } a^{i-k}_k = \frac{(i+k-1)!(p-i)!}{(i-1)!(p-k-i)!}$$ (A.9)
\(\mathbb{W}_k\) are \(n \times n\) matrices whose entries \(W^a_k\) \((a = 0, \ldots, n^2 - 1)\) are the \(\mathbb{W}\)-generators (with \(W^a_k\) related to \(C_k+1\) for \(k > 0\) and \(W^0_0 = 0\) by the traceless condition on \(\mathbb{J}\)). Note that the indices run from 0 to \(n^2 - 1\) because we are using \(gl(n)\) indices instead of \(sl(n)\) ones (see appendix 3 for precisions).

Using this notation and demanding that \(\delta_\varepsilon\mathbb{J}\) keeps the form (A.2), one can compute the commutator \([\varepsilon\mathbb{J}, \mathbb{J}]\) to get the relations defining the matrix \(\varepsilon\). These relations are quite awful, but, for our purpose, we just need to compute the matrix elements \([\varepsilon\mathbb{J}, \mathbb{J}]_{1,2}\) and \([\varepsilon\mathbb{J}, \mathbb{J}]_{1,3}\). A rather long calculation leads to:

\[
\{\text{tr}(\mu \mathbb{W}_0), \mathbb{W}_k\}_{PB} = \frac{1}{p}[\mathbb{W}_k, \mu] \quad k = 0, 1, 2, ...
\]

\[
\{\text{tr}(\lambda \mathbb{W}_1), \mathbb{W}_1\}_{PB} = \frac{6}{p(p^2 - 1)} \left( \frac{p^2 - 4}{5} [\mathbb{W}_2, \lambda] + \frac{1}{2} (\{\mathbb{W}_0, \{\mathbb{W}_1, \lambda\}\} + \{\mathbb{W}_1, [\mathbb{W}_0, \lambda]\}) + \right.
\left. - \frac{1}{2} [\mathbb{W}_0, [\mathbb{W}_0, \lambda]] \right)
\]

\[
\{\text{tr}(\lambda \mathbb{W}_1), \mathbb{W}_2\}_{PB} = \frac{6}{p(p^2 - 1)} \left( \frac{3(p^2 - 9)}{14} [\mathbb{W}_3, \lambda] + \{\mathbb{W}_2, [\mathbb{W}_0, \lambda]\} + \frac{1}{2} [\mathbb{W}_0, \{\mathbb{W}_2, \lambda\}] + 
\right.
\left. + \frac{1}{3} \{\mathbb{W}_1, [\mathbb{W}_1, \lambda]\} - \frac{1}{2} [\mathbb{W}_1, [\mathbb{W}_0, [\mathbb{W}_0, \lambda]]] + 
\right.
\left. - \frac{1}{4} [\mathbb{W}_0, [\mathbb{W}_1, [\mathbb{W}_1, \lambda]]] - \frac{1}{12} [\mathbb{W}_0, [\mathbb{W}_0, [\mathbb{W}_1, \lambda]]] \right)
\]

where \(\mu\) (resp. \(\lambda\)) is a \(n \times n\) matrix whose entries \(\mu^a\) (resp. \(\lambda^a\)) are the parameters of the infinitesimal transformations associated to \(W^a_0\) (resp. \(W^a_1\)):

\[
\mu = \mu^a t_a; \quad \lambda = \lambda^a t_a; \quad \mathbb{W}_k = W^a_k t_a \quad k = 0, 1, 2 \quad \text{with} \quad t_0 = \mathbb{I}_n
\]

**B Classical \(\mathcal{W}(sl(np), n.sl(p))\) algebras**

**B.1 Generalities**

As we are using heavily the isomorphism \(gl(np) \sim gl(n) \otimes gl(p)\) for our calculations, we are forced to make use of \(gl(n)\) indices instead of \(sl(n)\) ones. We denote the last index by \(a = 0\). It corresponds to the \(gl(1)\) generator that commutes with \(sl(n)\) in \(gl(n)\). We can consistently extend the definition of the totally (anti-)symmetric tensors \(f\) and \(d\) from \(sl(n)\) to \(gl(n)\) by

\[
d^{ab} = 2 \eta^{ab} \quad f^{ab} = 0 \quad \forall \ a, b = 0, 1, \ldots, n^2 - 1
\]

In the fundamental representation of \(gl(n)\), we have then the decomposition:

\[
t^a t^b = \frac{1}{2} (f^{ab}_c + d^{ab}_c) t^c \quad \text{with} \quad t^0 = \mathbb{I}_n
\]
Then, it is easy to show that the Jacobi identities

\begin{align}
    f^{ab}_c f^{cd}_e + f^{bd}_c f^{ca}_e + f^{da}_c f^{cb}_e &= 0 \tag{B.3} \\
    d^{ab}_c f^{cd}_e + d^{bd}_c f^{ca}_e + d^{da}_c f^{cb}_e &= 0 \tag{B.4}
\end{align}

are still valid for any values of $a, b, d, e = 0, 1, \ldots, n^2 - 1$. If we compute \{\{t^a, t^b\}, t^c\} − \{\{t^c, t^b\}, t^a\} = [[t^a, t^c], t^b]$, we get also the relation between $f$ and $d$ tensors:

\begin{equation}
    d^{ab}_d d^{dc}_e - d^{bc}_d d^{da}_e = f^{ac}_d f^{db}_e \tag{B.5}
\end{equation}

These identities will be the only one needed for our purpose. Note that the identity

\begin{equation}
    f^{ab}_c f_{ab} = c_a \eta_{cd} \tag{B.6}
\end{equation}

is not valid in $gl(n)$ since the left hand side is $0$ for $c = d = 0$.

As an aside comment, let us remark that the isomorphism $gl(np) \sim gl(n) \otimes gl(p)$ together with the above conventions allow us to construct the structure constants of $gl(np)$ from those of $gl(n)$ and $gl(p)$. Indeed let $t^a$ (resp. $\bar{t}^a$ and $T^{(a,q)} = t^a \otimes \bar{t}^q$) be the generators in the fundamental representation of $gl(n)$ (resp. $gl(p)$ and $gl(np)$); let $f^{ab}_c$ (resp. $\bar{f}^{qr}_s$ and $F^{(a,q)(b,r)}_{(c,s)}$) be their structure constants; and let $d^{ab}_c$ (resp. $\bar{d}^{qr}_s$ and $D^{(a,q)(b,r)}_{(c,s)}$) be their totally symmetric invariant tensor. The calculation of $[T^{(a,q)}, T^{(b,r)}]$ and $\{T^{(a,q)}, T^{(b,r)}\}$ show that

\begin{align}
    F^{(a,q)(b,r)}_{(c,s)} &= \frac{1}{2} \left( f^{ab}_c \bar{d}^{qr}_s + d^{ab}_c \bar{f}^{qr}_s \right) \tag{B.7} \\
    D^{(a,q)(b,r)}_{(c,s)} &= \frac{1}{2} \left( f^{ab}_c \bar{f}^{qr}_s + d^{ab}_c \bar{d}^{qr}_s \right) \tag{B.8}
\end{align}

which shows that e.g.

\begin{equation}
    D^{(a,q)(b,r)}_{(0,0)} = 2\eta^{(a,q)(b,r)} = 2\eta^{ab}\eta^{qr} \tag{B.9}
\end{equation}

in agreement with our conventions.

With these conventions and properties, we deduce from the soldering procedure result (A.10, A.12) the PBs

\begin{align}
    \{W^a_0, W^b_k\} &= \frac{1}{p} f^{ab}_c W^c_k \quad k = 0, 1, 2, \ldots \tag{B.10} \\
    \{W^a_1, W^b_1\} &= \frac{6}{p(p^2 - 1)} \left[ \frac{p^2 - 4}{5} f^{ab}_c W^c_2 + \frac{1}{2} (d^{ab}_{cu} f^{ub}_d - d^{ab}_{cu} f^{ua}_d) W^c_1 W^d_0 + \right. \\
    &\left. + \frac{1}{2} f^{ab}_{cu} f^{bd}_{dv} f^{uv}_e W^c_1 W^d_0 W^e_0 \right] \tag{B.11} \\
    \{W^a_1, W^b_2\} &= \frac{6}{p(p^2 - 1)} \left[ \frac{3(p^2 - 9)}{14} f^{ab}_c W^c_3 + \left( \frac{1}{2} f^{ab}_{cu} d^{ua}_d - f^{ab}_{cu} d^{ua}_d \right) W^c_0 W^d_2 + \frac{1}{6} f^{ab}_u d^{cd} e W^c_1 W^d_1 + \right. \\
    &\left. + \left( \frac{1}{2} f^{ab}_{cu} f^{bd}_{ev} f^{uv}_d + \frac{1}{4} f^{ab}_{du} f^{bd}_{ev} f^{uv}_d + \frac{1}{12} f^{ab}_{cu} f^{bd}_{ev} f^{uv}_d \right) W^c_0 W^d_0 W^e_1 \right] \tag{B.12}
\end{align}
We repeat that the indices run from 0 to \( n^2 - 1 \).

Noting the identity (proved using (B.5) and the commutativity of the product)

\[
f_{ab}c W_0^c W_0^d W_0^e = \left( \frac{1}{2} f_{ab}^c \delta_{ce} \delta_{de} - \frac{3}{4} (d_{uv} f_{ub}^c - d_{uv} f_{ua}^c) d_{de} \right) W_0^c W_0^d W_0^e
\]

and performing a change of basis

\[
\tilde{W}^a_1 = \frac{p(p^2 - 1)}{6} W_1^a + \frac{p}{4} d_{bc} W_0^b W_0^c
\]

\[
\tilde{W}^a_2 = \frac{p(p^2 - 1)(p^2 - 4)}{6} W_2^a + 5 d_{bc} \tilde{W}^b_1 \tilde{W}^c_1 + \frac{5p(p^2 - 4)}{24} d_{bu} d_{cd} W_0^b W_0^c W_0^d W_0^e
\]

\[
\tilde{W}^a_3 = \frac{p(p^2 - 1)(p^2 - 4)(p^2 - 9)}{6} W_3^a
\]

we obtain the PB:

\[
\{ W^a_1, W^b_1 \} = \frac{1}{5} f^{ab} c W_2^c - \frac{p^3}{16} (d_{uv} f_{bc} - d_{uv} f_{cu}) d_{de} W_0^c W_0^d W_0^e
\]

Note that in this basis, we have \( W^0_1 = C_2 \) (i.e. \( W^0_1 \) is central).

Now, as the relations that we have to verify are different if \( n \) is 2 or not, we specify both cases. We begin with the general case.

### B.2 The generic case \( n \neq 2 \)

One has to verify that the PB (B.17) obeys to the defining relations of the Yangian. We rewrite (B.3) as

\[
f^{bc} d \{ Q^a_1, Q^d_1 \} + \text{circ. perm.} (a, b, c) = f^{a}_{q_d f^{b}_{r_x} f^{c}_{s_y} f^{x y d}} Q^q_0 Q^r_0 Q^s_0
\]

Plugging the PB into the left hand side of (B.18) leads to

\[
\text{lhs} = -\beta_0^2 \beta_1 \frac{p^3}{16} f^{bc} d (d_{uv} f_{d \mu} - d_{uv} f_{a \mu}) + \text{circ. perm.} (a, b, c) d^{a \rho} W_0^\pi W_0^\nu W_0^\rho
\]

This has to be compared with

\[
\text{rhs} = \beta_0^3 f^{a}_{q_x f^{b}_{r_y} f^{c}_{s_z} f^{x y z}} W_0^q W_0^r W_0^s
\]

where we have used latin (resp. greek) letters for \( sl(n) \) (resp. \( gl(n) \)) indices.

---

\(^4\)We keep the notation \( W^a_j \) for \( \tilde{W}^a_j \): throughout the text it is \( W^a_j \) which is used, except in the equations (B.10-B.12) and the convention (A.13).
To prove the equality between lhs and rhs, we first remark, using the Jacobi identity for \( f \), that the index 0 can be dropped from lhs, or equivalently added to rhs. We choose to use \( gl(n) \) indices, and come back to latin letters to denote them.

\[
\text{lhs} = -\beta_0 \beta_1^2 \frac{D^2}{16} f_{ab}^d \left( d_{cy}^x f_{dq}^d - d_{dy}^x f_{qc}^d \right) d_{rs}^r W_0^q W_0^r W_0^s + \text{circ. perm.} \ (a, b, c)
\]

\[
= -\beta_0 \beta_1^2 \frac{D^2}{8} f_{ab} d_{xy} d_{fc}^c y_q d_{rs}^r W_0^q W_0^r W_0^s + \text{circ. perm.} \ (a, b, c)
\]

(B.19)

where we have used the Jacobi identities (B.3-B.4). With (B.3) and the symmetry in \( (q, r, s) \), one can rewrite rhs as:

\[
\frac{1}{\beta_0^3} \text{rhs} = f_{aq} f_{br} (d_{cx}^y d_{dy}^z - d_{cy}^x d_{yx}^z) W_0^q W_0^r W_0^s
\]

\[
= \left( \frac{1}{2} d_{xy}^c f_{br} (d_{dy}^z f_{cq}^b + d_{yz}^y f_{yc}^b) - \frac{1}{2} d_{xy}^c d_{sq} (d_{xq}^f f_{br} + d_{yx}^y f_{rs}^f) \right) W_0^q W_0^r W_0^s
\]

\[
= \frac{1}{2} R_{abc}^d d_{rs}^r W_0^q W_0^r W_0^s
\]

(B.20)

Using the Jacobi identity (B.4), we have

\[
R_{abc}^d = f_{by} (f_{ac} d_{dy}^y + f_{bc} d_{yz}^y d_{dy}^y) - (a \leftrightarrow b)
\]

\[
= (f_{ac} f_{by} - f_{bc} f_{ay}) y_d d_{qy}^y + (f_{by} f_{ac} y_d + f_{bc} f_{ay} y_d) d_{qy}^y
\]

\[
= (f_{ac} f_{by} - f_{bc} f_{ay}) y_d d_{qy}^y + f_{ab} f_{dx} d_{qy}^y
\]

(B.21)

Now, since rhs is invariant under cyclic permutations of \( (a, b, c) \), we can write

\[
6 \text{rhs} = \beta_0^3 \left( 2 f_{ab}^d f_{cy}^x d_{yq}^d - f_{ab}^y f_{dx} d_{qy}^y + \text{circ. perm.} \ (a, b, c) \right) d_{rs}^r W_0^q W_0^r W_0^s
\]

\[
= \beta_0^3 f_{ab}^d (-2 f_{cy}^x d_{yq}^d - f_{yq}^d d_{xy}^q + \text{circ. perm.} \ (a, b, c)) d_{rs}^r W_0^q W_0^r W_0^s
\]

\[
= -3 \beta_0^3 f_{ab}^d f_{cy}^x d_{yq}^d d_{rs}^r W_0^q W_0^r W_0^s + \text{circ. perm.} \ (a, b, c)
\]

(B.22)

From the normalisation \( \beta_0 = p \), we deduce that lhs and rhs are equal when \( \beta_1^2 = 4 \), i.e. for

\[
Q_0^a = p W_0^a \quad \text{and} \quad Q_1^a = 2 W_1^a
\]

(B.23)

which ends the proof for the generic case.

### B.3 The particular case of \( Y(sl(2)) \)

As a normalisation, we take for the fundamental representation of \( gl(2) \) the matrices:

\[
t^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad t^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad t^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ; \quad t^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

(B.24)
We have in that case
\[ a^{abc} = 2 \delta^a_0 \eta^{bc} + \text{circ. perm.} \ (a, b, c) \quad \text{and} \quad c_v = -8 \] (B.25)

Then, using the special property \[ f^{ij}_m f^m_{kl} = -4 (\eta^i_k \eta^j_l - \eta^i_l \eta^j_k) \] valid in \( sl(2) \) (i.e. when none of the index is 0), we get the PB

\[ \{ W^i_1, W^j_1 \} = f^{ij}_k \left[ \frac{1}{5} W^k_2 - \frac{p^3}{2} (\bar{W}^2_0 \cdot \bar{W}^2_0) W^k_0 \right] \] (B.26)

where \( \bar{x} \cdot \bar{y} = x^1 y_1 + x^2 y_2 + x^3 y_3 \) and the indices \( i, j, k \) now run from 1 to 3.

Note that for \( p = 2 \), once the constraint \( W^a_0 = 10 [W^0_0 W^1_1 + \delta^a_0 (\bar{W}^1 \cdot \bar{W}^1_0)] \) is applied, we recover the algebra presented in section 2.4, up to the normalisation \( J^i = W^i_0, S^i = 2 W^i_1, C_2 = W^0_1 \) and \( f^{ij}_k = 2 i \varepsilon^{ij}_k \).

After multiplication by \( f^{ij}_k f^m_{mn} \), the relation (3.4) can be rewritten as:

\[ f^{ij}_k \{ \{ W^i_1, W^j_1 \}, W^l_1 \} + f^{ij}_l \{ \{ W^i_1, W^j_1 \}, W^k_1 \} = 32 (f^{ij}_k \eta^l_m + f^{ij}_l \eta^k_m) W^m_0 W^j_0 W^i_1 \] (B.27)

Using the above PB and the normalisation \( \beta_0 = p \), we get:

\[ \begin{align*}
\text{lhs} & = c_v \beta_1^3 \left[ \frac{1}{5} \left( \{ W^k_2, W^l_1 \} + \{ W^l_2, W^k_1 \} \right) - p^2 \left( f^{ij}_k \eta^l_m + f^{ij}_l \eta^k_m \right) W^m_0 W^j_0 W^i_1 \right] \\
\text{rhs} & = 32 p^2 \beta_1 (f^{ij}_k \eta^l_m + f^{ij}_l \eta^k_m) W^m_0 W^j_0 W^i_1
\end{align*} \] (B.28)

Thus, we need to simplify the PBs (B.12) in the new basis (B.14-B.15). For \( sl(2) \) it takes the form:

\[ \{ W^i_1, W^j_2 \} = f^{ij}_k \left[ \frac{3}{14} W^k_3 + 3 W^0_2 W^k_0 + \frac{6(3 p^2 - 7)}{p(p^2 - 1)} W^k_1 W^0_1 + \frac{(p^2 - 9)(p^2 - 4)}{2(p^2 - 1)} W^k_1 \bar{W}^2_0 + \right. \\
\left. - 30 W^k_0 (W^1 \cdot \bar{W}^1_0) \right] \] (B.29)

so that it does not contribute to lhs. Hence, we have

\[ \text{lhs} = -c_v \beta_1^3 p^2 (f^{k\eta^l_m} + f^{k\eta^l_m}) W^m_0 W^j_0 W^i_1 \] (B.30)

The relation (B.27) is then satisfied for

\[ Q^0_0 = p W^a_0 \quad \text{and} \quad Q^i_0 = 2 W^i_1 \] (B.31)

which is the same normalisation as for the generic case.
C  Quantum $\mathcal{W}(sl(np), n.sl(p))$ algebras

In the quantum case, one has to check that the corrections to the leading terms in the $\mathcal{W}$-algebras do not perturb the defining relations of the Yangian\footnote{More exactly that the modification is the same as the one introduced in replacing the commutative product in $\mathcal{U}(G^\ast)$ by the (symmetrised) non Abelian product of $\mathcal{U}(G)$.}.

- **In the general case** $n \neq 2$, the calculation is quite easy. Indeed, the commutator takes the form

\[
[W^a_1, W^b_1] = -\frac{p^3}{16} \left( d^{mu}_v f^b_{cv} - d^{mu}_v f^a_{cv} \right) d^{\nu}_i d^{\rho}_{i\alpha} \eta_{\alpha c} c_3(W^c_0, W^d_0, W^e_0) + \]

\[
f^{ab}_{\mu} \left[ \frac{1}{5} W^2_2 + \mu_1 W^c_1 + \mu_2 d^{c}_{de} s_2(W^d_0, W^e_0) + \mu_3 W^c_0 \right]
\]

where $\mu_i$ ($i = 1, 2, 3$) are undetermined constants. However, one remarks that the terms containing $f^{ab}_{\mu}$ in $[W^a_1, W^b_1]$ do not contribute to (3.3). Since these are the only type of terms we add, the calculation is identical to the classical one (up to symmetrization of the products).

- **In the case of** $\mathcal{W}(sl(2p), 2.sl(p))$, we need a little more. Due to the calculations done in the classical case, we already know that proving (3.4) amounts to show that

\[
\mu_1 \left( [W^i_1, W^j_1] + [W^j_1, W^i_1] \right) + \mu_3 \left( [W^i_0, W^j_1] + [W^j_0, W^i_1] \right) + \frac{1}{5} \left( [W^i_2, W^j_1] + [W^j_2, W^i_1] \right) = 0 \quad (C.2)
\]

where the indices run from 1 to 3. The terms corresponding to $\mu_1$ and $\mu_3$ disappear because of the antisymmetry in $(i, j)$. Thus, we just need to compute the corrections to the commutator $[W^a_2, W^b_1]$. Using the results of appendix $[\ref{D}]$, we compute the most general form of this commutator:

\[
[W^a_1, W^b_2] = f^{ab}_{\mu} \left[ \frac{3}{14} W^c_5 + \frac{6(3p^2 - 7)}{p(p^2 - 1)} s_2(W^c_1, W^0_1) + 3 s_2(W^c_0, W^0_0) + \right.
\]

\[
\left. + \left( \frac{(p^2 - 9)(p^2 - 4)}{2(p^2 - 1)} \eta^{\mu}_{\nu} \eta^{\nu}_{\nu} - 30 \eta^{\mu}_{\nu} \eta^{\nu}_{\nu} \right) s_3(W^0_0, W^d_0, W^e_0) \right]
\]

\[
+ f^{ab}_{\mu} \left( \nu_1 W^c_2 + \nu_1' W^c_1 + \nu_1'' W^c_0 \right) + \nu_2 f^{ab}_{\mu} \eta_{de} s_3(W^c_0, W^d_0, W^e_0) + \]

\[
+ \left( \nu_3 f^{ab}_{\mu} \eta_{cd} + \nu_3' \eta^{ab}_{\mu} \eta_{cd} + \nu_3'' \eta^{ab}_{\mu} \eta_{de} \right) W^c_1 W^d_0 + \]

\[
+ \left( \nu_4 \eta^{ab}_{\mu} \eta_{cd} + \nu_4' \eta^{ab}_{\mu} \eta_{cd} + \nu_4'' \eta^{ab}_{\mu} \eta_{de} \right) s_2(W^c_0, W^d_0)
\]

with indices running from 0 to 3. Looking at (C.2), one sees that some of the new terms that may appear in the right hand side of the commutator do contribute to (C.2). Thus, one has to check that they are not in the true commutator. It is done thank to the Jacobi identity based on the results of appendix D.
on \((W^a, W^b, W^c)\) which shows (for \(a, b, c\) all different) that\(^6\) \(\nu_3 = \nu''_3 = \nu_4 = \nu'_4 = 0\). We deduce that the commutator takes the form:

\[
[W^a_1, W^b_2] = f^{abc} \left[ \frac{3}{14} W^c_3 + \frac{6(3p^2 - 7)}{p(p^2 - 1)} s_2(W^c, W^1) + 3 s_2(W^0_2, W^c) + \right. \\
+ \left. \left( \frac{(p^2 - 9)(p^2 - 4)}{2(p^2 - 1)} \eta_{de} \eta_e - 30 \eta_{de} \eta_g \right) s_3(W^g_0, W^d_0, W^e_1) + \right. \\
+ \left. \nu_1 W^c_2 + \nu'_1 W^c_1 + \nu''_1 W^c_0 + \nu_2 \eta_{de} s_3(W^r_0, W^d_0, W^e_1) + \nu_3 \eta_{de} W^d_0 W^e_1 \right]
\]

so that (C.2) and hence (3.4) are satisfied.

\section{Tensor products of some finite dimensional representations of \(sl(n)\)}

We want here to compute the tensor product of the \(G\)-adjoint representation by itself several times, for \(G = sl(n)\). We will also need to select the totally symmetric part of these products.

For such a purpose, we use Young diagrams, which allow us to determine the decompositions:

\[
G \otimes G = (2, 0, \ldots, 0, 2) \oplus (1, 0, \ldots, 0, 1) \oplus (2, 0, \ldots, 0, 1, 0) \oplus (0, 1, 0, \ldots, 0, 2) \oplus (0, 1, 0, \ldots, 0, 1, 0) \oplus (0, 0, \ldots, 0, 0)
\]

where we have denoted by \(G = (1, 0, \ldots, 0, 1)\) the adjoint representation.

It remains to select the (anti-)symmetric part of these products. For \(G \otimes G\), the calculation has already been done (see e.g. \(23\)) and reads:

\[
S_2(G) = (G \otimes G)_{sym} = (2, 0, \ldots, 0, 2) \oplus (1, 0, \ldots, 0, 1) \oplus (0, 1, 0, \ldots, 0, 1, 0) \oplus (0, 0, \ldots, 0) \quad (D.1)
\]

\[
\Lambda_2(G) = (G \otimes G)_{skew} = (1, 0, \ldots, 0, 1) \oplus (2, 0, \ldots, 0, 1, 0) \oplus (0, 1, 0, \ldots, 0, 2) \quad (D.2)
\]

As far as \(S_3(G)\) is concerned, we already now that this sum of representations belongs to \((S_2(G) \otimes G)_{sym}\), which decomposes as

\[
(S_2(G) \otimes G)_{sym} = (3, 0, \ldots, 0, 3) \oplus 3 (2, 0, \ldots, 0, 2) \oplus 3 (1, 0, \ldots, 0, 1) \oplus 3 (0, 1, 0, \ldots, 0, 1, 0) \oplus \\
\ominus 2 (1, 1, 0, \ldots, 0, 1, 0) \oplus 2 [(2, 0, \ldots, 0, 1, 0) \oplus (0, 1, 0, \ldots, 0, 2)] \oplus \\
\oplus (0, 0, 1, 0, \ldots, 0, 1, 0) \oplus [(3, 0, \ldots, 0, 1, 1) \oplus (1, 1, 0, \ldots, 0, 3)] \oplus \\
\oplus [(1, 1, 0, \ldots, 0, 1, 0, 0) \oplus (0, 0, 1, 0, \ldots, 0, 1, 1)] \oplus (0, 0, \ldots, 0) \quad (D.3)
\]

This implies that we must have

\[
S_3(G) = a (3, 0, \ldots, 0, 3) \oplus b (2, 0, \ldots, 0, 2) \oplus c (1, 0, \ldots, 0, 1) \oplus d (0, 1, 0, \ldots, 0, 1, 0) \oplus \]

\(^6\)Let us note en passant that the Jacobi identity has just removed in the new terms those which are symmetric in \(a, b\).
\[ \oplus e(1, 1, 0, \ldots, 0, 1, 1) \oplus f(0, 0, 1, 0, \ldots, 0, 1, 0, 0) \oplus m(0, \ldots, 0) \oplus \]
\[ \oplus g[(2, 0, \ldots, 0, 1, 0) \oplus (0, 1, 0, \ldots, 0, 2)] \oplus \]
\[ \oplus h[(3, 0, \ldots, 0, 1, 1) \oplus (1, 1, 0, \ldots, 0, 3)] \oplus \]
\[ \oplus i[(1, 1, 0, \ldots, 0, 1, 0, 0) \oplus (0, 0, 1, 0, \ldots, 0, 1, 1)] \]
\[ \text{(D.4)} \]

with each multiplicity in \( \text{(D.4)} \) lower or equal to the corresponding multiplicity in \( \text{(D.3)} \). But we know the dimension of \( S_3(\mathcal{G}) \): it is the dimension of a totally symmetric tensor with 3 indices in a space of dimension \( \dim \mathcal{G} = n^2 - 1 \), i.e. \( \frac{n^2(n^2-1)}{6} \). Computing this dimension with \( \text{(D.4)} \) leads to only two possible solutions for the parameters: \( a = e = f = 1 \), \( h = i = 0 \), \( b = d = m \), \( c = 3 - m \) and \( g = 2 - m \) with \( m = 0 \) or \( 1 \). As \( m \) is the multiplicity of the trivial representation in \( S_3(\mathcal{G}) \), we deduce that, for \( \mathcal{G} = sl(n) \), \( n \neq 2 \), we have \( m = 1 \) (since \( d_{abc} \) belongs to this space). Thus

\[ S_3(\mathcal{G}) = (3, 0, \ldots, 0, 3) \oplus (2, 0, \ldots, 0, 2) \oplus 2(1, 0, \ldots, 0, 1) \oplus (0, 1, 0, \ldots, 0, 1, 0) \oplus \]
\[ \oplus [(1, 1, 0, \ldots, 0, 1, 1) \oplus (0, 0, 1, 0, \ldots, 0, 1, 0)] \oplus \]
\[ \oplus [(2, 0, \ldots, 0, 1, 0) \oplus (0, 1, 0, \ldots, 0, 2)] \oplus (0, \ldots, 0) \]
\[ \text{(D.5)} \]

Finally, the multiplicity of the trivial representation in the tensor products occuring in section \( \text{[5]} \) is computed through the remark that, in \( sl(n) \), the tensor product of two finite dimensional irreducible representations \( R \) and \( R' \) contains the trivial representation if and only if \( R \) and \( R' \) are conjugate. In that case, the multiplicity is 1. This leads to the multiplicities given in \( \text{(5.4)} \). We give also a basis for the corresponding spaces. To be complete, let us mention the bases:

\[
\Lambda_2(\mathcal{G}) \otimes \Lambda_2(\mathcal{G}) = \begin{cases}
\lambda_{ab}^{cd} = f_{a}^{e} f_{c}^{d} & \\
\lambda_{ab}^{cd} = d_{a}^{e} d_{c}^{d} - d_{c}^{e} d_{a}^{d} & \\
\lambda_{ab}^{cd} = f_{a}^{e} d_{c}^{d} - f_{c}^{e} d_{a}^{d} & \\
\end{cases}
\text{(D.6)}
\]

\[
\Lambda_2(\mathcal{G}) \otimes S_3(\mathcal{G}) = \begin{cases}
\lambda_{ab}^{cde} = f_{a}^{e} c_{d}^{de} + \text{circ. perm.} (c, d, e) & \\
\lambda_{ab}^{cde} = f_{a}^{e} g_{d}^{b} d_{m}^{c} m_{de} + \text{circ. perm.} (c, d, e) & \\
\lambda_{ab}^{cde} = (\eta_{b}^{a} d_{de} - \eta_{d}^{a} c_{de}) + \text{circ. perm.} (c, d, e) & \\
\lambda_{ab}^{cde} = (f_{a}^{g} d_{gb}^{b} m - f_{b}^{g} d_{ga}^{c} m) m_{de} + \text{circ. perm.} (c, d, e) & \\
\end{cases}
\text{(D.7)}
\]

In the case of \( sl(2) \), we need more informations. Fortunately, the calculation is easier in that case, and we can go further. Indeed, we have \( \text{(with } \mathcal{D}_j \text{ the } (2j + 1)\text{-dimensional representation of } sl(2)) \):

\[ (\mathcal{D}_1 \times \mathcal{D}_1)_{\text{sym}} = \mathcal{D}_0 \oplus \mathcal{D}_2 \quad (\mathcal{D}_1 \times \mathcal{D}_1)_{\text{skew}} = \mathcal{D}_1 \quad S_3(\mathcal{D}_1) = \mathcal{D}_1 \oplus \mathcal{D}_3 \quad (\text{D.7}) \]

\( \text{The case } \mathcal{G} = sl(2) \text{ is treated below.} \)

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which leads to the multiplicities and tensors:

\[ M_0[(D_1 \times D_1)_{\text{skew}} \times (D_1 \times D_1)_{\text{skew}}] = 1 : \quad f^{ab}_u f^{u}_{cd} \sim \eta^a_c \eta^b_d - \eta^a_d \eta^b_c \]

\[ M_0[(D_1 \times D_1)_{\text{sym}} \times (D_1 \times D_1)_{\text{sym}}] = 2 \quad \left\{ \begin{array}{l}
\eta^{ab} \eta_{cd} \\
\eta^{a}_c \eta^b_d + \eta^{a}_d \eta^b_c
\end{array} \right. \]

\[ M_0[(D_1 \times D_1)_{\text{skew}} \times (D_1 \times D_1)_{\text{sym}}] = 0 \quad - \]

\[ M_0[(D_1 \times D_1)_{\text{skew}} \times S_3(D_1)] = 1 \quad : \quad f^{ab}_c \eta_{de} + \text{circ. perm.} \ (c, d, e) \]

\[ M_0[(D_1 \times D_1)_{\text{sym}} \times S_3(D_1)] = 0 \quad - \]

\[ (D.8) \]

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