Gauge invariant operators in field theories on non-commutative spaces

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Abstract

We review some selected aspects of the construction of gauge invariant operators in field theories on non-commutative spaces and their relation to the energy momentum tensor as well as to the non-commutative loop equations.

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1 Introduction

We consider $U(N)$ gauge theory on the simplest non-commutative space, flat space with the commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

among its coordinates. Throughout the following discussions use is made of the $\star$-product formulation. The gauge field transforms under gauge transformations as

$$A_\mu \to \Omega \star A_\mu \star \Omega^\dagger + i \Omega \star \partial_\mu \Omega^\dagger, \quad \Omega \star \Omega^\dagger = 1$$

and the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - iA_\mu \star A_\nu + iA_\nu \star A_\mu$ as

$$F_{\mu\nu} \to \Omega \star F_{\mu\nu} \star \Omega^\dagger.$$  

The non-commutative $\star$-product under integration over the whole space obeys

$$\int dx \ f \star g = \int dx \ fg = \int dx \ g \star f.$$  

Local operators like $\text{tr} F^2$ being gauge invariant in standard gauge theories are no longer gauge invariant, however due to (4) gauge invariance is restored after integration over the whole space.

The situation is similarly for the Wilson functionals

$$U[C, A] = P_* \exp \left( i \int_0^1 A_\mu(\xi(t)) \frac{d\xi^\mu}{dt} dt \right)$$

where $P_*$ denotes path ordering along the contour $C$ defined by $\xi(t)$ from right to left with respect to increasing $t$ of $\star$-products of functions. The star multiplication is performed with respect to a constant mode in $\xi(t).$ Under gauge transformations $U[C, A]$ transforms as

$$U[C, A] \to \Omega(\xi(1)) \star U[C, A] \star \Omega^\dagger(\xi(0)).$$

For closed contours $\xi(0) = \xi(1)$ besides the trace one has again to perform an integration over the whole space to get a gauge invariant quantity.

In addition there appears the new possibility to relate gauge invariant objects also to open contours $\int$

$$W[C, A] = \frac{1}{N} \int dx \ \text{tr} \ U[x + C, A] \star e^{-ik\xi} \ \text{with} \ \xi(1) - \xi(0) = \theta k \xi.$$ 

The gauge invariance of $W$ is a consequence of (3),(4) and

$$e^{ikx} \star f(x) \star e^{-ikx} = f(x + \theta k).$$

3If $\xi(t) = x + \eta(t) = x' + \eta'(t)$ taking the $\star$-products w.r.t. $x$ or $x'$ yields the same result.
2 Localized gauge invariant operators

Here one starts with local $\mathcal{O}(x)$ transforming under (2) in the adjoint, i.e. $\mathcal{O}(x) \rightarrow \Omega(x) \star \mathcal{O}(x) \star \Omega^\dagger$. Then the trace of the Fourier transform

$$\int dx \, \text{tr} \mathcal{O}(x) e^{-ikx}$$

is gauge invariant only for $k = 0$. To construct some generalization which is invariant for all $k$ we follow [2, 3] and repair the mismatch of the gauge functions $\Omega$ by inserting a suitable adapted Wilson functional, i.e.

$$\text{tr} \tilde{\mathcal{O}}(k) = \int dx \, \text{tr}(U[x + C, A] \star \mathcal{O}(x)) \star e^{-ikx}. \quad (9)$$

The above construction is gauge invariant for each contour $C$ with

$$\xi(1) = \theta k, \quad \xi(0) = 0. \quad (10)$$

Applying to $\text{tr} \tilde{\mathcal{O}}(k)$ the usual inverse Fourier transformation one arrives at the gauge invariant coordinate space operator

$$\text{tr} \hat{\mathcal{O}}(y) = \frac{1}{(2\pi)^D} \int dk \, \text{tr} \tilde{\mathcal{O}}(k) \, e^{iky}. \quad (11)$$

Among the contours $C$ obeying (10) the straight ones are distinguished. Only then the construction with $\mathcal{O}$ inserted at an endpoint can equally be replaced by a setup where $\mathcal{O}$ is inserted at an arbitrary point of the contour [3]. Another benefit of using straight contours in (9) is related to the use of covariant coordinates in the sense of ref. [4]

$$X^\mu = x^\mu + (\theta A(x))^\mu. \quad (12)$$

One can prove for the exponential $\star$-power series of $-ikX$ the remarkable identity [5]

$$e^{-ikX} = e^{-ikx} \star U(k, x), \quad (13)$$

with

$$U(k, x) = U[x, C], \quad C: \xi(t) = \theta kt. \quad (14)$$

Then the construction of $\hat{\mathcal{O}}(x)$ out of $\mathcal{O}(x)$ can be summarized by

$$\text{tr} \hat{\mathcal{O}}(y) = \int dx \, \text{tr} \mathcal{O}(x) \, \delta_\star(X - y). \quad (15)$$

Replacing the $\delta_\star$-function by some smooth regularization one gets the pseudo localized operators studied in more detail in ref. [6].

\textsuperscript{4}Here we understand local on a technical level as operators built out of $\star$ powers of the gauge field and its derivatives.
3 Energy momentum tensor

The energy momentum tensor of non-commutative gauge theories has been studied both from the string theoretical [7] and the field theoretical [9] point of view. The resulting expressions are different since the leading order in $\alpha'$ studied so far is not seen on the pure field theoretical level. Therefore it would be interesting to extend the string calculation to the next-leading contributions. We now comment the field theory analysis.

Applying the Noether procedure combined with a suitably adapted gauge transformation one gets

$$D_{\mu} T^{\mu\nu} = \partial_{\mu} T^{\mu\nu} - iA_{\mu} \star T^{\mu\nu} + iT^{\mu\nu} \star A_{\mu} = 0,$$

with

$$T^{\mu\nu} = 2\{F^{\mu\rho}, F_{\rho}^{\nu}\} - \eta^{\mu\nu} F^{\alpha\beta} \star F_{\alpha\beta}.$$  

(16)

This symmetric tensor $T^{\mu\nu}$ even after taking the trace is not gauge invariant. It also does not fulfill the standard local conservation law. With the technique presented in the last chapter we find the gauge invariant tensor

$$\hat{T}^{\mu\nu}(y) = \frac{1}{(2\pi)^D} \int dk dx e^{iky} e^{-ikx} \star \text{tr}(U(k, x) \star T^{\mu\nu}(x)).$$

(18)

$\partial_{\mu} \hat{T}^{\mu\nu}(y)$ turns out to be different from zero but equal to a derivative. Therefore after a straightforward redefinition we end up with

$$T^{\mu\nu}(y) = \frac{1}{(2\pi)^D} \int dk dx e^{iky} e^{-ikx} \star \text{tr}\left[U(k, x) \star T^{\mu\nu}(x) - \theta^{\mu\alpha} P_\star \left(\int_0^1 ds F_{\alpha\beta}(x + s\theta k) U(k, x) T^{\beta\nu}(x)\right)\right].$$

(19)

as our energy momentum tensor. $T^{\mu\nu}$ is gauge invariant and locally conserved

$$\partial_{\mu} T^{\mu\nu} = 0,$$

(20)

The price for enforcing (20) is the loss of the symmetry of the tensor. For more discussions of the interplay between local conservation and symmetry see [9].

4 Loop equations

In the previous sections we concentrated ourselves on the use of Wilson functionals as building blocks in the construction of localized gauge invariant operators. Now we look closer on the dependence of the Wilson functionals on the shape of the contours. In analogy to standard Yang-Mills gauge theories one expects that the dynamics can be encoded in equations containing second variational derivatives with respect to the contour. We now closely follow the steps known in the standard case [8]. The geometrical setting
to derive a formula for the derivative with respect to the insertion of an area derivative is unchanged, hence

$$
\frac{\partial^\mu \delta W[C]}{\delta \sigma^{\mu \nu}(\xi(t))} = \frac{i}{N} \int dx \text{tr} P \left( D^\mu F_{\mu \nu}(x + \xi(t)) \exp \left( i \int_C A_{\mu}(x + \xi(s)) d\xi^\mu(s) \right) \right).
$$

(21)

In the quantized version of the standard Yang-Mills theory the equation of motion gives rise to contact terms constituting a nontrivial r.h.s. of the loop equations [8]. Under the vacuum expectation value translation invariance is restored as usual. This implies a trivial divergence for the vev of (7) which we cancel by dividing out the space-time volume $V$. Furthermore, a careful analysis of the modifications introduced by the $\star$-product leads for the vev of $W$ for a closed contour to [9]

$$
\frac{1}{V} \frac{\partial^\mu \delta}{\delta \sigma^{\mu \nu}(\xi)} \langle W_c[C] \rangle = -\frac{g^2 N}{V} \int d\eta_\nu \delta(\xi - \eta) \langle W_c[C_\xi] \rangle \langle W_c[C_\eta] \rangle
$$

(22)

$C_{\eta \xi}$ denotes the part of $C$ between $\xi$ and $\eta$. Relative to the commutative case where on the r.h.s. only $\xi = \eta$ contributes, now the former contact terms become smeared in some sense, and we have contributions from all $\eta$. Separating connected and disconnected parts for the correlator on the r.h.s. one finally gets

$$
\frac{1}{V} \frac{\partial^\mu \delta}{\delta \sigma^{\mu \nu}(\xi)} \langle W_c[C] \rangle = -\frac{g^2 N}{V^2} \int d\eta_\nu \delta(\xi - \eta) \langle W_c[C_\xi] \rangle \langle W_c[C_\eta] \rangle
$$

$$
-\frac{g^2 N}{(2\pi)^D V} \int d\eta_\nu \langle W_o[C_\xi] W_o[C_\eta] \rangle_{\text{conn}}.
$$

(23)

It is remarkable that for finite $N$ the new gauge invariant objects for open contours appear to be necessary for the description of the dynamics of closed loops. In the 'tHooft limit $(N \to \infty, \ g^2 N \text{ fix})$ the second term on the r.h.s. is suppressed by a relative $1/N^2$ factor with respect to the first one resulting in just the same equation as in the commutative case.

For the two-point function of $W$’s for closed contours there appears of course on the r.h.s. one term in which the second contour has a pure spectator role. Due to the smearing of the contact term there contributes still another term, even if the two contours have no point in common [4]

$$
\frac{1}{V} \frac{\partial^\mu \delta}{\delta \sigma^{\mu \nu}(\xi)} \langle W_c[C^1] W_c[C^2] \rangle =
$$

$$
-\frac{g^2 N}{(2\pi)^D V} \int d\eta_\nu^1 \langle W_o[C^1_\xi] W_o[C^1_\eta] \rangle \langle W_c[C^2] \rangle
$$

$$
-\frac{g^2}{NV} \int d\eta_\nu^2 \langle W_c[C^1_\xi] \circ (C^2_\eta + \xi^1 - \eta^2) \rangle.
$$

(24)

5Below we use a subscript $c$ and $o$ for emphasizing the closed and open nature of the contour.
Above \( C_1^i \circ (C_2^2 + \xi - \eta) \) denotes the closed contour obtained by starting at \( \xi \) on \( C^1 \) going along the whole \( C^1 \) back to \( \xi \) and then along the with \((\xi - \eta)\) shifted version of \( C^2 \).

The one-point function of \( W \) for open contours vanishes since translation invariance leads to a factor \( \delta(k) \). The correlators of \( W \)'s for several open contours are nonvanishing if the related momenta sum up to zero. These correlation functions have been studied in ref. [11]. Let us first introduce some notation. By \( C_1^{(s)} = \{ \xi_s(t), \ 0 \leq t \leq 1 \} \) we denote the following contour tailored out of \( C = \{ \xi(t), \ 0 \leq t \leq 1 \} \)

\[
\begin{align*}
\xi_s(t) &= \xi(s + t) - \xi(1) + \xi(0), \ 0 \leq t \leq 1 - s \\
\xi_s(t) &= \xi(t - 1 + s), \ 1 - s \leq t \leq 1.
\end{align*}
\]

(25)

If \( C \) is closed \( C_1^{(s)} \) is equal to \( C_{\xi(s)} \) in the sense used in (24). In addition we use the notion \( C_{(st)} \) for the part of \( C \) between \( \xi(\min(s,t)) \) and \( \xi(\max(s,t)) \) as well as \( C_{(st)} \) for the contour obtained from \( C \) by cutting out \( C_{(st)} \) and gluing together the remaining two parts after a suitable translation of one of the partners. \( (k_1)_{(st)} \) stands for the momentum related to \( C_{(st)}^{1} \). Then the equation for the two-point function, after making use of the cyclic symmetry [11]

\[
W[C] = W[C_1^{(s)}],
\]

(26)

takes the form

\[
\begin{align*}
\frac{1}{V} \delta^\mu & \frac{\delta}{\delta \sigma^{\mu\nu}(\xi^1(t))} \langle W_\alpha[C^1]W_\alpha[C^2]\rangle = \\
&- \frac{g^2 N}{(2\pi)^2 V \det \theta} \int_{C^1} d\xi^1_s(s) \langle W_\alpha[C_{(st)}^{1}]W_\alpha[C^1/C_{(st)}^{1}]W_\alpha[C^2]\rangle e^{-\frac{i}{\theta}(\xi(t) - \xi^1(s))} \\
&- \frac{g^2}{NV} \int_{C^2} d\xi^2_s(s) \langle W_\alpha[C_{(st)}^{1}] \circ (C_2^2 + \xi^2(t) - \xi^2(s) - \xi^1(1) + \xi^1(0))\rangle e^{-ik_2(\xi^1(t) - \xi^2(s))}.
\end{align*}
\]

(27)

In the limit of closed contours \( C^1 \) and \( C^2 \) (27) equals (24). Note further that \( k_1 + k_2 = 0 \) always implies \( \xi^1_1(1) - \xi^1(0) = \xi^2(0) - \xi^2(1) \). Therefore the second term on the r.h.s. involves the Wilson functional for a closed contour.

The above equation exhibits a remarkable feature [11] in the t’Hooft limit. The leading contributions to both the l.h.s as well as to the first term of the r.h.s. are \( O(\frac{1}{N^2}) \). The connected parts to the first term of the r.h.s. are of order \( O(\frac{1}{N^2}) \). There is also an \( O(\frac{1}{N^2}) \) term built from a disconnected part which is proportional to the original correlation function if \( C^1 \) has no intersections. Then, at least in this restricted geometrical setting, the leading term to the open contour correlator can be expressed in terms of a closed contour functional.

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\[ ^6 \text{After adapting the notation: } \xi^1(t) \text{ is called } \xi^1 \text{ and } \xi^1(s) \eta^1 \text{ in [24].} \]
Further applications of non-commutative Wilson functionals

There are numerous further instances where Wilson functionals for closed and open contours play a crucial role. Among them are the construction of an explicit formula for the Seiberg-Witten map and effective actions for non-commutative gauge theories. We want to close this short note with a proposal for a constraint picking out non-commutative $SU(N)$ configurations out of general $U(N)$ configurations [12]. It has the same structure as (9), only the candidate gauge invariant operator $O$ is replaced by the gauge field $A$ itself

$$\int dx \, \text{tr} \left( U(k, x) \star A(x) \star e^{-ikx} \right) = 0, \quad \forall k.$$  (28)

The allowed gauge transformations $\Omega(x)$ then have to satisfy

$$\int dx \, \text{tr} \left( U(k, x) \star \Omega \star d\Omega^\dagger(x) \star e^{-ikx} \right) = 0, \quad \forall k.$$  (29)

The last condition closes under the composition of two gauge transformations.

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Note that in the present review we have chosen path ordering from right to left.
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