Simultaneous Embeddings with Vertices Mapping to Pre-Specified Points

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Abstract. We discuss the problem of embedding graphs in the plane with restrictions on the vertex mapping. In particular, we introduce a technique for drawing planar graphs with a fixed vertex mapping that bounds the number of times edges bend. An immediate consequence of this technique is that any planar graph can be drawn with a fixed vertex mapping so that edges map to piecewise linear curves with at most $3n + O(1)$ bends each. By considering uniformly random planar graphs, we show that $2n + O(1)$ bends per edge is sufficient on average.

To further utilize our technique, we consider simultaneous embeddings of $k$ uniformly random planar graphs with vertices mapping to a fixed, common point set. We explain how to achieve such a drawing so that edges map to piecewise linear curves with $O(n^{1 - \frac{1}{k}})$ bends each, which holds with overwhelming probability. This result improves upon the previously best known result of $O(n)$ bends per edge for the case where $k \geq 2$. Moreover, we give a lower bound on the number of bends that matches our upper bound, proving our results are optimal.

1 Introduction

Of fundamental importance to graph drawing is the problem of drawing graphs in the plane with restrictions on how vertices and edges are embedded. Indeed, discussions on planar embeddings, where vertices map to points and edges map to continuous non-crossing curves, were commensurate with the introduction of graph theory [5].

Bridges and Prussian cities aside, investigation into the properties of planar embeddings has been motivated by applications such as information visualization and VLSI circuit design (see [1], [9], [14]). These applications provide metrics for which certain embeddings become aesthetically or functionally preferable. For example, a situation might prefer that edges be drawn as straight lines.

A classic result of Fáry [11] showed that all planar graphs permit embeddings in the plane where each edge maps to a straight line segment (a result independently proven by Wagner [19] and Stein [17]). If we further restrict the vertices to map to points on an $(n-2) \times (n-2)$ grid, then a planar embedding can still be achieved with edges mapping to straight line segments [16].

On the other hand, if the vertex mapping is completely fixed, a straight-line embedding does not always exist. In fact, it was shown by Pach and Wenger [15] that if we require edges to be drawn as polygonal curves (piecewise linear
curves) then there does not always exist an embedding with $o(n^2)$ total bends. Their results went further to show that this lower bound holds \textit{asymptotically almost surely} for a uniformly random planar graph on $n$ vertices; that is, the lower bound holds with probability 1 as $n$ tends to infinity.

Kaufmann and Wiese [12] considered the case where the range of the vertex mapping is restricted to a fixed point set $P$ of size $n$. They showed that any planar graph can be embedded so that each vertex maps to a unique point in $P$ and each edge maps to a polygonal curve with at most 2 bends. This result is optimal in that there exists point sets (points on a line, for example) for which not all planar graphs can be drawn with edges bending at most once.

The problem of drawing graphs to minimize bends has also been discussed in regards to \textit{simultaneous embeddings}. A simultaneous embedding is a drawing in the plane of $k$ graphs $G_1, G_2, \ldots, G_k$, each over a common vertex set $V$, such that no two edges of one graph cross. The concept of a simultaneous embedding with this terminology was introduced in [6]. A related result of particular interest was discussed in [10] by Erten and Kobourov. They considered the special case of constructing a simultaneous embedding for when $k = 2$. Their results showed that 2 bends per edge suffice to construct a simultaneous embedding of two planar graphs.

One aim of our paper is to consolidate the above results on embedding graphs with restrictions on the vertex mapping into a single drawing technique. Lemma 3 establishes such a technique that optimally minimizes the number of bends (up to constant factors). Moreover, for the case where the vertex mapping is completely fixed, we give a result matching the constant factor of 3 on the number of bends per edge that was given in [2]. An advantage of our technique, however, is that it lends itself well to probabilistic analysis. Given a fixed vertex mapping, our technique gives at most $2n$ bends per edge on expectation for a uniformly random planar graph, by which we mean a graph sampled uniformly at random from the set of all planar graphs over the vertex set $V = \{1, 2, \ldots, n\}$.

Another aim of our paper is to generalize our results to simultaneous embeddings. Our goal is to simultaneously embed planar graphs $G_1, G_2, \ldots, G_k$, each over a common vertex set $V$, so that each vertex uniquely maps to one of $n = |V|$ pre-specified points. Using Lemma 3, we give a construction for which each edge bends $O(n^{1-\frac{1}{2k}})$ times with overwhelming probability if we assume that the $k$ graphs are sampled uniformly at random; that is, each edge bends $O(n^{1-\frac{1}{2k}})$ times with probability at least $1 - n^{-c}$ for any fixed constant $c$.

We go further to show that our result on simultaneous embeddings is optimal using information theory. That is, we use an encoding argument to prove a lower bound that matches our upper bound.

The drawing technique relies fundamentally on results related to book embeddings, which we introduce in Section 2. We describe the drawing technique in Section 3. Section 4 applies the drawing technique to the case of embedding a uniformly random planar graph with a fixed vertex mapping. The application of the drawing technique to simultaneous embeddings is described in Section 5. The proofs of the lower bounds are in Section 6.
2 Book Embeddings

A well-known result regarding book embeddings is that all Hamiltonian planar graphs have book thickness 2 (see [3]). A trivial consequence of this result is that a Hamiltonian planar graph can be embedded in the plane so that all vertices lie on a common line and all edges lie strictly above or below this line, except at their ends. Observe that in such an embedding, each edge can be drawn as a polygonal curve with at most 1 bend (see Fig. 1a for an example).

Fig. 1: The induced planar embedding of a graph from a book embedding.

Any planar graph can be augmented to become 4-connected by subdividing each edge at most once and by adding additional edges. A classic theorem of Tutte [18] showed that all 4-connected planar graphs are Hamiltonian. It follows that we can always construct a Hamiltonian supergraph \( G' \) of a subdivision of a planar graph \( G \) by subdividing each original edge at most once\(^1\). From the Hamiltonian graph \( G' \), we can construct a book embedding (as in Fig. 1a), which induces an embedding of the original graph \( G \) (as in Fig. 1b). Observation 1 summarizes this embedding. Note that this embedding and its construction has been frequently described in graph drawing literature (as early as [1]).

**Observation 1.** A planar graph \( G \) can be embedded in the plane so that

1. all vertices lie on a common line,
2. each edge bends at most once above the line, at most once below the line, and at most once on the line.

3 Overview of the Drawing Technique

Let \( G = (V, E) \) be a planar graph, and suppose that \( \gamma : V \rightarrow R^2 \) is a fixed vertex mapping. We define \( \delta \) to be any vector in \( R^2 \) such that \( \delta \cdot \gamma(u) = \delta \cdot \gamma(v) \), for \( \delta \) can also be constructed in linear time by combining results from [1] and [8].
Let \( u, v \in V \), only if \( u = v \) (here \( \cdot \) is the standard dot product over \( R^2 \)). That is, the vertices in \( V \) map under \( \gamma \) to points at unique distances along the direction of the vector \( \delta \). Such a direction can trivially be seen to always exist.

Suppose that \( G \) is embedded as per Observation 1. For convenience, we will refer to this embedding as the book embedding of \( G \) and the line on which the vertices lie as the spine. We can assume without loss of generality that \( \delta \) is aligned with the spine. Let \( v_1, v_2, \ldots, v_n \) be the vertices in \( V \) as they occur along the direction of \( \delta \) in the book embedding. We relate the mapping \( \gamma \) to this embedding of \( G \) using order-theoretic concepts.

**Definition 2.** Let \( \prec \) be a partial order over \( V \) such that \( v_a \prec v_b \) if and only if \( a \leq b \) and \( \delta \cdot \gamma(v_a) \leq \delta \cdot \gamma(v_b) \). Similarly, let \( \succ \) be a partial order over \( V \) such that \( v_a \succ v_b \) if and only if \( a \leq b \) and \( \delta \cdot \gamma(v_a) \geq \delta \cdot \gamma(v_b) \).

Thus, a chain with respect to \( \prec \) is a set of vertices that occur along \( \delta \) in the same order in both the book embedding of \( G \) and under the mapping \( \gamma \). On the other hand, the vertices in a chain with respect to \( \succ \) occur in the reversed order in the book embedding of \( G \) from their order under \( \gamma \). Using this notation, we can state the effect of our drawing technique as follows.

**Lemma 3.** Suppose that \( V \) is partitioned into \( V_1, V_2, \ldots, V_r \), so that \( v_a \in V_i \) and \( v_b \in V_j \) satisfy \( \delta \cdot \gamma(v_a) < \delta \cdot \gamma(v_b) \) if \( i < j \). Then, if \( V_i \) forms a chain with respect to \( \prec \) when \( i \) is odd and a chain with respect to \( \succ \) when \( i \) is even, we can embed \( G \) in the plane with the vertex mapping \( \gamma \) using at most \( 3r + O(1) \) bends per edge.

**Proof.** Without loss of generality, we can assume \( \delta \) is directed horizontally. Thus, we can assume that

1. \( v_1, v_2, \ldots, v_n \) are the vertices in \( G \) in the order they are mapped from left to right in the book embedding,
2. \( V_1, V_2, \ldots, V_r \) map under \( \gamma \) to the point sets \( P_1, P_2, \ldots, P_r \), respectively, such that all points in \( P_i \) occur left of all points in \( P_{i+1} \), for \( i = 1, 2, \ldots, r - 1 \),
3. for odd \( i \), the vertices in \( V_i \) map to points in \( P_i \) with the same relative left-to-right order as they occur along the spine of the book embedding,
4. for even \( i \), the vertices in \( V_i \) map to points in \( P_i \) with the reverse relative left-to-right order as they occur along the spine of the book embedding.

Thus, we can think of the vertex sets \( V_1, V_2, \ldots, V_r \) as mapping to disjoint intervals \( \Delta_1, \Delta_2, \ldots, \Delta_r \) along the x-axis, each (strictly) containing the points \( P_1, P_2, \ldots, P_r \) respectively. See Fig. 2 for an example of such a configuration. We will return to this idea to show how to partially embed the edges in \( G \) inside each interval, but before doing so, we first introduce some terminology.

All points in the book embedding of \( G \) that intersect with the spine either correspond to a vertex in \( G \) or a point at which an edge crosses the spine. Let \( \Gamma_1, \Gamma_2, \ldots, \Gamma_h \) correspond to these points in the order they occur along the spine from left to right. If \( \Gamma_i \) corresponds to a vertex \( v \), then we define \( \text{vertex}(\Gamma_i) = v \). Furthermore, we define \( \text{top}(\Gamma_i) \) to be the set of edges incident to \( v \) that were embedded on the top page and \( \text{bottom}(\Gamma_i) \) to be those embedded on the bottom.
Fig. 2: An example configuration for a graph on 6 vertices.
follows by the symmetric definition of $V_i$, for even $i$, why this construction can be achieved.

Thus, we can repeat the above process for each $\Delta$-interval $\Delta_1, \Delta_2, \ldots, \Delta_r$. After this procedure, each $\Delta$-interval will have a set of lines extending upwards and a set of lines extending downwards (some of which may correspond to the same edge). When $i$ is odd, we say that the lines extending upward in $\Delta_i$ are entering $\Delta_i$ and those extending downward are leaving. When $i$ is even, the definitions are reversed. We make a few observations about the configuration of these lines.

1. When $i$ is odd, the lines in $\Delta_i$ define subintervals along the $x$-axis that correspond to $\Gamma_1, \Gamma_2, \ldots, \Gamma_h$ from left to right.
2. When $i$ is even, the lines in $\Delta_i$ define subintervals along the $x$-axis that correspond to $\Gamma_h, \Gamma_{h-1}, \ldots, \Gamma_1$ from left to right.
3. For contiguous intervals $\Delta_i$ and $\Delta_{i+1}$, the lines leaving $\Delta_i$ for a particular $\Gamma_t$ correspond to the same set of edges as the lines entering $\Delta_{i+1}$ for $\Gamma_t$.

The last observation follows by construction. Clearly it holds for any $\Gamma_t$ that corresponded to an edge crossing the spine. Suppose instead that $v = \text{vertex}(\Gamma_t)$. If the lines leaving $\Delta_i$ corresponded to the edges in $\text{top}(\Gamma_t)$, then $v \notin V_j$ for all $j \leq i$, implying that the lines entering $\Delta_{i+1}$ for $\Gamma_t$ also correspond to $\text{top}(\Gamma_t)$. On the other hand, if the lines leaving $\Delta_i$ corresponded to the edges in $\text{bottom}(\Gamma_t)$, then $v \in V_j$, for some $j \leq i$, implying that the lines entering $\Delta_{i+1}$ for $\Gamma_t$ also correspond to $\text{bottom}(\Gamma_t)$.

We proceed to show how to join the vertical lines from contiguous $\Delta$-intervals. Let $B$ be an axis-aligned box containing all points in $P$ (where $P$ is the image of $V$ under $\gamma$). For even $i$, suppose we were to rotate all vertical lines in the interval $\Delta_i$ clockwise by a small angle $\epsilon$ so that the lines remain parallel and only leave the interval $\Delta_i$ outside of the box $B$. Eventually, the lines drawn for each $\Gamma_i$ in the interval $\Delta_i$ would intersect with the vertical lines drawn for $\Gamma_i$ in the interval $\Delta_{i-1}$. We can then terminate the lines drawn for $\Delta_{i-1}$ and $\Delta_i$. 

![Diagram](image-url)
at these intersection points, hence joining the lines drawn for $I_t$ in $\Delta_{i-1}$ and $\Delta_i$. Similarly, if we consider the intersection between the lines extending from $\Delta_i$ with the vertical lines extending upward from $\Delta_{i+1}$ (assuming $\Delta_{i+1}$ exists), we can again terminate these lines at the points the lines from a common $I_t$ intersect. See Fig. 5 for an example of this procedure.

By the previous observation, this procedure therefore joins the lines leaving $\Delta_{i-1}$ to those entering $\Delta_i$ and joins the lines leaving $\Delta_i$ to those entering $\Delta_{i+1}$. Moreover, since the lines in each odd-indexed $\Delta$-interval are left unrotated, we can repeat this procedure for each even-indexed $\Delta$-interval. That is, for each $I_t$ with $\text{vertex}(I_t) = v$, the edges in $\text{top}(I_t)$ are drawn from $\gamma(v)$ to a set of vertical lines entering $\Delta_1$ in the same left-to-right order as these edges were drawn incident to $v$ on the top page of the book embedding. Similarly, the edges
in \textbf{bottom}(\Gamma_r) are drawn from \gamma(v) to a set of lines leaving \Delta_r in the same left-to-right order as these edges were drawn incident to \(v\) on the bottom page. Furthermore, a line is drawn entering \Delta_1 and leaving \Delta_r for each edge that had crossed the spine in the book embedding.

To complete the desired embedding, we consider the vertical lines entering \Delta_1 and the lines leaving \Delta_r. The vertical lines entering \Delta_1 correspond to the ends of the edges drawn on the top page of the book embedding (either where they are incident to a vertex or where they cross the spine). To join the two vertical lines corresponding to the same edge, we can use the embedding of the top page of the book embedding. The procedure is as follows. First, truncate the vertical lines at some common \(y\)-coordinate. Then, draw the top page of the book embedding above the vertical lines, excluding the region within some small distance \(\epsilon\) from the spine (truncating the edges before they meet their incident vertices). We can then trivially connect the ends of the vertical lines to the ends of the truncated edges from the top page since they occur from left to right in the same order. See Fig 6 for a depiction of this procedure.

![Fig. 6: An example of how we join the vertical lines entering \Delta_1 by using the top page of the book embedding.](image)

To join the edges leaving \Delta_r with each other, we consider two cases. If \(r\) is odd, then the lines leave downwards, and we can connect them using the bottom page in the same manner as we did with the lines entering \Delta_1. If \(r\) is even, then the lines leave upwards and we can again join them by using the bottom page by simply rotating it to face the opposite direction.

We now bound the number of times an edge bends in the embedding. Recall that for each of the \(\Gamma_1, \Gamma_2, \ldots, \Gamma_h\) we drew a set of piecewise linear curves through the intervals \(\Delta_1, \Delta_2, \ldots, \Delta_r\). Each of these curves bent at most once for each of the intervals and \(O(1)\) times where they connected to a vertex or joined another curve where they entered \(\Delta_1\) or left \(\Delta_r\). By using the embedding from Observation 1, it follows that each edge in \(G\) is associated with at most three of \(\Gamma_1, \Gamma_2, \ldots, \Gamma_h\). Thus, it follows that each edge bends at most \(3r + O(1)\) times. \(\square\)
4 Drawing a Planar Graph with a Fixed Vertex Mapping

In Section 3, we established a technique for drawing a graph $G = (V, E)$ with a fixed vertex mapping $\gamma$, where the number of bends is proportional to the size of a partition $V_1, V_2, \ldots, V_r$ of $V$ satisfying the conditions of Lemma 3. In this section, we discuss how to construct such a vertex partition for an arbitrary fixed vertex mapping.

Recall the definition of $\prec$ and $\succ$ from Definition 2. Clearly, any singleton set forms a chain with respect to both $\prec$ and $\succ$. Thus, by Lemma 3, we can use the partition $V_1 = \{v_1\}, V_2 = \{v_2\}, \ldots, V_n = \{v_n\}$ to embed $G$ with any vertex mapping using $3n + O(1)$ bends per edge. This bound’s constant factor matches the best known result of Badent et al. described in [2]. Using average-case analysis, we can improve the bound.

A uniformly random planar graph on $n$ vertices is a graph sampled uniformly at random from the set of all planar graphs over the vertex set $V = \{1, 2, \ldots, n\}$. We consider two isomorphic graphs to be different if their vertex labelings differ. Suppose that we constructed a book embedding of such a graph as per Observation 1. If we do so in a manner that is independent from the labeling of the vertices, then we can assume that the vertices occur along the spine of the book embedding in a uniformly random order. To enforce independence, one could simply relabel the vertices uniformly at random before constructing the book embedding, reverting to the original labeling afterwards. Hence, we can make the following observation.

Observation 4. A uniformly random planar graph $G$ can be embedded in the plane so that

1. all vertices lie on a common line in a uniformly random order,
2. each edge bends at most once above the line, at most once below the line, and at most once on the line.

By Observation 4, our analysis on random planar graphs reduces to an analysis of random permutations. The proof of the next theorem delineates this point.

Theorem 5. A uniformly random planar graph $G = (V, E)$ can be embedded in the plane with a fixed vertex mapping using at most $2n + O(1)$ bends per edge on expectation.

Proof. Let $\gamma$ be a fixed vertex mapping, and let $\delta$ be a direction for which $\delta \cdot \gamma(u) = \delta \cdot \gamma(v)$, for $u, v \in V$, only if $u = v$. Let $p_1, p_2, \ldots, p_n$ be the points in the image of $V$ under $\gamma$ in the order they occur along $\delta$. Define $v_1 = \gamma^{-1}(p_1), v_2 = \gamma^{-1}(p_2), \ldots, v_n = \gamma^{-1}(p_n)$.

Embed $G$ as per Observation 4 so that the spine is aligned with the direction $\delta$. For $i = 1, \ldots, n$, define $\alpha(v_i)$ to be the index along $\delta$ at which $v_i$ occurs in this embedding. Thus, by Definition 2, $v_i \prec v_j$ if $i \leq j$ and $\alpha(v_i) \leq \alpha(v_j)$. Similarly, $v_i \succ v_j$ if $i \leq j$ and $\alpha(v_i) \geq \alpha(v_j)$.

By our choice of embedding, $\alpha(v_1), \alpha(v_2), \ldots, \alpha(v_n)$ is a uniformly random permutation of $1, 2, \ldots, n$. Construct a partition of $V$ as follows. Let $t_1$ be the
largest index such that $\alpha(v_1), \alpha(v_2), \ldots, \alpha(v_t)$ is increasing. Then, let $t_2$ be the largest index such that $\alpha(v_{t_1+1}), \alpha(v_{t_1+2}), \ldots, \alpha(v_{t_2})$ is decreasing. Repeat this process for $t = 3, \ldots, r$, maximizing increasing sequences when $i$ is odd and decreasing sequences when $i$ is even. The partition $V_1 = \{v_1, v_2, \ldots, v_t\}$, $V_2 = \{v_{t_1+1}, v_{t_1+2}, \ldots, v_{t_2}\}$, $\ldots$, $V_r = \{v_{t_{r-1}+1}, v_{t_{r-1}+2}, \ldots, v_t\}$ satisfies the conditions of Lemma 3 by construction. We can therefore construct the desired embedding of $G$ so long as $r$ is at most $\frac{2}{3} n + O(1)$.

Thus, to complete the proof we consider how large $r$ is on average. Let $X$ be the set of integers $1 < i < n$ for which $\alpha(v_i), \alpha(v_{i+1}) < \alpha(v_i)$ or $\alpha(v_{i-1}), \alpha(v_{i+1}) > \alpha(v_i)$. Clearly $r \leq |X| + 2$. Let $X_2, X_3, \ldots, X_{n-1}$ be indicator variables such that $X_i = 1$ if $i \in X$ and $X_i = 0$ otherwise. By linearity of expectation, it follows that
\[
\mathbb{E}|X| = \sum_{i=1}^{n} \mathbb{E}X_i
\]
and since $\mathbb{E}X_i = \mathbb{P}[X_i = 1] = 2/3$, it follows that $r \leq 2/3(n + 1)$.

5 Simultaneous Embeddings with a Fixed Vertex Range

In this section, we consider the problem of embedding $k$ uniformly random planar graphs $G_1, G_2, \ldots, G_k$ over a common vertex set $V$, where the range of the vertex mapping $\gamma$ is restricted to a fixed point set $P$ of size $n = |V|$. As in the proof of Theorem 5, our analysis relies on properties of uniformly random permutations.

Lemma 6 (7). Let $\pi_1, \pi_2, \ldots, \pi_k$ be uniformly random permutations over the set $S = \{1, 2, \ldots, n\}$. Then, there exists a partition $T_1, T_2, \ldots, T_r$ of $S$, where the elements in each part form an increasing subsequence in each of $\pi_1, \pi_2, \ldots, \pi_k$, such that $r$ is $O(n^{1-\frac{1}{t-r}})$ with overwhelming probability.

This bound was established by Brightwell in [7]. The following result follows by combining this bound with Lemma 3.

Theorem 7. If $G_1, G_2, \ldots, G_k$ are uniformly random planar graphs, then we can embed each graph in the plane with a common vertex mapping $\gamma : V \rightarrow P$ so that all edges have $O(n^{1-\frac{1}{t-r}})$ bends each with overwhelming probability.

Proof. Let $\delta$ be a direction for which $\delta \cdot p = \delta \cdot q$, for $p, q \in P$, only if $p = q$. Let $p_1, p_2, \ldots, p_n$ be the points in $P$ in the order they occur along $\delta$. Embed $G_1$ as per Observation 4 so that its spine is aligned with $\delta$. Let $v_1, v_2, \ldots, v_n$ be the vertices in $V$ in the order they occur along $\delta$ in this embedding. Embedding $G_2, \ldots, G_k$ in the same manner gives the corresponding vertex orders $\pi_2, \pi_3, \ldots, \pi_n$, where $\pi_i$ is a uniformly random permutation of $v_1, v_2, \ldots, v_n$. By Lemma 6, it follows that $V$ can be partitioned into $V_1, V_2, \ldots, V_r$ such that the vertices in $V_i$ occur along $\delta$ in the same order in the embeddings of each of $G_1, G_2, \ldots, G_k$. Furthermore, $r$ is $O(n^{1-\frac{1}{t-r}})$ with overwhelming probability.
Let \( \{u_1, \ldots, u_t\} = V_1, \{u_{t+1}, \ldots, u_{t+2}\} = V_2, \ldots, \{u_{t+r-1+1}, \ldots, u_n\} = V_r \) such that \( u_i \) occurs before \( u_{i+1} \) along \( \delta \) in the embedding of \( G_1 \), for all \( i \), if \( u_i, u_{i+1} \in V_j \) for some \( j \). Consider the vertex mapping \( \gamma \), defined such that \( \gamma(u_1) = p_1, \gamma(u_2) = p_2, \gamma(u_n) = p_n \). By construction, each \( V_i \) forms a chain with respect to \( \prec \) from Definition 2. Since Lemma 3 requires a partition that alternates between chains with respect to \( \prec \) and \( \succ \), we can introduce empty sets into our partition after each \( V_i \), at most doubling its size. This extension suffices since the empty set forms a chain with respect to both \( \prec \) and \( \succ \). Moreover, since \( r \) is \( O(n^{1-\frac{1}{k}}) \), so is this extension, and thus the claim follows by Lemma 3. \( \square \)

6 Lower Bounds on the Number of Bends

In this section we prove that Theorem 7 is optimal by using an encoding argument. The proof relies on the following lemma.

**Lemma 8.** If the planar graph \( G \) can be drawn in the plane with \( \beta \) total bends under a fixed vertex mapping that maps \( V \) to a convex point set, then \( G \) can be encoded using \( n \log \left( \frac{\beta + n}{n} \right) + O(n) \) bits.

The proof of this lemma is lengthy and is thus deferred to Appendix A. The following result follows by the information theoretic lower bound on the number of bits required to encode a planar graph.

**Theorem 9.** Let \( G_1, G_2, \ldots, G_k \) be uniformly random planar graphs over the vertex set \( V \), and let \( P \) be a convex point set of size \( |V| = n \). Then, in all simultaneous embeddings of \( G_1, G_2, \ldots, G_k \) that map \( V \) to \( P \), at least one of \( G_1, G_2, \ldots, G_k \) has \( \Omega(2^{1-\frac{1}{k}}) \) total bends with overwhelming probability.

**Proof.** Suppose that \( G_1, G_2, \ldots, G_k \) can be drawn on \( P \) with \( \beta \) total bends for some vertex mapping \( \gamma \). Since there are \( n! \) possible vertex mappings, \( \gamma \) can be encoded using \( \log n! \) bits. Thus, \( G_1, G_2, \ldots, G_k \) can be encoded using

\[
kn \log \left( \frac{\beta + n}{n} \right) + O(kn) + \log n!
\]

bits by Lemma 8. Since there are more than \( n! \) planar graphs on \( n \) vertices, it follows that at least \( \log n! - \Delta \) bits are required to encode a uniformly random planar graph with probability at least \( 1 - 2^{-\Delta} \). It follows that

\[
kn \log \left( \frac{\beta + n}{n} \right) + O(kn) + \log n! \geq k \log n! - \Delta
\]

with probability at least \( 1 - 2^{-\Delta} \). Thus, there exists a constant \( c \) for which

\[
k n \log \left( \frac{(\beta + n)^{2^{-c \cdot \frac{1}{k}}} n^\frac{1}{k}}{n} \right) \geq kn \log n
\]
with probability at least $1 - 2^{-\Delta}$ by Stirling’s approximation. Dividing a factor of $kn$ and exponentiating both sides shows that the inequality

$$\frac{(\beta + n)2^{c_1 \frac{1}{\pi^2}} n^{\frac{1}{2}}}{n} \geq n$$

or equivalently,

$$\beta \geq \frac{n^{\frac{1}{2}} - \frac{1}{n}}{2^{c_1 \frac{1}{\pi^2}} n^\alpha} - n$$

holds with probability at least $1 - 2^{-\Delta}$. □

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Proof of Lemma 8

In this section, we will demonstrate how to encode a planar graph $G$ with a fixed vertex mapping $\gamma$ under the assumption that $G$ can be drawn on a convex point set $P$ with $\beta$ bends. The encoding technique involves a series of decompositions of $G$. First, we reduce the problem to encoding a spanning tree of $G$. To encode this spanning tree, we will construct a Hamiltonian 3-regular graph from the union of the tree’s edges and the edges on the convex hull of $P$.

The encoding of this Hamiltonian 3-regular graph follows by its recursive structure; that is, we will describe a recursive method to encode it. This recursive procedure makes use of edge separators. In particular, we will use an edge separator that takes into account the crossing number of a graph.

**Lemma 10 ([13]).** Let $G = (V, E)$ be a graph with nonnegative vertex weights that sum to at most 1 and do not individually exceed $2/3$. Then, $G$ has an edge separator of size

$$1.58 \sqrt{16 \text{cr}(G) + \sum_{v \in V} \deg^2(v)}$$

where $\text{cr}(G)$ is the crossing number of $G$.

Using such a separator, we can effectively decompose a Hamiltonian 3-regular graph into 2 smaller Hamiltonian 3-regular graphs, separated by the edge separator. The proof of the following lemma is essentially an analysis of this recursion.

**Lemma 11.** Let $G = (V, E)$ be a 3-regular graph with a fixed Hamiltonian cycle $C = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. Then, $G$ can be encoded using

$$\frac{n}{2} \log \left( \frac{\text{cr}(G) + n}{n} \right) + O(n)$$

bits, where $\text{cr}(G)$ is the crossing number of $G$.

**Proof.** To prove the desired claim, we will prove the more precise bound

$$\frac{n}{2} \log \left( \frac{\sigma + n}{n} \right) + c_1n - c_2\sqrt{\sigma + n} \log \left( \frac{n}{\sqrt{\sigma + n}} \right)$$

by induction, where $\sigma$ is any parameter such that $\sigma \geq \text{cr}(G)$.

If $n \leq 2$, the claim holds trivially as no bits are required to encode $G$ and the bound we are trying to prove is nonnegative. Thus, we can proceed by induction on $n$.

Define $r = \sqrt{\sigma + n}$. We first make the assumption that $r \leq \frac{n}{2\sigma}$. By the separator theorem, we can find a separator for $G$ of size at most $1.58\sqrt{16\sigma + 9n}$ that partitions the vertices into two sets $V_1$ and $V_2$, such that $\frac{1}{4}n \leq |V_1| \leq |V_2| \leq \frac{3}{4}n$. Furthermore, we can assume that every edge in the separator was in $C$ by at most doubling the size of the separator. Indeed, if any edge $uv \in E \setminus C$ were such that $u \in V_1$ and $v \in V_2$, then we could add $v$ to $V_1$, removing the edge
uv from the separator and adding at most two edges incident to v in C. Thus we can assume that the edge separator contains only edges in C and has size at most $2(1.58\sqrt{16\sigma + 9n}) \leq 13r$.

We can then encode G recursively by encoding the two subgraphs induced by the vertices in $V_1$ and the vertices in $V_2$ (after re-establishing the canonical Hamiltonian cycle in each subgraph) and specifying how to combine these subgraphs to construct G. It suffices to encode the edges in the separator, the size of the separator, and a bit identifying which of $V_1$ or $V_2$ contained $v_1$. This additional information costs at most

$$\lg \left( \frac{n}{13r} \right) + \lg 13r + 2$$

bits, which by Stirling’s approximation is at most $13r \lg \frac{n}{r}$ bits. Thus, by induction, G can be encoded using

$$T(n_1, \sigma_1) + T(n_2, \sigma_2) + 13r \lg \frac{n}{r} \tag{1}$$

bits, where

$$T(n, \sigma) = \frac{n}{2} \lg \left( \frac{\sigma + n}{n} \right) + c_1 n - c_2 \sqrt{\sigma + n} \lg \left( \frac{n}{\sqrt{\sigma + n}} \right)$$

and $n_1, n_2, \sigma_1, \sigma_2$ satisfy

$$n_1 + n_2 = n, \quad \sigma_1 + \sigma_2 = \sigma, \quad \frac{1}{3} n \leq n_1 \leq n_2 \leq \frac{2}{3} n.$$ 

Our goal thus is to bound the size of (1). Define $\alpha$ and $\lambda$ such that $\alpha n = n_1$ and $\lambda(\sigma + n) = \sigma_1 + n_1$. Thus, it follows that $(1 - \alpha)n = n_2$ and $(1 - \lambda)(\sigma + n) = \sigma_2 + n_2$. We can therefore express (1) as

$$\frac{\alpha n}{2} \lg \left( \frac{\lambda(\sigma + n)}{\alpha n} \right) + c_1 \alpha n - c_2 \sqrt{\lambda r} \lg \left( \frac{\alpha n}{\sqrt{\lambda r}} \right)$$

$$+ \frac{(1 - \alpha)n}{2} \lg \left( \frac{(1 - \lambda)(\sigma + n)}{(1 - \alpha)n} \right) + c_1 (1 - \alpha)n - c_2 \sqrt{1 - \lambda r} \lg \left( \frac{(1 - \alpha)n}{\sqrt{1 - \lambda r}} \right) + 13r \lg \frac{n}{r}$$

bits or equivalently as

$$\frac{\alpha n}{2} \lg \left( \frac{\lambda(\sigma + n)}{\alpha n} \right) + \frac{(1 - \alpha)n}{2} \lg \left( \frac{1 - \lambda(\sigma + n)}{(1 - \alpha)n} \right) + c_1 n$$

$$- c_2 \left( \sqrt{\lambda r} \lg \left( \frac{\alpha n}{\sqrt{\lambda r}} \right) + \sqrt{1 - \lambda r} \lg \left( \frac{(1 - \alpha)n}{\sqrt{1 - \lambda r}} \right) - \frac{13}{c_2} r \lg \frac{n}{r} \right)$$

bits. To prove that this achieves the desired bound, we will consider 2 cases on the value of $\lambda$. First, assume that $\frac{1}{6} \leq \lambda \leq \frac{5}{6}$. Observe that the function

$$\frac{\alpha n}{2} \lg \left( \frac{\lambda(\sigma + n)}{\alpha n} \right) + \frac{(1 - \alpha)n}{2} \lg \left( \frac{(1 - \lambda)(\sigma + n)}{(1 - \alpha)n} \right)$$

achieves its maximum when $\lambda = \alpha$ and is therefore at most $\frac{n}{2} \lg \left( \frac{\sigma + n}{n} \right)$. On the other hand, the function

$$\sqrt{\lambda r} \lg \left( \frac{\alpha n}{\sqrt{\lambda r}} \right) + \sqrt{1 - \lambda r} \lg \left( \frac{(1 - \alpha)n}{\sqrt{1 - \lambda r}} \right)$$

achieves its minimum when $\lambda = \frac{1}{6}$ and $\alpha = \frac{1}{3}$ (without loss of generality). Thus, in this case, the bound (1) is at most

$$\frac{n}{2} \lg \left( \frac{\sigma + n}{n} \right) + c_1 n - c_2 r \lg \frac{n}{r}$$

which simplifies to

$$\frac{n}{2} \lg \left( \frac{\sigma + n}{n} \right) + c_1 n - c_2 r \lg \frac{n}{r}$$

$$+ 2 \left( \frac{13}{c_2} r \lg \frac{n}{r} - (\sqrt{1/6} \lg \sqrt{2/3}) + (\sqrt{5/6} \lg \sqrt{15/8}) - (\sqrt{1/6} + \sqrt{5/6} - 1)r \lg \frac{n}{r} \right)$$

which is at most

$$\frac{n}{2} \lg \left( \frac{\sigma + n}{n} \right) + c_1 n - c_2 r \lg \frac{n}{r}$$

and, by setting $c_2 = 41$ is just

$$\frac{n}{2} \lg \left( \frac{\sigma + n}{n} \right) + c_1 n - c_2 r \lg \frac{n}{r}$$

since we assumed that $r \leq \frac{n}{2} r$.

Next, we assume that $\lambda \leq \frac{1}{6}$. In this case, the function

$$\frac{an}{2} \lg \left( \frac{\lambda(\sigma + n)}{\alpha n} \right) + \frac{(1 - \alpha)n}{2} \lg \left( \frac{(1 - \lambda)(\sigma + n)}{(1 - \alpha)n} \right)$$

achieves its maximum when $\lambda = \frac{1}{6}$ and $\alpha = \frac{1}{3}$ and is therefore at most

$$\frac{n}{2} \lg \left( \frac{\sigma + n}{n} \right) - \frac{n}{18}.$$

Furthermore, the function

$$\sqrt{\lambda r} \lg \left( \frac{\alpha n}{\sqrt{\lambda r}} \right) + \sqrt{1 - \lambda r} \lg \left( \frac{(1 - \alpha)n}{\sqrt{1 - \lambda r}} \right)$$

is at least

$$r \lg \frac{n}{r} - 2r$$
since $\frac{1}{3} \leq \alpha \leq \frac{2}{3}$. Thus, it follows that the bound (1) is at most

$$\frac{n}{2} \lg \left( \frac{\sigma + n}{n} \right) + c_1 n - c_2 r \lg \frac{n}{r} + 2r + 13r \frac{n}{r} - \frac{n}{18}$$

which again is at most

$$\frac{n}{2} \lg \left( \frac{\sigma + n}{n} \right) + c_1 n - c_2 r \lg \frac{n}{r}$$

as we have assumed that $r \leq \frac{n}{2r}$.

To complete the proof, we consider the case that $r > \frac{n}{2r}$. That is, we can assume that $\frac{\sigma + \sigma}{n} > \frac{n}{2r}$. As $G$ can be encoded simply by encoding the edges in $E \setminus C$, it follows that $G$ can be encoded using $\frac{n}{2} \lg n$ bits or equivalently,

$$\frac{n}{2} \lg \frac{n}{2r} + 45n$$

which in this case is at most

$$\frac{n}{2} \lg \left( \frac{n + \sigma}{n} \right) + 45n$$

bits. Furthermore, since

$$r \lg \frac{n}{r} \leq n$$

for all values of $r$, it follows that $G$ can be encoded using

$$\frac{n}{2} \lg \left( \frac{n + \sigma}{n} \right) + (45 + c_2)n - c_2 n \leq \frac{n}{2} \lg \left( \frac{n + \sigma}{n} \right) + c_1 n - c_2 r \frac{n}{r}$$

bits for any $c_1 \geq 86$ as we set $c_2 = 41$. \hfill \Box

**Corollary 12.** Let $T$ be an ordered tree on $n$ vertices $V$, and let $P$ be a convex point set. If $T$ can be drawn with a fixed vertex mapping $\gamma : V \to P$ such that edges bend a total of $\beta$ times, then $T$ can be encoded with

$$n \lg \left( \frac{\beta + n}{n} \right) + O(n)$$

bits.

**Proof.** Consider the boundary of the convex hull of the convex point set to which $V$ is mapped by $\gamma$. In an embedding of $T$ with a total of $\beta$ bends under this vertex mapping, the edges in $T$ can cross this boundary at most $2\beta$ times. Indeed, each piece composing an edge in $T$ can cross the boundary at most twice, except those incident to a vertex, which can cross the boundary at most once. Thus, if we construct a graph $G$ by the union of $T$ and the cycle $C$ defined by the boundary of the convex hull of the point set, it follows that $G$ has crossing number at most $2\beta$. Replace each vertex $v$ in $G$ with a set of vertices, all of which are consecutive
along $C$ and each of which is incident to a unique edge that was incident to $v$ in $T$. This operation produces a Hamiltonian 3-regular graph on $2n - 2$ vertices. Furthermore, as this operation cannot increase the crossing number, it follows that the resulting graph can be encoded with

$$n \log \left( \frac{\beta + n}{n} \right) + O(n)$$

bits. To recover $G$ from the encoding of this graph, it suffices to encode the blocks of consecutive vertices along $C$ that corresponded to original vertices. This can trivially be done using $O(n)$ bits. We can then recover the tree $T$ from $G$ as the vertex mapping is fixed.

To conclude this section, we describe how to use Corollary 12 to encode an arbitrary planar graph $G$ drawn with a fixed vertex mapping into a convex point set $P$ with $\beta$ total bends. Observe that $G$ can be assumed to be connected. If not, we could make $G$ connected by introducing edges with at most $O(\beta + n)$ total bends. In which case, the embedding of $G$ has an ordered spanning tree $T$ as a subgraph that can be drawn with a fixed vertex mapping into a convex point set using at most $O(\beta + n)$ total bends. By Corollary 12, $T$ can be encoded using

$$n \log \left( \frac{\beta + n}{n} \right) + O(n)$$

bits.

We claim that only $O(n)$ bits are required to recover $G$ from the encoding of $T$. Using the technique from [15], we can introduce a Hamiltonian cycle in $G$ by adding edges along a walk of the spanning tree $T$, starting from an arbitrary vertex. The resulting Hamiltonian graph can be recovered from an encoding of the Hamiltonian cycle and the encoding of the two outerplanar graphs defining the edges inside and outside the Hamiltonian cycle. The outerplanar graphs can be encoded using $O(n)$ bends (by a bijection with well formed parenthesizations). Moreover, by construction, the order of the vertices in the Hamiltonian cycle is recoverable from $T$. To construct the Hamiltonian graph required introducing at most $O(n)$ subdivision vertices in $G$. Thus, we can encode which vertices these corresponded to using $O(n)$ bits and recover the original graph by removing them and the edges in the Hamiltonian cycle.