FOURIER-MUKAI TRANSFORMS AND THE WALL-CROSSING BEHAVIOR FOR BRIDGELAND’S STABILITY CONDITIONS

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0. Introduction.

Let $X$ be a $K3$ surface or an abelian surface over a field $k$. In [4], Bridgeland introduced the notion of stability condition for objects in the bounded derived category $\mathsf{D}(X)$ of coherent sheaves on $X$. It consists of a $t$-structure of $\mathsf{D}(X)$ and a stability function on the heart. Bridgeland showed that the set of stability conditions $\text{Stab}(X)$ has a structure of complex manifold. Then he studied $\text{Stab}(X)$. In particular, the non-emptiness of this space was shown by constructing interesting examples of stability conditions. For $\beta \in \text{NS}(X)_{\mathbb{Q}}$ and an ample $\mathbb{Q}$-divisor $\omega$, the example consists of an abelian category $\mathfrak{A}_{(\beta, \omega)}$ which is a tilting of Coh($X$) by a torsion pair and a stability function $Z_{(\beta, \omega)}$ on it. The structure of $\mathfrak{A}_{(\beta, \omega)}$ was studied further by Huybrechts [8]. In particular, it was shown that $\mathfrak{A}_{(\beta, \omega)}$ is useful to study the usual Gieseker semi-stable sheaves. In this paper, we shall slightly generalize Bridgeland’s construction. In particular, we shall relax the requirement on $\omega$ in [5]. Then we study some basic properties of the categories. We introduce a special parameter space of the category $\mathfrak{A}_{(\beta, \omega)}$ and study its chamber structure. Since $(\mathfrak{A}_{(\beta, \omega)}, Z_{(\beta, \omega)})$ is a stability condition, our parameter space is a subspace of $\text{Stab}(X)$.

In order to study the moduli space of Gieseker semi-stable sheaves, it is important to study the behavior of Gieseker semi-stable sheaves under Fourier-Mukai transforms. In [22] and [30, sect. 3.7], we studied this problem. In this paper, we shall translate our previous results into the theory of Bridgeland’s stability...
conditions. In particular, we shall discuss the relation between Bridgeland’s stability and the twisted stability under the so-called large volume limit, i.e., \( (\omega^2) \gg 0 \). This relation was already discussed by Bridgeland \[5\], Toda \[21\], Bayer \[2\], Okawa \[19\], and our result is regarded as a supplement of theirs. Independently similar supplements and generalizations are obtained by Kawatani \[11, 12\], Lo and Qin \[15\]. In particular, Lo and Qin investigated the relation for any surface.

By the definition of the stability condition, the Fourier-Mukai transform preserves a suitable stability condition, but \( (\omega^2) \) is replaced by \( 1/(\omega^2) \). Thus this stability condition is far from Gieseker’s stability, and we need to across walls for Bridgeland’s stability conditions. So we need to discuss the wall crossing behavior of stability conditions. The wall crossing behavior for Bridgeland stability condition was studied by Arcara-Bertram \[1\] and Toda \[21\] for \( K3 \) surfaces, and by Lo-Qin for arbitrary surfaces. In particular, Toda defined a counting invariant for moduli stack of semi-stable objects and studied the wall crossing behavior of the invariants. As an application, we shall explain previous results on the birational maps induced by the Fourier-Mukai transform on an abelian surface. In \[29\] and \[22\], we introduced operations to improve the unstability. We shall show that these operations correspond to crossing walls of Bridgeland stability conditions. In particular, we shall recover the birational transforms of moduli spaces of stable sheaves in \[29\].

1. Bridgeland’s stability conditions.

1.1. Preliminaries. Let \( X \) be an abelian surface or a \( K3 \) surface over a field \( k \). The base field \( k \) is arbitrary unless otherwise stated. Let \( \mathfrak{g}_X \in H^4(X, \mathbb{Z}) \) be the fundamental class of \( X \). We define a lattice structure \( \langle \quad , \quad \rangle \) on \( H^{ev}(X, \mathbb{Z}) := \bigoplus_{i=0}^{2} H^{2i}(X, \mathbb{Z}) \) by

\[
\langle x, y \rangle := (x_1, y_1) − (x_0 y_2 + x_2 y_0),
\]

where \( x = x_0 + x_1 + x_2 \mathfrak{g}_X \) and \( y = y_0 + y_1 + y_2 \mathfrak{g}_X \) with \( x_0, y_0 \in \mathbb{Z} = H^0(X, \mathbb{Z}), x_1, y_1 \in H^2(X, \mathbb{Z}) \) and \( x_2, y_2 \in \mathbb{Z} \). It is now called the Mukai lattice. For a coherent sheaf \( E \) on \( X \),

\[
v(E) := \text{ch}(E)\sqrt{td_X} = \text{rk}(E) + c_1(E) + (\chi(E) − \varepsilon \text{rk}(E))\mathfrak{g}_X \in H^{ev}(X, \mathbb{Z})
\]

is called the Mukai vector of \( E \), where \( \varepsilon = 0, 1 \) according as \( X \) is an abelian surface or a \( K3 \) surface. We also define

\[
H^*(X, \mathbb{Z})_{\text{alg}} := \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z}\mathfrak{g}_X.
\]

Then \( H^*(X, \mathbb{Z})_{\text{alg}} \) is a sublattice. \( H^*(X, \mathbb{Z})_{\text{alg}} \) is well-defined over any field \( k \).

For a ring extension \( R \to R' \) and an \( R \)-module \( M \), we set \( M_{R'} := M \otimes_R R' \). Let \( E \) be an object of \( D(X) \). \( E^\vee := \mathbb{R}\text{Hom}_{O_X}(E, O_X) \) denotes the derived dual of \( E \). We denote the rank of \( E \) by \( \text{rk}(E) \). For a fixed nef and big divisor \( H \) on \( X \), \( \deg(E) \) denotes the degree of \( E \) with respect to \( H \). For \( G \in K(X)_\mathbb{Q} \) with \( \text{rk}(G) > 0 \), we also define the twisted rank and degree by \( \text{rk}_G(E) := \text{rk}(G^\vee \otimes E) \) and \( \deg_G(E) := \deg(G^\vee \otimes E) \) respectively. We set \( \mu_G(E) := \deg_G(E) / \text{rk}_G(E) \), if \( \text{rk}(E) \neq 0 \). Finally we define \( v(E) := \text{ch}(E)\sqrt{td_X} \) and call it the Mukai vector of the object \( E \) of \( D(X) \).

For a \((-2)\)-vector \( u \in H^{ev}(X, \mathbb{Z}) \),

\[
(1.1) \quad R_u : \ H^{ev}(X, \mathbb{Z}) \to H^{ev}(X, \mathbb{Z}) \quad x \mapsto x + \langle u, x \rangle u
\]

is the reflection by \( u \).

Let \( U \) be a complex of coherent sheaves such that \( \text{Hom}(U, U) = \mathfrak{g} \) and \( \text{Hom}(U, U[p]) = 0 \) for \( p \neq 0, 2 \). Let \( p_i : X \times X \to X, \ i = 1, 2 \) be the \( i \)-th projection. We set

\[
E := \text{Cone}(p_1^*(U) \otimes p_2^*(U^\vee) \to O_\Delta).
\]

Then

\[
\Phi_U : \ D(X) \to D(X) \quad E \mapsto \mathbb{R}p_{1*}(E \otimes p_2^*(E))
\]

is an equivalence and the quasi-inverse \( \Phi^{-1}_U \) is given by

\[
\Phi^{-1}_U : \ D(X) \to D(X) \quad F \mapsto \mathbb{R}p_{2*}(E^\vee \otimes p_1^*(F))[2].
\]

Moreover \( \Phi_U \) induces the \((-2)\)-reflection \( R_{v(U)} \).
1.2. Several definitions. In this subsection, we shall explain several basic notions to define Bridgeland stability conditions. Let $X$ be an abelian surface or a $K3$ surface over $\mathfrak{k}$, and $\pi : X \to Y$ a contraction of $X$ to a normal surface $Y$ over $\mathfrak{k}$. We note that $\pi$ is an isomorphism for an abelian surface. If $\pi$ is not isomorphic, then $Y$ has rational double points as singularities. Let $H$ be the pull-back of an ample divisor on $Y$. We take $\beta \in \text{NS}(X)_\mathbb{Q}$ such that $(\beta, D) \notin \mathbb{Z}$ for all $(-2)$-curves $D$ with $(D, H) = 0$. Then the following proposition holds.

Proposition 1.2.1 ([30 Prop. 2.4.9]). Assume that $\beta \in \text{NS}(X)_\mathbb{Q}$ satisfies $(\beta, D) \notin \mathbb{Z}$ for all $(-2)$-curves $D$ with $(D, H) = 0$. Then there is a category of perverse coherent sheaves $\mathcal{C}$ such that $\langle e^\beta, v(E) \rangle < 0$ for all 0-dimensional objects $E$ of $\mathcal{C}$.

If $\pi$ is isomorphic, then $\mathcal{C}$ is nothing but $\text{Coh}(X)$.

Let $r_0$ be a positive integer such that $r_0e^\beta$ is a primitive element of $H^*(X, \mathbb{Z})$. Let $G$ be an element of $K(X)_\mathbb{Q}$ such that $v(G) = r_0e^\beta - aQX$, $a \in \mathbb{Q}$.

Definition 1.2.2 ([30]). (1) Let $E$ be an object of $\mathcal{C}$. We set $E(n) := E(nH)$ for the fixed $H$.
   (a) Assume that $\text{rk} E > 0$. Then $E$ is $G$-twisted semi-stable if
   \[
   \chi(G, F(n)) \leq (\text{rk} F) \frac{\chi(G, E(n))}{\text{rk} E}, \quad n \gg 0
   \]
   for all proper subobject $F$ of $E$. If the inequality is strict for every $F$, $E$ is $G$-twisted stable.
   (b) Assume that $\text{rk} E = 0$ and $(c_1(E), H) > 0$. Then $E$ is $G$-twisted semi-stable if
   \[
   \chi(G, F) \leq (c_1(F), H) \frac{\chi(G, E)}{(c_1(E), H)}
   \]
   for all proper subobject $F$ of $E$. If the inequality is strict for every $F$, $E$ is $G$-twisted stable.
   (2) Obviously the $G$-twisted semi-stability depends only on $\beta = c_1(G)/\text{rk} G$. We define the $\beta$-twisted semi-stability as the $O_X(\beta)$-twisted semi-stability.
   (3) For $v \in H^*(X, \mathbb{Z})_{\text{alg}}$, $M^H(v)^{ss}$ (resp. $M^H(v)^s$) is the moduli stack of $\beta$-twisted semi-stable (resp. stable) objects $E$ of $\mathcal{C}$ with $v(E) = v$. We also define $\mu$-semi-stability by using the slope $\mu_G$. $M^H(v)^{\mu ss}$ denotes the moduli stack of $\mu$-semi-stable objects $E$ with $v(E) = v$.

Definition 1.2.3. Let $E \neq 0$ be an object of $\mathcal{C}$.

(1) There is a (unique) filtration

\[
E \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_s = E
\]

such that each $E_j := F_j/F_{j-1}$ is a torsion object or a torsion free $G$-twisted semi-stable object and

\[
(\text{rk } E_{j+1}) \chi(G, E_j(n)) > (\text{rk } E_j) \chi(G, E_{j+1}(n)), \quad n \gg 0.
\]

We call it the Harder-Narasimhan filtration of $E$.

(2) In the notation of (1), we set

\[
\mu_{\text{max},G}(E) := \begin{cases} 
\mu_G(E_1), & \text{rk } E_1 > 0 \\
\infty, & \text{rk } E_1 = 0,
\end{cases}
\]

\[
\mu_{\text{min},G}(E) := \begin{cases} 
\mu_G(E_s), & \text{rk } E_s > 0 \\
\infty, & \text{rk } E_s = 0.
\end{cases}
\]

Remark 1.2.4. Let $\overline{\mathfrak{k}}$ be the algebraic closure of $\mathfrak{k}$. For an object $E$ of $\mathcal{C}$, $E$ is $G$-twisted semi-stable if and only if $E \otimes_\mathfrak{k} \overline{\mathfrak{k}}$ is $G \otimes_\mathfrak{k} \overline{\mathfrak{k}}$-twisted semi-stable. Hence 1.2.2 is invariant under the extension of the field.

We define several torsion pairs of $\mathcal{C}$.

Definition 1.2.5. (1) Let $\mathfrak{A}^\mu$ be the full subcategory of $\mathcal{C}$ such that $E \in \mathcal{C}$ belongs to $\mathfrak{A}^\mu$ if (i) $E$ is a torsion object or (ii) $\mu_{\text{min},G}(E) > 0$.

(2) Let $\mathfrak{A}^\mu$ be the full subcategory of $\mathcal{C}$ such that $E \in \mathcal{C}$ belongs to $\mathfrak{A}^\mu$ if $E \neq 0$ or $E$ is a torsion free object with $\mu_{\text{max},G}(E) \leq 0$.

Definition 1.2.6. (1) Let $\mathfrak{A}_G$ be the full subcategory of $\mathcal{C}$ such that $E \in \mathcal{C}$ belongs to $\mathfrak{A}_G$ if (i) $E$ is a torsion object or (ii) for the Harder-Narasimhan filtration 1.2 of $E$, $E_s$ satisfies $\mu_G(E_s) > 0$ or $\mu_G(E_s) = 0$ and $\chi(G, E_s) > 0$.

(2) Let $\mathfrak{A}_G$ be the full subcategory of $\mathcal{C}$ such that $E \in \mathcal{C}$ belongs to $\mathfrak{A}_G$ if $E$ is a torsion free object and for the Harder-Narasimhan filtration 1.2 of $E$, $E_1$ satisfies $\mu_G(E_1) < 0$ or $\mu_G(E_1) = 0$ and $\chi(G, E_1) \leq 0$.

Definition 1.2.7. ($\mathfrak{A}^\mu$, $\mathfrak{A}^\mu_\mathbb{G}$), and ($\mathfrak{A}_G$, $\mathfrak{A}_G$) are torsion pairs of $\mathcal{C}$. We denote the tiltings of $\mathcal{C}$ by $\mathfrak{A}^\mu$ and $\mathfrak{A}_G$ respectively.
Definition 1.2.8. If \( v(G)/ \text{rk } G = e^\beta \), then we set \((\Sigma, \mathfrak{Z}) := (\Sigma_G, \mathfrak{Z}_G)\) and \( \mathfrak{A} := \mathfrak{A}_G \).

For \( E \in \mathcal{D}(X) \), we can write
\[
(1.3) \quad v(E) = re^\beta + a_0X + (dH + D) + (dH + D, \beta)g_X,
\]
where \( D \in \text{NS}(X) \otimes \mathbb{Q} \cap H^1 \), \( dH + D = c_1(E) - r\beta \), \( a \in \mathbb{Q} \). We note that \( (dH + D) + (dH + D, \beta)g_X \in (e^\beta) \perp \) and
\[
d = \frac{\text{deg}_G(E)}{r_0(H^2)} = \frac{\text{deg}(E(-\beta))}{(H^2)},
a = -\chi(E(\beta)).
\]

Hereafter we take \( \omega \in \mathbb{Q}_{>0}H \), and for the pair \((\beta, \omega)\) we construct some functions and categories as a preliminary of stability conditions.

**Definition 1.2.9.** We define \( Z(\beta, \omega) : \mathcal{D}(X) \to \mathbb{C} \) by
\[
Z(\beta, \omega)(E) := \langle e^{\beta + \sqrt{-1}\omega}, v(E) \rangle
\]
\[
= \langle e^\beta - \frac{(\omega^2)}{2}g_X + \sqrt{-1}(\omega + (\omega, \beta)g_X), v(E) \rangle
\]
\[
= -a + r \frac{(\omega^2)}{2} + \sqrt{-1}d(H, \omega).
\]

If \( Z(\beta, \omega)(\mathfrak{A}_G) \subset \mathbb{H} \cup \mathbb{R}_{<0} \), then \( Z(\beta, \omega) \) is a stability function in the sense of [3, 4] on \( \mathfrak{A}_G \), where \( \mathbb{H} := \{ z \in \mathbb{C} | \text{Im} z > 0 \} \) is the upper half plane. In this case, we have a function \( \phi : \mathcal{D}(X) \to \mathbb{R} \) such that
\[
Z(\beta, \omega)(E) = |Z(\beta, \omega)(E)| e^{\pi \sqrt{-1}\phi(E)}
\]
and \( \phi(E) \in (n, n + 1) \) for \( E \in \mathfrak{A}_G[n] \).

**Remark 1.2.10.** For the category of twisted sheaves, we take a locally free twisted sheaf \( G \) with \( \chi(G, G) = 0 \). Then we replace the Mukai vector \( v(E) \) by
\[
v_G(E) := \frac{\text{ch}(G^\vee \otimes E)}{\sqrt{\text{ch}(G^\vee \otimes G)}} \sqrt{\text{td}_X} \in H^*(X, \mathbb{Q})_{\text{alg}}.
\]
Then \( v_G(G) = \text{rk } Ge^\beta \) with \( \beta = 0 \) and we have an expression
\[
v_G(E) = r + a_0X + (dH + D),
\]
since \( \beta = 0 \). In this case, \( Z(\beta, \omega) \) is also well-defined.

**Definition 1.2.11.** For \((\beta, \omega), G(\beta, \omega) \in K(X)_{\mathbb{Q}} \) is an element satisfying
\[
(1.5) \quad v(G(\beta, \omega)) = e^\beta - \frac{(\omega^2)}{2}g_X.
\]

Then
\[
Z(\beta, \omega)(E) = -\chi_G(\beta, \omega)(E) + \sqrt{-1}d(H, \omega).
\]

**Definition 1.2.12.** For \((\beta, \omega)\), we define \((\Sigma(\beta, \omega), \mathfrak{Z}(\beta, \omega))\) and \( \mathfrak{A}(\beta, \omega) \) to be the categories \((\Sigma_G(\beta, \omega), \mathfrak{Z}_G(\beta, \omega))\) and \( \mathfrak{A}_G(\beta, \omega) \) with [15].

### 1.3. Examples of Bridgeland’s stability conditions.

In this section, we shall generalize Bridgeland’s explicit construction of stability condition [3 sect. 7]. It is also a geometric construction of the stability condition in [7, Lem. 4.8]. We keep the notation in the last subsection and fix the pair \((\beta, \omega)\).

**Proposition 1.3.1.** Assume that there is no \( G(\beta, \omega) \)-twisted stable object \( E \) with \( \text{deg}_G(\beta, \omega)(E) = \chi_G(\beta, \omega)(E) = 0 \). Then \( \sigma(\beta, \omega) := (\mathfrak{A}(\beta, \omega), Z(\beta, \omega)) \) is an example of Bridgeland’s stability condition.

**Proof.** By the definition of \( \mathfrak{A}(\beta, \omega), Z(\beta, \omega) \) is a stability function on \( \mathfrak{A}(\beta, \omega) \). We prove that the stability function \( Z(\beta, \omega) \) satisfies the Harder-Narasimhan properties.

Let \( E \) be an object of \( \mathfrak{A}(\beta, \omega) \). By the same proof of [3 Prop. 7.1], there is no chain of monomorphisms in \( \mathfrak{A}(\beta, \omega) \)
\[
\cdots \subset E_{i+1} \subset E_i \subset \cdots \subset E_1 \subset E_0 = E
\]
with \( \phi(E_{i+1}) > \phi(E_i) \) for all \( i \).

Assume that there is a chain of epimorphisms
\[
E = E_0 \to E_1 \to \cdots \to E_i \to E_{i+1} \to \cdots
\]
with \( \phi(E_i) > \phi(E_{i+1}) \) for all \( i \). By the same proof of \([3]\) Prop. 7.1, we may assume that \( \text{Im}Z(\beta, \omega)(E) = \text{Im}Z(\beta, \omega)(E_i) \) and \( H^0(E) \to H^0(E_i) \) is an isomorphism for all \( i \). We set \( L_i := \ker(E \to E_i) \). Then there is a chain

\[
0 = L_0 \subset L_1 \subset \cdots \subset L_i \subset \cdots \subset E
\]

and \( \text{Im}Z(\beta, \omega)(L_i) = 0 \) for all \( i \). By the definition of \( \mathfrak{A}(\beta, \omega) \), \( H^{-1}(L_i) \) is a \( \mu \)-semi-stable object with \( \deg_{G(\beta, \omega)}(H^{-1}(L_i)) = 0 \), and \( H^0(L_i) \) is an extension

\[
0 \to T \to H^0(L_i) \to F \to 0
\]

of a \( \mu \)-semi-stable object \( F \) with \( \deg_{G}(F) = 0 \) by a 0-dimensional object \( T \).

We may assume that \( H^{-1}(L_i) \to H^{-1}(L_{i+1}) \) is an isomorphism for all \( i \). We set \( B_i := L_i/L_{i-1} \). Then we have an exact sequence

\[
0 \to H^{-1}(B_i) \to H^0(L_{i-1}) \to H^0(L_i) \to H^0(B_i) \to 0.
\]

Since \( \chi_{G(\beta, \omega)}(H^{-1}(B_i)) \leq 0 \) and \( \chi_{G(\beta, \omega)}(H^0(B_i)) \geq 0 \), \( \chi_{G(\beta, \omega)}(H^0(L_{i-1})) \leq \chi_{G(\beta, \omega)}(H^0(L_i)) \).

We have

\[
0 \geq \chi_{G(\beta, \omega)}(H^{-1}(E_i)) = \chi_{G(\beta, \omega)}(H^{-1}(E)) - \chi_{G(\beta, \omega)}(H^{-1}(L_i)) + \chi_{G(\beta, \omega)}(H^0(L_i)).
\]

Hence \( \chi_{G(\beta, \omega)}(H^0(L_i)) \) is bounded above. Therefore \( \chi_{G(\beta, \omega)}(H^0(L_i)) \) is constant for \( i \gg 0 \). Then we have \( \chi_{G(\beta, \omega)}(H^0(B_i)) = \chi_{G(\beta, \omega)}(H^{-1}(B_i)) = 0 \), which implies that \( H^0(B_i) = 0 \). Hence \( H^0(L_{i-1}) \to H^0(L_i) \) is surjective for \( i \gg 0 \). By the Noetherian properties of \( \mathfrak{C} \), \( H^0(L_{i-1}) \to H^0(L_i) \) is an isomorphism for \( i \gg 0 \). Therefore \( L_{i-1} \to L_i \) is an isomorphism for \( i \gg 0 \).

\[\square\]

Remark 1.3.2. If \( X \) is an abelian surface, then \( \mathfrak{A}(\beta, \omega) = \mathfrak{A}^\mu \) for any \( \omega \). Thus Definition 1.2.12 is meaningful only for a K3 surface.

Definition 1.3.3. For \( E, E' \in \mathfrak{A}(\beta, \omega) \), we set

\[
\Sigma(\beta, \omega)(E', E) := \det \begin{pmatrix} \text{Re}Z(\beta, \omega)(E') & \text{Re}Z(\beta, \omega)(E) \\ \text{Im}Z(\beta, \omega)(E') & \text{Im}Z(\beta, \omega)(E) \end{pmatrix}.
\]

Definition 1.3.4. Let \( \mathfrak{k} \) be an arbitrary field. Assume that \( \sigma(\beta, \omega) \) is a stability function. Then \( E \in \mathfrak{A}(\beta, \omega) \) is semi-stable (with respect to \( Z(\beta, \omega) \)), if

\[
\Sigma(\beta, \omega)(E', E) \geq 0
\]

for all subobject \( E' \) of \( E \).

Remark 1.3.5. Since

\[
\Sigma(\beta, \omega)(E', E) = |Z(\beta, \omega)(E')||Z(\beta, \omega)(E)|\sin(\pi(\phi(E) - \phi(E')))
\]

and \( 0 < \phi(E), \phi(E') \leq 1 \), \( \Sigma(\beta, \omega)(E', E) \geq 0 \) if and only if \( \phi(E) - \phi(E') \geq 0 \). Thus the above definition is equivalent to Bridgeland’s definition of stability.

For \( (\beta, \omega) \), let \( \overline{\beta, \omega} \) be the corresponding element on \( X \otimes \mathfrak{k} \overline{\mathfrak{k}} \), where \( \overline{\mathfrak{k}} \) is the algebraic closure of \( \mathfrak{k} \). Then we have a natural identification \( \mathfrak{A}(\overline{\beta, \omega}) = (\mathfrak{A}(\beta, \omega)) \otimes \mathfrak{k} \overline{\mathfrak{k}} \) by Remark 1.2.34. In the appendix section 3.4 we shall prove that the stability of \( E \) is equivalent to the stability of \( E \otimes \mathfrak{k} \overline{\mathfrak{k}} \).

Definition 1.3.6. Assume that \( \sigma(\beta, \omega) \) is a stability function. Then \( E \in \mathfrak{A}(\beta, \omega) \) is stable (with respect to \( Z(\beta, \omega) \)), if

\[
\Sigma(\beta, \omega)(E', E \otimes \mathfrak{k} \overline{\mathfrak{k}}) > 0
\]

for all proper subobject \( E' \neq 0 \) of \( E \otimes \mathfrak{k} \overline{\mathfrak{k}} \).

Definition 1.3.7. \( M(\beta, \omega)(v) \) denotes the moduli stack of semi-stable objects \( E \) with respect to \( Z(\beta, \omega) \) such that \( v(E) = v \). If there is a coarse moduli scheme of stable objects, then we denote it by \( M(\beta, \omega)(v) \).

Remark 1.3.8. If the moduli scheme \( M(\beta, \omega)(v) \) exists, then the deformation theory implies that \( M(\beta, \omega)(v) \) is smooth of \( \dim M(\beta, \omega)(v) = (v^2) + 2 \).

Lemma 1.3.9. Let \( E \) be an irreducible object of \( \mathfrak{A}(\beta, \omega) \) with \( \deg(E(-\beta)) = 0 \).

1. If \( \text{rk } E \geq 0 \), then \( H^{-1}(E) = 0 \) and \( H^0(E) \) is 0-dimensional or \( H^0(E) \) is a \( \beta \)-twisted stable object with \( \langle v(E)^2 \rangle = -2 \).
2. If \( \text{rk } E < 0 \), then \( H^0(E) = 0 \) and \( H^{-1}(E) \) is a \( \beta \)-twisted stable object of \( \mathfrak{C} \).
Proof. We set $G := G_{(\beta, \omega)}$. For $E \in \mathfrak{A}_{(\beta, \omega)}$, we have an exact sequence
$$0 \to H^{-1}(E)[1] \to E \to H^0(E) \to 0.$$  

(1) Assume that $\text{rk} E \geq 0$. If $H^{-1}(E) \neq 0$, then the irreducibility of $E$ implies that $H^0(E) = 0$ and $H^{-1}(E)$ is a torsion free object of $\mathfrak{C}$ with $\text{rk} H^{-1}(E) > 0$, which is a contradiction. Therefore $H^{-1}(E) = 0$. If $H^0(E)$ has a torsion subobject $T$, then we have an exact sequence in $\mathfrak{A}_{(\beta, \omega)}$:  
$$0 \to T \to E \to E/T \to 0.$$  
By the irreducibility of $E$, $E$ is a torsion object. If dim $E = 1$, then we have a non-trivial quotient $\varphi : E \to E_1$ in $\mathfrak{C}$, which gives a non-trivial quotient of $E$ in $\mathfrak{A}_{(\beta, \omega)}$, since dim ker $\varphi \leq 1$. Therefore dim $E = 0$. If $H^0(E)$ is torsion free, we take the Harder-Narasimhan filtration of $H^0(E)$:
$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = H^0(E).$$  
Since $H^0(E) \in \mathfrak{S}_{(\beta, \omega)}$ and $\text{deg}(H^0(E)(-\beta)) = 0$, we see that $\text{deg}(F_i/F_{i-1}(-\beta)) = 0$ for all $i$. Then we have
$$\frac{\chi_G(F_1)}{\text{rk} F_1} > \frac{\chi_G(F_2/F_1)}{\text{rk} F_2/F_1} > \cdots > \frac{\chi_G(F_s/F_{s-1})}{\text{rk} F_s/F_{s-1}} > 0.$$  
Thus $F_i/F_{i-1} \in \mathfrak{S}_{(\beta, \omega)}$ for all $i$. By the irreducibility of $E$, $s = 1$. Thus $H^0(E)$ is $\beta$-twisted semi-stable. By the irreducibility, we also see that $H^0(E)$ is $\beta$-twisted stable. Since $\chi(H^0(E)(-\beta)) > \chi_G(H^0(E)) > 0$, we see that $(\psi(E))^2 < 0$. Hence $(\psi(E))^2 = -2$.

(2) Since $\text{rk} E < 0$, we have $H^{-1}(E) \neq 0$. By the irreducibility of $E$, $H^0(E) = 0$. Let
$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = H^{-1}(E)$$  
be the Harder-Narasimhan filtration of $H^{-1}(E)$. Since $H^{-1}(E) \in \mathfrak{S}_{(\beta, \omega)}$ and $\text{deg}(H^{-1}(E)(-\beta)) = 0$, we see that $\text{deg}(F_i/F_{i-1}(-\beta)) = 0$ for all $i$. Then we have
$$0 \geq \frac{\chi_G(F_1)}{\text{rk} F_1} > \frac{\chi_G(F_2/F_1)}{\text{rk} F_2/F_1} > \cdots > \frac{\chi_G(F_s/F_{s-1})}{\text{rk} F_s/F_{s-1}}.$$  
Thus $F_i/F_{i-1} \in \mathfrak{S}_{(\beta, \omega)}$ for all $i$. By the irreducibility of $E$, $s = 1$. Thus $H^{-1}(E)$ is $\beta$-twisted semi-stable. By the irreducibility, we also see that $H^{-1}(E)$ is $\beta$-twisted stable. □

**Lemma 1.3.10.** Let $E$ be a $\beta$-twisted stable object of $\mathfrak{C}$ with $\text{deg}(E(-\beta)) = \chi(E(-\beta)) = 0$. Then $E[1]$ is an irreducible object of $\mathfrak{A}$.  

**Proof.** We set $G := G_{(\beta, \omega)}$. Assume that there is an exact sequence in $\mathfrak{A}_{(\beta, \omega)}$:
$$0 \to F_1 \to E[1] \to F_2 \to 0.$$  
Then we have an exact sequence in $\mathfrak{C}$:
$$0 \to H^{-1}(F_1) \to E \xrightarrow{\psi} H^{-1}(F_2) \to H^0(F_1) \to 0.$$  
Since $\text{deg}_{\mathfrak{S}}(H^{-1}(F_2)) \leq 0$ and $\text{deg}_{\mathfrak{S}}(H^0(F_1)), \text{deg}_{\mathfrak{S}}(H^0(F_2)) \geq 0$, we have $0 = \text{deg}_{\mathfrak{S}}(E) = \text{deg}_{\mathfrak{S}}(H^{-1}(F_1)) + \text{deg}_{\mathfrak{S}}(H^{-1}(F_2)) - \text{deg}_{\mathfrak{S}}(H^0(F_1)) \leq 0$. Hence $\text{deg}_{\mathfrak{S}}(H^{-1}(F_1)) = \text{deg}_{\mathfrak{S}}(H^{-1}(F_2)) = \text{deg}_{\mathfrak{S}}(H^0(F_1)) = 0$. Since $\chi_G(H^0(F_1)) > 0$ and $\chi_G(H^{-1}(F_2)) \leq 0$, $\chi_G(\text{im} \psi) < 0$. Since $\chi_G(E)$ is sufficiently close to 0, $\chi(\text{im} \psi(-\beta)) = 0$. By the $\beta$-stability of $E$, $E \to \psi$ is an isomorphism. Therefore $E[1]$ is irreducible. □

## 1.4. The wall and chamber for categories.

### 1.4.1. For the stability condition $\sigma_{(\beta, \omega)}$, the abelian category $\mathfrak{A}_{(\beta, \omega)}$ depends on the choice of $\beta$ and $\omega \in \mathbb{R}_{>0}H$. In this subsection, we shall study the dependence under fixing $b := (\beta, H)/(H^2) \in \mathbb{Q}$. So we assume that $X$ is a $K3$ surface (cf. Remark 1.3.2). We first note that $\eta := \beta - bH \in H^2$.

**Definition 1.4.1.** We set
$$\mathfrak{S} := \{(\eta, \omega) | \eta \in \text{NS}(X)_{\mathbb{Q}}, \langle \eta, H \rangle = 0, \omega \in \mathbb{R}_{>0}H\},$$  
$$\mathfrak{S}_{b} := \{(\eta, \omega) | \eta \in \text{NS}(X)_{\mathbb{R}}, \langle \eta, H \rangle = 0, \omega \in \mathbb{R}_{>0}H\}.$$  
We have an embedding of $\mathfrak{S}_{b}$ into $\text{NS}(X)_{\mathbb{C}}$ via $(\eta, \omega) \mapsto \eta + \sqrt{-1}\omega$. Thus we have an identification:
$$\mathfrak{S}_{b} \cong (\text{NS}(X)_{\mathbb{R}} \cap H^+) + \sqrt{-1}\mathbb{R}_{>0}H.$$  

**Remark 1.4.2.** We shall introduce an embedding of $\mathfrak{S}_{b}$ into a sphere. For the vector space
$$V_H := (\text{NS}(X)_{\mathbb{R}} \cap H^+) + \sqrt{-1}\mathbb{R}H,$$
the intersection pairing is a negative definite real form. We set
$$\mathfrak{I} := \{\mathbb{R}x \in \mathbb{P}(\mathbb{R} \oplus V_H) \cap (\eta^2) = 0\},$$  
$$\mathfrak{I}_{bH} := e^{bH}\mathfrak{I} := \{\mathbb{R}x \in \mathbb{P}(H^*(X, \mathbb{C}))|\mathbb{R}xe^{-bH} \in \mathfrak{I}\}.$$
We set $x = r + \xi + a q_X \in J_{BH}$. If $r \neq 0$, then $x = re^{bH + \eta + \sqrt{-1} \omega}, \eta + \sqrt{-1} \omega \in V_H$. If $r = 0$, then $x = a q_X$. Thus $J$ is identified with a compactification $V_H := V_H \cup \{\infty\}$ of $V_H$, where $R_{BX}$ corresponds to $\infty$. We shall prove that $J$ is diffeomorphic to $\rho$-dimensional sphere $S^\rho$, where $\rho = \text{rk} \, \text{NS}(X)$. For $x = r + \xi + a q_X$ with $\xi \in V_H, \langle x^2 \rangle = 0$ if and only if ($\xi^2 = 2ra$. We shall identify $R^\rho$ with $V_H$ by sending $(y_1, \ldots, y_{\rho-1}, y_\rho)$ to $\sum_{i=1}^{\rho-1} y_i \xi_i + \sqrt{-1} y_\rho H$, where $h = H/\sqrt{(H^2)}$ and $\xi_i \in H^1 (1 \leq i \leq \rho - 1)$ satisfy $-(\xi_i, \xi_j) = \delta_{ij}$. Let $S^\rho$ be a sphere in $R^\rho \times R$. Then we have a diffeomorphism:

$$J \rightarrow S^\rho$$

$$R(r + \xi + a q_X) \rightarrow \left(\frac{2\xi}{r - 2a}, \frac{2a + r}{2a - r}\right).$$

The correspondence $S^\rho \rightarrow V_H$ is nothing but the stereographic projection from $(0,1) \in S^\rho$, and we get a desired embedding $J_{RB} \hookrightarrow S^\rho$. We set

$$J_{RB} := J_R \cup \{\infty\}.$$ This embedding will be used in subsection 1.5 to describe the action of Fourier-Mukai transforms.

**Definition 1.4.3.** We set

$$\mathcal{R} := \{u \in H^*(X, Z)_{alg}| u \in (H + (H,bH)g_X)^+, \langle u^2 \rangle = -2\}.$$ For $u \in \mathcal{R}$, we define a wall $W_u$ of $\mathcal{R}$ as

$$\{(\eta, \omega) \in J_{RB}| \text{rk} u(\omega^2)/2 = -(e^{bH + \eta}, u)\}.$$ A connected component of $J_{RB} \setminus \cup_{u \in \mathcal{R}} W_u$ is called a chamber for categories.

**Remark 1.4.4.** Assume that $u = re^\beta + a q_X + D + (D, \beta)g_X$ belongs to $\mathcal{R}$. If $Z_{(\beta, \omega)}(E) = 0$, then $a = \rho(\omega^2)$ implies that $r > 0$ if and only if $a > 0$. If $r = 0$, then $u = D + (D, \beta)g_X$ implies that $(D, \beta) \in Z$. Hence $u \in \mathcal{R}$ with $\text{rk} u = 0$ are used to describe the dependence on the category $\mathcal{C}$.

**Remark 1.4.5.** For $u \in \mathcal{R}$ with $\text{rk} u > 0$, $W_u$ is the half sphere defined by

$$-(\eta - c_1(u)/\text{rk} u + bH)^2 + (\omega^2) = 2/\text{rk} u^2.$$ 

**Lemma 1.4.6.** The set of walls is locally finite.

**Proof.** Let $B$ be a compact subset of $J_{RB}$. We shall prove that

$$\{u \in \mathcal{R}| W_u \cap B \neq \emptyset\}$$

is a finite set. For $u = re^{bH} + a q_X + D + (D, bH)g_X \in \mathcal{R}, a = \frac{(D^2 + 2}{4r}$ and $W_u$ is the half sphere

$$-(\eta - D/r)^2 + (\omega^2) = 2/r^2.$$ Hence $r^2 < 2/(\omega^2)$ and

$$2 > -(r\eta - D)^2 \geq \left(\sqrt{2/(\omega^2)} - \sqrt{-(D^2)}\right)^2.$$ Since $B$ is a compact subset of $J_{RB}$, the choice of $r$ and $(D^2)$ are finite. We denote the denominator of $b$ by $b_0$. Since $D = c_1(u) - rbH \in \frac{1}{b_0}\text{NS}(X)$, the choice of $D$ is also finite. Hence the claim holds.

1.4.2. For a fixed $\beta := bH + \eta, \eta \in H^1$, we have an injection

$$i_\beta : \mathcal{R}_{bH} \rightarrow J_{RB}$$

$$\omega \rightarrow (\eta, \omega).$$ Then we also have the notion of walls and chambers on $\mathcal{R}_{bH}$. In this case, the category $\mathcal{C}$ is fixed.

**Lemma 1.4.7.** We set

$$\mathcal{R}_\beta := \{u \in \mathcal{R}| \text{rk} u > 0, \langle e^\beta, u \rangle > 0\}.$$ Then $\mathcal{R}_\beta$ is a finite set and $\text{rk} u \leq r_0$.

**Proof.** We set

$$u := re^\beta + a q_X + (D + (D, bH)g_X), D \in H^1.$$ By the assumption, $a = -(e^\beta, u) > 0$ and $-2 = (u^2) = -2ra + (D^2) \leq -2ra$. Hence $0 < r(r_0a) \leq r_0$. Since $-r_0a = (r_0e^\beta, u) \in Z$ and $r \in Z, r$ and $r_0a$ are positive integers with $r(r_0a) \leq r_0$. Thus the choices of $r$ and $a$ are finite. Since $D = (c_1(u) - r\beta) \in \frac{1}{r_0}\text{NS}(X) \cap H^1$ and $0 \leq -(D^2) \leq -2ra + 2$, the choice of $u$ is also finite.

**Definition 1.4.8.** For $u \in \mathcal{R}_\beta$, we define a wall $W_{\beta,u}$ of $\mathcal{R}_{bH}$ as

$$\{\omega \in \mathcal{R}_{bH}| (\omega^2)/2 = -(e^\beta, u)/\text{rk} u\}.$$ A connected component of $\mathcal{R}_{bH} \setminus \cup_{u \in \mathcal{R}_\beta} W_{\beta,u}$ is called a chamber for categories.

**Remark 1.4.9.** If $W_{\beta,u} \neq \emptyset$, then $W_{\beta,u} = i_\beta^{-1}(W_u)$, and $W_u$ intersects with $i_\beta(\mathcal{R}_{bH})$ transversely.
For \( u \in \mathfrak{R}_\beta \), there is a \( \beta \)-twisted semi-stable object \( E \) of \( \mathfrak{C} \) with \( v(E) = u \). Since \( \mathcal{W}_{\beta,u} \) depends only on \( u / \text{rk} u \), we introduce the following definition.

**Definition 1.4.10.** Let \( \text{Exc}_\beta \) be the set of \( \beta \)-twisted stable objects of \( \mathfrak{C} \) with

\[
v(E) = re^\beta + q_X + (D + (D, \beta)q_X), r, a > 0, D \in H^1.
\]

**Lemma 1.4.11.** (1) \( \text{Exc}_\beta \) is a finite set and \( \{ v(E) | E \in \text{Exc}_\beta \} \subset \mathfrak{R}_\beta \).

(2) For \( E \in \text{Exc}_\beta \), \( \text{rk} E \leq r_0 \) and \( \text{rk} E < r_0 \) unless \( r_0 = 1 \) and \( v(E) = e^\beta + q_X \).

(3) Let \( E_1, E_2, \ldots, E_s \) be the objects of \( \text{Exc}_\beta \) with \( \chi_{G_\beta}(E_i) > 0 \).

(a) For \( E \in \mathfrak{T}_{(\beta, \omega)} \), there is an exact sequence

\[
0 \to F_1 \to E \to F_2 \to 0
\]

such that \( F_1 \in \mathfrak{T}_\mu \) and \( F_2 \) is a successive extension of \( E_i \), \( 1 \leq i \leq s \).

(b) For \( E \in \mathfrak{T}_{(\beta, \omega)} \), there is an exact sequence

\[
0 \to F_1 \to E \to F_2 \to 0
\]

such that \( F_1 \) is a successive extension of \( E_i \), \( s + 1 \leq i \) and \( F_2 \in \mathfrak{T} \).

**Proof.** (1), (2) In the notation of \((\ref{1.3})\), if \( E \in \text{Exc}_\beta \), then \(-2 \leq \langle v(E)^2 \rangle = -2ra + (D^2) \leq -2ra < 0 \). Hence \( \langle v(E)^2 \rangle = -2 \), which implies that \( v(E) \in \mathfrak{R}_\beta \). In particular, \( E \) is an exceptional object:

\[
\text{Hom}(E, E) = \mathfrak{k}, \text{Ext}^1(E, E) = 0.
\]

Therefore (1) and by Lemma \([1.4.7] \) the first claim of (2) hold. If \( r = r_0 \), then we have \( r_0 a = 1 \) and \( D = 0 \).

Hence \( v(E) = r_0 e^\beta - q_X \). Since \( r_0 e^\beta \in H^1(X, \mathbb{Z}) \), \( a \in \mathbb{Z} \). Then \( ra = 1 \) implies that \( r = r_0 = 1 \) and \( a = 1 \).

(3) In the notation of \((\ref{1.3})\), if \( E \) is a \( G_\beta \)-stable object of \( \mathfrak{C} \) with \( \deg_{G_\beta}(E) = \chi_{G_\beta}(E) \geq 0 \), then \( d = 0 \) and \( a \geq \frac{\langle \omega^2 \rangle}{r_0^2} \). If \( r = 0 \), then \( E \) is a 0-dimensional object. Thus \( E \in \mathfrak{T}_\mu \). If \( r > 0 \), then \( a > 0 \), which implies that \( E \in \text{Exc}_\beta \). Then the claim follows from the definition of \( \mathfrak{T}_{(\beta, \omega)} \) and \( \text{Exc}_\beta \).

**Corollary 1.4.12.** We fix \( \beta \) and take \( \omega \in \mathbb{Q}_{>0}H \). Then \( \mathfrak{A}_{(\beta, \omega)} \) depends only on the chamber in \( \mathbb{R}_{>0}H \) where \( \omega \) belongs.

We note that \( \mathfrak{T} \supseteq \mathfrak{T}_{(\beta, \omega)} \supseteq \mathfrak{T}_\mu \).

**Corollary 1.4.13.**

(1) If \( \langle \omega^2 \rangle < \frac{2}{r_0^2} \), then \( \mathfrak{T} = \mathfrak{T}_{(\beta, \omega)} \). Thus \( (\mathfrak{A}, \mathfrak{Z}_{(\beta, \omega)}) \) is an example of Bridgeland’s stability condition.

(2) If \( \langle \omega^2 \rangle \geq 2 \), then \( \mathfrak{T}_{(\beta, \omega)} = \mathfrak{T}_\mu \). Thus \( (\mathfrak{A}_\mu, \mathfrak{Z}_{(\beta, \omega)}) \) is an example of Bridgeland’s stability condition.

**Proof.** By Lemma \([1.4.11] (3), \mathfrak{T} = \mathfrak{T}_{(\beta, \omega)} \) if \( \langle \omega^2 \rangle / 2 < -\langle e^\beta, v(E) \rangle / \text{rk} E \) for all \( E \in \text{Exc}_\beta \), and \( \mathfrak{T}_\mu = \mathfrak{T}_{(\beta, \omega)} \) if \( \langle \omega^2 \rangle / 2 > -\langle e^\beta, v(E) \rangle / \text{rk} E \) for all \( E \in \text{Exc}_\beta \). For \( E \in \text{Exc}_\beta \), we have

\[
\frac{1}{\langle \omega^2 \rangle} \geq \frac{\langle e^\beta, v(E) \rangle}{\text{rk} E} \geq \frac{1}{r_0 \text{rk} E}.
\]

By Lemma \([1.4.11] (2) \) and the definition of \( \text{Exc}_\beta \), we have \( 1 \leq \text{rk} E \leq r_0 \), which implies the claims.

**Example 1.4.14.** Let \( X \) be a K3 surface with \( \text{Pic}(X) = \mathbb{Z}H \) and \( E_0 \) be an exceptional vector bundle on \( X \). We set \( \beta := c_1(E_0) / \text{rk} E_0 \). Then \( r_0 = (\text{rk} E_0)^2 \) and \( v(E_0) = \sqrt{\text{rk} e^\beta + \sqrt{\text{rk} - 1}} q_X \). Hence \( \langle e^\beta - (\omega^2)q_X/2, v(E_0) \rangle = 0 \) if and only if \( \langle \omega^2 \rangle = 1/r_0 \). Therefore

\[
\mathfrak{A}_{(\beta, \omega)} = \begin{cases}
\mathfrak{A}, & \langle \omega^2 \rangle < 2/r_0 \\
\mathfrak{A}_\mu, & \langle \omega^2 \rangle > 2/r_0.
\end{cases}
\]

The following example is inspired by Washino \([31] \).

**Example 1.4.15.** Let \( \pi : X \to \mathbb{P}^1 \) be an elliptic K3 surface with a section \( \sigma \). Let \( f \) be a fiber of \( \pi \). We set \( H := \sigma + 4f \) and \( D := \sigma - 2f \). Then \( (H^2) = 6 \), \( (H, D) = 0 \) and \( (D^2) = -6 \). We set \( \beta + \sqrt{-1} \omega = x D + \sqrt{-1} y H \), \( x, y \in \mathbb{R} \). For \( \eta := x D, x < 1/2, \chi(O_X(\eta), O_X(D)) < \chi(O_X(\eta), O_X) \). Then a non-trivial extension

\[
0 \to O_X(D) \to E_1 \to O_X \to 0
\]

defines an \( \eta \)-twisted stable sheaf. We set \( u_1 := v(O_X) \), \( u_2 := v(O_X(D)) \) and \( u_3 := v(E_1) = u_1 + u_2 \). Then the equations for \( W_{u_i}, i = 1, 2, 3 \) are

\[
\begin{align*}
W_{u_1} & : x^2 + y^2 = \frac{1}{3} \\
W_{u_2} & : (x - 1)^2 + y^2 = \frac{1}{3} \\
W_{u_3} & : (x - \frac{1}{2})^2 + y^2 = \frac{1}{12}.
\end{align*}
\]
They pass through the point \( \left( \frac{1}{2}, \frac{1}{2\sqrt{3}} \right) \). By the action of \( R_{u_1}, W_{u_2} \) and \( W_{u_3} \) are exchanged. It is easy to see that

\[
\begin{align*}
\mathfrak{A}_{D/3} &= \{ u_1, u_3 \}, \\
\mathfrak{A}_{D/2} &= \{ u_1, u_2, u_3 \}.
\end{align*}
\]

For \( \beta = D/3 \), we have three categories \( \mathfrak{A}, \mathfrak{A}^\mu, \mathfrak{A}_3 \):

\[
\mathfrak{A}_{(\beta, \omega)} = \begin{cases} 
\mathfrak{A}, & (\omega^2)/2 < 1/6 \\
\mathfrak{A}_3, & 1/6 < (\omega^2)/2 < 2/3 \\
\mathfrak{A}^\mu, & (\omega^2)/2 > 2/3.
\end{cases}
\]

For \( \beta = D/2 \), we have

\[
\mathfrak{A}_{(\beta, \omega)} = \begin{cases} 
\mathfrak{A}, & (\omega^2)/2 < 1/4 \\
\mathfrak{A}^\mu, & (\omega^2)/2 > 1/4.
\end{cases}
\]

In this example, \( u_1, u_2 \) generate a negative definite lattice of type \( A_2 \).

**Definition 1.4.16.** Let \( W \) be a wall of \( \mathbb{R}_{>0}H \). Let \( \mathcal{S}_W \) be the category generated by all \( E \in \text{Exc}_\beta \) with \( W_{\beta, \tau(E)} = W \).

**Lemma 1.4.17.** \( \mathcal{S}_W \) is the category of \( \beta \)-twisted semi-stable objects \( E \) of \( \mathcal{C} \) with \( \text{rk} \, E > 0 \), \( \deg(E(-\beta)) = 0 \) and \( \chi(E(-\beta)) = \frac{(\omega^2)}{2} \text{rk} \, E \) (constant).

**Lemma 1.4.18.** Assume that \( \omega \) belongs to a wall \( W \). We take \( \omega_\pm \in \mathbb{Q}_{>0}H \) such that \( \omega_\pm \) are sufficiently close to \( \omega \) and \( (\omega^2) < (\omega^2) < (\omega^2) \).

1. Let \( E_1 \) be a subobject of \( E \in \mathcal{S}_W \) in \( \mathfrak{A}_{(\beta, \omega_-)} \). Then \( E_1 \in \mathcal{S}_W \). In particular, \( E \in \mathcal{S}_W \cap \text{Exc}_\beta \) is an irreducible object of \( \mathfrak{A}_{(\beta, \omega_-)} \).

2. Let \( E_1 \) be a subobject of \( E[1] \in \mathcal{S}_W[1] \) in \( \mathfrak{A}_{(\beta, \omega_+)} \). Then \( E_1 \in \mathcal{S}_W[1] \). In particular, \( E[1] \in (\mathcal{S}_W \cap \text{Exc}_\beta)[1] \) is an irreducible object of \( \mathfrak{A}_{(\beta, \omega_+)} \).

**Proof.** We set \( G := G_{(\beta, \omega_-)}, G_+ := G_{(\beta, \omega_+)} \in K(X) \).

1. Assume that there is an exact sequence in \( \mathfrak{A}_{(\beta, \omega_-)} \)

\[
0 \to E_1 \to E \to E_2 \to 0.
\]

Then we have an exact sequence

\[
0 \to H^{-1}(E_2) \xrightarrow{\varphi} H^0(E_1) \to E \to H^0(E_2) \to 0.
\]

As in the proof of Lemma 1.3.10 we see that \( \text{deg}_G(H^{-1}(E_2)) = \text{deg}_G(H^0(E_1)) = \text{deg}_G(H^0(E_2)) = 0 \). Assume that \( E_1 \neq 0 \). Then \( \chi_G(H^0(E_1)) > 0 \) and \( \chi_{G_+}(H^{-1}(E_2)) \leq 0 \). Since \( \omega_- \) is sufficiently close to \( \omega \), we have \( \chi_G(\text{im} \, \varphi) = \chi_G(H^0(E_1)) - \chi_{G_+}(H^{-1}(E_2)) \geq \chi_G(H^{-1}(E_2)). \) If \( H^{-1}(E_2) \neq 0 \), then \( \text{rk} \, H^{-1}(E_2) \neq 0 \), which implies that \( \chi_G(H^{-1}(E_2)) < \chi_{G_+}(H^{-1}(E_2)) \leq 0 \). Therefore \( \chi_G(\text{im} \, \varphi) > 0 \). By the semi-stability of \( E \) and \( \chi_G(E) = 0 \), \( \chi_G(\text{im} \, \varphi) \leq 0 \), which is a contradiction. Therefore \( H^{-1}(E_2) = 0 \) and \( E_1 \) is a \( \beta \)-twisted semi-stable object with \( \chi_G(E_1) = 0 \). Thus \( E_1 \in \mathcal{S}_W \).

2. Assume that there is an exact sequence in \( \mathfrak{A}_{(\beta, \omega_+)} \)

\[
0 \to E_1 \to E[1] \to E_2 \to 0.
\]

Then we have an exact sequence

\[
0 \to H^{-1}(E_1) \to E \to H^{-1}(E_2) \xrightarrow{\varphi} H^0(E_1) \to 0.
\]

Assume that \( E_2 \neq 0 \). If \( H^0(E_1) \neq 0 \), then \( \chi_G(H^0(E_1)) \geq \chi_G(H^0(E)) > 0 \). Since \( \omega_+ \) is sufficiently close to \( \omega \), we have \( \chi_G(H^{-1}(E_2)) < \chi_{G_+}(H^0(E)) \). Hence \( \chi_G(\text{ker} \, \varphi) < 0 \), which is a contradiction. Therefore \( H^0(E_1) = 0 \). Since \( \chi_G(H^{-1}(E_2)) \leq 0 \), \( H^{-1}(E_1) \) and \( H^{-1}(E_2) \) are \( \beta \)-twisted semi-stable objects with \( \chi_G(H^{-1}(E_1)) = \chi_G(H^{-1}(E_2)) = 0 \). Therefore \( E_1 \in \mathcal{S}_W[1] \).

**Lemma 1.4.19.** For a fixed \( G \), \( \{ E \in K(X) | \text{deg}_G(E) = \chi_G(E) = 0 \} \) is a negative definite sublattice of \( H^*(X, \mathbb{Z})_{\text{alg}} \). In particular, the sublattice \( \langle \mathcal{S}_W \rangle \) generated by \( \mathcal{S}_W \) is a direct sum of lattices of type \( ADE \).

**Proof.** The signature of \( H^*(X, \mathbb{Z})_{\text{alg}} \) is \( (2, \rho(X)) \). Since \( \langle v(G)^2 \rangle = v^2_0(\omega^2), (H^2) > 0 \) and \( H + (H, \beta) \mathcal{E}_X \perp v(G) \), we get the claim.
1.5. Relation with the Fourier-Mukai transforms. Let $X'$ be an abelian surface or a $K3$ surface. Let
\[
\Phi_{X \to X'}^\vee : \mathcal{D}(X) \to \mathcal{D}(X')
\]
be a Fourier-Mukai transform whose kernel is $E \in \mathcal{D}(X \times X')$. Assume that $v(E_{|X 	imes \{s\}}) = r_0 e_\beta$, $r_0 > 0$. For simplicity, we set $\Phi := \Phi_{X \to X'}^\vee$. For $C \in \text{NS}(X)_\mathbb{Q}$, we set $\tilde{C} := -c_1(\Phi(C + (C, \beta)q_X))) \in \text{NS}(X')_\mathbb{Q}$. Since $\Phi(e_\beta) = \frac{1}{r_0}q_{X'}$, and $\Phi(q_X) = r_0 e_\beta$, we have
\[
v(\Phi(E)[1]) = -r_0 a e_\beta - \frac{r}{r_0} q_{X'} + (d\tilde{H} + \tilde{D}) + (d\tilde{H} + \tilde{D}, \beta')q_{X'},
\]
where $v(E)$ is given by (1.3). We set $b' := (\beta', \tilde{H})/(\tilde{H}^2)$. Then we have a diffeomorphism
\[
\mathcal{H}_H \to \mathcal{H}_{b', \tilde{H}}
\]
induced by $\Phi$. Let $\mathcal{H}'_\mathbb{R}$ be the space for $(X', \tilde{H})$ in Definition 1.4.1. We shall study the relation of $\mathcal{H}'_\mathbb{R}$ and $\mathcal{H}'_\mathbb{R}$ explicitly. For $\eta + \sqrt{1-\omega}$ with $(\eta, H) = 0$, we set
\[
\bar{\eta} + \sqrt{1-\omega} := -\frac{2}{r_0((\eta + \sqrt{1-\omega})^2)}(\tilde{\eta} + \sqrt{1-\omega}).
\]
Then we have
\[
\Phi(e^{\beta + \eta + \sqrt{1-\omega}}) = \Phi(e^\beta(1 + (\eta + \sqrt{1-\omega}) + \frac{(\eta + \sqrt{1-\omega})^2}{2} q_X))
\]
\[
= \Phi(e^\beta + (\eta + \sqrt{1-\omega}) + (\eta + \sqrt{1-\omega}, \beta)q_X + \frac{(\eta + \sqrt{1-\omega})^2}{2} q_X)
\]
\[
= \frac{1}{r_0} q_{X'} + \frac{r_0}{r_0} \frac{(n + \sqrt{1-\omega})^2}{2} e_\beta' - ((\bar{\eta} + \sqrt{1-\omega}) + (\bar{\eta} + \sqrt{1-\omega}, \beta')q_{X'})
\]
\[
= \frac{r_0}{r_0} \frac{(n + \sqrt{1-\omega})^2}{2} e_\beta'.
\]
Hence
\[
Z_{(\beta', \bar{\eta}, \bar{\omega})}(\Phi(E)[1]) = (e^{\beta + \bar{\eta} + \sqrt{1-\omega}}, v(\Phi(E)[1]))
\]
\[
= -\frac{2}{r_0((\eta + \sqrt{1-\omega})^2)}(\Phi(e^{\beta + \eta + \sqrt{1-\omega}}, v(\Phi(E)[1]))
\]
\[
= -\frac{2}{r_0((\eta + \sqrt{1-\omega})^2)}(e^{\beta + \eta + \sqrt{1-\omega}, v(E))
\]
\[
= -\frac{2}{r_0((\eta + \sqrt{1-\omega})^2)} Z_{(\beta + \eta, \omega)}(E).
\]
Remark 1.5.1. Since $(\eta, H) = 0$, we have $-((\eta + \sqrt{1-\omega})^2 = (\omega^2) - (\eta^2) > 0$. We also have
\[
((\bar{\omega}^2) - (\bar{\eta}^2))((\omega^2) - (\eta^2)) = \frac{4}{r_0}.
\]
Lemma 1.5.2. Assume that $\bar{\omega}$ is nef and big.

(1) The correspondence $(\eta, \omega) \mapsto (\bar{\eta}, \bar{\omega})$ induced by $\Phi[1]$ preserves the structure of chamber.
(2) If $(\mathcal{H}_{(\beta + \eta, \omega)}, Z_{(\beta + \eta, \omega)})$ is a stability condition on $X$, then $\Phi[1]$ induces a stability condition $(\Phi[1])((\mathcal{H}_{(\beta + \eta, \omega)}, Z_{(\beta + \eta, \omega)}))$ on $X'$.

Proof. (1) Let $\mathcal{Y}'$ be the set in Definition 1.4.3 associated to $X'$. Then $-\Phi(\mathcal{Y}) = \mathcal{Y}'$. By (1.11), $(\eta, \omega) \in W_u$ if and only if $(\bar{\eta}, \bar{\omega}) \in W_{-\Phi[1]}$. Hence the claim holds. (2) is obvious. \qed

Let $U$ be a $\beta$-twisted stable object of $\mathcal{C}$ with $\text{rk} U = r$, $\text{deg}(U(-\beta)) = 0$ and $\langle v(U)^2 \rangle = -2$. We set $\gamma := c_1(U)/r$. Then $v(U) = re^\gamma + \frac{1}{2} q_X$. We set $E := \text{Cone}(U \boxtimes U^\vee \to \mathcal{O}_X)[-1]$. Then $v(E_{|X \times \{s\}}) = r^2 e^\gamma$. Since the Fourier-Mukai transform $\Phi[1] := \Phi_{X \to X'}^E$ induces a $(-2)$-reflection $R_{\nu}(U)$, we get $\tilde{\xi} = \xi$ for any $\xi \in H^2(X, \mathbb{Q})$. We write $\gamma = bH + v$, $v \in H^1$. Then we have a diffeomorphism
\[
\Phi_{\beta} : \frac{\mathcal{H}'_{\mathbb{R}}}{\mathcal{H}_{-\Phi[1]}} \quad \eta + \sqrt{-1-\omega} + v \mapsto \frac{2(n + \sqrt{1-\omega})}{r((-\eta^2) + (\omega^2))} + v
\]
1.6. A small perturbation of $\sigma_{(\beta,\omega)} = (\mathfrak{G}_{(\beta,\omega)}, Z_{(\beta,\omega)})$. We take $\eta \in \NS(X)_{\mathbb{Q}}$ with $(H, \eta) = 0$. We shall study the perturbed stability condition $\sigma_{(\beta + \eta,\omega)}$ of $\sigma_{(\beta,\omega)}$. Since $(c_1(E) - (\rk E)(\beta + \eta), H) = (c_1(E) - (\rk E)\beta, H)$ for $E \in D(X)$, $(\mathfrak{F}^a, \mathfrak{G}^p)$ and $\mathfrak{G}^a$ do not depend on the choice of $\eta$. Since
\[
ed + v = e^\beta + \sqrt{\omega} + \eta + (\eta, \beta + \sqrt{-1}\omega)g_X + \frac{(\eta^2)}{2}g_X,
\]
we have
\[
Z_{(\beta + \eta,\omega)}(E) = Z_{(\beta,\omega)}(E) + (\eta, \beta)g_X, v(E)) - \frac{(\eta^2)}{2} \rk E.
\]

**Lemma 1.6.1.** Assume that $\eta$ satisfies $-2(\eta^2) < 1/r_0^2$, $(\omega^2) > -3(\eta^2)$ and $\sqrt{-2(\eta^2)} - \frac{(\eta^2)}{2} < \frac{(\omega^2)}{2}$. Let $E$ be a $(\beta + \eta)$-twisted stable object of $\mathfrak{C}$ with $\deg(E(-\beta)) = 0$. If $\chi(E(-\beta)) \leq 0$, then $Z_{(\beta + \eta,\omega)}(E) \in \mathbb{R}_{>0}$.

**Proof.** We set
\[
v(E) := re^\beta + ag_X + (D + (D, \beta)g_X), D \in H^+.
\]
We first assume that $a < 0$. Then $-a \geq \frac{1}{r_0}$. Assume that $-2(\eta^2) < 1/r_0^2$ and $(\omega^2) > -3(\eta^2)$. Since $-2 \leq \langle v(E)^2 \rangle = -2ra + (D^2), -(D^2) \leq 2(1 - ra)$. Since $H^+$ is negative definite, the Schwarz inequality implies that
\[
\langle (\eta, D) \rangle \leq \sqrt{-2(\eta^2)} \sqrt{-2(D^2)} \leq \sqrt{-2(\eta^2)(1 - ra)}.
\]
Since $-(\eta^2) < (\omega^2)/2 + (\eta^2)/2$, we can take $\lambda \in \mathbb{Q}$ such that $-(\eta^2) < \lambda < (\omega^2)/2 + (\eta^2)/2$. Then
\[
(\lambda r - a)^2 - (2(\eta^2)(1 - ra)) = \lambda^2 r^2 + a^2 + 2(\eta^2) - (2(\lambda + 2(\eta^2))ra)
\]
\[
+ \lambda^2 r^2 + (1/r_0^2 + 2(\eta^2)) - (2(\lambda + 2(\eta^2))ra) > 0.
\]
Hence $(\eta, D)^2 < (\lambda r - a)^2$, which implies that $-(\lambda r - a) < (\eta, D) < (\lambda r - a)$. Then we have
\[
Z_{(\beta + \eta,\omega)}(E) = (-a + (\eta, D)) + \frac{(\eta^2)}{2}r + \frac{(\omega^2)}{2}r
\]
\[
= \frac{(\omega^2) + (\eta^2)}{2} - \lambda r + (\lambda r - a + (\eta, D)) > 0.
\]
Assume that $a = 0$. Since $d = 0$, we have $r > 0$. We further assume that $\sqrt{-2(\eta^2)} - \frac{(\eta^2)}{2} < \frac{(\omega^2)}{2}$. In this case, we have $-\sqrt{-2(\eta^2)} \leq (\eta, D) \leq -\sqrt{-2(\eta^2)}$ by (1.13). Hence
\[
Z_{(\beta + \eta,\omega)}(E) \geq -\sqrt{-2(\eta^2)} + \frac{(\eta^2)}{2} + \frac{(\omega^2)}{2} > 0.
\]
\[
\square
\]

**Lemma 1.6.2.** Under the assumption in Lemma 1.6.1, $\mathfrak{G} \subset \mathfrak{G}_{(\beta + \eta,\omega)}$.

**Proof.** Let $E$ be an object of $\mathfrak{G}$ with
\[
v(E) = re^\beta + ag_X + (D + (D, \beta)g_X), a \leq 0.
\]
We take a $(\beta + \eta)$-twisted stable subobject $E_1$ of $E$ such that
\[
\chi(E_1(-\beta - \eta))/\rk E_1 = \max\{\chi(F(-\beta - \eta))/\rk F| F \subset E, \deg(F(-\beta)) = 0\}.
\]
We set
\[
v(E_1) = r_1e^\beta + a_1g_X + (D_1 + (D_1, \beta)g_X).
\]
Since $E_1 \in \mathfrak{G}$ and $\deg(E_1(-\beta)) = 0$, $a_1 \leq 0$. Then
\[
Z_{(\beta + \eta,\omega)}(E_1) = (-a_1 + (\eta, D_1) + \frac{(\eta^2)}{2}r_1 + r_1\frac{(\omega^2)}{2} > 0,
\]
which implies that $E \in \mathfrak{G}_{(\beta + \eta,\omega)}$.
\[
\square
\]

**Lemma 1.6.3.** Let $E$ be a $\mu$-semi-stable object of $\mathfrak{C}$ with $v(E) = re^\beta + ag_X + (D + (D, \beta)g_X), a > 0$. Assume that $-(\eta^2) < \frac{1}{2r_0^2}$.

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(1) Let $E_1$ be a subobject of $E$. Then

$$\frac{\chi(E(-\beta - \eta))}{\rk E} \prec \frac{\chi(E_1(-\beta - \eta))}{\rk E_1}$$

if

$$\frac{\chi(E(-\beta))}{\rk E} \prec \frac{\chi(E_1(-\beta))}{\rk E_1}.$$

(2) $E$ is $(\beta + \eta)$-twisted semi-stable if and only if $E$ is $\beta$-twisted semi-stable and

$$-\langle v(F), \eta + (\eta, \beta)g_X \rangle \leq -\langle v(E), \eta + (\eta, \beta)g_X \rangle$$

for any subobject $F$ with $\chi_G(F(n))/\rk F = \chi_G(E(n))/\rk E$.

Proof. Let $E_1$ be a $\mu$-semi-stable object of $E$ with $v(E_1) = r_1e^\beta + a_1g_X + (D_1 + (D_1, \beta))g_X$. We set $E_2 := E/E_1$ and $v(E_2) := r_2e^\beta + a_2g_X + (D_2 + (D_2, \beta))g_X$. By [13] Lem. 1.1, we have

$$\frac{\langle v(E)^2 \rangle}{r} = \sum_i \frac{\langle v(E_i)^2 \rangle}{r_i} - \sum_i r_i \left( \frac{D_i}{r_i} - \frac{D}{r} \right)^2.$$

Since $\langle v(E_i)^2 \rangle \geq -2r_i^2$ and $\langle v(E)^2 \rangle < 0$, we get

$$2r \geq -\sum_i r_i \left( \frac{D_i}{r_i} - \frac{D}{r} \right)^2.$$

Since $\frac{D_i}{r_i} - \frac{D}{r} = -\frac{r_i}{r_2} \left( \frac{D_i}{r_i} - \frac{D}{r} \right)$, we get

$$2r \geq -\frac{r_i}{r_2} \left( \frac{D_i}{r_i} - \frac{D}{r} \right)^2.$$

Hence

$$\left| \left( \frac{D}{r} - \frac{D_1}{r_1} - \eta \right) \right| \leq \sqrt{\left( \frac{D_1}{r_1} - \frac{D}{r} \right)^2} \geq \sqrt{2r} \sqrt{-\langle \eta^2 \rangle} \leq \frac{1}{r_0r_2}.$$

Assume that $a/r - a_1/r_1 \neq 0$. Since $|a/r - a_1/r_1| \geq \frac{1}{r_0r_2}$, (1) holds. We note that

$$\chi(E(-\beta - \eta)) = \frac{-\langle v(E), e^\beta + \eta + (\eta, \beta)g_X \rangle}{r} - \frac{-\langle v(E_1), e^\beta + \eta + (\eta, \beta)g_X \rangle}{r_1}.$$

Then (2) follows from (1) and (1.14). \qed

We note that $R_\beta$ is a finite set (Lemma [1.4.7]).

Definition 1.6.4. We set $N := \min_{\eta \in R_\beta} \frac{\eta}{2r^\omega}$. Assume that $-\langle \eta^2 \rangle < N$. Let $\mathfrak{U}$ be the set of $(\beta + \eta)$-twisted stable objects $U$ of $\mathfrak{C}$ with $\deg(U(-\beta)) = 0$ and $\chi(U(-\beta)) > 0$.

$U \in \mathfrak{U}$ are $(\beta + t\eta)$-twisted stable for $0 < t \leq 1$ and

$$\frac{\chi(U(-\beta - \eta))}{\rk U} \leq \frac{\chi(U'(-\beta - \eta))}{\rk U'}$$

if and only if

$$\frac{\chi(U(-\beta))}{\rk U} \prec \frac{\chi(U'(-\beta))}{\rk U'}$$

or

$$\frac{\chi(U(-\beta))}{\rk U} = \frac{\chi(U'(-\beta))}{\rk U'} - \frac{-\langle v(U), \eta + (\eta, \beta)g_X \rangle}{\rk U'} \leq \frac{-\langle v(U'), \eta + (\eta, \beta)g_X \rangle}{\rk U'}$$

for $U, U' \in \mathfrak{U}$.

Corollary 1.6.5. Let $E$ be a $\mu$-semi-stable object of $\mathfrak{C}$ with $v(E) = re^\beta + a_1g_X + D + (D, \beta)g_X$. For $E \in \mathfrak{C}$, let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_r = E$$

be the Harder-Narasimhan filtration with respect to $(\beta + \eta)$-twisted semi-stability. Then $F_i/F_{i-1}$ are generated by $U \in \mathfrak{U}$ with $\chi(U(-\beta - \eta)) = \frac{\chi(F_i/F_{i-1}(-\beta - \eta))}{\rk F_i/F_{i-1}}$. 

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Proof. Since \( E \in \mathfrak{F} \), we have \( a > 0 \). We take the Harder-Narasimhan filtration

\[
0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E
\]

with respect to \((\beta + \eta)\)-twisted semi-stability. By Lemma 1.6.3 (1), \( \chi(F_1(-\beta))/\rk F_1 \geq a/r > 0 \). Since \( E \in \mathfrak{F} \), we have \( E/F_1 \in \mathfrak{F} \), which implies that \( \chi(E/F_1(-\beta)) > 0 \). Then inductively we see that \( \chi(F_i/F_{i-1}(-\beta)) > 0 \) for all \( i \). Hence the claim holds. \( \square \)

**Proposition 1.6.6.**

1. \( \mathfrak{F}(\beta + \eta, \omega) \) is generated by \( \mathfrak{F} \) and \( U \in \mathfrak{U} \) with \( Z(\beta + \eta, \omega)(U) > 0 \).
2. \( \mathfrak{T}(\beta + \eta, \omega) \) is generated by \( \mathfrak{T}^\mu \) and \( U \in \mathfrak{U} \) with \( Z(\beta + \eta, \omega)(U) < 0 \).

Proof. (1) For \( E \in \mathfrak{F}(\beta + \eta, \omega) \), we have an exact sequence in \( \mathcal{C} \)

\[
0 \to E_1 \to E \to E_2 \to 0
\]

such that \( E_1 \in \mathfrak{T} \cap \mathfrak{F}(\beta + \eta, \omega) \) and \( E_2 \in \mathfrak{F} \). Then \( E_1 \) is a \( \mu \)-semi-stable object with \( \deg_G(E_1) = 0 \). Since \( E_1 \in \mathfrak{T} \), Corollary 1.6.9 implies that \( E_1 \) is generated by \( U \in \mathfrak{U} \) with \( Z(\beta + \eta, \omega)(U) > 0 \).

(2) For \( E \in \mathfrak{T}(\beta + \eta, \omega) \), we have an exact sequence in \( \mathcal{C} \)

\[
0 \to E_1 \to E \to E_2 \to 0
\]

such that \( E_1 \in \mathfrak{T}^\mu \) and \( E_2 \in \mathfrak{T}^\mu \cap \mathfrak{T}(\beta + \eta, \omega) \). If there is a quotient \( E_2 \to F \) of \( E_2 \) with \( F \in \mathfrak{F} \), then Lemma 1.6.2 implies that \( F \in \mathfrak{F}(\beta + \eta, \omega) \). Hence we get \( F = 0 \), which implies that \( E_2 \in \mathfrak{T} \). By Corollary 1.6.5 \( E_2 \) is generated by \( U \in \mathfrak{U} \) with \( Z(\beta + \eta, \omega)(U) < 0 \). \( \square \)

By Lemma 1.6.3, we have \( \text{Exc}_\beta \subset \mathfrak{U} \) and every \( U \in \mathfrak{U} \) is \( \beta \)-twisted semi-stable. Hence \( \mathfrak{U} \subset \cup \mathcal{G}_W \).

**Proposition 1.6.7.** Let \( I := \{ tH | a \leq t \leq b \} \) be a closed interval of \( \mathbb{R}_{\geq 0} \cdot H \). Then for a sufficiently small \( \eta := \eta_t \in H^\perp \otimes \mathbb{Q} \), we have the following claims:

1. If \( I \) does not intersect with any wall, then \( \mathfrak{A}(\beta + \eta, \omega) = \mathfrak{A}(\beta, \omega) \).
2. Assume that the interior of \( I \) intersects with exactly one wall \( W \). We take \( \omega_\pm \in I \cap \mathbb{Q}_{\geq 0} \cdot H \) such that \( \omega_- \) is separated from \( W \) and \( \omega_- < \omega_+ \). Assume that \( \omega \in I \cap \mathbb{Q}_{\geq 0} \cdot H \) satisfies \( Z(\beta + \eta, \omega)(E) \neq 0 \) for all \( E \in \mathfrak{U} \). Then
   (a) \( \mathfrak{F}(\beta + \eta, \omega) \) is generated by \( \mathfrak{F}(\beta, \omega_-) \) and \( U \in \mathfrak{U} \cap \mathcal{G}_W \) with \( Z(\beta + \eta, \omega)(U) > 0 \).
   (b) \( \mathfrak{T}(\beta + \eta, \omega) \) is generated by \( \mathfrak{T}(\beta, \omega_+) \) and \( U \in \mathfrak{U} \cap \mathcal{G}_W \) with \( Z(\beta + \eta, \omega)(U) < 0 \).

Proof. We set \( \text{Exc}_\beta^t := \cap_{E \in \mathfrak{F}} \{ E \in \text{Exc}_\beta : Z(\beta, \omega)(E) \neq 0 \} \). Since \( |Z(\beta, \omega)(E)| > 0 \) on \( I \) for all \( E \in \text{Exc}_\beta^t \),

\[
N := \min\{ |Z(\beta, \omega)(E)| : E \in \text{Exc}_\beta^t \}
\]

is a positive number. Then we can take a sufficiently small \( \eta \) with \( N > \langle \nu(E), \eta + (\eta, \beta)\mathfrak{g}_X \rangle - \frac{\eta^2}{2} \rk E \) for all \( E \in \text{Exc}_\beta \). For \( E \in \text{Exc}_\beta^t \), we have

\[
\begin{cases}
  Z(\beta + \eta, \omega)(E) > 0, & \text{if } Z(\beta, \omega)(E) > 0, \\
  Z(\beta + \eta, \omega)(E) < 0, & \text{if } Z(\beta, \omega)(E) < 0
\end{cases}
\]

by \( \mathfrak{A}(\beta, \omega) \).

1. By the assumption, \( \text{Exc}_\beta^t = \text{Exc}_\beta \). Hence the claim follows from \( \mathfrak{A}(\beta, \omega) \).
2. In this case, \( \text{Exc}_\beta^t = \text{Exc}_\beta \setminus \mathcal{G}_W \). By \( \mathfrak{F}(\beta, \omega_-) \subset \mathfrak{F}(\beta + \eta, \omega) \) and \( \mathfrak{T}(\beta, \omega_+) \subset \mathfrak{T}(\beta + \eta, \omega) \). \( \square \)

**Corollary 1.6.8.** In the notation of Proposition 1.6.7 (2), we choose \( \eta \) such that \( Z(\beta + \eta, aH)(E) < 0 < Z(\beta + \eta, bH)(E) \) for \( E \in \text{Exc}_\beta \cap \mathcal{G}_W \). Then \( \mathfrak{A}(\beta + \eta, aH) = \mathfrak{A}(\beta, \omega_-) \) and \( \mathfrak{A}(\beta + \eta, bH) = \mathfrak{A}(\beta, \omega_+) \).

Proof. We first note that \( \text{Exc}_\beta \cap \mathcal{G}_W \subset \mathcal{U} \cap \mathcal{G}_W \subset \mathcal{G}_W \) and \( \text{Exc}_\beta \cap \mathcal{G}_W \) generate the category \( \mathcal{G}_W \). By Proposition 1.6.7 (2), \( \mathfrak{T}(\beta + \eta, aH) = \mathfrak{T}(\beta, \omega_-) \) if and only if \( Z(\beta + \eta, aH)(U) < 0 \) for \( U \in \mathcal{U} \cap \mathcal{G}_W \). Hence \( \mathfrak{T}(\beta + \eta, aH) = \mathfrak{T}(\beta, \omega_-) \), which implies that \( \mathfrak{A}(\beta + \eta, aH) = \mathfrak{A}(\beta, \omega_-) \). The proof of \( \mathfrak{A}(\beta + \eta, bH) = \mathfrak{A}(\beta, \omega_+) \) is similar. \( \square \)

**Corollary 1.6.9.** \( \mathfrak{A}(bH + \eta, \omega) \) depends only on the chamber where \( (\eta, \omega) \) belongs.

Proof. It is sufficient to prove that \( \mathfrak{A}(bH + \eta, \omega) \) is locally constant on \( \mathfrak{S}_R \setminus \bigcup_{u \in \mathbb{R}} \mathcal{W}_u \). By Proposition 1.6.7 (1), the claim holds. \( \square \)

**Proposition 1.6.10.** Assume that \( C_1 \) and \( C_2 \) are chambers separated by exactly one wall \( W_u \) with \( \rk u > 0 \).

1. \( R_u \) (cf. \( \mathfrak{S}_R \)) induces a bijection \( C_1 \to C_2 \).
2. For \( (\eta_i, \omega_i) \in C_i, \ i = 1, 2 \), we set \( \beta_i := bH + \eta_i \). Then the stability conditions \( (\mathfrak{A}(\beta_1, \omega_1), Z(\beta_1, \omega_1)) \) and \( (\mathfrak{A}(\beta_2, \omega_2), Z(\beta_2, \omega_2)) \) are equivalent.
Proof. (1) is a consequence of Lemma 1.5.2

(2) By Corollary 1.6.9 we may assume that \( \eta_2 = \eta_1, \ (\omega_1^2) < (\omega_2^2) \). We set \( \beta := bH + \eta_1 \). Let \( U \) be a \( \beta \)-twisted semi-stable object of \( \mathfrak{C} \) with \( v(U) = u \). Since \( W_u \) is the unique wall separating \( (\eta_1, \omega_1) \) and \( (\eta_2, \omega_2) \), \( U \) is \( \beta \)-twisted stable. Then \( U \) is an irreducible object of \( \mathfrak{A}_{(\beta, \omega_1)} \) and \( U[1] \) is an irreducible object of \( \mathfrak{A}_{(\beta, \omega_2)} \). We shall prove that \( \Phi_{U}^{-1} \) induces an equivalence \( \mathfrak{A}_{(\beta, \omega_1)} \to \mathfrak{A}_{(\beta, \omega_2)} \): We first note that \( \mathcal{O}_D = \mathcal{O}_{D[-2]} \). Let \( E \) be an object of \( \mathfrak{A}_{(\beta, \omega_1)} \). Since \( \Phi_{U}^{-1}(U) = U[1] \), \( \text{Hom}(\Phi_{U}^{-1}(E)[1],U) = \text{Hom}(E[1],U[-1]) = 0 \). We have an exact sequence in \( \mathfrak{C} \):

\[
0 \longrightarrow \text{Hom}(U,E) \otimes U \\
\longrightarrow H^{-1}(\Phi_{U}^{-1}(E)) \longrightarrow H^{-1}(E) \longrightarrow \text{Ext}^1(U,E) \otimes U \\
\longrightarrow H^0(\Phi_{U}^{-1}(E)) \longrightarrow H^0(E) \longrightarrow \psi \text{Ext}^2(U,E) \otimes U \\
\longrightarrow H^1(\Phi_{U}^{-1}(E)) \longrightarrow H^1(E) = 0.
\]

By Lemma 1.4.18 and \( \text{Hom}(H^1(\Phi_{U}^{-1}(E)),U) = \text{Hom}(\Phi_{U}^{-1}(E)[1],U) = 0 \), \( \psi \) is a surjective morphism in \( \mathfrak{A}_{(\beta, \omega_2)} \), which also implies the surjectivity of \( \psi \) in \( \mathfrak{C} \). Therefore \( H^1(\Phi_{U}^{-1}(E)) = 0 \). Then we see that \( H^{-1}(\Phi_{U}^{-1}(E)) \in \mathfrak{X}_{(\beta, \omega_2)} \) and \( H^0(\Phi_{U}^{-1}(E)) \in \mathfrak{X}_{(\beta, \omega_1)} \). Since \( \text{Hom}(\Phi_{U}^{-1}(E),U) = \text{Hom}(U,E,U[-1]) = 0 \), we also have \( \text{Hom}(H^0(\Phi_{U}^{-1}(E)),U) = 0 \). Hence \( H^0(\Phi_{U}^{-1}(E)) \in \mathfrak{X}_{(\beta, \omega_2)} \) and \( \Phi_{U}^{-1}(E) \) is an object of \( \mathfrak{A}_{(\beta, \omega_2)} \). Thus \( \Phi_{U}^{-1}(\mathfrak{A}_{(\beta, \omega_1)}) \subset \mathfrak{A}_{(\beta, \omega_2)} \), which implies the claim.

\[ \square \]

Corollary 1.6.11. Assume that neither \( (\beta_1, \omega_1) \) nor \( (\beta_2, \omega_2) \) belongs to any wall. Then \( \mathfrak{A}_{(\beta_1, \omega_1)} \) is equivalent to \( \mathfrak{A}_{(\beta_2, \omega_2)} \).

1.7. Moduli of stable objects for isotropic Mukai vectors. Let \( v \in \mathbb{Q}_{>0}bH + \eta \cap H^*(X,\mathbb{Z}) \) be a primitive isotropic Mukai vector. Assume that \( \langle v, u \rangle \neq 0 \) for all \( u \in \mathfrak{R} \). Then \( M_{H, bH+\eta}(v) \) is a projective K3 surface by [13]. Since \( H \) is not a general polarization, \( M_{H, bH+\eta}(v) \) may not contain slope stable objects. In this subsection, we shall relate \( M_{H, bH+\eta}(v) \) to the moduli of slope stable objects as an application of the chamber structure.

Let \( I_t := \eta + \sqrt{-1}\xi \), \( 0 \leq t \leq t_0 \) be a segment of \( V_H \) (see 1.6) such that \( I_{t_0} \), \( 0 < t < t_0 \) belongs to a chamber and \( I_{t_0} \) belongs to a wall. Let \( \{W_u|u \in \mathfrak{R}, 1 \leq i \leq n\} \) be the set of all walls containing \( I_{t_0} \). Then \( \langle e^{bH+\xi}, u \rangle < 0 \) for all \( 0 < t < t_0 \) and \( u \in \mathfrak{R} \), and \( \langle e^{bH+\xi}, u \rangle = 0 \), \( 1 \leq i \leq n \). We have an auto-equivalence \( \Phi : \mathcal{D}(X) \to \mathcal{D}(X) \) inducing an equivalence \( \mathfrak{A}_{bH+\eta}(\mathfrak{V}) \to \mathfrak{A}_{bH+\eta}(\mathfrak{V}) \), where \( t_+ - t_0 < t < t_+ \) and \( t_+ - t_0 < 1 \). For a sufficiently small \( \xi \in H^1 \), let \( U_i \), \( 1 \leq i \leq n \) be \( (bH+\eta+\xi) \)-twisted stable objects of \( v(U_i) = u_i \). Then \( \Phi \) is a composite of reflections \( \Phi_{t_0}^{-1}, 1 \leq i \leq n \). Hence \( \Phi \) induces an isomorphism \( \Phi_{t_0}^{-1} : \mathfrak{X}_{bH+\eta}(\mathfrak{V}) \to \mathfrak{X}_{bH+\eta}(\mathfrak{V}) \), where \( t_+ - t_0 < t < t_+ \). Then there is no wall between \( \eta' + \sqrt{-1}\xi \) and \( \eta' + \sqrt{-1}t_0 \xi \). If the two points are separated by a wall \( W_{u} \), then \( e^{bH+\eta' + \sqrt{-1}t_0 \xi} \) and \( e^{bH+\eta' + \sqrt{-1}t_0 \xi} \) have different signatures. Since the two points are connected by the curve \( I_s := \begin{cases} \Phi_{t_0}(I_{t-s}), & 0 \leq s \leq t \\ I_{t-s}, & t \leq s \leq t + t' \end{cases} \)

and \( \langle e^{bH+\eta'}, u \rangle \neq 0 \), \( \langle e^{bH+\eta' + \sqrt{-1}t_0 \xi} \Phi_{t_0}(\omega), u \rangle = 0 \), \( 0 < s < t \) or \( \langle e^{bH+\eta' + \sqrt{-1}t_0 \xi} \Phi_{t_0}(\omega), u \rangle = 0 \), \( 0 < s < t' \), which is a contradiction. Therefore the claim holds. Then \( \Phi \) induces an isomorphism \( M_{H, bH+\eta}(v) \to M_{H, bH+\eta}(\Phi(v)) \). Continuing this procedure, we get an isomorphism \( M_{H, bH+\eta}(v) \to M_{H, bH+\eta}(\Phi(v)) \) such that \( M_{H, bH+\eta}(\Phi(v)) \) parametrizes irreducible objects of \( \mathfrak{A}_{\mu} \). In particular, \( M_{H, bH+\eta}(\Phi(v)) \) consists of slope stable objects or 0-dimensional objects.

Example 1.7.1. Let \( X \) be the elliptic K3 surface in Example 1.4.13 and use the same notation. We shall describe the universal family for \( M_{H, bH+\eta}(4e^\mathbb{Z}) \): Let \( I_t := D/2 + tH \) (\( 0 \leq t \leq 1 \)) be a segment in \( V_H \). Then \( I_t \) meets the walls at \( t = \frac{1}{2}\sqrt{3} \). We take a small perturbation \( I_{t_0} = x(t)D + tH \) of \( I_t \) so that \( x(t) < 1/2 \) in a neighborhood of \( t = \frac{1}{2}\sqrt{3} \). Then \( \Phi_{C_{D}(t)} \Phi_{E_1} \Phi_{C_{D}(t)}(\Omega_{D}) \) is the universal family on \( X \times M_{H, bH+\eta}(4e^\mathbb{Z}) \). We also have the universal family \( \Phi_{E_1} \Phi_{C_{D}(t)}(\Omega_{D}) \) on \( X \times M_{H, bH+\eta}(3e^\mathbb{Z}) \).

2. Relation of Gieseker’s stability and Bridgeland’s stability.

2.1. Some numerical conditions. We shall discuss numerical conditions which relate Gieseker’s stability with Bridgeland’s stability.
For $E_1 \in \mathcal{D}(X)$, we set
\begin{align}
(2.1) \quad v(E_1) = r_1 e^\beta + a_1 g_X + (d_1 H + D_1 + (d_1 H + D_1, \beta) g_X).
\end{align}

We consider the following conditions for $v$:

1. $(1) \ r \geq 0, \ d > 0$ and $(2) \ dr_1 - d_1 r > 0$ implies $(dr_1 - d_1 r)(\omega^2)/2 - (da_1 - d_1 a) > 0$ for all $E_1$ with $0 < d_1 < d$ and $(v(E_1)^2) \geq -2$.

2. $(1) \ r \geq 0, \ d < 0$ and $(2) \ dr_1 - d_1 r < 0$ implies $(dr_1 - d_1 r)(\omega^2)/2 - (da_1 - d_1 a) < 0$ for all $E_1$ with $d < d_1 < 0$ and $(v(E_1)^2) \geq -2$.

3. $(1) \ r \geq 0, \ d < 0$ and $(2) \ dr_1 - d_1 r \leq 0$ implies $(dr_1 - d_1 r)(\omega^2)/2 - (da_1 - d_1 a) \leq 0$ and the inequality is strict if $dr_1 - d_1 r < 0$ for all $E_1$ with $d \leq d_1 \leq 0$ and $(v(E_1)^2) \geq -2$.

**Remark 2.1.1.** We note that $(dr_1 - d_1 r)(H^2) = (c_1(E \otimes E_1), H)$. The condition $(\ast 1)$ says that $\mu(E) > \mu(E_1)$ implies $\phi(E_1) > \phi(E)$. The condition $(\ast 2)$ says that $\mu(E) < \mu(E_1)$ implies $\phi(E_1) < \phi(E)$.

**Remark 2.1.2.** $(\ast 1)$ for $v$ is equivalent to $(\ast 2)$ for $v^\vee$.

**Remark 2.1.3.** Assume that $(\ast 3)$ holds for $v$. Let $E$ be an object of $\mathfrak{C}$ with $v(E) = v$. Then $E$ is $\mu$-semi-stable if and only if $E$ is $\beta$-twisted semi-stable. Moreover $E$ is a local projective object of $\mathfrak{C}$.

Assume that $E$ is $\mu$-semi-stable and take a Jordan-Hölder filtration with respect to the $\mu$-stability
\[ 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E. \]

We set
\[ v(F_i/F_{i-1}) = r_i e^\beta + a_i g_X + (d_i H + D_i + (d_i H + D_i, \beta) g_X). \]

Then $d_i/r_i = d/r$. By $(\ast 3)$, we get $a_i/r_i \leq a/r$ for all $i$. Hence $a_i/r_i = a/r$ for all $i$. Thus $E$ is $\beta$-twisted semi-stable. We set $E_i := F_i/F_{i-1}$ and assume that $\text{Ext}^1(E_i, A) \neq 0$ for an irreducible object $A$ of $\mathfrak{C}$. Since $\text{Ext}^1(A, E_i) \cong \text{Ext}^1(E_i, A)^\vee$, we have a non-trivial extension
\[ 0 \rightarrow E_i \rightarrow F \rightarrow A \rightarrow 0. \]

Then $F$ is a $\mu$-stable object. Applying $(\ast 3)$ to $v(F) = v(E_i) + b g_X + (d' + (D', \beta) g_X)$, $b > 0$, we get $(a_i + b)/r \leq a/r$, which is a contradiction. Therefore $\text{Ext}^1(E_i, A) = 0$ for all $i$. Then $\text{Ext}^1(E, A) = 0$ and $E$ is a local projective object.

**Lemma 2.1.4.** Let $N(v) := N$ be the number in [23] Proposition 2.8. Assume that $d > N(v)$. Then $(\ast 1)$ holds for $v$ and $(\ast 3)$ holds for $w = r_0 a e^\beta + r_0 g_X - (dH + D + (dH + D, \beta) g_X)$.

**Proof.** If $dr_1 - d_1 r > 0$, then [23] Lemma 2.9 (2) implies that $(da_1 - d_1 a) < 0$. Hence $(dr_1 - d_1 r)(\omega^2)/2 - (da_1 - d_1 a) > 0$. Thus $(\ast 1)$ holds.

We note that $(-d)a_1 - (-d_1)a = d_1 a - da_1$ and $(-d)r_1 - (-d_1)r = d_1 r - dr_1$. Hence $(-d)a_1 - (-d_1)a \leq 0$ implies that $((-d)a_1 - (-d_1)a)(\omega^2)/2 - ((-d)r_1 - (-d_1)r) \leq 0$. Moreover the inequality is strict, if $(-d)a_1 - (-d_1)a < 0$. Therefore $(\ast 3)$ holds.

**Definition 2.1.5.** We set
\[ d_{\min} := \frac{1}{(H^2)^2} \min\{\deg(E(-\beta)) > 0| E \in K(X)\} \in \mathbb{Z}, \]
where $d[x]$ is the denominator of $x \in \mathbb{Q}$. Then $d \in \mathbb{Z}d_{\min}$ for any $v(E)$ with [133].

**Lemma 2.1.6.** If $d = d_{\min}$, then $(\ast 1)$ holds for $v = v(E), E \in A$ and $(\ast 2)$ holds for $v^\vee$.

**Proof.** For any $E_1 \in \mathcal{D}(X)$ with (2.1), $d_1 \leq 0$ or $d_1 \geq d_{\min}$. Hence the claim follows.

**Lemma 2.1.7.** For a fixed Mukai vector $v$, if $(\omega^2) > 0$, then $(\ast 1)$ holds.

**Proof.** Assume that $(\ast 1)$ does not hold. Then there is an object $E_1$ such that $dr_1 - d_1 r > 0$ but $(dr_1 - d_1 r)(\omega^2)/2 \leq da_1 - d_1 a$, where
\[ v(E_1) = r_1 e^\beta + a_1 g_X + (d_1 H + D_1 + (d_1 H + D_1, \beta) g_X), \]
and $(v(E_1)^2) \geq -2\varepsilon$. Since $r \geq 0$ and $d, d_1 > 0$, $r_1 > rd_1/d \geq 0$. We set
\[ \delta := \frac{1}{(H^2)^2} \min\{(D, H) > 0| D \in \text{Pic}(X)\}. \]

Then $dr_1 - d_1 r \geq \delta$. Assume that $r > 0$. Then we get
\[ a = \frac{d^2(H^2) - (\langle \omega^2 \rangle - (D^2))}{2r}, \quad a_1 \leq \frac{d_1^2(H^2) + 2\varepsilon}{2d_1}. \]
Lemma 2.1.8. Hence

\[ \frac{(dr_1 - d_1 r)(\omega^2)}{2} - (d_1 a - d_1 a) \]

\[ \geq \frac{(dr_1 - d_1 r)(\omega^2)}{2} - \frac{d_1^2 (H^2)}{2r_1} + \frac{d_1^2 (H^2)}{2r_1} \]

\[ = \frac{(dr_1 - d_1 r)(\omega^2)}{2} + \frac{(dr_1 - d_1 r)(H^2)}{2r_1} \frac{d_1}{r_1} - \frac{1}{r_1} (dr_1 (\omega_1^2) - (D^2))/2). \]

Thus we have

\[ \delta \left( \frac{\omega^2}{2} \right) \leq \frac{d_1}{r_1} + \frac{d_1}{2r} (\omega^2 - (D^2)) \leq \max \{ d_1, d_1 + \frac{d_1 - d_{\text{min}}}{2} (\omega^2 - (D^2)) \}. \]

If \( r = 0 \), then

\[ \frac{(dr_1 - d_1 r)(\omega^2)}{2} - (d_1 a - d_1 a) \]

\[ \geq \frac{r_1 (\omega^2)}{2} - \frac{d_1^2 (H^2)}{2r_1} + d_1 a \]

\[ \geq \frac{r_1 (\omega^2)}{2} + (d - d_{\text{min}}) d_2 \frac{(H^2)}{2} - \frac{d}{r_1} - (d - d_{\text{min}}) a. \]

Hence

\[ \delta \left( \frac{\omega^2}{2} \right) \leq \frac{d_1}{r_1} + \frac{d - d_{\text{min}}}{2} (\omega^2 - (D^2)) + |a|. \]

These mean that \( (\omega^2) \) is bounded above. □

Lemma 2.1.8.

\[ \Sigma_{(\omega_1, \omega_2)}(E, E_1) = (H, \omega)((r d_1 - r_1 d)(\omega^2)/2 - (d_1 a - d_1 a)). \]

In particular, \( \phi(E_1) \geq \phi(E) \) if and only if

\[ ((r d_1 - r_1 d)(\omega^2)/2 - (d_1 a - d_1 a)) \geq 0. \]

Proof. The claim follows from the equalities

\[ \Sigma_{(\omega_1, \omega_2)}(E, E_1) = \det \left( \begin{array}{cc} -a + r_1 d_1 (\omega_1^2) & -d_1 r_1 (\omega_1^2) \\ d_1 (H, \omega) & d_1 (H, \omega) \end{array} \right) \]

\[ = (H, \omega)((r d_1 - r_1 d)(\omega^2)/2 - (d_1 a - d_1 a)). \]

2.2. Gieseker’s stability and Bridgeland’s stability.

Proposition 2.2.1. Assume that \( (\ast 1) \) holds. For an object \( E \) of \( \mathcal{D}(X) \) with \( v(E) = v, E \) is a \( \beta \)-twisted semi-stable object of \( \mathcal{C} \) if and only if \( E \) is a semi-stable object of \( \mathcal{A}^\mu \).

Proof. (1) Assume that \( E \in \mathcal{C} \) is \( \beta \)-twisted semi-stable. We first assume that \( r > 0 \). Let \( \varphi : E_1 \to E \) be a stable subobject of \( E \) and set \( v(E_1) = r_1 e^\beta + a_1 g_X + (d_1 H + D_1 + (d_1 H + D_1, \beta) g_X) \). Then \( E/E_1 \) belongs to \( \mathcal{A}^\mu \). Since \( H^{-1}(E/E_1) \) and \( E \) are torsion free objects of \( \mathcal{C}, r_1 > 0 \). Since \( \mathcal{A}_{(\beta, \omega_1)} = \mathcal{A}^\mu, d_1 > 0 \). We set \( v(H^{-1}(E/E_1)) = r_1 e^\beta + a_1 g_X + (d_1 H + D_1 + (d_1 H + D_1, \beta) g_X) \). Then \( d_1 \leq 0 \). Let \( E' \) be the image of \( \varphi \) in \( \mathcal{C} \). Then

\[ d_1 / r_1 \leq (d_1 - d') / (r_1 - r') \leq d / r \]

and \( d_1 \leq d_1 - d' \leq d / r \). If \( d_1 = d, \) then \( \text{deg}_{G_0}(E/E_1) = r_0 (H^2)(d - d_1) = 0 \). Hence \( \phi(E/E_1) = 1 > \phi(E) \), which implies that \( \phi(E_1) < \phi(E) \). So we assume \( d_1 < d \). If \( d_1 / r_1 = d / r, \) then \( (r_1, r') = (0, 0), \) which implies that \( H^{-1}(E/E_1) = 0 \). Assume that \( d_1 / r_1 < d / r \). Then \( (\ast 1) \) implies that

\[ (r d_1 - r_1 d)(\omega^2)/2 - (d_1 a - d_1 a) > 0. \]

If \( d_1 / r_1 = d / r, \) then \( \beta \)-twisted semi-stability of \( E \) implies that \( a_1 / r_1 \leq a / r, \) which also means that \( (r d_1 - r_1 d)(\omega^2)/2 - (d_1 a - d_1 a) \geq 0 \). By Lemma 2.1.8, \( \phi(E_1) \leq \phi(E) \).

Thus \( E \) is a semi-stable object of \( \mathcal{A}^\mu \).

We next assume that \( r = 0 \). Let \( \varphi : E_1 \to E \) be a stable subobject of \( E \) and set \( v(E_1) = r_1 e^\beta + a_1 g_X + (d_1 H + D_1 + (d_1 H + D_1, \beta) g_X) \). Then \( E/E_1 \) belongs to \( \mathcal{A}^\mu \). We set \( v(H^{-1}(E/E_1)) = r_1 e^\beta + a_1 g_X + (d_1 H + D_1 + (d_1 H + D_1, \beta) g_X) \). Then \( d_1 \leq 0 \). Let \( E' \) be the image of \( \varphi \) in \( \mathcal{C} \). Then \( d \geq d_1 - d' \geq d_1 \geq 0 \). If \( d = d_1, \) then we see that \( \phi(E/E_1) = 1 > \phi(E) \). Hence \( \phi(E_1) < \phi(E) \). If \( d_1 = 0, \) then \( E_1 \) is a 0-dimensional object. Since \( E \) is purely 1-dimensional and \( H^{-1}(E/E_1) \) is torsion free, \( E' = 0 \) and \( H^{-1}(E/E_1) = 0 \). Thus \( E_1 = 0 \). Assume that \( d_1 > 0 \). If \( r_1 > 0, \) then \( (\ast 1) \) implies that \( (r_1 - d_1)(\omega^2)/2 + d_1 a - d_1 a > 0 \). Therefore \( \phi(E_1) < \phi(E) \). If \( r_1 = 0, \) then \( H^{-1}(E/E_1) = 0 \) and the \( \beta \)-twisted semi-stability of \( E \) implies that \( a_1 / d_1 = a / d \). Therefore \( \phi(E_1) \leq \phi(E) \).
(2) Conversely assume that $E \in \mathfrak{A}$ is semi-stable. We first assume that $r > 0$. We set

$$v(H^0(E)) = r'e^\beta + a'g_X + (d'H + D' + (d'H + D', \beta)g_X),$$
$$v(H^{-1}(E)) = r''e^\beta + a''g_X + (d''H + D'' + (d''H + D'', \beta)g_X).$$

Then $d' \geq 0$ and $d'' \leq 0$. Let $H^0(E) \to E_2$ be a quotient in $\mathfrak{C}$ such that $E_2$ is a $\beta$-twisted stable object with $v(E_2) = r_2e^\beta + a_2g_X + (d_2H + D_2 + (d_2H + D_2, \beta)g_X)$, $d'/r' \geq d_2/r_2$ and ker$(H^0(E) \to E_2) \in \mathfrak{A}$. Since $H^0(E) \in \mathfrak{T}$, $d_2 > 0$. Then $E \to E_2$ is surjective in $\mathfrak{A}$. Let $E_1$ be the kernel of $E \to E_2$ in $\mathfrak{A}$. Since $E$ is semi-stable, $\phi(E) \leq \phi(E_2)$, which implies that $(d_2 - a_2)/r_2 \geq (d_2 - a_2)/(\beta(\omega^2)/2)$. We have $d/r = (d' - d'')/(r' - r'') \geq d/r' \geq d_2/r_2$ and $d = d' - d'' \geq d' \geq d_2/r_2 \geq d_2$. If $d_2 = d$, then $deg_{g}(E_1) = 0$ and $\phi(E_1) = 1 > \phi(E)$. Therefore $d_2 < d$. If $d > d_2/r_2$, then $(\ast)$ implies that $(d_2 - a_2)/r_2 < (d_2 - d_2)/(\beta(\omega^2)/2)$. Therefore $d_2/r_2 = d/r$ and $(r'', d'') = (0, 0)$. In particular, $H^{-1}(E) = 0$. Then $d_2 - a_2/\beta \geq 0$ implies that $a_2/\beta \geq a/r$. Therefore $E$ is $\beta$-twisted stable.

We next assume that $r = 0$. We set

$$v(H^0(E)) = r'e^\beta + a'g_X + (d'H + D' + (d'H + D', \beta)g_X),$$
$$v(H^{-1}(E)) = r''e^\beta + a''g_X + (d''H + D'' + (d''H + D'', \beta)g_X).$$

Then $d' \geq 0$ and $d'' \leq 0$. Assume that $r'' > 0$. Let $H^0(E) \to E_2$ be a quotient in $\mathfrak{C}$ such that $E_2$ is a $\beta$-twisted stable object with $v(E_2) = r_2e^\beta + a_2g_X + (d_2H + D_2 + (d_2H + D_2, \beta)g_X)$, $d'/r' \geq d_2/r_2$ and ker$(H^0(E) \to E_2) \in \mathfrak{A}$. Since $H^0(E) \in \mathfrak{T}$, $d_2 > 0$. Then $E \to E_2$ is surjective in $\mathfrak{A}$. Let $E_1$ be the kernel of $E \to E_2$ in $\mathfrak{A}$. Since $E$ is semi-stable, $\phi(E) \leq \phi(E_2)$, which implies that $(d_2 - a_2)/r_2 \geq (d_2 - d_2)/(\beta(\omega^2)/2)$. We have $d = d' - d'' \geq d' \geq d_2/r_2$. Since $d_2 - d_2r = d_2 > 0$, $(\ast)$ implies that $(d_2 - a_2)/(d_2 - d_2)/(\beta(\omega^2)/2)$, which is a contradiction. Therefore $r'' = 0$. In particular, $H^{-1}(E) = 0$. Let $E_2$ be a $\beta$-twisted stable quotient of $E$ with $v(E_2) = a_2g_X + (d_2H + D_2 + (d_2H + D_2, \beta)g_X)$, $a/d > a_2/d_2$. Then $\phi(E) \leq \phi(E_2)$ implies that $d_2 - a_2 \geq 0$, which is a contradiction. Therefore $E$ is $\beta$-twisted semi-stable. □

**Lemma 2.2.2.** Assume that $(\ast)$ holds. For an object $E$ of $D(X)$ with $v(E) = v$, $E$ is a $\beta$-twisted stable object of $\mathfrak{C}$ if and only if $E[1]$ is a $\beta$-twisted stable object of $\mathfrak{A}(\beta, \omega)$.

**Proof.** Assume that $E \in \mathfrak{C}$ is $\beta$-twisted semi-stable. Then $E[1] \in \mathfrak{A}(\beta, \omega)$. We consider an exact sequence in $\mathfrak{A}(\beta, \omega)$

$$0 \to F_1[1] \to E[1] \to F_2[1] \to 0$$

such that $F_2[1] \in \mathfrak{A}(\beta, \omega)$ is a quotient object of $E[1]$. Then $H^0(F_2[1]) = 0$. We set $v(F_2) := r_2e^\beta + a_2g_X + (d_2H + D_2 + (d_2H + D_2, \beta)g_X)$.

Let $F'$ be the image of $E \to H^{-1}(F_2[1]) = F_2$ in $\mathfrak{C}$. We set $v(F') := r'e^\beta + a'g_X + (d'H + D') + (d'H + D', \beta)g_X$. Then $d'/r' \leq d'/r' \leq d_2/r_2$. Thus $r_2 - r_2 \geq 0$. Moreover if $d'/r' = d_2/r_2$, then $H^0(F_1[1]) = F_2/F'$ is a 0-dimensional object. If $d_2 = 0$, then $Z(\beta, \omega)(F_2[1]) \in \mathfrak{R}_{<0}$. Hence $\phi(F_2[1]) = 1 < \phi(F_2[1])$. So we assume that $d_2 < 0$. We note that $d_2 \geq d' \geq d_2/r' \geq d_2$. If $r_2 - r_2 > 0$, then $(\ast)$ implies that $\phi(F_2[1]) > \phi(E[1])$. If $r_2 = r_2 = 0$, then $(\ast)$ implies that $d_2 - a_2 \geq 0$. Therefore $E[1]$ is semi-stable. Moreover we see that $E[1]$ is stable if $E$ is $\beta$-twisted stable.

Conversely let $E[1]$ be a $\beta$-twisted stable object of $\mathfrak{A}(\beta, \omega)$. We shall prove that $E$ is a $\beta$-twisted semi-stable object of $\mathfrak{C}$. We set

$$v(H^{-1}(E[1])) := r''e^\beta + a''g_X + (d''H + D'' + (d''H + D'', \beta)g_X),$$
$$v(H^0(E[1])) := r'e^\beta + a'g_X + (d'H + D' + (d'H + D', \beta)g_X).$$

Then $r = r'' - r'$, $d = d'' - d'$ and $a = a'' - a'$. We also have $d''/r'' \geq d/r$ and if the equality holds, then $r' = d' = 0$. Let $F_1$ be a $\beta$-twisted stable object of $H^{-1}(E[1])$ such that

$$v(F_1) = r_1e^\beta + a_1g_X + (d_1H + D_1 + (d_1H + D_1, \beta)g_X),$$
and $H^{-1}(E[1])/F_1 \in \mathfrak{A}(\beta, \omega)$. Then $d_1/r_1 \geq d/r$ and $F_1[1] \to E[1]$ is injective in $\mathfrak{A}(\beta, \omega)$. If $d_1 = 0$, then $Z(\beta, \omega)(F_1[1]) \in \mathfrak{R}_{<0}$, which implies that $\phi(F_1[1]) = 1 > \phi(E[1])$. Therefore $d_1 < 0$. We note that $d_1 \geq d''/r'' \geq d'' \geq d$. Moreover if $d_1 = d$, then $r_1 = r''$, $d_1 = d'' = d$ and $d'' = 0$. If $d_1 = d'', then$ $(\ast)$ implies that $\phi(F_1[1]) > \phi(E[1])$, which is a contradiction. Therefore $d_1/r_1 = d/r$. Then we have $r' = d' = 0$ and $a' \geq 0$. By $(\ast)$ and $d_1/r_1 = d/r$, we get $a_1d - a_1d \geq 0$. If $a_1/r_1 \geq a''/r'' = (a + a')/r''$, then we have $a_1a - a_1a \leq 0$. Hence $a_1a = (a + a')/r'' = a/r$', which implies that $a'' = 0$. Therefore $H^0(E[1]) = 0$ and $E$ is $\beta$-twisted semi-stable. □

**Lemma 2.2.3.** Let $F$ be an object of $\mathfrak{A}(\beta, \omega)$.

1. If $F$ is a semi-stable object with $\phi(F) < 1$, then $F$ does not contain a 0-dimensional subobject.
(2) $F \in \mathfrak{A}_{(\beta, \omega)}$ does not contain a non-trivial 0-dimensional subobject if and only if $F$ is represented by a complex $\varphi : U_1 \to U_0$ such that $U_1$ and $U_0$ are local projective objects of $\mathcal{C}$.

**Proof.** (2) Assume that $F \in \mathfrak{A}_{(\beta, \omega)}$ does not contain a non-trivial 0-dimensional object. Let $G$ be a local projective generator of $\mathcal{C}$. Since $\text{Ext}^1(G(-n), H^{-1}(F)[1]) = \text{Ext}^2(G(-n), H^{-1}(F)) = 0$ for $n \gg 0$, we have a morphism $G(-n)^{\oplus N} \to F$ which induces a surjective morphism $G(-n)^{\oplus N} \to H^0(F)$. We set $U_0 := G(-n)^{\oplus N}$ and set $U_1 := \text{Cone}(U_0 \to F)[-1]$. By the exact triangle

$$U_1 \xrightarrow{\varphi} U_0 \to F \to U_1[1],$$

$H^i(U_0) = 0$, $i \neq 0$ and $H^i(F) = 0$, $i \neq -1, 0$ imply that $H^i(U_1) = 0$ for $i \neq 0$. Hence $U_1 \in \mathcal{C}$. If $	ext{Ext}^1(U_1, A) \neq 0$ for an irreducible object $A$ of $\mathcal{C}$, then we have a non-trivial extension

$$0 \to U_1 \to U'_{-1} \to A \to 0.$$

Since $\text{Ext}^1(A, U_0) \cong \text{Ext}^1(U_0, A)^\vee = 0$, we have a morphism $\varphi' : U'_{-1} \to U_0$ such that $U_1 \to U'_{-1} \xrightarrow{\varphi'} U_0$ is $\varphi$. Hence we have a morphism $A \to F$ which induces an injection $A \to H^0(F)$. It contradicts the assumption on $F$. Therefore $\text{Ext}^1(U_1, A) = 0$ for all $A$, which implies that $U_1$ is a local projective object of $\mathcal{C}$.

Conversely assume that $F$ is represented by a complex $U_1 \to U_0$ such that $U_1, U_0$ are local projective objects. Let $A$ be a 0-dimensional object of $\mathcal{C}$. By the exact triangle

$$U_1 \to U_0 \to F \to U_1[1]$$

and $\text{Hom}(A, U_0) = \text{Ext}^1(A, U_1) = 0$, we have $\text{Hom}(A, F) = 0$. \qed

**Remark 2.2.4.** Assume that $\text{rk} E < 0$ and $(c_1(E), H) > 0$. Then $\phi(E) < 1$. If $E$ is semi-stable, then $H^{-1}(E)$ is a local projective object. Indeed for a 0-dimensional object $A$ of $\mathcal{C}$, by using the exact sequence

$$0 \to H^{-1}(E)[1] \to E \to H^0(E) \to 0,$$

we have an exact sequence

$$\text{Hom}(A, H^0(E)[-1]) \to \text{Hom}(A, H^{-1}(E)[1]) \to \text{Hom}(A, E).$$

Therefore $\text{Hom}(A, H^{-1}(E)[1]) = 0$.

**Definition 2.2.5 ([Ko] Lem. 1.1.9).** Let $G$ be a local projective generator of $\mathcal{C}$. Then we have an abelian category

$$\mathcal{C}^D := \{ E \in D(X) | H^i(E) = 0 (i \neq -1, 0), \pi_*(G \otimes H^{-1}(E)) = 0, R^1\pi_*(G \otimes H^0(E)) = 0 \}.$$

Since $G^\vee$ is a local projective generator of $\mathcal{C}^D$, $G^\vee$-twisted semi-stability and $(-\beta)$-twisted semi-stability are defined by Definition 1.2.2.

**Remark 2.2.6.** Let $F$ and $\varphi$ be the complexes in Lemma 2.2.3. Since $\ker(\varphi^\vee)$ and $H^0(F)^\vee$ are the same on $X \setminus \cup_i Z_i$, we have $\mu_{\text{max}, G^\vee}(\ker(\varphi)) \leq 0$. Since $(\text{coker}(\varphi^\vee))^\vee$ and $H^{-1}(F)^\vee$ are the same on $X \setminus \cup_i Z_i$, we also have $\mu_{\text{min}, G^\vee}(\text{coker}(\varphi)) \geq 0$.

**Proposition 2.2.7.** Assume that $(\ast 2)$ holds for $v$ (i.e., $(\ast 1)$ holds for $v^\vee$). Let $F$ be an object of $D(X)$ with $v(F) = -v$. Then $F$ is a semi-stable object of $\mathfrak{A}^\mu$ if and only if $F^\vee[1]$ is a $(-\beta)$-twisted semi-stable object of $\mathcal{C}^D$ with $v(F^\vee[1]) = v^\vee$.

**Proof.** Let $E$ be a $(-\beta)$-twisted semi-stable object of $\mathcal{C}^D$ with $v(E) = v^\vee$. There is a surjective morphism $\psi : (G^\vee(-n))^{\oplus N} \to E$, where $n \gg 0$. Since $E$ does not contain a 0-dimensional subobject of $\mathcal{C}^D$, we see that $\ker \psi$ is a local projective object of $\mathcal{C}^D$. Thus $E$ is represented by a complex $V_{-1} \to V_0$ such that $V_i$ are local projective objects of $\mathcal{C}^D$. Then we have an exact triangle

$$E^\vee \to V_0^\vee \to V_{-1}^\vee \to E^\vee[1].$$

Since $V_i^\vee$ are local projective objects of $\mathcal{C}$, we see that $H^i(E^\vee[1]) = 0$ for $i \neq -1, 0$. Moreover we get $H^{-1}(E^\vee[1])$ is a $\mu$-semi-stable object of $\mathfrak{A}$ with $\deg(G(H^{-1}(E^\vee[1]))) = d(H^2) < 0$ and $H^0(E^\vee[1])$ is a 0-dimensional object of $\mathcal{C}$. Thus $E^\vee[1] \in \mathfrak{A}^\mu$. We take an exact sequence in $\mathfrak{A}^\mu$

$$0 \to F_1 \to E^\vee[1] \to F_2 \to 0$$

such that $F_2$ is a stable object. By Lemma 2.2.3 implies that $E^\vee[1]$, and hence $F_1$ does not contain a 0-dimensional object. We have an exact sequence

$$0 \longrightarrow H^{-1}(F_1) \longrightarrow H^{-1}(E^\vee[1]) \longrightarrow H^{-1}(F_2) \longrightarrow 0.$$
Since $H^0(E^\vee[1])$ is 0-dimensional, $H^0(F_2)$ is also 0-dimensional. Since $F_2$ does not contain a 0-dimensional object, by using Lemma 2.2.3, we see that $F_2^\vee[1] \in \mathcal{C}^D$. So we have an exact sequence

$$0 \to H^{-1}(F_2^\vee[1]) \to F_2^\vee[1] \to E \to H^0(F_2^\vee[1]) \to 0.$$ 

We set

$$v(F_2^\vee[1]) := r_2 e^{-\beta} + a_2 \phi_X + (-d_2 H + D_2) + (d_2 H + D_2, \beta) \phi_X,$$

$$v(\text{im } \psi) := r'e^{-\beta} + a' \phi_X + (-d'H + D') + (d'H + D', \beta) \phi_X.$$ 

Since $-d_2 \leq -d'$, we have $-d_2/r_2 \leq -d'/r'$. Since $E$ is $(-\beta)$-twisted semi-stable, $-d'/r' \leq -d/r$. Thus we have $-d_2/r_2 \leq -d/r$. If the equality holds, then $r_2 = r'$ and $-d_2 = -d'$. If $d_2 = 0$, then $\phi(F_2) > \phi(E^\vee[1])$. Assume that $d_2 < 0$. We note that $-d_2 \leq -d' \leq -d_2/r \leq -d/r$. If $d = d_2$, then $d = d'$ and $d = r'$. Hence coker $\psi$ is a 0-dimensional object. We also have $	ext{deg}_{G^\vee}(\ker \psi) = (d_2 - d')(H^2) = 0$. Then $\text{rk} F_1 = \text{rk} H^{-1}(F_2^\vee[1]) \geq 0$ and $\text{deg}_{G}(F_1) = 0$. In particular, $\text{deg}_{G}(H^0(F_1)) = \text{deg}_{G}(H^{-1}(F_1)) = 0$. Since $H^0(F_1) \in \mathcal{X}^\mu$ with $\text{deg}_{G}(H^0(F_1)) = 0$, $H^0(F_1)$ is a 0-dimensional object and $H^{-1}(F_1) = 0$. This means that $F_1$ is a 0-dimensional object, which is a contradiction. Therefore $d_2 > d$. If $rd_2 - r_2 d > 0$, then $(\ast)$ implies that $\phi(F_2) > \phi(E^\vee[1])$. If $rd_2 - r_2 d = 0$, then $r_2 = r'$ implies that $H^{-1}(F_2^\vee[1])$ is a torsion free object of rank 0. Therefore $H^{-1}(F_2^\vee[1]) = 0$. By the $(-\beta)$-twisted semi-stability of $E$, $a_2/r_2 \leq a/r$. Then we get $da_2 - d_2 a \geq 0$, which implies that $\phi(F_2) \geq \phi(E^\vee[1])$. Therefore $E^\vee[1]$ is a semi-stable object of $\mathcal{X}^\mu$ with $v(E^\vee[1]) = -v$.

Let $F$ be a semi-stable object of $\mathcal{X}^\mu$ such that $v(F) = -v$. By Lemma 2.2.3, $F$ is represented by a complex $U_{-1} \to U_0$ such that $U_{-1}$ and $U_0$ are local projective objects of $\mathcal{C}$. Then $F^\vee[1]$ is represented by a complex $U_{-1}^\vee \to U_0^\vee$. By [20, Rem. 1.1.9], $U_0^\vee$ and $U_{-1}^\vee$ are local projective objects of $\mathcal{C}^D$. By the proof of Lemma 2.2.3, $H^0(F)$ is 0-dimensional and $H^{-1}(F)$ is a $\mu$-semi-stable object of $\mathcal{C}$. Hence $F^\vee[1] \in \mathcal{C}^D$ is a $\mu$-semi-stable object. Let $F^\vee[1] \to F_2$ be a quotient of $F^\vee[1]$ such that $F_2$ is $(-\beta)$-twisted stable and

$$v(F_2) = r_1 e^{-\beta} + a_1 \phi_X + (-d_1 H + D_1) + (d_1 H + D_1, \beta) \phi_X), -d_1/r_1 = -d/r.$$ 

Since $F_1 := \ker(F^\vee[1] \to F_2)$ is a $\mu$-semi-stable object of $\mathcal{C}^D$, $F_2^\vee[1]$ and $F_2^\vee[1]$ are object of $\mathcal{X}^\mu$ and we have an exact sequence in $\mathcal{X}^\mu$:

$$0 \to F_2^\vee[1] \to F \to F_1^\vee[1] \to 0.$$ 

By the semi-stability of $F$, $\phi(F_2^\vee[1]) \leq \phi(F)$. Then we have $-da_1 + d_1 a \geq 0$, which implies that $a_1/r_1 \geq a/r$. Therefore $F^\vee[1]$ is $(-\beta)$-twisted semi-stable.

By Lemma 2.1.7, Proposition 2.2.4 and Proposition 2.2.7, we have the following result of Toda [21, Prop. 6.4, Lem. 6.5].

Corollary 2.2.8. Let $E$ be an object of $\mathcal{D}(X)$ with

$$v(E) = re^\beta + a \phi_X + (dH + D + (dH + D, \beta) \phi_X), d > 0, D \in H^1.$$ 

Assume that $(\omega^2) \gg (v^2 - (D^2))$.

(1) If $r \geq 0$, then $E$ is a semi-stable object of $\mathcal{X}^\mu$ if and only if $E$ is a $\beta$-twisted semi-stable object of $\mathcal{C}$.

(2) If $r < 0$, then $E$ is a semi-stable object of $\mathcal{X}^\mu$ if and only if $E^\vee[1]$ is a $(-\beta)$-twisted semi-stable object of $\mathcal{C}^D$.

By this corollary, we also have the following.

Corollary 2.2.9. For

$$v = re^\beta + a \phi_X + (dH + D + (dH + D, \beta) \phi_X), d > 0, D \in H^1,$$

assume that $(\omega^2) \gg (v^2 - (D^2))$. Then there is a coarse moduli scheme $M_{(\beta, \omega)}(v)$, which is given by

$$M_{(\beta, \omega)}(v) = \begin{cases} M_{H}^\beta(v), & \text{rk } v \geq 0, \\ M_{H}^\beta(-v^\vee), & \text{rk } v < 0. \end{cases}$$

Corollary 2.2.10. Assume that $d = d_{\text{min}}$. Then there is a coarse moduli scheme $M_{(\beta, \omega)}(v)$.

Proof. If $(\omega^2) > 2$, then $\mathcal{X}_{(\beta, \omega)} = \mathcal{X}^\mu$. In this case, by Lemma 2.1.6 and Corollary 2.2.10, we get the moduli scheme. For a general case, the claim follows from Proposition 1.6.10. □
2.3. A relation of \([28]\) with Bridgeland’s stability. We take a sufficiently small element \(\alpha \in \text{NS}(X) \otimes \mathbb{Q}\) such that \(-\langle e^{\beta+\alpha}, v(A) \rangle > 0\) for all 0-dimensional objects \(A\) of \(\mathcal{C}\). Let \(X' := \mathcal{M}_{H}^{\beta+\alpha}(v_{0})\) be the moduli space of \((\beta+\alpha)\)-twisted semi-stable objects of \(\mathcal{C}\) and there is a projective morphism \(X' \to Y'\), where \(Y' := \mathcal{M}_{H}^{\beta}(v_{0})\). By choosing a general \(\alpha\), we may assume that \(X'\) is a smooth projective surface. Then there is a universal family \(\mathcal{E}\) on \(X \times X'\) as a twisted object. For simplicity, we assume that \(\mathcal{E}\) is not twisted. Let

\[
\Phi : \mathbf{D}(X) \to \mathbf{D}(X')
\]

be the Fourier-Mukai transform. Let \(\mathcal{C}'\) be the category of perverse coherent sheaves on \(X'\) associated to \(\beta'\), that is, \(-\langle e^{\beta'}, v(A) \rangle > 0\) for all 0-dimensional objects \(A\) of \(\mathcal{C}'\). For \(\omega\) with \((\omega^{2}) < 2/r_{0}^{2}\), we shall consider the Fourier-Mukai transform of \(\mathfrak{A} = \mathfrak{A}_{(\beta, \omega)}\). We take \(\eta + \sqrt{-1}\omega\) such that \(\eta \in H^{\perp}\) and \(-\langle \eta^{2} \rangle\) is sufficiently small. Then \(\mathfrak{A}_{(\beta + \eta, \omega)} = \mathfrak{A}\). Let \(\mathfrak{A}^{\mu}\) be the category in Definition \([1.2.7]\). Suppose that \((\beta', \omega')\) is a pair associated to the \((\beta', \omega')\) (cf. \([1.2.7]\)). By Lemma \([1.1.4]\),

\[
Z_{(\beta'+\eta, \omega)}(\Phi(E)[1]) = -\frac{2}{r_{0}(\langle \eta + \sqrt{-1}\omega \rangle^{2})} Z_{(\beta + \eta, \omega)}(E).
\]

Thus we have the following diagram:

\[
\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{\Phi[1]} & \mathfrak{A}^{\mu} \\
Z_{(\beta + \eta, \omega)} \downarrow & & \downarrow Z_{(\beta'+\eta, \omega)} \\
\mathbb{C} & \xrightarrow{r_{0}(\langle \eta + \omega \rangle^{2})^{-1}} & \mathbb{C}
\end{array}
\]

We note that \((\omega^{2}) < \frac{2}{r_{0}^{4}}\) if and only if \((\omega')^{2} > 2\). Since \(\Phi[1] : \mathfrak{A} \to \mathfrak{A}^{\mu}\) is an equivalence (\([30]\) Thm. 3.5.8) and \(Z_{(\beta', \omega)}(\Phi(E)[1]) = \frac{2}{r_{0}(\langle \eta + \omega \rangle^{2})} Z_{(\beta, \omega)}(E)\), Corollary \([1.4.13]\) (1) also follows from (2). The following is a refinement of \([28]\) Thm. 1.7 and \([30]\) Prop. 3.7.2 (1).

**Proposition 2.3.1.** We set \(w := re^{\beta'} + a\theta \chi' + (d \tilde{H} + \tilde{D} + (d \tilde{H} + \tilde{D}, \beta') \theta \chi')\). Assume that \(d > N(v)\) (see Lemma \([2.1.4]\)). Let \(E \in \mathbf{D}(X)\) be a complex with \(v(\Phi(E)[2]) = w\). Then the following conditions are equivalent.

(i) \(E\) is a \(\beta\)-twisted semi-stable object of \(\mathcal{C}\).

(ii) \(E[1]\) is a semi-stable object of \(\mathfrak{A}\) for \((\beta, \omega)\).

(iii) \(\Phi(E)[2]\) is a semi-stable object of \(\mathfrak{A}^{\mu}\) for \((\beta', \omega')\).

(iv) \(\Phi(E)[2]\) is a \(\beta'\)-twisted semi-stable object of \(\mathcal{C}'\).

**Proof.** By \([2.4]\), (ii) is equivalent to (iii). The equivalence between (i) and (ii) follows from Lemma \([2.3.3]\) (1). The equivalence between (iii) and (iv) also follows from Lemma \([2.3.3]\) (2). \(\square\)

**Remark 2.3.2.** If \((\omega^{2}) \ll 2/r_{0}^{2}\), then (ii), (iii) and (iv) are equivalent. Thus in order to compare the moduli spaces \(M_{H}(\Phi^{-1}(w))\) and \(M_{\tilde{H}}(w)\), it is sufficient to study the wall-crossing behavior in \(\mathfrak{A}_{(\beta, \omega)}\).

**Lemma 2.3.3.** Assume that \(d > N(v)\).

1. \(\ast(3)\) holds for \(\Phi^{-1}(w)\) and any \((\omega^{2}) > 0\).
2. \(\ast(1)\) holds for \(w\) and any \((\omega')^{2} > 0\).

**Proof.** We note that \(\Phi^{-1}(w) = r_{0}ae^{\beta'} + \frac{1}{r_{0}} \theta \chi' - (dH + D + (dH + D, \beta) \theta \chi')\). Hence the claims follow from Lemma \([2.1.4]\). \(\square\)

**Remark 2.3.4.** Assume that \(\mathfrak{A}' = \mathfrak{A}^{\mu}\). If the following two conditions hold, then the conditions (ii), (iii), (iv) are equivalent to the \((-\beta)\)-twisted semi-stability of \(E^{\vee}\) in \(\mathcal{C}^{D}\).

1. \(\ast(2)\) holds for \(\Phi^{-1}(w)\) and any \((\omega^{2}) > 0\)

2. \(\ast(1)\) holds for \(w\) and any \((\omega')^{2} > 0\).

By Lemma \([2.1.6]\) we have the following claim.

**Lemma 2.3.5.** Assume that \(\mathfrak{A} = \mathfrak{A}^{\mu}\). If \(d = d_{\text{min}}\), then the conditions (ii), (iii), (iv) are equivalent to the \((-\beta)\)-twisted semi-stability of \(E^{\vee}\) in \(\mathcal{C}^{D}\).
3. The wall crossing behavior.

3.1. The definition of wall and chamber for stability. In this section, we fix $H$ and $(H, \beta)$ and study the wall crossing behavior of Bridgeland’s stability condition in $\mathfrak{M}_R$ or $i_\beta(\mathbb{R}_{>0}H)$ on a $K3$ surface or an abelian surface from the view of Fourier-Mukai transforms.

We set
\[ v_i := r_i e^\beta + a_i g_X + (d_i H + D_i + (d_i H + D_i, \beta) g_X), \quad i = 1, 2, D_i \in H^\perp. \]

Lemma 3.1.1 Assume that $d_1, d_2 > 0$. Then
\[ \langle v_1, v_2 \rangle = \frac{1}{2}(d_1 - d_2)^2 + \frac{d_2}{d_1} \langle v_1^2 \rangle + \frac{d_1}{d_2} \langle v_2^2 \rangle + \frac{(d_2 r_1 - d_1 r_2)(d_2 a_1 - d_1 a_2)}{d_1 d_2}. \]

Proof. The claim follows by using the following equalities
\[ \langle (v_1, v_2)^2 \rangle = (H^2) + ((D_1, D_1)^2) - 2(r_1/d_1)(a_1/d_1), \]
\[ \langle v_1, v_2 \rangle = (H^2) + ((D_2, D_2)^2) - 2(r_2/d_2)(a_2/d_2), \]
\[ \langle v_1, v_2 \rangle = (H^2) + (D_1, D_2, D_2) - (r_1/d_1)(a_2/d_2) - (r_2/d_2)(a_1/d_1). \]

Lemma 3.1.1 implies that
\[ \langle v_1, v_2 \rangle = \frac{(d_2 D_1 - d_1 D_2)^2}{2d_1 d_2} + \frac{d_2}{d_1} \langle v_1^2 \rangle + \frac{d_1}{d_2} \langle v_2^2 \rangle + \frac{(d_2 r_1 - d_1 r_2)(d_2 a_1 - d_1 a_2)}{d_1 d_2}. \]

Then we see that
\[ \langle (v_1 + v_2)^2 \rangle = \frac{(v_1^2)^2}{d_1} + \frac{(v_2^2)^2}{d_2} + \frac{(d_2 D_1 - d_1 D_2)^2}{d_1 d_2 (d_1 + d_2)} + \frac{(d_2 r_1 - d_1 r_2)(d_2 a_1 - d_1 a_2)}{d_1 d_2 (d_1 + d_2)}. \]

We also have
\[ \langle (v_1 + v_2)^2 \rangle = \frac{(v_1^2 - (D_1^2)}{2d_1} + \frac{(v_2^2 - (D_2^2)}{2d_2} + \frac{(r_1 - r_2)(a_1 - a_2)}{d_1 d_2 (d_1 + d_2)} \]

and
\[ \langle (v_1 + v_2)^2 \rangle = \frac{(v_1^2 - (D_1^2)}{2d_1} + \frac{(v_2^2 - (D_2^2)}{2d_2} + \frac{(d_2 r_1 - d_1 r_2)(d_2 a_1 - d_1 a_2)}{d_1 d_2 (d_1 + d_2)}. \]

Assume that
\[ v = re^\beta + a g_X + (d H + D + (d H + D, \beta) g_X), \quad d > 0 \]
has a decomposition
\[ v = \sum_{i=1}^s v_i, \quad \phi(v_i) = \phi(v), \]
where
\[ v_i = r_i e^\beta + a_i g_X + (d_i H + D_i + (d_i H + D_i, \beta) g_X), d_i > 0. \]

We note that $d_i/d_{\min} \in \mathbb{Z}$ (Definition 2.4.5). By using (3.3) and (3.5), we have
\[ \sum_i \left( \frac{\langle v_i^2 \rangle}{d_i} + \frac{2d_i}{d_{\min}^2} \varepsilon \right) \leq \frac{\langle v^2 \rangle}{d} + 2 \frac{d}{d_{\min}^2} \varepsilon, \]
\[ \sum_i \left( \frac{\langle v_i^2 \rangle - (D_i^2)}{d_i} + \frac{2d_i}{d_{\min}^2} \varepsilon \right) \leq \frac{\langle v^2 \rangle - (D^2)}{d} + 2 \frac{d}{d_{\min}^2} \varepsilon. \]

Lemma 3.1.2 (Bogomolov inequality). Let $E$ be a semi-stable object of $\mathfrak{M}_{(\beta, \omega)}$ with the Mukai vector (3.3). Assume that $d > 0$. Then
\[ \langle v(E)^2 \rangle \geq -2(d/d_{\min})^2 \varepsilon. \]

Proof. We may assume that $\varepsilon$ is algebraically closed. Then for a stable object $E$, Hom($E, E$) = 0. Hence the claim holds. In particular, if $d = d_{\min}$, the claim holds. For a general case, we take a Jordan-Hölder filtration of $E$. Then by (3.3), we get the claim.

Definition 3.1.3. Let $C$ be a chamber for categories, that is, $\mathfrak{M}_{(bH + \eta, \omega)}$ is constant for $(\eta, \omega) \in C \cap \mathfrak{M}$. For a Mukai vector $v$, we take the expansion (3.4), where $\beta = b H + \eta$.

(1) For $v_1$ in (3.6) satisfying
(a) $0 < d_1 < d$,
(b) $\langle v_1^2 \rangle < (d_1/d)(v^2) + 2 (d d_{\min}) \varepsilon$. 

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(c) \( (v_1^2) \geq -2 \frac{d^2}{d_{\min}^2} \varepsilon \),
we define a wall of type \( v_1 \) as the set of
\[
W_{v_1} := \{ (\eta, \omega) \in \mathcal{H}_R | (\omega^2)(dr_1 - d_1 r) = 2(-d(e^{bH + \eta}, v_1) + d_1(e^{bH + \eta}, v)) \}.
\]

(2) A chamber for stability is a connected component of \( \mathcal{C} \setminus \bigcup_{v_i} W_{v_i} \).

Remark 3.1.4. For a fixed \( \beta := bH + \eta \), we have the injection \( \iota_\beta : \mathbb{R}_{\geq 0} H \to \mathcal{H}_R \) (see 1.8). Then we have the notion of walls and chambers for \( \mathbb{R}_{\geq 0} H \). In this case, we can require that \( v_1 \) also satisfies
\[
(3.9) \quad (v_1^2) - (D_1^2)^2 < (d_1/d)(v_1^2) - (D_1^2)) + \frac{d_1 d_{\min}}{d^2} \varepsilon.
\]

We shall prove that the candidates of \((r_1, d_1, a_1, (D_1^2))\) are also bounded. Since \( -2d_1^2d_{\min}^2 < 2d_1^2H^2 - 2r_1a_1 < d_1(H^2) + 2d_1^2d_{\min}^2 \), \((D_1^2)\) is also bounded. In particular, the candidates of \((r_1, d_1, a_1)\) and \((D_1^2)\) are determined by \((H^2), b, \eta, v\).

**Lemma 3.1.5.** The set of walls is locally finite.

**Proof.** The claim is a consequence of [5] Prop. 9.3. For a convenience sake of the reader, we give a proof.
Let \( B \) be a compact subset of \( \mathcal{H}_R \). We shall prove that
\[
\{ v_1 | v_1 \text{ satisfies (a), (b), (c), } W_{v_1} \cap B \neq \emptyset \}
\]
is a finite set. We take \( r_0 \in \mathbb{Z} \) with \( r_0 e^{bH} \in H^*(X, \mathbb{Z}) \). For \( \beta = bH, bH + \eta \), we write
\[
v = re^{bH} + a'g_X + dH + D + (dH + D, bH)g_X.
\]
\[
\begin{aligned}
v = re^{bH + \eta} + a'g_X + dH + D' + (dH + D', bH + \eta)g_X, \\
v_1 = re^{bH} + a_1g_X + d_1H + D_1 + (d_1H + D_1, bH)g_X, \\
v_1 = re^{bH + \eta} + a_1'g_X + d_1H + D_1' + (d_1H + D_1', bH + \eta)g_X.
\end{aligned}
\]

Then
\[
(3.10) \quad a' = a - (D, \eta) + r(\eta^2)/2, \quad a'_1 = a_1 - (D_1, \eta) + r_1(\eta^2)/2.
\]
We note that
\[
a - \sqrt{-(D_1^2)\sqrt{-(\eta^2)} + r(\eta^2)/2} \leq a' \leq a + \sqrt{-(D_1^2)\sqrt{-(\eta^2)} + r(\eta^2)/2}.
\]
Assume that \((\eta, \omega) \in W_{v_1}\), i.e.,
\[
(3.11) \quad da'_1 - d_1a' = \frac{(\omega^2)}{2}(r_1d - rd_1).
\]
We shall give a bound of \( r_1 \) by \( v_1, (\omega^2) \) and \( (\eta^2) \). We have \( 2r_1a_1 < d_1^2(H^2) + 2d_1^2/d_{\min}^2 \). If \( r_1 < 0 \), then
\[
a_1 > \frac{1}{2r_1}(d_1^2(H^2) + 2d_1^2/d_{\min}^2) \geq -d_1(H^2)/2 + d_1^2/d_{\min}^2.
\]
Hence
\[
d_1r_1 - d_1r = 2(da_1 - d_1a)/(\omega^2) > \frac{1}{(\omega^2)} (-d_1(H^2)/2 + d_1^2/d_{\min}^2 - 2d_1a).
\]
Therefore
\[
r_1 > \frac{d_1}{d} r - \frac{1}{(\omega^2)}(d_1^2(H^2) + 2d_1^2/d_{\min}^2) - 2d_1d/(\omega^2 - a).
\]
If \( r_1 > 0 \), then
\[
a_1 < \frac{1}{2}(d_1^2(H^2) + 2d_1^2/d_{\min}^2).
\]
Hence we see that
\[
r_1 < \frac{d_1}{d} r + \frac{1}{(\omega^2)}(d_1^2(H^2) + 2d_1^2/d_{\min}^2) - 2d_1d/(\omega^2 - a).
\]
Since \( r_1 \) is an integer, the choice of \( r_1 \) is finite.
We have
\[
r_1a'_1 = r_1\left(\frac{(\omega^2)}{2}(r_1 - rd_1/d) + a'd_1/d\right) \geq r_1\frac{d_1}{d}\left(\frac{a'}{2} - \frac{(\omega^2)}{2}r\right)
\]

Then by using (3.10), we get
\[ r_1a_1 \geq r_1 \frac{d_1}{d} \left( a' - \frac{(\omega^2)}{2} r \right) + r_1(D_1, \eta) - r_1^2(\frac{\eta^2}{2}). \]

We have
\[
d_1^2(H^2) + 2 \frac{dd_1}{d_{\min}} \geq 2r_1a_1 - (D_1^2)
\]
\[
= 2r_1a_1' + 2r_1(D_1, \eta) - r_1^2(\eta^2) - (D_1^2)
\]
\[
\geq 2r_1a_1' + (\sqrt{-D_1^2} - \sqrt{-r_1^2(\eta^2)})^2
\]
\[
\geq 2r_1 \frac{d_1}{d} \left( a' - \frac{(\omega^2)}{2} r \right) + (\sqrt{-D_1^2} - \sqrt{-r_1^2(\eta^2)})^2.
\]

Hence \(-D_1^2\) is bounded. Since \(D_1 = (c_1(v_1) - r_1bH) - (c_1(v_1) - r_1bH, H)H/(H^2) \in \frac{1}{\Theta(H^2)}NS(X)\), the choice of \(D_1\) is finite. Then \(r_1a_1\) is also bounded. Since \(r_0a_1 \in \mathbb{Z}\), the choice of \(v_1\) is finite.

By the same arguments in (3.10), we get the following claim.

**Lemma 3.1.6.** If \((\eta, \omega), (\eta', \omega')\) belong to the same chamber, then \(\mathcal{M}_{(bH+\eta, \omega)}(v) = \mathcal{M}_{(bH+\eta', \omega')}(v)\).

Assume that \(d > 0\). Then \(\langle e^{bH+\eta+\sqrt{\omega}}, v \rangle \not\in \mathbb{R}\). Since \(\langle e^{bH+\eta+\sqrt{\omega}}, v_1/d_1 - v/d \rangle \in \mathbb{R}, \phi(v) = \phi(v_1)\) if and only if \(\langle e^{bH+\eta+\sqrt{\omega}}, v_1/d_1 - v/d \rangle = 0\).

\[
\langle e^{bH+\eta+\sqrt{\omega}}, v_1/d_1 - v/d \rangle = -\left( \frac{r_1}{d_1} - \frac{r}{d} \right) \frac{(\omega^2)}{2} + \left( \frac{a_1}{d_1} - \frac{a}{d} \right) + \left( \frac{D_1}{d_1} - \frac{D}{d} \right).
\]

\(\mathcal{Q}(v_1/v_1 - v/d)\) is determined by the wall \(W_{v_1}\).

If \(dr_1 - d_1r \neq 0\), then \(W_{v_1}\) is a half sphere in \(\mathcal{D}_R\):

\[
(\omega^2) - \left( \eta - \frac{dD_1 - d_1D}{dr_1 - d_1r} \right)^2 = 2 \frac{dd_1}{d_{\min}} - \left( \frac{dD_1 - d_1D}{dr_1 - d_1r} \right)^2.
\]

If \(dr_1 - d_1r = 0\), then \(W_{v_1}\) is a hyperplane in \(\mathcal{D}_R\):

\[
(\eta, dD_1 - d_1D) = (da_1 - d_1a).
\]

In this case, \(W_{v_1}\) is the wall for \(\eta\)-twisted semi-stability.

We can easily prove the following lemma, which will be used later.

**Lemma 3.1.7.** \(\langle v_1 - d_1v/d, e^{bH+\sqrt{\omega}} \rangle = 0\) if and only if \(\langle v_1, \xi_v \rangle = 0\), where

\[
\xi_v := \text{Re} \left( e^{bH+\sqrt{-\omega}} - \frac{\langle v/d, e^{bH+\sqrt{-\omega}} \rangle (H + (H, \beta) \varphi_X)}{\sqrt{H^2}} \right)
\]

\[
= e^{bH} - \frac{(\omega^2)}{2} \varphi_X - \frac{\langle v/d, e^H \rangle (H + (H, \beta) \varphi_X)}{\sqrt{H^2}}.
\]

The relative cases. Let \(\mathcal{Y} \to S\) be a polarized family of normal K3 surfaces or abelian surfaces over \(S\). Assume that there is a smooth family of polarized surfaces \(X \to S\) with a family of contractions \(\pi : X \to \mathcal{Y}\) over \(S\) such that \(\text{Re} \pi_*(O_X) = O_Y\). Assume that there is a locally free sheaf \(G\) on \(X\) which defines a family of tiltings \(\mathcal{E}_s, s \in S\) and \(G_s\) is a local projective generator of \(\mathcal{E}_s\). Let \(H\) be a relative \(\mathbb{Q}\)-Cartier divisor on \(X\) which is the pull-back of a relatively ample \(\mathbb{Q}\)-divisor on \(\mathcal{Y}\). We assume that there is a section \(\sigma : X \to S\).

Then \(\sigma\) gives a family of fundamental classes \(\varphi_{X_s}, s \in S\). We denote it by \(\varphi\). We take \(\beta \in NS(X/S)\). Let \(v_1 \in \mathbb{Z} \oplus NS(X/S) \oplus \mathbb{Z}\varphi\) be a family of Mukai vectors. For \(v_1 \in H^*(X_s, \mathbb{Z})_{\text{alg}}\), we write

\[
v_1 = r_1 e^{bH} + a_1 \varphi_{X_s} + d_1 H_s + D_1 + (d_1 H_s + D_1, \beta) \varphi_{X_s}, D_1 \in H^2_X.
\]

If \(v_1\) satisfies Definition (3.1.3) (1), then Remark 3.1.4 implies that the candidates of \(r_1, d_1, a_1\) and \((D_1^2)\) are finite. Assume that \(r_0 \beta \in NS(X_s)\). Then \(\xi := r_0 (d_1 H_s + D_1) \in NS(X_s)\) satisfies \(\langle \xi, H_s \rangle = r_0 d_1\) \((H_s^2)\) and \((\xi^2) = r_0^2 (d_1^2 (H_s^2) + (D_1^2))\). Since

\[
\chi(O_{X_s}, (\xi + nH_s)) = n^2 (H_s^2) + n r_0 d_1 (H_s^2) + \frac{r_0^2 (d_1^2 (H_s^2) + (D_1^2))}{2} + \chi(O_{X_s}),
\]

the set of Hilbert polynomials \(\chi(O_{X_s}, (\xi + nH_s))\) of \(O_{X_s}\) is finite. Since the relative Picard scheme is a fixed Hilbert polynomial is of finite type, the equivalence class of \(\xi\) is also finite, where \(\xi \in NS(X_s)\) and \(\xi' \in NS(X_s)\) is equivalent if \(\xi\) and \(\xi'\) belong to the same connected component of the relative Picard scheme.

Thus we get the following lemma.
Lemma 3.1.8. There is a dominant morphism $S' \to S$ such that for any point $s \in S'$, $v_1 \in H^*(X_s, Z)_{\text{alg}}$ in Definition 3.1.3 (1) extends to a family of Mukai vectors $v_1 \in \mathbb{Z} \oplus \text{NS}(X'/S') \oplus \mathbb{Z}_\partial$, where $X' = X \times_S S'$. 

3.2. The wall crossing behavior under the change of categories. In this subsection, we assume that $X$ is a K3 surface and we fix $\beta = bH + \eta$. Assume that $\omega \in \mathbb{R}_{>0}H$ belongs to a wall $W := W_{\omega}(E)$, where $E \in \text{Exc}_\beta$. Let $\omega_\pm \in \mathbb{Q}_{>0}H$ be ample $\mathbb{Q}$-divisors which are sufficiently close to $\omega$ and $\omega_\pm^2 < (\omega^2) < (\omega_\pm^2)$. We shall study the wall crossing behavior of the moduli spaces $M_{(\beta, \omega)}(v)$. Let $\phi_\pm$ be the phase function for $Z(\beta, \omega)$. We set 

$$v := re^\beta + a g_X + (dH + D + (dH, \beta)g_X), D \in H^2.$$ 

First of all, we shall slightly generalize the definition of the stability.

Definition 3.2.1. $E \in \mathfrak{A}_{(\beta, \omega)}$ is semi-stable with respect to $Z(\beta, \omega)$, if 

$$\Sigma_{(\beta, \omega)}(E', E) \geq 0$$ 

for all subobject $E'$ of $E$ (cf. Definition 1.5.3). $M_{(\beta, \omega)}(v)$ denotes the moduli stack of semi-stable object $E$ with respect to $Z(\beta, \omega)$ such that $v(E) = v$.

We study the categories of complexes $E \in D(X)$ such that $E \in \mathfrak{A}_{(\beta, \omega)}$ and $E$ is semi-stable with respect to $Z(\beta, \omega)$.

Lemma 3.2.2. 

1. For an object $E$ of $\mathfrak{A}_{(\beta, \omega)}$, if $\text{Hom}(E_0[1], E) = 0$ for all $E_0 \in \mathcal{G}_W$, then $E \in \mathfrak{A}_{(\beta, \omega)}$. In particular, for a semi-stable object $E$ of $\mathfrak{A}_{(\beta, \omega)}$ with $\phi_+(E) < 1$, $E \in \mathfrak{A}_{(\beta, \omega)}$.

2. For an object $E$ of $\mathfrak{A}_{(\beta, \omega)}$, if $\text{Hom}(E, E_0) = 0$ for all $E_0 \in \mathcal{G}_W$, then $E \in \mathfrak{A}_{(\beta, \omega)}$.

3. Assume that $d > 0$. Then $M_{(\beta, \omega)}(v) \subset M_{(\beta, \omega)}(v)$.

Proof. (1) Assume that $\text{Hom}(E_0[1], E) = \text{Hom}(E_0, H^{-1}(E)) = 0$ for all $E_0 \in \mathcal{G}_W$. Then $H^{-1}(E) \in \mathfrak{T}_{(\beta, \omega)}$, which implies that $E \in \mathfrak{A}_{(\beta, \omega)}$. Since $\phi_+(E) < 1$ and $\phi_+(E_0[1]) = 1$ for $E_0 \in \mathcal{G}_W$, $\text{Hom}(E_0[1], E) = 0$. Hence $E \in \mathfrak{A}_{(\beta, \omega)}$.

(2) Assume that $\text{Hom}(E, E_0) = 0$ for all $E_0 \in \mathcal{G}_W$. By Lemma 1.4.11 (3), we have $H^0(E) \in \mathcal{T}_{(\beta, \omega)}$, which implies that $E \in \mathfrak{A}_{(\beta, \omega)}$.

(3) For an object $E$ of $M_{(\beta, \omega)}(v)$, $d > 0$ implies that $\phi_+(E) < 1$. Then (1) implies $E \in \mathfrak{A}_{(\beta, \omega)}$. Let $F$ be a subobject of $E$ in $\mathfrak{A}_{(\beta, \omega)}$. Then there is an exact sequence in $\mathfrak{A}_{(\beta, \omega)}$ 

$$0 \to F_1 \to F \to F_2 \to 0$$

such that $H^{-1}(F_1) = H^{-1}(F) \in \mathfrak{T}_{(\beta, \omega)}$. $H^0(F_1) \in \mathfrak{T}_{(\beta, \omega)}$ and $F_2 \in \mathcal{G}_W$. Then $F_1$ is a subobject of $E$ in $\mathfrak{A}_{(\beta, \omega)}$ such that the exact sequence in $\mathfrak{A}_{(\beta, \omega)}$ 

$$0 \to F_1 \to F \to E/F_1 \to 0$$

is an exact sequence in $\mathfrak{A}_{(\beta, \omega)}$. Indeed $E \in \mathfrak{A}_{(\beta, \omega)}$ and $\text{Hom}(E/F_1, E_0) \subset \text{Hom}(E, E_0) = 0$ for all $E_0 \in \mathcal{G}_W$ implies $E/F_1 \in \mathfrak{A}_{(\beta, \omega)}$ by (2). Since $Z_{(\beta, \omega)}(F) = Z_{(\beta, \omega)}(F_1)$ and $\omega_+$ is sufficiently close to $\omega$, we have $\Sigma_{(\beta, \omega)}(F, E) = \Sigma_{(\beta, \omega)}(F_1, E) \geq 0$. Therefore $E \in M_{(\beta, \omega)}(v)$.

Since the semi-stability with respect to $Z(\beta, \omega)$ is defined in the category $\mathfrak{A}_{(\beta, \omega)}$, $M_{(\beta, \omega)}(v) \subset M_{(\beta, \omega)}(v)$ is obvious.

Lemma 3.2.3. 

1. If $E \in \mathfrak{A}_{(\beta, \omega)}$ satisfies $Z_{(\beta, \omega)}(E) = 0$, then $E \in \mathcal{G}_W$.

2. If $E \in \mathfrak{A}_{(\beta, \omega)}$ satisfies $Z_{(\beta, \omega)}(E) = 0$, then $E \in \mathcal{G}_W[1]$.

Proof. We set $G := G_{(\beta, \omega)}, G_\pm := G_{(\beta, \omega)} \subset K(X) \otimes \mathbb{Q}$. For $E \in \mathfrak{A}_{(\beta, \omega)}$, $Z_{(\beta, \omega)}(E) = 0$ implies that $H^{-1}(E)$ and $H^0(E)$ are $\beta$-twisted semi-stable objects of $\mathcal{C}$ with $\deg_G(H^{-1}(E)) = \chi_G(H^{-1}(E)) = 0$ and $\deg_G(H^0(E)) = \chi_G(H^0(E)) = 0$.

(1) If $H^{-1}(E) \neq 0$, then $\chi_G(H^{-1}(E)) > \chi_G(H^{-1}(E))$. Hence $E \in \mathfrak{A}_{(\beta, \omega)}$ implies that $H^{-1}(E) = 0$. Therefore $E = H^0(E) \in \mathcal{G}_W$.

(2) If $H^0(E) \neq 0$, then $\chi_G(H^{-1}(E)) < \chi_G(H^{-1}(E))$. Hence $E \in \mathfrak{A}_{(\beta, \omega)}$ implies that $H^0(E) = 0$. Therefore $E = H^{-1}(E)[1] \in \mathcal{G}_W[1]$.

Corollary 3.2.4. 

1. If $E \in \mathfrak{A}_{(\beta, \omega)}$ satisfies $\phi_-(E) = 1$ and $Z_{(\beta, \omega)}(E) \in \mathbb{R}_{\geq 0}e^{\pi \sqrt{-1} \phi}$, $0 < \phi < 1$, then $E \in \mathcal{G}_W$.

2. If $E \in \mathfrak{A}_{(\beta, \omega)}$ satisfies $\phi_+(E) = 1$ and $Z_{(\beta, \omega)}(E) \in \mathbb{R}_{\geq 0}e^{\pi \sqrt{-1} \phi}$, $0 < \phi < 1$, then $E \in \mathcal{G}_W[1]$.

Proof. We note that $Z_{(\beta, \omega)}(E) \in \mathbb{R}_{\leq 0}$. Then $Z_{(\beta, \omega)}(E) \in \mathbb{R}_{\geq 0}e^{\pi \sqrt{-1} \phi}$ implies that $Z_{(\beta, \omega)}(E) = 0$. Hence the claims follow from Lemma 3.2.3.

Proposition 3.2.5. Assume that $d > 0$. 

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(1) \( E \in \mathcal{M}_{(\beta, \omega)}(v) \) if and only if there is a filtration
\[
0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E
\]
in \( \mathfrak{A}_{(\beta, \omega)} \) such that
(a) \( Z_{(\beta, \omega)}(F_i/F_{i-1}) \in \mathbb{R}_{\geq 0}Z_{(\beta, \omega)}(E) \),
(b) \( F_i/F_{i-1} \) are semi-stable with respect to \( Z_{(\beta, \omega)} \),
(c) \( 1 \geq \phi_{-}(F_1/F_0) > \phi_{-}(F_2/F_1) > \cdots > \phi_{-}(F_s/F_{s-1}) > 0 \).

(2) \( E \in \mathcal{M}_{(\beta, \omega)}(v) \) if and only if there is a filtration
\[
0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E
\]
in \( \mathfrak{A}_{(\beta, \omega)} \) such that
(a) \( Z_{(\beta, \omega)}(F_i/F_{i-1}) \in \mathbb{R}_{\geq 0}Z_{(\beta, \omega)}(E) \),
(b) \( F_i/F_{i-1} \) are semi-stable with respect to \( Z_{(\beta, \omega)} \) or \( F_i/F_{i-1} \in \mathcal{S}_W \).
(c) \( 1 > \phi_{+}(F_1/F_0) > \phi_{+}(F_2/F_1) > \cdots > \phi_{+}(F_s/F_{s-1}) \geq 0 \).

Proof. (1) If \( E \in \mathcal{M}_{(\beta, \omega)}(v) \) is not a semi-stable object of \( \mathfrak{A}_{(\beta, \omega)} \) with respect to \( Z_{(\beta, \omega)} \), then we have the Harder-Narasimhan filtration
\[
0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E
\]
such that \( 1 \geq \phi_{-}(F_1/F_0) > \phi_{-}(F_2/F_1) > \cdots > \phi_{-}(F_s/F_{s-1}) > 0 \). Since \( \omega_{-} \) is sufficiently close to \( \omega \), we have \( Z_{(\beta, \omega)}(F_i/F_{i-1}) \in \mathbb{R}_{\geq 0}Z_{(\beta, \omega)}(E) \) for all \( i \). Conversely if \( E \) has a filtration with (a), (b), (c), then \( F_i/F_{i-1} \) with \( \phi_{-}(F_i/F_{i-1}) < 1 \) are semi-stable with respect to \( Z_{(\beta, \omega)} \) by Lemma \[3.2.2\] (3). If \( \phi_{-}(F_i/F_{i-1}) = 1 \), then Corollary \[3.2.4\] (1) implies that \( F_i/F_{i-1} \in \mathcal{S}_W \). Hence \( E \in \mathcal{M}_{(\beta, \omega)}(v) \).

(2) If \( E \in \mathcal{M}_{(\beta, \omega)}(v) \) is not semi-stable with respect to \( \sigma_{(\beta, \omega)} \), then we have an exact sequence
\[
0 \to E' \to E \to E_0 \to 0
\]
in \( \mathfrak{A}_{(\beta, \omega)} \), where \( E_0 \in \mathcal{S}_W \) and \( \text{Hom}(E', F) = 0 \) for all \( F \in \mathcal{S}_W \). Thus \( E' \in \mathfrak{A}_{(\beta, \omega)} \) by Lemma \[3.2.2\] (2). We take the Harder-Narasimhan filtration of \( E' \) in \( \mathfrak{A}_{(\beta, \omega)} \):
\[
0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_t = E'.
\]
Since \( \omega_{+} \) is sufficiently close to \( \omega \), \( Z_{(\beta, \omega)}(F_i/F_{i-1}) \in \mathbb{R}_{\geq 0}Z_{(\beta, \omega)}(E) \). If \( \phi_{+}(F_1) = 1 \), then Corollary \[3.2.4\] (2) implies that \( F_1 \in \mathcal{S}_W[1] \). By \( E \in \mathfrak{A}_{(\beta, \omega)} \), we have \( \phi_{+}(F_1) < 1 \). Thus
\[
1 > \phi_{+}(F_1/F_0) > \phi_{+}(F_2/F_1) > \cdots > \phi_{+}(F_t/F_{t-1}) > \phi_{+}(E_0) = 0.
\]
By Lemma \[3.2.2\] (1), (3), \( F_i/F_{i-1} \in \mathfrak{A}_{(\beta, \omega)} \) and they are semi-stable with respect to \( Z_{(\beta, \omega)} \). In particular, \[3.1.2\] is a filtration in \( \mathfrak{A}_{(\beta, \omega)} \). We set \( F_s := E \), where \( s := t + 1 \) for \( E_0 \neq 0 \) and \( s := t \) for \( E_0 = 0 \). Then we get a desired filtration. Conversely for a filtration with (a), (b), (c), \( F_i/F_{i-1} \) are semi-stable with respect to \( Z_{(\beta, \omega)} \) by Lemma \[3.2.2\] (3) or \( F_i/F_{i-1} \in \mathcal{S}_W \). Hence \( E \in \mathcal{M}_{(\beta, \omega)}(v) \).

By the following lemma, the choice of the Mukai vectors \( v(F_i/F_{i-1}) \) in Proposition \[3.2.3\] is finite.

Lemma 3.2.6. Let \( E \) be a semi-stable object with respect to \( Z_{(\beta, \omega)} \). Assume that \( E \) is \( S \)-equivalent to \( \bigoplus_{i=0}^n E_i \) such that (i) \( Z_{(\beta, \omega)}(E_0) = 0 \) and (ii) \( E_i \) are semi-stable with respect to \( \sigma_{(\beta, \omega)} \) for all \( i > 0 \) or \( E_i \) are semi-stable with respect to \( \sigma_{(\beta, \omega)} \) for all \( i > 0 \). The choice of \( v(E_0) \) is finite.

Proof. We set \( v(E_0) := r_0e^\beta + a_0g_X + D_0 + (D_0, \beta)g_X \). By \( Z_{(\beta, \omega)}(E_0) = 0 \), we have \( a_0 = r_0(\omega^2)/2 \). Then we have
\[
-2(d/d_{\min})^2 \leq \langle v(\bigoplus_{i=1}^n E_i) \rangle^2 - (D - D_0)^2
= -2(r - r_0)(a - a_0) + d^2(H^2)
= -\omega^2 \left( r - \left( r + \frac{2}{(\omega^2)^2}a \right) \right)^2 + \frac{\omega^2}{4} \left( r + \frac{2}{(\omega^2)^2}a \right)^2 + \langle v(E) \rangle^2 - (D^2).
\]
Hence the choice of \( r_0 \) is finite. Since \( E_0 \) is a successive extension of objects in \( \text{Exc}_{\beta} \), the choice of \( E_0 \) is also finite. \( \square \)
3.3. The wall crossing behavior for \(d = d_{\text{min}}\). As an example of the wall crossing behavior, we shall study \(M_{(\beta, \omega)}(v)\) for \(d = d_{\text{min}}\). In this case, there is no wall for stability and the filtration in Proposition 3.2.5 is of length \(s = 2\). Assume that \(\omega\) belongs to a wall \(W\) for categories. In order to study the contribution of \(\mathcal{S}_W\), we set
\[
R_+ := \{v(E)|E \in \mathcal{S}_W, \langle v(E)^2 \rangle = -2\}.
\]
By Lemma 1.4.18, \(R_+\) is a finite set. For a sufficiently small general element \(\eta \in \text{NS}(X)_\mathbb{Q}\), we have
\[
\langle u_j/\text{rk} u_j, \eta + (\eta, \beta)g_X \rangle \neq 0
\]
for all \(u, u' \in R_+\) with \(u \neq u'\). We may assume that \(R_+ = \{u_1, u_2, \ldots, u_n\}\),
\[
-\langle u_i/\text{rk} u_i, \eta + (\eta, \beta)g_X \rangle < -\langle u_j/\text{rk} u_j, \eta + (\eta, \beta)g_X \rangle
\]
for \(i < j\).

**Lemma 3.3.1.** (1) There is a unique \((\beta + \eta)\)-twisted semi-stable object \(U_i\) of \(\mathcal{C}\) with \(v(U_i) = u_i\).

(2) \(\text{Hom}(U_j, U_i) = 0, \quad j > i\).

**Proof.** (1) By Cor. 3.7.3, there is a \((\beta + \eta)\)-twisted semi-stable object \(U_i\) of \(\mathcal{C}\) with \(v(U_i) = u_i\). If \(U_i\) contains a \((\beta + \eta)\)-twisted stable subobject \(U'_i\) such that \(U'_i \in \mathcal{S}_W\) and \(\langle v(U'_i)^2 \rangle, \eta + (\eta, \beta)g_X \rangle / \text{rk} U'_i = \langle v(U_i), \eta + (\eta, \beta)g_X \rangle / \text{rk} U_i\), then \(v(U'_i) \in R_+\). By our choice of \(\eta\), \(U'_i = U_i\). Thus \(U_i\) is a \((\beta + \eta)\)-twisted stable.

Since \(\langle v(U_i)^2 \rangle = -2\), \(U_i\) is the unique \((\beta + \eta)\)-twisted stable object of \(\mathcal{C}\) with \(v(U_i) = u_i\). (2) follows from 3.3.3.

**Lemma 3.3.2.** Let \(E\) be an object of \(\mathfrak{A}(\beta, \omega)\). Assume that \(E\) is semi-stable with respect to \(Z(\beta, \omega)\) such that \(\text{Hom}(E, U_i) = 0\) for \(i < k\).

(1) \(\varphi : E \to \text{Hom}(E, U_k)^\vee \otimes U_k\) is surjective in \(\mathcal{C}\) and ker \(\varphi\) is a semi-stable object with respect to \(Z(\beta, \omega)\) such that \(\text{Hom}(\text{ker} \varphi, U_i) = 0\) for \(i \leq k\). Moreover for the universal extension
\[
0 \to U_k \otimes \text{Ext}^1(\text{ker} \varphi, U_k)^\vee \to E' \to \ker \varphi \to 0,
\]
\(E'\) is a semi-stable object with respect to \(Z(\beta, \omega)\) such that \(\text{Hom}(E', U_i) = 0\) for \(i < k\).

(2) If \(\text{Hom}(U_i, E) = 0\) for \(i \geq k\), then \(\text{Hom}(U_k, \ker \varphi) = 0\) for \(i \geq k\) and \(\text{Hom}(U_i, E') = 0\) for \(i < k\).

**Proof.** (1) By Lemma 1.4.18 \(\text{im} \varphi \in \mathcal{S}_W\). Let \(F_1\) be a subobject of \(\text{im} \varphi\) in \(\mathcal{C}\) such that \(F_1 = (\beta + \eta)\)-twisted stable with \(-\langle v(F_1), \eta + (\eta, \beta)g_X \rangle / \text{rk} F_1 \geq -\langle \text{rk} \text{im} \varphi, \eta + (\eta, \beta)g_X \rangle / \text{rk} \text{im} \varphi\). Then \(-\langle v(F_1), \eta + (\eta, \beta)g_X \rangle / \text{rk} F_1 \leq \langle \text{rk} \text{im} \varphi, \eta + (\eta, \beta)g_X \rangle / \text{rk} \text{im} \varphi\). On the other hand, for a quotient object \(F_2\) of \(\text{im} \varphi\) in \(\mathcal{C}\) such that \(F_2 = (\beta + \eta)\)-twisted stable with \(-\langle v(F_2), \eta + (\eta, \beta)g_X \rangle / \text{rk} F_2 \leq -\langle \text{rk} \text{im} \varphi, \eta + (\eta, \beta)g_X \rangle / \text{rk} \text{im} \varphi\). Our assumption implies that \(-\langle v(F_2), \eta + (\eta, \beta)g_X \rangle / \text{rk} F_2 \leq \langle \text{rk} \text{im} \varphi, \eta + (\eta, \beta)g_X \rangle / \text{rk} \text{im} \varphi\). Therefore \(\text{im} \varphi\) is a \((\beta + \eta)\)-twisted semi-stable object with \(\text{rk} \text{im} \varphi = u_k \otimes u_k\).

Then we see that \(m = \dim \text{Hom}(E, U_k)\). Since \(\text{Ext}^1(U_k, U_k) = 0\), we get \(\text{Hom}(\text{ker} \varphi, U_k) = 0\). If there is a non-trivial homomorphism \(\psi : \ker \varphi \to U_i\), \(i < k\), then by Lemma 1.4.18 \(F = \psi \varphi\) belongs to \(\mathcal{S}_W\) and \(-\langle v(F), \eta + (\eta, \beta)g_X \rangle / \text{rk} F \leq \langle \text{rk} \text{im} \varphi, \eta + (\eta, \beta)g_X \rangle / \text{rk} \text{im} \varphi\).

Then we have a quotient object \(U\) of \(E\) fitting in an exact sequence
\[
0 \to F \to U \to U_k^\oplus m \to 0.
\]
Hence \(U \in \mathcal{S}_W\) and \(-\langle v(U), \eta + (\eta, \beta)g_X \rangle / \text{rk} U \leq \langle \text{rk} \text{im} \varphi, \eta + (\eta, \beta)g_X \rangle / \text{rk} \text{im} \varphi\). This means that there is a quotient \(E \to U\), \(j < k\). Therefore \(\text{Hom}(\text{ker} \varphi, U_i) = 0\) for all \(i \leq k\). By the exact sequence (3.15), we get an exact sequence
\[
0 \to \text{Hom}(\text{ker} \varphi, U_i) \to \text{Hom}(E', U_i) \to \text{Hom}(U_k, U_i) \otimes \text{Ext}^1(\text{ker} \varphi, U_k).
\]
Since \(\delta_{ij}\) is isomorphic for \(i = k\) and \(\text{Hom}(\text{ker} \varphi, U_k) = 0\) for \(i \leq k\), we get \(\text{Hom}(E', U_i) = 0\) for \(i \leq k\). Since \(Z(\beta, \omega)(U_k) = 0\), the semi-stability is a consequence of its definition. (2) easily follows from (1).

**Definition 3.3.3.** \(\mathcal{M}_{(\beta, \omega)}(v) := \{E \in \mathcal{M}_{(\beta, \omega)}(v)|\text{Hom}(U_i, E) = 0, i \geq k, \text{Hom}(E, U_j) = 0, j < k\}\).

We define the Brill-Noether locus by
\[
\mathcal{M}_{(\beta, \omega)}(v)_m := \{E \in \mathcal{M}_{(\beta, \omega)}(v)|\text{dim} \text{Hom}(E, U_k) = m\},
\]
\[
\mathcal{M}_{(\beta, \omega)}(v)^m := \{E \in \mathcal{M}_{(\beta, \omega)}(v)|\text{dim} \text{Hom}(U_{k-1}, E) = m\}.
\]

We note that \(\mathcal{M}_{(\beta, \omega)}(v)_0 = \mathcal{M}_{(\beta, \omega, k+1)}(v)^0\). As in [24], we have the following description of the Brill-Noether locus.

**Proposition 3.3.4.** (1) \(\mathcal{M}_{(\beta, \omega)}(v)_m\) is a \(Gr(2m + \langle v, u_k \rangle, m)\)-bundle over \(\mathcal{M}_{(\beta, \omega, k)}(v - mu_k)_0\).
Then $E \in \mathcal{M}(\beta,\omega,k)(v)$. 

(2) For $F \in \mathcal{M}(\beta,\omega,k+1)(v-mu_k)^0$, $\text{Hom}(U_k, F) = \text{Ext}^2(U_k, F) = 0$. Hence $\dim \text{Ext}^1(U_k, F) = \langle u_k, v - mu_k \rangle = 2m + \langle v, u_k \rangle$. For a subspace $V$ of $\text{Ext}^1(U_k, F)$ with $\dim V = m$, take the associated extension

$$0 \to F \to E \to U_k \otimes V \to 0,$$

then $E \in \mathcal{M}(\beta,\omega,k)(v)_m$.

Then $E \in \mathcal{M}(\beta,\omega,k+1)(v)$. 

\[ \square \]

**Proposition 3.3.5.** For $E \in \mathcal{M}(\beta,\omega,k)(v)$, if $E$ is $S$-equivalent to $E_0 \oplus E_1$ such that $E_0$ is stable with respect to $Z(\beta,\omega)$ and $E_1 \in \mathcal{G}_W$, then $\text{Hom}(E, E) \cong \mathfrak{t}$. In particular, if $d = d_{\text{min}}$, then $\text{Hom}(E, E) \cong \mathfrak{t}$ for all $E \in \mathcal{M}(\beta,\omega,k)(v)$.

**Proof.** Let $\varphi : E \to E$ be a homomorphism such that $\ker \varphi \neq 0$ and $\text{coker} \varphi \neq 0$. Then $\ker \varphi, \text{im} \varphi, \text{coker} \varphi$ are semi-stable objects with respect to $Z(\beta,\omega)$. By our assumption, $\ker \varphi, \text{coker} \varphi \in \mathcal{G}_W$ or $\text{im} \varphi \in \mathcal{G}_W$. In the first case, $E \in \mathcal{M}(\beta,\omega,k)(v)$ implies that

$$- \frac{\langle v, u_k \rangle + \langle \eta, \beta \rangle \rho_X}{\text{rk} \ker \varphi} \leq \frac{\langle u_{k-1}, \eta + \langle \eta, \beta \rangle \rho_X \rangle}{\text{rk} u_{k-1}},$$

$$- \frac{\langle v, \text{coker} \varphi \rangle, \eta + \langle \eta, \beta \rangle \rho_X}{\text{rk} \text{coker} \varphi} \geq \frac{\langle u_k, \eta + \langle \eta, \beta \rangle \rho_X \rangle}{\text{rk} u_k}.$$

Since $v, \ker \varphi = v, \text{coker} \varphi$, we have

$$- \frac{\langle u_k, \eta + \langle \eta, \beta \rangle \rho_X \rangle}{\text{rk} u_k} \leq - \frac{\langle u_{k-1}, \eta + \langle \eta, \beta \rangle \rho_X \rangle}{\text{rk} u_{k-1}},$$

which is a contradiction. In the second case, we also see that

$$- \frac{\langle v, \text{im} \varphi \rangle, \eta + \langle \eta, \beta \rangle \rho_X}{\text{rk} \text{im} \varphi} \leq \frac{\langle u_{k-1}, \eta + \langle \eta, \beta \rangle \rho_X \rangle}{\text{rk} u_{k-1}},$$

which is a contradiction. Therefore $\varphi$ is an isomorphism or $\varphi = 0$. Then we see that $\text{Hom}(E, E) \cong \mathfrak{t}$ by a standard argument. 

\[ \square \]

**Proposition 3.3.6.** Assume that $d = d_{\text{min}}$. Then $\mathcal{M}(\beta,\omega,k)(v)$, $1 \leq k \leq n$, are all birationally equivalent.

**Proof.** We set $m := \max\{-\langle v, u_k \rangle, \}$. Then $\mathcal{M}(\beta,\omega,k)(v)_m$ is an open dense substack of $\mathcal{M}(\beta,\omega,k)(v)$ and $\mathcal{M}(\beta,\omega,k+1)(v)_m$ is an open dense substack of $\mathcal{M}(\beta,\omega,k+1)(v)$. For the proof of the claim, it is sufficient to show $\mathcal{M}(\beta,\omega,k+1)(v)_n \cong \mathcal{M}(\beta,\omega,k)(v)$. If $m = 0$, then $\mathcal{M}(\beta,\omega,k)(v) = \mathcal{M}(\beta,\omega,k+1)(v)^0$. Assume that $m > 0$. Let $\Phi_{U_k}$ be the reflection functor. Then for $E \in \mathcal{M}(\beta,\omega,k+1)(v)_m$, we have

$$\Phi_{U_k}(E) = \text{coker}(U_k \otimes \text{Hom}(U_k, E) \to E)$$

and $E' := \Phi_{U_k} \circ \Phi_{U_k}(E)$ fits in an exact sequence

$$0 \to E \to E' \to \text{Ext}^1(U_k, \Phi_{U_k}(E)) \otimes U_k \to 0.$$ 

Then we see that $E' \in \mathcal{M}(\beta,\omega,k)(v)_m$. Hence we have an isomorphism $\mathcal{M}(\beta,\omega,k+1)(v)_m \to \mathcal{M}(\beta,\omega,k)(v)_m$. 

\[ \square \]

**Corollary 3.3.7.** Assume that $d = d_{\text{min}}$.

(1) The birational type of $\mathcal{M}(\beta,\omega)(v)$ does not depend on the choice of general $\omega$.

(2) $M_H(v)$ is birationally equivalent to $M_{H'}(\pm \Phi(v))$, where $\Phi$ is the Fourier-Mukai transform in Subsection 2.3.

**Proof.** (1) We note that $\mathcal{M}(\beta,\omega,1)(v) = \mathcal{M}(\beta,\omega,2)(v)$ and $\mathcal{M}(\beta,\omega,n+1)(v) = \mathcal{M}(\beta,\omega,k)(v)$. By Proposition 3.3.5, $\mathcal{M}(\beta,\omega,k)(v)$ is birationally equivalent to $\mathcal{M}(\beta,\omega,k+1)(v)$. Since the stability is independent of each chamber, we get our claim.

(2) The claim follows from (1) and Proposition 3.3.5. 

\[ \square \]

3.4. The wall crossing formula for the numbers of semi-stable objects over $\mathbb{F}_q$. 

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3.4.1. The wall crossing formula for counting invariants was obtained by Toda [21]. Here we only consider its specialization.

Let \( \mathbb{F}_q \) be the finite field with \( q \) elements.

**Definition 3.4.1.** We denote the set of semi-stable objects over \( \mathbb{F}_q \) by \( \mathcal{M}_{(\beta, \omega)}(v)(\mathbb{F}_q) \).

We shall study the weighted number of semi-stable objects over \( \mathbb{F}_q \):

\[
\sum_{E \in \mathcal{M}_{(\beta, \omega)}(v)(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)}.
\]

We start with the moduli of \( \beta \)-twisted semi-stable objects of \( C \). There is an open subset \( Q^{ss} \) of a Quot scheme Quot_{\mathcal{O}_X(-m)\oplus N}/\mathcal{X}/\mathbb{F}_q \) such that \( M^\beta_H(v)^{ss} = [Q^{ss}/GL(N)] \). Since every \( GL(N) \)-orbit \( O \) over \( \mathbb{F}_q \) contains a \( \mathbb{F}_q \)-rational point (cf. [14, Thm. 2]), we have \#(O(\mathbb{F}_q)) = #GL(N)(\mathbb{F}_q)\) and

\[
(3.16) \quad \sum_{E \in \mathcal{M}^\beta_H(v)(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} = \frac{\#Q^{ss}(\mathbb{F}_q)}{\#GL(N)(\mathbb{F}_q)}.
\]

**Remark 3.4.2.** Let \( Q^s \) be the open subset of \( Q^{ss} \) parametrizing stable sheaves. Since \( Q^s \to M^\beta_H(v) \) is a principal \( PGL(N) \)-bundle, \( [14, \text{Thm. 2}] \) also says that

\[
\#M^\beta_H(v)(\mathbb{F}_q) = (q-1)\frac{\#Q^s(\mathbb{F}_q)}{\#GL(N)(\mathbb{F}_q)}.
\]

Let us compute the wall crossing formula for \( \sum_{E \in \mathcal{M}_{(\beta, \omega)}(v)(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} \). We first treat the case where \( \omega \) belongs to a wall \( W \) for stability in Definition [3.1.2]. By using Desale and Ramanan [6], we see that

\[
(3.17) \quad \sum_{E \in \mathcal{M}_{(\beta, \omega)}(v)(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} = \sum_{E \in \mathcal{M}_{(\beta, \omega+)}(v)(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)}
\]

\[
+ \sum_{(v_1, \ldots, v_s)} q^{\sum_{i>j}(v_i, v_j)} \prod_{i=1}^s \left( \sum_{E \in \mathcal{M}_{(\beta, \omega+)}(v_i)(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} \right),
\]

where \( v_1, \ldots, v_s \) satisfies \( v = \sum_i v_i, \phi(v_i) = \phi(v) \) and \( \phi_+(v_1) > \phi_+(v_2) > \cdots > \phi_+(v_s) \).

We next treat the case where \( \omega \) belongs to a wall \( W \) for categories in Definition [3.1.3]. For \( v \) with \( Z_{(\beta, \omega)}(v) = 0 \), we set

\[
\mathcal{M}_{(\beta, \omega-)}(v) := \{ E \in \mathfrak{S}_W | v(E) = v \}, \quad \phi_-(v) = 1
\]

\[
\mathcal{M}_{(\beta, \omega+)}(v) := \{ E \in \mathfrak{S}_W | v(E) = v \}, \quad \phi_+(v) = 0.
\]

Then we get

\[
(3.18) \quad \sum_{E \in \mathcal{M}_{(\beta, \omega)}(v)(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} = \sum_{E \in \mathcal{M}_{(\beta, \omega+)}(v)(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)}
\]

\[
+ \sum_{(v_1, \ldots, v_s)} q^{\sum_{i>j}(v_i, v_j)} \prod_{i=1}^s \left( \sum_{E \in \mathcal{M}_{(\beta, \omega+)}(v_i)(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} \right),
\]

where \( v_1, \ldots, v_s \) satisfies

(i) \( v = \sum_i v_i \),

(ii) \( Z_{(\beta, \omega)}(v_1) \in \mathbb{R}_{\geq 0} Z_{(\beta, \omega)}(v) \),

(iii) \( 1 \geq \phi_-(v_1) > \phi_-(v_2) > \cdots > \phi_-(v_s) > 0 \) and \( 1 > \phi_+(v_1) > \phi_+(v_2) > \cdots > \phi_+(v_s) \geq 0 \).

For \( v_1, \ldots, v_s \) with \( v \) satisfying the condition

\[
1 \geq \phi_-(v_1) > \phi_-(v_2) > \cdots > \phi_-(v_s) > 0
\]

is equivalent to

\[
1 > \phi_+(v_1) > \phi_+(v_2) > \cdots > \phi_+(v_1) \geq 0.
\]

By the induction on \( d \), we get the following claim, which is a special case of [21].

**Proposition 3.4.3.**

\[
\sum_{E \in \mathcal{M}_{(\beta, \omega-)}(v)(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} = \sum_{E \in \mathcal{M}_{(\beta, \omega+)}(v)(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)}.
\]
Proof. We prove the claim by induction on \(d\). By the wall crossing formula (3.17) (3.18), it is sufficient to prove the claim for \(d = d_{\min}\). In this case, by Proposition 3.3.4 we get
\[
\sum_{E \in \mathcal{M}(\beta, \omega, k)(v)_{m}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} = \sum_{E \in \mathcal{M}(\beta, \omega, k+1)(v)_{m}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)}.
\]
Then we have
\[
\sum_{E \in \mathcal{M}(\beta, \omega, k)(v)(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} = \sum_{m} \sum_{E \in \mathcal{M}(\beta, \omega, k)(v)_{m}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} = \sum_{m} \sum_{E \in \mathcal{M}(\beta, \omega, k+1)(v)_{m}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)}.
\]
Since \(\mathcal{M}(\beta, \omega, 1)(v) = \mathcal{M}(\beta, \omega)(v)\) and \(\mathcal{M}(\beta, \omega, n+1)(v) = \mathcal{M}(\beta, \omega+1)(v)\), we get the claim. \(\square\)

**Proposition 3.4.4.** If \(\omega\) is general, then
\[
\sum_{E \in \mathcal{M}(\beta, \omega)(v)(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} = \sum_{E \in \mathcal{M}_H^0(\pm v)^{ss}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)}.
\]

Proof. By Proposition 3.3.3 and Corollary 2.2.8 we have
\[
\sum_{E \in \mathcal{M}(\beta, \omega)(v)(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} = \left\{ \begin{array}{ll}
\sum_{E \in \mathcal{M}_H^\beta(v)^{ss}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)}, & \text{rk } v \geq 0 \\
\sum_{E \in \mathcal{M}_H^\beta(-v)^{ss}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)}, & \text{rk } v < 0.
\end{array} \right.
\]
Then the claim follows from Proposition 3.4.6 below. \(\square\)

**Corollary 3.4.5.** Let \(\Phi : \mathcal{D}(X) \to \mathcal{D}(X')\) be a Fourier-Mukai transform associated to a moduli of stable sheaves. Then
\[
\sum_{E \in \mathcal{M}_H^\beta(v)^{ss}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} = \sum_{E \in \mathcal{M}_H^\beta(\pm \Phi(v))^{ss}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)}.
\]

Proof. The claim follows from Proposition 2.3.1 and Proposition 3.4.4. \(\square\)

### 3.4.2. We shall prove the following result.

**Proposition 3.4.6.** Assume that \(\text{rk } v > 0\).
\[
\sum_{E \in \mathcal{M}_H^\beta(v)^{ss}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} = \sum_{E \in \mathcal{M}_H^\beta(v')^{ss}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)}.
\]

By the Harder-Narasimhan filtration, we can write down
\[
\sum_{E \in \mathcal{M}_H(v)^{ss}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} = \sum_{E \in \mathcal{M}_H(v')^{ss}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)}
\]
by using \(\sum_{E \in \mathcal{M}_H(v')^{ss}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)}\) with \(\text{rk } v' < \text{rk } v\). By the induction on \(\text{rk } v\), the proof of Proposition 3.4.6 is reduced to show the following claim.

**Lemma 3.4.7.** Assume that \(\text{rk } v > 0\).
\[
\sum_{E \in \mathcal{M}_H(v)^{ss}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)} = \sum_{E \in \mathcal{M}_H(v^{\prime})^{ss}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(E)}.
\]

Indeed if \(\text{rk } v = 1\), then Lemma 3.4.7 implies Proposition 3.4.6.

**Definition 3.4.8.** Let \(\mathcal{M}(\beta, \infty)(v)\) be the stack consisting of \(E \in \mathcal{A}^\mu\) with \(v(E) = v\) such that \(H^{-1}(E)\) is a \(\mu\)-semi-stable object and \(H^0(E)\) is a 0-dimensional object.

**Lemma 3.4.9.** Let \(E\) be an object of \(\mathcal{A}^\mu\) with \(v(E) = v\). Then \(E \in \mathcal{M}(\beta, \infty)(v)\) if and only if we have a filtration
\[
0 \subset F_1 \subset F_2 \subset F_3 = E
\]
such that \(F_1\) and \(F_3/F_2\) are 0-dimensional, \(F_2/F_1[-1]\) is a local projective object of \(\mathcal{E}\), \(F_3/F_1\) is torsion free, \(\text{Hom}(F_2, A) = 0\) for any 0-dimensional object \(A\) of \(\mathcal{E}\).
Hence the claim holds.

Lemma 3.4.10. For a torsion free object $E$ of $\mathcal{C}$, we have a unique extension

$0 \to E \to F \to T \to 0$

in $\mathcal{C}$ such that $F$ is a local projective object of $\mathcal{C}$ and $T$ is a 0-dimensional object of $\mathcal{C}$.

Proof. For a torsion free object $E_1 := E$ of $\mathcal{C}$, if $\text{Hom}(A_1, E_1[1]) \neq 0$ for a 0-dimensional irreducible object $A_1$, then we take a non-trivial extension

$0 \to E_1 \to E_2 \to A_1 \to 0$.

Then $E_2$ is a torsion free object of $\mathcal{C}$. If $\text{Hom}(A_2, E_2[1]) \neq 0$ for 0-dimensional object $A_2$ of $\mathcal{C}$, then a non-trivial extension

$0 \to E_2 \to E_3 \to A_2 \to 0$.

gives a torsion free object $E_3$ of $\mathcal{C}$. Continuing this procedure, we get a sequence of torsion free objects $E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$ with $v(E_i) = v(E_1) + \sum_{j=1}^{i-1} c(A_j)$. Since Bogomolov’s inequality holds for $\mu$-semi-stable objects, we see that $v(E_i)^2 = -N$, where $N$ depends on $rk E$ and the Harder-Narasimhan filtration of $E_1$ with respect to the $\mu$-semi-stability. We set

$v(E_1) := re^\beta + a_0 g_X + (dH + D + (dH + D, \beta) g_X), \quad r > 0,$

$v(A_i) := b_i g_X + (D_i + (D_i, \beta) g_X), \quad b_i > 0.$

Then

$\langle v(E_i)^2 \rangle - \langle v(E_1)^2 \rangle = -2r \sum_{j=1}^{i-1} b_j + ((D + \sum_{j=1}^{i-1} D_j)^2) - (D^2) \leq -2r \sum_{j=1}^{i-1} b_j - (D^2).$

Therefore there is an $n$ such that $\text{Hom}(A, E_n[1]) = 0$ for all 0-dimensional object $A$. Thus we get a desired local projective object. The uniqueness follows by using $\text{Hom}(T, F[1]) = 0$.

By the proof of Proposition 3.2.7 we get the following result.

Lemma 3.4.11. $E$ is a $\mu$-semi-stable object of $\mathcal{C}^{\beta}$ with $\deg_{G^\beta}(E) > 0$ if and only if $E^{\vee}[1]$ is an object of $\mathcal{C}^{\beta}$ such that $H^{-1}(E^{\vee}[1])$ is a $\mu$-semi-stable object of $\mathcal{C}$, $H^0(E^{\vee}[1])$ is a 0-dimensional object and $\text{Hom}(A, E^{\vee}[1]) = 0$ for all $\theta$-dimensional objects of $\mathcal{C}$.

By Lemma 3.4.10 and Lemma 3.4.11 we have the following expressions:

$$\sum_{E \in \mathcal{M}_{\beta, \infty}(v)(\mathbb{P}_q)} \frac{1}{\# \text{Aut}(E)} = \sum_{v_1 + v_2 = v} q^{(v_2, v_1)} \left( \sum_{E \in \mathcal{M}_{\beta, \infty}(v)(\mathbb{P}_q)} \frac{1}{\# \text{Aut}(E)} \right) \left( \sum_{E \in \mathcal{M}_{\beta, \infty}(v)(\mathbb{P}_q)} \frac{1}{\# \text{Aut}(E)} \right)$$

with $\text{rk} v_2 = (c_1(v_2), H) = 0$. We also have

$$\sum_{E \in \mathcal{M}_{\beta, \infty}(v)(\mathbb{P}_q)} \frac{1}{\# \text{Aut}(E)} = \sum_{v_1 + v_2 \equiv v} q^{(v_2, v_1)} \left( \sum_{E \in \mathcal{M}_{\beta, \infty}(v)(\mathbb{P}_q)} \frac{1}{\# \text{Aut}(E)} \right) \left( \sum_{E \in \mathcal{M}_{\beta, \infty}(v)(\mathbb{P}_q)} \frac{1}{\# \text{Aut}(E)} \right)$$

where $\text{rk} v_2 = (c_1(v_2), H) = 0$. We set $v = -re^\beta + a_0 g_X + (dH + D + (dH + D, \beta) g_X)$ and $v_2 = a_2 g_X + (D_2 + (D_2, \beta) g_X)$. Then $-2r^2 \geq \langle v_2^2 \rangle - \langle (v_2)^2 \rangle - (D^2) \leq -2r a_2 + ((D - D_2)^2)$. Hence the choices of $v_2$ is finite, which implies the above equalities are well-defined. By the induction on $\text{rk} v$, we get Lemma 3.3.7.
4. The wall defined by an isotropic Mukai vector. In this subsection, we assume that $X$ is an abelian surface over a field $k$. Let $\overline{\mathfrak{k}}$ be the algebraic closure of $k$. We note that $\mathfrak{X} = \mathfrak{X}^\prime$ and all $\mathfrak{X}(\beta, \omega)$ are the same. We fix $\beta = bh + \eta$ and study the wall crossing behavior with respect to $\omega \in \mathbb{Q}_{>0}H$. By Definition $3.1.3$ we have the following proposition.

**Proposition 4.1.1.** Assume that $d > 0$. If $v$ is a primitive and isotropic Mukai vector, then the stability does not depend on the choice of $\omega$. In particular,

\[ M_{(\beta, \omega)}(v) = \{ E \mid E \text{ is a semi-homogeneous sheaf of } v(E) = v \}, \text{rk } v \geq 0 \]

\[ M_{(\beta, \omega)}(v) = \{ F[1] \mid F \text{ is a semi-homogeneous sheaf of } v(F) = -v \}, \text{rk } v < 0 \]

Let $w_1$ be a primitive and isotropic Mukai vector. We shall study stable objects for $\omega$ lying on the wall $W_{w_1}$.

We set $X_1 := M_{(\beta, \omega)}(w_1)$. Let $\Phi^E_{X \to X_1} : \mathbf{D}(X) \to \mathbf{D}^a(X_1)$ be the Fourier-Mukai transform defined by the universal family $E$ on $X \times X_1$ as a $(1_X \times \alpha)$-twisted sheaf, where $\alpha$ is a representative of $[\alpha] \in H^2_{\mathfrak{k}}(X_1, \mathcal{O}_{X_1}^\times)$. Let $\mathbf{Ext}^2(E, F)$ be a sheaf on $X$ such that $E$ is semi-homogeneous.

**Lemma 4.1.2.** Let $F$ be an irreducible $\alpha$-twisted sheaf on a closed point $W$ of $X_1$. Then $\Phi^E_{X_1 \to X}(F)$ is a stable sheaf on $X$.

**Proof.** Let $y$ be a point of $X_1 \otimes \overline{\mathfrak{k}}$. Since $H^2_{\mathfrak{k}}(y, \overline{\mathfrak{k}})$ is trivial, there is an $\alpha$-twisted sheaf $\mathcal{O}_y^\alpha$ on $y$ such that $\dim_{\mathfrak{k}}(\mathcal{O}_y^\alpha) = 1$. Then $F \otimes \overline{\mathfrak{k}}$ is a successive extension of $\mathcal{O}_y^\alpha, y_1, \ldots, y_n \in X_1 \otimes \overline{\mathfrak{k}}$, where $n := \dim_{\mathfrak{k}} W$. Hence $E := \Phi^E_{X_1 \to X}(F)$ is a sheaf on $X$ such that $E \otimes \overline{\mathfrak{k}}$ is a successive extension of stable sheaves $\Phi^E_{X_1 \to X}(\mathcal{O}_y^\alpha)$. If $E$ is not stable, then there is an exact sequence

\[ 0 \to E_1 \to E \to E_2 \to 0 \]

such that $E_1 \otimes \overline{\mathfrak{k}}$ and $E_2 \otimes \overline{\mathfrak{k}}$ are successive extensions of stable sheaves $\Phi^E_{X_1 \to X}(\mathcal{O}_y^\alpha)$. Then we see that $F_i := \Phi^E_{X_1 \to X}(E_i)$ $(i = 1, 2)$ are $0$-dimensional $\alpha$-twisted sheaves and we have an exact sequence of $\alpha$-twisted sheaves

\[ 0 \to F_1 \to F \to F_2 \to 0 \]

on $W$. It contradicts the irreducibility of $W$. Therefore $E$ is stable.

**Lemma 4.1.3.** Let $E$ be a stable object of $\mathfrak{X}$ and assume that $\phi(E) = \phi(w_1)$.

1. $\langle v(E), w_1 \rangle \geq 0$.
2. If $\langle v(E), w_1 \rangle = 0$, then there are a closed point $W$ of $X_1$ and an irreducible $\alpha$-twisted sheaf $F$ on $W$ such that $E = \Phi^E_{X_1 \to X}(F)$. In particular, if $\mathfrak{k}$ is an algebraically closed field, then $E \in M_{(\beta, \omega)}(w_1)$.
3. If $\langle v(E), w_1 \rangle > 0$, then $\text{Hom}(E_1, E) = \mathbb{F}^2(E_1, E) = 0$ for any stable object $E_1$ with $v(E_1) \in \mathbb{Z}w_1$.

**Proof.** Let $E_1$ be a stable object with $v(E_1) \in \mathbb{Z}w_1$. We first note that $\text{Hom}(E_1, E) \neq 0$ or $\text{Hom}(E, E_1) \neq 0$ implies that $E \cong E_1$.

Since $\dim X_1 > 0$, we have a closed point $W$ of $X_1$ and an irreducible $\alpha$-twisted sheaf $F$ on $W$ such that $\Phi^E_{X_1 \to X}(F) \neq E$. Then $\langle v(E), n w_1 \rangle = \langle v(E), v(\Phi^E_{X_1 \to X}(F)) \rangle = \chi(E, v(\Phi^E_{X_1 \to X}(F))) \geq 0$, where $n = \dim_W F$. Thus (1) holds. We also have the first part of (3).

Assume that $E \neq \Phi^E_{X_1 \to X}(F)$ for any irreducible $\alpha$-twisted sheaf $F$ on a closed point $W$ of $X_1$. Then $\Phi^E_{X_1 \to X}(E)[1]$ is a locally free $\alpha$-twisted sheaf of rank $v(E), w_1$ on $X_1$. If $v(E), w_1 = 0$, then we have $\Phi^E_{X_1 \to X}(E)[1] = 0$, which implies that $E = 0$. Therefore $E = \Phi^E_{X_1 \to X}(F)$ for an irreducible $\alpha$-twisted sheaf $F$ on a closed point $W$ of $X_1$. Thus (2) holds.

If $v(E), w_1 = 1$, then $\Phi^E_{X_1 \to X}(E)[1]$ is a line bundle on $X_1$. Hence $\langle v(E)^2 \rangle = \langle v(\Phi^E_{X_1 \to X}(E)[1])^2 \rangle = 0$. Thus the second part of (3) follows.

**Lemma 4.1.4.** Let $E$ be a semi-stable object with $\langle v(E), w_1 \rangle = 1$. Then $E$ is $\mathbb{S}$-equivalent to $E_0 \oplus \oplus_{i=1}^n E_i$, where $E_0$ is a stable object with $\langle v(E_0)^2 \rangle = 0$ and $\langle v(E_0), w_1 \rangle = 1$, and $E_i$, $i > 0$ are stable object with $v(E_i) \in \mathbb{Z}w_1$.

**Proposition 4.1.5.** Assume that $\omega_\pm \in \mathbb{Q}_{>0}H$ are sufficiently close to $\omega$ and $\langle \omega_\pm^2 \rangle < \langle \omega^2 \rangle < \langle \omega_\pm^2 \rangle$. For $v$, assume that $\langle v, w_1 \rangle = 1$ and $\deg_{\omega_\pm}(w_1) > 0$.

1. There are (fine) moduli spaces $M_{(\beta, \omega_\pm)}(v)$, which are isomorphic to $\text{Pic}^0(X_1) \times \text{Hilb}^{(\omega_\pm^2)/2}_X$.
2. The universal families on $X \times M_{(\beta, \omega_\pm)}(v)$ are the simple complexes in $[22]$ Thm. 4.9.
Lemma 4.2.2. \( E \) is surjective and ker \( \psi \) is semi-stable. Since \( \langle v, w_1 \rangle = 1 \), \( X_1 \) is a fine moduli space, i.e., \( E \) is a coherent sheaf. We first assume that \( \phi_\pm(w_1) \neq \phi_\pm(v) \). In this case, for \( E \in \mathcal{M}_{(\beta, \omega)}(v) \) and a closed point \( W \) of \( X_1 \), Ext\(^2\)\((E_{X_1 \times X_1}, E) = \text{Hom}(E, E_{X_1 \times X_1})^\vee = 0 \). If \( \psi : E \to E_{X_1 \times X_1} \) is a non-trivial morphism, then \( \psi \) is surjective and ker \( \psi \) is semi-stable. Since \( \text{Hom}(E, E_{X_1 \times X_1})^\vee \neq 0 \) if and only if \( W = W' \), we see that \( \text{Hom}(E, E_{X_1 \times X_1}) = 0 \) except for finitely many closed points of \( X \). Hence \( \Phi_1^{X_1 \to X_1}(E)[1] \) is a torsion free sheaf of rank 1. Replacing the universal family \( E \), we may assume that \( c_1(\Phi_1^{X_1 \to X_1}(E)[1]) = 0 \). Thus \( \Phi_1^{X_1 \to X_1}(E)[1] = \pi^* \mathcal{L}, L \in \mathbf{Pic}^0(X_1) \) and \( \pi_2 \in \mathcal{H}(\mathcal{L})^2 \).

Conversely for a torsion free sheaf \( \mathcal{L} \), we shall prove the stability of \( \Phi_1^{X_1 \to X_1}(\mathcal{L})[1] \). We note that \( \Phi_1^{X_1 \to X_1}(\mathcal{L})[1] \in \mathcal{M}_{(\beta, \omega)}(w_0) \) where \( w_0 \) is a Mukai vector such that \( \langle w_0^2 \rangle = 0 \) and \( \langle w_0, w_1 \rangle = 1 \), and \( \Phi_1^{X_1 \to X_1}(\mathcal{L}) \in \mathcal{H}(\mathcal{L})^2 \) is a semi-stable object with \( \langle \Phi_1^{X_1 \to X_1}(\mathcal{L}) \rangle = n \mathcal{O}_Z \). Hence we have an exact sequence in \( \mathcal{A} \):

\[
0 \to \Phi_1^{X_1 \to X_1}(\mathcal{L}) \to \Phi_1^{X_1 \to X_1}(\mathcal{L})[1] \to 0.
\]

Let \( E_1 \) be a stable quotient object of \( \Phi_1^{X_1 \to X_1}(\mathcal{L})[1] \) such that \( \phi_\pm(v) > \phi_\pm(E_1) \). Then \( E_1 \) is semi-stable with respect to \( \omega \) and \( \phi(E_1) = \phi(v) \). By Lemma 4.1.1 \( v(E_1) = n_0 w_0 + n_1 w_1 \) with \( n_0 \) or \( n_1 = 0 \) or 1. Since \( \phi_\pm(w_1) < \phi_\pm(v) \), we see that \( n_0 = 0 \). By Lemma 4.1.3 \( (2) \), \( E_1 = \Phi_1^{X_1 \to X_1}(F) \), where \( F \) is an irreducible \( \alpha \)-twisted sheaf on a closed point \( W \) of \( X_1 \). Since \( \text{Hom}(\Phi_1^{X_1 \to X_1}(\mathcal{L}), E_1) = \text{Hom}(\mathcal{L}, \mathcal{L}[1], F) = 0 \), we get a contradiction. Therefore \( \Phi_1^{X_1 \to X_1}(\mathcal{L})[1] \) is semi-stable. By Lemma 4.1.4 it is easy to see that \( \mathcal{M}_{(\beta, \omega)}(v) \) consists of stable objects. Hence we get that \( \Phi_1^{X_1 \to X_1}(E)[1] \) induces an isomorphism \( M_{(\beta, \omega)}(v) \to \mathbf{Pic}^0(X_1) \times \mathcal{H}(\mathcal{L})^2 \). Thus (1) holds.

We next assume that \( \phi_\pm(w_1) > \phi_\pm(v) \). In this case, we see that \( \Phi_1^{X_1 \to X_1}(E)[1] \) is a torsion free sheaf of rank 1. By this correspondence, we get (1).

The relation with \( [22] \) Thm. 4.9 follows from the proof of \( [22] \) Thm. 4.9.

Corollary 4.1.6. Under the same assumption of Proposition 4.1.5 we have an isomorphism \( M_{(\beta, \omega)}(v) \to M_{(\beta, \omega)}(v) \) by a contravariant Fourier-Mukai functor.

Proof. Assume that \( \phi_\pm(w_1) < \phi_\pm(v) \). Then the isomorphism \( M_{(\beta, \omega)}(v) \to M_{(\beta, \omega)}(v) \) is given by

\[
E \mapsto \Phi_1^{X_1 \to X_1}(\Phi_1^{X_1 \to X_1}(E))^\vee,
\]

where \( E \in M_{(\beta, \omega)}(v) \). \( \square \)

4.2. Estimates of the wall-crossing terms. We shall estimate the dimension of the wall-crossing terms by computing the weighted numbers. For this purpose, we quote the following result of Lang and Weil [15].

Proposition 4.2.1 (Lang-Weil). Let \( Z \) be an algebraic set over \( \mathbb{F}_q \) with \( \dim Z = k \). Then

\[
\#Z(\mathbb{F}_q^n) = N q^{nk} + O(q^{n(k-1/2)}),
\]

where \( N \) is the number of irreducible components of dimension \( k \). In particular,

\[
\frac{1}{q^{n \dim M_{(\beta, \omega)}(v)^{ss}}} \left( \sum_{E \in M_{(\beta, \omega)}(v)^{ss}(\mathbb{F}_q^n)} \frac{1}{\# \text{Aut}(E)} \right)
\]

is bounded as a function of \( n \).

Lemma 4.2.2. For a Mukai vector \( v \), we write \( v = lv' \), where \( l \) is primitive and \( l > 0 \). Assume that \( (H, \beta) \) is general with respect to \( v \), that is, for any \( E \in M_{H}^\beta(v)^{ss} \), the S-equivalence class \( [E] \) of \( E \) satisfies \( v(E) \in \mathbb{Q}v \). Then

\[
\dim M_{H}^\beta(v)^{ss} = \begin{cases} (v^2) + 1, & (v^2) > 0, \\ (v^2) + l, & (v^2) = 0, \\ (v^2) + l^2, & (v^2) = -2. \end{cases}
\]

Proof. The proof is similar to those for [13] Lem. 3.2, Lem. 3.3. \( \square \)

Remark 4.2.3. For an abelian surface, stronger results hold ([13] Lem. 3.8):

\[
\dim M_{H}^\beta(v)^{ss} = \langle v^2 \rangle + 1
\]

for \( \langle v^2 \rangle > 0 \).

Lemma 4.2.4. Let \( v = \sum_{i=1}^s v_i \) be the decomposition in (3.5) with (3.6). Assume that \( \langle v_i^2 \rangle \geq 0 \) for all \( i \) (e.g., \( X \) is an abelian surface).

1. (i) \( \langle v_i, v_j \rangle \geq 3 \) for \( i \neq j \), or (ii) \( \langle v_i, v_j \rangle = 1, 2 \) and \( \langle v_i^2 \rangle = 0 \) or \( \langle v_j^2 \rangle = 0 \).
(2) Assume that \((H, \beta)\) is general with respect to \(v\) and all \(v_i\). Then

(a) \[
\dim M^\beta_H(v)_{ss} - \sum_{i>j} \langle v_i, v_j \rangle - \sum_i \dim M^\beta_H(v_i)_{ss} \geq 1 \]

unless \(s = 2, \{v_1, v_2\} = \{lu_1, u_2\}, \langle u_1^2 \rangle = 0, \langle u_1, u_2 \rangle = 1\).

(b) If \[
\dim M^\beta_H(v)_{ss} - \sum_{i>j} \langle v_i, v_j \rangle - \sum_i \dim M^\beta_H(v_i)_{ss} = 1,
\]

then one of the following conditions holds:

(b1) \(s = 2, \{v_1, v_2\} = \{u_1, u_2\}, \langle u_1^2 \rangle = 0, \langle u_1, u_2 \rangle = 2, u_1\) is primitive,

(b2) \(s = 2, \{v_1, v_2\} = \{2v_1, 2u_2\}, \langle u_2^2 \rangle = 0, \langle u_1, u_2 \rangle = 1, \)

(b3) \(s = 3, \langle v_1^2 \rangle = 0, \langle v_1, v_2 \rangle = 1\) and \(\langle v^2 \rangle = 6\)

(b4) \(s = 3, \{v_1, v_2, v_3\} = \{u_1, u_2, u_1 + u_2\}, \langle u_1^2 \rangle = 0, \langle u_1, u_2 \rangle = 1\) and \(v = 2(u_1 + u_2)\).

**Proof.** (1) Since \((d_2r_1 - d_1r_2)(d_2a_1 - d_1a_2) > 0\), by \((3.11)\) and the Hodge index theorem, we have \[
\langle v_i, v_j \rangle > \frac{1}{2} \left( \frac{d_2}{d_1} \langle v_i^2 \rangle + \frac{d_1}{d_2} \langle v_j^2 \rangle \right) \]

\[
\geq \sqrt{\langle v_i^2 \rangle \langle v_j^2 \rangle} \geq 0.
\]

If \(\langle v_i^2 \rangle, \langle v_j^2 \rangle > 0\), then \(\langle v_i, v_j \rangle \geq 3\). Therefore (1) holds.

(2) We write \(v_i = l_i v'_i\). Assume that \(l_i = 1\). If \(\langle v_i^2 \rangle > 0\), then \(\langle v_i, v_j \rangle = 1\) for all \(i < j\). Hence (a) holds. Assume that \(\langle v_i^2 \rangle = 0\) and the equality holds if and only if \(u_3 = u_1 + u_2\). Then \(\langle u_3 - u_1 - u_2 \rangle^2 \geq 0\) and the equality holds if and only if \(u_3 = u_1 + u_2\).

Proposition 4.2.5. Assume that \(X\) is an abelian surface over \(\mathbb{R}\) and \((\beta, H)\) is general with respect to \(v\). Then codim\(M_{(\beta, \omega)}(v)\) is general with respect to \(\omega_i\) and \((\beta, \omega)\) is general with respect to \(v\). Let \(\omega_1\) and \(\omega_2\) be general members of \(\mathbb{Q}_{>0} H\) which are not separated by any wall \(W_{u_1}\) with \(\langle v, u_1 \rangle = 1\) and \(\langle u_1^2 \rangle = 0\). Assume that there are moduli schemes \(M_{(\beta, \omega_j)}(v), i = 1, 2\). We set \(Z_{ij} := M_{(\beta, \omega_j)}(v) \setminus M_{(\beta, \omega_j)}(v), \langle i, j \rangle = \{1, 2\}\). Then codim\(M_{(\beta, \omega_i)}(v)(Z_{ij}) \geq 1\). Moreover if there is no wall \(W_{u_1}\) with \(\langle v, u_1 \rangle = 2\) and \(\langle u_1^2 \rangle = 0\), then codim\(M_{(\beta, \omega_i)}(v)(Z_{ij}) \geq 2\). In particular, \(M_{(\beta, \omega_1)}(v) \cap M_{(\beta, \omega_2)}(v)\) is an open dense subset of each \(M_{(\beta, \omega_i)}(v), i = 1, 2\).
Proof. We may assume that $X$ is defined over a finitely generated ring $R$ over $\mathbb{Z}$. Then there is a smooth surface $X_R$ over $R$ such that $\mathfrak{t}$ is an extension field of the quotient field of $R$ and $X \cong X_R \otimes_{\mathfrak{t}} \mathbb{Q}$. Lemma 3.1.3 implies that by a suitable base change, we may assume that all $v_i \in H^*(\mathbb{P}(X_R)_s, \mathbb{Z})$, $s \in \text{Spec}(R)$ in Definition 3.1.3 (1) are defined over $R$.

We first assume that there is no wall $W_{v_1}$ with $\langle v, v_1 \rangle = 1$ and $\langle w_1^2 \rangle = 0$. In this case, we may assume that $\langle \omega_1^2 \rangle \gg 1$ and $\langle \omega_2^2 \rangle \ll 1$. Then $M_{(\beta, \omega_2)}(v)$ parametrizes semi-stable sheaves on $X$ or their duals, and $M_{(\beta, \omega_2)}(v)$ parametrizes semi-stable sheaves on $X' = M_H(\sigma_0 e^\beta)$ via the Fourier-Mukai transform $\Phi_{X', X}^{E_F}$. Now we take a reduction over $\mathbb{P}_q$. We note that $\sum_{E \in Z_i(\mathbb{P}_q^n)} \frac{1}{\# \text{Aut}(E)} = \frac{\# Z_i(\mathbb{P}_q^n)}{q^2}$. We set $\beta := bH + \eta$, $\eta \in H^\perp$. By Lemma 3.1.3, we can find an open neighborhood $U$ of the segment connecting $(\eta, \omega_1)$ and $(\eta, \omega_2)$ such that $U$ does not intersect $W_{v_1}$ with $\langle v, v_1 \rangle = 1$ and $\langle w_1^2 \rangle = 0$. We take a path $I_t := (\eta_t, \omega_t)$, $1 \leq t \leq 2$ in $U$ such that (i) $\eta_1 = \eta_2 = \eta$, (ii) $I_t$ is very close to the segment $I'_t := (\eta_1, \omega_t)$, $1 \leq t \leq 2$, (iii) $I_t \cap W_{v_2} \cap W_{v_2} = \emptyset$ for any $W_{v_1} \neq W_{v_2}$ satisfying the conditions (a),(b),(c) in Definition 3.1.3 and 3.9. Assume that $(\eta_1, \omega_1)$ belongs to a wall $W$. Then for $s_1$ with $s_- \leq s_1 \leq s_+ - s_- \ll 1$, we have the formula (3.17), where $((\beta, \omega)$ and $(\beta, \omega_2)$ are replaced by $(bH + \eta_1, \omega_1)$ and $(bH + \eta_2, \omega_2)$). Then $(H, bH + \eta_2)$ are general with respect to $v_1$. Indeed if $v_1 = v_{i1} + v_{i2}$, $v_{i1} = r_1 \eta_2 + a_1 \eta_2 + d_1 H + D_{i1} + (d_1 H + D_{i1}, bH + \eta_2)(X, H, d_1 > 0, r_1/d_{i1} = r_{i1}/d_{i1}, a_{i1}/d_{i1} = a_i/d_i, \eta_{i1}/d_{i1} - v/d$ and $v_1/d_1 - v/d$ define the same wall. By our assumption, we have $v_1/d_{i1} - v_1/d_1 = (r_1/d_{i1} - r_2/d_2) w$, $w \in H^*(X, \mathbb{Q})$. Thus $r_1/d_{i1} - r_2/d_2 = 0$ implies that $v_1/d_1 - v_2/d_2 = 0$. In particular, we can apply Lemma 4.2.3 to estimate the weighted numbers of $Z_i(\mathbb{P}_q^n)$. Thus by Lemma 4.2.1, Proposition 4.2.1 and 3.17, we see that

$$\lim_{n \to \infty} \frac{1}{\# \text{Aut}(E)} \left( \sum_{E \in Z_i(\mathbb{P}_q^n)} \frac{1}{\# \text{Aut}(E)} \right) = 0.$$ 

By Proposition 4.2.1, dim $Z_{ij} \leq \dim M_{ij}^H(v_1) - 1 = \langle v^2 \rangle + 1$. Moreover if there is no wall $W_{v_1}$ with $\langle v, v_1 \rangle = 2$ and $\langle w_1^2 \rangle = 0$, then dim $Z_{ij} \leq \dim M_{ij}^H(v_1) = \langle v^2 \rangle$.

We next assume that there is a wall $W_{v_1}$ with $\langle v, v_1 \rangle = 1$ and $\langle w_1^2 \rangle = 0$. If $\omega_i$ is close to a wall $W_{v_1}$, then Proposition 4.1.3 implies that $M_{(\beta, \omega_i)}(v)$ is isomorphic to the moduli scheme of torsion free sheaves of rank 1 on an abelian surface. Then the same argument gives the claim.

4.3. Application.

Theorem 4.3.1 (29 Thm. 1.1)). Assume that $X$ is an abelian surface over a field $\mathfrak{t}$. Let $\Phi_{X' \to X}^{E_F} : D(X) \to D(X')$ be a Fourier-Mukai transform. Let $v \in H^*(X, \mathbb{Z})$ be a primitive Mukai vector with $\langle v^2 \rangle > 0$. Let $H$ be an ample divisor on $X$ which is general with respect to $v$. We set $v' := \pm \Phi_{X' \to X}^{E_F}(v)$ and assume that $H'$ is very general with respect to $v'$. Then there is an autoequivalence $\Phi_{X' \to X}^{E_F} : D(X') \to D(X')$ such that for a general $E \in M_H(v)$, $E := \Phi_{X' \to X}^{E'}(v)$ and $\Phi_{X' \to X}^{E'}(E)$ is a stable sheaf with $v(F) = v'$ or $v(F^\vee)$ is a stable sheaf with $v(F^\vee) = v'$, up to shift.

Proof. If $\text{rk} E = 0$, then we shall decompose $\Phi_{X' \to X}^{E_F}$ into a composition of two Fourier-Mukai transforms: $\Phi_{X' \to X}^{E_F} = \Phi_{X' \to X}^{E_F} \circ \Phi_{X' \to X}^{E_F}$ such that $\text{rk} E_i > 0$ $(i = 1, 2)$. Thus we may assume that $\text{rk} E > 0$. Then $E_{(X, \langle x' \rangle)}$ is a $\mu$-stable vector bundle with respect to any polarization. We also note that the $\beta$-twisted semi-stability for $E$ with $v(F) = v'$ does not depend on the choice of $\beta$, since $H$ is a general polarization. We set

$$\beta := c_1(E_{(X, \langle x' \rangle)})/\text{rk} E_{(X, \langle x' \rangle)}$$

$$\beta' := c_1(E_{(X, \langle x' \rangle)})/\text{rk} E_{(X, \langle x' \rangle)} = c_1(E_{(X, \langle x' \rangle)})/\text{rk} E_{(X, \langle x' \rangle)}.$$ 

(1) We first assume that $d > 0$. By Corollary 2.2.9, $M_{ij}^H(v) \cong M_{ij}^H(v')$ for $\langle \omega_i^2 \rangle > 0$. Applying $\Phi_{X' \to X}^{E_F}$, we have an isomorphism $M_{ij}^H(v) \cong M_{ij}(\mathbb{t}_H, v')(v')$, where $\langle \omega_i^2 \rangle \ll 1$. We shall study the wall-crossing behavior between $\omega'$ and $\omega'$, $r, t, \beta > 0$. By Proposition 4.2.3, it is sufficient to study the wall $W_{v_1}$ in Proposition 4.1.3. Then we have an isomorphism $M_{ij}(\mathbb{t}_H, v'(v')) \to M_{ij}(\mathbb{t}_H, v')(v')$ by a contravariant Fourier-Mukai transform. Therefore the claim holds in this case.

(2) We next assume that $d < 0$. We shall construct a rational map $M_{ij}^H(v) \to M_{ij}(\mathbb{t}_H, v')(v')$. Then the same proof of case (1) implies the claim. We set $v := l(r + \xi) + a_2 \xi$, $l \in \mathbb{Z}_{>0}$, $\text{gcd}(r, \xi) = 1$. If $r = 1, 2$, then $\eta := 2\xi/r \in \text{NS}(X)$. Then Corollary 2.2.8 implies that $E_1 := E_1 \otimes L$ is a Bridgeland stable object with $v(E_1) = v$, where $c_1(E) = \eta$. Thus we have an isomorphism $M_{ij}^H(v) \to M_{ij}(\mathbb{t}_H, v)(v')$ by this correspondence. Assume that $r > 3$. Let $M_{H}^*(v)$ be the open subset of $M_{ij}^H(v)$ consisting of $\mu$-stable locally free sheaves. Then the proof of 13 Lem. 3.5 implies that $\dim M_{ij}^H(v) \setminus M_{H}^*(v) \leq \dim M_{ij}^H(v)$. By Corollary 2.2.8, we have a desired embedding $M_{ij}^H(v) \to M_{ij}(\mathbb{t}_H, v)$.

(3) If $d = 0$, then the stability is preserved by $E \to \Phi_{X' \to X}^{E_F}(E) \to 2$ by 20 Thm. 2.3].

\[\square\]
Let us explain the relation of this section with [29] and [22]. Let $w_1$ be a primitive isotropic Mukai vector. For the Mukai vector $v$ in Proposition 4.1.5 we set $w_2 := v - (\omega^1_2)w_1$. Then $\langle w_2^2 \rangle = 0$, $\langle w_1, w_2 \rangle = 1$ and $deg_G(w_2) > 0$. We set

$$w_i := r_i e^b + a_i \theta_X + (d_i H + D_i + (d_i H + D_i, \beta) \theta_X).$$

Since $\langle w_i^2 \rangle = 0$, we see that $a_i = \frac{(d_i H + D_i, \beta)}{d_i}$. Hence

$$(\omega^2) = 2 a_1/d_1 - a_2/d_2,$$

$$(\omega_2^2) = (H + D_1/d_1)^2 d_1/r_1 - ((H + D_2/d_2)^2 d_2/r_2,$$

$$(r_1/d_1 - r_2/d_2) = -d_1 d_2 r_1 r_2 (H^2) + d_2 r_2 (D_1^2) - r_1 d_1 (D_2^2),$$

$$(r_1 r_2 (d_1 d_2 - r_1 d_1).$$

Assume that $r_1 < 0$ and $r_2 > 0$. Then $\langle \omega^2 \rangle > \langle \omega_2^2 \rangle > \langle \omega^1_2 \rangle$ means that $\phi_+ (w_1) > \phi_+ (w_2)$ and $\phi_- (w_1) < \phi_- (w_2)$. For $\omega_2$, a general stable object $E$ of $v(E) = v$ fits in an exact sequence

$$0 \to A[1] \to E \to B \to 0,$$

where $A[1] \in M_{(\beta, \omega)}(w_1)$ and $B \in M_{(\beta, \omega)}(w_2)$. In particular, $E$ is an honest complex with $H^{-1}(E) = A$ and $H^0(E) = B$. On the other hand, for $\omega_1$, a general stable object $E'$ of $v(E') = v$ fits in an exact sequence

$$0 \to B \to E' \to A \to 0,$$

where $A[1] \in M_{(\beta, \omega)}(w_1)$ and $B \in M_{(\beta, \omega)}(w_2)$. Thus by crossing the wall $W_{w_1}$ from $\omega_-$ to $\omega_+$, $rk H^i(E, \omega_i)$ become smaller. This fact was a key in the proof of [29] Thm. 1.1.

Remark 4.3.2. If $NS(X) = \mathbb{Z}H$, then [22] implies that we always have $r_1 r_2 < 0$. Thus if there is a stable object $E$ which is a sheaf up to shift, then a general member is Gieseker stable.

In [22], we studied the set of pairs $(w_1, w_2)$ by using the theory of quadratic forms. Thus a computation in [22] can be regarded as a computation of $\{W_{w_1}\}$.

4.4. Relations of Picard groups. Let $X$ be an abelian surface over $\mathbb{C}$. For a primitive Mukai vector $v = r + \xi + a \theta_X$, $\xi \in NS(X)$, assume that $r > 0$ or $r = 0$ and $\xi$ is represented by an effective divisor. We assume that $H$ is general with respect to $v$. Then we have a locally trivial morphism $a : M_H(v) \to X \times \hat{X}$, which is the Albanese map of $M_H(v)$. Denote by $K_H(v)$ a fiber of $a$. Then $K_H(v)$ is an irreducible symplectic manifold of $dim K_H(v) = (\omega^2) - 2$ which is deformation equivalent to a generalized Kummer variety constructed by Beauville [3]. For an irreducible symplectic manifold, Beauville constructed an integral bilinear form on the second cohomology group. In our case, we have an isomorphism

$$\theta_\psi : v^\perp \to H^2(K_H(v), \mathbb{Z}),$$

which preserves the Hodge structure and the bilinear form [25]. Then we have an isomorphism

$$\theta_\psi : H^*(X, \mathbb{Z})_{alg} \cap v^\perp \to Pic(K_H(v)).$$

The same claims also hold for the moduli of stable twisted sheaves ([27] Thm. 3.19]).

For a Fourier-Mukai transform $\Phi : D(X) \to D(X')$, if there is an open subset $U \subset M_H(v)$ such that $dim(M_H(v) \setminus U) \leq dim M_H(v) - 2$ and $\Phi(E) \in M_{H'}(\Phi(v))$ for $E \in U$, then we have an identification $\Phi_* : H^2(M_H(v), \mathbb{Z}) \to H^2(M_{H'}(\Phi(v)), \mathbb{Z})$ and a commutative diagram

$$\begin{array}{ccc}
v^\perp & \xrightarrow{\Phi} & \Phi(v)^\perp \\
\theta_v \downarrow & & \downarrow \theta_{\Phi(v)} \\
H^2(K_H(v), \mathbb{Z}) & \xrightarrow{\Phi_*} & H^2(K_{H'}(\Phi(v)), \mathbb{Z})
\end{array}$$

Thus Proposition 4.2.5 enables us to study the Picard groups of $K_H(v)$ unless there is no wall of type (b1) in Lemma 4.2.4.

4.4.1. A study on the wall of type (b1). For the wall of type (b1), the universal families are different along a divisor. We shall study the relation of two families in order to compare the Picard groups. For this purpose, we start with the analysis of a family of torsion free sheaves of rank 2.

We set $v = 2 + \xi + a \theta_X$. Let $a$ be a representative of $[a] \in H^4_{et}(X, \mathbb{Q}_X^\vee)$. Let $M$ be an open subscheme of the moduli space of simple torsion free $a^{-1}$-twisted sheaves $E$ with $v(E) = v^\vee$ such that $E^{**}$ is simple and $dim(E^{**}/E) < 1$, where $E^{**} := Hom(E, \mathbb{O}_X)$. Let $E$ be a universal family as a $(\alpha^{-1} \times \alpha')$-twisted sheaf on $X \times M$, where $\alpha'$ is a representative of $[\alpha'] \in H^4_{et}(X', \mathbb{Q}_{X'})$. Then $E^\vee := RHom(E, \mathbb{O}_{X \times M})$ is not a family of torsion free $a$-twisted sheaves. We shall modify this complex to a flat family of torsion free $a$-twisted sheaves.
sheaves. Let $D$ be the divisor of $M$ parametrizing non-locally free sheaves. We shall prove that $D$ is smooth and $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_{X \times M})$ is an $\mathcal{O}_D$-module. We take a locally free resolution of $\mathcal{E}$:

$$0 \to U \to V \to \mathcal{E} \to 0.$$  

Then $\mathcal{E}^\vee$ is the complex which is represented by $V^\vee \to U^\vee$. The support of $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_{X \times M})$ is $X \times D$. Assume that $E \in D$. By the local-global spectral sequence, we have an exact sequence

$$\text{Ext}^1(E, E) \xrightarrow{\varphi} H^0(X, \mathcal{E}xt^1(\mathcal{E}, E)) \to H^2(X, \mathcal{H}om(E, E)) \xrightarrow{\psi} \text{Ext}^2(E, E).$$

Since $E^{**}$ is simple, $\psi$ is isomorphic and $\varphi$ is surjective. Since $\mathcal{E}xt^1(E, E) \cong \mathcal{F}$ is the Zariski tangent space of local deformation of $E$, $D$ is smooth in a neighborhood of $E$. Moreover

$$\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_{X \times M}) = i_*(F),$$

where $i: X \times D \to X \times M$ is the inclusion and $F$ is a flat family of skyscraper $\alpha$-twisted sheaves $\xi_x, x \in X$ parametrized by $D$. We have an exact triangle

$$(4.3) \quad \mathcal{E}^* \to \mathcal{R}\mathcal{H}om(\mathcal{E}, \mathcal{O}_{X \times M}) \to i_*(F)[-1] \to \mathcal{E}^*[1].$$

We set $\mathcal{G} := \ker(U^\vee \to i_*(F))$. Since $\mathcal{Tor}_2(i_*(F), \mathcal{O}_{X \times D}) = 0$, we get $\mathcal{Tor}_1(F, \mathcal{O}_{X \times D}) = 0$. Thus $\mathcal{G}$ is flat over $M$ and $\mathcal{E}^*_{|X \times D} \to V^\vee_{|X \times D}$ is injective. We have exact sequences

$$0 \to \mathcal{E}^*_{|X \times D} \to V^\vee_{|X \times D} \to \mathcal{G}_{|X \times D} \to 0,$$

$$0 \to F(-D) \to \mathcal{G}_{|X \times D} \to U^\vee_{|X \times D} \to F \to 0.$$  

For $t \in D$, $\mathcal{E}^*_{|X \times \{t\}}$ and $\mathcal{E}^\vee_{|X \times \{t\}}$ fits in exact sequences in $\mathfrak{A}(\beta', \omega')$:

$$0 \to F_{|X \times \{t\}}[1] \to \mathcal{E}^*_{|X \times \{t\}} \to (\mathcal{E}^*_{|X \times \{t\}})^{**} \to 0,$$

$$0 \to (\mathcal{E}^\vee_{|X \times \{t\}})^{**}[1] \to \mathcal{E}^\vee_{|X \times \{t\}}[1] \to F_{|X \times \{t\}} \to 0,$$

where the intersection number satisfies $(\beta', \omega') \gg (\xi, \omega')$. These are non-trivial extensions.

Let us study case (b1) in Lemma 4.2.4. Let $W_{v_1}$ be a ball for $v$ such that $\langle w_1^2 \rangle = 0$ and $\langle v, w_1 \rangle = 2$. In the notation of section 4.1, $\Phi_{X_1 \times X_1}(E)$ or its dual is a torsion free sheaf of rank 2 on $X_1$. We shall prove that each moduli space is represented by an open subscheme $M$ of $M_2^L(w)$ such that $\dim(M_2^L(w) \setminus M) \geq 2$, where $L$ is an ample line bundle on an abelian surface $Y$ and $w \in H^2(Y, \mathcal{O}_Y)$ is a primitive Mukai vector. We shall prove this claim inductively by crossing walls of type (b1). So we may assume that one of the $M_{(\beta, \omega_{\pm})}(v)$ is represented by a scheme $M$ up to codimension 1, and there is a universal family $\mathcal{E}_\pm$ as a twisted sheaf. Indeed we may choose $M_2^L(w)$ as the moduli space of rank 1 torsion free sheaves, $M_2^L(v)$ or $M_2^L(-v')$. Then $\Phi_{X_1 \times X_1}(E_{\pm})$ or its dual is a family of torsion free sheaves of rank 2. For simplicity, assume that $\mathcal{E} := \Phi_{X_1 \times X_1}(E_{\pm})^\vee$ is a family of torsion free sheaves. Applying the above observation to this situation, we get a family of torsion free sheaves $\mathcal{E}^* := (\Phi_{X_1 \times X_1}(E_{\pm})^\vee)^*$. Then $\mathcal{E}_\mp := \Phi_{X_1 \times X_1}((\Phi_{X_1 \times X_1}(E_{\pm})^\vee)^*)$ is a family of stable objects with respect to $(\beta', \omega_{\mp})$. Therefore we have an exact triangle

$$\mathcal{E}_\mp \to \mathcal{E}_\pm \to \mathcal{F} \to \mathcal{E}^*[1]$$

where $\mathcal{F}$ is a family of stable objects with the Mukai vector $w_1$ parametrized by a divisor $D$. Then we have an identification $M_{(\beta, \omega_{\pm})}(v) \cong M_{(\beta, \omega_{\mp})}(v)$ up to codimension 1. Thus $M_{(\beta, \omega_{\mp})}(v)$ is also represented by $M$. We set $K := M \cap K_2(w)$. Since $\text{codim}_{K_2(w)}(K_2(w) \setminus K) \geq 2$, $H^2(K_2(w), \mathbb{Z}) \cong H^2(K, \mathbb{Z})$. For $x \in v^\perp$, we set

$$\theta_{\pm}(x) := -[p_{K*}(\mathcal{E}_{\pm}^* \mathcal{E}^*_{\mp}) p^* (x)]_1 \in H^2(K, \mathbb{Z})$$

where $p_K : X \times K \to K$ and $p : X \times K \to X$ are the projections, and $[z]_1$ means the $H^2(K, \mathbb{Z})$-component of $z$. We shall compare two isometries $\theta_{\pm}$. We write $D = \theta_{\mp}(d)$.

Lemma 4.4.1.

$$d = v - \frac{\langle v^2 \rangle}{2} w_1.$$  

**Proof.** Let $\mathcal{E}$ be the family of torsion free sheaves as above. We set $v' := v(\mathcal{E}_{|X_1 \times \{s\}}), s \in K_2(w)$. It is sufficient to prove

$$(4.4) \quad 2c_1(D) = -[p_{K*}(\mathcal{E}^*) p^* (2v' - \langle v', v' \rangle) g(x_1)]_1,$$

where $p_K$ and $p$ are the projections from $X_1 \times K$. Let $\mathcal{L}$ be an $(\alpha^2 \times (\alpha')^2)$-twisted line bundle on $X_1 \times K_2(w)$ which is an extension of $\mathcal{E}^*$. Indeed since $[\alpha \times \alpha']$ is a 2-torsion element of the Brauer group, the category of $(\alpha^2 \times (\alpha')^2)$-twisted sheaves is equivalent to the category of coherent sheaves. Since $\text{Pic}^0(K_2(w))$ is trivial, we have a decomposition $\mathcal{L} \cong L_1 \otimes L_2$, where $L_1$ and $L_2$ are $(\alpha^2)$-twisted line bundle on $X_1$ and $(\alpha')^2$-twisted line bundle on $K_2(w)$ respectively. We note that $\mathcal{E}^*$ is reflexive and $\mathcal{E} \cong \mathcal{E}^* \otimes (\det \mathcal{E}^*)^\vee$. Hence
\( \mathcal{E} \cong \mathcal{E}^* \otimes (L_1 \boxtimes L_2)^\vee \). Let \( G \) be an \( \alpha \)-twisted vector bundle of rank 2 on \( X_1 \) such that \( v(G) = v' \). Then \( v(G^\vee \boxtimes L_1) = v(G) = v' \). We have \( \det p_{K^*}(\text{Ext}^1(\mathcal{E} \otimes G, \mathcal{O}_{X_1 \times K})) \cong \mathcal{O}_K(2D) \). By using the relative duality, we see that

\[
\det p_{K}(\text{Ext}^1(\mathcal{E} \otimes G, \mathcal{O}_{X_1 \times K})) = -\det p_{K_1}(\mathcal{E}^\vee \otimes G^\vee) + \det p_{K_1}(\mathcal{E}^* \otimes G^\vee)
\]

\[
= -\det p_{K_1}(\mathcal{E}^* \otimes (L_1 \boxtimes L_2)^\vee, G^\vee \boxtimes \mathcal{O}_K) + \det p_{K_1}(\mathcal{E}^* \otimes G^\vee)
\]

\[
= -\det p_{K_1}(\mathcal{E}^* \otimes L_1^\vee \otimes \mathcal{O}_{X_1 \times K} \otimes L_2 + \det p_{K_1}(\mathcal{E}^* \otimes G^\vee).
\]

Hence

\[
\mathcal{O}_K(2D) = \det p_{K}(\text{Ext}^1(\mathcal{E} \otimes G, \mathcal{O}_{X_1 \times K}))
\]

\[
= \det p_{K_1}(\mathcal{E}^* \otimes (L_1 \boxtimes L_2)^\vee \otimes L_2 \langle v', v(L_1 \otimes G) \rangle \otimes \det p_{K_1}(\mathcal{E}^* \otimes G^\vee)
\]

\[
= \det p_{K_1}(\mathcal{E}^* \otimes (L_1 \boxtimes G^\vee - \langle v', v(L_1 \otimes G) \rangle A + G))^{\vee},
\]

where \( A \) is an \( \alpha \)-twisted structure sheaf of a point \( x_1 \in X_1 \). Since

\[
v(L_1 \boxtimes G^\vee - \langle v', v(L_1 \otimes G) \rangle A + G) = v' - \langle v', v \rangle \partial_{X'} + v',
\]

we get (4.4).

\[
\text{Proposition 4.4.2.} \quad \text{We have } \theta_\pm(x) = \theta_\mp(x - \langle w_1, x \rangle d), \ x \in v^\perp \text{ and}
\]

\[
x - \langle w_1, x \rangle d = x - 2 \frac{(d, x)}{(d, d)} d, \ x \in v^\perp.
\]

Thus \( \theta_\mp \circ \theta_\pm \) is the reflection by \( d \). In particular, it is an isometry of \( v^\perp \).

\[
\text{Proof.} \quad \text{For } x \in v^\perp, \ (4.3) \ implies \ that
\]

\[
\theta_\pm(x) = \theta_\mp(x) - \langle w_1, x \rangle \theta_\mp(d).
\]

By Lemma 4.4.1, we have \( \langle d^2 \rangle = -\langle v^2 \rangle \). Since \( \frac{(x^2)}{2} \langle w_1, x \rangle = -\langle d, x \rangle \), we get the claim. \( \square \)

\[
\text{Remark 4.4.3.} \quad \text{If } \theta_+ \text{ and } \theta_- \text{ are isometries, then we can determine } d \text{ as follows. Since } \theta_{\pm} \text{ are isometries, (4.5) implies that}
\]

\[
\langle x^2 \rangle = (\theta_\pm(x))^2 = (\theta_\mp(x - \langle w_1, x \rangle d))^2 = (\langle x - \langle w_1, x \rangle d \rangle^2) = \langle x^2 \rangle - 2\langle x, \langle w_1, x \rangle d \rangle + \langle w_1, x \rangle^2 \langle d^2 \rangle
\]

for \( x \in v^\perp \). Hence we get

\[
-2\langle x, d \rangle + \langle w_1, x \rangle \langle d^2 \rangle = 0.
\]

Since \( \langle v^\perp \rangle = Zv \), we have \(-2d + \langle d^2 \rangle w_1 = n\langle v \rangle, n \in Z \). Then \( n\langle v^2 \rangle = \langle d^2 \rangle \langle v \rangle, v = 2\langle d^2 \rangle \), which implies that \( d = -(n/2)v + (n/4)\langle v^2 \rangle w_1 \). Then we have

\[
n\langle v^2 \rangle = 2\langle \langle \frac{v}{2} - \frac{v}{4} \rangle w_1 \rangle^2.
\]

Thus \( n = 0, -2 \). Since \( \langle v, w_1 \rangle \neq 0, n = -2 \). Therefore

\[
d = v - \frac{\langle v^2 \rangle}{2} w_1.
\]

\[
\text{Remark 4.4.4.} \quad \text{If } \langle w_1, u \rangle \in \mathbb{Z} \text{ for any } u \in H^* (X, \mathbb{Z})_{\mathrm{alg}}, \text{ then } M_{(\beta, \omega_\pm)}(v) \text{ are isomorphic to open subschemes of the moduli of stable twisted sheaves on } X'. \text{ Thus we have a moduli space } M_{(\beta, \omega_\pm)}(v) \text{ as a scheme.}
\]

We have the following refinement of [20].

\[
\text{Corollary 4.4.5.} \quad \text{Let } X \text{ be an abelian surface. Assume that there is a primitive Mukai vector } u \in H^* (X, \mathbb{Z})_{\mathrm{alg}} \text{ such that } \langle u, v \rangle = \langle u^2 \rangle = 0. \text{ Then we can explicitly find a Fourier-Mukai transform } \Phi \text{ which induces a birational map } M_H(v) \cdots \rightarrow M_H^u(v'), \text{ where } M_H^u(v') \text{ is a moduli space of } \alpha \text{-twisted sheaves of dimension } 1 \text{ on an abelian surface. Moreover this birational map induces an isomorphism of the second cohomology groups provided } \langle v^2 \rangle > 6. \text{ Thus we can explicitly describe the line bundle on } K_H(v) \text{ which induces a rational Lagrangian fibration } K_H(v) \cdots \rightarrow \mathbb{P}(v^2)/2-1.
\]
5. Appendix.

5.1. Stability condition and base change. Now we discuss on the behaviour of our stability condition \( \sigma_{(\beta,\omega)} = (\mathfrak{A}_{(\beta,\omega)}, Z_{(\beta,\omega)}) \) under a field extension \( L/\mathfrak{t} \). Let us write by \( X_L \) the base change of \( X \), and set standard morphisms as

\[
\begin{align*}
X_L & \xrightarrow{\pi'} X \\
\pi & \downarrow \\
\text{Spec } L & \xrightarrow{p} \text{Spec } \mathfrak{t}
\end{align*}
\]

For given \( \beta, H, \omega \) on \( X \), we denote by \( \beta', H', \omega' \) the pull-backs on \( X_L \). The stability function \( Z_{(\beta',\omega')}(E_L) := (e^{\beta' + \sqrt{-1} \omega'}, v(E_L)), \ E_L \in \mathcal{D}(X) \).

The function \( Z_{(\beta',\omega')}(E_L) \) is also defined in the same way:

\[
\Sigma_{(\beta',\omega')}(E', E) := \det \begin{pmatrix} \text{Re} Z_{(\beta',\omega')}(E'_L) & \text{Re} Z_{(\beta',\omega')}(E'_L) \\ \text{Im} Z_{(\beta',\omega')}(E'_L) & \text{Im} Z_{(\beta',\omega')}(E'_L) \end{pmatrix}, \ E_L, E'_L \in \mathcal{D}(X_L).
\]

**Lemma 5.1.1.** Assume that the extension \( L/\mathfrak{t} \) is finite.

1. For an object \( E_L \in \mathcal{D}(X_L) \) we have

\[
[L : \mathfrak{t}] Z_{(\beta',\omega')}(E_L) = Z_{(\beta,\omega)}(p^*_L E_L).
\]

2. For objects \( E_L, E'_L \in \mathcal{D}(X_L) \) we have

\[
\Sigma_{(\beta',\omega')}(E'_L, E_L) \geq 0 \iff \Sigma_{(\beta,\omega)}(p^*_L E'_L, p^*_L E_L) \geq 0.
\]

**Proof.** Note that for an object \( E_L \in \mathcal{D}(X) \) we have

\[
Z_{(\beta',\omega')}(E_L) = \dim_L [\pi'_* (v(E_L) \cdot e^{\beta' + \sqrt{-1} \omega'})],
\]

where \( \cdot \) in the right hand side means the intersection product on the Chow group \( A^*(X_L) \) and [ ]\(_0\) denotes the degree zero part of an element of \( A^*(X_L) \). Then we have

\[
[L : \mathfrak{t}] Z_{(\beta',\omega')}(E_L) = \dim [\pi'_* (v(E'_L) \cdot e^{\beta' + \sqrt{-1} \omega'})]_0 = \dim [\pi'_* Z_{(\beta',\omega')}(E'_L) v(E'_L)]_0 = \Sigma_{(\beta,\omega)}(p^*_L E'_L, p^*_L E_L).
\]

Here at the third equality we used the projection formula. Thus we have (1). Then (2) follows by the definition of \( \Sigma_{(\beta,\omega)} \).

We denote by \( \mathfrak{C}_L \) and \( \mathfrak{A}_{(\beta',\omega')},L \) the categories over \( L \), which are similarly defined as \( \mathfrak{C} \) and \( \mathfrak{A}_{(\beta,\omega)} \). Then the stability condition \( \sigma_{(\beta',\omega')},L := (\mathfrak{A}_{(\beta',\omega')},L, Z_{(\beta',\omega')}) \) is well-defined.

**Lemma 5.1.2.**

1. The derived pull-back \( p^* : \mathcal{D}(X) \to \mathcal{D}(X_L) \) induces an exact functor \( p^* : \mathfrak{A}_{(\beta,\omega)} \to \mathfrak{A}_{(\beta',\omega'),L} \).

2. The derived push-forward \( p'_* : \mathcal{D}(X_L) \to \mathcal{D}(X) \) induces an exact functor \( p'_* : \mathfrak{A}_{(\beta',\omega'),L} \to \mathfrak{A}_{(\beta,\omega)} \).

**Proof.** What should be shown is that the image \( p^*(\mathfrak{A}_{(\beta,\omega)}) \) (resp. \( p'_*(\mathfrak{A}_{(\beta',\omega'),L}) \)) is indeed in \( \mathfrak{A}_{(\beta',\omega'),L} \) (resp. in \( \mathfrak{A}_{(\beta,\omega)} \)). The case (1) is clear, since the twisted semi-stability is preserved under pull-back.

For the case (2), let \( E \) be a \( \beta' \)-twisted semi-stable object of \( \mathfrak{C}_L \). It is enough to prove that \( p'_*(E) \) is also \( \beta' \)-twisted semi-stable.

We may assume that \( L \) is a normal extension of \( \mathfrak{t} \). Indeed for a normal extension \( L' \) of \( \mathfrak{t} \) containing \( L, q'_*(q^*(E)) = E^{[q] : [L' : L]} \), where \( q' : X_{L'} \to X_L \) is the projection.

Let \( F \) be a subobject of \( p'_* E \). Then \( p^*(F) \) is a subobject of \( p^*(p'_*(E)) \). By Lemma 5.1.3 below, \( p^* p'_*(E) \) is a \( \beta' \)-twisted semi-stable object with \( \chi(p^* p'_*(E)(n)) = \chi(p'_*(E)(n)) \). Hence

\[
\frac{\chi(F(n))}{\text{rk } F} = \frac{\chi(p^*(F)(n))}{\text{rk } p^* F} \leq \frac{\chi(p'_*(E)(n))}{\text{rk } p'_* (E)}.
\]

Therefore \( p'_*(E) \) is \( \beta' \)-twisted semi-stable.

**Lemma 5.1.3.** \( p^* p'_*(E) \) is a successive extension of \( E \).

**Proof.** We note that \( p^* p'_*(E) = E \otimes_{L'} (L \otimes L) \). Since \( L \) is a normal extension of \( \mathfrak{t}, R := L \otimes \mathfrak{t} \) is a successive extension of \( \mathfrak{t} \)-modules \( R/m_i \), where \( m_i \) are maximal ideals of \( R \) and \( R/m_i \cong L \). Since \( E \otimes_L (R/m_i) \cong E, p^* p'_*(E) \) is a successive extension of \( E \).

**Proposition 5.1.4.** Assume that the extension \( L/\mathfrak{t} \) is finite. If \( E \) is a semi-stable object with respect to \( \sigma_{(\beta,\omega)}, \) then \( p^* E \) is semi-stable with respect to \( \sigma_{(\beta',\omega')},L \).
Proof. Assume that $p^*(E)$ is not semi-stable with respect to $\sigma(\beta', \omega')$. (Here we are implicitly using Lemma 5.1.2 (1).) Take a distabilizing subobject $F$ of $p^*(E)$, so that we have $\Sigma(\beta', \omega')(F, p^*(E)) < 0$. Then by Lemma 5.1.1 (2) we have $\Sigma(\beta, \omega)(p'_F, p'_F|p^*(E)) < 0$.

On the other hand, since $p'_FP'_{\sigma}E \simeq d$ with $d := [L : \mathfrak{t}]$, $p'_F|p^*(E)$ is a semi-stable object with respect to $\sigma(\beta, \omega)$. Then since $p'_FP'_{\sigma}E \in \mathfrak{X}(\beta, \omega)$ and it is a subobject of $p'_F|p^*(E)$ by Lemma 5.1.2, we have $\Sigma(\beta, \omega)(p'_F, p'_F|p^*(E)) \geq 0$. Therefore by contradiction the claim holds. □

Remark 5.1.6. This claim implies that our stability condition $\sigma(\beta', \omega'), L$ is a member of $\text{Stab}(X_L)_p$ in [20, Definition 3.1].

Corollary 5.1.7. Let $L = \mathfrak{t}$, the algebraic closure of $\mathfrak{t}$. If $E$ is a semi-stable object with respect to $\sigma(\beta, \omega)$, then $p^*(E)$ is semi-stable with respect to $\sigma(\beta', \omega'), L$.

Proof. Assume that $p^*(E)$ is not semi-stable with respect to $\sigma(\beta', \omega'), L$. Let $F$ be the distabilizing subobject of $p^*(E)$. Since $F$ is (a class of) a complex consisting of coherent sheaves, we may assume that $F$ is defined on a field $L'$ such that $\mathfrak{t} \subset L' \subset L = \mathfrak{t}$ and $[L' : \mathfrak{t}] < \infty$. Then by Proposition 5.1.4 $p'_F|L'/\mathfrak{t}$ is semi-stable with respect to $\sigma(\beta', \omega'), L'$, where $p'_F|L' : X_{L'} \to X$ and $\beta'', \omega''$ are pull-backs of $\beta, \omega$ via $\mathfrak{t} \to L'$. But it contradicts with the choice of $F$. □

REFERENCES

[1] Arcara, D., Bertram, A., Bridgeland-stable moduli spaces for $K$-trivial surfaces, arXiv:0708.2247
[2] Bayer, A., Polynomial Bridgeland stability conditions and the large volume limit, arXiv:0712.1063
[3] Beauville, A., Variétés Kähleriennes dont la première classe de Chern est nulle, J. Diff. Geom. 18 (1983), 755–782
[4] Bridgeland, T., Stability conditions on triangulated categories, Ann. of Math. (2) 166 (2007), no. 2, 317–345
[5] Bridgeland, T., Stability conditions on $K3$ surfaces, math.AG/0307164, Duke Math. J. 141 (2008), 241–291
[6] Desale, U. V., Ramanan, S. Poincaré polynomials of the variety of stable bundles, Math. Ann. 216 (1975) pp. 233–244
[7] Hartmann, H., Cusps of the Kähler moduli space and stability conditions on $K3$ surfaces, arXiv:1012.3121
[8] Huybrechts, D., Derived and abelian equivalence of $K3$ surfaces, math.AG/0604150, J. Algebraic Geom. 17 (2008), 375–400
[9] Huybrechts, D., Macri, E., Stellari, P., Stability conditions for generic $K3$ categories, arXiv:math/0608430
[10] Huybrechts, D., Macri, E., Stellari, P., Derived equivalences of $K3$ surfaces and orientation, Duke Math. J. 149 (2009), 461–507
[11] Kawatani, K., Stability conditions and $\mu$-stable sheaves on $K3$ surfaces with Picard number one, arXiv:1005.3877
[12] Kurihara, K., Stability of Gieseker stable sheaves on $K3$ surfaces in the sense of Bridgeland and some applications, arXiv:1103.3921
[13] Lang, S., Algebraic groups over finite fields, Amer. J. Math. 78 (1956), 555–563
[14] Lang, S., Weil, A., Number of points of varieties in finite fields, Amer. J. Math. 76 (1954), 819–827
[15] Lo, J., Qin, Z., Mini-walls for Bridgeland stability conditions on the derived category of sheaves over surfaces, arXiv:1103.3479
[16] Mukai, S., Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface, Invent. math. 77 (1984), 101–116
[17] Mukai, S., On the moduli space of bundles on $K3$ surfaces I, Vector bundles on Algebraic Varieties, Oxford, 1987, 341–413
[18] Okawa, R., Moduli of Bridgeland semistable objects on $F^2$, arXiv:0812.1470
[19] Ossona, P., Stability conditions under change of base field, arXiv:1004.4835
[20] Toda, Y., Moduli stacks and invariants of semistable objects on $K3$ surfaces, arXiv:math/0703590, Advances in Math 217 (2008), 2736–2781
[21] Yanagida, S., Yoshioka, K., Semi-homogeneous sheaves, Fourier-Mukai transforms and moduli of stable sheaves on abelian surfaces, arXiv:0906.4603
[22] Yoshioka, K., Chamber structure of polarizations and the moduli of stable sheaves on a ruled surface, Internat. J. Math. 7 (1996), 411–431
[23] Yoshioka, K., Some examples of Mukai’s reflections on $K3$ surfaces, J. Reine Angew. Math. 515 (1999), 97–123
[24] Yoshioka, K., Moduli spaces of stable sheaves on abelian surfaces, Math. Ann. 321 (2001), 817–884, math.AG/0009001
[25] Yoshioka, K., Twisted stability and Fourier-Mukai transform I, Compositio Math. 138 (2003), 261–288
[26] Yoshioka, K., Twisted stability and Fourier-Mukai transform II, Compositio Math. 145 (2009), 112–142
[27] Yoshioka, K., Fourier-Mukai transform on abelian surfaces, Math. Ann. 345 (2009), 493–524
[28] Yoshioka, K., Perverse coherent sheaves and Fourier-Mukai transforms on surfaces, arXiv:1003.2522
[29] Washino, T., A classification of $\mu$-semi-stable sheaves and the analysis of the twisted stability on an elliptic $K3$ surface, Master thesis 2009, (Japanese language)

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