On (non)integrability of classical strings in $p$-brane backgrounds

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Abstract

We investigate the question of possible integrability of classical string motion in curved $p$-brane backgrounds. For example, the D3-brane metric interpolates between the flat and the $\text{AdS}_5 \times S^5$ regions in which string propagation is integrable. We find that while the point-like string (geodesic) equations are integrable, the equations describing an extended string in the complete D3-brane geometry are not. The same conclusion is reached for similar brane intersection backgrounds interpolating between flat space and $\text{AdS}_k \times S^l$.

We consider, in particular, the case of the NS 5-brane—fundamental string background. To demonstrate non-integrability we make a special ‘pulsating string’ ansatz for which the string equations reduce to an effective one-dimensional system. Expanding near this simple solution leads to a linear differential equation for small fluctuations that cannot be solved in quadratures, implying non-integrability of the original set of string equations.

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1. Introduction

Strings in curved space are described by 2D sigma models with complicated nonlinear equations of motion. This precludes, in most cases, a detailed understanding of their dynamics. Integrability selects a subclass of models or target space backgrounds for which one may hope to come as close as possible to the situation in flat space, i.e. to have a complete quantitative description of classical string motions and then also of the quantum string spectrum.

A prominent maximally symmetric ten-dimensional example is provided by the superstring in the $\text{AdS}_5 \times S^5$ space-time: much progress towards its solution, that was achieved recently [1], is based on its integrability. One may wonder if integrability can also apply to
string motion in closely related but less symmetric p-brane backgrounds [2–4], e.g. the D3-brane background [4, 5] which is a one-parameter (D3-brane charge $Q$) interpolation between flat space ($Q = 0$) and the AdS$_5 \times S^5$ space ($Q = \infty$). While string theory is integrable in these two limiting cases, what happens at finite $Q$ is a priori an open question. This is the question we are going to address in this paper.

Since quantum integrability of a bosonic model or integrability of its superstring counterpart is essentially implied (leaving aside some exceptional cases or anomalies) by the integrability of the corresponding classical bosonic sigma model, the study of the latter is the first step. Following the same method as used previously in [6, 7], we will demonstrate non-integrability of classical extended string motion in the D3-brane background (particle, i.e. geodesic, motion is still integrable). We will also reach similar conclusions for other p-brane backgrounds that interpolate between flat space and integrable AdS$_n \times S^k$ backgrounds.

As there is no general classification of integrable 2D sigma models, let us start with a brief survey of some known classically integrable bosonic models describing string propagation in curved backgrounds. A major class of such models have a target space metric $G_{MN}$ of a symmetric space, including spheres [8] (and other related constant curvature spaces [9]), group spaces (i.e. principal chiral model) [10] and various other $G/H$ coset models [10–12]. Integrability is also preserved by some anisotropic deformations of the group space [13, 14], the 2-sphere [15] and the 3-sphere [16–18] models. Attempts to find ‘non-diagonal’ generalizations of the principal chiral model by studying conditions for the existence of the corresponding zero curvature (Lax) representation were made in [19, 20] and also in [21].

Allowing for the parity-odd antisymmetric tensor ($B_{MN}$) coupling leads, in particular, to the WZW and related gauged WZW models [22, 23]. The generalized principal chiral model with the WZ term having an arbitrary coefficient is also integrable [24–26]. Some other ‘anisotropic’ integrable models with 3D target space were constructed in [18] and new examples of integrable coset-type models with extra WZ coupling were found in [26].

There are also examples of integrable pp-wave models related to (massive) integrable 2D models in light-cone gauge, see e.g. [27–31].

Given an integrable model, one can generate a family of other integrable models (i.e. find new ‘integrable’ target space backgrounds) by performing transformations that preserve the classical 2D equations of motion, e.g. using 2D Abelian duality (T-duality) or non-Abelian duality, combined with field redefinitions (coordinate transformations). Examples include, in particular, various gaugings or marginal deformations of WZW and related models, see e.g. [32–36]; for a more recent example related to AdS$_5 \times S^5$ see [37]. For discussions of an interplay between T-duality and integrability see [40, 21].

Given a particular string sigma model, to prove its integrability, one is to find a zero curvature representation implying its equations of motion. There is no general method for accomplishing this and the full set of necessary conditions for the existence of a Lax pair is not known. For example, the presence of a non-Abelian isometry group is not required. Among the necessary conditions should, of course, be that any consistent truncation of the 2D equations of motion to a 1D set of equations has to represent an integrable mechanical system. For example, rigid string motion on a sphere or AdS space is described by an integrable Neumann–Rosochatius model [41].

In particular, the point-like string, i.e. geodesic, motion should be integrable.

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2 One may also find new models by taking orbifolds of known integrable models (see, e.g., [38]), though division over discrete subgroups may also lead to a breakdown of integrability (cf [39]).

3 Indeed, generic sigma models obtained from gauged WZW models by solving for the 2D gauge field do not have isometries (or have only Abelian isometries) and yet, following from a gauged WZW action, they must be integrable.
The geodesic motion in a general class of higher dimensional (rotating) black hole backgrounds is known to be integrable [42]. It is thus not surprising that geodesics are also integrable in $p$-brane backgrounds that are discussed below.

To disprove integrability of a given sigma model, it is therefore sufficient to show that there exists at least one 1D truncation of its equations of motion that is not an integrable system of ordinary second-order differential equations. This can be done by demonstrating that the corresponding motion is chaotic, i.e. by showing that the equations for small variations around solutions in phase space cannot be solved in quadratures [48]. We shall review this method in section 2 (and appendix A) and illustrate it on a simple example of the non-integrable two-particle system with the potential $V(x_1, x_2) = \frac{1}{2}x_1^2 x_2^2$.

This approach was recently used to show that string motion in $\text{AdS}_5 \times T^{p,q}$, $\text{AdS}_5 \times Y^{p,q}$ [6, 7] and ‘confining’ supergravity backgrounds [49] is not integrable. Here, we shall follow a similar method to study integrability of strings in curved backgrounds that interpolate between two integrable limits—flat space and $\text{AdS}_n \times S^m \times T^k$.

The main cases include the D3-brane, the D5–D1 background and the four D3-brane intersection that interpolate between flat space and $\text{AdS}_3 \times S^3 \times T^4$ and $\text{AdS}_2 \times S^2 \times T^6$, respectively. While geodesic motion in these backgrounds is integrable, making a special ansatz for string motion we will find that the resulting 1D system of equations is not integrable by applying the variational non-integrability technique for Hamiltonian systems as used in [7] (section 3).

In section 4, we shall study the NS5-F1 background and show that integrability is absent for generic values of the two charges $Q_1$ and $Q_5$, but it is restored in the limit when $Q_5 \to \infty$.

In appendix A, we also give a brief summary of the ideas from differential Galois theory on which the non-integrability techniques for Hamiltonian systems are based on. In appendix B, we shall demonstrate non-integrability of string propagation in the pp-wave geometry T-dual to the fundamental string background [51, 30].

2. On the non-integrability of classical Hamiltonian systems

Integrability of classical Hamiltonian systems is closely related to the behaviour of variations around phase space curves. Based on this observation, the authors of [52] obtained necessary conditions for the existence of additional functionally independent integrals of motion. These conditions are given in terms of the monodromy group properties of the equations for small variations around phase space curves. Using differential Galois theory, the authors of [53, 54, 48] improved these results by showing that integrability implies that the identity component of the differential Galois group of variational equations normal to an integrable plane of solutions must be Abelian (see appendix A for some details).

This necessary integrability condition allows one to show non-integrability from the properties of the Galois group. However, determining the Galois group can be difficult and
therefore one usually takes a slightly different route in analysing the normal variational equation (NVE).

One considers a special class of solutions of the NVE which are functions of exponentials, logarithms, algebraic expressions of the independent variables and their integrals and which are known as Liouvillian solutions [55]. The existence of such solutions is equivalent to the condition that the identity component $G_0$ of the Galois group is solvable (see, for example, theorem 25 in [56]). This in turn implies that if the NVE does not admit Liouvillian solutions then $G_0$ is non-solvable (and thus non-Abelian) leading to the conclusion of non-integrability.

For Hamiltonian systems, the NVE is a second-order linear homogeneous differential equation and for such equations with rational coefficients Liouvillian solutions can be determined by the Kovacic algorithm [57] which fails if and only if no such solutions exist. Therefore, whenever no Liouvillian solutions of the NVE are found by the Kovacic algorithm one can deduce non-integrability of a Hamiltonian system.

An important subtlety here is that we first need to algebraize the NVE, i.e. rewrite it as a differential equation with rational coefficients. This can be done by means of a Hamiltonian change of the independent variable and it was shown in [58] that the identity component of the Galois group is preserved under this procedure.

In summary, if the algebraized NVE is not solvable in terms of Liouvillian functions or, equivalently, if the component of the quadratic fluctuation operator normal to an integrable subsystem does not admit Liouvillian zero modes, then the original system is non-integrable.

This is consistent with the usual definition of integrability in the sense of Liouville which, for Hamiltonian systems, implies that the equations of motion should be solvable in quadratures, i.e. in terms of Liouvillian functions.

The main steps to prove non-integrability of a Hamiltonian system are thus the following.

- Choose an invariant plane of solutions.
- Obtain the NVEs, i.e. the variational equations normal to the invariant plane.
- Check that the algebraized NVEs have no Liouvillian solutions using the Kovacic algorithm.

This approach was recently used in [7, 49] to show the non-integrability of string motion in several curved backgrounds, and we shall also follow it here.

Let us first explain what the algebraization procedure of the last step means. A generic NVE is a second-order differential equation of the form

$$\ddot{\eta} + q(t)\dot{\eta} + r(t)\eta = 0. \quad (2.1)$$

A change of the variable $t \to x(t)$ is called Hamiltonian if $x(t)$ is a solution of the Hamiltonian system $H(p, x) = \frac{p^2}{2} + V(x)$, where $V(x)$ is some potential. Then $x$ satisfies the first integral equation $\frac{1}{2}\dot{x}^2 + V(x) = h$, implying that $\dot{x}$ (and also $\dot{\eta}$) is a function of $x$ only. Changing the variable $t \to x(t)$ in (2.1), we obtain

$$\eta'' + \left(\frac{\dot{x}}{x^2} + \frac{q(t(x))}{x}\right)\eta' + \frac{r(t(x))}{x^2}\eta = 0. \quad (2.2)$$

We now require that the coefficients in this equation are rational functions of $x$.

Let us consider the following simple example of a Hamiltonian system with the canonical variables $(x_i, p_i) (i = 1, 2)$ [59]:

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}x_1^2x_2^2. \quad (2.3)$$

A choice of an invariant plane, referred to above, is $P = \{(x_1, x_2, p_1, p_2) : x_2 = p_2 = 0\}$ along which the solution curves are $x_1(\tau) = \kappa \tau + \text{const}$, $p_1 = \kappa$, $\kappa = \sqrt{2H}$. The system
The Hamiltonian vector field on this solution plane is 
\[ X \bar{x} \]
where
\[ \text{Restricting to the invariant plane } \]
along the plane \( P \) with the coordinates \((x_1, p_1)\) is integrable since the only required constant of motion is provided by the Hamiltonian.

To determine the direction normal to the solution curves of this integrable subsystem, we can use the Hamiltonian vector field along the curves \( X^{\kappa} = \kappa \), \( X^{p_1} = 0 \), \( X^{x_1} = 0 \), \( X^{p_2} = 0 \). Thus the normal direction is along \( x_2 \) and \( p_2 \) and expanding around \( x_2 = 0 \) with \( \delta x_2 = \eta \), we obtain the following NVE:
\[
\eta''(\tau) = -\tilde{x}_1'(\tau)\eta(\tau), \quad \tilde{x}_1(\tau) \equiv \kappa \tau, \quad (2.4)
\]
which is solved by the parabolic cylinder functions \( D_{-1/2}((i - 1)\kappa \tau ) \) and \( D_{-1/2}((i + 1)\kappa \tau ) \). These are not Liouvillian functions and this implies the non-integrability of the original Hamiltonian \( (2.3) \).

To illustrate the previously discussed algebraization procedure, let us consider the Hamiltonian
\[
H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(x_1^2 - x_2^2)^2 \quad (2.5)
\]
which is equivalent to \( (2.3) \) by the trivial canonical transformation
\[
x_1 \rightarrow \frac{x_1 - x_2}{\sqrt{2}}, \quad x_2 \rightarrow \frac{x_1 + x_2}{\sqrt{2}}, \quad p_1 \rightarrow \frac{p_1 - p_2}{\sqrt{2}}, \quad p_2 \rightarrow \frac{p_1 + p_2}{\sqrt{2}}. \quad (2.6)
\]
Restricting to the invariant plane \( P = \{(x_1, x_2, p_1, p_2) : x_2 = p_2 = 0\} \), we obtain an integrable subsystem
\[
\dot{x}_1 = p_1, \quad \dot{p}_1 = -\frac{1}{2}x_1^3. \quad (2.7)
\]
The Hamiltonian vector field on this solution plane is \( X^{p_1} = -\frac{1}{2}\tilde{x}_1^3 \), \( X^{p_1} = 0 \), \( X^{x_1} = \tilde{x}_1 \), \( X^{x_1} = 0 \). The NVE is now obtained by expanding the Hamiltonian equations for the coordinates normal to the invariant plane, i.e. \( x_2 \) and \( p_2 \), along \( (2.7) \). Denoting \( \delta x_2 = \eta \) we thus obtain
\[
\dot{\eta} = \frac{1}{2}\tilde{x}_1'(t)\eta, \quad \tilde{x}_1(t) = 2^{3/4}h^{1/4}\text{sn}\left(\frac{h}{2} \right)^{1/4} t, -1 \quad (2.8)
\]
where \( \tilde{x}_1 \) is a solution of \( (2.7) \) representing the curves in \( P \) parametrized by the Hamiltonian \( h > 0 \). We can algebraize this NVE by changing the variable \( t \rightarrow x = \tilde{x}_1(t) \). This leads to \( (\tau = \frac{dt}{dx}) \)
\[
\eta'' = \frac{2x^3}{8h - x^4} \eta' - \frac{2x^2}{8h - x^4} \eta = 0. \quad (2.9)
\]
Further changing the variable \( \eta \rightarrow \xi = (8h - x^4)^{1/4} \eta \), one obtains the NVE in the normal form
\[
\xi'' + \frac{8hx^2 + 2x^6}{(8h - x^4)^2} \xi = 0. \quad (2.10)
\]
This equation is solved in terms of hypergeometric functions, i.e. it has no Liouvillian solutions. This implies non-integrability of \( (2.5) \), in agreement with the previous discussion of the equivalent system \( (2.3) \).

To summarize, proving non-integrability of a Hamiltonian system can be achieved by demonstrating that given an integrable subsystem defined by some invariant plane in the phase space, the corresponding NVEs are not integrable in quadratures and thus do not admit sufficiently many conserved quantities.
3. String motion in \( p \)-brane backgrounds

In what follows, we shall consider classical bosonic string motion in a curved background. In the conformal gauge, the action is

\[
I = -\frac{1}{4\pi\alpha'} \int d\sigma dr [G_{MN}(Y) \delta_{J}^{Y} \gamma^{J} + \epsilon^{ab} B_{MN}(Y) \delta_{\alpha}^{Y} \delta_{\beta}^{Y}].
\]  

(3.1)

and the corresponding equations of motion should be supplemented by the Virasoro constraints

\[
G_{MN} \dot{Y}^{M} \dot{Y}^{N} = 0,
\]

(3.2)

\[
G_{MN}(\dot{Y}^{M} \dot{Y}^{N} + \dot{\bar{Y}}^{M} \dot{\bar{Y}}^{N}) = 0.
\]

(3.3)

In this section, we shall consider the case of \( B_{MN} = 0 \) and the target space metric given by the following \( p \)-brane ansatz \( (D = 10) \):

\[
d\tilde{s}^2 = f^2(r) dr^2 + r^2 d\Omega_2^2 + \sum_{n, m = 0}^{k-1} \eta_{\mu\nu} dx^\mu dx^\nu,
\]

(3.4)

where \( d\Omega_2^2 \) is the metric on a \( k \)-sphere.

For \( k = 8 - p \), \( n = 7 - p > 0 \) and \( m = \frac{1}{2} \), this metric describes the standard single \( p \)-brane geometry in \( D = 10 \) supergravity. It may be supported by an R–R field strength and dilution backgrounds, but since they do not couple to the classical bosonic string we shall ignore them here.

Keeping \( n, m, p \) generic also allows one to describe lower dimensional backgrounds representing (up to a flat factor) some special brane intersection geometries. In particular, the metric (3.4) with \( nm = 1 \) describes a one-parameter interpolation \([60, 61]\) between flat space (for \( Q = 0 \)) and the AdS\(_{p+2} \times S^8 \) space (for \( Q \to \infty \)).

Special cases include:

(i) \( n = 4, m = \frac{1}{3}, k = 5, p = 3 \): D3-brane interpolating between flat space and AdS\(_{5} \times S^5 \);

(ii) \( n = 2, m = \frac{1}{2}, k = 3, p = 1 \): D5-D1 (or NS5-F1 with non-zero \( B_{MN} \), see the following section) background \([62, 63]\) with \( Q_S = Q_I = Q \) interpolating between flat 10D space and AdS\(_2 \times S^5 \times T^2 \);

(iii) \( n = 1, m = 1, k = 2, p = 0 \): four equal-charge D3-brane intersection \([64]\) (or U-duality related backgrounds, see, e.g. \([65]\)) interpolating between flat 10D space and AdS\(_2 \times S^5 \times T^4 \); this may be viewed as a generalized Bertotti–Robinson geometry.

Below we shall show that the point-like string (i.e., geodesic) motion in this background is integrable. We shall then demonstrate that extended string motion in the \( p \)-brane geometry with \( p = 0, \ldots, 6 \) and in the backgrounds (ii), (iii) is not integrable for generic values of \( Q \), despite integrability being present in the limits \( Q = 0 \) and \( Q = \infty \).

3.1. Complete integrability of geodesic motion

The symmetries of the metric (3.4) are shifts in the \( x^\mu \) coordinates giving \( p + 1 \) conserved quantities and spherical symmetry which gives \( k/2 \) conserved commuting angular momenta for even \( k \) and \((k + 1)/2 \) for odd \( k \) (generators of the Cartan subgroup of SO\((k + 1)\)). Thus for

\[\text{More precisely, the metric (3.4) interpolates between the flat-space region} \( (r^a \gg Q) \text{ and the AdS}_{p+2} \times S^8 \text{ region} \( (r^a \ll Q) \). \text{Moreover, since} f(r) = \left(1 + \frac{Q}{r^9}\right)^{\nu} \text{gives a solution for any constant} \nu \text{including} \nu = 0, \text{AdS}_{p+2} \times S^8 \text{is also an exact solution.}\]
k \geq 3$ the spherical symmetry does not, a priori, provide us with sufficiently many conserved quantities.

Parametrizing the $k$-sphere by $k + 1$ embedding coordinates $\mathbf{y}$ the effective Lagrangian for point-like string motion is
\[ L = f^{-2}(r)\mathbf{x}^a \dot{x}^a \eta_{uv} + f^2(r)(\dot{r}^2 + r^2 \dot{\theta}^2) + \Lambda (\mathbf{y}^2 - 1). \] (3.6)

The first integrals for $x^a$ and $r$ are $E^a = f^{-2}(r)\dot{x}^a$ and the Hamiltonian, respectively. The equation of motion for $\mathbf{y}$ reads
\[ \partial_\tau (f^2(r) r^2 \dot{\mathbf{y}}) + \Lambda \dot{\mathbf{y}} = 0. \] (3.7)

Taking the scalar product of this equation with $\mathbf{y}$ and with $\mathbf{y}^\ast$ and using $\dot{\mathbf{y}} \dot{\mathbf{y}}^\ast = 1$, $\dot{\mathbf{y}}^\ast \dot{\mathbf{y}} = 0$, we conclude that
\[ \Lambda = f^2(r) r^2 \mathbf{y}^2 = f^{-2}(r) r^{-2} m^2, \quad m^2 = f^4(r) r^4 \mathbf{y}^2 = \text{const.} \] (3.8)

Substituting this expression into (3.7) allows one to eliminate $\Lambda$ giving
\[ \partial_\tau (f^2(r) r^2 \dot{\mathbf{y}}^\ast) + m^2 f^{-2}(r) r^{-2} \mathbf{y}^\ast = 0. \] (3.9)

Multiplying this equation by $f^2(r) r^2 \dot{\mathbf{y}}$ for fixed $i$ and integrating once, we obtain the constants of motion $C_i$ (no summation over $i$)
\[ f^2(r) r^2 (\dot{\mathbf{y}}^\ast)^2 + m^2 (\dot{\mathbf{y}}^\ast)^2 = C_i, \quad \sum_i C_i = 2m^2. \] (3.10)

We can rewrite the $C_i$ in terms of phase space coordinates $(\dot{\mathbf{y}}, \mathbf{p}^i = 2 f^2(r) r^2 \dot{\mathbf{y}}^\ast)$ as
\[ C_i = \frac{1}{2} (\mathbf{p}^i)^2 + m^2 (\dot{\mathbf{y}}^\ast)^2. \] (3.11)

The integrals $C_i$ are easily checked to be in involution. Only $k$ of the $k + 1$ integrals $C_i$ are functionally independent and correspond to the angles on the $\Omega_k$ sphere. Altogether with the $p + 1$ constants for the $x^a$ coordinates and the Hamiltonian, this gives in total $k + p + 2$ constants of motion which is the same as the number of degrees of freedom.

### 3.2. Non-integrability of string motion

Let us now study extended string motion in the geometry (3.4), (3.5). We shall choose a particular ‘pulsating string’ ansatz for the dependence of the string coordinates on the world-sheet directions $(\tau, \sigma)$: we shall assume that (i) only $x^0$, $r$ and two angles $\phi, \theta$ of $S^2 \subset S^4$ (with $d\Omega^2_S = d\Omega^2 + \sin^2 \theta d\phi^2$) are non-constant, and (ii) $x^0, r, \theta$ depend only on $\tau$ while $\phi$ depends only on $\sigma$, i.e.
\[ x^0 = t(\tau), \quad r = r(\tau), \quad \phi = \phi(\sigma), \quad \theta = \theta(\tau). \] (3.12)

This ansatz is consistent with the conformal-gauge string equations of motion and the Virasoro constraint (3.2). The remaining string equations of motion and the Virasoro constraint give
\[ i = E f^2, \quad E = \text{const}, \] (3.13)
\[ \dot{\phi} = v = \text{const}, \quad \dot{\phi} = v \sigma, \] (3.14)
\[ 2 \partial_\tau (f^2 r) = \partial_\tau (f^2) E^2 + \partial_\tau (f^2)^2 \dot{r}^2 + \partial_\tau (r^2 f^2) (\dot{\theta}^2 - v^2 \sin^2 \theta), \] (3.15)
\[ \partial_\tau (f^2 r^2 \dot{\theta}) = - f^2 r^2 v^2 \sin \theta \cos \theta, \] (3.16)

\footnote{Note that when only one of the angles of $S^4$ is non-constant, 1D truncations are not sufficient to establish non-integrability, see below.}
Thus the string is wrapped (with winding number \( n \)) on a circle of \( S^2 \) whose position in \( r \) and \( \theta \) changes with time. The equations for \( r \) and \( \theta \) can be derived from the following effective Lagrangian:

\[
\mathcal{L} = f^2 (\dot{r}^2 + r^2 \dot{\theta}^2) - v^2 r^2 \sin^2 \theta + E^2,
\]

with the corresponding Hamiltonian restricted to be zero by (3.17). We shall show that this 1D Hamiltonian system is not integrable, implying non-integrability of string motion in the \( p \)-brane background (3.4).

Let us choose as an invariant plane \( \{ (r, \theta; p_r, p_\theta) : \theta = \frac{n}{2}, p_\theta = 0 \} \), corresponding to the a string wrapped on the equator of \( S^2 \) and moving only in \( r \). Then (3.17) gives \( \dot{r}^2 + v^2 \dot{\theta}^2 = E^2 \) which is readily solved by (assuming e.g. that \( r(0) = 0 \))

\[
r = \tilde{r} (\tau) = \frac{E}{v} \sin (\nu \tau).
\]

According to the general method described in section 2, we have to show that small fluctuations near this special solution are not integrable, or, more precisely, that the variational equation normal to this surface of solutions parametrized by \( E \) and \( v \) has no Liouvillean solutions along the curves inside the surface.

For the invariant subspace \( \{ \theta = \frac{n}{2}, p_\theta = 0, r = \tilde{r}, p_r = 2 f^2 (\tilde{r}) \dot{\tilde{r}} \} \) the Hamiltonian vector field is

\[
X^\tau = \tilde{r}, \quad X^{p_\tau} = \tilde{p}_r = 4 f f' \dot{\tilde{r}}^2 + 2 f^2 \tilde{r}, \quad X^\theta = 0, \quad X^{p_\theta} = 0,
\]

and thus the normal direction to this plane is along \( \theta \) and \( p_\theta \). Expanding the Hamiltonian equation for \( \theta \) around \( \theta = \frac{n}{2}, \dot{\theta} = 0, r = \tilde{r} \), one obtains the NVE \( (\delta \theta = \eta) \):

\[
\eta + 2 \nu \cot (\nu \tau (1 + \frac{E}{v} \sin (\nu \tau) \frac{f' (\tilde{r} \sin (\nu \tau))}{f (\tilde{r} \sin (\nu \tau))}) \eta - v^2 \eta = 0.
\]

If \( f' / f \) is a rational function (as is the case for (3.5)), then this equation can be algebraized through the change of variable \( \tau \rightarrow x = \sin (\nu \tau) \) giving

\[
\eta'' + \left( \frac{2}{x} - \frac{x}{1 - x^2} + 2 \frac{E f' (\tilde{r} x)}{v f (\tilde{r} x)} \right) \eta' - \frac{1}{1 - x^2} \eta = 0.
\]

Using the explicit form of \( f(r) \) in (3.5), this NVE reduces to

\[
\eta'' + \left( \frac{2}{x} - \frac{x}{1 - x^2} - 2 \frac{mn^2 Q}{x (mn Q + En x^4)} \right) \eta' - \frac{1}{1 - x^2} \eta = 0.
\]

We shall assume that \( \nu \neq 0 \) as otherwise the string is point-like and we go back to the case of geodesic motion.

Bringing (3.23) to the normal form by changing the variable to \( \xi (x) = g(x) \eta (x) \), where \( g(x) \) is a suitably chosen function, one obtains

\[
\xi'' + \left[ \frac{2 + x^2}{4(x^2 - 1)^2} - \frac{(m - 1)mn^2}{x^4 (1 + B x^6)} + \frac{mn (n - 1 - (n - 2)x^2)}{x^3 (x^2 - 1)(1 + B x^6)} \right] \xi = 0, \quad B = \frac{E}{Qv^4}.
\]

There are two special limits of (3.23) and (3.24):

\[
Q \rightarrow 0 : \quad \xi'' + \frac{2}{4(x^2 - 1)^2} \xi = 0, \quad (3.25)
\]

\[
Q \rightarrow \infty : \quad \xi'' + \frac{2x^2 + x^4 - 4(mn)^2(x^2 - 1)^2 + 4mn(1 - 3x^2 + 2x^4)}{4x^2(x^2 - 1)^2} \xi = 0, \quad (3.26)
\]

corresponding to the flat space-time and \( \text{AdS}_2 \times S^2 \) for \( mn = 1 \).
We can now apply the Kovacic algorithm\(^9\) to determine whether these NVs do not admit Liouvillean solutions and thus the identity component of their Galois group is not solvable which would imply non-integrability.

For the special cases \(Q \to 0\) and for \(Q \to \infty\) with \(nm = 1\) Liouvillean solutions are found which is consistent with the fact that string motion in flat space-time and in AdS\(_3\) \x S\(^2\) is integrable. However, for a finite value of \(Q\), Liouvillean solutions do not exist for generic values of \(E, v\) for the cases of a \(p\)-brane background with \(p = 0, \ldots, 6\) and the intersecting brane backgrounds of (ii) and (iii) mentioned above. This implies non-integrability of string motion in those special cases of the general background (3.4), (3.5).

Finally, let us comment on the special case of a 7-brane in ten dimensions when the transverse space is two dimensional, i.e. the string motion is described by

\[
\mathcal{L} = f^{-2}(r)\partial_a x^a \partial^a x^a \eta_{\mu\nu} + f^2(r)(\partial_i r \partial^i r + r^2 \partial_\theta \partial^\theta).
\]  

(3.27)

It is easy to see that if we truncate this model to a one-parameter system by taking each of the string coordinates to be a function of \(r\) or \(\sigma\), only then the corresponding equations always admit constants of motion for \(x^a\) and \(\theta\), and the Virasoro condition then provides the solution for \(r\). To test integrability in this case one has to consider some more non-trivial truncations. Assuming the spatial coordinates \(x^i\) are constant the corresponding effective metric is three dimensional: \(ds^2 = -f^{-2}(r)dr^2 + f^2(r)(dr^2 + r^2 d\theta^2)\). Integrability of string motion in such a background deserves further study.

### 4. String motion in NS5-F1 background

Let us now include the possibility of a non-zero \(B_{MN}\) coupling and consider the case of string motion in a background produced by fundamental strings delocalized inside NS5 branes [66, 62, 67] (here we use the notations \(x^0 = t, x^1 = z\) and \(i, j = 2, \ldots, 5\))

\[
dx^2 = H_i^{-1}(r)(-dr^2 + dz^2) + dx^i dx_i + H_5(r)(dr^2 + r^2 d\Omega_5^2),
\]  

(4.1)

\[
d\Omega_5^2 = d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\phi^2,
\]

\[
B = -H_i^{-1}(r) dt \wedge dz + Q_5 \sin^2 \theta d\phi \wedge d\phi,
\]  

(4.2)

\[
H_5 = 1 + \frac{Q_5}{r^2}, \quad H_i = 1 + \frac{Q_i}{r^2},
\]  

(4.3)

with the dilaton given by \(e^{-2\phi} = \frac{H_i(x)}{H_i(r)}\). The corresponding string Lagrangian in (3.1)

\[
\mathcal{L} = (G_M + B_{MN})\partial_a Y^M \partial_a Y^N \theta_a_0 \theta_a_1 \partial_a \theta_a_2 \partial_a \theta_a_3 \partial_a \theta_a_4 \partial_a \theta_a_5
\]  

\[
+ (r^2 + Q_5)(\partial_i r \partial^i r + \sin^2 \phi \partial_i \phi \partial^i \phi + \cos^2 \theta \partial_i \theta \partial^i \theta) + Q_5 \sin^2 \theta (\partial_i \phi \partial^i \phi - \partial_i \phi \partial^i \phi) + r^2 + Q_5\]

\[
\]  

(4.4)

It interpolates between the flat space model for \(Q_1, Q_5 = 0\) and the \(SL(2) \times SU(2)\) WZW model (plus four free directions) for \(Q_1, Q_5 \to \infty\). Both of these limits are obviously integrable.

Another integrable special case is \(Q_1 = 0, Q_5 \to \infty\) when we obtain the \(SU(2)\) WZW model plus flat directions.

\(^9\) We use Maple’s function kovacicsols.
The opposite limit of \( Q_5 \to 0, Q_1 \to \infty \) is described (after a rescaling of coordinates) by
\[
L = r^2 \partial_u u \partial_v v + \partial_u \partial_v r + r^2 (\partial_u \partial_v \theta + \sin^2 \theta \partial_u \varphi \partial_v \varphi + \cos^2 \theta \partial_u \phi \partial_v \phi) + \text{free directions}.
\]
Solving the equations for \( u, v \) as \((\sigma^\pm = \tau \pm \rho)\),
\[
r^2 \partial_u u = \tilde{f}(\sigma^+), \quad r^2 \partial_v v = \tilde{f}(\sigma^-),
\]
one finds the effective Lagrangian for \( r \) being
\[
L = \partial_r \partial_r r - \frac{\tilde{f}(\sigma^-) \tilde{f}(\sigma^+)}{r^2} + r^2 (\partial_u \partial_v \theta + \sin^2 \theta \partial_u \varphi \partial_v \varphi + \cos^2 \theta \partial_u \phi \partial_v \phi).
\]
As we shall find below, this model is not integrable.\(^{10}\)

Let us note that in the limit \( Q_5 \to \infty \) the action (4.4) reduces to a combination of the \( SU(2) \) WZW model, flat directions and the following three-dimensional sigma model
\[
L = \frac{1}{1 + Q_5 e^{-2\varphi}} \partial_u u \partial_v v + Q_5 \partial_\rho \partial_\rho, \quad \rho \equiv \ln r.
\]
This model is T-dual to a pp-wave model with an exponential potential function \([34, 51]\) and is thus related to the SL(2) WZW model by a combination of T-dualities and a coordinate transformation (it can be interpreted, upon a rescaling of \( u, v \), as an exactly marginal deformation of the SL(2, \( R \)) WZW model \([35]\)). It should thus be integrable. Indeed, solving the equations for \( z \pm t \) as in (4.5), we obtain the following effective Lagrangian for \( \rho \) (cf (4.6)):
\[
L = Q_5 \partial_\rho \partial_\rho - f(\sigma^-) \tilde{f}(\sigma^+) (1 + Q_5 e^{-2\varphi}).
\]
Since \( f(\sigma^-) \tilde{f}(\sigma^+) \) can be made constant by conformal redefinitions of \( \sigma^\pm \) (reflecting residual gauge freedom in the conformal gauge) this model is equivalent to the Liouville theory, i.e. it is integrable. Indeed, the results of our analysis below are consistent with this conclusion.

A point-like string does not couple to the \( B \)-field, so geodesic motion in the background (4.1) can be shown to be integrable in the same way as in the previous section. To study extended string motion let us consider the following ansatz describing the probe string being stretched along the fundamental string direction \( z \) (that may be assumed to be compactified to a circle) and the \( S^3 \) angle \( \varphi \), i.e.
\[
t = t(\tau), \quad z = z(\sigma), \quad x^i = 0, \quad r = r(\tau), \quad \theta = \theta(\tau), \quad \varphi = \varphi(\sigma), \quad \phi = \phi(\tau).
\]
Then the string equations are solved by
\[
i = \kappa_1 H_1(r(\tau)) + \kappa_2, \quad z = \kappa_2 \sigma, \quad \varphi = v \sigma, \quad \phi = \frac{\sqrt{Q_5}}{r^2(\tau) H_5(r(\tau))}.
\]
The resulting 1D subsystem of equations for \( r \) and \( \theta \) is described by the following effective Lagrangian:
\[
L = H_5(r) \left[ \dot{r}^2 + v^2 \dot{\sigma}^2 - v^2 \sin^2 \theta - Q_5^2 v^2 r^2 H_5^{-2}(r) \cos^2 \theta \right] + \kappa_1^2 H_1(r),
\]
which should be supplemented by the consequence of the Virasoro constraint
\[
H_5(r) \left[ \dot{r}^2 + v^2 \dot{\sigma}^2 + v^2 \sin^2 \theta + Q_5^2 v^2 r^2 H_5^{-2}(r) \cos^2 \theta \right] - \kappa_1^2 H_1(r) = 2\kappa_1 \kappa_2,
\]
implying that the Hamiltonian corresponding to (4.11) is equal to \( 2\kappa_1 \kappa_2 \). This system admits the special solution
\[
\theta = \frac{\pi}{2}, \quad r = \bar{r}(\tau), \quad H_5(\bar{r})(\dot{\bar{r}}^2 + v^2 \bar{r}^2) = 2\kappa_1 \kappa_2 + \kappa_1^2 H_1(\bar{r}).
\]
\(^{10}\) Let us note that the constant term in the harmonic function of the fundamental string background (1 in \( H_1 \)) can be changed by a combination of T-duality, coordinate shift and another T-duality \([51, 61]\), so that the fundamental string background is not integrable for any value of \( Q_1 \) if it is not integrable for large \( Q_1 \). A similar shift in the harmonic function can also be achieved for D-brane backgrounds but this involves S-duality (mapping e.g. F1 to D1) which is not a symmetry of the world-sheet string action; thus there is no contradiction with integrability of the D3-brane background at large \( Q \).
which may be chosen as an invariant plane in the phase space. The corresponding Hamiltonian vector field is

\[ X^r = \dot{r}, \quad X^{\psi} = 2 \delta_r (\dot{r} H_S(\vec{r})), \quad X^\vartheta = 0, \quad X^\nu = 0. \tag{4.14} \]

Expanding along the normal direction to this solution plane, we find for the NVE \((\vartheta = \frac{x}{2} + \eta)\),

\[ \ddot{\eta} + \left( \frac{H_S(\vec{r})}{H_S(\vec{r})} + 2 \frac{\eta}{r} \right) \left( k_1^2 H_S(\vec{r}) + 2 k_2 \frac{\kappa_2}{H_S(\vec{r})} - \nu^2 r^2 \right)^{1/2} \dot{\eta} - \nu^2 \left( 1 - \frac{Q_5^2}{r^2 H_S^2(\vec{r})} \right) \eta = 0. \tag{4.15} \]

Changing the variable \(\tau\) to \(x = \dot{r}(\tau)\), we obtain the NVE in the normal form \((\dot{\vartheta} = \frac{q}{\vartheta})\)

\[ \xi''(x) + U(x) \xi(x) = 0, \quad \gamma \equiv \frac{Q_5}{\xi}, \tag{4.16} \]

where

\[
U(x) = \frac{1}{4x^2} \left[ \frac{Q_5(Q_5 - 2x^2)}{(Q_5 + x^2)^2} + \frac{4Q_5x^4 - 2x^6 + 2Q_5q^2(2Q_5 + x^2)}{(Q_5 + x^2)(-Q_5q^2 + x^4(Q_5 + x^2 - q^2(1 + \alpha)))} \right] + \frac{3(1 + \alpha)}{Q_5q^2 + x^4[-Q_5 - x^2 + q^2(1 + \alpha)]^2} = 0, \quad q \equiv \frac{k_1}{\nu}, \quad \alpha \equiv \frac{k_2}{k_1}. \tag{4.17}
\]

One finds that already for \(\alpha = 0\) i.e. for \(k_2 = 0\), this NVE has no Liouvillian solutions for general values of \(Q_1\) and \(Q_5\), implying non-integrability of string motion in the NS5-F1 background with generic values of the charges. Our results are summarized in the table below.

| limit      | \(U(x)\) in NVE | \(\xi''(x) + U(x)\xi(x) = 0, \gamma \equiv \frac{Q_5}{\xi}\) | rescaling |
|-----------|----------------|-------------------------------------------------|-----------|
| \(Q_5 \to 0\) | \(Q_1 \to 0\) | \(\frac{4\gamma(1 + \alpha) + x^2}{4(\gamma^2 + x^2 - \alpha)}\) | -         |
| \(Q_5 \to \infty\) | \(Q_1 \to \infty\) | \(\frac{(\gamma - q^2)(\gamma^2 + x^2 - \alpha)}{4(\gamma^2 + x^2 - \alpha)}\) | \(\alpha \to Q_1\alpha\) |
| \(Q_5 \to 0\) | \(Q_1 \to \infty\) | \(\frac{3}{4x^2} + \frac{Q_5}{4}\) | -         |
| \(Q_5 \to 0\) | \(Q_1 \not\to 0\) | \(\frac{(\gamma - q^2)(\gamma^2 + x^2 - \alpha)}{4(\gamma^2 + x^2 - \alpha)}\) | -         |
| \(Q_5 \not\to 0\) | \(Q_1 \not\to 0\) | \(\frac{(\gamma - q^2)(\gamma^2 + x^2 - \alpha)}{4(\gamma^2 + x^2 - \alpha)}\) | -         |

| limit      | \(U(x)\) in spacetime | 2-form contributes to the truncated system | NVE has Liouvillian solutions |
|-----------|-------------------------|------------------------------------------|-------------------------------|
| \(Q_5 \to 0\) | \(Q_1 \to 0\) | \(\mathbb{R}^{1,1}\) | \(\alpha = 0\) | \(\alpha \not\equiv 0\) | \(\alpha = 0\) | \(\alpha \not\equiv 0\) |
| \(Q_5 \to \infty\) | \(Q_1 \to \infty\) | \(AdS_3 \times S^3 \times \mathbb{R}^4\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) |
| \(Q_5 \to 0\) | \(Q_1 \to \infty\) | \(\mathbb{R}^{1,1} \times S^3\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) |
| \(Q_5 \to \infty\) | \(Q_1 \not\to 0\) | \(\mathbb{R}^{1,1} \times S^3\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) |
| \(Q_5 \not\to 0\) | \(Q_1 \not\to 0\) | \(\mathbb{R}^{1,1} \times S^3\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) |
| \(Q_5 \to 0\) | \(Q_1 \not\to 0\) | \(\mathbb{R}^{1,1} \times S^3\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) |
| \(Q_5 \not\to 0\) | \(Q_1 \not\to 0\) | \(\mathbb{R}^{1,1} \times S^3\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) |
Let us discuss some special cases. Taking \(Q_1 = 0\) or \(Q_5 = 0\), one can absorb the remaining brane charge by rescaling \(x, q\) and \(\kappa_1\). The resulting equations do not admit Liouvillian solutions for arbitrary values of \(q\) which implies non-integrability of string motion in the NS5-brane or in the fundamental string backgrounds.

In the limit \(Q_1 \to \infty, Q_5 \to \infty\), in which the non-trivial part of the string action becomes that of the \(SL(2, R) \times SU(2)\) WZW model, one finds Liouvillian solutions in agreement with the expected integrability. The same applies to the case of \(Q_1 \to 0, Q_5 \to \infty\) described by the \(SU(2)\) WZW model plus free fields and, in fact, to the model with \(Q_5 \to \infty\) and arbitrary \(Q_1\) described by (4.7), (4.8).

In the opposite case of \(Q_5 \to 0, Q_1 \to \infty\) described by (4.6) integrability appears to be absent as we did not find Liouvillian solutions (see the table)\(^{11}\).

Since 2D duality transformations of string coordinates preserve (non)integrability, similar conclusions can be reached for string backgrounds related to this NS5-F1 background or to its limits via T-dualities (and coordinate transformations). In particular, this applies to the pp-wave background related to the fundamental string by T-duality [51] which is studied in appendix B.

5. Concluding remarks

We have shown that for various \(p\)-brane backgrounds, for which the string sigma model interpolates between integrable flat and coset or WZW models and which has integrable geodesics, the corresponding extended classical string motion is not integrable in general.

To demonstrate non-integrability, we considered particular extended string motion for which the dynamics reduces to an effective one-dimensional system. The latter has special integrable subclasses of solutions, but perturbing near them leads to linear second-order differential equations that cannot be solved in quadratures.

It would be interesting to understand better why switching on a non-zero D3-brane charge or moving away from the ‘throat’ (AdS5 \(\times S^5\)) region leads to a breakdown of string integrability.

Together with similar previous results [7, 6, 49], this supports the expectation that integrability of classical string motion is a rare phenomenon. String integrability is thus a much more restrictive constraint than integrability of particle motion. Still, in the absence of a general classification of integrable 2D sigma models, one cannot rule out the possibility that there are still interesting examples of integrable backgrounds (that cannot be obtained from known solvable cases by coordinate transformations combined with T-dualities, cf [30, 29]), that remain to be discovered.

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\(^{11}\) Let us note that the results about the existence of Liouvillian solutions turn out to be independent of whether the truncated solutions contain any contribution from the B-flux of the fundamental string solution (\(\alpha \neq 0\) or \(\alpha = 0\)), reflecting a special choice of our ansatz (4.9).
Appendix A. Integrability and differential Galois theory

Integrability in the sense of Liouville implies that the equations of motion are solvable in quadratures, i.e. the solutions are combinations of integrals of rational functions, exponentials, logarithms and algebraic expressions. Solutions of this type are referred to as Liouvillian functions. Thus the question of integrability is related to the question of when differential equations admit only Liouvillian solutions. This is answered by differential Galois theory and the main integrability theorem, which our analysis relies on, can be stated as follows [69].

Theorem. Let $H$ be a Hamiltonian defined on a phase space manifold $M$ with an associated Hamiltonian vector field $X$. Let $P$ be a submanifold of $M$ which is invariant under the flow of $X$ with the invariant curves $\Gamma$ and let $X|_{\Gamma}$ be completely integrable. Also, let $G$ be the differential Galois group of the variational equation of the flow of $X|_{\Gamma}$ normal to $P$.

If $X$ is completely integrable on $M$, then the largest connected algebraic subgroup of $G$ which contains the identity is Abelian.

Since differential Galois theory is not frequently used in physics, we shall give a brief introduction and illustrate some basic concepts on a simple example below. We shall follow [70]. First, let us give some basic definitions:

1. A differential field is a field $F$ with a derivation $D_F$, which is an additive map $D_F : F \to F$, thus satisfying $D_F(ab) = D_F(a)b + aD_F(b)$, $\forall a, b \in F$.
2. A differential homomorphism/isomorphism between two differential fields $F_1$ and $F_2$ is a homomorphism/isomorphism $f : F_1 \to F_2$ which satisfies $D_{F_2}(f(a)) = f(D_{F_1}(a))$, $\forall a \in F_1$.
3. A linear differential operator $L$ is defined by

   $$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y,$$

   where $y^{(\alpha)} = D_F^\alpha y$ and $a_\alpha \in F$.
4. A differential field $E$ with derivation $D_E$ is called a differential field extension of a differential field $F$ with derivation $D_F$ iff $E \supseteq F$ and the restriction of $D_E$ to $F$ coincides with $D_F$.
5. Elements $a \in F$ whose derivative vanishes are called constants of $F$. The subfield of constants of $F$ is denoted by $C_F$.

In order to investigate the types of solutions a differential equation admits, one has to encode the relations and symmetries of the independent solutions. Relations can be encoded in the Picard–Vessiot extension of a differential field which includes the solutions of $L(y) = 0$.

Definition. An extension $E \supseteq F$ is called a Picard–Vessiot extension of $F$ for $L(y) = 0$ iff:

- $E$ is generated over $F$ as a differential field by the set of solutions of $L(y) = 0$ in $E$;
- $E$ and $F$ have the same constants;
- $L(y) = 0$ has $n$ solutions in $E$ linearly independent over the constants.

One can show that the Picard–Vessiot extension always exists for differential fields with an algebraically closed field of constants and this extension is unique up to isomorphisms.

Symmetries of the solution space are linear transformations which, applied to a solution, give a new solution. The simplest formulation is in terms of automorphisms of the Picard–Vessiot extension and this is related to the notion of the differential Galois group.
Definition. For a differential field $F$ with the Picard–Vessiot extension $E \supseteq F$, the differential Galois group $G(E/F)$ is the group of differential automorphisms $\sigma : E \to E$ whose restriction to $F$ is the identity map, i.e.

$$G(E/F) = \{\sigma : E \to E \mid \sigma(a) = a \ \forall a \in F\}.$$ 

A solution $y(x)$ of a linear differential equation $L(y) = 0$ is called algebraic if it is a root of a polynomial over $F$;

primitive if $y(x) \in F$, i.e. $y(x) = \int f(z) \, dz$ for $f(x) \in F$;

exponential if $\frac{y(x)}{x} \in F$, i.e. $y(x) = \exp \left( \int f(z) \, dz \right)$ with $f(z) \in F$.

Definition. The solution $y(x)$ is called Liouvillian if there exist differential extensions $E_i$, $i = 0, \ldots, m$, 

$$F = E_0 \subset E_1 \subset \cdots \subset E_m = E$$

such that $y(x) = y_m(x) \in E_i, E_i = E_{i-1}(y_i(x))$ and $y_i(x)$ is algebraic, primitive or exponential over $E_{i-1}$.

We will be interested in second-order homogeneous linear ordinary differential equations

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0,$$  \hspace{1cm} (A.2)

where $a(x), b(x) \in F$ and $F = \mathbb{C}(x)$ is the differential field of rational functions over complex numbers. Equation (A.2) can be cast into the normal form

$$y''(x) + U(x)y(x) = 0,$$  \hspace{1cm} (A.3)

by the transformation $y(x) \to y(x) \exp \left( -\frac{1}{2} \int a(x) \, dx \right)$.

Let us identify the corresponding Galois group. Let $E$ be a Picard–Vessiot extension of $F = \mathbb{C}(x)$ and $z_1(x), z_2(x) \in E$ be two independent solutions of (A.3). The Galois group $G(E/F)$ consists of differential automorphisms $\sigma$ which act on the space of solutions of (A.3), i.e. map solutions to solutions. Thus

$$\sigma(z_i(x)) = C_{ij}z_j(x), \quad i, j = 1, 2.$$  \hspace{1cm} (A.5)

Defining the Wronskian $w(z_1(x), z_2(x)) = \det \begin{pmatrix} z_1'(x) & z_2'(x) \\ z_1(x) & z_2(x) \end{pmatrix}$ we obtain

$$\sigma \left( w(z_1(x), z_2(x)) \right) = \det C \ w(z_1(x), z_2(x)),$$  \hspace{1cm} (A.6)

where $\det C$ is the determinant of the $C_{ij}$ matrix in (A.5). The Wronskian is a constant since $w'(x) = z_1(x)z_2''(x) - z_2(x)z_1''(x) = 0$ by virtue of (A.3), i.e. $w(x) \in F$. Therefore, $\sigma(w(x)) = w(x)$, giving $\det C = 1$. Thus the Galois group for (A.3) is a subgroup of $S\text{L}(2, \mathbb{C})$. For more general equations the Galois group is a subgroup of $G\text{L}(n, \mathbb{C})$.

Let us now address the main question: Whether a differential equation can be solved in terms of Liouvillian functions? Since the minimal differential extension generated by all solutions is the Picard–Vessiot extension, we therefore require that the Picard–Vessiot extension is generated by a tower of Liouvillian extensions. By the fundamental theorem of differential Galois theory, this tower of extensions is related to subgroups of the Galois group through a bijective correspondence. One can show that for the Picard–Vessiot extension to be Liouvillian, the identity-connected component of the Galois group $G^0$ has to be solvable (see theorem 25 in [56]).

12 A group is solvable if it has a subnormal series whose factor groups are all Abelian, that is, if there are subgroups $\{1\} \leq G_1 \leq \cdots \leq G_n = G$ such that $G_{i-1}$ is normal in $G_i$ (i.e. invariant under conjugation) and $G_i/G_{i-1}$ is an Abelian group for all $k = 1, \ldots, n$.  


A simple example is provided by the equation \( y''(x) + 2xy'(x) = 0 \) which has the normal form
\[
y''(x) - (x^2 + 1)y(x) = 0. \tag{A.7}
\]
For this equation, \( F = \mathbb{C}(x) \) is the field of rational functions over complex numbers. The solution space may be represented as \( \mathbb{C}(e^{1/2}) \oplus \mathbb{C}(e^{1/2} \int e^{-x^2}) \). The Picard–Vessiot extension is \( E = \mathbb{C}(x, e^{1/2}, e^{-1/2}, \int e^{-x^2}) \) and the tower of extensions of \( E \) is
\[
\mathbb{C}(x) \subset \mathbb{C}(x, e^{1/2}) \subset \mathbb{C}(x, e^{1/2}, e^{-1/2}) \subset E = \mathbb{C}(x, e^{1/2}, e^{-1/2}, \int e^{-x^2}).
\]
Then the Galois group \( G(E/F) \) and its identity component \( G^0 \) are
\[
G = G^0 = \left\{ \begin{pmatrix} a & 0 \\ c & 1/a \end{pmatrix} \right\}, \quad a \neq 0, c \in \mathbb{C}. \tag{A.8}
\]
This is a solvable group since it is a subgroup of \( SL(2, \mathbb{C}) \) whose algebraic subgroups are all solvable except for \( SL(2, \mathbb{C}) \) itself.

**Appendix B. String motion in pp-wave background**

Here, we shall study string motion in a pp-wave metric \((i = 1, \ldots, d)\),
\[
dx^2 = du dv + H(u, x) dv^2 + dx_1 dx_i, \tag{B.1}
\]
In conformal gauge, the equation for \( v \) reads \( \partial_i \partial_i u = 0 \) and is solved by \( u = f(\sigma^+) + \tilde{f}(\sigma^-) \). We can fix the residual conformal symmetry by choosing \( u = \rho \tau \). Then \( v \) is determined from the Virasoro constraints and we obtain the following effective Lagrangian for \( x_i \):
\[
\mathcal{L} = \dot{x}_i \dot{x}_i - \ddot{x}_i \dot{x}_i + \rho^2 H(u, x), \tag{B.2}
\]
which describes light-cone gauge string motion in a potential.

One familiar example is the Ricci flat space with \( H(x) = \mu_{ij} x_i x_j \), \( \mu_i = 0 \). Another is the pp-wave limit of the \( AdS_3 \times S^3 \) background for which \( H(x) = x_1 x_i \) [44]. Here, the string motion takes place in a quadratic potential and is obviously integrable. Penrose limits of the brane backgrounds that we consider in this paper also take the pp-wave metric form
\[
H(u, x) = h_{ij}(u) x_i x_j \quad \text{(see e.g. [46])}.
\]
In the light-cone gauge string motion takes place in a quadratic potential with a \( \tau \)-dependent coefficient and it is solvable [47] but formally the resulting model is not integrable\textsuperscript{13}.

There are also other integrable examples with \( H(x) \) corresponding, for example, to the Liouville or Toda potential.

The case that we shall study below is the pp-wave solution T-dual to the fundamental string [66, 51] for which \( H \) is a harmonic function
\[
H(x) = 1 + \frac{Q}{r^{d-2}}, \quad r^2 = x_1.
\tag{B.3}
\]
Not surprisingly, the conclusions will be the same as for the fundamental string in section 4: the geodesic motion is integrable but an extended string motion is not.

In the point-like string limit \( (B.2) \) takes the form (we set \( x' = ry' \) with \( y^2 = 1 \))
\[
\mathcal{L} = \dot{r}^2 + r^2 \dot{y}^2 + p^2 H(r) + \Lambda(y^2 - 1). \tag{B.4}
\]
The corresponding geodesic motion is completely integrable since we have \( d - 1 \) constants of motion from the coordinates \( y' \) (see section 3.1) plus the Hamiltonian.

\textsuperscript{13} For the relevant case of \( h \sim \frac{1}{r^d}, \) the Kovacic algorithm gives no Liouvillian solutions for string modes that depend on \( \sigma \).
Let us now consider an extended string moving only in a 3-space $d\tau/dx_i = dr^2 + r^2(d\theta^2 + \sin^2 \theta
d\phi^2)$ and choose the following ansatz (similar to the one in (3.12)):

$$r = r(\tau), \quad \phi = \phi(\sigma) = \nu \sigma, \quad \theta = \theta(\tau).$$

Then the effective Lagrangian for $r$ and $\theta$ takes the form

$$L = \dot{r}^2 + r^2 \dot{\theta}^2 - v^2 r^2 \sin^2 \theta + p^2 H(r).$$

If we also assume that $v = 0$ (i.e. the string also moves in the light-cone direction), then the Virasoro condition for (B.1) is equivalent to the vanishing of the Hamiltonian corresponding to (B.6)

$$\dot{r}^2 + r^2 \dot{\theta}^2 - v^2 r^2 \sin^2 \theta = 0.$$  \hspace{1cm} (B.7)

Restricting motion to the invariant plane $\{(r, p_r, \theta, p_\theta) : \theta = \frac{\pi}{2}, p_\theta = 0\}$ gives a one-dimensional integrable system parametrized by $v$ and $p$ with the Hamiltonian as the constant of motion. Imposing the zero Hamiltonian constraint (B.7) gives the solution for $r$,

$$\dot{r}^2 = v^2 r^2 H(r), \quad r = \bar{r}(\tau).$$  \hspace{1cm} (B.8)

The Hamiltonian vector field on the plane of these solutions is

$$X^r = \bar{r}, \quad X^\theta = 2 \bar{r} - 2v^2 \bar{r} + p^2 H(\bar{r}), \quad X^\theta = 0, \quad X^{p_r} = 0$$

and the direction normal to this plane is along $\nu$ and $p_\theta$. Expanding the equation of motion for $\theta$ gives the NVE ($\eta = \delta \theta = \theta - \frac{\pi}{2}$):

$$\ddot{\eta} + \frac{2}{r} \dot{\eta} - v^2 \eta = 0.$$  \hspace{1cm} (B.10)

Changing the independent variable $\tau \rightarrow r = \bar{r}(\tau)$, one obtains ($' \equiv \frac{d}{dr}$)

$$\eta'' + \left(\frac{2}{r} + \frac{1}{2} \frac{p^2 H'(r) - v^2 r}{p^2 H(r) - v^2 r^2}\right) \eta' - \frac{v^2}{p^2 H(r) - v^2 r^2} \eta = 0.$$  \hspace{1cm} (B.11)

Bringing this equation to the normal form via the change of variable $\eta(r) = g(r)\xi(r)$, we obtain

$$\xi'' + U(r)\xi = 0,$$

$$U(r) = -(d-2)q^4Q((d-6)Qr^2 + 4(d-3)r^d) + 4q^2r^d[(d^2 - 2d + 2)Qr^2 + 2r^d] + 4r^{2d}$$

$$16q^4Q^2 + r^d - r^{2d})^2$$  \hspace{1cm} (B.12)

where $q = \frac{r}{\bar{r}}$. For $2 < d < 13$ the Kovacic algorithm does not yield Liouvillian solutions for general finite non-zero values of $Q$ and $q$ which implies non-integrability.

Note that in the limit $Q \rightarrow 0$, when the metric becomes flat, we obtain

$$\xi'' + \frac{2q^2 + r^2}{4(q^4 - r^4)} \xi = 0,$$  \hspace{1cm} (B.13)

which has Liouvillian solutions in agreement with flat space integrability.

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