OPTIMAL PARTITIONS
FOR ROBIN LAPLACIAN EIGENVALUES

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Abstract. We prove the existence of an optimal partition for the multiphase shape optimization problem which consists in minimizing the sum of the first Robin Laplacian eigenvalue of \( k \) mutually disjoint open sets which have a \( \mathcal{H}^{d-1} \)-countably rectifiable boundary and are contained into a given box \( D \) in \( \mathbb{R}^d \).

1. Introduction

Aim of this paper is to study a multiphase shape optimization problems with Robin boundary conditions. More precisely, given an open bounded subset \( D \) of \( \mathbb{R}^d \) with Lipschitz boundary, we consider the optimal partition problem

\[
(P) \quad \inf \left\{ \sum_{i=1}^{k} \lambda_1(\Omega_i, \beta) : (\Omega_1, \ldots, \Omega_k) \in \mathcal{A}(D) \right\},
\]

where \( \mathcal{A}(D) \) is the class of \( k \)-tuples of open domains contained into \( D \) such that \( \Omega_i \cap \Omega_j = \emptyset \) for \( i \neq j \), and \( \lambda_1(\Omega, \beta) \) denotes the first Robin Laplacian eigenvalue, for a fixed parameter \( \beta > 0 \). If \( \Omega \) is sufficiently smooth (say Lipschitz), it is defined by

\[
\lambda_1(\Omega, \beta) := \inf_{u \in H^1(\Omega) \backslash \{0\}} \frac{\int_{\Omega} |\nabla u|^2 \, dx + \beta \int_{\partial \Omega} u^2 \, d\mathcal{H}^{d-1}}{\int_{\Omega} u^2 \, dx}.
\]

The analogous to problem \((P)\) in which \( \lambda_1(\Omega, \beta) \) is replaced by the first Dirichlet eigenvalue \( \lambda_1(\Omega) \) of the Laplacian has been extensively studied in the last years. In that case, the starting point of the analysis was the existence result in the class of quasi-open sets given in [6], and then the main existence statements in the class of open sets were proved by Conti-Terracini-Verzini [17, 18, 19] and Caffarelli-Lin [15]; meanwhile and subsequently, regularity results for optimal partitions have been obtained in [15, 17, 24, 27]; without attempt of completeness, further related works are [4, 5, 14, 22, 23].

In contrast, problem \((P)\) that we view as a prototype of a multiphase free discontinuity problem, seems to be completely unexplored. The topic which drove our attention to problem \((P)\) was its connection with optimal Cheeger partitions. In fact, by combining the results about the asymptotics of optimal Cheeger clusters recently obtained in [8, 9] with the relationship between \( \lambda_1(\Omega, \beta) \) and the quotient \( \beta \frac{|\partial \Omega|}{|\Omega|} \), the first and second authors have proved in [7] that, in dimension \( d = 2 \) and when the number of cells \( k \) becomes very large, optimal partitions for problem \((P)\) form a very special “hexagonal” pattern. Namely,
if \( r_k(D, \beta) \) denotes the infimum of problem (P), when the cells \( \Omega_i \) are convex planar sets, it holds
\[
\lim_{k \to +\infty} \frac{|D|^{1/2}}{k^{3/2}} r_k(D, \beta) = \beta h(H),
\]
being \( h(H) \) the Cheeger constant of the unit area regular hexagon (see [26] for an overview about the Cheeger problem). Remarkably, an analogous honeycomb-like asymptotical behaviour of optimal partitions in the Dirichlet case, conjectured by Caffarelli and Lin in [15], is still unproved.

Being this the state of the art, we were led in a natural way to investigate existence and regularity issues for multiphase problems in the Robin case. As a first contribution in this direction, the present work concerns the existence of optimal partitions in the class of open sets. Due to the presence of the boundary term in (1), the shape functional \( \lambda_1(\Omega, \beta) \) behaves quite differently from \( \lambda_1(\Omega) \) (in particular, it lacks monotonicity under domain inclusion), and obtaining the existence of solutions requires a completely different approach. Our strategy moves along the way traced by the first and third authors in some previous works about variational problems for the Robin Laplacian (see [10, 11, 12]). In particular, we settle our partition problem in the class
\[
\mathcal{A}(D) := \left\{ (\Omega_1, \ldots, \Omega_k) : \Omega_i \subset D, \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j \right\}
\]
\( \Omega_i \) open, \( \partial \Omega_i \) \( H^{d-1} \)-countably rectifiable with \( H^{d-1}(\partial \Omega_i) < +\infty \).

Notice carefully that the sets \( \Omega_i \) do not need to be Lipschitz, and in particular they may contain inner cracks. Nevertheless, the definition of first Robin eigenvalue can be extended to any open set \( \Omega \) having a \( H^{d-1} \)-countably rectifiable boundary of finite \( H^{d-1} \) measure by setting
\[
\lambda_1(\Omega, \beta) := \inf_{u \in (H^1(\Omega) \cap L^\infty(\Omega)) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial \Omega} [(u^+)^2 + (u^-)^2] \, d\mathcal{H}^{d-1}}{\int_\Omega u^2 \, dx}.
\]
Here \( u^\pm \) denote the two traces of the \( BV \) function \( u \) (extended to zero outside \( \Omega \)) along \( \partial \Omega \) (see Section 2 for more details). In particular, if \( \Omega \) is Lipschitz, one of the traces is \( 0 \) and the other one coincides with the usual trace of \( u \) in \( H^1(\Omega) \), so that definition (3) gives back (1).

In this setting we obtain the following existence result:

**Theorem 1.** Let the class \( \mathcal{A}(D) \) be given by (2) and, for every \( (\Omega_1, \ldots, \Omega_k) \in \mathcal{A}(D) \), let \( \lambda_1(\Omega_1, \beta) \) be defined by (3). Then problem (P) admits a solution.

The idea of our proof follows the pioneering point of view introduced by Alt and Caffarelli in [4] in order to deal with a one phase free boundary problem with Dirichlet boundary conditions. In fact, we pass through a relaxed formulation of the problem which is a minimization on functions rather than on sets. In view of the weak definition of \( \lambda_1(\Omega, \beta) \) given in (3), the space \( SBV(\mathbb{R}^d) \) of special functions of bounded variation introduced by De Giorgi and Ambrosio in [20] appears as the ideal ambient to study our free discontinuity problem. More precisely, taking into account that we have to model multiple phases, we need to consider the class of vector fields
\[
\mathcal{F}(D) := \left\{ (u_1, \ldots, u_k) \in (SBV^+(\mathbb{R}^d))^k : \text{supp}(u_i) \subseteq \overline{D}, \ u_i \geq 0, \ u_i \cdot u_j = 0 \text{ in } D \right\},
\]
where \( SBV^+(\mathbb{R}^d) \) denotes the space of nonnegative functions \( u \in L^2(\mathbb{R}^d) \) such that \( u^2 \) is in \( SBV(\mathbb{R}^d) \).
Our relaxed functional form of problem \((P)\) reads
\[
\inf \left\{ \frac{\sum_{i=1}^{k} \int_{\mathbb{R}^d} |\nabla u_i|^2 \, dx + \beta \sum_{i=1}^{k} \left[ (u_i^+)^2 + (u_i^-)^2 \right] \, d\mathcal{H}^{d-1}}{\int_{\mathbb{R}^d} u_i^2 \, dx} : (u_1, \ldots, u_k) \in \mathcal{F}(D) \right\}.
\]

The proof of Theorem 1 starts with an existence result for problem \((\overline{P})\) and is carried over in Section 2. Its main steps are highlighted below. For the sake of clarity, we have collected in Section 2 some background material which should allow a self-contained comprehension of the statement as well as of the proof’s outline.

I. **Existence in \(\mathcal{F}(D)\).** Problem \((\overline{P})\) admits a solution. This follows immediately from the compactness and lower semicontinuity properties in \(SBV^d(\mathbb{R}^d)\) proved in [10, Theorem 3.3] for each addendum of the energy in \((\overline{P})\).

II. **Upper and lower bounds on the supports.** If \((u_1, \ldots, u_k) \in \mathcal{F}(D)\) is a solution to problem \((\overline{P})\) and \(\omega_i\) denotes the support of \(u_i\), for every \(i = 1, \ldots, k\) there exist constants \(M_i, \alpha_i > 0\) such that \(M_i \geq u_i \geq \alpha_i\) a.e. on \(\omega_i\). This follows from the optimality of \(u_i\) by taking respectively the competitor \(v_i = u_i 1_{\{u_i \geq \varepsilon\}}\) to get the lower bound, and \(v_i = u_i \wedge M\) to get the upper bound (leaving unchanged the phases \(u_j\) for \(j \neq i\)). In particular, for the lower bound we need to settle a careful energy estimate in order to make work in our framework an iteration scheme which can be traced back in [12] (and has been refined in [13]).

III. **SBV regularity and finite \(\mathcal{H}^{d-1}\) measure of the jump sets.** For every \(i = 1, \ldots, k\), the function \(u_i\) satisfies \(\mathcal{H}^{d-1}(J_{u_i}) < +\infty\) and belongs to \(SBV(\mathbb{R}^d)\). This is obtained as a quite direct consequence of the previous step.

IV. **Essential closedness of the jump sets.** For every \(i = 1, \ldots, k\), the set \(J_{u_i}\) is essentially closed in \(D\). Also the proof of this crucial regularity property exploits the bounds obtained Step II, but it is much more delicate. It relies on a uniform density estimate from below for the supports of \(u_i\) (Lemma 5), which in turn is obtained by applying the Faber-Krahn inequality for the first Robin Laplacian eigenvalue established in [11]. With the aid of the local isoperimetric inequality, such a density lower bound provides some regularity properties for the supports, and gives information concerning their interaction (Corollary 5). The closure property follows by combining these properties with the fact that \(u_i\) is an almost-quasi minimizer for the Mumford-Shah functional “well inside” its support.

V. **Identification of an optimal \(k\)-tuple in \(\mathcal{A}(D)\) and conclusion.** Denoting by \(\Omega_i\) the connected component of \(\mathbb{R}^d \setminus J_{u_i}\) where \(u_i\) does not vanish, thanks to Step IV it turns out that \((\Omega_1, \ldots, \Omega_k)\) belongs to \(\mathcal{A}(D)\) and solves problem \((P)\).

A natural open question which stems from Theorem 1 is to establish the regularity of the free boundaries in an optimal \(k\)-tuple. As a first step in this direction, we show that the jump sets are Ahlfors regular in \(D\) (see Proposition 7). For a one phase problem under Robin boundary conditions, some regularity properties have been recently obtained in [16, 25]; we think it would be interesting to investigate their validity in our multiphase context.

To conclude, let us mention that our results can be extended, with minor modifications in the proofs, to other shape functionals under Robin boundary conditions, such as for instance thermal insulation of multiple obstacles, torsional rigidity or \(p\)-Laplacian energies.
A more challenging target is to deal with optimal partitions for higher eigenvalues of the Robin Laplacian.

2. Preliminaries

2.1. Basic notation. Throughout the paper $B_\rho(x)$ will denote the open ball with center $x \in \mathbb{R}^d$ and radius $\rho > 0$. Given $E \subseteq \mathbb{R}^d$, $1_E$ will stand for its characteristic function, while $|E|$ will denote its Lebesgue measure. In particular we set $\omega_d := |B_1(0)|$. Finally, given $a, b \in \mathbb{R}$, we set $a \land b := \min\{a, b\}$.

2.2. Definition of $\lambda_1(\Omega, \beta)$ for general open sets. Let us recall how the notion of first Robin Laplacian eigenvalue can be extended to domains with possibly irregular boundary. Assume that $\Omega$ is an open set with a $\mathcal{H}^{d-1}$-countably rectifiable boundary of finite $\mathcal{H}^{d-1}$ measure. For $\mathcal{H}^{d-1}$-a.e. $x \in \partial \Omega$, let $\nu(x)$ denote the unit outer normal to $\partial \Omega$ and set $B_\rho^\pm(x, \nu(x)) := \{y \in B_\rho(x) : (y-x) \cdot \nu(x) \geq 0\}$. Given $u \in H^1(\Omega) \cap L^\infty(\Omega)$, extend it to zero outside $\Omega$, and let $u^\pm$ denote its two traces along $\partial \Omega$, defined by

$$u^\pm(x) := \lim_{\rho \to 0} \frac{1}{|B_\rho^\pm(x, \nu(x))|} \int_{B_\rho^\pm(x, \nu(x))} u(y) \, dy.$$  

These limits turn out to exist at $\mathcal{H}^{d-1}$-a.e. $x \in \partial \Omega$ because $u1_\Omega$ belongs to $BV(\mathbb{R}^d)$ (cf. [11 Proposition 4.4]). Then, for every $\beta > 0$ (assumption which will be kept throughout the paper with no further mention), following [11 (3.4)], we set

$$\lambda_1(\Omega, \beta) := \min_{u \in (H^1(\Omega) \cap L^\infty(\Omega)) \setminus \{0\}} \left( \frac{\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial \Omega} [(u^+)^2 + (u^-)^2] \, d\mathcal{H}^{d-1}}{\int_\Omega u^2 \, dx} \right).$$

We point out that, if $\Omega$ is Lipschitz, one of the traces of $u$ is 0 and the other one coincides with the usual trace in $H^1(\Omega)$, so that the above definition gives back [11].

2.3. The space $SBV^+(\mathbb{R}^d)$. In the functional formulation of minimization problems for the above defined Robin eigenvalue, the following space intervenes in a natural way (see [10, Definition 3.1]):

$$SBV^+(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d) : u \geq 0 \text{ a.e. in } \mathbb{R}^d, \ u^2 \in SBV(\mathbb{R}^d) \right\}.$$

Here $SBV(\mathbb{R}^d)$ denotes the space of special functions of bounded variation, namely functions $u \in L^1(\mathbb{R}^d)$ which have bounded variation (meaning that the distributional gradient $Du$ is a measure) and are such that the singular part $D^s u$ of the measure $Du$ is concentrated on the jump set $J_u$. By definition, $J_u$ is the set of points $x \in \mathbb{R}^d$ such that the approximate upper and lower limits $u^\pm(x)$ do not coincide:

$$u^+(x) := \inf \left\{ t \in \mathbb{R} : x \in \{u > t\}^0 \right\}, \quad u^-(x) := \sup \left\{ t \in \mathbb{R} : x \in \{u < t\}^0 \right\},$$

where $E^0$ stands for the set of points $y \in \mathbb{R}^d$ at which $E$ has a zero density, i.e.

$$\lim_{\rho \to 0} \frac{|E \cap B_\rho(y)|}{\rho^d} = 0.$$

Let us recall that every function $u \in SBV(\mathbb{R}^d)$ is approximately differentiable a.e. (with approximate gradient denoted by $\nabla u$), its jump set $J_u$ is $\mathcal{H}^{d-1}$-countably rectifiable, and the distributional gradient $Du$ is given by

$$Du(E) = \int_E \nabla u \, dx + \int_{J_u \cap E} (u^+ - u^-) \nu_u \, d\mathcal{H}^{d-1} \quad \text{for every Borel set } E.$$
Here $\nu_u : J_u \to S^{d-1}$ is a Borel unit normal vector field such that, for $\mathcal{H}^{d-1}$-a.e. $x \in J_u$, the approximate upper and lower limits defined by (5) agree with the traces defined by (4) (taking $\nu = \nu_u$).

A case of special relevance is when $u$ is the characteristic function of a set $E$ of finite perimeter: in that case, the measure $Du$ is purely singular, and the jump set agrees with the essential boundary $\partial^e E$, defined by

$$\partial^e E := \mathbb{R}^d \setminus \left( E^{(1)} \cup E^{(0)} \right),$$

where $E^{(1)}$ stands for the set of points $y \in \mathbb{R}^d$ at which $E$ has a density one.

We refer the reader to the monograph [3] for the theory of functions of bounded variation, and to [10] for fine properties of functions in $SBV^s(\mathbb{R}^d)$.

Here, in order to make sense of the relaxed energy in (4), we limit ourselves to recall that a function $u \in SBV^s(\mathbb{R}^d)$ is approximately differentiable a.e. (with approximate gradient still denoted by $\nabla u$) and that its jump set (still denoted by $J_u$) is $\mathcal{H}^{d-1}$-countably rectifiable; moreover, $\nabla u$ and $J_u$ (the latter endowed with a unit normal vector $\nu_u$) are related respectively to the absolutely continuous and to the singular part of $D(u^2)$ by the identities

$$|\nabla u^2| dx = (2u \nabla u) dx, \quad \text{and} \quad D^a(u^2) = [(u^+)^2 - (u^-)^2] \nu_u \mathcal{H}^{d-1} \mathbf{1}_{J_u}. $$

### 2.4. Faber-Krahn inequalities

The main results of [11] can be summarized in the equalities

$$\min_{u \in SBV^{1/2}(\mathbb{R}^d), \ |\text{supp}(u)|=m} \frac{\int_{\mathbb{R}^d} |\nabla u|^2 dx + \int_{J_u} (|u^+|^2 + |u^-|^2)\, d\mathcal{H}^{d-1}}{\int_{\mathbb{R}^d} u^2 dx} = \min_{|\Omega|=m} \lambda_1(\Omega, \beta) = \lambda_1(B, \beta),$$

where $m > 0$, $B$ is a ball such that $|B| = m$, and $\Omega$ varies in the class of open sets with a $\mathcal{H}^{d-1}$-countably rectifiable boundary of finite $\mathcal{H}^{d-1}$ measure. Minimizers of the functional problem are supported on balls, the function coinciding with the associated first eigenfunction of the Robin-Laplacian. Finally, open sets optimal for $\lambda_1$ coincide with a ball up to $\mathcal{H}^{d-1}$-negligible sets.

### 2.5. Almost-quasi minimizers for the Mumford-Shah functional

We will say that $u \in SBV(\Omega)$ is an almost-quasi minimizer for the Mumford-Shah functional if there exist $0 < \Lambda_1 \leq \Lambda_2$, $\alpha > 0$ and $c_\alpha \geq 0$ such that for every $B_\rho(x) \subset \Omega$ and for every $v \in SBV(\Omega)$ such that $\{v \neq u\} \subseteq B_\rho(x)$ we have

$$\int_{B_\rho(x)} |\nabla u|^2 dx + \Lambda_1 \mathcal{H}^{d-1}(J_u \cap \overline{B_\rho(x)}) \leq \int_{B_\rho(x)} |\nabla v|^2 dx + \Lambda_2 \mathcal{H}^{d-1}(J_v \cap \overline{B_\rho(x)}) + c_\alpha \rho^{d-1+\alpha}.$$

In the case $\Lambda_1 = \Lambda_2$, such a notion reduces to that of quasi-minimizer introduced by De Giorgi, Carriero and Leaci in [21]. Under the strict inequality sign, the notion has been introduced in [13], where it was already applied to analyse a (one phase) free discontinuity problem with Robin boundary conditions.

In our analysis, we will employ the following property [13, Theorem 3.1]: the jump set of an almost-quasi minimizers is essentially closed, i.e.,

$$\mathcal{H}^{d-1}\left( \overline{J_u} \setminus J_u \cap \Omega \right) = 0.$$
3. Proof of Theorem \[ \text{1} \]

3.1. Existence in \(|D|\).

Proposition 2. Problem \((\overline{P})\) admits a solution.

Proof. Let \(u^n = (u^n_i, \ldots, u^n_k)\) be a minimizing sequence in \(|D|\). It is not restrictive to assume that \(\int_{\mathbb{R}^d}(u^n_i)^2 = 1\) for every \(i \in \{1, \ldots, k\}\) and every \(n \in \mathbb{N}\). By comparing with \((1_D, 0, \ldots, 0)\), we get

\[
\sum_{i=1}^{k} \int_{\mathbb{R}^d} |\nabla u^n_i|^2 \, dx + \beta \int_{J_{u^n_i}} \left[ ((u^n_i)^+)^2 + ((u^n_i)^-)^2 \right] \, dH^{d-1} \leq C.
\]

Then, by \([10] \text{ Theorem 3.3}\), up to (not relabeled) sequences, for every \(i = 1, \ldots, k\), the sequence \(u^n_i\) converges strongly in \(L^2_{\text{loc}}(\mathbb{R}^d)\) to a function \(u_i \in \text{SBV}^2(\mathbb{R}^d)\), with

\[
\int_{\mathbb{R}^d} |\nabla u_i|^2 \, dx \leq \lim\inf_{n \to +\infty} \int_{\mathbb{R}^d} |\nabla u^n_i|^2 \, dx
\]

and

\[
\int_{J_{u_i}} \left[ (u_i^+)^2 + (u_i^-)^2 \right] \, dH^{d-1} \leq \lim\inf_{n \to +\infty} \int_{J_{u^n_i}} \left[ ((u^n_i)^+)^2 + ((u^n_i)^-)^2 \right] \, dH^{d-1}.
\]

Notice that the strong convergence of \(u^n_i\) to \(u_i\) in \(L^2_{\text{loc}}(\mathbb{R}^d)\) ensures that \(u\) still satisfies the conditions \(\text{supp}(u_i) \subseteq \overline{D}\), \(u_i \geq 0\), \(\int_{D} u_i^2 \, dx = 1\), and \(u_i \cdot u_j = 0\) in \(D\). Summing the above inequalities over \(i = 1, \ldots, k\), we see that \(u = (u_1, \ldots, u_k)\) is a solution to \((\overline{P})\). \(\square\)

3.2. Upper and lower bounds on the supports.

Proposition 3. Let \(u = (u_1, \ldots, u_k) \in |D|\) be a solution to problem \((\overline{P})\). For every \(i = 1, \ldots, k\), there exist constants \(M_i, \alpha_i\) such that

\[
M_i \geq u_i \geq \alpha_i > 0 \quad \text{a.e. on } \text{supp}(u_i).
\]

Proof. Let us derive the lower and upper bounds separately.

\textbf{Lower bound.} Let \(i\) be a fixed index in \(\{1, \ldots, k\}\). In order to obtain the lower bound in \(\text{[8]}\), it is enough to show that there exists \(\eta_0\) sufficiently small such that

\[
\left| \left\{ \frac{2}{3} \eta < u_i < \frac{5}{6} \eta \right\} \right| = 0 \quad \forall \eta < \eta_0.
\]

To that aim, we implement the same iteration scheme exploited in the proof of \([12] \text{ Theorem 3.5}\). As noticed in \([12] \text{ Remark 3.7}\), such scheme successfully leads to \(\text{[9]}\) as soon as one is able to prove the following key estimate

\[
E(\varepsilon) + c_1 \delta^2 \gamma(\delta, \varepsilon) \leq c_2 \varepsilon^2 h(\varepsilon) \quad \text{for a.e. } 0 < \delta < \varepsilon \leq \varepsilon_0,
\]

where

\[
E(\varepsilon) := \int_{\{u_i < \varepsilon\}} |\nabla u_i|^2 \, dx
\]

\[
\gamma(\delta, \varepsilon) := H^{d-1}(\partial^c \{\delta < u_i < \varepsilon\} \cap J_{u_i})
\]

\[
h(\varepsilon) := H^{d-1}(\partial^c \{u_i \geq \varepsilon\} \setminus J_{u_i}).
\]

Thus, we limit ourselves to show that \(\text{(10)}\) is fulfilled: once this is gained, to obtain \(\text{(9)}\) one can follow exactly the proof of \([12] \text{ Theorem 3.5}\).
In order to prove \([10]\), we compare \(u = (u_1, \ldots, u_k)\) with \(v = (v_1, \ldots, v_k)\) defined by
\[ v_j = u_j \text{ if } j \neq i \quad \text{and} \quad v_i = u_i 1_{\{u_i < \varepsilon\}}. \]
Assuming with no loss of generality that \(\int_{\mathbb{R}^d} u_i^2 = 1\), by the optimality of \(u\) we obtain
\[
\int_{\mathbb{R}^d} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i}} [(u_i^+)^2 + (u_i^-)^2] \, d\mathcal{H}^{d-1} \leq \frac{1}{1 - \int_{\{u_i < \varepsilon\}} u_i^2 \, dx}.
\]
Using the inequality \(\frac{1}{1 - \varepsilon} \leq 1 + 2\delta\) holding for \(\delta \in [0, \frac{1}{2}]\), we deduce that there exists \(\varepsilon_0 > 0\) such that, for \(\varepsilon < \varepsilon_0\),
\[
\int_{\mathbb{R}^d} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i}} [(u_i^+)^2 + (u_i^-)^2] \, d\mathcal{H}^{d-1} \leq \left[ 1 + 2 \int_{\{u_i < \varepsilon\}} u_i^2 \, dx \right] .
\]
In turn, since
\[
\int_{\{u_i \geq \varepsilon\}} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i} \cap \{\varepsilon \leq u_i < u_i^+\}} [(u_i^+)^2 + (u_i^-)^2] \, d\mathcal{H}^{d-1} + \beta \int_{J_{u_i} \cap \{\varepsilon < u_i < u_i^+\}} \frac{(u_i^+)^2}{u_i^2} \, d\mathcal{H}^{d-1} \leq \int_{\mathbb{R}^d} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i}} [(u_i^+)^2 + (u_i^-)^2] \, d\mathcal{H}^{d-1},
\]
it follows that, for a positive constant \(C\) (depending on \(u\) but independent of \(\varepsilon\)),
\[
\int_{\{u_i \leq \varepsilon\}} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i} \cap \partial \{u_i < \varepsilon\}} [(u_i^+)^2 + (u_i^-)^2] \, d\mathcal{H}^{d-1}
\leq \left[ 1 + 2 \int_{\{u_i < \varepsilon\}} u_i^2 \, dx \right] \cdot \left[ \beta \varepsilon^2 \mathcal{H}^{d-1}(\partial^e \{u_i \geq \varepsilon\} \setminus J_{u_i}) \right] + C \int_{\{u_i < \varepsilon\}} u_i^2 \, dx
\leq 3\beta \varepsilon^2 \mathcal{H}^{d-1}(\partial^e \{u_i \geq \varepsilon\} \setminus J_{u_i}) + C \int_{\{u_i < \varepsilon\}} u_i^2 \, dx.
\]
Next we observe that, thanks to the Faber-Krahn inequality \([9]\),
\[
\int_{\{u_i < \varepsilon\}} u_i^2 \, dx \leq \frac{1}{\lambda_1(\{u_i < \varepsilon\}^* \cup \beta)} \int_{\{u_i < \varepsilon\}} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i} \cap \partial \{u_i < \varepsilon\}} [(u_i^+)^2 + (u_i^-)^2] \, d\mathcal{H}^{d-1},
\]
In order to prove that \( u \) is a solution to problem (P), for every \( i = 1, \ldots, k \) the function \( u_i \) satisfies \( \mathcal{H}^{d-1}(\partial^e \{ u_i = \varepsilon \}) < +\infty \) and belongs to \( SBV(\mathbb{R}^d) \).

**Proof.** To prove this, we first define the function

\[
\nu = \begin{cases} 
1 & \text{if } u < \varepsilon \\
0 & \text{if } u \geq \varepsilon 
\end{cases}
\]

where \( \{u_i < \varepsilon\}^* \) denotes a ball having the same volume as \( \{u_i < \varepsilon\} \). We can choose \( \varepsilon \) so small that

\[
\frac{C}{\lambda_1(\{u_i < \varepsilon\}^*, \beta)} < \frac{1}{2}.
\]

Therefore, up to reducing \( \varepsilon_0 \), for \( 0 < \delta < \varepsilon < \varepsilon_0 \) it holds

\[
\int_{\{u_i < \varepsilon\}} |\nabla u_i|^2 \, dx + \beta \varepsilon^2 \mathcal{H}^{d-1}\left( J_{u_i} \cap \partial^e \{ \delta < u_i < \varepsilon \} \right) \leq 6 \beta \varepsilon^2 \mathcal{H}^{d-1}\left( \partial^e \{ u_i \geq \varepsilon \} \setminus J_{u_i} \right).
\]

We have thus shown the validity of (10).

**Upper bound.** Let \( i \) be a fixed index in \( \{1, \ldots, k\} \). Assume by contradiction that \( u_i \notin L^\infty(\mathbb{R}^d) \). For every \( M > 0 \), we compare \( u = (u_1, \ldots, u_k) \) with \( v = (v_1, \ldots, v_k) \in \mathcal{F}(D) \) defined by

\[
v_j = u_j \quad \text{if } j \neq i \quad \text{and} \quad v_i = u_i \wedge M.
\]

By the optimality of \( u \), we have

\[
\int_{\mathbb{R}^d} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i}} \left[ (u_i^+)^2 + (u_i^-)^2 \right] \, d\mathcal{H}^{d-1} \\
\leq \int_{\mathbb{R}^d} u_i^2 \, dx \left[ \int_{\{ u_i \geq M \}} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i} \cap \{ u_i^- < u_i^+ \}} \left[ (u_i^+)^2 + (u_i^-)^2 \right] \, d\mathcal{H}^{d-1} \\
+ \beta \int_{J_{u_i} \cap \{ u_i^- < M < u_i^+ \}} \left[ M^2 + (u_i^-)^2 \right] \, d\mathcal{H}^{d-1} \right].
\]

The above inequality leads to a contradiction by exploiting the assumption that \( |\{u_i > M\}| > 0 \) for every \( M \) and arguing as in the proof of [11, Theorem 6.11].

3.3. **SBV regularity and finite \( \mathcal{H}^{d-1} \) measure of the jump sets.**

**Proposition 4.** If \( u = (u_1, \ldots, u_k) \in \mathcal{F}(D) \) is a solution to problem (P), for every \( i = 1, \ldots, k \) the function \( u_i \) satisfies \( \mathcal{H}^{d-1}(J_{u_i}) < +\infty \) and belongs to \( SBV(\mathbb{R}^d) \).

**Proof.** Up to a normalization, we can assume without loss of generality that \( \int_{\mathbb{R}^d} u_i^2 \, dx = 1 \) for every \( i = 1, \ldots, k \). Since

\[
\sum_{i=1}^k \int_{\mathbb{R}^d} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i}} \left[ (u_i^+)^2 + (u_i^-)^2 \right] \, d\mathcal{H}^{d-1} < +\infty,
\]

and \( u_i \geq \alpha_i > 0 \) a.e. on \( \text{supp}(u_i) \) for every \( i = 1, \ldots, k \), we have

\[
\beta \alpha_i^2 \mathcal{H}^{d-1}(J_{u_i}) \leq \beta \int_{J_{u_i}} \left[ (u_i^+)^2 + (u_i^-)^2 \right] \, d\mathcal{H}^{d-1} < +\infty \quad \forall i = 1, \ldots, k.
\]

In order to prove that \( u_i \in SBV(\mathbb{R}^d) \), we consider the sequence \( u_i^\varepsilon := (u_i^2 + \varepsilon)^{1/2} \). If \( A \) is an open bounded subset of \( \mathbb{R}^d \), we have \( u_i^\varepsilon \in SBV(A) \) and

\[
\sup_{\varepsilon > \varepsilon_1} \left[ \int_A |\nabla u_i^\varepsilon|^2 \, dx + \mathcal{H}^{d-1}(J_{u_i^\varepsilon} \cap A) + \|u_i^\varepsilon\|_{L^\infty(A)} \right] \\
\leq \left[ \int_A |\nabla u_i|^2 \, dx + \mathcal{H}^{d-1}(J_{u_i} \cap A) + (\|u_i\|_{L^\infty(A)}^2 + 1)^{1/2} \right] < +\infty,
\]

for every \( \varepsilon > 0 \) and \( 0 < \varepsilon_1 \).
where the last inequality follows from the upper bound in Proposition 3. By Ambrosio’s compactness theorem \cite{Ambrosio}, since $u_i^0 \to u_i$ in $L^1(A)$, we deduce that $u_i \in SBV(A)$. Moreover, the estimate

$$|Du_i|(A) \leq \int_A |\nabla u_i| \, dx + 2\|u_i\|_{\infty} \mathcal{H}^{d-1}(J_{u_i} \cap A) \leq |D|^{1/2} \left( \int_{\mathbb{R}^d} |\nabla u_i|^2 \, dx \right)^{1/2} + 2\|u_i\|_{\infty} \mathcal{H}^{d-1}(J_{u_i})$$

ensures that $|Du_i|(\mathbb{R}^d) < +\infty$ and therefore $u_i \in SBV(\mathbb{R}^d)$. \qed

3.4. **Essential closedness of the jump sets.** In the following we denote by $\omega_i$ the support of $u_i$. We will refer to $\{\omega_1, \ldots, \omega_k\}$ as the phases of our problem.

**Lemma 5 (Density lower bound for the phases).** If $u = (u_1, \ldots, u_k) \in F(D)$ is a solution to problem (T), for every $i = 1, \ldots, k$ there exist a constant $c_i > 0$ and a radius $\rho_i > 0$ such that the following property holds true: for every $x \in \mathbb{R}^d$ such that $|B_{\rho}(x) \cap \omega_i| > 0$ for every $\rho > 0$, we have

$$\frac{|\omega_i \cap B_{\rho}(x)|}{\rho^d} \geq c_i \quad \text{for every } \rho \in (0, \rho_i).$$

**Proof.** Let $i \in \{1, \ldots, k\}$ be fixed, and let $x \in \mathbb{R}^d$ such that $|B_{\rho}(x) \cap \omega_i| > 0$ for every $\rho > 0$. We compare $u = (u_1, \ldots, u_k)$ with $v = (v_1, \ldots, v_k) \in F(D)$ defined by

$v_j = u_j$ if $j \neq i$ and \quad $u_i = u_i \mathbf{1}_{\mathbb{R}^d \setminus B_{\rho}(x)}$.

We assume without loss of generality that $\int_{\mathbb{R}^d} u_i^2 \, dx = 1$, and we write for brevity $B_{\rho}$ in place of $B_{\rho}(x)$. In order to compute the energy of $v$ we apply Theorem 3.84 in \cite{Ambrosio}, and in particular we denote by $(u_i^+)^2$ the outer trace of $u_i^2$ on $\partial B_{\rho}$ defined for $\mathcal{H}^{d-1}$-a.e. $x \in \partial B_{\rho}$ according to \cite{Ambrosio}. By the optimality of $u$ we obtain

$$\int_{\mathbb{R}^d} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i}} [(u_i^+)^2 + (u_i^-)^2] \, d\mathcal{H}^{d-1} \leq \frac{\int_{\mathbb{R}^d \setminus B_{\rho}} |\nabla u_i|^2 \, dx + \beta \int_{B_{\rho}} [(u_i^+)^2 + (u_i^-)^2] \, d\mathcal{H}^{d-1} + \beta \int_{\partial B_{\rho}} (u_i^+)^2 \, d\mathcal{H}^{d-1} + \beta \int_{\partial B_{\rho}} (u_i^-)^2 \, d\mathcal{H}^{d-1}}{1 - \int_{B_{\rho}} u_i^2 \, dx}.$$  

Using the inequality $\frac{1}{1-\delta} \leq 1 + 2\delta$ holding for $\delta \in [0, \frac{1}{2}]$, we deduce that there exists $\rho_0 > 0$ such that, for $\rho < \rho_0$,

$$\int_{\mathbb{R}^d} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i}} [(u_i^+)^2 + (u_i^-)^2] \, d\mathcal{H}^{d-1} \leq \left[1 + 2 \int_{B_{\rho}} u_i^2 \, dx\right].$$

$$\cdot \left[\int_{\mathbb{R}^d \setminus B_{\rho}} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i}} [(u_i^+)^2 + (u_i^-)^2] \, d\mathcal{H}^{d-1} + \beta \int_{\partial B_{\rho}} (u_i^+)^2 \, d\mathcal{H}^{d-1} + \beta \int_{\partial B_{\rho}} (u_i^-)^2 \, d\mathcal{H}^{d-1}\right].$$

In turn, since

$$\int_{\mathbb{R}^d \setminus B_{\rho}} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i}} [(u_i^+)^2 + (u_i^-)^2] \, d\mathcal{H}^{d-1} \leq \int_{\mathbb{R}^d} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i}} [(u_i^+)^2 + (u_i^-)^2] \, d\mathcal{H}^{d-1},$$
and in view of the upper bound in \(3\), it follows that, for a positive constant \(C\) independent of \(\rho\),

\[
\int_{B_\rho} |\nabla u_i|^2 \, dx + \beta \int_{B_\rho \cap J_{u_i}} \left[ (u_i^+)^2 + (u_i^-)^2 \right] \, d\mathcal{H}^{d-1} \leq C \int_{B_\rho} u_i^2 \, dx + 3\beta \int_{\partial B_\rho} (u_i^+)^2 \, d\mathcal{H}^{d-1}.
\]

Adding to both sides the term \(\beta \int_{\partial B_\rho} (u_i^-)^2 \, d\mathcal{H}^{d-1}\), where \(u_i^-\partial B_\rho\) denotes the inner trace on \(\partial B_\rho\), thanks to the Faber-Krahn inequality \(3\) we obtain the estimate

\[
\lambda_1(\{\omega_i \cap B_\rho\}^*, \beta) \int_{B_\rho} u_i^2 \, dx \leq \int_{B_\rho} |\nabla u_i|^2 \, dx + \beta \int_{B_\rho \cap J_{u_i}} \left[ (u_i^+)^2 + (u_i^-)^2 \right] \, d\mathcal{H}^{d-1} + \beta \int_{\partial B_\rho} (u_i^-)^2 \, d\mathcal{H}^{d-1} \leq C \int_{B_\rho} u_i^2 \, dx + 3\beta \int_{\partial B_\rho} (u_i^+)^2 \, d\mathcal{H}^{d-1},
\]

where \(\{\omega_i \cap B_\rho\}^*\) denotes a ball with the same volume as \(\omega_i \cap B_\rho\), that is with radius

\[
r_i := \frac{|\omega_i \cap B_\rho|^{1/d}}{\omega_d^d}. \tag{14}
\]

Notice that there exists a positive constant \(C'\) (independent of \(x\)) such that

\[
\lambda_1(B_{r_i}, \beta) \sim \frac{C'}{|\omega_i \cap B_\rho|^{1/d}} \quad \text{as } \rho \to 0^+ . \tag{15}
\]

Indeed, we have

\[
\lambda_1(B_{r_i}, \beta) = \lambda_1(r_i B_1, \beta) = \frac{1}{r_i^d} \lambda_1(B_1, \beta r_i) = \frac{\beta}{r_i} \frac{1}{\beta r_i} \lambda_1(B_1, \beta r_i) \sim \frac{\beta}{r_i} \frac{|\partial B_1|}{|B_1|}.
\]

For the latter asymptotic equivalence, see \(7\) Proposition 9 and references therein.

Recalling the upper and lower bounds in \(3\), we deduce from \(13\), \(14\), and \(15\) that there exists a positive constant \(C''\) (independent of \(x\)) such that the function \(\theta_i(\rho) := |\omega_i \cap B_\rho|\) satisfies, for \(\rho\) sufficiently small, the differential inequality

\[
\theta_i^{1-\frac{2}{d}}(\rho) = \frac{|\omega_i \cap B_\rho|^{1-\frac{2}{d}}}{\omega_d^{d-2}} \leq C'' |\omega_i \cap \partial B_\rho| = C'' \theta_i(\rho) .
\]

Hence \(\theta_i(\rho) \geq C'' \rho^d\) for \(\rho\) sufficiently small, and the proof of \(11\) is achieved.

\[\square\]

**Corollary 6.** If \(u = (u_1, \ldots, u_k) \in \mathcal{F}(D)\) is a solution to problem \((\mathcal{F})\), for every \(i = 1, \ldots, k\) the following items hold true.

- \((a)\) \(\partial^c \omega_i \subseteq J_{u_i}\).
- \((b)\) \(\omega_i^{(0)}\) is open.
- \((c)\) \(C_i := \mathbb{R}^d \setminus \omega_i^{(0)}\) is closed with \(J_{u_i} \subseteq C_i\) and such that for every \(i \neq j\)

\[
C_i \cap C_j = \partial^c \omega_i \cap \partial^c \omega_j = J_{u_i} \cap J_{u_j} . \tag{16}
\]

**Proof.**

- \((a)\) The inclusion \(\partial^c \omega_i \subseteq J_{u_i}\) comes from the lower bound in \(3\).
(b) Let \( x \in \omega_i^{(0)} \), and assume by contradiction that there exists \( x_n \notin \omega_i^{(0)} \) with \( x_n \to x \). Then \( x_n \) is a point which satisfies Lemma 4. But then by (11) we infer that for every \( \rho < \rho_i \)

\[
\left| \omega_i \cap B_\rho(x) \right| \rho^d = \lim_{n \to \infty} \left| \omega_i \cap B_\rho(x_n) \right| \rho^d \geq c_i,
\]

so that \( x \) cannot have zero density with respect to \( \omega_i \).

(c) Clearly \( J_{u_i} \subseteq C_i \) since jump points have positive density for \( \omega_i \). Moreover, if \( x \in C_i \cap C_j \), we have that \( x \) has positive density with respect to both \( \omega_i \) and \( \omega_j \) so that \( x \in \partial^r \omega_i \cap \partial^r \omega_j \subseteq J_{u_i} \cap J_{u_j} \). We thus obtain equality (16).

\[\square\]

**Lemma 7 (Almost-quasi minimality).** Let \( u = (u_1, \ldots, u_k) \in \mathcal{F}(D) \) be a solution to problem (P). Then for every \( i = 1, \ldots, k \), the function \( u_i \) is an almost quasi minimizer for the Mumford-Shah functional (see Section 2.5) in the open set \( D \cap \bigcap_{j \neq i} \omega_j^{(0)} \).

**Proof.** Let us set for brevity

\[\hat{\omega}_i := \bigcap_{j \neq i} \omega_j^{(0)} .\]

Let us show that there exists a constant \( c > 0 \) such that, for any \( y \in \hat{\omega}_i \cap D \) and any \( v_i \in SBV(\hat{\omega}_i \cap D) \) with \( \{ v_i \neq u_i \} \subseteq B_\rho(y) \subset \hat{\omega}_i \cap D \), there holds

\[\int_{B_\rho(y)} |\nabla u_i|^2 dx + \beta \alpha_i^2 \mathcal{H}^{d-1}(J_{u_i} \cap \overline{B_\rho(y)}) \leq \int_{B_\rho(y)} |\nabla v_i|^2 dx + 2 \beta \| u_i \|^2_{\infty} \mathcal{H}^{d-1}(J_{v_i} \cap \overline{B_\rho(y)}) + c\rho^d,\]

where \( \alpha_i \) is given in (8). To that aim we can assume without loss of generality that

\[\int_{B_\rho(y)} |\nabla v_i|^2 dx + 2 \beta \| u_i \|^2_{\infty} \mathcal{H}^{d-1}(J_{v_i} \cap \overline{B_\rho(y)}) \leq \int_{B_\rho(y)} |\nabla u_i|^2 dx + \beta \alpha_i^2 \mathcal{H}^{d-1}(J_{u_i} \cap \overline{B_\rho(y)}).\]

We consider the function \( \tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_k) \) defined on \( D \) by

\[\tilde{v}_j := \begin{cases} u_j & \text{if } j \neq i \\ u_i1_{B_\rho \setminus B_\rho(y)} + [\| v_i \| + \| u_i \|_{\infty}]1_{B_\rho(y)} & \text{if } j = i. \end{cases}\]

Since \( \{ v_i \neq u_i \} \subseteq B_\rho(y) \subset \hat{\omega}_i \cap D \), the function \( \tilde{v} \) defines an element of \( \mathcal{F}(D) \). Let us take it as a competitor in problem (P). Exploiting the optimality of \( u \) for problem (P), the fact that \( \tilde{v}_j = u_j \) for every \( j \neq i \), and recalling that \( \{ v_i \neq u_i \} \subseteq B_\rho(y) \), we obtain (assuming without restriction that \( \int_{\mathbb{R}^d} u_i^2 dx = 1 \))

\[
\int_{\mathbb{R}^d} |\nabla u_i|^2 dx + \beta \int_{J_{u_i}} [(u_i^+)^2 + (u_i^-)^2] d\mathcal{H}^{d-1} \leq \frac{\int_{\mathbb{R}^d} |\nabla \tilde{v}_i|^2 dx + \beta \int_{\tilde{J}_{\tilde{v}_i}} [(\tilde{v}_i^+)^2 + (\tilde{v}_i^-)^2] d\mathcal{H}^{d-1}}{1 - \int_{B_\rho(y)} (u_i^2 - \tilde{v}_i^2) dx}.
\]
Then, by using the estimate
\[ |\int_{B_\rho(y)} (u_i^2 - \tilde{v}_i^2) \, dx| \leq 2\|u_i\|_{\infty}^2 \omega_d \rho^d, \]
and the inequality \( \frac{1}{1-\delta} \leq 1 + 2\delta \) holding for \( \delta \in [0, \frac{1}{2}] \), we obtain, for \( \rho \) sufficiently small,
\[ \int_{\mathbb{R}^d} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i}} [(u_i^+)^2 + (u_i^-)^2] \, d\mathcal{H}^{d-1} \leq \left[ \int_{\mathbb{R}^d} |\nabla \tilde{v}_i|^2 \, dx + \beta \int_{J_{\tilde{v}_i}} [(\tilde{v}_i^+)^2 + (\tilde{v}_i^-)^2] \, d\mathcal{H}^{d-1} \right] \cdot \left[ 1 + 2\|u_i\|_{\infty}^2 \omega_d \rho^d \right]. \]
In view of (13) and of the definition of \( \tilde{v}_i \), we see that
\[ \int_{\mathbb{R}^d} |\nabla \tilde{v}_i|^2 \, dx + \beta \int_{J_{\tilde{v}_i}} [(\tilde{v}_i^+)^2 + (\tilde{v}_i^-)^2] \, d\mathcal{H}^{d-1} \leq C', \]
for some positive constant \( C' \) depending on \( u \).

From (19) and (20) we deduce that, provided \( \rho \) is sufficiently small, the required inequality (17) is satisfied for some positive constant \( C \) (depending on \( u \)). Since the left hand side of such inequality is bounded in \( \rho \), up to increasing \( C \) we infer that it continues to hold for every \( \rho > 0 \) such that \( B_{\rho}(y) \subset \tilde{\omega}_i \cap D \). The proof is thus concluded. \( \square \)

**Proposition 8 (Essential closedness of the jump sets).** If \( u = (u_1, \ldots, u_k) \in \mathcal{F}(D) \) is a solution to problem (\( \overline{P} \)), for every \( i = 1, \ldots, k \) the set \( J_{u_i} \) is essentially closed in \( D \), i.e.,
\[ \mathcal{H}^{d-1}(\overline{J_{u_i}} \setminus J_{u_i} \cap D) = 0. \]

**Proof.** The relation
\[ \mathcal{H}^{d-1}(\overline{J_{u_i}} \setminus J_{u_i} \cap D \cap \bigcap_{j \neq i} \omega_j^{(0)}) = 0 \]
is a consequence of the almost-quasi minimality of Lemma 7 (see Section 2.5). On the other hand, recalling point (c) of Corollary 6,
\[ \overline{J_{u_i}} \setminus \bigcap_{j \neq i} \omega_j^{(0)} = J_{u_i} \setminus \bigcup_{j \neq i} C_j \subseteq C_i \cap \bigcup_{j \neq i} C_j \subseteq J_{u_i} \]
which entails
\[ \bigcap_{j \neq i} \omega_j^{(0)} = \emptyset. \]
The conclusion follows gathering (22) and (23). \( \square \)

**Proposition 9 (Ahlfors regularity).** If \( u = (u_1, \ldots, u_k) \in \mathcal{F}(D) \) is a solution to problem (\( \overline{P} \)), for every \( i = 1, \ldots, k \) the set \( J_{u_i} \) is Ahlfors regularity in \( D \), that is there exist a constant \( k_i > 0 \) and a radius \( r_i > 0 \) such that, for every \( x \in J_{u_i} \) and every \( \rho \in (0, r_i] \) such that \( B_{\rho}(x) \subset D \), there holds
\[ k_i \rho^{d-1} \leq \mathcal{H}^{d-1}(J_{u_i} \cap B_{\rho}(x)) \leq \frac{1}{k_i} \rho^{d-1}. \]
Proof. In order to prove the upper bound inequality in (24), we proceed as in the proof of Lemma 3 until we arrive at inequality (23). Using such inequality and Proposition 1, we obtain

\[
\beta \alpha_i^2 \mathcal{H}^{d-1}(J_{u_i} \cap B_\rho(x)) \leq \int_{B_\rho(x)} |\nabla u_i|^2 \, dx + \beta \int_{B_\rho(x) \cap J_{u_i}} [(u_i^+)^2 + (u_i^-)^2] \, d\mathcal{H}^{d-1}
\]

\[
\leq C \int_{B_\rho} u_i^2 + 3 \beta \int_{\partial B_\rho(x)} (u_i^+)^2 \, d\mathcal{H}^{d-1}
\]

\[
\leq C M_i^2 \omega d \rho^d + 3 \beta M_i^2 d \omega d \rho^{d-1},
\]

which clearly implies the validity of the upper bound inequality in (24) for \( \rho \) sufficiently small.

Concerning the lower bound, let us employ the following notation:

\[
\hat{\omega}_i := \bigcap_{j \neq i} \omega_j^{(0)}, \quad \delta_i(x) := d(x, \partial \hat{\omega}_i).
\]

Let us fix \( x \in D \), and let us distinguish the two cases

- **Case 1.** Let \( x \in J_{u_i} \setminus \hat{\omega}_i \). By Corollary 3, we have \( x \in C_i \cap C_j \) for some \( j \neq i \). In view of the relative isoperimetric inequality

\[
(25) \quad \left( \min \left\{ |B_\rho(x) \cap \omega_i|, |B_\rho(x) \setminus \omega_i| \right\} \right)^{\frac{1}{d-1}} \leq c_d \mathcal{H}^{d-1}(\partial^e \omega_i \cap B_\rho(x)),
\]

together with the inequality \( |B_\rho(x) \setminus \omega_i| \geq |B_\rho(x) \cap \omega_j| \) and the density lower bound (11) for \( \omega_i \) and \( \omega_j \) at the point \( x \), we infer that there exist \( \rho_i'^{} > 0 \) and \( k_i' > 0 \) (independent of \( x \)) such that for every \( \rho < \rho_i'^{} \)

\[
\mathcal{H}^{d-1}(J_{u_i} \cap B_\rho(x)) \geq \mathcal{H}^{d-1}(\partial^e \omega_i \cap B_\rho(x)) \geq k_i' \rho^{d-1}.
\]

- **Case 2.** Let \( x \in J_{u_i} \cap \hat{\omega}_i \). Recall that by Lemma 7, the function \( u_i \) is an almost-quasi minimizer for the Mumford-Shah functional in the open set \( D \setminus \hat{\omega}_i \), which entails the Ahlfors regularity of its jump set (see [13, Section 3.2]). As a consequence, there exist a radius \( \rho_i'' > 0 \) and a constant \( k_i'' > 0 \) (independent of \( x \)) such that for every \( \rho < \rho_i'' \wedge \delta_i(x) \)

\[
(26) \quad \mathcal{H}^{d-1}(J_{u_i} \cap B_\rho(x)) \geq k_i'' \rho^{d-1}.
\]

Let us extend this lower bound on balls \( B_\rho(x) \) contained in \( D \) (and not only in \( D \setminus \hat{\omega}_i \)).

To that aim we first remark that, up to changing \( k_i'' \) into \( k_i''/m^{d-1} \), the validity of (26) can be extended to radii \( \rho \in (0, m(\rho_i'' \wedge \delta_i(x))) \) for any \( m \in \mathbb{N} \). Indeed, by applying (26) with \( m \) in place of \( \rho \) we get:

\[
(27) \quad \mathcal{H}^{d-1}(J_{u_i} \cap B_\rho(x)) \geq \mathcal{H}^{d-1}(J_{u_i} \cap B_{\frac{\rho}{m}}(x)) \geq \frac{k_i'}{m^{d-1}} \rho^{d-1}
\]

for every \( \rho \in (0, m(\rho_i'' \wedge \delta_i(x))) \). Since we can choose \( m \) large enough (independent of \( x \)) so that

\[
(28) \quad m(\rho_i'' \wedge \delta_i(x)) \geq 2 \delta_i(x),
\]

we are reduced to show the lower bound inequality in (24) for the radii \( \rho \geq 2 \delta_i(x) \) such that \( B_\rho(x) \subset D \). For such a radius, we can proceed in a similar way as done for points \( x \in J_{u_i} \setminus \hat{\omega}_i \), namely we prove the inequality with \( J_{u_i} \) replaced by \( \partial^e \omega_i \).
To such purpose, let \( x_j \in C_j \) be a point such that \( |x_j - x| = \delta_i(x) \). Then we may write thanks to Lemma \[ \Box \]

\[ |B_\rho(x) \setminus \omega_i| \geq |\omega_j \cap B_\rho(x)| \geq |\omega_j \cap B_{\rho-\delta_i(x)}(x_j)| \geq c_j(\rho - \delta_i(x))^d \geq \frac{c_j}{2d} \rho^d. \]

Thanks to the isoperimetric inequality \[ \Box \], we infer that, for any \( \rho \geq \delta_i(x), \mathcal{H}^{d-1}(\partial \omega_i \cap B_\rho(x)) \), and hence \( \mathcal{H}^{d-1}(J_{u_i} \cap B_\rho(x)) \) is bounded from below by a constant (independent of \( x \)) times \( \rho^{d-1} \). Combining this assertion with \[ \Box \]–\[\Box\], we have achieved the proof of the lower bound inequality in \[ \Box \] also for points \( x \in J_{u_i} \cap \hat{\omega}_i \).

\[ \Box \]

### 3.5. Identification of an optimal \( k \)-tuple in \( A(D) \) and conclusion.

**Proposition 10.** If \( u = (u_1, \ldots, u_k) \in \mathcal{F}(D) \) is a solution to problem \( \mathcal{P} \), there exists a \( k \)-tuple of open connected sets \( (\Omega_1, \ldots, \Omega_k) \in A(D) \) such that, for every \( i = 1, \ldots, k \),

\[ \partial \Omega_i = \overline{\mathcal{J}_{u_i}}, \quad \mathcal{H}^{d-1}(\partial \Omega_i \setminus J_{u_i}) \cap D = 0, \quad u_i = 0 \text{ a.e. on } \mathbb{R}^d \setminus \Omega_i. \]

Moreover, for every \( i = 1, \ldots, k \), the function \( u_i \) belongs to \( H^1(\Omega_i) \cap L^\infty(\Omega_i) \) and satisfies \( u_i \geq \alpha_i > 0 \) a.e. on \( \Omega_i \).

**Proof.** We define \( \Omega_i \) as the union of the connected components of \( \mathbb{R}^d \setminus \overline{\mathcal{J}_{u_i}} \) where \( u_i \) is not identically zero. Clearly, by construction, \( \Omega_i \subseteq D \) is open, it satisfies \( \partial \Omega_i = \overline{\mathcal{J}_{u_i}}, u_i = 0 \) a.e. on \( \mathbb{R}^d \setminus \Omega_i \), and \( \Omega_i \cap \Omega_j = \emptyset \) for \( j \neq i \) (the latter condition comes from \( u_i \cdot u_j = 0 \) for \( j \neq i \)).

The property that \( \partial \Omega_i \setminus J_{u_i} \cap D \) is \( \mathcal{H}^{d-1} \)-negligible follows from Proposition \[ \Box \] Moreover, since

\[ \partial \Omega_i \subseteq \partial D \cup (\overline{\mathcal{J}_{u_i}} \cap D), \]

we infer \( \Omega_i \in A(D) \).

The fact that \( \Omega_i \) is connected can be easily proved by contradiction. Namely, if \( \Omega_i = \Omega'_i \cup \Omega''_i \), with \( \Omega'_i \) and \( \Omega''_i \) nonempty disjoint open sets, letting \( u'_i := u_i \cdot 1_{\Omega'_i} \) and \( u''_i := u_i \cdot 1_{\Omega''_i} \), considering one of the two \( k \)-tuples \( (u_1, \ldots, u'_i, \ldots, u_k) \) and \( (u_1, \ldots, u''_i, \ldots, u_k) \), and arguing as in the proof of \[ \Box \] Theorem 6.15 leads to contradict the optimality of \( (u_1, \ldots, u_i, \ldots, u_k) \) for problem \( \mathcal{P} \).

Finally, from the definition of \( \Omega_i \) we deduce that \( u_i \in H^1(\Omega_i) \), and the remaining part of the statement follows from Proposition \[ \Box \]

**Conclusion of the proof of Theorem \[ \Box \]** Let \( u = (u_1, \ldots, u_k) \in \mathcal{F}(D) \) be a solution to problem \( \mathcal{P} \), and let \( (\Omega_1, \ldots, \Omega_k) \) be a \( k \)-tuple of open connected sets as in Proposition \[ \Box \] Let us show that \( (\Omega_1, \ldots, \Omega_k) \) solves problem \( \mathcal{P} \), namely, for every \( k \)-tuple \( (A_1, \ldots, A_k) \in \mathcal{A}(D) \), it holds

\[ \sum_{i=1}^k \lambda_1(\Omega_i, \beta) \leq \sum_{i=1}^k \lambda_1(A_i, \beta). \]

First we observe that, since \( u_i \in H^1(\Omega_i) \cap L^\infty(\Omega_i) \) with \( u_i \geq \alpha_i > 0 \) a.e. on \( \Omega_i \) (cf. Proposition \[ \Box \]), \( u_i \) is admissible in the minimization problem which defines \( \lambda_1(\Omega_i, \beta) \) according to \[ \Box \].
Notice that
\[
\int_{\partial \Omega_i \cap \partial D} \left[ (u_i^+)^2 + (u_i^-)^2 \right] d\mathcal{H}^{d-1} = \int_{J_{u_i} \cap \partial D} \left[ (u_i^+)^2 + (u_i^-)^2 \right] d\mathcal{H}^{d-1}
\]
\[= \int_{J_{u_i} \cap \partial D} \left[ (u_i^+)^2 + (u_i^-)^2 \right] d\mathcal{H}^{d-1},
\]
since \(u_i^+ = 0\) \(\mathcal{H}^{d-1}\)-a.e. on \((J_{u_i} \setminus J_{u_i}) \cap \partial D\) (thanks to the regularity assumed on \(D\)). We thus may write
\[
\int_{\partial \Omega_i} \left[ (u_i^+)^2 + (u_i^-)^2 \right] d\mathcal{H}^{d-1} = \int_{\partial \Omega_i} \left[ (u_i^+)^2 + (u_i^-)^2 \right] d\mathcal{H}^{d-1} + \int_{\partial \Omega_i \cap \partial D} \left[ (u_i^+)^2 + (u_i^-)^2 \right] d\mathcal{H}^{d-1}
\]
\[= \int_{J_{u_i}} \left[ (u_i^+)^2 + (u_i^-)^2 \right] d\mathcal{H}^{d-1} + \int_{J_{u_i} \cap \partial D} \left[ (u_i^+)^2 + (u_i^-)^2 \right] d\mathcal{H}^{d-1}
\]
\[= \int_{J_{u_i}} \left[ (u_i^+)^2 + (u_i^-)^2 \right] d\mathcal{H}^{d-1},
\]
where the last equality follows from (21) in Proposition 8. We deduce
\[
\sum_{i=1}^{k} \lambda_1(\Omega_i, \beta) \leq \sum_{i=1}^{k} \left[ \int_{\Omega_i} |\nabla u_i|^2 \, dx + \beta \int_{\partial \Omega_i} \left[ (u_i^+)^2 + (u_i^-)^2 \right] d\mathcal{H}^{d-1} \right] \frac{\int_{\Omega_i} u_i^2 \, dx}{\int_{\Omega_i} u_i^2 \, dx}
\]
\[= \sum_{i=1}^{k} \left[ \int_{\Omega_i} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i}} \left[ (u_i^+)^2 + (u_i^-)^2 \right] d\mathcal{H}^{d-1} \right] \frac{\int_{\Omega_i} u_i^2 \, dx}{\int_{\Omega_i} u_i^2 \, dx}
\]
\[= \sum_{i=1}^{k} \left[ \int_{\Omega_i} |\nabla u_i|^2 \, dx + \beta \int_{J_{u_i}} \left[ (u_i^+)^2 + (u_i^-)^2 \right] d\mathcal{H}^{d-1} \right] \frac{\int_{\Omega_i} u_i^2 \, dx}{\int_{\Omega_i} u_i^2 \, dx}
\]
\[\leq \sum_{i=1}^{k} \left[ \int_{A_i} |\nabla \tilde{w}_i|^2 \, dx + \beta \int_{\partial A_i} \left[ (\tilde{w}_i^+)^2 + (\tilde{w}_i^-)^2 \right] d\mathcal{H}^{d-1} \right] \frac{\int_{A_i} \tilde{w}_i^2 \, dx}{\int_{A_i} \tilde{w}_i^2 \, dx}
\]
\[\leq \sum_{i=1}^{k} \left[ \int_{A_i} |\nabla w_i|^2 \, dx + \beta \int_{\partial A_i} \left[ (w_i^+)^2 + (w_i^-)^2 \right] d\mathcal{H}^{d-1} \right] \frac{\int_{A_i} w_i^2 \, dx}{\int_{A_i} w_i^2 \, dx}
\]
where the first inequality follows from the optimality of \(u\) for problem (\(\overline{P}\)), and the second one from the definition of \(\tilde{w}_i\) (which ensures in particular that \(\mathcal{H}^{d-1}(J_{\tilde{w}_i} \setminus \partial A_i) = 0\)).

By combining (30) and (31) we obtain
\[
\sum_{i=1}^{k} \lambda_1(\Omega_i, \beta) \leq \sum_{i=1}^{k} \left[ \int_{A_i} |\nabla w_i|^2 \, dx + \beta \int_{\partial A_i} \left[ (w_i^+)^2 + (w_i^-)^2 \right] d\mathcal{H}^{d-1} \right] \frac{\int_{A_i} w_i^2 \, dx}{\int_{A_i} w_i^2 \, dx},
\]
and the required inequality (29) follows by passing to the infimum over \((w_1, \ldots, w_k)\).
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