Deflection angle and quasinormal modes of a de Sitter black hole in $f(\mathcal{T}, \mathcal{B})$ gravity

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This work is intended to investigate the influence of the boundary term on the bending angle of light for a static spherically symmetric black hole in $f(\mathcal{T}, \mathcal{B})$ modified gravity. To this end, we use the Ishihara et al. method which allows us to derive the deflection angle of light for an observer and source at finite distances from a lens object in a non-asymptotically flat spacetime. This method interprets the viewpoint of the receiver. The obtained deflection angle becomes divergent at far distances from the lens object, which is due to the non-asymptotically flat spacetime. However, the divergence of the deflection angle can be controlled with the boundary term parameter $c_0$. For small values of the parameter $c_0$ this divergence can be minimized within the finite range of the source and observer. We also calculate the quasinormal modes of axial gravitational perturbations in the background of the black hole using the Padé averaged sixth order WKB approximation method. We observed that the boundary term of the model has notable influence on the quasinormal modes of the black hole. It is seen that for the physically perceptible quasinormal mode frequencies from the black hole, the value of the boundary term parameter $c_0$ should be less than 0.08. This result in fact supports the outcome of our deflection angle analysis.

PACS numbers:
Keywords: Modified Gravity; Deflection angle; Quasinormal modes, Black holes

I. INTRODUCTION

The revolutionary idea of the theory of general relativity (GR) first struck in the great mind of Albert Einstein, which he published in 1915 [1]. After the time of publication of this theory, it has gained utmost importance in describing many gravitational phenomena. Among different predictions of GR, black holes and gravitational waves (GWs) are two most significant ones. However, it has shown some limitations while explaining the current accelerated expansion of the Universe [2, 3] as well as the observed rotational dynamics of the galaxies [4–8]. GR needs some exotic stuff called dark energy and dark matter [9–14] in order to overcome its limitations. Thus, in order to represent the observed facts of the present universe, it can be conjectured that either the universe is filled with mysterious dark matter and dark energy, or a plausible modified theory of gravity (MTG) should be used such that it can explain the correct outcomes from observations. At present there are lots of proposed MTGs that are introduced to resolve unexplained issues of the universe [15–24].

The first experimental verification of GR was observed in 1919. It was the observation of the gravitational bending of light, during a solar eclipse [25] as per the prediction of GR. Moreover, from the interpretation of gravitational bending of light in the framework of geometrical optics of a lens, a key idea, the gravitational lensing has emerged [26–29]. It was in 1979, when the first instance of gravitationally lensed object, the double quasar Q0957 + 561 A,B was identified for the first time [30]. In the modern observational cosmology for exploring the exoplanets and for measuring the distribution of dark matter and dark energy, an important parameter is the angle of gravitational bending of light. For different black holes, the gravitational lensing around them has been investigated in a plethora of articles. For the case of Schwarzschild black hole such investigation was carried out in the Ref. [31]. For black holes in presence of the cosmological constant the studies of such lensing are reported in Ref.s [32, 33]. Again, the gravitational lensing by the naked singularity and horizonless ultracompact objects are studied in the Ref.s [34, 35].

In 2008, an alternative method for the derivation of deflection angle of light in a spherically symmetric black hole, introduced by Gibbons and Werner, by implementing the Gauss-Bonnet theorem (GBT) [36, 37] was reported in the Ref. [38]. This method is found to be good to calculate the exact form of the deflection angle in the weak field limit for Schwarzschild black hole spacetime [38]. Quite a few studies show that this approach can be used to derive the deflection angle for different black hole solutions [39–47]. To obtain the deflection angle by a Kerr black hole the Gibbons-Werner method was then extended using the Kerr-Randers optical geometry [48, 49]. This method was further extended in 2016 by Ishihara et al. to consider finite distances between the source and the observer for static as well as stationary black hole solutions [50]. For stationary black hole spacetimes, extension of this method is reported in the Ref.s [51–55]. For the case of asymptotically non-flat spacetimes such
study can be found in the Ref.s [56–58]. Using different MTGs such studies on gravitational deflection angle using GBT are reported in [54, 58]. Using GBT the effect of dark matter on deflection angle is also reported earlier in a few articles [59–62].

In our work, we intend to explore the impact of the modification of spacetime curvature on the gravitational lensing of a static spherically symmetric black hole spacetime. For this purpose, we consider the black hole solution in the \( f(\mathcal{T}, B) \) modified gravity theory \([63]\). The \( f(\mathcal{T}, B) \) modified gravity has been investigated in various aspects \([64–68]\), but for the case of gravitational lensing, we study for the first time the effects of \( f(\mathcal{T}, B) \) gravity on the deflection angle of light for a static spherically symmetric black hole using the GBT. Here we shall implement the Ishihara method for the asymptotically non-flat spacetime to obtain the deflection angle.

In this work we also study the quasinormal modes of the considered black hole spacetime. In general these are the modes of emission of GWs from compact and massive perturbed objects in the universe and are represented by some complex numbers \([69–72]\). In quasinormal modes, the real part is related to the emission frequency, while the imaginary part is related to its damping. In the last few years a number of authors have investigated the properties of GWs and also quasinormal modes of black holes in different MTGs \([23, 73–79]\).

Our paper is organized as follows. In Sec.II, we briefly review the field equations related to the \( f(\mathcal{T}, B) \) gravity and a static spherically symmetric black hole solution. For this purpose, we consider the black hole solution in the \( f(\mathcal{T}, B) \) modified gravity theory \([63]\). The action of the \( f(\mathcal{T}, B) \) gravity is done by introducing a boundary term \( \mathcal{B} \) related to the divergence of the torsion vector \( \mathcal{T}_{\mu} = \mathcal{T}_a^{\alpha} \mu \), as \( B = \frac{2}{\kappa^2} \partial_\mu (h \mathcal{T}^\mu) \), where \( h = \text{det}(h^a_{\mu}) \) is the determinant of the tetrads \( h^a_{\mu} \) with \( g \) being determinant of the metric \( g_{\mu\nu} \). This theory was introduced in \([63]\). The action of the \( f(\mathcal{T}, B) \) gravity is

\[
S_f(\mathcal{T}, \mathcal{B}) = \int d^4x \left[ \frac{1}{2\kappa^2} f(\mathcal{T}, \mathcal{B}) + \mathcal{L}_m \right],
\]

with \( \kappa^2 = 8\pi G \) and \( \mathcal{L}_m \) being the Lagrangian density of matter. The field equations of this theory can be obtained as \([63, 68]\)

\[
\delta^\lambda_\mu f_B = \nabla_\lambda \nabla_\nu f_B + \frac{1}{2} B f_B \delta^\lambda_\nu + \left( (\partial_\mu f_B) + (\partial_\mu \mathcal{T}^\mu) \right) S^\nu_\lambda + h^{-1} h^a_\nu \partial_\mu \left( h S^\nu_\mu \right) f_T - f_T T^{\sigma}_{\mu\nu} S^\lambda_\sigma - \frac{1}{2} f T^\lambda_{\nu} = \kappa^2 T^\mu_{\nu}.
\]

In this expression the circle over the terms is used to denote the quantities which are determined using the Levi-Civita connection and the subscripts below the functional \( f \) denote the respective derivatives. The term \( T^\lambda_{\nu} \) represents the energy-momentum tensor and the term \( S^\nu_\mu \) is known as the superpotential tensor \([68]\) defined as

\[
S^\nu_\mu = K^\nu_\mu - \delta^\nu_\mu T^\sigma_\sigma + \delta^\lambda_\mu T^\nu_\lambda - S^\nu_\lambda,
\]

where \( K^\nu_\mu \) is the contortion tensor as given by

\[
K^\nu_\mu = \Gamma^\nu_\mu - \Gamma^\nu_\mu = \frac{1}{2} \left( T^\nu_\mu + T^\nu_\mu - T^\nu_\mu \right).
\]
In Ref. [68] different exact and spherically symmetric perturbative solutions of field Eqs. (5) have been obtained. One of such solutions of our interest is a black hole solution which is obtained for a special tetrad and it behaves as Schwarzschild-de Sitter black hole solution given by

\[ ds^2 = f(r) \, dt^2 - \frac{dr^2}{f(r)} - r^2 \, d\Omega^2, \]  

(8)

where \( d\Omega^2 \equiv d\theta^2 + \sin^2\theta \, d\phi^2 \) and the metric co-efficient

\[ f(r) = 1 - \frac{2M}{r} - (\Lambda + c_0 M) \, r^2. \]  

(9)

The functional form of the \( f(T, B) \) gravity model considered in this solution is

\[ f(T, B) = \frac{T^2}{2} - \frac{8c_0}{3\sqrt{B + 18(Mc_0 + \Lambda)}} - 3\Lambda. \]  

(10)

Here \( c_0 \) is a parameter of the model carrying the effect of the boundary term \( B \). It is a dimensional parameter having the dimension of Length \(^{-3}\). Thus the solution is found to have an effective cosmological constant due to the contribution from the boundary term as given by \( \Lambda_{eff} = \Lambda + c_0 M \). One important point to be noted here is that the black hole solution (8) is not asymptotically flat because of the presence of \( r^2 \) term in the last part of the metric function (9). In this work we shall use this black hole solution to study the gravitational deflection of light and the quasinormal modes as discussed in the following sections.

### III. GRAVITATIONAL DEFLECTION OF LIGHT

We follow the Ishihara et al. approach [50] to obtain the deflection angle of light in the weak field limit of the non-asymptotically flat black hole spacetime given by the solution (8). Here, as shown in Fig. 1, the black hole works as a lens (L), which is at a finite distance from the source (S) and the receiver (R). In the equatorial plane (\( \theta = \pi/2 \)), the deflection angle can be expressed as [50, 51]

\[ \hat{\Theta} = \Psi_R - \Psi_S + \phi_{RS}, \]  

(11)

where \( \Psi_R \) and \( \Psi_S \) are the angles of light that are measured with respect to the lens at the positions of the receiver and the source respectively. \( \phi_{RS} = \phi_R - \phi_S \) is the separation angle between the receiver and the source. Here, \( \phi_R \) and \( \phi_S \) are the longitudes of the receiver and the source respectively.

![FIG. 1: Schematic figure for the quadrilateral \( \tilde{R}\tilde{S} \) embedded in a curved space [57].](image)

Since the light rays follow the null geodesic for which \( ds^2 = 0 \), the metric Eq. (8) can be rewritten for this geodesic as

\[ dt^2 = \gamma_{ij} dx^i dx^j = \frac{1}{f(r)^2} \, dr^2 + \frac{r^2}{f(r)} \, d\Omega^2. \]  

(12)
where $\gamma_{ij}$ is usually known as the optical metric, which specifies a 3D Riemannian space. We will denote this space by $\mathcal{M}^{(3)}$, where a ray of light is considered as a spatial curve. By using the optical metric $\gamma_{ij}$, we can define the angles $\Psi_R$ and $\Psi_S$. The non-vanishing components of this metric are

$$\gamma_{rr} = \frac{1}{f(r)^2}, \quad \gamma_{\phi\phi} = \frac{r^2}{f(r)}.$$  

(13)

An important parameter in the study of gravitational deflection angle of light is the impact parameter of light in the black hole spacetime. This is usually defined as the ratio of the angular momentum ($L$) to the energy ($E$) of photons, which are the constant of motion in the equatorial plane of spacetime. For the spacetime of black hole (8) these two constants of motion are $E = f(r) \dot{t}$ and $L = r^2 \dot{\phi}$, where the over dot denotes the derivative with respect to the affine parameter $\lambda$ along the path of the light ray.

Thus the impact parameter of a light ray is

$$\xi \equiv \frac{L}{E} = \frac{r^2}{f(r)} \frac{d\phi}{dt}.$$  

(14)

The unit radial vector from the center of the lens can be obtained as $e_{\text{rad}} = (f(r), 0)$, and the unit angular vector along the angular direction can be found as $e_{\text{ang}} = (0, f(r)/r)$. Again, the components of the unit tangent vector $K \equiv dx/dt$ along the light ray are obtained as [50]

$$(K^r, K^\phi) = \frac{\xi f(r)}{r^2} \left( \frac{dr}{d\phi}, 1 \right).$$  

(15)

Here $dr/d\phi$ gives the orbital path variation of the rays of light known as the orbit equation, which can be expressed as

$$\left( \frac{dr}{d\phi} \right)^2 = -r^2 f(r) + \frac{r^4}{\xi^2}.$$  

(16)

Now, if $\Psi$ denotes the angle between the radial component of the tangent vector and the radial vector, i.e. the angle of the light ray which is measured from the radial direction, then we can write,

$$\cos \Psi = \frac{\xi}{r^2} \frac{dr}{d\phi}.$$  

(17)

It gives,

$$\sin \Psi = \frac{\xi \sqrt{f(r)}}{r}.$$  

(18)

Moreover, in a general form Eq. (16) can be expressed as

$$\left( \frac{d\mu}{d\phi} \right)^2 = F(u),$$  

(19)

where we have considered a new variable $u = 1/r$ and hence the function $F(u) = -u^2 f(u) + 1/\xi^2$.

As mentioned earlier, we will use the GBT to calculate the deflection angle $\hat{\Theta}$. The GBT has many formulations. The simplest one states that the total Gaussian curvature of an embedded triangle can be expressed in terms of the total geodesic curvature of the boundary and the jump angles at the corners. Thus, mathematically this simplest version of the GBT can be expressed as [36]

$$\int_T KdS + \sum_{a=1}^N \int_{\partial T_a} \kappa_g dl + \sum_{a=1}^N \theta_a = 2\pi,$$  

(20)

where $T$ is a two-dimensional orientable surface shown in Fig. 2. The boundaries of the surface $\partial T_a$ ($a = 1, 2, ..., N$) are differentiable curves. $\theta_a$ are the jump angles between the curves. $K$ is the Gaussian curvature of the orientable surface $T$. $\kappa_g$ is the geodesic curvature of $\partial T_a$, $dS$ is the infinitesimal area element of the surface and $dl$ is the infinitesimal line element along the boundary. Also, $dl > 0$ for prograde motion of photons and for retrograde motion $dl < 0$. The sign of $dl$ is chosen to be consistent with the orientation of the surface $T$.

At this stage, we should note that the quadrilateral $R \Box S$ shown in Fig. 1 is embedded in a curved space $\mathcal{M}^{(3)}$ and consists of a spatial curve for a light ray from the source to the receiver, two outgoing radial lines from $R$ and from $S$, and a circular
Hence, the angle $\phi$ for the metric (8) is obtained as $\hat{\Theta} = \Psi_R - \Psi_S + \phi_{RS} = - \int \int_{\mathcal{R} \cap S} K dS$.

The separation angle $\phi_{RS}$ for our system can be obtained by integrating Eq. (19) as given by

$$\phi_{RS} = 2 \int_0^{u_0} \frac{du}{\sqrt{F(u)}},$$

where $u_0$ is the inverse of the distance of closest approach. As in the Ishihara et al. method, if we consider that the source and the receiver are at finite distances from each other, then deflection angle can be written as

$$\hat{\Theta} = \Psi_R - \Psi_S + \int_{u_R}^{u_0} \frac{du}{\sqrt{F(u)}} + \int_{u_S}^{u_0} \frac{du}{\sqrt{F(u)}}.$$  (23)

Again, for the metric (8) using Eq. (18), we obtain,

$$\Psi_R - \Psi_S = \arcsin(\xi u_R) + \arcsin(\xi u_S) - \pi + \frac{\xi(\Lambda + c_0 M)}{2} \left[ \frac{u_R^{-1}}{\sqrt{1 - \xi^2 u_R^2}} + \frac{u_S^{-1}}{\sqrt{1 - \xi^2 u_S^2}} \right] + \xi M \left[ \frac{u_R}{\sqrt{1 - \xi^2 u_R^2}} + \frac{u_S}{\sqrt{1 - \xi^2 u_S^2}} \right]$$

$$+ \frac{\xi M \Lambda}{2} \left[ \frac{1 - 2\xi^2 u_R^2}{(1 - \xi^2 u_R^2)^{3/2}} + \frac{1 - 2\xi^2 u_S^2}{(1 - \xi^2 u_S^2)^{3/2}} \right] + \frac{\xi M c_0 \Lambda}{4} \left[ \frac{u_R^3}{(1 - \xi^2 u_R^2)^{3/2}} + \frac{u_S^3}{(1 - \xi^2 u_S^2)^{3/2}} \right]$$

$$- \frac{\xi^3 M c_0 \Lambda}{2} \left[ \frac{u_R^{-1}}{(1 - \xi^2 u_R^2)^{3/2}} + \frac{u_S^{-1}}{(1 - \xi^2 u_S^2)^{3/2}} \right] + O(M^2, M \Lambda^2, \Lambda^2).$$  (24)

Here, it can be seen that the expansion of $\Psi_R - \Psi_S$ becomes divergent at $u_R \to 0$ and $u_S \to 0$ due to the fact that our spacetime is non-asymptotically flat. Hence, this series Eq. (24) must be used only within a certain limit of finite radius of convergence. Moreover, for the metric (8) the function $F(u)$ can be expressed as

$$F(u) = \frac{1}{\xi^2} - u^2 + 2Mu^3 + (\Lambda + c_0 M).$$  (25)

Hence, the angle $\phi_{RS}$ for the metric (8) is obtained as

$$\phi_{RS} = \pi - \arcsin(\xi u_R) - \arcsin(\xi u_S) + \frac{\Lambda}{\xi} \left[ \frac{2 - \xi^2 u_R^2}{\sqrt{1 - \xi^2 u_R^2}} + \frac{2 - \xi^2 u_S^2}{\sqrt{1 - \xi^2 u_S^2}} \right]$$

$$+ \frac{\xi(\Lambda + c_0 M)}{2} \left[ \frac{u_R}{\sqrt{1 - \xi^2 u_R^2}} + \frac{u_S}{\sqrt{1 - \xi^2 u_S^2}} \right] + \frac{\Delta \xi M}{2} \left[ \frac{2 - 3\xi^2 u_R^2}{(1 - \xi^2 u_R^2)^{3/2}} + \frac{2 - 3\xi^2 u_S^2}{(1 - \xi^2 u_S^2)^{3/2}} \right]$$

$$- \frac{\Delta \xi^5 c_0 M}{4} \left[ \frac{3u_R - 2\xi^2 u_R^3}{(1 - \xi^2 u_R^2)^{3/2}} + \frac{3u_S - 2\xi^2 u_S^3}{(1 - \xi^2 u_S^2)^{3/2}} \right] + O(M^2, M \Lambda^2, \Lambda^2).$$  (26)
Using Eqs. (24) and (26), we finally get the deflection angle of light for our considered black hole as

\[
\hat{\Theta} = \frac{\xi(\Lambda + c_0 M)}{2} \left[ \frac{1 + \xi^2 u_R^2}{u_R \sqrt{1 - \xi^2 u_R^2}} + \frac{1 + \xi^2 u_S^2}{u_S \sqrt{1 - \xi^2 u_S^2}} \right] + \xi M \left[ \frac{u_R}{\sqrt{1 - \xi^2 u_R^2}} + \frac{u_S}{\sqrt{1 - \xi^2 u_S^2}} \right] \\
+ \frac{\xi M \Lambda}{2} \left[ \frac{3 - 5\xi^2 u_R^2}{(1 - \xi^2 u_R^2)^{3/2}} + \frac{3 - 5\xi^2 u_S^2}{(1 - \xi^2 u_S^2)^{3/2}} \right] + \frac{\xi c_0 M \Lambda}{4} \left[ \frac{u_R^3}{(1 - \xi^2 u_R^2)^{3/2}} + \frac{u_S^3}{(1 - \xi^2 u_S^2)^{3/2}} \right] \\
- \frac{\xi^3 c_0 M \Lambda}{2} \left[ \frac{u_R^1}{(1 - \xi^2 u_R^2)^{3/2}} + \frac{u_S^1}{(1 - \xi^2 u_S^2)^{3/2}} \right] + \frac{M}{\xi} \left[ \frac{2 - \xi^2 u_R^2}{\sqrt{1 - \xi^2 u_R^2}} + \frac{2 - \xi^2 u_S^2}{\sqrt{1 - \xi^2 u_S^2}} \right] + \mathcal{O}(M^2, M \Lambda^2, \Lambda^2).
\]

(27)

By virtue of Eq. (24) few terms in the above expression may diverge in the far distance limit, \(u_R \to 0, u_S \to 0\). As mentioned earlier, this is due to the reason that the spacetime we have considered here is non-asymptotically flat, similar to the Kottler spacetime [82] used by Ishihara et al. [50]. As discussed in [50], we can state that this divergence in the deflection angle in the far distance limit does not matter as the limit \(u_R \to 0, u_S \to 0\) is not applicable for astronomical observations. Also, the effect of the boundary term contribution \(\Lambda_{eff} = \Lambda + c_0 M\) on the deflection angle can be seen in Eq. (27). It is observed that the deflection angle will increase with an increase in the effective cosmological constant term. Thus, the boundary term coming from \(f(T, B)\) gravity has significant effect on the deflection angle. Further, from Eq. (27) for \(\Lambda = c_0 = 0\), we can arrive at

\[
\hat{\Theta} \simeq \frac{M}{\xi} \left[ \frac{2 - \xi^2 u_R^2}{\sqrt{1 - \xi^2 u_R^2}} + \frac{2 - \xi^2 u_S^2}{\sqrt{1 - \xi^2 u_S^2}} \right],
\]

(28)

which at the far distance limit \((u_R \to 0, u_S \to 0)\), reduces to the deflection angle in the Schwarzschild case,

\[
\hat{\Theta} \simeq \frac{4M}{\xi}.
\]

(29)

Now, as an example we will explore the deflection angle for M87*, the central black hole candidate of M87 [83, 84] in the light of the above analysis. For this purpose we consider different values of the distance from the source \(r_s\) and also different values of the model parameter \(c_0\). The mass of this black hole is \(\sim 6.5 \times 10^9 M_{\odot} \sim 10^{13}\) m. The distance of the receiver from M87* is its distance from the earth, which is \(\sim 16\) Mpc \(\sim 10^{23}\) m, and the recent cosmological data suggest the cosmological constant to be equal to \(\Lambda = 10^{-52}\) m\(^{-2}\) [85]. In Fig. 3, we plot the deflection angle as a function of the impact parameter, and compare our results with that of the Schwarzschild case. In the first panel of this figure, we have chosen a particular value of \(r_s = 1\) Mpc. In this plot the solid curves depict the behaviour of the deflection angle for three positive values of the model parameter \(c_0 = 0.01, 0.001, 0.0001\) and the dashed curves show the same for three negative values of the model parameter \(c_0 = -0.0001, -0.001, -0.01\). All values of \(c_0\) in this and rest of the calculations are used in the unit of kpc\(^{-3}\). It can be seen that for \(c_0 = 0.01\), the deflection angle decreases upto a certain value of the impact parameter and then abruptly becomes divergent. As the value of \(c_0\) decreases, the deflection angle decreases with the impact parameter and diverges slowly after a certain point. For \(c_0 = 0.0001\), the curve almost overlaps with the Schwarzschild case upto a certain value of the impact parameter and then slowly diverges. For the negative values of the model parameter, it is seen that the deflection angle decreases and becomes negative for a certain value of the impact parameter. For \(c_0 = -0.0001\), the curve almost overlaps with the Schwarzschild case and then slowly becomes negative. This negative deflection angle can be comprehended as the repulsive gravitational effect of the teleparallel gravity theory [86, 87]. This type of negative deflection angle has been found in various studies [88–93]. In the second and the last panel, we plot the deflection angle as a function of the impact parameter for \(c_0 = 0.0001\) and \(c_0 = -0.001\) respectively, with three values of the distance from the source \(r_s = 1\) Mpc, 10 kpc, 1 kpc. It is seen that if the source (say, a cluster of galaxies) is at a distance of 1 kpc, the deflection angle decreases till a certain point and then abruptly diverges for both the cases of \(c_0 = 0.0001\) and \(c_0 = -0.001\). However, for \(c_0 = 0.0001\) the divergent behaviour is seen at a higher deflection angle than for \(c_0 = -0.001\). If we consider a galaxy cluster further away, say at 10 kpc, there is a change in the way the deflection angle diverges for \(c_0 = 0.0001\). Upto a certain value of the impact parameter, the deflection angle decreases in a way similar to the Schwarzschild case, but slowly becomes divergent after a particular value of the impact parameter. On the other hand, for \(c_0 = -0.001\), the deflection angle decreases and eventually becomes negative after a certain value of the impact parameter. For both the values of \(c_0\), such similarities are also seen if we consider a cluster of galaxies at a distance of 1 Mpc. Thus, the divergent behaviour of the deflection angle is observed at low impact parameter value when the galaxy cluster is nearer to the lens.
IV. QUASINORMAL MODES

In this section, we shall use the axial gravitational perturbations in the background of the black hole solution defined by the metric (8) to find out a Schrödinger like equation with an effective potential $V$, which will allow us to calculate the quasinormal modes associated with the perturbation numerically. In the case of axial perturbation, the axial components of the perturbed energy-momentum tensor for an anisotropic fluid are zero. Hence, in the tetrad formalism we can write the generalised resulting axial perturbation equations as [94]

$$R_{(a)(b)} = 0.$$  \hfill (30)

It is to be noted that this equation is not equivalent to the vanishing of Ricci tensor in the coordinate basis, but it gives the equations of axial perturbations that are equivalent to those obtained from the linearized form of Einstein equations in an anisotropic matter fluid. The $\theta \phi$ and $r \phi$ components of this equation for the metric (8) are [95],

$$r^2 f(r) (a_{2,\theta} - a_{3,r})_{,r} = r^2 \frac{1}{f(r)} (a_{1,\theta} - a_{3,t})_{,t},$$  \hfill (31)

$$r^2 f(r) (a_{3,r} - a_{2,\theta}) \sin^3 \theta \right]_{,\theta} = r^4 \sin^3 \theta (a_{1,r} - a_{2,t})_{,t},$$  \hfill (32)

where $a_1$, $a_2$ and $a_3$ are the metric perturbation expansion coefficients for the $t$, $r$, and $\theta$ coordinates respectively. These two component equations can be used to obtain the required equation for the calculations of the quasinormal modes from the black
hole. For this reason we define a function \( F_g(r, \theta) \equiv r^2 f(r) \left( a_{2, \theta} - a_{3, \theta} \right) \sin^3 \theta \), which can be decomposed into the \( r \) and \( \theta \) dependent parts as \( F_g(r, \theta) = F_g(r) Y(\theta) \). Here the \( \theta \) dependent function \( Y(\theta) \), known as the Gegenbauer function, satisfies the equation \( \frac{d^2}{d\theta^2} (\sin^3 \theta \frac{dY}{d\theta}) = \{-[(l+1) - 2] Y \sin^3 \theta \}. \) In terms of these defined functions, we can rearrange and combine equations (31) and (32) to obtain the equation of the form of Schrödinger wave equation as

\[
\partial_r^2 \psi_g + \omega^2 \psi_g = V(r) \psi_g, \tag{33}
\]

where \( \psi_g r = F_g(r) \) and \( r_* \) is the tortoise coordinate defined by

\[
\frac{dr_*}{dr} = \frac{1}{f(r)}. \tag{34}
\]

The effective potential in this expression is

\[
V(r) = f(r) \left[ \frac{2}{r^2} (f(r) - 1) + \frac{l(l+1)}{r^2} - \frac{1}{r} \left( \frac{d}{dr} f(r) \right) \right]. \tag{35}
\]

We shall use this expression of potential to calculate the quasinormal modes of gravitational perturbation of the black hole defined by the metric (8).

In this study, we have used the Padé averaged sixth order Wentzel-Kramers-Brillouin (WKB) approximation method. In this sixth order WKB method, the expression of oscillation frequency \( \omega \) of GWs can be given by

\[
\omega = \sqrt{-i \left[ (n + 1/2) + \sum_{k=2}^{6} \tilde{A}_k \right] \sqrt{-2V''_0} + V_0}, \tag{36}
\]

where \( n = 0, 1, 2, \ldots, V_0 = V|_{r=r_{max}} \) and \( V''_0 = \frac{d^2V}{dr^2}|_{r=r_{max}} \). Here \( r_{max} \) is the position in which the potential \( V(r) \) has its maximum value. \( \tilde{A}_k \) are the correction terms and the explicit forms of these correction terms as well as the Padé averaging recipe can be found in Ref.s [96–99].

**TABLE I:** Quasinormal modes from the black hole defined by the metric (8) for different values of the multipole moment with \( n = 0 \). In this calculation we have chosen the model parameter \( c_0 = -0.001 \) kpc\(^{-3} \) and \( l = 4 \).

| \( l \) | \( M(\omega_R - i\omega_I) \) | \( \Delta_{rm,s} \) | \( \Delta_6 \) |
|------|-------------------------------|----------------|----------------|
| 2    | 0.37861 - i0.08993            | 2.2264 \times 10^{-06} | 2.4807 \times 10^{-05} |
| 3    | 0.60743 - i0.09386            | 3.7648 \times 10^{-08} | 6.0461 \times 10^{-07} |
| 4    | 0.81998 - i0.09537            | 4.7876 \times 10^{-08} | 1.9717 \times 10^{-08} |
| 5    | 1.02583 - i0.09611            | 1.7203 \times 10^{-09} | 1.0687 \times 10^{-08} |
| 6    | 1.22822 - i0.09652            | 8.7203 \times 10^{-10} | 5.6075 \times 10^{-09} |
| 7    | 1.42861 - i0.09677            | 4.1397 \times 10^{-10} | 2.7580 \times 10^{-09} |
| 8    | 1.62770 - i0.09694            | 2.1356 \times 10^{-10} | 1.5833 \times 10^{-09} |
| 9    | 1.82593 - i0.09706            | 1.1647 \times 10^{-10} | 9.2574 \times 10^{-10} |
| 10   | 2.02354 - i0.09714            | 6.5085 \times 10^{-11} | 5.5614 \times 10^{-10} |

In Table I, we have listed the quasinormal modes for different values of the multipole moment with overtone number \( n = 0 \). In this table, \( \Delta_{rm,s} \) represents the rms error associated with the Padé averaged 6th order WKB approximation method and \( \Delta_6 \) provides a measurement of the error from two nearby order approximations defined as

\[
\Delta_6 = \frac{|\omega_7 - \omega_5|}{2}, \tag{37}
\]

where \( \omega_7 \) represents the quasinormal modes calculated using the Padé averaged 7th order WKB approximation method and \( \omega_5 \) represents the quasinormal modes calculated using the Padé averaged 5th order WKB approximation method. One can see that with an increase in the multipole moment \( l \), the error associated with the quasinormal modes decreases. It is a characteristic of the WKB approximation method which fails to provide significant results when the overtone number \( n \) is greater than the
The quasinormal frequencies shown in Table I are in geometric units with $M = 1$. To convert them to physical units, we can use the following conversion formula [100]:

$$f = \frac{32.26}{\eta} (M \omega_R) \text{ kHz},$$

where $\eta = M/M_\odot$. As listed in the Table I, for $l = 2$ the real quasinormal mode frequency in physical units will be $1.87907 \times 10^{-6}$ Hz. Similarly, for $l = 3$, $\omega_R = 3.01472 \times 10^{-6}$ Hz; for $l = 4$, $\omega_R = 4.06962 \times 10^{-6}$ Hz etc. As mentioned above, in the WKB approximation method the errors decrease with an increase in $l - n$ and for $n > l$ the method fails to provide actual quasinormal frequencies with a reasonable accuracy [101]. Hence for the rest part of the study, we shall consider $n = 0$ and a higher value of $l$ for a better accuracy of the results of quasinormal frequency calculations.

\[ \text{(38)} \]

In Fig. 4, we have plotted the real quasinormal frequencies on the left panel and the imaginary quasinormal modes on the right panel with respect to the model parameter $c_0$. For both the cases, we have used the mass of the black hole $M = 1$, overtone number $n = 0$ and multipole moment $l = 4$. One can see that with an increase in the value of $c_0$, the quasinormal frequency decreases non-linearly and approaches zero towards the positive values of $c_0$. Near $c_0 = 0.04$, the oscillation frequency becomes zero and beyond that it again starts to increase slowly. Then again near $c_0 = 0.08$, oscillation frequency drops and becomes zero. So, it seems that the positive values of the model parameter $c_0$ permits ringdown GWs of very large wavelengths which will be difficult to detect experimentally. Similarly, the decay rate of the damping rate of the quasinormal modes also decreases non-linearly with an increase in the model parameter $c_0$. Near $c_0 = 0.04$, the decay rate also becomes zero. However, beyond this point, decay rate increases drastically upto $c_0 = 0.08$ representing highly damped GWs. But for $c_0 > 0.08$, imaginary quasinormal modes become positive with a drastic rise denoting unstable gravitational perturbation on the black hole spacetime. So, from the analysis of quasinormal modes, we see that for the values of $c_0 > 0.08$, gravitational perturbation becomes unstable and such a region is not physically interesting.

\[ \text{FIG. 4: Variation of real (on the left panel) and imaginary (on the right panel) quasinormal frequencies associated with the gravitational perturbation for the black hole metric (8) with respect to the model parameter } c_0. \text{ Values of } c_0 \text{ are in the unit of kpc}^{-3}. \]

In this work, we study a recently introduced static black hole solution in an extension of modified teleparallel gravity i.e., $f(T, B)$ modified gravity, which includes the function of the torsion scalar $T$ and a related boundary term. In this modified gravity theory, we compute the deflection angle of light by a non-asymptotically flat black hole. Then we study the quasinormal modes associated with the axial gravitational perturbations in the background of the black hole. 

Gibbons and Werner first introduced this alternative way to calculate the gravitational bending angle using the Gauss-Bonnet theorem. They first evaluated the deflection angle for a Schwarzschild black hole. Since then their work has been extended in various ways for different kinds of black holes. Few researchers have computed the deflection angle from the stationary black holes. Few have again considered finite distances between source and receiver and derived the deflection angle by the static as well as stationary black holes. Some recent studies also considered black holes in modified gravity theories to obtain the deflection angle of light. In this work, we implemented the Ishihara et al. method to evaluate the deflection angle of light from
the receiver point of view. This method does not depend on the asymptotic flatness. Hence, we have applied this method in the non-asymptotically flat black hole in \( f(T, B) \) gravity. We have computed the bending angle considering the source and the receiver at a finite distance. It is found that when the source and the receiver are considered near to the lens object, the deflection angle becomes divergent. The boundary term coming from \( f(T, B) \) gravity has significant effect on the deflection angle. In the near future, we wish to extend our work to obtain the deflection angle of massive particles using Ishihara et al. method in black hole as well as in wormhole backgrounds in different MTGs.

Quasinormal modes are some complex numbers related to the emission of GWs from the compact objects in the universe. The real part of these modes is related to the emission frequency and the imaginary part is related to the damping. In this work, we try to find a Schrödinger type equation containing an effective potential, which allows us to compute the quasinormal modes associated with the axial gravitational perturbations in the background of the considered black hole in the \( f(T, B) \) gravity. For this purpose, we have used Padé averaged sixth order Wentzel-Kramers-Brillouin (WKB) approximation method. We obtained the quasinormal modes for different values of the multipole moment with \( n = 0 \). It has been observed that for smaller values of \( l \), the error associated with the quasinormal frequencies are higher in magnitude. With an increase in the value of \( l \), the errors associated with the frequencies decrease significantly. For the model parameter \( c_0 > 0.08 \), the gravitational perturbation becomes unstable and the real quasinormal mode or oscillation frequency is very small, in fact approximately equal to zero. Hence from the quasinormal modes, we provide a constraint on the model parameter \( c_0 < 0.08 \). It is to be noted that this constraint on \( c_0 \) is in accordance with our deflection angle analysis that the values \( c_0 \) should be very small to avoid the divergence of the deflection angle of light to be observed by an observer situated at finite distance from the source. However, for the observational constraints on the model from the quasinormal modes, we might need to wait for LISA [78].

Acknowledgments

UDG is thankful to the Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune for hospitality during his visits to the institute under the Visiting Associateship program.

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