ADIABATIC LIMIT, WITTEN DEFORMATION
AND ANALYTIC TORSION FORMS

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ABSTRACT. We consider a smooth fibration equipped with a flat complex vector bundle and a hypersurface cutting the fibration into two pieces. Our main result is a gluing formula relating the Bismut-Lott analytic torsion form of the whole fibration to that of each piece. This result confirms a conjecture proposed in a conference in Göttingen in 2003. Our approach combines an adiabatic limit along the normal direction of the hypersurface and a Witten type deformation on the flat vector bundle.

CONTENTS

0. Introduction 2
1. Preliminaries 12
  1.1. Finite dimensional Hodge theory and some estimates 12
  1.2. Torsion forms and some estimates 14
  1.3. Analytic torsion forms 19
2. Finite dimensional model 21
  2.1. Chain complexes from a pair of linear maps 22
  2.2. A flat family of complexes 23
3. Gluing formula for analytic torsion forms 23
  3.1. A two-parameter deformation 24
  3.2. Several intermediate results 25
4. One-dimensional Witten type deformation 29
  4.1. Hodge theory for an interval 29
  4.2. Witten type deformation on an interval 30
  4.3. Witten type deformation on a cylinder 32
5. Adiabatic limit and Witten type deformation 35
  5.1. Kernel of $D_{Z^R}$ 35
  5.2. Eigenspace of $D_{Z^R}$ associated with small eigenvalues 42
  5.3. De Rham operator on $\mathcal{E}_{[−1,1]}^{0,R,T}$ 52
  5.4. $L^2$-metric on $\mathcal{E}_{[−1,1]}^{0,R,T}$ 54
6. Analytic torsion forms associated with a fibration 58
  6.1. Decomposition of analytic torsion forms 58
  6.2. Small time contributions 59
  6.3. Large time contributions 63
7. Torsion forms associated with the Mayer-Vietoris exact sequence 69
  7.1. A filtration of the Mayer-Vietoris exact sequence 70
  7.2. Estimating $\mathcal{F}_k^{\pm}$ and $\mathcal{F}_k^{\pm}$ 70

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0. INTRODUCTION

We consider a unitarily flat complex vector bundle \((F, \nabla^F)\) over a compact smooth manifold \(X\) without boundary whose cohomology with coefficients in \(F\) vanishes, i.e., \(H^\bullet(X, F) = 0\). Franz [22], Reidemeister [53] and de Rham [19] constructed a topological invariant associated with \((F, \nabla^F)\), known as Reidemeister-Franz topological torsion (RF-torsion). RF-torsion is the first algebraic-topological invariant which can distinguish the homeomorphism types of certain homotopy-equivalent topological spaces [22, 53]. RF-torsion could be extended to the case \(H^\bullet(X, F) \neq 0\) [19, 42, 58]. Both the original construction of RF-torsion and its extensions are based on a complex of simplicial chains in \(X\) with values in \(F\).

By replacing the complex of simplicial chains by the de Rham complex, Ray and Singer [52] obtained an analytic version of RF-torsion, known as Ray-Singer analytic torsion (RS-torsion). In the same paper, Ray and Singer conjectured that RF-torsion and RS-torsion are equivalent.

Ray-Singer conjecture was proved independently by Cheeger [18] and Müller [45]. Their result is now known as Cheeger-Müller theorem. Bismut, Zhang and Müller simultaneously considered its extension. Müller [45] extended Cheeger-Müller theorem to the unimodular case, i.e., the induced metric on the determinant line bundle \(\det F\) is flat. Bismut and Zhang [12] extended Cheeger-Müller theorem to arbitrary flat vector bundle. There are also various extensions to equivariant cases [13, 36, 37].

Wagoner [57] conjectured that RF-torsion and RS-torsion can be extended to invariants of a fiber bundle, i.e., a fibration \(\pi : M \to S\) together with a flat complex vector bundle \((F, \nabla^F)\) over \(M\). Bismut and Lott [11] confirmed the analytic part of Wagoner’s conjecture by constructing analytic torsion forms (BL-torsion), which are even differential forms on \(S\). Inspired by the work of Bismut and Lott, Igusa [33] constructed higher topological torsions, known as Igusa-Klein torsion (IK-torsion). Goette, Igusa and Williams [27, 26] used IK-torsion to detect the exotic smooth structure of fiber bundles. Dwyer, Weiss and Williams [21] constructed another version of higher topological torsion (DWW-torsion). Then a natural and important problem is to understand the relation among these higher torsion invariants.

Bismut and Goette [8] established a higher version of Cheeger-Müller/Bismut-Zhang theorem under the assumption that there exist a fiberwise Morse function \(f : M \to \mathbb{R}\) and a fiberwise Riemannian metric such that the fiberwise gradient of \(f\) is Morse-Smale [54]. Goette [23, 24] extended the results in [8] to fiberwise Morse functions whose gradient vector fields are not necessarily Morse-Smale. Bismut and Goette [8] also extended BL-torsion to the equivariant case. And there are related works [16, 9]. We refer to the survey by Goette [25] for an overview on higher torsions.

Igusa [34] axiomatized higher torsion invariants. His axiomatization consists of two axioms: additivity axiom and transfer axiom. Igusa proved that IK-torsion satisfies his axioms. Moreover, any higher torsion invariant satisfying Igusa’s axioms is a linear combination of IK-torsion and the higher Miller-Morita-Mumford class [48, 43, 41]. Badzioch, Dorabiala, Klein and Williams [2] showed that DWW-torsion satisfies Igusa’s
axioms. Ma [39] studied the behavior of BL-torsion under the composition of submersions. The result of Ma implies that BL-torsion satisfies the transfer axiom. The additivity of BL-torsion was proposed as an open problem in a conference on higher torsion invariants in 2003.

Igusa’s theory begins with higher torsion invariants for fibrations with closed fibers. In the additivity axiom [34, 3.1], Igusa used fiberwise double to avoid considering fibrations with boundaries. Assuming that the torsion invariant in question is also defined for fibrations with boundaries, Igusa [34, §5] stated a gluing axiom equivalent to the additivity axiom. More precisely, given a hypersurface cutting the fibration into two pieces, the gluing axiom basically says that the torsion invariant of the total fibration should be the sum of that of each piece.

The gluing formula for BL-torsion was first precisely formulated by Zhu [61]. Zhu constructed analytic torsion forms for fibrations with boundaries and formulated a precise gluing formula for BL-torsion. This gluing formula, once proved, will lead to the conclusion that BL-torsion satisfies the gluing axiom.

Now we briefly recall previous works on the gluing formula for RS-torsion and BL-torsion. The gluing formula for RS-torsion associated with unitarily flat vector bundles was proved by Lück [37]. The proof is based on Cheeger-Müller theorem and the work of Lott and Rothenberg [36]. Vishik [56] gave an alternative proof without using Cheeger-Müller theorem or the work of Lott and Rothenberg. The gluing formula for RS-torsion was proved by Brüning and Ma [15] in full generality. The proof is based on the work of Bismut and Zhang [13], which is the equivariant version of [12], and the work of Brüning and Ma [14]. In our earlier paper [50] (announced in [51]), we gave another proof by means of adiabatic limit along the normal direction of the hypersurface, which is also one of the key tools in the present paper. There are also related works [28, 35, 44]. Zhu [61] proved the gluing formula for BL-torsion under the same assumption as in [8]. Zhu [62] also proved the gluing formula for BL-torsion under the assumption that the fiberwise cohomology of the hypersurface vanishes. This vanishing condition yields a uniform spectral gap of the fiberwise Hodge de Rham operator as the metric on the normal direction tends to infinity, which considerably simplifies the analysis involved.

The method used in [50] cannot be directly generalized to the family case. In other words, it does not lead to a proof of the gluing formula for BL-torsion in full generality. The main reason is the lack of a good interpretation of the limit of the analytic torsion forms when the metric on the normal direction of the hypersurface tends to infinity.

The purpose of this paper is to prove a gluing formula for BL-torsion in full generality, i.e., to solve the problem proposed in the conference on higher torsion invariants mentioned above. The technical core of this paper consists of two analytic tools: the adiabatic limit [20, 47, 17, 49] along the normal direction of hypersurface, which is exactly the same as in our earlier paper [50], and a Witten type deformation [32, 12, 13, 60] on the flat vector bundle. By introducing the Witten type deformation, we overcome the difficulties mentioned in the previous paragraph. We will give a more detailed explanation by the end of this introduction.

1Smooth Fibre Bundles and Higher Torsion Invariants, http://www.uni-math.gwdg.de/wm03/, Göttingen, 2003.
Now we briefly recall previous works on the two analytic tools used in this paper. The adiabatic limit of $\eta$-invariant first appeared in the work of Bismut and Freed \[6\] and in the work of Bismut and Cheeger \[5\]. The adiabatic limit used in our paper first appeared in the work of Douglas and Wojciechowski \[20\] and was further developed in \[47, 17, 49\]. We refer to the introduction of \[50\] for more details on previous works on the adiabatic limit. The Witten deformation was introduced by Witten \[59\] in the language of physics. In a series of works \[29, 30, 31, 32\], Helffer and Sjöstrand showed that the Witten instanton complex, which arises from Witten deformation, is isomorphic to the Thom-Smale complex. Bismut and Zhang \[12, \S 8\] extended the result of Helffer and Sjöstrand to arbitrary flat vector bundles. Later they gave a simple proof in \[13, \S 6\] (cf. \[60, \S 6\]), where they did not use the work of Helffer and Sjöstrand.

Let us now give more details about the matter of this paper.

**Bismut-Lott’s Riemann-Roch-Grothendieck type formula and analytic torsion forms.** Let $M$ be a smooth manifold. Let $(F, \nabla F)$ be a flat complex vector bundle over $M$ with flat connection $\nabla F$, i.e., $(\nabla F)^2 = 0$. Let $h^F$ be a Hermitian metric on $F$. We will view $h^F$ as a map from $F$ to $F^*$. Following \[12, (4.1)\] and \[11, (1.31)\], set

\[
(0.1) \quad \omega(F, h^F) = (h^F)^{-1} \nabla F h^F \in \Omega^1(M, \text{End}(F)).
\]

Let $f$ be an odd polynomial, i.e., $f(-x) = -f(x)$. We fix a square root of $i$, which we denote by $i^{1/2}$. In what follows, the choice of square root will be irrelevant. Following \[11, (1.34)\], set

\[
(0.2) \quad f(\nabla F, h^F) = (2\pi i)^{1/2} \text{Tr} \left[ f \left( \frac{1}{2} (2\pi i)^{-1/2} \omega(F, h^F) \right) \right] \in \Omega^{\text{odd}}(M).
\]

Bismut and Lott \[11, \S 1\] showed that $f(\nabla F, h^F)$ is closed and its de Rham cohomology class

\[
(0.3) \quad f(\nabla F) := [f(\nabla F, h^F)] \in H^{\text{odd}}(M)
\]

is independent of $h^F$. For a $\mathbb{Z}$-graded flat complex vector bundle $(F^* = \bigoplus_k F^k, \nabla F^* = \bigoplus_k \nabla F^k)$ and a Hermitian metric $h^{F*} = \bigoplus_k h^{F^k}$ on $F^*$, we denote

\[
(0.4) \quad f(\nabla F^*, h^{F*}) = \sum_k (-1)^k f(\nabla F^k, h^{F^k}) \in \Omega^{\text{odd}}(M),
\]

\[
(0.4) \quad f(\nabla F^*) = \sum_k (-1)^k f(\nabla F^k) \in H^{\text{odd}}(M).
\]

If $f$ is an odd formal power series, the constructions still make sense. In the sequel, we take

\[
(0.5) \quad f(x) = xe^{x^2}.
\]

Now let $\pi: M \to S$ be a fibration with compact fiber $Z$. Let $o(TZ)$ be the orientation line of the fiberwise tangent bundle $TZ$. Let $e(TZ) \in H^{\text{dim}Z}(M, o(TZ))$ be the Euler class of $TZ$ (cf. \[12, (3.17)\]). Let $H^*(Z, F)$ be the fiberwise de Rham cohomology of $Z$ with coefficients in $F$. Then $H^*(Z, F)$ is a $\mathbb{Z}$-graded complex vector bundle over
$S$ equipped with a canonical flat connection $\nabla^{H^*(Z,F)}$ (see [11] Def. 2.4]). Bismut and Lott [11, Thm. 3.17] established the following Riemann-Roch-Grothendieck type formula

\begin{equation}
(0.6) \quad f(\nabla^{H^*(Z,F)}) = \int_Z e(TZ)f(\nabla F) \in H^{odd}(S) .
\end{equation}

Bismut and Lott [11] refined equation (0.6). We consider a connection of the fibration, i.e., a splitting

\begin{equation}
(0.7) \quad TM = T^HM \oplus TZ ,
\end{equation}

a metric $g^{TZ}$ on $TZ$ and a Hermitian metric $h^F$ on $F$. Let $\nabla^{TZ}$ be the Bismut connection associated with $T^HM$ and $g^{TZ}$ [3, Def. 1.6]. Let $e(TZ,\nabla^{TZ}) \in \Omega^{dimZ}(M,o(TZ))$ be the Euler form (cf. [12] (3.17)). Let $h^{H^*(Z,F)}$ be the $L^2$-metric on $H^*(Z,F)$ induced by the Hodge theory. Bismut and Lott [11, Def. 3.22] constructed a differential form $T \in Q^S$ depending on $(T^HM, g^{TZ}, h^F)$ and showed that

\begin{equation}
(0.8) \quad dT = \int_Z e(TZ,\nabla^{TZ})f(\nabla F, h^F) - f(\nabla^{H^*(Z,F)}, h^{H^*(Z,F)}) .
\end{equation}

The differential form $T$ is called the analytic torsion form of Bismut-Lott. Now we explain the setup of our gluing formula for analytic torsion forms of Bismut-Lott.

**Gluing formula.** Let $N \subseteq M$ be a hypersurface transversal to $Z$. We suppose that $\pi|_N : N \to S$ is surjective. Then $\pi|_N$ is a fibration over $S$ with fiber $Y := N \cap Z$. We suppose that $N$ cuts $M$ into two pieces, which we denote by $M_1'$ and $M_2'$. We identify a tubular neighborhood of $N$ with

\begin{equation}
(0.9) \quad IN := [-1,1] \times N .
\end{equation}

such that

\begin{equation}
(0.10) \quad IN \cap M_1' = [-1,0] \times N , \quad IN \cap M_2' = [0,1] \times N .
\end{equation}

Set $\pi_3 = \pi|_{IN} : IN \to S$. Then $\pi_3$ is a fibration over $S$ with fiber

\begin{equation}
(0.11) \quad IY := [-1,1] \times Y .
\end{equation}

For $j = 1, 2$, set $M_j = M_j' \cup IN$. Let $\pi_j : M_j \to S$ be the restriction of $\pi$. Then $\pi_j$ is a fibration over $S$ with fiber $Z_j := M_j \cap Z$. For convenience, we denote

\begin{equation}
(0.12) \quad \pi_0 = \pi , \quad M_0 = M , \quad Z_0 = Z , \quad M_3 = IN , \quad Z_3 = IY .
\end{equation}

Then, for $j = 0, 1, 2, 3$, we have a fibration $\pi_j : M_j \to S$ with fiber $Z_j$.

Let $(u, y) \in [-1,1] \times Y$ be coordinates on $IY$. We suppose that the splitting (0.7) on $IN$ is the pullback of a splitting

\begin{equation}
(0.13) \quad TN = T^HN \oplus TY .
\end{equation}

In particular, we have

\begin{equation}
(0.14) \quad T^HM|_{IN} = p^*(T^HN|_N) ,
\end{equation}

where \( p : IN \to N \) is the canonical projection. Let \( g^{TY} \) be the metric on \( TY \) induced by the canonical embedding \( Y \hookrightarrow Z \). We suppose that the metric \( g^{TZ} \) is product on \( IN \), i.e.,

\[
g^{TZ}|_{\{u\} \times Y} = du^2 + g^{TY}.
\]

We trivialize \( F|_{IN} \) using the parallel transport along the curve \([-1, 1] \ni u \mapsto (u, y)\) with respect to \( \nabla^F \). Since \( \nabla^F \) is flat, we have

\[
(F, \nabla^F)|_{IN} = p^*(F|_N, \nabla^F|_N),
\]

where \( p : IN \to N \) is the canonical projection. We assume that

\[
h^{F}|_{IN} = p^*(h^{F}|_N).
\]

For \( j = 0, 1, 2, 3 \), let \( d^{Z_j} \) be the fiberwise de Rham operator on \( Z_j \) with values in \( F \). Let \( d^{Z_j, *} \) be the formal adjoint of \( d^{Z_j} \) with respect to the \( L^2 \)-product (see (0.51)). The Hodge de Rham operator is defined as

\[
D^{Z_j} = d^{Z_j} + d^{Z_j,*}.
\]

We identify the normal bundle \( n \) of \( \partial Z_j \) with the orthogonal complement of \( T(\partial Z_j) \subseteq TZ_j|_{\partial Z_j} \). We denote by \( e_n \) the inward pointing unit normal vector field on \( \partial Z_j \). Let \( e^n \) be the dual vector field. We denote by \( i \) (resp. \( \wedge \)) the interior (resp. exterior) multiplication. Following [15, (1.11), (1.12)] and [50, (1.4), (1.5)], we denote

\[
\Omega^\bullet_{\text{abs}}(Z_j, F) = \{ \omega \in \Omega^\bullet(Z_j, F) : i_{e_n}\omega|_{\partial Z_j} = 0 \},
\]

\[
\Omega^\bullet_{\text{abs},D^{Z_j}}(Z_j, F) = \{ \omega \in \Omega^\bullet(Z_j, F) : i_{e_n}(d^{Z_j}\omega)|_{\partial Z_j} = 0 \}.
\]

The self-adjoint extensions of \( D^{Z_j} \) and \( D^{Z_j,2} \) with domains

\[
\text{Dom}(D^{Z_j}) = \Omega^\bullet_{\text{abs}}(Z_j, F), \quad \text{Dom}(D^{Z_j,2}) = \Omega^\bullet_{\text{abs},D^{Z_j}}(Z_j, F),
\]
will also be denoted by $D^{Z_j}$ and $D^{Z_j,2}$. In the sequel, the boundary condition as above will be called absolute boundary condition. By the Hodge theorem (cf. \cite[Thm. 1.11]{15}), we have an isomorphism
\begin{equation}
H^\bullet(Z_j, F) \cong \text{Ker} \left(D^{Z_j,2}\right) \subseteq \Omega^\bullet(Z_j, F).
\end{equation}
Let $h_{H^\bullet(Z_j, F)}$ be the Hermitian metric on $H^\bullet(Z_j, F)$ induced by the $L^2$-metric on $\Omega^\bullet(Z_j, F)$ via the identification \((0.21)\).

We have a Mayer-Vietoris exact sequence of flat complex vector bundles over $S$,
\begin{equation}
\cdots \to H^k(Z, F) \to H^k(Z_1, F) \oplus H^k(Z_2, F) \to H^k(Y, F) \to \cdots .
\end{equation}
Let $\mathcal{J}_F \in Q^S$ be the torsion form (\cite[Def. 2.20]{11}, cf. \S1.2) associated with the exact sequence \((0.22)\) equipped with Hermitian metrics $(h_{H^\bullet(Z_j, F)})_{j=0,1,2,3}$. By \cite[Def. 2.22]{11}, we have
\begin{equation}
d\mathcal{J}_F = \sum_{j=0}^3 (-1)^{(j-3)/2} f\left(\nabla H^\bullet(Z_j, F), h_{H^\bullet(Z_j, F)}\right) .
\end{equation}
We put the absolute boundary condition on the boundary of $Z_j$ (see \((0.19)\) and \((0.20)\)). The analytic torsion form for fibration with boundary equipped with absolute boundary condition was constructed by Zhu \cite[Def. 2.18]{61}. For $j = 0, 1, 2, 3$, let $\mathcal{J}_j \in Q^S$ be the analytic torsion form associated with
\begin{equation}
(\pi_j, T^HM|_{M_j}, g^{TZ}|_{M_j}, F|_{M_j}, \nabla^F|_{M_j}, h^F|_{M_j}).
\end{equation}
We denote
\begin{equation}
[\partial Z_j : Y] = \left\{ \begin{array}{ll}
0 & \text{if } j = 0; \\
1 & \text{if } j = 1, 2; \\
2 & \text{if } j = 3.
\end{array} \right.
\end{equation}
In other words, $\partial Z_j$ consists of $[\partial Z_j : Y]$ copies of $Y$. Let $\nabla^{TY}$ be the Bismut connection on $TY$ associated with $T^HN$ and $g^{TY}$ \cite[Def. 1.6]{3}. By \cite[Thm. 2.19]{61}, we have
\begin{equation}
d\mathcal{J}_j = \int_{Z_j} e(TZ, \nabla^{TZ}) f(\nabla^F, h^F) + \frac{[\partial Z_j : Y]}{2} \int_Y e(TY, \nabla^{TY}) f(\nabla^F, h^F)
\end{equation}
\begin{equation}
- f\left(\nabla H^\bullet(Z_j, F), h_{H^\bullet(Z_j, F)}\right) .
\end{equation}
Let $Q^{S,0} \subseteq Q^S$ be the vector subspace of exact real even differential forms on $S$. The main result in this paper is the following theorem.

**Theorem 0.1.** The following equation holds,
\begin{equation}
\mathcal{F} = \mathcal{F}_1 - \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{J}_F \in Q^{S,0} .
\end{equation}
For any closed oriented submanifold $\mathcal{O} \subseteq S$, the following map
\begin{equation}
\int_{\mathcal{O}} : Q^S \to \mathbb{R}
\end{equation}
may be viewed as a linear function on $Q^S/Q^{S,0}$. By the Stokes' formula and the de Rham theorem (cf. \cite[Thm. 1.1 (d)]{15}), these linear functions separate the elements of $Q^S/Q^{S,0}$. As a consequence, to prove Theorem 0.1 it is sufficient to show that the integration of the left hand side of \((0.27)\) on each $\mathcal{O}$ vanishes. Hence, without loss
of generality, we may and we will assume that \( S \) is a compact manifold without boundary in the whole paper.

We note that in Theorem 0.1, we only use the absolute boundary condition, whereas the relative boundary condition appears in the gluing formula for the RS-torsion in [15, 50] and in the Zhu’s formulation of the gluing formula for the BL-torsion [61]. In fact, Theorem 0.1 implies Zhu’s formula. In order to keep this paper to a reasonable length, this will be proved in a subsequent paper, in which we will also discuss more precisely the link between BL-torsion and IK-torsion resulting from the work of Igusa [34] combined with [39] and Theorem 0.1.

Now we briefly describe the strategy of our proof.

**A two-parameter deformation and anomaly formulas.** For \( j = 1, 2 \), set \( M_j'' = M_j \setminus IN \). For \( R \geq 1 \), set
\[
(0.29) \quad IN_R = [-R, R] \times N, \quad M_R = M_1'' \cup N \cup M_2'',
\]
where we identify \( \partial M_j'' = N \) with \( \{ (-1)^j R \} \times N \subseteq IN_R \) for \( j = 1, 2 \). Then \( M_R \) is a closed manifold. In particular, \( M_R |_{R=1} = M \). We construct a smooth fibration
\[
(0.30) \quad \pi_R : M_R \to S
\]
as follows: \( \pi_R |_{M_j''} = \pi |_{M_j''} \) for \( j = 1, 2 \) and \( \pi_R |_{IN_R} \) being the composition of the canonical projection \( IN_R \to N \) and \( \pi |_{N} : N \to S \).

For \( R \geq 1 \), let \( \phi_R : [-1, 1] \to [-R, R] \) be a smooth bijective map such that
\[
(0.31) \quad \phi_R(u) \begin{cases} \geq 1, & \text{for } u \in [-1, 1], \\ = -\phi(u), & \text{for } u \in [-1, -1/2]. \end{cases}
\]
We construct a diffeomorphism \( \varphi_R : M \to M_R \) as follows:
\[
(0.32) \quad \varphi_R |_{M_1'' \cup M_2''} = Id, \quad \varphi_R |_{IN} : (u, y) \mapsto (\phi_R(u), y).
\]
Then the following diagram commutes
\[
\begin{array}{ccc}
M & \xrightarrow{\varphi_R} & M_R \\
\downarrow{\pi} & & \downarrow{\pi_R} \\
S & & \end{array}
\]

Let \( Z_R \) be the fiber of \( \pi_R \). We construct a metric \( g^{TZ_R} \) on \( TZ_R \) as follows:
\[
(0.34) \quad g^{TZ} |_{M_1'' \cup M_2''} = g^{TZ} |_{M_1'' \cup M_2''}, \quad g^{TZ_R} |_{IN_R} = du^2 + g^{TY}.
\]
Set \( g^{TZ_R} = \varphi_R^*(g^{TZ_R}) \). It is obvious that \( (\pi : M \to S, g^{TZ}) \) and \( (\pi_R : M_R \to S, g^{TZ_R}) \) are isometric. We will work on one or another depending on the context.

Let \( f_\infty : [-1, 1] \to \mathbb{R} \) be a self-indexed Morse function such that
\[
(0.35) \quad \{ u \in [-1, 1] : f_\infty(u) = 0 \} = \{-1, 0, 1\}, \quad f_\infty(-1) = f_\infty(1) = 0, \quad f_\infty(0) = 1.
\]
We can construct a family smooth function \( (f_T : [-1, 1] \to \mathbb{R})_{T \geq 0} \) such that
\[
(0.36) \quad f_T(u) = 0, \quad \text{for } |u \pm 1| \leq e^{-T^2}; \quad f_T'(u) - f_\infty'(u) = \mathcal{O}(e^{-T^2}).
\]
We will view $f_T$ as a smooth function on $M_R$ as follows:

\begin{equation}
(0.37) \quad f_T|_{M \setminus I N_R} = 0, \quad f_T(u, y) = f_T(u/R) \text{ for } (u, y) \in I N_R.
\end{equation}

Then $\varphi_R(f_T)$ is a smooth function on $M$. Set

\begin{equation}
(0.38) \quad h^F_{R,T} = \exp \left( -2T \varphi_R^*(f_T) \right) h^F.
\end{equation}

Replacing $(g_T^Z, h^F)$ by $(g_T^Z, h^F_{R,T})$ and proceeding in the same way as before, we get analytic torsion forms $\mathcal{F}_{j,R,T} \in Q^S$ (j = 0, 1, 2, 3) and torsion form $\mathcal{F}_{\mathcal{F},R,T} \in Q^S$. By anomaly formulas [11, Thm. 3.24], [62, Thm. 1.5], the class

\begin{equation}
(0.39) \quad [\mathcal{F}_{R,T} - \mathcal{F}_{1,R,T} - \mathcal{F}_{2,R,T} + \mathcal{F}_{3,R,T} + \mathcal{F}_{\mathcal{F},R,T}] \in Q^S/Q^{S,0}
\end{equation}

is independent of $R, T$. As a consequence, to prove Theorem 0.1 it is sufficient to show that (0.39) tends to zero as $R, T \to +\infty$.

**Spectral gap and Witten type theorem.** For simplicity, the pushforward $\varphi_{R,*}(F, \nabla^F, h^F)$ will also be denoted by $(F, \nabla^F, h^F)$. We construct a family of Hermitian metrics on $F$ over $Z_R$ as follows:

\begin{equation}
(0.40) \quad h^F_T = e^{-2T f_T} h^F.
\end{equation}

Then we have $h^F_T = \varphi_{R,*}(h^F_{R,T})$. Replacing $(g_T^Z, h^F)$ by $(g_T^Z, h^F_{R,T})$ in the construction of the Hodge de Rham operator $D_T^R$ and identifying $(Z_j, g_T^Z)$ with $(Z_j, R, g_T^Z, h^F)$ via the isometry $\varphi_R|_{Z^j}$, we obtain $D_T^{Z_j,R}$ acting on $\Omega^*_R(Z_j, F)$. The operator $\tilde{D}_T^{Z_j,R}$ is self-adjoint with respect to the $L^2$-metric induced by $g_T^{Z_j,R}$ and $h^F_T$. For convenience, we consider the conjugated operator $D_T^{Z_j,R} = e^{-T f_T} \tilde{D}_T^{Z_j,R} e^{T f_T}$, which is self-adjoint with respect to the $L^2$-metric induced by $g_T^{Z_j,R}$ and $h^F_T$.

We fix a constant $\kappa \in [0, 1/3]$. The following result is crucial (see Theorem 3.1): there exists $\alpha > 0$ such that for $T = R^\kappa \gg 1$, we have

\begin{equation}
(0.41) \quad \text{Sp} \left( R D_T^{Z_j,R} \right) \subseteq [-\infty, -\alpha \sqrt{T}] \cup [-1, 1] \cup [\alpha \sqrt{T}, +\infty[,
\end{equation}

where $\text{Sp}(\cdot)$ is the spectrum. We call the eigenvalues of $R D_T^{Z_j,R}$ lying in $[1, 1]$ (resp. out of $[-1, 1]$) small eigenvalues (resp. large eigenvalues). Let $\mathcal{E}^{[-1,1]}_{j,R,T} \subseteq \Omega^*(Z_j, F)$ be the eigenspace of $R D_T^{Z_j,R}$ associated with small eigenvalues. Set

\begin{equation}
(0.42) \quad d_T^{Z_j,R} = e^{-T f_T} d_T^{Z_j,R} e^{T f_T}.
\end{equation}

Since $d_T^{Z_j,R}$ commutes with $D_T^{Z_j,R,2}$, we get a finite dimensional complex

\begin{equation}
(0.43) \quad \left( \mathcal{E}^{[-1,1]}_{j,R,T}, d_T^{Z_j,R} \right).
\end{equation}

We will show that $\dim \mathcal{E}^{[-1,1]}_{j,R,T}$ is independent of $R$ for $R \gg 1$ and explicitly construct a complex $(C^\bullet_j, \partial)$ and show that the complex (0.43) is 'asymptotic' to $(C^\bullet_j, \partial)$ as $T = R^\kappa \to +\infty$ (see Theorem 3.3). For instance, taking $j = 0$, we have

\begin{equation}
(0.44) \quad C^k_0 = 0 \quad \text{for } k \neq 0, 1, \quad C^1_0 = H^*(Z_1, F) \oplus H^*(Z_2, F), \quad C^1_0 = H^*(Y, F) = H^*(I Y, F)
\end{equation}
with $\partial : H^\bullet(Z_1, F) \oplus H^\bullet(Z_2, F) \to H^\bullet(IY, F)$ being the same map as in (0.22). This result may be viewed as a variation of the Witten deformation.

**Finite propagation speed.** By the finite propagation speed for solutions of hyperbolic equations (cf. [55, §2.6, Thm. 6.1], [40, Appendix D.2]), the contribution of large eigenvalues to (0.39) tends to 0 as $T = R^\kappa \to +\infty$. On the other hand, we can explicitly estimate the contribution of small eigenvalues by applying our Witten type theorem (Theorem 3.3). These estimates will lead to the conclusion that (0.39) tends to zero as $T = R^\kappa \to +\infty$.

If we take $T = 0$ and $R \to +\infty$, the situation on each fiber is exactly what was studied in our earlier paper [50]. We owe readers an explanation for introducing the second parameter $T$. Now we try to answer the following questions.

- Why we cannot prove the gluing formula for analytic torsion forms by simply taking $T = 0$ and $R \to +\infty$?
- How does the second parameter $T$ improve the situation?

Both in [50] and in this paper, the contribution of large eigenvalues can be controlled by means of the finite propagation speed method. The difficulties come from the small eigenvalues. In [50], the small eigenvalues are handled in a rather brutal way: we estimate the contribution of each eigenvalue and take the sum of them. Such a proof highly relies on the expression of the analytic torsion in terms of the zeta-function associated with the eigenvalues, which does not hold for analytic torsion forms. An alternative way is to build a model encoding the asymptotic limit of the small eigenvalues. However, with $T = 0$ and $R \to +\infty$, we find infinitely many small eigenvalues. It seems hopeless to find a reasonable model. This problem is solved by taking $T = R^\kappa \to +\infty$. With the new parameter $T$ introduced, there remain finitely many small eigenvalues (see (0.43)). Moreover, for $T = R^\kappa$ large enough, the dimension of the eigenspace associated with small eigenvalues is a constant. And a model $(C^\bullet \cF, \partial)$ is built accordingly (see (0.44)).

Now we explain the model in more detail. Recall that the eigenspace associated with small eigenvalues is denoted by $E_{j,R,T}^{[-1,1]}$ (see (0.43)). Since we work with a fibration over $S$, both $C^\bullet \cF$ and $E_{j,R,T}^{[-1,1]}$ are vector bundles over $S$. The vague word ‘model’ should be interpreted as follows: we construct a bijection (parameterized by $R, T$) between vector bundles $C^\bullet \cF \to E_{j,R,T}^{[-1,1]}$, which we denote by $\cF_{j,R,T}$ in this paper (see Theorem 3.3). We denote

\[(0.45)\]

$$F^1 E_{j,R,T}^{[-1,1]} = \cF_{j,R,T}(C^1 \cF) \subseteq E_{j,R,T}^{[-1,1]}.$$ Then we have induced bijections

\[(0.46)\]

$$C^0 \cF \to E_{j,R,T}^{[-1,1]} / F^1 E_{j,R,T}^{[-1,1]} , \quad C^1 \cF \to F^1 E_{j,R,T}^{[-1,1]} .$$

There is a canonical way to equip $C^\bullet \cF$ and $E_{j,R,T}^{[-1,1]}$ with superconnections (parameterized by $R, T$). As $T = R^\kappa \to \infty$, the maps in (0.46) tend to be compatible with the superconnections in certain sense. Similar phenomena appeared in various works on the analytic torsion forms (cf. [8, §10, §11], [38, Thm. 2.9] and [39, Thm. 4.4]).
The situation becomes more clear once we pass to the cohomology. We will construct a bijection (see (5.51), (5.56), (5.212) and (5.217))

\[(0.47) \quad \mathcal{H}^*_{j, R, T} : H^*(C^*_{j, \partial}) \rightarrow H^*(\Theta_{j, R, T}^{[-1, 1]}, F_T) \approx H^*(Z_j, F),\]

where the last isomorphism is induced by the Hodge theory. Here \([\mathcal{H}^*_{j, R, T}]_T\) is not directly induced by \(\mathcal{H}^*_{j, R, T}\). Necessary modification is required since \(\mathcal{H}^*_{j, R, T}\) is not a map between complexes (see (3.34)). Both \(H^*(C^*_{j, \partial})\) and \(H^*(Z_j, F)\) are flat vector bundles over \(S\). But \([\mathcal{H}^*_{j, R, T}]_T\) is not necessarily a map between flat vector bundles.

To properly interpret the flatness of \([\mathcal{H}^*_{j, R, T}]_T\), we consider the short exact sequence induced by (0.47),

\[(0.48) \quad 0 \rightarrow H^1(C^*_{j, \partial}) \rightarrow H^*(Z_j, F) \rightarrow H^0(C^*_{j, \partial}) \rightarrow 0.\]

This is indeed an exact sequence of flat vector bundles.

This paper is organized as follows.

In §1 we establish several technical results concerning the finite dimensional Hodge theory and torsion forms. We also recall the construction of analytic torsion forms.

In §2 we build up a finite dimensional model of the problem addressed in this paper.

In §3 we state several intermediate results and show that these results lead to Theorem 0.1. The proof of these results are delayed to §5, 6, 7.

In §4 we study a one-dimensional Witten type deformation.

In §5 we establish the crucial spectral gap (0.41) and study the asymptotics of the complex \((\Theta_{j, R, T}^{[-1, 1]}, F_T)\).

In §6 we study the asymptotics of the analytic torsion forms \(\mathcal{H}^*_{j, R, T}\) as \(T = R^n \rightarrow +\infty\).

In §7 we study the asymptotics of the torsion form \(\mathcal{H}_{j, R, T}\) as \(T = R^n \rightarrow +\infty\).

Notations. Hereby we summarize some frequently used notations and conventions.

For a manifold \(X\) and a flat complex vector bundle \((F, \nabla^F)\) over \(X\), we denote

\[(0.49) \quad \Omega^*(X, F) = \mathcal{C}^\infty(X, \Lambda^* T^* X \otimes F),\]

the vector space of smooth differential forms on \(X\) with values in \(F\). The de Rham operator on \(\Omega^*(X, F)\) is defined as follows:

\[(0.50) \quad d^X : \omega \otimes s \mapsto d\omega \otimes s + (-1)^{\operatorname{deg}\omega} \omega \wedge \nabla^F s, \quad \text{for } \omega \in \Omega^*(X), s \in \mathcal{C}^\infty(X, F).\]

Then \((\Omega^*(X, F), d^X)\) is the de Rham complex of smooth differential forms on \(X\) with values in \(F\). Its cohomology is denoted by \(H^*(X, F)\).

For a submanifold \(U \subseteq X\) and \(\omega \in \Omega^*(X, F)\), we denote by \(\omega|_U \in \mathcal{C}^\infty(U, \Lambda^* T^* X \otimes F)\) its restriction on \(U\). Let \(j : U \rightarrow X\) be the canonical embedding. For \(\omega \in \Omega^*(X, F)\) closed, we denote \(|\omega||_U = j^*|\omega| \in H^*(U, F)\). We remark that in general \(\omega|_U \notin |\omega||_U\), unless \(\dim U = \dim X\).

If \(TX\) is equipped with a Riemannian metric \(g^{TX}\), and \(F\) is equipped with a Hermitian metric \(h^F\), we denote by \(\|\cdot\|_X\) (resp. \(\langle \cdot, \cdot \rangle_X\)) the \(L^2\)-norm (resp. \(L^2\)-product) on \(\Omega^*(X, F)\). More precisely, for \(\omega, \mu \in \Omega^*(X, F)\), we have

\[(0.51) \quad \langle \omega, \mu \rangle_X = \int_X \langle \omega_x, \mu_x \rangle_{A^* \Lambda^*(T^*_x X) \otimes F_x} dv(x),\]
where $\langle \cdot , \cdot \rangle_{\Lambda^k (T^*_x X) \otimes F_x}$ is the scalar product on $\Lambda^k (T^*_x X) \otimes F_x$ induced by $g^{TX}_x$ and $h^F_x$, and $dv$ is the Riemannian volume form on $(X, g^{TX})$. For a submanifold $U \subseteq X$, we denote by $\| \cdot \|_U$ (resp. $\langle \cdot , \cdot \rangle_U$) the $L^2$-norm (resp. $L^2$-product) on $\mathcal{C}^\infty (U, \Lambda^k (T^*_x X) \otimes F_x)$ with respect to the induced Riemannian metric on $TU$. For simplicity, for $\omega, \mu \in \Omega^k (X, F)$, we will abuse the notations as follows,

\[
\| \omega \|_U = \| \omega \|_{U} \quad \text{and} \quad \langle \omega, \mu \rangle_U = \langle \omega \|_{U}, \mu \|_{U} \rangle_U .
\]

For any set $X$, we denote by $\text{Id}_X : X \to X$ the identity map.

For a self-adjoint operator $A$, we denote by $\text{Sp}(A)$ its spectrum.

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1. Preliminaries

This section is organized as follows. In §1.1, we state the finite dimensional Hodge theory and establish several estimates concerning the spectral decomposition of the Hodge Laplacian. In §1.2, we recall the definition of torsion forms and establish several estimates concerning the comparison of torsion forms. In §1.3, we recall the definition of analytic torsion forms [11, 61].

1.1. Finite dimensional Hodge theory and some estimates. Let

\[
(W^\bullet, \partial) : 0 \to W^0 \to \cdots \to W^n \to 0
\]

be a chain complex of finite dimensional complex vector spaces. Let $H^\bullet (W^\bullet, \partial)$ be the cohomology of $(W^\bullet, \partial)$. Let $h^{W^\bullet} = \bigoplus_{k=0}^n h^{W^k}$ be a Hermitian metric on $W^\bullet$. Let $\partial^*$ be the adjoint of $\partial$. Set

\[
D = \partial + \partial^*,
\]

which is self-adjoint with respect to $h^{W^\bullet}$.

Now we state the finite dimensional Hodge theorem without proof.

**Theorem 1.1.** The following orthogonal decomposition holds,

\[
W^\bullet = \text{Ker} D \oplus \text{Im} \partial \oplus \text{Im} \partial^* .
\]

We have

\[
\text{Ker} D = \text{Ker} D^2 = \partial \cap \text{Ker} \partial^* \subseteq W^\bullet .
\]
Moreover, the induced map

\[ \text{Ker } D^2 \to H^\bullet(W^\bullet, \partial) \]
\[ w \mapsto [w] \]

is an isomorphism.

Let

\[ W^\bullet = \bigoplus_{\lambda \geq 0} W^\bullet_\lambda \]

be the spectral decomposition with respect to \( D^2 \), i.e., \( D^2|_{W^\bullet_\lambda} = \lambda \text{Id} \). We denote

\[ W^\bullet = W^\bullet \cap \text{Ker } \partial, \quad W^\bullet'' = W^\bullet \cap \text{Ker } \partial^\ast. \]

The following orthogonal decomposition holds for \( \lambda > 0 \),

\[ W^\bullet_\lambda = W^\bullet \cap \text{Ker } \partial \]
\[ W^\bullet_\lambda' = W^\bullet \cap \text{Ker } \partial^\ast. \]

Let \( \| \cdot \| \) be the norm on \( W^\bullet \) induced by \( h_{W^\bullet} \). For \( w' \in W^\bullet \) and \( w'' \in W^\bullet'' \), we have

\[ \| \partial^\ast w' \|^2 = \lambda \| w' \|^2, \quad \| \partial w'' \|^2 = \lambda \| w'' \|^2. \]

Let \( \Lambda \subseteq \mathbb{R} \), let

\[ P^\Lambda : W^\bullet \to \bigoplus_{\lambda \in \Lambda} W^\bullet_\lambda \]

be the orthogonal projection.

We state a naive estimate without proof.

**Proposition 1.2.** Let \( \alpha, \beta \geq 0 \) and \( w \in W^\bullet \). If \( \| Dw \| \leq \alpha \beta \), then \( \| w - P^{[0,\beta]}w \| \leq \alpha \).

Now we establish a more sophisticated estimate.

**Proposition 1.3.** Let \( \alpha, \beta, \gamma \geq 0 \) and \( w, v \in W^\bullet \). If

\[ \| \partial w \|^2 \leq \alpha \gamma, \quad \| \partial^\ast v \|^2 \leq \alpha \gamma, \quad \| w - v \|^2 \leq \beta, \]

then

\[ \| w - P^{[0,\gamma]}w \|^2 \leq 3\alpha + 2\beta, \quad \| v - P^{[0,\gamma]}v \|^2 \leq 3\alpha + 2\beta. \]

**Proof.** Let

\[ w = \sum_{\lambda} w_\lambda, \quad v = \sum_{\lambda} v_\lambda \]

be the decompositions with respect to (1.6), i.e., \( w_\lambda, v_\lambda \in W^\bullet_\lambda \). For \( \lambda > 0 \), let

\[ w_\lambda = w_\lambda' + w_\lambda'', \quad v_\lambda = v_\lambda' + v_\lambda''. \]

be the decompositions with respect to (1.8), i.e., \( w_\lambda', v_\lambda' \in W^\bullet_\lambda \) and \( w_\lambda'', v_\lambda'' \in W^\bullet_\lambda''. \)

By (1.9), (1.13) and (1.14), we have

\[ \| \partial w \|^2 = \sum_{\lambda > 0} \lambda \| w_\lambda'' \|^2, \quad \| \partial^\ast v \|^2 = \sum_{\lambda > 0} \lambda \| v_\lambda' \|^2. \]
By (1.11) and (1.15), we have
\begin{equation}
\left\| \sum_{\lambda > \gamma} w^\alpha_\gamma \right\|^2 = \sum_{\lambda > \gamma} \left\| w^\alpha_\gamma \right\|^2 \leq \alpha , \quad \left\| \sum_{\lambda > \gamma} v^\alpha_\gamma \right\|^2 = \sum_{\lambda > \gamma} \left\| v^\alpha_\gamma \right\|^2 \leq \alpha .
\end{equation}

On the other hand, by the third inequality in (1.11), (1.13) and (1.14), we have
\begin{equation}
\left\| \sum_{\lambda > \gamma} w^\alpha_\gamma - \sum_{\lambda > \gamma} v^\alpha_\gamma \right\|^2 \leq \beta .
\end{equation}

By the first identity in (1.14), (1.16) and (1.17), we have
\begin{equation}
\left\| \sum_{\lambda > \gamma} w^\alpha_\gamma \right\|^2 = \left\| \sum_{\lambda > \gamma} w^\alpha_\gamma \right\|^2 + \left\| \sum_{\lambda > \gamma} w^\alpha_\gamma \right\|^2
\end{equation}
\begin{equation}
\leq \left( \sum_{\lambda > \gamma} \left\| w^\alpha_\gamma \right\|^2 + 2 \left( \sum_{\lambda > \gamma} \left\| v^\alpha_\gamma \right\|^2 + \sum_{\lambda > \gamma} \left\| w^\alpha_\gamma - v^\alpha_\gamma \right\|^2 \right) \right) \leq 3\alpha + 2\beta ,
\end{equation}
which implies the first inequality in (1.12). The second inequality in (1.12) can be proved in the same way. This completes the proof of Proposition 1.3. \hfill \square

For \( w \in W^* \), we define \( \| w \|^2 = \| w \|^2 + \| Dw \|^2 \).

**Corollary 1.4.** Propositions 1.2, 1.3 hold with \( \| \cdot \| \) replaced by \( \| \cdot \|_1 \).

**Proof.** All the properties concerning \( \| \cdot \| \) hold for \( \| \cdot \|_1 \). In particular, the adjoint of \( \partial \) with respect to \( \| \cdot \|_1 \) is still \( \partial^* \). \hfill \square

### 1.2. Torsion forms and some estimates

Let \( S \) be a compact manifold without boundary.

Let
\begin{equation}
(W^*, \partial) : 0 \rightarrow W^0 \rightarrow \cdots \rightarrow W^n \rightarrow 0
\end{equation}
be a chain complex of complex vector bundles over \( S \), i.e., \( \partial : W^* \rightarrow W^{*+1} \) is a linear map between complex vector bundles satisfying
\begin{equation}
\partial^2 = 0 .
\end{equation}

We extend the action of \( \partial \) to \( \Omega^* (S, W^*) \) as follows: for \( \tau \in \Omega^k (S) \) and \( w \in \mathcal{C}^\infty (S, W^*) \),
\begin{equation}
\partial (\tau \otimes w) = (-1)^k \tau \otimes \partial w .
\end{equation}

Let \( \nabla^W^* = \bigoplus_{k=0}^n \nabla^W^k \) be a connection on \( W^* \). We extend the action of \( \nabla^W^* \) to \( \Omega^* (S, W^*) \) in the same way as in (0.50). We assume that \( \nabla^W^* \) is a flat connection. Equivalently, we assume that
\begin{equation}
(\nabla^W^*)^2 = 0 .
\end{equation}

Now we assume that \( (W^*, \nabla^W^*, \partial) \) is a chain complex of flat complex vector bundles. Equivalently, we assume that
\begin{equation}
\partial \nabla^W^* + \nabla^W^* \partial = 0 .
\end{equation}

By (1.23), \( \partial \) is covariantly constant with respect to the connection \( \nabla^W^* \). Thus there is a \( \mathbb{Z} \)-graded complex vector bundle \( H^* \) over \( S \) whose fiber over \( s \in S \) is the cohomology of \( (W^*_s, \partial|_{W^*_s}) \) (see [111, p. 307]). Let \( \nabla^H^* \) be the connection on \( H^* \) induced
by $\nabla^{W^*}$ in the sense of [11, Def. 2.4]. By [11, Prop. 2.5], $(H^*, \nabla^{H^*})$ is a $\mathbb{Z}$-graded flat complex vector bundle.

Recall that $f(z) = z e^{z^2}$. Let $f(\nabla^{W^*}), f(\nabla^{H^*}) \in H^{\text{odd}}(S)$ be as in (0.4). By [11, Thm. 2.19], we have

$$(1.24) \quad f(\nabla^{W^*}) = f(\nabla^{H^*}).$$

Set

$$(1.25) \quad A'' = \partial + \nabla^{W^*}.$$ 

By (1.20), (1.22), (1.23) and (1.25), we have

$$(1.26) \quad (A'')^2 = 0,$$

i.e., $A''$ is a flat superconnection in the sense of [11, §1].

Let $h^{W^*} = \bigoplus_{k=0}^{\infty} h^{W_k}$ be a Hermitian metric on $W^*$. Let $\omega^{W*} \in \Omega^1(S, \text{End}(W^*))$ be as in (0.1) with $(\nabla^F, h^F)$ replaced by $(\nabla^{W*}, h^{W*})$, i.e.,

$$(1.27) \quad \omega^{W*} = (h^{W*})^{-1}\nabla^{W*} h^{W*}.$$ 

Let $\partial^*$ be the adjoint of $\partial$. Let $A'$ be the adjoint superconnection of $A''$ in the sense of [11, §1]. By [11, §2(b)], we have

$$(1.28) \quad A' = \partial^* + \nabla^{W^*} + \omega^{W*}.$$

Set

$$(1.29) \quad X = \frac{1}{2}(A' - A'') = \frac{1}{2}(\partial^* - \partial) + \frac{1}{2} \omega^{W*} \in \Omega^*(S, \text{End}(W^*)).$$

Let $N^{W^*}$ be the number operator on $W^*$, i.e., $N^{W^*}|_{W_k} = k \text{Id}$. For $t > 0$, set $h^{W^*}_t = t^{N^{W^*}} h^{W^*}$. Let $X_t$ be the operator $X$ associated with $h^{W^*}_t$. We have

$$(1.30) \quad X_t = \frac{1}{2}(t\partial^* - \partial) + \frac{1}{2} \omega^{W*}.$$ 

We define $\varphi : \Omega^{\text{even}}(S) \to \Omega^{\text{even}}(S)$ as follows,

$$(1.31) \quad \varphi \omega = (2\pi i)^{-k} \omega, \quad \text{for } \omega \in \Omega^{2k}(S).$$

We remark that $f'(z) = (1 + 2z^2) e^{z^2}$. Set

$$(1.32) \quad f^\wedge(A'', h^{W^*}_t) = \varphi \text{Tr} \left( (-1)^{N^{W^*}} \frac{N^{W^*}}{2} f'(X_t) \right) \in \Omega^{\text{even}}(S).$$

Set

$$(1.33) \quad X_t = t^{N^{W^*}/2} X_t t^{-N^{W^*}/2} = \frac{\sqrt{t}}{2}(\partial^* - \partial) + \frac{1}{2} \omega^{W^*}.$$ 

We have an alternative definition,

$$(1.34) \quad f^\wedge(A'', h^{W^*}_t) = \varphi \text{Tr} \left( (-1)^{N^{W^*}} \frac{N^{W^*}}{2} f'(X_t) \right).$$

We denote

$$(1.35) \quad \chi'(W^*) = \sum_k (-1)^k k \text{rk}(W^k), \quad \chi'(H^*) = \sum_k (-1)^k k \text{rk}(H^k).$$
Analytic Torsion Forms

The following definition is due to Bismut and Lott [11, Def. 2.20].

**Definition 1.5.** The torsion form associated with \((\nabla^{W*}, \partial, h^{W*})\) is defined by

\[
\mathcal{T}(\nabla^{W*}, \partial, h^{W*}) = -\int_{0}^{+\infty} \left[ f'(A'', h^{W*}_{t}) - \frac{1}{2} \chi'(H^{*}) \right] \frac{dt}{t}.
\]

By [11, Thm. 2.13, Prop. 2.18], the integrand in (1.36) is integrable.

Let \(h^{H*}\) be the Hermitian metric on \(H^{*}\) induced by \(h^{W*}\) via the identification \(H^{*} \simeq \text{Ker} (\partial + \partial^{*})^{2} \hookrightarrow W^{*}\) defined by (1.5). Let \(f(\nabla^{W*}, h^{W*}), f(\nabla^{H*}, h^{H*}) \in \Omega^{\text{odd}}(S)\) be as in (0.4). By [11, Thm. 2.22], we have

\[
d\mathcal{T}(\nabla^{W*}, \partial, h^{W*}) = f(\nabla^{W*}, h^{W*}) - f(\nabla^{H*}, h^{H*}).
\]

Let \((\widetilde{W}^{*} = \bigoplus_{k=0}^{n} \widetilde{W}^{k}, \nabla^{\widetilde{W}^{*}}, \widetilde{\partial})\) be another chain complex of flat complex vector bundles over \(S\). Let \(\tilde{H}^{*}\) be its cohomology. We assume that for \(k = 0, \ldots, n\),

\[
\text{rk}(W^{k}) = \text{rk}(\tilde{W}^{k}), \quad \text{rk}(H^{k}) = \text{rk}(\tilde{H}^{k}).
\]

Let \(h^{\tilde{W}^{*}} = \bigoplus_{k=0}^{n} h^{\tilde{W}^{k}}\) be a Hermitian metric on \(\tilde{W}^{*}\).

Let \(g^{TS}\) be a Riemannian metric on \(T.S\). Let \(\| \cdot \|\) be the norm on \(TS\) induced by \(g^{TS}\).

For \(\omega \in \Omega^{*}(S)\), we denote

\[
|\omega| = \sup_{k \in \mathbb{N}, x \in S, v_{1}, \ldots, v_{k} \in T_{x}S, |v_{1}|, \ldots, |v_{k}| \leq 1} |\omega(v_{1}, \ldots, v_{k})|.
\]

For an operator \(A \in W^{*}\), we denote by \(\|A\|\) its operator norm with respect to \(h^{W*}\). For \(A \in \Omega^{*}(S, \text{End}(W^{*}))\), we denote

\[
\|A\| = \sup_{k \in \mathbb{N}, x \in S, v_{1}, \ldots, v_{k} \in T_{x}S, |v_{1}|, \ldots, |v_{k}| \leq 1} \|A(v_{1}, \ldots, v_{k})\|.
\]

Let \(0 < \lambda_{\min} \leq \lambda_{\max}\) such that

\[
\text{Sp}((\partial^{*} + \partial)^{2}) \subseteq \{0\} \cup [\lambda_{\min}^{2}, \lambda_{\max}^{2}].
\]

Let \(l > 0\) such that

\[
\|\omega^{W*}\| \leq l.
\]

**Proposition 1.6.** There exists a function \(C : \mathbb{N} \times \mathbb{N} \times \mathbb{R}_{+} \times \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}\) such that for any \((W^{*}, \nabla^{W*}, \partial, h^{W*}), (\tilde{W}^{*}, \nabla^{\tilde{W}^{*}}, \widetilde{\partial}, h^{\tilde{W}^{*}}), \lambda_{\min}, \lambda_{\max}\) and \(l\) as above, if there exist an isomorphism of graded complex vector bundles \(\alpha : W^{*} \rightarrow \tilde{W}^{*}\) and \(0 < \delta < 13^{-1} \lambda_{\min}^{-1} \lambda_{\max}\) satisfying

\[
\|\alpha^{*} \widetilde{\partial} - \partial\| \leq \lambda_{\min} \delta, \quad -\delta h^{W*} \leq \alpha^{*} h^{\tilde{W}^{*}} - h^{W*} \leq \delta h^{W*}, \quad \|\alpha^{*} \omega^{\tilde{W}^{*}} - \omega^{W*}\| \leq \delta,
\]

then

\[
\mathcal{T}(\nabla^{W*}, \partial, h^{W*}) - \mathcal{T}(\nabla^{\tilde{W}^{*}}, \widetilde{\partial}, h^{\tilde{W}^{*}}) \leq C \left( \text{dim } S, \text{rk}(W^{*}), l, \lambda_{\max}/\lambda_{\min} \right) \delta^{1/2}.
\]
We denote and \( l \)
\[
(1.53)\frac{Sp((\partial^* + \partial)^2)}{\subseteq \{0\} \cup [1, \lambda_{\max}^2], \quad \|\alpha^*\beta - \partial\| \leq \delta.}
\]

By (1.45), we have
\[
(1.46)\|\alpha^*\beta\| \leq \delta + \|\partial\| \leq \delta + \lambda_{\max}.\]

Since \( \|A\| = \|A^*\| \) for any operator \( A \) on \( W^* \), we have
\[
(1.47)\|\alpha^*\beta\| = \|\alpha^*\beta - \partial\|.
\]

Let \( \partial^* \) be the adjoint of \( \partial \) with respect to \( h^W \). Note that \( \alpha^*\partial^* \) is the adjoint of \( \alpha^*\beta \) with respect to \( \alpha^*h^W \) and \( (\alpha^*\beta)^* \) is the adjoint of \( \alpha^*\beta \) with respect to \( h^W \), by the second inequality in (1.43), we have
\[
(1.48)\|\alpha^*\beta - (\alpha^*\beta)^*\| \leq 3\delta\|\alpha^*\beta\|.
\]

By (1.45)-(1.48) and the assumption \( 0 < \delta < 13^{-1}\lambda_{\max}^{-1} \), we have
\[
(1.49)\|\alpha^*\beta - (\alpha^*\beta)^*\| \leq \|\alpha^*\beta - (\alpha^*\beta)^*\| + \|\alpha^*\beta - \partial^*\|
\leq 3\delta(\delta + \lambda_{\max}) + \delta \leq 6\delta\lambda_{\max}.
\]

By (1.45), (1.49) and the assumption \( 0 < \delta < 13^{-1}\lambda_{\max}^{-1} \), we have
\[
(1.50)\|\partial^* + \partial - \alpha^*(\partial^* + \beta)\| \leq 6\delta\lambda_{\max} \leq \frac{6}{13},
\]
\[
(1.50)\|\partial^* - \partial - \alpha^*(\partial^* - \beta)\| \leq 6\delta\lambda_{\max} \leq \frac{6}{13}.
\]

By (1.45) and (1.50), we have
\[
(1.51)\frac{Sp((\partial^* + \partial)^2)}{\subseteq \left[0, \frac{6^2}{13^2}\right] \cup \left[\frac{7^2}{13^2}, 3\lambda_{\max}^2\right]}.
\]

Moreover, the dimension of the eigenspace of \( (\partial^* + \partial)^2 \) associated with eigenvalues in \( [0, \frac{6^2}{13^2}] \) equals the dimension of \( \text{Ker} ((\partial^* + \partial)^2) \). On the other hand, by (1.5) and the second identity in (1.38), we have
\[
(1.52)\text{dim Ker} ((\partial^* + \partial)^2) = \text{rk} H^* = \text{rk} H^* = \text{dim Ker} ((\partial^* + \partial)^2).
\]

As a consequence, the only possible eigenvalue of \( (\partial^* + \partial)^2 \) in \( [0, \frac{6^2}{13^2}] \) is zero, i.e.,
\[
(1.53)\frac{Sp((\partial^* + \partial)^2)}{\subseteq \{0\} \cup \left[\frac{7^2}{13^2}, 3\lambda_{\max}^2\right]}.
\]

In the sequel, we will use \( C_1, C_2, \ldots \) to denote constants depending on \( \text{dim} S, \text{rk} W^*, l \) and \( \lambda_{\max}/\lambda_{\min} \).

Let \( \omega^W \) be as in (1.27) with \( (W^*, \nabla^W, h^W) \) replaced by \( (\tilde{W}^*, \nabla^W, \tilde{h}^W) \). For \( t > 0 \), we denote
\[
(1.54)\frac{\chi_t = \sqrt{t}(\partial^* - \partial) + \frac{1}{2}\omega^W, \quad \tilde{\chi}_t = \alpha^*(\sqrt{t}(\partial^* - \partial) + \frac{1}{2}\omega^W)}{.}
\]
Set $U = \{ \lambda \in \mathbb{C} : -1 < \Re(\zeta) < 1 \}$. By (1.42), the third inequality in (1.43), (1.45), (1.50) and (1.53), for $\lambda \in \partial U$ and $t > 0$, we have
\begin{equation}
(1.55) \quad \| (\lambda - \lambda_i)^{-1} \| \leq C_1, \quad \| (\lambda - \overline{\lambda}_i)^{-1} \| \leq C_1, \quad \| \lambda_i - \overline{\lambda}_i \| \leq C_1(1 + \sqrt{t})\delta.
\end{equation}
Let $f^\wedge(\tilde{\Lambda}', h_t^\wedge\ast)$ be as in (1.32) with $(\nabla^W, \partial, h^W)$ replaced by $(\nabla^\wedge, \overline{\partial}, h^\wedge)$. For $t > 0$, by (1.34), we have
\begin{equation}
(1.56) \quad f^\wedge(\Lambda', h_t^W) - f^\wedge(\tilde{\Lambda}', h_t^\wedge\ast) = \frac{1}{2\pi i} \int_{\partial U} \phi \text{Tr} \left[ (-1)^{N^{\wedge}\ast} \frac{N_{\wedge}}{2} \left( (\lambda - \lambda_i)^{-1} - (\lambda - \overline{\lambda}_i)^{-1} \right) \right] f'(\lambda) d\lambda
= \frac{1}{2\pi i} \int_{\partial U} \varphi \text{Tr} \left[ (-1)^{N^{\wedge}\ast} \frac{N_{\wedge}^\ast}{2} (\lambda - \lambda_i)^{-1}(\lambda - \overline{\lambda}_i)^{-1} \right] f'(\lambda) d\lambda.
\end{equation}
By (1.55) and (1.56), we have
\begin{equation}
(1.57) \quad \left| f^\wedge(\Lambda', h_t^W) - f^\wedge(\tilde{\Lambda}', h_t^\wedge\ast) \right| \leq C_2(1 + \sqrt{t})\delta.
\end{equation}
Proceeding the same way as in the proofs of [11] Prop. 2.18, Thm. 2.13 and applying (1.42), the third inequality in (1.43), (1.45) and (1.53), we obtain the following estimates: for $0 < t < 1$,
\begin{equation}
(1.58) \quad \left| f^\wedge(\Lambda', h_t^W) - \frac{1}{2} \chi'(W^\ast) \right| \leq C_3t, \quad \left| f^\wedge(\tilde{\Lambda}', h_t^\wedge\ast) - \frac{1}{2} \chi'(W^\ast) \right| \leq C_3t;
\end{equation}
for $t > 1$,
\begin{equation}
(1.59) \quad \left| f^\wedge(\Lambda', h_t^W) - \frac{1}{2} \chi'(H^\ast) \right| \leq \frac{C_4}{\sqrt{t}}, \quad \left| f^\wedge(\tilde{\Lambda}', h_t^\wedge\ast) - \frac{1}{2} \chi'(H^\ast) \right| \leq \frac{C_4}{\sqrt{t}}.
\end{equation}
By (1.36), (1.38) and (1.57)–(1.59), we have
\begin{equation}
(1.60) \quad \left| \mathcal{T}(\nabla^W, \partial, h^W) - \mathcal{T}(\nabla^\wedge, \overline{\partial}, h^\wedge) \right|
\leq 2 \int_0^\delta C_3t \frac{dt}{t} + \int_{\delta}^{\delta^{-1}} C_2t(1 + \sqrt{t})\delta \frac{dt}{t} + 2 \int_{\delta^{-1}}^{+\infty} C_4t \frac{dt}{t} \leq C_5\delta^{1/2}.
\end{equation}
This completes the proof of Proposition 1.6. \qed

Remark 1.7. Let $\mu \in \Omega^1(S, \text{End}(\nabla^\wedge))$. We assume that $\mu$ preserves the degree, i.e.,
\begin{equation}
\mu\left( \mathcal{C}_\infty(S, W^k) \right) \subseteq \Omega^1(S, W^k) \text{ for } k = 0, \ldots, n.
\end{equation}
Set
\begin{equation}
(1.61) \quad f^\wedge(\overline{\partial}, h_t^\wedge)^\ast, \mu) = \varphi \text{Tr} \left[ (-1)^{N^\wedge}\ast \frac{N^\wedge}{2} f'(\sqrt{2}(\overline{\partial}^\ast - \overline{\partial}) + \frac{1}{2}h^\wedge) \right].
\end{equation}
Let $\mathcal{T}(\overline{\partial}, h^\wedge, \mu)$ be as in (1.36) with $f^\wedge(\Lambda', h_t^W)$ replaced by $f^\wedge(\overline{\partial}, h_t^\wedge\ast, \mu)$. Then Proposition 1.6 holds with $\omega^\wedge$ replaced by $\mu$ and $\mathcal{T}(\nabla^\wedge, \overline{\partial}, h^\wedge)$ replaced by $\mathcal{T}(\overline{\partial}, h^\wedge, \mu)$.

Let $(F, \nabla^F)$ be a flat complex vector bundle over $S$. Let $h_t^F$ and $h_t^F$ be Hermitian metrics on $F$. Let $\omega^F$ (resp. $\omega^F$) be as in (0.1) with $h^F$ replaced by $h_t^F$ (resp. $h_t^F$). We consider the chain complex $F \xrightarrow{\Id} F$, where the first $F$ is equipped with Hermitian
metric $h_1^F$ and the second $F$ is equipped with Hermitian metric $h_2^F$. Let $\mathcal{T}(\nabla^F, h_1^F, h_2^F)$ be its torsion form.

Let $l > 0$ such that

$$\|\omega_1^F\| \leq l.$$  

**Corollary 1.8.** There exists a function $C : N \times N \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $(F, \nabla^F, h_1^F, h_2^F)$ and $l$ as above, if there exists $\delta \in (0, 13^{-1})$ satisfying

$$-\delta h_1^F \leq h_2^F - h_1^F \leq \delta h_1^F,$$

then

$$\mathcal{T}(\nabla^F, h_1^F, h_2^F) \leq C(\dim S, \text{rk}(F), l)\delta^{1/2}.$$  

**Proof:** Note that $\mathcal{T}(\nabla^F, h_1^F, h_2^F) = 0$, the inequality (1.64) is a direct consequence of Proposition 1.6. \qed

### 1.3. Analytic torsion forms

Let $\pi : M \to S$ be a smooth fibration with compact fiber $Z$. Let $N = \partial M$. We assume that $\pi|_N : N \to S$ is a smooth fibration with fiber $Y$. Then we have $Y = \partial Z$.

We identify a tubular neighborhood of $N \subseteq M$ with $[-1, 0] \times N$ such that $N$ is identified with $\{0\} \times N$ and the following diagram commutes,

$$[-1, 0] \times N \xrightarrow{pr_2} M \xrightarrow{\pi} S,$$

where $pr_2 : [-1, 0] \times N \to N$ is the projection to the second factor.

Let $T^H M \subseteq TM$ be a horizontal sub bundle of $TM$, i.e.,

$$TM = T^H M \oplus TZ.$$  

Then we have

$$\Lambda^*(T^* M) = \Lambda^*(T^H^* M) \otimes \Lambda^*(T^* Z) \simeq \pi^*(\Lambda^*(T^* S)) \otimes \Lambda^*(T^* Z).$$

We assume that $T^H M$ is product on $[-1, 0] \times N$, i.e.,

$$T^H M|_{[-1, 0] \times N} = \text{pr}_2^*(T^H M|_N).$$

We remark that $T^H N := T^H M|_N \subseteq TN$ is a horizontal sub bundle of $TN$, i.e.,

$$TN = T^H N \oplus TY.$$  

Let $g^{TZ}$ be a Riemannian metric on $TZ$. Let $g^{TY}$ be the Riemannian metric on $TY$ induced by $g^{TZ}$ via the embedding $N = \partial M \hookrightarrow M$. Let $(u, y) \in [-1, 0] \times N$ be coordinates. We assume that $g^{TZ}$ is product on $[-1, 0] \times N$, i.e.,

$$g^{TZ}_{(u, y)} = du^2 + g^{TY}_y.$$  

Let $(F, \nabla^F)$ be a flat complex vector bundle over $M$. We trivialize $F|_{[-1, 0] \times N}$ along the curve $[-1, 0] \ni u \mapsto (u, y)$ using the parallel transport with respect to $\nabla^F$. We have

$$\mathcal{T}(F, \nabla^F)|_{[-1, 0] \times N} = \text{pr}_2^*(F|_N, \nabla^F|_N).$$
Let $h^F$ be a Hermitian metric on $F$. We assume that $h^F$ is product on $[-1,0] \times N$, i.e., under the identification (1.71), we have
\begin{equation}
(1.72) \quad h^F_{|[-1,0] \times N} = \text{pr}_2^* (h^F_N) .
\end{equation}
Set $\mathcal{F} = \Omega^\bullet (Z, F)$, which is a $\mathbb{Z}$-graded complex vector bundle of infinite dimension over $S$. By (1.67), we have the formal identity $\Omega^\bullet (M, F) = \Omega^\bullet (S, \mathcal{F})$.

For $U \in TS$, let $U^H \in T^H M$ be its horizontal lift, i.e., $\pi_* U^H = U$. For $U \in C^\infty (S, TS)$, let $L_{U^H}$ be the Lie differentiation operator acting on $\Omega^\bullet (M, F)$. For $U \in C^\infty (S, TS)$ and $s \in \Omega^\bullet (S, \mathcal{F}) = \Omega^\bullet (M, F)$, we define
\begin{equation}
(1.73) \quad \nabla_{U^H} s = L_{U^H} s .
\end{equation}
Then $\nabla_{U^H}$ is a connection on $\mathcal{F}$ preserving the grading.

Let $P^{TZ} : TM \to TZ$ be the projection with respect to (1.66). For $U, V \in C^\infty (S, TS)$, set
\begin{equation}
(1.74) \quad \mathcal{T} (U, V) = - P^{TZ} [U^H, V^H] \in C^\infty (M, TZ) .
\end{equation}
Then $\mathcal{T} \in C^\infty (M, \pi^* (\Lambda^2 (T^* S)) \otimes TZ)$. Let $\iota _\mathcal{T} \in C^\infty (M, \pi^* (\Lambda^2 (T^* S)) \otimes \text{End} (\Lambda^\bullet (T^* Z)))$ be the interior multiplication by $\mathcal{T}$ in the vertical direction.

The flat connection $\nabla^F$ (resp. $\nabla^F_{|2}$) naturally extends to an exterior differentiation operator on $\Omega^\bullet (M, F)$ (resp. $\Omega^\bullet (Z, F) = \mathcal{F}$), which we denote by $d^M$ (resp. $d^Z$). In the sense of [11, §2(a)], the operator $d^M$ is a superconnection of total degree 1 on $\mathcal{F}$. By [11, Prop. 3.4], we have
\begin{equation}
(1.75) \quad d^M = d^Z + \nabla^F + i_\mathcal{T} .
\end{equation}
Let $\mathcal{T}^*_\mathcal{F} \in C^\infty (M, \pi^* (\Lambda^2 (T^* S)) \otimes T^* Z)$ be the dual of $\mathcal{T}$ with respect to $g^{TZ}$. Let $h^\mathcal{F}$ be the $L^2$-metric on $\mathcal{F}$ with respect to $g^{TZ}$ and $h^F$. Let $d^{M,*}, d^{Z,*}, \nabla^{\mathcal{F},*}$ be the formal adjoints of $d^M, d^Z, \nabla^\mathcal{F}$ with respect to $h^\mathcal{F}$ in the sense of [11, Def. 1.6]. By [11, Prop. 3.7], we have
\begin{equation}
(1.76) \quad d^{M,*} = d^{Z,*} + \nabla^{\mathcal{F},*} - \mathcal{T}^* \wedge .
\end{equation}
Let $N^{TZ}$ be the number operator on $\Lambda^\bullet (T^* Z)$, i.e., $N^{TZ}|_{\Lambda^p (T^* Z)} = p \text{Id}$. Then $N^{TZ}$ acts on $\mathcal{F}$ in the obvious way. For $t > 0$, let $d_t^{M,*}$ be the formal adjoints of $d^M$ with respect to $h_t^\mathcal{F} := t^{N^{TZ}} h^\mathcal{F}$. We have
\begin{equation}
(1.77) \quad d_t^{M,*} = t d^{Z,*} + \nabla^{\mathcal{F},*} - \frac{1}{t} \mathcal{T}^* \wedge .
\end{equation}
Set
\begin{equation}
(1.78) \quad \mathcal{D}_t = t^{N^{TZ}/2} (d_t^{M,*} - d^M) t^{-N^{TZ}/2}
\end{equation}
\begin{equation}
\quad = \frac{\sqrt{t}}{2} (d^{Z,*} - d^Z) + \frac{1}{2} \left( \nabla^{\mathcal{F},*} - \nabla^\mathcal{F} \right) - \frac{1}{2 \sqrt{t}} (\mathcal{T}^* \wedge + i \mathcal{T}) .
\end{equation}
We denote
\begin{equation}
(1.79) \quad \omega^* = \nabla^{\mathcal{F},*} - \nabla^\mathcal{F} \in \Omega^1 (S, \text{End} (\mathcal{F})) .
\end{equation}
For $X \in TZ$, we denote by $X^* \in T^* Z$ its dual with respect to $g^{TZ}$. For $X \in TZ$, we denote
\begin{equation}
(1.80) \quad \partial (X) = X^* \wedge + i X \in \text{End} (\Lambda^\bullet (T^* Z)) .
\end{equation}
By (1.78)-(1.80), we have

\begin{equation}
D_t = \frac{\sqrt{t}}{2}(dZ,dZ) + \frac{1}{2} \omega^F - \frac{1}{2\sqrt{t}} \hat{c}(T).
\end{equation}

In particular,

\begin{equation}
D_t^2 = -\frac{t}{4}(dZ,dZ) + \omega^F - \frac{1}{2} \sqrt{t} \hat{c}(T).
\end{equation}

where \(dZ,dZ + dZ,dZ\) is the fiberwise Hodge Laplacian.

By (1.82), the operator \(D_t^2\) is fiberwise essentially self-adjoint with respect to the absolute boundary condition (see (0.19)). Its self-adjoint extension with respect to the absolute boundary condition will still be denoted by \(D_t^2\). Let \(\text{End}_a(F) \subseteq \text{End}(F)\) be the sub vector bundle of trace class operators. Recall that \(f'(z) = (1 + 2z^2)e^{z^2}\). By (1.82), we have

\begin{equation}
\phi' \in \Omega^\bullet(\text{End}_a(F))\right).
\end{equation}

Let \(H^\bullet(Z,F)\) be the fiberwise singular cohomology of \(Z\) with coefficients in \(F\). Then \(H^\bullet(Z,F)\) is a \(\mathbb{Z}\)-graded complex vector bundle over \(S\). We denote

\begin{equation}
\chi'(Z,F) = \sum_{p=0}^{\dim Z} (-1)^p \text{rk}(H^p(Z,F)).
\end{equation}

Now we recall the definition of analytic torsion forms \[61\, \text{Def. 2.18}, \, [11\, \text{Def. 3.22}].

**Definition 1.9.** The analytic torsion form associated with \((T^H M, g^{TZ}, h^F)\) is defined by

\begin{equation}
\mathcal{T}(T^H M, g^{TZ}, h^F) = -\int_0^{+\infty} \left\{ \phi \text{Tr} \left[ (-1)^{NTZ} \frac{N^{TZ}}{2} f'(D_t) \right] - \frac{\chi'(Z,F)}{2} \right\} dt.
\end{equation}

The convergence of the integral in (1.84) follows from the family local index theorem \[11\, \text{Thm 3.21}, \, [61\, \text{Thm 2.17}]. And \(d \mathcal{T}(T^H M, g^{TZ}, h^F)\) is given by (0.26) with \(j = 0\).

Recall that \(Q^S\) is the vector space of real even differential forms on \(S\) and \(Q^S, 0 \subseteq Q^S\) is the sub vector space of exact forms. The analytic torsion form \(\mathcal{T}(T^H M, g^{TZ}, h^F)\) is viewed as an element in \(Q^S/Q^{S,0}\).

### 2. Finite dimensional model

The construction in this section may be viewed as a model of the problem addressed in this paper, in which the fibration has zero-dimensional fibers. This section is organized as follows. In §2.1 we construct a short exact sequence of chain complexes from a pair of linear maps. In §2.2 we extend the constructions in §2.1 to flat complex vector bundles.
2.1. Chain complexes from a pair of linear maps. Let $V_1$, $V_2$ and $V$ be finite-dimensional complex vector spaces. Let $\tau_1 : W_1 \to V$ and $\tau_2 : W_2 \to V$ be linear maps. We define a chain complex $(C^\bullet(\tau_1, \tau_2), \partial)$ as follows,

\begin{equation}
0 \to C^0(\tau_1, \tau_2) := W_1 \oplus W_2 \overset{\partial}{\to} C^1(\tau_1, \tau_2) := V \to 0
\end{equation}

$$
\begin{array}{c}
(w_1, w_2) \mapsto \tau_2(w_2) - \tau_1(w_1).
\end{array}
$$

We denote

\begin{equation}
V_1 = \text{Im}(\tau_1) \subseteq V, \quad V_2 = \text{Im}(\tau_2) \subseteq V.
\end{equation}

We define a chain complex $(C^\bullet(\tau_1, \tau_2), \partial)$ as follows,

\begin{equation}
0 \to C^0(\tau_1, \tau_2) := V_1 \oplus V_2 \overset{\partial}{\to} C^1(\tau_1, \tau_2) := V \to 0
\end{equation}

$$
\begin{array}{c}
(v_1, v_2) \mapsto v_2 - v_1.
\end{array}
$$

We denote

\begin{equation}
K_1 = \text{Ker}(\tau_1) \subseteq W_1, \quad K_2 = \text{Ker}(\tau_2) \subseteq W_2.
\end{equation}

We have a short exact sequence of chain complexes,

\begin{equation}
\begin{array}{ccccccccc}
0 & \to & 0 & \to & C^1(\tau_1, \tau_2) & \overset{\text{Id}}{\to} & C^1(\tau_1, \tau_2) & \to & 0 \\
& & & \downarrow & & \downarrow & & & \\
0 & \to & K_1 \oplus K_2 & \to & C^0(\tau_1, \tau_2) & \overset{\tau_1 \oplus \tau_2}{\to} & C^0(\tau_1, \tau_2) & \to & 0,
\end{array}
\end{equation}

where $K_1 \oplus K_2 \to C^0(\tau_1, \tau_2) = W_1 \oplus W_2$ is the direct sum of the embeddings in (2.4).

Set

\begin{equation}
C_0^\bullet = C^\bullet(\tau_1, \tau_2), \quad C_1^\bullet = C^\bullet(\tau_1, \text{Id}_V), \quad C_2^\bullet = C^\bullet(\text{Id}_V, \tau_2), \quad C_3^\bullet = C^\bullet(\text{Id}_V, \text{Id}_V).
\end{equation}

We define

\begin{equation}
\alpha_1 : C_0^\bullet \to C_1^\bullet, \quad \alpha_2 : C_0^\bullet \to C_2^\bullet, \quad \beta_1 : C_1^\bullet \to C_3^\bullet, \quad \beta_2 : C_2^\bullet \to C_3^\bullet
\end{equation}

as follows,

\begin{equation}
\begin{array}{c}
\alpha_1|c_0^\bullet = \text{Id}_W \oplus \tau_2, \quad \alpha_2|c_0^\bullet = \tau_1 \oplus \text{Id}_W, \quad \alpha_1|c_3^\bullet = \alpha_2|c_3^\bullet = \text{Id}_V, \\
\beta_1|c_1^\bullet = \tau_1 \oplus \text{Id}_V, \quad \beta_2|c_2^\bullet = \text{Id}_V \oplus \tau_2, \quad \beta_1|c_3^\bullet = \beta_2|c_3^\bullet = \text{Id}_V.
\end{array}
\end{equation}

We have a short exact sequence of chain complexes,

\begin{equation}
\begin{array}{ccccccccc}
0 & \to & C_0^\bullet & \overset{\alpha_1 \oplus \alpha_2}{\to} & C_1^\bullet \oplus C_2^\bullet & \overset{\beta_2 - \beta_1}{\to} & C_3^\bullet & \to & 0.
\end{array}
\end{equation}

For $j = 0, 1, 2, 3$, let $H^k(C_j^\bullet, \partial)$ be the $k$-th cohomology group of $(C_j^\bullet, \partial)$, i.e.,

\begin{equation}
H^k(C_j^\bullet, \partial) = \frac{\text{Ker}(\partial : C_j^k \to C_j^{k+1})}{\text{Im}(\partial : C_j^{k-1} \to C_j^k)}.
\end{equation}

From (2.9), we get a long exact sequence of cohomology groups,

\begin{equation}
\cdots \to H^k(C_0^\bullet, \partial) \to H^k(C_1^\bullet \oplus C_2^\bullet, \partial) \to H^k(C_3^\bullet, \partial) \to \cdots
\end{equation}
We denote
\[(2.12) \quad W_{12} = \left\{ (w_1, w_2) \in W_1 \oplus W_2 : \tau_1(w_1) = \tau_2(w_2) \right\}. \]

A direct calculation yields
\[(2.13) \quad H^0(C^j_0, \partial) = W_{12}, \quad H^0(C^j_0 \oplus C^j_2, \partial) = W_1 \oplus W_2, \quad H^0(C^j_3, \partial) = V, \]
\[H^1(C^j_0, \partial) = V/(V_1 + V_2), \quad H^1(C^j_1 \oplus C^j_2, \partial) = H^1(C^j_3, \partial) = 0. \]

Thus the long exact sequence \((2.11)\) is
\[(2.14) \quad 0 \to W_{12} \to W_1 \oplus W_2 \xrightarrow{\tau_2 - \tau_1} V \to V/(V_1 + V_2) \to 0, \]
where \(W_{12} \hookrightarrow W_1 \oplus W_2\) is the direct sum of the obvious embeddings \(W_{12} \hookrightarrow W_1\) and \(W_{12} \hookrightarrow W_2\).

2.2. A flat family of complexes. Now let \(W_1, W_2\) and \(V\) be flat complex vector bundles over a smooth manifold \(S\). Let \(\tau_1 : W_1 \to V\) and \(\tau_2 : W_2 \to V\) be morphisms between flat complex vector bundles. Then the chain complexes \((C^j_\bullet, \partial)\) \((j = 0, 1, 2, 3)\) considered in §2.1 become chain complexes of flat complex vector bundles over \(S\).

Let \(h^{W_1}, h^{W_2}\) and \(h^V\) be Hermitian metrics on \(W_1, W_2\) and \(V\). For \(j = 0, 1, 2, 3\), we construct a Hermitian metric \(h^{C^0_j} = h^{C^0_j} \oplus h^{C^1_j}\) on \(C^*_j\) as follows,
\[(2.15) \quad h^{C^0_0} = h^{W_1} \oplus h^{W_2}, \quad h^{C^0_2} = \frac{1}{2} h^V \oplus \frac{1}{2} h^V; \quad h^{C^0_1} = h^{W_1} \oplus \frac{1}{2} h^V; \quad h^{C^0_3} = h^{W_2} \oplus \frac{1}{2} h^V \quad \text{for } j = 1, 2; \]
\[h^{C^1_1} = h^V \quad \text{for } j = 0, 1, 2, 3. \]

Recall that \(Q^{S,0} \subseteq Q^S \subseteq \Omega^*(S)\) were defined in the paragraph containing (0.8). Let \(\mathcal{F}_j \in Q^S\) be the torsion form (cf. §1.2) associated with \((C^*_j, \partial, h^{C^*_j})\).

The exact sequence \((2.11)\) becomes an exact sequence of flat complex vector bundles. Let \(\mathcal{F}_\infty \in Q^S\) be the torsion form (cf. §1.2) associated with the exact sequence \((2.11)\) equipped with Hermitian metrics induced by \(h^{C^*_j}\) via \((1.5)\).

The following theorem is a consequence of [24, Thm. 7.37], which may be viewed as an analogue of a result of Ma on the Bott-Chern forms [38, Thm 1.2], and may also be viewed as a finite dimensional version of [39, Thm 0.1].

**Theorem 2.1.** The following equation holds,
\[(2.16) \quad \mathcal{F} - \mathcal{F}_1 - \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_\infty \in Q^{S,0}. \]

3. Gluing formula for analytic torsion forms

This section is the heart of this paper. The central idea is to deform the metrics on \(TZ\) and \(F\) such that the gluing formula considered in Theorem 0.1 degenerates to the gluing formula given in Theorem 2.1. This section is organized as follows. In §3.1, we introduce a two-parameter deformation of the objects constructed in the introduction. In §3.2, we prove Theorem 0.1. The proof is based on several intermediate results. Their proofs are delayed to §3.3, §3.4.
3.1. A two-parameter deformation. Recall that \( \pi_R : M_R \to S \) was constructed in the paragraph containing (0.30). By the second identity in (0.29), we may view \( M'_1, M'_2 \) and \( \IN_R \) as subsets of \( M_R \). Set

\[
M_{1,R} = M'_1 \cup \IN_R, \quad M_{2,R} = M'_2 \cup \IN_R, \quad M_{3,R} = \IN_R.
\]

For convenience, we denote \( M_{0,R} = M_R \). For \( j = 0, 1, 2, 3 \), set

\[
\pi_{j,R} = \pi_R|_{M_{j,R}} : M_{j,R} \to S.
\]

Let \( Z_{j,R} \) be the fiber of \( \pi_{j,R} \). We denote \( Z_R = Z_{0,R} \).

Recall that the diffeomorphism \( \varphi_R : M \to M_R \) was constructed in (0.32). Recall that \( T^H M \subseteq TM \) was constructed in the paragraph containing (0.7). Set

\[
T^H M_R = \varphi_{R,*}(T^H M) \subseteq TM_R.
\]

Then we have

\[
TM_R = T^H M_R \oplus TZ_R.
\]

Recall that \( T^H N \subseteq TN \) was constructed in the paragraph containing (0.13). By (0.14), (0.32), and (3.3), we have

\[
T^H M_R|_{\IN_R} = \pr_2^*(T^H N),
\]

where \( \pr_2 : \IN_R = [-R, R] \times N \to N \) is the projection to the second factor. For \( j = 0, 1, 2, 3 \), set

\[
T^H M_{j,R} = T^H M_R|_{M_{j,R}} \subseteq TM_{j,R}.
\]

Recall that the metric \( g^{TZ_R} \) on \( TZ_R \) was constructed in (0.34). For \( j = 0, 1, 2, 3 \), set

\[
g^{TZ_{j,R}} = g^{TZ_R}|_{M_{j,R}}.
\]

Recall that the flat complex vector bundle \((F, \nabla^F)\) over \( Z_R \) and the Hermitian metric \( h^F \) on \( F \) were constructed in the paragraph containing (0.40). We remark that (0.16) and (0.17) hold with \( \IN \) replaced by \( \IN_R \).

Let \( f_\infty : [-1, 1] \to \mathbb{R} \) be as in (0.35). We further assume that

\[
f_\infty(s) = f_\infty(-s), \quad |f'_\infty(s)| \leq 2, \quad \text{for } |s| \leq 1;
\]

\[
f_\infty(s) = 1 - s^2/2, \quad \text{for } |s| \leq 1/4;
\]

\[
f_\infty(s) = (s - b)^2/2, \quad \text{for } b = \pm 1, \ |s - b| \leq 1/4.
\]

Let \( \chi : \mathbb{R} \to \mathbb{R} \) be a smooth function such that

\[
0 \leq \chi \leq 1, \quad |\chi|_{[-1/4, 1]} = 0, \quad |\chi|_{[1/2, \infty[} = 1, \quad 0 \leq \chi' \leq 8.
\]

As \( f'_\infty \) is an odd function, for \( T \geq 0 \), there exists a unique smooth function \( f_T : [-1, 1] \to \mathbb{R} \) satisfying

\[
f_T(-1) = f_T(1) = 0, \quad f''_T(s) = f'_\infty(s) \chi(e^{T^2(1 - |s|)}).
\]

By (3.8)-(3.10), the following uniform estimates hold,

\[
f_T(s) = f_\infty(s) + \mathcal{O}(e^{-T^2}), \quad f'_T(s) = f'_\infty(s) + \mathcal{O}(e^{-T^2}), \quad f''_T(s) = \mathcal{O}(1).
\]
Moreover, we have \( \text{supp}(f_T) \subseteq [-1 + e^{-T^2}/4, 1 - e^{-T^2}/4] \). We will view \( f_T \) as a smooth function on \( M_R \) in the sense of (3.37). Set
\[
(3.12) \quad h_T^F = e^{-2Tf_T}h_F^F.
\]

For \( j = 0, 1, 2, 3 \), let
\[
(3.13) \quad \mathcal{F}_{j,R,T} \in Q^S
\]
be the analytic torsion form (cf. Definition 1.9) associated with
\[
(3.14) \quad \left( \pi_{j,R}, T^H M_{j,R}; g^{T^2_{j,R}}, F_{M_{j,R}}; \nabla^F_{M_{j,R}}; h_T^F \big|_{M_{j,R}} \right).
\]

For \( j = 0, 1, 2, 3 \), let \( d^{Z_{j,R}}_T \) be the de Rham operator on \( \Omega^\bullet(Z_{j,R}, F) \). Let \( \| \cdot \|_{Z_{j,R}} \) be the \( L^2 \)-metric on \( \Omega^\bullet(Z_{j,R}, F) \) with respect to \( g^{T_{j,R}} \) and \( h_T^F \). Let \( d^{Z_{j,R}}_T \) be the formal adjoint of \( d^{Z_{j,R}}_T \) with respect to \( \| \cdot \|_{Z_{j,R}} \). Set
\[
(3.15) \quad d^{Z_{j,R}}_T = e^{-Tf_T}d^{Z_{j,R}}_T e^{Tf_T}, \quad d^{Z_{j,R}}_T \ast = e^{Tf_T}d^{Z_{j,R}}_T e^{-Tf_T}, \quad D^{Z_{j,R}}_T = d^{Z_{j,R}}_T + d^{Z_{j,R}}_T \ast.
\]

We remark that \( e^{Tf_T}D^{Z_{j,R}}_T e^{-Tf_T} \) is the Hodge de Rham operator with respect to \( g^{T_{j,R}} \) and \( h_T^F \). The self-adjoint extension of \( D^{Z_{j,R}}_F \) with domain \( \text{Dom}(D^{Z_{j,R}}_F) = \Omega_{abs}^\bullet(Z_{j,R}, F) \) (cf. [50, (1.4)]) will still be denoted by \( D^{Z_{j,R}}_F \). By the Hodge theorem (cf. [14, Thm 3.1][50, Thm. 1.1]), the following map is bijective,
\[
(3.16) \quad \text{Ker}(D^{Z_{j,R}}_F) \to H^\bullet(Z_{j,R}, F)
\]
\[
\omega \mapsto [e^{Tf_T} \omega].
\]

For \( j = 0, 1, 2, 3 \), let \( h_{H^\bullet(Z_{j,R})}^{Z_{j,R}} \) be the Hermitian metric on \( H^\bullet(Z_{j,R}, F) = H^\bullet(Z_{j,R}, F) \) induced by \( \| \cdot \|_{Z_{j,R}} \) via the identification (3.16). Let
\[
(3.17) \quad \mathcal{F}_{j,R,T} \in Q^S
\]
be the torsion form ([11, §2], cf. [11,2] associated with the exact sequence (0.22) equipped with Hermitian metrics \( (h_{H^\bullet(Z_{j,R})}^{Z_{j,R}})_{j=0,1,2,3} \).

3.2. Several intermediate results. We fix a constant \( 0 < \kappa < 1/3 \).

**Theorem 3.1.** There exists \( \alpha > 0 \) such that for \( j = 0, 1, 2, 3 \) and \( T = R^\alpha \gg 1 \), we have
\[
(3.18) \quad \text{Sp}(RD^{Z_{j,R}}_T) \subseteq ]-\infty, -\alpha\sqrt{T}] \cup [-1, 1] \cup [\alpha\sqrt{T}, +\infty[.
\]

Let \( \mathcal{E}[^{[-1,1]}_{j,R,T}] \subseteq \Omega^\bullet(Z_{j,R}, F) \) be eigenspace of \( RD^{Z_{j,R}}_T \) associated with eigenvalues in \([-1, 1]\). Since \( d^{Z_{j,R}}_T \) commutes with \( D^{Z_{j,R}}_T^2 \), we have a finite dimensional complex
\[
(3.19) \quad (\mathcal{E}[^{[-1,1]}_{j,R,T}], d^{Z_{j,R}}_T).
\]

Recall that the chain complexes of flat complex vector bundles \( (C^j, \partial) \) with \( j = 0, 1, 2, 3 \) were constructed in §2.2. Their construction depends on the morphisms \( \tau_1 : W_1 \to V \) and \( \tau_2 : W_2 \to V \). In the sequel, we take
\[
(3.20) \quad V = H^\bullet(Y, F), \quad W_j = H^\bullet(Z_j, F), \quad \tau_j([\alpha]) = [\alpha]_V \quad \text{for } j = 1, 2.
\]
Then \( W_1, W_2 \) and \( V \) are \( \mathbb{Z} \)-graded. We will use the notations \( W_1^*, W_2^*, V^* \) and \((C^{\bullet \bullet}, \partial)\) to emphasize the grading, i.e., \( C^{0,0} = W_1^1 \oplus W_2^1, C^{1,1} = V^1 \), etc.

Now we construct a Hermitian metric on \( C^{\bullet \bullet} \). Let \( D^Y \) be the Hodge de Rham operator on \( \Omega^*(Y, F) \) with respect to \( g^T_{ij} \) and \( h^F_{j} \). We denote \( \mathcal{H}^*(Y, F) \) as \( \text{Ker}(D^Y) \). For \( j = 1, 2 \), let

\[
(3.21)\quad \mathcal{H}^\bullet_{abs}(Z_{j,\infty}, F) \subseteq \left\{ \omega \in \Omega^*(Z_{j,\infty}, F) : d\bar{z}_{j,\infty}\omega = d\bar{z}_{j,\infty}\ast\omega = 0 \right\} \times \mathcal{H}^*(Y, F)
\]

be as in [50, (2.52)]. By [50] Prop. 3.16, Thm. 3.19, the map

\[
(3.22)\quad \mathcal{H}^\bullet_{abs}(Z_{j,\infty}, F) \rightarrow H^*(Z_{j,\infty}, F) = W_j^*
\]

is bijective. By [50], (2.39), (2.52)], the following diagram commutes,

\[
\begin{array}{ccc}
\mathcal{H}^\bullet_{abs}(Z_{j,\infty}, F) & \xrightarrow{(\omega, \hat{\omega}) \mapsto \hat{\omega}} & W_j^* \\
\downarrow & & \downarrow \\
\mathcal{H}^\bullet(Y, F) & \xrightarrow{\hat{\omega} \mapsto \hat{\omega}} & V^* \\
\end{array}
\]

Let \( D^Z_{j,\infty} \) be the Hodge de Rham operator on \( \Omega^*(Z_{j,\infty}, F) \) with respect to \( g^T_{Z_{j,\infty}} \) and \( h^F_{Z_{j,\infty}} \). By [50], (2.40), we have

\[
(3.23)\quad W_j^* = K_j^* \oplus K_j^{*\perp}
\]

with

\[
K_j^* = \left\{ [\omega] : (\omega, \hat{\omega}) \in \mathcal{H}^\bullet_{abs}(Z_{j,\infty}, F), \hat{\omega} = 0 \right\},
\]

\[
(3.25)\quad K_j^{*\perp} = \left\{ [\omega] : (\omega, \hat{\omega}) \in \mathcal{H}^\bullet_{abs}(Z_{j,\infty}, F), \omega \text{ generalized eigensection of } D^Z_{j,\infty} \text{ associated with } 0 \right\}.
\]

\textbf{Remark 3.2}. As a convention, a generalized eigenvalue (resp. eigensection) is always associated with the absolutely continuous spectrum. In other words, a generalized eigenvalue (resp. eigensection) is not an eigenvalue (resp. eigensection).

By (3.23), the definition of \( K_j^* \) in (3.25) is compatible with (2.4). We construct a Hermitian metric \( h^{K_j^*} \) on \( K_j^* \) as follows: for \((\omega, \hat{\omega}) \in \mathcal{H}^\bullet_{abs}(Z_{j,\infty}, F) \) with \( \hat{\omega} = 0 \),

\[
(3.26)\quad h^{K_j^*}( [\omega], [\omega] ) = \| \omega \|^2_{Z_{j,\infty}}.
\]

By [50], (2.53), we have \( \| \omega \|^2_{Z_{j,\infty}} < +\infty \). Hence \( h^{K_j^*} \) is well-defined. We construct a Hermitian metric \( h^{K_j^{*\perp}} \) on \( K_j^{*\perp} \) as follows: for \((\omega, \hat{\omega}) \in \mathcal{H}^\bullet_{abs}(Z_{j,\infty}, F) \) with \( \omega \) a generalized eigensection of \( D^Z_{j,\infty} \),

\[
(3.27)\quad h^{K_j^{*\perp}}( [\omega], [\omega] ) = \| \hat{\omega} \|^2_Y.
\]

Set

\[
(3.28)\quad h^{W_j^*}_{R,T} = h^{K_j^*} \oplus \sqrt{\frac{\pi}{2} RT^{-1/2} h^{K_j^{*\perp}}}.\]
Let $h^{V\bullet}$ be the Hermitian metric on $V\bullet$ induced by $\| \cdot \|_Y$ via the identification $V\bullet = \mathcal{H}^\bullet(Y, F)$ induced by the Hodge theory. Set
\begin{equation}
(3.29)
\hat{h}_{R,T}^{V\bullet} = \sqrt{\pi RT^{-1/2}} h^{V\bullet}.
\end{equation}

We construct a Hermitian metric $h_j^{C_j^{\bullet\bullet}}$ on $C_j^{\bullet\bullet}$ as follows,
\begin{equation}
(3.30)
\begin{align*}
h_{R,T}^{C_0^{\bullet\bullet}} &= h_{R,T}^{W^\bullet} \oplus h_{R,T}^{W_2^\bullet}, \\
h_{R,T}^{C_j^{\bullet\bullet}} &= h_{R,T}^{W^\bullet} \oplus \frac{1}{2} h_{R,T}^{V^\bullet} \text{ for } j = 1, 2, \\
h_{R,T}^{C_3^{\bullet\bullet}} &= h_{R,T}^{V^\bullet}, \\
h_{R,T}^{C_j^{\bullet\bullet}} &= h_{R,T}^{V^\bullet} \text{ for } j = 0, 1, 2, 3.
\end{align*}
\end{equation}

For a positive function $G(R, T)$ on $R, T$ and a two-parameter family of operators $A_{R,T} \in \text{End}(C_j^{\bullet\bullet})$, we denote $A_{R,T} = \mathcal{O}_{R,T}(G(R, T))$ if there exists $C > 0$ such that the operator norm of $A_{R,T}$ with respect to $h_{R,T}^{C_j^{\bullet\bullet}}$ is bounded by $CG(R, T)$.

**Theorem 3.3.** There exist linear maps
\begin{equation}
(3.31)
\mathcal{I}_{j,R,T} : C_j^{\bullet\bullet} \to \mathcal{E}_{j,R,T}^{[-1,1]}
\end{equation}
with $j = 0, 1, 2, 3$ such that
- the map $\mathcal{I}_{j,R,T}$ preserves the grading, i.e.,
\begin{equation}
(3.32)
\mathcal{I}_{j,R,T}(C_j^{p,q}) \subseteq \mathcal{E}_{j,R,T}^{[-1,1]} \cap \Omega^{p+q}(Z_{j,R}, F);
\end{equation}
- for $T = R^c \gg 1$, the map $\mathcal{I}_{j,R,T}$ is bijective (as a consequence, $\dim \mathcal{E}_{j,R,T}^{[-1,1]}$ is independent of $T = R^c$);
- for $T = R^c \gg 1$ and $\sigma \in C_j^{\bullet\bullet}$, we have
\begin{equation}
(3.33)
\left\| \mathcal{I}_{j,R,T}(\sigma) \right\|_{Z_{j,R}}^2 = h_{R,T}^{C_j^{\bullet\bullet}}(\sigma, \sigma) \left( 1 + \mathcal{O}(R^{-1/2 + \kappa/4}) \right);
\end{equation}
- for $T = R^c \gg 1$, we have
\begin{equation}
(3.34)
\mathcal{I}_{j,R,T}^{-1} \circ d_{j,R}^{Z_{j,R}} \circ \mathcal{I}_{j,R,T} = \pi^{-1/2} R^{-1/2} T^{1/2} e^{-T} \left( \partial + \mathcal{O}_{R,T}(R^{-1/2 + \kappa/4}) \right).
\end{equation}

For ease of notations, we denote
\begin{equation}
(3.35)
\partial_T = \pi^{-1/2} T^{1/2} e^{-T} \partial : C_j^{0\bullet} \to C_j^{1\bullet}.
\end{equation}

Let $\mathcal{F}_{k,R,T}^{j} \in Q_j^S$ be the torsion form (cf. §1.2) associated with $(C_j^{k\bullet}, R^{-1} \partial_T, h_{R,T}^{C_j^{k\bullet}})$. We view $(C_j^{\bullet\bullet}, R^{-1} \partial_T)$ as a complex, whose component of degree $k$ is given by $\bigoplus_{p+q=k} C_j^{p,q}$. Let $\mathcal{F}_{j,R,T}^{k} \in Q_j^S$ be the torsion form associated with $(C_j^{\bullet\bullet}, R^{-1} \partial_T, h_{R,T}^{C_j^{\bullet\bullet}})$. The following identity is a consequence of [24, Thm. 7.37], which may be viewed as an analogue of a result of Ma on the Bott-Chern forms [38, Thm 1.2], and may also be viewed a finite dimensional version of [39, Thm 0.1],
\begin{equation}
(3.36)
\mathcal{F}_{j,R,T} = \sum_{k=0}^{\dim Z} (-1)^k \mathcal{F}_{j,R,T}^{k}.
\end{equation}
For \( G(R, T) \) a positive function on \( R, T \geq 1 \) and \( (\tau_{R,T})_{R,T \geq 1} \) a family of differential forms on \( S \) with values in a Hermitian vector bundle \((E, \| \cdot \|_E)\), we write
\[
(3.37) \quad \tau_{R,T} = \mathcal{O}_E(G(R, T)),
\]
if there exists \( C > 0 \) such that the \( C^0 \)-norm of \( \tau_{R,T} \) is dominated by \( CG(R, T) \) for \( R, T \geq 1 \). We remark that \( \mathcal{O}_E(\cdot) \) is independent of the norm \( \| \cdot \|_E \). If \( E \) is a trivial line bundle, we abbreviate (3.37) as \( \tau_{R,T} = \mathcal{O}(G(R, T)) \).

We equip \( Q^S \) with the \( C^0 \)-norm. Then, by the de Rham theorem (cf. [15, Thm. 1.1 (d)]), \( Q^{S,0} \subseteq Q^S \) is closed. We equip \( Q^S/Q^{S,0} \) with quotient norm. For a family of elements in \( Q^S/Q^{S,0} \) parameterized by \( R, T \geq 1 \), we use the same notation as in (3.37) with the \( C^0 \)-norm replaced by the quotient norm.

**Theorem 3.4.** For \( T = R^\kappa \gg 1 \), the following identity holds in \( Q^S/Q^{S,0} \),
\[
(3.38) \quad \sum_{j=0}^3 (-1)^{(j-3)/2} \mathcal{F}_{j,R,T} = \sum_{j=0}^3 (-1)^{(j-3)/2} \mathcal{F}_{j,R,T} + \mathcal{O}(R^{-\kappa/4}) .
\]

We have a Mayer-Vietoris exact sequence of flat complex vector bundles over \( S \),
\[
(3.39) \quad 0 \to H^0(C_0^+, R^{-1}\partial T) \to H^0(C_1^+, C_2^+ \oplus R^{-1}\partial T) \to H^0(C_3^+, R^{-1}\partial T) \to H^1(C_0^+, R^{-1}\partial T) = 0 ,
\]
which is induced by (2.9) with \( \partial \) replaced by \( R^{-1}\partial T \). We equip the cohomology groups in (3.39) with Hermitian metrics induced by \( h \). Let \( \mathcal{F}_{\mathcal{F},R,T} \in Q^S \) be the torsion form (cf. §1.2) associated with the exact sequence (3.39). Set
\[
(3.40) \quad \mathcal{F}_{\mathcal{F},R,T} = \sum_k (-1)^k \mathcal{F}_{\mathcal{F},R,T} \in Q^S .
\]

**Theorem 3.5.** For \( T = R^\kappa \gg 1 \), the following identity holds in \( Q^S/Q^{S,0} \),
\[
(3.41) \quad \mathcal{F}_{\mathcal{F},R,T} = \mathcal{F}_{\mathcal{F},R,T} + \mathcal{O}(R^{-1/4+\kappa/8}) .
\]

**Proof of Theorem 3.1** By Theorem 2.1, (3.36) and (3.40), the following identity holds in \( Q^S/Q^{S,0} \),
\[
(3.42) \quad \sum_{j=0}^3 (-1)^{(j-3)/2} \mathcal{F}_{j,R,T} + \mathcal{F}_{\mathcal{F},R,T} = 0 .
\]

By Theorems 3.4, 3.5 and (3.42), the following identity holds in \( Q^S/Q^{S,0} \) as \( T = R^\kappa \to +\infty \),
\[
(3.43) \quad \sum_{j=0}^3 (-1)^{(j-3)/2} \mathcal{F}_{j,R,T} + \mathcal{F}_{\mathcal{F},R,T} = \mathcal{O}(R^{-\kappa/4}) .
\]

On the other hand, using the anomaly formula [11, Thm. 3.24] [62, Thm. 1.5] in the same way as in [62, §1.7], we can show that the left hand side of (3.43) is independent of \( R \) and \( T \). Hence, for any \( R \geq 1 \) and \( T \geq 0 \), we have
\[
(3.44) \quad \sum_{j=0}^3 (-1)^{(j-3)/2} \mathcal{F}_{j,R,T} + \mathcal{F}_{\mathcal{F},R,T} \in Q^{S,0} .
\]
Taking \( R = 1 \) and \( T = 0 \) in (3.44), we obtain (0.27). This completes the proof of Theorem 0.1. \( \square \)

4. One-dimensional Witten type deformation

The construction in this section may be viewed as a model of the problem addressed in this paper, in which the fibration has one-dimensional fibers. This one-dimensional model and the zero-dimensional model constructed in §2 are linked by a Witten deformation, i.e., to take \( T \to +\infty \). This section is organized as follows. In §4.1, we construct a sheaf \( \mathcal{V} \) on \([-1, 1]\) and establish a Hodge theorem for \( \mathcal{V} \). In §4.2, we consider a Witten type deformation of the Hodge Laplacian in §4.1. In §4.3, we consider a Witten type deformation of the Hodge Laplacian on a cylinder.

4.1. Hodge theory for an interval. We denote \( I = [-1, 1] \). Let \( u \in I \) be the coordinate. Let \( V \) be a finite dimensional complex vector space. Let \( V_1, V_2 \subseteq V \) be vector subspaces. We construct a sheaf \( \mathcal{V} \) on \( I \) as follows: for any open subset \( U \subseteq I \),

\[
\mathcal{V}(U) = \left\{ \text{locally constant function } \alpha : U \to V : \right. \\
\left. \alpha(-1) \in V_1 \text{ if } -1 \in U , \alpha(1) \in V_2 \text{ if } 1 \in U \right\}.
\]

(4.1)

We construct sheaves \( (\mathcal{R}^k)_{k=0,1} \) on \( I \) as follows: for any open subset \( U \subseteq I \),

\[
\mathcal{R}^0(U) = \left\{ s \in C^\infty(U, V) : s(-1) \in V_1 \text{ if } -1 \in U , s(1) \in V_2 \text{ if } 1 \in U \right\},
\]

\[
\mathcal{R}^1(U) = \Omega^1(U, V).
\]

(4.2)

Let \( i : \mathcal{V} \to \mathcal{R}^0 \) is the obvious injection. Let \( d : \mathcal{R}^0 \to \mathcal{R}^1 \) be the de Rham operator. Then

\[
\mathcal{V} \xrightarrow{i} \mathcal{R}^0 \xrightarrow{d} \mathcal{R}^1
\]

is a resolution of \( \mathcal{V} \) by fine sheaves. Let \( H^\bullet(I, \mathcal{V}) \) be the sheaf theoretic cohomology of \( I \) with coefficients in \( \mathcal{V} \). We have

\[
H^\bullet(I, \mathcal{V}) = H^\bullet(\mathcal{R}^\bullet(I), d).
\]

(4.4)

Then a direct calculation yields

\[
H^0(I, \mathcal{V}) = V_1 \cap V_2 , \quad H^1(I, \mathcal{V}) = V/(V_1 + V_2).
\]

(4.5)

Let \( h^V \) be a Hermitian metric on \( V \). We denote \( V[du] = V \oplus V du = V \otimes \Lambda^\bullet(T^*I) \). Let \( \| \cdot \|_{V[du]} \) be the norm on \( V[du] \) induced by \( h^V \) and the metric on \( \Lambda^\bullet(T^*I) \) such that \( |du| = 1 \). We introduce the following Clifford actions on \( V[du] \),

\[
c = du \wedge - i \frac{\partial}{\partial u}, \quad \hat{c} = du \wedge + i \frac{\partial}{\partial u}.
\]

(4.6)

Then \( c \) (resp. \( \hat{c} \)) is skew-adjoint (resp. self-adjoint) with respect to \( \| \cdot \|_{V[du]} \). Moreover,

\[
c^2 = -1, \quad \hat{c}^2 = 1, \quad c\hat{c} + \hat{c}c = 0.
\]

(4.7)
For $u \in I$ and $\omega \in \Omega^\bullet(I, V) = C^\infty(I, V[du])$, we denote by $\omega_u \in V[du]$ the value of $\omega$ at $u$. Let $\| \cdot \|_{[-1,1]}$ be the $L^2$-norm on $\Omega^\bullet(I, V)$, i.e.,

$$\| \omega \|^2_{[-1,1]} = \int_{-1}^{1} \| \omega_u \|^2_{V[du]} du .$$

Let $d^V$ be the de Rham operator on $\Omega^\bullet(I, V)$. Let $d^{V*}$ be its formal adjoint. Set

$$D^V = d^V + d^{V*} .$$

Then, by (4.6), we have

$$D^V = e^\frac{\partial}{\partial u} , \quad D^{V^2} = -\frac{\partial^2}{\partial u^2} .$$

For $j = 1, 2$, let $V_j^\perp \subseteq V$ be the orthogonal complement of $V_j \subseteq V$ with respect to $h^V$. Set

$$\Omega^\bullet_{bd}(I, V) = \left\{ \omega \in \Omega^\bullet(I, V) : \omega_{-1} \in V_1 \oplus V_1^\perp du , \omega_1 \in V_2 \oplus V_2^\perp du \right\} .$$

Let $D_{bd}^V$ be the self-adjoint extension of $D^V$ with domain $\text{Dom}(D_{bd}^V) = \Omega^\bullet_{bd}(I, V)$. We will also consider $D_{bd}^{V^2}$ with domain

$$\text{Dom}(D_{bd}^{V^2}) = \left\{ \omega \in \Omega^\bullet_{bd}(I, V) : D^V \omega \in \Omega^\bullet_{bd}(I, V) \right\} .$$

We have

$$\text{Ker}(D_{bd}^{V^2}) \big|_{\Omega^\bullet_{bd}(I,V)} = V_1 \cap V_2 , \quad \text{Ker}(D_{bd}^{V}) \big|_{\Omega^\bullet_{bd}(I,V)} = (V_1^\perp \cap V_2^\perp) du ,$$

where the right hand sides are viewed as constant functions on $I$ with values in $V[du]$. From (4.5) and (4.13), we get a natural isomorphism

$$\text{Ker}(D_{bd}^{V}) \simeq H^\bullet(I, V) .$$

### 4.2. Witten type deformation on an interval

For $T \geq 0$, set

$$d_T^V = e^{-Tf_T} d^V e^{Tf_T} , \quad d_T^{V*} = e^{Tf_T} d^{V*} e^{-Tf_T} , \quad D_T^V = d_T^V + d_T^{V*} ,$$

where $f_T$ was defined by (3.10). The operator $D_T^V$ is formally self-adjoint with respect to $\| \cdot \|_{[-1,1]}$. We have

$$D_T^V = D^V + T f'_T \hat{c} = e^\frac{\partial}{\partial u} + T f'_T \hat{c} , \quad D_T^{V^2} = -\frac{\partial^2}{\partial u^2} + T f'_T \hat{c} + T^2 |f_T'|^2 .$$

Let $D_{T,bd}^V$ be the self-adjoint extension of $D_T^V$ with domain $\text{Dom}(D_{T,bd}^V) = \Omega^\bullet_{bd}(I, V)$.

**Theorem 4.1.** There exist $\beta > \alpha > 0$ such that for $T \gg 1$, we have

$$\text{Sp}(D_{T,bd}^V) \subseteq [ - \infty, - \alpha \sqrt{T} ] \cup [ - \beta \sqrt{T}e^{-T}, \beta \sqrt{T}e^{-T} ] \cup [ \alpha \sqrt{T}, +\infty [ .$$

**Proof.** Recall that $f_\infty$ was defined by (3.8). For $T \geq 0$, set

$$\tilde{D}_T^V = D^V + T f'_\infty \hat{c} .$$
Let $\tilde{D}_{T,\text{bd}}^V$ be the self-adjoint extension of $\tilde{D}_T^V$ with domain $\text{Dom}(\tilde{D}_{T,\text{bd}}^V) = \Omega_{\text{bd}}^*(I, V)$. By $(3.11), (4.16)$ and $(4.18)$, the operator norm of $D_T^V - \tilde{D}_T^V$ is bounded by $O(T e^{-T^2})$. Hence it is sufficient to show that there exist $\beta > \alpha > 0$ such that

$$\text{(4.19)} \quad \text{Sp}(\tilde{D}_{T,\text{bd}}^V) \subseteq ]-\infty, -2\alpha\sqrt{T}] \cup [-\beta\sqrt{T} e^{-T}/2, \beta\sqrt{T} e^{-T}/2] \cup [2\alpha\sqrt{T}, +\infty [ .$$

Recall that $\chi$ was defined by $(3.9)$. For $T \geq 1$, we construct smooth functions $\phi_{1,T}, \phi_{2,T}, \phi_{3,T} : I \to \mathbb{R}$ as follows,

$$\phi_{1,T}(u) = \phi_{2,T}(-u) = (1 - \chi(4u + 4)) \exp(-T(u + 1)^2/2),$$

$$\phi_{3,T}(u) = (1 - \chi(4|u|)) \exp(-Tu^2/2), \quad \text{for } u \in I .$$

Let $(C_t^r, \partial)$ be the complex in $(2.3)$ associated with $V_1, V_2 \subseteq V$. For $T \geq 1$, we construct a linear map $J_T : C_t^r \to \Omega^*(I, V)$ as follows,

$$\text{(4.20)} \quad \text{for } (v_1, v_2) \in V_1 \oplus V_2 = C_t^0, \quad J_T(v_1, v_2) = \phi_{1,T}v_1 + \phi_{2,T}v_2 \in \mathcal{C}^\infty(I, V);$$

$$\text{for } v \in V = C_t^1, \quad J_T(v) = \phi_{3,T}du \otimes v \in \Omega^1(I, V) .$$

Proceeding in the same way as in [13, §6] with $J_T$ in [13, Def. 6.5] replaced by the $J_T$ constructed in $(4.21)$, we obtain $(4.19)$. This completes the proof of Theorem 4.1.

For $\Lambda \subseteq \mathbb{R}$, we denote by $E_T^\Lambda$ the eigenspace of $D_{T,\text{bd}}^V$ associated with eigenvalues in $\Lambda$.

**Theorem 4.2.** For $T \gg 1$, we have $\dim E_T^{[-1,1]} = \dim C_t^*.$

**Proof.** Let $\tilde{D}_{T,\text{bd}}^V$ be as in the proof of Theorem 4.1. Let $\tilde{E}_T^{[-1,1]}$ be the eigenspace of $\tilde{D}_{T,\text{bd}}^V$ associated with eigenvalues in $[-1,1]$. Proceeding in the same way as in [13, §6] with $J_T$ in [13, Def. 6.5] replaced by the $J_T$ constructed in $(4.21)$, we obtain $\dim \tilde{E}_T^{[-1,1]} = \dim C_t^*$. On the other hand, by the proof of Theorem 4.1 we have $\dim E_T^{[-1,1]} = \dim \tilde{E}_T^{[-1,1]}$. This completes the proof of Theorem 4.2.

For $j = 1, 2$, let

$$\text{(4.22)} \quad P_j : V[du] \to V_j \oplus V_j^j[du]$$

be orthogonal projections with respect to $\| \cdot \|_{V[du]}$. We denote $P_j^\perp = \text{Id} - P_j$. Let

$$\text{(4.23)} \quad P_T^\Lambda : \Omega^*(I, V) \to E_T^\Lambda$$

be the orthogonal projection with respect to $\| \cdot \|_{[-1,1]}$. For $\omega \in \Omega^*(I, V)$, we denote

$$\text{(4.24)} \quad \| \omega \|_{V[du],\text{max}} = \max \left\{ \| \omega_u \|_{V[du]} : u \in [-1,1] \right\} .$$

**Proposition 4.3.** For $T \gg 1$, $\varepsilon > 0$, $0 < \varepsilon < \sqrt{T}$, $-\sqrt{T} < \lambda < \sqrt{T}$ and $\omega \in \Omega^*(I, V)$ satisfying

$$\text{(4.25)} \quad D_T^V \omega = \lambda \omega, \quad \| P_j^\perp \omega_{-1} \|_{V[du]} + \| P_2^\perp \omega_1 \|_{V[du]} \leq \varepsilon \| \omega \|_{V[du],\text{max}} ;$$

we have

$$\text{(4.26)} \quad \| \omega - P_T^{[\lambda - \varepsilon, \lambda + \varepsilon]} \omega \|_{[-1,1]} = O(T^{3/2}) \varepsilon \| \omega \|_{[-1,1]} .$$
By Proposition 1.2 and (4.30), we have
\begin{equation}
\omega' = \begin{cases} 
\omega_u - \chi_T(-u)P_1^+\omega_{-1} & \text{if } u < 0 , \\
\omega_u - \chi_T(u)P_2^+\omega_1 & \text{if } u \geq 0 .
\end{cases}
\end{equation}
(4.27)

By the inequality in (4.25), (4.27) and the assumption $0 < \varepsilon < \sqrt{T}$, we have
\begin{equation}
\|\omega - \omega'\|_{[-1,1]} = \mathcal{O}(T^{-1/2})\varepsilon\|\omega\|_{V[du],max} = \mathcal{O}(1)\varepsilon\|\omega\|_{V[du],max}.
\end{equation}
(4.28)

A direct calculation yields
\begin{equation}
\left(D_{Y,bd}^T - \lambda\right)\omega' = \begin{cases} 
\chi_T(-u)eP_1^+\omega_{-1} + (\lambda - T f_T^e)\chi_T(-u)P_1^+\omega_{-1} & \text{if } u < 0 , \\
-\chi_T(u)eP_2^+\omega_1 + (\lambda - T f_T^e)\chi_T(u)P_2^+\omega_1 & \text{if } u \geq 0 .
\end{cases}
\end{equation}
(4.29)

By the inequality in (4.25), (4.29) and the construction of $\chi_T$, we get
\begin{equation}
\left\|\left(D_{Y,bd}^T - \lambda\right)\omega'\right\|_{[-1,1]} = \mathcal{O}(\sqrt{T})\varepsilon\|\omega'\|_{V[du],max}.
\end{equation}
(4.30)

By Proposition 1.2 and (4.30), we have
\begin{equation}
\left\|\omega' - P_T^{\lambda - \varepsilon, \lambda + \varepsilon}\omega'\right\|_{[-1,1]} = \mathcal{O}(T)\varepsilon\|\omega\|_{V[du],max}.
\end{equation}
(4.31)

By (4.7), (4.16) and the identity in (4.25), we have
\begin{equation}
\frac{\partial}{\partial u} \omega = (T f_T^e c\tilde{c} - \lambda c) \omega .
\end{equation}
(4.32)

Let $\|\cdot\|_{H^1([-1,1])}$ be the $H^1$-norm on $\Omega^*(I, V)$. By Sobolev inequality and (4.32),
\begin{equation}
\|\omega\|_{V[du],max} = \mathcal{O}(1)\|\omega\|_{H^1([-1,1])} = \mathcal{O}(T)\|\omega\|_{[-1,1]}.
\end{equation}
(4.33)

From (4.28), (4.31) and (4.33), we obtain (4.26). This completes the proof of Proposition 4.3.

4.3. Witten type deformation on a cylinder. Let $(Y, g^{TY})$ be a closed Riemannian manifold. For $R > 0$, we denote $I_R = [-R, R]$ and $IY_R = I_R \times Y$. Let $(u, y) \in [-R, R] \times Y$ be the coordinates. We will also use the coordinates $(s, y) = (u/R, y) \in [-1, 1] \times Y$. We equip $T(IY_R)$ with the Riemannian metric $du^2 + g^{TY}$.

Let $F$ be a flat complex vector bundle over $Y$. Let $h^F$ be a Hermitian metric on $F$. The pull-back of $F$ (resp. $h^F$) via the canonical projection $IY_R \to Y$ will still be denoted by $F$ (resp. $h^F$). Let $D^Y$ be the Hodge de Rham operator on $\Omega^*(Y, F)$. Under the identification
\begin{equation}
\Omega^*(I_R, \Omega^*(Y, F)) \to \Omega^*(IY_R, F)
\end{equation}
(4.34)
\begin{equation}
\sigma + du \otimes \tau \mapsto \sigma + du \wedge \tau , \quad \text{for } \sigma, \tau \in \mathcal{C}^\infty(I_R, \Omega^*(Y, F)) ,
\end{equation}
the Hodge de Rham operator on $\Omega^*(IY_R, F)$ is given by
\begin{equation}
D^{IY_R} = \tilde{c}cD^Y + c \frac{\partial}{\partial u} = R^{-1}\left(\tilde{c}cRD^Y + c \frac{\partial}{\partial s}\right) ,
\end{equation}
(4.35)
where the term $\hat{c}c = i_{\frac{\partial}{\partial u}} du \wedge -du \wedge i_{\frac{\partial}{\partial u}}$ comes from the fact that $D^Y$ anti-commutes with $du \wedge$.

Recall that the function $f_T : I \to \mathbb{R}$ was constructed in (3.10). We will view $f_T$ as a function on $IY_R$, i.e., $f_T(s, y) = f_T(s)$, $f_T(u, y) = f_T(u/R)$. We denote

$$f_T' = \frac{\partial}{\partial u} f_T = R \frac{\partial}{\partial u} f_T .$$

For $T \geq 0$, let $\tilde{D}_{IY_R}^T$ be the Hodge de Rham operator with respect to $du^2 + g^{TY}$ and $h_T^F := e^{-2 f_T} h^F$. Set $D_{IY_R}^T = e^{-T f_T} D_{IY_R}^T e^{T f_T}$. We have

$$RD_{IY_R}^T = \hat{c}c RD^Y + c \frac{\partial}{\partial s} + T f_T' \hat{c} .$$

(4.37)

$$R^2 D_{IY_R}^{T,2} = R^2 D^{T,2} - \frac{\partial^2}{\partial s^2} + T f_T' \hat{c} + T^2 |f_T'|^2 .$$

Let

(4.38)

$$\Omega^*(Y, F) = \bigoplus_{\mu} \Psi^\mu(Y, F)$$

be the spectral decomposition with respect to $D^Y$, i.e., $D^Y|_{\Psi^\mu(Y, F)} = \mu \text{Id}$. We denote $\Psi^\mu(Y, F) = \text{Ker}(D^Y) = \Psi^0(Y, F)$. We have the formal decomposition

(4.39)

$$\Omega^*(IY_R, F) = \bigoplus_{\mu} \Omega^*(I_R, \Psi^\mu(Y, F)) .$$

Let $D_T^{\Psi^\mu(Y, F)}$ be the operator $D^Y_T$ in §4.2 with $V$ replaced by $\Psi^\mu(Y, F)$. We have

(4.40)

$$RD_T^{IY_R}|_{\Omega^*(I_R, \Psi^\mu(Y, F))} = R \mu \hat{c}c + D_T^{\Psi^\mu(Y, F)} = R \mu \hat{c}c + c \frac{\partial}{\partial s} + T f_T' \hat{c} .$$

As a consequence, we have

(4.41)

$$R^2 D_{IY_R}^{T,2}|_{\Omega^*(I_R, \Psi^\mu(Y, F))} = R^2 \mu^2 - \frac{\partial^2}{\partial s^2} + T^2 |f_T'|^2 + T f_T' \hat{c} .$$

In particular, we have

(4.42)

$$RD_T^{IY_R}|_{\Omega^*(I_R, \Psi^\mu(Y, F))} = D_T^{\Psi^\mu(Y, F)} = c \frac{\partial}{\partial s} + T f_T' \hat{c} .$$

For $\alpha > 0$, we define $C_\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \times [-1, 1] \to \mathbb{R}$ as follows,

(4.43)

$$C_\alpha(a, b, s) = (1 - e^{-4 \alpha})^{-1} \left( (a - be^{-2 \alpha}) e^{\alpha(-s-1)} + (b - ae^{-2 \alpha}) e^{\alpha(s-1)} \right) .$$

Then the following identities hold,

(4.44)

$$C_\alpha(a, b, -1) = a , \quad C_\alpha(a, b, 1) = b , \quad \left( \frac{\partial^2}{\partial s^2} - \alpha^2 \right) C_\alpha(a, b, s) = 0 .$$

**Lemma 4.4.** There exists $\alpha > 0$ such that for $T = R^c \gg 1$, $\mu \in \text{Sp}(D^Y) \setminus \{0\}$, $\omega \in \Omega^*(I_R, \Psi^\mu(Y, F))$ and $-\sqrt{\lambda} \leq \lambda \leq \sqrt{\lambda}$ satisfying

(4.45)

$$\left(R \mu \hat{c}c + D_T^{\Psi^\mu(Y, F)}\right) \omega = \lambda \omega ,$$
we have

\[(4.46) \quad \|\omega_s\|_Y^2 \leq C_{\alpha R} \left( \|\omega_{-1}\|_Y^2, \|\omega_1\|_Y^2, s \right), \quad \text{for } s \in [-1, 1].\]

**Proof.** We assert that for \(g \in C^\infty([-1, 1], \mathbb{R}^+)\) satisfying \(\frac{\partial^2}{\partial s^2} - \alpha^2 R^2 \geq 0\), we have \(g(s) \leq C_{\alpha R}(g(-1), g(1), s)\) for \(s \in [-1, 1]\). To prove the assertion, we take \(s_0 \in [-1, 1]\) such that

\[(4.47) \quad h := g - C_{\alpha R}(g(-1), g(1), \cdot) \in C^\infty([-1, 1], \mathbb{R})\]

reaches its maximum value at \(s_0\). If \(h(s_0) > 0\), then \(s_0 \neq \pm 1\) and \(h''(s_0) > 0\), which is a contradiction.

Now it remains to show that

\[(4.48) \quad \left( \frac{\partial^2}{\partial s^2} - \alpha^2 R^2 \right) \|\omega_s\|_Y^2 \geq 0 .\]

By \((4.40), (4.41)\) and \((4.45)\), we have

\[(4.49) \quad \frac{\partial^2}{\partial s^2} \omega_s = \left( R^2 \mu^2 + T f''_T(s) \hat{c} \hat{c} + T^2 |f'_T|^2(s) - \lambda^2 \right) \omega_s .\]

Set \(\alpha = \min \{|\mu| : \mu \in \text{Sp}(D^Y) \setminus \{0\}\}\). For \(R, T, \mu, \lambda\) satisfying the hypothesis of Lemma 4.4, we have

\[(4.50) \quad R^2 \mu^2 + T f''_T(s) \hat{c} \hat{c} + T^2 |f'_T|^2(s) - \lambda^2 \geq \alpha^2 R^2 / 2 .\]

From \((4.49), (4.50)\) and the obvious identity

\[(4.51) \quad \frac{\partial^2}{\partial s^2} \|\omega_s\|_Y^2 = \langle \frac{\partial^2}{\partial s^2} \omega_s, \omega_s \rangle_Y + \langle \omega_s, \frac{\partial^2}{\partial s^2} \omega_s \rangle_Y + 2 \|\frac{\partial}{\partial s} \omega_s\|_Y^2 ,\]

we obtain \((4.48)\). This completes the proof of Lemma 4.4. \(\square\)

**Lemma 4.5.** For \(\omega \in \Omega^*(I, \mathcal{H}^*(Y, F))\) and \(\lambda \in \mathbb{R}\) satisfying

\[(4.52) \quad D^{\mathcal{H}^*(Y,F)}_T \omega = \lambda \omega ,\]

we have

\[(4.53) \quad \frac{\partial}{\partial s} \langle c \omega_s, \omega_s \rangle_Y = 0 , \quad \text{for } s \in [-1, 1] .\]

**Proof.** By \((4.42)\) and \((4.52)\), we have

\[(4.54) \quad \frac{\partial}{\partial s} \omega_s = (T f'_T(s) \hat{c} \hat{c} - \lambda c) \omega_s .\]

Note that \(c\) is skew-adjoint and that \(\hat{c}\) is self-adjoint, equation \((4.53)\) follows from \((4.7)\) and \((4.54)\). This completes the proof of Lemma 4.5. \(\square\)
5. Adiabatic limit and Witten type deformation

The purpose of this section is to prove Theorems 3.1 and 3.3. First we summarize the proof of Theorem 3.1 given in this section. Let $\lambda$ be a reasonably small eigenvalue of $RD_T^{Z_R}$. Let $\omega$ be an eigensection associated with $\lambda$. First, in Lemma 5.7, we show that $\omega^{\text{zm}}$ (the zero-mode of $\omega$, see (5.5)) is the principal contributor to the norm of $\omega$. Second, in Lemma 5.8, we show that $\omega^{\text{zm}}$ almost lies in the domain of $D_{T, \text{bd}}^{\pi^*(Y,F)}$, which is the operator in (4.16) with $V = \mathcal{H}^*(Y,F) := \text{Ker}(D^V)$. Combining the results above, we show that $\lambda$ is very close to an eigenvalue of $D_{T, \text{bd}}^{\pi^*(Y,F)}$. On the other hand, by Theorem 4.1, we apply Proposition 1.3 with $C = \Omega(Y,F)$. On the other hand, by Theorem 4.1, we construct a bijection $S_{\text{bd}}(5.44))$. We construct a bijection $\mathcal{J}_{R,T}^H : \Omega^*(C^0_{\text{bd}}, \partial) \to \text{Ker}(D_T^{Z_R})$ by composing $F_{R,T} \oplus I_{R,T}$ with the orthogonal projection to the kernel of $D_T^{Z_R}$. To show the injectivity of $\mathcal{J}_{R,T}^H$, we apply Proposition 1.3 with $w = F_R^+ + I_R$ and $v = G_{R,T}^+$ (see Proposition 5.9). The proof of the surjectivity of $\mathcal{J}_{R,T}$ highly relies on the results mentioned in the last paragraph: we reduce the problem to $D_T^{\pi^*(Y,F)}$ whose spectrum is studied in §4.2. For various reasons, the kernel of $D_T^{Z_R}$ needs to be studied separately. Here the strategy is exactly the same as in the last paragraph. We explicitly construct $F_{R,T}, G_{R,T} : H^0(C^0, \partial) \to \Omega^*(Z_R, F)$ and $I_{R,T}, J_{R,T} : H^1(C^0, \partial) \to \Omega^*(Z_R, F)$ (see (5.20) and (5.44)). We construct a bijection $\mathcal{J}_{R,T}^H : \Omega^*(C^0, \partial) \to \text{Ker}(D_T^{Z_R})$ by composing $F_{R,T} \oplus I_{R,T}$ with the orthogonal projection to the kernel of $D_T^{Z_R}$. To show the injectivity of $\mathcal{J}_{R,T}^H$, we apply Proposition 1.3 with $w = F_{R,T}^+ \oplus I_{R,T}$ and $v = G_{R,T}^+ \oplus J_{R,T}$. This section is organized as follows. In §5.1, we estimate the kernel of the Witten Laplacian. In §5.2, we estimate the eigenspace of the Witten Laplacian associated with small eigenvalues. Theorem 3.1 will be proved in this subsection. In §5.3, we estimate the action of the de Rham operator on the eigenspace associated with small eigenvalues. In §5.4, we estimate the $L^2$-metric on the eigenspace associated with small eigenvalues. Theorem 3.3 will be proved in this subsection.

5.1. Kernel of $D_T^{Z_R}$. Recall that $D_T^{Z_R}$ was defined in (3.15). For convenience, we denote $D_T^{Z_R} = D_T^{Z_T}|_{T=0}$. By elliptic estimate (see the proof of [50, Prop. 3.4]), we may define the $H^1$-norm on $\Omega^*(Z_R, F)$ as follows: for $\omega \in \Omega^*(Z_R, F)$,

\[ \|\omega\|_{H^1, Z_R}^2 = \|\omega\|_{Z_R}^2 + \|D_T^{Z_R} \omega\|_{Z_R}^2. \]

We fix $\kappa \in [0, 1/3]$ as in (0.41).

Proposition 5.1. For $T = R^\kappa \gg 1$ and $\omega \in \Omega^*(Z_R, F)$, we have

\[ \|\omega\|_{H^1, Z_R}^2 \leq 2 \|\omega\|_{Z_R}^2 + \|D_T^{Z_R} \omega\|_{Z_R}^2. \]
We define 

\[ D_T^{Z_{R,2}} + 1 \geq D^{Z_{R,2}}. \]

From (5.1) and (5.3), we obtain (5.2). This completes the proof of Proposition 5.1. □

We will always use the canonical isometric embeddings

\[ IY_R \subseteq Z_{j,R} \subseteq Z_{j,\infty}, \quad Z_{j,R} \subseteq Z_R, \quad \text{for } j = 1, 2. \]

Recall that the vector subspaces \( \mathcal{H}^\bullet(Y, F) \subseteq \Omega^\bullet(Y, F) \) and \( \Theta^\mu(Y, F) \subseteq \Omega^\bullet(Y, F) \) were defined in the paragraph containing (4.38). For \( \omega \in \Omega^\bullet(Z_{j,R}, F) \) with \( j = 0, 1, 2, 3 \), we have the orthogonal decomposition

\[ \omega|_{IY_R} = \omega_{\text{zm}} + \omega_{\text{nz}}, \]

with

\[ \omega_{\text{zm}} \in \Omega^\bullet(I_R, \mathcal{H}^\bullet(Y, F)), \quad \omega_{\text{nz}} \in \bigoplus_{\mu \neq 0} \Omega^\bullet(I_R, \Theta^\mu(Y, F)). \]

We call \( \omega_{\text{zm}} \) (resp. \( \omega_{\text{nz}} \)) the zero-mode (resp. non-zero-mode) of \( \omega \).

Recall that \( \mathcal{H}^\bullet_{\text{abs}}(Z_{j,\infty}, F) \subseteq \mathcal{H}^\bullet(Z_{j,\infty}, F) \) was defined in (5.21). Let

\[ R_d^F, R_{d^*} : \mathcal{H}^\bullet_{\text{abs}}(Z_{1,\infty}, F) \rightarrow \Omega^\bullet([0, +\infty) \times Y, F) \]

be as in \([50, (2.44)]\) with \( X_\infty \) replaced by \( Z_{1,\infty} \) and \( \mathcal{H}^\bullet(X_\infty, F) \) replaced by \( \mathcal{H}^\bullet_{\text{abs}}(Z_{1,\infty}, F) \). Here we recall their construction. We identify \( Z_{1,\infty} \) with \( Z_{1,0} \cup [0, +\infty[ \times Y \). By \([50, \text{Prop. 2.5}]\) and \([50, (2.10)]\), for \( (\omega, \hat{\omega}) \in \mathcal{H}^\bullet_{\text{abs}}(Z_{1,\infty}, F) \), we have

\[ \omega|_{[0, +\infty[ \times Y} = \hat{\omega} + \omega_{\text{nz}} = \hat{\omega} + \sum_{\mu \neq 0, \mu \in \text{Sp}(D^Y)} e^{-|\mu|u} (\hat{\tau}_{\mu,1} - du \wedge \tau_{\mu,2}), \]

where \( \hat{\omega} \in \mathcal{H}^\bullet(Y, F)[du] \) is viewed as a constant section in

\[ \Theta^{\infty}([0, +\infty[, \mathcal{H}^\bullet(Y, F)[du]) = \Omega^\bullet([0, +\infty[, \mathcal{H}^\bullet(Y, F)) \subseteq \Omega^\bullet([0, +\infty[ \times Y, F), \]

and \( \hat{\tau}_{\mu,1}, \hat{\tau}_{\mu,2} \in \Omega^\bullet(Y, F) \) satisfy

\[ d^Y \tau_{\mu,1} = d^Y \tau_{\mu,2} = 0, \quad d^Y \tau_{\mu,1} = \mu |\tau_{\mu,2}, \quad d^Y \tau_{\mu,2} = |\mu| \tau_{\mu,1}. \]

We define

\[ R_d^F(\omega, \hat{\omega}) = \sum_{\mu \neq 0, \mu \in \text{Sp}(D^Y)} \frac{1}{|\mu|} e^{-|\mu|u} \tau_{\mu,2}, \]

\[ R_{d^*}(\omega, \hat{\omega}) = \sum_{\mu \neq 0, \mu \in \text{Sp}(D^Y)} \frac{1}{|\mu|} e^{-|\mu|u} du \wedge \tau_{\mu,1}. \]

Let

\[ \Omega^\bullet([0, +\infty) \times Y, F) \rightarrow \Omega^\bullet(IY_R, F) \]

be induced by the isometric identifications \( IY_R = [-R, R] \times Y \simeq [0, 2R] \times Y \rightarrow [0, +\infty) \times Y \). Composing (5.7) and (5.12), we get

\[ R_d^F, R_{d^*} : \mathcal{H}^\bullet(Z_{1,\infty}, F) \rightarrow \Omega^\bullet(IY_R, F). \]

We construct

\[ R_d^F, R_{d^*} : \mathcal{H}^\bullet(Z_{2,\infty}, F) \rightarrow \Omega^\bullet(IY_R, F). \]
in the same way. For \( j = 1, 2 \) and \((\omega, \hat{\omega}) \in \mathcal{H}(Z_{j,\infty}, F)\), we have
\[
\begin{align*}
d^{Z_R} R^{d_F} (\omega, \hat{\omega}) &= d^{Z_R} R^{d_F} (\omega, \hat{\omega}) = \omega^\infty, \\
d^{Z_R} R^{d_F} (\omega, \hat{\omega}) &= d^{Z_R} R^{d_F} (\omega, \hat{\omega}) = 0, \\
i \frac{\partial}{\partial u} R^{d_F} (\omega, \hat{\omega}) &= du \wedge R^{d_F} (\omega, \hat{\omega}) = 0.
\end{align*}
\]
(5.15)

Set
\[
\mathcal{H}^*(Z_{1,\infty}, F) = \left\{ (\omega_1, \omega_2, \hat{\omega}) : (\omega_j, \hat{\omega}) \in \mathcal{H}(Z_{j,\infty}, F) \right\}.
\]
(5.16)

Recall that the smooth function \( \chi : \mathbb{R} \to \mathbb{R} \) was defined in (3.9). Set
\[
\chi_1(s) = 1 - \chi(4(s + 1)), \quad \chi_2(s) = 1 - \chi(4(1 - s)).
\]
(5.17)

We will view \( \chi_j \) \((j = 1, 2)\) as functions on \( IY_R\), i.e.,
\[
\chi_j(s, y) = \chi_j(s), \quad \chi_j(u, y) = \chi_j(u/R).
\]
(5.18)

Following [50, (3.26)], we define
\[
F_{R, T}, G_{R, T} : \mathcal{H}^*(Z_{1,\infty}, F) \to \Omega^*(Z_R, F)
\]
as follows: for \((\omega_1, \omega_2, \hat{\omega}) \in \mathcal{H}(Z_{1,\infty}, F)\),
\[
F_{R, T}(\omega_1, \omega_2, \hat{\omega}) \big|_{Z_{1,0}} = G_{R, T}(\omega_1, \omega_2, \hat{\omega}) \big|_{Z_{1,0}} = \omega_j, \quad \text{for } j = 1, 2,
\]
(5.20)

where we use the identifications in (5.4). By (5.15), \( F_{R, T} \) and \( G_{R, T} \) are well-defined. Similarly to [50, (3.28)], by (3.15) and the identities \( D^Y \omega_j = i \frac{\partial}{\partial u} \omega_j = 0 \) for \( j = 1, 2 \), we have
\[
d^{Z_R} R^{d_F} F_{R, T}(\omega_1, \omega_2, \hat{\omega}) = d^{Z_R} R^{d_F} G_{R, T}(\omega_1, \omega_2, \hat{\omega}) = 0.
\]
(5.21)

Let \( P_{R, T} : \Omega^*(Z_R, F) \to \text{Ker} (D^{Z_R}_T) \) be the orthogonal projection with respect the \( L^2\)-metric induced by \( g^{Z_R} \) and \( h^F \).

**Proposition 5.2.** For \( T = R^\varepsilon \gg 1 \) and \((\omega_1, \omega_2, \hat{\omega}) \in \mathcal{H}(Z_{1,\infty}, F)\), we have
\[
\left\| (\text{Id} - P_{R, T}) F_{R, T}(\omega_1, \omega_2, \hat{\omega}) \right\|_{H^1, Z_R} = O'(R^{-2 - \kappa}) \left( \left\| \omega_1 \right\|_{Z_{1,0}}^2 + \left\| \omega_2 \right\|_{Z_{2,0}}^2 \right).
\]
(5.22)

**Proof:** The proof follows closely the proof of [50, Prop. 3.5]. It consists of several steps.

**Step 1.** We calculate \((F_{R, T} - G_{R, T})(\omega_1, \omega_2, \hat{\omega})\) and \( D^{Z_R}_T(F_{R, T} - G_{R, T})(\omega_1, \omega_2, \hat{\omega})\).

Recall that \( IY_R = [-R, R] \times Y \). By (5.17) and (5.18), we have
\[
\chi_2 \big|_{[-R,0] \times Y} = 0.
\]
(5.23)

By (5.15), (5.20) and (5.23), we have
\[
(F_{R, T} - G_{R, T})(\omega_1, \omega_2, \hat{\omega}) \big|_{[-R,0] \times Y} = \chi_1 \left( e^{-Tf_T} - e^{Tf_T} \right)^\infty \omega_{12}^x
\]
(5.24)

\[
+ \frac{\partial \chi_1}{\partial u} \left( e^{-Tf_T} du \wedge R^{d_F} (\omega_1, \hat{\omega}) + e^{Tf_T} i \frac{\partial}{\partial u} R^{d_F} (\omega_1, \hat{\omega}) \right).
\]
By (5.8), (5.10), (5.11) and (5.15), we have
\[ dZ^R du \wedge \mathcal{A}_dF(\omega_1, \omega) = -du \wedge \omega_1^{nz}, \quad d^{Z^R,R} \left( \mathcal{A}_dF(\omega_1, \omega) \right) = -i \frac{\partial}{\partial u_1} \omega_1^{nz}, \]
(5.25)
\[ d^{Z^R,R} \left( i \frac{\partial}{\partial u_1} \mathcal{A}_dF(\omega_1, \omega) \right) = -i \frac{\partial}{\partial u_1} \omega_1^{nz}. \]

By (3.15), the third identity in (5.15) and (5.25), we have
\[ D^R_T \left( du \wedge \mathcal{A}_dF(\omega_1, \hat{\omega}) \right) = -du \wedge \omega_1^{nz} - i \frac{\partial}{\partial u_1} \omega_1^{nz} + T \frac{\partial f}{\partial u} \mathcal{A}_dF(\omega_1, \hat{\omega}), \]
(5.26)
\[ D^R_T \left( i \frac{\partial}{\partial u_1} \mathcal{A}_dF(\omega_1, \hat{\omega}) \right) = -du \wedge \omega_1^{nz} - i \frac{\partial}{\partial u_1} \omega_1^{nz} + T \frac{\partial f}{\partial u} \mathcal{A}_dF(\omega_1, \hat{\omega}). \]

By (5.24) and (5.26), we have
\[ D^R_T (F_{R,T} - G_{R,T})(\omega_1, \omega_2, \hat{\omega}) \bigg|_{-R,0} \]
\[ = 2T \frac{\partial f}{\partial u} \chi_1 \left( e^{-Tf} i \frac{\partial}{\partial u} - e^{Tf} du \wedge \right) \omega_1^{nz} - 2 \frac{\partial \chi_1}{\partial u} \left( e^{-Tf} i \frac{\partial}{\partial u} + e^{Tf} du \wedge \right) \omega_1^{nz} + 2T \frac{\partial f}{\partial u} \frac{\partial \chi_1}{\partial u} \left( e^{-Tf} \mathcal{A}_dF(\omega_1, \hat{\omega}) + e^{Tf} \mathcal{A}_dF(\omega_1, \hat{\omega}) \right) \]
\[ + \frac{\partial^2 \chi_1}{\partial u^2} \left( e^{-Tf} \mathcal{A}_dF(\omega_1, \hat{\omega}) + e^{Tf} \mathcal{A}_dF(\omega_1, \hat{\omega}) \right). \]
(5.27)

Step 2. We estimate \( \| (F_{R,T} - G_{R,T})(\omega_1, \omega_2, \hat{\omega}) \|_{Z_R} \) and \( \| D^R_T (F_{R,T} - G_{R,T})(\omega_1, \omega_2, \hat{\omega}) \|_{Z_R} \).

For
\[ \tau \in \left\{ \omega_1^{nz}, \mathcal{A}_dF(\omega_1, \hat{\omega}), \mathcal{A}_dF_1(\omega_1, \hat{\omega}) \right\} \]
and \(-R \leq u \leq 0\), by (5.8) and (5.11), we have
\[ \| \tau \|_{\{u\} \times Y} = \mathcal{O} \left( e^{-a(R+u)} \right) \| \omega_1^{nz} \|_{Z_1,0} = \mathcal{O} \left( e^{-a(R+u)} \right) \| \omega_1 \|_{Z_1,0}, \]
(5.29)
where \( a > 0 \) is a universal constant. Since \( \omega_1 \in \text{Ker} \left( D^{Z_1,\infty} \right) \), by the Trace theorem for Sobolev spaces, we have
\[ \| \omega_1 \|_{Z_1,0} = \mathcal{O} \left( 1 \right) \| \omega_1 \|_{Z_1,0}. \]
(5.30)

By (5.29) and (5.30), for \(-R \leq u \leq 0\), we have
\[ \| \tau \|_{\{u\} \times Y} = \mathcal{O} \left( e^{-a(R+u)} \right) \| \omega_1 \|_{Z_1,0}. \]
(5.31)

By (3.10), (5.17) and (5.18), for \(-R \leq u \leq 0\), we have
\[ |f_T| = \mathcal{O} \left( R^{-2} \right) (R + u)^2, \quad \left| \frac{\partial f_T}{\partial u} \right| = \mathcal{O} \left( R^{-2} \right) (R + u), \]
\[ |\chi_1| \leq 1, \quad \left| \frac{\partial \chi_1}{\partial u} \right| = \mathcal{O} \left( R^{-1} \right), \quad \left| \frac{\partial^2 \chi_1}{\partial u^2} \right| = \mathcal{O} \left( R^{-2} \right). \]
By (5.24), (5.27), (5.31), (5.32) and the assumption $T = R^\kappa$, we have

$$
(5.33) \quad \left\| (F_{R,T} - G_{R,T})(\omega_1, \omega_2, \hat{\omega}) \right\|_{[-R,0] \times Y}^2 + \left\| D^\ast_T (F_{R,T} - G_{R,T})(\omega_1, \omega_2, \hat{\omega}) \right\|_{[-R,0] \times Y}^2 = O(R^{-2+\kappa}) \left\| \omega_1 \right\|_{Z_{1,0}}^2 \cdot \left\| \omega_2 \right\|_{Z_{2,0}}^2 .
$$

The same argument also shows that (5.33) holds with $[-R,0] \times Y$ replaced by $[0, R] \times Y$ and $\left\| \omega_1 \right\|_{Z_{1,0}}^2$ replaced by $\left\| \omega_2 \right\|_{Z_{2,0}}^2$. On the other hand, by (5.20), we have

$$
(5.34) \quad (F_{R,T} - G_{R,T})(\omega_1, \omega_2, \hat{\omega}) \big|_{Z_{1,0} \cup Z_{2,0}} = 0 .
$$

By (5.33) and (5.34), we have

$$
(5.35) \quad \left\| (F_{R,T} - G_{R,T})(\omega_1, \omega_2, \hat{\omega}) \right\|_{Z_R}^2 + \left\| D^\ast_T (F_{R,T} - G_{R,T})(\omega_1, \omega_2, \hat{\omega}) \right\|_{Z_R}^2 = O(R^{-2+\kappa}) \left( \left\| \omega_1 \right\|_{Z_{1,0}}^2 + \left\| \omega_2 \right\|_{Z_{2,0}}^2 \right) .
$$

The estimates in (1.4) hold with $(\mathbb{W}^\ast, \partial, \| \cdot \|)$ replaced by $(\mathbb{W}^\ast(Z_R, F), d^\ast_T Z_R, \| \cdot \|_{Z_R})$. Applying (5.21), (5.35) and Corollary 1.4 with $\gamma = 0$, $w = F_{R,T}(\omega_1, \omega_2, \hat{\omega})$ and $v = G_{R,T}(\omega_1, \omega_2, \hat{\omega})$, we get

$$
(5.36) \quad \left\| \left( \text{Id} - P_{R,T} \right) F_{R,T}(\omega_1, \omega_2, \hat{\omega}) \right\|_{Z_R}^2 + \left\| D^\ast_T \left( \text{Id} - P_{R,T} \right) F_{R,T}(\omega_1, \omega_2, \hat{\omega}) \right\|_{Z_R}^2 = O(R^{-2+\kappa}) \left( \left\| \omega_1 \right\|_{Z_{1,0}}^2 + \left\| \omega_2 \right\|_{Z_{2,0}}^2 \right) .
$$

From Proposition 5.1 and (5.36), we obtain (5.22). This completes the proof of Proposition 5.2.

For $j = 1, 2$, let

$$
(5.37) \quad \mathcal{L}^\ast_{j,\text{abs}} \subseteq \mathcal{H}^\ast(Y, F) , \quad \mathcal{L}^\ast_{j,\text{rel}} \subseteq \mathcal{H}^\ast(Y, F)du , \quad \mathcal{L}^\ast_j = \mathcal{L}^\ast_{j,\text{abs}} \oplus \mathcal{L}^\ast_{j,\text{rel}}
$$

be as in [50, (2.47),(2.49)] with $X_\infty$ replaced by $Z_{j,\infty}$. More precisely, under the identification $\mathcal{H}^\ast(Y, F) = H^\ast(Y, F)$, we have

$$
(5.38) \quad \mathcal{L}^\ast_{j,\text{abs}} = \text{Im} \left( H^\ast(Z_j, F) \to H^\ast(Y, F) \right) ,
$$

where the map is induced by $Y = \partial Z_j \hookrightarrow Z_j$, and

$$
(5.39) \quad \mathcal{L}^\ast_{j,\text{rel}} = du \mathcal{L}^\ast_{j,\text{abs}} ,
$$

where $\mathcal{L}^\ast_{j,\text{abs}} \subseteq \mathcal{H}^\ast(Y, F)$ is the orthogonal complement of $\mathcal{L}^\ast_{j,\text{abs}}$ with respect to $\| \cdot \|_Y$. Hence we have

$$
(5.40) \quad \mathcal{L}^\ast_{1,\text{rel}} \cap \mathcal{L}^\ast_{2,\text{rel}} = du \left( \mathcal{L}^\ast_{1,\text{abs}} \cap \mathcal{L}^\ast_{2,\text{abs}} \right) .
$$

Let $\mathcal{H}^\ast_{\text{rel}}(Z_{j,\infty}, F)$ be as in [50] (2.52)] with $X_\infty$ replaced by $Z_{j,\infty}$. For $\hat{\omega} \in \mathcal{L}^\ast_{j,\text{rel}} \cap \mathcal{L}^\ast_{\text{rel}}$, let

$$
(5.41) \quad (\omega_1, du \land \hat{\omega}) \in \mathcal{H}^\ast_{\text{rel}}(Z_{1,\infty}, F) , \quad (\omega_2, du \land \hat{\omega}) \in \mathcal{H}^\ast_{\text{rel}}(Z_{2,\infty}, F)
$$

be the unique element such that $\omega_1$ (resp. $\omega_2$) is a generalized eigensection of $DZ_{1,\infty}$ (resp. $DZ_{2,\infty}$). The existence and uniqueness are guaranteed by [50] (2.40).
Similarly to (5.17), set
\[
\chi_3(s) = 1 - \chi(4|s|) .
\]
We will view \(\chi_3\) as a function on \(IY_{R}\) in the same way as \(\chi_1, \chi_2\) in (5.18). We define
\[
I_{R,T}, J_{R,T} : \mathcal{L}_{1,\text{abs}}^\perp \cap \mathcal{L}_{2,\text{abs}}^\perp \to \Omega^{*+1}(Z_R; F)
\]
as follows: for \(\hat{\omega} \in \mathcal{L}_{1,\text{abs}}^\perp \cap \mathcal{L}_{2,\text{abs}}^\perp\),
\[
I_{R,T}(\hat{\omega})|_{Z_{j,0}} = 0, \quad J_{R,T}(\hat{\omega})|_{Z_{j,0}} = e^{-T} \omega_j, \quad \text{for } j = 1, 2 ,
\]
\[
I_{R,T}(\hat{\omega})|_{IY_R} = \chi_3 e^{T J_{R,T} - T du} \wedge \hat{\omega} ,
\]
\[
J_{R,T}(\hat{\omega})|_{IY_R} = e^{T J_{R,T} - T du} \wedge \hat{\omega} + e^{T J_{R,T} - T d^Z_{R,*}} (\chi_{1R,T}^{d_{R,*}}(\omega_1, du \wedge \hat{\omega}) + \chi_{2R,T}^{d_{R,*}}(\omega_2, du \wedge \hat{\omega})) .
\]
By (3.10), (3.15) and (5.15), \(I_{R,T}\) and \(J_{R,T}\) are well-defined. Moreover, we have
\[
d^Z_{T} I_{R,T}(\hat{\omega}) = d^Z_{T} J_{R,T}(\hat{\omega}) = 0 .
\]

**Proposition 5.3.** There exists \(a > 0\) such that for \(T = R^\kappa \gg 1\) and \(\hat{\omega} \in \mathcal{L}_{1,\text{abs}}^\perp \cap \mathcal{L}_{2,\text{abs}}^\perp\), we have
\[
\left\| (\text{Id} - P_{R,T}) I_{R,T}(\hat{\omega}) \right\|_{H^1, Z_R}^2 = o(e^{-aT})\left\| \hat{\omega} \right\|_{Y}^2 .
\]

**Proof:** We proceed in the same way as in the proof of Proposition 5.2. The map \(I_{R,T}\) (resp. \(J_{R,T}\)) plays the role of \(F_{R,T}\) (resp. \(G_{R,T}\)). \(\square\)

By (2.12), (3.20), (3.23) and (5.16), we have
\[
W_1^\perp \simeq \mathcal{H}_{\text{abs}}^{*,*}(Z_{12,\infty}; F) .
\]

By (2.2), (3.20), (3.23) and (5.38), we have
\[
V_j^\perp \simeq \mathcal{L}_{j,\text{abs}}^{*,*}, \quad \text{for } j = 1, 2 .
\]

As a consequence, we have
\[
V^\perp / (V_1^\perp + V_2^\perp) \simeq \mathcal{L}_{1,\text{abs}}^{*,*} \cap \mathcal{L}_{2,\text{abs}}^{*,*} .
\]

Recall that the complex \((C^*_0, \partial)\) was defined by (2.1) and (3.20). By (2.13), (5.47) and (5.49), we have
\[
H^0(C^*_0, \partial) \simeq \mathcal{H}_{\text{abs}}^{*,*}(Z_{12,\infty}; F) , \quad H^1(C^*_0, \partial) \simeq \mathcal{L}_{1,\text{abs}}^{*,*} \cap \mathcal{L}_{2,\text{abs}}^{*,*} .
\]

We define a map
\[
\mathcal{I}_{R,T}^H : H^*(C^*_0, \partial) \to \text{Ker}(D^Z_{T}) , \quad \mathcal{I}_{R,T}^H|_{H^0(C^*_0, \partial)} = P_{R,T} F_{R,T} , \quad \mathcal{I}_{R,T}^H|_{H^1(C^*_0, \partial)} = P_{R,T} I_{R,T} .
\]

**Theorem 5.4.** For \(T = R^\kappa \gg 1\), the map \(\mathcal{I}_{R,T}^H\) is bijective.
Proof. By (5.19), (5.20), (5.43), (5.44) and the fact that \( \chi_1 \chi_3 = \chi_2 \chi_3 = 0 \), for \((\omega_1, \omega_2, \dot{\omega}) \in \mathcal{H}_{ab}(Z_{12, \infty}, F) \) and \( \dot{\tau} \in \mathcal{L}^{\perp}_{1,ab} \cap \mathcal{L}^{\perp}_{2,ab} \), we have
\[
\left\| F_{R,T}(\omega_1, \omega_2, \dot{\omega}) \right\|^2_{Z_R} \geq \left\| \omega_1 \right\|^2_{Z_{1,0}} + \left\| \omega_2 \right\|^2_{Z_{2,0}} ,
\]
\[
\left\| I_{R,T}(\dot{\tau}) \right\|^2_{Z_R} \geq \left\| \dot{\tau} \right\|^2_Y .
\]
(5.52)

By Propositions 5.2, 5.3 and (5.52), we have
\[
\| \| P_{R,T} F_{R,T}(\omega_1, \omega_2, \dot{\omega}) + P_{R,T} I_{R,T}(\dot{\tau}) \|_{Z_R}^2 \geq \frac{1}{2} \left( \| \omega_1 \|_{Z_{1,0}}^2 + \| \omega_2 \|_{Z_{2,0}}^2 + \| \dot{\tau} \|_Y^2 \right).
\]
(5.53)

By (5.50) - (5.51) and (5.53), the map \( \mathcal{S}_{R,T}^{H} \) is injective. On the other hand, by the exactness of (0.22), the construction of \( (C^0_{ab}, \partial) \) and the Hodge theorem, we have
\[
\dim H^\bullet(C^0_{ab}, \partial) = \dim H^\bullet(Z, F) = \dim \ker (D^Z_T).
\]
(5.54)

Hence the map \( \mathcal{S}_{R,T}^{H} \) is bijective. This completes the proof of Theorem 5.4. \( \square \)

For \( \omega \in \Omega^\bullet(Z_R, F) \) satisfying \( d^Z_T \omega = 0 \), i.e., \( d^Z_T (e^{Tf_T} \omega) = 0 \), we denote
\[
[\omega]_T = [e^{Tf_T} \omega] \in H^\bullet(Z_R, F) = H^\bullet(Z, F).
\]
(5.55)

Corollary 5.5. For \( T = R^\kappa \gg 1 \), the map
\[
[\mathcal{S}_{R,T}^{H}]_T : H^\bullet(C^0_{ab}, \partial) \to H^\bullet(Z, F)
\]
\[
\sigma \mapsto [\mathcal{S}_{R,T}^{H}(\sigma)]_T
\]
(5.56)
is bijective.

Proof. This is a direct consequence of the Hodge theorem and Theorem 5.4. \( \square \)

Remark 5.6. By Corollary 5.5 and (5.50), we have a bijection
\[
\mathcal{H}_{ab}(Z_{12, \infty}, F) \oplus \left( \mathcal{L}_{1,ab}^{\perp} \cap \mathcal{L}_{2,ab}^{\perp} \right) \cong H^\bullet(Z, F).
\]
(5.57)

Set
\[
\mathcal{H}^\bullet(Z_{12, \infty}, F) = \left\{ (\omega_1, \omega_2, \dot{\omega}) : (\omega_j, \dot{\omega}) \in \mathcal{H}^\bullet(Z_{j, \infty}, F) \text{ for } j = 1, 2 \right\}.
\]
(5.58)

By [50] Thm. 3.7 and the Hodge theorem, we have a bijection
\[
\mathcal{H}^\bullet(Z_{12, \infty}, F) \cong H^\bullet(Z, F).
\]
(5.59)
The vector spaces in (5.57) and (5.59) are linked by the short exact sequence
\[
0 \to \mathcal{H}_{ab}^\bullet(Z_{12, \infty}, F) \to \mathcal{H}^\bullet(Z_{12, \infty}, F) \to \mathcal{L}_{1,ab}^{\perp} \cap \mathcal{L}_{2,ab}^{\perp} \to 0,
\]
which follows from (5.16), (5.58) and [50] (2.49)-(2.53).

41
5.2. Eigenspace of $D_T^{Z_R}$ associated with small eigenvalues. For $\omega \in \Omega^\bullet(IY_R, F) = \mathcal{C}^\infty([-R, R], \mathcal{M}^\bullet(Y, F)(du))$, we denote
\[(5.61) \|\omega\|_{Y, \text{max}} = \max \left\{ \|\omega_u\|_V : u \in [-R, R] \right\} .\]

Lemma 5.7. For $T = R^c > 1$ and $\omega \in \Omega^\bullet(Z_R, F)$ an eigensection of $R\partial_T^{Z_R}$ associated with eigenvalue $\lambda \in [-\sqrt{R}, \sqrt{R}] \setminus \{0\}$, we have
\[(5.62) \|\omega\|_{Z_{1,0} \cup Z_{2,0}} = O(1) \|\omega^{zn}\|_{Y, \text{max}} .\]

Proof: We will follow [50, Lemma 3.10]. Suppose, on the contrary, that there exist $R_i \to \infty$, $T_i = R_i^c$, $\lambda_i \in [-\sqrt{R}, \sqrt{R}] \setminus \{0\}$ and $\omega_i \in \Omega^\bullet(Z_{R_i}, F)$ such that
\[(5.63) R_iD_{T_i}^{Z_{R_i}}\omega_i = \lambda_i\omega_i , \quad \|\omega_i\|_{Z_{1,0} \cup Z_{2,0}} = 1 , \quad \lim_{i \to \infty} \|\omega_i^{zn}\|_{Y, \text{max}} = 0 .\]

Without loss of generality, we may assume that
\[(5.64) \liminf_{i \to \infty} \|\omega_i\|_{Z_{1,0}} > 0 .\]

Step 1. We extract a convergent subsequence of $(\omega_i)_i$.

By the Trace theorem for Sobolev spaces and (5.63), we have
\[(5.65) \|\omega_i\|_{\partial Z_{1,0}} = O(1) , \quad \|\omega_i\|_{\partial Z_{2,0}} = O(1) .\]

By Lemma 4.4 (5.5) and (5.65), we have
\[(5.66) \|\omega_i^{zn}\|_{Y, \text{max}} = O(1) .\]

By (5.63) and (5.66), we have
\[(5.67) \|\omega_i\|_{Y, \text{max}} \leq \|\omega_i^{zn}\|_{Y, \text{max}} + \|\omega_i^{zn}\|_{Y, \text{max}} = O(1) .\]

For $r \in \mathbb{N}$ and $R \geq r$, let $IY_{r} \subseteq Z_{1,r} \subseteq Z_{1,\infty}$ and $Z_{1,r} \subseteq Z_{R}$ be the canonical isometric embeddings. By (4.36) and (4.37), we have
\[(5.68) R\partial_T^{Z_{r}}|_{Z_{1,0}} = R\partial_T^{Z_{1,\infty}}|_{Z_{1,0}} , \quad R\partial_T^{Z_{r}}|_{IY_{r}} = R\partial_T^{Z_{1,\infty}}|_{IY_{r}} + T f^r_{T_i} c|_{IY_{r}} .\]

By (5.63) and (5.68), we have
\[(5.69) D_{Z_{1,\infty}}\omega_i|_{Z_{1,0}} = \lambda R_i^{-1}\omega_i|_{Z_{1,0}} , \quad D_{Z_{1,\infty}}\omega_i|_{IY_{r}} = \left(\lambda R_i^{-1} - R_i^{-1}T_i f^r_{T_i} c\right)\omega_i|_{IY_{r}} .\]

Since $\lambda R_i^{-1} \to 0$ and $R_i^{-1}T_i \to 0$, by the second identity in (5.32), (5.63), (5.67) and (5.69), the series $(\omega_i|_{Z_{1,r}})_i$ is $H^1$-bounded. Using Rellich’s lemma, by extracting a subsequence, we may suppose that $\omega_i|_{Z_{1,r}}$ is $L^2$-convergent. Applying (5.69) once again, we see that $(\omega_i|_{Z_{1,r}})_i$ is $H^1$-Cauchy. Let $\omega_{\infty,r}$ be the limit of $(\omega_i|_{Z_{1,r}})_i$, which is at least a $H^1$-current on $Z_{1,r}$ with values in $F$. Taking the limit of (5.69), we get
\[(5.70) D_{Z_{1,\infty}}\omega_{\infty,r}|_{Z_{1,r}} = 0 .\]

Since $D_{Z_{1,\infty}}$ is elliptic, equation (5.70) implies $\omega_{\infty,r} \in \Omega^\bullet(Z_{1,r}, F)$.
The standard diagonal argument allows us to extract a subsequence \((\omega_{i_j})\) of \((\omega_i)\) such that for any \(r \in \mathbb{N}\), \(\omega_{i_j}|_{Z_{1,r}}\) converges to \(\omega_{r,\infty}\) in \(H^1\)-norm. Now we replace \((\omega_i)\) by \((\omega_{i_j})\). There exists \(\omega_{\infty} \in \Omega^*(Z_{1,\infty}, F)\) such that for any \(r \in \mathbb{N}\),
\[(5.71)\quad \omega_i|_{Z_{1,r}} \to \omega_{\infty}|_{Z_{1,r}} \text{ in } H^1\text{-norm}.
\]
Since \((5.70)\) holds for all \(r \in \mathbb{N}\), we have
\[(5.72)\quad D_{Z_{1,\infty}} \omega_{\infty} = 0.
\]

Step 2. We show that \(\omega_{\infty}\) is \(L^2\)-integrable.

By the Trace theorem for Sobolev spaces, \((5.67)\) and \((5.71)\), we have
\[(5.73)\quad \|\omega_{\infty}\|_{Y,\max} < +\infty.
\]
By \((5.63)\) and \((5.71)\), we have
\[(5.74)\quad \omega_{\infty}^{zm} = 0.
\]
By \[(5.74)\] and \((5.72)\)-\((5.74)\), there exists \(a > 0\) such that for \(r > 0\), we have
\[(5.75)\quad \|\omega_{\infty}\|_{\partial Z_{1,r}} = O(e^{-ar}).
\]
In particular, \(\omega_{\infty}\) is \(L^2\)-integrable.

Step 3. We look for a contradiction.

By \((3.21)\), \((5.16)\), \((5.72)\) and \((5.75)\), we have
\[(5.76)\quad (\omega_{\infty}, 0, 0) \in \mathcal{H}_{\text{abs}}(Z_{1,\infty}, F).
\]
Recall that \(P_{T}F_{T} : \mathcal{H}_{\text{abs}}(Z_{1,\infty}, F) \to \text{Ker} \left(D_{T}^{Z_{R_i}}\right)\) was constructed in \((5.1)\). Set
\[(5.77)\quad \mu_i = P_{T}F_{T}(\omega_{\infty}, 0, 0) \in \text{Ker} \left(D_{T}^{Z_{R_i}}\right) \subseteq \Omega^*(Z_{R_i}, F).
\]
We decompose \(Z_{R_i}\) into two pieces \(Z_{R_i} = Z_{1,R_i} \cup Z_{2,0}\). We have
\[(5.78)\quad \langle \mu_i, \omega_i \rangle_{Z_{R_i}} - \langle \omega_{\infty}, \omega_i \rangle_{Z_{1,R_i}} = \langle \mu_i - \omega_{\infty}, \omega_i \rangle_{Z_{1,R_i}} + \langle \mu_i, \omega_i \rangle_{Z_{2,0}}.
\]
By \((5.63)\) and \((5.67)\), we have
\[(5.79)\quad \|\omega_i\|^2_{Z_{1,R_i}} + \|\omega_i\|^2_{Z_{2,0}} = \|\omega_i\|^2_{Z_{R_i}} = O(R_i).
\]
By Proposition \((5.2)\) and \((5.77)\), we have
\[(5.80)\quad \|\mu_i - F_{T}(\omega_{\infty}, 0, 0)\|^2_{Z_{R_i}} = O(R_i^{-2+\kappa}).
\]
By \((5.15)\), \((5.17)\), \((5.18)\), \((5.20)\), \((5.31)\) and \((5.32)\), we have
\[(5.81)\quad \|F_{T}(\omega_{\infty}, 0, 0) - \omega_{\infty}\|^2_{Z_{1,R_i}} = O(R_i^{-2+\kappa}), \quad \|F_{T}(\omega_{\infty}, 0, 0)\|^2_{Z_{2,0}} = 0.
\]
By \((5.80)\) and \((5.81)\), we have
\[(5.82)\quad \|\mu_i - \omega_{\infty}\|^2_{Z_{1,R_i}} = O(R_i^{-2+\kappa}), \quad \|\mu_i\|^2_{Z_{2,0}} = O(R_i^{-2+\kappa}).
\]
By \((5.78)\), \((5.79)\) and \((5.82)\), we have
\[(5.83)\quad \langle \mu_i, \omega_i \rangle_{Z_{R_i}} - \langle \omega_{\infty}, \omega_i \rangle_{Z_{1,R_i}} = O(R_i^{-1/2+\kappa/2}).
\]
By the dominated convergence theorem, (5.67), (5.71) and (5.75), we have
\begin{equation}
\lim_{t \to +\infty} \left\langle \omega, \omega_t \right\rangle_{Z_{1, R_i}} = \left\| \omega \right\|_{Z_{1, \infty}}^2.
\end{equation}

Since \( \kappa \in ]0, 1/3[ \), by (5.83) and (5.84), we have
\begin{equation}
\lim_{t \to +\infty} \left\langle \mu, \omega \right\rangle_{Z_{R_i}} = \left\| \omega \right\|_{Z_{1, \infty}}^2.
\end{equation}

By (5.64) and (5.71), we have \( \left\| \omega \right\|_{Z_{1, \infty}}^2 > 0 \). Thus \( \left\langle \mu, \omega \right\rangle_{Z_{R_i}} \neq 0 \) for \( i \) large enough.

But, by (5.63), (5.77) and the assumption \( \lambda_i \neq 0 \), we have \( \left\langle \mu, \omega \right\rangle_{Z_{R_i}} = 0 \). Contradiction. This completes the proof of Lemma 5.7.

Recall that the operator \( c \) was defined in (4.6). For \( \sigma \in \mathcal{H}^\bullet(Y, F)[du] \), we denote \( \sigma = \sigma^+ + \sigma^- \) such that \( \sigma \sigma^\pm = \mp i \sigma^\pm \).

For \( j = 1, 2 \), let \( C_j(\lambda) \in \text{End}(\mathcal{H}^\bullet(Y, F)[du]) \) be the scattering matrix as in [50, (2.32)] with \( X_\infty \) replaced by \( Z_{j, \infty} \). By [50, (2.35)], we have
\begin{equation}
c C_j(\lambda) = -C_j(\lambda) c.
\end{equation}

**Lemma 5.8.** For \( T = R^n \gg 1 \) and \( \omega \in \Omega^\bullet(Z_R, F) \) an eigensection of \( RD_T^Z \) associated with eigenvalue \( \lambda \in \left[ -\sqrt{R}, \sqrt{R} \right] \setminus \{0\} \), we have
\begin{equation}
\left\| \omega_{zm,-} \right\|_{Z_{1, 0}}^2 = O\left( R^{-2+\kappa} \right) \left\| \omega \right\|_{Z_{1, 0}}^2.
\end{equation}

**Proof:** We will follow [50, Lemma 3.12]. We only prove (5.87) for \( j = 1 \).

Let \( \omega' \in \Omega^\bullet(Z_{1, \infty}, F) \) be the unique generalized eigensection of \( D_{Z_{1, \infty}} \) associated with eigenvalue \( \lambda / R \) satisfying
\begin{equation}
\left\| \omega' \right\|_{Z_{1, 0}} = \left\| \omega \right\|_{Z_{1, 0}} \quad \text{and} \quad \left\| \omega' \right\|_{Z_{1, 0}} = \lambda \omega' \quad \text{for} \quad j = 1, 2.
\end{equation}

By the theory of ordinary differential equation, there exists \( \omega'' \in \Omega^\bullet(Z_{1, R}, F) \) satisfying
\begin{equation}
\left\| \omega'' \right\|_{Z_{1, 0}} = \left\| \omega' \right\|_{Z_{1, 0}} \quad \text{and} \quad \left\| \omega'' \right\|_{Z_{1, 0}} = \lambda \omega''.
\end{equation}

Set
\begin{equation}
\mu = \omega \mid_{Z_{1, \infty}} - \omega'' \mid_{Z_R}.
\end{equation}

By (5.88)-(5.91), we have
\begin{equation}
c \mu \mid_{Z_{1, 0}} = (\omega' + C_1(\lambda / R) \omega_{zm,-}) \mid_{Z_{1, 0}}.
\end{equation}

By the construction of \( \mu \), we have \( RD_T^Z \mid_{Y_R} \mu = \lambda \mu \). By Lemma 4.5, (5.86) and (5.92), we have
\begin{equation}
\left\langle c \mu, \mu \right\rangle_{Z_{1, R/2}} = \left\langle c \mu, \mu \right\rangle_{Z_{1, 0}} = -i \left\| \mu \right\|_{Z_{1, 0}}^2 = -i \left\| \omega_{zm,+} - C_1(\lambda / R) \omega_{zm,-} \right\|_{Z_{1, 0}}^2.
\end{equation}
By the Trace theorem for Sobolev spaces and Proposition 5.1, we have
\[ \|\omega\|_{\partial Z_{1,0}} = \mathcal{O}(1) \|\omega\|_{Z_{1,0}}, \quad \|\omega\|_{\partial Z_{2,0}} = \mathcal{O}(1) \|\omega\|_{Z_{2,0}}. \]
Applying Lemma 4.4 to \(\omega^{nz}\), there exists a universal constant \(a > 0\) such that
\[ \|\omega^{nz}\|_{\partial Z_{1,r}} = \mathcal{O}(e^{-ar}) \left(\|\omega^{nz}\|_{\partial Z_{1,0}} + \|\omega^{nz}\|_{\partial Z_{2,0}}\right), \quad \text{for } 0 \leq r \leq R/2. \]
By [50, Prop. 2.4], we have \(\|\omega^{nz}\|_{Z_{1,\infty} \setminus Z_{1,0}} < +\infty\). Moreover, by [50] (2.10), (2.38), there exists a universal constant \(a > 0\) such that
\[ \|\omega^{nz}\|_{\partial Z_{1,0}} = \mathcal{O}(e^{-ar}) \|\omega^{nz}\|_{Z_{1,\infty} \setminus Z_{1,0}} = \mathcal{O}(e^{-ar}) \|\omega^{zm}\|_{\partial Z_{1,0}}, \quad \text{for } r \geq 0. \]
Since \(C_1(\lambda/R)\) is unitary, (5.88) and (5.89) imply
\[ \|\omega^{zm}\|_{\partial Z_{1,0}} = \|\omega^{zm, -}\|_{\partial Z_{1,0}} + \|\omega^{zm,+}\|_{\partial Z_{1,0}} = 2 \|\omega^{zm, -}\|_{\partial Z_{1,0}}. \]
By (5.94), (5.97), we have
\[ \|\omega^{nz}\|_{\partial Z_{1,r}} = \mathcal{O}(e^{-ar}) \|\omega^{nz}\|_{Z_{1,0} \cup Z_{2,0}}, \quad \|\omega^{nz}\|_{\partial Z_{1,r}} = \mathcal{O}(e^{-ar}) \|\omega^{nz}\|_{Z_{1,0} \cup Z_{2,0}}. \]
By the second identity in (5.90), (5.91) and (5.98), we have
\[ \|\mu^{nz}\|_{\partial Z_{1,r}} = \mathcal{O}(e^{-ar}) \|\omega^{nz}\|_{Z_{1,0} \cup Z_{2,0}}. \]
We identify \(IY_R \subseteq Z_{1,R}\) with \([0, 2R] \times Y\). By the construction of \(\mu\) and (5.68), we have
\[ RD_T Z_R \mu \big|_{Z_{1,0}} = \lambda \mu, \quad RD_T Z_R \mu \big|_{[0, 2R] \times Y} = \lambda \mu - T f'_T \hat{\omega}^{nz}. \]
By (5.5) and (5.100), we have
\[ \langle RD_T Z_R \mu, \mu \rangle_{Z_{1,R/2}} - \langle \mu, RD_T Z_R \mu \rangle_{Z_{1,R/2}} = 2i \text{Im} \left( \langle \mu^{nz}, f'_T \hat{\omega}^{nz} \rangle \right)_{[0, R] \times Y}. \]
On the other hand, by Green’s formula (cf. [50], (2.8)), we have
\[ \langle RD_T Z_R \mu, \mu \rangle_{Z_{1,R/2}} - \langle \mu, RD_T Z_R \mu \rangle_{Z_{1,R/2}} = R \langle c \mu, \mu \rangle_{\partial Z_{1,R/2}}. \]
By (5.101), (5.102) and the assumption \(T = R^\kappa\), we have
\[ \left| \left( c \mu^{zm}, \mu^{zm} \right)_{\partial Z_{1,R/2}} + \left( c \mu^{nz}, \mu^{nz} \right)_{\partial Z_{1,R/2}} \right| = \left| \left( c \mu, \mu \right)_{\partial Z_{1,R/2}} \right| \leq 2R^{-1+\kappa} \left| \left( \mu^{nz}, f'_T \hat{\omega}^{nz} \right) \right|_{[0, R] \times Y}. \]
By (3.8) and (3.11), we have
\[ f'_T \big|_{\partial Z_{1,r}} = \mathcal{O}(R^{-1}) r. \]
By (5.98), (5.99) and (5.104), we have
\[ \left| \left( \mu^{nz}, f'_T \hat{\omega}^{nz} \right) \right|_{[0, R] \times Y} \leq \int_0^R \left| f'_T \big|_{\partial Z_{1,r}} \right|^{\mu^{nz}} \left| \omega^{nz} \right|_{\partial Z_{1,r}} d\tau = \mathcal{O}(R^{-1}) \left( \left| \omega \right|_{Z_{1,0} \cup Z_{2,0}}^2 \right) \]
By (5.99), (5.103) and (5.105), we have
\[ \left( c \mu^{zm}, \mu^{zm} \right)_{\partial Z_{1,R/2}} = \mathcal{O}(R^{-2+\kappa}) \left( \left| \omega \right|_{Z_{1,0} \cup Z_{2,0}}^2 \right). \]
From (5.93) and (5.106), we obtain (5.87) with $j = 1$. This completes the proof of Lemma 5.8.

**Proof of Theorem 3.1.** First we consider the case $j = 0$.

Let $\omega \in \Omega^*(Z_R, F)$ be an eigensection of $RD_T^{Z_R}$ associated with eigenvalue $\lambda \in [-\sqrt{T}, \sqrt{T}] \setminus \{0\}$. By Lemmas 5.7, 5.8 we have $\omega_{zm} \neq 0$ and

\[
\|\omega_{zm} + C_j(\lambda/R)\omega_{zm}\|_{\partial Z_{j,0}} = O\left(R^{-1+\kappa/2}\right)\|\omega_{zm}\|_{Y_{\max}}, \text{ for } j = 1, 2. \tag{5.107}
\]

Since $\lambda \mapsto C_j(\lambda)$ is analytic (cf. [47, §4] [50, Prop. 2.3]), by (5.107) and the assumption $|\lambda| \leq T^{1/2} = R^{\kappa/2}$, we have

\[
\|\omega_{zm} - C_j(0)\omega_{zm}\|_{\partial Z_{j,0}} = O\left(R^{-1+\kappa/2}\right)\|\omega_{zm}\|_{Y_{\max}}, \text{ for } j = 1, 2. \tag{5.108}
\]

Moreover, as $C_j(0)$ is unitary and $(C_j(0))^2 = \text{Id}$ (cf. [50, Prop. 2.3]), we have

\[
\|\omega_{zm} - C_j(0)\omega_{zm}\|_{\partial Z_{j,0}} = O\left(R^{-1+\kappa/2}\right)\|\omega_{zm}\|_{Y_{\max}}, \text{ for } j = 1, 2. \tag{5.109}
\]

By (5.109) we get (cf. [50, Prop. 2.3]), we have

\[
\mathcal{L}_j^* = \text{Ker} \left(\text{Id} - C_j(0)\right). \tag{5.110}
\]

For $j = 1, 2$, let

\[
P_j : \mathcal{H}^*(Y, F)[du] \to \mathcal{L}_j^*
\]

be orthogonal projections with respect to $\|\cdot\|_Y$. We denote

\[
P_j^\perp = \text{Id} - P_j. \tag{5.112}
\]

By (5.109), the estimate (5.109) is equivalent to the follows,

\[
\|P_j^\perp \omega_{zm}\|_{\partial Z_{j,0}} = O\left(R^{-1+\kappa/2}\right)\|\omega_{zm}\|_{Y_{\max}}, \text{ for } j = 1, 2. \tag{5.113}
\]

Let $D_{T, bd}^{\mathcal{H}^*(Y, F)}$ be the operator $D_{T, bd}^V$ in (4.15) with

\[
V = \mathcal{H}^*(Y, F), \quad V_j = \mathcal{L}_{j, \text{abs}}, \quad \text{for } j = 1, 2. \tag{5.114}
\]

Applying Proposition 4.3 to (5.113) with $\epsilon = R^{-1+3\kappa}$ and using the assumption $T = R^\kappa$, we get

\[
\|\omega_{zm} - P_T^{(\lambda - \epsilon, \lambda + \epsilon)}\omega_{zm}\|_{Y_{\max}} = O\left(R^{-\kappa/2}\right)\|\omega_{zm}\|_{Y_{\max}}. \tag{5.115}
\]

By (5.115), for $T = R^\kappa \gg 1$, we have $P_T^{(\lambda - \epsilon, \lambda + \epsilon)}\omega_{zm} \neq 0$. As a consequence,

\[
[\lambda - \epsilon, \lambda + \epsilon] \cap \text{Sp}(D_{T, bd}^{\mathcal{H}^*(Y, F)}) \neq \emptyset. \tag{5.116}
\]

From Theorem 4.1 and (5.116), we obtain (3.18) with $j = 0$.

We turn to the cases $j = 1, 2, 3$. Proceeding in the same way as in [50, §3.5], we may replace $Z_{j, R}$ by its ’double’, which is a compact manifold without boundary. Then we apply (3.18) with $j = 0$. This completes the proof of Theorem 3.1. \qed
For convenience, we denote
\begin{equation}
(5.117) \quad \mathcal{H}_{\text{abs}}^\ast (Z_{1,\infty} \sqcup Z_{2,\infty}, F) = \mathcal{H}_{\text{abs}}^\ast (Z_{1,\infty}, F) \oplus \mathcal{H}_{\text{abs}}^\ast (Z_{2,\infty}, F) .
\end{equation}
Similarly to the constructions of $F_{R,T}$ and $G_{R,T}$ in (5.11), we define
\begin{equation}
(5.118) \quad F_{R,T}^+, G_{R,T}^+ : \mathcal{H}_{\text{abs}}^\ast (Z_{1,\infty} \sqcup Z_{2,\infty}, F) \to \Omega^\ast (Z_R, F)
\end{equation}
as follows: for $(\omega_1, \tilde{\omega}_1, \omega_2, \tilde{\omega}_2) \in \mathcal{H}_{\text{abs}}^\ast (Z_{1,\infty} \sqcup Z_{2,\infty}, F)$,
\begin{equation}
(5.119) \quad F_{R,T}^+ (\omega_1, \tilde{\omega}_1, \omega_2, \tilde{\omega}_2) \big|_{\Omega_{R,T}} = e^{-T^T R} \left( \chi_1 \tilde{\omega}_1 + \chi_2 \tilde{\omega}_2 \right) + e^{-T^T R} d^R \left( \chi_1 d^R_\omega (\omega_1, \tilde{\omega}_1) + \chi_2 d^R_\omega (\omega_2, \tilde{\omega}_2) \right) ,
\end{equation}
\begin{equation}
(5.120) \quad G_{R,T}^+ (\omega_1, \tilde{\omega}_1, \omega_2, \tilde{\omega}_2) \big|_{\Omega_{R,T}} = e^{-T^T R} \left( \chi_1 \tilde{\omega}_1 + \chi_2 \tilde{\omega}_2 \right) + e^{T^T R} d^{R^*} \left( \chi_1 d^{R^*}_\omega (\omega_1, \tilde{\omega}_1) + \chi_2 d^{R^*}_\omega (\omega_2, \tilde{\omega}_2) \right) .
\end{equation}
By (50), (2.52), (3.21) and (5.117), we have $d^Y \omega_j = i \partial \tilde{\omega}_j = 0$ for $j = 1, 2$. Then, by (3.15), we have
\begin{equation}
(5.121) \quad d^{Z_R^*} G_{R,T}^+ (\omega_1, \tilde{\omega}_1, \omega_2, \tilde{\omega}_2) = 0 .
\end{equation}
Let $P_{R,T}^{-[1,1]} : \Omega^\ast (Z_R, F) \to \mathcal{H}_{\text{abs}}^\ast (Z_{1,\infty} \sqcup Z_{2,\infty}, F)$ be the orthogonal projection with respect to $\| \cdot \|_{Z_R}$, where $\mathcal{H}_{\text{abs}}^\ast (Z_{1,\infty} \sqcup Z_{2,\infty}, F)$ was defined in (3.19).

**Proposition 5.9.** For $T = R^k \gg 1$ and $(\omega_1, \tilde{\omega}_1, \omega_2, \tilde{\omega}_2) \in \mathcal{H}_{\text{abs}}^\ast (Z_{1,\infty} \sqcup Z_{2,\infty}, F)$, we have
\begin{equation}
(5.122) \quad \left\| (\text{Id} - P_{R,T}^{-[1,1]}) G_{R,T}^+ (\omega_1, \tilde{\omega}_1, \omega_2, \tilde{\omega}_2) \right\|^2_{H^1, Z_R} = \mathcal{O} \left( R^{-2+\kappa} \right) \left( \| \omega_1 \|^2_{Z_{1,0}} + \| \omega_2 \|^2_{Z_{2,0}} \right) .
\end{equation}

**Proof:** Though the constructions of $F_{R,T}^+$ and $G_{R,T}^+$ are different from the constructions of $F_{R,T}$ and $G_{R,T}$ in (5.20), we can directly verify that $(F_{R,T}^+ - G_{R,T}^+) (\omega_1, \tilde{\omega}_1, \omega_2, \tilde{\omega}_2)$ satisfies (5.24). Then, similarly to (5.35), we have
\begin{equation}
(5.123) \quad \left\| (F_{R,T}^+ - G_{R,T}^+) (\omega_1, \tilde{\omega}_1, \omega_2, \tilde{\omega}_2) \right\|^2_{Z_R} = \mathcal{O} \left( R^{-2+\kappa} \right) \left( \| \omega_1 \|^2_{Z_{1,0}} + \| \omega_2 \|^2_{Z_{2,0}} \right) ,
\end{equation}
\begin{equation}
(5.124) \quad \left\| d^{Z_R} F_{R,T}^+ (\omega_1, \tilde{\omega}_1, \omega_2, \tilde{\omega}_2) \right\|^2_{Z_R} = \mathcal{O} \left( e^{-aT} \right) \left( \| \omega_1 \|^2_{Z_{1,0}} + \| \omega_2 \|^2_{Z_{2,0}} \right) ,
\end{equation}
where $a > 0$ is a universal constant.
By Corollary 1.4, (5.121), (5.123) and (5.124), we have

\[
\left\| (\text{Id} - P_{R,T}^{-1}) G_{R,T}^+ (\omega_1, \omega_1, \omega_2, \omega_2) \right\|_{Z_R}^2 + \left\| D_{Z_R} (\text{Id} - P_{R,T}^{-1}) G_{R,T}^+ (\omega_1, \omega_1, \omega_2, \omega_2) \right\|_{Z_R}^2 = O(R^{-2+\kappa}) \left( \|\omega_1\|^2_{Z_{1,0}} + \|\omega_2\|^2_{Z_{2,0}} \right).
\]

From Proposition 5.1 and (5.125), we obtain (5.122). This completes the proof of Proposition 5.9.

We define

\[
I^+_R : \mathcal{H}^* \to \Omega^{*+1}(Z_R, F)
\]
as follows: for \( \bar{\omega} \in \mathcal{H}^*(Y, F) \),

\[
I^+_R(\bar{\omega})|_{Z_{j,0}} = 0, \quad \text{for } j = 1, 2,
\]

and

\[
I^+_R(\bar{\omega})|_{Y_R} = \chi e^{T_{\bar{r}} - T} du \wedge \bar{\omega}.
\]

We have

\[
d_{Z_R} I^+_R(\bar{\omega}) = 0.
\]

**Proposition 5.10.** For \( T = R^\kappa \gg 1 \) and \( \bar{\omega} \in \mathcal{H}^*(Y, F) \), we have

\[
\left\| (\text{Id} - P_{R,T}^{-1}) I^+_R(\bar{\omega}) \right\|_{H^1, Z_R}^2 = O(e^{-aT}) \left\|\bar{\omega}\right\|_Y^2.
\]

**Proof.** By (3.8), (3.11), (3.15) and the construction of \( I^+_R \) (see (5.126)), we have

\[
\left\| D_{Z_R} I^+_R(\bar{\omega}) \right\|_{Z_R}^2 = O(e^{-aT}) \left\|\bar{\omega}\right\|_Y^2, \quad \left\| D_{Z_R}^2 I^+_R(\bar{\omega}) \right\|_{Z_R}^2 = O(e^{-aT}) \left\|\bar{\omega}\right\|_Y^2.
\]

By Corollary 1.4 and (5.130), we have

\[
\left\| (\text{Id} - P_{R,T}^{-1}) I^+_R(\bar{\omega}) \right\|_{Z_R}^2 + \left\| D_{Z_R} (\text{Id} - P_{R,T}^{-1}) I^+_R(\bar{\omega}) \right\|_{Z_R}^2 = O(e^{-aT}) \left\|\bar{\omega}\right\|_Y^2,
\]

where \( a > 0 \) is a universal constant. From Proposition 5.1 and (5.131), we obtain (5.129). This completes the proof of Proposition 5.10.

We identify \( C^0_{0} = W_1 \oplus W_2^* = H^*(Z_1, F) \oplus H^*(Z_2, F) \) with \( \mathcal{H}^*(Z_1, \infty \sqcup Z_2, \infty, F) \) via the map (3.22). We identify \( C^0_{0} = V^* = H^*(Y, F) \) with \( \mathcal{H}^*(Y, F) \) via the isomorphism \( H^*(Y, F) \cong \mathcal{H}^*(Y, F) \) given by the Hodge theorem. We define a map

\[
\mathcal{J}_{R,T} : C^{0}_{0} \to \mathcal{E}^{-1,1}_{0, R, T},
\]

\[
\mathcal{J}_{R,T} C^{0}_{0} = P_{R,T}^{-1} G_{R,T}^+, \quad \mathcal{J}_{R,T} C^{0}_{0} = P_{R,T}^{-1} I_{R,T}^+.
\]

**Proposition 5.11.** The vector subspaces \( \mathcal{J}_{R,T}(C^{0}_{0}), \mathcal{J}_{R,T}(C^{1}_{0}) \subseteq \Omega^*(Z_R, F) \) are orthogonal with respect to \( \langle \cdot, \cdot \rangle_{Z_R} \).
Proof. We consider $\sigma_0 \in C^0_0$ and $\sigma_1 \in C^1_0$. Since the supports of $G^+_{R,T}(\sigma_0)$ and $I^+_{R,T}(\sigma_1)$ are mutually disjoint, we have

$$\left\langle G^+_{R,T}(\sigma_0), I^+_{R,T}(\sigma_1) \right\rangle_{Z_R} = 0 .$$

On the other hand, by (5.121) and (5.128), we have

$$G^+_{R,T}(\sigma_0) \in \text{Ker} \left( d_T^{Z_{R,R}} \right) = \text{Ker} \left( D_T^{Z_R} \right) \oplus \text{Im} \left( d_T^{Z_{R,R}} \right),$$

$$I^+_{R,T}(\sigma_1) \in \text{Ker} \left( d_T^{Z_R} \right) = \text{Ker} \left( D_T^{Z_R} \right) \oplus \text{Im} \left( d_T^{Z_{R,R}} \right).$$

Since $P_{R,T}^{R[-1,1]} := \text{Id} - P_{R,T}^{[-1,1]}$ commutes with $d_T^{Z_{R,R}}$ and $d_T^{Z_{R,R}}$, we have

$$P_{R,T}^{R[-1,1]} G^+_{R,T}(\sigma_0) \in \text{Im} \left( d_T^{Z_{R,R}} \right), \quad P_{R,T}^{R[-1,1]} I^+_{R,T}(\sigma_1) \in \text{Im} \left( d_T^{Z_{R,R}} \right),$$

which implies

$$\left\langle P_{R,T}^{R[-1,1]} G^+_{R,T}(\sigma_0), P_{R,T}^{R[-1,1]} I^+_{R,T}(\sigma_1) \right\rangle_{Z_R} = 0 .$$

From (5.133), (5.136) and the obvious identity

$$\left\langle G^+_{R,T}(\sigma_0), I^+_{R,T}(\sigma_1) \right\rangle_{Z_R} = \left\langle \mathcal{S}_{R,T}(\sigma_0), \mathcal{S}_{R,T}(\sigma_1) \right\rangle_{Z_R} + \left\langle P_{R,T}^{R[-1,1]} G^+_{R,T}(\sigma_0), P_{R,T}^{R[-1,1]} I^+_{R,T}(\sigma_1) \right\rangle_{Z_R},$$

we obtain $\left\langle \mathcal{S}_{R,T}(\sigma_0), \mathcal{S}_{R,T}(\sigma_1) \right\rangle_{Z_R} = 0$. This completes the proof of Proposition 5.11.

Theorem 5.12. For $T = R^\infty \gg 1$, the map $\mathcal{S}_{R,T}$ is bijective.

Proof. We will use the identifications (5.50). We construct a vector subspace $U^{\bullet,\bullet} \subseteq H^{\bullet}(C^\bullet\bullet, \partial)$ as follows,

$$U^{0,\bullet} = \left\{ (\omega_1, \omega_2, \omega) \in \mathcal{H}_{\text{abs}}(Z_{12,\infty}, F) : \omega_j \text{ is a generalized eigensection} \right\},$$

$$U^{1,\bullet} = H^1(C^\bullet\bullet, \partial) = \mathcal{L}_{1,\text{abs}}^+ \cap \mathcal{L}_{2,\text{abs}}^+ .$$

Step 1. We show that for $\sigma \in \mathcal{H}_{R,T}^H(U^{0,\bullet})$ or $\sigma \in \mathcal{H}_{R,T}^H(U^{1,\bullet})$,

$$\|\sigma^{zn}\|_{Y,\text{max}}^2 = O \left( R^{-1+\kappa} \right) \|\sigma^{zn}\|^2_{Y_R} ,$$

$$\|\sigma\|^2_{Z_{1,0} \cup Z_{2,0}} + \|\sigma^{zn}\|^2_{Y_R} = O \left( 1 \right) \|\sigma^{zn}\|^2_{Y,\text{max}} .$$

By the construction of $\mathcal{H}$ (see (5.51)), for $\sigma \in \mathcal{H}_{R,T}^H(U^{0,\bullet})$, there exists $(\omega_1, \omega_2, \omega) \in U^{0,\bullet}$ such that $\sigma = F_{R,T} F_{R,T}(\omega_1, \omega_2, \omega)$, where $F_{R,T}$ was defined in (5.20). We denote $\tilde{\sigma} = F_{R,T}(\omega_1, \omega_2, \omega)$. By (5.20), we have

$$\tilde{\sigma}^{zn} = e^{Tf} \omega .$$

By (3.8), (3.11), (5.140) and the assumption $T = R^\infty$, we have

$$\|\tilde{\sigma}^{zn}\|^2_{Y,\text{max}} = O \left( R^{-1+\kappa} \right) \|\tilde{\sigma}^{zn}\|^2_{Y_R} .$$
By (5.15), (5.20), (5.31) and (50) (2.36)-(2.38], we have
\[
\|\omega_1\|^2_{Z_{1,0}} + \|\omega_2\|^2_{Z_{2,0}} = \|\sigma\|^2_{Z_{1,0}\cup Z_{2,0}} = \mathcal{O}(1) \|\tilde{\sigma}^{zm}\|^2_{Y_{\max}},
\]
(5.142)
\[
\|\tilde{\sigma}^{nz}\|^2_{IY_R} = \mathcal{O}(1) \|\tilde{\sigma}^{zm}\|^2_{Y_{\max}}.
\]
By (5.142), we have
\[
\|\tilde{\sigma}\|^2_{Z_{1,0}\cup Z_{2,0}} + \|\tilde{\sigma}^{nz}\|^2_{IY_R} = \mathcal{O}(1) \|\tilde{\sigma}^{zm}\|^2_{Y_{\max}}.
\]
(5.143)
From the Trace theorem for Sobolev spaces, Proposition 5.2 and (5.141)-(5.143), we obtain (5.139) with \(\sigma \in L^2_R(T(U^0)\).

By the construction of \(\mathscr{H}^H\) (see (5.51)), for \(\sigma \in L^2_R(T(U^1))\), there exists \(\tilde{\omega} \in H^1\) such that \(\sigma = P_{R,T}T(I\tilde{\omega})\), where \(I_{R,T}\) was defined in (5.44). We denote \(\tilde{\sigma} = I_{R,T}(\tilde{\omega})\).

By (5.44), we have
\[
\left(1 + \Theta(e^{-aT})\right)\|\tilde{\omega}\|^2_Y = \|\sigma^{zm}\|^2_{Y_{\max}} = \mathcal{O}(R^{-1+\kappa}) \|\tilde{\sigma}^{zm}\|^2_{Y_{\max}},
\]
(5.144)
\[
\|\tilde{\sigma}^{nz}\|^2_{IY_R} = \|\tilde{\sigma}\|^2_{Z_{1,0}\cup Z_{2,0}} = 0,
\]
where \(a > 0\) is a universal constant. From the Trace theorem for Sobolev spaces, Proposition 5.3 and (5.144), we obtain (5.139) with \(\sigma \in L^2_R(T(U^1))\).

Step 2. We show that for \(\sigma \in \mathcal{E}^{(\lambda)}_{R,T}\) with \(\lambda \in [-1, 1]\},
\[
\|\sigma^{zm}\|^2_{Y_{\max}} = \mathcal{O}(R^{-1+2\kappa}) \|\sigma^{zm}\|^2_{Y_{\max}},
\]
(5.145)
\[
\|\sigma\|^2_{Z_{1,0}\cup Z_{2,0}} + \|\sigma^{nz}\|^2_{IY_R} = \mathcal{O}(1) \|\sigma\|^2_{Z_{1,0}\cup Z_{2,0}}.
\]

By the Trace theorem for Sobolev spaces, Lemma 4.4 and Proposition 5.1, we have
\[
\|\sigma^{nz}\|^2_{IY_R} = \mathcal{O}(1) \|\sigma^{nz}\|^2_{Z_{1,0}\cup Z_{2,0}} = \mathcal{O}(1) \|\sigma\|^2_{Z_{1,0}\cup Z_{2,0}}.
\]
(5.146)
By (4.42), \(\sigma^{zm}\) is an eigensection of \(D_T^*\) associated with \(\lambda\), i.e.,
\[
\left(e^{\frac{\partial}{\partial S}} + T f_T e\right) \sigma^{zm} = D_T^* \sigma^{zm} = \lambda \sigma^{zm}.
\]
(5.147)
The first inequality in (5.145) follows from the Sobolev inequality, (5.147) and the assumption \(\lambda \in [-1, 1]\). The second inequality in (5.145) follows from Lemma 5.7 and (5.146).

Let \(\mathcal{E}_T^{(-1,1]} \subseteq \Omega([0, 1], \mathcal{H}(Y, F))\) be the eigenspace of \(D_T^*\) associated with eigenvalues in \([-1, 1]\]. Let \(I_T^{[-1,1]} : \Omega([-1, 1], \mathcal{H}(Y, F)) \to \mathcal{E}_T^{[-1,1]}\) be the orthogonal projection.

Step 3. We show that the map
\[
\pi_{R,T} : \mathcal{H}_{R,T}(U^{*}) \oplus \mathcal{E}_{0,R,T}^{(-1,1]} \to \mathcal{E}_T^{[-1,1]}
\]
(5.148)
\[
\sigma \mapsto P_T^{[-1,1]} \sigma^{zm}
\]
is injective.

Let
\[
\sigma_1, \cdots, \sigma_m \in \mathcal{H}_{R,T}(U^{*}) \oplus \mathcal{E}_{0,R,T}^{[-1,1]}
\]
(5.149)
be a basis such that each $\sigma_i$ belongs to one of the following vector spaces
\begin{equation}
\mathcal{H}^H_{R,T}(U^{0\cdot}), \quad \mathcal{H}^H_{R,T}(U^{1\cdot}), \quad \mathcal{E}_{R,T}^{(\lambda)} \quad \text{with } \lambda \in [-1, 1]\setminus\{0\}.
\end{equation}
We suppose that for $i \neq j$ with $\sigma_i, \sigma_j$ belonging to the same vector space in (5.150),
\begin{equation}
\langle \sigma_i, \sigma_j \rangle_{Z_R} = 0.
\end{equation}
By Propositions 5.2, 5.3 and (5.152), for $\sigma_i \in \mathcal{H}^H_{R,T}(U^{0\cdot})$ and $\sigma_j \in \mathcal{H}^H_{R,T}(U^{1\cdot})$, we have
\begin{equation}
\langle \sigma_i, \sigma_j \rangle_{Z_R} = \mathcal{O}(R^{-1+\kappa/2}) \|\sigma_i\|_{Z_R} \|\sigma_j\|_{Z_R}.
\end{equation}
Since $\mathcal{H}^H_{R,T}(U^{\bullet\bullet}) \subseteq \text{Ker}(\partial_{Z_R})$, for $\sigma_i \in \mathcal{H}^H_{R,T}(U^{\bullet\bullet})$ and $\sigma_j \in \mathcal{E}_{R,T}^{[-1,1] \setminus \{0\}}$, we have
\begin{equation}
\langle \sigma_i, \sigma_j \rangle_{Z_R} = 0.
\end{equation}
By (5.151), (5.153) and (5.154), we have
\begin{equation}
\langle \sigma_i, \sigma_j \rangle_{Z_R} = \left(\delta_{ij} + \mathcal{O}(R^{-1+\kappa/2})\right) \|\sigma_i\|_{Z_R} \|\sigma_j\|_{Z_R},
\end{equation}
where $\delta_{ij}$ is the Kronecker delta.
By Steps 1, 2 and the obvious identity
\begin{equation}
\langle \sigma_i, \sigma_j \rangle_{Z_R} = \langle \sigma_i^z, \sigma_j^z \rangle_{Y_R} + \langle \sigma_i^{nz}, \sigma_j^{nz} \rangle_{Y_R} + \langle \sigma_i, \sigma_j \rangle_{Z_{1,0} \cup Z_{2,0}},
\end{equation}
we have
\begin{equation}
\langle \sigma_i^z, \sigma_j^z \rangle_{Y_R} = \langle \sigma_i, \sigma_j \rangle_{Z_R} + \mathcal{O}(R^{-1+2\kappa}) \|\sigma_i\|_{Z_R} \|\sigma_j\|_{Z_R}.
\end{equation}
Recall that the maps $P_j^\perp$ with $j = 1, 2$ were defined by (5.111)-(5.112). By (5.113), for $\sigma \in \mathcal{E}_{R,T}^{(\lambda)}$ with $\lambda \in [-1, 1]\setminus\{0\}$, we have
\begin{equation}
\left\|P_j^\perp \sigma^z\right\|^2_{\partial Z_{j,0}} = \mathcal{O}(R^{2+\kappa}) \|\sigma^z\|^2_{Y_{\text{max}}}, \quad \text{for } j = 1, 2.
\end{equation}
By the constructions of $F_{R,T}$ and $I_{R,T}$, for $\tilde{\sigma} \in F_{R,T}(U^{0\cdot})$ or $\tilde{\sigma} \in I_{R,T}(U^{1\cdot})$, we have
\begin{equation}
P_j^\perp \tilde{\sigma}^z\big|_{\partial Z_{j,0}} = 0, \quad \text{for } j = 1, 2.
\end{equation}
By the Trace theorem for Sobolev spaces, Propositions 5.2, 5.3, (5.142), (5.144) and (5.159), for $\sigma \in \mathcal{H}^H_{R,T}(U^{0\cdot})$ or $\sigma \in \mathcal{H}^H_{R,T}(U^{1\cdot})$, we have
\begin{equation}
\left\|P_j^\perp \sigma^z\right\|^2_{\partial Z_{j,0}} = \mathcal{O}(R^{-2+\kappa}) \|\sigma^z\|^2_{Y_{\text{max}}}, \quad \text{for } j = 1, 2.
\end{equation}
Applying Proposition 4.3 to (5.158) and (5.160) with $\epsilon = 1$, we get
\begin{equation}
\left\|\sigma_i^z - P_{T}^{[-2,2]} \sigma_i^z\right\|_{Y_R} = \mathcal{O}(R^{-1+3\kappa}) \|\sigma_i^z\|_{Y_R}.
\end{equation}
By Theorem 4.1, we have $P_T^{[-1,1]} = P_T^{[-2,2]}$. Then equation (5.161) yields
\begin{equation}
\langle P_T^{[-1,1]} \sigma_i^z, P_T^{[-1,1]} \sigma_j^z \rangle_{Y_R} = \langle \sigma_i^z, \sigma_j^z \rangle_{Y_R} + \mathcal{O}(R^{-1+3\kappa}) \|\sigma_i^z\|_{Y_R} \|\sigma_j^z\|_{Y_R}.
\end{equation}
By (5.155), (5.157) and (5.162), we have

\[
\langle P_T^{-1,1} \sigma_i^{zm}, P_T^{-1,1} \sigma_j^{zm} \rangle_{IY_R} = \left( \delta_{ij} + \mathcal{O} \left( R^{-1+3\kappa \varepsilon} \right) \right) \lVert \sigma_i \rVert_{Z_R} \lVert \sigma_j \rVert_{Z_R} .
\]

By (5.148) and (5.163), the Gram matrix \( \left( \langle \pi_{R,T}(\sigma_i), \pi_{R,T}(\sigma_j) \rangle_{IY_R} \right)_{1 \leq i,j \leq m} \) is positive-definite. Hence the map \( \pi_{R,T} \) is injective.

**Step 4.** We show that the map \( \mathcal{J}_{R,T} \) is bijective.

By Theorems 4.2, 5.4 and Step 3, we have

\[
\dim \left( \mathcal{E}_{0,R,T}^{[-1,1] \setminus \{ 0 \}} \right) + \dim U^{\bullet \bullet} = \dim \left( \mathcal{E}_{0,R,T}^{[-1,1] \setminus \{ 0 \}} \right) + \dim \mathcal{J}_{R,T}^H (U^{\bullet \bullet}) \leq \dim \mathcal{E}_{0,R,T}^{[-1,1]} = \dim C_{t \bullet}^{\bullet \bullet}.
\]

By Theorem 5.4, we have

\[
\dim \ker \left( D_T^{Z_R} \right) = \dim H^\bullet (C_0^{\bullet \bullet}, \partial) .
\]

By (2.4), (2.5) and (3.20), we have

\[
\dim C_0^{\bullet \bullet} - \dim C_t^{\bullet \bullet} = \dim K_1^\bullet + \dim K_2^\bullet .
\]

By the construction of \( U^{\bullet \bullet} \) and (3.23)-(3.25), we have

\[
\dim H^\bullet (C_0^{\bullet \bullet}, \partial) - \dim U^{\bullet \bullet} = \dim K_1^\bullet + \dim K_2^\bullet .
\]

From (5.164)-(5.167), we obtain

\[
\dim \mathcal{E}_{0,R,T}^{[-1,1]} = \dim \mathcal{E}_{0,R,T}^{[-1,1] \setminus \{ 0 \}} + \dim \ker \left( D_T^{Z_R} \right) \leq \dim C_0^{\bullet \bullet} .
\]

By Propositions 5.9, 5.11 and (5.132), the map \( \mathcal{J}_{R,T} : \dim C_0^{\bullet \bullet} \to \mathcal{E}_{R,T}^{[-1,1]} \) is injective. Then, by (5.168), it is bijective. This completes the proof of Theorem 5.12.

### 5.3. De Rham operator on \( \mathcal{E}_{0,R,T}^{[-1,1]} \).

**Proposition 5.13.** For \( T = R^\infty \gg 1 \), we have

\[
a_T^{Z_R} \mathcal{J}_{R,T} (C_0^{1 \bullet}) = 0 , \quad d_T^{Z_R} \mathcal{J}_{R,T} (C_0^{0 \bullet}) \subseteq \mathcal{J}_{R,T} (C_0^{1 \bullet}) .
\]

**Proof.** Since \( d_T^{Z_R} \) commutes with \( P_T^{-1,1} \), (5.128) and (5.132) yield

\[
da_T^{Z_R} \mathcal{J}_{R,T} (C_0^{1 \bullet}) = d_T^{Z_R} P_T^{-1,1} I_{R,T}^+ (C_0^{1 \bullet}) = P_T^{-1,1} d_T^{Z_R} I_{R,T}^+ (C_0^{1 \bullet}) = 0 .
\]

Since \( d_T^{Z_R,*} \) commutes with \( P_T^{-1,1} \), (5.121) and (5.132) yield

\[
da_T^{Z_R,*} \mathcal{J}_{R,T} (C_0^{0 \bullet}) = d_T^{Z_R,*} P_T^{-1,1} G_R (C_0^{0 \bullet}) = P_T^{-1,1} d_T^{Z_R,*} G_R (C_0^{0 \bullet}) = 0 .
\]

Thus \( \mathcal{J}_{R,T} (C_0^{0 \bullet}) \) is perpendicular to the image of \( d_T^{Z_R} \). On the other hand, by Proposition 5.11 and Theorem 5.12, we have an orthogonal decomposition

\[
\mathcal{E}_{0,R,T}^{[-1,1]} = \mathcal{J}_{R,T} (C_0^{0 \bullet}) \oplus \mathcal{J}_{R,T} (C_0^{1 \bullet}) .
\]

Hence \( d_T^{Z_R} \mathcal{J}_{R,T} (C_0^{0 \bullet}) \) must lie in \( \mathcal{J}_{R,T} (C_0^{1 \bullet}) \). This completes the proof of Proposition 5.13. \( \square \)
For \( \omega \in \Omega^\ast(Z_R, F) \), we will view \( \omega^{zm} \) as an element in \( \Omega^\ast([R, R], \mathcal{H}^\ast(Y, F)) \). Set

\[
\tau_{R,T}(\omega) = \int_{-R}^{R} e^{Tf_T} \omega^{zm} \in \mathcal{H}^\ast(Y, F).
\]

**Lemma 5.14.** There exists \( a > 0 \) such that for \( T = R^\varkappa \gg 1 \) and \( \dot{\omega} \in \mathcal{H}^\ast(Y, F) \), we have

\[
\left\| e^{-T} \tau_{R,T}(\mathcal{F}_{R,T}(\dot{\omega})) - \sqrt{\pi} R^{1-\kappa/2} \dot{\omega} \right\|_Y = O(e^{-aT}) \left\| \dot{\omega} \right\|_Y.
\]

**Proof.** By (3.8), (3.11), (5.42) and (5.127) and the assumption \( T = R^\kappa \), we have

\[
\int_{-R}^{R} e^{Tf_T} (I_{R,T}(\dot{\omega}))^{zm} = \int_{-R}^{R} \chi_{3} e^{2Tf_{T-2T}} du e^{T \dot{\omega}}\left(1 + O(e^{-aT})\right) \sqrt{\pi} R^{1-\kappa/2} e^{T \dot{\omega}}.
\]

From Proposition 5.10, (5.132) and (5.175), we obtain (5.174). This completes the proof of Lemma 5.14.

We will use the notation in (3.37).

**Theorem 5.15.** For \( T = R^\kappa \gg 1 \), we have

\[
\mathcal{F}_{R,T}^{-1} \circ d_{T}^{Z_{R}} \circ \mathcal{F}_{R,T} = \pi^{-1/2} R^{-1+\kappa/2} e^{-T} \left( \partial + \mathcal{O}_{End(\mathcal{H}^\ast)}(R^{-1+\kappa/2}) \right).
\]

**Proof.** For \( \omega \in \mathcal{F}_{R,T}(C_{0}^{0,\ast}) \), by (3.10), (3.15) and (5.173), we have

\[
\tau_{R,T}(d_{T}^{Z_{R}} \omega) = \int_{-R}^{R} d(e^{Tf_{T}} \omega^{zm}) = i \frac{\eta}{\pi^{2}} du \wedge \left( \omega^{zm} |_{\partial Z_{2,0}} - \omega^{zm} |_{\partial Z_{1,0}} \right).
\]

By the Trace theorem for Sobolev spaces, Proposition 5.9 and (5.132), for \( j = 1, 2 \) and \( (\omega_1, \dot{\omega}_1, \omega_2, \dot{\omega}_2) \in \mathcal{H}^\ast_{abs}(Z_{1,\infty} \sqcup Z_{2,\infty}, F) \), we have

\[
\left\| \left( \mathcal{F}_{R,T}(\omega_1, \dot{\omega}_1, \omega_2, \dot{\omega}_2) - G_{R,T}^{+}(\omega_1, \dot{\omega}_1, \omega_2, \dot{\omega}_2) \right)^{zm} \right\|_{\partial Z_{j,0}} = O(R^{-1+\kappa/2}) \left( \left\| \omega_1 \right\|_{Z_{1,0}} + \left\| \omega_2 \right\|_{Z_{2,0}} \right).
\]

By (5.119), we have

\[
\left( G_{R,T}^{+}(\omega_1, \dot{\omega}_1, \omega_2, \dot{\omega}_2) \right)^{zm} |_{\partial Z_{j,0}} = \dot{\omega}_j \in \mathcal{H}^\ast(Y, F).
\]

By (5.177)-(5.179), we have

\[
\left\| \dot{\omega}_2 - \dot{\omega}_1 - \tau_{R,T}(d_{T}^{Z_{R}} \mathcal{F}_{R,T}(\omega_1, \dot{\omega}_1, \omega_2, \dot{\omega}_2)) \right\|_{Y} = O(R^{-1+\kappa/2}) \left( \left\| \omega_1 \right\|_{Z_{1,0}} + \left\| \omega_2 \right\|_{Z_{2,0}} \right).
\]

From Proposition 5.13, Lemma 5.14, (2.1), (3.20) and (5.180), we obtain (5.176). This completes the proof of Theorem 5.15.
5.4. $L^2$-metric on $\mathcal{E}^{[1,1]}_{0,R,T}$. Recall that the Hermitian metric $h^{C_0^{\bullet\bullet}}_{R,T}$ on $C_0^{\bullet\bullet}$ was constructed in (3.30). We denote by $\| \cdot \|_{R,T}$ the norm on $C_0^{\bullet\bullet}$ associated with $h^{C_0^{\bullet\bullet}}_{R,T}$.

**Proposition 5.16.** For $T \geq 1$ and $\sigma \in C_0^{\bullet\bullet}$, we have

$$
\| \mathcal{J}_{R,T}(\sigma) \|^2_{Z_R} = \| \sigma \|^2_{R,T} \left( 1 + \mathcal{O}(R^{-1/2+\kappa/4}) \right).
$$

**Proof.** Let $\| \cdot \|'_{R,T}$ be the norm on $C_0^{\bullet\bullet}$ defined as follows: for $\sigma_0 \in C_0^{\bullet\bullet}$ and $\sigma_1 \in C_0^{\bullet\bullet}$,

$$
\| \sigma_0 + \sigma_1 \|^2_{R,T} = \| G_{R,T}^+(\sigma_0) \|^2_{Z_R} + \| I_{R,T}^+(\sigma_1) \|^2_{Z_R},
$$

where $G_{R,T}^+$ and $I_{R,T}^+$ were constructed in (5.119) and (5.127). By (5.119) and (5.127), we have

$$
\left( G_{R,T}^+(\sigma_0), I_{R,T}^+(\sigma_1) \right)_{Z_R} = 0.
$$

By Propositions 5.9, 5.10, 5.132, (5.182) and (5.183), we have

$$
\| \mathcal{J}_{R,T}(\cdot) \|^2_{Z_R} = \| \cdot \|'_{R,T} \left( 1 + \mathcal{O}(R^{-1+\kappa/2}) \right).
$$

By (5.119) and (5.182), the decomposition $C_0^{\bullet\bullet} = W_1^* \oplus W_2^* \oplus V^*$ is orthogonal with respect to $\| \cdot \|'_{R,T}$. Thus it remains to show that

$$
\| \sigma \|^2_{R,T} = \| \sigma \|^2_{R,T} \left( 1 + \mathcal{O}(R^{-1/2+\kappa/4}) \right)
$$

for $\sigma$ belonging to $W_1^*$ or $W_2^*$ or $V^*$.

First we consider $\sigma \in V^* = \mathcal{H}^\bullet(Y,F)$. By (3.30), we have

$$
\| \sigma \|^2_{R,T} = \| \sigma \|^2_{Y,R^{1-\kappa/2}} \sqrt{\pi}.
$$

On the other hand, by (5.127), we have

$$
\| \sigma \|^2_{R,T} = \| I_{R,T}(\sigma) \|^2_{Z_R} = \| \sigma \|^2_{Y,R^{1-\kappa/2}} \sqrt{\pi} \left( 1 + \mathcal{O}(e^{-aT}) \right),
$$

where $a > 0$ is a universal constant. From (5.186) and (5.187), we obtain (5.185) for $\sigma \in V^*$.

Now we consider $\sigma \in W_1^*$. For $(\omega, \hat{\omega}) \in K_{1,1} \subset W_1^* = \mathcal{K}_{\text{alg}}(Z_1, F)$, where $K_{1,1}$ was constructed in (3.25), by (5.119), we have

$$
\left\| G_{R,T}^+(\omega, \hat{\omega}, 0, 0) \right\|^2_{Z_R} = \| \omega \|^2_{Z_{1,0}} + \left\| \chi_1 e^{-TfT} \hat{\omega} \right\|^2_{IY_R}
$$

$$
+ \left\| e^{TfT} dZ_{R,\ast} \left( \chi_1 \mathcal{R}_{dF,\ast}(\omega, \hat{\omega}) \right) \right\|^2_{IY_R}
$$

$$
+ 2\Re \left\langle \chi_1 e^{-TfT} \hat{\omega}, e^{TfT} dZ_{R,\ast} \left( \chi_1 \mathcal{R}_{dF,\ast}(\omega, \hat{\omega}) \right) \right\rangle_{IY_R}.
$$

Similarly to (5.175), we have

$$
\left\| \chi_1 e^{-TfT} \hat{\omega} \right\|^2_{IY_R} = \| \hat{\omega} \|^2_{I,Y,R^{1-\kappa/2}} \left( 1 + \mathcal{O}(e^{-aT}) \right),
$$
where \( a > 0 \) is a universal constant. By the Trace theorem for Sobolev spaces and Proposition 5.1, we have

\[
\|\hat{\omega}\|_Y \leq \|\omega\|_{\partial Z_{1,0}} = \mathcal{O}(1) \|\omega\|_{Z_{1,0}}.
\]

By (5.15), (5.31), (5.32) and (5.190), we have

\[
\|e^{Tf_T} dZ_{R,*} \left( \chi_1 \mathcal{R}_{dF,*}(\omega, \hat{\omega}) \right) \|^2_{\mathcal{I}_R} = \mathcal{O}(1) \|\omega\|^2_{Z_{1,0}}.
\]

(5.191)

\[
\langle \chi_1 e^{-Tf_T} \hat{\omega}, e^{Tf_T} dZ_{R,*} \left( \chi_1 \mathcal{R}_{dF,*}(\omega, \hat{\omega}) \right) \rangle_{\mathcal{I}_R} = \mathcal{O}(1) \|\omega\|^2_{Z_{1,0}}.
\]

By (3.25), \( \omega \) is a generalized eigensection of \( D^{Z_{1,\infty}} \). Then, by [50, (2.38)], we have

\[
\|\omega\|^2_{Z_{1,0}} = \mathcal{O}(1) \|\hat{\omega}\|^2_Y.
\]

By (5.188)-(5.192), we have

\[
\|G_{R,T}^+(\omega, \hat{\omega}, 0, 0)\|^2_{\mathcal{I}_R} = \|\hat{\omega}\|^2_Y \left( R^{1-\kappa/2\sqrt{\pi}}/2 + \mathcal{O}(1) \right).
\]

(5.193)

For \((\tau, 0) \in K_1^* \subseteq W_1^* = \mathcal{H}_{\text{ad}}(Z_{1,\infty}, F)\), where \( K_1^* \) was constructed in (3.25), by (5.119), we have

\[
\|G_{R,T}^+(\tau, 0, 0, 0, 0)\|^2_{Z_R} = \|\tau\|^2_{Z_{1,0}} + \|e^{Tf_T} dZ_{R,*} \left( \chi_1 \mathcal{R}_{dF,*}(\tau, 0) \right) \|^2_{\mathcal{I}_R}.
\]

(5.194)

We will use the canonical embedding \( Z_{1,R} \subseteq Z_{1,\infty} \). Since \( e^{Tf_T} dZ_{R,*} \left( \chi_1 \mathcal{R}_{dF,*}(\tau, 0) \right) \) vanishes near \( \partial Z_{1,R} \), it may be extended to a section on \([0, +\infty) \times Y \subseteq Z_{1,\infty}\). We use the identification \( \mathcal{I}_R = [0, 2R] \times Y \subseteq [0, +\infty) \times Y \). By (5.15), (5.17), (5.18), (5.31) and (5.32), we have

\[
\|\tau_{nz}\|_{[0, +\infty) \times Y} = \|\tau_{nz}\|_{\mathcal{I}_R} + \mathcal{O}(e^{-aR}) \|\tau\|_{Z_{1,0}} = \mathcal{O}(1) \|\tau\|_{Z_{1,0}},
\]

(5.195)

\[
\|e^{Tf_T} dZ_{R,*} \left( \chi_1 \mathcal{R}_{dF,*}(\tau, 0) \right) - \tau_{nz}\|_{\mathcal{I}_R} = \mathcal{O}(R^{-2+\kappa}) \|\tau\|_{Z_{1,0}},
\]

where \( a > 0 \) is a universal constant. By (5.195), we have

\[
\|e^{Tf_T} dZ_{R,*} \left( \chi_1 \mathcal{R}_{dF,*}(\tau, 0) \right) \|^2_{\mathcal{I}_R} = \|\tau_{nz}\|^2_{[0, +\infty) \times Y} + \mathcal{O}(R^{-2+\kappa}) \|\tau\|^2_{Z_{1,0}}.
\]

By (3.25), the zero-mode \( \tau_{zm} \) vanishes. As a consequence, we have

\[
\|\tau\|^2_{Z_{1,\infty}} = \|\tau\|^2_{Z_{1,0}} + \|\tau_{nz}\|^2_{[0, +\infty) \times Y}.
\]

(5.197)

By (5.194)-(5.197), we have

\[
\|G_{R,T}^+(\tau, 0, 0, 0)\|^2_{Z_R} = \|\tau\|^2_{Z_{1,\infty}} \left( 1 + \mathcal{O}(R^{-2+\kappa}) \right).
\]

(5.198)
For \((\omega, \hat{\omega}) \in K_1^{\nu} \) and \((\tau, 0) \in K_1^{\nu}\), we have
\[
\left\langle G_{R,T}^{+}(\omega, \hat{\omega}, 0, 0), G_{R,T}^{+}(\tau, 0, 0, 0) \right\rangle_{Z_R} = \left\langle \omega, \tau \right\rangle_{Z_{1,0}}
\]
\[
+ \left\langle e^{TF} dZ_R^{\nu} \left( \chi_1 R_{dF}^{\nu}(\omega, \hat{\omega}) \right), e^{TF} dZ_R^{\nu} \left( \chi_1 R_{dF}^{\nu}(\tau, 0) \right) \right\rangle_{IY_R}
\]
\[
+ \left\langle \chi_1 e^{-TF} \hat{\omega}, e^{TF} dZ_R^{\nu} \left( \chi_1 R_{dF}^{\nu}(\tau, 0) \right) \right\rangle_{IY_R}.
\]  
(5.199)

Similarly to (5.191), by (5.15), (5.31) and (5.32), we have
\[
\left\langle e^{TF} dZ_R^{\nu} \left( \chi_1 R_{dF}^{\nu}(\omega, \hat{\omega}) \right), e^{TF} dZ_R^{\nu} \left( \chi_1 R_{dF}^{\nu}(\tau, 0) \right) \right\rangle_{IY_R}
\]
\[
= \mathcal{O}(1) \left\| \omega \right\|_{Z_{1,0}} \left\| \tau \right\|_{Z_{1,0}}.
\]  
(5.200)

By (5.192), (5.199) and (5.200), we have
\[
\left\langle G_{R,T}^{+}(\omega, \hat{\omega}, 0, 0), G_{R,T}^{+}(\tau, 0, 0, 0) \right\rangle_{Z_R} = \mathcal{O}(1) \left\| \hat{\omega} \right\|_{Y} \left\| \tau \right\|_{Z_{1,0}}.
\]  
(5.201)

From (5.193), (5.198) and (5.201), we obtain (5.185) with \(\sigma \in W^{\nu}_1\). We can prove (5.185) with \(\sigma \in W^{\nu}_2\) in the same way. This completes the proof of Proposition 5.16.

We will use the following identifications,
\[
H^0(C_0^{\nu}, \partial) = \text{Ker} \left( \partial : C_0^{0, \ast} \to C_0^{1, \ast} \right) \subseteq C_0^{0, \ast},
\]
\[
H^1(C_0^{\nu}, \partial) = \left( \text{Im} \left( \partial : C_0^{0, \ast} \to C_0^{1, \ast} \right) \right)^\perp \subseteq C_0^{1, \ast},
\]  
(5.202)

where the orthogonal is taken with respect to the metric \(h_{R,T}^{\nu}\) in (3.29). Since all the \(h_{R,T}^{\nu}\) are mutually proportional, this is independent of \(R, T\).

**Corollary 5.17.** For \(T = R^\nu \gg 1\) and \(\sigma \in H^{\nu}(C_0^{\nu}, \partial)\), we have
\[
\left\| \mathcal{L}_{R,T}^{\nu}(\sigma) \right\|_{Z_R}^2 = \left\| \sigma \right\|_{R,T}^2 \left( 1 + \mathcal{O}(R^{-1/2+\kappa/4}) \right).
\]  
(5.203)

**Proof.** By (5.20) and (5.119), for \(\sigma \in H^0(C_0^{\nu}, \partial) = \mathcal{H}_\text{abs}(Z_{1,\infty}, F) \subseteq \mathcal{H}_\text{abs}(Z_{1,\infty} \cup Z_{2,\infty}, F) = C_0^{0, \ast}\), we have
\[
\left\| F_{R,T}(\sigma) - F_{R,T}^{+}(\sigma) \right\|_{Z_R} = \mathcal{O}(e^{-aT}) \left\| \sigma \right\|_{R,T},
\]  
(5.204)

where \(a > 0\) is a universal constant. By the first identity in (5.123) and (5.204), we have
\[
\left\| F_{R,T}(\sigma) - G_{R,T}^{+}(\sigma) \right\|_{Z_R} = \mathcal{O}(R^{-1+\kappa/2}) \left\| \sigma \right\|_{R,T}.
\]  
(5.205)

By (5.44) and (5.127), for \(\sigma \in H^1(C_0^{\nu}, \partial) = \mathcal{L}_1^{\nu} \cap \mathcal{L}_1^{\nu} \subseteq \mathcal{H}(Y, F) = C_0^{1, \ast}\), we have
\[
I_{R,T}(\sigma) = I_{R,T}^{+}(\sigma).
\]  
(5.206)
By Propositions 5.2, 5.3, 5.9, 5.10, (5.51), (5.132), (5.205) and (5.206), we have
\[
\| J_{R,T}^H(\sigma) - J_{R,T}(\sigma) \|_{Z_R} = o\left( R^{-1/2 + \kappa/4} \right) \| \sigma \|_{R,T}.
\]
From Proposition 5.16 and (5.207), we obtain (5.203). This completes the proof of Corollary 5.17.

Proof of Theorem 3.3

First we consider the case $j = 0$. We will show that the map $J_{R,T}$ constructed in this section satisfies the desired properties. The first property follows from (5.119), (5.127) and (5.132). The second property follows from Theorem 5.12. The third property follows from Proposition 5.16. The fourth property follows from Theorem 5.15 and Proposition 5.16.

For the cases $j = 1, 2, 3$, we will only give the constructions of $J_{j,R,T}^H$ and $J_{j,R,T}$, the proof of these properties is then essentially the same as in the case $j = 0$.

Concerning the cases $j = 1, 2$, we construct $F_{j,R,T} : \mathcal{H}_{\text{als}}(Z_{j,\infty}, F) \to \Omega^*(Z_{j,R}, F)$ as follows: for $(\omega, \hat{\omega}) \in \mathcal{H}_{\text{als}}(Z_{j,\infty}, F),$
\[
F_{j,R,T}(\omega, \hat{\omega})|_{Z_{j,0}} = \omega,
\]
\[
F_{j,R,T}(\omega, \hat{\omega})|_{IY_R} = e^{-T \text{tr} \hat{\omega}} + e^{-T \text{tr} \text{d} \hat{Z}_R} \left( \chi_j \mathcal{R}_d(\omega, \hat{\omega}) \right).
\]
Let $P_{j,R,T} : \Omega^*(Z_{j,R}, F) \to \text{Ker} \left( D_T^{Z_{j,R}} \right)$ be the orthogonal projection with respect to $\| \cdot \|_{Z_{j,R}}$. We identify $H^\bullet(C_j^{\bullet, \bullet}, \partial) = H^0(C_j^{\bullet, \bullet}, \partial)$ with $\mathcal{H}_{\text{als}}(Z_{j,\infty}, F)$.

We construct $G_{j,R,T}^+ : \mathcal{H}_{\text{als}}(Z_{j,\infty}, F) \oplus \mathcal{H}^\bullet(Y, F) \to \Omega^*(Z_{j,R}, F)$ as follows: for $(\omega, \hat{\omega}) \in \mathcal{H}_{\text{als}}(Z_{j,\infty}, F)$ and $\tilde{\mu} \in \mathcal{H}^\bullet(Y, F),$
\[
G_{j,R,T}^+(\omega, \hat{\omega}, \tilde{\mu})|_{Z_{j,0}} = \omega,
\]
\[
G_{j,R,T}^+(\omega, \hat{\omega}, \tilde{\mu})|_{IY_R} = e^{-T \text{tr} \hat{\omega}} \left( \chi_j \hat{\omega} + \chi_{3-j} \tilde{\mu} \right) + e^{T \text{tr} \text{d} \hat{Z}_R^*} \left( \chi_j \mathcal{R}_d(\omega, \hat{\omega}) \right).
\]
We also construct $I_{j,R,T}^+ : \mathcal{H}^\bullet(Y, F) \to \Omega^{\bullet+\bullet}(Z_{j,R}, F)$ as follows: for $\hat{\omega} \in \mathcal{H}^\bullet(Y, F),$
\[
I_{j,R,T}^+(\hat{\omega})|_{Z_{j,0}} = 0,
\]
\[
I_{j,R,T}^+(\hat{\omega})|_{IY_R} = \chi_{3e^{T \text{tr} - T} d u \wedge \hat{\omega}}.
\]
Let $P_{j,R,T}^{[-1,1]} : \Omega^*(Z_{j,R}, F) \to \mathcal{H}_{\text{als}}^{[-1,1]}$ be the orthogonal projection with respect to $\| \cdot \|_{Z_{j,R}}$. We identify $C_j^{\bullet, \bullet}$ with $\mathcal{H}_{\text{als}}(Z_{j,\infty}, F) \oplus \mathcal{H}^\bullet(Y, F)$. We identify $C_j^{1, \bullet}$ with $\mathcal{H}^\bullet(Y, F)$.

We define
\[
J_{j,R,T} \big|_{C_j^{\bullet, \bullet}} = P_{j,R,T}^{[-1,1]} G_{j,R,T}^+,
\]
\[
J_{j,R,T} \big|_{C_j^{1, \bullet}} = P_{j,R,T}^{[-1,1]} I_{j,R,T}^+.
\]
Now, concerning the case $j = 3$, we construct $F_{3,R,T} : \mathcal{H}^\bullet(Y, F) \to \Omega^*(I_{Y,R}, F)$ as follows: for $\hat{\omega} \in \mathcal{H}^\bullet(Y, F),$
\[
F_{3,R,T}(\hat{\omega})|_{IY_R} = e^{-T \text{tr} \hat{\omega}}.
\]
Let $P_{3,R,T} : \Omega^*(I_{Y,R}, F) \to \text{Ker} \left( D_T^{I_{Y,R}} \right)$ be the orthogonal projection with respect to $\| \cdot \|_{IY_R}$. We identify $H^\bullet(C_3^{\bullet, \bullet}, \partial) = H^0(C_3^{\bullet, \bullet}, \partial)$ with $\mathcal{H}^\bullet(Y, F)$.

We define
\[
J_{3,R,T}^H = P_{3,R,T} F_{3,R,T} : H^\bullet(C_3^{\bullet, \bullet}, \partial) \to \text{Ker} \left( D_T^{I_{Y,R}} \right).
\]
We construct \( G^+_{3,R,T} : \mathcal{H}^*(Y,F) \oplus \mathcal{H}^*(Y,F) \to \Omega^*(IY_R,F) \) as follows: for \((\hat{\mu}_1, \hat{\mu}_2) \in \mathcal{H}^*(Y,F) \oplus \mathcal{H}^*(Y,F)\),

\[
G^+_{3,R,T}(\hat{\mu}_1, \hat{\mu}_2) = e^{-Tf^+T} \left( \chi_1 \hat{\mu}_1 + \chi_2 \hat{\mu}_2 \right).
\]

(5.215)

We also construct \( I^+_{3,R,T} : \mathcal{H}^*(Y,F) \to \Omega^{*-1}(IY_R,F) \) as follows: for \( \hat{\omega} \in \mathcal{H}^*(Y,F)\),

\[
I^+_{3,R,T}(\hat{\omega}) = \chi_3 e^{Tf^+T} du \wedge \hat{\omega}.
\]

(5.216)

Let \( P_{3,R,T}^{[-1,1]} : \Omega^*(IY_R,F) \to \mathcal{E}_{3,R,T}^{[-1,1]} \) be the orthogonal projection with respect to \( \| \cdot \|_{IY_R} \).

We identify \( C_3^{0,*} \) with \( \mathcal{H}^*(Y,F) \oplus \mathcal{H}^*(Y,F) \), and identify \( C_3^{1,*} \) with \( \mathcal{H}^*(Y,F) \). We define

\[
\mathcal{I}_{3,R,T}\big|_{C_3^{1,*}} = P_{3,R,T}^{[-1,1]} F^+_{3,R,T}, \quad \mathcal{I}_{3,R,T}\big|_{C_3^{0,*}} = I_{3,R,T}^{[-1,1]},
\]

(5.217)

This completes the proof of Theorem 3.3.

\[\square\]

Remark 5.18. The proof of Theorem 3.3 may be summarized as follows: all the results hold with \( \mathcal{I}_{R,T}^H \) replaced by \( \mathcal{I}_{3,R,T}^H \) and \( \mathcal{I}_{R,T} \) replaced by \( \mathcal{I}_{3,R,T} \).

6. Analytic torsion forms associated with a fibration

The purpose of this section is to prove Theorem 3.4. Many arguments in this section follow [10]. This section is organized as follows. In §6.1 we decompose the analytic torsion form in question into two terms: small time contribution and large time contribution. In §6.2, we estimate the small time contribution. In §6.3, we estimate the large time contribution. Theorem 3.4 will be proved in this subsection.

In the whole section, we take \( T = R^\kappa \), where \( \kappa \in [0, 1/3] \) is a fixed constant. For ease of notations, we will systematically omit a parameter \((R \ or \ T) \ as \ long \ as \ there \ is \ no \ confusion).\n
6.1. Decomposition of analytic torsion forms. For \( j = 0, 1, 2, 3 \) and \( R > 0 \), we denote \( \mathcal{I}_{j,R} = \Omega^*(\mathcal{J}_{j,R}, F) \), which is a complex vector bundle of infinite dimension over \( S \). Let \( \nabla \mathcal{F}_{j,R} \) be the connection on \( \mathcal{F}_{j,R} \) defined in (1.73). Let \( h^{\mathcal{F}_{j,R}} \) be the \( L^2 \)-metric on \( \mathcal{F}_{j,R} \) with respect to \( g^{\mathcal{T}Z_{j,R}} \) and \( h^F \). Let \( \omega^{\mathcal{F}_{j,R}} \in \Omega^1(S, \text{End}(\mathcal{F}_{j,R})) \) be as in (1.79) with \( (\nabla \mathcal{F}_{j,R}, h^{\mathcal{F}_{j,R}}) \) replaced by \( (\nabla \mathcal{F}_{j,R}, h^{\mathcal{F}_{j,R}}) \). We may extend the construction above to \( R = +\infty \) as follows,

\[
\mathcal{F}_{j,\infty} = \{ \omega \in \Omega^*(\mathcal{J}_{j,\infty}, F) : \omega \text{ is } L^2\text{-integrable} \}.
\]

By [12, Prop. 4.15], \( \omega^{\mathcal{F}_{j,\infty}} \in \Omega^1(S, \text{End}(\mathcal{F}_{j,\infty})) \) is well-defined. Let \( \mathcal{D}_{j,R,t} \) be the operator in (1.81) with \( (\nabla \mathcal{F}_{j,R}, h^{\mathcal{F}_{j,R}}) \) replaced by \( (\nabla \mathcal{F}_{j,R}, h^{\mathcal{F}_{j,R}}) \). We have

\[
\mathcal{D}_{j,R,t} = \sqrt{1} \left( d^{Z_{j,R},*}_T - d^{Z_{j,R}}_T \right) + \omega^{\mathcal{F}_{j,R}} - \frac{1}{\sqrt{t}} \mathcal{E}(T) \in \Omega^*(S, \text{End}(\mathcal{F}_{j,R})).
\]

(6.2)

We remark that

\[
\left( d^{Z_{j,R},*}_T - d^{Z_{j,R}}_T \right)^2 = - \left( d^{Z_{j,R},*}_T + d^{Z_{j,R}}_T \right)^2 = - D^{Z_{j,R},2}_T.
\]

(6.3)
We denote \( \mathcal{J}_{j,R} = \mathcal{J}_{j,R,T}|_{T=R^*} \), where \( \mathcal{J}_{j,R,T} \) was defined in (3.13). By (1.84), we have
\[
\mathcal{J}_{j,R} = - \int_0^{+\infty} \left\{ \varphi \text{Tr} \left[ (-1)^{NTZ} N^{TZ} \frac{\chi'(Z_j,F)}{\sqrt{2}} \right] - \frac{\chi'(Z_j,F)}{\sqrt{2}} \right\} \frac{dt}{t} \tag{6.4}
\]

Let \( \mathcal{J}_{j,R}^S \) (resp. \( \mathcal{J}_{j,R}^L \)) be as in (6.4) with \( \int_0^{+\infty} \) replaced by \( \int_0^{R^2-\kappa/2} \) (resp. \( \int_0^{+\infty} \)). The following identity is obvious,
\[
\mathcal{J}_{j,R} = \mathcal{J}_{j,R}^S + \mathcal{J}_{j,R}^L \tag{6.5}
\]

6.2. Small time contributions. For \( r \geq 1 \) and an operator \( A \) on a Hilbert space, the Schauder \( r \)-norm of \( A \) is defined as follows,
\[
\| A \|_r = \left( \text{Tr} \left[ (A^* A)^{r/2} \right] \right)^{1/r} \tag{6.6}
\]
If \( A \) is orthogonally diagonalizable, we have
\[
\| A \|_r = \left( \sum_{\lambda \in \text{Sp}(A)} |\lambda|^r \right)^{1/r} \tag{6.7}
\]

Let \( \| A \|_\infty \) be the operator norm of \( A \). These norms satisfy the Hölder’s inequality: for \( r_1, r_2, r_3 \in [1, +\infty] \) with \( 1/r_1 + 1/r_2 = 1/r_3 \), we have
\[
\| AB \|_{r_3} \leq \| A \|_{r_1} \| B \|_{r_2} \tag{6.8}
\]
Moreover, if \( A \) is of finite rank, we have
\[
\| A \|_r \leq (\text{rk}(A))^{1/r} \| A \|_\infty \tag{6.9}
\]
The proofs in this subsection involve sophisticated estimate of Schauder norms, which follows \([10, \S 9]\).

We remark that \( \text{Sp}(\mathcal{D}_{j,R,t}) \subseteq i\mathbb{R} \).

**Lemma 6.1.** There exist \( \alpha, \beta > 0 \) such that for \( r \geq \dim Z + 1 \), \( R \gg 1 \), \( 0 < t \leq R^2 - \kappa/2 \) and \( \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) = \pm 1 \), we have
\[
\left\| (\lambda - \mathcal{D}_{j,R,t})^{-1} \right\|_r = O(R^\beta) |\lambda| t^{-\alpha}, \quad \text{for } j = 0, 1, 2, 3. \tag{6.10}
\]

**Proof.** We only consider the case \( j = 0 \).

We denote
\[
a = \left( \lambda - \sqrt{t}(d_{j,R,t}^Z - d_{T,R}^Z) \right)^{-1}, \quad b = \omega_{T,R}^Z - \frac{1}{\sqrt{t}} \hat{\epsilon}(T). \tag{6.11}
\]
Since \( b \in \Omega^{>0}(S, \text{End}(\mathcal{F}_R)) \), we have
\[
(\lambda - \mathcal{D}_{R,t})^{-1} = a + aba + \cdots + a(ba)^{\dim S}. \tag{6.12}
\]
The same technique as above was used in \([11, (2.45)]\). Note that \( \text{Sp}(d_{T,R}^Z - d_{T,R}^Z) \subseteq i\mathbb{R} \) and \( \text{Re}(\lambda) = \pm 1 \), by (6.11), we have
\[
\| a \|_\infty \leq 1, \quad \| b \|_\infty = O(1 + t^{-1/2}). \tag{6.13}
\]
We will temporarily treat \( R \) and \( T \) as independent parameters. Note that \( \text{Sp}(d_T^{Z_{R*,*}} - d_T^{Z_{R,*}}) \) depends continuously on \( R, T \), there exist \( \{ \mu_{k,R,T} \in i\mathbb{R} \}_{k \in \mathbb{N}} \) such that

- for \( R \geq 1 \) and \( T \geq 0 \), we have \( \{ \mu_{k,R,T} : k \in \mathbb{N} \} = \text{Sp}(d_T^{Z_{R*,*}} - d_T^{Z_{R,*}}) \);
- the function \( (R, T) \mapsto \mu_{R,T} \) is continuous.

We will estimate \( \mu_{k,R,T} \) with \( k \in \mathbb{N} \) fixed. For ease of notations, we will omit the index \( k \). By \([50] \ (3.95)\) and \((6.3)\), we have

\[
\begin{align*}
(6.14) \quad & |\mu_{R,0}| \geq R^{-1}|\mu_{1,0}|. \\
Similarly to the first identity in \((4.37)\), we have \\
(6.15) \quad & (d_T^{Z_{R*,*}} - d_T^{Z_{R,*}})|_{IY_R} = \hat{c}c(d_Y^{*} - d_Y) - \hat{c} \frac{\partial}{\partial u} - R^{-1}T f_{T,c}. \\
\end{align*}
\]

By \((3.11), (6.3), (6.15)\) and the identity \( f_T|_{Z_{1,0} \cup Z_{2,0}} = 0 \), there exists \( \delta > 0 \) independent of \( R, T, \mu_{R,T} \) such that

\[
(6.16) \quad |\mu_{R,T} - \mu_{R,0}| \leq \delta R^{-1}T.
\]

We consider the triangle spanned by \( \lambda, \sqrt{t}\mu_{R,T} \in \mathbb{C} \). Let \( A \) be its area. As \( \text{Re}(\lambda) = \pm 1 \), we have

\[
(6.17) \quad |\lambda| - \sqrt{t}\mu_{R,T}| \geq 2A = |\sqrt{t}\mu_{R,T}|.
\]

Equivalently, we have

\[
(6.18) \quad |\lambda - \sqrt{t}\mu_{R,T}|^{-1} \leq |\lambda| |\sqrt{t}\mu_{R,T}|^{-1}.
\]

If \( |\mu_{R,T}| \geq 1/R \), by \((6.14)-(6.18)\), we have

\[
(6.19) \quad |\lambda - \sqrt{t}\mu_{R,T}|^{-1} \leq |\lambda| t^{-1/2}|\mu_{R,0}|^{-1} \left( 1 + \frac{|\mu_{R,0} - \mu_{R,T}|}{|\mu_{R,T}|} \right) \\
\quad \leq |\lambda| t^{-1/2} |\mu_{1,0}|^{-1} R(1 + \delta T) = O(R^{1+\epsilon}) |\lambda| t^{-1/2} |\mu_{1,0}|^{-1},
\]

where \( O(R^{1+\epsilon}) \) is uniform, i.e., it is bounded by \( CR^{1+\epsilon} \) with \( C > 0 \) independent of \( R, T, \mu_{R,T} \). On the other hand, as \( \text{Re}(\lambda) = \pm 1 \) and \( \mu_{R,T} \in i\mathbb{R} \), we have the obvious estimate

\[
(6.20) \quad |\lambda - \sqrt{t}\mu_{R,T}|^{-1} \leq 1.
\]

By Theorem \([3.1]\) and \((6.3)\), there exists \( \alpha > 0 \) such that

\[
(6.21) \quad \text{Sp} \left( i(d_T^{Z_{R*,*}} - d_T^{Z_{R,*}}) \right) \subseteq \left( -\infty, -\alpha \sqrt{T/R} \right] \cup \left[ -1/R, 1/R \right] \cup \left[ \alpha \sqrt{T/R}, +\infty \right). \\
\]

Moreover, by Theorem \([3.3]\) the number of eigenvalues lying in \([-1/R, 1/R]\) is constant for \( R^\epsilon = T \gg 1 \). Let \( P_{R,T}^{[1]} : \mathcal{F}_R \rightarrow \left( \ker(D_T^{Z_{R,*}})^* \right)^\perp \) be the orthogonal projection. By \((6.7)\), the first identity in \((6.11)\) and \((6.19)-(6.21)\), we have

\[
(6.22) \quad \|a\|_r = \mathcal{O}(1) + \mathcal{O}(R^{1+\epsilon}) |\lambda| t^{-1/2} \left\| (D_T^{Z_{R,*}})^{-1} P_{R,T}^{[1]} \right\|_{R=1,T=0}.
\]

Since \( r \geq \dim Z + 1 \), by Weyl’s law, we have \( \left\| (D_T^{Z_{R,*}})^{-1} P_{R,T}^{[1]} \right\|_{R=1,T=0} < +\infty \). Then \((6.22)\) becomes

\[
(6.23) \quad \|a\|_r = \mathcal{O}(1) + \mathcal{O}(R^{1+\epsilon}) |\lambda| t^{-1/2}.
\]
By (6.8) and (6.12), we have
\begin{equation}
\left\| (\lambda - \mathcal{D}_{R,t})^{-1} \right\|_r \leq \left\| a \right\|_r \sum_{k=0}^{\dim S} \left\| b \right\|_\infty^k \left\| d \right\|_\infty^k.
\end{equation}

From (6.13), (6.23), (6.24) and the assumption (6.27), we obtain (6.10). This completes the proof of Lemma 6.1.

Let \( \rho : \mathbb{R} \to [0, 1] \) be a smooth even function such that
\begin{equation}
\rho(x) = 1 \quad \text{for} \quad |x| \leq 1/2, \quad \rho(x) = 0 \quad \text{for} \quad |x| \geq 1.
\end{equation}
For \( \varsigma > 0 \) and \( z \in \mathbb{C} \), set
\begin{align}
F_\varsigma(z) &= (1 + 2z^2) \int_{-\infty}^{+\infty} \exp \left( \sqrt{2}xz \right) \exp \left( -\frac{x^2}{2} \right) \rho(\sqrt{2}\varsigma x) \frac{dx}{\sqrt{2\pi}}, \\
G_\varsigma(z) &= (1 + 2z^2) \int_{-\infty}^{+\infty} \exp \left( \sqrt{2}xz \right) \exp \left( -\frac{x^2}{2} \right) (1 - \rho(\sqrt{2}\varsigma x)) \frac{dx}{\sqrt{2\pi}}.
\end{align}
The construction above follows [10, Def. 13.3]. We have
\begin{equation}
F_\varsigma(z) + G_\varsigma(z) = f'(z).
\end{equation}
Moreover, \( F_\varsigma|_{i\mathbb{R}} \) and \( G_\varsigma|_{i\mathbb{R}} \) take real values, and lie in the Schwartz space \( S(i\mathbb{R}) \).

**Proposition 6.2.** There exists \( \alpha > 0 \) such that for \( R \gg 1 \) and \( 0 < t \leq R^{2-\kappa/2} \), we have
\begin{equation}
\left\| G_{tR^{-2+\kappa/4}}(\mathcal{D}_{j,R,t}) \right\|_1 \leq \exp \left( -\alpha R^{2-\kappa/4}/t \right), \quad j = 0, 1, 2, 3.
\end{equation}

**Proof:** We only consider the case \( j = 0 \).

Due to the relation \( \frac{d}{dt} \int_{\mathbb{R}}^m \exp \left( \sqrt{2}xz \right) = 2^{m/2} \int_{\mathbb{R}}^m \exp \left( \sqrt{2}xz \right) \), we can integrate by parts in the expression of \( z^{m} G_\varsigma(z) \) and obtain that for \( m \in \mathbb{N} \), there exists \( C_m > 0 \) such that for \( z \in \mathbb{C} \) with \( |\text{Re}(z)| \leq 1 \), we have
\begin{equation}
|z|^m |G_\varsigma(z)| \leq C_m \exp \left( -\frac{1}{32\varsigma} \right).
\end{equation}

The function \( G_\varsigma(z) \) is an even holomorphic function. Therefore there exists a holomorphic function \( \tilde{G}_\varsigma(z) \) such that
\begin{equation}
G_\varsigma(z) = \tilde{G}_\varsigma(z^2).
\end{equation}
Set
\begin{equation}
U = \left\{ z \in \mathbb{C} : 4\text{Re}(z) + |\text{Im}(z)|^2 < 4 \right\}.
\end{equation}
We have
\begin{equation}
\sqrt{U} = \left\{ z \in \mathbb{C} : z^2 \in U \right\} = \left\{ z \in \mathbb{C} : |\text{Re}(z)| < 1 \right\}.
\end{equation}
By (6.29), (6.30) and (6.32), for \( z \in U \), we have
\begin{equation}
|z|^{m/2} |\tilde{G}_\varsigma(z)| \leq C_m \exp \left( -\frac{1}{32\varsigma} \right).
\end{equation}
The technique above follows [10, §13 c)].
For \( r \in \mathbb{N} \), let \( \tilde{G}_{r, \varsigma}(z) \) be the unique holomorphic function satisfying
\[
\frac{1}{r!} \frac{d^r}{dz^r} \tilde{G}_{r, \varsigma}(z) = \tilde{G}_{\varsigma}(z), \quad \lim_{z \to -\infty} \tilde{G}_{r, \varsigma}(z) = 0.
\]

By (6.33) and (6.34), for \( m > 2r \), there exists \( C_{m, \varsigma} > 0 \) such that for \( z \in U \),
\[
|\tilde{G}_{r, \varsigma}(z)| \leq C_{m, \varsigma} |z|^\nu \exp \left( -\frac{1}{32\varsigma} \right).
\]

In the rest of the proof, we fix \( \varsigma = tR^{-2+\kappa/4} \) and \( N \geq r \geq 1 + \dim \mathbb{Z}/2 \). We have
\[
G_{\varsigma}(\mathcal{D}_{R,t}) = \tilde{G}_{\varsigma}(\mathcal{D}_{R,t}^2) = \frac{1}{2\pi i} \int_{\partial U} \tilde{G}_{r, \varsigma}(\lambda \mathcal{D}_{R,t}^2)^{-\nu-1} d\lambda.
\]

By (6.8), we have
\[
\left\| (\lambda \mathcal{D}_{R,t}^2)^{-\nu-1} \right\|_1 \leq \frac{1}{2r+2} \left\| (\lambda \mathcal{D}_{R,t}^2)^{-\nu-1} \right\|_{2r+2}.
\]

From Lemma 6.1 and (6.35)-(6.37), we obtain (6.28). This completes the proof of Proposition 6.2.

Let \( \chi'(Z_j, F) \) be as in (1.83) with \( Z \) replaced by \( Z_j \). Set
\[
\chi' = \sum_{j=0}^3 (-1)^{j(j-3)/2} \chi'(Z_j, F).
\]

**Proposition 6.3.** There exists \( \alpha > 0 \) such that for \( R \gg 1 \), we have
\[
\sum_{j=0}^3 (-1)^{j(j-3)/2} \mathcal{F}_{j,R}^S = -\frac{\chi'}{2} \int_0^{R^{2-\kappa/2}} \left\{ f'(\frac{i\sqrt{t}}{2}) - 1 \right\} \frac{dt}{t} + \mathcal{O}\left( \exp(-\alpha R^{\kappa/4}) \right).
\]

**Proof.** Let
\[
F_{iR^{-2+\kappa/4}}(\mathcal{D}_{j,R,t})(x, y)
\]
be the integration kernel of the operator \( F_{iR^{-2+\kappa/4}}(\mathcal{D}_{j,R,t}) \) with respect to the Riemannian volume form associated with \( g^{TZ_{j,R}} \). Let \( d(\cdot, \cdot) \) be the distance function on \( Z_{j,R} \). By the finite propagation speed of the wave equation for \( \mathcal{D}_{j,R,t} \) (cf. [55] §2.6, Thm. 6.1, [40] Appendix D.2), we have
\[
F_i(\mathcal{D}_{j,R,t})(x, y) = 0 \quad \text{for} \quad d(x, y) \geq \sqrt{t/\varsigma}.
\]

In particular, we have
\[
F_{iR^{-2+\kappa/4}}(\mathcal{D}_{j,R,t})(x, y) = 0 \quad \text{for} \quad d(x, y) \geq R^{1-\kappa/8}, \quad t \leq R^{2-\kappa/2}.
\]

Using (6.42) in the same way as in [50] Thm. 4.5, we get
\[
\sum_{j=0}^3 (-1)^{j(j-3)/2} \text{Tr}\left[ (-1)^{NTZ} \frac{NTZ}{2} F_{iR^{-2+\kappa/4}}(\mathcal{D}_{j,R,t}) \right] = 0, \quad \text{for} \quad t \leq R^{2-\kappa/2}.
\]
By Proposition 6.2, we have

\[ \int_0^{R^{2-\kappa/2}} \text{Tr} \left[ (-1)^{NT^Z} \frac{N^{T^Z}}{2} G_{tR-2+\kappa/4} \left( \mathcal{O}_{j,R,t} \right) \right] \frac{dt}{t} = O \left( \exp \left( -\alpha R^{\kappa/4} \right) \right). \]

From (6.5), (6.27), (6.43), (6.44) and the identity

\[ \chi(Z) - \chi(Z_1) - \chi(Z_2) + \chi(\text{IY}) = 0, \]

we get (6.39). This completes the proof of Proposition 6.3.

\[ \square \]

6.3. Large time contributions. Set

\[ U_{j,R,t} = \left\{ \lambda \in \mathbb{C} : \mid \text{Re}(\lambda) \mid < 1, \mid \text{Im}(\lambda) \mid > t^{1/2} R^{-1+\kappa/4} \right\} \]

\[ \cup \bigcup_{\mu \in \{ R^{-1} - R^{-2} \}} \text{Sp} \left( d_T^{Z_{j,R,t}} - d_T^{Z_{j,R,t}} \right) \]

By Theorem 3.1 and (6.3), for \( R \gg 1 \), we have

\[ \text{Sp} \left( \sqrt{t} \left( d_T^{Z_{j,R,t}} - d_T^{Z_{j,R,t}} \right) \right) \subseteq U_{j,R,t}. \]

Set

\[ \widetilde{\mathcal{O}}_{j,R,t} = \sqrt{t} P_{j,R,T}^{[-1,1]} \left( d_T^{Z_{j,R,t}} - d_T^{Z_{j,R,t}} \right) P_{j,R,T}^{[-1,1]} + P_{j,R,T}^{[-1,1]} \omega_{j,R} P_{j,R,T}^{[-1,1]} \]

We fix \( p, q \in \mathbb{N} \) such that

\[ q > \dim Z, \quad 1 - \frac{\kappa}{4} + \frac{3kq}{2} - \frac{\kappa p}{4} \leq 0. \]

Lemma 6.4. There exists \( \alpha > 0 \) such that for \( R \gg 1, t \geq R^{2-\kappa/2} \) and \( \lambda \in \partial U_{j,R,t} \), we have

\[ \left\| (\lambda - \mathcal{O}_{j,R,t})^{-p} - P_{j,R,T}^{[-1,1]} (\lambda - \widetilde{\mathcal{O}}_{j,R,t})^{-p} P_{j,R,T}^{[-1,1]} \right\|_1 = O \left( R^{-1/2} \right) \mid \lambda \mid^\alpha t^{-1/2}. \]

Proof. The technique that we will apply is similar to [4, Theorems 9.30]. We only consider the case \( j = 0 \).

Set

\[ \mathcal{O}_{R,t}^{\oplus} = P_{R,T}^{[-1,1]} \mathcal{O}_{R,t} P_{R,T}^{[-1,1]} + P_{R,T}^{[-1,1]} \mathcal{O}_{R,t} P_{R,T}^{[-1,1]}. \]

Step 1. We show that

\[ \left\| P_{R,T}^{[-1,1]} (\lambda - \mathcal{O}_{R,t}^{\oplus})^{-p} P_{R,T}^{[-1,1]} \right\|_1 = O \left( 1 \right) \mid \lambda \mid^\alpha t^{-1/2}. \]

We denote

\[ a = P_{R,T}^{[-1,1]} \left( \lambda - \sqrt{t} \left( d_T^{Z_{R,t}} - d_T^{Z_{R,t}} \right) \right)^{-1} P_{R,T}^{[-1,1]}, \]

\[ b = P_{R,T}^{[-1,1]} \left( \omega_{j,R} - \frac{1}{\sqrt{t}} \mathcal{O} (T) \right) P_{R,T}^{[-1,1]} \]

By (6.2) and (6.53), we have

\[ P_{R,T}^{[-1,1]} (\lambda - \mathcal{O}_{R,t}^{\oplus})^{-1} P_{R,T}^{[-1,1]} = a + aba + \cdots + a(ba)^{\dim S}. \]
By Theorem 3.1 (6.18) and the first identity in (6.53), we have
\[ \|a\|_\infty = \mathcal{O}(R^{1-\kappa/2}) |\lambda| t^{-1/2} . \]
By the second identity in (6.53), we have
\[ \|b\|_\infty = \mathcal{O}(1) . \]
By (6.54)-(6.56) and the assumption \( t \geq R^{2-\kappa/2} \), we have
\[ \left\| P_{R,T} R^{[-1,1]} (\lambda - \mathcal{D}_{R,t})^{-1} P_{R,T} R^{[-1,1]} \right\|_\infty \]
\[ = \mathcal{O}(R^{1-\kappa/2}) |\lambda|^\dim S + 1 t^{-1/2} = \mathcal{O}(R^{-\kappa/4}) |\lambda|^\dim S + 1 . \]
Similarly to (6.22), by (6.7), (6.19) and the first identity in (6.53), we have
\[ \|a\|_q = \mathcal{O}(R^{1+\kappa}) |\lambda| t^{-1/2} . \]
By (6.8), (6.54)-(6.56) and (6.58), we have
\[ \left\| P_{R,T} R^{[-1,1]} (\lambda - \mathcal{D}_{R,t})^{-1} P_{R,T} R^{[-1,1]} \right\|_q = \mathcal{O}(R^{1+\kappa}) |\lambda|^\dim S + 1 t^{-1/2} . \]
By (6.8), we have
\[ \left\| P_{R,T} R^{[-1,1]} (\lambda - \mathcal{D}_{R,t})^{-1} P_{R,T} R^{[-1,1]} \right\|_1 \]
\[ \leq \left\| \left( \lambda - \mathcal{D}_{R,t} \right)^{-1} P_{R,T} R^{[-1,1]} \right\|_q \left\| P_{R,T} R^{[-1,1]} (\lambda - \mathcal{D}_{R,t})^{-1} P_{R,T} R^{[-1,1]} \right\|_\infty^{P-q} . \]
From (6.49), (6.57), (6.59) and (6.60), we obtain (6.52).
Step 2. We show that
\[ (\lambda - \mathcal{D}_{R,t})^{-p} - (\lambda - \mathcal{D}_{R,t})^{-p} \]
\[ = \mathcal{O}(R^{1-\kappa/2}) |\lambda|^\alpha t^{-1/2} . \]
Since \( d^Z R - d^Z R \) commutes with \( P_{R,T} R^{[-1,1]} \), we have
\[ P_{R,T} R^{[-1,1]} \mathcal{D}_{R,t} P_{R,T} R^{[-1,1]} = P_{R,T} R^{[-1,1]} \omega^R P_{R,T} R^{[-1,1]} - \frac{1}{\sqrt{t}} P_{R,T} R^{[-1,1]} \hat{c}(T) P_{R,T} R^{[-1,1]} , \]
\[ P_{R,T} R^{[-1,1]} \mathcal{D}_{R,t} P_{R,T} R^{[-1,1]} = P_{R,T} R^{[-1,1]} \omega^R P_{R,T} R^{[-1,1]} - \frac{1}{\sqrt{t}} P_{R,T} R^{[-1,1]} \hat{c}(T) P_{R,T} R^{[-1,1]} . \]
As a consequence, we have
\[ \left\| P_{R,T} R^{[-1,1]} \mathcal{D}_{R,t} P_{R,T} R^{[-1,1]} \right\| = \mathcal{O}(1) , \quad \left\| P_{R,T} R^{[-1,1]} \mathcal{D}_{R,t} P_{R,T} R^{[-1,1]} \right\| = \mathcal{O}(1) . \]
The same argument as in (6.54)-(6.57) yields
\[ \left\| P_{R,T} R^{[-1,1]} (\lambda - \mathcal{D}_{R,t})^{-1} P_{R,T} R^{[-1,1]} \right\|_\infty = \mathcal{O}(1) . \]
We denote
\[ A = \left\{ P_{R,T} R^{[-1,1]} (\lambda - \mathcal{D}_{R,t})^{-1} P_{R,T} R^{[-1,1]} , P_{R,T} R^{[-1,1]} (\lambda - \mathcal{D}_{R,t})^{-1} P_{R,T} R^{[-1,1]} \right\} , \]
\[ B = \left\{ P_{R,T} R^{[-1,1]} \mathcal{D}_{R,t} P_{R,T} R^{[-1,1]} , P_{R,T} R^{[-1,1]} \mathcal{D}_{R,t} P_{R,T} R^{[-1,1]} \right\} . \]
We have
\[ (\lambda - \mathcal{R}_{R,t})^{-1} - (\lambda - \mathcal{P}_{R,t})^{-1} = \sum_{k=1}^{\dim S} \sum_{a_i, b_i \in B} a_0 b_1 a_1 b_2 a_2 \cdots b_k a_k \]
(6.66)
\[ = \sum_{k=1}^{\dim S} \sum_{a_i, b_i \in B, a_0 \neq a_1} a_0 b_1 a_1 b_2 a_2 \cdots b_k a_k . \]

By (6.57) and (6.63)-(6.66), we have
\[ \left\| (\lambda - \mathcal{R}_{R,t})^{-1} - (\lambda - \mathcal{P}_{R,t})^{-1} \right\|_{\infty} = O(R^{1-\kappa/2})|\lambda|^{(\dim S+1)^2} t^{-1/2} . \]
By (6.67), (6.64) and (6.67), we have
\[ \left\| (\lambda - \mathcal{R}_{R,t})^{-p} - (\lambda - \mathcal{P}_{R,t})^{-p} \right\|_{\infty} = O(R^{1-\kappa/2})|\lambda|^p(\dim S+1)^2 t^{-1/2} . \]
By (6.66), we have
\[ \text{Im} \left( (\lambda - \mathcal{R}_{R,t})^{-p} - (\lambda - \mathcal{P}_{R,t})^{-p} \right) \leq \sum_{k=1}^{p} \sum_{a \in A, b \in B} \text{Im}(a^k b) , \]
whose dimension is bounded by \( 4p \dim \left( E_{0,R,T}^{[-1,1]} \right) \dim \left( \Lambda^*(T^*S) \right) \). Hence
\[ \text{rk} \left( (\lambda - \mathcal{R}_{R,t})^{-p} - (\lambda - \mathcal{P}_{R,t})^{-p} \right) \leq 4p \dim \left( E_{0,R,T}^{[-1,1]} \right) \dim \left( \Lambda^*(T^*S) \right) . \]

From (6.9), (6.68) and (6.70), we obtain (6.61).

Step 3. We show that
\[ \left\| P_{R,T}^{-1,1} (\lambda - \mathcal{R}_{R,t})^{-p} P_{R,T}^{-1,1} - P_{R,T}^{-1,1} (\lambda - \mathcal{P}_{R,t})^{-p} P_{R,T}^{-1,1} \right\|_1 = O(1) t^{-1/2} . \]

Using the identity
\[ P_{R,T}^{-1,1} (\mathcal{R}_{R,t} - \mathcal{P}_{R,t}) P_{R,T}^{-1,1} = -\frac{1}{t} P_{R,T}^{-1,1} \mathcal{c}(T) P_{R,T}^{-1,1} \]
and proceeding in the same way as (6.54)-(6.57), we can show that
\[ \left\| P_{R,T}^{-1,1} (\lambda - \mathcal{R}_{R,t})^{-p} P_{R,T}^{-1,1} - P_{R,T}^{-1,1} (\lambda - \mathcal{P}_{R,t})^{-p} P_{R,T}^{-1,1} \right\|_{\infty} = O(1) t^{-1/2} . \]
Since the rank of the operator in (6.73) is bounded by \( \dim \left( E_{0,R,T}^{[-1,1]} \right) \dim \left( \Lambda^*(T^*S) \right) \), from (6.9) and (6.73), we obtain (6.71).

By Steps 1-3, we have (6.50). This completes the proof of Lemma 6.4. \( \square \)

Recall that the complexes \( (C_j^{**}, \partial) \) with \( j = 0, 1, 2, 3 \) were defined by (2.1), (2.6) and (3.20). We denote
\[ \chi'\left( C_j^{**} \right) = \sum_{p=0}^{1} \sum_{q=0}^{\dim Z} (-1)^{p+q} (p + q) \dim C_j^{p,q} . \]
Set
\[ \mathcal{T}_{j,R} = -\int_{0}^{+\infty} \varphi \text{Tr} \left[ (-1)^{N_{T^{2}}} \frac{N_{T^{2}}}{2} f'(\tilde{\mathcal{D}}_{j,R,t}) \right] - \frac{1}{2} \chi'(Z_{j}, F) \]
(6.75)
\[- \frac{1}{2} \left( \chi'(C_{j}^{\bullet \bullet}) - \chi'(Z_{j}, F) \right) f'(\sqrt{t}) \right\} \frac{dt}{t}. \]

By [11, Remark 2.21], Theorem 3.3 and (6.47), \( \mathcal{T}_{j,R} \) is well-defined.

**Proposition 6.5.** For \( R \gg 1 \), we have
\[ \sum_{j=0}^{3} (-1)^{j(\gamma-3)/2} \mathcal{T}_{j,R} = \sum_{j=0}^{3} (-1)^{j(\gamma-3)/2} \mathcal{T}_{j,R} + O(R^{-\kappa/4}). \]
(6.76)

**Proof.** Let \( \mathcal{T}_{j,R}^{S} \) (resp. \( \mathcal{T}_{j,R}^{L} \)) be as in (6.75) with \( \int_{0}^{+\infty} \) replaced by \( \int_{0}^{R^{2-\kappa/2}} \) (resp. \( \int_{R^{2-\kappa/2}}^{+\infty} \)). The following identity is obvious,
\[ \mathcal{T}_{j,R} = \mathcal{T}_{j,R}^{S} + \mathcal{T}_{j,R}^{L}. \]
(6.77)

We fix \( N \ni R > \dim Z \). Let \( f_{r} : \overline{U}_{j,R,t} \to \mathbb{C} \) be a holomorphic function satisfying
\[ \frac{1}{r!} \frac{d^{r}}{dz^{r}} f_{r}(z) = f'(z), \quad \lim_{z \to \pm i\infty} f_{r}(z) = 0. \]
(6.78)

We further assume that for each bounded connected component \( V \subseteq U_{j,R,t} \), there exists \( z \in V \) such that \( f_{r}(z) = 0 \). Note that the bounded (resp. unbounded) connected components of \( U_{j,R,t} \) cover the small (resp. large) eigenvalues of \( \sqrt{t}(d^{2}_{j,R,t} - d_{T}^{2}) \), by Theorem 3.3 and (6.47), the total area of the bounded connected components of \( U_{j,R,t} \) is bounded by a universal constant. Then, there exists \( C > 0 \) such that for any \( z \in \overline{U}_{j,R,t} \), we have
\[ |f_{r}(z)| \leq Ce^{-|z|}. \]
(6.79)

By (6.79) and (6.78), for \( R \gg 1 \), we have
\[ f'(\mathcal{D}_{j,R,t}) = \frac{1}{2\pi i} \int_{\partial U_{j,R,t}} f_{r}(\lambda)(\lambda - \mathcal{D}_{j,R,t})^{-r-1} d\lambda, \]
(6.80)
\[ f'(\tilde{\mathcal{D}}_{j,R,t}) = \frac{1}{2\pi i} \int_{\partial U_{j,R,t}} f_{r}(\lambda)(\lambda - \tilde{\mathcal{D}}_{j,R,t})^{-r-1} d\lambda. \]

By Lemma 6.4, (6.79) and (6.80), for \( R \gg 1 \) and \( t \geq R^{2-\kappa/2} \), we have
\[ \left\| f'(\mathcal{D}_{j,R,t}) - f'(\tilde{\mathcal{D}}_{j,R,t}) \right\|_{1} = O(R^{1-\kappa/2}) t^{-1/2}. \]
(6.81)

On the other hand, by (0.22), (3.20), (6.4), (6.5), (6.75) and (6.77), we have
\[ \sum_{j=0}^{3} (-1)^{j(\gamma-3)/2} \left( \mathcal{T}_{j,R}^{L} - \mathcal{T}_{j,R}^{L} \right) \]
(6.82)
\[ = -\sum_{j=0}^{3} (-1)^{j(\gamma-3)/2} \int_{R^{2-\kappa/2}}^{+\infty} \varphi \text{Tr} \left[ (-1)^{N_{T^{2}}} \frac{N_{T^{2}}}{2} f'(\mathcal{D}_{j,R,t}) - f'(\tilde{\mathcal{D}}_{j,R,t}) \right] \frac{dt}{t}. \]
By (6.81) and (6.82), we have

\[(6.83)\]  \[\sum_{j=0}^{3} (-1)^{j(j-3)/2} \mathcal{F}_{j,R} = \sum_{j=0}^{3} (-1)^{j(j-3)/2} \mathcal{G}_{j,R} + \mathcal{O}(R^{-\kappa/4}).\]

By Theorem 3.3 and (6.3), for \(R \gg 1\), we have

\[(6.84)\]  \[\text{Sp} \left( (d^Z_{j,R} - d^Z_{T}) \left|_{\mathcal{E}_{j,R}} \right. \right) \subseteq \{ -\exp(-R^\kappa), \exp(-R^\kappa) \}, \]

where the exponential term comes from \(e^{-T} = e^{-R^\kappa}\) in (3.34). As a consequence, we have

\[(6.85)\]  \[\left\| \sqrt{t} P_{1,1}^{|j,R,T|} \left( d^Z_{j,R} - d^Z_{T} \right) \right\| \leq \exp(-R^\kappa)t^{1/2}.\]

By (6.48) and (6.85), we have

\[(6.86)\]  \[\text{Tr} \left[ (-1)^{NT_z} N^{NT_z \frac{NT_z}{2}} f' \left( \mathcal{G}_{j,R,t} \right) \right] - \text{Tr} \left[ (-1)^{NT_z} N^{NT_z \frac{NT_z}{2}} f' \left( P_{1,1}^{|j,R,T|} \omega_{j,R} P_{1,1}^{|j,R,T|} \right) \right] = \mathcal{O} \left( \exp(-R^\kappa) \right) t^{1/2}.\]

Note that \(f'\) is an even function, by the proof of [11] Prop. 1.3, the function

\[(6.87)\]  \[\mathbb{R} \ni s \mapsto \text{Tr} \left[ (-1)^{NT_z} N^{NT_z \frac{NT_z}{2}} f' \left( s P_{1,1}^{|j,R,T|} \omega_{j,R} P_{1,1}^{|j,R,T|} \right) \right] \in QS\]

is constant. Taking \(s = 0, 1\) in (6.87) and using the identity \(f'(0) = 1\), we get

\[(6.88)\]  \[\text{Tr} \left[ (-1)^{NT_z} N^{NT_z \frac{NT_z}{2}} f' \left( P_{1,1}^{|j,R,T|} \omega_{j,R} P_{1,1}^{|j,R,T|} \right) \right] = \text{Tr} \left[ (-1)^{NT_z} N^{NT_z \frac{NT_z}{2}} P_{1,1}^{|j,R,T|} \right].\]

By Theorem 3.3, (6.74) and (6.88), we have

\[(6.89)\]  \[\text{Tr} \left[ (-1)^{NT_z} N^{NT_z \frac{NT_z}{2}} f' \left( P_{1,1}^{|j,R,T|} \omega_{j,R} P_{1,1}^{|j,R,T|} \right) \right] = \frac{1}{2} \chi (C_{j,R}).\]

By (6.38), (6.75), (6.77), (6.86) and (6.89), we have

\[(6.90)\]  \[\sum_{j=0}^{3} (-1)^{(j-3)/2} \mathcal{G}_{j,R} = -\frac{1}{2} \int_{0}^{R^{2-\kappa/2}} \left\{ f' \left( \frac{i}{2} \right) - 1 \right\} \frac{dt}{t} + \mathcal{O}(R^{-\kappa/4}).\]

From Proposition 6.3, (6.5), (6.77), (6.83) and (6.90), we obtain (6.76). This completes the proof of Proposition 6.5. \(\square\)

We denote \(\mathcal{G} = \Omega^*(Y, F)\), which is a complex vector bundle of infinite dimension over \(S\). We define the connection \(\nabla^\mathcal{G}\) on \(\mathcal{G}\) in the same way as in (1.73). Let \(h^\mathcal{G}\) be the \(L^2\)-metric on \(\mathcal{G}\) with respect to \(h^F\) and \(g^{TY}\). Let \(\omega^\mathcal{G} \in \Omega^1(S, \text{End} (\mathcal{G}))\) be as in (1.79) with \((\nabla^\mathcal{G}, h^\mathcal{G})\) replaced by \((\nabla^F, h^F)\).

Let \(\nabla^V^*\) be the canonical flat connection on \(V^* = H^*(Y, F)\) (see [11] Def. 2.4). We identify \(V^*\) with \(\mathcal{H}^*(Y, F) \subseteq \mathcal{G}\) via the Hodge theorem. Recall that \(h^V^*\) is the \(L^2\)-metric on \(V^*\) defined after (3.28). Let \(\omega^V^* \in \Omega^1(S, \text{End}(V^*))\) be as in (0.11) with \((\nabla^F, h^F)\) replaced by \((\nabla^V, h^V)\). Let \(P^V^* : \mathcal{G} \to V^*\) be the orthogonal projection with respect to \(h^\mathcal{G}\). By [11] Prop. 3.14, we have

\[(6.91)\]  \[\omega^V^* = P^V^* \omega^\mathcal{G} P^V^*.\]
Recall that the flat sub vector bundles \( V_j^* = \text{Im}(\tau_j : W_j^* \to V^*) \subseteq V^* \) with \( j = 1, 2 \) were defined by (2.2) and (3.20). The identity (6.91) also holds with \( V^* \) replaced by \( V_j^* \). Let
\[
(6.92) \quad \tau_j^\perp : K_j^{\perp} \to V_j^*
\]
be the restriction of \( \tau_j : W_j^* \to V_j^* \subseteq V^* \) defined in (3.26) to \( K_j^{\perp} \subseteq W_j^* \) defined in (3.25), which is bijective. Set
\[
(6.93) \quad \omega_{K_j^{\perp}} = (\tau_j^\perp)^{-1} \circ \omega_{V_j^*} \circ \tau_j^\perp \in \Omega^1(S, \text{End}(K_j^{\perp})) .
\]

For \( j = 1, 2 \), let \( \nabla^{W_j^*} \) be the canonical flat connection on \( W_j^* = H^*(Z_j, F) \) (see [11, Def. 2.4]). The sub vector bundle \( K_j^{\perp} \subseteq W_j^* \) is preserved by \( \nabla^{W_j^*} \). Then \( \nabla^{K_j^\perp} := \nabla^{W_j^*}|_{K_j^\perp} \) is a flat connection on \( K_j^\perp \). The Hermitian metric \( h_{K_j^\perp} \) was constructed in (3.26). Let \( \omega_{K_j^\perp} \in \Omega^1(S, \text{End}(K_j^\perp)) \) be as in (0.1) with \( (\nabla^F, h^F) \) replaced by \( (\nabla^{K_j^\perp}, h^{K_j^\perp}) \).

We identify \( K_j^\perp \) with \( \mathcal{F}_{j, \infty} \cap \text{Ker}(D_{Z_j, \infty}) \) via the map (3.22). Let \( P^{K_j^\perp} : \mathcal{F}_{j, \infty} \to K_j^\perp \) be the orthogonal projection. By [11, Prop. 3.14], we have
\[
(6.94) \quad \omega_{K_j^\perp} = P^{K_j^\perp} \omega_{\mathcal{F}_{j, \infty}} P^{K_j^\perp} \cdot 
\]

Recall that \( C_{0}\ast^* = W_1^* \oplus W_2^* \oplus V^* = K^\ast \oplus K_1^{\perp} \oplus K_2^{\perp} \oplus V^* \). Set
\[
(6.95) \quad \omega_{C_{0}\ast^*} = \omega_{K^\ast} + \omega_{K_1^{\perp}} + \omega_{K_2^{\perp}} + \omega_{V^*} \in \Omega^1(S, \text{End}(C_{0}\ast^*)) .
\]

For \( j = 1, 2, 3 \), we may construct \( \omega_{C_j^\ast^*} \) in the same way.

Recall that the bijection \( \mathcal{J}_{j, R, T} : C_j^\ast^* \to \mathcal{O}_{j, R, T}^{-1/1} \subseteq \Omega^*(Z_{j, R}, F) \) was defined in (5.132), (5.212) and (5.217).

**Lemma 6.6.** For \( j = 0, 1, 2, 3 \) and \( R \gg 1 \), we have
\[
(6.96) \quad \mathcal{J}_{j, R, T}^{-1} \circ \left( P_{R, T}^{-1} \omega_{\mathcal{F}_{j, R, T}} P_{R, T}^{-1} \right) \circ \mathcal{J}_{j, R, T} = \omega_{C_j^\ast^*} + \Theta_{R, T}(R^{-1/2+\kappa/4}) ,
\]
where \( \Theta_{R, T} (\cdot) \) was defined in the paragraph above Theorem 3.3.

**Proof.** We only prove the case \( j = 0 \).

We consider \( \sigma \in C_{0}^\ast^* \). All the estimates in the proof of Proposition 5.16 hold with \( \| \sigma \|^2_{Z_R} \) replaced by \( \langle \sigma, \omega_{\mathcal{F}_{R, T}} \sigma \rangle_{Z_R} \). Hence (5.181) holds with \( \| \sigma \|^2_{Z_R} \) replaced by \( \langle \sigma, \omega_{\mathcal{F}_{R, T}} \sigma \rangle_{Z_R} \), i.e.,
\[
(6.97) \quad \left( \mathcal{J}_{R, T}(\sigma), \omega_{\mathcal{F}_{R, T}} \mathcal{J}_{R, T}(\sigma) \right)_{Z_R} = \langle \sigma, \omega_{\ast^*} \sigma \rangle_{R, T} + \Theta(R^{-1/2+\kappa/4}) \| \sigma \|^2_{R, T} .
\]

From the polarization identity, Proposition 5.16 and (6.97), we obtain (6.96) with \( j = 0 \). This completes the proof of Lemma 6.6. \( \square \)

For \( j = 0, 1, 2, 3 \), let \( \nabla^{C_{j}^\ast^*} \) be the flat connection on \( C_j^\ast^* \) induced by \( \nabla^{W_1^*}, \nabla^{W_2^*} \) and \( \nabla^V \). Let \( \omega_{C_{j}^\ast^*} \in \Omega^*(S, \text{End}(C_j^\ast)) \) be as in (0.1) with \( (\nabla^F, h^F) \) replaced by \( (\nabla^{C_j^\ast^*}, h_{R, T}^{C_j^\ast^*}) \), where the metric \( h_{R, T}^{C_j^\ast^*} \) was defined in (3.30).

**Lemma 6.7.** For \( j = 0, 1, 2, 3 \) and \( R \gg 1 \), we have
\[
(6.98) \quad \omega_{R, T}^{C_j^\ast} = \omega_{C_j^\ast^*} + \Theta_{R, T}(R^{-1/2+\kappa/4}) .
\]
Proof. We only prove the case \( j = 0 \).

Let \( \omega_{R,T} \in \Omega^1(S, \text{End}(W_j^*)) \) be as in \((0.1)\) with \((\nabla^F, h^F)\) replaced by \((\nabla^{W_j^*}, h_{R,T}^{W_j^*})\), where the metric \( h_{R,T}^{W_j^*} \) was defined in \((3.28)\). More precisely, we have
\[
(6.99) \quad \omega_{R,T}^{W_j^*} = (h_{R,T}^{W_j^*})^{-1} \nabla^{W_j^*} h_{R,T}^{W_j^*}.
\]
By \((6.99)\) and the paragraph above Lemma \(6.7\), we have
\[
(6.100) \quad \omega_{R,T}^{W_j^*} = \omega_{R,T}^{W_j^*} + \omega_{R,T}^{W_j^*} + \omega^{V_j^*}.
\]
Comparing \((6.95)\) with \((6.100)\), it remains to show that
\[
(6.101) \quad \omega_{R,T}^{W_j^*} = \omega_{R,T}^{W_j^*} + \omega_{R,T}^{W_j^*} + \omega^{V_j^*}.
\]
Let \( P_j : W_j^* \to K_j^* \) and \( P_j^\perp : W_j^* \to K_j^* \perp \) be the projections with respect to the decomposition \( W_j^* = K_j^* \oplus K_j^* \perp \). Set
\[
(6.102) \quad \nabla^{W_j^*} = P_j \nabla^{W_j^*} P_j + P_j^\perp \nabla^{W_j^*} P_j^\perp.
\]
Since \( K_j^* \subseteq W_j^* \) is a flat sub vector bundle, we have
\[
(6.103) \quad \nabla^{W_j^*} - \nabla^{W_j^*} = P_j \nabla^{W_j^*} P_j + P_j^\perp \nabla^{W_j^*} P_j^\perp \in \Omega^1(S, \text{Hom}(K_j^* \perp, K_j^*)).
\]
We denote by \( \| \cdot \|_{R,T} \) the operator norm on \( \text{Hom}(K_j^* \perp, K_j^*) \) with respect to \( h_{R,T}^{W_j^*} \). By \((3.28)\), we have
\[
(6.104) \quad \| \cdot \|_{R,T} = R^{-1/2} T^{1/4} \| \cdot \|_{1,1} = R^{-1/2 + \kappa/4} \| \cdot \|_{1,1}.
\]
By \((6.103)\) and \((6.104)\), we have
\[
(6.105) \quad \nabla^{W_j^*} = \nabla^{W_j^*} + \Theta_{R,T}(R^{-1/2 + \kappa/4}).
\]
By \((3.28)\) and \((6.102)\), we have
\[
(6.106) \quad \Theta_{R,T}^{W_j^*} = \Theta_{R,T}(R^{-1/2 + \kappa/4}).
\]
From \((6.99)\), \((6.105)\) and \((6.106)\), we obtain \((6.101)\). This completes the proof of Lemma \(6.7\). \(\square\)

Proof of Theorem \(3.4\) Applying Remark \(1.7\) to the map \( \mathcal{F}_{j,R} : C_j^\bullet \to \Theta_{j,R}^{[-1,1]} \) and using Theorem \(3.3\), Lemmas \(6.6\), \(6.7\), \(3.36\) and \(6.75\), we get
\[
(6.107) \quad \mathcal{F}_{j,R} - \mathcal{F}_{j,R} = \Theta(R^{-1/4 + \kappa/8}).
\]
From Proposition \(6.5\) and \((6.107)\), we obtain \((3.38)\). This completes the proof of Theorem \(3.4\). \(\square\)

7. Torsion forms associated with the Mayer-Vietoris exact sequence

The purpose of this section is to prove Theorem \(3.5\). This section is organized as follows. In \(\S7.1\) we introduce a filtration of the Mayer-Vietoris exact sequence in question. In \(\S7.2\) we estimate the torsion form associated with the Mayer-Vietoris exact sequence. Theorem \(3.5\) will be proved in this subsection. In the whole section, we take \( T = R^\kappa \), where \( \kappa \in ]0, 1/3[ \) is a fixed constant. For ease of notations, we will systematically omit a parameter \( (R \text{ or } T) \) as long as there is no confusion.
7.1. A filtration of the Mayer-Vietoris exact sequence. Recall that $W_1^*, W_2^*, V_1^*, V_1^*$ and $V_2^*$ were defined by (2.2) and (3.20). Recall that $W_{12} \subseteq W_1^* \oplus W_2^*$ was defined by (2.12). For convenience, we denote $V_{\text{quot}}^* = V^*/(V_1^* + V_2^*)$.

For $k \in \mathbb{N}$, we construct a truncation of the exact sequence (0.22) as follows,

$$
\begin{align*}
\cdots & \longrightarrow H^k(Z, F) \longrightarrow W_1^k \oplus W_2^k \longrightarrow V^k \longrightarrow V_{\text{quot}}^k \longrightarrow 0.
\end{align*}
$$

The truncations of (0.22) at degree $k - 1$ and $k$ fit into the following commutative diagram with exact rows and columns,

$$
\begin{align*}
\cdots & \longrightarrow V_{\text{quot}}^{k-1} \longrightarrow 0 \\
\downarrow & \\
\cdots & \longrightarrow H^k(Z, F) \longrightarrow W_1^k \oplus W_2^k \longrightarrow V^k \longrightarrow V_{\text{quot}}^k \longrightarrow 0 \\
\downarrow & \\
0 & \longrightarrow W_{12}^k \longrightarrow W_1^k \oplus W_2^k \longrightarrow V^k \longrightarrow V_{\text{quot}}^k \longrightarrow 0.
\end{align*}
$$

We equip $H^\ast(Z, F)$, $W_1^*$, $W_2^*$ and $V^*$ in (7.2) with Hermitian metrics induced by $\| \cdot \|_{Z, R}$ $(j = 0, 1, 2, 3)$ via the identification (3.16). We equip $W_{12}$ in (7.2) with the Hermitian metric induced by $h_{R, T}^{W_1} \oplus h_{R, T}^{W_2}$, which were defined in (3.28), via the embedding $W_{12} \rightarrow W_1^* \oplus W_2^*$. Set

$$
\begin{align*}
a_{R, T} = R \int_{-1}^{1} \chi_3(s) e^{2Tf_R(s)-T} ds,
\end{align*}
$$

where $\chi_3 : \mathbb{R} \rightarrow \mathbb{R}$ was defined in (5.42). We equip $V_{\text{quot}}^*$ in (7.2) with the quotient metric of $a_{R, T}^{-2} h_{R, T}^{V^*}$, where $h_{R, T}^{V^*}$ was defined in (3.29). Let $\mathcal{T}_{\text{vert}, R, T}^k$ be the torsion form associated with the third row in (7.2). Let $\mathcal{T}_{\text{hor}, R, T}^k$ be the torsion form associated with the (unique) non trivial column in (7.2).

**Proposition 7.1.** The following identity holds in $Q^S/Q^{S, 0}$,

$$
\begin{align*}
\mathcal{T}_{\mathcal{H}, R, T} = \sum_{k=0}^{n} (-1)^k \mathcal{T}_{\text{hor}, R, T}^k - \sum_{k=1}^{n} (-1)^k \mathcal{T}_{\text{vert}, R, T}^k.
\end{align*}
$$

**Proof.** Let $\mathcal{T}_{\mathcal{H}, R, T}(k)$ be the torsion form associated with (7.1). In particular, $\mathcal{T}_{\mathcal{H}, R, T}(-1) = 0$ and $\mathcal{T}_{\mathcal{H}, R, T}(\dim Z) = \mathcal{T}_{\mathcal{H}, R, T}$. Applying [11, Thm. A1.4] to (7.2), we get

$$
\begin{align*}
\mathcal{T}_{\mathcal{H}, R, T}(k - 1) - \mathcal{T}_{\mathcal{H}, R, T}(k) + (-1)^k \mathcal{T}_{\text{hor}, R, T}^k - (-1)^k \mathcal{T}_{\text{vert}, R, T}^k \in Q^{S, 0}.
\end{align*}
$$

Taking the sum of (7.5) for $k = 0, 1, \cdots, \dim Z$, we obtain (7.4). This completes the proof of Proposition 7.1. \hfill \Box

7.2. Estimating $\mathcal{T}_{\text{vert}, R, T}^k$ and $\mathcal{T}_{\text{hor}, R, T}^k$. For $j = 0, 1, 2, 3$, under the identification (5.202), we view $H^\ast(C_j^\ast, \partial)$ as a vector subspace of $C_j^\ast$. Let

$$
\begin{align*}
P_j^H : C_j^\ast \rightarrow H^\ast(C_j^\ast, \partial)
\end{align*}
$$

be the orthogonal projection with respect to $h_{R, T}^{C_j^\ast}$ (see (3.30)). Note that $K_j^* \subseteq H^0(C_j^\ast, \partial)$, by (3.28) and (3.30), $P_j^H$ is independent of $R, T$. Let $\omega_{H^\ast(C_j^\ast, \partial)}^\ast$ (resp.
\( \omega_{R,T}^H(C_j^{\ast \ast}, \partial) \) be the 1-form on \( S \) with values in \( \text{End}(C_j^{\ast \ast}) \) induced by \( \omega^{C_j^{\ast \ast}} \) in (6.95) (resp. \( \omega_{R,T}^\ast \) in (6.98)) via the projection \( P_j^H \). More precisely, we have

\[
\begin{align*}
\omega^H(C_j^{\ast \ast}, \partial) &= P_j^H \omega^{C_j^{\ast \ast}} P_j^H \in \Omega^1(S, \text{End}(H^*(C_j^{\ast \ast}, \partial))); \quad (\text{7.7}) \\
\omega_{R,T}^H(C_j^{\ast \ast}, \partial) &= P_j^H \omega_{R,T}^\ast P_j^H \in \Omega^1(S, \text{End}(H^*(C_j^{\ast \ast}, \partial))).
\end{align*}
\]

By (2.13), we have \( H^*(C_j^{\ast \ast}, \partial) = H^0(C_j^{\ast \ast}, \partial) = W_j^\ast \) for \( j = 1, 2 \). Then we have

\[
\omega_{R,T}^H(C_j^{\ast \ast}, \partial) = W_j^\ast
\]

for \( j = 1, 2 \), where \( \omega_{R,T}^\ast \) was defined in (6.99).

For \( j = 0, 1, 2, 3 \), let \( h_{R,T}^{H,*}(Z_j, F) \) be the Hermitian metric on \( H^*(Z_j, F) = H^*(Z_{j,R}, F) \) induced by \( \| \cdot \|_{Z_{j,R}} \) via the identification (3.16). Let \( \nabla^{H,*}(Z_j, F) \) be the canonical flat connection on \( H^*(Z_j, F) \) (see [11, Def. 2.4]). Let \( \omega_{R,T}^{H,*}(Z_j, F) \in \Omega^1(S, \text{End}(H^*(Z_j, F))) \) be as in (0.1) with \( (\nabla^F, h^F) \) replaced by \( (\nabla^{H,*}(Z_j, F), h_{R,T}^{H,*}(Z_j, F)) \).

Let \( (\mathcal{H}^R_{j,R,T})_T : H^*(C_j^{\ast \ast}, \partial) \to H^*(Z_j, F) \) be the map defined by (5.51), (5.55), (5.212) and (5.217). Recall that the notation \( \theta_{R,T}(\cdot) \) was defined in the paragraph above Theorem 3.3.

**Lemma 7.2.** For \( R \gg 1 \), we have

\[
\left( (\mathcal{H}^R_{j,R,T})_T \right)^{-1} \circ \omega_{R,T}^{H,*}(Z_j, F) \circ (\mathcal{H}^R_{j,R,T})_T = \omega^H(C_j^{\ast \ast}, \partial) + \theta_{R,T}(R^{-1/2 + \kappa/4}),
\]

\[
\omega_{R,T}^H(C_j^{\ast \ast}, \partial) = \omega^H(C_j^{\ast \ast}, \partial) + \theta_{R,T}(R^{-1/2 + \kappa/4}).
\]

**Proof.** Recall that \( P_{j,R,T} : \mathcal{F}_{j,R} \to \text{Ker} \left( D^2_{P_j^R, R} \right) \) was defined above Proposition 5.2 above (5.212) and above (5.217). By [11, Prop. 3.14], the following identity holds under the identification (3.16),

\[
\omega_{R,T}^{H,*}(Z_j, F) = P_{j,R,T} \omega_{\mathcal{F}_{j,R}} P_{j,R,T},
\]

where \( \omega_{\mathcal{F}_{j,R}} \) was defined in (6.1). By Lemma 6.6 and (5.207), we have

\[
(\mathcal{H}^R_{j,R,T})^{-1} \circ \left( P_{j,R,T} \omega_{\mathcal{F}_{j,R}} P_{j,R,T} \right) \circ (\mathcal{H}^R_{j,R,T}) = P_j^H \omega^{C_j^{\ast \ast}} P_j^H + \theta_{R,T}(R^{-1/2 + \kappa/4}).
\]

From (7.7), (7.10) and (7.11), we obtain the first identity in (7.9). The second identity in (7.9) is a direct consequence of Lemma 6.7 and (7.7). This completes the proof of Lemma 7.2. \( \square \)

**Proposition 7.3.** For \( R \gg 1 \), we have

\[
\mathcal{F}_{vert, R,T}^k = \mathcal{O}(R^{-1/4 + \kappa/8}).
\]
Proof. Recall that $H^0(C^{*k}_0, \partial) = W^k_{12}$ and $H^1(C^{*k-1}_0, \partial) = V^{k-1}_{\text{quot}}$. First we show that the following diagram commutes,

$$
\begin{array}{cccc}
V^{k-1}_{\text{quot}} & \longrightarrow & V^{k-1}_{\text{quot}} \oplus W^k_{12} & \longrightarrow & W^k_{12} \\
\downarrow \text{a}_{R,T} \text{Id} & & \left[ \mathcal{S}^{H}_{R,T} \right] \downarrow \text{Id} & & \\
V^{k-1}_{\text{quot}} & \delta & H^k(Z, F) & \alpha & W^k_{12}
\end{array}
$$

(7.13)

where $a_{R,T}$ was defined by (7.3), the first row consists of canonical injection and projection, the second row is the (unique) non trivial column in (7.2). We remark that (7.13) is not a commutative diagram of flat complex vector bundles over $S$.

Let $\eta : [-R, R] \to \mathbb{R}$ be a smooth function such that

$$
\eta|_{[-R,-R/2]} = 0, \quad \eta|_{[R/2,R]} = 1.
$$

(7.14)

We will view $\eta$ as a function on $IY_R$. Let $\sigma \in \mathcal{H}^{k-1}(Y, F) = V^{k-1}$. Let $\sigma' \in V^{k-1}_{\text{quot}}$ be the image of $\sigma$. Let $\omega \in \Omega^*(Z_R, F)$ such that

$$
\omega|_{Z_{1,0}} = 0, \quad \omega|_{Z_{2,0}} = 0, \quad \omega|_{IY_R} = d\eta \wedge \sigma.
$$

Then we have

$$
\delta(\sigma) = [\omega] \in H^k(Z_R, F) = H^k(Z, F).
$$

(7.15)

Let $\sigma' \in (V^{k-1}_1 + V^{k-1}_2)^{1/2} \subseteq V^{k-1}$. By (5.43)-(5.45), we have

$$
I_{R,T}(\sigma')|_{Z_{1,0}} = 0, \quad I_{R,T}(\sigma')|_{Z_{2,0}} = 0, \quad I_{R,T}(\sigma')|_{IY_R} = \chi_3 e^{T_{J_T}} d\sigma' + \sigma'.
$$

(7.16)

Let $\sigma' \in V^{k-1}_{\text{quot}}$ be the image of $\sigma'$. By (5.51), (5.55), (5.212) and (5.217), we have

$$
\left[ \mathcal{S}^{H}_{R,T} \right]_T(\sigma') = [e^{T_{J_T}} I_{R,T}(\sigma') \in H^k(Z_R, F) = H^k(Z, F).
$$

(7.17)

By (7.3) and (7.14)-(7.18), we have

$$
\left[ \mathcal{S}^{H}_{R,T} \right]_T(\sigma) = \left( \int_{-R}^{R} \chi_3(u) e^{T_{J_T}}(u)^{-T} d\sigma \right) = a_{R,T}(\sigma).
$$

(7.19)

Hence the left square in (7.13) commutes.

Let $(\omega_1, \omega_2, \hat{\omega}) \in \mathcal{H}^{k}_{\text{abs}}(Z_{12}, F)$. Its image in $W^k_{12}$ via the identification (5.47) is given by $(\omega_1|_{Z_{1,0}}, \omega_2|_{Z_{2,0}})$. By (5.51), (5.55), (5.212) and (5.217), we have

$$
\left[ \mathcal{S}^{H}_{R,T} \right]_T \left[ \omega_1|_{Z_{1,0}}, \omega_2|_{Z_{2,0}} \right] = [e^{T_{J_T}} F_{R,T}(\omega_1, \omega_2, \hat{\omega}) \in H^k(Z_R, F) = H^k(Z, F).
$$

(7.20)

By (5.19)-(5.21), we have

$$
F_{R,T}(\omega_1, \omega_2, \hat{\omega})|_{Z_{1,0}} = \omega_1|_{Z_{1,0}}, \quad F_{R,T}(\omega_1, \omega_2, \hat{\omega})|_{Z_{2,0}} = \omega_2|_{Z_{2,0}}.
$$

(7.21)

On the other hand, for $[\omega] \in H^k(Z_R, F) = H^k(Z, F)$, we have

$$
\alpha([\omega]) = \left[ \omega|_{Z_{1,0}}, \omega|_{Z_{2,0}} \right] \in W^k_{12} \subseteq W^k_1 \oplus W^k_2.
$$

(7.22)
By \((7.20)-(7.22)\), we have
\[
(7.23) \quad \alpha \circ [(\mathcal{F}^H_{R,T})_{Z_R}] \left( [\omega_1|_{Z_{1,0}}], [\omega_2|_{Z_{2,0}}] \right) = \left( [\omega_1|_{Z_{1,0}}], [\omega_2|_{Z_{2,0}}] \right).
\]

Hence the right square in \((7.13)\) commutes.

We equip \(H^k(Z, F) = H^k(Z_R, F)\) in \((7.13)\) with the Hermitian metric induced by \(\| \cdot \|_{Z_R}\) via the identification \((3.16)\). We equip \(W^k_{12}\) in \((7.13)\) with the Hermitian metric induced by \(h^*_{R,T} + h^*_{R,T}\) via the embedding \(W^k_{12} \hookrightarrow W^k_1 \oplus W^k_2\). We equip \(V^*_{\text{quot}}\) in the first row of \((7.13)\) with the quotient metric of \(h^*_{R,T}\). We equip \(V^*_{\text{quot}}\) in the second row of \((7.13)\) with the quotient metric of \(a_{R,T}^{-2}h^*_{R,T}\). Then the torsion form of the first row in \((7.13)\) vanishes, and the torsion form of the second row in \((7.13)\) equals \(\mathcal{F}^k_{\text{vert},R,T}\). Applying Proposition 1.6 to \((7.13)\) and using Corollary 5.17 and Lemma 7.2, we obtain \((7.12)\). This completes the proof of Proposition 7.3.

**Proposition 7.4.** For \(R \gg 1\), the following identity holds in \(Q^S/Q^{S,0}\),
\[
(7.24) \quad \mathcal{F}^k_{\text{hor},R,T} = \mathcal{T}^k_{\mathcal{F},R,T} + O\left(R^{-1/4+s/8}\right)
\]
with \(\mathcal{F}^k_{\text{hor},R,T}\) being as in \((7.4)\) and \(\mathcal{T}^k_{\mathcal{F},R,T}\) being as in \((3.40)\).

**Proof.** We denote \(b_{R,T} = \pi^{1/2}RT^{-1/2}e^T\). By \((7.3)\), there exists \(a > 0\) such that
\[
(7.25) \quad a_{R,T} = b_{R,T} \left(1 + O\left(e^{-aT}\right)\right).
\]

Let \(p : V^k \to V^k_{\text{quot}}\) be the canonical projection. The following commutative diagram is obvious,
\[
\begin{array}{cccc}
W^k_{12} & \to & W^k_1 \oplus W^k_2 & \to & V^k & \to & V^k_{\text{quot}} \\
\downarrow \text{Id} & & \downarrow \text{Id} & & \downarrow \text{id} & & \downarrow b_{R,T}\text{Id} \\
W^k_{12} & \to & W^k_1 \oplus W^k_2 & \to & V^k & \to & V^k_{\text{quot}}.
\end{array}
\]

We equip \(W^k_{12}\) in \((7.26)\) with the Hermitian metric induced by \(h^*_{R,T} \oplus h^*_{R,T}\) via the inclusion \(W^k_{12} \subseteq W^k_1 \oplus W^k_2\). We equip \(W^k_1 \oplus W^k_2\) in the first row of \((7.26)\) with the Hermitian metric \(h^*_{1R,T} \oplus h^*_{2R,T}\). We equip \(W^k_1 \oplus W^k_2 = H^k(Z_{1,R}; F) \oplus H^k(Z_{2,R}; F)\) in the second row of \((7.26)\) with the Hermitian metric induced by \(\| \cdot \|_{Z_{j,R}}\) for \(j = 1, 2\) via the identification \((3.16)\). We equip \(V^k\) in the first row of \((7.26)\) with the Hermitian metric \(h^*_{R,T}\). We equip \(V^k\) in the second row of \((7.26)\) with the Hermitian metric induced by \(\| \cdot \|_{Z_{j,R}}\) for \(j = 1, 2\) via the identification \((3.16)\). We equip \(V^*_{\text{quot}}\) in the first (resp. second) row of \((7.26)\) with the quotient metric of \(h^*_{R,T}\) (resp. \(a_{R,T}^{-2}h^*_{R,T}\)). The torsion form of the first row of \((7.26)\) is given by \(\mathcal{T}^k_{\mathcal{F},R,T}\), and the torsion form of the second row of \((7.26)\) is given by \(\mathcal{F}^k_{\text{hor},R,T}\). For \(j = 1, 2, 3, 4\), let \(\mathcal{F}_j \in Q^{S}\) be the torsion form of the \(j\)-th column in \((7.26)\). Applying Thm. A1.4 to \((7.26)\), we get
\[
(7.27) \quad \mathcal{T}^k_{\mathcal{F},R,T} - \mathcal{F}^k_{\text{hor},R,T} - \mathcal{F}_1 + \mathcal{F}_2 - \mathcal{F}_3 + \mathcal{F}_4 \in Q^{S,0}.
\]
Since the first vertical map is isometric, we have

\[ T_1 = 0. \]  

By Corollary 1.8, Corollary 5.17, Remark 5.18 and Lemma 7.2, we have

\[ T_2 = O\left(R^{-1/4 + \kappa/8}\right), \quad T_3 = O\left(R^{-1/4 + \kappa/8}\right). \]

By Corollary 1.8 and (7.25), we have

\[ T_4 = O\left(e^{-aT/2}\right). \]

From (7.27)-(7.30), we obtain (7.24). This completes the proof of Proposition 7.4. □

**Proof of Theorem 3.5** We combine Propositions 7.1, 7.3, 7.4 and (3.40). □

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