Edge exponent in the dynamic spin structure factor of the Yang-Gaudin model

M. B. Zvonarev, V. V. Cheianov, and T. Giamarchi

1DPMC-MaNEP, University of Geneva, 24 quai Ernest-Ansermet, 1211 Geneva 4, Switzerland

Physics Department, Lancaster University, Lancaster, LA1 4YB, UK

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The dynamic spin structure factor $\mathcal{S}(k, \omega)$ of a system of spin-1/2 bosons is investigated at arbitrary strength of interparticle repulsion. As a function of $\omega$ it is shown to exhibit a power-law singularity at the threshold frequency defined by the energy of a magnon at given $k$. The power-law exponent is found exactly using a combination of the Bethe Ansatz solution and an effective field theory approach.

The remarkable progress achieved by the theory of one-dimensional (1D) quantum fluids is rooted in the fact that dimensionality imposes severe constraints on the fluid’s low energy excitation spectrum. Due to these constraints the investigation of the low-energy dynamics of the fluid reduces to choosing the effective field theory from a limited number of universality classes. Perhaps the most ubiquitous (and most thoroughly investigated) is the universality class called the Luttinger Liquid [4]. Other non-trivial examples include states with non-abelian currents, spin-incoherent [2, 3, 4] and ferromagnetic liquids [5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. For all such cases there exist well developed analytical methods allowing one to calculate infrared asymptotics of dynamical correlation function, spectral properties, and scaling dimensions of local observables.

In a series of recent papers [5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] it has been found that at infinite point-like repulsion and for arbitrary strength of interparticle repulsion $\Delta(k)$ implied by symmetries of microscopic Hamiltonian were discussed in Refs. [8, 11]. Constraints on $\Delta(k)$ were established. It was shown that for infinite point-like repulsion $\Delta(k)$ is the universality class called the Luttinger Liquid [1]. Other non-trivial examples include states with

$$\Delta(k) \sim -1 + \frac{K}{2} \left( \frac{k}{k_F} \right)^2,$$

where $K$ is the Luttinger parameter and $k_F = \pi \rho_0$ with $\rho_0$ being average particle density. Assuming the validity of Eq. (2) at a large but finite repulsion, a crossover between trapped and open regimes of spin propagation was characterized. A different approach to the study of the same system proposed in Ref. [4] confirmed Eq. (2). The approach of Ref. [4] was further developed in Ref. [6], demonstrating that for infinite point-like repulsion $\Delta(k)$ has the form (2) for arbitrary $k$. In Ref. [8] the small $k$ expansion of $\Delta(k)$ was shown to have the form (2) for arbitrary interparticle repulsion. However, the case of arbitrary $k$ and interparticle repulsion remains unexplored.

In this paper we investigate the behavior of $\Delta(k)$ and $\omega_-(k)$ for the dynamic spin structure factor of a strongly repulsive ferromagnetic Bose gas observable phenomena such as spin trapping and gaussian damping of spin waves were predicted and a link between these phenomena and the singular behavior, Eq. (1), of the dynamic spin structure factor was established. It was shown that at infinite point-like repulsion and for $k \to 0$,

$$\omega_-(k) \sim \frac{K_0}{2} \left( \frac{k}{k_F} \right)^2,$$

where $K_0$ is the Luttinger parameter and $k_F = \pi \rho_0$ with $\rho_0$ being average particle density. Assuming the validity of Eq. (3) at a large but finite repulsion, a crossover between trapped and open regimes of spin propagation was characterized. A different approach to the study of the same system proposed in Ref. [4] confirmed Eq. (3). The approach of Ref. [4] was further developed in Ref. [6], demonstrating that for infinite point-like repulsion $\Delta(k)$ has the form (2) for arbitrary $k$. In Ref. [8] the small $k$ expansion of $\Delta(k)$ was shown to have the form (2) for arbitrary interparticle repulsion. However, the case of arbitrary $k$ and interparticle repulsion remains unexplored.

The Hamiltonian of the Yang-Gaudin model is

$$H = \int_0^L dx \left[ \partial_x \psi^{(1)}_\uparrow \partial_x \psi^{(1)}_\downarrow + \partial_x \psi^{(1)}_\downarrow \partial_x \psi^{(1)}_\uparrow + g \rho^2 \right],$$

where $\psi^{(1)}_\uparrow(x)$, $\psi^{(1)}_\downarrow(x)$ are canonical Bose fields satisfying periodic boundary conditions on a ring of circumference $L$, and $\rho(x)$ is the total particle density operator.
We consider the dynamic spin structure factor
\[ S(k, \omega) = \int dx dt e^{i(\omega t - kx)} \langle \uparrow | s_+(x, t) s_-(0, 0) | \uparrow \rangle. \] (4)

Here \( s_+(x) = \psi^+_1(x) \psi_1(x) \) is the local spin raising operator, and \( s_-(x) = [s_+(x)]^\dagger \). The average in Eq. (4) is taken with respect to a fully polarized ground state \( | \uparrow \rangle \) of the Hamiltonian satisfying \( s_+(x) | \uparrow \rangle = 0 \) for all \( x \). In the spectral representation Eq. (4) takes the form
\[ S(k, \omega) = \sum_f \delta(\omega - E_f(k)) |(f, k | s_-(k) | \uparrow \rangle|^2, \] (5)

where the sum is taken over the eigenstates \( |f, k\rangle \) of the Hamiltonian \( H \) carrying the momentum \( k \). The energies \( E_f(k) \) are defined by \( H |f, k\rangle = E_f(k) |f, k\rangle \). The frequency \( \omega_-(k) \) in Eq. (1) is given by \( \omega_-(k) = \min_f E_f(k) \). Thus the calculation of \( \omega_-(k) \) reduces to the analysis of the energy spectrum of excitations. The calculation of \( \Delta(k) \) directly from the formula (5) is a far more difficult task. It requires the knowledge of the matrix element and their resummation procedure. For most integrable models, including Yang-Gaudin, such calculation is beyond the reach of the existing theory. A way to bypass this problem is to combine the BA with an effective field theory. This is the route we take in our calculations.

We begin our analysis with a brief description of BA and a calculation of \( \omega_-(k) \). All the states \( |f, k\rangle \) in Eq. (5) lie in the sector with the \( z \) projection of the total spin given by \( S_z = N/2 - 1 \). In this sector Bethe’s wave functions are characterized by a set of quasimomenta \( \{\lambda_1, \ldots, \lambda_N, \xi\} \) which satisfy the BA equations
\[ L\lambda_j + \sum_{k=1}^N \theta(\lambda_j - \lambda_k) = 2\pi I_j + 2(\lambda_j - 2\xi) + \pi. \] (6)

Here \( \theta(\lambda) = 2 \arctan(\lambda/g) \) is the two-particle phase shift, and \( I_j = n_j - (N + 1)/2 \), where \( n_j \) are a set of distinct integers. The branch of \( \theta(\lambda) \) is chosen so that \( \theta(\pm \infty) = \pm \pi \). The total energy \( E \) and momentum \( P \) of a system are given by \( E = \sum_{j=1}^N \lambda_j^2 \) and \( P = \sum_{j=1}^N \lambda_j \), respectively. The quasimomentum \( \xi \) enters in \( E \) and \( P \) indirectly, through the solution of Eqs. (6). In the limit \( \xi = \infty \) Bethe’s equations (6) are identical to Bethe’s equations of the fully polarized system \( S_z = N/2 \), which is equivalent to the Lieb-Liniger model \( \Xi \). The distribution of \( I_j \) in the ground state of the model is
\[ I_j = j - \frac{N + 1}{2}, \quad j = 1, \ldots, N. \] (7)

Introducing the quasimomenta density \( \rho(\lambda_j) = 1/[L(\lambda_{j+1} - \lambda_j)] \) and taking the thermodynamic limit \( 0 < \rho_0 < \infty \) as \( N, L \to \infty \) one gets the integral equation
\[ \rho(\lambda) - \frac{1}{2\pi} \int_{-\Lambda}^\Lambda d\nu \rho(\nu) K(\lambda, \nu) = \frac{1}{2\pi} \] (8)

for the quasimomenta in the state (7) and \( \xi = \infty \). The kernel \( K(\lambda, \nu) \equiv K(\lambda - \nu) = \partial \rho(\lambda) / \partial \lambda = 2g/(g^2 + \lambda^2) \). Note that \( \rho(\lambda) \) should satisfy \( \int_{-\Lambda}^\Lambda d\lambda \rho(\lambda) = \rho_0 \). This formula together with Eq. (5) is used to get the value of the Fermi quasimomentum \( \Lambda \) as a function of the particle density \( \rho_0 \). The ground state energy in the thermodynamic limit is
\[ E_0 = L \int_{-\Lambda}^\Lambda d\lambda \lambda^2 \rho(\lambda) \] (9)

and the momentum of the ground state is zero.

Consider now the state characterized by a finite value of \( \xi \) and \( I_j \) given by their ground state values, Eq. (7). This state is an excitation above the vacuum, which we shall call a magnon. Introducing the so-called shift function \( F(\lambda, \xi) \) by
\[ F(\lambda_j | \xi) = (\lambda_j - \lambda_j)/(\lambda_{j+1} - \lambda_j), \] where \( \lambda_j \) are ground state quasimomenta, and \( \lambda_j \) are those of the excited state, we get the following integral equation for \( F \) in the thermodynamic limit:
\[ F(\lambda | \xi) - \frac{1}{2\pi} \int_{-\Lambda}^\Lambda d\nu K(\lambda, \nu) F(\nu | \xi) = -\pi \rho(\lambda) \] (10)

The momentum of the excited state is
\[ k = \int_{-\Lambda}^\Lambda d\lambda \rho(\lambda) |\pi + \theta(2\lambda - 2\xi)|, \] (11)

and its energy above the ground state is
\[ \omega_-(k) = -\frac{1}{\pi} \int_{-\Lambda}^\Lambda d\lambda \epsilon(\lambda) K(2\lambda - 2\xi). \] (12)

Here \( \omega_-(k) \) is written as a function of the physical (observable) momentum \( k \), which is related to the quasimomentum \( \xi \) by the integral equation (11). The quasienergy \( \epsilon(\lambda) \) is given by the solution of the integral equation
\[ \epsilon(\lambda) - \frac{1}{2\pi} \int_{-\Lambda}^\Lambda d\nu \epsilon(\nu) K(\lambda, \nu) = \lambda^2 - \mu \] (13)

satisfying a condition \( \epsilon(\pm \Lambda) = 0 \). The parameter \( \mu \) entering Eq. (13) is the chemical potential, defined by \( \mu = (\partial E_0 / \partial N)_L \), where \( E_0 \) is found from Eq. (9). One can show that at small \( k \) the dispersion law (12) is parabolic \( \omega_-(k) = k^2/2m_\ast \), with the effective mass satisfying \( m_\ast^{-1} = -(\pi g \rho_0)^{-1} \int_{-\Lambda}^\Lambda d\lambda \epsilon(\lambda) \).

Another way to excite the system is to create a particle-hole pair by moving one of the quantum numbers \( I_j \) outside of the ground state distribution (7). Such excitations are analyzed in detail in \( \Xi \) (see also Ref. [26]). In particular, at small momentum they are shown to be equivalent to sound waves propagating at velocity
\[ v_s = \frac{1}{2\pi \rho(\Lambda)} \frac{\partial \epsilon(\lambda)}{\partial \lambda} \bigg|_{\lambda=\Lambda}. \] (14)
The energy of the magnon is proportional to \( k \).

We introduce an auxiliary function \( \tilde{S} \) we need to combine the BA solution with a continuum \( \tilde{H} \) with the following properties: (i) it conserves the total momentum, which will be denoted by \( q \). (ii) its excitation spectrum at \( q = k \) is gapless. (iii) its structure factor \( \tilde{S} \) satisfies

\[
\frac{\tilde{S}(q, \omega)}{\tilde{S}(q, \omega_-(k) + \omega)} \to 1, \quad q = k, \quad \omega \to 0. \tag{15}
\]

In integrable models \( \tilde{H} \) can be constructed as a linear combination of a finite number of mutually commuting local integrals of motion. The eigenstates \( |f, q\rangle \) of \( \tilde{H} \) are at the same time the eigenstates of \( \tilde{H} \), therefore \( \tilde{S}(q, \omega) = \sum f \delta(\omega - \tilde{E}_f(q))|\langle f, q|s_-(q)|t\rangle|^2 \), where \( \tilde{H}|f, q\rangle = \tilde{E}_f(q)|f, q\rangle \). Like for \( H \), the low energy spectrum of \( \tilde{H} \) consists of sound waves and a magnon. The energy of the magnon is proportional to \( (k - q)^2 \) as \( q \to k \). The condition \( \Delta(k) \) requires that the velocities of the right- and left-moving sound waves be different and given by

\[
v_\pm = v_s \pm \partial \omega_-(k)/\partial k, \tag{16}
\]

where \( v_s \) is given by Eq. \( \text{[14]} \).

The dynamics of sound waves is governed by the Luttinger Hamiltonian

\[
H_0 = \sum_{r=\pm} H_r, \quad H_r = \frac{v_r}{4\pi} \int_0^L dx : [\partial_x \varphi_r(x)]^2 :, \tag{17}
\]

where the operators \( \varphi_r \) are chiral boson fields, \([\varphi_r(x), \varphi_r(x')] = i\pi r \delta_{x, r}\) related to the microscopic particle density by

\[
\rho(x) = \rho_0 + (2\pi)^{-1} \sqrt{K} [\partial_x \varphi_+(x) - \partial_x \varphi_-(x)] \tag{18}
\]

and the symbol \( : \) stands for the boson normal ordering.

In order to describe the low-energy magnon excitation we introduce the spin density field \( \tilde{s}(x) \), related to the microscopic spin density by \( s_\pm(x) = \tilde{s}_\pm(x) + \rho_0/2 \) and \( s_\pm(x) = e^{\pm i k x} \tilde{s}_\pm(x) \), where \( s_\pm = s_s \pm is_y \) are the local spin-ladder operators of Eq. \( \text{[3]} \).

Within the effective theory the operators \( \tilde{s}_\pm \) are smooth spin flip fields. Since a local spin flip may excite sound waves, an effective theory should contain a coupling between \( \tilde{s} \) and \( \varphi_\pm \).

The minimal local coupling respecting the \( SU(2) \) symmetry of the microscopic theory and vanishing in the absence of magnon excitations \( \text{[31]} \) is

\[
H_i = -\sum_{r=\pm} \frac{v_r \beta_r}{2\pi} \int_0^L dx \partial_x \varphi_r(x) \tilde{s}_\pm(x). \tag{19}
\]

Other possible couplings involve higher gradient terms, which do not contribute to the critical exponents. The kinetic energy density of the spin field is represented by a higher gradient term \( \partial_x s_+ \) \( \partial_x \tilde{s}_- \) that can also be neglected in the calculation of the critical exponents \( \text{[28]} \).

The total Hamiltonian of the effective theory describing the dynamics near the threshold is thus given by \( H_{\text{eff}} = H_0 + H_i \). This Hamiltonian is diagonalized by a unitary transformation \( e^{iS} H_{\text{eff}} e^{-iS} \) with

\[
S = (2\pi)^{-1} \int_0^L dx [\beta_+ \varphi_+(x) - \beta_- \varphi_-(x)] \tilde{s}_\pm(x). \tag{20}
\]

What remains is to determine the coupling constants \( \beta_\pm \) in terms of the parameters of the microscopic theory. This is done by the comparison of the low-energy spectrum of the microscopic Hamiltonian \( \tilde{H} \), found from the BA solution, and the spectrum of the effective Hamiltonian \( H_{\text{eff}} \). This procedure yields

\[
\beta_r = 2\pi r F(r \Lambda |\xi|), \quad r = \pm 1, \tag{21}
\]

where \( F \) is defined by the solution of the integral equation \( \text{[10]} \). We solve this equation and find \( \Delta(k) \) numerically for different values of the coupling constant.
γ = g/ρ₀. For easier comparison with Eq. (2) we represent our result in the form
\[
\Delta(k) = -1 + \frac{K}{2} \left( \frac{k}{k_F} \right)^2 + \frac{(K - 1)^2}{K} \alpha(k),
\]
(22)

where k_F = πρ₀ and K = k_F/v_s is the Luttinger parameter calculated using Eq. (14). The function α(k) for different values of the dimensionless coupling constant γ. The values of the Luttinger parameter K are indicated for each curve and correspond in increasing order to γ = ∞, 1.65, 0.56, 0.238 and 0.109 respectively.

The problem considered in the present work is directly related to the X-ray edge problem in the theory of the mobile impurity. In this context, the model (23) was investigated in Ref. [22]. The approach of Ref. [22] exploits a transformation to the co-moving reference frame and combines BA with an effective field theory similar to ours. The method of Ref. [22] has recently been successfully applied to the Heisenberg model and later was shown [19] to produce results equivalent to the method of Ref. [17] used here. A direct comparison of the present work with Ref. [22] is however not possible, because the latter used an incorrect BA solution of the model (23).

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[29] This fact can be explained by symmetry considerations: The Hamiltonian (19) commutes with the total spin lowering operator S... By applying S... to the fully polarized eigenstates of H one gets the eigenstates of the same energy in the sector with S_z = N/2 - 1.
[30] This condition is analogous to Eq. (13) of Ref. [17].
[31] The vanishing of the Hamiltonian (19) in the absence of magnon excitations to the leading order in gradient expansion is ensured by the operator identity ρ(x) = 2s_z(x) valid in the fully polarized sector of the system’s Hilbert.
space and by Eq. (18).