Research Article
Continuous Dependence on a Parameter of Exponential Attractors for Nonclassical Diffusion Equations

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1. Introduction

In this paper, we study the existence and robustness of exponential attractors for the following nonclassical diffusion equation:

$$u_t^\epsilon - \epsilon \Delta u_t^\epsilon - \Delta u^\epsilon + f (u^\epsilon) = g, \quad x \in \Omega, t > 0,$$

(1)

with the initial-boundary value conditions

$$\begin{cases}
u_t^\epsilon (x, 0) = u_0 (x), & x \in \Omega, \\
u_t = 0, & \text{on } \partial \Omega,
\end{cases}$$

(2)

where $\epsilon \in [0, 1]$ and $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded open set with smooth boundary $\partial \Omega$. When $\epsilon = 0$, it turns out to be the classical reaction-diffusion equation. We assume that $g \in L^2 (\Omega)$ and the nonlinearity $f \in C^1 (\mathbb{R}, \mathbb{R})$ satisfies the following (see, e.g., [1]):

(F1): there exists $\nu > 0$ such that $f'' (s) \geq - \nu, \forall s \in \mathbb{R}$

(F2): there exists $k_1 > 0$ such that $|f'' (s)| \leq k_1 (1 + |s|^{2N-2}), \forall s \in \mathbb{R}$

(F3): $\liminf_{s \to -\infty} F(s)/s^2 \geq 0$, where $F(s) = \int_0^s f(r)dr$

(F4): there exists $k_2 > 0$ such that $\liminf_{s \to -\infty} s f (s) - k_2 F(s)/s^2 \geq 0$

Nonclassical diffusion equations appear in fluid mechanics, soil mechanics, and heat conduction theory (see, e.g., [2]). The long-time behavior of solutions to nonclassical diffusion equations has been extensively studied by many authors for both autonomous and nonautonomous cases [3–9].

The global attractor plays an important role in the study of long-time behavior of infinite dimension systems arising from physics and mechanics. It is a compact invariant set and attracts uniformly the bounded sets of the phase space. However, the rate of attraction may be arbitrary, and it may be sensible to perturbations. These drawbacks can be overcome by creating the notion of the exponential attractor [10], which is a compact, positively invariant set of finite dimension and exponentially attracts each bounded set. The existence of the exponential attractor has been extensively studied since 1994, see e.g., [5, 10–17].

As discussed in [12], exponential attractors are to be more robust objects under perturbations than global attractors. In general, global attractors are only upper semicontinuous with respect to perturbations, and the lower semicontinuity property is much more delicate and can be established only for some particular cases. However, one can prove the continuity of exponential attractors under perturbations in many cases [5, 13]. In particular, for problems (1) and (2), the existence of a pullback attractor was shown...
by Anh and Bao in [3] for the subcritical case in $H^1_0(\Omega)$. They also proved the upper semicontinuity of the pullback attractors. However, this upper semicontinuity of the pullback attractor was established only in $L^2(\Omega)$, and the upper semicontinuity in $H^1_0(\Omega)$ remains an open problem. In this paper, we not only prove the upper and lower semicontinuity of the exponential attractor but also show these continuities in a stronger space, i.e., $H^1_0(\Omega)$ when the initial value only belongs to $H^1_0(\Omega)$.

In [12], see also [17], Efendiev et al. gave an abstract result about the robustness of exponential attractors (Theorem 4.4 in [12]). A main assumption called “compact Lipschitz condition” was proposed in that theorem. The main difficulty when we apply this result to problems (1) and (2) is that the solution to problems (1) and (2) has no regularity as $\epsilon > 0$ [3]. For example, if the initial datum $u_0$ belongs to $H^1_0(\Omega)$, the solution with initial $u(0) = u_0$ is always in $H^1_0(\Omega)$ and has no higher regularity. Thus, it is impossible to verify the “compact Lipschitz condition” when we want to prove the continuity of exponential attractors in $H^1_0(\Omega)$. Motivated by [18], we modify the result in [12] to adapt to our case. Moreover, some of the coefficients are allowed to be dependent on the parameter $\epsilon$ which relax the conditions, see Theorem 1 in the following.

The rest of this paper is organized as follows. In Section 2, we formulate and prove the main abstract result, i.e., Theorem 1. In Section 3, we apply Theorem 1 to the dynamical system generated by (1) and (2) to prove the continuity of the exponential attractors, and we consider two cases according to the constant $\nu$ in this section.

Throughout this paper, we denote by $\| \cdot \|_X$ the norm of a Banach space $X$. The inner product and norm of $L^2(\mathbb{R}^n)$ are written as $(\cdot, \cdot)$ and $\| \cdot \|$, respectively. We also use $\| u \|_r$ to denote the norm of $u \in L^r(\mathbb{R}^n)$ ($r \geq 1, r \neq 2$) and $|u|$ to denote the modular of $u$. Letter $c$ is a generic positive constant independent of $\epsilon$ which may change its values from line to line even in the same line (sometimes for the special case, we also denote different positive constants by $c_i (i = 1, 2, \ldots)$).

### 2. The Abstract Result and Its Proof

In this section, we modify Theorem 4.4 in [12] to adapt to problems (1) and (2). We start with the definition of exponential attractors.

**Definition 1.** Let $E$ be a metric space, $B$ is a bounded set in $E$, and let $S : B \rightarrow B$ be a map. We define a discrete semigroup $\{S^n, n \in \mathbb{Z}^+\}$ by $S^n : x = S^{n-1} \circ \cdots \circ S(x)$ $(n \text{ times})$. A set $\mathcal{M} \subset B$ is an exponential attractor for the map if the following properties hold:

1. The set $\mathcal{M}$ is compact in $E$ and has finite fractal dimension.
2. The set $\mathcal{M}$ is positively invariant with respect to $S$, i.e., $S(\mathcal{M}) \subset \mathcal{M}$.
3. There exist positive constants $\alpha_0$ and $\beta_0$ such that

$$\text{dist}_E(S^nB, \mathcal{M}) \leq \alpha_0 \epsilon^{-\beta_0 n},$$

where $\text{dist}_E(C_1, C_2)$ denotes the Hausdorff semidistance between $C_1$ and $C_2$ in $E$ given by

$$\text{dist}_E(C_1, C_2) = \sup_{x \in C_1, y \in C_2} \| x - y \|_{E},$$

for $C_1, C_2 \subset E$. (4)

**Definition 2.** Let $X$ be a complete metric space endowed with the metric $d$ and $M$ be a bounded closed set in $X$. Assume that $\varrho$ is a pseudometric [19] defined on $M$. Let $B \subset M$ and $\epsilon > 0$.

1. A subset $\mathcal{H}$ in $B$ is said to be $(\epsilon, \varrho)$-distinguishable if $\varrho(x, x') > \epsilon$ for any $x, x' \in \mathcal{H}, x \neq x'$. We denote by $m_\varrho(B, \epsilon)$ the maximal cardinality of an $(\epsilon, \varrho)$-distinguishable subset of $B$.
2. Pseudometric $\varrho$ is said to be compact on $M$ iff $m_\varrho(M, \epsilon)$ is finite for every $\epsilon > 0$.
3. For any $r > 0$, we define a local $(r, \epsilon, \varrho)$-capacity of set $M$ by the formula

$$\mathcal{C}_\varrho(M; r, \epsilon) = \sup \{ \ln m_\varrho(B, \epsilon) : B \subset M, \text{diam}B \leq 2r \}.$$  

(5)

We now state and prove the main abstract theorem. The proof is essentially a combination of that in [18] and that of [12].

**Theorem 1.** Let $X$ be a Banach space and $B$ be a bounded set in $X$. We assume that there exists a family of operators $S_\epsilon : B \rightarrow B, \epsilon \in [0, \epsilon_0]$, which satisfy, for any $\epsilon \in [0, \epsilon_0]$,

1. $S_\epsilon$ is Lipschitz on $B$, i.e., there exists $L_\epsilon > 0$ such that

$$\|S_\epsilon x_1 - S_\epsilon x_2\|_X \leq L_\epsilon \|x_1 - x_2\|_X, \quad \forall x_1, x_2 \in B.$$  

(6)

2. There exist constants $\eta$ and $K_\epsilon$ and compact seminorms $n_\epsilon^1(x)$ and $n_\epsilon^2(x)$ on $B$ such that

$$\|S_\epsilon x_1 - S_\epsilon x_2\|_X \leq \eta \|x_1 - x_2\|_X + K_\epsilon [n_\epsilon^1(x_1 - x_2) + n_\epsilon^2(S_\epsilon x_1 - S_\epsilon x_2)].$$  

(7)

for any $x_1, x_2 \in B$, where $0 < \eta < 1$ is independent of $\epsilon$ and $K_\epsilon > 0$ is a constant which may be dependent on $\epsilon$ (seminorm $n(x)$ on $B$ is said to be compact iff for any $\epsilon > 0$ there exists a sequence $\{x_n\} \subset B'$ such that $n(x_m - x_n) \rightarrow 0$ as $n, m \rightarrow \infty$).

3. $\forall \epsilon \in [0, \epsilon_0], \forall \nu \in \mathbb{N}, \forall x \in B$,

$$\|S_\epsilon^\nu x - S_\epsilon^\nu y\|_X \leq \kappa \epsilon^\alpha,$$

(8)

where $\kappa > 0$ and $\alpha \in (0, 1)$ are constants independent of $\epsilon$ and for mapping $V$, $V' = V \circ \cdots \circ V$ ($\nu$ times).

Then, $\forall \epsilon \in [0, \epsilon_0]$, and the discrete dynamical system generated by the iterations of $S_\epsilon$ possesses an exponential attractor $\mathcal{M}_\epsilon$ on $B$ such that
(1) The fractal dimension of $\mathcal{M}_\epsilon$ is bounded in $X$:

$$\dim_{\mathcal{X}} \mathcal{M}_\epsilon \leq c(\epsilon) := \left[ \frac{2}{\ln 2} \ln m_\epsilon \left( \frac{4K_\epsilon (1 + L_\epsilon^2)^{1/2}}{1 - \eta} \right) \right]^{-1},$$

where $\dim_{\mathcal{X}} \mathcal{A}$ denotes the fractal dimension of $\mathcal{A}$ in $X$ and $m_\epsilon(R)$ is the maximal number of pairs $(x_i, y_i)$ in $X \times X$ possessing the properties

$$\|x_i\|_X^2 + \|y_i\|_X^2 \leq R^2, n_1^2(\|x_i - x_j\|) + n_2^2(\|y_i - y_j\|) > 1, \quad i \neq j.$$

(10)

(2) $\mathcal{M}_\epsilon$ attracts $B$ in $X$, uniformly with respect to $\epsilon$,

$$\text{dist}_X(S^n_\epsilon B, \mathcal{M}_\epsilon) \leq c_1 \epsilon^{-c_2}, \quad c_2 > 0, i \in \mathbb{N},$$

where $c_1$ and $c_2$ are independent of $\epsilon$.

(3) The family $\{\mathcal{M}_\epsilon, \epsilon \in [0, \epsilon_0]\}$ is continuous at 0:

$$\text{dist}_{\text{sym}, X}(\mathcal{M}_\epsilon, \mathcal{M}_0) \leq c_3 \epsilon^{c_4},$$

where $c_3$ and $c_4 \in (0, 1)$ are independent of $\epsilon$ and $\text{dist}_{\text{sym}, X}$ denotes the symmetric Hausdorff distance in $X$ between sets defined by

$$\text{dist}_{\text{sym}, X}(A, B) := \max \{\text{dist}_X(A, B), \text{dist}_X(B, A)\}.$$

Proof. For any fixed $\epsilon \in [0, \epsilon_0]$, we set $q_\epsilon(x, y) = K_\epsilon[n_1^2(x - y) + n_2^2(S_\epsilon x - S_\epsilon y)]$; then, $q_\epsilon$ is compact on $B$ in the sense of Definition 2. From [18], we see that the local $(r, \rho, q_\epsilon)$-capacity of the set $B$ admits the estimate

$$\mathcal{E}(B; r, \rho) \leq m_\epsilon \left( \frac{2K_\epsilon (1 + L_\epsilon^2)^{1/2}}{\rho} \right),$$

where $m_\epsilon(R)$ is the maximal number of pairs $(x_i, y_i)$ in $X \times X$ possessing the properties

$$\|x_i\|_X^2 + \|y_i\|_X^2 \leq R^2, n_1^2(\|x_i - x_j\|) + n_2^2(\|y_i - y_j\|) > 1, \quad i \neq j.$$

(15)

We assume $\text{diam} B = 2R$. Let $\delta = 1 - \eta/2$ and $\{x_i: i = 1, \ldots, n_1\}$ be a maximal $(\delta R, q_\epsilon)$-distinguishable subset of $B$. Then, from (14), we have

$$n_1 = m_\epsilon(B; \delta R) \leq \exp \left[ \mathcal{E}(B; R, \delta R) \right] \leq m_\epsilon \left( \frac{2K_\epsilon (1 + L_\epsilon^2)^{1/2}}{\delta} \right) := P_\epsilon,$$

(16)

$$B = \bigcup_{i=1}^{n_1} B_{i,1},$$

$$B_{i,1} = \{v \in B: q_\epsilon(v, x_i) \leq \delta R\}.$$

Therefore,

$$S_\epsilon B = \bigcup_{i=1}^{n_1} S_\epsilon B_{i,1}. \quad \quad (17)$$

If $y_1, y_2 \in B_{i,1}$, then from (7), we have

$$\|S_\epsilon y_1 - S_\epsilon y_2\|_X \leq \eta \|y_1 - y_2\|_X + \rho_\epsilon(y_1, x_i) + \rho_\epsilon(y_2, x_i),$$

(18)

where $\eta = \delta + 1 + \eta/2 < 1$. We set $E_\epsilon^0 = \{x_i\}_{i=1}^{n_1}$ and $E_\epsilon = S_\epsilon E_\epsilon^0$, then,

$$E_\epsilon^1 \subset S_\epsilon B, S_\epsilon E_\epsilon^0 \subset E_\epsilon^1,$$

$$\mathcal{E}(E_\epsilon^1; P_\epsilon, \delta R) \leq 2R \delta R.$$  \quad \quad (19)

Next, for any fixed $i_1 \in \{1, \ldots, n_1\}$, we assume that $\{x_{i_1,i_2}, i_2 = 1, \ldots, n_{i_2}\}$ is a maximal $(\delta R, q_\epsilon)$-distinguishable subset of $S_\epsilon B_{i_1,1}$. Then,

$$n_{i_1,i_2} = m_\epsilon(B_{i_1,1}; \delta R) \leq \exp \left[ \mathcal{E}(B_{i_1,1}; \delta R, \delta R) \right] \leq \exp \left[ \mathcal{E}(B; \delta R, \delta R) \right] \leq P_\epsilon,$$

(20)

$$S_{i_1,i_2} B = \bigcup_{i_1=1}^{n_1} \bigcup_{i_2=1}^{n_{i_2}} B_{i_1,i_2} = \{v \in S_{i_1,i_2} B: q_\epsilon(v, x_{i_1,i_2}) \leq \delta R\}.$$  \quad \quad (13)

Therefore,

$$S_{i_1,i_2}^2 B = \bigcup_{i_1=1}^{n_1} \bigcup_{i_2=1}^{n_{i_2}} S_{i_1,i_2} B_{i_1,i_2}. \quad \quad (21)$$

If $y_1, y_2 \in B_{i_1,i_2}$, then from (7), we have

$$\|S_{i_1,i_2} y_1 - S_{i_1,i_2} y_2\|_X \leq \eta \|y_1 - y_2\|_X + \rho_\epsilon(y_1, x_{i_1,i_2}) + \rho_\epsilon(y_2, x_{i_1,i_2}) \leq 2qR \delta R + 2\delta q R = 2q_\epsilon R.$$  \quad \quad (22)

We set $E_{i_1}^2 = S_{i_1,i_2} E_{i_1}^0 \bigcup \{S_{i_1,i_2} x_{i_1,i_2}\}$, and then we have

$$E_{i_1}^2 \subset S_{i_1,i_2}^2 B,$$

$$S_{i_1,i_2} E_{i_1}^0 \subset E_{i_1}^2,$$

$$\mathcal{E}(E_{i_1}^2; P_{i_1}^2, \delta R) \leq 2q_\epsilon R.$$  \quad \quad (23)

By the induction procedure, we can find sets $E_{i_1}^k$, $i \in \mathbb{N}$, enjoy the following properties:

$$E_{i_1}^k \subset S_{i_1,i_2}^k B,$$

$$S_{i_1,i_2}^k E_{i_1}^0 \subset E_{i_1}^{k+1},$$

$$\mathcal{E}(E_{i_1}^{k+1}; P_{i_1}^{k+1}, \delta R) \leq 2q_\epsilon R.$$  \quad \quad (26)

We now define the exponential attractor for the map $S_0 : B \to B$ as follows:
\[ M_0' = \bigcup_{i=1}^{\infty} E_i \subseteq M_0 \rightarrow M_0'. \]

Then, from (24)–(27), we see that \( M_0 \) is indeed an exponential attractor for the map \( S_0: B \rightarrow B \) (see [12]). For \( \epsilon \in (0, \epsilon_0) \), one can also construct exponential attractors for \( S_\epsilon: B \rightarrow B \) as above. However, they are not the ones we needed. Totally similar to [12], one can construct exponential attractors \( M_\epsilon \) based on \( E_0 \). We note that the only difference between our construction procedure and that of in [12] is that the number \( P_\epsilon \) may be dependent on \( \epsilon \). However, (26) only contributes to the fractal dimension of \( M_\epsilon \). Therefore, (11) and (12) in Theorem 1 hold true. Finally, assertion (9) is a direct result of [18]. The proof is completed.

Remark 1. Similar to [12], we can give an explicit expression for \( \epsilon_1 \), i.e., \( \epsilon_1 = -\alpha \ln q/\ln \kappa - \ln q \). If \( L_\epsilon, N_\epsilon, n_\epsilon^j(x) \), and \( n_\epsilon^h(x) \) are all independent \( \epsilon \), we can obtain the uniform bound with respect to \( \epsilon \) of the fractal dimension, and we will apply this abstract result in the last section.

3. Application to the Nonclassical Diffusion Equation

3.1. Some Useful Estimates of the Solution. Since \((-\Delta)^{-1}\) is a continuous compact operator in \( L^2(\Omega) \), by the classical spectral theorem, there exist a sequence \( \{\lambda_i\}_{i=1}^{\infty}, \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_j \rightarrow \infty \) as \( j \rightarrow \infty \) and a family of elements \( \{e_j\}_{j=1}^{\infty} \) of \( D(-\Delta) \), which forms an orthogonal basis in both \( L^2(D) \) and \( H^1_0(D) \) such that \(-\Delta e_j = \lambda_j e_j, \forall j \in N \).

Given \( m \), let \( X_m = \text{span}\{e_1, \ldots, e_m\} \) and \( P_m: L^2(D) \rightarrow X_m \) be the projection operator. For any \( \nu \in H_0^1(D) \), we write \( \nu = P_m \nu + (I - P_m)\nu =: \nu_1 + \nu_2 \).

Lemma 1 (see [1]). Let \( \epsilon > 0 \). Assume (F1)–(F4) hold and \( g \in L^1(\Omega) \). Then, for each \( \nu_0 \in H_0^1(\Omega) \), problems (1) and (2) have a unique solution \( \nu_\epsilon = \nu_\epsilon(t) = \nu_\epsilon(t; \nu_0) \) with \( \nu_\epsilon \in C[0,T], H^1_0(\Omega) \) and \( \nu_\epsilon^2 \in L^1[0,T; H^1_0(\Omega)] \) for any \( T > 0 \). Moreover, for any fixed \( t > 0 \), \( \nu_\epsilon \) is continuous in \( \nu_0 \).

Lemma 2 (see [1]). Let \( \epsilon = 0 \). Assume (F1)–(F4) hold and \( g \in L^1(\Omega) \). Then, for each \( \nu_0 \in H_0^1(\Omega) \), problems (1) and (2) have a unique solution \( \nu_0 = \nu_0(t) = \nu_0(t; \nu_0) \) which satisfies \( \nu_0 \in C[0,T], H^1_0(\Omega) \) and \( \nu_0^2 \in L^2[0,T; H^1_0(\Omega)] \) for any \( T > 0 \).

Also for any fixed \( t > 0 \), \( \nu_0 \) is continuous in \( \nu_0 \).

From Lemmas 1 and 2, we can define a semigroup \( S_\epsilon(t): H^1_0(\Omega) \rightarrow H^1_0(\Omega) \) by the expression

\[ S_\epsilon(t)\nu_0 = \nu_\epsilon(t), \quad t \geq 0, \]

where \( \nu_\epsilon(t) \) is the solution of (1) and (2).

Lemma 3 (see [1]). Assume (F1)–(F4) hold and \( g \in L^2(\Omega) \). Then, for any bounded set \( D \subset H^1_0(\Omega) \), there exist positive constants \( E(D), c_0 \), and \( T_1(D) \) such that, for any solution \( \nu_\epsilon \) of problems (1) and (2),

\[ \|\nabla \nu_\epsilon(t)\| \leq E(D), \quad t \geq 0, \]

\[ \|\nabla \nu_\epsilon(t)\| \leq c_0, \quad t \geq T_1(D), \]

provided \( \nu_0 \in D \), where \( E(D), c_0 \), and \( T_1(D) \) are independent of \( \epsilon \).

Set \( B = \{ \nu \in H^1_0(\Omega): \|\nu\|_{H^1_0(\Omega)} \leq c_0 \} \), where \( c_0 \) is the constant in Lemma 3; then, \( B \) is a uniformly bounded absorbing set for \( S_\epsilon(t) \) in \( H^1_0(\Omega) \). We note that the absorbing time is independent of \( \epsilon \), and we can choose \( T_1 = T_1(B) \) large enough such that \( S_\epsilon(t)B \subset B \) for any \( t \geq T_0 \) and any \( \epsilon \in [0,1] \).

The following lemma gives several priori estimates for the derivative \( \nu_\epsilon'(t) \) of the solution to (1) and (2).

Lemma 4. Under the assumptions (F1)–(F4), for any \( D \subset H^1_0(\Omega) \), \( D \) is bounded, and there exists \( T_2(D) \), which is independent of \( \epsilon \) such that, for all \( \epsilon \in [0,1] \),

\[ \|\nu_\epsilon'(t)\|^2 + \epsilon\|\nabla \nu_\epsilon'(t)\|^2 \leq c, \]

\[ \int_0^{T} \|\nabla \nu_\epsilon(s)\|^2 \, ds \leq c, \]

for any \( \nu_0 \in D \) and any \( t \geq T_2(D) \), where \( c \) is a constant independent of \( \epsilon \).

Proof. We first take the inner product of (1) with \( \nu_\epsilon'(t) \) in \( L^2(\Omega) \) to get

\[ \frac{1}{2} \frac{d}{dt} \|\nabla \nu_\epsilon\|^2 + \epsilon\|\nabla \nu_\epsilon'\|^2 + \|\nabla \nu_\epsilon\|^2 + \|f(\nu_\epsilon), \nu_\epsilon'\| \leq (g, \nu_\epsilon'). \]

Putting (35) into (34), it yields that

\[ \frac{d}{dt} \|\nabla \nu_\epsilon\|^2 + \epsilon\|\nabla \nu_\epsilon'\|^2 + \|\nabla \nu_\epsilon\|^2 + \|f(\nu_\epsilon), \nu_\epsilon'\| \leq 2(g, \nu_\epsilon') + 2\theta\|\nabla \nu_\epsilon\|^2 + 2\epsilon\theta. \]

By condition (F3), for any \( \gamma > 0 \), there exists \( c_\gamma > 0 \) satisfying

\[ \int_\Omega F(u) + \gamma\|u\|^2 + c_\gamma \geq 0, \quad \forall u \in H^1_0(\Omega). \]

Combining (36) and (37), we get

\[ \frac{d}{dt} \|\nabla \nu_\epsilon'\|^2 + \epsilon\|\nabla \nu_\epsilon''\|^2 + 2\|\nabla \nu_\epsilon''\|^2 \leq 2(g, \nu_\epsilon') + 2\theta\|\nabla \nu_\epsilon'\|^2 + 2\epsilon\theta. \]
We choose $\alpha_1 > 0$ small enough such that $\alpha_1 \lambda_1 < 1$ and $1 - \alpha_1 > 1/2$, and apply Poincare inequality in (39) to get

$$\frac{d}{dt} \left[ \|u_t\|^2 + \epsilon \|\nabla u_t\|^2 \right] + \|u_t\|^2 \leq 2(g, u_t) + 2\theta \|u_t\|^2 + 2c_\theta \|u_t\|^2 + 2c_\theta \epsilon. \quad (39)$$

Choosing $\theta, \gamma$ small enough and using H"{o}lder inequality, we can get from (40) that

$$\frac{d}{dt} \left[ \|u_t\|^2 + \epsilon \|\nabla u_t\|^2 \right] + \alpha_1 \lambda_1 \|u_t\|^2 + (1 - \alpha_1) \|u_t\|^2 \leq 2(g, u_t) + 2\theta \|u_t\|^2 + 2c_\theta + 2\kappa_2 \gamma \|u_t\|^2 + 2\kappa_2 \epsilon. \quad (40)$$

We set $\sigma = \alpha_1 \lambda_1 / 2$; then, $0 < \sigma < 1/2$. From (41), we obtain

$$\frac{d}{dt} \left[ \|u_t\|^2 + \epsilon \|\nabla u_t\|^2 \right] + \frac{\alpha_1 \lambda_1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u_t\|^2 \leq c. \quad (41)$$

Integrating the above inequality from $0$ to $t$, it yields that

$$\epsilon_t \left[ \|u(t)\|^2 + \epsilon \|\nabla u(t)\|^2 \right] \leq \left[ \|u_0\|^2 + \epsilon \|\nabla u_0\|^2 \right] + ce^{\sigma t}. \quad (43)$$

Now, we consider (36) again. If $0 < \kappa_2 \leq 1$, we can get from (36) that

$$\frac{d}{dt} \left[ \|u_t\|^2 + \epsilon \|\nabla u_t\|^2 \right] + \|\nabla u_t\|^2 + \kappa_2 \|\nabla u_t\|^2 + 2 \int_\Omega F(u_t) \leq 2(g, u_t) + 2\theta \|u_t\|^2 + 2c_\theta. \quad (45)$$

Using a similar process as (39)–(42), we deduce that, for $\theta$ small enough,

$$\frac{d}{dt} \left[ \|u_t\|^2 + \epsilon \|\nabla u_t\|^2 \right] + \|\nabla u_t\|^2 + \kappa_2 \|\nabla u_t\|^2 + 2 \int_\Omega F(u_t) \leq c. \quad (46)$$

Therefore,

$$\frac{d}{dt} e^{\sigma t} \left[ \|u_t\|^2 + \epsilon \|\nabla u_t\|^2 \right] + \|\nabla u_t\|^2 + \kappa_2 \|\nabla u_t\|^2 + 2 \int_\Omega F(u_t) \leq c e^{\sigma t}. \quad (47)$$

If $\kappa_2 > 1$, we can get from (36) that

$$\frac{d}{dt} \left[ \|u_t\|^2 + \epsilon \|\nabla u_t\|^2 \right] + \|\nabla u_t\|^2 + 2 \int_\Omega F(u_t) \leq c e^{\sigma t} + 2 \int_\Omega F(u_t) \leq c.$$

Applying H"{o}lder inequality to the above we get

$$\|u_t\|^2 + \epsilon \|\nabla u_t\|^2 + \frac{d}{dt} \|\nabla u_t\|^2 + 2 \int_\Omega F(u_t) \leq c, \quad t > 0. \quad (56)$$

Thus,

$$\frac{d}{dt} \left[ \|u_t\|^2 + \epsilon \|\nabla u_t\|^2 \right] + \|\nabla u_t\|^2 + 2 \int_\Omega F(u_t) \leq (g, u_t) + 2\theta \|u_t\|^2 + 2c_\theta. \quad (48)$$

Applying (37) to the above, we obtain

$$\frac{d}{dt} \left[ \|u_t\|^2 + \epsilon \|\nabla u_t\|^2 \right] + \|\nabla u_t\|^2 + \left[ \|u_t\|^2 + 2 \int_\Omega F(u_t) \right] \leq 2(g, u_t) + 2\theta \|u_t\|^2 + 2c_\theta + (2\kappa_2 - 2) \|u_t\|^2 + (2\kappa_2 - 2) c_\gamma. \quad (49)$$

Therefore,

$$\frac{d}{dt} e^{\sigma t} \left[ \|u_t\|^2 + \epsilon \|\nabla u_t\|^2 \right] + \|\nabla u_t\|^2 + 2 \int_\Omega F(u_t) \leq c e^{\sigma t}. \quad (51)$$

From (47) and (51), we see that, for any $\kappa_2 > 0$,

$$\frac{d}{dt} e^{\sigma t} \left[ \|u_t\|^2 + \epsilon \|\nabla u_t\|^2 \right] + \kappa_2 e^{\sigma t} \left[ \|u_t\|^2 + 2 \int_\Omega F(u_t) \right] \leq c e^{\sigma t}.$$

where $c_5 = \kappa_2$ when $0 < \kappa_2 \leq 1$ and $c_5 = 1$ when $\kappa_2 > 1$. For any $t > 0$, we integrate (52) over $[t, t + 1]$ to get

$$\int_t^{t+1} e^{\sigma s} \left[ \|u_t\|^2 + 2 \int_\Omega F(u_t) \right] \leq c e^{\sigma t} \left[ \|u_t\|^2 + \epsilon \|\nabla u_t\|^2 \right] + c e^{\sigma t}. \quad (53)$$

Putting (44) into (53), we have, for any $t > 0$,

$$\int_t^{t+1} e^{\sigma s} \left[ \|u_t\|^2 + 2 \int_\Omega F(u_t) \right] \leq c \left[ \|u_0\|^2 + \epsilon \|\nabla u_0\|^2 \right] + c e^{\sigma t}. \quad (54)$$

Next, we multiply (1) with $u_t$ and integrate it in $\Omega$ to get

$$\|u_t\|^2 + \epsilon \|\nabla u_t\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla u_t\|^2 + 2 \int_\Omega F(u_t) = (g, u_t). \quad (55)$$

Thus,
\[
\begin{align*}
&\epsilon^2 \left[ \|u^\epsilon_t\|^2 + \epsilon \|\nabla u^\epsilon_t\|^2 \right] + \frac{d}{dt} \epsilon^2 \left[ \|\nabla u^\epsilon_t\|^2 + 2 \int_\Omega F(u^\epsilon_t) \right] \\
&\leq c \epsilon^2 + c \epsilon^2 \left[ \|\nabla u^\epsilon_t\|^2 + 2 \int_\Omega F(u^\epsilon_t) \right], \quad t > 0.
\end{align*}
\]

Dropping the first term in the left-hand side of (57), from (54) and using uniform Gronwall inequality, we conclude that
\[
\epsilon^2 \left[ \|u^\epsilon_t(t)\|^2 + 2 \int_\Omega F(u^\epsilon_t(t)) \right] \leq c \left[ \|u_0\|^2 + \epsilon \|\nabla u_0\|^2 \right] + c \epsilon^2, \quad t \geq 1.
\]

We now differentiate equation (1) with respect to \(t\) to get
\[
u^\epsilon_t - c \Delta u^\epsilon_t - \Delta u^\epsilon_t + f'(u^\epsilon) u^\epsilon_t = 0.
\]

Taking the inner product of (59) with \(u^\epsilon_t\) in \(L^2(\Omega)\), we obtain
\[
\frac{d}{dt} \left[ \|u^\epsilon_t\|^2 + \epsilon \|\nabla u^\epsilon_t\|^2 \right] + 2 \|\nabla u^\epsilon_t\|^2 \leq c \|u^\epsilon_t\|^2.
\]

Since \(0 < \sigma < 1/2\), we have
\[
\frac{d}{dt} \left[ \|u^\epsilon_t\|^2 + \epsilon \|\nabla u^\epsilon_t\|^2 \right] + \sigma \left[ \|u^\epsilon_t\|^2 + \epsilon \|\nabla u^\epsilon_t\|^2 \right] \leq c \|u^\epsilon_t\|^2.
\]

Hence,
\[
\frac{d}{dt} \epsilon^2 \left[ \|u^\epsilon_t\|^2 + \epsilon \|\nabla u^\epsilon_t\|^2 \right] \leq c \epsilon^2 \left[ \|u^\epsilon_t\|^2 + \epsilon \|\nabla u^\epsilon_t\|^2 \right].
\]

Integrating (57) over \((t, t + 1)\) and using (58), we deduce that
\[
\left\| u^\epsilon_t(t) \right\| + \epsilon \left\| \nabla u^\epsilon_t(t) \right\| \leq c_\epsilon \left\| u_0 \right\| + \epsilon \left\| \nabla u_0 \right\| + c \epsilon^2, \quad t \geq 1.
\]

Combining (62) and (63) and using the uniform Gronwall inequality, we obtain that
\[
\left\| u^\epsilon_t(t) \right\|^2 + \epsilon \left\| \nabla u^\epsilon_t(t) \right\|^2 \leq c \left[ \|u_0\|^2 + \epsilon \|\nabla u_0\|^2 \right] + c, \quad t \geq 2.
\]

Thus, for any bounded \(D \subset H^1_0(\Omega)\), there exists \(T_2(D) \geq 2\) which is independent of \(\epsilon\) such that, for any \(t \geq T_2(D)\) and any \(u_0 \in D\),
\[
\left\| u^\epsilon_t(t) \right\|^2 + \epsilon \left\| \nabla u^\epsilon_t(t) \right\|^2 \leq c,
\]
where \(c\) is independent of \(\epsilon\). This proves assertion (32) in Lemma 4.

Finally, integrating (60) over \((t, t + 1)\), then assertion (33) can be easily verified by using (32). The proof is completed.

**Lemma 5.** Under conditions (F1)–(F4), for any two solutions of (1) and (2) \(u^{\epsilon_1}(t), u^{\epsilon_2}(t)\) (corresponding to \(\epsilon_1, \epsilon_2 \in [0, 1]\), respectively), with initials starting from \(B\) and for any \(v \in L^2(\Omega)\), we have
\[
\left\| \int_\Omega \left[ f[u^{\epsilon_1}(t)] - f[u^{\epsilon_2}(t)] \right] v \right\| \leq c \left\| \nabla u^{\epsilon_1, \epsilon_2}(t) \right\| \|v\|, \quad t \geq 0,
\]
where \(u^{\epsilon_1, \epsilon_2}(t) = u^{\epsilon_1}(t) - u^{\epsilon_2}(t)\) and \(c_\epsilon\) is a constant independent of \(\epsilon\).

**Proof.** By (F2), we have
\[
\left\| \int_\Omega \left[ f[u^{\epsilon_1}(t)] - f[u^{\epsilon_2}(t)] \right] v \right\| \leq c \left\| \nabla u^{\epsilon_1, \epsilon_2}(t) \right\| \|v\|,
\]
where \(l(t, \epsilon_1, \epsilon_2) = \int_0^t f'[\epsilon u^{\epsilon}(t) + (1 - s)u^{\epsilon_2}(t)] ds\).

From (31) and the Sobolev embedding \(H^1_0(\Omega) \subset L^{2N/2}(\Omega)\), we have that, for \(t \geq 0\),
\[
\left\| \int_\Omega \left[ f[u^{\epsilon_1}(t)] - f[u^{\epsilon_2}(t)] \right] v \right\| \leq c \left\| \nabla u^{\epsilon_1, \epsilon_2}(t) \right\| \|v\|
\]

\[
\leq c \epsilon \left\| \nabla u^{\epsilon_1, \epsilon_2} \right\| \|v\| + c \left\| \nabla u^{\epsilon_1} \right\| \left\| \nabla u^{\epsilon_2} \right\| \|v\| + c \left\| \nabla u^{\epsilon_1} \right\| \left\| \nabla u^{\epsilon_2} \right\| \|v\| \leq c \epsilon \left\| \nabla u^{\epsilon_1, \epsilon_2} \right\| \|v\|.
\]
The proof is completed.

3.2. The Main Result for a General Case: \( n \in \mathbb{R}^+ \). Now, we verify the conditions in Theorem 1 for \( S_e(t) \) in this case. We first verify condition (i), i.e., the Lipschitz continuity in \( H^1 \) (actually uniform Lipschitz continuity in \( H^1 \) since the coefficient is independent of \( e \)).

**Lemma 6.** Under assumptions (F1)–(F4), we have, for any \( e \in [0, 1] \) and any \( x_1, x_2 \in B \),

\[
\|S_e(t)x_1 - S_e(t)x_2\|_{H^1(\Omega)} \leq c_6(t)\|x_1 - x_2\|_{H^1(\Omega)}, \quad t \geq 0,
\]

where \( c_6 \) is the constant in Lemma 5.

**Proof.** Assume that \( u^e_1(t) \) and \( u^e_2(t) \) are two solutions of (1) and (2) starting from \( x_1, x_2 \in B \), respectively. We consider the difference \( w^e(t) = u^e_1(t) - u^e_2(t) \), and then \( w^e(t) \) satisfies

\[
w^e(t) - \varepsilon \Delta w^e(t) - \Delta w^e(t) + l(t, e)w^e(t) = 0.
\]

Multiplying (70) with \( w^e(t) \) and integrating it in \( \Omega \), we get

\[
\|w^e(t)\|^2 + \frac{1}{2} \frac{d}{dt}\|w^e(t)\|^2 \leq \int_{\Omega} |l(t, e)w^e(t)|^2 dx.
\]

Applying Lemma 5, we obtain

\[
\|w^e(t)\|^2 + \frac{1}{2} \frac{d}{dt}\|w^e(t)\|^2 \leq c_6\|w^e(t)\|_{H^1(\Omega)}^2, \quad t \geq 0.
\]

We use Hölder inequality to get

\[
\frac{d}{dt}\|w^e(t)\|^2 \leq c_6^2\|w^e(t)\|^2.
\]

Then, the result follows immediately from Gronwall inequality. The proof is completed.

**Lemma 7.** Under assumptions (F1)–(F4), we have, for any \( x_1, x_2 \in B \) and any \( \eta \in (0, 1) \),

(i) For \( e \in (0, e_0] \): there exist \( T_3 = T_3(\eta) \), \( K > 0 \), and compact seminorm \( n_{1}^{\eta} \) on \( B \) such that

\[
\|S_e(t)x_1 - S_e(t)x_2\|_{H^1(\Omega)} \leq \eta\|x_1 - x_2\|_{H^1(\Omega)} + Kn_{1}^{\eta}(x_1 - x_2), \quad t \geq T_3.
\]

(ii) For \( e = 0 \): there exist a positive integer \( M \), \( T_3 = T_3(\eta) \), and \( K > 0 \) such that

\[
\|S_0(t)x_1 - S_0(t)x_2\|_{H^1(\Omega)} \leq \eta\|x_1 - x_2\|_{H^1(\Omega)} + Kn_{1}(S_0(t)x_1 - S_0(t)x_2), \quad t \geq T_3,
\]

for some \( e_0 > 0 \), where \( n_{1}(x) = \|P_{Mx}\|_{H^1(\Omega)} \). \( M \) and \( K \) are independent of \( e, \eta, \) and \( t \) and \( n_{1}^{\eta} \) depend on \( e \) and \( t \).

**Proof.** We first take the inner product of (70) with \( kw^e \) in \( L^2(\Omega) \) (\( k \) is a constant which will be fixed later) to get

\[
k\langle w^e, w^e \rangle + k\Delta \|w^e\|^2 + k\int_{\Omega} l(t, e)|w^e|^2 = 0.
\]

Using (F1), we get

\[
k\|w^e\|^2 \leq k\|\nabla w^e\|^2 + k\|\nabla w^e\|^2 + k\|\nabla w^e\|^2.
\]

Combining (72) and (77), we get

\[
\|w^e\|^2 \leq k\|\nabla w^e\|^2 + k\|\nabla w^e\|^2 + c_6\|\nabla w^e\|\|w^e\|
\]

For the right-hand side of (78), we use Hölder inequality to get

\[
k\|\nabla w^e\|^2 + \Delta \|w^e\|^2 + k\|\nabla w^e\|^2 + c_6\|\nabla w^e\|\|w^e\|
\]

Putting the above inequality into (78), we obtain

\[
\frac{d}{dt}\|w^e\|^2 + 2k\|\nabla w^e\|^2 \leq (2k + k^2)\|w^e\|^2 + (k^2 + c_6^2)\|\nabla w^e\|^2.
\]

Fix \( k = c_6^2 \), and then choose \( e_0 \) such that \( e < 1/c_6^2 \), \( \forall e \leq e_0 \). Then, we get from (80) that

\[
\frac{d}{dt}\|w^e\|^2 + c_6\|\nabla w^e\|^2 \leq c_6\|w^e\|^2, \quad e \in [0, e_0], c_7 > 0,
\]

that is,

\[
\frac{d}{dt}e^{c_6t}\|w^e\|^2 \leq ce^{c_6t}\|w^e\|^2, \quad e \in [0, e_0].
\]

We integrate the above inequality over \( (0, t) \) to get

\[
\|\nabla w^e(t)\|^2 \leq e^{-c_6t}\|\nabla w^e(0)\|^2 + c\int_0^t \|\nabla w^e(s)\|^2 ds, \quad e \in [0, e_0].
\]

For any \( \eta \in (0, 1) \), we can choose \( T_3 = T_3(\eta) > 0 \) such that \( e^{-c_6T_3} \leq \eta^2 \). Therefore, from (83),

\[
\|\nabla w^e(t)\|^2 \leq \eta\|\nabla w^e(0)\|^2 + c\int_0^t \|\nabla w^e(s)\|^2 ds, \quad t \geq T_3, e \in [0, e_0].
\]

Thus,
\[ \|\nabla w^\epsilon(t)\| \leq \eta \|\nabla w^\epsilon(0)\| + K \left[ \int_0^t \|\nabla w^\epsilon(s)\|^2 \, ds \right]^{1/2}, \quad t \geq T_2, \quad \epsilon \in [0, \epsilon_0]. \]  

Now, we set \( \eta^2(t) = [\int_0^t \|w^\epsilon(s)\|^2 \, ds]^{1/2} \). We need to show that \( \eta^2(t) \) is compact on \( B \) for any \( t > 0 \) and any \( \epsilon \in [0, 1] \). From condition (F2), we deduce that, for any \( s \in \mathbb{R} \),

\[ f(s) \leq c \left( |s| + |s|^3 + |s|^{2N-2N/2}\right). \]

Hence,

\[ [f(w), u] \leq \left( \|u\|^2 + \|u\|^{2N-2N/2} \right) \leq c \left( \|u\|^2 + \|u\|^{2N-2N/2} \right) \leq c \left( \|w\|^2 + \|w\|^{2N-2N/2} \right), \quad \forall u \in H^1_0(\Omega). \]

where we have used the Sobolev embedding \( H^1_0(\Omega) \subset L^{2N/2}(\Omega) \) in the above inequality. From (35) and (87), we obtain

\[ x_0 \int_\Omega F(u) \leq \theta \|u\|^2 + c_0 + c \left( \|\nabla u\|^2 + \|\nabla u\|^{2N-2N/2} \right), \quad \forall u \in H^1_0(\Omega), \]

for any \( \theta > 0 \). For any \( t \geq 0 \), we can see from (32) that

\[ \int_0^t \|\nabla u^\epsilon_0(s)\|^2 \, ds \leq ct, \quad \epsilon \in [0, 1], \]

for any \( u_0 \in B \) and \( c \) is independent of \( \epsilon \). Next, we integrate (56) from 0 to \( t \), and we get

\[ c \int_0^t \|\nabla u^\epsilon_0(s)\|^2 \, ds \leq \epsilon + c \left( \|\nabla u^\epsilon_0\|^2 + 2 \right). \]

We first fix \( \theta \) small enough, and then put (88) into (90) to get, for all \( u_0 \in B \),

\[ \epsilon \int_0^t \|\nabla u^\epsilon_0(s)\|^2 \, ds \leq \epsilon + c, \quad \epsilon \in [0, 1]. \]

From (89) and (91), we can easily see that, for any \( t > 0 \) and any fixed \( \epsilon \in (0, 1) \), \( \mathcal{B}_{\epsilon^2} = \{u^\epsilon(s); s \in [0, t], u^\epsilon(0) = u^\epsilon_0 \} \) is bounded in \( W^{1,2}[0, t; H^1_0(\Omega)] \). Therefore, from the compact embedding \( W^{1,2}[0, t; H^1_0(\Omega)] \subset L^2[0, t; L^2(\Omega)] \), we deduce that \( \eta^2(t) \) is compact in \( B \). Thus, assertion (i) is proved.

Assertion (ii) can be easily proved by multiplying (70) (with \( \epsilon = 0 \)) with \( -\Delta u^\epsilon_0 \), and this procedure is elementary; here, we omit it (see, e.g., [20] for details). The proof is completed.

**Lemma 8.** Under assumptions (F1)–(F4), we have, for any \( \epsilon \in [0, 1] \), any \( i \in \mathbb{N} \), and any \( x \in B \), there exists a positive function \( \ell(t) \) independent of \( \epsilon \) such that, for all \( t \geq T_2 + 1 \),

\[ \|S^\epsilon_1(t)x - S^\epsilon_0(t)x\|_{H^1_0(\Omega)} \leq \ell(t) \epsilon^{1/4}. \]

where \( T_2 = T_2(B) \) (see Lemma 4).

**Proof.** We assume that \( u^\epsilon(t) \) (\( \epsilon \in (0, 1] \)) and \( u^\epsilon_0(t) \) are the solutions for the following equations:

\[ \begin{cases} u^\epsilon_t - \epsilon \Delta u^\epsilon - \Delta u^\epsilon + f(u^\epsilon) = g, & t > 0, x \in \Omega, \\ u^\epsilon|_{\partial \Omega} = 0, u^\epsilon(x, 0) = x_1(x) \in B, & x \in \Omega, \end{cases} \]

\begin{equation}
\begin{cases}
u^\epsilon_t - \Delta \nu^\epsilon + f(\nu^\epsilon) = g, & t > 0, x \in \Omega, \\ \nu^\epsilon|_{\partial \Omega} = 0, \nu^\epsilon(x, 0) = x_2(x) \in B, & x \in \Omega,
\end{cases}
\end{equation}

respectively. Set \( w = u^\epsilon - \nu^\epsilon \), and then \( w \) satisfies

\[ w_t - \epsilon \Delta u^\epsilon - \Delta w + l(t)w = 0, \]

where \( l(t) = \int_0^t f(\sigma u^\epsilon(t) + (1 - s)u^\epsilon_0(t)) \, ds \). Multiplying (94) with \( w \) and using Lemma 5, it yields

\[ \|w\|^2 + e(\nabla u^\epsilon, \nabla w) + \frac{1}{2} \frac{d}{dt} \|w\|^2 \leq c_0 \|w\| \|w\|, \quad t \geq 0. \]

We choose \( \sigma_1 = \max\{c_0, 2\epsilon\} \); then, the above inequality implies

\[ 2e \|\nabla u^\epsilon\|^2 + \frac{1}{2} \frac{d}{dt} \|w\|^2 \leq \sigma_1 \|w\|^2 + 2c \|\nabla u^\epsilon\| \|\nabla u^\epsilon\|. \]

Therefore,

\[ \frac{d}{dt} e^{-\sigma_1 t} \|w\|^2 \leq 2ce^{-\sigma_1 t} \|\nabla u^\epsilon\| \|\nabla u^\epsilon\|. \]

By Lemma 4, we see that, for \( t \geq T_2 \),

\[ \frac{d}{dt} e^{-\sigma_1 t} \|w\|^2 \leq e^{-\sigma_1 t} \|w\|^2 \leq \sigma_1 \|w\|^2 + 2c \|\nabla u^\epsilon\| \|\nabla u^\epsilon\|. \]

Multiplying (94) with \( w \) and integrating it in \( \Omega \), we obtain

\[ \frac{1}{2} \frac{d}{dt} \|w\|^2 + e(\nabla u^\epsilon, \nabla w) + \|w\|^2 \leq \eta \|w\|^2, \]

that is,

\[ \frac{d}{dt} \|w\|^2 + 2e(\nabla u^\epsilon, \nabla w) + \|w\|^2 \leq \sigma_1 \|w\|^2 + 2e \|\nabla u^\epsilon\| \|\nabla w\|. \]

The above inequality implies

\[ \frac{d}{dt} e^{-\sigma_1 t} \|w\|^2 + 2e^{-\sigma_1 t} \|w\|^2 \geq 2e \|\nabla u^\epsilon\| \|\nabla w\|. \]

Dropping the second term in the left-hand side of (101) and integrating it over \( (0, t) \), we get

\[ e^{-\sigma_1 t} \|w(t)\|^2 \leq \|w(0)\|^2 + c \left[ \int_0^t \|\nabla u^\epsilon(s)\|^2 \, ds \right]^{1/2} \]

\[ \cdot \left[ \int_0^t \|\nabla w(s)\|^2 \, ds \right]^{1/2}. \]

From (91), we see that, for any \( \epsilon \in (0, 1) \),

\[ \int_0^t \|\nabla u^\epsilon_0(s)\|^2 \, ds \leq \frac{1}{\epsilon} c(t + 1). \]

Combining (31), (102), and (103), we deduce that

\[ e^{-\sigma_1 t} \|w(t)\|^2 \leq \|w(0)\|^2 + c(t + 1)^{1/2}, \quad t \geq 0. \]

Now, we integrate (101) over \( (t, t + 1) \) to get
Let assumptions (F1)–(F4) hold, and then \( \forall e \in [0, e_0] \), and the discrete dynamical system \( \{S^d_e\}_n \) defined above possesses an exponential attractor \( \mathcal{M}_e^d \) on \( B \) such that

\[ \dim_{f,H^d_0(\Omega)}\mathcal{M}_e^d \leq c(e) := \left[ \ln 2 \over 1 + \eta \right]^{-1} \ln m_c \left( \frac{4K (1 + L^2)^{1/2}}{1 - \eta} \right), \]

where \( L = e^{cT_0} \) (see Lemma 6) and \( m_c(R) \) is the maximal number of pairs \( (x_i, y_j) \) in \( H^d_0(\Omega) \times H^d_0(\Omega) \) possessing the properties

\[ e \in (0, e_0]: \|x_i\|^2_{H^d_0(\Omega)} + \|y_j\|^2_{H^d_0(\Omega)} \leq R', n_i^1(x_i - x_j) > 1, \quad i \neq j, \]

\[ e = 0: \|x_i\|^2_{H^d_0(\Omega)} + \|y_j\|^2_{H^d_0(\Omega)} \leq R', n_i^1(y_i - y_j) > 1, \quad i \neq j. \]

The family \( \{\mathcal{M}_e^d, e \in [0, e_0]\} \) is continuous at 0 in \( H^d_0(\Omega) \):

\[ \text{dist}_{\text{sym},H^d_0(\Omega)}(\mathcal{M}_0^d, \mathcal{M}_e^d) \leq c_{10} e^{c_{10}}, \]

where \( c_{10} = -1/4 \ln 1 + \eta/2 \ln \ell(T_0) - \ln 1 + \eta/2 \) and \( c_{10} \) are independent of \( e \).

To obtain the corresponding result for continuous system \( \{S_e(t)\} \) defined in (30), we need to show the Hölder continuity with respect to time \( t \) and the initial conditions. In general, it is difficult to verify the uniform (with respect to \( e \)) Hölder continuity with respect to time \( t \) when \( t \) is small. However, when \( t \) is large enough, we have the following.

**Lemma 9.** Let assumptions (F1)–(F4) hold. Then, for any \( T \geq T_0 \), the semigroup \( S_e(t) \) defined in (30) is uniformly Hölder continuous on \( [0, T] \times B \), i.e.,

\[ \|S_e(t_1) x_1 - S_e(t_2) x_2\|_{H^d_0(\Omega)} \leq c \left( \|x_1 - x_2\|_{H^d_0(\Omega)} + |t_1 - t_2|^{1/4} \right), \]

\[ e \in [0, 1], \]

for \( x_1, x_2 \in B \) and \( T_0 \leq t_1 \leq t_2 \leq T \) and \( c \) is independent of \( e \).

**Proof.** The Lipschitz continuity with respect to the initial conditions is an immediate corollary of Lemma 6. It remains to prove the continuity with respect to time \( t \). From Lemmas 1 and 2, we know that the solution \( u^e(t) \in C(0, T, H^d_0(\Omega)) \) for the initial value belongs to \( B \). Therefore, for any \( T_0 \leq t_1 \leq t_2 \leq T \),
Let assumptions (F1)–(F4) hold. Then, for every setting independent of \(c\) where \(\varepsilon\) is independent of \(M_1\) and \(M_2\), and (2) possesses an exponential attractor \(\mathcal{M}_\varepsilon\) is also an exponential attractor for the discrete dynamical system \(\mathcal{M}_\varepsilon^d\). Thus, \(\mathcal{M}_\varepsilon^d\) is the set we needed (for details, see, e.g., [12]). The proof is completed.

3.3. The Main Result for a Special Case: 0 < \(\gamma_0 < \lambda_1\). If, in addition, \(\gamma < \lambda_1\), we can get the uniform bound (with respect to \(\varepsilon\)) for the fractal dimension. To this end, we need show that the constants \(L_0\) and \(L_1\) and the compact seminorms \(n_1^1\) and \(n_2^1\) in Theorem 1 are all independent of \(\varepsilon\) for the discrete system generated by problems (1) and (2). From the above, we have proved the constant \(L_0\) is independent of \(\varepsilon\); thus, we only need the following lemma:

**Lemma 10.** Let (F1)–(F4) hold; in addition, we assume that \(\gamma < \lambda_1\), and then we have for any \(x_1, x_2 \in B\) and any \(\eta \in (0, 1)\), the constants \(c_\eta \in (0, 1), K > 0, T_0(\eta) > T_0\) (\(T_0\) is defined above) and a compact seminorm \(n(x)\) on \(B\) such that

\[
\|S_\varepsilon(T_0)x_1 - S_\varepsilon(T_0)x_2\|_{H_\varepsilon^1(\Omega)}
\]

\[
\leq \eta\|x_1 - x_2\|_{H_\varepsilon^1(\Omega)} + K\|n(x_1 - x_2)\|_{H_\varepsilon^1(\Omega)}
\]

\[
+ n\[S_\varepsilon(T_0)x_1 - S_\varepsilon(T_0)x_2]\|, \quad \varepsilon \in [0, \varepsilon_0],
\]

where \(T_0(\eta), K,\) and the compact seminorm \(n(x)\) are all independent of \(\varepsilon\).

**Proof.** We assume that \(u_\varepsilon(t)\) and \(u_\varepsilon^d(t)\) are two solutions of (1) starting from \(x_1, x_2 \in B\), respectively. We consider the difference \(u_\varepsilon(t) - u_\varepsilon^d(t)\); then, \(u_\varepsilon(t)\) satisfies

\[
u_\varepsilon(t) - \varepsilon \Delta u_\varepsilon(t) - \Delta u_\varepsilon(t) + l(t, \varepsilon)u_\varepsilon(t) = 0.
\]

We first take the inner product of (127) with \(Ku_\varepsilon\) in \(L^2(\Omega)\) (\(k\) is a constant which will be fixed later) to get

\[
k\|\nabla w_\varepsilon\|^2 + k\|\nabla w_\varepsilon^d\|^2 + k\|\nabla w_\varepsilon^d\|\|w_\varepsilon\|^2 + k\int_{\Omega} l(t, \varepsilon)u_\varepsilon \cdot w_\varepsilon^d = 0.
\]

Using Lemma 5, we get

\[
k\|\nabla w_\varepsilon\|^2 + k\|\nabla w_\varepsilon^d\|^2 + k\|\nabla w_\varepsilon^d\|\|w_\varepsilon\|^2 + k\|\nabla w_\varepsilon^d\|\|w_\varepsilon\|^2.
\]

Next, we take the inner product of (127) with \(u_\varepsilon^{2d}\) in \(L^2(\Omega)\) to get

\[
\|\nabla w_\varepsilon^{2d}\|^2 + \varepsilon\|\nabla w_\varepsilon^{2d}\|^2 + \frac{1}{2} \frac{d}{dt}\|\nabla w_\varepsilon\|^2 + \int_{\Omega} l(t, \varepsilon)u_\varepsilon^{2d} = 0.
\]

We apply Lemma 5 again to get

\[
\|\nabla w_\varepsilon^{2d}\|^2 + \varepsilon\|\nabla w_\varepsilon^{2d}\|^2 + \frac{1}{2} \frac{d}{dt}\|\nabla w_\varepsilon\|^2 \leq c_\varepsilon\|\nabla w_\varepsilon\|^2 + \|\nabla w_\varepsilon\|^2.
\]

Combining (129) and (131), we get

\[
\|\nabla w_\varepsilon\|^2 + \varepsilon\|\nabla w_\varepsilon\|^2 + \frac{1}{2} \frac{d}{dt}\|\nabla w_\varepsilon\|^2 \leq c_\varepsilon\|\nabla w_\varepsilon\|^2 + \|\nabla w_\varepsilon\|^2.
\]
For the right-hand side of (132), we use H"{o}lder inequality to get
\begin{align}
\leq k\|\nabla w_0\|^2 + \frac{k^2}{2}\|\nabla^2 w_0\|^2 + \frac{k^2}{2}\|\nabla w_0\|^2 + \frac{k^2}{2}\|\nabla^2 w_0\|^2
\end{align}
(132)
Putting the above inequality into (132), we obtain
\begin{align}
\frac{d}{dt}\|\nabla w_0\|^2 + k\|\nabla w_0\|^2 \leq \epsilon(k^2 + k)\|\nabla w_0\|^2 + (c + k)\|\nabla^2 w_0\|^2
+ (c^2 + k^2\epsilon)\|\nabla w_0\|^2.
\end{align}
(134)
Fix \( k = 2c_1^2 \), and then choose \( \epsilon_0 \) such that \( \epsilon < 1/4c_1^2 \), \( \forall \epsilon \leq \epsilon_0 \). In the following, we assume \( \epsilon \in [0, \epsilon_0] \). Then, we get from (134) that
\begin{align}
\frac{d}{dt}\|\nabla w_0\|^2 + c_{15}\|\nabla w_0\|^2 \leq \epsilon\|\nabla^2 w_0\|^2 + c\|\nabla w_0\|^2,
\end{align}
(135)
\( c_{15} > 0 \).
Using Poincare inequality, we obtain
\begin{align}
\frac{d}{dt}\|\nabla w_0\|^2 + c_{15}\|\nabla w_0\|^2 \leq c_{15}\|\nabla w_0\|^2 + c\|\nabla w_0\|^2,
\end{align}
(136)
\( c_{15} > 0 \).
From Lemma 6, it yields that
\begin{align}
\frac{d}{dt}\epsilon^{-\alpha}\|\nabla w_0\|^2 \leq c_{15}\|\nabla w_0\|^2 + c\|\nabla w_0\|^2 + \epsilon\|\nabla^2 w_0\|^2.
\end{align}
(137)
We integrate the above inequality over \((0, t)\) to get
\begin{align}
\|\nabla w_0(t)\|^2 & \leq \epsilon^{-\alpha}\|\nabla w_0(0)\|^2 + c_{15}\|\nabla w_0(0)\|^2 + \epsilon\|\nabla^2 w_0(0)\|^2
+ \epsilon\int_0^t\|\nabla w_0(s)\|^2ds
\leq \left(\epsilon^{-\alpha} + c_{15}\right)\|\nabla w_0(0)\|^2 + \epsilon\int_0^t\|\nabla w_0(s)\|^2ds.
\end{align}
(138)
For any \( \eta \in (0, 1) \), we first choose \( T'_0 = T'_0(\eta) > T_0(\eta) \) such that \( e^{-c_{15}\alpha T'_0} \leq \eta^2/2 \), and then fix a positive integer \( M = M(\eta) \) such that \( \lambda_{M+1} \leq \eta^2/2 \). Therefore, from (138),
\begin{align}
\|\nabla w_0(T'_0)\|^2 & \leq \epsilon\|\nabla w_0(0)\|^2 + \epsilon\int_0^{T'_0}\|\nabla w_0(s)\|^2ds.
\end{align}
(139)
Therefore,
\begin{align}
\|\nabla w_0(T'_0)\|^2 & \leq \epsilon\|\nabla w_0(0)\|^2 + \epsilon\int_0^{T'_0}\|\nabla w_0(s)\|^2ds + \|\nabla w_0(T'_0)\|^2
\leq \epsilon\|\nabla w_0(0)\|^2 + \epsilon\int_0^{T'_0}\|\nabla w_0(s)\|^2ds + \|\nabla w_0(T'_0)\|^2,
\end{align}
(140)
that is,
\begin{align}
\|\nabla w_0(T'_0)\|^2 \leq \epsilon\|\nabla w_0(0)\|^2 + \epsilon\int_0^{T'_0}\|\nabla w_0(s)\|^2ds + \|\nabla w_0(T'_0)\|^2
\leq \epsilon\|\nabla w_0(0)\|^2 + \epsilon\|\nabla w_0(T'_0)\|^2
+ 2\|\nabla u_0(0)\|^2 + 2\|\nabla w_0(T'_0)\|^2.
\end{align}
(141)
By using (F1) and Poincare inequality, we can get from the above inequality
\begin{align}
\frac{d}{dt}\|\nabla w_0(t)\|^2 + \|\nabla w_0(t)\|^2 \leq \epsilon\|\nabla w_0(t)\|^2 + \epsilon\|\nabla w_0(t)\|^2
\leq 2\|\nabla w_0(t)\|^2 \leq 2\lambda_{1}^{-1}\|\nabla w_0(t)\|^2.
\end{align}
(142)
Therefore,
\begin{align}
\frac{d}{dt}\|\nabla w_0(t)\|^2 + \|\nabla w_0(t)\|^2 \leq \beta\|\nabla w_0(t)\|^2 \leq 0,
\end{align}
(143)
where \( \beta = 2(1 - \lambda_1^{-1}) > 0 \). Integrating (143) from 0 to \( T'_0 \), we get
\begin{align}
\int_0^{T'_0}\|\nabla w_0(s)\|^2ds \leq c\|\nabla w_0(0)\|^2 + \epsilon\|\nabla w_0(0)\|^2 \leq c\|\nabla w_0(0)\|^2.
\end{align}
(144)
Thus, for any \( \epsilon \in [0, \epsilon_0] \),
\begin{align}
\left(\int_0^{T'_0}\|\nabla w_0(s)\|^2ds\right)^{1/2} \leq cn(x_1 - x_2).
\end{align}
(145)
Putting (146) into (141), one can obtain the result. The proof is completed.

**Theorem 5.** Let assumptions (F1)–(F4) hold and \( \nu < \lambda_1 \). Then, for every \( \epsilon \in [0, \epsilon_0] \), the semigroup \( S_t \) generated by equations (1) and (2) possesses an exponential attractor \( \mathcal{M}_\nu^\epsilon \) in \( H_0^1(\Omega) \). Moreover, these exponential attractors can be constructed such that (2) and (3) in Theorem 3 hold and the
fractal dimension of $\mathcal{M}_\epsilon$ is uniformly (with respect to $\epsilon$) bounded in $H^1_0(\Omega)$, i.e.,
\[
\dim_{H^1_0(\Omega)} \mathcal{M}_\epsilon \leq c,
\]
where $c$ is a constant independent of $\epsilon$.

**Data Availability**
No data were used to support this study.

**Conflicts of Interest**
The authors declare that they have no conflicts of interest.

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