Log abelian varieties over a log point I

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Abstract. We study (weak) log abelian varieties with constant degeneration in the log flat topology. If the base is a log point, we further study the endomorphism algebras of log abelian varieties.

Introduction

As stated in [KKN08a], degenerating abelian varieties can not preserve group structure, properness, and smoothness at the same time. Log abelian variety is a construction aimed to make the impossible possible in the world of log geometry. The idea dates back to Kato's construction of log Tate curve in [Kat89] Sec. 2.2, in which he conjectured the existence of a general theory of log abelian varieties. The theory finally comes true in [KKN08b] and [KKN08a].

Log abelian varieties are defined as certain sheaves in the classical étale topology in [KKN08a], while the log flat topology is needed for studying some problems, for example finite group sub-objects of log abelian varieties, p-adic realisations of log abelian varieties, logarithmic Dieudonné theory of log abelian varieties and so on. In the first section of this paper, we prove various classical étale sheaves from [KKN08a] are also sheaves for the log flat topology, in particular we prove (weak) log abelian varieties with constant degeneration are sheaves for the log flat topology, see Theorem 1.1. We compute the first direct image sheaves of étale locally finite rank free constant sheaves, for changing to the log flat site from the classical étale site, in Lemma 1.1. This lemma can be viewed as a supplement or generalisation of [Kat91] Thm. 4.1. We also reformulate some results from [KKN08a] §2, §3 and §7 in the context of the log flat topology.

In the second section, we focus on the case in which the base is a log point. In this case, a log abelian variety is automatically a log abelian variety with constant degeneration. And only in this case, log abelian variety is the counterpart of abelian variety. While for general base, log abelian variety corresponds to abelian scheme. Now one may wonder if various results for abelian variety also hold for

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log abelian variety. We study isogenies and general homomorphisms between log abelian varieties over a log point. More precisely, we give several equivalent characterisations of isogeny in Proposition 2.2 and prove the dual short exact sequence in Theorem 2.1, Poincaré complete reducibility theorem for log abelian varieties in Theorem 2.2, and the finiteness of homomorphism group of log abelian varieties in Theorem 2.4, Corollary 2.3, Corollary 2.4 and Corollary 2.5.

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1. Log abelian varieties with constant degeneration in the log flat topology

When dealing with finite subgroup schemes of abelian varieties, one needs to work with the flat topology. Similarly, the log flat topology is needed in the study of log finite group subobjects of log abelian varieties. However, log abelian varieties in [KKN08a] are defined in the classical étale topology. In this section, we are going to reformulate some results from [KKN08a, §2, §3 and §7], which are in the context of classical étale topology, in the context of log flat topology.

Throughout this section, let $S$ be any fs log scheme, let $(fs/S)$ be the category of fs log schemes over $S$, and log schemes in this section will always be fs log schemes unless otherwise stated. Let $S_{\text{cl}}^{\text{ét}}$ (resp. $S_{\text{fl}}^{\text{cl}}$, resp. $S_{\text{fl}}^{\text{log}}$) be the classical étale site (resp. classical flat site, resp. log flat site) associated to the category $(fs/S)$, and let $\delta = m \circ \varepsilon_{\text{fl}} : S_{\text{fl}}^{\text{log}} \xrightarrow{\varepsilon_{\text{fl}}} S_{\text{fl}}^{\text{cl}} \xrightarrow{m} S_{\text{cl}}^{\text{ét}}$ be the canonical map of sites. For any inclusion $F \subset G$ of sheaves on $S_{\text{fl}}^{\text{log}}$, $G/F$ will denote the quotient sheaf in the category of sheaves on $S_{\text{fl}}^{\text{log}}$ by convention, unless otherwise stated.

Firstly we recall some definitions from [KKN08a]. Let $G$ be a commutative group scheme over the underlying scheme of $S$ which is an extension of an abelian scheme $B$ by a torus $T$. Let $X$ be the character group of $T$ which is a locally constant sheaf of finite generated free $\mathbb{Z}$-modules for the classical étale topology. The sheaf $G_{m,\text{log}}$ on $S_{\text{cl}}^{\text{ét}}$ is defined by

$$G_{m,\text{log}}(U) = \Gamma(U, M^\text{gp}_U),$$

the sheaf $T_{\text{log}}$ on $S_{\text{cl}}^{\text{ét}}$ is defined by

$$T_{\text{log}} := \mathcal{H}om_{S_{\text{fl}}^{\text{cl}}}(X, G_{m,\text{log}}),$$

and the sheaf $G_{\text{log}}$ is defined as the push-out of $T_{\text{log}} \leftarrow T \rightarrow G$ in the category of sheaves on $S_{\text{cl}}^{\text{ét}}$, see [KKN08a] 2.1.
Proposition 1.1. The sheaves \( \mathcal{G}_{m, \log}, X, T_{\log} \) and \( G_{\log} \) on \( S^1_{\text{Et}} \) are also sheaves for the log flat topology. Moreover, \( T_{\log} \) can be defined as \( \text{Hom}_{S^1_{\text{fl}}} (X, \mathcal{G}_{m, \log}) \), and \( G_{\log} \) can be defined as the push-out of \( T_{\log} \leftarrow T \rightarrow G \) in the category of sheaves on \( S^1_{\text{fl}} \).

Proof. The statement for \( \mathcal{G}_{m, \log} \) is just \([\text{Kat91}, \text{Thm. 3.2}]\). Being representable by a group scheme, \( X \) is a sheaf on \( S^1_{\text{fl}} \) by \([\text{Kat91}, \text{Thm. 3.1}]\). It follows then \( T_{\log} = \text{Hom}_{S^1_{\text{fl}}} (X, \mathcal{G}_{m, \log}) = \text{Hom}_{S^1_{\text{fl}}} (X, \mathcal{G}_{m, \log}) \) is also a sheaf on \( S^1_{\text{fl}} \). By its definition \( G_{\log} \) fits into a short exact sequence \( 0 \rightarrow T_{\log} \rightarrow G_{\log} \rightarrow B \rightarrow 0 \) of sheaves on \( S^1_{\text{et}} \). Consider the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & T_{\log} & \rightarrow & G_{\log} & \rightarrow & B & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & T_{\log} & \rightarrow & \delta_* \delta^* G_{\log} & \rightarrow & B & \rightarrow & 0 \\
\end{array}
\]

with exact rows in the category of sheaves on \( S^1_{\text{et}} \), where the vertical maps come from the adjunction \((\delta^*, \delta_*)\). The sheaf \( R^1 \delta_* T_{\log} \) is zero by Kato’s logarithmic Hilbert 90, see \([\text{Kat91, Cor. 5.2}]\) and \([\text{Niz08, Thm. 3.20}]\). It follows that the canonical map \( G_{\log} \rightarrow \delta_* \delta^* G_{\log} \) is an isomorphism, whence \( G_{\log} \) is a sheaf on \( S^1_{\text{fl}} \). Since \( G_{\log} \) as a push-out of \( T_{\log} \leftarrow T \rightarrow G \) in the category of sheaves on \( S^1_{\text{fl}} \), it coincides with the push-out of \( T_{\log} \leftarrow T \rightarrow G \) in the category of sheaves on \( S^1_{\text{fl}} \). □

Proposition 1.2. We have canonical isomorphisms

\( \text{Hom}_{S^1_{\text{fl}}} (X, \mathcal{G}_{m, \log} / \mathcal{G}_{m}) \cong T_{\log} / T \cong G_{\log} / G \).

Proof. By \([\text{Kat91}]\) \( G_{\log} \) is the the push-out of \( T_{\log} \leftarrow T \rightarrow G \) in the category of sheaves on \( S^1_{\text{fl}} \), so we get a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & T & \rightarrow & G & \rightarrow & B & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & T_{\log} & \rightarrow & G_{\log} & \rightarrow & B & \rightarrow & 0 \\
\end{array}
\]

with exact rows. Then the isomorphism \( T_{\log} / T \cong G_{\log} / G \) follows. Applying the functor \( \text{Hom}_{S^1_{\text{fl}}} (X, -) \) to the short exact sequence

\[
0 \rightarrow \mathcal{G}_{m} \rightarrow \mathcal{G}_{m, \log} \rightarrow \mathcal{G}_{m, \log} / \mathcal{G}_{m} \rightarrow 0
\]

we get a long exact sequence

\[
0 \rightarrow T \rightarrow T_{\log} \rightarrow \text{Hom}_{S^1_{\text{fl}}} (X, \mathcal{G}_{m, \log} / \mathcal{G}_{m}) \rightarrow \text{Ext}_{S^1_{\text{fl}}} (X, \mathcal{G}_{m})
\]
of sheaves on $S^\log_H$. Since $X$ is classical étale locally represented by a finite rank free abelian group, the sheaf $\mathcal{E}xt_{S^\log_H}(X, \mathbb{G}_m)$ is zero. It follows that the sheaf $\mathcal{H}om_{S^\log_H}(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ is canonically isomorphic to $T_{\log}/T$. □

It is obvious that the association of $G_{\log}$ to $G$ is functorial in $G$. Hence we have a natural map $\mathcal{H}om_{S^\log_H}(G, G') \to \mathcal{H}om_{S^\log_H}(G_{\log}, G'_{\log})$, where $G'$ is another commutative group scheme which is an extension of an abelian scheme by a torus over the underlying scheme of $S$. The following proposition describes some properties of this map.

**Proposition 1.3.** (1) The association of $G_{\log}$ to $G$ is functorial in $G$.
(2) The canonical map $\mathcal{H}om_{S^\log_H}(G, G') \to \mathcal{H}om_{S^\log_H}(G_{\log}, G'_{\log})$ is an isomorphism.
(3) For a group scheme $H$ of multiplicative type with character group $X_H$ over the underlying scheme of $S$, we define $H_{\log}$ to be $\mathcal{H}om_{S^\log_H}(X_H, \mathbb{G}_{m,\log})$. Let $0 \to H' \to H \to H'' \to 0$ be a short exact sequence of group schemes of multiplicative type over the underlying scheme of $S$ such that their character groups are étale locally finite rank constant sheaves, then the sequences

$$0 \to H'_{\log} \to H_{\log} \to H''_{\log} \to 0$$

and

$$0 \to \mathcal{H}om_{S^\log_H}(X_{H'}, \mathbb{G}_{m,\log}/\mathbb{G}_m) \to \mathcal{H}om_{S^\log_H}(X_H, \mathbb{G}_{m,\log}/\mathbb{G}_m)$$

$$\to \mathcal{H}om_{S^\log_H}(X_{H''}, \mathbb{G}_{m,\log}/\mathbb{G}_m) \to 0$$

are both exact.
(4) If $G \to G'$ is injective, so is $G_{\log} \to G'_{\log}$.
(5) If $G \to G'$ is surjective such that the induced map on the torus parts is also surjective (this is always the case if the underlying scheme of $S$ is a field), so is $G_{\log} \to G'_{\log}$.
(6) Let $0 \to G' \to G \to G'' \to 0$ be a short exact sequence of semi-abelian schemes over the underlying scheme of $S$, such that $G'$ (resp. $G$, resp. $G''$) is an extension of an abelian scheme $B'$ (resp. $B$, resp. $B''$) by a torus $T'$ (resp. $T$, resp. $T''$). Then we have a short exact sequence $0 \to G'_{\log} \to G_{\log} \to G''_{\log} \to 0$.

**Proof.** Part (1) is clear. The isomorphism of part (2) follows from [KKN08a, Prop. 2.5].

We prove part (3). Since we have a long exact sequence

$$0 \to H'_{\log} \to H_{\log} \to H''_{\log} \to \mathcal{E}xt_{S^\log_H}(X_{H'}, \mathbb{G}_{m,\log}),$$

it suffices to show $\mathcal{E}xt_{S^\log_H}(X_{H'}, \mathbb{G}_{m,\log}) = 0$. Since $\mathcal{E}xt_{S^\log_H}(\mathbb{Z}, \mathbb{G}_{m,\log}) = 0$, we are further reduced to show $\mathcal{E}xt_{S^\log_H}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_{m,\log}) = 0$ for any positive integer $n$. The short exact sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ gives rise to a long exact sequence

$$0 \to \mathcal{H}om_{S^\log_H}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_{m,\log}) \to \mathbb{G}_{m,\log} \xrightarrow{n} \mathbb{G}_{m,\log} \to \mathcal{E}xt_{S^\log_H}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_{m,\log}) \to 0.$$

Since $\mathbb{G}_{m,\log} \xrightarrow{n} \mathbb{G}_{m,\log}$ is surjective, the sheaf $\mathcal{E}xt_{S^\log_H}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_{m,\log})$ must be zero. The other short exact sequence is proved similarly.
We prove part (4). Since $G \to G'$ is injective, then the corresponding map $T \to T'$ on the torus parts is also injective and the corresponding map $X' \to X$ on the character groups is surjective. It follows that the induced map
$$G_{\log}/G = \text{Hom}_{S_n}(X, \mathcal{G}_{m, \log}/\mathcal{G}_m) \to \text{Hom}_{S_n}(X', \mathcal{G}_{m, \log}/\mathcal{G}_m) = G'_{\log}/G'$$
is injective. Hence $G_{\log} \to G'_{\log}$ is injective.

Now we prove part (5). By assumption we have a short exact sequence $0 \to X' \to X \to X/X' \to 0$ of étale locally constant sheaves. Applying the functor $\text{Hom}_{S_n}(-, \mathcal{G}_{m, \log}/\mathcal{G}_m)$ to this short exact sequence, we get a long exact sequence
$$\to G_{\log}/G \to G'_{\log}/G' \to \text{Ext}_{S_n}(X/X', \mathcal{G}_{m, \log}/\mathcal{G}_m).$$
Let $Z_{\text{tor}}$ be the torsion part of $X/X'$, let $n$ be a positive integer such that $nZ_{\text{tor}} = 0$. Since the multiplication-by-$n$ map on $\mathcal{G}_{m, \log}/\mathcal{G}_m$ is an isomorphism, we get the sheaf $\text{Ext}_{S_n}(Z_{\text{tor}}, \mathcal{G}_{m, \log}/\mathcal{G}_m)$ is zero. The torsion-free nature of $(X/X')/Z_{\text{tor}}$ implies $\text{Ext}_{S_n}(X/X', \mathcal{G}_{m, \log}/\mathcal{G}_m) = 0$, hence $\text{Ext}_{S_n}(X/X', \mathcal{G}_{m, \log}/\mathcal{G}_m) = 0$. It follows that $G_{\log}/G \to G'_{\log}/G'$ is surjective, hence $G_{\log} \to G'_{\log}$ is surjective.

At last, we prove part (6). Consider the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & G' \\
\downarrow & & \downarrow \\
G'_{\log} & \to & \text{Hom}_{S_n}(X', \mathcal{G}_{m, \log}) \to 0 \\
\downarrow & & \downarrow \\
0 & \to & G \\
\downarrow & & \downarrow \\
G_{\log} & \to & \text{Hom}_{S_n}(X, \mathcal{G}_{m, \log}) \to 0 \\
\downarrow & & \downarrow \\
0 & \to & G'' \\
\downarrow & & \downarrow \\
G''_{\log} & \to & \text{Hom}_{S_n}(X'', \mathcal{G}_{m, \log}) \to 0 \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\]

with the first column and all rows exact, where $\mathcal{G}_{m, \log}$ denotes $\mathcal{G}_{m, \log}/\mathcal{G}_m$. The maps $G' \to G \to G''$ induce $T' \to T \to T''$, furthermore $X' \leftarrow X \leftarrow X''$, lastly the third column of the diagram. Although $0 \to X'' \to X \to X' \to 0$ is not necessarily exact, it gives two exact sequences $0 \to Z \to X \to X' \to 0$ and $0 \to X'' \to Z \to Z/X'' \to 0$, where $Z := \text{Ker}(X \to X')$ is étale locally a finite rank free constant sheaf and $Z/X''$ is étale locally a finite torsion constant sheaf. By part (3), we get two short exact sequences
$$0 \to \text{Hom}_{S_n}(X', \mathcal{G}_{m, \log}) \to \text{Hom}_{S_n}(X, \mathcal{G}_{m, \log}) \to \text{Hom}_{S_n}(Z, \mathcal{G}_{m, \log}) \to 0$$
and
$$0 \to \text{Hom}_{S_n}(Z/X'', \mathcal{G}_{m, \log}) \to \text{Hom}_{S_n}(Z, \mathcal{G}_{m, \log}) \to \text{Hom}_{S_n}(X'', \mathcal{G}_{m, \log}) \to 0.$$
But $\mathcal{H}om_{cl^{\log}}(Z/X'', \mathcal{G}_{m, log}) = 0$, it follows that the third column of the diagram is exact. So is the middle column. □

Recall that in [KKN08a Def. 2.2], a log 1-motive $M$ over $S_{\text{Et}}^{cl}$ is defined as a two-term complex $[Y \xrightarrow{m} G_{\log}]$ in the category of sheaves on $S_{\text{Et}}^{cl}$, with the degree -1 term $Y$ an étale locally constant sheaf of finitely generated free abelian groups and the degree 0 term $G_{\log}$ as above. Since both $Y$ and $G_{\log}$ are sheaves on $S_{\eta}^{log}$, $M$ can also be defined as a two-term complex $[Y \xrightarrow{m} G_{\log}]$ in the category of sheaves on $S_{\eta}^{log}$. Parallel to [KKN08a 2.3], we have a natural pairing

\[<, > : X \times Y \rightarrow X \times (G_{\log}/G) = X \times \mathcal{H}om^{cl}_{cl^{\log}}(X, \mathcal{G}_{m, log}/\mathcal{G}_{m}) \rightarrow \mathcal{G}_{m, log}/\mathcal{G}_{m}.\]

By our convention, $T_{\log}/T$ denotes the quotient in the category of sheaves on $S_{\log}^{cl}$. For the quotient of $T \subset T_{\log}$ in the category of sheaves on $S_{\text{Et}}^{cl}$, we use the notation $(T_{\log}/T)_{S_{\text{Et}}^{cl}}$. Now we assume that the pairing (1.1) is admissible (see [KKN08a 7.1] for the definition of admissibility), in other words the log 1-motive $M$ is admissible. Recall that in [KKN08a 3.1], a subgroup sheaf $\mathcal{H}om_{S_{\text{Et}}^{cl}}(X, (\mathcal{G}_{m, log}/\mathcal{G}_{m})_{S_{\text{Et}}^{cl}})\mathcal{G}_{log}$ of the sheaf $\mathcal{H}om_{S_{\text{Et}}^{cl}}(X, (\mathcal{G}_{m, log}/\mathcal{G}_{m})_{S_{\text{Et}}^{cl}})$ on $S_{\text{Et}}^{cl}$ is defined by

\[\mathcal{H}om_{S_{\text{Et}}^{cl}}(X, (\mathcal{G}_{m, log}/\mathcal{G}_{m})_{S_{\text{Et}}^{cl}})\mathcal{G}_{log}(U) := \{ \varphi \in \mathcal{H}om_{S_{\text{Et}}^{cl}}(X, (\mathcal{G}_{m, log}/\mathcal{G}_{m})_{S_{\text{Et}}^{cl}})(U) \mid \text{for every } u \in U \text{ and } x \in X_u, \text{there exist } y_{u,x}, y'_{u,x} \in Y_u \text{ such that } < x, y_{u,x} > > u, |\varphi(x)|_u < x, y'_{u,x} > > u \}.\]

Here, for $u \in U$ and $a, b \in (\mathcal{M}_{\text{Fr}}^{\log}/\mathcal{O}_{\text{Fr}}^{\log})_{\bar{a}}$, $ab$ means $a^{-1}b \in (\mathcal{M}_{\text{Fr}}^{\log}/\mathcal{O}_{\text{Fr}}^{\log})_{\bar{a}}$. Similarly, we can define a subgroup sheaf $\mathcal{H}om_{S_{\eta}^{log}}(X, (\mathcal{G}_{m, log}/\mathcal{G}_{m})^{(Y)})$ of the sheaf $\mathcal{H}om_{S_{\eta}^{log}}(X, (\mathcal{G}_{m, log}/\mathcal{G}_{m})^{(Y)})$.

**Remark 1.1.** In [KKN08a 7.1], admissibility and non-degeneracy are defined for pairings into $(\mathcal{G}_{m, log}/\mathcal{G}_{m})_{S_{\text{Et}}^{cl}}$ in the classical étale site on $(\text{fs}/S)$. We can define admissibility and non-degeneracy for pairings into $\mathcal{G}_{m, log}/\mathcal{G}_{m}$ on the log flat site by the same way. Since both $X$ and $Y$ are classical étale locally constant sheaves of finite rank free abelian groups, the definitions of admissibility and non-degeneracy are independent of the choice of the topology.

Recall that in [KKN08a 3.2], the sheaf $G_{\log}^{(Y)} \subset G_{\log}$ on $S_{\text{Et}}^{cl}$ is defined to be the inverse image of $\mathcal{H}om_{S_{\text{Et}}^{cl}}(X, (\mathcal{G}_{m, log}/\mathcal{G}_{m})_{S_{\text{Et}}^{cl}})\mathcal{G}_{log}(Y)$ under the map

\[G_{\log} \rightarrow (G_{\log}/G)_{S_{\text{Et}}^{cl}} \simeq \mathcal{H}om_{S_{\text{Et}}^{cl}}(X, (\mathcal{G}_{m, log}/\mathcal{G}_{m})_{S_{\text{Et}}^{cl}}).\]

We could also consider the inverse image sheaf of $\mathcal{H}om_{S_{\eta}^{log}}(X, (\mathcal{G}_{m, log}/\mathcal{G}_{m})^{(Y)})$ under the map

\[G_{\log} \rightarrow G_{log}/G \simeq \mathcal{H}om_{S_{\eta}^{log}}(X, (\mathcal{G}_{m, log}/\mathcal{G}_{m})^{(Y)}).\]

The following proposition says that these two constructions coincide.

**Proposition 1.4. (1)** The sheaf $G_{\log}^{(Y)}$ on $S_{\text{Et}}^{cl}$ is also a sheaf on $S_{\eta}^{log}$.
(2) The sheaf $G_{\log}^{(Y)}$ fits into a canonical short exact sequence

\[(1.2) \quad 0 \to G \to G_{\log}^{(Y)} \to \mathcal{H}om_{S_{\text{cl}}}^{(Y)}(X, (\mathbb{G}_{m, \log}/\mathbb{G}_m)^{(Y)}) \to 0\]

of sheaves on $S_{\text{cl}}$. Part (1) and (2) follow.

(3) The association of $G_{\log}^{(Y)}$ to a log 1-motive $M = [Y \to G_{\log}]$ is functorial.

**Proof.** By [KKNT 2.7, 5.3], $G_{\log}^{(Y)}$ is a union of some representable sheaves on $S_{\text{cl}}^{\log}$; hence it is also a sheaf on $S_{\text{cl}}$. By the definition of $G_{\log}^{(Y)}$, we have a pullback diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & G & \longrightarrow & G_{\log}^{(Y)} & \longrightarrow & \mathcal{H}om_{S_{\text{cl}}}^{(Y)}(X, (\mathbb{G}_{m, \log}/\mathbb{G}_m)^{(Y)}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G & \longrightarrow & G_{\log} & \longrightarrow & \mathcal{H}om_{S_{\text{cl}}}^{(Y)}(X, (\mathbb{G}_{m, \log}/\mathbb{G}_m)^{(Y)}) & \longrightarrow & 0
\end{array}
\]

in the category of sheaves on $S_{\text{cl}}$. Since $G$, $G_{\log}^{(Y)}$ and $G_{\log}$ are all sheaves on $S_{\text{cl}}^{\log}$, applying the functor $\delta^*$ to the above commutative diagram, we get the following commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & G & \longrightarrow & G_{\log}^{(Y)} & \delta^* \mathcal{H}om_{S_{\text{cl}}}^{(Y)}(X, (\mathbb{G}_{m, \log}/\mathbb{G}_m)^{(Y)}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G & \longrightarrow & G_{\log} & \delta^* \mathcal{H}om_{S_{\text{cl}}}^{(Y)}(X, (\mathbb{G}_{m, \log}/\mathbb{G}_m)^{(Y)}) & \longrightarrow & 0
\end{array}
\]

Since we have canonical isomorphisms

\[
\delta^* \mathcal{H}om_{S_{\text{cl}}}^{(Y)}(X, (\mathbb{G}_{m, \log}/\mathbb{G}_m)^{(Y)}) \cong \mathcal{H}om_{S_{\text{cl}}}^{(Y)}(X, (\mathbb{G}_{m, \log}/\mathbb{G}_m)^{(Y)})
\]

and

\[
\delta^* \mathcal{H}om_{S_{\text{cl}}}^{(Y)}(X, (\mathbb{G}_{m, \log}/\mathbb{G}_m)^{(Y)}) \cong \mathcal{H}om_{S_{\text{cl}}}^{(Y)}(X, (\mathbb{G}_{m, \log}/\mathbb{G}_m)^{(Y)})
\]

part (1) and (2) follow.

Now we prove part (3). It is enough to prove that for a given homomorphism $(f_0, f_0) : M = [Y \to G_{\log}] \to M' = [Y' \to G'_{\log}]$, the composition $G_{\log} \hookrightarrow G_{\log}^{(Y)} \hookrightarrow G_{\log}^{(Y')} \hookrightarrow G_{\log}'$ factors through $G_{\log}^{(Y')} \hookrightarrow G_{\log}'$. Let $X, X'$ be the character groups of $G$ and $G'$ respectively, let $f_1 : X' \to X$ be the map induced from $f_0$, and let $f_4 : \mathcal{H}om_{S_{\text{log}}}^{(Y)}(X, (\mathbb{G}_{m, \log}/\mathbb{G}_m) \to \mathcal{H}om_{S_{\text{log}}}^{(Y')}\delta^*(X, (\mathbb{G}_{m, \log}/\mathbb{G}_m)$ be the map induced from $f_1$. By the definition of $G_{\log}^{(Y)}$ and $G_{\log}^{(Y')}$, we are reduced to show the composition

\[
\mathcal{H}om_{S_{\text{log}}}^{(Y)}(X, (\mathbb{G}_{\log})) \hookrightarrow \mathcal{H}om_{S_{\text{log}}}^{(Y)}(X, (\mathcal{G}_\log)) \hookrightarrow \mathcal{H}om_{S_{\text{log}}}^{(Y')}\delta^*(X', (\mathbb{G}_{\log}))
\]

factors through $\mathcal{H}om_{S_{\text{log}}}^{(Y')}\delta^*(X, (\mathbb{G}_{\log})) \hookrightarrow \mathcal{H}om_{S_{\text{log}}}^{(Y')}\delta^*(X', (\mathbb{G}_{\log}))$, where $\mathbb{G}_{\log}$ denotes $\mathbb{G}_{m, \log}/\mathbb{G}_m$. Let $\langle, \rangle : \times \to \mathbb{G}_{m, \log}/\mathbb{G}_m$ (resp. $\langle, \rangle' : \times \times \to \mathbb{G}_{m, \log}/\mathbb{G}_m$) be
the pairing associated to $M$ (resp. $M'$), then we have $\langle f_1(x'), y \rangle = \langle x', f_{-1}(y) \rangle$ for any $x' \in X'$, $y \in Y$. For any $U \in (\mathcal{S}/\mathcal{S})$, $\varphi \in \text{Hom}_{\mathcal{S}_{\text{log}}}(X, \mathcal{G}_{\text{log}}{(Y)}(U))$, we need to show $\phi := \varphi \circ f_1 \in \text{Hom}_{\mathcal{S}_{\text{log}}}(X', \mathcal{G}_{\text{log}}{(Y)}(U))$. For every $u \in U$ and $x' \in X'_u$, there exist $y_{u,x,1}, y_{u,x,2} \in Y_u$ such that $\langle f_1(x'), y_{u,x,1} \rangle > \langle \varphi(f_1(x')) \rangle \mid \langle f_1(x'), y_{u,x,2} \rangle >$. The relation could be rewritten as $\langle x', f_{-1}(y_{u,x,1}) \rangle > \langle \phi(x') \rangle \mid \langle x', f_{-1}(y_{u,x,2}) \rangle >$, which in turn implies that $\phi \in \text{Hom}_{\mathcal{S}_{\text{log}}}(X', \mathcal{G}_{\text{log}}{(Y)}(U))$. This finishes the proof of part (3).

**Remark 1.2.** Clearly, the image of $u : Y \rightarrow G_{\text{log}}$ is contained in $G_{\text{log}}{(Y)}$.

We further assume the pairing (1.1) is non-degenerate (see [KKN08a] 7.1 and Remark 11 for the definition of non-degenerate pairings), then the two induced maps $X \rightarrow \text{Hom}_{\mathcal{S}_{\text{log}}}(Y, \mathcal{G}_{\text{m,log}}/\mathcal{G}_m)$ and $Y \rightarrow \text{Hom}_{\mathcal{S}_{\text{log}}}(X, \mathcal{G}_{\text{m,log}}/\mathcal{G}_m)$ are both injective. Recall that in [KKN08a] Def. 3.3. (1)] (resp. [KKN15] 1.7]) a log abelian variety with constant degeneration (resp. weak log abelian variety with constant degeneration) over $S$ is defined to be a sheaf of abelian groups on $S_{\text{cl}}$ which is isomorphic to the quotient sheaf $(G_{\text{log}}{(Y)}/S_{\text{cl}})$ for a pointwise polarisable (resp. non-degenerate) log 1-motive $M = [Y \xrightarrow{u} G_{\text{log}}]$. Here a log 1-motive is said to be non-degenerate if its associated pairing (1.1) is non-degenerate. Since the polarisability implies the the non-degeneracy, a log abelian variety with constant degeneration over $S$ is in particular a weak log abelian variety with constant degeneration over $S$.

**Theorem 1.1.** Let $A$ be a weak log abelian variety with constant degeneration over $S$, suppose $A = (G_{\text{log}}{(Y)}/S_{\text{cl}})$ for a log 1-motive $M = [Y \xrightarrow{u} G_{\text{log}}]$. Then

1. $A$ is a sheaf on $S_{\text{cl}}$;
2. $A = G_{\text{log}}{(Y)}/Y$, in other words $A$ fits into a canonical short exact sequence

$$0 \rightarrow Y \rightarrow G_{\text{log}}{(Y)} \rightarrow A \rightarrow 0$$

in the category of sheaves of abelian groups on $S_{\text{cl}}$;
3. $A$ fits into a canonical short exact sequence

$$0 \rightarrow G \rightarrow A \rightarrow \text{Hom}_{\mathcal{S}_{\text{cl}}}(X, \mathcal{G}_{\text{m,log}}/\mathcal{G}_m){(Y)}/Y \rightarrow 0$$

in the category of sheaves of abelian groups on $S_{\text{cl}}$.

**Proof.** Part (2) follows from part (1). Since the log 1-motive $M$ is non-degenerate, the map $Y \rightarrow \text{Hom}_{\mathcal{S}_{\text{cl}}}(X, \mathcal{G}_{\text{m,log}}/\mathcal{G}_m){(Y)}$ is injective. Then the short exact sequence in part (3) is the one induced from the short exact sequence. We are left with part (1). The proof of part (1) is similar to that for the log étale case in [KKN15] 5.5, and we adopt some notations there for the proof.

Consider the short exact sequence $0 \rightarrow Y \rightarrow \tilde{A} \rightarrow A \rightarrow 0$ of [KKN15] 5.3. Note that $\tilde{A}$ is nothing but $G_{\text{log}}{(Y)}$ in our situation, however we stick to the notation $\tilde{A}$ for the sake of coherence with [KKN15] 5.3. The argument showing that
\( \tilde{A} \) is a log étale sheaf, also shows that \( \tilde{A} \) is a log flat sheaf, since representable functors are sheaves for the log flat topology by \([KKN15]\text{ Thm. } 5.2\). We have the canonical map \( \delta := m \circ \varepsilon : S^1_{\log} \to S^1_{\text{Et}} \) of sites. Applying \( \delta^* \) and \( \delta_* \) to \( 0 \to Y \to \tilde{A} \to A \to 0 \), we get a commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & Y & \to & \tilde{A} & \to & A & \to & 0 \\
0 & \to & Y & \to & \tilde{A} & \to & \delta_* \delta^* A & \to & R^1 \delta_* Y \\
\end{array}
\]

with exact rows, where the vertical maps are the ones given by the adjunction \((\delta^*, \delta_*)\). To prove that \( A \) is a sheaf for the log flat topology, it is enough to show the canonical map \( A \to \delta_* \delta^* A \) is an isomorphism. This follows from the above commutative diagram with the help of the lemma below.

\begin{lemma}
The sheaf \( R^1 \delta_* Y \) is zero.
\end{lemma}

\begin{proof}
Since \( Y \) is étale locally isomorphic to a finite rank free abelian group, we are reduced to the case \( Y = \mathbb{Z} \). Note that \( Y \) is a smooth group scheme over \( S \). The proof here is the same as the proof of \([Kat91]\text{ Thm. } 4.1\) (see also the proof of \([Niz08]\text{ Thm. } 3.12\]) except the very last part where the condition \( G \) being affine is used. The reason why the proof there can be generalised to our case lies in the fact that \( Y \) is étale over \( S \). We adopt some notations from \([Kat91]\text{ the second half of page } 22\) or \([Niz08]\text{ the first paragraph of page } 15\), and these two parallel parts are the very parts needed to modify. Let \( A \) be a strict local ring, \( \hat{A} \) its completion, and let \( \alpha \in H^1((\text{Spec}A)_{\log}^{\text{rig}}, \mathbb{Z}) \) such that it vanishes in \( H^1((\text{Spec} \hat{A})_{\log}^{\text{rig}}, \mathbb{Z}) \). By fpqc descent, \( \alpha \) is a class of a representable \( \mathbb{Z} \)-torsor over \( \text{Spec} \hat{A} \) such that its structure morphism is étale. Since \( A \) is a strict local ring, the torsor admits a section by \([Gro67]\text{ Prop. } 18.8.1\), so \( \alpha \) is zero. It follows \([Kat91]\text{ Thm. } 4.1\) also holds for the case \( G = \mathbb{Z} \), so \( R^1 m_* \mathbb{Z} = 0 \). The Leray spectral sequence gives a short exact sequence \( 0 \to R^1 m_* \mathbb{Z} \to R^1 \delta_* \mathbb{Z} \to m_* R^1 \varepsilon_{\mathbb{R}} \mathbb{Z} \). The sheaf \( R^1 m_* \mathbb{Z} = 0 \) by \([Gro68]\text{ Thm. } 11.7\), it follows that \( R^1 \delta_* \mathbb{Z} = 0 \).
\end{proof}

\begin{remark}
Lemma 1.1 can be viewed as a generalisation of Kato’s theorem \([Kat91]\text{ Thm. } 4.1\) to étale locally constant finitely generated torsion-free group schemes.
\end{remark}

The following lemma, which will be used repeatedly in this paper, relates the Hom sheaves in the classical étale topology to the Hom sheaves in the log flat topology.

\begin{lemma}
Let \( F, G \) be two sheaves on \( S^1_{\text{Et}} \) which are also sheaves on \( S^1_{\log} \) (this is the case for representable sheaves, \( G_{\log} \) by Proposition 1.1, \( G^{(Y)}_{\log} \) by Proposition 1.4 and weak log abelian varieties with constant degeneration by Theorem 1.1). Then we have \( \text{Hom}_{S^1_{\text{Et}}}(F, G) = \text{Hom}_{S^1_{\log}}(F, G) \), in particular \( \text{Hom}_{S^1_{\log}}(F, G) \) is a sheaf on \( S^1_{\log} \).
\end{lemma}
Proof. This is clear. \[\]

Now we give a reformulation of [KKN08a, Thm. 7.4] in the context of the log flat topology.

Theorem 1.2. Let \([Y \to G_{\text{log}}]\) be a log 1-motive over \(S\) of type \((X,Y)\) (see [KKN08a, Def. 2.2]) such that the induced paring \(X \times Y \to \mathbb{G}_{m,\text{log}}/\mathbb{G}_{m}\) is non-degenerate, and let \([X \to G_{\text{log}}]\) be its dual. Let \(A = G_{\text{log}}(Y)/Y\). Then we have:

1. \(\mathcal{E}xt_{S^\text{cl}_n}(A, \mathbb{Z}) \cong \mathcal{H}om_{S^\text{cl}_n}(Y, \mathbb{Z})\);
2. the sheaf \(\delta_*\mathcal{E}xt_{S^\text{cl}_n}(A, \mathbb{G}_m)\) fits into an exact sequence
   \[0 \to G^* \to \delta_*\mathcal{E}xt_{S^\text{cl}_n}(A, \mathbb{G}_m) \to \mathcal{H}om_{S^\text{cl}_n}(A, R^1\delta_*\mathbb{G}_m)\);
3. \(\mathcal{E}xt_{S^\text{cl}_n}(A, \mathbb{G}_{m,\text{log}}) \cong G_{\text{log}}^*/X\);
4. \(\mathcal{H}om_{S^\text{cl}_n}(A, \mathbb{Z}) = \mathcal{H}om_{S^\text{log}_n}(A, \mathbb{G}_m) = \mathcal{H}om_{S^\text{log}_n}(A, \mathbb{G}_{m,\text{log}}) = 0\).

Proof. Part (4) follows from [KKN08a, Thm. 7.4 (4)] with the help of Lemma [12].

Before going to the rest of the proof, we first introduce two spectral sequences. Let \(F_1\) (resp. \(F_2\)) be a sheaf on \(S_n^\text{cl}\) (resp. \(S_n^{\text{log}}\)), then we have
\[
\delta_*\mathcal{H}om_{S_n^{\text{log}}}(\delta^*F_1, F_2) = \mathcal{H}om_{S_n^\text{cl}}(F_1, \delta_*F_2).
\]
Let \(\theta\) be the functor sending \(F_2\) to \(\delta_*\mathcal{H}om_{S_n^{\text{log}}}(\delta^*F_1, F_2) = \mathcal{H}om_{S_n^\text{cl}}(F_1, \delta_*F_2)\), we get two Grothendieck spectral sequences
\[
E^{q, p}_2 = R^p\delta_*R^q\mathcal{H}om_{S_n^{\text{log}}}(\delta^*F_1, \theta) \Rightarrow R^{p+q}\theta,
\]
and
\[
E^{p, q}_2 = R^q\mathcal{H}om_{S_n^\text{cl}}(F_1, \delta_*\theta) \Rightarrow R^{p+q}\theta.
\]

These two spectral sequences give two exact sequences
\[
0 \to R^1\delta_*\mathcal{H}om_{S_n^{\text{log}}}(\delta^*F_1, F_2) \to R^1\theta(F_2) \to \delta_*\mathcal{E}xt_{S_n^{\text{log}}}(\delta^*F_1, F_2)
\]
and
\[
0 \to \mathcal{E}xt_{S_n^\text{cl}}(F_1, \delta_*F_2) \to R^1\theta(F_2) \to \mathcal{H}om_{S_n^\text{cl}}(F_1, R^1\delta_*F_2).
\]
Let \(F_1 = A\), and let \(F_2\) be \(\mathbb{Z}\), \(\mathbb{G}_m\) or \(\mathbb{G}_{m,\text{log}}\), part (4) implies
\[
R^1\theta(F_1) \cong \delta_*\mathcal{E}xt_{S_n^{\text{log}}}(A, F_2),
\]
so we get an exact sequence
\[
0 \to \mathcal{E}xt_{S_n^\text{cl}}(A, \delta_*F_2) \to \delta_*\mathcal{E}xt_{S_n^{\text{log}}}(A, F_2) \to \mathcal{H}om_{S_n^{\text{log}}}(A, R^1\delta_*F_2).
\]
Since \(\mathcal{E}xt_{S_n^\text{cl}}(A, \mathbb{G}_m) \cong G^*\) by [KKN08a, Thm. 7.4 (2)], the case \(F_2 = \mathbb{G}_m\) gives part (2). Since \(R^1\delta\mathbb{Z} = 0\) by Lemma [11] and \(\mathcal{E}xt_{S_n^{\text{log}}}(A, \mathbb{Z}) \cong \mathcal{H}om_{S_n^{\text{log}}}(Y, \mathbb{Z})\) by [KKN08a, Thm. 7.4 (1)], the case \(F_2 = \mathbb{Z}\) gives part (1). The sheaf \(R^1\delta_*\mathbb{G}_{m,\text{log}}\)}
equals zero by Kato’s logarithmic Hilbert 90 [Kat91 Cor. 5.2]. And we have \( \text{Ext}_{S\log}(A, G_{m, \log}) \cong G_{\log}^*/X \) by [KKN08a Thm. 7.4 (3)]. Then part (3) follows from the case \( F_2 = G_{m, \log} \).

Let \( A \) be a weak log abelian variety with constant degeneration over \( S \), let \( M = [Y \to G_{\log}] \) be the log 1-motive of type \((X, Y)\) defining \( A \). Then the pairing \( \langle \cdot, \cdot \rangle : X \times Y \to G_{m, \log}/G_m \) associated to \( M \) is non-degenerate. Let \( M^* = [X \to G_{\log}^*] \) be the dual log 1-motive of \( M \), then the pairing associated to \( M^* \) is the same (up to switching the positions of \( X \) and \( Y \)) as the paring associated to \( M \), hence it is automatically non-degenerate. If \( A \) is further a log abelian variety with constant degeneration, i.e. the log 1-motive \( M \) is pointwise polarisable, then so is \( M^* \).

**Definition 1.1.** Let \( A \) be a weak log abelian variety with constant degeneration (resp. log abelian variety with constant degeneration) over \( S \). The dual weak log abelian variety with constant degeneration (resp. dual log abelian variety with constant degeneration) of \( A \) is the weak log abelian variety with constant degeneration (resp. log abelian variety with constant degeneration) \( G_{\log}^*(X) \) associated to the log 1-motive \( M^* = [X \to G_{\log}^*] \). We denote the dual of \( A \) by \( A^* \).

Let \( \text{WLAV}_{S}^{\text{CD}} \) (resp. \( \text{LAV}_{S}^{\text{CD}} \)) denote the category of weak log abelian varieties with constant degeneration (resp. log abelian varieties with constant degeneration) over \( S \). Then we have the following proposition.

**Proposition 1.5.** The association of \( A^* \) to \( A \) gives rise to a contravariant functor

\[ (-)^* : \text{WLAV}_{S}^{\text{CD}} \to \text{WLAV}_{S}^{\text{CD}} \]

which restricts to a contravariant functor

\[ (-)^* : \text{LAV}_{S}^{\text{CD}} \to \text{LAV}_{S}^{\text{CD}}. \]

Moreover the functor is a duality functor, i.e. there is a natural isomorphism from the identity functor to \((-)^{**}\).

**Proof.** This follows from [KKN15 1.7] and [KKN08a Thm. 3.4], and the corresponding duality theory of log 1-motives over \( S \). \( \square \)

**Remark 1.4.** Given an abelian scheme \( A \) over the underlying scheme of \( S \), the dual abelian scheme \( A^* \) can also be interpreted as \( \text{Ext}_{S_{\log}}(A, G_m) \). One may wonder if something similar happen in the case of (weak) log abelian varieties with constant degeneration. Note that in the log world, \( G_{m, \log} \) plays the role of \( G_m \) in the non-log world. Part (2) and part (3) of Theorem 1.2 indicates that \( \text{Ext}_{S_{\log}}(A, G_{m, \log}) \cong G_{\log}^*/X \) is not \( A^* = G_{\log}^*/X \) but closely related to \( A^* \).

The following is a partial reformulation of [KKN08a Thm. 7.3].

**Theorem 1.3.** Let \( X \) and \( Y \) be two finitely generated free \( \mathbb{Z} \)-modules, and let \( \langle \cdot, \cdot \rangle : X \times Y \to G_{m, \log}/G_m \) be a non-degenerate paring on \( S_{\log}^2 \).

Let \( G \) be a commutative group scheme over the underlying scheme of \( S \) which is an extension of an abelian scheme \( B \) by a torus \( T \) over \( S \). Let \( X \) be the character
group of $T$. Let $T_{\log}^{(Y)} = \text{Hom}_{S_n^\text{log}}(X, \mathcal{G}_m, \log)^{(Y)} \subset T_{\log} = \text{Hom}_{S_n^\text{log}}(X, \mathcal{G}_m, \log)$ (resp. $G_{\log}^{(Y)} \subset G_{\log}$ be the inverse image of 
$\text{Hom}_{S_n^\text{log}}(X, \mathcal{G}_m, \log)^{(Y)} \subset \text{Hom}_{S_n^\text{log}}(X, \mathcal{G}_m, \log) \cong T_{\log}/T \cong G_{\log}/G$.

(1) Let $H$ be a commutative group scheme over the underlying scheme of $S$. Then we have 
$\text{Hom}_{S_n^\text{log}}(G_{\log}^{(Y)}, H) \cong \text{Hom}_{S_m^\text{log}}(B, H), \quad \text{Hom}_{S_n^\text{log}}(G_{\log}^{(Y)}/G, H) = 0.$

If $H$ satisfies the condition $R^1\delta_*H = 0$, we also have 
$\mathcal{E}xt_{S_m^\text{log}}(G_{\log}^{(Y)}, H) \cong \mathcal{E}xt_{S_m^\text{log}}(B, H), \quad \mathcal{E}xt_{S_m^\text{log}}(G_{\log}^{(Y)}/G, H) \cong \text{Hom}_{S_m^\text{log}}(T, H).$ 

In particular, since $R^1\delta_*\mathbb{Z} = 0$, we have 
$\mathcal{E}xt_{S_m^\text{log}}(G_{\log}^{(Y)}, \mathbb{Z}) \cong \mathcal{E}xt_{S_m^\text{log}}(B, \mathbb{Z}) = 0, \quad \mathcal{E}xt_{S_m^\text{log}}(G_{\log}^{(Y)}/G, \mathbb{Z}) \cong \text{Hom}_{S_m^\text{log}}(T, \mathbb{Z}) = 0.$

(2) We have $\text{Hom}_{S_m^\text{log}}(T_{\log}^{(Y)}, \mathcal{G}_m, \log) \cong X, \quad \text{Hom}_{S_m^\text{log}}(T_{\log}^{(Y)}/T, \mathcal{G}_m, \log) = 0, \quad \mathcal{E}xt_{S_m^\text{log}}(T_{\log}^{(Y)}/T, \mathcal{G}_m, \log) = 0,$
and $\delta_*\mathcal{E}xt_{S_m^\text{log}}(T_{\log}^{(Y)}, \mathcal{G}_m, \log) \subset R^2\delta_*X.$

(3) Let $G'$ be another commutative group scheme over the underlying scheme of $S$ which is an extension of an abelian scheme $B'$ by a torus $T'$ over $S$. Let $X'$ be the character group of $T'$. Then we have 
$\text{Hom}_{S_n^\text{log}}(G, G') \xrightarrow{\cong} \text{Hom}_{S_n^\text{log}}(G_{\log}^{(Y)}, G_{\log}^{(Y)}/G').$

and 
$\text{Hom}_{S_n^\text{log}}(X, X') \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \text{Hom}_{S_n^\text{log}}(G_{\log}^{(Y)}/G, G_{\log}^{(Y)}/G').$

**Proof.** We are going to follow the structure of the proof [KKN08a, Thm. 7.3]. Parallel to [KKN08a, 7.17], we first prove

$\text{Hom}_{S_n^\text{log}}(T_{\log}^{(Y)}, H) = \text{Hom}_{S_n^\text{log}}(T_{\log}^{(Y)}/T, \mathcal{G}_m, \log) = 0, \quad \mathcal{E}xt_{S_n^\text{log}}(T_{\log}^{(Y)}, H) = 0$ 
if $R^1\delta_*H = 0$, and $\delta_*\mathcal{E}xt_{S_n^\text{log}}(T_{\log}^{(Y)}, \mathcal{G}_m, \log) \subset R^2\delta_*X$. We have 
$\text{Hom}_{S_n^\text{log}}(T_{\log}^{(Y)}, H) = 0$

by the corresponding result in the classical étale topology of [KKN08a, 7.17], with the help of Lemma 7.2. The proof of $\text{Hom}_{S_n^\text{log}}(T_{\log}^{(Y)}/T, \mathcal{G}_m, \log) = 0$ in [KKN08a, 7.17] still works in the log flat topology, so we also have 
$\text{Hom}_{S_n^\text{log}}(T_{\log}^{(Y)}/T, \mathcal{G}_m, \log) = 0.$

We make use of the exact sequences (7.7) and (7.8) from the proof of Theorem 7.2 to investigate the sheaves $\mathcal{E}xt_{S_n^\text{log}}(T_{\log}^{(Y)}, H)$ and $\mathcal{E}xt_{S_n^\text{log}}(T_{\log}^{(Y)}, \mathcal{G}_m, \log)$. If $R^1\delta_*H = 0$, then the vanishing of $\text{Hom}_{S_n^\text{log}}(T_{\log}^{(Y)}, H)$ and $\mathcal{E}xt_{S_n^\text{log}}(T_{\log}^{(Y)}, H)$ imply 
$\mathcal{E}xt_{S_n^\text{log}}(T_{\log}^{(Y)}, H) = 0$ via (7.7) and (7.8). We have $\mathcal{E}xt_{S_n^\text{log}}(T_{\log}^{(Y)}, \mathcal{G}_m, \log) = 0$ by
Hence we have $R^1\delta_*G_{m,\log} = 0$ by Kato’s logarithmic Hilbert 90, then the inclusion $\delta_*E\text{xt}^*_n(S_{\text{et}}^{G/(G, H)}/G_{m,\log}) \subset R^2\delta_*X$ follows from \([\text{1.7}]\) and \([\text{1.8}]\).

Now we prove part (1). The isomorphism $\text{Hom}_{S_{\text{et}}^{G/(G, H)}} \cong \text{Hom}_{S_{\text{et}}^{G/(B, H)}}$ comes from the corresponding isomorphism of \([\text{KKN08a}]\) Thm. 7.3 (1) for the classical étale topology with the help of Lemma 1.2. We postpone the proof of the rest after the following lemma \([\text{1.3}]\).

Since $\text{Hom}_{S_{\text{et}}^{G/(G, H)}}(T_{\text{log}}^{(Y)}/G, H) = 0$, we get $\text{Hom}_{S_{\text{et}}^{G/(G, H)}}(T_{\text{log}}^{(Y)}/G, H) = 0$. Now we assume $R^1\delta_*H = 0$, then $\text{Ext}^*_{S_{\text{et}}^{G/(G, H)}}(T_{\text{log}}^{(Y)}, H) = 0$. Hence we have

$$\text{Ext}^*_{S_{\text{et}}^{G/(G, H)}}(T_{\text{log}}^{(Y)}, H) \cong \text{Ext}^*_{S_{\text{et}}^{G/(B, H)}}(T_{\text{log}}^{(Y)}, H), \quad \text{Ext}^*_{S_{\text{et}}^{G/(G, H)}}(T_{\text{log}}^{(Y)}, H) \cong \text{Hom}_{S_{\text{et}}^{G/(T, H)}}(T, H).$$

Next we prove part (2). The exact sequence $0 \to T \to T_{\text{log}}^{(Y)} \to T_{\text{log}}^{(Y)}/T \to 0$ gives rise to a long exact sequence $0 \to \text{Hom}_{S_{\text{et}}^{G/(G, H)}}(T_{\text{log}}^{(Y)}/T, G_{m,\log}) \to \text{Hom}_{S_{\text{et}}^{G/(G, H)}}(T_{\text{log}}^{(Y)}, G_{m,\log}) \to \text{Hom}_{S_{\text{et}}^{G/(T, H)}}(T, G_{m,\log}) \to \text{Ext}^*_{S_{\text{et}}^{G/(G, H)}}(T_{\text{log}}^{(Y)/T}, G_{m,\log}).$

Since the map $\text{Hom}_{S_{\text{et}}^{G/(G, H)}}(T_{\text{log}}^{(Y)}/T, G_{m,\log}) \to \text{Hom}_{S_{\text{et}}^{G}}(T, G_{m,\log})$ is canonically identical to the identity map $1_X : X \to X$ by \([\text{KKN08a}]\) 7.20, so is the map $\alpha$. Hence we have $\text{Hom}_{S_{\text{et}}^{G/(G, H)}}(T_{\text{log}}^{(Y)/T}, G_{m,\log}) = 0$. Since $\text{Hom}_{S_{\text{et}}^{G/(G, H)}}(T_{\text{log}}^{(Y)}), G_{m,\log}) = 0$, $R^1\delta_*G_{m,\log} = 0$ and $\text{Ext}^*_{S_{\text{et}}^{G/(G, H)}}(T_{\text{log}}^{(Y)/T}, G_{m,\log}) = 0$, we get $\text{Ext}^*_{S_{\text{et}}^{G/(G, H)}}(T_{\text{log}}^{(Y)/T}, G_{m,\log}) = 0$ by \([\text{KKN08a}]\) Thm. 7.3 (2), we get $\text{Ext}^*_{S_{\text{et}}^{G/(G, H)}}(T_{\text{log}}^{(Y)/T}, G_{m,\log}) = 0$ by the exact sequences \([\text{1.7}]\) and \([\text{1.8}]\).

At last, we show part (3). The isomorphism

$$\text{Hom}_{S_{\text{et}}^{G}}(G, G) \cong \text{Hom}_{S_{\text{et}}^{G}}(G_{m,\log}), G_{m,\log})$$

comes from the corresponding isomorphism of \([\text{KKN08a}]\) Thm. 7.4 (3) with the help of Lemma 1.2. We postpone the proof of the rest after the following lemma which is going to be used in the proof. □

**Remark 1.5.** In the proof of $\delta_*\text{Ext}^*_n(T_{\text{et}}^{G/(G, H)}/G_{m,\log}) \subset R^2\delta_*X$ in the proof of Theorem \([\text{1.3}]\), we have

$$\text{Hom}_{S_{\text{et}}^{G/(G, H)}}(T_{\text{et}}^{G/(G, H)}, G_{m,\log}) \cong \text{Hom}_{S_{\text{et}}^{G/(G, H)}}(T_{\text{et}}^{G/(G, H)}, G_{m,\log}) \cong X$$

by \([\text{KKN08a}]\) Thm. 7.3 (2). Since We have $\text{Ext}^*_n(T_{\text{et}}^{G/(G, H)}, G_{m,\log}) = 0$ and $R^1\delta_*G_{m,\log} = 0$, we have $R^1\delta_*X = 0$ by the exact sequences \([\text{1.7}]\) and \([\text{1.8}]\). This gives rise to an alternative proof of Lemma 1.3.

**Lemma 1.3.** (1) $\delta_*G_{m,\log}/G_m) = (G_{m,\log}/G_m)_{S_{\text{et}}^{G}} \otimes \mathbb{Z}/\mathbb{Q}$. 
(2) Let $H$ be a commutative group scheme over the underlying scheme of $S$ with connected fibres. Then $\text{Hom}_{S_{fl}}(H, G_{m, \log}/G_{m}) = 0$.

**Proof.** Applying the functor $\varepsilon_{fl}$ to the short exact sequence

$$0 \to G_{m} \to G_{m, \log} \to G_{m, \log}/G_{m} \to 0,$$

we get a long exact sequence

$$0 \to G_{m} \to G_{m, \log} \to \varepsilon_{fl}(G_{m, \log}/G_{m}) \to R^1\varepsilon_{fl}G_{m} \to R^1\varepsilon_{fl}G_{m, \log}.$$

Since $R^1\varepsilon_{fl}G_{m, \log} = 0$ and

$$R^1\varepsilon_{fl}G_{m} = \varprojlim_{n \neq 0} \text{Hom}_{S_{fl}}(\mathbb{Z}/n(1), G_{m}) \otimes_{\mathbb{Z}} (G_{m, \log}/G_{m})_{S_{fl}^1} = (G_{m, \log}/G_{m})_{S_{fl}^1} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z},$$

by [Kat91] Thm. 4.1, we get $\varepsilon_{fl}(G_{m, \log}/G_{m}) = (G_{m, \log}/G_{m})_{S_{fl}^1} \otimes_{\mathbb{Z}} \mathbb{Q}$. Then

$$\delta_{*}(G_{m, \log}/G_{m}) = m_{*}\varepsilon_{fl}(G_{m, \log}/G_{m}) = m_{*}((G_{m, \log}/G_{m})_{S_{fl}^1} \otimes_{\mathbb{Z}} \mathbb{Q}) = (G_{m, \log}/G_{m})_{S_{fl}^1} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

This proves part (1).

Now we prove part (2) which corresponds to [KKN08a] Lem. 6.1.1. We have

$$\text{Hom}_{S_{fl}}(H, G_{m, \log}/G_{m}) = \text{Hom}_{S_{fl}}(H, (G_{m, \log}/G_{m})) = \text{Hom}_{S_{fl}}(H, (G_{m, \log}/G_{m})_{S_{fl}^1} \otimes_{\mathbb{Z}} \mathbb{Q}).$$

By the same argument of the proof of [KKN08a] Lem. 6.1.1, we have

$$\text{Hom}_{S_{fl}}(H, (G_{m, \log}/G_{m})_{S_{fl}^1} \otimes_{\mathbb{Z}} \mathbb{Q}) = 0.$$

Hence part (2) is proved. 

**Proof of Theorem 1.3 (continued):** The short exact sequence

$$0 \to T \to T_{log}^{(Y)} \to G_{log}^{(Y)}/G \to 0$$

gives an exact sequence

$$0 \to \text{Hom}_{S_{fl}}(G_{log}^{(Y)}/G, G_{log}^{(Y)}/G') \to \text{Hom}_{S_{fl}}(T_{log}^{(Y)}, G_{log}^{(Y)}/G') \to \text{Hom}_{S_{fl}}(T, G_{log}^{(Y)}/G').$$

The sheaf $\text{Hom}_{S_{fl}}(T, G_{log}^{(Y)}/G')$ is zero by part (2) of Lemma 1.3 hence we are reduced to compute $\text{Hom}_{S_{fl}}(T_{log}^{(Y)}, G_{log}^{(Y)}/G')$. We have

$$\delta_{*}\text{Hom}_{S_{fl}}(T_{log}^{(Y)}, G_{log}^{(Y)}/G') = \text{Hom}_{S_{fl}^1}(T_{log}^{(Y)}, \delta_{*}(G_{log}^{(Y)}/G')) = \text{Hom}_{S_{fl}^1}(T_{log}^{(Y)}, (G_{log}^{(Y)}/G')_{S_{fl}^1} \otimes_{\mathbb{Z}} \mathbb{Q}).$$
where the second equality comes from part (1) of Lemma \[\text{(3)}\] By [KKN08a] Thm. 7.4 (3),
\[
\text{Hom}_{S_{\text{cl}}^{\text{et}}}((G^{(Y)}/G)_{S_{\text{cl}}^{\text{et}}}, (G_{\log}^{G'}/G')_{S_{\text{cl}}^{\text{et}}}) \\
\cong \text{Hom}_{S_{\text{cl}}^{\text{et}}}((G^{(Y)}/G)_{S_{\text{cl}}^{\text{et}}}, (G_{\log}^{G'}/G')_{S_{\text{cl}}^{\text{et}}}).
\]

It follows that
\[
\text{Hom}_{S_{\text{log}}^{\text{fl}}}(T^{(Y)}, G_{\log}^{G'}) \cong \text{Hom}_{S_{\text{cl}}^{\text{et}}}(X, X') \otimes \mathbb{Q} = \text{Hom}_{S_{\text{log}}^{\text{fl}}}(X, X') \otimes \mathbb{Q}.
\]

Since weak log abelian varieties with constant degeneration are defined in terms of log 1-motives, it is natural to try to relate every aspect of weak log abelian varieties with constant degeneration to the corresponding aspect of log 1-motives. In particular, we are keen on the relation on the homomorphisms. The following theorem is a generalisation of [KKN08a] Thm. 3.4 to the case of weak log abelian varieties with constant degeneration, see also [KKN15, 1.7].

**Theorem 1.4.** The functor \([Y \to G_{\log}] \mapsto G^{(Y)}/Y\) induces an equivalence from the category of non-degenerate log 1-motives (resp. pointwise polarisable log 1-motives) over \(S\) to that of weak log abelian varieties with constant degeneration (resp. log abelian varieties with constant degeneration) over \(S\).

**Proof.** Firstly we observe that the objects appeared in the proof [KKN08a] Thm. 8.1 of [KKN08a] Thm. 3.4, which are all sheaves for the classical étale topology, are also sheaves for the log flat topology. So Lemma \[\text{(2)}\] applies. Secondly the proof [KKN08a] Thm. 8.1 used [KKN08a] Thm. 7.4 (1) (4), the first part of Thm. 7.3 (3), and these results remain true in the log flat topology by Theorem \[\text{(3)}\] and Theorem \[\text{(3)}\]. Hence the proof [KKN08a] Thm. 8.1 also works identically here. \[\square\]

Let \(f : A \to A'\) be a homomorphism between two weak log abelian varieties over \(S\), let \(M = [Y \to G_{\log}], M' = [Y' \to G'_{\log}]\) be the log 1-motives defining \(A\) and \(A'\) respectively. By Theorem \[\text{(3)}\] \(f\) comes from a homomorphism from \(M\) to \(M'\), we denote it by \((f_{-1}, f_0)\). The proof of [KKN08a] Thm. 8.1 actually shows that \(f_0\) comes from a homomorphism from \(G\) to \(G'\), and we denote it by \(f_c\) by convention (here the subscript \(c\) stands for connected). The homomorphism can also be obtained from the following diagram

\[
\begin{array}{cccccc}
0 & \to & G & \to & A & \to & Q/Y & \to & 0 \\
\downarrow f_c & & \downarrow f & & \downarrow f_a & & \\
0 & \to & G' & \to & A' & \to & Q'/Y' & \to & 0
\end{array}
\]

with exact rows, combining with the fact \(\text{Hom}_{S_{\text{log}}^{\text{fl}}}(G, Q'/Y') = 0\). Here the exact rows come from part (3) of Theorem \[\text{(4)}\] \(Q\) (resp. \(Q'\)) denotes the sheaf \(\text{Hom}_{S_{\text{log}}^{\text{fl}}}(X, G_{m, \log}/G_m)(Y)\) (resp. \(\text{Hom}_{S_{\text{log}}^{\text{fl}}}(X', G_{m, \log}/G_m)(Y')\)). Furthermore the
diagram gives a homomorphism $\mathbb{Q}/Y \to \mathbb{Q}'/Y'$ which we denotes by $f_d$ (here the subscript stands for discrete). The procedure of getting $f$ from $(f^{-1}, f_0)$ also gives a homomorphism $\tilde{A} := G_{\log}^{(Y)} \to G_{\log}^{(Y')} =: \tilde{A}'$, which we denote by $\tilde{f}$. The homomorphism induces a homomorphism $\mathbb{Q} \to \mathbb{Q}'$ which we denote by $\tilde{f}_d$.

For practical reason, we make the following proposition which is nothing but a tedious summary of the above.

**Proposition 1.6.** Let $f : A \to A'$ be a homomorphism of weak log abelian varieties with constant degeneration over $S$. Then $f$ induces the following four commutative diagrams

\[
\begin{align*}
(1.9) & \quad 0 \to Y \to G_{\log}^{(Y)} \to A \to 0 \\
& \quad \downarrow f^{-1} \quad \downarrow \tilde{f} \quad \downarrow f \\
0 \to Y' \to G_{\log}^{(Y')} \to A' \to 0
\end{align*}
\]

\[
\begin{align*}
(1.10) & \quad 0 \to G \to A \to \mathbb{Q}/Y \to 0 \\
& \quad \downarrow f_c \quad \downarrow f \quad \downarrow f_d \\
0 \to G' \to A' \to \mathbb{Q}'/Y' \to 0
\end{align*}
\]

\[
\begin{align*}
(1.11) & \quad 0 \to G \to G_{\log}^{(Y)} \to \mathbb{Q} \to 0 \\
& \quad \downarrow f_c \quad \downarrow \tilde{f} \quad \downarrow f_d \\
0 \to G' \to G_{\log}^{(Y')} \to \mathbb{Q}' \to 0
\end{align*}
\]

\[
\begin{align*}
(1.12) & \quad 0 \to Y \to \mathbb{Q} \to \mathbb{Q}/Y \to 0 \\
& \quad \downarrow f^{-1} \quad \downarrow f_d \quad \downarrow f_d \\
0 \to Y' \to \mathbb{Q}' \to \mathbb{Q}'/Y' \to 0
\end{align*}
\]

with exact rows.

2. Isogeny

In this section we study log abelian varieties over a log point. Note in this case, log abelian varieties are necessarily log abelian varieties with constant degeneration by [KKN08a, Thm. 4.6 (2)].

Let $k$ be a field, and $S = (\text{Spec} k, M_S)$ be an fs log point with log structure induced by a chart $P \to k$, where $P$ is a sharp fs monoid so that $P \to (M_S/O_S^x/x)$ is an isomorphism (here $x$ is the underlying point of $S$). Let $(\text{fs}/S)$ be the category of fs log schemes over $S$, and log schemes in this section will always be fs log schemes unless otherwise stated. Let $S_{\log}^{\text{flat}}$ (resp. $S_{\text{cl}}^{\text{flat}}$) be the log flat (resp. classical flat) site
over $S$ on $(\text{fs}/S)$, and let $S^\log_{\text{Et}}$ (resp. $S^\log_{\text{Et}}$) be the log étale (resp. classical étale) site over $S$ on $(\text{fs}/S)$.

2.1. Isogeny. First we recall several kinds of finite group objects on $S^\log$ defined by Kato, see [Kat92 §1] or [MP] 2.1

**Definition 2.1.** The category $(\text{fin}/S)^c$ is the full subcategory of the category of sheaves of finite abelian groups over $S^\log$ consisting of objects which are representable by a classical finite flat group scheme over $S$. Here classical means the log structure of the representing log scheme is the one induced from $S$.

The category $(\text{fin}/S)^f$ is the full subcategory of the category of sheaves of finite abelian groups over $S^\log$ consisting of objects which are representable by a classical finite flat group scheme over a log flat cover of $S$. Let $F \in (\text{fin}/S)^f$, let $U \to S$ be a log flat cover of $S$ such that $F_U := F \times_S U \in (\text{fin}/S)^c$, the rank of $F$ is defined to be the rank of $F_U$ over $U$.

The category $(\text{fin}/S)^r$ is the full subcategory of $(\text{fin}/S)^f$ consisting of objects which are representable by a log scheme over $S$.

The category $(\text{fin}/S)^d$ is the full subcategory of $(\text{fin}/S)^r$ consisting of objects whose Cartier duals also lie in $(\text{fin}/S)^r$.

First we show that as a subcategory of $(\text{fin}/S)^r$ the category $(\text{fin}/S)^c$ is closed under taking subobject in $(\text{fin}/S)^r$.

**Lemma 2.1.** Let $F_1 \in (\text{fin}/S)^c$ and $F_2$ be a subobject of $F_1$ in $(\text{fin}/S)^r$. Then we have $F_2 \in (\text{fin}/S)^c$.

**Proof.** Let $0 \to F_2^\circ \to F_2 \to F_2^{\text{et}} \to 0$ be the connected-étale short exact sequence of $F_2$, see [Kat92] 2.6 or [MP] Lem. 2.1.6. Then we have $F_2^\circ, F_2^{\text{et}} \in (\text{fin}/S)^c$, and $F_2^\circ, F_2^{\text{et}} \in (\text{fin}/S)$. (resp. $F_2^\circ \in (\text{fin}/S)^r$) by [Kat92] Prop. 2.7 (2) or [MP] Prop. 2.1.7 (resp. [Kat92] Prop. 2.7 (1)). The inclusion $i : F_2 \hookrightarrow F_1$ gives a commutative diagram

\[
\begin{array}{ccc}
0 & \to & F_2^\circ \to F_2 \to F_2^{\text{et}} \to 0 \\
\downarrow i & & \downarrow i & \downarrow i_{\text{et}} \\
0 & \to & F_1^\circ \to F_1 \to F_1^{\text{et}} \to 0 
\end{array}
\]

with exact rows and injective vertical homomorphisms. As a subobject of $F_1^{\text{et}}$ which is classical étale, $F_2^{\text{et}}$ is classical étale by considering the theory of logarithmic fundamental groups, see [Kat91] §10 or [MP] Thm. 1.4.5. Let $E$ be the pullback
of the extension $F_1$ along $i^*: F_2^{et} \hookrightarrow F_1^{et}$, then we have a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & F_2 & \rightarrow & F_2^{et} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & F_1 & \rightarrow & F_1^{et} & \rightarrow & 0 \\
\end{array}
$$

with exact rows. Note that $E$ lies in $(\text{fin}\, S)^c$, and $E$ is also the pushout of $F_2$ along $i^*: F_2^{et} \rightarrow F_1^{et}$. Now we make use of Kato’s description of the category of extensions of two classical finite group schemes in $S_{\log}^{\text{et}}$; see [MP] 2.3.0.1, Thm. 2.3.1 or [Kat92] Thm. 3.3. Note the functor $\Phi$ of [MP] 2.3.0.1 is compatible with pushout along the second argument, hence we have a commutative diagram

$$
\begin{array}{cccccc}
\text{Ext}_{\text{et}}(F_2^{et}, F_2^{et}) & \times & \text{Hom}(F_2^{et}(1), F_2^{et}) & \otimes & \text{Pr}\text{esp} & \cong & \text{Ext}_{\text{et}}(F_2^{et}, F_2^{et}) \\
\downarrow i^* & & \downarrow i^* & & \downarrow i^* & & \\
\text{Ext}_{\text{et}}(F_2^{et}, F_2^{et}) & \times & \text{Hom}(F_2^{et}(1), F_2^{et}) & \otimes & \text{Pr}\text{esp} & \cong & \text{Ext}_{\text{et}}(F_2^{et}, F_2^{et})
\end{array}
$$

with rows equivalences of categories. Let $[E] \in \text{Ext}_{\text{et}}(F_2^{et}, F_2^{et})$ (resp. $[F_2] \in \text{Ext}_{\text{et}}(F_2^{et}, F_2^{et})$) denotes the class represented by $E$ (resp. $F_2$), with the help of the above commutative diagram, $[E] \in \text{Ext}_{\text{et}}(F_2^{et}, F_2^{et})$ implies $[F_2] \in \text{Ext}_{\text{et}}(F_2^{et}, F_2^{et})$.

\begin{proof}
Let $n$ be a positive integer such that $nF = 0$, then we have $F \subset A[n] := \text{Ker}(A \xrightarrow{\times n} A)$. Since the map $G \xrightarrow{\times n} G$ is an isogeny, we get a short exact sequence $0 \rightarrow G[n] \rightarrow A[n] \rightarrow (\mathbb{Q}/\mathbb{Z})[n] \rightarrow 0$ by diagram (1.10), where $\mathbb{Q}$ denotes the sheaf $\text{Hom}(X, \mathbb{G}_{m, \log}/\mathbb{G}_{m})^{(\ast)}$. Since the map $\mathbb{Q} \xrightarrow{\times n} \mathbb{Q}$ is an isomorphism, we get $(\mathbb{Q}/\mathbb{Z})[n] \cong Y/nY$ by diagram (1.12). We also have $A[n] \in (\text{fin}\, S)^c$ by [KKN15] Prop. 18.1 (1). Now let $F'$ be the kernel of the composition $F \rightarrow A[n] \rightarrow (\mathbb{Q}/\mathbb{Z})[n] \cong Y/nY$, $F''$ be the image of $F$ in $Y/nY$, then we have a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & F' & \rightarrow & F & \rightarrow & F'' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & G[n] & \rightarrow & A[n] & \rightarrow & Y/nY & \rightarrow & 0
\end{array}
$$

Proposition 2.1. Let $A$ be a log abelian variety over $S$, and $F \in (\text{fin}\, S)^c$ be a subsheaf of $A$. Then:

1. $F$ is an extension of objects of $(\text{fin}\, S)^c$.
2. $F \in (\text{fin}\, S)^c_d$.
3. The quotient $A/F$ is also a log abelian variety over $S$.

Proof. Let $n$ be a positive integer such that $nF = 0$, then we have $F \subset A[n] := \text{Ker}(A \xrightarrow{\times n} A)$. Since the map $G \xrightarrow{\times n} G$ is an isogeny, we get a short exact sequence $0 \rightarrow G[n] \rightarrow A[n] \rightarrow (\mathbb{Q}/\mathbb{Z})[n] \rightarrow 0$ by diagram (1.10), where $\mathbb{Q}$ denotes the sheaf $\text{Hom}(X, \mathbb{G}_{m, \log}/\mathbb{G}_{m})^{(\ast)}$. Since the map $\mathbb{Q} \xrightarrow{\times n} \mathbb{Q}$ is an isomorphism, we get $(\mathbb{Q}/\mathbb{Z})[n] \cong Y/nY$ by diagram (1.12). We also have $A[n] \in (\text{fin}\, S)^c$ by [KKN15] Prop. 18.1 (1). Now let $F'$ be the kernel of the composition $F \rightarrow A[n] \rightarrow (\mathbb{Q}/\mathbb{Z})[n] \cong Y/nY$, $F''$ be the image of $F$ in $Y/nY$, then we have a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & F' & \rightarrow & F & \rightarrow & F'' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & G[n] & \rightarrow & A[n] & \rightarrow & Y/nY & \rightarrow & 0
\end{array}
$$

\end{proof}
with exact rows and injective vertical homomorphisms. As a kernel of a homomorphism between two representable objects, \( F' \in (\text{fin}/S)_c \); as a subobject of \( Y/nY \) which is a classical finite étale group scheme, \( F'' \in (\text{fin}/S)_c \). Applying Lemma \([2.1]\) to the inclusion \( F' \subset G[n] \), we conclude \( F' \in (\text{fin}/S)_c \), hence part (1) is proven. Part (2) follows from part (1) and [Kat92 Prop. 2.3].

Now we show part (3). It suffices to find a polarisable log 1-motive such that \( A/F \) is isomorphic to its associated quotient. Consider the pullback \( E \) of \( G_{\log}^{(Y)} \) along \( A \)

\[
\begin{array}{ccccccc}
0 & \rightarrow & Y & \rightarrow & E & \rightarrow & 0 \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \rightarrow & Y & \rightarrow & G_{\log}^{(Y)} & \rightarrow & 0,
\end{array}
\]

let \( E_{\text{tor}} \) be the torsion subsheaf of \( E \), \( Y' := E/E_{\text{tor}} \). Since the sheaf

\[
G_{\log}^{(Y)}/G = \mathcal{H}om(X, G_{m,\log}/G_m)^{(Y)}
\]

is torsion-free, \( E_{\text{tor}} \) maps into \( G \). So we get \( E_{\text{tor}} = F' \) and \( Y'/Y = F'' \subset Y/nY \), and \( Y' \) is étale locally constant. Let \( G' = G/E_{\text{tor}} = G/F' \), the inclusion \( E \hookrightarrow G_{\log}^{(Y)} \hookrightarrow G_{\log} \) gives a homomorphism \( Y' \to G_{\log}/F' = G'_{\log} \) by taking quotient by \( F' \). In this way, we get a log 1-motive \( M' := [Y' \to G'] \) together with a homomorphism \((f_{-1}, f_0) : M := [Y \to G] \to M' \). By the construction of the homomorphism \((f_{-1}, f_0) \), it is clear that the multiplication by \( n \) maps on \( M \) factors through \((f_{-1}, f_0) \). Let \((g_{-1}, g_0) : M' \to M \) be the homomorphism such that \( n_M = (g_{-1}, g_0) \circ (f_{-1}, f_0) \), let \((h_{-1}, h_0) : M \to M^* = [X \to G_{\log}] \) be a polarisation of \( M \), then \((g^*_{-1} \circ h_{-1} \circ g_{-1}, g^*_0 \circ h_0 \circ g_0) \) gives rise to a polarisation of \( M' \), where \((g^*_{-1}, g^*_0) : M^* \to M'^* = [X' \to G_{\log}^{(Y)}]^* \) is the dual of \((g_{-1}, g_0) \). By [KK08a Thm. 3.4], the homomorphism \((f_{-1}, f_0) \) gives rise to a homomorphism \( f : A \to A' \) of log abelian varieties with constant degeneration, where \( A' \) is the associated log abelian variety of \( M' \). It is easy to see that \( A' = A/F \).

**Definition 2.2.** Let \( A, A' \) be two log abelian varieties over \( S \). An isogeny from \( A \) to \( A' \) is a homomorphism \( f \) from \( A \) to \( A' \) such that \( f \) is surjective for the log flat topology and \( \text{Ker}(f) \in (\text{fin}/S)_r \).

**Remark 2.1.** Let \( f : A \to A' \) be an isogeny between two log abelian varieties over \( S \). By Proposition \([2.1]\) (2), we have \( \text{Ker}(f) \in (\text{fin}/S)_d \). Hence we can replace the condition \( \text{Ker}(f) \in (\text{fin}/S)_r \), by the a priori stronger condition \( \text{Ker}(f) \in (\text{fin}/S)_d \) in the definition of isogeny.

**Example 2.1.** By Proposition \([2.1]\) a subsheaf \( F \in (\text{fin}/S)_r \) of a log abelian variety \( A \) over \( S \) gives an isogeny \( A \to A/F \).

Isogenies between abelian varieties can be defined by several equivalent conditions, and some of the conditions concern the dimension. Here we show the same thing happens for log abelian varieties over \( S \).
\textbf{Lemma 2.2.} Let $M = [Y \to G_{\log}]$, $M' = [Y' \to G'_{\log}]$ be two non-degenerate log 1-motives over $S$, let $(f_{-1}, f_0) : M \to M'$ be a homomorphism of log 1-motives, and let $f_c : G \to G'$ be the map induced by $f_0$. If $f_c$ is an isogeny, then the map $\tilde{f} : G(Y) \to G'^{(Y)}$ induced by $(f_{-1}, f_0)$ is surjective with kernel $\text{Ker}(f_c)$, and the induced map $\tilde{f}_d : \text{Hom}_{\mathcal{S}^m}(X, \mathbb{G}_{m, \log}/\mathbb{G}_m)(Y) \to \text{Hom}_{\mathcal{S}^m}(X', \mathbb{G}_{m, \log}/\mathbb{G}_m)(Y')$ is bijective.

\textbf{Proof.} Let $X$ (resp. $X'$) be the character group of the torus part $T$ (resp. $T'$) of $G$ (resp. $G'$), let $f_1 : X' \to X$ be the map induced by $f_c$, then the map $T \to T'$ induced by $f_c$ is an isogeny and $f_1$ is injective and of finite cokernel. Let

$$Q := \text{Hom}_{\mathcal{S}^m}(X, \mathbb{G}_{m, \log}/\mathbb{G}_m)(Y), \quad Q' := \text{Hom}_{\mathcal{S}^m}(X', \mathbb{G}_{m, \log}/\mathbb{G}_m)(Y'),$$

and let $\tilde{f}_d : Q \to Q'$ be the map induced by $f_1$. We consider the following commutative diagram

$$\begin{array}{cccccc}
0 & \rightarrow & G & \rightarrow & G^{(Y)}_{\log} & \rightarrow & Q & \rightarrow & 0 \\
\downarrow{f_c} & & \downarrow{f} & & \downarrow{\tilde{f}_d} & & \\
0 & \rightarrow & G' & \rightarrow & G'^{(Y)}_{\log} & \rightarrow & Q' & \rightarrow & 0
\end{array}$$

with exact rows. To show the surjectivity of $\tilde{f}$, it is enough to show the surjectivity of $\tilde{f}_d$. The induced map $T_{\log} \rightarrow T'_{\log}$ is surjective by Proposition 3.3 (5). Furthermore we have the surjectivity of the map

$$\text{Hom}_{\mathcal{S}^m}(X, \mathbb{G}_{m, \log}/\mathbb{G}_m) \rightarrow \text{Hom}_{\mathcal{S}^m}(X', \mathbb{G}_{m, \log}/\mathbb{G}_m).$$

Thus for any $\varphi' \in Q'$, there exists some $\varphi \in \text{Hom}_{\mathcal{S}^m}(X, \mathbb{G}_{m, \log}/\mathbb{G}_m)$ mapping to $\varphi'$, i.e. $\varphi' = \varphi \circ f_1$. In order to show the surjectivity of $\tilde{f}_d$, it suffices to show that $\varphi \in Q$. Let $n$ be a positive integer killing $X/f_1(X')$, let $<, > : X \times Y \rightarrow \mathbb{G}_{m, \log}/\mathbb{G}_m$ and $<, >' : X' \times Y' \rightarrow \mathbb{G}_{m, \log}/\mathbb{G}_m$ be the pairings associated to $M$ and $M'$ respectively. Given any $U \in (\mathcal{S}/S), u \in U, x \in X_0$, there exists $x' \in X_0'$ such that $nx = f_1(x')$. By the definition of $Q'$, there exist $y'_{u,x,1}, y'_{u,x,2} \in Y_0'$ such that

$$< x', y'_{u,x,1} >' | \varphi'(x') | < x', y'_{u,x,2} >'.$$

The map $f_1$ being injective with finite cokernel, together with the non-degeneracy of $M$ and $M'$, forces $f_{-1}$ to be injective with finite cokernel. If necessary we enlarge $n$ such that it also kills the cokernel of $f_{-1}$. Then there exist $y_{u,x,1}', y_{u,x,2}' \in Y_0'$ such that $ny'_{u,x,1}' = f_{-1}(y_{u,x,1})$ and $ny'_{u,x,2}' = f_{-1}(y_{u,x,2})$. Raising the relation (2.1) to $n$-th power, we get a new relation

$$< x', f_{-1}(y_{u,x,1}) >' | \varphi'(x')^n | < x', f_{-1}(y_{u,x,2}) >'.$$

Since $< f_1(-), >' = < - , f_{-1}(-) >'$, the relation (2.2) can be rewritten as

$$< x, y_{u,x,1} >^n | \varphi(x)^n | < x, y_{u,x,2} >^n.$$
By [KKN15 18.10], there exist \( y_1, y_2 \in Y_A \) such that \( \langle x, y_1^n \rangle > \langle x, y_1, y_1' \rangle \) and \( \langle x, y_1, y_2' \rangle > \langle x, y_2^n \rangle \). Therefore relation (2.3) gives another relation
\[
\langle x, y_1 \rangle > n^2 | \varphi(x) n^2 | < x, y_2 > n^2 .
\]
Removing the exponents from (2.4), we get \( < x, y_1 > | \varphi(x) | < x, y_2 > \), hence \( \varphi \in \mathcal{Q} \).

The injectivity of \( \tilde{f}_d \) follows from the injectivity of \( \text{Hom}_{S_Y}^\log(X, \mathbb{G}_{m, \log}/\mathbb{G}_m) \to \text{Hom}_{S_Y}^\log(X', \mathbb{G}_{m, \log}/\mathbb{G}_m) \). The identification \( \text{Ker}(f_c) = \text{Ker}(\tilde{f}) \) follows from the injectivity of \( \tilde{f}_d \).

Recall that the dimension of a log abelian variety is defined to be the dimension of its semi-abelian part, see [KKN08a 4.4].

**Proposition 2.2.** Let \( f : A \to A' \) be a homomorphism of log abelian varieties over \( S \). Consider the following conditions:

1. \( f \) is an isogeny;
2. \( f \) is surjective for the log flat topology and \( \dim A = \dim A' \);
3. \( \text{Ker} f \in (\text{fin}/S)_r \) and \( \dim A = \dim A' \).

Then we have (2) \( \iff \) (1) \( \iff \) (3).

**Proof.** Let \( M = [Y \to \mathbb{G}_{log}] \) (resp. \( M' = [Y' \to \mathbb{G}'] \)) be the log 1-motive defining \( A \) (resp. \( A' \)), let \( T \) (resp. \( T' \)) be the torus part of \( G \) (resp. \( G' \)), and let \( X \) (resp. \( X' \)) be the character group of \( T \) (resp. \( T' \)). By Theorem 1.4, \( f \) comes from a homomorphism \( (f_1, f_0) : M \to M' \). Consider the short exact sequences
\[
0 \to G \to A \to \mathcal{Q}/Y \to 0
\]
and
\[
0 \to G' \to A' \to \mathcal{Q}'/Y' \to 0
\]
associated to \( A \) and \( A' \) respectively, where \( \mathcal{Q} := \text{Hom}(X, \mathbb{G}_{m, \log}/\mathbb{G}_m)^{\vee} \) and \( \mathcal{Q}' := \text{Hom}(X', \mathbb{G}_{m, \log}/\mathbb{G}_m)^{\vee} \). Since \( \text{Hom}(G, \mathcal{Q}/Y) = 0 \) by [KKN08a 9.2], \( f \) induces a commutative diagram
\[
\begin{array}{ccc}
0 & \longrightarrow & G \\
\downarrow f_c & & \downarrow f \\
A & \longrightarrow & \mathcal{Q}/Y \\
\downarrow f_\mathcal{A} & & \downarrow f_\mathcal{A} \\
0 & \longrightarrow & A'
\end{array}
\]
with exact rows, where the subscripts of \( f_c \) and \( f_\mathcal{A} \) stands for “connected” and “discrete” respectively. If \( f \) is an isogeny, we have \( \text{Ker} f \in (\text{fin}/S)_r \) and \( A' = A'/\text{Ker} f \). By the construction of \( A'/\text{Ker} f \) as a log abelian variety in the proof of part (3) of Proposition 2.1, we have \( f_c \) is an isogeny of semi-abelian varieties, hence \( \dim G = \dim G' \). This shows that (1) implies both (2) and (3).

Now we show (3) implies (1). The condition \( \text{Ker} f \in (\text{fin}/S)_r \) implies \( \text{Ker} f_c \) is a finite group scheme. Since \( \dim G = \dim A = \dim A' = \dim G' \), \( f_c \) is an isogeny.
By Lemma 2.2, the map $\tilde{f} : G_{\log}^{(Y)} \to G_{\log}^{(Y')} = 0$ is surjective. Then the surjectivity of $f$ follows from the following commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & Y \\
\downarrow f^{-1} & & \downarrow \tilde{f} \\
0 & \longrightarrow & G_{\log}^{(Y)} \\
\end{array}
\begin{array}{ccc}
& \longrightarrow & A \\
& \downarrow f & \downarrow \\
& 0 & \longrightarrow 0 \\
\end{array}
$$

with exact rows. Hence $f$ is an isogeny. 

\[ \square \]

**Example 2.2.** Let $A$ be a log abelian variety over $S$. Let $M = [Y \to G_{\log}]$ be the log 1-motive defining $A$, let $M^* = [X \to G_{\log}^*]$ be the dual of $M$ and $(\lambda_-, \lambda_0) : M \to M^*$ be a polarisation, see [KKN08a, Def. 2.8] for the definition of polarisation. Then the map $\lambda : A \to A^*$ induced by $(\lambda_-, \lambda_0)$ is an isogeny. One calls $\lambda$ a polarisation of the log abelian variety $A$.

**Proposition 2.3.** Let $A$ be a log abelian variety over $S$, let $g$ be the dimension of $A$, and let $n$ be a positive integer.

1. The multiplication-by-$n$ map $n_A : A \to A$ is an isogeny.
2. The rank of $A[n] := \text{Ker}(n_A)$ is $n^{2g}$.
3. $A[n] \in \text{fin}(S)_d$.
4. Let $(n_A)^*$ be the dual of the map $(n_A)$, then $(n_A)^* = n_{A^*}$.
5. If $n$ is coprime to the characteristic of $k$, Kummer étale locally on $S$, $A[n]$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$.

**Proof.** By [KKN15, Prop. 18.1], we have $A[n] \in \text{fin}(S)_r$, hence part (1) is a corollary of Proposition 2.2. For part (2) and part (5), we refer to [KKN15, Prop. 18.1]. Part (3) follows from Remark 2.1. We are left with part (4). Let $M = [Y \to G_{\log}]$ be the log 1-motive defining $A$, then $n_A$ is the map induced by the map $n_M$. Since the dual of $n_M$ is the map $n_{M^*}$, where $M^*$ denote the dual of $M$, the dual of $n_A$ is nothing but $n_{A^*}$. 

\[ \square \]

**2.2. The dual short exact sequence.** Recall for an isogeny $f : A \to A'$ between two abelian varieties over a field, we have that the dual $f^*$ of $f$ is an isogeny with kernel $(\text{Ker} f)^*$. In this subsection we show the same thing holds for log abelian varieties over $S$.

Let $f : A \to A'$ be an isogeny between two log abelian varieties over $S$, let $F$ be the kernel of $f$, then we get a short exact sequence $0 \to F \to A \xrightarrow{f} A' \to 0$. Applying the functor $\mathcal{E}xt_{S^{\log}_{n}}(\_ , \mathbb{G}_{m, \log})$ to this short exact sequence, we get a long exact sequence

\[ \begin{align*}
\to & \mathcal{E}xt_{S^{\log}_{n}}(A, \mathbb{G}_{m, \log}) \rightarrow \mathcal{E}xt_{S^{\log}_{n}}(F, \mathbb{G}_{m, \log}) \\
\rightarrow & \mathcal{E}xt_{S^{\log}_{n}}(A', \mathbb{G}_{m, \log}) \\
\end{align*} \]

(2.5)

By Theorem 1.2 (4), $\mathcal{E}xt_{S^{\log}_{n}}(A, \mathbb{G}_{m, \log}) = 0$. By Theorem 1.2 (3), the map $\mathcal{E}xt_{S^{\log}_{n}}(A', \mathbb{G}_{m, \log}) \to \mathcal{E}xt_{S^{\log}_{n}}(A, \mathbb{G}_{m, \log})$ is just the map $G_{\log}^{a*}/X' \to G_{\log}^{a}/X$. The
torsion-free nature of $G_{m, \log}/G_m$ implies $\text{Hom}_{S_{\log}}(F, G_{m, \log}/G_m) = 0$, hence we have $\text{Hom}_{S_{\log}}(F, G_{m, \log}) = F^*$, where $F^* = \text{Hom}_{S_{\log}}(F, G_m)$ is the Cartier dual of $F$ (see [MP 2.1.1]).

We have $A^* = G^*(X)/X$ and $A'^* = G'^*(X')/X'$, and the map $X' \to G'^*(X')$ gives a short exact sequence

$$0 \to A'^* \to G'^*(X')/X' \to R \to 0$$

where $R$ denotes the quotient sheaf $\text{Hom}_{S_{\log}}(Y', G_{m, \log}/G_m) / \text{Hom}_{S_{\log}}(Y', G_{m, \log}/G_m)(X')$.

Putting all these ingredients together, we get a commutative diagram

(2.6) $\begin{pmatrix}
0 & A'^* & A^* \\
A'^* & F^* & A^* \\
0 & G'^*(X')/X' & G^*(X)/X \\
& G'^*(X')/X' & G^*(X)/X \\
& R & R \\
& 0 & 0
\end{pmatrix}$

with exact rows and columns.

**Lemma 2.3.** The sheaf $R$ is torsion free.

**Proof.** We consider the following short exact sequence

$$0 \to Q^* \to \text{Hom}_{S_{\log}}(Y', G_{m, \log}/G_m) \to R \to 0$$

where $Q^*$ denotes $\text{Hom}_{S_{\log}}(Y', G_{m, \log}/G_m)(X')$. To show $R$ is torsion-free, it is enough to show that any section $\varphi \in \text{Hom}_{S_{\log}}(Y', G_{m, \log}/G_m)$, satisfying $\varphi^n \in Q^*$, actually lies in $Q^*$. For any $U \in (\text{fs}/S), u \in U, y' \in Y'_u$, there exist $x'_{u, y', 1}, x'_{u, y', 2} \in X'_u$ such that $<x'_{u, y', 1}, y'> < \varphi^n(y') < x'_{u, y', 2}, y'>$. By [KKN15 18.10], there exist $x_1, x_2 \in X'_u$ such that

$$<x_1, y'> < x'_{u, y', 1}, y'> < x'_{u, y', 2}, y'> < x_2, y' >.$$

Then we get a relation $<x_1, y'> < x'_{u, y', 1}, y'> < x_2, y' >$. Removing the exponent, we further get $<x_1, y'> < \varphi^n(y') < x_2, y' >$, which shows that $\varphi \in Q^*$.

**Theorem 2.1.** We have a canonical short exact sequence

$$0 \to F^* \to A'^* \overset{f^*}{\to} A^* \to 0,$$

in other words $f^*$ is an isogeny with kernel the Cartier dual of $F$. 
Proof. Since \( F \in (\text{fin}/S)_A \) by Proposition 2.1 (2), \( F^* \in (\text{fin}/S)_A \). The sheaf \( R \) is torsion-free by Lemma 2.3 hence we have \( \text{Hom}_{\text{fin}/S}(F^*, R) = 0 \). It follows that the map \( F^* \to G^*_1/X' \) in the diagram (2.6) factors through \( A^* \). Furthermore, \( F^* \) is actually the kernel of \( f^* \). Since \( \dim A^* = \dim A^* \), \( f^* \) is an isogeny by Proposition 2.2.

2.3. The Poincaré complete reducibility theorem. The Poincaré complete reducibility theorem for abelian varieties plays a very important role in the theory of abelian varieties. In this subsection, we formulate a Poincaré complete reducibility theorem for log abelian varieties admitting a polarisation over \( S \).

Lemma 2.4. Let \( M = [Y \to G_{\text{log}}] \) be a log 1-motive over \( S \) with a polarisation \((\lambda_{-1}, \lambda_0) : M = [Y \to G_{\text{log}}] \to [X \to G^*_{\text{log}}] = M^* \), let \( A \) be the log abelian variety associated to \( M \). Let \( M_1 = [Y_1 \to G^*_{\text{log}}] \) be another log 1-motive with \( \text{rank}_2 Y_1 = \text{rank}_2 X_1 \), where \( X_1 \) is the character group of the torus part of \( G_1 \). Let \((i_{-1}, i_0) : M_1 \to M \) be a homomorphism of log 1-motives with both \( i_{-1} \) and \( i_0 \) injective, and let \( \gamma_{-1} := i_{-1}^* \circ \lambda_{-1} \circ i_{-1} \) and \( \gamma_0 := i_0^* \circ \lambda_0 \circ i_0 \). Then \((\gamma_{-1}, \gamma_0) : M_1 = [Y_1 \to G^*_{\text{log}}] \to [X_1 \to G^*_{\text{log}}] = M^*_1 \) is a polarisation, and \((i_{-1}, i_0) \) induces a homomorphism \( i : A_1 \to A \), where \( A_1 \) is the log abelian variety associated to \( M_1 \).

Proof. To prove \((\gamma_{-1}, \gamma_0) \) is a polarisation, we need to verify the conditions (a), (b), (c) and (d) of [KKN08, Def. 2.8]. We have the following commutative diagram

\[
\begin{array}{ccc}
Y_1 & \xrightarrow{i_{-1}} & Y \\
\downarrow G_{\text{log}} & \text{iota} & \downarrow G_{\text{log}} \\
X & \xrightarrow{\lambda_{-1}} & X_1 \\
\downarrow G^*_{\text{log}} & \text{iota} & \downarrow G^*_{\text{log}} \\
X_1 & \xrightarrow{i'_0} & X_1 \\
\end{array}
\]

We also have the following commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{\lambda_{-1}} & G_{\text{log}} \\
\downarrow \lambda_{-1} & & \downarrow \lambda_{-1} \\
0 & \xrightarrow{\lambda_{0}} & G^*_{\text{log}} \\
\downarrow \lambda_{0} & & \downarrow \lambda_{0} \\
0 & \xrightarrow{\lambda_{-1}} & G^*_{\text{log}} \\
\downarrow \lambda_{-1} & & \downarrow \lambda_{-1} \\
0 & \xrightarrow{\lambda_{0}} & G^*_{\text{log}} \\
\end{array}
\]

with exact rows, where the rows are the torus-abelian variety decomposition exact sequences of semi-abelian varieties.

By the construction of the duality theory of log 1-motives, we have that \( i'_{-1} \) (resp. \( i'_0 \)) is induced by \( i_{-1} \) (resp. \( i_{-1} \)). Then condition (d) follows.
Let \( <, > : X \times Y \to G_{m, \log}/G_m \) (resp. \( <, > : X_1 \times Y_1 \to G_{m, \log}/G_m \)) be the pairing associated to \( M \) (resp. \( M_1 \)). For \( y \in Y_{1S}\setminus\{0\} \), where \( s \) denotes the only point of \( S \), we have \( i_{-1}(y) \neq 0 \) by the injectivity of \( i_{-1} \). Hence we have

\[
< \gamma_{-1}(y), y >_{1s} = \langle \lambda_{-1} \circ i_{-1}(y), i_{-1}(y) >_{S} \in (M_{S,s}/\mathcal{O}_{S,s}^\times)\setminus\{1\}
\]

which gives condition (c).

For condition (b), it suffices to show the injectivity of \( \gamma_{-1} \) because of \( \text{rank}_2 Y_1 = \text{rank}_2 X_1 \). But this already follows from condition (c).

At last, we show condition (a). The injectivity of \( i_{0} \) implies that \( i_{c} \) is injective.

By diagram chasing, the map \( i_{ab} \) must have finite kernel. Let \( B_1 := B_1/\text{Ker}(i_{ab}) \), let \( B_1 \xrightarrow{q} B_1 \xrightarrow{i_{ab}} B \) be the image decomposition of \( i_{ab} \), and let \( \gamma_{ab} := i_{ab} \circ \lambda_{ab} \circ i_{ab} \).

The duality theory of abelian varieties tells us \( \gamma_{ab} \) is a polarisation of \( B_1 \). We want to show \( \gamma_{ab} = q^* \circ \gamma_{ab} \circ q \) is a polarisation of \( B_1 \).

Without loss of generality, we may assume \( \gamma_{ab} = \varnothing \circ \varnothing \) for an ample line bundle \( \mathcal{L} \) on \( B_1 \), where \( \varnothing \circ \varnothing \) is defined by \( \varnothing \circ \varnothing := \iota \mathcal{L} \circ \mathcal{L}^{-1} \) for \( b \in B_1 \). Then we have \( \gamma_{ab} = q^* \circ \varnothing \circ q = \varnothing \circ \varnothing \). Clearly \( \varnothing \circ \varnothing \) is an isogeny, so \( \gamma_{ab} \) is a polarisation on \( B_1 \). This finishes the verification of condition (a). Hence \( (\gamma_{-1}, \gamma_{0}) \) is a polarisation of \( M_1 \).

**Proposition 2.4.** Let \( f : A \to A' \) be a homomorphism of log abelian varieties over \( S \). Then there exists a log abelian subvariety \( j : A_1 \to A \) such that \( f|_{A_1} = 0 \), and \( A_1 \) possesses the following universal property: for any homomorphism \( g : A_2 \to A \) of log abelian varieties over \( S \) such that \( f \circ g = 0 \), \( g \) factors through \( A_1 \) uniquely.

In other words, \( A_1 \) is the kernel of \( f \) in the category of log abelian varieties over \( S \).

**Proof.** Let \( M = [Y \to G_{\text{log}}] \) (resp. \( M' = [Y' \to G'_{\text{log}}] \)) be the log 1-motive defining \( A \) (resp. \( A' \)), let \( (f_{-1}, f_{0}) : M \to M' \) be the homomorphism defining \( f \).

We first construct the log 1-motive defining \( A_1 \). The homomorphism \( f \) induces a homomorphism \( f_c : G \to G' \). Let \( G_1 \) be the reduced neutral component of \( \text{Ker}(f_c) \), then \( G_1 \) is a semi-abelian variety by [Liu Prop. 5]. Let \( j_c \) be the inclusion \( G_1 \subset G \), and let \( j_0 : G_{\text{log}} \to G_{\text{log}} \) be the map induced by \( j_c \). We consider the following commutative diagram

\[
\begin{array}{cccccc}
0 & \to & G_1 & \to & G_{1\text{log}} & \to & \text{Hom}_{\mathcal{G}_{\text{log}}}(X_1, \mathcal{G}_{m, \text{log}}) & \to & 0 \\
\downarrow j_c & & \downarrow j_0 & & & & j_0 & & \\
0 & \to & G & \to & G_{\text{log}} & \to & \text{Hom}_{\mathcal{G}_{\text{log}}}(X, \mathcal{G}_{m, \text{log}}) & \to & 0 \\
\downarrow f_c & & \downarrow f_0 & & & & f_0 & & \\
0 & \to & G' & \to & G'_{\text{log}} & \to & \text{Hom}_{\mathcal{G}_{\text{log}}}(X', \mathcal{G}_{m, \text{log}}) & \to & 0
\end{array}
\]

with exact rows. By part (3) of Proposition 1.3, we have

\[
\text{Ker}(f_0) = \text{Hom}_{\mathcal{G}_{\text{log}}}(\text{Coker}(f_1), \mathcal{G}_{m, \text{log}}),
\]

...
where \( f_t : X' \to X \) is the map induced by \( f_t : T \to T' \). The above diagram gives rise to another commutative diagram

\[
\begin{array}{ccc}
0 & \to & G_1 \\
\downarrow{j_v} & & \downarrow{j_n} \\
0 & \to & \text{Ker}(f_v) \\
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow{j_v} & & \downarrow{j_n} \\
\downarrow{f} & & \downarrow{f} \\
0 & \to & \text{Ker}(f_0) \\
\end{array}
\]

with exact rows. Since the map \( \text{Coker}(f) \to X_1 \) induced by \( G_1 \to G \to G' \) is an isomorphism up to torsion, the map \( j_0 \) in the above diagram is an isomorphism. Hence \( G_{\log} \) is canonically embedded into \( \text{Ker}(f_0) \) with cokernel \( \text{Ker}(f_0)/G_1 \).

Now let \( Y_1 \) be the pullback of \( G_{\log} \) along \( \text{Ker} f_{-1} \hookrightarrow \text{Ker}(f_0) \), let \( j_{-1} \) be the canonical inclusion \( Y_1 \subset Y \), we get a log 1-motive \( M_1 := [Y_1 \to G_{\log}] \) together with a canonical map \( (j_{-1}, j_0) : M_1 \to M \). In order to apply Lemma 2.4, we need to show \( \text{rank}_2 Y_1 = \text{rank}_2 X_1 \). Without loss of generality, we may assume both \( M \) and \( M' \) admit a polarisation, in particular we have homomorphisms \( h : Y \to X \) and \( h' : Y' \to X' \) which are both injective with finite cokernel. Consider the following diagram

\[
\begin{array}{ccc}
\text{Coker}(f_{-1}) \times \text{Ker}(f_{-1}) & \to & \text{Coker}(f_{-1}) \\
\uparrow{\delta} & \downarrow{\alpha} & \\
X \times X' & \to & \mathbb{G}_{m, \log}/\mathbb{G}_m \\
\uparrow{f_{-1}} & \downarrow{f_{-1}} & || \\
X' \times Y' & \to & \mathbb{G}_{m, \log}/\mathbb{G}_m \\
\uparrow{c} & \downarrow{\beta} & \\
\text{Ker}(f_{-1}) \times \text{Coker}(f_{-1}) & \to & \\
\end{array}
\]

in which the parings \(<,>\) and \(<,>\) are compatible with the maps \( f_{-1} \) and \( f_{-1} \), we have the following relations

1. the composition \( \text{Ker}(f_{-1}) \xrightarrow{\alpha} Y \xrightarrow{\beta} X \xrightarrow{\delta} \text{Coker}(f_{-1}) \) is injective, whence \( \text{rank}_2 \text{Ker}(f_{-1}) \leq \text{rank}_2 \text{Coker}(f_{-1}) \);

2. the composition \( \text{Ker}(f_{-1}) \otimes \mathbb{Q} \xrightarrow{\alpha} X' \otimes \mathbb{Q} \xrightarrow{\beta} \text{Coker}(f_{-1}) \otimes \mathbb{Q} \) is injective, whence \( \text{rank}_2 \text{Ker}(f_{-1}) \leq \text{rank}_2 \text{Coker}(f_{-1}) \);

3. \( \text{rank}_2 X = \text{rank}_2 Y_1 \), \( \text{rank}_2 X' = \text{rank}_2 Y' \);

4. \( \text{rank}_2 \text{Ker}(f_{-1}) - \text{rank}_2 \text{Coker}(f_{-1}) = \text{rank}_2 Y - \text{rank}_2 Y' \);

5. \( \text{rank}_2 \text{Coker}(f_{-1}) - \text{rank}_2 \text{Ker}(f_{-1}) = \text{rank}_2 X - \text{rank}_2 X' \).

The relations (3), (4) and (5) are trivial. For \( y \in \text{Ker}(f_{-1}) \) such that \( d \circ h(y) = 0 \), we have \( h'(y') = f_{-1}(x') \) for some \( x' \in X' \), hence

\[
0 = < x', f_{-1}(y)' >= f_{-1}(x'), y >= < h(y), y >
\]

implies \( y = 0 \), which shows relation (1). Relation (2) can be shown by a similar argument. Now these five relations together force \( \text{rank}_2 \text{Ker}(f_{-1}) = \text{rank}_2 \text{Coker}(f_{-1}) \), so we get \( \text{rank}_2 Y_1 = \text{rank}_2 X_1 \).
Applying Lemma 2.4 to $M_1$ (if necessary we take base change to $k$ in order to get a polarisation on $M$), we have that $M_1$ defines a log abelian variety $A_1$ and $(j_{-1}, j_0)$ gives a homomorphism $j : A_1 \to A$. We leave the tedious proof of the injectivity of $j$ in Lemma 2.5.

Now we are left with checking the universal property. Let $\varphi : (g_{-1}, g_0) : M_2 = [Y_2 \to G_{2\log}] \to M$ be the homomorphism defining $g$. We have that $f \circ g = 0$ implies $f_c \circ g_c = 0$, hence $g_c$ factors through $j_c$. Furthermore, we get $g_0$ factors through $j_0$, then $f_0 \circ g_0 = 0$, further $f_{-1} \circ g_{-1} = 0$, lastly $g_{-1}(Y_2) \subset \ker(f_{-1})$. Since $Y_1$ is defined as the pullback of $G_{1\log}$ along $\ker f_{-1} \hookrightarrow \ker(f_0)$, the homomorphism $Y_2 \to \ker(f_{-1})$ factors through $Y_1$. It follows that $(g_{-1}, g_0)$ factors through $(j_{-1}, j_0)$, and $g$ factors through $j$. □

**Lemma 2.5.** The homomorphism $j : A_1 \to A$ in the proof of Proposition 2.4 is injective.

**Proof.** It suffices to prove the injectivity of $j_4 : Q_1/Y_1 \to Q/Y$, where $Q_1$ and $Q$ denote $\text{Hom}_{\text{aff}}(X_1, \mathbb{G}_{m, \log})^Y$ and $\text{Hom}_{\text{aff}}(X, \mathbb{G}_{m, \log})^Y$ respectively. We consider the following commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & Y_1 \\
\downarrow j_{-1} & & \downarrow j_4 \\
0 & \longrightarrow & Y \\
\end{array}
\begin{array}{ccc}
0 & \longrightarrow & Q_1/Y_1 \\
\downarrow j_4 & & \downarrow j_a \\
0 & \longrightarrow & Q/Y \\
\end{array}
$$

with exact rows. Let $j_1 : \mathbb{X} \to X_1$ be the map induced by $j_1 : T_1 \hookrightarrow T$. Let $\varphi_1 \in Q_1$ such that $j_4(\varphi_1) = 0$, then $j_4(\varphi_1) = \varphi_1 \circ j_1 = < -, y_0 >$ for some $y_0 \in Y$. To prove the injectivity of $j_4$, we are reduced to show that $y_0$ actually lies in $Y_1$. We have the following diagram

$$
\begin{array}{ccc}
X_1 & \times & Y_1 \\
\uparrow j_{\text{lin}} & & \downarrow j_{-1} \\
X & \times & Y \\
\uparrow & & \downarrow \\
\ker(j_1) & \times & Y/Y_1
\end{array}
$$

in which the pairings $<, >_{1}$ and $<, >$ are compatible with the maps $j_1$ and $j_{-1}$. We have $< \ker(j_1), y_0 > = \varphi_1 \circ j_1(\ker(j_1)) = 1$. We are going to deduce $ny_0 \in Y_1$ for some $n \in \mathbb{N}$ from $< \ker(j_1), y_0 > = 1$.

The composition $\ker(j_1) \otimes Q \hookrightarrow X \otimes Q \xrightarrow{h_{-1}} Y \otimes Q \to (Y/Y_1) \otimes Q$ is injective by the same argument as in the proof of Proposition 2.4. We know $\text{rank}_\mathbb{Z}(\ker(j_1)) = \text{rank}_\mathbb{Z}(Y/Y_1)$, hence there exists $z \in Y$ and $n \in \mathbb{N}$ such that $z - ny_0 \in Y_1$ and
h(z) \in \text{Ker}(j_1). \text{ But we have}

\begin{align*}
<h(z), z> &= <h(z), ny_0> < h(z), z - ny_0>
\ &= <h(z), y_0>^n \cdot <j_1(h(z)), z - ny_0>
\ &= 0, \\
\text{hence } z &= 0 \text{ and } ny_0 \in Y_1. \text{ This shows } \text{Ker}(j_1) \in (\text{fin}/S)_c, \text{ hence } \text{Ker}(j) \in (\text{fin}/S)_r. \text{ By the universal property of } j : A_1 \to A \text{ and part (3) of Proposition 2.1, } \text{Ker}(j)
\end{align*}

has to be zero.

**Theorem 2.2 (Poincaré complete reducibility theorem).** Let A be a log abelian variety over S with a polarisation \( \lambda : A \to A^* \), and A_1 is log abelian subvariety of A. Then there is another log abelian subvariety A_2 such that \( \lambda \mid A_1 \times A_2 \) is isogenous to A.

**Proof.** Let \( M = [\text{Y} \to G_{\log}] \) and \( M_1 = [\text{Y}_1 \to G_{\log}] \) be the log 1-motives defining A and A_1 respectively. Let i be the inclusion \( A_1 \subset A \), let \( i^* \) be the dual of i, and let \( f = i^* \circ \lambda \). Let \( (f_1, f_0) : M \to M_1^* \) be the homomorphism defining f. By Proposition 1.6, we have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & G & \xrightarrow{\lambda_c} & A & \xrightarrow{\lambda} & Q/Y & \longrightarrow & 0 \\
\phantom{0} & \downarrow{\lambda_c} & \phantom{0} & \downarrow{\lambda} & \phantom{0} & \downarrow{\lambda} & \phantom{0} & \downarrow{\lambda} & \phantom{0} \\
0 & \longrightarrow & G^* & \xrightarrow{i^*} & A^* & \xrightarrow{i} & Q^*/X & \longrightarrow & 0 \\
\phantom{0} & \downarrow{i^*} & \phantom{0} & \downarrow{i} & \phantom{0} & \downarrow{i} & \phantom{0} & \downarrow{i} & \phantom{0} \\
0 & \longrightarrow & G_1^* & \xrightarrow{i^*_c} & A_1^* & \xrightarrow{i_c} & Q_1^*/X_1 & \longrightarrow & 0
\end{array}
\]

with exact rows.

Firstly we study the homomorphism \( f_c \) via studying \( \lambda_c \) and \( i^*_c \). We have \( \text{Ker}(\lambda) \in (\text{fin}/S)_r \) by Proposition 2.2 and \( \dim G = \dim G^* \), hence \( \lambda_c \) is an isogeny. The construction of \( i^*_c \) gives a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & T^* & \xrightarrow{i^*_c} & G^* & \xrightarrow{i^*_c} & B^* & \longrightarrow & 0 \\
\phantom{0} & \downarrow{i} & \phantom{0} & \downarrow{i^*_c} & \phantom{0} & \downarrow{i^*_c} & \phantom{0} & \downarrow{i^*_c} & \phantom{0} \\
0 & \longrightarrow & T_1^* & \xrightarrow{i^*_c} & G_1^* & \xrightarrow{i^*_c} & B_1^* & \longrightarrow & 0
\end{array}
\]

with exact rows, where the map \( T^* \to T_1^* \) is induced by \( i_1 : Y_1 \to Y \) and the map \( B^* \to B_1^* \) is induced by the abelian variety part \( i_{ab} \) of \( i_c \). The injectivity of i implies the injectivity of \( i_{ab} \), hence the injectivity of \( \tilde{i} : G_{1Y_{\log}} \to G_{Y_{\log}} \) by part (4) of Proposition 1.3. Furthermore, we get the injectivity of \( i_{1} : Y_1 \to Y \) from the injectivity of \( \tilde{i} \). It follows that \( i^*_c \) is surjective. The injectivity of \( i^*_c \) implies that \( i_{ab} : B_1 \to B \) has finite kernel. The duality theory of abelian varieties tells us \( \dim(\text{Ker}(i^*_{ab})) = \dim B^* - \dim B_1^* \). Since the map \( \text{Ker}(i^*_{ab}) \to \text{Coker}(i^*_c) \) has finite image, we get \( \dim(\text{Ker}(i^*_c)) = \dim G^* - \dim G_1^* \). On the other hand, \( G_1^* \) is smooth and connected over k, hence \( \text{Coker}(i^*_c) \) is smooth connected and of dimension 0, in other words \( i^*_c \) is surjective. It follows that \( f_c \) is surjective.
Let $G_2$ be the reduced neutral component of $\ker(f_e)$, then $G_2$ is a semi-abelian variety by [Liu] 5. Let $j_2$ be the inclusion $G_2 \subset G$, and let $j_0 : G_{2\log} \to G_{\log}$ be the map induced by $j_2$. Proposition 2.4 and its proof supply us a log 1-motive $M_2 = [Y_2 \to G_{2\log}]$ together with a homomorphism $(j_{-1}, j_0) : M_2 \to M$, such that both $j_{-1}$ and $j_0$ are injective, and $(j_{-1}, j_0)$ induces a log abelian subvariety $j : A_2 \hookrightarrow A$ which is the kernel of $f : A \to A_1$ in the category of log abelian varieties over $S$.

Let $Y' = Y_1 \times Y_2, G' = G_1 \times G_2$, then we have a natural log 1-motive $M' = [Y' \to G'_{\log}]$ and a homomorphism $(\alpha_{-1}, \alpha_0) : M' \to M$, where $\alpha_{-1}$ is the map $Y_1 \times Y_2 \to Y, (y_1, y_2) \mapsto y_1 + y_2$, and $\alpha_0$ is the map induced by $\alpha_c : G_1 \times G_2 \to G, (g_1, g_2) \mapsto g_1 + g_2$. We claim $\alpha_{-1}$ is injective and of finite cokernel. Note that $(\gamma_1, \gamma_0) := (i_{c1}^* \circ \lambda_{-1} \circ i_{1c}, i_{c1}^* \circ \lambda_0 \circ i_0)$ is a polarisation on $M_1$ by Lemma 2.4. For $y \in Y_{18} \cap Y_{25}$, $\gamma_1(y) = f_{-1}(y) = 0$, hence $0 = < 0, y >_{18} = < \gamma_1(y), y >_{18}$ implies $y = 0$, and $\ker(\alpha_{-1}) = Y_1 \cap Y_2 = 0$. Hence $\alpha_{-1}$ has finite cokernel due to rank reason. We also claim that the map $\alpha_c$ is an isogeny. Since the $\lambda_c$ is an isogeny, the composition $i_c^* \circ \lambda_c \circ i_c$ is also an isogeny by Lemma 2.4. Hence for any $g \in G$, there exist $g_1 \in G$ such that $i_c^* \circ \lambda_c \circ i_c(g_1) = i_c^* \circ \lambda_c(g)$, it follows then $g - i_c(g_1) \in \ker(f_e)$. Since $\ker(f_e)/G_2$ is a finite group scheme, the map $\alpha_c$ is an isogeny. Now consider the following commutative diagram

$$
\begin{array}{cccccc}
0 & \to & Y' & \to & G^{(Y')} & \to & A_1 \times A_2 & \to & 0 \\
0 & \to & Y & \downarrow{\alpha_{-1}} & \downarrow{i_c} & \downarrow{\alpha} & \to & A & \to & 0
\end{array}
$$

with exact rows, we know $\tilde{\alpha}$ is surjective with kernel $\ker(\alpha_c)$ by Lemma 2.2, hence we get a short exact sequence $0 \to \ker(\alpha_c) \to \ker(\alpha) \to \coker(\alpha_{-1}) \to 0$. Then we get $\ker(\alpha) \in (\text{fin}/S)_e$ by [Kai92] Prop. 2.3. It follows then $\alpha : A_1 \times A_2 \to A$ is an isogeny of log abelian varieties by Proposition 2.2.

Remark 2.2. Since abelian varieties over a field are always projective, they carry an ample line bundle, hence they are always polarisable. For log abelian varieties over a log point, they admit a polarisation after base change to the algebraic closure of the base field by their definition. The author doesn’t know if they actually carry a polarisation over the base log point. But he does think, over a log point, log abelian variety admitting a polarisation serves as the right counterpart of abelian variety for at least two reasons. Firstly, the canonical 1-parameter log abelian variety degeneration ([Zha14]) of an abelian variety transports a polarisation of the generic fibre to the special fibre. In other words the special fibre (which is a log abelian variety over a log point) as the degeneration of the generic fibre (which is an abelian variety over a trivial log point) is necessarily polarisable. Secondly, a polarisation is needed in the proof of Poincaré complete reducibility theorem (see Theorem 2.2), and we know Poincaré complete reducibility theorem for abelian varieties plays a very important role in the theory of abelian varieties.
Let $A$ be a log abelian variety over $S$, a log abelian subvariety of $A$ is a subsheaf of $A$ which is also a log abelian variety. The log abelian variety $A$ is simple if it has no non-zero proper log abelian subvariety. In other words if $A_1$ is a log abelian variety contained in $A$, then $A_1$ is zero.

**Corollary 2.1.** Let $A$ be a log abelian variety over $S$ admitting a polarisation. Then $A$ is isogenous to a product $A_1^{n_1} \times \cdots \times A_r^{n_r}$, where the $A_i$'s are simple log abelian varieties and not isogenous to each other. The isogeny type of the $A_i$ and the integers $n_i$'s are uniquely determined.

**Definition 2.4.** Let $A, A'$ be two log abelian varieties over $S$, we abbreviate $\text{Hom}_{\text{log}}^0(A, A') = \text{Hom}_{\text{log}}(A, A')$ as $\text{Hom}^0(A, A')$. We define $\text{Hom}^0(A, A')$ as $\text{Hom}(A, A') \otimes \mathbb{Q}$, and $\text{End}^0(A)$ as $\text{End}(A) \otimes \mathbb{Q} = \text{Hom}(A, A) \otimes \mathbb{Q}$. We define the category $\text{LAV}^0_S$ of log abelian varieties up to isogeny over $S$, by localising the category $\text{LAV}_S$ of log abelian varieties over $S$ at the class of isogenies.

**Corollary 2.2.** Let $A$ be a log abelian variety over $S$ admitting a polarisation. If $A$ is simple, the ring $\text{End}^0(A)$ is a division ring. In general, if $A$ is isogenous to $A_1^{n_1} \times \cdots \times A_r^{n_r}$ with $A_i$ simple and not isogenous to each other, and $D_i = \text{End}^0(A_i)$, then $\text{End}^0(A) = M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$.

**Proof.** For $A$ simple, let $f$ be an endomorphism of $A$. If $f$ is not an isogeny, then there exists a non-zero log abelian subvariety of $A$ which is killed by $f$. Since $A$ is simple, $f$ has to be zero. Hence the ring $\text{End}^0(A)$ is a division ring. By the same reason, we have $\text{Hom}^0(A, A') = 0$ for two non-isogenous simple log abelian varieties. Hence the second part follows.

**Lemma 2.6.** The abelian group $\text{Hom}(A, A')$ is torsion-free.

**Proof.** Let $f \in \text{Hom}(A, A')$ such that $nf = 0$ for some positive integer $n$. Since $0 = nf = f \circ n_A$ and $n_A$ is surjective, $f$ must be zero. Hence $\text{Hom}(A, A')$ is torsion-free.

**Definition 2.5.** (1) Let $f : A \to A'$ be an isogeny between two log abelian varieties over $S$. The degree $\deg(f)$ of $f$ is defined to be the rank of the finite log group object $\text{Ker}(f)$. By convention, if $f$ is not an isogeny, we let $\deg(f) = 0$.

(2) Let $f_{-1} : Y \to Y'$ be an monomorphism with finite cokernel between two étale locally finite rank free constant sheaf, the degree $\deg(f_{-1})$ of $f_{-1}$ is defined to be the determinant of $f_{-1}$. By convention, if $f$ is not injective of finite cokernel, we let $\deg(f) = 0$.

(3) Let $f_c : G \to G'$ be an isogeny between semi-abelian varieties, the degree $\deg(f_c)$ of $f_c$ is defined to be the rank of the finite group scheme $\text{Ker}(f_c)$. By convention, if $f_c$ is not an isogeny, we let $\deg(f_c) = 0$.

**Lemma 2.7.** Let $f : A \to A'$ be an isogeny between two log abelian varieties over $S$. Let $f_{-1} : Y \to Y'$ and $f_c : G \to G'$ be the homomorphisms induced by $f$ as in Proposition [1.6] and let $f_t : T \to T'$ and $f_{ab} : B \to B'$ be the homomorphisms induced by $f_c$ on torus parts and abelian variety parts respectively. Then:
(1) \( \deg(f) = \deg(f_{-1})\deg(f_t)\deg(f_{ab}) \);
(2) let \( g : A \to A' \) be another isogeny, let \( h = f + g \), then \( h_{-1} = f_{-1} + g_{-1} \), \( h_t = f_t + g_t \) and \( h_{ab} = f_{ab} + g_{ab} \).

**Proof.** Since \( f \) is an isogeny, so is \( f_t, f_{ab} \) and \( f_{ab} \). Also we have \( f_{-1} \) is injective and of finite cokernel. By diagram (1.10), we get a short exact sequence

\[
0 \to \text{Ker}(f_t) \to \text{Ker}(f) \to \text{Ker}(f_{ab}) \to 0.
\]

Similarly, we have another short exact sequence

\[
0 \to \text{Ker}(f_t) \to \text{Ker}(f) \to \text{Ker}(f_{ab}) \to 0.
\]

By diagram (1.12), we get \( \text{Ker}(f_{ab}) \cong \text{Coker}(f_{-1}) \). Then

\[
\deg(f) = \deg(f_{-1})\deg(f_t)\deg(f_{ab}).
\]

This shows part (1). Part (2) is obvious. \( \square \)

**Theorem 2.3.** The function \( f \mapsto \deg(f) \) on \( \text{End}(A) \) extends to a homogeneous polynomial function of degree \( 2g \) on \( \text{End}^0(A) \), where \( g \) is the dimension of \( A \).

**Proof.** Since for any \( f \in \text{End}(A) \) and \( n \in \mathbb{Z} \),

\[
\deg(nf) = \deg(nA) \cdot \deg(f) = n^{2g} \cdot \deg(f),
\]

it suffices to show for \( f, g \in \text{End}(A) \), the function \( P(n) = \deg(nf + g) \) is a polynomial function. By Lemma 2.7, we are reduced to show the functions \( \deg(nf_{1} + g_{1}) \), \( \deg(nf_{ab} + g_{ab}) \) and \( \deg(nf_{-1} + g_{-1}) \) are all polynomial functions. The case for \( \deg(nf_{1} + g_{1}) \) is a standard result for abelian varieties, see [Mum70 §19, Thm. 2]. And \( \deg(nf_{-1} + g_{-1}) \) as a determinant function is clearly a polynomial function. The case for \( \deg(nf_{ab} + g_{ab}) \) is reduced to the case for \( \deg(nf_{-1} + g_{-1}) \) by taking the character groups of the tori. \( \square \)

**Definition 2.6.** Let \( l \) be prime number which is coprime to the characteristic of \( k \). The \( l \)-adic Tate module of \( A \) is defined to be

\[
T_l(A)_{s(k\acute{e}t)} = \varprojlim_n A[l^n]_{s(k\acute{e}t)},
\]

where \( s(k\acute{e}t) \) denotes a log geometric point of the Kummer étale site over \( S \).

Let \( \pi_1^\log \) be the log fundamental group of \( S \). By Proposition 2.3 we have \( T_l(A)_{s(k\acute{e}t)} \) is a free \( \mathbb{Z}_l \)-module of rank \( 2g \) endowed with a continuous \( \pi_1^\log \)-action. Any homomorphism \( f : A \to A' \) induces a homomorphism

\[
T_l(f) : T_l(A)_{s(k\acute{e}t)} \to T_l(A')_{s(k\acute{e}t)}
\]

which is \( \pi_1^\log \)-equivariant. It follows we have a functor

\[
T_l : \text{LAV}_S \longrightarrow (\pi_1^\log, \mathbb{Z}_l) - \text{Mod}
\]

from the category of log abelian varieties over the log point \( S \) to the category of finite rank \( \mathbb{Z}_l \)-modules with continuous \( \pi_1^\log \)-action. In particular, the functor \( T_l \) gives rise to a homomorphism \( \text{Hom}(A, A') \to \text{Hom}(\pi_1^\log, \mathbb{Z}_l) - \text{Mod}(T_l(A)_{s(k\acute{e}t)}, T_l(A')_{s(k\acute{e}t)}) \). The
later is clearly a $\mathbb{Z}_l$-submodule of $\text{Hom}_{\mathbb{Z}_l}(T_l(A)_{k(\kappa)}, T_l(A')_{k(\kappa)})$ which is of finite $\mathbb{Z}_l$-rank. Moreover, we have the following canonical homomorphism

$$T_l : \text{Hom}(A, A') \otimes_{\mathbb{Z}} \mathbb{Z}_l \to \text{Hom}_{\mathbb{Z}_l}(T_l(A)_{k(\kappa)}, T_l(A')_{k(\kappa)})$$

We are going to use this map to investigate the finiteness of $\text{Hom}(A, A')$.

**Theorem 2.4.** Let $A, A'$ be two log abelian varieties over $S$ admitting a polarisation. Then the canonical map

$$T_l : \text{Hom}(A, A') \otimes_{\mathbb{Z}} \mathbb{Z}_l \to \text{Hom}_{\mathbb{Z}_l}(T_l(A)_{k(\kappa)}, T_l(A')_{k(\kappa)})$$

is injective.

**Proof.** We have already proven the degree function on $\text{End}^0(A)$ is a homogeneous polynomial function of degree $2g$ in Theorem 2.3. Now the proof of [Mum70] §19, Thm. 3 works verbatim here. □

**Corollary 2.3.** Let $A, A'$ be two log abelian varieties over $S$. Then the canonical map

$$T_l : \text{Hom}(A, A') \otimes_{\mathbb{Z}} \mathbb{Z}_l \to \text{Hom}_{\mathbb{Z}_l}(T_l(A)_{k(\kappa)}, T_l(A')_{k(\kappa)})$$

is injective.

**Proof.** This follows from Theorem 2.4. □

**Corollary 2.4.** Let $A, A'$ be two log abelian varieties over $S$. Then we have $\text{Hom}(A, A') \cong \mathbb{Z}^r$ with $r \leq 4\text{dim}A \cdot \text{dim}A'$.

**Corollary 2.5.** Let $A$ be a log abelian variety over $S$ admitting a polarisation. Then $\text{End}^0(A)$ is finite-dimensional semisimple algebra over $\mathbb{Q}$.

**Proof.** This follows from Corollary 2.2 and Corollary 2.4. □

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