PIECEWISE SEMIPRIME RINGS AND SOME APPLICATIONS

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Received: 20 November 2012; Revised: 28 September 2013
Communicated by Pınar Aydoğu

Dedicated to the memory of Professor Efraim P. Armendariz

ABSTRACT. We define piecewise semiprime (PWSP) rings $R$ in terms of a set of triangulating idempotents in $R$. The class of PWSP rings properly contains both the class of semiprime rings and the class of piecewise prime rings. The PWSP property is Morita invariant and it is shared by some important ring extensions. A ring is PWSP if and only if it has a generalized upper triangular matrix representation with semiprime rings on the main diagonal. Another characterization of PWSP rings involves a generalization of the concept of $m$-systems and is similar to the description of a semiprime ring in terms of the prime radical. Finally we use the PWSP property to determine (right) weak quasi-Baer rings. These are rings in which the right annihilator of every nilpotent ideal is generated as a right ideal by an idempotent.

Mathematics Subject Classification 2010: 16P60, 16P99, 16S50
Keywords: Piecewise prime, piecewise semiprime, triangulating idempotents, weak quasi-Baer ring

1. Introduction

All rings are associative and $R$ denotes a ring with unity 1. The word ideal without the adjective right or left means two sided ideal. A ring $R$ is quasi-Baer (Baer) if the right annihilator of every right ideal (nonempty subset) of $R$ is generated as a right ideal by an idempotent. We now recall a few definitions and results from [1] which motivated our study and serve as the background material for the present work. An idempotent $e \in R$ is a left semicentral idempotent if $exe = xe$, for all $x \in R$. Similarly right semicentral idempotent can be defined. The set of all left (right) semicentral idempotents of $R$ is denoted by $S_l(R)$ ($S_r(R)$). An idempotent $e \in R$ is semicentral reduced if $S_l(eRe) = \{0, e\}$. If 1 is semicentral reduced, then $R$ is called semicentral reduced. An ordered set $\{e_1, ..., e_n\}$ of nonzero distinct idempotents of $R$ is called a set of left triangulating idempotents of $R$ if all the following hold.

(1) $e_1 + \cdots + e_n = 1$,
(2) $e_1 \in S_l(R)$,
(3) $e_{k+1} \in S_l(c_kRc_k)$, where $c_k = 1 - (e_1 + \cdots + e_k)$ for $1 \leq k \leq n$. 
From part (3) of the above definition, it can be seen that a set of left triangulating idempotents is a set of pairwise orthogonal idempotents. A set \( E = \{ e_1, ..., e_n \} \) of left triangulating idempotents of \( R \) is complete, if each \( e_i \) is semicentral reduced. A (complete) set of right triangulating idempotents is defined similarly. The cardinalities of complete sets of left triangulating idempotents of \( R \) are the same and is denoted by \( \tau \dim(R) \) [1, Theorem 2.10]. According to [1, Proposition 1.3] \( R \) has a (complete) set of left triangulating idempotents if and only if there exists an isomorphism of rings between \( R \) and an \( n \times n \) upper triangular matrix ring with the \((i,j)\)-entry \( R_{ij} \) where each \( R_{ii} \) is a (semicentral reduced) ring with identity, and each \( R_{ij} \) is a left \( R_{ii} \) right \( R_{jj} \) bimodule for \( i < j \). In this case, the ring \( R \) is said to have a (complete) generalized triangular matrix representation [3]. Following [1, p. 591], a ring \( R \) is called piecewise prime (abbreviated PWP) if there exists a complete set of left triangulating idempotents \( E = \{ e_1, ..., e_n \} \) of \( R \) such that \( xRy = 0 \) implies \( x = 0 \) or \( y = 0 \) where \( x \in e_iR e_j \), and \( y \in e_jR e_k \) for \( 1 \leq i, j, k \leq n \) (in this case, we say that \( R \) is PWP with respect to \( E \)). In [1, Theorem 4.11], it is shown that if \( R \) is a PWP ring then \( R \) is PWP with respect to any complete set of left triangulating idempotents of \( R \); furthermore if \( \tau \dim(R) = n \), then \( R \) is a PWP ring if and only if it is a quasi-Baer ring. The factor ring of a quasi-Baer ring by its prime radical is considered in [3]. For a comprehensive study of these concepts and results the reader is referred to [1-5]. It should be mentioned that in [7], modules whose endomorphism rings are of finite triangulating dimension are thoroughly investigated and the triangulating dimension is generalized from rings to modules.

In this paper, we define (in Section 2 to follow) piecewise semiprime (abbreviated PWSP) rings with respect to sets of left triangulating idempotents. We show that if \( R \) is PWSP with respect to a set of left triangulating idempotents of \( R \), then it is PWSP with respect to any complete set of left triangulating idempotents of \( R \). We prove that \( R \) is PWSP if and only if \( R \) has a generalized triangular matrix representation with semiprime rings on the main diagonal. The PWSP property is Morita invariant and it is shared between a ring and some of the more important extensions of that ring. Another characterization of PWSP rings is given using the concept of generalized m-systems with respect to sets of triangulating idempotents. Accordingly, a ring \( R \) is PWSP with respect to a set of triangulating idempotents \( E \) if and only if for every \( e \in E \), \( eNe = 0 \) where \( N \) is the prime radical of \( R \). In the final section of this paper the concept of weak quasi Baer ring is defined and investigated in terms of PWSP property. Throughout the paper, we shall only deal with rings of finite triangulating dimension in all results.
2. Piecewise Semiprime Rings

In this section, we define a piecewise semiprime (ideal) ring and show that this property is shared between a ring and some of the more important extensions of that ring. Also, piecewise semiprime endomorphism rings are investigated. We begin with the following definitions.

**Definition 2.1.** Let $I$ be a proper ideal of a ring $R$.

1. The ideal $I$ is called piecewise prime (PWP ideal) if there is a complete set of left triangulating idempotents $E = \{e_1, \ldots, e_n\}$ such that $xRy \subseteq I$ implies $x \in I$ or $y \in I$, where $x \in e_iRe_j$, and $y \in e_jRe_k$ for $1 \leq i, j, k \leq n$.

2. If $E = \{e_1, \ldots, e_n\}$ is a set of left triangulating idempotents then the ideal $I$ is called a piecewise semiprime (PWSP) ideal with respect to $E$ if for any $x \in R$ and $i = 1, \ldots, n$,

$$e_i xe_i Re_i xe_i \subseteq I \Rightarrow e_i xe_i \in I$$

The ring $R$ is piecewise semiprime with respect to $E$ if $0$ is a PWSP ideal with respect to $E$. Also an ideal $J$ of a ring $R$ is said to be PWSP if there exists a set of left triangulating idempotents $E$ of $R$ such that $J$ is PWSP with respect to $E$. The ring $R$ is then called PWSP if $0$ is a PWSP ideal of $R$.

**Remarks 2.2.** If $E = \{e_1, \ldots, e_n\}$ is a set of left triangulating idempotents of a ring $R$, then $I$ is a PWSP ideal of $R$ with respect to $E$ if and only if $R/I$ is a PWSP ring with respect to $\{e_1 + I, \ldots, e_n + I\}$. Any semiprime ring is PWSP with respect to any set of left triangulating idempotents. If $R$ is PWSP with respect to the set of left triangulating idempotent $\{1_R\}$, then it is semiprime. In general if $R$ is a PWSP ring with respect to a set of left triangulating idempotents then $R$ is not necessarily semiprime. The following non reversible implications hold for rings; see Examples and Remarks 2.12.

$$\begin{align*}
\text{prime} &\Rightarrow \text{semiprime} \\
\downarrow &\downarrow \\
\text{piecewise prime} &\Rightarrow \text{piecewise semiprime}
\end{align*}$$

**Theorem 2.3.** Let $E = \{e_1, \ldots, e_n\}$ be a set of left triangulating idempotents of a ring $R$ and $I$ an ideal of $R$. If $I$ is a PWSP ideal with respect to $E$, then it is PWSP with respect to any complete set of left triangulating idempotents.

**Proof.** Let $F = \{f_1, \ldots, f_m\}$ be a complete set of left triangulating idempotents of $R$ and $f_x f_i R f_x f_i \subseteq I$ for some $f_i \in F$ and $x \in R$. Let $d$ be the smallest in $\{1, \ldots, n\}$ such that $f_i e_d \neq 0$. Since $f_i x f_i R f_x f_i \subseteq I$, $e_d(f_i x f_i) e_d R e_d(f_i x f_i) e_d \subseteq I$ which
implies that $e_d f_i x f_d e_d = 1$ and then $f_i e_d f_i x f_d e_d f_i = 1$. If $c_d = e_1 + \cdots + e_d$, then $f_i c_d = f_i e_d \neq 0$. Since $c_d \in S_l(R)$ by [1, Proposition 1.6], we have $f_i c_d = f_i c_d f_i e_d$. Thus $f_i c_d f_i = f_i e_d f_i \neq 0$. By [1, Lemma 1.4(ii)], $f_i c_d f_i \in S_l(f_i R f_i)$. Since $f_i$ is a semicentral reduced idempotent, $f_i e_d f_i = f_i c_d f_i = f_i$. Hence $f_i x f_i = 1$ as desired.

Corollary 2.4. Let $E = \{e_1, \ldots, e_n\}$ be a set of left triangulating idempotents of $R$. If $R$ is PWSP with respect to $E$, then it is PWSP with respect to any complete set of left (right) triangulating idempotents of $R$.

Proof. This is evident by Theorem 2.3.

Proposition 2.5. Let $R$ be a PWSP ring with respect to a set of left triangulating idempotents $E = \{e_1, \ldots, e_n\}$. Then $R$ is semiprime if and only if $E \subseteq S_l(R)$.

Proof. If $R$ is semiprime then each semicentral left idempotent is central. Since for each $1 \leq i \leq n, e_i = e_1 + \cdots + e_i \in S_l(R)$ by [1, Proposition 1.6], we deduce that for each $1 \leq i \leq n, e_i$ is central. This implies that for each $1 \leq i \leq n$, $e_i$ is central in particular $E \subseteq S_l(R)$. Conversely if $E \subseteq S_l(R)$, then $R = \oplus_{i=1}^n e_i R e_i = \oplus_{i=1}^n e_i R e_i$. Since each $e_i R e_i$ is semiprime by definition, we deduce that $R$ is a semiprime ring.

Proposition 2.6. Let $E = \{e_1, \ldots, e_n\}$ be a set of left triangulating idempotents and $I$ be an ideal of $R$. Then the following are equivalent.

1. $I$ is a PWSP ideal with respect to $E$.
2. If $J$ is any ideal of $R$ such that $Je_i J \subseteq I$ for some $e_i \in E$, then $e_i J e_i \subseteq I$.
3. If $J$ is any right ideal of $R$ such that $Je_i J \subseteq I$ for some $e_i \in E$, then $e_i J e_i \subseteq I$.
4. If $J$ is any left ideal of $R$ such that $Je_i J \subseteq I$ for some $e_i \in E$, then $e_i J e_i \subseteq I$.
5. If $(a)$ is any principal right (left) ideal of $R$ such that $(a)e_i (a) \subseteq I$ for some $e_i \in E$, then $e_i a e_i \subseteq I$.

Proof. (1) $\Rightarrow$ (2) Let $x \in J$. Since $e_i x e_i R e_i x e_i \subseteq Je_i J \subseteq I$, $e_i x e_i \subseteq I$. Hence $e_i J e_i \subseteq I$.

(2) $\Rightarrow$ (1) Assume $e_i x e_i R e_i x e_i \subseteq I$ for some $x \in R$ and $e_i \in E$. Thus $(Re_i x e_i R) e_i (Re_i x e_i R) \subseteq I$ which implies that $e_i x e_i \subseteq I$.

(2) $\Rightarrow$ (3) If $Je_i J \subseteq I$ for some $e_i \in E$ and a right ideal $J$, then $(Re_i J) e_i (Re_i J) \subseteq I$ which implies $e_i J e_i \subseteq I$.

(3) $\Rightarrow$ (1) Assume $e_i x e_i R e_i x e_i \subseteq I$ for some $x \in R$ and $e_i \in E$. Then $(e_i x e_i R) e_i (e_i x e_i R) \subseteq I$ which implies that $e_i x e_i \subseteq I$.

(1) $\Leftrightarrow$ (4) The proof is similar to the proof of (1) $\Leftrightarrow$ (3).

(2) $\Leftrightarrow$ (5) It is routine.
Proposition 2.7. Let $R \subseteq S$ be rings, $aS$ be free with a multiplicatively closed basis set $X$ such that $rx = xr$ for all $x \in X$ and $r \in R$. If $E = \{e_1, \ldots, e_n\}$ is a set of left triangulating idempotents of $R$, then

1. $E$ is a set of left triangulating idempotents of $S$.
2. $R$ is PWSP with respect to $E$ if and only if $S$ is PWSP with respect to $E$.

Proof. By hypothesis $aRb = 0$ if and only if $aSb = 0$ where $a, b \in R$. This proves (1). Also, if $(e_i^{*} s e_i) S (e_i^{*} s e_i) = 0$ for some $e_i \in E$, $s \in S$ and $s = \sum_{j=1}^{m} r_j x_j$ with $x_1, \ldots, x_m \in X$, then since $X$ is a multiplicatively closed set, we can deduce that $(e_i^{*} r_j e_i) R (e_i^{*} r_j e_i) = 0$ for each $j = 1, \ldots, m$, the proof is complete. \hfill $\square$

Suppose that $R$ is a ring. The $n$ by $n$ matrix ring on $R$ will be denoted by $M_n(R)$. Let $X$ be a non empty set of not necessarily commuting indeterminates on $R$ and $x, y \in X$. By $R[[x]]$ and $R < X >$ we mean the ring of formal power series in $x$ and the free $R$-ring generated by $X$, respectively. Also if $xy = yx$ and $I$ is the ideal generated by $xy - 1$ in $R[x, y]$, then the ring $R[x, y]/I$, as usual, be denoted by $R[x, x^{-1}]$; see [8, 1.2, p. 6] for an excellent reference for these rings.

Proposition 2.8. Let $E = \{e_1, \ldots, e_n\}$ be a set of left triangulating idempotents of $R$, $x \in X$ be a non empty set of not necessarily commuting indeterminates on $R$, and $I_m$ be an identity matrix of size $m$. Then the following statements are equivalent.

(i) $R$ is PWSP with respect to $E$.
(ii) $M_n(R)$ is PWSP with respect to $\overline{E} = \{e_i I_m\}_{i=1}^{n}$.
(iii) $R < X >$ is PWSP with respect to $E$.
(iv) $R[[x]]$ is PWSP with respect to $E$.
(v) $R[x, x^{-1}]$ is PWSP with respect to $E$.

Proof. (i) $\Leftrightarrow$ (ii). Let $e_i = e_i I_m$ for each $i$. If $J$ is an ideal of $R$, then it is seen that $M_m(J e_i M_m(J) = M_m(J e_i) J$ and $e_i M_m(J) e_i = M_m(J e_i) e_i$. Thus the equivalence (i) $\Leftrightarrow$ (ii) is true by Proposition 2.6 and the fact that any ideal of $M_n(R)$ has the form $M_m(J)$ for some ideal $J$ of $R$. The proof is now completed by Proposition 2.7. \hfill $\square$

Lemma 2.9. Let $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, where $R$ and $S$ are two rings and $R M_S$ is a nonzero bimodule. If $\{e_i, m_i \} \begin{pmatrix} e_i & m_i \\ 0 & f_i \end{pmatrix}_{i=1}^{n}$ is a set of left triangulating idempotents of $T$, then $E_0 = \{e_1, \ldots, e_n\} \setminus \{0\}$, and $F_0 = \{f_1, \ldots, f_n\} \setminus \{0\}$ are sets of left triangulating idempotents of $R$ and $S$ respectively.
Proof. It is easy to see that for each $1 \leq i \leq n$, $e_i$ and $f_i$ are idempotents with $\sum e_i = 1_R$ and $\sum f_i = 1_S$. Thus [1, Lemma 1.2], implies that $E_0$ and $F_0$ are sets of left triangulating idempotents of $R$ and $S$ respectively. \hfill $\square$

**Proposition 2.10.** Let $T$ be a formal triangular matrix ring as in 2.9. Then $T$ is PWSP if and only if $R$ and $S$ are PWSP.

**Proof.** Let $R$ and $S$ be PWSP with respect to the sets of left triangulating idempotents $E = \{e_1, \ldots, e_n\}$ and $F = \{f_1, \ldots, f_m\}$ respectively. Then $T$ is PWSP with respect to $\{(e_i \ 0 \ 0 \ 0 \ 0 \ f_i) \ 0 \ 0 \ 0 \ 0 \ f_i)\}_{i=1}^n$. Conversely if $T$ is PWSP with respect to $\{(e_i \ m_i \ 0 \ 0 \ f_i) \ 0 \ 0 \ 0 \ 0 \ f_i)\}_{i=1}^n$, then $R$, and $S$ are PWSP with respect to $E_0$ and $F_0$ as in 2.9. To see this let $e_i xe_i Re_i xe_i = 0$ where $e_i \in E_0$, and $x \in R$. Since $e_i xe_i Re_i xe_i = 0$, if

$$
\begin{pmatrix}
e_i & m_i \\
0 & f_i
\end{pmatrix}
\begin{pmatrix}
x e_i & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
e_i & m_i \\
0 & f_i
\end{pmatrix}
= 
\begin{pmatrix}
e_i xe_i & e_i xe_i m_i \\
0 & 0
\end{pmatrix}
$$

then $tTt = 0$. This implies $t = 0$. Hence $e_i xe_i = 0$. A similar argument implies that $S$ is PWSP with respect to $F_0$. \hfill $\square$

**Corollary 2.11.** Let $R = \bigoplus_{i=1}^n R_i$ be a ring decomposition. Then $R$ is PWSP (PWP) if and only if each $R_i$ is PWSP (PWP).

**Proof.** For the PWSP case use induction and Proposition 2.10. The PWP case has a routine argument. \hfill $\square$

**Examples and Remarks 2.12.** (1) If $R_1$ and $R_2$ are PWP rings, then by Corollary 2.11, $R_1 \oplus R_2$ is a PWP ring but not a prime ring.

(2) If $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ with a nonzero bimodule $A \ M, B$, then $R$ is never a semiprime ring but it is a PWSP ring provided that $A, B$ are so (Proposition 2.10). Also if $A$ is a prime ring, $X$ is a non-trivial ideal of $A$ and $R = \begin{pmatrix} A & X \\ 0 & A \end{pmatrix}$, then $R$ is PWSP with respect to the complete set of left triangulating idempotents $\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$. But the right annihilator of $\begin{pmatrix} 0 & A/X \\ 0 & 0 \end{pmatrix}$ is $\begin{pmatrix} A & A/X \\ 0 & X \end{pmatrix}$ which is not a direct summand of $R$. This shows that $R$ is not quasi-Baer. Hence by [1, Theorem 4.11], $R$ is not PWP.

(3) A subring of a PWSP ring need not be PWSP. To see this let $R$ be a PWSP ring with respect to a set of left triangulating idempotents $E = \{e_1, \ldots, e_n\}$. Then
$S = \{ \begin{pmatrix} r & a \\ 0 & r \end{pmatrix} \mid r, a \in R \}$ is a subring of $T = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$. Now $T$ is PWSP but $S$ is not. For if $E^* = \{ \begin{pmatrix} e_i \\ r_i \\ 0 \end{pmatrix} \mid e_i \in E, r_i \in R, i = 1, \ldots, n \}$ is a set of left triangulating idempotents of $S$, then \[ \begin{pmatrix} e_i \\ r_i \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e_i \\ r_i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & e_i \\ 0 & 0 \end{pmatrix}, \]

and so it can be seen that $\begin{pmatrix} 0 & e_i \\ 0 & 0 \end{pmatrix} S \begin{pmatrix} 0 & e_i \\ 0 & 0 \end{pmatrix} = 0$.

(4) Part (2) above shows that if $R$ is PWSP with respect to a set of left triangulating idempotents $E = \{e_1, \ldots, e_n\}$, then $R$ need not be PWSP with respect to another set of left triangulating idempotents $E'$ of $R$ say $E' = \{1_R\}$.

(5) If $R$ is a PWSP ring and $\alpha$ is a ring endomorphism of $R$, then the skew polynomial ring $S = R[x, \alpha]$ of $R$ need not be PWSP. Note that by definition, $S$ is a ring extension of $R$ such that $R S$ is free with a basis of the form $1, x, x^2, \ldots$ and $x r = \alpha(r)x$ for all $r \in R$. Now consider $S = R[x, \alpha]$, where $R = k[y]$ for some field $k$ and $\alpha : R \to R$ is the evaluation map at zero. If $S$ were PWSP for a set of left triangulating idempotents then by Corollary 2.4, it would be PWSP with respect to any complete set of left triangulating idempotents. However the singleton $\{1_R\}$, by [2, Theorem 4.4], is such a set, hence $S$ would be semiprime. But $S$ is not semiprime because $yxR[x, \alpha]yx = 0$. It implies that $S$ is not PWSP.

Following [9], an $R$-module $M$ is called duo if every submodule of $M$ is fully invariant in $M$.

**Proposition 2.13.** Let $M$ be an $R$-module, $S = \text{End}_R(M)$ and $F = \{b_1, \ldots, b_n\}$ be a set of left triangulating idempotents of $S$.

1. $S$ is PWSP with respect to $F$ if and only if $b_i g b_i N = 0$ implies $b_i g b_i = 0$ where $b_i \in F$, $N$ any fully invariant submodule of $M$, and $g \in \text{Hom}_R(M, N)$.
2. If $F$ is a duo module then $S$ is PWSP with respect to $F$ if and only if for each $g \in S$, $(b_i g)^2 = 0$ implies $b_i g b_i = 0$.

**Proof.** Let $S$ be PWSP with respect to $F$. If $b_i g b_i N = 0$, then we have $b_i g b_i S b_i g b_i M \subseteq b_i g b_i N = 0$, which implies $b_i g b_i = 0$. Conversely let $b_i g b_i S b_i g b_i = 0$. Then $N = S b_i g b_i M$ is a fully invariant submodule of $M$ where $b_i g b_i N = 0$. Thus $b_i g b_i = 0$. For second part let $N = g(M)$ for $g \in S$.

Let $M$ be a nonzero $R$-module. A decomposition $M = \oplus_{i=1}^n M_i$ is a left (right) orthogonal decomposition if $\text{Hom}_R(M_j, M_i) = 0$ for each $i < j$ $(j < i)$ $(i, j = 1, \ldots, n)$ [7].
Proposition 2.14. Let $M$ be an $R$-module and $S = \text{End}_R(M)$. Then the following statements are equivalent.

1. $S$ is PWSP.
2. There exists a left orthogonal decomposition $M = \oplus_{i=1}^n M_i$ such that for each $i$, $\text{End}_R(M_i)$ is semiprime.

Proof. (1) $\Rightarrow$ (2) Let $S$ be PWSP with respect to a set of left triangulating idempotents $F = \{f_1, \ldots, f_n\} \subseteq S$. Then $M = \oplus_{i=1}^n f_i M$ is an orthogonal decomposition for $M$ and for each $i$, $\text{End}_R(f_i M) \simeq f_i S f_i$ which is semiprime by Definition 2.1.

(2) $\Rightarrow$ (1) Let $X = \oplus_{i=2}^n M_i$. Since $M$ has a left orthogonal decomposition $M = M_1 \oplus X$, the ring $S$ is isomorphic to \[
\begin{pmatrix}
A & N \\
0 & C
\end{pmatrix}
\] where $A = \text{End}_R(M_1)$, $N = \text{Hom}_R(X, M_1)$ and $C = \text{End}_R(X)$. Thus the result follows by Proposition 2.10, using an induction argument on $n$.

Corollary 2.15. The following statements are equivalent.

1. $R$ is a PWSP ring.
2. There exists a set of left triangulating idempotents $E = \{e_1, \ldots, e_n\}$ of $R$ such that for each $i$, $e_i Re_i$ is semiprime.
3. $R$ has a triangular matrix ring representation with semiprime rings on the main diagonal.

Proposition 2.16. If $R$ is a PWSP ring and $e$ an idempotent in $R$, then $eRe$ is a PWSP ring.

Proof. Let $R$ be a ring and $e$ an idempotent in $R$. Suppose that $R$ is a PWSP ring. By Corollary 2.15, there exists a ring isomorphism $\phi$ from $R$ to a generalized triangular matrix ring $T$ with semiprime rings on its main diagonal. Since $\phi(e)$ is an idempotent in $T$ then its main diagonal elements are idempotent and the restriction of $\phi$ is a ring isomorphism from $eRe$ onto $\phi(e)T\phi(e)$, where the rings on the main diagonal of $\phi(e)T\phi(e)$ are also semiprime. Thus again by Corollary 2.15, $eRe$ is PWSP.

Theorem 2.17. The property PWSP is Morita invariant.

Proof. If $R$ is Morita equivalent to a ring $S$, then it is known that $S$ is isomorphic to the ring $eM_n(R)e$ for some natural number $n$ and suitable idempotent $e$ in $M_n(R)$. The result is now obtained by Propositions 2.8(2) and 2.16.

Proposition 2.18. Let $R$ be a PWSP ring with respect to a set of left triangulating idempotents $E = \{e_1, \ldots, e_n\}$. Then $R = A \oplus B$ such that $A = \oplus_{i=1}^r A_i$ is a direct
sum of semiprime rings and there is a ring isomorphism

\[
B \cong \begin{pmatrix}
B_1 & B_{12} & \cdots & B_{1l} \\
0 & B_2 & \cdots & B_{2l} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_l
\end{pmatrix}
\]

such that each \( B_i \) is a semiprime ring, \( B_{ij} \) is a left \( B_i \)-right \( B_j \) bimodule, and \( k + l = n \).

**Proof.** Let \( \{a_1, \ldots, a_k\} \) be the set of central idempotents of \( E \). Since \( e = a_1 + \ldots + a_k \) is a central idempotent, \( R \cong eR \oplus (1 - e)R = A \oplus B \) is a ring decomposition in which \( A = \bigoplus_{i=1}^k A_i = \bigoplus_{i=1}^k a_i R a_i \). By definition, for each \( i \), \( A_i \) is semiprime. It can be seen that \( \{b_1, \ldots, b_l\} = E \setminus \{a_1, \ldots, a_k\} \) is a set of left triangulating idempotents for \( B \). Thus there is an isomorphism as in (*) such that for each \( 1 \leq j \leq l \), \( B_j \) is semiprime. Also \( k + l = n \). □

3. Piecewise m-Systems

It is common knowledge that m-systems play a useful role in the study of prime and semiprime rings. A nonempty subset \( S \) of \( R \) is called an m-system if for every \( x \) and \( y \) in \( S \) there exists \( r \in R \) such that \( xry \in S \). It is well known that for every ideal of \( R \), the set \( \{r \in R \mid \text{ every m-system containing } r \text{ meets } I\} \) is equal to \( \sqrt{I} := \cap \mathcal{P} \) where \( \mathcal{P} \) is the set of all prime ideals of \( R \) containing \( I \). The purpose of this section is to develop an analogous concept for the study of PWSP rings.

**Definition 3.1.** Let \( R \) be a ring and \( E = \{e_1, \ldots, e_n\} \) be a set of left triangulating idempotents of \( R \). A nonempty subset \( S \) of \( R \) is a piecewise m-system with respect to \( E \), if for each \( e_i a e_j \) and \( e_j b e_k \in S \), where \( e_i, e_j, e_k \in E \) and \( a, b \in R \), there...
exists \( r \in R \) such that \( e_i a e_j r e_j b e_k \in S \). The intersection of all PWP ideals of \( R \) containing \( I \) will be denoted by \( \sqrt{p} \).

**Lemma 3.2.** Let \( R \) be a ring and \( E \) be a complete set of left triangulating idempotents of \( R \).

1. If \( P \) is an ideal of \( R \) then \( R \setminus P \) is a piecewise m-system with respect to \( E \), if and only if \( P \) is a PWP ideal.

2. Let \( P \) be an ideal of \( R \) such that \( e_i P e_i \cap S = \emptyset \) for some \( e_i \in E \) and some piecewise m-system \( S \). If \( P \) is maximal with respect to this property then \( P \) is prime.

**Proof.** (1) It is easy to check.

(2) Let \( I_1 \) and \( I_2 \) be ideals such that \( I_1 I_2 \subseteq P \) and \( a_j \in I_j \setminus P \) (\( j = 1, 2 \)). Since \( P \) is maximal with respect to the mentioned property, there exist \( s_j \in (e_i(P + (a_j))e_i) \cap S \) (\( j = 1, 2 \)). Thus \( e_i s_1 e_i r e_i s_2 e_i \in (e_i P e_i) \cap S \) which is a contradiction. Thus \( P \) is prime.

The following example shows that m-systems and piecewise m-systems are different in general.

**Example 3.3.** Let \( R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} \) where \( k \) is a field. Then \( R \) is a PWP ring. Thus by Lemma 3.2, \( R \setminus \{0\} \) is a piecewise m-system with respect to the complete set of left triangulating idempotents \( \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \). However that set is not an m-system.

**Proposition 3.4.** Let \( R \) be a ring, \( E = \{e_1, ..., e_n\} \) a set of left triangulating idempotents, and \( I \) an ideal in \( R \). If

\[ I_p = \{ e_i r e_i \mid r \in R, \text{every piecewise m-system containing } e_i r e_i \text{ meets } e_i I e_i \} \]

then \( I_p = \sqrt{p} = \sqrt{I} \).

**Proof.** Suppose that \( e_i r e_i \in I_p \). We shall show that \( e_i r e_i \in \sqrt{p} \). If there exists a PWP ideal \( P \) of \( R \) containing \( I \) such that \( e_i r e_i \notin P \), then \( e_i r e_i \in R \setminus P := S \) and \( S \cap e_i I e_i = \emptyset \). On the other hand, \( S \) is a piecewise m-system by Lemma 3.2 and by definition of \( I_p \), \( S \) meets \( e_i I e_i \), a contradiction. Thus \( e_i r e_i \in P \). This shows that \( I_p \subseteq \sqrt{I} \). Since each prime ideal of \( R \) is PWP, we have \( \sqrt{p} \subseteq \sqrt{I} \). It is enough to show that \( \sqrt{I} \subseteq I_p \). Let \( e_i r e_i \notin I_p \). Then there exists a piecewise m-system \( T \) such that \( e_i r e_i \in T \) and \( T \cap e_i I e_i = \emptyset \). Let \( A = \{ J \subseteq R \mid I \subseteq J \text{ and } e_i J e_i \cap T = \emptyset \} \). By Zorn’s Lemma, \( A \) has a maximal member \( Q \). By Lemma 3.2, \( Q \) is a prime ideal of \( R \). Thus \( e_i r e_i \notin \sqrt{I} \). \( \square \)
Theorem 3.5. Let $E = \{e_1, ..., e_n\}$ be a set of left triangulating idempotents of a ring $R$. Then for any ideal $I$ in $R$ the following are equivalent.

1. $I$ is a PWSP ideal with respect to $E$.
2. For each $e_i \in E$, $e_1I = e_i\sqrt{I}e_i$ where $\sqrt{I}$ is the prime radical of $I$.
3. For each $e_i \in E$, $e_1I = e_i(p\sqrt{I})e_i$.

Proof. (1) $\Rightarrow$ (2) Let $x_0 = e_i^2e_i \notin e_1I$. Then $e_1x_0e_iRe_1x_0e_i \not\subseteq I$. Now let $x_1 \in e_1x_0e_iRe_1x_0e_i \setminus I$. Continuing this method will yield a set $S = \{x_0, x_1, \ldots\}$. By Zorn's Lemma there exists an ideal $P$ containing $I$ and disjoint from $S$. It can be seen that $P$ is prime. Hence $x_0 \notin e_i\sqrt{I}e_i$.

(2) $\Rightarrow$ (1) Let $e_ixe_iRe_1xe_i \subseteq I$ for some $e_i \in E$ and $x \in R$. Then by (2), $e_ixe_iRe_1xe_i \subseteq e_1Ie_i = e_i\sqrt{I}e_i \subseteq \sqrt{I}$. Thus $e_ixe_i \in e_i\sqrt{I}e_i = e_i\sqrt{I}e_i \subseteq I$.

(2) $\Leftrightarrow$ (3) This follows from Proposition 3.4.

Proposition 3.6. Let $P$ be a PWP (resp. PWSP) ideal of $R$ with respect to a complete set of left triangulating idempotents $E = \{e_1, ..., e_n\}$. Then $P$ contains a minimal PWP (resp. PWSP) ideal with respect to $E$.

Proof. Let $\mathcal{A}$ be the set of PWP (resp. PWSP) ideals with respect to $E$ which are contained in $P$. By Zorn’s Lemma it can be seen that $\mathcal{A}$ has a minimal member which is a PWP (resp. PWSP) ideal. Thus $P$ contains a minimal PWP (resp. PWSP) ideal.

4. Weak Quasi-Baer Rings

As mentioned in Section 1, for a ring $R$ of finite triangulating dimension, it is shown in [1, Theorem 4.11] that $R$ is PWP if and only if $R$ is quasi-Baer. This important result naturally motivates the following question. What modification of the concept of a Baer ring is equivalent to the PWSP property? Although we do not know the precise answer to the above question, here we introduce the concept of right weak quasi Baer rings and then find a description of these rings in terms of the PWSP property. If $M_R$ is a module and $X$ is a nonempty subset of $M$, then the right annihilator of $X$ in $R$ is denoted by $r_R(X)$.

Definition 4.1. A ring $R$ is called right weak quasi-Baer, if the right annihilator of every nilpotent ideal of $R$ is generated as a right ideal by an idempotent.

Similarly, the left weak quasi-Baer property for a ring $R$ is defined (i.e. the left annihilator of every nilpotent ideal of $R$ is generated as a left ideal by an idempotent). Examples 4.8(3) shows that the weak quasi-Baer property is not left-right symmetric.
We recall from the introduction that \( \tau \dim(R) \) is assumed to be finite in the following results.

**Proposition 4.2.** If \( R \) is a right weak quasi-Baer ring, then \( R \) is PWSP.

**Proof.** Let \( E = \{e_1, ..., e_n\} \) be a complete set of left triangulating idempotents of \( R \) and let \( e_i x e_i R e_i x e_i = 0 \) for some \( i \) and element \( x \in R \). Then \( I = Re_i x e_i R \) is an ideal of \( R \) such that \( I^2 = 0 \). Since \( R \) is right weak quasi-Baer, \( r_R(I) = f R \) for some nonzero idempotent \( f \in R \). Since \( f R \) is an ideal in \( R \), \( zf = f zf \) for all \( z \in R \). It follows that \( e_i f e_i \) is an idempotent element in \( e_i R e_i \), and since \( e_i y = ye_i = y \) for all \( y \in e_i R e_i \), we can deduce that \( e_i f e_i \in S_i(e_i R e_i) \). Therefore \( e_i f e_i = e_i \) or \( 0 \). If \( e_i f e_i = e_i \) then \( e_i x e_i = e_i x(e_i f e_i) = 0 \) because \( I f = 0 \). Suppose that \( e_i f e_i = 0 \). Since \( e_i x e_i \in f R \), we have \( e_i x e_i \in (e_i f) R = (f e_i) R \). Hence \( e_i x e_i \in (e_i f e_i) e_i R = 0 \). The proof is complete. \( \square \)

We are now going to find conditions under which a PWSP ring is right weak quasi-Baer. Since a PWSP ring is isomorphic to a formal triangular matrix ring \( T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \) for suitable bimodule \( AMB \), we first state when \( T \) is a right weak quasi-Baer ring. In [4], the characterization of \( T \) as a quasi-Baer ring is provided. A characterization of formal triangular matrix rings that are weak quasi-Baer is then obtained by methods as in [4, Theorem 3.2] using the fact that nilpotent ideals of \( T \) have the form \( \begin{pmatrix} I & N \\ 0 & J \end{pmatrix} \) where \( I \) and \( J \) are nilpotent ideals in \( A \) and \( B \) respectively, and \( N \) is a sub-bimodule of \( AMB \). We record this below.

**Proposition 4.3.** Let \( M \) be an \( R \)-\( S \) bimodule and \( T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \). Then the following statements are equivalent.

1. \( T \) is right weak quasi-Baer.
2. (a) \( R \) is right weak quasi-Baer.
   (b) For each nilpotent ideal \( I \) of \( R \), \( r_M(I) = r_R(I)M \).
   (c) If \( r_N S \subseteq r_M S \) and \( J \) is any nilpotent ideal of \( S \), then \( r_S(N) \cap r_S(J) \) is a direct summand of \( S \).

**Corollary 4.4.** Let \( M \) be an \( R \)-\( S \) bimodule, \( R \) be a semiprime ring and \( T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \). Then \( T \) is right weak quasi-Baer if and only if for any \( r_N S \subseteq r_M S \) and nilpotent ideal \( J \) of \( S \), \( r_S(N) \cap r_S(J) \) is a direct summand of \( S \).

**Proof.** This follows from Proposition 4.3. \( \square \)

Motivated by Proposition 4.3, we give the following definition.
Definition 4.5. Let $e$ be a semicentral idempotent in a ring $R$, $T = eRe$, and $S = (1 - e)R(1 - e)$. We say that $R$ is right $e$-weak Baer if for every left-$T$, right-$S$ sub-bimodule $N$ of $eR(1 - e)$ and nilpotent ideal $J$ of $S$, the ideal $r_S(J) \cap r_S(N)$ is generated as a right ideal by an idempotent. Clearly, right 0-weak quasi-Baer means right weak quasi-Baer.

Lemma 4.6. Let $e$ be a semicentral reduced idempotent in a ring $R$. Then there exists a complete set of left triangulating idempotents $E = \{e_1, ..., e_n\}$ such that $e = e_1$.

Proof. We have $R = eR \oplus (1 - e)R$ such that $\text{Hom}_R(eR, (1 - e)R) = 0$. Since $\tau\text{dim}(R)$ is assumed to be finite, the $R$-module $(1 - e)R$ has finite triangulating dimension [7, Theorem 2.4]. Thus $(1 - e)R$ has a left orthogonal decomposition $(1 - e)R = \oplus_{i=1}^{k} M_i$ such that for each $i = 1, ..., k$, $\tau\text{dim}(\text{End}_R(M_i)) = 1$ [7, Proposition 2.7]. It follows that there exists a complete set of left triangulating idempotents $E = \{e_1, ..., e_n\}$ such that $e = e_1$. □

Theorem 4.7. The following statements are equivalent for a ring $R$.

(1) $R$ is right weak quasi-Baer.

(2) $R$ is PWSP and right $e$-weak quasi-Baer for each $e \in S_l(R)$.

(3) $R$ is PWSP and right $e$-weak quasi-Baer for some reduced idempotent $e$ in $S_l(R)$.

Proof. (1) $\Rightarrow$ (2) This follows from Propositions 4.2 and 4.3 using the fact that if $e \in S_l(R)$, then $R \cong \begin{pmatrix} eRe & eR(1 - e) \\ 0 & (1 - e)R(1 - e) \end{pmatrix}$.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) Suppose $R$ is PWSP and $e$ is a reduced idempotent in $S_l(R)$. By Lemma 4.6, there exists a complete set of left triangulating idempotents $E = \{e_1, ..., e_n\}$ with $e = e_1$. By Corollary 2.4, $R$ is PWSP with respect to $E$. Thus $R \cong \begin{pmatrix} eRe & eR(1 - e) \\ 0 & (1 - e)R(1 - e) \end{pmatrix}$ where $eRe$ is a semiprime ring. The result is now obtained by e-weak quasi-Baer condition on $R$ and Corollary 4.4. □

Examples 4.8. (1) Any semiprime ring is clearly right weak quasi-Baer. Further examples can be readily constructed by taking $S$ to be a semisimple ring in Corollary 4.4.

(2) Let $F$ be a field, $F_n = F$ for $n = 1, 2, ...$ and let

$$R = \{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} F_n \mid (a_n)_{n=1}^{\infty} \text{ is eventually constant}\}$$

Then $R$ is a subring of $\prod_{n=1}^{\infty} F_n$ and $R[[x]]$ is a semiprime ring which is not quasi-Baer; see [2, Example 1.6]. Let now $I$ be a maximal ideal of $R[[x]]$. Let now I be a maximal ideal of $R[[x]]$. Then $R/I$ is a semiprime ring which is not quasi-Baer.
and $S = R[[x]]/I$. Then by [4, Theorem 3.2], $T = \begin{pmatrix} R[[x]] & S \\ 0 & S \end{pmatrix}$ is not quasi-Baer and clearly it is not semiprime but by Corollary 4.4, $T$ is a right weak quasi-Baer ring.

(3) The ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ is right weak quasi-Baer, but it is not left weak quasi-Baer because the left annihilator of $\begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$ is not a direct summand of $R R$.

We end by recording a noteworthy fact:

**Remark 4.9.** Being right weak quasi-Baer is a Morita invariant for rings. To see this, in [5, Lemma 2], let $I$ be a nilpotent ideal.

**Acknowledgment.** The authors would like to thank the referees for the valuable suggestions and comments.

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