On uniform estimates for \((n-1)\)-form fully nonlinear partial differential equations on compact Hermitian manifolds

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Abstract

We obtain a priori \(L^\infty\) estimate for a general class of \((n-1)\)-form fully nonlinear partial differential equations on compact Hermitian manifolds. Our method relies on the local version of comparison with auxiliary Monge-Ampère equations, developed earlier by B. Guo and D. H. Phong. The key is to find the appropriate elliptic operator such that the maximum principle applies.

1 Introduction

A priori estimates have always been fundamental in the study of partial differential equations. Among them, \(L^\infty\) estimates are especially important. As the most prominent example, \(L^\infty\) estimate for the Monge-Ampère equation obtained through Moser iteration was crucial in Yau’s solution to Calabi conjecture [25]. Kolodziej [20] improved it to a sharp version using pluripotential theory. A recent development initiated by Guo, Phong and Tong [6] provided a PDE-based proof to Kolodziej’s result using comparison with solutions to auxiliary Monge-Ampère equations. Moreover, the method extends to a wide class of non-linear equations satisfying a structural condition, which has been shown by Harvey and Lawson [19] to be quite large. This flexible method can also be applied to obtain stability estimates for Monge-Ampère and Hessian equations [7]; \(L^\infty\) estimates for Monge-Ampère equations on nef classes rather than just Kähler classes [8]; sharp modulus of continuity for non-Hölder solutions [9]; extensions to parabolic equations [3]; Regularization of \(m\)-subharmonic functions and Hölder continuity [4]; lower bounds for the Green’s function [10]; uniform entropy estimates [12]; and diameter estimates and convergence theorems in Kähler geometry not requiring bounds on the Ricci curvature [14, 15]. Sroka [21] applied the same method to obtain a sharp uniform bound for the quaternionic Monge-Ampère equation on hyperhermitian manifolds. For an exposition of these topics, we refer the readers to the recent survey paper by Guo and Phong [13].

Recently, Guo and Phong [11] developed a local version of the comparison method to extend the \(L^\infty\) bound to equations on Hermitian manifolds. It also applies to \((n-1)\)-form Monge-Ampère equations [11], which was first solved by Tosatti-Weinkove [23] with regard to Calabi-Yau equation for Gauduchon metrics. A natural question is whether the comparison method in [11] applies to more general \((n-1)\)-form fully nonlinear equations other than Monge-Ampère equations. In this paper, we generalize the \(L^\infty\) estimate to a wide class of \((n-1)\)-form fully nonlinear equations satisfying structure conditions proposed in
The key is to find the appropriate elliptic operator similar to the one defined in [23, p.17] and [11, p.20], so that the maximum principle applies to the test function.

We now describe our results in detail.

Consider a positive function $f : \Gamma \subset \mathbb{R}^n \to \mathbb{R}_+$ such that

1. the domain $\Gamma \subset \mathbb{R}^n$ is an open symmetric cone satisfying
   \[
   \Gamma_n \subset \Gamma \subset \Gamma_1;
   \]  
2. $f(\lambda)$ is symmetric in $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma$ and it satisfies
   \[
   \sum_j \frac{\partial f}{\partial \lambda_j} \lambda_j \leq \beta f
   \]  
for some positive constant $\beta > 0$;
3. $\frac{\partial f}{\partial \lambda_j} > 0$ for all $\lambda \in \Gamma$, $j = 1, \ldots, n$;
4. There exists $\gamma > 0$ such that
   \[
   \prod_{j=1}^{n} \frac{\partial f}{\partial \lambda_j} \geq \gamma.
   \]

for all $\lambda \in \Gamma$.

Remark that (1.2) slightly relaxes the conditions in [6, 7, 8, 9, 10, 11], which require $f$ to be homogeneous of degree one.

Suppose $(X, \omega)$ is a compact Hermitian manifold without boundary and $\omega_\hbar$ is another Hermitian metric on $X$. For any $\varphi \in C^2(X)$, we set

\[
\tilde{\omega} = \omega_\hbar + \frac{1}{n-1}((\Delta_\omega \varphi) \omega - i\partial \bar{\partial} \varphi) + \chi[\varphi],
\]

where

\[
\Delta_\omega \varphi = n \frac{i\partial \bar{\partial} \varphi \wedge \omega^{n-1}}{\omega^n}
\]

is the complex Laplacian of $\varphi$ with respect to $\omega$ and $\chi[\varphi] = x(z, \varphi, \partial \varphi, \bar{\partial} \varphi)$ is the gradient term which may depend on the $\varphi$ and its first order derivatives.

Let $h_\varphi : TX \to TX$ be the relative endomorphism from $\omega$ to $\tilde{\omega}$. In local coordinates, $(h_\varphi)^i_j = g^{jk} \tilde{g}_{ki}$, where $(\tilde{g}_{ki})$ denotes the components of $\tilde{\omega}$ and $(g^{jk})$ denotes the inverse of $\omega$. Let $\lambda[h_\varphi]$ be the (unordered) vector of eigenvalues of $h_\varphi$.

For a smooth function $F$ on $X$, we consider the fully nonlinear partial differential equation

\[
f(\lambda[h_\varphi]) = e^F, \text{ and } \lambda[h_\varphi] \in \Gamma.
\]

with $\sup_X \varphi = 0$ and $\lambda[h_\varphi] \in \Gamma$ on $X$.
We remark that our definition of relative endomorphism $h_\varphi := \omega^{-1}\tilde{\omega}$ is different from those in [6, 7, 8, 9, 10, 11] due to a different form of unknown metric (1.4). So the condition $\lambda \in \Gamma$ means $\lambda[\omega^{-1} \cdot \tilde{\omega}] \in \Gamma$, instead of $\lambda[\omega^{-1} \cdot (\omega + i\partial\bar{\partial}\varphi)] \in \Gamma$, which imposes different condition on $\varphi$ as in [6, 7, 8, 9, 10, 11]. For example, when $\chi = 0$, $\Gamma = \Gamma_n$, this requires $\varphi$ being $(n-1)$–plurisubharmonic in the sense of [17, 18], instead of being plurisubharmonic.

The following two special cases is of particular interests. When $f(\lambda[h_\varphi]) = \left(\frac{\tilde{\omega}^n}{\omega^m}\right)^{1/n}$, $\chi = 0$ and $\Gamma = \Gamma_n$, the equation is the $(n-1)$–form Monge-Ampère equation considered by Tosatti-Weinkove [23]. They showed when $\omega$ is Kähler, the equation is solvable if $F$ is modified by a suitable additive constant, in which the $C^0$ estimate is an essential step of their proof. When $f(\lambda[h_\varphi]) = \left(\frac{\tilde{\omega}^n}{\omega^m}\right)^{1/n}$, $\chi[h_\varphi] = *\text{Re}\{\sqrt{-1}\partial\bar{\partial}\varphi \wedge \bar{\partial}\omega^{n-2}\}$ and $\Gamma = \Gamma_n$, the equation is considered by Šekelyhidi-Tosatti-Weinkove [22] in their celebrated solution to the Gauduchon conjecture.

Later in Guo-Phong [11], a priori $C^0$ estimate of $\varphi$ can be obtained for the $(n-1)$–form Monge-Ampère equation even when $\omega$ is not Kähler. Here we generalize their results to arbitrary equations $f$ satisfying the conditions (1-4) above with the assumption the gradient term $\chi$ being semi-positive.

**Theorem 1** Assume $\chi \geq 0$, let $\varphi$ be a $C^2$ solution on a compact Hermitian manifold $(X, \omega)$ of the equation

$$f(\lambda[h_\varphi]) = e^F$$

where the operator $f(\lambda)$ satisfies the conditions (1-4) spelled out in Section §1.

Fix any $p > n$, we have

$$||\varphi||_{L^\infty(X)} \leq C$$

where $C$ is a constant depending only on $X, \omega, \omega_h, n, p,$ and $\|e^F\|_{L^1(\log L)^p}$. Here the $L^1(\log L)^p$ norms are with respect to the volume form $\omega^n$.

Recently, Guo-Phong manage to prove $C^0$ estimates for the gradient term $\chi$ of a specific form without being semi-positive. Their method uses real auxiliary Monge-Ampère equation, we refer the interested readers to the most recent version of [11].

## 2 The local estimate and proof of Theorem 1

We denote by $G^{ij} = \frac{\partial \log f(\lambda[h])}{\partial h_{ij}} = \frac{\partial f(\lambda[h])}{f \partial h_{ij}}$ the coefficients of the linearization of the operator $\log f(\lambda[h])$.

It follows from the structure conditions of $f$ that $G^{ij}$ is positive definite at $h_\varphi$ and

$$\text{det}(G^{ij}) = \frac{1}{f^n} \text{det}(\frac{\partial f(\lambda[h])}{\partial h_{ij}}) \geq \frac{\gamma}{f(\lambda)^n}.$$
Fix any point \( z_0 \in X \), by simple linear algebra there is a smooth local frame \( \{ e^i \} \) for the holomorphic cotangent bundle \( T^{*1,0}X \) in a neighborhood of \( z_0 \), such that \( \omega = \sqrt{-1} \sum_j e^j \wedge e^j \) and \( \tilde{\omega} = \sqrt{-1} \sum_j \lambda_j e^j \wedge e^j \). In particular, \( (h_\lambda)_{ji} = \lambda_j \delta_{ij} \) and \( G^{ij} = \frac{1}{f} \partial f / \partial x_j \delta_{ij} \) in this frame. Therefore, we have in the neighborhood of \( z_0 \)

\[
\text{tr}_{G} \tilde{\omega} = \sum_{i,j} G^{ij} \tilde{g}_{ji} = \sum_{j} \frac{1}{f} \frac{\partial f}{\partial x_j} \lambda_j \leq \beta. \tag{2.2}
\]

The last equality follows from (1.2). As \( z_0 \) is arbitrary, we know that \( \text{tr}_{G} \tilde{\omega} \leq C \) holds everywhere on \( X \).

Define the following tensor \( \Theta^{ij} \) by

\[
\Theta^{ij} = \frac{1}{n-1} \left( (G^{kl} g_{lk}) g^{ij} - G^{ij} \right).
\]

**Remark.** This tensor is similar to the tensor \( \Theta^{ij} \) defined in [23, p.17] and [11, p.20], except that we put \( G^{ij} \) in place of \( \tilde{g}^{ij} \) in order to deal with more general fully nonlinear equations.

We summarize the key properties of \( \Theta^{ij} \) in the following lemma.

**Lemma 1** The tensor \( \Theta^{ij} \) then satisfies the following: for \( \lambda \in \Gamma \),

(a) \( \Theta^{ij} \varphi_{ji} \leq \beta - G^{ij} (g_h)_{ji} - G^{ij} \chi_{ji} \) and \( G^{ij} (g_h)_{ji}, G^{ij} \chi_{ji} \geq 0 \);

(b) \( \Theta^{ij} \) is positive definite, and \( \det(\Theta^{ij}) \geq \frac{2}{f(\lambda)^n} \).

**Proof.** We have

\[
\beta \geq \text{tr}_{G} \tilde{\omega} = G^{ij} \left( (g_h)_{ji} + \frac{1}{n-1} \left( (G^{kl} g_{lk}) g^{ij} - G^{ij} \right) \right) + \chi_{ji} = G^{ij} (g_h)_{ji} + G^{ij} \chi_{ji} + \frac{1}{n-1} \left( (G^{kl} g_{lk}) g^{ij} - G^{ij} \right) \varphi_{ji} + \Theta^{ij} \varphi_{ji}
\]

where the first equality follows from the equation (2.2), the second equality follows from definition of \( \tilde{\omega} \) and the last equality follows from the definition of \( \Theta \). Since \( G^{ij} \) is positive definite, we know \( G^{ij} (g_h)_{ji} \geq 0 \). This proves (a).

We can choose holomorphic coordinates at a given point \( p \in X \) such that \( g_{ij}|_p = \delta_{ij} \) and \( G_{ij}|_p = \mu_i \delta_{ij} \). Note that \( \mu_i > 0 \) for each \( i = 1, \ldots, n \) since \( G^{ij} \) is positive definite. Then we have

\[
\Theta^{ij}|_p = \frac{1}{n-1} \left( \sum_{k \neq i} \mu_k \right) \delta_{ij}
\]

which is clearly positive definite. Moreover,

\[
\det(\Theta^{ij})|_{x_0} = \frac{1}{(n-1)^n} \prod_{i=1}^{n} \left( \sum_{k \neq i} \mu_k \right) \geq \prod_{i=1}^{n} \left( \prod_{k \neq i} \mu_k \right)^{1/(n-1)} = \prod_{i=1}^{n} \mu_i = \det(G^{ij}) \geq \frac{\gamma}{f(\lambda)^n}.
\]
where the middle inequality follows from the arithmetic-geometric (AG) inequality and the last inequality follows from the equation \((2.1)\). This proves (b). Q.E.D.

Let \(x_0\) be the point where \(\varphi\) attains its minimum. Without loss of generality, we assume \(-\varphi(x_0) \geq 2\). We then fix a local holomorphic coordinate \(z\) centered at \(x_0\) such that

\[
\frac{1}{2} i \partial \bar{\partial} |z|^2 \leq \omega \leq 2i \partial \bar{\partial} |z|^2 \text{ in } B(x_0, 2r_0)
\]  \(2.3\)

where \(B(x_0, 2r_0)\) is the Euclidean ball of radius \(2r_0\) in this coordinate. We denote \(\Omega := B(x_0, 2r_0)\) for simplicity.

Choose a small constant \(\epsilon' > 0\) such that

\[
\omega_h \geq \frac{2\epsilon'}{n-1} (\text{tr}_\omega \omega_h) \omega \text{ in } \Omega.
\]  \(2.4\)

Define \(s_0 := 4\epsilon' r_0^2\). Then for any \(s \in (0, s_0)\), we consider the following comparison function \(u_s : \Omega \to \mathbb{R}\) given by

\[
u_s(z) := \varphi(z) - \varphi(x_0) + \epsilon' |z|^2 - s,
\]  \(2.5\)

Let

\[
\Omega_s = \{z \in \Omega; u_s(z) < 0\}.
\]  \(2.6\)

be the sublevel set of \(u_s\). It’s easy to see from the definition that \(u_s\) is positive on \(\partial \Omega\), so sub-level sets \(\Omega_s\) is relatively compact in \(\Omega\).

Set

\[
A_s = \int_{\Omega_s} (-u_s) e^{nF} \omega^n.
\]  \(2.7\)

To make the right hand side of our auxiliary Monge-Ampère equation smooth, we choose the following sequence of smooth positive functions \(\tau_k : \mathbb{R} \to \mathbb{R}_+\) such that

\[
\tau_k(x) = x + \frac{1}{k}, \quad \text{when } x \geq 0,
\]  \(2.8\)

and

\[
\tau_k(x) = \frac{1}{2k}, \quad \text{when } x \leq -\frac{1}{k},
\]

and \(\tau_k(x)\) lies between \(1/2k\) and \(1/k\) for \(x \in [-1/k, 0]\). Clearly \(\tau_k\) converge pointwise to \(\tau_\infty(x) = x \cdot \chi_{\mathbb{R}_+}(x)\) as \(k \to \infty\), where \(\chi_{\mathbb{R}_+}\) denotes the characteristic function of \(\mathbb{R}_+\).

The auxiliary Monge-Ampère equation we consider is the following

\[
(i \partial \bar{\partial} \psi_{s,k})^n = \frac{\tau_k(-u_s)}{A_{s,k}} e^{nF} \omega^n \text{ in } \Omega, \quad \psi_{s,k} = 0 \text{ on } \partial \Omega
\]  \(2.9\)
with \(i\partial\bar{\partial}\psi_{s,k} \geq 0\), and \(A_{s,k}\) is defined by
\[
A_{s,k} = \int_{\Omega} \tau_k(-u_s)e^{n_F}\omega^n. \tag{2.10}
\]

By Caffarelli-Kohn-Nirenberg-Spruck [1], this Dirichlet problem admits a unique solution \(\psi_{s,k}\) which is of class \(C^\infty(\bar{\Omega})\), with \(\psi_{s,k} \leq 0\). By the definition of \(A_{s,k}\), we have \(A_{s,k} \to A_s\) as \(k \to \infty\), and
\[
\int_{\Omega} (i\partial\bar{\partial}\psi_{s,k})^n = 1. \tag{2.11}
\]

Now we are ready to establish the following key comparison lemma

**Lemma 2** Let \(u_s\) be a \(C^2\) solution of the fully non-linear equation (1.6) and \(\psi_{s,k}\) be the solutions of the complex Monge-Ampère equation (2.9) as defined above. Then we have
\[
-u_s \leq \varepsilon(-\psi_{s,k})^{\frac{n}{n+1}} - u_s \tag{2.12}
\]
where \(\varepsilon\) is the constant defined by \(\varepsilon^{n+1} = A_{s,k}^{\frac{1}{n+1}}\beta^{n+1}\frac{(n+1)^n}{n^{2n}}\).

**Proof.** We show that the function
\[
\Phi = -\varepsilon(-\psi_{s,k})^{\frac{n}{n+1}} - u_s \tag{2.13}
\]
is always \(\leq 0\) on \(\bar{\Omega}\). Let \(x_{\text{max}} \in \bar{\Omega}\) be a maximum point of \(\Phi\). If \(x_{\text{max}} \in \bar{\Omega}\setminus\Omega_s\), clearly \(\Phi(x_{\text{max}}) \leq 0\) by the definition of \(\Omega_s\) and the fact that \(\psi_{s,k} < 0\) in \(\Omega\). If \(x_{\text{max}} \in \Omega_s\), then we have \(i\partial\bar{\partial}\Phi(x_{\text{max}}) \leq 0\) by the maximum principle. Since \(\Theta^\bar{\partial}\) is positive definite, we calculate at \(x_{\text{max}}\):

\[
0 \geq \Theta^\bar{\partial}\Phi_{ji} = \frac{n\varepsilon}{n+1}(-\psi_{s,k})^{-\frac{1}{n+1}}\partial^\bar{\partial}(-\psi_{s,k})_{ij} + \frac{\varepsilon n}{(n+1)^2}(-\psi_{s,k})^{-\frac{2}{n+1}}\partial^\bar{\partial}(-\psi_{s,k})_{ij}\psi_{s,k}^{\frac{1}{n+1}} + \beta + \text{tr}_G\omega_h - \frac{\varepsilon'}{n-1}(\text{tr}_G\omega\cdot\text{tr}_\omega\omega\omega - \text{tr}_G\omega\cdot\omega) \\
\geq \frac{n\varepsilon}{n+1}(-\psi_{s,k})^{-\frac{1}{n+1}}\partial^\bar{\partial}(-\psi_{s,k})_{ij} - \beta + \text{tr}_G[\omega_h - \frac{2\varepsilon'}{n-1}(\text{tr}_\omega\omega_h)\omega] \\
\geq \frac{n^2\varepsilon}{n+1}(-\psi_{s,k})^{-\frac{1}{n+1}}\partial^\bar{\partial}(\det\Theta^\bar{\partial})^{1/n}\partial(\psi_{s,k})_{ij}^{1/n} - \beta \\
\geq \frac{n^2\varepsilon}{n+1}(-\psi_{s,k})^{-\frac{1}{n+1}}\gamma^{1/n}\frac{(-u_s)^{1/n}e^{n_F}}{A_{s,k}^{1/n}} - \beta \\
\geq \frac{n^2\varepsilon}{n+1}\gamma^{1/n}(-\psi_{s,k})^{-\frac{1}{n+1}}\frac{(-u_s)^{1/n}}{A_{s,k}^{1/n}} - \beta.
\]
Here the first equality follows from the definition of $\Theta^{ij}$ and part (a) of Lemma 1

$$\Theta^{ij} \varphi^{ij} \leq \beta - G^{ij}(g_{li})_j - G^{ij} \chi_{ji}. $$

The third line follows from the choice of $\epsilon'$ in (2.4). In the fourth line, we applied the standard arithmetic-geometric inequality. The fifth line follows from part (b) of Lemma 1 and the definition of $\psi_{s,k}$. The last line follows from the equation (1.5). By the choice of $\epsilon$, this implies that $\Psi(x_0) \leq 0$. Hence $\sup_{\Omega} \Psi \leq 0$. Q.E.D.

Along the same spirit in [11], as long as we establish the comparison between $u_s$ and $\psi_{s,k}$ as in Lemma 2, we could derive the $L^\infty$ estimate, without referring to the differential equations satisfied by $u_s$ and $\psi_{s,k}$. We cite the lemma here without repeating the proof from [11, Lemma 2].

**Lemma 3** Assume the functions $u_s = \varphi - \varphi(x_0) + q(z) - s$ satisfies $u_s > 0$ on $\partial \Omega$ and $u_s(x_0) < 0$. Let $\Omega_s$, $A_s$, and $A_{s,k}$ be the corresponding notions defined above.

Assume that

$$ -u_s \leq C(n, \gamma, \beta) \frac{1}{A_{s,k}^{\frac{1}{2}}} (-\psi_{s,k})^\frac{1}{\pi + 1} \text{ on } \Omega $$

for some constant $C(n, \gamma, \beta)$, where $\psi_{s,k}$ are plurisubharmonic functions on $\Omega$ such that $\int_{\Omega}(i \partial \bar{\partial} \psi_{s,k})^n = 1$ and $\psi_{s,k} = 0$ on $\partial \Omega$. Then for any $p > n$, we have

$$ -\varphi(x_0) \leq C(n, \omega, \gamma, \beta, p, \|\varphi\|_{L^1(\Omega, \omega^m)}). $$

Now, Theorem 1 follows from Lemma 2 and Lemma 3, except that the a priori $L^\infty$ bound of $\varphi$ may rely on the $L^1$ norm of $\varphi$. To remove this dependence, we cite the following lemma from [11, Lemma 8].

**Lemma 4** Let $\varphi \in C^2(X)$ so that $\lambda[\omega^{-1} \cdot (\omega + i \partial \bar{\partial} \varphi)] \in \Gamma_1$ with $\sup_X \varphi = 0$, then

$$ \int_X (-\varphi) \omega^n \leq C(n, \omega). $$

**Remark.** We stress again that our definition of relative endomorphism $h_\varphi := \omega^{-1} \tilde{\omega}$ is different from those in [6, 7, 8, 9, 10, 11] due to a different form of unknown metric (1.4). So the condition $\lambda \in \Gamma$ means $\lambda[\omega^{-1} \cdot \tilde{\omega}] \in \Gamma$, instead of $\lambda[\omega^{-1} \cdot (\omega + i \partial \bar{\partial} \varphi)] \in \Gamma$.

Since our assumption for $\varphi$ is that $\lambda[\omega^{-1} \cdot \tilde{\omega}]$ lies in $\Gamma \subset \Gamma_1$, i.e. $\text{tr}_\omega \tilde{\omega} \geq 0$. This implies that $\Delta_\omega \varphi \geq -\text{tr}_\omega \omega_h \geq -C'$ by (1.4), for some positive constant $C'$. Therefore $\text{tr}_\omega (\omega + i \partial \bar{\partial} (\frac{\varphi}{C'})) \geq 0$. In other words, $\lambda[\omega^{-1} \cdot (\omega + i \partial \bar{\partial} (\frac{\varphi}{C'}))]$ is in $\Gamma_1$. Lemma 4 then yields the desired $L^1$ bound of $\varphi$ given our normalization $\sup_X \varphi = 0$. The proof of Theorem 1 is complete.
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