The Effect of Anisotropy on Holographic Entanglement Entropy and Mutual Information

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Abstract

We study the effect of anisotropy on holographic entanglement entropy (HEE) and holographic mutual information (MI) in Q-lattice model, by exploring the HEE and MI for infinite strips along arbitrary direction. We find that lattice always enhances HEE. The MI, however, is enhanced by lattice for large sub-regions, while for small sub-regions, the MI is suppressed by lattice. We also discuss how these phenomena result from the deformation of geometry caused by Q-lattices.
I. INTRODUCTION

Anisotropy is universal and results in many rich phenomena in nature, such as magnetic systems, latticed systems, and so on [1]. In some strongly correlated systems, the anisotropy is associated with some entanglement structures, and have novel applications in measuring instruments. For example, the quantum entanglement can be exploited to design magnetic compass sensors [2–4]. Moreover, the entanglement has been associated with certain observables for some anisotropic quantum phase transitions [5]. The effects of anisotropy on entanglement structure for strongly correlated systems are useful for practical uses and are worthy of further investigation. Strongly correlated systems, however, are long-standing hard problems in physics; entanglement is also hard to study. Gauge/gravity duality can bring together the strongly correlated systems and entanglement and offer a good platform to study the anisotropy of entanglement in strongly correlated systems.

Gauge/gravity duality has been proved powerful tools to study strongly correlated systems and quantum information properties [6–8]. Anisotropy is also ubiquitous in holographic systems, such as systems with lattices, anisotropic axions, massive gravity and so on [9–11].
All these models realize the anisotropy by explicitly breaking the isotropic symmetry. The anisotropy can also be introduced by spontaneous symmetry breaking, such as holographic charge density wave models [12, 13]. Especially, the latticed structure plays a crucial role in obtaining finite direct current transportation coefficients, Mott insulator, metal-insulator transitions [14, 15].

Another huge advantage of gauge/gravity duality is that many information related quantities have been associated with geometrical quantities in holographic theories. The entanglement entropy (EE), which is a widely accepted entanglement measure, has been proposed to be proportional to the area of the minimal surface. This geometric prescription is dubbed as the holographic entanglement entropy (HEE), which has been widely studied and applied in studying the phase transitions, and so on [16–27]. In addition to that, more information-related quantities have been associated with some geometrical objects. The mutual information (MI), whose definition follows from the definition of the HEE, discloses more details of entanglement structures of quantum systems. Also, the Rényi entropy has been proposed as proportional to the minimal area of the cosmic brane [28]. The entanglement of purification, which involves the purification of the mixed states, have been associated with the area of the minimal cross section of the entanglement wedge [29]. Information related quantities are becoming one of the cores of the holographic theories.

Despite its power in measuring the pure state entanglement, EE is unsuitable for measuring the mixed state entanglement. Many other measures have been proposed to measure the mixed state entanglement, such as mutual information, entanglement of purification, non-negativity, and so on - all these have holographic duals [29–31]. The geometric interpretations of entanglement measures greatly simplify the study of entanglement structures in strongly correlated systems.

We study the effect of anisotropy on HEE and MI (for infinite strip) in the holographic Q-lattice model. We find that the Q-lattice always enhances the HEE, in despite of the system parameters and the size of the subregion. The Q-lattice effects on MI, however, depend on the size of the subregions: for small subregions the Q-lattice enhance the MI; when the subregion enlarges the Q-lattice effect become non-monotonic; for large enough subregions the Q-lattice suppresses the MI. We also discuss how these phenomena result from the deformation of the geometry caused by Q-lattices. These results deepen our understanding of how the anisotropy affects entanglement measures and may inspire further investigations.
on this new topic.

This paper is organized as follows: we review the Q-lattice model and deduce the anisotropic HEE in II. Then we study the anisotropic HEE and MI in II B, and we conclude and discuss in V.

II. HOLOGRAPHIC Q-LATTICE MODEL AND ANISOTROPIC HOLOGRAPHIC ENTANGLEMENT ENTROPY

A. Holographic Q-lattice model

The holographic Q-lattice model is a concise realization of the periodic structure. Previous holographic lattice models, such as the ionic lattices model and the scalar lattices model, introduces spatially periodic structures on scalar fields or the chemical potential (see [9] for a recent review). The resultant equations of motion are a group of highly nonlinear partial differential equations, which poses a challenge to solve them numerically. By contrast, the Q-lattice model introduces a complex scalar field, and the resultant equations of motion are only ordinary differential equations. Therefore the Q-lattice model is an easier realization of lattice structures. The Q-lattice model is useful in modeling the Mott insulator and metal-insulator transitions [15, 32].

The Lagrangian of the Q-lattice model is [14, 21, 33, 34],

\[ \mathcal{L} = R + 6 - F_{ab}F_{ab} + \partial_a \Phi^\dagger \partial^a \Phi + \frac{1}{2} m^2 |\Phi|^2. \] (1)

System (1) can be solved with ansatz,

\[ \begin{align*}
   ds^2 &= \frac{1}{z^2} \left[ -(1 - z)p(z)U dt^2 + \frac{dz^2}{(1 - z)p(z)U} + V_1 dx^2 + V_2 dy^2 \right], \\
   A &= \mu(1 - z)adt, \\
   \Phi &= e^{ikx} z^{3-\Delta} \phi,
\end{align*} \] (2)

where \( p(z) = 1 + z + z^2 - \mu^2 z^3/2 \) and \( \Delta = 3/2 \pm (9/4 + m^2)^{1/2} \). We set \( m^2 = -2 \) for concreteness, and hence \( \Delta = 2 \). The horizon and the boundary locate at \( z = 1 \) and \( z = 0 \) respectively. The \( A_\mu \) is the Maxwell field, and \( \phi \) is the complex scalar field mimicking the lattice structures. Consequently, the functions to solve are \( (a, \phi, U, V_1, V_2) \).

In order to solve the system (1), we need to specify the boundary conditions and system parameters. We set \( a(0) = 1 \), then \( A_t(0) = \mu \) becomes the chemical potential of the
dual system. The boundary condition $\tilde{\lambda} = \phi(0)$ is the strength of the lattice deformation, and $\tilde{k}$ is the wave vector of the periodic structure. The asymptotic AdS$_4$ requires that $U(0) = 1, V_1(0) = 1, V_2(0) = 1$. Other boundary conditions at the horizon ($z = 1$) can be fixed by regularity. The Hawking temperature reads $\tilde{T} = (6 - \mu^2) U(1)/(8\pi)$. The black brane solutions can be categorized by 3 dimensionless parameters $(T, \lambda, k) \equiv (\tilde{T}/\mu, \tilde{\lambda}/\mu, \tilde{k}/\mu)$, where we adopt the chemical potential $\mu$ as the scaling unit.

B. Anisotropic Holographic entanglement: HEE over arbitrary direction

The HEE of subregion $A$ is

$$S_A = \frac{\text{Area}(\Sigma_A)}{4G_N^{d+2}}.$$  \hspace{1cm} (3)

where $\Sigma_A$ is the minimal surface satisfying $\partial A = \partial \Sigma_A$ [8]. The HEE of many different subregions, such as disks, infinite strips and cusps, has been widely studied in holographic systems. Apparently, only HEE of non-circular subregions is sensitive to the anisotropy.

For simplicity, we consider the HEE of infinite strip partition for 4-dimensional homogeneous geometry[1],

$$ds^2 = g_{tt} dt^2 + g_{zz} dz^2 + g_{xx} dx^2 + g_{yy} dy^2.$$  \hspace{1cm} (4)

Previous researches on HEE usually adopt the infinite strip along a fixed direction, of which the HEE does not capture the anisotropy effect. It is important to study how anisotropy affect the entanglement structure. To this end, we study the HEE of infinite strips along arbitrary directions (see Fig. 1), which we call as anisotropy HEE.

For an infinite strip pointing at direction $\vec{\theta} = (\cos \theta, \sin \theta)$ on $(x, y)$-plane, the minimal surface will be invariant along $\vec{\theta}$. It is then more convenient to work in a new coordinate

$$\tilde{t} = t, \quad \tilde{z} = z, \quad \tilde{x} = x \cos(\theta) + y \sin(\theta), \quad \tilde{y} = y \cos(\theta) - x \sin(\theta),$$  \hspace{1cm} (5)

where the direction $\vec{\theta}$ is along $\tilde{x}$ in coordinate system (5). The minimal surface is invariant along $\tilde{x}$, and hence the minimal surface can be described by $\tilde{z}(\tilde{y})$. In coordinate system (5) the geometry (4) is written as,

$$ds^2 = g_{tt} d\tilde{t}^2 + g_{zz} d\tilde{z}^2 - (g_{xx} - g_{yy}) \sin(2\theta) d\tilde{x} d\tilde{y}$$

$$+ (g_{xx} \cos^2(\theta) + g_{yy} \sin^2(\theta)) \, d\tilde{x}^2 + (g_{xx} \sin^2(\theta) + g_{yy} \cos^2(\theta)) \, d\tilde{y}^2.$$  \hspace{1cm} (6)

1 Our deduction can also be applied to systems with off-diagonal metric.
The induced metric on the hypersurface $\tilde{z}(\tilde{y})$ at $\tilde{t} = \text{const}$ reads,

$$d\hat{s}^2 = g_{\tilde{x}\tilde{x}} d\tilde{x}^2 + g_{\tilde{z}\tilde{z}} d\tilde{z}^2 + g_{\tilde{y}\tilde{y}} d\tilde{y}^2 + g_{\tilde{x}\tilde{y}} d\tilde{x} d\tilde{y}$$

$$= g_{\tilde{x}\tilde{x}} d\tilde{x}^2 + (g_{zz} z'(\tilde{y})^2 + g_{\tilde{y}\tilde{y}}) d\tilde{y}^2 + g_{\tilde{x}\tilde{y}} d\tilde{x} d\tilde{y}.$$  (7)

Therefore the area of the minimal surface is

$$A = \int_{\Sigma} \mathcal{L} d\tilde{x} d\tilde{y} = L_x \int_{\Sigma} \mathcal{L} d\tilde{y}. \quad (8)$$

where $\mathcal{L} = \sqrt{g} = \sqrt{g_{xx} g_{yy} + g_{zz} z'(\tilde{y})^2 (g_{xx} \cos^2(\theta) + g_{yy} \sin^2(\theta))}$, and $L_x \equiv \int d\tilde{x}$ is the length of the infinite strip along $\tilde{x}$. For simplicity we ignore some common factors and denote the HEE as,

$$\hat{S} \equiv \int_{\Sigma} \mathcal{L} d\tilde{y}. \quad (9)$$

Adopting the $\mu$ as the scaling unit, the scale invariant width and HEE is given by $l \equiv \hat{l} \mu$ and $S \equiv \hat{S}/\mu$ with $\hat{l} = \int d\tilde{y}$.

Treating the $(9)$ as a Lagrangian independent of $\tilde{y}$, the corresponding Hamiltonian is a constant along the minimal surface $\tilde{z}(\tilde{y})$. The homogeneity of the background requires that a minimal surface shall reach a local bottom at some $\tilde{z}_*$, with which the width and $l$ the HEE $S$ can be uniquely decided $l(\tilde{z}_*)$ and $S(\tilde{z}_*)$.  

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Given the algorithm to compute the $l$ and $S$ we turn to study the anisotropy effects on HEE and MI.

III. ANISOTROPIC HOLOGRAPHIC ENTANGLEMENT ENTROPY IN Q-LATTICE MODEL

The HEE is dictated by the background geometry, which spans over $(\lambda, k, T)$ for the Q-lattice model. From the physical interpretation of Q-lattice, the $(\lambda, k)$ represents the lattice strength and the lattice wavelength respectively. For $\lambda = 0$ the lattice is absent, the system reduces to AdS-RN black hole; while for $k = 0$, the translational invariance and isotropy are recovered. Therefore, the anisotropy effect is only significant for large enough values of $\lambda$ and $k$.

First, we reveal the entanglement structure by studying the relation between the HEE and parameter $(\lambda, k, \theta)$, at a typical temperature $T = 0.1^2$. As depicted in Fig. 2, at small values of $\lambda$ or $k$ the HEE at arbitrary direction $\theta$ are all the close to each other$^3$. This is the reflection of the fact that the lattice effect is only significant for large enough values of $\lambda$ and $k$.

Moreover, the left plot of Fig. 2 shows that the HEE exhibits some extremal behavior near the quantum critical points of the metal-insulator transition [21]. This phenomenon indicates that, the HEE along arbitrary direction, rather than the HEE along the direction perpendicular to the lattice (see [21]), can characterize metal-insulator transitions. This phenomenon is expected since the HEE along arbitrary direction is mainly contributed by the thermal entropy for relatively large subregions, whose HEE reads,

$$S = \int_{\Sigma} Ld\hat{y} = \int_{\Sigma} \sqrt{g_{xx}g_{yy} + g_{zz}(\hat{y})^2 (g_{xx} \cos^2(\theta) + g_{yy} \sin^2(\theta))}d\hat{y} \simeq \int_{\Sigma} \sqrt{g_{xx}g_{yy}}d\hat{y}. \quad (10)$$

Therefore, the fact that thermal entropy characterizes the metal-insulator transitions results in the phenomenon that HEE along arbitrary direction characterizes the metal-insulator transitions. We also remark that the relation $S$ vs $\lambda$ can also exhibits extremal behavior near the quantum critical points, as long as the temperature is low enough.

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$^2$ Similar phenomena are found for low-temperature regions.

$^3$ From the parity of (8) we see that only $\theta \in [0, \pi/2]$ is needed to be explored.
Fig. 2 also shows that the HEE decreases with $\theta$, regardless of the values of $\lambda, k$ and $T$. This phenomenon indicates that the lattice always enhances the HEE, noticing that the lattice points at $x$-direction (corresponds to $\theta = 0$). To demonstrate more comprehensive content of anisotropic effect of HEE, we show the angular dependence of HEE in Fig. 3. We also see from Fig. 3 that the $\partial_\theta S|_{\theta=0,\pi/2} = 0$, this is a simple reflection of the fact that $\partial_\theta L \sim \sin(2\theta)$.

Another important feature of the anisotropy effect on HEE is that the monotonic behavior of $S(\theta)$ is independent of values of $l$. This feature could be understood from the geometry deformed by Q-lattices. The monotonically decreasing behavior of $S(\theta)$ means positive $\partial_\theta L = \sin(2\theta) (g_{yy} - g_{xx})$, whose sign is defined by $g_{yy} - g_{xx}$. We find $g_{yy} - g_{xx} < 0$ for all values of $(\lambda, k, T)$ (see Fig. 4). Therefore it is the geometry deformed by the Q-lattice that leads to the phenomena that Q-lattice always enhance the HEE. Notice also that $(g_{yy} - g_{xx})|_{z \to 0} \to 0$ is due to the boundary condition $g_{xx}(0) = g_{yy}(0) = 1$. The monotonic behavior of $S(\theta)$ is apparently model-dependent, the scenario could be more diverse for other holographic models.

The HEE is a good measure of pure state entanglement, while it is not suitable to characterize mixed state entanglement. Especially, for large subregions, the thermal entropy starts to contribute to the HEE [35] and subordinate the quantum entanglement. In order to further understand the entanglement, we study the MI structure over the anisotropic Q-lattice model.
IV. ANISOTROPIC MUTUAL INFORMATION IN Q-LATTICE MODEL

The mutual information $I(A; C)$ measures the entanglement between two separate sub-regions $A$ and $C$.

$$I(A; C) := S(A) + S(C) - S(A \cup C).$$

There are two configurations of $S(A \cup C)$ with locally minimal area, the blue ones and red ones (see Fig. 5 for demonstration). The definition of HEE requires the global minimum, i.e., $S(A \cup C) = \min\{S(A) + S(C), S(B) + S(A \cup B \cup C)\}$. Therefore the MI is,

$$I(A; C) = \begin{cases} 0, \\ S(A) + S(C) - S(B) - S(A \cup B \cup C). \end{cases}$$

FIG. 3: $S(\theta)$ at different values of $(\lambda, k)$ specified as the plot label in each plot. All plots are obtained at $l = 0.748, 1.20, 1.65, 2.09, 2.54, 2.99$ from bottom to top in each plot.

FIG. 4: $g_{yy} - g_{xx}$ are all negative at $k = 0.1, T = 0.1$. 
The definition of MI (11) not only cancels out the area law divergence, but also the volume law from the thermal contribution [35]. We study the MI structure of the parallel infinite strips. Given a two-party system with $A \cup C$ with the separation $B$, we demonstrate the MI structure over the angle $\theta$.

For simplicity, we fix the length of the $A$ and $C$ to be equal. We find that the angular behavior of MI depends on the size of the subregions (see Fig. 6). At first, the MI is monotonically increasing, which is opposite with respect to HEE. With the increase of $l$, the MI starts to become non-monotonic and finally becomes monotonic again at large enough subregions. In other words, the entanglement between small subregions is suppressed by the lattice, while for large subregions, the lattice enhances the entanglement.

The above size-dependence of the angular behavior $I(\theta)$ can be explained by the $\partial_\theta S$ at different ranges of $l$. The derivatives of MI with respect to $\theta$ reads,

$$\partial_\theta I = \partial_\theta S(A) + \partial_\theta S(C) - \partial_\theta S(A \cup C).$$

(13)

From (13) we see that for small values of $a$ and $c$ at some fixed $b$, the $S(A \cup C)$ is more dominating than $S(A)$ and $S(C)$. Therefore, the MI is dominated by $-\partial_\theta S(A \cup C)$ for small values of $a$ and $c$, which explains the opposite angular behavior as that of the HEE. At large values of $a$ and $c$, however, the angular behavior will be the same as that of the HEE. For large value of $a$ and $c$, the $S(A \cup C)$ is dominated by the near horizon geometry, i.e.
FIG. 6: The plot is at \((\lambda, k) = (3, 0.5)\) (insulating phase) and at \(T = 0.00892\), we also remark that the phenomena do not depend on temperature, therefore we demonstrate the phenomena at a specific value of \(T\). We also remark that similar behaviors can be observed for metallic phases.

the thermal entropy density \(\sqrt{g_{xx}g_{yy}}\). Therefore for large enough values of \(a\) and \(c\) we have \(\partial_\theta S(A \cup C)\) is close to 0 since \(\partial_\theta \sqrt{g_{xx}g_{yy}} = 0\). Therefore the angular behavior of MI will be the same as that of the HEE.

For small values of \(a\) and \(c\), it can be easily understood. \(S(A \cup C)\) dominates over \(S(A) + S(C)\), which means that \(S(A \cup C)\) decreases more rapidly than that of \(S(A) + S(C)\). Subregion \(C\) needs to downsize to maintain a nonzero MI. For large values of \(a\) and \(c\), however, the monotonic behavior of \(c_c\) results from a very different origin. For nonzero MI, the \(S(A \cup C)\) is equal to \(S(B) + S(A \cup B \cup C)\), where \(S(A \cup B \cup C)\) will be almost independent of \(\theta\) for large values of \(a\) and \(c\). Therefore, the critical behavior of MI with \(\theta\) is determined by \(S(A) + S(C)\) and \(S(B)\). The monotonic behavior of \(c_c\) means that \(S(B)\) always decreases more rapidly with the angle \(\theta\) than that of \(S(A) + S(B)\) near the critical regions of MI. This phenomenon still ask for further analytical understanding.
\[(\lambda, k) = (2, 2), (a, b) = (0.8, 0.5), \theta=0\]

FIG. 7: The critical values of \(c\). The red dots are the location of \(c_c\) for different temperatures.

\[(\lambda, k) = (2, 2)\]

FIG. 8: The \(c_c\) for \((a, b) = (2, 0.6)\).

V. DISCUSSION

We studied the effect of anisotropy on HEE and MI in anisotropic Q-lattice model. We find that the lattice always enhances the HEE, which reflects how Q-lattice deforms the background geometry. We also find that the anisotropic HEE always characterizes the QPT. For MI we find the angular behavior is size-dependent, which can be understood from the angular behavior of HEE. Next, we point out several directions worthy of further investigations.

First, more general anisotropic models are worthy of considerations to deepen our un-
derstanding of how anisotropy affects the HEE and the MI. For example, the axion models, Q-lattice models with two-dimensional lattice, scalar lattice models and ionic lattice models. We also point out that for scalar lattice models and ionic lattice models, the background could be both anisotropic and inhomogeneous. Consequently, the study of entanglement measures could be much harder. Moreover, the anisotropic entanglement properties are tied to responses of quantum systems [2–4]. Therefore we may study the DC conductivity, and figure out its connection to the anisotropic HEE and MI.

In addition to studying more anisotropic models, we could explore more comprehensive content of the entanglement structure by studying the more entanglement measures. For example, the entanglement of purification, complexity, negativity, Rènyi entropy, and so on. We could expect that all these information-related quantities could reveal more novel properties of the anisotropic systems.

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