Topological bifurcation structure of
one-parameter families of $C^1$ unimodal
maps

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Abstract
We consider the bifurcation structure of one-parameter families of unimodal
maps whose differentiability is only $C^1$. The structure of its bifurcation dia-
gram can be a very wild one in such case. However we prove that in a certain
topological sense, the structure is the same as that of the standard family of
quadratic polynomials. In the case of families of polynomials, irreducible com-
ponent of the bifurcation diagram can be defined naturally by dividing by the
polynomials corresponding to lower periods. We show that such an irreducible
component can be defined even if the maps and the family satisfy only a very
mild differentiability condition. By removing components of lower periods, the
structure of the bifurcation diagram becomes a considerably simplified one. We
prove that the symbolic condition for the irreducible components is exactly the
same as that of the standard family of quadratic polynomials. When we con-
sider families of maps without strict conditions, the bifurcation diagrams may
have infinitely many wild components. We show that such a situation does not
affect the irreducible component essentially by proving a separation theorem
for compact set in the plane which asserts that a given connected component
can be cut out from the compact set by a curve.

Keywords: unimodal map, bifurcation diagram, kneading theory, symbolic
dynamics, continua theory
1. Introduction

One parameter families of unimodal maps are the most widely studied object in the theory of nonlinear dynamical systems. Their bifurcation and the dynamics are beautifully described by the kneading theory and the theory of one-dimensional dynamical systems [MT], [G], [CE], [JR1], [JR2], [dMvS]. However, most neat results are obtained for ‘good’ families of unimodal maps such as a family of quadratic polynomials, a family of maps having a negative Schwartzian derivative or a family of maps having only regular bifurcations.

In many practical problems where a family of unimodal maps appears, we might not expect the maps to have such nice properties. In this paper, we show that even if we do not impose any strict conditions on the family of maps, the bifurcation structure of one-parameter families of $C^1$ unimodal maps is, in a certain topological sense, the same as that of the standard family of quadratic polynomials.

We are interested in the bifurcation structure of one-parameter family of unimodal maps which starts from a map having no non-wandering point and ends up with a map with full dynamics. By full dynamics, we mean that the non-wandering set is conjugate to the full shift of one-sided sequences of two symbols. We shall call such a unimodal map a horseshoe. More precisely,

**Definition 1.** A $C^1$ unimodal map $f$ is a horseshoe if $f$ satisfies the following (a) and (b) (as shown in figure 1).

(a) $f$ has two fixed points.
(b) Let $a$ be the left fixed point. There exist $a_1$, $b_1$ and $b$ such that

- $a < a_1 < b_1 < b$,
- $f(b) = a$,
- $f(a_1) = f(b_1) = b$ and
- $|f'(x)| > 1$ $\forall x \in [a, a_1] \cup [b_1, b]$.

**Remark 1.** It is easy to see that $a < a_1 < b_1 < b$.

We suppose that the one-parameter families of $C^1$ maps we deal with in this paper satisfy the following mild continuity and differentiability conditions.

**Definition 2.** Let $\{ f_t \}_{t \in [0,1]}$ be a one-parameter family of $C^1$ maps on $\mathbb{R}$. We say that $\{ f_t \}_{t \in [0,1]}$ is a continuous family of $C^1$ maps if $f^0(t, x) = f_t(x)$ and $f^t(x) = \frac{df}{dt}(x)$ are continuous functions on $[0, 1] \times \mathbb{R}$.

**Remark 2.**

(a) It is easy to see that $\{ f_t \}_{t \in [0,1]}$ is a continuous family of $C^1$ maps if and only if $t \mapsto f_t$ is a continuous curve in the space of $C^1$ maps on $\mathbb{R}$ endowed with the $C^1$ topology.
(b) We do not assume that $f^0(t, x)$ is differentiable with respect to $t$. Thus, ‘continuous family of $C^1$ maps’ is weaker than ‘$f^0(t, x)$ is $C^1$ on $[0, 1] \times \mathbb{R}$’.

**Definition 3.** Let $\{f_t\}_{t \in [0, 1]}$ be a continuous family of $C^1$ unimodal maps on $\mathbb{R}$. We say that $\{f_t\}_{t \in [0, 1]}$ is a continuous full family if it satisfies the following properties.

(a) $f_0$ has no fixed point (therefore no non-wandering points).
(b) $f_1$ is a horseshoe.
(c) There exists an $M > 0$ such that fixed points of $f_t$ are in $[-M, M]$ for any $0 \leq t \leq 1$.

**Remark 3.** Since our definition of unimodal map is so general that $f_t$ could have infinitely many fixed points diverging to $-\infty$. To avoid such a situation, we impose the property (c). It is easy to see that no non-wandering point exists outside of $[-M, M]$ for any $f_t$.

In this paper, we use the following terminology for the periodicity of points and sequences. Let $f : X \to X$ be a map on a set $X$. When $f^n(p) = p$, we say that $p$ is a periodic point of $f$ of period $n$. Moreover, if $p$ is not a periodic point of period $k$ for any $0 < k < n$, then we say that the minimal period of $p$ is $n$. The minimal period of $p$ is denoted by $\text{per}(p)$.

Let $\{f_t\}_{t \in [0, 1]}$ be a continuous full family of $C^1$ unimodal maps. We are interested in the topological structure of the bifurcation diagram of periodic points of $\{f_t\}$. Let $n$ be a positive integer. The bifurcation diagram of periodic points of period $n$ is the set $\{(t, x) \in [0, 1] \times \mathbb{R} | f_t^n(x) = x\}$ (refer figure 2). Occasionally we simply call it the bifurcation diagram of period $n$.

By the definition, $f_1$ is a horseshoe. Therefore the dynamics of $f_1$ on its non-wandering set is equivalent to the full shift of one-sided sequences of two symbols, and we know rigorously what kind of periodic points $f_1$ has.

Our question in this paper is: ‘what types of periodic points of $f_1$ are on the same connected component of the bifurcation diagram?’

If there exists even one point where $f_t$ is not differentiable, then this question is not interesting at all. For example, $f_t(x) = (2t - 1) - 3|x|$ has only one connected component of the bifurcation diagram for any $n \geq 1$ ($f_t(x)$ is just a reparametrization of $g_t(x) = t - 3|x|$).
Figure 2. The bifurcation diagram of period 6 of the quadratic family \( q_t(x) = t - x^2 \). \( f_t(x) = (7/2)t - 1 - x^2 \), a reparametrization of \( q_t(x) \), satisfies the condition of continuous full family. By an easy calculation, we can see that \( q_t \) is a horseshoe for \( t > (5 + 2\sqrt{5})/4 \approx 2.368 \).

The purpose of this paper is to show that for continuous full family of \( C^1 \) unimodal maps, such an equivalence relation of being contained in the same connected component of the bifurcation diagram is exactly the same as that of the standard family of quadratic polynomials.

We had thought that such a result had been almost obvious or it had been known already as a folklore theorem. However, we could not find any previous work proving such a statement rigorously. And also we found that a considerable amount of non-trivial arguments were necessary in the proof. So we decided to write it down rigorously.

There are two important points on the bifurcation of continuous full family of \( C^1 \) unimodal maps. One is the existence of irreducible component of the bifurcation diagram. If we consider connected component of the bifurcation diagram itself, then some components of different periods have intersection because of period doubling bifurcation, and the condition of being contained in the same component would become a very confusing one. To avoid such a confusing situation and make the statement clearer, we show that we can divide iterated maps and define irreducible component.

In the case of polynomial maps, irreducible component of the bifurcation diagram can be defined naturally. For example, let \( g_t(x) = (4t - 1) - x^2 \) be the standard family of quadratic maps and set \( G_2(t, x) = g_t^2(x) - x \) (we just reparametrized \( q_t(x) = t - x^2 \) to get a family consistent with the definition of continuous full family). Then \( G_2^{-1}(0) \) is a bifurcation diagram of period 2. However it also contains a component of fixed points. \( g_t^2(x) - x \) can be divided by
Nonlinearity 34 (2021) 7991 A Sannami and T Yokoyama

g_t(x) − x and G_2(t, x) = \tilde{G}_2(t, x) / (g_t(x) − x) = x^2 − x + 2(1 − 2t) is also a polynomial of x and t. G_2^{-1}(0) is the bifurcation diagram of minimal period 2 except one point of bifurcation, and it does not contain the component of fixed points. In this paper, we show that such a division is possible, and we can define irreducible component of the bifurcation diagram even if it satisfies only a very mild differentiability condition as in definition 3.

Another important point is a separation theorem for compact set in the plane. Our family of unimodal maps is so general that its bifurcation diagram may have infinitely many wild components. We have to guarantee a certain kind of separation of those components and show that such a confusing situation does not affect the irreducible components topologically. We use a recently developed technique ‘filling’ which was devised to analyse homeomorphisms on two-dimensional manifolds [JKP], [KT1], [KT2].

On bifurcation diagram of one-dimensional maps, we should mention the works of Guckenheimer, Jonker and Rand. A study of the bifurcation diagram of a family of unimodal maps is first attempted in a paper of Guckenheimer [G]. Assuming certain regularity for maps and families, he obtained some qualitative properties of such families. [JR2] of Jonker and Rand is a remarkable work about a family of C^1 unimodal maps. Based on the works of Jonker [J] and Jonker and Rand [JR1], they investigated family of C^1 unimodal maps thoroughly, and obtained a universal property of the changing process of kneading invariant of such a family. Some arguments in the symbolic dynamics part of our paper are similar to some arguments in [JR2], but their result is not about the bifurcation diagram itself and our arguments are mainly about the symbolic properties of irreducible component which is not defined in [JR2]. For this reason, we wrote our paper in a self-contained way.

2. Irreducible component

In order to remove components corresponding to periodic points of the half period, we define a quotient map. To formulate it rigorously, we need the following proposition.

Proposition 1. Let f : \mathbb{R} \to \mathbb{R} be a C^1 map and n a positive integer. Define a map F_{2n} : \mathbb{R} \to \mathbb{R} as follows.

\[
F_{2n}(x) = \begin{cases} 
\frac{f^{2n}(x) − x}{f^n(x) − x} & \text{on } \{ x|f^n(x) − x \neq 0 \} \\
(f^n)'(x) + 1 & \text{on } \{ x|f^n(x) − x = 0 \}
\end{cases}
\]

Then F_{2n} is continuous on \mathbb{R}.

Proof. Since f is C^1, clearly F_{2n} is continuous on \{ x|f^n(x) − x \neq 0 \} and on the interior of \{ x|f^n(x) − x = 0 \}. We show that F_{2n} is continuous at any point in \{ x|f^n(x) − x \neq 0 \} \cap \{ x|f^n(x) − x = 0 \}.

Let p be an arbitrary point in \{ x|f^n(x) − x \neq 0 \} \cap \{ x|f^n(x) − x = 0 \}. We write \( g(x) = f^n(x) − x \). Then for x ∈ \{ x|f^n(x) − x \neq 0 \},

\[
F_{2n}(x) = \frac{g(x + g(x)) − g(x)}{g(x)} + 2.
\]
Since \( g(x) \neq 0 \) and \( g \) is \( C^1 \) on \( \mathbb{R} \), by the mean value theorem, there is a \( c_x \in (x, x + g(x)) \) or \( c_x \in (x + g(x), x) \) such that
\[
\frac{g(x + g(x)) - g(x)}{g(x)} = g'(c_x).
\]

When \( x \to p \), we have \( c_x \to p \) for \( g(p) = 0 \). Since \( g'(x) = (f^n)'(x) - 1 \) and it is continuous,
\[
\lim_{f^n(x) \to x} F_{2n}(x) = \lim_{f^n(x) \to x} \left\{ \frac{g(x + g(x)) - g(x)}{g(x)} + 2 \right\} = g'(p) + 2 = (f^n)'(p) + 1 = F_{2n}(p).
\]

In what follows, we fix a continuous full family of \( C^1 \) unimodal maps \( \{f_t\}_{t \in [0,1]} \). Similarly as in proposition 1, we define its quotient map as follows.

**Definition 4.** For any \( m \in \mathbb{N} \), we define a map \( G_m : [0,1] \times \mathbb{R} \to \mathbb{R} \) as follows. For odd \( m = 2n + 1 \), define \( G_{2n+1}(t,x) = f_{2n+1}^m(t,x) - x \). For even \( m = 2n \),
\[
G_{2n}(t,x) = \begin{cases} 
\frac{f_{2n}^m(x) - x}{f_p^m(x) - x} & \text{on } \{x|f_p^m(x) - x \neq 0\} \\
(f_p^m)'(x) + 1 & \text{on } \{x|f_p^m(x) - x = 0\}
\end{cases}
\]

We call \( G_m \) the quotient map of \( \{f_t\}_{t \in [0,1]} \) (although we do not divide \( f_t^m(x) - x \) for odd \( m \)).

Using this quotient map, we can define irreducible component of the bifurcation diagram as follows.

**Definition 5.**

(a) Let \( m \) be a positive integer. We call each connected component \( V \) of \( G_m^{-1}(0) \) a periodic point component of \( \{f_t\}_{t \in [0,1]} \), and the number \( \max \{ \text{per}(x)(t, x) \in V \} \) the period of \( V \).

(b) When \( p \) is a periodic point of \( f_t \) of minimal period \( m \), we sometimes write that \( (t, p) \) is a periodic point of minimal period \( m \).

**Remark 4.**

(a) It is clear that \( G_m^{-1}(0) \) is a subset of the bifurcation diagram of \( \{f_t\} \) of period \( m \).

(b) By the definition of \( G_m \), \( G_m^{-1}(0) \) contains all periodic points of the minimal period \( m \), and possibly certain periodic points of lower period, but the period must be a divisor of \( m \).

(c) If \( m \) is odd, for any divisor \( k \) of \( m \) (including 1), components of period \( k \) are contained in \( G_m^{-1}(0) \). When \( m \) is even, \( G_m^{-1}(0) \) can contain a periodic point \( (t, x) \) of period \( m/2 \) (and its divisors) only when \( (f_t^{m/2})'(x) = -1 \).

(d) Note that \( G_m^{-1}(0) \subset [0,1] \times [-M, M] \), by definition 3 (c). Therefore, \( G_m^{-1}(0) \) is a bounded set. By proposition 1, \( G_m \) is continuous as a function of \( x \). We will show that \( G_m \) is continuous on \( [0,1] \times \mathbb{R} \) in proposition 2. Therefore, in \( [0,1] \times \mathbb{R} \), \( G_m^{-1}(0) \) and each periodic point component are compact.

(e) ‘Periodic point component’ in definition 5 (a) is exactly what we called irreducible component. We did not define the term ‘irreducible component’ rigorously, but used it in the
similar meaning in algebraic geometry. The reason why the periodic point component is suitable for being called irreducible is the following.

Let $V$ be a periodic point component of period $m$. We will see in proposition 11 that if the period of a point on $V$, say $k$, is smaller than $m$ then $k = m/2$. When $m$ is odd, the period of any point on $V$ is $m$, and $V$ does not intersect with components of lower periods.

When $m$ is even, if the period of a component is not $m/2$, then it cannot intersect with $V$. And also the branches of period $m/2$ are almost removed by the division (except for points $x$ such that $(f^n_t)'(x) = -1$). Thus, the periodic point component may be called irreducible.

But there is still a question. Is there a possibility that period $m$ components of ‘different types’ intersect? That is exactly the main theme of this paper. We show that such intersections do not occur.

(f) There is a possibility of existence of many small periodic point components. If the shape of a part of the graph of $f^n_t$ changes as in figure 3 when the parameter $t$ increases, then a small closed component appears as in figure 4.

Suppose that the graph of $f^n_t$ has a wave intersecting to the line $y = x$ and converging to a point as in figure 5(a). When $t$ increases, if the shape of this wave changes in a certain way, then there is a possibility of existence of converging small component to some component $V$ as in figure 5(b). Note that the shapes of those small components may be a more deformed one, and there may be more and more such small components in many places.

Our definition of unimodal map is so general that it allows an interval of periodic points. In such a case, we cannot exclude non-pathwise connected periodic point component as shown in figure 6. Thus, the bifurcation diagram can be a very wild one. That is why we need a quite complicated argument of general topology to prove our main theorem.

The continuity of $G_m$ is essential.

**Proposition 2.** $G_m$ is continuous.

**Proof.** It is clear that $G_{2n+1}$ is continuous. We prove that $G_{2n}$ is continuous. We write $W = \{(t, x) | f^n_t(x) - x = 0\} \subset [0, 1] \times \mathbb{R}$. It is clear that on $W' = \{(t, x) | f^n_t(x) - x \neq 0\}$, $G_{2n}$ is continuous by the definition of $G_{2n}$. Also on the interior of $W$, $G_{2n}|_W$ is continuous, because $G_{2n}(t, x) = (f^n_t)'(x) + 1$ on $W$ and $f_t$ is $C^1$. We shall prove that $G_{2n}$ is continuous at $(s, p) \in W \cap \overline{W'}$. We have to show that when $(t, x) \to (s, p)$ for $(t, x) \in W'$, $G_{2n}(t, x) \to G_{2n}(s, p) = (f^n_s)'(p) + 1$.

Let $h_t(x) = f^n_t(x) - x$. Then
Figure 4. A closed component.

Figure 5. Converging infinite waves and converging infinite components.

\[ G_{2n}(t, x) = \frac{f^n_t(x) - x}{f^n(x) - x} \]
\[ = \frac{f^n_t(x - x + x) - f^n_t(x) + f^n_t(x) - x}{h_t(x)} \]
\[ = \frac{f^n_t(x + h_t(x)) - f^n_t(x)}{h_t(x)} + 1 \]

for \((t, x) \in W^c\). By the mean value theorem, there exists a \(c(t, x) \in (x, x + h_t(x))\) or \(c(t, x) \in (x + h_t(x), x)\) such that

\[ \frac{f^n_t(x + h_t(x)) - f^n_t(x)}{h_t(x)} = (f^n_t)'(c(t, x)) \]

for \(f^n_t\) is \(C^1\). When \((t, x) \to (s, p)\), \((f^n_t)'(c(t, x)) \to (f^n_s)'(p)\), because \(h_t(p) = 0\) and \(g^t(t, x) := (f^n_t)'(x)\) is continuous by definition 2. That means \(G_{2n}(t, x) \to G_{2n}(s, p) = (f^n_s)'(p) + 1\). ∎
The following are important properties of points of the half period of periodic point component.

**Proposition 3.** Let $V$ be a periodic point component of period $2n$, and $V_n = \{(t,x) \in V | f^n_t(x) - x = 0\}$. Then,

(a) $(f^n_t)'(x) = -1$ for any $(t,x) \in V_n$.
(b) For any $t \in [0,1]$, $V_n \cap \{(t) \times \mathbb{R}\}$ is a finite set.

**Proof.** (a) Follows from the definition of $G_{2n}$.

By proposition 2, $V$ is compact. Since $V_n$ is a closed subset of $V$, $V_n$ is also compact. We fix a $t \in [0,1]$. If $V_n \cap \{(t) \times \mathbb{R}\}$ has an infinite number of points, then there is an accumulate point $(t,p) \in V_n$. However from (a), $(f^n_t)'(p) = -1$ and therefore $p$ is isolated in the set $\{x \in \mathbb{R} | f^n_t(x) - x = 0\}$. That is a contradiction. □

3. **Symbolic condition and the statement of theorem 1**

In order to formulate our results rigorously, we need to give the symbolic condition for periodic points to be contained in the same periodic point component. First, let us recall some definitions and results on the kneading theory. There are two languages for kneading theory. One is Milnor–Thurston’s invariant coordinate [MT], and another is Collet–Eckmann’s RL-method [CE] which is originated in [MSS]. They are essentially equivalent. In this paper, we employ Collet–Eckmann’s method because it is easier to imagine.

Let $f$ be a unimodal map and $x \in \mathbb{R}$. The *itinerary* $I(x,f) = A_0 A_1 A_2 \ldots$ of $x$ for $f$ is the sequence of symbols $L$, $R$ and $C$ such that,

$$A_i = \begin{cases} 
L & \text{if } f^i(x) < 0 \\
R & \text{if } f^i(x) > 0 \\
C & \text{if } f^i(x) = 0
\end{cases}$$

When the symbol $C$ appears, the subsequent sequence is just the itinerary of 0. We shall omit the sequence after $C$. An infinite sequence of $R$ and $L$, or a finite sequence of $R$ and $L$ followed by $C$ is called an *admissible sequence*. Thus, an itinerary is an admissible sequence. In particular, the itinerary of the critical value $I(f(0),f)$ is called the *kneading sequence* of $f$ and denoted...
by $K(f)$. As in [CE], we will write just $I(x)$ instead of $I(x, f)$, when the map acting on $x$ is clear in the context.

A finite sequence of symbols $R$ and $L$ is called even (resp. odd) if it has an even (resp. odd) number of $R$’s. We define a natural ordering among admissible sequences. Let $A = A_0A_1A_2...$ and $B = B_0B_1B_2...$ be admissible sequences. We say $A < B$ if either $A_0...A_{n-1} = B_0...B_{n-1}$ is even and $A_n < B_n$, or it is odd and $A_n > B_n$, where we define $L < C < R$. Let $S(A_0A_1A_2...) = A_1A_2...$ be the shift map for admissible sequences. $S(C)$ is not defined. It is clear that for any $x \in \mathbb{R}$, $S(I(x)) = I(f(x))$.

A sequence $A$ is called maximal if $S^n(A) \leq A$ for all $n \geq 0$. The order of itineraries as admissible sequences is closely related to the order of corresponding points.

**Proposition 4 [CE].** Let $f$ be a unimodal map.

(a) If $I(x) < I(x')$, then $x < x'$.
(b) If $x < x'$, then $I(x) \leq I(x')$.

By this proposition, we see that if $I(x)$ is maximal then $x \geq f^i(x)$ for any $i \in \mathbb{N}$, namely, $x$ is the biggest among the points of its orbit.

Let $f : [a, b] \to \mathbb{R}$ be a horseshoe as in definition 1, i.e. $a$ is a fixed point, $f(a) = f(b) = a$ and there are $a_1$ and $b_1$ such that $a < a_1 < b_1 < b$, $f(a_1) = f(b_1) = b$ and $|f'(x)| > 1$ for any $x \in [a, a_1] \cup [b_1, b]$. Then, it is a basic fact that $I : A(f) \to \Pi$ gives a topological conjugacy between $f : A(f) \to A(f)$ and $S : \Pi \to \Pi$, where $A(f) = \bigcap_{n \geq 0} f^{-n}([a, b])$ and $S : \Pi \to \Pi$ is the one-sided full shift of symbols $L$ and $R$. In our case, $f_1$ is a horseshoe. Therefore, there is a one-to-one correspondence between periodic points of $f_1$ and periodic sequences of $S : \Pi \to \Pi$.

Now we define a map $\mu$ from the set of all periodic admissible sequences to itself. It plays a key role in our argument.

**Definition.** Let $A = (A_1A_2...A_n)^\infty$ be a periodic admissible sequence of minimal period $n$. By some shifts, $S^m(A) = (A_{m+1}A_{m+2}...A_{m+n})^\infty$ ($0 \leq m \leq n - 1$) is a maximal sequence. We define

$$\mu(A) = S^{n-m}(A_{m+1}A_{m+2}...A_{m+n-1}A_{m+n})^\infty$$

where $R = R$ and $L = L$, and we assume that $A_{n+i} = A_i$ for any $i$.

Note that in the definition of $\mu$, minimal period of $A$ is important. Since there is a unique $m$ ($0 \leq m \leq n - 1$) such that $(A_{m+1}A_{m+2}...A_{m+n})^\infty$ is maximal, this definition is well-defined. It is clear that;

**Proposition 5.** Let $A$ be a periodic admissible sequence. For any $k \in \mathbb{N}$, $\mu(S^k(A)) = S^k(\mu(A))$.

If $A = (A_1...A_n)^\infty$ is maximal, then $\mu(A) = (A_1...A_{n-1}A_n)^\infty$. As mentioned above, if $I(x) = A = (A_1...A_n)^\infty$ for a periodic point $x$ of $f$ is maximal, then $x$ is the biggest among its orbit. The symbol $A_n$ corresponds to the previous point of the orbit of $x$ i.e. $f^{n-1}(x)$. It is located near the critical point $0$ and it moves either from left to right or from right to left passing through $0$ after either saddle-node bifurcation or period doubling bifurcation. That is the meaning of the map $\mu$. If we identify those periodic sequences with periodic points of a unimodal map having the sequences as their itineraries, then $\mu$ maps the periodic point to the twin which was born at the same time through a saddle-node bifurcation, in the case where $\text{per}(\mu(A)) = \text{per}(A)$. In some cases, $\text{per}(\mu(A))$ can be smaller than $\text{per}(A)$. It is clear that it must
be a divisor of $n$, but actually it must be exactly $n/2$ when it is really smaller than $n$. The following is the proof, but we give a more general statement for later use.

**Proposition 6.** Let $A = (A_1 \ldots A_n)^\infty$ be a periodic admissible sequence of period $n$ (not necessarily of minimal period $n$).

(a) If $A = (A_1 \ldots A_n)^\infty$ is maximal, then the minimal period of $\hat{A} = (A_1 \ldots A_{n-1}A^n_n)^\infty$ is either $n$ or $n/2$.

(b) If $A = (A_1 \ldots A_n)^\infty$ is maximal, then either per($A$) or per($\hat{A}$) is $n$.

(c) If per($A$) = $n$, then per($\mu(A)$) is either $n$ or $n/2$.

**Proof.**

(a) Suppose that the minimal period of $\hat{A} = (A_1 \ldots A_{n-1}A^n_n)^\infty$ is smaller than $n/2$. Then it is written as $A_1 \ldots A_{n-1}\overline{A}_n = B \ldots B$, where $B$ is a finite sequence of $R$ and $L$. We denote the length of $B$ and the number of $B$’s by $m$ and $k$ respectively, then $k \geq 3$ and $km = n$. Let $B = B_1 \ldots B_m$ and $B = B_1 \ldots B_{m-1}\overline{B}_m$. Then $A_1 \ldots A_n = B \ldots BB$, where the number of $B$’s is $k - 1$. Since $A$ is maximal and $k - 1 \geq 2$, by shifting $A$, we have $BB > BB$ and $B > B$. The second inequality means that $B$ is odd. However that contradicts the first inequality.

(b) Suppose that per($\hat{A}$) ≠ $n$. Then by (a), per($\hat{A}$) = $n/2$, and therefore we can write $A_1 \ldots A_{n-1}\overline{A}_n = BB$ for some finite sequence $B$ of $R$ and $L$. Note that the $n$ is even in this case.

Assume that per($A$) = $k ≠ n$. Firstly, we claim that $n/k$ is odd. Because if $n/k$ is even, we can write $A = D \ldots D$ where $D$ is a sequence of $R$ and $L$ of length $k$, and the number of $D$’s is even. However that contradicts $A = BB$. Now suppose that $n/k$ is odd. Note that $n/k \geq 3$ because $k ≠ n$. Since the number of $D$’s is odd and $n$ is even, the length of $D$ is even. Therefore, we can write $D = D' D''$ where the lengths of $D'$ and $D''$ are the same. Since $A_1 \ldots A_{n-1}\overline{A}_n = BB$, the first $B$ must start with $D'$, and the second $B$ must start with $D''$. Therefore $D' = D'$ and $D = D' D'$. Then $B$ is a sequence of odd number of $D'$’s, and $A = D \ldots D$ is a double of $B$. That means $A = BB = \hat{A}$ and leads to a contradiction.

(c) Let per($A$) = $n$. By some shift, we can assume that $A$ is maximal. Then by (a), per($\mu(A)$) = $n/2$ if per($\mu(A)$) < $n$. \hfill \Box

By the previous proposition, we have two situations when we apply $\mu$ on a periodic sequence. One is that we get a sequence whose minimal period is the same as the original one. In section 4, we will see that $\mu(\mu(A)) = A$ in this case. Another case is that the minimal period of $\mu(A)$ is exactly the half. In this case, $\mu(A)$ indicates the itinerary of the point which from the periodic point corresponding to $A$ was born through a period-doubling bifurcation. What we want to get by applying $\mu$ is the itinerary of the point which is on the same periodic point component of the bifurcation diagram. In case of period-doubling bifurcation, that is a point which is in the same orbit and obtained from the first one by applying the map the half times of the period. Thus, in this case, we need another treatment of the sequence.

**Definition 7.** We define a map $\nu$ from the set of all periodic admissible sequences to itself as follows. Let $A = (A_1 A_2 \ldots A_n)^\infty$ be a periodic admissible sequence of minimal period $n$. When per($\mu(A)$) = $n$, define $\nu(A) = \mu(A)$. When per($\mu(A)$) is smaller than $n$ (and is $n/2$), define $\nu(A) = S^{n/2}(A)$.
For the standard family of quadratic maps, by the monotonicity of kneading sequences [MT] and the non-degeneracy of bifurcation [DH], we can see that expanding periodic points \( p \) and \( q \) of minimal period \( n \) are on the same periodic point component if and only if \( \pi(p) = \pi(q) \).

Our main theorem in this paper asserts that the same is true for any continuous full family of \( C^1 \) unimodal maps.

**Theorem 1.** Let \((1, p), (1, q) \in G^{-1}_n(0) \cap \{ t = 1 \} \) \((p \neq q)\) and \( \text{per}(p) = \text{per}(q) = n \). Then, \((1, p), (1, q)\) are on the same periodic point component if and only if \( \pi(p, q) = \pi(q, p) \).

**Example 1.** Let us consider the standard family of quadratic maps \( q_t(x) = t - x^2 \) and its bifurcation diagram of period 6. Refer figure 2. Figure 7 is the \([1.4, 2.1] \times [-2, 2]\) part of figure 2. The component of fixed points and the component of period 2 are removed.

When \( t \) is approximately 1.47, a saddle-node bifurcation occurs and the first period orbit of period 6 appears. That produces the components \( C_1, \ldots, C_6 \). Let \( q(t, x) = (t, q_t(x)) \). Then \( \pi(C_i) = C_{i+1} \) for \( 1 \leq i \leq 5 \) and \( \pi(C_6) = C_1 \). The itineraries of two points at the right end of \( C_i \) are \((RLRRRR)^\infty\) and \((RLRRRL)^\infty\). \( C_1 \) is too thin and we cannot see the shape very well. But the shape is basically the same as that of other \( C_i \)'s.) It is natural that those two itineraries are maximal, because any point on \( C_1 \) has the biggest \( x \)-value among its orbit.

It is known that for any \( t > 2 \), any periodic point of \( q_t \) is expanding and its itinerary does not change when \( t \) increases. Therefore, we can discuss the itineraries of the periodic points of \( q_{2.1} \) on the right end of figure 7 although it is not a horseshoe yet.

Let \( x_1 > x_2 \) be two right end points on \( C_1 \). Then \( I(x_1) = (RLRRRR)^\infty \) and \( \mu(I(x_1)) = (RLRRRR)^\infty = (RLRRRL)^\infty = I(x_2) \). Since the period of \( \mu(I(x_2)) \) is also 6, by the definition 7, we have \( \nu(I(x_1)) = \mu(I(x_1)) = I(x_2) \). Also for any other component \( C_i \), the situation is the same. Since \( q'(C_i) = C_{i+1} \) for \( 1 \leq i \leq 5 \), the itineraries of the right end two points of \( C_i \) are \( I(x) \) and \( \mu(I(x)) \) each other.

The components \( D_1, D_2, D_3 \) appear when \( t \) is approximately 1.75. They are components of the periodic orbit of period 3. Note that \( q(D_1) = D_2, q(D_2) = D_3 \) and \( q(D_3) = D_1 \). The itineraries of two right end points of \( D_1 \) are \((RLL)^\infty\) and \((RLR)^\infty\), and they are maximal. (Also \( D_1 \) is too thin, but the shape is basically the same as \( D_3 \) and \( D_2 \).) Note that the period 3 part of \( D_i \) \((i = 1, 2, 3)\) is removed by the division. Therefore, it is not contained in \( G^{-1}_n(0) \).

A period-doubling bifurcation occurs when \( t \) is approximately 1.77. On \( D_1 \), it occurs on \((RLL)^\infty\) side, and as a result, a period 6 branch is born. Its itinerary is \((RLRRLL)^\infty\) and the period is 6. The periodic point corresponding to the three-times shift \( S^3((RLRRLL)^\infty) = (RLRRLL)^\infty \) is on the same period 6 component. \((RLRRLL)^\infty\) is maximal. If we apply \( \mu \) to \((RLRRLL)^\infty\), the period of \( \mu((RLRRLL)^\infty) = (RLRRLL)^\infty = (RLL)^\infty \) becomes 3. In this case, by the definition 7, \( \nu((RLRRLL)^\infty) = S^3((RLRRLL)^\infty) = (RLRRLL)^\infty \) and the corresponding point is on the same period 6 branch. On \( D_2, D_3 \), the situation is similar because they are the mapped images of \( D_1 \).

In the following sections, we prove some preliminary results. The proof of theorem 1 will be given in section 7.

4. Basic properties of the map \( \mu \)

The proof of the following proposition is straightforward. Also this proposition is essentially the same as lemma 3 of [JR2]. Although some translation work is necessary, we can see that the definition of ‘admissible’ in [JR2] is the same as our ‘maximal’.

**Proposition 7.** Let \( A = (A_1 \ldots A_n)^\infty \) be a maximal sequence of minimal period \( n \). Then \( \hat{A} = (A_1 \ldots A_{n-1} \hat{A}_n)^\infty \) is also a maximal sequence.
Figure 7. A magnified bifurcation diagram.

Note that if $\text{per}(A) < n$, then this proposition does not hold. For example, $A = (RLLRLLRLL)^\infty$ is maximal. However $\tilde{A} = (RLLRLLRL)^\infty$ is not maximal.

From this proposition, it follows easily that;

**Proposition 8.** Let $A$ be a periodic sequence of minimal period $n$. If $\text{per}(\mu(A)) = n$, then $\mu(\mu(A)) = A$.

The following is one of the important properties of the map $\mu$.

**Proposition 9.** Let $A$ and $B$ be periodic sequences of minimal period $n$. If $\mu(A) = \mu(B)$, then either $A = B$ or, $n$ is even and $S^{n/2}(A) = B$.

**Proof.** If $\text{per}(\mu(A)) = \text{per}(\mu(B)) = n$, then by proposition 8, $A = \mu(\mu(A)) = \mu(\mu(B)) = B$.

So, we assume that $\text{per}(\mu(A)) = \text{per}(\mu(B)) = n/2$.

Let $A = (A_1 \ldots A_n)^\infty$ and $B = (B_1 \ldots B_n)^\infty$. Let $k$ and $h$ be integers such that $1 \leq k \leq n$, $1 \leq h \leq n$ and

$$\mu(A) = (A_1 \ldots \bar{A}_k \ldots A_n)^\infty$$

$$\mu(B) = (B_1 \ldots \bar{B}_h \ldots B_n)^\infty.$$ 

If $k = h$, then $A = B$ for $\mu(A) = \mu(B)$. So, without loss of generality, we assume that $k > h$.

By the definition of $\mu$ and proposition 7, $S^k(\mu(A)) = (A_{k+1} \ldots A_nA_1 \ldots \bar{A}_k)^\infty$ and $S^h(\mu(B)) = (B_{h+1} \ldots B_nB_1 \ldots \bar{B}_h)^\infty$ are maximal. Since the maximal sequence of periodic sequence is
unique and $\mu(A) = \mu(B)$, we have $S^k(\mu(A)) = S^h(\mu(B))$. Since $\mu(A) = \mu(B)$ again, we have,

$$S^k(\mu(A)) = S^h(\mu(B)) = S^h(\mu(A)).$$

Therefore, $S^{k-h}(\mu(A)) = \mu(A)$. Then $k - h = n/2$, because $1 \leq k - h < n$ and $\text{per}(\mu(A)) = n/2$. Also $(\overline{A}_k + \ldots + \overline{A}_2 \overline{A}_1 \ldots \overline{A}_h)^\infty = (B_{h+1} \ldots B_h B_1 \ldots B_n)^\infty$ means that $S^h(A) = S^h(B)$. So we get

$$S^h(B) = S^h(A) = S^{n/2 + h}(A)$$

and this means $S^{n/2}(A) = B$. \qed

5. Periodic point components

In the definition of periodic point components, we removed components corresponding to the period $n/2$ from $G_{n-1}(0)$ when $n$ is an even number. Note that for a divisor $k$ of $n$ which is not a divisor of $n/2$, $G_{n-1}(0)$ is not removed from $G_{n-1}(0)$. However, we can show that such components do not have any intersection with components of minimal period $n$. In fact, we prove the following proposition which asserts that for any continuous family of $C^1$ maps on $\mathbb{R}$ (not only of unimodal maps), on the bifurcation diagrams, minimal period changes only to the half when it decreases.

**Proposition 10.** Let $g_n (n = 1, 2, \ldots)$ be a sequence of $C^1$ maps converging to a $C^1$ map $g$ in the $C^1$ topology. Let $\mathcal{O}_n = \{q_1(n) < q_2(n) < \cdots < q_m(n)\}$ be a periodic orbit of $g_n$ of minimal period $m$ such that $q_i(n)$ converges to a periodic point $q_i$ of $g$ as $n \to \infty$ for each $i$. We write $\mathcal{O} = \{q_1 \leq q_2 \leq \cdots \leq q_m\}$. If the minimal period of $\mathcal{O}$, say $k$, is smaller than $m$, then $k = m/2$.

**Proof.** First of all, we claim that $\mathcal{O}$ does not contain any critical point. If one of those points is a critical point, then $\mathcal{O}$ is a hyperbolic sink of minimal period $k < m$. Since it is stable under small $C^1$ perturbation, for sufficiently large $n$, $g_n$ turns out to have a sink of the same period near $\mathcal{O}$, and its basin is very close to that of $\mathcal{O}$. That means $g_n$ cannot have a periodic point of period $m$ near $\mathcal{O}$. However that is a contradiction.

We write $\mathcal{O} = \{q_1 \leq \cdots \leq q_m\} = \{p_1 < \cdots < p_k\}$. If $n$ is sufficiently large, there are numbers $1 = i_1 < i_2 < \cdots < i_k < i_{k+1} = m + 1$ such that if $i_j \leq h < i_{j+1}$ then $q_{i_j}(n)$ is very close to $p_j$ for all $1 \leq j \leq k$.

We write $I_j(n) = [q_{i_j}(n), q_{i_{j+1}}(n)]$ for all $1 \leq j \leq k$. Since there is no critical point in $\mathcal{O}$, for sufficiently large $n$, $g_n$ is a local homeomorphism near $\mathcal{O}$, in particular, $g_n$ must map each interval $I_j(n)$ onto one of others homeomorphically. That means the set of all the boundary points of $I_j(n)$’s is invariant under $g_n$. Since $\mathcal{O}_n$ is a periodic orbit of minimal period $m$, $\mathcal{O}_n$ must consist of only the boundary points. Therefore, we have $k = m/2$. \qed

**Proposition 11.** Let $V$ be a periodic point component. If the period of $V$ is $n$ and $k = \min \{\text{per}(t, x) | (t, x) \in V\}$ is smaller than $n$, then $k = n/2$.

**Proof.** Suppose that $k \neq n/2$. Since $k$ must be a divisor of $n$, we have $k < n/2$. We write $V' = \{(s, x) \in V | \text{per}(x) = n \text{ or } n/2\}$ and $V'' = V - V'$. $V''$ is a closed subset of $V$, because $\text{per}(x) < n/2$ for any $(s, x) \in V''$.

It is clear that there exists a point $(s_0, q_0) \in V''$ such that $(s_0, q_0) \in \overline{V'}$, otherwise $V''$ is a closed and open subset of $V$, and $V$ turns out to be non-connected. Therefore, there is a sequence of points $(s_i, p_i) \in V' (i = 1, 2, \ldots)$ such that $(s_i, p_i) \to (s_0, q_0)$ as $i \to \infty$. If $\text{per}(p_i) = n$ for
ininitely many \( i \)'s, then by proposition 10, \( \text{per}(q_0) \) must be \( n/2 \), and that contradicts the definition of \( V^n \). So, we can assume that \( \text{per}(p_i) = n/2 \) for any \( i \geq 1 \). Then by proposition 10 again, we get \( \text{per}(q_0) = n/4 \). Since \( f^{n/4}_0(q_0) = q_0 \), we have \( (f^{n/4}_0)'(q_0) = ((f^{n/4}_0)'(q_0))^2 > 0 \). However that contradicts proposition 3 (a).

One of the keys to the proof of our main theorem is proposition 13 which claims that every periodic point component has two and more intersection points with \( \{ t = 1 \} \) line. (In fact, we will see later that the number of intersection points is exactly two.) The essence of the proof of proposition 13 is the following proposition 12 whose proof needs a quite involved argument of general topology.

**Proposition 12.** Let \( M > 0 \) be a positive number, \( D = [0, 1] \times [-M, M] \) a rectangle and \( G : D \to \mathbb{R} \) a continuous map such that \( G^{-1}(0) \cap \partial D \) is a finite set on \( \{ 1 \} \times (-M, M) \).

If there is a connected component \( F_C \) of \( G^{-1}(0) \) which intersects exactly once to \( \{ 1 \} \times (-M, M) \) at a point \( p_0 = (1, y_0) \), then there is a path in \( D - G^{-1}(0) \) connecting points \( y_- \) and \( y_+ \) in \( \{ 1 \} \times (-M, M) \) which are arbitrary close to \( y_0 \) with \( y_- < y_0 < y_+ \).

This proposition might be seemingly obvious. But actually, we need a quite complicated argument for the proof, because there is a possibility of the existence of infinitely many components accumulating to other components as mentioned in remark 4(f). Such accumulation can occur in many places and also the shapes of such components can be a quite deformed one. For those reasons, the existence of a continuous curve as stated in proposition 12 is not obvious.

We will give the proof in section 8. In our situation, \( G^{-1}(0) \cap \partial D \) is a finite set on \( \{ 1 \} \times (-M, M) \) for any \( n \), because \( f_1 \) is a horseshoe and \( f \) does not have any periodic point on \( \partial D \) except them by hypothesis.

**Proposition 13.** Let \( V \) be a periodic point component of \( G^{-1}_n(0) \) of period \( n \). If \( V \cap \{ t = 1 \} \neq \emptyset \), then \( V \cap \{ t = 1 \} \) contains at least two points.

**Proof.** Suppose that \( V \cap \{ t = 1 \} = \{ (1, p) \} \) is only one point. Note that the minimal period of \( p \) is \( n \), because by proposition 11, \( \text{per}(p) \) is \( n \) or \( n/2 \), and if \( \text{per}(p) = n/2 \), then by proposition 3(a), \( \left( f^{n/2}_1 \right)'(p) = -1 \). However \( f_1 \) is horseshoe and periodic points must be hyperbolic. Therefore \( \text{per}(p) = n \).

Let \( J \) be an open interval in \( \{ t = 1 \} \) such that \( p \in J \) and \( J \cap G_n^{-1}(0) = \{ p \} \). Then by proposition 12, there are two points \( p_0 < p < p_1 \) in \( J \) and a continuous curve \( \gamma : [0, 1] \to [0, 1] \times [-M, M] \) such that \( \gamma(0) = p_0, \gamma(1) = p_1 \) and \( \gamma([0, 1]) \cap G^{-1}(0) = \emptyset \).

When \( n \) is odd, \( G_n(t, x) = f_1^n(x) - x \) and

\[
\frac{dG_n}{dx}(1, p) = (f_1^n)'(p) - 1.
\]

When \( n \) is even, note that \( f_1^{n/2}(p) - p \neq 0 \) because the minimal period of \( p \) is \( n \). By an easy calculation, we have

\[
\frac{dG_n}{dx}(1, p) = \frac{(f_1^n)'(p) - 1}{f_1^{n/2}(p) - p}.
\]

Since \( p \) is a hyperbolic periodic point of \( f_1 \), \( (f_1^n)'(p) \neq 1 \). Therefore, in both cases, \( \frac{dG_n}{dx}(1, p) \neq 0 \), and the signs of \( G_n(1, p_0) \) and \( G_n(1, p_1) \) are different. However, since \( \gamma \) does not have any intersection with \( G_n^{-1}(0) \), \( G_n(\gamma(s)) \) is non-zero for any \( 0 \leq s \leq 1 \). That is a contradiction.
6. Shuffle of periodic point

For a positive integer \( n \), we denote the set of all permutations of \( \{0, 1, \ldots, n-1\} \) by \( S_n \).

**Definition 8.** Let \((i_1, i_2, \ldots, i_n) \in S_n \) and \( f : \mathbb{R} \to \mathbb{R} \) be a map. For a periodic point \( p \) of \( f \) with minimal period \( n \), when

\[
 f^{i_1}(p) < f^{i_2}(p) < \cdots < f^{i_n}(p),
\]

we denote \( \sigma(p) = (i_1, i_2, \ldots, i_n) \) and call it the shuffle of \( p \).

**Proposition 14.** Let \( f \) and \( g \) be unimodal maps. Let \( p \) and \( q \) be periodic points of \( f \) and \( g \) respectively such that

(a) \( \text{per}(p) = \text{per}(q) = n \),

(b) the orbits of \( p \) and \( q \) do not contain the critical point,

(c) \( \sigma(p) = \sigma(q) \).

Then at least one of the following holds:

(a) \( I(p) = I(q) \),

(b) \( \text{per}(I(q)) = n \) and \( I(p) = \mu(I(q)) \),

(c) \( \text{per}(I(p)) = n \) and \( I(q) = \mu(I(p)) \).

**Proof.** Assume that \( \sigma(p) = \sigma(q) = (i_1, \ldots, i_n) \). Let \( x \) stands for \( p \) or \( q \), and \( h \) stands for \( f \) or \( g \) corresponding to \( x \).

If \( n = 1 \), then \( I(x) = R^\infty \) or \( L^\infty \) because neither \( p \) nor \( q \) is the critical point. So, the statement clearly holds. We assume \( n \geq 2 \).

By the definition, \( h^u(x) \) is the biggest among \( h^i(x) \)'s. We write \( i_k = i_n - 1 \). When \( i_n = 0 \), we define \( i_k = i_n - 1 \). Then \( h(h^u(x)) = h^u(x) \).

Since \( i_k \neq i_n \), \( h^{k+1}(x) \) exists. Therefore, by the definition of shuffle,

\[
 h^{k+1}(x) < h^u(x) < h^{k+1}(x)
\]

if \( h^{k+1}(x) \) exists.

We have \( h^{k+1}(x) > 0 \), because if \( h^{k+1}(x) < 0 \) then \( h^k(x) < h^{k+1}(x) < 0 \), and since \( h \) is orientation preserving on \( \{x < 0\} \), that contradicts the fact that \( h^u(x) \) is the biggest among \( h^i(x) \)'s. If \( k \neq 1 \), \( h^{k+1}(x) \) exists and \( h^{k+1}(x) < h^u(x) \). We have \( h^{k+1}(x) < 0 \), because if \( h^{k+1}(x) > 0 \), then \( 0 < h^{k+1}(x) < h^u(x) \), and since \( h \) is orientation reversing on \( \{x > 0\} \), that contradicts the fact that \( h^u(x) \) is the biggest among \( h^i(x) \)'s.

Those facts mean that only \( h^u(x) \) has the freedom of \( R \) or \( L \). Note that the orbits of \( p \) and \( q \) do not have the critical point, and symbol \( C \) does not appear.

If \( I(p) \neq I(q) \), then only the symbols corresponding to the points \( f^{i_n}(p) \) and \( g^{i_n}(q) \) are different. By proposition 4, \( I(f^{i_n}(p)) \) and \( I(g^{i_n}(q)) \) are maximal sequences. If \( I(f^{i_n}(p)) \neq I(g^{i_n}(q)) \), then by proposition 6 (b), either \( \text{per}(I(f^{i_n}(p))) \) or \( \text{per}(I(g^{i_n}(q))) \) is \( n \). Therefore we have that \( \text{per}(I(q)) = n \) and \( I(p) = \mu(I(q)) \), or \( \text{per}(I(p)) = n \) and \( I(q) = \mu(I(p)) \) by the definition of \( \mu \).

Note that \( \text{per}(I(p)) \) or \( \text{per}(I(q)) \) can be \( n/2 \). The property ‘same shuffle’ is an open property on components of constant period, namely;
Proposition 15.

(a) Let \((t, p)\) be a periodic point of minimal period \(n\). Then there exists a \(\delta > 0\) such that if \((s, q)\) is a periodic point of minimal period \(n\) and \((s, q) \in B(\delta, (t, p))\) then the shuffles of \((t, p)\) and \((s, q)\) are the same, where \(B(\delta, (t, p))\) is the \(\delta\)-disk in \(\mathbb{R}^2\) centered at \((t, p)\).

(b) Let \(W\) be a connected subset of a periodic point component. If \(\text{per}(x)\) is the same for any \((t, x) \in W\), then the shuffle \(\sigma(x)\) is the same for any \((t, x) \in W\).

Proof.

(a) Let \(p_1 < \cdots < p_n\) be the orbit of \(p\) and \(B(\epsilon, p_i)\) the \(\epsilon\)-neighborhood of \(p_i\) in \(\mathbb{R}\). It is clear that there is an \(\epsilon > 0\) satisfying the following properties.

1. \(B(\epsilon, p_i)\)'s are disjoint.
2. For any \(B(\epsilon, p_i), f_i(B(\epsilon, p_i))\) has an intersection with only \(B(\epsilon, f_i(p_i))\).

Then, also it is clear that if we take a \(\delta > 0\) small enough, any periodic point \((s, q)\) of period \(n\) contained in \(B(\delta, (t, p))\) satisfies the following.

1. Let \(q_1 < \cdots < q_n\) be the orbit of \(q\). Then \(q_i \in B(\epsilon/2, p_i)\) for any \(i\).
2. \(f_i(q_i) \in B(\epsilon/2, f_i(p_i))\) for any \(i\).

That means that the shuffles of \((s, q)\) and \((t, p)\) are the same.

(b) Since there are only a finite number of shuffles for a fixed period, \(W\) can be divided into a finite number of disjoint subsets \(W_1, \ldots, W_k\) such that the shuffle is the same on each \(W_i\). (a) means that each \(W_i\) is an open set because the period is the same on \(W\). Therefore, \(W\) is a disjoint union of open sets \(W_i\)'s. For \(W\) is connected, we have \(k = 1\), namely, the shuffle is constant on \(W\).

The main point in the proof of our theorem is that if a periodic point component \(V\) of period \(n\) contains only points of period \(n\) and \(n/2\), then shuffles of points of period \(n\) in \(V\) must be either the same or \(n/2\)-shift more precisely.

Definition 9. Let \(\sigma = (i_1, \ldots, i_n) \in S_n\) and \(n\) be even. We define \(n/2\)-shift of \(\sigma\) by \(\gamma(\sigma) = ([i_1 + n/2], \ldots, [i_n + n/2])\), where \([k] = k \mod n\).

Note that if \(x\) is a periodic point of period \(n\) and \(n\) is even, then \(\gamma(\sigma(x)) = \sigma(f^{n/2}(x))\). Also, it is clear that \(\gamma(\gamma(\sigma)) = \sigma\). Therefore, we can define that \(\sigma\) is equivalent to \(\rho\) if either \(\sigma = \rho\) or \(\gamma(\sigma) = \rho\). We denote the equivalence class of \(\sigma\) by \([\sigma]\), and call it the shuffle class of \(\sigma\). The following proposition is essential in the proof of our theorem.

Proposition 16. Let \(n\) be even, \(V\) a periodic point component of period \(n\) and \(\min\{\text{per}(x)| (t, x) \in V\} = n/2\).

(a) For any \((t_1, p_1), (t_2, p_2) \in V\), if \(\text{per}(p_1) = \text{per}(p_2) = n\) then \([\sigma(p_1)] = [\sigma(p_2)]\).
(b) If the orbit of \(p\) for \((s, p) \in V\) does not contain the critical point and \(\text{per}(I(p)) = n\), then \(\text{per}(\mu(I(p))) = n/2\).

Proof. (a) We write \(V_{n/2} = \{(t, x) \in V|\text{per}(x) = n/2\}\). Note that \(V_{n/2} \subset \{t < 1\}\), because by proposition 3(a), \((f^{n/2})^t(x) = -1\) for \((t, x) \in V_{n/2}\), and for \(t = 1\), any periodic point must be hyperbolic. Since \(V_{n/2}\) is a bounded closed set, it is compact.

Let \(V_{n/2} = K_1 \cup \cdots \cup K_n\) be the decomposition of \(V_{n/2}\) into the sets of the same shuffle. Namely, the shuffles of points in each \(K_i\) are the same, and for \(K_i \neq K_j\), the shuffles of points
in \( K_i \) and \( K_j \) are different. Since there are only a finite number of shuffles of period \( n/2 \), \( V_{n/2} = \bigcup K_i \) is a finite decomposition. Each \( K_i \) is a compact set, because by proposition 15(a), each \( K_i \) is open in \( V_{n/2} \) and \( \bigcup \bigcap_i \neq K_i \) is open in \( V_{n/2} \). Therefore, each \( K_i \) is closed in \( V_{n/2} \) and since \( V_{n/2} \) is compact, \( K_i \) is compact.

On the other hand, since \( V_{n/2} \) is closed in \( V \), \( V - V_{n/2} \) is an open set in \( V \). Since the minimal period of any point in \( V - V_{n/2} \) is \( n \) and there are only a finite number of shuffles of length \( n \), \( V - V_{n/2} \) is divided into a finite number of disjoint subsets \( \{ E_j \} \) of the same shuffle class. By proposition 15(a), each \( E_j \) is an open set in \( V \). We claim the following lemma.

**Lemma 1.** For any \( K_i \), there exists a unique \( E_j \) such that \( K_i \cap \overline{E_j} \neq \emptyset \).

If this lemma is true, then for each \( K_i \) there is a unique \( E_j \) such that \( K_i \cap \overline{E_j} \neq \emptyset \). We define \( H_i = K_i \cup E_i \) for all \( 1 \leq i \leq v \). And let \( H_0 = E_{j_1} \cup \cdots \cup E_{j_v} \) be the union of \( E_{j_i} \)’s which do not have a \( K_i \) such that \( K_i \cap \overline{E_j} \neq \emptyset \). Then \( V = H_0 \cup H_1 \cup \cdots \cup H_v \). We claim the following lemma. We give the proofs of those lemmas at the end of this section.

**Lemma 2.** For any \( 0 \leq i \leq v \) and \( 0 \leq j \leq v \), if \( i \neq j \) then \( \overline{E_i} \cap H_j = \emptyset \).

Since \( V \) is connected, this lemma says that only one \( H_i \) exists and \( V = H_i \). Note that \( V \) is not \( H_0 \) because \( V_{n/2} \) is not empty. \( V = H_i = K_i \cup E_i \), which means \( [\sigma(p_1)] = [\sigma(p_2)] \) for any \((t_1, p_1), (t_2, p_2) \) \( V \) of \( \text{per}(p_1) = \text{per}(p_2) = n \), and this proves (a).

(b) As mentioned above, \( f_n^{(i/2)}(x) = −1 \) at any \((t, x) \in V_{n/2} \). Since \( V_{n/2} \) is compact, there is a neighborhood of \( U \) of \( V_{n/2} \) in \( \mathbb{R}^2 \) such that the orbit of any point in \( V \cap U \) does not contain the critical point. Since \( V \) is connected, there exists a \((t, x) \in (V - V_{n/2}) \cap U \) sufficiently close to \( V_{n/2} \) such that \( \text{per}(x) = n \) and \( \text{per}(I(x)) = n/2 \). Since \( \text{per}(I(p)) = n \), \( \text{per}(p) = n \). It follows from (a) that \([\sigma(p)] = [\sigma(x)]. \) Thus either \( \sigma(p) = \sigma(x) \) or \( \sigma(p) = \gamma(\sigma(x)). \) If \( \sigma(p) = \sigma(x) \), then by proposition 14, either \( I(p) = I(x) \), or \( \text{per}(I(x)) = n \) and \( I(p) = \mu(I(x)) \), or \( \text{per}(I(p)) = n \) and \( I(x) = \mu(I(p)) \). Since \( \text{per}(I(x)) = n/2 \) and \( \text{per}(I(p)) = n \), we have \( I(x) = \mu(I(p)) \). Therefore \( \text{per}(\mu(I(p))) = n/2 \) in this case. If \( \sigma(p) = \gamma(\sigma(x)) = \sigma(f_n^{(i/2)}(x)) \), then by proposition 14 again, either \( I(p) = I(f_n^{(i/2)}(x)) \), or \( \text{per}(I(f_n^{(i/2)}(x))) = n \) and \( I(p) = \mu(I(f_n^{(i/2)}(x))) \), or \( \text{per}(I(p)) = n \) and \( I(f_n^{(i/2)}(x)) = \mu(I(p)) \). Since \( \text{per}(I(f_n^{(i/2)}(x))) = n/2 \) and \( \text{per}(I(p)) = n \), we have \( I(f_n^{(i/2)}(x)) = \mu(I(p)) \). Therefore \( \text{per}(\mu(I(p))) = n/2 \) in this case too.

**Proof of Lemma 1.** First of all, we prove that for any \( K_i \), there exists an \( E_j \) such that \( K_i \cap \overline{E_j} \neq \emptyset \).

Suppose that there exists a \( K_i \) such that it does not have an intersection with any \( \overline{E_j} \). Let \( K = K_{i_1} \cup \cdots \cup K_{i_v} \) be the union of all such \( K_i \)’s, and \( K' = K_{i_1} \cup \cdots \cup K_{i_v} \) the union of all \( K_i \)’s which have an intersection with some \( E_j \). Then, \( (\bigcup \overline{E_j}) \cup K' \) and \( K \) are disjoint closed sets, and the union is \( V \). That contradicts the connectivity of \( V \).

Secondly, we show that such \( E_j \) is unique. If there is a \((t, x) \in K_i \cap \overline{E_j} \), then a sequence of periodic points of period \( n \) in \( E_j \) converges to \((t, x) \in K_i \). In such case, as mentioned in the proof of proposition 10, the way of convergence is joining the points of the orbit two by two from the smallest one. Since \( f_n^{(i/2)}(x) = −1 \) for \((t, x) \in V_{n/2} \), there is not the critical point in the orbit of \( x \). Therefore, the itinerary of periodic point of \( E_j \) converging to \( x \) is the same as \( I(x) \) if it is sufficiently close to \( x \). Moreover, \( f_n \) is a local homeomorphism on a neighborhood of each point of the orbit of \( x \), and whether \( f_n \) and nearby \( f_n \) are orientation-preserving or not on them is defined uniquely by \( I(x) \). That means that the shuffle class of \( E_j \) is uniquely defined by \( K_i \).

**Proof of Lemma 2.** Suppose that \( \overline{E_i} \cap H_j \neq \emptyset \). This means that there is a sequence of points \( x_t \) in \( H_i \) converging to a point \( p \in H_j \).
\textbf{7. Proof of theorem 1}

(a) First, we shall prove that if \( p, q \in G_n^{-1}(0) \cap \{ t = 1 \} \) (\( p \neq q \)) are on the same periodic point component and \( \text{per}(p) = \text{per}(q) = n \), then \( I(p) = \nu(I(q)) \).

Let \( V \) be the connected component of \( G_n^{-1}(0) \) containing \( p \) and \( q \). Write \( k = \min\{\text{per}(x)(t, x) \in V\} \). Then either \( k = n \) or \( k < n \).

\textbf{Case 1.} \( k = n \).

By proposition 15(b), the shuffle \( \sigma(x) \) is the same for any \( (t, x) \in V \). The orbits of \( p \) and \( q \) do not contain the critical point of \( f_1 \), because \( f_1 \) is a horseshoe. Therefore, by proposition 14, either \( I(p) = I(q) \) or \( I(p) = \mu(I(q)) \) or \( I(q) = \mu(I(p)) \). If \( I(p) = I(q) \), then \( p \) and \( q \) must be identical because \( f_1 \) is a horseshoe again. We have \( I(p) = \mu(I(q)) \) or \( I(q) = \mu(I(p)) \). Since \( f_1 \) is a horseshoe, \( \text{per}(x) \) and \( \text{per}(I(x)) \) must be identical for any periodic point \( x \). Therefore, by proposition 8, ‘\( I(p) = \mu(I(q)) \)’ and ‘\( I(q) = \mu(I(p)) \)’ are equivalent. By the definition of \( \nu \), we get \( I(p) = \nu(I(q)) \).

\textbf{Case 2.} \( k < n \).

In this case, by proposition 11, \( n \) is even and \( k = n/2 \). Then by proposition 16(a), \( \lfloor \sigma(p) \rfloor = \lfloor \sigma(q) \rfloor \).

If \( \sigma(p) = \sigma(q) \), then since \( f_1 \) is a horseshoe, there is no critical point in the orbits of \( p \) and \( q \). By proposition 14, either \( I(p) = I(q) \), or \( \text{per}(I(q)) = n \) and \( I(p) = \mu(I(q)) \), or \( \text{per}(I(p)) = n \) and \( I(q) = \mu(I(p)) \). However all those cases are impossible. Because first of all \( I(p) \neq I(q) \). Secondly, by proposition 16(b), we have \( \text{per}(\mu(I(p))) = n/2 \) and \( \text{per}(\mu(I(q))) = n/2 \), because \( \text{per}(I(p)) = n \) and \( \text{per}(I(q)) = n \). Thus, both \( I(p) = \mu(I(q)) \) and \( I(q) = \mu(I(p)) \) are impossible.

Hence we have only the case \( \sigma(p) = \gamma(\sigma(q)) = \sigma \left( f_1^{n/2}(q) \right) \). By proposition 14, either \( I(p) = I \left( f_1^{n/2}(q) \right) \), or \( \text{per}(I \left( f_1^{n/2}(q) \right)) = n \) and \( I(p) = \mu \left( I \left( f_1^{n/2}(q) \right) \right) \), or \( \text{per}(I(p)) = n \) and \( I \left( f_1^{n/2}(q) \right) = \mu(I(p)) \). But the second and the third cases do not hold, because by proposition 16(b) again, we have \( \text{per} \left( \mu \left( I \left( f_1^{n/2}(q) \right) \right) \right) = n/2 \) and \( \text{per}(\mu(I(p))) = n/2 \) because \( \text{per} \left( I \left( f_1^{n/2}(q) \right) \right) = n \) and \( \text{per}(I(p)) = n \). Therefore, we have only the case \( I(p) = I \left( f_1^{n/2}(q) \right) \). Since \( f_1 \) is a horseshoe, we have \( p = f_1^{n/2}(q) \). This means \( I(p) = \nu(I(q)) \) and \( I(q) = \nu(I(p)) \) by the definition of \( \nu \).

(b) Suppose that \( I(p) = \nu(I(q)) \). Let \( V \) be the periodic point component containing \( (1, q) \). By proposition 13, \( V \cap \{ t = 1 \} \) must have at least two points. Let \( (1, q') \) be one of other points. Note that \( \text{per}(q') = n \) because of the same reason as mentioned in the first part of the proof of proposition 13. Since \( (1, q) \) and \( (1, q') \) are on the same periodic point component \( V \), by
the above argument, \( I(q') = ν(I(q)) = I(p) \). Then \( q' \) must be \( p \), because \( f_i \) is a horseshoe and there is only one point whose itinerary is \( I(p) \). ⌣

8. The proof of proposition 12

Recall some definitions of dimensions. For a topological space \( X \), the Lebesgue covering dimension \( \dim X \) of \( X \) is less than or equal to \( n \) if each finite open cover of \( X \) has a refinement \( \mathcal{V} \) such that no point is included in more than \( n + 1 \) elements of \( \mathcal{V} \). A small inductive dimension \( \text{ind}(X) \) of \( X \) is defined as follows: \( \text{ind}(\emptyset) = -1 \). By induction, \( \text{ind}(X) = 0 \) if for any point \( x \in X \) and any open neighborhood \( U \) of \( x \), there is an open neighborhood \( V \) of \( x \) such that \( \text{ind}(\partial V) \leq n - 1 \), where the boundary \( \partial V \) of \( V \) is defined by \( \partial V := \overline{V} \setminus \text{int} V \). A large inductive dimension \( \text{ind}(X) \) of \( X \) is defined as follows: \( \text{ind}(\emptyset) = -1 \). By induction, \( \text{ind}(X) = 0 \) if for any open subset \( U \) of \( X \) and for any closed subset \( F \subseteq U \), there is a boundary neighborhood \( V \) of \( F \) with \( \overline{V} \subseteq U \) such that \( \text{ind}(\partial V) \leq n - 1 \).

By dimension, we mean the small inductive dimension. Recall that Urysohn’s theorem says \( \dim \mathbb{R}^n = n \). By induction, \( \text{ind} X \leq n \) if for any point \( x \in X \) and any open neighborhood \( U \) of \( x \), there is an open neighborhood \( V \) of \( x \) such that \( \text{ind}(\partial V) \leq n - 1 \), where the boundary \( \partial V \) of \( V \) is defined by \( \partial V := \overline{V} \setminus \text{int} V \). A large inductive dimension \( \text{ind}(X) \) of \( X \) is defined as follows: \( \text{ind}(\emptyset) = -1 \). By induction, \( \text{ind}(X) = 0 \) if for any open subset \( U \) of \( X \) and for any closed subset \( F \subseteq U \), there is an open neighborhood \( V \) of \( F \) with \( \overline{V} \subseteq U \) such that \( \text{ind}(\partial V) \leq n - 1 \).

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By a decomposition, we mean a family \( F \) of pairwise disjoint nonempty subsets of a set \( X \) such that \( X = \sqcup F \), where \( \sqcup \) denotes a disjoint union. Let \( F \) be a decomposition of a topological space \( X \). For any \( x \in X \), denote by \( F(x) \) the element of \( F \) containing \( x \). For a subset \( V \) of \( X \), write the saturation \( F(V) := \bigcup_{x \in V} F(x) \). A subset \( V \subseteq X \) is saturated if \( V = F(V) \). The decomposition \( F \) of \( X \) is upper semicontinuous if each element of \( F \) is both closed and compact and for any \( L \in F \) and for any open neighborhood \( U \subseteq L \) there is a saturated neighborhood of \( L \) contained in \( U \). Epstein has shown the following equivalence.

**Lemma 3 (Remark after theorem 4.1 [E]).** The following are equivalent for a decomposition \( F \) into connected compact elements of a locally compact Hausdorff space \( X \):

(a) \( F \) is upper semicontinuous.

(b) The quotient space \( X/F \) is Hausdorff.

(c) The canonical projection \( p : X \to X/F \) is closed (i.e. the saturation of a closed subsets is closed).

By a continuum, we mean a nonempty compact connected metrizable space. A subset \( C \) in a topological space \( X \) is separating if the complement \( X - C \) is disconnected. Define a filling \( \text{Fill}_{\mathbb{R}^2}(W) \subseteq \mathbb{R}^2 \) of a continuum \( W \subseteq \mathbb{R}^2 \) as follows: \( p \in \text{Fill}_{\mathbb{R}^2}(W) \) if either \( p \in W \) or the connected component of \( \mathbb{R}^2 - W \) containing \( p \) is an open disk whose boundary is contained in \( W \). Here an open disk means a nonempty simply connected open subset in a plane or a sphere.

Recall that the boundary of an open disk in a plane can be disconnected in general. On the other hand, boundedness implies the following observation for the connectivity of the boundaries of a bounded open disk in a plane and an open disk in a sphere.

**Lemma 4.** The boundaries of a bounded open disk in a plane and an open disk in a sphere are connected.

**Proof.** Let \( D \) be either a bounded open disk in a plane \( \mathbb{R}^2 \) or an open disk in \( S^2 \) and \( p : D_0 \to D \) a homeomorphism from the unit open disk \( D_0 \) in a plane. Then the boundary \( \partial D \) is compact. Suppose that the boundary \( \partial D \) is disconnected. There are disjoint nonempty closed subsets \( A \)
and $B$ whose union is $\partial D$. Put $d := \min\{d(a, b)|a \in A, b \in B\} > 0$. Then $U_A := B_{d/2}(A)$ and $U_B := B_{d/2}(B)$ are open neighborhoods of $A$ and $B$ respectively such that $U_A \cap D \neq \emptyset$ and $U_B \cap D \neq \emptyset$. Denote by $C(\theta)$ the image of a curve $p((r \cos \theta, r \sin \theta)|r \in [0, 1])$ for any $\theta \in [0, 2\pi)$. For any $\theta \in [0, 2\pi)$, since the curve $C(\theta)$ in the open disk $D$ is closed as a subspace of $D$ (i.e. $C(\theta) \cap D = C(\theta)$), the difference $C(\theta) - C(\theta)$ is contained in $\partial D$. Since any curve from a point in $D$ to a point in the boundary $\partial D$ intersects $U_A \cup U_B$, compactness of $\partial D$ implies that the preimage $p^{-1}((U_A \cup U_B) \cap D)$ contains an annulus $A := \{(r \cos t, r \sin t)|t \in \mathbb{R}, r \in (s, 1)\}$ for some $s \in (0, 1)$. This implies that $p^{-1}((U_A \cap D) \cap A)$ and $p^{-1}((U_B \cap D) \cap A)$ form an open covering of $A$ and so that $A$ is disconnected. That contradicts that $A$ is annular. Thus $\partial D$ is connected.

Notice that the previous lemma is true for not only two dimensional open disk but also $n$-dimensional open ball. The previous lemma implies the following observation.

Corollary 1. \textit{The boundary of an unbounded open disk in a plane is unbounded unless the disk is $\mathbb{R}^2$.}

Proof. Let $S^2$ be the one-point compactification of $\mathbb{R}^2$, $\infty$ the point at infinity (i.e. $\{\infty\} = S^2 - \mathbb{R}^2$), and $D$ a bounded open disk which is a proper subset of $\mathbb{R}^2$. Unboundedness of $D$ implies that the boundary $\partial_{S^2} D$ of $D$ in $S^2$ contains $\infty$. Since $D \subseteq \mathbb{R}^2$, the boundary $\partial_{S^2} D$ contains a point in $\mathbb{R}^2$. Lemma 4 implies that the boundary $\partial_{S^2} D$ is connected and so the boundary $\partial_{S^2} D = \partial_{S^2} D - \{\infty\}$ of $D$ in $\mathbb{R}^2$ is unbounded.

We show the following property of the complement of a filling.

Lemma 5. \textit{The complement of a continuum $W$ on $\mathbb{R}^2$ consists of one unbounded open annulus and bounded open disks, and the complement of the filling $\text{Fill}_{S^2}(W)$ is an unbounded open annulus on $\mathbb{R}^2$. In other words, the complement of $W$ in the one-point compactification $S^2$ of $\mathbb{R}^2$ consists of open disks, and the complement of the filling $\text{Fill}_{S^2}(W)$ is an open disk on $S^2$ containing the point at infinity.}

Proof. Since $W$ is bounded and closed, the complement $\mathbb{R}^2 - W$ is the union of bounded open disks with or without punctures and exactly one unbounded connected component $A$. We show that each connected component of $\mathbb{R}^2 - W$ which is bounded is simply connected. Indeed, assume that there is a connected component $D \subseteq \mathbb{R}^2$ of the complement $\mathbb{R}^2 - W$ which is bounded but is not an open disk. Since each connected component of $\mathbb{R}^2 - W$ is closed in $\mathbb{R}^2 - W$, we have $\partial D \subseteq W$. The Riemann mapping theorem states that each nonempty open simply connected proper subset of $C$ is conformally equivalent to the unit disk, and so the component $D$ is not simply connected. Then there is a simple closed curve $\gamma \subseteq D$ which is not contractible in $D$. By the Jordan curve theorem, the complement $\mathbb{R}^2 - \gamma$ consists of an unbounded open annulus $A_\gamma$ and an open disk $D_\gamma$ each of which intersects a connected component of the boundary $\partial D \subseteq W$. Since $\gamma \subseteq D \subseteq \mathbb{R}^2 - W$, the disjoint union $A_\gamma \sqcup D_\gamma$ of open subsets is an open neighborhood of $W$, which contradicts the connectivity of $W$. Similarly, each connected component of $\mathbb{R}^2 - W$ is simply connected and so the connected component $A$ is an open annulus because $A$ is an open disk minus the point at infinity. By corollary 1, the boundary of any unbounded open disk in $\mathbb{R}^2$ is unbounded unless the disk is $\mathbb{R}^2$. Since $W$ is bounded, each open disk whose boundary is contained in $W$ is bounded and so does not intersect $A$ because the boundary of any bounded disk intersecting $A$ intersects $A \subseteq \mathbb{R}^2 - W$. This implies that $A \cap \text{Fill}_{S^2}(W) = \emptyset$. Since the boundary of any connected component of $\mathbb{R}^2 - W$ is contained in $W$, we have $\text{Fill}_{S^2}(W) = \mathbb{R}^2 - A$ and so $\mathbb{R}^2 - \text{Fill}_{S^2}(W) = A$. In other words, the complement $\mathbb{R}^2 - \text{Fill}_{S^2}(W)$ is the unbounded open annulus $A$. \hfill $\square$
We show that the filling of a continuum is a non-separating continuum.

**Lemma 6.** The filling of a continuum in a plane is a non-separating continuum.

**Proof.** Let $W$ be a continuum in $\mathbb{R}^2$. By lemma 5, the filling $F_W := \text{Fill}_{\mathbb{R}^2}(W) \subseteq \mathbb{R}^2$ is the complement of an unbounded open annulus, and so is bounded, closed, and non-separating. Moreover, the filling $F_W$ is a disjoint union of $W$ and open disks $U_\lambda (\lambda \in \Lambda)$ whose boundaries are contained in $W$. Then it suffices to show that $F_W$ is connected. Indeed, assume that $F_W$ is disconnected. Then there are two disjoint open subsets $U$ and $V$ whose union is a neighborhood of $F_W$ such that $F_W \cap U \neq \emptyset$ and $F_W \cap V \neq \emptyset$. Then $F_W \not\subseteq U$ and $F_W \not\subseteq V$. The connectivity of $W$ implies either $W \subseteq U$ or $W \subseteq V$. We may assume that $W \subseteq U$. Fix any $\lambda \in \Lambda$. Since $\partial U_\lambda \subset W \subset U$, we have $U_\lambda \cap U \neq \emptyset$. Since $U$ and $V$ are disjoint nonempty open subsets with $U_\lambda \subset U \cup V$, the connectivity of $U_\lambda$ implies that $U_\lambda \subset U$. This means that $F_W = W \cup \bigcup_{\lambda \in \Lambda} U_\lambda \subset U$, which contradicts $F_W \not\subseteq U$. \qed

We state an observation which is a generalization of a part of the Jordan curve theorem.

**Lemma 7.** A bounded open disk in a plane which contains the boundary of another bounded open disk in the plane contains another open disk.

**Proof.** Let $D \subset \mathbb{R}^2$ be a bounded open disk and $D' \subset \mathbb{R}^2$ a bounded open disk with $\partial D \subset D'$. By lemma 4, the boundary $\partial D \subset \mathbb{R}^2$ is a bounded closed connected subset and so is a continuum. By lemma 5, the complement of $\text{Fill}_{\mathbb{R}^2}(\partial D)$ is an unbounded open annulus $\Lambda$ on $\mathbb{R}^2$ and the complement of the difference $\text{Fill}_{\mathbb{R}^2}(\partial D) - D$ consists of the unbounded open annulus $\Lambda$ and the open disk $D$. Moreover, there is an open neighborhood $U$ of $\partial D$ which does not contain $D$ but is a finite union of open balls of finite radii such that $\overline{U} \subset D'$ and that each pair of boundaries of such two distinct open balls intersects transversely if it intersects. Then each connected component of $\partial U$ consists of finitely many arcs and so is a simple closed curve. This implies that $U$ is a punctured disk whose boundary is a finite union of simple closed curves and that the filling $\text{Fill}_{\mathbb{R}^2}(U)$ of $U$ is a bounded open disk whose boundary is a simple closed curve.

Since $\partial U$ consists of finitely many simple closed curves contained in the bounded open disk $D'$, the Jordan curve theorem to the open disk $D'$ implies that any simple closed curve which is a connected component of $\partial U \subset D'$ bounds an open disk on $D'$. This means that $\text{Fill}_{\mathbb{R}^2}(U) \subset D'$. Since $\partial D \subset U$, we have $\overline{U} \subset \text{Fill}_{\mathbb{R}^2}(\partial D) \subset \text{Fill}_{\mathbb{R}^2}(U) \subset D'$. \qed

We show that the inclusion relation on the set of fillings of elements of a decomposition is a partial order.

**Lemma 8.** For any continua $F \neq F'$ in $\mathbb{R}^2$ with $\text{Fill}_{\mathbb{R}^2}(F) \cap \text{Fill}_{\mathbb{R}^2}(F') \neq \emptyset$, we have either $\text{Fill}_{\mathbb{R}^2}(F) \subset D_F \subset \text{int}(\text{Fill}_{\mathbb{R}^2}(F'))$ or $\text{Fill}_{\mathbb{R}^2}(F') \subset D_F \subset \text{Fill}_{\mathbb{R}^2}(F)$, where $D_F$ is some bounded open disk which is a connected component of $\mathbb{R}^2 - F'$ and $D_F$ is some bounded open disk which is a connected component of $\mathbb{R}^2 - F$.

**Proof.** By lemma 5, the complements of $\text{Fill}_{\mathbb{R}^2}(F)$ and $\text{Fill}_{\mathbb{R}^2}(F')$ (resp. $F$ and $F'$) are unbounded open annuli (resp. unbounded open annuli and bounded open disks) on $\mathbb{R}^2$. Since $\text{Fill}_{\mathbb{R}^2}(F) \cap \text{Fill}_{\mathbb{R}^2}(F') \neq \emptyset$, fix a point $p \in \text{Fill}_{\mathbb{R}^2}(F) \cap \text{Fill}_{\mathbb{R}^2}(F')$. Suppose that $p \in F$. Then there is a bounded open disk $D_F$ which is the connected component of $\mathbb{R}^2 - F'$ containing $p$. Hence $p \in F \cap D_F$. Since $F \cap F' = \emptyset$ and $\partial D_F \subset \partial F' \subset F'$, we have $F \cap \partial D_F = \emptyset$. Since the disjoint union $D_F \cup (\mathbb{R}^2 - D_F) = \mathbb{R}^2 - \partial D_F$ is an open neighborhood of $F$, the connectivity of $F$ implies $F \subset D_F$. By lemma 7, we obtain $\text{Fill}_{\mathbb{R}^2}(F) \subset D_F \subset \text{int}(\text{Fill}_{\mathbb{R}^2}(F'))$. By symmetry, we may assume that $p \notin F \cup F'$. Then there are bounded open disks $D_F$ and $D_F'$ such that $D_F$ (resp. $D_F'$) is the connected component of $\mathbb{R}^2 - F$ (resp. $\mathbb{R}^2 - F'$) containing $p$. Then $\partial D_F \subset \partial F \subset F$ and $\partial D_F' \subset \partial F' \subset F'$. Since $F \cap F' = \emptyset$, we have $\partial D_F \cap \partial D_F' = \emptyset$. \qed
Define a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ as follows: $f(x) = -\min_{y \in \partial D_F} d(x, y)$ if $x \notin D_F$, and $f(x) = \min_{y \in \partial D_F} d(x, y)$ if $x \in D_F$, where $d$ is the Euclidean distance on $\mathbb{R}^2$. We show that $\partial D_F \subset D_F$ or $D_F \supset \partial D_F$. Indeed, since $\partial D_F \cap \partial D_F' = \emptyset$, we have $0 \notin f(\partial D_F)$. By lemma 4, the boundary $\partial D_F$ is connected and so we obtain either $f(\partial D_F) \subset \mathbb{R}_{<0}$ or $f(\partial D_F) \subset \mathbb{R}_{>0}$. This means either that $\partial D_F \subset D_F$ or $\partial D_F \subset \mathbb{R}^2 - D_F$. Suppose that $\partial D_F \subset D_F$. By lemma 7, we have $\partial D_F \subset D_F$. Thus we may assume that $\partial D_F \subset \mathbb{R}^2 - D_F$. Since $f(p) < 0$, we have $0 \in f(D_F)$ and so $D_F \cap \partial D_F = \emptyset$. Similarly, define a continuous function $f' : \mathbb{R}^2 \to \mathbb{R}$ as follows: $f'(x) = -\min_{y \in \partial D_F} d(x, y)$ if $x \in D_F$ and $f'(x) = \min_{y \in \partial D_F} d(x, y)$ if $x \notin D_F$. As the same argument, we may assume that $\partial D_F \subset D_F$ or $\partial D_F \subset \mathbb{R}^2 - D_F$. Therefore $\partial D_F \subset D_F$. Lemma 7 implies $\partial D_F \subset D_F$. Since either $\partial D_F \subset D_F$ or $D_F \supset \partial D_F$, by symmetry, we may assume that $\partial D_F \subset D_F$. Then $F \cap D_F \neq \emptyset$. Since $F \cap F' = \emptyset$, we have $F \cap \partial D_F = \emptyset$ and so $0 \notin f(F)$. The connectivity of $F$ implies that $f(F) \subset \mathbb{R}_{<0}$. This means that $F \subset D_F$ and so $\text{Fill}_{S^2}(F) \subset D_F \subset \text{int}(\text{Fill}_{S^2}(F'))$. \hfill \square

We recall the following tool.

**Lemma 9 (Moore’s theorem (cf p 3 in [D])).** Let $S$ be a plane or a sphere. The quotient space $S/F$ of an upper semicontinuous decomposition $F$ of $S$ into non-separating continua is homeomorphic to $S$ unless $F$ is the singleton of the sphere.

Fix a square $D := [0, 1] \times [-M, M]$ and a continuous function $G : D \to \mathbb{R}$ such that $G^{-1}(0) \cap \partial D \subset \{1\} \times [-M, M]$ is finite. Let $R : [0, 1] \times [-M, M] \to [1, 2] \times [-M, M]$ be the reflection with respect to $\{1\} \times [-M, M]$ and with respect to $\{1\} \times [-M, M]$ (i.e. $R(x, y) = (2 - x, y)$). For a subset $B \subset \mathbb{R}^2$, define the union $R(B) := B \cup R(B)$. Consider the double $[0, 2] \times [-M, M]$ of an upper semicontinuous decomposition $F$ of $S$ into non-separating continua.

**Lemma 10.** Let $C_0 := \{C \in \mathcal{G}_0 \mid \text{Fill}_{S^2}(C) \subseteq \text{Fill}_{S^2}(C)\}$ for a continuum $C_0 \in \mathcal{G}_0$. Then the family $\mathcal{F}_{C_0} := \{\text{Fill}_{S^2}(C) \mid C \in C_0\}$ is a totally ordered set with respect to the inclusion relation and has a maximal element.

**Proof.** By lemma 8, the set $\mathcal{F}_{C_0}$ with the inclusion is a totally ordered set. Put $D_0 := \bigcup \mathcal{F}_{C_0} = \bigcup \{\text{Fill}_{S^2}(C) \mid C \in C_0\} \subset [0, 2] \times [-M, M]$. Assume that $\mathcal{F}_{C_0}$ has no maximal element. By lemma 8, there is a family $\mathcal{D}_0 := \{D_F\}_{F \in \mathcal{F}_{C_0}}$ of bounded open disks with
\[ D_0 = \bigcup D_0 \] which is total ordered with respect to the inclusion relation. We show that each closed curve \( \gamma \) on \( D_0 \) is null homotopic in \( D_0 \). Indeed, since \( \gamma \) is compact and the family \( D_0 \) consists of open disks and is an open covering of \( \gamma \) and the totally ordered set with respect to the inclusion relation, there is an element \( D \in D_0 \) such that \( \gamma \subset D \) and so the curve \( \gamma \) is null homotopic. Therefore the union \( D_0 \) is a bounded open disk on \( \mathbb{R}^2 \). By lemma 4, the boundary \( \partial D_0 \) is connected. Moreover we have \( \partial D_0 \subseteq \bigcup (\partial \text{Fill}_{22}(C) | C \in C_0) \subseteq \partial C | C \in C_0 \subseteq G^{-1}(0) \). Since \( \partial D_0 \subseteq G^{-1}(0) \) is connected, there is an element \( C_\infty \in G_0 \) such that \( \partial D_0 \subseteq C_\infty \) and so \( D_0 \subseteq \text{Fill}_{22}(C_\infty) \). Since \( \text{Fill}_{22}(C_\infty) \subseteq D_0 \), we obtain \( \text{Fill}_{22}(C_\infty) \subseteq \text{Fill}_{22}(C_\infty) \) and so \( C_\infty \subseteq C_0 \). This means that \( \text{Fill}_{22}(C_\infty) \) is the maximal element of \( \mathcal{F}_{C_\infty} \), which contradicts the assumption.

We will show the following statement which is a statement of proposition 12 for the extended continuous map \( G \) on the double \([0, 2] \times [-M, M] \), and which is equivalent to proposition 12.

**Lemma 11.** If there is a connected component of \( G^{-1}(0) \) which intersects exactly once to \( \{1\} \times [-M, M] \) at a point \( p_0 = (1, y_0) \), then there is a path in \( D \) \( - \) \( G^{-1}(0) \) connecting points \( y_- \) and \( y_+ \) in \( \{1\} \times [-M, M] \) which are arbitrary close to \( y_0 \) with \( y_- < y_0 < y_+ \).

**Proof.** Since \( \partial \hat{R}(D) \cap G^{-1}(0) = \emptyset \), we may assume that \( G \) is constant on \( \partial \hat{R}(D) = \partial ([0, 2] \times [-M, M]) \). Collapsing the boundary \( \partial \hat{R}(D) \) into a point, denoted by \( \infty \), the resulting surface is a sphere, denoted by \( \mathbb{S}^2 \) (see the left figure in figure 9). Then the induced map \( G : \mathbb{S}^2 \to \mathbb{R} \) is a well-defined continuous map. Suppose that there is a connected component \( L_\infty \) of \( G^{-1}(0) \) which intersects exactly once to \( \{1\} \times [-M, M] \) at a point \( (1, y_0) \). Let \( L_1, \ldots, L_k \) be the connected components except \( L_\infty \) of \( G^{-1}(0) \) intersecting \( \{1\} \times [-M, M] \). Write \( F_\mathcal{C} := \text{Fill}_{22}(L_\infty) \). Then the complement \( D := \mathbb{S}^2 - F_\mathcal{C} \) is an open disk. Put \( F_i := \text{Fill}_{22}(L_i) \) for \( i = 1, \ldots, k \). Lemma 6 implies that the fillings \( F_\mathcal{C}, F_1, \ldots, F_k \) are non-separating continua. By lemma 8, we have that either \( F_i \cap F_j = \emptyset \) or \( F_i \subseteq F_j \) or \( F_j \subseteq F_i \) for any pair \( i \neq j \). By lemma 10, we may assume that \( F_1, \ldots, F_k \) are the maximal elements with respect to the inclusion relation.

Define a decomposition \( \mathcal{F}_0 \) on \( \mathbb{S}^2 \) which consists of connected components of \( F_\mathcal{C}, F_1, \ldots, F_k \) and singletons of the points of the complement of \( F_\mathcal{C} \cup F_1 \cup \cdots \cup F_k \). In other words,

\[
\mathcal{F}_0 = \{ F_\mathcal{C}, F_1, \ldots, F_k \} \cup \{ x \in \mathbb{S}^2 - F_\mathcal{C} \cup F_1 \cup \cdots \cup F_k \}
\]

Since \( F_\mathcal{C}, F_1, \ldots, F_k \) are closed and \( \mathbb{S}^2 \) is normal, the quotient space \( \mathbb{S}^2 / \mathcal{F}_0 \) is Hausdorff. Lemma 3 implies that the decomposition \( \mathcal{F}_0 \) is upper semicontinuous. Since each element of \( \mathcal{F}_0 \) is non-separating, applying the Moore’s theorem to a decomposition \( \mathcal{F}_0 \) of \( \mathbb{S}^2 \), the quotient space \( \mathbb{S}^2 / \mathcal{F}_0 \) is a sphere (see the middle figure in figure 9). This means that there are finitely many connected components of \( G^{-1}(0) / \mathcal{F}_0 \) intersecting \( \{1\} \times [-M, M] / \mathcal{F}_0 \), which are singletons. Recall that \( \mathcal{G}_0 \) is the set of connected components of \( G^{-1}(0) \) and that max \( P \) is the subset of maximal elements of a partial order set \( P \). Putting \( \mathcal{M} := \text{max} \{ \text{Fill}_{22}(L) | L \in \mathcal{G}_0 - \{ F_\mathcal{C}, F_1, \ldots, F_k \} \} \cup \{ F_\mathcal{C}, F_1, \ldots, F_k \} \) with respect to the inclusion relation, lemma 10 implies that the family \( \mathcal{M} \) and the points of the complement \( \mathbb{S}^2 - \bigcup \mathcal{M} \) form a decomposition \( \mathcal{F}_1 \). In other words, the decomposition \( \mathcal{F}_1 \) is defined by

\[
\mathcal{F}_1 := \mathcal{M} \cup \{ x \in \mathbb{S}^2 - \bigcup \mathcal{M} \}
\]

Note that \( \mathcal{M} = \text{max} \{ \text{Fill}_{22}(L) | L \in \mathcal{G}_0 - \{ F_\mathcal{C}, F_1, \ldots, F_k \} \} \cup \{ F_\mathcal{C}, F_1, \ldots, F_k \} = \text{max} \{ \text{Fill}_{22}(L) | L \in \mathcal{G}_0 - \{ F_\mathcal{C}, F_1, \ldots, F_k \} \} \cup \{ F_\mathcal{C}, F_1, \ldots, F_k \} \). Lemma 6 implies that each element of \( \mathcal{M} \) is a non-separating continuum. Since each element of \( \mathcal{F}_1 \) is closed and \( \mathbb{S}^2 \) is normal, the quotient space \( \mathbb{S}^2 / \mathcal{F}_1 \) is Hausdorff. Lemma 3 implies that the decomposition \( \mathcal{F}_1 \) is upper semicontinuous. Since each element of \( \mathcal{F}_1 \) is non-separating, applying the
Moore’s theorem to a decomposition $\mathcal{F}_1$, the quotient space $S^2 / \mathcal{F}_1$ is a sphere (see the right figure in figure 9). Since a locally compact Hausdorff space is zero-dimensional if and only if it is totally disconnected, a compact totally disconnected subset $G^{-1}(0) / \mathcal{F}_1$ of a sphere is zero-dimensional. Since a sphere is a Cantor manifold, the complement $(S^2 - \mathcal{F}_1(G^{-1}(0))) / \mathcal{F}_1$ is path-connected. Since $S^2 - \mathcal{F}_1(G^{-1}(0))$ consists of two connected components such that the connected component of $D^2 - \text{Im} I$ containing $C$ contains no other connected components of $K$ intersecting $\partial D^2$ (i.e. $C \cap \partial D^2 = B \cap K \cap \partial D^2$, where $B$ is the connected component of the complement $D^2 - \text{Im} I$ intersecting $C$). Finally, we state an existence of separating chord as follows.

**Theorem 2.** Let $K$ be a compact subset of the unit disk $D^2$ such that $K \cap \partial D^2$ is finite and $C$ a connected component of $K$ intersecting the boundary $\partial D^2$ exactly once. Then there is a separating chord from $C$. 

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Appendix

Roughly speaking, proposition 12 asserts the existence of separating chord. To state this, we define a separating chord as follows: let $D^2$ be the unit disk in $\mathbb{R}^2$. An injective continuous curve $I : [0, 1] \to D^2$ is a chord if $\partial D^2 \cap \text{Im} I = \{l(0), l(1)\}$. For a compact subset $K \subset D^2$ such that $K \cap \partial D^2$ is finite and for a connected component $C$ of $K$ intersecting the boundary $\partial D^2$, a chord $I$ is a separating chord from $C$ if $K \cap \text{Im} I = \emptyset$ and the complement $D^2 - \text{Im} I$ consists of two connected components such that the connected component of $D^2 - \text{Im} I$ containing $C$ contains no other connected components of $K$ intersecting $\partial D^2$ (i.e. $C \cap \partial D^2 = B \cap K \cap \partial D^2$, where $B$ is the connected component of the complement $D^2 - \text{Im} I$ intersecting $C$). Finally, we state an existence of separating chord as follows.

**Theorem 2.** Let $K$ be a compact subset of the unit disk $D^2$ such that $K \cap \partial D^2$ is finite and $C$ a connected component of $K$ intersecting the boundary $\partial D^2$ exactly once. Then there is a separating chord from $C$. 

8015
Proof. Define a continuous function \( g : D^2 \to \mathbb{R}_{\geq 0} \) by \( g(x) := \min\{d(x, y) | y \in K\} \), where \( d \) is the Euclidean metric. Then \( K = g^{-1}(0) \). Let \( h : D = [0, 1] \times [-M, M] \to D^2 \) be a homeomorphism such that \( h^{-1}(K) \cap \partial D \) is a finite set on \( \{1\} \times (-M, M) \). The composition \( G := g \circ h : D \to \mathbb{R} \) is a continuous function such that \( G^{-1}(0) \cap \partial D \) is a finite set on \( \{1\} \times (-M, M) \). The inverse image \( F_C := h^{-1}(C) \) is a connected component of \( G^{-1}(0) \) which intersects exactly once on \( \{1\} \times (-M, M) \) at a point \( p_0 = (1, y_0) \). Applying proposition 12, then there is a path \( J : [0, 1] \to D \) in \( D - G^{-1}(0) \) connecting points \( y_- \) and \( y_+ \) in \( \{1\} \times (-M, M) \) with \( y_- < y_0 < y_+ \) such that \( G^{-1}(0) \cap \{1\} \times (y_-, y_+) = \{p_0\} \). Then the composition \( h \circ J : [0, 1] \to D^2 \) is a separating chord from \( C \).

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