Weakly maximal subgroups of branch groups

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Let $G$ be a branch group acting by automorphisms on a rooted tree $T$. Stabilizers of infinite rays in $T$ are examples of weakly maximal subgroups of $G$ (subgroups that are maximal among subgroups of infinite index), but in general they are not the only examples.

In this note we describe two families of weakly maximal subgroups of branch groups. We show that, for the first Grigorchuk group as well as for the torsion GGS groups, every weakly maximal subgroup belongs to one of these families. The first family is a generalization of stabilizers of rays, while the second one consists of weakly maximal subgroups with a block structure.

We obtain different equivalent characterizations of these families in terms of finite generation, the existence of a trivial rigid stabilizer, the number of orbit-closures for the action on the boundary of the tree or by the means of sections.

1 Introduction

Let $T$ be a locally finite spherically regular rooted tree. Among groups that act on $T$ by automorphisms, branch groups are of particular interest (see Section 2 for all the relevant definitions). The class of branch groups contains finitely generated groups with interesting properties, such as being infinite torsion, or having intermediate growth. Branch groups have interesting subgroup structure and some of them, but not all, do not have maximal subgroups of infinite index. This is the case, for example, of the first Grigorchuk group and of the torsion GGS groups.

The next step in understanding the subgroup structure of branch groups is to study weakly maximal subgroups, that is the maximal elements among the subgroups of infinite index. The study of such subgroups began with the following result of Bartholdi and Grigorchuk.

Proposition 1.1 ([3, 21]). If $G \leq \text{Aut}(T)$ is weakly branch, then all the $\text{Stab}_G(\xi)$ for $\xi \in \partial T$ are infinite and pairwise distinct. Moreover, if $G$ is branch, then all these subgroups are weakly maximal.
The stabilizers of rays, $\text{Stab}_G(\xi)$, are called \textit{parabolic subgroups}.

The study of weakly maximal subgroups continued with an example of Pervova exhibiting a non-parabolic weakly maximal subgroup of the first Grigorchuk group, answering a question of Grigorchuk about the existence of such subgroups. After that, the author together with Bou-Rabee and Nagnibeda proved the following two results.

\textbf{Theorem 1.2} (\cite{8}). \textit{Let $T$ be a regular rooted tree and $G \leq \text{Aut}(T)$ be a finitely generated branch group. Then, for any finite subgroup $F \leq G$ there exists uncountably many weakly maximal subgroups of $G$ containing $F$.}

\textbf{Theorem 1.3} (\cite{8}). \textit{Let $G$ be the first Grigorchuk group or a branch GGS group. For any vertex $v$, the subgroup $\text{Stab}_G(v)$ contains a weakly maximal subgroup $W$ that does not stabilize any vertex of level greater than the level of $v$.}

The flavour of Theorem 1.2 is small subgroups are contained in many weakly maximal subgroups while Theorem 1.3 can be thought as big subgroups (vertex stabilizers) contain many weakly maximal subgroup. Indeed, Theorem 1.3 implies that $\text{Stab}_G(v)$ contains a weakly maximal subgroup that is not a parabolic subgroup. On the other hand, one important corollary of Theorem 1.2 is the existence for some branch groups of uncountably many weakly maximal subgroups that are not parabolic. This result hold for any $G$ that admits a unique branch action and such that $G$ contains a finite subgroup $F$ that does not fix any point in $\partial T$; two conditions that hold in a lot of branch groups.

The aim of this paper is twofold: a better understanding of weakly maximal subgroups in general branch groups as well as a full description of weakly maximal subgroups for the particular case of the first Grigorchuk group and of the GGS torsion groups. In doing this, we will particularly focus on two special families of weakly maximal subgroups.

We first have the following general result.

\textbf{Lemma 1.4.} \textit{Let $G \leq \text{Aut}(T)$ be a group with all $\text{Rist}_G(v)$ infinite. Then every weakly maximal subgroup of $G$ is infinite.}

We then study \textit{generalized parabolic subgroups}. These are the setwise stabilizers $\text{SStab}_G(C)$ of closed but not clopen subsets $C \subset \partial T$ such that the action $\text{SStab}_G(C) \curvearrowright C$ is minimal. For this special class of subgroups we are able to prove the following generalization of Bartholdi and Grigorchuk’s results.

\textbf{Theorem 1.5.} \textit{Let $T$ be a rooted tree and $G \leq \text{Aut}(T)$ be a branch group. Then all generalized parabolic subgroups of $G$ are weakly maximal and pairwise distinct.}

Here pairwise distinct is to be understood as $\text{SStab}_G(C_1) \neq \text{SStab}_G(C_2)$ if they are both generalized parabolic subgroups with $C_1 \neq C_2$.

We then provide stronger versions of the results of \cite{8}. Observe that this time we do not require $G$ to be finitely generated, the results apply to subgroups that are not necessarily finite and we are able to show that finite (and other subgroups) subgroups are contained

\footnote{The result in \cite{8} is only stated for regular branch groups, but the proof can easily be adapted for general branch groups.}
a continuum of weakly maximal subgroups. Moreover, for torsion groups we are able to obtain infinitely many tree-equivalence classes of weakly maximal subgroups. Where two generalized parabolic subgroups \( \text{SStab}_G(C_1) \) and \( \text{SStab}_G(C_2) \) are tree-equivalent if and only if there exists an automorphism \( \varphi \) of \( \text{Aut}(T) \) such that \( \varphi(C_1) = C_2 \), see Definition 4.14 for a general definition. The important point to keep in mind is that it is a coarser equivalence relation than conjugation of subgroups and that parabolic subgroups form one class for this relation. Finally, our proof does not use the axiom of choice.

**Theorem 1.6.** Let \( G \leq \text{Aut}(T) \) be a branch group and \( F \leq G \) be any subgroup. Let \( \mathcal{C}_F \) be the set of all non-open orbit-closures of the action \( F \curvearrowright \partial T \). Then the function \( \text{SStab}_G(C) : \mathcal{C}_F \to \text{Sub}(G) \) is injective and has values in generalized parabolic subgroups (which are all weakly maximal).

**Corollary 1.7.** Let \( G \leq \text{Aut}(T) \) be a branch group and \( F \leq G \) be any subgroup. Let \( v \) be a vertex of \( T \).

1. If all orbit-closures of \( \text{Stab}_F(v) \curvearrowright \partial T \) are non-open, then \( F \) is contained in uncountably many generalized parabolic subgroup,
2. If all orbits of \( \text{Stab}_F(v) \curvearrowright \partial T \) are at most countable and closed, then \( F \) is contained in a continuum of generalized parabolic subgroup. If moreover \( \text{Stab}_F(v) \) is non-trivial, then \( F \) is contained in a continuum of generalized parabolic subgroups that are not parabolic.

**Corollary 1.8.** Let \( G \leq \text{Aut}(T) \) be a branch group.

1. If \( G \) is not torsion-free, it contains a continuum of generalized parabolic subgroups that are not parabolic.
2. If \( G \) has elements of arbitrarily high finite order, there are infinitely many tree-equivalence classes of generalized parabolic subgroups that each contains a continuum of subgroups.
3. If \( G \) is torsion, there is a continuum of tree-equivalence classes of generalized parabolic subgroups that each contains infinitely many subgroups.

An important observation at this point is that by [4] Theorem 6.9, if \( G \) is a branch group which is torsion, then it satisfies the hypothesis of the second part of Corollary 1.8. In particular, it has both infinitely many tree-equivalence classes of generalized parabolic subgroups that each contains a continuum of subgroups and a continuum of tree-equivalence classes of generalized parabolic subgroups that each contains infinitely many subgroups.

We also provide a version of Theorem 1.3 that holds in any self-replicating branch group.

**Theorem 1.9.** Let \( G \leq \text{Aut}(T) \) be a self-replicating branch group and let \( W \) be a weakly maximal subgroup of \( G \). For any vertex \( v \), there exists a weakly maximal subgroup \( W^v \) contained in \( \text{Stab}_G(v) \) such that the section of \( W^v \) at \( v \) is equal to \( W \).
We then turn our attention on another families of weakly maximal subgroups, the so-called subgroups with a block structure, see Definition 6.2. Informally, $H \leq G$ has a block structure if it is, up to finite index, a product of copies of $G$, some of them embedded diagonally. If $G$ is finitely generated, the subgroups with a block structure are also finitely generated and therefore, they are at most countably many of them. In the special case of the first Grigorchuk as well as for the Gupta-Sidki 3-group, these subgroups coincide with the finitely generated subgroups, [24]. Subgroups with a block structure are a source of new examples of weakly maximal subgroups, and once again there are “as much” of them as possible.

**Proposition 1.10.** Let $G$ be either the first Grigorchuk group, or a torsion GGS group. Then there exists infinitely many distinct tree-equivalence classes, each of them containing infinitely many weakly maximal subgroups with a block structure.

The above results, following [8], are mostly quantitative results. We now give more qualitative results. The main idea here is that a weakly maximal subgroup of a branch group should retain some properties of the full group. There is, a priori, two distinct directions for this. On one hand, we can ask if $W$ is “nearly branch” and on the other hand if $W$ is “almost of finite index” in some sense. Recall that a subgroup $G \leq \text{Aut}(T)$ is weakly branch if and only if $G \curvearrowright \partial T$ has exactly one orbit-closure and all the subgroups $\text{Rist}_G(v)$ are infinite. We first show that a weakly maximal subgroup $W$ of a branch group is always “nearly branch” in the following sense.

**Proposition 1.11.** Let $G \leq \text{Aut}(T)$ be a branch group and let $W$ be a weakly maximal subgroup of $G$. Then at least one of the following holds:

1. $W \curvearrowright \partial T$ has a finite number of orbit-closures,
2. All the subgroups $\text{Rist}_W(v)$ are infinite.

On the other hand, the action on $\partial T$ of a finite index subgroup $H$ of a branch group $G$ has always finitely many orbit-closures and for every $v$ in $T$, the section $\pi_v(\text{Stab}_H(v))$ has finite index in $\pi_v(\text{Stab}_G(v))$. In nice branch groups, weakly maximal subgroups will always be “almost of finite index” in the following sense.

**Proposition 1.12.** Let $G \leq \text{Aut}(T)$ be a branch group that is just infinite, self-replicating and such that for every vertex of the first level $\text{Stab}_G(v) = \text{Stab}_G(L_1)$. Let $W$ be a weakly maximal subgroup of $G$. Then at least one of the following holds:

1. $W \curvearrowright \partial T$ has a finite number of orbit-closures,
2. There exists a level $n$ such that for every $v$ in $L_n$, the section $\pi_v(\text{Stab}_W(v))$ has finite index in $G = \pi_v(\text{Stab}_G(v))$.

For the particular case of the first Grigorchuk group and of torsion GGS groups (which encompass Gupta-Sidki $p$-groups), we even have a full description of the weakly maximal subgroups. They split into two classes: generalized parabolic subgroups and subgroups with a block structure. More precisely, we have the following theorem that is summarized in Table 1.
Theorem 1.13. Let $G$ be either the first Grigorchuk group or a torsion GGS group. Let $W$ be a weakly maximal subgroup of $G$. Then the following properties are equivalent:

1. $W$ has a block structure,

2. There exists $n$ such that $\pi_v(\text{Stab}_W(v))$ has finite index in $\pi_v(\text{Stab}_G(v))$ for every vertex of level $n$,

3. There exists a vertex $v$ with $\text{Rist}_W(v) = \{1\}$,

4. $W \curvearrowright \partial T$ has finitely many orbit-closures,

5. $W$ is not generalized parabolic.

and they all imply that $W$ is finitely generated.

Moreover, if $G$ is either the first Grigorchuk group or the Gupta-Sidki 3-group, then being finitely generated is equivalent to any of the above 5 properties.

| Generalized parabolic | Weakly maximal with a block structure |
|-----------------------|---------------------------------------|
| $\forall v : \text{Rist}_W(v)$ is infinite | $\exists v : \text{Rist}_W(v) = \{1\}$ |
| $W \curvearrowright \partial T$ has infinitely many orbit-closures | $W \curvearrowright \partial T$ has finitely many orbit-closures |
| $\forall n \exists v \in L_n : [\pi_v(G) : \pi_v(W)]$ is infinite | $\exists n \forall v \in L_n : [\pi_v(G) : \pi_v(W)]$ is finite |

Table 1: The two classes of weakly maximal subgroups of $G$, where $G$ is either the first Grigorchuk group or a torsion GGS group. If $G$ is either the first Grigorchuk group of the Gupta-Sidki 3-group, then generalized parabolic subgroups are not finitely generated.

Finally, we investigate further the particular case of the first Grigorchuk group. Among other things, we gave a particular attention to sections and we prove that parabolic subgroups, as well as some generalized parabolic subgroups, behave well under taking the closure in the profinite topology. More precisely

**Proposition 1.14.** Let $\mathfrak{G} \curvearrowright T$ be the branch action of the first Grigorchuk group. For any finite subset $C$ of $\partial T$, we have

$$\overline{\text{Stab}_\mathfrak{G}(C)} = \text{Stab}_\mathfrak{G}(C).$$

If moreover $C$ is contained in one $\mathfrak{G}$-orbit, then we also have

$$\overline{\text{SStab}_\mathfrak{G}(C)} = \text{SStab}_\mathfrak{G}(C).$$

We also gave the first example of a weakly maximal subgroup of $\mathfrak{G}$ that acts level transitively on $T$.

This paper is organized as follow. The next section contains the definitions and some useful reminders as well as preliminary results on weakly maximal subgroups in branch...
groups. Section 3 is devoted to the study of generalized parabolic subgroups. It contains proofs of Theorems 1.5 and 1.6 and of Corollary 1.7. In Section 4 we introduce the notion of the non-rigidity tree of a (weakly maximal) subgroup and study its properties. This tool turns out to be of great interest for the study of (weakly maximal) subgroups of branch groups. Among other results, we prove there Lemma 1.4, Corollary 1.8 and give a characterization of generalized parabolic subgroups in terms of the non-rigidity tree. The next section is about sections and “lifting” and contains the proof of Theorem 1.9. Section 6 concerns the study of subgroups with a block structure. It contains a general structural result about weakly maximal subgroups of branch groups which encompass Propositions 1.11 and 1.12 as well as Theorem 1.13 and which implies Proposition 1.10. In Section 7 we turn our attention to specific examples of weakly maximal subgroups with block structure, specifically the ones that acts minimally on $\partial T$. The last section deals in more details with the specific case of the first Grigorchuk group and contains the proof of Proposition 1.14.

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2 Definitions and preliminaries

First of all, we insist on the fact that, unless explicitly specified, we will not assume that groups under consideration are finitely generated, or even countable. Recall that a subgroup $H < G$ is weakly maximal if it is maximal among subgroup of infinite index. In particular, every maximal subgroup of infinite index is weakly maximal, but in general they are far to be the only examples. If $G$ is finitely generated, then, supposing the axiom of choice, every infinite index subgroup is contained in at least one weakly maximal subgroup.

A rooted tree $T$ is an unoriented connected graph without cycles and with a distinguished vertex, the root. We will often identify $T$ and its vertex set. Vertices of a rooted tree can be partitioned into levels, where a vertex $v$ is in $L_n$ if and only if its distance to the root is $n$. This naturally endows the vertices of $T$ with a partial order, where $v \leq w$ if there is a path $v = v_1, \ldots, v_n = w$ such that $v_{i+1}$ is the unique neighbour of $v_i$ of level strictly less than the level of $v_i$. The terms parent, ancestor, child, descendant and sibling have their obvious meaning in relation with this order. For example, every vertex has a unique parent, except for the root that has none. A rooted tree is spherically regular if the degree (equivalently the number of children) of a vertex depends only of its level. It is $d$-regular if every vertex has $d$ children. If $(m_i)_{i \in \mathbb{N}}$ is a sequence of cardinals greater than 1, we will denote by $T_{(m_i)}$ an infinite spherically regular rooted tree such that a vertex of level $i$ has exactly $m_i$ children (that is, has degree $m_i + 1$). Unless stated otherwise, in the following we will always assume our trees to be spherically rooted. In general,
we will not assume that \( T \) is locally finite (i.e. all the \( m_i \)'s are finite). Nevertheless, if \( G \leq \text{Aut}(T) \) is branch, or more generally, an almost level transitive rigid group, then \( T \) is automatically locally finite, see Definition 2.2 and the discussion after it. We will identify the locally finite rooted tree \( T_{(m_i)} \) with its realization as words \( x_0x_1x_2 \ldots x_n \) such that \( x_i \) is in \( \{0, 1, \ldots, m_i - 1\} \); in particular the root is the empty word. There is a natural notion of rays (infinite paths emanating from the root, or equivalently right infinite words \( x_0x_1x_2 \ldots \)) and of boundary \( \partial T \). The space \( \partial T \) is a metric space, where the distance between two distinct rays \( x_0x_1x_2 \ldots \) and \( y_0y_1y_2 \ldots \) is equal to \( 2^{-i_0} \) where \( i_0 \) is the lowest index such that \( x_{i_0} \neq y_{i_0} \). When \( T \) is locally finite, \( \partial T \) is homeomorphic to a Cantor space.

For a vertex \( v \) of \( T \), we will denote by \( T_v \) the subtree of \( T \) consisting of all descendants of \( v \). It is naturally rooted at \( v \).

The group \( \text{Aut}(T) \) consists of all graph automorphisms of \( T \) sending the root onto itself. That is, an element of \( \text{Aut}(T) \) is a bijection of the vertex set that preserves the adjacency relation and fixes the root. The group \( \text{Aut}(T) \) acts on \( \partial T \) by isometries, in fact \( \text{Aut}(T) = \text{Isom}(\partial T) \). On the other hand, elements of \( \text{Aut}(T) \) naturally preserve the levels. This can be used to put a metric on \( \text{Aut}(T) \), where the distance between two distinct automorphisms \( \varphi \) and \( \psi \) is equal to \( 2^{-i_0} \) where \( i_0 \) is the lowest index such that there is a vertex \( v \) of level \( i_0 \) with \( \varphi(v) \neq \psi(v) \). If \( T \) is locally finite, this induces a compact Hausdorff topology on \( \text{Aut}(T) \), turning \( \text{Aut}(T) \) into a profinite group.

Let \( \{v_0, \ldots, v_{n-1}\} \) be the vertices of the first level of \( T \). Since \( T \) is spherically regular, all the \( T_{v_i} \) are isomorphic and we have a natural isomorphism

\[
\varphi: \text{Aut}(T) \to \text{Aut}(T_{v_0}) \wr \text{Sym}(\mathcal{L}_1) \cong \text{Aut}(T_{v_0}) \wr \text{Sym}(\mathcal{L}_1)
\]

where \( \text{Sym}(\mathcal{L}_1) \) is the group of all bijections of vertices of the first level. That is, for any \( g \in \text{Aut}(T) \), there exists a unique permutation \( \sigma \) of \( \mathcal{L}_1 \) and unique \( q_{v_i} \)'s in \( \text{Aut}(T_i) \) such that \( \varphi(g) = (q_{v_0}, \ldots, q_{v_{n-1}})\sigma \). The element \( q_{v_0} \) is called the \textit{section} of \( g \) at \( v \) and can be defined inductively for any \( v \) in \( T \). In practice, we will often write \( g = (g_{v_0}, \ldots, g_{v_{n-1}})\sigma \) as a shorthand for \( \varphi(g) \). For any vertex \( v \), we have a surjective homomorphism

\[
\pi_v: \text{Stab}_{\text{Aut}(T)}(v) \to \text{Aut}(T_v) \\
g \mapsto q_{v_0} \cdot g_{v_0}
\]

The \textit{portrait} \( \mathcal{P}(g) \) of an automorphism \( g \) of \( T \) is a labelling of vertices of \( T = T_{(m_i)} \) by elements of \( \text{Sym}(m_i) \) constructed inductively as following. Label the root by \( \sigma \), where \( g = (q_{v_0}, \ldots, q_{v_{n-1}})\sigma \) and decorated the subtrees \( T_{v_i} \) by the portrait of \( q_{v_0} \). There is a natural bijection between elements of \( \text{Aut}(T) \) and portraits. An element \( g \) is \textit{finitary} if its portrait has only finitely many non-trivial labels. It is \textit{finitary along rays} if any ray in the portrait of \( g \) has only finitely many non-trivial labels. A subgroup \( G \) of \( \text{Aut}(T) \) is said to be \textit{finitary}, respectively \textit{finitary along rays}, if all its elements have the desired property. We have

\[
\text{Aut}_f(T) < \text{Aut}_{fr}(T) < \text{Aut}(T)
\]
where \( \text{Aut}_f(T) \), respectively \( \text{Aut}_{fr}(T) \), is the subgroup of all finitary, respectively finitary along ray, automorphisms of \( T \). If \( T \) is locally finite, the subgroup \( \text{Aut}_f(T) \) is countable, while the other two have the cardinality of the continuum.

**Definition 2.1.** Let \( T \) be a regular rooted tree. A subgroup \( G \) of \( \text{Aut}(T) \) is self-similar if for every vertex \( v \in T \) we have \( \pi_v(\text{Stab}_G(v)) \leq G \), or equivalently if all the \( g_v \) belong to \( G \).

It is self-replicating (or fractal) if for every vertex \( v \in T \) we have \( \pi_v(\text{Stab}_G(v)) = G \).

When \( T \) is a regular rooted tree, the subgroups \( \text{Aut}_f(T) \), \( \text{Aut}_{fr}(T) \) and \( \text{Aut}(T) \) are all self-replicating.

Given \( G \) a subgroup of \( \text{Aut}(T) \), it is enlightening to look at stabilizers of various subsets of \( T \) or of \( \partial T \). For example it is natural to look at stabilizers of rays \( \text{Stab}_G(\xi) \) for \( \xi \in \partial T \), also-called parabolic subgroups, and we well see later that setwise stabilizers of closed subsets of \( \partial T \) will also play an important role. We will also be interested in stabilizers of vertices \( \text{Stab}_G(v) \) and in pointwise stabilizers of levels \( \text{Stab}_G(L_n) \), the latter ones being normal subgroups of finite index. More generally, if \( X \) is any subset of \( T \cup \partial T \), \( \text{Stab}_G(X) \) denotes the pointwise stabilizer of \( X \), that is \( \text{Stab}_G(X) = \bigcap_{v \in X} \text{Stab}_G(v) \), while \( \text{SStab}_G(X) \) denotes its setwise stabilizer.

We will also look at rigid stabilizers of vertices, where \( \text{Rist}_G(v) \) is the pointwise stabilizer of \( T \setminus T_v \), that is the subgroup of elements acting trivially outside \( T_v \). Finally, rigid stabilizer of levels, \( \text{Rist}_G(L_n) \), are defined as the product of all \( \text{Rist}_G(v) \) for \( v \) of level \( n \).

**Definition 2.2.** A subgroup \( G \) of \( \text{Aut}(T) \) is level transitive if it acts transitively on \( L_n \) for every \( n \), or equivalently if \( G \rhd \partial T \) is minimal. It is almost level transitive if the number of orbits for the actions \( G \rhd L_n \) is bounded, or equivalently if \( G \rhd \partial T \) has a finite number of orbit-closures.

The group \( G \) is said to be weakly rigid if all the \( \text{Rist}_G(v) \) are infinite and rigid if all the \( \text{Rist}_G(L_n) \) have finite index in \( G \).

A level transitive subgroup is said to be weakly branch if it is weakly rigid and branch if it is rigid.

Observe that if \( G \leq \text{Aut}(T) \) is both almost level transitive and rigid (for example, \( G \) branch), then \( T \) is locally finite. In particular, \( T \) is locally finite if and only if the three subgroups \( \text{Aut}_f(T) \), \( \text{Aut}_{fr}(T) \) and \( \text{Aut}(T) \) are all branch.

While in the introduction we stated our results for (weakly) branch groups, most of these statements admit generalizations to groups that are not necessarily level transitive but only almost level transitive. The detailed versions are found in the next sections.

**Remark 2.3.** We will sometimes say that an abstract group \( G \) is (weakly) branch, meaning that it admits a faithful (weakly) branch action. The existence of such an action can be characterized algebraically from the subgroup lattice of \( G \), see [23, 38]. Moreover, some striking examples of branch groups (as the first Grigorchuk group [23] or branch generalized multi-edge spinal groups [27]) admit a unique branch action.
Before going further, we recall the following fact that will greatly help for the study of infinite index subgroups of branch groups.

**Fact 2.4.** Let $G$ be a group acting transitively on a set $X$. For every subgroup $H \leq G$, the number of $H$-orbits for the action $H \rhd X$ is bounded above by $[G : H]$.

While this is stated for transitive action, we will often use it for almost transitive action, that is for actions with a finite number of orbits. In this case, we have that the number of $H$-orbits is bounded above by the number of $G$-orbits times $[G : H]$. As a nice consequence of it, we have the following result.

**Lemma 2.5.** Let $G \leq \text{Aut}(T)$ be an almost level transitive rigid group and $n$ be an integer. Then there exists $m \geq n$ such that for every $v$ of level $n$, $\text{Rist}_G(v)$ acts level transitively on $T_w$ for all $w$ descendant of $v$ of level $m$.

**Proof.** Since the action is almost level transitive, there exists an integer $d$ such that for every $n$, the number of orbits of $G \rhd \mathcal{L}_n$ is bounded above by $d$. Let $v$ be any vertex of level $n$. If $\text{Rist}_G(v)$ does not act level transitively on $T_v$, there exists some level $n_1 > n$ where the restriction of $\text{Rist}_G(v) \rhd T_v$ has at least two orbits. If $\text{Rist}_G(v)$ does not act level transitively on $T_w$ for all descendant $w$ of $v$ of level $n_1$, there exists a level $n_2 > n_1$ where the restriction of $\text{Rist}_G(v) \rhd T_v$ has at least four orbits, and so on. Now, the number of $\text{Rist}_G(\mathcal{L}_n)$-orbits for the action $\text{Rist}_G(\mathcal{L}_n) \rhd \mathcal{L}_m$ is bounded by $[G : \text{Rist}_G(\mathcal{L}_n)] \cdot d$ which is finite. In particular, there exists $m_v$ such that $\text{Rist}_G(\mathcal{L}_n)$, and hence $\text{Rist}_G(v)$, acts level transitively on $T_w$ for all $w$ descendant of $v$ of level $m_v$. We finish the proof by taking $m$ to be the maximum of all the $m_v$ for $v$ of level $n$. \qed

While the proof of the following fact is an easy exercise, it will be of great help to find subgroups that are branch.

**Fact 2.6.** Let $G \leq \text{Aut}(T)$ be a branch group and $H \leq G$ be a finite index subgroup. Then the action of $H$ on $T$ is branch if and only if it is level transitive.

Be careful that even if the action $H \rhd T$ is not branch, the subgroup $H$ may still be branch for some other action.

Let $T$ be a locally finite rooted tree. Every subgroup $G \leq \text{Aut}(T)$ comes with the following 3 natural topologies. The **profinite topology**, where a basis of neighbourhood at 1 is given by subgroups of finite index, the **congruence topology**, where a basis of neighbourhood at 1 is given by $(\text{Stab}_G(n))_{n \in \mathbb{N}}$ and finally the **branch topology** where a basis of neighbourhood at 1 is given by the $(\text{Rist}_G(\mathcal{L}_n))_{n \in \mathbb{N}}$. The corresponding completion are denoted by $\hat{G}$ (profinite completion), $\hat{G}$ (congruence completion, which coincides with the closure of $G$ in the profinite group $\text{Aut}(T)$) and $\hat{G}$ (branch completion). Since $\text{Stab}_G(n)$ is always of finite index and contains $\text{Rist}_G(\mathcal{L}_n)$, we have two natural epimorphisms $\hat{G} \to G$ and $\hat{G} \to G$. If moreover $G$ is rigid, then we have $\hat{G} \to \hat{G} \to G$. Therefore, for branch groups we have three kernels: the **congruence kernel** $\ker(\hat{G} \to G)$, the **branch kernel** $\ker(\hat{G} \to \hat{G})$ and the **rigid kernel** $\ker(\hat{G} \to G)$. A branch group $G$ is said to have the **congruence subgroup property** if the congruence kernel is trivial; this
implies that the branch kernel is also trivial. See [5] for more details on the congruence subgroup property. An important result in this subject, due to Garrido [17], is the fact that for a branch group $G$ the congruence and branch topology are intrinsic properties of $G$ and do not depend on the chosen branch action of $G$ on a tree. In particular, the congruence, branch and rigid kernels are intrinsic properties of $G$.

The congruence subgroup property for branch group has attracted a lot of attentions these last years and was used, among other things, to describe the structure of maximal subgroups in many branch groups. Earlier work from Pervova showed that the first Grigorchuk group, [32], as well as torsion Gupta-Sidki groups, [33], do not have maximal subgroups of infinite index. This result was later generalized to all torsion multi-edge spinal groups [27]. Moreover, if $G$ has all its maximal subgroups of finite index, then the same is true for any group that is abstractly commensurable to $G$, in particular this property passes to subgroups of finite index. One consequence of this result is that if $G$ is a finitely generated branch group without maximal subgroup of infinite index, then every weakly maximal subgroup of $G$ is closed in the profinite topology.

Another interesting property that a branch group may possess is the fact to be just infinite.

Definition 2.7. A group $G$ is just infinite if it is infinite and every of its proper quotient is finite, equivalently if every non-trivial normal subgroup is of finite index.

Building upon the work of Wilson, Grigorchuk showed in [18] that the class of just infinite groups consists of groups of three distinct types that are, roughly speaking, finite powers of a simple group, finite power of an hereditarily just infinite groups (just infinite groups whose finite index subgroups are also just infinite), and the just infinite branch groups.

The following criterion will help to find just infinite subgroups in branch groups.

Lemma 2.8. Let $G \leq \text{Aut}(T)$ be a branch group and $H$ a finite index subgroup. Then $H$ is just infinite if and only if it acts level transitively.

Proof. For any vertex $v$ of level $n$, the subgroup $L := \prod_{w \in H.v} \text{Rist}_H(w)$ is always a normal in $H$. Since $\text{Rist}_H(v) = \text{Rist}_G(v) \cap H$ has finite index in $\text{Rist}_G(v)$, it is not trivial. If $H.v$ is not equal to $L_n$, then $L$ acts trivially on some $T_u$ for $u$ in $L_n \setminus H.v$ and is thus of infinite index in $G$ and also in $H$.

The other direction is Lemma 8.5 of [21].

Finally, we record the following fact about centralizer of elements in a branch group.

Lemma 2.9. Let $G \leq \text{Aut}(T)$ be a branch group with trivial branch kernel. For any $g \neq 1$ in $\text{Aut}(T)$, the centralizer $C_G(g)$ has infinite index in $G$.

Proof. Since $g$ is non-trivial, it moves a vertex $v$. Let $f, h$ be two distinct elements of $\text{Rist}_G(v)$ and $w$ be a descendent of $v$ such that $f(w) \neq h(w)$. We have $gf(w) \neq gh(w)$ while $hg(w) = fg(w) = g(w)$, hence at most one of $f$ or $h$ belongs to $C_G(g)$. As a consequence, $C_G(g)$ contains no rigid stabilizers of levels and is of infinite index. 

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2.1 Some examples of branch groups

The first Grigorchuk group $G$ is probably the most well-known and most studied branch group. This was the first example of a group of intermediate growth, [19].

**Definition 2.10.** The first Grigorchuk group $G = \langle a, b, c, d \rangle$ is the subgroup of $\text{Aut}(T_2)$ generated by $a = (1,1) \varepsilon$ where $\varepsilon$ is the cyclic permutation $(12)$ of the first level and by the three elements $b$, $c$ and $d$ of $\text{Stab}_{\text{Aut}(T)}(L_1)$ that are recursively defined by

\[
b = (a, c) \quad c = (a, d) \quad d = (1, b).
\]

This is a 2-group of rank 3, all generators have order 2 and $b$, $c$ and $d$ pairwise commute. By definition, for every vertex $v$ of the first level we have $\text{Stab}_G(v) = \text{Stab}_G(L_1)$. Moreover, $G$ is branch, self-replicating, just infinite, has the congruence subgroup property and possesses a unique branch action in the sense of [23]. Finally, all maximal subgroups of $G$ have finite index, [32]. See [10, 20] for references and Section 8 for more details.

Other well-studied examples of branch groups are the Gupta-Sidki groups, [26], as well as their generalizations.

**Definition 2.11.** Let $p$ be a prime, $T$ the $p$-regular tree and let $e = (e_0, \ldots, e_{p-2})$ be a vector in $(F_p)^{p-1} \setminus \{0\}$. The GGS group $G_e = \langle a, b \rangle$ with defining vector $e$ is the subgroup of $\text{Aut}(T)$ generated by the two automorphisms

\[
a = (1, \ldots, 1) \cdot \varepsilon \\
b = (a^{e_0}, \ldots, a^{e_{p-2}}, b)
\]

where $\varepsilon$ is the cyclic permutation $(12 \ldots p)$.

The name GGS stands for Grigorchuk-Gupta-Sidki as these groups generalize both the Gupta-Sidki groups (where $e = (1, -1, 0, \ldots, 0)$) and the second Grigorchuk group (where $p = 4$ is not prime and $e = (1, 0, 1)$).

A wide generalization of GGS groups is provided by the *generalised multi-edge spinal groups*, where, for $p$ an odd prime and $T$ the $p$-regular rooted tree,

\[G = \langle \{a\} \cup \{b_i^{(j)}\} | 1 \leq j \leq p, 1 \leq i \leq r_j \rangle \leq \text{Aut}(T_p)\]

is a subgroup of $\text{Aut}(T)$ that is generated by one automorphism $a = (1, \ldots, 1) \varepsilon$ and $p$ families $b_1^{(j)}, \ldots, b_{r_j}^{(j)}$ of directed automorphisms, each family sharing a common directed path disjoint from the paths of the other families. See [27] for a precise definition.

It is shown in [27] that generalized multi-edge spinal groups enjoy a lot of interesting properties. First of all, they always are residually-(finite $p$) group that are self-replicating. By definition, they also satisfy that for every vertex $v$ of the first level we have $\text{Stab}_G(v) = \text{Stab}_G(L_1)$. Moreover, Klopsch and Thillaisundaram showed

**Theorem 2.12** ([27]). Let $G$ be a generalized multi-edge spinal group.

1. If $G$ is branch, then it admits a unique branch action in the sense of [23],
2. If $G$ is torsion, then it is just infinite and branch,

3. If $G$ is torsion, then $G$ does not have a maximal subgroup of infinite index. The same holds for groups commensurable with $G$.

In particular, if $G$ is a torsion generalized multi-edge spinal group, then its maximal subgroups are normal and of index $p$. Since all maximal subgroups are of finite index, then by a result of Myropolska [30], the derived subgroup $G'$ of $G$ is contained in the Frattini subgroup (that is, the intersection of all the maximal subgroups) of $G$.

Some of these groups have the congruence subgroup property, for example torsion GGS groups (Pervova [31]), but this is not always the case. [27].

For the special case of GGS groups, there is an easy criterion, due to Vovkivsky [36], to decide whenever $G$ is torsion. Indeed, a GGS group with defining vector $e$ is torsion if and only if $\sum_{i=0}^{p-2} e_i = 0$.

For GGS groups we will also make use of the following facts, see [13] for example for a proof.

**Proposition 2.13.** Let $G$ be a GGS group a $G'$ be its derived subgroup. Then,

1. $\text{Stab}_G(L_1) = \langle b \rangle^G = \langle b, aba^{-1} \ldots, a^{p-1}ba^{-(p-1)} \rangle$,

2. $G = \langle a \rangle \rtimes \text{Stab}_G(L_1)$,

3. $G/G' = \langle aG', bG' \rangle \cong C_p \times C_p$,

4. $\text{Stab}_G(L_2) \leq G' \leq \text{Stab}_G(L_1)$.

Finally, we will use the existence of a normal form for elements in $\text{Stab}_G(L_1)$. See [16] for the case of Gupta-Sidki groups. This normal form is best described by the use of the circulant matrix

$$\text{Circ}(e, 0) = \begin{pmatrix}
e_0 & e_1 & \ldots & e_{p-2} & 0 \\
e_0 & e_1 & \ldots & e_{p-2} \\
e_{p-2} & 0 & e_1 & \ldots & e_{p-3} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
e_1 & \ldots & e_{p-2} & 0 & e_0
\end{pmatrix},$$

where $e = (e_0, \ldots, e_{p-1})$ is the defining vector of $G$.

The following is a generalization of a similar result for Gupta-Sidki groups that was originally obtained by Garrido in [10].

**Lemma 2.14.** Let $g$ be an element of the GGS group $G$. Then $g$ is in $\text{Stab}_G(L_1)$ if and only if there exists $(n_i)_{i=0}^{p-1}$ in $\mathbb{F}_p$ and $(d_i)_{i=0}^{p-1}$ in $G'$ such that

$$g = (a^{\alpha_0}b^{\beta_0}d_0, \ldots, a^{\alpha_{p-1}}b^{\beta_{p-1}}d_{p-1}) \quad \text{(*)}$$

where

$$(\alpha_0, \ldots, \alpha_{p-1}) = (n_0, \ldots, n_{p-1}) \text{Circ}(e, 0) \quad \text{and} \quad (\beta_0, \ldots, \beta_{p-1}) = (n_0, \ldots, n_{p-1})P$$

with $P$ the matrix representation of the permutation $(12 \ldots p)$. Moreover, when $g$ is in $\text{Stab}_G(L_1)$, the representation (*) is unique.
Proof. It is trivial that if \( g \) is of the form \([g]_1\) then it is in the stabilizer of the first level. Let \( g \) be in \( B = \text{Stab}_G(L_1) = \langle b_0, \ldots, b_{p-1} \rangle \). Using the fact that
\[
B/B' = \{ b_0^{r_0} b_1^{r_1} \cdots b_{p-1}^{r_{p-1}} | r_i \in \mathbb{F}_p \}
\]
there exists some \( n_i \) in \( \mathbb{F}_p \) and \( c \) in \( B' \) with
\[
g = b_0^{n_0} b_1^{n_1} \cdots b_{p-1}^{n_{p-1}} c
\]
\[
= (a_0^{n_0 \epsilon_0} a_1^{n_1 \epsilon_1} \cdots a_{p-1}^{n_{p-1} \epsilon_1}) c
\]
where \((\alpha_0, \ldots, \alpha_{p-1}) = (n_0, \ldots, n_{p-1}) \text{Circ}(e, 0)\) and the \( d_i \) belong to \( G' \).
The \( n_i \) as well as \( c \) are uniquely determined, and therefore so are the \( \alpha_i, \beta_i \) and \( d_i \). \( \square \)

3 Generalized parabolic subgroups

In this section, we study setwise stabilizers of closed subsets in the setting of weakly maximal subgroups. The main motivation for this is Theorem 1.5. Recall that since the action of \( \text{Aut}(T) \) on \( \partial T \) is continuous, if \( C \) is a closed subset of \( \partial T \), then \( \text{SStab}_G(C) \) is closed for the congruence topology. In particular, \( \text{SStab}_G(C) \) is closed for the profinite topology and also closed for the branch topology.

The following definition is motivated by the forthcoming Proposition 3.9.

Definition 3.1. A generalized parabolic subgroup of a group \( G \leq \text{Aut}(T) \) is the setwise stabilizer \( \text{SStab}_G(C) \) of some closed non-open subset \( C \) of \( \partial T \) such that \( \text{SStab}_G(C) \) acts minimally on \( C \).

It directly follows from the definition that if \( \text{SStab}_G(C) \) is a generalized parabolic subgroup, then \( C \) is closed and nowhere dense.

Before going further, recall that the set of (non necessarily spherically regular) subtrees of \( T \) containing the root and without leaf is in bijection with \( C \) the set of non-empty closed subset of \( \partial T \). This bijection is given by \( S \mapsto \partial S \) and \( C \mapsto T_C := \{ v \in T | \exists \xi \in C : v \in \xi \} \) that are \( G \)-equivariant maps.

For every non-empty closed subset \( C \) of \( \partial T \), the action \( \text{SStab}_G(C) \) is minimal if and only if \( \text{SStab}_G(T_C) = \text{SStab}_G(C) \text{Circ}(e, 0) \) is level transitive. In particular, if \( \text{SStab}_G(C) \) is a generalized parabolic subgroup, then \( T_C \subset T \) is a spherically regular rooted tree.

Generalized parabolic subgroups of weakly branch groups retain some of the branch structure. More precisely, let \( W = \text{SStab}_G(C) \) be a generalized parabolic subgroup. Since \( C \) is nowhere dense, for every vertex \( v \) in \( T \), there exists \( w \leq v \) that is not in \( T_C \). But then \( W \) contains \( \text{Rist}_G(w) \) and hence \( \text{Rist}_W(v) \geq \text{Rist}_W(w) = \text{Rist}_G(w) \) is infinite. We just proved

Lemma 3.2. Let \( W = \text{SStab}_G(C) \) be a generalized parabolic subgroup of a weakly rigid group \( G \). Then, for every vertex \( v \) the rigid stabilizer \( \text{Rist}_W(v) \) is infinite, that is \( W \) is weakly rigid.

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Example 3.3. Let \((m_i)_{i \in \mathbb{N}}\) be a sequence of integers greater than 1 and \(T = T_{(m_i)}\) the corresponding spherically regular rooted tree. Let \(G\) be one of the group \(\text{Aut}(T)\), \(\text{Aut}_t(T)\) or \(\text{Aut}_t(T)\). Let \((n_i)_{i \in \mathbb{N}}\) be a sequence such that \(1 \leq n_i \leq m_i\) for all \(i\) and such that \(n_i < m_i\) for infinitely many \(i\). For any \(S \subseteq T\) spherically regular subtree that is isomorphic to \(T_{(n_i)}\), the subgroup \(\text{SSstab}_G(\partial S)\) is generalized parabolic. Moreover, all generalized parabolic subgroups of \(G\) arise in this form.

Indeed, as we have seen, the condition that \(S\) is spherically regular is always necessary for \(\text{SSstab}_G(S) = \text{SSstab}_G(\partial S)\) to acts minimally on \(\partial S\). On the other hand, for \(G = \text{Aut}_t(T)\) (and also for \(G = \text{Aut}_t(T)\) or \(\text{Aut}(T)\)), it is also a sufficient condition. Finally, the condition that infinitely many \(n_i\) are strictly less than \(m_i\) is equivalent to the fact that \(\partial S\) is nowhere dense.

A corollary of this example is that if \(G \leq \text{Aut}(T)\) with \(T\) locally finite, then \(G\) has at most a continuum of generalized parabolic subgroups, even if \(G\) itself has the cardinality of the continuum.

We observe that setwise stabilizers are always infinite.

Lemma 3.4. Let \(G \leq \text{Aut}(T)\) be a weakly rigid group. Then, for every closed subset \(C \subseteq \partial T\), \(\text{SSstab}_G(C)\) is infinite.

Proof. If \(C = \partial T\), then \(\text{SSstab}_G(C) = G\) is infinite.

On the other hand, if \(C \neq \partial T\), then \(\partial T \setminus C\) is a non-empty open set an therefore has at least one interior point \(\xi = (v_i)_{i \geq 1}\). Then there is \(i\) such that \(C\) contains no rays passing through \(v_i\). This implies that \(\text{SSstab}_G(C)\) contains \(\text{Rist}_G(v_i)\) which is infinite.

While being infinite, setwise stabilizers of non-open closed subsets of \(\partial T\) are still "small" subgroups in the sense that they are of infinite index and have infinitely many orbit-closures.

Lemma 3.5. Let \(G\) be any subgroup of \(\text{Aut}(T)\) and \(C\) be a non-open closed subset of \(\partial T\). Then the number of orbit-closures for the action \(\text{SSstab}_G(C) \rtimes \partial T\) is infinite.

Moreover, if \(G\) is almost level transitive, then \(\text{SSstab}_G(C)\) has infinite index in \(G\).

Proof. For \(\xi = (v_i)_{i \geq 1}\) in \(\partial T\) let \(d(\xi, C)\) be the distance in \(\partial T\) between \(\xi\) and \(C\). If \(\xi\) and \(\eta\) are in the same \(\text{SSstab}_G(C)\)-orbit, then \(d(\xi, C) = d(\eta, C)\). Moreover, if \(\xi_i\) is a sequence converging to \(\eta\), then the \(d(\xi_i, C)\) converge to \(d(\eta, C)\). Therefore, in order to prove the infinity of orbit-closures of \(\text{SSstab}_G(C) \rtimes \partial T\), it is enough to find infinitely many \(\xi\) with distinct \(d(\xi, C)\).

Since \(C\) is not open, there exists \(\eta\) in \(C\) that is not an interior point. That is, there exists \((\xi_i)\) such that \(d(\xi_i, C)\) is a strictly decreasing sequence.

Moreover, the infinity of orbit-closures on \(\partial T\) is equivalent to the fact that the number of orbits of \(\text{SSstab}_G(C)\) on \(\mathcal{L}_n\) is not bounded. If \(G\) acts almost level transitively on \(T\), Fact 2.4 implies that \(\text{SSstab}_G(C)\) has infinite index.

We are now able to prove that generalized parabolic subgroups in branch groups are pairwise distinct.
Lemma 3.6. Let \( G \leq \text{Aut}(T) \) be an almost level transitive rigid group. Let \( C_1 \neq C_2 \) be two distinct closed nowhere dense subsets of \( \partial T \). Then \( \text{Stab}_G(C_1) \neq \text{Stab}_G(C_2) \).

Proof. Let \( \xi = (v_i)_{i \geq 1} \) be in \( C_1 \) but not in \( C_2 \). Since \( C_2 \) is closed, there exists \( i \) such that the intersection of \( C_2 \) with \( \partial T_{v_i} \) is empty. This implies that \( \text{Stab}_G(C_2) \) contains \( \text{Rist}_G(v_i) \). On the other hand, we claim that if \( G \) is almost level transitive and rigid, then \( \text{Stab}_G(C_1) \) does not contain \( \text{Rist}_G(v_i) \). Indeed, if \( \text{Stab}_G(C_1) \) contains \( \text{Rist}_G(v_i) \), then by Lemma 2.5, \( C \) would contain a small neighbourhood of \( \xi \), which is impossible. \( \square \)

Observe that while \( G \) weakly branch is sufficient for showing that parabolic subgroups are distinct (Proposition 1.1), we suppose in Lemma 3.6 that \( G \) is branch. However, a similar result holds for weakly branch groups if we restrict a little bit which kind of closed subsets we are allowed to look at. Recall that \( \text{Acc}(C) \) is the set of accumulation points of \( C \). For example, \( (C_1 \Delta C_2) \setminus (\text{Acc}(C_1) \cup \text{Acc}(C_2)) \) is non-empty as soon as \( C_1 \neq C_2 \) are both finite.

Lemma 3.7. Let \( G \leq \text{Aut}(T) \) be an almost level transitive weakly rigid group. Let \( C_1 \neq C_2 \) be two distinct closed subsets of \( \partial T \) such that \( (C_1 \Delta C_2) \setminus (\text{Acc}(C_1) \cup \text{Acc}(C_2)) \) is non-empty. Then \( \text{Stab}_G(C_1) \neq \text{Stab}_G(C_2) \).

Proof. Let \( \xi \) be in \( (C_1 \Delta C_2) \setminus (\text{Acc}(C_1) \cup \text{Acc}(C_2)) \). For example, \( \xi \) in \( C_1 \). Thus \( \xi \) is not in \( C_2 \) and since \( C_2 \) is closed there exists \( v \) a vertex of \( \xi \) such that \( \text{Stab}_G(C_2) \) contains \( \text{Rist}_G(v) \). On the other hand, \( \text{Stab}_G(C_1) \) does not contain \( \text{Rist}_G(v) \) as otherwise \( C_1 \) would contains \( \text{Rist}_G(v) \). \( \xi \) which, by Lemma 2.5 implies that \( \xi \) is an accumulation point of \( C_1 \). \( \square \)

Lemma 3.8. Let \( T \) be locally finite, \( G \leq \text{Aut}(T) \) be a rigid group and \( C \) be a closed subset of \( \partial T \). If \( C \) is open, then \( \text{Stab}_G(C) \) has finite index in \( G \).

Proof. By hypothesis, \( C \) is a clopen subset of \( \partial T \). Since \( T \) is locally finite, \( \partial T \) is compact and so is \( C \). On the other hand, since \( C \) is open, it is a union of \( \partial T_v \) and thus \( C = \bigcup_{i=1}^{\infty} \partial T_{v_i} \).

Let \( m \) be the maximal level of the \( v_i \)'s and \( v \) any vertex of level \( m \). Either \( v \) is not in \( T_C \) and hence \( \text{Rist}_G(v) \) is contained in \( \text{Stab}_G(C) \), or \( v \) is in \( T_C \) and hence below some \( v_i \). But in this case \( \text{Rist}_G(v_i) \) and hence \( \text{Rist}_G(v) \) is also contained in \( \text{Stab}_G(C) \). Therefore, \( \text{Stab}_G(C) \) contains the rigid stabilizer of the \( m \)th level and is of finite index. \( \square \)

Using Lemmas 3.5 and 3.8 we obtain.

Proposition 3.9. Let \( G \leq \text{Aut}(T) \) be an almost level transitive rigid group and \( C \) be a closed subset of \( \partial T \).

Then \( \text{Stab}_G(C) \) is generalized parabolic if and only if it is weakly maximal.

Proof. If \( \text{Stab}_G(C) \) is weakly maximal, it is of infinite index and hence \( C \) is not open. Let \( \xi \) be any boundary point of \( C \) and let \( C' \) be the closure of the \( \text{Stab}_G(C) \)-orbit of \( \xi \). Then \( C' \subseteq C \) is closed but not open, and \( \text{Stab}_G(C) \leq \text{Stab}_G(C') \) which is of
infinite index. Therefore they are equal and since SStab\(_G(C')\) acts minimally on \(C'\) it is a generalized parabolic subgroup.

On the other hand, let \(C'\) be a non-open closed subset of \(\partial T\) such that the action SStab\(_G(C)\) \(\cdot\) \(C'\) is minimal. Since \(C\) is not open, SStab\(_G(C)\) is an infinite index subgroup of \(G\). Moreover, it contains Rist\(_G(v)\) for all \(v\) outside \(T_C\). Now, let us take \(g\) not in SStab\(_G(C)\). There exists \(v \notin T_C\) such that \(g.v\) is in \(T_C\). In particular, \(\langle g, SStab\(_G(C)\) \rangle\) contains Rist\(_G(g.v)\) and by transitivity also Rist\(_G(w)\) for all \(w\) in \(T_C\) of the same level as \(v\). That is, \(\langle g, SStab\(_G(C)\) \rangle\) contains a rigid stabilizer of a level and is of finite index. \(\square\)

Together with Lemmas 3.4 and 3.6 we obtain the following corollary, which is a strong version of Theorem 1.5.

**Corollary 3.10.** Let \(G\) be a subgroup of Aut\(_T\).

If \(G\) is weakly rigid, then all generalized parabolic subgroups are infinite.

If \(G\) is almost level transitive and rigid, then all generalized parabolic subgroups are weakly maximal and pairwise distinct.

We now have all ingredients to prove generalizations of Theorem 1.6, Corollary 1.7 and the first part of Corollary 1.8.

**Proposition 3.11.** Let \(G \leq\) Aut\(_T\) be an almost level transitive rigid group and \(F \leq G\) be any subgroup. Let \(C_F\) be the set of all non-open orbit-closures of the action \(F \acts \partial T\). Then the function SStab\(_G(C)\) \(\cdot\) \(C_F\) is injective and has values in generalized parabolic subgroups, which are all weakly maximal.

**Proof.** For any \(C\) in \(C_F\), \(F\) is a subgroup of SStab\(_G(C)\) and acts minimally on \(C\). Therefore, SStab\(_G(C)\) is a generalized parabolic subgroup that is weakly maximal.

On the other hand, by Lemma 3.6, if \(C_1 \neq C_2\), the subgroups SStab\(_G(C_1)\) and SStab\(_G(C_2)\) are distinct. \(\square\)

**Corollary 3.12.** Let \(G \leq\) Aut\(_T\) be an almost level transitive rigid group and \(F \leq G\) be any subgroup. Let \(v\) be a vertex of \(T\).

1. If all orbit-closures of Stab\(_F(v)\) \(\acts \partial T\) are non-open, then \(F\) is contained in uncountably many generalized parabolic subgroup that are all weakly maximal,

2. If all orbit-closures of Stab\(_F(v)\) \(\acts \partial T\) are at most countable, then \(F\) is contained in a continuum of generalized parabolic subgroup. If moreover Stab\(_F(v)\) is non-trivial, then \(F\) is contained in a continuum of generalized parabolic subgroups that are not parabolic.

**Proof.** For \(F\) a subgroup of \(G\) and \(v\) a vertex of \(T\), let \(C_{F,v}\) be the set of non-open orbit-closures of Stab\(_F(v)\) \(\acts \partial T\). In particular, \(C_{F,\emptyset} = C_F\) with the notation of Proposition 3.11.

We claim that for any \(F\) and \(v\), the map \(\varphi: C_{F,v} \to\) Sub\(_G\) sending a Stab\(_F(v)\) orbit-closure \(C\) onto SStab\(_G(F,C)\) is injective and has values onto generalized parabolic subgroups containing \(F\). Indeed, \(\varphi\) is the composition of \(\psi: C_{F,v} \to C_F\) sending \(C\) onto \(F.C\) an the map SStab\(_G(\cdot): C_F \to\) Sub\(_G\). By Proposition 3.11, SStab\(_G(\cdot)\) is injective.
and has values in generalized parabolic subgroups. It remains to show that \( \psi \) is well-defined (i.e., that \( F.C \) is closed, non-open and \( F \) acts minimally on it) and injective. But this follows directly from the topology of \( \partial T \).

Suppose now that all \( \text{Stab}_F(v) \) orbit-closures are not open, and hence have empty-interior. In particular, for any ray \( \xi \) passing through \( v \), the subgroup \( \text{Stab}_G(F.\xi) \) is generalized parabolic. Since \( \partial T \) is a Cantor space, it is also a Baire space. That is, the union of countably many closed sets with empty interior has empty interior. In particular, \( \partial T_v \) cannot be covered by a countable union of \( \text{Stab}_F(v) \)-orbit-closures.

If all \( \text{Stab}_F(v) \) orbit-closures are countable, they are never open and there must be a continuum of them to cover \( \partial T_v \). Suppose moreover that \( \text{Stab}_F(v) \) is not trivial. Then there is a descendant \( w \) of \( v \) that is moved by \( \text{Stab}_F(v) \). For any ray \( \xi \) going through \( w \), \( F.\xi \) is at most countable and contains at least 2 elements. Therefore, all the \( \text{Stab}_G(F.\xi) \) for \( \xi \) passing through \( w \) are generalized parabolic subgroups that are not parabolic and there is a continuum of them by the last point.

By taking \( F = \langle g \rangle \) with \( g \) a non-trivial element of finite order, we have

**Corollary 3.13.** Let \( G \) be an almost level transitive rigid group. If \( G \) is not torsion-free, it contains a continuum of generalized parabolic subgroups that are not parabolic.

Observe that unlike the result of [8], Corollary 3.13 does not require the existence of a finite subgroup which a specific action on the tree, but requires only the existence of a non-trivial element of finite order — a fact that is independent of the chosen branch action. Moreover, by Proposition 3.11, the conclusion of Corollary 3.13 holds as soon as \( G \) has a subgroup \( F \) such that there is a continuum of orbit-closures for the action \( F \acts \partial T \) that are neither open nor reduced to a point. In fact, if the cardinality of \( G \) is strictly less than \( 2^{\aleph_0} \), this is also a necessary condition. In particular, if \( G \) is a countable branch group that does not have a continuum of generalized parabolic subgroups that are not parabolic, then for every \( 1 \neq g \) in \( G \), almost all (that is except for a countable number) orbit-closures of \( Z \cong \langle g \rangle \acts \partial T \) are either open or reduced to a point. A fact that seems unlikely. This led us to ask the following

**Question 3.14.** Let \( G \leq \text{Aut}(T) \) be a countable branch group. Does \( G \) have a continuum of generalized parabolic subgroups that are not parabolic?

Finally, we provide some examples of generalized parabolic subgroups. By definition, parabolic subgroups, that is stabilizers of rays, are generalized parabolic. If \( \xi \) is a ray of \( T \) and \( g \in G \) is an element of finite order, then \( C = \langle g \rangle.\xi \) is finite and hence \( \text{Stab}_G(C) \) is generalized parabolic. More generally, if \( g \) is such that \( \text{Stab}_G(v) \) does not act level transitively on \( T_v \) for every \( v \), then \( C = \langle g \rangle.\xi \) is nowhere dense (and closed) and thus \( \text{Stab}_G(C) \) is generalized parabolic. In general, more complicated situations may happen. Let \( T \) be the \( d \)-regular rooted tree. Recall that a subgroup \( G \leq \text{Aut}(T) \) is a *regular branch group over \( K \)* if \( G \) is level transitive, self-similar and \( K \) is a finite index subgroup of \( G \) such that \( K^d \) is contained in \( \text{Stab}_K(L_1) \) as a finite index subgroup.
Example 3.15. Let $G$ be a regular branch group over $K$. If $G$ is torsion, then there exists a continuum of generalized parabolic subgroups of $G$ of the form $\text{Stab}_G(C)$ where $C$ is a nowhere dense Cantor subset of $\partial T$.

To construct such a subgroup start with some $k \in K \setminus \text{Stab}_G(0^\infty)$. Such a $k$ always exists since $K$ has finite index while $\text{Stab}_G(0^\infty)$ has infinite index in $G$. Since $G$ is regular branch over $K$, for every vertex $v$ of $T$, there exists an element $k_v$ of $\text{Rist}_G(v)$ that acts on $T_v$ as $k$ on $T$. Let $(m_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of integers and $C := \langle k_{0^{m_n}} \mid n \in \mathbb{N} \rangle.0^\infty$. The sequence of rays $k_{0^{m_n}}.0^\infty$ is in $C \setminus \{0^\infty\}$ and converges to $0^\infty$, hence $C$ has no isolated points and is a Cantor space. It remains to show that for some sequence $(m_n)$ the subset $C$ is not open. Since $k$ has finite order, it does not act level transitively on $T$. Therefore, there exists an integer $i$ and a vertex $v_i = x_0 \ldots x_i$ such that $v_i$ is not in the $\langle k \rangle$ orbit of $0^i$. By taking $(n \cdot i)$ for the sequence $(m_n)$, we have that none of the vertices $0^{n(i-1)}x_0 \ldots x_i$ are on a ray of the $\langle k_0^{m_n} \mid n \in \mathbb{N} \rangle$ orbit of $0^\infty$. This proves that $C$ is not open. If instead of taking the sequence $(n \cdot i)$ we take a subsequence of it, we still end with a $C$ that is not open. Two different subsequences give raises to distinct subsets of $\partial T$ and there is a continuum of such sequences.

Finally, we study the rank of generalized parabolic subgroups. The aim is to answer the following question.

Question 3.16. Which condition on $G$ ensures that generalized parabolic subgroups are not finitely generated.

A first answer comes from Francoeur’s thesis [14] where he proved that in many weakly branch self-similar groups, parabolic subgroups are never finitely generated.

Theorem 3.17 ([14]). Let $T$ be a locally finite regular rooted tree and let $G \leq \text{Aut}(T)$ be a weakly branch group. If there exists $N \in \mathbb{N}$ such that for any $v \in T$, $\text{Rist}_{\pi_v(G)}(\mathcal{L}_1)$ acts non-trivially on level $N$, then parabolic subgroups are not finitely generated.

For groups with the subgroup induction property, see Definition 6.3, we have a stronger result.

Proposition 3.18. Let $G \leq \text{Aut}(T)$ be a self-replicating branch group with the subgroup induction property. Then generalized parabolic subgroups of $G$ are not finitely generated.

Proof. In [24] it is shown that if $G$ is a self-replicating branch group with the subgroup induction property, and $H$ is a finitely generated subgroup of $G$, then there exists a transversal $X$ of $T$ and a finite index subgroup $L$ of $H$ such that the sections of $\text{Stab}_L(X)$ along $X$ are either trivial or equal to $G$. If $W$ is a generalized parabolic subgroup, any transversal contains at least one vertex $v$ of $\text{NR}(W)$. But then, by Lemma 5.3, $\pi_v(W)$ is a weakly maximal subgroup of $G$ and hence an infinite proper subgroup. This implies that if $L$ is a finite index subgroup of $W$, the section of $\text{Stab}_L(X)$ at $v$ is neither trivial, nor equal to $G$. □

Proposition 3.18 will only be use once in the rest of the text, namely in the proof of Theorem 6.10. Therefore the use of Lemma 5.3 in its proof does not create a circular argument.
Finally, in order to decide if generalized parabolic subgroups are finitely generated, it may be useful to look at specific subgroups of them. Recall that for \( C \subset \partial T \) closed, \( T_C \) denotes the vertices of \( T \) above \( C \). In particular, \( T_C \cap \mathcal{L}_n \) is what is called sometimes the shadow of \( C \) on level \( n \).

**Definition 3.19.** Let \( C \) be a closed subset of \( \partial T \). The *neighbourhood stabiliser* of \( C \) is

\[
\text{SSStab}_G(C)^0 := \{ g \in \text{Stab}_G(C) \mid \exists n \forall v \in \mathcal{L}_n \cap T_C \text{ the portrait of } g \text{ is trivial on } T_v \}
\]

This is a normal subgroup of \( \text{Stab}_G(C) \).

If \( C \) is a ray \( \xi \), this coincides with the usual definition of the neighbourhood stabiliser, as the set of elements that fixes a small neighbourhood of \( \xi \). If we fix \( n \) in the above definition, we obtain a subgroup \( \text{SSStab}_G(C)^0_n \) with the property that \( \text{SSStab}_G(C)^0_n \) is the increasing union of the \( \text{SSStab}_G(C)^0_n \). If the sequence \( \text{SSStab}_G(C)^0_n \) is not eventually constant, then \( \text{SSStab}_G(C)^0 \) is not finitely generated. But this is the case if \( W \) is a generalized parabolic subgroup. Indeed, \( \text{Rist}_W(v) \) is never trivial, while \( \text{Rist}_{\text{SSStab}_G(C)^0_n}(v) = \{1\} \) for \( v \) in \( \mathcal{L}_n \cap T_C \). We just proved

**Lemma 3.20.** Let \( G \leq \text{Aut}(T) \) be a weakly rigid group and let \( W = \text{SSStab}_G(C) \) be a generalized parabolic subgroup. If \( \text{SSStab}_G(C)/\text{SSStab}_G(C)^0 \) is finite, then \( W \) is not finitely generated.

When \( C = \{\xi\} \), the group \( \text{SSStab}_G(C)/\text{SSStab}_G(C)^0 \) is known as the group of germs of \( G \) at \( \xi \). A ray \( \xi \) is said to be regular if the group of germs is trivial and singular otherwise. This definition extends mutatis mutandis to closed subset. We hence have

**Corollary 3.21.** Let \( G \leq \text{Aut}(T) \) be a weakly rigid group and let \( W = \text{SSStab}_G(C) \) be a generalized parabolic subgroup with \( C \) regular. Then \( W \) is not finitely generated.

### 4 Non-rigidity tree

An important tool to study a (weakly maximal) subgroup \( H \) of a branch group \( G \) acting on \( T \) is the knowledge of the vertices \( v \) of \( T \) such that a finite index subgroup of \( \text{Rist}_G(v) \) is contained in \( H \). This motivates the following definition.

**Definition 4.1.** Let \( G \) be a subgroup of \( \text{Aut}(T) \) and \( H \) be a subgroup of \( G \). The *non-rigidity tree* of \( H \), written \( \text{NR}(H) \) is the subgraph of \( T \) generated by all \( v \) such that \( \text{Rist}_H(v) \) has infinite index in \( \langle \text{Rist}_G(v) \rangle \).

It directly follows from the definition that \( v \) is in \( \text{NR}(H) \) if and only if \( H \) does not contain a finite index subgroup of \( \text{Rist}_G(v) \).

The following useful lemma describes the behaviour of subgroups under the addition of some rigid stabilizer.

**Lemma 4.2.** Let \( T \) be a locally finite rooted tree. Let \( G \leq \text{Aut}(T) \) be a rigid group and let \( H \) be a subgroup of \( G \). If \( v \) is a vertex of \( T \) with children \( \{w_1, \ldots, w_d\} \) such that \( H \) contains all the \( \text{Rist}_G(w_i) \), then \( H \) has finite index in \( \langle H, \text{Rist}_G(v) \rangle \).
Proof. Let $H' = \langle H, \text{Rist}_G(v) \rangle$. We have $H = \langle \text{Rist}_G(w_1) \times \cdots \times \text{Rist}_G(w_d) \rangle^H \cdot H$ while $H' = \langle H, \text{Rist}_G(v) \rangle = \langle \text{Rist}_G(v) \rangle^H \cdot H$. In particular, the index of $H$ in $H'$ is bounded by the index of $\langle \text{Rist}_G(w_1) \times \cdots \times \text{Rist}_G(w_d) \rangle^H = \prod_{w \in H, \{w_1, \ldots, w_d\}} \text{Rist}(w)$ in $\langle \text{Rist}_G(v) \rangle^H = \prod_{w \in H, v} \text{Rist}(v)$. This latter index is itself bounded by $[\text{Rist}_G(v) : \text{Rist}_G(w_1) \times \cdots \times \text{Rist}_G(w_d)]^{\langle H, v \rangle}$ which is finite. \hfill $\Box$

Corollary 4.3. Let $T$ be a locally finite rooted tree. Let $G \leq \text{Aut}(T)$ be a rigid group and $W$ a weakly maximal subgroup of $G$. If $v$ is a vertex of $T$ with children $\{w_1, \ldots, w_d\}$ such that $W$ contains all the $\text{Rist}_G(w_i)$, then $W$ contains $\text{Rist}_G(v)$.

The following lemma justifies the name and shows some elementary properties of non-rigidity trees.

Lemma 4.4. Let $G \leq \text{Aut}(T)$ and let $H$ be a subgroup of $G$. The non-rigidity tree of $H$ is a tree. Moreover, it is non-empty if and only if $H$ has infinite index in $G$. In this case,

1. $\text{NR}(H)$ is a rooted subtree of $T$, rooted at $\emptyset$ (the root of $T$).
2. If $H \leq K$, then $\text{NR}(K) \subseteq \text{NR}(H)$, with equality if $H$ has finite index in $K$,
3. Suppose that $T$ is locally finite and that $G$ is rigid. If $W$ is weakly maximal subgroup, then $\text{NR}(W)$ has no leaf.

Proof. Let $v$ be a vertex outside $\text{NR}(H)$ and $w$ be one of its descendent. By hypothesis, $\text{Rist}_H(v)$ has finite index in $\text{Rist}_G(v)$ which implies that $\text{Rist}_H(w)$ has finite index in $\text{Rist}_G(w)$ and hence that $w$ is not in $\text{NR}(H)$. That is, $\text{NR}(H)$ is always a tree. It is empty if and only if it does not contain the root, that is if $H$ has finite index in $G$.

The second assertion is a direct consequence of the definition while the third assertion follows from Corollary 4.3. \hfill $\Box$

For groups with trivial branch kernel we also have an alternative description of the non-rigidity tree of weakly maximal subgroups.

Lemma 4.5. Let $G$ be a branch group with trivial branch kernel, $W$ be a weakly maximal subgroup of $G$ and let $v$ be a vertex of $T$. Then $W$ contains $\text{Rist}_G(v)$ if and only if $v$ is not in $\text{NR}(W)$.

Proof. The only if direction is trivial. For the other direction, suppose that $\text{Rist}_W(v)$ has finite index in $\text{Rist}_G(v)$. Then, by the trivial branch kernel property, there exists an integer $n$ such that $W$ contains $\text{Rist}_G(w)$ for every $w$ in $L_n \cap T_v$. By using multiple times Corollary 4.3 we obtain that $W$ contains $\text{Rist}_G(v)$. \hfill $\Box$

For every subgroup $H$ we have $H \leq \text{SStab}_G(\text{NR}(H)) = \text{SStab}_G(\partial \text{NR}(H))$, with $\partial \text{NR}(H)$ being a closed subset of $\partial T$. If $W$ is a generalized parabolic subgroup of $G$, then $\partial \text{NR}(H)$ is not open. As an application of Proposition 3.9 we obtain
Lemma 4.6. Let $G \leq \text{Aut}(T)$ be an almost level transitive rigid group and $W$ be a weakly maximal subgroup. Then $W = \text{SStab}_G(\text{NR}(W))$ if and only if $W$ is generalized parabolic.

More precisely, the maps $W \mapsto \partial \text{NR}(W)$ and $C \mapsto \text{SStab}_G(C)$ are inverse bijections from the set of generalized parabolic subgroups of $G$ and the set of closed non-open subset of $\partial T$ such that the action $\text{SStab}_G(C) \ltimes C$ is minimal.

We obtain the following characterization of generalized parabolic subgroups.

Corollary 4.7. Let $G$ be an almost level transitive rigid group and $W \leq G$ be a weakly maximal subgroup. Then, the followings are equivalent

1. $W$ is a generalized parabolic subgroup,
2. $\text{SStab}_G(\text{NR}(W))$ is of infinite index,
3. $\partial \text{NR}(W)$ is not open,
4. The number of orbit-closures of the action $W \curvearrowright \partial T$ is infinite.

Proof. Observe that $\text{SStab}_G(\text{NR}(W)) = \text{SStab}_G(\partial \text{NR}(W))$. Lemmas 3.5 and 3.8 imply the equivalence of Properties 2 and 3.

Lemma 4.6 and the fact that $W \leq \text{SStab}_G(\text{NR}(W))$ imply the equivalence of the first two properties.

It remains to show that a weakly maximal subgroup is generalized parabolic if and only if the number of orbit-closures of the action $W \curvearrowright \partial T$ is infinite. The left-to-right implication is Lemma 3.5. Finally, suppose that $W$ is not generalized parabolic. We claim that in this case every-orbit-closure $C$ of the action $W \curvearrowright \partial T$ is open. Indeed, by definition $W$ is contained in $\text{SStab}_G(C)$ and acts minimally on $C$. If $C$ was not open, then by Lemma 3.5, $\text{SStab}_G(C)$ would have infinite index in $G$ and hence equal to $W$, which is absurd. Since all orbit-closures are open, by compacity of $\partial T$ there are only a finite number of them.

By Lemma 4.6, generalized parabolic subgroups act minimally on the boundary of their non-rigidity tree. We conjecture that this fact generalizes to every weakly maximal subgroup.

Conjecture 4.8. Let $G$ be a branch group and let $W \leq G$ be a weakly maximal subgroup. Then the action $W \curvearrowright \partial \text{NR}(W)$ is minimal.

Remark 4.9. As a consequence of Lemma 4.6, distinct generalized parabolic subgroups have distinct non-rigidity trees. This does not generalize to distinct weakly maximal subgroups. Indeed, if $W$ is a weakly maximal subgroup then for every $g$ in $\text{SStab}_G(\partial \text{NR}(W))$, the non-rigidity tree of $W^g$ coincide with the one for $W$. But if $W$ is not generalized parabolic, then $\text{SStab}_G(\partial \text{NR}(W))$ is a finite index subgroup of $G$, while if $G$ is torsion $W$ is self-normalizing by [8]. That is, in a torsion branch group, weakly maximal subgroups that are not generalized parabolic admit infinitely many distinct conjugates with the same non-rigidity tree.
This remark raises the following question.

**Question 4.10.** Let $G$ be a branch group and $W$ and $M$ two weakly maximal subgroups with the same non-rigidity tree $\text{NR}(W) = \text{NR}(M)$. Does this necessarily imply that $W$ and $M$ are conjugated?

If $W$ is level transitive, then it contains no rigid stabilizer and hence $\text{NR}(W) = T$.

**Question 4.11.** Let $G$ be a branch group and $W$ be a weakly maximal subgroup. Is it true that $W$ is level transitive if and only if $\text{NR}(W) = T$?

Observe that if Conjecture 4.8 is true, then it implies a positive answer to Question 4.11.

We have the following alternative for the action of $W$ onto $\text{NR}(W)$, which shows that possible counterexamples to Conjecture 4.8 can arise only among a special kind of weakly maximal subgroups.

**Lemma 4.12.** Let $G$ be any subgroup of $\text{Aut}(T)$ and $W$ be a weakly maximal subgroup of $G$. For every integer $n$, if $W$ does not act transitively on $\text{NR}(W) \cap \mathcal{L}_n$, then for every $v$ of level $n$ the subgroup $\pi_n(\text{Stab}_W(v))$ has finite index in $\pi_n(\text{Stab}_G(v))$.

**Proof.** Let $v$ be any vertex of $\text{NR}(W) \cap \mathcal{L}_n$ and $W' := \langle W, \text{Rist}_G(v) \rangle$. This is a finite index subgroup of $G$, and hence for every $u$ of level $n$ the subgroup $H_u = \pi_n(\text{Stab}_{W'}(u))$ has finite index in $\pi_n(\text{Stab}_G(u))$. Now, let $\{v_\alpha\}_{\alpha \in I}$ be the $W$-orbit of $v$. We have $W' = W \prod_{\alpha \in I} \text{Rist}_G(v_\alpha)$. In particular, for every vertex $u$ of level $n$ and for every $g$ in $\text{Stab}_{W'}(u)$ there exists $w \in W$ and $g_\alpha$ in $\text{Rist}_G(v_\alpha)$ (with all the $g_\alpha$ trivial except for a finite number) such that $g = w \prod_{\alpha \in I} g_\alpha$. Equivalently, we have $w = g \prod_{\alpha \in I} g_\alpha^{-1}$. If $u$ was not in the $W$-orbit of $v$, then $w^u_\alpha = g^u_\alpha$ which implies that $\pi_n(\text{Stab}_{W'}(u)) = H_u$.

Finally, if the action of $W$ on $\text{NR}(W) \cap \mathcal{L}_n$ is not transitive, then for every $v$ of level $n$ there exists $u$ of level $n$ with $u$ not in the $W$-orbit of $v$ and we are done. \qed

As a corollary, we prove Lemma 1.4

**Corollary 4.13.** Let $T$ be a non necessarily locally finite tree and $G \leq \text{Aut}(T)$ be a weakly rigid group. Then every weakly maximal subgroup of $G$ is infinite.

**Proof.** If $\text{NR}(W)$ is not equal to $T$, then $W$ contains a finite index subgroup of some $\text{Rist}_G(v)$ and is hence infinite. If $\text{NR}(W) = T$ and $W$ acts level transitively on it, then it is infinite. Finally, the last case is $\text{NR}(W) = T$ but $W$ does not act level transitively on it. By the last lemma, $W$ contains a subgroup that projects onto some infinite group and we are done. \qed

In the first part of Corollary 1.8 we showed that a branch group $G$ with a torsion element has a continuum of generalized parabolic subgroups that are not parabolic. If $G$ is countable, then there is still a continuum of such subgroups, even up to conjugation or up to $\text{Aut}(G)$. Equivalence of subgroups up to conjugation, or up to $\text{Aut}(G)$ were already considered in [8]. The problem with these two equivalence relations is that each class contains at most countably elements and hence parabolic subgroups split into many distinct classes. This motivates the following definition.
**Definition 4.14.** Let $G$ be a subgroup of $\text{Aut}(T)$. Two weakly maximal subgroups $W_1$ and $W_2$ of $G$ are said to be *tree-equivalent* if there is an automorphism of $T$ sending $\text{NR}(W_1)$ to $\text{NR}(W_2)$.

Observe that this notion a priori depends on the chosen action.

It is possible to consider other equivalence relations on the set of weakly maximal subgroups, for example the fact that $\text{NR}(W_1)$ and $\text{NR}(W_2)$ are conjugated by an element of the profinite completion of $G$, or by an element of the normalizer of $G$ in $\text{Aut}(T)$. Among these possibilities, being tree-equivalent is the coarser equivalence relation, that is the one with the biggest classes.

It follows from Lemma 3.5 that

**Lemma 4.15.** Let $G \leq \text{Aut}(T)$ be an almost level transitive group and $W$ be a weakly maximal subgroup of $G$. Then $W$ is tree-equivalent to a parabolic, respectively generalized parabolic, subgroup if and only if $W$ is parabolic, respectively generalized parabolic.

**Example 4.16.** Let $(m_i)_{i \in \mathbb{N}}$ be a sequence of integers greater than 1 and $T = T_{(m_i)}$ the corresponding spherically regular rooted tree. Let $G$ be one of the group $\text{Aut}(T)$, $\text{Aut}_f(T)$ or $\text{Aut}_r(T)$. Then tree-equivalence classes of generalized parabolic subgroups of $G$ are in bijection with sequences $(n_i)_{i \in \mathbb{N}}$ of integers such that $1 \leq n_i \leq m_i$ for all $i$ and such that $n_i < m_i$ for infinitely many $i$. In particular, there is a continuum of such classes, each containing a continuum of distinct subgroups.

Indeed, this is the last lemma applied to Example 3.3. Lemma 3.6 ensures that each equivalence class contains a continuum of subgroups.

The above example shows that for $\text{Aut}(T)$, $\text{Aut}_f(T)$ and $\text{Aut}_r(T)$ we have “a lot” of “big” classes of generalized parabolic subgroups. But these groups are not finitely generated. Corollary 1.8 asserts that we still have this kind of result for more general groups if we allow either “a lot” or “big” to be weakened a little and means infinitely many instead of a continuum.

In order to finish the proof of Corollary 1.8 we will use the fact that in torsion groups weakly maximal subgroups are self-normalizing, [8]; that is, they are equal to their normalizer.

We now show this slight generalization of Corollary 1.8.

**Proposition 4.17.** Let $G \leq \text{Aut}(T)$ be an almost level transitive rigid group.

1. If $G$ is not torsion-free, it contains a continuum of generalized parabolic subgroups that are not parabolic.

2. If $G$ has elements of arbitrarily high finite order, there are infinitely many tree-equivalence classes of generalized parabolic subgroups that each contains a continuum of subgroups.

3. If $G$ is torsion, there is a continuum of tree-equivalence classes of generalized parabolic subgroups that each contains infinitely many subgroups.
Proof. The first part is Corollary 3.13. Let \( (g_i)_{i=1}^n \) be a sequence of elements of \( G \) with finite but increasing order. Let \( C_i \) be an orbit of maximal size for \( \langle g_i \rangle \rhd \partial T \). Up to extracting a subsequence, \( (|C_i|)_i \) is a strictly increasing sequence of finite numbers. All the \( \text{Stab}_G(C_i) \) are generalized parabolic subgroups and since the \( C_i \) have pairwise distinct cardinalities, the \( \text{Stab}_G(C_i) \) are in distinct tree-equivalence classes.

On the other hand, since \( C_i \) is finite, there exists a vertex \( v_i \) of \( T \) such that \( \langle g_i \rangle \{v_i\} \) contains \( |C_i| \) elements. Let \( \{v_{i,j}\}_{j=1}^{|C_i|} \) be these elements. By maximality, for every ray \( \xi \) passing through \( v_i \), its orbit under \( \langle g_i \rangle \) consists of exactly one ray under each \( v_{i,j} \) for \( 1 \leq j \leq C_i \). That is, for every \( \xi \) passing through \( v_i \), the subgroup \( \text{Stab}_G(\langle g_i \rangle) \cdot \{\xi\} \) is a generalized parabolic subgroup that is tree-equivalent to \( \text{Stab}_G(C_i) \). There is a continuum of such \( \xi \) giving raise to pairwise distinct subgroups by Lemma 3.6. This finishes the proof of the second part of the proposition.

Suppose now that \( G \) is torsion. Then weakly maximal subgroups are self-normalizing. In consequence, any weakly maximal subgroup of \( G \) has infinitely many conjugates, which are all tree-equivalent.

Now, let \( \xi = (v_i)_i \) be a ray in \( \partial T \). By Lemma 2.5 there exists an element \( g_1 \) in \( \text{Rist}_G(v_1) \) that moves \( \xi \). Since this element is torsion, the orbit \( \langle g_1 \rangle \cdot \xi \) is finite. In particular, there exists \( i_2 \) such that \( \text{Stab}_{\langle g_1 \rangle}(v) \) acts trivially on \( T_v \) for every \( v \) of level \( i_2 \) in the \( \langle g_1 \rangle \)-orbit of \( \xi \). Let \( g_2 \) be an element of \( \text{Rist}_G(v_{i_2+1}) \) moving \( \xi \). Again, \( g_2 \) being torsion, there exists \( i_3 \) such that \( \text{Stab}_{\langle g_2 \rangle}(v) \) acts trivially on \( T_v \) for every \( v \) of level \( i_3 \) in the \( \langle g_2 \rangle \)-orbit of \( \xi \). Since \( \langle g_1, g_2 \rangle = \langle g_1 \rangle \cdot \langle g_2 \rangle \) and the fact that \( g_2 \) is in \( \text{Rist}_G(v_{i_2+1}) \), we obtain that \( \text{Stab}_{\langle g_1, g_2 \rangle}(v) \) acts trivially on \( T_v \) for every \( v \) of level \( i_3 \) in the \( \langle g_1, g_2 \rangle \)-orbit of \( \xi \). By induction, we obtain a sequence of integers \( (i_j)_{j \geq 1}, i_1 = 1 \), and a sequence of elements \( (g_j) \) of \( G \) such that \( g_j \) belongs to \( \text{Rist}_G(v_{i_j+1}) \), moves \( \xi \) and such that the subgroup \( \text{Stab}_{\langle g_1, \ldots, g_j \rangle}(v) \) acts trivially on \( T_v \) for every \( v \) of level \( i_{j+1} \) in the \( \langle g_1, \ldots, g_j \rangle \)-orbit of \( \xi \).

Let \( b = (b_j)_{j \geq 1} \) be a binary sequence. We associate to it the subset \( C_b := \langle g_j, \cdot \rangle \cdot \xi \) of \( \partial T \) and the subgroup \( W_b := \text{Stab}_G(C_b) \). We claim that all the \( W_b \) are generalized parabolic subgroups of \( G \) and that if \( b \neq b' \), then \( W_b \) and \( W_{b'} \) are not tree-equivalent. Since there is a continuum of binary sequences, the claim implies the last part of the proposition.

By definition, \( C_b \) is closed and \( W_b \) acts minimally on it. In order to prove that \( W_b \) is generalized parabolic, it remains to show that \( C_b \) is not open. For every \( v_{i_j+1} \) and every sibling \( w \) of \( v_{i_j+1} \), no element of \( \partial T_w \) is in \( \langle g_j, \cdot \rangle \cdot \xi \). This implies that \( \langle g_j, \cdot \rangle \cdot \xi \) does not contain \( \partial T_v \) for every \( v \) and so neither does \( C_b \). Since \( \xi \) is in \( C_b \), this proves that this subset of \( \partial T \) is not open.

Finally, \( W_b \) and \( W_{b'} \) are not tree-equivalent if and only if there is no automorphism of \( T \) sending \( C_b \) onto \( C_{b'} \). Let \( k \) be the smallest integer such that \( b_k \neq b'_k \). By construction, the cardinality of \( L_{i_{k+1}} \cap C_b \cdot \xi \) is equal to the cardinality of \( L_{i_{k+1}} \cap \langle g_1^{b_1}, \ldots, g_k^{b_k} \rangle \cdot \xi \). That is, we have \( |L_{i_{k+1}} \cap C_b \cdot \xi| \neq |L_{i_{k+1}} \cap C_{b'} \cdot \xi| \) which implies that \( C_b \) cannot be send by an automorphism of \( T \) onto \( C_{b'} \).

\[ \square \]
5 Sections of weakly maximal subgroups

The aim of this section is to better understand sections of weakly maximal subgroups. We will also introduce an operation of “lifting” subgroups of $\pi_v(G)$ to subgroups of $G$ and show that, for weakly maximal subgroup, it is the inverse of the section operation.

Before going further, we insist on the fact that when we write $\pi_v(G)$, this will always be a shorthand for $\pi_v(\text{Stab}_G(v))$.

First of all, we remark that sections behave nicely with respect to the properties of being branch and just infinite.

**Lemma 5.1.** Let $P$ be any property of the following list:

- almost level transitive, level transitive, weakly rigid, rigid, just infinite and branch.

Let $G$ be a subgroup of $\text{Aut}(T)$. If $G$ has $P$, then for every vertex $v$ the group $\pi_v(G) \leq \text{Aut}(T_v)$ has also $P$.

**Proof.** It is obvious that sections of (almost) level transitive groups are (almost) level transitive. Similarly, sections of (weakly) rigid groups are also (weakly) rigid and hence sections of branch groups are branch.

By Grigorchuk [13], a branch group $G$ is just infinite if and only if the derived subgroup $\text{Rist}_G(v)'$ has finite index in $\text{Rist}_G(v)$ for every vertex $v$.

Suppose now that $G$ is branch and just infinite. Let $w$ be a vertex of $T_v$. Then $\text{Rist}_G(w)'$ has finite index in $\text{Rist}_G(w)$ by hypothesis. Therefore $[\pi_v(\text{Rist}_G(w)) : \pi_v(\text{Rist}_G(w)')]$ is finite. We have $\pi_v(\text{Rist}_G(w)) \leq \text{Rist}_{\pi_v(G)}(w)$ and hence also $\pi_v(\text{Rist}_G(w)') \leq \text{Rist}_{\pi_v(G)}(w)'$. All it remains to do is to show that $\pi_v(\text{Rist}_G(w))$ has finite index in $\text{Rist}_{\pi_v(G)}(w)$. Let $n$ be the level of $v$ and $d$ the index of $\text{Rist}_G(\mathcal{L}_n)$ in $G$. Then $\pi_v(\text{Rist}_G(w))$ is equal to $\pi_v(\text{Rist}_{\text{Rist}_G(\mathcal{L}_n)}(w)) = \text{Rist}_{\pi_v(\text{Rist}_G(\mathcal{L}_n))}(w)$ has index at most $d$ in $\text{Rist}_{\pi_v(G)}(w)$. \qed

**Lemma 5.2.** Let $G$ be a rigid subgroup of $\text{Aut}(T)$. Let $W$ be a weakly maximal subgroup of $G$ that is contained in the first level stabilizer. Then at most one of the first level sections of $G$ if of infinite index.

More precisely, let $\{v_\alpha\}_{\alpha \in I}$ be the collection of the first level vertices. If $[\pi_{v_\alpha}(G) : \pi_{v_\alpha}(W)]$ is of infinite index for some $\alpha$, then $\pi_{v_\alpha}(W)$ is a weakly maximal subgroup of $\pi_{v_\alpha}(G)$ and for $\beta \neq \alpha$ the section $\pi_{v_\beta}(\text{Rist}_G(v_\beta))$ contains the section $\pi_{v_\alpha}(\text{Rist}_G(v_\beta))$.

**Proof.** Suppose that $\pi_{v_\alpha}(W) := \pi_{v_\alpha}(W)$ is of infinite index in $\pi_{v_\alpha}(G)$. If it were not weakly maximal, then we would have an infinite index subgroup $L$ of $\pi_{v_\alpha}(G)$ with $\pi_{v_\alpha}(W) < L$. For $h \in L \setminus \pi_{v_\alpha}(W)$, there exists $g \in \text{Stab}_G(v_\alpha)$ such that $g$ projects onto $h$. Then we have $W < W' := \langle W, g \rangle \leq \text{Stab}_G(v_\alpha)$. On the other hand, $\pi_{v_\alpha}(W') \leq L$ is of infinite index in $\pi_{v_\alpha}(G)$ which implies that $W'$ is itself of infinite index in $G$, which is absurd.

On the other hand, let $h_\beta$ be in $\pi_{v_\beta}(\text{Rist}_G(v_\beta))$ and let $g_\beta$ be the only element of $\text{Rist}_G(v_\beta)$ projecting to $h_\beta$. Let $W' := \langle W, g_\beta \rangle$. We will show that $W'$ is equal to $W$. Since $W$ is weakly maximal, it is enough to show that $W'$ is of infinite index. But the section of $W'$ at $v_\alpha$ is equal to the section of $W$, and hence of infinite index, which finishes the proof. \qed
For generalized parabolic subgroups, we even have a characterization of the non-rigidity tree in terms of sections.

**Lemma 5.3.** Let $G \leq \text{Aut}(T)$ be an almost level transitive rigid group. Let $W$ be a generalized parabolic subgroup of $G$. Then $\pi_v(W)$ has infinite index in $\pi_v(G)$ if and only if $v$ is in $\text{NR}(W)$. Moreover, if $v$ is in $\text{NR}(W)$, then $\pi_v(W)$ is a weakly maximal subgroup of $\pi_v(G)$.

**Proof.** If $v$ is not in $\text{NR}(W)$, then $\text{Rist}_W(v)$ has finite index in $\text{Rist}_G(v)$. In this case, $\pi_v(W)$ contains $\pi_v(\text{Rist}_W(v))$ which has finite index in $\pi_v(\text{Rist}_G(v))$ which itself has finite index in $\pi_v(G)$ by rigidity of $G$. On the other hand, if $v$ is in $\text{NR}(W)$, then $\pi_v(W)$ is equal to $\text{Stab}_{\pi_v(G)}(C \cap \partial T_v)$ that is weakly maximal by Proposition 3.9. Indeed, since $G$ is almost level transitive and rigid, so is $\pi_v(G)$, while $C$ closed and nowhere dense implies the same properties for $C \cap T_v$ and $\pi_v(W)$ acts minimally on $C \cap T_v$ since $W$ does so on $C$. 

The first Grigorchuk group as well as the torsion GGS groups behave particularly well with respect to sections. This is the content of Lemma 5.4 as well as of Corollary 5.6.

The following lemma is due to Dominik Francoeur and plays a key role in the proof of the assertion 7 of Theorem 6.10. We are grateful for his help. Its proof uses the description made by Grigorchuk and Wilson in [22] of subgroups of the first Grigorchuk group $\mathfrak{G}$.

**Lemma 5.4 (Francoeur).** Let $H \leq \mathfrak{G}$ be a subgroup acting level transitively on $T$. Then there exists a vertex $v$ such that $\pi_v(H) = \mathfrak{G}$.

**Proof.** Suppose that this is not the case. Then, for all $v \in T$, we have $\pi_v(H) \neq \mathfrak{G}$. As $\mathfrak{G}'$ is the Frattini subgroup of $\mathfrak{G}$, we must also have that $\pi_v(H) \mathfrak{G}' \neq \mathfrak{G}$. It follows from Lemma 6 of [22] that if $\pi_w(H) \mathfrak{G}' \leq \langle a,x \rangle \mathfrak{G}'$ for some $x \in \{b,c,d\}$, then there exists some $w \leq v$ such that $\pi_w(H) \leq \text{Stab}_{\mathfrak{G}}(\mathcal{L}_1)$. This is of course absurd, since $H$ acts level transitively on $T$. We conclude that $\pi_v(H) \mathfrak{G}'$ must be either $\langle b,ad \rangle \mathfrak{G}'$, $\langle c,ab \rangle \mathfrak{G}'$ or $\langle d,ac \rangle \mathfrak{G}'$. Then, by Lemma 5 of [22], we have that $\pi_v(H)$ is actually equal to $\langle b,ad \rangle$, $\langle c,ab \rangle$ or $\langle d,ac \rangle$ respectively. It is easy to check that the sections of the stabilisers of each of these groups is eventually $\mathfrak{G}$, which contradicts our hypothesis. 

The following lemma is a generalization to all torsion GGS groups of a result that was originally obtained by Garrido, [16], for Gupta-Sidki groups.

**Lemma 5.5.** Let $G$ be a torsion GGS group and $H$ be a subgroup of $G$ that is not contained in $\text{Stab}_G(\mathcal{L}_1)$. Then either all first level sections of $H$ are equal to $G$, or they are all contained in $\text{Stab}_G(\mathcal{L}_1)$ so that $\text{Stab}_H(\mathcal{L}_1) = \text{Stab}_H(\mathcal{L}_2)$.

**Proof.** Denote by $H_0 = \pi_0(H), \ldots, H_{p-1}$ the first level sections of $H$. Since $H$ does not fix pointwise the first level, they are all conjugated in $G$ and we may thus assume that no $H_i$ is equal to $G$. Being conjugated in $G$, all the $H_i$ have the same image modulo $G'$, the derived subgroup of $G$. The possibilities for these are $\langle ab^k \rangle$ for $k \in \mathbb{F}_p$, or $\langle b \rangle$ or $\langle a \rangle \langle b \rangle$ or $\{1\}$. 

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If $H_iG' = G'$, then $H_i \leq G' \leq \text{Stab}_G(L_1)$ and $\text{Stab}_H(L_1) = \text{Stab}_H(L_2)$ as desired.

We claim that $H_i = G$ if $H_iG' = \langle a \rangle \langle b \rangle G'$. Suppose not, then $H_i$ is contained in some maximal subgroup $M < G$. Then $M \geq G'$ and $G = \langle a \rangle \langle b \rangle G' = H_iG' \leq MG' = M$, a contradiction.

So either $H_iG' = \langle b \rangle G'$ for all $i$, or there exists $k \in F_p$ such that $H_iG' = \langle ab^k \rangle G'$ for all $i$. Let $h = (h_0, \ldots, h_{p-1})$ be an element of $\text{Stab}_H(L_1)$. If there exist $i \neq j$ such that $h_iG'$ and $h_jG'$ lie in different cyclic subgroups of $G/G'$ then, conjugating by suitable elements of $H \setminus \text{Stab}_H(L_1)$ we would obtain $H_iG' = H_jG' = \langle a \rangle \langle b \rangle G'$, a contradiction to the above. Thus all $h_iG'$ lie in the same cyclic subgroup of $G/G'$ and we need to show that this cyclic subgroup is $\langle b \rangle G'$. Suppose for a contradiction that there is some $k$ in $F_p \setminus \{0\}$ and some $r_0, \ldots, r_{p-1}$ in $F_p$ such that $$h_0G', \ldots, h_{p-1}G' = ((ab^k)^{r_0}G', \ldots, (ab^k)^{r_{p-1}}G').$$

By Lemma 2.14 there exists $n_i$ in $F_p$ such that $$(n_0, \ldots, n_{p-1})C = (r_0, \ldots, r_{p-1}) \quad \text{and} \quad (n_0, \ldots, n_{p-1})P = k(r_0, \ldots, r_{p-1})$$

where $P$ is the permutation matrix associated to $(12 \ldots p)$ and $C$ is the circulant matrix $\text{Circ}(e_0, \ldots, e_{p-2}, 0)$. These equations are equivalent to $$(n_0, \ldots, n_{p-1})(kCP^{-1} - \text{Id}) = (0, \ldots, 0).$$

By [13], if $D = \text{Circ}(a_0, \ldots, a_{p-1})$ is a circulant matrix with entries in $F_p$, then $D$ is invertible as soon as $\sum_{i=0}^{p-1} a_i \neq 0$. Since $G$ is of torsion we have $\sum_i e_i = 0$ which implies that the matrix $kCP^{-1} - \text{Id} = \text{Circ}(ke_1 - 1, ke_2, \ldots, ke_{p-2}, 0, ke_0)$ is invertible by the above criterion. Therefore, $(n_0, \ldots, n_{p-1}) = (0, \ldots, 0)$ is the only solution to our equation and all the $h_i$ are in $G'$, which finishes the proof.

This directly implies

**Corollary 5.6.** Let $G$ be a torsion GGS group and $H \leq G$ be a subgroup acting level transitively on $T$. Then, for every vertex $v$ of the first level, $\pi_v(H) = G$.

Theorem 1.6 is a result about weakly maximal subgroups containing a given "small" subgroup of $G$. We now turn our attention to weakly maximal subgroups contained in a given "big" subgroup.

**Definition 5.7.** Let $G$ be a subgroup of Aut($T$), $v$ a vertex of $T$ and $H$ a subgroup of $\pi_v(G)$. The subgroup $H^v$ is defined by $$H^v := \{ g \in \text{Stab}_G(v) \mid g|_{_L} \in H \}.$$ This is the biggest subgroup of Stab$_G(v)$ such that $\pi_v(H^v) = H$.

Observe that $H^v$ contains Rist$_G(u)$ for every $u$ not in $T_v$ and that the rank of $H$ is at most the rank of $H^v$. 

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Since \( \pi_{d,v}(G) = \pi_v(G)^g \), if \( v \) and \( w \) are of the same level and \( G \) is level transitive, the subgroups \( \pi_v(G) \) and \( \pi_w(G) \) are conjugated and hence abstractly isomorphic. We can define a partial map

\[
\eta: T \times \{ H \leq \pi_v(G) \mid v \in T \} \to \{ H \leq G \}
\]

that is defined for couples \( (v, H) \) with \( H \leq \pi_v(G) \) and in this case has value \( H^v \). Observe that if \( G \) is self-replicating (that is, \( \pi_v(G) = G \) for every vertex), then the domain of definition of \( \eta \) is exactly \( T \times \{ H \leq G \} \).

The main application of the following proposition is for self-replicating branch group in which case it implies [1, 9] Nevertheless it can also be useful for self-replicating families of groups.

**Proposition 5.8.** Let \( G \leq \text{Aut}(T) \) be a rigid group, \( v \) a vertex of \( T \) of level \( n \) and \( H \) a subgroup of \( \pi_v(G) \). Then

1. \( [G : H^v] \) is finite if and only if \( [\pi_v(G) : H] \) is finite. More precisely, \( [\pi_v(G) : H] \leq [G : H^v] \leq [\pi_v(G) : H] \cdot [G : \text{Rist}_G(n)] \).
2. \( H^v \) is weakly maximal if and only if \( H \) is weakly maximal,
3. If \( G \) is finitely generated, then \( H^v \) is finitely generated if and only if \( H \) is finitely generated,
4. For every \( v \in T \), the function \( \eta(v, \cdot) \) is injective on \( \{ H \leq \pi_v(G) \} \),
5. If \( W \) is a weakly maximal subgroup with no fix-point except for the root, then \( \eta(\cdot, W) \) is injective on its domain of definition,
6. If \( W \) is a weakly maximal subgroup with no fix-point except for the root, then for every \( g \in G \), when defined, \( W^v = g(W^u)g^{-1} \) if and only if \( g.u = v \),
7. \( \pi_v(H^v) = H \) and \( (\pi_v(H))^v \geq H \). Moreover, if \( H \) is weakly maximal and \( v \) is the unique vertex of level \( n \) in \( \text{NR}(H) \), then \( (\pi_v(H))^v = H \).

**Proof.** We begin by proving a claim that directly implies the first assertion of the proposition, but will also be useful for the rest of the proof. Let \( K \) be a subgroup of \( G \) such that \( \pi_v(K) = H \) and \( K \) contains \( \text{Rist}_G(w) \) for every \( w \in \mathcal{L}_n \setminus \{ v \} \). We claim that \( [G : K] \) is finite if and only if \( [\pi_v(G) : H] \) is finite and that in this case \( [\pi_v(G) : H] \leq [G : K] \leq [\pi_v(G) : H] \cdot [G : \text{Rist}_G(n)] \).

Since \( \pi_v(K) = H \), we have \( \pi_v(K) \leq \pi_v(G) \). On the other hand, let \( d = [G : \text{Rist}_G(n)] \) and \( d' = [K : K \cap \text{Rist}_G(n)] \). Since \( G \) is rigid, both \( d \) and \( d' \) are finite and \( [G : K] = \frac{d}{d'} [\text{Rist}_G(n) : K \cap \text{Rist}_G(n)] \). The assumption on \( K \) implies that \( [\text{Rist}_G(n) : K \cap \text{Rist}_G(n)] = [\pi_v(\text{Rist}_G(n)) : \pi_v(K \cap \text{Rist}_G(n))] \leq [\pi_v(G) : \pi_v(K \cap \text{Rist}_G(n))] \). Finally, \( [\pi_v(K) : \pi_v(K \cap \text{Rist}_G(n))] \leq d' \). Altogether, this gives us

\[
[G : K] \leq d \cdot \frac{\pi_v(K \cap \text{Rist}_G(n))}{\pi_v(K \cap \text{Rist}_G(n))} = d \cdot [\pi_v(G) : \pi_v(K)] = d \cdot [\pi_v(G) : H].
\]
We now prove that $H^v$ is weakly maximal if $H$ is. Let $g$ be an element of $G$ that is not in $H^v$. If $g$ is not in $\text{Stab}_G(v)$, then $(g,H^v)$ contains $\text{Rist}_G(L_n)$ and is of finite index. If $g$ is in $\text{Stab}_G(v)$, the subgroup $(g,H^v)$ stabilizes $v$ and thus $\pi_v((g,H^v)) = \langle g^v,H^v \rangle$ is a finite index subgroup of $\pi_v(G)$. By the above claim, this implies that $\langle g, H^v \rangle$ is a finite index subgroup of $G$, that is that $H^v$ is weakly maximal.

Suppose now that $H^v$ is a weakly maximal subgroup of $G$ and let $g$ be in $\pi_v(G) \setminus H$. Then there exists $f \in \text{Stab}_G(v) \setminus H^v$ such that $f^v_v = g$. Therefore, the subgroup $\langle H^v, f \rangle$ is of finite index in $G$ and so is $\pi_v(\langle H^v, f \rangle) = \langle H, g \rangle$.

We already know that $\text{rank}(H) \leq \text{rank}(H^v)$. It remains to show that if $H$ is finitely generated, then so is $H^v$. For that, we turn our attention to the homomorphism

$$\pi_v : \text{Stab}_G(v) \to \pi_v(G) \quad g \mapsto g^v_v$$

and more particularly at its kernel. By definition, $\ker(\pi_v)$ is the set of all elements $g$ in $\text{Stab}_G(v)$ acting trivially on $T_v$. Let $R := \prod_{v \neq u \in L_n} \text{Rist}_G(w)$. Then $\ker(\pi_v) \cap \text{Rist}_G(L_n) = R$. Since $R$ is a quotient of $\text{Rist}_G(L_n)$, a finite index subgroup of $G$, $R$ is finitely generated if $G$ is. On the other hand, since $\text{Rist}_G(L_n)$ is a finite index subgroup, $R$ is a finitely generated subgroup of finite index in $\ker(\pi_v)$. Therefore, $\ker(\pi_v)$ is finitely generated. We now look at the restriction of $\pi_v$ to $H^v$. We have $\pi_v|_{H^v} : H^v \to H$ and $\ker(\pi_v|_{H^v}) = \ker(\pi_v) \cap H^v$. But $\ker(\pi_v)$ is contained in $H^v$, and therefore $\ker(\pi_v|_{H^v}) = \ker(\pi_v)$ is finitely generated. Since $\pi_v|_{H^v}$ is onto $H$, this implies that if $H$ is finitely generated, so is $H^v$.

It follows from $\pi_v(H^v) = H$, that $H^v = K^v$ if and only if $H = K$.

On the other hand, let $W$ be a weakly maximal subgroup and $M := \langle W^v, W^u \rangle$ for $u \neq v$ two vertices of $T$. Then $M$ contains every $\text{Rist}_G(w)$ for $w$ not in $T_v$, but also for every $w$ not in $T_u$. That is, if $u$ and $v$ are incomparable in $T$ ($v$ is not a descendant of $u$ nor $u$ is a descendant of $v$), $M$ contains a rigid stabilizer of a level and is of finite index. In particular, $W^v \neq W^u$. It remains to treat the case where $u \in T_v$ (the case $v \in T_u$ is done by symmetry). But then, $M$ contains all $\text{Rist}_G(w)$ for $w \neq u$ of the same level of $u$, and also $\text{Rist}_G(u)$ since the action of $W^v$ on $T_v$ is the same as the action of $W$ and hence moves $u$.

It is clear that $g(W^u)g^{-1} = W^{g.u}$. The other direction follows from the injectivity of $\eta(\cdot,W)$.

Finally, it follows from the definitions that $\pi_v(H^v) = H$ and $(\pi_v(H))^v \geq H$. Suppose now that $H$ is weakly maximal and that $v$ is the unique vertex of level $n$ in $\text{NR}(H)$. Since $H$ projects onto $\pi_v(H)$, we may use the claim of the beginning of the proof. Applying this to the pairs $(H, \pi_v(H))$ and $((\pi_v(H))^v, \pi_v(H))$ we obtain that $(\pi_v(H))^v$ is an infinite index subgroup of $G$ containing the weakly maximal subgroup $H$. They are thus equal.

As a direct corollary, we obtain this generalization of Theorem 1.9.

**Corollary 5.9.** Let $G$ be a self-replicating rigid group. Then for any weakly maximal subgroup $W$ of $G$ and any vertex $v$ of $G$, there exists a weakly maximal subgroup $W^v$ of $G$ that is contained in $\text{Stab}_G(v)$ and with $\pi_v(W^v) = W$. 

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More generally, one useful application of Proposition 5.8 is the possibility to take a weakly maximal subgroup $W$ with non-rigidity tree $S$ and to cut or add some trunk to $S$ to obtain a new subtree $\tilde{S}$ in $T$ and to automatically have a weakly maximal subgroup $\tilde{W}$ with $\text{NR}(\tilde{W}) = \tilde{S}$.

The procedure applies to a family of self-replicating groups, but it is easier to describe it for a single self-replicating rigid group $G$. The trunk of a rooted subtree $S$ of $T$ is the maximal subtree $B$ of $S$ containing the root and such that every vertex of $B$ has exactly 1 children. The trunk is equal to the whole subtree $S$ if and only if $S$ is a ray, otherwise it is finite (possibly empty). The crown $C$ of the subtree $S$ is $S$ minus its trunk. Now, let $W$ be any weakly maximal subgroup of $G$, and let $B$ and $C$ be respectively the trunk and the crown of $\text{NR}(W)$. Since adding or removing an initial segment to a ray still produces a ray and that parabolic subgroups are always weakly maximal, we may suppose that $C$ is not empty. Let $v$ be the vertex of $C$ of smallest level. Then $\pi_v(W)$ is a weakly maximal subgroup of $\pi_v(G) = G$ with empty trunk and with crown isomorphic to $C$ (via the identification of $T$ and $T_v$). Now, for every vertex $w$ in $T$, the subgroup $(\pi_v(W))_w$ is a weakly maximal subgroup of $G$ with trunk the unique segment from the root to $w$ and with crown isomorphic to $C$ (via the identification of $T_w$ and $T_v$).

6 Block subgroups

Another interesting subgroups of groups acting on rooted trees are subgroups with block structure. The definition first appeared in [25] and recently it was proved in [24] that for the first Grigorchuk group and the Gupta-Sidki 3-group they coincide with the finitely generated subgroups.

**Definition 6.1.** Let $G$ be group and $I$ a set. We say that a subgroup $H$ of $G^I$ is diagonal if there exists automorphisms $(\varphi_i)_{i \in I}$ of $G$ such that

$$H = \text{diag}(\prod_{i \in I} \varphi_i(G)) := \{(\psi_i(g))_{i \in I} \mid g \in G\}.$$  

**Definition 6.2.** Let $G$ be a subgroup of $\text{Aut}(T)$. A subgroup $H$ of $G$ has a block structure, if there exists a transversal $X$ of $T$, a partition of $X$ onto subsets $X_\alpha$ and subgroups $D_\alpha$ of $\prod_{v \in X_\alpha} \text{Rist}_G(v)$ such that

1. If $X_\alpha$ contains only one element, then $D_\alpha$ is either trivial or a finite index subgroup of $\text{Rist}_G(v)$,

2. If $X_\alpha$ has at least two elements, then there exists a group $L$ and embeddings $L \hookrightarrow \text{Rist}_G(v)$ as finite index subgroup for every $v \in X_\alpha$ such that the subgroup $D_\alpha$ is a diagonal subgroup of $L^{X_\alpha}$,

3. $H$ contains $\prod_\alpha D_\alpha$ as a finite index subgroup.

The subgroup $D_\alpha$ is said to be a trivial block if it is trivial, a full block if $|X_\alpha| = 1$ and $D_\alpha$ has finite index in $G_\alpha$ and a diagonal block if $X_\alpha$ as at least two elements.
Figure 1: A subgroup of $\mathfrak{S}$ with block structure. The transversal is $\{000, 001, 01, 1\}$. There is one full block at 1, one trivial block at 01 and one diagonal block on $\{000, 001\}$.

See Figure 1 for an example. Every finite index subgroup $H$ of $G$ has a block structure, by simply taking $n = 0$. It follows from the definition that if $T$ is locally finite and $G \leq \text{Aut}(T)$ is a finitely generated rigid group, then every $H \leq G$ with a block structure is finitely generated. It is natural to ask whether the converse is true.

The following definition appears to be central in this context.

**Definition 6.3.** Let $G \leq \text{Aut}(T)$ be a self-similar group. A class $\mathcal{X}$ of subgroups of $G$ is said to be **inductive** if

1. Both $\{1\}$ and $G$ belong to $\mathcal{X}$,
2. If $H \in \mathcal{X}$, then $L \in \mathcal{X}$ for any $L \leq G$ for which $H$ is a finite index subgroup of $L$,
3. If $H$ is a finitely generated subgroup of $\text{Stab}_G(1)$ and all first level sections of $H$ are in $\mathcal{X}$, then $H \in \mathcal{X}$.

The group $G$ has the **subgroup induction property** if for any inductive class of subgroups $\mathcal{X}$, each finitely generated subgroup of $G$ is contained in $\mathcal{X}$.

In [24] it is shown that if $G$ is a self-replicating, just infinite branch group with the subgroup induction property, then every finitely generated subgroup of $G$ has a block structure. Until now, the only known examples of such groups are the first Grigorchuk group, [22], and the Gupta–Sidki 3-group, [16].

On the other hand, in rigid groups, subgroups with a block structure behave nicely with respect to the topology.

**Lemma 6.4.** Let $G \leq \text{Aut}(T)$ be a self-replicating rigid group with the congruence subgroup property. Then every subgroup with a block structure is closed for the profinite topology.

*Proof.* Since every transversal is finite, it is enough to prove that all the trivial blocks, the full blocks and the diagonal blocks are closed in the profinite topology. The trivial blocks are obviously closed.

If $L$ is a finite index subgroup of $G$, it is closed for the profinite topology and hence for the congruence topology. But then, full blocks as well as diagonal blocks on $L$ are obviously closed in the congruence topology and hence also in the profinite topology.  $\Box$
Observe that if $G$ is not self-replicating, then Lemma 6.4 holds under the additional assumption that all the sections $\pi_v(\text{Stab}_G(v))$ have the congruence subgroup property.

Some weakly maximal subgroups have a block structure, but not every subgroup with a block structure is weakly maximal. For example, if $G$ is rigid, a block subgroup with no trivial blocks and no diagonal blocks is of finite index. As a consequence, if $W$ is a weakly maximal subgroup with a block structure there exists $v$ such that $\text{Rist}_W(v)$ is trivial. We hence obtain

**Lemma 6.5.** Let $G$ be an almost level transitive rigid group. Let $W$ be a weakly maximal subgroup of $G$ with block structure. Then $W$ is not generalized parabolic, has no trivial block and at least one diagonal block. Moreover, there exists $v$ such that $\text{Rist}_W(v)$ is trivial.

**Proof.** We already know that $W$ has at least one trivial or one diagonal block and that it has one trivial branch stabilizer.

If $W$ is generalized parabolic, then by Lemma 3.2 its rigid stabilizers are all infinite and therefore $W$ does not have a block structure.

If $W$ is not generalized parabolic, by Corollary 4.7 it acts on $\partial T$ with finitely many orbit-closures. The same remains true for every $\text{Stab}_W(L_n)$. In particular, $W$ has no trivial blocks. \hfill $\square$

**Corollary 6.6.** Let $G$ be an almost level transitive rigid group. Let $W$ be a weakly maximal subgroup of $G$ with block structure. Then there exists $n$ such that for every vertex $v$ of level $n$, the section $\pi_v(W)$ has finite index in $\pi_v(G)$.

We have seen that generalized parabolic subgroups $W$ are characterized, among weakly maximal subgroup, by the fact that the action $W \curvearrowright \partial T$ has infinitely many orbit-closures. Moreover, they have all their rigid stabilizers infinite and for every $n$ there exists a vertex $v$ of level $n$ with $\pi_v(W)$ of infinite index in $\pi_v(G)$. In view of the last lemma and corollary, we may ask if it is possible to have a similar characterization of subgroups with a block structure. The first step in this direction is

**Proposition 6.7.** Let $G \leq \text{Aut}(T)$ be a branch group that is just infinite, self-replicating and such that for every vertex of the first level $\text{Stab}_G(v) = \text{Stab}_G(L_1)$.

Let $H$ be a subgroup of $G$. Suppose that there exists $n$ such that for all vertex $v$ of level $n$, the section $\pi_v(H)$ has finite index in $\pi_v(G) = G$. Then $H$ has a block structure.

**Proof.** By [24], if $A \leq \prod_{i=1}^n G_i$ is a subdirect product with all the $G_i$ just infinite and not virtually abelian, then $A$ has a block structure. Therefore, it is sufficient to find a transversal $X$ such that the sections of $\text{Stab}_H(X)$ have finite index in $G$ and such that for every $v \in X$, the section $\pi_v(\text{Stab}_H(X))$ acts level transitively on $T_v$. Indeed, in this case, $\text{Stab}_H(X)$ is a finite index subgroup of $H$ and a subdirect product of the $\pi_v(\text{Stab}_H(X))$. On the other hand, $\pi_v(\text{Stab}_H(X))$ has finite index in $G$ and is therefore not virtually abelian since $G$, being branch, is not. Since $\pi_v(\text{Stab}_H(X))$ acts level transitively on $T_v$, it is just infinite by Lemma 2.8.
Since there exists a transversal $S = \mathcal{L}_n$ such that all the sections of $H$ along $S$ have finite index, there exists a transversal $X \leq S$ such that all the sections of $H$ along $X$ have finite index and acts level transitively on the corresponding subtrees. We have to show that the same remains true for the sections of $\text{Stab}_H(X)$. It is obvious that the sections of $\text{Stab}_H(X)$ still have finite index in $G$, the fact that they act level transitively follows from Lemma 6.9. \hfill \Box \\

Observe that $\text{Stab}_G(v) = \text{Stab}_G(\mathcal{L}_1)$ for every first level vertex is equivalent to the fact that the action of $G/\text{Stab}_G(\mathcal{L}_1)$ on the first level is free. In practice, this hypothesis is not too restrictive. Indeed, for most constructions of self-replicating branch groups $G$, the group $G/\text{Stab}_G(\mathcal{L}_1)$ acts cyclically and hence freely on $\mathcal{L}_1$.

As a consequence of Proposition 6.7 and of Lemma 4.12, we obtain the following that shows that if a counterexample to Conjecture 4.8 exists, it lies in the realm of subgroups with a block structure.

**Corollary 6.8.** Let $G \leq \text{Aut}(T)$ be a branch group that is just infinite, self-replicating and such that for every vertex of the first level $\text{Stab}_G(v) = \text{Stab}_G(\mathcal{L}_1)$.

Let $W$ be a weakly maximal subgroup of $G$. If the action $W \curvearrowright \partial \text{NR}(W)$ is not minimal, then $W$ has a block structure.

The following lemma finishes the proof of Proposition 6.7.

**Lemma 6.9.** Let $T$ be a locally finite rooted tree. Let $G \leq \text{Aut}(T)$ be a group that is self-similar and such that for every vertex of the first level $\text{Stab}_G(v) = \text{Stab}_G(\mathcal{L}_1)$. Let $H$ be any subgroup of $G$. Then for every transversal $X$, and for every vertex $v$ of $X$, the groups $\text{Stab}_H(X)$ and $\text{Stab}_H(v)$ have the same orbits on $T_v$.

**Proof.** If $X$ consist of only the root, then the assertion is trivially true. Assume that the assertion holds for some $X$ and let $v$ be any vertex of $X$ and $\{v_1, \ldots, v_d\}$ be its children. We will show that the assertion still holds for $X' := (X \setminus \{v\}) \cup \{v_1, \ldots, v_d\}$. If $\text{Stab}_H(X)$ already stabilizes $X'$, there is nothing to prove. Otherwise, we have that $\text{Stab}_H(X') = \text{Stab}_H(X) \cap \bigcap_v \text{Stab}_H(v) = \text{Stab}_H(X) \cap \text{Stab}_H(v_1)$ and this is an index $d$ subgroup of $\text{Stab}_H(X)$. By Fact 2.4 for every transversal of $\bigcup_{w \in X} T_w$, the number of orbits of $\text{Stab}_H(X')$ is at most $d$-times the number of orbits of $\text{Stab}_H(X)$ and every $\text{Stab}_H(X)$-orbit is a union of $\text{Stab}_H(X')$-orbits. Since $\text{Stab}_H(X)$ acts on $\{v_1, \ldots, v_d\}$ transitively, while $\text{Stab}_H(X')$ fixes it pointwise, we are done. \hfill \Box

Both Proposition 6.7 and Lemma 6.9 admit straightforward generalizations to groups that are not necessarily self-replicating. In this case, we need to ask that for every vertex, the group $\pi_v(G)$, which is not necessarily equal to $G$, satisfies the hypothesis we asked for $G$. That is, to be such that $\pi_v(G)/\text{Stab}_{\pi_v(G)}(\mathcal{L}_1)$ acts freely on the first level of $T_v$. Indeed, by Lemma 5.1 $\pi_v(G)$ is automatically branch and just infinite if $G$ is.

Examples of groups that satisfy all the hypothesis of Proposition 6.7 are the first Grigorchuk group as well as torsion generalized multi-edge spinal groups. We are now able to prove a structural theorem for weakly maximal subgroups of branch groups. This theorem encompass Propositions 1.11 and 1.12 as well as Theorem 1.13.
Theorem 6.10. Let $G \leq \text{Aut}(T)$ be an almost level transitive rigid group and $W$ be a weakly maximal subgroup of $G$. For the following properties that $W$ may have,

(a) $W$ has a block structure,

(b) There exists $n$ such that $\pi_v(W)$ has finite index in $\pi_v(G)$ for every vertex of level $n$,

(c) $W$ is not weakly rigid,

(d) $W$ is almost level transitive,

(e) $W$ is not generalized parabolic,

(f) $W$ is finitely generated.

we have the implications

1. In general

\[ (a) \implies (b) \]
\[ (c) \implies (d) \]

2. If $G$ is finitely generated, then $(a) \implies (f)$,

3. If $G$ is self-replicating, branch, just infinite group with $\text{Stab}_G(v) = \text{Stab}_G(L_1)$ for every first-level vertex, then $(b) \implies (a)$,

4. If $G$ is self-replicating, branch and has the subgroup induction property, then $(f) \implies (e)$,

5. If $G$ is self-replicating, branch, just infinite and has the subgroup induction property, then $(f) \implies (a)$,

6. If $G$ is a torsion GGS group, then properties (a) to (e) are equivalent and imply (f),

7. If $G$ is either the first Grigorchuk group, or the Gupta-Sidki 3-group, then properties (a) to (f) are equivalent.

Proof. In every branch group, $(d)$ and $(e)$ are equivalent by Corollary 4.7, and they are both implied by $(c)$ since generalized parabolic subgroups are weakly rigid. It is also clear that $(b)$ always implies $(d)$. On the other hand, $(a)$ implies both $(c)$ by Lemma 6.5 and $(b)$ by Corollary 6.6.

If $G$ is finitely generated, then $(a)$ also implies (f) by the remark after Definition 6.2. The third item is exactly Proposition 6.7. The fourth item is Proposition 3.18.

If $G$ is self-replicating, just infinite and has the subgroup induction property, then for any subgroup $W$ of $G$, property (f) implies (a) by [24].
Finally, suppose that $G$ is either the first Grigorchuk group or a torsion GGS group and let $W$ be a weakly maximal subgroup of $G$. Since all these groups have all the desired properties, it remains to show that (d) implies (b). Suppose that (d) holds, that is that $W$ is almost level transitive. Then there exists a transversal $X$ such that $\pi_v(W)$ acts level transitively on $T_v$ for every $v$ in $X$. By Lemma 5.4 for the first Grigorchuk group or Proposition 5.6 for the torsion GGS groups, we have another transversal $X'$ such that every section is equal to $G$, which implies (b). □

Let $G$ be either the first Grigorchuk group, or a torsion GGS group. We already know that $G$ has a continuum of weakly maximal subgroups. On the other hand, $G$ is finitely generated and thus has at most countably many weakly maximal subgroups that are finitely generated. As a corollary of Theorem 6.10 we prove Proposition 1.10 that says that, in some sense, $G$ has also as much finitely generated weakly maximal subgroups as it is possible.

**Proof of Proposition 1.10.** Let $\{v_1, \ldots, v_p\}$ be the vertices of the first level and let $H$ be the diagonal subgroup $\text{diag}(\text{Rist}_G(v_1) \times \cdots \times \text{Rist}_G(v_p))$. This is an infinite index subgroup and it is hence contained in a weakly maximal subgroup $W$. The subgroup $H$, and hence also $W$, acts on $\partial T$ with finitely many orbit-closures. In particular, $W$ is not generalized parabolic.

For $v$ a vertex in $T$, let $W^v$ be the subgroup of $G$ acting like $W$ on $T_v$, see Definition 5.7. By Proposition 5.8 all the subgroups $W^v$ are weakly maximal and none of them is generalized parabolic. Moreover, it follows from the description of $\text{NR}(W)$ and from the discussion after Corollary 5.9 that if $v$ and $w$ are vertices of different level, then $W^v$ and $W^w$ are not tree equivalent. Finally, since the $W^v$ are weakly maximal, they are self-normalizing by [8] and hence have infinitely many conjugate. Every conjugate of $W^v$ is still a non generalized parabolic weakly maximal subgroup that is tree-equivalent to $W^v$ since $\text{NR}(gW^v g^{-1}) = \text{NR}(W^v)^g$. Since none of these subgroups are generalized parabolic, they all have a block structure. □

It is natural to ask if all properties listed in Theorem 6.10 are always equivalent. In [15] Francoeur and Garrido classified maximal subgroups of non-torsion Šunić groups acting on the binary rooted tree and that are not equal to the infinite dihedral group. All groups in this family are finitely generated, have trivial congruence kernel (and hence trivial branch kernel), are just infinite, branch and self-replicating. Francoeur and Garrido exhibited maximal subgroups of infinite index that are finitely generated, are weakly rigid (that is, do not have property (c)) and do not have property (b) of Theorem 6.10. Such subgroups are not closed in the profinite topology and hence are not generalized parabolic (that is, have property (d)). To summarize this, even when restricted to finitely generated branch group that are just infinite, self-replicating and have the congruence subgroup property, (c) does not imply (d) or (e), and nor does (d) imply (c).

We call a weakly maximal subgroup that is both weakly rigid and almost level transitive (that is, with property (d) but not property (b) of Theorem 6.10) exotic. Every maximal subgroup of infinite index is exotic, and at our knowledge, all known examples of branch...
groups $G$ that have an exotic weakly maximal subgroups also have a maximal subgroup of infinite index.

**Remark 6.11.** If $G$ is such that all weakly maximal subgroups that are not weakly rigid have a block structure, then we have a trichotomy for weakly maximal subgroups of $G$. They are either generalized parabolic (if and only if they are weakly rigid but not almost level transitive), or have a block structure (if and only if they are almost level transitive but not weakly rigid) or they are exotic (both weakly rigid and almost level transitive).

On the other hand, we are inclined to believe that for a large class of groups, properties (b) and (c) of Theorem 6.10 are equivalent. The heuristic behind this is that a subgroup with (b) but not (c) should be “too big” (that is, of finite index), while a subgroup with (c) but not (b) should be “too small” (that is, not weakly maximal).

In view of the above discussion we formulate the following questions.

**Question 6.12.** Is it true that in any branch group, properties (b) and (c) of Theorem 6.10 are equivalent?

**Question 6.13.** Let $G$ be a branch group. Suppose that $G$ does not have maximal subgroup of infinite index. Does this imply that $G$ does not have exotic (both almost level transitive and weakly rigid) weakly maximal subgroup?

For both questions, if the answer happens to be negative, it would be nice to both have a counterexample and to find a “large” class of branch groups in which the answer is true.

If Question 6.12 admits a positive answer, then Remark 6.11 apply to all self-replicating branch group $G$ that are just infinite and such that for every vertex of the first level $\text{Stab}_G(v) = \text{Stab}_G(L_1)$.

### 7 Level transitive weakly maximal subgroups

The aim of this section is to exhibit examples of weakly maximal subgroups that are level transitive. Recall that a subgroup $A$ in a branch group $G$ is dense for the congruence topology if and only if $(A\text{Stab}_G(n))/\text{Stab}_G(n) = G/\text{Stab}_G(n)$ for every $n$. On the other hand, $A$ is level transitive if and only if, for every $n$ the subgroup $(A\text{Stab}_G(n))/\text{Stab}_G(n) \leq G/\text{Stab}_G(n)$ acts transitively on the $n^{th}$ level of the tree. This directly implies

**Fact 7.1.** In a branch group, every maximal subgroup of infinite index is a level transitive weakly maximal subgroup.

As we will see, some branch groups have level transitive weakly maximal subgroups that are not maximal. We will first outline a method to search level transitive weakly maximal subgroups in branch groups acting on the 2-regular rooted tree. We then illustrate this method in the particular example of the Grigrochuk group. Finally, for every branch group $G$ we will construct some extension $\hat{G}$ which will retains a lot of the properties of $G$ and show that $\hat{G}$ always has a level transitive weakly maximal subgroup.

All these constructions are variations of the following simple example.
Example 7.2. Let $T$ be a locally finite tree. Let $G$ be a subgroup of $\text{Aut}(T)$ such that $\text{Rist}_G(L_1) = G \times \cdots \times G$ and such that the subgroup $R$ of rooted automorphisms of $G$ acts transitively on the first level. For example, $G$ is one of $\text{Aut}(T)$, $\text{Aut}_r(T)$ or $\text{Aut}_t(T)$, or is the Bondarenko’s example from [7]. Then $\langle R, \text{diag}(G \times \cdots \times G) \rangle$ is an infinite index subgroup of $G$ acting level transitively on $G$.

The condition on $\text{Rist}_G(L_1)$ in the above example is pretty restrictive. Nevertheless, while the groups $\text{Aut}(T)$, $\text{Aut}_r(T)$ and $\text{Aut}_t(T)$ are never finitely generated, the Bondarenko’s example is finitely generated. However, it is possible to adapt a little bit the construction in order to deal with other branch groups.

Lemma 7.3. Let $G$ be a weakly branch group acting on a $2$-regular tree. Suppose that $G$ contains $a = (01)$ the automorphism of $T$ permuting rigidly $T_0$ and $T_1$. Suppose moreover that there exists a subgroup $A$ of $G$ and an automorphism $\varphi$ of $G$ such that:

1. $\varphi^2(A) = A$ and the restriction of $\varphi^2$ on $A$ is the identity,
2. $\text{diag}(A \times \varphi(A)) = \{(g, \varphi(g))\}_{g \in A}$ is a subgroup of $G$,
3. $\langle A, \varphi(A) \rangle$ acts level transitively on $T$.

Then the subgroup $L := \langle a, \text{diag}(A \times \varphi(A)) \rangle$ is an infinite index subgroup of $G$ that acts level transitively on $T$.

Proof. The second condition implies that $L$ is a subgroup of $G$. The third condition and the fact that $a$ acts transitively on the first level imply that $L$ acts level transitively. Finally, by definition of $a$, the rigid stabilizer in $L$ of the vertex $0$ consists of elements of the form

$$(g_1\varphi(h_1) \cdots g_n\varphi(h_n), \varphi(g_1)h_1 \cdots \varphi(g_n)h_n)$$

where the $g_i$ and $h_i$ belong to $A$ and $\varphi(g_1)h_1 \cdots \varphi(g_n)h_n = 1$. Since $\varphi^2$ is the identity on $A$, this implies that $\text{Rist}_L(0) = \{1\}$ and therefore that $L$ is of infinite index in $G$ as soon as $G$ is weakly branch.

We now give the first example of a level transitive weakly maximal subgroup for the particular case of the first Grigorchuk group.

Lemma 7.4. The subgroup

$$W_L := \langle a, \text{diag}((b, ac) \times (b, ac)^a) \rangle = \langle a, bab, cadab \rangle$$

is a level transitive weakly maximal subgroup of the first Grigorchuk group $\mathcal{G}$.

Proof. We first show that $W_L$ is an infinite index subgroup that is level transitive as a consequence of the last lemma. Hence, it is sufficient to verify the hypothesis of the said lemma. The conjugation by $a$ is an involution, which directly implies the first hypothesis. It is enough to verify the second condition on the generators of $(b, ac)$, that is to check that $(b, aba)$ and $(ac, aaca) = (ac, ca)$ are elements of $\mathcal{G}$. We have indeed $acadaba = (b, aba)$
and $baba = (ca, ac)$. This also gives the equality between the 2 subgroups of the lemma. Finally, $(b, ac)$ is a normal subgroup of index 2 ($J_{0,5}$ in the notation of [9]) that contains $K = ((ab)^2, (bd)^2, (b^s d)^2)$, the subgroup of index 16 on which $\mathfrak{S}$ is branch. The subgroup $K$ itself is not branch, but contains $K \times K$ and acts transitively on the first level of $T_0$ (by $(ab)^2 = (ca, ac)$ for example). Since $(b, ac)$ acts transitively on the first level of $T$ and contains $K$, it acts level transitively. Hence $W_L$ is of infinite index and level transitive. In particular, it is contained in a weakly maximal subgroup that is level transitive.

The proof of the weak maximality of $W_L$ is done in the next section, in Lemma \[8.5\] \(\square\)

We now give a general construction which, given a branch group $G$, produces a branch group $\hat{G}$ in which $G$ embeds diagonally and that has an infinite index subgroup that is level transitive. Moreover, if $G$ is finitely generated, so is $\hat{G}$, which implies that $\hat{G}$ contains a level transitive weakly maximal subgroup.

Let $T = T_{(m_i)}$ be a locally finite rooted tree. Let $G$ be any subgroup of $\text{Aut}(T)$ and let $A$ be the quotient $G/\text{Stab}(L_1)$. Let $m'_i = m_0$ and $m'_i = m_{i-1}$ for $i \geq 1$. Let $\hat{G}$ be the subgroup of the automorphism group of $T' = T_{(m'_i)}$ generated by $A$ (viewed as a group of rooted automorphisms) and $G \times \cdots \times G \leq \text{Stab}_{\text{Aut}(T)}(L_1)$ (each copies of $G$ acts on a subtree rooted at a vertex of the first level, all this subtrees are isomorphic to $T$).

**Proposition 7.5.** For every property $P$ in the following list, $\hat{G}$ has $P$ if and only if $G$ has $P$.

1. Finitely generated,
2. Almost level transitive, respectively level transitive,
3. Weakly rigid, respectively rigid,
4. The rigid (respectively branch, respectively congruence) kernel is trivial,
5. Branch and just infinite,
6. Polynomial (respectively intermediate, respectively exponential) growth,
7. Is torsion,
8. The tree is $p$-regular and the group is a $p$-group ($p$ prime).

**Proof.** The rank of $\hat{G}$ is at most the rank of $A$ (a finite group) plus the rank of $G$. On the other hand, $G$ is the section of the first level stabilizer of $\hat{G}$. In particular, rank($G$) is finite if rank($\hat{G}$) is finite.

It directly follows from the definition of $A$ and the fact that $\pi_u(\hat{G}) = G$ for any first level vertex that $G$ is (almost) level transitive if and only if $\hat{G}$

We have $\text{Rist}_{\hat{G}}(\emptyset) = \text{Rist}_{\hat{G}}(L_0) = \text{Stab}_{\hat{G}}(L_0) = \hat{G}$. On the other hand, for any vertex $v = u_1u_2 \ldots u_n$ we have

$$\text{Rist}_{\hat{G}}(v) = \{1\} \times \cdots \times \text{Rist}_{\hat{G}}(u_2 \ldots u_n) \times \cdots \times \{1\},$$
where the non-trivial factor appears in position $u_1$. Finally, for any $n \geq 1$ we have

$$Rist_{\hat{G}}(\mathcal{L}_n) = Rist_{\hat{G}}(\mathcal{L}_{n-1}) \times \cdots \times Rist_{\hat{G}}(\mathcal{L}_{n-1})$$

$$Stab_{\hat{G}}(\mathcal{L}_n) = Stab_{\hat{G}}(\mathcal{L}_{n-1}) \times \cdots \times Stab_{\hat{G}}(\mathcal{L}_{n-1}).$$

This directly implies the assumption for the weakly rigid property. Since $G \times \cdots \times G$ has finite index in $\hat{G}$, $Rist_{\hat{G}}(\mathcal{L}_n)$ has finite index in $\hat{G}$ if and only if $Rist_{\hat{G}}(\mathcal{L}_{n-1})$ has finite index in $G$, which finishes the proof for the rigid property.

Suppose that $G$ has a trivial branch kernel and let $H$ be a finite index subgroup of $\hat{G}$. Then the index of $H \cap (G \times \{1\} \times \cdots \times \{1\})$ in $G \times \{1\} \times \cdots \times \{1\}$ is $G$ is also finite. By assumption, there exists $n_1$ such that $Rist_{\hat{G}}(\mathcal{L}_{n_1}) \leq H \cap (G \times \{1\} \times \cdots \times \{1\})$. Repeating this argument on other coordinates, we obtain $Rist_{\hat{G}}(\mathcal{L}_{n_1}) \times \cdots \times Rist_{\hat{G}}(\mathcal{L}_{n_0}) \leq H$ and therefore $Rist_{\hat{G}}(\mathcal{L}_{\text{max}(n_1)+1}) \leq H$. A similar argument shows that if $G$ has a trivial rigid kernel, so does $\hat{G}$. On the other hand, suppose that $\hat{G}$ has trivial branch kernel and let $H$ be a finite index subgroup of $G$. Then $H \times G \times \cdots \times G$ is a finite index subgroup of $\hat{G}$ and hence contains $Rist_{\hat{G}}(\mathcal{L}_n) = Rist_{\hat{G}}(\mathcal{L}_{n-1}) \times \cdots \times Rist_{\hat{G}}(\mathcal{L}_{n-1})$ for some $n$. In particular $H$ contains $Rist_{\hat{G}}(\mathcal{L}_{n-1})$, and we have proved the triviality of the branch kernel for $G$. As before, a similar argument takes care of the rigid kernel. Finally, the triviality of the congruence kernel is equivalent to the simultaneous triviality of both the branch and the rigid kernels.

By [13], a branch group acting on a locally finite tree is just infinite if and only if for every vertex $v$, the derived subgroup $Rist_{\hat{G}}(v)'$ has finite index in $Rist_{\hat{G}}(v)$. The description of rigid stabilizers of $\hat{G}$ directly implies that if $\hat{G}$ is just infinite, so is $G$. For the other direction, we only need to check that the derived subgroup of $\hat{G}$ has finite index in $\hat{G}$. But the derived subgroup of $\hat{G}$ contains the derived subgroup of $Rist_{\hat{G}}(\mathcal{L}_1)$ which has finite index in $Rist_{\hat{G}}(\mathcal{L}_1)$ and hence finite index in $\hat{G}$.

Abstractly, the group $\hat{G}$ is the semi-direct product of $A$ (a finite group) and of $G^{m_0}$ (a finite product of copies of $G$). In particular, $G$ and $\hat{G}$ have the same type (polynomial, intermediate or exponential) of growth rate.

Since $G$ embeds into $\hat{G}$, if $\hat{G}$ is torsion or a $p$-group so is $G$. On the other hand, every element $g$ of $\hat{G}$ is of the form $(g_1, \ldots, g_{m_0})a$ for some $a \in A$ and $g_i$’s in $G$. In particular, $g^{m_0} = (h_1, \ldots, h_{m_0})$ for some $h_i$’s in $G$. If all the $h_i$’s have finite order, denoted $o_i$, then the order of $g$ divides $m_0 \text{lcm}(o_i)$.

**Lemma 7.6.** If $T$ is $d$-regular and $G$ is self-similar, so is $\hat{G}$.

**Proof.** Suppose that $G$ is self-similar and let $g = (g_1, \ldots, g_d)a$ be an element of $\hat{G}$. Since $g_1$ is in $G$, we have $g_1 = (h_1, \ldots, h_{d})b$ with $b$ in $A$ and the $h_i$ in $G$ by self-similarity. Therefore, $g_1$ belongs to $\hat{G}$ and the same is true for the others $g_i$.

**Question 7.7.** Suppose that $\hat{G}$ is self-similar. Does this implies that $G$ is also self-similar?

What can be said about the self-replicacity of $G$ and $\hat{G}$? 

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The above results shows that the groups \( \hat{G} \) and \( G \) look alike for a lot of purposes. The main difference is that the construction of \( \hat{G} \) allows to easily find a level transitive subgroup of infinite index.

**Lemma 7.8.** If \( G \) is weakly branch, then \( \langle A, \text{diag}(G \times \cdots \times G) \rangle \) is an infinite index level transitive subgroup of \( \hat{G} \).

**Proof.** The proof is similar to the proof of Lemma 7.3, with the automorphism \( \varphi \) being the identity. \( \square \)

## 8 Weakly maximal subgroups of the first Grigorchuk group

We now take a special look at what is probably the most studied branch group: the first Grigorchuk group \( \mathfrak{G} \).

This group is branch over \( K := (ab)^2 \mathfrak{G} \) and we have \( K < B := \langle b \rangle \mathfrak{G} \), where both \( K \) and \( B \) are normal in \( \mathfrak{G} \). There is exactly 10 subgroups of \( \mathfrak{G} \) containing \( B \), \[29\]: the group \( \mathfrak{G} \) itself, 3 subgroups of index 2, 5 subgroups of index 4 (on which \( J_{1,5} = \langle B, (ad)^2 \rangle \) is the only normal subgroup) and \( B \) itself, see Figure 2 and Table 2.

Finally, let us recall that \( H = \text{Stab}_\mathfrak{G}(\mathcal{L}_1) = \langle b, c, d, aba, aca, ada \rangle \) is the stabilizer of the first level and that rigid stabilizers have the following description

\[
\text{Rist}_\mathfrak{G}(\mathcal{L}_1) = B \times B \quad \text{Rist}_\mathfrak{G}(\mathcal{L}_n) = K \times K \times \cdots \times K \underbrace{\times \cdots \times K}_{2^n \text{ factors}}.
\]

![Figure 2: The lattice of subgroups of \( \mathfrak{G} \) containing \( B \). Each subgroup has index 2 in the one above it.](image)


| Subgroup | Generators | Conjugates |
|----------|------------|------------|
| $\mathfrak{G}$ | $a, b, c$ | $\mathfrak{G}$ |
| $J_{0,2}$ | $b, a, a^c$ | $J_{0,2}$ |
| $J_{0,5}$ | $b, ac$ | $J_{0,5}$ |
| $H$ | $c, c^a, d, d^a$ | $H$ |
| $J_{1,5}$ | $b, b^a, d, d^a$ | $J_{1,5}$ |
| $S_{2,3,0,0}$ | $b, c, b^a, b^a b^a c, c^a c$ | $S_{2,3,0,0}$ |
| $S_{2,4,0,0}$ | $a, b, d^a, a^d d^a$ | $S_{2,4,0,0}$ |
| $B$ | $b, b^a, b^a d^a$ | $B$ |

Table 2: The 10 subgroups of $\mathfrak{G}$ containing $B$ and their generators; 6 of them are normal.

### 8.1 More examples of weakly maximal subgroups of $\mathfrak{G}$

In this subsection we investigate two explicit examples of weakly maximal subgroups that have a block structure. The first example, due to Pervova and firstly mentioned in \cite{21}, is

$$W_P := \langle a, \text{diag}(J_{1,5} \times J_{1,5}), \{1\} \times K \times \{1\} \times K \rangle.$$  

The second example is the subgroup

$$W_L = \langle a, \text{diag}(J_{0,5} \times J_{0,5}^a) \rangle = \langle a, bab, cadab \rangle = \langle a, bab, cac \rangle$$

of Lemma \ref{lem:exa1}.

**The subgroup** $W_P$  

The first thing we need to do is to show that the group $W_P$ is a subgroup of $\mathfrak{G}$. By definition $a$ belongs to $\mathfrak{G}$ and it is well known that $\{1\} \times K \times \{1\} \times K$ is a subgroup of $\mathfrak{G}$. Therefore, it remains to check that $\text{diag}(J_{1,5} \times J_{1,5})$ is also a subgroup of $\mathfrak{G}$. But we have $J_{1,5} = \langle b, aba, dada \rangle$ and an easy verification gives us $\text{diag}(a, b) = (b, b)$, $d^c d^c = (aba, aba)$ and $(ac)^4 = (dada, adad) = (dada, dada)$.

The non-rigidity tree $\text{NR}(W_P)$ of $W_P$ is the subtree $S$ of $T$ generated by $T_{00}$ and $T_{10}$ and $W_P$ acts level transitively on it. Indeed, since $W_P$ contains $\{1\} \times K \times \{1\} \times K$ we have $\text{NR}(W) \subseteq S$. On the other hand, $W_P$ contains $a$ which sends $T_{00}$ to $T_{10}$ and $\text{diag}(J_{1,5} \times J_{1,5})$. Since $J_{1,5} = \langle b, aba, dada \rangle$ its left section contains $\langle a, c, b \rangle = \mathfrak{G}$. Therefore, $\text{diag}(J_{1,5} \times J_{1,5})$ acts level transitively on $T_{00}$ and $W_P$ acts level transitively on $S$ which implies $\text{NR}(W) = S$.

We will now prove several results in order to better understand the structure of $W_P$ and of $W_P$, its closure in $\hat{\mathfrak{G}}$ the profinite completion of $\mathfrak{G}$. For the weak maximality of $W_P$, we will mainly give the proof of \cite{21} but with more details.

**Lemma 8.1** (\cite{21}). Let $A$ be a subgroup of $\mathfrak{G}$ containing $J_{1,5}$. Let $1 \neq x \in \text{Stab}_\mathfrak{G}(1)$. Then $\langle x \rangle^A$ has infinite index in $\mathfrak{G}$ if and only if $x^i_1 = 1$ for some $i \in \{0, 1\}$.

**Proof.** If $x^i_1 = 1$, then $\langle x \rangle^A \leq \{1\} \times \mathfrak{G}$ is of infinite index by Fact \ref{f:inf}. The case $x^i_1 = 1$ is similar.
On the other hand, suppose that $x = (x_0, x_1)$ with $x_i \neq 1$ for $i \in \{0, 1\}$. In this case, the centralizer $C_{\mathfrak{S}}(x)$ has infinite index in $\mathfrak{S}$. This implies that for $i \in \{0, 1\}$ there exists $y_i \in K \setminus C_{\mathfrak{S}}(x_i)$. We then have $1 \neq [x_i, y_i]$ belongs to $K$. Since $A$ contains $J_{1,5}$, it contains $K \times K$ and $(1, [x_1, y_1]) = [x_1, (1, y_1)] = x_1 \cdot (1, y_1)x^{-1}(1, y_1)^{-1}$ is in $\langle x \rangle^A$. On the other hand, $\pi_i(A) \geq \pi_i(J_{1,5}) = \mathfrak{S}$. All together, we have $\langle x \rangle^A \geq \langle [x_0, y_0] \rangle^\mathfrak{S} \times \langle [x_1, y_1] \rangle^\mathfrak{S}$. Both $\langle [x_i, y_i] \rangle^\mathfrak{S}$ are non-trivial normal subgroups of $\mathfrak{S}$ and therefore of finite index since $\mathfrak{S}$ is just infinite. This shows that $(\langle x \rangle^A)^i$ itself is of finite index in $\mathfrak{S}$.

Lemma 8.2 ([21]). $W_P$ is a finitely generated weakly maximal subgroup of $\mathfrak{S}$.

Proof. The subgroups $K$ and $J_{1,5}$ being finitely generated, so is $W_P$. Now, if $g$ is an element of $W_P \cap (K \times \{1\} \times \{1\})$, we have $g = (k, 1, 1)$ and also $g = (g_0, g_1, g_0, g_3)$ which implies $g = 1$. We have shown that $W_P \cap (K \times \{1\} \times \{1\}) = \{1\}$ and therefore that $W_P$ is of infinite index.

We now want to prove that $W_P$ is weakly maximal. That is, for all $x \in \mathfrak{S} \setminus W_P$, the subgroup $\tilde{W} := \langle W_P, x \rangle$ is of finite index in $\mathfrak{S}$. Since $a$ belongs to $W_P$, we can assume that $x$ belongs to $H$ and $x = (x_0, x_1)$. We have $\mathfrak{S}/B = \{1, a, d, ad, ada, \ldots, (ad)^3a\} \cong D_{24}$, the dihedral group of order 8, and hence $\mathfrak{S}/J_{1,5} = \{1, a, ad, a^2\}$. By factorizing the first coordinate by $\pi_0(W_P) \geq J_{1,5}$ we can assume that $x_0$ belongs to $\{1, a, d, ad\}$ which leave us with four cases to check. If $x_0$ is not in $H$, then $\tilde{W}$ contains $(1 \times K \times \{1\} \times \{1\})x_0 = K \times \{1\} \times \{1\} \times \{1\}$ and $a$. In this case, $\tilde{W}$ contains $K \times K \times K \times K$ and is of finite index. We can hence suppose that $x_0$ is in $H$, and by symmetry, that $x_1$ is also in $H$. It thus remains to check two cases: $\{(1, x_1)\}$ and $(d, x_1)$ with $x_1$ in $H$.

If $x_0 = 1$, then $x_1 \neq 1$. In this case, $\tilde{W}$ contains $\{1\} \times \langle x_1 \rangle^{J_{1,5}}$ and $\mathrm{diag}(\langle x_1 \rangle^{J_{1,5}} \times \langle x_1 \rangle^{J_{1,5}}) \leq \mathrm{diag}(J_{1,5} \times J_{1,5})$. This implies $\tilde{W} \geq \langle x_1 \rangle^{J_{1,5}} \times \langle x_1 \rangle^{J_{1,5}}$. If $\langle x_1 \rangle^{J_{1,5}}$ has finite index in $\mathfrak{S}$, then $\tilde{W}$ has also finite index in $\mathfrak{S}$. We can therefore assume that $\langle x_1 \rangle^{J_{1,5}}$ has infinite index in $\mathfrak{S}$, which implies by Lemma 8.1 that $x_1 = (1, z)$ or $x_1 = (z, 1)$, $z \neq 1$. In both cases, we have $x = (1, x_1)$ is an element of $\mathrm{Rist}_\mathfrak{S}(2) = K \times K \times K \times K$ which implies that $z$ is in $K$. This rules out the case $x_1 = (1, z)$, since in this case we would have $x = (1, 1, 1, z) \in \{1\} \times K \times \{1\} \times K \leq W$. On the other hand, if $x_1 = (z, 1)$, the subgroup $\tilde{W}$ contains $(\langle 1, 1, z, 1 \rangle)^{\mathrm{diag}(J_{1,5} \times J_{1,5})} = \{1\} \times \{z\}^{\mathfrak{S}} \times \{1\}$ since $\pi_0(J_{1,5}) = \mathfrak{S}$. The group $\mathfrak{S}$ being just infinite and $z \neq 1$, the subgroup $\langle z \rangle^\mathfrak{S}$ has finite index in $\mathfrak{S}$ and $\langle z \rangle^\mathfrak{S} \cap K$ has finite index in $K$. Therefore, $\tilde{W}$ contains $(\langle z \rangle^\mathfrak{S} \cap K) \times \{\langle z \rangle^\mathfrak{S} \cap K\} \times K$ which has finite index in $K \times K \times K \times K = \mathrm{Rist}_\mathfrak{S}(L_2)$ and thus in $\mathfrak{S}$.

We will now show that the case $x_0 = d$ cannot happen if $x_1$ is in $H$. Indeed, $(d, x_1)$ belongs to $\mathfrak{S}$ if and only if $(1, ax_1) = c^a \cdot (d, x_1)$ is in $\mathfrak{S}$. But in this case, $(1, ax_1)$ belongs to $\mathrm{Rist}_\mathfrak{S}(1) = B \times B$ and so $ax_1$ is in $B \leq H$, which is impossible if $x_1 \in H$.

Lemma 8.3. The subgroup $W_P \cap H$ is a weakly maximal subgroup of $H$.

Proof. Let $x$ be in $H \setminus W_P$ and look at $\tilde{W} := \langle x, W_P \cap H \rangle$. Then $x = (x_0, x_1)$ and factorizing the first factor by $J_{1,5}$, we can assume that $x_0$ is either 1 or $d$. The rest of the proof is the same as the proof of the weak maximality of $W_P$ in $\mathfrak{S}$.

Let $G$ be a topological group. A subgroup $W$ of $G$ is said to be weakly maximal closed if it is maximal among all closed subgroups of infinite index.
Lemma 8.4.

1. \( \overline{W}_P \) is a (topologically) finitely generated weakly maximal closed subgroup of \( \hat{\mathfrak{G}} \);

2. There are uncountably many distinct conjugates of \( \overline{W}_P \) in \( \hat{\mathfrak{G}} \).

Proof. Let \( 1 \neq x \in \text{Stab}_{\hat{\mathfrak{G}}}(1) \). We claim that \( (x)^{J_{1.5}} \) has infinite index in \( \hat{\mathfrak{G}} \) if and only if \( x_i = 1 \) for some \( i \in \{0, 1\} \). The proof of this claim is the same as the proof of Lemma 8.1 and we only need to show that for every \( 1 \neq x \in \hat{\mathfrak{G}} \), there exists \( y \in \mathfrak{K} \) such that \( [x, y] \neq 1 \). But this follows from the fact that if \( 1 \neq x \) is any element of \( \hat{\mathfrak{G}} \) then \( [\mathfrak{G} : C_{\mathfrak{G}}(x)] = \infty \) which is Lemma 2.9. Indeed, \([\hat{\mathfrak{G}} : C_{\hat{\mathfrak{G}}}(x)] = \infty \) if and only if \([\mathfrak{G} : C_{\mathfrak{G}}(x) \cap \mathfrak{G}] = [\mathfrak{G} : C_{\mathfrak{G}}(x)] = \infty \). In this case, the finite index subgroup \( \mathfrak{K} \) cannot be contained in \( C_{\hat{\mathfrak{G}}}(x) \).

The application \( \overline{\text{Sub}} : \text{Sub}_{\mathfrak{G}}(G) \to \text{Sub}_{\hat{\mathfrak{G}}}(\hat{\mathfrak{G}}) \) that sends a close (in the profinite topology) subgroup of \( \mathfrak{G} \) to its closure in \( \hat{\mathfrak{G}} \) sends infinite index subgroups to infinite index subgroups. See [29] for more details. Since \( W_P \) is weakly maximal in \( \mathfrak{G} \), it is close and we have that \( \overline{W}_P \) is an infinite index closed subgroup of \( \hat{\mathfrak{G}} \). The element \( a \) normalizes both \( \text{diag}(J_{1.5} \times J_{1.5}) \) and \( \{1\} \times K \times \{1\} \times K \) and since \( K \) is normal in \( J_{1.5} \leq H \), the subgroup \( \{1\} \times K \times \{1\} \times K \) is normalized by \( \text{diag}(J_{1.5} \times J_{1.5}) \). Therefore, \( W_P = \{1, a\} \cdot \text{diag}(J_{1.5} \times J_{1.5}) \cdot \{1\} \times K \times \{1\} \times K \) and \( \overline{W}_P = \{1, a\} \cdot \overline{\text{diag}(J_{1.5} \times J_{1.5}) \cdot \{1\} \times K \times \{1\} \times K} \). The subgroup \( \{1, a\} \) is closed and we have \( \{1\} \times K \times \{1\} \times K \) = \( \{1\} \times K \times \{1\} \times K \) and \( \overline{\text{diag}(J_{1.5} \times J_{1.5})} = \text{diag}(J_{1.5} \times J_{1.5}) \). Altogether, we have

\[
\overline{W}_P = \langle a, \text{diag}(J_{1.5} \times J_{1.5}), \{1\} \times K \times \{1\} \times K \rangle
\]

is a (topologically) finitely generated subgroup of \( \hat{\mathfrak{G}} \).

For all \( n \) we have \( \text{Stab}_{\mathfrak{G}}(n) = \text{Stab}_{\hat{\mathfrak{G}}}(n) \cap \mathfrak{G} \) and \( \text{Rist}_{\mathfrak{G}}(n) = \text{Rist}_{\hat{\mathfrak{G}}}(n) \cap \mathfrak{G} \). Since these subgroups are of finite index, for all \( n \) we have

\[
\text{Stab}_{\mathfrak{G}}(n) = \text{Stab}_{\hat{\mathfrak{G}}}(n) \quad \text{Rist}_{\mathfrak{G}}(n) = \text{Rist}_{\hat{\mathfrak{G}}}(n)
\]

In particular, we have

\[
\text{Rist}_{\hat{\mathfrak{G}}}(\mathcal{L}_1) = \overline{B} \times \overline{B} \quad \text{Rist}_{\hat{\mathfrak{G}}}(\mathcal{L}_2) = \overline{K} \times \overline{K} \times \overline{K} \times \overline{K}
\]

On the other hand, the closure preserves transversals for finite index subgroups. In particular, \( \hat{\mathfrak{G}}/H = \{1, a\} \) and \( \hat{\mathfrak{G}}/J_{1.5} = \{1, a, ad, d\} \). Let \( x \) be an element from \( \mathfrak{G} \setminus \overline{W}_P \) and look at \( \overline{W} := \overline{(W_P, x)} \). The proof that \( \overline{W} \) is of finite index is the same as the one for \( \overline{W} \), where Lemma 8.1 is replaced by the claim at the beginning of this proof.

Since \( W_P \) is weakly maximal, it is self-normalizing by [3]. On the other hand, \( \hat{\mathfrak{G}} \) is topologically just infinite and hence contains no subgroup with countable index, [28]. In particular, \( W_P \) has uncountably many distinct conjugates.

Using more convoluted methods, it is showed in [29] that \( \overline{W}_P \) has a continuum of conjugates in \( \hat{\mathfrak{G}} \).
The subgroup $W_L$ We have seen in the last section that $W_L$ is an infinite index subgroup of $G$ that is finitely generated and acts level transitively. It remains to show that it is weakly maximal in order to finish the proof of Lemma 7.4.

Lemma 8.5. The subgroup $W_L$ is weakly maximal.

Proof. Let $x$ be an element of $G \setminus W_L$ and $\tilde{W}_L := \langle W_L, x \rangle$. Since $a$ belongs to $W_L$, we may suppose that $x = (x_0, x_1)$ is in $H$. On the other hand, $\mathfrak{G} = J_{0,5} \cup aJ_{0,5}$ and $W_L$ contains $\text{diag}(J_{0,5} \times J_{0,5}^a)$ and thus we may suppose that $x = (x_0, 1)$ or $x = (x_0, a)$.

Firstly suppose that $x = (x_0, 1)$. In particular, $x$ belongs to $\text{Rist}_G(0) = B \times \{1\}$ and therefore $x_0$ is in $B$ and thus in $H$ and $x_0 = (z, t)$. If both $z$ and $t$ are not trivial, then by Lemma 8.1, the subgroup $\langle x_0 \rangle J_{0,5}$ has finite index in $\mathfrak{G}$. But $\tilde{W}_L$ contains $\langle x_0 \rangle J_{0,5} \times \{1\}$ and, as before, $\tilde{W}_L$ contains $\langle x_0 \rangle J_{0,5} \times \{1\}$, which is a finite index subgroup of $\mathfrak{G}$ and we are done. On the other hand, suppose that at least one of $z$ or $t$ is trivial, say $t$. Then $x = (z, 1, 1, 1)$ with $z \neq 1$. In this case, $\tilde{W}_L$ contains

$$\langle x \rangle^{\text{diag}(J_{0,5} \times J_{0,5}^a)} = \langle (z, 1) \rangle J_{0,5} \times \{1\} = \langle z \rangle^\mathfrak{G} \times \{1\} \times \{1\} \times \{1\}$$

Once again, $\mathfrak{G}$ being just infinite and $z$ non-trivial implies that $\langle z \rangle^\mathfrak{G}$ is a finite index subgroup $N$ of $\mathfrak{G}$ and since $\tilde{W}_L$ acts transitively on the second level it contains $N \times N \times N$ which is a finite index subgroup of $\mathfrak{G}$.

Let us now look at the case $x = (x_0, a) \notin W_L$. Observe that $\pi_1(\tilde{W}_L) = \langle \pi_1(W_L), x_1 \rangle = \langle J_{0,5}, a \rangle = \mathfrak{G}$. Since $x$ belongs to $\mathfrak{G}$, so is $xaba = (x_0c, 1)$ which implies that $x_0 = yc$ with $y \in B$. Both $x^{-1} = (cy^{-1}, a)$, and $(y, aya) \in \text{diag}(J_{0,5} \times J_{0,5}^a)$ belong to $\tilde{W}_L$. In particular, $\tilde{W}_L$ contains $\langle x^{-1} \cdot (y, aya) \rangle^2 = (1, yaya)$ and therefore $\tilde{W}_L$ contains $\{1\} \times \langle yaya \rangle^\mathfrak{G}$. If $yaya$ is not trivial, then the same argument as before shows that $\tilde{W}_L$ is of finite index. Since $y$ is in $B$, so is $yb$ and therefore $(yb, ayba)$ belongs to $W_L$. In particular, $(x^{-1} \cdot (yb, ayba))^2 = (1, ybayba)$ is in $\tilde{W}_L$. As before, if $ybayba$ is not trivial, $\tilde{W}_L$ is of finite index. All we have to do is to show that $yaya$ and $ybayba$ cannot be both trivial. Since $y$ is in $B$ it is equal to $(t_1, t_2)$. Then $yaya = (t_1t_2, t_2t_1)$ and $ybayba = (t_1at_2c, t_2ct_1a)$. If they are both trivial, then $t_1 = t_2^{-1}$ and $t_2^{-1}at_2c = 1$. But $t_2^{-1}at_2c$ is never in $H$ independently of the value of $t_2$ and we have the desired contradiction. □

8.2 Sections of weakly maximal subgroups of $\mathfrak{G}$

While general results about sections of weakly maximal subgroups are given in Section 5, we give here more details for the particular case of the first Grigorchuk group.

Let $W < \mathfrak{G}$ be a weakly maximal subgroup. If $W$ is contained in the stabilizer of the first level, then by Lemma 5.2, if one of the first-level sections is of infinite index, then it is a weakly maximal subgroup of $\mathfrak{G}$ and the other section contains $B$. It is natural to ask if this result can be extended to weakly maximal subgroups that do not stabilize the first level. In fact, the same proof show that if $W \cap H$ is weakly maximal in $H$ and one of the
first-level sections of $W \cap H$ is of infinite index, then it is a weakly maximal subgroup of $\mathfrak{G}$ and the other section contains $B$. This remark and Lemma 5.3 imply the following.

**Corollary 8.6.** If $W$ is a generalized parabolic subgroup of $\mathfrak{G}$ not contained in $H$, then $W \cap H$ is not weakly maximal in $H$.

Let $\xi \in \partial T$ be any ray. Then $W = \text{SSStab}_G(\{\xi, a.\xi\})$ satisfies the hypothesis of the above corollary and show the existence of weakly maximal subgroup of $\mathfrak{G}$ such that $W \cap H$ is not a weakly maximal subgroup of $H$. This answer by the positive question 6.5.6 of [29].

By Proposition 5.8, for every weakly maximal subgroup $L$ of $G$, there exists a weakly maximal subgroup $W$ of $G$ that stabilizes the first level and such that $\pi_1(W) = L$. In particular, by taking $L$ any weakly maximal subgroup that is not (generalized) parabolic, we obtain infinitely many examples of weakly maximal subgroups that are not (generalized) parabolic, but still contained in $H$, hence answering question 6.5.3 of [29].

In [29], the author also asked the following question about sections of weakly maximal subgroups.

**Question 8.7.** Let $W$ be a weakly maximal subgroup of $\mathfrak{G}$ contained in $H$ and $\pi_0(W)$ and $\pi_1(W)$ its left and right sections. If $\pi_1(W)$ is of infinite index, which of the subgroups of $G$ containing $B$ could appear as $\pi_0(W)$?

We will show that both $\mathfrak{G}$, Lemma 8.8, and $J_{0,2}$, Proposition 8.14, can be obtained as $\pi_0(W)$.

**Lemma 8.8.** There exists a continuum of generalized parabolic subgroup $W$ of $\mathfrak{G}$ such that $W$ is contained in $H$, $\pi_1(W)$ is of infinite index in $\mathfrak{G}$ and $\pi_0(W) = \mathfrak{G}$.

**Proof.** Let $F = \langle c, a \rangle$. This is a finite subgroup of $\mathfrak{G}$ and hence contained in a continuum of pairwise distinct generalized parabolic subgroup $W$, Corollary 1.7. For each of these $W$, let $W^1 = \{ g \in \text{Stab}_G(1) \mid g_{1_{\mathfrak{H}}} \in W \}$. By Proposition 5.8, the $W^1$ are weakly maximal subgroup of $\mathfrak{G}$ contained in $\text{Stab}_G(1) = H$, with $\pi_1(W^1) = W$ that are pairwise distinct. Moreover, since $W$ is generalized parabolic, so is $W^1$. Finally, since $W$ contains $c$ and $a$, both $b = (a, c)$ and $aba = (c, a)$ are in $W^1$. Thus, $\pi_0(W^1)$ contains $\langle B, a, c \rangle = \mathfrak{G}$. 

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**8.3 Generalized parabolic subgroups of $\mathfrak{G}$**

In this subsection, we show that parabolic subgroups of $\mathfrak{G}$ behave nicely under the closure in the Aut($T$) topology and give a full description of their sections.

In order to study parabolic subgroups, it is of great help to have a precise description of vertex stabilizer and of one particular parabolic subgroup. This was done in [3], where $\text{Stab}_\mathfrak{G}(1^\infty)$ is described as an iterated semi-direct product, as depicted in Figure 3. In particular, for $P := \text{Stab}_\mathfrak{G}(1^\infty)$ we have

$$P = \left( B \times \left( (K \times ((K \times \ldots) \rtimes ((ac)^4))) \rtimes (a, (ac)^4) \right) \rtimes (c, (ac)^4) \right).$$

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As already said, $\mathfrak{G}$ can be endowed with the congruence topology coming from $\text{Aut}(T)$, which is the same as the profinite topology since $\mathfrak{G}$ has a trivial congruence kernel. Recall that for a subgroup $G$ of $\text{Aut}(T)$, we denote by $\tilde{G}$ its closure in $\text{Aut}(T)$. Then we naturally have $\text{Stab}_G(\xi) \leq \text{Stab}_{\tilde{G}}(\xi)$ for any ray in $\partial T$. The first Grigorchuk group satisfies the interesting properties that this inequality is in fact always an equality:

**Proposition 8.9.** Let $\mathfrak{G} \curvearrowright T$ be the branch action of the first Grigorchuk group. For any finite subset $C$ of $\partial T$, we have

$$\overline{\text{Stab}_\mathfrak{G}(C)} = \text{Stab}_{\tilde{\mathfrak{G}}}(C).$$

If moreover $C$ is contained in one $\mathfrak{G}$-orbit, then we also have

$$\overline{\text{SSStab}_\mathfrak{G}(C)} = \text{SSStab}_{\tilde{\mathfrak{G}}}(C).$$

Before giving the proof of this proposition, let us take a look at some of its consequences. For $C$ a closed and nowhere dense subset of $\partial T$, we know that $\text{SSStab}_\mathfrak{G}(C)$ is weakly maximal if and only if it acts minimally on $C$. An obvious necessary condition for the action $\text{SSStab}_\mathfrak{G}(C) \curvearrowright C$ to be minimal is that both the action $\text{SSStab}_{\tilde{\mathfrak{G}}}(C) \curvearrowright C$ and $G \curvearrowright C$ are minimal. Proposition 8.9 implies that for finite $C$, this is also a sufficient condition:

**Corollary 8.10.** Let $C$ be a finite subset of $\partial T$. Then $\text{SSStab}_\mathfrak{G}(C)$ is a generalized parabolic subgroup if and only if $C$ is contained in one $\mathfrak{G}$-orbit and in one $\text{SSStab}_{\tilde{\mathfrak{G}}}(C)$-orbit.

The first Grigorchuk group, as well as other groups with all maximal subgroups of finite index, has the nice property that every weakly maximal subgroup is closed in the profinite topology. In [29], the author studied closed subgroups of finitely generated branch groups, and more particularly the maps between the set of closed subgroups of $G$ and of the set of closed subgroups of its profinite completion.

$$\Theta : \text{Sub}_{\text{cl}}(G) \to \text{Sub}_{\text{cl}}(\hat{G})$$

$$H \mapsto \hat{H}$$

$$\Psi : \text{Sub}_{\text{cl}}(\hat{G}) \to \text{Sub}_{\text{cl}}(G)$$

$$M \mapsto M \cap G$$
When restricted to finite index subgroups, these maps are lattice-isomorphisms (that also preserve normality and the index) and $\Theta$ is the inverse of $\Psi$. The author thus asked if the image by one of this map of a weakly maximal subgroup was still a weakly maximal subgroup. This is not the case as shown by the following corollary of Propositions 8.9 and 3.9.

**Corollary 8.11.** Let $\Theta$ and $\Psi$ be the above maps for $G = \mathfrak{G}$ the first Grigorchuk group. Let $C$ be the set of finite subsets $C$ of $\partial T$ such that $\text{SStab}_\mathfrak{G}(C)$ acts minimally on $C$. When restricted to generalized parabolic subgroups of the form $\text{SStab}(C)$ with $C$ in $C$, the maps $\Theta$ and $\Psi$ are bijective and $\Theta$ is the inverse of $\Psi$.

On the other hand, for all $C = \{\xi, \eta\}$ in $\partial T$ of cardinality 2, the subgroup $W_{\xi,\eta} := \text{SStab}_\mathfrak{G}(\{\xi, \eta\})$ is weakly maximal and closed, while for a fixed $\xi$, the subgroup $\Psi(W_{\xi,\eta}) = \text{SStab}_\mathfrak{G}(\{\xi, \eta\})$ is weakly maximal only for countably many $\eta$.

Let $T_{[3]}$ be the finite subtree of $T$ consisting of all vertices of level at most 3. A labelling of $T_{[3]}$ by elements of $\text{Sym}(2)$ is called an allowed pattern if it occurs as the top of the portrait $P(g)$ of some $g$ in $\mathfrak{G}$. Otherwise it is a forbidden pattern. A portrait $P(g)$ of an element $g \in \text{Aut}(T)$ is said to contain a forbidden pattern if there is a labelled subtree in it that is a forbidden pattern. Observe that looking at labelling of vertices of $T_{[3]}$ is equivalent to look at labelling of inner vertices of $T_{[4]}$, that is at automorphisms of $T_{[4]}$. The following result about portrait will be one of the two ingredients in the proof of Proposition 8.9.

**Proposition 8.12 (23).** Let $T$ be the 2 regular rooted tree and $g$ an element of $\text{Aut}(T)$. Then

1. $g$ is in $\mathfrak{G} = \hat{\mathfrak{G}}$ if and only if $P(g)$ contains no forbidden patterns,
2. $g$ is in $G$ if and only if $P(g)$ contains no forbidden patterns and there is a transversal $X$ for $\partial T$ such that for every $v$ in $X$, the portrait of $g$ below $T_v$ is the portrait of one element in $\{1,a,b,c,d\}$.

The following technical lemma about stabilizers of vertices of level 4 is the second main ingredient of the proof of Proposition 8.9.

**Lemma 8.13.** Let $v$ be a vertex of level 4. For any $g \in \text{Stab}_\mathfrak{G}(v)$, there exists $h$ in $\text{Stab}_\mathfrak{G}(T_v)$, the pointwise stabilizer of $T_v$, such that $P(g)$ and $P(h)$ coincide on $T_{[3]}$.

**Proof.** Up to conjugating by an element of $\mathfrak{G}$, it is enough to prove the lemma for $v = 1^4$. On the other hand, it is also sufficient to prove the lemma for elements of some generating set of $\text{Stab}_\mathfrak{G}(1^4)$. Recall that $\text{Stab}_\mathfrak{G}(1^4)$ is the iterated semi-direct product depicted in Figure 3 for $n = 4$. That is, $\text{Stab}_\mathfrak{G}(1^4)$ is generated by

$$S = B_0 \cup K_{00} \cup K_{010} \cup K_{0110} \cup K_{01110} \cup K_{01111} \cup \{c, (ac)^4, b_{01}, (ac)^4_{01}, (ac)^4_{011}, (ab)^2_{0111}\}$$

where $B_0 := B^0 \cap \text{Rist}_\mathfrak{G}(0) = \{g \in \text{Rist}_\mathfrak{G}(0) | g_{00} \in B\}$ and similarly for the other elements. Observe that $(ac)^4 = (b, b, b, b)$ and $(ab)^2 = aba^{-1} \cdot b$ belong to $K$ and that $b_{01} = d$. Hence every elements of $S$ is indeed in $\mathfrak{G}$.
It now remains to construct a function $\psi: S \to \text{Stab}_G(1^4) = \langle S \rangle$ such that $s$ and $\psi(s)$ agree on $T_{[3]}$ and $\psi(s)$ belongs to $\text{Stab}_G(T_{14})$. If $s$ belongs to $B_{g00} \cup K_{g10} \cup K_{g110} \cup K_{g1110}$, it fixes pointwise $T_{14}$ and we can take $\psi(s) = s$. If $s$ belongs to $K_{g1111} \cup \{(ac)^4_{g1111}, (ac)^3_{g1111}, (ab)^2_{g1111}\}$ then $P(s)$ is trivial on $T_{[3]}$ and we can take $\psi(s) = 1$. Finally, we define $\psi(c) := c(ac)^4_{g1111}$, $\psi(b_{g11}) := b_{g11}(ac)^4_{g1111}$ and $\psi((ac)^4) := (ac)^4_{g1111}$. Since the portrait of $(ac)^4_{g1111}$ and $(ac)^3_{g1111}$ are trivial on $T_{[3]}$ we each time have that the portrait of $s$ and of $\psi(s)$ coincide on $T_{[3]}$ and that $\psi(c) = c_{[4]}^{14} = (ac)^3_{g1111} = (ac)^4_{g1111}$ and similarly for $\psi(b_{g11})$ and $\psi((ac)^4)$. We have
\[
c_{[14]} = d \quad b_{[14]} = b \quad (ac)^4_{[14]} = d \quad (ac)^3_{[14]} = c \quad (ac)^2_{[14]} = b
\]
and direct computations give us that $\psi(c) = \psi(b_{g11}) = \psi((ac)^4) = 1$ and hence that $\psi(c), \psi(b_{g11})$ and $\psi((ac)^4)$ all fix pointwise $T_{14}$ as desired. □

We can now prove Proposition 8.9

**Proof of Proposition 8.9.** Since $C$ is finite, there exists a level $n_0$ such that every vertex $v$ of level at least $n_0$ belongs to at most one ray in $G$. For the following, we will always look at levels $n > n_0$.

We begin by proving the equality for pointwise stabilizers. Let $g$ be in $\text{Stab}_G(C)$. We will first define a sequence $(h_n)_n$ of elements of $G$ converging to $g$ and then explain how to modify it to a sequence $(g_n)_n$ of elements of $\text{Stab}_G(C)$ (respectively $\text{SSStab}_G(C)$) that still converges to $g$.

In order to define $h_n$, we start with $P(g)$ and modify it a little bit. Let $v$ be a vertex of level $n$. By Proposition 8.12, it is possible to replace the labelling of $T_v$ by the labelling of an element $g_v$ of $G$ such that the new labelling on $T$ is still the portrait of an element of $G$. Moreover, we have a transversal for $\partial T_v$ satisfying the "$\{1, a, b, c, d\}$-portrait" condition. If we do that for all vertices of level $n$, we obtain an element $h_n$ in $G$ that is at distance at most $2^{-n}$ of $g$. Moreover, the union of transversals for the $\partial T_v$ gives a transversal for $\partial T$ and thus $h_n$ is in $G$.

Now, let $\{v_1, \ldots, v_d\}$ be the intersection of $C$ with level $n$ and $\{w_1, \ldots, w_d\}$ be the intersection of $C$ with level $n + 4$. By assumption on $n \geq n_0$, we can choose $w_i$ to be the unique vertex below $v_i$, of level $n + 4$ that is in $C$. By Lemma 8.13 in the above construction, we can replace the portrait of $g$ below $v_i$ by the portrait of some $h_{v_i}$ fixing $T_{w_i}$. The element $g_n$ obtained in this way is in $G$ (Proposition 8.12), still at distance at most $2^{-n}$ of $g$ and the restriction of its portrait to the $T_{w_i}$'s is trivial. Hence, $g_n$ pointwise stabilizes $C$.

We will show that the same strategy takes care of the setwise stabilizer.

Firstly, we claim that if $C$ is contained in one $G$-orbit, then for every $g \in \text{SSStab}_G(C)$, there exists $n_1$ such that the portrait of $g$ contains only 1 on vertices of level at least $n_1$ that are above $C$. This is due to the fact that $G$ is finitary along rays. Indeed, let $g \in \text{SSStab}_G(C)$ and $\xi$ be in $C$. Then $g.\xi$ belongs to $C$ and since $C$ is contained in one $G$-orbit, there exists $h$ in $G$ such that the portrait of $g$ and $h$ coincide along the ray $\xi$. 48
Therefore, there exists \( n_\xi \) such that the portrait of \( g \) contains only 1 on vertices above \( \xi \) that are of level at least \( n_\xi \). Since \( C \) is finite, we can take \( n_1 \) to be the max of the \( n_\xi \)'s.

Now, let \( n \geq \max\{n_0, n_1\} \) and \( g \) be in \( \text{SStab}_\Phi(C) \). As for the pointwise stabilizer, let \( \{v_1, \ldots, v_d\} \) be the intersection of \( C \) with level \( n \) and \( \{w_1, \ldots, w_d\} \) be the intersection of \( C \) with level \( n + 4 \). By the above remark, the portrait of \( g \) contains only 1 on vertices between \( v_i \) and \( w_i \) included. Hence, we can still apply Lemma 8.13 to obtain an element \( g_n \) at distance at most \( 2^{-n} \) of \( g \) such that the portrait of \( g_n \) contains only 1's on vertices below the \( w_i \). Since the portraits of \( g \) and \( g_n \) coincide on the \( n \)th level and that they both have only 1 on vertices of level at least \( n + 1 \) lying above elements of \( C \), the action of \( g_n \) on \( C \) coincide with the action of \( g \) on \( C \). In particular, \( g_n \) belongs to \( \text{SStab}_G(C) \).

We finally provide a full description of the sections of parabolic subgroups.

**Proposition 8.14.** Let \( W = \text{Stab}_\Phi(\xi) \), with \( \xi = (v_i)_{i \geq 0} \) and let \( \sigma: \{0, 1\}^N \rightarrow \{0, 1\}^N \) be the shift operator: \( \sigma(v_i)_{i \geq 0} = (v_{i+1})_{i \geq 0} \). Then for all \( j \) we have
\[
\pi_{\bar{v}_j}(W) = \text{Stab}_\Phi(\sigma^j(\xi)) \quad \pi_{\bar{v}_j}(W) = J_{0,2} < 2 \hat{\mathcal{G}}
\]
where \( \bar{v}_j \) is the only sibling of \( v_j \).

Moreover, if \( v \in T \) is not equal to any \( v_j \) or \( \bar{v}_j \), then \( \pi_v(W) = \mathcal{G} \).

**Proof.** The description of \( P \) immediately implies that for all \( j \) we have
\[
\pi_{\bar{v}_j}(P) = P \quad \pi_{1^j 1^{j+1}}(P) = \langle B, (ad)^2, a \rangle = J_{0,2} < 2 \hat{\mathcal{G}}
\]
For the general case, let \( \xi \) be any ray in \( T \) and \( W = \text{Stab}_\Phi(\xi) \). There exists \( g \in \hat{\mathcal{G}} \) sending \( \xi \) onto \( 1^\infty \). Therefore, for all \( j \), we have
\[
\overline{\pi_{\bar{v}_j}(W)} = \pi_{\bar{v}_j}(\overline{W}) = \pi_{v_j}(\text{Stab}_\Phi(\xi)) = \pi_{1^j 1^{j+1}}(\text{Stab}_\Phi(1^\infty))^g = J_{0,2}^g = J_{0,2}.
\]
Where the last equality follows from the fact that the closure preserves normality. Finally we have \( \pi_{\bar{v}_j}(W) = \overline{\pi_{\bar{v}_j}(W)} \cap G = J_{0,2} \).

For the last part, if \( v \) is not equal to one off the \( v_j \) or \( \bar{v}_j \), then it is a descendant of some \( \bar{v}_j \). In this case, \( \pi_v(W) \) is equal to the section \( \pi_w(J_{0,2}) \) for some vertex \( w \) distinct from the root. All these sections are equal to \( \mathcal{G} \). \( \square \)

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