Two-step Shape Invariance in the Framework of $\mathcal{N}$-fold Supersymmetry

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Abstract

We extensively investigate two-step shape invariance in the framework of $\mathcal{N}$-fold supersymmetry. We first show that any two-step shape-invariant system possesses type A 2-fold supersymmetry with an intermediate Hamiltonian and thus has second-order parasupersymmetry as well. Employing the general form of type A 2-fold supersymmetry, we systematically construct two-step shape-invariant potentials. In addition to the well-known ordinary shape-invariant potentials, we obtain several new and novel two-step shape-invariant ones which are not ordinary shape invariant. Furthermore, some of the latter potentials are conditionally two-step shape invariant and thus are conditionally solvable.

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I. INTRODUCTION

In the formalism of supersymmetric quantum mechanics (SUSY QM) [1, 2] potentials with unbroken SUSY and shape invariance (SI) [3] can be exactly solved by a well-known standard procedure [4, 5]. The potential algebras of these systems have also been identified [6–9] providing an alternate method of finding exact solutions. Mainly three classes of SI have been studied in the literature, namely, (1) translational class [10–13] - when the parameters differ only by a constant i.e. \( b^{(1)} = b^{(0)} + \alpha \), \( \alpha \) being a constant, (2) scaling class [14, 15] - when the parameters are related by a scaling factor i.e. \( b^{(1)} = q b^{(0)} \) with \( 0 < q < 1 \), and (3) exotic class [14] - when the parameters are related by other equalities such as \( b^{(1)} = q (b^{(0)})^p \) and \( b^{(1)} = q b^{(0)}/(1 + p b^{(0)}) \). It should be mentioned here that SI is neither a necessary nor a sufficient condition for exact solvability. In fact, some exactly solvable potentials are shown not to be SI [15]. Recently a complete set of additive SI potentials have been generated from an Euler equation [16]. The SI condition is also studied in the context of fractional SUSY QM of order \( k \) (\( k = 3, 4, \ldots \)) generalizing the \( Z_2 \) grading of the relevant Hilbert space to a \( Z_k \) grading [17–20].

The idea of SI can be extended to the more general concept of SI in two and even multi-steps. While there are quite a number of works on one-step (namely, ordinary) SI, the literature dealing with SI in two or more steps is rather scanty. In [14] the authors first introduced the concept of SI in two steps to enlarge the class of exactly solvable Hamiltonians. The two-step SI approach was utilized for dealing with SUSY QM problems with spontaneously broken SUSY [21]. For a quantum mechanical system with position-dependent mass the same approach was used to handle broken SUSY problem [22]. Recently a class of solvable potentials of translational SI in two steps were obtained [23, 24]. It was found that discontinuity at some points was a characteristic of the two-step superpotentials, therefore giving rise to Dirac delta-function singularities in the corresponding potentials if they are considered in the whole line \( x \in (-\infty, +\infty) \). The translational SI potentials were shown to possess a potential algebra involving three generators of angular momentum type. The potential algebra for the case of SI in \( k \) steps (\( k \) being an arbitrary positive integer) was found [25] to be equivalent to the generalized deformed oscillator algebra that had a built-in \( Z_4 \) grading structure and was constructed in terms of the generators of the deformed harmonic oscillator \((I, A, A^\dagger, N)\) as well as the grading generator \( T \) of the cyclic group of order \( k \). The obtained potentials included the cyclic SI potentials of period \( k \) as a special case.

On the other hand, the framework of \( N \)-fold SUSY [26–28] has gained much interest during the last few years. It is possibly one of the most powerful frameworks among various existing methods for finding or constructing quantum systems which admit exact solutions in a certain sense. In particular, a few remarkable features are in order here:

1) \( N \)-fold SUSY includes all the ordinary SUSY methods as its special cases and is equivalent to weak quasi-solvability [27] which is less restrictive concept than quasi-exact solvability [29, 30].

2) An arbitrary one-body quantum Hamiltonian which admits two (local) solutions in closed form belongs to a special class of \( N \)-fold SUSY, namely, type A 2-fold SUSY [31, 32] irrespective of whether or not it is Hermitian, \( \mathcal{PT} \) symmetric [33], pseudo-Hermitian [34] and so on.

3) Many of the so far constructed (quasi-)solvable position-dependent mass Hamiltonians are also realized as special cases of type A \( N \)-fold SUSY [35].

We note that almost all the models having essentially the same symmetry as \( N \)-fold SUSY
but called with other terminologies in the literature, such as Pöschl–Teller and Lamé potentials, are actually also particular cases of type A $\mathcal{N}$-fold SUSY. For a review of $\mathcal{N}$-fold SUSY, see Ref. [36]. Recently in Refs. [37, 38], the necessary and sufficient conditions for type A $\mathcal{N}$-fold SUSY Hamiltonians to possess intermediate Hamiltonians were derived for $\mathcal{N} = 2$ and 3. As a by-product, some well-known translational SI potentials were naturally obtained. It indicates that one can in principle go one step further and consider two-step SI also in the framework of $\mathcal{N}$-fold SUSY.

In this paper, we extensively investigate two-step SI in the framework of $\mathcal{N}$-fold SUSY. A key ingredient of our approach relies on the observation that any two-step SI system possesses type A 2-fold SUSY with an intermediate Hamiltonian whose analytic and algebraic structures are both well understood. The latter fact indeed enables us to make a sophisticated analysis on general aspects of two-step SI without recourse to any specific assumption or model. Employing the general form of type A 2-fold SUSY, we systematically construct two-step SI potentials without relying on any ad hoc ansatz. In addition to the well-known ordinary SI potentials, we successfully obtain several novel two-step SI ones which have not been reported in the existing literature. Furthermore, some of the latter two-step SI are conditional and thus the corresponding systems are conditionally solvable.

We organize this paper as follows. In Section II, we review the concept of two-step SI and discuss its general aspects. In particular, we show that any two-step SI system has type A 2-fold SUSY with an intermediate Hamiltonian and second-order paraSUSY and that two-step SI always means solvability and actually lies between solvability and ordinary SI. In Section III, we investigate two-step SI for particular models which can be realized in the cases of type A $\mathcal{N}$-fold SUSY for $\mathcal{N} > 2$. The obtained two-step SI potentials include the well-known ones such as (radial) harmonic oscillators, Morse, Scarf, Pöschl–Teller potentials, and so on. In addition, we also obtain some novel potentials which are expressible in terms of elliptic, exponential (including hyperbolic and trigonometric), and rational functions. In Section IV, we construct more general two-step SI potentials under a less restrictive condition. We obtain more novel two-step SI potentials some of which admit analytic expressions only implicitly. Most of the new potentials obtained in Sections III and IV do not have ordinary SI and some of their two-step SI are conditional. Finally in Section V, we summarize the results and discuss perspectives and future issues.

II. TWO-STEP SHAPE INVARIANCE AND TYPE A 2-FOLD SUSY

First of all, let us introduce the concept of two-step SI. Suppose we have a system composed of two sets of SUSY QM

$$2H_i^\pm(c) = A_i^+(c)A_i^-(_c), \quad A_i^\pm(c) = \mp \frac{d}{dx} + W_i(x; c), \quad (i = 0, 1),$$

(2.1)

where $c$ denotes a set of all the parameters involved in the system, which satisfies the following constraint

$$H_0^+(c) = H_1^-(c) + R(c),$$

(2.2)

where $R(c)$ is a constant depending only on $c$. According to Ref. [14], the system is said to have two-step shape invariance if $H_0^-$ and $H_1^+$ satisfy

$$H_1^+(c^{(0)}) = H_0^-(c^{(2)}) + \tilde{R}_2(c^{(0)}),$$

(2.3)
where $\tilde{R}_2(c^{(0)})$ is another constant and $c^{(2)} = c^{(2)}(c^{(0)})$ is another set of parameters both of which depend only on $c^{(0)}$.

To begin with, we shall show in what follows that any such a system has type A 2-fold SUSY with an intermediate Hamiltonian studied in Ref. [37]. We first note from (2.1) and (2.2) that $H_0^{-}(c)$ and $H_1^{+}(c) + R(c)$ are intertwined by the second-order linear differential operator $A_1^{-} A_0^{-}$ as

$$ A_1^{-}(c) A_0^{-}(c) H_0^{-}(c) = (H_1^{+}(c) + R(c)) A_1^{-}(c) A_0^{-}(c), \quad (2.4) $$

and thus form a pair of 2-fold SUSY. It is evident from (2.2) that the latter 2-fold SUSY system possesses an intermediate Hamiltonian. In addition, we can have an expression of the kernel of $A_1^{-}(c) A_0^{-}(c)$ in closed form as

$$ \ker A_1^{-}(c) A_0^{-}(c) = \left\{ 1, \int^x dx' \exp \left( \int^{x'} dx'' (W_0(x''; c) - W_1(x''; c)) \right) \right\} \times \exp \left( - \int^x dx' W_0(x'; c) \right), \quad (2.5) $$

which is preserved by $H_0^{-}(c)$ due to the intertwining relation (2.4). Hence, $H_0^{-}(c)$ admits two linearly independent local solutions in closed form. The latter fact is a necessary and sufficient condition for a one-dimensional Hamiltonian to belong to type A 2-fold SUSY proved in Ref. [32]. Therefore, we finally conclude that any two-step SI system has type A 2-fold SUSY with an intermediate Hamiltonian.

It is now clear from the above process of the proof that any two-step SI system (2.1)–(2.3) has one-to-one correspondence with a type A 2-fold SUSY system with an intermediate Hamiltonian $(H^+, H^{11}, P^-_2 = P_{21}^- P_{22}^1)$ in Ref. [37] by

$$ H_0^- \leftrightarrow H^-, \quad H_0^+ \leftrightarrow H^{11}, \quad H_1^+ + R \leftrightarrow H^+, \quad A_0^\pm \leftrightarrow P_{22}^\pm, \quad A_1^\pm \leftrightarrow P_{21}^\pm. \quad (2.6) $$

To investigate general structure of two-step SI systems in the framework of type A 2-fold SUSY with an intermediate Hamiltonian, it is convenient to make a ‘gauge’ transformation from the physical $x$-space to a gauged $z$-space defined by

$$ \tilde{H}^\pm = e^{W_z^\pm} H^\pm e^{-W_z^\pm}, \quad \tilde{P}_2^\pm = e^{W_z^\pm} P_2^\pm e^{-W_z^\pm}. \quad (2.7) $$

According to Ref. [37], the general form of a gauged type A 2-fold SUSY system with an intermediate Hamiltonian is given by

$$ \tilde{H}^-(c) = -A(z) \frac{d^2}{dz^2} - Q(z; b) \frac{d}{dz} + \frac{1}{2} Q'(z; b) - R, \quad (2.8) $$

$$ \tilde{H}^+(c) = -A(z) \frac{d^2}{dz^2} - Q(z; b) \frac{d}{dz} - \frac{3}{2} Q'(z; b) + \frac{Q(z; b) A'(z)}{A(z)} - R, \quad (2.9) $$

$$ \tilde{P}_2^-(c) = (z')^2 \frac{d^2}{dz'^2}, \quad \tilde{P}_2^+(c) = -\int dz' P_2^+(c) e^{\int dz' \frac{Q(z; b)}{A(z)}} = (z')^2 \left( \frac{d}{dz} + \frac{Q(z; b)}{A(z)} \right)^2, \quad (2.10) $$

with the set of parameters $c = \{b, R\}$, where $A(z)$ is an arbitrary function of $z$ at present and $Q(z; b)$ is a polynomial of at most first-degree in $z$

$$ Q(z; b) = b_1 z + b_0. \quad (2.11) $$
The function \( z' = \frac{dz}{dx} \) which connects the variable \( z \) in the gauged space and the coordinate \( x \) in the physical space is determined by

\[
z'(x)^2 = 2A(z) \bigg|_{z=z(x)}; \tag{2.12}\]

It can be easily checked by a direct calculation that \( \tilde{H}^\pm \) are intertwined by \( \tilde{P}_2^\pm \) as

\[
\tilde{P}_2^-(c)\tilde{H}^-(c) = \tilde{H}^+(c)\tilde{P}_2^-(c), \quad \tilde{P}_2^+(c)\tilde{H}^+(c) = \tilde{H}^-(c)\tilde{P}_2^+(c). \tag{2.13}\]

With a factorized form of the 2-fold supercharges

\[
\tilde{P}_2^-(c) = \tilde{P}_{21}^-(c)\tilde{P}_{22}^-(c), \quad \tilde{P}_2^+(c) = \tilde{P}_{21}^+(c)\tilde{P}_{22}^+(c), \tag{2.14}\]

the three component Hamiltonians \( \tilde{H}^-, \tilde{H}^1, \) and \( \tilde{H}^+ \) are written as

\[
2\tilde{H}^- = \tilde{P}_{22}^-(c)\tilde{P}_{22}^+(c) + 2C_{22}(c), \tag{2.15}\]
\[
2\tilde{H}^1 = \tilde{P}_{22}^-(c)\tilde{P}_{22}^+(c) + 2C_{22}(c) = \tilde{P}_{21}^+(c)\tilde{P}_{21}^-(c) + 2C_{21}(c), \tag{2.16}\]
\[
2\tilde{H}^+ = \tilde{P}_{21}^+(c)\tilde{P}_{21}^-(c) + 2C_{21}(c). \tag{2.17}\]

The type A gauged 2-fold supercharges (2.10) admit a factorization (2.14) with

\[
\begin{align*}
\tilde{P}_{21}^-(c) &= z' \left( \frac{d}{dz} - \frac{A'(z)}{2A(z)} \right), \quad \tilde{P}_{22}^-(c) = z' \frac{d}{dz}, \\
\tilde{P}_{22}^+(c) &= -z' \left( \frac{d}{dz} + \frac{2Q(z;b) - A'(z)}{2A(z)} \right), \quad \tilde{P}_{21}^+(c) = -z' \left( \frac{d}{dz} + \frac{Q(z;b)}{A(z)} \right). \tag{2.18}\end{align*}
\]

It is easy to check that the pair of Hamiltonians \( \tilde{H}^\pm \) (2.15) and (2.17) obtained from (2.18) actually coincides with the pair of type A gauged Hamiltonians (2.8) and (2.9) with

\[
2C_{22} = b_1 - 2R, \quad 2C_{21} = -b_1 - 2R. \tag{2.19}\]

The intermediate gauged Hamiltonian \( \tilde{H}^1 \) (2.16) with respect to the factorization (2.18) is calculated as

\[
\tilde{H}^1(c) = -A(z)\frac{d^2}{dz^2} - Q(z;b) \frac{d}{dz} + \frac{A''(z)}{2} + \frac{(2Q(z;b) - A'(z))A'(z)}{4A(z)} - \frac{b_1}{2} - R. \tag{2.20}\]

As was shown in Ref. [37], the type A gauged 2-fold supercharges (2.10) admit, in addition to (2.18), a one-parameter family of factorizations, denoted by symbols with wide hat, with

\[
\begin{align*}
\hat{P}^-_{21} &= z' \left( \frac{d}{dz} - \frac{A'(z)}{2A(z)} + \frac{1}{z + z_0} \right), \quad \hat{P}^-_{22} = z' \left( \frac{d}{dz} - \frac{1}{z + z_0} \right), \\
\hat{P}^+_{22} &= -z' \left( \frac{d}{dz} + \frac{2Q(z;b) - A'(z)}{2A(z)} + \frac{1}{z + z_0} \right), \\
\hat{P}^+_{21} &= -z' \left( \frac{d}{dz} + \frac{Q(z;b)}{A(z)} - \frac{1}{z + z_0} \right), \tag{2.21}\end{align*}
\]

where \( z_0 \in \mathbb{C} \) is a parameter. Then, a type A 2-fold SUSY system admits another intermediate gauged Hamiltonian \( \hat{H}^{12} \) with respect to one of the above one-parameter family of
Another remarkable aspect of type A 2-fold SUSY systems with an intermediate Hamiltonian factorizations if and only if $b_1 \neq 0$. In fact, if it is the case, the pair of gauged Hamiltonians $\hat{H}^\pm$ (2.15) and (2.17) calculated with the second factorization (2.21) coincides with the pair of type A gauged Hamiltonians (2.8) and (2.9) with

$$2\hat{C}_{22} = -b_1 - 2R, \quad 2\hat{C}_{21} = b_1 - 2R,$$

provided that $z_0 = b_0/b_1$. The second intermediate gauged Hamiltonian $\hat{H}^{i2}$ with respect to the second factorization (2.21) defined via (2.16) with the corresponding hatted quantities is calculated as

$$\hat{H}^{i2} = \hat{H}^{i1} - \frac{b_1 A(z)}{Q(z;b)} + \frac{2(b_1)^2 A(z)}{Q(z;b)^2}.$$  

(2.23)

Another remarkable aspect of type A 2-fold SUSY systems with an intermediate Hamiltonian is that they possess second-order paraSUSY as well [37]. Introducing a triple of operators $(H_P, Q^\pm_P)$ as

$$H_P = H^- (\psi_P^-)^2 (\psi_P^+)^2 + H^{i1} (\psi_P^- \psi_P^-) (\psi_P^+ \psi_P^-) + H^+ (\psi_P^+)^2 (\psi_P^-)^2,$$

$$Q_P = P_{22}^- (\psi_P^-)^2 + P_{21}^+ (\psi_P^+)^2,$$

(2.24)

$$Q_P^- = P_{22}^- (\psi_P^-)^2 + P_{21}^+ (\psi_P^+)^2 \psi_P^-,$$

where $H^\pm$ are a pair of type A 2-fold SUSY Hamiltonians, $H^{i1}$ is (one of) its intermediate Hamiltonians, and $\psi^\pm_P$ are second-order parafermions satisfying

$$(\psi_P^\pm)^2 \neq 0, \quad (\psi_P^\pm)^3 = 0, \quad \{\psi_P^-, \psi_P^+\} + \{(\psi_P^-)^2, (\psi_P^+)^2\} = 2I,$$

(2.25)

we see that the triple $(H_P, Q^\pm_P)$ satisfies the second-order paraSUSY relations in [39]

$$(Q_P^\pm)^2 \neq 0, \quad (Q_P^\pm)^3 = 0, \quad [Q_P^\pm, H_P] = 0,$$

(2.26)

$$(Q_P^\pm)^2 Q_P^\mp + Q_P^\pm Q_P^\mp Q_P^\pm + Q_P^\mp (Q_P^\pm)^2 = 4Q_P^\pm H_P,$$

as well as the generalized 2-fold superalgebra

$$(Q_P^-)^2 (Q_P^\pm)^2 + Q_P^\pm (Q_P^\pm)^2 Q_P^\pm + (Q_P^\pm)^2 (Q_P^-)^2 = 4 (H_P)^2 - (b_1)^2,$$

(2.27)

where $b_1$ is the same parameter as the one appeared in (2.11).

To see what are significant consequences of two-step SI, let us first show that if a pair of type A 2-fold SUSY gauged Hamiltonians $\hat{H}^\pm$ satisfies the two-step SI condition (2.3), that is,

$$\hat{H}^+(c^{(2k)}) = \hat{H}^-(c^{(2k+2)}) + \hat{R}_2(c^{(2k)}),$$

(2.28)

where we have generalized $c^{(0)}$ and $c^{(2)}$ in (2.3) to $c^{(2k)}$ and $c^{(2k+2)}$ ($k = 0, 1, 2, \ldots$), and if, in addition, $\hat{H}^-(c^{(2k+2)})$ preserves a linear space $\tilde{\mathcal{V}}^-(c^{(2k+2)})$, then $\hat{H}^-(c^{(2k)})$ preserves the space $\tilde{P}_2^+ (c^{(2k)}) \tilde{\mathcal{V}}^- (c^{(2k+2)})$. In a mathematical language, we have

$$\hat{H}^-(c^{(2k+2)}) \tilde{\mathcal{V}}^- (c^{(2k+2)}) \subset \tilde{\mathcal{V}}^- (c^{(2k+2)})$$

$$\Rightarrow \hat{H}^-(c^{(2k)}) \tilde{P}_2^+ (c^{(2k)}) \tilde{\mathcal{V}}^- (c^{(2k+2)}) \subset \tilde{P}_2^+ (c^{(2k)}) \tilde{\mathcal{V}}^- (c^{(2k+2)}).$$

(2.29)
Indeed, it follows from (2.14)–(2.17) that

\[ 2\tilde{H}^-(c^{(2k)})\tilde{P}^+_2(c^{(2k)})\tilde{V}^-(c^{(2k+2)}) = \tilde{P}^+_2(c^{(2k)})\left[\tilde{H}^-(c^{(2k+2)}) + \tilde{R}_2(c^{(2k)})\right]\tilde{V}^-(c^{(2k+2)}), \]  

and thus conclude (2.29). On the other hand, it is evident from (2.13) that \( \tilde{H}^- \) preserves the kernel of \( \tilde{P}^-_2 \) which is at most two dimensional:

\[ \tilde{H}^-(c)\tilde{V}^-_2(c) \subset \tilde{V}^-_2(c), \quad \tilde{V}^-_2(c) = \ker \tilde{P}^-_2(c) = \langle 1, z \rangle, \]  

and thus it is quasi-solvable [36]. Combining (2.29) and (2.31), we see that the Hamiltonian \( \tilde{H}^-(c^{(0)}) \) preserves an infinite flag of finite-dimensional linear spaces

\[ \tilde{V}^-_2(c^{(0)}) \subset \tilde{V}^-_4(c^{(0)}) \subset \cdots \subset \tilde{V}^-_{2n}(c^{(0)}) \subset \cdots, \]  

where \( \tilde{V}^-_{2n}(c^{(0)}) \) is defined by

\[ \tilde{V}^-_{2n}(c^{(0)}) = \tilde{V}^-_2(c^{(0)}) + \tilde{P}^+_2(c^{(0)})\tilde{V}^-_2(c^{(2)}) + \cdots + \tilde{P}^+_2(c^{(0)})\tilde{P}^+_2(c^{(2)})\cdots \tilde{P}^+_2(c^{(2n-4)})\tilde{V}^-_2(c^{(2n-2)}). \]  

Hence, any two-step SI Hamiltonian is solvable [36] unless there is a natural number \( n \) such that \( \tilde{V}^-_{2n}(c^{(0)}) \subset \tilde{V}^-_2(c^{(0)}) \). We also note that if an intermediate Hamiltonian \( \tilde{H}^{11} \) satisfies

\[ \tilde{H}^{11}(c^{(2k+1)}) = \tilde{H}^-(c^{(2k+2)}) + \tilde{R}_1(c^{(2k+1)}), \]  

where \( \tilde{R}_1(c^{(2k+1)}) \) is a constant depending only on \( c^{(2k+1)} \), then the three Hamiltonians \( \tilde{H}^- \), \( \tilde{H}^{11} \), and \( \tilde{H}^+ \) constitute a sequence of ordinary SI with

\[ \tilde{H}^+(c^{(2k)}) = \tilde{H}^{11}(c^{(2k+1)}) + \tilde{R}_1(c^{(2k)}), \]  

where \( \tilde{R}_2(c^{(2k)}) = \tilde{R}_1(c^{(2k+1)}) + \tilde{R}_1(c^{(2k)}) \). Conversely, it is evident from the definition that any ordinary SI Hamiltonian is two-step SI. It means that ordinary SI can be regarded as a sufficient condition for two-step SI. Summarizing the above arguments, we have the following relation:

\[ \text{(Solvability)} \supset \text{(Two-step SI)} \supset \text{(Ordinary SI)}. \]
We shall say that two-step SI is \textit{irreducible} if it is \textit{not} ordinary SI simultaneously, and is \textit{reducible} otherwise.

The pair of type A 2-fold SUSY potentials $V^\pm$ with an intermediate Hamiltonian is given by \cite{37},

$$V^\pm(x, c) = -\frac{1}{4A(z)} \left[ A(z)A''(z) - \frac{3}{4} A'(z)^2 - (b_1 z + b_0)^2 \right. \left. \mp 2((b_1 z + b_0)A'(z) - 2b_1 A(z)) \right] \bigg|_{z=z(x)} - R, \tag{2.36}$$

where the set of parameters is $c = \{b_1, b_0, R\}$. The potential terms in the first and second intermediate Hamiltonians $H^{i1}$ and $H^{i2}$, the latter of which exists only when $b_1 \neq 0$, are calculated as

$$V^{i1}(x, c) = \left. \frac{A''(z)}{4} - \frac{A'(z)^2 - 4(b_1 z + b_0)^2}{16 A(z)} \right|_{z=z(x)} - R, \tag{2.37}$$

$$V^{i2}(x; c) = V^{i1}(x; c) - \frac{b_1 A'(z)}{b_1 z + b_0} + \frac{2(b_1)^2 A(z)}{(b_1 z + b_0)^2} \bigg|_{z=z(x)} - R. \tag{2.38}$$

In what follows, we put $R = 0$ to remove an irrelevant parameter without any loss of generality. From the form of a pair of potentials (2.36), we immediately see that there is always two-step SI of the following type

$$V^+(x; b_1, b_0) = V^-(x; -b_1, -b_0), \tag{2.39}$$

that is, the relation (2.3) holds with $b_1^{(2)} = -b_1^{(0)}$, $b_0^{(2)} = -b_0^{(0)}$, and $R_2(b_1^{(0)}, b_0^{(0)}) = 0$, irrespective of the form of $A(z)$. The latter two-step SI usually changes the normalizability of the corresponding solvable sectors and thus can be employed to convert a solvable model without normalizable eigenfunctions into an exactly solvable one. A similar tactics was already demonstrated, although with a different type from (2.39), in Ref. \cite{21}. We shall say that the two-step SI under consideration is \textit{reflective} if it is characterized by (2.39). In general, reflective two-step SI is irreducible. Our central concern is now what kinds of $A(z)$ can admit non-reflective two-step SI.

\section*{III. POLYNOMIAL $A(z)$}

To begin with, we recall the fact that in type A $N$-fold SUSY for $N > 2$ the function $A(z)$ must be a polynomial of at most fourth degree:

$$A(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0. \tag{3.1}$$

Hence, it is natural to examine first the case of this polynomial $A(z)$ of at most fourth degree. In the followings, we present the forms of potentials resulting from a polynomial $A(z)$ of from zeroth to fourth degree.

\textbf{Case 1-1:} $a_4 = a_3 = a_2 = a_1 = 0$ \& $a_0 \neq 0$

Let us first consider the case of a zeroth degree $A(z)$. In this case, the pair of potentials
(2.36) are given by

\[ V^{\pm}(x; b_1, b_0) = \frac{(b_1)^2 z^2 + 2b_1 b_0 z + (b_0)^2 \mp 4a_0 b_1}{4a_0} \bigg|_{z=z(x)}. \]  

(3.2)

In addition to (2.39), they have non-reflective two-step SI as

\[ V^+(x; b_1, b_0) = V^-(x; b_1, b_0) - 2b_1, \]  

(3.3)

that is, the relation (2.3) holds with \( b_1^{(2)} = b_1^{(0)}, \) \( b_0^{(2)} = b_0^{(0)}, \) and \( \tilde{R}_2(b_1^{(0)}, b_0^{(0)}) = -2b_1^{(0)} \). The first intermediate potential (2.37) is calculated as

\[ V^{i1}(x; b_1, b_0) = \frac{(b_1)^2 z^2 + 2b_1 b_0 z + (b_0)^2}{4a_0} \bigg|_{z=z(x)}, \]  

(3.4)

and thus it together with \( V^{\pm} \) constitutes a sequence of ordinary SI as

\[ V^{i1}(x; b_1, b_0) = V^-(x; b_1, b_0) - b_1, \]  
\[ V^+(x; b_1, b_0) = V^{i1}(x; b_1, b_0) - b_1. \]  

(3.5)

Therefore, the system has reducible two-step SI. On the other hand, the second intermediate potential (2.38) which exists only when \( b_1 \neq 0 \) reads as

\[ V^{i2}(x; b_1, b_0) = V^{i1}(x; b_1, b_0) + \frac{2a_0 (b_1)^2}{(b_1 z + b_0)^2} \bigg|_{z=z(x)}, \]  

(3.6)

and has no SI with the other potentials. The equation (2.12) in the present case is integrated as

\[ z = \sqrt{2a_0} x. \]  

(3.7)

Substituting (3.7) into (3.2), (3.4), and (3.6), we finally obtain the potentials in the \( x \)-space. The system is well-known harmonic oscillators.

**Case 1-2:** \( a_4 = a_3 = a_2 = 0 \) \& \( a_1 \neq 0 \)

Next, we shall consider a first degree \( A(z) \) with \( a_4 = a_3 = a_2 = 0 \) and \( a_1 \neq 0 \). In this case, the pair of potentials (2.36) are given by

\[ V^{\pm}(x; b_1, b_0) = \frac{(b_1)^2 z}{4a_1} \bigg( \frac{b_1 a_0}{a_1} - \frac{b_0 \pm 3a_1}{2} \bigg) \bigg|_{z=z(x)} \]  

(3.8)

In addition to (2.39), they have non-reflective two-step SI as

\[ V^+(x; b_1, b_0) = V^-(x; b_1, b_0 + 2a_1) - 2b_1, \]  

(3.9)

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that is, the relation (2.3) holds with \( b^{(2)}_1 = b^{(0)}_1, \ b^{(2)}_0 = b^{(0)}_0 + 2a_1, \) and \( \tilde{R}_2(b^{(0)}_1, b^{(0)}_0) = -2b^{(0)}_1. \)

The first intermediate potential (2.37) is calculated as

\[
V^{11}(x; b_1, b_0) = \frac{(b_1)^2}{4a_1} z + \frac{1}{4(a_1 z + a_0)} \left( \frac{b_1 a_0}{a_1} - \frac{2b_0 + a_1}{2} \right) \left( \frac{b_1 a_0}{a_1} - \frac{2b_0 - a_1}{2} \right) \left. \right|_{z = z(x)},
\]

and thus it together with \( V^\pm \) constitutes a sequence of ordinary SI as

\[
V^{11}(x; b_1, b_0) = V^-(x; b_1, b_0 + a_1) - b_1, \quad V^+(x; b_1, b_0) = V^{11}(x; b_1, b_0 + a_1) - b_1.
\]

Therefore, the system has reducible two-step SI. On the other hand, the second intermediate potential (2.38) which exists only when \( b_1 \neq 0 \) reads as

\[
V^{12}(x; b_1, b_0) = V^{11}(x; b_1, b_0) + \frac{a_1 b_1}{b_1 z + b_0} + \frac{2(a_0 b_1 - a_1 b_0) b_1}{(b_1 z + b_0)^2} \left. \right|_{z = z(x)},
\]

and has no SI with the other potentials. The equation (2.12) in the present case is integrated as

\[
z = \frac{a_1}{2} x^2 - \frac{a_0}{a_1}.
\]

Substituting (3.13) into (3.8), (3.10), and (3.12), we finally obtain the potentials in the \( x \)-space. The system consists of well-known radial harmonic oscillators.

**Case 1-3: \( a_4 = a_3 = 0 \ \& \ a_2 \neq 0 \)**

Next, we shall consider the case of a second degree \( A(z) \). In this case, the pair of potentials (2.36) are given by

\[
V^{\pm}(x; b_1, b_0) = \frac{(b_1)^2 + (a_2)^2}{4a_2} + \frac{(2a_2 b_0 - a_1 b_1)(b_1 \pm 2a_2)}{4a_2(a_2 z^2 + a_1 z + a_0)} z
+ \frac{a_2(2b_0 \pm a_1)(2b_0 \pm 3a_1) - 4a_0(b_1 \pm a_2)(b_1 \pm 3a_2)}{16a_2(a_2 z^2 + a_1 z + a_0)} \left. \right|_{z = z(x)}.
\]

In addition to (2.39), they have non-reflective two-step SI as

\[
V^+(x; b_1, b_0) = V^-(x; b_1 + 4a_2, b_0 + 2a_1) - 2(b_1 + 2a_2),
\]

that is, the relation (2.3) holds with \( b^{(2)}_1 = b^{(0)}_1 + 4a_2, \ b^{(2)}_0 = b^{(0)}_0 + 2a_1, \) and \( \tilde{R}_2(b^{(0)}_1, b^{(0)}_0) = -2(b^{(0)}_1 + 2a_2). \) The first intermediate potential (2.37) is calculated as

\[
V^{11}(x; b_1, b_0) = \frac{(b_1)^2 + (a_2)^2}{4a_2} + \frac{(2a_2 b_0 - a_1 b_1)b_1}{4a_2(a_2 z^2 + a_1 z + a_0)} z
+ \frac{a_2(2b_0 + a_1)(2b_0 - a_1) - 4a_0(b_1 + a_2)(b_1 - a_2)}{16a_2(a_2 z^2 + a_1 z + a_0)} \left. \right|_{z = z(x)},
\]
and thus it together with \( V^\pm \) constitutes a sequence of ordinary SI as

\[
V^{ii}(x; b_1, b_0) = V^-(x; b_1 + 2a_2, b_0 + a_1) - b_1 - a_2, \\
V^+(x; b_1, b_0) = V^{ii}(x; b_1 + 2a_2, b_0 + a_1) - b_1 - a_2.
\] (3.17)

Therefore, the system has reducible two-step SI. On the other hand, the second intermediate potential (2.38) which exists only when \( b_1 \neq 0 \) reads as

\[
V^{i2}(x; b_1, b_0) = V^{ii}(x; b_1, b_0) + \frac{a_1 b_1 - 2a_2 b_0}{b_1 z + b_0} + 2 \frac{a_0(b_1)^2 - a_1 b_1 b_0 + a_2(b_0)^2}{(b_1 z + b_0)^2} \Bigg|_{z = z(x)},
\] (3.18)

and has no SI with the other potentials. The equation (2.12) in the present case is integrated as

\[
z + \frac{a_1}{2a_2} = \frac{1}{2} e^{\sqrt{2a_2} x} - \left( \frac{a_0}{2a_2} - \frac{(a_1)^2}{8(a_2)^2} \right) e^{-\sqrt{2a_2} x}.
\] (3.19)

Substituting (3.19) into (3.14), (3.16), and (3.18), we finally obtain the potentials in the \( x \)-space. The system consists of well-known Morse, Scarf, or Pöschl–Teller potentials, depending on the relative relation among given values of the parameters \( a_i \) (\( i = 0, 1, 2 \)).

**Case 1-4: \( a_4 = 0 \) & \( a_3 \neq 0 \)**

Next, we shall consider the case of a third degree \( A(z) \). In this case, the pair of potentials (3.36) are given by

\[
V^+(x; b_1, b_0) = \frac{3a_3}{16} z + \frac{a_2 + 8b_1}{16} - \frac{3a_3(3a_1 \mp 8b_0) - a_2(3a_2 \mp 8b_1) - 4(b_1)^2}{16A(z)} z^2 \\
- \frac{27a_3 a_0 - a_1(3a_2 \mp 16b_1) - 8b_0(b_1 \pm 2a_2)}{16A(z)} z \\
- \frac{a_0(9a_2 \mp 24b_1) - a_1(3a_1 \mp 8b_0) - 4(b_0)^2}{16A(z)} \Bigg|_{z = z(x)}.
\] (3.20)

The first intermediate potential (2.37) is calculated as

\[
V^{ii}(x; b_1, b_0) = \frac{15a_3}{16} z + \frac{5a_2}{16} + \frac{4(b_1)^2 + 3a_3 a_1 - (a_2)^2}{16A(z)} z^2 \\
+ \frac{8b_1 b_0 + 9a_3 a_0 - a_2 a_1}{16A(z)} z + \frac{4(b_0)^2 + 3a_2 a_0 - (a_1)^2}{16A(z)} \Bigg|_{z = z(x)}.
\] (3.21)

On the other hand, the second intermediate potential (2.38) which exists only when \( b_1 \neq 0 \) reads as

\[
V^{i2}(x; b_1, b_0) = V^{ii}(x; b_1, b_0) - a_3 z - \frac{a_3 b_0}{b_1} + \frac{a_1(b_1)^2 - 2a_2 b_1 b_0 + 3a_3(b_0)^2}{b_1(b_1 z + b_0)} \\
+ 2 \frac{a_0(b_1)^3 - a_1(b_1)^2 b_0 + a_2 b_1(b_0)^2 - a_3(b_0)^3}{b_1(b_1 z + b_0)^2} \Bigg|_{z = z(x)}.
\] (3.22)

Although \( V^+ \) and \( V^- \) have the same functional dependence on the variable \( z \), it does not necessarily mean that there is a transformation of the parameters \( b_1 \) and \( b_0 \) which lead to
non-reflective two-step SI. In fact, they satisfy the condition for two-step SI (2.3) if and only if

\[
\begin{pmatrix}
-(b_1^- + 2a_2) & 6a_3 \\
 b_0^- + 4a_1 & b_1^- - 4a_2 \\
 6a_0 & b_0^- - 2a_1
\end{pmatrix}
\begin{pmatrix}
b_1^+ \\
b_0^+
\end{pmatrix} = 0,
\]

(3.23)

where \(b_i^\pm = b_i^{(2)} \pm b_i^{(0)}\). The trivial solution \(b_1^+ = b_0^+ = 0\) just produces reflective two-step SI (2.39). To have a non-trivial solution, the parameters must satisfy

\[
6a_3(b_0^- + 4a_1) = -(b_1^- + 2a_2)(b_1^- - 4a_2),
\]

(3.24)

\[
(b_1^- + 2a_2)(b_0^- - 2a_1) = -36a_3a_0.
\]

(3.25)

We first consider the particular case when \(b_1^- = -2a_2\). In the latter case, the set of equations (3.24) and (3.25) reduces by the assumption \(a_3 \neq 0\) to

\[
b_0^+ = b_0^- + 4a_1 = a_0 = 0.
\]

(3.26)

Hence, we have

\[
A(z) = a_3z^3 + a_2z^2 + a_1z, \quad b_1^{(2)} = b_1^{(0)} - 2a_2, \quad b_0^{(2)} = -b_0^{(0)} = -2a_1.
\]

(3.27)

In this case, the pair of potentials (3.20) reads as

\[
V^\pm(x; b_1, b_0) = \frac{3a_3}{16}z + \frac{a_2 \pm 8b_1}{16} - \frac{3a_3(3a_1 \pm 8b_0) - a_2(3a_2 \pm 8b_1) - 4(b_1)^2}{16(a_3z^2 + a_2z + a_1)}z
\]

\[
+ \frac{a_1(3a_2 \pm 16b_1) + 8b_0(b_1 \pm 2a_2)}{16(a_3z^2 + a_2z + a_1)} + \frac{a_1(3a_1 \pm 8b_0) + 4(b_0)^2}{16(a_3z^2 + a_2z + a_1)}z\big|_{z = z(x)}.
\]

(3.28)

We can easily check that the latter pair satisfies

\[
V^+(x; b_1, b_0)A(z) = V^-(x; b_1 - 2a_2, -b_0)A(z) + (b_1 - a_2)\left[a_3z^3 + a_2z^2 + (b_0 - a_1)z\right],
\]

(3.29)

and thus has two-step SI if and only if \(b_0 = 2a_1\), as has been indicated by the second equality in (3.27). It is quite intriguing that a system has SI only when a parameter fulfills a particular condition. This situation is reminiscent of conditional exact solvability [40]. Indeed, as was proved in the previous section, two-step SI always means solvability, and thus the present system is solvable only when \(b_0 = 2a_1\). That is, it is conditionally solvable. Hence, we shall call such SI conditional.

The first and second intermediate potentials (3.21) and (3.22) in this case of conditional two-step SI read respectively as

\[
V_{i1}(x; b_1, b_0) = \frac{15a_3}{16}z + \frac{5a_2}{16} + \frac{4(b_1)^2 + 3a_3a_1 - (a_2)^2}{16(a_3z^2 + a_2z + a_1)}z
\]

\[
+ \frac{8b_1b_0 - a_2a_1}{16(a_3z^2 + a_2z + a_1)} + \frac{4(b_0)^2 - (a_1)^2}{16(a_3z^2 + a_2z + a_1)}z\big|_{z = z(x)}.
\]

(3.30)
and
\[
V^{12}(x; b_1, b_0) = V^{11}(x; b_1, b_0) - a_3 z - \frac{a_3 b_0}{b_1} + a_1 (b_1)^2 - 2a_2 b_1 b_0 + 3a_3 (b_0)^2 \frac{b_1 (b_1 z + b_0)}{b_1 (b_1 z + b_0)} - 2b_0 \frac{a_1 (b_1)^2 - a_2 b_1 b_0 + a_3 (b_0)^2}{b_1 (b_1 z + b_0)^2} \bigg|_{z=z(x)}. \tag{3.31}
\]

Both of them do not have ordinary SI with $V^{\pm}$ given in (3.28) irrespective of whether $b_0 = 2a_1$ or not. Hence, the conditional two-step SI of the system (3.28) is irreducible.

The general solution to equation (2.12) with $A(z)$ given in (3.27) is expressed in terms of elliptic functions when $a_1 \neq 0$. The potentials are thus of deformed Lamé or Heun type. We omit its involved general form here. When $a_1 = 0$, the system automatically has two-step SI since $b_0^{(2)} = b_0^{(0)} = 0$ from the third equality in (3.27). In addition, the form of $A(z)$ reduces to the one given in (3.36) with $d_1 = -2a_2$, and the solution to (2.12) is given by (3.41). Substituting (3.41) into (3.28), (3.30), and (3.31), we finally obtain the potentials in the $x$-space.

When $b_1^- \neq -2a_2$, we see from (3.24) and (3.25) that $b_1^-$ and $b_0^-$ are functions of $a_i$ ($i = 0, \ldots, 3$) and thus are denoted by
\[
b_1^- = b_1^{(2)} - b_1^{(0)} = 2d_1(a), \quad b_0^- = b_0^{(2)} - b_0^{(0)} = 2d_0(a). \tag{3.32}
\]
Substituting (3.32) into a general solution $6a_3 b_0^+ = (b_1^- + 2a_2) b_1^+$ to the homogeneous equation (3.23), we obtain
\[
b_0^+ = \frac{a_2 + d_1}{3a_3} (b_1^+ + d_1) - d_0, \tag{3.33}
\]
where and hereafter we omit the argument of $d_i$ for the brevity. The equality (3.33) shows that $b_1^{(0)}$ and $b_0^{(0)}$ are dependent parameters. Then, $b_1^{(2)}$ and $b_0^{(2)}$ must fulfill the same relation as theirs for the consistency. The latter requirement results in
\[
d_0 = \frac{a_2 + d_1}{3a_3} d_1, \quad b_0^{(2)} = \frac{a_2 + d_1}{3a_3} b_1^{(2)}, \quad b_0^{(0)} = \frac{a_2 + d_1}{3a_3} b_1^{(0)}. \tag{3.34}
\]
Substituting (3.34) back into (3.24) and (3.25), we have
\[
a_1 = \frac{(a_2)^2 - (d_1)^2}{3a_3}, \quad a_0 = \frac{(a_2 + d_1)^2 (a_2 - 2d_1)}{27(a_3)^2}. \tag{3.35}
\]
Under the latter parameter relations, the function $A(z)$ is factorized as
\[
A(z) = a_3 \left( z + \frac{a_2 + d_1}{3a_3} \right)^2 \left( z + \frac{a_2 - 2d_1}{3a_3} \right), \tag{3.36}
\]
and the pair of potentials (3.20) reads as
\[
V^{\pm}(x; b_1) := V^{\pm} \left( x; b_1, \frac{a_2 + d_1}{3a_3} \right) \bigg|_{a_1=\frac{(a_2)^2-(d_1)^2}{3a_3}}, \quad a_0=\frac{(a_2+d_1)^2(a_2-2d_1)}{27(a_3)^2}

= \frac{3a_3}{16} z + \frac{a_2 \pm 8b_1}{16} + \frac{3(d_1 \pm 2b_1)(3d_1 \pm 2b_1)}{16(3a_3 z + a_2 - 2d_1)} \bigg|_{z=z(x)}. \tag{3.37}
\]
We now easily check that the latter pair of potentials satisfies
\[ V^+(x; b_1) = V^-(x; b_1 + 2d_1) + b_1 + d_1, \] (3.38)
which is consistent with the formula for \( b_3^{(2)} \) in (3.32), and thus has two-step SI. The first and second intermediate potentials (3.21) and (3.22) in this two-step SI case read respectively as
\[
V^{i1}(x; b_1) := V^{i1}(x; b_1, \frac{a_2 + d_1}{3a_3}b_1) \bigg|_{a_1 = \frac{(a_2)^2 - (d_1)^2}{3a_3^2}, \quad a_0 = \frac{(a_2 + d_1)^2(a_2 - 2d_1)}{27(a_3)^2}} \\
= \frac{15a_3}{16} + \frac{5a_2}{16} + \frac{3(2b_1 + d_1)(2b_1 - d_1)}{16(3a_3z + a_2 - 2d_1)} \\
\bigg|_{z = z(x)}, \quad (3.39)
\]
and
\[
V^{i2}(x; b_1) := V^{i2}(x; b_1, \frac{a_2 + d_1}{3a_3}b_1) \bigg|_{a_1 = \frac{(a_2)^2 - (d_1)^2}{3a_3^2}, \quad a_0 = \frac{(a_2 + d_1)^2(a_2 - 2d_1)}{27(a_3)^2}} \\
= -\frac{a_3}{16}z - \frac{a_2 + 16d_1}{48} + \frac{3(2b_1 + d_1)(2b_1 - d_1)}{16(3a_3z + a_2 - 2d_1)} \\
\bigg|_{z = z(x)}. \quad (3.40)
\]
Although both of them have the same functional dependence on the variable \( z \) as \( V^\pm \), neither has ordinary SI with \( V^\pm \). Hence, the system has irreducible two-step SI. The equation (2.12) with \( A(z) \) given in (3.36) is integrated as
\[
z + \frac{a_2 + d_1}{3a_3} = \begin{cases} 
\frac{d_1}{a_3} \text{sech}^2 \sqrt{-\frac{d_1}{2}} x & \text{for} \quad d_1 \neq 0, \\
\frac{2}{\sqrt{a_3}x^2} & \text{for} \quad d_1 = 0,
\end{cases} \tag{3.41}
\]
Substituting (3.41) into (3.37), (3.39), and (3.40), we finally obtain the potentials in the \( x \)-space.

**Case 1-5: \( a_4 \neq 0 \)**

In the last, we shall consider the case of a fourth degree \( A(z) \). In this case, the pair of potentials (2.36) are given by
\[
V^\pm(x; b_1, b_0) = \frac{1}{4A(z)} \left[ \left( 3a_3a_2 - 6a_4a_1 - \frac{3(a_3)^3}{4a_4} \right) z^3 + 3a_3b_1 \pm 8a_4b_0 \right] \\
+ \left( \frac{3(a_2)^2}{2} - 12a_4a_0 - \frac{3a_3a_1}{2} - \frac{3(a_3)^2a_2}{4a_4} \right) z^2 \\
+ \left( 3a_2a_1 - 6a_3a_0 - \frac{3(a_3)^2a_1}{4a_4} \pm 6a_3b_1 \pm 4a_2b_0 + (b_1)^2 \right) z \\
+ \left( \frac{3}{4} - \frac{3(a_3)^2a_0}{4a_4} \right) \pm 8a_0b_1 \pm 2a_1b_0 + (b_0)^2 \right] - \frac{a_2}{2} + \frac{3(a_3)^2}{16a_4} \pm b_1 \bigg|_{z = z(x)}. \tag{3.42}
\]
The first intermediate potential (2.37) is calculated as

\[
V^{i1}(x; b_1, b_0) = \frac{1}{16A(z)} \left[ \left( 8a_4a_1 - 4a_3a_2 + \frac{(a_3)^3}{a_4} \right) z^3 + \left( 4(b_1)^2 + 16a_4a_0 + 2a_3a_1 \right)
- 4(a_2)^2 + \frac{(a_3)^2a_2}{a_4} \right] z^2 + \left( 8b_1b_0 + 8a_3a_0 - 4a_2a_1 + \frac{(a_3)^2a_1}{a_4} \right) z
+ 4(b_0)^2 - (a_1)^2 + \frac{(a_3)^2a_0}{a_4} \right] + 2a_4z^2 + a_3z + \frac{a_2}{2} - \frac{(a_3)^2}{16a_4} \right]_{z = z(x)}, \tag{3.43}
\]

while the second intermediate potential (2.38) which exists only when \( b_1 \neq 0 \) is as

\[
V^{i2}(x; b_1, b_0) = V^{i1}(x; b_1, b_0) - 2a_4z^2 - a_3z - \frac{a_3b_1b_0 - 2a_4(b_0)^2}{(b_1)^2} \left( b_1 \right)^3 - 2a_2(b_1)^2b_0 + 3a_3b_1(b_0)^2 - 4a_4(b_0)^3
+ 2a_0(b_1)^4 - (b_1)^3b_0 + 2a_2(b_1)^2(b_0)^2 - a_3b_1(b_0)^3 + a_4(b_0)^4 \right] \left( b_1 \right)^2(b_1z + b_0)^2 \right] \left( z = z(x) \right). \tag{3.44}
\]

Although \( V^+ \) and \( V^- \) have the same functional dependence on the variable \( z \), it does not necessarily mean that there is a transformation of the parameters \( b_1 \) and \( b_0 \) which lead to non-reflective two-step SI. In fact, they satisfy the condition for two-step SI (2.3) if and only if

\[
\begin{pmatrix}
a_3 & -4a_4 \\
b_1 - 4a_2 & -6a_3 \\
b_0 + 6a_1 & b_1 - 4a_2 \\
8a_0 & b_0 - 2a_1
\end{pmatrix}
\begin{pmatrix}
b_1^+ \\
b_0^+
\end{pmatrix}
= 0, \tag{3.45}
\]

where \( b_1^\pm = b_1^{(2)} \pm b_1^{(0)} \). The trivial solution \( b_1^+ = b_0^+ = 0 \) just produces reflective two-step SI (2.39). To have a non-trivial solution, the parameters must satisfy

\[
\begin{align*}
2a_4(b_1^+ + 4a_2) &= 3(a_3)^2, \tag{3.46} \\
a_3(b_1^+ - 4a_2) &= -4a_4(b_0^+ + 6a_1), \tag{3.47} \\
a_3(b_0^+ + 2a_1) &= -32a_4a_0. \tag{3.48}
\end{align*}
\]

We first consider the particular case when \( a_3 = 0 \). In the latter case, the set of equations (3.46)–(3.48) reduces by the assumption \( a_4 \neq 0 \) to

\[
b_0^+ = b_1^- + 4a_2 = b_0^- + 6a_1 = a_0 = 0. \tag{3.49}
\]

Hence, we have

\[
b_1^{(2)} = b_1^{(0)} - 4a_2, \quad b_0^{(0)} = -b_0^{(0)} = -3a_1, \quad A(z) = a_4z^4 + a_2z^2 + a_1z. \tag{3.50}
\]

In this case, the pair of potentials (3.42) reads as

\[
V^\pm(x; b_1, b_0) = \frac{1}{4A(z)} \left[ -2a_4(3a_1 \mp 4b_0)z^3 + (3a_2 \mp b_1)(a_2 \mp b_1)z^2
+ (3a_2a_1 \mp 6a_1b_1 \pm 4a_2b_0 + 2b_1b_0)z + \frac{3(a_1)^2}{4}
\mp 8a_0b_1 \pm 2a_1b_0 + (b_0)^2 \right] - \frac{a_2}{2} \pm b_1. \tag{3.51}
\]

We can easily check that the latter pair satisfies

\[ V^+(x; b_1, b_0)A(z) = V^-(x; b_1 - 4a_2, -b_0)A(z) + (b_1 - 2a_2) \left[ 2a_4 z^4 + 2a_2 z^2 + (b_0 - a_1) z \right], \] (3.52)

and thus has two-step SI if and only if \( b_0 = 3a_1 \), as has been indicated by the second equality in (3.50). Hence, the present system has conditional two-step SI.

The first and second intermediate potentials (3.43) and (3.44) in this case of conditional two-step SI read respectively as

\[ V^{i1}(x; b_1, b_0) = \frac{1}{4A(z)} \left[ 2a_4 a_1 z^3 + (b_1^2 - (a_2)^2) z^2 + (2b_1 b_0 - a_2 a_1) z \right. \\
+ \left. (b_0)^2 - \frac{(a_1)^2}{4} + 2a_4 z^2 + \frac{a_2}{2} \right]_{z = z(x)}, \] (3.53)

and

\[ V^{i2}(x; b_1, b_0) = V^{i1}(x; b_1, b_0) - 2a_4 z^2 + \frac{2a_4 (b_0)^2}{(b_1)^2} + \frac{a_1 (b_1)^3 - 2a_2 (b_1)^2 b_0 - 4a_4 (b_0)^3}{(b_1)^2 (b_1 z + b_0)} - 2b_0 \frac{a_1 (b_1)^3 - a_2 (b_1)^2 b_0 - a_4 (b_0)^2}{(b_1)^2 (b_1 z + b_0)^2} \right]_{z = z(x)}. \] (3.54)

Both of them do not have ordinary SI with \( V^\pm \) given in (3.51) irrespective of whether \( b_0 = 3a_1 \) or not. Hence, the conditional two-step SI of the system (3.51) is irreducible.

The general solution to equation (2.12) with \( A(z) \) given in (3.50) is expressed in terms of elliptic functions when \( a_1 \neq 0 \). The potentials are thus of deformed Lamé or Heun type. We omit its involved general form here. When \( a_1 = 0 \), the system automatically has two-step SI since \( b_0^{(2)} = b_0^{(0)} = 0 \) from the second equality in (3.50). In addition, the form of \( A(z) \) reduces to the one given in (3.60) with \( a_3 = 0 \), and the solution to (2.12) is given by (3.66). Substituting (3.66) into (3.51), (3.53), and (3.54), we finally obtain the potentials in the \( x \)-space.

When \( a_3 \neq 0 \), we immediately obtain from (3.46) and (3.48)

\[ b_1^{-} = -4a_2 + \frac{3(a_3)^3}{2a_4}, \quad b_0^{-} = -\frac{32a_4 a_0}{a_3} + 2a_1. \] (3.55)

Substituting them into (3.47), we have the following constraint

\[ 64(a_4)^2 (4a_4 a_0 - a_3 a_1) = -(a_3)^2 \left[ 16a_4 a_2 - 3(a_3)^2 \right]. \] (3.56)

On the other hand, the homogeneous equations (3.45) under the conditions (3.46)–(3.48) lead to a single equation \( a_3 b_1^{+} = 4a_4 b_0^{+} \). Eliminating \( b_1^{(2)}, b_0^{(2)} \), and \( a_0 \) in it by using (3.55) and (3.56), we obtain

\[ a_3 b_1^{(0)} - 4a_4 b_0^{(0)} + 12a_4 a_1 - 6a_3 a_2 + \frac{3(a_3)^3}{2a_4} = 0. \] (3.57)
This relation means that \( b_1 \) and \( b_0 \) cannot be independent parameters. It is evident that it must hold not only between \( b_1^{(0)} \) and \( b_0^{(0)} \) but also between \( b_1^{(2)} \) and \( b_0^{(2)} \) for the consistency. From the latter requirement, we have another constraint:

\[
8a_4a_1 - 4a_3a_2 + \left( \frac{(a_3)^3}{a_4} \right) = 0.
\]

(3.58)

Combining (3.56)–(3.58), we finally obtain

\[
b_0 = \frac{a_3}{4a_4} b_1, \quad b_1^{(2)} = b_1^{(0)} - 4a_2 + \frac{3(a_3)^2}{2a_4},
\]

(3.59)

\[
a_1 = \frac{a_3 a_2}{2a_4} - \frac{(a_3)^3}{8(a_4)^2}, \quad a_0 = \frac{(a_3)^2 a_2}{16(a_4)^2} - \frac{5(a_3)^4}{256(a_4)^2}.
\]

Under the latter parameter relations, the function \( A(z) \) is factorized as

\[
A(z) = a_4 \left( z + \frac{a_3}{4a_4} \right)^2 \left( z^2 + \frac{a_3}{2a_4} z + \frac{a_2}{a_4} - \frac{5(a_3)^2}{16(a_4)^2} \right),
\]

(3.60)

and the pair of potentials (3.42) reads as

\[
V^\pm(x; b_1) := V^\pm \left( x; b_1, \frac{a_3}{4a_4} b_1 \right) \bigg|_{a_1 = a_3 a_2 / 2a_4, a_0 = (a_3)^2 a_2 / 16(a_4)^2 - 5(a_3)^4 / 256(a_4)^2, a_1 = a_3 a_2 / 2a_4, a_0 = (a_3)^2 a_2 / 16(a_4)^2 - 5(a_3)^4 / 256(a_4)^2}.
\]

(3.61)

\[
= \frac{4a_4}{16(a_4)^2 z^2 + 8a_4 a_3 z + 16a_4 a_2 - 5(a_3)^2}
\]

\[
\times \left( a_2 - \frac{3(a_3)^2}{8a_4} \mp b_1 \right) \left( 3a_2 - \frac{9(a_3)^2}{8a_4} \mp b_1 \right) - \frac{a_2}{2} + \frac{3(a_3)^2}{16a_4} \pm b_1 \bigg|_{z = z(x)}.
\]

We now easily check that the latter pair of potentials satisfies

\[
V^+(x; b_1) = V^-(x; b_1 - 4a_2 + \frac{3(a_3)^2}{2a_4}) + 2 \left[ b_1 - 2a_2 + \frac{3(a_3)^2}{4a_4} \right],
\]

(3.62)

which is consistent with the formula for \( b_1^{(2)} \) in (3.59), and thus has two-step SI. The first and second intermediate potentials (3.43) and (3.44) in this two-step SI case read respectively as

\[
V^{i1}(x; b_1) := V^{i1} \left( x; b_1, \frac{a_3}{4a_4} b_1 \right) \bigg|_{a_1 = a_3 a_2 / 2a_4, a_0 = (a_3)^2 a_2 / 16(a_4)^2 - 5(a_3)^4 / 256(a_4)^2, a_1 = a_3 a_2 / 2a_4, a_0 = (a_3)^2 a_2 / 16(a_4)^2 - 5(a_3)^4 / 256(a_4)^2}.
\]

(3.63)

\[
= \frac{4a_4}{16(a_4)^2 z^2 + 8a_4 a_3 z + 16a_4 a_2 - 5(a_3)^2}
\]

\[
\times \left[ (b_1)^2 - \left( a_2 - \frac{3(a_3)^2}{8a_4} \right)^2 \right] + 2a_4 z^2 + a_3 z + \frac{a_2}{2} - \frac{(a_3)^2}{16a_4} \bigg|_{z = z(x)}.
\]
and
\[
V^{i2}(x; b) := V^{i2}\left(x; b, \frac{a_3}{4a_4}b_1\right)_{a_3=\frac{a_2a_4}{8a_4}} = \frac{a_2}{4a_4}, \quad a_0=\frac{(a_2)^2a_3}{16(a_4)^2} - \frac{3(a_3)^2}{256(a_4)^2}
\]


\[
= \frac{16(a_4)^2z^2 + 8a_4a_3z + 16a_4a_2 - 5(a_3)^2}{(b_1)^2 - \left(a_2 - \frac{3(a_3)^2}{8a_4}\right)^2 + \frac{a_2}{2} - \frac{3(a_3)^2}{16a_4}} \bigg|_{z=z(x)}.
\]

Hence, the second intermediate potential \(V^{i2}\) together with \(V^\pm\) constitutes a sequence of ordinary SI as

\[
V^{i2}(x; b) = V^-(x; b, -2a_2 + \frac{3(a_3)^2}{4a_4}) + b_1 - a_2 + \frac{3(a_3)^2}{8a_4},
\]

\[
V^+(x; b) = V^{i2}(x; b, -2a_2 + \frac{3(a_3)^2}{4a_4}) + b_1 - a_2 + \frac{3(a_3)^2}{8a_4}.
\]

Therefore, the system has reducible two-step SI. The equation (2.12) with \(A(z)\) given in (3.60) is integrated as

\[
z + \frac{a_3}{4a_4} = \begin{cases} 
4\sqrt{c}e^{\sqrt{2a_4}c}x & \text{for } c \neq 0, \\
\frac{1}{\sqrt{2a_4}x} & \text{for } c = 0,
\end{cases}
\]

where

\[
c = \frac{a_2}{a_4} - \frac{3(a_3)^2}{8(a_4)^2}.
\]

Substituting (3.66) into (3.61), (3.63), and (3.64), we finally obtain the potentials in the \(x\)-space.

To summarize, we have found that a type A 2-fold SUSY system with an intermediate Hamiltonian resulting from a polynomial \(A(z)\) of at most fourth degree can possess non-reflective two-step SI, always when the degree is less than or equal to two, and under certain conditions when it is three or four. Another remarkable result is that all the type A 2-fold SUSY systems which have two-step SI as well in Cases 1-1 to 1-3 turn to have also ordinary SI, that is, they have reducible two-step SI, and they are all well-known SI potentials. On the other hand, all the other two-step SI potentials in Cases 1-4 and 1-5 are, to the best of our knowledge, new and except for the last model in Case 1-5 they do not possess ordinary SI, that is, they have irreducible two-step SI. In the next section, we shall examine more general cases where the function \(A(z)\) is not given by a polynomial.

**IV. NON-POLYNOMIAL \(A(z)\)**

To investigate the possibility of other irreducible two-step SI, let us coming back to the form of potentials (2.36). In general, the function \(A(z)\) can depend on the parameters \(b_i\).
this case, the change of variables \( z = z(x) \) also depends on them via (2.12), and analysis of the two-step SI condition (2.3) with the pair of potentials (2.36) gets quite involved. In this work, we shall restrict our analysis to the case where \( A(z) \), and thus \( z = z(x) \) as well, does not depend on the parameters \( b_i \). Under this restriction, we easily see from (2.36) that the two-step SI condition (2.3) holds if and only if the function \( A(z) \) satisfies

\[
\frac{1}{4A(z)} \left[ (b_1^{(0)} z + b_0^{(0)})^2 + 2((b_1^{(0)} z + b_0^{(0)}) A'(z) - 2b_1^{(0)} A(z)) \right] = \frac{1}{4A(z)} \left[ (b_1^{(2)} z + b_0^{(2)})^2 - 2((b_1^{(2)} z + b_0^{(2)}) A'(z) - 2b_1^{(2)} A(z)) \right] + \tilde{R}_2(b_1^{(0)}, b_0^{(0)}). \tag{4.1}
\]

The latter condition is identical to the first-order linear differential equation

\[
(b_1^+ z + b_0^+) (b_1^- z + b_0^-) - 2(b_1^+ z + b_0^+) A'(z) + 4\tilde{R} A(z) = 0, \tag{4.2}
\]

where \( b_i^\pm = b_i^{(2)} \pm b_i^{(0)} \) and \( \tilde{R} = \tilde{R}_2 + b_1^+ \). It is apparent that its most general solution \( A(z) \) depends on the parameters \( b_i^{(0)} \). Hence, to obtain a solution without such dependence, we must impose the following additional condition:

\[
\frac{\partial A(z)}{\partial b_i^{(0)}} = 0. \tag{4.3}
\]

As we will show in what follows, the latter condition completely determines the admissible form of the function \( A(z) \) and the parameters \( b_i^{(2)} \) and \( \tilde{R}_2 \).

To solve the differential equation (4.2), we first note that we can assume without any loss of generality that either \( b_1^+ \) or \( b_0^+ \) is non-zero. Otherwise, we have

\[
b_1^{(2)} = -b_1^{(0)}, \quad b_0^{(2)} = -b_0^{(0)}, \quad \tilde{R}_2 = 0, \tag{4.4}
\]

which exactly leads to the two-step SI of the type (2.39). Thus, \( b_1^+ z + b_0^+ \) is not identically zero, and (4.2) is integrated as

\[
2A(z) = \exp \left( \int dz \frac{2\tilde{R}}{b_1^+ z + b_0^+} \right) \int dz (b_1^+ z + b_0^-) \exp \left( - \int z' \frac{2\tilde{R}}{b_1^+ z' + b_0^+} \right). \tag{4.5}
\]

It requires separate treatments according to whether \( b_1^+ \) is zero or not.

**Case 2: \( b_1^+ \neq 0 \)**

Let us first study the case of non-zero \( b_1^+ \). In this case, equation (4.5) reads as

\[
2A(z) = (b_1^+ z + b_0^+)^\mu \int dz (b_1^+ z + b_0^-) (b_1^+ z + b_0^+)^{-\mu}
\]

\[
= (b_1^+ z + b_0^+)^\mu \int dz \left[ \frac{b_1^+ b_0^- - b_1^- b_0^+}{b_1^+} (b_1^+ z + b_0^+) \right], \tag{4.6}
\]

where \( \mu = 2\tilde{R}/b_1^+ \). Hence, we have the following three inequivalent cases.

**Case 2-1: \( \mu \neq 1, 2 \)**
In this case, we obtain from (4.6)

$$A(z) = a_2 z^2 + a_1 z + a_0 + c (b_1^+ z + b_0^-)^\mu,$$  \hspace{1cm} (4.7)

where $c$ is an integral constant and

$$a_2 = \frac{b_1^-}{2(2 - \mu)}, \quad a_1 = -\frac{\mu b_1^- b_0^+}{2(1 - \mu)(2 - \mu)b_1^+} + \frac{b_0^-}{2(1 - \mu)},$$

$$a_0 = -\frac{b_1^- (b_1^+)^2}{2(1 - \mu)(2 - \mu)(b_1^+)^2} + \frac{b_1^- b_0^-}{2(1 - \mu)b_1^+}.$$  \hspace{1cm} (4.8)

The condition (4.3) in this case is equivalent to the following set of equations:

$$\frac{\partial a_2}{\partial b_i^{(0)}} = \frac{\partial a_1}{\partial b_i^{(0)}} = \frac{\partial a_0}{\partial b_i^{(0)}} = b_1^+ \frac{\partial c}{\partial b_i^{(0)}} + c \mu \frac{\partial b_i^+}{\partial b_i^{(0)}} = b_0^+ \frac{\partial c}{\partial b_i^{(0)}} + c \mu \frac{\partial b_0^+}{\partial b_i^{(0)}} = \frac{\partial \mu}{\partial b_i^{(0)}} = 0. \hspace{1cm} (4.9)$$

We assume $c \neq 0$ to avoid duplication of Case 1-3. The last equality just means that $\mu$ does not depend on $b_i^{(0)}$. Substituting the expression of the parameters (4.8) into (4.9), we see that the following equations must hold:

$$\frac{\partial b_1^-}{\partial b_i^{(0)}} = 0, \quad \frac{\partial b_0^-}{\partial b_i^{(0)}} = 0, \quad b_0^+ \frac{\partial b_1^+}{\partial b_i^{(0)}} - b_1^+ \frac{\partial b_0^+}{\partial b_i^{(0)}} = 0. \hspace{1cm} (4.10)$$

From the first and second equalities in (4.10), we immediately have

$$b_1^{(2)} = b_1^{(0)} + 2d_1, \quad b_0^{(2)} = b_0^{(0)} + 2d_0,$$  \hspace{1cm} (4.11)

where $d_1$ and $d_0$ are constants which do not depend on $b_i^{(0)}$.

Until now, we have not made any assumption on the relation between $b_1^{(0)}$ and $b_0^{(0)}$, that is, they can be independent or dependent parameters. So, let us first assume that they are independent. In this case, we shall first show that $b_0^+ = 0$. For this purpose, we note that so long as $b_1^+ = 2b_0^{(0)} + 2d_1 \neq 0$ together with the assumption $b_1^+ = 2b_0^{(0)} + 2d_1 \neq 0$

$$\frac{\partial b_1^+}{\partial b_1^{(0)}} = \frac{\partial b_0^+}{\partial b_0^{(0)}} = 2, \quad \frac{\partial b_1^+}{\partial b_0^{(0)}} = \frac{\partial b_0^+}{\partial b_0^{(0)}} = 0. \hspace{1cm} (4.12)$$

Then, the third equation in (4.10) results in $b_1^+ = b_0^+ = 0$, which is obviously contradictory. Hence, we must have $b_0^+ = 0$, which together with (4.11) means

$$b_1^{(2)} = b_1^{(0)} + 2d_1, \quad b_0^{(2)} = -b_0^{(0)} = d_0,$$  \hspace{1cm} (4.13)

and the third condition in (4.10) is trivially satisfied. All the remaining equations in (4.9) to be satisfied now read as

$$\frac{\partial c}{\partial b_i^{(0)}} + 2\mu \frac{c}{b_1^+} = \frac{\partial c}{\partial b_0^{(0)}} = 0,$$  \hspace{1cm} (4.14)

and their general solution is given by

$$c = c_0/(b_1^+)^\mu,$$ \hspace{1cm} (4.15)
where $c_0$ and $\mu$ are constants which do not depend on $b_1^{(0)}$. Substituting (4.13) back into (4.8), we have

$$d_1 = -(\mu - 2)a_2, \quad d_0 = -(\mu - 1)a_1, \quad a_0 = 0. \quad (4.16)$$

With the aid of the solutions (4.13), (4.15), and (4.16), we finally obtain the admissible form of the function $A(z)$ and the parameters $b_1^{(2)}$ and $\bar{R}_2$ as

$$A(z) = a_2 z^2 + a_1 z + c_0 z^\mu, \quad b_1^{(2)} = b_1^{(0)} - 2(\mu - 2)a_2, \quad b_0^{(2)} = -b_0^{(0)} = -(\mu - 1)a_1, \quad \bar{R}_2 = (\mu - 2)[b_1^{(0)} - (\mu - 2)a_2]. \quad (4.17)$$

The pair of potentials (2.36) in this case are given by

$$16V^\pm(x; b_1, b_0)A(z(x)) = 4 \left[ (a_2)^2 + (b_1)^2 \right] z^2 + 4(a_2a_1 + 2a_1b_1 \pm 4a_2b_0 + 2b_1b_0)z + 3(a_1)^2 \pm 8a_1b_0 + 4(b_0)^2 - 4 \left[ (\mu^2 - 4\mu + 2)a_2 \pm 2(\mu - 2)b_1 \right] c_0 z^\mu - 2\mu \left[ (2\mu - 5)a_1 \mp 4b_0 \right] c_0 z^{\mu - 1} - \mu(\mu - 4)(c_0)^2 z^{2\mu - 2} \bigg|_{z=z(x)}. \quad (4.18)$$

We can easily check that the latter pair satisfies

$$V^+(x; b_1, b_0)A(z) = V^-(x; b_1 - 2(\mu - 2)a_2, -b_0)A(z) + [b_1 - (\mu - 2)a_2] \left[ (\mu - 2)a_2z^2 + (b_0 - a_1)z + (\mu - 2)c_0 z^\mu \right], \quad (4.19)$$

and thus has two-step SI if and only if $b_0 = (\mu - 1)a_1$, as has been indicated by the third equality in (4.17). Hence, the present system has conditional two-step SI.

We note that $b_1 \neq 0$ in this case and the system always admits two intermediate Hamiltonians. The first and second intermediate potentials (2.37) and (2.38) are respectively calculated as

$$16V^{ii}(x; b_1, b_0)A(z(x)) = 4 \left[ (a_2)^2 + (b_1)^2 \right] z^2 + 4(a_2a_1 + 2b_1b_0)z - (a_1)^2 + 4(b_0)^2 + 4(\mu^2 - 2\mu + 2)a_2c_0 z^\mu + 2\mu(2\mu - 3)a_1c_0 z^{\mu - 1} + \mu(3\mu - 4)(c_0)^2 z^{2\mu - 2} \bigg|_{z=z(x)}, \quad (4.20)$$

and

$$V^{ii}(x; b_1, b_0) = V^{ii}(x; b_1, b_0) + \frac{a_1 b_1 - 2a_2b_0 - (\mu - 2)c_0 b_1 z^{\mu - 1}}{b_1z + b_0} \bigg|_{z=z(x)} - 2b_0 \frac{a_1 b_1 - a_2b_0 + c_0 b_1 z^{\mu - 1}}{(b_1z + b_0)^2} \bigg|_{z=z(x)}. \quad (4.21)$$

Both of them do not have ordinary SI with $V^\pm$ given in (4.18) irrespective of whether $b_0 = (\mu - 1)a_1$ or not. Hence, the conditional two-step SI of the system (4.18) is irreducible. We note that the present system (4.17)–(4.21) reduces to the one (3.27)–(3.31) in Case 1-4 when we put $\mu = 3$ with $c_0 = a_3$ and to the one (3.50)–(3.54) in Case 1-5 when we put $\mu = 4$ with $c_0 = a_4$.

The equation (2.12) with $A(z)$ given in (4.17) cannot be integrated analytically unless $a_1 = 0$. When $a_1 = 0$, the system automatically has two-step SI since $b_0^{(2)} = b_0^{(0)} = 0$ from the third equality in (4.17). In addition, the form of $A(z)$ reduces to the one given in (4.24) with $a_1 = 0$, and the solution to (2.12) is given by (4.31). Substituting (4.31) into (4.18),
of the function $A$.

With the aid of the solutions (4.11), (4.22), and (4.23), we finally obtain the admissible form of the system and obtain

$$b_0^{(0)} = z_1 b_1^{(0)}, \quad d_0 = z_1 d_1.$$  \hspace{1cm} (4.22)

All the remaining equations in (4.9) are satisfied with the solution (4.15). Substituting (4.11) and (4.22) back into (4.8), we find that $a_2 \neq 0$ is necessary for the non-triviality of the system and obtain

$$b_1^{(2)} = b_1^{(0)} - 2(\mu - 2)a_2, \quad 2a_2 z_1 = a_1, \quad 4a_2 a_0 = (a_1)^2.$$  \hspace{1cm} (4.23)

With the aid of the solutions (4.11), (4.22), and (4.23), we finally obtain the admissible form of the function $A(z)$ and the parameters $b_1^{(2)}$ and $\tilde{R}_2$ as

$$A(z) = a_2 \left(z + \frac{a_1}{2a_2}\right)^2 + c_0 \left(z + \frac{a_1}{2a_2}\right)^\mu, \quad b_1^{(2)} = b_1^{(0)} - 2(\mu - 2)a_2,$$

$$\tilde{R}_2 = (\mu - 2)[b_1^{(0)} - (\mu - 2)a_2].$$  \hspace{1cm} (4.24)

The pair of potentials (2.36) in this case with the definition $V(x; b_1) := V(x; b_1, a_1 b_1/(2a_2))$ are calculated as

$$16V^\pm(x; b_1)A(z(x)) = 4 \left[(a_2)^2 + (b_1)^2\right] \left(z + \frac{a_1}{2a_2}\right)^2 - 4 \left[(\mu^2 - 4\mu + 2)a_2^2 + 2(\mu - 2)b_1\right]c_0 \left(z + \frac{a_1}{2a_2}\right)^\mu - \mu(\mu - 4)(c_0)^2 \left(z + \frac{a_1}{2a_2}\right)^{2\mu-2} \bigg|_{z=z(x)}.$$  \hspace{1cm} (4.25)

We can easily check that the latter pair satisfies

$$V^+(x; b_1) = V^-(x; b_1 - 2(\mu - 2)a_2) + (\mu - 2)[b_1 - (\mu - 2)a_2],$$  \hspace{1cm} (4.26)

which is consistent with the formulas for $b_1^{(2)}$ and $\tilde{R}_2$ in (4.24), and thus has two-step SI. The first and second intermediate potentials (2.37) and (2.38), which always exist thanks to $b_1 \neq 0$, are respectively given by

$$16V^{i1}(x; b_1)A(z) = 4 \left[(a_2)^2 + (b_1)^2\right] \left(z + \frac{a_1}{2a_2}\right)^2 + 4(\mu^2 - 2\mu + 2)a_2 c_0 \left(z + \frac{a_1}{2a_2}\right)^\mu + \mu(3\mu - 4)(c_0)^2 \left(z + \frac{a_1}{2a_2}\right)^{2\mu-2},$$  \hspace{1cm} (4.27)

and

$$V^{i2}(x; b_1) = V^{i1}(x; b_1) - (\mu - 2)c_0 \left(z + \frac{a_1}{2a_2}\right)^{\mu-2}.$$  \hspace{1cm} (4.28)
Both of them in general do not have ordinary SI with $V^\pm$ given in (4.25). Hence, the present system has irreducible two-step SI in general. It is worth noting, however, that the system (4.25)–(4.28) reduces to the reducible two-step SI system (3.61)–(3.64) in Case 1-5 when $\mu = 4$. Indeed, we can check that with the following substitution

$$c_0 = \bar{a}_4, \quad a_2 = \bar{a}_2 - \frac{3(\bar{a}_3)^2}{8\bar{a}_4}, \quad a_1 = \frac{\bar{a}_3\bar{a}_2}{2\bar{a}_4} - \frac{3(\bar{a}_3)^3}{16(\bar{a}_4)^2},$$

(4.29)

the system (4.25)–(4.28) coincides with (3.61)–(3.64) with $a_i$ replaced by $\bar{a}_i$ when $\mu = 4$.

On the other hand, when $\mu = 3$, the system (4.25)–(4.28) reduces to the one (3.37)–(3.40) in Case 1-4 by the following substitution

$$c_0 = \bar{a}_3, \quad a_2 = -\bar{d}_1, \quad a_1 = -\frac{2\bar{d}_1(\bar{a}_2 + \bar{d}_1)}{3\bar{a}_3},$$

(4.30)

keeping the irreducibility intact. Therefore, the reducibility of two-step SI in the system (3.61)–(3.64) in Case 1-5 is rather exceptional.

The equation (2.12) with $A(z)$ given in (4.24) is integrated as

$$\left(z + \frac{a_1}{2a_2}\right)^{2-\mu} = \frac{c_0}{a_2} \tanh^2(\mu - 2) \sqrt{\frac{a_2}{2} x}.$$  

(4.31)

Substituting (4.31) into (4.25), (4.27), and (4.28), we finally obtain the potentials in the $x$-space.

**Case 2-2: $\mu = 1$**

In this case, we obtain from (4.6)

$$A(z) = a_2 z^2 + a_1 z + a_0 + \bar{c}(b_1^+ z + b_0^+) \ln |b_1^+ z + b_0^|,$$

(4.32)

where

$$a_2 = \frac{b_1^-}{2}, \quad a_1 = \frac{b_1^- b_0^+ + cb_1^+}{2b_1^+}, \quad a_0 = cb_0^+, \quad \bar{c} = \frac{b_1^+ b_0^- - b_1^- b_0^+}{2(b_1^+)^2},$$

(4.33)

and $c$ is an integral constant. The condition (4.3) in this case is equivalent to the following set of equations:

$$\frac{\partial a_2}{\partial b_i^{(0)}} + \frac{\partial a_1}{\partial b_i^{(0)}} + \frac{\partial cb_1^+}{\partial b_i^{(0)}} = 0, \quad \frac{\partial a_0}{\partial b_i^{(0)}} + \frac{\partial cb_0^+}{\partial b_i^{(0)}} = 0, \quad \frac{\partial (\bar{c}b_1^+)}{\partial b_i^{(0)}} = 0.$$  

(4.34)

We assume $\bar{c} \neq 0$ to avoid duplication of Case 1-3. Substituting the expression of the parameters (4.33) into (4.34), we see that the condition (4.10) must hold also in the present case. So, let us first investigate the case when $b_1^{(0)}$ and $b_0^{(0)}$ are independent. In this case, by following the same argument below (4.10), we arrive at the same solution (4.13). All the remaining equations in (4.34) to be satisfied now read as

$$\frac{\partial c}{\partial b_i^{(0)}} + 2\frac{b_1^+ c + d_0}{(b_1^+)^2} = \frac{\partial c}{\partial b_i^{(0)}} = 0,$$

(4.35)
and their general solution is given by
\[ b_1^+ c = c_0 - d_0 \ln |b_1^+|, \] (4.36)
where \( c_0 \) is a constant which does not depend on \( b_1^{(0)} \). Substituting (4.13) and (4.36) back into (4.33), we have
\[ d_1 = a_2, \quad a_1 = c_0 - d_0 \ln |b_1^+|, \quad a_0 = 0, \quad \bar{c} = d_0/b_1^+. \] (4.37)

With the aid of the solutions (4.13) and (4.37), we finally obtain the admissible form of the function \( A(z) \) and the parameters \( b_i^{(2)} \) and \( \bar{R}_2 \) as
\[ A(z) = a_2 z^2 + c_0 z + d_0 z \ln |z|, \quad b_1^{(2)} = b_1^{(0)} + 2a_2, \]
\[ b_0^{(2)} = -b_0^{(0)} = d_0, \quad \bar{R}_2 = -b_0^{(0)} - a_2. \] (4.38)

The pair of potentials (2.36) in this case are given by
\[ 16V^\pm(x; b_1, b_0)A(z(x)) = 4[(a_2)^2 + (b_1)^2]z^2 + 4(a_2 c_0 + 2a_2 d_0 + 2 c_0 b_1 \pm 2d_0 b_1 \pm 4a_2 b_0 + 2b_1 b_0)z + 3(c_0)^2 + 2c_0 d_0 + 3(d_0)^2 \pm 8(c_0 + d_0)b_0 + 4(b_0)^2 \]
\[ + 4d_0(a_2 \mp 2b_1)z \ln |z| + 2d_0(3c_0 + d_0 \pm 4b_0) \ln |z| + 3(d_0)^2(\ln |z|)^2 \bigg|_{z=z(x)}. \] (4.39)

We can easily check that the latter pair satisfies
\[ V^+(x; b_1, b_0)A(z) = V^-(x; b_1 + 2a_2, -b_0)A(z) \]
\[ - (b_1 + a_2) [a_2 z^2 + (c_0 - d_0 - b_0)z + d_0 z \ln |z|], \] (4.40)
and thus has two-step SI if and only if \( b_0 = -d_0 \), as has been indicated by the third equality in (4.38). Hence, the present system has conditional two-step SI.

We note that \( b_1 \neq 0 \) in this case and the system always admits two intermediate Hamiltonians. The first and second intermediate potentials (2.37) and (2.38) are respectively calculated as
\[ 16V_{11}(x; b_1, b_0)A(z(x)) = 4[(a_2)^2 + (b_1)^2]z^2 + 4(a_2 c_0 + 2b_1 b_0)z - (c_0 - d_0)^2 \]
\[ + 4(b_0)^2 + 4a_2 d_0 z \ln |z| - 2d_0(c_0 - d_0) \ln |z| - (d_0)^2(\ln |z|)^2 \bigg|_{z=z(x)}, \] (4.41)
and
\[ V_{12}(x; b_1, b_0) = V_{11}(x; b_1, b_0) + \frac{(c_0 - d_0)b_1 - 2a_2 b_0 + d_0 b_1 \ln |z|}{b_1 z + b_0} \]
\[ - 2b_0 \frac{c_0 b_1 - a_2 b_0 + d_0 b_1 \ln |z|}{(b_1 z + b_0)^2} \bigg|_{z=z(x)}. \] (4.42)

Both of them do not have ordinary SI with \( V^\pm \) given in (4.39) irrespective of whether \( b_0 = -d_0 \) or not. Hence, the conditional two-step SI of the system (4.39) is irreducible. Unfortunately, the equation (2.12) with \( A(z) \) given in (4.38) cannot be integrated analytically in general for \( d_0 \neq 0 \).
In the next, let us consider the case when \( b_1^{(0)} \) and \( b_0^{(0)} \) are dependent. In this case, the general solutions to (4.10) are given by (4.11) with (4.22). However, with the latter solutions \( \tilde{c} \) will be zero by the last equality in (4.33), and we have a quadratic \( A(z) \) as a result. Hence, the system with the dependent \( b_1^{(0)} \) and \( b_0^{(0)} \) just reduces to a system in Case 1-3.

**Case 2-3: \( \mu = 2 \)**

In this case, we obtain from (4.6)

\[
A(z) = a_2 z^2 + a_1 z + a_0 + \tilde{c}(b_1^+ z + b_0^+)^2 \ln |b_1^+ z + b_0^+|,
\]

where

\[
a_2 = c(b_1^+)^2, \quad a_1 = 2c b_1^+ b_0^+ - \frac{b_1^+ b_0^- - b_1^- b_0^+}{2b_1^+}, \quad a_0 = c(b_0^+)^2 - \frac{b_1^+ b_0^- - b_1^- b_0^+}{2(b_1^+)^2}b_0^+,
\]

and \( c \) is an integral constant. The condition (4.3) in this case is equivalent to the following set of equations:

\[
\frac{\partial a_2}{\partial b_i^{(0)}} + \tilde{c} b_i^+ \frac{\partial b_i^+}{\partial b_i^{(0)}} = \frac{\partial a_1}{\partial b_i^{(0)}} + \tilde{c} b_i^+ \frac{\partial b_0^+}{\partial b_i^{(0)}} + \tilde{c} b_i^+ \frac{\partial b_1^+}{\partial b_i^{(0)}} = \frac{\partial a_0}{\partial b_i^{(0)}} + \tilde{c} b_i^+ \frac{\partial b_0^+}{\partial b_i^{(0)}}
\]

\[
= b_i^+ \frac{\partial \tilde{c}}{\partial b_i^{(0)}} + 2\tilde{c} \frac{\partial b_i^+}{\partial b_i^{(0)}} = b_i^+ \frac{\partial \tilde{c}}{\partial b_i^{(0)}} + 2\tilde{c} \frac{\partial b_0^+}{\partial b_i^{(0)}} = 0.
\]

We assume \( \tilde{c} \neq 0 \) to avoid duplication of Case 1-3. Substituting the expression of the parameters (4.44) into (4.45), we see that the condition (4.10) must hold also in the present case. So, let us first investigate the case when \( b_1^{(0)} \) and \( b_0^{(0)} \) are independent. In this case, by following the same argument below (4.10), we arrive at the same solution (4.13). All the remaining equations in (4.34) to be satisfied now read as

\[
\frac{\partial c}{\partial b_1^{(0)}} + \frac{2(b_1^+)^2 c + d_1}{(b_1^+)^2} = \frac{\partial c}{\partial b_0^{(0)}} = 0,
\]

and their general solution is given by

\[
(b_1^+)^2 c = c_0 - d_1 \ln |b_1^+|,
\]

where \( c_0 \) is a constant which does not depend on \( b_1^{(0)} \). Substituting (4.13) and (4.47) back into (4.44), we have

\[
a_2 = c_0 - d_1 \ln |b_1^+|, \quad a_1 = -d_0, \quad a_0 = 0, \quad \tilde{c} = d_1/(b_1^+)^2.
\]

With the aid of the solutions (4.13) and (4.48), we finally obtain the admissible form of the function \( A(z) \) and the parameters \( b_1^{(2)} \) and \( \tilde{R}_2 \) as

\[
A(z) = c_0 z^2 + a_1 z + d_1 z^2 \ln |z|, \quad b_1^{(2)} = b_1^{(0)} + 2d_1, \quad b_0^{(2)} = -b_0^{(0)} = -a_1, \quad \tilde{R}_2 = 0.
\]
The pair of potentials (2.36) in this case are given by

\[
16V^\pm (x; b_1)A(z(x)) = \left[ 4(c_0)^2 + 3(d_1)^2 \pm 8d_1b_1 + 4(b_1)^2 \right] z^2 + 2a_1(2c_0 - 3d_1) \\
\mp 4a_1b_1 \pm 4(2c_0 + d_1)b_0 + 4b_1b_0)z + 3(a_1)^2 \pm 8a_1b_0 + 4(b_0)^2 \\
+ 8c_0d_1z^2 \ln |z| + 4d_1(a_1 \pm 4b_0)z \ln |z| + 4(d_1)^2z^2(\ln |z|)^2 \big|_{z=z(x)}. \tag{4.50}
\]

We can easily check that the latter pair satisfies

\[
V^+(x; b_1, b_0)A(z) = V^-(x; b_1 + 2d_1, -b_0)A(z) + (b_0 - a_1)(b_1 + d_1)z, \tag{4.51}
\]

and thus has two-step SI if and only if \( b_0 = a_1 \), as has been indicated by the third equality in (4.49). Hence, the present system has conditional two-step SI.

We note that if \( b_1 \neq 0 \) in this case and the system always admits two intermediate Hamiltonians. The first and second intermediate potentials (2.37) and (2.38) are respectively calculated as

\[
16V^{i1}(x; b_1, b_0)A(z(x)) = \left[ 4(c_0)^2 + 8c_0d_1 - (d_1)^2 + 4(b_1)^2 \right] z^2 \\
+ 2(2a_1c_0 + 5a_1d_1 + 4b_1b_0)z - (a_1)^2 + 4(b_0)^2 \\
+ 8d_1(c_0 + d_1)z^2 \ln |z| + 4a_1d_1z \ln |z| + 4(d_1)^2z^2(\ln |z|)^2 \big|_{z=z(x)}, \tag{4.52}
\]

and

\[
V^{i2}(x; b_1, b_0) = V^{i1}(x; b_1, b_0) - d_0 + \frac{a_1b_1 - (2c_0 - d_1)b_0 - 2d_1b_0 \ln |z|}{b_1z + b_0} \\
\quad - 2b_0 \frac{a_1b_1 - c_0b_0 - d_1b_0 \ln |z|}{(b_1z + b_0)^2} \big|_{z=z(x)}. \tag{4.53}
\]

Both of them do not have ordinary SI with \( V^\pm \) given in (4.50) irrespective of whether \( b_0 = a_1 \) or not. Hence, the conditional two-step SI of the system (4.50) is irreducible.

The equation (2.12) with \( A(z) \) given in (4.49) cannot be integrated analytically unless \( a_1 = 0 \). When \( a_1 = 0 \), the system automatically has two-step SI since \( b_0^{(2)} = b_0^{(0)} = 0 \) from the third equality in (4.49). In addition, the form of \( A(z) \) reduces to the one given in (4.55) with \( a_1 = 0 \), and the solution to (2.12) is given by (4.59). Substituting (4.59) into (4.50), (4.52), and (4.53), we finally obtain the potentials in the \( x \)-space.

In the next, let us consider the case when \( b_0^{(0)} \) and \( b_0^{(0)} \) are dependent. In this case, the general solutions to (4.10) are given by (4.11) with (4.22). All the remaining equations in (4.9) are satisfied with the solution (4.47). Substituting (4.11), (4.22), and (4.47) back into (4.44), we find that \( a_2 \neq 0 \) is necessary for the non-triviality of the system and obtain

\[
a_2 = c_0 - d_1 \ln |b^+_1|, \quad 2a_2z_1 = a_1, \quad 4a_2a_0 = (a_1)^2. \tag{4.54}
\]

With the aid of the solutions (4.11), (4.22), and (4.54), we finally obtain the admissible form of the function \( A(z) \) and the parameters \( b_1^{(0)} \) and \( \tilde{R}_2 \) as

\[
A(z) = \left( z + \frac{a_1}{2a_2} \right) \left( c_0 + d_1 \ln \left| z + \frac{a_1}{2a_2} \right| \right), \quad b_1^{(0)} = b_1^{(1)} + 2d_1, \tag{4.55}
\]

\[
b_0^{(2)} = \frac{a_1}{2a_2} b_1^{(2)}, \quad b_0^{(0)} = \frac{a_1}{2a_2} b_1^{(0)}, \quad \tilde{R}_2 = 0.
\]
The pair of potentials \((2.36)\) in this case with the definition
\[
V(x; b_1) := V(x; b_1, a_1 b_1/(2a_2))
\]
are calculated as

\[
16V^\pm(x; b_1)A(z(x)) = \left( z + \frac{a_1}{2a_2} \right)^2 \left[ 4(c_0)^2 + (d_1 \pm 2b_1)(3d_1 \pm 2b_1) \\
+ 8c_0d_1 \ln|z + \frac{a_1}{2a_2}| + 4(d_1)^2 \left( \ln|z + \frac{a_1}{2a_2}| \right)^2 \right] \bigg|_{z=x(x)}. \tag{4.56}
\]

We can easily check that the latter pair satisfies

\[
V^+(x; b_1) = V^-(x; b_1 + 2d_1), \tag{4.57}
\]
which is consistent with the formulas for \(b_1^{(2)}\) and \(R_2\) in (4.55), and thus has two-step SI. The first intermediate potential \((2.37)\) is given by

\[
16V_{i1}^+(x; b_1)A(z) = \left( z + \frac{a_1}{2a_2} \right)^2 \left[ 4(c_0)^2 + 8c_0d_1 - (d_1)^2 + 4(b_1)^2 \\
+ 8(c_0 + d_1)d_1 \ln|z + \frac{a_1}{2a_2}| + 4(d_1)^2 \left( \ln|z + \frac{a_1}{2a_2}| \right)^2 \right] \bigg|_{z=x(x)}. \tag{4.58}
\]

Intriguingly, the second intermediate potential \((2.38)\) which always exist thanks to \(b_1 \neq 0\) differs from the first one only by a constant, \(V_{i2}^+(x; b_1) = V_{i1}^+(x; b_1) - d_1\). Both of them do not have ordinary SI with \(V^\pm\) given in (4.56). Hence, the present system has irreducible two-step SI.

The equation \((2.12)\) with \(A(z)\) given in (4.55) is integrated as

\[
\ln|z + \frac{a_1}{2a_2}| = \frac{d_1}{2} x^2 - \frac{c_0}{d_1}, \tag{4.59}
\]
Substituting (4.59) into (4.56) and (4.58), we finally obtain the potentials in the \(x\)-space.

**Case 3: \(b_1^+ = 0\)**

In this case, we obtain from (4.6)

\[
2A(z) = e^{\nu z} \int dz \left( -2b_1^{(0)} z + b_0 \right) e^{-\nu z}, \tag{4.60}
\]
where \(\nu = 2R/b_0^+\). We can assume without any loss of generality that \(\nu \neq 0\). Indeed, if \(\nu = 0\), we have

\[
2A(z) = -b_1^{(0)} z^2 + b_0^+ z + 2c, \tag{4.61}
\]
where \(c\) is an integral constant, and thus it just reproduces the case of quadratic \(A(z)\). When \(\nu \neq 0\), we obtain

\[
A(z) = a_1 z + a_0 + c e^{\nu z}, \tag{4.62}
\]
where $c$ is an integral constant and

$$a_1 = b^{(0)}_1 / \nu, \quad a_0 = \frac{2b^{(0)}_1 - b^{(0)}_0 \nu}{2\nu^2}. \quad (4.63)$$

The condition (4.3) in this case is equivalent to the following set of equations:

$$\frac{\partial a_1}{\partial b^{(0)}_1} = \frac{\partial a_0}{\partial b^{(0)}_1} = \frac{\partial c}{\partial b^{(0)}_1} = \frac{\partial \nu}{\partial b^{(0)}_1} = 0. \quad (4.64)$$

The last two equalities just mean that both $c$ and $\nu$ do not depend on $b^{(0)}_1$. We assume $c \neq 0$ to avoid duplication of Case 1-2. Substituting the expression of the parameters (4.63) into the first two equalities in (4.64), we have

$$\frac{\partial b^{(0)}_1}{\partial b^{(0)}_1} = 0, \quad \frac{\partial b^{(0)}_0}{\partial b^{(0)}_1} = 0. \quad (4.65)$$

The first equation in (4.65) is inconsistent if $b^{(0)}_1$ is an independent variable. Thus, the system has only one independent variable $b^{(0)}_0$, and $b^{(0)}_1$ is a function of it. Then, we immediately have $b^{(0)}_1 = d_1$ where $d_1$ is a constant which does not depend on $b^{(0)}_0$. The same equality must hold also for $b^{(2)}_1$ for the consistency; $b^{(2)}_1 = d_1$. On the other hand, $b^{+}_1 = b^{(2)}_1 + b^{(0)}_1 = 0$ in the present case and thus $b^{(2)}_1 = b^{(0)}_1 = 0$ is the only permissible solution. The general solution to the second in (4.65) is obviously $b^{(2)}_0 = b^{(0)}_0 + 2d_0$ where $d_0$ does not depend on $b^{(0)}_0$. Substituting these solutions back into (4.63), we obtain

$$a_1 = 0, \quad d_0 = -a_0 \nu. \quad (4.66)$$

With the aid of the latter solutions, we finally obtain the admissible form of the function $A(z)$ and the parameters $b^{(2)}_i$ and $\tilde{R}_2$ as

$$A(z) = a_0 + ce^{\nu z}, \quad b^{(2)}_0 = b^{(0)}_0 - 2a_0 \nu, \quad \tilde{R}_2 = (b^{(0)}_0 - a_0 \nu) \nu. \quad (4.67)$$

The pair of potentials (2.36) in this case are given by

$$16V^\pm(x; b_0)A(z(x)) = -c^2 \nu^2 e^{2\nu z} - 4c\nu (a_0 \nu \mp 2b_0) e^{\nu z} + 4(b_0)^2 z = z(x). \quad (4.68)$$

We can easily check that they satisfy the two-step SI condition (2.3) with the parameters $b^{(2)}_0$ and $\tilde{R}_2$ given in (4.67). In this case, $b_1 = 0$ and the system admits only one intermediate Hamiltonian. The intermediate potential (2.37) is calculated as

$$16V^{il}(x; b_0)A(z(x)) = 3c^2 \nu^2 e^{2\nu z} + 4a_0 c \nu^2 e^{\nu z} + 4(b_0)^2. \quad (4.69)$$

Although it has the same functional dependence on the variable $z$ as $V^\pm$ but is not ordinary SI. Hence, the system is irreducible two-step SI.

The equation (2.12) with $A(z)$ given in (4.67) is integrated as

$$ce^{\nu z} = \begin{cases} a_0 \left( \tanh \sqrt{\frac{a_0}{2} \nu x} - 1 \right) & \text{for } a_0 \neq 0, \\ \frac{2}{\nu^2 x^2} & \text{for } a_0 = 0. \end{cases} \quad (4.70)$$

Substituting (4.70) into (4.68) and (4.69), we finally obtain the potentials in the $x$-space.
| Case 1-1 | Reducible |
| Case 1-2 | Reducible |
| Case 1-3 | Reducible |
| Case 1-4 | Irreducible, Conditional ($b_1 = -2a_2$) |
| Case 1-4 | Irreducible ($b_1 = -2a_2$) |
| Case 1-5 | Irreducible, Conditional ($a_3 = 0$) |
| Case 1-5 | Reducible ($a_3 \neq 0$) |
| Case 2-1 | Irreducible, Conditional (2 independent parameters) |
| Case 2-1 | Irreducible (1 independent parameter, $\mu \neq 4$) |
| Case 2-2 | Irreducible, Conditional |
| Case 2-3 | Irreducible, Conditional (2 independent parameters) |
| Case 2-3 | Irreducible (1 independent parameter) |
| Case 3  | Irreducible |

TABLE I: List of the obtained two-step SI potentials. Cases 1-4 and 1-5 can be included in Case 2-1 ($\mu = 3$ and $\mu = 4$, respectively).

V. DISCUSSION AND SUMMARY

In this paper, we have studied and constructed two-step SI potentials by utilizing the framework of $\mathcal{N}$-fold SUSY. Recognizing the crucial fact that two-step SI always means type A 2-fold SUSY with an intermediate Hamiltonian, we have successfully revealed its general aspects and made the systematic construction of such systems. The essential point of our analysis resides in the fact that type A $\mathcal{N}$-fold SUSY systems possess the simple general form (2.36) in terms of the variable $z$. It enables us to make the model-independent analysis without recourse to any specific assumption or ansatz. Furthermore, we have obtained even such two-step SI systems that admit an analytical expression only in terms of $z$, namely, in Case 2-1 with $a_1 \neq 0$, Case 2-2, and Case 2-3 with $a_1 \neq 0$. In Table I, we summarize the inequivalent two-step SI systems and their properties obtained in this paper.

It is remarkable that all the obtained two-step SI as well as ordinary SI potentials are of translational classes, that is, parameter relations are characterized by constant shifts $b_i^{(2)} = b_i^{(0)} + 2d_i$. In the traditional approaches, one assumes the latter relation from the beginning and then tries to construct an SI potential which meets it. A significant point of our analysis is that it is a consequence of the differential equation $\partial b_i^{-} / \partial b_i^{(0)} = 0$ originated from the requirement (4.3) that $A(z)$ should not depend on the parameters $b_i^{(0)}$. It in particular means that another class of SI potentials which is different from translational ones could be obtainable only from such a $A(z)$ that depends explicitly on $b_i^{(0)}$. Then, to obtain a scaling SI potential which has parameter relations $b_i^{(2)} = q_i b_i^{(0)}$, for instance, SI conditions are expected to lead to a differential equation like

$$\frac{\partial}{\partial b_i^{(0)}} (\ln b_i^{(2)} - \ln b_i^{(0)}) = 0.$$ 

In this way, we would be able to make a systematic analysis also on non-translational SI potentials.

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In [14], the concept of multi-step SI was also introduced as a natural extension of two-step SI. Following a similar argument in Section II, we easily conclude that multi-step SI is a special case of $\mathcal{N}$-fold SUSY with intermediate Hamiltonians at every intermediate positions for $\mathcal{N} > 2$. Until now on, there has been only one systematic study on such systems in the case of type A 3-fold SUSY [38]. It was shown in the latter reference that every type A 3-fold SUSY systems with intermediate Hamiltonians at every intermediate positions, which were dubbed Class (1,1), were ordinary SI as well. That is, any three-step SI in type A 3-fold SUSY is reducible. Hence, irreducible three-step SI can be realized, if it exists, only in other types of 3-fold SUSY.

Another interesting aspect of our present results which can be immediately read from Table I is that all the conditional two-step SI are irreducible. To the best of our knowledge, there have been no decisive mathematical understanding of conditional exact solvability. The result thus indicates the possibility that the latter concept might be well characterized in a context of irreducible two- and multi-step SI.

Although we have restricted our investigation to ordinary scalar Schrödinger operators, the concepts of ordinary, two-step, and multi-step SI are applicable to much wider systems. In fact, $\mathcal{N}$-fold SUSY was successfully formulated also for Schrödinger operators with position-dependent mass [35], matrix ones [41], and ones with reflection operators [42]. Hence, we would be able to make systematic studies on various SI in these systems with the framework of $\mathcal{N}$-fold SUSY, as have been done in this work.

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