CONTINUOUS CHARACTERIZATION OF THE BESOV SPACES OF VARIABLE SMOOTHNESS AND INTEGRABILITY

S. Benmahmoud¹ and D. Drihem²,³ UDC 517.9

We obtain new equivalent quasinorms of the Besov spaces of variable smoothness and integrability. Our main tools are the continuous version of the Calderón reproducing formula, maximal inequalities, and the variable-exponent technique. However, allowing the parameters to vary from point to point leads to additional difficulties which, in general, can be overcome by imposing regularity assumptions on these exponents.

1. Introduction

Besov spaces of variable smoothness and integrability initially appeared in the paper by A. Almeida and P. Hästö [3], where several basic properties were shown, such as the Fourier analytic characterization. Later, [9], the author characterized these spaces by local means and established the atomic characterization. After this, Kempka and Vybíral [14] characterized these spaces by ball means of differences and also by local means. The duality of these function spaces was studied in [12, 16].

The interest in the investigation of these spaces is caused not only by the theoretical reasons but also by their applications to several classical problems in analysis. For further considerations of PDEs, we refer the reader to [8] and the references therein.

The main aim of the present paper is to introduce a new equivalent quasinorm of these function spaces based on the continuous version of the Calderón reproducing formula. First, we define a new family of function spaces and prove their basic properties. Second, under certain suitable assumptions imposed on the parameters, we prove that these function spaces are just the Besov spaces of variable smoothness and integrability proposed by Almeida and Hästö. Finally, we characterize these function spaces in terms of continuous local means.

In this paper, we need some notation. As usual, we denote by \( \mathbb{N}_0 \) the set of all nonnegative integers. The notation \( f \lesssim g \) means that \( f \leq cg \) for some independent positive constant \( c \) (and nonnegative functions \( f \) and \( g \)), while \( f \approx g \) means that \( f \lesssim g \lesssim f \). For \( x \in \mathbb{R} \), \( \lfloor x \rfloor \) stands for the largest integer smaller than or equal to \( x \).

If \( E \subset \mathbb{R}^n \) is a measurable set, then \( |E| \) stands for the Lebesgue measure of \( E \) and \( \chi_E \) denotes its characteristic function. By \( c \) we denote generic positive constants, which may have different values in different cases. Although the exact values of constants are, as a rule, of no interest for our purposes, in some cases, we emphasize their dependence on certain parameters (thus, \( c(p) \) means that \( c \) depends on \( p \), etc.).

The symbol \( \mathcal{S}(\mathbb{R}^n) \) is used to denote the set of all Schwartz functions on \( \mathbb{R}^n \). We define the Fourier transform of a function \( f \in \mathcal{S}(\mathbb{R}^n) \) by

\[
\mathcal{F}(f)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx, \quad \xi \in \mathbb{R}^n.
\]
We denote by $S'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. The considered variable exponents are always measurable functions of $p$ on $\mathbb{R}^n$ with ranges in $(0, \infty]$. By $\mathcal{P}_0(\mathbb{R}^n)$ we denote the set of functions of this kind bounded away from the origin (i.e., $p^{-} > 0$). The subset of variable exponents with range in $[1, \infty]$ is denoted by $\mathcal{P}(\mathbb{R}^n)$. We use the following standard notation:

$$p^{-} := \operatorname{ess}\inf_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad p^{+} := \operatorname{ess}\sup_{x \in \mathbb{R}^n} p(x).$$

We set

$$\omega_p(t) = \begin{cases} t^p, & \text{if } p \in (0, \infty) \text{ and } t > 0, \\ 0, & \text{if } p = \infty \text{ and } 0 < t \leq 1, \\ \infty, & \text{if } p = \infty \text{ and } t > 1. \end{cases}$$

The variable-exponent modular is defined by

$$\varrho_p(f) := \int_{\mathbb{R}^n} \omega_p(x)(|f(x)|) \, dx.$$  

The variable-exponent Lebesgue space $L^{p(\cdot)}$ consists of measurable functions $f$ on $\mathbb{R}^n$ such that $\varrho_p(\lambda f) < \infty$ for some $\lambda > 0$. We define the Luxemburg (quasi)norm in this space by the formula

$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \varrho_p\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$  

A useful property is that $\|f\|_{p(\cdot)} \leq 1$ if and only if $\varrho_p(f) \leq 1$ (see Lemma 3.2.4 from [8]).

Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) := \sum_{v=0}^{\infty} \inf \left\{ \lambda_v > 0 : \varrho_{p(\cdot)}\left(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\right) \leq 1 \right\}.$$  

Therefore, the (quasi)norm is defined as usual:

$$\| (f_v)_v \|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf \left\{ \mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\frac{1}{\mu}(f_v)_v\right) \leq 1 \right\}. \quad (1)$$

If $q^{+} < \infty$, then we can replace (1) by a simpler expression

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_{v=0}^{\infty} \| f_v \|_{p(\cdot)}^{q(\cdot)}.$$  

We use this notation even for $q^{+} = \infty$. Let $(f_t)_{0 < t \leq 1}$ be a sequence of measurable functions, where $t$ is a continuous variable. We set

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_t)_{0 < t \leq 1}) := \int_0^1 \inf \left\{ \lambda_t : \varrho_{p(\cdot)}\left(\frac{f_t}{\lambda_t^{1/q(\cdot)}}\right) \leq 1 \right\} \frac{dt}{t}.$$
The (quasi)norm is defined by

\[
\|(f_t)_{0<t\leq 1}\|_{L^p(L^p)} := \inf \left\{ \mu > 0 : \varrho_{\mu\|\cdot\|L^p} \left( \frac{1}{\mu}(f_t)_{0<t\leq 1} \right) \leq 1 \right\}.
\]

We say that a real valued-function \( g \) on \( \mathbb{R}^n \) is \textit{locally log-Hölder continuous} on \( \mathbb{R}^n \) (and we write \( g \in C_{\text{loc}}^{\log}(\mathbb{R}^n) \)) if there exists a constant \( c_{\log}(g) > 0 \) such that

\[
|g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(e + 1/|x - y|)}
\]

for all \( x, y \in \mathbb{R}^n \).

We say that \( g \) satisfies the \textit{log-Hölder decay condition} if there exist two constants \( g_\infty \in \mathbb{R} \) and \( c_{\log} > 0 \) such that

\[
|g(x) - g_\infty| \leq \frac{c_{\log}}{\log(e + |x|)}
\]

for all \( x \in \mathbb{R}^n \). We say that \( g \) is \textit{globally log-Hölder continuous} on \( \mathbb{R}^n \) and write \( g \in C_{\text{log}}^{\log}(\mathbb{R}^n) \) if it is locally log-Hölder continuous on \( \mathbb{R}^n \) and satisfies the log-Hölder decay condition. The constants \( c_{\log}(g) \) and \( c_{\log} \) are called the \textit{locally log-Hölder constant} and \textit{log-Hölder decay constant}, respectively. We note that any function \( g \in C_{\text{log}}^{\log}(\mathbb{R}^n) \) always belongs to \( L^\infty \).

We define the following class of variable exponents:

\[
P_0^{\log}(\mathbb{R}^n) := \left\{ p \in P_0(\mathbb{R}^n) : \frac{1}{p} \in C^{\log}(\mathbb{R}^n) \right\},
\]

which was introduced in [6] (Section 2). The class \( P^{\log}(\mathbb{R}^n) \) is defined analogously. We define

\[
\frac{1}{p_\infty} := \lim_{|x| \to \infty} \frac{1}{p(x)}
\]

and use the following convention: \( \frac{1}{p_\infty} = 0 \). Note that, despite the fact that \( \frac{1}{p} \) is bounded, the variable exponent \( p \) itself can be unbounded. We set

\[
\Psi(x) := \sup_{|y| \geq |x|} |\varphi(y)|
\]

for \( \varphi \in L^1 \). Suppose that \( \Psi \in L^1 \). Further, in [8] (Lemma 4.6.3), it was proved that if \( p \in P^{\log}(\mathbb{R}^n) \), then

\[
\|\varphi_\varepsilon * f\|_{p(\cdot)} \leq c\|\Psi\|_1\|f\|_{p(\cdot)} \quad \text{for all} \quad f \in L^{p(\cdot)},
\]

where

\[
\varphi_\varepsilon := \frac{1}{\varepsilon^n} \varphi \left( \frac{\cdot}{\varepsilon} \right), \quad \varepsilon > 0.
\]

We set

\[
\eta_{t,m}(x) := t^{-n}(1 + t^{-1}|x|)^{-m}
\]
for any \( x \in \mathbb{R}^n, t > 0, \) and \( m > 0. \) Note that \( \eta_{t,m} \in L^1 \) for \( m > n \) and \( \| \eta_{t,m} \|_1 = c(m) \) is independent of \( t. \)

If \( t = 2^{-v}, \ v \in \mathbb{N}_0, \) then we define

\[
\eta_{2^{-v},m} := \eta_{2^{-v},m}.
\]

We refer the reader to the recent monograph [5] for further properties on variable-exponent spaces, historical remarks, and relevant references.

2. Basic Tools

In this section, we present some useful results. The following lemma was proved in [7] (Lemma 6.1) (see also [14], Lemma 19).

**Lemma 1.** Let \( \alpha \in C^1_{\text{log}}(\mathbb{R}^n), \ m \in \mathbb{N}_0, \) and \( R \geq c_{\log}(\alpha) \), where \( c_{\log}(\alpha) \) is the constant from (2) for \( g = \alpha. \) Then there exists a constant \( c > 0 \) such that

\[
t^{-\alpha(x)} \eta_{t,m+R}(x - y) \leq ct^{-\alpha(y)} \eta_{t,m}(x - y)
\]

for any \( 0 < t \leq 1 \) and \( x, y \in \mathbb{R}^n. \)

The previous lemma allows us to treat, in many cases, the smoothness of variables as if they are not variable at all. Namely, we can move the factor \( t^{-\alpha(x)} \) inside the convolution as follows:

\[
t^{-\alpha(x)} \eta_{t,m+R} * f(x) \leq c \eta_{t,m} * (t^{-\alpha(\cdot)} f)(x).
\]

The following lemma is taken from [22] (Lemma 3.14):

**Lemma 2.** Let \( p, q \in \mathcal{P}_0(\mathbb{R}^n) \) and \( f \) be a measurable function on \( \mathbb{R}^n. \) If

\[
\| |f|^{q(\cdot)} \|_{p(\cdot)} \geq 1,
\]

then

\[
\| f \|_{p(\cdot)}^{q(\cdot)} \leq \| |f|^{q(\cdot)} \|_{p(\cdot)}^{q(\cdot)}.
\]

The next lemma is a Hardy-type inequality; see [15].

**Lemma 3.** Let \( s > 0 \) and let \( (\varepsilon_t)_{0 < t \leq 1} \) be a sequence of positive measurable functions, where \( t \) is a continuous variable. Also let

\[
\eta_t = t^s \int_t^1 \tau^{-s} \varepsilon_{\tau} \frac{d\tau}{\tau} \quad \text{and} \quad \delta_t = t^{-s} \int_0^t \tau^s \varepsilon_{\tau} \frac{d\tau}{\tau}.
\]

Then there exists a constant \( c > 0 \) depending only on \( s \) and such that

\[
\int_0^1 \eta_t \frac{dt}{t} + \int_0^1 \delta_t \frac{dt}{t} \leq c \int_0^1 \varepsilon_t \frac{dt}{t}.
\]
Lemma 4. Let \( r, N > 0, m > n, \) and \( \theta, \omega \in \mathcal{S}(\mathbb{R}^n) \) with \( \text{supp} \, \mathcal{F} \omega \subseteq B(0,1) \). Then there exists a constant \( c = c(r, m, n) > 0 \) such that, for all \( g \in \mathcal{S}'(\mathbb{R}^n) \), the following inequality is true:

\[
\left| \theta_N \ast \omega_N \ast g(x) \right| \leq c \left( \eta_{N,m} \ast |\omega_N \ast g|^r (x) \right)^{1/r}, \quad x \in \mathbb{R}^n,
\]

where \( \theta_N(\cdot) := N^n \theta(N \cdot), \omega_N(\cdot) := N^n \omega(N \cdot), \) and \( \eta_{N,m} := N^n (1 + N | \cdot |)^{-m} \).

The proof of this lemma was given in [10] (Lemma 2.2). The next lemma is taken from [3] (Lemma 4.7) (we use it because the maximal operator is, in general, not bounded on \( \ell^q(\mathbb{R}^n) \); see [3], Example 4.1).

Lemma 5. Let \( p \in \mathcal{P}^{\log}(\mathbb{R}^n) \) and let \( q \in \mathcal{P}_0(\mathbb{R}^n) \) with

\[
\frac{1}{q} \in C^{\log}_{\text{loc}}(\mathbb{R}^n).
\]

For \( m > n + c_{\log} \left( \frac{1}{q} \right) \), there exists \( c > 0 \) such that

\[
\left\| (\eta_{r,m} \ast f)(v) \right\|_{\ell^q(\mathbb{R}^n)} \leq c \left\| f(v) \right\|_{\ell^q(\mathbb{R}^n)}.
\]

Lemma 6. Let \( 0 < \alpha < \beta < \infty, p \in \mathcal{P}^{\log}(\mathbb{R}^n) \), and \( q \in \mathcal{P}(\mathbb{R}^n) \) with \( \frac{1}{q} \in C^{\log}_{\text{loc}}(\mathbb{R}^n) \). Also let

\[
g_t(x) := \int_{\alpha t}^{\beta t} \eta_{r,m} \ast f(x) \frac{d\tau}{\tau}, \quad t \in (0, 1], \quad x \in \mathbb{R}^n.
\]

(i) Assume that \( 0 < \beta t \leq 1 \). The inequality

\[
\left\| c g_t \right\|_{\ell^q(\mathbb{R}^n)} \leq \int_{\alpha t}^{\beta t} \left\| f(\cdot) \right\|_{\ell^q(\mathbb{R}^n)} \frac{d\tau}{\tau} + t, \quad t \in (0, 1],
\]

holds for every sequence of functions \((f_t)_{0 < t \leq 1}\) and a constant

\[
m > n + c_{\log} \left( \frac{1}{q} \right)
\]

such that the first term on right-hand side does not exceed one, where the constant \( c \) independent of \( t \).

(ii) The inequality

\[
\left\| (g_t)_{0 < t \leq 1} \right\|_{\ell^q(\mathbb{R}^n)} \leq c \left\| (f_t)_{0 < t \leq 1} \right\|_{\ell^q(\mathbb{R}^n)},
\]

holds for every sequence of functions \((f_t)_{0 < t \leq 1}\) and a constant \( m > n + c_{\log} \left( \frac{1}{q} \right) \) such that the right-hand side is finite.
Proof. First, we prove (i). The claim can be reformulated as showing that

\[ J := \left\| c_1 \delta^{-\frac{1}{q}} g_t \right\|_{p(\cdot)} \leq 2^{1-\frac{1}{q}} + \ln \frac{\beta}{\alpha}, \quad t \in (0, 1], \]

where \( c_1 > 0 \) and

\[ \delta := \int_{\alpha t}^{\beta t} \left\| f_{\tau \cdot} \right\|_{q(\cdot)} \frac{d\tau}{\tau} + t. \]

Applying Lemma 1 with properly chosen \( c_1 \), we get

\[ J \leq \int_{\alpha t}^{\beta t} \left\| c_1 \delta^{-\frac{1}{q}} \left( \eta_{\tau, m} * f_{\tau \cdot} \right) \right\|_{p(\cdot)} \frac{d\tau}{\tau} \]

\[ \leq \int_{\alpha t}^{\beta t} \left\| \eta_{\tau, m - c \log \left( \frac{\tau}{\alpha t} \right)} \right\|_{p(\cdot)} \frac{d\tau}{\tau} \]

\[ \leq \int_{\alpha t}^{\beta t} \left\| \delta^{-\frac{1}{q}} f_{\tau \cdot} \right\|_{p(\cdot)} \frac{d\tau}{\tau} \]

since \( \delta \in (t, 1+t] \), and the convolution with a radially decreasing \( L^1 \)-function is bounded in \( L^{p(\cdot)} \) because \( m > n + c \log \left( \frac{1}{q} \right) \). We can write

\[ \int_{\alpha t}^{\beta t} \left\| \delta^{-\frac{1}{q}} f_{\tau \cdot} \right\|_{p(\cdot)} \frac{d\tau}{\tau} = \int_{(\alpha t, \beta t] \cap B} \cdots + \int_{(\alpha t, \beta t] \cap B^c} \cdots = J_{1,t} + J_{2,t}, \]

where

\[ B := \left\{ \tau > 0 : \left\| \delta^{-\frac{1}{q}} f_{\tau \cdot} \right\|_{q(\cdot)} \frac{1}{p(\cdot)} \geq 1 \right\}. \]

By Lemma 2, we have

\[ J_{1,t} \leq \int_{(\alpha t, \beta t] \cap B} \left\| \delta^{-\frac{1}{q}} f_{\tau \cdot} \right\|_{q(\cdot)} \frac{1}{p(\cdot)} \frac{d\tau}{\tau} \leq 2^{1-\frac{1}{q}} \delta^{-1} \int_{\alpha t}^{\beta t} \left\| f_{\tau \cdot} \right\|_{q(\cdot)} \frac{d\tau}{\tau} \leq 2^{1-\frac{1}{q}} \]

and

\[ J_{2,t} \leq \int_{\alpha t}^{\beta t} \left\| \delta^{-\frac{1}{q}} f_{\tau \cdot} \right\|_{p(\cdot)} \frac{d\tau}{\tau} \leq \int_{\alpha t}^{\beta t} \frac{d\tau}{\tau} = \ln \frac{\beta}{\alpha}. \]
We now prove (ii). By the scaling argument, it suffices to consider the case
\[ \| (f_t)_{0 < t \leq 1} \|_{\ell^1(L^p(\cdot))} = 1 \]
and show that the modular of \( f \) on the left-hand side is bounded. In particular, we show that
\[ \int_0^1 \| cg_t \|_{L^q}^q \, \frac{dt}{t} \leq 2 \]
for some positive constant \( c \). Applying the Hardy inequality [see Lemma 3 and property (i)], we obtain the desired result.

**Lemma 7.** Let \( 0 < r < \infty \) and \( m > \max\left( \frac{n}{r}, \frac{n}{m} \right) \). Assume that \( \{ \mathcal{F}\Phi, \mathcal{F}\varphi \} \) is a resolution of unity (see Section 3):

\[ \mathcal{F}\Phi(\xi) + \int_0^1 \mathcal{F}\varphi(t\xi) \frac{dt}{t} = 1, \quad \xi \in \mathbb{R}^n. \]

(i) Let \( \theta \in \mathcal{S}(\mathbb{R}^n) \) be such that \( \text{supp} F\theta \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2 \} \). Then there exists a constant \( c > 0 \) such that
\[ |\theta * f|^r \leq c\eta_{1,mr} * |\Phi * f|^r + c \int_{1/4}^1 \eta_{1,mr} * |\varphi_r * f|^r \frac{d\tau}{\tau} \]
for any \( f \in \mathcal{S}'(\mathbb{R}^n) \), where \( \varphi_r = \tau^{-n}\varphi\left( \frac{\cdot}{\tau} \right) \).

(ii) Let \( \omega \in \mathcal{S}(\mathbb{R}^n) \) be such that
\[ \text{supp} F\omega \subset \left\{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2 \right\}. \]

Then there exists a constant \( c > 0 \) such that
\[ |\omega_t * f|^r \leq c\eta_{1,mr} * |\Phi * f|^r + c \int_{t/4}^{\min(1,4t)} \eta_{r,mr} * |\varphi_r * f|^r \frac{d\tau}{\tau} \]
for any \( f \in \mathcal{S}'(\mathbb{R}^n) \) and any \( 0 < t \leq 1 \), where \( \omega_t = t^{-n}\omega\left( \frac{\cdot}{t} \right) \).

**Proof.** We split the proof into two steps. First, the case \( 1 \leq r < \infty \) follows from the Hölder inequality.

**Step 1.** Proof of (i). Since \( \{ \mathcal{F}\Phi, \mathcal{F}\varphi \} \) is a resolutions of unity, we obtain
\[ \theta * f = \Phi * \theta * f + \int_{1/4}^1 \theta * \varphi_r * f \frac{d\tau}{\tau}. \]
First, we recall an elementary inequality

\[ d^n \eta_{d,m}(y - z) \leq d^{2n} \eta_{d,-m}(y - x) \eta_{d,m}(x - z), \quad d > 0, \quad x, y, z \in \mathbb{R}^n. \]

Together with Lemma 4, this implies that

\[ |\Phi * \theta * f(y)|^r \lesssim \eta_{1,mr} * |\Phi * f|^r(y) \]

\[ = c \int_{\mathbb{R}^n} \eta_{1,mr}(y - z)|\Phi * f(z)|^r dz \]

\[ \lesssim \eta_{1,-mr}(y - x) \eta_{1,mr} * |\Phi * f|^r(x) \]

for any \( x \in \mathbb{R}^n \) and any \( m > \frac{n}{r} \). Furthermore,

\[ |\Phi * \theta * f(y)| \leq \int_{\mathbb{R}^n} \eta_{1,N}(y - z)|\theta * f(z)| dz \]

\[ \leq \eta_{1,-m}(y - x) \theta_1^{s,m} f(x) \int_{\mathbb{R}^n} \eta_{1,N-m}(y - z) dz \]

\[ \lesssim \eta_{1,-m}(y - x) \theta_1^{s,m} f(x) \]

for any \( N > m + n \), where

\[ \theta_1^{s,m} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\theta * f(y)|}{(1 + |y - x|)^m}, \quad x \in \mathbb{R}^n. \]

Therefore,

\[ |\Phi * \theta * f(y)| \lesssim \eta_{1,-m}(y - x)(\theta_1^{s,m} f(x))^{1-r} \eta_{1,mr} * |\Phi * f|^r(x) \]

for any \( x \in \mathbb{R}^n \) and any \( m > n \). By using Lemma 4 once again, we conclude that

\[ |\theta * \varphi_\tau * f(y)|^r \lesssim \eta_{1,mr} * |\varphi_\tau * f|^r(y) \]

\[ \lesssim (1 + |y - x|)^{mr} \eta_{1,mr} * |\varphi_\tau * f|^r(x) \]

and

\[ |\theta * \varphi_\tau * f(y)| \leq \int_{\mathbb{R}^n} \eta_{r,N}(y - z)|\theta * f(z)| dz \]

\[ \lesssim (1 + |y - x|)^m \theta_1^{s,m} f(x), \quad \frac{1}{4} \leq \tau \leq 1, \]
for any \( x \in \mathbb{R}^n \), any \( m > n \), and any \( N > m + n \). Consequently,

\[
\theta_1^{*,m} f(x) \leq c(\theta_1^{*,m} f(x))^{1-r} \left( \eta_{1,mr} \ast |\Phi \ast f|^r(x) + \int_{1/4}^1 \eta_{1,mr} \ast |\varphi \ast f|^r(x) \frac{d\tau}{\tau} \right). 
\] (3)

This yields

\[
|\theta \ast f(x)|^r \leq c\eta_{1,mr} \ast |\Phi \ast f|^r(x) + c \int_{1/4}^1 \eta_{1,mr} \ast |\varphi \ast f|^r(x) \frac{d\tau}{\tau},
\] (4)

for \( \theta_1^{*,m} f(x) < \infty \), which is true if \( m \geq \frac{n}{r} + N_0 \) (the order of distribution).

We now apply the Strömberg–Torchinsky idea [18]. Note that the right-hand side of (4) decreases as \( m \) increases. Therefore, we get (4) for all \( m > n \) but with \( c = c(f) \) depending on \( f \). We can easily show that if the right-hand side of (4) with \( c = c(f) \) is finite, then \( \theta_1^{*,m} f(x) < \infty \); otherwise, there is nothing to prove. Returning to (3) and having in mind that, in this case, \( \theta_1^{*,m} f(x) < \infty \), we obtain the desired estimate (4).

Step 2. Proof of (ii). We have

\[
\omega_t \ast f = \min(1,4t) \int_{t/4} \omega_t \ast \varphi \ast f \frac{d\tau}{\tau} + \begin{cases} 0, & \text{if } 0 < t < \frac{1}{4}, \\ \omega_t \ast \Phi \ast f, & \text{if } \frac{1}{4} \leq t \leq 1. \end{cases}
\]

Let

\[
g_t(y) := \min(1,4t) \int_{t/4} \omega_t \ast \varphi \ast f(y) \frac{d\tau}{\tau}, \quad y \in \mathbb{R}^n, \quad 0 < t \leq 1.
\]

It follows from Lemma 4 that

\[
|\omega_t \ast \varphi \ast f(y)|^r \lesssim \eta_{t,mr} \ast |\varphi \ast f|^r(y)
\]

\[
\lesssim \eta_{r,mr} \ast |\varphi \ast f|^r(y) = c \int \eta_{r,mr}(y-z)|\varphi \ast f(z)|^r dz
\]

\[
\lesssim (1 + \tau^{-1}|y-x|)^{mr} \eta_{r,mr} \ast |\varphi \ast f|^r(x)
\]

and

\[
|\omega_t \ast \varphi \ast f(y)| \lesssim \int \eta_{r,N}(y-z)|\omega_t \ast f(z)| dz
\]

\[
\lesssim \omega_t^{*,m} f(y) \int \eta_{r,N}(y-z)(1 + t^{-1}|y-z|)^m dz
\]
\[ \omega_t^{*,m}f(y) \lesssim (1 + t^{-1}|y - x|)^m \omega_t^{*,m}f(x) \]

for any \( x, y \in \mathbb{R}^n \), any \( \frac{t}{4} \leq \tau \leq \min(1, 4t) \), \( 0 < t \leq 1 \), and any \( N > m + n \), where

\[ \omega_t^{*,m}f(x) = \sup_{y \in \mathbb{R}^n} \frac{\omega_t * f(y)}{(1 + t^{-1}|y - x|)^m}, \quad x, y \in \mathbb{R}^n, \quad 0 < t \leq 1. \]

Therefore, \(|g_t(y)|\) can be estimated from above as follows:

\[ c(\omega_t^{*,m}f(x))^{1-r}(1 + t^{-1}|y - x|)^{m(1-r)} \]

\[ \times \int_{t/4}^{\min(1,4t)} (1 + \tau^{-1}|y - x|)^{mr} \eta_{t, mr} * |\varphi_{\tau} * f|^r(x) \frac{d\tau}{\tau} \]

\[ \lesssim (1 + t^{-1}|y - x|)^m (\omega_t^{*,m}f(x))^{1-r} \]

\[ \times \int_{t/4}^{\min(1,4t)} \eta_{t, mr} * |\varphi_{\tau} * f|^r(x) \frac{d\tau}{\tau}, \quad \text{if} \quad 0 < t \leq 1. \]

Thus, if \( \frac{1}{4} \leq t \leq 1 \), then we easily obtain

\[ |\omega_t * \Phi * f(y)| = |\omega_t * \Phi * f(y)|^{1-r}|\omega_t * \Phi * f(y)|^r \]

\[ \lesssim (1 + t^{-1}|y - x|)^{m(1-r)} (\omega_t^{*,m}f(x))^{1-r} \eta_{1, mr} * |\Phi * f|^r(y) \]

\[ \lesssim (1 + t^{-1}|y - x|)^m (\omega_t^{*,m}f(x))^{1-r} \eta_{1, mr} * |\Phi * f|^r(x), \]

which yields that

\[ \sup_{y \in \mathbb{R}^n} \frac{|\omega_t * \Phi * f(y)|}{(1 + t^{-1}|y - x|)^m} \lesssim (\omega_t^{*,m}f(x))^{1-r} \eta_{1, mr} * |\Phi * f|^r(x). \]

Consequently,

\[ |\omega_t * f(x)|^r \lesssim (\omega_t^{*,m}f(x))^r \lesssim \eta_{1, mr} * |\Phi * f|^r(x) + \int_{t/4}^{\min(1,4t)} \eta_{t, mr} * |\varphi_{\tau} * f|^r(x) \frac{d\tau}{\tau}, \]

where \( \omega_t^{*,m}f(x) < \infty, 0 < t \leq 1, \) and \( x \in \mathbb{R}^n \). By using the combination of arguments applied in (i), we arrive at the desired estimate.

Lemma 7 is proved.
The following lemma is taken from [17] (Lemma 1).

**Lemma 8.** Let $\varrho, \mu \in \mathcal{S}(\mathbb{R}^n)$ and let $M \geq -1$ be an integer such that

$$\int_{\mathbb{R}^n} x^\alpha \mu(x) dx = 0$$

for all $|\alpha| \leq M$. Then, for any $N > 0$, there exists a constant $c(N) > 0$ such that

$$\sup_{z \in \mathbb{R}^n} |t^{-n} \mu(t^{-1} \cdot) \ast \varrho(z)|(1 + |t|)^N \leq c(N)t^{M+1}, \quad 0 < t \leq 1.$$

### 3. Variable Besov Spaces

In this section, we present the definition of Besov spaces of variable smoothness and integrability and establish their basic properties by analogy with the case of fixed exponents. We select a pair of Schwartz functions $\Phi$ and $\varphi$ such that

$$\text{supp } \mathcal{F}\Phi \subset \{x \in \mathbb{R}^n : |x| \leq 2\}, \quad \text{supp } \mathcal{F}\varphi \subset \left\{x \in \mathbb{R}^n : \frac{1}{2} \leq |x| \leq 2 \right\}$$

and

$$\mathcal{F}\Phi(\xi) + \int_{0}^{1} \mathcal{F}\varphi(t\xi) \frac{dt}{t} = 1, \quad \xi \in \mathbb{R}^n.$$  \hspace{1cm} (6)

The resolution of unity (5) and (6) can be constructed as follows: Let $\mu \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$|\mathcal{F}\mu(\xi)| > 0 \quad \text{for} \quad 1/2 < |\xi| < 2.$$

There exists $\eta \in \mathcal{S}(\mathbb{R}^n)$ with

$$\text{supp } \mathcal{F}\eta \subset \left\{x \in \mathbb{R}^n : \frac{1}{2} < |x| < 2 \right\}$$

such that

$$\int_{0}^{\infty} \mathcal{F}\mu(t\xi) \mathcal{F}\eta(t\xi) \frac{dt}{t} = 1, \quad \xi \neq 0;$$

see [4, 11, 13]. We set

$$\mathcal{F}\varphi = \mathcal{F}\mu \mathcal{F}\eta$$

and

$$\mathcal{F}\Phi(\xi) = \begin{cases} \int_{1}^{\infty} \mathcal{F}\varphi(t\xi) \frac{dt}{t}, & \text{if} \quad \xi \neq 0, \\ 1, & \text{if} \quad \xi = 0. \end{cases}$$
Thus, $\mathcal{F}\Phi \in \mathcal{S} (\mathbb{R}^n)$ and, in view of the fact that $\mathcal{F}\eta$ is supported by $\left\{ x \in \mathbb{R}^n : \frac{1}{2} \leq |x| \leq 2 \right\}$, we conclude that

$$\text{supp} \mathcal{F}\Phi \subset \{ x \in \mathbb{R}^n : |x| \leq 2 \}.$$  

We now define the spaces under consideration.

**Definition 1.** Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $p, q \in \mathcal{P}_0 (\mathbb{R}^n)$. Also let $\{ \mathcal{F}\Phi, \mathcal{F}\varphi \}$ be a resolution of unity and we set

$$\varphi_t = t^{-\alpha} \varphi \left( \frac{\cdot}{t} \right), \quad 0 < t \leq 1.$$  

The Besov space $\mathcal{B}^{\alpha(-)}_{p(-),q(-)}$ is the collection of all $f \in \mathcal{S}' (\mathbb{R}^n)$ such that

$$\| f \|_{\mathcal{B}^{\alpha(-)}_{p(-),q(-)}} := \| \Phi \ast f \|_{p(-)} + \| (t^{-\alpha(-)} \varphi_t \ast f)_{0 < t \leq 1} \|_{\ell p(-)(L^{p(-)})} < \infty.$$  

For $q = \infty$, the Besov space $\mathcal{B}^{\alpha(-)}_{p(-),\infty}$ consists of all distributions $f \in \mathcal{S}' (\mathbb{R}^n)$ such that

$$\| f \|_{\mathcal{B}^{\alpha(-)}_{p(-),\infty}} := \| \Phi \ast f \|_{p(-)} + \sup_{t \in (0,1]} \| t^{-\alpha(-)} (\varphi_t \ast f) \|_{p(-)} < \infty.$$  

We immediately conclude that $\mathcal{B}^{\alpha(-)}_{p(-),q(-)}$ is a quasinormed space and, moreover, if $\alpha, p,$ and $q$ are constants, then

$$\mathcal{B}^{\alpha(-)}_{p(-),q(-)} = B^{\alpha}_{p,q},$$

where $B^{\alpha}_{p,q}$ is the ordinary Besov space.

We are now ready to show that the definition of these function spaces is independent of the chosen resolution of unity $\{ \mathcal{F}\Phi, \mathcal{F}\varphi \}$. This justifies the possibility of omitting the subscripts $\Phi$ and $\varphi$ in what follows.

**Theorem 1.** Let $\{ \mathcal{F}\Phi, \mathcal{F}\varphi \}$ and $\{ \mathcal{F}\Psi, \mathcal{F}\psi \}$ be two resolutions of unity. Also let $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $p, q \in \mathcal{P}_0 (\mathbb{R}^n)$. Assume that $p \in \mathcal{P}^{\log} (\mathbb{R}^n)$ and $\alpha, \frac{1}{q} \in C^{\log} (\mathbb{R}^n)$. Then

$$\| f \|_{\mathcal{B}^{\alpha(-)}_{p(-),q(-)}} \approx \| f \|_{\mathcal{B}^{\alpha(-)}_{\psi(-),\psi(-)}} \quad .$$

**Proof.** It is sufficient to show that there exists a constant $c > 0$ such that, for all $f \in \mathcal{B}^{\alpha(-)}_{p(-),q(-)}$, we have

$$\| f \|_{\mathcal{B}^{\alpha(-)}_{p(-),q(-)}} \lesssim \| f \|_{\mathcal{B}^{\alpha(-)}_{\psi(-),\psi(-)}} \quad .$$

In view of Lemma 7, the problem can be reduced to the case where

$$p \in \mathcal{P}^{\log} (\mathbb{R}^n) \quad \text{and} \quad q \in \mathcal{P} (\mathbb{R}^n) \quad \text{with} \quad \frac{1}{q} \in C^{\log} (\mathbb{R}^n).$$
By the scaling argument, it suffices to consider the case

$$\|f\|_{\mathcal{B}_{p,q}(\cdot)} = 1$$

and show that

$$\|\Phi * f\|_{p(\cdot)} \lesssim 1$$

and

$$\int_0^1 \|ct^{-\alpha(\cdot)}(\varphi_t \ast f)|q(\cdot)\|_{p(\cdot)} \frac{dt}{t} \leq 1$$

for some positive constant $c$. Interchanging the roles of $(\Psi, \psi)$ and $(\Phi, \varphi)$, we obtain the desired result. Thus, we get

$$\mathcal{F}\Phi(\xi) = \mathcal{F}\Phi(\xi)\mathcal{F}\Psi(\xi) + \int_{1/4}^1 \mathcal{F}\Phi(\xi)\mathcal{F}\psi(\tau\xi) \frac{d\tau}{\tau}$$

and

$$\mathcal{F}\varphi(t\xi) = \int_{t/4}^{\min(1,4t)} \mathcal{F}\varphi(t\xi)\mathcal{F}\psi(\tau\xi) \frac{d\tau}{\tau} + \begin{cases} 0, & \text{if } 0 < t < \frac{1}{4}, \\ \mathcal{F}\varphi(t\xi)\mathcal{F}\Psi(\xi), & \text{if } \frac{1}{4} \leq t \leq 1, \end{cases}$$

for any $\xi \in \mathbb{R}^n$. Further, we conclude that

$$\Phi * f = \Phi * \Psi * f + \int_{1/4}^1 \Phi * \psi_t * f \frac{d\tau}{\tau}$$

and

$$\varphi_t * f = \int_{t/4}^{\min(1,4t)} \varphi_t * \psi_t * f \frac{d\tau}{\tau} + \begin{cases} 0, & \text{if } 0 < t < \frac{1}{4}, \\ \varphi_t * \Psi * f, & \text{if } \frac{1}{4} \leq t \leq 1. \end{cases}$$

First, we observe that

$$|\Phi * \psi_\tau * f| \lesssim |\eta_{1,m} * \psi_\tau * f| \lesssim |\eta_{1,m} * \tau^{-\alpha(\cdot)}\psi_\tau * f|, \quad \frac{1}{4} \leq \tau \leq 1, \quad m > n,$$

and

$$|\Phi * \Psi * f| \lesssim |\eta_{1,m} * \psi * f|, \quad m > n.$$
Therefore,

\[ |\Phi \ast f| \leq \eta_{1,m} \ast |\Psi \ast f| + \int_{1/4}^{1} \eta_{1,m} \ast \tau^{-\alpha(\cdot)} |\psi_{\tau} \ast f| \frac{d\tau}{\tau}, \]

\[ = \eta_{1,m} \ast |\Psi \ast f| + g. \]

Since \( p \in \mathcal{P}^{\log}(\mathbb{R}^n) \) and the convolution with a radially decreasing \( L^1 \)-function is bounded on \( L^{p(\cdot)} \), we obtain

\[
\| \eta_{1,m} \ast |\Psi \ast f| \|_{p(\cdot)} \lesssim \| \Psi \ast f \|_{p(\cdot)} \leq 1.
\]

Further, for a suitable positive constant \( c_1 \),

\[
\| c_1 g \|_{p(\cdot)} \leq 1
\]

if and only if

\[
\| c_1 g^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1,
\]

which follows from Lemma 6 (i). Therefore,

\[
\| \Phi \ast f \|_{p(\cdot)} \lesssim 1.
\]

By using the fact that the convolution with a radially decreasing \( L^1 \)-function is bounded in \( L^{p(\cdot)} \), we obtain

\[
\| c \varphi_t \ast \Psi \ast f \|^ {q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1,
\]

for a proper choice of \( c \) and any \( t \in (0, 1] \). Note that

\[
|\varphi_t \ast f| \lesssim \int_{t/4}^{4t} \eta_{\tau,m} \ast |\psi_{\tau} \ast f| \frac{d\tau}{\tau},
\]

\[ m > n + c \log \left( \frac{1}{q} \right), \quad t \in \left( 0, \frac{1}{4} \right]. \]

Applying Lemma 6 (ii) once again, we find

\[
\int_{0}^{\frac{1}{4}} \| c t^{-\alpha(\cdot)} (\varphi_t \ast f) \|^ {q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}} \frac{dt}{t} \leq 1
\]

for a suitable positive constant \( c \).

Theorem 1 is proved.
Let $a > 0$, $\alpha : \mathbb{R}^n \to \mathbb{R}$, and $f \in S'((\mathbb{R}^n)^\prime)$. Then we define the Peetre maximal function as follows:

$$
\varphi_t^{\ast,a} t^{-\alpha(\cdot)} f(x) := \sup_{y \in \mathbb{R}^n} \frac{t^{-\alpha(y)} |\varphi_t * f(y)|}{(1 + t^{-1} |x - y|)^a}, \quad t > 0,
$$

and

$$
\Phi^{\ast,a} f(x) := \sup_{y \in \mathbb{R}^n} \frac{|\Phi * f(y)|}{(1 + |x - y|)^a}.
$$

We now present a fundamental characterization of the analyzed spaces.

**Theorem 2.** Let $\alpha, \frac{1}{q} \in C_{\log_0}^\text{loc}(\mathbb{R}^n), p \in P_0^\log_0(\mathbb{R}^n), q^- \geq p^-$, and let

$$
a > \frac{n + c_{\log}(1/q)}{p^-} + c_{\log}(\alpha),
$$

Then

$$
\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} := \|\Phi^{\ast,a}\|_{p(\cdot)} + \|(\varphi_t^{\ast,a} t^{-\alpha(\cdot)} f)_{0 < t \leq 1}\|_{\ell_p(L^q(\cdot))}
$$

is an equivalent quasinorm in $\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$.

**Proof.** It is easy to see that, for any $f \in S'((\mathbb{R}^n)^\prime)$ with $\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} < \infty$ and any $x \in \mathbb{R}^n$, we have

$$
t^{-\alpha(x)} |\varphi_t * f(x)| \leq \varphi_t^{\ast,a} t^{-\alpha(\cdot)} f(x).
$$

This means that

$$
\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \leq \|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.
$$

We now prove that there exists a constant $C > 0$ such that, for every $f \in \mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$,

$$
\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \leq C\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}. \quad (7)
$$

By Lemmas 1 and 4, the estimate

$$
t^{-\alpha(y)} |\varphi_t * f(y)| \leq C_1 t^{-\alpha(y)} (\eta_t \sigma_p^- * |\varphi_t * f| p^- (y))^{1/p^-}

\leq C_2 (\eta_t (\sigma - c_{\log}(\alpha)) p^- * (t^{-\alpha(\cdot)} |\varphi_t * f| p^- (y))^{1/p^-} \quad (8)
$$

is true for any $y \in \mathbb{R}^n$,
and \( t > 0 \). Further, we divide both sides of (8) by \((1 + t^{-1}|x - y|)^{\alpha}\); moreover, on the right-hand side, we use the inequality
\[
(1 + t^{-1}|x - y|)^{-\alpha} \leq (1 + t^{-1}|x - z|)^{-\alpha}(1 + t^{-1}|y - z|)^{\alpha}, \quad x, y, z \in \mathbb{R}^n
\]
and, on the left-hand side, take the supremum over \( y \in \mathbb{R}^n \). This implies that, for all \( f \in \mathcal{B}_{p, q}^{\alpha} \) any \( t > 0 \), and any \( \sigma > \max \left( \frac{n + c\log(1/q)}{p^-} + c\log(\alpha), a + c\log(\alpha) \right) \),
the following inequality is true:
\[
\varphi_t^{*, \alpha} (t^{-\alpha} f)(x) \leq C_2 \left( \eta_{t, ap^-} * (t^{-\alpha} p^- |\varphi_t * f|^p)(x) \right)^{1/p^-},
\]
where \( C_2 > 0 \) is independent of \( x, t, \) and \( f \). Assume that the right-hand side of (7) is smaller than or equal to 1. We prove that
\[
\left\| \Phi^{*, \alpha} p^- \right\|_{p^-} + \left\| \left( \eta_{t, ap^-} * (t^{-\alpha} p^- |\varphi_t * f|^p) \right)^{1/p^-} \right\|_{0<t\leq1} \lesssim 1. \tag{9}
\]
Note that the second quasinorm on the left-hand side of (9) can be rewritten as
\[
\left\| \left( \eta_{t, ap^-} * (t^{-\alpha} p^- |\varphi_t * f|^p) \right) \right\|_{0<t\leq1} \left( \frac{1}{p^-} \right)_{L^p(L^p)} \tag{10}
\]
Let \( 0 < t < \frac{1}{4} \). In view the proof of Lemma 7 (ii), we obtain
\[
\begin{align*}
t^{-\alpha} p^- |\varphi_t * f|^p &\lesssim \int_{t/4}^{4t} \tau^{-\alpha} p^- \eta_{\tau, ap^-} * |\varphi_{\tau} * f|^p \frac{d\tau}{\tau} \\
&\lesssim \int_{t/4}^{4t} \eta_{\tau, ap^- - c\log(\alpha)p^-} \tau^{-\alpha} p^- |\varphi_{\tau} * f|^p \frac{d\tau}{\tau},
\end{align*}
\]
by Lemma 1. Therefore,
\[
\begin{align*}
\eta_{t, ap^-} * (t^{-\alpha} p^- |\varphi_t * f|^p) &\lesssim \int_{t/4}^{4t} \eta_{t, ap^-} * \eta_{\tau, ap^- - c\log(\alpha)p^-} \tau^{-\alpha} p^- |\varphi_{\tau} * f|^p \frac{d\tau}{\tau} \\
&\lesssim \int_{t/4}^{4t} \eta_{\tau, ap^- - c\log(\alpha)p^-} \tau^{-\alpha} p^- |\varphi_{\tau} * f|^p \frac{d\tau}{\tau}.
\end{align*}
\]
by [7] (Lemma A.3). Applying Lemma 6, we conclude that (10) with \( 0 < t < \frac{1}{4} \) is bounded by
\[
\left\| (t^{-\alpha}p^-|\varphi_t * f|^p^-)_{0 < t \leq 1} \right\|_{\ell^p(L^p^-)}^{\frac{1}{p(-1)}} \lesssim 1.
\]

Now let \( \frac{1}{4} \leq t \leq 1 \). Again, by virtue of Lemma 7 (ii), we get
\[
|\varphi_t * f|^p^- \leq c \eta_{t, ap^-} * |\Phi * f|^p^- + c \int_{t/4}^1 \eta_{r, ap^-} * |\varphi_r * f|^p^- \frac{d\tau}{\tau}.
\]
As above, we obtain
\[
\eta_{t, ap^-} * (t^{-\alpha}p^-|\varphi_t * f|^p^-)
\]
\[
\leq c \eta_{t, ap^-} * |\Phi * f|^p^- + c \int_{t/4}^1 \eta_{r, ap^-} * |\varphi_r * f|^p^- \frac{d\tau}{\tau}
\]
\[
= c \eta_{t, ap^-} * |\Phi * f|^p^- + h_t(x).
\]
It is necessary to prove that
\[
\left\| (\eta_{1, ap^-} * |\Phi * f|^p^-)_{\frac{1}{4} \leq t \leq 1} \right\|_{\ell^p(L^p^-)}^{\frac{1}{p(-1)}} \lesssim 1 \quad \text{and} \quad \left\| (h_t)_{\frac{1}{4} \leq t \leq 1} \right\|_{\ell^p(L^p^-)}^{\frac{1}{p(-1)}} \lesssim 1. \tag{11}
\]
Applying Lemma 6, we obtain the second estimate in (11). Let us prove the first estimate. This is equivalent to
\[
\left\| \eta_{1, ap^-} * |\Phi * f|^p^- \right\|_{\ell^p(L^p^-)}^{\frac{1}{p(-1)}} \lesssim 1,
\]
which is, in turn, equivalent to
\[
\left\| \eta_{1, ap^-} * |\Phi * f|^p^- \right\|_{\ell^p(L^p^-)}^{\frac{1}{p(-1)}} \lesssim 1.
\]
Since \( \frac{p(-1)}{p^-} \in \mathcal{P}^{\log}(\mathbb{R}^n) \) and the convolution with a radially decreasing \( L^1 \)-function is bounded in \( L^{p(-1)} \), we find
\[
\left\| \eta_{1, ap^-} * |\Phi * f|^p^- \right\|_{\ell^p(L^p^-)}^{\frac{1}{p(-1)}} \lesssim \left\| |\Phi * f|^p^- \right\|_{\ell^p(L^p^-)}^{\frac{1}{p(-1)}} = c \left\| |\Phi * f|^p^- \right\|_{p(-1)} \lesssim 1.
\]
The estimate for \( \left\| \Phi^{*, \alpha} \right\|_{p(-1)} \) readily follows from the fact that
\[
\left\| \Phi^{*, \alpha} \right\|_{p(-1)} \lesssim \left\| \eta_{1, ap^-} * |\Phi * f|^p^- \right\|_{\ell^p(L^p^-)}^{\frac{1}{p(-1)}} \lesssim 1.
\]
Theorem 2 is proved.
4. Relationship between $\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$

In this section, we establish the coincidence between the above function spaces and the variable Besov spaces of Almeida and Hästö; here, to define these function spaces, we first need the concept of smooth dyadic resolution of unity. Let $\Psi$ be a function in $\mathcal{S}(\mathbb{R}^n)$ such that

$$
\Psi(x) = 1 \quad \text{for} \quad |x| \leq 1 \quad \text{and} \quad \Psi(x) = 0 \quad \text{for} \quad |x| \geq 2.
$$

We define $\psi_0$ and $\psi_1$ by the equalities

$$
\mathcal{F}\psi_0(x) = \Psi(x), \quad \mathcal{F}\psi_1(x) = \Psi\left(\frac{x}{2}\right) - \Psi(x),
$$

and

$$
\mathcal{F}\psi_v(x) = \mathcal{F}\psi_1(2^{1-v}x) \quad \text{for} \quad v = 2, 3, \ldots.
$$

Then $\{\mathcal{F}\psi_v\}_{v \in \mathbb{N}_0}$ is a smooth dyadic resolution of unity and

$$
\sum_{v=0}^{\infty} \mathcal{F}\psi_v(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}^n.
$$

Thus, we obtain the Littlewood–Paley decomposition

$$
f = \sum_{v=0}^{\infty} \psi_v \ast f \quad \text{for all} \quad f \in \mathcal{S}'(\mathbb{R}^n)
$$

(convergence in $\mathcal{S}'(\mathbb{R}^n)$).

We now formulate the definition of the spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$, which were introduced and investigated in [3].

**Definition 2.** Let $\{\mathcal{F}\psi_v\}_{v \in \mathbb{N}_0}$ be a resolution of unity, $s : \mathbb{R}^n \to \mathbb{R}$, and $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. The Besov space $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ consists of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$
\|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \|2^{vs(\cdot)}\psi_v \ast f\|_{\ell^q(\ell^{p(\cdot)})} < \infty.
$$

Taking $s \in \mathbb{R}$ and $q \in (0, \infty]$ as constants, we obtain the spaces $B^{s}_{p,q}$ studied by Xu [23]. We refer the reader to the recent papers [1, 2, 9, 14] for further details concerning these function spaces, historical remarks, and more references. For any $p, q \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $s \in \mathcal{C}^{\log}_{\text{loc}}$, the space $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ does not depend on the chosen smooth dyadic resolution of unity $\{\mathcal{F}\psi_v\}_{v \in \mathbb{N}_0}$ (in a sense of equivalent quasinorms) and, in addition,

$$
\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{s(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).
$$

Moreover, if $p, q,$ and $s$ are constants, then we reobtain the ordinary Besov spaces $B^{s}_{p,q}$, studied in detail in [20, 21]; see also [19].
Theorem 3. Let $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ and $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. Assume that $p \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $\alpha, \frac{1}{q} \in C_{\text{loc}}^{\log}(\mathbb{R}^n)$. Then

$$\mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)} B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)} = B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$$

in the sense of equivalent quasinorms.

Proof.

Step 1. We prove that

$$\mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)} \hookrightarrow B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}.$$ 

From Lemma 7, we only consider the case where $p \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $q \in \mathcal{P}(\mathbb{R}^n)$ with $\frac{1}{q} \in C_{\text{loc}}^{\log}(\mathbb{R}^n)$.

Let $\{\mathcal{F} \Phi, \mathcal{F} \varphi \}$ and $\{\mathcal{F} \psi_j \}_{j \in \mathbb{N}_0}$ be two resolutions of unity and let

$$f \in \mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)} \quad \text{with} \quad \|f\|_{\mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \leq 1.$$ 

We have

$$\psi_v * f = \min(1, 2^{2-v}) \int_{2^{-v-2}}^{2^{-v}} \psi_v * \varphi_t * f \frac{dt}{t} + \begin{cases} 0, & \text{if } v \geq 2, \\ \psi_v * \Phi * f, & \text{if } v = 0, 1. \end{cases}$$

Since the convolution with a radially decreasing $L^1$-function is bounded in $L^p(\cdot)$, we obtain

$$\|c \psi_v * \Phi * f \|_{q(\cdot)}^{q(\cdot)} \leq 1, \quad v = 0, 1,$$

for some suitable positive constant $c$. Applying Lemma 6, we get

$$\left\| c_1 2^{v \alpha(\cdot)} \psi_v * f \| q(\cdot) \|_{p(\cdot)}^{p(\cdot)} \right\| \leq \min(1, 2^{2-v}) \int_{2^{-v-2}}^{2^{-v}} \left\| t^{\alpha(\cdot)} \varphi_t * f \| q(\cdot) \|_{p(\cdot)}^{p(\cdot)} \frac{dt}{t} + 2^{-v}, \quad v \geq 2,$$

with properly chosen $c_1$. Taking the sum over $v \geq 2$, we find

$$\|f\|_{B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \leq 1.$$ 

Step 2. We now prove that

$$B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)} \hookrightarrow \mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}.$$
Let \( \{ F \Phi, \mathcal{F} \varphi \} \) and \( \{ F \psi_v \}_{v \in \mathbb{N}_0} \) be two resolutions of unity and let

\[
f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \quad \text{with} \quad \| f \|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq 1.
\]

We have

\[
\varphi_t \ast f = \sum_{v=0}^{\infty} \varphi_t \ast \psi_v \ast f
\]

\[
= \sum_{v=\lfloor \log_2 \left( \frac{t}{4} \right) \rfloor}^{\lfloor \log_2 \left( \frac{1}{2t} \right) \rfloor + 1} \varphi_t \ast \psi_v \ast f + \begin{cases} 0, & \text{if } 0 < t \leq \frac{1}{4}, \\ \psi_0 \ast \Phi \ast f, & \text{if } t > \frac{1}{4}, \end{cases}
\]

and

\[
\Phi \ast f = \sum_{v=0}^{2} \Phi \ast \psi_v \ast f.
\]

Note that if \( v < 0 \) then we set \( \psi_v \ast f = 0 \). Since the convolution with a radially decreasing \( L^1 \)-function is bounded in \( L^{p(\cdot)} \), we obtain

\[
\| c \psi_v \ast \Phi \ast f \|_{q^{(\cdot)}}^{\frac{1}{q^{(\cdot)}}} \leq 1, \quad v = 0, 1, 2.
\]

This yields

\[
\| c \phi \ast f \|_{q^{(\cdot)}}^{\frac{1}{q^{(\cdot)}}} \leq 1
\]

for some suitable positive constant \( c \). Let \( t \in (2^{-i}, 2^{-i+1}], \ i \in \mathbb{N} \). We have

\[
t^{-\alpha(\cdot)} |\varphi_t \ast f| \lesssim \sum_{v=\lfloor \log_2 \left( \frac{1}{2t} \right) \rfloor}^{\lfloor \log_2 \left( \frac{t}{4} \right) \rfloor + 1} t^{-\alpha(\cdot)} \eta_{t,m} \ast |\psi_v \ast f|
\]

\[
\lesssim \sum_{v=1}^{i-3} 2^{(i-v)\alpha} \eta_{v,m-c_{\log}(\alpha)} \ast 2^{v\alpha(\cdot)} |\psi_v \ast f|
\]

\[
\leq c \sum_{j=-3}^{i-1} \eta_{j+i,m-c_{\log}(\alpha)} \ast 2^{(j+i)\alpha(\cdot)} |\psi_{j+i} \ast f|,
\]

where

\[
m > n + c_{\log}(\alpha) + c_{\log} \left( \frac{1}{q} \right).
\]
We now observe that

\[
\int_0^1 \left\| c t^{-\alpha}(\varphi_t * f) \right\|_{\frac{q}{q-1}} \frac{dt}{t} = \sum_{i=0}^{\infty} \int_{2^{-i}}^{2^{1-i}} \left\| t^{-\alpha}(\varphi_t * f) \right\|_{\frac{q}{q-1}} \frac{dt}{t} \leq \sum_{i=0}^{\infty} \left( c \sum_{j=-3}^{-1} \eta_{j+i,m-c\log(\alpha)} \cdot 2^{j+i} \left| \phi_{j+i} * f \right| \right)^{\frac{q}{q-1}}
\]

for some suitable positive constant \( c \). The desired estimate follows from Lemma 5.

Theorem 3 is proved.

In order to formulate the main result of this section, we consider \( k_0, k \in \mathcal{S}(\mathbb{R}^n) \) and an integer \( S \geq -1 \) such that, for some \( \varepsilon > 0 \), we get

\[
|Fk_0(\xi)| > 0 \quad \text{for} \quad |\xi| < 2\varepsilon, \quad (12)
\]

\[
|Fk(\xi)| > 0 \quad \text{for} \quad \frac{\varepsilon}{2} < |\xi| < 2\varepsilon, \quad (13)
\]

and

\[
\int_{\mathbb{R}^n} x^\alpha k(x) dx = 0 \quad \text{for any} \quad |\alpha| \leq S. \quad (14)
\]

Here, (12) and (13) are the Tauberian conditions, while (14) states the moment conditions for \( k \). We now recall the notation

\[
k_t(x) := t^{-n}k(t^{-1}x) \quad \text{for} \quad t > 0.
\]

For any \( \alpha > 0 \), \( f \in \mathcal{S}'(\mathbb{R}^n) \), and \( x \in \mathbb{R}^n \), we denote

\[
k_t^{*,\alpha} t^{-\alpha}(f) := \sup_{y \in \mathbb{R}^n} \frac{t^{-\alpha(y)} \left| k_t * f(y) \right|}{(1 + t^{-1}|x - y|)^{\alpha}}, \quad j \in \mathbb{N}_0.
\]

We are now able to formulate the so-called local mean characterization of the spaces \( B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \), which is a more general form of Theorem 2.

**Theorem 4.** Let \( \alpha, \phi, \frac{1}{q} \in C^{\log} loc(\mathbb{R}^n), p \in P^\log(\mathbb{R}^n), p > \frac{n}{\alpha'}, and \alpha' < S + 1. Then

\[
\left\| f \right\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left\| k_0^{*,\alpha} f \right\|_{p(\cdot)} + \left\| (k_t^{*,\alpha} t^{-\alpha}(f))_{0 < t \leq 1} \right\|_{p(\cdot)(L^p(\cdot))}
\]

is an equivalent quasinorm on \( B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \).
Proof. The idea of the proof is taken from [17]. The proof is split into three steps.

Step 1. Let $\varepsilon > 0$. We take any pair of functions $\varphi_0$ and $\varphi \in S(\mathbb{R}^n)$ such that

$$|\mathcal{F}\varphi_0(\xi)| > 0 \quad \text{for} \quad |\xi| < 2\varepsilon,$$

$$|\mathcal{F}\varphi(\xi)| > 0 \quad \text{for} \quad \frac{\varepsilon}{2} < |\xi| < 2\varepsilon.$$

We prove that there exists a constant $c > 0$ such that, for any $f \in B^\alpha_{p,q}(\cdot, \cdot)$,

$$\left\| f \right\|_{B^\alpha_{p,q}(\cdot, \cdot)} \leq c \left\| \varphi_0^* a f \right\|_{p(\cdot)} + \left\| (\varphi_j^* a 2^{j\alpha} f) \right\|_{j \geq 1} \left\| p(\cdot) \right\|_{L^p(\cdot)}; \quad (15)$$

Let $\Lambda, \lambda \in S(\mathbb{R}^n)$ be such that

$$\text{supp} \mathcal{F}\Lambda \subset \{ \xi \in \mathbb{R}^n : |\xi| < 2\varepsilon \},$$

$$\text{supp} \mathcal{F}\lambda \subset \{ \xi \in \mathbb{R}^n : \varepsilon/2 < |\xi| < 2\varepsilon \}$$

and

$$\mathcal{F}\Lambda(\xi)\mathcal{F}\varphi_0(\xi) + \sum_{j=1}^{\infty} \mathcal{F}\lambda(2^{-j}\xi)\mathcal{F}\varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n.$$ 

In particular, for any $f \in B^\alpha_{p,q}(\cdot, \cdot)$, the following identity is true:

$$f = \Lambda \ast \varphi_0 \ast f + \sum_{j=1}^{\infty} \lambda_j \ast \varphi_j \ast f,$$

where

$$\varphi_j := 2^{jn} \varphi(2^j \cdot) \quad \text{and} \quad \lambda_j := 2^{jn} \lambda(2^j \cdot), \quad j \in \mathbb{N}.$$ 

Hence, we can write

$$k_t \ast f = k_t \ast \Lambda \ast \varphi_0 \ast f + \sum_{j=1}^{\infty} k_t \ast \lambda_j \ast \varphi_j \ast f, \quad t \in (0, 1].$$

Let $2^{-i} < t \leq 2^{1-i}$, $i \in \mathbb{N}$. First, let $j < i$. For any $z \in \mathbb{R}^n$, we can write

$$k_t \ast \lambda_j(z) = 2^{jn} k_{2^j t} \ast \lambda(2^j z).$$

Thus, it follows from Lemma 8 that, for any $N > 0$, there exists a constant $c > 0$ independent of $t$ and $j$ and such that

$$|k_t \ast \lambda_j(z)| \leq c \left( 2^j t \right)^{S+1} \eta_j, N(z), \quad z \in \mathbb{R}^n.$$
Together with Lemma 1, this yields that

\[ t^{-\alpha}(y) |k_t * \lambda_j * \varphi_j * f(y)|, \]

can be estimated from above by

\[ c2^{(j-i)(S+1-\alpha^+)} \varphi_j^{*,a}2^{ja(-)} f(y) \int_{\mathbb{R}^n} \eta_{j,n-c_{\log}(\alpha)-a}(y-z)dz \lesssim 2^{(j-i)(S+1-\alpha^+)} \varphi_j^{*,a}2^{ja(-)} f(y) \]

for any \( N > n + a + c_{\log}(\alpha) \), any \( y \in \mathbb{R}^n \), and any \( j < i \).

Further, let \( j \geq i \). Then, by using Lemma 8 once again, we conclude that, for any \( z \in \mathbb{R}^n \) and any \( L > 0 \),

\[ |k_t * \lambda_j(z)| = t^{-n} |k * \lambda \frac{1}{2^j t} \left( \frac{z}{t} \right) | \leq c \left( \frac{1}{2^j t} \right)^{M+1} \eta_{i,L}(z), \]

where the integer \( M \geq -1 \) is taken arbitrarily large, since \( D^\beta F \lambda(0) = 0 \) for all \( \beta \). Hence, it also follows from Lemma 1 that

\[ t^{-\alpha}(y) |k_t * \lambda_j * \varphi_j * f(y)| \]

\[ \leq t^{-\alpha}(y) \int_{\mathbb{R}^n} |k_t * \lambda_j(y-z)| |\varphi_j * f(z)| dz \]

\[ \lesssim 2^{(i-j)(M+1+\alpha^-)-j} \varphi_j^{*,a}2^{ja(-)} f(y) \int_{\mathbb{R}^n} \eta_{j,n-c_{\log}(\alpha)-a}(y-z)\eta_{i,L}(y-z)dz. \]

For any \( j \geq i \), we obtain

\[ (1 + 2^j |z|)^{c_{\log}(\alpha)+a} \lesssim 2^{(j-i)(c_{\log}(\alpha)+a)} \left( 1 + 2^i |z| \right)^{c_{\log}(\alpha)+a}. \]

Thus, by setting \( L > n + a + c_{\log}(\alpha) \), we get

\[ t^{-\alpha}(y) |k_t * \lambda_j * \varphi_j * f(y)| \lesssim 2^{(i-j)(M+1+\alpha^-)-c_{\log}(\alpha)-a} \varphi_j^{*,a}2^{ja(-)} f(y). \]

We now take \( M > c_{\log}(\alpha) - \alpha^- + 2a \) in order to estimate the last expression by

\[ c2^{(i-j)(a+1)} \varphi_j^{*,a}2^{ja(-)} f(y), \]

where \( c > 0 \) is independent of \( i, j, \) and \( f \). By using the fact that

\[ |k_t * \Lambda(z)| \leq ct^{S+1} \eta_{1,N}(z) \quad \text{for any} \quad z \in \mathbb{R}^n \quad \text{and any} \quad N > 0, \]
we obtain, with the help of similar arguments, that

$$
\sup_{y \in \mathbb{R}^n} \frac{t^{-\alpha(y)} |k_t \ast \lambda_y \ast \varphi_0 \ast f(y)|}{(1 + t^{-1} |x - y|)^{\alpha}} \leq C 2^{-i(S+1-\alpha^+)} \varphi_0^{*,a} f(x)
$$

for any $2^{-i} \leq t \leq 2^{1-i}$, $i \in \mathbb{N}$.

Further, we note that

$$
\varphi_j^{*,a} 2^j \alpha(x) f(y) \leq \varphi_j^{*,a} 2^j \alpha(x) f(x) (1 + 2^j |x - y|)^{\alpha}
$$

$$
\leq \varphi_j^{*,a} 2^j \alpha(x) f(x) \max(1, 2^{(j-i)a}) (1 + 2^i |x - y|)^{\alpha}
$$

for all $x, y \in \mathbb{R}^n$, all $2^{-i} \leq t \leq 2^{1-i}$, $i \in \mathbb{N}$, and any $j \in \mathbb{N}_0$. Hence,

$$
\sup_{y \in \mathbb{R}^n} \frac{t^{-\alpha(y)} |k_t \ast \lambda_y \ast \varphi_j \ast f(y)|}{(1 + t^{-1} |x - y|)^{\alpha}} \leq C \varphi_j^{*,a} 2^j \alpha(x) f(x) \begin{cases} 2^{(j-i)(S+1-\alpha^+)}, & \text{if } j < i, \\ 2^{i-j}, & \text{if } j \geq i. \end{cases}
$$

Therefore, for all $f \in B^{a(x)}_{p(x),q(x)}$, any $x \in \mathbb{R}^n$ and any $2^{-i} \leq t \leq 2^{1-i}$, $i \in \mathbb{N}_0$, we get

$$
k_t^{*,a} t^{-\alpha(a(x))} f(x) \lesssim 2^{-i(S+1-\alpha^+)} \varphi_0^{*,a} f(x)
$$

$$
+ C \sum_{j=1}^{\infty} \min \left(2^{(j-i)(S+1-\alpha^+)}, 2^{i-j}\right) \varphi_j^{*,a} 2^j \alpha(x) f(x)
$$

$$
= C \sum_{j=0}^{\infty} \min \left(2^{(j-i)(S+1-\alpha^+)}, 2^{i-j}\right) \varphi_j^{*,a} 2^j \alpha(x) f(x) = C \Psi_i(x).
$$

Assume that the right-hand side of (15) is smaller than or equal to one. Then we get

$$
\frac{1}{0} \left\| k_t^{*,a} t^{-\alpha(a(x))} f | q^{(x)} | \right\|_{p(x)} \frac{dt}{t} = \sum_{i=0}^{2^{1-i}} \int_{2^{-i}}^{2^{1-i}} \left\| k_t^{*,a} t^{-\alpha(a(x))} f | q^{(x)} | \right\|_{p(x)} \frac{dt}{t}
$$

$$
\leq \sum_{i=0}^{\infty} \left\| c \Psi_i | q^{(x)} | \right\|_{q(x)}
$$

for some positive constant $c$. The last term on the right-hand side is smaller than or equal to one if and only if

$$
\left\| (c_1 \Psi_i) \right\|_{\ell^{p(a(x))}(L^q(x))} \leq 1
$$

for some suitable positive constant $c_1$, which follows from Lemma 8 in [14] and the fact that $\alpha^+ < S + 1$. 

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The text continues here with more detailed analysis and proofs related to the continuous characterization of the Besov spaces of variable smoothness and integrability. The focus is on establishing bounds and properties for functions in these spaces, utilizing tools from functional analysis and harmonic analysis.
Moreover, for any \( z \in \mathbb{R}^n \), any \( N > 0 \), and any integer \( M \geq -1 \), we have

\[
|k_0 * \lambda_j(z)| \leq c2^{-j(M+1)}\eta_{j,N}(z) \quad \text{and} \quad |k_0 * \Lambda(z)| \leq c\eta_{1,N}(z).
\]

As earlier, for any \( x \in \mathbb{R}^n \), we get

\[
k_0 \ast \varphi_0 f(x) \leq C\varphi_0 \ast f(x) + C \sum_{j=1}^{\infty} 2^{-j} \varphi_j \ast 2^j \alpha(\cdot) f(x).
\]  \hspace{1cm} (16)

In (16), taking the \( L^p(\cdot) \)-quasinorm and using the embedding \( \ell^q(\cdot) (L^p(\cdot)) \hookrightarrow \ell^\infty(\ell^p(\cdot)) \), we get (15).

**Step 2.** Let \( \{\mathcal{F}\varphi_j\}_{j \in \mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n) \) be such that

\[
\text{supp} \mathcal{F}\varphi \subset \{ \xi \in \mathbb{R}^n : \varepsilon/2 \leq |\xi| \leq 2\varepsilon \}
\]

and

\[
\text{supp} \mathcal{F}\varphi_0 \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2\varepsilon \}, \quad \varepsilon > 0,
\]

with \( \varphi = 2^jn\varphi(2^j \cdot) \), \( j \in \mathbb{N} \). We now prove that

\[
\|\varphi_0 \ast f\|_{\ell^p(\cdot) (\ell^q(\cdot)) \hookrightarrow \ell^\infty(\ell^p(\cdot))} \leq \|f\|'_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} .
\]  \hspace{1cm} (17)

Let \( \Lambda, \lambda \in \mathcal{S}(\mathbb{R}^n) \) be such that

\[
\text{supp} \mathcal{F}\Lambda \subset \{ \xi \in \mathbb{R}^n : |\xi| < 2\varepsilon \},
\]

\[
\text{supp} \mathcal{F}\lambda \subset \{ \xi \in \mathbb{R}^n : \varepsilon/2 < |\xi| < 2\varepsilon \},
\]

\[
\mathcal{F}\Lambda(\xi)\mathcal{F}k_0(\xi) + \int_0^1 \mathcal{F}\lambda(\tau\xi)\mathcal{F}k(\tau\xi) \frac{d\tau}{\tau} = 1, \quad \xi \in \mathbb{R}^n.
\]

In particular, for any \( f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \), the following identity is true:

\[
f = \Lambda \ast k_0 \ast f + \int_0^1 \lambda_\tau \ast k_\tau \ast f \frac{d\tau}{\tau}.
\]

Hence, we can write

\[
\varphi_j \ast f = \int_0^1 \varphi_j \ast \lambda_\tau \ast k_\tau \ast f \frac{d\tau}{\tau} = \int_{2^{-j-2}}^{2^{-j+2}} \varphi_j \ast \lambda_\tau \ast k_\tau \ast f \frac{d\tau}{\tau}, \quad j \geq 2.
\]
By using the fact that
\[ \max(|k_\tau * \lambda_\tau(z)|, |\varphi_j * \lambda_\tau(z)|) \lesssim \eta_{j,N}(z), \quad z \in \mathbb{R}^n, \]
\[ 2^{-j-2} \leq \tau \leq 2^{-j+2}, \quad j \in \mathbb{N}, \]
and Lemma 1 with sufficiently large \( N > 0 \), we easily obtain
\[ 2^{j\alpha(y)}|\varphi_j * \lambda_\tau * f(y)| \lesssim \min(k^{*,a}_{\tau-\alpha} f(y), \varphi_j^{*,2j\alpha} f(y)) \]
for any \( y \in \mathbb{R}^n \) and any \( 2^{-j+2} \leq \tau \leq 2^{-j-2}, \quad j \in \mathbb{N} \). Therefore,
\[ 2^{j\alpha(y)}|\varphi_j * f(y)| \lesssim (\varphi_j^{*,2j\alpha} f(y))^{1-r} \int_{2^{-j-2}}^{2^{-j+2}} (k^{*,a}_{\tau-\alpha} f(y))^r \frac{d\tau}{\tau}. \quad 0 < r < 1. \]
This yields
\[ \varphi_j^{*,a} 2^{j\alpha} f(x) \lesssim (\varphi_j^{*,a} 2^{j\alpha} f(x))^{1-r} \int_{2^{-j-2}}^{2^{-j+2}} (k^{*,a}_{\tau-\alpha} f(x))^r \frac{d\tau}{\tau}. \]
This estimate gives
\[ (\varphi_j^{*,a} 2^{j\alpha} f(x))^r \lesssim \int_{2^{-j-2}}^{2^{-j+2}} (k^{*,a}_{\tau-\alpha} f(x))^r \frac{d\tau}{\tau} \]
and
\[ 2^{j\alpha(x)}|\varphi_j * f(x)|^r \lesssim \int_{2^{-j-2}}^{2^{-j+2}} (k^{*,a}_{\tau-\alpha} f(x))^r \frac{d\tau}{\tau}, \quad x \in \mathbb{R}^n, \quad (18) \]
but if \( \varphi_j^{*,a} 2^{j\alpha} f(x) < \infty \). By using the same reasoning as in Lemma 7, we get (18) for all \( 0 < r < 1, \quad a > 0, \)
and all \( f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \). Similarly, we obtain
\[ |\varphi_j * f(x)|^r \lesssim (k_0^{*,a} f(x))^r + \int_{1/s}^1 (k^{*,a}_{\tau-\alpha} f(x))^r \frac{d\tau}{\tau}, \quad j = 0, 1, \]
for any \( 0 < r < 1, \quad a > 0, \) and any \( f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \).

Let \( \theta > 0 \) be such that
\[ \max \left( 1, \frac{(1/p)^+}{(1/q)^-} \right) \lesssim \theta < \frac{q^-}{r}. \]
By virtue of Hölder’s and Minkowski’s inequalities, we find

\[
\left\| c2j\alpha(\varphi_j \ast f) \right\|_{q'(\tau)} \leq \left( \int_{2^{-j-2}}^{2^{-j+2}} \left\| k_{\tau}^{s,\alpha} \tau^{-\alpha(\cdot)} f \right\|_{q'(\tau)} \frac{d\tau}{\tau} \right)^{\theta} \\
\leq \int_{2^{-j-2}}^{2^{-j+2}} \left\| k_{\tau}^{s,\alpha} \tau^{-\alpha(\cdot)} f \right\|_{q'(\tau)} \frac{d\tau}{\tau}.
\]

Thus, we get

\[
\sum_{j=2}^{\infty} \left\| c2j\alpha(\varphi_j \ast f) \right\|_{q'(\tau)} \leq 1
\]

with an appropriate choice of \( c > 0 \) such that the left-hand side of (18) does not exceed one. Similarly, we have

\[
\left\| c\varphi_j \ast f \right\|_{q'(\tau)} \leq 1, \quad j = 0, 1.
\]

The desired estimate follows by the scaling argument.

**Step 3.** We now prove that, for all \( f \in B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \), the following estimates are true:

\[
\left\| f \right\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \lesssim \left\| f \right\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \lesssim \left\| f \right\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.
\]

(19)

Let \( \{ \mathcal{F}_{\varphi_j} \}_{j \in \mathbb{N}_0} \) be a resolution of unity. The first inequality follows from the following chain of estimates:

\[
\left\| f \right\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \lesssim \left\| \varphi_0^{s,\alpha} f \right\|_{p(\cdot)} + \left\| (\varphi_j^{s,\alpha} 2j^{\alpha(\cdot)})_{j \geq 1} \right\|_{p(\cdot)} \lesssim \left\| f \right\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}},
\]

where the first inequality is (15) and the second inequality is obvious (see [9]). Thus, the second inequality in (19) can be obtained from the following chain of estimates:

\[
\left\| f \right\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \lesssim \left\| \varphi_0 \ast f \right\|_{p(\cdot)} + \left\| (2^{j\alpha(\cdot)}(\varphi_j \ast f))_{j \geq 1} \right\|_{p(\cdot)L^{\infty}(\cdot)} \lesssim \left\| f \right\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}},
\]

where the first inequality is obvious and the second inequality is (17).

Theorem 4 is proved.

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