While Q-balls have been investigated intensively for many years, another type of nontopological solutions, Q-tubes, have not been understood very well. In this paper we make a comparative study of Q-balls and Q-tubes. First, we investigate their equilibrium solutions for four types of potentials. We find, for example, that in some models the charge-energy relation is similar between Q-balls and Q-tubes while in other models the relation is quite different between them. To understand what determines the charge-energy relation, which is a key of stability of the equilibrium solutions, we establish an analytical method to obtain the two limit values of the energy and the charge. Our prescription indicates how the existing domain of solutions and their stability depends on their shape as well as potentials, which would also be useful for a future study of Q-objects in higher-dimensional spacetime.
A. Q-balls

For a Q-ball, we assume spherical symmetry and homogeneous phase rotation,
\[ \phi = \phi(r)(\cos \omega t, \sin \omega t). \]  
(3)
One has a field equation,
\[ \frac{d^2 \phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} + \omega^2 \phi = \frac{dV}{d\phi}. \]  
(4)
This is equivalent to the field equation for a single static scalar field with an effective potential
\[ V_\omega = V - \frac{1}{2} \omega^2 \phi^2. \]  
(5)
Equilibrium solutions \( \phi(r) \) with a boundary condition
\[ \frac{d\phi}{dr}(r = 0) = 0, \quad \phi(r \to \infty) = 0, \]  
(6)
exist if \( \min(V_\omega) < V_\omega(0) \) and \( d^2V_\omega/d\phi^2(0) > 0 \). This condition is rewritten as
\[ \min \left[ \frac{2V}{\phi^2} \right] < \omega^2 < m^2 \equiv \frac{d^2V}{d\phi^2}(0), \]  
(7)
where we have put \( V(0) = 0 \) without loss of generality.

For a Q-ball solution, we can define the energy and the charge, respectively, as
\[ E = 4\pi \int_0^\infty r^2 dr \left\{ \frac{1}{2} \frac{d\phi}{dr} \phi^2 + \frac{1}{2} \left( \frac{d\phi}{dr} \right)^2 + V \right\}, \]
\[ Q = 4\pi \omega \int_0^\infty r^2 \phi^2 dr. \]  
(8)
The Q-E relation is a key to understand stability of equilibrium solutions in terms of catastrophe theory \[8\].

B. Q-tubes

For a Q-tube, we suppose a string-like configuration,
\[ \phi = \phi(R)(\cos(n \varphi + \omega t), \sin(n \varphi + \omega t)), \]  
(9)
where \( n \) is nonnegative integer and \((R, \varphi, z)\) is the cylindrical coordinate system. The field equation becomes
\[ \frac{d^2 \phi}{dR^2} + \frac{1}{R} \frac{d\phi}{dR} - \frac{n^2 \phi}{R^2} + \omega^2 \phi = \frac{dV}{d\phi}. \]  
(10)
In the case of \( n = 0 \), the field equation is the same as \[11\] except for a numerical coefficient. Therefore, Q-ball-like solutions of \( \phi(R) \) exist if the condition \[17\] is satisfied.

In the case of \( n \geq 1 \), there is no regular solution which satisfies \( \phi(0) \neq 0 \). However, if we adopt a different boundary condition,
\[ \phi(R = 0) = \phi(R \to \infty) = 0, \]  
(11)
there is a new type of regular solutions. We introduce an auxiliary variable \( \psi \) which is defined by \( \phi(R) = R^n \psi(R) \), Then, Eq. \[10\] becomes
\[ \frac{d^2 \psi}{dR^2} + \frac{2n + 1}{R} \frac{d\psi}{dR} + \omega^2 \psi = R^{-n} \frac{dV}{d\phi} \bigg|_{\phi=R^n \psi}. \]  
(12)

FIG. 1: Interpretation of (a) Q-balls and \( n = 0 \) solutions in Q-tubes and (b) \( n \geq 1 \) solutions in Q-tubes by analogy with a particle motion in Newtonian mechanics.
If we choose $\psi(0)$ appropriately, we obtain a solution $\psi(R)$ which is expressed in the Maclaurin series without odd powers in the neighborhood of $R = 0$. In terms of the original variable $\phi(R)$, the $n$th differential coefficient $\phi^{(n)}(0) = \psi(0)$ should be determined by the shooting method, while any lower derivative vanishes at $R = 0$.

In the same way as for Q-balls, existence of Q-tube solutions can be interpreted as follows. If one regards the radius $R$ as ‘time’ and the scalar amplitude $\phi(R)$ as ‘the position of a particle’, one can understand $n = 0$ solutions in words of Newtonian mechanics, as shown in Fig. 1(a).

Equation (10) describes a one-dimensional motion of a particle under the conserved force due to the potential $-V_\omega(\phi)$ and the ‘time’-dependent friction $-(1/R)d\phi/dR$. If one chooses the ‘initial position’ $\phi(0)$ appropriately, the static particle begins to roll down the potential slope, climbs up and approaches the origin over infinite time.

Similarly, we can also understand $n \geq 1$ solutions as shown in Fig. 1(b). In this case, there are two non-conserved forces, the friction $-(1/R)d\phi/dR$ and the repulsive force $n^2\phi^2/R^2$. If $n = 1$, by choosing the ‘initial velocity’ $d\phi/dR(0)$ appropriately, the particle goes down and up the slope, and at some point $\phi = \phi_{\text{max}}$ it turns back and approaches the origin over infinite time. If $n \geq 2$, $d\phi/dR(0)$ vanishes; instead, the $n$th derivative $\phi^{(n)}(0)$ gently pushes the particle at $\phi = 0$. Therefore, with the appropriate choice of $\phi^{(n)}(0)$, the particle moves along a similar trajectory to that of $n = 1$. This argument also indicates that the existence condition of $n \geq 1$ solutions are the same as that of $n = 0$ solutions. Solutions with the same behavior as the $n = 1$ solutions were obtained by Kim et al., who studied the SO(3)-symmetric scalar field without Q-charge.

Because our Q-ball solutions are infinitely long, the energy and the charge (8) diverge. We therefore define the energy and the charge per unit length, respectively, as

$$ e = 2\pi \int_0^\infty R dR \left\{ \frac{1}{2} \omega^2 \phi^2 + \frac{1}{2} \left( \frac{d\phi}{dR} \right)^2 + \frac{n^2 \phi^2}{2R^2} + V \right\}, $$

$$ q = 2\pi \omega \int_0^\infty R d\phi dR. \quad (13) $$

### C. Two types and two limits

The existence condition (8) indicates that both Q-balls and Q-tubes are classified into two types of solutions, according to the sign of $\min[V(\phi)]$.

| Type | $\min[V] = 0$ | $\min[2V/\phi^2]$ (thin) | $m^2$ (thick) |
|------|---------------|-------------------------|--------------|
| Type I: $\min[V] = 0$ | $\min[2V/\phi^2]$ (thin) | $m^2$ (thick) |
| Type II: $\min[V] < 0$ | $0$ | $m^2$ (thick) |

**TABLE I:** Two types of Q-balls/Q-tubes solutions and two limits of $\omega^2$.

**Type I: $\min[V(\phi)] = V(0) = 0$.** In this case $\min[2V/\phi^2]$ is also positive and the lower limit of $\omega^2$ is $m^2$. The two limits $\omega^2 \rightarrow \min[2V/\phi^2]$ and $\omega^2 \rightarrow m^2$ correspond to the thin-wall limit and the thick-wall limit, respectively.

**Type II: $\min[V(\phi)] < 0$.** In this case $\min[2V/\phi^2]$ is negative. Because $\omega^2 > 0$, there is no thin-wall limit, $\omega^2 \rightarrow \min[2V/\phi^2]$. The thick-wall limit, $\omega^2 \rightarrow m^2$, still exists.

The two limits of $\omega^2$ for the two types of solutions are summarized in Table I.

**FIG. 2:** The field configurations of the scalar field for Q-tubes in the $V_3$ model with $\tilde{m}^2 = 0.6$ (Type I): (a) $\epsilon^2 = 0.01$ (thick-wall) and $\epsilon^2 = 0.48$ (thick-wall).

### III. Solutions in various potentials

Here we investigate equilibrium solutions of Q-balls and Q-tubes for four types of potentials.
A. $V_3$ model

First, we summarize the previous results in the $V_3$ model. We rescale the quantities as

\[ \tilde{\phi} \equiv \frac{\lambda}{\mu} \phi, \quad \tilde{m} \equiv \sqrt{\lambda} m, \quad \tilde{\omega} \equiv \frac{\sqrt{\lambda}}{\mu} \omega, \]

\[ \tilde{r} \equiv \frac{\mu}{\sqrt{\lambda}} r, \quad \tilde{E} \equiv \frac{\lambda^{3/2}}{\mu} E, \quad \tilde{Q} \equiv \lambda Q, \]

\[ \tilde{R} \equiv \frac{\mu}{\sqrt{\lambda}} R, \quad \tilde{c} \equiv \frac{\lambda^2}{\mu^2} c, \quad \tilde{\eta} \equiv \frac{\lambda^{3/2}}{\mu} q, \]

and define a parameter,

\[ \epsilon^2 \equiv \tilde{m}^2 - \tilde{\omega}^2. \]

Then, the existing condition for the two types becomes

\[ 0 < \epsilon^2 < \frac{1}{2} \quad \text{for} \quad \tilde{m}^2 > \frac{1}{2} \quad \text{(Type I)} \]

\[ 0 < \epsilon^2 < \tilde{m}^2 \quad \text{for} \quad \tilde{m}^2 < \frac{1}{2} \quad \text{(Type II).} \]

The limits $\epsilon^2 \to 1/2$ and $\epsilon^2 \to 0$ correspond to the thin-wall limit and the thick-wall limit, respectively. As we discussed in the last section, however, in Type II solutions there is no thin-wall limit and the upper limit of $\epsilon^2$ is $\tilde{m}^2$ instead of $1/2$.

Figure 2 shows examples of the field configurations of Q-tubes. We fix $\tilde{m}^2 = 0.6$ (Type I), and choose $\epsilon^2 = 0.01$ (thick-wall) in (a) and $\epsilon^2 = 0.48$ (thin-wall) in (b). In each diagram we show the three solutions $n = 0, 1$ and 2, which indicates that the maximum amplitude of the scalar field $\phi_{\text{max}}$ for $n = 0$ is largest among them. We can understand it by analogy with the Newtonian mechanics in Fig. 1. For $n \geq 1$, the particle must make a
FIG. 5: (a) $\tilde{Q}-\tilde{E}$ and (b) $\tilde{Q}-\epsilon^2$ relations for Type I Q-balls in the $V_3$ model: $\tilde{m}^2 = 0.3$.

FIG. 6: (a) $\tilde{q}-\tilde{e}$ and (b) $\tilde{q}-\epsilon^2$ relations for Type I Q-tubes in the $V_3$ model: $\tilde{m}^2 = 0.3$.

round trip while it goes an one-way for $n = 0$. Nevertheless, $\phi_{\text{max}}$ in all cases are qualitatively unchanged which means that the conservation law of energy approximately holds in words of the Newtonian mechanics. Of course, the behavior of a Q-ball is similar to that of a Q-tube for $n = 0$. These properties are independent of potentials, which is important in understanding Q-balls and Q-tubes in an unified way as we shall see in Sec. IV.

We show the charge-energy-$\epsilon$ relations for Type I ($\tilde{m}^2 = 0.6$): Q-balls in Fig. 5 and Q-tubes in Fig. 6. As for Q-tubes, we show results for $n = 0, 1$ and 2. Similarity between Q-balls and Q-tubes is quite remarkable. In the thin-wall limit ($\epsilon^2 \to 1/2$), we confirm that $\tilde{Q}, \tilde{E}, \tilde{q}$ and $\tilde{e}$ diverge. In the thick-wall limit ($\epsilon^2 \to 0$), on the other hand, these quantities approach zero.

We also show the same relations for Type II ($\tilde{m}^2 = 0.3$): Q-balls in Fig. 5 and Q-tubes in Fig. 6. The crucial difference from Type I is that $\tilde{Q}$ and $\tilde{q}$ approach zero in the upper limit $\epsilon^2 \to \tilde{m}^2$ while $\tilde{E}$ and $\tilde{e}$ have nonzero finite values corresponding to the points $C$. As a result, $\tilde{Q}$, $\tilde{E}$, $\tilde{q}$ and $\tilde{e}$ have maximum values for intermediate value of $\epsilon^2$ corresponding to the points $B$ where cusp structures appear in Figs. 5 and 6 (a). The stability of Q-balls and Q-tubes can be understood using catastrophe theory [17]. Solutions from the point $A$ to $B$ is stable while $B$ to $C$ unstable.

The extreme values of the energy and the charge of Q-balls and Q-tubes in the $V_3$ model are summarized in Table II.
which we call the $V_4$ model. We rescale the quantities as

\[
\tilde{\phi} \equiv \frac{\phi}{\sqrt{\lambda M}}, \quad \tilde{m} \equiv \frac{m}{\lambda M}, \quad \tilde{\omega} \equiv \frac{\omega}{\lambda M},
\]

\[
\tilde{r} \equiv \lambda Mr, \quad \tilde{E} \equiv \frac{E}{M}, \quad \tilde{Q} \equiv \lambda Q,
\]

\[
\tilde{R} \equiv \lambda MR, \quad \tilde{e} \equiv \frac{e}{\lambda M^2}, \quad \tilde{q} \equiv \frac{q}{M},
\]

and again define a parameter $\epsilon$ by (15).

Then the existing condition is identical to (16) in the $V_3$ case. We show the charge-energy-$\epsilon$ relations in Figs. 7-10. Type I Q-balls in Fig. 7, Type I Q-tubes in Fig. 8, Type II Q-balls in Fig. 9, and Type II Q-tubes in Fig. 10. Contrary to the case of the $V_3$ model, qualitative difference between Q-tubes and Q-balls appears. The extreme values of the energy and the charge of Q-balls and Q-tubes in the $V_4$ model are summarized in Table III.

**B. the $V_4$ model**

Second, we consider another simple potential,

\[
V_4(\phi) := \frac{m^2}{2} \phi^2 - \lambda \phi^4 + \frac{\phi^6}{M^2}
\]

with $m^2$, $\lambda$, $M^2 > 0$,

\[(17)\]

| Type I: $\tilde{m}^2 > 1/2$ | $\tilde{E}, \tilde{Q}, \tilde{e}, \tilde{q} \to \infty$ | $\epsilon^2 \to 0$ (thick) |
|-----------------------------|---------------------------------|-------------------|
| Type II: $\tilde{m}^2 < 1/2$ | $\tilde{E}, \tilde{Q}, \tilde{e}, \tilde{q} \to 0$ | $\epsilon^2 \to \min[1/2, \tilde{m}^2]$ |

**TABLE II: Extreme values of the energy and the charge of Q-balls and Q-tubes in the $V_3$ model.**

![FIG. 7](image_url) (a) $\tilde{Q}$-$\tilde{E}$ and (b) $\tilde{Q}$-$\epsilon^2$ relations for Type I Q-balls in the $V_4$ model: $\tilde{m}^2 = 0.6$.

![FIG. 8](image_url) (a) $\tilde{q}$-$\tilde{e}$ and (b) $\tilde{q}$-$\epsilon^2$ relations for Type I Q-tubes in the $V_4$ model: $\tilde{m}^2 = 0.6$. 

![FIG. 9](image_url) (a) $\tilde{Q}$-$\tilde{E}$ and (b) $\tilde{Q}$-$\epsilon^2$ relations for Type II Q-balls in the $V_4$ model: $\tilde{m}^2 = 0.6$.

![FIG. 10](image_url) (a) $\tilde{q}$-$\tilde{e}$ and (b) $\tilde{q}$-$\epsilon^2$ relations for Type II Q-tubes in the $V_4$ model: $\tilde{m}^2 = 0.6$. 

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FIG. 9: (a) $\tilde{Q}$-$\tilde{E}$ and (b) $\tilde{Q}$-$\epsilon^2$ relations for Type II Q-balls in the $V_4$ model: $\tilde{m}^2 = 0.4$.

The structures of the solution series of Type II Q-balls and Q-tubes are not simple. In the case of Q-balls, there are two cusps in the $Q$-$E$ diagram, $B$ and $C$. Only the solutions between these two points represent stable solutions. In the case of Q-tubes, a cusp appears for $n = 0$, while no cusp appears for $n \geq 1$.

| Type     | $\epsilon^2 \to \min[1/2, \tilde{m}^2]$ | $\epsilon^2 \to 0$ (thick) |
|----------|---------------------------------------|-----------------------------|
| Type I: $\tilde{m}^2 > 1/2$ | $E, Q, \tilde{e}, \tilde{q} \to \infty$ | $E, Q \to \infty$ |
|           | $\tilde{e}, \tilde{q} \to \text{nonzero finite}$ | $\tilde{e}, \tilde{q} \to 0$ |
| Type II: $\tilde{m}^2 < 1/2$ | $E, \tilde{e} \to \text{nonzero finite}$ | $E, Q \to \infty$ |
|           | $\tilde{Q}, \tilde{q} \to 0$ | $\tilde{e}, \tilde{q} \to \text{nonzero finite}$ |

TABLE III: Extreme values of the energy and the charge of Q-balls and Q-tubes in the $V_4$ model.

C. AD gravity-mediation type

From the theoretical point of view, it is important to investigate Q-tubes as well as Q-balls in the AD mechanism. There are two types of potentials: gravity-mediation type and gauge-mediation type. Here we consider the former type,

$$V_{\text{grav}}(\phi) := \frac{m_{\text{grav}}^2}{2} \phi^2 \left[ 1 + K \ln \left( \frac{\phi}{M} \right)^2 \right]$$

with $m_{\text{grav}}^2, M > 0$.

We rescale the quantities as

$$\tilde{\phi} \equiv \frac{\phi}{M}, \quad \tilde{\omega} \equiv \frac{\omega}{m_{\text{grav}}},$$

$$\tilde{r} \equiv m_{\text{grav}} r, \quad \tilde{E} \equiv \frac{m_{\text{grav}} E}{M^2}, \quad \tilde{Q} \equiv \frac{m_{\text{grav}}^2 Q}{M^2},$$

FIG. 10: (a) $\tilde{q}$-$\tilde{e}$ and (b) $\tilde{q}$-$\epsilon^2$ relations for Type II Q-tubes in the $V_4$ model: $\tilde{m}^2 = 0.4$. 

From the theoretical point of view, it is important to investigate Q-tubes as well as Q-balls in the AD mechanism. There are two types of potentials: gravity-mediation type and gauge-mediation type. Here we consider the former type,

$$V_{\text{grav}}(\phi) := \frac{m_{\text{grav}}^2}{2} \phi^2 \left[ 1 + K \ln \left( \frac{\phi}{M} \right)^2 \right]$$

with $m_{\text{grav}}^2, M > 0$.

We rescale the quantities as

$$\tilde{\phi} \equiv \frac{\phi}{M}, \quad \tilde{\omega} \equiv \frac{\omega}{m_{\text{grav}}},$$

$$\tilde{r} \equiv m_{\text{grav}} r, \quad \tilde{E} \equiv \frac{m_{\text{grav}} E}{M^2}, \quad \tilde{Q} \equiv \frac{m_{\text{grav}}^2 Q}{M^2},$$
FIG. 11: (a) $\tilde{Q}-\tilde{E}$ and (b) $\tilde{Q}-\varepsilon^2$ relations for $V_{\text{grav}}$: $K = -0.1$.

$$
\tilde{R} \equiv m_{\text{grav}} R, \quad \tilde{\varepsilon} \equiv \frac{\varepsilon}{M^2}, \quad \tilde{q} \equiv \frac{m_{\text{grav}} q}{M^2},
$$

(20)

and define a parameter $\varepsilon$ as

$$
\varepsilon^2 = 1 - \tilde{\omega}^2.
$$

(21)

The existing condition (7) becomes

$$
K < 0, \quad \varepsilon^2 < 1.
$$

(22)

Thus, $\varepsilon^2$ is not bounded below, which is in contrast to the $V_3$ and $V_4$ models. Only Type II solutions exist in this model unless we introduce additional terms in the potential. We show the charge-energy-$\varepsilon$ relations: Q-balls in Fig. 11 and Q-tubes in Fig. 12. The extreme values of the energy and the charge of Q-balls and Q-tubes in the gravity-mediation type are summarized in Table IV. There is no qualitative difference in the charge-energy relation between Q-balls and Q-tubes. These properties are common to Type II solutions in the $V_3$ model.

FIG. 12: (a) $\tilde{q}-\tilde{e}$ and (b) $\tilde{q}-\varepsilon^2$ relations for $V_{\text{grav}}$: $K = -0.1$.

| Type II | $\varepsilon^2 \to +\infty$ (thick) | $\varepsilon^2 \to \infty$ (thick) |
|---------|-----------------------------------|-----------------------------------|
| $E, \tilde{\varepsilon} \to$ nonzero finite | $\tilde{E}, \tilde{\varepsilon}$, $\tilde{q} \to 0$ |
| $\tilde{Q}, \tilde{q} \to 0$ |

TABLE IV: Extreme values of the energy and the charge of Q-balls and Q-tubes in the AD gravity-mediation type.

D. AD gauge-mediation type

Finally, we consider the gauge-mediation type in the AD mechanism,

$$
V_{\text{gauge}}(\phi) := m_{\text{gauge}}^4 \ln \left(1 + \frac{\phi^2}{m_{\text{gauge}}^2}\right) \quad \text{with} \quad m_{\text{gauge}}^2 > 0.
$$

(23)

We rescale the quantities as

$$
\phi \equiv \frac{\phi}{m_{\text{gauge}}}, \quad \tilde{\omega} \equiv \frac{\omega}{m_{\text{gauge}}},
$$
FIG. 13: (a) $\tilde{Q}$-$E$ and (b) $\tilde{Q}$-$\epsilon^2$ relations for $V_{\text{gauge}}$.  

$\tilde{r} \equiv m_{\text{gauge}} r$,  
$\tilde{E} \equiv \frac{E}{m_{\text{gauge}}}$,  
$\tilde{Q} \equiv Q$,  
$\tilde{R} \equiv m_{\text{gauge}} R$,  
$\tilde{c} \equiv \frac{c}{m_{\text{gauge}}^2}$,  
$\tilde{q} \equiv \frac{q}{m_{\text{gauge}}}$,  
(24) 

and define a parameter $\epsilon$ as 

$\epsilon^2 = 2 - \tilde{\omega}^2$.  
(25) 

Then the existing condition becomes 

$0 < \epsilon^2 < 2$.  
(26) 

Only Type I solutions exist in this model. We show the charge-energy-$\epsilon$ relation: Q-balls in Fig. 13 and Q-tubes in Fig. 14. The extreme values of the energy and the charge of Q-balls and Q-tubes in the gravity-mediation type are summarized in Table V. These properties are common to the Type I solutions in the $V_4$ model. 

![Diagram](image1.png)  

![Diagram](image2.png)  

FIG. 14: (a) $\tilde{q}$-$\tilde{c}$ and (b) $\tilde{q}$-$\epsilon^2$ relations for $V_{\text{gauge}}$. 

| Type | $\epsilon^2 \rightarrow 2$ (thin) | $\epsilon^2 \rightarrow 0$ (thick) | $\tilde{E}, \tilde{Q} \rightarrow \infty$ | $\tilde{E}, \tilde{Q} \rightarrow \infty$ | $\tilde{c}, \tilde{q} \rightarrow \text{nonzero finite}$ |
|---|---|---|---|---|---|
| I | $\epsilon^2 \rightarrow 2$ (thin) | $\epsilon^2 \rightarrow 0$ (thick) | $\tilde{E}, \tilde{Q} \rightarrow \infty$ | $\tilde{E}, \tilde{Q} \rightarrow \infty$ | $\tilde{c}, \tilde{q} \rightarrow \text{nonzero finite}$ |

TABLE V: Extreme values of the energy and the charge of Q-balls and Q-tubes in the AD gauge-mediation type. 

IV. UNIFIED PICTURE OF Q-BALLS AND Q-TUBES 

Our numerical results in the last section indicate that the charge-energy relation of equilibrium solutions depends greatly on functional forms of the potential $V(\phi)$. In this section we discuss what determines the extreme values of the energy and the charge by analytical methods. As we explained in Sec. II, we can understand Q-balls and Q-tubes in words of a particle motion in Newtonian mechanics. In Fig. 1 if we ignore ‘non conserved force’ the maximum of $\phi$, $\phi_{\text{max}}$, is determined by
the nontrivial solution of $V_w = 0$. Using this $\tilde{\phi}_{\text{max}}$, we can evaluate the order of magnitude of the energy and the charge, (8) and (13), as

$$\tilde{E} \sim \tilde{r}_{\text{max}}^3 \left\{ \frac{1}{2} \tilde{\omega}_{\text{max}}^2 \tilde{\phi}_{\text{max}}^2 + \frac{1}{2} \left( \frac{d\tilde{\phi}}{d\tilde{R}} \right)^2 + \tilde{V} \right\},$$

$$\tilde{Q} \sim \tilde{\omega} \tilde{R}_{\text{max}}^2 \tilde{\phi}_{\text{max}}^2,$$

$$\tilde{\epsilon} \sim \tilde{R}_{\text{max}}^2 \frac{1}{2} \tilde{\omega}_{\text{max}}^2 \tilde{\phi}_{\text{max}}^2 + \frac{1}{2} \left( \frac{d\tilde{\phi}}{d\tilde{R}} \right)^2 + \frac{n^2 \tilde{\phi}_{\text{max}}^2}{2 \tilde{R}_{\text{max}}^2} + \tilde{V},$$

$$\tilde{\epsilon} \sim \tilde{\omega} \tilde{R}_{\text{max}}^2 \tilde{\phi}_{\text{max}}^2,$$

(27)

where the subscript “max” denote the values at which $\phi = \phi_{\text{max}}$. As for $\tilde{R}_{\text{max}}$ for $n = 0$ or $\tilde{r}_{\text{max}}$, it is reasonable to take $\tilde{R}$ or $\tilde{r}$ where $\phi$ becomes about $0.5\tilde{\phi}_{\text{max}}$.

What we want to discuss is whether $\tilde{\phi}_{\text{max}}$ and $\tilde{\epsilon}$ approach zero, infinity or nonzero finite values as $\epsilon^2$ approaches the upper or lower limit. The approximate expression (26) is appropriate for this purpose.

First, we discuss the upper limit of $\epsilon^2$, or equivalently, the lower limit of $\omega^2$. In Type I solutions, where $\min[V] = V(0) = 0$, in the limit of $\omega \to \min[2V/\phi^n]$, the minimum of $V_w$ approaches zero. In this case, in the Newtonian-mechanics picture of Fig. 1, a particle rolls down from the top of the hill over infinite time, i.e., $\tilde{R}_{\text{max}}$ diverges. This limit corresponds to the thin-wall limit. From the expression (27), we see that $\tilde{Q}$, $\tilde{E}$, $\tilde{q}$ and $\tilde{\epsilon}$ diverge.

On the other hand, in the Type II solutions, where $\min[V_w] < 0$, because $V_w < V$, there is no limit of $\min[V_w] \to 0$. Therefore, $\tilde{Q}$, $\tilde{E}$, $\tilde{q}$ and $\tilde{\epsilon}$ must have their upper limits.

Next, we investigate the lower limit of $\epsilon^2$, or equivalently, the upper limit of $\omega^2$. This limit corresponds to the thick-wall limit. Except for the $V_{\text{grav}}$ model, $\epsilon$ satisfies

$$\epsilon^2 = \frac{1}{m^2} \frac{d^2V_w}{d\phi^2}(0),$$

(28)

which means that $\epsilon$ is the mass scale of $V_w$ normalized by $m$. Therefore, the wall thickness normalized by $m$ is of order of $1/\epsilon$. Because the radius and the wall thickness are of the same order in the thick-wall limit, except for the $V_{\text{grav}}$ model, we obtain

$$\tilde{r}_{\text{max}}, \quad \tilde{R}_{\text{max}} \sim \frac{1}{\epsilon}.$$  

(29)

In the following, from the approximate expression (27) and (29) we evaluate the limits of the charge and the energy as $\epsilon$ approaches the lower limit.

(A) $V_3$ case

The solution of $V_w = 0$ is

$$\tilde{\phi}_{\text{max}} = \frac{1 - \sqrt{1 - 2\epsilon^2}}{2}.$$  

(30)

In the lower limit $\epsilon^2 \to 0$, we have $\tilde{\phi}_{\text{max}} \sim \epsilon^2/2$. Therefore, from (27)-(29), we find

$$\tilde{Q}, \quad \tilde{E}, \quad \tilde{q}, \quad \tilde{\epsilon} \to 0,$$

(31)

which agree with the numerical results in Table II.

(B) $V_4$ case

From $V_w = 0$, we obtain

$$\tilde{\phi}_{\text{max}}^2 = \frac{1 - \sqrt{1 - 2\epsilon^2}}{2}.$$  

(32)

In the lower limit $\epsilon^2 \to 0$, we have $\tilde{\phi}_{\text{max}} \sim \epsilon$. Substituting this and (29) into (27), we have

$$\tilde{E} \sim \frac{1}{\epsilon^2} \tilde{\omega} \epsilon \epsilon \rightarrow \infty, \quad \tilde{Q} \sim \frac{1}{\epsilon^2} \tilde{\omega} \epsilon \epsilon \rightarrow \infty,$$

$$\tilde{\epsilon} \sim \frac{1}{\epsilon^2} \tilde{\omega} \epsilon \epsilon \rightarrow \text{const.}, \quad \tilde{q} \sim \frac{1}{\epsilon^2} \tilde{\omega} \epsilon \epsilon \rightarrow \text{const.},$$

(33)

which agree with the numerical results in Table III. This explains why the results between Q-tubes and Q-balls are different in this model while no qualitative difference appears in the $V_3$ model.

(C) $V_{\text{grav}}$ case

The solution of $V_w = 0$ is

$$\tilde{\phi}_{\text{max}} = e^{-\pi}.$$  

(34)

We note that dependence on $K$ is extremely large. $\tilde{\phi}_{\text{max}}$ approaches zero in the lower limit $\epsilon^2 \to -\infty$. Since $\tilde{R}_{\text{max}}$ does not diverge,

$$\tilde{Q}, \quad \tilde{E}, \quad \tilde{q}, \quad \tilde{\epsilon} \to 0,$$

(35)

which agree with the numerical results in Table IV.

In a realistic situation, we anticipate that $V_{\text{grav}}$ has also the nonrenormalization term $V_{\text{NR}} = \beta \tilde{\phi}^n$ where $\beta > 0$ and $n > 2$. This does not change the qualitative behavior in the lower limit. However, in the upper limit, $V_w = 0$ has degenerate solutions as in Type I models. Therefore, we anticipate that the charge-energy relation for $V_{\text{grav}}$ with $V_{\text{NR}}$ is similar to that for Type I solutions in the $V_3$ model.

(D) $V_{\text{gauge}}$ case

We should solve

$$\ln(1 + \tilde{\phi}_{\text{max}}^2) = \frac{\tilde{\phi}_{\text{max}}^2}{2}.$$  

(36)

In the lower limit $\epsilon^2 \to 0$, if we use Maclaurin expansion and neglect higher order terms $O(\tilde{\phi}_{\text{max}}^5)$, we have

$$\tilde{\phi}_{\text{max}}^2 \epsilon^2 \sim \tilde{\phi}_{\text{max}}^4.$$  

(37)

Then, we obtain

$$\tilde{\phi}_{\text{max}} \sim \epsilon,$$

(38)

as in the $V_4$ model. Therefore, the limit values are identical to (33), which agree with the numerical results in Table V. We also understand why the results for $V_4$ with $\tilde{m}^2 > 1/2$ and for $V_{\text{gauge}}$ are qualitatively the same.
V. SUMMARY AND DISCUSSIONS

We have made a comparative study of Q-balls and Q-tubes. First, we have investigated their equilibrium solutions for four types of potentials. The charge-energy relation depends on potential models. We have also noted that in some models the charge-energy relation is similar between Q-balls and Q-tubes while in other models the relation is quite different between them. To understand what determines the charge-energy relation, which is a key of stability of the equilibrium solutions, we have established an analytical method to obtain the two limit values of the energy and the charge. Our results have indicated how the existent domain of solutions and their stability depends on their shape as well as potentials. This method would also be useful for other Q-objects or those in higher-dimensional spacetime. These are our next subjects.

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