Four-dimensional conformal field theory using quaternions

Sergio Giardino

Departamento de Física & Centro de Matemática e Aplicações, Universidade da Beira Interior
Rua Marquês D’Ávila e Bolama 6200-001 Covilhã, Portugal

We build a four-dimensional quaternion-parametrized conformal field theory (QCFT) using quaternion holomorphic functions as the generators of quaternionic conformal transformations. Taking the two-dimensional complex-parametrized conformal field theory (CCFT) as our model, we study the stress tensor, the conserved charge, the symmetry generators, the quantization conditions and several operator product expansions (OPE’s). Future applications are also addressed.

PACS numbers:

I. INTRODUCTION

A conformal field theory is a framework used in many applications in physics. These models may be built for any dimension, but the two-dimensional conformal field theory [1,2] is the most widely used, and it involves many features that explain this widespread use. The infinite-dimensional conformal algebra and its central extension are certainly important applicability factors. Another feature is the parametrization of two-dimensional CFT using complex numbers, and thus the employment of the full machinery of complex analysis in order to build a complex-parametrized conformal field theory (CCFT).

On the other hand, four-dimensional conformal field theories have attracted considerable attention for recent applications in high-spin models [3–5], but there also are developments in other areas, such as electro-dynamical applications [6] and mathematical consistency [7]. The four-dimensional CFT considered in this article begins with the question of whether it is possible to develop a quaternion-parametrized four-dimensional conformal field theory (QCFT). A quaternionic scalar field theory is already known [8], and so it would be seem a good idea to elaborate a four-dimensional model by preserving many of the properties of the two-dimensional theory and increasing the number of degrees of freedom. In fact, the quaternionic four-dimensional case has more constraints than the complex two-dimensional case, meaning that several results that may be obtained in the two-dimensional CFT based on symmetry arguments cannot be obtained in QCFT. Examples include correlation functions, creation and annihilation operators and the Hilbert space. However, the concept of generalizing two-dimensional CFT in terms of quaternion-parametrized four-dimensional CFT was not originally ours. [9] reviews several attempts to achieve this aim for self-dual Yang-Mills theories, and another example of this kind is provided by [10]. One serious restriction for constructing a quaternionic four-dimensional theory is the severely limiting quaternion analyticity, which only admits affine quaternion functions in the left-derivative class [11]. A less restrictive quaternionic analyticity has been used for building a four-dimensional \( \sigma \)–model with applications in gauge theories [12].

In fact, the use of quaternions for generalizing complex-based theories is not straightforward, and quaternionic theories may not necessarily recover complex results by canceling the pure quaternionic variables. There are solutions for the Dirac equation for the quaternionic step potential that have no complex limit [13,14], and the quaternionic Dirac square well also has no counterpart within the complex limit [15]. These questions enable us to suppose that using quaternions as a substitute for complex numbers may generate theories that are significantly different from the complex parametrized theories that have inspired them.

This article presents another attempt to test the feasibility of using quaternions to build a four-dimensional CFT. The novelty here is that the conformal transformations are quaternion holomorphic functions (QHF). We believe that this proposal is the closest QCFT to a CCFT that has ever been built. The resulting quaternionic theory is more restrictive than the complex theory, first of all because it has a more restrictive finite dimensional symmetry group, while the CCFT has an infinite dimensional Lie algebra that parametrizes its symmetries [16]. We decided to avoid a trivial theory by imposing quaternionic analyticity only on conformal symmetry transformations, and not on quaternionic functions that suffer the transformation. This choice permits a wider class of quaternionic functions to be employed in the theory, and thus a QCFT has been formally built for holomorphic symmetry transformations comprising the quantization of the fields and the operator product expansions.

The article is organized as follows, Section II presents the QHF’s and the constraints they impose on conformal transformations and on the stress tensor; in Section III the quantization of quaternionic primary fields is discussed, and in Section IV the operator product expansions (OPE’s) are obtained for the stress tensor and for the quaternionic primary field. Section IV

*Electronic address: p12@ubi.pt
II. CONFORMAL INVARIANCE

A conformal transformation on a $(b + d)$-dimensional flat space $\mathbb{R}^{b,d}$ is achieved through a coordinate change $x \rightarrow x'$ so that the metric tensor $\eta_{\mu\nu}$ transforms by a global scale factor $H(x)$, so that

$$\eta_{\kappa\lambda}(x') \partial_{\mu}x^\kappa \partial_{\nu}x^\lambda = H(x) \eta_{\mu\nu}(x),$$

and the summation of indexes is implicit. In order to determine the symmetry algebra of conformal transformations, we perform an infinitesimal transformation

$$x'^\mu = x^\mu + \varepsilon^\mu(x),$$

which applied on (1) leads to

$$\partial_\mu \varepsilon^\nu + \partial_\nu \varepsilon_\mu = \frac{2}{D}(\partial \cdot \varepsilon) \eta_{\mu\nu} \quad \text{so that} \quad \partial \cdot \varepsilon = \eta^{\mu\nu} \partial_\mu \varepsilon_\nu. \quad (3)$$

Higher order terms on $\varepsilon_\mu$ were discarded, and $D = b + d$ is the dimension of the space. A four dimensional Euclidean metric, where $\eta_{\mu\nu} = \delta_{\mu\nu}$ and $D = 4$, implies that an Euclidean infinitesimal conformal transformation $\varepsilon$ changes to

$$\partial_\mu \varepsilon_\nu = \frac{1}{4} \partial \cdot \varepsilon \quad \text{for} \quad \mu = \nu \quad \text{and} \quad \partial_\mu \varepsilon_\nu = - \partial_\nu \varepsilon_\mu \quad \text{for} \quad \mu \neq \nu. \quad (4)$$

Then we impose that a four dimensional conformal transformation may be organized as a quaternion valued function $\varepsilon$ so that

$$\varepsilon = \varepsilon_0 + \varepsilon_1 i + \varepsilon_2 j + \varepsilon_3 k, \quad (5)$$

where $i, j$ and $k$ are the associative and anti-commutative complex units of quaternions, which obey

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ijk = -1. \quad (6)$$

Using the symplectic notation, quaternions are written in complex components, so that

$$q = z + \zeta j \quad \text{for} \quad z = x_0 + x_1 i \quad \text{and} \quad \zeta = x_2 + x_3 i. \quad (6)$$

Consequently, the quaternion evaluated function $\varepsilon$ will be

$$\varepsilon = \varepsilon_0 + \varepsilon_1 j, \quad \text{so that} \quad \varepsilon_0 = \varepsilon_0 + i \varepsilon_1 \quad \text{and} \quad \varepsilon_1 = \varepsilon_2 + i \varepsilon_3. \quad (7)$$

Now, we will assume a further constraint, that $\varepsilon$ is a holomorphic quaternion function. This class of function is very restricted, and admits only affine quaternion functions, so that

$$\varepsilon = qa + b \quad (8)$$

for $a, b$ and $q$ quaternionic. In symplectic notation $\varepsilon$, $\varepsilon$ must obey some constraints in order to be holomorphic $\varepsilon$, namely

$$\partial_\mu \varepsilon_0 = -i \partial_1 \varepsilon_0 = \partial_2 \varepsilon_1 = i \partial_3 \varepsilon_1, \quad \partial_0 \varepsilon_1 = i \partial_1 \varepsilon_0 = - \partial_2 \varepsilon_0 = i \partial_3 \varepsilon_0 \quad (9)$$

and furthermore

$$\partial_\varepsilon \varepsilon_0 = \partial_\varepsilon \varepsilon_1 \quad \text{and} \quad \partial_\varepsilon \varepsilon_0 = - \partial_\varepsilon \varepsilon_1. \quad (10)$$

From (9) and (7), we see that the quaternionic function $\varepsilon$ will be holomorphic if constrained by the previous condition (4) and by the imposition of

$$\partial_0 \varepsilon_1 = - \partial_2 \varepsilon_3, \quad \partial_0 \varepsilon_2 = \partial_1 \varepsilon_3, \quad \partial_0 \varepsilon_3 = - \partial_1 \varepsilon_2 \quad \text{and} \quad \partial_\mu \varepsilon_\mu = \partial_\nu \varepsilon_\nu. \quad (11)$$
The situation is quite analogous to the two-dimensional case, where the Cauchy-Riemann conditions were generated instead of \( (1) \) and \( (11) \). There, holomorphic complex valued functions are the natural choice for two-dimensional conformal field theories. The quaternion case, on the other hand, seems not to be most general case, because of the additional constraint \( (11) \). We might imagine that some class of quaternion function could generate \( (4) \) either without further assumptions or by adopting less restrictive conditions. At this moment, we cannot say whether this is possible or not, and the quaternion holomorphic function appears to be the only feasible way to develop a quaternion parametrized conformal field theory. By way of example, an holomorphic function \( \mathcal{F}(q) = q + \mathcal{E}(q) \) applied on a four dimensional metric leads to a conformal transformation

\[
\frac{ds^2}{dq} = \frac{d\mathcal{F}}{dq} \biggl| \frac{dq d\bar{q}}{d\mathcal{F}} ,
\]

with \( \frac{d\mathcal{F}}{dq} \) as scale factor and where \( d\bar{q} \rightarrow \frac{d\bar{q}}{d\mathcal{F}} = \frac{d\bar{q}}{d\mathcal{F}} \) were used.

The symmetry algebra \( \mathcal{g} \) of the group \( \mathcal{G} \) of transformations on left-derivative quaternion holomorphic functions like \( (8) \) were studied at \([16]\). The \( \text{su}(2, \mathbb{C}) \) algebra is generated by the \( \{ g_1, g_2, g_3 \} \) sub-algebra of \( \mathcal{g} \). We associate these operators with rotations on a two sphere, the operator \( g_4 \) generates dilations and \( g_5 \) and \( g_6 \) generate translations. This algebra has an important difference when compared to the conformal algebra of two-dimensional theories, because the two dimensional case has an infinite dimensional algebra, the Witt algebra. In the two-dimensional case, the rôle of \( \mathcal{g} \) is played by the algebra of the \( \text{SL}(2, \mathbb{C})/\mathbb{Z}_2 \) group. The Witt algebra is infinite dimensional and admits a central extension. On the other hand, \( \mathcal{g} \) is finite dimensional, and consequently does not admit a central extension.

## A. Quaternion primary fields

In two-dimensional complex CFT (CCFT), there exists a terminology for the physical fields according to their conformal properties. Complex holomorphic fields \( \Phi(z) \) are called chiral and complex anti-holomorphic fields \( \Phi(\bar{z}) \) are called anti-chiral. For quaternion functions, we will not use the term holomorphic and anti-holomorphic, because these definitions are too restrictive for quaternions, but we can loosely call \( \Lambda(z, \bar{\zeta}) \) chiral and \( \bar{\Lambda}(\bar{\zeta}, \zeta) \) anti-chiral. We define a conformal transformation on a quaternionic field \( \Lambda(q, \bar{q}) \) in terms of a transformation \( \mathcal{F} \) on its coordinates as

\[
\Lambda(q, \bar{q}) = \left( \mathcal{F}^\ell_{\partial q} \right)^\ell \Lambda \left( \mathcal{F}(q), \mathcal{F}(\bar{q}) \right),
\]

so that the pair \( (\ell, \bar{\ell}) \) is the conformal weight, which are supposed to be real. Complex conformal weights are found for particle generating models \([18–20]\), and we will not consider here because of the non-commutativity of quaternions. Furthermore, we see that if \( \ell \neq \bar{\ell} \), then there is an ambiguity in the ordering of the factors, and then we avoid this problem imposing \( \ell = \bar{\ell} \), and thus the product of the quaternion factors is real, precluding any ordering problem. Note that \( \Lambda \) needs not to be quaternion holomorphic. It may be a real function parametrized by quaternions, like the quaternionic norm. On the other hand, we are dealing with quaternion holomorphic \( \mathcal{F} \), and then the conformal factors on \( (13) \) are constant.

Now we want to determine the change suffered on the field \( \Lambda \) caused by an infinitesimal quaternionic transformation

\[
\mathcal{F} = q + \mathcal{E} \quad \text{so that} \quad \mathcal{E} = q \varepsilon + \delta,
\]

where \( |\mathcal{E}| \ll 1 \), \( \varepsilon = \varepsilon_\ell + \varepsilon_{\bar{\ell}} j \) and \( \delta = \delta_\ell + \delta_{\bar{\ell}} j \). We want to expand \( (13) \) in a power series, but remembering that \( \Lambda \) is not holomorphic, we cannot obtain such a series using \( q \) and \( \bar{q} \) as variables. In the symplectic coordinates, we get

\[
\mathcal{E} = \varepsilon_\ell + \varepsilon_{\bar{\ell}} j = z \varepsilon_\ell - \bar{\zeta} \varepsilon_{\bar{\ell}} + \delta_\ell + (\bar{\zeta} \varepsilon_\ell + z \varepsilon_{\bar{\ell}} + \delta_{\bar{\ell}}) j,
\]

where \( \varepsilon_\ell = \varepsilon_{\bar{\ell}} \). In order to obtain the expansion, we write the operators

\[
\hat{\delta} = (z \varepsilon_\ell - \bar{\zeta} \varepsilon_{\bar{\ell}} + \delta_\ell) \partial_\ell + (\bar{\zeta} \varepsilon_\ell + z \varepsilon_{\bar{\ell}} + \delta_{\bar{\ell}}) \partial_{\bar{\ell}},
\]

where \( j \rightarrow \partial_\ell \) and \( 1 \rightarrow \partial_{\bar{\ell}} \) have been used, and \( \hat{\delta} \) may be obtained from the complex conjugate. Thus, using \( (13) \) and the approximated expansions

\[
|\partial_q \mathcal{F}|^{2\ell} \approx 1 + \ell (\varepsilon_\ell + \varepsilon_{\bar{\ell}}) \Lambda \approx \left( 1 + \hat{\delta} + \hat{\delta}^* \right) \Lambda,
\]

(17)
we obtain
\[ \delta \Lambda \approx \left[ (z \epsilon_z - \zeta \epsilon_z + \delta_z) \partial_z + (\bar{z} \epsilon_z - \bar{\zeta} \epsilon_z + \bar{\delta}_z) \partial_{\bar{z}} + \ell (\epsilon_z + \bar{\epsilon}_z) \right] \Lambda. \]  
(18)

This result has important consequences, because there are no derivatives depending neither on \( \zeta \) nor on \( \bar{\zeta} \), and thus the symmetry operators depending on these derivatives does not contribute to the conserved charge. In order to build the conserved charge, we now discuss the stress tensor.

**B. The stress tensor**

The conserved charge associated to a symmetry is obtained from a conserved current \( j_\mu \), so that
\[ j_\mu = T_{\mu \nu} \delta^\nu, \]  
(19)
with \( T_{\mu \nu} \) the symmetric stress tensor. For conformal theories, the conformal stress tensor is always traceless. In four-dimensional symplectic coordinates, we have
\[ T_\mu^\mu = T_{\bar{z} \bar{z}} + T_{\bar{\zeta} \bar{\zeta}} = 0, \]  
(20)
where the quaternion symplectic metric (A4) has been used. Another property of the stress tensor in symplectic coordinates is
\[ T_{\mu \nu} = T_{\bar{\nu} \bar{\mu}}, \]
which follows directly from \( T_{\mu \nu} = \partial_\mu x^\alpha \partial_\nu \lambda^\beta T_{\alpha \beta} \) with complex \( \mu, \nu \) and real \( \alpha, x^\alpha \) and \( T_{\alpha \beta} \). These conditions mean that from the ten components of \( T_{\mu \nu} \), there are only five independent components, namely \( T_{\bar{z} \bar{z}}, T_{\bar{\zeta} \bar{\zeta}}, T_{\bar{z} \bar{\zeta}} \) and \( T_{\bar{\zeta} \bar{z}} \). The \( T_{\bar{z} \bar{z}} \) component is identical to its complex conjugate, and consequently it is real. Then \( T_{\mu \nu} \) has the nine degrees of freedom coming from the real stress tensor \( T_{\alpha \beta} \), as expected. We may obtain the conserved charge in symplectic coordinates (A2) from the conserved current, so that
\[ j_0 = j_z + j_{\bar{z}} \quad \text{where} \quad j_z = 2 \left( T_{\bar{z} \bar{z}} \epsilon_z + T_{\bar{\zeta} \bar{\zeta}} \epsilon_{\bar{\zeta}} + T_{\bar{\zeta} \bar{z}} \epsilon_{\bar{\zeta}} + T_{\bar{\zeta} \bar{z}} \epsilon_{\bar{\zeta}} \right) \quad \text{and} \quad j_{\bar{z}} = \bar{j}_z. \]  
(21)
Using (15) we obtain
\[ j_z + j_{\bar{z}} = 2 \left[ (z \epsilon_z - \zeta \epsilon_z + \delta_z) T_1 + (\bar{z} \epsilon_{\bar{z}} + \bar{\zeta} \epsilon_{\bar{z}} + \bar{\delta}_z) T_2 + C.C. \right] \quad \text{where} \quad T_1 = T_{\bar{z} \bar{z}} + T_{\bar{\zeta} \bar{\zeta}} \quad \text{and} \quad T_2 = T_{\bar{\zeta} \bar{z}} + T_{\bar{\zeta} \bar{z}}. \]  
(22)
The conserved current is obtained from the volume integral
\[ Q = \int dv j_0 = \int dv (j_z + j_{\bar{z}}). \]  
(23)
The symmetry transformation on \( \Lambda \) is given by
\[ \delta \Lambda = [Q, \Lambda] = \int dv \left[ j_z + j_{\bar{z}}, \Lambda \right], \]  
(24)
where the volume element \( dv \) is real am may be factored out. In order to calculate the volume integral, we avail ourselves of the quaternion parametrization from appendix[A] so that
\[ q = \cos \theta e^{\tau + i\phi} + \sin \theta e^{\tau + i\phi} j, \]  
(25)
where \( \theta \in [0, \pi/2] \) maintains the positivity of the radii of each complex component of (23). This parametrization maps a four-dimensional cylinder to a quaternionic hyper-plane. We identify \( \tau \) as a time coordinate, and we further consider \( q = e^\tau + e^{i\phi} j \), so that
\[ z = \tau + \ln \cos \theta + i\phi, \quad \zeta = \tau + \ln \sin \theta + i\phi, \]  
(26)
and the radii of the complex parts in \( q \) are not identical, namely \( |e^\tau| \neq |e^{i\phi}| \). Thus, the volume element in the quaternionic plane for a constant time is calculated as
\[ dv = \cot \theta d\theta d\phi d\varphi. \]  
(27)
From (23) we observe the relationship between the conserved charge $Q$ and the conserved current $j_{\mu}$, and consequently the correspondence between the stress tensor and the conserved charge. Now we want to determine the relation between the conserved charge and the symmetry operators of the QHT. The symmetry operators $x_i$ that generate the QHT \cite{16} are related to the infinitesimal parameters as

$$
\delta \rightarrow \partial_z = x_1, \quad \epsilon \rightarrow z \partial_z = x_2, \quad \epsilon_x \rightarrow \zeta \partial_z = x_3, \quad \delta \rightarrow \partial_{\bar{z}} = x_4, \quad \epsilon \rightarrow \zeta \partial_{\bar{z}} = x_5, \quad \epsilon_x \rightarrow z \partial_{\bar{z}} = x_6,
$$

and then we want to express $Q$ in terms of $x_i$ as

$$
Q = \sum_i Q_i = \sum_i \epsilon_i x_i.
$$

However, from (18) we see that $x_4$, $x_5$ and $x_6$ do not contribute for $\delta \Lambda$, and then we set $T_2 = 0$ in (22). In order to employ the current to determine the conserved charge, we define

$$
u = k + \kappa j, \quad \text{so that} \quad k = e^{\tau + i\phi} \quad \text{and} \quad \kappa = e^{\tau + i\phi},
$$

and using (28) we write

$$
T_1 = \frac{\tan \theta}{2} T \quad \text{so that} \quad T = \frac{2}{k \kappa} \frac{x_1}{\kappa} + \frac{x_2}{k} + \frac{x_3}{k \kappa}.
$$

$T$ depend on $k$ and $\kappa$, but not on their complex conjugates, and thus it may be considered analogue of chiral fields of the CT. We eliminate the dependence on $\theta$ through the integration

$$
J + J = \int \frac{d\theta}{\cos \theta} \left( j_z + j_{\bar{z}} \right) = \left( k \epsilon_z - \kappa \epsilon_{\bar{z}} + \delta \right) T + C.C.
$$

The $\phi$ and $\varphi$ integrals may be changed to $k$ and $\kappa$ integrals over the radius $|k| = |\kappa| = e^\gamma$, and then

$$
Q = \int \frac{dk}{2\pi i} \int \frac{d\kappa}{2\pi i} \left( J + J \right) = \epsilon \epsilon_2 + \epsilon_x \epsilon_3 + \delta \epsilon_1.
$$

The $x_i$ generators may be obtained by inverting (31), so that

$$
x_1 = \int \frac{dk \, d\kappa}{(2\pi i)^2} T, \quad x_2 = \int \frac{dk \, d\kappa}{(2\pi i)^2} k T, \quad x_3 = \int \frac{dk \, d\kappa}{(2\pi i)^2} \kappa T.
$$

The results show a very constrained model, where the quaternionic coordinate $\zeta$ has no effect on the transformation of the primary field $\Lambda$. In fact, this is coherent with the transformation law \cite{18}, where the $\partial_{\bar{z}}$ does not generate transformations.

III. QUANTIZED FIELDS

In this section, we outline several formal aspects of quantizing a QCFT. The discussion is formal because every specific case to be quantized must be considered independently, and the discussion below only presents general aspects that must be followed.

As in the two-dimensional case, the quaternion parametrization \cite{25} relates dilations to time translations, and $q \rightarrow 0$ is associated to the infinite past, where $\tau \rightarrow -\infty$, or equivalently $q \rightarrow 0$. We note that the infinite past is valid only for both the complex variables going to zero. They may independently go to zero for specific values of $\theta$, and this is not associated to the infinite past. Using $e^\gamma = \rho$ in \cite{25}, we obtain the operators

$$
\partial_z = \frac{e^{-i\rho}}{2} \left( \cos \theta \partial_\rho - \frac{\sin \theta}{\rho} \partial_\theta - \frac{i}{\rho \sin \theta} \partial_\varphi \right), \quad \partial_{\bar{z}} = \frac{e^{-i\rho}}{2} \left( \sin \theta \partial_\rho - \frac{\cos \theta}{\rho} \partial_\theta - \frac{i}{\rho \sin \theta} \partial_\varphi \right),
$$

which permit us to obtain

$$
\rho \partial_\rho = z \partial_z + \zeta \partial_{\bar{z}} + \bar{z} \partial_z + \bar{\zeta} \partial_{\bar{z}}, \quad \partial_\theta = \sqrt{z \bar{z}} \left( \frac{1}{z} \partial_z + \frac{1}{\zeta} \partial_{\bar{z}} - \frac{1}{\bar{z}} \partial_z - \frac{1}{\bar{\zeta}} \partial_{\bar{z}} \right), \quad \partial_\varphi = i(z \partial_z - \bar{z} \partial_{\bar{z}}), \quad \partial_{\varphi} = i \left( \zeta \partial_z - \bar{\zeta} \partial_{\bar{z}} \right).
$$

In terms of the generators \cite{33} of the algebra of $G$, we can write

$$
\rho \partial_\rho = -(g_4 + \bar{g}_4) \quad \partial_\varphi = i(g_3 - g_4 - \bar{g}_3 + \bar{g}_4) \quad \partial_\varphi = i(\bar{g}_3 + \bar{g}_4 - g_3 - g_4)
$$

(36)
At this point, we observe a difference between the two-dimensional CCFT and the four-dimensional QCFT. In two dimensions the time operator is associated to the real dilation and the space translations may be written in terms of a linear combination of symmetry operators. In four dimensions, this is no longer observed because the \( \theta \) coordinate cannot be written in terms of the symmetry operators of \( \mathfrak{g} \). Furthermore, the direct sum of the symmetry algebra \( \mathfrak{g} \) and the algebra generated by the complex conjugates of its operators, \( \mathfrak{g} \oplus \bar{\mathfrak{g}} \), is not an algebra. In two-dimensions, this direct sum is an algebra, and this very fact presents a big difference between the two cases, indicating that the four-dimensional case is more restrictive because it cannot avail itself of the full symmetry algebra. We note that these differences raise due to the complex conjugacy of quaternions, were \((z, \bar{\zeta}) \to (\bar{z}, -\bar{\zeta})\).

The second aspect to be considered is defining a primary field to be quantized, and thereto we propose the expansion

\[
\Lambda(z, \bar{z}, \zeta, \bar{\zeta}) = (q\bar{q})^{-\ell} \sum_{\hat{m}, n, r, \hat{r} \in \mathbb{Z}} z^{-n} \bar{z}^{-\hat{m}} \zeta^{-r} \bar{\zeta}^{-\hat{r}} \Lambda_{\hat{m}, n, r, \hat{r}},
\]

(37)

where \( \Lambda_{\hat{m}, n, r, \hat{r}} \) are quaternionic constants, and quantization is obtained promoting the primary field (37) to an operator \( \hat{\Lambda} \). We define asymptotic past states using \( \tau \to -\infty \) in (25), so that \( q \to 0 \) and then asymptotic past states may be written

\[
|\Lambda\rangle_{\infty} = \lim_{q \to 0} \hat{\Lambda}|0\rangle.
\]

(38)

Considering that all coordinates have the real factor \( e^{z} \) which governs the approaching to zero, we eliminate negative powers on the variables of \( \hat{\Lambda} \) by imposing

\[
\hat{\Lambda}_{\hat{m}, n, r, \hat{r}}|0\rangle = 0 \quad \text{for} \quad \sigma + 2\ell > 0,
\]

(39)

so that \( \sigma = n + \hat{m} + r + \hat{r} \). The asymptotic state is then generated by the independent term of \( \hat{\Lambda} \), so that \( \sigma = 0 \).

A third important formal aspect of the quantized primary field is the Hermitian conjugate \( \Lambda^\dagger \) of the primary field. Considering that Hermitian conjugacy changes the sign of the time coordinate, then \( \tau \to -\tau \) in (25). Remembering that \( e^z = \sqrt{q\bar{q}} \), then

\[
z^\dagger = \frac{z}{q\bar{q}}, \quad \zeta^\dagger = \frac{\zeta}{q\bar{q}} \quad \text{and consequently} \quad q^\dagger = \frac{q}{q\bar{q}}.
\]

(40)

We then see that Hermitian conjugacy is not a quaternion conformal transformation as defined in [16] because it implies in the inversion of the quaternionic coordinate. We thus define

\[
\Lambda^\dagger(z, \bar{z}, \zeta, \bar{\zeta}) = (q\bar{q})^{\ell+\sigma} \sum_{\hat{m}, n, r, \hat{r} \in \mathbb{Z}} \Lambda^\dagger_{\hat{m}, n, r, \hat{r}} z^{-n} \bar{z}^{-\hat{m}} \zeta^{-r} \bar{\zeta}^{-\hat{r}},
\]

(41)

such that the negative powers on the variables of \( \hat{\Lambda}^\dagger \) are eliminated by imposing

\[
\hat{\Lambda}^\dagger_{\hat{m}, n, r, \hat{r}}|0\rangle = 0 \quad \text{for} \quad \sigma + 2\ell < 0,
\]

(42)

and then we have well-defined primary fields for quantizing.

IV. OPERATOR PRODUCTS

In two-dimensional CFT, the time coordinate is parametrized by the radial direction, so that \( z = e^{i\theta} \). Thus, the time ordering of an operator product like \( \hat{X}(z)\hat{Y}(w) \) is determined by the relation between \( |z| \) and \( |w| \). In the quaternion parametrization (25) this is also true because \( |q| = e^z \) and \( \tau \) is identified with a time coordinate. However, as \( q = z + \zeta j \), the time direction cannot be identified neither with of \( |z| \) nor with \( |\zeta| \). This is an important detail in order to calculate the conserved charge and consequently the symmetry transformations. Using (33), we rewrite (24) as

\[
\delta \Lambda = \oint \frac{dk}{2\pi i} \oint \frac{d\kappa}{2\pi i} [J + \hat{J}, \Lambda].
\]

(42)

We want to get some physical insight from the equality between (18) and (24), namely

\[
\int dv[j_z + j_{\bar{z}}, \Lambda] = \left[ \varepsilon_z (\ell + z \partial_z) - \varepsilon_{\bar{z}} (\ell + \bar{z} \partial_{\bar{z}}) + \delta_z \partial_z + \text{C.C.} \right] \Lambda.
\]

(43)

The \( \theta \)-dependence may be eliminated from the right hand side of (43) using (32), and the integral on \( \phi \) and \( \varphi \) may be turned into \( k \) and \( \kappa \) integrals. However, the right hand side depends on \( z \) and \( \zeta \) variables, which explicitly depend on \( \theta \). Our interpretation
Finally, we use the OPE (46) and (34) to calculate the commutators among the symmetry operators $J, J$. Using (34) and the $\Theta$ algebra (B2), we calculate the OPE for $T, \Lambda$.

\[
[ J + J, \Lambda ] \mapsto \Theta \left[ ( J + J ) \Lambda \right] = \left( k \epsilon_z - \kappa \epsilon_z + \delta_z \right) \Theta [ T \Lambda ] + \left( \bar{k} \epsilon_z - \bar{\kappa} \epsilon_z + \bar{\delta}_z \right) \Theta [ T \Lambda ]
\]

(44)

We may determine the operator product expansion by substitution (44) back in (43) and using the identities

\[
f(w) = \frac{1}{2\pi i} \int dk \frac{f(k)}{k - w} \quad \text{and} \quad \partial_w f(w) = \frac{1}{2\pi i} \int dk \frac{f(k)}{(k - w)^2},
\]

(45)

where $f(k)$ is a complex holomorphic function. Thus, we obtain

\[
\Theta [ T \Lambda ] = \frac{1}{(k - w)(\kappa - \omega)} \partial_w \Lambda + \frac{\ell}{(k - w)^2(\kappa - \omega)} \Lambda.
\]

(46)

Using (34) and the $\Theta$ algebra (B2), we calculate the OPE for $T, \Lambda$.

\[
\Theta [ T(k, \kappa) T(w, \omega) ] = 2 \frac{T(w, \omega)}{(k - w)^2 \kappa} - \frac{T(w, \omega)}{(k - w)^2(\kappa - \omega)}.
\]

(47)

If we make $\Lambda = T$ in (46) we see that $T$ is not a primary field. Indeed, using (47) and $\Lambda = T$ in (44) and (32) $\Lambda = T$, we get

\[
\delta T = \epsilon_z T.
\]

(48)

Finally, we use the OPE (44) and (34) to calculate the commutators among the symmetry operators $x_i$ and the primary field $\Lambda$.

\[
[ x_1, \Lambda ] = \Lambda, \quad [ x_2, \Lambda ] = \ell \Lambda - w \partial_n \Lambda, \quad [ x_3, \Lambda ] = \omega \partial_n \Lambda.
\]

(49)

In CCFT the commutation relations between primary fields and the symmetry operators are useful for defining creation and annihilation operators. In the QCFT discussed here this must still be the case, but as we were unable to find a general expression for them, we leave this task as a direction of research to be studied in specific examples in the future. Another quantity that is calculated in general in CCFT are correlation functions, but these computations may not be trivial in four dimensional [4, 5], and then we postpone this issue for a future study.

V. CONCLUSION

In this article we presented a four-dimensional CFT parametrized with quaternions (QCFT) and whose conformal transformations are restricted to quaternion holomorphic functions (QHF). We have shown that such a theory may be quantized and that operator product expansions (OPE’s) may also be defined. This theory must of course now be tested in order to see its properties in building concrete examples of physical theories, and the well known examples for CCFT are the most important candidates. The study of scalar and fermionic fields are certainly interesting future directions for research, but even applications to string theory, the Thirring model and super-symmetric theories may be tried. Only after these applications is that the utility of QCFT may be correctly evaluated. At this point, the construction presented in this article is only the outline of a mathematical theory.

ACKNOWLEDGEMENTS

Sergio Giardino receives the financial grant number 206383/2014-2 from the CNPq for his research and is grateful for the hospitality of Professor Paulo Vargas Moniz and the Centre for Mathematics and Applications of the Beira Interior University.

Appendix A: Metric tensor

A four-dimensional Euclidean metric $g_{\mu \nu}$ has different expressions using different quaternion variables. For quaternion extended variables,

\[
q = x_0 + x_1 i + x_2 j + x_3 k \quad \Rightarrow \quad g_{\mu \nu} = \delta_{\mu \nu}.
\]

(A1)
Different coordinate systems lead to changes in the metric tensor. Using the Euclidean coordinates in terms of the symplectic ones, we get

\[ x_0 = \frac{q + \bar{q}}{2} = \frac{z + \bar{z}}{2}, \quad x_1 = \frac{q \bar{i} + i q}{2} = \frac{z - \bar{z}}{2i}, \quad x_2 = \frac{q \bar{j} - j q}{2} = \frac{\zeta + \bar{\zeta}}{2}, \quad x_3 = \frac{q \bar{k} - k q}{2} = \frac{\zeta - \bar{\zeta}}{2i}. \tag{A2} \]

The metric tensor shall be calculated using

\[ g_{\mu \nu} = \partial_{\mu} x^\lambda \partial_{\nu} x^\kappa G_{\kappa \lambda}. \tag{A3} \]

Using the Euclidean metric, so that \( G_{\mu \nu} = \delta_{\mu \nu} \), and defining \( g_{\mu \nu} = h_{\mu \nu} \), we obtain

\[ h_{\mu \nu} = \partial_{\mu} x^\lambda \partial_{\nu} x^\kappa \delta_{\kappa \lambda} \quad \text{so that} \quad h_{\mu \nu} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{A4} \]

We call \( h_{\mu \nu} \) the quaternion symplectic metric (QSM). Let us see some other ways of obtaining it using different coordinate systems. In polar coordinates,

\[ q = \rho \left( \cos \theta e^{i \phi} + \sin \theta e^{i \Psi} \right) \Rightarrow g_{\mu \nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho^2 & 0 & 0 \\ 0 & 0 & \rho^2 \cos^2 \theta & 0 \\ 0 & 0 & 0 & \rho^2 \sin^2 \theta \end{bmatrix} = \begin{bmatrix} g_{\rho \rho} & 0 & 0 & 0 \\ 0 & g_{\theta \theta} & 0 & 0 \\ 0 & 0 & g_{\phi \phi} & 0 \\ 0 & 0 & 0 & g_{\psi \psi} \end{bmatrix}, \tag{A5} \]

and

\[ \rho = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}, \quad \cos \theta = \frac{x_0}{\rho}, \quad \cos \phi = \frac{x_0}{\sqrt{x_0^2 + x_1^2}}, \quad \cos \varphi = \frac{x_2}{\sqrt{x_2^2 + x_3^2}}. \tag{A6} \]

Rewriting (A7) in symplectic coordinates,

\[ \rho = \sqrt{z \bar{z} + \zeta \bar{\zeta}}, \quad \phi = \frac{1}{2i} \ln \frac{z}{\bar{z}}, \quad \phi = \frac{1}{2i} \ln \frac{\zeta}{\bar{\zeta}}, \quad \theta = \frac{1}{2} \arccos \frac{z \bar{z} - \zeta \bar{\zeta}}{z \bar{z} + \zeta \bar{\zeta}}, \tag{A7} \]

we obtain the quaternion symplectic metric \( h_{\mu \nu} \) using (A3), (A5) and (A7). The important case for conformal field theory is a variant of (A5), where \( \rho = e^{\tau} \).

\[ q = \cos \theta e^{\tau + i \phi} + \sin \theta e^{\tau + i \Psi} \Rightarrow g_{\mu \nu} = e^{2 \tau} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos^2 \theta & 0 \\ 0 & 0 & 0 & \sin^2 \theta \end{bmatrix} = \begin{bmatrix} g_{\tau \tau} & 0 & 0 & 0 \\ 0 & g_{\theta \theta} & 0 & 0 \\ 0 & 0 & g_{\phi \phi} & 0 \\ 0 & 0 & 0 & g_{\psi \psi} \end{bmatrix}. \tag{A8} \]

where \( z = \cos \theta e^{\tau + i \phi} \) and \( \zeta = \sin \theta e^{\tau + i \Psi} \). We note the conformal symmetry of this metric, where \( e^{2 \tau} \) is the conformal factor. On the other hand, from (A8) and using symplectic coordinates, we get

\[ \tau = \frac{1}{2} \ln \left( z \bar{z} + \zeta \bar{\zeta} \right), \quad \phi = \frac{1}{2i} \ln \frac{z}{\bar{z}}, \quad \phi = \frac{1}{2i} \ln \frac{\zeta}{\bar{\zeta}}, \quad \theta = \arctan \sqrt{\frac{\zeta \bar{\zeta}}{z \bar{z}}}. \tag{A9} \]

Using this coordinate system, we obtain the metric given by (A4). It is important to note that always when a system of coordinates is changed to the symplectic coordinates, the QSM will be obtained. However, we see that (A8) has conformal properties, and because of this it must be chosen for QCFT. As a last example, using

\[ q = e^{\chi + i \phi} + e^{\xi + i \Psi} \]  \quad \text{where} \quad r = z \bar{z} \quad \text{and} \quad \rho = \zeta \bar{\zeta}, \quad \Rightarrow \quad g_{\mu \nu} = \begin{bmatrix} e^{2\chi} & 0 & 0 & 0 \\ 0 & e^{2\chi} & 0 & 0 \\ 0 & 0 & e^{2\xi} & 0 \\ 0 & 0 & 0 & e^{2\xi} \end{bmatrix}, \tag{A10} \]

we again obtain the QSM in symplectic coordinates, but in (A10) the conformal symmetry is not evident, and then it will not be used to study the QCFT.
Appendix B: Quaternion holomorphic transformation representations

The quaternion holomorphic transformation \([16]\) has two important representations. The first is \(\chi\), given by

\[
x_1 = \partial_z, \quad x_2 = z\partial_z, \quad x_3 = \zeta\partial_z, \quad x_4 = \partial\zeta, \quad x_5 = \zeta\partial\zeta, \quad x_6 = z\partial\zeta.
\]

(B1)

The Lie algebra for \(\chi\) is written through the commutation relations

\[
[x_1, x_2] = x_1 \\
[x_1, x_3] = 0 \quad [x_2, x_3] = -x_3 \\
[x_1, x_4] = 0 \quad [x_2, x_4] = 0 \quad [x_3, x_4] = -x_1 \\
[x_1, x_5] = 0 \quad [x_2, x_5] = 0 \quad [x_3, x_5] = -x_3 \quad [x_4, x_5] = x_4 \\
[x_1, x_6] = x_4 \quad [x_2, x_6] = x_5 \quad [x_3, x_6] = x_5 - x_2 \quad [x_4, x_6] = 0 \quad [x_5, x_6] = -x_6
\]

(B2)

The second representation used in this article is \(g\), whose generators

\[
g_1 = x_3 + x_6, \quad g_2 = i(x_6 - x_3), \quad g_3 = x_2 - x_5, \quad g_4 = -(x_2 + x_3), \quad g_5 = x_1, \quad g_6 = -x_4
\]

(B3)

obey the algebra

\[
[g_1, g_2] = 2ig_3 \\
[g_1, g_3] = 2ig_2 \quad [g_2, g_3] = 2ig_1 \\
[g_1, g_4] = 0 \quad [g_2, g_4] = 0 \quad [g_3, g_4] = 0 \\
[g_1, g_5] = g_6 \quad [g_2, g_5] = ig_6 \quad [g_3, g_5] = -g_5 \quad [g_4, g_5] = g_5 \\
[g_1, g_6] = g_5 \quad [g_2, g_6] = -ig_5 \quad [g_3, g_6] = g_6 \quad [g_4, g_6] = g_6 \quad [g_5, g_6] = 0
\]

(B4)

This representation is interesting because we see the \(su(2)\) algebra generated by \(g_1, g_2\) and \(g_3\) and the complex dilation generated by \(g_4, g_5\) and \(g_6\) are associated with complex translations.

[1] P. H. Ginsparg. “Applied conformal field theory”. Les Houches summer school, (1988) hep-th/9108028.
[2] R. Blumenhagen; E. Plauschinn. “Introduction to conformal field theory”. Lect. Notes Phys., 779:1–256, (2009).
[3] R. Rattazzi; V. S. Rychkov; E. Tonni; A. Vichi. “Bounding scalar operator dimensions in 4D CFT”. JHEP, 12:031, (2008) arXiv:0807.0004[hep-th].
[4] G. Vos. “Generalized Additivity in Unitary Conformal Field Theories”. Nucl. Phys., B899:91–111, (2015) arXiv:1405.7941[hep-th].
[5] E. Elkhidir; D. Karateev; M. Serone. “General Three-Point Functions in 4D CFT”. JHEP, 01:133, (2015) arXiv:1412.1796[hep-th].
[6] S. Moon; S.-J. Lee; J. Lee; J.-H. Oh. “Electric-magnetic duality implies (global) conformal invariance”. J. Korean Phys. Soc., 67(3):427–432, (2015) arXiv:1405.4934[hep-th].
[7] M. Bischoff; D. Meise; K.-H. Rehren; I. Wagner. “Conformal quantum field theory in various dimensions”. Bulg. J. Phys., 36:170–185, (2009) arXiv:0908.3391[math-ph].
[8] S. Giardino; P. Teotonio-Sobrinho. “A non-associative quaternion scalar field theory”. Mod. Phys. Lett., A28(35):1350163, (2013) arXiv:1211.5049[math-ph].
[9] M. Evans; F. Gursey; V. Gligorovska. “From 2-D conformal to 4-D selfdual theories: Quaternionic analyticity”. Phys. Rev., D47:3496–3508, (1993) hep-th/9207089.
[10] A. D. Popov. “Holomorphic Chern-Simons-Witten theory: From 2-D to 4-D conformal field theories”. Nucl. Phys., B550:585–621, (1999) hep-th/9806239.
[11] A. Sudbery. “Quaternionic Analysis”. Math. Proc. Camb. Phil. Soc., 85:199–225, (1979).
[12] F. Gursey; H. C. Tze. “Complex and Quaternionic Analyticity in Chiral and Gauge Theories. Part I”. Annals Phys., 128:29, (1980).
[13] S. De Leo; S. Giardino. “Dirac solutions for quaternionic potentials”. J.Math.Phys., 55:022301, (2014) arXiv:1311.6673[math-ph].
[14] S. De Leo; G. Ducati; S. Giardino. “Quaternionic Dirac Scattering”. J. Phys. Math., 6:1000130, (2015) arXiv:1505.01807[math-ph].
[15] S. Giardino. “Quaternionic particle in a relativistic box”. (2015) arXiv:1504.00643[quantu-ph].
[16] S. Giardino. “Möbius transformation for left-derivative quaternion holomorphic functions”. (2015) arXiv:1508.01933[math-ph].
[17] C. A. Beavours. “Quaternion Calculus”. Am. Math. Monly, 80:995–1008, (1973).
[18] A. Chatterjee; D. A. Lowe. “Holographic operator mapping in dS/CFT and cluster decomposition”. (2015) arXiv:1503.07482[hep-th].
[19] C.-H. Chen ; S. P. Kim; I-C. Lim; J-R. Sun; M-F. Wu. “Spontaneous Pair Production in Reissner-Nordstrom Black Holes”. Phys. Rev., D85:124041, (2012) arXiv:1202.3224[hep-th].
[20] E. Pomoni; L. Rastelli. “Large N Field Theory and AdS Tachyons”. JHEP, 04:020, (2009) arXiv:0805.2261[hep-th].