The sharp second order Caffarelli-Kohn-Nirenberg inequality and stability estimates for the sharp second order uncertainty principle

Anh Tuan Duong* and Van Hoang Nguyen†

February 25, 2022

Abstract

In this paper we prove a class of second order Caffarelli-Kohn-Nirenberg inequalities which contains the sharp second order uncertainty principle recently established by Cazacu, Flynn and Lam [13] as a special case. We also show the sharpness of our inequalities for several classes of parameters. Finally, we prove a stability version of the sharp second order uncertainty principle of Cazacu, Flynn and Lam by showing that the difference of both sides of the inequality controls the distance to the set of extremal functions in $L^2$ norm of gradient of functions.

1 Introduction

The Heisenberg uncertainty principle in quantum mechanics states that the position and the momentum of a given particle cannot both be determined exactly at the same time (see [31]). The rigorous mathematical formulation of this principle is established by Kennard [33] and Weyl [49] (who attributed it to Pauli) stating that the function itself and its Fourier transform cannot be sharply localized at the origin simultaneously. Mathematically, the Heisenberg-Pauli-Weyl uncertainty principle is described by the following inequality

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \int_{\mathbb{R}^n} |x|^2 |u|^2 dx \geq \frac{n^2}{4} \left( \int_{\mathbb{R}^n} |u|^2 dx \right)^2$$

(1.1)
for any function $u \in H^1(\mathbb{R}^n)$ (the first order Sobolev space in $\mathbb{R}^n$) such that $\int_{\mathbb{R}^n} |x|^2 |u|^2 dx < \infty$. It is well known that the constant $n^2/4$ is sharp and is attained only by Gaussian functions (see [28]).

In [50], Xia extends the inequality (1.1) and obtain the following inequality

$$\frac{n - t\gamma}{t} \int_{\mathbb{R}^n} \frac{|u|^t}{|x|^t} dx \leq \left( \int_{\mathbb{R}^n} \frac{|
abla u|^p}{|x|^\alpha} dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^n} \frac{|u|^\frac{n(t-1)}{t}}{|x|^\beta} dx \right)^{\frac{n-1}{p}}$$

where $n \geq 2$, $1 < p < t$, and $\alpha, \beta, \gamma$ satisfying the following conditions

$$n - \alpha p > 0, \quad n - \beta > 0, \quad n - t\gamma > 0,$$

and the balanced condition

$$\gamma = \frac{1 + \alpha}{t} + \frac{\beta}{tp}.$$  
Moreover, when $1 + \alpha - \frac{\beta}{r} > 0$ and

$$n - \beta < \left( 1 + \alpha - \frac{\beta}{r} \right) \frac{p(t-1)}{t-p},$$

then the inequality (1.2) is sharp and the extremal functions are given by

$$u(x) = (\lambda + |x|^{1+\alpha-\frac{\beta}{r}})^{-\frac{1-p}{p}}, \quad \lambda > 0.$$  
The inequality (1.2) is extended to the Riemannian manifolds in [42], the Finsler manifolds in [32] and the stratified Lie groups in [39].

Both the Heisenberg-Pauli-Weyl principle (1.1) and the Xia inequality (1.2) belong to a larger class of the first order interpolation inequalities which are called Caffarelli-Kohn-Nirenberg (CKN) inequalities established in [8] to study the Navier-Stokes equation and the regularity of particular solutions [7]. The class of CKN inequalities contains many well-known inequalities such as the Sobolev inequality, the Hardy inequality, the Hardy-Sobolev inequality, the Gagliardo-Nirenberg inequality, etc. They play an important role in theory of partial differential equations and have been extensively studied in many settings. Concerning to the sharp version of CKN inequalities, we refer the reader to the papers [2, 10, 11, 17, 18, 20, 21, 34, 46].

The higher order CKN inequalities were established by Lin [35]. In contrast to the first order inequalities, much less is known on the sharp version of the higher order CKN inequalities except the Rellich inequality [44] and the sharp higher order Sobolev inequality [19, 34]. In recent paper [13], Cazacu, Flynn and Lam proved the following sharp second order uncertainty principle which is a special case of the second order CKN inequalities

$$\int_{\mathbb{R}^n} |\Delta u|^2 dx \int_{\mathbb{R}^n} |x|^2 |\nabla u|^2 dx \geq \left( \frac{n + 2}{2} \right)^2 \left( \int_{\mathbb{R}^n} |\nabla u|^2 dx \right)^2.$$  

(1.3)
The constant \((n + 2)^2/4\) above is sharp and is attained by Gaussian functions. In fact, the inequality (1.3) without the sharp constant can be derived from (1.1) and Cauchy-Schwartz inequality (see the comment in the introduction in [13]). The inequality (1.3) is motivated by an open question of Maz’ya [36] on finding the sharp constant in (1.1) when we replace \(u\) by a divergence-free vector field \(U\). In particular, the inequality (1.3) answers affirmatively the question of Maz’ya in the case \(n = 2\).

The first aim in this paper is to extend the inequality (1.3) to a larger class of parameters in spirit of (1.2). For \(\alpha, \beta\) satisfying the conditions \(n - 2\alpha > 0\) and \(n - \beta > 0\), we denote by \(H^2_{\alpha, \beta}(\mathbb{R}^n)\) the second order Sobolev space which is completion of \(C_{0}^{\infty}(\mathbb{R}^n)\) under the norm

\[
\|u\|_{H^2_{\alpha, \beta}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2\alpha} dx + \int_{\mathbb{R}^n} |\nabla u|^2 |x|^{-\beta} dx \right)^{1/2}, \quad u \in C_{0}^{\infty}(\mathbb{R}^n).
\]

Then the first main result in this paper reads as follows.

**Theorem 1.1.** Let \(n \geq 1\) and \(\alpha \in \mathbb{R}\) satisfy \(n - 2\alpha > 0\), \(n + 2\alpha > 0\) and \(n + 2 + 4\alpha > 0\). Then the following inequality

\[
\int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2\alpha} dx \int_{\mathbb{R}^n} |x|^{2\alpha} |\nabla u \cdot x|^2 dx \geq \left( \frac{n + 2 + 4\alpha}{2} \right)^2 \left( \int_{\mathbb{R}^n} |\nabla u|^2 dx \right)^2
\]

holds true for any function \(u \in H^2_{n-2-2\alpha}(\mathbb{R}^n)\). Furthermore, if \(1 + \alpha > 0\) then the inequality (1.4) is sharp and is attained by function

\[
U_0(x) = \exp \left( - \frac{|x|^{2(1+\alpha)}}{2(1+\alpha)} \right).
\]

In particular, when \(\alpha = 0\), Theorem 1.1 implies the following.

**Corollary 1.2.** Let \(n \geq 1\). Then there holds

\[
\int_{\mathbb{R}^n} |\Delta u|^2 dx \int_{\mathbb{R}^n} |\nabla u \cdot x|^2 dx \geq \left( \frac{n + 2}{2} \right)^2 \left( \int_{\mathbb{R}^n} |\nabla u|^2 dx \right)^2
\]

for any \(u \in H^2_{n-2}(\mathbb{R}^n)\). This inequality is sharp and is attained by the Gaussian function \(\exp(-|x|^2/2)\).

Evidently, we have \(|\nabla u \cdot x| \leq |\nabla u||x|\). Hence, our inequality (1.5) is still stronger than (1.3).

Recall that the proof of (1.3) in [13] is quite long and complicated. The authors have used the decomposition of function \(u\) into spherical harmonic and integral estimates for radial functions.

In order to prove (1.4), we develop a new approach which is completely different with the one of Cazacu, Flynn and Lam. Indeed, our approach is based on establishing a new identity

\[
\int_{\mathbb{R}^n} \frac{|\Delta u + \nabla u \cdot x|^2}{|x|^{2\alpha}} dx = \int_{\mathbb{R}^n} |\Delta u|^2 dx + \int_{\mathbb{R}^n} |x|^{2\alpha} |\nabla u \cdot x|^2 dx - (n - 2) \int_{\mathbb{R}^n} |\nabla u|^2 dx.
\]
Then, using a factorization of $u$ as $u = vU_0$, we are able to show that
\[
\int_{\mathbb{R}^n} \frac{\Delta u + \nabla u \cdot x|x|^{2\alpha}}{|x|^{2\alpha}} dx \geq 2(n + 2\alpha) \int_{\mathbb{R}^n} |\nabla u|^2 dx.
\]
Combining two estimates above and a simple minimizing argument, we obtain (1.4). More details in the proof are given in Section §2 below.

For more general on the parameters as in (1.2), we obtain the following inequality for radial functions.

**Theorem 1.3.** Let $n \geq 1$, $t \geq 2$ and $\alpha, \beta, \gamma$ be such that
\[
n - 2\alpha > 0, \quad n - \beta > 0, \quad n - t\gamma > 0
\]
and
\[
\gamma = \frac{1 + \alpha}{t} + \frac{\beta}{2t}.
\]
Then the following inequality
\[
\int_{\mathbb{R}^n} \frac{|\Delta u|^2}{|x|^{2\alpha}} dx \int_{\mathbb{R}^n} \frac{|\nabla u|^{2(t-1)}}{|x|^{\beta}} dx \geq \left( \frac{n + t(1 + 2\alpha - \gamma)}{t} \right)^2 \left( \int_{\mathbb{R}^n} |\nabla u|^t dx \right)^2
\]
holds true for any radial function $u \in H^2_{\alpha,\beta}(\mathbb{R}^n)$. Moreover, under the following conditions
\[
(1 + 2\alpha)(t - 2) + 1 + \alpha - \frac{\beta}{2} > 0, \quad t < 3 + \alpha - \frac{\beta}{2},
\]
\[
n + 2\alpha > 0,
\]
and
\[
n - \beta < \frac{2(t - 1)}{t - 2}(1 + \alpha - \frac{\beta}{2}),
\]
then the constant $(n + t(1 + 2\alpha - \gamma))^2/t^2$ is sharp and is attained only up to a dilation and a multiplicative constant by the function the form
\[
U_1(x) = \int_{|x|}^\infty s^{1+2\alpha} \exp \left( - \frac{s^{1+\alpha-\frac{\beta}{2}}}{1+\alpha-\frac{\beta}{2}} \right) ds,
\]
if $t = 2$, and
\[
U_2(x) = \int_{|x|}^\infty r^{1+2\alpha} \left( 1 + (t - 2) \frac{r^{(1+2\alpha)(t-2)+1+\alpha-\frac{\beta}{2}}}{(1+2\alpha)(t-2) + 1 + \alpha - \frac{\beta}{2}} \right)^{\frac{1}{t-1}} dr,
\]
if $t > 2$.  


In the special case where $\alpha = \beta = 0$, $t = 2$ and $\gamma = 1/2$, we recover the second result of Cazacu, Flynn and Lam in [13] for radial functions in $H^2_{0,0}(\mathbb{R}^n)$ which is the sharp second order Hydrogen uncertainty principle

$$\int_{\mathbb{R}^n} |\Delta u|^2 dx \int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \frac{(n + 1)^2}{4} \left( \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|} dx \right)^2. \tag{1.12}$$

In fact, in that paper, Cazacu, Flynn and Lam proved the inequality (1.12) for any function $u$ (without radiality assumption) for any $n \geq 5$. The condition $n \geq 5$ appears due to the technical restrictions of decomposing a smooth function into spherical harmonics. They conjectured that the inequality (1.12) still holds for $n = 2, 3, 4$. The inequality (1.12) is sharp and an extremal function is given by $u(x) = c(1 + a|x|)e^{-a|x|}$ with $c \in \mathbb{R}$ and $a > 0$.

In the general case of parameters, comparing (1.8) with the first order inequality (1.2), the sharp constant changes from $((n - \gamma t)/t)^2$ to $((n + t(1 + 2\alpha - \gamma))/t)^2$. Moreover, to obtain the sharpness and the attainability of constant, we need some more conditions on the parameters (see (1.9) and (1.10)). Indeed, these conditions ensure that the function $U_1$ and $U_2$ is well-defined, and $\Delta U_1$ and $\Delta U_2$ exists in the distributional sense and belongs to $H^2_{\alpha,\beta}(\mathbb{R}^n)$.

In the non-radial case, following the approach of Cazacu, Flynn, and Lam by using spherical harmonics, we are able to establish the sharp second order CKN inequalities (see Theorem 1.4 below) which extends (1.8) to any function in $H^2_{\alpha,\beta}(\mathbb{R}^n)$ but with some restrictions on the dimension $n$ (as the case of the second order hydrogen uncertainty principle (1.12)). However, this approach works only for $t = 2$.

To state our next result, let us define for $n, \alpha, \beta, \gamma$ as in Theorem 1.3 with $t = 2$ and $k \geq 0$

$$A_{n,\alpha,k}(g) = \int_0^\infty (g'(r)^2 r^{n+2k-2\alpha-1} dr + (1 + 2\alpha)(n + 2k - 1) \int_0^\infty (g'(r)^2 r^{n+2k-2\alpha-3} dr \tag{1.13}$$

$$B_{n,\beta,k}(g) = \int_0^\infty (g'(r)^2 r^{n+2k-2\beta-1} dr + \beta k \int_0^\infty g^{2} r^{n+2k-2\beta-3} dr \tag{1.14}$$

$$C_{n,\gamma,k}(g) = \int_0^\infty (g'(r)^2 r^{n+2k-2\gamma-1} dr + 2\gamma k \int_0^\infty g^{2} r^{n+2k-2\gamma-3} dr \tag{1.15}$$

and

$$A_k(n, \alpha, \beta) = \inf_{g \not\equiv 0} \frac{A_{n,\alpha,k}(g)B_{n,\beta,k}(g)}{C_{n,\gamma,k}(g)^2} \tag{1.16}$$

where the infimum is taken on all function $g \in C^2([0, \infty))$ such that $A_{n,\alpha,k}(g)$ and $B_{n,\beta,k}(g)$ are finite. Our next result is given in the following theorem.

**Theorem 1.4.** Let $n \geq 1$ and $\alpha, \beta, \gamma$ satisfy the conditions of Theorem 1.3 with $t = 2$, then we have

$$\int_{\mathbb{R}^n} |\Delta^2 f|^2 dx \int_{\mathbb{R}^n} |\nabla f|^2 dx \geq \min_{k \in \mathbb{N}} A_k(n, \alpha, \beta) \left( \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{|x|^2} dx \right)^2, \tag{1.17}$$

5
for any function \( u \in H^2_{\alpha,\beta}(\mathbb{R}^n) \). Moreover, the constant \( \min_{k \in \mathbb{N}} A_k(n, \alpha, \beta) \) in (1.17) is sharp and satisfies the estimate

\[
\min_{k \in \mathbb{N}} A_k(n, \alpha, \beta) \geq \min_{k \in \mathbb{N}} \frac{1 + \min \left\{ 0, \frac{4\beta k}{(n+2k-\beta-2)^2} \right\}}{1 + \max \left\{ 0, \frac{8k}{(n+2k-2\gamma-2)^2} \right\}} \left( \frac{n + 2k + 4\alpha - 2\gamma + 2}{2} \right)^2. \tag{1.18}
\]

The inequality (1.18) was proved by Cazacu, Flynn and Lam in [13] corresponding to the case \( \alpha = \gamma = 0, \beta = -2 \) and \( \alpha = \beta = 0, \gamma = \frac{1}{2} \). It plays an important role in their proof of the sharp second uncertainty principle (1.3) and the second order hydrogen uncertainty principle (1.12). Indeed, by considering the right-hand side as a function of \( k \), they show that the infimum of the right-hand side is attained at \( k = 0 \) in the case \( \alpha = \gamma = 0, \beta = -2 \) for any \( n \geq 2 \) and in the case \( \alpha = \beta = 0, \gamma = \frac{1}{2} \) for any \( n \geq 5 \). This implies (1.3) and (1.12). We believe that this argument together with (1.18) provides the same conclusion for general \( \alpha, \beta, \gamma \) when \( n \) large enough. Nevertheless, we do not pursue this direction in this paper.

The last aim of this paper is to establish the stability estimates for the sharp second order uncertainty principle (1.3) of Cazacu, Flynn and Lam. Let us define

\[
\delta(u) = \left( \int_{\mathbb{R}^n} |\Delta u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} |x| |\nabla u|^2 dx \right)^{\frac{1}{2}} - 1
\]

for \( u \in H^2_{0,-2}(\mathbb{R}^n) \setminus \{0\} \) and \( \delta(0) = 0 \). Evidently, we always have \( \delta(u) \geq 0 \). Establishing the stability estimates for the sharp second order uncertainty principle (1.3) means that we use \( \delta(u) \) to control the distance from \( u \) to the set of extremal functions, i.e., the set

\[ E = \left\{ c e^{-a|x|^2} : c \in \mathbb{R}, a > 0 \right\}. \]

More precisely, we will prove the following result.

**Theorem 1.5.** Given \( n \geq 2 \). Then the following inequality

\[
\delta(u) \geq \frac{1}{384(n+2)} \inf \left\{ \frac{\int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 dx}{\int_{\mathbb{R}^n} |\nabla u|^2 dx} : \varphi \in E, \int_{\mathbb{R}^n} |\nabla u|^2 dx = \int_{\mathbb{R}^n} |\nabla \varphi|^2 dx \right\}, \tag{1.19}
\]

holds true for any \( u \in H^2_{0,-2}(\mathbb{R}^n) \setminus \{0\} \).

In recent years, there has been an enormous attention on establishing the stability version of the sharp inequalities in analysis and geometry, especially the Sobolev type inequality. The question on the stability estimate for the sharp Sobolev inequality was posed by Brézis and Lieb [6]. This question was affirmatively answered by Bianchi and Egnell [4] for functions in \( H^1(\mathbb{R}^n) \) by exploiting the Hilbert structure of this space. For the case of \( W^{1,p}(\mathbb{R}^n) \) with \( p \neq 2 \), the stability estimates for the Sobolev inequality were established in [14–16,26,27,29,38,45]. We also refer the readers to the papers [5,9,22–24,40].
for the stability version of the Gagliardo-Nirenberg inequality and logarithmic Sobolev inequality. In recent paper [37], McCurdy and Venkatraman exploit the approach of Bianchi and Egnell [4] to prove a stability version of the classical Heisenberg-Pauli-Weyl inequality (1.1)

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \int_{\mathbb{R}^n} |x|^2 |u|^2 dx - \frac{n^2}{4} \left( \int_{\mathbb{R}^n} |u|^2 dx \right)^2 \geq \tilde{C} \inf_{v \in E} \int_{\mathbb{R}^n} |u - v|^2 dx$$

for any $u \in H^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} |u|^2 dx = 1$ where $\tilde{C} > 0$ in an implicit constant depending on $n$. In [25], Fathi gave a new and simple proof of the above inequality by using Poincaré inequality for Gaussian measure. In [43], the first author establishes the stability version of the inequality (1.2) generalized the result of McCurdy and Venkatraman to a larger family of parameters.

Let us finish this introduction by some comment on the proofs of Theorem 1.5. First, we prove an improvement of (1.3) for functions that are orthogonal to radial functions. More precisely, we prove that the inequality (1.3) still holds true with an explicit larger constant for such functions. As an application, we show that $\delta(u)$ provides an upper bound for $\int_{\mathbb{R}^n} |\nabla u|^2 dx$ where $u_o(x) = (u(x) - u(-x))/2$ denotes the odd part of $u$. Hence, when $\delta(u)$ is small, the function $u$ is almost even. The second step is to establish a stability estimate of (1.3) for even functions (see Lemma 3.8 below). In order to prove this result, we shall prove an important identity

$$\int_{\mathbb{R}^n} |\Delta u|^2 - (n + 2) \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} |\nabla u|^2 |x|^2 dx = \int_{\mathbb{R}^n} \|\nabla^2 v - x \otimes \nabla v\|^2_{HS} e^{-|x|^2} dx$$

with $u = ve^{-|x|^2/2}$, where $\nabla^2 v$ denotes the Hessian matrix of $v$, $\nabla v \otimes x$ denotes matrix $(\partial_i v x_j)_{i,j=1}^n$ and $\|A\|_{HS}$ denotes the Hilbert-Schmidt norm of an $n \times n$ matrix $A$, i.e., $\|A\|_{HS} = \left( \text{Tr}(A^t A) \right)^{1/2}$ with $A^t$ being the transpose of $A$. Using spectral analysis of the Ornstein-Uhlenbeck type operator associated with the Gaussian measure and Hermite polynomials, we arrive at the following estimate for every function $v \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \|\nabla^2 v - x \otimes \nabla v\|^2_{HS} e^{-|x|^2} dx \geq \frac{4n}{n + 2} \left( \int_{\mathbb{R}^n} |\nabla ((v - c)e^{-|x|^2/2})|^2 dx \right)$$

where

$$c = \frac{\int_{\mathbb{R}^n} ve^{-|x|^2} dx}{\int_{\mathbb{R}^n} e^{-|x|^2} dx}.$$  

Combining the estimates for the odd and even functions, we obtain Theorem 1.5.

The rest of this paper is organized as follows. In Section §2, we prove the second order CKN inequalities given in Theorem 1.1, Theorem 1.3, and Theorem 1.4. Section §3 is devoted to prove the stability version of the second order uncertainty principle of Cazacu, Flynn and Lam given in Theorem 1.5.
2 The second order CKN inequalities: Proof of Theorems 1.1, 1.3, and 1.4

In this section, we provide the proof of the second order CKN inequalities in Theorems 1.1, 1.3 and 1.4. We also show that under the conditions of parameters in these theorems, the obtained inequalities are sharp and we exhibit a class of extremal functions. We begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. For any function \( u \in C_0^\infty(\mathbb{R}^n) \), we have

\[
\int_{\mathbb{R}^n} |\Delta u + \nabla u \cdot x| x^{2\alpha} |x|^{-2\alpha} dx
= \int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2\alpha} dx + |\nabla u \cdot x|^2 |x|^{2\alpha} dx + 2 \int_{\mathbb{R}^n} \Delta u \nabla u \cdot x dx.
\]

(2.1)

Using integration by parts, we have

\[
\int_{\mathbb{R}^n} \Delta u \nabla u \cdot x dx = -\int_{\mathbb{R}^n} \nabla^2 u (\nabla u) \cdot x dx - \int_{\mathbb{R}^n} |\nabla u|^2 dx
= -\frac{1}{2} \int_{\mathbb{R}^n} \nabla(|\nabla u|^2) \cdot x dx - \int_{\mathbb{R}^n} |\nabla u|^2 dx
= \frac{n-2}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx.
\]

Inserting this equality in (2.1), we get

\[
\int_{\mathbb{R}^n} |\Delta u + \nabla u \cdot x| x^{2\alpha} |x|^{-2\alpha} dx
= \int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2\alpha} dx + \int_{\mathbb{R}^n} \left| \nabla u \cdot \frac{x}{|x|} \right|^2 |x|^{2+2\alpha} dx
+ (n-2) \int_{\mathbb{R}^n} |\nabla u|^2 dx.
\]

(2.2)

We first consider the case \( \alpha \neq -1 \). Setting \( u = vU_0(x) \) with

\[
U_0(x) = \exp \left( -\frac{|x|^{2+2\alpha}}{2 + 2\alpha} \right),
\]

we have

\[
\nabla u = \nabla v U_0(x) + v(x) \nabla U_0(x), \quad \Delta u = \Delta v U_0 + 2 \nabla v \nabla U_0 + v \Delta U_0
\]

and

\[
\nabla U_0(x) = -x|x|^{2\alpha} e^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}}, \quad \Delta U_0(x) = -(n+2\alpha)|x|^{2\alpha} e^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}} + |x|^{2+4\alpha} e^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}}.
\]

Plugging this expression into \( \Delta u + \nabla u \cdot x |x|^{2\alpha} \) and using simple computations imply

\[
\Delta u + \nabla u \cdot x |x|^{2\alpha} = \left( \Delta v - \nabla v \cdot x |x|^{2\alpha} - (n+2\alpha)v |x|^{2\alpha} \right) e^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}}.
\]
Hence, it holds
\[
\int_{\mathbb{R}^n} |\Delta u + \nabla u \cdot x| |x|^{2\alpha} |x|^{-2\alpha} dx \\
= \int_{\mathbb{R}^n} |\Delta v - \nabla v \cdot x| |x|^{2\alpha} |x|^{-2\alpha} e^{-2|x|^{2+2\alpha}} dx + (n + 2\alpha)^2 \int_{\mathbb{R}^n} v^2 |x|^{2\alpha} e^{-2|x|^{2+2\alpha}} dx \\
- 2(n + 2\alpha) \int_{\mathbb{R}^n} (\Delta v - \nabla v \cdot x) ve^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} dx.
\]

We next compute the last integral in the right-hand side of the preceding equality. Notice that the function \( v = u e^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}} \) is not \( C^2 \) at origin in general. So we can not use integration by parts directly. To overcome this difficulty, we first notice that under the assumption \( u \in C_0^\infty(\mathbb{R}^n) \), \( n + 2\alpha > 0 \) and \( n + 2 + 4\alpha > 0 \), we have
\[
v \Delta ve^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} = u(\Delta u + 2\nabla u \cdot x| |x|^{2\alpha} + (n + 2\alpha)|x|^{2\alpha} u + |x|^{2+4\alpha} u) \in L^1(\mathbb{R}^n)
\]
and
\[
v \nabla v \cdot x| |x|^{2\alpha} e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} = u(\nabla u \cdot x| |x|^{2\alpha} - u|x|^{2+4\alpha}) \in L^1(\mathbb{R}^n).
\]

Therefore it holds
\[
\int_{\mathbb{R}^n} (\Delta v - \nabla v \cdot x| |x|^{2\alpha}) ve^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} dx = \lim_{\epsilon \to 0^+} \int_{B_\epsilon^c} (\Delta v - \nabla v \cdot x| |x|^{2\alpha}) ve^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} dx,
\]
where \( B_\epsilon^c = \{x \in \mathbb{R}^n : |x| \geq \epsilon\} \). Using integration by parts we have
\[
\int_{B_\epsilon^c} (\Delta v - \nabla v \cdot x| |x|^{2\alpha}) ve^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} dx \\
= \int_{B_\epsilon^c} \text{div}(ve^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}}) ve^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}} dx \\
= -\int_{B_\epsilon^c} \nabla v \cdot ve^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}} dx + \int_{\{|x| = \epsilon\}} \nabla v \cdot \frac{x}{|x|} ve^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} ds \\
= -\int_{B_\epsilon^c} |\nabla v|^2 e^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}} dx + \frac{1}{2} \int_{B_\epsilon^c} \nabla v \cdot x| |x|^{2\alpha} e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} dx + \int_{\{|x| = \epsilon\}} \nabla v \cdot \frac{x}{|x|} ve^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} ds \\
= -\int_{B_\epsilon^c} |\nabla v|^2 e^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}} dx + \frac{n + 2\alpha}{2} \int_{B_\epsilon^c} v^2 |x|^{2\alpha} e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} dx + \int_{B_\epsilon^c} v^2 |x|^{2+4\alpha} e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} dx \\
+ \epsilon^{1+2\alpha} e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} \int_{\{|x| = \epsilon\}} v^2 ds + \int_{\{|x| = \epsilon\}} \nabla v \cdot \frac{x}{|x|} ve^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} ds.
\]

Letting \( \epsilon \to 0^+ \) and using the assumptions \( n + 2\alpha > 0 \) and \( n + 2 + 4\alpha > 0 \), we obtain
\[
\lim_{\epsilon \to 0^+} \int_{B_\epsilon^c} (\Delta v - \nabla v \cdot x| |x|^{2\alpha}) ve^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} dx
\]
Consequently, we arrive at

\[
\int_{\mathbb{R}^n} |\Delta u + \nabla u \cdot x|^{2|a|} |x|^{-2a} dx
\]

\[
= \int_{\mathbb{R}^n} |\Delta v - \nabla v \cdot x|^{2|a|} |x|^{-2a} e^{-2|a|x^{2+2a}} dx + 2(n + 2\alpha) \int_{\mathbb{R}^n} |\nabla v|^{2} e^{-2|a|x^{2+2a}} dx
\]

\[
+ 2(n + 2\alpha)^2 \int_{\mathbb{R}^n} v^2 |x|^{2a} e^{-2|a|x^{2+2a}} dx - 2(n + 2\alpha) \int_{\mathbb{R}^n} v^2 |x|^{2+4a} e^{-2|a|x^{2+2a}} dx. \quad (2.3)
\]

Again by using integration by parts on \(B^\epsilon\) and letting \(\epsilon \to 0^+\), we have

\[
\int_{\mathbb{R}^n} |\nabla u|^2 dx = \int_{\mathbb{R}^n} |\nabla v|^2 U_0^2 dx + \int_{\mathbb{R}^n} v^2 |\nabla U_0|^2 dx + \int_{\mathbb{R}^n} \nabla v \cdot U_0 \nabla U_0 dx
\]

\[
= \int_{\mathbb{R}^n} |\nabla v|^2 U_0^2 dx - \int_{\mathbb{R}^n} v^2 U_0 \Delta U_0 dx
\]

\[
= \int_{\mathbb{R}^n} |\nabla v|^2 e^{-2|a|x^{2+2a}} dx + (n + 2\alpha) \int_{\mathbb{R}^n} v^2 |x|^{2a} e^{-2|a|x^{2+2a}} dx
\]

\[
- \int_{\mathbb{R}^n} v^2 |x|^{2+4a} e^{-2|a|x^{2+2a}} dx. \quad (2.4)
\]

Combining (2.2), (2.3) and (2.4), we obtain

\[
\int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2a} dx + \int_{\mathbb{R}^n} \left| \nabla u \cdot \frac{x}{|x|} \right|^2 |x|^{2+2a} dx + (n - 2) \int_{\mathbb{R}^n} |\nabla u|^2 dx
\]

\[
= \int_{\mathbb{R}^n} |\Delta v - \nabla v \cdot x|^{2|a|} |x|^{-2a} e^{-2|a|x^{2+2a}} dx + 2(n + 2\alpha) \int_{\mathbb{R}^n} |\nabla u|^2 dx
\]

which is equivalent to

\[
\int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2a} dx + \int_{\mathbb{R}^n} \left| \nabla u \cdot \frac{x}{|x|} \right|^2 |x|^{2+2a} dx
\]

\[
= \int_{\mathbb{R}^n} |\Delta v - \nabla v \cdot x|^{2|a|} |x|^{-2a} e^{-2|a|x^{2+2a}} dx + (n + 4\alpha + 2) \int_{\mathbb{R}^n} |\nabla u|^2 dx. \quad (2.5)
\]

By density argument, (2.5) still holds for any functions \(u \in H^2_{\alpha,-2-2a}(\mathbb{R}^n)\). It follows from (2.5) that

\[
\int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2a} dx + \int_{\mathbb{R}^n} \left| \nabla u \cdot \frac{x}{|x|} \right|^2 |x|^{2+2a} dx \geq (n + 4\alpha + 2) \int_{\mathbb{R}^n} |\nabla u|^2 dx
\]
for any \( u \in H^2_{\alpha -2, \alpha -2}(\mathbb{R}^n) \). Replacing \( u \) by function \( u_\lambda(x) = \lambda^{\frac{\alpha -2}{2}} u(\lambda x) \) with \( \lambda > 0 \), we get
\[
\lambda^{2+2\alpha} \int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2\alpha} dx + \lambda^{-2-2\alpha} \int_{\mathbb{R}^n} \left| \nabla u \cdot \frac{x}{|x|} \right|^2 |x|^{2+2\alpha} dx \geq (n + 4\alpha + 2) \int_{\mathbb{R}^n} |\nabla u|^2 dx.
\]
The left-hand side of the preceding inequality is minimized by
\[
\lambda_0 = \left( \frac{\int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2\alpha} dx}{\int_{\mathbb{R}^n} \left| \nabla u \cdot \frac{x}{|x|} \right|^2 |x|^{2+2\alpha} dx} \right)^{\frac{1}{1+4\alpha}}.
\]

Hence, by taking \( \lambda = \lambda_0 \), we obtain (1.4) for \( \alpha \neq -1 \). The case \( \alpha = -1 \) follows by letting \( \alpha \to -1 \).

It remains to check the sharpness of (1.4) when \( 1 + \alpha > 0 \). Taking \( u = U_0 \) implies \( \nu \equiv 1 \). Hence, (2.5) becomes
\[
\int_{\mathbb{R}^n} |\Delta U_0|^2 |x|^{-2\alpha} dx + \int_{\mathbb{R}^n} \left| \nabla U_0 \cdot \frac{x}{|x|} \right|^2 |x|^{2+2\alpha} dx = (n + 4\alpha + 2) \int_{\mathbb{R}^n} |\nabla U_0|^2 dx.
\]
Furthermore, by the direct computations and integration by parts, we have
\[
\int_{\mathbb{R}^n} |\Delta U_0|^2 |x|^{-2\alpha} dx = (n + 2\alpha)^2 \int_{\mathbb{R}^n} |x|^{2\alpha} U_0^2 dx - 2(n + 2\alpha) \int_{\mathbb{R}^n} |x|^{2+4\alpha} U_0^2 dx
\]
\[
+ \int_{\mathbb{R}^n} |x|^{4+6\alpha} U_0^2 dx
\]
\[
= (n + 2\alpha)^2 \int_{\mathbb{R}^n} |x|^{2\alpha} U_0^2 dx + (n + 2\alpha) \int_{\mathbb{R}^n} \nabla U_0^2 \cdot x |x|^{-2\alpha} dx
\]
\[
+ \int_{\mathbb{R}^n} \left| \nabla U_0 \cdot \frac{x}{|x|} \right|^2 |x|^{2+2\alpha} dx
\]
\[
= \int_{\mathbb{R}^n} \nabla U_0 \cdot \frac{x}{|x|}^2 |x|^{2+2\alpha} dx.
\]
This implies that the equality occurs in (1.4) with \( u = U_0 \). Hence, the inequality (1.4) is sharp and \( U_0 \) is an extremal function. This completes the proof of Theorem 1.1.

We next prove Theorem 1.3.

Proof of Theorem 1.3. By density argument, it is enough to prove (1.8) for radial functions \( u \in C^\infty_0(\mathbb{R}^n) \). Let \( u \in C^\infty_0(\mathbb{R}^n) \) be a radial function, by using integration by parts, we have
\[
\int_{\mathbb{R}^n} \frac{|\nabla u|^t}{|x|^{t\gamma}} dx = \left| S^{n-1} \right| \int_0^\infty |u'|^t r^{n-t\gamma-1} dr
\]
\[
= \frac{1}{n-t\gamma} \left| S^{n-1} \right| \int_0^\infty |u'|^t (r^{n-t\gamma})' dr
\]

11
Furthermore, using integration by parts, we have
\[ -\frac{t}{n-t\gamma}\left|S^{n-1}\right| \int_0^\infty |u'|^{t-1}u''r^{n-t\gamma}dr \]
\[ = -\frac{t}{n-t\gamma}\left|S^{n-1}\right| \int_0^\infty |u'|^{t-2}u''u'' + \frac{n-1}{r}u' - \frac{n+2\alpha}{r}u'\right) r^{n-t\gamma}dr \]
\[ - \frac{t(1+2\alpha)}{n-t\gamma}\left|S^{n-1}\right| \int_0^\infty |u'|^{n-t\gamma-1}dr. \]

This gives
\[ n + t(1+2\alpha - \gamma) \int_{\mathbb{R}^n} \frac{|
abla u|^t}{|x|^{t\gamma}} dx \]
\[ = -\left|S^{n-1}\right| \int_0^\infty |u'|^{t-2}u''\left(u'' + \frac{n-1}{r}u' - \frac{n+2\alpha}{r}u'\right) r^{n-2\gamma}dr \]
\[ = -\left|S^{n-1}\right| \int_0^\infty |u'|^{t-2}u'r^{\frac{\gamma}{2}}\left(u'' + \frac{n-1}{r}u' - \frac{n+2\alpha}{r}u'\right) r^{-\alpha}r^{n-1}dr; \quad (2.6) \]

here we use (1.7). By density argument, (2.6) still holds for radial function \( u \in H^{2,\alpha,\beta}(\mathbb{R}^n) \).

Using Hölder inequality, we arrive at
\[ \left|\frac{n + t(1+2\alpha - \gamma)}{t} \int_{\mathbb{R}^n} \frac{|
abla u|^t}{|x|^{t\gamma}} dx \right| \leq \left(\left|S^{n-1}\right| \int_0^\infty \left(u'' + \frac{n-1}{r}u' - \frac{n+2\alpha}{r}u'\right)^2 r^{n-2\alpha-1}dr \right)^{\frac{1}{2}} \]
\[ \times \left(\left|S^{n-1}\right| \int_0^\infty |u'|^{2(t-1)}r^{n-\beta-1}dr \right)^{\frac{1}{2}} \]
\[ = \left(\left|S^{n-1}\right| \int_0^\infty \left(u'' + \frac{n-1}{r}u' - \frac{n+2\alpha}{r}u'\right)^2 r^{n-2\alpha-1}dr \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \frac{|
abla u|^{2(t-1)}}{|x|^\beta} dx \right)^{\frac{1}{2}}. \quad (2.7) \]

Furthermore, using integration by parts, we have
\[ \int_0^\infty \left(u'' + \frac{n-1}{r}u' - \frac{n+2\alpha}{r}u'\right)^2 r^{n-2\alpha-1}dr \]
\[ = \int_0^\infty (\Delta u(r))^2 r^{n-2\alpha-1}dr + (n+2\alpha)^2 \int_0^\infty (u')^2 r^{n-2\alpha-3}dr \]
\[ - 2(n+2\alpha) \int_0^\infty \left(u'' + \frac{n-1}{r}u'\right) u' r^{n-2\alpha-1}dr \]
\[ = \int_0^\infty (\Delta u(r))^2 r^{n-2\alpha-1}dr - (n+2\alpha)(n-2\alpha-2) \int_0^\infty (u')^2 r^{n-2\alpha-3}dr \]
\[ - (n+2\alpha) \int_0^\infty ((u')^2)' r^{n-2\alpha-2}dr \]
\[ = \int_0^\infty (\Delta u(r))^2 r^{n-2\alpha-1}dr. \]
Consequently, it holds

\[
|S^{n-1}| \int_0^\infty \left( u'' + \frac{n-1}{r} u' - \frac{n+2\alpha}{r} u \right)^2 r^{n-2\alpha-1} dr = \int_{\mathbb{R}^n} (\Delta u)^2 |x|^{2\alpha} dx.
\]

Inserting this equality into (2.7), we obtain (1.8).

Suppose that a nonzero radial function \( u \in H^{2,\alpha,\beta}_{a,b}(\mathbb{R}^n) \) is an extremal function for (1.8). Notice that under the conditions (1.9) and (1.10), we have

\[
n + t(1 + 2\alpha - \gamma) = n + 2\alpha + (1 + 2\alpha)(t - 2) + 1 + \alpha - \frac{\beta}{2} > 0.
\]

Hence, the equation holds when applying the Hölder inequality to (2.6) if and only if

\[
u'' + \frac{n-1}{r} u' - \frac{n+2\alpha}{r} u' = -\lambda |u'|^{t-2} u r^{\alpha - \frac{\beta}{2}}
\]

for some \( \lambda > 0 \), which is equivalent to

\[
u'' - \frac{1 + 2\alpha}{r} u' + \lambda |u'|^{t-2} u r^{\alpha - \frac{\beta}{2}} = 0.
\]

Denote \( u' = r^{1+2\alpha} w \), then \( w \) satisfies the equation

\[
w' + \lambda r^{(1+2\alpha)(t-2)+\alpha-\frac{\beta}{2}} |w|^{t-2} w = 0.
\]

We have following two cases:

**Case 1: \( t = 2 \).** In this case, we have \( w' + \lambda r^{\alpha-\frac{\beta}{2}} w = 0 \) which implies \( w(r) = c \exp(-\lambda r^{1+\alpha-\frac{\beta}{2}}/(1 + \alpha - \beta/2)) \) for some \( c \in \mathbb{R} \). Hence,

\[
u'(r) = cr^{1+2\alpha} \exp \left( -\lambda \frac{r^{1+\alpha-\beta/2}}{1 + \alpha - \beta/2} \right)
\]

and

\[
u(x) = c \int_{|x|}^\infty r^{1+2\alpha} \exp \left( -\lambda \frac{r^{1+\alpha-\beta/2}}{1 + \alpha - \beta/2} \right) dr.
\]

**Case 2: \( t > 2 \).** In this case, we have \((|w|^{2-t})' = \lambda(t-2)\lambda r^{(1+2\alpha)(t-2)+\alpha-\frac{\beta}{2}} \) which implies

\[
|w(r)| = \left( c + \lambda(t-2) \frac{r^{(1+2\alpha)(t-2)+1+\alpha-\frac{\beta}{2}}}{(1 + 2\alpha)(t - 2) + 1 + \alpha - \frac{\beta}{2}} \right)^{\frac{1}{t-2}},
\]

for some \( c > 0 \), here we use (1.9). From this expression, up to a multiplicative constant 1 or \(-1\), we can assume that

\[
w(r) = \left( c + \lambda(t-2) \frac{r^{(1+2\alpha)(t-2)+1+\alpha-\frac{\beta}{2}}}{(1 + 2\alpha)(t - 2) + 1 + \alpha - \frac{\beta}{2}} \right)^{\frac{1}{t-2}}.
\]
Therefore, the extremal function has the form
\[ u(x) = \int_{|x|}^{\infty} r^{1+2\alpha} \left( c + \lambda(t-2) \right) \frac{r^{(1+2\alpha)(t-2)+1+\alpha - \frac{\alpha}{2}}}{(1+2\alpha)(t-2)+1+\alpha - \frac{\alpha}{2}} \frac{1}{r} dr \]
as desired.

We finish this section by proving Theorem 1.4. Our proof follows the approach of Cazacu, Flynn and Lam to prove (1.3) by using the decomposition of \( u \) into spherical harmonics. It is well known that the technique of decomposing a function into spherical harmonics is a very useful method to prove the Hardy-Rellich type inequalities (see [12, 30, 41, 47, 48] and references therein). Let us recall some facts on this method which we borrow from [47, Section 2.2]. A function \( f \in C_0^\infty(\mathbb{R}^n) \) can be decomposed into spherical harmonics as
\[ f(x) = \sum_{k=0}^{\infty} f_k(r)\phi_k(\omega), \quad x = rw, |x| = r, \omega \in S^{n-1} \quad (2.8) \]
where \( \phi_k \) are the orthogonal eigenfunctions of the Laplace-Beltrami operator on \( S^{n-1} \) with the corresponding eigenvalue \( c_k = k(n+k-2) \). Notice that \( \phi_0 \equiv 1 \), \( \phi_k \) is restriction of the \( k \) order homogeneous, harmonic polynomials in \( \mathbb{R}^n \) to \( S^{n-1} \) and \( f_k(r) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f \phi_k ds \) with \( k \geq 0 \). Hence \( f_k \in C_0^\infty(\mathbb{R}^n) \) with \( f_k(r) = O(r^k) \) as \( r \to 0 \).

We have
\[ \Delta f(x) = \sum_{k=0}^{\infty} \left( f_k''(r) + \frac{n-1}{r} f_k'(r) - c_k f_k(r) \right) \phi_k(\omega) \]
and
\[ |\nabla f(x)|^2 = \sum_{k=0}^{\infty} \left( |\nabla f_k|^2 \phi_k^2 + \frac{f_k^2}{r^2} |\nabla S^{n-1} \phi_k|^2 \right). \]
Following Cazacu, Flynn and Lam, we set \( f_k(r) = r^k g_k(r) \). By the simple computations, we have
\[
\begin{align*}
f_k''(r) + \frac{n-1}{r} f_k'(r) - c_k \frac{f_k(r)}{r^2}
&= r^k g_k'' + 2kr^{k-1}g_k' + k(k-1)r^{k-2}g_k + (n-1)r^{k-1}g_k' + k(n-1)r^{k-2}g_k - c_k r^{k-2}g_k \\
&= r^k g_k'' + (n+2k-1)r^{k-1}g_k'
\end{align*}
\]
and
\[ |\nabla f_k|^2 = k^2 r^{2(k-1)}g_k^2 + r^{2k}(g_k')^2 + 2kr^{2k-1}g_k g_k'. \]
Therefore, using integration by parts and the definitions (1.13), (1.14) and (1.15), we get
\[
\int_{\mathbb{R}^n} \frac{|\Delta f|^2}{|x|^{2\alpha}} dx = \sum_{k=0}^{\infty} \int_0^{\infty} \left( f_k''(r) + \frac{n-1}{r} f_k'(r) - c_k \frac{f_k(r)}{r^2} \right)^2 r^{n-2\alpha-1} dr
\]
\[
= \sum_{k=0}^{\infty} \int_{0}^{\infty} \left( g''_k(r) + \frac{n+2k-1}{r} g'_k(r) \right)^2 r^{n-2\alpha+2k-1} dr
\]
\[
= \sum_{k=0}^{\infty} \left( \int_{0}^{\infty} (g''_k)^2 r^{n+2k-2\alpha-1} dr + (n+2k-1)^2 \int_{0}^{\infty} (g'_k)^2 r^{n+2k-2\alpha-3} dr \right.
\]
\[
+ 2(n+2k-1) \int_{0}^{\infty} g_k g'_k r^{n+2k-2\alpha-2} dr \right)
\]
\[
= \sum_{k=0}^{\infty} \left( \int_{0}^{\infty} (g''_k)^2 r^{n+2k-2\alpha-1} dr
\]
\[
+ (1+2\alpha)(n+2k-1) \int_{0}^{\infty} (g'_k)^2 r^{n+2k-2\alpha-3} dr \right)
\]
\[
= \sum_{k=0}^{\infty} A_{n,\alpha,k}(g_k),
\]

(2.9)

\[
\int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{|x|^{2\gamma}} \, dx = \sum_{k=0}^{\infty} \left( \int_{0}^{\infty} (g'_k)^2 r^{n+2k-2\gamma-1} dr + 2\gamma k \int_{0}^{\infty} g_k^2 r^{n+2k-2\gamma-3} dr \right)
\]
\[
= \sum_{k=0}^{\infty} B_{n,\beta,k}(g_k)
\]

(2.10)

and

\[
\int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{|x|^{2\gamma}} \, dx = \sum_{k=0}^{\infty} \left( \int_{0}^{\infty} (g'_k)^2 r^{n+2k-2\gamma-1} dr + 2\gamma k \int_{0}^{\infty} g_k^2 r^{n+2k-2\gamma-3} dr \right)
\]
\[
= \sum_{k=0}^{\infty} C_{n,\gamma,k}(g_k).
\]

(2.11)

With these preparations, we are ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** It follows from (2.9), (2.10) and (2.11) that

\[
\left( \int_{\mathbb{R}^n} \frac{\|\Delta f\|^2}{|x|^{2\alpha}} \, dx \right) \left( \int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{|x|^{\beta}} \, dx \right) = |S^{n-1}|^2 \left( \sum_{k=0}^{\infty} A_{n,\alpha,k}(g_k) \right) \left( \sum_{k=0}^{\infty} B_{n,\beta,k}(g_k) \right)
\]

15
\[ \int_{\mathbb{R}^n} |\nabla f|^2 \frac{1}{|x|^{2\gamma}} \, dx = |S^{n-1}| \sum_{k=0}^{\infty} C_{n,\gamma,k}(g_k). \]

By Minkowski inequality and the definition (1.16) of \( A_k(n, \alpha, \beta) \), we have
\[
\left( \sum_{k=0}^{\infty} A_{n,\alpha,k}(g_k) \right) \left( \sum_{k=0}^{\infty} B_{n,\beta,k}(g_k) \right) \geq \left( \sum_{k=0}^{\infty} \sqrt{A_{n,\alpha,k}(g_k) B_{n,\beta,k}(g_k)} \right)^2 \geq \inf_{k \in \mathbb{N}} A_k(n, \alpha, \beta) \left( \sum_{k=0}^{\infty} C_{n,\gamma,k}(g_k) \right)^2,
\]
which yields
\[
\left( \int_{\mathbb{R}^n} |\Delta f|^2 \frac{1}{|x|^{2\alpha}} \, dx \right) \left( \int_{\mathbb{R}^n} |\nabla f|^2 \frac{1}{|x|^{2\beta}} \, dx \right) \geq \inf_{k \in \mathbb{N}} A_k(n, \alpha, \beta) \left( \int_{\mathbb{R}^n} |\nabla f|^2 \frac{1}{|x|^{2\gamma}} \, dx \right)^2.
\]

Furthermore, by using one dimensional Hardy inequality, we have
\[
\int_0^\infty (g')^2 r^{n+2k-\beta-1} \, dr \geq \left( \frac{n+2k-2-\beta}{2} \right)^2 \int_0^\infty g^2 r^{n+2k-\beta-3} \, dr,
\]
and
\[
\int_0^\infty (g')^2 r^{n+2k-2\gamma-1} \, dr \geq \left( \frac{n+2k-2\gamma-2}{2} \right)^2 \int_0^\infty g^2 r^{n+2k-2\gamma-3} \, dr
\]
which imply
\[
B_{n,\beta,k}(g) \geq \left( 1 + \min \left\{ 0, \frac{4\beta k}{(n+2k-\beta-2)^2} \right\} \right) \int_0^\infty (g')^2 r^{n+2k-\beta-1} \, dr
\]
and
\[
C_{n,\gamma,k}(g) \leq \left( 1 + \max \left\{ 0, \frac{8\gamma k}{(n+2k-2\gamma-2)^2} \right\} \right) \int_0^\infty (g')^2 r^{n+2k-2\gamma-1} \, dr.
\]
Notice that
\[
1 + \min \left\{ 0, \frac{4\beta k}{(n+2k-\beta-2)^2} \right\} > 0
\]
and
\[
A_{n,\alpha,k}(g) = \int_0^\infty \left( g''(r) + \frac{n+2k-1}{r} g'(r) - \frac{4\beta k}{(n+2k-\beta-2)^2} \right) r^{n+2k-\alpha-1} \, dr.
\]
Hence, applying the inequality (1.8) for radial functions in dimension \( n+2k \) we get
\[
A_k(n, \alpha, \beta) \geq \frac{1 + \min \left\{ 0, \frac{4\beta k}{(n+2k-\beta-2)^2} \right\}}{1 + \max \left\{ 0, \frac{8\gamma k}{(n+2k-2\gamma-2)^2} \right\}} \left( \frac{n+2k+4\alpha-2\gamma+2}{2} \right)^2. \tag{2.12}
\]
This gives
\[ \lim_{k \to \infty} A_k(n, \alpha, \beta) = \infty \]
and hence
\[ \inf_k A_k(n, \alpha, \beta) = \min_k A_k(n, \alpha, \beta). \]
Consequently, we get
\[ (\int_{\mathbb{R}^n} |\Delta f|^2 \, dx) (\int_{\mathbb{R}^n} |\nabla f|^2 \, dx) \geq \min_k A_k(n, \alpha, \beta) \left( \int_{\mathbb{R}^n} |\nabla f|^2 \, dx \right)^2 \]
as wanted (1.17).

It is easy to see that the constant \( \min_k A_k(n, \alpha, \beta) \) is sharp in (1.17). Indeed, there exists \( k_0 \) and a sequence of function \( h_k \) such that \( \min_k A_k(n, \alpha, \beta) = A_{k_0}(n, \alpha, \beta) \) and
\[ \lim_{k \to \infty} \frac{A_{n,\alpha,k_0}(h_k) B_{n,\beta,k_0}(h_k)}{C_{n,\gamma,k_0}(h_k)^2} = A_{k_0}(n, \alpha, \beta). \]

Testing (1.17) by functions \( h_k \varphi_{k_0} \) implies the sharpness of \( \min_k A_k(n, \alpha, \beta) \).

Finally, the estimate (1.18) immediately follows from (2.12).

\[ \square \]

3 The proof of Theorem 1.5

In this section, we prove the stability version of the second order uncertainty principle (1.3) due to Cazacu, Flynn and Lam given in Theorem 1.5. We divided the proof into two parts. In the first part, we show that \( \delta(u) \) gives an upper bound for the odd part of the function \( u \), hereafter for a function \( u \) we call the function \( u_o(x) = (u(x) - u(-x))/2 \) as its odd part. This task is done by establishing an improvement of (1.3) on the odd functions (even more general, on functions that are orthogonal to all radial functions). Consequently, when \( \delta(u) \) is small, \( u \) is almost an even function. In the second part, we prove the stability estimate (1.19) for even functions. This is done by using spectral analysis of the Ornstein-Uhlenbeck operators on the Gaussian space. Combining these estimates, we prove theorem 1.5.

3.1 Estimate for the odd functions

We start this subsection by proving an improvement of (1.3) for compactly supported smooth functions that are orthogonal to all radial functions. A function \( u \) is called to be orthogonal to all radial functions if
\[ \int_{\mathbb{R}^n} u(x) \varphi(x) \, dx = 0 \]
for any radial function \( \varphi \). More precisely, we will prove the following result.
Theorem 3.1. Given $n \geq 2$. For any function $u \in C_0^\infty(\mathbb{R}^n)$ that is orthogonal to all radial functions, it holds

\[
\left( \int_{\mathbb{R}^n} |\Delta u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} |x|^2 |\nabla u|^2 dx \right)^{\frac{1}{2}} \geq C_1(n) \int_{\mathbb{R}^n} |\nabla u|^2 dx
\]

(3.1)

where

\[
C_1(n) = \frac{n + 4}{2} \left( 1 - \frac{8}{(n + 2)^2} \right)^{\frac{1}{2}}.
\]

Proof. We follow the proof of (1.3) given in [13] by using the spherical harmonic decomposition (2.8) (see also the proof of Theorem 1.4). Since $u$ is orthogonal to all radial function, then

\[
u(x) = \sum_{k=1}^\infty u_k(r) \phi_k(\omega), \quad r = |x|, \ x = r \omega.
\]

Let $u_k(r) = r^k v_k(r)$, then we have from (2.9), (2.10) and (2.11) that

\[
\int_{\mathbb{R}^n} (\Delta u)^2 dx = \sum_{k=1}^\infty A_{n,0,k}(v_k),
\]

\[
\int_{\mathbb{R}^n} |x|^2 |\nabla u|^2 dx = \sum_{k=1}^\infty B_{n,-2,k}(v_k),
\]

and

\[
\int_{\mathbb{R}^n} |\nabla u|^2 dx = \sum_{k=1}^\infty C_{n,0,k}(v_k).
\]

It was proved in [13, Proof of Theorem 2.1, Case $N \geq 2$] that

\[
A_{n,0,k}(v_k)B_{n,-2,k}(v_k) = \left( \int_0^\infty r^{n+2k-1} (v''_k)(r) \right)^2 + (n + 2k - 1) \int_0^\infty r^{n+2k-3} (v'_k)(r) \right) + \left( \int_0^\infty r^{n+2k+1} (v'_k)(r) \right)^2 - 2k \int_0^\infty r^{n+2k-1} v_k(r) \right) + \left( 1 - \frac{8k}{(n + 2k)^2} \right) \frac{(n + 2k + 2)^2}{4} \int_0^\infty r^{n+2k-1} (v'_k)(r) \right) + \left( 1 - \frac{8k}{(n + 2k)^2} \right) \frac{(n + 2k + 2)^2}{4} C_{n,0,k}(v_k).
\]

Using Cauchy-Schwarz inequality, we have

\[
\left( \int_{\mathbb{R}^n} |\Delta u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} |x|^2 |\nabla u|^2 dx \right)^{\frac{1}{2}} \geq \sum_{k=1}^\infty \sqrt{A_{n,0,k}(v_k)B_{n,-2,k}(v_k)}
\]

18
Using the Cauchy-Schwarz inequality, we have
\[
\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq \inf_{k \geq 1} \left( 1 - \frac{8k}{(n+2k)^2} \right) \frac{n+2k+2}{2} \sum_{k=1}^{\infty} C_{n,k}(v_k)
\]
\[
= \inf_{k \geq 1} \left( 1 - \frac{8k}{(n+2k)^2} \right) \frac{n+2k+2}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx.
\]
Since \( n \geq 2 \), it was show in the proof of Lemma 3.3 in [13] that
\[
\inf_{k \geq 1} \left( 1 - \frac{8k}{(n+2k)^2} \right) \frac{n+2k+2}{2} = \frac{n+4}{2} \left( 1 - \frac{8}{(n+2)^2} \right)^{\frac{1}{2}}.
\]
This completes the proof of this theorem. \( \square \)

It is easy to see that \( \sqrt{1 - 2t^2} \geq 1 - (2 - \sqrt{2})t \) for any \( 0 \leq t \leq \frac{1}{2} \), and hence
\[
(1+t)\sqrt{1 - 2t^2} \geq (1+t)(1 - (2 - \sqrt{2})t) = 1 + t(\sqrt{2} - 1 - (2 - \sqrt{2})t) \geq 1 + \frac{t}{10}
\]
for any \( 0 \leq t \leq \frac{1}{2} \). Using this estimate, we get
\[
C_1(n) = \frac{n+4}{2} \left( 1 - \frac{8}{(n+2)^2} \right)^{\frac{1}{2}} \geq \frac{n+2}{2} + \frac{1}{10}. \tag{3.2}
\]
Since \( C_1(n) > (n+2)/2 \), then the inequality (3.1) provides an improvement of (1.3) for functions that are orthogonal to radial functions.

Let \( u \in C_0^\infty(\mathbb{R}^n) \) be a function such that \( \int_{\mathbb{R}^n} |\nabla u|^2 \, dx = 1 \), we define \( u_o(x) = (u(x) - u(-x))/2 \) and \( u_e(x) = (u(x) + u(-x))/2 \) the odd part and even part of \( u \) respectively. Note that \( u_o \) is an odd function, \( u_e \) is an even function and \( u = u_o + u_e \). It is easy to check that
\[
\int_{\mathbb{R}^n} (\Delta u)^2 \, dx = \int_{\mathbb{R}^n} (\Delta u_o)^2 \, dx + \int_{\mathbb{R}^n} (\Delta u_e)^2 \, dx,
\]
\[
\int_{\mathbb{R}^n} |\nabla u|^2 |x|^2 \, dx = \int_{\mathbb{R}^n} |\nabla u_o|^2 |x|^2 \, dx + \int_{\mathbb{R}^n} |\nabla u_e|^2 |x|^2 \, dx
\]
and
\[
1 = \int_{\mathbb{R}^n} |\nabla u|^2 \, dx = \int_{\mathbb{R}^n} |\nabla u_o|^2 \, dx + \int_{\mathbb{R}^n} |\nabla u_e|^2 \, dx.
\]
Using the Cauchy-Schwarz inequality, we have
\[
\delta(u) := \frac{2}{n+2} \left( \left( \int_{\mathbb{R}^n} |\Delta u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} |x|^2 |\nabla u|^2 \, dx \right)^{\frac{1}{2}} - \frac{n+2}{2} \right)
\]
\[
= \frac{2}{n+2} \left( \left( \int_{\mathbb{R}^n} |\Delta u_o|^2 \, dx + \int_{\mathbb{R}^n} |\Delta u_e|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} |x|^2 |\nabla u_o|^2 \, dx + \int_{\mathbb{R}^n} |x|^2 |\nabla u_e|^2 \, dx \right)^{\frac{1}{2}}
\]
\[
- \frac{n+2}{2} \int_{\mathbb{R}^n} |\nabla u_o|^2 \, dx - \frac{n+2}{2} \int_{\mathbb{R}^n} |\nabla u_e|^2 \, dx \right). \tag{3.3}
\]
\[ \geq \frac{2}{n+2} \left( \left( \int_{\mathbb{R}^n} |\Delta u_o|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} |x|^2 |\nabla u_o|^2 dx \right)^{\frac{1}{2}} \right. \]
\[ \left. \quad + \left( \int_{\mathbb{R}^n} |\Delta u_e|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} |x|^2 |\nabla u_e|^2 dx \right)^{\frac{1}{2}} \right) \]
\[ \quad - \frac{n+2}{2} \int_{\mathbb{R}^n} |\nabla u_o|^2 dx - \frac{n+2}{2} \int_{\mathbb{R}^n} |\nabla u_e|^2 dx \right) \]
\[ \geq \frac{1}{5(n+2)} \int_{\mathbb{R}^n} |\nabla u_o|^2 dx + \delta(u) \int_{\mathbb{R}^n} |\nabla u_e|^2 dx. \quad (3.3) \]

Here we used (3.2) for the last inequality. As consequences of (3.3), we have the following results

**Theorem 3.2.** Given \( n \geq 2 \). For any function \( u \in C^\infty_0(\mathbb{R}^n) \) with \( \int_{\mathbb{R}^n} |\nabla u|^2 dx = 1 \) and \( \delta(u) \leq \frac{1}{10(n+2)} \), it holds
\[ \int_{\mathbb{R}^n} |\nabla u_o|^2 dx \leq 5(n+2) \delta(u), \quad (3.4) \]
and
\[ \delta(u_e) \leq 2 \delta(u). \quad (3.5) \]

**Proof.** Since \( \delta(u_e) \geq 0 \), we have from (3.3)
\[ \delta(u) \geq \frac{1}{5(n+2)} \int_{\mathbb{R}^n} |\nabla u_o|^2 dx. \]
This proves (3.4). We have
\[ \int_{\mathbb{R}^n} |\nabla u_e|^2 dx = 1 - \int_{\mathbb{R}^n} |\nabla u_o|^2 dx \geq 1 - 5(n+2) \delta(u) \geq \frac{1}{2} \]
here we use the assumption \( \delta(u) \leq \frac{1}{10(n+2)} \). Using again (3.3), we get
\[ \delta(u) \geq \frac{1}{2} \delta(u_e). \]
This proves (3.5). \( \square \)

### 3.2 Estimate for the even functions

We start this subsection by establishing an identity that provides an alternative proof of the second order uncertainty principle of Cazacu, Flynn and Lam.

**Lemma 3.3.** For any function \( u \in C^\infty_0(\mathbb{R}^n) \), it holds
\[ \int_{\mathbb{R}^n} |\Delta u|^2 dx + \int_{\mathbb{R}^n} |x|^2 u^2 dx = (n+2) \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} \|\nabla^2 v - x \otimes \nabla v\|_{HS}^2 e^{-|x|^2} dx, \quad (3.6) \]
where \( u = ve^{-\frac{|x|^2}{2}} \).
Proof. Firstly, by using integration by parts we can easily check that
\[
\int_{\mathbb{R}^n} \|
abla^2 u + \nabla u \otimes x\|_{HS}^2 \, dx = \int_{\mathbb{R}^n} \|\nabla^2 u\|_{HS}^2 \, dx + 2 \int_{\mathbb{R}^n} \nabla^2 u(\nabla u) \cdot x \, dx + \int_{\mathbb{R}^n} |\nabla u|^2 |x|^2 \, dx
\]
which yields
\[
\int_{\mathbb{R}^n} \|\nabla^2 u + \nabla u \otimes x\|_{HS}^2 \, dx = \int_{\mathbb{R}^n} |\Delta u|^2 + \int_{\mathbb{R}^n} \nabla(|\nabla u|^2) \cdot x \, dx + \int_{\mathbb{R}^n} |\nabla u|^2 |x|^2 \, dx
\]
and
\[
= \int_{\mathbb{R}^n} |\Delta u|^2 - n \int_{\mathbb{R}^n} |\nabla u|^2 \, dx + \int_{\mathbb{R}^n} |\nabla u|^2 |x|^2 \, dx.
\] \tag{3.7}

Next, we set \( u = ve^{-|x|^2/2} \). Then, \( v \in C_0^\infty(\mathbb{R}^n) \) and we have
\[
\nabla^2 u = [\nabla^2 v - \nabla v \otimes x - x \otimes \nabla v - v(I_n - x \otimes x)]e^{-|x|^2/2}
\]
and
\[
\nabla u \otimes x = (\nabla v \otimes x - x \otimes x v)e^{-|x|^2/2}.
\]
Consequently, it holds
\[
\nabla^2 u + \nabla u \otimes x = [\nabla^2 v - x \otimes \nabla v - vI_n]e^{-|x|^2/2}
\]
which yields
\[
\|\nabla^2 u + \nabla u \otimes x\|^2_{HS} = \|\nabla^2 v - x \otimes \nabla v\|^2_{HS} e^{-|x|^2} - 2(\Delta v - \nabla v \cdot x)v e^{-|x|^2} + nv^2 e^{-|x|^2} \quad \text{. (3.8)}
\]
By using again integration by parts, we get
\[
\int_{\mathbb{R}^n} (\Delta v - \nabla v \cdot x)v e^{-|x|^2} \, dx = -\int_{\mathbb{R}^n} |\nabla v|^2 e^{-|x|^2} \, dx + \int_{\mathbb{R}^n} \nabla v \cdot x v e^{-|x|^2} \, dx
\]
which yields
\[
= -\int_{\mathbb{R}^n} |\nabla v|^2 e^{-|x|^2} \, dx + \frac{1}{2} \int_{\mathbb{R}^n} \nabla v^2 \cdot x e^{-|x|^2} \, dx
\]
and
\[
= -\int_{\mathbb{R}^n} |\nabla v|^2 e^{-|x|^2} \, dx - \frac{n}{2} \int_{\mathbb{R}^n} v^2 e^{-|x|^2} \, dx + \int_{\mathbb{R}^n} v^2 |x|^2 e^{-|x|^2} \, dx.
\]
Integrating both sides of (3.8) in \( \mathbb{R}^n \) and using the preceding equality, we arrive at
\[
\int_{\mathbb{R}^n} \|\nabla^2 u + \nabla u \otimes x\|^2_{HS} \, dx = \int_{\mathbb{R}^n} \|\nabla^2 v - x \otimes \nabla v\|^2_{HS} e^{-|x|^2} \, dx + 2 \int_{\mathbb{R}^n} |\nabla v|^2 e^{-|x|^2} \, dx
\]
which yields
\[
+ 2n \int_{\mathbb{R}^n} v^2 e^{-|x|^2} \, dx - 2 \int_{\mathbb{R}^n} v^2 |x|^2 e^{-|x|^2} \, dx. \quad \text{(3.9)}
\]
Finally, by using integration by parts again, we have
\[
\int_{\mathbb{R}^n} |\nabla u|^2 \, dx = \int_{\mathbb{R}^n} |\nabla v - x v|^2 e^{-|x|^2} \, dx
\]
which yields
\[
= \int_{\mathbb{R}^n} |\nabla v|^2 e^{-|x|^2} \, dx - 2 \int_{\mathbb{R}^n} \nabla v \cdot x v e^{-|x|^2} \, dx + \int_{\mathbb{R}^n} |x|^2 v^2 e^{-|x|^2} \, dx.
\]
\[
\int_{\mathbb{R}^n} |\nabla v|^2 e^{-|x|^2} \, dx - \int_{\mathbb{R}^n} \nabla v \cdot xe^{-|x|^2} \, dx + \int_{\mathbb{R}^n} |x|^2 v^2 e^{-|x|^2} \, dx
\]

\[
= \int_{\mathbb{R}^n} |\nabla v|^2 e^{-|x|^2} \, dx + n \int_{\mathbb{R}^n} v^2 e^{-|x|^2} \, dx - \int_{\mathbb{R}^n} |x|^2 v^2 e^{-|x|^2} \, dx. \tag{3.10}
\]

The identity (3.6) is now followed from (3.7), (3.9) and (3.10).

As a remark, we show that (3.6) provides another proof of (1.3). Indeed, from (3.6) we have

\[
\int_{\mathbb{R}^n} |\Delta u|^2 \, dx + \int_{\mathbb{R}^n} |x|^2 u^2 \, dx \geq (n + 2) \int_{\mathbb{R}^n} |\nabla u|^2 \, dx, \tag{3.11}
\]

for any function \(u \in C_0^\infty(\mathbb{R}^n)\). Applying (3.11) for function \(u_\lambda(x) = \lambda^{n/2-1} u(\lambda x)\) with \(\lambda > 0\), we get

\[
\lambda^2 \int_{\mathbb{R}^n} |\Delta u|^2 \, dx + \lambda^2 \int_{\mathbb{R}^n} |x|^2 u^2 \, dx \geq (n + 2) \int_{\mathbb{R}^n} |\nabla u|^2 \, dx
\]

for any \(\lambda > 0\). Choosing \(\lambda = \left(\frac{\int_{\mathbb{R}^n} |\Delta u|^2 \, dx}{\int_{\mathbb{R}^n} |x|^2 u^2 \, dx}\right)^{\frac{1}{2}}\) implies (1.3).

Our next task is to use (3.6) to prove a stability estimate for (1.3) on even functions.

We first prove the following result

**Theorem 3.4.** Let \(v \in C_0^\infty(\mathbb{R}^n)\) be an even function. Then it holds

\[
\int_{\mathbb{R}^n} \|\nabla^2 v - x \otimes \nabla v\|_{HS}^2 e^{-|x|^2} \, dx \geq \frac{4n}{n+2} \left(\int_{\mathbb{R}^n} |\nabla((v - c)e^{-\frac{|x|^2}{2}})|^2 \, dx\right), \tag{3.12}
\]

where

\[
c = \frac{\int_{\mathbb{R}^n} ve^{-|x|^2} \, dx}{\int_{\mathbb{R}^n} e^{-|x|^2} \, dx}.
\]

Before proving Theorem 3.4, we provide some preparations. Note that

\[
\nabla[(v - c)e^{-\frac{|x|^2}{2}}] = [\nabla(v - c) - x(v - c)]e^{-\frac{|x|^2}{2}},
\]

then by using integration by parts, we get

\[
\int_{\mathbb{R}^n} |\nabla((v - c)e^{-\frac{|x|^2}{2}})|^2 \, dx = \int_{\mathbb{R}^n} |\nabla(v - c)|^2 e^{-|x|^2} \, dx + n \int_{\mathbb{R}^n} (v - c)^2 e^{-|x|^2} \, dx - \int_{\mathbb{R}^n} (v - c)^2 |x|^2 e^{-|x|^2} \, dx.
\]

Notice that by making the change of variable \(x = y/\sqrt{2}\) and making the change of function \(w(y) = v(y/\sqrt{2})\), then the inequality (3.12) is equivalent to

\[
\int_{\mathbb{R}^n} \|2\nabla^2 w - y \otimes \nabla w\|_{HS}^2 \, dx
\]

22
\[ \geq \frac{2n}{n+2} \left( 4 \int_{\mathbb{R}^n} |\nabla (w - c)|^2 d\gamma_n + 2n \int_{\mathbb{R}^n} (w - c)^2 d\gamma_n - \int_{\mathbb{R}^n} |y|^2 (w - c)^2 d\gamma_n \right), \tag{3.13} \]

where \( \gamma_n \) denotes the standard Gaussian measure on \( \mathbb{R}^n \), i.e.,
\[ d\gamma_n(y) = e^{-\frac{|y|^2}{2}} \frac{1}{(2\pi)^{\frac{n}{2}}} dy. \]

The proof of (3.13) is based on the spectral analysis of the Ornstein-Uhlenbeck operator on the Gaussian space. Let \( L_n \) denote the Ornstein-Uhlenbeck operator on the Gaussian space, i.e., \( L_n w(y) = \Delta w(y) - \nabla w(y) \cdot y \). It holds that
\[ \int_{\mathbb{R}^n} L_n w_1 \cdot w_2 d\gamma_n = -\int_{\mathbb{R}^n} \nabla w_1 \cdot \nabla w_2 d\gamma_n. \tag{3.14} \]

It is well-known that \( L^2(\gamma_1) \) has an orthogonal basis consisting of the Hermite polynomials \( H_0, H_1, H_2, \ldots \) with
\[ H_i(t) = (-1)^i e^{t^2/2} \frac{d^i}{dt^i}(e^{-t^2/2}), \quad i = 0, 1, 2, \ldots. \]

Note that \( H_i \) is a polynomial of degree \( i \), \( L_1 H_i = -i H_i \) and \( \int_{\mathbb{R}} H_i^2 d\gamma_1 = i! \). For examples, \( H_0 \equiv 1, H_1(t) = t, H_2(t) = t^2 - 1, H_3(t) = t^3 - 3t, \ldots. \) The Hermite polynomials have the following properties (see [1, formulas 22.7.14 and 22.8.8])
\[ H_{i+1}(t) = t H_i(t) - i H_{i-1}(t), \quad i \geq 1, \]
and
\[ H'_i(t) = i H_{i-1}(t), \quad i \geq 1. \]

These properties imply the following results,

**Proposition 3.5.** There holds
\[ t^2 H_0(t) = H_2(t) + H_0(t), \quad t^2 H_1(t) = H_3(t) + 3H_1(t) \]
and
\[ t^2 H_i(t) = H_{i+2}(t) + (2i+1)H_i(t) + i(i-1)H_{i-2}(t), \quad i \geq 2. \]

Using Proposition 3.5, we can easily to show for any \( i \leq j \) that
\[ \int_{\mathbb{R}} t^2 H_i(t) H_j(t) d\gamma_1(t) = \begin{cases} 0 & \text{if } j = i + 1 \text{ or } j > i + 2 \\ (2i+1)! & \text{if } i = j \\ (i+2)! & \text{if } j = i + 2. \end{cases} \tag{3.15} \]

For each \( I \in \mathbb{Z}_+^n, I = (i_1, \ldots, i_n), i_k \geq 0, k = 1, \ldots, n, \) we denote
\[ H_I(y) = H_{i_1}(y_1) \cdots H_{i_n}(y_n). \]

23
So, \( \{H_I\}_{I \in \mathbb{Z}_n^+} \) forms an orthogonal basis of \( L^2(\gamma_n) \) and
\[
L_n H_I = -|I| H_I, \quad \text{with} \quad |I| = i_1 + i_2 + \cdots + i_n.
\]

Notice that \( \int_{\mathbb{R}^n} H_I^2 d\gamma_n = i_1! i_2! \cdots i_n! =: |I|! \) for \( I = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}_n^+ \). For \( I, J \in \mathbb{Z}_n^+ \), we say that \( I \) is a neighborhood of \( J \) (denote by \( I \sim J \)) if and only if there exists uniquely a \( 1 \leq k \leq n \) such that \( |j_k - i_k| = 2 \) and \( j_l = i_l \) for any \( l \neq k \). We say that \( I \leq J \) if \( i_k \leq j_k \) for any \( k = 1, 2, \ldots, n \). We shall use (3.15) to prove the next lemma.

**Lemma 3.6.** For any \( I, J \in \mathbb{Z}_n^+ \), it holds,
\[
\int_{\mathbb{R}^n} H_I(y) H_J(y)|y|^2 d\gamma_n(y) = \begin{cases} 
(2|I| + n)! & \text{if } I = J \\
|J|! & \text{if } I \leq J \text{ and } I \sim J \\
|I|! & \text{if } J \leq I \text{ and } J \sim I \\
0 & \text{otherwise.}
\end{cases}
\] (3.16)

**Proof.** We first consider the case \( I = J \). Using (3.15), we have
\[
\int_{\mathbb{R}^n} H_I^2(y)|y|^2 d\gamma_n = \sum_{k=1}^{n} \int_{\mathbb{R}^n} H_{i_k}^2(y_k) y_k^2 d\gamma_n = \sum_{k=1}^{n} \left( \int_{\mathbb{R}} H_{i_k}^2 y_{i_k}^2 d\gamma_{i_k} \right) \prod_{l \neq k} \int_{\mathbb{R}} H_{i_l}^2 d\gamma_{i_l}
\]
\[
= \sum_{k=1}^{n} (2i_k + 1)i_k! \prod_{l \neq k} i_l! = (2|I| + n)!.
\]

We next consider the case \( I \leq J \) and \( I \sim J \). So there is an index \( l \) such that \( j_l = i_l + 2 \) and \( j_k = i_k \) for any \( k \neq l \). Using (3.15) and \( \int_{\mathbb{R}} H_i H_{i+2} d\gamma_{i+1} = 0 \), we have
\[
\int_{\mathbb{R}^n} H_I H_J |y|^2 d\gamma_n = \sum_{p=1}^{n} \int_{\mathbb{R}^n} H_{i_p}(y_p) H_{i_p+2}(y_l) \left( \prod_{k \neq l} H_{i_k}^2 \right) y_p^2 d\gamma_n
\]
\[
= \int_{\mathbb{R}} H_{i_l}(y_l) H_{i_l+2}(y_l) y_l^2 d\gamma_l \prod_{k \neq l} \int_{\mathbb{R}} H_{i_k}^2 d\gamma_k
\]
\[
+ \sum_{p \neq l} \left( \int_{\mathbb{R}} H_{i_p} H_{i_p+2} d\gamma_{i_p+1} \right) \left( \int_{\mathbb{R}} \left( \prod_{k \neq l} H_{i_k}^2 \right) y_{p_l}^2 d\gamma_{n-1} \right)
\]
\[
= (i_l + 2)! \prod_{k \neq l} i_k! = J!.
\]

The case \( J \leq I \) and \( J \sim I \) is treated similarly.

Finally, we consider the case \( I \neq J \) and \( I \neq J \). In this case, there are two subcases:
- If there exist at least two indices \( i_k \neq j_k \) and \( i_l \neq j_l \) \( (k \neq l) \), then
\[
\int_{\mathbb{R}^n} H_I H_J |y|^2 d\gamma_n = \sum_{p=1}^{n} \int_{\mathbb{R}^n} H_I H_J y_p^2 d\gamma_n = 0
\]
since for each \( p \), in the decomposition of \( \int_{\mathbb{R}^n} H_I H_J y_p \, d\gamma_n \), there is always at least one factor of the form \( \int_{\mathbb{R}} H_{i_k} H_{j_k} \, d\gamma_1 = 0 \) or \( \int_{\mathbb{R}} H_{i_k} H_{j_k} \, d\gamma_1 = 0 \).

\[ \bullet \text{If there is only one index } i_k \neq j_k, \text{ then } |i_k - j_k| \neq 2 \text{ since } I \not\sim J. \] Using (3.15), we have \( \int_{\mathbb{R}} H_{i_k} H_{j_k} y_p \, d\gamma_1 = 0 \). Remark that \( \int_{\mathbb{R}} H_{i_k} H_{j_k} \, d\gamma_1 = 0 \). From these facts, we get

\[
\int_{\mathbb{R}^n} H_I H_J |g|^2 \, d\gamma_n = \sum_{p=1}^{n} \int_{\mathbb{R}^n} H_I H_J y_p \, d\gamma_n \\
= \int_{\mathbb{R}} H_{i_k} H_{j_k} y_k^2 \, d\gamma_1 \left( \prod_{l \neq k} \int_{\mathbb{R}} H_l^2 \, d\gamma_1 \right) \\
+ \sum_{p \neq k} \left( \int_{\mathbb{R}} H_{i_k} H_{j_k} \, d\gamma_1 \right) \left( \int_{\mathbb{R}^{n-1}} \left( \prod_{l \neq k} H_l^2 \right) y_p^2 \, d\gamma_{n-1} \right) \\
= 0.
\]

\[ \square \]

Let \( \{e_1, e_2, \ldots, e_n\} \) be the canonical basis of \( \mathbb{R}^n \). The next lemma is an application of Fubini theorem for sums.

**Lemma 3.7.** Let \( \{c_I\} \) be a nonnegative sequence on \( \mathbb{Z}_+^n \) and \( f, g \) be two nonnegative functions on \( \mathbb{Z} \). Then we have

\[
\sum_{|I|\geq 2} f(|I|) \sum_{l=1}^{n} c_{I+2e_l} g(i_l + 2) = \sum_{|I|\geq 4} c_I f(|I| - 2) \sum_{k=2}^{n} g(i_k).
\]

**Proof.** Note that for each \( J \geq I \) and \( J \sim I \), there exists a unique \( l \in \{1, 2, \ldots, n\} \) such that \( J = I + 2e_l \). Hence, it holds

\[
\sum_{l=1}^{n} c_{I+2e_l} g(i_l + 2) = \sum_{J} \sum_{h=1}^{n} c_J \chi_{\{K: K \geq I, K \sim I\}}(J) \chi_{\{k: j_k - i_k = 2\}}(h) g(j_h)
\]

here, \( \chi_A \) denotes the characteristic function of the set \( A \). Using Fubini theorem, we have

\[
\sum_{|I|\geq 2} f(|I|) \sum_{l=1}^{n} c_{I+2e_l} g(i_l + 2) = \sum_{|I|\geq 2} \sum_{J} \sum_{h=1}^{n} f(|I|) c_J \chi_{\{K: K \geq I, K \sim I\}}(J) \chi_{\{k: j_k - i_k = 2\}}(h) f(j_h)
\]

\[
= \sum_{J} \sum_{|I|\geq 2} \sum_{h=1}^{n} f(|I|) c_J \chi_{\{K: K \geq I, K \sim I\}}(J) \chi_{\{k: j_k - i_k = 2\}}(h) f(j_h)
\]

\[
= \sum_{J} \sum_{|I|\geq 2, I \sim J} \sum_{h=1}^{n} f(|I|) c_J \chi_{\{k: j_k - i_k = 2\}}(h) f(j_h)
\]

\[
= \sum_{|J|\geq 4} f(|J| - 2) c_J \sum_{|I|\geq 2} \sum_{I \sim J, h=1}^{n} \chi_{\{k: j_k - i_k = 2\}}(h) f(j_h),
\]

25
here we use the fact that if $I \leq J$ and $I \sim J$ then $|I| = |J| - 2$. Note that for a fixed $J$ with $|J| \geq 4$, if there is an index $l$ such that $j_l \geq 2$ then $I = (j_1, \ldots, j_l-2, \ldots, j_n)$ satisfies $|I| \geq 2$, $I \leq J$ and $I \sim J$. Thus, we have

$$\sum_{|I| \geq 2, I \leq J, I \sim J} \sum_{h=1}^{n} \chi_{\{k: j_k = 2\}}(h) f(j_h) = \sum_{h: j_h \geq 2} f(j_h).$$

Inserting this equality in the preceding one, we complete the proof of this lemma.

We are now ready to prove Theorem 3.4

**Proof of Theorem 3.4.** As mentioned before, it is enough to prove (3.13). We remark that

$$c = \frac{\int_{\mathbb{R}^n} v e^{-|x|^2} \, dx}{\int_{\mathbb{R}^n} e^{-|x|^2} \, dx} = \int_{\mathbb{R}^n} w(y) d\gamma_n(y).$$

We decompose $w$ in the basis $\{H_I\}_{I \in \mathbb{Z}_+^n}$ as

$$w = \sum_{I \in \mathbb{Z}_+^n} a_I H_I,$$

with

$$a_I = \frac{1}{I!} \int_{\mathbb{R}^n} w(y) H_I(y) d\gamma_n(y).$$

Notice that $c = a_{(0, \ldots, 0)}$ and $a_I = 0$ for any $I$ with $|I| = 1$ since $w$ is even.

We first compute

$$4 \int_{\mathbb{R}^n} |\nabla(w - c)|^2 d\gamma_n + 2n \int_{\mathbb{R}^n} (w - c)^2 d\gamma_n - \int_{\mathbb{R}^n} |y|^2 (w - c)^2 d\gamma_n.$$

Denote $\tilde{w} = w - c$, we have

$$\tilde{w} = \sum_{I: |I| \geq 2} a_I H_I.$$

This implies that

$$\int_{\mathbb{R}^n} |\nabla \tilde{w}|^2 d\gamma_n = \sum_{|I| \geq 2} |I| b_I^2,$$

$$\int_{\mathbb{R}^n} \tilde{w}^2 d\gamma_n = \sum_{|I| \geq 2} b_I^2,$$

(3.17)

here we denote $b_I = a_I \sqrt{I!}$. Using Lemma 3.6, we have

$$\int_{\mathbb{R}^n} \tilde{w}^2 |y|^2 d\gamma_n = \sum_{|I| \geq 2} a_I^2 \int_{\mathbb{R}^n} H_I^2 |y|^2 d\gamma_n + \sum_{I \neq J, |I|, |J| \geq 2} a_I a_J \int_{\mathbb{R}^n} H_I H_J |y|^2 d\gamma_n$$
\[
\begin{align*}
&= \sum_{|I| \geq 2} (2|I| + n) b_I^2 + 2 \sum_{|I| \geq 2} a_I \sum_{J \sim I, J \geq I} a_J J!.
\end{align*}
\] (3.18)

From (3.17) and (3.18) and Lemma 3.7, we obtain

\[
4 \int_{\mathbb{R}^n} |\nabla \tilde{w}|^2 d\gamma_n + 2 n \int_{\mathbb{R}^n} \tilde{w}^2 d\gamma_n - \int_{\mathbb{R}^n} |y|^2 \tilde{w}^2 d\gamma_n
\]

\[
= \sum_{|I| \geq 2} (2|I| + n) b_I^2 - 2 \sum_{|I| \geq 2} a_I \sum_{J \sim I, J \geq I} a_J J!
\]

\[
= \sum_{|I| \geq 2} (2|I| + n) b_I^2 - 2 \sum_{|I| \geq 2} b_I \sum_{l=1}^n b_{I+2e_l} \sqrt{(i_l + 1)(i_l + 2)}
\]

\[
= \sum_{|I| \geq 2} \sum_{l=1}^n ((i_l + 1) b_I^2 - 2 b_I \sqrt{i_l + 1} b_{I+2e_l} \sqrt{i_l + 2} + b_{I+2e_l}^2 (i_l + 2))
\]

\[
+ \sum_{|I| \geq 2} |I| b_I^2 - \sum_{|I| \geq 2} \sum_{l=1}^n b_{I+2e_l}^2 (i_l + 2)
\]

\[
= \sum_{|I| \geq 2} \sum_{l=1}^n \left( \sqrt{i_l + 1} b_I - \sqrt{i_l + 2 b_{I+2e_l}} \right)^2 + \sum_{|I| \geq 2} |I| b_I^2 - \sum_{|I| \geq 4} b_I^2 \sum_{k,i_k \geq 2} i_k
\]

\[
= \sum_{|I| \geq 2} \sum_{l=1}^n \left( \sqrt{i_l + 1} b_I - \sqrt{i_l + 2 b_{I+2e_l}} \right)^2 + \sum_{|I| \geq 2} |I| b_I^2 - \sum_{|I| \geq 4} b_I^2 \left( |I| - \sum_{k,i_k \leq 2} i_k \right)
\]

\[
= \sum_{|I| \geq 2} \sum_{l=1}^n \left( \sqrt{i_l + 1} b_I - \sqrt{i_l + 2 b_{I+2e_l}} \right)^2 + \sum_{|I| \geq 2} |I| b_I^2 + \sum_{|I| \geq 4} b_I^2 \sum_{k,i_k \leq 2} i_k. \quad \text{(3.19)}
\]

We next compute the left hand side of (3.13). Using (3.14) and the commutator relation \(L_n \partial_j f = \partial_j L_n f + (\partial_j f)^2\) we have

\[
\int_{\mathbb{R}^n} \|2 \nabla^2 w - y \otimes \nabla w\|^2_{HS} d\gamma_n
\]

\[
= \int_{\mathbb{R}^n} \|2 \nabla^2 \tilde{w} - y \otimes \nabla \tilde{w}\|^2_{HS} d\gamma_n
\]

\[
= 4 \sum_{j=1}^n \int_{\mathbb{R}^n} \|\nabla \partial_j \tilde{w}\|^2 d\gamma_n - 4 \int_{\mathbb{R}^n} \nabla^2 \tilde{w} (\nabla \tilde{w}) \cdot y d\gamma_n + \int_{\mathbb{R}^n} |y|^2 |\nabla \tilde{w}|^2 d\gamma_n
\]

\[
= -4 \sum_{j=1}^n \int_{\mathbb{R}^n} L(\partial_j \tilde{w}) \partial_j \tilde{w} d\gamma_n - 2 \int_{\mathbb{R}^n} \nabla (|\nabla \tilde{w}|^2) \cdot y d\gamma_n + \int_{\mathbb{R}^n} |\nabla \tilde{w}|^2 |y|^2 d\gamma_n
\]

\[
= \sum_{j=1}^n \left( -4 \int_{\mathbb{R}^n} \partial_j L_n \partial_j \tilde{w} d\gamma_n - 4 \int_{\mathbb{R}^n} (\partial_j \tilde{w})^2 d\gamma_n \right) + 2 n \int_{\mathbb{R}^n} |\nabla \tilde{w}|^2 d\gamma_n
\]

\[
- \int_{\mathbb{R}^n} |y|^2 |\nabla \tilde{w}|^2 d\gamma_n
\]

27
\[
\int_{\mathbb{R}^n} |y|^2 |\nabla \bar{w}|^2 d\gamma_n = -\int_{\mathbb{R}^n} \bar{w} L_n \bar{w} |y|^2 d\gamma_n + n \int_{\mathbb{R}^n} \bar{w}^2 d\gamma_n - \int_{\mathbb{R}^n} |y|^2 \bar{w}^2 d\gamma_n.
\]

An elementary computation gives
\[
\int_{\mathbb{R}^n} |y|^2 |\nabla \bar{w}|^2 d\gamma_n = \int_{\mathbb{R}^n} \bar{w} L_n \bar{w} |y|^2 d\gamma_n + n \int_{\mathbb{R}^n} \bar{w}^2 d\gamma_n - \int_{\mathbb{R}^n} |y|^2 \bar{w}^2 d\gamma_n.
\]

Substituting this into previous equality, we arrive at
\[
\int_{\mathbb{R}^n} \|2\nabla^2 w - y \otimes \nabla w\|_{H^s}^2 d\gamma_n = 4 \int_{\mathbb{R}^n} (L_n \bar{w})^2 d\gamma_n + 2(n - 2) \int_{\mathbb{R}^n} |\nabla \bar{w}|^2 d\gamma_n + \int_{\mathbb{R}^n} \bar{w} L_n \bar{w} |y|^2 d\gamma_n
- n \int_{\mathbb{R}^n} \bar{w}^2 d\gamma_n + \int_{\mathbb{R}^n} |y|^2 \bar{w}^2 d\gamma_n.
\] (3.20)

It holds that
\[
L_n \bar{w} = \sum_{I: |I| \geq 2} -a_I |I| H_I.
\]

This implies
\[
\int_{\mathbb{R}^n} (L_n \bar{w})^2 d\gamma_n = \sum_{|I| \geq 2} |I|^2 b_I^2.
\] (3.21)

Using again Lemma 3.6, we have
\[
\int_{\mathbb{R}^n} \bar{w}(-L_n \bar{w}) |y|^2 d\gamma_n = \sum_{|I|, |J| \geq 2} a_I |J| a_J \int_{\mathbb{R}^n} H_I H_J |y|^2 d\gamma_n
= \sum_{|I| \geq 2} a_I^2 |I| \int_{\mathbb{R}^n} H_I^2 |y|^2 d\gamma_n + \sum_{I \neq J} a_I a_J |J| \int_{\mathbb{R}^n} H_I H_J |y|^2 d\gamma_n
= \sum_{|I| \geq 2} |I| (2|I| + n) b_I^2 + \sum_{|I| \geq 2} a_I \sum_{|J| \geq 2} a_J |J| J! + \sum_{|I| \geq 2} a_I \sum_{|J| \geq 2} a_J |J| J!
= \sum_{|I| \geq 2} |I| (2|I| + n) b_I^2 + 2 \sum_{|I| \geq 2} a_I (|I| + 1) \sum_{|J| \geq 2} a_J J!.
\] (3.22)

Plugging (3.21), (3.22) and (3.18) into (3.20), we get
\[
\int_{\mathbb{R}^n} \|2\nabla^2 w - y \otimes \nabla w\|_{H^s}^2 d\gamma_n
= 4 \sum_{|I| \geq 2} |I|^2 b_I^2 + 2(n - 2) \sum_{|I| \geq 2} |I| b_I^2 - \sum_{|I| \geq 2} |I| (2|I| + n) b_I^2 - 2 \sum_{|I| \geq 2} a_I (|I| + 1) \sum_{J \sim I, J \geq I} a_J J!
- n \sum_{|I| \geq 2} b_I^2 + \sum_{|I| \geq 2} (2|I| + n) b_I^2 + 2 \sum_{|I| \geq 2} a_I \sum_{J \sim I, J \geq I} a_J J!.
\]
\[
\sum_{|I| \geq 2} |I|(2|I| + n - 2)b_I^2 - 2 \sum_{|I| \geq 2} a_I|I| \sum_{J \geq I, J \sim I} a_J J!.
\]

Note that
\[
\sum a_I|I| \sum_{J \geq I, J \sim I} a_J J! = \sum b_I|I| \sum_{l=1}^n b_{l+2e_I} \sqrt{(i_l + 1)(i_l + 2)}.
\]

Using Lemma 3.7, we obtain
\[
\int_{\mathbb{R}^n} \|2\nabla^2 w - y \otimes \nabla w\|_{HS}^2 d\gamma_n
\]
\[
= \sum_{|I| \geq 2} |I|(2|I| + n - 2)b_I^2 - 2 \sum_{|I| \geq 2} b_I|I| \sum_{l=1}^n b_{l+2e_I} \sqrt{(i_l + 1)(i_l + 2)}
\]
\[
= \sum_{|I| \geq 2} |I| \sum_{l=1}^n (\sqrt{i_l + 1}b_I - \sqrt{i_l + 2b_{l+2e_I}})^2 + \sum_{|I| \geq 2} |I|(|I| - 2)b_I^2
\]
\[- \sum_{|I| \geq 2} b_I^2(|I| - 2) \sum_{k:i_k \geq 2} i_k
\]
\[
= \sum_{|I| \geq 2} |I| \sum_{l=1}^n (\sqrt{i_l + 1}b_I - \sqrt{i_l + 2b_{l+2e_I}})^2 + \sum_{|I| \geq 2} |I|(|I| - 2)b_I^2
\]
\[- \sum_{|I| \geq 4} b_I^2(|I| - 2)(|I| - \sum_{k:i_k \geq 2} i_k)
\]
\[
= \sum_{|I| \geq 2} |I| \sum_{l=1}^n (\sqrt{i_l + 1}b_I - \sqrt{i_l + 2b_{l+2e_I}})^2 + 3 \sum_{|I| = 2} b_I^2 + \sum_{|I| \geq 4} b_I^2(|I| - 2) \sum_{k:i_k \geq 2} i_k
\]
\[\geq 2 \sum_{|I| \geq 2} \sum_{l=1}^n (\sqrt{i_l + 1}b_I - \sqrt{i_l + 2b_{l+2e_I}})^2 + 2 \sum_{|I| \geq 4} b_I^2 \sum_{k:i_k \geq 2} i_k \quad (3.23)
\]

On the other hand, we have
\[
\int_{\mathbb{R}^n} \|2\nabla^2 w - y \otimes \nabla w\|_{HS}^2 d\gamma_n
\]
\[
= \sum_{|I| \geq 2} |I|(2|I| + n - 2)b_I^2 - 2 \sum_{|I| \geq 2} b_I|I| \sum_{l=1}^n b_{l+2e_I} \sqrt{(i_l + 1)(i_l + 2)}
\]

29
\[
= \sum_{|I| \geq 2} \left( |I| b_I - \sum_{l=1}^{n} b_{I+2e_l} \sqrt{(i_l + 1)(i_l + 2)} \right)^2 + \sum_{|I| \geq 2} |I|(|I| + n - 2)b_I^2 \\
- \sum_{|I| \geq 2} \left( \sum_{l=1}^{n} b_{I+2e_l} \sqrt{(i_l + 1)(i_l + 2)} \right)^2. \tag{3.24}
\]

Using the Cauchy-Schwarz inequality, we get
\[
\left( \sum_{l=1}^{n} b_{I+2e_l} \sqrt{(i_l + 1)(i_l + 2)} \right)^2 \leq \left( \sum_{l=1}^{n} (i_l + 1) \right) \left( \sum_{l=1}^{n} b_{I+2e_l}^2 (i_l + 2) \right) = (|I| + n) \sum_{l=1}^{n} b_{I+2e_l}^2 (i_l + 2).
\]

This and Lemma 3.7 imply that
\[
\sum_{|I| \geq 2} \left( \sum_{l=1}^{n} b_{I+2e_l} \sqrt{(i_l + 1)(i_l + 2)} \right)^2 \leq \sum_{|I| \geq 2} (|I| + n) \sum_{l=1}^{n} b_{I+2e_l}^2 (i_l + 2) \\
= \sum_{|J| \geq 4} b_J^2 (|J| + n - 2) \sum_{j_k \geq 2} j_k \\
= \sum_{|J| \geq 4} b_J^2 (|J| + n - 2) \left( |J| - \sum_{j_k < 2} j_k \right). \tag{3.25}
\]

Substituting (3.25) into (3.24), we obtain
\[
\int_{\mathbb{R}^n} \|2\nabla^2 w - y \otimes \nabla w\|_{HS}^2 d\gamma_n \geq \sum_{|I| \geq 2} \left( |I| b_I - \sum_{l=1}^{n} b_{I+2e_l} \sqrt{(i_l + 1)(i_l + 2)} \right)^2 \\
+ \sum_{2 \leq |I| \leq 3} |I|(|I| + n - 2)b_I^2 + \sum_{|J| \geq 4} b_J^2 \left( \sum_{j_m < 2} j_m \right) \\
\geq n \sum_{2 \leq |I| \leq 3} |I| b_I^2. \tag{3.26}
\]

The desired estimate (3.13) follows from (3.23), (3.26) and (3.19). \qed

The next lemma gives a stability estimate of (1.3) for even functions.

**Lemma 3.8.** Let \( u \in C_0^\infty (\mathbb{R}^n) \) be a non-zero even function, then
\[
\delta(u) \geq \frac{4n}{(n + 2)^2} \inf_{\varphi \in E} \left\{ \frac{\int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 dx}{\int_{\mathbb{R}^n} |\nabla u|^2 dx} \right\}. \tag{3.27}
\]
Proof of Lemma 3.8. Since both sides of the inequality (3.27) are invariant under the dilation and multiplication by a non-zero constant, then without loss of generality we can assume that \( \int_{\mathbb{R}^n} |\nabla u|^2 dx = 1 \) and
\[
\int_{\mathbb{R}^n} |\Delta u|^2 dx = \int_{\mathbb{R}^n} |x|^2 |\nabla u|^2 dx.
\]

Using the identity (3.6), we have
\[
\delta(u) = \frac{\int_{\mathbb{R}^n} |\Delta u|^2 dx + \int_{\mathbb{R}^n} |x|^2 |\nabla u|^2 dx - (n + 2) \int_{\mathbb{R}^n} |\nabla u|^2 dx}{n + 2} = \frac{1}{n + 2} \int_{\mathbb{R}^n} \|\nabla^2 v - x \otimes \nabla v\|_{HS}^2 e^{-|x|^2} dx
\]
with \( u = ve^{-|x|^2} \). Note that \( v \) is also even. Let \( c = \int_{\mathbb{R}^n} vd\mu_n \). Applying Theorem 3.4, we obtain
\[
\delta(u) \geq \frac{4n}{(n + 2)^2} \int_{\mathbb{R}^n} |\nabla[(v - c)e^{-|x|^2}]|^2 dx = \frac{4n}{(n + 2)^2} \int_{\mathbb{R}^n} |\nabla u - (c e^{-|x|^2})|^2 dx \geq \frac{4n}{(n + 2)^2} \inf_{\varphi \in E} \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 dx.
\]

\[
3.3 \quad \text{Proof of Theorem 1.5}
\]

In this subsection, we given the proof of Theorem 1.5 by using the results from two preceding subsections.

Proof of Theorem 1.5. By the density argument, it is enough to prove Theorem 1.5 for function \( u \in C_0^\infty(\mathbb{R}^n) \). Without loss of generality, we can assume that \( \int_{\mathbb{R}^n} |\nabla u|^2 dx = 1 \). Let
\[
u_o(x) = \frac{u(x) - u(-x)}{2}, \quad u_e(x) = \frac{u(x) + u(-x)}{2}
\]
be the odd part and even part of \( u \) respectively.

We first consider the case \( \delta(u) \leq \frac{1}{96(n+2)} \). It follows from Theorem 3.2 that
\[
\int_{\mathbb{R}^n} |\nabla u_o|^2 dx \leq 5(n + 2)\delta(u)
\]
and \( \delta(u_e) \leq 2\delta(u) \). Applying Lemma 3.8 for \( u_e \), we have
\[
\inf_{\varphi \in E} \int_{\mathbb{R}^n} |\nabla u_e - \nabla \varphi|^2 dx \leq \frac{(n + 2)^2}{4n} \delta(u_e) \int_{\mathbb{R}^n} |\nabla u_e|^2 dx \leq \frac{(n + 2)^2}{2n} \delta(u) \leq (n + 2)\delta(u),
\]
Since $n \geq 2$. Using the inequality
\[
\int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx \leq 2 \left( \int_{\mathbb{R}^n} |\nabla u_0|^2 \, dx + \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx \right),
\]
we obtain
\[
\inf_{\varphi \in E} \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx \leq 12(n + 2) \delta(u).
\]
From the previous estimate, we get
\[
\inf_{v \in E} \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx \leq \frac{1}{8}.
\]
We now show that the infimum above is attained by a non-zero function $\varphi \in E$. Indeed, let $\varphi_i = c_i e^{-\lambda_i |x|^2/2}$ be a sequence in $E$ such that
\[
\lim_{i \to \infty} \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi_i|^2 \, dx = \inf_{\varphi \in E} \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx \leq \frac{1}{8}.
\]
Using triangle inequality, we have
\[
\left( \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi_i|^2 \, dx \right)^{\frac{1}{2}} \geq \left( \int_{\mathbb{R}^n} |\nabla \varphi_i|^2 \, dx \right)^{\frac{1}{2}} - 1.
\]
Thus for $i$ large enough, we get
\[
\frac{1}{4} \leq \int_{\mathbb{R}^n} |\nabla \varphi_i|^2 \, dx \leq \frac{9}{4}.
\]
A simple computation gives
\[
\int_{\mathbb{R}^n} |\nabla \varphi_i|^2 \, dx = c_i^2 \lambda_i^{-\frac{n}{2} + 1} \int_{\mathbb{R}^n} |x|^2 e^{-|x|^2/2} \, dx.
\]
So there are $a, A > 0$ such that $a^2 \leq c_i^2 \lambda_i^{-\frac{n}{2} + 1} \leq A^2$ for $i$ large enough. We also note that
\[
\int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx = 1 + c_i^2 \lambda_i^{-\frac{n}{2} + 1} \int_{\mathbb{R}^n} |x|^2 e^{-|x|^2} \, dx + 2c_i \lambda_i^{-\frac{1}{2}} \int_{\mathbb{R}^n} \nabla u \cdot (\lambda_i^{\frac{1}{2}} x) e^{-|\lambda_i^{\frac{1}{2}} x|^2/2} \, dx.
\] (3.28)
We next claim that $\lambda_i$ is bounded from above and below by positive constants. Indeed, suppose, up to extract a subsequence, that
\[
\lim_{i \to \infty} \lambda_i = \infty.
\]
For any $\epsilon > 0$, there exists $R > 0$ small enough such that $\int_{B_R} |\nabla u|^2 \, dx < \epsilon^2$. Hence, it holds
\[
\left| \int_{\mathbb{R}^n} \nabla u \cdot (\lambda_i^{\frac{1}{2}} x) e^{-|\lambda_i^{\frac{1}{2}} x|^2/2} \, dx \right| \leq \int_{B_R} |\nabla u| \lambda_i^{\frac{1}{2}} x \, dx e^{-|\lambda_i^{\frac{1}{2}} x|^2/2} + \int_{B_R^c} |\nabla u|^2 \lambda_i^{\frac{1}{2}} x e^{-|\lambda_i^{\frac{1}{2}} x|^2/2} \, dx
\]
32
\[
\begin{align*}
&\leq \left( \int_{B_R} |\nabla u|^2 dx \right)^\frac{1}{2} \left( \int_{B_R} |\frac{1}{\lambda_i} x|^2 e^{-|\frac{1}{\lambda_i} x|^2} dx \right)^\frac{1}{2} \\
&\quad + \epsilon \left( \int_{B_R} |\lambda_i^\frac{1}{2} x|^2 e^{-|\lambda_i^\frac{1}{2} x|^2} dx \right)^\frac{1}{2} \\
&\leq \lambda_i^{-\frac{n}{2}} \left( \int_{B_R \sqrt{\lambda_i}} |x|^2 e^{-|x|^2} dx \right)^\frac{1}{2} + \lambda_i^{-\frac{n}{4}} \epsilon \left( \int_{B_R \sqrt{\lambda_i}} |x|^2 e^{-|x|^2} dx \right)^\frac{1}{2},
\end{align*}
\]

which implies
\[
\left| c_i \lambda_i^\frac{1}{2} \int_{\mathbb{R}^n} \nabla u \cdot (\lambda_i^\frac{1}{2} x) e^{-|\lambda_i^\frac{1}{2} x|^2/2} dx \right| \leq A \left( \int_{B_R \sqrt{\lambda_i}} |x|^2 e^{-|x|^2} dx \right)^\frac{1}{2} + A \epsilon \left( \int_{B_R \sqrt{\lambda_i}} |x|^2 e^{-|x|^2} dx \right)^\frac{1}{2}.
\]

Let \( i \to \infty \), we obtain
\[
\limsup_{i \to \infty} \left| c_i \lambda_i^\frac{1}{2} \int_{\mathbb{R}^n} \nabla u \cdot (\lambda_i^\frac{1}{2} x) e^{-|\lambda_i^\frac{1}{2} x|^2/2} dx \right| \leq A \left( \int_{\mathbb{R}^n} |x|^2 e^{-|x|^2} dx \right)^\frac{1}{2} \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, then
\[
\limsup_{i \to \infty} \left| c_i \lambda_i^\frac{1}{2} \int_{\mathbb{R}^n} \nabla u \cdot (\lambda_i^\frac{1}{2} x) e^{-|\lambda_i^\frac{1}{2} x|^2/2} dx \right| = 0.
\]

This together with (3.28) implies
\[
\frac{1}{8} \geq \lim_{i \to \infty} \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi_i|^2 dx > 1
\]

which is impossible. Concerning the lower bound of \( \lambda_i \), suppose, up to extract a subsequence, that
\[
\lim_{i \to \infty} \lambda_i = 0.
\]

For any \( \epsilon > 0 \), there exists \( R > 0 \) large enough such that \( \int_{B_R} |\nabla u|^2 dx < \epsilon^2 \). Hence, it holds
\[
\left| \int_{\mathbb{R}^n} \nabla u \cdot (\lambda_i^\frac{1}{2} x) e^{-|\lambda_i^\frac{1}{2} x|^2/2} dx \right| \leq \int_{B_R} |\nabla u| |\lambda_i^\frac{1}{2} x| e^{-|\lambda_i^\frac{1}{2} x|^2/2} dx + \int_{B_R} |\nabla u| |\lambda_i^\frac{1}{2} x| e^{-|\lambda_i^\frac{1}{2} x|^2/2} dx
\]
\[
\leq \left( \int_{B_R} |\nabla u|^2 dx \right)^\frac{1}{2} \left( \int_{B_R} |\lambda_i^\frac{1}{2} x|^2 e^{-|\lambda_i^\frac{1}{2} x|^2} dx \right)^\frac{1}{2}
\[
\quad + \epsilon \left( \int_{B_R} |\lambda_i^\frac{1}{2} x|^2 e^{-|\lambda_i^\frac{1}{2} x|^2} dx \right)^\frac{1}{2}
\]
\[
\leq \lambda_i^{-\frac{n}{2}} \left( \int_{B_R \sqrt{\lambda_i}} |x|^2 e^{-|x|^2} dx \right)^\frac{1}{2} + \lambda_i^{-\frac{n}{4}} \epsilon \left( \int_{B_R \sqrt{\lambda_i}} |x|^2 e^{-|x|^2} dx \right)^\frac{1}{2}
\]

33
which implies
\[ |c_i \lambda_i^{\frac{1}{2}} \int_{\mathbb{R}^n} \nabla u \cdot (\lambda_i^{\frac{1}{2}} x) e^{-|\lambda_i^{\frac{1}{2}} x|^2/2} \, dx| \leq A \left( \int_{B_R \sqrt{n}} |x|^2 e^{-|x|^2} \, dx \right)^{\frac{1}{2}} + A \epsilon \left( \int_{B_R \sqrt{n}} |x|^2 e^{-|x|^2} \, dx \right)^{\frac{1}{2}}. \]

Let \( i \to \infty \), we obtain
\[ \limsup_{i \to \infty} |c_i \lambda_i^{\frac{1}{2}} \int_{\mathbb{R}^n} \nabla u \cdot (\lambda_i^{\frac{1}{2}} x) e^{-|\lambda_i^{\frac{1}{2}} x|^2/2} \, dx| \leq A \left( \int_{\mathbb{R}^n} |x|^2 e^{-|x|^2} \, dx \right)^{\frac{1}{2}} + A \epsilon \left( \int_{\mathbb{R}^n} |x|^2 e^{-|x|^2} \, dx \right)^{\frac{1}{2}}. \]

Since \( \epsilon > 0 \) is arbitrary, then
\[ \limsup_{i \to \infty} |c_i \lambda_i^{\frac{1}{2}} \int_{\mathbb{R}^n} \nabla u \cdot (\lambda_i^{\frac{1}{2}} x) e^{-|\lambda_i^{\frac{1}{2}} x|^2/2} \, dx| = 0. \]

This together with (3.28) implies
\[ \frac{1}{8} \geq \lim_{i \to \infty} \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx > 1 \]
which is again impossible. So the claim is proved. As a consequence, there are \( a_1, a_2, a_3 > 0 \) such that
\[ |c_i| \leq a_1, \quad 0 < a_2 \leq \lambda_i \leq a_3 \]
for any \( i \). Extracting a subsequence, we can assume that \( c_i \to c \) and \( \lambda_i \to \lambda \in [a_2, a_3] \).

Denote \( \varphi = ce^{-\lambda|x|^2/2} \in E \). We then have \( \nabla \varphi_i \to \nabla \varphi \) in \( L^2 \) and hence
\[ \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx = \lim_{i \to \infty} \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi_i|^2 \, dx = \inf_{\varphi \in E} \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx. \]

Remark that
\[ \int_{\mathbb{R}^n} |\nabla u|^2 \, dx = 1, \quad \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx = \inf_{\varphi \in E} \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx \leq \frac{1}{8} \]
then \( \varphi \neq 0 \). So we can choose a positive constant \( a \) such that
\[ \int_{\mathbb{R}^n} |\nabla (a \varphi)|^2 \, dx = \int_{\mathbb{R}^n} |\nabla u|^2 \, dx = 1 \]
or, equivalently \( a = \left( \int_{\mathbb{R}^n} |\nabla \varphi|^2 \, dx \right)^{-\frac{1}{2}}. \) By the triangle inequality, we have
\[ \left| \left( \int_{\mathbb{R}^n} |\nabla \varphi|^2 \, dx \right)^{\frac{1}{2}} - 1 \right| \leq \left( \int_{\mathbb{R}^n} |\nabla \varphi|^2 \, dx \right)^{\frac{1}{2}} - \left( \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx \right)^{\frac{1}{2}} \leq \sqrt{12(n+2)\delta(u)}. \]
Using this estimate and the Cauchy-Schwartz inequality, we obtain
\[
\int_{\mathbb{R}^n} |\nabla u - \nabla (a\varphi)|^2 \, dx = \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi + (1-a)\nabla \varphi|^2 \, dx \\
\leq 2 \int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx + 2(a-1)^2 \int_{\mathbb{R}^n} |\nabla \varphi|^2 \, dx \\
\leq 24(n+2)\delta(u) + 2 \left( \int_{\mathbb{R}^n} |\nabla \varphi|^2 \, dx \right)^{\frac{1}{2}} - 1^2 \\
\leq 48(n+2)\delta(u).
\]
This gives
\[
\delta(u) \geq \frac{1}{48(n+2)} \inf \left\{ \frac{\int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx}{\int_{\mathbb{R}^n} |\nabla u|^2 \, dx} : \varphi \in E, \int_{\mathbb{R}^n} |\nabla u|^2 \, dx = \int_{\mathbb{R}^n} |\nabla \varphi|^2 \, dx \right\}. \quad (3.29)
\]
when \( \delta(u) \leq \frac{1}{96(n+2)} \).

We next prove the theorem for \( \delta(u) > \frac{1}{96(n+2)} \). Since we always have
\[
\inf \left\{ \frac{\int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx}{\int_{\mathbb{R}^n} |\nabla u|^2 \, dx} : \varphi \in E, \int_{\mathbb{R}^n} |\nabla u|^2 \, dx = \int_{\mathbb{R}^n} |\nabla \varphi|^2 \, dx \right\} \leq 4
\]
then
\[
\delta(u) \geq \frac{1}{384(n+2)} \inf \left\{ \frac{\int_{\mathbb{R}^n} |\nabla u - \nabla \varphi|^2 \, dx}{\int_{\mathbb{R}^n} |\nabla u|^2 \, dx} : \varphi \in E, \int_{\mathbb{R}^n} |\nabla u|^2 \, dx = \int_{\mathbb{R}^n} |\nabla \varphi|^2 \, dx \right\}. \quad (3.30)
\]
The inequality (1.19) follows immediately from (3.29) and (3.30).

\section*{Acknowledgments}
This work was initiated and done when the second author visit Vietnam Institute for Advanced Study in Mathematics (VIASM) in 2020. He would like to thank the institute for hospitality and support during the visit.

\section*{References}

[1] M. Abramowitz and I. A. Stegun. \textit{Handbook of mathematical functions with formulas, graphs, and mathematical tables}, volume 55 of \textit{National Bureau of Standards Applied Mathematics Series}. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] T. Aubin. Problèmes isopérimétriques et espaces de Sobolev. \textit{J. Differential Geometry}, 11(4):573–598, 1976.
[3] W. Beckner. A generalized Poincaré inequality for Gaussian measures. Proc. Amer. Math. Soc., 105(2):397–400, 1989.

[4] G. Bianchi and H. Egnell. A note on the Sobolev inequality. J. Funct. Anal., 100(1):18–24, 1991.

[5] M. Bonforte, J. Dolbeault, B. Nazaret, and N. Siminov. Stability in gagliardo-nirenberg inequalities. preprint, arXiv:2007.03674, 2020.

[6] H. Brezis and E. H. Lieb. Sobolev inequalities with remainder terms. J. Funct. Anal., 62(1):73–86, 1985.

[7] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the Navier-Stokes equations. Comm. Pure Appl. Math., 35(6):771–831, 1982.

[8] L. Caffarelli, R. Kohn, and L. Nirenberg. First order interpolation inequalities with weights. Compositio Math., 53(3):259–275, 1984.

[9] E. A. Carlen, R. L. Frank, and E. H. Lieb. Stability estimates for the lowest eigenvalue of a Schrödinger operator. Geom. Funct. Anal., 24(1):63–84, 2014.

[10] F. Catrina and D. G. Costa. Sharp weighted-norm inequalities for functions with compact support in \( \mathbb{R}^N \setminus \{0\} \). J. Differential Equations, 246(1):164–182, 2009.

[11] F. Catrina and Z.-Q. Wang. On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions. Comm. Pure Appl. Math., 54(2):229–258, 2001.

[12] C. M. Cazacu. A new proof of the Hardy-Rellich inequality in any dimension. Proc. Roy. Soc. Edinburgh Sect. A, 150(6):2894–2904, 2020.

[13] C. M. Cazacu, J. Flynn, and N. Lam. Sharp second order uncertainty principles. preprint, arXiv:2012.12667, 2020.

[14] S. Chen, R. L. Frank, and T. Weth. Remainder terms in the fractional Sobolev inequality. Indiana Univ. Math. J., 62(4):1381–1397, 2013.

[15] A. Cianchi. A quantitative Sobolev inequality in BV. J. Funct. Anal., 237(2):466–481, 2006.

[16] A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli. The sharp Sobolev inequality in quantitative form. J. Eur. Math. Soc. (JEMS), 11(5):1105–1139, 2009.

[17] D. Cordero-Erausquin, B. Nazaret, and C. Villani. A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. Adv. Math., 182(2):307–332, 2004.
[18] D. G. Costa. Some new and short proofs for a class of Caffarelli-Kohn-Nirenberg type inequalities. *J. Math. Anal. Appl.*, 337(1):311–317, 2008.

[19] A. Cotsiolis and N. K. Tavoularis. Best constants for Sobolev inequalities for higher order fractional derivatives. *J. Math. Anal. Appl.*, 295(1):225–236, 2004.

[20] M. Del Pino and J. Dolbeault. Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl. (9)*, 81(9):847–875, 2002.

[21] M. Del Pino and J. Dolbeault. The optimal Euclidean $L^p$-Sobolev logarithmic inequality. *J. Funct. Anal.*, 197(1):151–161, 2003.

[22] J. Dolbeault and G. Toscani. Improved interpolation inequalities, relative entropy and fast diffusion equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 30(5):917–934, 2013.

[23] J. Dolbeault and G. Toscani. Stability results for logarithmic Sobolev and Gagliardo-Nirenberg inequalities. *Int. Math. Res. Not. IMRN*, (2):473–498, 2016.

[24] M. Fathi, E. Indrei, and M. Ledoux. Quantitative logarithmic Sobolev inequalities and stability estimates. *Discrete Contin. Dyn. Syst.*, 36(12):6835–6853, 2016.

[25] M. Fathi. A short proof of quantitative stability for the Heisenberg-Pauli-Weyl inequality. *Nonlinear Anal.*, 210:112403, 2021.

[26] A. Figalli, F. Maggi, and A. Pratelli. Sharp stability theorems for the anisotropic Sobolev and log-Sobolev inequalities on functions of bounded variation. *Adv. Math.*, 242:80–101, 2013.

[27] A. Figalli and R. Neumayer. Gradient stability for the Sobolev inequality: the case $p \geq 2$. *J. Eur. Math. Soc. (JEMS)*, 21(2):319–354, 2019.

[28] G. B. Folland and A. Sitaram. The uncertainty principle: a mathematical survey. *J. Fourier Anal. Appl.*, 3(3):207–238, 1997.

[29] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative Sobolev inequality for functions of bounded variation. *J. Funct. Anal.*, 244(1):315–341, 2007.

[30] N. Ghoussoub and A. Moradifam. *Functional inequalities: new perspectives and new applications*, volume 187 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013.

[31] W. Heisenberg. Über quantentheoretische Kinematik und Mechanik. *Math. Ann.*, 95(1):683–705, 1926.

[32] L. Huang, A. Kristály, and W. Zhao. Sharp uncertainty principles on general Finsler manifolds. *Trans. Amer. Math. Soc.*, 373(11):8127–8161, 2020.
[33] E. H. Kennard. Zur quantenmechanik einfacher bewegungstypen. *Zeit. Phys.*, 44(4-5):326–352, 1927.

[34] E. H. Lieb. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. of Math. (2)*, 118(2):349–374, 1983.

[35] C. S. Lin. Interpolation inequalities with weights. *Comm. Partial Differential Equations*, 11(14):1515–1538, 1986.

[36] V. Maz’ya. Seventy five (thousand) unsolved problems in analysis and partial differential equations. *Integral Equations Operator Theory*, 90(2):Paper No. 25, 44, 2018.

[37] S. McCurdy and R. Venkatraman. Quantitative stability for the Heisenberg-Pauli-Weyl inequality. *Nonlinear Anal.*, 202:112147, 13, 2021.

[38] R. Neumayer. A note on strong-form stability for the Sobolev inequality. *Calc. Var. Partial Differential Equations*, 59(1):Paper No. 25, 8, 2020.

[39] V. H. Nguyen. Sharp Caffarelli-Kohn-Nirenberg inequalities on stratified Lie groups. *Ann. Acad. Sci. Fenn. Math.*, 43(2):1073–1083, 2018.

[40] V. H. Nguyen. The sharp Gagliardo-Nirenberg-Sobolev inequality in quantitative form. *J. Funct. Anal.*, 277(7):2179–2208, 2019.

[41] V. H. Nguyen. New sharp Hardy and Rellich type inequalities on Cartan–Hadamard manifolds and their improvements. *Proc. Roy. Soc. Edinburgh Sect. A*, 150(6):2952–2981, 2020.

[42] V. H. Nguyen. Sharp Caffarelli-Kohn-Nirenberg inequalities on riemannian manifolds: the influence of curvature. *Proc. Roy. Soc. Edinburgh Sect. A*, to appear, 2020.

[43] V. H. Nguyen. Stability version of the sharp uncertainty principles. *in preparation*, 2021.

[44] F. Rellich. Halbbeschränkte Differentialoperatoren höherer Ordnung. In *Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III*, pages 243–250. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956.

[45] F. Seuffert. An extension of the Bianchi-Egnell stability estimate to Bakry, Gentil, and Ledoux’s generalization of the Sobolev inequality to continuous dimensions. *J. Funct. Anal.*, 273(10):3094–3149, 2017.

[46] G. Talenti. Best constant in Sobolev inequality. *Ann. Mat. Pura Appl. (4)*, 110:353–372, 1976.

[47] A. Tertikas and N. B. Zographopoulos. Best constants in the Hardy-Rellich inequalities and related improvements. *Adv. Math.*, 209(2):407–459, 2007.
[48] J. L. Vazquez and E. Zuazua. The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential. *J. Funct. Anal.*, 173(1):103–153, 2000.

[49] H. Weyl. *The theory of groups and quantum mechanics*. Dover Publications, Inc., New York, 1950. Translated from the second (revised) German edition by H. P. Robertson, Reprint of the 1931 English translation.

[50] C. Xia. The Caffarelli-Kohn-Nirenberg inequalities on complete manifolds. *Math. Res. Lett.*, 14(5):875–885, 2007.