MEDIANS IN MEDIAN GRAPHS IN LINEAR TIME

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Abstract. The median of a graph $G$ is the set of all vertices $x$ of $G$ minimizing the sum of distances from $x$ to all other vertices of $G$. It is known that computing the median of dense graphs in subcubic time refutes the APSP conjecture and computing the median of sparse graphs in subquadratic time refutes the HS conjecture.

In this paper, we present a linear time algorithm for computing medians of median graphs, improving over the existing quadratic time algorithm. Median graphs constitute the principal class of graphs investigated in metric graph theory, due to their bijections with other discrete and geometric structures (CAT(0) cube complexes, domains of event structures, and solution sets of 2-SAT formulas). Our algorithm is based on the known majority rule characterization of medians in a median graph $G$ and on a fast computation of parallelism classes of edges ($\Theta$-classes) of $G$. The main technical contribution of the paper is a linear time algorithm for computing the $\Theta$-classes of a median graph $G$ using Lexicographic Breadth First Search (LexBFS). Namely, we show that any LexBFS ordering of the vertices of a median graph $G$ has the following fellow traveler property: the fathers of any two adjacent vertices of $G$ are also adjacent. Using the fast computation of the $\Theta$-classes of a median graph $G$, we also compute the Wiener index (total distance) of $G$ in linear time.

1. Introduction

The median problem (also called the Fermat-Torricelli problem or the Weber problem) is one of the oldest optimization problems in Euclidean geometry [34]. The median problem can be defined for any metric space $(X,d)$: given a finite set $P \subset X$ of points with positive weights, compute the set of points $x$ (or a point $x$) of $X$ minimizing the sum of the distances from $x$ to the points of $P$ multiplied by their weights. The median problem in graphs is one of the principal models in network location theory [31, 53] and is equivalent to finding nodes with largest closeness centrality in network analysis [12, 13, 48]. It also occurs in social group choice under the name of Kemeny median. In the consensus problem in social group choice, given $n$ individual rankings of $d$ candidates, one has to compute a consensual group decision. By classical Arrow impossibility theorem, there is no consensus function that satisfies the three natural “fairness” axioms. It is also well-known that the majority rule is the subject of Condorcet’s paradox, i.e., to the existence of cycles in the majority relation. In this respect, the Kemeny median [35, 36] is an important consensus function and corresponds to the median problem on the $d$-dimensional permutahedron (the graph whose vertices are all $d!$ permutations of the candidates and whose edges are the pairs of permutations differing by adjacent transpositions). Other classical algorithmic problems on graphs related to distances are the diameter and center problems. Yet another such problem comes from chemistry and consists in the computation of the Wiener index of a graph. This is a topological index of a molecule, defined as the sum of the lengths of the shortest paths between all pairs of vertices in the chemical graph representing the non-hydrogen atoms in the molecule [55].

The median problem in Euclidean spaces can be solved numerically, by a convergent iterative algorithm [44] using the convexity of the distance function. If instead of the $\ell_2$-metric one consider the $\ell_1$-metric, then the median problem becomes much easier and can be solved by performing the majority rule on each coordinate, i.e., by taking as median a point whose $i$th coordinate is the median of the list consisting of $i$th coordinates of the points of $P$. This majority rule was used by C. Jordan [33] to define centroids of trees (which in fact coincide...
with their medians \([28, 53]\), and can be viewed as a particular instance of the majority rule in social choice theory. In the case of graphs with \(n\) vertices, \(m\) edges, and standard graph distance, the median problem can be trivially solved in \(O(nm)\) time by running an algorithm for the All Pairs Shortest Paths problem (APSP). One may ask if solving APSP is necessary to compute the median. However, it was shown in [1] Theorem 1.1 that APSP and median problem are equivalent under subcubic reductions (and are equivalent to radius and betweenness centrality problems). Moreover, it was shown in [2] that computing the median of sparse graphs in subquadratic time refutes the HS (Hitting Set) conjecture. It was also mentioned in [16] that computing the Wiener index (the sum of the pairwise distances) of a sparse graph in subquadratic time will refute the Exponential time (SETH) hypothesis. Finally notice that the Kemeny median problem is NP-hard [25] when the input is the list of individual preferences.

In this paper, we show that the median problem in median graphs can be solved in optimal linear \(O(m)\) time (i.e., without solving APSP). Median graphs are the graphs in which each triplet \(u, v, w\) of vertices has a unique median, i.e., a vertex \(m(u, v, w)\) metrically lying between \(u\) and \(v\), \(v\) and \(w\), and \(w\) and \(u\). Median graphs originally arise in universal algebra [4,14], and their properties have been first investigated in [39,42]. Median graphs are closely related to hypercubes: median graphs can be isometrically embedded into hypercubes and they also are obtained from hypercubes by amalgamations. It was shown in [21,46] that the cube complexes of median graphs are exactly the CAT(0) cube complexes, i.e., cube complexes of global non-positive curvature. CAT(0) cube complexes, introduced and nicely characterized by Gromov [29], in a local-to-global way, are now one of the principal objects of investigation in geometric group theory [50]. Median graphs also occur in Computer Science: by [3,11] they are exactly the domains of event structures (one of the basic abstract models of concurrency) [43] and median-closed subsets of hypercubes are exactly the solution sets of 2-SAT formulas [41,51]. The bijections between median graphs, CAT(0) cube complexes, and event structures have been used by two authors of this paper in [17,18,22] to disprove three conjectures in concurrency and to establish a bijection between 1-safe Petri nets and special cube complexes. Finally, median graphs, viewed as median closures of sets of vertices of the hypercube, contain all most parsimonious (Steiner) trees [7] and as such have been extensively applied in human genetics. Median graphs are also at the origin of several other graph classes investigated in metric graph theory. For a survey of the properties of median graphs and their connections with other discrete and geometric structures, see the book [32], the survey [9], and the recent paper [19].

As we noticed above, median graphs have strong structural properties. First, median graphs are bipartite and contain at most \(O(n \log n)\) edges. Second, for median problem the concepts of \(\Theta\)-classes and halfspaces are essential. Two edges of a median graph \(G\) are called opposite if they are opposite edges of a common square (4-cycle) of \(G\). The relation \(\Theta\) is the equivalence relation which is the reflexive and transitive closure of this oppositeness relation. Each equivalence class of \(\Theta\) is called a \(\Theta\)-class (\(\Theta\)-classes correspond to hyperplanes in CAT(0) cube complexes and to events in event structures). Removing the edges of a \(\Theta\)-class, the graph \(G\) will be split into two connected components, called halfspaces. Halfspaces of a median graph are convex and gated (the latter meaning that each vertex \(v\) outside a halfspace \(H\) has a unique projection \(v'\) in \(H\) and \(v'\) belongs to a shortest path between any vertex \(u\) of \(H\) and \(v\)). The convexity of halfspaces implies (via Dikovic’s theorem [24]) that median graphs are partial cubes, i.e., graphs that are isometrically embeddable into hypercubes. The dimension \(q\) of a smallest hypercube into which a median graph \(G\) embeds is equal to the number of \(\Theta\)-classes of \(G\).

1.1. Our results. In this paper, we show that the \(\Theta\)-classes of a median graph \(G\) with \(n\) vertices and \(m\) edges can be computed in linear \(O(m)\) time (the previous best algorithm for this problem has complexity \(O(m \log n)\) [30]). Namely, we prove that a simplified version of Lexicographic Breadth First Search (LexBFS) of Rose, Tarjan, and Lueker [47] produces an ordering of the vertices of a median graph \(G\) satisfying the following friend traveler property: the fathers of any two adjacent vertices of \(G\) are also adjacent. This property allows to compute for each edge its \(\Theta\)-class in constant time. With \(\Theta\)-classes of a median graph \(G\) at hand and the majority rule for halfspaces in median graphs established in [6,52], we can compute the median...
of $G$ in optimal time $O(m)$. The previous best algorithm for median problem in median graphs has complexity $O(qn)$ under the assumption that an isometric embedding in a $q$-hypercube is given. Notice that $q$ may be linear in $n$ as in the case of trees and is always at least $d(\sqrt[n]{n} - 1)$ as we show below (where $d$ is the largest dimension of a hypercube included in $G$). Notice also that computing an isometric embedding in a $q$-hypercube requires $O(qn)$ time just to output the embedding and all known algorithms start by computing the $\Theta$-classes of $G$. Finally, using the fast computation of $\Theta$-classes of a median graph $G$, we also compute the Wiener index (total distance) of $G$ in linear time.

1.2. Related work. The investigation of medians in median graphs originated in the papers [6, 52] and continued in the papers [5, 40, 15]. Using different techniques and extending the majority rule for trees [28], the following majority rule have been established in [6, 52]: a halfspace $H$ of a median graph $G$ contains at least one median if and only if $H$ contains at least one half of the total weight of $G$; moreover, the median of $G$ coincides with the intersection of majoritary halfspaces of $G$ [52], i.e., of halfspaces containing strictly more than one half of the total weight. Hence the median is a convex/gated subgraph of $G$. It was shown in [6] that the median is always an interval of $G$. It was shown in [52] that the median function of a median graph is weakly peakless (which can be viewed as an analog of a discrete convex function), thus its local minima are global minima. Later it was proven in [8] that this property of the median function characterizes the graphs with connected medians and the graphs in which all local medians are global. A nice axiomatic characterization of medians of median graphs via three basic axioms has been obtained in [40]. More recently, the paper [45] characterized median graphs as closed Condorcet domains. Condorcet domains are sets of linear orders with the property that, whenever the preferences of all voters belong to this set, their majority relation has no cycles. Every such domain is closed in the sense that it contains the majority relation of every profile with an odd number of voters whose preferences belong to this domain. It is shown in [45] that every closed Condorcet domain can be endowed with the structure of a median graph and that, conversely, every median graph is associated with a closed Condorcet domain. Finally, as mentioned above, the paper [5] describes an algorithm with complexity $O(qn)$ (which maybe of order of $O(n^2)$) for computing the median set of a median graph $G$ with $n$ vertices and $q$ $\Theta$-classes.

As noticed above, the $\Theta$-classes of a median graph $G$ correspond to coordinates of the hypercube in which $G$ isometrically embeds. Thus one can define $\Theta$-classes for all partial cubes. Eppstein [26] performed an efficient computation of $\Theta$-classes as a main step of his $O(n^2)$ algorithm for recognizing partial cubes. For this, he runs several Breadth First Searches (BFS) on the input graph. The computation of $\Theta$-classes of a median graph in $O(m \log n)$ time by Hagauer et al., [30] was used in their subquadratic recognition of median graphs. The fellow-traveler property (which is essential in our computation of $\Theta$-classes) is a notion coming from geometric group theory [27] and is one of the principal tool used to prove the biautomaticity of a group. In a slightly stronger form it allows to establish dismantlability of graphs (see, for example, [15, 21] and references therein for classes of graphs in which such fellow traveler order can be obtained by BFS or LexBFS).

There exists an extensive literature on Wiener index in graphs [32, 37]. Notice only that the Wiener index of a tree can be computed in linear time [38]. Using this and the fact that benzenoids isometrically embed in the product of three trees, [23] proposes a linear time algorithm for the Wiener index of benzenoids. Finally, in a recent breakthrough [16], Cabello presented a subquadratic algorithm for the Wiener index and the diameter of all planar graphs.

2. Preliminaries

All graphs $G = (V, E)$ in this paper are finite, undirected, simple, and connected; $V$ is the vertex-set and $E$ is the edge-set of $G$. We write $u \sim v$ if two vertices $u$ and $v$ are adjacent. The distance $d(u, v) = d_G(u, v)$ between two vertices $u$ and $v$ is the length of a shortest $(u, v)$-path, and the interval $I(u, v) = \{x \in V : d(u, x) + d(x, v) = d(u, v)\}$ consists of all the vertices on shortest $(u, v)$-paths. A subgraph $H$ of $G$ is called isometric if $d_H(u, v) = d_G(u, v)$ for any two
vertices \(u, v\) of \(H\). A subgraph \(H\) is called convex if \(I(u, v) \subseteq H\) for any two vertices \(u, v\) of \(H\). Finally, a subgraph \(H\) of \(G\) is called gated if for every vertex \(v \in V(G)\), there exists a vertex \(v' \in V(H)\) such that for all \(u \in V(H)\), \(d_G(v, u) = d_G(v', u) + d_G(v', u)\) (\(v'\) is called the gate of \(v\) in \(H\)). For a vertex \(x\) of a gated subgraph \(H\) of \(G\), the set \(P(x) = \{v \in V : x\) is the gate of \(v\) in \(H\}\) is called the fiber of \(x\) with respect to \(H\). The fibers \(\{P(x) : x \in H\}\) define a partition of \(V(G)\).

The \(k\)-dimensional hypercube \(Q_k\) has all subsets of \(\{1, \ldots, k\}\) as the vertex-set and \(A \sim B\) iff \(|A \Delta B| = 1\).

A graph \(G\) is called median if the intersection \(I(x, y) \cap I(y, z) \cap I(z, x)\) is a singleton for each triplet \(x, y, z\) of vertices; this unique intersection vertex \(m(x, y, z)\) is called the median of \(x, y, z\). Median graphs are bipartite and do not contain induced \(K_{2,3}\). Basic examples of median graphs are trees, hypercubes, rectangular grids, and Hasse diagrams of distributive lattices and of median semilattices. The dimension \(d = \dim(G)\) of a median graph \(G\) is the largest dimension of a hypercube of \(G\). We call squares all 4-cycles and cubes all hypercube subgraphs of \(G\).

A map \(w : V \to \mathbb{R}^+ \cup \{0\}\) is called a weight function. For a vertex \(v \in V\), \(w(v)\) denotes the weight of \(v\). Then \(F_w(x) = \sum_{v \in V} w(v)d(x, v)\) is called the median function of the graph \(G\) for the weight function \(w\). A vertex \(x\) minimizing \(F_w\) is called a median vertex of \(G\) for the weight function \(w\). Finally, \(\text{Med}_w(G) = \{x \in V : x\) is a median of \(G\}\) is called the median set (or simply, the median) of \(G\) with respect to the weight function \(w\). The Wiener index \(W(G)\) (called also the total distance) of a graph \(G = (V, E)\) is the sum of all pairwise distances between the vertices of \(G\). Given a weight function \(w : V \to \mathbb{R}^+ \cup \{0\}\), the Wiener index of \(G\) with respect to \(w\) is the sum \(W_w(G) = \sum_{u, v \in V} w(u)w(v)d(u, v)\).

3. Properties of median graphs

In this section we recall the principal properties of median graphs used in our algorithms. These properties are not new and some of them are well-known, but in several cases it is difficult to find the appropriate references to them. Therefore, in the appendix we provide the proofs of such results. Throughout this section, \(G = (V, E)\) is median graph. We start with three simple properties of median graphs, which follow immediately from the definition.

**Lemma 1** (Quadrangle Condition). For any vertices \(u, v, w, z\) of \(G\) such that \(d(u, z) = k + 1\), \(v, w \sim z\), and \(d(u, v) = d(u, w) = k\), there is a unique vertex \(x \sim v, w\) such that \(d(u, x) = k - 1\).

**Lemma 2** (Cube Condition). Any three squares of \(G\), pairwise intersecting in three edges and all three intersecting in a single vertex, belong to a 3-dimensional cube of \(G\).

**Lemma 3** (Convex=Gated). Convex and gated subgraphs of \(G\) are the same.

We say that two edges \(uv\) and \(u'v'\) of \(G\) are in relation \(\Theta_0\) if \(u'v'\) is a square of \(G\) and \(uv\) and \(u'v'\) are opposite edges of this square. Let \(\Theta\) denotes the reflexive and transitive closure of \(\Theta_0\). Denote by \(E_1, \ldots, E_\Theta\) the equivalence classes of the equivalence relation \(\Theta\) and call them \(\Theta\)-classes.

**Lemma 4** (Halfspaces). For any \(\Theta\)-class \(E_i\) of \(G\), the graph \(G_i = (V, E \setminus E_i)\) consists of exactly two connected components \(H_i'\) and \(H_i''\) that are gated subgraphs of \(G\); \(H_i'\) and \(H_i''\) are halfspaces of \(G\). If \(uv\) is any edge of \(E_i\), then \(H_i'\) and \(H_i''\) are the subgraphs of \(G\) induced by the sets \(W(u, v) = \{x \in V : d(u, x) < d(v, x)\}\) and \(W(v, u) = \{x \in V : d(v, x) < d(u, x)\}\).

The boundary \(\partial H_i'\) of a halfspace \(H_i'\) is the subgraph of \(H_i'\) induced by all vertices \(v'\) of \(H_i'\) having a neighbor \(v''\) in \(H_i''\); \(\partial H_i''\) is defined analogously and \(\partial H_i'\) and \(\partial H_i''\) are isomorphic by Lemma 4.

**Lemma 5** (Boundaries). For any \(\Theta\)-class \(E_i\) of \(G\), the boundaries \(\partial H_i'\) and \(\partial H_i''\) are gated.

A halfspace \(H_i'\) of \(G\) is called a peripheral halfspace if \(\partial H_i' = H_i'\). In a finite tree \(T\), the \(\Theta\)-classes are the edges of \(T\), the complementary halfspaces are the two subtrees obtained by removing an edge of \(T\), and the peripheral halfspaces are exactly the leaves of \(T\). In a rectangular grid \(\Gamma\), the \(\Theta\)-classes are the edges of \(\Gamma\) intersected by the same vertical or horizontal line, and
peripheral halfspaces are the two bounding vertical paths and the two bounding horizontal paths of $\Gamma$. By the next lemma, all median graphs have peripheral halfspaces.

From now on we suppose that the median graph $G$ is rooted at an arbitrary but fixed basepoint $v_0$. For any $\Theta$-class $E_i$, we assume that $v_0$ belongs to the halfspace $H'_i$. Let $d(v_0, H'_i) = \min\{d(v_0, x) : x \in H'_i\}$ denote the distance from $v_0$ to $H'_i$. Since $H'_i$ is gated by Lemma 4, the gate of $v_0$ in $H'_i$ is the unique vertex of $H'_i$ at distance $d(v_0, H'_i)$ from $v_0$.

**Lemma 6** (Peripheral Halfspaces). For any basepoint $v_0$ of $G$, any halfspace $H'_i$ maximizing the distance to $v_0$ is peripheral.

Since median graphs are bipartite, the choice of a basepoint $v_0$ defines a canonical basepoint orientation of the edges of $G$: an edge $uv$ is oriented from $u$ to $v$ (notation $\overrightarrow{uv}$) if $d(v_0, u) < d(v_0, v)$.

**Lemma 7** (Orientation). The basepoint orientation defines an orientation of all edges of $G$.

We denote the resulting oriented pointed graph by $\overrightarrow{G}_{v_0}$. For a vertex $v$, all vertices $u$ such that $\overrightarrow{uv}$ is an edge of $\overrightarrow{G}_{v_0}$ are called parents of $v$ and are denoted by $\Lambda(v)$. Equivalently, $\Lambda(v)$ consists of all neighbors of $v$ in the interval $I(v_0, v)$.

A median graph $G$ with a basepoint $v_0$ satisfies the downward cube property if for any vertex $v$, $v$ and all its parents $\Lambda(v)$ belong to a single cube of $G$.

**Lemma 8** (Downward Cube Property). $G$ satisfies the downward cube property.

Lemma 8 immediately implies the following upper bound on the number of edges of $G$.

**Corollary 1.** If $G$ has $n$ vertices, $m$ edges, and dimension $d$, then $m \leq dn \leq n \log n$.

Finally, we provide a lower bound on the number $q$ of $\Theta$-classes of $G$ which is new to the best of our knowledge.

**Proposition 1.** If $G$ has $n$ vertices, $q$ $\Theta$-classes, and dimension $d$, then $q \geq d(\sqrt[n]{m} - 1)$. This lower bound is realized for products of $d$ paths of length $(\sqrt[n]{m} - 1)$.

**Proof.** We consider the crossing graph $\Gamma(G)$ of $G$, where $V(\Gamma(G))$ is the set of $\Theta$-classes of $G$ and where two $\Theta$-classes are adjacent if there exists a square of $G$ with edges in both $\Theta$-classes. Observe that $|V(\Gamma(G))| = q$. Let $X(\Gamma(G))$ be the clique complex of $\Gamma(G)$. By the characterization of median graphs among ample classes [10, Proposition 4], the number of vertices of $G$ is equal to the number $|X(\Gamma(G))|$ of simplices of $X(\Gamma(G))$. Since $G$ is of dimension $d$, by [10, Proposition 4], $\Gamma(G)$ does not contain cliques of size $d + 1$. Consequently, by Zykov's theorem [57] (see also [56]), the number of simplices of size $k$ in $X(\Gamma(G))$ is at most $\binom{d}{k} \binom{2^{k}}{2}^k$. Consequently, $n = |V(G)| = |X(\Gamma(G))| \leq \sum_{k=0}^{d} \binom{d}{k} \binom{q^k}{2} k = (1 + \frac{q}{2})^d$ and thus $q \geq d(\sqrt[n]{m} - 1)$.

Assume now that $G$ is the Cartesian product of $d$ paths of length $(\sqrt[n]{m} - 1)$. Then $G$ has $(\sqrt[n]{m} - 1 + 1)^d = n$ vertices and $d(\sqrt[n]{m} - 1)$ $\Theta$-classes (since each $\Theta$-class of $G$ corresponds to an edge of one of the paths).

4. Computation of the $\Theta$-classes

In this section we describe two algorithms for computing the $\Theta$-classes of a median graph $G$: one with complexity $O(dm)$, uses BFS and the second, with optimal complexity $O(m)$, uses LexBFS.

4.1. $\Theta$-classes via BFS. The Breadth-First Search (BFS) is a classical level-by-level graph traversal algorithm. BFS refines the basepoint order and defines the same orientation $\overrightarrow{G}_{v_0}$ of $G$. BFS uses a queue $Q$ and the insertion in this queue defines a total order $<$ on the vertices of $G$: $x < y$ if and only if $x$ is inserted before $y$ in $Q$. When a vertex $u$ arrives at the head of $Q$, it is removed from $Q$ and all not yet discovered neighbors $v$ of $u$ are inserted in $Q$ (breaking ties arbitrarily); $u$ becomes the father $f(v)$ of such $v$: $f(v)$ is the smallest parent of $v \neq v_0$. The arcs $f(v)v$ define the BFS-tree of $G$. For each vertex $v$, BFS produces the list $\Lambda(v)$ of parents.
of \( v \) ordered by \( <; \) denote this ordered list by \( \Lambda_<(v) \). By Lemma 8, each list \( \Lambda_<(v) \) has size at most \( d := \text{dim}(G) \). Notice also that the total order \( < \) on vertices of \( G \) give raise to a total order on the edges of \( G \): for two edges \( uv \) and \( u'v' \) with \( u < v \) and \( u' < v' \) we have \( uv < u'v' \) if and only if \( u < u' \) or if \( u = u' \) and \( v < v' \).

Now we show how to use a BFS rooted at \( v_0 \) to compute, for each edge \( uv \) of a median graph \( G \), the unique \( \Theta \)-class \( E(uv) \) containing the edge \( uv \). Suppose that \( uv \) is oriented by BFS from \( u \) to \( v \), i.e., \( d(v_0, u) < d(v_0, v) \). There are only two possibilities: either the edge \( uv \) is the first edge of the \( \Theta \)-class \( E(uv) \) discovered by BFS or the \( \Theta \)-class of \( uv \) already exists. The following lemma shows how to distinguish between these two cases (compare with the definition of prime geodesic traces from [18, Subsection 5.2]):

**Lemma 9.** An edge \( uv \) with \( d(v_0, u) < d(v_0, v) \) of a median graph \( G \) is the first edge of a \( \Theta \)-class \( E_i \) of \( G \) if and only if \( u \) is the unique parent of \( v \), i.e., \( \Lambda_<(v) = \{ u \} \).

**Proof.** First suppose that \( uv \) is the first edge of \( E_i \) discovered by BFS. Since \( H'_i \) is gated, necessarily \( v \) is the gate of \( v_0 \) in \( H'_i \) and \( u \) is the unique neighbor of \( v \) in \( H'_i \). We assert that \( v \) has only \( u \) as a neighbor in \( I(v_0, v) \). Suppose by way of contradiction that \( v \) contains a second neighbor \( u'' \) in \( I(v_0, v) \). Since \( v \) is the gate of \( v_0 \) in \( H'_i \) and \( u'' \) is closer to \( v_0 \) than \( v \), necessarily \( u'' \) belong to \( H''_i \). But then \( v \) has two nonadjacent neighbors \( u \) and \( u'' \) in \( H''_i \), contrary to the convexity of \( H''_i \). Conversely, suppose that \( v \) has only \( u \) as a neighbor in \( I(v_0, v) \) but \( uv \) is not the closest to \( v_0 \) edge of \( E_i \). This implies that the gate \( x \) of \( v_0 \) in \( H'_i \) is different from \( v \). Let \( u' \) be a neighbor of \( v \) in \( I(x, v) \subseteq I(v_0, v) \). Since \( x, v \in H'_i \) and \( H'_i \) is convex, \( u' \) belongs to \( H'_i \). Since \( u \) belongs to \( H''_i \), we conclude that \( u \) and \( u' \) are two different neighbors of \( v \) in \( I(v_0, v) \), a contradiction. \( \Box \)

If \( uv \) is not the first edge of its \( \Theta \)-class, the following lemma shows how to find its \( \Theta \)-class:

**Lemma 10.** Let \( uv \) be an edge of a median graph with \( u \in \Lambda_<(v) \). If \( v \) has a second parent \( v' \), then there exists a square \( u'uvv' \) in which \( uv \) and \( u'v' \) are opposite edges and \( u'v' \in \Lambda_<(u) \cap \Lambda_<(v') \).

**Proof.** Indeed, by the quadrangle condition, the vertices \( u \) and \( v' \) have a unique common neighbor \( u' \) such that \( u'uvv' \) is a square of \( G \) and \( u' \) is closer to \( v_0 \) than \( u \) and \( v' \). Consequently, \( u' \in \Lambda_<(u) \cap \Lambda_<(v') \) and \( uv \) and \( u'v' \) are opposite edges of \( u'uvv' \). \( \Box \)

From Lemmas 9 and 10 we deduce the following algorithm for computing the \( \Theta \)-classes of \( G \). First, run a BFS and return a BFS-ordering of vertices and edges of \( G \) and the ordered lists \( \Lambda_<(v), v \in V \). Then consider the edges of \( G \) in the BFS-order. Pick a current edge \( uv \) and suppose that \( u \in \Lambda_<(v) \). If \( \Lambda_<(v) = \{ u \} \), by Lemma 9, \( uv \) is the first edge of its \( \Theta \)-class, thus create a new \( \Theta \)-class \( E_i \) and insert \( uv \) in \( E_i \). Otherwise, if \( v \) has a second parent \( v' \), then traverse the ordered lists \( \Lambda_<(u) \) and \( \Lambda_<(v') \) to find their unique common parent \( u' \) (which exists by Lemma 10). Then insert the edge \( uv \) in the \( \Theta \)-class of the edge \( u'v' \). Since the two sorted lists \( \Lambda_<(u) \) and \( \Lambda_<(v') \) are of size at most \( d \), their intersection (that contains only \( u' \)) can be computed in time \( O(d) \), and thus the \( \Theta \)-class of each edge \( uv \) of \( G \) can be computed in \( O(d) \) time. Consequently, we obtain:

**Proposition 2.** The \( \Theta \)-classes of a median graph \( G \) with \( n \) vertices, \( m \) edges, and dimension \( d \) can be computed in \( O(dm) = O(d^2n) \) time.

### 4.2. \( \Theta \)-classes via LexBFS

The **Lexicographic Breadth-First Search (LexBFS)**, proposed by Rose, Tarjan, and Lueker [47], is another classical graph traversal algorithm, refining the Breadth-First Search. In the standard BFS, if two vertices \( v \) and \( w \) have the same earliest predecessor, then the algorithm will order them arbitrarily. Instead, the LexBFS will choose between \( v \) and \( w \) by considering the ordering of their second-earliest predecessors. If only one of them has a second-earliest predecessor, then that one is chosen. If both \( v \) and \( w \) have the same second-earliest predecessor, then the tie is broken by considering their third-earliest predecessor, and so on. Applying this rule directly would lead to an inefficient algorithm. Instead, the LexBFS uses a set partitioning data structure in order to produce the same ordering more efficiently and can be implemented in linear time [47]. In median graphs, the next lemma shows that it is enough to consider only the earliest and second-earliest predecessors of the vertices:
Lemma 11. If $v$ and $w$ are two vertices of a median graph $G$, then $|\Lambda(v) \cap \Lambda(w)| \leq 1$.

Proof. Let $u \neq u'$ be two parents of $v$ and $w$. Since $u, u' \in \Lambda(v) \cap \Lambda(w)$, we conclude that $d(v_0, v) = d(v_0, w) = d(v_0, u) + 1 = d(v_0, u') + 1 = k + 1$. By the quadrangle condition, there exists a vertex $x$ adjacent to $u$ and $u'$ at distance $k-1$ from $v_0$. But then $u, u', v, w, x$ induce a forbidden $K_{3,3}$.

By Lemma 11 to implement LexBFS on median graphs, it suffices to keep for each vertex $v$ only its earliest parent, i.e., its father, and its second-earliest parent (if it exists). Therefore if two vertices $v$ and $w$ have the same earliest parent, then LexBFS will order $v$ before $w$ if and only if either the second-earliest parent of $v$ is ordered before the second earliest parent of $w$ or if $v$ has a second-earliest parent and $w$ does not. Similarly to BFS, LexBFS in median graphs can be implemented using a single queue $Q$. Additionally, each already labeled vertex $u$ must store the position $\pi(u)$ in $Q$ of the earliest vertex of $Q$ having $u$ as a single parent. In a BFS queue, all vertices having $u$ as their father occur consecutively. Additionally, among these vertices, the ones having a second parent must occur before the vertices having only $u$ as a parent and the vertices having a second parent must be ordered according to that second parent. To ensure this property, we use the following rule: if a vertex $v$ in $Q$, currently having only $u$ as a parent, discovers yet another parent $u'$, then $v$ is swapped in $Q$ with the vertex $\pi(u)$, and $\pi(u)$ is updated. Clearly this implementation gives a linear $O(m)$ time algorithm.

We say that a graph $G$ satisfies the fellow-traveler property if for any LexBFS ordering of vertices of $G$, for any edge $uv$ the fathers $f(u)$ and $f(v)$ are adjacent.

Theorem 1. Any median graph $G$ satisfies the fellow-traveler property.

Proof. Let $<$ be an arbitrary LexBFS order of the vertices of $G$ and $f$ be its father map. Since any LexBFS order is a BFS order, $<$ and $f$ satisfy the following elementary properties of BFS:

(BFS1) if $u < v$, then $f(u) \leq f(v)$;

(BFS2) if $f(u) < f(v)$, then $u < v$;

(BFS3) if $v \neq v_0$, then $f(v) = \min_{<} \{u : u \sim v\}$;

(BFS4) if $u < v$ and $v \sim f(u)$, then $f(v) = f(u)$.

Notice also the following simple but useful property:

Claim 1. Let $abcd$ be a square of $G$ with $d(v_0, c) = k$, $d(v_0, b) = d(v_0, d) = k+1$, and $d(v_0, a) = k+2$. If $f(a) = b$ and the edge $ad$ satisfies the fellow-traveler property, then $f(d) = c$.

Proof. Suppose $f(d) \neq c$. By the fellow-traveler property for $ad$, $b \sim f(d)$. Consequently, $a, b, c, d$, and $f(d)$ induce a forbidden $K_{3,3}$, a contradiction.

Now, we prove the fellow-traveler property by induction on the total order on the edges of $G$ defined by $<$ (in a similar way as for BFS). The proof is illustrated by several figures (the arcs of the father map are represented in bold). We will use the following convention: all vertices having the same distance to the basepoint $v_0$ will be labeled by the same letter but will be indexed differently; for example, $w_1$ and $w_2$ are two vertices having the same distance to $v_0$.

Suppose by way of contradiction that $e = u_1v_2$ with $v_3 < u_1$ is the first edge in the order $<$ such that the fathers $f(u_1)$ and $f(v_3)$ of $u_1$ and $v_3$ are not adjacent. Then necessarily $f(u_1) \neq v_3$. Set $v_1 = f(u_1)$ and $w_3 = f(v_3)$ (Fig. 1a). Since $d(v_0, v_1) = d(v_0, v_3)$ and $u_1 \sim v_1, v_3$, by the quadrangle condition $v_1$ and $v_3$ are a common neighbor at distance $d(v_0, v_1) - 1$ from $v_0$. This vertex cannot be $w_3$, otherwise $f(u_1)$ and $f(v_3)$ would be adjacent. Therefore there exists a vertex $v_4 \sim v_1, v_3$ at distance $d(v_0, v_1) - 1$ from $v_0$ (Fig. 1b). By induction hypothesis, the father $x_3 = f(w_3)$ of $w_3$ is adjacent to $w_3 = f(v_3)$. Since $u_1 \sim v_1 = f(u_1)$, $v_3$ and $v_4 \sim v_3 = f(v_3)$, $w_4$, by (BFS3) we conclude that $v_3 < v_4$ and $w_3 < w_4$. By (BFS2), $f(v_1) \leq f(v_3)$, whence $f(v_1) \leq w_3$ and since $f(v_1) \neq f(v_3)$ (otherwise, $f(u_1) \sim f(v_3)$), we deduce that $f(v_1) < w_3 < w_4$. Hence $f(v_1) \neq w_4$. Set $w_1 = f(v_1)$. By induction hypothesis, $f(v_1) = w_1$ is adjacent to $f(w_4) = x_3$ (Fig. 1b). By the cube condition applied to the squares $w_4v_1w_1x_3$, $w_4v_1w_1v_3$, and $w_4v_3w_3x_3$ there exists a vertex $v_2$ adjacent to $w_4$, $v_1$, and $w_3$. Since $u_1 \sim v_2$ and $f(u_1) = v_1$, by (BFS3) we obtain $v_1 < v_2$. Since $v_2$ is adjacent to $w_1$ and $w_1 = f(v_1)$, by (BFS4) we obtain
Claim 2. Let $H = (V', E')$ (Fig. 2a) be an induced graph of $G$, where $d(v_0, w_1) = d(v_0, x_2) + 1 = \ldots = d(v_0, x_5) + 1 = d(v_0, y_4) + 2 = d(v_0, y_5) + 2 = d(v_0, z_3) + 3$ and $f(x_5) = y_5$ and $f(x_4) = y_4$, such that $x_2 < x_3 < x_4 < x_5$ and $y_4 < y_5$. If $G$ satisfies the fellow-traveler property up to distance $d(v_0, w_1)$, then there exists a vertex $x_0$ such that $x_0 < x_2$ and $x_0 \sim w_1, y_4$ (Fig. 2b).
Since $G$ contains a subgraph $H$ satisfying the conditions of Claim 2, there exists a vertex $x_0$ such that $x_0 < x_2$ and $x_0 \sim w_1, y_4$ (Fig. 1). By the cube condition applied to the squares $x_3w_1x_0y_4, x_3w_1v_2w_3$, and $x_3w_3x_4y_4$, there exists $w_0 \sim x_0, v_2, x_4$ (Fig. 1). Since $x_0$ is adjacent to $w_0$, by (BFS3) $f(w_0) \leq x_0 < x_2 = f(w_2)$. By (BFS2), $w_0 < w_2$. Recall that $f(v_1) = w_1 = f(v_2)$ and that $w_2$ is the second-earliest parent of $v_1$. Since $w_0 < w_2$ and $w_0$ is a parent of $v_2$, by LexBFS we deduce that $v_2 < v_1$. Since $v_1$ and $v_2$ are both adjacent to $u_1$ we obtain a contradiction with $f(u_1) = v_1$. This contradiction shows that any median graph $G$ satisfies the fellow-traveller property. This finishes the proof of Theorem 1. \hfill $\square$

Finally, we prove Claim 2.

**Proof of Claim 2.** We proceed by contradiction and consider a median graph $G$ for which Claim 2 does not hold. Among all induced subgraphs of $G$ satisfying the conditions of the claim but for which there does not exist a vertex $x_0 \neq x_3 \sim u_1, y_4$ with $x_0 < x_2$, we select a copy of $H$ minimizing the distance $d(v_0, w_1)$. First, suppose $f(w_1) = x_2$. Applying Claim 4 to the square $w_1x_2y_5x_3$, we deduce $f(x_3) = y_5$. Then, by (BFS1), we get $y_5 = f(x_3) \leq f(x_4) \leq f(x_5) = y_5$. Hence, $f(x_4) = y_5$, a contradiction. Therefore $f(w_1) \neq x_2$. Since $G$ satisfies the fellow-traveler property up to distance $d(v_0, w_1)$, we get $f(x_2) \sim f(w_1)$. Let $x_1$ be the father of $w_1$ (Fig. 3a) and let $y_2 = f(x_2)$ be the father of $x_2$. To avoid an induced $K_{2,3}$, $y_2$ cannot coincide with $y_5$. Moreover, $y_2$ does not coincide with $y_4$ because otherwise $x_1$ would be the common neighbor of $w_1$ and $y_4$ required by Claim 2. Let $z_5$ be the father of $y_5$. By the fellow-traveler property, $z_5 = f(y_5)$ is adjacent to $y_2 = f(x_2)$. Applying the cube condition applied to the squares $x_2w_1x_1y_2, x_2w_1x_3y_5$, and $x_2y_2z_5y_5$, we find a neighbor $y_3$ of $x_3, x_1$, and $z_5$. If $z_5 = z_3$, then $y_3 = y_4$ (otherwise we get a $K_{2,3}$) and $x_1$ is the neighbor of $u_1$ and $y_4$ required by Claim 2, a contradiction. Therefore $y_3 \neq y_4$ and $z_5 \neq z_3$. Moreover, by Claim 1 applied to the square $w_1x_1y_3x_3, y_3 = f(x_3)$ (see Fig. 3b). Let $t$ be the father of $z_3$. By induction hypothesis, $z_5 = f(y_5) \sim t = f(z_3)$. Applying the cube condition to the squares $y_5z_3t_5, y_5x_3y_3z_5$, and $y_5x_4y_3z_3$, we find a neighbor $z_4$ of $t$, $y_2$ and $y_4$. By Claim 1 applied to the square $x_3y_3z_4y_4, f(y_4) = z_4$ (Fig. 3c) and by (BFS1), $x_2 < x_3 < x_4 < x_5$ implies $y_2 = f(x_2) < y_3 = f(x_3) < y_4 = f(x_4) < y_5 = f(x_5)$. Since $d(x_1, v_0) < d(w_1, v_0)$, our choice of $H$ implies the existence of a neighbor $y_0$ of $x_1$ and $z_4$ such that $y_0 < y_2$ (Fig. 3d). Applying the cube condition to the squares $y_3x_1y_0z_4, y_3x_1w_1x_3$ and

**Figure 2.** The induced subgraph $H$ of Claim 2

**Figure 3.** To the proof of Claim 2
Algorithm 1: Θ-classes via LexBFS

Data: $G = (V, E), v_0 \in V$
Result: The Θ-classes Θ of $G$ ordered by increasing distance from $v_0$
begin
  $\Theta \leftarrow \emptyset$
  $E, \Lambda_\prec, f \leftarrow \text{LexBFS}(V, E)$
  // $E$: the list of edges ordered by LexBFS
  // $\Lambda_\prec: V \mapsto 2^V$ such that $\Lambda_\prec(v)$ is the list of the parents of $v$
  // $f: V \mapsto V$ such that $f(v)$ is the father of $v$
  for $uv \in E$ do
    if $|\Lambda_\prec[v]| = 1$ then
      Add a new Θ-class $\{uv\}$ to $\Theta$  // first edge in the Θ-class
    else if $f(v) = u$ then
      Pick any $x$ in $\Lambda_\prec(v) \setminus \{u\}$
      Add the edge $uv$ to the Θ-class of the edge $f(x)x$
    else
      Add the edge $uv$ to the Θ-class of the edge $f(u)f(v)$
  return $\Theta$
end

Now we show how to use Theorem 1 to compute the Θ-classes of $G$ is $O(m)$ time. As in the case of BFS, we run a LexBFS and return a LexBFS-ordering of vertices and edges of $G$ and the ordered lists $\Lambda_\prec(v), v \in V$. Then consider the edges of $G$ in the LexBFS-order. Pick a current edge $uv$ and suppose that $u \in \Lambda_\prec(v)$. If $\Lambda_\prec(v) = \{u\}$, by Lemma 2, $uv$ is the first edge of its Θ-class, thus we create a new Θ-class $E_i$ and insert $uv$ as the first edge of $E_i$. We call $uv$ the root of $E_i$ and we keep $d(v_0, v)$ as the distance from $v_0$ to $H'$. Otherwise, if $u \neq f(v)$, by Theorem 1 the vertices $u, v, f(v)$, and $f(u)$ define a square of $G$ and $uv$ and $f(u)f(v)$ are opposite edges of this square. Therefore $uv$ belongs to the Θ-class of $f(u)f(v)$ (which was already computed because $f(u)f(v) < uv$ in the LexBFS order). In order to recover the Θ-class of the edge $f(u)f(v)$ in constant time, we use a (non-initialized) matrix $M$ whose rows and columns correspond to the vertices of $G$ such that $M[x, y]$ contains the Θ-class of the edge $xy$ when $x$ and $y$ are adjacent and the Θ-class of $xy$ has already been computed and $M[x, y]$ is undefined if $x$ and $y$ are not adjacent or if the Θ-class of $xy$ has not been computed yet. Finally, if $|\Lambda_\prec(v)| \geq 2$ and $u = f(v)$, then pick $x \in \Lambda_\prec(v), x \neq u$. By Theorem 1 the vertices $u = f(v), v, x$, and $f(x)$ define a square of $G$ and $uv = f(v)v$ and $f(x)x$ are opposite edges of this square. Since $f(x)x$ appears before $uv$ in the LexBFS order, the Θ-class of $f(x)x$ has already been computed, and the algorithm inserts $uv$ in the Θ-class of $f(x)x$. Notice that each Θ-class $E_i$ is totally ordered by the order in which the edges are inserted in $E_i$. Consequently, we obtain (see Algorithm 1 for the pseudo-code):

**Theorem 2.** The Θ-classes of a median graph $G$ with $m$ edges can be computed in linear $O(m)$ time.

5. Computation of the median and of the Wiener index

In this section, we use Theorem 2 to compute the median set $\text{Med}_w(G)$ and the Wiener index $W_w(G)$ of a median graph $G$ in $O(m)$ time. In both algorithms, we use the existence of peripheral halfspaces, their contraction (or the contraction of their complements) and the majority rule for the median problem. In both algorithms, the vertices of the contracted halfspaces transmit their weight to their gates in the complements. We use the following notation: for a weight function $w: V \mapsto \mathbb{R}^+ \cup \{0\}$ and a set $S \subseteq V$, let $w(S) = \sum_{x \in S} w(x)$. 

$y_3 x_3 y_4 z_4$, we find a neighbor $x_0$ of $w_1, y_4$, and $y_0$. By (BFS3), $f(x_0) \leq y_0 < y_2 = f(x_2)$ and thus, by (BFS2), $x_0 < x_2$ (Fig. 3d), a contradiction with our choice of $H$. □
5.1. Peripheral halfspaces. We start with the use and the detection of peripheral halfspaces. Notice that the order \(E_1, E_2, \ldots, E_q\) in which the \(\Theta\)-classes \(E_i\) of \(G\) are constructed correspond to the distances from \(v_0\) to \(H_i^r\), i.e., if \(i < j\), then \(d(v_0, H_i^r) \leq d(v_0, H_j^r)\) (recall that in all our results we suppose that \(v_0\) belongs to \(H_q^r\), \(i = 1, \ldots, q\)). By Lemma 6 the halfspace \(H_q^r\) of the last discovered \(\Theta\)-class \(E_q\) is a peripheral halfspace. If we contract all edges of \(E_q\) of graph \(G_q := G\) (i.e., we identify the vertices of \(H_q^r = \partial H_q^r\) with their neighbors in \(\partial H_q^r\)) we get a smaller median graph \(G_{q-1} = H_{q-1}^m\), which is a gated subgraph of \(G_q\). The median graph \(G_{q-1}\) has \(q - 1\) \(\Theta\)-classes \(E_1', \ldots, E_{q-1}'\), where \(E_i'\) consists of the edges of \(E_i\) belonging to \(G_{q-1}\). Analogously, \(E_1', \ldots, E_{q-1}'\) correspond to the ordering of halfspaces \(H_1', \ldots, H_{q-1}'\) of \(G_{q-1}\) by their distances to \(v_0\). Therefore the last halfspace \(H_q^r\) of \(G_q\) is a peripheral halfspace of \(G_{q-1}\). Therefore the ordering \(E_1, \ldots, E_{q-1}, E_q\) of the \(\Theta\)-classes of \(G\) provide us at each iteration \(i\) with the \(\Theta\)-class \(E_i\) defining a peripheral halfspace in the median graph \(G_i\) obtained after the successive contractions of the peripheral halfspaces of the graphs \(G_1, G_2, \ldots, G_i\). Since each vertex of \(G\) and each \(\Theta\)-class is contracted only once, we do not need to compute explicitly the restriction of each \(\Theta\)-class of \(G\) to the current median graph \(G_i\). For this it is enough to keep for each vertex \(v\) a variable, indicating if this vertex was already contracted or not. Using this, when the restriction of \(E_i\) on \(G_i\) must be contracted, we simply traverse the edges of \(E_i\) and select those edges whose both ends are not yet contracted.

5.2. The Wiener index \(W_w(G)\). Since the algorithm for computing the Wiener index \(W_w(G)\) is simpler, we present it first. Similarly to the algorithms for trees \[38\] or for benzenoids \[23\], we have to show how are related the Wiener indices of a median graph \(G\) and of the median graph \(G' = H''\) obtained after the contraction of a peripheral halfspace \(H'\) of \(G\). On \(G'\) we define the following weight function \(w'\): for each vertex \(v \in H'' \setminus \partial H''\), set \(w'(v) = w(v)\) and for each vertex \(v'' \in \partial H''\) adjacent to the vertex \(v'\) of \(\partial H' = H'\) we set \(w'(v'') = w(v'') + w(v')\). The key ingredient is the following simple lemma:

**Lemma 12.** \(W_w(G) = W_w'(G') + w(H')w(H'')\).

**Proof.** The contraction of \(G\) to \(G'\) affects the contributions to \(W_w(G)\) and to \(W_w'(G')\) only of the pairs of vertices \(x', y\) with \(x' \in H'\) and \(y \in H''\). For every such pair \(x', y\), assigning the weight of \(x'\) to its neighbor \(x''\) in \(H''\) (which is the gate of \(x'\) in \(H''\)) decreases the distance by one and the weighted distance by \(w(x')w(y)\). Summing over all such pairs \(x', y\), this operation decreases the Wiener index by \(w(H')w(H'')\).

**Algorithm 2: Wiener(G, w, \Theta)**

**Data:** \(G = (V, E)\) a median graph, \(w : V \to \mathbb{R}\), \((E_1, \ldots, E_q)\) : the \(\Theta\)-classes ordered by increasing distance to the basepoint \(v_0\)

**Result:** The Wiener index \(W_w(G)\) of \(G\)

1 begin
2 \[W \leftarrow 0\]
3 \[\text{for } i = q, \ldots, 1 \text{ do}\]
4 \[w(H'_i) \leftarrow \sum_{v \in H'_i} w(v)\]
5 \[w(H''_i) \leftarrow w(V) - w(H'_i)\]
6 \[W \leftarrow W + w(H'_i)w(H''_i)\]
7 \[\text{for } v' \in H'_i \text{ do}\]
8 \[\text{Let } v'' \text{ be the neighbor of } v' \text{ in } H''_i\]
9 \[w(v'') \leftarrow w(v') + w(v'')\]
10 \[\text{Remove from } G \text{ the vertices of } H'_i \text{ and set } G \leftarrow H''_i\]
11 return \(W\)

By Lemma 12 the Wiener index \(W_w(G)\) can be computed in the following way. The algorithm traverses the \(\Theta\)-classes in the order \(E_q, \ldots, E_1\). By our previous discussion, we know that after
contracting the classes \(E_i, \ldots, E_{i+1}\), the halfspace \(H'_i\) of the \(\Theta\)-class \(E_i\) is peripheral in the current graph. Initially, we set \(W = 0\). If the current median graph \(G\) contains a single vertex, then return \(W\) as the Wiener index \(W_w(G)\) of \(G\). Otherwise, pick the peripheral halfspace \(H'_i = H'_1\) of \(G\). Traverse the vertices of \(H'_i\) (by considering the edges of \(E_i\)) to compute the weight \(w(H'_i)\) of \(H'_i\). Set \(w(H'_i) = w(G) - w(H'_i)\) and update \(W\) by setting \(W \leftarrow W + w(H'_i)w(H'_i)\). Update \(G\) by setting \(G \leftarrow H''\) and update the weight function \(w\) as follows: traverse the vertices of \(H'\) and for each vertex \(v' \in H'\) and its neighbor \(v''\) in \(H''\), set \(w(v'') \leftarrow w(v') + w(v'')\). We obtain the following result (see Algorithm 2 for the pseudo-code):

**Proposition 3.** The Wiener index of a median graph \(G\) with \(m\) edges can be computed in linear \(O(m)\) time.

### 5.3. The median \(\text{Med}_w(G)\)

We continue with the computation of medians of median graphs. We start with a simple property of the median function \(F_w\), which directly follows from the second assertion of Lemma 1.

**Lemma 13.** If \(xy\) is an edge of \(G\) and \(xy\) belong to the \(\Theta\)-class \(E_i\) with \(x \in H'_i\) and \(y \in H''_i\), then \(F_w(x) - F_w(y) = w(H'_i) - w(H''_i)\).

We also restate the majority rule in the way we will use it. A halfspace \(H\) of a median graph \(G\) is called a majority halfspace if \(w(H) > \frac{1}{2}w(G)\), a minority halfspace if \(w(H) < \frac{1}{2}w(G)\), and a egalitarian halfspace if \(w(H) = \frac{1}{2}w(G)\).

**Proposition 4.** Let \(E_i\) be a \(\Theta\)-class of a median graph \(G\) and let \(H'_i, H''_i\) be the two halfspaces defined by \(E_i\). If \(H'_i\) (or \(H''_i\)) is a majority halfspace, then \(\text{Med}_w(G) \subseteq H'_i\) (respectively \(\text{Med}_w(G) \subseteq H''_i\)). If \(H'_i\) and \(H''_i\) are egalitarian halfspaces, then \(\text{Med}_w(G)\) intersects the boundaries of both \(H'_i\) and \(H''_i\) and \(x' \in \partial H'_i\) is in \(\text{Med}_w(G)\) if and only if its neighbor \(x''\) in \(\partial H''_i\) is also in \(\text{Med}_w(G)\).

**Proof.** Let us first prove an auxiliary claim from which the different statements of Proposition 4 easily follow.

**Claim 3.** Let \(E_i\) be a \(\Theta\)-class of a median graph \(G\) and let \(H'_i, H''_i\) be the two halfspaces defined by \(E_i\). If \(x'' \in H''_i\) and \(x'\) is its gate in \(H'_i\), then \(F_w(x'') \geq F_w(x) + d(x'', x')(w(H'_i) - w(H''_i))\).

**Proof.** By definition of the median function,

\[
F_w(x'') - F_w(x') = \sum_{u \in V} d(x'', u)w(u) - \sum_{u \in V} d(x', u)w(u) = \sum_{u \in V} (d(x'', u) - d(x', u))w(u).
\]

Then, we decompose the sum over the complementary halfspaces \(H'_i\) and \(H''_i\):

\[
F_w(x'') - F_w(x') = \sum_{u' \in H'_i} (d(x'', u') - d(x', u'))w(u') + \sum_{u'' \in H''_i} (d(x'', u'') - d(x', u''))w(u'').
\]

Since \(x'\) is the gate of \(x''\) on \(H'_i\), for any \(u' \in H'_i\), \(d(x'', u') - d(x', u') = d(x'', x')\). By triangle inequality, \(d(x'', u'') - d(x', u'') \geq -d(x'', x')\) for every \(u'' \in H''_i\). We get

\[
F_w(x'') - F_w(x') \geq \sum_{u \in H'_i} d(x'', x')w(u') - \sum_{u'' \in H''_i} d(x'', x')w(u'')
\]

and conclude that \(F_w(x'') \geq F_w(x') + d(x'', x')(w(H'_i) - w(H''_i))\). 

Let \(H''_i\) and \(H'_i\) be two complementary halfspaces such that \(w(H'_i) > w(H''_i)\). Pick any vertex \(x'' \in H''_i\) and its gate \(x'\) in \(H'_i\). By Claim 3, \(F_w(x'') > F_w(x')\) and therefore \(x''\) cannot be a median. This shows that the complement of a majority halfspace does not contain any median vertex.

Now, consider two egalitarian complementary halfspaces \(H''_i\) and \(H'_i\). Suppose that a median vertex \(x'\) belongs to \(H'_i\) and let \(x''\) be its gate on \(H''_i\). By Claim 3, \(F_w(x'') \leq F_w(x')\). Therefore, \(x''\) is also median. By symmetry, we conclude that both \(H'_i\) and \(H''_i\) contain a median vertex. It remains to show that \(x' \in \partial H'_i \cap \text{Med}_w(G)\) if and only if its neighbor \(x''\) in \(H''_i\) also belongs to \(\text{Med}_w(G)\). By definition, \(x'\) is the gate of \(x''\) in \(H'_i\) and \(x''\) is the gate of \(x'\) in \(H''_i\). Therefore,
applying Claim 3 to \( x' \) and \( x'' \) in both ways, we get that \( F_w(x') = F_w(x'') \). This concludes the proof of Proposition 4. \( \Box \)

Let \( H' \) and \( H'' \) be complementary halfspaces of \( G \). By Lemma 13, \( H' \) and \( H'' \) are gated. For a vertex \( v \in H'' \), recall that the fiber \( P(v) \) of \( v \) consists of all vertices of \( G \) having \( v \) as a gate in \( H'' \). Set \( G' = H'' \) and define a weight function \( w' \) on \( G' \) in the following way: for any \( v \) in \( G' \) let \( w'(v) = w(P(v)) \). The next lemma establishes a relationship between \( \text{Med}_w(G) \) and \( \text{Med}_w(G') \):

**Lemma 14.** If \( H'' \) is a majority halfspace, then \( \text{Med}_w(G') = \text{Med}_w(G) \). If \( H'' \) is an egalitarian halfspace, then \( \text{Med}_w(G') = \text{Med}_w(G) \cap H'' \). Additionally, if \( H' \) is a peripheral halfspace of \( G \) and \( M' = \{ v' \in H' : v' \sim v'' \text{ and } v'' \in \text{Med}_w(G') \cap H'' \} \), then \( \text{Med}_w(G) = \text{Med}_w(G') \cup M' \).

**Proof.** Let \( x \) be an edge of \( G' \), say \( xy \) belongs to the \( \Theta \)-class \( E_i \) with \( x \in H'_i \) and \( y \in H''_i \). Then \( H'_i \cap H''_i \) and \( H''_i \cap H''_i \) are the halfspaces of \( G' \) defined by \( E_i \). By definitions of fibers, if \( v \) belongs to \( H'_i \cap H''_i \) (respectively, to \( H''_i \cap H''_i \)), then the fiber \( P(v) \) belongs to \( H'_i \) (respectively, \( H''_i \)). Applying Lemma 13 to median functions \( F_w \) and \( F_w' \), we have that \( F_w(x) - F_w(y) = w(H''_i) - w(H'_i) \) and \( F_w'(x) - F_w'(y) = w'(H''_i \cap H''_i) - w'(H'_i \cap H''_i) \). Therefore, the right-hand sides of both expressions coincide, yielding that \( F_w'(x) - F_w'(y) = F_w'(x) - F_w'(y) \). This implies that the functions \( F_w \) and \( F_w' \) have the same sets of minima on \( G' \). By Proposition 4 we are done. \( \Box \)

**Algorithm 3: Median(G, w, Θ)**

**Data:** A median graph \( G = (V, E) \), a weight function \( w: V \to \mathbb{R} \), the \( \Theta \)-classes \( \Theta = (E_1, \ldots, E_q) \) ordered by increasing distance to the basepoint \( v_0 \).

**Result:** The median set \( \text{Med}_w(G) \)

```plaintext
begin
    if G contains a single vertex v then
        return \{v\}
    else
        Let \( H' \) and \( H'' \) be two complementary halfspaces defined by \( E_q \) (as usual \( v_0 \in H'' \))
        Set \( w(H') \leftarrow \sum_{v \in H'_i} w(v) \) and \( w(H''') \leftarrow w(V) - w(H') \)
        EgalitarianHalfspaces \( \leftarrow (w(H') = w(H'')) \)
        if \( w(H') \leq w(H'') \) then
            for \( v' \in H' \) do
                \( v'' \leftarrow \) the neighbor of \( v' \) in \( H'' \)
                \( w(v'') \leftarrow w(v') + w(v'') \)
                \( \Theta' \leftarrow (E_1, \ldots, E_{q-1}) \)
                \( M \leftarrow \text{Median}(H'', w, \Theta') \)
                if EgalitarianHalfspaces then
                    Set \( M' \leftarrow \{ v' \in H' : v' \sim v'' \text{ and } v'' \in M \} \)
                    return \( M \cup M' \)
                else
                    return \( M \)
            else
                Using a BFS from \( H' \), compute the fiber \( P(v) \) of every vertex \( v \in H' \)
                for \( v \in H' \) do
                    \( w'(v) \leftarrow w(P(v)) \)
                    \( \Theta' \leftarrow (E_1, \ldots, E_{q-1}) \) reordered with respect to a BFS order from a vertex \( v_0 \in H' \)
                return Median(H', w', \Theta')
end
```

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As in the case of the Wiener index, the algorithm computing Med\(_w(G)\) traverses the \(\Theta\)-classes in order \(E_q, \ldots, E_1\) so that after contracting the classes \(E_q, \ldots, E_{i+1}\), the halfspace \(H_i^r\) of the \(\Theta\)-class \(E_i\) is peripheral in the current graph. If the current median graph \(G\) contains at least two vertices, then pick the peripheral halfspace \(H' = H_i^r\) of \(G\) (corresponding to the \(\Theta\)-class \(E_i\)). Traverse the vertices of \(H'\) (by considering the edges of \(E_i\)) to compute the weight \(w(H')\) of \(H'\). Set \(w(H'') = w(G) - w(H')\). We have three possibilities.

**Case 1.** \(H'\) is a minority halfspace of \(G\).

By Lemma \([14]\) Med\(_w(G)\) = Med\(_{w'}(G')\), where \(G' = H''\) and the weight function \(w'\) on \(H''\) is easily computable in the following way: for each vertex \(v' \in H'\) having the neighbor \(v''\) in \(H''\), set \(w'(v'') \leftarrow w(v') + w(v'')\). Then we recursively call the algorithm to the median graph \(G' = H''\) endowed with the weight function \(w'\). The complexity of this step is \(O(|V(H'')|) = O(|E_i|)\) and the algorithm does not consider the halfspace \(H'\) again.

**Case 2.** \(H'\) is a majority halfspace of \(G\).

Again, by Lemma \([14]\) Med\(_w(G)\) = Med\(_{w'}(G')\), where \(G' = H'\) and \(w'(v) = w(P(v))\) for each \(v \in H'\). Therefore, in this case we project the minority halfspace \(H''\) on the peripheral halfspace \(H'\) and to each \(v \in H'\) we assigns the weight of its fiber \(P(v)\) in \(G\) as its weight. Therefore, we have to find for all \(v \in H'\) its fiber \(P(v)\) and its weight. This can be done in time linear in the number of edges of the halfspace \(H''\) by running a simultaneous BFS from each vertex of \(H'\). This can be implemented by a single BFS, by inserting at the beginning all vertices of \(H'\) in a queue \(Q\). Therefore, the complexity of this step is \(O(|E(H'')|)\) and the algorithm does not consider the halfspace \(H''\) again. Consequently, we have to recursively call the algorithm to the median graph \(G' = H''\) endowed with the weight function \(w'\). Since the basepoint \(v_0\) does no longer belong to \(H'\), we have to reorder the \(\Theta\)-classes of \(G'\). Since the \(\Theta\)-classes of \(G'\) are already known, we run a BFS from an arbitrary vertex \(v'_0\) of \(G'\) to reorder them. This will take \(O(|E(G'')|) = O(|E(H'')|)\) additional time. Since \(H'\) is peripheral, \(H'\) is isomorphic to \(\partial H''\), thus \(O(|E(G'')|)\) is at most \(O(|E(H'')|)\).

**Case 3.** \(H'\) and \(H''\) are egalitarian halfspaces of \(G\).

By Lemma \([14]\) Med\(_w(G)\) = Med\(_{w'}(G')\) \(\cup M'\), where \(G' = H''\) and the weight function \(w'\) on \(H''\) is defined and computed as in Case 1. In this case, we recursively compute Med\(_{w'}(G')\) by calling the algorithm to the median graph \(G' = H''\) endowed with the weight function \(w'\). Now, suppose that the algorithm returned Med\(_{w'}(G')\). To compute Med\(_w(G)\) we have to traverse again all vertices of \(H'\) and to include in \(M'\) those vertices \(v'\) of \(H'\) whose neighbors in \(H''\) belong to Med\(_{w'}(G')\). This will take additional \(O(|E_i|) = O(|V(H')|)\) time.

Consequently, we obtained an algorithm (see Algorithm \([3]\) for the pseudo-code) that correctly computes Med\(_w(G)\). Given the \(\Theta\)-classes of \(G\), this algorithm traverses each vertex and each edge of \(G\) a constant number of times. We obtain the main result of the paper:

**Theorem 3.** For any weight function \(w\) on a median graph \(G\) with \(m\) edges, the median Med\(_w(G)\) can be computed in linear \(O(m)\) time.

**Question 1.** Is it possible to use the fast computation of \(\Theta\)-classes of median graphs to efficiently compute other metric parameters such as the diameter or the center of median graphs?
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Appendix A. Proofs from Section 3

We present the proofs of all auxiliary results from Section 3. Some of those results are well known to the people working in metric graph theory, other results were also known to us, but it is difficult to give references first presenting them.

Proof of Lemma 3. Let $x$ be the median of the triplet $u, v, w$. Then $x$ must be adjacent to $v, w$ and must have distance $k - 1$ to $u$. If there exists yet another such vertex $x'$, then the triplet $u, v, w$ will have two medians (or alternatively, the vertices $v, w, x, x'$ will define a $K_{2,3}$).

Proof of Lemma 4. Pick any three squares $xutv, utvy,$ and $vtwz$ of $G$, pairwise intersecting in three edges and all three intersecting in a single vertex. Since $G$ is bipartite and $K_{2,3}$-free, $d(x, w) = d(y, v) = d(z, u) = 3$ and $d(x, y) = d(y, z) = d(z, x) = 2$. Therefore, the median of vertices $x, y, z$ is a new vertex $r$ adjacent to $x, y$, and $z$ and having distance 3 to $t$. Consequently, the 8 vertices induce an isometric 3-cube of $G$.

Proof of Lemma 5. That gated sets $S$ are convex holds for all graphs (and metric spaces). Indeed, pick $x, y \in S$ and any $z \in I(x, y)$. Let $z'$ be the gate of $z$ in $S$. Then $z' \in I(z, x) \cap I(z, y)$. Since $z \in I(x, y)$ this is possible only if $z' = z$, i.e., $z \in S$. Conversely, let $S$ be a convex subgraph of a median graph $G$ and pick any vertex $x \notin S$. Let $v$ be a closest to $x$ vertex of $S$. Pick any vertex $y \in S$ and let $u$ be the median of the triplet $x, y, v$. Since $u \in I(y, v)$ and $S$ is convex, we deduce that $u \in S$. Since $u \in I(x, v)$, from the choice of $v$ we have $u = v$. Thus $v \in I(x, y)$, i.e., $v$ is the gate of $x$ in $S$.

The convexity of halfspaces of median graphs and of their boundaries was first established by Mulder [39]. We give a different and shorter proof of this result.

Proof of Lemma 4. For an edge $uv$ of a median graph $G$, recall that $W(u, v) = \{x \in V : d(u, x) < d(v, x)\}$ and $W(v, u) = \{x \in V : d(v, x) < d(u, x)\}$. We assert that $W(u, v)$ and $W(v, u)$ are convex. We use the local characterization of convexity of median graphs (and more general classes of graphs, see [20]): a connected subgraph $H$ of a median graph $G$ is convex if and only if $I(x, y) \subseteq H$ for any two vertices $x, y$ of $H$ with $d_H(x, y) = 2$. Since both sets $W(u, v)$ and $W(v, u)$ induce connected subgraphs, we can use this result. Pick $x, y \in W(u, v)$ such that $x$ and $y$ have a common neighbor $z$ in $W(u, v)$. Suppose by way of contradiction that there exists a vertex $t \sim x, y$ belonging to $W(v, u)$. Then $d(u, x) = d(u, y) = d(v, t) = k$ and $d(v, x) = d(v, y) = d(u, t) = k + 1$. Since $G$ is bipartite, $d(u, z)$ is either $k - 1$ or $k + 1$. If $d(u, z) = k + 1$, by the quadrangle condition we will find a vertex $s \sim x, y$ at distance $k - 1$ from $u$. But then the vertices $x, y, z, s, t$ induce a forbidden $K_{2,3}$. Thus $d(u, z) = k - 1$, i.e., $d(v, z) = k$. Therefore $z, t \in I(x, v), x \sim z, t$, and by the quadrangle condition there exists a vertex $r \sim t, z$ with $d(r, v) = k - 1$. But then again $x, y, z, t, r$ induce a forbidden $K_{2,3}$. This contradiction establishes that for each edge $uv$ of $G$, $W(u, v)$ and $W(v, u)$ induce convex and thus gated subgraphs of $G$.

Next, define another binary relation $\Psi$ on edges of $G$: for two edges $uv$ and $xy$ we write $uv\Psi xy$ if and only if $x \in W(u, v)$ and $y \in W(v, u)$. It can be easily seen that the relation $\Psi$ is reflexive and symmetric. Next we will prove that $\Psi$ is transitive and that $\Psi$ and $\Theta$ coincide. For transitivity of $\Psi$ it suffices to show that if $uv\Psi xy$, then $W(u, v) = W(x, y)$. Suppose by way of contradiction that there exists a vertex $z \in W(u, v) \setminus W(x, y)$. This implies that $z \in W(y, x)$, i.e., $y \in I(x, z)$. Since $x, z \in W(u, v)$ and $y \in W(v, u)$, this contradicts the convexity of $W(u, v)$. Consequently, $\Psi$ is an equivalence relation.

To conclude the proof of the lemma it remains to show that $\Theta = \Psi$. It is obvious that $\Theta_0 \subseteq \Psi$. Since $\Theta$ is the transitive closure of $\Theta_0$, we conclude that $\Theta \subseteq \Psi$. To show the converse inclusion $\Psi \subseteq \Theta$, pick any two edges $uv$ and $xy$ with $uv\Psi xy$. We proceed by induction on $k = d(u, x) = d(v, y)$. Let $x'$ be a neighbor of $x$ in $I(x, u) \subseteq W(u, v)$. Then $d(x', v) = d(y, v) = k$ and $d(x, v) = k + 1$. By the quadrangle condition, there exists a vertex $y' \sim x', y$ at distance $k - 1$ from $v$. Since $y' \in I(y, v) \subseteq W(v, u)$, we conclude that $uv\Psi x'y'$. Since $d(u, x') = d(v, y') = k - 1$, by induction hypothesis $uv\Theta x'y'$. Since $x'y'$ and $xy$ are opposite edges of a square, $x'y'\Theta_0 xy$, yielding $uv\Theta xy$. This finishes the proof of Lemma 4. □
Proof of Lemma 6. The last argument of the proof of Lemma 4 implies that the boundaries \( \partial H'_i, \partial H''_i \) are connected subgraphs of \( G \). Thus it suffices to show that they are locally convex. Pick \( x', y' \in \partial H'_i \) having a common neighbor \( z' \in \partial H'_i \). Let \( x'', y'', z'' \) be the neighbors of respectively \( x', y', z' \) in \( \partial H''_i \) (they are unique because \( H''_i \) is gated). Pick any common neighbor \( t \) of \( x', y' \) different from \( z' \). Since \( H'_i \) is convex, \( t \) belongs to \( H'_i \). By the cube condition, there exists a vertex \( s \) adjacent to \( t, x'', y'' \). But then obviously \( s \in H''_i \), whence \( t \in \partial H'_i \). \( \square \)

Proof of Lemma 7. We have to prove that any furthest from the basepoint \( v_0 \) halfspace \( H'_i \) of \( G \) is peripheral. Let \( x \) be the gate of \( v_0 \) in \( H'_i \). Then \( x \in \partial H'_i \) and \( d(v_0, x) = d(v_0, H'_i) \). Suppose by way of contradiction that the boundary \( \partial H'_i \) is a proper subset of \( H'_i \) and let \( v \) be a closest to \( v_0 \) vertex in \( H'_i \setminus \partial H'_i \) which is adjacent to a vertex \( u \) of \( \partial H'_i \) (such a vertex exists because \( H'_i \) is convex and thus connected). Let \( E_j \) be the \( \Theta \)-class of the edge \( uv \). Since \( u \) is the gate of \( v \) in \( \partial H'_i \), \( u \in I(x, v) \). Since \( x \in I(v_0, v) \), we conclude that there exists a shortest \((v_0, v)\)-path \( P(v_0, v) \) passing via \( x \) and \( u \). Consequently, \( v_0, u \in H''_i \) and \( v \in H'_i \).

Let \( y \) be the gate of \( v_0 \) in \( H'_i \). Clearly, \( y \in \partial H'_i \) and \( d(v_0, y) = d(v_0, H'_i) \). From the choice of \( H'_i \) as a furthest from \( v_0 \) halfspace, \( d(v_0, y) \leq d(v_0, x) \). Since \( x \) is the gate of \( v_0 \) in \( H'_i \), this implies that \( y \) cannot be located in \( H'_i \). Therefore \( y \) belongs to \( H''_i \). Let \( z \) be the gate of \( y \) in \( H'_i \) (and \( \partial H'_i \)) and note that \( z \in I(y, v) \). Since \( u \) is the gate of \( v \) in \( \partial H'_i \), we also have \( u \in I(v, z) \). Consequently, \( u \in I(v, y) \) and belongs to \( H'_i \) since \( H'_i \) is convex, which is impossible. This shows that \( H'_i \) is peripheral and finishes the proof of Lemma 7. \( \square \)

Proof of Lemma 8. This result is trivial. \( \square \)

Proof of Lemma 9. To prove the downward cube property, pick any vertex \( v \) and let \( u_1, \ldots, u_d \) be the parents of \( v \), i.e., the neighbors of \( v \) in \( I(v_0, v) \). Since all cubes of \( G \) are locally convex, they are convex and gated, if we prove that \( v \) and any subset of \( k \) parents of \( v \) belong to a cube of dimension \( k \), then this \( k \)-cube is unique. We proceed by induction on \( d(v_0, v) \) and on the number \( k \) of chosen parents. By the quadrangle condition, this assertion holds for \( v \) and any two parents of \( v \). Now pick \( v \), and the \( k \) parents \( u_1, \ldots, u_k \) and assume that \( v \) and any \( 2 \leq k' < k \) parents of \( v \) belong to a unique \( k' \)-cube. By the quadrangle condition, there exist common neighbors \( z_i \) of \( u_1 \) and \( u_i, i = 2, \ldots, k \) one step closer to \( v_0 \). The vertices \( z_2, \ldots, z_k \) are distinct parents of \( u_1 \). By induction hypothesis, \( u_1 \) and \( z_2, \ldots, z_k \) belong to a unique \((k - 1)\)-cube \( R' \). Analogously, \( v \) and its parents \( u_2, \ldots, u_k \) belong to a unique \((k - 1)\)-cube \( R'' \). We assert that \( R' \cup R'' \) is a \( k \)-cube of \( G \). Any vertex \( u_i, i = 2, \ldots, k \) has as parents \( z_i \) and the \( k - 2 \) neighbors in \( R'' \) different from \( v \). By induction hypothesis, \( u_i, z_i \), and the parents of \( u_i \) in \( R'' \) define a \((k - 1)\)-cube \( R_i \) of \( G \). This implies that there exists an isomorphism between the facet of \( R' \) containing \( z_i \) and not \( u_1 \) and the facet of \( R'' \) containing \( u_i \) and not containing \( v \). Since \( R' \) and \( R'' \) are convex, this defines an isomorphism between \( R' \) and \( R'' \) which maps vertices of \( R' \) to their neighbors in \( R'' \). Hence \( R' \cup R'' \) defines a \( k \)-cube of \( G \), finishing the proof of Lemma 9. \( \square \)