On a Strong Robust-Safety Notion for Differential Inclusions

Mohamed Maghenem and Diana Karaki

Abstract—A dynamical system is strongly robustly safe provided that it remains safe in the presence of a continuous and positive perturbation, named robustness margin, added to both the argument and the image of the right-hand side (the dynamics). Therefore, in comparison with existing robust-safety notions, where the continuous and positive perturbation is added only to the image of the right-hand side, the proposed notion is shown to be relatively stronger in the context of set-valued right-hand sides. Furthermore, we distinguish between strong robust safety and uniform strong robust safety, which requires the existence of a constant robustness margin. The first part of the article proposes sufficient conditions for strong robust safety in terms of barrier functions. The proposed conditions involve only the barrier function and the system’s right-hand side. Furthermore, we establish the equivalence between strong robust safety and the existence of a smooth barrier certificate. The second part of the article proposes scenarios, under which, strong robust safety implies uniform strong robust safety. Finally, we propose sufficient conditions for the latter notion in terms of barrier functions.

Index Terms—Constrained control, robust safety, robust control, stability of nonlinear systems.

I. INTRODUCTION

SAFETY is one of the most important properties to ensure for a dynamical system, as it requires the solutions starting from a given set of initial conditions to never reach a given unsafe region [1]. Depending on the considered application, reaching the unsafe set can correspond to the impossibility of applying a predefined feedback law due to saturation or loss of fidelity in the model, or simply, colliding with an obstacle or another system. Ensuring safety is in fact a key step in many engineering applications, such as traffic regulation [2], aerospace [3], and human–robot interactions [4].

A. Motivation

In this work, we consider the case where the dynamical system is given by the differential inclusion

\[ \Sigma : \dot{x} \in F(x) \quad x \in \mathbb{R}^n. \]  

Furthermore, we introduce two subsets \((X_o, X_u) \subset \mathbb{R}^n \times \mathbb{R}^n\), where \(X_o\) represents the subset of initial conditions and \(X_u\) represents the unsafe set. Hence, we necessarily have \(X_o \cap X_u = \emptyset\).

In the following, we recall the definition of safety as in classical literature [1, 5].

**Definition 1 (Safety):** System \(\Sigma\) is safe with respect to \((X_o, X_u)\) if, for each solution \(\phi\) starting from \(x_o \in X_o\), we have \(\phi(t) \in \mathbb{R}^n \setminus X_u\) for all \(t \in \text{dom} \phi\).

It is worth noting that we usually show safety by showing the existence of a subset \(K \subset \mathbb{R}^n\), with \(X_o \subset K\) and \(K \cap X_u = \emptyset\), that is, forward invariant; namely, the solutions to \(\Sigma\) starting from \(K\) remain in \(K\) [6].

Since the safety notion is not robust in nature, inspired by the works in [1, 7], and [8], the following robust and uniform robust safety notions are introduced in [9] and [10].

**Definition 2 (Robust safety):** System \(\Sigma\) is robustly safe with respect to \((X_o, X_u)\) if there exists a continuous function \(\varepsilon: \mathbb{R}^n \to \mathbb{R}_{>0}\) such that the system \(\Sigma_{\varepsilon}\) given by

\[ \Sigma_{\varepsilon} : \dot{x} \in F(x) + \varepsilon(x) \mathbb{B} \quad x \in \mathbb{R}^n \]  

where \(\mathbb{B}\) is the closed unit ball centered at the origin, whose dimension is understood from the context, is safe with respect to \((X_o, X_u)\). Such a function \(\varepsilon\) is called robust-safety margin. Furthermore, \(\Sigma\) is uniformly robustly safe with respect to \((X_o, X_u)\) if it admits a constant robust-safety margin. By definition, uniform robust safety implies robust safety, which, in turn, implies safety. The opposite directions are not always true; see [11, Examples 1 and 2].

B. Background

To analyze safety and robust safety without computing solutions, barrier functions have been used extensively in recent years, and different types of conditions are derived; see [7], [8], [12], [13], [14], [15]. The barrier functions of interest here are scalar functions with distinct signs on \(X_o\) and \(X_u\). Any scalar function verifying such a property is called barrier function candidate.

**Definition 3 (Barrier function candidate):** A function \(B: \mathbb{R}^n \to \mathbb{R}\) is a barrier function candidate with respect to \((X_o, X_u)\) if

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if
\[
B(x) > 0 \quad \forall x \in X_u \quad \text{and} \quad B(x) \leq 0 \quad \forall x \in X_u.
\] (3)

A barrier function candidate \( B \) defines the zero-sublevel set
\[
K := \{ x \in \mathbb{R}^n : B(x) \leq 0 \}.
\] (4)

Note that the set \( K \) is closed provided that \( R_{\epsilon} \) defines the zero-sublevel set for other possible characterizations of robust and safe starting from \( \epsilon \in 0 \in X_u \). As a result, we can show safety and robust safety by, respectively, showing the following:

1) forward invariance of the set \( K \) for \( \Sigma \);
2) forward invariance of the set \( K \) for \( \Sigma_{\epsilon} \).

When \( B \) is at least continuous, the set \( K \) is forward invariant for \( \Sigma \) if [16] the following statement holds.

\[ \exists \) There exists \( U(K) \), an open neighborhood of the set \( K \), such that, along each solution \( \phi \) to \( \Sigma \), starting from \( U(K) \setminus K \) and remaining in \( U(K) \setminus K \), the map \( t \mapsto B(\phi(t)) \) is nonincreasing.

The monotonicity condition \((\ast)\) can be characterized using infinitesimal conditions depending on the regularity of \( B \). In particular, when \( B \) is continuously differentiable, \((\ast)\) holds provided that
\[
\langle \nabla B(x), \eta \rangle \leq 0 \quad \forall (\eta, x) \in (F(x), U(K) \setminus K).
\] (5)

Note that condition \((5)\) certifies robust safety if we replace \( F(x) \) therein by \( F(x) + \varepsilon(x)B \), for a continuous and positive function \( \varepsilon \) to be found. Nonetheless, it is more interesting to characterize robust safety without searching for a robustness margin, i.e., using a condition that does not involve \( \varepsilon \). As a result, the following condition is used in [7], [8], and [17]:
\[
\langle \nabla B(x), \eta \rangle < 0 \quad \forall (\eta, x) \in (F(x), \partial K).
\] (6)

See also [9] for other possible characterizations of robust and uniform robust safety.

Note that many works use numerical tools such as SOS and learning-based algorithms to certify safety by numerically searching an appropriate barrier certificate; see [18], [19], [20]. Hence, it is natural to raise the following questions.

Converse safety: Given a safe system \( \Sigma \).

1) Is there a continuous barrier candidate \( B \) satisfying \((\ast)\)?
2) Is there a continuously differentiable barrier function candidate \( B \) satisfying \((5)\)?

Converse robust safety: Given a robustly-safe system \( \Sigma \), is there a continuously differentiable barrier function candidate \( B \) satisfying \((6)\)?

The converse safety problem has been studied in [16], [21], and [22], where the answer is shown to be negative unless extra assumptions on the solutions or on the sets \( (X_o, X_u) \) are made, or time-varying barrier functions are used; see the discussion in the latter reference. Whereas, the converse robust-safety problem is studied in [8], [7], [10], and [17]. To the best of our knowledge, the latter reference proposed the following least restrictive assumptions for the answer to be positive.

Assumption 1: The map \( F \) is continuous with nonempty, convex, and compact images, and \( \text{cl}(X_o) \cap \text{cl}(X_u) = \emptyset \).

As an application, this converse result is used in [10] to discuss the existence of a self-triggered control strategy, with interevent times larger than a constant, such that the self-triggered closed-loop system is safe, provided that the continuous implementation of the controller guarantees robust safety.

C. Contributions

In this article, we introduce a new robust-safety notion, called strong robust safety. This notion is stronger than the robust-safety notion used in the existing literature. In particular, strong robust safety for \( \Sigma \) corresponds to safety for a perturbed version of \( \Sigma \), denoted \( \Sigma_{\epsilon} \). Different from \( \Sigma_{\epsilon} \), the perturbation term \( \epsilon \) in \( \Sigma_{\epsilon} \) is added both to the argument and the image of the map \( F \); thus, the right-hand side of \( \Sigma_{\epsilon} \) is \( \text{co}\{F(x + \epsilon(x)B)\} + \epsilon(x)B \), where \( \text{co} \) denotes the convex hull, which we use to avoid ill-posedness issues. Furthermore, when \( \Sigma_{\epsilon} \) is safe with \( \epsilon \) constant, we recover the uniform strong robust-safety notion. Note that robustness with respect to sufficiently-small perturbations affecting both the argument and the image of the dynamics is well studied in the context of asymptotic stability; see [23] and [24]. Furthermore, several applications, in which guaranteeing robustness with respect to sufficiently small perturbations is useful, are presented in [9]. Those examples are reminiscent of the contexts of self-triggered control, singularly perturbed systems, and assume-guarantee contracts.

A motivation to study strong robust safety comes from the context of control loops subject to measurement and actuation noises. That is, without loss of generality, given a control system of the form
\[
\Sigma_u : \dot{x} = u \quad x \in \mathbb{R}^n
\]
and a feedback law \( u \in F(x) \). We can see that the nominal closed-loop system has the form of \( \Sigma \). As shown in Fig. 1, having the nominal closed-loop system strongly robustly safe implies that the control loop tolerates small perturbations in both sensing and actuation while classical robust safety induces that the control loop tolerates perturbations in actuation only. Clearly, uniform strong robust safety implies strong robust safety, which implies robust safety. However, as we show via counterexamples, the opposite directions are not always true.

Following the spirit of [9] and [10], the main contributions of the article can be divided into two parts. In the first part, we show that, under mild regularity assumptions on \( F \), the sufficient conditions for robust safety in [9, Th. 1] are strong enough
to guarantee strong robust safety. Furthermore, we are able to establish the equivalence between strong robust safety and the existence of a barrier function candidate \( F \) satisfying (6) under very mild assumptions on \( \Sigma \) and the sets \( \{X_o, X_u\} \). Different from the converse robust-safety theorem in [10], the proposed converse strong robust safety theorem does not require \( F \) to be continuous. In the second part of the article, we study uniform strong robust safety. That is, as we show via a counterexample, the sufficient conditions for uniform robust safety in [9] are not strong enough to guarantee uniform strong robust safety. As a result, a set of sufficient conditions for uniform strong robust safety is derived based on the smoothness of \( B \) and \( F \), and depending on whether \( X_o \) and \( X_u \) are bounded or not. Those sufficient conditions are more restrictive than those certifying robust safety and uniform robust safety, which is natural as we want to ensure a relatively stronger property.

The rest of the article is organized as follows. Preliminaries are in Section II. The problem formulation is in Section III. The characterization of strong robust safety is in Section IV. Sufficient conditions for uniform strong robust safety are in Section V. Finally, Section VI concludes this article.

Notation: For \( x, y \in \mathbb{R}^n \), \( x^T \) denotes the transpose of \( x \), \( |x| \) the Euclidean norm of \( x \), and \( (x, y) := x^T y \) the inner product between \( x \) and \( y \). For a set \( K \subset \mathbb{R}^m \), we use \( \text{int}(K) \) to denote its interior, \( \partial K \) its boundary, \( U(K) \) any open neighborhood around \( K \), and \( |x|_K := \text{inf}\{ |x - y| : y \in K \} \) to denote the distance between \( x \) and the set \( K \). For \( O \subset \mathbb{R}^m \), \( K \) denotes the subset of elements of \( K \) that are not in \( O \) and \( |O - K|_H \) denotes the Hausdorff distance between \( O \) and \( K \). For a function \( \phi : \mathbb{R}^m \to \mathbb{R}^m \), \( \partial \phi \) denotes the domain of \( \phi \).

By \( F : \mathbb{R}^m \to \mathbb{R}^m \), we denote a set-valued map associating each element \( x \) in \( \mathbb{R}^m \) into a subset \( F(x) \subset \mathbb{R}^m \). For a set \( D \subset \mathbb{R}^m \), \( F(D) := \{y \in F(x) : x \in D\} \) we denote the graph of \( F \). For a differentiable map \( x \to B(x) \subset \mathbb{R}, \nabla_x B \) denotes the gradient of \( B \) with respect to \( x \), \( i \in \{1, 2, \ldots, n\} \), and \( \nabla B \) denotes the gradient of \( B \) with respect to \( x \).

II. PRELIMINARIES

A. Set-Valued Versus Single-Valued Maps

Let set-valued map \( F : K \ni \mathbb{R}^n \), where \( K \subset \mathbb{R}^m \).

1) The map \( F \) is outer semicontinuous at \( x \in K \) if, for every sequence \( \{x_i \}_{i=0}^\infty \subset K \) and for each \( y_i \in \mathbb{R}^n \) with \( \lim_{i \to \infty} x_i = x \), \( \lim_{i \to \infty} y_i = y \), \( y \in \mathbb{R}^n \), and \( y_i \in F(x_i) \) for all \( i \in \mathbb{N} \), we have \( y \in F(x) \); see [25].

2) The map \( F \) is lower semicontinuous at \( x \in K \) if, for each \( y \in F(x) \) and for each sequence \( \{x_i \}_{i=0}^\infty \subset K \) converging to \( x \), there exists a sequence \( \{y_i \}_{i=0}^\infty \subset \mathbb{R}^n \) with \( y_i \in F(x_i) \) for all \( i \in \mathbb{N} \), that converges to \( y \); see [26, Definition 1.4.2].

3) The map \( F \) is upper semicontinuous at \( x \in K \) if, for each \( \varepsilon > 0 \), there exists a neighborhood of \( x \), denoted \( U(x) \), such that for each \( y \in U(x) \cap K \), \( F(y) \subset F(x) + \varepsilon \mathbb{B} \); see [26, Definition 1.4.1].

4) The map \( F \) is said to be continuous at \( x \in K \) if it is both upper and lower semicontinuous at \( x \).

5) The map \( F \) is locally bounded at \( x \in K \), if there exists a neighborhood of \( x \), denoted \( U(x) \), and \( \beta > 0 \) such that \( |x| \leq \beta \) for all \( x \in F(y) \) and for all \( y \in U(x) \cap K \).

Furthermore, the map \( F \) is upper, lower, outer semicontinuous, continuous, or locally bounded if, respectively, so it is for all \( x \in K \).

Let a single-valued map \( B : K \to \mathbb{R} \), where \( K \subset \mathbb{R}^m \).

1) \( B \) is lower semicontinuous at \( x \in K \) if, for every sequence \( \{x_i \}_{i=0}^\infty \subset K \) such that \( \lim_{i \to \infty} x_i = x \), we have \( \lim \inf_{i \to \infty} B(x_i) \geq B(x) \).

2) \( B \) is upper semicontinuous at \( x \in K \) if, for every sequence \( \{x_i \}_{i=0}^\infty \subset K \) such that \( \lim_{i \to \infty} x_i = x \), we have \( \lim \sup_{i \to \infty} B(x_i) \leq B(x) \).

3) \( B \) is continuous at \( x \in K \) if it is both upper and lower semicontinuous at \( x \).

Furthermore, \( B \) is upper, lower semicontinuous, or continuous if, respectively, so it is for all \( x \in K \).

B. Differential Inclusions: Concept of Solutions and Well-Posedness

Consider the differential inclusion \( \Sigma \) in (1), with the right-hand side \( F : \mathbb{R}^n \to \mathbb{R}^n \) being a set-valued map [27].

Definition 4 (Concept of solutions): \( \phi : \text{dom} \phi \to \mathbb{R}^n \), where \( \text{dom} \phi \) is of the form \([0, T] \) or \([0, T] \) for some \( T \in \mathbb{R}_{>0} \cup \{+\infty\} \), is a solution to \( \Sigma \) starting from \( x_o \in \mathbb{R}^n \) if \( \phi(0) = x_o \), then the map \( t \to \phi(t) \) is locally absolutely continuous, and \( \phi(t) \in F(\phi(t)) \) for almost all \( t \in \text{dom} \phi \).

A solution \( \phi \) to \( \Sigma \) starting from \( x_o \) is forward complete if \( \text{dom} \phi \) is unbounded. It is maximal if there is no solution \( \psi \) to \( \Sigma \) starting from \( x_o \) such that \( \psi(t) = \phi(t) \) for all \( t \in \text{dom} \phi \) and \( \text{dom} \psi \) strictly included in \( \text{dom} \phi \).

Remark 1: We say that \( \phi : \text{dom} \phi \to \mathbb{R}^n \) is a backward solution to \( \Sigma \) if \( \text{dom} \phi \) is of the form \([-T, 0] \) or \((-\infty, -T] \) for some \( T \in \mathbb{R}_{>0} \cup \{+\infty\} \) and \( \phi(-\cdot) : \text{dom} \phi \to \mathbb{R}^n \) is a solution to \( \Sigma \). \( x \in F(\phi(t)) \).

The differential inclusion \( \Sigma \) is said to be well posed if the right-hand side \( F \) satisfies the following assumption.

Assumption 2: The map \( F : \mathbb{R}^n \to \mathbb{R}^n \) is upper semicontinuous and \( F(x) \) is nonempty, compact, and convex for all \( x \in \mathbb{R}^n \).

Assumption 2 guarantees the existence of solutions from any \( x \in \mathbb{R}^n \), as well as adequate structural properties for the set of solutions to \( \Sigma \); see [27], [28]. Furthermore, when \( F \) is single valued, Assumption 2 reduces to just continuity of \( F \).

Remark 2: We recall that having \( F \) upper semicontinuous with compact images is equivalent to having \( F \) is outer semicontinuous and locally bounded; see [25, Th. 5.19] and [24, Lemma 5.15].

III. PROBLEM FORMULATION

Given the differential inclusion \( \Sigma \), we introduce its “strongly” perturbed version \( \Sigma_\varepsilon \) given by

\[
\Sigma_\varepsilon : \dot{x}(t) \in \text{co}(F(x + \varepsilon(t)B)) \cap \varepsilon(t)B \quad x \in \mathbb{R}^n.
\]
Next, we introduce the proposed strong robust-safety notion.

Definition 5 (Strong robust safety): System $\Sigma$ is said to be strongly robustly safe with respect to $(X_o, X_u)$ if there exists a continuous function $\varepsilon : \mathbb{R}^n \to \mathbb{R}_{>0}$ such that $\Sigma_{\varepsilon}$ is safe with respect to $(X_o, X_u)$. Such a function $\varepsilon$ is called a strong robust-safety margin.

We also introduce the uniform strong robust-safety notion.

Definition 6 (Uniform strong robust safety): System $\Sigma$ is said to be uniformly strongly robustly safe with respect to $(X_o, X_u)$ if it is strongly robustly safe and admits a constant strong robust-safety margin.

Clearly, strong robust safety implies robust safety. However, as we show in the following example, the opposite is not always true.

Example 1: Consider the system $\Sigma$ with $n = 1$ and

$$F(x) := \begin{cases} 2 & \text{if } x \leq 0 \\ -1 & \text{if } x = 0 \\ -2(x + 1) & \text{if } x > 0. \end{cases}$$

Consider the sets [see Fig. 2(a)]

$$X_o := \{ x \in \mathbb{R} : x \in [-2, 0] \} \quad \text{and} \quad X_u := \mathbb{R} \setminus X_o.$$

Let the barrier function candidate $B(x) := x(x + 2)$. As a result, it follows that $K = [-2, 0]$ and

$$U(K) \setminus K = (-2 - \sigma, -2) \cup (0, \sigma)$$

for some $\sigma > 0$.

We start noting that $F$ satisfies Assumption 2 and that

$$\langle \nabla B(x), F(x) \rangle = \begin{cases} -2(x + 1) & x \in (0, \sigma) \\ 4(x + 1) & x \in (-2 - \sigma, -2). \end{cases}$$

Hence,

$$\langle \nabla B(x), F(x) \rangle \leq 0 \quad \forall x \in U(\partial K) \setminus K.$$

Thus, using Lemma 1, we conclude that $\Sigma$ is safe with respect to $(X_o, X_u)$.

Now, we consider the perturbed version $\Sigma_{\varepsilon}$ when $\varepsilon(x) = 1$ for all $x \in \mathbb{R}$ [see Fig. 2(b)].

We note that, for all $\mu \in \mathbb{B}$

$$\langle \nabla B(x), F(x) + \mu \rangle = \begin{cases} 2(x + 1)(-1 + \mu) & \forall x \in (0, \sigma) \\ 2(x + 1)(2 + \mu) & \forall x \in (-2 - \sigma, -2). \end{cases}$$

The latter implies that

$$\langle \nabla B(x), F(x) + \mu \rangle \leq 0 \quad \forall x \in U(\partial K) \setminus K, \forall \mu \in \mathbb{B}.$$

Therefore, using Lemma 1, we conclude that $\Sigma_{\varepsilon}$ is safe with respect to $(X_o, X_u)$. Hence, $\Sigma$ is robustly safe with respect to $(X_o, X_u)$.

At this point, we show that $\Sigma$ is not strongly robustly safe with respect to $(X_o, X_u)$. Namely, we show that for any constant perturbation $\varepsilon$, there is a solution to $\Sigma_{\varepsilon}$ that starts from and leaves $X_o$.

Note that we can reason on constant perturbations since $X_o$ is compact.

For each $\varepsilon > 0$, we introduce the system

$$\dot{x} \in \begin{cases} 2 + \varepsilon \mathbb{B} & \text{if } x + \varepsilon \mathbb{B} \leq 0 \\ [-1, 2] + \varepsilon \mathbb{B} & \text{if } x + \varepsilon \mathbb{B} = 0 \\ -1 + \varepsilon \mathbb{B} & \text{if } x + \varepsilon \mathbb{B} \geq 0. \end{cases}$$

The previous system can be further expressed as [see Fig. 2(c)]

$$\dot{x} \in \begin{cases} [2 - \varepsilon, 2 + \varepsilon] & \text{if } x \leq -\varepsilon \\ [-1 - \varepsilon, 2 + \varepsilon] & \text{if } x \in [-\varepsilon, \varepsilon] \\ [-1 - \varepsilon, -1 + \varepsilon] & \text{if } x \geq \varepsilon. \end{cases} \quad (8)$$

Now, for any $\varepsilon > 0$, we consider the function $\phi_{\varepsilon} : [0, \varepsilon] \to \mathbb{R}^n$ satisfying $\phi_{\varepsilon}(0) = 0$ and $\phi_{\varepsilon}(t) = 1 \in F(\phi_{\varepsilon}(t) + \varepsilon(\phi_{\varepsilon}(t)) \mathbb{B}) + \varepsilon(\phi_{\varepsilon}(t)) \mathbb{B}$. Hence, $\phi_{\varepsilon}$ is a solution to (8), starts from $\partial X_o$ and leaves $X_o$. Hence, for any $\varepsilon > 0$, $\Sigma_{\varepsilon}$ is not safe. $\square$

IV. STRONG ROBUST SAFETY

In the next section, we propose sufficient conditions, in terms of barrier functions, to guarantee strong robust safety. Furthermore, we address the converse problem.

We start recalling a tool that allows us to use locally-Lipschitz barrier functions [29].

Definition 7 (Clarke generalized gradient): Let $B : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. Let $\Omega \subset \mathbb{R}^n$ be any null-measure set, and let $\Omega_B \subset \mathbb{R}^n$ be the set of points at which $B$ fails to be differentiable. The Clarke generalized gradient of $B$ at $x$ is the set-valued map $\partial C B : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by

$$\partial C B(x) := \text{co} \left\{ \lim_{i \to \infty} \nabla B(x_i) : x_i \to x, x_i \in \Omega \setminus \Omega_B \right\}$$

Fig. 2. Initial and unsafe sets $(X_o, X_u)$ and the different maps used in Example 1. (a) Map of $F()$. (b) Map of $F() + \mathbb{B}$. (c) Map of $(F() + \varepsilon(x) \mathbb{B}) + \varepsilon(x) \mathbb{B}$. 
where \(\text{co}(\cdot)\) is the convex hull of the elements in (\·\).

Remark 3: When \(B\) is locally Lipschitz, then \(\partial B\) is upper semicontinuous and its images are nonempty, compact, and convex.

A. Sufficient Conditions

In the following result, we show that the sufficient conditions for robust safety in [9] are strong enough to guarantee strong robust safety under Assumption 2.

Theorem 1: Consider system \(\Sigma\) such that Assumption 2 holds. Let \(B : \mathbb{R}^n \to \mathbb{R}\) be a barrier function candidate with respect to \((X_0, X_u) \subset \mathbb{R}^n \times \mathbb{R}^n\). Then, \(\Sigma\) is strongly robustly safe with respect to \((X_0, X_u)\) if one of the following conditions holds.

C11. There exists a continuous function \(\varepsilon : \mathbb{R}^n \to \mathbb{R}_{>0}\) such that the set \(K\) in (4) is forward invariant for \(\Sigma^\varepsilon\).

C12. \(\Sigma\) is robustly safe with respect to \((X_0, X_u)\) and \(F\) is continuous.

C13. The function \(B\) is locally Lipschitz and

\[
\langle \partial B(x), \eta \rangle \in \mathbb{R}_{<0} \quad \forall \eta \in F(x), \forall x \in \partial K. \tag{9}
\]

C14. \(B\) is continuously differentiable and (6) holds.

Proof: We start noting that establishing strong robust safety under C14. would follow straightforwardly if we prove robust safety under C13. Indeed, when \(B\) is continuously differentiable, \(\partial B = \nabla B\).

We now prove strong robust safety under each of the statements in C11–C13.

1) Under C11., we conclude the existence of a continuous function \(\varepsilon : \mathbb{R}^n \to \mathbb{R}_{>0}\) such that the solutions to \(\Sigma^\varepsilon\) starting from \(X_0\) never reach \(X_u\), which by definition implies strong robust safety of \(\Sigma\) with respect to \((X_0, X_u)\).

2) Under C12., there exists \(\varepsilon : \mathbb{R}^n \to \mathbb{R}_{>0}\) such that \(\Sigma^\varepsilon\) is safe. Using Lemma 2, we conclude the existence of \(\delta : \mathbb{R}^n \to \mathbb{R}_{>0}\) such that

\[
F(x + \delta(x)B) \subset F(x) + (\varepsilon(x)/2)B \quad \forall x \in \mathbb{R}^n
\]

which implies that

\[
F(x + \delta(x)B) + (\varepsilon(x)/2)B \subset F(x) + \varepsilon(x)B \quad \forall x \in \mathbb{R}^n.
\]

Now, since \(F\) is convex, we conclude that

\[
\text{co}(F(x + \delta(x)B) + (\varepsilon(x)/2)B) \subset F(x) + \varepsilon(x)B \quad \forall x \in \mathbb{R}^n.
\]

Hence, \(\Sigma^\varepsilon_{\min}\) defined as in (7), with \(\varepsilon_{\min} := \min(\varepsilon, \varepsilon/2)\), is safe.

3) Under C13., we propose to grid the boundary \(\partial K\) using a sequence of nonempty compact subsets \(\{D_i\}_{i=1}^N\), where \(N \in \{1, 2, 3, \ldots, \infty\}\). We assume that

\[
\bigcup_{i=1}^N D_i = \partial K, \quad D_i \subset \partial K \quad \forall i \in \{1, 2, \ldots, N\}.
\]

Furthermore, we choose the sequence \(\{D_i\}_{i=1}^N\) such that, for each \(i \in \{1, 2, \ldots, N\}\), we can find a finite set \(N_i \subset \{1, 2, \ldots, N\}\) such that

\[
D_i \cap D_j = \emptyset \quad \forall j \notin N_i \quad \text{int}_{\partial K}(D_i \cap D_j) = \emptyset \quad \forall j \in N_i
\]

where \(\text{int}_{\partial K}(\cdot)\) denotes the interior of (\·\) relative to \(\partial K\). Such a decomposition always exists according to Whitney covering lemma [30].

Next, we introduce the following claim.

Claim 1: For each \(i \in \{1, 2, \ldots, N\}\), there exists \(\delta_i > 0\) such that, for each \(x \in D_i\)

\[
\langle \partial B(x), \eta \rangle \subset (-\infty, 0) \quad \forall \eta \in H(x, \delta_i) \tag{10}
\]

where \(H(x, \delta) := \text{co}(F(x + \delta B)) + \delta B\).

To complete the proof, under Claim 1, we let

\[
\varepsilon_o(x) := \min\{\delta_i : x \in D_i\}. \tag{11}
\]

Clearly, since the sequence \(\{\delta_i\}_{i=1}^\infty\) is positive, and by definition of the sequence \(\{D_i\}_{i=1}^\infty\), we conclude that \(\varepsilon_o\) is strictly positive and lower semicontinuous. Finally, using [31, Th. 1], we conclude the existence of a continuous function \(\varepsilon : \partial K \to \mathbb{R}_{>0}\) such that \(\varepsilon(x) \leq \varepsilon_o(x)\) for all \(x \in \partial K\). As a result, we conclude that

\[
\langle \partial B(x), \eta \rangle \subset (-\infty, 0) \quad \forall \eta \in H(x, \varepsilon(x)) \forall x \in \partial K. \tag{12}
\]

As a result, using [32, Th. 5], we conclude that the system \(\dot{x} \in H(x, \varepsilon(x))\), with \(x \in \mathbb{R}^n\), is robustly safe; hence, \(\Sigma\) is strongly robustly safe.

Finally, we prove Claim 1 using contradiction. That is, we assume the existence of a positive sequence \(\{\sigma_j\}_{j=1}^{\infty}\) that converges to zero such that, for each \(j \in \{1, 2, \ldots\}\), we can find \(x_j \in D_i\), \(\eta_j \in \partial B(x_j)\), and \(\zeta_j \in H(x_j, \sigma_j)\) such that

\[
\langle \eta_j, \zeta_j \rangle \geq 0.
\]

Since \(D_i\) is compact, by passing to an appropriate subsequence, we conclude the existence of \(x^* \in D_i\) such that \(\lim_{j \to \infty} x_j = x^*\).

Next, since the set-valued map \(\partial B\) is upper semicontinuous with nonempty and compact images, we conclude that by passing to an appropriate subsequence

\[
\lim_{j \to \infty} \eta_j = \eta^* \in \partial B(x^*).
\]

Furthermore, we note that \(H\) is the convex hull of the composition of \(F\) and \((x, \delta) \mapsto x + \delta B\) added to the map \(\delta \to \delta B\). The latter map is continuous and \(F\) verifies Assumption 2. Hence, \(H\) enjoys the same properties as \(F\). As a result, we have

\[
\lim_{j \to \infty} \zeta_j = \zeta^* \in H(x^*, 0) = F(x^*).
\]

Now, since \(\langle \eta_j, \zeta_j \rangle \geq 0\) for all \(j \in \{1, 2, \ldots\}\), then \(\lim_{j \to \infty} \langle \eta_j, \zeta_j \rangle \geq 0\). However, this contradicts (9).

Remark 4: Note that we can show strong robust safety by showing robust safety for the system \(\dot{x} \in \text{co}(F(x + \varepsilon(x)B))\), or by showing safety for \(\Sigma^\varepsilon\), for some \(\varepsilon\) continuous and positive. However, existing conditions to guarantee the latter two statements must involve the term \(\varepsilon\); hence, one needs to search for such a valid candidate \(\varepsilon\), for which, the condition holds. As opposed to the conditions proposed in this article, which involve the nominal system \(\Sigma\) only, and induce the existence of a robustness margin \(\varepsilon\).

B. Converse Problem

In this section, we establish the equivalence between strong robust safety and the existence of a continuously differentiable
barrier function candidate $B$ satisfying (6). Thanks to the following intermediate result, which establishes that we can always squeeze a continuous set-valued map between an upper semicontinuous map and its strongly perturbed version.

**Proposition 1:** Consider a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfying Assumption 2. Then, for each continuous function $\varepsilon : \mathbb{R}^n \to \mathbb{R}_{>0}$, there exists a continuous set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with convex and compact images such that

$$F(x) \subset G(x) \subset \text{co}\{F(x + \varepsilon(x)B)\} + \varepsilon(x)B \quad \forall x \in \mathbb{R}^n.$$ 

The latter proposition allows us to prove the converse strong robust-safety theorem, using the converse robust-safety theorem established in [10], under the following mild assumption.

**Assumption 3:** $\text{cl}(X_0) \cap \text{cl}(X_u) = \emptyset$.

Different from the converse robust safety theorem in [10], in this case, we do not require $F$ to be continuous.

**Theorem 2:** Consider the differential inclusion $\Sigma$ in (1), with $F$ satisfying Assumption 2. Consider the initial and unsafe sets $(X_0, X_u) \subset \mathbb{R}^n \times \mathbb{R}^n$ such that Assumption 3 holds. Then, the following statements are equivalent:

- **S21.** $\Sigma$ is strongly robustly safe with respect to $(X_0, X_u)$.
- **S22.** There exists a continuously differentiable barrier function candidate $B$ such that (6) holds.

**Proof:** The proof that S22. implies S21. follows from Theorem 1.

To prove that S21. implies S22., we assume that $\Sigma$ is strongly robustly safe with respect to $(X_0, X_u)$ and we let $\varepsilon$ be a strong robust-safety margin. By virtue of Proposition 1, we conclude that the existence of a continuous set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with convex and compact images such that

$$F(x) \subset G(x) \subset \text{co}\{F(x + \varepsilon(x)B)\} + (\varepsilon(x)/2)B \quad \forall x \in \mathbb{R}^n.$$ 

Note that strong robust safety of $\Sigma$ implies robust safety of $\Sigma_g : \dot{x} \in G(x) \quad x \in \mathbb{R}^n$

with $\varepsilon/2$ as a robustness margin; namely, the system

$$\Sigma_g : \dot{x} \in G(x) + (\varepsilon(x)/2)B \quad x \in \mathbb{R}^n$$

is safe. Hence, using the work in [10], we conclude that the existence of a continuously differentiable barrier function candidate $B$ such that $\langle \nabla B(x), \eta \rangle < 0$ for all $(\eta, x) \in (G(x), \partial K)$. The latter completes the proof since $F \subset G$.

**V. UNIFORM STRONG ROBUST SAFETY**

In this section, we strengthen the sufficient conditions in Theorem 1 to conclude uniform strong robust safety. In the following theorem, we analyze particular situations, where uniform strong robust safety is equivalent to strong robust safety.

**Theorem 3:** Consider the differential inclusion $\Sigma$ such that Assumption 2 holds. Let $B : \mathbb{R}^n \to \mathbb{R}$ be a barrier function candidate with respect to $(X_0, X_u)$. Then, $\Sigma$ is uniformly strongly robustly safe with respect to $(X_0, X_u)$ if one of the following conditions holds.

C31. There exists a continuous function $\varepsilon : \mathbb{R}^n \to \mathbb{R}_{>0}$ such that the set $K$ is forward invariant for $\Sigma^\varepsilon$ and the set $\partial K$ is bounded.

C32. $\Sigma$ is strongly robustly safe with respect to $(X_0, X_u)$ and either $\mathbb{R}^n \setminus X_u$ or $\mathbb{R}^n \setminus X_0$ is bounded.

**Proof:**

1) Under C31, and when $\partial K$ is bounded, we show that $\varepsilon^* := \inf\{\varepsilon(x) : x \in U(\partial K)\} > 0$, for $U(\partial K)$ a neighborhood of $\partial K$, is a strong robust-safety margin using contradiction. Indeed, assume that there exists a solution $\phi$ to $\Sigma^\varepsilon$, starting from $x_0 \in X_0$, that reaches the set $X_u$ in finite time. Namely, there exists $t_1, t_2 \in \text{dom} \phi$ such that $t_2 > t_1 \geq 0$, $\phi(t_2) \in U(\partial K) \setminus K$, $\phi(t_1) \in U(\partial K) \cap K$, $\phi(0) \in X_0$, and $\phi([t_1, t_2]) \subset U(\partial K)$. The latter implies that $\phi$, restricted to the interval $[t_1, t_2]$, is also a solution to $\Sigma^\varepsilon$. However, since the set $K$ is forward invariant for $\Sigma^\varepsilon$, we conclude that the solution $\phi([t_1, t_2])$ must lie within the set $K$, which yields to a contradiction.

2) Under C32, we conclude the existence of a continuous function $\varepsilon : \mathbb{R}^n \to \mathbb{R}_{>0}$ such that $\Sigma^\varepsilon$ is safe with respect to $(X_0, X_u)$. Hence, the set $K_\varepsilon := \{\phi(t) : t \in \text{dom} \phi, \phi \in S\Sigma^\varepsilon(x), x \in X_0\}$ where $S\Sigma^\varepsilon(x)$ is the set of maximal solutions to $\Sigma^\varepsilon$ starting from $x$, is forward invariant for $\Sigma^\varepsilon$, $K_\varepsilon \cap X_u = \emptyset$, and $X_0 \subset K_\varepsilon$. Furthermore, when either the complement of $X_0$ is bounded or the complement of $X_u$ is bounded, we conclude that $\partial K_\varepsilon$ is bounded. Hence, the rest of the proof follows as in the proof of C31.

In the next theorem, we propose a set of infinitesimal conditions to guarantee uniform strong robust safety without using the boundedness assumptions in C31. and C32. Before doing so, we recall a tool that allows us to consider semicontinuous barrier functions [29].

**Definition 8 (Proximal subdifferential):** The proximal subdifferential of $B : \mathbb{R}^n \to \mathbb{R}$ at $x$ is the set-valued map $\partial_p B : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by

$$\partial_p B(x) := \{\zeta \in \mathbb{R}^n : [\zeta^\top - 1]^\top \in N^p_{\text{epi} B}(x, B(x))\}$$

where $\text{epi} B := \{(x, r) : x \in \mathbb{R}^n \times \mathbb{R} : r \geq B(x)\}$ and, given a closed subset $S \subset \mathbb{R}^{n+1}$ and $y \in \mathbb{R}^{n+1}$

$$N^p_S(y) := \{\zeta \in \mathbb{R}^{n+1} : \exists r > 0 \text{ s.t. } |y + r\zeta|_S = r|\zeta|\}.$$

**Remark 5:** When $B$ is twice continuously differentiable at $x \in \mathbb{R}^n$, then $\partial_p B(x) = \{\nabla B(x)\}$. Moreover, the latter equality is also true if $B$ is continuously differentiable at $x$ and $\partial_p B(x) \neq \emptyset$.

Furthermore, we consider the following assumption.

**Assumption 4:** There exists $\lambda_1 : \mathbb{R} \to \mathbb{R}_{>0}$ continuous and there exists $\lambda_2 : \mathbb{R}^n \to \mathbb{R}_{>0}$ continuous with $\lambda_1(0) = 0$ such that

$$F(x + \delta B) \subset F(x) + \lambda_1(\delta)\lambda_2(x)B \quad \forall x \in \partial K, \forall \delta > 0.$$ 

Note that Assumption 4 implies the continuity of $F$ in the set-valued sense. Furthermore, we will show in the following
result that the continuity of $F$ implies the existence of continuous functions $\lambda_1 : \mathbb{R} \to \mathbb{R}_{\geq 0}$ and $\lambda_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, with $\lambda_1(0) = 0$, such that (13) holds. However, we do not guarantee the explicit knowledge of $\lambda_2$ unless the following extra assumption is made.

**Assumption 5:** There exists $g_0 : (-\infty, \bar{a}] \to [0, +\infty)$ nonincreasing and continuous, such that $g(x) = \max\{g(y), s : x \in \partial K\}$ and $g(y, s) := |F(y + sB) - F(y)|_H - |F(sB) - F(0)|_H$. ±

The following result summarizes the aforementioned discussion. The proof is in the Appendix.

**Proposition 2:** For every continuous set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, there exist two continuous functions $\lambda_1 : \mathbb{R} \to \mathbb{R}_{\geq 0}$ and $\lambda_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, with $\lambda_1(0) = 0$, such that (13) holds. Moreover, when Assumption 5 holds. Then, we can take $\lambda_2(x) := \exp(\max\{g, h\}(\ln|x|)) + 1$, where $h(\ln|x|) := \sup\{c(a, \ln|x|) - g(a) : a \in \mathbb{R}\}$

$$g(a) := \begin{cases} -\frac{1}{2}g_0(a) & \text{if } a \leq \bar{a} \\ c(a, a) - c(\bar{a}, \bar{a}) + a - \bar{a} & \text{if } a > \bar{a}. \end{cases}$$

At this point, we present the main result of this section.

**Theorem 4:** Consider the differential inclusion $\Sigma$ such that Assumptions 2 and 4 hold. Let $B : \mathbb{R}^n \rightrightarrows \mathbb{R}$ be a barrier function candidate with respect to $(X_n, X_u)$. Then, $\Sigma$ is uniformly strongly robust with respect to $(X_n, X_u)$ if one of the following conditions holds.

C41. $B$ is continuously differentiable and

$$\inf \left\{ \frac{\langle \nabla B(x), -F(x) \rangle}{1 + \lambda_2(x)} : x \in \partial K \right\} > 0.$$  

C42. $B$ is locally Lipschitz and

$$\inf \left\{ \frac{\langle \zeta, -F(x) \rangle}{1 + \lambda_2(x)} : (\zeta, x) \in \partial B(x) \times \partial K \right\} > 0.$$  

C43. $B$ is lower semicontinuous, $F$ is locally Lipschitz, and

$$\inf \left\{ \frac{\langle \zeta, -F(x) \rangle}{1 + \lambda_2(x)} : (\zeta, x) \in \partial B(x) \times \partial K \right\} > 0.$$  

C44. $B$ is upper semicontinuous, $F$ is locally Lipschitz, $cl(K) \cap X_n = \emptyset$, and

$$\inf \left\{ \frac{\langle \zeta, F(x) \rangle}{1 + \lambda_2(x)} : (\zeta, x) \in \partial B(-B(x)) \times \partial K \right\} > 0.$$  

Proof: The proof under C41. would follow straightforwardly if we prove strong robust safety under C42. Indeed, when $B$ is continuously differentiable, $\partial B = \nabla B$.

1) Under C42., we let $\varepsilon > 0$ and

$$\sigma := \inf \left\{ \frac{\langle \zeta, -F(x) \rangle}{\nabla B(x)} : x \in \partial K, \zeta \in \partial B(x) \right\}. $$

Then, for all $(x, \mu) \in (\partial K, B)$, we use (13) to conclude that, for each $\eta_1 \in F(x + \varepsilon \mu)$, we can find $\eta_2 \in F(x)$ such that $|\eta_1 - \eta_2| \leq \lambda_1(\varepsilon)\lambda_2(x)$. As a result, for each $\zeta \in \partial B(x)$, we have

$$\langle \zeta, \eta_1 + \varepsilon \mu \rangle = \langle \zeta, \eta_2 + \varepsilon \mu \rangle + \langle \zeta, \eta_1 - \eta_2 \rangle \leq -\sigma |\zeta| (1 + \lambda_2(x)) + |\zeta| (\varepsilon + \lambda_1(\varepsilon)\lambda_2(x)) \leq (-\sigma + \max(\varepsilon, \lambda_1(\varepsilon))) |\zeta| (1 + \lambda_2(x)).$$

Hence, by choosing $\varepsilon > 0$ sufficiently small, we guarantee that $-\sigma + \max(\varepsilon, \lambda_1(\varepsilon)) < 0$. Therefore, for each $\zeta \in \partial B(x)$, we have

$$\langle \zeta, F(x + \varepsilon B) + \varepsilon B \rangle \subset \mathbb{R}_{< 0} \quad \forall x \in \partial K.$$  

and, thus, for each $\zeta \in \partial B(x)$, we have

$$\langle \zeta, \co\{F(x + \varepsilon B) + \varepsilon B\} \subset \mathbb{R}_{< 0} \quad \forall x \in \partial K.$$  

This implies forward invariance of the set $K$ for $\Sigma$ using [32, Th. 6], which is enough to conclude uniform strong robust safety.

2) Under C43., when $B$ is lower semicontinuous, we show the existence of a constant $\varepsilon > 0$ such that

$$\langle \partial B(x), \co\{F(x + \varepsilon B) + \varepsilon B\} \subset \mathbb{R}_{< 0} \forall x \in U(\partial K).$$

(14)

Now, using Lemma 1, we conclude that (14) implies that, along each solution $\phi$ to $\Sigma$ starting from $U(\partial K)$ and remaining in $U(\partial K)$, the map $t \mapsto B(\phi(t))$ is nonincreasing and that $\Sigma$ is safe with respect $(X_n, X_u)$.  

3) Under C44., we follow the same computations as when $B$ is smooth, to conclude that, for some constant $\varepsilon > 0$ sufficiently small, we have

$$\langle \zeta, -\co\{F(x + \varepsilon B) + \varepsilon B\} \subset \mathbb{R}_{< 0} \forall x \in \mathbb{R}^n$$

for all $(\zeta, x) \in \partial B(-B(x)) \times U(\partial K)$. Since $-B$ is lower semicontinuous, we use Lemma 1 to conclude that, along each solution $\psi$ to

$$\Sigma^\gamma_{\varepsilon} : \dot{x} \in -\co\{F(x + \varepsilon B) + \varepsilon B \} \quad x \in \mathbb{R}^n$$

starting from $U(\partial K)$ and remaining in $U(\partial K)$, the map $t \mapsto -B(\psi(t))$ is nonincreasing. Thus, the map $t \mapsto B(\psi(t))$ is nondecreasing. Now, we pick a solution $\phi$ to $\Sigma^\gamma_{\varepsilon}$ with $\dom \phi = [0, T]$ and $\phi(\dom \phi) \subset U(\partial K)$. Note that the map $t \mapsto \psi(t) := \phi(T - t)$ is a solution to $\Sigma^\gamma_{\varepsilon}$ satisfying $\psi(\dom \phi) \subset U(\partial K)$. Thus, $t \mapsto B(\phi(T - t))$ is nondecreasing, which implies that $t \mapsto B(\phi(t))$ is nonincreasing. The latter implies, using Lemma 1, that $\Sigma^\gamma_{\varepsilon}$ is safe with respect $(X_n, X_u)$. □

**Remark 6:** To guarantee uniform robust safety, in [9], the inequalities in C41., C42., C43., and C44. are used while replacing the $\lambda_2$ therein by $0$. Thus, Assumption 4 is not needed to characterize uniform robust safety. However, as we show in the next example, for the case where $B$ is continuously differentiable, the sufficient inequality used in [9] to guarantee
uniform robust safety, which is
\[
\inf \left\{ \frac{\nabla B(x), -F(x)}{\nabla B(x)} : x \in \partial K \right\} > 0
\]  
(16)
is not strong enough to guarantee uniform strong robust safety.

**Example 2:** Consider the differential inclusion \( \Sigma \) with
\[
F(x) = \begin{bmatrix} F_1(x) \\ -1 + x_1^2 x_2 \end{bmatrix}.
\]
Consider the initial and unsafe sets \( X_0 = \{ x \in \mathbb{R}^2 : x_2 \leq 0 \} \) and \( X_u = \mathbb{R}^2 \setminus X_0 \). Next, we choose \( B(x) = x_2 \) as a barrier function candidate with respect to \((X_0, X_u)\). Note that in this case \( K = X_0 \) and \( \partial K = \{ x \in \mathbb{R}^2 : x_2 = 0 \} \). We start showing that the system is strongly robustly safe with respect to \((X_0, X_u)\). To do so, we note that
\[
\nabla B(x), F(x) \geq 0 \quad \forall x \in \partial K.
\]
Now, we show that system \( \Sigma \) is uniformly robustly safe by verifying (16). Indeed, we note that
\[
\frac{\nabla B(x), -F(x)}{\nabla B(x)} = 1 - x_1^2 x_2 = 1 > 0 \quad \forall x \in \partial K.
\]
Hence, (16) holds, which proves uniform robust safety.

Now, we show that the considered system cannot be uniformly strongly robustly safe by showing that, for each constant \( \varepsilon > 0 \), \( \Sigma_\varepsilon \) admits a solution starting from \( x_o := (1/\sqrt{\varepsilon}, 0) \in \partial K \) that enters \( X_u \). To do so, we start noting that, for \( \mu := [0 1]^T \), we have
\[
\nabla B(x_o), F(x_o + \varepsilon \mu + \varepsilon \mu) = -1 + \varepsilon + x_1^2 \varepsilon
\]
\[
= -1 + \varepsilon + \left( \frac{1}{\sqrt{\varepsilon}} \right)^2 \varepsilon = \varepsilon > 0.
\]
As a consequence, with [32, Lemma 4] with \( B(x) = -x_2 \), we conclude that \( F(x_o + \varepsilon \mu + \varepsilon \mu \subset D_{\text{aff}}(x_o) \), where \( D_K \) is the Dubovitsky–Miliutin cone of \( K \) defined as
\[
D_K(x) := \{ v \in \mathbb{R}^n : \exists \varepsilon > 0 : x + \delta(v + w) \in K \forall \delta \in (0, \varepsilon), \forall w \in \varepsilon B \}.
\]
Finally, using [28, Th. 4.3.4], we conclude that the solution starting from \( x_o \) to the system \( \dot{x} = F(x + \varepsilon \mu) + \varepsilon \mu \) enters \( X_u \).

\[\square\]

**VI. CONCLUSION**

We introduced a strong robust-safety notion for continuous-time systems inspired by the scenario of control loops subject to perturbations in both sensing and actuation. We demonstrated that this notion is stronger than the existing safety and robust-safety notions available in the literature. Furthermore, we showed that the existing sufficient conditions for robust safety in [9] are strong enough to guarantee strong robust safety under mild regularity assumptions on \( F \). Following that, we showed that strong robust safety is equivalent to the existence of a smooth barrier certificate. Unlike the converse robust safety theorem in [10], we do not require \( F \) to be continuous. Furthermore, we introduced the uniform strong robust-safety notion, which requires constant strong robust-safety margins, and we showed that the existing sufficient conditions for uniform robust safety in [9] are not strong enough to guarantee uniform strong robust safety. Hence, new sufficient conditions are derived. In future work, it would be interesting to analyze robust safety and strong robust safety for constrained and hybrid systems.

**APPENDIX I**

**PROOF OF PROPOSITIONS**

A. **Proof of Proposition 1**

The proof follows five steps.

1) **1st step:** Consider a compact subset \( I \subset \mathbb{R}^n \) and a continuous function \( \varepsilon : \mathbb{R}^n \to [0, \infty). \) Let
\[
\varepsilon_I := \min \{ \varepsilon(x) : x \in I \} > 0.
\]
We will show the existence of \( \delta_I > 0 \) such that
\[
gph (F) + \delta_I \mathbb{B} \subset gph (F_{\varepsilon_I})
\]
(18)
where \( F_{\varepsilon_I}(x) := \text{co} \{ F(x + \varepsilon \mathbb{B}) \} + \varepsilon \mathbb{B}, \) \( gph \left( F \right) \subset \mathbb{R}^n \times \mathbb{R}^n \) denotes the graph of \( F \), and the dimension of \( \mathbb{B} \) in (18) is \( n \times n \). In other words, we show the existence \( \delta_I > 0 \) such that, for each \( (x, y) \in (I \times \mathbb{R}^n) \not\subset gph (F_{\varepsilon_I}) \), we have \( \| (x, y) \|_{gph (F)} > \delta_I \).

To find a contradiction, we suppose that there exits a sequence \( \{ (x_i, y_i) \}_{i=0}^{\infty} \subset (I \times \mathbb{R}^n) \setminus gph (F_{\varepsilon_I}) \) such that \( \lim_{i \to \infty} \| (x_i, y_i) \|_{gph (F)} = 0 \). Now, since \( I \) is compact, \( F \) is outer semicontinuous and locally bounded, we conclude that \( gph (F) \) restricted to \( I \) is compact, which implies that the sequence \( \{ (x_i, y_i) \}_{i=0}^{\infty} \) is uniformly bounded. Hence, by passing to an appropriate subsequence, we conclude the existence of \( (x^*, y^*) \in \text{cl}((I \times \mathbb{R}^n) \setminus gph (F_{\varepsilon_I})) \) with \( x^* \in I \) such that \( \lim_{i \to \infty} (x_i, y_i) = (x^*, y^*) \), which implies, using the continuity of the distance function, that
\[
\lim_{i \to \infty} \| (x_i, y_i) \|_{gph (F)} = \| (x^*, y^*) \|_{gph (F)} = 0.
\]
As a result, we have
\[
(x^*, y^*) \in gph (F) \subset \text{int} (gph (F_{\varepsilon_I}))
\]
which yields to a contradiction.

2) **2nd step:** We will show the existence of a sequence of continuous set-valued maps with convex and compact images, defined on \( I \), that converges graphically to \( F \). To do so, we start using [33, Th. 2] to conclude the existence of a sequence \( \{ F_k(x) \}_{k=1}^{\infty} \) of continuous set-valued maps with convex and compact images, defined on \( I \), such that, for each \( x \in I \), the following properties hold.

(\text{p1}) \( F_{k+1}(x) \subset F_k(x) \quad \forall k \in \{1, 2, \ldots\}. \)

(\text{p2}) \( F(x) \subset \text{int} (F_k(x)) \quad \forall k \in \{1, 2, \ldots\}. \)

(\text{p3}) \( F(x) = \lim_{k \to \infty} F_k(x) = \bigcap_{k=1}^{\infty} F_k(x). \)

Next, we show that the aforementioned sequence converges graphically to \( F \). To find a contradiction, we assume that the opposite holds; namely, there exists \( \sigma > 0 \) and a sequence \( \{ (x_k, y_k) \}_{k=1}^{\infty} \) such that \( (x_k, y_k) \in gph (F_k) \) and
\[
\| (x_k, y_k) \|_{gph (F)} > \sigma \quad \forall k \in \{1, 2, \ldots\}.
\]
Since $I$ is compact, $\{x_k\}_{k=1}^{\infty} \subset I$, and, under (p1)

$$y_k \in F_k(x_k) \subset F_1(x_k) \subset F_1(I) \quad \forall k \in \{1, 2, \ldots\}$$

we conclude that the sequence $\{(x_k, y_k)\}_{k=1}^{\infty}$ is uniformly bounded. Thus, by passing to an appropriate subsequence, we conclude the existence of $(x^*, y^*) \in I \times \mathbb{R}^n$ such that

$$\lim_{k \to \infty} (x_k, y_k) = (x^*, y^*).$$

The latter plus the continuity of the distance function allows us to conclude that $||x^*, y^*)||_{\text{gph}(F)} > \sigma$, which means that $y^* \notin F(x^*) + \sigma \mathbb{B}$.

Now, using (p3), we can choose $k^* \in \{1, 2, \ldots\}$ sufficiently large to have

$$F_k(x^*) \subset F(x^*) + \sigma \mathbb{B}. \quad (19)$$

Next, using (p1), we conclude that

$$y_k \in F_k(x_k) \subset F_k(x_k) \quad \forall k \geq k^*. \quad (20)$$

Furthermore, since $F_k$ is continuous, we conclude the existence of $k^{**} \geq k^*$ such that

$$F_k(x_k) \subset F_k(x^*) + \sigma \mathbb{B} \quad \forall k \geq k^{**}. \quad (21)$$

Moreover, since $\lim_{k \to \infty} y_k = y^*$, we conclude that we can take $k^{**}$ even larger to obtain

$$y^* \in y_k + \sigma \mathbb{B} \quad \forall k \geq k^{**}. \quad (22)$$

Finally, combining (19)–(21), and (22), we obtain

$$y^* \in F(x^*) + (3\sigma/4) \mathbb{B}$$

and the contradiction follows.

3) **3rd step:** Using the previous two steps, we show that there exists a continuous set-valued map $G_I : I \rightrightarrows \mathbb{R}^n$ with convex and compact images such that

$$F(x) \subset G_I(x) \subset F_{\delta_1}(x) \quad \forall x \in I.$$ 

Indeed, we showed the existence of a sequence of continuous set-valued maps $\{F_k\}_{k=1}^{\infty}$ that converges graphically to $F$. In particular, there exists $k_0 \in \{1, 2, \ldots\}$ such that

$$\text{gph}(F_{k_0}) \subset \text{gph}(F) + (\delta_1/2) \mathbb{B} \subset \text{gph}(F_{\delta_1}).$$

Moreover, under (p2), we have $\text{gph}(F) \subset \text{gph}(F_{k_0})$. Hence, the proof is completed by taking $G_I = F_{k_0}$.

4) **4th step:** At this point, we propose to grid $\mathbb{R}^n$ using a sequence of nonempty compact subsets $\{I_i\}_{i=1}^{N}$, where $N \in \{1, 2, 3, \ldots, \infty\}$. That is, we assume that $I_i \subset \mathbb{R}^n$ for all $i \in \{1, 2, \ldots, N\}$ and $\bigcup_{i=1}^{\infty} I_i = \mathbb{R}^n$. Furthermore, for each $i \in \{1, 2, \ldots, N\}$, there exists a finite set $N_i \subset \{1, 2, \ldots, N\}$ such that

$$I_i \cap I_j = \emptyset \quad \forall j \notin N_i \quad (23)$$

$$\text{int}(I_i \cap I_j) = \emptyset \quad \forall j \in N_i \setminus \{i\} \quad (24)$$

where $\text{int}(\cdot)$ denotes the interior of $(\cdot)$.

Using the previous steps, we conclude that, on every subset $I_i$, there exists a continuous set-valued map $G_{I_i} : I_i \rightrightarrows \mathbb{R}^n$ with convex and compact images such that

$$F(x) \subset G_{I_i}(x) \subset F_{\delta_1}(x) \quad \forall x \in I_i.$$ 

Now, we let $G_o : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined as

$$G_o(x) := \begin{cases} G_{I_i}(x) & \text{if } x \in \text{int}(I_i) \\ \{y \in G_{I_j}(x) : x \in I_j, j \in N_i\} & \text{otherwise}. \end{cases}$$

The function $G_o$ is, by definition, lower semicontinuous and satisfies

$$F(x) \subset G_o(x) \subset \text{co}\{F(x + \varepsilon(x) \mathbb{B})\} + \varepsilon \mathbb{B} \quad \forall x \in \mathbb{R}^n.$$ 

Now, using [33, Th. 4], we conclude the existence of a continuous set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with convex and compact images such that

$$F(x) \subset G(x) \subset \text{co}\{F(x + \varepsilon(x) \mathbb{B})\} + \varepsilon(x) \mathbb{B} \quad \forall x \in \mathbb{R}^n.$$ 



\section*{B. Proof of Propostion 2}

We start introducing the continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ given by

$$\beta(r, \delta) := \max\{g(y, s) : |y| \leq r, 0 \leq s \leq \delta\}$$

where $g : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ is given by

$$g(y, s) := |\|F(y + s \mathbb{B}) - F(y)\|_H - |\|F(s \mathbb{B}) - F(0)\|_H|.$$ 

It is clear to see that

$$g(x, \delta) \leq \beta(|x|, \delta) \quad \forall (x, \delta) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}.$$ 

Furthermore, we note that the function $\beta$ enjoys the following properties.

1) $\beta(\cdot, \delta)$ is continuous and nondecreasing for each $\delta \in \mathbb{R}_{\geq 0}$.  
2) $\beta(r, \cdot)$ is continuous and nondecreasing for each $r \in \mathbb{R}_{\geq 0}$.  
3) $\beta(0, \delta) = \beta(r, 0) = 0$ for all $(r, \delta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$.

Next, we will construct a continuous and nondecreasing function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that $\alpha(0) = 0$ and

$$\beta(r, \delta) \leq \alpha(r) \alpha(\delta) \quad \forall (r, \delta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}.$$ 

Indeed, the latter would imply that

$$g(x, \delta) \leq \alpha(|x|) \alpha(\delta) \quad \forall (x, \delta) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}.$$ 

Hence, for each $(x, \delta) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, we have

$$\|F(x + \delta \mathbb{B}) - F(x)\|_H \leq \alpha(|x|) \alpha(\delta) + |\|F(\delta \mathbb{B}) - F(0)\|_H| \leq \alpha(|x|) + 1 \max\{\alpha(\delta), |\|F(\delta \mathbb{B}) - F(0)\|_H\}.$$ 

Finally, we take $\lambda_1(\delta) := \max\{\alpha(\delta), |\|F(\delta \mathbb{B}) - F(0)\|_H\}$ and $\lambda_2(x) := |\|x\|_1| + 1$.

The proposed construction of $\alpha$ follows the same steps as in the proof of [34, Propositions IV.4 and IV.5]. The only difference is that, in the aforementioned reference, $\beta$ is assumed to be strictly increasing in each of its arguments. In the following, we recall such a proof since $\beta$ is only nondecreasing in our case, and to have an explicit form of the function $\alpha$. To this end, we introduce a set of intermediate functions.

1) We define the function $c : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$c(a, b) := \log\left(\beta\left(e^a, e^b\right)\right).$$

Note that the function $c$ enjoys the following properties:
a) $c$ is continuous;
b) $c$ nondecreasing in each of its arguments;
c) $\lim_{a \to -\infty} c(a, b) = -\infty$ for all $b \in \mathbb{R}$;
d) $\lim_{b \to -\infty} c(a, b) = -\infty$ for all $a \in \mathbb{R}$.

We will assume, without loss of generality, that $c(0, 0) > 0$ and we let
\[ \tilde{a} := \sup\{a \in \mathbb{R} : c(a, 0) = 0\}. \]

Note that such $\tilde{a}$ always exists since $c(\cdot, 0)$ is continuous, $c(0, 0) > 0$, and $c(-\infty, 0) = -\infty$.

2) Now, we introduce a nonincreasing and continuous function $\phi : (-\infty, \tilde{a}] \to [0, +\infty)$ such that $\phi(x) := \begin{cases} -\frac{g_o(a)}{2} & \text{if } a \leq \tilde{a} \\ c(a, a) - c(\tilde{a}, \tilde{a}) + a - \tilde{a} & \text{if } a > \tilde{a} \end{cases}$

By construction, $\phi$ is continuous, nondecreasing, and we have that $\lim_{a \to -\infty} \phi(a) = -\infty$.

At this point, we show that $\lim_{a \to -\infty} \phi(a) = -\infty$.

Indeed, we pick $b \in \mathbb{R}$ and we let $a \geq \max\{\tilde{a}, b\}$, we note that $c(a, b) - g(a) = c(a, b) - c(a, a) - a + \tilde{a} \leq \tilde{a} - a$ where $\tilde{a} := a + c(a, \tilde{a})$. As a result, it follows that $\lim_{a \to -\infty} \phi(a) = -\infty$.

On the other hand, for any $a \leq \tilde{a}$ such that $g_o(a) > b$, we have
\[
\begin{align*}
c(a, b) - g(a) &= c(a, b) + \frac{1}{2} g_o(a) \\
&\leq c(a, g_o(a)) + g_o(a) - \frac{1}{2} g_o(a) = -\frac{1}{2} g_o(a)
\end{align*}
\]
and $\lim_{a \to -\infty} g_o(a) = +\infty$. Hence, for each $b \in \mathbb{R}$, we have $\lim_{a \to -\infty} \phi(a) = -\infty$.

4) We introduce the function $h : \mathbb{R} \to \mathbb{R}$ given by
\[ h(b) := \sup_{a \in \mathbb{R}} [c(a, b) - g(a)]. \]

As $a \to c(a, b) - g(a)$ is continuous, and is negative for large $a$, it follows that $h$ is well defined. Since $h(\cdot)$ is the supremum of a family
\[ \{c(a, \cdot) - g(\cdot)\}_{a \in \mathbb{R}} \]
of continuous functions, $h$ is itself continuous, and since, each member of this family is nondecreasing, it follows that $h$ is also nondecreasing.

We prove, now, that $\lim_{b \to -\infty} h(b) = -\infty$. To do so, we show that, for each $v < 0$, we can find $l \leq 0$ such that, whenever $b < l$, then $h(b) < v$.

Indeed, given $v < 0$, we pick $\rho > 0$ so that $c(a, 0) - g(a) < v$ for all $a$ such that $|a| > \rho$.

Next, we pick $l \leq 0$ so that $c(\rho, l) - g(-\rho) < v$.

Such an $l$ exists since $c(\rho, \cdot)$ is unbounded ahead.

Consider first the case $|a| \leq \rho$; then $c(a, b) - g(a) \leq c(\rho, l) - g(-\rho) < v$.

If instead $|a| > \rho$, then also $c(a, b) - g(a) \leq c(a, 0) - g(a) < v$ using the fact that $b < l \leq 0$.

So, we have constructed $g$ and $h$ such that $c(a, b) \leq g(a) + h(b)$.

Hence, we obtain
\[
c(a, b) \leq \max\{g, h\}(a) + \max\{g, h\}(b) \quad \forall (a, b) \in \mathbb{R} \times \mathbb{R}.
\]

Note that, the function $\max\{g, h\}$ is also continuous, nondecreasing, and unbounded ahead. Hence, the function $\alpha : \mathbb{R} \to \mathbb{R}$ given by $s \mapsto \alpha(s) := \exp(\max\{g, h\}(\ln(s)))$ is continuous, nondecreasing and satisfies $\alpha(0) = 0$. 

**Appendix II**

**Intermediate Results**

**Lemma 1:** Given initial and unsafe sets $(X_o, X_u) \subset \mathbb{R}^n \times \mathbb{R}^n$, $\Sigma$ is safe with respect to $(X_o, X_u)$ if there exists a barrier function candidate $B : \mathbb{R}^n \to \mathbb{R}$ such that the following holds.

$(\ast)$ Along each solution $\phi$ to $\Sigma$ with $\phi(\text{dom } \phi) \subset U(\partial K)$, the map $t \mapsto B(\phi(t))$ is nonincreasing.

If $B$ is continuous, we can replace $U(\partial K)$ in $(\ast)$ by $U(\partial K) \setminus K$. In turn, the following holds.

1) When $B$ is lower semicontinuous and $F$ is locally Lipschitz, $(\ast)$ is satisfied if and only if $(\partial_P B(x), \eta) \subset \mathbb{R}_{<0}$ for all $x \in F(x)$, $\forall \eta \in U(\partial K)$.

2) When $B$ is locally Lipschitz, $(\ast)$ is satisfied, while replacing $U(\partial K)$ therein by $U(\partial K) \setminus K$, if $(\partial_C B(x), \eta) \subset \mathbb{R}_{\leq 0}$ for all $x \in F(x)$, $\forall \eta \in U(\partial K) \setminus K$.

**Proof:** To find a contradiction, we assume the existence of a solution $\phi$ starting from $x_o \in X_o$ that reaches the set $X_u$ in finite time, and we note that the (zero-sublevel) set $K$ satisfies $X_o \subset K$ and $X_u \subset \mathbb{R}^n \setminus K$. Using the continuity of $\phi$, we conclude the existence of $0 \leq t_1 < t_2$ such that $\phi(t_2) \in U(\partial K) \setminus K$, $\phi(t_1) \in K$, and $\phi(t_2) \in U(\partial K) \setminus K$. As a result, having $B(\phi(t_1)) \leq 0$ and $B(\phi(t_2)) > 0$ contradicts $(\ast)$.

When $B$ is continuous, we conclude that the (zero-sublevel) set $K$ is closed. Hence, there exists $0 \leq t_1 < t_2$
such that \( \phi(t_2) \in U(\partial K) \setminus K \), \( \phi(t_1) \in \partial K \), and \( \phi((t_1, t_2)) \subset U(\partial K) \setminus K \). Now, having \( t \mapsto B(\phi(t)) \) nonincreasing on \((t_1, t_2)\) allows us to conclude, using the continuity of both \( B \) and \( \phi \), that \( t \mapsto B(\phi(t)) \) is nonincreasing on \([t_1, t_2]\). But at the same time \( B(\phi(t_1)) \leq 0 \) and \( B(\phi(t_2)) > 0 \), which yields to a contradiction.

The proof of the remaining two statements follows using [29, Th. 6.3] and [35, Th. 3], respectively.

**Lemma 2:** Consider a set-valued map \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) satisfying Assumption 2 and such that \( F \) is continuous. Then, for each continuous function \( \epsilon : \mathbb{R}^n \rightarrow \mathbb{R}_{>0} \), there exists a continuous function \( \delta : \mathbb{R}^n \rightarrow \mathbb{R}_{>0} \) such that

\[
F(x + \delta(x)B) \subset F(x) + \epsilon(x)B \quad \forall x \in \mathbb{R}^n.
\]

(25)

**Proof:** As in the proof of Proposition 1, we grid \( \mathbb{R}^n \) using a sequence of nonempty compact subsets \( \{I_i\}_{i=1}^N \subset \mathbb{R}^n \), where \( N \in \{1, 2, 3, \ldots, \infty \} \), such that, for each \( i \in \{1, 2, \ldots, N\} \), there exists \( N' \subset \{1, 2, \ldots, N\} \) finite such that \( I_i \cap I_j = \emptyset \) for all \( j \notin N' \), and (23) holds.

Since \( F \) is continuous on each \( x \in I_i \), \( i \in \{1, 2, \ldots, N\} \), then, for each \( \epsilon_i > 0 \), there exists \( \delta > 0 \) such that

\[
F(x + \delta B) \subset F(x) + \epsilon_i B.
\]

(26)

Furthermore, since every continuous \( F \) is locally uniformly continuous, we conclude the existence of \( \delta_i > 0 \) such that

\[
F(x + \delta_i B) \subset F(x) + \epsilon_i B \quad \forall x \in I_i.
\]

(27)

Now, if we let \( \epsilon_i := \min_{x \in I_i} \epsilon(x) \), we conclude that

\[
F(x + \delta_i B) \subset F(x) + \epsilon(x)B \quad \forall x \in I_i.
\]

(28)

Next, we introduce the function \( \delta_0 : \mathbb{R}^n \rightarrow \mathbb{R}_{>0} \) given by

\[
\delta_0(x) := \min\{\delta_i : x \in I_i \}.
\]

By definition, \( \delta_0 \) is lower semicontinuous; hence, using Lemma 3 in Appendix II to conclude the existence of the function \( \delta : \mathbb{R}^n \rightarrow \mathbb{R}_{>0} \) continuous and satisfying

\[
\delta(x) \leq \delta_0(x) \quad \forall x \in \mathbb{R}^n.
\]

(29)

Hence, (25) follows.

**Lemma 3:** Consider a closed subset \( K \subset \mathbb{R}^n \) and a set-valued map \( G : K \rightrightarrows \mathbb{R}^n \) that is upper semicontinuous with \( G(x) \) nonempty, compact, and convex for all \( x \in K \). Then, there exists a continuous function \( g : K \rightarrow \mathbb{R} \) such that

\[
0 > g(x) = \sup\{\eta : \eta \in G(x)\} \quad \forall x \in K.
\]

(30)

**Proof:** We start introducing the function \( f : K \rightarrow \mathbb{R}_{>0} \) given by \( f(x) := \sup\{\eta : \eta \in G(x)\} \). Using [26, Th. 1.4.16], we conclude that \( f \) is upper semicontinuous. Finally, using [31, Th. 1], we conclude that there exists a continuous function \( g : K \rightarrow \mathbb{R}_{>0} \) such that

\[
g(x) \geq f(x) \quad \forall x \in K.
\]

\[ \square \]

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[31] M. Katětov, “On real-valued functions in topological spaces,” Fundamenta Mathematicae, vol. 1, no. 38, pp. 85–91, 1951.

[32] M. Maghenem and R. G. Sanfelice, “Sufficient conditions for forward invariance and contractivity in hybrid inclusions using barrier functions,” Automatica, vol. 124, 2020, Art. no. 109328.

[33] S. M. Aseev, “Approximation of semicontinuous multivalued mappings by continuous ones,” Math. USSR-Izvestiya, vol. 20, 1983, Art. no. 435.

[34] D. Angeli, E. D. Sontag, and Y. Wang, “A characterization of integral input-to-state stability,” IEEE Trans. Autom. Control, vol. 45, no. 6, pp. 1–36, Jun. 2000.

[35] M. Maghenem, A. Melis, and R. G. Sanfelice, “Necessary and sufficient conditions for the nonincrease of scalar functions along solutions to constrained differential inclusions,” ESAIM. Control, Optim., Calculus Variations, vol. 28, 2022, Art. no. 13.

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