Classical Pole Placement Adaptive Control Revisited: Exponential Stabilization

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Abstract—While the original classical parameter adaptive controllers did not handle noise or unmodelled dynamics well, redesigned versions were proven to have some tolerance; however, exponential stabilization and a bounded gain on the noise was rarely proven. Here we consider a classical pole placement adaptive controller using the original projection algorithm rather than the commonly modified version; we impose the assumption that the plant parameters lie in a convex, compact set. We demonstrate that the closed-loop system exhibits very desirable closed-loop behaviour: there are linear-like convolution bounds on the closed loop behaviour, which confers exponential stability and a bounded noise gain, and can be leveraged to prove tolerance to unmodelled dynamics and plant parameter variation. We emphasize that there is no persistent excitation requirement of any sort; the improved performance arises from the vigilant nature of the parameter estimator.

Keywords: Adaptive control, Projection algorithm, Exponential stability, Bounded gain.

I. INTRODUCTION

Adaptive control is an approach used to deal with systems with uncertain or time-varying parameters. The classical adaptive controller consists of a linear time-invariant (LTI) compensator together with a tuning mechanism to adjust the compensator parameters to match the plant. The first general proofs that adaptive controllers could work came around 1980, e.g. see [2], [19], [4], [25], and [26]. However, such controllers were typically not robust to unmodelled dynamics, did not tolerate time-variations well, and did not handle noise or disturbances well, e.g. see [27]. During the following two decades a great deal of effort was made to address these shortcomings. The most common approach was to make small controller design changes, such as the use of signal normalization, deadzones, and $\sigma-$modification, to ameliorate these issues, e.g. see [13], [12], [28], [11], [8]. Indeed, simply using projection (onto a convex set of admissible parameters) has proved quite powerful, and the resulting controllers typically provide a bounded-noise bounded-state property, as well as tolerance of some degree of unmodelled dynamics and/or time-variations, e.g. see [34], [35], [22], [33], [32] and [9]. Of course, it is clearly desirable that the closed-loop system exhibit LTI-like system properties, such as a bounded gain and exponential stability. As far as the author is aware, in the classical approach to adaptive control a bounded gain on the noise is proven only in [35]; however, a crisp exponential bound on the effect of the initial condition is not provided, and a minimum phase assumption is imposed. While it is possible to prove a form of exponential stability if the reference input is sufficiently persistently exciting, e.g. see [23], this places a stringent requirement on an exogenous input.

There are several non-classical approaches to adaptive control which provide LTI-like system properties. First of all, in [3] and [18] a logic-based switching approach was used to switch between a predefined list of candidate controllers; while exponential stability is proven, the transient behaviour can be quite poor and a bounded gain on the noise is not proven. A more sophisticated logic-based approach, labelled Supervisory Control, was proposed by Morse; here a supervisor switches in an efficient way between candidate controllers - see [20], [21], [6], [30] and [7]. In certain circumstances a bounded gain on the noise can be proven - see [31] and the Concluding Remarks section of [21]. A related approach, called localization-based switching adaptive control, uses a falsification approach to prove exponential stability as well as a degree of tolerance of disturbances, e.g. see [36].

Another non-classical approach, proposed by the author, is based on periodic estimation and control: rather than estimate the plant or controller parameters, the goal is to estimate what the control signal would be if the plant parameters and plant state were known and the 'optimal controller' were applied. Exponential stability and a bounded gain on the noise is achieved, as well as near optimal performance, e.g. see [14], [15], and [29]; a degree of unmodelled dynamics and time variations can be allowed. The cost of these desirable features is that the noise gain increases dramatically the closer that one gets to optimality.

In this paper we consider the discrete-time setting and we propose an alternative approach to obtaining LTI-like system properties. We return to a common approach in classical adaptive control - the use of the projection algorithm together with the Certainty Equivalence Principle. In the literature it is the norm to use a modified version of the ideal

1Since the closed-loop system is nonlinear, a bounded-noise bounded-state property does not automatically imply a bounded gain on the noise.
Projection Algorithm in order to avoid division by zero; \footnote{An exception is the work of Ydstie \cite{34}, \cite{35}, who considers the ideal Projection Algorithm as a special case; however, a crisp bound on the effect of the initial condition is not proven and a minimum phase assumption is imposed.} it turns out that an unexpected consequence of this minor adjustment is that some inherent properties of the scheme are destroyed. Here we use the original version of the Projection Algorithm coupled with a pole placement Certainty Equivalence based controller. We obtain linear-like convolution bounds on the closed-loop behaviour, which immediately confers exponential stability and a bounded gain on the noise; such convolution bounds are, as far as the author is aware, a first in adaptive control, and it allows us to use a modular approach to analyse robustness and tolerance to time-varying parameters. To this end, the results will be presented in a very pedagogically desirable fashion: we first deal with the ideal plant (with disturbances); we then leverage that result to prove that a large degree of time-variations is tolerated; we then demonstrate that the approach tolerates a degree of unmodelled dynamics, in a way familiar to those versed in the analysis of LTI systems.

In a recent short paper we consider the first order case \cite{16}. Here we consider the general case, which requires much more sophisticated analysis and proofs. Furthermore, in comparison to \cite{16}, here we (i) present a more general estimation algorithm, which alleviates the classical concern about dividing by zero, (ii) prove that the controller achieves the objective in the presence of a more general class of time-variations, and (iii) prove robustness to unmodelled dynamics. An early version of this paper has been submitted to a conference \cite{17}.

Before proceeding we present some mathematical preliminaries. Let $\mathbb{Z}$ denote the set of integers, $\mathbb{Z}^+$ the set of non-negative integers, $\mathbb{N}$ the set of natural numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{R}^+$ the set of non-negative real numbers. We let $\mathbb{D}^0$ denote the open unit disk of the complex plane. We use the Euclidean 2-norm for vectors and the corresponding induced norm for matrices, and denote the norm of a vector or matrix by $\| \cdot \|$. We let $l_\infty(\mathbb{R}^n)$ denote the set of $\mathbb{R}^n$-valued bounded sequences; we define the norm of $u \in l_\infty(\mathbb{R}^n)$ by $\|u\|_\infty := \sup_{k \in \mathbb{Z}} \|u(k)\|$. Occasionally we will deal with a map $F : l_\infty(\mathbb{R}^n) \to l_\infty(\mathbb{R}^m)$; the gain is given by $\sup_{u \neq 0} \frac{\|F u\|_\infty}{\|u\|_\infty}$ and denoted by $\|F\|$. With $T \in \mathbb{Z}$, the truncation operator $P_T : l_\infty(\mathbb{R}^n) \to l_\infty(\mathbb{R}^n)$ is defined by

$$
(P_T x)(t) = \begin{cases}
  x(t) & t \leq T \\
  0 & t > T.
\end{cases}
$$

We say that the map $F : l_\infty(\mathbb{R}^n) \to l_\infty(\mathbb{R}^n)$ is causal if $P_T F P_T = P_T F$ for every $T \in \mathbb{Z}$.

If $S \subset \mathbb{R}^p$ is a convex and compact set, we define $\|S\| := \max_{x \in S} \|x\|$ and the function $\pi_S : \mathbb{R}^p \to S$ denotes the projection onto $S$; it is well-known that $\pi_S$ is well-defined.

\section{The Setup}

In this paper we start with an $n^{th}$ order linear time-invariant discrete-time plant given by

$$
y(t + 1) = - \sum_{i=0}^{n-1} a_i y(t - i) + \sum_{i=0}^{n-1} b_i u(t - i) + d(t)
$$

$$
= \begin{bmatrix}
y(t) \\
y(t - n + 1) \\
u(t) \\
u(t - n + 1)
\end{bmatrix}^T
\begin{bmatrix}
a_1 \\
\vdots \\
a_n \\
b_1 \\
\vdots \\
b_n
\end{bmatrix}
+ d(t),
$$

where $y(t) \in \mathbb{R}$ the measured output, $u(t) \in \mathbb{R}$ the control input, and $d(t) \in \mathbb{R}$ the disturbance (or noise) input. We assume that $\theta^*$ is unknown but belongs to a known set $S \subset \mathbb{R}^{2n}$. Associated with this plant model are the polynomials

$$
A(z^{-1}) := 1 + a_1 z^{-1} + \cdots + a_n z^{-n},
$$

$$
B(z^{-1}) := b_1 z^{-1} + \cdots + b_n z^{-n}
$$

and the transfer function $\frac{B(z^{-1})}{A(z^{-1})}$.

\textbf{Remark 1:} It is straight-forward to verify that if the system has a disturbance at both the input and output, then it can be converted to a system of the above form.

We impose an assumption on the set of admissible plant parameters.

\begin{assumption}

$S$ is convex and compact, and for each $\theta^* \in S$, the corresponding pair of polynomials $A(z^{-1})$ and $B(z^{-1})$ are coprime.

\end{assumption}

The convexity part of the above assumption is common in a branch of the adaptive control literature - it is used to facilitate parameter projection, e.g. see \cite{5}. The boundedness part is less common, but it is quite reasonable in practical situations; it is used here to ensure that we can prove uniform bounds and decay rates on the closed-loop behaviour.

The main goal here is to prove a form of stability, with a secondary goal that of asymptotic tracking of an exogenous reference signal $y^*(t)$; since the plant may be non-minimum phase, there are limits on how well the plant can be made to track $y^*(t)$. To proceed we use a parameter estimator together with an adaptive pole placement control law. At this point, we discuss the most critical aspect - the parameter estimator.
A. Parameter Estimation

We can write the plant as

\[ y(t+1) = \phi(t)^T \theta^* + d(t). \]

Given an estimate \( \hat{\theta}(t) \) of \( \theta^* \) at time \( t \), we define the prediction error by

\[ e(t+1) := y(t+1) - \phi(t)^T \hat{\theta}(t); \]

this is a measure of the error in \( \hat{\theta}(t) \). The common way to obtain a new estimate is from the solution of the optimization problem

\[ \arg\min_{\theta} \{ ||\theta - \hat{\theta}(t)|| : y(t+1) = \phi(t)^T \theta \}, \]

yielding the ideal (projection) algorithm

\[
\hat{\theta}(t + 1) = \begin{cases} 
\hat{\theta}(t) + \frac{\phi(t)^T}{\phi(t)^T \phi(t)} e(t+1) & \text{if } \phi(t) = 0 \\
\hat{\theta}(t) + \frac{\phi(t)}{\phi(t)^T \phi(t)} e(t+1) & \text{otherwise.} 
\end{cases}
\]

Of course, if \( \phi(t) \) is close to zero, numerical problems can occur, so it is the norm in the literature (e.g. [4] and [5]) to replace this by the following classical algorithm: with

\[ 0 < \alpha < 2 \text{ and } \beta > 0, \text{ define} \]

\[
\hat{\theta}(t + 1) = \hat{\theta}(t) + \frac{\alpha \phi(t)}{\beta + \phi(t)^T \phi(t)} e(t+1) \tag{3}
\]

This latter algorithm is widely used, and plays a role in many discrete-time adaptive control algorithms; however, when this algorithm is used, all of the results are asymptotic, and exponential stability and a bounded gain on the noise are never proven. It is not hard to guess why - a careful look at the estimator shows that the gain on the update law is small if \( \phi(t) \) is small. A more mathematically detailed argument is given in the following example.

Remark 2: Consider the simple first order plant

\[ y(t + 1) = -a_1 y(t) + b_1 u(t) + d(t) \]

with \( a_1 \in [-2, -1] \) and \( b_1 \in [1, 2] \). For simplicity, we assume that in the estimator [5] we have \( \alpha = \beta = 1 \), and, as in [34], [35], [22], [33] and [9], we use projection to keep the parameters inside \( S \) so as to guarantee a bounded-input bounded-state property. Further suppose \( y^* = d = 0 \), and that a classical pole placement adaptive controller places the closed-loop pole at zero: \( u(t) = \frac{a_1(t)}{b_1(t)} y(t) =: f(t) y(t) \). Suppose that

\[ y(0) = y_0 = \varepsilon \in (0, 1), \]

\[ \hat{\theta}(0) = \begin{bmatrix} -\frac{a_1(0)}{b_1(0)} \\ \frac{1}{2} \end{bmatrix}, \quad \theta^* = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]

so that \( \hat{f}(0) = -0.5 \) and \( -a_1 + b_1 \hat{f}(0) = 1.5 \), i.e. the system is initially unstable. An easy calculation verifies that \( \hat{f}(t) \in [-2, -0.5] \) and \( -a_1 + b_1 \hat{f}(t) \in [0, 1.5] \) for \( t \geq 0 \), which leads to a crude bound on the closed loop behaviour:

\[ |y(t)| \leq (1.5)^t \varepsilon \quad \text{for } t \geq 0. \]

With \( N(\varepsilon) := \text{int}(\frac{1}{2 \ln(1.5)} \ln(\frac{1}{\varepsilon})) \), it follows that

\[ |y(t)| \leq \varepsilon^{1/2}, \quad t \in [0, N(\varepsilon)]. \]

A careful examination of the parameter estimator shows that

\[ \|\hat{\theta}(t) - \theta_0\| \leq 10(2)^{1/2} \varepsilon, \quad t \in [0, N(\varepsilon)]. \]

From the form of \( \hat{f}(t) \), it follows that for small \( \varepsilon \) we have

\[ |\varepsilon| a + b_1 \hat{f}(0)| \geq 1.25 \forall t \in [0, N(\varepsilon)], \]

in which case

\[ |y(N(\varepsilon))| \geq 1.25^N(\varepsilon) \varepsilon \Rightarrow \left| y(N(\varepsilon)) \right| = (1.25)^N(\varepsilon); \]

since \( N(\varepsilon) \to \infty \) as \( \varepsilon \to 0 \), we see that exponential stability is unachievable. A similar kind of analysis can be used to prove that a bounded gain on the noise is not achievable either.

Now we return to the problem as hand - analysing the ideal algorithm [3]. We will be using the ideal algorithm with projection to ensure that the estimate remains in \( S \) for all time. With an initial condition of \( \hat{\theta}(t_0) = \theta_0 \in S \), for \( t \geq t_0 \) we set

\[
\hat{\theta}(t + 1) = \begin{cases} 
\hat{\theta}(t) + \frac{\phi(t)}{\phi(t)^T \phi(t)} e(t+1) & \text{if } \phi(t) = 0 \\
\hat{\theta}(t) + \frac{\phi(t)^T}{\phi(t)^T \phi(t)} e(t+1) & \text{otherwise,} 
\end{cases}
\]

which we then project onto \( S \):

\[
\hat{\theta}(t + 1) := \pi_S(\hat{\theta}(t+1)).
\]

Because of the closed and convex property of \( S \), the projection function is well-defined; furthermore, it has the nice property that, for every \( \theta \in \mathbb{R}^{2n} \) and every \( \theta^* \in S \), we have

\[ ||\pi_S(\theta) - \theta^*|| \leq ||\theta - \theta^*||, \]

i.e. projecting \( \theta \) onto \( S \) never makes it further away from the quantity \( \theta^* \).

B. Revised Parameter Estimation

Some readers may be concerned that the original problem of dividing by a number close to zero, which motivates the use of classical algorithm, remains. Of course, this is balanced against the soon-to-be-proved benefit of using [4]-[5]. We propose a middle ground as follows. A straightforward analysis of \( e(t+1) \) reveals that

\[ e(t+1) = -\phi(t)^T \hat{[\hat{\theta}(t) - \theta^*]} + d(t), \]

which means that

\[ |e(t+1)| \leq 2 ||S|| \times ||\phi(t)|| + ||d(t)||. \]

Therefore, if

\[ |e(t+1)| > 2 ||S|| \times ||\phi(t)||, \]

then the update to \( \hat{\theta}(t) \) will be greater than \( 2 ||S|| \), which means that there is little information content in \( e(t+1) \) - it is dominated by the disturbance. With this as motivation, and with \( \delta \in (0, \infty) \), let us replace [4] with

\[
\hat{\theta}(t + 1) = \begin{cases} 
\hat{\theta}(t) + \frac{\phi(t)}{\phi(t)^T \phi(t)} e(t+1) & \text{if } |e(t+1)| < (2 ||S|| + \delta) ||\phi(t)|| \\
\hat{\theta}(t) & \text{otherwise.} 
\end{cases}
\]

[It is common to make this more general by letting \( \alpha \) be time-varying.]
in the case of $\delta = \infty$, we will adopt the understanding that $\infty \times 0 = 0$, in which case the above formula collapses into the original one (3). In the case that $\delta < \infty$, we can be assured that the update term is bounded above by $2\|S\| + \delta$, which should alleviate concern about having infinite gain. We would now like to rewrite the update to make it more concise. To this end, we now define $\rho_3: \mathbb{R}^{2n} \times \mathbb{R} \to \{0,1\}$ by

$$\rho_3(\phi(t), e(t+1)) :=
\begin{cases}
1 & \text{if } |e(t+1)| < (2\|S\| + \delta)\|\phi(t)\| \\
0 & \text{otherwise},
\end{cases}$$

yielding a more concise way to write the estimation algorithm update:

$$\hat{\theta}(t+1) = \hat{\theta}(t) + \rho_3(\phi(t), e(t+1))\frac{\phi(t)}{\|\phi(t)\|^2} e(t+1); \quad (7)$$

once again, we project this onto $S$:

$$\hat{\theta}(t+1) := \pi_S(\hat{\theta}(t+1)). \quad (8)$$

C. Properties of the Estimation Algorithm

Analysing the closed-loop system will require a careful analysis of the estimation algorithm. We define the parameter estimation error by

$$\hat{\theta}(t) := \hat{\theta}(t) - \theta^*,$$

and the corresponding Lyapunov function associated with $\hat{\theta}(t)$, namely $V(t) := \hat{\theta}(t)^T \hat{\theta}(t)$. In the following result we list a property of $V(t)$; it is a generalization of what is well-known for the classical algorithm [3], and (8) is applied to the plant (11), the following holds:

$$\|\hat{\theta}(t+1) - \hat{\theta}(t)\| \leq \rho_4(\phi(t), e(t+1)) \frac{|e(t+1)|}{\|\phi(t)\|}, \quad t \geq t_0,$$

$$V(t) \leq V(t_0) + \sum_{j=t_0}^{t-1} \rho_3(\phi(j), e(j+1)) \times \left[ \frac{\|\phi(j)\|^2}{2\|\phi(j)\|^2} + \frac{\|d(j)\|^2}{2\|\phi(j)\|^2}, \quad t \geq t_0 + 1. \quad (9) \right.$$}

**Proposition 1:** For every $t_0 \in \mathbb{Z}$, $\phi_0 \in \mathbb{R}^{2n}$, $\theta_0 \in S$, $\theta^* \in S$, $d \in l_\infty$, and $\delta \in (0, \infty]$, when the estimator (7) and (8) is applied to the plant (11), the following holds:

$$\|\hat{\theta}(t+1) - \hat{\theta}(t)\| \leq \rho_4(\phi(t), e(t+1)) \frac{|e(t+1)|}{\|\phi(t)\|}, \quad t \geq t_0,$$

$$V(t) \leq V(t_0) + \sum_{j=t_0}^{t-1} \rho_3(\phi(j), e(j+1)) \times \left[ \frac{\|\phi(j)\|^2}{2\|\phi(j)\|^2} + \frac{\|d(j)\|^2}{2\|\phi(j)\|^2}, \quad t \geq t_0 + 1. \quad (9) \right.$$}

**Proof:** See the Appendix. □

D. The Control Law

The elements of $\hat{\theta}(t)$ are partitioned in a natural way as

$$\hat{\theta}(t) = [ -\hat{a}_1(t) \ \cdots \ -\hat{a}_n(t) \ \hat{b}_1(t) \ \cdots \ \hat{b}_n(t) ]^T$$

Associated with $\hat{\theta}(t)$ are the polynomials

$$\hat{\mathcal{A}}(t, z^{-1}) := 1 + \hat{a}_1(t)z^{-1} + \cdots + \hat{a}_n(t)z^{-n},$$

$$\hat{\mathcal{B}}(t, z^{-1}) := \hat{b}_1(t)z^{-1} + \cdots + \hat{b}_n(t)z^{-n}.$$ 

While we can use an $n-1^{th}$ order **proper** controller to carry out pole placement, it will be convenient to follow the lead of [33] and use an $n^{th}$ order **strictly proper** controller. In particular, we first choose a $2n^{th}$ order monic polynomial

$$A^*(z^{-1}) = 1 + a_1^s z^{-1} + \cdots + a_{2n}^s z^{-2n}$$

so that $z^{2n}A^*(z^{-1})$ has all of its zeros in $D^o$. Next, we choose two polynomial

$$\hat{\mathcal{L}}(t, z^{-1}) = 1 + \hat{\mathcal{L}}_1(t)z^{-1} + \cdots + \hat{\mathcal{L}}_n(t)z^{-n}$$

and

$$\hat{\mathcal{P}}(t, z^{-1}) = \hat{\mathcal{P}}_1(t)z^{-1} + \cdots + \hat{\mathcal{P}}_n(t)z^{-n}$$

which satisfy the equation

$$\hat{\mathcal{A}}(t, z^{-1})\hat{\mathcal{L}}(t, z^{-1}) + \hat{\mathcal{B}}(t, z^{-1})\hat{\mathcal{P}}(t, z^{-1}) = A^*(z^{-1}); \quad (10)$$

given the assumption that the $\hat{\mathcal{A}}(t, z^{-1})$ and $\hat{\mathcal{B}}(t, z^{-1})$ are coprime, it is well known that there exist unique $\hat{\mathcal{L}}(t, z^{-1})$ and $\hat{\mathcal{P}}(t, z^{-1})$ which satisfy this equation. Indeed, it is easy to prove that the coefficients of $\hat{\mathcal{L}}(t, z^{-1})$ and $\hat{\mathcal{P}}(t, z^{-1})$ are analytic functions of $\hat{\theta}(t) \in S$.

In our setup we have an exogenous signal $y^*(t)$. At time $t$ we choose $u(t)$ so that

$$u(t) = -\hat{\mathcal{L}}_1(t-1)u(t-1) - \cdots - \hat{\mathcal{L}}_n(t-1)u(t-n) - \hat{\mathcal{P}}_1(t-1)y(t-1) - y^*(t-1) - \cdots - \hat{\mathcal{P}}_n(t-1)y(t-n) - y^*(t-n). \quad (11)$$

So the overall controller consists of the estimator (7)-(8) together with (11).

It turns out that we can write down a state-space model of our closed-loop system with $\phi(t) \in \mathbb{R}^{2n}$ as the state. Only two elements of $\phi$ have a complicated description:

$$\phi_1(t+1) = y(t+1) = e(t+1) + \hat{\theta}(t)^T \phi(t),$$

$$\phi_{n+1}(t+1) = u(t+1) = -\sum_{i=1}^{n} \hat{\mathcal{L}}_i(t)u(t+1-i) - \hat{\mathcal{P}}_i(t-1)\{y(t+1-i) - y^*(t+1-i)] \} =\left[ -\hat{\mathcal{L}}_1(t) \ \cdots \ -\hat{\mathcal{L}}_n(t) \ -\hat{\mathcal{P}}_1(t) \ \cdots \ -\hat{\mathcal{P}}_n(t) \right] \phi(t) + \sum_{i=1}^{n} \hat{\mathcal{P}}_i(t) y^*(t+1-i).$$

With $e_i \in \mathbb{R}^{2n}$ the $i^{th}$ normal vector, if we now define

$$\hat{\mathcal{A}}(t) := \begin{bmatrix}
-\hat{a}_1(t) & -\hat{a}_2(t) & \cdots & -\hat{a}_n(t) \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0
\end{bmatrix}$$

$$\hat{\mathcal{A}}(t) := \begin{bmatrix}
-\hat{p}_1(t) & -\hat{p}_2(t) & \cdots & -\hat{p}_n(t) \\
0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0
\end{bmatrix}$$

\footnote{We also implicitly use a pole placement procedure to obtain the controller parameters from the plant parameter estimates; this entails solving a linear equation.}
than zero, but it is trivial to extend this to allow for zero as well. 

\[ \hat{b}_1(t) \cdots \cdots \hat{b}_n(t) \]

\[ \begin{array}{cccc}
0 & \cdots & 0 \\
\vdots & \ddots & & \vdots \\
0 & \cdots & 0 \\
-\hat{a}_1(t) & -\hat{a}_2(t) & \cdots & -\hat{a}_n(t) \\
1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
1 & 0 & \cdots & 0 \\
\end{array} \] ,

then the following key equation holds:

\[ \phi(t + 1) = \hat{A}(t)\phi(t) + B_1e(t + 1) + B_2r(t); \] (13)

notice that the characteristic equation of \( \hat{A}(t) \) always equals

\[ z^{2n}A^*(z^{-1}). \]

Before proceeding, define

\[ \hat{a} := \max\{\|\hat{A}(\hat{\theta})\| : \hat{\theta} \in S\}. \]

III. PRELIMINARY ANALYSIS

The closed-loop system given in (13) arises in classical adaptive control approaches in slightly modified fashion, so we will borrow some tools from there. More specifically, the following result was proven by Kreisselmeier [10], in the context of proving that a slowly time-varying adaptive control system is stable (in a weak sense); we are providing a special case of his technical lemma to minimize complexity.

Proposition 2: [10] Consider the discrete-time system

\[ x(t + 1) = [A_{nom}(t) + \Delta(t)]x(t) \]

with \( \Phi(t, \tau) \) denoting the corresponding state transition matrix. Suppose that there exist constants \( \sigma, \gamma_1 > 1, \alpha_1 \geq 0, \) and \( \beta_1 \geq 0 \) so that

(i) for all \( t \geq t_0 \), we have \( \|A_{nom}(t)\| \leq \gamma_1\sigma^t, i \geq 0; \)

(ii) for all \( t \geq \tau \) we have

\[ \sum_{i=\tau}^{t-1} \|A_{nom}(i + 1) - A_{nom}(i)\| \leq \alpha_0 + \alpha_1(t - \tau)^{1/2} + \alpha_2(t - \tau) \]

and \( \sum_{i=\tau}^{t-1} \|\Delta(i)\| \leq \beta_0 + \beta_1(t - \tau)^{1/2} + \beta_2(t - \tau); \)

(iii) there exists a \( \mu \in (\sigma, 1) \) and \( N \in \mathbb{N} \) satisfying \( \alpha_2 + \frac{\beta_2}{N} \leq \frac{1}{N} \mu^N \gamma_1^N \sigma - \sigma \).

Then there exists a constant \( \gamma_2 \) so that the transition matrix satisfies

\[ \|\Phi(t, \tau)\| \leq \gamma_2\mu^{t-\tau}, t \geq \tau. \]

Remark 3: We apply the above proposition in the following way. Suppose that \( \sigma, \gamma_1 > 1, \alpha_1 \geq 0, \beta_1 \geq 0 \) are such conditions (i) and (ii) hold. If \( \mu \in (\sigma, 1) \), then it follows that \( \frac{\mu}{\gamma_1^N} - \sigma > 0 \) for large enough \( N \in \mathbb{N} \), so condition (iii) will hold as well as long as \( \alpha_2 \) and \( \beta_2 \) are small enough.

In applying Proposition 2, the matrix \( \hat{A}(t) \) will play the role of \( A_{nom}(t) \). A key requirement is that Condition (i) holds: the following provides relevant bounds. Before proceeding, let

\[ \Delta := \max\{\|\lambda\| : \lambda \text{ is a root of } z^{2n}A^*(z^{-1})\}. \]

Lemma 1: For every \( \delta \in (0, \infty) \) and \( \sigma \in (\Delta, 1) \) there exists a constant \( \gamma \geq 1 \) so that for every \( t_0 \in \mathbb{Z}, \theta_0 \in \mathcal{S}, \theta^* \in \mathcal{S}, \) and \( \gamma^*, d \in l_\infty, \) when the adaptive controller (7), (8) and (11) is applied to the plant (1), the matrix \( \hat{A}(t) \) satisfies, for every \( t \geq t_0 \):

\[ \|\hat{A}(t)\| \leq \gamma \sigma^k, k \geq 0, \]

and for every \( t > k \geq t_0 \):

\[ \|\hat{A}(j + 1) - \hat{A}(j)\| \leq \gamma \times \]

\[ \sum_{j=k}^{t-1} \rho_\ell(\phi(j), e(j + 1)) \frac{e(j + 1)^2}{\|\phi(j)\|^2}^{1/2}(t - k)^{1/2}. \]

Proof: See the Appendix. \( \square \)

IV. THE MAIN RESULT

Theorem 1: For every \( \delta \in (0, \infty) \) and \( \lambda \in (\Delta, 1) \) there exists a \( c > 0 \) so that for every \( t_0 \in \mathbb{Z}, \theta_0 \in \mathcal{S}, \theta^* \in \mathcal{S}, \phi_0 \in \mathbb{R}^{2n} \), and \( y^*, d \in \ell_\infty \), when the adaptive controller (7), (8) and (11) is applied to the plant (1), the following bound holds:

\[ \|\phi(k)\| \leq c\lambda^{k-t_0}\|\phi_0\| + \sum_{j=t_0}^{k-1} c\lambda^{k-1-j}\|d(j)\| \]

\[ + \sum_{j=t_0-n+1}^{k-1} c\lambda^{k-1-j}\|y(j)\|, \quad k \geq t_0. \] (14)

Remark 4: We see from (13) that \( r(t) \) is a weighted sum of \( \{y^*(t), ..., y^*(t - n + 1)\} \). Hence, there exists a constant \( \tilde{c} \) so that the bound (14) can be rewritten as

\[ \|\phi(k)\| \leq c\lambda^{k-t_0}\|\phi_0\| + \sum_{j=t_0}^{k-1} c\lambda^{k-1-j}\|d(j)\| + \sum_{j=t_0-n+1}^{k-1} c\lambda^{k-1-j}\|y(j)\|, \quad k \geq t_0. \]

Remark 5: Theorem 1 implies that the system has a bounded gain (from \( d \) to \( y \)) in every \( p \)-norm. More specifically, for \( p = \infty \) we see immediately from (14) that

\[ \|\phi(k)\| \leq c\|\phi_0\| + \frac{c}{1 - \lambda^N} \sup_{\tau \in [t_0, k]} \|r(\tau) + |d(\tau)|\|, \quad k \geq t_0. \]

Furthermore, for \( 1 \leq p < \infty \) it follows from Young’s Inequality applied to (14) that

\[ \|\sum_{j=t_0}^{k} \|\phi(j)\|^p \|^{1/p} \leq \frac{c}{(1 - \lambda^N)^{1/p}} \|\phi_0\| + \]

\[ \sum_{j=t_0}^{k} \|\phi(j)\|^p \|^{1/p} \leq \frac{c}{(1 - \lambda^N)^{1/p}} \|\phi_0\| + \]

\[ \sum_{j=t_0-n+1}^{k-1} c\lambda^{k-1-j}\|y(j)\|, \quad k \geq t_0. \]
\[
\frac{c}{1-\lambda}\{\sum_{j=t_0}^k \|r(j)\|^{1/p} + \sum_{j=t_0}^k \|d(j)\|^{1/p}\}, \ k \geq t_0.
\]

**Remark 6:** Most pole placement adaptive controllers are proven to yield a weak form of stability, such as boundedness (in the presence of a zero disturbance) or asymptotic stability (in the case of a zero disturbance), which means that details surrounding initial conditions can be ignored. Here the goal is to prove a stronger, linear-like, convolution bound, so it requires more detailed analysis.

**Remark 7:** With \( \hat{G}(t, z^{-1}) = \sum_{i=1}^{2n} \hat{g}_i(t)z^{-i} = B(t, z^{-1})\hat{P}(t, z^{-1}) \) it is possible to use arguments like those in [5] to prove, when the disturbance \( d \) is identically zero, a weak tracking result of the form

\[
\lim_{t \to \infty} \sum_{i=0}^{2n} a_i^*y(t-i) - \sum_{i=1}^{2n} \hat{g}_i(t)y^*(t-i) = 0.
\]

Since the main goal of the paper is on stability issues, we omit the proof. However, we do discuss step tracking in a later section.

**Proof:** Fix \( \delta \in (0, \infty) \) and \( \lambda \in \{0, 1\} \). Let \( t_0 \in \mathbb{Z} \), \( \theta_0 \in \mathcal{S} \), \( \theta^* \in \mathcal{S}, \phi_0 \in \mathbb{R}^{2n} \), and \( y^*, d \in l_{\infty} \) be arbitrary. Define \( r \) via (12). Now choose \( \lambda_1 \in \{0, 1\} \).

We have to be careful in how to apply Proposition 2 to (13) - we need the \( \Delta(t) \) term to be something which we can bound using Proposition 1. So define

\[
\Delta(t) := \rho_s(\phi(t), e(t+1) + B_1\phi(t)^T; \ (15)
\]

it is easy to check that

\[
\Delta(t)\phi(t) = \rho_s(\phi(t), e(t+1))B_1\phi(t)
\]

and that

\[
||\Delta(t)|| = \rho_s(\phi(t), e(t+1)) \frac{|e(t+1)|}{||\phi(t)||},
\]

which is a term which plays a key role in Proposition 1. We can now rewrite (13) as

\[
\phi(t+1) = [\hat{A}(t) + \Delta(t)]\phi(t) + B_1[1 - \rho_s(\phi(t), e(t+1))]e(t+1) + B_2r(t).
\]

If \( \rho_s(\phi(t), e(t+1)) = 1 \) then \( \eta(t) = 0 \), but if \( \rho_s(\phi(t), e(t+1)) = 0 \) then

\[
|e(t+1)| \geq (2||S|| + 2)||\phi(t)||;
\]

but we also know that

\[
|e(t+1)| \leq 2||S|| \times ||\phi(t)|| + |d(t)|;
\]

combining these equations we have

\[
(2||S|| + \delta)||\phi(t)|| \leq 2||S|| \times ||\phi(t)|| + |d(t)|,
\]

which implies that \( ||\phi(t)|| \leq \frac{1}{2}|d(t)| \); it is easy to check that this holds even when \( \delta = \infty \). Using (17) we conclude that

\[
|\eta(t)| \leq \frac{2||S||}{\delta} + 1|d(t)|, \ t \geq t_0.
\]

We now analyse (16). We let \( \Phi(t, r) \) denote the transition matrix associated with \( \hat{A}(t) + \Delta(t) \); this matrix clearly implicitly depends on \( \theta_0, \theta^*, d \) and \( r \). From Lemma 1 there exists a constant \( \gamma_1 \) so that

\[
||\hat{A}(t)^i|| \leq \gamma_1 \lambda_i^i, \ i \geq 0, \ t \geq t_0,
\]

and for every \( t > k \geq t_0 \), we have

\[
\gamma_1 \sum_{j=k}^{t-1} ||\hat{A}(j+1) - \hat{A}(j)|| \leq \gamma_1 \sum_{j=k}^{t-1} \rho_\delta(\phi(j), e(j+1)) \frac{|e(j+1)|}{||\phi(j)||^2} \frac{|e(j+1)|}{1/2(t-k)^{1/2}}.
\]

Using the definition of \( \Delta \) given in (15) and the Cauchy-Schwarz inequality we also have

\[
\gamma_1 \sum_{j=k}^{t-1} ||\Delta(j)|| \leq \sum_{j=k}^{t-1} \rho_\delta(\phi(j), e(j+1)) \times \frac{|e(j+1)|}{||\phi(j)||^2} \frac{|e(j+1)|}{1/2(t-k)^{1/2}}, \ t > k \geq t_0.
\]

At this point we consider two cases: the easier case in which there is no noise, and the harder case in which there is noise.

**Case 1:** \( d(t) = 0, \ t \geq t_0 \).

Using the bound on \( \eta(t) \) given in (18), in this case (16) becomes

\[
\phi(t+1) = [\hat{A}(t) + \Delta(t)]\phi(t) + B_2r(t), \ t \geq t_0.
\]

The bound on \( V(t) \) given by Proposition 1 simplifies to

\[
V(t) \leq V(t_0) - \frac{1}{2} \sum_{j=t_0}^{t-1} \rho_\delta(\phi(j), e(j+1)) \frac{|e(j+1)|}{||\phi(j)||^2}, \ t \geq t_0 + 1.
\]

Since \( V(\cdot) \geq 0 \) and \( V(t_0) = ||\theta_0 - \theta^*||^2 \leq 4||S||^2 \), this means that

\[
\sum_{j=t_0}^{t-1} \rho_\delta(\phi(j), e(j+1)) \frac{|e(j+1)|}{||\phi(j)||^2} \leq 2V(t_0) \leq 8||S||^2.
\]

Hence, from (20) and (21) we conclude that

\[
\sum_{j=k}^{t-1} ||\hat{A}(j+1) - \hat{A}(j)|| \leq 8^{1/2}\gamma_1||S||(t-k)^{1/2},
\]

\[
\sum_{j=k}^{t-1} ||\Delta(j)|| \leq 8^{1/2}||S||(t-k)^{1/2}, \ t > k \geq t_0.
\]

Now we apply Proposition 2: we set

\[
\alpha_0 = \beta_0 = \alpha_2 = \beta_2 = 0,
\]

\[
\alpha_1 = 8^{1/2}\gamma_1||S||, \ \beta_1 = 8^{1/2}||S||, \ \mu = \lambda.
\]
Following Remark 3 it is now trivial to choose $N \in \mathbb{N}$ so that $-\frac{\ln(\gamma_1)}{\gamma_1 N} - \lambda_1 > 0$, namely

$$N = \text{int}[\frac{\ln(\gamma_1)}{\ln(\lambda) - \ln(\lambda_1)}] + 1,$$

(23)

which means that

$$0 = \alpha_2 + \frac{\beta_2}{N} < \frac{1}{N \gamma_1}(-\frac{\ln(\gamma_1)}{\gamma_1 N} - \lambda_1).$$

From Proposition 2 we see that there exists a constant $\gamma_2$ so that the state transition matrix $\Phi(t, \tau)$ corresponding to $\bar{A}(t) + \Delta(t)$ satisfies

$$\|\Phi(t, \tau)\| \leq \gamma_2 \lambda^{1-\tau}, \quad t \geq \tau \geq t_0.$$

If we now apply this to (22), we end up with the desired bound:

$$\|\phi(k)\| \leq \gamma_2 \lambda^{k-t_0} \|\phi(t_0)\| + \sum_{j=t_0}^{k-1} \gamma_2 \lambda^{k-1-j} |r(j)|, \quad k \geq t_0.$$

**Case 2:** $d(t) \neq 0$ for some $t \geq t_0$.

This case is much more involved since noise can radically affect parameter estimation. Indeed, even if the parameter estimate is quite accurate at a point in time, the introduction of a large noise signal (large relative to the size of $\phi(t)$) can create a highly inaccurate parameter estimate. To proceed we partition the timeline into two parts: one in which the noise is small versus $\phi$ and one where it is not; the actual choice of the line of division will become clear as the proof progresses. To this end, with $\varepsilon > 0$ to be chosen shortly, partition $\{j \in \mathbb{Z} : j \geq t_0\}$ into two sets:

$$\begin{align*}
S_{\text{good}} &:= \{j \geq t_0 : \phi(j) \neq 0 \text{ and } \frac{|d(j)|^2}{\|\phi(j)\|^2} < \varepsilon\}, \\
S_{\text{bad}} &:= \{j \geq t_0 : \phi(j) = 0 \text{ or } \frac{|d(j)|^2}{\|\phi(j)\|^2} \geq \varepsilon\};
\end{align*}$$

clearly $\{j \in \mathbb{Z} : j \geq t_0\} = S_{\text{good}} \cup S_{\text{bad}}$. Observe that this partition clearly depends on $\theta_0$, $\theta^*$, $\phi_0$, $d$ and $r/y^*$. We will apply Proposition 2 to analyse the closed-loop system behaviour on $S_{\text{good}}$; on the other hand, we will easily obtain bounds on the system behaviour on $S_{\text{bad}}$. Before doing so, we partition the time index $\{j \in \mathbb{Z} : j \geq t_0\}$ into intervals which oscillate between $S_{\text{good}}$ and $S_{\text{bad}}$. To this end, it is easy to see that we can define a (possibly infinite) sequence of intervals of the form $[k_i, k_{i+1}]$ satisfying:

(i) $k_1 = t_0$, and

(ii) $[k_i, k_{i+1}]$ either belongs to $S_{\text{good}}$ or $S_{\text{bad}}$, and

(iii) if $k_{i+1} \neq \infty$ and $[k_i, k_{i+1}]$ belongs to $S_{\text{good}}$ (respectively, $S_{\text{bad}}$), then the interval $[k_{i+1}, k_{i+2}]$ must belong to $S_{\text{bad}}$ (respectively, $S_{\text{good}}$).

Now we turn to analysing the behaviour during each interval.

**Sub-Case 2.1:** $[k_i, k_{i+1}]$ lies in $S_{\text{bad}}$.

Let $j \in [k_i, k_{i+1}]$ be arbitrary. In this case $\phi(j) = 0$ or $\frac{|d(j)|^2}{\|\phi(j)\|^2} \geq \varepsilon$ holds. In either case we have

$$\|\phi(j)\| \leq \frac{1}{\varepsilon^{1/2}} |d(j)|, \quad j \in [k_i, k_{i+1}).$$

(24)

From (13) and (17) we see that

$$\begin{align*}
\|\phi(j+1)\| &\leq \bar{a}|\phi(j)| + \frac{1}{\varepsilon}\|\phi(j)\|^2 + |d(j)| + |r(j)| \\
&\leq [1 + \bar{a} + \varepsilon] |d(j)| + \frac{1}{\varepsilon^{1/2}} |d(j)| + |r(j)|, \quad j \in [k_i, k_{i+1}).
\end{align*}$$

(25)

If we combine this with (24) we conclude that

$$\|\phi(j)\| \leq \begin{cases} 
\frac{1}{\varepsilon^{1/2}} |d(j)| & \text{if } j = k_i \\
(1 + \bar{a} + \varepsilon)|d(j-1)| + |r(j-1)| & \text{if } j = k_i + 1, \ldots, k_{i+1}.
\end{cases}$$

(26)

**Sub-Case 2.2:** $[k_i, k_{i+1})$ lies in $S_{\text{good}}$.

Let $j \in [k_i, k_{i+1})$ be arbitrary. In this case $\phi(j) \neq 0$ and

$$\frac{|d(j)|^2}{\|\phi(j)\|^2} < \varepsilon, \quad j \in [k_i, k_{i+1}),$$

which implies that

$$\rho_3(\phi(j), e(j+1)) \frac{|d(j)|^2}{\|\phi(j)\|^2} < \varepsilon, \quad j \in [k_i, k_{i+1}).$$

(27)

From Proposition 1 we have that

$$V(\bar{k}) \leq V(\bar{k}) + \sum_{j=\bar{k}}^{\bar{k}-1} \rho_3(\phi(j), e(j+1)) \frac{|d(j)|^2}{\|\phi(j)\|^2} + \frac{2 |e(j+1)|^2 + 2 |d(j)|^2}{\|\phi(j)\|^2}, \quad k_i \leq \bar{k} \leq k_{i+1};$$

using (27) and the fact that $0 \leq V(\cdot) \leq 4 \|S\|^2$, we obtain

$$\sum_{j=\bar{k}}^{\bar{k}-1} \rho_3(\phi(j), e(j+1)) \frac{e(j+1)}{\|\phi(j)\|^2} \leq \frac{2 V(\bar{k}) + \sum_{j=\bar{k}}^{\bar{k}-1} \rho_3(\phi(j), e(j+1)) \frac{|d(j)|^2}{\|\phi(j)\|^2}}{\|\phi(j)\|^2} \leq 8 \|S\|^2 + 4 \varepsilon(\bar{k} - \bar{k}), \quad k_i \leq \bar{k} \leq k_{i+1}.$$

Hence, using this in (20) and (21) yields

$$\begin{align*}
\sum_{j=\bar{k}}^{\bar{k}-1} \|\bar{A}(j+1) - \bar{A}(j)\| &\leq \gamma_1 \|S\|^2 + 4 \varepsilon(\bar{k} - \bar{k}) \|S\|^2(\bar{k} - \bar{k})^{1/2} \\
&\leq \gamma_1 2 \|S\|^2 \|S\|(\bar{k} - \bar{k})^{1/2} + 2 \gamma_1 \varepsilon(\bar{k} - \bar{k}), \quad k_i \leq \bar{k} \leq k_{i+1},
\end{align*}$$

where $\gamma_1 = \max(\rho_3, 1)$. 


as well as
\[
{\sum_{j=k}^{k-1} \|\Delta(j)\|} \leq \left(8\|S\|^2 + 4\varepsilon(\bar{k} - k)^{1/2}(\bar{k} - k)^{1/2}\right) \\
\leq 8^{1/2}\|S\|^{1/2}(\bar{k} - k)^{1/2} + 2\varepsilon^{1/2}(\bar{k} - k), \\
k_i \leq \bar{k} < \bar{k} \leq k_{i+1}.
\]
Now we will apply Proposition 2: we set
\[
\alpha_0 = \beta_0 = 0, \quad \alpha_1 = \gamma_1 8^{1/2}\|S\|, \quad \beta_1 = 8^{1/2}\|S\|, \\
\alpha_2 = 2\gamma_1 \varepsilon^{1/2}, \quad \beta_2 = 2\varepsilon^{1/2}, \quad \mu = \lambda.
\]
With \(N\) chosen as in Case 1 via (23), we have that \(\delta := \frac{1}{\gamma_1^{1/2}} - \lambda_1 > 0\); we need
\[
\alpha_2 + \frac{\beta_2}{N} < \frac{1}{N\gamma_1\delta},
\]
which will certainly be the case if we set \(\varepsilon := \frac{\delta^2}{8\gamma_1^{1/2}(\Gamma N)^{1/2}}\).
From Proposition 2 we see that there exists a constant \(\gamma_4\) so that the state transition matrix \(\Phi(t, \tau)\) corresponding to \(A(t) + \Delta(t)\) satisfies
\[
\|\Phi(t, \tau)\| \leq \gamma_4^{1-\tau}, \quad k_1 \leq \tau \leq t \leq k_{i+1}.
\]
If we now apply this to (16) and use (18) to provide a bound on \(\eta(t)\), we end up with
\[
\|\phi(k)\| \leq \gamma_4^{1-\tau(k)}\|\phi(k)\| + (2\|S\|\delta + 1)\times \sum_{j=k}^{k-1} \gamma_4^{1-j}(\|r(j)\| + |d(j)|), \quad k_i \leq k \leq k_{i+1}.
\]
This completes Sub-Case 2.2.
Now we combine Sub-Case 2.1 and Sub-Case 2.2 into a general bound on \(\phi(t)\). Define
\[
\gamma_5 := \max\{1, 1 + \frac{\gamma_3}{\varepsilon^{1/2}}, \gamma_4, 2\|S\|\delta + 2 + \frac{\gamma_3}{\varepsilon^{1/2}}\}.
\]
It remains to prove
Claim: The following bound holds:
\[
\|\phi(k)\| \leq \gamma_5^{1-k-t_0}\|\phi(t_0)\| + \sum_{j=t_0}^{k-1} \gamma_5^{1-j}(\|r(j)\| + |d(j)|), \quad k \geq t_0.
\]
Proof of the Claim:
If \([k_1, k_2) = [t_0, k_2) \subset S_{good}\), then (29) holds for \(k \in [t_0, k_2)\) by (28). If \([t_0, k_2) \subset S_{bad}\), then from (26) we obtain
\[
\|\phi(j)\| \leq \left\{ \begin{array}{ll}
\|\phi(k_1)\| = \|\phi(t_0)\| \\
\|\phi(k_1)\| = (1 + \frac{c_0}{\varepsilon^{1/2}})\|r(1)\| + |d(1)| \\
\|\phi(j)\| = (1 + \frac{c_0}{\varepsilon^{1/2}})\|r(j)\| + |d(j)| \\
\end{array} \right.
\]
which means that (29) holds for \(k \in [t_0, k_2)\) for this case as well.
We now use induction - suppose that (29) holds for \(k \in [k_1, k_i]\); we need to prove that it holds for \(k \in (k_i, k_{i+1})\) as well. If \([k_i, k_{i+1}) \subset S_{bad}\) then from (26) we have
\[
\|\phi(k)\| \leq (1 + \frac{\gamma_3}{\varepsilon^{1/2}})\|d(k-1)\| + |r(k-1)|, \quad j = k_i + 1, ..., k_{i+1},
\]
which means that (29) holds for \(k \in (k_i, k_{i+1})\). On the other hand, if \([k_i, k_{i+1}) \subset S_{good}\), then \(k_{i+1} \in S_{bad}\); from (26) we have that
\[
\|\phi(k)\| \leq (1 + \frac{\gamma_3}{\varepsilon^{1/2}})\|d(k-1)\| + |r(k-1)|.
\]
Using (28) to analyse the behaviour on \([k_i, k_{i+1})\), we have
\[
\|\phi(k)\| \leq \gamma_4^{1-k}\|\phi(k_{i+1})\| + (2\|S\|\delta + 1)\gamma_4 \times \sum_{j=k_i}^{k-1} \gamma_4^{1-j}(\|r(j)\| + |d(j)|) \\
\leq \gamma_4^{1-k}[(1 + \frac{\gamma_3}{\varepsilon^{1/2}})\|d(k_i-1)\| + |r(k_i-1)|] + \gamma_4(2\|S\|\delta + 1)\sum_{j=k_i}^{k-1} \gamma_4^{1-j}(\|r(j)\| + |d(j)|) \\
\leq [\gamma_4(1 + \frac{\gamma_3}{\varepsilon^{1/2}})] + \gamma_4(2\|S\|\delta + 1)\sum_{j=k_i}^{k-1} \gamma_4^{1-j}(\|r(j)\| + |d(j)|) \\
\leq \gamma_5 \sum_{j=t_0}^{k-1} \gamma_4^{1-j}(\|r(j)\| + |d(j)|), \quad k = k_i + 1, ..., k_{i+1},
\]
as desired. □
This completes the proof.
□

V. TOLERANCE TO TIME-VARIATIONS

The linear-like bound proven in Theorem 1 can be leveraged to prove that the same behaviour will result even in the presence of slow time-variations with occasional jumps. So suppose that the actual plant model is
\[
y(t + 1) = \phi(t)^T \theta^* + d(t), \quad \phi(t_0) = \phi_0,
\]
with \(\theta^*(t) \in S\) for all \(t \in R\). We adopt a common model of acceptable time-variations used in adaptive control: with \(c_0 \geq 0\) and \(\varepsilon > 0\), we let \(s(S, c_0, \varepsilon)\) denote the subset of \(L_{\infty}(R^{2n})\) whose elements \(\theta^*\) satisfy \(\theta^*(t) \in S\) for every \(t \in Z\) as well as
\[
\sum_{t=t_1}^{t_2-1} \|\theta^*(t + 1) - \theta^*(t)\| \leq c_0 + \varepsilon(t_2 - t_1), \quad t_2 > t_1
\]
for every \(t_1 \in Z\). We will now show that, for every \(c_0 \geq 0\), the approach tolerates time-varying parameters in \(s(S, c_0, \varepsilon)\) if \(\varepsilon\) is small enough.
Theorem 2: For every $\delta \in (0, \infty]$, $\lambda_1 \in (\Lambda_1, 1)$ and $c_0 \geq 0$, there exists a $c_1 > 0$ and $\varepsilon > 0$ so that for every $t_0 \in \mathbb{Z}$, $\theta_0 \in \mathcal{S}$, $\theta^* \in s(\mathcal{S}, c_0, \varepsilon)$, $\phi_0 \in \mathbb{R}^{2n}$, and $y^*, \hat{d} \in \ell_\infty$, when the adaptive controller (7), (8) and (11) is applied to the time-varying plant (30), the following holds:

$$
\|\phi(k)\| \leq c_1 \lambda_1^{k-t_0} \|\phi_0\| + \sum_{j=0}^{k-1} c_1 \lambda_1^{k-1-j} (|r(j)| + |d(j)|),
$$

$k \geq t_0$.

**Proof:**

Fix $\delta \in (0, \infty]$, $\lambda_1 \in (\Lambda_1, 1)$, $\lambda \in (\Lambda_1, 1)$ and $c_0 \geq 0$. Let $t_0 \in \mathbb{Z}$, $\theta_0 \in \mathcal{S}$, $\phi_0 \in \mathbb{R}^{2n}$, and $y^*, \hat{d} \in \ell_\infty$ be arbitrary.

With $m \in \mathbb{N}$, we will consider $\phi(t)$ on intervals of the form $[t_0 + im, t_0 + (i+1)m]$; we will be analysing these intervals in groups of $m$ (to be chosen shortly); we set $\varepsilon = \frac{\varepsilon_m}{m^c}$, and let $\theta^* \in s(\mathcal{S}, c_0, \varepsilon)$ be arbitrary.

First of all, for $i \in \mathbb{Z}^+$ we can rewrite the plant equation as

$$
y(t+1) = \phi(t)^T \theta^*(t_0 + im) + d(t) + \phi(t)^T [\theta^*(t) - \theta^*(t_0 + im)],
$$

$$
t \in [t_0 + im, t_0 + (i+1)m]. \quad (32)
$$

Theorem 1 applied to (32) says that there exists a constant $c > 0$ so that

$$
\|\phi(t)\| \leq c \lambda^{t-t_0-im} \|\phi(t_0 + im)\| + \sum_{j=t_0+im}^{t-1} c \lambda^{t-j-1} (|r(j)| + |d(j)|) + |\hat{n}(j)|),
$$

$$
t \in [t_0 + im, t_0 + (i+1)m].
$$

The above is a difference inequality associated with a first order system; using this observation together with the fact that $c \geq 1$, we see that if we define

$$
\psi(t+1) = \lambda \psi(t) + |r(t)| + |d(t)| + |\hat{n}(t)|,
$$

$$
t \in [t_0 + im, t_0 + (i+1)m - 1],
$$

with $\psi(t_0 + im) = \|\phi(t_0 + im)\|$, then

$$
\|\phi(t)\| \leq c \psi(t), \quad t \in [t_0 + im, t_0 + (i+1)m].
$$

Now we analyse this equation for $i = 0, 1, \ldots, m-1$.

**Case 1:** $|\hat{n}(t)| \leq \frac{1}{2c} (\lambda_1 - \lambda) \|\phi(t)\|$ for all $t \in [t_0 + im, t_0 + (i+1)m]$.

In this case

$$
\psi(t+1) \leq \lambda \psi(t) + |r(t)| + |d(t)| + |\hat{n}(t)|
$$

$$
\leq \lambda \psi(t) + |r(t)| + |d(t)| + \frac{1}{2c} (\lambda_1 - \lambda) \psi(t)
$$

$$
\leq \left(\frac{\lambda + \lambda_1}{2}\right) \psi(t) + |r(t)| + |d(t)|,
$$

$$
t \in [t_0 + im, t_0 + (i+1)m],
$$

which means that

$$
\psi(t) \leq \left(\frac{\lambda + \lambda_1}{2}\right)^{t-t_0-im} \psi(t_0 + im) + \sum_{j=t_0+im}^{t-1} \left(\frac{\lambda + \lambda_1}{2}\right)^{t-j-1} (|r(j)| + |d(j)|),
$$

$$
t = t_0 + im, \ldots, t_0 + (i+1)m.
$$

This, in turn, implies that

$$
\|\phi(t_0 + (i+1)m)\| \leq c \left(\frac{\lambda + \lambda_1}{2}\right)^m \|\phi(t_0 + im)\| + \sum_{j=t_0+im}^{t_0+(i+1)m-1} c \left(\frac{\lambda + \lambda_1}{2}\right)^{t_0+(i+1)m-1-j} (|r(j)| + |d(j)|).
$$

(33)

**Case 2:** $|\hat{n}(t)| > \frac{1}{2c} (\lambda_1 - \lambda) \|\phi(t)\|$ for some $t \in [t_0 + im, t_0 + (i+1)m]$.

Since $\theta^*(t) \in \mathcal{S}$ for $t \geq t_0$, we see

$$
|\hat{n}(t)| \leq 2\|S\| \times \|\phi(t)\|, \quad t \in [t_0 + im, t_0 + (i+1)m].
$$

This means that

$$
\psi(t+1) \leq \lambda \psi(t) + |r(t)| + |d(t)| + |\hat{n}(t)|
$$

$$
\leq \lambda \psi(t) + |r(t)| + |d(t)| + 2\|S\| \psi(t)
$$

$$
\leq (1 + 2\|S\|) \psi(t) + |r(t)| + |d(t)|,
$$

$$
t \in [t_0 + im, t_0 + (i+1)m],
$$

which means that

$$
\|\psi(t)\| \leq \gamma_1^{t-t_0-im} \|\psi(t_0 + im)\| + \sum_{j=t_0+im}^{t-1} \gamma_1^{t-j-1} (|r(j)| + |d(j)|),
$$

$$
t = t_0 + im, \ldots, t_0 + (i+1)m.
$$

This, in turn, implies that

$$
\|\phi(t_0 + (i+1)m)\| \leq c \gamma_1^m \|\phi(t_0 + im)\| + \sum_{j=t_0+im}^{t_0+(i+1)m-1} (\gamma_1)^{t_0+(i+1)m-j-1} (|r(j)| + |d(j)|)
$$

$$
\leq c \gamma_1^m \|\phi(t_0 + im)\| + c \left(\frac{2 \gamma_1}{\lambda + \lambda_1}\right)^m \times \sum_{j=t_0+im}^{t_0+(i+1)m-1} \left(\frac{\lambda + \lambda_1}{2}\right)^{t_0+(i+1)m-j-1} (|r(j)| + |d(j)|). \quad (34)
$$

On the interval $[t_0, t_0 + m^2]$ there are $m$ sub-intervals of length $m$; furthermore, because of the choice of $\varepsilon$ we have that

$$
\sum_{j=t_0}^{t_0+m^2-1} \|\theta^*(j+1) - \theta^*(j)\| \leq c_0 + \varepsilon m^2 \leq 2c_0.
$$

A simple calculation reveals that there are at most $N_1 := \frac{4m \sqrt{c_0}}{\lambda + \lambda_1}$ sub-intervals which fall into the category of Case 2, with the remaining number falling into the category of Case 1. Henceforth we assume that $m > N_1$. If we use (33) and (34) to analyse the behaviour of the closed-loop system
on the interval $[t_0, t_0 + m^2]$, we end up with a crude bound of
\[
\|\phi(t_0 + m^2)\| \leq e^{m \gamma_1 \gamma_1 m \left( \frac{\lambda_1 + \lambda}{2} \right)(m - N_i)} \|\phi(t_0)\| +
\sum_{j=t_0}^{t_0 + m^2 - 1} \left( \frac{2 \gamma_1}{\lambda + \lambda} \right)^m \left( \frac{2}{\lambda + \lambda} \right)^{m^2} \gamma_2.
\]
(35)

At this point we would like to choose $m$ so that
\[
 e^{m \gamma_1 \gamma_1 m \left( \frac{\lambda_1 + \lambda}{2} \right)(m - N_i)} \leq \lambda^{m^2 - N_i}
\]
\[
\Leftrightarrow e^{m \gamma_1 \gamma_1 m \left( \frac{2}{\lambda + \lambda} \right)^{m N_i}} \leq \left( \frac{2 \lambda_1}{\lambda_1 + \lambda} \right)^{m^2};
\]
notice that $\frac{2 \lambda_1}{\lambda_1 + \lambda} > 1$, so if we take the log of both sides, we see that we need
\[
m \ln(c) + N_1 m \ln(\gamma_1) + N_1 m \ln\left( \frac{2}{\lambda + \lambda} \right),
\]
which will clearly be the case for large enough $m$, so at this point we choose such an $m$. It follows from (35) that there exists a constant $\gamma_2$ so that
\[
\|\phi(t_0 + m^2)\| \leq \lambda^{m^2} \|\phi(t_0)\| +
\sum_{j=t_0}^{t_0 + m^2 - 1} \lambda_1^{t_0 + m^2 - j - 1} (|r(j)| + |d(j)|).
\]

Indeed, by time-invariance of the closed-loop system we see that
\[
\|\phi(t + m^2)\| \leq \lambda^{m^2} \|\phi(t)\| +
\sum_{j=t}^{t + m^2 - 1} \lambda_1^{t + m^2 - j - 1} (|r(j)| + |d(j)|),
\]
\[
\gamma_2 \sum_{j=t}^{t + m^2 - 1} \lambda_1^{t + m^2 - j - 1} (|r(j)| + |d(j)|), \quad \bar{t} \geq t_0.
\]

Solving iteratively yields
\[
\|\phi(t_0 + im^2)\| \leq \lambda^{im^2} \|\phi(t_0)\| +
\sum_{j=t_0}^{t_0 + im^2 - 1} \lambda_1^{t_0 + im^2 - j - 1} (|r(j)| + |d(j)|), \quad i \geq 0.
\]
(36)

We now combine this bound with the bounds which hold on the good intervals (33) and the bad intervals (34), and conclude that there exists a constant $\gamma_3$ so that
\[
\|\phi(t)\| \leq \gamma_3 \lambda_1^{-t_0} \|\phi(t_0)\| +
\sum_{j=t_0}^{t-1} \lambda_1^{-j-1} (|r(j)| + |d(j)|), \quad t \geq t_0,
\]
as desired. \(\square\)

VI. TOLERANCE TO UNMODELLLED DYNAMICS

Due to the linear-like bounds proven in Theorems 1 and 2, we can use the Small Gain Theorem to good effect to prove the tolerance of the closed-loop system to unmodelled dynamics. However, since the controller, and therefore the closed-loop system, is nonlinear, handling initial conditions is more subtle: in the linear-time invariant case we can separate out the effect of initial conditions from that of the forcing functions ($r$ and $d$), but in our situation they are intertwined. We proceed by looking at two cases - with and without initial conditions. In all of the cases we consider the time-varying plant (30) with $d_{\Delta}(t)$ added to represent the effect of unmodelled dynamics:
\[
y(t + 1) = \phi(t)^T \theta^*(t) + d(t) + d_{\Delta}(t), \quad \phi(t_0) = \phi_0.
\]
(37)

To proceed, fix $\delta \in (0, \infty]$, $\lambda_1 \in (\Delta, 1)$ and $c_0 \geq 0$; from Theorem 2 there exists a $c_1 > 0$ and $\varepsilon > 0$ so that for every $t_0 \in \mathbb{Z}$, $0 \leq t_0 \leq \tau$, and $\theta^* \in S(\mathcal{C}, c_0, \varepsilon)$, when the adaptive controller (7), (8) and (11) is applied to the time-varying plant (37), the following bound holds:
\[
\|\phi(k)\| \leq c_1 \lambda_1^{-k-t_0} \|\phi_0\| +
\sum_{j=t_0}^{k-1} c_1 \lambda_1^{-k-j-1} (|r(j)| + |d(j)| + |d_{\Delta}(j)|),
\]
\[
k \geq t_0.
\]
(38)

A. Zero Initial Conditions

In this case we assume that $\phi(t) = 0$ for $t \leq t_0$; we derive a bound on the closed-loop system behaviour in the presence of unmodelled dynamics. Suppose that the unmodelled dynamics is of the form $d_{\Delta}(t) = (\Delta \phi)(t)$ with $\Delta : l_\infty(\mathbb{R}^{2n}) \rightarrow l_\infty(\mathbb{R}^{2n})$ a (possibly nonlinear time-varying) causal map with a finite gain of $\|\Delta\|$. It is easy to prove that if $\|\Delta\| < \frac{1}{c_1}$, then
\[
\|\phi(k)\| \leq \frac{c_1}{1 - \lambda_1 - c_1 \|\Delta\|} (\sup_{t \geq t_0} \|r(t)\| + \sup_{t \geq t_0} \|d(t)\|),
\]
\[
k \geq t_0,
\]
i.e., a form of closed-loop stability is attained. Following the approach of Remark 5, we could also analyse the closed-loop system using $l_p$-norms with $1 \leq p < \infty$.

B. Non-Zero Initial Conditions

Now we allow unmodelled LTI dynamics with non-zero initial conditions, and we develop convolution-like bounds on the closed-loop system. To this end suppose that the unmodelled dynamics are of the form
\[
d_{\Delta}(t) := \sum_{j=0}^{\infty} \Delta_j \phi(t - j),
\]
(39)
with $\Delta_j \in \mathbb{R}^{1 \times 2n}$; the corresponding transfer function is $\Delta(z^{-1}) := \sum_{j=0}^{\infty} \Delta_j z^{-j}$. It is easy to see that this model subsumes the classical additive uncertainty, multiplicative uncertainty, and uncertainty in a coprime factorization, which is common in the robust control literature, e.g. see
[37], with the only constraint being that the perturbations correspond to strictly causal terms. In order to obtain linear-like bounds on the closed-loop behaviour, we need to impose more constraints on \( \Delta(z) \) than in the previous subsection: after all, if \( \Delta(z^{-1}) = \Delta_p z^{-p} \), it is clear that \( \| \Delta \| = \| \Delta_p \| \) for all \( p \), but the effect on the closed-loop system varies greatly - a large value of \( p \) allows the behaviour in the far past to affect the present. To this end, with \( \mu > 0 \) and \( \beta \in (0, 1) \), we shall restrict \( \Delta(z^{-1}) \) to a set of the form

\[
B(\mu, \beta) := \{ \sum_{j=0}^{\infty} \Delta_j z^{-j} : \Delta_j \in \mathbb{R}^{1 \times 2n} \text{ and } \| \Delta_j \| \leq \mu \beta^j, \; j \geq 0 \}.
\]

It is easy to see that every transfer function in \( B(\mu, \beta) \) is analytic in \( \{ z \in \mathbb{C} : |z| > \beta \} \), so it has no poles in that region.

Now we fix \( \mu > 0 \) and \( \beta \in (0, 1) \) and let \( \Delta(z^{-1}) \) belong to \( B(\mu, \beta) \); the goal is to analyse the closed-loop behaviour of \( \text{(37)} \) for \( t \geq t_0 \) when \( d_\Delta \) is given by \( \text{(39)} \). We first partition \( d_\Delta(t) \) into two parts - that which depends on \( \phi(t) \) for \( t \geq t_0 \) and that which depends on \( \phi(t) \) for \( t < t_0 \):

\[
d_\Delta(t) = \sum_{j=0}^{t_0-1} \Delta_j \phi(t-j) + \sum_{j=t_0}^{\infty} \Delta_j \phi(t-j).
\]

It is clear that

\[
\| d_\Delta^+(t) \| \leq \sum_{j=t_0}^{t_0-1} \mu \beta^{t-j} \| \phi(j) \|,
\]

\[
\| d_\Delta^-(t) \| \leq \sum_{j=-\infty}^{t_0-1} \mu \beta^{t-j} \| \phi(j) \|
\]

\[
= \mu \beta^{t-t_0} \sum_{j=1}^{\infty} \beta^j \| \phi(t_0 - j) \|, \; t \geq t_0.
\]

If \( \phi(t) \) is bounded on \( \{ t \in \mathbb{Z} : t < t_0 \} \) then \( \sum_{j=1}^{\infty} \beta^j \| \phi(t_0 - j) \| \) is finite, in which case we see that \( d_\Delta^-(t) \) goes to zero exponentially fast; henceforth, we make the reasonable assumption that this is the case. It turns out that we can easily bound \( d_\Delta(t) \) with a difference equation. To this end, consider

\[
m(t+1) = \beta m(t) + \beta \| \phi(t) \|, \; t \geq t_0,
\]

with \( m(t_0) = m_0 := \sum_{j=1}^{\infty} \beta^j \| \phi(t_0 - j) \| \); it is straightforward to prove that

\[
|d_\Delta(t)| \leq |d_\Delta^+(t)| + |d_\Delta^-(t)| \leq \mu m(t) + \mu \| \phi(t) \|, \; t \geq t_0.
\]

This model of unmodelled dynamics is similar to that used in the adaptive control literature, e.g. see [11].

**Theorem 3:** For every \( \beta \in (0, 1) \) and \( \lambda_2 \in (\max\{1, \beta\}, 1) \), there exist \( \tilde{\mu} > 0 \) and \( c_0 > 0 \) so that for every \( t_0 \in \mathbb{Z} \), \( \phi_0 \in \mathbb{R}^{2n} \), \( m_0 \in \mathbb{R} \), \( \theta_0 \in \mathcal{S} \), \( y^*, d \in \ell_\infty \), \( \theta^* \in \mathcal{S}(\mathbb{C}, \alpha, \varepsilon) \) and \( \mu \in (0, \tilde{\mu}) \), when the adaptive controller \( \text{(37)}, \text{(39)} \) and \( \text{(11)} \) is applied to the time-varying plant \( \text{(37)} \) with \( d_\Delta \) satisfying \( \text{(40)} \) and \( \text{(41)} \), the following bound holds:

\[
\| \phi(k) \| \leq c_2 \lambda_2^{k-t_0} (\| \phi_0 \| + |m_0|) + \sum_{j=t_0}^{k-1} c_2 \lambda_2^{k-j-1} (|d(j)| + |r(j)|), \; k \geq t_0.
\]

**Proof:**

Fix \( \beta \in (0, 1) \) and \( \lambda_2 \in (\max\{1, \beta\}, 1) \). The first step is to convert difference inequalities to difference equations. To this end, consider the difference equation

\[
\tilde{\phi}(t+1) = \lambda_1 \tilde{\phi}(t) + c_1 |r(t)| + c_1 |d(t)| + c_1 \mu \tilde{m}(t) + c_1 \mu \tilde{\phi}(t), \; \tilde{\phi}(t_0) = c_1 \| \phi(t_0) \|,
\]

\[
\text{(42)}
\]

together with the difference equation based on \( \text{(40)} \):

\[
\tilde{m}(t+1) = \beta \tilde{m}(t) + \beta \tilde{\phi}(t), \; \tilde{m}(t_0) = |m_0|,
\]

\[
\text{(43)}
\]

It is easy to use induction together with \( \text{(38), (40)}, \text{and (41)} \) to prove that

\[
\| \phi(t) \| \leq \tilde{\phi}(t), \; |m(t)| \leq \tilde{m}(t), \; t \geq t_0.
\]

\[
\text{(44)}
\]

If we combine the difference equations \( \text{(42)} \) with \( \text{(43)} \), we end up with

\[
\begin{bmatrix}
\tilde{\phi}(t+1) \\
\tilde{m}(t+1)
\end{bmatrix} =
\begin{bmatrix}
\lambda_1 + c_1 \mu & c_1 \mu \\
\beta & \beta
\end{bmatrix}
\begin{bmatrix}
\tilde{\phi}(t) \\
\tilde{m}(t)
\end{bmatrix} +
\begin{bmatrix}
c_1 \\
0
\end{bmatrix}
(|d(t)| + |r(t)|), \; t \geq t_0.
\]

\[
\text{(45)}
\]

Now we see that \( A_{cl}(\mu) \to \begin{bmatrix} \lambda_1 & 0 \\ \beta & \beta \end{bmatrix} \) as \( \mu \to 0 \), and this matrix has eigenvalues of \( \{ \lambda_1, \beta \} \). Now choose \( \tilde{\mu} > 0 \) so that all eigenvalues are less than \( \left( \frac{2 \beta}{1 - \beta} + \frac{\beta}{2} \max\{\lambda_1, \beta\} \right) \) in magnitude for \( \mu \in (0, \tilde{\mu}) \), and define \( \varepsilon := \frac{\beta}{2} - \frac{\lambda_1}{\lambda_2} \). Using the proof technique of Desoer in [1], we can conclude that for \( \mu \in (0, \tilde{\mu}) \), we have

\[
\| A_{cl}(\mu) \|^k \leq \left( \frac{3 + 2 \beta + 2 c_1 \mu}{\varepsilon^2} \right) \lambda_k^{\gamma_1}, \; k \geq 0;
\]

if we use this in \( \text{(45)} \) and then apply the bounds in \( \text{(44)} \), it follows that

\[
\| \phi(k) \| \leq c_1 \gamma_1 \lambda_2^{k-t_0} (\| \phi_0 \| + |m_0|) + \sum_{j=t_0}^{k-1} c_1 \gamma_1 \lambda_2^{k-j-1} (|d(j)| + |r(j)|), \; k \geq t_0,
\]

as desired. \( \square \)
VII. Step Tracking

If the plant is non-minimum phase, it is not possible to track an arbitrary bounded reference signal using a bounded control signal. However, as long as the plant does not have a zero at \( z = 1 \), it is possible to modify the controller design procedure to achieve asymptotic step tracking if there is no noise/disturbance. So at this point assume that the corresponding plant polynomial \( B(z^{-1}) \) has no zero at \( z = 1 \) for any plant model \( \theta \in \mathcal{S} \). To proceed, we use the standard trick from the literature, e.g. see [5]: we still estimate \( A(z^{-1}) \) and \( B(z^{-1}) \) as before, but we now design the control law slightly differently. To this end, we first define

\[
\hat{A}(t, z^{-1}) := (1 - z^{-1}) \tilde{A}(t, z^{-1}),
\]

and then let \( A^*(z^{-1}) \) be a \( 2(n + 1)^{th} \) monic polynomial (rather than a \( 2n^{th} \) one) of the form

\[
A^*(z^{-1}) = 1 + a_1^* z^{-1} + \cdots + a_{2n+2}^* z^{-2n-2}
\]

so that \( z^{2(n+1)} A^*(z^{-1}) \) has all of its zeros in \( D^0 \). Next, we choose two polynomial

\[
\hat{L}(t, z^{-1}) = 1 + \hat{l}_1(t) z^{-1} + \cdots + \hat{l}_{n+1}(t) z^{-n-1}
\]

and

\[
\hat{P}(t, z^{-1}) = \hat{p}_1(t) z^{-1} + \cdots + \hat{p}_{n+1}(t) z^{-n-1}
\]

which satisfy the equation

\[
\hat{A}(t, z^{-1}) \hat{L}(t, z^{-1}) + \hat{B}(t, z^{-1}) \hat{P}(t, z^{-1}) = A^*(z^{-1});
\]

since \( \hat{A}(t, z^{-1}) \) and \( \hat{B}(t, z^{-1}) \) are coprime, there exist unique \( \hat{L}(t, z^{-1}) \) and \( \hat{P}(t, z^{-1}) \) which satisfy this equation. We now define

\[
\hat{L}(t, z^{-1}) = 1 + \hat{l}_1(t) z^{-1} + \cdots + \hat{l}_{n+1}(t) z^{-n-2}
\]

\[
:= (1 - z^{-1}) \hat{L}(t, z^{-1});
\]

at time \( t \) we choose \( u(t) \) so that

\[
u(t) = -\hat{l}_1(t - 1) u(t - 1) - \cdots
\]

\[
\hat{l}_{n+2}(t - 1) u(t - n - 2) - \hat{p}_1(t - 1) [y(t - 1) - y^*(t - 1)] - \cdots
\]

\[
- \hat{p}_{n+1}(t - 1) [y(t - n - 1) - y^*(t - n - 1)].
\]

We can use a modified version of the argument used in the proof of Theorem 1 to conclude that a similar type of result holds here; we can also prove that asymptotic step tracking will be attained if the noise is zero and the reference signal \( y^* \) is constant. The details are omitted due to space considerations.

VIII. A Simulation Example

Here we provide an example to illustrate the benefit of the proposed adaptive controller. To this end, consider the second order plant

\[
y(t + 1) = -a_1(t) y(t) - a_2(t) y(t - 1) + b_1(t) u(t) + b_2(t) u(t - 1) + d(t)
\]

with \( a_1(t) \in [0, 2], a_2(t) \in [1, 3], b_1(t) \in [0, 1], \) and \( b_2(t) \in [-5, -2] \). So every admissible model is unstable and non-minimum phase, which makes this a challenging plant to control. We set \( \delta = \infty \).

A. Stability

In this sub-section we consider the problem of stability only - we set \( y^* = 0 \). First we compare the ideal algorithm (4)-(5) (with projection onto \( S \)) with the classical one (3) (suitably modified to have projection onto \( S \)); in both cases we couple the estimator with the adaptive pole placement controller (11) where we place all closed-loop poles at zero. In the case of the classical estimator (3) we arbitrarily set \( \alpha = \beta = 1 \). Suppose that the actual value of \( (a_1, a_2, b_1, b_2) \) is \((2, 3, 1, -2)\) and the initial estimate is set to the midpoint of the interval. In the first simulation we set \( y(0) = y(-1) = 0.01 \) and \( u(-1) = 0 \) and set the noise \( d(t) \) to zero - see the top plot of Figure 1. In the second simulation we set \( y(0) = y(-1) = u(-1) = 0 \) and the noise \( d(t) = 0.01 * \sin(5t) \) - see the bottom plot of Figure 1. In both cases the controller based on the ideal algorithm (4)-(5) is clearly superior to the one based on the revised classical algorithm (3).

Fig. 1. A comparison of the ideal algorithm (solid) and the classical algorithm (dashed) with a non-zero initial condition and no noise (top plot) and a zero initial condition and noise (bottom plot).

Now we further examine the case of the proposed controller when it is applied to the time-varying plant with unmodelled dynamics, a zero initial condition, and a non-zero noise. More specifically, we set

\[
a_1(t) = 1 + \sin(0.001t), \quad a_2(t) = 2 + \cos(0.001t),
\]
$b_1(t) = 0.5 + 0.5 \sin(0.005t)$,  
$b_2(t) = -3.5 - 1.5 \sin(0.005t)$,  
$d(t) = 0.01 \sin(5t)$.

For the unmodelled part of the plant we use a term of the form discussed in Section VI.B:

$$m(t + 1) = 0.75m(t) + 0.75||\phi(t)||, \quad m(0) = 0,$$

$$d_\Delta(t) = \begin{cases} 
0 & t = 0, 1, \ldots, 4999 \\
0.025m(t) + 0.025||\phi(t)|| & t \geq 5000.
\end{cases}$$

We plot the result in Figure 2; we see that the parameter estimator approximately follows the system parameters, and the effect of the noise is small on average, even in the presence of unmodelled dynamics.

![Fig. 2. The system behaviour with time-varying parameters and unmodelled dynamics; the parameters are dashed and the estimates are solid.](image)

**B. Step Tracking**

The plant in the previous sub-section has a large amount of uncertainty, as well as a wide range of unstable poles and non-minimum phase zeros, which means that there are limits on the quality of the transient behaviour even if the parameters were fixed and known. Hence, to illustrate the tracking ability we look at a sub-class of systems: one with parameters were fixed and known. Hence, to illustrate the tracking we look at a sub-class of systems: one with $a_1$ and $b_1$ as before, namely $a_1(t) \in [0, 2]$ and $b_1(t) \in [0, 1]$, but now with $a_2 = 1$ and $b_2 = -3.5$. With fixed parameters the corresponding system is still unstable and non-minimum phase.

We simulate the closed-loop pole placement step tracking controller of Section VII with a zero initial condition, initial parameter estimates at the midpoints of the admissible intervals, and with time-varying parameters:

$$a_1(t) = 1 + \sin(0.002t), \quad b_1(t) = 0.5 + 0.5 \cos(0.005t),$$

with a non-zero disturbance:

$$d(t) = \begin{cases} 
0.01 \sin(5t) & t = 0, 1, \ldots, 2499 \\
0.05 \sin(5t) & t = 2500, \ldots, 4999,
\end{cases}$$

and a square wave reference signal of $y^*(t) = \text{sgn}[\sin(0.01t)]$. We plot the result in Figure 3; we see that the parameter estimates crudely follows the system parameters, with less accuracy than in the previous sub-section, partly due to the fact that the constant setpoint dominates the estimation process and leads to higher inaccuracy. As a result, $y(t)$ does a good job of following $y^*$ on average, but with the occasional flurry of activity when the parameter estimates are highly inaccurate. When the noise is increased five-fold at $k = 2500$, the behaviour degrades only slightly.

![Fig. 3. The pole placement tracking controller with time-varying parameters and small noise; the parameters are dashed and the estimates are solid.](image)

**IX. Summary and Conclusions**

Here we show that if the original, ideal, projection algorithm is used in the estimation process (subject to the assumption that the plant parameters lie in a convex, compact set), then the corresponding pole placement adaptive controller guarantees linear-like convolution bounds on the closed loop behaviour, which confers exponential stability and a bounded noise gain (in every $p$-norm with $1 \leq p \leq \infty$), unlike almost all other parameter adaptive controllers. This can be leveraged to prove tolerance to unmodelled dynamics and plant parameter variation. We emphasize that there is no persistent excitation requirement of any sort; the improved performance arises from the vigilant nature of the ideal parameter estimation algorithm.

As far as the author is aware, the linear-like convolution bound proven here is a first in parameter adaptive control. It allows a modular approach to be used in analysing time-varying parameters and unmodelled dynamics. This approach avoids all of the fixes invented in the 1980s, such as signal normalization and deadzones, used to deal with the lack of robustness to unmodelled dynamics and time-varying parameters.

We are presently working on extending the approach to the model reference adaptive control setup. It will be interesting to see if the convexity assumption can be removed by using multi-estimators, i.e. cover the the set of admissible
parameters by a finite number of convex sets, and then use an estimator for each such set. Extending the approach to
the continuous-time setting may prove challenging, since a direct application would yield a non-Lipschitz continuous
estimator, which brings with it mathematical solveability issues.

X. APPENDIX

Proof of Proposition 1:
Since projection does not make the parameter estimate worse, it follows from (7) that
\[
\|\tilde{\theta}(t + 1) - \hat{\theta}(t)\| \leq \|\tilde{\theta}(t + 1) - \hat{\theta}(t)\|
\]
so the first inequality holds.

We now turn to energy analysis. We first define \(\tilde{\theta}(t) := \tilde{\theta}(t) - \theta^*\) and \(V(t) := \tilde{\theta}(t)^T\tilde{\theta}(t)\). Next, we subtract \(\theta^*\) from each side of (7), yielding
\[
\tilde{\theta}(t + 1) = \tilde{\theta}(t) + \rho_s(\phi(t), e(t + 1)) \frac{\phi(t)}{\phi(t)^T\phi(t)} + \rho_s(\phi(t), e(t + 1)) \frac{\phi(t)^T\phi(t)}{\phi(t)^T\phi(t)} \tilde{\theta}(t) + [\theta^* - \rho_s(\phi(t), e(t + 1)) \frac{\phi(t)}{\phi(t)^T\phi(t)}] \theta^* + W_z(t)
\]
and
\[
\rho_s(\phi(t), e(t + 1)) \frac{\phi(t)}{\phi(t)^T\phi(t)} d(t).
\]

Then
\[
\tilde{\theta}(t + 1) = \theta^* - \rho_s(\phi(t), e(t + 1)) \frac{\phi(t)}{\phi(t)^T\phi(t)} \theta^* + W_z(t)
\]
and
\[
\rho_s(\phi(t), e(t + 1)) \frac{\phi(t)}{\phi(t)^T\phi(t)} d(t).
\]

Now let us analyse the three terms on the RHS: the fact that \(W_1(t)^2 = W_1(t)\) allows us to simplify the first term; the fact that \(W_1(t)^2 = W_2(t)^2\) means that the second term is zero; \(W_2(t)^2 W_2(t) = \rho_s(\phi(t), e(t + 1)) \frac{1}{\phi(t)^T\phi(t)}\), which simplifies the third term. We end up with
\[
\tilde{\theta}(t + 1) = \tilde{\theta}(t) + \rho_s(\phi(t), e(t + 1)) \frac{d(t)}{\phi(t)^T\phi(t)}
\]
\[
= V(t) - \rho_s(\phi(t), e(t + 1)) \frac{\tilde{\theta}(t)^T\phi(t)}{\phi(t)^T\phi(t)} + \rho_s(\phi(t), e(t + 1)) \frac{d(t)}{\phi(t)^T\phi(t)}
\]
\[
\leq V(t) + \rho_s(\phi(t), e(t + 1)) \frac{d(t)^2}{\phi(t)^T\phi(t)}.
\]

Since projection never makes the estimate worse, it follow that
\[
V(t + 1) \leq V(t) + \rho_s(\phi(t), e(t + 1)) \frac{-\frac{1}{2}e(t + 1)^2 + 2d(t)^2}{\phi(t)^T\phi(t)}.
\]

□

Proof of Lemma 1: Fix \(\delta \in (0, \infty)\) and \(\sigma \in (\Delta, 1]\). First of all, it is well known that the characteristic polynomial of \(\tilde{\theta}(t)\) is exactly \(z^{2n}A^*(z^{-1})\) for every \(t \in Z\). Furthermore, it is well known that the coefficients of \(L(t, z^{-1})\) and \(P(t, z^{-1})\) are the solution of a linear equation, and are analytic functions of \(\theta(t) \in S\). Hence, there exists a constant \(\gamma_1\) so that, for every set of initial conditions, \(y^* \in l_\infty\) and \(d \in l_\infty\), we have \(\|\tilde{\theta}(t)\| \leq \gamma_1\).

To prove the first bound we now invoke the argument used in [1], who considered a more general time-varying situation but with more restrictions on \(\sigma\). By making a slight adjustment to the first part of the proof given there, we can prove that with \(\gamma_2 := \sigma^{(\sigma+n)\frac{n}{2}}\), then for every \(t \geq t_0\) we have \(\|\tilde{\theta}(t)^k\| \leq \gamma_2\sigma^k, k \geq 0\), as desired.

Now we turn to the second bound. From Proposition 1 and the Cauchy-Schwarz inequality we obtain
\[
\sum_{j=k}^{t-1} \|\tilde{\theta}(j + 1) - \hat{\theta}(j)\| \leq \sum_{j=k}^{t-1} \rho_s(\phi(j), e(j + 1)) \frac{|e(j + 1)|}{\|\phi(j)\|} \|\phi(j)\|^2
\]
\[
\leq \left(\sum_{j=k}^{t-1} \rho_s(\phi(j), e(j + 1)) \frac{|e(j + 1)|}{\|\phi(j)\|} \|\phi(j)\|^2\right)^{1/2} (t - k)^{1/2}.
\]

Now notice that
\[
\|\tilde{\theta}(t + 1) - \hat{\theta}(t)\| = \|\tilde{\theta}(t + 1)\| - \|\hat{\theta}(t)\| + \sum_{i=1}^n (|\tilde{l}_i(t + 1) - \hat{l}_i(t)| + |\tilde{p}_i(t + 1) - \hat{p}_i(t)|).
\]

The fact that the coefficients of \(\tilde{L}(t, z^{-1})\) and \(\tilde{P}(t, z^{-1})\) are analytic functions of \(\theta(t) \in S\) means that there exists a constant \(\gamma_3 \geq 1\) so that
\[
\sum_{j=k}^{t-1} \|\tilde{\theta}(j + 1) - \hat{\theta}(j)\| \leq \gamma_3 \sum_{j=k}^{t-1} \|\tilde{\theta}(j + 1) - \hat{\theta}(j)\|
\]
so we conclude that the second bound holds as well. □.
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