On teaching sets of \( k \)-threshold functions

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Abstract
Let \( f \) be \( \{0, 1\} \)-valued function over integer \( d \)-dimensional cube \( \{0, 1, \ldots, n - 1\}^d \), for \( n \geq 2 \) and \( d \geq 1 \). The function \( f \) is called threshold if there exists a hyperplane which separates 0-valued points from 1-valued points. Let \( C \) be a class of functions and \( f \in C \). A point \( x \) is essential for the function \( f \) with respect to \( C \) if there exists a function \( g \in C \) such that \( x \) is unique point on which \( f \) differs from \( g \). A set of points \( X \) is called teaching for the function \( f \) with respect to \( C \) if no function in \( C \setminus \{f\} \) agrees with \( f \) on \( X \). It is known that the minimal teaching set for a threshold function is unique and coincides with the set of its essential points. In this paper we study teaching sets of \( k \)-threshold functions, i.e. functions that can be represented as a conjunction of \( k \) threshold functions. We reveal connection between essential points of \( k \) threshold functions and essential points of the corresponding \( k \)-threshold function. We note that, in general, a \( k \)-threshold function is not specified by its essential points and can have more than one minimal teaching sets. We show that for \( d = 2 \) the number of minimal teaching sets for a 2-threshold function can grow as \( \Omega(n^2) \). We also consider the class of polytopes with vertices in \( d \)-dimensional cube. Each polytope from this class can be defined by \( k \)-threshold function for some \( k \). In terms of \( k \)-threshold functions we prove that a polytope with vertices in \( d \)-dimensional cube has the unique minimal teaching set which is equal to the set of its essential points. For \( d = 2 \) we describe structure of minimal teaching set for a polytope and show that cardinality of this set is either \( \Theta(n^2) \) or \( O(n) \) and depends on perimeter and minimum angle of the polytope.

Keywords: machine learning, threshold function, essential point, teaching set, learning complexity, \( k \)-threshold function

1. Introduction

Let \( n \) and \( d \) be integers such that \( n \geq 2 \) and \( d \geq 1 \) and let \( E^d_n \) denote the \( d \)-dimensional cube, that is \( E^d_n = \{0, 1, \ldots, n - 1\}^d \). A function \( f \) that maps \( E^d_n \) to \( \{0, 1\} \) is threshold, if there exist real numbers \( a_0, a_1, \ldots, a_d \) such that

\[
M_1(f) = \left\{ x \in E^d_n : \sum_{j=1}^{d} a_j x_j \leq a_0 \right\}.
\]

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where $M_\nu(f)$ is the set of points $x \in E_n^d$ for which $f(x) = \nu$. The inequality $\sum_{j=1}^d a_j x_j \leq a_0$ is called \textit{threshold}. We denote by $\mathfrak{T}(d,n)$ the class of all threshold functions over $E_n^d$.

Let $k$ be a natural number. A function $f$ that maps $E_n^d$ to $\{0,1\}$ is called $k$-\textit{threshold} if there exist real numbers $a_{10}, a_{11}, \ldots, a_{kd}$ such that

$$M_1(f) = \left\{ x \in E_n^d : \sum_{j=1}^d a_{ij} x_j \leq a_{i0}, \quad i = 1, \ldots, k \right\}. \quad (1)$$

The system of inequalities $\sum_{j=1}^d a_{ij} x_j \leq a_{i0}, \quad i = 1, \ldots, k$ is called \textit{threshold} and defines the $k$-threshold function $f$. Let $\mathfrak{T}(d,n,k)$ be the class of $k$-threshold functions over $E_n^d$. By definition $\mathfrak{T}(d,n,1) = \mathfrak{T}(d,n)$. Note that a $k$-threshold function is also a $j$-threshold function for $j > k$. Denote by $\mathfrak{T}(d,n,*)$ the class of all $k$-threshold functions over $E_n^d$ for all natural $k$, that is $\mathfrak{T}(d,n,*) = \bigcup_{k \geq 1} \mathfrak{T}(d,n,k)$.

For any $k$-threshold function $f$ there exist threshold functions $f_1, \ldots, f_k$ such that

$$f(x) = f_1(x) \lor \ldots \lor f_k(x).$$

We will say that $f$ is \textit{defined} by $f_1, \ldots, f_k$ and $\{f_1, \ldots, f_k\}$ is \textit{defining set} for $f$.

A convex hull of a set of points $X \subseteq \mathbb{R}^d$ is denoted by $\text{Conv}(X)$. For a function $f : E_n^d \to \{0,1\}$ denote by $P(f)$ the convex hull of $M_1(f)$, that is $P(f) = \text{Conv}(M_1(f))$. For any polytope $P$ with vertices in $E_n^d$ there exists a unique $k$-threshold function $f$, such that $P = P(f)$. Therefore there is one-to-one correspondence between functions in the class $\mathfrak{T}(d,n,*)$ and polytopes with vertices in $E_n^d$, and we can say that $\mathfrak{T}(d,n,*)$ is \textit{class of polytopes with vertices in $E_n^d$}.

In [1] Angluin considered model of concept learning with membership queries. In this model a \textit{domain} $X$ and a \textit{concept class} $\mathcal{S} \subseteq 2^X$ are known to both the learner (or \textit{learning algorithm}) and the oracle. The goal of the learner is to identify an unknown \textit{target concept} $S_T \in \mathcal{S}$ that has been fixed by the oracle. To this end, learner may ask the oracle membership queries “does an element $x$ belong to $S_T$?”, to which the oracle returns “yes” or “no”. The learning complexity of a learning algorithm $A$ with respect to a concept class $\mathcal{S}$ is the minimum number of membership queries sufficient for $A$ to identify any concept in $\mathcal{S}$. The \textit{learning complexity of a concept class} $\mathcal{S}$ is defined as the minimum learning complexity of a learning algorithm with respect to $\mathcal{S}$ over all learning algorithms which learn $\mathcal{S}$ using membership queries.

In terms of Angluin’s model $\{0,1\}$-valued functions over $E_n^d$ can be considered as characteristic functions of concepts. Here $E_n^d$ is the domain and a function $f : E_n^d \to \{0,1\}$ defines concept $M_1(f)$. Concept learning with membership queries for classes of threshold functions, $k$-threshold functions, and polytopes with vertices in $E_n^d$ corresponds to the problem of identifying geometric objects in $E_n^d$ with certain properties.

From results of [2] and [3] it follows that learning complexity of the class of threshold functions $\mathfrak{T}(d,n)$ is $O\left(\frac{\log^{d-1} n}{\log \log n}\right)$. In [4] Maass and Bultman studied learning complexity of the class $k$-HALFSPACES$_n^{2^p}$, where $0 < p \leq \frac{\log d}{d}$. The class $k$-HALFSPACES$_n^{2^p}$ is the subclass of $k$-threshold functions over $E_n^d$ with restrictions that for any $f$ in this...
subclass $P(f)$ has edges with length at least $16 \cdot \left\lceil \frac{1}{p} \right\rceil$ and an angle $\alpha$ between a pair of adjacent edges satisfies $p \leq \alpha \leq \pi - p$. The learning algorithm proposed in [2] for identification a function $f$ in $k$-HALFSPACES $S^{d}_{n,p}$ requires a vertex of the polygon $P(f)$ as input and uses $O(k(\frac{1}{p} + \log n))$ membership queries.

Let $\mathcal{C}$ be a class of $\{0,1\}$-valued functions over $E^d_n$ and $f \in \mathcal{C}$. **Teaching set** for a function $f$ with respect to $\mathcal{C}$ is a set of points $T \subseteq E^d_n$ such that the only function in $\mathcal{C}$ which agrees with $f$ on $T$ is $f$ itself. The teaching set $T$ is **minimal** if no its proper subset is teaching for $f$. Note that teaching set for a function $f \in \mathcal{T}(d,n,k)$ with respect to $\mathcal{T}(d,n,+)\mathcal{g}$ is a teaching set with respect to $\mathcal{T}(d,n,k)$. A point $x \in E^d_n$ is called **essential** for a function $f \in \mathcal{C}$ if there exists a function $g \in \mathcal{C}$ such that $f(x) \neq g(x)$ and $f$ agrees with $g$ on $E^d_n \setminus \{x\}$. Let us denote the set of essential points for a function $f$ with respect to a class $\mathcal{C}$ by $S(f,\mathcal{C})$ or by $S(f)$ when $\mathcal{C}$ is clear. Let $S_{v}(f) = S(f) \cap M_{v}(f)$. By $J(f,\mathcal{C})$ we denote the number of minimal teaching sets for a function $f$ with respect to a class $\mathcal{C}$ and by $\sigma(f,\mathcal{C})$ the minimum cardinality of a teaching set for $f$ with respect to $\mathcal{C}$. **Teaching dimension** for the class $\mathcal{C}$ is denoted by $\sigma(\mathcal{C})$, where

$$\sigma(\mathcal{C}) = \max_{f \in \mathcal{C}} \sigma(f,\mathcal{C}).$$

The main aim of a learning algorithm with membership queries is to find any teaching set for a target function $f$ with respect to a concept class $\mathcal{C}$. The algorithm succeeds if it asked queries in all points of some teaching set for the function. Therefore teaching dimension for the class $\mathcal{C}$ is the lower bound on learning complexity of this class.

It is known (see, for example, [3] and [4]), that the set of essential points of a threshold function is a teaching set for this function. Together with the simple observation that any teaching set for a function should contains all its essential points, this imply that any threshold function have the unique minimal teaching set, that is $J(f,\mathcal{T}(d,n)) = 1$. In addition, it follows from [2] and [3] that

$$\sigma(\mathcal{T}(d,n)) = \Theta(\log_{2}^{d-2} n).$$

In this paper we study combinatorial and structural properties of teaching sets for $k$-threshold functions for $k \geq 2$. In particular, we show that 2-threshold functions from $\mathcal{T}(2,n,2)$, in contrast with threshold functions, can have more than one minimal teaching sets with respect to $\mathcal{T}(2,n,2)$. Moreover, we construct a sequence of functions from $\mathcal{T}(2,n,2)$ for which number of minimal teaching sets grows as $\Omega(n^2)$. On the other hand, we show that any $k$-threshold function $f$ (or polytope with vertices in $E^d_n$) has the unique minimal teaching set with respect to $\mathcal{T}(d,n,\cdot)$ coinciding with the set of essential points for $f$ with respect to $\mathcal{T}(d,n,\cdot)$. In addition, we give a general description of minimal teaching sets for such functions. For functions from $\mathcal{T}(2,n,\cdot)$ we refine the given structure and derive a bound on the cardinality of the minimal teaching sets.

The organization of the paper is as follows. In Section 2, we consider essential points of $k$-threshold function $f$ and their connection with essential points of thershold functions $f_1, \ldots, f_k$ which define $f$. In the beginning of Section 3 we show that $k$-threshold functions in general case can have more than one minimal teaching sets. The main result of Subsection 3.1 is Theorem [3] which states that the minimal teaching set for a $k$-threshold function with respect to $\mathcal{T}(d,n,\cdot)$ is unique and coincides with $S(f,\mathcal{T}(d,n,\cdot))$. The structure of $S(f,\mathcal{T}(d,n,\cdot))$ is given as well. In Subsection 3.2 we consider the class $\mathcal{T}(2,n,\cdot)$
and for $f \in \mathfrak{T}(2, n, \ast)$ prove an upper bound on the cardinality of $S(f, \mathfrak{T}(2, n, \ast))$. Finally, in Subsection 3.3 we consider functions in $\mathfrak{T}(2, n, 2)$ with special properties and show that each of these functions has a minimal teaching set with cardinality at most 9 and there are functions with $\Omega(n^2)$ minimal teaching sets with respect to $\mathfrak{T}(2, n, 2)$.

2. The set of essential points of $k$-threshold functions

Since $k$-threshold function is a conjunction of $k$ threshold functions, it is interesting to investigate connection between essential points of threshold functions $f_1, \ldots, f_k$ and essential points of their conjunction. In this section we prove several propositions that establish this relationship.

**Proposition 1.** Let $f_1, \ldots, f_k$ be threshold functions over $E_n^d$ defining a $k$-threshold function $f$. Then the following inclusions hold:

$$S_1(f, \mathfrak{T}(d, n)) \cap M_1(f_j) \subseteq S_1(f, \mathfrak{T}(d, n, k)) \quad (i = 1, \ldots, k).$$

**Proof.** Let $x \in S_1(f_i, \mathfrak{T}(d, n)) \cap M_1(f)$ for some $i \in \{1, \ldots, k\}$. Since $x$ is essential point of $f_i$ and $f_i(x) = 1$, there exists a threshold function $f'_i$ which differs from $f_i$ at the unique point $x$. The set $\{f_1, \ldots, f_{i-1}, f'_i, f_{i+1}, \ldots, f_k\}$ defines the $k$-threshold function $f'$ which differs from $f$ in the unique point $x$, namely $f'(x) = 0 \neq f(x)$. It means that $x$ is essential for $f$, i.e. $x \in S_1(f, \mathfrak{T}(d, n, k))$.

**Proposition 2.** Let $f_1, \ldots, f_k$ be threshold functions over $E_n^d$ defining a $k$-threshold function $f$. Then the following inclusions hold:

$$S_0(f, \mathfrak{T}(d, n)) \cap \bigcap_{j \neq i} M_1(f_j) \subseteq S_0(f, \mathfrak{T}(d, n, k)) \quad (i = 1, \ldots, k).$$

**Proof.** Let $x \in S_0(f_i, \mathfrak{T}(d, n)) \cap \bigcap_{j \neq i} M_1(f_j)$ for some $i \in \{1, \ldots, k\}$. Then $f(x) = 0$ and there exists a threshold function $f'_i$ such that $f'_i(x) = 1$ and $f'_i(y) = f_i(y)$ for every $y \in E_n^d \setminus \{x\}$. Since $x \in \bigcap_{j \neq i} M_1(f_j)$, the set $\{f_1, \ldots, f_{i-1}, f'_i, f_{i+1}, \ldots, f_k\}$ defines the $k$-threshold function $f'$ which differs from $f$ in the unique point $x$, namely $f'(x) = 1 \neq f(x)$. Therefore $x$ is essential for $f$ and $x \in S_0(f, \mathfrak{T}(d, n, k))$.

**Proposition 3.** Let $f$ be a $k$-threshold function over $E_n^d$ which defining set $\{f_1, \ldots, f_k\}$ is unique. Then

$$S(f_i, \mathfrak{T}(d, n)) \subseteq \bigcap_{j \neq i} M_1(f_j) \quad (i = 1, \ldots, k).$$

**Proof.** Suppose to the contrary that there exists $x \in E_n^d$ such that $x \in S(f_i, \mathfrak{T}(d, n))$ and $f_j(x) = 0$ for some distinct indices $i, j \in \{1, \ldots, k\}$. It means that $f(x) = 0$. Since $x$ is essential for $f_i$, there exists a threshold function $f'_i$ which differs from $f_i$ in the unique point $x$. Clearly, $\{f_1, \ldots, f_{i-1}, f'_i, f_{i+1}, \ldots, f_k\}$ defines $f$, which contradicts the uniqueness of defining set $\{f_1, \ldots, f_k\}$. 

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Corollary 4. Let \( f \) be a \( k \)-threshold function over \( E^d_n \) which defining set \( \{f_1, \ldots, f_k\} \) is unique. Then
\[
\bigcup_{i=1}^{k} S_{\nu}(f_i, \mathfrak{T}(d, n)) \subseteq S_{\nu}(f, \mathfrak{T}(d, n, k)) \quad (\nu = 1, 2).
\]

Proof. Since the function \( f \) satisfies conditions of Proposition 3,
\[
S_1(f_i, \mathfrak{T}(d, n)) \subseteq M_1(f) \quad (i = 1, \ldots, k)
\]
and
\[
S_0(f_i, \mathfrak{T}(d, n)) \subseteq \bigcap_{j \neq i} M_1(f_j) \quad (i = 1, \ldots, k).
\]
By Propositions 1 and 2 we get
\[
\bigcup_{i=1}^{k} S_{\nu}(f_i, \mathfrak{T}(d, n)) \subseteq S_{\nu}(f, \mathfrak{T}(d, n, k)).
\]

3. Teaching sets for \( k \)-threshold functions

Recall that the minimal teaching set for a threshold function is unique and equal to the set of its essential points. The situation becomes different for \( k \)-threshold functions, where \( k \geq 2 \). In order to show this we consider the following example.

Example 5. Let \( f \) be a function from \( \mathfrak{T}(2, 4, 2) \) with
\[
M_1(f) = \{(1, 2), (1, 3), (2, 2), (2, 3)\}.
\]
The set of essential points \( S(f) \) is
\[
\{(1, 1), (1, 2), (2, 1), (2, 2), (0, 3), (3, 3)\}.
\]
This set is not a teaching set because there exists a function \( g \in \mathfrak{T}(2, 4, 2) \) with \( M_1(g) = \{(1, 2), (2, 2)\} \), which agrees with \( f \) on \( S(f) \) (see Fig. 1). Though if we add any of the two points \((1, 3)\) or \((2, 3)\) to \( S(f) \), then we get a minimal teaching set for the function \( f \) (see Fig. 2) with respect to \( \mathfrak{T}(2, 4, 2) \), therefore \( J(f, \mathfrak{T}(2, 4, 2)) \geq 2 \).

3.1. Teaching sets for functions in \( \mathfrak{T}(d, n, *) \)

In this section we prove that for \( k \geq 2 \) and \( d \geq 2 \) the teaching dimension of \( \mathfrak{T}(d, n, k) \) is \( n^d \). Then we consider the class \( \mathfrak{T}(d, n, *) \) and show that for a function \( f \in \mathfrak{T}(d, n, *) \) the set of its essential points with respect to \( \mathfrak{T}(d, n, *) \) is also a teaching set, and therefore it is the unique minimal teaching set for \( f \) with respect to \( \mathfrak{T}(d, n, *) \).

Lemma 6. Let \( f : E^d_n \to \{0, 1\} \) be a function such that all points in \( M_1(f) \) lie on a straight line \( l \) and
\[
\{\lambda x + (1 - \lambda)y\} \cap E^d_n \subseteq M_1(f)
\]
for any points \( x, y \in M_1(f) \) and for any real \( \lambda \in [0, 1] \). Then \( f \in \mathfrak{T}(d, n, k) \) for any \( k \) such that \( k \geq 2 \).
Figure 1: Stars denote essential points. Black elements denote the points from $M_1(f)$. Empty elements denote points from $M_0(f)$. Functions $f$ (left panel) and $g$ (right panel) agree on $S(f)$.

Figure 2: Stars denote points of minimal teaching sets $S(f) \cup \{1,3\}$ (left panel) and $S(f) \cup \{2,3\}$ (right panel).
Theorem 9. Let \( f : \mathbb{R}^d \rightarrow \{0, 1\} \) be a function with the unique 1-valued point. Then \( f \) is a \( k \)-threshold function for any \( k \) such that \( k \geq 2 \).

**Proof.** It is sufficient to show that \( f \) is a 2-threshold function. Let \( x \) and \( y \) be the two vertices of \( P(f) \). Note that if \( |M_1(f)| = 1 \), then \( x = y \).

Clearly, it is possible to choose two parallel hyperplanes \( H' \) and \( H'' \) sufficiently close to each other such that \( E_d^m \cap H' = \{x\} \), \( E_d^m \cap H'' = \{y\} \), and there are no points between \( H' \) and \( H'' \) in \( E_d^m \setminus M_1(f) \). These hyperplanes can be used to define a 2-threshold function, which coincides with \( f \).

**Corollary 7.** Let \( f : E_d^m \rightarrow \{0, 1\} \) be a function with the unique 1-valued point. Then \( f \) is a \( k \)-threshold function for any \( k \) such that \( k \geq 2 \).

Corollary 7 gives us the teaching dimension for \( d(n, k) \), \( k \geq 2 \):

**Lemma 8.** \( \sigma(d, n, k) = n^d \) for any \( k \) such that \( k \geq 2 \).

**Proof.** Let us consider function \( f \), which is identically equal to zero and obviously belongs to \( d(n, k) \) for any \( k \) such that \( k \geq 2 \). From Corollary 7 it follows that the minimal teaching set for \( f \) with respect to \( d(n, k) \) coincides with \( E_d^m \).

For a polytope \( P \) denote by \( \text{Vert}(P) \) the set of vertices of \( P \), by \( B(P) \) the set of integer points on the border of \( P \) and by \( \text{Int}(P) \) the set of internal integer points of \( P \). For \( f \in d(n, *, *) \) denote by \( D(f) \) the set \( \{x \in M_0(f) : \text{Conv}(P(f) \cup \{x\}) \cap M_0(f) = \{x\}\)\).

**Theorem 9.** Let \( f \in d(n, *, *) \), \( d \geq 2 \), \( n \geq 2 \). Then

\[
S(f, d(n, *)) = \begin{cases} 
E_d^m, & M_1(f) = \emptyset; \\
\text{Vert}(P(f)) \cup D(f), & M_1(f) \neq \emptyset;
\end{cases}
\]

and \( S(f, d(n, *)) \) is the unique minimal teaching set for \( f \).

**Proof.** If \( M_1(f) = \emptyset \) then \( f \equiv 0 \) and therefore \( S(f) = E_d^m \). Clearly, in this case \( S(f) \) is the unique minimal teaching set for \( f \).

Now let \( M_1(f) \neq \emptyset \). We split the proof of this case into two parts. At first we show that all points from \( \text{Vert}(P(f)) \cup D(f) \) are essential and then we prove that this set is the unique minimal teaching set.

1. Let \( f' : E_d^m \rightarrow \{0, 1\} \) be a function which differs from \( f \) in the unique point \( x \in \text{Vert}(P(f)) \). Obviously \( M_0(f) = \emptyset \) and \( f' \) belongs to \( d(n, *, *) \). Therefore \( x \) is essential for \( f \) with respect to \( d(n, *, *) \). Now let \( f' : E_d^m \rightarrow \{0, 1\} \) be a function which differs from \( f \) in the unique point \( x \in D(f) \). The choice of \( x \) implies that the function \( f' \) belongs to \( d(n, *, *) \) and hence \( x \) is essential point for \( f \) with respect to \( d(n, *, *) \).

2. Since \( f \) belongs to the class \( d(n, *, *) \), knowing values of the function in \( \text{Vert}(P(f)) \) is sufficient to recover \( f \) in \( M_1(f) \). Further, for every point \( x \in M_0(f) \) such that \( |\text{Conv}(P(f) \cup \{x\}) \cap M_0(f)| > 1 \) the set \( \text{Conv}(P(f) \cup \{x\}) \cap M_0(f) \) necessarily contains at least one point from \( D(f) \). Therefore, the knowledge of values of the function in points from \( D(f) \) and \( P(f) \) is sufficient to recover \( f \) in \( M_0(f) \). This leads us to the conclusion that \( \text{Vert}(P(f)) \cup D(f) \) is a teaching set. Moreover, since all points in this set are essential and any teaching set contains all essential points, we conclude that \( \text{Vert}(P(f)) \cup D(f) \) is the unique minimal teaching set and coincides with \( S(f) \).
Lemma 10. Let $f \in \mathcal{I}(d, n, k), d \geq 2, k \geq 2$ and $M_{1}(f) = \{x'\}$. Then

$$S(f, \mathcal{I}(d, n, k)) = \{x'\} \cup \{x \in E_{n}^{d} : \gcd(|x_{1} - x'_{1}|, \ldots, |x_{d} - x'_{d}|) = 1\},$$

and $S(f, \mathcal{I}(d, n, k))$ is the unique teaching set for $f$ with respect to $\mathcal{I}(d, n, k)$ and

$$|S(f, \mathcal{I}(d, n, k))| = \Theta(n^{d}).$$

Proof.

Let $S = \{x \in E_{n}^{d} : \gcd(|x_{1} - x'_{1}|, \ldots, |x_{d} - x'_{d}|) = 1\}$. According to Lemma 5 any function $g : E_{n}^{d} \rightarrow \{0, 1\}$, such that $M_{1}(g) = \{x', x''\}$ and $\gcd(|x'_{1} - x''_{1}|, \ldots, |x'_{d} - x''_{d}|) = 1$, belongs to the class $\mathcal{I}(d, n, k)$ for any $k \geq 2$. Therefore all points in $S$ are essential for $f$. On the other hand $S \cup \{x'\}$ is a teaching set for $f$ because for any point $y \in M_{0}(f) \setminus S$ there exists a point $y' \in S$ such that $y$, $y'$, $x'$ are collinear and $y'$ is between $y$ and $x'$.

Let $\varphi(i)$ be Euler function. It is well known (see, for example, [9]) that

$$\sum_{i \leq n} \varphi(i) = \frac{3}{\pi^{2}} n^{2} + O(n \ln n).$$

Using this formula we can get a lower bound on the cardinality of the minimal teaching set:

$$|S \cup \{x'\}| \geq |\{x = (x_{1}, \ldots, x_{d}) \in E_{n}^{d} : \gcd(|x_{1} - x'_{1}|, \ldots, |x_{d} - x'_{d}|) = 1\}| + 1 \geq \left(\sum_{i \leq n/2} \varphi(i)\right) n^{d-2} = \left(\frac{3}{\pi^{2}} \left(\frac{n}{2}\right)^{2} + O\left(\frac{n}{2} \ln \frac{n}{2}\right)\right) n^{d-2} = \Theta(n^{d}).$$

So the lower bound is equal to the upper bound because $|E_{n}^{d}| = n^{d}$. And we get $|S(f, \mathcal{I}(d, n, k))| = \Theta(n^{d})$. ■

3.2. Teaching sets for functions in $\mathcal{I}(2, n, \ast)$

In the previous section we proved that for a function from $\mathcal{I}(d, n, \ast), d \geq 2$ the set of its essential points is also the unique minimal teaching set. In this section we consider the class $\mathcal{I}(2, n, \ast)$ and describe the structure of the set of essential points for a function in this class. We also give an upper bound on the cardinality of this set.

Let us consider an arbitrary function $f \in \mathcal{I}(2, n, \ast)$. Note that $P(f)$ can be the empty set, a point, a segment or a polygon. Let $P(f)$ be a segment or a polygon, that is $|M_{1}(f)| > 1$, and let $a_{1}x_{1} + a_{2}x_{2} = 0$ be the edge equality for an edge $e$ of $P(f)$. Without loss of generality we may assume that $\gcd(a_{1}, a_{2}) = 1$. That of two inequalities $a_{1}x_{1} + a_{2}x_{2} \leq 0$ and $a_{1}x_{1} + a_{2}x_{2} \geq 0$ which is satisfied by all points of $P(f)$ we call edge inequality. Note that if $P(f)$ is a segment, then it has one edge but two edge inequalities corresponding to the edge. If $P(f)$ is a polygon, then it has exactly one edge inequality for each edge. Hence the number of edge inequalities for $P(f)$ is equal to the number of its vertices.
Let $f$ be a function from $\mathcal{T}(2, n, \ast)$ such that $|M_1(f)| > 1$ and let
\[
a_{i1}x_1 + a_{i2}x_2 \leq a_{i0}, i = 1, \ldots, |\text{Vert}(P(f))|
\]
be edge inequalities for $P(f)$. Let $P'(f)$ be the following extension of $P(f)$
\[
\{x = (x_1, x_2) : a_{i1}x_1 + a_{i2}x_2 \leq a_{i0} + 1\},
\]
and let
\[
\Delta P(f) = P'(f) \setminus P(f).
\]
If $P$ is a polygon then denote by $\mathcal{P}(P)$ the perimeter of $P$, by $S(P)$ the area of $P$ and by $q_{\text{min}}(P)$ the minimum angle between neighboring edges in $P$.

**Proposition 11.** Let $f \in \mathcal{T}(2, n, \ast)$ and $S(P(f)) > 0$. Then $D(f) = \Delta P(f) \cap M_0(f)$.

**Proof.** Note that by construction all integer points in $\Delta P(f)$ lie on the border of $P'(f)$, which implies that $\Delta P(f) \cap M_0(f) \subseteq D(f)$. On the other hand, using Pick’s formula (see, for example, [10]) it is easy to show that $|\text{Conv}(P(f) \cup \{x\}) \cap M_0(f)| > 1$ for any $x \in M_0(f) \setminus \Delta P(f)$, and therefore $D(f) \subseteq \Delta P(f) \cap M_0(f)$.

**Lemma 12.** Let $f \in \mathcal{T}(2, n, \ast)$ and $S(P(f)) > 0$. Then
\[
S(f) = (\Delta P(f) \cap M_0(f)) \cup \text{Vert}(P(f))
\]
and
\[
|S(f)| = O \left( \min \left( n, \mathcal{P}(P(f)) + \frac{1}{q_{\text{min}}(P(f))} \right) \right).
\]

**Proof.**
From Theorem 9 and Proposition 11 it easily follows that $(\Delta P(f) \cap M_0(f)) \cup \text{Vert}(P(f))$ is the set of essential points $S(f)$ for $f$ with respect to $\mathcal{T}(2, n, \ast)$.

Since every point of $S(f)$ is integer and either belongs to the border of $P(f)$ or to the border of $P'(f)$, the cardinality of $S(f)$ can be bounded from above by the sum of perimeters $\mathcal{P}(P(f))$ and $\mathcal{P}(P'(f))$. Taking into account the fact that the distance between straight lines, defined by the equations $a_{i1}x_1 + a_{i2}x_2 = a_{i0}$ and $a_{i1}x_1 + a_{i2}x_2 = a_{i0} + 1$, is at most 1, we have:
\[
|S(f)| \leq \mathcal{P}(P(f)) + \mathcal{P}(P'(f)) \leq 2\mathcal{P}(P(f)) + \sum_{i=1}^{\lfloor \text{Vert}(P(f)) \rfloor} 2 \cot \frac{q_i}{2},
\]
where $q_i$ for $i \in \{1, \ldots, \lfloor \text{Vert}(P(f)) \rfloor \}$ are angles between neighboring edges of $P(f)$ if $P(f)$ is a polygon and zero if $P(f)$ is a segment.

The number of integer vertices of a convex polygon is not more than perimeter of this polygon, so $|\text{Vert}(P(f))| \leq \mathcal{P}(P(f))$. Obviously, only 2 angles of a convex polygon can be less than $\frac{\pi}{3}$. So we have:
\[
2\mathcal{P}(P(f)) + \sum_{i=1}^{\lfloor \text{Vert}(P(f)) \rfloor} 2 \cot \frac{q_i}{2} \leq \frac{\pi}{3} |\text{Vert}(P(f))|.
\]
so we can make final conclusion:

Example 13. Consider function $f \in \mathcal{X}(2, 12, \ast)$ (see Fig. 2). Gray set is $\Delta P(f)$. Black stars are points from $\text{Vert}(P(f))$ and white stars are points from $\Delta P(f) \cap M_0(f)$.

Corollary 14. For every function $f \in \mathcal{X}(2, n, \ast)$ there exists $k_{\text{min}} \geq 2$ such that the function $f$ is $k_{\text{min}}$-threshold and $S(f, \mathcal{X}(2, n, \ast))$ is also the set of essential points and the minimal teaching set for $f$ with respect to $\mathcal{X}(2, n, k)$ for $k \geq k_{\text{min}}$.

Example 15. For some integer $m$ such that $2 \leq m \leq n - 2$ consider function $f^{(m)} \in \mathcal{X}(2, n, \ast)$ such that $M_1(f^{(m)}) = \{(x, y) \in E_n^2 : x \leq m, y \leq m\}$. Vert$(P(f^{(m)})) = \{(0, 0), (m, 0), (0, m)\}$ and $\Delta P(f^{(m)}) \cap E_n^2 = \{(x, y) \in E_n^2 : x + y = m + 1\}$. $S(f^{(m)})$ with respect to $\mathcal{X}(2, n, \ast)$ is $\text{Vert}(P(f^{(m)})) \cup \Delta P(f^{(m)}) \cap E_n^2$. This is also the set of essential points and the minimal teaching set with respect to $\mathcal{X}(2, n, k)$ for any $k \geq 2$.

3.3. The teaching set for functions from $\mathcal{X}(2, n, 2)$ with the unique defining set of threshold functions

In this section we consider the subset of 2-threshold functions over $E_n^2$, for which the cardinality of minimal teaching set can be bounded by a constant. Also we show that for such 2-threshold functions the number of minimal teaching sets can grow as $\Omega(n^2)$.

Lemma 16. Let $f \in \mathcal{X}(2, n, 2)$ and for some defining set of threshold functions $\{f_1, f_2\}$ the following is true:

\[
\begin{align*}
S(f_1) \cap M_0(f_2) &= \emptyset; \\
S(f_2) \cap M_0(f_1) &= \emptyset.
\end{align*}
\]

If $M_1(f) \cap B(\text{Conv}(E_n^2)) \neq \emptyset$, then $\{f_1, f_2\}$ is the unique defining set for $f$ and $\sigma(f, \mathcal{X}(2, n, 2)) \leq 9$.
Figure 3: Gray set is $\Delta P(f)$.

Figure 4: $f^{(2)} \in \exists(2, 4, 2)$. Gray set is $\Delta P(f^{(2)})$. 

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Proof. Note, that

$$B(\text{Conv}(E_{n}^{2})) = \{ x \in E_{n}^{2} : x_{1} = 0 \lor x_{2} = 0 \lor x_{1} = n - 1 \lor x_{2} = n - 1 \}. $$

It is known \([1]\) that \(|S(g)| \in \{3, 4\}\) and \(|S_{1}(g)|, |S_{0}(g)| \in \{1, 2\}\) for any threshold function \(g\) over \(E_{n}^{2}\). Let us consider two cases:

1. \(|S_{0}(f_{i})| = 1\) for some \(i \in \{1, 2\}\). Assume, without loss of generality, that \(S_{0}(f_{1}) = \{x_{1}\}\) and \(S_{1}(f_{1}) = \{y_{1}, y_{2}\}\). We claim, that in this case \(S(f_{1}) \cup S(f_{2})\) is a teaching set for \(f\). Suppose, \(f'\) is a 2-threshold function which agrees with \(f\) on \(S(f_{1}) \cup S(f_{2})\). Let \(F'\) be a set of threshold functions defining \(f'\) and \(F' = \{f_{1}', f_{2}'\}\). We will show that \(\{f_{1}, f_{2}\} = F'\) and therefore \(f'\) coincides with \(f\) and \(\{f_{1}, f_{2}\}\) is the unique defining set for \(f\). Since \(S(f_{1}) \cap M_{0}(f_{2}) = \emptyset, f_{1}(x) = f(x) = f'(x)\) for every \(x \in S(f_{1})\). Hence one of the functions from \(F'\), say \(f_{1}'\), should take the value 0 on \(x_{1}\) and the value 1 on \(y_{1}\) and \(y_{2}\), which means that

$$f_{1}' = f_{1}. $$

(2)

Since \(S(f_{2}) \cap M_{0}(f_{1}) = \emptyset, f_{2}(x) = f(x) = f'(x)\) for every \(x \in S(f_{2})\). This together with \([2]\) imply that \(f_{2}'\) agrees with \(f_{2}\) on \(S(f_{2})\) and therefore \(f_{2}' = f_{2}\). Since \(|S(f_{1})| = 3\) and \(|S(f_{2})| \leq 4\), the set \(S(f_{1}) \cup S(f_{2})\) has at most 7 points.

2. \(S_{0}(f_{i}) = \{x_{1}, x_{2}\}, S_{0}(f_{j}) = \{x_{3}, x_{4}\}\). Denote by \(G \subseteq \mathbb{T}(2, n, 2)\) a set of 2-threshold functions, which agree with \(f\) on \(S(f_{i}) \cup S(f_{j})\). From conditions of lemma it follows that \(S_{0}(f_{i}) \cap S_{0}(f_{j}) = \emptyset\) and that the points from \(S_{0}(f_{i})\) should lie on the one side of the plane with respect to the line passing through the points of \(S_{0}(f_{j})\) if \(i \neq j\). Therefore \(S_{0}(f_{i}) \cup S_{0}(f_{j})\) is a set of vertices of convex quadrilateral \(P = (x_{1}, x_{2}, x_{3}, x_{4})\), and for each of the threshold functions vertices from its teaching set are neighboring (see Fig. \([2]\)). This implies that \(G\) is the disjoint union of two sets:

$$G_{1} = \{ g \mid g \in G, \{x_{1}, x_{2}\} \subseteq M_{0}(g_{1}) \text{ and } \{x_{3}, x_{4}\} \subseteq M_{0}(g_{2}) \}$$

and

$$G_{2} = \{ g \mid g \in G, \{x_{1}, x_{4}\} \subseteq M_{0}(g_{1}) \text{ and } \{x_{2}, x_{3}\} \subseteq M_{0}(g_{2}) \},$$

where \(g_{1}\) and \(g_{2}\) are some threshold functions defining \(g\). Applying the same arguments as in Case 1 it could be shown that \(G_{1} = \{f\}\) and \(\{f_{1}, f_{2}\}\) is the unique defining set for \(f\). Now we will show that in order to distinguish \(f\) from each of the functions from \(G_{2}\) it is sufficient to add at most one more point to \(S(f_{1}) \cup S(f_{2})\). Note that \((S_{1}(f_{1}) \cup S_{1}(f_{j})) \setminus \text{Int}(P) \neq \emptyset\) if and only if \(G_{2} = \emptyset\). So we may assume that all the points from \(S_{1}(f_{1}) \cup S_{1}(f_{2})\) are inside \(P\). We are interested in a point \(x'\) such that

$$f(x') \neq g(x') \text{ for all } g \in G_{2}. $$

(3)

In order to ensure the existence of such a point let us note that

$$M_{1}(f) \cap \bigcup_{g \in G_{2}} M_{1}(g) \subseteq \text{Int}(P).$$

This means that \(g(x) = 0\) for all \(g \in G_{2}\) and \(x \in M_{1}(f) \setminus \text{Int}(P)\). Since \(B(\text{Conv}(E_{n}^{2})) \cap \text{Int}(P) = \emptyset\) and \(B(\text{Conv}(E_{n}^{2})) \cap M_{1}(f) \neq \emptyset\), we can take any point from \(B(\text{Conv}(E_{n}^{2})) \cap \text{Int}(P) = \emptyset, M_{1}(f) \neq \emptyset\).
$M_1(f)$ as $x'$ and obtain a teaching set $T$ for $f$ which is equal to $S(f_1) \cup S(f_2) \cup \{x'\}$. Note that any such a teaching set $T$ that $T = S(f_1) \cup S(f_2) \cup \{x'\}$ and $x' \in B(Conv(E_n^2)) \cap M_1(f)$ is minimal and $|T| \leq 9$. 

\[\sigma(f, \Sigma(2, n, 2)) \leq 9.\]

**Proof.** By proposition 3 for $f_1$ and $f_2$ the following is true:

\[
\begin{align*}
S(f_1) \cap M_0(f_2) &= \emptyset; \\
S(f_2) \cap M_0(f_1) &= \emptyset.
\end{align*}
\]

Therefore $f$ satisfies conditions of Lemma 16.

**Remark 17.** Lemma 16 is also true when domain is a convex subset of $E_n^2$.

**Corollary 18.** Let $f \in \Sigma(2, n, 2)$ and the set of threshold functions $\{f_1, f_2\}$ defining $f$ is unique. If $M_1(f) \cap B(Conv(E_n^2)) \neq \emptyset$, then

\[
\sigma(f, \Sigma(2, n, 2)) \leq 9.
\]

**Proof.** By proposition 3 for $f_1$ and $f_2$ the following is true:

\[
\begin{align*}
S(f_1) \cap M_0(f_2) &= \emptyset; \\
S(f_2) \cap M_0(f_1) &= \emptyset.
\end{align*}
\]

Therefore $f$ satisfies conditions of Lemma 16.

Recall that by $J(f, C)$ we denote the number of minimal teaching sets for a function $f$ with respect to a class $C$. Using functions from Lemma 16 next lemma proves that number of minimal teaching sets for 2-threshold functions can grow as $\Omega(n^2)$.

**Lemma 19.**

\[
\max_{f \in \Sigma(2, n, 2)} J(f, \Sigma(2, n, 2)) = \Omega(n^2).
\]

**Proof.** Let

\[
m = m(n) = \max_{i \leq n, 4| i} i.
\]

For $n \geq 21$ let $f^{(n)} \in \Sigma(2, n, 2)$ be defined by threshold functions $f_1^{(n)}$ and $f_2^{(n)}$ with the corresponding inequalities:

\[
\begin{align*}
-3x_1 - 4x_2 &\leq -25, \\
3x_1 + 42x_2 &\leq 12m - 1.
\end{align*}
\]
The functions \( f_1^{(n)} \) and \( f_2^{(n)} \) have the following essential points:

\[
\begin{align*}
S_1(f_1) &= \{y_1 = (7, 1), y_2 = (3, 4)\}, \\
S_0(f_1) &= \{y_3 = (0, 6), y_4 = (8, 0)\}, \\
S_1(f_2) &= \{x_1 = (4m - 3, 2), x_2 = (1, 3m - 1)\}, \\
S_0(f_2) &= \{x_3 = (0, 3m), x_4 = (4m, 0)\}.
\end{align*}
\]

Note that the function \( f^{(n)} \) satisfies conditions of Corollary 18 and Lemma 16 and quadrilateral \( P \) from Lemma 16 is \((y_3, x_3, x_4, y_4)\). Denote by \( G \subseteq \mathcal{S}(2, n) \) the set of functions such that for every \( g \in G \) and for some threshold functions \( g_1, g_2 \) that define \( g \) the following is true:

\[
g(x) = 1 \text{ for every } x \in S_1(f_1^{(n)}) \cup S_1(f_2^{(n)})
\]

and

\[
\{y_4, x_4\} \subset M_0(g_1), \\
\{y_3, x_3\} \subset M_0(g_2).
\]

The set \( G \) corresponds to the set \( G_2 \) from the proof of Lemma 16, therefore all functions from \( G \) and only them agree with \( f \) on \( S(f_1^{(n)}) \cup S(f_2^{(n)}) \). Note that \( S_1(f_1^{(n)}) \cup S_1(f_2^{(n)}) \) is subset of \( \text{Int}(P) \) and hence \( G \neq \emptyset \). Let us lower bound the number of such points \( x' \) that

\[
f^{(n)}(x') \neq g(x') \text{ for all } g \in G.
\]

Denote by \( R(n) \) the triangle with vertices \( x_1, x_2 \) and \((n-1, n-1)\) and by \( L(n) \) the segment between \( x_1 \) and \( x_2 \). It is clear that \( R(n) \cap M_1(f^{(n)}) = L(n) \cap E_n^2 \). By construction of the set \( G \), for any \( g \in G \) the next inclusion is true: \( R(n) \cap E_n^2 \subset M_1(g) \). It means that any point in \( (R(n) \setminus L(n)) \cap E_n^2 \) separates \( f^{(n)} \) from any function in \( G \). Therefore the number of minimal teaching sets for \( f^{(n)} \) can be lower bounded by the cardinality of \( (R(n) \setminus L(n)) \cap E_n^2 \), which is equal to \( |R(n) \cap E_n^2| - |L(n) \cap E_n^2| \).

The number of integer points in \( L(n) \) can be calculated through the GCD of the differences between coordinates of \( x_1 \) and \( x_2 \):

\[
|L(n) \cap E_n^2| = 2 + \gcd((4m - 3) - 1, (3m - 1) - 2) = m + 1.
\]

The number of integer points in \( R(n) \) can be calculated by means of Pick’s formula. Indeed, since \( R(n) \) is a polygon with vertices in \( E_n^2 \), we have

\[
S(R(n)) = |\text{Int}(R(n))| + \frac{|B(R(n))|}{2} - 1
\]

and therefore

\[
|R(n) \cap E_n^2| = |\text{Int}(R(n))| + |B(R(n))| = S(R(n)) + \frac{|B(R(n))|}{2} + 1.
\]

Now since

\[
|B(R(n))| \geq |L(n) \cap E_n^2| + 1 \geq m + 2
\]

and

\[
S(R(n)) = \frac{|(m - 1)(12m - 7n + 6)|}{14},
\]
we conclude that

\[ |(R(n) \setminus L(n)) \cap E_n^2| \geq \frac{(m - 1)(12m - 7n + 6)}{2} + \frac{m + 2}{2} + 1 - (m + 1) = \Theta(n^2). \]

That is the number of minimal teaching sets for the function \( f^{(n)} \) grows as \( \Omega(n^2) \).

\[ \blacksquare \]

4. Open problems

In this paper, we investigated structural and quantitative properties of sets of essential points and minimal teaching sets for \( k \)-threshold functions.

We proved that a function in the class \( \mathcal{T}(d, n, *) \) has the unique minimal teaching set which is equal to the set of essential points for this function with respect to the class. For a function in the class \( \mathcal{T}(2, n, *) \) we estimated the cardinality of the set of essential points for the function. It would be interesting to find analogous bounds on the cardinality of the set of essential points for a function in \( \mathcal{T}(d, n, *) \) for \( d > 2 \).

We considered \( \mathcal{T}(2, n, 2) \) and proved that the set of essential points for a function in this class is not necessary a minimal teaching set. Moreover we showed that \( J(\mathcal{T}(2, n, 2)) = \Omega(n^2) \). Also in the class \( \mathcal{T}(2, n, 2) \) we identified functions with minimal teaching sets of cardinality at most 9. It would be interesting to estimate the proportion of functions with this property in the class \( \mathcal{T}(2, n, 2) \).

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