Weak Resilience of Networked Control Systems

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\textbf{Abstract—} In this paper, we propose a method to establish a networked control system that maintains its stability in the presence of certain undesirable incidents on local controllers. We call such networked control systems weakly resilient. We first derive a necessary and sufficient condition for the weak resilience of networked systems. Networked systems do not generally satisfy this condition. Therefore, we provide a method for designing a compensator which ensures the weak resilience of the compensated system. Finally, we illustrate the efficiency of the proposed method by a power system example based on the IEEE 14-bus test system.

I. INTRODUCTION

Many infrastructure and industrial processes, e.g., power networks [1], [2], transportation networks [3] and fabrication plants [4], are integrations of computer-based cyber systems and physical processes. By emerging advanced technologies, the level of integration of the cyber and physical systems has intensified. Along with this, several challenging problems in control system design arise.

Resilient system design is one of the most challenging problems for cyber-physical systems. The concept of resilient system design, which means control system design in an adversarial and uncertain cyber environment, has been introduced in [5]. Furthermore, in [6], the authors have discussed a conceptual property of resilient control systems. Moreover, in [7], the authors have proposed resilient controller design for cyber-physical networks under Denial of Service (DoS) attacks which lead to severe time-delays and degradation of control performance. However, it is still an open problem to design resilient systems maintaining an acceptable level of operation or service in face of undesirable incidents on cyber systems, e.g., adversarial attacks and faults caused by human errors.

On the other hand, in [8], the authors have proposed a method for constructing systems whose stability is maintained against any modification of local controllers, which stabilize local subsystems disconnected in the networked system. In this method, we design a supervisory compensator such that the compensated networked system has the property that the stability of the overall closed-loop system is guaranteed against any modification of locally stabilizing controllers. However, no characterization of compensated networked systems having this property has been shown.

This paper continues the research of [8] and establishes its connection to resilient control design, for the first time. First, we define weakly resilient networked systems such that the overall closed-loop system maintains its stability in the presence of any undesirable incidents on local controllers that maintain local stability (to be defined in Section \textbf{II}). To clarify the class of networked systems which are weakly resilient against undesirable incidents on local controllers, we provide a necessary and sufficient characterization of weakly resilient networked systems. However, networked systems do not generally satisfy the shown necessary condition. Thus, we provide a design method to make a given networked system weakly resilient. Finally, we show the efficiency of the proposed system design through a power system example based on the IEEE 14-bus test system [9].

This paper is organized as follows. In Section \textbf{II} we introduce and characterize weakly resilient networked systems. In Section \textbf{III} we consider compensator design such that the networked system is weakly resilient. In Section \textbf{IV} we show the efficiency of the proposed system design through a numerical example. Finally, concluding remarks are provided in Section \textbf{V}.

\textbf{Notation:} Denote the set of real numbers by \(\mathbb{R}\), the set of complex numbers by \(\mathbb{C}\). Denote the \(n\)-dimensional identity matrix by \(I_n\), where we omit the subscript \(n\) when no confusion occurs. For \(\mathbb{N} := \{1, \ldots, N\}\), denote the block-diagonal matrix having matrices \(M_1, \ldots, M_N\) on its diagonal by \(dg(M_i)_{i \in \mathbb{N}}\). We omit the subscript \(i \in \mathbb{N}\) when no confusion occurs. Given signals \(x_1(t) \cdots x_N(t)\), denote \(x(t) := [x_1^T(t), \ldots, x_N^T(t)]^T\), where we omit the time variable \(t\) when no confusion occurs. Denote by \(\Sigma : u(t) \mapsto y(t)\) a finite-dimensional linear time-invariant system. Given \(\kappa : y_1 \mapsto u_1\) and \(\Sigma : \{u_1, u_2\} \mapsto \{y_1, y_2\}\), \((\Sigma, \kappa)\) denotes the (well-posed) interconnected system with the external input \(u_2\) and external output \(y_2\). For example, given

\[
\kappa : \begin{cases}
\dot{\xi} = K \xi + Hy_1 \\
u_1 = M \xi
\end{cases}
\]

and

\[
\Sigma : \begin{cases}
\dot{x} = Ax + B_1 u_1 + B_2 u_2 \\
y_1 = C_1 x \\
y_2 = C_2 x,
\end{cases}
\]

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For each $i \in \{1, 2\}$, consider $\Sigma_i$ in (1) and $\kappa_i$ in (4). Define $\Sigma$ in (2) and the set of locally stabilizing controllers as

$$K_i := \{ \kappa_i | (\Sigma_i, \kappa_i) \text{ is stable} \} \quad (5)$$

for $i \in \{1, 2\}$. The system $\Sigma$ is said to be weakly resilient if $(\Sigma, \{\kappa_i\})$ is stable for any $\kappa_i \in K_i$, $i \in \{1, 2\}$.

The reason why we adopt the terminology weak resilience for this condition is that there does not exist a locally stabilizing controller that destabilizes the overall system $\Sigma$. Hence, weak resilience, in this sense, appears to be a minimum requirement for the resilience of networked systems.

In the next subsection, we will provide a characterization of weakly resilient networked systems.

### B. Characterization of Weakly Resilient Networked Systems

In this subsection, we show a necessary and sufficient condition for the resilience of networked systems in the sense of Definition (1). For simplicity, we assume that the input and output signals of $\Sigma_i$ in (1) are scalar, i.e.,

$$z_i \in \mathbb{R}, \quad u_i \in \mathbb{R}, \quad y_i \in \mathbb{R}, \quad i \in \{1, 2\}.$$

In this setting, we give the following theorem:

**Theorem 1:** For each $i \in \{1, 2\}$, consider $\Sigma_i$ in (1) and $\kappa_i$ in (4). Define $\Sigma$ in (2). Suppose $(A_i, B_i)$ is controllable and $(A_i, C_i)$ is observable for each $i \in \{1, 2\}$. The system $\Sigma$ is weakly resilient if, and only if, $\Sigma$ is a cascade system, i.e.,

$$J_iS_j = 0, \quad D_iS_j = 0 \quad (6)$$

where $D_i$ and $S_i$ are defined in (3).
for either $i = 1$ or $i = 2$ and with $j \neq i$.

**Proof:** See Appendix.

We emphasize that the cascade property of the system is not only a sufficient condition, but also necessary. In other words, if the system does not have any cascade realization, the system is not weakly resilient. However, in general, networked systems are not necessarily cascade. Thus, in the next section, let us consider designing a compensator to make networked systems weakly resilient.

### III. COMPENSATOR DESIGN FOR WEAK RESILIENCE

In this section, instead of $\Sigma$ in (2), we deal with networked systems with additional input signals described as

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Rv \\ y = Cx + r, \end{cases}$$

where $A$, $B$, and $C$ are defined as in (3), and $v \in \mathbb{R}^p$ and $r \in \mathbb{R}^q$ are the additional input signals from the compensator introduced next. We suppose that $(A, R)$ is controllable.

For this system, we consider designing a compensator described by

$$\Phi : \begin{cases} \dot{\phi} = \Lambda \phi + \Gamma z \\ r = \Xi \phi \\ v = \Theta \phi, \end{cases}$$

where $\phi \in \mathbb{R}^n$. Denote the compensated system by $\Sigma_\phi := (\Sigma, \Phi)$. The network structure of this compensated system is shown in Fig. 2

In this setting, the following corollary follows from Theorem I

**Corollary 1:** Given $\Sigma$ in (7), consider $\Phi$ in (8). Define the interconnected system $\Sigma_\phi := (\Sigma, \Phi)$. Then, $\Sigma_\phi$ is weakly resilient if, and only if, $\Sigma_\phi(s)$ satisfies

$$\Sigma_\phi(s) = \begin{bmatrix} C_1(sI - A_1)^{-1}B_1 & \sigma(s) \\ 0 & C_2(sI - A_2)^{-1}B_2 \end{bmatrix}$$

with a proper transfer function $\sigma(s)$, or $\Sigma_\phi(s)$ has a similar lower-triangular form.

Next, we consider designing a compensator such that the transfer matrix $\Sigma_\phi(s)$ has the form (9). As a related work, noninteracting control based on geometric control theory has been proposed in the literature, e.g., [10], [11], where several off-diagonal elements of the transfer matrix are canceled. However, in general, the diagonal elements of the transfer matrix cannot be arbitrarily designed by the existing methods. Thus, existing methods do not enable us to construct $\Sigma_\phi(s)$ having the form (9) because the $i$-th diagonal element of $\Sigma_\phi(s)$ in (9) must be $C_i(sI - A_i)^{-1}B_i$.

To overcome this difficulty, in this paper, we consider designing a compensator by taking another approach, which was recently developed in [8]. For simplicity, we assume that $D_i = 0$ in (1). Note that the system $\Sigma$ in (7) is not a cascade. In this setting, we provide the following compensator on the basis of the state-space expansion technique proposed in [8]:

**Proposition 1:** Given $\Sigma$ in (7), consider $\Phi$ in (8) with

$$\Lambda = \begin{bmatrix} A_1 & J_1S_2 \\ 0 & A_2 \end{bmatrix} + R\Theta, \quad \Gamma = \begin{bmatrix} 0 & 0 \\ J_2 \end{bmatrix}, \quad \Xi = -dg(C_i),$$

where $\Theta$ is given such that it stabilizes $A + R\Theta$. Then, $\Sigma_\phi := (\Sigma, \Phi)$ is weakly resilient.

**Proof:** The compensated system $\Sigma_\phi$ is described by

$$\Sigma_\phi : \begin{cases} \dot{\phi} = \Lambda \phi + \Gamma dg(S_i) \\ y = -dg(C_i)\phi + dg(C_i)x. \end{cases}$$

Taking the coordinate transformation $\chi = x - \phi$, we have

$$\begin{cases} \dot{\chi} = (A + R\Theta) \chi + \Gamma dg(S_i) \\ y = dg(C_i)\chi, \end{cases}$$

with

$$A := \begin{bmatrix} A_1 & J_1S_2 \\ 0 & A_2 \end{bmatrix}.$$
overall closed-loop system is bounded by that of the local closed-loop systems. In general, it is not clear to what extent the performance of the whole closed-loop system is deteriorated under attacks on local controllers. In contrast, the compensated system $\Sigma_{\Phi}$ has an advantage that the performance deterioration of the overall closed-loop system can be evaluated by that of the local closed-loop systems.

Remark 1: In [8], we have dealt with a similar compensator $\Phi$ but it made the transfer matrix $\Sigma_{\Phi}$ diagonal. In this case, it has been shown that the rank of $\Gamma$ in (8) coincides with the sum of the rank of $J_i$ for $i \in \{1, 2\}$. Compared to this, $\Gamma$ in (10) is a lower-rank matrix. Note that the low-rankness of $\Gamma$ has a direct relationship to the decay rate of Hankel singular values of $\Phi$. Thus, the compensator provided in this paper has a potential to be approximated by a lower-dimensional system as compared to the compensator considered in [8].

Remark 2: Even if $z_i$ in (1) is not measureable, we can construct a compensator such that the compensated system is weakly resilient by using an observer as follows: We design an observer using

$$w = Sx$$

as a measureable output signal of $\Sigma$ in (7). Define

$$O: \begin{cases} \dot{x} = (A - HS)\dot{x} + dg(B_i)u + Hw + Rw \\ \dot{\tilde{z}} = \Gamma \tilde{\dot{x}}, \end{cases}$$

(14)

where $H$ is given such that $A - HS$ is Hurwitz. Let $\Phi$ be given by (10) using $\tilde{\dot{x}}$ instead of $\dot{x}$. Then, $(\Sigma, \Phi, O)$ is weakly resilient.

IV. NUMERICAL SIMULATION

A. Power Network Model

In this section, we show the efficiency of the proposed weakly resilient system design through a numerical example. We deal with the IEEE 14-bus power test system provided by [9], where the system involves five generators and 11 loads. The power system is shown in Fig. 3. For $k \in \{1, \ldots, 5\}$, the $k$-th generator dynamics is described by

$$G_{[k]}: \begin{cases} \dot{\zeta}_{[k]} = A_{[k]}\zeta_{[k]} + b_{[k]}v_{[k]} + b_{[k]}^T\tau_{[k]} \\ \delta_{[k]} = c\zeta_{[k]}, \end{cases}$$

(15)

where the states of $\zeta_{[k]} \in \mathbb{R}^d$ represent the phase angle difference, angular velocity difference, mechanical input difference, and valve position difference. In addition, $u_{[k]} \in \mathbb{R}$ and $v_{[k]} \in \mathbb{R}$ are the angular velocity difference command, $\tau_{[k]} \in \mathbb{R}$ is the electric torque difference from the connected generators, and $\delta_{[k]} \in \mathbb{R}$ is the phase angle difference.

Furthermore, the system matrices in (15) are given by

$$A_{[k]} := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -D_{[k]}/M_{[k]} & -1/M_{[k]} & 0 \\ 0 & 0 & -1/T_{[k]} & 1/T_{[k]} \\ 0 & 1/K_{[k]} & 0 & -R_{[k]}/K_{[k]} \end{bmatrix}, \quad b_{[k]} := \frac{1}{K_{[k]}} e_{[k]}^4, \quad b_{[k]}^T := \frac{1}{M_{[k]}} e_{[k]}^4, \quad c := (e_{[k]}^1)^T,$$

(16)

where $e_{[k]}^i \in \mathbb{R}^n$ is the $i$-th column of $I_n$ and $M_{[k]}, D_{[k]}, T_{[k]}, K_{[k]}$ and $R_{[k]}$ are an inertia constant, damping coefficient, turbine time constant, governor time constant, and droop characteristic, respectively. These parameters are randomly chosen from the intervals $[0.01, 1]$, $[0.4, 11]$, $[0.01, 0.02]$, $[0.03, 0.7]$ and $[0.01, 0.05]$, respectively. Note that the unit of all physical variables is [p.u.] unless otherwise stated. Furthermore, all loads are modeled as constant power loads, see [9].

We give the interconnection structure among generators by

$$\tau = -Y \delta,$$

(17)

where $\tau := [\tau_{[1]}, \ldots, \tau_{[5]}]^T$ and $\delta := [\delta_{[1]}, \ldots, \delta_{[5]}]^T$. In (17), $Y$ compatible with the interconnection structure among generators is calculated by using MATPOWER [9].

Finally, the first to third generators are clustered as the first subsystem, and the others are clustered as the second subsystem. Interconnecting these two subsystems, we have a system $\Sigma$ in (7) where the state variable is defined as $x = [\zeta_{[1]}^T, \ldots, \zeta_{[5]}^T]^T$, and input signals are defined as $u = [u_{[1]}, \ldots, u_{[5]}]^T$ and $v = [v_{[1]}, \ldots, v_{[5]}]^T$. Furthermore, the measurement signal is taken as the angle differences, i.e., $y = [\delta_{[1]}, \ldots, \delta_{[5]}]^T$. For the system matrices of $\Sigma$ in (7), $A$ is given by

$$A = dg(A_{[k]}) - dg(b_{[k]}^T)Y(\mathcal{I}_5 \otimes c),$$

where $\otimes$ denotes the Kronecker product. In addition, $B$, $R$ and $C$ are given as the matrices compatible with $u$, $v$ and $y$.

B. Demonstration of Compensator Design

In this section, we show the efficiency of the compensator design for the power network given in the previous section.

First, we design the local controllers such that the power flow of the whole closed-loop system is desirable when no adversarial attacks occur in the local controllers. Since the power flow depends on the angle differences among generators, we construct the local controllers such that the angle difference $y \in \mathbb{R}^5$ tracks a given reference signal,
denoted by $y^d \in \mathbb{R}^5$. More specifically, given $\Sigma$ in (7), we consider an augmented system whose states are $\hat{x}$ and the error between $y$ and $y^d$. For this augmented system, the local controllers $\{\kappa_i\}$ in (4) are designed by LQR design techniques.

To calculate the transient responses of the closed-loop system, we give an initial state of the system and that of the controllers as zero. Furthermore, we give the same reference signal in each subsystem, and each reference signal is taken as a random signal.

In Fig. 4, the blue solid (resp. red dotted) lines show the transient responses (resp. reference signals) of the angle differences of all generators when no attacks occur. We can see from this figure that the transient responses track the reference signals. Furthermore, suppose that the local controllers are modified such that the tracking performance of individual local closed-loop systems gets worse, even though the local closed-loop systems are stable. In Fig. 5, the yellow dash-dotted lines depict the transient responses in this case. We can see from this figure that the instability of the closed-loop system is induced by the attack on the local controllers.

For the augmented networked system, we design $\Phi$ in (8) and (10) by minimizing $\gamma$ in (13), and construct a compensated system $(\Sigma, \Phi)$. The transient responses of the angle differences of all generators in the case of $(\Sigma, \Phi)$ are depicted in Fig. 5 where the legends are the same as those in Fig. 4. Furthermore, the (attacked) local controllers are the same as those shown above. From Fig. 5, even though the performance of the closed-loop system becomes worse when the local controllers are attacked, it should be emphasized that the stability of the whole system is preserved under attacks on local controllers by compensating the networked system by $\Phi$.

Finally, we numerically demonstrate the operation of the compensated power system under attacks on local controllers. To simulate this, we suppose a situation where local controllers are attacked while operating the whole system. We plot transient responses of the angle difference of all generators by the blue solid lines in Fig. 6 during $t \in [0, 200)$. Subsequently, we suppose that an attack occurs in the two local controllers at $t = 200$ such that the tracking performance of individual local closed-loop systems gets worse. We can see from this figure around $t \in [200, 1000)$ that the stability of the whole system is preserved even though the tracking performance gets worse. Finally, we suppose that the controllers are recovered at $t = 1000$. As a result, the tracking performance is recovered. As shown in this numerical demonstration, the guarantee of the whole system stability against attacks on local controllers enables us to recover the controller while operating the whole power system.

V. CONCLUSION

In this paper, we have proposed a method to establish a networked control system that maintains its stability in the presence of certain undesirable incidents on local controllers. We call such networked control systems weakly resilient. To
clarify the class of weakly resilient networked systems, we have
provided a necessary and sufficient condition of weakly
resilient networked systems. However, networked systems do
not generally satisfy the necessary condition shown here in
general. Thus, we have provided a method for designing a
compensator such that the compensated networked system
is weakly resilient. Finally, we have shown the efficiency of
the proposed method through a power system example of the
IEEE 14-bus test system.

In this paper, we have dealt with network systems com-
posed of two subsystems, and shown a necessary and suf-
cient characterization of weakly resilient network systems.

The generalization of this characterization to networked sys-
tems composed of an arbitrary number of subsystems is un-
der investigation. Furthermore, we have shown a fundamental
result of weakly resilient system design under undesirable
incidents on local controllers preserving the stability of the
local closed-loop system. The extension of this result to
incidents destabilizing the system, e.g., the stuxnet attack
[12], is amongst the topics of future works.

APPENDIX

Proof of Theorem 1: The sufficiency is obvious. We show
the necessity, i.e., \( \Sigma \) in (20) is cascade if \( (\Sigma, \{ \kappa_i \}) \) is stable
for any \( \kappa_i \in \mathcal{K}_i \), where \( \Sigma_i \) and \( \kappa_i \) are defined as in (1) and
(4), and \( \mathcal{K}_i \) is defined as in (5).

We first parametrize the \( i \)-th local closed-loop system
\((\Sigma_i, \kappa_i)\) based on the Youla-parametrization in [13] as fol-
lows. Let the input \( u_i \) be composed of \( u_i \) and \( \tilde{u}_i \), satisfying

\[
u_i = u_i + \tilde{u}_i,\]

where \( u_i \) is generated by a controller \( \kappa_i \) stabilizing \((\Sigma_i, \kappa_i)\), i.e.,

\[
\kappa_i : \begin{cases} 
\dot{\xi}_i = (A_i + B_i F_i - H_i C_i) \xi_i + H_i y_i \\
u_i = F_i \xi_i
\end{cases}
\]

with \( F_i \) and \( H_i \) such that \( A_i + B_i F_i \) and \( A_i - H_i C_i \) are
Hurwitz. In addition, \( \tilde{u}_i \) is generated by a controller \( \tilde{\kappa}_i : y_i \rightarrow \tilde{u}_i \),
which corresponds to the free parameter introduced below.
The schematic depiction of the \( i \)-th local closed-loop system
\( \delta_i := (\Sigma_i, \kappa_i) \) is shown in Fig. 7. Let

\[
d_i := z_j, \quad j \neq i
\]

and \( X_i = [x_i^T, \xi_i^T]^T \), we have

\[
(\Sigma_i, \kappa_i) : \begin{cases}
\dot{X}_i = A_i X_i + J_i d_j + B_i \tilde{u}_i \\
y_i = C_i X_i + D_i d_j,
\end{cases}
\]

where

\[
A_i := \begin{bmatrix} A_i & B_i F_i \\ H_i C_i & A_i + B_i F_i - H_i C_i \end{bmatrix}, \quad B_i := \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad J_i := \begin{bmatrix} J_i \\ 0 \end{bmatrix}, \quad S_i := [S_i | 0], \quad C_i := [C_i | 0].
\]

Hence, the closed-loop system \( \delta_i \) can be parametrized as

\[
\delta_i(s; q_i) = \Sigma_i^{dz}(s) + q_i(s) \Sigma_i^{d2}(s) \Sigma_i^{dy}(s),
\]

where

\[
\Sigma_i^{dz}(s) = S_i (sI - A_i)^{-1} J_i, \quad \Sigma_i^{d2}(s) = S_i (sI - A_i)^{-1} B_i, \quad \Sigma_i^{dy}(s) = C_i (sI - A_i)^{-1} J_i + D_i
\]

and \( q_i(s) := \kappa_i(s)(1 - C_i(sI - A_i)^{-1} B_i \kappa_i(s))^{-1} \). Note
that there exists \( \kappa_i(s) \) for any \( q_i(s) \in \mathcal{R} \mathcal{H}_\infty \). Thus, we can
regard \( q_i(s) \in \mathcal{R} \mathcal{H}_\infty \) as a free parameter.

Next, we show a necessary condition on \( q_i \) to guarantee
the stability of the closed-loop system \((\delta_1, \delta_2)\) as shown in
Fig. 8 for any \( q_i \) stabilizing \( \delta_1(s; q_i) \). Since \( \delta_i \) is supposed to
be stable by the assumptions of Theorem 1, it follows from the
Nyquist stability theorem in [13] that the interconnection
\((\delta_1, \delta_2)\) as shown in Fig. 8 is stable if, and only if, all roots of

\[
1 - \delta_1(s; q_1) \delta_2(s; q_2) = 0
\]

are in the open left half complex plane. Based on this fact,
we show the following lemma:

Lemma 1: The closed-loop system \((\delta_1(s; q_1), \delta_2(s; q_2))\) is
stable for any \( q_i \) stabilizing \( \delta_i(s; q_i) \) only if

\[
f(q_1, q_2) := 1 - \delta_1(j \omega; q_1) \delta_2(j \omega; q_2)
\]

is independent of \( q_1(s) \in \mathcal{R} \mathcal{H}_\infty \) and \( q_2(s) \in \mathcal{R} \mathcal{H}_\infty \).

Proof: We first show that there exist \( \omega \geq 0 \) satisfying

\[
f(q_1, q_2) = 0,
\]

if there exist \( \{ q_i \} \) such that \( (\delta_1(s; q_i), \delta_2(s; q_i)) \) is unstable,
i.e., the roots of \( f(q_1, q_2) \) are in the right half plane. Note that
there exist \( \{ q_i \} \) such that the roots of \( f(q_1, q_2) \) are in the open
left half plane because there exist \( \{ \kappa_i \} \), which corresponds
to \( \{ q_i \} \), stabilizing \((\Sigma, \{ \kappa_i \})\). Hence, due to the continuity
of roots, there exists \( s = j \omega \) satisfying \( f(q_1, q_2) = 0 \).

Fig. 7. The \( i \)-th local closed-loop system

Fig. 8. Networked system with two local controllers
Next, we show the claim by showing the contraposition. Namely, supposing that \( f(q_1, q_2) \) depends on \( q_1 \), we show that there exist \( \omega \geq 0 \) and \( q_1(s) \in RH_\infty \) such that \( (\delta_1, \delta_2) \) is unstable, i.e., \( f(q_1, q_2) = 0 \). It follows from the fundamental theorem of algebra that there exist \( \omega \geq 0 \) and \( \tilde{q}_i \in \mathbb{C} \) satisfying

\[
1 - \tilde{\delta}_1 \tilde{\delta}_2 = 0
\]

where \( \tilde{\delta}_i \in \mathbb{C} \) is defined as

\[
\tilde{\delta}_i := \Sigma_i dz(j\omega) + \tilde{q}_i \Sigma_i \tilde{z}(j\omega) \Sigma_i du(j\omega), \quad \tilde{q}_i \in \mathbb{C}.
\]

Hence, it suffices to show that there exists a function \( q(s) \in RH_\infty \) satisfying \( q(j\omega) = \tilde{q} \) for a given \( \tilde{q} \in \mathbb{C} \) and \( \omega \geq 0 \). Give

\[
q(s) = k \left( \frac{s - a}{s + a} \right)^2
\]

with \( k > 0 \) and \( a > 0 \). Since the \( q(s) \) is an all-pass filter rotating \( 2\pi \) rad while having gain \( k \), there exist \( k \) and \( a \) satisfying \( q(j\omega) = \tilde{q} \). This completes the proof.

It follows from Lemma 11 that we have

\[
\frac{\partial f}{\partial q_2}(0, q_2) = \Sigma_1 dz(j\omega) \Sigma_2 \tilde{z}(j\omega) \Sigma_2^d(j\omega) = 0,
\]

where the first equality follows from the definition of \( f \) in (21) and (22). Thus, one of the transfer functions \( \Sigma_1 dz(j\omega) \), \( \Sigma_2 \tilde{z}(j\omega) \), and \( \Sigma_2^d(j\omega) \) is zero. Finally, we show the analysis of three cases.

**Case 1:** Suppose \( \Sigma_1 dz(j\omega) = 0 \), i.e.,

\[
S_1(j\omega I - A_1)^{-1}J_1 = 0.
\]

Now, we introduce the following lemma:

**Lemma 2:** Give \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^n \) such that \((A, B)\) is controllable, and \( C \in \mathbb{R}^{1 \times n}, R \in \mathbb{R}^n \) and \( D \in \mathbb{R} \). Then,

\[
C(j\omega I - (A + BF))^{-1}R + D = 0
\]

holds for any \( \omega \geq 0 \) and \( F \in \mathbb{R}^{1 \times n} \) such that \( A + BF \) is Hurwitz if, and only if, \( D = 0 \), and \( C = 0 \) or \( R = 0 \).

**Proof:** We only show the only if part. First, \( D = 0 \) follows from (25) because

\[
\lim_{\omega \to \infty} (j\omega I - A)^{-1} = 0.
\]

Next, we show \( C = 0 \) or \( R = 0 \). Note that \( F = F - \tilde{F} + \tilde{F} \) for any \( \tilde{F} \in \mathbb{R}^{1 \times n} \). Since \((A, B)\) is controllable if and only if \((A + B(F - \tilde{F}), B)\) is controllable, no assumptions on stabilizability of \( F \in \mathbb{R}^{1 \times n} \) are required without loss of generality. It follows from the Laurent expansion that

\[
(j\omega I - A)^{-1} = \sum_{k=1}^{\infty} (j\omega)^{-k} A^{k-1}.
\]

Hence, taking the derivative of (25) with respect to \( \omega \), we have

\[
C(A + BF)^{-k-1}R = 0
\]

for all \( k \geq 1 \) and \( F \in \mathbb{R}^{1 \times n} \). Taking \( F = 0 \), we have \( CA^{k-1}R = 0 \) for all \( k \geq 1 \). Furthermore, let \( k = 2 \) in (26).

We have

\[
CBFR = 0
\]

for all \( F \in \mathbb{R}^{1 \times n} \). Taking the derivative of (27) with respect to \( F \), we have \( RCB = 0 \). Let \( k = 3 \) in (26). It follows from \( CBF = tr(CBFAR) = tr(RCBFA) = 0 \) and \( CA^2R = 0 \) that

\[
CBF + (BF)^2R = 0
\]

Taking the derivative with respect to \( F \) and \( F = 0 \), we have

\[
RCAB = 0.
\]

Taking a similar procedure for \( k > 3 \), we have

\[
RC[B, AB, \ldots, A^{n-1}B] = 0.
\]

Since the pair of \((A_1, B_1)\) is controllable, we have \( RC = 0 \), which is equivalent to \( R = 0 \) or \( C = 0 \). This completes the proof.

Note that \( S_1(j\omega I - A_1)^{-1}J_1 \) in (24) is transformed into

\[
\left| S_1 - S_1 \right| \left( J_1 \right) = \left( \left| S_1 \right| - \left| S_1 \right| \right)^{-1} \left( I \right) \left( J_1 \right).
\]

Taking \( F_1 = 0 \) and using Lemma 2, (24) shows that \( S_1 = 0 \) or \( J_1 = 0 \). Note that \( S_1 = 0 \) yields (6). Thus, in what follows, we focus on the case that \( J_1 = 0 \). Now, we have

\[
f(q_1, q_2) = 1 - q_1 \Sigma_1^{a}D_1 \Sigma_2^{d} - q_1 q_2 \Sigma_1^{a}D_1 \Sigma_2^{d} \Sigma_2^{d}.
\]

Hence, the independency of \( f(q_1, q_2) \) from \( q_1 \) and \( q_2 \) is equivalent to

\[
\Sigma_1^{a}(j\omega)D_1 \Sigma_2^{d}(j\omega) = 0
\]

and

\[
\Sigma_1^{a}(j\omega)D_1 \Sigma_2^{d}(j\omega) \Sigma_2^{d}(j\omega) = 0.
\]

We show the based on the case analysis as follows:

**Case 1a:** Suppose \( \Sigma_1^{a}(j\omega) = 0 \). Note that \( B_1 \neq 0 \) follows because \((A_1, B_1)\) is controllable. Thus, \( \Sigma_1^{a}(j\omega) = 0 \) is equivalent to \( S_1 = 0 \), which yields (6).

**Case 1b:** Suppose \( D_1 = 0 \), which yields (6).

**Case 1c:** Suppose \( \Sigma_2^{d}(j\omega)D_1 \neq 0 \). Then, (28) yields.

**Case 2:** Suppose \( \Sigma_2^{d}(j\omega) = 0 \). Similar to case 1.

**Case 3:** Suppose \( \Sigma_2^{d}(j\omega) = 0 \). Similar to case 1.

Therefore, we complete the proof of Theorem 1.

**REFERENCES**

[1] A. Giani, S. Sastry, K. H. Johansson, and H. Sandberg, "The VIKING project: an initiative on resilient control of power networks," in Proc. of International Symposium on Resilient Control Systems, 2009, pp. 31–35.

[2] P. Kundur, Power system stability and control. McGraw-Hill Education, 1994.

[3] M. G. H. Bell and Y. Iida, Transportation network analysis. Wiley, 1997.

[4] E. D. Knapp and J. T. Langill, Industrial Network Security: Securing critical infrastructure networks for smart grid, SCADA, and other Industrial Control Systems. Springer, 2014.

[5] C. G. Rieger, D. Gertman, and M. McQueen, “Resilient control systems: next generation design research,” in Proc. of International Conference on Human System Interactions, 2009, pp. 632–636.
[6] D. Wei and K. Ji, “Resilient industrial control system (RICS): Concepts, formulation, metrics, and insights,” in *Proc. of International Symposium on Resilient Control Systems*, 2010, pp. 15–22.

[7] Y. Yuan, Q. Zhu, F. Sun, Q. Wang, and T. Basar, “Resilient control of cyber-physical systems against denial-of-service attacks,” in *Proc. of International Symposium on Resilient Control Systems*, 2013, pp. 54–59.

[8] T. Sadamoto, T. Ishizaki, and J. Imura, “Hierarchical distributed control for networked linear systems,” in *Proc. Conference on Decision and Control*, 2014, pp. 2447–2452.

[9] R. D. Zimmerman, C. E. Murillo-Sánchez, and R. J. Thomas, “Matpower: Steady-state operations, planning, and analysis tools for power systems research and education,” *IEEE Transactions on Power Systems*, vol. 26, no. 1, pp. 12–19, 2011.

[10] A. S. Morse and W. Wonham, “Status of noninteracting control,” *IEEE Transactions on Automatic Control*, vol. 16, no. 6, pp. 568–581, 1971.

[11] J. C. Willems and C. Commault, “Disturbance decoupling by measurement feedback with stability or pole placement,” *SIAM Journal on Control and Optimization*, vol. 19, no. 4, pp. 490–504, 1981.

[12] R. Langner, “Stuxnet: Dissecting a cyberwarfare weapon,” *Security & Privacy, IEEE*, vol. 9, no. 3, pp. 49–51, 2011.

[13] K. Zhou, J. C. Doyle, and K. Glover, *Robust and optimal control*. Prentice Hall New Jersey, 1996, vol. 40.