TOEPLITZ ALGEBRAS OVER FOCK AND BERGMAN SPACES

SHENGKUN WU¹ AND XIANFENG ZHAO²

ABSTRACT. In this paper, we investigate Toeplitz subalgebras generated by certain class of Toeplitz operators on the $p$-Fock space and the $p$-Bergman space with $1 < p < \infty$. Let $\text{BUC}(\mathbb{C}^n)$ and $\text{BUC}(\mathbb{B}_n)$ denote the collections of bounded uniformly continuous functions on $\mathbb{C}^n$ and $\mathbb{B}_n$ (the unit ball in $\mathbb{C}^n$), respectively. On the $p$-Fock space, we show that the Toeplitz algebra which has a translation invariant closed subalgebra of $\text{BUC}(\mathbb{C}^n)$ as its set of symbols is linearly generated by Toeplitz operators with the same space of symbols. This answers an open question recently raised by Fulsche [3]. On the $p$-Bergman space, we study Toeplitz algebras with symbols in some translation invariant closed subalgebras of $\text{BUC}(\mathbb{B}_n)$. In particular, we obtain that the Toeplitz algebra generated by all Toeplitz operators with symbols in $\text{BUC}(\mathbb{B}_n)$ is equal to the closed linear space generated by Toeplitz operators with such symbols. This generalizes the corresponding result for the case of $p = 2$ obtained by Xia [12], which was proven by a different approach.

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1. INTRODUCTION

We begin with some basic notations and knowledge about Fock spaces, Bergman spaces and Toeplitz operators on such spaces. For a positive parameter $t$, let

$$d\mu_t(z) = \frac{1}{(\pi t)^n} e^{-|z|^2 t} dV(z)$$

be the Gaussian measure on $\mathbb{C}^n$, where $dV(z)$ denotes the Lebesgue measure on $\mathbb{C}^n$. For $1 < p < \infty$, the $p$-Fock space $F^p_t$ is the space of entire functions $f$ on $\mathbb{C}^n$ which are $p$-integrable with respect to the Gaussian measure with a parameter $\frac{2t}{p} > 0$, i.e.,

$$\|f\|_{F^p_t} = \left[ \int_{\mathbb{C}^n} |f(z)|^p d\mu_{2t/p}(z) \right]^{\frac{1}{p}} < \infty.$$

Let $q$ be the conjugate number of $p$, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual space of $F^p_t$ is $F^q_t$ and the duality pairing is given by

$$\langle f, g \rangle_t = \int_{\mathbb{C}^n} f(z) \overline{g(z)} d\mu_t(z).$$
For $p = 2$, it is well-known that $F^2_t$ is a reproducing kernel Hilbert space with the reproducing kernel

$$K^t_z(w) = e^{\frac{w \cdot \overline{z}}{t}}, \quad z, w \in \mathbb{C}^n,$$

where

$$w \cdot \overline{z} = w_1 \overline{z_1} + \cdots + w_n \overline{z_n}$$

for $w = (w_1, \ldots, w_n)$ and $z = (z_1, \ldots, z_n)$. The normalized reproducing kernel is given by

$$k^t_z(w) = \frac{K^t_z(w)}{\|K^t_z\|_{F^2_t}} = e^{\frac{w \cdot \overline{z} - |z|^2}{t}}, \quad z, w \in \mathbb{C}^n.$$

Moreover, one can check easily that

$$\|f\|_{F^p_t} = \left[ \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} |\langle f, k^t_z \rangle_t|^p dV(z) \right]^{\frac{1}{p}}$$

for $1 \leq p < \infty$.

With the normalized reproducing kernel for the Fock space $F^2_t$, the Berezin transform of an operator $T$ on $F^p_t$ is defined by

$$\tilde{T}(z) = \langle T k^t_z, k^t_z \rangle_t, \quad z \in \mathbb{C}^n.$$

For $\varphi \in \mathcal{L}^\infty(\mathbb{C}^n, dV)$, the Toeplitz operator $T_\varphi$ with symbol $\varphi$ on $F^p_t$ is defined by

$$T_\varphi f(z) = \int_{\mathbb{C}^n} \varphi(w) f(w) K^t_z(w) d\mu_t(w).$$

Then $T_\varphi$ is bounded on $F^p_t$, i.e., $T_\varphi$ is bounded from $F^p_t$ to $F^p_t$.

Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ denote the open unit ball in $\mathbb{C}^n$. Let $dv$ denote the normalized Lebesgue measure on $\mathbb{B}_n$ with the normalization $v(\mathbb{B}_n) = 1$. For $1 < p < \infty$, the $p$-Bergman space $L^p_a$ is the collection of all holomorphic functions that are $p$-integrable with respect to $dv$. The norm on the Banach space $L^p_a$ is given by

$$\|f\|_p = \left[ \int_{\mathbb{B}_n} |f(z)|^p dv(z) \right]^{\frac{1}{p}}.$$

Furthermore, the dual space of $L^p_a$ is $L^q_a$ under the standard duality pairing:

$$\langle f, g \rangle = \int_{\mathbb{B}_n} f(z) \overline{g(z)} dv(z),$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Recall that the reproducing kernel for the Bergman space $L^2_a$ is given by

$$K_z(w) = \frac{1}{(1 - w \cdot \overline{z})^{n+1}}, \quad z, w \in \mathbb{B}_n.$$

Similar to the setting of $p$-Fock space, the Toeplitz operator $T_\varphi$ with symbol $\varphi \in \mathcal{L}^\infty(\mathbb{B}_n, dv)$ on $L^p_a$ can also be defined via the reproducing kernel:

$$T_\varphi f(z) = \int_{\mathbb{B}_n} \varphi(w) f(w) K_z(w) dv(w),$$

which is bounded on $L^p_a$.

Toeplitz algebras over Bergman spaces and Fock spaces have been investigated by many authors for a long time, see for example [2], [9], [10], [1], [11], [5], [12], [13], [14], [3] and [4]. Let us fix more notations before going to review the background about the investigation of Toeplitz algebras over these two function spaces.

By the full Toeplitz algebra, we mean that the Banach algebra generated by all Toeplitz operators with (essentially) bounded symbols. Let $\mathcal{J}$ be a subset of $\mathcal{L}^\infty(\mathbb{C}^n, dv)$ and $\mathcal{T}(\mathcal{J})$ be the Banach algebra generated by $\{T_\varphi : \varphi \in \mathcal{J}\}$. In this paper, we shall call $\mathcal{T}(\mathcal{J})$ the Toeplitz algebra generated by all Toeplitz operators with symbols in $\mathcal{J}$. Moreover, we use $\mathcal{T}_{\text{lin}}(\mathcal{J})$ to denote the closed linear space generated by Toeplitz operators with symbols in $\mathcal{J}$. Let $\text{BUC}(\mathbb{C}^n)$ denote the set of bounded functions on $\mathbb{C}^n$ which are
uniformly continuous with respect to the Euclidean metric. For \( f \in F^p_t \) and \( z \in \mathbb{C}^n \), the translation \( \alpha_z \) is defined by
\[
(\alpha_z f)(w) := f(w + z), \quad w \in \mathbb{C}^n.
\]
We say that \( \mathcal{J} \subset \text{BUC}(\mathbb{C}^n) \) is translation invariant on \( \mathbb{C}^n \) if \( \alpha_z f \in \mathcal{J} \) for all \( f \in \mathcal{J} \) and \( z \in \mathbb{C}^n \). Note that \( \text{BUC}(\mathbb{C}^n) \) is translation invariant. In addition, the Berezin transform of any bounded linear operator on \( F^p_t \) is also in \( \text{BUC}(\mathbb{C}^n) \), see [3, Lemma 2.8] if necessary.

In the setting of the \( p \)-Bergman space, let \( \mathcal{I} \) be a subset of \( L^\infty(\mathbb{B}_n, dv) \) and \( \mathcal{T}^b(\mathcal{I}) \) be the Banach algebra generated by \( \{ T_\varphi : \varphi \in \mathcal{I} \} \). We denote \( \mathcal{T}^b(\mathcal{I}) \) by the closed linear space generated Toeplitz operators with symbols in \( \mathcal{I} \). We use \( \text{BUC}(\mathbb{B}_n) \) to denote the set of bounded functions on \( \mathbb{B}_n \) which are uniformly continuous with respect to the Bergman metric (which will be introduced in Section 3). For \( f \in \text{BUC}(\mathbb{B}_n) \) and \( z \in \mathbb{B}_n \), we define \( \tau_z \) on \( \text{BUC}(\mathbb{B}_n) \) as
\[
(\tau_z f)(u) = f(\varphi_u(z)), \quad u \in \mathbb{B}_n,
\]
where \( \varphi_u \) is the Möbius transform of \( \mathbb{B}_n \) that interchanges 0 and \( u \). It is worth noting that \( \tau_z f \) is not defined by \( \tau_z f = f \circ \varphi_z \). However, this “unnatural” definition of \( \tau_z \) in (1.1) plays an important role in the study of some Toeplitz subalgebras over the Bergman space of \( \mathbb{B}_n \). For a subset \( \mathcal{I} \) of \( \text{BUC}(\mathbb{B}_n) \), we say that \( \mathcal{I} \) is translation invariant on \( \mathbb{B}_n \) if \( \tau_z f \in \mathcal{I} \) for all \( f \in \mathcal{I} \) and \( z \in \mathbb{B}_n \).

In 2007, Suárez [10] obtained that the full Toeplitz algebra over the \( p \)-Bergman space \( L^p_\alpha \) equals the Toeplitz algebra with symbols in \( \text{BUC}(\mathbb{B}_n) \) via the \( n \)-Berezin transform. Then Bauer and Isralowitz [1] established a similar result for the full Toeplitz algebra over weighted Fock spaces. In [11], Xia and Zheng defined the sufficiently localized operator on \( F^2_t \) and characterized the compactness of operators in the \( C^* \)-algebra generated by sufficiently localized operators. Furthermore, they obtained that the \( C^* \)-algebra generated by sufficiently localized operators contains the full Toeplitz algebra over the Fock space \( F^2_t \). Later, Isralowitz, Mitkovski and Wick [5] introduced the weakly localized operators on \( L^p_\alpha \) (and also \( F^p_t \)) and showed that such class of operators forms an algebra and its closure also contains the full Toeplitz algebra over \( L^p_\alpha \) (\( F^p_t \)). Based on the study of the \( C^* \)-algebra generated by weakly localized operators, Xia [12] showed that the full Toeplitz algebra over \( L^2_\alpha \) is equal to the norm closure of \( \{ T_\varphi : \varphi \in L^\infty(\mathbb{B}_n, dv) \} \). Indeed, in the cases of the Bergman space \( L^2_\alpha \) and the Fock space \( F^2_t \), Xia [12] obtained that the full Toeplitz algebra coincides with the \( C^* \)-algebra generated by the class of weakly localized operators. In 2020, using a correspondence theory of translation invariant symbol on \( \mathbb{C}^n \) and operator spaces, Fulsche [3] showed that the full Toeplitz algebra over the \( p \)-Fock space \( F^p_t \) is the norm closure of all Toeplitz operators with symbols in \( \text{BUC}(\mathbb{C}^n) \), which generalizes the result obtained by Xia [12] in the case of \( p = 2 \). Moreover, Fulsche [3] proved that every Toeplitz algebra which has a translation and \( U \)-invariant \( C^* \)-subalgebra of \( \text{BUC}(\mathbb{C}^n) \) as its set of symbols is linearly generated by Toeplitz operators with the same space of symbols. Recently, by using a characterization of Toeplitz algebras over the \( p \)-Fock space \( F^p_t \) [3], Hagger [4] established that the full Toeplitz algebra over \( F^p_t \) coincides with each of the algebras generated by band-dominated, sufficiently localized and weakly localized operators, respectively.

In this paper, we will mainly consider the Toeplitz algebras generated by Toeplitz operators with symbols in some translation invariant closed subalgebras of \( \text{BUC}(\mathbb{C}^n) \) and \( \text{BUC}(\mathbb{B}_n) \) on the \( p \)-Fock space and the \( p \)-Bergman space, respectively. Recall that the orthonormal basis of \( L^2_\alpha \) and the integral representation in terms of certain sum of rank-one operators over a lattice for a class of Toeplitz operators are the crucial ingredients in Xia’s approach [12]. Although the techniques used in [12] can not be applied directly to treat Toeplitz operators with translation invariant symbols on the Banach spaces \( L^p_\alpha \) and \( F^p_t \), some useful ideas (such as the usage of separated sets in the unit ball, weakly localized operators and integral representations for Toeplitz operators) provide a great inspiration to us. Noting that the automorphism group of \( \mathbb{C}^n \) is commutative, thus the convolution of two functions (or two operators) has some nice properties on Fock spaces and which can be used to approximate an operator in the Toeplitz algebra by a sequence of Toeplitz operators [3]. However, the automorphism group of \( \mathbb{B}_n \) is not commutative, so the techniques used in [3] may not do work for the case of Bergman spaces over the unit ball.
In order to characterize Toeplitz algebras with symbols in a subset of $\text{BUC}(\mathbb{C}^n)$ ($\text{BUC}(\mathbb{B}_n)$) over the $p$-Fock space ($p$-Bergman space), we will first establish an integral representation for weakly localized operators on the $p$-Fock space ($p$-Bergman space). Based on this integral representation, we are able to further study weakly localized operators on $F_t^p$ and $L_u^p$ via the Berezin transform. We now give a short outline of the rest of the paper. In Section 2, we show that the Toeplitz algebra generated by Toeplitz operators on the $p$-Fock space $F_t^p$ with symbols in a translation invariant closed subalgebra of $\text{BUC}(\mathbb{C}^n)$ is linearly generated by Toeplitz operators with the same space of symbols, see Theorem 2.1. This answers an open question posed in [3] by using different methods. In Section 3, we obtain an integral representation for $s$-weakly localized operators (see Definition 3.2) on the $p$-Bergman space $L_u^p$, see Theorem 3.3. Then we apply this integral representation to study Toeplitz algebras with symbols in some translation invariant subalgebra $\mathcal{I} \subset \text{BUC}(\mathbb{B}_n)$ in Section 4. In particular, we obtain in Theorem 4.9 that $\mathcal{T}^b[\text{BUC}(\mathbb{B}_n)]$ and $\mathcal{T}_{\text{lin}}[\text{BUC}(\mathbb{B}_n)]$ are both equal to the norm closure of the collection of $s$-weakly localized operators on $L_u^p$; $\mathcal{T}^b[C_0(\mathbb{B}_n)]$ and $\mathcal{T}_{\text{lin}}[C_0(\mathbb{B}_n)]$ are both equal to the ideal of compact operators on $L_u^p$, where $C_0(\mathbb{B}_n)$ is a (translation invariant) subalgebra of $\text{BUC}(\mathbb{B}_n)$ consisting of functions $f$ with $f(z) \to 0$ as $|z| \to 1$. This generalizes a result obtained by Xia in the case $p = 2$, see [12, Theorem 1.5].

In the following, $T^*$ denotes the Banach space adjoint of the bounded linear operator $T$ on $F_t^p$ or $L_u^p$. In addition, the notation $A \lesssim B$ for two nonnegative quantities $A$ and $B$ means that there is some inessential constant $C > 0$ such that $A \leq CB$.

2. Toeplitz algebras over the $p$-Fock space

This section is devoted to the study of the Toeplitz algebra $\mathcal{T}(\mathcal{J})$ over $F_t^p$ with $1 < p < \infty$, where $\mathcal{J}$ is a translation invariant closed subalgebra of $\text{BUC}(\mathbb{C}^n)$. The main result of this section is the following theorem, which shows the assumption that the $U$-invariance of $\mathcal{J}$ (i.e., $U\mathcal{J} \subset \mathcal{J}$, where $Uf(z) = f(-z)$ for $f \in \mathcal{J}$ and $z \in \mathbb{C}^n$) and the self-adjointness of $\mathcal{J}$ are in fact unnecessary. This answers an open problem recently raised by Fulsche in [3, page 39].

Theorem 2.1. Let $\mathcal{J}$ be a translation invariant closed subalgebra of $\text{BUC}(\mathbb{C}^n)$, then

$$\mathcal{T}(\mathcal{J}) = \mathcal{T}_{\text{lin}}(\mathcal{J})$$

holds on $F_t^p$. Moreover, if $\mathcal{J}_1$ is a translation invariant closed ideal of $\mathcal{J}$, then $\mathcal{T}(\mathcal{J}_1)$ is a two-sided ideal in $\mathcal{T}(\mathcal{J})$.

The proof of the above theorem relies on the integral representation for weakly localized operators on $F_t^p$. Let us recall the definition of this class of operators on the $p$-Fock space $F_t^p$, which was first introduced by Isralowitz, Mitkovski and Wick [5, Definition 1.1].

Definition 2.2. Let $T$ be a bounded linear operator on $F_t^p$ and $T^*$ be the Banach space adjoint of $T$. Then $T$ is said to be weakly localized if it satisfies the following four conditions:

$$\sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} |\langle Tk_z^t, k_w^t \rangle| dV(w) \leq \infty, \quad \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} |\langle T^*k_z^t, k_w^t \rangle| dV(w) \leq \infty,$$

$$\lim_{r \to \infty} \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n \setminus B(z,r)} |\langle Tk_z^t, k_w^t \rangle| dV(w) = 0,$$

where $B(z,r)$ denotes the ball $\{ w \in \mathbb{C}^n : |w-z| < r \}$ in $\mathbb{C}^n$ with center $z$ and radius $r$. We denote the collection of weakly localized operators on $F_t^p$ by $\mathcal{A}^p$.

As the proof of Theorem 2.1 is long, we will divide the proof into several steps and complete the details in the end of this section. The main three steps of our approach are the following.

(1) First, we establish an integral representation for weakly localized operators on $F_t^p$ (Theorem 2.5);
(2) Based on the integral representation mentioned above, we show that a weakly localized operator belongs to the space \( T_{\text{lin}}(J) \) if its Berezin transform is in \( J \) (Proposition 2.8);

(3) For a finite product of Toeplitz operators with symbols in \( J \), we show that its Berezin transform belongs to \( J \) (Proposition 2.9).

Noting that the finite sum of finite products of Toeplitz operators with symbols in \( J \) is weakly localized, and \( T_{\text{lin}}(J) \subset T(J) \), then Theorem 2.1 will follow immediately from (1)–(3).

In order to establish an integral representation for weakly localized operators on \( F^p_t \), first we need to recall some properties of Weyl operators on Fock spaces. For \( z \in \mathbb{C}^n \), the Weyl operator \( W_z \) is defined by

\[
W_z f(w) = k_z^t(w)f(w - z), \quad w \in \mathbb{C}^n,
\]

where \( f \in F^p_t \). It is easy to check that the following hold for all \( z, w \in \mathbb{C}^n \): \( W_z^* = W_{-z} \) and

\[
W_w W_z = e^{-i\frac{\Im(w - z)}{t}} W_{w+z}, \quad W_w k_z^t = e^{-i\frac{\Im(w - z)}{t}} k_{w+z}^t, \quad \|W_z f\|_{F^p_t} = \|f\|_{F^p_t}.
\]

(2.1) For more information on Weyl operators, one can consult Chapter 2 of [15].

Let us begin with the following lemma.

Lemma 2.3. Let \( T \) be a bounded linear operator on \( F^p_t \). Suppose that \( h_0 \in L^\infty(\mathbb{C}^n, dV) \) and \( h_1, h_2 \in F^1_t \), then we have

\[
\sup_{u \in \mathbb{C}^n} |\langle TW_u h_1, W_u h_2 \rangle_t| \lesssim \|T\| \|h_1\|_{F^1_t} \|h_2\|_{F^1_t}
\]

and

\[
\left| \int_{\mathbb{C}^n} h_0(u) \langle f, W_u h_1 \rangle_t \langle W_u h_2, g \rangle_t dV(u) \right| \lesssim \|h_0\|_{L^\infty} \|h_1\|_{F^1_t} \|h_2\|_{F^1_t} \|f\|_{F^p_t} \|g\|_{F^p_t}
\]

for all \( f \in F^p_t \) and \( g \in F^p_t \).

Proof. First, we have

\[
|\langle TW_u h_1, W_u h_2 \rangle_t| \leq \|T\| \|W_u h_1\|_{F^p_t} \|W_u h_2\|_{F^p_t}
= \|T\| \|h_1\|_{F^1_t} \|h_2\|_{F^1_t}
\lesssim \|T\| \|h_1\|_{F^1_t} \|h_2\|_{F^1_t},
\]

where the last inequality follows from [15, Theorem 2.10]. Recall that the identity operator can be written as

\[
I = \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} (k_z^t \otimes k_z^t) dV(z)
\]

on \( F^p_t \), where

\[
(k_z^t \otimes k_z^t)f = \langle f, k_z^t \rangle_t k_z^t
\]

for \( f \in F^p_t \). Then we have

\[
\langle f, W_u h_1 \rangle_t = \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} \langle f, k_z^t \rangle_t \langle k_z^t, W_u h_1 \rangle_t dV(z).
\]

This gives us that

\[
\left| \int_{\mathbb{C}^n} h_0(u) \langle f, W_u h_1 \rangle_t \langle W_u h_2, g \rangle_t dV(u) \right|
\lesssim \int_{\mathbb{C}^n} |h_0(u)| \int_{\mathbb{C}^n} |\langle f, k_z^t \rangle_t| |\langle k_z^t, W_u h_1 \rangle_t dV(z)| \int_{\mathbb{C}^n} |\langle W_u h_2, k_w^t \rangle_t| |\langle k_w^t, g \rangle_t dV(w)dV(u)|
\leq \|h_0\|_{L^\infty} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle f, k_z^t \rangle_t| \left( \int_{\mathbb{C}^n} |\langle k_z^t, W_u h_1 \rangle_t| |\langle W_u h_2, k_w^t \rangle_t dV(u)| \right) |\langle k_w^t, g \rangle_t dV(w)dV(z).
\]
Denoting
\[ F(z, w) := \int_{\mathbb{C}^n} |\langle k_z, W_u h_1 \rangle_t| |\langle W_u h_2, k_w \rangle_t| dV(u), \]
then Hölder’s inequality gives
\[
\int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle f, k_z \rangle_t| \left( \int_{\mathbb{C}^n} |\langle k_z, W_u h_1 \rangle_t| |\langle W_u h_2, k_w \rangle_t| dV(u) \right) |\langle k_w, g \rangle_t| dV(w) dV(z)
\]
\[
= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle f, k_z \rangle_t| F(z, w) |\langle k_w, g \rangle_t| dV(w) dV(z)
\]
\[
= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle f, k_z \rangle_t| F(z, w) \frac{1}{p} F(z, w)^{\frac{1}{p}} dV(z) |\langle k_w, g \rangle_t| dV(w)
\]
\[
\leq \left[ \int_{\mathbb{C}^n} \left( \int_{\mathbb{C}^n} |\langle f, k_z \rangle_t| F(z, w)^{\frac{1}{p}} F(z, w)^{\frac{1}{q}} dV(z) \right)^p dV(w) \right]^{\frac{1}{p}} \|g\|_{F_t^q}^q
\]
\[
\leq \left[ \int_{\mathbb{C}^n} \left( \int_{\mathbb{C}^n} |\langle f, k_z \rangle_t|^p F(z, w) dV(z) \right) \left( \int_{\mathbb{C}^n} F(z, w) dV(z) \right)^{\frac{q}{p}} dV(w) \right]^{\frac{1}{q}} \|g\|_{F_t^q}^q
\]
\[
\leq \|f\|_{F_t^p} \|g\|_{F_t^q} \left[ \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} F(z, w) dV(w) \right]^{\frac{1}{p}} \left[ \sup_{w \in \mathbb{C}^n} \int_{\mathbb{C}^n} F(z, w) dV(z) \right]^{\frac{1}{q}}.
\]
Furthermore, we have that
\[
\sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} F(z, w) dV(w) \leq \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} |\langle k_z, W_u h_1 \rangle_t| |\langle W_u h_2, k_w \rangle_t| dV(u) dV(w)
\]
\[
= \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle k_z, h_1 \rangle_t| |\langle h_2, k_w \rangle_t| dV(u) dV(w)
\]
\[
= \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} |\langle k_z, h_1 \rangle_t| \int_{\mathbb{C}^n} |\langle h_2, k_w \rangle_t| dV(u) dV(w)
\]
\[
\leq \|h_1\|_{F_t^1} \|h_2\|_{F_t^1},
\]
where the last inequality follows from that
\[
\int_{\mathbb{C}^n} |\langle \psi, k_z \rangle_t| dV(\lambda) = (\pi t^n \|\psi\|_{F_t^1}
\]
for \(\psi \in F_t^1\) and \(\lambda \in \mathbb{C}^n\). This yields that
\[
\left| \int_{\mathbb{C}^n} h_0(u) \langle f, W_u h_1 \rangle_t \langle W_u h_2, g \rangle_t dV(u) \right| \lesssim \|h_0\|_{\infty} \|h_1\|_{F_t^1} \|h_2\|_{F_t^1} \|f\|_{F_t^p} \|g\|_{F_t^q}.
\]
This completes the proof of the lemma. \(\square\)

For each \(h_0 \in L^\infty(\mathbb{C}^n, dV)\) and \(h_1, h_2 \in F_t^1\), we define the operator
\[
\int_{\mathbb{C}^n} h_0(u)(W_u h_1 \otimes W_u h_2) dV(u)
\]
on \(F_t^p\) by
\[
\left\langle \int_{\mathbb{C}^n} h_0(u)(W_u h_1 \otimes W_u h_2) dV(u), g \right\rangle_t = \int_{\mathbb{C}^n} h_0(u) \langle f, W_u h_2 \rangle_t \langle W_u h_1, g \rangle_t dV(u),
\]
where \(f \in F_t^p\) and \(g \in F_t^q\). Note that Lemma 2.3 guarantees that this operator is bounded.

The following lemma is elementary, but we include a proof here for the sake of completeness.
Lemma 2.4. For any \( r > 0 \) and \( w, w' \in B(0, r) \), we have that
\[
\| k^t_w - k^t_{w'} \|_{F^1_t} \leq C_r |w - w'|
\]
for some positive constant \( C_r \) depending only on \( r \).

Proof. Using the definition of \( W_w \) and (2.1), we have
\[
\| k^t_w - k^t_{w'} \|_{F^1_t} = \| W_w 1 - W_w e^{\frac{\text{Im} (w' - w)}{t}} k^t_{w' - w} \|_{F^1_t}
\]
\[
\leq \| 1 - e^{\frac{\text{Im} (w' - w)}{t}} k^t_{w' - w} \|_{F^1_t} + \| 1 - k^t_{w' - w} \|_{F^1_t}
\]
\[
\leq \| 1 - e^{\frac{\text{Im} (w' - w)}{t}} \|_{F^1_t} + \| 1 - k^t_{w' - w} \|_{F^1_t}
\]
\[
\leq \| 1 - e^{\frac{\text{Im} (w' - w)}{t}} \|_{F^1_t} + \| 1 - k^t_{w' - w} \|_{F^2_{2t}}
\]
\[
\leq \| 1 - e^{\frac{\text{Im} (w' - w)}{t}} \|_{F^1_t} + \| 1 - k^t_{w' - w} \|_{F^2_{2t}}
\]
\[
= \| 1 - e^{\frac{\text{Im} (w' - w)}{t}} \|_{F^1_t} + \| 1 - k^t_{w' - w} \|_{F^2_{2t}}
\]
\[
\leq C_r |w - w'|,
\]
where the third inequality follows from that \( k^t_{w' - w} = k^t_{w' - w} \).

Based on the previous two lemmas, we are able to establish an integral representation for weakly localized operators on \( F^p_t \).

Theorem 2.5. Let \( A \) be a bounded linear operator on \( F^p_t \). Then for each \( r > 0 \), the mapping
\[
w \mapsto \int_{C^n} \langle AW_z k^t_0, W_z k^t_{w'} \rangle_t (W_z k^t_0) \otimes (W_z k^t_{w'}) dV(z) = \int_{C^n} \langle Ak^t_z, k^t_{z+w} \rangle_t (k^t_{z+w} \otimes k^t_z) dV(z)
\]
is uniformly continuous and uniformly bounded from \( B(0, r) \) to the set of bounded linear operators on \( F^p_t \). Moreover, the integral
\[
\int_{B(0,r)} \int_{C^n} \langle Ak^t_z, k^t_{z+w} \rangle_t (k^t_{z+w} \otimes k^t_z) dV(z) dV(w)
\]
is convergent in the norm topology. Furthermore, if \( A \) is weakly localized on \( F^p_t \), then
\[
\frac{1}{(\pi t)^{2n}} \int_{B(0,r)} \int_{C^n} \langle Ak^t_z, k^t_{z+w} \rangle_t (k^t_{z+w} \otimes k^t_z) dV(z) dV(w)
\]
converges to \( A \) in norm as \( r \to \infty \).

Proof. We obtain by (2.1) that
\[
\int_{C^n} \langle AW_z k^t_0, W_z k^t_{w'} \rangle_t (W_z k^t_0) \otimes (W_z k^t_{w'}) dV(z) = \int_{C^n} \langle Ak^t_z, k^t_{z+w} \rangle_t (k^t_{z+w} \otimes k^t_z) dV(z).
\]
Then the first conclusion follows from Lemmas 2.3 and 2.4. Thus the integral
\[
\int_{B(0,r)} \int_{C^n} \langle Ak^t_z, k^t_{z+w} \rangle_t (k^t_{z+w} \otimes k^t_z) dV(z) dV(w)
\]
is convergent in the norm topology.
Let $A$ be weakly localized on $F^p_t$. For any $f \in F^p_t$ and $g \in F^q_t$, we have
\[
\langle Af, g \rangle_t = \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} \langle Af, k^t_w \rangle_t \langle k^t_w, g \rangle_t dV(w)
\]
\[
= \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} \langle f, A^* k^t_w \rangle_t \langle k^t_w, g \rangle_t dV(w)
\]
\[
= \frac{1}{(\pi t)^{2n}} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \langle f, k^t_z \rangle_t \langle k^t_z A^* k^t_w \rangle_t \langle k^t_w, g \rangle_t dV(z) dV(w)
\]
\[
= \frac{1}{(\pi t)^{2n}} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \langle f, k^t_z \rangle_t \langle Ak^t_z k^t_{w+z}, k^t_{w+z}, g \rangle_t dV(w) dV(z).
\]
Using the same method as in the proof of Lemma 2.3, we get
\[
\left| \langle Af, g \rangle_t - \frac{1}{(\pi t)^{2n}} \int_{B(0,r)} \int_{\mathbb{C}^n} \langle Ak^t_z, k^t_{z+w} \rangle_t \langle k^t_{z+w}, g \rangle_t dV(z) dV(w) f, g \rangle_t \right|
\]
\[
= \frac{1}{(\pi t)^{2n}} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \langle f, k^t_z \rangle_t \langle |Ak^t_z, k^t_{w+z} \rangle_t \langle k^t_{w+z}, g \rangle_t dV(z) dV(w)
\]
\[
= \frac{1}{(\pi t)^{2n}} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \langle f, k^t_z \rangle_t \langle Ak^t_z, k^t_{w+z} \rangle_t \langle k^t_{w+z}, g \rangle_t dV(w) dV(z)
\]
\[
\lesssim \|f\|_{F^p_t} \|g\|_{F^q_t} \left[ \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} |\langle Ak^t_z, k^t_{w+z} \rangle_t dV(w) \right]^\frac{1}{r} \left[ \sup_{w \in \mathbb{C}^n} \int_{\mathbb{C}^n} |\langle Ak^t_z, k^t_{w+z} \rangle_t dV(z) \right]^\frac{1}{q}.
\]
Now the desired conclusion follows from the definition of a weakly localized operator. \(\Box\)

Next, we will establish a sufficient condition for a weakly localized operator to be in the space $T_{ lin}(J)$ via the Berezin transform. Before going further, we still need some preparations.

Let $L$ be a bilinear map from $F^1_t \times F^1_t$ to some Banach space $B$. Suppose that $L(f, g)$ is linear with respect to $f$ and conjugate linear with respect to $g$. We say $L$ is bounded if
\[
\|L(f, g)\|_B \lesssim \|f\|_{F^1_t} \|g\|_{F^1_t}
\]
for all $f, g \in F^1_t$. For any multi-index $a = (a_1, \cdots, a_n)$ with $a_j \geq 0$ and $z \in \mathbb{C}^n$, we denote
\[
z^a = z_1^{a_1} \cdots z_n^{a_n}, \quad a! = a_1! \cdots a_n!
\]
and $|a| = a_1 + a_2 + \cdots + a_n$.

With the notations above, we have the following proposition.

**Proposition 2.6.** Let $L$ be a bounded bilinear map from $F^1_t \times F^1_t$ to a Banach space $B$. Let $B_1$ be a closed subspace of $B$. If $L(k^t_z, k^t_w) \in B_1$ for any $z \in \mathbb{C}^n$, then $L(k^t_z, k^t_w) \in B_1$ for all $z, w \in \mathbb{C}^n$.

**Proof.** We only need to show that $L(K^t_z, K^t_w) \in B_1$ if
\[
L(K^t_z, K^t_w) \in B_1
\]
for all $z, w \in B(0, r)$ with $r > 0$.

For any multi-index $a = (a_1, \cdots, a_n)$, let $g_a(\xi) = \xi^a$, which is in $F^1_t$. We know that $K^t_z(\xi)$ has a series expansion
\[
K^t_z(\xi) = \sum_a c_a z^a g_a(\xi)
\]
with $c_a > 0$. Noting that
\[
\lim_{m \to \infty} \sum_{|a| \leq m} c_a z^a g_a(\xi) = K^t_z(\xi)
\]
and 
\[
\left| K_z^t(\xi) - \sum_{|a| \leq m} c_a z^a g_a(\xi) \right| \leq e^{\frac{r|\xi|}{l}} + e^{\frac{r|\xi|+r|a|}{l}},
\]
we conclude by the dominated convergence theorem that 
\[
\lim_{m \to \infty} \left\| K_z^t - \sum_{|a| \leq m} c_a z^a g_a \right\|_{F^t_1} = 0
\]
for each \( z \in \mathbb{C}^n \). Thus, it is enough to show that \( L(g_a, g_b) \in B_1 \) for any multi-indices \( a \) and \( b \). Let 
\[
K_{z,a}^t := \frac{\partial^a K_t}{\partial \xi^a} \bigg|_{u=z}.
\]
Since 
\[
\frac{\partial^a K_t}{\partial \xi^a} \bigg|_{z=0} = c_a a! g_a,
\]
it is sufficient to show that \( L(K_{0,a}^t, K_{0,b}^t) \in B_1 \) for any two multi-indices \( a, b \). Let us prove this by induction.

But to get the induction argument to work, we need to show that \( L(K_{z,a}^t, K_{z,b}^t) \in B_1 \) for all multi-indices \( a \) and \( b \), and \( z \in \mathbb{C}^n \).

First, when \( a = b = 0 \), we have 
\[
L(K_{z,0}^t, K_{z,0}^t) = L(K_{z}^t, K_{z}^t) \in B_1.
\]
Suppose that \( L(K_{z,a}^t, K_{z,b}^t) \in B_1 \) if \( |a| + |b| \leq m \). Now we are going to show that \( L(K_{z,a}^t, K_{z,b}^t) \in B_1 \) if \( |a| + |b| = m + 1 \). Without loss of generality, we may assume that \( a_1 \geq 1 \). Let \( e = (1, 0, \cdots, 0) \). Then there is a multi-index \( a' \) such that \( a' + e = a \). For any \( s > 0 \) satisfying \( z + se \in B(0, r) \), we have 
\[
\frac{1}{s} \left[ L(K_{z+a'}^t, K_{z+se,b}^t) - L(K_{z+a'}^t, K_{z,b}^t) \right]
\]
\[
= L \left( \frac{K_{z+a'}^t - K_{z,b}^t}{s}, K_{z+se,b}^t \right) = L \left( K_{z,a'}, \frac{K_{z+a'}^t - K_{z,b}^t}{s} \right).
\]
By the dominated convergence theorem again, we have 
\[
\lim_{s \to 0} \left\| \frac{K_{z+a'}^t - K_{z,b}^t}{s} \right\|_{F^t_1} = 0
\]
and 
\[
\lim_{s \to 0} \left\| K_{z+se,b}^t - K_{z,b}^t \right\|_{F^t_1} = 0,
\]
to obtain 
\[
L(K_{z,a}^t, K_{z,b}^t) + L(K_{z,a'}^t, K_{z,b+e}^t) = \lim_{s \to 0} L \left( \frac{K_{z+a'}^t - K_{z,b}^t}{s}, K_{z+se,b}^t \right) + \lim_{s \to 0} L \left( K_{z,a'}, \frac{K_{z+a'}^t - K_{z,b}^t}{s} \right)
\]
\[
= \lim_{s \to 0} \frac{1}{s} \left[ L(K_{z+a'}^t, K_{z+se,b}^t) - L(K_{z+a'}^t, K_{z,b}^t) \right] \in B_1.
\]

On the other hand, we have 
\[
\frac{1}{is} \left[ L(K_{z+i se,a'}^t, K_{z+i se,b}^t) - L(K_{z,a'}, K_{z,b}^t) \right]
\]
\[
= L \left( \frac{K_{z+i se,a'}^t - K_{z,a'}^t}{is}, K_{z+i se,b}^t \right) - L \left( K_{z,a'}, \frac{K_{z+i se,b}^t - K_{z,b}^t}{is} \right).
\]
Similarly, we have 
\[
L(K_{z,a'}^t, K_{z,b}^t) - L(K_{z,a'}^t, K_{z,b+e}^t) \in B_1.
\]
Therefore, we obtain that 
\[
2L(K_{z,a}^t, K_{z,b}^t) = L(K_{z,a}^t, K_{z,b}^t) - L(K_{z,a'}^t, K_{z,b+e}^t) + L(K_{z,a}^t, K_{z,b}^t) + L(K_{z,a'}^t, K_{z,b+e}^t)
\]
belongs to \( B_1 \). This finishes the proof of Proposition 2.6. \( \square \)
In view of Proposition 2.6, we obtain that \( \langle AW_{\alpha}k^l_z, W_{\alpha}k^l_w \rangle_t \in \mathcal{J} \) when the Berezin transform of the operator \( A \) is in \( \mathcal{J} \), where \( W_{\alpha} \) is a Weyl operator.

**Corollary 2.7.** Let \( A \) be a bounded linear operator on \( F^p_t \), \( z, w \in \mathbb{C}^n \) and \( \mathcal{J} \) be a translation invariant closed subspace of \( \text{BUC}(\mathbb{C}^n) \). If the Berezin transform of \( A \) is in \( \mathcal{J} \), then
\[
\langle AW_{\alpha}k^l_z, W_{\alpha}k^l_w \rangle_t \in \mathcal{J}.
\]
In addition, if \( h \in \mathcal{J} \), then
\[
\int_{\mathbb{C}^n} h(u)(W_u k^l_z) \otimes (W_u k^l_w) dV(u) \in \mathcal{T}_{\text{lin}}(\mathcal{J}).
\]

**Proof.** Since \( \mathcal{J} \) is translation invariant and \( \tilde{A} \in \mathcal{J} \), we have
\[
\langle AW_{\alpha}k^l_z, W_{\alpha}k^l_w \rangle_t = \langle AK_{\alpha}k^l_z, k^l_w \rangle_{t} = \tilde{A}(\cdot + z) \in \mathcal{J}.
\]
According to Lemma 2.3 and Proposition 2.6, we get that
\[
\langle AW_{\alpha}k^l_z, W_{\alpha}k^l_w \rangle_t \in \mathcal{J}.
\]

Let \( h \) be in \( \mathcal{J} \). Then for any \( f \in F^p_t \) and \( g \in F^q_t \), we have
\[
\int_{\mathbb{C}^n} h(u)(W_u k^l_z) \otimes (W_u k^l_w) dV(u)f,g \rangle_t = \int_{\mathbb{C}^n} h(u)\langle f, k^l_{z+u} \rangle_t \langle k^l_{z+u}, g \rangle_t dV(u)
\]
\[
= \int_{\mathbb{C}^n} h(u-z)\langle f, k^l_u \rangle_t \langle k^l_u, g \rangle_t dV(u)
\]
\[
= (\pi t)^n \langle T_{\alpha_{-z}} h, f, g \rangle_t,
\]
where the translation \( \alpha_{-z} \) is defined in Section 1. Since \( \mathcal{J} \) is translation invariant, we have that \( \alpha_{-z}h \in \mathcal{J} \) and
\[
\int_{\mathbb{C}^n} h(u)(W_u k^l_z) \otimes (W_u k^l_w) dV(u) = (\pi t)^n T_{\alpha_{-z}}h \in \mathcal{T}_{\text{lin}}(\mathcal{J}).
\]
Using Lemma 2.3 and Proposition 2.6 again, we obtain
\[
\int_{\mathbb{C}^n} h(u)(W_u k^l_z) \otimes (W_u k^l_w) dV(u) \in \mathcal{T}_{\text{lin}}(\mathcal{J}),
\]
to complete the proof of this corollary.

Combining Theorem 2.5 with Corollary 2.7 yields that a weakly localized operator on \( F^p_t \) belongs to \( \mathcal{T}_{\text{lin}}(\mathcal{J}) \) if its Berezin transform is in \( \mathcal{J} \).

**Proposition 2.8.** Let \( A \) be a weakly localized operator on \( F^p_t \) and \( \mathcal{J} \) be a translation invariant closed subspace of \( \text{BUC}(\mathbb{C}^n) \). If the Berezin transform of \( A \) is in \( \mathcal{J} \), then
\[
A \in \mathcal{T}_{\text{lin}}(\mathcal{J}).
\]

**Proof.** Since \( A \) is weakly localized, we have by Theorem 2.5 that
\[
A = \lim_{t \to \infty} \frac{1}{(\pi t)^{2n}} \int_{B(0,r)} \int_{\mathbb{C}^n} \langle AW_z k^l_{t0}, W_z k^l_{tw} \rangle_t (W_z k^l_{tw}) dV(z) dV(w).
\]
Now Corollary 2.7 gives us that
\[
\int_{\mathbb{C}^n} \langle AW_z k^l_{t0}, W_z k^l_{tw} \rangle_t (W_z k^l_{tw}) dV(z) dV(w) \in \mathcal{T}_{\text{lin}}(\mathcal{J}).
\]
This completes the proof.

Finally, we show in the following that the Berezin transform of the finite product of Toeplitz operators with symbols in \( \mathcal{J} \) also belongs to \( \mathcal{J} \).
Proposition 2.9. Let $J$ be a translation invariant closed subalgebra of $BUC(\mathbb{C}^n)$. Suppose that $\varphi_1, \ldots, \varphi_m \in J$ and $A = T_{\varphi_1} T_{\varphi_2} \cdots T_{\varphi_m}$. Then the Berezin transform $\tilde{A}$ is in $J$. Moreover, if $J_1$ is a translation invariant closed ideal of $J$, $\psi_1, \psi_2, \ldots, \psi_k \in J_1$ and $B = T_{\psi_1} T_{\psi_2} \cdots T_{\psi_k}$, then $AB \in J_1$ and $BA \in J_1$.

Proof. For a single Toeplitz operator $T_{\varphi_1}$, using $W_z T_{\varphi_1} W_z = T_{\alpha_z \varphi_1}$ we have

$$
\langle T_{\varphi_1} k_1^t, k_1^t \rangle_t = \langle W_z T_{\varphi_1} W_z, 1 \rangle_t = \langle T_{\alpha_z \varphi_1}, 1 \rangle_t = \int_{\mathbb{C}^n} \alpha_z \varphi_1(\xi) d\mu_t(\xi).
$$

Since $\varphi_1 \in J$ and $J$ is translation invariant, we have that $\alpha(\cdot) \varphi_1(\xi) \in J$ for any $\xi \in \mathbb{C}^n$, and moreover, $\xi \mapsto \alpha(\cdot) \varphi_1(\xi)$

is uniformly continuous and uniformly bounded from $\mathbb{C}^n$ to $J$ with respect to the $L^\infty$-norm. Thus we have

$$
\langle T_{\varphi_1} k_1^t, k_1^t \rangle_t = \int_{\mathbb{C}^n} \alpha(\cdot) \varphi_1(\xi) d\mu_t(\xi) \in J.
$$

To show that the Berezin transform of the product of $m$ Toeplitz operators belongs to $J$, we suppose that the conclusion holds for $m \leq k - 1$. Then we are going to prove the conclusion for the case of $m = k$. Noting that $T_{\varphi_1} = T_{\varphi_1^t}$, we have by (2.2) that

$$
\langle T_{\varphi_1} T_{\varphi_2} \cdots T_{\varphi_k} k_1^t, k_1^t \rangle_t = \langle T_{\varphi_2} \cdots T_{\varphi_k} k_1^t, T_{\varphi_1^t} k_1^t \rangle_t
$$

and

$$
\langle T_{\varphi_2} \cdots T_{\varphi_k} W_{\varphi_1} k_0^t, W_{\varphi_1} k_1^t \rangle_t = \langle T_{\varphi_1} W_{\varphi_1} k_0^t, W_{\varphi_1} k_1^t \rangle_t
$$

for any $w \in \mathbb{C}^n$. Since $J$ is a translation invariant subalgebra, we obtain by Corollary 2.7 that

$$
\langle T_{\varphi_2} \cdots T_{\varphi_k} W_{\varphi_1} k_0^t, W_{\varphi_1} k_1^t \rangle_t \in J.
$$

Using that $W_z T_{\varphi_1} W_z = T_{\alpha_z \varphi_1}$ on $F^2_t$ for $\varphi \in L^\infty(\mathbb{C}^n, dV)$, we have

$$
\left\| \langle T_{\varphi_2} \cdots T_{\varphi_k} W_{\varphi_1} k_0^t, W_{\varphi_1} k_1^t \rangle_t \right\|_{\infty}
$$

$$
= \sup_{z \in \mathbb{C}^n} \left| \langle T_{\varphi_2} \cdots T_{\varphi_k} W_z k_0^t, W_z k_1^t \rangle_t \langle T_{\varphi_1} W_z k_0^t, W_z k_1^t \rangle_t \right|
$$

$$
\leq \sup_{z \in \mathbb{C}^n} \left| \langle T_{\alpha_z \varphi_2} \cdots T_{\alpha_z \varphi_k} 1, k_1^t \rangle_t \langle k_1^t, T_{\alpha_z \varphi_1} 1 \rangle_t \right|
$$

$$
\leq \left\| \varphi_1 \mathbb{1} \cdots \varphi_k \mathbb{1} \right\| \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} \sup_{z \in \mathbb{C}^n} \left| e^{i z \cdot \overline{\xi}} - e^{-i z \cdot \overline{\xi}} \right| \left| e^{-|\xi|^2/2} \right| dV(\xi)
$$

where the last inequality comes from [15, Corollary 2.5]. By Lemmas 2.3 and 2.4, the mapping

$$
w \mapsto \langle T_{\varphi_2} \cdots T_{\varphi_k} W_{\varphi_1} k_0^t, W_{\varphi_1} k_1^t \rangle_t \langle T_{\varphi_1} W_{\varphi_1} k_0^t, W_{\varphi_1} k_1^t \rangle_t
$$

is uniformly continuous from any compact subset of $\mathbb{C}^n$ to $J$. Thus (2.3) gives that

$$
\langle T_{\varphi_1} T_{\varphi_2} \cdots T_{\varphi_k} (\cdot) \rangle_t = \langle T_{\varphi_1} T_{\varphi_2} \cdots T_{\varphi_k} k_1^t, k_1^t \rangle_t \in J.
$$

This implies that $\tilde{A} \in J$ when $A = T_{\varphi_1} T_{\varphi_2} \cdots T_{\varphi_m}$ with $m \geq 1$, as desired.
By the definition of the Berezin transform of $AB$, we have
\[
\langle ABk_z^t,k_z^t \rangle_t = \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} \langle BW_z(k^t_0,W_zk^t) \rangle_t \langle AW_z(k^t_0,W_zk^t) \rangle_t dV(w).
\]
From the arguments above, it follows that
\[
\langle AW_z(k^t_0,W_zk^t) \rangle_t \in J \quad \text{and} \quad \langle BW_z(k^t_0,W_zk^t) \rangle_t \in J.
\]
Since $J_1$ is an ideal, we get that
\[
\langle BW_z(k^t_0,W_zk^t) \rangle_t \langle AW_z(k^t_0,W_zk^t) \rangle_t \in J_1,
\]
to obtain $\tilde{AB} \in J_1$. Similarly, we can show that $\tilde{BA} \in J_1$. This completes the proof of Proposition 2.9. \qed

Now we are ready to present the proof of Theorem 2.1.

\textbf{Proof of Theorem 2.1.} Let $J$ be a translation invariant closed subalgebra of $\text{BUC}(\mathbb{C}^n)$. Let us first show that
\[
\mathcal{T}(J) = \mathcal{T}_{\text{lin}}(J).
\]
For $\varphi_1, \cdots, \varphi_m \in J$, let $A = T\varphi_1 T\varphi_2 \cdots T\varphi_m$. Proposition 2.9 implies that $\tilde{A} \in J$. Using Proposition 2.8 and the fact that $A$ is weakly localized, we have
\[
A \in \mathcal{T}_{\text{lin}}(J).
\]
Observing that $\mathcal{T}_{\text{lin}}(J) \subset \mathcal{T}(J)$ is obvious, so we have $\mathcal{T}(J) = \mathcal{T}_{\text{lin}}(J)$.

To complete the proof of Theorem 2.1, it remains to show that $\mathcal{T}(J_1)$ is a two-sided ideal in $\mathcal{T}(J)$ if $J_1$ is a translation invariant closed ideal of $J$. To do so, we let $B = T\psi_1 T\psi_2 \cdots T\psi_k$ with $\psi_1, \psi_2, \cdots, \psi_k \in J_1$. Then we have by Proposition 2.9 that
\[
\tilde{AB} \in J_1 \quad \text{and} \quad \tilde{BA} \in J_1.
\]
Since $AB$ and $BA$ both are weakly localized operators, we deduce by Proposition 2.8 that
\[
AB \in \mathcal{T}_{\text{lin}}(J_1) = \mathcal{T}(J_1) \quad \text{and} \quad BA \in \mathcal{T}_{\text{lin}}(J_1) = \mathcal{T}(J_1).
\]
This completes the proof of Theorem 2.1. \qed

3. Integral representations on the $p$-Bergman space

The main purpose of this section is to establish an integral representation for $s$-weakly localized operators on the $p$-Bergman space $L^p_a$ with $1 < p < \infty$. First, let us review some basic knowledge about the reproducing kernel for the Bergman space $L^2_a$, the M"{o}bius transform and the Bergman metric on the unit ball $\mathbb{B}_n$.

Recall that the reproducing kernel for the Bergman space $L^2_a$ is given by
\[
K_z(w) = \frac{1}{(1 - \overline{w} \cdot z)^{n+1}}, \quad z, w \in \mathbb{B}_n.
\]
A simple calculation shows that
\[
c'_{p,q}(1 - |z|^2)^{-\frac{n+1}{q}} \leq \|K_z\|_p \leq c_{p,q}(1 - |z|^2)^{-\frac{n+1}{q}}
\]
for some positive constants $c_{p,q}$ and $c'_{p,q}$ depending only $p$ and $q$, where $\frac{1}{p} + \frac{1}{q} = 1$. Letting
\[
k_z^{(p)} := (1 - |z|^2)^{-\frac{n+1}{q}} K_z,
\]
then we have that
\[
c_p^{-1} \leq \|k_z^{(p)}\|_p \leq c_p
\]
for some constant $c_p > 0$ depending only on $p$. Recall that the Berezin transform $\tilde{T}$ of a bounded linear operator $T$ on $L^p_a$ is defined by
\[
\tilde{T}(z) = \langle Tk_z, k_z \rangle, \quad z \in \mathbb{B}_n.
\]
where $k_z = k_z^{(2)}$ is the normalized reproducing kernel for $L_a^2$.

Let $\varphi_z$ be the Möbius transform of $\mathbb{B}_n$ that interchanges $0$ and $z$. Then we have

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a} \cdot z|^2},$$

see [8, pages 25-26] for the details. The Bergman metric on $\mathbb{B}_n$ is defined by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbb{B}_n,$$

which is Möbius invariant. For each $z \in \mathbb{B}_n$ and $0 < r < \infty$, the corresponding $\beta$-ball is given by

$$D(z, r) = \{ w \in \mathbb{B}_n : \beta(z, w) < r \},$$

see pages 22-28 in [16]. Recall that the formula

$$d\lambda(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}}$$

gives us the standard Möbius-invariant measure on the unit ball. It is well-known that on $L_a^p$ we have

$$I = \int_{\mathbb{B}_n} (k_z \otimes k_z) d\lambda(z).$$

Recall that the pseudo-hyperbolic metric on $\mathbb{B}_n$ is defined by

$$\rho(u, v) = |\varphi_u(v)|, \quad u, v \in \mathbb{B}_n.$$

For the pseudo-hyperbolic metric, we have

$$\rho(\varphi_u(z), \varphi_v(z)) \leq \frac{G}{(1 - |z|^2)^2} \rho(u, v)$$  \hspace{1cm} (3.2)

holds for some positive constant $G$, which was obtained in [10, Lemma 6.2].

The relationship between the Bergman metric and the pseudo-hyperbolic metric on $\mathbb{B}_n$ is given by the following:

$$\beta(u, v) = \frac{1}{2} \log \frac{1 + |\rho(u, v)|}{1 - |\rho(u, v)|} = \tanh^{-1}(\rho(u, v))$$  \hspace{1cm} (3.3)

see [16, Corollary 1.22] if required.

For a subset $\mathcal{I}$ of $\text{BUC}(\mathbb{B}_n)$, recall that $\mathcal{I}$ is translation invariant if $\tau_z f \in \mathcal{I}$ for all $f \in \mathcal{I}$ and $z \in \mathbb{B}_n$, where $\tau_z$ is defined by (1.1):

$$(\tau_z f)(u) = f(\varphi_u(z)), \quad u \in \mathbb{B}_n.$$

We will show in the next lemma that $\text{BUC}(\mathbb{B}_n)$ is translation invariant.

**Lemma 3.1.** Let $z, u$ and $v$ be in $\mathbb{B}_n$. If $\beta(u, v) < \tanh^{-1} \left[ \frac{(1 - |z|^2)}{G} \right]$, then we have

$$\beta(\varphi_u(z), \varphi_v(z)) \leq \tanh^{-1} \left[ \frac{G}{(1 - |z|^2)^2} \tanh(\beta(u, v)) \right],$$

where $G$ is constant in (3.2). As a consequence, $\text{BUC}(\mathbb{B}_n)$ is translation invariant.

**Proof.** Note that the assumption $\beta(u, v) < \tanh^{-1} \left[ \frac{(1 - |z|^2)}{G} \right]$ implies that $\frac{G}{(1 - |z|^2)^2} \rho(u, v) < 1$. Thus

$$\tanh^{-1} \left[ \frac{G}{(1 - |z|^2)^2} \rho(u, v) \right]$$
is well-defined. It follows that
\[
\beta(L_u(z), L_v(z)) = \tanh^{-1}(\rho(L_u(z), L_v(z))) \leq \tanh^{-1}\left[ \frac{G}{(1 - |z|)^2} \rho(u, v) \right]
\]
\[
= \tanh^{-1}\left[ \frac{G}{(1 - |z|)^2} \tanh(\beta(u, v)) \right]
\]
since the function \(\tanh^{-1}(x)\) is monotone increasing for \(x \in (-1, 1)\). This proves the lemma. \(\Box\)

Let \(z \in \mathbb{B}_n\), the operator \(U_z\) is defined by
\[
U_z f(w) = f(L_z(w))k_z(w), \quad w \in \mathbb{B}_n, \ f \in L^p_a.
\] (3.4)

In particular, \(U_z k_z = \eta(u, z)k_{L_u(z)}\), where \(u, z \in \mathbb{B}_n\) and \(\eta(u, z) = \frac{|1 - u - z|^{n+1}}{(1 - u z)^n}\). Moreover, one can check readily that \(U_z^* = U_z\) and
\[
U_z^* T \varphi U_z = T \varphi L_z \quad (3.5)
\]
on \(L^p_a\) for \(\varphi \in L^\infty(\mathbb{B}_n, dv)\) and \(z \in \mathbb{B}_n\).

The following definition of the s-weakly localized operator was first introduced in [5, Definition 1.4], which plays an important role in the characterization of Toeplitz algebras over the Bergman space \(L^2_a\) [12, 13, 14].

**Definition 3.2.** For any real number \(s\) such that \(0 < s < \min\{p, q\}\), we say that a bounded linear operator \(T\) on \(L^p_a\) is s-weakly localized if it satisfies
\[
\sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|\langle Tk_z, k_{\nu}\rangle|}{\|K_z\|_2^{-\frac{2s}{q(n+1)}}\|K_{\nu}\|_2^{\frac{2s}{q(n+1)}}} d\lambda(w) < \infty,
\]
\[
\sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|\langle T^* k_z, k_{\nu}\rangle|}{\|K_z\|_2^{-\frac{2s}{p(n+1)}}\|K_{\nu}\|_2^{\frac{2s}{p(n+1)}}} d\lambda(w) < \infty,
\]
\[
\lim_{r \to \infty} \sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n \backslash D(z, r)} \frac{|\langle Tk_z, k_{\nu}\rangle|}{\|K_z\|_2^{-\frac{2s}{q(n+1)}}\|K_{\nu}\|_2^{\frac{2s}{q(n+1)}}} d\lambda(w) = 0,
\]
\[
\lim_{r \to \infty} \sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n \backslash D(z, r)} \frac{|\langle T^* k_z, k_{\nu}\rangle|}{\|K_z\|_2^{-\frac{2s}{p(n+1)}}\|K_{\nu}\|_2^{\frac{2s}{p(n+1)}}} d\lambda(w) = 0.
\]
The collection of s-weakly localized operators on \(L^p_a\) is denoted by \(\mathcal{A}^p_s\).

In the rest of this paper, we fix an \(s\) such that \(0 < s < \min\{p, q\}\). For an s-weakly localized operator \(T\) acting on \(L^p_a\), we define
\[
E_r(T, s) = \sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n \backslash D(z, r)} \frac{|\langle Tk_z, k_{\nu}\rangle|}{\|K_z\|_2^{-\frac{2s}{q(n+1)}}\|K_{\nu}\|_2^{\frac{2s}{q(n+1)}}} d\lambda(w)
\]
and
\[
E'_r(T, s) = \sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n \backslash D(z, r)} \frac{|\langle T^* k_z, k_{\nu}\rangle|}{\|K_z\|_2^{-\frac{2s}{p(n+1)}}\|K_{\nu}\|_2^{\frac{2s}{p(n+1)}}} d\lambda(w).
\]
Define \(D(z, 0) = \emptyset\) for \(z \in \mathbb{B}_n\). Then we have that \(E_0(T, s) < \infty, E'_0(T, s) < \infty\) and
\[
\lim_{r \to \infty} E_r(T, s) = \lim_{r \to \infty} E'_r(T, s) = 0.
\]
The main theorem of this section is the following integral representation for \( s \)-weakly localized operators on the \( p \)-Bergman space \( L^p_a \), which is parallel to Theorem 2.5 in the previous section.

**Theorem 3.3.** Let \( T \) be a bounded linear operator on \( L^p_a \) and \( r > 0 \). Then the mapping 
\[
v \mapsto \int_{\mathbb{B}_n} \langle TU_u k_0, U_u k_v \rangle (U_u k_v) \otimes (U_u k_0) d\lambda(u)
\]

is uniformly continuous (with respect to the Bergman metric) and uniformly bounded on \( D(0, r) \). Moreover, the integral 
\[
\int_{D(0, r)} \int_{\mathbb{B}_n} \langle TU_u k_0, U_u k_v \rangle (U_u k_v) \otimes (U_u k_0) d\lambda(u) d\lambda(v)
\]
is convergent in the norm topology. Furthermore, if \( T \in \mathcal{A}^p_v \), then 
\[
\int_{D(0, r)} \int_{\mathbb{B}_n} \langle TU_u k_0, U_u k_v \rangle (U_u k_v) \otimes (U_u k_0) d\lambda(u) d\lambda(v)
\]
converges to \( T \) in norm as \( r \to \infty \).

The following estimation related to the Bergman metric is useful for us to obtain the integral representation for \( s \)-weakly localized operators on \( L^p_a \), see [16, Lemma 2.20] and [16, Lemma 2.27] if needed.

**Lemma 3.4.** For any \( R > 0 \) and any \( b \in \mathbb{R} \), there exists a constant \( C > 0 \) such that 
\[
\left| \frac{(1 - z \cdot \overline{v})}{(1 - z \cdot \overline{v})^b} - 1 \right| \leq C \beta(u, v), \quad \left| \frac{(1 - |u|^2)}{(1 - |u|^2)^b} - 1 \right| \leq C \beta(u, v),
\]
\[
C^{-1} (1 - |u|^2) \leq 1 - |v|^2 \leq C (1 - |u|^2)
\]
and 
\[
C^{-1} |1 - z \cdot \overline{v}| \leq |1 - z \cdot \overline{v}| \leq C |1 - z \cdot \overline{v}|
\]
for all \( z, u, v \in \mathbb{B}_n \) with \( \beta(u, v) \leq R \).

The next definition of *separated set* in the unit ball is quoted from [12, Definition 2.1].

**Definition 3.5.** Let \( \Gamma \) be a nonempty subset of \( \mathbb{B}_n \). We say that \( \Gamma \) is \( \delta \)-separated (or separated), if there exists a \( \delta > 0 \) depending only on \( \Gamma \) such that \( \beta(u, v) \geq \delta \) for all \( u \neq v \) in \( \Gamma \).

Then the proof of Theorem 3.3 begins with the following lemma.

**Lemma 3.6.** Let \( h \) be in \( L^\infty(\mathbb{B}_n, dv) \) and \( \Gamma \) be a \( \delta \)-separated set. Let \( \{h_{1,u}\} \) and \( \{h_{2,u}\} \) be two families of bounded analytic functions on \( \mathbb{B}_n \). Then for each \( f \in L^p_a \) and \( g \in L^q_a \), we have 
\[
\sum_{u \in \Gamma} |h(u) \langle f, U_u h_{1,u} \rangle \langle U_u h_{2,u}, g \rangle| \leq C \|h\|_{\infty} \sup_{u \in \mathbb{B}_n} \|h_{1,u}\|_{\infty} \sup_{u \in \mathbb{B}_n} \|h_{2,u}\|_{\infty} \|f\|_p \|g\|_q
\]
and 
\[
\int_{\mathbb{B}_n} |h(u) \langle f, U_u h_{1,u} \rangle \langle U_u h_{2,u}, g \rangle| d\lambda(u) \leq C' \|h\|_{\infty} \sup_{u \in \mathbb{B}_n} \|h_{1,u}\|_{\infty} \sup_{u \in \mathbb{B}_n} \|h_{2,u}\|_{\infty} \|f\|_p \|g\|_q,
\]
where the constant \( C \) depends only on \( \delta \) and \( C' \) is an absolute constant. Moreover, for any bounded linear operator \( T \) on \( L^p_a \), we have 
\[
\sup_{u \in \mathbb{B}_n} |\langle TU_u h_{1,u}, U_u h_{2,u} \rangle| \leq C'' \|T\| \sup_{u \in \mathbb{B}_n} \|h_{1,u}\|_{\infty} \sup_{u \in \mathbb{B}_n} \|h_{2,u}\|_{\infty},
\]
where \( C'' \) is an absolute constant.
Proof. Since $U_u h_{1,u} = (h_{1,u} \circ \varphi_u) k_u$ and $U_u h_{2,u} = (h_{2,u} \circ \varphi_u) k_u$ are bounded analytic functions, $\langle f, U_u h_{1,u} \rangle$ and $\langle U_u h_{2,u}, g \rangle$ are both well-defined. Then we have

$$
\sum_{u \in \Gamma} |h(u)\langle f, U_u h_{1,u} \rangle\langle U_u h_{2,u}, g \rangle|
= \sum_{u \in \Gamma} |h(u) \int_{\mathbb{B}_n} f(\xi) \langle K_\xi, U_u h_{1,u} \rangle dv(\xi) \int_{\mathbb{B}_n} \langle U_u h_{2,u}, K_\xi \rangle g(\xi) dv(\xi)|
\leq \|h\|_{\infty} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} |f(\xi)| \sum_{u \in \Gamma} |\langle K_\xi, U_u h_{1,u} \rangle\langle U_u h_{2,u}, K_\xi \rangle| g(\xi) dv(\xi)
$$

for each $f \in L^p_u$ and $g \in L^q_d$. Denote

$$F(\xi, \zeta) := \sum_{u \in \Gamma} |\langle K_\xi, U_u h_{1,u} \rangle\langle U_u h_{2,u}, K_\xi \rangle|.$$  \hfill (3.6)

Then for any $b > 0$, we have by Hölder’s inequality that

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} |f(\xi)|F(\xi, \zeta) dv(\xi)|g(\zeta)| dv(\zeta)
\leq \left[ \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} |f(\xi)|F(\xi, \zeta) dv(\xi) \right)^p dv(\zeta) \right]^\frac{1}{p} \|g\|_q
$$

and

$$\leq \left[ \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} |f(\xi)|F(\xi, \zeta)^\frac{1}{p} \frac{|K_\xi|_2^b}{|K_\xi|_2^b} F(\xi, \zeta) \frac{1}{q} \frac{|K_\xi|_2^b}{|K_\xi|_2^b} dv(\xi) \right)^p dv(\zeta) \right]^\frac{1}{p} \|g\|_q.$$  

This yields that

$$\sum_{u \in \Gamma} |h(u)\langle f, U_u h_{1,u} \rangle\langle U_u h_{2,u}, g \rangle|
\leq \|h\|_{\infty} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} |f(\xi)|F(\xi, \zeta) dv(\xi)|g(\zeta)| dv(\zeta)
\leq \|h\|_{\infty} \|g\|_q \left[ \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} |f(\xi)|^p F(\xi, \zeta)^\frac{1}{p} \frac{|K_\xi|_2^b}{|K_\xi|_2^b} dv(\xi) \left( \int_{\mathbb{B}_n} F(\xi, \zeta)^\frac{1}{q} \frac{|K_\xi|_2^b}{|K_\xi|_2^b} dv(\zeta) \right)^\frac{q}{p} \right]^\frac{1}{p}
$$

and

$$\leq \|h\|_{\infty} \|g\|_q \|f\|_p \left[ \sup_{\zeta \in \mathbb{B}_n} \int_{\mathbb{B}_n} F(\xi, \zeta)^\frac{1}{p} \frac{|K_\xi|_2^b}{|K_\xi|_2^b} \frac{1}{|K_\xi|_2^b} dv(\xi) \right]^\frac{1}{p} \left[ \sup_{\zeta \in \mathbb{B}_n} \int_{\mathbb{B}_n} F(\xi, \zeta)^\frac{1}{q} \frac{|K_\xi|_2^b}{|K_\xi|_2^b} \frac{1}{|K_\xi|_2^b} dv(\zeta) \right]^\frac{1}{q},$$

where the second inequality is due to Hölder’s inequality.

Next, we estimate the first term of the above inequality:

$$\sup_{\zeta \in \mathbb{B}_n} \int_{\mathbb{B}_n} F(\xi, \zeta)^\frac{1}{p} \frac{|K_\xi|_2^b}{|K_\xi|_2^b} dv(\xi).$$

According to the definition of $U_u$ gives us that

$$|\langle K_\xi, U_u h_{1,u} \rangle| = |\langle U_u K_\xi, h_{1,u} \rangle| = \left| \int_{\mathbb{B}_n} (U_u K_\xi)(w) h_{1,u}(w) dv(w) \right|
= \left| \int_{\mathbb{B}_n} K_\xi(\varphi_u(w)) \frac{(1 - |u|^2)^{\frac{n+1}{2}}}{(1 - w \cdot \overline{u})^{n+1}} h_{1,u}(w) dv(w) \right|.$$  \hfill (3.7)
Applying the change of variables formula in [15, Proposition 1.13] to the last integral in (3.7), we obtain that

\[ |\langle K_\xi, U_h u_1, u \rangle| = \left| \int_{\mathbb{B}_n} K_\xi(w) \frac{(1 - |u|^2)^{\frac{n+1}{2}}}{1 - \varphi_u(w) \cdot \bar{w}^n+1} h_{1,u}(\varphi_u(w)) \frac{(1 - |u|^2)^{n+1}}{1 - w \cdot \bar{w}^{2n+2}} dv(w) \right| \]

\[ \leq \int_{\mathbb{B}_n} |K_\xi(w)| \frac{(1 - |u|^2)^{\frac{n+1}{2}}}{1 - \varphi_u(w) \cdot \bar{w}^n+1} \left| h_{1,u}(\varphi_u(w)) \frac{(1 - |u|^2)^{n+1}}{1 - w \cdot \bar{w}^{2n+2}} \right| dv(w) \]

\[ \leq \sup_{u \in \mathbb{B}_n} \|h_{1,u}\| \int_{\mathbb{B}_n} |K_\xi(w)| \frac{(1 - |u|^2)^{\frac{n+1}{2}}}{1 - w \cdot \bar{w}^{n+1}} dv(w). \]

Similarly, we have

\[ |\langle U_h h_{2,u}, K_\xi \rangle| \leq \sup_{u \in \mathbb{B}_n} \|h_{2,u}\| \int_{\mathbb{B}_n} |K_\xi(z)| \frac{(1 - |u|^2)^{\frac{n+1}{2}}}{1 - z \cdot \bar{w}^{n+1}} dv(z). \]

Thus we obtain by (3.6) that

\[ F(\xi, \zeta) \leq \sup_{u \in \mathbb{B}_n} \|h_{1,u}\| \sup_{u \in \mathbb{B}_n} \|h_{2,u}\| \times I(\xi, \zeta), \]

where

\[ I(\xi, \zeta) := \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} |K_\xi(w)| |K_\xi(z)| \sum_{u \in \Gamma} \frac{(1 - |u|^2)^{n+1}}{1 - w \cdot \bar{w}^{n+1} 1 - z \cdot \bar{w}^{n+1}} dv(z) dv(w). \]

To estimate the above integral \( I(\xi, \zeta) \), we need to deal with the summation first. For any \( u \in \Gamma \) and \( \xi \in D(u, \frac{\delta}{2}) \), we have by Lemma 3.4 that

\[ \frac{(1 - |u|^2)^{n+1}}{1 - z \cdot \bar{w}^{n+1} 1 - w \cdot \bar{w}^{n+1}} \leq C_1 \frac{(1 - |\xi|^2)^{n+1}}{1 - z \cdot \bar{\xi}^{n+1} 1 - w \cdot \bar{\xi}^{n+1}} \]

for some constant \( C_1 > 0 \) depending only on \( \delta \). This gives that

\[ \sum_{u \in \Gamma} \frac{(1 - |u|^2)^{n+1}}{1 - z \cdot \bar{w}^{n+1} 1 - w \cdot \bar{w}^{n+1}} \leq \sum_{u \in \Gamma} C_1 \lambda(D(u, \frac{\delta}{2})) \int_{D(u, \frac{\delta}{2})} \frac{(1 - |\xi|^2)^{n+1}}{1 - z \cdot \bar{\xi}^{n+1} 1 - w \cdot \bar{\xi}^{n+1}} d\lambda(\xi) \]

\[ \leq C_2 \int_{\mathbb{B}_n} \frac{d\lambda(u)}{1 - z \cdot \bar{w}^{n+1} 1 - w \cdot \bar{w}^{n+1}} \]

\[ = C_2 \int_{\mathbb{B}_n} |K_\xi(u)| |K_u(w)| dv(u), \]

where \( C_2 \) is a positive constant depending only on \( \delta \). Therefore,

\[ I(\xi, \zeta) \leq C_2 \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} |K_\xi(w)| |K_\xi(z)| \int_{\mathbb{B}_n} |K_\xi(u)| |K_u(w)| dv(u) dv(z) dv(w). \]

Denoting the integral on the right hand side of the above inequality by \( II(\xi, \zeta) \), then we have that \( II(\xi, \zeta) = II(\zeta, \xi) \) and

\[ F(\xi, \zeta) \leq C_2 \sup_{u \in \mathbb{B}_n} \|h_{1,u}\| \sup_{u \in \mathbb{B}_n} \|h_{2,u}\| \times II(\xi, \zeta). \]
This gives us that
\[
\sup_{\zeta \in \mathbb{B}, n} \int_{\mathbb{B}, n} F(\xi, \zeta) \frac{\|K_{\xi}\|^{bq}}{\|K_{\xi}\|^{bq}} d\nu(\xi)
\leq C_2 \sup_{u \in \mathbb{B}, n} \|h_{1,u}\|_\infty \sup_{u \in \mathbb{B}, n} \|h_{2,u}\|_\infty \left( \sup_{\zeta \in \mathbb{B}, n} \int_{\mathbb{B}, n} \Pi(\xi, \zeta) \frac{\|K_{\xi}\|^{bq}}{\|K_{\xi}\|^{bq}} d\nu(\xi) \right),
\]
where
\[
\int_{\mathbb{B}, n} \Pi(\xi, \zeta) \frac{\|K_{\xi}\|^{bq}}{\|K_{\xi}\|^{bq}} d\nu(\xi) = \int_{\mathbb{B}, n} \int_{\mathbb{B}, n} \int_{\mathbb{B}, n} |K_{\xi}(w)K_{\xi}(z)K_{\xi}(u)K_{\xi}(w)| \frac{\|K_{\xi}\|^{bq}}{\|K_{\xi}\|^{bq}} d\nu(\xi)d\nu(u)d\nu(z)d\nu(w).
\]
(3.8)
Note that the integrand in (3.8) can be written as
\[
|K_{\xi}(w)K_{\xi}(z)K_{\xi}(u)K_{\xi}(w)| \frac{\|K_{\xi}\|^{bq}}{\|K_{\xi}\|^{bq}} = |K_{\xi}(\zeta)| \frac{\|K_{\xi}\|^{bq}}{\|K_{\xi}\|^{bq}} \frac{\|K_{\xi}\|^{bq}}{\|K_{\xi}\|^{bq}} \frac{\|K_{\xi}\|^{bq}}{\|K_{\xi}\|^{bq}} \frac{\|K_{\xi}\|^{bq}}{\|K_{\xi}\|^{bq}}.
\]
By [16, Theorem 1.12], we have
\[
\sup_{\zeta \in \mathbb{B}, n} \int_{\mathbb{B}, n} |K_{\xi}(\zeta)| \frac{\|K_{\xi}\|^{bq}}{\|K_{\xi}\|^{bq}} d\nu(z) = \sup_{\zeta \in \mathbb{B}, n} \int_{\mathbb{B}, n} (1 - |z|^2)^{-\frac{bq(n+1)}{2}} (1 - |\zeta|^2)^{-\frac{bq(n+1)}{2}} d\nu(z)
= C_3 < \infty
\]
if \( \frac{bq(n+1)}{2} < 1 \). Now we choose \( b > 0 \) such that
\[
\frac{bq(n+1)}{2} < 1 \quad \text{and} \quad \frac{bp(n+1)}{2} < 1.
\]
In this case, we obtain that the multiple integral in (3.8) is less than some positive constant \( C'_3 \). This leads to
\[
\sup_{\zeta \in \mathbb{B}, n} \int_{\mathbb{B}, n} F(\xi, \zeta) \frac{\|K_{\xi}\|^{bq}}{\|K_{\xi}\|^{bq}} d\nu(\xi) \leq C_4 \sup_{u \in \mathbb{B}, n} \|h_{1,u}\|_\infty \sup_{u \in \mathbb{B}, n} \|h_{2,u}\|_\infty.
\]
Similarly, we have
\[
\sup_{\zeta \in \mathbb{B}, n} \int_{\mathbb{B}, n} F(\xi, \zeta) \frac{\|K_{\xi}\|^{bq}}{\|K_{\xi}\|^{bq}} d\nu(\xi) \leq C_5 \sup_{u \in \mathbb{B}, n} \|h_{1,u}\|_\infty \sup_{u \in \mathbb{B}, n} \|h_{2,u}\|_\infty.
\]
Combining the two inequalities above gives that
\[
\sum_{u \in \Gamma} \left| h(u) \langle f, U_{a,1,u} \rangle \langle U_{a,2,u}, g \rangle \right| \leq C \|h\|_\infty \sup_{u \in \mathbb{B}, n} \|h_{1,u}\|_\infty \sup_{u \in \mathbb{B}, n} \|h_{2,u}\|_\infty \|f\|_p \|g\|_q
\]
for some constant \( C \) depending only on \( \delta \), which is the first conclusion of this lemma.

Observe that the proof of the above inequality implies that
\[
\int_{\mathbb{B}, n} \left| h(u) \langle f, U_{a,1,u} \rangle \langle U_{a,2,u}, g \rangle \right| d\nu(u) \leq C' \|h\|_\infty \sup_{u \in \mathbb{B}, n} \|h_{1,u}\|_\infty \sup_{u \in \mathbb{B}, n} \|h_{2,u}\|_\infty \|f\|_p \|g\|_q
\]
for some absolute constant \( C' > 0 \). This proves the second inequality in the lemma.

To obtain the last conclusion in our lemma, we first recall that
\[
\|k_u\|_p \leq (1 - |u|^2)^{\frac{q-p}{4}} \quad , \quad u \in \mathbb{B}.n.
\]
Then we have by (3.4) that
\[
|\langle TU_u h_{1,u}, U_u h_{2,u} \rangle| \leq |\langle Th_{1,u} \circ \varphi_u k_u, h_{2,u} \circ \varphi_u k_u \rangle| \\
\leq \|T\| \|h_{1,u} \circ \varphi_u k_u\|_p \|h_{2,u} \circ \varphi_u k_u\|_q \\
\leq \|T\| \sup_{u \in \mathbb{B}_n} \|h_{1,u}\|_\infty \sup_{u \in \mathbb{B}_n} \|h_{2,u}\|_\infty \|k_u\|_p \|k_u\|_q \\
\lesssim \|T\| \sup_{u \in \mathbb{B}_n} \|h_{1,u}\|_\infty \sup_{u \in \mathbb{B}_n} \|h_{2,u}\|_\infty (1 - |u|^2)^{\frac{p+1}{2} - \frac{n+1}{q}} (1 - |u|^2)^{\frac{p+1}{2} - \frac{n+1}{p}} \\
= \|T\| \sup_{u \in \mathbb{B}_n} \|h_{1,u}\|_\infty \sup_{u \in \mathbb{B}_n} \|h_{2,u}\|_\infty,
\]
as desired. This finishes the proof of Lemma 3.6.

Using the same notations as in Lemma 3.6, \(\Gamma\) denotes a separated set, \(h\) is a bounded function on \(\mathbb{B}_n\), \(\{h_{1,u}\}\) and \(\{h_{2,u}\}\) are two families of bounded analytic functions on \(\mathbb{B}_n\) with the property that
\[
\sup_{u \in \mathbb{B}_n} \|h_{1,u}\|_\infty \sup_{u \in \mathbb{B}_n} \|h_{2,u}\|_\infty < \infty.
\]
With the help of Lemma 3.6, now we can define two bounded linear operators
\[
\sum_{u \in \Gamma} h(u)(U_u h_{1,u}) \otimes (U_u h_{2,u}) \quad \text{and} \quad \int_{\mathbb{B}_n} h(u)(U_u h_{1,u}) \otimes (U_u h_{2,u})d\lambda(u)
\]
as follows:
\[
\left\langle \sum_{u \in \Gamma} h(u)(U_u h_{1,u}) \otimes (U_u h_{2,u})f, g \right\rangle = \sum_{u \in \Gamma} h(u)\langle f, U_u h_{2,u} \rangle \langle U_u h_{1,u}, g \rangle
\]
and
\[
\left\langle \int_{\mathbb{B}_n} h(u)(U_u h_{1,u}) \otimes (U_u h_{2,u})d\lambda(u)f, g \right\rangle = \int_{\mathbb{B}_n} h(u)\langle f, U_u h_{2,u} \rangle \langle U_u h_{1,u}, g \rangle d\lambda(u),
\]
where \(f \in L^p_a\) and \(g \in L^q_a\). Noting that if \(h_{1,u} = h_{2,u} = 1\), we have
\[
\int_{\mathbb{B}_n} h(u)(U_u h_{1,u}) \otimes (U_u h_{2,u})d\lambda(u) = \int_{\mathbb{B}_n} h(u)(k_u \otimes k_u)d\lambda(u),
\]
and hence
\[
\left\langle \int_{\mathbb{B}_n} h(u)(k_u \otimes k_u)d\lambda(u)f, g \right\rangle = \int_{\mathbb{B}_n} h(u)\langle f, k_u \rangle \langle k_u, g \rangle d\lambda(u) = \langle Thf, g \rangle.
\]
This implies that the operator
\[
\int_{\mathbb{B}_n} h(u)(k_u \otimes k_u)d\lambda(u)
\]
is actually a Toeplitz operator with symbol \(h\).

To establish Theorem 3.3, we need one more simple lemma.

**Lemma 3.7.** For each \(u\) and \(v\) in \(D(0, r)\), we have
\[
\|k_u - k_v\|_\infty \leq C_r \beta(u, v),
\]
where \(C_r\) is a positive constant depending only on \(r\).
Proof. Let \( r > 0 \), \( z \in \mathbb{B}_n \) and \( u, v \in D(0, r) \). We have by Lemma 3.4 that
\[
|k_u(z) - k_v(z)| = \left| \frac{(1 - |u|^2)^{n+1}}{(1 - z \cdot \overline{u})^{n+1}} - \frac{(1 - |v|^2)^{n+1}}{(1 - z \cdot \overline{v})^{n+1}} \right|
\]
\[
= \left| \frac{(1 - |u|^2)^{n+1} - (1 - |v|^2)^{n+1}}{(1 - z \cdot \overline{u})^{n+1}} - \frac{(1 - |v|^2)^{n+1}}{(1 - z \cdot \overline{v})^{n+1}} \right|
\]
\[
\leq \left| \frac{(1 - |u|^2)^{n+1} - (1 - |v|^2)^{n+1}}{1 - z \cdot \overline{u}} \right| + \left| \frac{(1 - |v|^2)^{n+1}}{1 - z \cdot \overline{v}} \right|
\]
\[
\leq C_r [\beta(u, v)|k_u(z)| + \beta(u, v)|k_v(z)|]
\]
\[
\leq C_r \beta(u, v),
\]
where the last inequality follows from that
\[
|k_u(z)| + |k_v(z)| \leq 2\frac{1}{2} \left( \frac{(1 - |u|)^{n+1}}{(1 - |u|)^n} + \frac{(1 - |v|)^{n+1}}{(1 - |v|)^n} \right) \leq C_r''
\]
for \( u, v \in D(0, r) \) and \( z \in \mathbb{B}_n \). Here, \( C_r', C_r \) and \( C_r'' \) denote the positive constants depending only on \( r \). This proves the lemma.

We are now in position to prove the main result of this section.

Proof of Theorem 3.3. First, combining Lemmas 3.6 and 3.7 gives us that the mapping
\[
v \mapsto \int_{\mathbb{B}_n} \langle TU_uk_0, U_uk_v \rangle (U_uk_v) \otimes (U_uk_0) d\lambda(u)
\]
is uniformly continuous and uniformly bounded on each \( D(0, r) \). This yields that the integral
\[
\int_{D(0, r)} \int_{\mathbb{B}_n} \langle TU_uk_0, U_uk_v \rangle (U_uk_v) \otimes (U_uk_0) d\lambda(u) d\lambda(v)
\]
is convergent in the norm topology.

Let \( T \) be in \( \mathcal{A}_p^0 \). Then the Möbius-invariance of \( d\lambda \) gives that
\[
\langle Tf, g \rangle = \int_{\mathbb{B}_n} \langle Tf, k_v \rangle \langle k_v, g \rangle d\lambda(v)
\]
\[
= \int_{\mathbb{B}_n} \langle f, T^*k_v \rangle \langle k_v, g \rangle d\lambda(v)
\]
\[
= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \langle f, k_u \rangle \langle k_u, T^*k_v \rangle d\lambda(u) \langle k_v, g \rangle d\lambda(v)
\]
\[
= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \langle f, k_u \rangle \langle Tk_u, k_v \rangle d\lambda(u) \langle k_v, g \rangle d\lambda(v)
\]
\[
= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \langle f, k_u \rangle \langle Tk_u, k \varphi_u(v) \rangle \langle k \varphi_u(v), g \rangle d\lambda(u) d\lambda(v)
\]
\[
= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \langle f, U_uk_0 \rangle \langle TU_uk_0, U_uk_v \rangle \langle U_uk_v, g \rangle d\lambda(u) d\lambda(v)
\]
for each $f \in L_p^B$ and $g \in L_q^B$. This implies that

$$\langle Tf, g \rangle = \int_{D(0, r)} \left\langle T(U_uk_0, U_uk_v)(U_uk_v) \otimes (U_uk_0) \psi \lambda(u) \right\rangle \psi \lambda(v) f \psi \lambda(g)$$

$$= \langle Tf, g \rangle - \int_{D(0, r)} \left\langle (U_uk_0)(TU_uk_0, U_uk_v)(U_uk_v, g) \psi \lambda(u) \psi \lambda(v) \right\rangle \psi \lambda(g)$$

$$= \int_{B_n \setminus D(0, r)} \left\langle (f, U_uk_0)(TU_uk_0, U_uk_v)(U_uk_v, g) \psi \lambda(u) \right\rangle \psi \lambda(v) \psi \lambda(g)$$

$$= \int_{B_n \setminus D(0, r)} \left\langle f, U_uk_0 \right\rangle \left\langle T(U_uk_0, U_uk_v)(U_uk_v, g) \right\rangle \psi \lambda(u) \psi \lambda(v) \psi \lambda(g)$$

$$= \int_{B_n \setminus D(0, r)} \left\langle f, U_uk_0 \right\rangle \left\langle T(U_uk_0, U_uk_v)(U_uk_v, g) \right\rangle \psi \lambda(u) \psi \lambda(v) \psi \lambda(g)$$

$$= \int_{B_n \setminus D(0, r)} \left\langle f, U_uk_0 \right\rangle \left\langle T(U_uk_0, U_uk_v)(U_uk_v, g) \right\rangle \psi \lambda(u) \psi \lambda(v) \psi \lambda(g)$$

$$= \int_{B_n \setminus D(0, r)} \left\langle f, U_uk_0 \right\rangle \left\langle T(U_uk_0, U_uk_v)(U_uk_v, g) \right\rangle \psi \lambda(u) \psi \lambda(v) \psi \lambda(g)$$

$$= \int_{B_n \setminus D(0, r)} \left\langle f, U_uk_0 \right\rangle \left\langle T(U_uk_0, U_uk_v)(U_uk_v, g) \right\rangle \psi \lambda(u) \psi \lambda(v) \psi \lambda(g)$$

$$= \int_{B_n \setminus D(0, r)} \left\langle f, U_uk_0 \right\rangle \left\langle T(U_uk_0, U_uk_v)(U_uk_v, g) \right\rangle \psi \lambda(u) \psi \lambda(v) \psi \lambda(g)$$

where the last equality follows from (3.1):

$$k_z^{(p)}(w) = \frac{(1 - |z|^2)^{n + 1}}{(1 - w \cdot \overline{z})^{n + 1}}$$

and

$$k_z^{(q)}(w) = \frac{(1 - |z|^2)^{n + 1}}{(1 - w \cdot \overline{z})^{n + 1}}$$

for $z, w \in B_n$. Therefore,

$$\int_{B_n \setminus D(0, r)} \left\langle f, U_uk_0 \right\rangle \left\langle T(U_uk_0, U_uk_v)(U_uk_v, g) \right\rangle \psi \lambda(u) \psi \lambda(v) \psi \lambda(g)$$

$$= \|f\|_p \int_{B_n \setminus D(0, r)} \left\langle T(U_uk_0, U_uk_v)(U_uk_v, g) \right\rangle \psi \lambda(u) \psi \lambda(v) \psi \lambda(g)$$

$$= \|f\|_p \int_{B_n \setminus D(0, r)} \left\langle T(U_uk_0, U_uk_v)(U_uk_v, g) \right\rangle \psi \lambda(u) \psi \lambda(v) \psi \lambda(g)$$

$$= \|f\|_p \int_{B_n \setminus D(0, r)} \left\langle T(U_uk_0, U_uk_v)(U_uk_v, g) \right\rangle \psi \lambda(u) \psi \lambda(v) \psi \lambda(g)$$

where the last equality follows from that

$$\int_{B_n \setminus D(0, r)} \left\langle f, U_uk_0 \right\rangle \left\langle T(U_uk_0, U_uk_v)(U_uk_v, g) \right\rangle \psi \lambda(u) \psi \lambda(v) \psi \lambda(g)$$

$$= \|f\|_p \int_{B_n \setminus D(0, r)} \left\langle T(U_uk_0, U_uk_v)(U_uk_v, g) \right\rangle \psi \lambda(u) \psi \lambda(v) \psi \lambda(g)$$

Furthermore, we note that

$$\int_{B_n \setminus D(0, r)} \left\langle T(U_uk_0, U_uk_v)(U_uk_v, g) \right\rangle \psi \lambda(u) \psi \lambda(v) \psi \lambda(g)$$

$$= \int_{B_n \setminus D(0, r)} \left\langle T(U_uk_0, U_uk_v)(U_uk_v, g) \right\rangle \psi \lambda(u) \psi \lambda(v) \psi \lambda(g)$$

$$= \int_{B_n \setminus D(0, r)} \left\langle T(U_uk_0, U_uk_v)(U_uk_v, g) \right\rangle \psi \lambda(u) \psi \lambda(v) \psi \lambda(g)$$

where

$$b = \frac{2(n + 1) - 2s}{pq(n + 1)}.$$
Using (3.1) again, we observe that
\[ I' = \left( \sup_{u \in \mathbb{B}_n} \int_{\mathbb{B}_n \setminus D(u, r)} |\langle Tk_u, k_v \rangle|^q \frac{||K_u||_2^{bp+\frac{2}{q}}}{||K_v||_2^{bp+\frac{2}{q}}} d\lambda(v) \right)^\frac{2}{p} \]
and
\[ II' = \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} |\langle Tk_u, k_v \rangle|^q \frac{||K_v||_2^{bp+\frac{2}{p}}}{||K_u||_2^{bp+\frac{2}{p}}} |\langle k_v^{(p)}, g \rangle|^q d\lambda(v) d\lambda(u). \]
Elementary calculations give us that
\[ bp + 1 - \frac{2}{q} = 1 - \frac{2s}{q(n+1)} \quad \text{and} \quad bq + 1 - \frac{2}{p} = 1 - \frac{2s}{p(n+1)}. \]
Combining the above expressions for I' and II', we obtain that
\[ \int_{\mathbb{B}_n} \int_{\mathbb{B}_n \setminus D(u, r)} \left| \langle Tk_u^{(p)}, k_v^{(q)} \rangle \langle k_v^{(p)}, g \rangle d\lambda(v) \right|^q d\lambda(u) \leq (E_r(T, s))^{\frac{2}{p}} \sup_{v \in \mathbb{B}_n} \int_{\mathbb{B}_n} \left| \langle k_u, T^*k_v \rangle \right|^q \frac{||K_v||_2^{bp+\frac{2}{p}}}{||K_u||_2^{bp+\frac{2}{p}}} d\lambda(u) \int_{\mathbb{B}_n} \left| \langle k_v^{(p)}, g \rangle \right|^q d\lambda(v) = (E_r(T, s))^{\frac{2}{p}} E'_0(T, s) \|g\|_q^q. \]
This yields
\[ \left| \langle Tf, g \rangle - \left( \int_{D(0, r)} \int_{\mathbb{B}_n} \langle TU_u k_0, U_u k_v \rangle (U_u k_0) \otimes (U_u k_0) d\lambda(u) d\lambda(v) f, g \right) \right| \leq \|f\|_p \left( \int_{\mathbb{B}_n} \int_{\mathbb{B}_n \setminus D(u, r)} \left| \langle Tk_u^{(p)}, k_v^{(q)} \rangle \langle k_v^{(p)}, g \rangle d\lambda(v) \right|^q d\lambda(u) \right)^\frac{1}{q} \leq (E_r(T, s))^{\frac{2}{p}} \left( E'_0(T, s) \right)^{\frac{2}{q}} \|f\|_p \|g\|_q. \]
According to our assumption that \( T \in \mathcal{A}_p^b \), we have
\[ \lim_{r \to \infty} \left( E_r(T, s) \right)^{\frac{2}{p}} \left( E'_0(T, s) \right)^{\frac{2}{q}} = 0, \]
to complete the proof of Theorem 3.3.

4. Toeplitz algebras over the \( p \)-Bergman space

In the final section, we will study Toeplitz algebras with symbols in a translation invariant subalgebra \( \mathcal{I} \subset \text{BUC}(\mathbb{B}_n) \) in terms of the integral representation obtained in the previous section. More specifically, the following will be established in Theorem 4.9 for the Toeplitz algebras \( \mathcal{T}^b[\text{BUC}(\mathbb{B}_n)] \) and \( \mathcal{T}^b[\text{C}_0(\mathbb{B}_n)] \) over the \( p \)-Bergman space \( L^p_\mathfrak{b} \).

\[ \mathcal{T}^b[\text{BUC}(\mathbb{B}_n)] = \mathcal{T}^b[\text{BUC}(\mathbb{B}_n)] = \text{clos}(\mathcal{A}_p^b) \quad \text{and} \quad \mathcal{T}^b[\text{C}_0(\mathbb{B}_n)] = \mathcal{T}^b[\text{C}_0(\mathbb{B}_n)] = \mathcal{K}, \]
where \( \text{clos}(\mathcal{A}_p^b) \) denotes the closure of \( \mathcal{A}_p^b \) with respect to the operator norm, and \( \mathcal{K} \) denotes the ideal of compact operators on \( L^p_\mathfrak{b} \). The basic strategy used to prove the above conclusions is similar to the strategy used in Section 2 to prove the corresponding results for the case of \( F^p_1 \).

Let \( H(\mathbb{B}_n) \) be the set of functions which are holomorphic on the open unit ball \( \mathbb{B}_n \). Let \( G \) be a bilinear map from \( H(\mathbb{B}_n) \times H(\mathbb{B}_n) \) to a Banach space \( \mathcal{B} \). We suppose that \( G(f, g) \) is linear with respect to \( f \) and conjugate linear with respect to \( g \). We say that \( G \) is bounded if
\[ \|G(f, g)\|_\mathcal{B} \lesssim \|f\|_\infty \|g\|_\infty. \]
for all \( f, g \in H(\mathbb{B}_n) \).

**Proposition 4.1.** Let \( G \) be a bounded bilinear map from \( H(\mathbb{B}_n) \times H(\mathbb{B}_n) \) to some Banach space \( B \). Let \( B_1 \) be a closed subspace of \( B \). If \( G(k_z, k_w) \in B_1 \) for each \( z \in \mathbb{B}_n \), then \( G(k_w, k_z) \in B_1 \) for all \( z, w \in \mathbb{B}_n \).

**Proof.** Note that \( G(k_z, k_w) \in B_1 \) is equivalent to \( G(K_z, K_w) \in B_1 \) for \( z, w \in \mathbb{B}_n \). Thus, we only need to show that \( G(K_z, K_w) \in B_1 \) if \( G(k_z, k_w) \in B_1 \).

Fix a multi-index \( a = (a_1, a_2, \ldots, a_n) \) with \( a_j \geq 0 \). Let \( e_a \) be the function on \( \mathbb{B}_n \) of the form:

\[
e_a(v) = v^{a_1}v_2^{a_2} \cdots v_n^{a_n}.
\]

For any \( r > 0 \) and \( z \in D(0, r) \), the reproducing kernel can be represented as:

\[
K_z = \sum_a c_a z^a e_a,
\]

where \( c_a \) is a nonzero constant. Moreover, the summation above is uniformly convergent on each \( D(0, r) \), i.e.,

\[
\lim_{m \to \infty} \sup_{z \in D(0, r)} \left\| K_z - \sum_{|a| \leq m} c_a z^a e_a \right\|_\infty = 0,
\]

where \( |a| = a_1 + a_2 + \cdots + a_n \) and \( 0 < r < \infty \).

As \( G \) is bounded, we only need to show that \( G(e_a, e_b) \in B_1 \) for any two multi-indices \( a \) and \( b \). Denote

\[
K_{z,a} := \left. \frac{\partial^a K_z}{\partial u^a} \right|_{u=z}.
\]

It is sufficient to show that \( G(K_{0,a}, K_{0,b}) \in B_1 \) for all multi-indices \( a \) and \( b \), since \( K_{z,a}|_{z=0} = c_a e_a \). In a similar way as in the proof of Proposition 2.6, we will use induction and show that \( G(K_{z,a}, K_{z,b}) \in B_1 \) for all multi-indices \( a \) and \( b \), and \( z \in \mathbb{B}_n \).

First, we have by the assumption that

\[
G(K_{z,0}, K_{z,0}) \in B_1.
\]

For any positive integer \( m \), we suppose that

\[
G(K_{z,a}, K_{z,b}) \in B_1
\]

for any \( a \) and \( b \) with \( |a + b| \leq m \). Let \( a', b' \) be two indices such that

\[
|a' + b'| = m + 1.
\]

Without loss of generality, we may assume that there exist multi-indices \( a \) and \( b \) such that

\[
a' = a + e, \quad b' = b,
\]

where \( e = (1, 0, \ldots, 0) \). Since \( |a + b| \leq m \), we have

\[
G(K_{z,a}, K_{z,b}) \in B_1.
\]

On one hand, we have

\[
\frac{1}{t} \left( G(K_{z+a}, K_{z+b}) - G(K_{z,a}, K_{z,b}) \right)
\]

\[
= G \left( \frac{K_{z+a} - K_{z,b}}{t}, K_{z+b} \right) + G \left( K_{z,a}, \frac{K_{z+a} - K_{z,b}}{t} \right).
\]

Since

\[
\lim_{t \to 0} \left\| \frac{K_{z+a} - K_{z,b}}{t} - K_{z,a+e} \right\|_\infty = 0 \quad \text{and} \quad \lim_{t \to 0} \left\| K_{z+a} - K_{z,b} \right\|_\infty = 0,
\]

we have

\[
G(K_{z,a}, K_{z,b}) \in B_1.
\]
we have by the boundedness of $G$ that
\[
G(K_{z,a+e}, K_{z,b}) + G(K_{z,a}, K_{z,b+e})
= \lim_{t \to 0} G \left( \frac{K_{z+te,a} - K_{z,a}}{t}, \frac{K_{z+te,b} - K_{z,b}}{t} \right) + \lim_{t \to 0} G \left( \frac{K_{z,a} - K_{z+te,a}}{it}, \frac{K_{z,a} - K_{z+te,b}}{it} \right)
= \lim_{t \to 0} \frac{1}{t} \left[ G(K_{z+te,a}, K_{z+te,b}) - G(K_{z,a}, K_{z,b}) \right] \in \mathcal{B}_1.
\]

On the other hand,
\[
G(K_{z,a+e}, K_{z,b}) - G(K_{z,a}, K_{z,b+e})
= \lim_{t \to 0} G \left( \frac{K_{z+ite,a} - K_{z,a}}{it}, \frac{K_{z+ite,b} - K_{z,b}}{it} \right) - \lim_{t \to 0} G \left( \frac{K_{z,a} - K_{z+ite,a}}{it}, \frac{K_{z,a} - K_{z+ite,b}}{it} \right)
= \lim_{t \to 0} \frac{1}{it} \left[ G(K_{z+ite,a} - K_{z,a}, K_{z+ite,b} - K_{z,b}) + G(K_{z,a}, K_{z+ite,b} - K_{z,b}) \right] - \lim_{t \to 0} \frac{1}{it} \left[ G(K_{z,a} - K_{z+ite,a}, K_{z,a} - K_{z+ite,b}) \right]
= \lim_{t \to 0} \frac{1}{it} \left[ G(K_{z+ite,a}, K_{z+ite,b}) - G(K_{z,a}, K_{z,b}) \right] \in \mathcal{B}_1.
\]

Thus we conclude that
\[
2G(K_{z,a+e}, K_{z,b}) = G(K_{z,a+e}, K_{z,b}) + G(K_{z,a}, K_{z,b+e}) + G(K_{z,a+e}, K_{z,b}) - G(K_{z,a}, K_{z,b+e})
\]
belongs to $\mathcal{B}_1$. This completes the proof. \qed

Applying Proposition 4.1 to the bilinear form $G(f, g) = \langle TU(\cdot)f, U(\cdot)g \rangle$, we deduce that $\langle TU(\cdot)k_z, U(\cdot)k_w \rangle$ belongs to a translation invariant closed subalgebra of $\text{BUC}(\mathbb{B}_n)$ if the Berezin transform of $T$ does.

**Corollary 4.2.** Let $T$ be a bounded linear operator on $L^p_\mathcal{I}$ and $\mathcal{I}$ be a translation invariant closed subspace of $\text{BUC}(\mathbb{B}_n)$. If the Berezin transform of $T$ is in $\mathcal{I}$, then
\[
\langle TU(\cdot)k_z, U(\cdot)k_w \rangle \in \mathcal{I}
\]
for all $z, w \in \mathbb{B}_n$.

**Proof.** Since $\mathcal{I}$ is translation invariant and $\overline{T} \in \mathcal{I}$, we have
\[
\langle TU(\cdot)k_z, U(\cdot)k_z \rangle = \langle Tk_{\varphi(\cdot)(z)}, k_{\varphi(\cdot)(z)} \rangle = \overline{T}(\varphi(\cdot)(z)) = \tau_z(\overline{T}) \in \mathcal{I}
\]
for any $z \in \mathbb{B}_n$. By Lemma 3.6 and Proposition 4.1, we get that
\[
\langle TU(\cdot)k_z, U(\cdot)k_w \rangle \in \mathcal{I}
\]
for all $z, w \in \mathbb{B}_n$. This completes the proof. \qed

Since the Bergman metric is Möbius invariant, $\{\varphi(z) : u \in \Gamma\}$ is separated for any $z$ when $\Gamma$ is separated. Although the set $\{\varphi(u) : u \in \Gamma\}$ may not be separated, we have the following lemma in the case that $\Gamma$ is $c$-separated (see Definition 3.5).

**Lemma 4.3.** Let $r > 0$, $0 \leq \rho < 1$ and $\Gamma$ be a c-separated set in $\mathbb{B}_n$. Then there exist a positive integer $m$ and a finite partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ such that $\{\varphi_{\rho}^j(z) : u \in \Gamma_j\}$ is r-separated for any $|z| \leq \rho$ and $j \in \{1, 2, \cdots, m\}$. Moreover, $m$ depends only on $r$, $\rho$ and $c$.

**Proof.** Observing that this lemma is essentially comes from [12, Lemma 2.2 (b)], we shall include a proof here for the sake of completeness. Let $R = r + \log \frac{1+r}{1-\rho}$ and
\[
m = \left\lceil \frac{\Lambda(D(0, R + \frac{c}{2}))}{\Lambda(D(0, \frac{c}{2}))} \right\rceil + 2,
\]
where $\Lambda$ is the Loewner function.
where \([x]\) denotes the greatest integer less than or equal to \(x\).

From the proof of [12, Lemma 2.2 (a)], we have that

\[
\text{card}\{v \in \Gamma : \beta(u, v) \leq R\} \leq \frac{\lambda(D(0, R + \frac{\rho}{2}))}{\lambda(D(0, \frac{\rho}{2}))} \leq \left[ \frac{\lambda(D(0, R + \frac{\rho}{2}))}{\lambda(D(0, \frac{\rho}{2}))} \right] + 1 = m - 1
\]

for any \(u \in \Gamma\).

Next, we will show that there is a finite partition \(\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m\) such that each \(\Gamma_j\) is \(R\)-separated. If this is correct, then we have that

\[
\beta(\varphi_u(z), \varphi_v(z)) \geq \beta(u, v) - \beta(\varphi_u(z), u) - \beta(v, \varphi_v(z)) = \beta(u, v) - 2\beta(z, 0)
\]

\[
\geq R - \log \frac{1 + |z|}{1 - |z|} \geq r + \log \frac{1 + \rho}{1 - \rho} - \log \frac{1 + |z|}{1 - |z|} \geq r
\]

for \(\varphi_u(z)\) and \(\varphi_v(z)\) with \(u, v \in \Gamma_j\) and \(|z| \leq \rho\). It follows that \(\{\varphi_u(z) : u \in \Gamma_j\}\) is \(R\)-separated for any \(|z| \leq \rho\) and \(1 \leq j \leq m\).

Thus, it is sufficient to prove that there exists a partition \(\Gamma = \bigcup_{j=1}^{m} \Gamma_j\) such that \(\Gamma_j\) is \(R\)-separated for any \(j\). To this end, we need a maximality argument. Let us consider the set \(G_i = \{\Gamma_1, \cdots, \Gamma_m\}\), where \(\Gamma_1, \Gamma_2, \cdots, \Gamma_m\) are subsets of \(\Gamma\). We say that \(G_i\) is \(R\)-separated if \(\Gamma_j \cap \Gamma_k = \emptyset\) when \(\ell \neq k\) and each \(\Gamma_j\) is \(R\)-separated. For any \(G_{i1}\) and \(G_{i2}\), if \(\Gamma_{i1} \subset \Gamma_{i2}\) for all \(\ell\), then we denote this by \(G_{i1} \prec G_{i2}\). This is a partial order defined on \(\{G_i\}_i\) (the collection of all \(R\)-separated sets).

For a chain \(\{G_i\}_{i \in A}\) in \(\{G_i\}_i\), its maximal element is given by

\[
\left\{ \bigcup_{i \in A} \Gamma_{i1}, \bigcup_{i \in A} \Gamma_{i2}, \cdots, \bigcup_{i \in A} \Gamma_{im} \right\}.
\]

According to Zorn’s lemma, \(\{G_i\}_i\) has a maximal element and we denote it by \(G_0 = \{\Gamma_1^0, \cdots, \Gamma_m^0\}\). Then we must have that

\[
\bigcup_{j=1}^{m} \Gamma_j^0 = \Gamma.
\]

Otherwise, we can choose \(u \in \Gamma\) such that \(u \notin \bigcup_{j=1}^{m} \Gamma_j^0\). If there exists a \(v_j \in \Gamma_j^0\) such that \(\beta(v_j, u) \leq R\) for any \(j\), then we would have

\[
\text{card}\{v \in \Gamma : \beta(v, u) \leq R\} \geq m,
\]

which is a contradiction. Thus there is an \(\ell\) such that \(\beta(v, u) > R\) for all \(v \in \Gamma_\ell\). Let \(\Gamma_\ell = \Gamma_\ell^0 \cup \{u\}\) and

\[G' = \{\Gamma_1^0, \cdots, \Gamma_\ell^0, \cdots, \Gamma_m^0\}\].

It follows that \(G'\) is also \(R\)-separated, \(G_0 \prec G'\) and \(G' \neq G_0\). But this contradicts the fact that \(G_0\) is maximal. Therefore, we conclude that there exists a partition \(\Gamma = \bigcup_{j=1}^{m} \Gamma_j^0\) such that \(\Gamma_j^0\) is \(R\)-separated for every \(j\). This finishes the proof of Lemma 4.3.\(\Box\)

In view of the above lemma, we are able to construct a partition of the unity on the unit ball. Let \(c > 0\), \(l > 0\) and \(\Gamma\) be a \(c\)-separated set. Without loss of generality, we may assume that \(0 \in \Gamma\). Let \(\Phi_0^c\) be a radial nonnegative smooth function on \(\mathbb{B}_n\) such that \(0 \leq \Phi_0^c \leq 1\) and

\[
\Phi_0^c(\xi) = \begin{cases} 1, & \text{if } \xi \in D(0, l), \\ 0, & \text{if } \xi \notin D(0, 2l). \end{cases}
\]

(4.1)

For \(\zeta \in \Gamma\), we define \(\Phi_\zeta^c\) by

\[
\Phi_\zeta^c(\xi) = \Phi_0^c(\varphi_\zeta(\xi)), \quad \xi \in \mathbb{B}_n.
\]

(4.2)
If $\Gamma$ is a subset of the unit ball $\mathbb{B}_n$ such that \(\{D(\zeta, \frac{r}{2}) : \zeta \in \Gamma\}\) are mutually disjoint and \(\{D(\zeta, c) : \zeta \in \Gamma\}\) covers $\mathbb{B}_n$, we define

\[
\Psi_\zeta(\xi) := \sum_{\zeta \in \Gamma} \Phi_\zeta^c(\xi), \quad \xi \in \mathbb{B}_n. \tag{4.3}
\]

Clearly, $\Psi_\zeta(\varphi_\zeta(\xi)) = \Psi_0(\xi)$, $\sum_{\zeta \in \Gamma} \Psi_\zeta(\xi) = 1$ and $\text{supp}(\Psi_\zeta) \subset D(\zeta, 2c)$.

Let $\mathcal{I}$ be a translation invariant closed subalgebra of $\text{BUC}(\mathbb{B}_n)$. Then we define $\mathcal{I}'$ to be the subspace of $L^\infty(\mathbb{B}_n, dv)$ generated by

\[
\left\{ \sum_{u \in \Gamma} f(u)\Phi_{\varphi_u(z)}^l : l > 0, \ \Gamma \text{ is a separated set, } z \in \mathbb{B}_n \text{ and } f \in \mathcal{I} \right\}. \tag{4.4}
\]

Let $\psi = \varphi_{z} \circ \varphi_u$. Since $\psi(\varphi_u(z)) = 0$, we have by [8, Theorem 2.2.5] that there is a rotation $R$ of $\mathbb{B}_n$ such that

\[
\varphi_z \circ \varphi_u = \psi = R\varphi_{\varphi_u(z)}. \tag{4.5}
\]

Since $\Phi_0^l$ is radial, we obtain that

\[
\Phi_{\varphi_u(z)}^l(\zeta) = \Phi_0^l[\varphi_{\varphi_u(z)}(\zeta)] = \Phi_0^l[\varphi_z(\varphi_u(\zeta))] = \Phi_z^l(\varphi_u(\zeta)). \tag{4.5}
\]

Let

\[
C_0(\mathbb{B}_n) = \left\{ f \in \text{BUC}(\mathbb{B}_n) : \lim_{|z| \rightarrow 0} f(z) = 0 \right\}.
\]

With the discussion above, we have the following result.

**Lemma 4.4.** Let $\mathcal{I}$ be a translation invariant closed subalgebra of $\text{BUC}(\mathbb{B}_n)$. Then the space $\mathcal{I}'$ defined by (4.4) is contained in $\text{BUC}(\mathbb{B}_n)$. Moreover, $\mathcal{I}' \subset C_0(\mathbb{B}_n)$ if $\mathcal{I} = C_0(\mathbb{B}_n)$.

**Proof.** Let $l > 0$, $z \in \mathbb{B}_n$ and $\Gamma$ be a $c$-separated set. By Lemma 4.3, for any $r > 0$ there is a finite partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ such that $\{\varphi_u(z) : u \in \Gamma_j\}$ is $r$-separated. Thus,

\[
\sum_{u \in \Gamma} f(u)\Phi_{\varphi_u(z)}^l = \sum_{j=1}^m \sum_{u \in \Gamma_j} f(u)\Phi_{\varphi_u(z)}^l
\]

for any $f \in \text{BUC}(\mathbb{B}_n)$. Now we choose $r$ big enough such that the support sets of $\Phi_{\varphi_u(z)}^l$ are $c$-separated, i.e.,

\[
\beta\left(\text{supp}(\Phi_{\varphi_u(z)}^l), \text{supp}(\Phi_{\varphi_v(z)}^l)\right) \geq c
\]

for $u, v \in \Gamma_j$.

Next, we are going to show that $\sum_{u \in \Gamma_j} f(u)\Phi_{\varphi_u(z)}^l \in \text{BUC}(\mathbb{B}_n)$. Let $w_1$ and $w_2$ be two points in $\mathbb{B}_n$ with $\beta(w_1, w_2) < c$. Since the support sets of $\Phi_{\varphi_u(z)}^l$ are $c$-separated for $u \in \Gamma_j$, there exists $u_0 \in \Gamma_j$ such that

\[
\Phi_{\varphi_u(z)}^l(w_1) = \Phi_{\varphi_u(z)}^l(w_2) = 0
\]
for all \( u \in \Gamma_j \setminus \{u_0\} \). It follows from (4.2) that
\[
\lim_{\beta(w_1, w_2) \to 0} \left| \sum_{u \in \Gamma_j} f(u) \Phi^l_{\varphi_u(z)}(w_1) - \sum_{u \in \Gamma_j} f(u) \Phi^l_{\varphi_u(z)}(w_2) \right| 
\leq \|f\|_{\infty} \lim_{\beta(w_1, w_2) \to 0} \left| \Phi^l_{\varphi_{u_0}(z)}(w_1) - \Phi^l_{\varphi_{u_0}(z)}(w_2) \right|
= \|f\|_{\infty} \lim_{\beta(w_1, w_2) \to 0} \left| \Phi^l_0[\varphi_{u_0}(z)](w_1) - \Phi^l_0[\varphi_{u_0}(z)](w_2) \right|
= \|f\|_{\infty} \lim_{d \to 0} \left| \Phi^l_0[\varphi_{u_0}(z)](w_1) - \Phi^l_0[\varphi_{u_0}(z)](w_2) \right|
= 0,
\]
where
\[ d = \beta(\varphi_{u_0}(z), \varphi_{u_0}(z)) = \beta(w_1, w_2) \]
and the last equality is due to that \( \Phi^l_0 \) is continuous with compact support. This means that
\[
\sum_{u \in \Gamma_j} f(u) \Phi^l_{\varphi_u(z)} \in \text{BUC}(\mathbb{B}_n),
\]
to obtain
\[
\sum_{u \in \Gamma_j} f(u) \Phi^l_{\varphi_u(z)} \in \text{BUC}(\mathbb{B}_n).
\]

Finally, we show that \( \sum_{u \in \Gamma_j} f(u) \Phi^l_{\varphi_u(z)} \in C_0(\mathbb{B}_n) \) for each \( j \in \{1, 2, \ldots, m\} \) if \( f \in \mathcal{I} = C_0(\mathbb{B}_n) \). Let \( w \) be a point in \( \mathbb{B}_n \). If \( w \notin \text{supp}(\Phi^l_{\varphi_u(z)}) \) for any \( u \), then we have
\[
\sum_{u \in \Gamma_j} f(u) \Phi^l_{\varphi_u(z)}(w) = 0.
\]
Otherwise, suppose that \( w \in \text{supp}(\Phi^l_{\varphi_u(z)}) \) for some \( u \in \Gamma_j \). Then we have
\[
\left| \sum_{u \in \Gamma_j} f(u) \Phi^l_{\varphi_u(z)}(w) \right| = \left| f(u_w) \Phi^l_{\varphi_{u_w}(z)}(w) \right| \leq |f(u_w)|.
\]
Since \( \beta(w, \varphi_{u_w}(z)) < l \), we have \( \beta(z, u_w) = \beta(0, \varphi_{u_w}(z)) > \beta(w, 0) - l \). This yields that
\[
\beta(u_w, 0) > \beta(w, 0) - l - \beta(z, 0).
\]
Given an \( \epsilon > 0 \). Since \( f \in C_0(\mathbb{B}_n) \), there is a \( \delta > 0 \) such that \( |f(u)| < \epsilon \) whenever \( \beta(u, 0) > \delta \). If \( \beta(w, 0) > \delta + l + \beta(z, 0) \), then we have that \( \beta(u_w, 0) > \delta \) and
\[
\left| \sum_{u \in \Gamma_j} f(u) \Phi^l_{\varphi_u(z)}(w) \right| \leq |f(u_w)| < \epsilon.
\]
This implies that
\[
\sum_{u \in \Gamma_j} f(u) \Phi^l_{\varphi_u(z)} \in C_0(\mathbb{B}_n),
\]
and so is \( \sum_{u \in \Gamma} f(u) \Phi^l_{\varphi_u(z)} \). Thus we conclude that \( \mathcal{I}' \subset C_0(\mathbb{B}_n) \), to finish the proof of Lemma 4.4. \( \square \)

Let \( \mathcal{I} \) be a translation invariant closed subalgebra of \( \text{BUC}(\mathbb{B}_n) \) and \( \mathcal{I}' \) be defined in (4.4). Before showing that an \( s \)-weakly localized operator belongs to \( T^l_{\text{lin}}(\mathcal{I}') \) when its Berezin transform is in \( \mathcal{I} \), the following key proposition is required.
Proposition 4.5. Let $\mathcal{I}$ be a translation invariant closed subalgebra of $\text{BUC}(\mathbb{B}_n)$ and $\Gamma$ be $c$-separated. For each $f \in \mathcal{I}$ and $z, w \in \Gamma$, we have
\[
\sum_{u \in \Gamma} f(u)(U_u k_z) \otimes (U_u k_w) \in \mathcal{T}_{\text{lin}}^b(\mathcal{I}')
\]
and
\[
\int_{\mathbb{B}_n} f(u)(U_u k_z) \otimes (U_u k_w) d\lambda(u) \in \mathcal{T}_{\text{lin}}^b(\mathcal{I}').
\]

Proof. Based on Lemma 3.6 and Proposition 4.1, we need only to show that
\[
\sum_{u \in \Gamma} f(u)(U_u k_z) \otimes (U_u k_z) \in \mathcal{T}_{\text{lin}}^b(\mathcal{I}')
\]
and
\[
\int_{\mathbb{B}_n} f(u)(U_u k_z) \otimes (U_u k_z) d\lambda(u) \in \mathcal{T}_{\text{lin}}^b(\mathcal{I}').
\]
for every $z \in \Gamma$.

Using
\[
U_u k_z = \frac{1 - u \cdot z}{(1 - u \cdot z)^{n+1}} k_{\varphi_u(z)},
\]
we have
\[
\sum_{u \in \Gamma} f(u)(U_u k_z) \otimes (U_u k_z) = \sum_{u \in \Gamma} f(u) k_{\varphi_u(z)} \otimes k_{\varphi_u(z)}
\]
and
\[
\int_{\mathbb{B}_n} f(u)(U_u k_z) \otimes (U_u k_z) d\lambda(u) = \int_{\mathbb{B}_n} f(u) k_{\varphi_u(z)} \otimes k_{\varphi_u(z)} d\lambda(u).
\]
By Lemmas 3.6 and 3.7, we obtain that the following two mappings:
\[
z \mapsto \sum_{u \in \Gamma} f(u)(U_u k_z) \otimes (U_u k_z)
\]
and
\[
z \mapsto \int_{\mathbb{B}_n} f(u)(U_u k_z) \otimes (U_u k_z) d\lambda(u)
\]
are both uniformly continuous and uniformly bounded from $D(0, r)$ to the set of bounded linear operators on $L^p$ in the norm topology, where $r > 0$.

Let us first show that
\[
\sum_{u \in \Gamma} f(u) k_{\varphi_u(z)} \otimes k_{\varphi_u(z)} \in \mathcal{T}_{\text{lin}}^b(\mathcal{I}'),
\]
where $f \in \mathcal{I}$ and $\Gamma$ is $c$-separated. To do this, let
\[
a_l = \int_{\mathbb{B}_n} \Phi_l^i(\xi) d\lambda(\xi)
\]
and
\[
A_l = \int_{\mathbb{B}_n} \sum_{u \in \Gamma} f(u) \frac{\Phi_l^i(\xi)}{a_l} k_{\xi} \otimes k_{\xi} d\lambda(\xi),
\]
where the functions of the form \( \Phi^l u \) are defined by (4.1) and (4.2). Since \( A_l \) is the Toeplitz operator with symbol \( \sum_{u \in \Gamma} f(u) \frac{\Phi^l u(z)}{a_l} \), we have that \( A_l \in \mathcal{T}_{lim}(I') \). Furthermore, we have by (4.1) and (4.5) that

\[
\int_{B_n} \sum_{u \in \Gamma} f(u) \frac{\Phi^l u(z)}{a_l} k_\zeta \otimes k_\zeta d\lambda(\zeta) = \int_{B_n} \sum_{u \in \Gamma} f(u) \frac{\Phi^l u(z)}{a_l} k_\zeta \otimes k_\zeta d\lambda(\zeta)
\]

\[
= \int_{B_n} \sum_{u \in \Gamma} f(u) \frac{\Phi^l u(z)}{a_l} k_{\varphi u(z)} \otimes k_{\varphi u(z)} d\lambda(\zeta)
\]

\[
= \int_{D(z,2l)} \frac{\Phi^l u(z)}{a_l} \sum_{u \in \Gamma} f(u) k_{\varphi u(z)} \otimes k_{\varphi u(z)} d\lambda(\zeta).
\]

We may assume that \( 0 < l < 1 \). Then we have that \( z, \zeta \in D(0, 2 + \tanh^{-1}(|z|)) \) if \( \zeta \in D(z,2l) \). Using Lemmas 3.6 and 3.7 again, we obtain

\[
\left\| \sum_{u \in \Gamma} f(u) k_{\varphi u(z)} \otimes k_{\varphi u(z)} - \sum_{u \in \Gamma} f(u) k_{\varphi u(z)} \otimes k_{\varphi u(z)} \right\| \leq C_{|z|} l,
\]

where \( C_{|z|} \) is a constant depending only on \(|z|\). This implies that

\[
\left\| A_l - \sum_{u \in \Gamma} f(u) k_{\varphi u(z)} \otimes k_{\varphi u(z)} \right\|
\]

\[
= \left\| \int_{D(z,2l)} \frac{\Phi^l u(z)}{a_l} \left( \sum_{u \in \Gamma} f(u) k_{\varphi u(z)} \otimes k_{\varphi u(z)} - \sum_{u \in \Gamma} f(u) k_{\varphi u(z)} \otimes k_{\varphi u(z)} \right) d\lambda(\zeta) \right\|
\]

\[
\leq C'_{|z|} l
\]

for some constant \( C'_{|z|} \) depending only on \(|z|\). It follows that

\[
\sum_{u \in \Gamma} f(u) k_{\varphi u(z)} \otimes k_{\varphi u(z)} \in \mathcal{T}_{lim}^b (I')
\]

(4.6)

if \( f \in I \) and \( \Gamma \) is c-separated.

Next, we are going to show that

\[
\int_{B_{\frac{n}{2}}} f(u) k_{\varphi u(z)} \otimes k_{\varphi u(z)} d\lambda(u) \in \mathcal{T}_{lim}^b (I').
\]

Let \( \Gamma' \) be a subset of \( B_{\frac{n}{2}} \) such that \( \{ D(\zeta, \frac{\zeta}{2}) : \zeta \in \Gamma' \} \) are mutually disjoint and \( \{ D(\zeta,c) : \zeta \in \Gamma' \} \) covers \( B_{\frac{n}{2}} \). Recalling that \( \Psi_{\zeta}(\xi) \) is the function constructed in (4.3), then we have that

\[
\left\langle \int_{B_n} f(u) k_{\varphi u(z)} \otimes k_{\varphi u(z)} d\lambda(u) g, h \right\rangle
\]

\[
= \int_{B_n} f(u) \langle h, k_{\varphi u(z)} \rangle \langle k_{\varphi u(z)}, g \rangle d\lambda(u)
\]

\[
= \int_{B_{\frac{n}{2}}} \sum_{\zeta \in \Gamma'} \Psi_{\zeta}(u) f(u) \langle h, k_{\varphi u(z)} \rangle \langle k_{\varphi u(z)}, g \rangle d\lambda(u)
\]

\[
= \int_{B_{\frac{n}{2}}} \Psi_{\zeta}(u) \sum_{\zeta \in \Gamma'} f(\varphi_{\zeta}(u)) \langle h, k_{\varphi_{\zeta}(u)} \rangle \langle k_{\varphi_{\zeta}(u)}, g \rangle d\lambda(u)
\]

for any \( h \in L^2 I \) and \( g \in L^2 I \).

By Lemma 4.3, there is a partition \( \Gamma' = \Gamma'_1 \cup \cdots \cup \Gamma'_{m} \) such that \( \{ \varphi_{\zeta}(u) : \zeta \in \Gamma'_j \} \) is c-separated for any \( j \in \{1, 2, \ldots, m\} \). Applying Lemma 4.3 to each of the c-separated sets \( \{ \varphi_{\zeta}(u) : \zeta \in \Gamma'_j \} \) (which is denoted
by \( G_j \), we can get a partition \( G_j = \bigcup_{j=1}^{N_j} G_{j,\ell} \) such that each \( \{ \varphi \varphi(z) : \varphi(u) \in G_{j,\ell} \} \) is also \( c \)-separated.

Let \( \Gamma_{j,\ell} = \{ \zeta : \varphi(z) \in G_{j,\ell} \} \). Then we see that \( \{ \varphi \varphi(z) : \zeta \in \Gamma_{j,\ell} \} \) is \( c \)-separated. This means that

\[
\Gamma = \bigcup_{j=1}^{m} \Gamma_j = \bigcup_{j=1}^{m} \bigcup_{\ell=1}^{N_j} \Gamma_{j,\ell}.
\]

From the above arguments we may assume that \( \{ \varphi \varphi(z) : \zeta \in \Gamma_j \} \) is \( c \)-separated for \( j \in \{1, 2, \ldots, m\} \).

Furthermore, we have

\[
\left\langle \int_{\mathbb{B}_n} f(u)k_{\varphi \varphi(z)} \otimes k_{\varphi \varphi(z)} d\lambda(u)h, g \right\rangle
= \sum_{j=1}^{m} \int_{\mathbb{B}_n} \Psi_0(u) \sum_{\zeta \in \Gamma_j} f(\varphi(z)) \langle h, k_{\varphi \varphi(z)} \rangle \langle k_{\varphi \varphi(z)}, g \rangle d\lambda(u)
= \sum_{j=1}^{m} \int_{D(0,2c)} \Psi_0(u) \sum_{\zeta \in \Gamma_j} f(\varphi(z)) k_{\varphi \varphi(z)} \otimes k_{\varphi \varphi(z)} h, g \rangle d\lambda(u),
\]

where the last equality follows from that \( \text{supp}(\Psi_0) \subset D(0,2c) \). Next, we will show that the mapping

\[
u \mapsto \sum_{\zeta \in \Gamma_j} f(\varphi(z)) k_{\varphi \varphi(z)} \otimes k_{\varphi \varphi(z)}
\]

is uniformly continuous and uniformly bounded from \( D(0,2c) \) to the set of bounded linear operators on \( L^p \) in the norm topology.

Let \( u \) and \( u' \) be two points in \( D(0,2c) \) with \( \beta(u, u') < \tanh^{-1} \left( \frac{(1-|z|)^2}{2c} \right) \). Then we have by Lemma 3.1 and the Möbius invariance of \( \beta \) that

\[
\beta(\varphi \varphi(z), \varphi \varphi(u') (z)) \leq \tanh^{-1} \left[ \frac{G}{(1-|z|)^2} \tanh (\beta(u, u')) \right]
\]

for all \( \zeta \in \mathbb{B}_n \). Letting \( a(u, u', z) = \varphi \varphi(z)(\varphi \varphi(u') (z)) \), we have

\[
\beta(a(u, u', z), 0) \leq \tanh^{-1} \left[ \frac{G}{(1-|z|)^2} \tanh (\beta(u, u')) \right]
\]

and

\[
\varphi \varphi(z) \varphi(a(u, u', z), z) = \varphi \varphi(u') (z).
\]

Then it follows that

\[
\left\| \sum_{\zeta \in \Gamma_j} f(\varphi(z)) k_{\varphi \varphi(z)} \otimes k_{\varphi \varphi(z)} - \sum_{\zeta \in \Gamma_j} f(\varphi(z)) k_{\varphi \varphi(z)} \otimes k_{\varphi \varphi(z)} \right\|
\]

\[
\leq \left\| \sum_{\zeta \in \Gamma_j} \left[ f(\varphi(z)) U_{\varphi \varphi(z)} k_{a(u, u', z)} \otimes U_{\varphi \varphi(z)} k_{a(u, u', z)} - f(\varphi(z)) U_{\varphi \varphi(z)} k_0 \otimes U_{\varphi \varphi(z)} k_0 \right] \right\|
\]

\[
\leq \left\| \sum_{\zeta \in \Gamma_j} \left[ f(\varphi(z)) U_{\varphi \varphi(z)} k_{a(u, u', z)} \otimes U_{\varphi \varphi(z)} k_{a(u, u', z)} \right] \right\|
\]

\[
+ \left\| \sum_{\zeta \in \Gamma_j} \left[ f(\varphi(z)) U_{\varphi \varphi(z)} k_{a(u, u', z)} \otimes U_{\varphi \varphi(z)} (k_{a(u, u', z)} - k_0) \right] \right\|
\]

\[
+ \left\| \sum_{\zeta \in \Gamma_j} \left[ f(\varphi(z)) U_{\varphi \varphi(z)} (k_{a(u, u', z)} - k_0) \otimes U_{\varphi \varphi(z)} k_0 \right] \right\|.\]
As \( \{ \varphi_{\zeta}(u) : \zeta \in \Gamma' \} \) is \( c \)-separated, we have that

\[
\left\| \sum_{\zeta \in \Gamma'_j} f(\varphi(\zeta)) U_{\varphi_{\zeta}(u)}(z) (k_{a(u,u',\zeta,z)} - k_0) \otimes U_{\varphi_{\zeta}(u)}(z) k_0 \right\|
\]

\[
= \sup_{g_1 \in L^p_\infty \parallel g_1 \parallel = 1} \left\| \sum_{\zeta \in \Gamma'_j} f(\varphi(\zeta)) U_{\varphi_{\zeta}(u)}(z) (k_{a(u,u',\zeta,z)} - k_0) \otimes U_{\varphi_{\zeta}(u)}(z) k_0 g_1, g_2 \right\|
\]

\[
\leq \sup_{g_1 \in L^p_\infty \parallel g_1 \parallel = 1} \sum_{\zeta \in \Gamma'_j} |f(\varphi(\zeta))| \left\| U_{\varphi_{\zeta}(u)}(z) (k_{a(u,u',\zeta,z)} - k_0) \right\| \left\| g_1, U_{\varphi_{\zeta}(u)}(z) k_0 \right\|
\]

\[
\lesssim \|f\|_\infty \sup_{\zeta \in \mathbb{B}_n} \|k_{a(u,u',\zeta,z)} - k_0\|_\infty \lesssim \|f\|_\infty \sup_{\zeta \in \mathbb{B}_n} \beta(a(u,u',\zeta,z),0)
\]

\[
\leq \|f\|_\infty \tanh^{-1} \left[ \frac{G}{(1 - |z|)^2} \tanh (\beta(u,u')) \right]
\]

where the second inequality is due to Lemma 3.6 (note that \( |f(\varphi(\zeta))| \) is dominated by \( \|f\|_\infty \), so Lemma 3.6 still can be applied here), and the third inequality comes from Lemma 3.7. Moreover, we have that

\[
|a(u,u',\zeta,z)| = \rho(a(u,u',\zeta,z),0) = \tanh \left[ \beta(a(u,u',\zeta,z),0) \right] = \frac{G}{(1 - |z|)^2} \tanh (\beta(u,u')) < \frac{1}{2}
\]

since \( \beta(u,u') < \tanh^{-1} \left( \frac{(1-|z|)^2}{2G} \right) \). This gives \( \|k_{a(u,u',\zeta,z)}\|_\infty \lesssim 1 \). Then using the same method as used in (4.8), we obtain that

\[
\left\| \sum_{\zeta \in \Gamma'_j} \left[ f(\varphi(\zeta'))(f(\varphi(\zeta))) U_{\varphi_{\zeta}(u)}(z) k_{a(u,u',\zeta,z)} \otimes U_{\varphi_{\zeta}(u)}(z) k_{a(u,u',\zeta,z)} \right] \right\|
\]

\[
\lesssim \sup_{\zeta \in \mathbb{B}_n} |f(\varphi(\zeta)) - f(\varphi(\zeta'))|
\]

and

\[
\left\| \sum_{\zeta \in \Gamma'_j} \left[ f(\varphi(\zeta)) U_{\varphi_{\zeta}(u)}(z) k_{a(u,u',\zeta,z)} \otimes U_{\varphi_{\zeta}(u)}(z) (k_{a(u,u',\zeta,z)} - k_0) \right] \right\|
\]

\[
\lesssim \|f\|_\infty \tanh^{-1} \left[ \frac{G}{(1 - |z|)^2} \tanh (\beta(u,u')) \right].
\]

Since \( f \in \text{BUC}(\mathbb{B}_n) \) and \( \beta \) is Möbius invariant, we deduce that the mapping defined in (4.7) is uniformly continuous. Similarly, we can show that this mapping is also uniformly bounded.

Finally, from

\[
\int_{\mathbb{B}_n} f(u) k_{\varphi_{\zeta}(u)}(z) \otimes k_{\varphi_{\zeta}(u)}(z) d\lambda(u) = \sum_{j=1}^{m} \int_{D(0,2c_0)} \Psi_{0}(u) \sum_{\zeta \in \Gamma'_j} f(\varphi(\zeta)) k_{\varphi_{\zeta}(u)}(z) \otimes k_{\varphi_{\zeta}(u)}(z) d\lambda(u),
\]

we conclude that the above integral is convergent in the norm topology. Since \( \{ \varphi_{\zeta}(u) : \zeta \in \Gamma'_j \} \) is \( c \)-separated for any \( j \in \{1,2,\cdots ,m\} \), it follows from (4.6) that

\[
\int_{\mathbb{B}_n} f(u) k_{\varphi_{\zeta}(u)}(z) \otimes k_{\varphi_{\zeta}(u)}(z) d\lambda(u) \in \mathcal{T}^b_{\text{lin}}(\mathcal{I}').
\]

This completes the proof of Proposition 4.5. \( \square \)

**Theorem 4.6.** Let \( \mathcal{I} \) be a translation invariant closed subalgebra of \( \text{BUC}(\mathbb{B}_n) \) and \( \mathcal{I}' \) be defined in (4.4). Let \( T \) be a bounded linear operator on \( L^p_\alpha \). If \( T \in \mathcal{A}_b^p \) and \( \bar{T} \in \mathcal{I} \), then \( T \in \mathcal{T}^b_{\text{lin}}(\mathcal{I}') \).
Proof. From our assumption and Corollary 4.2, we have that \( \langle TU_{(\cdot)}k_w, U_{(\cdot)}k_z \rangle \in \mathcal{I} \) for all \( z, w \in \mathbb{B}_n \). By Proposition 4.5, we obtain

\[
\int_{\mathbb{B}_n} \langle TU_{u}k_0, U_{u}k_u \rangle (U_{u}k_u) \otimes (U_{u}k_0) d\lambda(u) \in \mathcal{T}_{\text{lin}}^b(\mathcal{I}').
\]

Therefore, we conclude by Theorem 3.3 that

\[
T = \lim_{r \to \infty} \int_{D(0, r)} \int_{\mathbb{B}_n} \langle TU_{u}k_0, U_{u}k_u \rangle (U_{u}k_u) \otimes (U_{u}k_0) d\lambda(u) d\lambda(v) \in \mathcal{T}_{\text{lin}}^b(\mathcal{I}').
\]

This completes the proof of Theorem 4.6. \( \square \)

We next show that the Berezin transform of the finite product of Toeplitz operators with translation invariant symbols is also translation invariant on \( \mathbb{B}_n \).

**Proposition 4.7.** Let \( \mathcal{I} \) be a translation invariant closed subalgebra of \( \text{BUC}(\mathbb{B}_n) \). For any \( \varphi_1, \varphi_2, \cdots, \varphi_m \) in \( \mathcal{I} \), let \( A = T_{\varphi_1}T_{\varphi_2} \cdots T_{\varphi_m} \). Then we have

\[
\overline{A} \in \mathcal{I}.
\]

**Proof.** We prove this proposition by induction. First, let us consider the case of \( m = 1 \). Recall that

\[
\overline{T_{\varphi_1}(z)} = \langle T_{\varphi_1}, k_z \rangle = \int_{\mathbb{B}_n} \varphi_1 \circ \varphi_z(w) dv(w).
\]

Since \( \varphi_1 \in \text{BUC}(\mathbb{B}_n) \) and

\[
\beta(\varphi_z(w), \varphi_z(w')) = \beta(w, w'),
\]

we obtain that the mapping

\[
w \mapsto \varphi_1 \circ \varphi_{(\cdot)}(w)
\]

is uniformly continuous on \( \mathbb{B}_n \). Note that \( \varphi_1 \circ \varphi_{(\cdot)}(w) \in \mathcal{I} \) as \( \varphi_1 \in \mathcal{I} \). Moreover, since

\[
\| \varphi_1 \circ \varphi_{(\cdot)}(w) \|_{\mathcal{I}} = \sup_{z \in \mathbb{B}_n} |\varphi_1 \circ \varphi_z(w)| = \| \varphi_1 \|_{\infty} < \infty,
\]

we have that the integral

\[
\int_{\mathbb{B}_n} \varphi_1 \circ \varphi_{(\cdot)}(w) dv(w)
\]

is convergent in the norm topology of \( \mathcal{I} \). It follows that

\[
\overline{T_{\varphi_1}(\cdot)} = \int_{\mathbb{B}_n} \varphi_1 \circ \varphi_{(\cdot)}(w) dv(w) \in \mathcal{I}.
\]

This shows that our conclusion holds for \( m = 1 \).

Now we suppose that the conclusion holds for \( m = k - 1 \). Let us consider the case of \( m = k \).

\[
\langle T_{\varphi_1}T_{\varphi_2} \cdots T_{\varphi_k}k_z, k_z \rangle = \langle T_{\varphi_1} \cdots T_{\varphi_k}k_z, T_{\varphi_{(\cdot)}}k_z \rangle
\]

\[
= \int_{\mathbb{B}_n} \langle T_{\varphi_2} \cdots T_{\varphi_k}k_z, k_w \rangle \langle k_w, T_{\varphi_{(\cdot)}}k_z \rangle d\lambda(w)
\]

\[
= \int_{\mathbb{B}_n} \langle T_{\varphi_2} \cdots T_{\varphi_k}k_z, k_{\varphi_z(w)} \rangle \langle k_{\varphi_z(w)}, T_{\varphi_{(\cdot)}}k_z \rangle d\lambda(w)
\]

\[
= \int_{\mathbb{B}_n} \langle T_{\varphi_2} \cdots T_{\varphi_k}U_zk_0, U_zk_w \rangle \langle T_{\varphi_1}U_zk_w, U_zk_0 \rangle d\lambda(w).
\]

By our hypothesis, \( \langle T_{\varphi_2} \cdots T_{\varphi_k}k_{(\cdot)}, k_{(\cdot)} \rangle \in \mathcal{I} \) and \( \tau_w \langle T_{\varphi_2} \cdots T_{\varphi_k}k_{(\cdot)}, k_{(\cdot)} \rangle \in \mathcal{I} \). It follows that

\[
\langle T_{\varphi_2} \cdots T_{\varphi_k}U_zk_w, U_zk_w \rangle = \langle T_{\varphi_2} \cdots T_{\varphi_k}k_{\varphi_z(w)}, k_{\varphi_z(w)} \rangle \in \mathcal{I}, \quad w \in \mathbb{B}_n,
\]

since \( \mathcal{I} \) is translation invariant. Using Corollary 4.2, we obtain

\[
\langle T_{\varphi_2} \cdots T_{\varphi_k}U_{(\cdot)}k_0, U_{(\cdot)}k_w \rangle \in \mathcal{I}.
\]
Similarly, we have
\[ \langle T_{\varphi_1}U(\cdot)k_w, U(\cdot)k_0 \rangle \in \mathcal{I}, \]
to obtain
\[ \langle T_{\varphi_2}\cdots T_{\varphi_k}U(\cdot)k_0, U(\cdot)k_w \rangle \langle T_{\varphi_1}U(\cdot)k_w, U(\cdot)k_0 \rangle \in \mathcal{I}. \]

It follows from Lemmas 3.6 and 3.7 that the mapping
\[ w \mapsto \langle T_{\varphi_2}\cdots T_{\varphi_k}U(\cdot)k_0, U(\cdot)k_w \rangle \langle T_{\varphi_1}U(\cdot)k_w, U(\cdot)k_0 \rangle \]
is uniformly continuous from each \( D(0, r) \) to \( \mathcal{I} \).

Choosing \( t > 2 > t' > 1 \) such that \( \frac{1}{t} + \frac{1}{t'} = 1 \), then we have by (3.5) that
\[
\left\| \langle T_{\varphi_2}\cdots T_{\varphi_k}U(\cdot)k_0, U(\cdot)k_w \rangle \langle T_{\varphi_1}U(\cdot)k_w, U(\cdot)k_0 \rangle \right\|
\leq \left\| \langle T_{\varphi_2}\cdots T_{\varphi_k}U(\cdot)1, U(\cdot)k_w \rangle \langle T_{\varphi_1}U(\cdot)1, U(\cdot)k_0 \rangle \right\|
\leq (1 - |w|^2)^{(\frac{1}{t'} - \frac{1}{t})(n+1)},
\]
where the function \((1 - |w|^2)^{(\frac{1}{t'} - \frac{1}{t})(n+1)}\) is integrable with respect to the measure \( d\lambda \) on \( \mathbb{B}_n \). Therefore,
\[
\langle T_{\varphi_1}T_{\varphi_2}\cdots T_{\varphi_k}k(\cdot), k(\cdot) \rangle = \int_{\mathbb{B}_n} \langle T_{\varphi_2}\cdots T_{\varphi_k}U(\cdot)k_0, U(\cdot)k_w \rangle \langle T_{\varphi_1}U(\cdot)k_w, U(\cdot)k_0 \rangle d\lambda(w)
\]
is also in \( \mathcal{I} \), as desired. This completes the proof of Proposition 4.7. \( \square \)

Given a translation invariant closed subalgebra \( \mathcal{I} \subset \text{BUC}(\mathbb{B}_n) \), we recall that \( \mathcal{I}' \) is defined by (4.4). Then combining Theorem 4.6 with Proposition 4.7 gives that the Toeplitz algebra \( \mathcal{T}^b(\mathcal{I}) \) is contained in the closed linear space \( \mathcal{T}^b_{\text{lin}}(\mathcal{I}') \).

**Proposition 4.8.** Suppose that \( \mathcal{I} \) is a translation invariant closed subalgebra of \( \text{BUC}(\mathbb{B}_n) \). Then
\[ \mathcal{T}^b(\mathcal{I}) \subset \mathcal{T}^b_{\text{lin}}(\mathcal{I}') \]
holds on the \( p \)-Bergman space \( L^p_b \).

**Proof.** Let \( A = T_{\varphi_1}T_{\varphi_2}\cdots T_{\varphi_m} \in \mathcal{T}^b(\mathcal{I}) \) with \( \varphi_1, \varphi_2, \ldots, \varphi_m \in \mathcal{I} \). It follows from Proposition 4.7 that \( \hat{A} \in \mathcal{I} \). By Propositions 2.2 and 2.3 in [5], we deduce that \( A \in \mathcal{A}^p \). Now we conclude by Theorem 4.6 that \( A \in \mathcal{T}^b_{\text{lin}}(\mathcal{I}') \). This finishes the proof of Proposition 4.8. \( \square \)

The next theorem generalizes the corresponding result obtained by Xia [12, Theorem 1.5] for the case of \( p = 2 \).

**Theorem 4.9.** On the \( p \)-Bergman space \( L^p_b \), we have that
\[ \mathcal{T}^b_{\text{lin}}[\text{BUC}(\mathbb{B}_n)] = \mathcal{T}^b[\text{BUC}(\mathbb{B}_n)] = \text{clos}(\mathcal{A}^p) \]
and
\[ \mathcal{T}^b_{\text{lin}}[C_0(\mathbb{B}_n)] = \mathcal{T}^b[C_0(\mathbb{B}_n)] = \mathcal{K}, \]
where \( \text{clos}(\mathcal{A}^p) \) denotes the norm closure of \( \mathcal{A}^p \) and \( \mathcal{K} \) denotes the set of compact operators on \( L^p_b \).

Before giving the proof of Theorem 4.9, we mention here that the Berezin transform of any bounded linear operator on \( L^p_b \) is in \( \text{BUC}(\mathbb{B}_n) \), see [6, Proposition 4.9] and its proof, or Proposition 8.3 and Lemma 9.1 in [9] if needed.
Proof of Theorem 4.9. First, we note that
\[ T_{\text{lin}}^b[\text{BUC}(\mathbb{B}_n)] \subset T^b[\text{BUC}(\mathbb{B}_n)] \quad \text{and} \quad T_{\text{lin}}^b[C_0(\mathbb{B}_n)] \subset T^b[C_0(\mathbb{B}_n)]. \]

On the other hand, it follows from Lemma 4.4 and Proposition 4.8 that
\[ T^b[\text{BUC}(\mathbb{B}_n)] \subset T_{\text{lin}}^b[\text{BUC}(\mathbb{B}_n)'] \subset T_{\text{lin}}^b[\text{BUC}(\mathbb{B}_n)] \]
and
\[ T^b[C_0(\mathbb{B}_n)] \subset T_{\text{lin}}^b[C_0(\mathbb{B}_n)'] \subset T_{\text{lin}}^b[C_0(\mathbb{B}_n)]. \]

Thus we obtain
\[ T_{\text{lin}}^b[\text{BUC}(\mathbb{B}_n)] = T^b[\text{BUC}(\mathbb{B}_n)] \quad \text{and} \quad T_{\text{lin}}^b[C_0(\mathbb{B}_n)] = T^b[C_0(\mathbb{B}_n)]. \]

Moreover, it follows from [5, Propositions 2.2 and 2.3] that
\[ T^b[\text{BUC}(\mathbb{B}_n)] \subset \text{clos}(A^p_s). \]

Recall that the Berezin transform \( \tilde{T} \in \text{BUC}(\mathbb{B}_n) \) if \( T \in A^p_s \) (see the paragraph before the proof of Theorem 4.9). Therefore, we deduce by Theorem 4.6 that
\[ T^b[\text{BUC}(\mathbb{B}_n)] = \text{clos}(A^p_s). \]

In order to finish the proof of this theorem, it remains to show that
\[ \mathcal{K} = T_{\text{lin}}^b[C_0(\mathbb{B}_n)]. \]

Let
\[ \mathcal{F} = \text{span}\{k_x \otimes k_y : x, y \in \mathbb{B}_n\}. \]

Since the linear span of the normalized reproducing kernels is dense in \( L^p_a \) and \( L^q_a \), we have that \( \mathcal{F} \) is dense in \( \mathcal{K} \). Now we are going to show \( \mathcal{F} \subset A^p_s \). To do this, we need only to show that \( k_x \otimes k_y \) is \( s \)-weakly localized for all \( x, y \in \mathbb{B}_n \).

In fact, for every \( k_x \otimes k_y \) we have
\[ |\langle (k_x \otimes k_y)k_z, k_w \rangle| = |\langle k_z, k_y \rangle| |\langle k_x, k_w \rangle|. \]

Suppose that \( x, y \in D(0, a) \) for some \( a > 0 \), then we have by Lemma 3.4 that
\[ |\langle k_z, k_y \rangle| |\langle k_x, k_w \rangle| \leq C_a |\langle k_z, k_\xi \rangle| |\langle k_\xi, k_w \rangle| \]
for all \( \xi \in D(0, a) \), where \( C_a \) is a positive constant depending only on \( a \). This gives that
\[
|\langle k_z, k_y \rangle| |\langle k_\xi, k_w \rangle| \leq \frac{C_a}{\lambda(D(0, a))} \int_{D(0, a)} |\langle k_\xi, k_z \rangle| |\langle k_\xi, k_w \rangle| d\lambda(\xi) \\
\leq \frac{C_a}{\lambda(D(0, a))} \int_{\mathbb{B}_n} |\langle k_z, k_\xi \rangle| |\langle k_\xi, k_w \rangle| d\lambda(\xi),
\]
where \( \lambda(D(0,a)) = \int_{D(0,a)} d\lambda \). Thus we have

\[
\sup_{z \in \mathbb{B}_n} \int_{B_n \setminus D(z,r)} |\langle k_z \otimes k_{y}\rangle k_z, k_w| \frac{\|K_z\|^2_2}{2^{\frac{q}{q+1}}} \|K_w\|^2_2 \frac{d\lambda(w)}{1 - 2^{\frac{2q}{q+1}}} \] 

\[
\leq \sup_{z \in \mathbb{B}_n} \int_{B_n \setminus D(z,r)} \int_{B_n \setminus D(z,z/2)} |\langle k_z, k_{\xi}\rangle| \|K_z\|^2_2 \frac{d\lambda(\xi)}{1 - 2^{\frac{2q}{q+1}}} \frac{d\lambda(w)}{1 - 2^{\frac{2q}{q+1}}} 
\]

\[
\leq \sup_{z \in \mathbb{B}_n} \int_{D(z,z/2)} \int_{B_n \setminus D(z/2)} |\langle k_z, k_{\xi}\rangle| \|K_z\|^2_2 \frac{d\lambda(\xi)}{1 - 2^{\frac{2q}{q+1}}} \frac{d\lambda(w)}{1 - 2^{\frac{2q}{q+1}}} 
\]

\[
= I_1(r) + I_2(r).
\]

For \( I_1(r) \), Fubini’s theorem gives us

\[
I_1(r) = \sup_{z \in \mathbb{B}_n} \int_{B_n \setminus D(z,r)} \int_{D(z,z/2)} |\langle k_z, k_{\xi}\rangle| \|K_z\|^2_2 \frac{d\lambda(\xi)}{1 - 2^{\frac{2q}{q+1}}} \frac{d\lambda(w)}{1 - 2^{\frac{2q}{q+1}}} 
\]

\[
= \sup_{z \in \mathbb{B}_n} \int_{D(z,z/2)} \int_{B_n \setminus D(z/2)} |\langle k_z, k_{\xi}\rangle| \|K_z\|^2_2 \|K_{\xi}\|^2_2 \frac{d\lambda(\xi)}{1 - 2^{\frac{2q}{q+1}}} 
\]

\[
\leq \left[ \sup_{\xi \in \mathbb{B}_n} \int_{B_n \setminus D(\xi,z/2)} |\langle k_{\xi}, k_{\xi}\rangle| \|K_{\xi}\|^2_2 \frac{d\lambda(\xi)}{1 - 2^{\frac{2q}{q+1}}} \right] \left[ \sup_{z \in \mathbb{B}_n} \int_{B_n} |\langle k_z, k_{\xi}\rangle| \|K_z\|^2_2 \frac{d\lambda(w)}{1 - 2^{\frac{2q}{q+1}}} \right] ,
\]

where the first inequality follows from that \( D(\xi, z/2) \subset D(z, r) \) for \( \xi \in D(z, z/2) \).

To estimate \( I_2(r) \), we note that

\[
I_2(r) = \sup_{z \in \mathbb{B}_n} \int_{B_n \setminus D(z,r)} \int_{B_n \setminus D(z,z/2)} |\langle k_z, k_{\xi}\rangle| \|K_z\|^2_2 \frac{d\lambda(\xi)}{1 - 2^{\frac{2q}{q+1}}} \frac{d\lambda(w)}{1 - 2^{\frac{2q}{q+1}}} 
\]

\[
\leq \sup_{z \in \mathbb{B}_n} \int_{B_n \setminus D(z,z/2)} \int_{B_n \setminus D(z, z/2)} |\langle k_z, k_{\xi}\rangle| \|K_z\|^2_2 \frac{d\lambda(\xi)}{1 - 2^{\frac{2q}{q+1}}} \frac{d\lambda(w)}{1 - 2^{\frac{2q}{q+1}}} 
\]

\[
\leq \left[ \sup_{z \in \mathbb{B}_n} \int_{B_n \setminus D(z,z/2)} |\langle k_z, k_{\xi}\rangle| \|K_z\|^2_2 \frac{d\lambda(\xi)}{1 - 2^{\frac{2q}{q+1}}} \right] \left[ \sup_{\xi \in \mathbb{B}_n} \int_{B_n} |\langle k_{\xi}, k_{\xi}\rangle| \|K_{\xi}\|^2_2 \frac{d\lambda(w)}{1 - 2^{\frac{2q}{q+1}}} \right] .
\]
By [5, Lemma 2.1], we get that
\[
\lim_{r \to \infty} \sup_{z \in \mathbb{B}_r} \int_{\mathbb{B}_r \setminus D(z,r)} |(k_x \otimes k_y)z, k_w)\|K_x\|_2\|K_w\|_2 \leq \lim_{r \to \infty} I_1(r) + \lim_{r \to \infty} I_2(r) = 0.
\]
Using the same arguments as above, one can verify that \(k_x \otimes k_y\) satisfies the other three conditions in Definition 3.2. Thus we obtain that
\[
k_x \otimes k_y \in \mathcal{A}_s^p,
\]
and hence \(\mathfrak{F} \subset \mathcal{A}_s^p\).

For any \(T \in \mathfrak{F} \subset \mathcal{K}\), we have that \(\tilde{T} \in C_0(\mathbb{B}_n)\). Since \(T \in \mathfrak{F} \subset \mathcal{A}_s^p\), Lemma 4.4 and Theorem 4.6 imply that
\[
T \in \mathcal{T}_{\text{lin}}^b[C_0(\mathbb{B}_n)],
\]
to get \(\mathfrak{F} \subset \mathcal{T}_{\text{lin}}^b[C_0(\mathbb{B}_n)]\). This gives that
\[
\mathcal{K} = \text{clos}(\mathfrak{F}) \subset \mathcal{T}_{\text{lin}}^b[C_0(\mathbb{B}_n)].
\]
Since \(\mathcal{T}_{\text{lin}}^b[C_0(\mathbb{B}_n)] \subset \mathcal{K}\), we finally obtain that
\[
\mathcal{K} = \mathcal{T}_{\text{lin}}^b[C_0(\mathbb{B}_n)].
\]
This completes the proof of Theorem 4.9. \(\square\)

Let us make a remark for the above theorem.

**Remark.** As mentioned in Section 1, it was proved by Xia [12, Theorem 1.5] that
\[
\mathcal{T}^b[L^\infty(\mathbb{B}_n, dv)] = \text{clos}(\mathcal{A}_s^2) = \mathcal{T}_{\text{lin}}^b[L^\infty(\mathbb{B}_n, dv)]
\]
and \(\text{clos}(\mathcal{A}_s^2)\) equals the \(C^*\)-algebra generated by \(\mathcal{A}_s^2\) for the case of \(L^2_a\). But Suárez obtained in [10, Theorem 7.3] that
\[
\mathcal{T}_{\text{lin}}^b[\text{BUC}(\mathbb{B}_n)] = \mathcal{T}_{\text{lin}}^b[L^\infty(\mathbb{B}_n, dv)] \quad \text{and} \quad \mathcal{T}^b[\text{BUC}(\mathbb{B}_n)] = \mathcal{T}^b[L^\infty(\mathbb{B}_n, dv)]
\]
hold on the \(p\)-Bergman space \(L^p_a\). Therefore, (4.9) can be rewritten as:
\[
\mathcal{T}^b[\text{BUC}(\mathbb{B}_n)] = \text{clos}(\mathcal{A}_s^2) = \mathcal{T}_{\text{lin}}^b[\text{BUC}(\mathbb{B}_n)] \quad \text{on the Bergman space } L^2_a.
\]
This means that Theorem 4.9 extends Xia’s result to case of the \(p\)-Bergman space with \(1 < p < \infty\).

Let \(1 < p < \infty\) and \(\text{VMO}^p(\mathbb{B}_n)\) denote the collection of vanishing mean oscillation functions on the unit ball (see Section 5 of [7]). Denote
\[
\text{VMO}^p_\infty(\mathbb{B}_n) = \text{VMO}^p(\mathbb{B}_n) \cap L^\infty(\mathbb{B}_n, dv).
\]
Furthermore, [7, Theorem 6.1] tells us that the Hankel operator \(H_\varphi\) is compact on \(L^p_a\) (\(1 < p < \infty\)) whenever \(\varphi \in \text{VMO}^p_\infty(\mathbb{B}_n)\). It follows that
\[
T_{\varphi_1}T_{\varphi_2} - T_{\varphi_1\varphi_2} = -H_{\varphi_1}^*H_{\varphi_2}
\]
is compact on \(L^p_a\) if \(\varphi_1\) and \(\varphi_2\) are both in \(\text{VMO}^p_\infty(\mathbb{B}_n)\).

We end this section by the following corollary, which gives a characterization for the Toeplitz algebra with symbols in an algebra lying between \(C_0(\mathbb{B}_n)\) and \(\text{VMO}^p_\infty(\mathbb{B}_n)\).

**Corollary 4.10.** Let \(1 < p < \infty\) and \(\mathcal{I}\) be an algebra such that
\[
C_0(\mathbb{B}_n) \subset \mathcal{I} \subset \text{VMO}^p_\infty(\mathbb{B}_n).
\]
Then we have
\[
\mathcal{T}^b(\mathcal{I}) = \text{clos}\{T_\varphi + K : \varphi \in \mathcal{I} \text{ and } K \in \mathcal{K}\} = \mathcal{T}_{\text{lin}}^b(\mathcal{I}),
\]
where \(\text{clos}\{T_\varphi + K : \varphi \in \mathcal{I} \text{ and } K \in \mathcal{K}\}\) denotes the norm closure of \(\{T_\varphi + K : \varphi \in \mathcal{I} \text{ and } K \in \mathcal{K}\}\).
Proof. First, we claim that
\[ \mathcal{T}^b(\mathcal{I}) \subset \text{clos}\{T_\varphi + K : \varphi \in \mathcal{I} \text{ and } K \in \mathcal{K}\}. \]
Let us assume this for the moment and we will give its proof later. As \( \mathcal{T}^b_{\text{lin}}(\mathcal{I}) \subset \mathcal{T}^b(\mathcal{I}) \), it is enough to show that
\[ \text{clos}\{T_\varphi + K : \varphi \in \mathcal{I} \text{ and } K \in \mathcal{K}\} \subset \mathcal{T}^b_{\text{lin}}(\mathcal{I}). \]
According to Theorem 4.9, we have
\[ \mathcal{K} = \mathcal{T}^b_{\text{lin}}[C_0(\mathbb{B}_n)], \]
to obtain
\[ \text{clos}\{T_\varphi + K : \varphi \in \mathcal{I} \text{ and } K \in \mathcal{K}\} = \text{clos}\{T_\varphi + T_\psi : \varphi \in \mathcal{I} \text{ and } \psi \in C_0(\mathbb{B}_n)\} \subset \text{clos}\{T_\varphi : \varphi \in \mathcal{I}\} \quad (\text{since } C_0(\mathbb{B}_n) \subset \mathcal{I}) \]
\[ = \mathcal{T}^b_{\text{lin}}(\mathcal{I}). \]
In order to complete the proof, we shall prove the above claim by induction. First, we have
\[ T_{\varphi_1} \in \text{clos}\{T_\varphi + K : \varphi \in \mathcal{I} \text{ and } K \in \mathcal{K}\} \]
if \( \varphi_1 \in \mathcal{I} \). Then we suppose that
\[ T_{\varphi_1} \cdots T_{\varphi_{m-1}} \in \text{clos}\{T_\varphi + K : \varphi \in \mathcal{I} \text{ and } K \in \mathcal{K}\} \]
for any functions \( \varphi_1, \cdots, \varphi_{m-1} \in \mathcal{I} \). Let \( \varphi_m \) be in \( \mathcal{I} \), then we have
\[ T_{\varphi_1} \cdots T_{\varphi_{m-1}} T_{\varphi_m} = T_{\varphi_1} \cdots T_{\varphi_{m-2}} T_{\varphi_{m-1} \varphi_m} + T_{\varphi_1} \cdots T_{\varphi_{m-2}} (T_{\varphi_{m-1} \varphi_m} - T_{\varphi_{m-1} \varphi_m}). \]
Note that \( T_{\varphi_{m-1} \varphi_m} - T_{\varphi_{m-1} \varphi_m} \) is compact on \( L^\alpha_b \), since \( \varphi_{m-1} \) and \( \varphi_m \) are in \( \mathcal{I} \subset \text{VMO}^\alpha_{\text{loc}}(\mathbb{B}_n) \). By the induction hypothesis, we have
\[ T_{\varphi_1} \cdots T_{\varphi_{m-1}} T_{\varphi_m} \in \text{clos}\{T_\varphi + K : \varphi \in \mathcal{I} \text{ and } K \in \mathcal{K}\}. \]
This yields that
\[ T_{\varphi_1} \cdots T_{\varphi_{m-1}} T_{\varphi_m} \in \text{clos}\{T_\varphi + K : \varphi \in \mathcal{I} \text{ and } K \in \mathcal{K}\}, \]
to get
\[ \mathcal{T}^b(\mathcal{I}) \subset \text{clos}\{T_\varphi + K : \varphi \in \mathcal{I} \text{ and } K \in \mathcal{K}\}. \]
This proves the claim and so the proof of Corollary 4.10 is finished. \( \square \)

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1 College of Mathematics and Statistics, Chongqing University, Chongqing, 401331, P. R. China  
Email address: shengkunwu@foxmail.com

2 College of Mathematics and Statistics, Chongqing University, Chongqing, 401331, P. R. China  
Email address: xianfengzhao@cqu.edu.cn