On the Verification of Belief Programs

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Abstract

In a recent paper, Belle and Levesque proposed a framework for a type of program called belief programs, a probabilistic extension of GOLOG programs where every action and sensing result could be noisy and every test condition refers to the agent’s subjective beliefs. Inherited from GOLOG programs, the action-centered feature makes belief programs fairly suitable for high-level robot control under uncertainty. An important step before deploying such a program is to verify whether it satisfies properties as desired. At least two problems exist in doing verification: how to formally specify properties of a program and what is the complexity of verification. In this paper, we propose a formalism for belief programs based on a modal logic of actions and beliefs. Among other things, this allows us to express PCTL-like temporal properties smoothly. Besides, we investigate the decidability and undecidability for the verification problem of belief programs.

1 Introduction

The GOLOG (Levesque et al. 1997) family of agent programming language has been proven to be a powerful means to express high-level agent behavior. Combining GOLOG with probabilistic reasoning, Belle and Levesque (2015) proposed an extension called belief programs, where every action and sensing result could be noisy. Along with the feature that test conditions refer to the agent’s subjective beliefs, belief programs are fairly suitable for robot control in an uncertain environment.

For safety and economic reasons, verifying such a program to ensure that it meets certain properties as desired before deployment is essential and desirable. As an illustrative example, consider a robot searching for coffee in a one-dimensional world as in Fig 1. Initially, the horizontal position $h$ of the robot is at 0 and the coffee is at 2. Additionally, the robot has a knowledge base about its own location (usually a belief distribution, e.g. a uniform distribution among two points $\{0, 1\}$). The robot might perform noisy sensing $\text{sencfe}$ to detect whether its current location has the coffee or not and an action $\text{east}(1)$ to move 1 unit east. A possible belief program is given in Table 1. The robot continuously uses its sensor to detect whether its current location has the coffee or not (line 2-3). When it is confident enough,$^1$ it tries to move 1 unit east (line 5). If it still does not fully believe it reached the coffee, i.e. position at 2 (line 1), it repeats the above process. The program is an online program as its execution depends on the outcome of sensing.

Some interesting properties of the program are:

1. $\mathbf{P1}$: whether the probability that within 2 steps of the program the robot believes it reached the coffee with certainty is higher than 0.05;
2. $\mathbf{P2}$: whether it is almost certain that eventually the robot believes it reached the coffee with certainty.

Often, the above program properties are specified by temporal formulas via Probabilistic Computational Tree Logic (PCTL) in model checking. Obtaining the answers is non-trivial as the answers depend on both the physical world (like the robot’s position and action models of actuators and sensors) and the robot’s epistemic state (like the robot’s beliefs about its position and action models). There are at least two questions in verifying belief programs: 1. how can we formally specify temporal properties as above; 2. what is the complexity of the verification problem?

\begin{table}[h]
\centering
\begin{tabular}{l}
\hline
1 \textbf{while} $B(h = 2) < 1$ \textbf{do} \\
\hline
2 \textbf{while} $\text{Conf}(h, 0.5) \leq 0.5$ \textbf{do} \\
3 \hspace{1cm} $\text{sencfe}$; \\
4 \hspace{1cm} \textbf{endWhile} \\
5 \hspace{1cm} $\text{east}(1)$; \\
6 \hspace{1cm} \textbf{endWhile} \\
\hline
\end{tabular}
\caption{A online belief program for the robot.}
\end{table}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{robot.png}
\caption{A coffee searching robot.}
\end{figure}

\footnote{$^1$The agent’s confidence $\text{Conf}(h, n)$ of a random variable $h$ wrt a number $n$ is defined as its belief that $h$ is somewhere in the interval $[\text{Exp}(h) - n, \text{Exp}(h) + n]$, here $\text{Exp}(h)$ is the expectation of $h$.}
The semantics of belief programs proposed by Belle and Levesque (2015) is based on the well-known BHL logic (Bacchus, Halpern, and Levesque 1999) that combines situation calculus and probabilistic reasoning in a purely axiomatic fashion. While verification has been studied in this fashion in the non-probabilistic case (De Giacomo, Ternovska, and Reiter 2019), it is somewhat cumbersome as it relies heavily on the use of second-order logic and the μ-calculus. For instance, consider a domain where the robot is programmed to serve coffee for guests on request (Claßen 2013). An interesting property of the program is whether every request will eventually be served. Such a property is then expressed as follows:

\[(\forall x, \delta, s)Trans^{s}(\delta_0, \delta, do(requestCoffee(x), s) ) \supset EventuallyServed(x, \delta, do(requestCoffee(x), s) )\]

where Trans\(^s\) refers to the transitive closure of the Trans predicate (a predicate axiomatically defining the transitions among program configurations) and EventuallyServed is defined by

\[EventuallyServed(x, \delta_1, s_1) :=\]

\[\mu_{\delta, s_1}\{((\exists s'')s = do(selectRequest(x), s'') \lor ((\exists \delta', \delta)Trans(\delta, \delta, \delta', s') \land (\forall \delta', s') Trans(\delta, s, \delta', s') \supset P(\delta', s' ))\} (\delta_1, s_1).\]

Here the notion \(\mu_{\delta, s_1}\) denotes a least fixpoint according to the formula \(\nu \delta \{\mu_{\delta, s_2}\Phi(P, y)(\delta) \equiv (\forall y)\Phi(P, y) \supset (P(y) \supset P(\delta))\}\). We do not go into more details here but refer interested readers to (Claßen 2013).

In this paper, we propose a new semantics for belief programs based on the logic DS\(_p\) (Liu and Feng 2021), a modal version of the BHL logic with a possible-world semantics. Such a modal formalism makes it smoother than axiomatic approaches to express temporal properties like eventually and globally by using the usual modalities F and G in temporal logic. Subsequently, we study the boundary of decidability of the verification problem. As it turns out, the result is strongly negative. However, we also investigate a case where the problem is decidable.

The rest of the paper is organized as follows. In section 2, we introduce the logic DS\(_p\). Subsequently, we present the proposed semantics and specification of temporal properties for belief programs in section 3. In section 4, we study the boundary of decidability of the verification problem in a specific dimension. Section 5 considers a special case where the problem is decidable. In section 6 and 7, we review related work and conclude.

2 Logical Foundation

2.1 The Logic DS\(_p\)

The logic DS\(_p\) is a modal variant of the epistemic situation calculus. There are two sorts: object and action. Implicitly, we assume that number is a sub-sort of object and refers to the computable numbers \(\mathbb{C}\).\(^2\)

\(^2\)We use the computable numbers as they are still enumerable and allow us to refer to certain real numbers such as \(\sqrt{2}\) and Euler’s number e.

The Language We use DS\(_p\)’s first-order fragment with equality. The logic features a countable set of so-called standard names \(N\), which are isomorphic with a fixed universe of discourse. Roughly, this amounts to having an infinite domain closure axiom together with the unique name assumption, \(N = N_0 \cup N_A\) where \(N_0\) and \(N_A\) are standard object names and standard action names, respectively. Function symbols are divided into fluent function symbols and rigid function symbols. For simplicity, all action functions are rigid and we do not include predicate symbols. Fluents vary as the result of actions, yet denotations of rigid functions are fixed. The language includes modal operators \(B\) and \(O\) for degrees of belief and only-believing, respectively. Finally, there are two special fluent functions: a function \(l(a)\) specifies action \(a\)’s likelihood and a binary function \(oi\) encodes the observational indistinguishability among actions. The idea is that in an uncertain setting, instead of saying an action might have non-deterministic effects, we say the action is stochastic and has non-deterministic alternatives, which are observationally indistinguishable by the agent and each of which has deterministic effects.

The terms of the language are formed in the usual way from variables, standard names and function symbols. A term is said to be rigid if it does not mention fluents. Ground terms are terms without variables. Primitive terms are terms of the form \(f(n_1, \ldots, n_k)\), where \(f\) is a function symbol and \(n_i\) are standard object names. We denote the sets of primitive terms of sort object and action as \(P_O\) and \(P_A\), respectively. While standard object names are syntactically like constants, we require that standard action names are all the primitive action terms, i.e. \(N_A = P_A\). For example, the sensing action \(sense(1)\), where the robot receives a positive signal, is considered as a standard action name. Furthermore, \(Z\) refers to the set of all finite sequences of standard action names, including the empty sequence \(\langle\rangle\). We reserve standard names \(\top\), \(\bot\) in \(N_O\) for truth values (to simulate predicates).

Atomic formulas are expressions of the form \(t_1 = t_2\) for terms \(t_1, t_2\). Arbitrary formulas are formed with the usual logical operators \(\neg, \land, \lor\), the quantifier \(\forall\), and modal operators \([t]\), where \([t]\) is an action term, \(Bt_1 = t_2\) and \(Ot_1 = t_2\), where \(t_1\) are formulas and the \(r_i\) rigid terms of sort number.

\([t]\alpha\) should be read as “\(\alpha\) holds after action \(t\)”, \(\exists t\alpha\) as “\(\alpha\) holds after any sequence of actions,” \(B(\alpha) : \tau\) as “\(\alpha\) is believed with a probability \(\tau\),” \(O(\alpha_1 : r_1, \ldots, \alpha_k : r_k)\) may be read as “the acting \(\alpha_i\) with a probability \(r_i\) are all that is believed”. Similarly, \(O\alpha\) means “\(\alpha\) is only known” and is an abbreviation for \(O(\alpha) : 1\). For action sequence \(z : t_1 \cdots t_k\), we write \([z]\alpha\) to mean \([t_1]\cdots[t_k]\alpha\), which is the formula obtained by substituting all free occurrences of \(x\) in \(\alpha\) by \(t\). As usual, we treat \(\alpha \lor \beta, \alpha \land \beta, \alpha \equiv \beta,\) and \(\exists t\alpha\) as abbreviations.

A sentence is a formula without free variables. We use \(\forall x\) as an abbreviation for \(\forall x(a \equiv x)\), and \(\exists x\) for its negation. A formula with no \(\Box\) is called bounded. A formula with no \(\Box\) or \([t]_a\) is called static. A formula with no \(B\) or \(O\) is called objective. A formula with no fluent, \(\Box\) or \([t]_a\) outside \(B\) or \(O\) is called subjective. A formula with no \(B\), \(O\), \(\Box\), \([t]_a\), \(l\), \(oi\) is called a fluent formula. A fluent formula
without fluent functions is called a rigid formula.

The Semantics The semantics is given in terms of possible worlds. A world $w$ is a mapping from the primitive terms $(P_O \cup P_P)$ and $Z$ to $\mathcal{N}$ of the right sort, satisfying rigidity and arithmetical correctness. We denote the set of all such worlds as $W$. Given $w \in W$, $z \in Z$, and a ground term $t$, we define $|t|^w$ (the denotation for $t$ given $w$, $z$) by:

1. If $t \in N$, then $|t|^w = t$;
2. $|f(t_1, \ldots, t_k)|^w = w[f(|t_1|^w, \ldots, |t_k|^w), z]$.

For a rigid ground term $t$, we use $|t|$ instead of $|t|^w$. We will require that $l(a)$ is of sort number, and $oi(a, a')$ only takes values $\top$ or $\bot$, and $oi$ is an equivalence relation (reflexive, symmetric, and transitive). Intuitively, $l(a)$ denotes the likelihood of action $a$, while $oi(a, a')$ means $a$ and $a'$ are mutual alternatives. In the example of Fig. 1, the robot might perform a stochastic action east($x, y$), where $x$ is its intended moving distance and $y$ is the actual outcome selected by nature. Then, $oi$ (east(1,0), east(1,1)) says that nature can non-deterministically select 0 or 1 as a result for the intended value 1.

A distribution $d$ is a mapping from $W$ to $\mathbb{R}^{\geq 0}$ and an epistemic state $e$ is any set of distributions. By a model, we mean a triple $(e, w, z)$. To account for $B$ and $O$ after actions, we need to extend the fluents $l$, $oi$ from actions to action sequences:

**Definition 1.** Given a world $w$, we define:

1. $l^*(w) : \mathcal{W} \times Z \mapsto \mathbb{R}^{\geq 0}$ as
   
   $l^*(w, \langle \rangle) = 1;
   
   l^*(w, z \cdot a) = l^*(w, z) \times n$ where $w[l(a), z] = n.$

2. $z \sim_w z'$ as
   
   $\langle \rangle \sim_w z' \iff z' = \langle \rangle;$

   $\sim_w z' \iff z' = z^a \cdot z^w, z \sim_w z^w, w[oi(a, a^w), z] = \top.$

To obtain a well-defined sum over uncountably many worlds, some conditions are used for $B$ and $O$:

**Definition 2.** We define $\text{BND}$, $\text{EQ}$, $\text{NORM}$ for any distribution $d$ and any set $\mathcal{V} = \{(w_1, z_1), (w_2, z_2), \ldots\}$ as follows:

1. $\text{BND}(d, V, r)$ iff $\exists k: (w_1, z_1), \ldots, (w_k, z_k) \in \mathcal{V}$ such that $\sum_{i=1}^{k} d(w_i, z_i) > r$.

2. $\text{EQ}(d, V, r)$ iff $\text{BND}(d, V, r)$ and there is no $r' < r$ such that $\text{BND}(d, V, r')$ holds.

3. For any $\mathcal{U} \subseteq \mathcal{V}$, $\text{NORM}(d, \mathcal{U}, V, r)$ iff $\exists b \neq 0$ such that $\text{EQ}(d, U, b \times r) \land \text{EQ}(d, V, b)$.

Intuitively, given $\text{NORM}(d, U, V, r)$, $r$ can be viewed as the normalized sum of the weights of worlds in $U$ wrt $d$ in relation to $V$. Here $\text{EQ}(d, V, r)$ expresses that the weight of the worlds wtr $d$ in $V$ is $b$, and finally $\text{BND}(d, V, b)$ ensures the weights of worlds in $V$ is bounded by $b$. In essence, even if $W$ is uncountable, the condition $\text{NORM}$ ensures $d$ is in fact discrete, i.e. only countably many worlds have non-zero weight wtr $d$ (Belle, Lakemeyer, and Levesque 2016).

The truth of sentences in $\mathcal{D}S_p$ is defined as:

$$e, w, z \models t_1 = t_2 \iff |t_1|^w = |t_2|^w$$

...Rigidity: If $t$ is rigid, then for all $(w, z), (w', z'), w[t, z] = w'[t, z']$. Arithmetical Correctness: Any arithmetical expression is rigid and has its standard value.

- $e, w, z \models \neg \alpha \iff e, w, z \not\models \alpha$;
- $e, w, z \models \alpha \land \beta \iff e, w, z \models \alpha$ and $e, w, z \models \beta$;
- $e, w, z \models \forall x. \alpha \iff e, w, z \models \alpha_n$ for every standard name $n$ of the right sort;
- $e, w, z \models \square \alpha \iff e, w, z \models \alpha$ for all $z \in Z$.

To prepare the semantics of epistemic operators, let $\mathcal{W}_\alpha^z = \{ (w', z') \mid z' \sim_w w, e, w', \langle \rangle \models \{ z' \} \}$.

**Definition 3.** Given $w \in W$, $d \in D$, $z \in Z$, we define:

- $w_z$ as a world such that for all primitive terms $t$ and $z' \in Z$, $w_z[t, z'] = w[t, z' \cdot z]$

- $d_z$ a mapping such that for all $w \in W$, $d_z(w) = \sum_{w' : d(w') > 0} \sum_{z' : z' \sim_w z, w'_z = w} d(w') \times l^*(w', z')$.

$d_z$ is called the progressed world of $w$ while $d_z$ is called the progressed distribution wtr $z$. A remark is that the $d_z$ might not be regular for a regular $d$. For example, if the likelihood of a ground sensing action $t_{sen}$ is zero in all worlds with non-zero weights, then $\text{EQ}(d_{t_{sen}}, \mathcal{W}_{t_{sen}}^\{true\})$, and hence we define:

**Definition 4.** A distribution $d$ is compatible with action sequence $z$, $d \sim_{comp} z$ iff $d_z \in D$; given an epistemic state $e$, the set $e_z = cl(\{ d_z \mid d \in e \cap D, d \sim_{comp} z \})$ is called the progressed epistemic state of $e$ wtr $z$, here $cl(\cdot)$ is a closure operator.

Intuitively, $d \sim_{comp} z$ ensures $z$ has non-zero likelihood in at least one world whose weight is non-zero in $d$. As a consequence, $d \sim_{comp} \langle \rangle$ iff $d \in D$. Note that the progressed epistemic state of $e$ is only about its regular subset $e \cap D$ and $e_z \subseteq D$, therefore $e \neq e_\emptyset$ in general.

The truth of $B$ and $O$ is given by:

- $e, w, z \models B(\alpha : r) \iff \forall d \in e_z, \text{NORM}(d, W^{\alpha} \cdot \text{TRUE}, n)$ for $n \in \mathbb{C}$ and $n = |r|$

- $e, w, z \models O(\alpha_1 : r_1, \ldots, \alpha_k : r_k) \iff \forall d \in e_z \text{ for all } 1 \leq i \leq k, \text{NORM}(d, W^{\alpha_i} \cdot \text{TRUE}, n_i)$ for $n_i \in \mathbb{C}$ and $n_i = |r_i|$

For any sentence $\alpha$, we write $e, w, z \models \alpha$ instead of $e, w, z \models \alpha$. When $\Sigma$ is a set of sentences and $\alpha$ is a sentence, we write $\Sigma \models \alpha$ (read: $\Sigma$ logically entails $\alpha$) to mean that for every set of regular distributions $e$ and $w$, if

\footnote{More precisely, $cl(\cdot)$ is the closure operator of the metric space $(\mathcal{D}, \rho)$ where $\rho(d, d') = \sum_{w \in W} |d(w) - d'(w)|$ for $d, d' \in D$. The closure operator is important to ensure a correct semantic of progression in $\mathcal{D}S_p$ as Liu and Feng (2021) shows that the set of discrete distributions that satisfies a given belief is a closed set in $(\mathcal{D}, \rho)$.}
is a formula of the form \( \vec{h} \) is not necessarily the same as the BAT believed by the agent. Similarly, we write \( e \models \alpha \) instead of \( e, w \models \alpha \) if \( \alpha \) is subjective.

2.2 Basic Action Theories and Projection

Besides the usual +, \( \times \), it is desirable to include some usual mathematical functions as logical terms. We achieve this by axioms. We call these axioms definitional axioms,\(^5\) such functions as definitional functions, and terms constructed by definitional functions as definitional terms. E.g. the following axiom specifies the uniform distribution \( U_{\{0,1\}} \):

\[
\forall v, \forall u, U_{\{0,1\}}(u) = v \equiv (u = 0 \lor u = 1) \land v = 0.5 \\
\lor \neg (u = 0 \lor u = 1) \land v = 0 \tag{1}
\]

Basic Action Theories \quad BATs were first introduced by Reiter (2001) to describe the dynamics of an application domain. Given a finite set of fluents \( \mathcal{H} \), a BAT \( \Sigma \) over \( \mathcal{H} \) consists of the union of the following sets:

- \( \Sigma_{\text{post}} \): A set of successor state axioms (SSAs), one for each fluent \( h \) in \( \mathcal{H} \), of the form \( \Box h(p) = u \equiv \gamma h \) to characterize action effects, also providing a solution to the frame problem (Reiter 2001). Here \( \gamma h \) is a fluent formula with free variables \( \vec{p} \), \( u \) and it is functional in \( u \),

- \( \Sigma_{\text{no}} \): A single axiom of the form \( \Box oi(a, a') = T \equiv \psi \) to represent the observational indistinguishability relation among actions. Here \( \psi \) is a rigid formula.\(^7\)

- \( \Sigma_{\text{l}} \): A single likelihood axiom (LA) of the form \( \Box \mathcal{L}(a) = \mathcal{L}(a) \), here \( \mathcal{L}(a) \) is a definitional term with \( a \) free.

Besides BATs, we need to specify what holds initially. This is achieved by a set of fluent sentences \( \Sigma_0 \). By belief distribution, we mean the joint distribution of a finite set of random variables. Formally, assuming all fluents in \( \mathcal{H} \) are nullary,\(^8\) \( \mathcal{H} = \{ h_1, \ldots, h_m \} \), a belief distribution \( B^f \) of \( \mathcal{H} \) is a formula of the form \( \forall \vec{u}, B(h = \vec{u}) = \gamma h \) where \( \vec{u} \) is a set of variables, \( h = \vec{u} \) stands for \( \bigwedge h_i = u_i \), and \( f \) is a definitional function of sort number with free variables \( \vec{u} \). Finally, by a knowledge base (KB), we mean a sentence of the form \( \gamma (B^f \land \Sigma) \). Note that the BAT of the actual world is not necessarily the same as the BAT believed by the agent.

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\(^{5}\)In the rest of the paper, whenever we write logical entailment \( \Sigma \models \alpha \), we implicitly mean \( \Sigma \cup \Delta \models \alpha \), where \( \Delta \) is the set of all definitional axioms of functions involved in \( \Sigma \) and \( \alpha \).

\(^{6}\)Free variables are implicitly universally quantified from the outside. The \( \Box \) modality has lower syntactic precedence than the connectives, and \( [\cdot] \) has the highest priority.

\(^{7}\)The rigidity here is crucial for properties like introspection and regression, see (Liu and Lakemeyer 2021).

\(^{8}\)Allowing fluents with arguments would result in joint distribution over infinitely many random variables, which is generally problematic in probability theory (Belle and Levesque 2018).

Example 1. The following is a BAT \( \Sigma \) for our coffee robot:

\[
\Box \lnot h = u \equiv \exists x, y.a = east(x, y) \land h = u \\
\lor \forall x, y.a \neq east(x, y) \land h = u \\
\Box \lnot o_i(a, a') = T \equiv \exists x, y, g'.a = east(x, y) \\
\land a' = east(x, y) \lor \exists y.a = sencef(y) \land a' = a \\
\Box (a) = \mathcal{L}(a) \text{ with } \\
\mathcal{L}(a) = \{ \begin{array}{ll}
U_{\{x,x-1\}}(y) & \exists x, y.a = east(x, y) \\
\theta_{\text{noisy}}(h, y) & \exists y.a = sencef(y)
\end{array}
\}
\]

where \( \theta_{\text{noisy}}(x, y) \) is defined as \(^9\)

\[
\theta_{\text{noisy}}(x, y) = \begin{cases}
\theta(x) & y = 1 \\
1 - \theta(x) & y = 0
\end{cases}
\]

In English, the robot’s position \( h \) can only be affected by \( east(x, y) \) and the value is determined by nature’s choice \( y \), not the intended value \( x \). If the robot is 1 unit away from the position 2 (\( x = 2 \)), it has a low likelihood (0.1), sensing returns 1. Initially, the robot is 1 unit away from the position 2 and when the robot is at 2 (\( x = 2 \)) the coffee is located, with a high likelihood (0.8), sensing returns 1 and when the robot is 1 unit away from the position 2 (\( x \in \{1, 3\} \)), with a low likelihood (0.1), sensing returns 0. Initially, the robot is at a certain non-positive position and it believes its position distributes uniformly among \( \{0, 1\} \). Furthermore, although its sensor is noisy, it believes the sensor is accurate (\( \theta_{\text{acc}}(x, y) \)).

Projection by Progression \quad Projection in general is to decide what holds after actions. Progression is a solution to projection and the idea is to change the initial state according to the effects of actions and then evaluate queries against the updated state. Lin and Reiter (1997) showed that progression is only second order definable in general. However, Liu and Feng (2021) showed that if all fluents are nullary, for the objective fragment, progression is first-order definable.

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\(^{9}\)Here, \( ^{\infty} \) should be understood as a finite disjunction. For readability, we write the definitional functions in this form, they should be understood as logical formulas as Eq. (1).
where $I$ is a definitional function given by
\[ I(u, u', a, t) = \begin{cases} 1 & \text{Pro}(h = u', a) \\ 0 & \text{o.w.} \end{cases} \]
Here $\psi$ is the RHS of $\Sigma_{a_{i}}$. If $t$ is a sensing action, then $f'$ is given by
\[ f'(u) = \frac{1}{2} f(u) \times L(t)_{u} \]
and $\eta$ is a normalizer as $\sum \in (\Sigma_{a_{i}}) \times L(t)_{u} = 1$.

Example 2. Let $O(B^{j} \land \Sigma')$ be as in Example 1, then its progression wrt the stochastic action $\text{east}(1, 1)$ is $O(B^{j} \land \Sigma')$, the progression of $O(B^{j} \land \Sigma')$ wrt the sensing action $\text{sence}(1)$ is $O(B^{j'} \land \Sigma')$ where $f'$ and $f''$ are given by:
\[ f'(u) = \begin{cases} 1 & u \in \{0, 2\} \\ 0 & u = 1 \text{ and } f''(u) = \begin{cases} 1 & u = 2 \\ 0 & \text{o.w.} \end{cases} \end{cases} \]

Avoiding Infinite Summation A notable point above is that progression requires infinite summation. $\mathcal{DS}_{p}$ treats summation as a rigid logical term just like $+, \times$ and disregards the computational issues therein. Nevertheless, to ensure decidability of the logic, one needs to avoid infinite summation.

Consequently, we have the following restrictions. Firstly, we assume that only two types of action symbol are used: stochastic actions $s_{a_{1}}, \ldots, s_{a_{k}}$ and sensing $s_{e_{1}}, \ldots, s_{e_{k}}$. Moreover, parameters of stochastic action $s_{a}(\vec{x}, \vec{y})$ are divided into two parts, where $\vec{x}$ is a set of controllable and observable parameters and $\vec{y}$ is a set of uncontrollable and unobservable parameters. Parameters of sensing $s_{e}(\vec{y})$ are all observable yet uncontrollable by the agent. Additionally, we require:

1. $\phi \in \Sigma_{a_{i}}$ has the form $\psi \equiv \psi_{sa} \lor \psi_{sen}$ with $\psi_{sa} \equiv \bigvee_{v} \exists \vec{x}. \exists \vec{y}. \exists \vec{z}. \phi_{a}. s_{a}(\vec{x}, \vec{y})$ and $\psi_{sen} \equiv \bigvee_{v} \exists \vec{x}. \exists \vec{y}. s_{e}(\vec{y}) \land v = \sum_{\vec{x}, \vec{y}}. \phi_{sen}(\vec{y})$
2. $\Sigma_{l}$ is of the form $\square(l(a) = v) \equiv \bigvee_{v} \exists \vec{x}. \exists \vec{y}. s_{a}(\vec{x}, \vec{y}) \land v = \sum_{\vec{x}, \vec{y}}. \phi_{sen}(\vec{y})$,

where $\Sigma_{sa}$ and $\Sigma_{sen}$ are given by: (free variables are implicitly universally quantified from the outside)
\[ L_{sa}(\vec{x}, \vec{y}) = v \equiv \bigvee_{j} \phi_{sa}(\vec{x}, \vec{y}) \land v = \sum_{j} \phi_{sa}(\vec{x}, \vec{y}) \]
\[ L_{sen}(\vec{y}) = v \equiv \bigvee_{j} \phi_{sen}(\vec{y}) \land v = \sum_{j} \phi_{sen}(\vec{y}) \]

here $\phi_{sa}(\vec{x})$ and $\phi_{sa}(\vec{x})$ are rigid terms with variables $\vec{x}$; $\phi_{sa}(\vec{x})$, the likelihood contexts, are fluent formulas with free variables among $\vec{x}$; $\phi_{sa}$ and $\phi_{sen}$ are rigid terms, $\phi_{sa}$ are fluent without variables.

Besides, we require that likelihood contexts are disjoint and complete: 1) for all $i$ and distinct $j_{1}, j_{2}$ $|\forall \vec{x}. (\phi_{sa}(\vec{x}) \lor \sum_{j} \phi_{sa}(\vec{x}))$; 2) $|\forall \vec{x}. \bigvee_{j} \phi_{sa}(\vec{x})$ for all $i$; 3) $|\sum_{j} \phi_{sa}(\vec{x}) = 1$ for all $i, j$.

3. $B^{j}$ in $KB$ is finite, namely, of the form $f(u) = v \equiv \bigvee_{j} \vec{u} = n_{i} \lor v = r_{i}$ and $t_{i} \equiv 1$.

Intuitively, the first two conditions ensure that for any $s_{a}(\vec{x}, \vec{y})$, only finitely many alternatives, which satisfy $\vec{y} = \phi_{sa}(\vec{x})$, have nonzero likelihood; similarly, sensing only has finitely many outcomes: $\vec{y} = \phi_{sen}(\vec{x})$. The third item says that only finitely many fluent values are believed with non-zero degree. With these restrictions, $\sum_{a} f(u)$ can be replaced by the finite sum $\sum_{a} f(n_{i})$ and $\sum_{aw} L(a)$ can be replaced by the finite sum $\sum_{a} \phi_{sen} c_{i} j_{i}$. The BAT and KB in Example 1 satisfy all the above conditions. A remark is that given a KB with a finite belief distribution and a BAT satisfying the above conditions, the belief distribution of its progression is still finite.

3 The Proposed Framework

3.1 Belief Programs

The atomic instructions of our belief programs are the so-called primitive programs which are actions that suppress their uncontrollable parameters. A primitive program $\rho$ can be instantiated by a ground action $t_{a}$, i.e. $\rho \rightarrow t_{a}$, if $\Sigma_{a_{i}} \models \exists \vec{y}. o_{i}(\vec{y}) \land t_{a} = T$, where $o_{i}(\vec{y})$ is the action that restores its suppressed parameters by $\vec{y}$. For instance, $\text{east}(1, 1) \rightarrow (\text{east}(1, 1), \text{sence} \rightarrow \text{sence}(1))$.

Definition 5. A program expression $\delta$ is defined as:
\[ \delta ::= \rho[\alpha?; \delta][\delta]; \delta \]

Namely, a program expression can be a primitive program $\rho$, a test $\alpha$ where $\alpha$ is a static subjective formula without $O$, or constructed from sub-program by sequence $\delta$, choice $\delta$, nondeterministic choice $\delta$, and non-deterministic iteration $\delta^{*}$.

Furthermore, if statements and while loops can be defined as abbreviations in terms of these constructs:

if $\alpha$ then $\delta_{1}$ else $\delta_{2}$ endif $::= [\alpha?; \delta_{1}][\neg \alpha?; \delta_{2}]

while $\alpha$ do $\delta$ endwhile $::= \alpha?; \delta^{*}; \neg \alpha$?

Given BATs $\Sigma, \Sigma'$, the initial state axioms $\Sigma_{0}$, a KB $O(B^{j} \land \Sigma')$, and a program expression $\delta$, a belief program $P$ is a pair $P = (\Sigma_{0} \cup \Sigma \cup O(B^{j} \land \Sigma'), \delta)$. An example of a belief program is where $\delta$ is given by Table 1 and $\Sigma_{0}, \Sigma_{k}, KB$ are given by Example 1.

In order to handle termination and failure, we reserve two nullary fluent $\text{Final}$ and $\text{Fail}$. Moreover, $[\Box[a] \text{Final} = u \equiv a = \epsilon \land u = T \lor \text{Final} = u$ (likewise for $\text{Fail}$ with action $f$) is implicitly assumed to be part of $\Sigma$ and $\Sigma'$. Additionally, $\Sigma_{0} \models \text{Final} = \bot \land \text{Fail} = \bot$, and actions $\epsilon, f$ do not occur in $\delta$. A configuration $(z, \delta)$ consists of an action sequence $z$ and a program expression $\delta$.

Definition 6 (program semantics). Let $P = (\Sigma_{0} \cup \Sigma \cup O(B^{j} \land \Sigma'), \delta)$ be a belief program, the transition relation $\rightarrow_{\sigma}$ among configurations, given $e$ s.t. $e \models O(B^{j} \land \Sigma')$, is defined inductively:

\[ \text{We use } B(h = 2) < 1 \text{ to denote } \exists u. B(h = 2; u) \land u < 1. \]

The confidence $\text{Conf}(h, u)$ of a fluent $h$ of sort number wrt $u$ is defined as: $\Box \text{Conf}(h, u) = u \equiv B(h \rightarrow \text{Exp}(h)) < u < v$ while the expectation $\text{Exp}(h)$ is defined as $\Box \text{Exp}(h) = v \equiv v = \sum_{u \in L} u \times (\text{if } u \neq u' \text{ then } v \text{ else } 0)$.}
The set of final configuration Fin(e) wrt e is the smallest set such that:

1. \((z, \langle \rangle) \in \text{Fin}(e)\);
2. \((z, \alpha?) \in \text{Fin}(e)\) if \(e, w, z \models \alpha\);
3. \((z, \delta_1; \delta_2) \in \text{Fin}(e)\) if \((z, \delta_1) \in \text{Fin}(e)\) and \((z, \delta_2) \in \text{Fin}(e)\);
4. \((z, \delta_1; \delta_2) \in \text{Fin}(e)\) if \((z, \delta_1) \in \text{Fin}(e)\) or \((z, \delta_2) \in \text{Fin}(e)\);
5. \((z, \delta^*) \in \text{Fin}(e)\).

The set of failing configurations is given by: \(\text{Fail}(e) = \{ (z, \delta) | (z, \delta) \notin \text{Fin}(e), \text{there is no } (z \cdot t, \delta') \text{ s.t. } (z, \delta) \xrightarrow{e} (z \cdot t, \delta') \}\).

We extend final and failing configurations with addition transitions. This is achieved by defining an extension of \(\xrightarrow{e}\).

The extended transition relation \(\xrightarrow{e}\) among configurations is defined as the least set such that:

1. \((z, \langle \rangle) \xrightarrow{e} (z \cdot t, \langle \rangle)\) if \((z, \delta) \xrightarrow{e} (z \cdot t, \delta')\);
2. \((z, \delta) \xrightarrow{e} (z \cdot e, \langle \rangle)\) if \((z, \delta) \in \text{Fin}(e)\);
3. \((z, \delta) \xrightarrow{e} (z \cdot f, \langle \rangle)\) if \((z, \delta) \in \text{Fail}(e)\).

The execution of a program \(P\) yields a countably infinite \(^{11}\) Markov Decision Process \(M^e_{\delta, w} = (S, A, P, s_0)\) wrt \(e, w, t\).

1. \(S\) is the set of configurations reachable from \(\langle \rangle, \delta\) under \(\xrightarrow{e}\) (transitive and reflexive closure of \(\xrightarrow{e}\));
2. \(A\) is the finite set of primitive programs in \(\delta\);
3. \(P\) is the transition function \(P : S \times A \times S \rightarrow \mathbb{C}\); with \(P((z, \delta), \varrho, (z \cdot t, \delta'))\) given by:

\[
P(\cdot) = \begin{cases} 
p & \varrho \rightarrow t, w, z \models t(l(t) = p), \\
1 & (z, \delta) \in \text{Fin}(e) \text{ and } \varrho = t = \delta' = \epsilon \\
0 & \text{otherwise.} 
\end{cases}
\]

4. \(s_0\) is the initial state \(\langle \rangle, \delta\).

\(^{11}\)Our restrictions on \(\Sigma_{oc}\) and \(\Sigma\) ensure a bounded branching for the MDP; therefore its states are countable.

such path is denoted by \(\pi[j]\). The set of all \(\sigma\)-paths starting in \(s\) is denoted by \(\text{Path}^\sigma(s, M^e_{\delta, w})\).

Every policy \(\sigma\) induces a probability space \(Pr^\sigma_{\delta, w}\) on the set of infinite paths starting in \(s\), using the cylinder set construction: For any finite path prefix \(\tau_{\text{fin}} = s_0 \xrightarrow{e_1} s_1 \cdots s_n\), we define the probability measure:

\[
Pr^\sigma_{s_0, \text{fin}} = Pr^\sigma_{s_0, \text{fin}} = P(s_0, q_1, s_1) \times P(s_1, q_2, s_2) \cdots P(s_{n-1}, q_n, s_n)
\]

### 3.2 Temporal Properties of Programs

We use a variant of PCTL to specify program properties. The syntax is given as:

\[
\Phi ::= \beta | \neg \Phi | \Phi \land \Phi | \Phi [I] \tag{A}
\]
\[
\Psi ::= X\Phi | (\Phi U \Phi) | (\Phi U^{\leq k} \Phi) \tag{B}
\]

where \(\beta\) is a static subjective \(DS_p\) formula without \(O\). We call formulas according to (A) state formulas and according to (B) trace formulas. Here \(I \subseteq \{0, 1\}\) is an interval. \(\Phi U^{\leq k} \Phi\) is the step-bound version of the until operator. Some useful abbreviations are: \(F \Phi\) (eventually \(\Phi\)) for \(\text{true} \cup \Phi\) and \(G \Phi\) (globally \(\Phi\)) for \(\neg F \neg \Phi\).

Let \(\Phi\) be a temporal state formula, \(\Psi\) a temporal trace formula, \(M^e_{\delta, w}\) the infinite-state MDP of a program \(P = (\Sigma_0 \cup \Sigma \cup O(B^f \land \Sigma'), \delta)\) wrt \(e, w, t\). s.t. \(e, w \models \Sigma_0 \cup \Sigma \cup O(B^f \land \Sigma')\), and \(s \in S\). Truth of state formula \(\Phi\) is given as:

1. \(M^e_{\delta, w}, s \models \beta\) iff \(s = (z, \delta)\) and \(e, w, z \models \beta\);
2. \(M^e_{\delta, w}, s \models \neg \Phi \iff M^e_{\delta, w}, s \not\models \Phi\);
3. \(M^e_{\delta, w}, s \models \Phi_1 \land \Phi_2 \iff M^e_{\delta, w}, s \models \Phi_1\) and \(M^e_{\delta, w}, s \models \Phi_2\);
4. \(M^e_{\delta, w}, s \models \Phi[I] \iff \forall \text{proper policies } \sigma, \Phi^\sigma[\{\pi \models I\}]\).

Furthermore, let \(\pi \in \text{Path}^\sigma(s, M^e_{\delta, w})\) be an infinite path for some proper policy \(\sigma\), truth of trace formula \(\Psi\) is as:

1. \(M^e_{\delta, w}, s \models X \Phi \iff M^e_{\delta, w}, [1] \models \Phi\);
2. \(M^e_{\delta, w}, s \models \Phi_1 \cup \Phi_2 \iff \exists i.0 \leq i \leq k. M^e_{\delta, w}, s \models [i] \iff M^e_{\delta, w}, s \models \Phi_1\) and \(\forall j.0 \leq j \leq i. M^e_{\delta, w}, s \models [j]\);
3. \(M^e_{\delta, w}, s \models \Phi_1 \cup^{\leq k} \Phi_2 \iff \exists i.0 \leq i \leq k. M^e_{\delta, w}, s \models [i] \iff M^e_{\delta, w}, s \models \Phi_2\) and \(\forall j.0 \leq j \leq i. M^e_{\delta, w}, s \models [j]\).

**Definition 7** (Verification Problem). A temporal state formula \(\Phi\) is valid in a program \(P\), \(P \models \Phi\), iff for all \(e, w\) with \(e, w \models \Sigma_0 \cup \Sigma \cup O(B^f \land \Sigma')\), it holds that \(M^e_{\delta, w}, s_0 \models \Phi\).

E.g. \(P_{>0.05}[F^{\leq k} B(h = 2: 1)]\) and \(P_{=1}[F B(h = 2: 1)]\) specify the two properties \(P1\) and \(P2\) in the introduction respectively.

### 4 Undecidability

The verification problem is undecidable because belief programs are probabilistic variants of GOLOG programs with sensing, for which undecidability was shown in (Zarrinie and Claßen 2016). Claßen et al. (2013) observed that many dimensions affect the complexity of the GOLOG program.
verification including the underlying logic, the program constructs, and the domain specifications. Since then, efforts have been made to find decidable fragments. Arguably, the dimension of domain specification is less well-studied. Here we study the boundary of decidability from this dimension. Hence, in this paper, we set the other two dimensions to a known decidable status.\footnote{Formally, we assume our logic only contains $+, \times$ as rigid function symbols and whenever we write logical entailment $\Sigma \models \alpha$, we mean $\Sigma \cup \Delta \cup \mathcal{R}_C \models \alpha$ where $\Delta$ is as before and $\mathcal{R}_C$ is the theory of the reals, where validity is decidable (Tarski 1998). In terms of program constructs, we disallow non-deterministic pick of program parameters, $\pi.x.\delta(x)$, which is proven to be a source of undecidability in (Claßen, Liebenberg, and Lakemeyer 2013).}

In deterministic settings, domain specifications mainly refer to SSAs. Nevertheless, in our case, the likelihood axiom (LA) plays an important role as well. Some relevant variants of SSAs are context-free (Reiter 2001) and local-effect SSAs (Liu and Levesque 2005).

**Definition 8.** A set of SSAs is called:
1. context-free, if for all fluents $h$, $\gamma_h$ is rigid;
2. local-effect, if for all fluents $h$, $\gamma_h$ is a disjunction of the form $\exists! a = as(\vec{v}) \land \nabla$, where as is an action symbol, $\vec{v}$ contains $u$ and $\mu$, and $\nabla$ is a fluent formula with free variables in $\vec{v}$.

Intuitively, context-free means that effects of actions are independent of the state while for local-effect, effects might depend on the state specified by effect context $\nabla$ but only locally. An example of local-effect SSAs is the blocks-world domain, where the action $move(x, y, z)$, i.e. moving object $x$ from $y$ to $z$, only affects properties of objects $x, y, z$. The SSA in Example 1 is not local-effect. A context-free SSA is also local-effect.

Since $\gamma_h$ is functional in $u$ and only finitely many action symbols are used: $sa_1, \ldots, sa_k$ for stochastic actions or $sen_1, \ldots, sen_k$ for sensing, $\gamma_h$ can be written in the form (sensing does not change fluents):

$$\gamma_h \equiv \bigwedge_i \exists \bar{x}, \vec{y}, a = sa_i(\bar{x}, \vec{y}) \land u = t^a_{sa}(\bar{x}, \vec{y})$$

$$\land \forall \bar{x}, \vec{y}. \bigwedge_i a \neq sa_i(\bar{x}, \vec{y}) \land h = u$$

(3)

where $t^a_{sa}(\bar{x}, \vec{y})$ are definition terms with variables $\bar{x}, \vec{y}$. After such rewrite, a SSA is context-free iff $t^a_{sa}(\bar{x}, \vec{y})$ are rigid. To ensure a SSA to be local-effect, we require that the $t^a_{sa}(\bar{x}, \vec{y})$ in Eq. (3) are of the form:

$$t^a_{sa}(\bar{x}, \vec{y}) = \begin{cases} v_1 & \nabla_1(\bar{x}, \vec{y}) \\ \vdots \\ v_k & \nabla_k(\bar{x}, \vec{y}) \end{cases}$$

where $v_i$ are variables among $\bar{x} \cup \vec{y}$ and $\nabla_i(\bar{x}, \vec{y})$, the effect contexts, are fluent formulas with free variables among $\bar{x} \cup \vec{y}$. Obviously, this restriction is sufficient to ensure the SSA to be local-effect: since $u = v_i$ for some $v_i \in \bar{x} \cup \vec{y}$, the variable $v_i$ can be eliminated by replacing it with $u$ directly.

Table 2: Decidability of the verification problem

| # | LA | SSA | Decidable |
|---|----|-----|-----------|
| 1 | -  | context-free | Decidable |
| 2 | context-free | local-effect | Decidable |
| 3 | context-free | context-free | Decidable |

which further ensures the SSA fulfills the definition of local-effect.

We call a LA context-free if the RHS of $\Sigma_i$ is rigid. Obviously, context-free LA excludes sensing since sensing always involves fluents.

Table 2 lists the decidability of the belief program verification problem. Dashes mean no constraint. The result is arranged as follows. We first explore decidability for the case with no restriction on the LA. As it turns out, the problem is undecidable even if SSAs are context-free (1). Therefore, we set the LA to be context-free, which results in undecidability for the case of local-effect SSAs (2). The case with question mark remains open (3).

**Theorem 1.** The verification problem is undecidable for programs with context-free SSAs.\footnote{For more details on the proof sketch see the original paper.}

**Proof sketch.** We show the undecidability by a reduction of the undecidable emptiness problem of probabilistic automata (Paz 2014). A probabilistic finite automaton (PA) is a quintuple $A = (Q, L, (M_l)_{l \in L}, q_1, F)$ where $Q$ is a finite set of states, $L$ is a finite alphabet of letters, $(M_l)_{l \in L}$ are the stochastic transition matrices, $q_1 \in Q$ is the initial state and $F \subseteq Q$ is a set of accepting states. For each letter $l \in L$, $M_l \in [0, 1]^{Q \times Q}$ defines transition probabilities: $0 \leq M_l(q_i, q_j) \leq 1$ is the probability from state $q_i$ to $q_j$ when reading a letter $l$. The emptiness problem is that given a PA $A$ and $\xi \in [0, 1]$, deciding whether there exists a word $w$ (a sequence of letters) such that $P_A(q_1, w, F) \geq \xi$, namely, the probability of reaching accepting states from the initial state upon reading $w$ is no less than $\xi$. The emptiness problem is known to be undecidable. The following is a belief program with context-free SSAs to simulate the run of a given probabilistic finite automaton $A$ and threshold $\xi$.

Formally, we have a single fluent $h_s$ to record the current state, a set of standard names $N_Q = \{n_1, n_2, \ldots, n_{|Q|}\}$ to represent the states in $Q$, a set of stochastic actions $\phi_i(y)$ to simulate the read of letter $l_i \in L$. For the BAT $\Sigma$, we have

$$\Box[a]h_s = u \equiv \bigvee_i \exists y.a = \phi_i(y) \land \nabla \equiv \bigwedge_i \forall y.a \neq \phi_i(y) \land h_s = u$$

$$\Box[l](a) = v \equiv \bigvee_i \exists y.a = \phi_i(y) \land v = L_{\phi_i}(y)$$

where $L_{\phi_i}(y)$ is given by

$$L_{\phi_i}(y) = \begin{cases} M_l(h_s, y) & h_s, y \in N_Q \\ 0 & o.w. \end{cases}$$

(4)

Intuitively, the BAT says that fluent $h_s$ can only be changed by action $\phi_i(y)$ and the unobservable parameter $y$
determines the new state; the likelihood of \( g_i(y) \) depends on the current state \( h_s \) and equals the transition probability \( M_k(h_s, y) \). Now let \( \Sigma_0 = \{ h_s = n \} \), \( B^1 = B(h_s = n_1 \mid 1) \), then the program \( P = (\Sigma_0 \cup \Sigma \cup O(B^f \wedge \Sigma), \delta) \) simulates the run of \( PA \) where

\[
\delta := \text{while } B(h_s \in \mathcal{N}_F) < \xi \text{ do } g_1 \mid g_2, \ldots \mid g_{|L|} \text{ endwhile.}
\]

Here \( \mathcal{N}_F \) is the set of standard names representing the accepting states \( F \) in \( A \) and \( g_i \rightarrow g_i(y) \). This is sound in the sense that for any action sequence \( z \in P \) produced by ground actions in \( g_i(y) \) s.t. \( y \in \mathcal{N}_G \) and any number \( r \),

\[
O(B^f \wedge \Sigma) = \{ z \} B(h_s \in \mathcal{N}_F : r) \text{ iff } P_A(q_1 \models w F) = r,
\]

where \( w \) is the corresponding word of \( z \). Hence,

\[
P = P > 0[F(B(h_s \in \mathcal{N}_F) \geq \xi)] \text{ iff } \exists w. P_A(q_1 \rightarrow F) \geq \xi
\]

A crucial point in the above reduction is that the RHS of \( \Box \) will entail that the verification problem is decidable if we set the LA to be rigid. The following theorem provides a negative answer for this when the SSAs are local-effect.

**Theorem 2.** The verification problem is undecidable for programs with local-effect SSAs and context-free LA.

Since the LA is restricted to be context-free, the previous reduction breaks as transition probabilities of probabilistic automata might depend on states in general. Nevertheless, we reduce the emptiness problem of the simple probabilistic automata (SPA), i.e. PA whose transition probabilities are among \( \{0, \frac{1}{2}, 1\} \), to the verification problem with context-free LA and local-effect SSAs. More precisely, the simple probabilistic automata we considered are super simple probabilistic automata (SSPA), SPA with a single probabilistic transition and every transition has a unique letter. Fijalkow et al. (2012) showed that the emptiness problem of the SPA with even a single probabilistic transition is undecidable. Their result can be easily extended to SSPA.

The idea of the reduction is to shift the likelihood context in LAs to the context formula in SSAs. More concretely, instead of saying an action’s likelihood depends on the state and the action’s effect is fixed, which is the view of the BAT in the previous reduction, we say the action’s effect depends on the state and the action’s likelihood is fixed. This is better illustrated by an example. Consider a SSPA consisting of a single probabilistic transition with \( q \xrightarrow{0.5,1} q' \) and \( q \xrightarrow{0.5,2} q'' \). Clearly, one can construct a BAT as in the previous reduction to simulate this, nevertheless, the following BAT with a local-effect SSA and a context-free LA can simulate it as well:

\[
\square(a)h_s = u \equiv \exists y.a = g(y) \wedge u = y \wedge h_s = n \\
\square y.a \neq g(y) \vee h_s \neq n \wedge h_s = u
\]

\[
\square y.a \equiv \exists y.a = g(y) \wedge v = \begin{cases} 1 & y \in \{n, n''\} \\
0 & \text{o.w.}
\end{cases}
\]

Here, \( n, n', n'' \) are standard names corresponding to the states \( q, q', q'' \). The SSA is local-effect as it complies with our conditions for local-effect SSAs: \( t_{\chi}^h(y) = y \) and \( \nabla(y) \equiv h_s = n \). The simulation is sound in the sense that the belief distribution of fluent \( h_s \) corresponds to the probability distribution among states, as in the previous reduction.

## 5 A Decidable Case

Another source of undecidability comes from the property specification, more precisely, the unbounded until operators. In fact, in our program semantics, the MDP \( M_{\mathcal{O}}^{\mathcal{U}} \) is indeed an infinite partially observable MDP (POMDP) where the set of observations is just the set of possible KBs that can be progressed to from the initial KB regarding a certain possible action sequence of the program. Verifying belief programs against specifications with unbounded \( \mathcal{U} \) requires verification of indefinite-horizon POMDPs, which is known to be undecidable. This motivates us to focus on the case with only bounded until operators. In contrast to the previous section, we now allow arbitrary domain specifications.

A state formula \( \Phi' \) is called bounded iff it contains no \( \mathcal{U} \) and no nested \( P \), namely, \( \Phi' := \beta[P_1 \mid \Phi'] \) with \( \Phi' := \chi \beta[\mathcal{U} \leq k \beta] \).

For example, the property \( P_1 \mathcal{P}_{>0.5}[\mathcal{F} \leq 2 \mathcal{B}(h = 2 \mid 1)] \) is bounded while the property \( P_2 \) is not. For bounded state formulas, we only need to consider action sequences with a bounded length, namely, only a finite subset of \( \mathcal{M}_{\mathcal{O}}^{\mathcal{U}} \)'s states and observations needs to be considered. Although model-checking the finite subset of \( \mathcal{M}_{\mathcal{O}}^{\mathcal{U}} \) against PCTL formulas without unbounded \( \mathcal{U} \) operators is decidable, this does not entail that the verification problem is decidable as infinitely many such subsets exist. This is because there are infinitely many models \((e, w)\) satisfying the initial state axioms. Our solution is to abstract them into finitely many equivalence classes (Zarrifi and Claßen 2016).

First, we need to identify the so-called program context \( \mathcal{C}(P) \) of a given program \( P \), which contains: 1) all sentences in \( \Sigma_0 \); 2) all likelihood conditions \( \phi_{\tau}(x, \alpha) \) and \( \phi_{\tau}^\text{env}(x) \); 3) all test conditions in the program expression; 4) all \( \mathcal{DS}_p \) sub-formulas in the temporal property; 5) the negation of formulas from 1) - 4). We then define types of models as follows:

**Definition 9 (Types).** Given a belief program \( P \) and a bounded state formula \( \Phi' \), let \( \mathcal{A}_P \) be the set of all ground actions with non-zero likelihood in \( \mathcal{P} \), \( \mathcal{A}_P^k \) be the set of all action sequences by actions in \( \mathcal{A}_P \) with length no greater than \( k \).

The set of all type elements is given by:

\[
\mathcal{T}(\mathcal{P}, \Phi') = \{(z, \alpha) \mid z \in (\mathcal{A}_P)^k, \alpha \in \mathcal{C}(\mathcal{P})\}
\]

A type wrt \( \mathcal{P}, \Phi' \) is a set \( \tau \subseteq \mathcal{T}(\mathcal{P}, \Phi') \) that satisfies:

1. \( \forall \alpha \in \mathcal{C}(\mathcal{P}), \forall z \in (\mathcal{A}_P)^k, (z, \alpha) \in \tau \), or \( (z, -\alpha) \in \tau \);
2. there exists \( e, w \) s.t. \( e, w \models \Sigma_0 \cup O(B^f \wedge \Sigma) \cup \{[z] \alpha \mid (z, \alpha) \in \tau\}\).

Let \( \text{Types}(\mathcal{P}, \Phi') \) denote the set of all types wrt \( \mathcal{P} \) and \( \Phi' \). The type of a model \((e, w)\) is given by \( \text{type}(e, w) := \{(z, \alpha) \in \mathcal{T}(\mathcal{P}, \Phi') \mid e, w \models [z] \alpha\} \). Types(\( \mathcal{P}, \Phi' \)) partitions \( e, w \) into equivalence classes in the sense that if

\[
^1\text{Verifying properties with nested } \mathcal{P} \text{ is known to be considerably more difficult (Norman, Parker, and Zou 2017).}
\]

\[
^2\text{If } \Phi' \text{ does not contain bounded until operators, we set } k = 0 \text{ for } \Phi' \equiv \beta \text{ and } k = 1 \text{ for } \Phi' \equiv P_1[\mathcal{X} \beta].
\]
type(e, w) = type(e′, w′), then e, w |= \{z\}α iff e′, w′ |= \{z\}α for z ∈ (AP)k and α ∈ C(P).

Thirdly, we use a representation similar to the characteristic program graph (Claßen and Lakemeyer 2008) where nodes are the reachable subprograms Sub(δ), each of which is associated with a termination condition Fin(δ′) (the initial node v0 corresponds to the overall program δ), and where an edge δ1 → δ2 represents a transition from δ1 to δ2 by the primitive program p if test condition α holds. Moreover, failure conditions are given by Fail(δ′) := ¬(Fin(δ′) ∨ ∨ p′(δ, α)).

Lastly, we define a set of atomic propositions AP = \{pα | α ∈ C(P) and α is subjective\} one for each subjective α ∈ C(P).

The finite POMDP for a type τ of a program P is a tuple $M_φ^τ = (S_φ, s_φ^0, A_φ, P_φ, O_φ, Ω_φ, L_φ)$ consisting of:
1. the set of states $S_φ = (AP)k × Sub(δ)$;
2. the initial state $s_φ^0 = (∅, δ)$;
3. the set of primitive programs $A_φ = A$;
4. the transition function $P_φ(\cdot) : (z_1, \delta_1, q, (z_2, \delta_2))$
   - $P_φ(\cdot) = c_{t, j}^{α_i}(\bar{n})$ if $|z_1| < k$, $δ_1$ $\vdash$ $δ_2$, $\sigma_i \in τ$, and for some $s_i, \bar{n}, \bar{r}_{i}^{s_i}, φ_{i}^{t, s_i}(\bar{n})$, it holds that (likewise for sensing)
   $q \rightarrow s_i(\bar{n}, \bar{r}_{i}^{s_i}), z_2 = z_1 | s_i(\bar{n}, \bar{r}_{i}^{s_i}), (z_1, φ_{i}^{t, s_i}(\bar{n}))) ∈ τ$;
   - $P_φ(\cdot) = 1$ if $|z_1| = k$, $q = f$, $z_1 = z_2$, $δ_2 = δ_i$;
   - $P_φ(\cdot) = 1$ if $|z_1|, Fin(δ_1) ∈ τ, q = δ_2 = ε$;
   - $P_φ(\cdot) = 1$ if $|z_1|, Fail(δ_1) ∈ τ, q = δ_2 = f$;
5. the observations $O_φ = \{Pro(O(B^f ∧ Σ'), z) | z ∈ A_P\}$;
6. the state to observation mapping $Ω_φ$ as $Ω_φ(\cdot) = Pro(O(B^f ∧ Σ'), z)$;
7. the labeling $L_φ(\cdot) = \{p_i | p_α \in AP, o \vdash α\}$.  

Lemma 1. Given a program P and a bounded state formula $Φ_p$, for all e, w, s.t. e, w |= $Σ_0 ∪ ∪ O(B^f ∧ Σ'), M_P^{σ_e, w} ⊨ Φ_p$ iff $M_P^{σ_e, w, p} ⊨ Φ_p$, where τ is the type of e, w, $Φ_p$ a PCTL formula obtained from $Φ_p$ by replacing all its $DS_P$ sub-formula with the counter-part atomic proposition, and $\models_p$ is defined in the standard way (Norman, Parker, and Zhou 2017).

Since there are only finitely many type elements, there are only finitely many types for a given program. Hence, we can exploit existing model-checking tools like PRISM (Kwiatkowska, Norman, and Parker 2011) or STORM (Hensel et al. 2021) to verify the PCTL properties against these finitely many POMDPs. Consequently, we have the following theorem.

Theorem 3. The verification problem is decidable for temporal properties specified by bounded state formulas.

In our coffee robot example, we obtain three types τ1, τ2, and τ3 for worlds satisfying $h = 0$, $\{h = -1\}$, and $\{h < 0 ∧ h \neq -1\}$ in the initial state respectively. This is because $Σ_0$ only says $\{h \leq 0\}$. The corresponding finite POMDPs are depicted in Fig. 2. Note that the POMDPs for τ2, τ3 are the same. The observations of states are indicated by colors. Black, blue and green represent the observations of the KBS with belief distribution $f, f′$, and $f''$ in Example 2 respectively, while red stands for the observation of the KB with belief distribution $f'''$ as $f'''(u) = \{if u = 0 then \frac{1}{3} else if u = 1 then \frac{2}{3} else 0\}$. Clearly, only the POMDP of τ1 can reach the observation $O(B^{\varepsilon} ∧ Σ': 1)$ which satisfies the label $p_B(h = 2: 1)$, and the probability of reaching it is $0.5 × 0.1 = 0.05$, therefore $P ≠ P ≥ 0.05[FS^2B(h = 2 : 1)]$ (recall that in Def. 7 the satisfiability of a property for a program requires all the underlying POMDPs to satisfy the property).

6 Related Work

Our formalism extends the modal logic $DS_P$ (Liu and Feng 2021), a variant of (Belle and Lakemeyer 2017). The idea of using the same modal logic to specify the program and its properties is inspired by the work of Claßen and Zarrieß (2017). Similar approaches on the verification of CTL*, LTL, and CTL properties of GOLOG programs include (Claßen and Lakemeyer 2008; Zarrieß and Claßen 2015; Zarrieß and Claßen 2016). Axiomatic approaches to the verification of GOLOG programs can be found in (De Giacomo, Ternovska, and Reiter 2019; De Giacomo et al. 2016).

While the verification of arbitrary GOLOG programs is clearly undecidable due to the underlying first-order logic, Claßen et al. (2013) established decidability in case the underlying logic is restricted to the two-variable fragment, the program constructs disallow non-deterministic pick of action parameters, and the BATs are restricted to be local-effect. Later, the constraints on BATs are relaxed to acyclic and flat BATs in (Zarrieß and Claßen 2016). Under similar settings, (Zarrieß and Claßen 2015; Claßen and Zarrieß
show that the verification of \textit{ALCOK-GOLOG} programs, where the underlying logic is a description logic, and \textit{DT-GOLOG} programs against LTL and PRCTL specification, respectively, is decidable. What distinguishes our work from the above is that we assume the environment is partially observable to the agent while they assume full observability.

Verifying temporal properties under partial observation has been studied extensively in model checking (Chatterjee, Chmelik, and Tracol 2016; Chatterjee et al. 2016; Norman, Parker, and Zou 2017; Bork et al. 2020; Bork, Katoen, and Quatmann 2022), in planning (Madani, Hanks, and Condon 2003), and in stochastic games (Kwiatkowska, Norman, and Parker 2009). Notably the work on probabilistic planning (Madani, Hanks, and Condon 2003) is closely related to our belief program verification as belief programs can be viewed as a compact representation of a plan. Moreover, it suggested that probabilistic planning is undecidable under different restrictions. Perhaps, the most relevant restriction is that probabilistic planning is undecidable even without observations, which essentially corresponds to our restriction on context-free likelihood axioms, which excludes sensing. However, our results go beyond this as we show the problem remains undecidable when restricting actions to be local-effect. Another proposal on compact representation of plans is the belief program by (Lang and Zanuttini 2015). Nevertheless, the proposal is primitive as the underlying logic is propositional, i.e., beliefs are only about propositions. Hence, verification there reduces to regular model-checking. In contrast, our framework based on the logic $\mathcal{DS}_p$, which allows us to express incompleteness about the underlying model. Therefore, to verify a belief program, one has to perform model-checking for potentially infinitely many POMDPs. Other virtues of our belief program, to name but a few, include that 1) tests of the program can refer to beliefs about belief, i.e. meta-beliefs, and beliefs with quantifying-in 2) we can express that dynamics of a domain that holds in the real world are different from what the agent believes (Our coffee robot is an example of this kind). Hence, although (Lang and Zanuttini 2015) showed that the verification problem is decidable when restricting to finite horizon, our result on decidability goes beyond them since our problem is more general than theirs.

7 Conclusion

We reconsider the proposal of belief programs by Belle and Levesque based on the logic $\mathcal{DS}_p$. Our new formalism allows, amongst others, to define the transition system and specify the temporal properties like \textit{eventually} and \textit{globally} more smoothly. Besides, we study the complexity of the verification problem. As it turns out, the problem is undecidable even in very restrictive settings. We also show a case where the problem is decidable.

As for future work, there are two promising directions. On the complexity of verification, whether it is decidable or not remains open for the case where the SSAs and LA are context-free. Our sense is that, under such a setting, belief programs in general cannot simulate arbitrary probabilistic automata, but only a subset. Since the emptiness problem of probabilistic automata is a special case of the verification problem, evidence showing undecidability of emptiness problem for such a subset could prove the undecidability for the verification problem for programs with context-free SSAs and LA. Besides, (Chatterjee and Tracol 2012; Fijalkow, Gimbert, and Ouahlad 2012) show a set of decidable decision problems in related to special types of probabilistic automata. It is interesting to see how these problem can be transformed to the verification problem and hence find decidable cases. Another direction is more practical. It is desirable to design a general algorithm to perform verification of arbitrary belief programs, even if the algorithm might not terminate. In this regard, symbolic approaches in solving first-order MDP and first-order POMDP (Sanner and Boutilier 2009; Sanner and Kersting 2010), compact representations of (infinite) (PO)MDPs, are relevant.

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