REAL ENUMERATIVE GEOMETRY AND EFFECTIVE
ALGEBRAIC EQUIVALENCE

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1. Introduction

Determining the common zeroes of a set of polynomials is further complicated over non-algebraically closed fields such as the real numbers. A special case is whether a problem of enumerative geometry can have all its solutions be real. We call such a problem fully real.

Little is known about enumerative geometry from this perspective. A standard proof of Bézout’s Theorem shows the problem of intersecting hypersurfaces in projective space is fully real. Khovanskii [9] considers intersecting hypersurfaces in a torus defined by few monomials and shows the real zeros are at most a fraction of the complex zeroes. Fulton, and more recently, Ronga, Tognoli and Vust [14] have shown the problem of 3264 plane conics tangent to five given conics is fully real. The author [17] has shown all problems of enumerating lines incident on linear subspaces of projective space are fully real.

There are few methods for studying this phenomenon. We ask: How can the knowledge that one enumerative problem is fully real be used to infer that a related problem is fully real? We give several procedures to accomplish this inference and examples of their application, lengthening the list of enumerative problems known to be fully real.

We study intersections of any dimension, not just the zero dimensional intersections of enumerative problems. Our technique is to deform general intersection cycles into simpler cycles. This modification of the classical method of degeneration was used by Chiavacci and Escamilla-Castillo [4] to investigate these questions.

Let $\alpha_1, \ldots, \alpha_a$ be cycle classes spanning the Chow ring of a smooth variety. For cycle classes $\beta_1, \ldots, \beta_b$, there exist integers $c_i$ for $i = 1, \ldots, a$ such that

$$\beta_1 \cdots \beta_b = \sum_{i=1}^{a} c_i \cdot \alpha_i.$$ 

When the $c_i \geq 0$, this product formula has a geometric interpretation. Suppose $Y_1, \ldots, Y_b$ are cycles representing the classes $\beta_1, \ldots, \beta_b$ which meet generically transversally in a cycle $Y$. Then $Y$ is algebraically equivalent to $Z := Z_1 \cup \cdots \cup Z_a$, where $Z_i$ has $c_i$ components, each representing the cycle class $\alpha_i$. This algebraic equivalence is effective if there is a family of cycles containing both $Y$ and $Z$ whose general member is a generically transverse intersection of cycles representing the classes $\beta_1, \ldots, \beta_b$. If the cycles $Y_1, \ldots, Y_b$ and each component of $Z$ are defined over $\mathbb{R}$ and both $Y$ and $Z$ are in the same connected component of the real points of that family, then the effective algebraic equivalence is real.

Date: 5 February 1996.

1991 Mathematics Subject Classification. 14M15, 14N10, 14P99.

Key words and phrases. Grassmannian, flag variety, real enumerative geometry.

Research supported in part by NSERC grant # OGP0170279.
Real effective algebraic equivalence can be used to show an enumerative problem is fully real, or more generally, to obtain lower bounds on the maximal number of real solutions. Suppose the cycles $Y_1, \ldots, Y_b, W_1, \ldots, W_c$ give an enumerative problem and the problem obtained by substituting $Z$ for $Y_1, \ldots, Y_b$ has at least $d$ real solutions. Then there exist real cycles $Y'_1, \ldots, Y'_b$ such that the original problem (with $Y'_i$ in place of $Y_i$) has at least $d$ real solutions, since the number of real solutions is preserved by small real deformations.

Sections 2 through 5 introduce and develop our basic notions and techniques. Subsequent sections are devoted to elaborations and applications of these ideas.

In Section 6, we prove that any enumerative problem on a flag variety involving five Schubert varieties, three of which are special Schubert varieties, is fully real. Given a map $\pi : Y \to X$ with equidimensional fibres, real effective algebraic equivalence on $X$ and $Y$ is compared in Section 7 and used in Sections 8 and 9 to show that many Schubert-type enumerative problems in two classes of flag varieties are fully real.

A proof of Bézout’s Theorem in Section 10 suggests another method for obtaining fully real enumerative problems. This is applied in Section 11 to show the enumerative problem of $(n-2)$-planes in $\mathbb{P}^n$ meeting $2n-2$ rational normal curves is fully real.

The author thanks Bernd Sturmfels for encouraging these investigations.

2. Intersection Problems

2.1. Conventions. Varieties are reduced, complex, and defined over the real numbers $\mathbb{R}$. Let $X$ and $Y$ denote smooth projective varieties and $U$, $V$, and $W$ smooth quasi-projective varieties. Equip the real points $X(\mathbb{R})$ of $X$ with the classical topology. Let $A^*X$ be the Chow ring of cycles modulo algebraic equivalence.

Two subvarieties meet generically transversally if they meet transversally along a dense subset of each component of their intersection. Such an intersection scheme is reduced at the generic point of each component, or generically reduced. A subvariety $\Xi \subset U \times X$ (or $\Xi \to U$) with generically reduced equidimensional fibres over a smooth base $U$ is a family of multiplicity free cycles on $X$ over $U$. All fibres of $\Xi$ over $U$ are algebraically equivalent, and we say $\Xi \to U$ represents that algebraic equivalence class.

2.2. Chow varieties. Positive cycles on $X$ of a fixed dimension and degree are parameterized by the Chow variety of $X$. We suppress the dependence on dimension and degree and write $\text{Chow} X$ for any Chow variety of $X$. The open Chow variety $\text{Chow}^\circ X$ is the open subset of $\text{Chow} X$ parameterizing multiplicity free cycles on $X$. There is a tautological family $\Phi \to \text{Chow}^\circ X$ of cycles on $X$ with the property that $\zeta \in \text{Chow}^\circ X$ represents the fundamental cycle of the fibre $\Phi_\zeta$.

Let $\Xi \to U$ be a family of multiplicity free cycles on $X$. The association of a point $u$ of $U$ to the fundamental cycle of the fibre $\Xi_u$ defines the fibre function $\phi$, which is algebraic on a dense open subset $U'$ of $U$. If $U$ is a curve, then $U = U'$.

2.3. Proposition. $\phi(U')$ is dense in the set $\phi(U)$.

**Proof:** Let $u \in U$ and $C \subset U$ be a smooth curve with $u \in C$ and $C - \{u\} \subset U'$. Such a curve is not necessarily closed in $U$, but is the smooth points of a closed subvariety. The fibre function $\phi|_C$ of $\Xi|_C \to C$ is algebraic, hence $\phi(u) \in \overline{\phi(C - \{u\})} \subset \overline{\phi(U')}$. $lacksquare$

Two families $\Xi \to U$ and $\Psi \to V$ of multiplicity free cycles on $X$ are equivalent if $\overline{\phi(U)} = \overline{\phi(V)}$, that is, if they have essentially the same cycles. Our results remain valid when one
family of cycles is replaced by an equivalent family, perhaps with the additional assumption that 
\(\overline{\phi(U(\mathbb{R}))} = \overline{\phi(V(\mathbb{R}))}\).

The varieties \(Chow X\) and \(Chow^\phi X\) as well as \(U'\) and the morphism \(\phi : U' \to Chow X\) are defined over \(\mathbb{R}\) ([5], §I.9). We use \(\phi\) to denote all fibre functions. Any ambiguity may be resolved by context.

2.4. Intersection Problems. For \(1 \leq i \leq b\), let \(\Xi_i \to U_i\) be a family of multiplicity free cycles on \(X\). Let \(U \subset \prod_{i=1}^b U_i\) be the locus where the fibres of the product family \(\prod_{i=1}^b \Xi_i\) meet the (small) diagonal \(\Delta_X^b\) of \(X^b\) generically transversally. Equivalently, \(U\) is the locus where fibres of \(\Xi_1, \ldots, \Xi_b\) meet generically transversally in \(X\). If \(U\) is nonempty, then \(\Xi_1, \ldots, \Xi_b\) give a \((\text{well-posed})\) intersection problem.

Given an intersection problem as above, let \(\delta : X \looparrowright \Delta_X^b \subset X^b\) and set \(\Xi\) to be

\[
\Xi := (1_U \times \delta)^* \prod_{i=1}^b \Xi_i \subset U \times X,
\]

a family of multiplicity free cycles on \(X\) over \(U\). We often suppress the dependence on the original families and write \(\Xi \to U\) for this intersection problem.

Not all collections of families of cycles give well-posed intersection problems, some transversality is needed to guarantee \(U\) is nonempty. When a reductive group acts transitively on \(X\), Kleiman’s Transversality Theorem [10] has the following consequence.

2.5. Proposition. Suppose a reductive group acts transitively on \(X\), \(\Xi_1\) is a constant family, and for \(2 \leq i \leq b\), \(\Xi_i\) is equivalent to a family of multiplicity free cycles stable under that action. Then \(\Xi_1, \ldots, \Xi_b\) give a well-posed intersection problem.

Grassmannians and flag varieties have such an action. For these, we suppose all families of cycles are stable under that action, and thus give well-posed intersection problems.

Suppose a reductive group acts on \(X\) with a dense open orbit \(X'\). For example, if \(X\) is a toric variety, or more generally, a spherical variety [3] [11] [13]. Each family may be stable under that action, but the collection need not give a well-posed intersection problem as Kleiman’s Theorem [10] only guarantees transversality in \(X'\). However, it is often the case that only points of intersection in \(X'\) are desired, and suitable blow up of \(X\) or a different equivariant compactification of \(X'\) exists on which the corresponding intersection problem is well-posed (see [3], §I.4 or [3], §9 and §10.4).

3. EFFECTIVE ALGEBRAIC EQUIVALENCE

Let \(\alpha_1, \ldots, \alpha_a\) be distinct additive generators of \(A^*X\), and for \(1 \leq i \leq a\), suppose \(\Psi(\alpha_i) \to V(\alpha_i)\) is a family of multiplicity free cycles on \(X\) representing the cycle class \(\alpha_i\). When \(X\) is a Grassmannian or flag variety, \(\alpha_1, \ldots, \alpha_a\) will be the Schubert classes, and \(\Psi(\alpha_i) \to V(\alpha_i)\) the corresponding families of Schubert varieties.

A family of multiplicity free cycles \(\Xi \subset U \times X\) has an effective algebraic equivalence with witness \(Z \in \overline{\phi(U)} \cap Chow^\phi X\) if each (necessarily multiplicity free) component of \(Z\) is a fibre of some family \(\Psi(\alpha_i)\). This effective algebraic equivalence is real if \(Z \in \overline{\phi(U(\mathbb{R}))}\) and each component of \(Z\) is a fibre over a real point of some \(V(\alpha_i)\). An intersection problem \(\Xi_1, \ldots, \Xi_b\) has (real) effective algebraic equivalences if its family of intersection cycles \(\Xi \to U\) has (real) effective algebraic equivalences.
3.1. **Products in** $A^*X$. Suppose $\beta_1, \ldots, \beta_b$ are classes from $\{\alpha_1, \ldots, \alpha_a\}$ and the families $\Psi(\beta_1), \ldots, \Psi(\beta_b)$ give an intersection problem $\Psi \to V$. We say $\Psi \to V$ is given by $\beta_1, \ldots, \beta_b$. Suppose $\Psi \to V$ has an effective algebraic equivalence with witness $Z$. Fibres of $\Psi \to V$ are generically transverse intersections of fibres of $\Psi(\beta_1), \ldots, \Psi(\beta_b)$, and so have cycle class $\beta_1 \cdots \beta_b$. As $Z \in \phi(V)$, this equals the cycle class of $Z$, which is $\sum_{i=1}^a c_i \alpha_i$, where $c_i$ counts the components of $Z$ lying in the family $\Psi(\alpha_i)$. Thus in $A^*X$, we have

$$\beta_1 \cdots \beta_b = \sum_{i=1}^a c_i \alpha_i. \quad (3.1)$$

To compute products in $A^*X$, classical geometers would try to understand a generically transverse intersection of degenerate cycles in special position, as a generic intersection cycle is typically too difficult to describe. Effective algebraic equivalence extends this method of degeneration by also considering limiting positions of such intersection cycles as the subvarieties degenerate further, attaining excess intersection.

3.2. **Pieri-type intersection problems.** A Schubert subvariety of a flag variety is determined by a complete flag $F$ and a coset $w$ of a parabolic subgroup in the symmetric group. Thus Schubert classes $\sigma_w$ are indexed by these cosets and families $\Psi_w$ of Schubert varieties have base $\mathbb{F}\ell$, the variety of complete flags.

A **special Schubert subvariety** of a Grassmannian is the locus of planes meeting a fixed linear subspace non-trivially, or the image of such a subvariety in the dual Grassmannian. More generally, a special Schubert subvariety of a flag variety is the pullback of a special Schubert subvariety from a Grassmannian projection. If $m$ is the index of a special Schubert class, then the Pieri-type formulas of [12, 16] show that for any $w$, there exists a subset $I_{m,w}$ of these cosets such that

$$\sigma_m \cdot \sigma_w = \sum_{v \in I_{m,w}} \sigma_v. \quad (3.2)$$

3.3. **Theorem.** The intersection problem $\Xi \to U$ given by the classes $\sigma_m$ and $\sigma_w$ has real effective algebraic equivalences.

**Proof:** The Borel subgroup $B$ of $GL_n \mathbb{C}$ stabilizing a real complete flag $F$ acts on the Chow variety with fixed points the $B$-stable cycles, which are sums of Schubert varieties determined by $F$. As Hirschowitz [8] observed, $\overline{\phi(U)}$ is $B$-stable, and must contain a fixed point ([2], III.10.4). In fact, if $F'$ is a real flag in linear general position with $F$, then the $B(\mathbb{R})$-orbit of $\Omega_m F \cap \Omega_w F'$ is a subset of $\phi(U(\mathbb{R}))$. Moreover its closure has a $B(\mathbb{R})$-fixed point, as the proof in [2] may be adapted to show that complete $B(\mathbb{R})$-stable real analytic sets have fixed points. Since the coefficients of the sum (3.2) are all 1, $\sum_{v \in I_{m,w}} \Omega_w F'$ is the only $B(\mathbb{R})$-stable cycle in its algebraic equivalence class, and therefore

$$\sum_{v \in I_{b,w}} \Omega_w F' \in \overline{\phi(U(\mathbb{R}))}.$$  

4. **Fully real enumerative problems**

An **enumerative problem of degree** $d$ is an intersection problem $\Xi \to U$ with zero-dimensional fibres of cardinality $d$. An enumerative problem is fully real if there exists $u \in U(\mathbb{R})$ with all points in the fibre $\Xi_u$ real. In this case, $u = (u_1, \ldots, u_b)$ with $u_i \in U_i(\mathbb{R})$ and the cycles $(\Xi_1)_{u_1}, \ldots, (\Xi_b)_{u_b}$ meet transversally with all points of intersection real.
4.1. **Theorem.** An enumerative problem $\Xi \to U$ is fully real if and only if it has real effective algebraic equivalences. That is, if and only if there exists a point $\zeta \in \phi(U(\mathbb{R}))$ representing distinct real points.

**Proof:** The forward implication is a consequence of the definition. For the reverse, let $d$ be the degree of $\Xi \to U$. Then $\phi : U \to S^dX$, the Chow variety of effective degree $d$ zero cycles on $X$. The real points $S^dX(\mathbb{R})$ of the Chow variety represent degree $d$ zero cycles stable under complex conjugation. Its dense set of multiplicity free cycles have an open subset $\mathcal{M}$ parameterizing cycles of distinct real points, and $\zeta \in \mathcal{M}$. Thus $\phi(U(\mathbb{R})) \cap \mathcal{M} \neq \emptyset$, which implies $\Xi \to U$ is fully real. \(\square\)

The set of witnesses to $\Xi \to U$ being fully real contains an open subset $\phi^{-1}(\mathcal{M}) \cap U(\mathbb{R})$.

5. **Curve selection**

Subsequent sections use real effective algebraic equivalence for one or more families to infer results about related families. While intuition supports the claim that the functions we define between Chow varieties are algebraic (or at least continuous), we are unaware of general results verifying this intuition. An obvious obstruction is that the Chow variety does not represent a functor. However, weaker claims suffice. Our tool is the Curve Selection Lemma [1] of real semi-algebraic geometry, in the following guise:

5.1. **Curve Selection Lemma.** Let $V$ be a real variety and $R \subset V(\mathbb{R})$ a semi-algebraic subset. If $\zeta \in \overline{R}$, then there is a real algebraic map $f : C \to V$ with $C$ a smooth curve, and a point $s$ on a connected arc $S$ of $C(\mathbb{R})$ such that $f(S - \{s\}) \subset R$ and $f(s) = \zeta$.

**Proof:** By the Curve Selection Lemma ([1], 2.6.20), there exists a semi-algebraic function $g : [0, 1] \to \overline{R}$ with $g(0) = \zeta$, $g(0, 1] \subset R$, and $g$ a real analytic homeomorphism onto its image in $\overline{R}$. Let $C^o$ be the Zariski closure of $g[0, 1]$ in $V$, and $f : C \to C^o$ its normalization. Let $S \subset C(\mathbb{R})$ be a connected arc of $f^{-1}(g[0, 1])$ whose image contains $g(0)$ and let $s \in S \cap f^{-1}(g(0))$. \(\square\)

6. **Pieri-type enumerative problems**

6.1. **Theorem.** Any enumerative problem in any flag variety involving five Schubert varieties, three of which are special, is fully real.

This generalizes Theorem 5.2 of [18], the analogous result for Grassmannians. It requires an additional transversality result.

6.2. **Lemma.** Let $(w_1, w_2)$ and $(v_1, v_2)$ be indices of Schubert subvarieties of a flag variety, with $w_1$ and $w_2$ (respectively $v_1$ and $v_2$) defining defining Schubert varieties of the same dimension. Suppose $m$ is the index of a special Schubert subvariety such that $(w_1, v_1, m)$ gives an enumerative problem. If $(w_1, v_1) \neq (w_2, v_2)$, and $F, F'$ are complete flags in linear general position, then there is an open set $V$ of the variety $\mathbb{F} \ell$ of complete flags consisting of flags $E_\ell$ such that

$$\Omega_{w_1}F \cap \Omega_{v_1}F' \cap \Omega_mE_\ell \quad \text{and} \quad \Omega_{w_2}F \cap \Omega_{v_2}F' \cap \Omega_mE_\ell$$

are transverse intersections which coincide only when empty.

If the three flags are real, then a nonempty intersection as above is a single real point.

**Proof:** By Kleiman’s Theorem [11], there is an open subset $U$ of $\mathbb{F} \ell \times \mathbb{F} \ell \times \mathbb{F} \ell$ consisting of triples $(F, F', E_\ell)$ such that each intersection is transverse and so is either empty or a single point, by the Pieri-type formulas of [12, 14]. Suppose neither is empty.
Similarly, there is an open subset $V$ of triples for which

\[
\left( \Omega_{w_1} F \cap \Omega_{v_2} F \right) \cap \left( \Omega_{v_1} F' \cap \Omega_{v_2} F' \right) \cap \Omega_b E
\]

is proper. Since $(w_1, v_1) \neq (w_2, v_2)$, it is empty. Thus for triples $(F, F', E)$ in $U \cap V$,

\[
\Omega_{w_1} F \cap \Omega_{v_1} F' \cap \Omega_b E \neq \Omega_{w_2} F \cap \Omega_{v_2} F' \cap \Omega_b E.
\]

The lemma follows, as $U \cap V$ is stable under the diagonal action of $GL_n \mathbb{C}$ and the set of pairs $(F, F')$ in linear general position is the open $GL_n \mathbb{C}$-orbit in $\mathbb{P} \times \mathbb{P}$.

6.3. **Proof of Theorem 6.1**: Let $\Xi_1, \Xi_2, \Xi_3, \Gamma_1, \Gamma_2$ be families of Schubert varieties representing the classes $\sigma_{m_1}, \sigma_{m_2}, \sigma_{m_3}, \sigma_{w_1}, \sigma_{w_2}$. Suppose $\sigma_{m_1}, \sigma_{m_2}, \sigma_{m_3}$ are special Schubert classes, and these families give an enumerative problem $\Xi \to U$.

By §3.2, for each $i = 1, 2$, the intersection problem $\Psi_i \to V_i$ given by the families $\Xi_i$ and $\Gamma_i$ has a real effective algebraic equivalence with witness \(\sum_{v_i \in I_{m_i, w_i}} \Omega_{v_i} F\) for any real flag $F$. Let $F_i$ and $F'_i$ be real flags in linear general position and set

\[
Z_1 := \sum_{v_1 \in I_{m_1, w_1}} \Omega_{v_1} F \quad \text{and} \quad Z_2 := \sum_{v_2 \in I_{m_2, w_2}} \Omega_{v_2} F'.
\]

For $i = 1, 2$, let $\phi_i$ be the fibre function for $\Psi_i \to V_i$. Then $Z_i \in \phi_i(V_i(\mathbb{R}))$ and by Lemma 6.1, there is a map $f_i : C_i \to \phi_i(V_i)$ with $C_i$ a smooth curve, and a point $s_i$ on a connected arc $S_i$ of $C_i(\mathbb{R})$ such that $f(S_i - \{s_i\}) \subset \phi_i(V_i(\mathbb{R}))$ and $f_i(s_i) = Z_i$. Then $f_1^* \Phi \to C_1, f_2^* \Phi \to C_2$, and $\Xi_3 \to \mathbb{P} \mathbb{L}$ give a well-posed fully real enumerative problem $\Psi \to V$, as $(s_1, s_2, E_i) \in V(\mathbb{R})$.

Let $\mathcal{M}$ be the open subset of the real points of the Chow variety parameterizing cycles consisting entirely of real points. Then $\phi(s_1, s_2, E_i) \in \mathcal{M}$ and so $\phi^{-1}(\mathcal{M})$ meets $R := (S_1 - \{s_1\}) \times (S_2 - \{s_2\}) \times \{E_i\}$. However, fibres of $\Psi$ over points of $R$ are fibres of $\Xi$ over points of $U(\mathbb{R})$, showing $\Xi \to U$ to be fully real.

7. **Fibrations**

Suppose $\pi : Y \to X$ has equidimensional fibres. If $\Xi \to U$ is a family of multiplicity free cycles on $X$ representing the cycle class $\alpha$, its pullback $\pi^* \Xi := (\pi \times 1_U)^{-1} \Xi \to U$ is a family of multiplicity free cycles on $Y$ representing the cycle class $\pi^* \alpha$.

Suppose $\alpha_1, \ldots, \alpha_n$ generate $A^* X$ additively and $\Psi(\alpha_1), \ldots, \Psi(\alpha_n)$ are families of cycles representing these generators. The classes $\pi^* \alpha_1, \ldots, \pi^* \alpha_n$ generate the image of $A^* X$ in $A^* Y$ and are represented by the families $\pi^* \Psi(\alpha_1), \ldots, \pi^* \Psi(\alpha_n)$. Effective algebraic equivalence is preserved by pullbacks:

7.1. **Theorem.** If $\Xi \to U$ is a family of multiplicity free cycles on $X$ having effective algebraic equivalences with witness $Z$, then $\pi^* \Xi \to U$ is a family of multiplicity free cycles on $X$ having effective algebraic equivalences with witness $\pi^{-1} Z$. Likewise, if $\Xi \to U$ has real effective algebraic equivalences, then so does $\pi^* \Xi \to U$.

Associating a cycle $Z$ on $X$ to $\pi^{-1} Z \subset Y$ defines a function $\pi^* : Chow X \to Chow Y$. If $\phi$ is the fibre function of $\Xi \to U$, then $\pi^* \circ \phi$ is the fibre function of $\pi^* \Xi \to U$. Letting $W = \phi(U')$ and $R = \phi(U'(\mathbb{R}))$, we see that Theorem 7.1 is a consequence of the following lemma.
7.2. Lemma. Let $W \subset \text{Chow}^\delta X$ be constructible and $V := \overline{W \cap \text{Chow}^\delta X}$. Then $\pi^*(V) \subset \overline{\pi^*(W)}$ in $\text{Chow}Y$. Likewise, if $R \subset \text{Chow}^\delta X(\mathbb{R})$ is semi-algebraic and $Q := \overline{R \cap \text{Chow}^\delta X(\mathbb{R})}$, then $\pi^*(Q) \subset \overline{\pi^*(R)}$ in $\text{Chow}Y(\mathbb{R})$.

Proof: For the first part, let $\zeta \in V$. We show $\pi^*(\zeta) \in \overline{\pi^*(W)}$.

Let $C^o \subset \text{Chow}^\delta X$ be an irreducible curve with $\zeta \in C^o$ and $C^o - \{\zeta\} \subset W$. Let $f : C \to C^o$ be its normalization and let $s \in f^{-1}(\zeta)$. Let $\Phi \subset \text{Chow}^\delta X \times X$ be the tautological family. Then $f^*\Phi$ is a family of multiplicity free cycles on $X$ with fibre function $f$. Similarly, $\pi^* f$ is the fibre function of the family $\pi^*(f^*\Phi)$ of multiplicity free cycles on $Y$ over the smooth curve $C$. As noted in §2, this implies $\pi^* f$ is algebraic, and so $\pi^*(\zeta) \in \pi^*(f(C)) \subset \overline{\pi^*(W)}$, since $\pi^*(f(C - \{f^{-1}(\zeta)\})) \subset \overline{\pi^*(W)}$.

For the second part, suppose $R \subset \text{Chow}^\delta X(\mathbb{R})$ and $\zeta \in Q = \overline{R \cap \text{Chow}^\delta X}$. By Lemma 5.1, there is a smooth curve $C$, a connected arc $S \subset C(\mathbb{R})$, a point $s \in S$, and an algebraic map $f : C \to \text{Chow}^\delta X$ such that $f(s) = \zeta$ and $f(S - \{s\}) \subset R$. Arguing as above shows $\pi^*(\zeta) \in \pi^*(f(S)) \subset \overline{\pi^*(R)}$.  

8. Schubert-type enumerative problems in $\mathbb{F}l_{0,1}\mathbb{P}^n$ are fully real

The variety $\mathbb{F}l_{0,1}\mathbb{P}^n$ of partial flags $q \subset l \subset \mathbb{P}^n$ with $q$ a point and $l$ a line has projections

$$p : \mathbb{F}l_{0,1}\mathbb{P}^n \longrightarrow \mathbb{P}^n$$

and

$$\pi : \mathbb{F}l_{0,1}\mathbb{P}^n \longrightarrow \mathbb{G}_1\mathbb{P}^n,$$

where $\mathbb{G}_1\mathbb{P}^n$ is the Grassmannian of lines in $\mathbb{P}^n$.

A Schubert subvariety of $\mathbb{G}_1\mathbb{P}^n$ is determined by a partial flag $F \subset P$ of $\mathbb{P}^n$:

$$\Omega(F, P) := \{l \in \mathbb{G}_1\mathbb{P}^n | l \cap F \neq \emptyset \text{ and } l \subset P\}.$$

If $F$ is a hyperplane of $P$, then $\Omega(F, P) = \mathbb{G}_1P$, the Grassmannian of lines in $P$.

In addition to $\pi^{-1}\Omega(F, P)$, there is another Schubert subvariety of $\mathbb{F}l_{0,1}\mathbb{P}^n$ which projects onto $\Omega(F, P)$ in $\mathbb{G}_1\mathbb{P}^n$:

$$\widehat{\Omega}(F, P) := \{(q, l) \in \mathbb{F}l_{0,1}\mathbb{P}^n | q \in F \text{ and } l \subset P\}.$$

Any Schubert subvariety of $\mathbb{F}l_{0,1}\mathbb{P}^n$ is one of $\Omega(F, P)$ or $\widehat{\Omega}(F, P)$, for suitable $F \subset P$. The varieties $\widehat{\Omega}(F, P)$ have another description, which is straightforward to verify:

8.1. Lemma. Let $N, P$ be subspaces of $\mathbb{P}^n$. Then

$$p^{-1}N \cap \pi^{-1}\mathbb{G}_1P = \widehat{\Omega}(N \cap P, P),$$

and, if $N$ and $P$ meet properly, this intersection is generically transverse.

8.2. Corollary. Any Schubert-type enumerative problem on $\mathbb{F}l_{0,1}\mathbb{P}^n$ is equivalent to one involving only pullbacks of Schubert subvarieties of $\mathbb{P}^n$ and $\mathbb{G}_1\mathbb{P}^n$.

The next lemma, an exercise in linear algebra, describes Poincaré duality for Schubert subvarieties of $\mathbb{F}l_{0,1}\mathbb{P}^n$.

8.3. Lemma. Suppose a linear subspace $N$ meets a partial flag $F \subset P$ properly in $\mathbb{P}^n$. If $\pi^{-1}\Omega(F, P)$ and $p^{-1}N$ have complimentary dimension in $\mathbb{F}l_{0,1}\mathbb{P}^n$, then their intersection is empty unless $F$ and $N \cap P$ are points. In that case, they meet transversally in a single point and $\pi^{-1}\Omega(F, P) \cap p^{-1}N = (N \cap P, \langle F, N \cap P \rangle)$.
8.4. **Theorem.** Any Schubert-type enumerative problem in $\mathbb{F}l_{0,1}\mathbb{P}^n$ is fully real.

**Proof:** By Corollary 8.2, it suffices to consider enumerative problems involving only pullbacks of Schubert subvarieties of $\mathbb{P}^n$ and $G_1\mathbb{P}^n$. Since the intersection of linear subspaces in $\mathbb{P}^n$ is another linear subspace, we may further suppose the enumerative problem $\Xi \to U$ is given by families $p^*\Xi_1, \pi^*\Xi_2, \ldots, \pi^*\Xi_b$, where $\Xi_1$ is the family of subspaces of a fixed dimension in $\mathbb{P}^n$ and $\Xi_2, \ldots, \Xi_b$ are families of Schubert subvarieties of $G_1\mathbb{P}^n$.

By Theorem C′ of [17], the intersection problem $\Psi \to V$ on $G_1\mathbb{P}^n$ given by $\Xi_2, \ldots, \Xi_b$ has real effective algebraic equivalences. Let $Z$ be a witness. By Theorem 7.4, $p^*\Psi \to V$ has a real effective algebraic equivalence with witness $\pi^*Z$.

By Lemma 5.1, there is a real algebraic map $f : C \to \phi(W) \cap \text{Chow}^p\mathbb{F}l_{0,1}\mathbb{P}^n$ with $C$ a smooth curve, and a point $s$ on a connected arc $S$ of $C(\mathbb{R})$ such that $f(s) = \pi^{-1}Z$ and $f(S - \{s\}) \subset \phi(V(\mathbb{R}))$. Let $\Phi \to \text{Chow}^p\mathbb{F}l_{0,1}\mathbb{P}^n$ be the tautological family and consider the family $f^*\Phi \to C$. The fibre over $s$ of $f^*\Phi$ is $\pi^{-1}Z$.

Let $\mathcal{L}$ be the lattice of subspaces of $\mathbb{P}^n$ generated by the (necessarily real) subspaces defining components of $Z$, and let $N$ be a real subspace from the family $\Xi_1$ meeting all subspaces of $\mathcal{L}$ properly. By Lemma 8.3, $p^{-1}N \cap \pi^{-1}Z$ is transverse with all points of intersection real. Thus there is a Zariski open subset $C'$ of $\mathcal{L}$ such that fibres of $f^*\Phi$ over $C'$ meet $p^{-1}N$ transversally. Then $s \in (S - \{s\}) \cap C'(\mathbb{R})$, so there is a point $t \in S - \{s\}$ such that $p^{-1}N \cap (f^*\Phi)_t$ is transverse and consists entirely of real points. But $f(t) \in \phi(V(\mathbb{R}))$, so $(f^*\Phi)_t = \Phi_{f(t)}$ is a fibre of $\pi^*\Phi$ over $V(\mathbb{R})$, and hence a generically transverse intersection of real Schubert varieties from the families $\pi^*\Xi_2, \ldots, \pi^*\Xi_b$. Thus $\Xi \to U$ is fully real. $\blacksquare$

8.5. **Effective algebraic equivalence for $\mathbb{F}l_{0,1}\mathbb{P}^n$.** Any Schubert-type intersection problem on $\mathbb{F}l_{0,1}\mathbb{P}^n$ has real effective algebraic equivalences. We give an outline, as a complete analysis is lengthy and involves no new ideas beyond [17].

By Corollary 8.2, it suffices to consider intersection problems $\Xi \to U$ given by families $p^*\Xi_1, \pi^*\Xi_2, \ldots, \pi^*\Xi_b$, where $\Xi_1$ is a family of subspaces of a fixed dimension in $\mathbb{P}^n$ and $\Xi_2, \ldots, \Xi_b$ are families of Schubert subvarieties of $G_1\mathbb{P}^n$.

In [17], the intersection problem given by $\Xi_2, \ldots, \Xi_b$ is shown to have real effective algebraic equivalences with witness $Z$. Let $\Psi \to V$ be the intersection problem given by $p^*\Xi_1$ and the constant family $\pi^{-1}Z$. Using Theorem 7.4 and Lemma 5.1 one may show

$$\phi(V) \subset \phi(U) \quad \text{and} \quad \phi(V(\mathbb{R})) \subset \phi(U(\mathbb{R})).$$

It suffices to show $\Psi \to V$ has real effective algebraic equivalences.

A proof that $\Psi \to V$ has real effective algebraic equivalences mimics the proof of Theorem E of [17], with the following Lemma playing the role of Lemma 2.4 of [17].

8.6. **Lemma.** Let $F, P, N,$ and $H$ be linear subspaces of $\mathbb{P}^n$ and suppose that $H$ is a hyperplane containing neither $P$ nor $N$, $F$ is a proper subspace of $P \cap H$, and $N$ meets $F$, and hence $P$ properly. Set $L = N \cap H$. Then $\pi^{-1}\Omega(F, P)$ and $p^{-1}L$ meet generically transversally,

$$\pi^{-1}\Omega(F, P) \cap p^{-1}L = \hat{\Omega}(N \cap F, P) + \pi^{-1}\Omega(F, P \cap H) \cap p^{-1}N,$$

and the second term is itself an irreducible generically transverse intersection.

The proof of this statement is almost identical to the proof of Lemma 2.4 of [17].
9. Some Schubert-type enumerative problems in $\mathbb{F}_l^{1,n-2}\mathbb{P}^n$

The variety $\mathbb{F}_l^{1,n-2}\mathbb{P}^n$ of partial flags $l \subset \Lambda \subset \mathbb{P}^n$, where $l$ is a line and $\Lambda$ an $(n-2)$-plane has projections

$$\pi : \mathbb{F}_l^{1,n-2}\mathbb{P}^n \to \mathbb{G}_1\mathbb{P}^n \text{ and } p : \mathbb{F}_l^{1,n-2}\mathbb{P}^n \to \mathbb{G}_{n-2}\mathbb{P}^n,$$

where $\mathbb{G}_{n-2}\mathbb{P}^n$ is the Grassmannian of $(n-2)$-planes in $\mathbb{P}^n$.

9.1. Theorem. Any enumerative problem in $\mathbb{F}_l^{1,n-2}\mathbb{P}^n$ given by pullbacks of Schubert subvarieties of $\mathbb{G}_1\mathbb{P}^n$ and $\mathbb{G}_{n-2}\mathbb{P}^n$ is fully real.

Proof: Suppose $\pi^*\Xi_1, \ldots, \pi^*\Xi_b, p^*\Gamma_1, \ldots, p^*\Gamma_c$ give an enumerative problem on $\mathbb{F}_l^{1,n-2}\mathbb{P}^n$ where, for $1 \leq i \leq b$, $\Xi_i$ is a family of Schubert subvarieties of $\mathbb{G}_1\mathbb{P}^n$ and for $1 \leq j \leq c$, $\Gamma_i$ is a family of Schubert subvarieties of $\mathbb{G}_{n-2}\mathbb{P}^n$.

By Theorem 7.1 and Lemma 7.1, $\pi^*\Xi_1, \ldots, \pi^*\Xi_b$ give an intersection problem $\Psi_1 \to V_1$ which has a real algebraic equivalence with witness $Z_1$. Identifying $\mathbb{P}^n$ with its dual projective space gives an isomorphism $\mathbb{G}_{n-2}\mathbb{P}^n \cong \mathbb{G}_1\mathbb{P}^n$, mapping Schubert subvarieties to Schubert subvarieties. It follows that $p^*\Gamma_1, \ldots, p^*\Gamma_c$ give an intersection problem $\Psi_2 \to V_2$ which has a real algebraic equivalence with witness $Z_2$. It suffices to show the enumerative problem $\Psi \to V$ given by $\Psi_1$ and $\Psi_2$.

Since $Z_1$ and $Z_2$ may be replaced by any translate by elements of $\text{PGL}_{n+1}\mathbb{R}$, we assume $Z_1$ and $Z_2$ intersect transversally. Components of $Z_1$ and $Z_2$ are Schubert varieties defined by real flags. Moreover, each component of $Z_1$ has complementary dimension to each component of $Z_2$. In a flag variety, Schubert varieties of complementary dimension which meet transversally and are defined by real flags either have empty intersection, or meet in a single real point. Thus $Z_1 \cap Z_2$ consists entirely of real points.

By Lemma 7.1 for each $i = 1, 2$, there is a real algebraic map $f_i : C_i \to \phi(V_i)$ where $C_i$ is a smooth curve, and a point $s_i$ on a connected arc $S_i$ of $C_i(\mathbb{R})$ such that $f(S_i - \{s_i\}) \subset \phi(V_i(\mathbb{R}))$ and $f_i(s_i) = Z_i$.

The enumerative problem $\Psi' \to V'$ given by $f_i^*\Psi_1 \to C_1$ and $f_2^*\Psi_2 \to C_2$ is fully real, as $\Psi'_{(s_1,s_2)} = Z_1 \cap Z_2$. Since $(s_1, s_2) \in (S_1 - \{s_1\}) \times (S_2 - \{s_2\}) \cap V'(\mathbb{R})$, there is a point $(t_1, t_2) \in (S_1 - \{s_1\}) \times (S_2 - \{s_2\})$ such that $\Psi'_{(t_1,t_2)} = (f_1^*\Psi_1)_{t_1} \cap (f_2^*\Psi_2)_{t_2}$ is transverse and consists entirely of real points. Since $f_i(t_i) \in \phi(V_i(\mathbb{R}))$, we see that $(f_i^*\Psi_i)_{t_i} = (\Psi_i)_{f_i(t_i)}$ is a fibre of $\Psi_i$ over a point of $V_i(\mathbb{R})$. This shows $\Psi \to V$ is fully real.

10. Powers of Enumerative Problems

A method to construct a new fully real enumerative problem out of a given one is illustrated by a proof of Bézout’s Theorem in the plane. We will formalize this method.

10.1. Bézout’s Theorem. Let $d_1$ and $d_2$ be positive integers. Then there exist smooth real plane curves $D_1$ and $D_2$ of degrees $d_1$ and $d_2$ meeting transversally in $d_1 \cdot d_2$ real points.

Proof: Two distinct real lines meet in a single real point. Thus if $D_1$ consists of $d_1$ distinct real lines, $D_2$ of $d_2$, and if $D_1$ and $D_2$ meet transversally, then $D_1 \cap D_2$ is $d_1 \cdot d_2$ real points.

The family of real reduced degree $d$ plane curves has general member a smooth curve and contains all cycles of $d$ distinct real lines. Thus the enumerative problem of intersecting reduced curves $D_1$ and $D_2$ of respective degrees $d_1$ and $d_2$ is fully real of degree $d_1 \cdot d_2$. Moreover, pairs of smooth real curves are dense in the set of pairs of reduced real curves, showing the enumerative problem of intersecting two smooth plane curves of respective degrees $d_1$ and $d_2$ is fully real of degree $d_1 \cdot d_2$. 


10.2. Powers of intersection problems. Suppose $\Xi \to U$ is a family of multiplicity free cycles on $X$ and $d$ is a positive integer. If the locus of $d$-tuples $(u_1, \ldots, u_d)$ such that no two of $\Xi_{u_1}, \ldots, \Xi_{u_d}$ share a component is dense in $U^d$, then let $U^{(d)}$ be an open subset of that locus. Let $\Xi^{\otimes d} \to U^{(d)}$ be the family of multiplicity free cycles whose fibre over $(u_1, \ldots, u_d) \in U^{(d)}$ is $
abla_{j=1}^d \Xi_{u_j}$.

Suppose $\Xi_1 \to U_1, \ldots, \Xi_b \to U_b$ are families of multiplicity free cycles on $X$ giving an intersection problem $\Xi \to U$ and $d_1, \ldots, d_b$ is a sequence of positive integers. Then the families $\Xi_i^{\otimes d_i} \to U_i^{(d_i)}$, $\Xi_{\otimes d_b} \to U_b^{(d_b)}$ give a well-posed intersection problem if general members of the families $\Xi \to U$ and $\Xi_i \to U_i$ meet properly, for $1 \leq i \leq b$.

When a reductive group $G$ acts transitively on $X$ and the families of cycles are $G$-stable, $\Xi_i^{\otimes d_i}, \ldots, \Xi_b^{\otimes d_b}$ give an intersection problem. Moreover, if $\Xi \to U$ is fully real, then so is that intersection problem. We produce a witness with a particular form.

10.3. Lemma. Suppose $\Xi_1 \to U_1, \ldots, \Xi_b \to U_b$ give a fully real enumerative problem of degree $d$. Let $d_1, \ldots, d_b$ be a sequence of positive integers and suppose that for $1 \leq i \leq b$, $V_i$ is a $G$-stable subset of $U_i^{(d_i)}$ such that $\Delta_i^{d_i} U_i(\mathbb{R}) \subset \overline{U_i(\mathbb{R})}$, as subsets of $U_i(\mathbb{R})^{d_i}$.

Then for $1 \leq i \leq b$, there exists $v_i \in V_i(\mathbb{R})$ such that $(\Xi_1^{\otimes d_1})_{v_1}, \ldots, (\Xi_b^{\otimes d_b})_{v_b}$ intersect transversally in $d_1 \cdot \ldots \cdot d_b$ real points.

Proof: The restriction $\Psi_i$ of $\Xi_i^{\otimes d_i}$ to $V_i$ is $G$-stable. Thus $\Psi_1, \ldots, \Psi_b$ give a well-posed enumerative problem $\Psi \to V$. We show this is fully real and compute its degree.

Since $\Xi \to U$ is fully real, there is an open subset $R$ of points $u \in U(\mathbb{R})$ such that $\Xi_u$ is a $d$ distinct real points. Since $U(\mathbb{R}) \subset \prod_{i=1}^b U_i(\mathbb{R})$, for $1 \leq i \leq b$ there exists an open subset $R_i$ of $U_i(\mathbb{R})$ such that $\prod_{i=1}^b R_i \subset R$. Then $V_i(\mathbb{R}) \cap R_i \neq \emptyset$, as $\Delta_i^{d_i} R_i \subset \Delta_i U_i(\mathbb{R}) \subset \overline{V_i(\mathbb{R})}$. Thus $R':=V(\mathbb{R}) \cap \prod_{i=1}^b R_i^{d_i}$ is nonempty, as $V(\mathbb{R})$ is dense in $\prod_{i=1}^b V_i(\mathbb{R})$.

Let $w = (w_{11}, \ldots, w_{1d_1}, \ldots, w_{bd_b}) \in R'$. Here, $w_{ij} \in R_i$ and $(w_{11}, \ldots, w_{bd_b}) \in V_i(\mathbb{R})$. If $1 \leq j \leq d_i$, then $(w_{1j}, \ldots, w_{bj}) \in U(\mathbb{R})$. Furthermore, $\Psi_w = \bigcap_{i=1}^b (\Psi_i)_{(w_{1i}, \ldots, w_{id_i})}$ is a transverse intersection, as $R' \subset V$. Since $(\Psi_i)_{(w_{1i}, \ldots, w_{id_i})} = \sum_{j=1}^{d_i} (\Xi_i)_{w_{ij}}$, we have

\[\Psi_w = \sum_{i=1}^b \sum_{j=1}^{d_i} (\Xi_i)_{w_{ij}} = \sum_{1 \leq j_1 \leq d_1, \ldots, 1 \leq j_b \leq d_b} \bigcap_{i=1}^b (\Xi_i)_{w_{ij_i}} = \sum (\Xi_{w_{1j_1}, \ldots, w_{bj_b}}).
\]

Since this intersection is transverse, it consists of $d_1 \cdot \ldots \cdot d_b$ real points.

10.4. Real Bézout’s Theorem. Let $d_1, \ldots, d_b$ be positive integers. Then there exist smooth real hypersurfaces $H_1, \ldots, H_b$ in $\mathbb{P}^b$ of respective degrees $d_1, \ldots, d_b$ which intersect transversally in $d_1 \cdot \ldots \cdot d_b$ real points.

Proof: Let $\Xi \to U$ be the family of hyperplanes in $\mathbb{P}^b$. Since $b$ real hyperplanes in general position meet in a real point, either simple checking or Lemma 10.3 with $V := U^{(d_i)}$ shows that $\Xi^{\otimes d_1}, \ldots, \Xi^{\otimes d_b}$ give a fully real enumerative problem of degree $d_1 \cdot \ldots \cdot d_b$. Note that $\Xi^{\otimes d_i} \to U^{(d_i)}$ is the family of hypersurfaces composed of $d_i$ distinct hyperplanes.

Let $W_i \subset \mathbb{P}(\text{Sym}^{d_i} \mathbb{C}^{b+1})$ be the space of forms of degree $d_i$ with no repeated factors and $\Gamma_i \to W_i$ the family of reduced degree $d_i$ hypersurfaces. Let $W_i' \subset W_i$ be the dense subset of forms determining smooth hypersurfaces. Note that $U^{(d_i)} \subset W_i'$ and $\Xi^{\otimes d_i} = \Gamma_i U^{(d_i)}$.

It follows that $\Gamma_1, \ldots, \Gamma_b$ give a fully real enumerative problem of degree $d_1 \cdot \ldots \cdot d_b$. Let $R$ be an open set of witnesses. Since $U^{(d_i)}(\mathbb{R}) \subset \overline{W_i'(\mathbb{R})}$ and $R$ meets $\prod_{i=1}^b U^{(d_i)}(\mathbb{R})$, we see that $R$
meets $\prod_{i=1}^{b} W'_i(\mathbb{R})$. That is, there exist smooth real hypersurfaces $H_1, \ldots, H_b$ in $\mathbb{P}^b$ of respective degrees $d_1, \ldots, d_b$ which intersect transversally in $d_1 \cdots d_b$ real points. \quad \square

11. (n - 2)-PLANES MEETING RATIONAL NORMAL CURVES IN $\mathbb{P}^n$

Let $G_{n-2}\mathbb{P}^n$ be the Grassmannian of $(n - 2)$-planes in $\mathbb{P}^n$, a variety of dimension $2n - 2$. Those $(n - 2)$-planes which meet a curve form a hypersurface in $G_{n-2}\mathbb{P}^n$. We synthesize ideas of previous sections to prove the following theorem.

11.1. Theorem. The enumerative problem of $(n - 2)$-planes meeting $2n - 2$ general rational normal curves in $\mathbb{P}^n$ is fully real and has degree $\binom{2n-2}{n-1} n^{2n-3}$.

Proof: Identifying $\mathbb{P}^n$ with its dual projective space gives an isomorphism $G_{n-2}\mathbb{P}^n \sim G_1\mathbb{P}^n$, mapping Schubert subvarieties to Schubert subvarieties. By Theorem C of [17], any enumerative problem involving Schubert subvarieties of $G_{n-2}\mathbb{P}^n$ is fully real. In particular, the enumerative problem given by $2n - 2$ copies of the family $\Xi \to U$ is fully real, where $U = G_1\mathbb{P}^n$ and the fibre of $\Xi$ over $l \in U$ is the Schubert variety $\Omega_l$ of $(n - 2)$-planes meeting $l$.

We compute its degree, $d$. The image of $\Omega_l$ under the isomorphism $G_{n-2}\mathbb{P}^n \sim G_1\mathbb{P}^n$ is the Schubert subvariety of all lines meeting a fixed $(n - 2)$-plane. Thus $d$ is the number of lines meeting $2n - 2$ general $(n - 2)$-planes in $\mathbb{P}^n$. By Corollary 3.3 of [17], this is the number of (standard) Young tableaux of shape $(n - 1, n - 1)$, which is $\frac{1}{n} \binom{2n-2}{n-1}$, by the hook length formula of Frame, Robinson, and Thrall [5].

Let $e_0, \ldots, e_n$ be real points spanning $\mathbb{P}^n$. For $1 \leq i \leq n$, let $l_i := (e_{i-1}, e_i)$. Then $\Omega_{l_1} + \cdots + \Omega_{l_n}$ is the fibre of $\Xi^{\oplus n}$ over $(l_1, \ldots, l_n) \in U^{(n)}(\mathbb{R})$. Let $V = PGL_{n+1} \mathbb{C} \cdot (l_1, \ldots, l_n) \subset U^{(n)}$. For $t \in [0, 1]$ and $1 \leq i \leq n$, let

$$l_i(t) := (te_{i-1} + (1-t)e_i, te_i + (1-t)e_i),$$

where $\overline{j} \in \{0, 1\}$ is congruent to $j$ modulo $2$. Let $\gamma(t) := (l_1(t), \ldots, l_n(t))$. If $t \in (0, 1]$, then $\gamma(t) \in V(\mathbb{R})$. Since $\gamma(0) = (l_1, \ldots, l_n)$ and $\Delta^n U(\mathbb{R}) = PGL_{n+1} \mathbb{R} \cdot \gamma(0)$, it follows that $\Delta^n U(\mathbb{R}) \subset \overline{V}(\mathbb{R})$. Then, by Lemma 10.3, there exist points $v_1, \ldots, v_{2n-2} \in \overline{V}(\mathbb{R})$ such that $\Xi^{\oplus n_1}, \ldots, \Xi^{\oplus n_{2n-2}}$ meet transversally in $\frac{1}{n} \binom{2n-2}{n-1} n^{2n-3}$ points.

Let $p(m) := n \cdot m + 1$, the Hilbert polynomial of a rational normal curve in $\mathbb{P}^n$. Let $\mathcal{H}$ be the open subset of the Hilbert scheme parameterizing reduced schemes with Hilbert polynomial $p$. Let $\Psi \subset \mathcal{H} \times G_{n-2}\mathbb{P}^n$ be the family of multiplicity free cycles on $G_{n-2}\mathbb{P}^n$ whose fibre over a curve $C \in \mathcal{H}$ is the hypersurface of $(n - 2)$-planes meeting $C$.

Note that $p$ is also the Hilbert polynomial of $l_1 \cup \cdots \cup l_n$. If $V'$ is the $PGL_{n+1}\mathbb{C}$-orbit of $l$ in $\mathcal{H}$, then $\Psi |_{V'} \to V'$ is isomorphic to the family $\Xi^{\oplus n} \to V$, under the obvious isomorphism between $V$ and $V'$. It follows that the enumerative problem given by $2n - 2$ copies of $\Psi \to \mathcal{H}$ is fully real.

Let $W$ be the subset of $\mathcal{H}$ representing rational normal curves. We claim $V'(\mathbb{R}) \subset \overline{W}(\mathbb{R})$, from which it follows that the enumerative problem of $(n - 2)$-planes meeting $2n - 2$ rational normal curves in $\mathbb{P}^n$ is fully real and has degree $\binom{2n-2}{n-1} n^{2n-3}$.

Let $[x_0, \ldots, x_n]$ be homogeneous coordinates for $\mathbb{P}^n$ dual to the basis $e_0, \ldots, e_n$. For $t \in \mathbb{C}$, define the ideal $\mathcal{I}_t$ by

$$\mathcal{I}_t := (x_ix_j - tx_{i+1}x_{j-1} | 0 \leq i < j \leq n \text{ and } j - i \geq 2).$$

For $t \neq 0$, $\mathcal{I}_t$ is the ideal of a rational normal curve and $\mathcal{I}_0$ is the ideal of $l_1 \cup \cdots \cup l_n$. 
This family of ideals is flat. Let $\varphi : \mathbb{C} \to \mathcal{H}$ be the map representing this family. Then $\varphi(\mathbb{R} - \{0\}) \subset W(\mathbb{R})$. Noting $\varphi(0) = \lambda$ shows $\lambda \in W(\mathbb{R})$. Since $W(\mathbb{R})$ is $PGL_{n+1}\mathbb{R}$-stable, we conclude that $V'(\mathbb{R}) \subset W(\mathbb{R})$. 

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