Mathematical theory of analysis and synthesis of control systems based on the general principle of isomorphism

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Abstract. A new paradigm in the theory of control systems is presented, in which all the basic concepts and properties of systems are formalized on the basis of a single principle - the general principle of isomorphism. The principle made it possible to formulate and prove the realization theorem, on the basis of which the basic concepts (model, abstract system, simple, complex, large system) have been mathematically strictly defined and the fundamental properties of systems and criteria for their provision have been formalized: in a new way, up to isomorphic models of a rather general form, the concepts and criteria of observability, controllability, checkability, identifiability of systems have been defined, and new formalized methods of analysis and synthesis of regulators, observers and integrated information-control systems have been developed, the examples of applications of the theory have been given.

Introduction
One of applied 'technical' and, however, fundamental problems is the problem of existence and uniqueness of solution of the systems theory tasks, the analysis of which started in [1, 2] where it is formulated as the principle of uniqueness in the realization (identification) problem. The essence of the realization problem is to build a model that explains (implements) some data about the system.

The principle of uniqueness in the realization problem states [1]: if the data are accurate and complete, then there is one and only one minimal system (mathematical model) that reproduces these data. Corollary: all minimal explanations of the same data are isomorphic models. Note that "the concept of a minimal realization for which the state space has minimal dimension" [2] is one of the fundamental concepts of "the abstract realization theory" [2] and, in general, "the algebraic theory of linear systems" [2]. In [1], it is concluded that the general realization theorem corresponding to the uniqueness principle, has not been formulated and proved, since the concepts such as "data", "system", "data completeness" and "data accuracy" have not been mathematically strictly defined, i.e. the problem of formalization of concepts has not been solved.

The topic of uniqueness of solving the problems of the system theory was discussed in [3-7]. However, these and other works did not solve the problem of formalization of concepts, and, in general, do not contain the formulation and proof of the general realization theorem and do not allow to obtain unique solutions, in the sense of [1,6,7], of classical problems of analysis and synthesis of systems in their original formulation (without reducing the models, using, for example, embedding in simply-connected isomorphic image systems).

Here, on the basis of [8-13], a new formulation of the uniqueness principle in the form of the general isomorphism principle is considered, which, in our opinion, allows us to solve both
fundamental problems of the system theory and problems caused by difficulties in solving frequently encountered applied problems in their original formulation. As in [1, 2, 6, 7, 11-13], the presentation is conducted in the language of mappings and morphisms. Based on the general provisions of the theory of realization [1,2], on the basis of the general principle of isomorphism in [8-13], a theorem on the realization of isomorphism by a composition of maps has been formulated and proved, which asserts the existence of inverses in maps that form this composition. The novelty lies in the fact that the maps that make up the composition, in general, outside the commutative diagram, are not isomorphisms in the usual sense. In this regard, the problem of determining the inverses becomes nontrivial. The realization theorem allows us to solve this problem: in [11-13] it is shown that the inverses to maps that make up a composition that implements a certain isomorphism can be determined up to this isomorphism in the corresponding commutative diagram. A method for solving this fundamental problem is given and illustrated with examples. An overview of applications of the realization theorem and the specified method to solving problems of modeling and identification of systems, synthesis and integration of regulators and observers in an information and control system, system design and integration of systems of general form, construction and calibration of intelligent sensors is given.

In general, the work is devoted to the review of a new paradigm in the field of analysis and synthesis of control systems based on the general principle of isomorphism proposed by the author. The paradigm contains an integral system of views on systems and methods of their research and synthesis, as well as a general formal mathematical approach to the analysis of their properties and is based on a transparent, minimally necessary axiomatics strictly justified in terms of practice.

1. The general principle of isomorphism and its applications to the analysis and synthesis of control systems

From a practical standpoint, the formulation of the uniqueness principle given in [1] should be clarified. The fact is that, observing the behavior of the system as a "black box" (input and output), it is not known a priori whether we have complete and accurate data about the system. In addition, as mentioned in [1], it is very difficult to formally define the concepts of "completeness" and "accuracy" of data. It is therefore advisable to introduce a new formulation of the uniqueness principle in the form the following general isomorphism principle.

**The general principle of isomorphism in the system theory:** if there is a minimal mapping (model) \( f \) in the sense of [1,2] that uniquely (isomorphically) explains some data about the system, then

a) up to the model-isomorphism of \( f \), these data are complete and accurate – there is an isomorphism of the data;

b) the model \( f \) is unique in the sense that if there are other minimal isomorphic models-explanations of the same data, then these models-explanations are isomorphic to model \( f \) – there is an isomorphism of the models.

In other words, isomorphism is observed in both directions: first, as a one-to-one transformation of input and output data into each other (an isomorphism of the model-mapping \( f \) itself, which is called a data isomorphism); second, as a one-to-one transformation of any number of "minimal explanations" (models) built on these data (such an isomorphism is called a model isomorphism). The general principle of isomorphism allowed us to formulate and prove the realization theorem.

**The realization theorem.** The commutative diagram shown in Fig. 1 is considered. Commutative diagrams in many problems of the algebraic systems theory are brought into such a diagram [8-13].

For the diagram, the realization theorem [12-13] is valid, which is presented here without proof.

**The realization theorem:** If the map \( f:X \rightarrow Y \) is an ordinary isomorphism and there are maps \( g:X \rightarrow Z \) and \( h:Z \rightarrow Y \) such that the composition \( f=hg \) is satisfied, then the maps \( g \) and \( h \) are isomorphisms up to the isomorphism \( f \).

As for the composition of \( hg \) of maps \( g \) and \( h \), it can be said that it realizes, models the mapping-isomorphism \( f \).
map $f$ must be isomorphic outside the commutative diagram, that is, it must be an ordinary, "unconditional" isomorphism.

As applied to the maps $g$ and $h$, the theorem states that even if they are not isomorphisms separately, outside the composition and the commutative diagram, then in the composition $f=hg$, closed by an isomorphism $f$ into a commutative diagram, they become *isomorphisms up to the isomorphism* $f$, that is, they become some "relative isomorphisms" – isomorphisms relative to (up to) the isomorphism $f$. That means that each of the maps $g$ and $h$ is an isomorphism only within the framework of the commutative map diagram shown in Fig. 1. Under these conditions, the maps $g$ and $h$ in the commutative diagram satisfy all the properties of isomorphic maps defined in algebra [14]. The proof is given in [12].

The complexity lies in the fact that the maps $g$ and $h$ that make up the composition, in general, outside the commutative diagram, are not isomorphisms in the usual sense. In this regard, the problem of determining the inverses becomes non-trivial. The realization theorem allows us to solve this problem. In [12,13], the method of this problem solution is analyzed and illustrated with examples. It follows from the realization theorem [12,13] that the inverses $g^{-1}$ and $h^{-1}$ for the maps $g$ and $h$ are unique, and relevant to them, the following relations must be satisfied in the commutative diagram

\[
\begin{align*}
    e_y^{\text{right}} &= f^{-1}f, \quad e_y^{\text{left}} = ef^{-1}, \quad e_x^{\text{left}} = g^{-1}g, \quad e_y^{\text{right}} = hh^{-1}, \quad f^{-1} = ef^{-1}, \quad f^{-1} = f^{-1}e_y, \\
    e_y &= e_x^{\text{right}} = e_y^{\text{left}} = e_y^{\text{right}} = e_y = e, \\
    h^{-1} &= e_z h^{-1}, \quad h^{-1} = h^{-1}e_y, \quad g^{-1} = g^{-1}e_y, \quad g^{-1} = e_z g^{-1}, \\
    f^{-1} &= g^{-1}h^{-1}, \quad e_x^{\text{left}} = g g^{-1}, \quad e_y^{\text{right}} = h^{-1}h, \\
    e_y^{\text{left}} &= g g^{-1}, \quad e_y^{\text{right}} = h^{-1}h, \quad e_x^{\text{left}} = e_y^{\text{right}} = e_z, \quad e_x^{\text{left}} = e_y^{\text{right}} = e_z, \quad e_z^{\text{left}} = e_z, \quad (e_z)^{-1} = e_z, \quad e_z = e_z.
\end{align*}
\]

(1)

where $e_y^{\text{right}}$, $e_y^{\text{left}}$ are right and left units on $X$; $e_y^{\text{right}}$, $e_y^{\text{left}}$ are right and left units on $Y$; $e_z^{\text{left}}$, $e_z^{\text{right}}$ are left and right units on $Z$. By virtue of the uniqueness of the map $f^{-1}$ (since by condition $f$ is an ordinary isomorphism), the maps $e_y$ and $e_y$ are also unique and equal to $e$, where $e$ is an ordinary unit map such that $e = ee = ee = e = e^{-1}$. Note that in general, $e$ is some unit on $Z$ and is not a usual unit map and is not equal to $e$. The inverses are easily calculated using the formulas

\[
\begin{align*}
    g^{-1} &= f^{-1}, \quad h^{-1} = gf^{-1},
\end{align*}
\]

which result directly from the commutative diagram. One can verify that the maps $g^{-1}$ and $h^{-1}$ are indeed inverses by substituting them in the equalities (1).

2. Applications of the realization theorem to the analysis and synthesis of control systems

Due to the limited scope of the report, only references for those works are provided that describe the new paradigm in the system theory and some of its applications.

The application of the realization theorem to the formalization of the basic concepts of system theory is considered in [12]. Here the consequences of the realization theorem are also discussed: "prime systems", measurability of systems, the "postulate" causality up to isomorphism, determinacy of "randomness", the principle of relativity in systems theory, the fundamental properties of the systems, and the principle of superposition in the systems theory. Algebraic formalism of the systems general properties (observability, controllability, verifiability and identifiability of the systems up to isomorphism, duality of controllability and observability) is presented in [8, 9, 11, 12]. Integration, integration paradoxes and properties of integrated systems, as well as the separation theorem for the synthesis of information and control systems are presented in [12, 15, 16]. Applications to modeling and identification of systems, as well as to the development and calibration of smart sensors are considered in [10, 17-20]. The methods and examples of synthesis of isomorphic regulators are considered in [21-26], and isomorphic observers are considered in [27-30]. Applications to the system design are considered in [31-36], to integration of information and control systems - in [15,n16].
Conclusion

The paper presents an overview of the main provisions of the new mathematical theory of analysis and synthesis of control systems, developed on the basis of simple axiomatics, based on a single and unique principle, justified in terms of practice – the general principle of isomorphism. The practical feasibility and significance of the theory is illustrated by examples of its applications to solving the typical problems of analysis and design of information and control systems. The effectiveness of applied synthesis methods developed on the basis of the theory is demonstrated in comparison with the known and traditionally used methods.

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