On the number of outer automorphisms of the automorphism group of a right-angled Artin group

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Abstract

We show that there is no uniform upper bound on $|\text{Out}(\text{Aut}(A_\Gamma))|$ when $A_\Gamma$ ranges over all right-angled Artin groups. This is in contrast with the cases where $A_\Gamma$ is free or free abelian: for all $n$, Dyer-Formanek and Bridson-Vogtmann showed that $|\text{Out}(\text{Aut}(F_n))| = 1$, while Hua-Reiner showed $|\text{Out}(\text{Aut}(\mathbb{Z}^n))| = |\text{Out}(\text{GL}(n,\mathbb{Z}))| \leq 4$. We also prove the analogous theorem for $|\text{Out}(\text{Out}(A_\Gamma))|$. We establish our results by giving explicit examples; one useful tool is a new class of graphs called *austere graphs*.

1 Overview

A finite simplicial graph $\Gamma$ with vertex set $V$ and edge set $E \subset V \times V$ defines the *right-angled Artin group* $A_\Gamma$ via the presentation

$$\langle v \in V \mid [v, w] = 1 \text{ if } (v, w) \in E \rangle.$$ 

The class of right-angled Artin groups contains all finite rank free and free abelian groups, and allows us to interpolate between these two classically well-studied classes of groups.

A centreless group $G$ is *complete* if the natural embedding $\text{Inn}(G) \hookrightarrow \text{Aut}(G)$ is an isomorphism. Dyer-Formanek [6] showed that $\text{Aut}(F_n)$ is complete for $F_n$ a free group of rank $n \geq 2$, giving $|\text{Out}(\text{Aut}(F_n))| = 1$. Bridson-Vogtmann [2] later proved this for $n \geq 3$ using more geometric methods, and showed that $\text{Out}(F_n)$ is also complete. Although $\text{GL}(n,\mathbb{Z})$ is not complete (its centre is $\mathbb{Z}_2$), we observe similar behaviour for free abelian groups. Hua-Reiner [9] explicitly determined $|\text{Out}(\text{GL}(n,\mathbb{Z}))|$; in particular, $|\text{Out}(\text{GL}(n,\mathbb{Z}))| \leq 4$ for all $n$. In other words, for free or free abelian $A_\Gamma$, the orders of $\text{Out}(\text{Aut}(A_\Gamma))$ and $\text{Out}(\text{Out}(A_\Gamma))$ are both uniformly bounded above. The main result of this paper is that no such uniform upper bounds exist when $A_\Gamma$ ranges over the entire class of right-angled Artin groups.

**Theorem A.** For any $N \in \mathbb{N}$, there exists a right-angled Artin group $A_\Gamma$ such that $|\text{Out}(\text{Aut}(A_\Gamma))| > N$. Moreover, we can take $A_\Gamma$ to have either trivial or non-trivial centre.

We also prove the analogous result regarding the order of $|\text{Out}(\text{Out}(A_\Gamma))|$.

**Theorem B.** For any $N \in \mathbb{N}$, there exists a right-angled Artin group $A_\Gamma$ such that $|\text{Out}(\text{Out}(A_\Gamma))| > N$. 
We remark that neither Theorem A nor B follows from the other, since in general, given a quotient $G/N$, the groups $\text{Aut}(G/N)$ and $\text{Aut}(G)$ may behave very differently.

We prove both theorems by exhibiting classes of right-angled Artin groups over which the groups in question grow without bound. We introduce the notions of an *austere graph* and an *austere graph with star cuts* in Sections 2 and 4, respectively. These lead to tractable decompositions of $\text{Out}(A_{\Gamma})$ and $\text{Out}(A_{\Gamma})$, which then yield numerous members of $\text{Out}(\text{Aut}(A_{\Gamma}))$ and $\text{Out}(\text{Out}(A_{\Gamma}))$. Our methods do not obviously yield infinite order elements of $\text{Out}(\text{Aut}(A_{\Gamma}))$; we discuss this further in Section 5.

**Outline of paper.** In Section 2, we recall the finite generating set of $\text{Aut}(A_{\Gamma})$ and give the proof of Theorem B. Sections 3 and 4 contain two proofs of Theorem A; first for right-angled Artin groups with non-trivial centre, then for those with trivial centre. In Section 5, we discuss generalisations of this work, including the question of extremal behaviour of $\text{Out}(\text{Aut}(A_{\Gamma}))$. The Appendix contains a calculation used in the proof of Proposition 3.2.

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## 2 Proof of Theorem B

Let $\Gamma$ be a finite simplicial graph with vertex set $V$ and edge set $E \subseteq V \times V$. We write $\Gamma = (V, E)$. We abuse notation and consider $v \in V$ as both a vertex and a generator of $A_{\Gamma}$. We will also often consider a subset $S \subseteq V$ as the full subgraph of $\Gamma$ which it spans. For a vertex $v \in V$, we define its *link*, $\text{lk}(v)$, to be the set of vertices in $V$ adjacent to $v$, and its *star*, $\text{st}(v)$, to be $\text{lk}(v) \cup \{v\}$.

**The LS generators.** Laurence [10] and Servatius [12] gave a finite generating set for $\text{Aut}(A_{\Gamma})$, which we now recall. We specify the action of the generator on the elements of $V$. If a vertex $v \in V$ is omitted, it is assumed to be fixed. There are four types of generators:

1. *Inversions*, $\iota_v$: for each $v \in V$, $\iota_v$ maps $v$ to $v^{-1}$.
2. *Graph symmetries*, $\phi$: each $\phi \in \text{Aut}(\Gamma)$ induces an automorphism of $A_{\Gamma}$, which we also denote by $\phi$, mapping $v \in V$ to $\phi(v)$.
3. *Dominated transvections*, $\tau_{xy}$: for $x, y \in V$, whenever $\text{lk}(y) \subseteq \text{st}(x)$, we write $y \leq x$, and say $y$ is *dominated* by $x$ (see Figure 1a). In this case, $\tau_{xy}$ is well-defined, and maps $y$ to $yx$. The vertex $x$ may be adjacent to $y$, but it need not be.
4. *Partial conjugations*, $\gamma_{c,D}$: fix $c \in V$, and select a connected component $D$ of $\Gamma \setminus \text{st}(c)$ (see Figure 1b). The partial conjugation $\gamma_{c,D}$ maps every $d \in D$ to $cdc^{-1}$.

We refer to the generators on this list as the *LS generators* of $\text{Aut}(A_{\Gamma})$.
Austere graphs. We say that a graph $\Gamma = (V,E)$ is *austere* if it has trivial symmetry group, no dominated vertices, and for each $v \in V$, the graph $\Gamma \setminus \text{st}(v)$ is connected. We use examples of austere graphs to prove Theorem B.

**Proof of Theorem B.** For an austere graph $\Gamma = (V,E)$, the only well-defined LS generators of $\text{Aut}(A_\Gamma)$ are the inversions and the partial conjugations. Let $n = |V|$. Note that each partial conjugation is an inner automorphism. We have the decomposition

$$\text{Aut}(A_\Gamma) \cong \text{Inn}(A_\Gamma) \rtimes I_\Gamma,$$

where $I_\Gamma \cong \mathbb{Z}_2^n$ is the group generated by the inversions. The inversions act on $\text{Inn}(A_\Gamma) \cong A_\Gamma$ in the obvious way, either inverting or fixing (conjugation by) each $v \in V$. We have $\text{Out}(A_\Gamma) \cong I_\Gamma$, and so $\text{Aut}(\text{Out}(A_\Gamma)) \cong \text{Out}(\text{Out}(A_\Gamma)) \cong \text{GL}(n,\mathbb{Z}_2)$. If we can find austere graphs for which $n$ is as large as we like, then we will have proved Theorem B.

The Frucht graph, seen in Figure 2, was constructed by Frucht [7] as an example of a 3-regular graph with trivial symmetry group. In fact, the Frucht graph has no dominated vertices and removing the star of any vertex leaves it connected; hence, it is austere. Baron-Imrich [1] generalised the Frucht graph to produce a family of finite, 3-regular graphs with trivial symmetry groups, over which $n = |V|$ is unbounded. Like the Frucht graph, these graphs have no dominated vertices and remain connected when the star of any vertex is
removed. They are therefore austere, and so define a class of right-angled Artin groups which proves Theorem B.

3 Proof of Theorem A: right-angled Artin groups with non-trivial centre

In this section, we assume that $A_{\Gamma}$ has non-trivial centre. Let $\{\Gamma_i\}$ be a collection of graphs. The join, $J\{\Gamma_i\}$, of $\{\Gamma_i\}$ is the graph obtained from the disjoint union of $\{\Gamma_i\}$ by adding an edge $(v_i, v_j)$ for all vertices $v_i$ of $\Gamma_i$ and $v_j$ of $\Gamma_j$, for all $i \neq j$. Observe that for a finite collection of finite simplicial graphs $\{\Gamma_i\}$, we have

$$A_{J\{\Gamma_i\}} \cong \prod_i A_{\Gamma_i}.$$ 

When we take the join of only two graphs, $\Gamma$ and $\Delta$, we write $J(\Gamma, \Delta)$ for their join.

3.1 Decomposing $\text{Aut}(A_{\Gamma})$

A vertex $s \in V$ is said to be social if it is adjacent to every vertex of $V \setminus \{s\}$. Let $S$ denote the set of social vertices of $\Gamma$ and set $k = |S|$. Let $\Delta = \Gamma \setminus S$. By The Centralizer Theorem of Servatius [12], we have $\Gamma = J(S, \Delta)$, and

$$A_{\Gamma} \cong \mathbb{Z}^k \times A_{\Delta}.$$ 

No vertex $v \in \Delta$ can dominate any vertex of $S$ (otherwise $v$ would be social), and any $\phi \in \text{Aut}(\Gamma)$ must preserve $S$ and $\Delta$ as sets. Determining the LS generators, we see that $\text{Aut}(A_{\Gamma})$ has $\text{GL}(k, \mathbb{Z}) \times \text{Aut}(A_{\Delta})$ as a proper subgroup. The only LS generators not contained in this proper subgroup are of the form $\tau_{sa}$, where $s \in S$ and $a \in \Delta$. Note that this dominated transvection is defined for any pair $(s, a) \in S \times \Delta$. We will refer to this type of transvection as a lateral transvection, as they occur ‘between’ the two graphs, $S$ and $\Delta$.

**Proposition 3.1.** Let $\Gamma = J(S, \Delta)$ define a right-angled Artin group, $A_{\Gamma}$, with non-trivial centre. The group $L$ generated by the lateral transvections is isomorphic to $\mathbb{Z}^{k|\Delta|}$.

**Proof.** It is clear the lateral transvections $\tau_{sa}$ and $\tau_{tb}$ commute if $a \neq b$. The only case left to check is $\tau_{sa}$ and $\tau_{ta}$, for $s, t \in S$ and $a \in \Delta$. We see that

$$\tau_{ta}\tau_{sa}\tau_{ta}^{-1}(a) = \tau_{ta}\tau_{sa}(at^{-1}) = \tau_{ta}(ast^{-1}) = ast^{-1} = as,$$

since $s$ and $t$ commute. Therefore $\tau_{ta}\tau_{sa}\tau_{ta}^{-1} = \tau_{sa}$, and hence $L$ is abelian. That it has no torsion follows from the fact that $\mathbb{Z}^k$ has no torsion. A straightforward calculation verifies that the lateral transvections form a $\mathbb{Z}$-basis for $L$. To deduce the rank, observe there is a bijection between $\{\tau_{sa} \mid S \in S, a \in \Delta\}$ and $S \times \Delta$. 

We now show that $L$ is the kernel of a split product decomposition of $\text{Aut}(A_{\Gamma})$. This is an $\text{Aut}(A_{\Gamma})$ version of a decomposition of $\text{Out}(A_{\Gamma})$ given by Charney-Vogtmann [5].
Proposition 3.2. Let $\Gamma = J(S, \Delta)$ define a right-angled Artin group, $A_\Gamma$, with non-trivial centre. The group $\text{Aut}(A_\Gamma)$ splits as the product

$$\mathbb{Z}^{k|\Delta|} \times [\text{GL}(k, \mathbb{Z}) \times \text{Aut}(A_\Delta)].$$

Proof. Standard computations show that $\mathcal{L} \cong \mathbb{Z}^{k|\Delta|}$ is closed under conjugation by the LS generators: these calculations are summarised in the Appendix. We observe that the intersection of $\mathcal{L}$ and $\text{GL}(k, \mathbb{Z}) \times \text{Aut}(A_\Delta)$ is trivial: the elements of $\mathcal{L}$ transvect vertices of $\Delta$ by vertices of $S$, whereas the elements of $\text{GL}(k, \mathbb{Z}) \times \text{Aut}(A_\Delta)$ carry $\mathbb{Z}^k$ and $A_\Delta$ back into themselves. Thus, $\text{Aut}(A_\Gamma)$ splits as in the statement of the proposition. \qed

We look to the $\mathbb{Z}^{k|\Delta|}$ kernel as a source of automorphisms of $\text{Aut}(A_\Gamma)$. We must however ensure that the split product action is preserved; this is achieved using the theory of automorphisms of split products, which we now recall.

**Automorphisms of split products.** Let $G = N \rtimes H$ be a split product, where $N$ is abelian, with the action of $H$ on $N$ being encoded by a homomorphism $\alpha : H \to \text{Aut}(N)$, writing $h \mapsto \alpha_h$. We will often write $(n, h) \in G$ simply as $nh$. Let $\text{Aut}(G, N) \leq \text{Aut}(G)$ be the subgroup of automorphisms which preserve $N$ as a set. For each $\gamma \in \text{Aut}(G, N)$, we get an induced automorphism $\phi$, say, of $G/N$, and an automorphism $\theta$, say, of $N$, by restriction. The map $P : \text{Aut}(G, N) \to \text{Aut}(N) \times \text{Aut}(H)$ given by $P(\gamma) = (\theta, \phi)$ is a homomorphism.

An element $(\theta, \phi) \in \text{Aut}(N) \times \text{Aut}(H)$ is said to be a **compatible pair** if

$$\theta\alpha_h\theta^{-1} = \alpha_{\phi(h)},$$

for all $h \in H$. Let $C \leq \text{Aut}(N) \times \text{Aut}(H)$ be the subgroup of all compatible pairs. This is a special (split, abelian kernel) case of the notion of compatibility for group extensions $[11, 15]$. Notice that the image of $P$ is contained in $C$, since $\gamma \in \text{Aut}(G, N)$ must preserve the relation $hnh^{-1} = \alpha_h(n)$ for all $h \in H$, $n \in N$. We therefore restrict the codomain of $P$ to $C$. Note that while $P$ (with its new codomain) is surjective, it need not be injective. We map $C$ back into $\text{Aut}(G, N)$ using the homomorphism $R$, defined by

$$R(\theta, \phi)(nh) = \theta(n)\phi(h).$$

Let $\text{Aut}_H(G, N)$ be the subgroup of $\text{Aut}(G, N)$ of maps which induce the identity on $H$. This group is mapped via $P$ onto

$$C_1 := \{\theta \in \text{Aut}(N) \mid \theta\alpha(h)\theta^{-1} = \alpha(h) \quad \forall h \in H\}.$$ 

Note $C_1$ is the centraliser of $\text{im}(\alpha)$ in $\text{Aut}(N)$. We determine $C_1$ for the split decomposition of $\text{Aut}(A_\Gamma)$ given by Proposition 3.2 and use $R$ to map $C_1$ into $\text{Aut}(\text{Aut}(A_\Gamma))$.

### 3.2 Ordering the lateral transvections

In order to determine the image of $\alpha$ for our split product, $\mathbb{Z}^{k|\Delta|} \times [\text{GL}(k, \mathbb{Z}) \times \text{Aut}(A_\Delta)]$, we specify an ordering on the lateral transvections. Let $s_1 \leq \ldots \leq s_k$ be a total order on the vertices of $S$. For lateral transvections $\tau_{s_1, a}, \tau_{s_j, b}$, we say $\tau_{s_i, a} \leq \tau_{s_j, b}$ if $s_i \leq s_j$. For a fixed $i$, we refer to the set $\{\tau_{s_i, a} \mid a \in \Delta\}$ as a $\Delta$-block.
We now use properties of the graph \( \Delta \) to determine the rest of the ordering on the lateral transvections. Recall that for vertices \( x, y \in V \), \( x \) dominates \( y \) if \( \text{lk}(y) \subseteq \text{st}(x) \), and we write \( y \preceq x \). Charney-Vogtmann \([5]\) show that \( \preceq \) is a pre-order (that is, a reflexive, transitive relation) on \( V \), and use it to define the following equivalence relation. Let \( v, w \in V \). We say \( v \) and \( w \) are domination equivalent if \( v \preceq w \) and \( w \preceq v \). If this is the case, we write \( v \sim w \), and let \([v]\) denote the domination equivalence class of \( v \).

The pre-order on \( V \) descends to a partial order on \( V/\sim \). We also denote this partial order by \( \preceq \). The group \( \text{Aut}(\Delta) \) acts on the set of domination classes of \( \Delta \). Let \( O \) be the set of orbits of this action, writing \( O_v \) for the orbit of the class \([v]\). We wish to define a partial order \( \ll \) on \( O \) which respects the partial order on the domination classes. That is, if \([v] \preceq [w] \), then \( O_v \ll O_w \), for domination classes \([v]\) and \([w]\).

We achieve this by defining a relation \( \ll \) on \( O \) by the rule \( O_v \ll O_w \) if and only if there exists \([w'] \in O_w \) such that \([v] \preceq [w'] \). This is well-defined, since \( \text{Aut}(\Delta) \) acts transitively on each \( O_v \in O \). The properties of \( \preceq \) discussed above give us the following proposition.

**Proposition 3.3.** The relation \( \ll \) on \( O \) is a partial order.

**Proof.** We utilise the transitive action of \( \text{Aut}(\Delta) \) on each \( O_v \in O \). The only work lies in establishing the anti-symmetry of \( \ll \). This can be achieved by noting that if \([v] \preceq [w] \), then \( |\text{st}(v)| \leq |\text{st}(w)| \), and if \([v] \preceq [w] \) with \( |\text{st}(v)| = |\text{st}(w)| \) then \([v] = [w] \). \( \Box \)

We use \( \ll \) to define a total order on the vertices of \( \Delta \), by first extending \( \ll \) to a total order on \( O \). We also place total orders on the domination classes within each \( O_v \in O \), and on the vertices within each domination class. Now each vertex is relabelled \( T(p,q,r) \) to indicate its place in the order: \( T(p,q,r) \) is the \( r \)th vertex of the \( q \)th domination class of the \( p \)th orbit. When working with a given \( \Delta \)-block, we can identify the lateral transvections with the vertices of \( \Delta \), allowing us to think of \( T(p,q,r) \) as a lateral transvection. Thus, we may think of a specific \( \Delta \)-block as inheriting an order from the ordering on \( \Delta \).

**The centraliser of the image of \( \alpha \).** We now explicitly determine the image of \( \alpha \), and its centraliser, in \( \text{GL}(k|\Delta|, \mathbb{Z}) \). Looking at how \( \text{GL}(k, \mathbb{Z}) \rtimes \text{Aut}(A_\Delta) \) acts on \( \mathbb{Z}^{|\Delta|} \), we see that the image of \( \alpha \) is

\[
Q := \text{GL}(k, \mathbb{Z}) \times \Phi_\Delta,
\]

where \( \Phi_\Delta \) is the image of \( \text{Aut}(A_\Delta) \) under the homomorphism induced by abelianising \( A_\Delta \). The action on \( \mathbb{Z}^{|\Delta|} \) is given in the Appendix.

The matrices in \( Q \) have a natural block decomposition given by the \( \Delta \)-blocks: each \( M \in Q \) may be partitioned into \( k \) horizontal blocks and \( k \) vertical blocks, each of which has size \( |\Delta| \times |\Delta| \). We write \( M = (A_{ij}) \), where \( A_{ij} \) is the block entry in the \( i \)th row and \( j \)th column. Under this decomposition, we see that the \( \text{GL}(k, \mathbb{Z}) \) factor of \( Q \) is embedded as

\[
\text{GL}(k, \mathbb{Z}) \cong \{ (a_{ij} \cdot I_{|\Delta|}) \mid (a_{ij}) \in \text{GL}(k, \mathbb{Z}) \},
\]

where \( I_{|\Delta|} \) is the identity matrix in \( \text{GL}(|\Delta|, \mathbb{Z}) \). We write \( \text{Diag}(D_1, \ldots, D_k) \) to denote the block diagonal matrix \((B_{ij})\) where \( B_{ii} = D_i \) and \( B_{ij} = 0 \) if \( i \neq j \). The \( \Phi_\Delta \) factor of \( Q \) embeds as

\[
\Phi_\Delta \cong \{ \text{Diag}(M, \ldots, M) \mid M \in \Phi_\Delta \} \leq Q.
\]
We now determine the centraliser, \( C(Q) \), of \( Q \) in \( \GL(k|\Delta|,\mathbb{Z}) \). The proof is similar to the standard computation of \( Z(\GL(k,\mathbb{Z})) \).

**Lemma 3.4.** The centraliser \( C(Q) \) is a subgroup of \( \{ \Diag(M,\ldots,M) | M \in \GL(|\Delta|,\mathbb{Z}) \} \).

**Proof.** Clearly an element of \( C(Q) \) must centralise the \( \GL(k,\mathbb{Z}) \) factor of \( Q \). Let \( D \) be the subgroup of diagonal matrices in \( \GL(k,\mathbb{Z}) \), and define

\[
\hat{D} := \{ (\epsilon_{ij} \cdot I_{|\Delta|}) | (\epsilon_{ij}) \in D \} \leq Q.
\]

Suppose \((A_{ij}) \in C(Q)\) centralises \( \hat{D} \). Then for each \((\epsilon_{ij} \cdot I_{|\Delta|}) \in \hat{D}, \) we must have

\[
(A_{ij}) = (\epsilon_{ij} \cdot I_{|\Delta|})(A_{ij})(\epsilon_{ij} \cdot I_{|\Delta|}) = (\epsilon_{ii} \epsilon_{jj} A_{ij}),
\]

since \((\epsilon_{ij} \cdot I_{|\Delta|})\) is block diagonal. Since \( \epsilon_{ii} \in \{-1,1\} \) for \( 1 \leq i \leq k \), we must have \( A_{ij} = 0 \) if \( i \neq j \), so \((A_{ij})\) is block diagonal. By considering which block diagonal matrices centralise \((E_{ij} \cdot I_{|\Delta|})\), where \((E_{ij}) \in \GL(k,\mathbb{Z})\) is an elementary matrix, we see that any block diagonal matrix centralising the \( \GL(k,\mathbb{Z}) \) factor of \( Q \) must have the same matrix \( M \in \GL(|\Delta|,\mathbb{Z}) \) in each diagonal block. It is then a standard calculation to verify that any choice of \( M \in \GL(|\Delta|,\mathbb{Z}) \) will centralise the \( \GL(k,\mathbb{Z}) \) factor of \( Q \).

The problem of determining \( C(Q) \) has therefore been reduced to determining the centraliser of \( \Phi_{\Delta} \) in \( \GL(|\Delta|,\mathbb{Z}) \). The total order we specified on the vertices of \( \Delta \) gives a block lower triangular decomposition of \( M \in \Phi_{\Delta} \), which we utilise in the proof of Proposition 3.5. This builds upon a matrix decomposition given by Wade [14].

Observe that \( \Phi_{\Delta} \) contains the diagonal matrices of \( \GL(|\Delta|,\mathbb{Z}) \). As in the above proof, anything centralising \( \Phi_{\Delta} \) must be a diagonal matrix. For a diagonal matrix \( E \in \GL(|\Delta|,\mathbb{Z}) \), we write \( E(p,q,r) \) for the diagonal entry corresponding to the vertex \( T(p,q,r) \) of \( \Delta \).

**Proposition 3.5.** A diagonal matrix \( E \in \GL(|\Delta|,\mathbb{Z}) \) centralises \( \Phi_{\Delta} \) if and only if the following conditions hold:

1. If \( p = p' \), then \( E(p,q,r) = E(p',q',r') \), and,
2. If \( T(p,q,r) \) is dominated by \( T(p',q',r') \), then \( E(p,q,r) = E(p',q',r') \)

**Proof.** We define a block decomposition of the matrices in \( \GL(|\Delta|,\mathbb{Z}) \) using the sizes of the orbits, \( O_{[v]} \ll \ldots \ll O_{[v]} \). Let \( m_i = |O_{[v]}| \). We partition \( M \in \GL(|\Delta|,\mathbb{Z}) \) into \( l \) horizontal blocks and \( l \) vertical blocks, writing \( M = (M_{ij}) \), where \( M_{ij} \) is an \( m_i \times m_j \) matrix. Observe that due to the ordering on the lateral transvections, if \( i < j \), then \( M_{ij} = 0 \).

Let \( E \in \GL(|\Delta|,\mathbb{Z}) \) satisfy the conditions in the statement of the proposition. We may write \( E = \Diag(\epsilon_1 \cdot I_{m_1 \times m_1}, \ldots, \epsilon_l \cdot I_{m_l \times m_l}) \), where each \( \epsilon_i \in \{-1,1\} \) \((1 \leq i \leq l)\). Then \( EM = (\epsilon_i \cdot M_{ij}) \) and \( ME = (\epsilon_j \cdot M_{ij}) \). We see that \( ME \) and \( EM \) agree on the diagonal blocks, and on the blocks where \( M_{ij} = 0 \). If \( i > j \) and \( M_{ij} \neq 0 \), then there must be a vertex \( T(j,q,r) \) being dominated by a vertex \( T(i,q',r') \). By assumption, \( \epsilon_i = \epsilon_j \). Therefore \( EM = ME \) and \( E \in C(Q) \).
Suppose now that $E \in \text{GL}(|\Delta|, \mathbb{Z})$ fails the first condition. Without loss of generality, suppose $E(p, q, 1) \neq E(p, q', 1)$. Since, by definition, $\text{Aut}(\Delta)$ acts transitively on the elements of $O_{[p]}$, there is some $P \in \text{GL}(|\Delta|, \mathbb{Z})$ induced by some $\phi \in \text{Aut}(\Delta)$ which acts by exchanging the $q$th and $q'$th domination classes. A standard calculation shows that $[E, P] \neq 1$.

Finally, suppose $E \in \text{GL}(|\Delta|, \mathbb{Z})$ fails the second condition. Assume that $T(p, q, r)$ is dominated by $T(p', q', r')$, but that $E(p, q, r) \neq E(p', q', r')$. In this case, $E$ fails to centralise the elementary matrix which is the result of transvecting $T(p, q, r)$ by $T(p', q', r')$.

**Extending elements of $C(Q)$ to automorphisms of $\text{Aut}(A_\Gamma)$**. Using the map $R$ from section 3.1, for $A \in C(Q) = C_1$ we obtain $R(A) \in \text{Aut}(\text{Aut}(A_\Gamma))$ which acts as $A$ on $\mathbb{Z}^{k|\Delta|}$ and as the identity on $\text{GL}(k, \mathbb{Z}) \times \text{Aut}(A_\Delta)$. If there are $d$ domination classes in $\Delta$, then $|C_1| \leq 2^d$. We now determine $\hat{R}(C_1)$, the image of $R(C_1)$ in $\text{Out}(\text{Aut}(A_\Gamma))$.

Let $nh \in \mathbb{Z}^{k|\Delta|} \times [\text{GL}(k, \mathbb{Z}) \times \text{Aut}(A_\Delta)]$, with $h \neq 1$. Conjugating $\text{Aut}(A_\Gamma)$ by $nh$ fixes $\text{GL}(k, \mathbb{Z}) \times \text{Aut}(A_\Delta)$ pointwise only if $h$ is central in $\text{GL}(k, \mathbb{Z}) \times \text{Aut}(A_\Delta)$. The only such non-trivial central element is $\iota$, the automorphism inverting each generator of $\mathbb{Z}^k$. Given that $\alpha_i(n) = -n$ for each $n \in \mathbb{Z}^{k|\Delta|}$, we see that for any $m \in \mathbb{Z}^{k|\Delta|}$, we have $(m, 1)^{(n, \iota)} = (-m, 1)$.

So, regardless of which $n$ we choose, conjugation by $nu$ is equal to $R(-I_{k|\Delta|})$. In other words, when we conjugate by $nu$, we map each lateral transvection to its inverse. Thus, for $A, B \in C_1$, $R(AB^{-1})$ is inner if and only if $A(p, q, r) = -B(p, q, r)$ for every $p, q$, and $r$. This means $|\hat{R}(C_1)| = 2|\hat{R}(C_1)|$.

**First proof of Theorem A**. We are now able to prove Theorem A for right-angled Artin groups with non-trivial centre.

**Proof (1) of Theorem A.** By Proposition 3.2, we have a split decomposition of $\text{Aut}(A_\Gamma)$, whose kernel is $\mathbb{Z}^{k|\Delta|}$. The structure of $C_1 = C(Q)$ is given by Proposition 3.5. We have fewest constraints on $C_1$ if $\Delta$ is such that domination occurs only between vertices in the same domination class, and when each domination class lies in an $\text{Aut}(\Delta)$-orbit by itself. This is achieved, for example, if $\Delta = X$, a disjoint union of pairwise non-isomorphic complete graphs, each of rank at least two. Suppose $X$ has $d$ connected components. For $A \in C(Q)$, Proposition 3.5 implies $A$ is entirely determined by the entries $A(p, 1, 1)$ $(1 \leq p \leq d)$. This gives $|C(Q)| = 2^d$, and so the image of $C(Q)$ in $\text{Out}(\text{Aut}(A_\Gamma))$ has order $2^{d-1}$. As we may choose $d$ to be as large as we like, the result follows.

**4 Proof of Theorem A: centreless right-angled Artin groups**

In this section, we demonstrate that Theorem A also holds for classes of centreless right-angled Artin groups. From now on, we assume that the graph $\Gamma$ has no social vertices, so that $A_\Gamma$, has trivial centre. A simplicial graph $\Gamma = (V, E)$ is said to have no separating intersection of links (‘no SILs’) if for all $v, w \in V$ with $v$ not adjacent to $w$, each connected component of $\Gamma \setminus (\text{lk}(v) \cap \text{lk}(w))$ contains either $v$ or $w$. We have the following theorem.

**Theorem 4.1** (Charney-Ruane-Stambaugh-Vijayan [3]). Let $\Gamma$ be a finite simplicial graph with no SILs. Then $\text{PC}(A_\Gamma)$, the subgroup of $\text{Aut}(A_\Gamma)$ generated by partial conjugations,
is a right-angled Artin group, whose defining graph has vertices in bijection with the partial conjugations of $A_\Gamma$.

We restrict ourselves to looking at certain no SILs graphs, to obtain a nice decomposition of $\text{Aut}(A_\Gamma)$. We say a graph $\Gamma$ is \textit{austere with star cuts} if it has trivial symmetry group and no dominated vertices. Note that this is a loosening of the definition of an austere graph: removing a vertex star need no longer leave the graph connected.

**Lemma 4.2.** Let $\Gamma = (V, E)$ be austere with star cuts and have no SILs. For $c \in V$, let $K_c = |\pi_0(\Gamma \setminus \text{st}(c))|$. Then
\[|\text{Out}(\text{Aut}(A_\Gamma))| \geq 2^{K_c - 1}.\]

**Proof.** Since $\Gamma$ is austere with star cuts, the only LS generators which are defined are the inversions and the partial conjugations. Letting $I_\Gamma$ denote the finite subgroup generated by the inversions $\iota_v$ ($v \in V$), we obtain the decomposition
\[\text{Aut}(A_\Gamma) \cong \text{PC}(A_\Gamma) \rtimes I_\Gamma,\]
where the inversions act by inverting partial conjugations in the obvious way. Since $\Gamma$ has no SILs, it follows from Theorem 4.1 that $\text{PC}(A_\Gamma) \cong A_\Delta$ for some simplicial graph $\Delta$ whose vertices are in bijection with the partial conjugations of $A_\Gamma$.

Fix $c \in V$ and let $\{\gamma_{c,D_i} \mid 1 \leq i \leq K_c\}$ be the set of partial conjugations by $c$. Let $\eta_{c,j}$ be the LS generator of $\text{Aut}(A_\Delta)$ which inverts $\gamma_{c,D_j}$, but fixes the other vertex-generators of $A_\Delta$. This extends to an automorphism of $\text{Aut}(A_\Gamma)$, by specifying that $I_\Gamma$ is fixed pointwise: all that needs to be checked is that the action of $I_\Gamma$ on $\text{PC}(A_\Gamma)$ is preserved, which is a straightforward calculation. We abuse notation, and write $\eta_{c,j} \in \text{Aut}(\text{Aut}(A_\Gamma))$.

If $K_c > 1$, we see $\eta_{c,j}$ is not inner. Assume $\eta_{c,j}$ is equal to conjugation by $p\kappa \in \text{PC}(A_\Gamma) \rtimes I_\Gamma$. For $\gamma \in \text{PC}(A_\Gamma)$, we have $(\gamma, 1)^{p\kappa} = (p\gamma\kappa p^{-1}, 1)$. Since $\eta_{c,j}(\gamma_{c,D_j}) = \gamma_{c,D_j}^{-1}$, an exponent sum argument tells us that $\kappa$ must act by inverting $\gamma_{c,D_j}$, and so $\kappa$ must invert $c$ in $A_\Gamma$. However, $\eta_{c,j}$ fixes $\gamma_{c,D_i}$ for all $i \neq j$, by definition, and a similar exponent sum argument implies that $\kappa$ \textit{cannot} invert $c$ in $A_\Gamma$. Thus, by contradiction, $\eta_{c,j}$ cannot be inner. This establishes that $|\text{Out}(\text{Aut}(A_\Gamma))| \geq K_c$.

As above, we may choose a subset of $\{\gamma_{c,D_i}\}$ to invert, and extend this to an automorphism of $\text{Aut}(A_\Gamma)$. Take two distinct such automorphisms, $\eta_1$ and $\eta_2$. Their difference $\eta_1\eta_2^{-1}$ is inner if and only if it inverts every element of $\{\gamma_{c,D_i}\}$. Otherwise, we will get the same contradiction as before. A counting argument gives the desired lower bound of $2^{K_c - 1}$. \hfill \Box

Observe that if $\Gamma$ is austere, we cannot find a vertex $c$ with $K_c > 1$. This is the reason we loosen the definition and consider austere with star cuts graphs.

**Second proof of Theorem A.** By exhibiting an infinite family of graphs over which the size of $|\{\gamma_{c,D_i}\}|$ is unbounded, applying Lemma 4.2 will give a second proof of Theorem A.

**Proof (2) of Theorem A.** Fix $t \in \mathbb{Z}$ with $t \geq 3$. Define $e_0 = 0$ and choose $\{e_1 < \ldots < e_t\} \subset \mathbb{Z}^+$ subject to the conditions:
(1) For each $0 < i \leq t$, we have $e_i - e_{i-1} > 2$, and  
(2) If $i \neq j$, then $e_i - e_{i-1} \neq e_j - e_{j-1}$.

We use the set $E := \{e_i\}$ to construct a graph. Begin with a cycle on $e_t$ vertices, labelled $0, 1, \ldots, e_t - 1$ in the natural way. Join one extra vertex, labelled $c$, to those labelled $e_i$, for $0 \leq i < t$. We denote the resulting graph by $\Gamma_E$. Figure 3 shows an example of such a $\Gamma_E$.

![Figure 3: The graph $\Gamma_E$, for $E = \{3, 7, 12\}$.](image)

For $E \subset \mathbb{Z}^+$ satisfying the above conditions, we see that $\Gamma_E$ is austere with star cuts and has no SILs. Condition (1) ensures that no vertex is dominated by another. Observe that $c$ is fixed by any $\phi \in \text{Aut}(\Gamma_E)$. Since each connected component of $\Gamma \setminus \text{st}(c)$ has $e_i - e_{i-1} - 1$ elements (for some $1 \leq i \leq t$), condition (2) implies that $\text{Aut}(\Gamma_E) = 1$. To see that $\Gamma_E$ has no SILs, observe that the intersection of the links of any two vertices has order at most 1. When a single vertex is removed, $\Gamma_E$ remains connected, and so it has no SILs.

Lemma 4.2 applied to the family of graphs $\{\Gamma_E\}$ proves the theorem.

5 Extremal behaviour and generalisations

In Sections 3 and 4, we gave examples of $A_{\Gamma}$ for which $\text{Out}(\text{Aut}(A_{\Gamma}))$ was non-trivial, but not necessarily infinite. Currently, there are very few known $A_{\Gamma}$ for which $\text{Out}(\text{Aut}(A_{\Gamma}))$ exhibits ‘extremal behaviour’, that is, $A_{\Gamma}$ for which $\text{Out}(\text{Aut}(A_{\Gamma}))$ is trivial or infinite. In this final section, we discuss the possibility of such behaviour, and generalisations of the current work to automorphism towers.

Complete automorphisms groups. Recall that a group $G$ is said to be complete if it has trivial centre and every automorphism of $G$ is inner. Our proofs of Theorems A and B relied upon us being able to exhibit large families of right-angled Artin groups whose automorphisms groups are not complete. It is worth noting that if $A_{\Gamma}$ is not free abelian, then $\text{Aut}(A_{\Gamma})$ has trivial centre, and so a priori, $\text{Aut}(A_{\Gamma})$ could be complete.

**Proposition 5.1.** Let $A_{\Gamma}$ be a right-angled Artin group. Then $Z(\text{Aut}(A_{\Gamma}))$ has order at most two. In particular, if $A_{\Gamma}$ is not free abelian, then $\text{Aut}(A_{\Gamma})$ is centreless.
Proof. For brevity of proof, we assume that \( A_{\Gamma} \cong \mathbb{Z}^k \times A_{\Delta} \), taking \( k = 0 \), and \( \mathbb{Z}^k = 1 \) if \( A_{\Gamma} \) is centreless. If \( A_{\Gamma} \) is free abelian of rank \( k \), then \( Z(\text{Aut}(A_{\Gamma})) \cong Z(\text{GL}(k, \mathbb{Z})) \cong \mathbb{Z}_2 \). From now on, we assume the centre of \( A_{\Gamma} \) is proper.

We now adapt the standard proof that a centreless group has centreless automorphism group. Suppose that \( \phi \in \text{Aut}(A_{\Gamma}) \) is central. We know that \( \text{Inn}(A_{\Gamma}) \cong A_{\Gamma}/Z^k \cong A_{\Delta} \). For any \( \gamma_w \in \text{Inn}(A_{\Gamma}) \), we must have \( \gamma_w = \phi \gamma_w \phi^{-1} = \gamma_{\phi(w)} \). So, for \( \phi \) to be central, it must fix every element of \( A_{\Delta} \).

Observe that if \( k = 0 \), then \( \phi \) must be trivial, and we are done.

Assume now that \( k \geq 1 \). For any \( \phi \in \text{Aut}(A_{\Gamma}) \), we also have \( \phi(u) \in Z^k \), for all \( u \in Z^k \). So, a central \( \phi \) must simply be an element of \( \text{GL}(k, \mathbb{Z}) \), since it must be the identity on \( A_{\Delta} \), and take \( Z^k \) into itself.

In particular, we have that \( Z(\text{Aut}(A_{\Gamma})) \leq Z(\text{GL}(k, \mathbb{Z})) = \{1, \iota\} \), where \( \iota \) is the automorphism inverting each generator of \( Z^k \). However, lateral transvections are not centralised by \( \iota \), and so the centre of \( \text{Aut}(A_{\Gamma}) \) is trivial.

In this paper, we have focused on finding right-angled Artin groups whose automorphism groups are not complete: an equally interesting question is which right-angled Artin groups do have complete automorphism groups, beyond the obvious examples of ones built out of direct products of free groups. We conjecture the following.

**Conjecture 5.2.** When \( \Gamma \) is austere, \( \text{Aut}(A_{\Gamma}) \) is complete.

It might also be possible to adapt Bridson-Vogtmann’s geometric proof \cite{bridson_vogtmann} of the completeness of \( \text{Out}(F_n) \) to find examples of \( A_{\Gamma} \) for which \( \text{Out}(A_{\Gamma}) \) is complete, using Charney-Stambaugh-Vogtmann’s newly developed outer space for right-angled Artin groups \cite{charney_stambaugh_vogtmann}.

**Infinite order automorphisms.** At the other extreme, we might wonder which \( A_{\Gamma} \), if any, have \( \text{Out}(\text{Aut}(A_{\Gamma})) \) of infinite order. An obvious approach to this problem is to exhibit an element \( \alpha \in \text{Out}(\text{Aut}(A_{\Gamma})) \) of infinite order. The approach taken in Section 4, involving graphs \( \Gamma \) with no SILs, might seem hopeful, as we certainly know of infinite order non-inner elements of \( \text{Aut}(\text{PC}(A_{\Gamma})) \): in particular, dominated transvections and partial conjugations. A key property that allowed us to extend \( \eta_{c,j} \in \text{Aut}(\text{PC}(A_{\Gamma})) \) to an element of \( \text{Aut}(\text{Aut}(A_{\Gamma})) \) was that it respected the natural partition of the partial conjugations by their conjugating vertex. More precisely, \( \eta_{c,j} \) sent a partial conjugation by \( v \in V \) to a string of partial conjugations, also by \( v \). This ensured that the action of \( I_{\Gamma} \) on \( \text{PC}(A_{\Gamma}) \) was preserved when we extended \( \eta_{c,j} \) to be the identity on \( I_{\Gamma} \).

It might be hoped that we could find a transvection \( \tau \in \text{Aut}(\text{PC}(A_{\Gamma})) \) which also respected this partition, as \( \tau \) could then easily be extended to an infinite order element of \( \text{Aut}(\text{Aut}(A_{\Gamma})) \). However, it is not difficult to verify that whenever \( \Gamma \) has no dominated vertices, as in Section 4, no such \( \tau \) will be well-defined. Similarly, the only obvious way to extend a partial conjugation \( \gamma \in \text{PC}(\text{PC}(A_{\Gamma})) \) is to an element of \( \text{Inn}(\text{Aut}(A_{\Gamma})) \). This leads us to formulate the following open question.

**Question:** Does there exist a simplicial graph \( \Gamma \) such that \( \text{Out}(\text{Aut}(A_{\Gamma})) \) is infinite?

It seems possible that such a \( \Gamma \) could exist, however the methods used in this paper do not find one. Our main approach was to find elements of \( \text{Aut}(\text{Aut}(A_{\Gamma})) \) which preserve some
nice decomposition of $\text{Aut}(A_\Gamma)$. To find infinite order elements of $\text{Aut}(\text{Aut}(A_\Gamma))$, it may be necessary to loosen this constraint. This would be analogous to the situation where we find only two field automorphisms of $\mathbb{C}$ which preserve $\mathbb{R}$, but uncountably many which do not.

**Automorphism towers.** Let $G$ be a centreless group. Then $G$ embeds into its automorphism group, $\text{Aut}(G)$, as the subgroup of inner automorphisms, $\text{Inn}(G)$, and $\text{Aut}(G)$ is also centreless. We inductively define

$$\text{Aut}^i(G) = \text{Aut}(\text{Aut}^{i-1}(G))$$

for $i \geq 0$, with $\text{Aut}^0(G) = G$. This yields the following chain of normal subgroups:

$$G \triangleleft \text{Aut}(G) \triangleleft \text{Aut}(\text{Aut}(G)) \triangleleft \ldots \triangleleft \text{Aut}^i(G) \triangleleft \ldots,$$

which we refer to as the *automorphism tower of* $G$. An automorphism tower is said to *terminate* if there exists an $i \in \mathbb{N}$ such that the embedding $\text{Aut}^i(G) \hookrightarrow \text{Aut}^{i+1}(G)$ is an isomorphism. Observe that a complete group’s automorphism tower terminates at the first step. Thomas [13] showed that any centreless group has a terminating automorphism tower, although it may not terminate after a finite number of steps (direct limits are needed). Hamkins [8] showed that the automorphism tower of *any* group terminates, although in the above definition, we have only considered automorphism towers of centreless groups.

**Problem:** Determine the automorphism tower of $A_\Gamma$ for an arbitrary $\Gamma$.

This seems a difficult problem in general. A first approach might be to find $A_\Gamma$ for which $\text{Out}(\text{Aut}(A_\Gamma))$ is finite. It would then perhaps be easier to study the structure of $\text{Aut}^2(A_\Gamma)$.

### A Appendix: Conjugating the lateral transvections

In this appendix, we demonstrate the effect of conjugating the lateral transvections $\tau_{sa}$ by the LS generators of $\text{Aut}(A_\Gamma)$, to show that $\mathcal{L} = \langle \tau_{sa} \mid a \in \Delta, s \in S \rangle$ is normal in $\text{Aut}(A_\Gamma)$. Let $S$ be the set of LS generators of $\text{Aut}(A_\Gamma)$. Table 1 displays the conjugates necessary for us to draw this conclusion. Note that we have used a classically observed generating set of $\text{GL}(k, \mathbb{Z})$ consisting of inversions and transvections, rather than the full list of LS generators. Also, we decompose any $\phi \in \text{Aut}(\Gamma)$ into its actions on $S$ and $\Delta$.

| $\lambda \in S \cup S^{-1}$ | $\lambda \cdot \tau_{sa} \cdot \lambda^{-1}$ | $\lambda \in S \cup S^{-1}$ | $\lambda \cdot \tau_{sa} \cdot \lambda^{-1}$ |
|-----------------------------|------------------------------------------|-----------------------------|------------------------------------------|
| $t_t$                        | $\tau_{sa}$                              | $i_b$                       | $\tau_{sa}$                              |
| $l_s$                        | $-\tau_{sa}$                             | $l_a$                       | $-\tau_{sa}$                             |
| $\tau_{st}$                  | $\tau_{sa}$                              | $\tau_{bd}$                 | $\tau_{sa}$                              |
| $\tau_{rt}$                  | $\tau_{sa}$                              | $\tau_{ab}$                 | $\tau_{sa}$                              |
| $\tau_{ts}$                  | $\tau_{sa} + \tau_{a}$                   | $\tau_{ab}^{-1}$            | $\tau_{sa} - \tau_{sb}$                  |
| $\tau_{ts}^{-1}$             | $\tau_{sa} - \tau_{a}$                   | $\phi \in \text{Aut}(\Delta)$ | $\tau_{sa}$                              |
| $\gamma_{c,D}$               |                                         |                             |                                          |

Table 1: The effect of conjugating a lateral transvection $\tau_{sa}$ by elements of the set $S \cup S^{-1}$. The vertices $a, b, d, r, s$ and $t$ are taken to be distinct, with $c$ being any vertex in $\Delta$ and $D$ being any connected component of $\Gamma \setminus \text{st}(c)$. 

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