SOME NEW STRICHARTZ ESTIMATES FOR THE SCHRÖDINGER EQUATION

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Abstract. We deal with fixed-time and Strichartz estimates for the Schrödinger propagator as an operator on Wiener amalgam spaces. We discuss the sharpness of the known estimates and we provide some new estimates which generalize the classical ones. As an application, we present a result on the wellposedness of the linear Schrödinger equation with a rough time dependent potential.

1. Introduction

Consider the Cauchy problem for the Schrödinger equation

\[
\begin{cases}
  i\partial_t u + \Delta u = 0 \\
  u(0, x) = u_0(x),
\end{cases}
\]

with \( x \in \mathbb{R}^d, d \geq 1 \). Several estimates have been obtained for the solution \( u(t, x) = (e^{it\Delta}u_0)(x) \) of (1) in terms of Lebesgue spaces, with important applications to wellposedness and scattering theory for nonlinear Schrödinger equations, possibly with potentials \([7, 14, 18, 19, 24, 27, 28, 34]\). Among them, we recall the important fixed-time estimates

\[
\|e^{it\Delta}u_0\|_{L^r(\mathbb{R}^d)} \lesssim |t|^{-d\left(\frac{1}{2} - \frac{1}{r}\right)}\|u_0\|_{L^{r'}(\mathbb{R}^d)}, \quad 2 \leq r \leq \infty,
\]

as well as the homogeneous Strichartz estimates: for \( q \geq 2, r \geq 2 \), with \( 2/q + d/r = d/2 \), \((q, r, d) \neq (2, \infty, 2)\), i.e., for \((q, r)\) Schrödinger admissible,

\[
\|e^{it\Delta}u_0\|_{L^q_tL^r_x} \lesssim \|u_0\|_{L^2_x},
\]

where, as usual, \( \|F\|_{L^q_tL^r_x} = \left( \int \|F(t, \cdot)\|_{L^r_x}^q \, dt \right)^{1/q} \). As a matter of fact, these estimates express a gain of local \( x \)-regularity of the solution \( u(t, \cdot) \), and a decay of its \( L^r_x \) norm, both in some \( L^q_t \)-averaged sense.

Recently several authors \([11, 12, 51, 61, 31, 32, 33]\) have turned their attention to fixed-time and space-time estimates of the Schrödinger propagator between spaces...
widely used in Time-Frequency Analysis [15], known as modulation spaces and Wiener amalgam spaces. These spaces were first introduced by H. Feichtinger in [8] (now available in textbooks [15]) and have recently become very popular in the framework of signal analysis. Loosely speaking, given a Banach space $B$, like $B = L^p$ or $B = \mathcal{F}L^p$, the Wiener amalgam space $W(B, L^q)$ consists of functions which are locally in $B$ but display an $L^q$-decay at infinity. In particular, $W(L^p, L^q) = L^p$, $W(L^p, L^q) \hookrightarrow W(L^{p_2}, L^{q_2})$ if $p_1 \geq p_2$, $q_1 \leq q_2$, and $W(\mathcal{F}L^{p_1}, L^{q_1}) \hookrightarrow W(\mathcal{F}L^{p_2}, L^{q_2})$ if $p_1 \leq p_2$, $q_1 \leq q_2$.

As explained in [5, 6], one of the main motivations for considering estimates in these spaces is that they control the local regularity of a function and its decay at infinity separately. Hence, they highlight and distinguish between the local properties and the global behaviour of the solution $u(t, x) = e^{it\Delta}u_0$ and therefore, as far as Strichartz estimates concern, they are natural candidates to perform an analysis of the solution which is finer than the classical one. Actually, Wiener amalgam spaces have already appeared as technical tool in PDE’s. In particular, the space $W(L^p, L^q)$ coincides with the space $X_{q,p}^0$ introduced in [26].

Some Strichartz estimates in this environment are in [5, 6], where the following fixed-time estimates are presented: for $2 \leq r \leq \infty$,

(4) $\|e^{it\Delta}u_0\|_{W(\mathcal{F}L^{r'}, L^r)} \lesssim (|t|^{-2} + |t|^{-1})^{d(1/2 - 1/r)} \|u_0\|_{W(\mathcal{F}L^{r'}, L^r)}$.

The related homogeneous Strichartz estimates, obtained by combining (4) with orthogonality arguments, read

(5) $\|e^{it\Delta}u_0\|_{W(L^{q/2}, L^q)_{t}} \|W(\mathcal{F}L^{r'}, L^r)_{x} \lesssim \|u_0\|_{L^2}$,

for $4 < q \leq \infty$, $2 \leq r \leq \infty$, with $2/q + d/r = d/2$. When $q = 4$ the same estimate holds with the Lorentz space $L^{r';2}$ in place of $L^r$. Dual homogeneous and retarded estimates hold as well.

For comparison, the classical estimates (3) can be rephrased in terms of Wiener amalgam spaces as follows:

(6) $\|e^{it\Delta}u_0\|_{W(L^q, L^q)_{t}} \|W(\mathcal{F}L^{r'}, L^r)_{x} \lesssim \|u_0\|_{L^2}$.

Thereby, the new estimates (5) contain the following insight: the classical estimates (3) can be modified by (conveniently) moving local regularity from the time variable to the space variable. Indeed, $\mathcal{F}L^{r'} \subset L^r$ if $r \geq 2$, but the bound in (4) is worse than the one in (2), as $t \to 0$; consequently, in the estimate (5) we average locally in time by the $L^{q/2}$ norm, which is rougher than the $L^q$ norm in (3) or, equivalently, in (6).

In the present paper we perform the converse approach, by showing that it is possible to move local regularity in (3) from the space variable to the time variable. As a result, we obtain new estimates involving the Wiener amalgam spaces $W(L^p, L^q)$ that generalize (3). This requires some preliminary steps.
First, we show that (4) can be enlarged to more general pairs \((s', r)\), rather than only conjugate-exponent pairs \((r', r)\), (see Theorem (3.3)). In particular, if we choose \(s' = 2\), the related fixed-time estimates read
\[
\|e^{it\Delta} u_0\|_{W(L^2, L^r)} \lesssim (1 + t^2)^{-\frac{d}{2}} (\frac{1}{2} - \frac{1}{r}) \|u_0\|_{W(L^2, L^{r'})}, \quad 2 \leq r \leq \infty.
\]
Using techniques similar to \([5, 19]\), the related Strichartz estimates are achieved in Theorem 4.1. Finally, the complex interpolation with the classical estimates (3) yields our main result:

**Theorem 1.1.** Let \(1 \leq q_1, r_1 \leq \infty, 2 \leq q_2, r_2 \leq \infty\) such that \(r_1 \leq r_2\),

\[
\frac{2}{q_1} + \frac{d}{r_1} \geq \frac{d}{2},
\]

\(\text{(7)}\)

\[
\frac{2}{q_2} + \frac{d}{r_2} \leq \frac{d}{2},
\]

\(\text{(8)}\)

\(r_1, d \neq (\infty, 2), (r_2, d) \neq (\infty, 2)\) and, if \(d \geq 3, r_1 \leq 2d/(d - 2)\). The same for \(\tilde{q}_1, \tilde{q}_2, \tilde{r}_1, \tilde{r}_2\). Then, we have the homogeneous Strichartz estimates

\[
\|e^{it\Delta} u_0\|_{W(L^{q_1}, L^{q_2}), W(L^{r_1}, L^{r_2})_x} \lesssim \|u_0\|_{L^2},
\]

\(\text{(9)}\)

the dual homogeneous Strichartz estimates

\[
\| \int e^{-is\Delta} F(s) \, ds \|_{L^2} \lesssim \|F\|_{W(L^q, L^r)_x, W(L^\tilde{q}, L^\tilde{r})_x},
\]

\(\text{(10)}\)

and the retarded Strichartz estimates

\[
\| \int_{s< t} e^{i(t-s)\Delta} F(s) \, ds \|_{W(L^{q_1}, L^{q_2}), W(L^{r_1}, L^{r_2})_x} \lesssim \|F\|_{W(L^{q_1}, L^{q_2}), W(L^{r_1}, L^{r_2})_x},
\]

\(\text{(11)}\)

Figure 1 illustrates the range of exponents for the homogeneous estimates when \(d \geq 3\). Notice that, if \(q_1 \leq q_2\), these estimates follow immediately from (6) and the inclusion relations of Wiener amalgam spaces recalled above. So, the issue consists in the cases \(q_1 > q_2\).
Figure 1: When $d \geq 3$, (9) holds for all pairs $(1/q_1, 1/r_1) \in I_1$, $(1/q_2, 1/r_2) \in I_2$, with $1/r_2 \leq 1/r_1$.

Since there are no relations between the pairs $(q_1, r_1)$ and $(q_2, r_2)$ other than $r_1 \leq r_2$, these estimates tell us, in a sense, that the analysis of the local regularity of the Schrödinger propagator is quite independent of its decay at infinity.

We then discuss the sharpness of the results above, as well as those in [5]. Indeed, in Section 5 we first focus on the fixed-time estimates, proving that the range $r \geq 2$ in (4) is sharp, and the same for the decay $t^{-d(\frac{d}{2} - \frac{1}{4})}$ at infinity, and the bound $t^{-2d(\frac{d}{2} - \frac{1}{2})}$, when $t \to 0$. Then, we present the sharpness of the Strichartz estimates (5), except for the threshold $q \geq 4$, which seems quite hard to obtain. Next, we turn our attention to the new estimates in Theorem 1.1 and show that, for $d \geq 3$, all the constraints on the range of exponents in Theorem 1.1 are necessary, except for $r_1 \leq r_2$, $r_1 \leq 2d/(d-2)$, which remain an open problem. However, we prove the weaker result (Proposition 5.3):

**Assume** $r_1 > r_2$ and $t \neq 0$. **Then the propagator** $e^{it\Delta}$ **does not map** $W(L^{r_1'}, L^{r_2'})$ **continuously into** $W(L^{r_1}, L^{r_2})$.

This shows that estimates (9) for exponents $r_1 > r_2$, if true, cannot be obtained from fixed-time estimates and orthogonality arguments.
The arguments here employed for the necessary conditions differ from the classical setting of Lebesgue spaces, where necessary conditions are usually obtained by general scaling considerations (see, for example, [28, Exercises 2.35, 3.42], and [20] for the interpretation in terms of Gaussian curvature of the characteristic manifold). In our framework this method does not work directly. Indeed, the known bounds for the norm of the dilation operator \( f(x) \mapsto f(\lambda x) \) between Wiener amalgam spaces ([25, 29]), yield constraints which are weaker than the desired ones.

Our idea is to consider families of rescaled Gaussians as initial data, for which the action of the operator \( e^{it\Delta} \) and the involved norms can be computed explicitly.

In the last section, we present an application to the linear Schrödinger equation with time-dependent potential (see [7] and the references therein for the existent literature). Our result extends that in [5] to the dimension \( d \geq 1 \) (instead of \( d > 1 \)) and to potentials \( V(t, x) \) in Wiener amalgam spaces rather than the classical \( L^p \) spaces. Precisely, we prove the wellposedness in \( L^2 \) of the Cauchy problem

\[
\begin{aligned}
\frac{i}{t}u + \Delta u &= V(t, x)u, \quad t \in [0, T] = I_T, \quad x \in \mathbb{R}^d, \\
u(0, x) &= u_0(x),
\end{aligned}
\]

for the class of potentials

\[
V \in L^\alpha(I_T; W(\mathcal{F}L^p', L^p)_x), \quad \frac{1}{\alpha} + \frac{d}{p} \leq 1, \quad 1 \leq \alpha < \infty, \quad d < p \leq \infty.
\]

This result seems of interest especially in dimension \( d = 1 \), where, if we choose \( 1 < p < 2 \), \( V(t, x) \) is allowed to be locally in rough spaces of temperate distributions \( \mathcal{F}L^p' \), with respect to the space variable \( x \) (see Remark [5,2]).

Estimates similar to those proved here should hold for other dispersive equations, like the wave equation, too. Our plan is to investigate these issues in a subsequent paper.

**Notation.** We define \( |x|^2 = x \cdot x \), for \( x \in \mathbb{R}^d \), where \( x \cdot y = xy \) is the scalar product on \( \mathbb{R}^d \). The space of smooth functions with compact support is denoted by \( C_0^\infty(\mathbb{R}^d) \), the Schwartz class is \( \mathcal{S}(\mathbb{R}^d) \), the space of tempered distributions \( \mathcal{S}'(\mathbb{R}^d) \). The Fourier transform is normalized to be \( \hat{f}(\xi) = \mathcal{F}f(\xi) = \int f(t)e^{-2\pi i \xi t} dt \). Translation and modulation operators (time and frequency shifts) are defined, respectively, by

\[
T_x f(t) = f(t - x) \quad \text{and} \quad M_\xi f(t) = e^{2\pi i \xi t} f(t).
\]

We have the formulas \((T_x f)' = M_{-x} \hat{f}, \quad (M_\xi f)' = T_\xi \hat{f}, \) and \( M_\xi T_x = e^{2\pi ix\xi} T_x M_\xi \). The notation \( A \lesssim B \) means \( A \leq cB \) for a suitable constant \( c > 0 \), whereas \( A \asymp B \) means \( c^{-1} A \leq B \leq cA \), for some \( c \geq 1 \). The symbol \( B_1 \hookrightarrow B_2 \) denotes the continuous embedding of the linear space \( B_1 \) into \( B_2 \).
2. WIENER AMALGAM SPACES

We briefly recall the definition and the main properties of Wiener amalgam spaces. We refer to [8, 9, 10, 11, 12] for details.

Let \( g \in C_0^\infty \) be a test function that satisfies \( \| g \|_{L^2} = 1 \). We will refer to \( g \) as a window function. Let \( B \) one of the following Banach spaces: \( L^p, \mathcal{F}L^p, 1 \leq p \leq \infty, L^{p,q}, 1 < p < \infty, 1 \leq q \leq \infty \), possibly valued in a Banach space, or also spaces obtained from these by real or complex interpolation. Let \( C \) be one of the following Banach spaces: \( L^p, 1 \leq p \leq \infty \), or \( L^{p,q}, 1 < p < \infty, 1 \leq q \leq \infty \), scalar-valued. For any given function \( f \) which is locally in \( B \) (i.e. \( gf \in B, \forall g \in C_0^\infty \)), we set \( f_B(x) = \| fT_x g \|_B \).

The Wiener amalgam space \( W(B, C) \) with local component \( B \) and global component \( C \) is defined as the space of all functions \( f \) locally in \( B \) such that \( f_B \in C \). Endowed with the norm \( \| f \|_{W(B, C)} = \| f_B \|_C \), \( W(B, C) \) is a Banach space. Moreover, different choices of \( g \in C_0^\infty \) generate the same space and yield equivalent norms.

If \( B = \mathcal{F}L^1 \) (the Fourier algebra), the space of admissible windows for the Wiener amalgam spaces \( W(\mathcal{F}L^1, C) \) can be enlarged to the so-called Feichtinger algebra \( W(\mathcal{F}L^1, L^1) \). Recall that the Schwartz class \( \mathcal{S} \) is dense in \( W(\mathcal{F}L^1, L^1) \).

We use the following definition of mixed Wiener amalgam norms. Given a measurable function \( F \) of the two variables \((t, x)\) we set
\[
\| F \|_{W(L^{q_1}, L^{q_2}); W(\mathcal{F}L^{r_1}, L^{r_2})_x} = \| \| F(t, \cdot) \|_{W(\mathcal{F}L^{r_1}, L^{r_2})_x} \|_{W(L^{q_1}, L^{q_2})_t}.
\]

Observe that (5)
\[
\| F \|_{W(L^{q_1}, L^{q_2}); W(\mathcal{F}L^{r_1}, L^{r_2})_x} = \| F \|_{W(L^{q_1}(\mathcal{F}L^{r_1}), L^{q_2}(\mathcal{F}L^{r_2}))}.
\]

The following properties of Wiener amalgam spaces will be frequently used in the sequel.

**Lemma 2.1.** Let \( B_i, C_i, i = 1, 2, 3 \), be Banach spaces such that \( W(B_i, C_i) \) are well defined. Then,

(i) Convolution. If \( B_1 \hookrightarrow B_2 \hookrightarrow B_3 \) and \( C_1 \ast C_2 \hookrightarrow C_3 \), we have
\[
W(B_1, C_1) \ast W(B_2, C_2) \hookrightarrow W(B_3, C_3).
\]
In particular, for every \( 1 \leq p, q \leq \infty \), we have
\[
\| f \ast u \|_{W(\mathcal{F}L^p, L^q)} \leq \| f \|_{W(\mathcal{F}L^\infty, L^1)} \| u \|_{W(\mathcal{F}L^p, L^q)}.
\]

(ii) Inclusions. If \( B_1 \hookrightarrow B_2 \) and \( C_1 \hookrightarrow C_2 \),
\[
W(B_1, C_1) \hookrightarrow W(B_2, C_2).
\]
Moreover, the inclusion of \( B_1 \) into \( B_2 \) need only hold “locally” and the inclusion of \( C_1 \) into \( C_2 \) “globally”. In particular, for \( 1 \leq p_i, q_i \leq \infty, i =
we have

\[ p_1 \geq p_2 \text{ and } q_1 \leq q_2 \implies W(L^{p_1}, L^{q_1}) \hookrightarrow W(L^{p_2}, L^{q_2}). \]

(iii) Complex interpolation. For \( 0 < \theta < 1 \), we have

\[ [W(B_1, C_1), W(B_2, C_2)]_{\theta} = W \left( [B_1, B_2]_{\theta}, [C_1, C_2]_{\theta} \right), \]

if \( C_1 \) or \( C_2 \) has absolutely continuous norm.

(iv) Duality. If \( B', C' \) are the topological dual spaces of the Banach spaces \( B, C \) respectively, and the space of test functions \( C_0^\infty \) is dense in both \( B \) and \( C \), then

\[ W(B, C)' = W(B', C'). \]

The proof of all these results can be found in (8, 9, 10, 16).

3. Fixed-time estimates

In this section we study estimates for the solution \( u(t, x) \) of the Cauchy problem (1), for fixed \( t \). We take advantage of the explicit formula for the solution

\[ u(t, x) = (K_t \ast u_0)(x), \]

where

\[ K_t(x) = \frac{1}{(4\pi it)^{d/2}} e^{-i|x|^2/(4t)}. \]

We already know that (19) is in the Wiener amalgam space \( W(FL^1, L^\infty) \) see [1, 5, 33]. This is the finest Wiener amalgam space-norm for (19) which, consequently, gives the worst behaviour in the time variable. We aim at improving the latter, at the expenses of a rougher \( x \)-norm. This is achieved in the following result. Indeed, \( W(FL^1, L^\infty) \subset W(FL^p, L^\infty) \) for \( 1 \leq p \leq \infty \) and the case \( p = 1 \) recaptures [5, Proposition 3.2].

**Proposition 3.1.** For \( a \in \mathbb{R}, a \neq 0 \), let \( f_{ai}(x) = (ai)^{-d/2} e^{-|x|^2/(a^2)} \). Then, for \( 1 \leq p \leq \infty \), \( f_{ai} \in W(FL^p, L^\infty) \) and

\[ \|f_{ai}\|_{W(FL^p, L^\infty)} \asymp |a|^{-d/p} (1 + a^2)^{(d/2)(1/p - 1/2)}. \]

**Proof.** It follows the footsteps of [5, Proposition 3.2]. We consider the Gaussian \( g(t) = e^{-\pi t^2} \) as window function to compute

\[ \|f_{ai}\|_{W(FL^p, L^\infty)} \asymp \sup_{x \in \mathbb{R}^d} \| \hat{f}_{ai} * M_{-x} g \|_{L^p}. \]
Using $\hat{f}_a(\xi) = e^{-\pi a|\xi|^2}$, we have, for $p < \infty$,
\[
\|\hat{f}_a \ast M_{-x} g\|_{L^p} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} e^{-\pi a|\xi-y|^2} e^{-\pi|y|^2} dy \right)^p d\xi \right)^{1/p} = (1 + a^2)^{-d/4} |a|^{-d/p} (\int_{\mathbb{R}^d} e^{-\pi|z|^2/(1+a^2)} dz)^{1/p} = (1 + a^2)^{(d/2)(1/p-1/2)} |a|^{-d/p} (1+a^2)^{-d/(2p)}.
\]

Since the right-hand side does not depend on $x$, taking the supremum on $\mathbb{R}^d$ with respect to the $x$-variable we attain the desired estimate.

If $p = \infty$,
\[
\|f_a\|_{W(FL^\infty, L^\infty)} \asymp \sup_{\xi \in \mathbb{R}^d} |1 + ai|^{-d/2} e^{-\pi(x-a\xi)^2/(1+ai)} = (1 + a^2)^{-d/4}
\]
and we are done.  

For $a = 4\pi t$, $t \neq 0$, we infer:

**Corollary 3.2.** Let $K_t(x)$ be the kernel in (13). Then, if $1 \leq p \leq \infty$,
\[(21) \quad \|K_t\|_{W(FL^p, L^\infty)} \asymp |t|^{-d/p} (1 + t^2)^{(d/2)(1/p-1/2)}.
\]

**Lemma 3.1.** Let $1 \leq p, q, r \leq \infty$, with $1/r = 1/p + 1/q$, then
\[(22) \quad W(FL^p, L^\infty) * W(FL^q, L^1) \hookrightarrow W(FL^r, L^\infty).
\]

**Proof.** This is a consequence of the convolution relations for Wiener amalgam spaces in Lemma 2.1 (i), being $FL^p * FL^q = FL^r$ by Hölder’s Inequality with $1/r = 1/p + 1/q$, and $L^\infty * L^1 \hookrightarrow L^\infty$.  

**Proposition 3.2.** It turns out, for $2 \leq q \leq \infty$,
\[(23) \quad \|e^{it\Delta} u_0\|_{W(FL^q, L^\infty)} \lesssim |t|^{d(2/q-1)} (1 + t^2)^{(d(1/4-1/q))} \|u_0\|_{W(FL^q, L^1)}.
\]

**Proof.** We use the explicit representation of the Schrödinger evolution operator $e^{it\Delta} u_0(x) = (K_t * u_0)(x)$. Let $1 \leq p, q \leq \infty$, satisfying
\[(24) \quad \frac{1}{p} + \frac{2}{q} = 1.
\]

Then the kernel norm (21) and the convolution relations (22) yield the desired result.

**Theorem 3.3.** For $2 \leq q, r, s \leq \infty$ such that
\[(25) \quad \frac{1}{s} = \frac{1}{r} + \frac{2}{q} \left( \frac{1}{2} - \frac{1}{r} \right),
\]
we have
\[ \|e^{it\Delta}u_0\|_{W(\mathcal{F}L^q, L^p)} \lesssim |t|^{d\left(\frac{1}{q} - 1\right)\left(1 + t^2\right)\left(\frac{d}{2} - \frac{1}{q}\right)} \|u_0\|_{W(\mathcal{F}L^q, L^p)}. \]

In particular, for \(2 \leq r \leq \infty\),
\[ \|e^{it\Delta}u_0\|_{W(L^2, L^r)} \lesssim (1 + t^2)^{-\frac{d}{2} + \frac{1}{q} - \frac{1}{r}} \|u_0\|_{W(L^2, L^r)}. \quad (26) \]

**Proof.** Estimates (26) follow by complex interpolation of (23), which corresponds to \(r = \infty\), with the \(L^2\) conservation law
\[ \|e^{it\Delta}u_0\|_{L^2} = \|u_0\|_{L^2}, \quad (27) \]
which corresponds to \(r = 2\).

Indeed, \(L^2 = W(\mathcal{F}L^2, L^2) = W(L^2, L^2)\). By Lemma 2.1, item (iii), for \(0 < \theta = 2/r < 1\), and \(1/s' = (1 - 2/r)/q' + 1/r\), so that relation (25) holds,
\[ \left[W(\mathcal{F}L^q, L^\infty), W(\mathcal{F}L^2, L^2)\right]_{\theta} = W\left([\mathcal{F}L^q, \mathcal{F}L^2]_{\theta}, [L^\infty, L^2]_{\theta}\right) = W(\mathcal{F}L^q, L^r) \]
and
\[ \left[W(\mathcal{F}L^q, L^1), W(\mathcal{F}L^2, L^2)\right]_{\theta} = W\left([\mathcal{F}L^q, \mathcal{F}L^2]_{\theta}, [L^1, L^2]_{\theta}\right) = W(\mathcal{F}L^q, L^r), \]
so that the estimate (26) is attained. \(\square\)

### 4. Strichartz Estimates

In this section we prove Theorem 4.1. Precisely, we first study the case \(q_1 = \tilde{q}_1 = \infty, r_1 = \tilde{r}_1 = 2\). In view of the inclusion relation of Wiener amalgam spaces, it suffices to prove it for the pairs \((q_2, r_2)\) scale invariant (i.e. satisfying (2) with equality), namely the following result.

**Theorem 4.1.** Let \(2 \leq q \leq \infty\), \(2 \leq r \leq \infty\), such that
\[ \frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \]
\((q, r, d) \neq (2, \infty, 2)\), and similarly for \(\tilde{q}, \tilde{r}\). Then we have the homogeneous Strichartz estimates
\[ \|e^{it\Delta}u_0\|_{W(L^\infty, L^q)_x W(L^2, L^r)} \lesssim \|u_0\|_{L^2_x}, \quad (28) \]
the dual homogeneous Strichartz estimates
\[ \| \int e^{-i\Delta} F(s) \, ds \|_{L^2} \lesssim \|F\|_{W(L^1, L^{q'})_x W(L^2, L^{r'})_x}, \quad (29) \]
and the retarded Strichartz estimates
\[ \| \int_{s<t} e^{i(t-s)\Delta} F(s) \, ds \|_{W(L^\infty, L^q)_x W(L^2, L^r)} \lesssim \|F\|_{W(L^1, L^{q'})_x W(L^2, L^{r'})_x}. \quad (30) \]
Proof in the non-endpoint case. Here we prove Theorem 1.1 in the non-endpoint case, namely for $q > 2, \tilde{q} > 2$. The techniques are quite similar to those in [5, Theorem 1.1]. We shall sketch the proof of the homogeneous and dual Strichartz estimates. We first show the estimate (28). The case: $q = \infty$, $r = 2$, follows at once from the conservation law (27). Indeed, $W(L^\infty, L^\infty)_t = L^\infty_t$ and $W(L^2, L^2)_x = L^2_x$, so that, taking the supremum over $t$ in $\|e^{it\Delta}u_0\|_{L^2_x} = \|u_0\|_{L^2_x}$, we attain the claim.

To prove the remaining cases, we can apply the usual $TT^*$ method (or “orthogonality principle”, see [14, Lemma 2.1] or [21, page 353]), because of Hölder’s type inequality

$$\langle F, G \rangle_{L^1_t L^1_x} \leq \|F\|_{W(L^\infty, L^q)_t W(L^2, L^r)_x} \|G\|_{W(L^q, L^r), W(L^2, L^{q'})_x}, \quad (31)$$

which can be proved directly from the definition of these spaces.

As a consequence, it suffices to prove the estimate

$$\| \int e^{i(t-s)\Delta}F(s) \, ds \|_{W(L^\infty, L^q)_t W(L^2, L^r)_x} \lesssim \|F\|_{W(L^q, L^r), W(L^2, L^{q'})_x}. \quad (32)$$

Recall the Hardy-Littlewood-Sobolev fractional integration theorem (see e.g. [21, page 119]) in dimension 1:

$$L^p(\mathbb{R}) \ast L^{1/\alpha, \infty}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R}), \quad (33)$$

for $1 \leq p < q < \infty$, $0 < \alpha < 1$, with $1/p = 1/q + 1 - \alpha$ (here $L^{1/\alpha, \infty}$ is the weak $L^{1/\alpha}$ space, see e.g. [23]). Now, set $\alpha = d(1/2 - 1/r) = 2/q$ ($q > 2$) so that, for $p = q'$, $L^q \ast L^{1/\alpha, \infty} \hookrightarrow L^q$. Moreover, observe that $(1 + |t|)^{-\alpha} \in W(L^\infty, L^{1/\alpha, \infty})(\mathbb{R})$. The convolution relations (14) then give

$$W(L^1, L^{q'})(\mathbb{R}) \ast W(L^\infty, L^{1/\alpha, \infty})(\mathbb{R}) \hookrightarrow W(L^\infty, L^q)(\mathbb{R}).$$

The preceding relations, together with the fixed-time estimates (26) and Minkowski’s inequality allow to write

$$\| \int e^{i(t-s)\Delta}F(s) \, ds \|_{W(L^\infty, L^q)_t W(L^2, L^r)_x} \leq \| \int \|e^{i(t-s)\Delta}F(s)\|_{W(L^2, L^r)_x} \, ds \|_{W(L^\infty, L^q)_t} \lesssim \| \int \|F(s)\|_{W(L^2, L^{q'})_x} (1 + |t - s|)^{-\alpha} \, ds \|_{W(L^\infty, L^q)_t} \lesssim \|F\|_{W(L^1, L^{q'}), W(L^2, L^{q'})_x}. \quad (30)$$

The estimate (29) follows from (28) by duality.

The same arguments as in [5, page 13] then give the retarded Strichartz estimate (30). \(\square\)
Proof in the endpoint case. We are going to prove Theorem 4.1 in the endpoint case \((q, r) = P := (2, 2d/(d - 2))\) or \((\bar{q}, \bar{r}) = P, d > 2\). Hence, we prove the estimates
\[
\|e^{it\Delta} u_0\|_{W(L^\infty, L^2)_{t} W(L^2, L^r)_{x}} \lesssim \|u_0\|_{L^2}, \quad r = 2d/(d - 2),
\]
\[
\|\int e^{-is\Delta} F(s) \, ds\|_{L^2} \lesssim \|F\|_{W(L^3, L^{\bar{r}})_{t} W(L^2, L^{r'})_{x}}, \quad \bar{r} = 2d/(d - 2),
\]
and
\[
\|\int_{s < t} e^{i(t-s)\Delta} F(s) \, ds\|_{W(L^\infty, L^2)_{t} W(L^2, L^r)_{x}} \lesssim \|F\|_{W(L^3, L^2)_{t} W(L^2, L^{r'})_{x}},
\]
with \((q, r), (\bar{q}, \bar{r})\) Schrödinger admissible, and \((q, r) = P\) or \((\bar{q}, \bar{r}) = P\).

We follow the pattern in \([5, 19]\), with several changes according to our setting. Hence, we study bilinear form estimates via a Whitney decomposition in time (see \((33)\)). We estimate each dyadic contribution in \((42)\) and \((43)\). Finally we conclude by a lemma of real interpolation theory to sum these estimates.

Precisely, by the same duality arguments as the ones used in the previous part of the proof, we observe that it suffices to prove \((34)\). This is equivalent to the bilinear estimate
\[
|\int \langle e^{-is\Delta} F(s), e^{-it\Delta} G(t) \rangle \, ds \, dt| \lesssim \|F\|_{W(L^3, L^2)_{t} W(L^2, L^{r'})_{x}} \|G\|_{W(L^3, L^2)_{t} W(L^2, L^{r'})_{x}}.
\]
By symmetry, it is enough to prove
\[
|T(F, G)| \lesssim \|F\|_{W(L^3, L^2)_{t} W(L^2, L^{r'})_{x}} \|G\|_{W(L^3, L^2)_{t} W(L^2, L^{r'})_{x}},
\]
where
\[
T(F, G) = \int_{s < t} \langle e^{-is\Delta} F(s), e^{-it\Delta} G(t) \rangle \, ds \, dt.
\]
Here the critical exponent \(q = 2\) appears in the global component, which control the decay in the \(t\)-variable at infinity, hence the form \(T(F, G)\) is decomposed dyadically as
\[
T = \tilde{T} + \sum_{j \geq 0} T_j,
\]
with
\[
\tilde{T}(F, G) = \int_{t-1 < s < t} \langle e^{-is\Delta} F(s), e^{-it\Delta} G(t) \rangle \, ds \, dt
\]
and
\[
T_j(F, G) = \int_{t-2^j+1 < s \leq t-2^j} \langle e^{-is\Delta} F(s), e^{-it\Delta} G(t) \rangle \, ds \, dt.
\]
In the sequel we shall study the behaviour of \(\tilde{T}\) and \(T_j\) separately. We shall use repeatedly the following fact.
Lemma 4.1. Let $1 \leq p, q, r \leq \infty$ be such that $1/p + 1/q = 1/r$. For every $f \in W(L^1, L^q)(\mathbb{R})$ supported in any interval $I$ of length $L \geq 1$, it turns out

$$\|f\|_{W(L^1, L^r)} \leq C_p L^{1/p} \|f\|_{W(L^1, L^q)}. \tag{41}$$

Proof. To compute the $W(L^1, L^r)$ norm of $f$ we choose $g = \chi_{[0,1]}$, the characteristic function of the interval $[0,1]$. Then, $\|f\|_{W(L^1, L^r)} \lesssim \|fT_y g\|_{L^1}$. Since $f$ is supported in an interval $I$ of length $L$, the mapping

$$y \mapsto \|fT_y g\|_{L^1}$$

is supported in an interval $\tilde{I}$ of length $L + 2$. Hence, for $1/p + 1/q = 1/r$, Hölder’s inequality yields

$$\|fT_y g\|_{L^1} \|f\|_{L^r} = \left( \int_I \|fT_y g\|_{L^1}^r \, dy \right)^{1/r} \leq (L + 2)^{1/p} \|fT_y g\|_{L^1} \|f\|_{L^q} \leq C_p L^{1/p} \|f\|_{W(L^1, L^q)}$$

as desired.

Lemma 4.2. We have

$$|\tilde{T}(F, G)| \lesssim \|F\|_{W(L^1, L^2), W(L^2, L^r)_x} \|G\|_{W(L^1, L^2), W(L^2, L^r)_x}. \tag{42}$$

Proof. We assume $F$ and $G$ compactly supported, with respect to the time variable, in intervals of duration 1 (indeed, in (39), $F$ and $G$ can be replaced by $\chi_{[t-1, t]}(s)F(s)$ and $\chi_{[s, s+1]}(t)G(t)$, respectively).

Since $|t - s| \leq 1$, it follows from the duality properties (17) and the fixed-time estimate (26) that

$$|\langle e^{-is\Delta} F(s), e^{-it\Delta} G(t) \rangle| = \|F(s) \|e^{i(s-t)\Delta} G(t)\|_{W(L^2, L^r)_x} \lesssim \|F(s)\|_{W(L^2, L^r)_x} \|e^{i(s-t)\Delta} G(t)\|_{W(L^2, L^r)_x} \lesssim \|F(s)\|_{W(L^2, L^r)_x} \|(1 + |s - t|)^{-d(\frac{1}{2} - \frac{1}{q})}\|G(t)\|_{W(L^2, L^r)_x} \lesssim \|F(s)\|_{W(L^2, L^r)_x} \|G(t)\|_{W(L^2, L^r)_x} \$$

Integrating with respect to the variables $s$ and $t$ we then obtain

$$|\tilde{T}(F, G)| \lesssim \|F\|_{L^1,L^2,L^r,x} \|G\|_{L^1,L^2,L^r,x} \|G(t)\|_{W(L^2, L^r)_x},$$

Lemma 4.1 with $p = q = 2$ and $r = 1$, applied to each function $\|F(t)\|_{W(L^2, L^r)_x}$ and $\|G(t)\|_{W(L^2, L^r)_x}$, gives the result.

The estimates of the pieces $T_j(F, G)$ follow the techniques of [19] Lemma 4.1 and [5] Lemma 5.2, adapted to our context.
Lemma 4.3. We have
\[
\|T_j(F,G)\| \lesssim 2^{-j\beta(a,b)}\|F\|_{W(L^1,L^2),W(L^2,L^r)}\|G\|_{W(L^1,L^2),W(L^2,L^r)},
\]
for \((1/a,1/b)\) in a neighborhood of \((1/r,1/r)\).

Proof. Observe that here \(d \geq 2\) hence \(r < \infty\). Then, the result follows by complex interpolation (Lemma 2.1 (iii)) from the following cases:

(i) \(a = \infty, b = \infty\),
(ii) \(2 \leq a < r, b = 2\),
(iii) \(a = 2, 2 \leq b < r\).

Case (i). We need to show the estimate
\[
|T_j(F,G)| \lesssim 2^{-j(\frac{d}{2}-1)}\|F\|_{W(L^1,L^2),W(L^2,L^1)}\|G\|_{W(L^1,L^2),W(L^2,L^1)}.
\]

Here the fixed-time estimate (26) and \(|t-s| \lesssim 2^j\) yield
\[
|\langle e^{-is\Delta}F(s), e^{-it\Delta}G(t)\rangle| \lesssim 2^{-j\frac{d}{2}}\|F(s)\|_{W(L^2,L^1)}\|G(t)\|_{W(L^2,L^1)}.
\]

Integrating with respect to the variables \(s\) and \(t\),
\[
|T_j(F,G)| \lesssim 2^{-j\frac{d}{2}}\|F\|_{L^1_t(W(L^2,L^1)_x)}\|G\|_{L^1_t(W(L^2,L^1)_x)}.
\]

Again, we can assume \(F\) and \(G\) compactly supported, with respect to the time variable, in intervals of duration \(2^j\). Applying Lemma 4.1 with \(p = q = 2\) and \(r = 1\), to both functions \(\|F(t)\|_{W(L^2,L^1)_x}\) and \(\|G(t)\|_{W(L^2,L^1)_x}\) we attain (44).

Case (ii). We have to show
\[
|T_j(F,G)| \lesssim 2^{-j(\frac{d}{4}-1)}\|F\|_{W(L^1,L^2),W(L^2,L^r)}\|G\|_{W(L^1,L^2),L^2},
\]

Using similar arguments to the previous case we obtain
\[
|T_j(F,G)| \lesssim \sup_t \| \int_{t-2^j+1 \leq s \leq t-2^j} e^{-is\Delta}F(s)\ ds\|_{L^2_t} \|G\|_{L^1_tL^2},
\]

and
\[
\|G\|_{L^1_tL^2} \lesssim 2^{j/2}\|G\|_{W(L^1,L^2),L^2}.
\]

For \(a \geq 2\), let now \(\tilde{q} = \tilde{q}(a)\) be defined by
\[
\frac{2}{\tilde{q}(a)} + \frac{d}{a} = \frac{d}{2},
\]

The non-endpoint case of (29), written for \(\tilde{r} = a\) and the \(\tilde{q}\) above, gives
\[
\sup_t \| \int_{t-2^j+1 \leq s \leq t-2^j} e^{-is\Delta}F(s)\ ds\|_{L^2_t} = \sup_t \| \int_{\mathbb{R}} e^{-is\Delta}(T_{-t}(\chi_{[-2^j+1,\leq t]}))F(s)\ ds\|_{L^2_t}
\lesssim \|F\|_{W(L^1,L^\tilde{r})W(L^2,L^r)},
\]
for every $2 \leq a < r$. Since the support of $F$ with respect to the time is contained in an interval of duration $2^j$, Lemma 4.1 with $q = 2$, $r = \tilde{q}'$, and

$$\frac{1}{p} = \frac{1}{\tilde{q}'} - \frac{1}{2} = \frac{1}{2} - \frac{1}{q} = \frac{1}{2} - \frac{d}{4} + \frac{d}{2a},$$

gives

$$\|F\|_{W(L^1, L^{\tilde{q}'})W(L^2, L^{q})} \lesssim 2^{j(\frac{1}{2} + \frac{d}{4} - \frac{d}{4})} \|F\|_{W(L^1, L^2)W(L^2, L^q')}.$$  

This estimate, together with (46) and (47), yields the estimate (45).

Case (iii). Use the same arguments as in case (ii).

It remains to show

$$\sum_{j \geq 0} |T_j(F, G)| \lesssim \|F\|_{W(L^1, L^2)_{x}W(L^2, L^{q'})_{x}} \|G\|_{W(L^1, L^2)_{x}W(L^2, L^{q'})_{x}}. \quad (49)$$

Now, (49) can be achieved from (43) and some real interpolation results (collected in Appendix A below), as in [5, 19].

In details, we single out $a_0, a_1, b_0, b_1$ such that $(1/r, 1/r)$ is inside a small triangle with vertices $(1/a_0, 1/b_0)$, $(1/a_1, 1/b_0)$ and $(1/a_0, 1/b_1)$ (see Figure 2), so that

$$\beta(a_0, b_1) = \beta(a_1, b_0) \neq \beta(a_0, b_0).$$

![Figure 2](image-url)
Then, we apply Lemma A.1 with \( T = \{ T_j \} \), (after setting \( T_j = 0 \) for \( j < 0 \)) \( C_0 = l^\beta_{(a_0, b_0)}, C_1 = l^\beta_{(a_0, b_1)} \) and, for \( k = 0, 1 \), we take

\[
A_k = W(L^1(W(L^2, L^{a_k'})(x)), L^2), \quad B_k = W(L^1(W(L^2, L^{b_k'})(x)), L^2).
\]

Here we choose \( \theta_0, \theta_1 \), so that

\[
1/r = (1 - \theta_0)/a_0 + \theta_0/a_1, \quad 1/r = (1 - \theta_1)/b_0 + \theta_1/b_1.
\]

The assumptions are satisfied in view of (43). Moreover, with \( \theta = \theta_0 + \theta_1 \), we have

\[
(1 - \theta)\beta(a_0, b_0) + \theta\beta(a_0, b_1) = \beta(r, r) = 0.
\]

Hence we attain the desired estimate (49), because

\[
\begin{align*}
(A_0, A_1)_{\theta_0, 2} &\quad \leftrightarrow \quad W((L^1(W(L^2, L^{a_0'})(x)), L^1(W(L^2, L^{a_1'})(x)))_{\theta_0, 2}, L^2) \\
&\quad \leftrightarrow \quad W(L^1(W(L^2, L^{a_0'})(x), W(L^2, L^{a_1'})(x))_{\theta_0, 2}), L^2 \\
&\quad \leftrightarrow \quad W(L^1(W(L^2, L^{a_0'})(x)), L^2) = W(L^1, L^2), W(L^2, L^{a_0'})(x),
\end{align*}
\]

where we used Proposition A.2 for the first and third embedding and Proposition A.1 for the second one (the same holds for \((B_0, B_1)_{\theta_0, 2}\)).

Similarly to [5, page 19] one can obtain the retarded estimates. This concludes the proof of Theorem 4.1.

The Strichartz estimates in Theorem 4.1 can be combined with the classical ones (3) to obtain the estimates in Theorem 1.1.

**Proof of Theorem 1.1.** We prove the homogeneous estimates (9), the other ones follow by similar arguments. Recall that the classical homogeneous Strichartz estimates can be written as

\[
\|e^{it \Delta} u_0\|_{W(L^{q_1}(L^{r_1}), W(L^{q_2}, L^{r_2}))} \lesssim \|u_0\|_{L^2_x}.
\]

for every Schrödinger admissible pair \((\tilde{q}, \tilde{r})\), i.e. \( \tilde{q}, \tilde{r} \geq 2 \), with \( 2/\tilde{q} + d/\tilde{r} = d/2 \), \( (\tilde{q}, \tilde{r}, d) \neq (2, \infty, 2) \). By complex interpolation between (50) and (28) one has

\[
\|e^{it \Delta} u_0\|_{W(L^{q_1}, L^{\theta q_2}) W(L^{q_1'}, L^{\theta q_2'})} \lesssim \|u_0\|_{L^2_x},
\]

with \((q_i, r_i), i = 1, 2, \) Schrödinger admissible pairs.

Here

\[
\frac{1}{q_1} = \frac{1 - \theta}{\tilde{q}} + \frac{\theta}{\infty}, \quad \frac{1}{q_2} = \frac{1 - \theta}{\tilde{q}} + \frac{\theta}{q},
\]

so that

\[
\frac{1}{q_2} = \frac{1}{q_1} + \frac{\theta}{q}.
\]

---

\(^1\) We set \( L^q_s = L^q(\mathbb{Z}, 2^js \cdot dj) \), where \( dj \) is the counting measure. Recall (see [3, Section 5.6]) that \((L^q_s, L^q_s)_{\theta, 1} = L^q_t\) whenever \( s_0 \neq s_1 \) and \( s = (1 - \theta)s_0 + \theta s_1, 0 < \theta < 1.\)
Hence $q_1 \geq q_2$, i.e., $r_1 \leq r_2$. This shows (28) when $(q_1, r_1)$ and $(q_2, r_2)$ satisfy (7) and (8) with equality. The general case follows from the inclusion relations of Wiener amalgam spaces, which allow us to increase $q_2, r_2$, and diminish $q_1, r_1$ (see Figure 1).

5. Sharpness of fixed-time and Strichartz estimates

In this section we prove the sharpness of the estimates (4), (5) and (9). To this end we need the following three lemmata.

We recall from [13, page 257] the following well-known formula for Gaussian integrals.

**Lemma 5.1.** Let $A$ be a $d \times d$ complex matrix such that $A = A^*$ and $\text{Re} A$ is positive definite. Then for every $z \in \mathbb{C}^n$,

$$
\int e^{-\pi Ax - 2\pi i zx} dx = (\det A)^{-1/2} e^{-\pi z A^{-1} z},
$$

where the branch of the square root is determined by the requirement that $(\det A)^{-1/2} > 0$ when $A$ is real and positive definite.

**Lemma 5.2.** For $c \in \mathbb{C}$, $c \neq 0$, consider the function $\phi^{(c)}(x) = e^{-\pi c|x|^2}$, $x \in \mathbb{R}^d$. For every $c_1, c_2 \in \mathbb{C}$, with $\text{Re} c_1 \geq 0$, $\text{Re} c_2 > 0$, we have

$$
\phi^{(c_1)} \ast \phi^{(c_2)} = (c_1 + c_2)^{-d/2} \phi^{\left(\frac{c_1 c_2}{c_1 + c_2}\right)}.
$$

**Proof.** Using the equality (52),

$$
\left(\phi^{(c_1)} \ast \phi^{(c_2)}\right)(x) = \int e^{-\pi c_1 |x - t|^2 - \pi c_2 |t|^2} dt
$$

$$
= e^{-\pi c_1 |x|^2} \int e^{-\pi (c_1 + c_2)|t|^2 - 2\pi c_1 x t} dt
$$

$$
= (c_1 + c_2)^{-d/2} e^{-\pi c_1 |x|^2} \frac{\pi c_1^2 |x|^2}{e^{c_1 + c_2}},
$$

$$
= (c_1 + c_2)^{-d/2} \phi^{\left(\frac{c_1 c_2}{c_1 + c_2}\right)}(x),
$$

as desired.  

In particular, the solution of the Cauchy problem (1), with initial datum $u_0(x) = e^{-\pi |x|^2}$, is given by the formula

$$
(54) \quad u(t, x) = (1 + 4\pi it)^{-d/2} e^{-\frac{\pi |x|^2}{1 + 4\pi it}}.
$$

This follows at once from Lemma 5.2, since the solution $u(t, x)$ can be rephrased as

$$
(54) \quad u(t, x) = (K_t \ast u_0)(x) = \frac{1}{(4\pi it)^{d/2}} (\phi^{(4\pi it)^{-1}} \ast \phi^{(1)})(x).
$$
Lemma 5.3. For $a, b \in \mathbb{R}$, $a > 0$, set $f_{a+ib}(x) = (a + ib)^{-d/2}e^{-\frac{\pi |x|^2}{a+ib}}$. Then, for every $1 \leq q, r \leq \infty$,

$$
(55) \quad \|f_{a+ib}\|_{W(FL^q, L^r)} \asymp \frac{((a + 1)^2 + b^2)^\frac{d}{2}(\frac{1}{a} - \frac{1}{r})}{a^\frac{d}{2} (a(a + 1) + b^2)^\frac{d}{2}(\frac{1}{a} - \frac{1}{r})}.
$$

Proof. We use the Gaussian $g(y) = e^{-\pi |y|^2}$ as a window function, so that the Wiener amalgam norm $W(FL^q, L^r)$ reads

$$
\|f_{a+ib}\|_{W(FL^q, L^r)} \asymp \|f_{a+ib}T_x g\|_{FL^q} \|L^r_x\|.
$$

Now,

$$
(56) \quad f_{a+ib}T_x g(\omega) = \left(\hat{f}_{a+ib} * M_{-x} g\right)(\omega) = \int e^{-\pi(a+ib)|\omega-y|^2} e^{-2\pi ixy} e^{-\pi |y|^2} dy
$$

$$
= e^{-\pi(a+ib)|\omega|^2} \int e^{-\pi(a+1+ib)|\omega|^2-2\pi i(\omega + (a+ib)\omega)g} dy
$$

$$
= (a + 1 + ib)^{-d/2} e^{-\pi(a+ib)|\omega|^2} e^{-\pi \left(\frac{\omega + (a+ib)\omega}{a+1+ib}\right)}
$$

where we used (52). Hence, after a simple computation,

$$
|f_{a+ib}T_x g(\omega)| = ((a + 1)^2 + b^2)^{-d/4} e^{-\frac{\pi}{(a+1)^2+b^2}[(a+1)+b^2]|\omega|^2+2b\omega+(a+1)|x|^2}.
$$

It follows that

$$
\|f_{a+ib}T_x g\|_{FL^q} = ((a + 1)^2 + b^2)^{-d/4} e^{-\frac{\pi}{(a+1)^2+b^2}[(a+1)+b^2]|\omega|^2+2b\omega+(a+1)|x|^2} \left(\int e^{-\frac{\pi}{(a+1)^2+b^2}[(a+1)+b^2]|\omega|^2+2b\omega+(a+1)|x|^2} d\omega\right)^\frac{1}{q}
$$

$$
= ((a + 1)^2 + b^2)^{-d/4} e^{-\frac{\pi a |x|^2}{a(a+1)+b^2}} \left(\int e^{-\frac{\pi}{(a+1)^2+b^2}[(a+1)+b^2]|\omega|^2+2b\omega+(a+1)|x|^2} d\omega\right)^\frac{1}{q}
$$

$$
= ((a + 1)^2 + b^2)^{-d/4} e^{-\frac{\pi a |x|^2}{a(a+1)+b^2}} (a(a+1)+b^2)^{-d/2} \left((a + 1)^2 + b^2\right)^{d/2} \left((a + 1)^2 + b^2\right)^{d/2} e^{-\frac{\pi a |x|^2}{a(a+1)+b^2}}.
$$

By taking the $L^r$ norm of this expression one obtains (55).

Proposition 5.1. (Sharpness of (4)). Suppose that, for some fixed $t_0 \in \mathbb{R}$, $1 \leq r \leq \infty$, $C > 0$, the following estimate holds:

$$
(57) \quad \|e^{it_0 \Delta} u_0\|_{W(FL^r, L^r)} \leq C \|u_0\|_{W(FL^r, L^r)}, \quad \forall u_0 \in \mathcal{S}(\mathbb{R}^d).
$$

Then $r \geq 2$. 

Assume now that, for some \( \alpha \in \mathbb{R}, C > 0, \delta > 0, 1 \leq r \leq \infty, \) the estimate
\[
\|e^{it\Delta}u_0\|_{W(\mathcal{F}L^r',L^r')} \leq Ct^\alpha\|u_0\|_{W(\mathcal{F}L^r',L^r')}, \quad \forall u_0 \in \mathcal{S}(\mathbb{R}^d),
\]
holds for every \( t \in (0, \delta). \) Then
\[
\alpha \leq -2d \left( \frac{1}{2} - \frac{1}{r} \right).
\]

Finally, if \( u_0(x) = e^{-\pi|x|^2} \) then
\[
\|e^{it\Delta}u_0\|_{W(\mathcal{F}L^r',L^r')} \sim t^{-d\left(\frac{1}{2} - \frac{1}{r}\right)}, \quad \text{as } t \to +\infty.
\]

Proof. We consider the one parameter family of initial data \( u_0(\lambda x) = e^{-\pi\lambda^2|x|^2}, \lambda > 0. \) If \( u \) is the function in (51), the corresponding solutions will be
\[
u(\lambda^2 t, \lambda x) = (1 + 4\pi i t \lambda^2)^{-d/2} e^{-\frac{\pi \lambda^2 |x|^2}{1 + 4\pi i t \lambda^2}}
\]
\[= \lambda^{-d} f_{\lambda^{-2} + 4\pi i t}(x),
\]
where we used the notation in Lemma 5.3.

Now, (55) gives
\[
\|u(\lambda^2 t, \lambda \cdot)\|_{W(\mathcal{F}L^r',L^r')} = \lambda^{-d} \|f_{\lambda^{-2}}\|_{W(\mathcal{F}L^r',L^r')} \sim c \lambda^{-d/r'}
\]
both for \( \lambda \to 0^+ \) and \( \lambda \to +\infty, \) for some \( c > 0. \) On the other hand, again an application of (55) yields
\[
\|u(\lambda^2 t, \lambda \cdot)\|_{W(\mathcal{F}L^r',L^r')} = \lambda^{-d} \|f_{\lambda^{-2} + 4\pi i t}\|_{W(\mathcal{F}L^r',L^r')}
\]
\[\lesssim \frac{\lambda^{-d/r'} [(1 + \lambda^{-2})^2 + t^2]^{\frac{d}{2} \left(\frac{1}{2} - \frac{1}{r'}\right)}}{[\lambda^{-2}(\lambda^{-2} + 1) + t^2]^{d\left(\frac{1}{2} - \frac{1}{r}\right)}}.
\]
Now, for fixed \( t = t_0, \) the expression in (63) is asymptotically equivalent to
\[c_0 \lambda^{-d/r' + 2d\left(\frac{1}{2} - \frac{1}{r'}\right)} (c_0 > 0), \text{ as } \lambda \to 0^+,
\]
which, combined with the estimates (62) and (67), yields \( r \geq 2. \)

Similarly, if the inequality (58) holds for \( t \in (0, \delta), \) one must have
\[
\frac{[(1 + \lambda^{-2})^2 + t^2]^{\frac{d}{2} \left(\frac{1}{2} - \frac{1}{r'}\right)}}{[\lambda^{-2}(\lambda^{-2} + 1) + t^2]^{d\left(\frac{1}{2} - \frac{1}{r}\right)}} \leq Ct^\alpha,
\]
for every \( t \in (0, \delta), \lambda > 0. \) Choosing \( t = \lambda^{-1}, \) we see that, when \( \lambda \to +\infty, \) the left-hand side of (64) is asymptotically equivalent to \( c_1 \lambda^{2d\left(\frac{1}{2} - \frac{1}{r'}\right)} (c_1 > 0), \) and this proves (59).

Finally, choosing \( \lambda = 1 \) and letting \( t \to +\infty, \) we see that the expression in (63) is asymptotically equivalent to \( c_2 t^{-d\left(\frac{1}{2} - \frac{1}{r}\right)}, \) which is (60).
\[ \square \]
Proposition 5.2. (Sharpness of (5)). Assume that for some $1 \leq \alpha, \beta, r \leq \infty$, $C > 0$, the following estimate holds:

$$\|e^{it\Delta}u_0\|_{W(L^\alpha, L^r), W(FL', L')} \leq C\|u_0\|_{L^2}, \quad \forall u_0 \in S(\mathbb{R}^d).$$

Then $r \geq 2$. Moreover, if $q$ is defined by the scaling relation $2/q + d/r = d/2$, then

$$\alpha \leq \frac{q}{2}, \quad (66)$$

$$\beta \geq q. \quad (67)$$

Proof. We first prove that $r \geq 2$ and (67) holds. We use the family of initial data $u_0(\lambda x) = e^{-\pi \lambda^2 |x|^2}$, $\lambda > 0$. The $W(FL', L')$ norm of the corresponding solutions (61) is computed in (63). We use that expression to estimate from below the norm in the left hand side of (65). We single out $g = \chi_{[-1,1]}$, the characteristic function of the interval $[-1,1]$, as window function to compute the $W(FL', L')$ norm. Then,

$$\|\|u(\lambda^2 t, \lambda \cdot)\|_{W(FL', L')} T_y g\|_{L^\gamma_t} \gtrsim \lambda^{-\frac{d}{2}} (\lambda^{-4} + y^2)^{-\frac{q}{4}(\frac{1}{2} - \frac{1}{r})}, \quad 0 < \lambda \leq 1. \quad (68)$$

Since the left-hand side of (65) is finite, this estimate for $\lambda = 1$ already implies $r \geq 2$. Now, to compute the $L^\beta$ norm of the expression in (68) we apply the formula (69)

$$\int_{-\infty}^{+\infty} (\mu + y^2)^\gamma dy = c_\gamma \mu^{\frac{1}{2} + \gamma}, \quad \mu > 0,$n$$

for every $\gamma$, and for a convenient $c_\gamma \in (0, \infty]$, independent of $\mu$ ($c_\gamma < \infty$ if $\gamma < -\frac{1}{2}$).

We deduce

$$\|u(\lambda^2 t, \lambda \cdot)\|_{W(L^\alpha, L^r), W(FL', L')} \gtrsim \lambda^{-\frac{d}{2}} (\lambda^{-4} + y^2)^{-\frac{q}{4}(\frac{1}{2} - \frac{1}{r})}. \quad (70)$$

We write the assumption (65) for the solution $u(\lambda^2 t, \lambda x)$, corresponding to the initial datum $u_0(\lambda x)$. Using the minorization (70), the trivial equality

$$\|u_0(\lambda \cdot)\|_{L^2} = \lambda^{-\frac{d}{2}} \|u_0\|_{L^2}, \quad (71)$$

and letting $\lambda \to 0^+$, we infer

$$-\frac{d}{r'} = \frac{2}{\beta} + 2d \left(\frac{1}{2} - \frac{1}{r}\right) \geq -\frac{d}{2},$$

that is $\frac{2}{\beta} \leq \frac{d}{2} - \frac{d}{r} = \frac{2}{q}$, i.e., the estimate (67).

We now prove (66). Again we use the formula (63) to estimate

$$\|\|u(\lambda^2 t, \lambda \cdot)\|_{W(FL', L')} T_y g\|_{L^\gamma_t} \gtrsim \lambda^{-\frac{d}{2}} (1 + y^2)^\frac{d}{4}(\frac{1}{2} - \frac{1}{r}) \left(\int_{y-1}^{y+1} (\lambda^{-2} + t^2)^{-\alpha d(\frac{1}{2} - \frac{1}{r})} dt\right)^\frac{1}{\alpha}, \quad \lambda \geq 1. \quad (72)$$
An application of the inequality
\[ \int_{-1/2}^{1/2} (\mu + t^2)^\gamma \, dt \gtrsim \mu^{1/2+\gamma}, \quad 0 < \mu \leq 1, \]
allows us to estimate the expression in (72) as
\[ \|u(\lambda^2 t, \lambda \cdot)\|_{W(L^{r_1},L^{r_2})} \gtrsim \lambda^{-\frac{d}{r_2}} \frac{1}{\alpha} + 2d \left( \frac{1}{2} - \frac{1}{r} \right), \quad |y| \leq \frac{1}{2}, \quad \lambda \geq 1. \]

Taking the $L^\beta_y$ norm, we obtain
\[ (73) \quad \|u(\lambda^2 \cdot, \lambda \cdot)\|_{W(L^{r_1},L^{r_2})} \gtrsim \lambda^{-\frac{d}{r_2}} \frac{1}{\alpha} + 2d \left( \frac{1}{2} - \frac{1}{r} \right), \quad \lambda \geq 1. \]

As above, using the assumption (65) for the solution $u(\lambda^2 t, \lambda x)$, corresponding to the initial datum $u_0(\lambda x)$, and the estimates (71) and (73), we let $\lambda \to +\infty$ and infer
\[ -\frac{d}{r_2} - \frac{1}{\alpha} + 2d \left( \frac{1}{2} - \frac{1}{r} \right) \leq -\frac{d}{2}, \]

namely, \( \frac{1}{\alpha} \geq \frac{d}{2} - \frac{d}{r} = \frac{2}{q} \), that is (66).

We now discuss the sharpness of the estimates (7). The following two auxiliary results are needed.

**Lemma 5.4.** Let $a > 0, b \in \mathbb{R}$. With the notation in Lemma 5.2, we have
\[ (74) \quad \|\phi(a+ib)\|_{W(L^1,L^2)^2} \asymp a^{-\frac{d}{2}} \left( a + 1 \right)^{\frac{d}{2} \left( \frac{1}{r_2} - \frac{1}{r_1} \right)}. \]

**Proof.** We take the Gaussian function $g(x) = e^{-\pi|x|^2}$ as window in the definition of $W(L^1,L^2)$. Then
\[ |\phi(a+ib)(x)T_yg(x)| = e^{-\pi[(a+1)|x|^2 - 2xy + |y|^2]}, \]
so that (52) gives
\[ \|\phi(a+ib)T_yg\|_{L^1} = [r_1(a+1)]^{-\frac{d}{r_2}} e^{-\frac{\pi a}{r_2} |y|^2}. \]
Now we compute the $L^{r_2}$ norm of this expression (again by using (52)) and we obtain (74).

**Lemma 5.5.** Let $u(t,x)$ be the solution of (1) with the initial datum $u_0(x) = e^{-\pi|x|^2}$. Then, for the solution $u(\lambda^2 t, \lambda x)$ corresponding to the initial datum $u_0(\lambda x)$, $\lambda > 0$, we have
\[ (75) \quad \|u(\lambda^2 t, \lambda \cdot)\|_{W(L^{r_1},L^{r_2})} \asymp \lambda^{-\frac{d}{r_2}} \left( 1 + (t\lambda^2)^2 \right)^{\frac{d}{2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right)} \left( 1 + \lambda^2 + (t\lambda^2)^2 \right)^{\frac{d}{2} \left( \frac{1}{r_2} - \frac{1}{r_1} \right)}. \]
Proof. We use the explicit formula (61), which can be written as

\[ u(\lambda^2t, \lambda x) = (1 + 4\pi t\lambda^2)^{-d/2} \phi^{(a+ib)}, \]

with

\[ a = \frac{\lambda^2}{1 + (4\pi t\lambda^2)^2}, \quad b = -\frac{4\pi t\lambda^2}{1 + (4\pi t\lambda^2)^2}. \]

Hence (75) follows from the estimate (74).

Proposition 5.3. Let \( u_0(x) = e^{-\pi|x|^2} \). Then

\[ \|e^{it\Delta}u_0\|_{W(L^{r_1},L^{r_2})} \sim C t^{d \left( \frac{1}{r_2} - \frac{1}{2} \right)}, \quad \text{as } t \to +\infty, \]

for some \( C > 0 \).

Moreover, suppose that for some \( t_0 \neq 0, 1 \leq r_1, r_2 \leq \infty, C > 0 \), the following estimate holds:

\[ \|e^{it_0\Delta}u_0\|_{W(L^{r_1},L^{r_2})} \leq C \|u_0\|_{W(L^{r_1'},L^{r_2'})}, \quad \forall u_0 \in \mathcal{S}(\mathbb{R}^d). \]

Then \( r_1 \leq r_2 \).

Proof. The formula (76) is an immediate consequence of the norm (75), with \( \lambda = 1 \).

To prove the remaining part of the statement, we apply the assumption (77) to the initial data \( u_0(\lambda x) = e^{-\pi\lambda^2|x|^2}, \lambda > 0 \). As a consequence of the norm expressions computed in (74) and (75), we see that, when \( \lambda \to +\infty \), the left-hand side and right-hand side of (77) are asymptotically equivalent to \( c_1 \lambda^{\frac{d}{r_2} - \frac{d}{2}}, \) and \( c_2 \lambda^{\frac{d}{r_1'}} \) respectively, for some \( c_1, c_2 > 0 \). This implies \( \frac{d}{r_2} - d \leq \frac{d}{r_1'}, \) namely \( r_1 \leq r_2 \).

Proposition 5.4. (Sharpness of (9)). Suppose that, for some \( 1 \leq q_1, q_2, r_1, r_2 \leq \infty, C > 0 \), the following estimate holds:

\[ \|e^{it\Delta}u_0\|_{W(L^{q_1},L^{q_2}),W(L^{r_1},L^{r_2})} \leq C \|u_0\|_{L^2}, \quad \forall u_0 \in \mathcal{S}(\mathbb{R}^d). \]

Then we have

\[ \frac{2}{q_1} + \frac{d}{r_1} \geq \frac{d}{2}, \]

\[ \frac{2}{q_2} + \frac{d}{r_2} \leq \frac{d}{2}. \]

In particular, (because of (80)), \( r_2 \geq 2 \). Finally, it must be \( q_2 \geq 2 \).
Proof. We apply the estimate (78) to the family of initial data $u_0(\lambda x) = e^{-\pi \lambda^2 |x|^2}$, $\lambda > 0$. Choose $g = \chi_{[-1,1]}$ as window function in the definition of the space $W(L^{q_1}, L^{q_2})_t$. Then, for $y \in \mathbb{R}$, $\lambda \geq 1$,

\begin{equation}
\|u(\lambda^2 t, \lambda \cdot )\|_{W(L^{q_1}, L^{q_2})_t} \leq \lambda \frac{d}{q_1} \left( \int_{y-1}^{y+1} (\lambda^{-4} + t^2)^{\frac{d}{2}} (\lambda^{-2} + t^2)^{\frac{d}{4}} \left( \lambda^{-\frac{d}{4}} + t \right)^{-\frac{d}{2}} dt \right)^{\frac{1}{q_1}}.
\end{equation}

To estimate this expression from below, we apply the easily verified formula

\begin{align*}
\int_{-\frac{1}{2}}^{1} (\mu^2 + t^2)^{\gamma_1} (\mu + t^2)^{\gamma_2} dt \gtrsim \mu^{1 + 2 \gamma_1 + \gamma_2}, & \quad 0 < \mu \leq 1,
\end{align*}

with $\mu = \lambda^{-2}$. Thereby,

\begin{equation}
\|u(\lambda^2 t, \lambda \cdot )\|_{W(L^{q_1}, L^{q_2})_t} \gtrsim \lambda \frac{d}{q_1} \frac{d}{q_1} \quad \text{for } |y| \leq \frac{1}{2}, \lambda \geq 1.
\end{equation}

Taking the $L^{q_2}$ norm of this expression yields

\begin{equation}
\|u(\lambda^2 t, \lambda \cdot )\|_{W(L^{q_1}, L^{q_2})} \gtrsim \lambda \frac{d}{q_1} \frac{d}{q_1}, \quad \lambda \geq 1.
\end{equation}

On the other hand, the right-hand side of (78) is equal to $\lambda^{-\frac{d}{2}}$, therefore letting $\lambda \to +\infty$, we obtain the index relation (79).

We now prove (80). If $0 < \lambda \leq 1$ it follows from the norm estimate (75) that

\begin{equation}
\|u(\lambda^2 t, \lambda \cdot )\|_{W(L^{q_1}, L^{q_2})} \leq \lambda \frac{d}{q_1} \frac{d}{q_1} (\lambda^{-4} + t^2)^{\frac{d}{2}} \left( \lambda^{-\frac{d}{4}} + t \right)^{-\frac{d}{2}}.
\end{equation}

Hence, we have

\begin{equation}
\|u(\lambda^2 t, \lambda \cdot )\|_{W(L^{q_1}, L^{q_2})_t} \leq \lambda \frac{d}{q_1} \frac{d}{q_1} (\lambda^{-4} + t^2)^{\frac{d}{2}} \left( \lambda^{-\frac{d}{4}} + t \right)^{-\frac{d}{2}}, \quad 0 < \lambda \leq 1.
\end{equation}

Taking the $L^{q_2}$ norm of this expression and applying the formula (69), with $\mu = \lambda^{-4}$, we obtain the minorization

\begin{equation}
\|u(\lambda^2 \cdot, \lambda \cdot )\|_{W(L^{q_1}, L^{q_2})} \gtrsim \lambda \frac{d}{q_2} \frac{d}{q_2}, \quad 0 < \lambda \leq 1.
\end{equation}

Since the right-hand side of (78) is equal to $\lambda^{-\frac{d}{2}}$, the estimate (80) follows by letting $\lambda \to 0^+$.

We are left to prove the condition $q_2 \geq 2$. We follow the pattern outlined in [28, Exercise 2.42]. Precisely, we take as initial datum

\begin{equation}
u_0(x) = \sum_{j=1}^{N} e^{-it_j \Delta} f,
\end{equation}
where $f$ is a fixed test function, normalized so that the function $v(t, x)$, defined by $v(t, x) = (e^{it\Delta} f)(x)$, satisfies

$$\|v\|_{W(L^{q_1}, L^{q_2})_t W(L^{r_1}, L^{r_2})_x} = 1.$$  

Here $t_1, \ldots, t_N$ are widely separated times that will be chosen later on. Notice that the corresponding solution will be

$$u(t, x) = (e^{it\Delta} u_0)(x) = \sum_{j=1}^{N} v(t - t_j, x).$$

We claim that

$$(83) \quad \|u_0\|_{L^2} = \|\sum_{j=1}^{N} e^{-it_j \Delta} f\|_{L^2} \leq (N + 1)^{\frac{1}{2}} \|f\|_{L^2},$$

if $t_1, \ldots, t_N$ are suitable separated.

Indeed,

$$\| \sum_{j=1}^{N} e^{-it_j \Delta} f \|^2_{L^2} = \sum_{j=1}^{N} \|e^{-it_j \Delta} f\|^2_{L^2} + \sum_{j \neq k} \langle e^{i(t_j - t_k) \Delta} f, f \rangle \leq N \|f\|^2_{L^2} + C \sum_{j \neq k} |t_j - t_k|^{-d/2} \|f\|^2_{L^1},$$

where we used Cauchy-Schwarz inequality and the classical dispersive estimate $\|e^{it\Delta} f\|_{L^\infty} \leq C |t|^{-d/2} \|f\|_{L^1}$. Hence (83) follows if

$$|t_j - t_k|^{-d/2} \leq [C(N^2 - N) \|f\|^2_{L^1}]^{-1} \|f\|^2_{L^2}.$$  

We now estimate from below the left-hand side of (78). To this end, let $\tilde{v}(t, x) = v(t, x) \chi_R(t)$, where $\chi_R(t)$ is the characteristic function of the interval $[-R, R]$. Moreover, we assume $q_2 < \infty$ and choose $R$ large enough so that

$$\|v - \tilde{v}\|_{W(L^{q_1}, L^{q_2})_t W(L^{r_1}, L^{r_2})_x} \leq \frac{1}{N}.$$  

We claim that, if $|t_j - t_k| \geq 2R + 2$, for every $j \neq k$, then

$$\|u(t, x)\|_{W(L^{q_1}, L^{q_2})_t W(L^{r_1}, L^{r_2})_x} = \|\sum_{j=1}^{N} v(t - t_j, x)\|_{W(L^{q_1}, L^{q_2})_t W(L^{r_1}, L^{r_2})_x} \geq N^{\frac{1}{q_2}} \left(1 - \frac{1}{N}\right) - 1.$$  

This, together with the assumption (78) and the $L^2$-estimate of the initial datum (83), gives the condition $q_2 \geq 2$, for $N$ large enough.
In order to prove the minorization (85), observe that, by the assumption (84),
\[ \| \sum_{j=1}^{N} v(t-t_j, x) - \sum_{j=1}^{N} \tilde{v}(t-t_j, x) \|_{W(L^{q_1}, L^{q_2})_x W(L^{r_1}, L^{r_2})_x} \leq 1. \]
Hence it suffices to prove
\[ (86) \quad \| \sum_{j=1}^{N} \tilde{v}(t-t_j, x) \|_{W(L^{q_1}, L^{q_2})_x W(L^{r_1}, L^{r_2})_x} \geq N^{\frac{1}{q_2}} \left( 1 - \frac{1}{N} \right). \]
Now,
\[ \| \sum_{j=1}^{N} \tilde{v}(t-t_j, \cdot) \|_{W(L^{r_1}, L^{r_2})} = \sum_{j=1}^{N} \| \tilde{v}(t-t_j, \cdot) \|_{W(L^{r_1}, L^{r_2})}, \]
since, for every fixed \( t \), there is at most one function in the sum which is not identically zero. Hence, upon setting \( h_j(t) := \| \tilde{v}(t-t_j, \cdot) \|_{W(L^{r_1}, L^{r_2})} \), we have
\[ \| \sum_{j=1}^{N} \tilde{v}(t-t_j, \cdot) \|_{W(L^{q_1}, L^{q_2})_x W(L^{r_1}, L^{r_2})_x} = \| \| \sum_{j=1}^{N} h_j T_y g \|_{L^{q_1}} \|_{L^{q_2}}. \]
Choosing the window function \( g \) supported in \([0, 1]\), since the \( h_j \)'s are supported in intervals separated by a distance \( \geq 2 \), we see that the last expression is equal to
\[ \| \sum_{j=1}^{N} \| h_j T_y g \|_{L^{q_1}} \|_{L^{q_2}}. \]
In turn, since the functions \( y \mapsto \| h_j T_y g \|_{L^{q_1}}, j = 1, \ldots, N \), have disjoint supports, the norm above can be written as
\[ \left( \sum_{j=1}^{N} \| \| h_j T_y g \|_{L^{q_1}} \|_{L^{q_2}}^{q_2} \right)^{\frac{1}{q_2}} = N^{\frac{1}{q_2}} \| \tilde{v} \|_{W(L^{q_1}, L^{q_2})_x W(L^{r_1}, L^{r_2})_x}. \]
Hence, the minorization (86) follows from the assumptions (82) and (84).

6. An application to Schrödinger equations with time-dependent potentials

In this section we prove the wellposedness in \( L^2 \) of following Cauchy problem in any dimension \( d \geq 1 \):
\[ (87) \quad \begin{cases} i \partial_t u + \Delta u = V(t, x) u, & t \in [0, T] = I_T, \quad x \in \mathbb{R}^d, \\ u(0, x) = u_0(x). \end{cases} \]
for the class of potentials

\[(88) \quad V \in L^\alpha (I_T; W(\mathcal{F}L^{p'}, L^p)_x), \quad \frac{1}{\alpha} + \frac{d}{p} \leq 1, \quad 1 \leq \alpha < \infty, \quad d < p \leq \infty.\]

Precisely, we generalize [5, Theorem 6.1] by treating the one dimensional case as well and allowing the potential to belong to Wiener amalgam space with respect to \(x\), rather than simply \(L^p\) spaces.

To this end, let us prove directly a simple point-wise multiplication property of Wiener amalgam spaces (it is a special case of the modulation space property [4, Proposition 2.4]).

**Lemma 6.1.** Let \(1 \leq p, q, r \leq \infty\). If

\[(89) \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r},\]

then

\[(90) \quad W(\mathcal{F}L^{p'}, L^p)(\mathbb{R}^d) \cdot W(\mathcal{F}L^{q'}, L^q)(\mathbb{R}^d) \subset W(\mathcal{F}L^r, L^{r'})(\mathbb{R}^d)\]

with norm inequality

\[\|fh\|_{W(\mathcal{F}L^r, L^{r'})} \lesssim \|f\|_{W(\mathcal{F}L^{p'}, L^p)} \|h\|_{W(\mathcal{F}L^{q'}, L^q)}.\]

**Proof.** We measure the Wiener amalgam norm with respect to the window \(g(x) = g_1(x)g_2(x)\), \(g_1, g_2 \in C_0^\infty(\mathbb{R}^d)\), with \(\|g\|_2 = \|g_1\|_2 = \|g_2\|_2 = 1\). (Different windows yield equivalent norms for the Wiener amalgam spaces).

Using \(\hat{g} = \hat{g}_1 * \hat{g}_2\), \(\widehat{T_x g} = M_{-x} \hat{g}\) and

\[M_{-x} \hat{g} = M_{-x} \hat{g}_1 * M_{-x} \hat{g}_2,\]

we can write

\[
\|fh\|_{W(\mathcal{F}L^r, L^{r'})} \asymp \|\hat{f} \ast \hat{h} \ast \widehat{T_x g}\|_{L^r} \|L^{r'}
\]

\[
= \|\|\| \hat{f} \ast (M_{-x} \hat{g}_1) \| \ast \| \hat{h} \ast (M_{-x} \hat{g}_2) \| \|_{L^r} \|L^{r'}
\]

\[
= \|\|\| \hat{f} T_x g_1 \ast h T_x g_2\|_{L^r} \|_{L^{r'}}
\]

\[
\lesssim \|\|\| \hat{f} T_x g_1\|_{\mathcal{F}L^{p'}} \|h T_x g_2\|_{\mathcal{F}L^{q'}} \|_{L^r} \|_{L^{r'}}
\]

\[
\lesssim \|\|\| \hat{f} T_x g_1\|_{\mathcal{F}L^{p'}} \|\|\| \hat{h} T_x g_2\|_{\mathcal{F}L^{q'}} \|_{L^q}
\]

\[
= \|\|\| \hat{f}\|_{W(\mathcal{F}L^{p'}, L^p)} \|h\|_{W(\mathcal{F}L^{q'}, L^q)}\|_{L^r} \|_{L^{r'}}.
\]

where the former inequality is the consequence of Young’s Inequality with \(1/p' + 1/q' = 1 + 1/r\), which follows from the assumption (89), and the latter is Hölder’s inequality with index relation (89).

We have now the instruments to prove the following result.
Theorem 6.1. Consider the class of potentials (88). Then, for all \((q, r)\) such that \(2/q + d/r = d/2\), \(q > 4, r \geq 2\), the Cauchy problem (87) has a unique solution

(i) \(u \in \mathcal{C}(I_T; L^2(\mathbb{R}^d)) \cap L^{q/2}(I_T; W(\mathcal{F}L^r, L^r))\), if \(d = 1\);

(ii) \(u \in \mathcal{C}(I_T; L^2(\mathbb{R}^d)) \cap L^{q/2}(I_T; W(\mathcal{F}L^r, L^r)) \cap L^2(I_T; W(\mathcal{F}L^{2d/(d+1)}, L^{2d/(d-1)}))\), if \(d > 1\).

Proof. It is enough to prove the case \(d = 1\). Indeed, for \(d \geq 2\), condition (88) implies \(p > 2\), so that \(\mathcal{F}L^p \hookrightarrow L^p\) and the inclusion relations of Wiener amalgam spaces yield \(W(\mathcal{F}L^p, L^p) \hookrightarrow W(L^p, L^p) = L^p\). Hence our class of potentials is a subclass of those of \([5\) Theorem 6.1], for which the quoted theorem provides the desired result.

We now turn to the case \(d = 1\). The proof follows the ones of \([7\) Theorem 1.1, Remark 1.3] and \([5\) Theorem 6.1] (see also \([34\]).

First of all, since the interval \(I_T\) is bounded, by Hölder’s inequality it suffices to assume \(1/\alpha + d/p = 1\).

We choose a small time interval \(J = [0, \delta]\) and set, for \(q \geq 2, q \neq 4, r \geq 1\),

\[
Z_{q/2, r} = L^{q/2}(J; W(\mathcal{F}L^r, L^r)_x).
\]

Now, fix an admissible pair \((q_0, r_0)\) with \(r_0\) arbitrarily large (hence \((1/q_0, 1/r_0)\) is arbitrarily close to \((1/4, 0)\)) and set \(Z = \mathcal{C}(J; L^2) \cap Z_{q_0/2, r_0}\), with the norm \(\|v\|_Z = \max\{\|v\|_{\mathcal{C}(J; L^2)}, \|v\|_{Z_{q_0/2, r_0}}\}\). We have \(Z \subset Z_{q/2, r}\) for all admissible pairs \((q, r)\) obtained by interpolation between \((\infty, 2)\) and \((q_0, r_0)\). Hence, by the arbitrary of \((q_0, r_0)\) it suffices to prove that \(\Phi\) defines a contraction in \(Z\).

Consider now the integral formulation of the Cauchy problem, namely \(u = \Phi(v)\), where

\[
\Phi(v) = e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta}V(s)v(s)\, ds.
\]

By the homogeneous and retarded Strichartz estimates in \([5\) Theorems 1.1, 1.2] the following inequalities hold:

\[
\|\Phi(v)\|_{Z_{q/2, r}} \leq C_0\|u_0\|_{L^2} + C_0\|Vv\|_{Z_{q/2, r'}}.
\]

for all admissible pairs \((q, r)\) and \((\tilde{q}, \tilde{r})\), \(q > 4, \tilde{q} > 4\).

Consider now the case \(1 \leq \alpha < 2\). We choose \(((\tilde{q}/2)', \tilde{r}) = (\alpha, 2p/(p+2))\). Since \(v \in L^\infty(J; L^2)\) applying (90) for \(q = 2\) we get

\[
\|Vv\|_{W(\mathcal{F}L^r, L^r')} \lesssim \|V\|_{W(\mathcal{F}L^r, L^r')}\|v\|_{L^2},
\]

whereas Hölder’s Inequality in the time-variable gives

\[
\|Vv\|_{Z_{(q/2)', r'}} \lesssim \|V\|_{L^\alpha(J; W(\mathcal{F}L^r, L^r))}\|v\|_{L^\infty(J; L^2)}.
\]

The estimate (91) then becomes

\[
\|\Phi(v)\|_{Z_{q/2, r}} \leq C_0\|u_0\|_{L^2} + C_0\|V\|_{L^\alpha(J; W(\mathcal{F}L^r, L^r))}\|v\|_{L^\infty(J; L^2)}.
\]
By taking \((q, r) = (\infty, 2)\) or \((q, r) = (q_0, r_0)\) one deduces that \(\Phi : Z \to Z\) (the fact that \(\Phi(u)\) is continuous in \(t\) when valued in \(L^2_x\) follows from a classical limiting argument [7, Theorem 1.1, Remark 1.3]). Also, if \(J\) is small enough, \(C_0 \|V\|_{L^\infty_t L^2_x} < 1/2\), and \(\Phi\) is a contraction. This gives a unique solution in \(J\). By iterating this argument a finite number of times one obtains a solution in \([0, T]\).

The case \(2 \leq \alpha < \infty\) is similar. We again consider the inequality (91) for all admissible pairs \((q, r)\) and \((\tilde{q}, \tilde{r})\), \(q > 4, \tilde{q} > 4\). Since \(\alpha \geq 2\) we can find an admissible pair \((\tilde{q}, \tilde{r})\), \(\tilde{q} > 4\), such that

\[
\frac{1}{\tilde{q}/2} = \frac{1}{q_0/2} + \frac{1}{\alpha}
\]

(93)

\[
\frac{1}{1/r'} = \frac{1}{r_0} + \frac{1}{p}
\]

(94)

\[
(\tilde{q}/2)'/\alpha = (1/2) + \left(\frac{\alpha}{\tilde{q}}\right)'
\]

(95)

Using the Wiener point-wise property (90) with index relation (94) we have

\[
\|Vv\|_{W((\mathcal{F}L^{\tilde{q}}, L^{r'})} \lesssim \|V\|_{W((\mathcal{F}L^{q}, L^{r})} \|v\|_{W((\mathcal{F}L^{q_0}, L^{r_0})}
\]

Finally, Hölder’s Inequality with index relation (93) gives

\[
\|Vv\|_{L^{(\tilde{q}/2)'}(J; W((\mathcal{F}L^{\tilde{q}}, L^{r'})} \lesssim \|V\|_{L^{\alpha}(J; W((\mathcal{F}L^{q}, L^{r})} \|v\|_{L^{\alpha}(J; W((\mathcal{F}L^{q_0}, L^{r_0})}
\]

The final part of the proof is analogous to the previous case. \(\square\)

**Remark 6.2.** The most interesting case in Theorem 6.1 is when \(d = 1\). Indeed, choosing \(p > 2\), so that \(1 < p' < 2\), we have the embedding \(H_s \hookrightarrow \mathcal{F}L^{p'}\), for \(s > 1/p' - 1/2\) (see, e.g., [17, Theorem 7.9.3]). Whence, examples of potentials \(V(t, x)\) satisfying (88) are given by tempered distributions locally in \(H_s\) as above in the \(x\)-variable, conveniently localized in \(x\), and belonging to \(L^\alpha_t, \alpha \geq p'\), with respect to the \(t\)-variable.

**Appendix A. Some results in real interpolation theory**

Here we collected some results in real interpolation theory which are used in the proof of the Strichartz estimates.

Let \((X, \mathcal{B}, \mu)\) a measure space, where \(X\) is a set, \(\mathcal{B}\) a \(\sigma\)-algebra and \(\mu\) a positive \(\sigma\)-finite measure. If \(A\) is a Banach space, \(1 \leq p \leq \infty\), then we shall write \(L^p(A)\) for the usual vector-valued \(L^p\) spaces in the sense of the Bochner integral. The first result is a generalization of [5, Proposition 2.3]. The proof uses arguments similar to those in [30, Section 1.18.4].

**Proposition A.1.** Let \(\{A_0, A_1\}\) be an interpolation couple. For every \(1 \leq p_0, p_1 < \infty\), \(0 < \theta < 1\), \(1/p = (1 - \theta)/p_0 + \theta/p_1\) and \(p \leq q\) we have

\[
L^p ((A_0, A_1)_\theta,q) \hookrightarrow (L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q}.
\]
Proposition A.2. Given two local components $B_0, B_1$, for every $1 \leq p_0, p_1 < \infty$, $0 < \theta < 1$, $1/p = (1 - \theta)/p_0 + \theta/p_1$, and $p \leq q$ we have

$$W((B_0, B_1)_{\theta, q}, L^p) \hookrightarrow (W(B_0, L^{p_0}), W(B_1, L^{p_1}))_{\theta, q}.$$
\[ T : A_1 \times B_0 \rightarrow C_1, \]

then, if \(0 < \theta_i < \theta < 1, i = 0, 1, \theta = \theta_0 + \theta_1,\) one has

\[ T : (A_0, A_1)_{\theta_0,2} \times (B_0, B_1)_{\theta_1,2} \rightarrow (C_0, C_1)_{\theta,1}. \]

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