NOTES ON MVW-EXTENSIONS

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ABSTRACT. We review certain basic geometric and analytic results concerning MVW-extensions of classical groups, following Moeglin-Vigneras-Waldspurger. The related results for Jacobi groups, metaplectic groups, and special orthogonal groups are also included.

1. Preliminaries

Let $k$ be a field, say, of characteristic zero. It is well known that for every square matrix $x$ with coefficients in $k$, its transpose $x^t$ is conjugate to $x$ (namely, there is an invertible matrix $g$ with coefficients in $k$ such that $x^t = gxg^{-1}$). In this note, we review some basic results on classical groups and other related groups, which are closely related to this simple fact.

1.1. $\epsilon$-Hermitian modules. We introduce the following terminologies and notations in order to treat all classical groups uniformly. We find that this general setting is very convenient when we apply Harish-Chandra descent in the proof. Let $A$ be commutative semisimple finite-dimensional $k$-algebra. It is thus a finite product of finite field extensions of $k$. Let $\tau$ be a $k$-algebra involution on $A$. We call such a pair $(A, \tau)$ a commutative involutive algebra (over $k$). It is said to be simple if it is nonzero, and every $\tau$-stable ideal of $A$ is either zero or $A$. Denote by $A^+$ the algebra of $\tau$-invariant elements in $A$. Then $(A, \tau)$ is simple if and only if $A^+$ is a field. When this is the case, either $A = A^+$, or $(A, \tau)$ is isomorphic to one of the followings:

$(A^+ \times A^+, \tau_{A^+})$, (a quadratic field extension of $A^+$, the nontrivial Galois element),

where $\tau_{A^+}$ is the coordinate exchanging map. Every commutative involutive algebra is uniquely (up to ordering) a product of simple ones.

Let $\epsilon = \pm 1$ and let $E$ be an $\epsilon$-Hermitian $A$-module, namely it is a finitely generated $A$-module, equipped with a non-degenerate $k$-bilinear map

$\langle \cdot , \cdot \rangle_E : E \times E \to A$

satisfying

$\langle u, v \rangle_E = \epsilon \langle v, u \rangle_E^r, \quad \langle au, v \rangle_E = a \langle u, v \rangle_E, \quad a \in A, \ u, v \in E.$

1
Denote by $U(E)$ the group of all $A$-module automorphisms of $E$ which preserve the form $\langle , \rangle_E$. When $(A, \tau)$ is simple, $E$ is free as an $A$-module, and $U(E)$ is a classical group as in the following table:

| $\epsilon$ | $A^+$  | $A^+ \times A^+$ | quadratic filed extension |
|-----------|--------|-----------------|-------------------------|
| $\epsilon = 1$ | orthogonal group | general linear group | unitary group |
| $\epsilon = -1$ | symplectic group | general linear group | unitary group |

In general, write

$$(A, \tau) = (A_1, \tau_1) \times (A_2, \tau_2) \times \cdots \times (A_r, \tau_r)$$

as a product of simple commutative involutive algebras. Then $E_i := A_i \otimes_A E$ is obviously an $\epsilon$-Hermitian $A_i$-module. We have that

$$(2) \quad E = E_1 \times E_2 \times \cdots \times E_r,$$

and

$$U(E) = U(E_1) \times U(E_2) \times \cdots U(E_r).$$

We say that $E$ is simple if it is nonzero, and every non-degenerate $A$-submodule of it is either zero or $E$. Every $\epsilon$-Hermitian $A$-module is isomorphic to an orthogonal sum of simple ones.

For every $a \in (A^\times)^{\tau=\epsilon} := \{a \in A^\times \mid a^\tau = \epsilon a\}$,
write $A(a) := A$ as an $A$-module, equipped with the form

$$\langle u, v \rangle_{A(a)} = auv^\tau \in A, \quad u, v \in A(a).$$

Then $A(a)$ is an $\epsilon$-Hermitian $A$-module, and $A(a)$ is isomorphic to $A(a')$ if and only if

$$a'a^{-1} = bb^\tau \quad \text{for some } b \in A^\times.$$

The following classification of simple $\epsilon$-Hermitian $A$-modules is obvious.

**Proposition 1.1.** Assume that $(A, \tau)$ is simple.

(a) If $A = A^+ \times A^+$, then there is a unique simple $\epsilon$-Hermitian $A$-module (up to isomorphism). It has rank 1.

(b) If $A = A^+$ and $\epsilon = -1$, then there is a unique simple $\epsilon$-Hermitian $A$-module. It has rank 2.

(c) If $A = A^+$ and $\epsilon = 1$, or $A$ is a quadratic field extension of $A^+$, then every simple $\epsilon$-Hermitian $A$-module is of the form $A(a)$ ($a \in (A^\times)^{\tau=\epsilon}$).
Write $E_\tau := E$ as a $k$-vector space, and for every $v \in E$, write $v_\tau := v$, viewed as a vector in $E_\tau$. Then $E_\tau$ is an $\epsilon$-Hermitian $A$-module under the scalar multiplication
\[ av_\tau := (a^\tau v), \quad a \in A, v \in E, \]
and the form
\[ \langle u_\tau, v_\tau \rangle_{E_\tau} := \langle v, u \rangle_E, \quad u, v \in E. \]

**Proposition 1.2.** The $\epsilon$-Hermitian $A$-modules $E_\tau$ and $E$ are isomorphic to each other.

**Proof.** Without loss of generality, assume that $(A, \tau)$ is simple and $E$ is simple. Then the proposition follows trivially from Proposition 1.1. □

1.2. **Harish-Chandra descent.** Let $E$ be an $\epsilon$-Hermitian $A$-modules. Define an involution on $\text{End}_A(E)$, which is still denoted by $\tau$, by requiring that
\[ \langle xu, v \rangle_E = \langle u, x^\tau v \rangle_E, \quad x \in \text{End}_A(E), u, v \in E. \]

Let $s$ be a semisimple element of $\text{End}_A(E)$ (that is, it is semisimple as a $k$-linear operator). Assume it is normal in the sense that $s^\tau s = ss^\tau$. Denote by $A_s$ the subalgebra of $\text{End}_A(E)$ generated by $s$, $s^\tau$ and scalar multiplications by $A$. It is $\tau$-stable and $(A_s, \tau)$ is a commutative involutive algebra. Write $E_s := E$, viewed as an $A_s$-module. Define a $k$-bilinear map
\[ \langle \cdot, \cdot \rangle_{E_s} : E_s \times E_s \to A_s \]
by requiring that
\[ \text{tr}_{A_s/k}(a\langle u, v \rangle_{E_s}) = \text{tr}_{A/k}(<au, v>_{E}), \quad u, v \in E, \quad a \in A_s. \]
Then $E_s$ becomes an $\epsilon$-Hermitian $A_s$-modules. When $s \in U(E)$, geometric and analytic problems on $U(E)$ around $s$ are often reduced to that on $U(E_s)$. The procedure is called Harish-Chandra descent.

1.3. **$\epsilon$-Hermitian $\mathfrak{sl}_2$-modules.** Let $A$, $\tau$, $A^+$, $\epsilon$ be as before. Let $E$ be an $\epsilon$-Hermitian $(\mathfrak{sl}_2, A)$-module, namely it is an $\epsilon$-Hermitian $A$-module, equipped with a Lie algebra action
\[ \mathfrak{sl}_2(k) \times E \to E, \quad x, v \mapsto xv, \]
which is $k$-linear on the first factor, $A$-linear on the second factor, and satisfies
\[ \langle xu, v \rangle_E + <u, xv>_{E} = 0, \quad x \in \mathfrak{sl}_2(k), u, v \in E. \]
We say that $E$ is irreducible if it is nonzero, and every $\mathfrak{sl}_2(k)$-stable $A$-submodule of it is either zero or $E$. We say that $E$ is simple if it is nonzero, and every $\mathfrak{sl}_2(k)$-stable non-degenerate $A$-submodule of it is either zero or $E$. Every $\epsilon$-Hermitian $(\mathfrak{sl}_2, A)$-module is isomorphic to an orthogonal sum of simple ones.
Write

\[ h := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \]

which form a basis of the Lie algebra \( sl_2(k) \). The following lemma is also obvious.

**Lemma 1.3.** Assume that \((A, \tau)\) is simple. If \( A = A^+ \) and \( \epsilon = 1 \), or \( A \) is a quadratic field extension of \( A^+ \), then for every positive odd integer \( 2d-1 \) and every \( a \in (A^\times)_{\tau=\epsilon} \), there is a unique (up to isomorphism) simple \( \epsilon \)-Hermitian \((sl_2, A)\)-module with the following property: it has rank \( 2d-1 \) and its \( h \)-invariant vectors form an \( \epsilon \)-Hermitian \( A \)-module which is isomorphic to \( A(a) \).

Write \( A^{2d-1}(a) \) for this simple \( \epsilon \)-Hermitian \((sl_2, A)\)-module. Then \( A^{2d-1}(a) \) is isomorphic to \( A^{2d-1}(a') \) if and only if

\[ a'a^{-1} = bb^* \quad \text{for some} \quad b \in A^\times. \]

The following proposition, which follows easily from the representation theory of \( sl_2 \), classifies simple \( \epsilon \)-Hermitian \((sl_2, A)\)-modules.

**Proposition 1.4.** Assume that \((A, \tau)\) is simple.

(a) Further assume that \( A = A^+ \times A^+ \). Then for every positive integer \( d \), there is a unique simple \( \epsilon \)-Hermitian \((sl_2, A)\)-module of rank \( d \).

(b) Further assume that \( A \) is a quadratic field extension of \( A^+ \). Then for every positive even integer \( 2d \), there is a unique simple \( \epsilon \)-Hermitian \((sl_2, A)\)-module of rank \( 2d \); every simple \( \epsilon \)-Hermitian \((sl_2, A)\)-module of odd rank is of the form \( A^{2d-1}(a) \) \((a \in (A^\times)_{\tau=\epsilon})\).

(c) Further assume that \( A = A^+ \) and \( \epsilon = -1 \). Then for every positive even integer \( d \), there is a unique simple \( \epsilon \)-Hermitian \((sl_2, A)\)-module of rank \( 2d \); for every positive odd integer \( d \), there are exactly two simple \( \epsilon \)-Hermitian \((sl_2, A)\)-module of rank \( 2d \), one is irreducible and the other is reducible.

(d) Further assume that \( A = A^+ \) and \( \epsilon = 1 \). Then for every positive integer \( d \), there is a unique simple \( \epsilon \)-Hermitian \((sl_2, A)\)-module of rank \( 4d \), and there is no simple \( \epsilon \)-Hermitian \((sl_2, A)\)-module of rank \( 4d - 2 \); every simple \( \epsilon \)-Hermitian \((sl_2, A)\)-module of odd rank is of the form \( A^{2d-1}(a) \) \((a \in A^\times)\).

Recall that \( E \) is an \( \epsilon \)-Hermitian \((sl_2, A)\)-module. We define an \( \epsilon \)-Hermitian \((sl_2, A)\)-module \( E_\tau \) as follows. As an \( \epsilon \)-Hermitian \( A \)-module, it is as in Section 1.1. The \( sl_2(k) \)-action is given by

\[ hv_\tau := (hv)_\tau, \quad ev_\tau := -(ev)_\tau, \quad fv_\tau := -(fv)_\tau, \quad v \in E. \]

**Proposition 1.5.** The \( \epsilon \)-Hermitian \((sl_2, A)\)-modules \( E_\tau \) and \( E \) are isomorphic to each other.

**Proof.** This follows trivially from Proposition 1.4. \( \square \)
2. Geometric results

2.1. Classical groups. Following Moeglin-Vigneras-Waldspurger ([MVW, Proposition 4.I.2]), we extend $U(E)$ to a larger group, which is denoted by $\hat{U}(E)$, and is defined to be the subgroup of $GL(E_k) \times \{\pm 1\}$ consisting of pairs $(g, \delta)$ such that either

$$\delta = 1 \quad \text{and} \quad \langle gu, gv \rangle_E = \langle u, v \rangle_E, \quad u, v \in E,$$

or

$$\delta = -1 \quad \text{and} \quad \langle gu, gv \rangle_E = \langle v, u \rangle_E, \quad u, v \in E.$$  

Here $E_k := E$, viewed as a $k$-vector space. Every $g \in GL(E_k)$ is automatically $A$-linear if $(g, 1) \in \hat{U}(E)$, and is conjugate $A$-linear (with respect to $\tau$) if $(g, -1) \in \hat{U}(E)$.

We call $\hat{U}(E)$ the MVW-extension of $U(E)$. Proposition 1.2 amounts to saying that the projection map $\hat{U}(E) \to \{\pm 1\}$ is surjective. Therefore, we have a short exact sequence

$$1 \to U(E) \to \hat{U}(E) \to \{\pm 1\} \to 1.$$

The following basic fact of classical groups is a part of [MVW, Proposition 4.I.2]. With the preparation of Section 1, we sketch a short proof here.

Theorem 2.1. For every $x \in U(E)$, there is an element $\tilde{g} \in \hat{U}(E) \setminus U(E)$ such that $\tilde{g}x\tilde{g}^{-1} = x^{-1}$.

Proof. By using Jordan decomposition and Harish-Chandra descent, we may (and do) assume that $x$ is unipotent. By Jacobson-Morozov Theorem, we choose an action of $sl_2(k)$ on $E$ such that it makes $E$ an $\varepsilon$-Hermitian $(sl_2, A)$-module, and that the exponential of the action of $e$ coincides with $x$. Then the theorem follows from Proposition 1.4. □

Moeglin-Vigneras-Waldspurger also prove the Lie algebra analog of Theorem 2.1, namely, for every $x$ in the Lie algebra of $U(E)$, there is an element $\tilde{g} \in \hat{U}(E) \setminus U(E)$ such that $\text{Ad}_{\tilde{g}}x = -x$. When $U(E)$ is a general linear group, this is just the simple fact mentioned at the beginning of this note.

Similar to Theorem 2.1, the following Theorems 2.2-2.5 (as well as [MVW, Proposition 4.I.2]) can be proved by using Harish-Chandra descent and representation theory of $sl_2$. We leave the details to the interested reader.

2.2. Jacobi groups. In the remaining part of this note, we assume for simplicity that $(A, \tau)$ is simple. Denote by $L$ a free $A$-submodule of $E$ of rank one (if $E$ is nonzero), and by $L^+$ a cyclic $A^+$-subspace of $L$ which generates $L$ as an $A$-module.

Write $U_L(E)$ for the subgroup of $U(E)$ fixing $L$ point-wise, and write

$$\hat{U}_{L^+}(E) := \{(g, \delta) \in \hat{U}(E) \mid g \text{ fixes } L^+ \text{ point-wise}\}.$$
It contains $U_L(E)$ as a subgroup of index two.

Similar to Theorem 2.1 we have

**Theorem 2.2.** Assume that $L$ is totally isotropic. Then for every $x \in U_L(E)$, there is an element $\tilde{g} \in \tilde{U}_{L^+}(E) \setminus U_L(E)$ such that $\tilde{g}x\tilde{g}^{-1} = x^{-1}$.

It seems that Theorem 2.2 (and Theorem 2.5 of Section 2.4) are new. If $L$ is not totally isotropic, then it is non-degenerate and Theorem 2.2 also holds (which is a restatement of Theorem 2.1).

2.3. Special orthogonal groups. Assume that $A = A^+$ and $\epsilon = 1$. Following Waldspurger ([Wald]), we define

$$S\tilde{O}(E) := \{(g, \delta) \in O(E) \times \{\pm 1\} \mid \det(g) = \delta^{\left\lceil \frac{\dim E + 1}{2} \right\rceil}\}.$$ 

It contains the special orthogonal group $SO(E)$ as a subgroup of index two. By convention, $S\tilde{O}(E) := O(E) \times \{\pm 1\} = \{\pm 1\}$ if $E = \{0\}$.

**Theorem 2.3.** (cf. [Wald]) For every $x \in SO(E)$, there is an element $\tilde{g} \in S\tilde{O}(E) \setminus SO(E)$ such that $\tilde{g}x\tilde{g}^{-1} = x^{-1}$.

2.4. Metaplectic groups. In the remaining part of this note, we assume that $k$ is a local field of characteristic zero. In this subsection further assume that $\epsilon = -1$. Write $E_k := E$, viewed as a $k$-symplectic space under the form

$$\langle u, v \rangle_{E_k} := \text{tr}_{A/k}(\langle u, v \rangle_E).$$

Denote by

$$1 \to \{\pm 1\} \to \tilde{\text{Sp}}(E_k) \to \text{Sp}(E_k) \to 1$$

the metaplectic cover of the symplectic group $\text{Sp}(E_k)$. This is a (topologically) exact sequence of locally compact topological groups. It splits when either $E = 0$ or $k = \mathbb{C}$. Otherwise, this is the unique non-split (topological) central extension of $\text{Sp}(E_k)$ by $\{\pm 1\}$ (cf. [Moor, Theorem 10.4]).

Note that $\tilde{\text{Sp}}(E_k) := \tilde{U}(E_k)$ equals to the subgroup of $\text{GSp}(E_k)$ with similitudes $\pm 1$. It is shown in [MVW, Page 36] that there is a unique action

$$\text{Ad} : \tilde{\text{Sp}}(E_k) \times \text{Sp}(E_k) \to \tilde{\text{Sp}}(E_k)$$

of $\tilde{\text{Sp}}(E_k)$ as group automorphisms on $\tilde{\text{Sp}}(E_k)$ which lifts the adjoint action

$$\text{Ad} : \text{Sp}(E_k) \times \text{Sp}(E_k) \to \text{Sp}(E_k)$$

and fixes the central element $-1 \in \tilde{\text{Sp}}(E_k)$. 
Denote by $\tilde{\Gamma}(E)$ the double cover of $\Gamma(E) \subset \text{Sp}(E_k)$ induced by the cover (3). Then the action (4) restricts to an action

$$\text{Ad} : \tilde{\Gamma}(E) \times \tilde{\Gamma}(E) \to \tilde{\Gamma}(E).$$

**Theorem 2.4.** For every $x \in \tilde{\Gamma}(E)$, there is an element $\check{g} \in \tilde{\Gamma}(E) \setminus \Gamma(E)$ such that $\text{Ad}_{\check{g}}x = x^{-1}$.

The most interesting case is when $\Gamma(E)$ is a symplectic group. Then Theorem 2.4 is proved for semisimple elements in [MVW, Proposition 4.1.8] and for general elements in [FS, Theorem 1.1].

Recall $L$ and $L^+$ from Section 2.2. Denote by $\tilde{\Gamma}_L(E)$ the double cover of $\Gamma_L(E) \subset \text{Sp}(E_k)$ induced by the cover (3). The action (4) restricts to an action

$$\text{Ad} : \tilde{\Gamma}_L^+(E) \times \tilde{\Gamma}_L(E) \to \tilde{\Gamma}_L(E).$$

**Theorem 2.5.** Assume that $L$ is totally isotropic. Then for every $x \in \tilde{\Gamma}_L(E)$, there is an element $\check{g} \in \tilde{\Gamma}_L^+(E) \setminus \Gamma_L(E)$ such that $\text{Ad}_{\check{g}}x = x^{-1}$.

### 3. Analytic results

Recall that $k$ is a local field of characteristic zero, $(A, \tau)$ is simple, and $E$ is an $\epsilon$-Hermitian $A$-module.

#### 3.1. Contragredient representations

Let $G$ denote one of the following groups:

- $\Gamma(E), \text{SO}(E)$ (when $\Lambda = A^+$ and $\epsilon = -1$), $\Gamma(E)$ (when $\epsilon = -1$),
- or one of the following groups when there is a totally isotropic rank one free $A$-submodule $L$ of $E$:

$$\Gamma_L(E), \quad \tilde{\Gamma}_L(E) \text{ (when } \epsilon = -1).$$

Let $\check{g}$ be respectively an element of

$$\tilde{\Gamma}(E) \setminus \Gamma(E), \quad \tilde{\Gamma}(E) \setminus \text{SO}(E), \quad \tilde{\Gamma}(E) \setminus \Gamma(E),$$

or

$$\tilde{\Gamma}_L^+(E) \setminus \Gamma_L(E), \quad \tilde{\Gamma}_L^+(E) \setminus \Gamma_L(E).$$

Here and as before, $L^+$ is a one-dimensional $A^+$-subspace which generates $L$ as an $A$-module. In all cases, we have a group automorphism $\text{Ad}_{\check{g}} : G \to G$.

**Theorem 3.1.** For every invariant (under the adjoint action of $G$) generalized function $f$ on $G$, one has that $f(\text{Ad}_{\check{g}}x) = f(x^{-1})$ (as generalized functions on $G$).
For the usual notion of generalized functions, see Sun [Section 2] (non-archimedean case) and JSZ [Section 2.1] (archimedean case), for example. By the localization principle of Bernstein and Zelevinsky ([BZ, Theorem 6.9]), Theorem 2.1-2.5 implies Theorem 3.1 in the non-archimedean case. In both archimedean and non-archimedean cases, Theorem 3.1 is implied by Theorems 3.2 and 3.3 of the next two subsections.

When \( k \) is non-archimedean, Theorem 3.1 implies that for every irreducible admissible smooth representation \( \pi \) of \( G \), its contragredient \( \pi^\vee \) is isomorphic to its twist \( x \mapsto \pi(\text{Ad}_{\tilde{g}}x) \). Certain archimedean analog of this fact also holds. But it is less satisfactory due to the lack of a suitable notion of “admissible representations” for non-reductive real groups.

### 3.2. Multiplicity one theorem I

Only in this subsection, assume that \( L \) is non-degenerate. (This is imposable in the symplectic case.) Denote by \( E_0 \) the orthogonal complement of \( L \) in \( E \).

Let \( G \) denote one of the following groups:

\[
U(E), \quad SO(E) \quad \text{(when } A = A^+ \text{ and } \epsilon = -1), \quad \tilde{U}(E) \quad \text{(when } \epsilon = -1),
\]

and let \( G_L \) denote its respective subgroup

\[
U(E_0), \quad SO(E_0), \quad \tilde{U}(E_0).
\]

When \( G = SO(E) \), let \( \tilde{g} \) be an element

\[
SO(E_0) \setminus SO(E_0) \subset O(E_0) \times \{\pm 1\} \subset O(E) \times \{\pm 1\} \supset G,
\]

and in all other cases, let \( \tilde{g} \) be an element of of

\[
\tilde{U}(E_0) \setminus U(E_0) = \tilde{U}_L^+(E) \setminus U_L(E) \subset \tilde{U}(E).
\]

As before, we have a group automorphism \( \text{Ad}_{\tilde{g}} : G \to G \).

**Theorem 3.2.** For every generalized function \( f \) on \( G \) which is invariant under the adjoint action of \( G_L \), one has that \( f(\text{Ad}_{\tilde{g}}x) = f(x^{-1}) \).

In the non-archimedean case, Theorem 3.2 is proved by Aizenbud-Gourevitch-Rallis-Schifffmann in [AGRS] (except for the case of special orthogonal groups, which is proved by Waldspurger in [Wald]). In the archimedean case, it is proved by Sun-Zhu in [SZ] (and independently by Aizenbud-Gourevitch in [AG] for general linear groups).

By Gelfand-Kazhdan criteria, Theorem 3.2 implies that \( (G, G_L) \) is a “multiplicity one pair” (see [GGP, AGRS, AG, SZ]). This multiplicity one theorem has been expected by Bernstein and Rallis since 1980’s.
3.3. **Multiplicity one theorem II.** Assume that $L$ is totally isotropic in this subsection. Let $L'$ be another totally isotropic rank one free $A$-submodule of $E$ which is dual to $L$ under the form $\langle \cdot, \cdot \rangle_E$. Then

$$E := L \oplus E_0 \oplus L',$$

where $E_0$ is the orthogonal complement of $L \oplus L'$ in $E$.

Let $G_L$ denote one of the following groups

$$U_L(E), \quad \tilde{U}_L(E) \text{ (when } \epsilon = -1).$$

Let $G_0$ denote its respective subgroup $U(E_0)$ or $\tilde{U}(E_0)$. Let $\tilde{g} = (g, -1) \in \tilde{U}(E_0) \setminus U(E_0)$. View it as an element of $\tilde{U}(E) \setminus U(E)$ by extending $g$ to a $\tau$-conjugate linear automorphism of $E$, preserving the decomposition (5) and fixing $L^+$ point-wise. Again, we have a group automorphism $\text{Ad}_{\tilde{g}} : G_L \to G_L$.

**Theorem 3.3.** For every generalized function $f$ on $G_L$ which is invariant under the adjoint action of $G_0$, one has that $f(\text{Ad}_{\tilde{g}}x) = f(x^{-1})$.

For a proof of Theorem 3.3 see [Sun, Dijk, SZ]. Similar to Theorem 3.2, Theorem 3.3 implies that the pair $(G_L, G_0)$ is a “multiplicity one pair”. This multiplicity one theorem was expected by Prasad, at least for symplectic groups ([Pras, Page 20]).

**Remarks:** In fact, the metaplectic cases of Theorem 3.2 and Theorem 3.3 were not treated in the literature. However, the available method works as well.

3.4. **The group** $S\tilde{O}(E) \ltimes E$. Assume that $A = A^+$ and $\epsilon = 1$. Let $S\tilde{O}(E)$ act on $E$ by $(g, \delta)v := gv$, and we form the semidirect product $S\tilde{O}(E) \ltimes E$. In general, the desired geometric property does not hold for this group. For example, if $E$ is split and has dimension 2, and $x \in E \subset SO(E) \ltimes E$ is a nonzero isotropic vector, then there is no $\tilde{g} \in (S\tilde{O}(E) \ltimes E) \setminus (SO(E) \ltimes E)$ such that $\tilde{g}x\tilde{g}^{-1} = x^{-1}$. However, the corresponding analytic result still holds:

**Theorem 3.4.** For every invariant (under the adjoint action of $SO(E) \ltimes E$) generalized function $f$ on $SO(E) \ltimes E$, and for every element of $\tilde{g}$ of $(S\tilde{O}(E) \ltimes E) \setminus (SO(E) \ltimes E)$, one has that $f(\tilde{g}x\tilde{g}^{-1}) = f(x^{-1})$.

This is much weaker than the following

**Theorem 3.5.** Let $\tilde{g} \in S\tilde{O}(E) \setminus SO(E)$. Then for every generalized function $f$ on $SO(E) \ltimes E$ which is invariant under the adjoint action of $SO(E)$, one has that $f(\tilde{g}x\tilde{g}^{-1}) = f(x^{-1})$.

Theorem 3.5 can be proved by using the methods and results of Aizenbud-Gourevitch-Rallis-Schiffmann and Sun-Zhu (cf. [Dijk] [Sun] [Wald] [SZ]).
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