COMPOSING GENERIC LINEARLY PERTURBED MAPPINGS
AND IMMersions/INJECTIONS

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Dedicated to Professor Takashi Nishimura on the occasion of his 60th birthday

Abstract. Let \( N \) (resp., \( U \)) be a manifold (resp., an open subset of \( \mathbb{R}^m \)). Let \( f : N \to U \) and \( F : U \to \mathbb{R}^\ell \) be an immersion and a \( C^\infty \) mapping, respectively. Generally, the composition \( F \circ f \) does not necessarily yield a mapping transverse to a given subfiber-bundle of \( J^1(N, \mathbb{R}^\ell) \). Nevertheless, in this paper, for any \( A^1 \)-invariant fiber, we show that composing generic linearly perturbed mappings of \( F \) and the given immersion \( f \) yields a mapping transverse to the subfiber-bundle of \( J^1(N, \mathbb{R}^\ell) \) with the given fiber. Moreover, we show a specialized transversality theorem on crossings of compositions of generic linearly perturbed mappings of a given mapping \( F : U \to \mathbb{R}^\ell \) and a given injection \( f : N \to U \). Furthermore, applications of the two main theorems are given.

1. Introduction

Throughout this paper, let \( \ell, m \) and \( n \) stand for positive integers. In this paper, unless otherwise stated, all manifolds and mappings belong to class \( C^\infty \) and all manifolds are without boundary. Let \( \pi : \mathbb{R}^m \to \mathbb{R}^\ell \), \( U \) and \( F : U \to \mathbb{R}^\ell \) be a linear mapping, an open subset of \( \mathbb{R}^m \) and a mapping, respectively. Set

\[
F_\pi = F + \pi.
\]

Here, the mapping \( \pi \) in \( F_\pi \) is restricted to \( U \).

Let \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \) be the space consisting of all linear mappings of \( \mathbb{R}^m \) into \( \mathbb{R}^\ell \). Remark that we have the natural identification \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell \). An \( n \)-dimensional manifold is denoted by \( N \). For a given mapping \( f : N \to U \), a property of mappings \( F_\pi \circ f : N \to \mathbb{R}^\ell \) will be said to be true for a generic mapping if there exists a subset \( \Sigma \) with Lebesgue measure zero of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \) such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma \), the mapping \( F_\pi \circ f : N \to \mathbb{R}^\ell \) has the property. In the case \( F = 0 \), by John Mather, for a given embedding \( f : N \to \mathbb{R}^m \), a generic mapping \( \pi \circ f : N \to \mathbb{R}^\ell \) \((m > \ell)\) is investigated in the celebrated paper \([10]\). The main theorem in \([10]\) yields many applications. On the other hand, in this paper, for a given immersion or a given injection \( f : N \to U \), a generic mapping \( F_\pi \circ f : N \to \mathbb{R}^\ell \) is investigated, where \( \ell \) is an arbitrary positive integer which may possibly satisfy \( m \leq \ell \).

The main purpose of this paper is to show two main theorems (Theorems \([1]\) and \([2]\) in Section \([2]\) and to give some of their applications. The first main theorem (Theorem \([1]\) as follows. Let \( f : N \to U \) (resp., \( F : U \to \mathbb{R}^\ell \)) be an immersion
Definition 1. Let $g : N \to P$ be a mapping. Then, generally, the composition $F \circ f$ does not necessarily yield a mapping transverse to a given subfiber-bundle of the jet bundle $J^1(N, \mathbb{R}^\ell)$. Nevertheless, Theorem 1 asserts that for any $\mathcal{A}^1$-invariant fiber, a generic mapping $F_\pi \circ f$ yields a mapping transverse to the subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with the given fiber. The second main theorem (Theorem 2) is a specialized transversality theorem on crossings of a generic mapping $F_\pi \circ f$, where $f : N \to U$ is a given injection and $F : U \to \mathbb{R}^\ell$ is a given mapping.

For a given immersion (resp., injection) $f : N \to U$, the following (1)-(4) (resp., (5)) are obtained as applications of Theorem 1 (resp., Theorem 2).

1. If $(n, \ell) = (n, 1)$, then a generic function $F_\pi \circ f : N \to \mathbb{R}$ is a Morse function.
2. If $(n, \ell) = (n, 2n - 1)$ and $n \geq 2$, then any singular point of a generic mapping $F_\pi \circ f : N \to \mathbb{R}^{2n-1}$ is a singular point of Whitney umbrella.
3. If $\ell \geq 2n$, then a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an immersion.
4. A generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ has corank at most $k$ singular points (for the definition of corank at most $k$ singular points, see Subsection 5.1), where $k$ is the maximum integer satisfying $(n - v + k)(\ell - v + k) \leq n \ (v = \min\{n, \ell\})$.
5. If $\ell > 2n$, then a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is injective.

Moreover, by combining the assertions (3) and (5), for a given embedding $f : N \to U$, the following assertion (6) is obtained.

6. If $\ell > 2n$ and $N$ is compact, then a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an embedding.

In Section 2 some standard definitions are reviewed, and the two main theorems (Theorems 1 and 2) are stated. Section 3 (resp., Section 4) is devoted to the proof of Theorem 1 (resp., Theorem 2). In Section 5 the assertions (1)-(6) above are shown. Moreover, in Section 6 as further applications, the two main theorems are adapted to quadratic mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$ of a special type called "generalized distance-squared mappings" (for the precise definition of generalized distance-squared mappings, see Section 6). Since some corollaries in this paper (the assertion (6) in Section 1 Corollary 7 in Section 5 and Corollary 9 in Section 6) are also obtained by using the main theorem in 4, which is an improvement of the main theorem in 10, for the sake of readers’ convenience, Section 7 explains the main theorems in 4 and 10 as an appendix.

2. Preliminaries and the Statements of Theorems 1 and 2

Let $N$ and $P$ be manifolds. Firstly, we recall the definition of transversality.

**Definition 1.** Let $W$ be a submanifold of $P$. Let $g : N \to P$ be a mapping.

1. We say that $g : N \to P$ is transverse to $W$ at $q$ if $g(q) \notin W$ or in the case of $g(q) \in W$, the following holds:
   
   \[ dg_q(T_q N) + T_{g(q)} W = T_{g(q)} P. \]

2. We say that $g : N \to P$ is transverse to $W$ if for any $q \in N$, the mapping $g$ is transverse to $W$ at $q$.

We say that $g : N \to P$ is $\mathcal{A}$-equivalent to $h : N \to P$ if there exist diffeomorphisms $\Phi : N \to N$ and $\Psi : P \to P$ such that $g = \Psi \circ h \circ \Phi^{-1}$. 

Let \( J^r(N, P) \) be the space of \( r \)-jets of mappings of \( N \) into \( P \). For a given mapping \( g : N \to P \), the mapping \( j^r g : N \to J^r(N, P) \) is defined by \( q \mapsto j^r g(q) \) (for details on the space \( J^r(N, P) \) or the mapping \( j^r g : N \to J^r(N, P) \), see for example, [3]).

For the statement and the proof of Theorem [1] it is sufficient to consider the case of \( r = 1 \) and \( P = \mathbb{R}^l \). Let \( \{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda} \) be a coordinate neighborhood system of \( N \). Let \( \Pi : J^1(N, \mathbb{R}^l) \to N \times \mathbb{R}^l \) be the natural projection defined by \( \Pi(j^1 g(q)) = (q, g(q)) \). Let \( \Phi_\lambda : \Pi^{-1}(U_\lambda \times \mathbb{R}^l) \to \varphi_\lambda(U_\lambda) \times \mathbb{R}^l \times J^1(n, \ell) \) be the homeomorphism defined by

\[
\Phi_\lambda \left( j^1 g(q) \right) = \left( \varphi_\lambda(q), g(q), j^1 \left( \psi_\lambda \circ g \circ \varphi_\lambda^{-1} \circ \tilde{\varphi}_\lambda \right)(0) \right),
\]

where \( J^1(n, \ell) = \{ j^1 g(0) \mid g : (\mathbb{R}^n, 0) \to (\mathbb{R}^l, 0) \} \) and \( \tilde{\varphi}_\lambda : \mathbb{R}^n \to \mathbb{R}^n \) (resp., \( \psi_\lambda : \mathbb{R}^m \to \mathbb{R}^m \)) is the translation defined by \( \tilde{\varphi}_\lambda(0) = \varphi_\lambda(g)(0) \) (resp., \( \psi_\lambda(g(q)) = 0 \)). Then, \( \{(\Pi^{-1}(U_\lambda \times \mathbb{R}^l), \Phi_\lambda)\}_{\lambda \in \Lambda} \) is a coordinate neighborhood system of \( J^1(N, \mathbb{R}^l) \).

A subset \( X \) of \( J^1(n, \ell) \) is said to be \( \mathcal{A}^1 \)-invariant if for any \( j^1 g(0) \in X \), and for any two germs of diffeomorphisms \( H : (\mathbb{R}^l, 0) \to (\mathbb{R}^l, 0) \) and \( h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \), we have \( j^1(H \circ g \circ h^{-1})(0) \in X \). Let \( X \) be an \( \mathcal{A}^1 \)-invariant submanifold of \( J^1(n, \ell) \). Set

\[
X(N, \mathbb{R}^l) = \bigcup_{\lambda \in \Lambda} \Phi_\lambda^{-1}(\varphi_\lambda(U_\lambda) \times \mathbb{R}^l \times X).
\]

Then, the set \( X(N, \mathbb{R}^l) \) is a subfiber-bundle of \( J^1(N, \mathbb{R}^l) \) with the fiber \( X \) such that

\[
\text{codim } X(N, \mathbb{R}^l) = \dim J^1(N, \mathbb{R}^l) - \dim X(N, \mathbb{R}^l) = \dim J^1(n, \ell) - \dim X = \text{codim } X.
\]

Then, the first main theorem in this paper is the following.

**Theorem 1.** Let \( N = \mathbb{R}^m \) be a manifold of dimension \( n \). Let \( f \) be an immersion of \( N \) into an open subset \( U \) of \( \mathbb{R}^m \). Let \( F : U \to \mathbb{R}^l \) be a mapping. If \( X \) is an \( \mathcal{A}^1 \)-invariant submanifold of \( J^1(n, \ell) \), then there exists a subset \( \Sigma \) with Lebesgue measure zero of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^l) \) such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^l) - \Sigma \), the mapping \( j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^l) \) is transverse to the submanifold \( X(N, \mathbb{R}^l) \).

Now, in order to state the second main theorem (Theorem 2), we will prepare some definitions. Set \( N^{(s)} = \{(q_1, q_2, \ldots, q_s) \in N^s \mid q_i \neq q_j \ (i \neq j)\} \). Notice that \( N^{(s)} \) is an open submanifold of \( N^s \). For any mapping \( g : N \to P \), let \( g^{(s)} : N^{(s)} \to P^s \) be the mapping defined by

\[
g^{(s)}(q_1, q_2, \ldots, q_s) = (g(q_1), g(q_2), \ldots, g(q_s)).
\]

Set \( \Delta_s = \{(y, \ldots, y) \in P^s \mid y \in P\} \). It is clearly seen that \( \Delta_s \) is a submanifold of \( P^s \) such that

\[
\text{codim } \Delta_s = \dim P^s - \dim \Delta_s = (s - 1) \dim P.
\]

**Definition 2.** Let \( g \) be a mapping of \( N \) into \( P \). Then, \( g \) is called a mapping with normal crossings if for any positive integer \( s \ (s \geq 2) \), the mapping \( g^{(s)} : N^{(s)} \to P^s \) is transverse to the submanifold \( \Delta_s \).
For any injection \( f : N \rightarrow \mathbb{R}^m \), set

\[
s_f = \max \left\{ s \mid \forall (q_1, q_2, \ldots, q_s) \in N^{(s)}, \dim \sum_{i=2}^{s} \mathbb{R}f(q_i)f(q_1) = s - 1 \right\}.
\]

Since the mapping \( f \) is injective, we get \( 2 \leq s_f \). Since \( f(q_1), f(q_2), \ldots, f(q_{s_f}) \) are points of \( \mathbb{R}^m \), it follows that \( s_f \leq m + 1 \). Thus, we have

\[
2 \leq s_f \leq m + 1.
\]

Furthermore, in the following, for a set \( X \), we denote the number of its elements (or its cardinality) by \( |X| \). Then, the second main theorem in this paper is the following.

**Theorem 2.** Let \( N \) be a manifold of dimension \( n \). Let \( f \) be an injection of \( N \) into an open subset \( U \) of \( \mathbb{R}^m \). Let \( F : U \rightarrow \mathbb{R}^\ell \) be a mapping. Then, there exists a subset \( \Sigma \) of \( L(\mathbb{R}^m, \mathbb{R}^\ell) \) with Lebesgue measure zero such that for any \( \pi \in L(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma \), and for any \( s \) \((2 \leq s \leq s_f)\), the mapping \((F_\pi \circ f)^{(s)}: N^{(s)} \rightarrow (\mathbb{R}^\ell)^s\) is transverse to the submanifold \( \Delta_s \). Moreover, if the mapping \( F_\pi \) satisfies that \(|F_\pi^{-1}(y)| \leq s_f\) for any \( y \in \mathbb{R}^\ell \), then \( F_\pi \circ f : N \rightarrow \mathbb{R}^\ell \) is a mapping with normal crossings.

The following well known result is important for the proofs of Theorems 1 and 2

**Lemma 1** ([1], [10]). Let \( N, P, Z \) be manifolds, and let \( W \) be a submanifold of \( P \). Let \( \Gamma : N \times Z \rightarrow P \) be a mapping. If \( \Gamma \) is transverse to \( W \), then there exists a subset \( \Sigma \) of \( Z \) with Lebesgue measure zero such that for any \( p \in Z - \Sigma \), the mapping \( \Gamma_p : N \rightarrow P \) is transverse to \( W \), where \( \Gamma_p(q) = \Gamma(q, p) \).

**Remark 1.**

1. We explain the advantage that the domain of the mapping \( F \) is an arbitrary open set. Suppose that \( U = \mathbb{R} \). Let \( F : \mathbb{R} \rightarrow \mathbb{R} \) be the mapping defined by \( x \mapsto |x| \). Since \( F \) is not differentiable at \( x = 0 \), we cannot apply Theorems 1 and 2 to the mapping \( F : \mathbb{R} \rightarrow \mathbb{R} \).

2. On the other hand, if \( U = \mathbb{R} - \{0\} \), then Theorems 1 and 2 can be applied to the restriction \( F|_U \).

3. There is a case of \( s_f = 3 \) as follows. If \( n + 1 \leq m \), \( N = S^n \) and \( f : S^n \rightarrow \mathbb{R}^m \) is the inclusion \( f(x) = (x, 0, \ldots, 0) \), then it is easily seen that \( s_f = 3 \). Indeed, suppose that there exists a point \((q_1, q_2, q_3) \in (S^n)^{(3)}\) such that \( \sum_{i=2}^{3} \mathbb{R}f(q_i)f(q_1) = 1 \). Then, since the number of the intersections of \( f(S^n) \) and a straight line of \( \mathbb{R}^m \) is at most two, this contradicts the assumption. Thus, we get \( s_f \geq 3 \). From \( S^1 \times \{0\} \subset f(S^n) \), it follows that \( s_f < 4 \), where \( 0 = (0, \ldots, 0) \). Hence, we have \( s_f = 3 \).

4. The essential idea for the proofs of Theorems 1 and 2 is to apply Lemma 1 and it is almost similar to the idea of the proofs of main results in [8]. Nevertheless, the two main theorems in this paper are drastically improved. As an effect of the improvement, many applications are obtained by the two main theorems (for the applications, see Sections 5 and 6).
3. Proof of Theorem \[ \text{[1]} \]

Let \((\alpha_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}\) be a representing matrix of a linear mapping \(\pi : \mathbb{R}^m \to \mathbb{R}^\ell\). Set \(F_\alpha = F_{\pi}\), and we have

\[
F_\alpha(x) = \left( F_1(x) + \sum_{j=1}^{m} \alpha_{1j} x_j, F_2(x) + \sum_{j=1}^{m} \alpha_{2j} x_j, \ldots, F_\ell(x) + \sum_{j=1}^{m} \alpha_{\ell j} x_j \right),
\]

where \(F = (F_1, F_2, \ldots, F_\ell)\), \(\alpha = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1m}, \ldots, \alpha_{\ell 1}, \alpha_{\ell 2}, \ldots, \alpha_{\ell m}) \in (\mathbb{R}^m)^\ell\) and \(x = (x_1, x_2, \ldots, x_m)\). For a given immersion \(f : N \to U\), the mapping \(F_\alpha \circ f : N \to \mathbb{R}^\ell\) is given as follows:

\[
F_\alpha \circ f = \left( F_1 \circ f + \sum_{j=1}^{m} \alpha_{1j} f_j, F_2 \circ f + \sum_{j=1}^{m} \alpha_{2j} f_j, \ldots, F_\ell \circ f + \sum_{j=1}^{m} \alpha_{\ell j} f_j \right),
\]

where \(f = (f_1, f_2, \ldots, f_m)\). Since we have the natural identification \(\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell\), in order to prove Theorem \[ \text{[1]} \] it is sufficient to show that there exists a subset \(\Sigma\) with Lebesgue measure zero of \((\mathbb{R}^m)^\ell\) such that for any \(\alpha \in (\mathbb{R}^m)^\ell - \Sigma\), the mapping \(j^1(F_\alpha \circ f) : N \to J^1(N, \mathbb{R}^\ell)\) is transverse to the given submanifold \(X(N, \mathbb{R}^\ell)\).

Now, let \(\Gamma : N \times (\mathbb{R}^m)^\ell \to J^1(N, \mathbb{R}^\ell)\) be the mapping defined by

\[
\Gamma(q, \alpha) = j^1(F_\alpha \circ f)(q).
\]

If the mapping \(\Gamma\) is transverse to the submanifold \(X(N, \mathbb{R}^\ell)\), then from Lemma \[ \text{[1]} \] it follows that there exists a subset \(\Sigma\) of \((\mathbb{R}^m)^\ell\) with Lebesgue measure zero such that for any \(\alpha \in (\mathbb{R}^m)^\ell - \Sigma\), the mapping \(\Gamma_\alpha : N \to J^1(N, \mathbb{R}^\ell)\) \((\Gamma_\alpha = j^1(F_\alpha \circ f))\) is transverse to the submanifold \(X(N, \mathbb{R}^\ell)\). Thus, in order to finish the proof of Theorem \[ \text{[1]} \] it is sufficient to show that if \(\Gamma(\tilde{q}, \tilde{\alpha}) \in X(N, \mathbb{R}^\ell)\), then the following holds:

\[
d\Gamma(\tilde{q}, \tilde{\alpha})(T_{(\tilde{q}, \tilde{\alpha})}(N \times (\mathbb{R}^m)^\ell)) + T_{\Gamma(\tilde{q}, \tilde{\alpha})}X(N, \mathbb{R}^\ell) = T_{\Gamma(\tilde{q}, \tilde{\alpha})}J^1(N, \mathbb{R}^\ell).
\]

As in Section \[ \text{[2]} \] let \(\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}\) (resp., \(\{(\Pi^{-1}(U_\lambda \times \mathbb{R}^\ell), \Phi_\lambda)\}_{\lambda \in \Lambda}\)) be a coordinate neighborhood system of \(N\) (resp., \(J^1(N, \mathbb{R}^\ell)\)). There exists a coordinate neighborhood \((U_\chi \times (\mathbb{R}^m)^\ell, \varphi_\chi \times id)\) containing the point \((\tilde{q}, \tilde{\alpha})\) of \(N \times (\mathbb{R}^m)^\ell\), where \(id\) is the identity mapping of \((\mathbb{R}^m)^\ell\) into \((\mathbb{R}^m)^\ell\), and the mapping \(\varphi_\chi \times id : U_\chi \times (\mathbb{R}^m)^\ell \to \varphi_\chi(U_\chi) \times (\mathbb{R}^m)^\ell \subset \mathbb{R}^n \times (\mathbb{R}^m)^\ell\) is defined by \((\varphi_\chi \times id)(q, \alpha) = (\varphi_\chi(q), id(\alpha))\). There exists a coordinate neighborhood \((\Pi^{-1}(U_\tilde{\chi} \times \mathbb{R}^\ell), \Phi_\tilde{\chi})\) containing the point \(\Gamma(\tilde{q}, \tilde{\alpha})\) of \(J^1(N, \mathbb{R}^\ell)\). Let \(t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n\) be a local coordinate on \(\varphi_\chi(U_\chi)\).
containing $\varphi_\lambda^{-1}(\bar{q})$. Then, the mapping $\Gamma$ is locally given by the following:

$$(\Phi_\lambda \circ \Gamma \circ (\varphi_\lambda \times id)^{-1})(t, \alpha) = (\Phi_\lambda \circ j^1(F_\alpha \circ f) \circ \varphi_\lambda^{-1})(t)$$

$$= \left( t, (F_\alpha \circ f \circ \varphi_\lambda^{-1})(t), \frac{\partial (F_{\alpha,1} \circ f \circ \varphi_\lambda^{-1})}{\partial t_1}(t), \frac{\partial (F_{\alpha,1} \circ f \circ \varphi_\lambda^{-1})}{\partial t_2}(t), \ldots, \frac{\partial (F_{\alpha,1} \circ f \circ \varphi_\lambda^{-1})}{\partial t_n}(t), \frac{\partial (F_{\alpha,2} \circ f \circ \varphi_\lambda^{-1})}{\partial t_1}(t), \frac{\partial (F_{\alpha,2} \circ f \circ \varphi_\lambda^{-1})}{\partial t_2}(t), \ldots, \frac{\partial (F_{\alpha,2} \circ f \circ \varphi_\lambda^{-1})}{\partial t_n}(t), \ldots \right)$$

$$= \left( t, (F_\alpha \circ f \circ \varphi_\lambda^{-1})(t), \frac{\partial F_1 \circ \bar{f}}{\partial t_1}(t) + \sum_{j=1}^m \alpha_{1j} \frac{\partial \bar{f}_j}{\partial t_1}(t) + \frac{\partial F_1 \circ \bar{f}}{\partial t_2}(t) + \sum_{j=1}^m \alpha_{1j} \frac{\partial \bar{f}_j}{\partial t_2}(t) + \ldots + \frac{\partial F_\ell \circ \bar{f}}{\partial t_1}(t) + \sum_{j=1}^m \alpha_{\ell j} \frac{\partial \bar{f}_j}{\partial t_1}(t), \frac{\partial F_2 \circ \bar{f}}{\partial t_1}(t) + \sum_{j=1}^m \alpha_{2j} \frac{\partial \bar{f}_j}{\partial t_1}(t) + \frac{\partial F_2 \circ \bar{f}}{\partial t_2}(t) + \sum_{j=1}^m \alpha_{2j} \frac{\partial \bar{f}_j}{\partial t_2}(t) + \ldots + \frac{\partial F_\ell \circ \bar{f}}{\partial t_1}(t) + \sum_{j=1}^m \alpha_{\ell j} \frac{\partial \bar{f}_j}{\partial t_1}(t), \ldots \right)$$

where $F_\alpha = (F_{\alpha,1}, F_{\alpha,2}, \ldots, F_{\alpha,\ell})$ and $\bar{f} = (\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_m) = (f_1 \circ \varphi_\lambda^{-1}, f_2 \circ \varphi_\lambda^{-1}, \ldots, f_m \circ \varphi_\lambda^{-1}) = f \circ \varphi_\lambda^{-1}$. The Jacobian matrix of the mapping $\Gamma$ at $(\bar{q}, \bar{\alpha})$ is the following:

$$J\Gamma(\bar{q}, \bar{\alpha}) = \begin{pmatrix} E_n & 0 & \cdots & 0 \\ * & \ddots & \cdots & * \\ * & \ddots & \ddots & \ddots \\ * & \ddots & \ddots & \ddots \end{pmatrix}$$

where $E_n$ is the $n \times n$ unit matrix and $Jf_{\bar{q}}$ is the Jacobian matrix of the mapping $f$ at $\bar{q}$. Note that $t(Jf_{\bar{q}})$ is the transpose of the matrix $Jf_{\bar{q}}$ and that there are $\ell$ copies of $t(Jf_{\bar{q}})$ in the above description of $J\Gamma(\bar{q}, \bar{\alpha})$. Since $X(N, \mathbb{R}^\ell)$ is a subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with the fiber $X$, it is clear that in order to show (3.3), it suffices to
prove that the matrix $M_1$ given below has rank $n + \ell + n\ell$:

$$M_1 = \begin{pmatrix}
E_{n+\ell} & * & \cdots & * \\
0 & (Jf_{\tilde{q}}) & 0 & \cdots \\
0 & 0 & \ddots & \cdots \\
(t,\alpha) & (\tilde{\varphi}_\lambda(q),\tilde{a})
\end{pmatrix},$$

where $E_{n+\ell}$ is the $(n + \ell) \times (n + \ell)$ unit matrix. Note that there are $\ell$ copies of $(Jf_{\tilde{q}})$ in the above description of $M_1$. Notice that for any $i$ ($1 \leq i \leq n\ell$), the $(n + \ell + i)$-th column vector of $M_1$ coincides with the $(n + i)$-th column vector of $Jf_{\tilde{q},\tilde{a}}$. Since the mapping $f$ is an immersion ($n \leq m$), we have that the rank of the matrix $M_1$ is equal to $n + \ell + n\ell$. Hence, we have (3.3). \hfill \Box

4. PROOF OF THEOREM 2

By the same method as in the proof of Theorem 1 set $F_\alpha = F_\tau$, where $F_\alpha$ is given by (3.1) in Section 3. For a given injection $f : N \to U$, the mapping $F_\alpha \circ f : N \to \mathbb{R}^\ell$ is given by the same expression as (3.2). Since we have the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$, in order to show that there exists a subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, and for any $s$ ($2 \leq s \leq s_f$), the mapping $(F_\tau \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to the submanifold $\Delta_s$, it is sufficient to show that there exists a subset $\Sigma$ of $(\mathbb{R}^m)^\ell$ with Lebesgue measure zero such that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$, and for any $s$ ($2 \leq s \leq s_f$), the mapping $(F_\tau \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to $\Delta_s$.

Now, let $s$ be a positive integer satisfying $2 \leq s \leq s_f$. Let $\Gamma : N^{(s)} \times (\mathbb{R}^m)^\ell \to (\mathbb{R}^\ell)^s$ be the mapping defined by

$$\Gamma(q_1, q_2, \ldots, q_s, \alpha) = ((F_\alpha \circ f)(q_1), (F_\alpha \circ f)(q_2), \ldots, (F_\alpha \circ f)(q_s)).$$

If for any positive integer $s$ ($2 \leq s \leq s_f$), the mapping $\Gamma$ is transverse to $\Delta_s$, then from Lemma 1 it follows that for any positive integer $s$ ($2 \leq s \leq s_f$), there exists a subset $\Sigma_s$ of $(\mathbb{R}^m)^\ell$ with Lebesgue measure zero such that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma_s$, the mapping $\Gamma_\alpha : N^{(s)} \to (\mathbb{R}^\ell)^s$ ($\Gamma_\alpha = (F_\alpha \circ f)^{(s)}$) is transverse to $\Delta_s$. Then, set $\Sigma = \bigcup_{s=2}^{s_f} \Sigma_s$. It is clearly seen that $\Sigma$ is a subset of $(\mathbb{R}^m)^\ell$ with Lebesgue measure zero. Therefore, it follows that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$, and for any $s$ ($2 \leq s \leq s_f$), the mapping $\Gamma_\alpha : N^{(s)} \to (\mathbb{R}^\ell)^s$ ($\Gamma_\alpha = (F_\alpha \circ f)^{(s)}$) is transverse to $\Delta_s$.

Hence, for the proof, it is sufficient to show that for any positive integer $s$ ($2 \leq s \leq s_f$), if $\Gamma(\tilde{q}, \tilde{a}) \in \Delta_s$ ($\tilde{q} = (\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_s)$), then the following holds:

$$d\Gamma_{(\tilde{q},\tilde{a})}(T_{(\tilde{q},\tilde{a})}(N^{(s)} \times (\mathbb{R}^m)^\ell)) + T_{\Gamma(\tilde{q},\tilde{a})}\Delta_s = T_{\Gamma(\tilde{q},\tilde{a})}(\mathbb{R}^\ell)^s. \quad (4.1)$$

Let $\{(\varphi_\lambda)\}_{\lambda \in \Lambda}$ be a coordinate neighborhood system of $N$. There exists a coordinate neighborhood $(U_{\lambda_1} \times U_{\lambda_2} \times \cdots \times U_{\lambda_s} \times (\mathbb{R}^m)^\ell, \varphi_{\lambda_1} \times \varphi_{\lambda_2} \times \cdots \times \varphi_{\lambda_s} \times \text{id})$ containing the point $(\tilde{q},\tilde{a})$ of $N^{(s)} \times (\mathbb{R}^m)^\ell$, where $\text{id}$ is the identity mapping of $(\mathbb{R}^m)^\ell$ into $(\mathbb{R}^m)^\ell$, and the mapping $\varphi_{\lambda_1} \times \varphi_{\lambda_2} \times \cdots \times \varphi_{\lambda_s} \times \text{id} : U_{\lambda_1} \times U_{\lambda_2} \times \cdots \times U_{\lambda_s} \times (\mathbb{R}^m)^\ell \to (\mathbb{R}^m)^s \times (\mathbb{R}^m)^\ell$ is defined by $(\varphi_{\lambda_1} \times \varphi_{\lambda_2} \times \cdots \times \varphi_{\lambda_s} \times \text{id})(q_1, q_2, \ldots, q_s, \alpha) = ((\varphi_{\lambda_1}(q_1), \varphi_{\lambda_2}(q_2), \ldots, \varphi_{\lambda_s}(q_s), \text{id}(\alpha)))$. Let $t_i = (t_{i1}, t_{i2}, \ldots, t_{in})$ be a local coordinate around $\varphi_{\lambda_i}(\tilde{q}_i)$ ($1 \leq i \leq s$). Then, the mapping $\Gamma$ is locally given by the
following:

$$
\Gamma \circ \left( \varphi_{\lambda_1} \times \varphi_{\lambda_2} \times \ldots \times \varphi_{\lambda_s} \times \text{id} \right)^{-1}(t_1, t_2, \ldots, t_s, \alpha)
$$

$$
= \left( (F_\alpha \circ f \circ \varphi_{\lambda_1}^{-1})(t_1), (F_\alpha \circ f \circ \varphi_{\lambda_2}^{-1})(t_2), \ldots, (F_\alpha \circ f \circ \varphi_{\lambda_s}^{-1})(t_s) \right)
$$

$$
= \left( F_1 \circ \tilde{f}(t_1) + \sum_{j=1}^{m} \alpha_{1j} \tilde{f}_j(t_1), F_2 \circ \tilde{f}(t_2) + \sum_{j=1}^{m} \alpha_{2j} \tilde{f}_j(t_2), \ldots, F_\ell \circ \tilde{f}(t_\ell) + \sum_{j=1}^{m} \alpha_{\ell j} \tilde{f}_j(t_\ell), \right)
$$

$$
F_1 \circ \tilde{f}(t_1) + \sum_{j=1}^{m} \alpha_{1j} \tilde{f}_j(t_1), F_2 \circ \tilde{f}(t_2) + \sum_{j=1}^{m} \alpha_{2j} \tilde{f}_j(t_2), \ldots, F_\ell \circ \tilde{f}(t_\ell) + \sum_{j=1}^{m} \alpha_{\ell j} \tilde{f}_j(t_\ell), \right)
$$

where $\tilde{f}(t_i) = (\tilde{f}_1(t_i), \tilde{f}_2(t_i), \ldots, \tilde{f}_m(t_i)) = (f_1 \circ \varphi_{\lambda_1}^{-1}(t_i), f_2 \circ \varphi_{\lambda_2}^{-1}(t_i), \ldots, f_m \circ \varphi_{\lambda_s}^{-1}(t_i))$ ($1 \leq i \leq s$). For simplicity, set $t = (t_1, t_2, \ldots, t_s)$ and $z = (\varphi_{\lambda_1} \times \varphi_{\lambda_2} \times \ldots \times \varphi_{\lambda_s})(\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_s)$.

The Jacobian matrix of the mapping $\Gamma$ at $(\tilde{q}, \tilde{a})$ is the following:

$$
JT_{(\tilde{q}, \tilde{a})} = \begin{pmatrix}
* & B(t_1) \\
* & B(t_2) \\
* & \vdots \\
* & B(t_s)
\end{pmatrix}_{(t, \alpha) = (\tilde{z}, \tilde{\alpha})}
$$

where

$$
B(t_i) = \begin{pmatrix}
b(t_i) & 0 \\
0 & \ddots \\
0 & b(t_i)
\end{pmatrix}_{\ell \text{ rows}}
$$

and $b(t_i) = (\tilde{f}_1(t_i), \tilde{f}_2(t_i), \ldots, \tilde{f}_m(t_i))$. By the construction of $T_{\Gamma(\tilde{q}, \tilde{a})} \Delta_s$, in order to show (4.1), it is sufficient to show that the rank of the following matrix $M_2$ is equal to $\ell s$:

$$
M_2 = \begin{pmatrix}
E_\ell & B(t_1) \\
E_\ell & B(t_2) \\
\vdots & \vdots \\
E_\ell & B(t_s)
\end{pmatrix}_{t = \tilde{z}}
$$

There exists an $\ell s \times \ell s$ regular matrix $Q_1$ such that

$$
Q_1 M_2 = \begin{pmatrix}
E_\ell & B(t_1) \\
0 & B(t_2) - B(t_1) \\
\vdots & \vdots \\
0 & B(t_s) - B(t_1)
\end{pmatrix}_{t = \tilde{z}}
$$
There exists an \((\ell + m\ell) \times (\ell + m\ell)\) regular matrix \(Q_2\) such that

\[
Q_1M_2Q_2 = \begin{pmatrix}
E_\ell & 0 \\
0 & B(t_2) - B(t_1) \\
\vdots & \vdots \\
0 & B(t_s) - B(t_1)
\end{pmatrix}_{t=z}
\]

where \(\vec{f}(t_1)f(t_2)\) and \(\vec{f}(t_1)f(t_s)\) are columns of the matrix \(Q_2\), \((2 \leq i \leq s)\) and \(t = z\). From \(s - 1 \leq s_f - 1\) and the definition of \(s_f\), it follows that

\[
\dim \sum_{i=2}^{s} \mathbb{R}\vec{f}(t_1)f(t_i) = s - 1,
\]

where \(t = z\). Thus, by the construction of the matrix \(Q_1M_2Q_2\) and \(s - 1 \leq m\), we have that the rank of the matrix \(Q_1M_2Q_2\) is equal to \(\ell s\). Hence, the rank of the matrix \(M_2\) must be equal to \(\ell s\). Therefore, we have (4.1). Thus, there exists a subset \(\Sigma\) of \(\mathcal{L}(\mathbb{R}^m, \mathbb{R}\ell)\) with Lebesgue measure zero such that for any \(\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}\ell) - \Sigma\), and for any \(s\) \((2 \leq s \leq s_f)\), the mapping \((F_{\pi} \circ f)^{(s)} : N^{(s)} \rightarrow (\mathbb{R}\ell)^s\) is transverse to the submanifold \(\Delta_s\).

Moreover, suppose that the mapping \(F_{\pi}\) satisfies that \(|F_{\pi}^{-1}(y)| \leq s_f\) for any \(y \in \mathbb{R}\ell\). Since \(f : N \rightarrow \mathbb{R}^m\) is injective, it follows that \(|(F_{\pi} \circ f)^{-1}(y)| \leq s_f\) for any \(y \in \mathbb{R}\ell\). Hence, it follows that for any positive integer \(s\) with \(s \geq s_f + 1\), we have \((F_{\pi} \circ f)^{(s)}(N^{(s)}) \cap \Delta_s = \emptyset\). Namely, for any positive integer \(s\) with \(s \geq s_f + 1\), the mapping \((F_{\pi} \circ f)^{(s)}\) is transverse to \(\Delta_s\). Thus, \(F_{\pi} \circ f : N \rightarrow \mathbb{R}\ell\) is a mapping with normal crossings.

\[\square\]

5. Applications of Theorems 1 and 2

In Subsection 5.1 (resp., Subsection 5.2), applications of Theorem 1 (resp., Theorem 2) are stated and proved. In Subsection 5.2, applications obtained by combining Theorems 1 and 2 are also given.
5.1. Applications of Theorem 1

Set

\[ \Sigma^k = \{ j^1 g(0) \in J^1(n, \ell) \mid \text{corank } Jg(0) = k \} , \]

where \( \text{corank } Jg(0) = \min\{n, \ell\} - \text{rank } Jg(0) \) and \( k = 1, 2, \ldots, \min\{n, \ell\} \). Then, \( \Sigma^k \) is an \( A^1 \)-invariant submanifold of \( J^1(n, \ell) \). Set

\[ \Sigma^k(N, \mathbb{R}^\ell) = \bigcup_{\lambda \in \Lambda} \Phi^{-1}_\lambda (\varphi_\lambda(U_{\lambda}) \times \mathbb{R}^\ell \times \Sigma^k) , \]

where the mappings \( \Phi_\lambda \) and \( \varphi_\lambda \) are as defined in Section 2. Then, the set \( \Sigma^k(N, \mathbb{R}^\ell) \) is a subfiber-bundle of \( J^1(N, \mathbb{R}^\ell) \) with the fiber \( \Sigma^k \) such that

\[ \text{codim } \Sigma^k(N, \mathbb{R}^\ell) = \dim J^1(N, \mathbb{R}^\ell) - \dim \Sigma^k(N, \mathbb{R}^\ell) = (n - v + k)(\ell - v + k) , \]

where \( v = \min\{n, \ell\} \). (For details on \( \Sigma^k \) and \( \Sigma^k(N, \mathbb{R}^\ell) \), see for example [3], pp. 60–61).

As applications of Theorem 1, we have the following Proposition 1, Corollaries 1, 2, 3 and 4.

**Proposition 1.** Let \( N \) be a manifold of dimension \( n \). Let \( f \) be an immersion of \( N \) into an open subset \( U \) of \( \mathbb{R}^m \). Let \( F : U \to \mathbb{R}^\ell \) be a mapping. Then, there exists a subset \( \Sigma \) of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma \), the mapping \( j^1(F_{\pi} \circ f) : N \to J^1(N, \mathbb{R}^\ell) \) is transverse to the submanifold \( \Sigma^k(N, \mathbb{R}^\ell) \) for any positive integer \( k \) satisfying \( 1 \leq k \leq v \). Especially, in the case of \( \ell \geq 2 \), we have \( k_0 + 1 \leq v \) and it follows that the mapping \( j^1(F_{\pi} \circ f) \) satisfies that \( j^1(F_{\pi} \circ f)(N) \nabla \Sigma^k(N, \mathbb{R}^\ell) = \emptyset \) for any positive integer \( k \) satisfying \( k_0 + 1 \leq k \leq v \), where \( k_0 \) is the maximum integer satisfying \( (n - v + k_0)(\ell - v + k_0) \leq n \) \((v = \min\{n, \ell\})\).

**Proof.** By Theorem 1, for any positive integer \( k \) satisfying \( 1 \leq k \leq v \), there exists a subset \( \Sigma_k \) of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma_k \), the mapping \( j^1(F_{\pi} \circ f) : N \to J^1(N, \mathbb{R}^\ell) \) is transverse to \( \Sigma^k(N, \mathbb{R}^\ell) \). Set \( \Sigma = \bigcup_{k=1}^v \Sigma_k \). Then, it is clearly seen that \( \Sigma \) is a subset of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \) with Lebesgue measure zero. Hence, it follows that there exists a subset \( \Sigma \) of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma \), the mapping \( j^1(F_{\pi} \circ f) : N \to J^1(N, \mathbb{R}^\ell) \) is transverse to the submanifold \( \Sigma^k(N, \mathbb{R}^\ell) \) for any positive integer \( k \) satisfying \( 1 \leq k \leq v \).

Now, we will consider the case of \( \ell \geq 2 \). First, we will show that \( k_0 + 1 \leq v \) in the case. Suppose that \( v \leq k_0 \). Then, by \( (n - v + k_0)(\ell - v + k_0) \leq n \), we have \( n\ell \leq n \). This contradicts the assumption \( \ell \geq 2 \).

Secondly, we will show that in the case of \( \ell \geq 2 \), the mapping \( j^1(F_{\pi} \circ f) : N \to J^1(N, \mathbb{R}^\ell) \) satisfies that \( j^1(F_{\pi} \circ f)(N) \nabla \Sigma^k(N, \mathbb{R}^\ell) = \emptyset \) for any positive integer \( k \) satisfying \( k_0 + 1 \leq k \leq v \). Suppose that there exist a positive integer \( k \) \((k_0 + 1 \leq k \leq v) \) and a point \( q \in N \) such that \( j^1(F_{\pi} \circ f)(q) \in \Sigma^k(N, \mathbb{R}^\ell) \). Since the mapping \( j^1(F_{\pi} \circ f) : N \to J^1(N, \mathbb{R}^\ell) \) is transverse to \( \Sigma^k(N, \mathbb{R}^\ell) \) at the point \( q \), the following holds:

\[ d(j^1(F_{\pi} \circ f))(q)T_q N + T_j^1(F_{\pi} \circ f)(q)\Sigma^k(N, \mathbb{R}^\ell) = T(j^1(F_{\pi} \circ f))(q)J^1(N, \mathbb{R}^\ell) . \]
Corollary 2. There exists a subset \( H \) of \( \pi \Sigma \) such that for any mapping \( F \) in Proposition 1. Hence, there exists a subset \( \Sigma \) of \( J \mathbb{R} \) and the mapping \( F \pi \) satisfies that for any \( n \in \mathbb{R}^\ell \). Since the given integer \( k_0 \) is the maximum integer satisfying \( n \geq (n-v+k)(\ell-v+k) \), it follows that \( k \leq k_0 \). This contradicts the assumption \( k_0 + 1 \leq k \).

**Remark 2.**
1. In Proposition \( \Pi \) by \((n-v+k)(\ell-v+k) \leq n \), it is clearly seen that \( k_0 \geq 0 \).
2. In Proposition \( \Pi \) in the case of \( \ell = 1 \), we have \( k_0 + 1 > v \). Indeed, in the case, by \( v = 1 \), we get \((n-1+k_0)k_0 \leq n \). Hence, we have \( k_0 = 1 \).

A mapping \( g : N \to \mathbb{R} \) is called a **Morse function** if all of the singularities of the mapping \( g \) are nondegenerate (for details on Morse functions, see for example, \( [3] \), p.63). In the case of \((n, \ell) = (n, 1) \), we have the following.

**Corollary 1.** Let \( N \) be a manifold of dimension \( n \). Let \( f \) be an immersion of \( N \) into an open subset \( U \) of \( \mathbb{R}^m \). Let \( F : U \to \mathbb{R} \) be a mapping. Then, there exists a subset \( \Sigma \) of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) - \Sigma \), the mapping \( F \pi \circ f : N \to \mathbb{R} \) is a Morse function.

**Proof.** By Proposition \( \Pi \) there exists a subset \( \Sigma \) with Lebesgue measure zero of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}) \) such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) - \Sigma \), the mapping \( j^1(F \pi \circ f) : N \to J^1(N, \mathbb{R}) \) is transverse to the submanifold \( \Sigma^1(N, \mathbb{R}) \). Hence, if \( q \in N \) is a singular point of the mapping \( F \pi \circ f \), then the point \( q \) is nondegenerate. \( \square \)

For a given mapping \( g : N \to \mathbb{R}^{2n-1} (n \geq 2) \), a singular point \( q \in N \) is called a **singular point of Whitney umbrella** if there exist two germs of diffeomorphisms \( H : (\mathbb{R}^{2n-1}, g(q)) \to (\mathbb{R}^{2n-1}, 0) \) and \( h : (N, q) \to (\mathbb{R}^n, 0) \) such that \( h \circ g \circ h^{-1} : (x_1, x_2, \ldots, x_n) = (x_1^2, x_1 x_2, \ldots, x_1 x_n, x_2, \ldots, x_n) \) where \((x_1, x_2, \ldots, x_n)\) is a local coordinate around the point \( h(q) = 0 \in \mathbb{R}^n \). In the case of \((n, \ell) = (n, 2n-1) (n \geq 2) \), we have the following.

**Corollary 2.** Let \( N \) be a manifold of dimension \( n \) \((n \geq 2) \). Let \( f \) be an immersion of \( N \) into an open subset \( U \) of \( \mathbb{R}^m \). Let \( F : U \to \mathbb{R}^{2n-1} \) be a mapping. Then, there exists a subset \( \Sigma \) with Lebesgue measure zero of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1}) \) such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1}) - \Sigma \), any singular point of the mapping \( F \pi \circ f : N \to \mathbb{R}^{2n-1} \) is a singular point of Whitney umbrella.

**Proof.** By, for example, \( [3] \), p.179, we see that a point \( q \in N \) is a singular point of Whitney umbrella of the mapping \( F \pi \circ f \) if \( j^1(F \pi \circ f)(q) \in \Sigma^1(N, \mathbb{R}^{2n-1}) \) and the mapping \( j^1(F \pi \circ f) \) is transverse to the submanifold \( \Sigma^1(N, \mathbb{R}^{2n-1}) \) at \( q \). Set \( \ell = 2n-1 \) and \( v = n \) in Proposition \( \Pi \) Then, it is clearly seen that we have \( k_0 = 1 \) in Proposition \( \Pi \). Hence, there exists a subset \( \Sigma \) of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1}) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1}) - \Sigma \), the mapping \( F \pi : N \to \mathbb{R}^{2n-1} \) is transverse to \( \Sigma^k(N, \mathbb{R}^{2n-1}) \) for any positive integer \( k \) satisfying \( 1 \leq k \leq n \), and the mapping satisfies that \( j^1(F \pi \circ f)(N) \cap \Sigma^k(N, \mathbb{R}^{2n-1}) = \emptyset \) for any positive integer \( k \) satisfying \( 2 \leq k \leq n \). Thus, if a point \( q \in N \) is a singular point of the mapping \( F \pi \circ f \), then it follows that \( j^1(F \pi \circ f)(q) \in \Sigma^1(N, \mathbb{R}^{2n-1}) \) and \( j^1(F \pi \circ f) \) is transverse to \( \Sigma^1(N, \mathbb{R}^{2n-1}) \) at \( q \). \( \square \)
In the case of $\ell \geq 2n$, the immersion property of a given mapping $f : N \to U$ is preserved by composing generic linearly perturbed mappings as follows:

**Corollary 3.** Let $N$ be a manifold of dimension $n$. Let $f$ be an immersion of $N$ into an open subset $U$ of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping ($\ell \geq 2n$). Then, there exists a subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an immersion.

**Proof.** It is clearly seen that the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an immersion if and only if $j^1(F_\pi \circ f)(N) \bigcap \bigcup_{k=1}^m \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$. Set $v = n$ and $\ell \geq 2n$ in Proposition 1. Then, it is clearly seen that $k_0 \leq 0$. By Remark 2 we get $k_0 = 0$. Hence, there exists a subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ satisfies that $j^1(F_\pi \circ f)(N) \bigcap \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$ for any positive integer $k$ ($1 \leq k \leq n$). \qed

A mapping $g : N \to \mathbb{R}^\ell$ has corank at most $k$ singular points if 

$$\sup \{ \text{corank} \, dg_q \mid q \in N \} \leq k,$$

where $\text{corank} \, dg_q = \min\{n, \ell\} - \text{rank} \, dg_q$. By Proposition 1 we have the following corollary.

**Corollary 4.** Let $N$ be a manifold of dimension $n$. Let $f$ be an immersion of $N$ into an open subset $U$ of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. Let $k_0$ be the maximum integer satisfying $(n - v + k_0)(\ell - v + k_0) \leq n$ ($v = \min\{n, \ell\}$). Then, there exists a subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ has corank at most $k_0$ singular points.

### 5.2. Applications of Theorem 2

**Proposition 2.** Let $N$ be a manifold of dimension $n$. Let $f$ be an injection of $N$ into an open subset $U$ of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If $(s_f - 1)\ell > ns_f$, then there exists a subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $F_\pi \circ f : N \to \mathbb{R}^\ell$ is a mapping with normal crossings satisfying $(F_\pi \circ f)^{(s_f)}(N^{(s_f)}) \bigcap \Delta_{s_f} = \emptyset$.

**Proof.** By Theorem 2 there exists a subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, and for any $s$ ($2 \leq s \leq s_f$), the mapping $(F_\pi \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to the submanifold $\Delta_s$. Hence, in order to show Proposition 2 it is sufficient to show that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $(F_\pi \circ f)^{(s_f)}$ satisfies that $(F_\pi \circ f)^{(s_f)}(N^{(s_f)}) \bigcap \Delta_{s_f} = \emptyset$.

Suppose that there exists an element $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ such that there exists a point $q \in N^{(s_f)}$ satisfying $(F_\pi \circ f)^{(s_f)}(q) \in \Delta_{s_f}$. Since $(F_\pi \circ f)^{(s_f)}$ is transverse to $\Delta_{s_f}$, we have the following:

$$d((F_\pi \circ f)^{(s_f)})(T_qN^{(s_f)}) + T_{(F_\pi \circ f)^{(s_f)}(q)} \Delta_{s_f} = T_{(F_\pi \circ f)^{(s_f)}(q)}(\mathbb{R}^\ell)^{s_f}.$$

Hence, we have

$$\dim d((F_\pi \circ f)^{(s_f)})(T_qN^{(s_f)}) \leq \dim T_{(F_\pi \circ f)^{(s_f)}(q)}(\mathbb{R}^\ell)^{s_f} - \dim T_{(F_\pi \circ f)^{(s_f)}(q)} \Delta_{s_f}\ 
= \text{codim} \, T_{(F_\pi \circ f)^{(s_f)}(q)} \Delta_{s_f}.\]
Thus, we get \( ns_f \geq (s_f - 1)\ell \). This contradicts the assumption \((s_f - 1)\ell > ns_f\). \(\square\)

In the case of \( \ell > 2n \), the injection property of a given mapping \( f : N \rightarrow U \) is preserved by composing generic linearly perturbed mappings as follows:

**Corollary 5.** Let \( N \) be a manifold of dimension \( n \). Let \( f \) be an injection of \( N \) into an open subset \( U \) of \( \mathbb{R}^m \). Let \( F : U \rightarrow \mathbb{R}^\ell \) be a mapping. If \( \ell > 2n \), then there exists a subset \( \Sigma \) of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma \), the mapping \( F_\pi \circ f : N \rightarrow \mathbb{R}^\ell \) is injective.

**Proof.** Since \( s_f \geq 2 \) and \( \ell > 2n \), it is easily seen that the dimension pair \((n, \ell)\) satisfies the assumption \((s_f - 1)\ell > ns_f\) of Proposition 2. Indeed, from \( \ell > 2n \), it follows that \((s_f - 1)\ell > 2n(s_f - 1)\). By \( s_f \geq 2 \), we get \( 2n(s_f - 1) \geq ns_f \).

Hence, by Proposition 2 there exists a subset \( \Sigma \) of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma \), the mapping \( (F_\pi \circ f)^{(2)} : N^{(2)} \rightarrow (\mathbb{R}^\ell)^2 \) is transverse to \( \Delta_2 \). In order to show Corollary 5 it is sufficient to show that the mapping \( (F_\pi \circ f)^{(2)} \) satisfies that \( (F_\pi \circ f)^{(2)}(N^{(2)}) \cap \Delta_2 = \emptyset \).

Suppose that there exists a point \( q \in N^{(2)} \) such that \( (F_\pi \circ f)^{(2)}(q) \in \Delta_2 \). Then, we have the following:

\[
\dim d((F_\pi \circ f)^{(2)})(q)(T_qN^{(2)}) + T_{(F_\pi \circ f)^{(2)}(q)}\Delta_2 = T_{(F_\pi \circ f)^{(2)}(q)}(\mathbb{R}^\ell)^2.
\]

Hence, we have

\[
\dim d((F_\pi \circ f)^{(2)})(q)(T_qN^{(2)}) \\
\geq \dim T_{(F_\pi \circ f)^{(2)}(q)}(\mathbb{R}^\ell)^2 - \dim T_{(F_\pi \circ f)^{(2)}(q)}\Delta_2 \\
= \operatorname{codim} T_{(F_\pi \circ f)^{(2)}(q)}\Delta_2.
\]

Thus, we get \( 2n \geq \ell \). This contradicts the assumption \( \ell > 2n \). \(\square\)

By combining Corollaries 4 and 5 we have the following.

**Corollary 6.** Let \( N \) be a manifold of dimension \( n \). Let \( f \) be an injective immersion of \( N \) into an open subset \( U \) of \( \mathbb{R}^m \). Let \( F : U \rightarrow \mathbb{R}^\ell \) be a mapping. If \( \ell > 2n \), then there exists a subset \( \Sigma \) of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma \), the mapping \( F_\pi \circ f : N \rightarrow \mathbb{R}^\ell \) is an injective immersion.

In Corollary 6 suppose that the mapping \( F_\pi \circ f : N \rightarrow \mathbb{R}^\ell \) is proper. Then, an injective immersion \( F_\pi \circ f \) is necessarily an embedding (see 3, p. 11). Thus, we get the following.

**Corollary 7.** Let \( N \) be a compact manifold of dimension \( n \). Let \( f \) be an embedding of \( N \) into an open subset \( U \) of \( \mathbb{R}^m \). Let \( F : U \rightarrow \mathbb{R}^\ell \) be a mapping. If \( \ell > 2n \), then there exists a subset \( \Sigma \) of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma \), the mapping \( F_\pi \circ f : N \rightarrow \mathbb{R}^\ell \) is an embedding.

### 6. Further Applications

#### 6.1. Introduction of generalized distance-squared mappings.

Let \( p_i = (p_{i1}, p_{i2}, \ldots, p_{im}) \) \((1 \leq i \leq \ell)\) (resp., \( A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m} \)) be points of \( \mathbb{R}^m \) (resp., an \( \ell \times m \) matrix with all entries being non-zero real numbers). Set \( p = (p_1, p_2, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell \). Let \( G_{(p, A)} : \mathbb{R}^m \rightarrow \mathbb{R}^\ell \) be the mapping defined by

\[
G_{(p, A)}(x) = \left( \sum_{j=1}^m a_{1j}(x_j - p_{1j})^2, \sum_{j=1}^m a_{2j}(x_j - p_{2j})^2, \ldots, \sum_{j=1}^m a_{\ell j}(x_j - p_{\ell j})^2 \right).
\]
where \( x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \). The mapping \( G_{(p,A)} \) is called a generalized distance-squared mapping, and the \( \ell \)-tuple of points \( p = (p_1, p_2, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell \) is called the central point of the generalized distance-squared mapping \( G_{(p,A)} \). A distance-squared mapping \( D_p \) (resp., Lorentzian distance-squared mapping \( L_p \)) is the mapping \( G_{(p,A)} \) satisfying that each entry of \( A \) is equal to 1 (resp., \( a_{i1} = -1 \) and \( a_{ij} = 1 \) (\( j \neq 1 \)).

In [5] (resp., [6]), a classification result of distance-squared mappings (resp., Lorentzian distance-squared mappings) is given.

In [9], a classification result of generalized distance-squared mappings of the plane into the plane is given. If the rank of \( A \) is equal to one, then a generalized distance-squared mapping having a generic central point is a mapping of which any singular point is a fold point except one cusp point. The singular set is a rectangular hyperbola. If the rank of \( A \) is equal to one, then a generalized distance-squared mapping having a generic central point is \( A \)-equivalent to the normal form of fold singularity \( (x_1, x_2) \mapsto (x_1, x_2^2) \).

In [7], a classification result of generalized distance-squared mappings of \( \mathbb{R}^{m+1} \) into \( \mathbb{R}^{2m+1} \) is given. If the rank of \( A \) is equal to \( m + 1 \), then a generalized distance-squared mapping having a general central point is \( A \)-equivalent to the normal form of Whitney umbrella \( (x_1, x_2, \ldots, x_{m+1}) \mapsto (x_1^2, x_1 x_2, \ldots, x_1 x_{m+1}, x_2, \ldots, x_{m+1}) \). If the rank of \( A \) is strictly smaller than \( m + 1 \), then a generalized distance-squared mapping having a generic central point is \( A \)-equivalent to the inclusion \( (x_1, x_2, \ldots, x_{m+1}) \mapsto (x_1, x_2, \ldots, x_{m+1}, 0, \ldots, 0) \).

Namely, in [5], [6], [7] and [9], the properties of generic generalized distance-squared mappings are investigated. Hence, it is natural to investigate the properties of compositions with generic generalized distance-squared mappings.

We have another original motivation. Height functions and distance-squared functions have been investigated in detail so far, and they are useful tools in the applications of singularity theory to differential geometry (for instance, see [2]). A mapping in which each component is a height function is nothing but a projection. Projections as well as height functions or distance-squared functions have been investigated so far. In [10], compositions of generic projections and embeddings are investigated.

On the other hand, a mapping in which each component is a distance-squared function is a distance-squared mapping. In addition, the notion of a generalized distance-squared mapping is an extension of that of a distance-squared mapping. Therefore, it is natural to investigate compositions with generic generalized distance-squared mappings as well as projections.

6.2. Applications of Theorem 1 to \( G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell \).

**Proposition 3.** Let \( N \) be a manifold of dimension \( n \). Let \( f : N \to \mathbb{R}^m \) be an immersion. Let \( A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m} \) be an \( \ell \times m \) matrix with all entries being non-zero real numbers. If \( X \) is an \( A^1 \)-invariant submanifold of \( J^1(N, \mathbb{R}) \), then there exists a subset \( \Sigma \) of \( (\mathbb{R}^m)^\ell \) with Lebesgue measure zero such that for any \( p \in (\mathbb{R}^m)^\ell - \Sigma \), the mapping \( j^1(G_{(p,A)} \circ f) : N \to J^1(N, \mathbb{R}^\ell) \) is transverse to the submanifold \( X(N, \mathbb{R}^\ell) \).
Proof. Let \( H : \mathbb{R}^\ell \to \mathbb{R}^\ell \) be a diffeomorphism of the target for deleting constant terms. The composition \( L \) for any \( i \) and \( \pi \) have \( \psi \) with \( \Delta \), such that for any \( p \in \mathbb{R}^{m} \) we get analogies of Proposition 2, Corollaries 5, 6 and 7.

Proposition 4. Let \( f : N \to \mathbb{R}^m \) be an injection. Let \( A = (a_{ij})_{1 \leq i, j \leq m} \) be an \( \ell \times m \) matrix with all entries being non-zero real numbers. Then, there exists a subset \( \Sigma \) of \( \mathbb{R}^m \) with Lebesgue measure zero such that for any \( p \in (\mathbb{R}^m)^\ell - \Sigma \), the mapping \( j^1((G \circ f) : N \to J^1(N, \mathbb{R}^\ell)) \) is transverse to \( X(N, \mathbb{R}^\ell) \). Since \( \psi^{-1} : \mathbb{R}^m, \mathbb{R}^\ell \) is a \( C^\infty \) mapping, \( \psi^{-1}(\Sigma) \) is a subset of \( \mathbb{R}^m \) with Lebesgue measure zero. For any \( p \in (\mathbb{R}^m)^\ell - \psi^{-1}(\Sigma) \), we have \( \psi(p) \in (\mathbb{R}^m)^\ell - \Sigma \). Hence, for any \( p \in (\mathbb{R}^m)^\ell - \psi^{-1}(\Sigma) \), the mapping \( j^1((G \circ f) : N \to J^1(N, \mathbb{R}^\ell)) \) is transverse to \( X(N, \mathbb{R}^\ell) \). Then, since \( H : \mathbb{R}^\ell \to \mathbb{R}^\ell \) is a diffeomorphism, the mapping \( j^1((G \circ f) : N \to J^1(N, \mathbb{R}^\ell)) \) is transverse to \( X(N, \mathbb{R}^\ell) \).

Remark 3. As applications of Proposition 3 regarding generalized distance-squared mappings, we get analogies of Proposition 2 and Corollaries 1, 2, 3 and 4.

6.3. Applications of Theorem 2 to \( G(p, A) : \mathbb{R}^m \to \mathbb{R}^\ell \). By Theorem 2, we get the following proposition, which can be proved by the same argument as in the proof of Proposition 3 and we omit the proof.

Proposition 4. Let \( N \) be a manifold of dimension \( n \). Let \( f : N \to \mathbb{R}^m \) be an injection. Let \( A = (a_{ij})_{1 \leq i, j \leq m} \) be an \( \ell \times m \) matrix with all entries being non-zero real numbers. Then, there exists a subset \( \Sigma \) of \( \mathbb{R}^m \) with Lebesgue measure zero such that for any \( p \in (\mathbb{R}^m)^\ell - \Sigma \), and for any \( s \leq s_f \), the mapping \( (G \circ f)^{h(s)} : N(s) \to (\mathbb{R}^\ell)^s \) is transverse to the submanifold \( \Delta_s \). Moreover, if the mapping \( G(p, A) \) satisfies \( |G^{-1}(y)| \leq s_f \) for any \( y \in \mathbb{R}^\ell \), then \( G(p, A) \circ f : N \to \mathbb{R}^\ell \) is a mapping with normal crossings.

Remark 4. As applications of Proposition 3 regarding generalized distance-squared mappings, we get analogies of Proposition 2 and Corollaries 5, 6 and 7.

As the special case of the classification result of distance-squared mappings (resp., Lorentzian distance-squared mappings) in [5] (resp., [6]), we have Lemma 2.

Lemma 2 ([5, 6]). We have the following.

(1) For any \( p \in \mathbb{R} \), the mappings \( D_p : \mathbb{R} \to \mathbb{R} \) and \( L_p : \mathbb{R} \to \mathbb{R} \) are \( A \)-equivalent to \( x \mapsto x^2 \).
(2) For \( m \geq 2 \), there exists a subset \( \Sigma_D \) (resp., \( \Sigma_L \)) of \((\mathbb{R}^m)^m\) with Lebesgue measure zero such that for any \( p \in (\mathbb{R}^m)^m - \Sigma_D \) (resp., \( p \in (\mathbb{R}^m)^m - \Sigma_L \)), the mapping \( D_p : \mathbb{R}^m \to \mathbb{R}^m \) (resp., \( L_p : \mathbb{R}^m \to \mathbb{R}^m \)) is \( \mathcal{A} \)-equivalent to the normal form of definite fold mappings \((x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_{m-1}, x_m^2)\).

(3) In the case of \( 1 \leq m < \ell \), there exists a subset \( \Sigma_D \) (resp., \( \Sigma_L \)) of \((\mathbb{R}^m)^\ell\) with Lebesgue measure zero such that for any \( p \in (\mathbb{R}^m)^\ell - \Sigma_D \) (resp., \( p \in (\mathbb{R}^m)^\ell - \Sigma_L \)), the mapping \( D_p : \mathbb{R}^m \to \mathbb{R}^\ell \) (resp., \( L_p : \mathbb{R}^m \to \mathbb{R}^\ell \)) is \( \mathcal{A} \)-equivalent to the normal form of definite fold mappings \((x_1, x_2, \ldots, x_{\ell-1}, x_{\ell}) \mapsto (x_1, x_2, \ldots, x_{\ell-1}, 0)\).

**Proposition 5.** Let \( N \) be a manifold of dimension \( n \). Let \( f : N \to \mathbb{R}^m \) be an injection. Then, the following holds:

1. For \( m \geq 1 \), there exists a subset \( \Sigma_D \) (resp., \( \Sigma_L \)) of \((\mathbb{R}^m)^m\) with Lebesgue measure zero such that for any \( p \in (\mathbb{R}^m)^m - \Sigma_D \) (resp., \( p \in (\mathbb{R}^m)^m - \Sigma_L \)), \( D_p \circ f : N \to \mathbb{R}^m \) (resp., \( L_p \circ f : N \to \mathbb{R}^m \)) is a mapping with normal crossings.
2. For \( m < \ell \), there exists a subset \( \Sigma_D \) (resp., \( \Sigma_L \)) of \((\mathbb{R}^m)^\ell\) with Lebesgue measure zero such that for any \( p \in (\mathbb{R}^m)^\ell - \Sigma_D \) (resp., \( p \in (\mathbb{R}^m)^\ell - \Sigma_L \)), the mapping \( D_p \circ f : N \to \mathbb{R}^\ell \) (resp., \( L_p \circ f : N \to \mathbb{R}^\ell \)) is an injective immersion.

**Proof.** The proof for distance-squared mappings is the same as that for Lorentzian distance-squared mappings. Hence, it is sufficient to give the proof for distance-squared mappings.

Firstly, we will show the assertion 1. From Lemma 2 there exists a subset \( \Sigma_1 \) of \((\mathbb{R}^m)^m\) with Lebesgue measure zero such that for any \( p \in (\mathbb{R}^m)^m - \Sigma_1 \), the mapping \( D_p : \mathbb{R}^m \to \mathbb{R}^m \) satisfies \( |D_p^{-1}(y)| \leq 2 \) for any \( y \in \mathbb{R}^m \). On the other hand, from Proposition 4 there exists a subset \( \Sigma_2 \) of \((\mathbb{R}^m)^m\) with Lebesgue measure zero such that for any \( p \in (\mathbb{R}^m)^m - \Sigma_2 \), if \( D_p \) satisfies \( |D_p^{-1}(y)| \leq s_f \) for any \( y \in \mathbb{R}^m \), then \( D_p \circ f : N \to \mathbb{R}^m \) is a mapping with normal crossings. Set \( \Sigma_D = \Sigma_1 \cup \Sigma_2 \). It is clearly seen that \( \Sigma_D \) is a subset of \((\mathbb{R}^m)^m\) with Lebesgue measure zero. Then, for any \( p \in (\mathbb{R}^m)^m - \Sigma_D \), \( D_p \circ f : N \to \mathbb{R}^m \) is a mapping with normal crossings.

In the case of \( m < \ell \), since from Lemma 2 there exists a subset \( \Sigma_D \) of \((\mathbb{R}^m)^\ell\) with Lebesgue measure zero such that for any \( p \in (\mathbb{R}^m)^\ell - \Sigma_D \), the mapping \( D_p : \mathbb{R}^m \to \mathbb{R}^\ell \) is \( \mathcal{A} \)-equivalent to the inclusion, the assertion 2 holds.

By combining Proposition 4 and the analogy of Corollary 3 in Remark 3 we have the following.

**Corollary 8.** Let \( N \) be a manifold of dimension \( n \). Let \( f : N \to \mathbb{R}^m \) be an injective immersion \((2n \leq m)\). Then, there exists a subset \( \Sigma_D \) (resp., \( \Sigma_L \)) of \((\mathbb{R}^m)^m\) with Lebesgue measure zero such that for any \( p \in (\mathbb{R}^m)^m - \Sigma_D \) (resp., \( p \in (\mathbb{R}^m)^m - \Sigma_L \)), the mapping \( D_p \circ f : N \to \mathbb{R}^m \) (resp., \( L_p \circ f : N \to \mathbb{R}^m \)) is an immersion with normal crossings.

In Corollary 3 if \( m = 2n \) and the mapping \( D_p \circ f : N \to \mathbb{R}^{2n} \) (resp., \( L_p \circ f : N \to \mathbb{R}^{2n} \)) is proper, then the immersion with normal crossings \( D_p \circ f : N \to \mathbb{R}^{2n} \) (resp., \( L_p \circ f : N \to \mathbb{R}^{2n} \)) is necessarily stable (see [9], p. 86). Thus, we get the following.

**Corollary 9.** Let \( N \) be a compact manifold of dimension \( n \). Let \( f : N \to \mathbb{R}^{2n} \) be an embedding. Then, there exists a subset \( \Sigma_D \) (resp., \( \Sigma_L \)) of \((\mathbb{R}^{2n})^{2n}\) with Lebesgue
measure zero such that for any \( p \in (\mathbb{R}^{2n})^{2n} - \Sigma_D \) (resp., \( p \in (\mathbb{R}^{2n})^{2n} - \Sigma_L \)), the mapping \( D_p \circ f : N \to \mathbb{R}^{2n} \) (resp., \( L_p \circ f : N \to \mathbb{R}^{2n} \)) is stable.

Remark that the dimension of the target space in Corollary \( \text{[3]} \) is smaller than that in Corollary \( \text{[7]} \).

7. Appendix

In this section, the main theorems in \( \text{[4]} \) and \( \text{[10]} \) are stated. For this, we prepare some notions.

Let \( N \) and \( P \) be manifolds. Let \( sJ^r(N, P) \) be the space consisting of elements \((j^rg(q_1), j^rg(q_2), \ldots, j^rg(q_s)) \in J^r(N, P)^s\) satisfying \((q_1, q_2, \ldots, q_s) \in N^s\). Since \( N^s \) is an open submanifold of \( N \), the space \( sJ^r(N, P) \) is also an open submanifold of \( J^r(N, P)^s \).

For a given mapping \( g : N \to P \), the mapping \( sJ^r g : N^s \to sJ^r(N, P) \) is defined by \((q_1, q_2, \ldots, q_s) \mapsto (j^rg(q_1), j^rg(q_2), \ldots, j^rg(q_s))\).

Let \( W \) be a submanifold of \( sJ^r(N, P) \). A mapping \( g : N \to P \) will be said to be transverse with respect to \( W \) if \( sJ^r g : N^s \to sJ^r(N, P) \) is transverse to \( W \).

Following Mather \( \text{[10]} \), we can partition \( P^n \) as follows. Given any partition \( \Pi \) of \( \{1, 2, \ldots, s\} \), let \( P^n_\Pi \) denote the set of \( s \)-tuples \((y_1, y_2, \ldots, y_s) \in P^n\) such that \( y_i = y_j \) if and only if the two positive integers \( i \) and \( j \) are in the same member of the partition \( \Pi \).

Let \( \text{Diff} N \) denote the group of diffeomorphisms of \( N \). We have the natural action of \( \text{Diff} N \times \text{Diff} P \) on \( sJ^r(N, P) \) such that for a mapping \( g : N \to P \), the equality \((h, H) \cdot sJ^r g(q) = sJ^r (H \circ g \circ h^{-1})(q') \) holds, where \( q = (q_1, q_2, \ldots, q_s) \) and \( q' = (h(q_1), h(q_2), \ldots, h(q_s)) \). A subset \( W \) of \( sJ^r(N, P) \) is said to be invariant if it is invariant under this action.

We recall the following identification (7.1) from \( \text{[10]} \). For \( q = (q_1, q_2, \ldots, q_s) \in N^s \), let \( g : U \to P \) be a mapping defined in a neighborhood \( U \) of \((q_1, q_2, \ldots, q_s) \) in \( N \), and let \( z = sJ^r g(q) \), \( q' = (g(q_1), g(q_2), \ldots, g(q_s)) \). Let \( sJ^r(N, P)_q \) and \( sJ^r(N, P)_{q, q'} \) denote the fibers of \( sJ^r(N, P) \) over \( q \) and over \((q, q')\) respectively. Let \( J^r(N) \) denote the \( \mathbb{R} \)-algebra of \( r \)-jets at \( q \) of functions on \( N \). Namely, \( J^r(N)_q = sJ^r(N, \mathbb{R})_q \).

Set \( g^*TP = \bigcup_{q \in U} T_{g(q)}P \), where \( TP \) is the tangent bundle of \( P \). Let \( J^r(g^*TP)_q \) denote the \( J^r(N)_q \)-module of \( r \)-jets at \( q \) of sections of the bundle \( g^*TP \). Let \( m_q \) be the ideal in \( J^r(N)_q \) consisting of jets of functions which vanish at \( q \). Namely, \( m_q = \{ sJ^r h(q) \in sJ^r(N, \mathbb{R})_q \mid h(q_1) = h(q_2) = \cdots = h(q_s) = 0 \} \).

Let \( m_q J^r(g^*TP)_q \) be the set consisting of finite sums of products of an element of \( m_q \) and an element of \( J^r(g^*TP)_q \). Namely, we set \( m_q J^r(g^*TP)_q = J^r(g^*TP)_q \cap \{ sJ^r \xi(q) \in sJ^r(N, T^*P)_q \mid \xi(q_1) = \xi(q_2) = \cdots = \xi(q_s) = 0 \} \).

Then, it is easily seen that we have the following canonical identification of \( \mathbb{R} \)-vector spaces:

\[
T(sJ^r(N, P)_{q, q'}) = m_q J^r(g^*TP)_q.
\]

Let \( W \) be a non-empty submanifold of \( sJ^r(N, P) \). Choose \( q = (q_1, q_2, \ldots, q_s) \in N^s \) and \( g : N \to P \), and set \( z = sJ^r g(q) \) and \( q' = (g(q_1), g(q_2), \ldots, g(q_s)) \). Suppose that the choice is made so that \( z \in W \). Set \( W_{q, q'} = \pi^{-1}(q, q') \), where \( \pi : W \to N^s \times P^s \) is defined by \( \pi(sJ^r \bar{g}(\bar{q})) = (\bar{q}, (\bar{g}(\bar{q}_1), \bar{g}(\bar{q}_2), \ldots, \bar{g}(\bar{q}_s))) \) and \( \bar{q} = (\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_s) \in N^s \).
Then, under the identification (7.1), the tangent space $T(W_{q,q'})_z$ can be identified with a vector subspace of $m_q J^r(g^*TP)_q$. We denote this vector subspace by $E(g,q,W)$.

Definition 3. The submanifold $W$ is said to be modular if conditions ($\alpha$) and ($\beta$) below are satisfied.

($\alpha$) The set $W$ is an invariant submanifold of $sJ^r(N,P)$, and lies over $P\Pi$ for some partition $\Pi$ of $\{1,2,\ldots,s\}$.

($\beta$) For any $q \in N^{(s)}$ and any mapping $g : N \to P$ such that $s_j^r g(q) \in W$, the subspace $E(g,q,W)$ is a $J^r(N)_q$-submodule.

Now, suppose that $P = \mathbb{R}^\ell$. The main theorem in [10] is the following.

Theorem 3 ([10]). Let $N$ be a manifold of dimension $n$. Let $f$ be an embedding of $N$ into $\mathbb{R}^m$. If $W$ is a modular submanifold of $sJ^r(N,\mathbb{R}^\ell)$ and $m > \ell$, then there exists a subset $\Sigma$ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell) - \Sigma$, $\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to $W$.

Then, the main theorem in [4] is the following.

Theorem 4 ([4]). Let $N$ be a manifold of dimension $n$. Let $f$ be an embedding of $N$ into an open subset $U$ of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If $W$ is a modular submanifold of $sJ^r(N,\mathbb{R}^\ell)$, then there exists a subset $\Sigma$ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell) - \Sigma$, $F_\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to $W$.

The assertion (6) in Section 1, Corollary 7 in Section 5 and Corollary 9 in Section 6 of the present paper are obtained as corollaries of Theorems 1 and 2 in this paper. On the other hand, they are also corollaries of Theorem 4.

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