On general sums involving the floor function with applications to $k$-free numbers

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Abstract In this paper, we consider sums related to the floor function. We can improve previous results for some special arithmetic functions considered by Bordellès [4], Stucky [9] and Liu-Wu-Yang [11]. It is worth emphasizing that we use much simpler methods to give much better results than previous.

Keywords Asymptotic formulas, Exponential sums, Sequences and sets

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1. Introduction

Recently, the sum

$$S_f(x) = \sum_{n \leq x} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right)$$

has attracted many experts’ special attention (for example, see [2, 4, 7, 11, 12]), where $f$ is a complex-valued arithmetic function and $\lfloor \cdot \rfloor$ denotes the floor function (i.e. the greatest integer function). One can call $S_f(x)$ the fractional sum of $f$ (see [9]).

Specially, for some fixed $\eta \in (0, 1)$ and

$$f(n) \ll n^\eta,$$

independently, Wu [11] and Zhai [12] showed that

$$S_f(x) = C_f x + O \left( x^{(1+\eta)/2} \right),$$

where $f$ is a complex-valued arithmetic function and

$$C_f = \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)}.$$

This formula improves the recent result obtained by Bordellès, Dai, Heyman, Pan and Shparlinski [2].

On the other hand, for some fixed $\eta \in (0, 2)$ and

$$\sum_{n \leq x} |f(n)|^2 \ll x^\eta,$$

in [2], Bordellès, Dai, Heyman, Pan and Shparlinski proved that

$$S_f(x) = C_f x + O \left( x^{(1+\eta)/3} (\log x)^{(1+\eta)(2+\varepsilon_2(x))/6} \right),$$

where $f$ is a complex-valued arithmetic function,

$$\varepsilon_2(x) = \left( \frac{2 \log \log \log x}{\log \log x} \right)^{1/2} \left( 1 + \frac{30}{\log \log \log x} \right)$$

and

$$C_f = \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)}.$$
Recently, for some fixed $\eta \in (0, 2)$ and
$$\sum_{n \leq x} |f(n)|^2 \ll x^\eta,$$
Wu [11] showed that
$$S_f(x) = C_f x + O \left( x^{(1+\eta)/3} (\log x)^{(1+\eta)\varepsilon_2(x)/6} \right),$$
where here and throughout, $C_f$ denotes the constant
$$C_f = \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)},$$
$f$ is a complex-valued arithmetic function and
$$\varepsilon_2(x) = \left( \frac{2 \log \log \log x}{\log \log x} \right)^{1/2} \left( 1 + \frac{30}{\log \log \log x} \right).$$
This formula improves the recent result obtained by Bordellès, Dai, Heyman, Pan and Shparlinski [2] for the aspect of powers of $\log x$.

Recently, in [14], Zhao and Wu showed that
$$S_f(x) = C_f x + O \left( x^{(2+3\eta)/8+\epsilon} \right),$$
by using the following result of [1]
$$\sum_{n \geq 1} \left( \left\{ \frac{x}{n+1} \right\} - \left\{ \frac{x}{n} \right\} \right)^2 = \frac{\zeta(3/2)\sqrt{x}}{\pi} + O(x^{3/7}).$$
We point out that combining the main result of Balazard [1], we can obtain a slight better result.

The first aim of this paper is to show the following. We can improve previous results and give non-trivial estimate under the assumption
$$\sum_{n \leq x} |f(n)|^2 \ll x^\eta,$$
with any fixed $\eta \in (0, 2)$. We can improve (1.3) by eliminating the $x^\epsilon$ term. Moreover, the proofs of us are much more elementary. The main idea is a basic observation such that
$$\sum_{n \leq x} |f(n)|^2 \ll x^\eta$$
implies $|f(n)| \ll x^{\eta/2}$. Hence we can avoid using the so-called $r$-th Hooley divisor function used in [2, 11, 14].

**Theorem 1.1.** Let $f$ be a complex-valued arithmetic function. Assume that
$$\sum_{n \leq x} |f(n)|^2 \ll x^\eta,$$
with any fixed $\eta \in (0, 2)$. Then we have
$$S_f(x) = C_f x + O \left( x^{(2+3\eta)/8} \right),$$
where
\[ C_f = \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)}. \]

In general, for \( f \) a positive real-valued arithmetic function, one can obtain some much better results by using the theory of Fourier series and exponential sums. For example, one can refer to [4, 7, 8, 9]. In this paper, we consider the following general result. We can improve previous results for some special arithmetic functions considered by Bordellès [4], Stucky [9] and Liu-Wu-Yang [11].

**Theorem 1.2.** Let \( (\kappa, \lambda) \neq (0, 1) \) be an exponent pair. Let \( f \) be a positive real-valued arithmetic function such that \( f(n) = \sum_{d|n} g(d) \), and for any sufficiently large \( x \)

\[ \sum_{n \leq x} |g(n)| \ll x^{\lambda/(1+\kappa)}. \]

Then we have
\[ S_f(x) = C_f x + \begin{cases} O \left( x^{\lambda/(1+\lambda)} (\log x)^{\alpha} \right) & \text{if } f(n) \ll 1, \\ O \left( x^{\lambda/(1+\lambda) + \epsilon} \right) & \text{if } f(n) \ll n^\epsilon, \end{cases} \]

where
\[ C_f = \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)}. \]

and \( \alpha = 1 \) if \( (\kappa, \lambda) = (1/2, 1/2) \) and 0 otherwise.

Let \( \mu_k(n) \) be the indicator function of the \( k \)-free numbers. Bordellès [4] considered the sum
\[ S_{\mu_2}(x) = \sum_{n \leq x} \mu_2 \left( \left[ \frac{x}{n} \right] \right) \]
and proved that
\[ S_{\mu_2}(x) = \sum_{n=1}^{\infty} \frac{\mu_2(n)}{n(n+1)} x + O \left( x^{1919/4268 + \epsilon} \right). \]  
(1.5)

Recently, Liu-Wu-Yang [7] improved the result of Bordellès [4] by showing that
\[ S_{\mu_2}(x) = \sum_{n=1}^{\infty} \frac{\mu_2(n)}{n(n+1)} x + O \left( x^{2/5 + \epsilon} \right). \]  
(1.6)

In [9], for \( k \geq 3 \), Stucky remarked that
\[ S_{\mu_k}(x) = \sum_{n \leq x} \mu_k \left( \left[ \frac{x}{n} \right] \right) = \sum_{n=1}^{\infty} \frac{\mu_k(n)}{n(n+1)} x + O \left( x^{\theta_k} \right), \]  
(1.7)

where
\[ \theta_k = \left( 1 + \frac{1}{k} \right) \left( 3 + \frac{1}{k} \right)^{-1}. \]

By using Theorem 1.2, we can obtain the following corollary by choosing suitable exponent pairs.
Corollary 1.3. We have

\[ S_{\mu_k}(x) = \begin{cases} 
C_{\mu_k} x + O\left(x^{11/29}(\log x)^2\right) & \text{if } k = 2, \\
C_{\mu_k} x + O\left(x^{1/3}(\log x)\right) & \text{if } k \geq 3,
\end{cases} \]

where

\[ C_{\mu_k} = \sum_{n=1}^{\infty} \frac{\mu_k(n)}{n(n+1)} \]

and \( \mu_k(n) \) is the indicator function of the \( k \)-free numbers.

One can find that \( 11/29 < 0.3794 \) and \( 2/5 = 0.4 \). Hence according to (1.5) and (1.6), we can give much better results. For \( k \geq 3 \), we find that

\[ \theta_k = \left(1 + \frac{1}{k}\right) \left(3 + \frac{1}{k}\right)^{-1} > 1/3. \]

Hence, our result is better than (1.7) for fixed \( k \geq 2 \).

On the other hand, Stucky [9] also considered the fractional sum of \( f(n) = \sum_{d|n} \frac{1}{d^\beta} \). For \( \beta \in (2/3, 1] \) (the result is trivial for \( 0 < \beta \leq 2/3 \), see [9] for details), it is proved that

\[ S_f(x) = C_f x + O\left(x^{2-\beta}/\log x\right). \]

For this special situation, we can obtain the following result.

Corollary 1.4. Let

\[ f(n) = \sum_{d|n} \frac{1}{d^\beta}. \]

For \( \beta \in (2/3, 1] \), we have

\[ S_f(x) = C_f x + O\left(x^{1/3}(\log x)\right). \]

Remark 1. It is believable that one can only get a best possible error term \( O(x^{1/3}) \) for such type sums. Hence the error term of the square-free case may be improved further by using some deep results related to exponential sums.

2. Proof of Theorem 1.1

Let

\[ x^\varepsilon \leq N \leq x^{1-\varepsilon} \]

be a parameter to be chosen later. We can write

\[ S_f(x) := S_{f,1}(x, N) + S_{f,2}(x, N), \]

where

\[ S_{f,1} = \sum_{n \leq N} f\left(\left[\frac{x}{n}\right]\right) \]

and

\[ S_{f,2} = \sum_{N < n \leq x} f\left(\left[\frac{x}{n}\right]\right). \]
By the assumption in Theorem 1.1, one has
\[ |f(n)|^2 \leq \sum_{n \leq x} |f(n)|^2 \ll x^\eta. \quad (2.1) \]
Then we have
\[ |f(n)| \ll x^{\eta/2}. \]
Then we can obtain that
\[ S_{f,1} = \sum_{n \leq N} f\left(\left[\frac{x}{n}\right]\right) = \sum_{n \leq N} \frac{(x/n)^\eta}{n/2} \ll \frac{x^{\eta/2} N^{1-\eta/2}}{x^{(2+3\eta)/8}}, \]
where \( N = x^{1/4}. \)

Note that by Cauchy’s inequality, the estimates (2.1) implies that
\[ \sum_{n \leq x} |f(n)| \ll x^{(1+\eta)/2}. \]
For \( \eta \in (0, 2), \) this gives that
\[ \sum_{n \leq N} \frac{f(n)}{n(n+1)} \ll 1. \]

Hence by using Cauchy’s inequality and (1.4), we can get
\[ S_{f,2} = \sum_{N < n \leq x} f\left(\left[\frac{x}{n}\right]\right) \]
\[ = \sum_{d \leq x/N} f(d) \sum_{x/(d+1) < n \leq x/d} 1 \]
\[ = \sum_{d \leq x/N} f(d) \left( \frac{x}{d} - \frac{x}{d+1} + \left\{ \frac{x}{d+1} \right\} - \left\{ \frac{x}{d} \right\} \right) \]
\[ = x \sum_{d=1}^\infty \frac{f(d)}{d(d+1)} - x \sum_{d > x/N} \frac{f(n)}{d(d+1)} \]
\[ + \sum_{d \leq x/N} |f(d)| \left( \left\{ \frac{x}{d+1} \right\} - \left\{ \frac{x}{d} \right\} \right) \]
\[ \ll x \sum_{d=1}^\infty \frac{f(d)}{d(d+1)} + x \sum_{d > x/N} \frac{f(n)}{d(d+1)} \]
\[ + O \left( \sum_{d \leq x/N} |f(d)|^2 \sum_{d \leq x/N} \left( \left\{ \frac{x}{d+1} \right\} - \left\{ \frac{x}{d} \right\} \right)^2 \right)^{1/2} \]
\[ = x \sum_{d=1}^\infty \frac{f(d)}{d(d+1)} + O \left( \frac{x^{1+\frac{1+\eta}{2}} - x^{1/2}}{N^{1+\eta/2} + x^{1/4} \left( \frac{x}{N} \right)^2} \right). \]

Observing that \( N = x^{1/4}, \) we obtain the desired result.
3. Proof of Theorem 1.2

We will start the proof for Theorem 1.2 with some necessary lemmas. The following lemma can be seen in Theorem A.6 in [6]. Let \( \psi(t) = t - \lfloor t \rfloor - 1/2 \) for \( t \in \mathbb{R} \) and \( \delta \geq 0 \). Define

\[
G(x, D) := \sum_{D < d \leq 2D} f(d) \psi \left( \frac{x}{d + \delta} \right).
\]

**Lemma 3.1.** For \( 0 < |t| < 1 \), let

\[
W(t) = \pi t (1 - |t|) \cot \pi t + |t|.
\]

For \( x \in \mathbb{R}, H \geq 1 \), we define

\[
\psi^*(x) = \sum_{1 \leq |h| \leq H} (2\pi ih)^{-1} W \left( \frac{h}{H + 1} \right) e(hx)
\]

and

\[
R_H(x) = \frac{1}{2H + 2} \sum_{|h| \leq H} \left( 1 - \frac{|h|}{H + 1} \right) e(hx).
\]

Then \( \delta(x) \) is non-negative, and we have

\[
|\psi^*(x) - \psi(x)| \leq R_H(x).
\]

In fact, by using some ideas related to Lemma 4.3 in [6] or Corollary 6.7 in [3], we can obtain the following lemma.

**Lemma 3.2.** Let \( \delta \geq 0 \) be a fixed constant. Let \( f \) be a positive real-valued arithmetic function such that \( f(n) = \sum_{d \mid n} g(d) \) and for any sufficiently large \( x \)

\[
\sum_{n \leq x} |g(n)| \ll x^{\lambda/(1+\kappa)}.
\]

We have

\[
G(x, D) \ll \left( x^\kappa D^{1+\lambda} \right)^{1/(1+\kappa)} + x^{-1} D^2,
\]

uniformly for \( 1 \leq D \leq x \), where \( (\kappa, \lambda) \) is an exponent pair.

**Remark 2.** If we choose

\[
f(n) = \frac{\phi(n)}{n},
\]

in [11], it is proved that

**Theorem A.** (Wu [11]) Let \( \delta \geq 0 \) be a fixed constant. We have

\[
G(x, D) \ll \left( x^\kappa D^{1+\lambda} \right)^{1/(1+\kappa)} + x^\kappa D^{-2\kappa+\lambda}(\log x) + x^{-1} D^2,
\]

uniformly for \( 1 \leq D \leq x \), where \( (\kappa, \lambda) \) is an exponent pair. Furthermore, if \( (\kappa, \lambda) \neq (1/2, 1/2) \), then the factor \( \log x \) can be omitted.

Our result eliminates one term and the possible \( \log x \) in [11].

**Proof.** Now we will prove Lemma 3.2. The proof relies on the relation

\[
f(d) = \sum_{mn=d} g(m).
\]
Hence one can write
\[ G(x, D) = \sum_{m \leq 2D} g(m) \sum_{D/m < n \leq 2D/m} \psi \left( \frac{x}{mn + \delta} \right). \]

If \( D \leq 100x^{1+2\kappa-\lambda}m^{1-\lambda+\kappa} \), then we have
\[ (x^nD^{\lambda-\kappa})^{1/(1+\kappa)} \gg D \gg G(x, D). \]

Hence, we may always assume that \( D > 100x^{1+2\kappa-\lambda}m^{1-\lambda+\kappa} \). By Lemma 3.1, for \( x \geq 1 \) and \( H \geq 1 \), we have
\[ G(x, D) \ll \sum_{m \leq 2D} |g(m)| \left( \frac{D/m}{H} + \sum_{h \leq H} \frac{1}{\bar{h}} \sum_{D/m < n \leq 2D/m} e \left( \frac{hx}{mn + \delta} \right) \right) \]
with \( H \geq 1 \). We also need the following well-known lemma (for example, one can refer to page 441 of [3] or page 34 of [6]).

**Lemma 3.3.** Let \( s^{(k)}(x) \asymp YX^{1-k} \) for \( 1 < X \leq x \leq 2X \) and \( k = 1, 2, \cdots \). Then one has
\[ \sum_{X < n \leq 2X} e(s(n)) \ll Y^\kappa X^\lambda + Y^{-1}, \]
where \((\kappa, \lambda) \neq (0, 1)\) is any exponent pair.

By Lemma 3.3, we have
\[ G(x, D) \ll \sum_{m \leq 2D} |g(m)| \left( \frac{D/m}{H} + \sum_{h \leq H} \frac{1}{\bar{h}} \left( \frac{hx}{D^2/m} \right)^\kappa (D/m)^\lambda + \frac{D^2/m}{hx} \right) \]
\[ \ll \sum_{m \leq 2D} |g(m)| \left( (D/m)H^{-1} + x^\kappa H^{-\lambda} (D^2/m)^{-\kappa} (D/m)^\lambda + x^{-1}(D^2/m) \right). \]

Choosing
\[ H = \left[ D^{\frac{1+2\kappa-\lambda}{1+\kappa}}x^{-\frac{\kappa}{1+\kappa}}m^{1-\lambda+\kappa} \right] \]
gives the following result.

**Lemma 3.4** (Stucky). Let \( \delta \geq 0 \) be a fixed constant. We have
\[ G(x, D) \ll \sum_{d \leq 2D} |g(d)| \left( (x^nD^{-\kappa+\lambda}d^{-\lambda})^{1/(1+\kappa)} + x^{-1}d^{-1}D^2 \right), \]
uniformly for \( 1 \leq D \leq x \), where \((\kappa, \lambda) \neq (0, 1)\) is an exponent pair.

**Remark.** This lemma can be obtained by equation (4.2) of [9] by partial summation and the assumption about exponent pairs. As Stucky sketched the proof of equation (4.2) in [9], here we give a supplement for this.

Then the desired conclusion can be obtained by Lemma 3.4 and the assumption that
\[ \sum_{n \leq x} |g(n)| \ll x^{\lambda/(1+\kappa)}. \]

□
Now we begin the proof of Theorem 1.2. Let
\[ x^\varepsilon \leq N \leq x^{1-\varepsilon} \]
be a parameter to be chosen later. We can write
\[ S_f(x) := S_{f,1} + S_{f,2}, \]
where
\[ S_{f,1} = \sum_{n \leq N} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right) \]
and
\[ S_{f,2} = \sum_{N < n \leq x} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right). \]
Obviously, by the assumption that \( f(n) \ll n^\varepsilon \), we have
\[ S_{f,1} = \sum_{n \leq N} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{n \leq N} (x/n)^\varepsilon \ll N^{1+\varepsilon} \ll x^{\lambda/(1+\lambda)+\varepsilon}, \]
where \( N = x^{\lambda/(1+\lambda)}. \)
As to \( S_{f,2} \), firstly, by the assuming that \( f(n) = \sum_{d|n} g(d) \) and
\[ \sum_{n \leq x} |g(n)| \ll x^{\lambda/(1+\kappa)}, \]
we have
\[ \sum_{n \leq x} |f(n)| \ll x. \]
Hence we can get
\[ S_{f,2} = \sum_{N < n \leq x} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right) \]
\[ = \sum_{d \leq x/N} f(d) \sum_{x/(d+1) < n \leq x/d} 1 \]
\[ = \sum_{d \leq x/N} f(d) \left( \frac{x}{d} - \frac{x}{d+1} - \frac{x}{d} \psi \left( \frac{x}{d} \right) + \psi \left( \frac{x}{d+1} \right) \right) \]
\[ = x \sum_{d=1}^{\infty} \frac{f(d)}{d(d+1)} + O \left( N^{1+\varepsilon} \right) + O \left( \sum_{d \leq x/N} f(d) \psi \left( \frac{x}{d+\delta} \right) \right). \]
Let \( x/N = 2^k \). Then we have
\[ \sum_{d \leq x/N} f(d) \psi \left( \frac{x}{d+\delta} \right) = \sum_{D=2^k, 0 \leq l < k} G(x, D) + O(1) \]
\[ \ll 1 + \sum_{D=2^k, 0 \leq l < k} \left( x^{\kappa/(1+\kappa)} D^{\lambda-\kappa}/(1+\kappa) + D^2/x \right), \]
where we have used Lemma 3.2 and the assumption that
\[ \sum_{n \leq x} |g(n)| \ll x^{\lambda/(1+\kappa)}. \]

Then by the estimates of \( S_{f,1}, S_{f,2}, \) and choosing \( N = x^{\lambda/(1+\lambda)} \), we get
\[ S_{f}(x) = x \sum_{d=1}^{\infty} \frac{f(d)}{d(d+1)} + O \left( x^{\lambda/(1+\lambda)+\varepsilon} + (\log x)^{\alpha} x^{1-2\lambda/(1+\lambda)} \right). \]

Recall the fact such that \( \lambda \geq 1/2 \), we can finish the proof of Theorem 1.2 for \( |f(n)| \ll n^\varepsilon \).

For \( |f(n)| \ll 1 \), similar arguments give that
\[ S_{f}(x) = x \sum_{d=1}^{\infty} \frac{f(d)}{d(d+1)} + O \left( x^{\lambda/(1+\lambda)} + (\log x)^{\alpha} x^{1-2\lambda/(1+\lambda)} \right). \]

4. Remarks on Corollary 1.3

Let \( \mu_k(n) \) be the indicator function of the \( k \)-free numbers. Then we have
\[ \mu_k(n) = \sum_{d|n} g(d) \]
with
\[ g(d) = \begin{cases} 
\mu(l) & \text{if } d = l^k, \\
0 & \text{otherwise.} 
\end{cases} \]

Then we have
\[ \sum_{d \leq x} g(d) = \begin{cases} 
O \left( x^{1/2} \right) & \text{if } k = 2, \\
O \left( x^{1/3} \right) & \text{if } k \geq 3. 
\end{cases} \]

Choosing \( (\kappa, \lambda) = (1/2, 1/2) \) in Theorem 1.2 gives the case of \( k \geq 3 \) in Corollary 1.3. In fact, as \( \mu_3(n) \ll 1 \), we have
\[ S_{\mu_3}(x) = \sum_{n=1}^{\infty} \frac{\mu_3(n)}{n(n+1)} x + O \left( x^{1/3} (\log x)^2 \right). \]

And one log can be cancelled by partial summation for \( k > 3 \). Hence for \( k > 3 \), we have
\[ S_{\mu_k}(x) = \sum_{n=1}^{\infty} \frac{\mu_k(n)}{n(n+1)} x + O \left( x^{1/3} (\log x) \right). \]

Choosing \( (\kappa, \lambda) = BABAAB(0, 1) = (4/18, 11/18) \) in Theorem 1.2, we can obtain Corollary 1.3 for \( k = 2 \). However, for \( k = 2 \), the log term in the error term is not really important since the exponent is probably not the best possible one.

In fact, inspired by the recent work of the author [13], we can obtain a slightly better result for the case of \( k = 2 \) in Corollary 1.3. Assuming that \( (a, 1/2 + a) \) is an exponent pair, then by
\[ BABA(a, 1/2 + a) = \left( \frac{2a + 1}{6a + 5}, \frac{4a + 3}{6a + 5} \right), \]
we can get
\[ S_{\mu_2}(x) = \sum_{n=1}^{\infty} \frac{\mu_2(n)}{n(n+1)} x + O\left(x^{\vartheta(a)+\varepsilon}\right) \]
such that
\[ \vartheta(a) = \frac{4a + 3}{10a + 8}. \]
Choosing \( a = 1/6 \), we have
\[ S_{\mu_2}(x) = \sum_{n=1}^{\infty} \frac{\mu_2(n)}{n(n+1)} x + O\left(x^{11/29+\varepsilon}\right). \]
This gives the result of \( k = 2 \). By the work of Bourgain [5], we can choose \( a = 13/84 + \varepsilon \).
Then we have
\[ S_{\mu_2}(x) = \sum_{n=1}^{\infty} \frac{\mu_2(n)}{n(n+1)} x + O\left(x^{152/401+\varepsilon}\right), \]
which gives an improvement of 11/29.

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