On the existence of attractors

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Abstract

On every compact 3-manifold, we build a non-empty open set $U$ of $\text{Diff}^1(M)$ such that, for every $r \geq 1$, every $C^r$-generic diffeomorphism $f \in U \cap \text{Diff}^r(M)$ has no topological attractors. On higher dimensional manifolds, one may require that $f$ has neither topological attractors nor topological repellers. Our examples have finitely many quasi attractors. For flows, we may require that these quasi attractors contain singular points. Finally we discuss alternative definitions of attractors which may be better adapted to generic dynamics.

1 Introduction

The aim of dynamical systems is to describe the asymptotic behavior of the orbits when the time tends to infinity. For simple dynamical systems, the behavior of the orbits looks like the gradient flow of a Morse function: most of the orbits tend to a sink, and the union of the basins of the sink is a dense open set in the ambient manifold.

However, many dynamical systems present a more complicated behavior and many orbits do not tend to periodic orbits; their $\omega$-limit set may be chaotic. In the sixties and seventies, many people tried to give a definition of attracting sets, allowing to describe most of the possible behaviors of dynamical systems. An attractor $\Lambda$ of a diffeomorphism $f$ needs to satisfy two kinds of properties:

• it attracts “many orbits”. According to the authors, this means: the basin of $\Lambda$ contains a neighborhood of $\Lambda$, an open set, a residual subset of an open set, a set with positive Lebesgue measure, . . . .

• it is indecomposable, that is, it cannot split into the union of smaller attractors; many notions of indecomposability are used: transitivity (generic orbits of the attractor are dense in the attractor), chain recurrence (for every $\delta > 0$, one can go from any point of the attractor to any point of the attractor by $\delta$-pseudo orbits inside the attractor), uniqueness of the SRB measure, . . . .

None of these notions can cover all the possible behaviors of dynamical systems. For every notion of (indecomposable) attractors, one can find examples of dynamical systems without attractors. 1

A natural idea for bypassing this difficulty is to restrict the study to generic dynamical systems, in order to avoid the most pathological and fragile behaviors. A property is $C^r$-generic if it holds on a residual subset of the space of $C^r$ diffeomorphisms $\text{Diff}^r(M)$ endowed with the $C^r$ topology.

This viewpoint has been considered very early by Smale and Thom, with the hope that generic dynamical systems would have a simple behavior. For instance one can read in [T, Chapter 4.1 B]: Il n’est pas certain qu’un champ $X$ donné dans $M$ présente des attracteurs, a fortiori des attracteurs structurellement stables. Toutefois, selon certaines idées récentes de Smale, si la variété $M$ est

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1If one removes the indecomposability hypothesis, Conley shows that attractors exist for any homeomorphism of compact metric space. More precisely, given any points $x, y$, either one can join $x$ to $y$ by $\delta$-pseudo orbits, for every $\delta > 0$, or there is an attracting region $U$ containing $x$ but not $y$. Hence the dynamics admits attracting regions, or the chain recurrent set is the whole space. Conley calls attractors the maximal invariant sets in the attracting regions. The attractors in Conley theory are not assumed to be indecomposable: an attractor can contain smaller attractors.
compacte, presque tout champ présenterait un nombre fini d’attracteurs isolément structurellement stables;(...). Thom’s idea was renewed and formalized in 1975 as Thom’s conjecture by Palis and Pugh [PP, Problem 26]: There is a dense open set in Diff$^r(M)$ such that for almost every point $x \in M$, the $\omega$-limit set $\omega(x)$ is a topological attractor, and each attractor is topologically stable.

After thirty years of progress in the field, this conjecture can look naively optimistic. Indeed, Thom’s original idea was disproved in most of its aspects: finiteness, stability.

- there are open sets of systems without structurally stable attractors, as the robustly transitive non-hyperbolic diffeomorphisms built by Shub in [Sh];
- there are $C^r$-locally generic diffeomorphisms having infinitely many sinks (see [N1, N2] for $r \geq 2$ and [BD] for $r = 1$).

However, the existence of at least one attractor remains an open question. In this paper, we will give a negative answer to this question, showing that the usual notion of topological attractor is too strong and not adapted to generic dynamical systems. Let us be now somewhat more precise.

A topological attractor of a diffeomorphism $f : M \to M$ is a compact subset $\Lambda \subset M$ with the following properties

- $\Lambda$ is invariant (i.e. $f(\Lambda) = \Lambda$);
- $\Lambda$ admits a compact neighborhood $U$ which is an attracting region (i.e. the image $f(U)$ is contained in the interior of $U$) such that all the orbits in $U$ converge to $\Lambda$: $\Lambda = \bigcap_{n \in \mathbb{N}} f^n(U)$;
- $\Lambda$ is transitive (i.e. the positive orbits of generic points in $\Lambda$ are dense in $\Lambda$), or at least chain recurrent$^4$ (i.e. any two points in $\Lambda$ can be joined by $\varepsilon$-pseudo orbits, for all $\varepsilon > 0$). We will speak on transitive topological attractor, and on chain recurrent topological attractor, if we need to emphasize the kind of indecomposability we require.

A topological repeller of $f$ is, by definition, a topological attractor of $f^{-1}$. In 2004 [BC] and in 2005 [BDV, Problem 10.35] still asked:

**Question 1.** Do $C^1$-generic diffeomorphisms admit at least one (chain recurrent or transitive$^4$) topological attractor? Does the union of the basins of the topological attractors cover a dense open subset of the manifold?

The answer is “no”: The fact that $C^0$-generic homeomorphisms have no attractors is known from Hurley’s work [H]: every attracting region of a $C^0$-generic homeomorphism contains infinitely many repelling regions and infinitely many disjoint attracting regions. Theorems A and B show that, for every $r \geq 1$, the property of having (at least) one topological attractor is not $C^r$-generic.

Our results use the notion of quasi attractors, introduced by Hurley: a chain recurrence class of a homeomorphism is a quasi attractor$^5$ if it is the intersection of a sequence of attracting regions. A quasi repeller for $h$ is a quasi attractor for $h^{-1}$.

**Theorem A.** For every three-dimensional manifold $M$, there is a non-empty $C^1$-open subset $U \subset \text{Diff}^1(M)$ such that:

- there are hyperbolic periodic saddles $p_1, ..., p_k$ varying continuously with $f \in U$, whose chain recurrence classes $\Lambda_1, ..., \Lambda_k$ are the unique quasi attractors of $f$;
- the set $\{f \in U, f \text{ has no attractors}\}$ is $C^r$-residual in $U \cap \text{Diff}^r(M)$ for every $r \geq 1$.

The $C^r$-generic diffeomorphisms $f$ in the open set $U$ have no attractors but infinitely many repellers. This motivates the following problem:

$^2$“It is not clear if a given vector field in $M$ has an attractor, a fortiori a structurally stable attractor; however, according to recent ideas by Smale, if the manifold is compact, almost all vector fields would admit finitely many attractors, each of them structurally stable;(...).”

$^3$some authors say “chain transitive”.

$^4$Chain recurrent topological attractors of $C^1$-generic diffeomorphisms are homoclinic classes, hence are transitive.

$^5$Some authors use the terminology “weak attractor” instead of quasi attractor.
Problem. For three-dimensional manifold $M^3$, is there a dense (open and dense, residual) set $\mathcal{D} \subset \text{Diff}^r(M)$, such that any $f \in \mathcal{D}$ has neither attractors nor repellers?

Theorem B. For every compact manifold $M$, with $\dim M \geq 4$ there is a non-empty open set $\mathcal{U} \subset \text{Diff}^1(M)$ such that:

- there are hyperbolic periodic saddles $p_{1,f}, \ldots, p_{k,f}$ varying continuously with $f \in \mathcal{U}$, whose chain recurrence classes $\Lambda_{1,f}, \ldots, \Lambda_{k,f}$ are the unique quasi attractors of $f$;
- there are hyperbolic periodic saddles $q_{1,f}, \ldots, q_{l,f}$ varying continuously with $f \in \mathcal{U}$, whose chain recurrence classes $\Sigma_{1,f}, \ldots, \Sigma_{l,f}$ are the unique quasi repellers of $f$;
- the set $\{f \in \mathcal{U}, f \text{ has neither attractors nor repellers}\}$ is $C^r$-residual in $\mathcal{U} \cap \text{Diff}^r(M)$ for every $r \geq 1$.

Our results can be easily adapted for vector fields, building locally generic vector fields having finitely many (non-singular) quasi attractors but no attractors. However, one of the main differences between diffeomorphisms and flows is the existence of singularities, in particular when these singularities are not isolated from the regular part of the limit set of the flow.

This phenomenon has first been suspected experimentally by Lorenz [Lo], and then proved rigorously in [Gu, ABS, GuW], where the authors exhibited, in dimension 3, a $C^1$-open set of vector fields having a robust attractor containing infinitely many periodic orbits accumulating on a saddle singularity. Their construction (known as geometric model of Lorenz attractor) leads to the notion of singular attractors, which have been studied in extends on 3-manifolds: for instance, if the presence of a singularity inside the attractor prevents the usual definition of hyperbolicity, robust singular attractors in dimension 3 always present a kind of weak hyperbolicity called singular hyperbolicity, see [MPP1, MPP2]. In particular, they satisfy the star condition: $C^1$-robustly all the periodic orbits are hyperbolic. [LGW, GWZ, MM] show that in any dimension, robust singular attractors satisfying the star condition are singular hyperbolic. Recent examples [BKR] [BLY] show that robust singular attractors may satisfy neither the star condition nor the singular hyperbolicity. However even these new examples admit a strong stable direction, invariant by the flow and dominated by a center-unstable bundle.

Hence it is natural to ask:

Question 2. Does every $C^1$-robust singular attractor admit a strong stable bundle?

Indeed, this question has been our first motivation for this work. Before presenting our results, let us make a comment on this question. Examples of (non-singular) robustly transitive attractor whose flow does not admit any dominated splitting are already known (just consider the suspension flows of robustly transitive diffeomorphisms without invariant hyperbolic bundles in [BV]). For this reason one considers the linear Poincaré flow on the normal bundle and this flow admits a dominated splitting. However, the linear Poincaré flow is not defined on the singularity: for this reason, it is not clear what kind of hyperbolicity will satisfy the singular attractors.

Now we state our result for flows. Our construction can be adapted in order to build a robust singular quasi attractor whose tangent bundle doesn’t have any dominated splitting with respect to the tangent flow.

Theorem C. There is a non-empty open set $\mathcal{U}$ of the space $\mathcal{X}^r(B^4)$ of $C^r$ vector fields on the 4-ball, such that:

- any $X \in \mathcal{U}$ is transverse to the boundary and entering inside the ball;
- any $X \in \mathcal{U}$ has a unique zero $0_X$ in $B^4$; one denotes by $\Lambda_X$ the chain recurrence class of $0_X$;
- any $X \in \mathcal{U}$ has a unique quasi attractor in $B^4$ which is $\Lambda_X$;
- the subset $\{X \in \mathcal{U}, \Lambda_X \text{ is not an attractor}\}$ is $C^r$-residual in $\mathcal{U}$;
- for $X \in \mathcal{U}$, there is no dominated splitting for the tangent flow of $X$ on $\Lambda_X$. 

3
1.1 Organization of the paper

Our main result is the construction in Section 3 of an example of locally generic diffeomorphisms of the solid torus $S^1 \times D^2$, without attractors.

Putting the solid torus in a ball $B^3$, we get a model of an attracting ball without attractors, which allows us, in Section 4 to replace the sinks of a gradient like diffeomorphism by these attracting balls without attractors, proving Theorem A.

Multiplying this ball $B^3$ by a normal contraction, one gets in Section 4.3 an attracting ball $B^n$, for $n > 3$, without attractors and repellers. This section ends the proof of Theorem B.

Section 5 considers the case of vector fields and shows that our construction in Section 3 leads to locally generic vector fields $X$ on 4-manifolds having a unique quasi attractor $\Lambda_X$ and no attractors; furthermore $\Lambda_X$ is the chain recurrence class of a singularity of $X$.

Section 6 concludes this paper by discussing alternative notions of attractors which could be better adapted to generic dynamical systems.

2 Notations, definitions and preliminaries

2.1 Disks and balls

For every $d \in \mathbb{N}$ and $r \in \mathbb{R}$, we denote by $D^d(r)$ the closed ball in $\mathbb{R}^d$ centered at 0 and with radius $r$, i.e., $D^d(r) = \{ x \in \mathbb{R}^d : \|x\| \leq r \}$. For simplicity, we denote $D^d = D^d(1)$. Given a compact Riemannian manifold $M$, a point $x \in M$, and a real number $\delta > 0$, we denote $B_\delta(x) = \{ y \in M : d(x,y) \leq \delta \}$, the compact ball centered at $x$ and with radius $r$.

Recall that every orientation preserving diffeomorphism of $D^2$ is smoothly isotopic to the identity.

An essential disk in $S^1 \times D^2$ is an embedding $D : D^2 \hookrightarrow S^1 \times D^2$ whose boundary $\partial D = D(\partial D^2)$ is contained in $\partial (S^1 \times D^2) = S^1 \times S^1$, and is not homotopic to a point in $S^1 \times S^1$.

2.2 Hyperbolicity, partial hyperbolicity, dominated splitting

Let $f$ be a diffeomorphism of a manifold $M$ of dimension $d$, a periodic point of $f$, and $\pi$ its period. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$ be the moduli of the eigenvalues of the differential $Df^n(x)$. The point $x$ is hyperbolic if $\lambda_i \neq 1$ for all $i \in \{1, \ldots, d\}$. The point $x$ is sectionally area expanding (or sectionally expanding) if

$$\lambda_i \lambda_j > 1,$$

for all $i, j \in \{1, \ldots, d\}, i \neq j$.

We say a compact invariant set $\Lambda$ of $f$ is hyperbolic if there are $Df$-invariant splitting

$$TM|_\Lambda = E^s \oplus E^u,$$

and constants $C > 0$, $\lambda \in (0,1)$ such that for any $x \in \Lambda$ and $n \in \mathbb{N}$

$$\|Df^n|_{E^s(x)}\| \leq C\lambda^n, \quad \|Df^{-n}|_{E^u(x)}\| \leq C\lambda^n.$$

The bundles $E^s$ and $E^u$ are called the stable and unstable bundle of $\Lambda$, respectively. They are always continuous bundles, so that the dimensions $\dim E^s(x)$ and $\dim E^u(x)$ are locally constant. If $\dim E^u(x)$ is independent on $x \in \Lambda$, then we call $E^u$ the index of the hyperbolic set $\Lambda$.

We say $\Lambda$ is a basic set if $\Lambda$ is an isolated hyperbolic transitive set: there is an open neighborhood $U$ of $\Lambda$ such that $\Lambda = \bigcap_{U \in F(U)}$.

Given a compact invariant set $\Lambda$, a $Df$-invariant splitting $TM|_\Lambda = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is dominated, and we denote $E_1 \oplus \cdots \oplus E_k$, if the dimensions $\dim(E_i)$ are constant over $\Lambda$, and if there are constants $C > 0$, $\lambda \in (0,1)$ such that for any $x \in \Lambda$, $n \in \mathbb{N}$ and $i \in \{1, \ldots, k-1\}$, we have

$$\|Df^n|_{E_i(x)}\| \|Df^{-n}|_{E_{i+1}+(f^n(x))}\| \leq C\lambda^n.$$

A $Df$-invariant bundle $E$ is called (uniformly) contracting if there are constants $C > 0$, $\lambda \in (0,1)$ such that for any $x \in \Lambda$ and $n \in \mathbb{N}$, we have $\|Df^n|_{E(x)}\| \leq C\lambda^n$; it is called (uniformly) expanding if it is contracting for $f^{-1}$.

A dominated splitting $E_1 \oplus \cdots \oplus E_k$ is partially hyperbolic if $E_1$ is uniformly contracting or $E_k$ is uniformly expanding.
2.3 Cone fields associated to a dominated splitting

Given a compact set \( V \subset M \), a continuous (not necessary invariant) bundle \( F \subset TM \), and a positive number \( \alpha > 0 \), the cone field on \( V \) associated to \( F \) of size \( \alpha > 0 \) is

\[
C_\alpha^F(x) = \{ v \in T_xM : \exists v_F \in F, v_{F^\perp} \in F^\perp, \text{ s.t. } v = v_F + v_{F^\perp}, |v_{F^\perp}| \leq \alpha |v_F| \}
\]

for \( x \in V \), where \( F^\perp \) is the orthogonal subbundle of \( F \).

We say a cone field \( C_\alpha^F \) is strictly \( Df \)-invariant, if there is \( \beta \in (0, \alpha) \) such that, for any \( x \in V \) such that \( f(x) \in V \), we have

\[
Df(C_\alpha^F(x)) \subset C_\beta^F(f(x)).
\]

If an invariant compact set \( \Lambda \) has a dominated splitting \( T_\Lambda M = E \oplus F \), then there is \( \alpha_0 > 0 \), such that for any \( \alpha \in (0, \alpha_0) \), there is \( N \in \mathbb{N} \) such that the cone field \( C_\alpha^F \) is strictly \( Df^{-N} \)-invariant and the cone field \( C_\alpha^F \) is strictly \( Df^N \)-invariant.

2.4 Conley theory and quasi attractors

Let \( (X, d) \) be a compact metric space and \( f: X \to X \) a homeomorphism.

For any \( x, y \in X \), we denote \( x \sim y \) if for every \( \epsilon > 0 \) there is a \( \epsilon \)-pseudo orbit joining \( x \) to \( y \), that is: there are \( n > 0 \) and a sequence of points \( \{ x = x_0, x_1, \cdots, x_n = y \} \) verifying \( d(f(x_i), x_{i+1}) < \epsilon \) for \( 0 \leq i \leq n - 1 \).

We say that \( x \) is chain recurrent if \( x \sim x \), and we denote by \( \mathcal{R}(f) \) the set of chain recurrent points of \( f \), called the chain recurrent set of \( f \).

An invariant compact set \( K \) of \( X \) is chain recurrent (or chain transitive) if every point \( x \in K \) is chain recurrent for the restriction \( f|_K \); in other words, \( K = \mathcal{R}(f|_K) \).

We say \( x \) and \( y \) are chain equivalent if \( x \sim y \) and \( y \sim x \). The chain equivalence is an equivalence relation on \( \mathcal{R}(f) \). For any \( x \in \mathcal{R}(f) \), the equivalence class of \( x \) is called the chain recurrence class of \( x \), and denoted by \( C(x) \).

A quasi attractor \( \Lambda \) is a chain transitive set which admits a base of neighborhood which are attracting regions (this implies that \( \Lambda \) is a chain recurrence class).

2.5 Plykin attractor

Let \( \text{Diff}^1_0(\mathbb{D}^2, \text{Int}(\mathbb{D}^2)) \) denote the space of orientation preserving \( C^1 \)-embeddings \( \phi: \mathbb{D}^2 \to \text{Int}(\mathbb{D}^2) \). Notice that the elements of \( \text{Diff}^1_0(\mathbb{D}^2, \text{Int}(\mathbb{D}^2)) \) are all isotopic, in particular are isotopic to any linear contraction of \( \mathbb{D}^2 \).

In [Pl] Plykin built a non-empty open subset \( \mathcal{P} \subset \text{Diff}^1_0(\mathbb{D}^2, \text{Int}(\mathbb{D}^2)) \) such that for any \( \phi \in \mathcal{P} \) the chain recurrent set of \( \phi \) consists in the union of a non-trivial hyperbolic attractor \( A_\phi \) and a finite set of periodic sources.

We denote by \( \mathcal{P}_0 \subset \mathcal{P} \) the non-empty open subset of diffeomorphisms such that the hyperbolic attractor \( A_\phi \) contains a fixed point \( x_\phi \) which is an area expanding saddle point:

\[
\text{Det}(D\phi(x_\phi)) > 1.
\]

2.6 Solenoid maps associated to a braid in \( S^1 \times \mathbb{D}^2 \)

A connected braid \( \gamma \) of \( S^1 \times \mathbb{D}^2 \) is (the isotopy class of) an embedding of the circle \( S^1 \) in \( S^1 \times \mathbb{D}^2 \), transverse to the fibers \( \{ \emptyset \} \times \mathbb{D}^2 \), for \( \emptyset \in S^1 \). The projection \( S^1 \times \mathbb{D}^2 \to S^1 \) induces on \( \gamma \) a finite covering of the circle; we denote by \( n_\gamma \neq 0 \) the order of this finite cover.

For any braid \( \gamma \), we denote by \( \mathcal{U}_\gamma \) the (non-empty) open subset of diffeomorphisms \( f: S^1 \times \mathbb{D}^2 \to \text{Int}(S^1 \times \mathbb{D}^2) \) such that \( f(S^1 \times \{ \emptyset \}) \) is isotopic to the braid \( \gamma \).

We call canonical solenoid maps associated to a braid \( \gamma \) the maps built as follows: denote \( n = n_\gamma \); we choose a representative \( \gamma: S^1 \to S^1 \times \mathbb{D}^2 \) of the braid having the following form:

\[
\gamma(t) = (n.t, z(t)).
\]

We fix \( \delta > 0 \) such that

for all \( t \in S^1 \), \( d(z(t), \{ n.t \} \times \partial \mathbb{D}^2) > 2\delta \);

for any \( t_1, t_2 \in S^1 \), \( (t_1 \neq t_2 \text{ and } n.t_1 = n.t_2 \in S^1) \Rightarrow d(z(t_1), z(t_2)) > 2\delta \).

Now the map \( f_{\gamma,\delta} \) defined on \( S^1 \times \mathbb{D}^2 \) by \( f_{\gamma,\delta}(t, z) = (n.t, \delta.z + z(t)) \) belongs to \( \mathcal{U}_\gamma \) and is called a canonical solenoid map associated to a braid \( \gamma \).
2.7 Partially hyperbolic solenoid maps

For every $\alpha > 0$ we denote by $C_\alpha$ the cone field on $S^1 \times \mathbb{D}^2$ defined by

$$C_\alpha(x) = \{ u = (u_1, u_2) \in T_x(S^1 \times \mathbb{D}^2) = \mathbb{R} \times \mathbb{R}^2 : |u_2| \leq \alpha |u_1| \}.$$  

We denote by $\mathcal{U}_\gamma^{\text{part.hyp}}$ the set of diffeomorphisms $f \in \mathcal{U}_\gamma$ such that there are $\alpha > 0$ and $\ell \in \mathbb{N} \setminus \{0\}$ such that:

- the cone field $C_\alpha$ is strictly invariant under $Df^\ell$;
- there is $\lambda > 1$ such that for every $x \in S^1 \times \mathbb{D}^2$ and every vector $u = (u_1, u_2) \in C_\alpha(x)$ one has:
  $$|v_1| \geq \lambda |u_1|,$$
  where $Df^\ell(u) = (v_1, v_2) \in T_{f^\ell(x)}(S^1 \times \mathbb{D}^2)$.

The set $\mathcal{U}_\gamma^{\text{part.hyp}}$ is a $C^1$-open subset of $\mathcal{U}_\gamma$. Moreover, one easily verifies:

**Lemma 2.1.** Let $f \in \mathcal{U}_\gamma$ be of the form $(t, z) \mapsto (\gamma(t), \phi_t(z))$. Assume that:

- for every $t \in S^1$ one has
  $$\left| \frac{d}{dt} \gamma(t) \right| > 1;
  $$
- for every $(t, z) \in S^1 \times \mathbb{D}^2$ one has
  $$\| Dz(\phi_t) \| < \left| \frac{d}{dt} \gamma(t) \right|. $$

Then $f$ is partially hyperbolic; more precisely, $f \in \mathcal{U}_\gamma^{\text{part.hyp}}$.

As a direct consequence one gets:

**Corollary 2.2.** For every braid $\gamma$ with $|n_\gamma| \geq 2$ every canonical solenoid map $f$ associated to $\gamma$ belongs to $\mathcal{U}_\gamma^{\text{part.hyp}}$.

In particular the open set $\mathcal{U}_\gamma^{\text{part.hyp}}$ is non-empty.

**Corollary 2.3.** Let $f \in \mathcal{U}_\gamma$ satisfying the hypotheses of Lemma 2.1, and $h : S^1 \times \mathbb{D}^2 \to S^1 \times \mathbb{D}^2$ be a diffeomorphism of the form $(t, z) \mapsto (t, h_t(z))$ where $h_t$ is an orientation preserving diffeomorphism of $\mathbb{D}^2$.

Then the map $g = h^{-1}fh$ belongs to $\mathcal{U}_\gamma^{\text{part.hyp}}$.

2.8 Realization of a map $\varphi \in \text{Diff}_0^1(\mathbb{D}^2, \text{Int}(\mathbb{D}^2))$ by a solenoid map $f \in \mathcal{U}_\gamma^{\text{part.hyp}}$

The aim of this section is to prove:

**Proposition 2.4.** Given any $\varphi \in \text{Diff}_0^1(\mathbb{D}^2, \text{Int}(\mathbb{D}^2))$ and any braid $\gamma$ with $|n_\gamma| \geq 2$, there is a diffeomorphism $f \in \mathcal{U}_\gamma^{\text{part.hyp}}$ such that the disk $\{0\} \times \mathbb{D}^2$ is positively invariant (and normally hyperbolic), and the restriction of $f$ to $\{0\} \times \mathbb{D}^2$ is $\varphi$.

**Proof:** We denote $n = n_\gamma$. We choose a representative $\gamma : S^1 \to S^1 \times \mathbb{D}^2$, $\gamma(t) = (n.t, z(t))$. Consider a canonical solenoid map $f_{\gamma, \delta}$, associated to the braid $\gamma$, for some $0 < \delta < 1$. Recall that $f_{\gamma, \delta}(t, z) = (n.t, z(t) + h_\delta(z))$ where $h_\delta : \mathbb{D}^2 \to \text{Int}(\mathbb{D}^2)$ is the homothety of ration $\delta$.

Consider $\varphi \in \text{Diff}_0^1(\mathbb{D}^2, \text{Int}(\mathbb{D}^2))$. By using Corollary 2.3, one just needs to prove Proposition 2.4 for a conjugate of $\varphi$ by an orientation preserving diffeomorphism of $\mathbb{D}^2$.

This allows us to assume that $\varphi(\mathbb{D}^2)$ is contained in the disk $\mathbb{D}^2(\delta)$ of radius $\delta$ and that there is a differentiable isotopy from $\varphi$ to the homothety $h_\delta$, whose image remains contained in $\mathbb{D}^2(\delta)$.

More precisely, there is a $C^1$-map $\Phi : \mathbb{D}^2 \times [-1, 1] \to \text{Int}(\mathbb{D}^2)$ of the form $\Phi(x, t) = \varphi_t(x)$ where:

- for every $t \in [-1, 1]$ one has $\varphi_t \in \text{Diff}_0^1(\mathbb{D}^2, \text{Int}(\mathbb{D}^2))$;
- for every $t \in [-1, 1]$ one has $\varphi_t(\mathbb{D}^2) \subset \mathbb{D}^2(\delta)$. 

6
• $\varphi_0 = \varphi$, and
• $\varphi_t = h_\delta$ if $|t| \geq \frac{1}{2}$.

We denote $C = \max_{x \in [-1, 1], z \in \mathbb{D}^2} \|D_z(\varphi_t)\|$. Let $\psi : [-\frac{1}{|n|}, \frac{1}{|n|}] \to [-1, 1]$ be a diffeomorphism such that there is $0 < \varepsilon < \frac{C}{2|n|}$ with the following properties:

• $\psi(\frac{1}{n}) = 1$ and $\psi(-\frac{1}{n}) = -1$;
• for every $t \in [-\frac{1}{|n|}, \frac{1}{|n|}]$ one has $\left|\frac{d}{dt}\psi(t)\right| > 1$;
• $\left|\frac{d}{dt}\psi(t)\right| > 2C$ for every $t \in [-\varepsilon, \varepsilon]$;
• $\left|\frac{d}{dt}\psi(t)\right| = |n|$ for $|t| \leq \frac{1}{|n|} - \varepsilon$.

We define $f : S^1 \times \mathbb{D}^2 \to \text{Int}(S^1 \times \mathbb{D}^2)$ as follows:

• $f(t, z) = (\psi(t), z(\psi(t)) + \varphi_\frac{1}{n}(z))$ if $t \in [-\varepsilon, \varepsilon]$,
• $f(t, z) = (\psi(t), z(\psi(t)) + h_\delta(z))$ if $|t| \in \varepsilon, \frac{1}{|n|}$,
• $f(t, z) = (n, t, z + h_\delta(z))$ if $t \not\in [-\frac{1}{|n|}, \frac{1}{|n|}]$.

Notice that $f(0) \times \mathbb{D}^2 \subset \{0\} \times \mathbb{D}^2$, the disk is normally hyperbolic and the restriction of $f$ to that disk induces $\varphi$. One concludes the proof of Proposition 2.4 by proving:

**Claim 1.** The map $f$ defined above belongs to $\mathcal{U}_\gamma^{\text{part.hyp}}$.

**Proof:** We first notice that the image $f(t, 0)$ belongs to the curve $\gamma(S^1)$; in other words $f(t, 0) = \gamma(\tau_t)$, where $t \mapsto \tau_t$ is a diffeomorphism of the circle. So the image of $\{t\} \times \mathbb{D}^2$ is contained in a disc of radius $\delta$ in $\{n, \tau_t\} \times \mathbb{D}^2$ centered at $\gamma(\tau_t)$. As a consequence, if $r \neq s$ then $f(\{r\} \times \mathbb{D}^2) \cap f(\{s\} \times \mathbb{D}^2) = \emptyset$. One deduces that $f$ is injective, hence is a diffeomorphism from $S^1 \times \mathbb{D}^2$ onto its image contained in $f_{\gamma, \delta}(S^1 \times \mathbb{D}^2)$. One deduces that $f$ belongs to $\mathcal{U}_\gamma$.

In order to get the partial hyperbolicity, we will verify that $f$ satisfies the hypotheses of Lemma 2.1 in each of the possible expressions. We first notice that $f$ keeps invariant the trivial foliation of $S^1 \times \mathbb{D}^2$ by the disks $\{t\} \times \mathbb{D}^2$. It remains to get the control of the derivative of $f$.

The map $f$ coincides with $f_{\gamma, \delta}$ out of $[-\frac{1}{|n|}, \frac{1}{|n|}] \times \mathbb{D}^2$, giving the condition in this region. On $\left([-\frac{1}{|n|}, -\frac{1}{|n|}] + \varepsilon \cup [\frac{1}{|n|} - \varepsilon, \frac{1}{|n|}]\right) \times \mathbb{D}^2$, one notices that the derivative of the restriction of $f$ to each disk $\{t\} \times \mathbb{D}^2$ is the homothety of ratio $0 < \delta < 1$; hence the conclusion holds because $\left|\frac{d}{dt}\psi(t)\right| > 1$.

Finally, for $t \in \varepsilon, \frac{1}{|n|}$ the derivative of the restriction of $f$ to the disk $\{t\} \times \mathbb{D}^2$ is bounded by the constant $C$, and $\left|\frac{d}{dt}\psi(t)\right| > C$, by assumption. \hfill $\Box$

\hfill $\Box$

3 \textbf{An attracting solid torus} $S^1 \times \mathbb{D}^2$ \textbf{without attractors.}

Our main results are consequences of a construction in the solid torus $S^1 \times \mathbb{D}^2$, that we explain in this section.

3.1 \textbf{Plykin attractors on normally hyperbolic disks, for solenoid maps}

Recall that $\mathcal{P}_0$ is the open set of structurally stable diffeomorphisms in $\text{Diff}^1_0(\mathbb{D}^2, \text{Int}(\mathbb{D}^2))$, defined at Section 2.5, whose non-wandering set consists exactly in the union of a non-trivial hyperbolic attractor (a Plykin attractor) and a finite set of periodic sources.

Given any braid $\gamma$ with $|n_\gamma| \geq 2$, let $\mathcal{U}_\gamma^{\text{Ply}}$ denote the set of diffeomorphisms $f \in \mathcal{U}_\gamma^{\text{part.hyp}}$ such that $f$ leaves positively invariant a normally hyperbolic essential disk $D_f$, and such that the restriction $\phi_f$ of $f$ to $D_f$ is $C^1$-conjugate to an element $\phi \in \mathcal{P}_0$.

As a corollary of Proposition 2.4 one gets:
Corollary 3.1. Given any braid $\gamma$ with $|n_\gamma| \geq 2$, the set $U^{\text{Ply}}_{\gamma}$ is a non-empty $C^1$-open subset of $U^{\text{part.hyp}}_{\gamma}$.

Proof: Proposition 2.4 implies that $U^{\text{Ply}}_{\gamma}$ is non-empty. It is open because the disk $D_f$ is normally hyperbolic, hence persists by perturbation and vary $C^1$-continuously with $f$; hence the restriction $\phi_f$ varies $C^1$-continuously with $f$; one concludes by recalling that $\mathcal{P}_0$ is an open subset of $\text{Diff}_0^1(\mathbb{D}^2, \text{Int}(\mathbb{D}^2))$. □

Let $\gamma \subset S^1 \times \mathbb{D}^2$ be a braid with $|n_\gamma| \geq 2$. Consider $f \in U^{\text{Ply}}_{\gamma}$. By definition of $U^{\text{Ply}}_{\gamma}$ and $\mathcal{P}_0$, one has the following properties:

- there are $\alpha_f > 0$ and $\ell > 0$ such that the cone field $C_{\alpha_f}$ is strictly invariant by $Df^\ell$;
- the disk $D_f$ is positively invariant and normally hyperbolic; hence the disk $D_f$ is transverse to the cone field $C_{\alpha_f}$;
- the restriction $\phi_f$ of $f$ to $D_f$ belongs to $\mathcal{P}_0$; hence, the disk $D_f$ contains a Plykin attractor $A_f$ of $\phi_f$; as the disk $D_f$ is normally hyperbolic, $A_f$ is a hyperbolic basic set for $f$;
- the Plykin attractor $A_f$ contains a hyperbolic fixed point $p_f = x_{\phi_f}$ such that $\text{Det}(D\phi_f(p_f)) > 1$; as a consequence, the product of any two eigenvalues of $Df(p_f)$ has modulus larger than 1; hence, the point $p_f$ is a sectionally expanding fixed point of $f$;
- we denote by $\Lambda_f$ the chain recurrence class of $A_f$; in an equivalent way, $\Lambda_f$ is the chain recurrence class of the fixed point $p_f$.

3.2 Statement of our main result

Theorem 1. Given any braid $\gamma$ with $|n_\gamma| \geq 2$,

1. For every $f \in U^{\text{Ply}}_{\gamma}$, the chain recurrence class $\Lambda_f$ is the unique quasi attractor of $f$.

2. We denote

$$U_{\text{wild}, \gamma} = \{ f \in U^{\text{Ply}}_{\gamma}, \Lambda_f \cap \{\text{sources of } f\} \neq \emptyset \}.$$ 

In particular, $\Lambda_f$ is not an attractor for $f \in U_{\text{wild}, \gamma}$.

Then, for every $r \geq 1$, the subset $U_{\text{wild}, \gamma}$ is residual in $U^{\text{Ply}}_{\gamma}$ for the $C^r$ topology.

According to [BC], for $C^1$-generic diffeomorphisms, the $\omega$-limit set $\omega(x)$ of any generic point $x$ of the manifold is a quasi attractor. Hence the item 1 of Theorem 1 implies:

Corollary 3.2. There is a $C^1$-residual subset of $U^{\text{Ply}}_{\gamma}$ of diffeomorphisms $f$ for which the basin of $\Lambda_f$ is residual in $S^1 \times \mathbb{D}^2$.

We don’t know if Corollary 3.2 holds for $C^r$-topology, $r > 1$. However, we think that it is possible to prove:

Conjecture 1. There is a $C^2$-open subset of $U^{\text{Ply}}_{\gamma}$ of diffeomorphisms for which $\Lambda_f$ carries an SRB-measure whose basin has total Lebesgue measure in $S^1 \times \mathbb{D}^2$.

3.3 First step of the proof of Theorem 1: uniqueness of the quasi attractor

Proof: Let $U \subset S^1 \times \mathbb{D}^2$ be an open attracting region of $f$: $f(U) \subset U$. Consider a segment $\sigma \subset U$ which is tangent to $C_{\alpha_f}$. As $Df^\ell$ leaves strictly invariant the cone field $C_{\alpha_f}$ and expands the vectors in that cone field, the forward iterates $f^n(\sigma)$, $n > 0$, remain tangent to $C_{\alpha_f}$ and their length tends to $\infty$. One deduces that there is $n > 0$ such that $f^n(\sigma) \cap D_f \neq \emptyset$. Hence $f^n(U) \cap D_f$ contains a non-empty open set. By definition of $\mathcal{P}_0$ the basin of the Plykin attractor $A_f$ of $\phi_f$ is a dense open subset of $D_f$. As a consequence, $f^n(U)$ contains a point $x$ in this basin. So $\omega(x, f) \subset A_f$. However $\omega(x) \subset U$ because $U$ is by definition an attracting region. So $U \cap A_f \neq \emptyset$. 

8
As $A_f$ is transitive and $U$ is an attracting region, this implies $A_f \subset U$. In the same way, the chain recurrence class $\Lambda_f$ of $A_f$ is contained in $U$.

Recall that a quasi attractor is a chain recurrence class which is the intersection of a decreasing sequence of attracting regions. This implies that every quasi attractor of $f$ contains $\Lambda_f$, hence is equal to $\Lambda_f$.

On the other hand, as $S^1 \times \mathbb{D}^2$ is an attracting region, it contains at least one quasi attractor. This concludes the proof. \hfill $\square$

### 3.4 Robust homoclinic tangencies

Now our construction consists in proving:

**Proposition 3.3.** For every $f \in U_{\gamma}^{Ply}$ and every point $x$ of the hyperbolic basic set $A_f$, there is $y \in A_f$ such that $W^u(x)$ and $W^s(y)$ meet tangentially at one point.

**Idea of the proof of Proposition 3.3:** Proposition 3.3 is completely analogous to [As, Proposition 3.1]. We just recall the ideas of the proof for completeness.

The hyperbolic set $A_f$ is a hyperbolic attractor of $\phi_f$. Furthermore, by hypothesis, the basin $W^s(A_f)$ contains the whole disk $D_f$, punctured by the finite set $R_f$ of repelling periodic points (contained in $f(D_f)$). Hence $D_f \setminus R_f$ is foliated by the stable manifolds of the point in $A_f$. Let us denote $F^s$ this foliation.

On the other hand, as $D_f$ is an essential disk, transverse to the cone field $C_{\alpha_f}$, for every circle $S^1 \times \{z\}$ the image $f(S^1 \times \{z\})$ cuts transversely $D_f$ in exactly $|n_\alpha| > 1$ points, (always with the same orientation). In particular, $f(S^1 \times \mathbb{D}^2)$ cuts $D_f$ in exactly $|n_\gamma|$ connected components, and one of them is $f(D_f)$. Let $\Delta(f)$ be another component. Notice that $F^s$ induces a foliation of the disk $\Delta(f)$, already denoted by $F^s$.

Recall that $A_f$ is a lamination whose leaves are the unstable leaves for $\phi_f$ of the points $z \in A_f$; these leaves are tangent to the center-unstable direction of $A_f$ considered as a basic set for $f$, and we denote them $L^u(z)$.

The unstable leaves of the points of $A_f$ are $C^1$ surfaces. More precisely, for every point $z \in A_f$, the unstable manifold $W^u(z)$ for $f$ is the union of all the strong unstable leaves $L^{uu}(x)$ for $x \in L^c(z)$.

Each strong unstable leaf is a curve tangent to the cone field $C_{\alpha_f}$, contained in $f(S^1 \times \mathbb{D}^2)$ and of infinite length. In particular, every sufficiently large segment of strong unstable leaf cuts the disk $\Delta(f)$. We endow the strong unstable leaves with the orientation induced by the orientation of the circle $S^1$. Hence, for any point $z \in A_f$ one has a well defined point $h(z) \in \Delta(f)$ which is the first intersection point of $L^{uu}(z)$ with $\Delta(f)$. Notice that the map $h$ is continuous.

Now $L_f = h(A_f)$ is a regular 1-dimensional compact lamination contained in $\Delta(f)$. Moreover, the leaves are $C^1$ curves varying $C^1$-continuously, because they are obtained as the (transverse) intersection of $\Delta(f)$ with the unstable manifolds of the points $z \in A_f$.

Given any compact 1-dimensional lamination by uniformly $C^1$ curves of a 2-disk endowed with a non-singular foliation, every leaf of the lamination admits tangency points with the foliation. So every leaf of the lamination $L_f$ admits tangency points with $F^s$, ending the proof. \hfill $\square$

### 3.5 Proof of Theorem 1

For proving Theorem 1 we will show:

**Proposition 3.4.** Given any braid $\gamma$ with $|n_\gamma| \geq 2$, and any $\varepsilon > 0$, the set

$$U_{n,\gamma} = \{f \in U_{n}^{Ply}, \exists q_{n,f} \text{ hyperbolic periodic source, } d(p_f, q_{n,f}) < \frac{1}{n}\}$$

is open for the $C^1$ topology and is dense in $U_{n}^{Ply} \cap \text{Diff}^r(S^1 \times \mathbb{D}^2, \text{Int}(S^1 \times \mathbb{D}^2))$ for the $C^r$ topology, for every $r \geq 1$.

Notice that $\bigcap_{n \in \mathbb{N}} U_{n,\gamma} \subset U_{\text{wild},\gamma}$. Hence Proposition 3.4 implies that $U_{\text{wild},\gamma}$ is residual in $U_{n}^{Ply}$ for the $C^r$ topology, for any $r \geq 1$, ending the proof of Theorem 1.
The fact that $\mathcal{U}_{n,\gamma}$ is $C^1$-open is a simple consequence of the continuous dependence of hyperbolic periodic points, for the $C^1$ topology. The difficulty is to prove the $C^r$-density. As the set of $C^{r+1}$ diffeomorphisms is dense in the set of $C^r$ diffeomorphisms for the $C^r$ topology, the $C^r$-density of $\mathcal{U}_{n,\gamma}$ is implied by the $C^{r+1}$-density: hence it is enough to prove the $C^r$-density of $\mathcal{U}_{n,\gamma}$ for $r$ large enough.

However, the $C^1$-density can be proved by an argument of different nature, involving specific $C^1$-perturbations lemmas. We present both argument in the next sections.

### 3.6 $C^1$-density of $\mathcal{U}_{n,\gamma}$

Recall that $\mathcal{U}_{\gamma}^{Pl}$ is contained in $\mathcal{U}_r^{part.hyp}$. Hence every $f \in \mathcal{U}_{\gamma}^{Pl}$ is partially hyperbolic on $S^1 \times \mathbb{D}^2$: there is a dominated splitting $T_x(S^1 \times \mathbb{D}^2) = E^{cs}(x) \oplus u(x)$, for $x \in \bigcap_{n \in \mathbb{Z}} f^n(S^1 \times \mathbb{D}^2)$, where $\dim E^{cs} = 2$, $\dim u = 1$, and the vectors in $u$ are uniformly expanded.

For every $f \in \mathcal{U}_{\gamma}^{Pl}$, consider the set $\Sigma_f$ of hyperbolic periodic saddle points which are homoclinically related with the point $p_f$ (i.e. whose stable and unstable manifolds cut transversally to the unstable and stable manifold of $p_f$, respectively). Let $\Sigma_{f,0} \subset \Sigma_f$ be the set of saddle points $p \in \Sigma_f$ which are sectionally expanding; in other words, $p \in \Sigma_f$ belongs to $\Sigma_{f,0}$ if,

$$\left| \text{Det}(Df^{\pi(p)}|_{E^{cs}(p)}) \right| > 1,$$

where $\pi(p)$ is the period of $p$ and $Df^{\pi(p)}|_{E^{cs}(p)}$ is the restriction of the derivative at the period to the center stable bundle at $p$.

Recall that the point $p_f$ is sectionally expanding (i.e. $p_f \in \Sigma_{f,0}$). A classical argument, formalized by using the notion of transitions in [BDP] and used by many authors, implies that for every $f$ the set $\Sigma_{f,0}$ is dense in the homoclinic class $H(p_f, f)$ of $p_f$ (i.e. the closure of $\Sigma_f$). More precisely, there is a sequence $p_{f,i} \in \Sigma_{f,0}$, $i \in \mathbb{N}$, such that, for every $\delta > 0$ and for every $i$ large enough, the orbit of $p_{f,i}$ is $\delta$-dense in $H(p_f, f)$. We denote by $\pi_i$ the period of $p_{f,i}$.

Now, according to [BC], for $C^1$-generic $f \in \mathcal{U}_{\gamma}^{Pl}$, the homoclinic class $H(p_f, f)$ coincides with the chain recurrence class $\Lambda_f$. As a consequence one gets:

**Lemma 3.5.** For $C^1$-generic $f \in \mathcal{U}_{\gamma}^{Pl}$, the closure of $\Sigma_{f,0}$ contains $\Lambda_f$.

According to Proposition 3.3, for any $f \in \mathcal{U}_{\gamma}^{Pl}$ the chain recurrence class $\Lambda_f$ contains a tangency point $q_f$ of $W^u(p_f)$ with $W^s(A_f)$. One deduces:

**Lemma 3.6.** For every $f \in \mathcal{U}_{\gamma}^{Pl}$ the $2$-dimensional bundle $E^{cs}$ does not admit any dominated splitting along $\Lambda_f$.

**Proof:** We argue by contradiction, assuming that there is a dominated splitting $E^{cs} = E_1 \oplus E_2$ on $\Lambda_f$: this splitting defines a dominated splitting $T_{\Lambda_f}(S^1 \times \mathbb{D}^2) = E_1 \oplus u \oplus E_2 \oplus u$ on $\Lambda_f$. Then the stable manifold $W^s(p_f)$ is tangent to $E_2 \oplus u$. Furthermore, for every $x \in A_f$ and every $y \in W^s(x) \cap \Lambda_f$, the stable manifold $W^s(x)$ is tangent to $E_1(y)$ at $y$. This prevents $W^u(p_f)$ to have a tangency point with $W^s(x)$ for $x \in A_f$, hence contradicts Proposition 3.3.

As a direct corollary one gets:

**Corollary 3.7.** For every $C^1$-generic $f \in \mathcal{U}_{\gamma}^{Pl}$, the $2$-dimensional bundle $E^{cs}$ does not admit any dominated splitting along $\Sigma_{f,0}$.

Now, an argument of Mañé in [Ma] (see also [BDP]) shows that, for every $\varepsilon > 0$ and every $i$ large enough, there is an $\varepsilon$-$C^1$-perturbation $g_i \in \mathcal{U}_{\gamma}^{Pl}$ of $f$ which coincides with $f$ on the orbit of $p_{f,i}$ and out an arbitrarily small neighborhood of this orbit, and such that the (real or complex) eigenvalues of $Dg_i^{\pi_i}(p_{f,i})$ corresponding to the center-stable bundle $E^{cs}$ have the same modulus; furthermore, as $p_{f,i}$ was sectionally expanding for $f$, this modulus can be taken larger than $1$; as the eigenvalue corresponding to the unstable bundle is also larger than $1$ one gets that the orbit of $p_{f,i}$ is a hyperbolic source for $g_i$. Hence, choosing $\varepsilon > 0$ small enough (so that the continuation $g_i$ of $p_f$ remains arbitrarily close to $p_f$) and $i$ large enough (so that the orbit of $p_{f,i}$ is passing arbitrarily close to $p_f$) one gets $g_i \in \mathcal{U}_{n,\gamma}$, ending the proof of the density of $\mathcal{U}_{n,\gamma}$ in $\mathcal{U}_{\gamma}^{Pl}$ for the $C^1$ topology.
3.7 \( C^r \)-density of \( \mathcal{U}_{n,\gamma} \) for \( r \geq 2 \)

We consider now \( \mathcal{U}_{n,\gamma}^{PL} = \mathcal{U}_{n,\gamma} \cap \text{Diff}^r(S^1 \times \mathbb{D}^2, \text{Int}(S^1 \times \mathbb{D}^2)) \) endowed with the \( C^r \)-topology, for \( r \geq 2 \).

According to Proposition 3.3 for every \( f \in \mathcal{U}_{n,\gamma}^{PL} \), the unstable manifold \( W^u(p_f) \) presents a tangency point \( q_f \) with the stable manifold of a point \( z_f \in A_f \). Notice that \( W^u(p_f) \) and \( W^s(z_f) \) are \( C^r \)-immersed submanifold, and \( r \geq 2 \). By performing an arbitrarily small \( C^r \) perturbation of \( f \), one may assume that the tangency point \( q_f \) is a quadratic tangency point.

Then, for every \( q \) in a small \( C^2 \)-neighborhood \( \mathcal{V} \) of \( f \) the tangency point \( q_f \) of \( W^u(p_f) \) with the stable foliation of \( A_f \) has a unique continuation \( q_g \), quadratic tangency point of \( W^s(p_g) \) with the stable foliation of \( A_g \). This tangency point varies continuously with \( g \).

Notice that the positive orbit of \( q_f \) is contained in the invariant normally hyperbolic disk \( D_f \) containing \( A_f \). The negative orbit of \( q_f \) is not contained in \( D_f \); by construction, \( q_f \) belongs to the lamination \( \mathcal{L}_f = h(A_f) \), hence, is the first return map on \( D_f \) of the strong unstable leaf of a point \( y_f \in A_f \) (i.e. \( q_f = h(y_f) \)); so for \( n > 0 \) large \( f^{-n}(q_f) \) is a point contained in the local strong unstable leaf of \( f^{-n}(y_f) \in A_f \subset D_f \). So, one can perform small \( C^r \)-perturbation of \( f \) in a neighborhood of \( f^{-n}(q_f) \) without modifying the restriction of \( f \) to the disk \( D_f \), hence without modifying the stable foliation \( \mathcal{F}_f \) of \( A_f \) in \( D_f \). So we get:

Lemma 3.8. There is a \( C^r \) arc \( \{f_t\}, t \in [0, 1] \) of \( C^r \) diffeomorphisms \( f_t \in \mathcal{V} \) such that:

- \( f_0 = f \);
- for every \( t \in [0, 1] \), \( f_t \) coincides with \( f \) on the disk \( D_f \) (in particular the stable foliation \( \mathcal{F}_f \), of \( A_f \) is \( \mathcal{F}_f \));
- the tangency point \( q_t = q_{f_t} \) defines an arc transverse to the stable foliation \( \mathcal{F}_f \).

Recall that \( A_f \) is a (transitive) hyperbolic attractor for the restriction on \( f \) to \( D_f \), and the fixed point \( p_f \) belongs to \( A_f \). Hence the stable manifold of \( p_f \) is a dense leaf of the foliation \( A_f \). As a consequence one gets:

Corollary 3.9. There is a sequence \( t_n > 0 \) tending to 0 such that, for every \( n \in \mathbb{N} \), the point \( q_{t_n} \) is a quadratic tangency point of the stable manifold of \( p_f \) with the unstable manifold of \( p_f \), and \( p_f \) is a hyperbolic sectionally expanding point of \( f_{t_n} \).

Hence \( g = f_{t_n} \) is an arbitrarily small \( C^r \)-perturbation of \( f \) having a quadratic homoclinic tangency point associated to a sectionally expanding fixed point \( p_g = p_f \). This situation has been studied in [PV]:

Theorem 2. [PV] If \( \{g_s\}_{s \in [0, 1]} \) is a generic arc of \( C^r \) diffeomorphisms \( (r \geq 2) \), and there is a periodic hyperbolic point \( p \) of \( g_0 \) which is sectionally expanding, and such that \( W^s(p, g_0) \cap W^u(p, g_0) \) contains a quadratic tangency point \( q \). Then there are a sequence \( s_i \) converging to 0 and periodic sources \( q_i \) of \( g_{s_i} \) converging to \( q \).

Notice that, for large \( i \), the orbits of the periodic sources \( q_i \) are passing arbitrarily close to the point \( p \). As a consequence, for large \( i \) the diffeomorphism \( g_{s_i} \) belongs to \( \mathcal{U}_{n,\gamma} \), and is an arbitrarily \( C^r \)-small perturbation of \( g \) which is an arbitrarily \( C^r \)-small perturbation of \( f \). This proves the \( C^r \)-density of \( \mathcal{U}_{n,\gamma} \) in \( \mathcal{U}_{n,\gamma}^{PL} \), ending the proof of Proposition 3.4.

4 Non-existence of attractors for diffeomorphisms

4.1 An attracting ball \( B^3 \) without attractors

Theorem A is obtained from Theorem 1 by building locally generic diffeomorphisms of an attracting ball \( B^3 \) without topological attractors and with a unique quasi attractor:

Theorem 3. There is a non-empty \( C^1 \)-open subset \( \mathcal{U} \subset \text{Diff}^1(\mathbb{D}^3, \text{Int}(\mathbb{D}^3)) \) and, for \( f \in \mathcal{U} \), a hyperbolic periodic point \( p_f \) varying continuously with \( f \) such that:

1. the diffeomorphism \( f \) is \( C^1 \) conjugated with the homothety \( z \rightarrow \frac{1}{2}z \) in a neighborhood of the sphere \( \partial \mathbb{D}^3 \);
2. for every $f \in \mathcal{U}$, the chain recurrence class $\Lambda_f = C(p_f)$ is the unique quasi attractor of $f$;
3. for every $r \geq 1$, the subset
\[ \mathcal{U}_{\text{wild}} = \{ f \in \mathcal{U}, \Lambda_f \cap \{ \text{sources of } f \} \neq \emptyset \} \]
is residual for the $C^r$ topology.

Next lemma can be easily proved by using the same kind of perturbations used for the derived from Anosov diffeomorphisms in [Sm]. We leaves the details of the construction to the reader.

**Lemma 4.1.** Let $f : S^1 \times \mathbb{D}^2 \to \text{Int}(S^1 \times \mathbb{D}^2)$ be a solenoid map such that $\bigcap_{n \in \mathbb{N}} f^n(S^1 \times \mathbb{D}^2)$ is a hyperbolic attractor. Then, there is $g$ isotopic to $f$, which coincides with $f$ in a neighborhood of the boundary $\partial(S^1 \times \mathbb{D}^2)$, and such that the chain recurrent set in $S^1 \times \mathbb{D}^2$ consists in exactly one fixed hyperbolic sink $\omega$ and a hyperbolic basic set of saddle type (i.e. neither attracting nor repelling). Moreover, if $f$ is orientation preserving, one may require that the derivative $Dg(\omega)$ is the homothety of ratio $\frac{1}{2}$.

**Proof of Theorem 3 :** According to [Gi] there is a diffeomorphism $f_0$ of the 3 sphere $S^3$ admitting a torus $T$ with the following properties:

- the torus $T$ bounds two solid tori $\Delta_1$ and $\Delta_2$;
- $f_0(\Delta_1)$ is contained in the interior of $\Delta_1$ and the restriction $f_0|_{\Delta_1}$ is a hyperbolic Smale-solenoid attractor corresponding to a 2-braid $\gamma$;
- $f_0^{-1}(\Delta_2)$ is contained in the interior of $\Delta_2$ and the restriction $f_0^{-1}|_{\Delta_2}$ is a hyperbolic Smale-solenoid attractor corresponding to a 2-braid $\gamma$.

We now modify $f_0$ by surgery in both solid tori $\Delta_1$ and $\Delta_2$, in order to get a diffeomorphism $f_1$ with the following properties:

- $f_1$ coincides with $f_0$ in the neighborhood of the torus $T$; as a consequence $f_1(\Delta_1) \subset \text{Int}(\Delta_1)$ and $f_1^{-1}(\Delta_2) \subset \text{Int}(\Delta_2)$;
- the restriction of $f_1$ to the solid torus $\Delta_1$ belongs to the $C^1$-open set $f \in \mathcal{U}_{\text{Pl}_r}$;
- the intersection of the chain recurrent set $\mathcal{R}(f_1)$ with $\Delta_2$ consists exactly in a hyperbolic fix source $\alpha_1$ and a non-trivial hyperbolic set $K_1$ of saddle type (this is obtained by applying Lemma 4.1 to the restriction of $f^{-1}$ to the solid torus $f(\Delta_2)$).

Now one removes from $S^3$ the interior of a small ball $B$ centered at $\alpha_1$. Then $B = S^3 \setminus \text{Int}(B)$ is a compact ball diffeomorphic to $\mathbb{D}^3$. Furthermore $f_1(B)$ is contained in the interior of $B$. Now there is a $C^1$ neighborhood $\mathcal{U}$ of $f_1$ such that every $f \in \mathcal{U}$ satisfies the following properties:

- there is a diffeomorphism $\varphi : B \to \mathbb{D}^3$ such that $\varphi f \varphi^{-1} : \mathbb{D}^3 \to \mathbb{D}^3$ coincides with the homothety $z \mapsto \frac{1}{2}z$ in a neighborhood of $S^2 = \partial \mathbb{D}^3$;
- the image of the solid torus $\Delta_1 \subset B$ is contained in its interior and the restriction $f|_{\Delta_1}$ belongs to $\mathcal{U}_{\text{Pl}_r}$, one denotes by $\Lambda_f$ the unique quasi attractor of $f$ contained in $\Delta_1$, and by $p_f$ the hyperbolic sectionally expanding saddle point in $\Lambda_f$ associated to $f|_{\Delta_1} \in \mathcal{U}_{\text{Pl}_r}$;
- the intersection of the chain recurrent set $\mathcal{R}(f)$ with $B \setminus \text{Int}(\Delta_1)$ is a hyperbolic basic set of saddle type.

One concludes by noticing that $C^r$-generic diffeomorphisms $f \in \mathcal{U}$ induce by restriction on $\Delta_1$ $C^r$-generic diffeomorphisms in $\mathcal{U}_{\text{Pl}_r}$; as a consequence, there is a sequence of hyperbolic sources converging to a point in $\Lambda_f$, preventing $\Lambda_f$ to be an attractor. \(\square\)
4.2 End of the proof of Theorem A

For getting Theorem A one considers the time one map of the flow of a gradient vector field of a Morse function on $M$. Then one replaces the diffeomorphism in a neighborhood of each sink by a diffeomorphism in the open set $U$ built in Theorem 3.

**Remark 4.2.** Let $M$ be a compact orientable 3-manifold. Using the fact that $M$ admits a Heegaard splitting in two handlebodies, one easily verifies that $M$ admits a gradient like diffeomorphism having a unique sink. As a consequence, we can assume that $k = 1$ in the statement of Theorem A.

4.3 Non existence of attractors and repellers in higher dimensions: proof of Theorem B

Multiplying our construction in $B^3$ by a transverse contraction allows us to get

**Lemma 4.3.** Given any $d > 3$, there is a non-empty $C^1$-open subset $U_d \subset \text{Diff}^1(D^d, \text{Int}(D^d))$ such that every $f \in U_d$ satisfies the following properties:

1. the diffeomorphism $f$ is $C^1$ conjugated with the homothety $z \mapsto \frac{1}{2}z$ in a neighborhood of the sphere $\partial D^d$;
2. the chain recurrent set of $f$ is contained in a normally hyperbolic 3-disc $D_f$;
3. the restriction $f|_{D_f}$ belongs to the open subset $U$ given by Theorem 3; in particular for every $f \in U_d$, the chain recurrence class $\Lambda_f$ of the fixed point $p_f$ is the unique quasi attractor of $f$.

As a consequence, for every $r \geq 1$, the subset $U_{\text{wild}} = \{ f \in U_d, \Lambda_f \cap \{ \text{sources of the restriction } f|_{D_f} \} \neq \emptyset \}$ is residual for the $C^r$ topology. Then $f \in U_{\text{wild}}$ has neither attractors nor repellers in $D^d$.

Given any manifold $M$ with $\text{dim}(M) > 3$, one considers a diffeomorphism $f_0$ which is the time one map of a Morse function. Now, one builds a diffeomorphism $f_1$ obtained from $f_0$ as follows:

- one replaces $f_0$, in a small ball centered to each sink, by a diffeomorphism in the open set $U_d$ built at Lemma 4.3;
- one replaces $f_0^{-1}$, in a small ball centered to each source, by the inverse of a diffeomorphism in the open set $U_0$ built at Lemma 4.3.

Now the open set announced in Theorem B is obtained by considering a small neighborhood of the diffeomorphism $f_1$ above.

5 Singular flows: proof of Theorem C

Our example for flow is very similar to the examples built for Theorem 1, so that we will just sketch the construction.

We consider an open set $U$ of vector fields on $\mathbb{R}^4$, such that every $X \in U$ satisfies the following properties:

- the vector field $X$ admits a transverse cross section $\Sigma$ diffeomorphic to a solid torus $S^1 \times D^2$;
- the vector field $X$ has a unique singular point $0_X$ which is a saddle with $\text{dim}(\text{W}^s(0_X)) = 3$; the eigenvalues of the derivative $D_{0_X}X$ are $\lambda_1 < \lambda_2 < \lambda_3 < 0 < \lambda_4$, with $\lambda_4 + \lambda_1 > 0$;
- there is an essential disc $D_0 \subset \Sigma$, transverse to all the circles $S^1 \times \{z\}, z \in D^2$, and contained in the local stable manifold of the saddle point $0_X$;
the first return map on $\Sigma$ is well defined on $\Sigma \setminus D_0$ and the image is contained in the interior of $\Sigma$; we denote it $P : \Sigma \setminus D_0 \to \text{Int}(\Sigma)$;

the first return map $P$ leaves invariant a splitting $T\Sigma = E^{cs} \oplus E^u$ which is a dominated splitting with $\dim E^{cs} = 2$ and $\dim E^u = 1$; moreover, $E^u$ is transverse to the discs $\{t\} \times \mathbb{D}^2$, $t \in S^1$;

the bundle $E^u$ is uniformly expanding by a factor larger than 3; more precisely, given any non-zero vector $u$ tangent to $E^u(x)$, $x \in \Sigma$, denote $u = u_h + u_v$ where $u_h$ is tangent to the $S^1$ fiber through $x$ and $u_v$ is tangent to the $\mathbb{D}^2$ fiber through $x$; assume $x \in \Sigma \setminus D_0$ and let $w = D_x P(u) = w_h + w_v$; then we require:

$$|w_h| > 3|u_h|;$$

there is an essential disc $D_1 \subset \Sigma \setminus D_0$, invariant by $P$ (i.e. $P(D_1) \subset \text{Int}(D_1)$), normally hyperbolic, and such that the restriction $P|_{D_1}$ is smoothly conjugated to an element of the open set $\mathcal{P}_0$ of structurally stable diffeomorphisms in $\text{Diff}^1_c(\mathbb{D}^2, \text{Int}(\mathbb{D}^2))$, defined at Section 2.5; in particular, the chain recurrent set of $P|_{D_1}$ consists of a Plykin attractor $A_X$ and finitely many repelling points, and the Plykin attractor $A_X$ contains a fixed point $p_X$ which is sectionally expanding;

there is an essential disc $D_2 \subset \Sigma \setminus D_0$, invariant by $P$, normally hyperbolic, and such that the restriction $P|_{D_2}$ has a unique fixed point $q_X$; the disc $D_2$ is contained in the stable manifold of $q_X$ (for the map $P$); finally, the derivative $D_{q_X}(P)$ of $P$ at $q_X$ has a complex (non-real) eigenvalue, corresponding to the tangent space $T_{q_X}D_2$.

It is not hard to build a non-empty open set $U$ of vector fields satisfying all the properties above (see also [BLY] which contains the details of a analogous construction).

As in the proof of Theorem 1, one verifies that, for any open subset $O \subset \Sigma$ there is $n > 0$ such that $f^n(O)$ meets $D_0$, $D_1$ and $D_2$; this implies that every attracting region for $X$ which meets $\Sigma$ contains the singular point $0_X$, the Plykin attractor $A_X$ (and hence its orbits by the flow of $X$) and the orbit $\gamma_X$ of the point $q_X$. Hence there is a unique quasi attractor $\Lambda_X$ for the orbits of $X$ through $\Sigma$ and this quasi attractor contains $0_X$, $A_X$ and $\gamma_X$. An analogous argument shows that, for every $X \in U$, the invariant manifolds of $A_X$ for $P$ present a tangency point. This implies that $C^r$-generic paths in $U$ unfold generic homoclinic bifurcations associated to $p_X$, implying that, for $C^r$-generic $X \in U$ the quasi attractor $\Lambda_X$ is accumulated by periodic sources, which prevents $\Lambda_X$ to be an attractor.

One concludes the proof of Theorem C by proving

**Lemma 5.1.** For any $X \in U$, the tangent flow of $X$ on $\Lambda_X$ does not admit any dominated splitting.

**Proof:** Assume that there is a dominated splitting $TM|_{\Lambda_X} = E \oplus F$, for the tangent flow of $X$. This dominated splitting induces on $\Sigma \cap \Lambda_X$ a dominated splitting $T\Sigma|_{\Sigma \cap \Lambda_X} = E_\Sigma \oplus F_\Sigma$ invariant by $P$ (just consider $E_\Sigma = (E + \mathbb{R} X) \cap T\Sigma$ and $F_\Sigma = (F + \mathbb{R} X) \cap T\Sigma$).

The fact that $q_X$ belongs to $\Lambda_X \cap \Sigma$ implies that $\dim E_\Sigma = 2$. One deduces that $E_\Sigma = E^{cs}$ and $F_\Sigma = E^u$. As a consequence one gets two possibilities for the splitting $T_x M = E(x) \oplus F(x)$ at $x \in \Lambda_X \cap \Sigma$:

- either $E = E^{cs} \oplus \mathbb{R} X$ and $F \subset E^u \oplus \mathbb{R} X$,
- or $E \subset E^{cs} \oplus \mathbb{R} X$ and $F = E^u \oplus \mathbb{R} X$.

In the first case, $X$ is tangent to $E$ along $\Lambda_X$. However, $\Lambda_X$ contains the unstable manifold of $0_X$ (a quasi attractor always contains its unstable manifold). This manifold consists in $0_X$ and 2 orbits of $X$. Hence $W^u(0_X)$ is tangent to $E$. This implies that $E(0_X)$ contains the eigenspace corresponding to $\lambda_1$, which contradicts the fact that $E$ is dominated by $F$.

In the second case, $X$ is tangent to $F$ along $\Lambda_X$. However, for $x \in A_X$, the space $E^{cs}(x)$ contains vectors tangent to the hyperbolic attractor $A_X \subset D_1$, hence contains vectors which are
exponentially expanded by the derivative $DP^n$, for $n \to +\infty$. This implies that the space $E(x)$ contains vectors $u \in E(x)$ and a sequence of times $t_n \to +\infty$ such that $X_{t_n}(x) = P^n(x) \in \Sigma$ and

$$\lim_{n \to +\infty} |(X_{t_n})_*(u)| = +\infty,$$

where $(X_t)_*$ denotes the derivative of the time $t$ of the flow of $X$.

On the other hand, $X(x) \in F(x)$ but $|(X_{t_n})_*(X(x))|$ remains bounded, contradicting the fact that $F$ dominates $E$.

Hence both cases lead to contradiction, ending the proof. □

6 Changing the definition of attractors

With the better understanding of the complexity of generic dynamics, people tried the definition of attractors in order to ensure their existence.

6.1 Palis approach from the point of view of ergodic theory

From the ergodic viewpoint, an attractor $\Lambda$ of $f$ should satisfy the following

- “indecomposable property”: there is an ergodic invariant probability measure $\mu$ such that $\text{supp}(\mu) = \Lambda$;
- “attracting property”: its basin $B(\Lambda)$ has positive Lebesgue measure, where

$$x \in B(\Lambda) \iff \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} = \mu$$

(here $\delta_z$ stands the Dirac measure at the point $z$).

**Conjecture** (Palis [P1, P2, P3]). There is a dense set $D \subset \text{Diff}^r(M)$ such that for any $f \in D$, $f$ has only finitely many (ergodic) attractors, and the union of the basins of attractors forms a full Lebesgue measure set in $M$.

Palis completed his conjecture by continuity properties of the basins of the attractors with respect to the diffeomorphism.

6.2 A topological approach

In [H], Hurley proved that, for generic homeomorphisms of a compact manifold, the $\omega$-limit set of every generic point is a quasi attractor, and he stated the conjecture

**Conjecture** (Hurley). For $C^r$-generic diffeomorphisms the $\omega$-limit sets of generic points in $M$ are quasi attractors.

This conjecture has been proved in [MP, BC] for the $C^1$-topology and remains open in more regular topologies.

However, the information given by Hurley’s conjecture is very weak: every $C^0$-generic homeomorphism $h$ has uncountably many quasi attractors, and the closure of the basin of each quasi attractor has empty interior$^6$. In the setting of the $C^1$ topology, [BD] shows that there are locally generic diffeomorphisms having an uncountable family of quasi attractors which are at the same time quasi repellers; in particular the basin of each of them is reduced to the quasi-attractor itself (which is a Cantor set).

Let us define a new notion of attractor which will allow us to propose a new conjecture.

$^6$The proof of this last fact (the closure of each basin has empty interior) was found by the first author of this present paper, writing this conclusion; Hurley kindly wrote us that he did not notice this fact.
Definition 6.1. • A residual attractor of a diffeomorphism $f$ is a chain recurrence class admitting a neighborhood $U$ which is an attracting region and such that the $\omega$-limit set of the generic points in $U$ is $\Lambda$. 

• A locally residual attractor of a diffeomorphism $f$ is a chain recurrence class admitting an open set $U$ such that the $\omega$-limit set of the generic points in $U$ is $\Lambda$. Notice here $U$ may not be a neighborhood of $\Lambda$.

Remark 6.2. • For $C^1$-generic diffeomorphisms, one can deduce from [BC] that the residual attractors are exactly the quasi attractors which are isolated in the set of quasi attractors: they admit a neighborhood disjoint from any other quasi attractor.

• For $C^1$-generic diffeomorphisms, our notion of locally residual attractor coincides with the notion of generic attractor introduce by Milnor in [Mi]. More precisely, Milnor first defines a minimal attractor for the ergodic point of view: the basin has positive Lebesgue measure and every proper subset’s basin only has zero Lebesgue measure; then, on the topological generic setting he writes: “There is an analogous concept of generic-attractor. The definition will be left to the reader”. Hence, a generic attractor is an invariant set whose basin is a locally residual set, and such that every proper subset’s basin is meager. As Hurley’s conjecture is proved for $C^1$-generic diffeomorphisms, Milnor’s generic attractors of a $C^1$-generic diffeomorphism are its locally residual attractors.

The locally generic examples built in Theorem A have finitely many residual attractors and the union of their basin is a residual subset of the whole manifold $M$. This motivates the following problem:

Problem 1. 1. Is it true that $C^r$-generic diffeomorphisms have at least one (locally) residual attractor?

2. For any $C^r$-generic diffeomorphism, is it true that the $\omega$-limit set of every generic point is a (locally) residual attractor?

(A positive answer to these questions is known for locally residual attractors of $C^1$-generic diffeomorphisms: see the next section, devoted to the $C^1$-topology).

We would like to understand better these residual attractors, in particular to understand if their are associated to periodic orbits. Recall that the homoclinic class of a periodic orbit is the closure of the transverse intersection of its invariant manifolds. It is an invariant compact set canonically associated to the periodic orbit. [BC] shows that, for $C^1$-generic diffeomorphisms, the chain recurrence class of a periodic orbit is its homoclinic class; as a consequence, isolated chain recurrence classes of $C^1$-generic diffeomorphisms are homoclinic classes (in particular, this holds for topological attractors). As we noticed above, the residual attractors are the quasi attractors which are isolated in the set of quasi attractors. It seems natural to ask:

Problem 2. Let $\Lambda$ be a residual attractor of a generic diffeomorphism. Is $\Lambda$ the homoclinic class of a periodic orbit?

6.3 Remarks on the $C^1$ topology

For $C^1$ generic non-critical (i.e. far from homoclinic tangencies) diffeomorphisms, [Y] gave a positive answer to Problems 1 and 2 proving that every quasi attractor is a homoclinic class. Since for $C^1$ generic diffeomorphism, we can have only countably many homoclinic classes, together with the results in [MP, BC], there is at least one locally residual attractor; furthermore, the (countable) union of the basins of the locally residual attractors is a residual subset of the manifold.

In a forthcoming work, we can get more precise results for the $C^1$ topology.

• On the contrary of Theorem A, we can prove that for two dimensional manifold $M^2$, there is a $C^1$ dense open set $\mathcal{U} \subset \text{Diff}^1(M^2)$, such that for any $f \in \mathcal{U}$, $f$ has a hyperbolic attractor; 7

• As a complement of Theorem A, for any compact three dimensional manifold $M^3$ without boundary, we can construct a $C^1$ open set $\mathcal{U} \subset \text{Diff}^1(M^3)$, such that $C^1$-generic $f \in \mathcal{U}$ have neither attractors nor repellers;

7 For the examples built in Theorem A, there are infinitely many repellers.
Together with S. Gan, we give a positive answer to Problems 1 and 2 in the setting of partially hyperbolic splitting with 1-dimensional center bundle. In these setting, we prove that for $C^1$ generic diffeomorphism, every quasi attractor is a residual attractor.

References

[ABS] V. Afraimovič, V. Bykov, and L. Sil’nikov, The origin and structure of the Lorenz attractor, Dokl. Akad. Nauk SSSR, 234 (1977), 336–339.

[As] M. Asaoka, Hyperbolic sets exhibiting $C^1$-persistent homoclinic tangency for higher dimensions, Proc. Amer. Math. Soc., 136 (2008), no. 2, 677–686.

[BKR] R. Bamon, J. Kiwi, and J. Rivera, Wild Lorenz like attractors, preprint.

[BC] C. Bonatti and S. Crovisier, Récurrence et généricité (French), Invent. Math., 158 (2004), 33–104.

[BD] C. Bonatti and L. Díaz, On maximal transitive sets of generic diffeomorphisms, Inst. Hautes études Sci. Publ. Math., 96 (2002), 171–197.

[BDP] C. Bonatti, L. Diaz, and E. Pujals, A $C^1$-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources, Ann. of Math., 158 (2003), 355–418.

[BDV] C. Bonatti, L. Diaz, and M. Viana, Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective. Encyclopaedia of Mathematical Sciences, 102. Mathematical Physics, III. Springer-Verlag, Berlin, 2005, xviii+384 pp.

[BLY] C. Bonatti, M. Li, and D. Yang, Robustly chain transitive attractor with singularities of different indices, preprint, 2008.

[BV] C. Bonatti and M. Viana, SRB measures for partially hyperbolic systems whose central direction is mostly contracting, Israel J. Math., 115 (2002), 157–193.

[Co] C. Conley, Isolated invariant sets and Morse index, CBMS Regional Conference Series in Mathematics, 38, AMS Providence, R.I., 1978.

[GWZ] S. Gan, L. Wen, and S. Zhu, Indices of singularities of robustly transitive sets, Disc. Cont. Dynam. Syst., 21 (2008), 945–957.

[Gi] J. C. Gibbons, One-dimensional basic sets in the three-sphere, Transactions of the American Mathematical Society, 164 (1972), 163–178.

[Gu] J. Guckenheimer, A strange, strange attractor, The Hopf bifurcation theorems and its applications (Applied Mathematical Series, 19), Springer-Verlag, 1976, pp. 368–381.

[GuW] J. Guckenheimer and R. Williams, Structural stability of Lorenz attractors, Inst. Hautes Études Sci. Publ. Math., 50 (1979), 59–72.

[H] M. Hurley, Attractors: persistence, and density of their basins, Trans. Am. Math. Soc., 269 (1982), 247–271.

[LGW] M. Li, S. Gan, and L. Wen, Robustly transitive singular sets via approach of an extended linear Poincaré flow, Discrete Contin. Dyn. Syst., 13 (2005), 239–269.

[Lo] E. N. Lorenz, Deterministic nonperiodic flow, J. Atmosph. Sci., 20 (1963), 130–141.

[Ma] R. Mañé, An ergodic closing lemma, Ann. Math., 116 (1982), 503–540.

[MM] R. Metzger and C. Morales, Sectional-hyperbolic systems, Ergodic Theory Dynam. Systems, 28 (2008), no. 5, 1587–1597.

[Mi] J. Milnor, On the concept of attractor, Commun. Math. Phys., 99 (1985), 177–195.
[MP] C. Morales and M. Pacifico, Lyapunov stability of $\omega$-limit sets, *Disc. Cont. Dyn. Sys.*, 8 (2002), 671–674.

[MPP1] C. Morales, M. Pacifico, and E. Pujals, On $C^1$ robust singular transitive sets for three-dimensional flows, *C. R. Acad. Sci. Paris*, 326 (1998), 81–86.

[MPP2] C. Morales, M. Pacifico, and E. Pujals, Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers, *Ann. of Math.*, 160 (2004), 375–432.

[N1] S. Newhouse, Nondensity of axiom A(a) on $S^2$, *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)* Amer. Math. Soc., Providence, R.I., (1970), 191–202.

[N2] S. Newhouse, Diffeomorphisms with infinitely many sinks, *Topology*, 13 (1974), 9–18.

[N3] S. Newhouse, The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms, *Inst. Hautes Études Sci. Publ. Math.*, 50 (1979), 101–151.

[P1] J. Palis, A global view of dynamics and a conjecture of the denseness of finitude of attractors, *Astérisque*, 261 (2000), 335–347.

[P2] J. Palis, A global perspective for non-conservative dynamics, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 22 (2005), 485–507.

[P3] J. Palis, Open questions leading to a global perspective in dynamics, *Nonlinearity*, 21 (2008), 37–43.

[PP] J. Palis and C. Pugh, Fifty problems in dynamical systems, *Lect. Notes Math.*, 468 (1975), 345–353, Springer-Verlag.

[Pl] R. V. Plykin, Hyperbolic attractors of diffeomorphisms, *Usp. Math. Nauk*, 35 (1980), no. 3, 94–104. [English Transl.: Russ. Math. Survey, 35 (1980), no. 3, 109–121.]

[PV] J. Palis and M. Viana, High dimension diffeomorphisms displaying infinitely many periodic attractors, *Ann. of Math.*, 140 (1994), 207–250.

[Sh] M. Shub, *Topological transitive diffeomorphisms in $T^4$*, Lecture Notes in Math. Vol. 206, Springer Verlag, 1971.

[Sm] S. Smale, Differentiable dynamical systems, *Bull. Amer. Math. Soc.*, 73 (1967), 747–817.

[T] R. Thom, *Structurally stability and morphogenesis*, Benjamin, Reading, Mass., 1976.

[W] L. Wen, Homoclinic tangencies and dominated splittings, *Nonlinearity*, 15 (2002), 1445–1469.

[Y] J. Yang, Lyapunov stable chain recurrent class, *preprint*, 2007.

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