A CONDITIONAL LIMIT THEOREM
FOR A BIVARIATE REPRESENTATION
OF A UNIVARIATE RANDOM VARIABLE
AND CONDITIONAL EXTREME VALUES

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Abstract. We consider a real random variable \(X\) represented through a random pair \((R, T)\) in \(\mathbb{R}^2\) and a deterministic function \(u\) as \(X = Ru(T)\). Under some additional assumptions, we prove a limit theorem for \((R, T)\) given \(X > x\), as \(x\) tends to infinity. As a consequence, we derive conditional limit theorems for random pairs \((X, Y) = (Ru(T), Rv(T))\) given that \(X\) is large. These results imply earlier ones which were obtained in the literature under stronger assumptions.

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1. Introduction. The purpose of this paper is to clarify some conditional limit theorems on bivariate vectors given that one of the component is large. The significance of such limit theorems stems from their applications in multivariate extreme value theory, where one is interested in both making statistical inference on a system given that a component has an extreme behavior and understanding the dependence structure between extreme events. These conditional theorems provide the theoretical support in the study of extremal behavior of random vectors in the conditional extreme value models introduced by Heffernan and Tawn (2004) and Heffernan and Resnick (2007), Das and Resnick (2011), as well as for studying estimators in statistical applications as done by Fougères and Soulier (2012).

Following these authors we are interested in a generalization of elliptically distributed random vectors, namely, random vectors \((X, Y)\) with representation \((Ru(T), Rv(T))\), where \(u\) and \(v\) are deterministic functions, \(R\) and \(T\) are independent real random variables, and the distribution of \(R\) is in the Gumbel max-domain of attraction (see Berman, 1983; Fougères and Soulier, 2010; Hashorva, 2012; Seifert, 2012). For elliptical random variables, \(R\) is the radial component and \(T\) the angular distribution. However, in our more general setting, the map \((R, T) \mapsto (X, Y)\) may not be one-to-one.
Beyond immediate applications to extreme value theory, our results have bearing to the description of the convex hull of samples and related problems which are in part driven by extreme value theory.

The novelty of our paper is to show that a conditional limit theorem for \((X, Y)\) given that \(X\) is large is not intrinsically about the pair \((X, Y)\) but about the representation of the single variable \(X\) in terms of the pair \((R, T)\). This approach allows us to recover previous results under minimal assumptions, to provide a better understanding of the earlier work, and, through more versatile assumptions, to widen the applicability of this model.

Throughout the paper, \(R\) is a real random variable, so that \(R(u(T), v(T))\) means \((Ru(T), Rv(T))\).

2. Main result. In this section, we are interested in random variables \(X\) which are represented as \(X = Ru(T)\), and conditional limit theorems for properly normalized \((R, T)\) given \(X > x\) as \(x\) tends to infinity. In the next section, equipped with such a conditional limit theorem we will use some continuous mapping argument to derive a conditional limit theorem for properly normalized \(Y = Rv(T)\) given \(X > x\) as \(x\) tends to infinity.

We write \(H\) for the cumulative distribution function of \(R\), and \(\overline{H}\) for the survival function \(1 - H\). We assume that \(T\) has a density \(g(t)\).

We will use the following assumptions.

**Assumption 1.** The survival function \(\overline{H}\) of \(R\) is in the class \(\Gamma(\psi)\), meaning that there exists an ultimately positive function \(\psi\) such that for any fixed real number \(\lambda\),

\[
\lim_{x \to \infty} \frac{\overline{H}(x + \psi(x)\lambda)}{\overline{H}(x)} = e^{-\lambda}.
\]

This property is equivalent to \(H\) belonging to the max-domain of attraction of the Gumbel distribution (de Haan, 1970; Resnick, 2007). The function \(\psi\) is unique up to asymptotic equivalence, and, necessarily, \(\psi(x) = o(x)\) at infinity.

**Assumption 2.** There exists a \(t_0\) such that \(u(t_0) = 1\) and for any \(\epsilon\) positive, \(\sup_{t \geq t_0 > \epsilon} u(t) < 1\). Moreover, the function \(\tilde{u}(s) = u(t_0) - u(t_0 + s)\)
is regularly varying at $0+$ with positive index $\kappa$,

meaning that for any positive $\lambda$,

$$\lim_{s \to 0^+} \frac{\tilde{u}(\lambda s)}{\tilde{u}(s)} = \lambda^\kappa.$$ 

The first part of assumption 2 asserts that on the right of $t_0$, the function $u$ has a unique maximum at $t_0$ and that for $u(t)$ to be close to 1, we must have $t$ close to $t_0$.

Since $\psi(x) = o(x)$ at infinity and $\tilde{u}$ is regularly varying with positive index, there exists an ultimately positive function $\phi$ such that

$$\tilde{u} \circ \phi(x) \sim \frac{\psi(x)}{x}$$

as $x$ tends to infinity, and $\lim_{x \to \infty} \phi(x) = 0$.

We will also use the notation

$$\tilde{g}(s) = g(t_0 + s),$$

and assume that

**Assumption 3.** The density $\tilde{g}$ of $T - t_0$ is regularly varying at $0+$ with index $\tau > -1$.

Since $\tilde{g}$ is locally integrable, its index of regular variation must be at least $-1$. Furthermore, if $\tilde{g}$ is positive and continuous at 0, then $\tau$ vanishes.

To keep track of the notation, note that whenever a function has a tilde, it means that it is regularly varying at 0.

Our main result is the following conditional limit theorem for $(R,T)$ given $X > x$ and $T > t_0$, as $x$ tends to infinity. We will see in the next section how the conditioning by $T > t_0$ may be removed under additional assumptions.

**Theorem 2.1.** Let $X = Ru(T)$. Under assumptions 1, 2 and 3, the conditional distribution of

$$\left( \frac{R - x}{\psi(x)}, \frac{T - t_0}{\phi(x)} \right)$$

is regularly varying at $0+$ with positive index $\kappa$, meaning that for any positive $\lambda$,

$$\lim_{s \to 0^+} \frac{\tilde{u}(\lambda s)}{\tilde{u}(s)} = \lambda^\kappa.$$
given $X > x$ and $T > t_0$ converges weakly*, as $x$ tends to infinity, to the measure whose density with respect to the Lebesgue measure is

$$
\frac{\kappa}{\Gamma\left(\frac{1+\tau}{\kappa}\right)} t^\tau e^{-r} 1\{0 < t < r^{1/\kappa}\}
$$

as $x$ tends to infinity. Furthermore,

$$
P\{X > x; T > t_0\} \sim \phi(x) \tilde{g} \circ \phi(x) \Pi(x) \frac{1}{\kappa} \Gamma\left(\frac{1+\tau}{\kappa}\right)
$$

as $x$ tends to infinity.

Some heuristic arguments explaining why Theorem 2.1 may be true are given at the beginning of section 5.

As a function defined on some right neighborhood of the origin, $\tilde{u}$ has an asymptotic inverse $\tilde{u}^*$ such that $\tilde{u} \circ \tilde{u}^*(s) \sim s$ as $s$ tends to 0 (see Bingham, Goldie and Teugels, 1989, §1.5.7). Thus, $\phi(x) \sim \tilde{u}^*(\psi(x)/x)$ and

$$(\phi g \circ \phi)(x) \sim (\tilde{u}^* \tilde{g} \circ \tilde{u}^*)(x)$$

as $x$ tends to infinity. Therefore, we may view $\tilde{\phi} \tilde{g} \circ \phi$ as a function of $\psi(x)/x$ which is then regularly varying of index $(1 + \tau)/\kappa$ in terms of the argument $\psi(x)/x$.

3. Two-sided extensions. In some applications it is desirable to have analogues of Theorem 2.1 when the conditioning involves only the event $X > x$. Under two-sided conditions on the behavior of $\tilde{u}$ and $\tilde{g}$ near $t_0$, such extensions present no conceptual difficulty. To illustrate this assertion, we present two such extensions, relying on the following two-sided versions of assumptions 2 and 3.

**Assumption 4.** There exists a $t_0$ such that $u(t_0) = 1$ and for any $\epsilon$ positive, sup$_{|t-t_0| > \epsilon} u(t) < 1$. Moreover, the function $\tilde{u}$ is regularly varying at $0-$ and $0+$ with respective positive indices $\kappa_-$ and $\kappa_+$. The second part of assumption 4 signifies that for any given sign $\sigma$ in $\{-, +\}$ and any positive $\lambda$

$$
\lim_{s \to 0^+} \frac{\tilde{u}(\sigma \lambda s)}{\tilde{u}(\sigma s)} = \lambda^{\kappa_{\sigma}}.
$$
Similarly, we strengthen assumption 3 as follows.

**Assumption 5.** \( \tilde{g} \) is regularly varying at 0− and 0+ with respective indices \( \tau_{-} \) and \( \tau_{+} \), both these indices being greater than \(-1\).

Equipped with these two-sided hypotheses, we define, as in (2.1), for each sign \( \sigma \), an ultimately positive function \( \phi_{\sigma} \) such that

\[
\tilde{u}(\sigma \phi_{\sigma}(x)) \sim \frac{\psi(x)}{x}
\]

as \( x \) tends to infinity, and \( \lim_{x \to \infty} \phi_{\sigma}(x) = 0 \). In order to describe the contributions of both sides of \( t_0 \) to the asymptotic behavior of \( (R, T) \), we further suppose the following.

**Assumption 6.** For any sign \( \sigma \),

\[
p_{\sigma} = \lim_{x \to \infty} \frac{\phi_{\sigma} \tilde{g}(\sigma \phi_{\sigma})}{\phi_{-} \tilde{g}(-\phi_{-}) + \phi_{+} \tilde{g}(\phi_{+})}(x)
\]

exists.

Both \( p_{-} \) and \( p_{+} \) are nonnegative and their sum is 1. They represent the contribution of the events \( T < t_0 \) and \( T > t_0 \) to the limiting conditional distribution of \( T - t_0 \) given \( X > x \).

Considering \( \phi_{\sigma} \tilde{g}(\sigma \phi_{\sigma}) \) as a regularly varying function of \( \psi(x)/x \) of index \((1 + \tau_{\sigma})/\kappa_{\sigma}\), we see that if both \( p_{-} \) and \( p_{+} \) do not vanish, then \((1 + \tau_{+})/\kappa_{+} = (1 + \tau_{-})/\kappa_{-}\).

To state our results, we introduce the random sign

\[ S = \text{sign}(T - t_0) . \]

We consider also a random sign \( S \) whose distribution is

\[
P\{S = \sigma\} = \frac{p_{\sigma} \Gamma\left(\frac{1 + \tau_{\sigma}}{\kappa_{\sigma}}\right)}{p_{-} \Gamma\left(\frac{1 + \tau_{-}}{\kappa_{-}}\right) + p_{+} \Gamma\left(\frac{1 + \tau_{+}}{\kappa_{+}}\right)}, \quad \sigma \in \{-, +\} .
\]

Central to our two-sided extension is the following consequence of Theorem 2.1. This result is also of importance to understand how the results in the next section, stated under one-sided assumptions
and an extra conditioning on $T > t_0$, can be extended with two-sided assumptions and no conditioning on $T > t_0$.

**Proposition 3.1.** Under assumptions 1, 4, 5 and 6, the conditional distribution of $S$ given $X > x$ converges weakly* to that of $S$.

**Proof.** The second assertion of Theorem 2.1 implies that for any sign $\sigma$,

$$P\{ X > x ; S = \sigma \} \sim \phi_\sigma(x)\tilde{g}(\sigma\phi_\sigma(x))\mathcal{P}(x)\frac{1}{\kappa_\sigma}\Gamma\left(\frac{1 + \tau_\sigma}{\kappa_\sigma}\right)$$

as $x$ tends to infinity. The proposition then follows from the formula

$$P\{ S = \sigma \mid X > x \} = \frac{P\{ X > x ; S = \sigma \}}{P\{ X > x ; S = -\} + P\{ X > x ; S = +\}}.$$

We then define a random pair $(R, T_S)$ whose conditional distribution given $S = \sigma$ has density with respect to the Lebesgue measure

$$\frac{\kappa_\sigma}{\Gamma\left(\frac{1 + \tau_\sigma}{\kappa_\sigma}\right)}t^{r_\sigma}e^{-r_\sigma}1\{0 < t < r_\sigma^{1/\kappa_\sigma}\}.$$

**Theorem 3.2.** Under assumptions 1, 4, 5 and 6, the conditional distribution of

$$\left(\frac{R - x}{\psi(x)}, \frac{T - t_0}{\phi_S(x)}\right)$$

given $X > x$ converges weakly* as $x$ tends to infinity to the distribution of $(R, ST_S)$.

The density of the limiting distribution can be written explicitly as

$$\sum_{\sigma \in \{-,+\}} |t|^{r_\sigma}e^{-r_\sigma}\frac{p_\sigma 1\{|t|^{\kappa_\sigma} < r ; \sigma t > 0\}}{p_-\Gamma\left(\frac{1 + \tau_-}{\kappa_-}\right) + p_+\Gamma\left(\frac{1 + \tau_+}{\kappa_+}\right)}.$$

**Proof.** For any Borel subset $A$ of $\mathbb{R}^2$, we have

$$P\left\{ \left(\frac{R - x}{\psi(x)}, \frac{T - t_0}{\phi_S(x)}\right) \in A \mid X > x \right\}$$

$$= \sum_{\sigma \in \{-,+\}} P\left\{ \left(\frac{R - x}{\psi(x)}, \frac{T - t_0}{\phi_S(x)}\right) \in A \mid X > x; S = \sigma \right\}P\{ S = \sigma \mid X > x \}.$$  (3.1)
Theorem 2.1 implies that the conditional distribution of
\[
\left( \frac{R - x}{\psi(x)}, \sigma \frac{T - t_0}{\phi_\sigma(x)} \right)
\]
given \(X > x\) and \(S = \sigma\) converges weakly* to that of a random variable \((R, T_\sigma)\) whose density with respect to the Lebesgue measure is
\[
\frac{\kappa_{\sigma}}{\Gamma \left( \frac{1 + \tau_{\sigma}}{\kappa_{\sigma}} \right)} t^{-\tau_{\sigma}} e^{-r \{0 < t < r^{1/\kappa_{\sigma}}\}}.
\]
Combining Proposition 3.1 and (3.1), we obtain that the conditional distribution of
\[
\left( \frac{R - x}{\psi(x)}, \frac{T - t_0}{\phi_S(x)} \right)
\]
given \(X > x\) converges weakly* to that of \((R, ST_S)\).  

One may argue that the random norming of \(T - t_0\) by \(1/\phi_S(x)\) in Theorem 3.2 would be better replaced by a deterministic one. This can be done, defining
\[
\phi_* = \phi_+ + \phi_-
\]
and assuming

**Assumption 7.** For any sign \(\sigma\), the limit \(q_\sigma = \lim_{x \to \infty} \frac{\phi_\sigma}{\phi_*}(x)\) exists.

We then have the following.

**Theorem 3.3.** Under assumptions 1, 4–7, the conditional distribution of
\[
\left( \frac{R - x}{\psi(x)}, \frac{T - t_0}{\phi_*(x)} \right)
\]
given \(X > x\) converges weakly* as \(x\) tends to infinity to the distribution of \((R, q_SST_S)\).

Again, the limiting density can be made explicit if needed.

**Proof.** Given Proposition 3.1 and the definition of \(p_\sigma\), the conditional distribution of the random variable \(\phi_S(x)/\phi_*(x)\) given \(X > x\)
converges weakly* to that of \( q_S \), and this convergence holds jointly with the conditional convergence of

\[
\left( \frac{R - x}{\psi(x)}, \frac{T - t_0}{\phi_S(x)} \right).
\]

The result follows.

4. Bivariate conditional limit theorems. The purpose of this section is to use Theorem 2.1 to shed a new light on previous results dealing with conditional bivariate distributions given one extreme component.

To do so, we consider another random variable, \( Y = Rv(T) \), under the conditional distribution given \( X > x \) and \( T > t_0 \). Below, we will make precise why we condition on both \( X > x \) and \( T > t_0 \). However, the conditioning by \( T > t_0 \) can be easily removed by imposing the proper two-sided condition and using the same arguments used to extend Theorem 2.1 to Theorems 3.2 and 3.3. In particular, removing the conditioning by \( T > t_0 \) does not seem to add any insight on the problem. Thus, we choose to keep this conditioning to keep the exposition concise. We set

\[
R_x = \frac{R - x}{\psi(x)} \quad \text{and} \quad T_x = \frac{T - t_0}{\phi(x)}.
\]

Under the conditional distribution given \( X > x \) and \( T > t_0 \), Theorem 2.1 asserts that \((R_x, T_x)\) converges in distribution to some \((R, T)\) whose density with respect to the Lebesgue measure is given by (2.2).

Similarly to \( \tilde{u} \), define \( \tilde{v}(s) = v(t_0) - v(t_0 + s) \). Let us assume that

**Assumption 8.** \( \rho = v(t_0) \) is well defined and \( \tilde{v} \) is regularly varying at \( 0+ \), with nonnegative index \( \delta \).

Note that \( \delta = 0 \) is allowed; one could also look at what happens if \( \delta \) is negative, using the same technique but working directly with \( v(t_0 + s) \) instead of \( \tilde{v}(s) \); so the sign of \( \delta \) does not really matter, but we will take it nonnegative in order to see how some known results follow from Theorem 2.1.

We have

\[
Y = Rv(T) = (x + \psi(x)R_x)v(t_0 + \phi(x)T_x) = (x + \psi(x)R_x)(\rho - \tilde{v}(\phi(x)T_x)) = \rho x + \rho \psi(x)R_x - (x + \psi(x)R_x)\tilde{v}(\phi(x)T_x)
\]
Since \( \psi(x) = o(x) \) and \( \tilde{v} \) is regularly varying and both \( R_x \) and \( T_x \) remain bounded in probability, we obtain, when \( T_x \) is nonnegative,

\[
Y = \rho x + \rho \psi(x) R_x - T_x^\delta \tilde{v} \circ \phi(x) \left( 1 + o(1) \right).
\]

(4.1)

Using the Skorokhod-Dudley-Wichura theorem (see e.g. Dudley, 1989, sections 11.6 and 11.7), we can assume that we have versions of \( R_x \) and \( T_x \) which converge almost surely to \((R, T)\) on the events \( \{ X > x; T > t_0 \} \). We then obtain, under the conditional distribution given \( X > x \) and \( T > t_0 \),

\[
Y \overset{d}{=} \rho x + \rho \psi(x) R(1 + o(1)) - T^\delta \tilde{v} \circ \phi(x) \left( 1 + o(1) \right)
\]

as \( x \) tends to infinity. Given (2.1), this means

\[
Y \overset{d}{=} \rho x + \rho x \tilde{u} \circ \phi(x) R(1 + o(1)) - T^\delta x \tilde{v} \circ \phi(x) \left( 1 + o(1) \right).
\]

(4.2)

Recall that \( \tilde{u} \) is regularly varying with index \( \kappa \) and \( \tilde{v} \) is regularly varying with index \( \delta \), and that we have \( R > T^\kappa \) almost surely. We can now vary the assumptions in several ways, which we state as examples.

**Remark.** We can now see what happens if we do not wish to condition on \( T > t_0 \). We need to introduce the random sign \( S = \text{sign}(T - t_0) \) and follow what was done in section 3. Identity (4.1) becomes, with rather obvious notation,

\[
Y = \rho x + \rho \psi(x) R_x \left( 1 + o(1) \right) - |T_x|^\delta x \tilde{v} \circ \left( S \phi_S(x) \right) \left( 1 + o(1) \right).
\]

One then needs to discuss the behavior of \( \tilde{v} \) on both sides of \( 0 \), both in terms of regular variation and sign, and one can also discuss the possible replacement of \( \phi_S \) by \( \phi_* \). Such a discussion requires to distinguish very many cases and does not appear to bring further understanding. Thus we choose to state results that seems to be the most useful to specialize in applications.

**Example 1.** We assume that

\[
\lim_{s \to \overline{0}^+} \rho \tilde{u}(s) / \tilde{v}(s) = 0.
\]

(4.3)

This is implied by Fougères and Soulier’s assumption that \( \delta < \kappa \), and it is also satisfied whenever \( \rho \) is 0. Theorem 2.1 implies the following
result which was proved under stronger assumptions in Fougères and Soulier (2010), up to the conditioning by $T > t_0$ which can be removed in using the same arguments as in the previous section. Our proof shows that while this result looks like a truly two-dimensional result, it is really two-dimensional in $(R, T)$ but one-dimensional in $(X, Y)$.

**Corollary 4.1.** Under the assumptions of Theorem 2.1, assumption 8 and (4.3), the conditional distribution of

\[
\left( \frac{X - x}{\psi(x)}, \frac{Y - \rho x}{x \tilde{v} \circ \phi(x)} \right)
\]

given $X > x$ and $T > t_0$ converges weakly* to that of $(R - T^\kappa, -T^\delta)$ as $x$ tends to infinity.

**Proof.** (4.2) gives $Y \overset{d}= \rho x - T^\delta x \tilde{v} \circ \phi(x) (1 + o(1))$, and we have the convergence in distribution

\[
\frac{Y - \rho x}{x \tilde{v} \circ \phi(x)} \to -T^\delta.
\]

This is the result. \[\square\]

Corollary 4.1 makes it quite clear why the function $H_{\eta, \tau}$ come up in Fougères and Soulier (2010): this is what one gets from Theorem 2.1 and the continuous mapping theorem, and it occurs because of what the joint distribution of $(R, T)$ is.

**Example 2.** Assume that

\[
\lim_{s \to 0^+} |\tilde{u}(s)/\tilde{v}(s)| = +\infty. \quad (4.4)
\]

This is the case if $\delta > \kappa$ for instance.

**Corollary 4.2.** Under the assumptions of Theorem 2.1, assumption 8 and (4.4), the conditional distribution of

\[
\left( \frac{X - x}{\psi(x)}, \frac{Y - \rho x}{\psi(x)} \right)
\]

given $X > x$ and $T > t_0$ converges weakly* to that of $(R - T^\kappa, \rho R)$ as $x$ tends to infinity.
Proof. It follows from (4.2).

Note that when $\rho$ vanishes, Corollary 4.2 yields a limiting distribution with degenerate second marginal. This means that in the conditional distribution $Y = o_P(\psi(x))$ as $x$ tends to infinity.

Example 3. Assume that
\[
\lim_{s \to 0^+} \frac{\tilde{u}(s)}{\tilde{v}(s)} = C \in \mathbb{R}.
\] (4.5)

When $C = 0$, this is example 1.

Corollary 4.3. Under the assumption of Theorem 2.1, assumption 8 and (4.5), the conditional distribution of
\[
\left( \frac{X - x}{\psi(x)}, \frac{Y - \rho x}{x \tilde{v} \circ \phi(x)} \right)
\]
given $X > x$ and $T > t_0$ converges weakly* to that of $(R - T^\kappa, C \rho R - T^\delta)$.

Proof. It follows from (4.2).

If $C \neq 0$, then Corollary 4.3 asserts as well that the conditional distribution of
\[
\left( \frac{X - x}{\psi(x)}, \frac{Y - \rho x}{\psi(x)} \right)
\]
given $X > x$ and $T > t_0$ converges weakly* to that of $(R - T^\kappa, \rho R - T^\delta/C)$. This restatement gives example 2 at the limit when $C$ tends to infinity.

Example 4. Assume now that
\[
v(t) = (t - t_0 + \rho)u(t) \text{ in a neighborhood of } t_0.
\] (4.6)

Note that $\rho = v(t_0)$ as required in assumption 8. In this case, $\tilde{v}(s) = (\rho + s)\tilde{u}(s) - su(t_0)$. This identity shows that $\delta = \kappa \wedge 1$ if $\rho \neq 0$ and $\delta = 1$ if $\rho = 0$; therefore, we can be in any of the cases covered by examples 1, 2 or 3: for instance, $\kappa > 1$ or $\rho = 0$ yield (4.3); $\kappa < 1$ and $\rho \neq 0$ yield (4.5); and $\kappa = 1$ and $\rho \neq 0$ may yield any of (4.3),(4.4) or (4.5). The question arises as to whether it is possible
to have a unified normalization for $Y$ for its conditional distribution to converge. The following result shows that with assumption (4.6), we cannot anymore normalize $Y$ by some deterministic quantities independent of $\kappa$ and $\rho$. However, we can use a normalization which involves $X$, as for instance Heffernan and Resnick (2007) did. Up to the conditioning on $T > t_0$, the following result was obtained by Seifert (2012) under stronger conditions.

**Corollary 4.4.** Under the assumptions of Theorem 2.1, the conditional distribution of

$$\left(\frac{X - x}{\psi(x)}, \frac{(Y/X) - \rho}{\phi(x)}\right)$$

given $X > x$ and $T > t_0$ converges weakly* to that of $(R - T^\kappa, T)$ as $x$ tends to infinity.

**Proof.** We have

$$Y = Rv(T) = Xv\left(\frac{v(T)}{u(T)}\right) = X(T - t_0 + \rho).$$

Thus

$$\frac{(Y/X) - \rho}{\phi(x)} = T_x.$$

The result follows since $T_x$ converges in distribution to $T$ when $X > x$. $\blacksquare$

In typical situations, $v$ is continuous and monotone on a neighborhood of $t_0$, while $u$ is continuous and monotone on a punctured neighborhood of $t_0$ and $\kappa > \delta$, as assumed in Fougères and Soulier (2010). As shown in Seifert (2012), a suitable reparametrisation of $T$ yields (4.6).

**Example 5.** The previous example can be generalized in the following way, yielding a somewhat exotic limiting behavior. Define the function $\theta(t)$ by the relation

$$v(t) = \theta(t) u(t). \tag{4.7}$$

and assume that for some nonnegative integer $n$, $\theta$ is $n$ times differentiable and

$$\theta^{(j)}(t_0) = 0 \text{ if } j = 1, 2, \ldots, n - 1, \quad \text{and} \quad \theta^{(n)}(t_0) \neq 0. \tag{4.8}$$
Put differently, \( n \) corresponds to the first nonvanish Taylor coefficient of \((v/u)(t) - (v/u)(t_0)\).

**Corollary 4.5.** Under the assumptions of Theorem 2.1 and (4.7), (4.8), the conditional distribution of

\[
\left( \frac{X - x}{\psi(x)}, \frac{(Y/X) - \theta(t_0)}{\phi(x)^n} \right)
\]

given \( X > x \) and \( T > t_0 \) converges weakly* to that of \((R - T^n, T^n \theta(n)(t_0)/n!)/n!\) as \( x \) tends to infinity.

**Proof.** Using (4.7),

\[
Y/X = \theta(T) = \theta(t_0 + \phi(x)T_x)
\]
as \( x \) tends to infinity. Since \( \phi(x) \) tends to 0 as \( x \) tends to infinity, Taylor formulas and the convergence in distribution of \( T_x \) to \( T \) yield

\[
Y/X \xrightarrow{d} \theta(t_0) + \phi(x)^nT^n\theta^{(n)}(t_0)/n!(1 + o(1))
\]
as \( x \) tends to infinity, which is the result.

Of course, one could extend this example further in assuming that \( \theta(t) - \theta(t_0) \) is regularly varying at \( t_0 \), and numerous other variations are possible.

To conclude, since all the results of this paper use basic regular variation theory, it is certain that a truly multivariate extension is possible. Such extension is not unique for there exists various theories of multivariate regular variation, beyond what is popular in extreme value theory; see for instance the works of Mershaert and Scheffer (2001), the book by Vladimirov, Drozzinov and Zvialov (1988), and some pointers in Bingham, Goldie and Teugels (1989). Which one is the most relevant seems application dependent.

**5. Proof of Theorem 2.1.** Before giving a formal proof, it is enlightening to give an intuition on how this result was found and why it might be true. We have \( X = Ru(T) \). If \( X > x \) and \( x \) is large, since \( u \) is at most 1 and \( R \) has a light tail, we should expect \( R \) to be about \( x \) and \( u(T) \) about 1, that is, \( T \) about \( t_0 \); more precisely, since
\( \overline{H} \) is in \( \Gamma(\psi) \), we should have, \( R \approx x + \psi(x)R \) for some \( R \) of order 1, and, hopefully, \( T \approx t_0 + \phi(x)T \) for some function \( \phi \) which tends to 0 at infinity, and some \( T \) of order 1. Moreover, if \( T > t_0 \) then \( T \) should be nonnegative. That would give

\[
X = Ru(T) \approx (x + \psi(x)R)u(t_0 + \phi(x)T) \quad (5.1)
\]

One should then look at \( u \) near \( t_0 \), and so we define

\[
\tilde{u}(s) = u(t_0) - u(t_0 + s).
\]

If this function is regularly varying at 0+ with index \( \kappa \), and since \( u(t_0) = 1 \), we expect

\[
u(t_0 + \phi(x)T) = u(t_0) - \tilde{u}(\phi(x)T) \approx 1 - T^\kappa \tilde{u} \circ \phi(x).
\]

Thus, given (5.1) and that \( \psi(x) = o(x) \),

\[
X = Ru(T) \approx x + \psi(x)R - T^\kappa x \tilde{u} \circ \phi(x).
\]

We see that for \( R \) and \( T \) to contribute to \( X \) (that is, to find the limiting behavior of \( R \) and \( T \) conditioned on \( X > x \)), we should have \( \psi(x) \) and \( x \tilde{u} \circ \phi(x) \) of the same order of magnitude (otherwise, one of the terms would dominate the other one, and either \( R \) or \( T \) would be lost in the asymptotic). Therefore, we should define \( \phi \) by requiring \( \tilde{u} \circ \phi(x) \sim \psi(x)/x \) as \( x \) tends to infinity. We would then obtain

\[
X \approx x + \psi(x)(R - T^\kappa)
\]

and the condition that \( X > x \) translates into \( R > T^\kappa \). It remains us to formalize this sketch and turn it into a proof.

As most of the time with asymptotic analysis of integrals involving regularly varying functions, we will need a little more than just the definition, namely Potter’s bounds. To say that \( \tilde{u} \) is regularly varying at 0 with positive index \( \kappa \) means that \( u(1/t) \) is regularly varying with index \(-\kappa\) at infinity. Potter’s bounds are that \( \tilde{u}(1/t)/\tilde{u}(1/s) \) is sandwiched between quantities of the form \( A^{\pm 1}(s/t)^{\pm \eta} \) where the real number \( A \) can be chosen as close to 1 as one wants, \( \eta \) is positive and we take is less than \( \kappa \), and the sandwich is good whenever \( t \) and \( s \) are large enough (see Bingham, Goldie and Teugels, 1989, §1.5). Consequently, given an \( A \) greater than 1, and a positive \( \eta \), the ratio
\( \tilde{u}(s)/\tilde{u}(t) \) is sandwiched between quantities of the form \( A^{\pm 1}(s/t)^{\kappa \pm \eta} \) whenever \( s \) and \( t \) are small enough — say less than some \( \epsilon_0 \).

The proof of Theorem 2.1 has two steps, tightness and convergence, which are disguised as asymptotic analysis of some integrals.

We will use repeatedly that, since \( u(t_0) = 1 \),

\[ u(t) = 1 - \tilde{u}(t - t_0). \]

**Step 1. Convergence.** Let \( f \) be a nonnegative continuous function on \( \mathbb{R}^2 \), whose support is a compact subset of \( (\mathbb{R} \setminus \{0\})^2 \). Consider the integral

\[
I(x) = \int f \left( \frac{r - x}{\psi(x)}, \frac{t - t_0}{\phi(x)} \right) \mathbb{1}\{ru(t) > x; t > t_0\} g(t) \, dH(r) \, dt.
\]

This integral is

\[
E\left( f \left( \frac{R - x}{\psi(x)}, \frac{T - t_0}{\phi(x)} \right) \mathbb{1}\{X > x; T > t_0\} \right),
\]

that is, the conditional expectation given \( X > x \) and \( T > t_0 \) multiplied by \( P\{X > x; T > t_0\} \). The change of variables consisting in substituting \( r \) for \( (r - x)/\psi(x) \) and \( t \) for \( (t - t_0)/\phi(x) \) yields

\[
I(x) = \int f(r, t) \mathbb{1}\{(x + r\psi(x))u(t_0 + t\phi(x)) > x; t > 0\} \tilde{g}(t\phi(x)) \, dH(x + r\psi(x)) \, dt. \tag{5.2}
\]

Since \( f \) has compact support which excludes the 0-coordinates, this integral is in fact an integral over a compact subset of \( \mathbb{R}^2 \) which excludes \( r = 0 \) and \( t = 0 \). Since \( r \) and \( t \) are now in a compact set which excludes 0, the regular variation properties of the various functions yield

\[
u(t_0 + t\phi(x)) = 1 - \tilde{u}(t\phi(x)) = 1 - t^\kappa \tilde{u} \circ \phi(x)(1 + o(1))
\]

and

\[
\tilde{g}(t\phi(x)) = t^\tau \tilde{g} \circ \phi(x)(1 + o(1))
\]
as \( x \) tends to infinity, and both \( o(1) \) are uniform in \( t \) such that \((r,t)\) is in the support of \( f \) — again, because we excluded the axis of \( \mathbb{R}^2 \).

Thus, since \( \psi(x) = o(x) \), we have

\[
(x + r\psi(x))u(t_0 + t\phi(x)) = (x + r\psi(x))\left(1 - \tilde{u}(t\phi(x))\right) = x + r\psi(x) - x(1 + o(1))\tilde{u}(t\phi(x)) = x + r\psi(x) - xt^\kappa \tilde{u} \circ \phi(x)(1 + o(1)).
\]

Thus, referring to part of the integrand in (5.2), and using the definition of \( \phi \),

\[
1 \{ (x + r\psi(x))u(t_0 + t\phi(x)) > x; \ t > 0 \}
\]

\[
= 1 \{ r\psi(x) - t^\kappa x\tilde{u} \circ \phi(x)(1 + o(1)) > 0; \ t > 0 \}
\]

\[
= 1 \{ \psi(x)(r - t^\kappa (1 + o(1))) > 0; \ t > 0 \}
\]

\[
= 1 \{ r > t^\kappa (1 + o(1)); \ t > 0 \}.
\]

If \( x \) is large, the previous display shows that the indicator function in (5.3) can be sandwiched between functions

\[
1 \{ r > (1 - \epsilon)t^\kappa; \ t > 0 \},
\]

(take \( \epsilon \) positive for an upper bound, \( \epsilon \) negative for an lower bound).

That allows us to sandwich \( I(x) \) between integrals of the form

\[
I_\epsilon(x) = \int f(r,t) 1 \{ r > (1 - \epsilon)t^\kappa; \ t > 0 \} \tilde{g} \circ \phi(x)\phi(x) t^\tau \ dt \ dH(x + \psi(x)r),
\]

provided \( x \) is large enough; thus for \( \epsilon \) positive and \( x \) large enough,

\[
(1 - \epsilon)I_{-\epsilon}(x) \leq I(x) \leq (1 + \epsilon)I_\epsilon(x).
\]

The measure \( dH(x + \psi(x)r)/\mathcal{H}(x) \) converges vaguely to a measure with density \( e^{-r} \) with respect to the Lebesgue measure — note that we are using vague convergence of measure, so that \( r \) has to remain in a compact set, which is why we took \( f \) having a compact support with respect to both variables \( r \) and \( t \). Consequently, we obtain

\[
\lim_{x \to \infty} \frac{I_\epsilon(x)}{\phi(x)\tilde{g} \circ \phi(x)\mathcal{H}(x)}
= \int f(r,t) 1 \{ r > (1 - \epsilon)t^\kappa; \ t > 0 \} t^\tau e^{-r} \ dr
\]

\[
= \int f(r,t) 1 \{ r > (1 - \epsilon)t^\kappa; \ t > 0 \} t^\tau e^{-r} \ dr.
\]

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as \( x \) tends to infinity. Since \( \epsilon \) is arbitrary, combining (5.4) and (5.5) yield
\[
\lim_{x \to \infty} \frac{I(x)}{\phi(x)\tilde{g} \circ \phi(x)\overline{H}(x)} = \int f(r,t)1\{r > t^\kappa; t > 0\}t^\tau dt e^{-r} dr.
\]

**Step 1+1/2. Refinement.** In step 1, the function \( f \) is supported in \((\mathbb{R} \setminus \{0\})^2\). To prove vague convergence of the distribution as a distribution on \( \mathbb{R}^2 \), we need to allow for compact support in the entire \( \mathbb{R}^2 \), not excluding the axes. To make this extension, it suffices to show that there is no mass accumulation along the axes \( \{0\} \times \mathbb{R} \) and \( \mathbb{R} \times \{0\} \). Thus, setting
\[
J_{1,\epsilon}(x) = P\left\{ \frac{|R - x|}{\psi(x)} \leq \epsilon; Ru(T) > x; T > t_0 \right\}
\]
and
\[
J_{2,\epsilon}(x) = P\left\{ \frac{|T - t_0|}{\phi(x)} \leq \epsilon; Ru(T) > x; T > t_0 \right\},
\]
we need to prove that for \( j = 1, 2, \)
\[
\lim_{\epsilon \to 0} \limsup_{x \to \infty} \frac{J_{j,\epsilon}}{\phi(x)\tilde{g} \circ \phi(x)\overline{H}(x)} = 0. \tag{5.6}
\]
To do this, we have, for \( x \) large enough, that \( J_{1,\epsilon}(x) \) is at most
\[
P\left\{ |R - x| \leq \epsilon \psi(x); u(T) > \frac{x}{x + \epsilon \psi(x)}; T > t_0 \right\}
\leq \left( \overline{H}(x - \epsilon \psi(x)) - \overline{H}(x + \epsilon \psi(x)) \right) P\left\{ u(T) > 1 - 2\epsilon \frac{\psi(x)}{x}; T > t_0 \right\}
\leq \overline{H}(x)(e^\epsilon - e^{-\epsilon})(1 + o(1)) P\left\{ \tilde{u}(T - t_0) < \frac{2\epsilon \psi(x)}{x}; T > t_0 \right\}
\]
the last inequality coming from \( \overline{H} \in \Gamma(\psi) \), the definition of \( \tilde{u} \) and that \( u(t_0) = 1 \). But since \( \tilde{u} \) is regularly varying with index \( \kappa \),
\[
\tilde{u}\left((2\epsilon)^{1/\kappa}\phi(x)\right) \sim 2\epsilon \tilde{u} \circ \phi(x) \\
\sim 2\epsilon \psi(x)/x
\]
as \( x \) tends to infinity. Consequently, for \( x \) large enough,
\[
P\left\{ \tilde{u}(T - t_0) < \frac{2\epsilon \psi(x)}{x}; T > t_0 \right\}
\leq P\left\{ \tilde{u}(T - t_0) < \tilde{u}\left((4\epsilon)^{1/\kappa}\phi(x)\right); T > t_0 \right\}
\leq P\left\{ |T - t_0| < (8\epsilon)^{1/\kappa}\phi(x); T > t_0 \right\},
\]

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the last inequality coming from the fact that a regularly varying function of positive index is asymptotically equivalent to a monotone function — see Bingham, Goldie and Teugels (1989, §1.5.2).

Note that for any \( \theta \) positive,

\[
P\{ 0 < T - t_0 \leq \theta \phi(x) \} \sim \phi(x) \tilde{g} \circ \phi(x) \int_0^{\theta} y^{-\tau} \, dy,
\]

as \( x \) tends to infinity, because this probability is

\[
\int_0^{\theta \phi(x)} \tilde{g}(s) \, ds = \phi(x) \tilde{g} \circ \phi(x) \int_0^{\theta} \frac{\tilde{g}(s \phi(x))}{\tilde{g} \circ \phi(x)} \, ds
\]

and \( \tilde{g} \) is regularly varying with index \( \tau > -1 \). Thus, combining the various bounds, we have, for \( x \) large enough,

\[
J_{1, \epsilon}(x) \leq 2 \Phi(x) (e^\epsilon - e^{-\epsilon}) \phi(x) \tilde{g} \circ \phi(x) \int_0^{(16\epsilon)^{1/\kappa}} y^{-\tau} \, dy
\]

and this proves (5.6) for \( j = 1 \).

To prove (5.6) for \( j = 2 \), we see that for \( x \) large enough,

\[
J_{2, \epsilon}(x) \leq P\{ 0 < T - t_0 \leq \epsilon \phi(x) ; R > x \}
= P\{ 0 < T - t_0 \leq \epsilon \phi(x) \} \Phi(x).
\]

Then, we use (5.7) to bound \( J_{2, \epsilon}(x) \), establishing (5.6) for \( j = 2 \).

Combined with Step 1, this shows that for any nonnegative continuous compactly supported function \( f \) on \( \mathbb{R}^2 \)

\[
E\left( f\left( \frac{R-x}{\psi(x)}, \frac{T-t_0}{\phi(x)} \right) \mathbb{1}\{ X > x ; T > t_0 \} \right)
\sim \phi(x) \tilde{g} \circ \phi(x) \Phi(x) \int f(r,t) \mathbb{1}\{ r > t^\kappa ; t > 0 \} t^\tau \, dt \, e^{-r} \, dr
\]

as \( x \) tends to infinity. By writing any continuous function as the sum of its positive and negative part, this still holds for any continuous and compactly supported function on \( \mathbb{R}^2 \).

Step 2. Tightness. We now show that \( (R-x)/\psi(x) \) and \( (T-t_0)/\phi(x) \) are tight random variables under the conditional probability given \( X > x \) and \( T > t_0 \). For this purpose, given step 1 and anticipating the conclusion of the proof, we need to show that

\[
\lim_{r \to \infty} \limsup_{x \to \infty} \frac{P\{ \left| \frac{R-x}{\psi(x)} \right| > r ; Ru(T) > x ; T > t_0 \}}{\phi(x) \tilde{g} \circ \phi(x) \Phi(x)} = 0,
\]

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and
\[
\lim_{t \to \infty} \limsup_{x \to \infty} \frac{P\left\{ \left| \frac{T - t_0}{\phi(x)} \right| > t ; Ru(T) > x ; T > t_0 \right\}}{\phi(x) \tilde{g} \circ \phi(x) \mathcal{H}(x)} = 0 , \quad (5.8)
\]

This is a bit painful, because of the absolute values involved. We will examine the three cases obtained when ‘removing’ the absolute values.

Case 1. Let \( r \) be positive and let us bound
\[
P_{1,r}(x) = P\{ R > x + r\psi(x) ; Ru(T) > x ; T > t_0 \} .
\]

For \( x \) large enough, this is at most
\[
\mathcal{H}(x + r\psi(x)) P\left\{ u(T) > \frac{x}{x + r\psi(x)} ; T > t_0 \right\}
\sim \mathcal{H}(x)e^{-r} P\left\{ \tilde{u}(T - t_0) < r \frac{\psi(x)}{x} (1 + o(1)) ; T > t_0 \right\}.
\]

As in step 1+1/2, using (5.7), this is of order at most
\[
\mathcal{H}(x) \phi(x) g \circ \phi(x) e^{-r} \int_0^{(4r)^{1/\kappa}} y^{-\kappa} dy.
\]

Thus,
\[
\lim_{r \to \infty} \limsup_{x \to \infty} \frac{P_{1,r}(x)}{\phi(x) \tilde{g} \circ \phi(x) \mathcal{H}(x)} = 0 .
\]

Case 2. For \( r \) positive, define
\[
P_{2,r}(x) = P\{ R < x - r\psi(x) ; Ru(T) > x ; T > t_0 \} .
\]

When \( x \) is large enough, \( \psi(x) \) is well defined and positive. In that range of \( x \), since \( |u| \leq 1 \), we cannot have \( Ru(T) > x \) while having \( R < x - r\psi(x) \). Thus \( P_{2,r}(x) = 0 \) whenever \( x \) is large enough.

Case 3. The probability involved in the numerator of (5.8) is
\[
P_{3,t}(x) = P\{ T > t_0 + t\phi(x) ; Ru(T) > x ; T > t_0 \} .
\]

We see that
\[
P_{3,t}(x) = \int_t^\infty \mathcal{H}\left( \frac{x}{u(t_0 + \phi(x)s)} \right) \phi(x) \tilde{g}(s\phi(x)) ds . \quad (5.9)
\]
We write
\[ u(t_0 + s\phi(x)) = 1 - \tilde{u}(s\phi(x)) \]
and we use the usual arguments to bound the integral: Potter’s bound whenever we can, and ad hoc argument elsewhere. This goes as follows. We may assume that \( t \) is greater than 1. Let \( \eta \) be a (small) positive real number. Let \( \epsilon \) be small enough so that Potter’s bounds
\[ \tilde{u}(s\phi(x)) \geq \frac{1}{2} \tilde{u} \circ \phi(x)s^{\kappa-\eta} \]
and
\[ \tilde{g}(s\phi(x)) \leq 2\tilde{g} \circ \phi(x)s^{\tau+\eta} \quad (5.10) \]
apply on the range \( 1 \leq t \leq s \leq \epsilon/\phi(x) \). We then have, on that range of \( s \) (provided \( \epsilon \) was chosen small enough),
\[ \frac{1}{u(t_0 + s\phi(x))} = \frac{1}{1 - \tilde{u}(s\phi(x))} \geq 1 + \frac{1}{4} \tilde{u}(s\phi(x)) \]
\[ \geq 1 + \frac{1}{8} \tilde{u} \circ \phi(x)s^{\kappa-\eta}. \quad (5.11) \]
Referring to part of the integral (5.9), using the definition of \( \phi \), (5.10) and (5.11), we have then
\[
\int_{t}^{\epsilon/\phi(x)} \Pi\left(\frac{x}{u(t_0 + s\phi(x))}\right)\tilde{g}(s\phi(x))\phi(x)\,ds
\leq 2\phi(x)\tilde{g} \circ \phi(x) \int_{t}^{\epsilon/\phi(x)} \Pi\left(x + \frac{1}{16}\psi(x)s^{\kappa-\eta}\right)s^{\tau+\eta}\,ds
\]
Using the first statement of Lemma 5.1 in Fougères and Soulier (2010) (note we can take \( C = 2 \) in that Lemma, which we do here), this upper bound is at most
\[
4\phi(x)\tilde{g} \circ \phi(x)\Pi(x) \int_{t}^{\infty} \frac{s^{\tau+\eta}}{(1+(s^{\kappa-\eta}/16))^{p}}\,ds, \quad (5.12)
\]
where \( p \) is taken large enough so that the integral converges.

We now work on the easy part of the integral (5.9), namely, that for \( s \) between \( \epsilon/\phi(x) \) and \( \infty \). Given how this integral was obtained, this part corresponds to \( T > t_0 + \epsilon \), and it is at most (again, provided we choose \( \epsilon \) small enough)
\[
P\{ Ru(t_0 + \epsilon/2) > x \} = \Pi\left(\frac{x}{u(t_0 + \epsilon/2)}\right). \quad (5.13)
\]
We now claim that if $c > 1$ (think of $c$ as $1/u(t_0 + \epsilon/2)$), then
\[
\overline{H}(cx) = o\left(\overline{H}(x)\phi(x)\hat{g} \circ \phi(x)\right).
\] (5.14)

Indeed, using the second statement of Lemma 5.1 in Fougères and Soulier (2010), for any positive $p$ we have
\[
\overline{H}(cx) \leq \left(\frac{\psi(x)}{x}\right)^p \overline{H}(x)
\]
provided $x$ is large enough (note that we can take $C = 1$ in their inequality: it suffices to divide their $p$ by 2 and see that their $C$ times $(\psi(x)/x)^{p/2}$ tends to 0 and is less than 1 for $x$ large enough).

Thus, to prove (5.14), we have to show that for any $p$ large enough
\[
\left(\frac{\psi(x)}{x}\right)^p = o\left(\phi(x)\hat{g} \circ \phi(x)\right).
\]

But this comes from viewing $\phi(x)\hat{g} \circ \phi(x)$ has a function of $\psi(x)/x$ which is then regularly varying of index $(\tau + 1)/\kappa$ in that argument.

Now, combining (5.13) and (5.14), we obtain that, referring to part of (5.9)
\[
\int_1^\infty \overline{P}\left(\frac{x}{u(t_0 + \phi(x)s)}\right)\hat{g}(\phi(x)s)\phi(x)\,ds = o\left(\phi(x)\hat{g} \circ \phi(x)\overline{H}(x)\right)
\]
as $x$ tends to infinity. Combined with (5.12), and referring to (5.9) this shows that
\[
\limsup_{x \to \infty} \frac{P_{3,t}(x)}{\phi(x)\hat{g} \circ \phi(x)\overline{H}(x)} \leq 4 \int_t^\infty \frac{s^{\tau+\eta}}{(1 + (s^{\kappa-\eta}/16))^\kappa} ds,
\]
and, therefore,
\[
\lim_{t \to \infty} \limsup_{x \to \infty} \frac{P_{3,t}(x)}{\phi(x)\hat{g} \circ \phi(x)\overline{H}(x)} = 0.
\]

To conclude the proof, combining steps 1, 1+1/2 and 2, we obtain that
\[
P\{X > x; T > t_0\}
\sim \phi(x)\hat{g} \circ \phi(x)\overline{P}(x) \int_1^\infty \{r > t^\kappa; t > 0\} t^\tau e^{-r} \,dt \,dr
\sim \phi(x)\hat{g} \circ \phi(x)\overline{P}(x) \frac{1}{\kappa} \Gamma\left(\frac{1+\tau}{\kappa}\right)
\]
as \( x \) tends to infinity. Then, step 2 implies that the conditional distribution of

\[
\left( \frac{R - x}{\psi(x)} , \frac{T - t_0}{\phi(x)} \right)
\]

given \( X > x \) and \( T > t_0 \) is tight, and step 1 proves that it converges to the limit given in Theorem 2.1.

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