TOTALLY REAL SUBMANIFOLDS OF (LCS)\(n\)-MANIFOLDS.

SHYAMAL KUMAR HUI AND TANUMOY PAL

Abstract. The present paper deals with the study of totally real submanifolds and \(C\)-totally real submanifolds of \((LCS)_{n}\)-manifolds with respect to Levi-Civita connection as well as quarter symmetric metric connection. It is proved that scalar curvature of \(C\)-totally real submanifolds of \((LCS)_{n}\)-manifold with respect to both the said connections are same.

1. Introduction

As a generalization of LP-Sasakian manifold, recently Shaikh [11] introduced the notion of Lorentzian concircular structure manifolds (briefly, \((LCS)_{n}\)-manifolds) with an example. Such manifolds has many applications in the general theory of relativity and cosmology ([12], [13]).

The notion of semisymmetric linear connection on a smooth manifold is introduced by Friedmann and Schouten [3]. Then Hayden [5] introduced the idea of metric connection with torsion on a Riemannian manifold. Thereafter Yano [16] studied semisymmetric metric connection on a Riemannian manifold systematically. As a generalization of semisymmetric connection, Golab [4] introduced the idea of quarter symmetric linear connection on smooth manifolds. A linear connection \(\nabla\) in an \(n\)-dimensional smooth manifold \(\widetilde{M}\) is said to be a quarter symmetric connection [4] if its torsion tensor \(T\) is of the form

\[
T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]
\]

\[
= \eta(Y)\phi X - \eta(X)\phi Y,
\]

where \(\eta\) is an 1-form and \(\phi\) is a tensor of type \((1, 1)\). In particular, if \(\phi X = X\) then the quarter symmetric connection reduces to semisymmetric connection. Further if the quarter symmetric connection \(\nabla\) satisfies the condition \((\nabla_X g)(Y, Z) = 0\) for all \(X, Y, Z \in \chi(\widetilde{M})\), the set of all smooth vector fields on \(\widetilde{M}\), then \(\nabla\) is said to be a quarter symmetric metric connection.

Due to important applications in applied mathematics and theoretical physics, the geometry of submanifolds has become a subject of growing interest. Analogous to almost Hermitian manifolds, the invariant and anti-invariant submanifolds are depend on the behaviour of almost contact metric structure \(\phi\). A submanifold of a contact metric manifold \(\widetilde{M}\) is said to be anti-invariant if for any \(X\) tangent to \(M\), \(\phi X\) is
normal to \( M \), i.e., \( \phi(TM) \subset T^\perp M \) at every point of \( M \), where \( T^\perp M \) is the normal bundle of \( M \). So, if a submanifold \( M \) of a contact metric manifold \( \tilde{M} \) is normal to the structure vector field \( \xi \), then it is anti-invariant. A submanifold \( M \) in a contact metric manifold \( \tilde{M} \) is called a \( C \)-totally real submanifold if every tangent vector of \( M \) belongs to the contact distribution \([15]\). Thus a submanifold \( M \) in a contact metric manifold is a \( C \)-totally real submanifold if \( \xi \) is normal to \( M \). Consequently \( C \)-totally real submanifolds in a contact metric manifold are anti-invariant, as they are normal to \( \xi \). Recently Hui et al. \([1], [6], [7], [14]\) studied submanifolds of \((LCS)_n\)-manifolds. The present paper deals with the totally real submanifolds and \( C \)-totally real submanifolds of \((LCS)_n\)-manifolds with respect to Levi-Civita as well as quarter symmetric metric connection. It is shown that the scalar curvature of a \( C \)-totally real submanifold of \((LCS)_n\)-manifolds with respect to Levi-Civita connection and quarter symmetric metric connection are same. However in case of totally real submanifolds of \((LCS)_n\)-manifolds with respect to Levi-Civita connection and quarter symmetric metric connection they are different. An inequality for the square length of the shape operator in case of totally real submanifold of \((LCS)_n\)-manifold is derived. The equality case is also considered.

2. Preliminaries

Let \( \tilde{M} \) be an \( n \)-dimensional Lorentzian manifold \([10]\) admitting a unit timelike concircular vector field \( \xi \), called the characteristic vector field of the manifold. Then we have

\[
\phi(X) = \frac{1}{\alpha} \tilde{\nabla}_X \xi,
\]

from (2.4) and (2.6) we have

\[
\phi X = X + \eta(X) \xi,
\]
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(2.8) \[ g(\phi X, Y) = g(X, \phi Y), \]

from which it follows that $\phi$ is a symmetric (1,1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold $\tilde{M}$ together with the unit timelike concircular vector field $\xi$, its associated 1-form $\eta$ and an (1,1) tensor field $\phi$ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold). Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [9]. In a $(LCS)_n$-manifold ($n > 2$), the following relations hold [11]:

(2.9) \[ \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X, \phi Y) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \]

(2.10) \[ \phi^2 X = X + \eta(X)\xi, \]

(2.11) \[ \tilde{R}(X, Y)Z = \phi \tilde{R}(X, Y)Z + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi \]

for all $X, Y, Z \in \Gamma(T\tilde{M})$ Using (2.8) in (2.11) we get,

(2.12) \[ \tilde{R}(X, Y, Z, W) = \tilde{R}(X, Y, Z, \phi W) + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\eta(W) \]

Let $M$ be a submanifold of dimension $m$ of a $(LCS)_n$-manifold $\tilde{M}$ ($m < n$) with induced metric $g$. Also let $\nabla$ and $\nabla^\perp$ be the induced connection on the tangent bundle $TM$ and the normal bundle $T^\perp M$ of $M$ respectively. Then the Gauss and Weingarten formulae are given by

(2.13) \[ \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \]

and

(2.14) \[ \tilde{\nabla}_X V = -A_V X + \nabla^\perp_X V \]

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where $h$ and $A_V$ are second fundamental form and the shape operator (corresponding to the normal vector field $V$) respectively for the immersion of $M$ into $\tilde{M}$ and they are related by [18]

(2.15) \[ g(h(X, Y), V) = g(A_V X, Y) \]

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$. And the equation of Gauss is given by

(2.16) \[ \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)) \]

for any vectors $X, Y, Z, W$ tangent to $M$.

Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal basis of the tangent space $\tilde{M}$ such that, refracting to $M^m$, $\{e_1, e_2, \cdots, e_m\}$ is the orthonormal basis to the tangent space $T_x M$ with respect to induced connection.

We write

\[ h^r_{ij} = g(h(e_i, e_j), e_r). \]
Then the square length of $h$ is

$$||h||^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j))$$

and the mean curvature of $M$ associated to $\nabla$ is

$$H = \frac{1}{m} \sum_{i=1}^{m} h(e_i, e_i).$$

The quarter symmetric metric connection $\tilde{\nabla}$ and Riemannian connection $\nabla$ on a $(LCS)_n$-manifold $\tilde{M}$ are related by

(2.17) $\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi.$

If $\tilde{R}$ and $\tilde{R}$ are the curvature tensors of a $(LCS)_n$-manifold $\tilde{M}$ with respect to quarter symmetric metric connection $\tilde{\nabla}$ and Riemannian connection $\nabla$, then

(2.18) $\tilde{R}(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) + (2\alpha - 1)[g(\phi X, Z)g(\phi Y, W)

- g(\phi Y, Z)g(\phi X, W)] + \alpha[\eta(Y)g(X, W)

- \eta(X)g(Y, W)]\eta(Z) + \alpha[g(Y, Z)\eta(X)

- g(X, Z)\eta(Y)]\eta(W)$

for all $X, Y, Z$ and $W$ on $\chi(\tilde{M})$.

Let $L$ be a $k$-plane section of $T_x M$ and $X$ be a unit vector in $L$. We choose an orthonormal basis $\{e_1, e_2, \ldots, e_k\}$ of $L$ such that $e_1 = X$. Then the Ricci curvature $\text{Ric}_L$ of $L$ at $X$ is defined by

(2.19) $\text{Ric}_L(X) = K_{12} + K_{13} + \cdots + K_{1k},$

where $K_{ij}$ denotes the sectional curvature of the 2-plane section spanned by $e_i, e_j$. Such a curvature is called a $k$-Ricci curvature.

The scalar curvature $\tau$ of the $k$-plane section $L$ is given by

(2.20) $\tau(L) = \sum_{i \leq i < j \leq k} K_{ij}.$

For each integer $k$, $2 \leq k \leq n$, the Riemannian invariant $\Theta_k$ on an $n$-dimensional Riemannian manifold $M$ is defined by

(2.21) $\Theta_k(x) = \frac{1}{k-1} \inf_{L, \text{X}} \text{Ric}_L(X), \quad x \in M,$

where $L$ runs over all $k$-plane sections in $T_x M$ and $X$ runs over all unit vectors in $L$. The relative null space for a submanifold $M$ of a Riemannian manifold at a point $x \in M$ is defined by

(2.22) $N_x = \{X \in T_x M | h(X, Y) = 0, Y \in T_x M\}.$
3. Main results

This section deals with the study of totally real submanifolds of $(LCS)_n$-manifolds with respect to Levi-Civita as well as quarter symmetric metric connection. We prove the following:

**Theorem 3.1.** Let $M$ be a totally real submanifold of dimension $m$ ($m < n$) of a $(LCS)_n$-manifold $\tilde{M}$. Then

$$m^2||H||^2 = 2\tau + ||h||^2 + (m - 1)(\alpha^2 - \rho),$$

where $\tau$ is the scalar curvature of $M$.

**Proof.** Let $M$ be a totally real submanifold of a $(LCS)_n$-manifold $\tilde{M}$. Now from (2.12) and (2.16), we get

$$R(X, Y, Z, W) = \tilde{R}(X, Y, Z, \phi W) + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\eta(W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W))$$

for any $X, Y, Z, W \in \Gamma(TM)$.

Since $M$ is totally real submanifold i.e., anti-invariant so $\tilde{R}(X, Y, Z, \phi W) = g(\tilde{R}(X, Y)Z, \phi W) = 0$

and hence (3.2) yields

$$R(X, Y, Z, W) = (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\eta(W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W))$$

for any $X, Y, Z, W \in \Gamma(TM)$. Putting $X = W = e_i$ and $Y = Z = e_j$ in (3.3) and taking summation over $1 \leq i < j \leq m$, we get

$$\sum_{1 \leq i < j \leq m} R(e_i, e_j, e_j, e_i) = (\alpha^2 - \rho) \sum_{1 \leq i < j \leq m} [g(e_j, e_j)\eta(e_i)\eta(e_i) - g(e_i, e_j)\eta(e_j)\eta(e_j)]$$

$$+ \sum_{1 \leq i < j \leq m} g(h(e_i, e_i), h(e_j, e_j)) - \sum_{1 \leq i < j \leq m} g(h(e_i, e_j), h(e_j, e_i))$$

i.e.,

$$2\tau = -(m - 1)(\alpha^2 - \rho) + m^2||H||^2 - ||h||^2,$$

which implies (3.1). \qed

**Corollary 3.1.** Let $M$ be a $C$-totally real submanifold of dimension $m$ ($m < n$) of a $(LCS)_n$-manifold $\tilde{M}$. Then

$$m^2||H||^2 = 2\tau + ||h||^2,$$

where $\tau$ is the scalar curvature of $M$. 

Proof. In a $C$-totally real submanifold, it is known that $\eta(X) = 0$ for all $X \in \Gamma(TM)$ and $\xi \in T^\perp M$. Then (3.3) yields

$$R(X, Y, Z, W) = g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),$$

from which as similar in above we can prove that $m^2||H||^2 = 2\tau + ||h||^2$. □

Now let $M$ be a submanifold of dimension $m$ ($m < n$) of a $(LCS)_n$-manifold $\tilde{M}$ with respect to quarter symmetric metric connection $\overline{\nabla}$ and $\nabla$ be the induced connection of $M$ associated to the quarter symmetric metric connection. Also let $\overline{h}$ be the second fundamental form of $M$ with respect to $\nabla$. Then the Gauss formula can be written as

$$(3.5) \overline{\nabla}_X Y = \nabla_X Y + \overline{h}(X, Y)$$

and hence by virtue of (2.13) and (2.17) we get

$$(3.6) \overline{\nabla}_X Y + \overline{h}(X, Y) = \nabla_X Y + h(X, Y) + \eta(Y)\phi X - g(\phi X, Y)\xi$$

If $M$ is totally real submanifold of $\tilde{M}$ then $\phi X \in T^\perp M$ for any $X \in TM$ and hence $g(\phi X, Y) = 0$ for $X, Y \in TM$. So, equating the normal part from (3.6) we get

$$(3.7) \overline{h}(X, Y) = h(X, Y) + \eta(Y)\phi X.$$  

Further, if $M$ is $C$-totally real submanifold of $\tilde{M}$ then $\xi \in T^\perp M$ and hence $\eta(Y) = 0$ for all $Y \in TM$. So, (3.7) yields

$$(3.8) \overline{h}(X, Y) = h(X, Y).$$

Let $U$ be a unit tangent vector at $x \in \tilde{M}$ and $\{e_i : i = 1, 2, \ldots, n\}$ be an orthonormal basis of the tangent space $\tilde{M}$ such that $e_1 = U$ refracting to $M^m$, $\{e_1, e_2, \ldots, e_m\}$ is the orthonormal basis to the tangent space $T_x M$ with respect to induced quarter symmetric metric connection. Then we have the following:

**Theorem 3.2.** Let $M$ be a totally real submanifold of a $(LCS)_n$-manifold $\tilde{M}$ with respect to quarter symmetric metric connection then

$$(3.9) m^2||H||^2 = 2\tau + ||h||^2 + (2m - 1)\alpha + m\alpha\eta^2(U),$$

where $\tau$ is the scalar curvature of $M$ with respect to induced connection associated to the quarter symmetric metric connection.

**Proof.** In case of $(LCS)_n$-manifold $\tilde{M}$ with respect to quarter symmetric metric connection, the relation (2.16) becomes

$$(3.10) \overline{R}(X, Y, Z, W) = \overline{R}(X, Y, Z, W) + g(\overline{h}(X, Z), \overline{h}(Y, W)) - g(\overline{h}(X, W), \overline{h}(Y, Z)).$$
In view of (2.7) and (2.8) (3.10) yields from which, by similar as above (3.14) follows. □

(3.12) \( \tilde{R}(X, Y, Z, W) = (\alpha^2 - \rho) \{ g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \} \eta(W) \\
+ \alpha \{ \eta(X)g(Y, W) - \eta(Y)g(X, W) \} \eta(Z) \\
+ \alpha \{ g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \} \eta(W) \\
+ g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)) \\
- \eta(Z)g(h(X, W), \phi Y) - \eta(W)g(\phi X, h(Y, Z)) \\
+ \eta(Z)g(\phi X, h(Y, W)) + \eta(W)g(h(X, Z), \phi Y). \)

Putting \( X = W = e_i \) and \( Y = Z = e_j \) in (3.12) and taking summation over \( 1 \leq i < j \leq m \) we get

(3.13) \( 2\tau = -(m - 1)(\alpha^2 - \rho) - \alpha(1 + \eta^2(U))m - \alpha(m - 1) \\
+ m^2 ||H||^2 - ||h||^2, \)

from which (3.9) follows. □

**Corollary 3.2.** Let \( M \) be a \( C \)-totally real submanifold of a \((LCS)_n\)-manifold \( \tilde{M} \) with respect to quarter symmetric metric connection. Then

(3.14) \( m^2 ||H||^2 = 2\tau + ||h||^2, \)

where \( \tau \) is the scalar curvature of \( M \) with respect to induced quarter symmetric metric connection.

**Proof.** If \( M \) is \( C \)-totally real submanifold then \( \eta(Y) = 0 \) for all \( Y \in TM \) and hence (3.12) implies that

(3.15) \( \tilde{R}(X, Y, Z, W) = g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)) \)

from which, by similar as above (3.14) follows. □

From Corollary 3.1 and Corollary 3.2 we get \( \tau = \tau \) i.e., the scalar curvature of \( C \)-totally real submanifold of a \((LCS)_n\)-manifold with respect to induced Levi-Civita connection and induced quarter symmetric metric connection are identical. Thus we can state the following:
Theorem 3.3. Let $M$ be a $C$-totally real submanifold of a $(LCS)_n$-manifold $\tilde{M}$. Then the scalar curvature of $M$ with respect to induced Levi-Civita connection and induced quarter symmetric metric connection are same.

Next we prove the following:

Theorem 3.4. Let $\tilde{M}$ be a $(LCS)_n$-manifold and $M$ be a totally real submanifold of $\tilde{M}$ of dimension $m$ ($m < n$). Then

(i) for each unit vector $X \in T_xM$,

$$4\text{Ric}(X) \leq m^2\|H\|^2 + 2(\alpha^2 - \rho)(m - 2) + 4(m - 2)(\alpha^2 - \rho)\eta^2(X);\tag{3.16}$$

(ii) in case of $H(x) = 0$, a unit tangent vector $X$ at $x$ satisfies the equality case of (3.16) if and only if $X$ lies in the relative null space $N_x$ at $x$.

(iii) the equality case of (3.16) holds identically for all unit tangent vectors at $x$ if and only if either $x$ is a totally geodesic point or $m = 2$ and $x$ is a totally umbilical point.

Proof. Let $X \in T_xM$ be a unit tangent vector at $x$. We choose an orthonormal basis $\{e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_n\}$ such that $\{e_1, \ldots, e_m\}$ are tangent to $M$ at $x$ and $e_1 = X$. Then from (3.1) we have

$$m^2\|H\|^2 = 2\tau + \sum_{r=m+1}^n \{(h^r_{11})^2 + (h^r_{22} + \cdots + h^r_{mm})^2\}$$

$$-2\sum_{r=m+1}^n \sum_{2i<j\leq n}^n h^r_{ii}h^r_{jj} + (m - 1)(\alpha^2 - \rho)$$

$$= 2\tau + \frac{1}{2} \sum_{r=m+1}^n \{(h^r_{11} + \cdots + h^r_{mm})^2 + (h^r_{11} - h^r_{22} - \cdots - h^r_{mm})^2\}$$

$$+ 2\sum_{r=m+1}^n \sum_{i<j}^n (h^r_{ij})^2 - 2\sum_{r=m+1}^n \sum_{2i<j\leq n}^n h^r_{ii}h^r_{jj} + (m - 1)(\alpha^2 - \rho).\tag{3.17}$$

From the equation of Gauss, we find

$$K_{ij} = \sum_{r=m+1}^n \left[h^r_{ii}h^r_{jj} - (h^r_{ij})^2\right] + (\alpha^2 - \rho)\eta^2(e_i),$$

and consequently

$$\sum_{2i<j\leq m} K_{ij} = \sum_{r=m+1}^n \sum_{2i<j\leq m} \left[h^r_{ii}h^r_{jj} - (h^r_{ij})^2\right] + (\alpha^2 - \rho)[m - 2 + \eta^2(X)].\tag{3.18}$$
Using (3.18) in (3.17) we get

\[ m^2 \|H\|^2 \geq 2\tau + \frac{m^2}{2} \|H\|^2 + 2 \sum_{r=m+1}^{n} \sum_{j=2}^{m} (h_{rj})^2 - 2 \sum_{2 \leq i < j \leq m} K_{ij} \]

\[ - (\alpha^2 - \rho)(m - 3) - 2(m - 2)(\alpha^2 - \rho)\eta^2(X). \]

Therefore,

\[ m^2 \|H\|^2 \geq 2\text{Ric}(X) - (\alpha^2 - \rho)(m - 3) - 2(m - 2)(\alpha^2 - \rho)\eta^2(X), \]

from which we get (3.16)

Let us assume that \( H(x) = 0 \). Then the equality holds in (3.16) if and only if

\[ h_{r1}^r = h_{r2}^r = \cdots = h_{rm}^r = 0, \text{ and } h_{11}^r = h_{22}^r + \cdots + h_{mm}^r, \quad r \in \{m + 1, \cdots, n\}. \]

Then \( h_{rj}^r = 0 \) for every \( j \in \{1, \cdots, m\}, r \in \{m + 1 \cdots n\} \), i.e., \( X \in \mathbb{N}_x \).

(iii) The equality case of (3.16) holds for every unit tangent vector at \( x \) if and only if

\[ h_{rj}^r = 0, i \neq j \text{ and } h_{11}^r + h_{22}^r + \cdots + h_{mm}^r - 2h_{ii}^r = 0. \]

We distinguish two cases:

(a) \( m \neq 2 \), then \( x \) is a totally geodesic point;
(b) \( m = 2 \), it follows that \( x \) is a totally umbilical point.

The converse is trivial. \( \square \)

Next we obtain the following:

**Theorem 3.5.** Let \( M \) be a totally real submanifold of dimension \( m \) of a \((LCS)_n\) manifold \( \tilde{M}(m < n) \). Then we have

\[ ||H||^2 \geq \frac{2\tau}{m(m-1)} + \frac{1}{m}(\alpha^2 - \rho). \]

**Proof.** We choose an orthonormal basis \( \{e_1, \cdots e_m, e_{m+1}, \cdots e_n\} \) at \( x \) such that \( e_{m+1} \) is parallel to the mean curvature vector \( H(x) \), and \( e_1, \cdots e_m \) diagonalise the shape operator \( A_{e_{m+1}} \). Then the shape operator takes the form

\[ A_{e_{m+1}} = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}, \]

\[ A_{e_r} = (h_{rj}^r), \quad i, j = 1, \cdots, m; r = m + 2, \cdots, n, \quad \text{trace}A_{e_r} = \sum_{i=1}^{m} h_{ii}^r = 0 \]

and from (3.11) we get

\[ m^2 ||H||^2 = 2\tau + \sum_{i=1}^{m} a_i^2 + \sum_{r=m+2}^{n} \sum_{i,j=1}^{m} (h_{rj}^r)^2 + (m - 1)(\alpha^2 - \rho). \]
On the other hand, since
\[(3.23) \quad 0 \leq \sum_{i<j} (a_i - a_j)^2 = (m - 1) \sum_i a_i^2 - 2 \sum_{i<j} a_i a_j,\]
we obtain
\[(3.24) \quad m^2 \|H\|^2 = \left( \sum_{i=1}^{m} a_i \right)^2 + 2 \sum_i a_i^2 - 2 \sum_{i<j} a_i a_j \leq m \sum_{i=1}^{m} a_i^2,\]
which implies that
\[(3.25) \quad \sum_i a_i^2 \geq m \|H\|^2.\]
In view of (3.25), (3.22) yields
\[(3.26) \quad m^2 \|H\|^2 \geq 2 \tau + m \|H\|^2 + (m - 1)(\alpha^2 - \rho),\]
which implies (3.20). \(\square\)

**Theorem 3.6.** Let \(M\) be a totally real submanifold of dimension \(m\) of a \((LCS)_n\) manifold \(\widetilde{M}(m < n)\). Then for any integer \(k\), \(2 \leq k \leq m\), and any point \(x \in M\), we have
\[(3.27) \quad \|H\|^2(x) \geq \Theta_k(x) + \frac{1}{m}(\alpha^2 - \rho).\]

**Proof.** Let \(\{e_1, e_2, \cdots, e_m\}\) be an orthonormal basis of \(T_x M\). Denote by \(L_{i_1,\cdots,i_k}\) the \(k\)-plane section spanned by \(e_{i_1}, \cdots, e_{i_k}\). Then we have [2]
\[(3.28) \quad \tau(x) \geq \frac{m(m-1)}{2} \Theta_k(x).\]
Using (3.28) in (3.20), (3.27) follows. \(\square\)

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S. K. Hui
Department of Mathematics, The University of Burdwan, Golapbag, Burdwan – 713104, West Bengal, India
E-mail: shyamal_hui@yahoo.co.in, skhui@math.buruniv.ac.in

T. Pal
A. M. J. High School, Mankhamar, Bankura – 722144, West Bengal, India
E-mail: tanumoypalmath@gmail.com