It is known that for a prime \( p \neq 2 \), there is the following natural description of the homology algebra of an Abelian group \( H_*(A, F_p) \cong \Lambda(A/p) \otimes \Gamma_p(A) \), and for finitely generated Abelian groups there is the following description of the cohomology algebra of \( H^*(A, F_p) \cong \Lambda((A/p)^\vee) \otimes \operatorname{Sym}((pA)^\vee) \). It is proved that for \( p = 2 \), there are no such descriptions “depending” on \( A/2 \) and \( 2A \) only. Moreover, natural descriptions of \( H_*(A, F_2) \) and \( H^*(A, F_2) \), “depending” on \( A/2, 2A \), and a linear map \( \beta : 2A \to A/2 \) are presented. It is also proved that there is a filtration by subfunctors on \( H_*(A, F_2) \), whose quotients are \( \Lambda^{n-2i}(A/2) \otimes \Gamma_i(2A) \), and there is a natural filtration on \( H^n(A,F_2) \) for finitely generated Abelian groups, whose quotients are \( \Lambda^{n-2i}((A/2)^\vee) \otimes \operatorname{Sym}^i((2A)^\vee) \).

Bibliography: 14 titles.

Dedicated to Alexander Generalov on the occasion of his 70th birthday

**Introduction**

Let \( K \) be a field and \( A \) an Abelian group. Then the summation homomorphism \( A \oplus A \to A, (a, b) \mapsto a+b \), induces a map \( H_*(A, K) \otimes H_*(A, K) \to H_*(A, K) \) called the Pontryagin product, which gives a structure of a graded supercommutative algebra on \( H_*(A, K) \) (see [3, Chap. V]). Moreover, the diagonal map \( A \to A \oplus A \) and sign map \( A \to A \) induce a comultiplication and an antipode on \( H_*(A, K) \) which give a structure of a graded Hopf algebra on \( H_*(A, K) \). For a prime \( p \), we set

\[
A/p := A/pA \cong A \otimes \mathbb{Z}/p, \quad pA := \{a \in A \mid pa = 0\} \cong \operatorname{Tor}(A, \mathbb{Z}/p).
\]

For a vector space \( V \), we denote by \( V[n] \) the graded vector space concentrated in the degree \( n \). It was proved by H. Cartan in [5] (and exposed later in details in [4], see also [3, Chap. V, Theorem 6.6]) that for a prime \( p \neq 2 \), there is a natural graded algebra isomorphism

\[
H_*(A, F_p) \cong \Lambda(A/p[1]) \otimes \Gamma_p(A[2]), \tag{0.1}
\]

where \( \Lambda \) is the exterior algebra over \( F_p \) and \( \Gamma \) is the divided power algebra over \( F_p \). In fact, this is a natural isomorphism of Hopf algebras. Since cohomology of finitely generated Abelian groups is dual to homology, there is an isomorphism

\[
H^*(A, F_p) \cong \Lambda((A/p)^\vee[1]) \otimes \operatorname{Sym}((pA)^\vee[2]),
\]

where \((-)^\vee\) is the dual vector space. In particular, there are natural isomorphisms

\[
H_n(A, F_p) \cong \bigoplus_{i=0}^{[n/2]} \Lambda^{n-2i}(A/p) \otimes \Gamma_i(pA) \tag{0.2}
\]

for each homology group, and in the case of finitely generated groups

\[
H^n(A, F_p) \cong \bigoplus_{i=0}^{[n/2]} \Lambda^{n-2i}((A/p)^\vee) \otimes \operatorname{Sym}^i((pA)^\vee)
\]
for each cohomology group.

The isomorphism (0.1) means that the homology algebra $H_\ast(A, \mathbb{F}_p)$ can naturally be recovered as a graded Hopf algebra from the two vector spaces $A/p$ and $pA$. Informally, the couple $(A/p, pA)$ is the "minimum information" about $A$, required to describe the homology algebra in the case $p \neq 2$ naturally. More formally, denote by $\text{ Vect}^2$ and $t: \text{ Vect} \to \text{ Vect}$ the category of couples of $\mathbb{F}_p$-vector spaces and the functor $A \mapsto (A/p, pA)$, respectively. Then there is a factorization of the functor $H_\ast(\ast, \mathbb{F}_p)$ to the category of graded Hopf algebras via the functor $t$. In particular, there is a factorization

$$\begin{align*}
\text{ Vect}^2 & \xrightarrow{H_\ast(\ast, \mathbb{F}_p)} \text{ Vect} \\
t & \downarrow \\
\text{ Vect} & \xrightarrow{t}
\end{align*}$$

for each homology group. Isomorphism (0.2) gives a simple formula for $H_2(A, \mathbb{F}_p)$ and $p \neq 2$:

$$H_2(A, \mathbb{F}_p) \cong \Lambda^2(A/p) \oplus pA.$$ 

However, for $p = 2$ the only we have is the short exact sequence

$$0 \to \Lambda^2(A/2) \to H_2(A, \mathbb{F}_2) \to 2A \to 0,$$

which does not split naturally [10, Sec. 3]. Thus, the behavior of $H_\ast(A, \mathbb{F}_p)$ for $p = 2$ is more complicated.

The goal of the present paper is to study the situation for $p = 2$ and, in particular, to get a natural description of the graded Hopf algebra $H_\ast(A, \mathbb{F}_2)$ for arbitrary Abelian group $A$, and the algebra $H^\ast(A, \mathbb{F}_2)$ for arbitrary finitely generated group $A$. Moreover, our informal goal is to get the description using "minimum information" about $A$ to recover the homology algebra. We also obtain a natural filtration on $H_n(A, \mathbb{F}_2)$, whose quotients are $\Lambda^i(A/2) \otimes \Gamma^j(2A)$. Dually, in the case of finitely generated group $A$ we obtain a natural filtration on $H^n(A, \mathbb{F}_2)$, whose quotients are $\Lambda^i((A/2)^\vee) \otimes \text{ Sym}^j((2A)^\vee)$. This is an analog of decomposition (0.2) in the case $p = 2$.

We prove that there is no factorization similar (0.3) for any $n \geq 2$ and $p = 2$ (Proposition 5.1). So the couple $(A/2, 2A)$ does not provide sufficient information to recover the Hopf algebras $H_\ast(A, \mathbb{F}_2)$ and $H^\ast(A, \mathbb{F}_2)$, or even the group $H_n(A, \mathbb{F}_2)$ for some $n \geq 2$. However, adding a linear map between these two vector spaces turns out to be sufficient to obtain information to recover the homology algebra naturally. Namely, we need to add the map

$$\bar{\beta}: 2A \to A/2$$

which is the composition of the embedding $2A \hookrightarrow A$ and the projection $A \to A/2$.

We describe the cohomology of a finitely generated Abelian group $A$ as follows. We set

$$T(A) = (A/2)^\vee[1] \oplus (2A)^\vee[2]$$

and prove that there is a natural isomorphism

$$H^\ast(A, \mathbb{F}_2) \cong \text{ Sym}(T(A))/I,$$

where $I$ is the ideal generated by the set $\{x^2 - \bar{\beta}^\vee(x) \mid x \in (A/2)^\vee\}$. We also prove that there is a unique structure of unstable $\mathcal{A}$-algebra (where $\mathcal{A}$ is the Steenrod algebra) on $\text{ Sym}(T(A))/I$, such that $Sq^1((2A)^\vee[2]) = 0$ and the isomorphism is an isomorphism of unstable $\mathcal{A}$-algebras. Moreover, there is a short exact sequence of bicommutative Hopf algebras

$$\mathbb{F}_2 \to \text{ Sym}((2A)^\vee) \to H^\ast(A, \mathbb{F}_2) \to \Lambda((A/2)^\vee) \to \mathbb{F}_2.$$
Furthermore the group $H^n(A, \mathbb{F}_2)$ is naturally isomorphic to the cokernel of the map

$$\bigoplus_{2k+l+2m=n; \quad k \geq 1} \text{Sym}^{k}(A/2^\vee) \otimes \text{Sym}^{l}(A/2^\vee) \otimes \text{Sym}^{m}(2A^\vee) \longrightarrow \bigoplus_{i+2j=n} \text{Sym}^{i}(A/2^\vee) \otimes \text{Sym}^{j}(2A^\vee)$$

and there is a natural filtration of $H^n(A, \mathbb{F}_2)$ by subfunctors $\Phi^i$ such that

$$\Phi^i/\Phi^{i+1} \cong \Lambda^{n-2i}(A/2^\vee) \otimes \text{Sym}^{i}(2A^\vee).$$

All the results can be generalized to the case of arbitrary Abelian group $A$ if we consider the cohomology $(A/2)^\vee$ and $(2A)^\vee$ as profinite vector spaces and replace all the constructions to their profinite versions.

We describe the homology of an Abelian group $A$ (not necessary finitely generated) as follows. The following square is a pullback in the category of bicommutative Hopf algebras:

$$\begin{array}{ccc}
H_*(A, \mathbb{F}_2) & \longrightarrow & \Gamma(2A[2]) \\
\downarrow & & \downarrow \\
\Gamma(A/2[1]) & \longrightarrow & \Gamma(A/2[2]),
\end{array}$$

where $\mathcal{V}$ denotes the Verschiebung. We also prove that there is a short exact sequence of graded bicommutative Hopf algebras

$$\mathbb{F}_2 \longrightarrow \Lambda(A/2) \longrightarrow H_*(A, \mathbb{F}_2) \longrightarrow \Gamma(2A) \longrightarrow \mathbb{F}_2.$$ 

Moreover $H_n(A, \mathbb{F}_2)$ is naturally isomorphic to the kernel of the natural transformation

$$\bigoplus_{i+2j=n} \Gamma^{i}(A/2) \otimes \Gamma^{j}(2A) \longrightarrow \bigoplus_{2k+l+2m=n; \quad k \geq 1} \Gamma^{k}(A/2) \otimes \Gamma^{l}(A/2) \otimes \Gamma^{m}(2A),$$

and there is a natural filtration of $H_n(A, \mathbb{F}_2)$ by subfunctors $\Psi_i$ such that

$$\Psi_i/\Psi_{i-1} \cong \Lambda^{n-2i}(A/2) \otimes \Gamma^{i}(2A).$$

As an auxiliary result, we prove that for a short exact sequence of graded connected bicommutative Hopf algebras

$$K \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow K$$

over a field $K$ and any $n \geq 0$, there is an isomorphism

$$\mathcal{A}_n^\oplus B/\mathcal{A}_n^{n+1} B \cong \mathcal{A}_n^\oplus /\mathcal{A}_n^{n+1} \otimes C$$

which depends on the short exact sequence (without a linear splitting) only. Moreover, the isomorphism is natural by the short exact sequence. Note that the quotients depend only on $A$ and $C$ and do not depend on the short exact sequence.

The first author believes that using the results of the present paper would have made his paper [9] half the length. We also think that there are some gaps in arguments of some papers that are filled by our results. For example, Bousfield uses in [2] the following statement without any reference: if $A$ is a module over a group $G$ such that $A/p$ and $pA$ are nilpotent $G$-modules, then $H_n(A, \mathbb{F}_p)$ is a nilpotent $G$-module (see the proof of [2, Proposition 2.8]). This statement is an obvious corollary of the result of Cartan for $p \neq 2$. Apparently, Bousfield missed the case of $p = 2$, and our results fill the gap.
1. Filtration of Hopf algebra associated with short exact sequence

In this section we prove that a short exact sequence of graded connected bicommutative Hopf algebras \( K \rightarrow A \rightarrow B \rightarrow C \rightarrow K \) over a field \( K \) provides a natural filtration on \( B \), whose quotients depend only on \( A \) and \( C \) and do not depend on the short exact sequence.

Let \( \alpha: A \rightarrow \widetilde{A} \) be a morphism of augmented algebras. Assume that \( V \) is a vector space and \( M \) is an \( \widetilde{A} \)-module. Then for a linear map \( f: V \rightarrow M \), we denote by \( \bar{f} \) the \( A \)-module homomorphism

\[
\bar{f}: A \otimes V \rightarrow M, \quad \bar{f}(a \otimes v) = \alpha(a)f(v).
\]

It is easy to see that \( \bar{f}(A_n^+ \otimes V) \subseteq \widetilde{A}_+^n M \). Then \( f \) induces a map on the quotients,

\[
f'_n: A_n^+ / A_{n+1}^+ \otimes V \rightarrow \widetilde{A}_+^n M / \widetilde{A}_+^{n+1} M.
\]

**Lemma 1.1.** If \( \text{Im}(f) \subseteq \widetilde{A}_+ M \), then \( f'_n = 0 \) for any \( n \geq 0 \).

**Proof.** Since \( \text{Im}(f) \subseteq \widetilde{A}_+^n M \), we have \( \text{Im}(\bar{f}) \subseteq \widetilde{A}_+^n M \). The fact that \( \bar{f} \) is a homomorphism of \( A \)-modules, implies that \( \bar{f}(\widetilde{A}_+^n \otimes V) \subseteq \alpha(\widetilde{A}_+^n) \cdot \text{Im}(\bar{f}) \subseteq \widetilde{A}_+^{n+1} M \). The assertion follows. \( \Box \)

Now assume that there is a short exact sequence in the category bicommutative Hopf algebras

\[
K \rightarrow A \overset{i}{\rightarrow} B \overset{\pi}{\rightarrow} C \rightarrow K
\]

and \( \tilde{A} = A \) and \( \alpha = \text{id}_A \). Then \( B \) has a natural \( A \)-module structure and any linear map \( f: C \rightarrow B \) induces an \( A \)-module homomorphism

\[
\bar{f}: A \otimes C \rightarrow B, \quad a \otimes c \mapsto a \cdot f(c),
\]

**Lemma 1.2.** Let (1.1) be a short exact sequence of graded connected commutative and cocommutative Hopf algebras. Assume that \( f: C \rightarrow B \) is a graded linear map such that \( \pi f = \text{id}_C \). Then

\[
\bar{f}: A \otimes C \cong B, \quad a \otimes c \mapsto a \cdot f(c),
\]

is an isomorphism of \( A \)-modules.

**Proof.** Follows from [11, Proposition 1.7], see also the proof of Theorem 4.4 and Proposition 4.9 in [11]. \( \Box \)

**Theorem 1.3.** For a short exact sequence of graded connected bicommutative Hopf algebras

\[
K \rightarrow A \overset{i}{\rightarrow} B \overset{\pi}{\rightarrow} C \rightarrow K
\]

and any \( n \geq 0 \), there is an isomorphism

\[
A_n^+ B / A_{n+1}^+ B \cong A_n^+ / A_{n+1}^+ \otimes C
\]

which depends on the short exact sequence (without linear splitting) only. Moreover, the isomorphism is natural by the short exact sequence.

**Proof.** If \( f: C \rightarrow B \) is a graded linear splitting of the epimorphism \( \pi: B \rightarrow C \), then by Lemma 1.2, \( \bar{f}: A \otimes C \cong B \) is an isomorphism of \( A \)-modules. It follows that \( f'_n \) is an isomorphism of the form (1.2).

Let us prove that this isomorphism does not depend on the choice of the splitting \( f \). The difference of two such splittings is a map \( g: C \rightarrow B \) such that \( \text{Im} g \subseteq \text{Ker} \pi \). Then it suffices to prove that \( g'_n = 0 \) for any linear map \( g: C \rightarrow B \) such that \( \text{Im} g \subseteq \text{Ker} \pi = A_+ B \). This follows from Lemma 1.1. Thus, the isomorphism (1.2) does not depend on the choice of \( f \).
Let us prove that this isomorphism is natural by the short exact sequence. Assume that there is a morphism of short exact sequences,

\[
\begin{array}{cccccc}
K & \rightarrow & A & \rightarrow & B & \rightarrow ^\pi C & \rightarrow & K \\
\downarrow^{\alpha_A} & & \downarrow^{\alpha_B} & & \downarrow^{\alpha_C} & & \downarrow & \\
K & \rightarrow & \widetilde{A} & \rightarrow & \widetilde{B} & \rightarrow ^{\widetilde{\pi}} \widetilde{C} & \rightarrow & K.
\end{array}
\]

We fix a splitting \(f\) of \(\pi\) and a splitting \(g\) of \(\widetilde{\pi}\). Set \(h = \alpha_B f - g \alpha_C\). We claim that \(\text{Im}(h) \subseteq \text{Ker} \; \widetilde{\pi}\). Indeed, this follows from \(\text{Im}(h) = \text{Im}(h \pi)\) and \(\widetilde{\pi} h \pi = \widetilde{\pi} \alpha_B f \pi - \widetilde{\pi} g \alpha_C \pi = \alpha_C \pi h \pi - \pi \widetilde{\pi} g \alpha_B = \alpha_C \pi - \widetilde{\pi} \alpha_B = 0\). Then \(\text{Im}(h) \subseteq \widetilde{A}, \widetilde{B}\). Lemma 1.1 implies that \(h' = 0\). It follows that the diagram

\[
\begin{array}{cc}
(A_n/A_{n+1}) \otimes C & \longrightarrow^f A_n/B/A_{n+1}B \\
\downarrow & \downarrow \\
(\widetilde{A}_n/\widetilde{A}_{n+1}) \otimes \widetilde{C} & \longrightarrow^g \widetilde{A}_n/B/\widetilde{A}_{n+1}B
\end{array}
\]

is commutative. This completes the proof. \(\square\)

2. Generalities about Bockstein homomorphism and universal coefficient theorem

For a space \(X\), we denote by

\[
\beta : H_n(X, \mathbb{Z}/p) \longrightarrow H_{n-1}(X, \mathbb{Z}/p)
\]

the Bockstein homomorphism. The short exact sequence \(\mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \mathbb{Z}/p\) induces a boundary map \(\beta' : H_n(X, \mathbb{Z}/p) \rightarrow H_{n-1}(X, \mathbb{Z})\) whose composition with the map \(H_{n-1}(X, \mathbb{Z}) \rightarrow H_{n-1}(X, \mathbb{Z}/p)\) is \(\beta [8, \text{Sec. 3E}]\). The universal coefficient theorem says that there is a short exact sequence

\[
0 \rightarrow H_n(X, \mathbb{Z})/p \overset{\theta_n}{\longrightarrow} H_n(X, \mathbb{Z}/p) \overset{\tau_n}{\longrightarrow} pH_{n-1}(X, \mathbb{Z}) \rightarrow 0,
\]

where the map \(\tau_n\) is constructed as the restriction of the image of \(\beta'\),

\[
\begin{array}{ccc}
H_n(X, \mathbb{Z}/p) & \xrightarrow{\beta'} & H_{n-1}(X, \mathbb{Z}) & \xrightarrow{p} & H_{n-1}(X, \mathbb{Z}) \\
\downarrow^{\tau_n} & & & & \\
pH_{n-1}(X, \mathbb{Z})
\end{array}
\]

It follows that \(\beta\) equals the composition

\[
\beta : H_n(X, \mathbb{Z}/p) \xrightarrow{\tau_n} pH_{n-1}(X, \mathbb{Z}) \leftrightarrow H_{n-1}(X, \mathbb{Z}) \rightarrow H_{n-1}(X, \mathbb{Z}/p).
\]

The last homomorphism can also be decomposed as

\[
H_{n-1}(X, \mathbb{Z}) \xrightarrow{\theta_{n-1}} H_{n-1}(X, \mathbb{Z}/p) \xrightarrow{\beta_{n-1}} H_{n-1}(X, \mathbb{Z}/p).
\]

This implies the following lemma.
Lemma 2.1. The Bockstein homomorphism $\beta$ decomposes via homomorphisms from the universal coefficient theorem as follows:

$$
H_n(X, \mathbb{Z}/p) \xrightarrow{\beta} H_{n-1}(X, \mathbb{Z}/p) \\
\downarrow \tau_n \quad \quad \theta_{n-1} \uparrow \\
pH_{n-1}(X, \mathbb{Z}) \xrightarrow{\tilde{\beta}} H_{n-1}(X, \mathbb{Z})/p,
$$

where $\tilde{\beta}$ is the composition of the embedding $pH_{n-1}(X, \mathbb{Z}) \hookrightarrow H_{n-1}(X, \mathbb{Z})$ and projection $H_{n-1}(X, \mathbb{Z}) \rightarrow H_{n-1}(X, \mathbb{Z})/p$.

3. Cohomology

For an Abelian group $A$, we set $T(A) = (A/2)^{\vee}[1] \oplus (2A)^{\vee}[2]$.

Theorem 3.1. Let $A$ be a finitely generated Abelian group. Then

(a) There is a natural isomorphism of graded Hopf algebras

$$
H^*(A, \mathbb{F}_2) \cong \text{Sym}(T(A))/I,
$$

where $I$ is the ideal generated by the set $\{x^2 - \tilde{\beta}^{\vee}(x) \mid x \in (A/2)^{\vee}\}$.

(b) The following square is pushout and pullback in the category of graded bicommutative Hopf algebras:

$$
\begin{array}{ccc}
\text{Sym}((A/2)^{\vee}[2]) & \xrightarrow{\mathcal{F}} & \text{Sym}((A/2)^{\vee}[1]) \\
\downarrow \text{Sym}(\tilde{\beta}^{\vee}) & & \downarrow \\
\text{Sym}((2A)^{\vee}[2]) & \longrightarrow & H^*(A, \mathbb{F}_2),
\end{array}
$$

where $\mathcal{F}$ is the Frobenius homomorphism.

(c) There exists a uniquely determined structure of unstable $\mathbb{A}$-algebra on the algebra $\text{Sym}(T(A))/I$, such that $Sq^1(x) = 0$ for any $x \in (2A)^{\vee}$. The morphism (3.1) is an isomorphism of unstable $\mathbb{A}$-algebras with respect to this structure.

Proof of Theorem 3.1. Dualizing the universal coefficient theorem, we obtain a short exact sequence

$$
0 \longrightarrow (2H_{n-1}(A, \mathbb{Z}))^{\vee} \xrightarrow{\tau_n} H^n(A, \mathbb{F}_2) \xrightarrow{\theta_n} (H_n(A, \mathbb{Z})/2)^{\vee} \longrightarrow 0.
$$

Identifying $H_1(A, \mathbb{Z}) = A$, we get the morphisms

$$
(\theta^1)^{-1}: (A/2)^{\vee} \xrightarrow{\cong} H^1(A, \mathbb{F}_2),
\quad \tau^2: (2A)^{\vee} \rightarrow H^2(A, \mathbb{F}_2).
$$

These maps are natural by $A$. The diagram of naturality of these maps with respect to the addition homomorphism $A \oplus A \rightarrow A$ implies that the images of the maps (3.3) consist of primitive elements. Then they give morphisms of Hopf algebras from symmetric algebras,

$$
\text{Sym}((A/2)^{\vee}[1]) \rightarrow H^*(A, \mathbb{F}_2),
\text{Sym}((2A)^{\vee}[2]) \rightarrow H^*(A, \mathbb{F}_2).
$$

The Bockstein homomorphism $\beta: H^1(A, \mathbb{F}_2) \rightarrow H^2(A, \mathbb{F}_2)$ equals the Frobenius homomorphism. Then Lemma 2.1 implies that the diagram

$$
\begin{array}{ccc}
(A/2)^{\vee} & \xrightarrow{\tilde{\beta}^{\vee}} & (2A)^{\vee} \\
\downarrow \cong & & \downarrow \\
H^1(A, \mathbb{F}_2) & \xrightarrow{x \mapsto x^2} & H^2(A, \mathbb{F}_2)
\end{array}
$$

is commutative.
It follows that the square of Hopf algebras (3.2) is commutative. It is known from [12, Theorem 4.4] and [14, Corollary 4.16] that the category of bicommutative Hopf algebras is Abelian, the direct sum is given by the tensor product, and the cokernel of the morphism $\alpha: C \to D$ is given by $D/\alpha(C_+)$, where $C_+$ is the augmentation ideal of $C$. It is easy to check that a commutative square in the category of bicommutative graded Hopf algebras is pushout if and only if it is pushout in the category of bicommutative (nongraded) Hopf algebras. Therefore the pushout of the morphisms $F$ and $\text{Sym}(\beta)$ in the diagram (3.2) is $\text{Sym}(T(A))/I$ (here we use the fact that in an Abelian category the pushout of two maps is the obvious quotient of their direct sum, and $\text{Sym}(T(A)) \cong \text{Sym}((A/2)^{\mathbb{N}}[1]) \otimes \text{Sym}((2A)^{\mathbb{N}}[2])$).

Then they give a natural morphism of graded Hopf algebras,

$$\text{Sym}(T(A))/I \to H^*(A, \mathbb{F}_2),$$

and the square (3.2) is pushout if and only if the morphism is an isomorphism.

All functors in diagram (3.2) are additive as functors from the category of Abelian groups to the category of bicommutative Hopf algebras (they send direct sums to tensor products). It follows that in order to prove that the square is pushout for finitely generated Abelian groups, it suffices to verify that the square is pushout for cyclic groups $A = \mathbb{Z}, \mathbb{Z}/p^n$, where $p$ is prime and $n \geq 1$. If $p \neq 2$ and $n \geq 1$, then all the algebras in the square are trivial. The case $A = \mathbb{Z}$ follows from the standard fact that $H^*(\mathbb{Z}, \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^2)$. The other cases $\mathbb{Z}/2$ and $\mathbb{Z}/2^n$ for $n \geq 2$ follow from the isomorphisms $H^*(\mathbb{Z}/2, \mathbb{F}_2) \cong \mathbb{F}_2[x]$ and $H^*(\mathbb{Z}/2^n, \mathbb{F}_2) \cong \mathbb{F}_2[x,y]/(x^2)$, where $\deg(x) = 1$ and $\deg(y) = 2$ (see [6, Sec. 3.2]).

Recall that the square in an Abelian category is pushout (pullback) if and only if its totalization is a right (left) exact sequence. Thus, if a square is pushout, then it is pullback if and only if the first arrow of its totalization is a monomorphism. Therefore the fact that our square is also pullback follows from the fact that the Frobenius homomorphism $\text{Sym}((A/2)^{\mathbb{N}}) \to \text{Sym}((A/2)^{\mathbb{N}})$ is monomorphism.

We have proved (a) and (b). Now we prove (c). The Verschiebung on the Steenrod algebra $v: \mathcal{A} \to \mathcal{A}$ is an endomorphism that acts as follows: $v(Sq^{2n}) = Sq^n$, $v(Sq^{2n+1}) = 0$ (see [13, Chap. II, Proposition 3.5]). For any vector space $V$, there is a unique structure of unstable $\mathcal{A}$-algebra on the algebra $\text{Sym}(V[1])$, which comes from the isomorphism $\text{Sym}(V[1]) \cong H^*(V, \mathbb{F}_2)$. The action via Verschiebung gives a structure of unstable $\mathcal{A}$-algebra on $\text{Sym}(V[2])$. Using the natural Hopf algebra structure on $\mathcal{A}$, we conclude that there is a natural structure of unstable $\mathcal{A}$-algebra on the tensor product $\text{Sym}(V[1]) \otimes \text{Sym}(U[2]) \cong \text{Sym}(V[1] \oplus U[2])$ for any two vector spaces $V, U$. This defines the structure of $\mathcal{A}$-module on $\text{Sym}(T(A))$. Let us prove that the ideal $I$ is closed under this action. It suffices to verify that the image of the generating set with respect to the action of $Sq^n$ lies in $I$. This follows from the equations $Sq^1(x^2 - \tilde{\beta}^V(x)) = 0$ and $Sq^2(x^2 - \tilde{\beta}^V(x)) = (x^2 - \tilde{\beta}^V(x))^2$. Thus, we obtain the required structure of unstable $\mathcal{A}$-algebra on $\text{Sym}(T(A))/I$. It is easy to see that this structure is unique. In order to prove that the isomorphism $\text{Sym}(T(A))/I \cong H^*(A, \mathbb{F}_2)$ respects the action of $\mathcal{A}$, it suffices to verify this for the elements from $T(A)$. Since $\text{Sym}(T(A))/I$ and $H^*(A, \mathbb{F}_2)$ are unstable $\mathcal{A}$-algebras, the actions of $Sq^1$ on $(A/2)^{\mathbb{N}}[1]$ and $Sq^2$ on $(2A)^{\mathbb{N}}[2]$ are respected by the isomorphism. Thus the only we need to prove is that the isomorphism respects the action of $Sq^1$ on $(2A)^{\mathbb{N}}[2]$, which is trivial. Therefore we need to prove that $Sq^1(\text{Im}(\tau_A^2: (2A)^{\mathbb{N}} \to H^2(A, \mathbb{F}_2))) = 0$. Consider the vector space $V = 2A$. Then the embedding $V \hookrightarrow A$ induces a commutative square

$$
\begin{array}{ccc}
V^{\vee} & \xrightarrow{id} & 2A^{\vee} \\
\downarrow\tau_V^2 & & \downarrow\tau_A^2 \\
H^2(V, \mathbb{F}_2) & \xrightarrow{} & H^2(A, \mathbb{F}_2).
\end{array}
$$

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Hence the assertion follows from the equality $Sq^1(\tau_2^2(x)) = Sq^1 Sq^1(x') = 0$, where $x'$ is the image of $x$ under the isomorphism $V^\vee \cong H^1(V, \mathbb{F}_2)$.

**Corollary 3.2.** For a finitely generated Abelian group $A$, there is a natural short exact sequence of Hopf algebras,

$$
\mathbb{F}_2 \longrightarrow \text{Sym}((2A)^\vee) \longrightarrow H^*(A, \mathbb{F}_2) \longrightarrow \Lambda((A/2)^\vee) \longrightarrow \mathbb{F}_2.
$$

**Proof.** This follows from the fact that the sequence $\mathbb{F}_2 \rightarrow \text{Sym}(V) \xrightarrow{F} \text{Sym}(V) \rightarrow \Lambda(V) \rightarrow \mathbb{F}_2$ is short exact in the category of bicommutative Hopf algebras for any vector space $V$ and the pushout from Theorem 3.1. □

**Corollary 3.3.** For a finitely generated Abelian group $A$, there is a natural filtration of $H^n(A, \mathbb{F}_2)$ by subfunctors $H^n(A, \mathbb{F}_2) = \Phi^0 \supseteq \Phi^1 \supseteq \cdots \supseteq \Phi^{[n/2]+1} = 0$ such that

$$
\Phi^i/\Phi^{i+1} = \Lambda^{n-2i}((A/2)^\vee) \otimes \text{Sym}^i((2A)^\vee).
$$

**Proof.** This follows from Corollary 3.2 and Theorem 1.3. □

**Corollary 3.4.** Let $A$ be a finitely generated Abelian group. Then the group $H^n(A, \mathbb{F}_2)$ is naturally isomorphic to the cokernel of the map

$$
\bigoplus_{2k+l+2m=n; \ k \geq 1} \text{Sym}^k(A/2^\vee) \otimes \text{Sym}^l(A/2^\vee) \otimes \text{Sym}^m(2A^\vee) \longrightarrow \bigoplus_{i+2j=n} \text{Sym}^i(A/2^\vee) \otimes \text{Sym}^j(2A^\vee).
$$

In particular, there is a short exact sequence

$$
0 \longrightarrow (A/2)^\vee \longrightarrow \text{Sym}^2(A/2^\vee) \oplus 2A^\vee \longrightarrow H^2(A, \mathbb{F}_2) \longrightarrow 0,
$$

where the first map is $x \mapsto (x^2, \tilde{\beta}^2(x))$, and a short exact sequence

$$
0 \longrightarrow (A/2)^\vee \otimes (A/2)^\vee \longrightarrow \text{Sym}^3((A/2)^\vee) \oplus (A/2)^\vee \otimes (2A)^\vee \longrightarrow H^3(A, \mathbb{F}_2) \longrightarrow 0,
$$

where $x \otimes y \mapsto (x^2 y, y \otimes \tilde{\beta}^2(x))$.

**Proof.** Theorem 3.1 implies that in the category of bicommutative Hopf algebras, $H^*(A, \mathbb{F}_2)$ is the cokernel of the morphism $\text{Sym}((A/2)^\vee[2]) \rightarrow \text{Sym}((A/2)^\vee[1]) \otimes \text{Sym}((2A)^\vee[2])$. The assertion follows. □

### 4. Homology

In this section dual results for homology are presented. They hold in a more general setting, namely, for arbitrary Abelian group not necessarily finitely generated. The reason why the results hold in the general setting is that homology, divided powers, tensor products, and all the other functors that occur in the statements commute with filtered colimits. The proof for all the statements is the following: first we prove the results for finitely generated Abelian groups, dualize the results for cohomology, and then use the fact that all the functors commute with filtered colimits and present an Abelian group as the colimit of its finitely generated subgroups.

For a vector space $V$, we denote by $\Gamma(V) = \bigoplus \Gamma^n(V)$ the divided power algebra of $V$. It is a subalgebra of the shuffle algebra $\text{Sh}(V)$, such that $\Gamma^n(V)$ is the space of invariants of $V^\otimes n$ under the action of the symmetric group. It is well known that if $V$ is finite-dimensional, then $\Gamma(V) = \text{Sym}(V^\vee)^\vee$. 801
Theorem 4.1. Let $A$ be an Abelian group (not necessarily finitely generated). Then the following square is pullback in the category of graded bicommutative Hopf algebras:

\[
\begin{array}{ccc}
H_\ast(A, \mathbb{F}_2) & \longrightarrow & \Gamma(2A[2]) \\
\downarrow & & \downarrow \Gamma(\beta) \\
\Gamma(A/2[1]) & \longrightarrow & \Gamma(A/2[2]),
\end{array}
\]

where $\mathcal{V}$ denotes the Verschiebung.

Corollary 4.2. For an Abelian group $A$, there is a short exact sequence of graded bicommutative Hopf algebras

\[
\mathbb{F}_2 \longrightarrow \Lambda(A/2) \longrightarrow H_\ast(A, \mathbb{F}_2) \longrightarrow \Gamma(2A) \longrightarrow \mathbb{F}_2.
\]

Corollary 4.3. For an Abelian group $A$, there is a natural filtration of $H_n(A, \mathbb{F}_2)$ by subfunctors $0 = \Psi_{-1} \subseteq \Psi_0 \subseteq \cdots \subseteq \Psi_{[n/2]} = H_n(A, \mathbb{F}_2)$ such that

\[
\Psi_i/\Psi_{i-1} \cong \Lambda^{n-2i}(A/2) \otimes \Gamma^i(2A).
\]

Corollary 4.4. For an Abelian group $A$, the group $H_n(A, \mathbb{F}_2)$ is naturally isomorphic to the kernel of the map

\[
\bigoplus_{i+2j=n} \Gamma^i(A/2) \otimes \Gamma^j(2A) \longrightarrow \bigoplus_{2k+l+2m=n; k \geq 1} \Gamma^k(A/2) \otimes \Gamma^l(A/2) \otimes \Gamma^m(2A).
\]

In particular, there are short exact sequences

\[
0 \longrightarrow H_2(A, \mathbb{F}_2) \longrightarrow \Gamma^2(A/2) \oplus 2A \longrightarrow A/2 \longrightarrow 0
\]

and

\[
0 \longrightarrow H_3(A, \mathbb{F}_2) \longrightarrow \Gamma^3(A/2) \oplus A/2 \otimes 2A \longrightarrow A/2 \otimes A/2 \longrightarrow 0.
\]

5. Nonexistence of factorization via pairs of vector spaces

Proposition 5.1. Let $\text{Vect}^2$ be the category of pairs of vector spaces over $\mathbb{F}_2$, and let $t: \text{Ab} \to \text{Vect}^2$ be the functor such that $t(A) = (A/2, 2A)$. Assume that $n \geq 2$. Then there is no functor $\Phi: \text{Vect}^2 \to \text{Vect}$ such that $\Phi \circ t \cong H_n(-, \mathbb{F}_2)$,

\[
H_n(A, \mathbb{F}_2) \neq \Phi(A/2, 2A).
\]

Moreover, there is no such a functor even if we restrict the functors to the category of finitely generated Abelian groups and the category of finite-dimensional vector spaces.

Proof. Here we use the theory of quadratic functors from the category of free finitely generated Abelian groups $\text{fAb} \to \text{Ab}$ and quadratic modules; we use the equivalence of these two categories (see [1, 7, 10]).

Let $n = 2$. Assume on the contrary that there exists a functor $\Phi: \text{Vect}^2 \to \text{Vect}$ such that there is a natural isomorphism $H_2(A, \mathbb{F}_2) \cong \Phi(A/2, 2A)$. Let $F$ be a free finitely generated Abelian group. Then there is an isomorphism $\Phi(F/2, 0) \cong H_2(F, \mathbb{F}_2) \cong \Lambda^2(F/2)$ which is natural by $F$. Because of the standard isomorphism $\text{Sym}((F/2)^\vee) \cong H^4(F/2, \mathbb{F}_2)$, there is natural isomorphism $H_\ast(F/2, \mathbb{F}_2) \cong \Gamma^2(F/2)$. It follows that $H_2(F/2, \mathbb{F}_2) \cong \Gamma^2(F/2)$. Therefore, $\Phi(F/2, F/2) \cong \Gamma^2(F/2)$. Since $(F/2, F/2) = (F/2, 0) \oplus (0, F/2)$ in the category $\text{Vect}^2$, we see that $\Phi(F/2, 0) \cong \Lambda^2 F/2$ is natural retract $\Phi(F/2, F/2) \cong \Gamma^2(F/2)$. On the other hand, it is easy to compute the quadratic $\mathbb{Z}$-module of $\Lambda^2(F/2) : 0 \to \mathbb{Z}/2 \to 0$ and the quadratic $\mathbb{Z}$-module of $\Gamma^2(F/2) : \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z}/2$. However, the first one is not a retract of the second one, a contradiction. This proves the statement for $n = 2$. 

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We prove the statement for \( n > 2 \) by induction. The Künneth theorem implies that \( H_n(A \times \mathbb{Z}, \mathbb{F}_2) = H_n(A, \mathbb{F}_2) \oplus H_{n-1}(A, \mathbb{F}_2) \). Then

\[
H_{n-1}(A, \mathbb{F}_2) = \text{Coker}(H_n(A, \mathbb{F}_2) \to H_n(A \times \mathbb{Z}, \mathbb{F}_2)).
\]

It follows that the existence of such a functor \( \Phi \) for \( H_n \) implies the existence of such a functor for \( H_{n-1} \). The assertion follows. \( \square \)

This research is supported by the RSF grant No. 16-11-10073.

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