Jacobi Fields on Statistical Manifolds of Negative Curvature

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Abstract. Two entropic dynamical models are considered. The geometric structure of the statistical manifolds underlying these models is studied. It is found that in both cases, the resulting metric manifolds are negatively curved. Moreover, the geodesics on each manifold are described by hyperbolic trajectories. A detailed analysis based on the Jacobi equation for geodesic spread is used to show that the hyperbolicity of the manifolds leads to chaotic exponential instability. A comparison between the two models leads to a relation among statistical curvature, stability of geodesics and relative entropy-like quantities. Finally, the Jacobi vector field intensity and the entropy-like quantity are suggested as possible indicators of chaoticity in the ED models due to their similarity to the conventional chaos indicators based on the Riemannian geometric approach and the Zurek-Paz criterion of linear entropy growth, respectively.

Keywords: inductive inference, information geometry, statistical manifolds, entropy, nonlinear dynamics and chaos.

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1. INTRODUCTION

Entropic Dynamics (ED) [1] is a theoretical framework constructed on statistical manifolds to explore the possibility that laws of physics, either classical or quantum, might be laws of inference rather than laws of nature. It is known that thermodynamics can be obtained by means of statistical mechanics which can be considered a form of statistical inference [2] rather than a pure physical theory. Indeed, even some features of quantum physics can be derived from principles of inference [3]. Finally, recent research considers the possibility that Einstein’s theory of gravity is derivable from general principles of inductive inference [4]. Unfortunately, the search for the correct variables that encode relevant information about a system is a major obstacle in the description and understanding of its evolution. The manner in which relevant variables are selected is not straightforward. This selection is made, in most cases, on the basis of intuition guided by experiment. The Maximum relative Entropy (ME) method [5, 6, 7] is used to construct ED models. The ME method is designed to be a tool of inductive inference. It is used for updating from a prior to a posterior probability distribution when new information in the form of constraints becomes available. We use known techniques [1] to show that this principle leads to equations that are analogous to equations of motion. Information is processed using ME methods in the framework of Information Geometry (IG) [8] that

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is, Riemannian geometry applied to probability theory. In our approach, probability theory is a form of generalized logic of plausible inference. It should apply in principle, to any situation where we lack sufficient information to permit deductive reasoning.

In this paper, we focus on two special entropic dynamical models. In the first model (ED1), we consider an hypothetical system whose microstates span a 2D space labelled by the variables \( x_1 \in \mathbb{R}^+ \) and \( x_2 \in \mathbb{R} \). We assume that the only testable information pertaining to the quantities \( x_1 \) and \( x_2 \) consists of the expectation values \( \langle x_1 \rangle \), \( \langle x_2 \rangle \) and the variance \( \Delta x_2 \). In the second model (ED2), we consider a 2D space of microstates labelled by the variables \( x_1 \in \mathbb{R} \) and \( x_2 \in \mathbb{R} \). In this case, we assume that the only testable information pertaining to the quantities \( x_1 \) and \( x_2 \) consists of the expectation values \( \langle x_1 \rangle \) and \( \langle x_2 \rangle \) and of the variances \( \Delta x_1 \) and \( \Delta x_2 \). Our models may be extended to more elaborate systems (highly constrained dynamics) where higher dimensions are considered. However, for the sake of clarity, we restrict our considerations to the above relatively simple cases. Given two known boundary macrostates, we investigate the possible trajectories of systems on the manifolds. The geometric structure of the manifolds underlying the models is studied. The metric tensor, Christoffel connections coefficients, Ricci and Riemann curvature tensors are calculated in both cases and it is shown that in both cases the dynamics takes place on negatively curved manifolds. The geodesics of the dynamical models are hyperbolic trajectories on the manifolds. A detailed study of the stability of such geodesics is presented using the equation of geodesic deviation (Jacobi equation). The notion of statistical volume elements is introduced to investigate the asymptotic behavior of a one-parameter family of neighboring geodesics. It is shown that the behavior of geodesics on such manifolds is characterized by exponential instability that leads to chaotic scenarios on the manifolds. These conclusions are supported by the asymptotic behavior of the Jacobi vector field intensity. Finally, a relation among entropy-like quantities, instability and curvature in the two models is presented.

2. CURVED STATISTICAL MANIFOLDS

In the case of ED1, a measure of distinguishability among the states of the system is achieved by assigning a probability distribution \( p\left(\bar{x} | \bar{\theta}\right) \) to each state defined by expected values \( \theta_1^{(1)}, \theta_1^{(2)}, \theta_2^{(2)} \) of the variables \( x_1, x_2 \) and \( (x_2 - \langle x_2 \rangle)^2 \). In the case of ED2, one assigns a probability distribution \( p\left(\bar{x} | \bar{\theta}\right) \) to each state defined by expected values \( \theta_1^{(1)}, \theta_2^{(1)}, \theta_1^{(2)}, \theta_2^{(2)} \) of the variables \( x_1, (x_1 - \langle x_1 \rangle)^2, x_2 \) and \( (x_2 - \langle x_2 \rangle)^2 \). The process of assigning a probability distribution to each state provides the statistical manifolds of the ED models with a metric structure. Specifically, the Fisher-Rao information metric \([9, 10, 11, 12]\) defined in (7) is used to quantify the distinguishability of probability distributions \( p\left(\bar{x} | \bar{\theta}\right) \) that live on the manifold (the family of distributions \( \{p^{(\text{tot})}\left(\bar{x} | \bar{\theta}\right)\} \) is as a manifold, each distribution \( p^{(\text{tot})}\left(\bar{x} | \bar{\theta}\right) \) is a point with coordinates \( \theta^i \) where \( i \) labels the macrovariables). As such, the Fisher-Rao metric assigns an IG to the space of states.
2.1. The Statistical Manifold $\mathcal{M}_{S1}$

Consider an hypothetical physical system evolving over a two-dimensional space. The variables $x_1 \in \mathbb{R}^+$ and $x_2 \in \mathbb{R}$ label the 2D space of microstates of the system. We assume that all information relevant to the dynamical evolution of the system is contained in the probability distributions. For this reason, no other information (such as external fields) is required. We assume that the only testable information pertaining to the quantities $x_1$ and $x_2$ consists of the expectation values $\langle x_1 \rangle$, $\langle x_2 \rangle$ and the variance $\Delta x_2$. Therefore, these three expected values define the 3D space of macrostates $\mathcal{M}_{S1}$ of the ED1 model. Each macrostate may be thought as a point of a three-dimensional statistical manifold with coordinates given by the numerical values of the expectations $\theta_1^{(1)}$, $\theta_1^{(2)}$, $\theta_2^{(2)}$. The available information can be written in the form of the following constraint equations,

$$
\langle x_1 \rangle = \int_0^{+\infty} dx_1 x_1 p_1 \left( x_1 | \theta_1^{(1)} \right), \quad \langle x_2 \rangle = \int_{-\infty}^{+\infty} dx_2 x_2 p_2 \left( x_2 | \theta_1^{(2)}, \theta_2^{(2)} \right),
$$

$$
\Delta x_2 = \sqrt{\left( \langle x_2 - \langle x_2 \rangle \rangle \right)^2} = \left[ \int_{-\infty}^{+\infty} dx_2 (x_2 - \langle x_2 \rangle)^2 p_2 \left( x_2 | \theta_1^{(2)}, \theta_2^{(2)} \right) \right]^{\frac{1}{2}},
$$

where $\theta_1^{(1)} = \langle x_1 \rangle$, $\theta_1^{(2)} = \langle x_2 \rangle$ and $\theta_2^{(2)} = \Delta x_2$. The probability distributions $p_1$ and $p_2$ are constrained by the conditions of normalization,

$$
\int_0^{+\infty} dx_1 p_1 \left( x_1 | \theta_1^{(1)} \right) = 1, \quad \int_{-\infty}^{+\infty} dx_2 p_2 \left( x_2 | \theta_1^{(2)}, \theta_2^{(2)} \right) = 1.
$$

Information theory identifies the exponential distribution as the maximum entropy distribution if only the expectation value is known. The Gaussian distribution is identified as the maximum entropy distribution if only the expectation value and the variance are known. ME methods allow us to associate a probability distribution $p^{(tot)} \left( \vec{x} | \vec{\theta} \right)$ to each point in the space of states. The distribution that best reflects the information contained in the prior distribution $m(\vec{x})$ updated by the constraints $(\langle x_1 \rangle, \langle x_2 \rangle, \Delta x_2)$ is obtained by maximizing the relative entropy

$$
\left[ S(\vec{\theta}) \right]_{\text{ED1}} = - \int_0^{+\infty} \int_{-\infty}^{+\infty} dx_1 dx_2 p^{(tot)} \left( \vec{x} | \vec{\theta} \right) \log \left[ \frac{p^{(tot)} \left( \vec{x} | \vec{\theta} \right)}{m(\vec{x})} \right],
$$

where $m(\vec{x}) \equiv m$ is the uniform prior probability distribution. The prior $m(\vec{x})$ is set to be uniform since we assume the lack of initial available information about the system (postulate of equal a priori probabilities). Upon maximizing (3), given the constraints (1) and (2), we obtain

$$
p^{(tot)} \left( \vec{x} | \vec{\theta} \right) = p_1 \left( x_1 | \theta_1^{(1)} \right) p_2 \left( x_2 | \theta_1^{(2)}, \theta_2^{(2)} \right) = \frac{1}{\mu_1} e^{-\frac{x_1^2}{2\mu_1}} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}},
$$
where \( \theta_1^{(1)} = \mu_1, \theta_1^{(2)} = \mu_2 \) and \( \theta_2^{(2)} = \sigma_2 \). The probability distribution (4) encodes the available information concerning the system and \( \mathcal{M}_{s_1} \) becomes,

\[
\mathcal{M}_{s_1} = \left\{ p^{(\text{tot})}(\vec{x} | \vec{\theta}) = \frac{1}{\mu_1} e^{-\frac{\vec{x}^2}{2\mu_1}} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}} : \vec{x} \in \mathbb{R}^+ \times \mathbb{R} \text{ and } \vec{\theta} \equiv (\mu_1, \mu_2, \sigma_2) \right\}.
\]

Note that we have assumed uncoupled constraints between the microvariables \( x_1 \) and \( x_2 \). In other words, we assumed that information about correlations between the microvariables needed not to be tracked. This assumption leads to the simplified product rule in (4). Coupled constraints however, would lead to a generalized product rule in (4) and to a metric tensor (7) with non-trivial off-diagonal elements (covariance terms). Correlation terms may be fictitious. They may arise for instance from coordinate transformations. On the other hand, correlations may arise from interaction of the system with external fields. Such scenarios would require more delicate analysis.

2.1.1. The Metric Tensor on \( \mathcal{M}_{s_1} \)

We cannot determine the evolution of microstates of the system since the available information is insufficient. Instead we can study the distance between two total distributions with parameters \((\mu_1, \mu_2, \sigma_2)\) and \((\mu_1 + d\mu_1, \mu_2 + d\mu_2, \sigma_2 + d\sigma_2)\). Once the states of the system have been defined, the next step concerns the problem of quantifying the notion of change in going from the state \( \vec{\theta} \) to the state \( \vec{\theta} + d\vec{\theta} \). For our purpose a convenient measure of change is distance. The measure we seek is given by the dimensionless "distance" \( ds \) between \( p(\vec{x} | \vec{\theta}) \) and \( p(\vec{x} | \vec{\theta} + d\vec{\theta}) \):

\[
ds^2 = g_{ij} d\theta^i d\theta^j,
\]

where

\[
g_{ij} = \int d\vec{x} p(\vec{x} | \vec{\theta}) \frac{\partial \log p(\vec{x} | \vec{\theta})}{\partial \theta^i} \frac{\partial \log p(\vec{x} | \vec{\theta})}{\partial \theta^j}
\]

is the Fisher-Rao information metric. Substituting (4) into (7), the metric \( g_{ij} \) on \( \mathcal{M}_{s_1} \) becomes,

\[
(g_{ij})_{\mathcal{M}_{s_1}} = \begin{pmatrix}
\frac{1}{\mu_1^2} & 0 & 0 \\
0 & \frac{1}{\sigma_2^2} & 0 \\
0 & 0 & \frac{2}{\sigma_2^2}
\end{pmatrix}.
\]

Substituting (8) into (6), the "length" element reads,

\[
(ds^2)_{\mathcal{M}_{s_1}} = \frac{1}{\mu_1^2} d\mu_1^2 + \frac{1}{\sigma_2^2} d\mu_2^2 + \frac{2}{\sigma_2^4} d\sigma_2^2.
\]
Notice that the metric structure of $\mathcal{M}_{s_1}$ is an emergent structure and is not itself fundamental. It arises only after assigning a probability distribution $p(\vec{x} | \vec{\theta})$ to each state $\vec{\theta}$.

### 2.1.2. The Curvature of $\mathcal{M}_{s_1}$

In this paragraph we calculate the statistical curvature $R_{\mathcal{M}_{s_1}}$. This is achieved via application of standard differential geometry methods to the space of probability distributions $\mathcal{M}_{s_1}$. Recall the definitions of the Ricci tensor $R_{ij}$ and Riemann curvature tensor $R_{\alpha \mu \nu \rho}$,

$$R_{ij} = g^{ah} R_{aibj} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{ij} \Gamma^m_{km} - \Gamma^m_{ik} \Gamma^k_{jm},$$  \hspace{1cm} \text{(10)}

and

$$R^\alpha_{\mu \nu \rho} = \partial_\nu \Gamma^\alpha_{\mu \rho} - \partial_\rho \Gamma^\alpha_{\mu \nu} + \Gamma^\alpha_{\beta \nu} \Gamma^\beta_{\mu \rho} - \Gamma^\alpha_{\beta \rho} \Gamma^\beta_{\mu \nu}.$$

(11)

The Ricci scalar $R_{\mathcal{M}_{s_1}}$ is obtained from (10) or (11) via appropriate contraction with the metric tensor $g_{ij}$ in (8), namely

$$R = R_{ij} g^{ij} = R_{\alpha \beta \gamma \delta} g^{\alpha \gamma} g^{\beta \delta},$$

(12)

where $g^{ik} g_{kj} = \delta^i_j$ so that $g^{ij} = (g_{ij})^{-1} = \text{diag}(\mu_1^2, \sigma_2^2, \sigma_3^2)$. The Christoffel symbols $\Gamma^k_{ij}$ appearing in (10) and (11) are defined by,

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \left( \partial_i g_{mj} + \partial_j g_{im} - \partial_m g_{ij} \right).$$

(13)

Substituting (8) into (13), we calculate the non-vanishing components of the connection coefficients,

$$\Gamma^1_{11} = -\frac{1}{\mu_1}, \Gamma^3_{22} = \frac{1}{2\sigma_2}, \Gamma^3_{33} = -\frac{1}{\sigma_2}, \Gamma^2_{23} = \Gamma^3_{32} = -\frac{1}{\sigma_2}.$$  \hspace{1cm} \text{(14)}

By substituting (14) in (10) we determine the Ricci tensor components,

$$R_{11} = 0, R_{22} = -\frac{1}{2\sigma_2^2}, R_{33} = -\frac{1}{\sigma_2^2}.$$  \hspace{1cm} \text{(15)}

The non-vanishing Riemann tensor component is,

$$R_{2323} = -\frac{1}{\sigma_2^4}.$$  \hspace{1cm} \text{(16)}

Finally, by substituting (15) or (16) into (12) and using $(g_{ij})^{-1}$ we obtain the Ricci scalar,

$$R_{\mathcal{M}_{s_1}} = -1 < 0.$$  \hspace{1cm} \text{(17)}

From (17) we conclude that $\mathcal{M}_{s_1}$ is a manifold of constant negative ($-1$) curvature.
2.2. The Statistical Manifold $\mathcal{M}_{S_2}$

In this case we assume that the 2D space of microstates of the system is labelled by the variables $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$. We assume, as in subsection (2.1), that all information relevant to the dynamical evolution of the system is contained in the probability distributions. Moreover, we assume that the only testable information pertaining to the quantities $x_1$ and $x_2$ consist of the expectation values $\langle x_1 \rangle$ and $\langle x_2 \rangle$ and of the variances $\Delta x_1$ and $\Delta x_2$. Therefore, these four expected values define the $4D$ space of macrostates $\mathcal{M}_{S_2}$ of the ED2 model. Each macrostate may be thought as a point of a four-dimensional statistical manifold with coordinates given by the numerical values of the expectations $\theta_1^{(1)}$, $\theta_2^{(1)}$, $\theta_1^{(2)}$, $\theta_2^{(2)}$. The available information can be written in the form of the following constraint equations,

$$
\langle x_1 \rangle = \int_{-\infty}^{+\infty} dx_1 x_1 p_1 \left( x_1 | \theta_1^{(1)} \right), \quad \langle x_2 \rangle = \int_{-\infty}^{+\infty} dx_2 x_2 p_2 \left( x_2 | \theta_1^{(2)} , \theta_2^{(2)} \right),
$$

$$
\Delta x_1 = \sqrt{\left( \langle x_1 \rangle - \langle x_1 \rangle \right)^2} = \left[ \int_{-\infty}^{+\infty} dx_1 \left( x_1 - \langle x_1 \rangle \right)^2 p_1 \left( x_1 | \theta_1^{(1)} , \theta_2^{(1)} \right) \right]^{\frac{1}{2}},
$$

$$
\Delta x_2 = \sqrt{\left( \langle x_2 \rangle - \langle x_2 \rangle \right)^2} = \left[ \int_{-\infty}^{+\infty} dx_2 \left( x_2 - \langle x_2 \rangle \right)^2 p_2 \left( x_2 | \theta_1^{(2)} , \theta_2^{(2)} \right) \right]^{\frac{1}{2}},
$$

where $\theta_1^{(1)} = \langle x_1 \rangle$, $\theta_2^{(1)} = \Delta x_1$, $\theta_1^{(2)} = \langle x_2 \rangle$ and $\theta_2^{(2)} = \Delta x_2$. The probability distributions $p_1$ and $p_2$ are constrained by the conditions of normalization,

$$
\int_{-\infty}^{+\infty} dx_1 p_1 \left( x_1 | \theta_1^{(1)} , \theta_2^{(1)} \right) = 1, \quad \int_{-\infty}^{+\infty} dx_2 p_2 \left( x_2 | \theta_1^{(2)} , \theta_2^{(2)} \right) = 1.
$$

The distribution that best reflects the information contained in the uniform prior distribution $m \left( \bar{x} \right) \equiv m$ updated by the constraints $\left( \langle x_1 \rangle, \Delta x_1, \langle x_2 \rangle, \Delta x_2 \right)$ is obtained by maximizing the relative entropy

$$
\left[ S \left( \bar{\theta} \right) \right]_{\text{ED2}} = - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx_1 dx_2 p^{\text{tot}} \left( \bar{x} | \bar{\theta} \right) \log \left[ \frac{p^{\text{tot}} \left( \bar{x} | \bar{\theta} \right)}{m \left( \bar{x} \right)} \right].
$$

Upon maximizing (20), given the constraints (18) and (19), we obtain

$$
p^{\text{tot}} \left( \bar{x} | \bar{\theta} \right) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{\left( x_1 - \mu_1 \right)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{\left( x_2 - \mu_2 \right)^2}{2\sigma_2^2}}.
$$
The probability distribution (21) encodes the available information concerning the system and $\mathcal{M}_{s_2}$ becomes,

$$
\mathcal{M}_{s_2} = \left\{ p^{(\text{tot})}(\mathbf{x}|\tilde{\theta}) = \frac{1}{\sqrt{2\pi \sigma_1^2}} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi \sigma_2^2}} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}} : \mathbf{x} \in \mathbb{R} \times \mathbb{R} \text{ and } \tilde{\theta} \equiv (\mu_1, \sigma_1, \mu_2, \sigma_2) \right\}.
$$

(22)

2.2.1. The Metric Tensor on $\mathcal{M}_{s_2}$

Proceeding as in (2.1.1), we determine the metric on $\mathcal{M}_{s_2}$. Substituting (21) into (7), the metric $g_{ij}$ on $\mathcal{M}_{s_2}$ becomes,

$$
(g_{ij})_{\mathcal{M}_{s_2}} = \begin{pmatrix}
\frac{1}{\sigma_1^2} & 0 & 0 & 0 \\
0 & \frac{2}{\sigma_1^2} & 0 & 0 \\
0 & 0 & \frac{1}{\sigma_2^2} & 0 \\
0 & 0 & 0 & \frac{2}{\sigma_2^2}
\end{pmatrix}.
$$

(23)

Finally, substituting (23) into (6), the "length" element reads,

$$
(ds^2)_{\mathcal{M}_{s_2}} = \frac{1}{\sigma_1^2} d\mu_1^2 + \frac{2}{\sigma_1^2} d\sigma_1^2 + \frac{1}{\sigma_2^2} d\mu_2^2 + \frac{2}{\sigma_2^2} d\sigma_2^2.
$$

(24)

2.2.2. The Curvature of $\mathcal{M}_{s_2}$

Proceeding as in (2.1.2), we calculate the statistical curvature $R_{\mathcal{M}_{s_2}}$ of $\mathcal{M}_{s_2}$. Notice that $g^{ij} = (g_{ij})^{-1} = \text{diag}(\frac{\sigma_1^2}{2}, \frac{\sigma_1^2}{2}, \frac{\sigma_2^2}{2}, \frac{\sigma_2^2}{2})$. Substituting (23) into (13), the non-vanishing components of the connection coefficients become,

$$
\Gamma_{12}^1 = \Gamma_{21}^1 = -\frac{1}{\sigma_1^2}, \quad \Gamma_{22}^2 = -\frac{1}{\sigma_1^2}, \quad \Gamma_{11}^2 = \frac{1}{2\sigma_1^2}, \quad \Gamma_{34}^3 = \Gamma_{43}^3 = -\frac{1}{\sigma_2^2}, \quad \Gamma_{33}^4 = \frac{1}{2\sigma_2^2}, \quad \Gamma_{44}^4 = -\frac{1}{\sigma_2^2}.
$$

(25)

By substituting (25) in (10) we determine the Ricci tensor components,

$$
R_{11} = -\frac{1}{2\sigma_1^2}, \quad R_{22} = -\frac{1}{\sigma_1^2}, \quad R_{33} = -\frac{1}{2\sigma_2^2}, \quad R_{44} = -\frac{1}{\sigma_2^2}.
$$

(26)

The non-vanishing Riemann tensor components are,

$$
R_{1212} = -\frac{1}{\sigma_1^2}, \quad R_{3434} = -\frac{1}{\sigma_2^2}.
$$

(27)
Finally, by substituting (26) or (27) into (12) and using \((g_{ij})^{-1}\), we obtain the Ricci scalar,

\[
R_{\mathcal{M}_{s_2}} = -2 < 0. \tag{28}
\]

From (28) we conclude that \(\mathcal{M}_{s_2}\) is a manifold of constant negative \((-2)\) curvature.

3. THE ED MODELS

The ED models can be derived from a standard principle of least action (Maupertuis-Euler-Lagrange-Jacobi-type) [1, 13]. The main differences are that the dynamics being considered here is defined on a space of probability distributions \(\mathcal{M}_s\), not on an ordinary linear space \(V\). Also, the standard coordinates \(q_j\) of the system are replaced by statistical macrovariables \(\theta_j\).

Given the initial macrostate and that the system evolves to a fixed final macrostate, we investigate the expected trajectories of the ED models on \(\mathcal{M}_{s_1}\) and \(\mathcal{M}_{s_2}\). It is known [13] that the classical dynamics of a particle can be derived from the principle of least action in the Maupertuis-Euler-Lagrange-Jacobi form,

\[
\delta J_{\text{Jacobi}} [q] = \delta \int_{s_i}^{s_f} ds \mathcal{F} \left( q_j, \frac{dq_j}{ds}, s, H \right) = 0, \tag{29}
\]

where \(q_j\) are the coordinates of the system, \(s\) is an affine parameter along the trajectory and \(\mathcal{F}\) is a functional defined as

\[
\mathcal{F} \left( q_j, \frac{dq_j}{ds}, s, H \right) \equiv \left[ 2 (H - U) \right]^{\frac{1}{2}} \left( \sum_{j,k} a_{jk} \frac{dq_j}{ds} \frac{dq_k}{ds} \right)^{\frac{1}{2}}. \tag{30}
\]

For a non-relativistic system, the energy \(H\) is,

\[
H = T + U (q) = \frac{1}{2} \sum_{j,k} a_{jk} (q) \dot{q}_j \dot{q}_k + U (q) \tag{31}
\]

where the coefficients \(a_{jk}\) are the reduced mass matrix coefficients and \(\dot{q} = \frac{dq}{ds}\) is the time derivative of the canonical coordinate \(q\). We now seek the expected trajectory of the system assuming it evolves from \(\theta^\mu_{\text{old}} = \theta^\mu \equiv (\mu_1 (s_i), \mu_2 (s_i), \sigma_2 (s_i))\) to \(\theta^\mu_{\text{new}} = \theta^\mu + d\theta^\mu \equiv (\mu_1 (s_f), \mu_2 (s_f), \sigma_2 (s_f))\). It is known [1] that such a system moves along a geodesic in the space of states, which is a curved manifold. Since the trajectory of the system is a geodesic, the ED-action is itself the length; that is,

\[
J_{\text{ED}} [\theta] = \int (ds)^2 \frac{1}{2} = \int (g_{ij} d\theta^i d\theta^j)^{\frac{1}{2}} = \int_{s_i}^{s_f} ds \left( g_{ij} \frac{d\theta^i (s)}{ds} \frac{d\theta^j (s)}{ds} \right)^{\frac{1}{2}} \equiv \int_{s_i}^{s_f} ds \mathcal{L} (\theta, \dot{\theta}) \tag{32}
\]

where \(\dot{\theta} = \frac{d\theta}{ds}\) and \(\mathcal{L} (\theta, \dot{\theta})\) is the Lagrangian of the system,

\[
\mathcal{L} (\theta, \dot{\theta}) = (g_{ij} \dot{\theta}^i \dot{\theta}^j)^{\frac{1}{2}}. \tag{33}
\]
A useful choice for $s$ is one satisfying the condition $g_{ij} \frac{d\theta^i}{ds} \frac{d\theta^j}{ds} = 1$. Therefore, we formally identify the affine parameter $s$ with the temporal evolution parameter $\tau$, $s \equiv \tau$. Performing a standard calculus of variations with $s \equiv \tau$, we obtain

$$\delta J_{ED}[\theta] = \int d\tau \left( \frac{1}{2} \frac{\partial g_{ij}}{\partial \dot{\theta}^k} \dot{\theta}^i \dot{\theta}^j - \frac{d\theta^k}{d\tau} \right) \delta \theta^k = 0, \forall \delta \theta^k. \quad (34)$$

Note that from (34), $\frac{d\theta^k}{d\tau} = \frac{1}{2} \frac{\partial g_{ij}}{\partial \theta^k} \theta^i \theta^j$. This differential equation shows that if $\frac{\partial g_{ij}}{\partial \theta^k} = 0$ for a particular $k$ then the corresponding $\dot{\theta}_k$ is conserved. This suggests to interpret $\theta_k$ as momenta. Equations (34) and (13) lead to the geodesic equation,

$$\frac{d^2 \theta^k(\tau)}{d\tau^2} + \Gamma_{ij}^k \frac{d\theta^i(\tau)}{d\tau} \frac{d\theta^j(\tau)}{d\tau} = 0. \quad (35)$$

Observe that (35) are nonlinear, second order coupled ordinary differential equations. These equations describe a dynamics that is reversible and their solution is the trajectory between an initial and a final macrostate. The trajectory can be equally well traversed in both directions.

### 3.1. The ED1: Geodesics on $\mathcal{M}_{s_1}$

We seek the explicit form of (35) for ED1. Substituting (14) in (35), we obtain,

$$\frac{d^2 \mu_1}{d\tau^2} = \frac{1}{\mu_1} \left( \frac{d\mu_1}{d\tau} \right)^2, \frac{d^2 \mu_2}{d\tau^2} = \frac{2}{\mu_2} \frac{d\mu_2}{d\tau} \frac{d\sigma_2}{d\tau} \frac{d\sigma_2}{d\tau} = \frac{1}{\sigma_2} \left[ \left( \frac{d\sigma_2}{d\tau} \right)^2 - \frac{1}{2} \left( \frac{d\mu_2}{d\tau} \right)^2 \right]. \quad (36)$$

Integrating this set of differential equations, we obtain

$$\mu_1(\tau) = A_1 \left[ \cosh(\alpha_1 \tau) - \sinh(\alpha_1 \tau) \right],$$

$$\mu_2(\tau) = B_1 \frac{\alpha_1}{2 \beta_1} \frac{1}{\cosh(2\beta_1 \tau) - \sinh(2\beta_1 \tau) + \frac{\beta_1^2}{s^2 \beta_1}} + C_1, \quad \sigma_2(\tau) = B_1 \frac{[\cosh(\beta_1 \tau) - \sinh(\beta_1 \tau)]}{\cosh(2\beta_1 \tau) - \sinh(2\beta_1 \tau) + \frac{\beta_1^2}{s^2 \beta_1}}, \quad (37)$$

where $A_1, B_1, C_1, \alpha_1$ and $\beta_1$ are real integration constants ($5 = 6 - 1, \left( \dot{\theta}_j \dot{\theta}_j \right)^{1/2} = 1$). The set of equations (37) parametrizes the evolution surface of the statistical submanifold $m_{s_1}$ of $\mathcal{M}_{s_1}$,

$$m_{s_1} = \left\{ p^{(tot)} \left( \bar{\theta} \right) \in \mathcal{M}_{s_1} : \dot{\bar{\theta}} \text{ satisfy (36)} \right\}. \quad (38)$$
3.2. The ED2: Geodesics on $\mathcal{M}_{s2}$

We seek the explicit form of (35) for ED2. Substituting (25) in (35), we obtain,

$$
\frac{d^2 \mu_1}{d \tau^2} = \frac{2}{\sigma_1} \frac{d \mu_1}{d \tau} \frac{d \sigma_1}{d \tau}, \quad \frac{d^2 \sigma_1}{d \tau^2} = \frac{1}{\sigma_1} \left[ \left( \frac{d \sigma_1}{d \tau} \right)^2 - \frac{1}{2} \left( \frac{d \mu_1}{d \tau} \right)^2 \right],
$$

and

$$
\frac{d^2 \mu_2}{d \tau^2} = \frac{2}{\sigma_2} \frac{d \mu_2}{d \tau} \frac{d \sigma_2}{d \tau}, \quad \frac{d^2 \sigma_2}{d \tau^2} = \frac{1}{\sigma_2} \left[ \left( \frac{d \sigma_2}{d \tau} \right)^2 - \frac{1}{2} \left( \frac{d \mu_2}{d \tau} \right)^2 \right].
$$

Integrating this set of differential equations, we obtain

$$
\mu_1 (\tau) = \frac{A_1^2}{2 \alpha_2} \frac{1}{\cosh(2\alpha_2 \tau) - \sinh(2\alpha_2 \tau) + \frac{A_2^2}{8\alpha_2^2}} + C_1, \quad \sigma_1 (\tau) = A_2 \frac{[\cosh(\alpha_2 \tau) - \sinh(\alpha_2 \tau)]}{\cosh(2\alpha_2 \tau) - \sinh(2\alpha_2 \tau) + \frac{A_2^2}{8\alpha_2^2}},
$$

and

$$
\mu_2 (\tau) = \frac{B_1^2}{2 \beta_2} \frac{1}{\cosh(2\beta_2 \tau) - \sinh(2\beta_2 \tau) + \frac{B_2^2}{8\beta_2^2}} + C_2, \quad \sigma_2 (\tau) = B_2 \frac{[\cosh(\beta_2 \tau) - \sinh(\beta_2 \tau)]}{\cosh(2\beta_2 \tau) - \sinh(2\beta_2 \tau) + \frac{B_2^2}{8\beta_2^2}},
$$

where $A_2, B_2, C_1, C_2, \alpha_2$ and $\beta_2$ are real integration constants. The set of equations (40) parametrizes the evolution surface of the statistical submanifold $m_{s2}$ of $\mathcal{M}_{s2}$,

$$
\left\{ \rho^{(\text{tot})} \left( \tilde{x}, \tilde{\theta} \right) \in \mathcal{M}_{s2} : \tilde{\theta} \text{ satisfy (39)} \right\}.
$$

4. CHAOTIC INSTABILITY IN THE ED MODELS

It is known that [13] the Riemannian curvature of a manifold is closely connected with the behavior of the geodesics on it, i.e., with the motion of the corresponding dynamical system. If the Riemannian curvature of a manifold is positive (as on a sphere or ellipsoid), then the nearby geodesics oscillate about one another in most cases; whereas if the curvature is negative (as on the surface of a hyperboloid of one sheet), geodesics rapidly diverge from one another.

4.1. Instability in ED1

In this subsection, the stability of ED1 is considered. It is shown that neighboring trajectories are exponentially unstable under small perturbations of initial conditions. In the rest of the paper, for the sake of simplicity, we assume that $A_1 = B_1 = A_2 = B_2 \equiv A, A_1 = B_1 = \alpha_2 = \beta_2 \equiv \alpha$. Our conclusions do not depend on the particular initial conditions chosen.
4.1.1. The Geodesic Length $\Theta_{\mathcal{M}_{s_1}}$

Consider the one-parameter family of geodesics $\mathcal{F}_{G_{\mathcal{M}_{s_1}}} (\alpha) \equiv \left\{ \theta^\mu_{\mathcal{M}_{s_1}} (\tau; \alpha) \right\}_{\alpha \in \mathbb{R}^+}$, where $\theta^\mu_{\mathcal{M}_{s_1}}$ are solutions of (36). The length of geodesics in $\mathcal{F}_{G_{\mathcal{M}_{s_1}}} (\alpha)$ is defined as,

$$
\Theta_{\mathcal{M}_{s_1}} (\tau; \alpha) \overset{\text{def}}= \int_0^\tau \left[ \frac{1}{\mu_1^2} \left( \frac{d\mu_1}{d\tau'} \right)^2 + \frac{1}{\sigma_2^2} \left( \frac{d\mu_2}{d\tau'} \right)^2 + 2 \left( \frac{d\sigma_2}{d\tau'} \right)^2 \right]^{\frac{1}{2}} d\tau'.
$$

Substituting (37) in (42) and considering the asymptotic expression of $\Theta_{\mathcal{M}_{s_1}} (\tau; \alpha)$, we obtain

$$
\Theta_{\mathcal{M}_{s_1}} (\tau \to \infty; \alpha) \equiv \Theta_1 (\tau; \alpha) \approx \sqrt{3} \alpha \tau. \tag{43}
$$

In order to investigate the asymptotic behavior of two neighboring geodesics labelled by the parameters $\alpha$ and $\alpha + \delta \alpha$, we consider the following difference,

$$
\Delta \Theta_1 \equiv |\Theta_1 (\tau; \alpha + \delta \alpha) - \Theta_1 (\tau; \alpha)| = \sqrt{3} |\delta \alpha| \tau. \tag{44}
$$

It is clear that $\Delta \Theta_1$ diverges, that is, the lengths of two neighboring geodesics with slightly different parameters $\alpha$ and $\alpha + \delta \alpha$ differ in a remarkable way as the evolution parameter $\tau \to \infty$. This hints at the onset of instability of the hyperbolic trajectories on $\mathcal{M}_{s_1}$.

4.1.2. The Statistical Volume Elements $V_{\mathcal{M}_{s_1}}$

The instability of ED1 can be further explored by studying the behavior of the one-parameter family of statistical volume elements $\mathcal{F}_{V_{\mathcal{M}_{s_1}}} (\alpha) \equiv \left\{ V_{\mathcal{M}_{s_1}} (\tau; \alpha) \right\}_{\alpha}$. Recall that $\mathcal{M}_{s_1}$ is the space of probability distributions $p^{(\text{tot})} (\vec{x} | \vec{\theta})$ labeled by parameters $\theta_1^{(1)}, \theta_1^{(2)}, \theta_2^{(2)}$. These parameters are the coordinates of the point $p^{(\text{tot})}$, and in these coordinates a 3D volume element $dV_{\mathcal{M}_{s_1}}$ reads

$$
dV_{\mathcal{M}_{s_1}} = \sqrt{g} d\theta_1^{(1)} d\theta_1^{(2)} d\theta_2^{(2)} \equiv \sqrt{g} d\mu_1 d\mu_2 d\sigma_2, \tag{45}
$$

where in the ED1 model here presented, $g = |\det (g_{ij})_{\mathcal{M}_{s_1}}| = \frac{2}{\mu_1 \sigma_2^2}$. Hence, the volume element $dV_{\mathcal{M}_{s_1}}$ is given by,

$$
dV_{\mathcal{M}_{s_1}} = \frac{\sqrt{2}}{\mu_1 \sigma_2^2} d\mu_1 d\mu_2 d\sigma_2. \tag{46}
$$
The volume of an extended region of $\mathcal{M}_{s_1}$ is defined by,

$$
\Delta V_{\mathcal{M}_{s_1}} \equiv V_{\mathcal{M}_{s_1}}(\tau) - V_{\mathcal{M}_{s_1}}(0) \equiv \int_{V_{\mathcal{M}_{s_1}}(0)}^{V_{\mathcal{M}_{s_1}}(\tau)} \mu_1(\tau) \mu_2(\tau) \sigma_2(\tau) \frac{\sqrt{2}}{\mu_1 \sigma_2^2} d\mu_1 d\mu_2 d\sigma_2.
$$

Integrating (47) using (37), we obtain

$$
\Delta V_{\mathcal{M}_{s_1}} = \frac{\tau}{\sqrt{2}} e^{\alpha \tau} - \frac{\ln A}{\sqrt{2} \alpha} e^{\alpha \tau} + \frac{\ln A}{\sqrt{2} \alpha}.
$$

The quantity that actually encodes relevant information about the stability of neighboring volume elements is the average volume

$$
\langle \Delta V_{\mathcal{M}_{s_1}} \rangle_\tau \equiv \frac{1}{\tau} \int_0^\tau \Delta V_{\mathcal{M}_{s_1}}(\tau'; \alpha) d\tau' = \frac{1}{\tau} \left\{ \frac{1}{\sqrt{2} \alpha^2} (\alpha \tau - 1) e^{\alpha \tau} - \frac{\ln A}{\sqrt{2} \alpha^2} e^{\alpha \tau} + \frac{\ln A}{\sqrt{2} \alpha} \right\}.
$$

For convenience, let us rename $\langle \Delta V_{\mathcal{M}_{s_1}} \rangle_\tau \equiv \Delta V_1$. Therefore, the asymptotic expansion of $\Delta V_1$ for $\tau \to \infty$ reads,

$$
\Delta V_1 \approx \frac{1}{\sqrt{2} \alpha} e^{\alpha \tau}.
$$

This asymptotic regime of diffusive evolution in (50) describes the exponential increase of average volume elements on $\mathcal{M}_{s_1}$. The exponential instability characteristic of chaos forces the system to rapidly explore large areas (volumes) of the statistical manifolds. It is interesting to note that this asymptotic behavior appears also in the conventional description of quantum chaos [14] where the entropy increases linearly at a rate determined by the Lyapunov exponents [15]. The linear entropy increase as a quantum chaos criterion was introduced by Zurek and Paz. In our information-geometric approach a relevant variable that will be useful for comparison of the two different degrees of instability characterizing the two ED models is the relative entropy-like quantity defined as,

$$
S_1 \equiv \ln (\Delta V_1).
$$

Substituting (50) in (51) and considering the asymptotic limit $\tau \to \infty$, we obtain

$$
S_1 \approx \alpha \tau.
$$

The entropy-like quantity $S_1$ in (52) may be interpreted as the asymptotic limit of the natural logarithm of a statistical weight $\langle \Delta V_{\mathcal{M}_{s_1}} \rangle_\tau$ defined on $\mathcal{M}_{s_1}$. Equation (52) is the information-geometric analog of the Zurek-Paz chaos criterion.
4.1.3. The Jacobi Vector Field \( J_{\mathcal{M}_{s_1}} \)

We study the behavior of the one-parameter family of neighboring geodesics \( \mathcal{F}_{G,\mathcal{M}_{s_1}}(\alpha) \equiv \{ \theta^\mu_{\mathcal{M}_{s_1}}(\tau; \alpha) \}_{\alpha \in \mathbb{R}^+} \) where,

\[
\begin{align*}
\theta^1(\tau; \alpha) &= \mu_1(\tau; \alpha) = Ae^{\alpha \tau}; \\
\theta^2(\tau; \alpha) &= \mu_2(\tau; \alpha) = A^2 e^{-2\alpha \tau} + \frac{A^2}{8\alpha^2}, \\
\theta^3(\tau; \alpha) &= \sigma_2(\tau; \alpha) = A e^{-\alpha \tau} e^{-2\alpha \tau} + \frac{A^2}{8\alpha^2}.
\end{align*}
\]

The relative geodesic spread is characterized by the Jacobi equation \([16, 17]\),

\[
\frac{D^2 (\delta \theta^i)}{D\tau^2} + R^i_{kml} \frac{\partial \theta^k}{\partial \tau} \frac{\partial \theta^l}{\partial \tau} \delta \theta^m = 0 \quad (54)
\]

where \( i = 1, 2, 3 \) and,

\[
\delta \theta^i \equiv \delta_\alpha \theta^i \overset{\text{def}}{=} \left( \frac{\partial \theta^i(\tau; \alpha)}{\partial \alpha} \right)_\tau \delta \alpha. \quad (55)
\]

Equation (54) forms a system of three coupled ordinary differential equations linear in the components of the deviation vector field (55) but nonlinear in derivatives of the metric (8). It describes the linearized geodesic flow: the linearization ignores the relative velocity of the geodesics. When the geodesics are neighboring but their relative velocity is arbitrary, the corresponding geodesic deviation equation is the so-called generalized Jacobi equation \([18]\). The nonlinearity is due to the existence of velocity-dependent terms in the system.

Neighboring geodesics accelerate relative to each other with a rate directly measured by the curvature tensor \( R_{\alpha\beta\gamma\delta} \). Multiplying both sides of (54) by \( g_{ij} \) and using the standard symmetry properties of the Riemann curvature tensor, the geodesic deviation equation becomes,

\[
\frac{D^2 (\delta \theta^i)}{D\tau^2} + R^i_{jkl} \frac{\partial \theta^k}{\partial \tau} \frac{\partial \theta^l}{\partial \tau} \delta \theta^m = 0. \quad (56)
\]

Recall that the covariant derivative \( \frac{D^2 (\delta \theta^\mu)}{D\tau^2} \) in (54) is defined as,

\[
\frac{D^2 \delta^\mu}{D\tau^2} = \frac{d^2 \delta^\mu}{d\tau^2} + 2\Gamma^\mu_{\alpha\beta} \frac{d \theta^\alpha}{d\tau} \frac{d \theta^\beta}{d\tau} \delta \theta^\mu + \Gamma^\mu_{\alpha\beta} \delta \theta^\alpha \frac{d^2 \theta^\beta}{d\tau^2} + \Gamma^\mu_{\alpha\beta\gamma} \frac{d \theta^\alpha}{d\tau} \frac{d \theta^\beta}{d\tau} \delta \theta^\gamma + \Gamma^\mu_{\alpha\beta\gamma} \delta \theta^\alpha \frac{d \theta^\beta}{d\tau} \frac{d \theta^\gamma}{d\tau} \quad (57)
\]
and that the only non-vanishing Riemann tensor component is $R_{2323} = -\frac{1}{\sigma^2}$. Therefore, the three differential equations for the geodesic deviation are,

\[ \frac{d^2(\delta \theta^1)}{d\tau^2} + 2\Gamma^1_{11} \frac{d\theta^1}{d\tau} \frac{d(\delta \theta^1)}{d\tau} + \partial_1 \Gamma^1_{11} \left( \frac{d\theta^1}{d\tau} \right)^2 \delta \theta^1 = 0, \quad (58) \]

\[ \frac{d^2(\delta \theta^2)}{d\tau^2} + 2 \left[ \Gamma^2_{23} \frac{d\theta^3}{d\tau} \frac{d(\delta \theta^2)}{d\tau} + \Gamma^2_{32} \frac{d\theta^2}{d\tau} \frac{d(\delta \theta^3)}{d\tau} \right] + \partial_3 \Gamma^2_{23} \left( \frac{d\theta^3}{d\tau} \right)^2 \delta \theta^2 + \Gamma^2_{32} \frac{d\theta^2}{d\tau} \frac{d(\delta \theta^3)}{d\tau} = 0, \quad (59) \]

\[ \frac{d^2(\delta \theta^3)}{d\tau^2} + 2 \left[ \Gamma^3_{22} \frac{d\theta^2}{d\tau} \frac{d(\delta \theta^3)}{d\tau} + \Gamma^3_{33} \frac{d\theta^3}{d\tau} \frac{d(\delta \theta^2)}{d\tau} \right] + \partial_3 \Gamma^3_{33} \left( \frac{d\theta^3}{d\tau} \right)^2 \delta \theta^3 + \Gamma^3_{22} \frac{d\theta^2}{d\tau} \frac{d(\delta \theta^3)}{d\tau} = 0, \quad (60) \]

Substituting (14), (16) and (53) in equations (58), (59) and (60) and considering the asymptotic limit $\tau \rightarrow \infty$, the geodesic deviation equations become,

\[ \frac{d^2(\delta \theta^1)}{d\tau^2} + 2\alpha \frac{d(\delta \theta^1)}{d\tau} + \alpha^2 \delta \theta^1 = 0, \quad (61) \]

\[ \frac{d^2(\delta \theta^2)}{d\tau^2} + 2\alpha \frac{d(\delta \theta^2)}{d\tau} + 16\alpha^2 \frac{d(\delta \theta^3)}{d\tau} + \left( \alpha^2 - \frac{8\alpha^3}{A} e^{-\alpha \tau} \right) \delta \theta^3 = 0, \quad (62) \]

\[ \frac{d^2(\delta \theta^3)}{d\tau^2} + 2\alpha \frac{d(\delta \theta^3)}{d\tau} + \left( \alpha^2 - \frac{32\alpha^4}{A^2} e^{-2\alpha \tau} \right) \delta \theta^3 - \frac{8\alpha^2}{A} e^{-\alpha \tau} \frac{d(\delta \theta^2)}{d\tau} - \frac{8\alpha^3}{A} e^{-\alpha \tau} \delta \theta^2 = 0. \quad (63) \]

Neglecting the exponentially decaying terms in $\delta \theta^3$ in (62) and (63) and assuming that,

\[ \lim_{\tau \rightarrow \infty} \left( 16\alpha^2 \frac{d(\delta \theta^3)}{d\tau} \right) = 0, \quad \lim_{\tau \rightarrow \infty} \left( \frac{8\alpha^2}{A} e^{-\alpha \tau} \frac{d(\delta \theta^2)}{d\tau} \right) = 0, \quad \lim_{\tau \rightarrow \infty} \left( \frac{8\alpha^3}{A} e^{-\alpha \tau} \delta \theta^2 \right) = 0, \quad (64) \]

the geodesic deviation equations finally become,

\[ \frac{d^2(\delta \theta^1)}{d\tau^2} + 2\alpha \frac{d(\delta \theta^1)}{d\tau} + \alpha^2 \delta \theta^1 = 0, \quad \frac{d^2(\delta \theta^2)}{d\tau^2} + 2\alpha \frac{d(\delta \theta^2)}{d\tau} + \alpha^2 \delta \theta^2 = 0, \quad (65) \]
Note that in order to prove that our assumptions in (64) are correct, we will check \textit{a posteriori} its consistency. Integrating the system of differential equations (65), we obtain

\[
\delta \mu_1 (\tau) = (a_1 + a_2 \tau) e^{-\alpha \tau}, \quad \delta \mu_2 (\tau) = (a_3 + a_4 \tau) e^{-\alpha \tau} - \frac{1}{2\alpha} a_5 e^{-2\alpha \tau} + a_6, \quad (66)
\]

\[
\delta \sigma_2 (\tau) = (a_3 + a_4 \tau) e^{-\alpha \tau},
\]

where \(a_i, \, i = 1, \ldots, 6\) are integration constants. Note that conditions (64) are satisfied and therefore our assumption are compatible with the solutions obtained. Finally, consider the vector field components \(J^k \equiv \delta \theta^k\) defined in (55) and its magnitude \(J\),

\[
J^2 = g_{ij} J^i J^j. \quad (67)
\]

The magnitude \(J\) is called the Jacobi field intensity. In our case (67) becomes,

\[
J^2 \mathcal{M}_1 = \frac{1}{\mu_1^2} (\delta \mu_1)^2 + \frac{1}{\sigma_2^2} (\delta \mu_2)^2 + \frac{2}{\sigma_2^2} (\delta \sigma_2)^2. \quad (68)
\]

Substituting (53) and (66) in (68), and keeping the leading term in the asymptotic expansion in \(J^2 \mathcal{M}_1\), we obtain

\[
J \mathcal{M}_1 \approx C \mathcal{M}_1 e^{\alpha \tau}, \quad (69)
\]

where the constant coefficient \(C \mathcal{M}_1 = \frac{Aa_3}{2\sqrt{2\alpha}}\) encodes information about initial conditions and depends on the model parameter \(\alpha\). We conclude that the geodesic spread on \(\mathcal{M}_1\) is described by means of an \textit{exponentially divergent} Jacobi vector field intensity \(J \mathcal{M}_1\). It is known that classical chaotic systems exhibit exponential sensitivity to initial conditions. This characterization, quantified in terms of Lyapunov exponents, is an important ingredient in any conventional definition of classical chaos. In our approach, the quantity \(\lambda_J \approx \frac{1}{\tau} \lim_{\tau \to \infty} \ln \left[ \frac{|J(\tau)|}{|J(0)|} \right]\) with \(J\) given in (69) would play the role of the conventional Lyapunov exponents.

\section*{4.2. Instability in RED2}

In this subsection, the instability of the geodesics on \(\mathcal{M}_2\) is studied. We proceed as in subsection (4.1).
4.2.1. The Geodesic Length $\Theta_{\mathcal{M}_2}$

Consider the one-parameter family of geodesics $\mathcal{F}_{G_{\mathcal{M}_2}}(\alpha) \equiv \{\theta_{\mathcal{M}_2}^\mu(\tau; \alpha)\}_{\alpha \in \mathbb{R}^+}$ where $\theta_{\mathcal{M}_2}^\mu$ are solutions of (39). The length of geodesics in $\mathcal{F}_{G_{\mathcal{M}_2}}(\alpha)$ is defined as,

$$
\Theta_{\mathcal{M}_2}(\tau; \alpha) \equiv \int_0^\tau \sqrt{g_{ij}(d\theta^i d\theta^j)} = \int_0^\tau \left[ \frac{1}{\sigma_1^2} \left( \frac{d\mu_1}{d\tau'} \right)^2 + \frac{2}{\sigma_1^2} \left( \frac{d\sigma_1}{d\tau'} \right)^2 + \frac{1}{\sigma_2^2} \left( \frac{d\mu_2}{d\tau'} \right)^2 + \frac{2}{\sigma_2^2} \left( \frac{d\sigma_2}{d\tau'} \right)^2 \right]^\frac{1}{2} d\tau'.
$$

(70)

Substituting (40) in (70) and considering the asymptotic limit of $\Theta_{\mathcal{M}_2}(\tau; \alpha)$ when $\tau \to \infty$, we obtain,

$$
\Theta_{\mathcal{M}_2}(\tau \to \infty; \alpha) \equiv \Theta_2(\tau; \alpha) \approx 2\alpha \tau.
$$

(71)

In order to investigate the asymptotic behavior of two neighboring geodesics labelled by the parameters $\alpha$ and $\alpha + \delta \alpha$, we consider the following difference,

$$
\Delta \Theta_2 \equiv |\Theta_2(\tau; \alpha + \delta \alpha) - \Theta_2(\tau; \alpha)| = 2|\delta \alpha| \tau.
$$

(72)

It is clear that $\Delta \Theta_2$ diverges, that is the lengths of two neighboring geodesics with slightly different parameters $\alpha$ and $\alpha + \delta \alpha$ differ in a significant way as the evolution parameter $\tau \to \infty$. This hints at the onset of instability of the hyperbolic trajectories on $\mathcal{M}_2$.

4.2.2. The Statistical Volume Elements $V_{\mathcal{M}_2}$

The instability of ED2 can be explored by studying the behavior of the one-parameter family of statistical volume elements $\mathcal{F}_{V_{\mathcal{M}_2}}(\alpha) \equiv \{V_{\mathcal{M}_2}(\tau; \alpha)\}_\alpha$. Recall that $\mathcal{M}_2$ is the space of probability distributions $p^{(\text{tot})}(\vec{x}|\vec{\theta})$ labeled by parameters $\theta_1^{(1)}$, $\theta_2^{(1)}$, $\theta_1^{(2)}$, $\theta_2^{(2)}$. These parameters are the coordinates of the point $p^{(\text{tot})}$, and in these coordinates a 4D infinitesimal volume element $dV_{\mathcal{M}_2}$ reads,

$$
dV_{\mathcal{M}_2} = \sqrt{g} d\theta_1^{(1)} d\theta_2^{(1)} d\theta_1^{(2)} d\theta_2^{(2)} \equiv \sqrt{g} d\mu_1 d\sigma_1 d\mu_2 d\sigma_2,
$$

(73)

where in the ED2 model here presented, $g = |\det(g_{ij})_{\mathcal{M}_2}| = \frac{4}{\sigma_1^2 \sigma_2^2}$. Hence, the infinitesimal volume element $dV_{\mathcal{M}_2}$ is given by,

$$
dV_{\mathcal{M}_2} = \frac{2}{\sigma_1^2 \sigma_2^2} d\mu_1 d\sigma_1 d\mu_2 d\sigma_2.
$$

(74)
The volume of an extended region of $\mathcal{M}_2$ is defined by,

\[
\Delta V_{\mathcal{M}_2} \equiv V_{\mathcal{M}_2}(\tau) - V_{\mathcal{M}_2}(0) \equiv \int_{V_{\mathcal{M}_2}(0)}^{V_{\mathcal{M}_2}(\tau)} \left( \frac{\mu_1(\tau)\sigma_1(\tau) + \mu_2(\tau)\sigma_2(\tau)}{\sigma_1^2 \sigma_2^2} \right) d\mu_1 d\sigma_1 d\mu_2 d\sigma_2.
\]

Integrating (75) and using (40), we obtain

\[
\Delta V_{\mathcal{M}_2} = \frac{A^2}{2\alpha^2} e^{2\alpha \tau} - \frac{A^2}{2\alpha^2}. \tag{76}
\]

The average volume on $\mathcal{M}_2$ is $\langle \Delta V_{\mathcal{M}_2} \rangle_\tau$,

\[
\langle \Delta V_{\mathcal{M}_2} \rangle_\tau \equiv \frac{1}{\tau} \int_0^\tau \Delta V_{\mathcal{M}_2}(\tau'; \alpha) d\tau' = \frac{A^2}{4\alpha^4} e^{2\alpha \tau} - \frac{A^2}{2\alpha^2}. \tag{77}
\]

For convenience, let us rename $\langle \Delta V_{\mathcal{M}_2} \rangle_\tau \equiv \Delta V_2$. Therefore, the asymptotic expansion of $\Delta V_2$ for $\tau \to \infty$ reads,

\[
\Delta V_2 \approx \frac{A^2}{4\alpha^4} e^{2\alpha \tau}. \tag{78}
\]

In analogy to (51) we introduce,

\[
S_2 \equiv \ln(\Delta V_2). \tag{79}
\]

Substituting (78) in (79) and considering its asymptotic limit, we obtain

\[
S_2 \approx 2\alpha \tau. \tag{80}
\]

4.2.3. The Jacobi Vector Field $J_{\mathcal{M}_2}$

We proceed as in (4.1.3). Study the behavior of the one-parameter ($\alpha$) family of neighboring geodesics on $\mathcal{M}_2$, $\{\theta^i(\tau; \alpha)\}_{i=1,2,3,4}$ with

\[
\theta^3(\tau; \alpha) \equiv \theta^1(\tau; \alpha) = \mu_1(\tau; \alpha) = \frac{A^2}{2\alpha} e^{-2\alpha \tau} + \frac{A^2}{8\alpha^2}, \tag{81}
\]

\[
\theta^4(\tau; \alpha) \equiv \theta^2(\tau; \alpha) = A \frac{e^{-\alpha \tau}}{e^{-2\alpha \tau} + \frac{A^2}{8\alpha^2}}. \tag{82}
\]

Note that because we will compare the two Jacobi fields $J_{\mathcal{M}_1}$ on $\mathcal{M}_1$ and $J_{\mathcal{M}_2}$ on $\mathcal{M}_2$, we assume the same initial conditions as considered in (4.1.3). Recall that the non-vanishing Riemann tensor components are $R_{1212} = -\frac{1}{\sigma_1^2}$ and $R_{3434} = -\frac{1}{\sigma_2^2}$ given in (27).
Therefore two of the four differential equations describing the geodesic spread are,

\[
\frac{d^2(\delta \theta^1)}{d\tau^2} + 2 \left[ \Gamma^{12}_{12} \frac{d^2(\delta \theta^1)}{d\tau^2} + \Gamma^{11}_{21} \frac{d\theta^1}{d\tau} \frac{d(\delta \theta^1)}{d\tau} \right] + \partial_2 \Gamma^{12}_{12} \left( \frac{d\theta^2}{d\tau} \right)^2 \delta \theta^1 + \Gamma^{12}_{22} \left( \frac{d\theta^2}{d\tau} \right)^2 \delta \theta^1 = 0,
\]

(83)

\[
\frac{d^2(\delta \theta^2)}{d\tau^2} + 2 \left[ \Gamma^{22}_{12} \frac{d^2(\delta \theta^2)}{d\tau^2} + \Gamma^{22}_{22} \frac{d\theta^2}{d\tau} \frac{d(\delta \theta^2)}{d\tau} \right] + \partial_2 \Gamma^{22}_{22} \left( \frac{d\theta^1}{d\tau} \right)^2 \delta \theta^2 + \Gamma^{22}_{12} \frac{d\theta^1}{d\tau} \frac{d\theta^2}{d\tau} \delta \theta^1 = 0.
\]

(84)

The other two equations can be obtained from (83) and (84) substituting the index 1 with 3 and 2 with 4. Thus, we will limit our considerations just to the above two equations. Using equations (25), (27), (81) and (82) in (83) and (84) and considering the asymptotic limit \( \tau \to \infty \), the two equations of geodesic deviation become,

\[
\frac{d^2(\delta \theta^1)}{d\tau^2} + 2\alpha \frac{d(\delta \theta^1)}{d\tau} + \frac{16\alpha^2}{A} e^{-\alpha \tau} \frac{d(\delta \theta^2)}{d\tau} + \left( \alpha^2 - \frac{8\alpha^3}{A} e^{-\alpha \tau} \right) \delta \theta^2 = 0,
\]

(85)

\[
\frac{d^2(\delta \theta^2)}{d\tau^2} + 2\alpha \frac{d(\delta \theta^2)}{d\tau} + \left( \alpha^2 - \frac{32\alpha^4}{A^2} e^{-2\alpha \tau} \right) \delta \theta^2 - \frac{8\alpha^2}{A} e^{-\alpha \tau} \frac{d(\delta \theta^1)}{d\tau} - \frac{8\alpha^3}{A} e^{-\alpha \tau} \delta \theta^1 = 0.
\]

(86)

Neglecting the exponentially decaying terms in \( \delta \theta^2 \) in (85) and (86) and assuming

\[
\lim_{\tau \to \infty} \left( \frac{16\alpha^2}{A} e^{-\alpha \tau} \frac{d(\delta \theta^2)}{d\tau} \right) = 0, \quad \lim_{\tau \to \infty} \left( \frac{8\alpha^2}{A} e^{-\alpha \tau} \frac{d(\delta \theta^1)}{d\tau} \right) = 0, \quad \lim_{\tau \to \infty} \left( \frac{8\alpha^3}{A} e^{-\alpha \tau} \delta \theta^1 \right) = 0
\]

(87)

the geodesic deviation equations in (85) and (86) become,

\[
\frac{d^2(\delta \theta^1)}{d\tau^2} + 2\alpha \frac{d(\delta \theta^1)}{d\tau} + \alpha^2 \delta \theta^2 = 0, \quad \frac{d^2(\delta \theta^2)}{d\tau^2} + 2\alpha \frac{d(\delta \theta^2)}{d\tau} + \alpha^2 \delta \theta^2 = 0.
\]

(88)

The consistency of the assumptions in (87) will be checked \( a \ posteriori \) after integrating equations in (88). It follows that the geodesics spread on \( M_2 \) is described by the temporal evolution of the following deviation vector components,

\[
\delta \mu_1 (\tau) = (a_1 + a_2 \tau) e^{-\alpha \tau} - \frac{1}{2\alpha} a_3 e^{-2\alpha \tau} + a_4, \quad \delta \sigma_1 (\tau) = (a_1 + a_2 \tau) e^{-\alpha \tau}
\]

(89)

\[
\delta \mu_2 (\tau) = (a_5 + a_6 \tau) e^{-\alpha \tau} - \frac{1}{2\alpha} a_7 e^{-2\alpha \tau} + a_8, \quad \delta \sigma_2 (\tau) = (a_5 + a_6 \tau) e^{-\alpha \tau}
\]
where $a_i$, $i = 1, \ldots, 8$ are integration constants. Note that $a_4$ and $a_8$ in (89) equal $a_6$ in (66). Furthermore, note that these solutions above are compatible with the assumptions in (87). Finally, consider the Jacobi vector field intensity $J_{\mathcal{M}_2}$ on $\mathcal{M}_2$,

$$J^2_{\mathcal{M}_2} = \frac{1}{\sigma_1^2} (\delta \mu_1)^2 + \frac{2}{\sigma_1^2} (\delta \sigma_1)^2 + \frac{1}{\sigma_2^2} (\delta \mu_2)^2 + \frac{2}{\sigma_2^2} (\delta \sigma_2)^2. \quad (90)$$

Substituting (81), (82) and (89) in (90) and keeping the leading term in the asymptotic expansion in $J^2_{\mathcal{M}_2}$, we obtain

$$J_{\mathcal{M}_2} \approx C_{\mathcal{M}_2} e^{\alpha \tau}. \quad (91)$$

where the constant coefficient $C_{\mathcal{M}_2} = \frac{Aa_3}{\sqrt{2\alpha}} \equiv 2C_{\mathcal{M}_1}$ encodes information about initial conditions and it depends on the model parameter $\alpha$. We conclude that the geodesic spread on $\mathcal{M}_2$ is described by means of an exponentially divergent Jacobi vector field intensity $J_{\mathcal{M}_2}$.

5. STATISTICAL CURVATURE, JACOBI FIELD INTENSITY AND ENTROPY-LIKE QUANTITIES

It is known that statistical manifolds allow differential geometric methods to be applied to problems in mathematical statistics, information theory and in stochastic processes. For instance, an important class of statistical manifolds is that arising from the so-called exponential family [8] and one particular family is that of gamma probability distributions. These distributions have been shown [19] to have important uniqueness properties for near-random stochastic processes. In this paper, two statistical manifolds of negative curvature $\mathcal{M}_1$ and $\mathcal{M}_2$ have been considered. They are representations of smooth families of probability distributions (exponentials and Gaussians for $\mathcal{M}_1$, Gaussians for $\mathcal{M}_2$). They represent the "arena" where the entropic dynamics takes place. The instability of the trajectories of the ED1 and ED2 on $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, has been studied using the statistical weight $\langle \Delta V_{\mathcal{M}_1} \rangle_\tau$ defined on the curved manifold $\mathcal{M}_1$ and the Jacobi vector field intensity $J_{\mathcal{M}_1}$. Does our analysis lead to any possible further understanding of the role of statistical curvature in physics and statistics? We argue that it does.

The role of curvature in physics is fairly well understood. It encodes information about the field strengths for all the four fundamental interactions in nature. The curvature plays a key role in the Riemannian geometric approach to chaos [20]. In this approach, the study of the Hamiltonian dynamics is reduced to the investigation of geometrical properties of the manifold on which geodesic flow is induced. For instance, the stability or local instability of geodesic flows depends on the sectional curvature properties of the suitable defined metric manifold. The sectional curvature brings the same qualitative and quantitative information that is provided by the Lyapunov exponents in the conventional approach. Furthermore, the integrability of the system is connected with existence of Killing vectors on the manifold. However, a rigorous relation among curvature, Lyapunov exponents and Kolmogorov-Sinay entropy [21] is still under investigation. In addition, there does not exist a well defined unifying characterization of
chaos in classical and quantum physics [22] due to fundamental differences between the two theories. In addition, the role of curvature in statistical inference is even less understood. The meaning of statistical curvature for a one-parameter model in inference theory was introduced in [23]. Curvature served as an important tool in the asymptotic theory of statistical estimation. The higher the scalar curvature at a given point on the manifold, the more difficult it is to do estimation at that point [24].

Recall that the entropy-like quantity $S$ is the asymptotic limit of the natural logarithm of the average portion of the statistical volume $\langle \Delta V_{M_s} \rangle_\tau$ associated to the evolution of the geodesics on $M_s$. Considering equations (52) and (80), we obtain,

\[ S_2 \approx 2S_1. \]  

Furthermore, the relationship between the statistical curvatures on the curved manifolds $M_{s_1}$ and $M_{s_2}$ is,

\[ R_{M_{s_2}} = 2R_{M_{s_1}}. \]  

In view of (92) and (93), it follows that there is a direct proportionality between the curvature $R_{M_s}$ and the asymptotic expression for the entropy-like quantity $S$ characterizing the ED on manifolds $M_{s_i}$ with $i = 1, 2$, namely

\[ \frac{R_{M_{s_2}}}{R_{M_{s_1}}} = \frac{S_2}{S_1}. \]

Moreover, from (69) and (91), we obtain the following relation,

\[ J_{M_{s_2}} \approx 2J_{M_{s_1}}. \]

The two manifolds $M_{s_1}$ and $M_{s_2}$ are exponentially unstable and the intensity of Jacobi vector field $J_{M_{s_2}}$ of manifold $M_{s_2}$ with curvature $R_{M_{s_2}} = -2$ is asymptotically twice the intensity of the Jacobi vector field $J_{M_{s_1}}$ of manifold $M_{s_1}$ with curvature $R = -1$. Considering (93) and (95), we obtain

\[ \frac{R_{M_{s_2}}}{R_{M_{s_1}}} = \frac{J_{M_{s_2}}}{J_{M_{s_1}}}. \]

It seems there exists a direct proportionality between the curvature $R_{M_s}$ and the intensity of the Jacobi field $J_{M_s}$ characterizing the degree of chaoticity of a statistical manifold of negative curvature $M_s$. Finally, comparison of (94) and (96) leads to the formal link between curvature, entropy and chaoticity:

\[ R \sim S \sim J. \]

Though several points need deeper understanding and analysis, we hope that our work convincingly shows that this information-geometric approach may be useful in providing a unifying criterion of chaos of both classical and quantum varieties, thus deserving further research and developments.
6. CONCLUSIONS

Two entropic dynamical models have been considered: a $3D$ and $4D$ statistical manifold $\mathcal{M}_{s1}$ and $\mathcal{M}_{s2}$ respectively. These manifolds serve as the stage on which the entropic dynamics takes place. In the former case, macro-coordinates on the manifold are represented by the expectation values of microvariables associated with Gaussian and exponential probability distributions. In the latter case, macro-coordinates are expectation values of microvariables associated with two Gaussians distributions. The geometric structure of $\mathcal{M}_{s1}$ and $\mathcal{M}_{s2}$ was studied in detail. It was shown that $\mathcal{M}_{s1}$ is a curved manifold of constant negative curvature $-1$ while $\mathcal{M}_{s2}$ has constant negative curvature $-2$. The geodesics of the ED models are hyperbolic curves on $\mathcal{M}_{si}$ ($i = 1, 2$). A study of the stability of geodesics on $\mathcal{M}_{s1}$ and $\mathcal{M}_{s2}$ was presented. The notion of statistical volume elements was introduced to investigate the asymptotic behavior of a one-parameter family of neighboring volumes $\mathcal{F}_{\mathcal{M}_{si}}(\alpha) \equiv \{V_{\mathcal{M}_{si}}(\tau; \alpha)\}_\alpha$. An information-geometric analog of the Zurek-Paz chaos criterion was presented. It was shown that the behavior of geodesics is characterized by exponential instability that leads to chaotic scenarios on the curved statistical manifolds. These conclusions are supported by a study based on the geodesic deviation equations and on the asymptotic behavior of the Jacobi vector field intensity $J_{\mathcal{M}_{si}}$ on $\mathcal{M}_{s1}$ and $\mathcal{M}_{s2}$. A Lyapunov exponent analog similar to that appearing in the Riemannian geometric approach was suggested as an indicator of chaos.

On the basis of our analysis a relationship among an entropy-like quantity, chaoticity and curvature in the two models ED1 and ED2 is proposed, suggesting to interpret the statistical curvature as a measure of the entropic dynamical chaoticity.

The implications of this work is twofold. Firstly, it helps understanding possible future use of the statistical curvature in modelling real processes by relating it to conventionally accepted quantities such as entropy and chaos. On the other hand, it serves to cast what is already known in physics regarding curvature in a new light as a consequence of its proposed link with inference. Finally we remark that based on the results obtained from the chosen ED models, it is not unreasonable to think that should the correct variables describing the true degrees of freedom of a physical system be identified, perhaps deeper insights into the foundations of models of physics and reasoning (and their relationship to each other) may be uncovered.

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