The Convergence of Yang-Mills Integrals

Peter Austing

Department of Physics, University of Oxford
Theoretical Physics,
1 Keble Road,
Oxford OX1 3NP, UK
E-mail: p.austing@physics.ox.ac.uk

John F. Wheater

Department of Physics, University of Oxford
Theoretical Physics,
1 Keble Road,
Oxford OX1 3NP, UK
E-mail: j.wheater@physics.ox.ac.uk

Abstract: We prove that $SU(N)$ bosonic Yang-Mills matrix integrals are convergent for dimension (number of matrices) $D \geq D_c$. It is already known that $D_c = 5$ for $N = 2$; we prove that $D_c = 4$ for $N = 3$ and that $D_c = 3$ for $N \geq 4$. These results are consistent with the numerical evaluations of the integrals by Krauth and Staudacher.

Keywords: Yang-Mills, supersymmetry, D-brane.
1. Introduction

The discovery of D-branes and the realization of their importance in string theory and M-theory has led to a number of exciting conjectures relating these very complicated theories to (at least technically) much simpler M(atrix) theories [1, 2] (for a review see [3]). This has generated renewed interest in supersymmetric Yang-Mills quantum mechanics [4, 5, 6, 7, 8] which is obtained by the the dimensional reduction of $D = 10$ supersymmetric Yang-Mills gauge theory (SSYM) to 1 remaining space-time dimension. Further dimensional reduction to 0 dimensions leads to the supersymmetric Yang-Mills matrix integrals which are an important component in calculating the Witten index for the quantum mechanics. Both the quantum mechanics and the matrix integrals exist for any dimension $D$ in which the original SSYM exists, ie $D = 3, 4, 6,$ and $10$. Dropping the supersymmetry requirement leads to the bosonic Yang-Mills matrix integrals which exist for any $D$ and are the main subject of this paper.

A couple of years ago Moore, Nekrasov and Shatashvili [9] found a way of calculating the partition function by deforming the Yang-Mills matrix integrals to a
cohomological theory. Their method does not allow the calculation of arbitrary correlation functions in the original Yang-Mills picture where it is not known how to do exact calculations unless the gauge group is $SU(2)$ [10]. However it is possible to do numerical calculations provided the gauge group is not too big [11, 12, 13]. Partition functions, some correlation functions, and eigenvalue distributions have been found for $SU(N)$ with $N = 3, 4, 5$; where comparison is possible the results agree with the cohomological calculations in [9]. In the course of this work Krauth and Staudacher [12] also investigated the properties of purely bosonic Yang-Mills integrals (these are defined below). It had been believed that the flat directions in the action would cause these integrals to diverge (their supersymmetric cousins being saved by the Pfaffian arising from the integration of the fermions which vanishes along the flat directions). Simple analytic calculations of the partition functions in the case of $SU(2)$, and delicate numerical computations for $SU(3)$, $SU(4)$ and $SU(5)$ (and subsequently for other gauge groups as well [14]) showed that this is not necessarily the case. Unfortunately up to now an analytic demonstration of the convergence of these integrals for $SU(N > 2)$ has been lacking. Our purpose here is to provide such a demonstration.

The bosonic Yang-Mills partition function for gauge group $G$ in $D$ “space-time” dimensions $^1$ dimensionally reduced to 0 is given by

$$Z_{D,G} = \prod_{\mu=1}^{D} \int_{-\infty}^{\infty} dX_{\mu} \exp \left( \sum_{\mu > \nu} \text{Tr} \left[ X_{\mu}, X_{\nu} \right]^2 \right)$$

(1.1)

where the matrices $\{X_{\mu}, \mu = 1, \ldots D\}$, which are traceless and hermitian, take values in the Lie algebra of $G$ and can be written

$$X_{\mu} = \sum_{a=1}^{g} X_{\mu}^{a} t^{a}.$$  

(1.2)

The $\{t^{a}, a = 1, \ldots g\}$ are the generators in the fundamental representation satisfying

$$\text{Tr} t^{a} t^{b} = 2\delta^{ab}$$

(1.3)

and we shall use $l$ to denote the rank of the Lie algebra. In this paper we will restrict ourselves to the groups $SU(N)$. The measure and the integrand in (1.1) are then invariant under the $SU(N)$ gauge symmetry

$$X_{\mu} \rightarrow U^{\dagger} X_{\mu} U, \quad U \in SU(N)$$

(1.4)

and the $SO(D)$ symmetry

$$X_{\mu} \rightarrow \sum_{\nu} Q_{\mu\nu} X_{\nu}, \quad Q \in SO(D).$$

(1.5)

$^1$In the bosonic case there is no requirement of supersymmetry to restrict $D$ so it is possible to consider $D$ as a continuous variable by analytic continuation; however in this paper it is to be taken strictly as an integer.
We will prove that $Z_{D,SU(N)}$ is convergent for dimension $D \geq D_c$ and divergent for $D < D_c$. It is already known from exact calculation that $D_c = 5$ for $N = 2$ (although we will show that our methods reproduce this almost trivially). We prove that $D_c = 4$ for $N = 3$ and that $D_c = 3$ for $N \geq 4$.

The body of this paper is concerned with establishing which integrals converge. In section 2 we set up our procedure and establish some results that are useful in every case. Section 3 deals with $SU(2)$, section 4 with $SU(3)$, and section 5 with $SU(N > 3)$. In Appendix A we show which integrals diverge. We conclude with a brief discussion of the implications of our results for the supersymmetric theories and other gauge groups.

2. Preliminaries

The dangerous regions which might cause the integral (1.1) to diverge are where all the commutators almost vanish but the magnitude of $X_\mu$ goes to infinity. Hence we let

$$X_\mu = Rx_\mu, \quad \text{Tr} x_\mu x_\mu = 1 \quad (2.1)$$

where, as from now on, we use the summation convention for Greek indices. Then we have

$$Z_{D,G} = \int_0^{\infty} R^{Dg-1} X_{D,G}(R) dR \quad (2.2)$$

where

$$X_{D,G}(R) = \prod_{\nu=1}^D \int dx_\nu \delta(1 - \text{Tr} x_\mu x_\mu) \exp \left( -R^4 S \right) \quad (2.3)$$

and

$$S = -\frac{1}{2} \text{Tr} [x_\mu, x_\nu] [x_\mu, x_\nu]$$

$$= \frac{1}{2} \sum_{i,j,\mu,\nu} \left| [x_\mu, x_\nu]_{i,j} \right|^2. \quad (2.4)$$

We note that for any finite $R$ the integral $X_{D,G}(R)$ is bounded by a constant (since every term in the argument of the exponential is negative semi-definite) and therefore if for large $R$

$$X_{D,G}(R) < \frac{\text{const}}{R^\alpha}, \quad \text{where } \alpha > Dg, \quad (2.5)$$

then the partition function $Z_{D,G}$ is finite. Our tactic for proving convergence of $Z_{D,G}$ is therefore to find a bound of the form (2.5) on $X_{D,G}(R)$.

Now we split the integration region in (2.3) into two

$$\mathcal{R}_1 : S < (R^{-2-\eta})^2$$

$$\mathcal{R}_2 : S \geq (R^{-2-\eta})^2 \quad (2.6)$$
where $\eta$ is small but positive. We see immediately that the contribution to $I_{D,G}(R)$ from $R^2$ is bounded by $A_1 \exp(-R^{2\eta})$ (we will use the capital letters $A$, $B$ and $C$ to denote constants throughout this paper) and thus automatically satisfies $\frac{2.3}{2.3}$. Thus we can confine our efforts to the contribution from $R_1$ in which we replace the exponential function by unity to get the bound

\[ X_{D,G}(R) < A_1 \exp(-R^{2\eta}) + I_{D,G}(R) \quad (2.7) \]

where

\[ I_{D,G}(R) = \prod_{\nu=1}^{D} \int_{R_1} dx_\nu \delta(1 - \text{Tr} x_\mu x_\mu). \quad (2.8) \]

Since $S$ is a sum of squares it follows that the region $R_1'$ defined by

\[ \left| [x_\mu, x_\nu]_{ij} \right| < R^{-(2-\eta)}, \quad \forall \mu, \nu, i, j \quad (2.9) \]

is larger than the region $R_1$ which we can therefore replace in $\frac{2.7}{2.7}$ by $R_1'$.

Now we utilise the $SU(N)$ symmetry to diagonalise $x_1$ which we may therefore write as $x_1 = \text{diag}(\lambda_1, \ldots, \lambda_N)$. The constraint $\frac{2.9}{2.9}$ then becomes

\[ |(\lambda_i - \lambda_j)(x_\nu)_{ij}| < R^{-(2-\eta)} \quad (2.10) \]

This immediately leads to the generic case which is when the eigenvalues of $x_1$ are not degenerate

\[ |\lambda_i - \lambda_j| > \epsilon \quad (2.11) \]

where $\epsilon$ is a constant which we may choose but will always be finite. Then all the off-diagonal elements of $\{x_\nu, \nu = 2, \ldots, D\}$ are bounded by $\frac{2.10}{2.10}$ and $I_{D,G}(R)$ $\frac{2.8}{2.8}$ has a contribution $I_{D,G}^{\text{gen}}(R)$ which is bounded

\[ I_{D,G}^{\text{gen}}(R) < A_2 R^{-(2-\eta)(D-1)(g-1)} \quad (2.12) \]

where the constant $A_2$ comes from the integral over all the diagonal elements (and does of course depend on $\epsilon$). Note that once the off-diagonal elements are bounded by $\frac{2.10}{2.10}$ and $\frac{2.11}{2.11}$ all commutators are constrained to be $O(R^{-(2-\eta)})$ although the coefficient may be more than 1. Enforcing $\frac{2.9}{2.9}$ implies constraints on the diagonal elements which lower $A_2$ in $\frac{2.12}{2.12}$ but does not affect the power of $R$.

3. $SU(2)$

In the case of $SU(2)$ each matrix $x_\mu$ has eigenvalues $\lambda_\mu$ and $-\lambda_\mu$. However the constraint in $\frac{2.11}{2.11}$ implies that

\[ 2\lambda_\mu \lambda_\mu = 1 \quad (3.1) \]
and it follows that there must always be one matrix with eigenvalues of magnitude at least \((2D)^{-\frac{1}{2}}\). We can choose this matrix to be \(x_1\) and hence by taking \(\epsilon < 2(2D)^{-\frac{1}{2}}\) we always have the generic case. Hence

\[
\mathcal{X}_{D,SU(2)}(R) < A_1 \exp(-R^{2\eta}) + A_2 R^{-2(2-\eta)(D-1)}
\]  

(3.2)

and it follows immediately that \(\mathcal{Z}_{D,SU(2)}\) is finite for \(D \geq 5\). We show in Appendix A that \(\mathcal{Z}_{D,SU(2)}\) is divergent for smaller \(D\). Of course these results are well known because \(\mathcal{Z}_{D,SU(2)}\) can be calculated exactly.

4. \(SU(3)\)

From the constraint in 2.1 it follows that there must always be one matrix with an eigenvalue of magnitude at least \((3D)^{-\frac{1}{2}}\); as before we choose this matrix to be \(x_1\). When all the eigenvalues of \(x_1\) are separated by at least \(\epsilon\) (2.11) we get the generic contribution 2.12 to the integral. However now we have the new possibility that two of the eigenvalues of \(x_1\) become degenerate (it is not possible for all three eigenvalues to become degenerate provided we choose \(\epsilon < \frac{1}{2}(3D)^{-\frac{1}{2}}\)) in which case the condition 2.11 does not apply and we have to proceed more carefully; \(I_{D,SU(3)}(R)\) is made up of a piece where 2.11 applies to all eigenvalues plus a new piece \(I^{\text{deg}}_{D,SU(3)}(R)\) from the region of integration where 2.11 is not satisfied by all eigenvalues,

\[
I_{D,SU(3)}(R) = I^{\text{gen}}_{D,SU(3)}(R) + I^{\text{deg}}_{D,SU(3)}(R).
\]  

(4.1)

First we write

\[
x_1 = \frac{\rho_1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} x_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}
\]  

(4.2)

where \(x_1 = \text{diag}(\xi, -\xi)\); we know that \(|2\rho_1/\sqrt{3}| > (3D)^{-\frac{1}{2}}\) and, because we are just interested in the case of degenerate eigenvalues, \(2|\xi| < \epsilon\). Because only two of the three eigenvalues of \(x_1\) are degenerate the constraint 2.10 bounds some of the off-diagonal elements of \(x_2,...,D\), so we can write these matrices in the form

\[
x_\nu = \frac{\rho_\nu}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} \tilde{x}_\nu & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & O(R^{-2-\eta}) \\ 0 & 0 & O(R^{-2-\eta}) \\ O(R^{-2-\eta}) & O(R^{-2-\eta}) & 0 \end{pmatrix}
\]  

(4.3)

where \(\tilde{x}_\nu\) is a \(2 \times 2\) traceless hermitian matrix (ie it lives in the \(su(2)\) sub-algebra of the \(su(3)\) algebra inhabited by \(x_\nu\)), and by \(O(R^{-2-\eta})\) we mean that the elements are bounded by 2.10. At this stage the off-diagonal elements in the third row and column of the \(x_{2,...,D}\) are innocuous and can be integrated out to get

\[
I^{\text{deg}}_{D,SU(3)}(R) < B_1 R^{-(2-\eta)(4(D-1)-2)} \left( \int_{|\rho_1|>4D} \prod_{\mu=1}^{D} d\rho_\mu \right) \left( \int_{\tilde{R}} \prod_{\nu=2}^{D} d\tilde{x}_\nu \right) (2\xi)^2
\]
\[ \times \Omega_{SU(2)}(\sqrt{3}\rho_1 - \xi)^2(\sqrt{3}\rho_1 + \xi)^2 \theta(1 - (\text{Tr} \tilde{x}_\alpha \tilde{x}_\alpha + 2\rho_\alpha \rho_\alpha)) \times \theta\left((\text{Tr} \tilde{x}_\alpha \tilde{x}_\alpha + 2\rho_\alpha \rho_\alpha) - \left(1 - 4(\epsilon R^{2-\eta})^{-2}\right)\right) \]  

(4.4)

where \( \theta \) denotes the step function, we have included the Vandermonde determinant for \( x_1 \), and \( \Omega_{SU(2)} \) is the volume of \( SU(2) \). The region \( \tilde{\mathcal{R}} \) is defined by \( |\xi| < \epsilon/2 \) and \( |\rho_1| > (4D)^{-\frac{1}{2}} \). Doing the \( \rho_\mu \) integrals we are left with

\[ I_{D, SU(2)}^{\text{deg}}(R) < B_2 R^{-(2-\eta)4(D-1)} F_{D, SU(2)}^{\text{deg}}(R) \]  

(4.6)

where

\[ F_{D, SU(2)} = \left( \int_{\tilde{\mathcal{R}}} d\xi \prod_{\nu=2}^{D} d\tilde{\mu} \right) (2\xi)^2 \Omega_{SU(2)} \theta(1 - \text{Tr} \tilde{x}_\mu \tilde{x}_\mu) \]

\[ = (2-\eta) R^{-(2-\eta)} \int_0^R du u^{1-\eta} \left( \int_{\tilde{\mathcal{R}}} \prod_{\nu=1}^{D} d\tilde{\mu} \right) \delta \left( \left[ \frac{u}{R} \right]^{2-\eta} - \text{Tr} \tilde{x}_\mu \tilde{x}_\mu \right) \]  

(4.7)

Making the rescaling \( \tilde{x}_\mu = \gamma_\mu [u/R]^{1-\eta/2} \) we find that

\[ F_{D, SU(2)} = R^{-(2-\eta)3D/2} \int_0^R u^{3D(1-\eta/2)-1} I_{D, SU(2)}^{\text{deg}}(u) du. \]  

(4.8)

As \( R \to \infty \) we use the results from section 3 to bound the remaining integral giving

\[ F_{D, SU(2)} < B_3 R^{-(2-\eta)3D/2}, \quad D \geq 5 \]

\[ B_4 R^{-(2-\eta)3D/2} \log R, \quad D = 4 \]

\[ B_5 R^{-(2-\eta)(3D-1)/2}, \quad D = 3 \]  

(4.9)

and so we find that

\[ I_{D, SU(3)}^{\text{deg}}(R) < B_6 \left( \frac{1}{R^{2-\eta}} \right)^{4(D-1)+(3D-\delta_{D,3})/2} \] \[ \times (\log R)^{\delta_{D,4}}. \]  

(4.10)

From 2.12 we know that

\[ I_{D, SU(3)}^{\text{gen}}(R) < B_7 R^{-(2-\eta)6(D-1)}. \]  

(4.11)

Applying the criterion 2.3 we see that both \( I_{D, SU(3)}^{\text{deg}}(R) \) and \( I_{D, SU(3)}^{\text{gen}}(R) \) make a finite contribution to \( Z_{D, SU(3)} \) only for \( D \geq 4 \). It is straightforward to check that \( Z_{D=3, SU(3)} \) diverges (see appendix). Thus we conclude that \( Z_{D, SU(3)} \) is finite for \( D \geq 4 \) only.
5. $N > 3$

The argument for higher $N$ has the same structure as for $SU(3)$. Again there is a generic contribution which is bounded as shown in 2.12 and a degenerate contribution. As usual $x_1$ can be chosen so that at least one of its eigenvalues has magnitude $(ND)^{-\frac{1}{2}}$ or more. The difference is that $x_1$ can have not only two degenerate but $2, 3, \ldots N-1$ and two or more sets of them. To deal with this we note that whenever $x_1$ has some configuration of exactly degenerate eigenvalues there is a sub-algebra of $su(N)$ which commutes with $x_1$. (As we saw in section 4 for the case of $SU(3)$ the only possibility is to have two degenerate eigenvalues and the sub-algebra is $su(2)$.) Suppose that $x_1$ has $K$ sets of degenerate eigenvalues with degeneracies $\{N_1, \ldots N_K\}$; then the sub-algebra which commutes with $x_1$ is

$$H = \bigoplus_{k=1}^{K} su(N_k)$$

(5.1)

Now let $\mathcal{P}^a$ denote those generators of $G$ that do not lie in $H$, and decompose $x_\mu$ into

$$x_1 = \text{diag} \left( \rho_1 I_{N_1}, \ldots, \rho_K I_{N_K}, \sigma_1^1 \ldots \sigma_1^M \right) + \text{diag} \left( \tilde{x}_1^1, \ldots, \tilde{x}_1^K, 0 \ldots 0 \right)$$

$$x_{\mu>1} = \text{diag} \left( \rho_1 I_{N_1}, \ldots, \rho_K I_{N_K}, \sigma_\mu^1 \ldots \sigma_\mu^M \right) + \text{diag} \left( \tilde{x}_\mu^1, \ldots, \tilde{x}_\mu^K, 0 \ldots 0 \right)$$

$$+ \sum_a \tilde{x}_\mu^a \mathcal{P}^a.$$

(5.2)

Here $I_N$ denotes the $N \times N$ Identity matrix, $\tilde{x}_\mu^k$ lies in the Lie algebra $su(N_k)$ with $\tilde{x}_1^k$ diagonal, $M$ is given by

$$M = N - \sum_{k=1}^{K} N_k$$

(5.3)

and the tracelessness condition is

$$0 = \sum_{k=1}^{M} \sigma_\mu^k + \sum_{k=1}^{K} N_k \rho_\mu^k.$$

(5.4)

Note that the ordering of $\rho$s and $\sigma$s in (5.2) is not significant; for example we could chose to take $x_1$ so that the elements are in decreasing order down the diagonal and then degenerate and non-degenerate eigenvalues would be all mixed up in general. If $\tilde{x}_1^{1\ldots K} = 0$ then the eigenvalues of $x_1$ are exactly degenerate; we have $K$ blocks of eigenvalues $\rho_1^{1\ldots K}$ with degeneracy $N_k$ and singleton eigenvalues $\sigma_1^{1\ldots M}$. When the exact degeneracy is relaxed slightly we have blocks of eigenvalues $\lambda_m^k, m = 1, \ldots N_k$ with each block having central value

$$\frac{1}{N_k} \sum_{m=1}^{N_k} \lambda_m^k = \rho_1^k.$$

(5.5)
together with the singleton eigenvalues \( \lambda_1^{(K+j)} = \sigma_j^i, j = 1, \ldots, M. \)

For each sub-algebra \( H \) there is a contribution \( \mathcal{I}_{D,SU(N)}^H(R) \) to \( \mathcal{I}_{D,SU(N)}^{deq}(R) \). This comes from the integration region \( \mathcal{P} \) where the \( \rho_k^i, \sigma_1^j, \bar{x}_k^i \) are such that the eigenvalues of \( x_1 \) satisfy

\[
\mathcal{P} : \begin{align*}
|\lambda_m^k - \lambda_{m'}^{k'}| &> \epsilon, \quad k \neq k', \ m = 1, \ldots, N_k, \ m' = 1, \ldots, N_{k'} \\
|\lambda_m^k - \lambda_{m+1}^k| &\leq \epsilon, \quad m = 1, \ldots, N_k - 1.
\end{align*}
\tag{5.6}
\]

We note that any sequence of eigenvalues can be arranged in the manner implied by (5.6) for some \( H \). Therefore by considering the \( \mathcal{I}_{D,SU(N)}^H(R) \) for all possible \( H \) we exhaust all possible nearly degenerate eigenvalue configurations for \( x_1 \). That is to say the degenerate term in \( \mathcal{I}_{D,SU(N)}(R) \) is now the sum of contributions from all the possible \( H \)s

\[
\mathcal{I}_{D,SU(N)}^{deq}(R) = \sum_H \mathcal{I}_{D,SU(N)}^H(R). \tag{5.7}
\]

Now we bound the \( \mathcal{I}_{D,SU(N)}^H(R) \). The Vandermonde determinant for \( x_1 \) is easily bounded from above through

\[
\Delta = \prod_{k > k', m, m'} (\lambda_m^k - \lambda_{m'}^{k'})^2 \prod_{k, m > m'} (\lambda_m^k - \lambda_{m'}^k)^2 < 2^{n_H} \prod_{k, m > m'} (\lambda_m^k - \lambda_{m'}^k)^2
\]

\[
n_H = N(N - 1) - \sum_{k=1}^K N_k(N_k - 1) \tag{5.8}
\]

where we have used the fact that none of the eigenvalues can have magnitude more than 1 on account of the constraint (2.1). Of course this bound is simply the product of the Vandermonde determinants for the constituent \( su(N_k) \) factors of the sub-algebra \( H \). Now we note that the \( \tau_\mu^a \), which are those off-diagonal elements of \( x_{2,\ldots,D} \) that do not lie in \( H \), are constrained by (2.10) and, following our procedure in the \( SU(3) \) case, we integrate them out to get

\[
\mathcal{I}_{D,SU(N)}^H(R) < C_1 R^{-(2-\eta)((D-1)n_H - 2)} \int_{\mathcal{P}} \prod_{\mu=1}^D \left( \prod_{k=1}^K d\rho_k^\mu \prod_{j=1}^{M-1} d\sigma_j^\mu \right) \times \prod_{k=1}^K \left( \int_{\tilde{R}^k} \prod_{\nu=1}^D d\bar{x}_\nu^k \right) \theta \left( 1 - \sum_{k=1}^K (\text{Tr} \bar{x}_\mu^k \bar{x}_\mu^k + N_k \rho_\mu^k \rho_\mu^k) - \sum_{j=1}^M \sigma_j^\mu \sigma_j^\mu \right) \times \theta \left( \sum_{k=1}^K \text{Tr} \bar{x}_\mu^k \bar{x}_\mu^k + N_k \rho_\mu^k \rho_\mu^k + \sum_{j=1}^M \sigma_j^\mu \sigma_j^\mu - (1 - n_H (\epsilon R^{(2-\eta)^{-2}})) \right) \tag{5.9}
\]

where \( \sigma^M_\mu \) is given by (5.4), the region \( \tilde{R}^k \) is defined by

\[
\left| \begin{bmatrix} \bar{x}_\mu^k & \bar{x}_\nu^k \end{bmatrix}_{ij} \right| < R^{-(2-\eta)}, \quad \forall \mu, \nu, i, j. \tag{5.10}
\]
The right hand side of 5.9 is now bounded above by dropping the \( P \) constraint and integrating out the \( \rho^k_\mu \) and \( \sigma^j_\mu \) which leaves us with

\[
I_{H}^{D,SU(N)}(R) < C_2 R^{-(2-\eta)(D-1)n_H} \prod_{k=1}^{K} \left( \int_{\tilde{R}^k}^{D} \prod_{\nu=1}^{D} d\tilde{x}_\nu^k \right) \theta \left( 1 - \sum_{k=1}^{K} \text{Tr} \tilde{x}_\mu^k \tilde{x}_\mu^k \right)
\]

\[
< C_2 R^{-(2-\eta)(D-1)n_H} \prod_{k=1}^{K} F_{D,SU(N_k)}(N)
\]

where

\[
F_{D,SU(N_k)} = \left( \int_{\tilde{R}^k}^{D} \prod_{\nu=1}^{D} d\tilde{x}_\nu^k \right) \theta \left( 1 - \text{Tr} \tilde{x}_\mu^k \tilde{x}_\mu^k \right).
\]

We note in passing that if \( H \) is empty then 5.11 simply reduces to the generic case 2.12. We now repeat the steps 4.7 to 4.9 to find that

\[
F_{D,SU(N_k)} = R^{-(2-\eta)D(N_k^2-1)/2} \int_{0}^{R} u^{D(N_k^2-1)(1-\eta/2)-1} I_{D,SU(N_k)}(u) du
\]

The final step is by induction on \( N \):

1. \( F_{D,SU(2)} \) is given in 4.3.

2. From our results for \( SU(3) \) 4.10 and 4.11 we deduce that

\[
F_{D,SU(3)} < C_3 R^{-4(2-\eta)D}, \quad D \geq 4
\]

\[
C_4 R^{-4(2-\eta)D \log R}, \quad D = 3
\]

3. For gauge group \( SU(4) \) the possible sub-algebras, \( H \), are \( su(2) \), \( su(2) \oplus su(2) \) and \( su(3) \). For \( D = 3 \) we find that

\[
I_{3,SU(4)}(R) < C_5 R^{-48 \log R}
\]

and hence the integral for \( Z_{3,SU(4)} \) converges. For \( D \geq 4 \) it is simple to check that \( Z_{D,SU(4)} \) converges and the dominant term in \( I_{D,SU(N)}(R) \) comes from \( H = su(3) \) (we will give a general formula for the behaviour of \( I_{D,SU(N)}(R) \) below).

4. We now assume that convergence is established for \( D \geq 3 \) for \( N < N^* \). As a consequence we have that

\[
F_{D,SU(N)} < C R^{-(2-\eta)D(N^2-1)/2}, \quad D \geq 3, \quad \text{and} \quad 3 < N < N^*
\]

(5.16)

together with the slightly different bounds 4.9 and 5.14 for \( N = 2 \) and 3. This is enough information to bound \( I_{D,SU(N^*)}^{H}(R) \) using 5.11 for any sub-algebra \( H \)

\[
I_{D,SU(N^*)}^{H}(R) < C_6 R^{-(2-\eta)((D-1)n^*_H+\frac{1}{2}D\sum_{k}(N_k^2-1))} \left[ R^{n_{2\delta_{D,3}} \log R} n_{2\delta_{D,3}+n_{3\delta_{D,4}}} \right]
\]

(5.17)
where \( n_{2,3} \) denotes the number of \( su(2) \) and \( su(3) \) factors respectively in \( H \). It is straightforward to check that for \( D \geq 3 \) the slowest decay at large \( R \) occurs when \( H = su(N-1); \) basically this minimises the number of \( R^{-2} \) factors coming from off-diagonal elements not in the sub-algebra and maximises the number of \( R^{-1} \) factors coming from elements in the sub-algebra. Thus for \( D \geq 3 \) and \( N \geq 4 \) we have

\[
I_{deg}^{D,SU(N)}(R) < CR^{-(4(D-1)(N-1)+DN(N-2))} (\log R)^{\delta_{N,4} \delta_{D,3}}
\]  

(5.18)

Applying the criterion 2.5 shows that both generic 2.12 and degenerate terms make a finite contribution to \( Z_{D,SU(N)} \) for \( D \geq 3, N \geq 4 \).

This completes our proof.

6. Discussion

6.1 Correlation Functions

We can extend the definition of the partition function 1.1 to correlation functions so that

\[
< . > = \prod_{\mu = 1}^{D} \int_{-\infty}^{\infty} dX_{\mu}(.) \exp \left( \sum_{\mu > \nu} \text{Tr} \left[ X_{\mu}, X_{\nu} \right]^2 \right)
\]  

(6.1)

where (.) represents some kind of product of the \( X_\mu \) with \( P \) factors. Making the change of variables 2.1, realising that the absolute value of the corresponding product of the \( x_\mu \) must be bounded by a constant, and using 5.18 we get that

\[
|< . >| < C_8 \int_0^R dR R^{-2N(D-2)+3D-5} R^P.
\]  

(6.2)

Thus correlators with fewer than

\[
P_c = 2N(D-2) - 3D + 4
\]  

(6.3)

factors are guaranteed to be finite; of course correlators with more factors than this may be finite but then they must have some special property so that the leading divergences cancel. The authors of [13] “guessed” on the basis of reasonable arguments that the eigenvalue density \( \rho(\lambda) \) for \( X_\mu \) behaves like

\[
\rho(\lambda) \sim \text{const} \lambda^{-2N(D-2)+3D-5}
\]  

(6.4)

at large \( \lambda \). This is completely consistent with our results.
6.2 Supersymmetric Integrals

The supersymmetric partition functions are given by

\[ Z_{SS}^{D,G} = \prod_{\mu=1}^{D} \int_{-\infty}^{\infty} dX_{\mu} \, \mathcal{P}_{D,G}(X_{\mu}) \exp \left( \sum_{\mu > \nu} \text{Tr} \left[ X_{\mu}, X_{\nu} \right]^2 \right) \]  

(6.5)

where the Pfaffian \( \mathcal{P}_{D,G} \) arises from integrating out the fermionic degrees of freedom and is a homogeneous polynomial of degree \((D - 2)(N^2 - 1)\). We can of course regard \( Z_{SS}^{D,G} \) as being a correlation function in the bosonic theory and apply the considerations of section 6.1 to it for \( P = (D - 2)(N^2 - 1) \); we find immediately that all the supersymmetric partition functions are naively divergent. However the Pfaffian contains many terms with different signs and we expect many cancellations. The simplest example is for \( SU(2) \) where the Pfaffians are known explicitly \([11]\) and (except for \( D = 3 \) where \( Z_{3,SU(2)}^{SS} = 0 \) because \( \mathcal{P}_{3,SU(2)} \) is an odd function) can be expressed as sums of powers of commutators \([X_{\mu}, X_{\nu}]\). This is particularly convenient with our method because the rescaling in 2.1 followed by the restriction to the region \( \mathcal{R}_1 \) in 2.6 means that we can bound \( Z_{SS}^{D,SU(2)} \) simply by setting all commutators to a constant. It follows that if the bosonic partition function converges so does the supersymmetric one; thus the \( D = 6 \) and 10 partition functions are convergent but \( D = 4 \) is marginal and we would need to work harder (in fact it is known to converge).

The situation with bigger \( N \) is more complicated mainly because relatively little is known about the Pfaffians and we will return to this problem in a separate paper.

6.3 Other Gauge Groups

In this paper we have concentrated on \( SU(N) \) gauge groups. However all the results we obtain depend in a well defined way on simple group theoretical properties such as the order, rank and sub-algebras of Lie algebras. It is therefore tempting to suppose that expressed in this form our results would carry over directly to any Lie group. However some of the steps we have made such as the diagonalization of \( x_1 \) and the inductive argument in section 5 do depend on the group being \( SU(N) \) and we will deal with the other groups in a separate paper.

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A. Divergent Matrix Integrals

The \( D = 2 \) integral

\[ Z_{2,G} = \int_{-\infty}^{\infty} dX_1 dX_2 \exp \left( \text{Tr} \left[ X_1, X_2 \right]^2 \right) \]  

(A.1)
is divergent for all $SU(N)$. This is easily seen by diagonalizing $X_1$; the integrand then does not depend upon the diagonal elements of $X_2$ and so the integral over them diverges.

Some other low $N$ and low $D$ integrals are divergent. To see this we go back to (1.1), diagonalize $X_1$, and separate out the diagonal elements for $\nu > 1$,

$$X_\nu = \text{diag}(\lambda_{\nu 1}, \ldots, \lambda_{\nu N}) + X_\nu^\perp. \quad (A.2)$$

We then change variables from the $X_\nu^\perp$ to $(D-1)(g-l)$ dimensional polar coordinates with radial variable $r$ and angular variables $\{\theta_i\}$. The integral over all the diagonal elements is gaussian so we do it and are left with

$$Z_{D,G} = \int_0^\infty r^{D(g-2l)-2g+2l-1} dr \int d\Omega F_1(\{\theta_i\}) \exp(-r^4 F_2(\{\theta_i\})) \quad (A.3)$$

where $F_1$ and $F_2$ are horrible but positive semi-definite functions and $\Omega$ is the $(D-1)(g-l)$ dimensional solid angle. We see immediately that the integral over $r$ diverges at $r = 0$ if

$$D \leq \frac{2(g-l)}{g-2l} \quad (A.4)$$

so we deduce that the $D = 3, 4$ integrals diverge for $SU(2)$ and that the $D = 3$ integral diverges for $SU(3)$.

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