Projections in free product $C^*$–algebras, II

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Abstract

Let $(A, \varphi)$ be the reduced free product of infinitely many $C^*$–algebras $(A_\iota, \varphi_\iota)$ with respect to faithful states. Assume that the $A_\iota$ are not too small, in a specific sense. If $\varphi$ is a trace then the positive cone of $K_0(A)$ is determined entirely by $K_0(\varphi)$. If, furthermore, the image of $K_0(\varphi)$ is dense in $\mathbb{R}$, then $A$ has real rank zero. On the other hand, if $\varphi$ is not a trace then $A$ is simple and purely infinite.

Introduction

Let $I$ be a set having at least two elements and, for every $\iota \in I$, let $A_\iota$ be a unital $C^*$–algebra with a state, $\varphi_\iota$, whose GNS representation is faithful. Their reduced free product,

$$(A, \varphi) = \ast_{\iota \in I} (A_\iota, \varphi_\iota)$$

was introduced by Voiculescu [20] and independently (in a more restricted way) by Avitzour [1]. Thus $A$ is a unital $C^*$–algebra with canonical, injective, unital $*$–homomorphisms, $\pi_\iota: A_\iota \to A$, and $\varphi$ is a state on $A$ such that $\varphi \circ \pi_\iota = \varphi_\iota$ for all $\iota$. It is the natural construction in Voiculescu’s free probability theory (see [21]), and Voiculescu’s theory has been vital to the study of these $C^*$–algebras.

In [12], for reduced free product $C^*$–algebras $A$ as in (1), when all the $\varphi_\iota$ are faithful, we investigated projections in $A$ and the related topic of positive elements in $K_0(A)$. The behaviour we discovered, under mild conditions specifying that the $A_\iota$ are not too small, depended broadly on whether $\varphi$ is a trace, (i.e. on whether all the $\varphi_\iota$ are traces). If $\varphi$ is a not trace then by [12] $A$ is properly infinite. It remained open whether $A$ must be purely infinite. (Some special classes of reduced free product $C^*$–algebras have in [13] and [1] been shown to be purely infinite.) When $\varphi$ is a trace, then it follows from [12] that
for every element, \( x \), of the subgroup, \( G \), of \( K_0(A) \) generated by \( \bigcup_{i \in I} K_0(\pi_i)(K_0(A_i)) \), if \( K_0(\varphi)(x) > 0 \) then \( x \geq 0 \) and if \( 0 < K_0(\varphi)(x) < 1 \) then there is a projection \( p \in A \) such that \( x = [p]_0 \). By work of E. Germain [14], [15], [16], if each \( A_i \) is an amenable \( C^* \)-algebra then \( K_0(A) = G \) and \( G \) can be found from the groups \( K_0(A_i) \) by using exact sequences, (and by taking inductive limits if \( I \) is infinite); hence under the hypothesis of amenability, we used Germain’s results to give a complete characterization of the positive cone of \( K_0(A) \) and of its elements corresponding to projections in \( A \).

In the present paper we investigate similar questions for reduced free product \( C^* \)-algebras, (1), when \( I \) is infinite and when, for infinitely many \( i \in I \), there is a unitary, \( u \in A_i \) such that \( \varphi_i(u) = 0 \). We show that in this case, if \( \varphi \) is not a trace then \( A \) is purely infinite and simple. If \( \varphi \) is a trace then, although we do not know in general if the subgroup \( G \) described above exhausts \( K_0(A) \), we nonetheless show that for every \( x \in K_0(A) \), if \( K_0(\varphi)(x) > 0 \) then \( x \geq 0 \); furthermore, we show that if \( x \in K_0(A) \) and if \( 0 < K_0(\varphi)(x) < 1 \) then there is a projection \( p \in A \) such that \( x = [p]_0 \). We also show that if the image of \( K_0(\varphi) \) is dense in \( \mathbb{R} \) then \( A \) has real rank zero.

The real rank of a \( C^* \)-algebra, \( A \), is denoted \( RR(A) \) and was invented by L.G. Brown and G.K. Pedersen [4]. Of particular interest is the case \( RR(A) = 0 \), which is defined, for a unital \( C^* \)-algebra \( A \), to mean that the invertible self–adjoint elements are dense in the set of all self–adjoint elements of \( A \). If \( \varphi \) is a faithful state on an infinite dimensional, simple \( C^* \)-algebra \( A \), then a necessary condition for \( RR(A) = 0 \) is that there be projections, \( p \), in \( A \) such that \( \varphi(p) \) is arbitrarily small and positive; hence in particular, the image of \( K_0(\varphi) \) must be dense in \( \mathbb{R} \). We show that this condition is sufficient when \( A \) is a reduced free product of infinitely many algebras, as above, and when \( \varphi \) is a trace. (Moreover, when \( \varphi \) is not a trace then \( A \) is purely infinite, so by a result of S. Zhang [22], \( A \) has real rank zero.)

1 Comparison between positive elements and projections

Most of this section is a reformulation of results from [19]. The proof of Theorem 1.5 below is almost identical to the proof of [19, Theorem 7.2], but the statements of these two theorems are quite different.

We recall the notion of comparison of positive elements as introduced by J. Cuntz in [3] and [4] (see also [19]). Let \( A \) be a \( C^* \)-algebra, and let \( a, b \) be positive elements in \( A \).
Then $a \precsim b$ will mean that there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A$ with

$$\lim_{n \to \infty} \|a - x_nbx_n^*\| = 0.$$ 

If $p, q \in A$ are projections, then the definition above of $p \precsim q$ agrees with the usual definition: $p = vv^*$ and $v^*v \leq q$ for some partial isometry $v \in A$.

If $A$ is unital, and if $\varphi$ is a state on $A$, then define $D_\varphi: A^+ \to [0, 1]$ by

$$D_\varphi(a) = \lim_{\varepsilon \to 0^+} \varphi(f_\varepsilon(a)),$$

where $f_\varepsilon: \mathbb{R}^+ \to [0, 1]$ is the continuous function, which is zero on $[0, \varepsilon/2]$, linear on $[\varepsilon/2, \varepsilon]$, and equal to 1 on $[\varepsilon, \infty)$. If $\varphi$ is a trace, then $D_\varphi$ is a dimension function (in the sense of Cuntz, \[6\]). Notice that $D_\varphi(p) = \varphi(p)$ for all projections $p \in A$.

We shall use the following facts:

$$f_{2\varepsilon}(a) \leq f_\delta(f_\varepsilon(a)) \leq f_{\varepsilon/2}(a), \quad f_{\varepsilon/2}(a)f_\varepsilon(a) = f_\varepsilon(a), \quad (2)$$

when $\varepsilon > 0$ and $0 < \delta \leq 1/2$, and, consequently, $D_\varphi(f_\varepsilon(a)) \leq \varphi(f_{\varepsilon/2}(a))$. Moreover, if $0 \leq a \leq 1$, then $\varphi(a) \leq D_\varphi(a)$. Recall from \[18\] that the stable rank of a unital C*-algebra $A$ is equal to 1 if and only if the set invertible elements of $A$ is dense in $A$.

**Lemma 1.1** Let $A$ be a C*-algebra of stable rank one, let $B$ be a hereditary subalgebra of $A$, let $a$ be a positive element in $B$, and let $q$ be a projection in $B$ such that $a \precsim q$. Then for each $\varepsilon > 0$ there is a projection $p \in B$ such that $f_\varepsilon(a) \leq p \sim q$.

**Proof:** Observe first that the comparisons $a \precsim p$ and $p \sim q$ are independent of whether they are relative to $A$, $B$, or $\tilde{B}$, where $\tilde{B}$ denotes the C*-algebra obtained by adjoining a unit to $B$. It follows from \[18\] and \[8\] that if $A$ has stable rank one, then so do $B$ and $\tilde{B}$. By \[19\], Proposition 2.4], there is for each $\varepsilon > 0$ a unitary $u$ in $\tilde{B}$ with $uf_\varepsilon(a)u^* \in q\tilde{B}q \subseteq qBq$. Put $p = u^*qu$. Then $p$ is as desired. □

**Lemma 1.2** Let $A$ be a C*-algebra, let $a$ be a positive element in $A$, and let $p$ be a projection in $A$. Then the following are equivalent:

(i) $p \precsim a$,

(ii) $p = xax^*$ for some $x \in A$, 

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(iii) $p$ is equivalent to some projection in the hereditary subalgebra of $A$ generated by $a$.

Proof: (i) → (ii). If $p \preceq a$, then $\|p - yay^*\| < 1/2$ for some $y \in A$. Hence $pyay^*p$ is invertible (and positive) in $pAp$, and therefore $p = zpyay^*pz^*$ for some $z \in pAp$.

(ii) → (iii). Put $u = xa^{1/2}$. Then $uu^* = p$, and hence $u$ is a partial isometry. Put $q = u^*u = a^{1/2}x^*xa^{1/2}$. Then $q$ is a projection in the hereditary subalgebra generated by $a$, and $q \sim p$.

(iii) → (i). Assume $q$ is a projection in the hereditary subalgebra generated by $a$, and that $q \sim p$. Then there exists an $n \in \mathbb{N}$ such that $\|q - a^{1/n}qa^{1/n}\| < 1/2$. It follows that $qa^{1/n}qa^{1/n}q$ and, consequently, $qa^{2/n}q$, are invertible in $qAq$. Therefore $q = rqa^{2/n}qr^*$ for some $r \in qAq$. This shows that $p \sim q \preceq a^{2/n} \preceq a$. □

Lemma 1.3 ([19, Proposition 2.2]) Let $A$ be a unital $C^*$-algebra, let $a, b$ be positive elements in $A$, and let $\varepsilon > 0$. If $\|a - b\| < \varepsilon$, then $f_\varepsilon(a) \preceq b$.

Lemma 1.4 Let $A$ be a unital $C^*$-algebra, and let $\mathfrak{A}$ be a dense unital *-subalgebra of $A$. Suppose that for each positive element $a \in \mathfrak{A}$ and each $\varepsilon > 0$ there is a projection $p \in A$ and $0 < \delta < \varepsilon$ such that $f_\varepsilon(a) \leq p \leq f_\delta(a)$. Then $RR(A) = 0$.

Proof: To show that $RR(A) = 0$ it will suffice (by [4]) to show that all self-adjoint elements in the dense *–subalgebra $\mathfrak{A}$ can be approximated by invertible self-adjoint elements.

Let $a$ be a self-adjoint element in $\mathfrak{A}$, and write $a = a_+ - a_-$. For each $n \in \mathbb{N}$ find $\delta_n > 0$ and projections $p_n, q_n$ in $A$ such that

$$f_{1/n}(a_+) \leq p_n \leq f_{\delta_n}(a_+), \quad f_{1/n}(a_-) \leq q_n \leq f_{\delta_n}(a_-).$$

Then $p_n \perp q_n$, $p_na_+p_n \to a_+$, and $q_na_-q_n \to a_-$. Set

$$b_n = (p_na_+p_n + \frac{1}{n}p_n) - (q_na_-q_n + \frac{1}{n}q_n) + \frac{1}{n}(1 - p_n - q_n).$$

Then each $b_n$ is invertible and self-adjoint, and $b_n \to a$. □

Let $Q$ be a compact convex subset of the state space of a unital $C^*$-algebra $A$, and let $\text{Aff}(Q)$ denote the real vector space of all affine continuous functions $Q \to \mathbb{R}$. Equip this space with the strict ordering, i.e., $f \geq 0$ if $f = 0$ or if $f(\varphi) > 0$ for all $\varphi \in Q$, and, in turn, with the topology induced by this ordering. All self-adjoint elements $a \in A$ induce
an element $\hat{a} \in \text{Aff}(Q)$ through the formula $\hat{a}(\varphi) = \varphi(a)$. We will consider the interval $[0, 1]$ of $\text{Aff}(Q)$, defined by

$$[0, 1] = \{ f \in \text{Aff}(Q) \mid 0 \leq f \leq 1 \}.$$

**Theorem 1.5** Let $A$ be a unital $C^*$-algebra, let $Q$ be a compact convex subset of the state space of $A$, let $\Pi$ be a subset of of the set of projections in $A$, and let $\mathfrak{A}$ be a dense $*$-subalgebra of $A$, which is closed under continuous function calculus (on its normal elements).

Assume that each state in $Q$ is faithful on $\mathfrak{A}$, and that the following comparison properties hold for all positive elements $a \in \mathfrak{A}$ and for all projections $p \in \Pi$:

(i) if $D_\varphi(a) < \varphi(p)$ for all $\varphi \in Q$, then $a \preceq p$,

(ii) if $\varphi(p) < D_\varphi(a)$ for all $\varphi \in Q$, then $p \preceq a$, and

(iii) the subset of $\text{Aff}(Q)$ induced by $\Pi$ is dense in the interval $[0, 1]$ of $\text{Aff}(Q)$.

It follows that

(i) if $s_r(A) = 1$, then $RR(A) = 0$, and

(ii) if all nonzero projections in $\Pi$ are infinite and full, then $A$ is simple and purely infinite.

**Proof:** (i). We show that the conditions in Lemma 1.4 are satisfied. So let $a \in \mathfrak{A}$ be a positive element, and let $\varepsilon > 0$. We must find $0 < \delta < \varepsilon$ and a projection $p \in A$ with $f_\varepsilon(a) \leq p \leq f_\delta(a)$.

If $\text{sp}(a) \cap (\varepsilon/8, \varepsilon/4) = \emptyset$, then $p = f_{\varepsilon/8}(a)$ and $\delta = \varepsilon/8$ will be as desired.

Assume now that $\text{sp}(a) \cap (\varepsilon/8, \varepsilon/4) \neq \emptyset$. Then $0 \leq \varphi(f_{\varepsilon/4}(a)) < \varphi(f_{\varepsilon/8}(a)) \leq 1$ for all $\varphi \in Q$ because each such $\varphi$ is assumed to be faithful on $\mathfrak{A}$. Since $\Pi$ is dense in the interval $[0, 1]$ of $\text{Aff}(Q)$ there is $q \in \Pi$ with $\varphi(f_{\varepsilon/4}(a)) < \varphi(q) < \varphi(f_{\varepsilon/8}(a))$ for all $\varphi \in Q$. By (2),

$$D_\varphi(f_{\varepsilon/2}(a)) \leq \varphi(f_{\varepsilon/4}(a)) < \varphi(q) < \varphi(f_{\varepsilon/8}(a)) \leq D_\varphi(f_{\varepsilon/8}(a)).$$

By assumptions (a) and (b) this implies that $f_{\varepsilon/2}(a) \preceq q \preceq f_{\varepsilon/8}(a)$.

From Lemma 1.2 there is a projection $r$ in the hereditary subalgebra, $B$, generated by $f_{\varepsilon/8}(a)$ such that $q \sim r$. By Lemma 1.1 there is a projection $p \in B$ such that $f_{1/2}(f_{\varepsilon/2}(a)) \preceq p \preceq f_{\varepsilon/8}(a)$. By Lemma 1.3 we have $f_{\varepsilon/8}(a) \preceq \varphi(p) = D_\varphi(f_{\varepsilon/8}(a))$ for each $\varphi \in Q$.
(and \( p \sim r \)). By (3), this entails that \( f_\varepsilon(a) \leq p \leq f_\varepsilon/16(a) \). The claim is therefore proved with \( \delta = \varepsilon/16 \).

(ii). If each non-zero hereditary subalgebra of \( A \) contains a full element, then \( A \) must be simple. If, moreover, each such hereditary subalgebra contains an infinite projection, then \( A \) is purely infinite and simple (c.f. Cuntz’ definition of purely infinite simple C\(^*\)-algebras in [7]). It therefore suffices to show that each non-zero hereditary subalgebra of \( A \) contains an infinite full projection.

Let \( B \) be a non-zero hereditary subalgebra of \( A \), and let \( b \) be a positive element in \( B \) with \( \|b\| = 1 \). Find a positive element \( a \in A \) with \( \|a - b\| < 1/2 \). Since each \( \varphi \in \mathcal{Q} \) is faithful, since \( f_{1/2}(a) \neq 0 \), and since \( \Pi \) is dense in the interval \([0,1]\) of \( \text{Aff}(\mathcal{Q}) \), there is \( q \in \Pi \) with \( \varphi(q) < \varphi(f_{1/2}(a)) \) for all \( \varphi \in \mathcal{Q} \). This implies that \( \varphi(q) < D_\varphi(f_{1/2}(a)) \) for all \( \varphi \in \mathcal{Q} \), and by assumption (\( \beta \)) we get \( q \lesssim f_{1/2}(a) \). Using Lemma 1.3 we obtain that \( q \lesssim b \), and Lemma 1.2 finally implies that there is a projection \( p \in \) the hereditary subalgebra of \( A \) generated by \( b \) (which is contained in \( B \), so that \( p \in B \)) such that \( p \sim q \). Since \( q \) is infinite and full, so is \( p \), and the proof is complete.

\[ \square \]

### 2 Application to reduced free products of C\(^*\)-algebras

Throughout this section, we consider a reduced free product of C\(^*\)-algebras,

\[
(A, \varphi) = \bigast_{i \in I} (A_i, \varphi_i),
\]

where \( I \) is an infinite set, where each \( \varphi_i \) is a faithful state and where for infinitely many \( i \in I \) there is a unitary \( u \in A_i \) with \( \varphi_i(u) = 0 \). It follows from [8] that \( \varphi \) is faithful on \( A \).

Avitzour’s result [1] gives that \( A \) is simple if, for example, \( \varphi \) is a trace. Indeed, by partitioning the set \( I \) into two suitable subsets, \( A \) can be viewed as a reduced free product,

\[
(A, \varphi) = (B_1, \psi_1) \ast (B_2, \psi_2),
\]

such that there are unitaries, \( u \in B_1 \) and \( v, w \in B_2 \) satisfying that \( \psi_1(u) = 0 = \psi_2(v) = \psi_2(w) \) and that \( v \) and \( w \) are \( * \)-free; hence also \( \psi_2(v^*w) = 0 \). (Avitzour’s result also applies in somewhat more general instances.) In addition, if \( \varphi \) is a trace then by [10] the stable rank of \( A \) is equal to 1.

The \( K_0 \)-group, \( K_0(D) \), of a C\(^*\)-algebra \( D \) is equipped with a positive cone and a scale.
defined respectively by

\[ K_0(D)^+ = \{ [p]_0 \mid p \in \text{Proj}(D \otimes K) \}, \]
\[ \Sigma(D) = \{ [p]_0 \mid p \in \text{Proj}(D) \}, \]

where \( \text{Proj}(D) \) is the set of projection in \( D \), and where \( [\cdot]_0 : \text{Proj}(D \otimes K) \to K_0(D) \) is the canonical map from which \( K_0 \) is defined. The positive cone gives rise to an ordering on \( K_0(D) \) by \( x \leq y \) if \( y - x \in K_0(D)^+ \), and \( x < y \) if \( y - x \in K_0(D)^+ \setminus \{0\} \). Each (positive) trace \( \varphi \) on \( D \) induces a positive group-homomorphism \( K_0(\varphi) : K_0(D) \to \mathbb{R} \) which satisfies \( K_0(\varphi)([p]_0) = \varphi(p) \) for \( p \in \text{Proj}(D) \), and \( K_0(\varphi)([p]_0) = (\varphi \otimes \text{Tr}_n)(p) \) for \( p \in \text{Proj}(D \otimes M_n(\mathbb{C})) \), where \( \text{Tr}_n \) is the (unnormalized) trace on \( M_n(\mathbb{C}) \). The ordered abelian group \( (K_0(D), K_0(D)^+) \) is called weakly unperforated if \( nx > 0 \) for some \( n \in \mathbb{N} \) and some \( x \in K_0(D) \) implies that \( x \geq 0 \).

**Theorem 2.1** Let \( (A, \varphi) = \ast_{i \in I} (A_i, \varphi_i) \) be the reduced free product C*-algebra, where each \( A_i \) is a unital C*-algebra, \( \varphi_i \) is a faithful state on \( A_i \), the index set \( I \) is infinite, and infinitely many \( A_i \) contain a unitary in the kernel of \( \varphi_i \).

If \( \varphi \) is a trace (which is the case if all \( \varphi_i \) are traces), then

(i) whenever \( p, q \in A \otimes M_n(\mathbb{C}) \) are projections such that \( (\varphi \otimes \text{Tr}_n)(p) < (\varphi \otimes \text{Tr}_n)(q) \), it follows that \( p \preceq q \);

(ii) the positive cone and the scale of \( K_0(A) \) are given by

\[ K_0(A)^+ = \{ 0 \} \cup \{ x \in K_0(D) \mid 0 < K_0(\varphi)(x) \}, \]
\[ \Sigma(A) = \{ 0, 1 \} \cup \{ x \in K_0(D) \mid 0 < K_0(\varphi)(x) < 1 \}, \]

and, as a consequence, \( (K_0(A), K_0(A)^+) \) is weakly unperforated;

(iii) \( \text{RR}(A) = 0 \) if and only if \( K_0(\varphi)(K_0(A)) \) is dense in \( \mathbb{R} \).

If \( \varphi \) is not a trace (i.e., if at least one \( \varphi_i \) is not a trace), then \( A \) is simple and purely infinite.

**Proof:** We consider, for every finite subset \( F \subseteq I \), the C*-subalgebra, \( \mathfrak{A}_F \), of \( A \) generated by \( \bigcup_{i \in F} \pi_i(A_i) \), and we let \( \mathfrak{A} = \bigcup_{F \subseteq \mathbb{I}} \mathfrak{A}_F \), where the union is over all finite subsets of
I. Note that $\mathfrak{A}$ is a dense, unital $*$-subalgebra of $\mathcal{A}$ that is closed under the continuous functional calculus.

Suppose that $\varphi$ is a trace, let $n \in \mathbb{N}$ and let $p, q \in A \otimes M_n(\mathbb{C})$ be projections with $(\varphi \otimes \text{Tr}_n)(p) < (\varphi \otimes \text{Tr}_n)(q)$. Using the density of $\mathfrak{A}$ in $A$ and continuous functional calculus, we find a finite subset $F$ of $I$ and projections $\tilde{p}, \tilde{q} \in \mathfrak{A}_F \otimes M_n(\mathbb{C})$ such that $\|\tilde{p} - p\| < 1$ and $\|\tilde{q} - q\| < 1$. This implies $\tilde{p} \sim p$ and $\tilde{q} \sim q$. There are $n^2$ distinct elements $\iota(1), \iota(2), \ldots, \iota(n^2) \in I \setminus F$ with unitaries $u_k \in A_{\iota(k)}$ such that $\varphi_{\iota(k)}(u_k) = 0$. Let $B$ be the $C^*$-algebra generated by $\{u_1, u_2, \ldots, u_{n^2}\}$. Note that $B$ and $\mathfrak{A}_F$ are free. Then as in the proof of Proposition 3.3 of [12], from the unitaries $u_1, u_2, \ldots, u_{n^2}$ we can construct a Haar unitary, $v \in B \otimes M_n(\mathbb{C})$ such that $\{v\}$ and $\mathfrak{A}_F \otimes M_n(\mathbb{C})$ are $*$-free (with respect to the tracial state $\varphi \otimes (\frac{1}{n} \text{Tr}_n)$). Now $\tilde{q} \sim v^*\tilde{q}v$ and the pair $\tilde{p}$ and $v^*\tilde{q}v$ is free; moreover, $(\varphi \otimes \text{Tr}_n)(v^*\tilde{q}v) = (\varphi \otimes \text{Tr}_n)(\tilde{q})$. So by Proposition 1.1 of [12], $v^*\tilde{q}v$ is equivalent to a subprojection, $r$, of $\tilde{p}$; hence $q \lesssim p$. We have thus proved (i).

The inclusions $\subseteq$ in (ii) are easy consequences of the fact that $\varphi$ is faithful. Assume $x \in K_0(A)$ and that $K_0(\varphi)(x) > 0$. Since $A$ is unital, there are $n \in \mathbb{N}$ and projections $p, q \in A \otimes M_n(\mathbb{C})$ such that $x = [p]_0 - [q]_0$. Now,

$$(\varphi \otimes \text{Tr}_n)(p) - (\varphi \otimes \text{Tr}_n)(q) = K_0(\varphi)(x) > 0.$$ 

Hence, by (i), $q$ is equivalent to a subprojection $\tilde{q}$ of $p$. Thus $x = [p - \tilde{q}]_0 \in K_0(A)^+$. 

Assume next that $x \in K_0(A)$ and that $0 < K_0(\varphi)(x) < 1$. Then, by the argument above, $x = [p]_0$ for some projection $p \in A \otimes M_n(\mathbb{C})$. Let $1_A$ denote the unit of $A$, and let $e \in A \otimes M_n(\mathbb{C})$ be the diagonal projection whose upper left corner is $1_A$ and with all other entries equal to 0. Then $(\varphi \otimes \text{Tr}_n)(p) = K_0(\varphi)(x) < 1 = (\varphi \otimes \text{Tr}_n)(e)$. By (i), this implies that $p$ is equivalent to a subprojection $\tilde{p}$ of $e$. Hence $x = [\tilde{p}]_0$, and it is easily seen that $[\tilde{p}]_0 \in \Sigma(A)$.

Finally, to see that $(K_0(A), K_0(A)^+)$ is weakly unperforated, assume that $x \in K_0(A)$ and that $nx > 0$ for some $n \in \mathbb{N}$. Then $K_0(\varphi)(x) = \frac{1}{n} K_0(\varphi)(nx) > 0$. Hence $x > 0$. We have thus shown (ii).

Let

$$\Pi = \bigcup_{F \ll I} \text{Proj}(\mathfrak{A}_F).$$

For the set $\mathcal{Q}$ used in Theorem 1.3 we take the singleton $\{\varphi\}$. We now show that, regardless of whether $\varphi$ is a trace or not, conditions $(\alpha)$ and $(\beta)$ of Theorem 1.3 hold for every $p \in \Pi$ and every positive element, $a \in \mathfrak{A}$. Given a positive element $a \in \mathfrak{A}$ and given $p \in \Pi$, there
is a finite subset $F$ of $I$ such that $a, p \in \mathcal{A}_F$. Let $u \in A_i$, for some $i \in I \setminus F$, be a unitary such that $\varphi_i(u) = 0$. Then $\{a, p\}$ and $\{u\}$ are $*$-free with respect to $\varphi$. Now it follows that $u^*pu$ is a projection with $\varphi(u^*pu) = \varphi(p)$, and that $u^*pu$ and $a$ are free. Hence by Lemma 5.3 of [12], it follows that $a \precsim u^*pu$ if $D_\varphi(a) < \varphi(p)$ and $u^*pu \precsim a$ if $\varphi(p) < D_\varphi(a)$. But $u^*pu \sim p$, so $(\alpha)$ and $(\beta)$ hold.

Suppose now that $\varphi$ is a trace and that the image of $K_0(\varphi)$ is dense in $\mathbb{R}$, and let us show that $\text{RR}(A) = 0$. We will show that $\{\varphi(p) \mid p \in \Pi\}$ is dense in $[0, 1]$, which will imply that condition $(\gamma)$ of Theorem 1.3 holds. Since the image of $K_0(\varphi)$ is dense in $\mathbb{R}$, the intersection of this image with $[0, 1]$ is dense in $[0, 1]$. By (ii), it follows that $\{\varphi(p) \mid p \in \text{Proj}(A)\}$ is dense in $[0, 1]$. Since $\mathcal{A}$ is dense in $A$, and using continuous functional calculus, we find for every $p \in \text{Proj}(A)$, a projection, $\tilde{p} \in \mathcal{A}$ such that $\varphi(\tilde{p}) = \varphi(p)$. But $\tilde{p} \in \Pi$. We have shown that condition $(\gamma)$ of Theorem 1.3 holds, and we have already shown that conditions $(\alpha)$ and $(\beta)$ hold. Now using the fact that $\text{sr}(A) = 1$, we get from Theorem 1.3(i) that $\text{RR}(A) = 0$. This implies one direction of (iii), but the other direction follows from more general results. Indeed, the image of $K_0(\varphi)$ will be dense in $\mathbb{R}$ if $A$ contains at least one projection and if $A$ has no minimal projections. Both of these conditions hold if $\text{RR}(A) = 0$, and if $A$ is simple and infinite dimensional, as in our case.

Now suppose that $\varphi$ is not a trace, and let us show that $A$ is purely infinite and simple. Let $F$ be a finite subset of $I$ such that for at least three distinct $i \in F$ there is a unitary $u \in A_i$, satisfying $\varphi_i(u) = 0$, and such that for some $i \in F$, $\varphi_i$ is not a trace. Then by Theorem 4 of [12], the unit is a properly infinite projection in $\mathcal{A}_F$. Let $\Pi' \subseteq \Pi$ be the set of all full, properly infinite projections in $\mathcal{A}$. We have already shown that conditions $(\alpha)$ and $(\beta)$ are satisfied for every $a \in \mathcal{A}$ and every $p \in \Pi'$. Since 1 is a properly infinite projection in some $\mathcal{A}_F$, using Lemma 2.2 below we get that $\{\varphi(p) \mid p \in \Pi'\}$ is dense in $[0, 1]$, so condition $(\gamma)$ is satisfied. Tautologically, each $p \in \Pi'$ is infinite and full. Hence by Theorem 1.3(ii), $A$ is purely infinite and simple. $\square$

**Lemma 2.2** Let $A$ be a unital $C^*$-algebra in which 1 is properly infinite and let $\varphi$ be a state on $A$. Then for every $t \in \mathbb{R}$, $0 < t \leq 1$, there is a projection $p \in A$ such that $p \sim 1$ and $\varphi(p) = t$.

**Proof:** Using that 1 is properly infinite, we find isometries, $v_1, v_2, \ldots$ in $A$ whose range projections are mutually orthogonal. These generate a unital $C^*$-subalgebra of $A$ isomorphic to the Cuntz algebra $\mathcal{O}_\infty$. By Cuntz’s paper [4], it follows that if $p, q \in \text{Proj}(\mathcal{O}_\infty) \setminus \{0, 1\}$ and if $[p]_0 = [q]_0$ in $K_0(\mathcal{O}_\infty)$ then $p$ is homotopic to $q$. (Indeed, it follows that $p \sim q$ and $1 - p \sim 1 - q$, hence $p$ is unitarily similar to $q$. But the unitary group of $\mathcal{O}_\infty$ is connected.)
Now let $\varepsilon > 0$. For some $n$ we must have $\varphi(v_n v_n^*) < \varepsilon$; let $p = v_n v_n^*$. Then $q \overset{\text{def}}{=} 1 - v_n (1 - p) v_n^*$ is a projection in $O_\infty$ with $[q]_0 = [1]_0$, and $\varphi(q) > 1 - \varepsilon$. Thus there is a continuous path $r_t$ in $\text{Proj}(O_\infty)$ such that $r_0 = p$ and $r_1 = q$. We have that $r_t \sim 1$ for all $t$ and $\{\varphi(r_t) \mid t \in [0, 1]\} \supseteq (\varepsilon, 1 - \varepsilon)$. □

Let us now state a straightforward application of Theorem 2.1 to reduced group $C^*$-algebras. For a group, $G$, taken with the discrete topology, the reduced group $C^*$-algebra of $G$, denoted $C^*_\text{red}(G)$, is the $C^*$-algebra generated by the left regular representation of $G$. The canonical tracial state, $\tau_G$, is the vector state for the characteristic function of the identity element of $G$. The following corollary was cited in [11], where also a partial converse was included.

**Corollary 2.3** Let $I$ be an infinite set and let

$$G = *_{\iota \in I} G_{\iota}$$

be the free product of nontrivial groups, $G_{\iota}$. Suppose that $G$ has finite subgroups of arbitrarily large order. Then

$$\text{RR}(C^*_\text{red}(G)) = 0.$$  

**Proof:** We have

$$(C^*_\text{red}(G), \tau_G) \cong *_{\iota \in I} (C^*_\text{red}(G_{\iota}), \tau_{G_{\iota}}).$$

If $x$ is a nontrivial element of $G_{\iota}$ then the left translation operator, $\lambda_x \in C^*_\text{red}(G_{\iota})$ is a unitary and $\tau_{G_{\iota}}(\lambda_x) = 0$. In order to apply Theorem 2.1, it is thus sufficient to show that $C^*_\text{red}(G)$ contains projections whose traces (under $\tau_G$) are arbitrarily small and positive. But this is clear, since for a finite group $H$, $C^*_\text{red}(H)$ contains projections of trace $1/|H|$. □

It follows easily from the Kurosh Subgroup Theorem for free products of groups, (see page 178 of [17]), that $G$ has finite subgroups of arbitrarily high order only if for every positive integer $n$ there is $\iota \in I$ such that $G_{\iota}$ has a finite subgroup of order greater than $n$.

**Example 2.4** If

$$G = \underset{n=2}{\overset{\infty}{*}} (\mathbb{Z}/n\mathbb{Z}),$$

then $C^*_\text{red}(G)$ has real rank zero. Moreover, this $C^*$-algebra is simple, has unique tracial state, has stable rank one and is not approximately divisible.
Proof: It has real rank zero by the above corollary. It is simple and has unique tracial state by Avitzour [1]. It has stable rank one by [10]. That it is not approximately divisible follows from the argument in Example 4.8 of [2], because $C^*_\text{red}(G)$ is weakly dense in the group von Neumann algebra $L(G)$, which does not have central sequences. □

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