Ab Initio Wall-Crossing

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Abstract

We derive supersymmetric quantum mechanics of \(n\) BPS objects with \(3n\) position degrees of freedom and \(4n\) fermionic partners with \(SO(4)\) R-symmetry. The potential terms, essential and sufficient for the index problem for non-threshold BPS states, are universal, and \(2(n - 1)\) dimensional classical moduli spaces \(M_n\) emerge from zero locus of the potential energy. We emphasize that there is no natural reduction of the quantum mechanics to \(M_n\), contrary to the conventional wisdom. Nevertheless, via an index-preserving deformation that breaks supersymmetry partially, we derive a Dirac index on \(M_n\) as the fundamental state counting quantity. This rigorously fills a missing link in the “Coulomb phase” wall-crossing formula in literature. We then impose Bose/Fermi statistics of identical centers, and derive the general wall-crossing formula, applicable to both BPS black holes and BPS dyons. Also explained dynamically is how the rational invariant \(\sim \Omega(\beta)/p^2\), appearing repeatedly in wall-crossing formulae, can be understood as the universal multiplicative factor due to \(p\) identical, coincident, yet unbound, BPS particles of charge \(\beta\). Along the way, we also clarify relationships between field theory state countings and quantum mechanical indices.

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1 Introduction

Wall-crossing refers to a phenomenon where BPS spectrum [1, 2] changes abruptly as either a parameter or a vacuum of a supersymmetric field theory is smoothly changed. In the four dimensional setting, it was first discovered in the context of $SU(2)$ Seiberg-Witten theory [3, 4], where the asymptotic spectrum consists of massive vector mesons and an infinite tower of dyons with unit magnetic charge while the spectrum at the center of the vacuum moduli space is composed of a monopole and a dyon [5]. This apparent “decay” or “disappearance” of BPS states occurs across a wall of marginal stability in the moduli space, and this phenomenon turned out to be generic in all of $N \geq 2$ $D = 4$ theories.

The mechanism of the disappearance was clarified a few years later. It was found that a BPS one-particle state is generically a bound state consisting of more than one charge centers, which are spatially distributed according to balance of classical forces [6]. Although initial studies were done for 1/4 BPS states in $N = 4$ theories and in classical setting, this was quickly elevated to a quantum statement [7, 8] and then extended to $N = 2$ theories [9, 10, 11]. From this new viewpoint, the wall-crossing occurs simply because the size of such bound states become infinitely large as a marginal stability wall is approached. Across the wall, these charge centers become mutually repulsive, precluding any bound state [7, 9].1 Thus, what was initially thought to be a problem of ”decay” is actually a problem of bound state formation, if viewed backward.

For the simplest $SU(2)$ example, the infinite tower of dyons and the massive vector meson which exists in the weak coupling side only should be all realized as loose bound states of the two BPS particles inside, monopoles and dyons of unit electric charge. At least this must be the right picture just outside the wall.

The most interesting aspect of this problem is that rules for the formation or the dissociation of the bound state, whichever way one view it, seems quite universal and does not depend on detailed dynamics, which became more clear when the multi-center aspect was rediscovered later in the $N = 2$ black hole context [13, 14]. With $N = 2$ supersymmetry, BPS states are characterized by the charge $\gamma$, the central charge $Z_\gamma$, and the supermultiplet. Thanks to the partially preserved supersymmetry, the multiplet structure has a reduced form, which can be written as

$$[j] \otimes ([1/2] \oplus [0] \oplus [0])$$

(1.1)

#1These development were, however, restricted to weakly coupled theories, even though it generalized greatly the previous decades of monopole/dyon studies in the fully supersymmetric setting of $N = 2$ and $N = 4$ Yang-Mills theories. Much of these findings, and their relation to the more conventional monopole dynamics from 1980’s and 1990’s, was summarized in a review article [12].
The index that counts degeneracy of such BPS state is the 2nd helicity trace
\[ \Omega = -\frac{1}{2} \text{tr} \left( (-1)^{2J_3}(2J_3)^2 \right) = (-1)^{2j}(2j + 1). \] (1.2)

For example, when a state \( \gamma_1 + \gamma_2 \) disappears across a marginal stability wall, and dissociates into \( \gamma_1 \) and \( \gamma_2 \) on the other side, the indices of these three kind of BPS particles are known to obey a universal formula [15],
\[ \Omega^- (\gamma_1 + \gamma_2) = (-1)^{|\langle \gamma_1, \gamma_2 \rangle|} |\langle \gamma_1, \gamma_2 \rangle| \Omega^+ (\gamma_1) \Omega^+ (\gamma_2), \] (1.3)

where \( \pm \) denote the two sides of the wall. The wall itself is naturally defined by the condition of vanishing binding energy that \( |Z_1| + |Z_2| = |Z_1 + Z_2| \).

This simplest wall-crossing formula has been studied in many examples, generalized to the so-called semi-primitive cases for \( \gamma_1 + k\gamma_2 \) states [15], and most recently embedded into an algebraic reformulation by Konsevitch and Soibelmann [16], which in turn was explained in more physical basis [17, 18, 19, 20]. Despite successes of these, later and more comprehensive, developments, much of the literature remain mathematical and sometimes, from physics viewpoint, opaque. More direct approaches to the problem, based on physical mechanism of the bound states and the dissociation thereof, do exist but until very recently were confined to special examples and situation.

In Ref. [21], a new approach to the low energy dynamics of dyons in generic \( N = 2 \) Seiberg-Witten theory was proposed. Assuming that bound states of interest are large, which is always true whenever the theory is near a wall of marginal stability, the authors showed how a \( \mathcal{N} = 4 \) supersymmetric dynamics can be explicitly written from the special Kähler data of the vacuum moduli space only. When applied in the limit of a single dynamical probe dyon in the presence of another (very massive) BPS state, the bound states can be constructed explicitly and counted, again confirming the above (so-called primitive) wall-crossing formula. It is abundantly clear that this method can be used for an arbitrary number and varieties of dyons, as well, as long as the proximity to a marginal stability wall is satisfied. In this note, we wish to set up dynamics of arbitrary number of dyons near such a wall, with \( \mathcal{N} = 4 \) supersymmetry,\(^2\) and generate wall-crossing formula via index theorem.

Among the universal formulations that were previously attempted, Denef’s quiver dynamics picture [14] gives a very similar picture in the so-called Coulomb phase description. A recent work by Manschot et.al. [22, 23]\(^3\) fully took advantage of the latter pictures and wrote down a wall-crossing formula. At the end of the day, our
computation will lead to the same final wall-crossing formula. Since we start from Seiberg-Witten theory and derive the wall-crossing formula from scratch, we offer several improvements.

The first improvement concerns the question of what is the relevant index theorem. In the Denef’s Coulomb phase approach, the most comprehensive studies to date involve a truncation of dynamics where one ends up with a geometric quantization problem on the classical moduli space of charge centers, which are typically compact. In this paper, we denote such moduli spaces for \( n \) centers as \( \mathcal{M}_n \). For two-center case, this manifold is always \( S^2 \). The Lagrangian has no kinetic term, but a minimal coupling to certain magnetic field induces a symplectic structure on the moduli space, making it a phase space. In turns out, however, the naive low energy dynamics on this classical moduli space on \( \mathcal{M}_n \) end up with too many fermionic degrees of freedom. The anticipated and empirically correct answer, which is a Dirac index [25], results only if one can somehow remove half of the fermions. This deficiency has remained unresolved until now.

In this note, we will explain why the naive truncation to \( \mathcal{M}_n \) was ill-motivated. It turns out that there is no separation of scales, and all 3\( n \) bosons and 4\( n \) fermions are of equal massgap. Instead, one can choose to reduce the index problem to \( \mathcal{M}_n \) by deforming the theory with supersymmetry partially broken. As long as there is one supersymmetry left unbroken and since the quantum mechanics has a gap, the index is left invariant under the deformation. At the end of the day, we will thus have provided an ab initio derivation of the anticipated Dirac index on \( \mathcal{M}_n \), for the first time.

The second concerns the physical interpretation of certain rational invariants, defined and extensively used by Manschot et.al. [22], of the form

\[
\tilde{\Omega}(\gamma) = \sum_{p|\gamma} \frac{\Omega(\gamma/p)}{p^2},
\]

where the sum is over divisors of \( \gamma \). The expression naturally appears in other formulation of the wall-crossing, most notably in Konsevitch-Soibelmann. In the course of enumerating the bound states of bosonic or fermionic statistics, we will encounter \( \Omega(\beta)/p^2 \) as a universal effective degeneracy of \( p \) identical particles of charge \( \beta \). It appears as the multiplicative factor from the normal bundle as one computes contributions from a submanifold fixed by the permutation group of order \( p \).\footnote{This same numerical factor \( 1/p^2 \) had appeared before in the context of the D-brane bound state problems of 1990's [26, 27], where identical nature of the D-branes were also of some importance.}

Along the way, our work also clarifies relation between the field theory indices, namely the second helicity trace and the protected spin character, and the quantum
mechanical ones. Quantum mechanical index usually suffers from ambiguity over the definition of $(-1)^{F}$. Usual index formulae relies on certain (mathematically) canonical choice of $(-1)^{F}$. Retaining three bosonic coordinates per dyons allow us to inherit both the spatial rotation group, denoted by $SU(2)_L$, and the R-symmetry of $N = 2$ field theory, $SU(2)_R$. The supersymmetries belong to $(2, 2)$ representation, so both $(-1)^{2J_3}$ of $SU(2)_L$ and $(-1)^{2I_3}$ of $SU(2)_R$ are chirality operators. The second helicity trace is then computed unambiguously by $\text{Tr}(-1)^{2J_3}$. We in turn relate the latter to $\text{Tr}(-1)^{2I_3}$ which turns out to be equivalent to the canonical choice leading to the usual Dirac index formula. This derives, for the first time, the well-known sign pre-factors in the wall-crossing formulae universally. In addition, we also explain why the protected spin character of the field theory is actually computed by equivariant index, by showing that the quantum mechanical “angular momentum” operator that appears in the latter is actually a diagonal sum, $J_3 + I_3$, from the spacetime viewpoint.

The paper is organized as follows. Section 2 reviews Ref. [21] and generalize the low energy dynamics to the case of arbitrary number of dynamical charge centers, and note the universal nature of the potential terms. Section 3 defines the index as a method of BPS bound state counting, and in particular makes contact with the field theory indices, commonly known as the 2nd helicity trace and its generalization known as the protected spin character. It turns out that the quantum mechanics found have $SU(2)_L \times SU(2)_R$ R-symmetry, each of which defines chirality operators $(-1)^{2J_3}$ and $(-1)^{2I_3}$. The field theory index corresponds to the former, while mathematical index formulae are more directly related to the latter. We discuss a universal relationship between the two, and conjecture that all BPS bound states in our quantum mechanics are all $SU(2)_R$ singlets.

Section 4 sets up index theorem for this dynamics and show how reduction to the classical moduli manifold may be achieved. Here we show why the naive derivative truncation leading to the geometric quantization is unjustified by demonstrating that there is no natural separation of scales between classically massive directions and classically massless directions. The main point is that $\mathcal{M}_n$ is of finite size, and the quantum gaps due to this are always equal to those along the classically massive directions. We show, nevertheless, how one can deform the theory while preserving the index, such that classically massive modes are decoupled from the evaluation of the index, at the cost of partially broken supersymmetry. We also observe that the reduction process keeps a diagonal subgroup $SU(2)_J$, and identify the generator $J_3 = J_3 + I_3$ as the operator usually used for equivariant index computations. This way, we show that the equivariant index of quantum mechanics on $\mathcal{M}_n$ actually computes the protected spin character of $N = 2$ field theory.

After the derivation of Dirac index in section 4, we go on to evaluate in section 5 the wall-crossing formula by taking into account the bosonic or the fermionic stas-
tics. Projection operators are introduced for the purpose, and the index formula is decomposed into additive contributions from various fixed submanifolds associated with coincident identical particles. The reduced index problems on the fixed submanifolds appears in the full index with a universal degeneracy factor $\sim 1/p^2$, which arises from orbifolding action of the $p$-th order permutation group $S(p)$. Summing up all relevant contributions, we find an expression identical to Manschot et.al.'s wall-crossing formula. Section 6 further supports computation in section 5, by studying general orbifold index theorem. We close with summary and comments in section 7.

\section{$\mathcal{N} = 4$ Moduli Mechanics for $n$ BPS Objects}

In Ref. [21], a general framework for deriving moduli dynamics of dyons of Seiberg-Witten theory was given under the assumption that one works in the field theory vacuum where the central charge are almost aligned in terms of the phases of the respective central charges; in other words, very near the marginal stability wall. This program was then carried out explicitly when one can treat only one dyon as dynamical, with other dyons as external objects. In this note, we wish to generalize this to arbitrary number of charge centers, be they field theory dyons or charged black holes. For this, all dyons should be treated as dynamical, and we will denote their charges as $\gamma_A$’s. For the above derivation of one dynamical center, the proximity to a marginal stability wall played an essential role, allowing the nonrelativistic approximation and thus the moduli space approximation possible, so we need to retain this assumption.

While the moduli dynamics should have $\mathcal{N} = 4$ supersymmetry, as demanded by the BPS nature of the dyons, simple off-shell $\mathcal{N} = 4$ descriptions fail to accommodate key interaction terms. Furthermore, as we will see in section 4 where we compute the supersymmetric index, it is more convenient to take one of the four supersymmetries, say $Q_4$, and give up others. For these reasons, we employ the $\mathcal{N} = 1$ superspace [28] where this supersymmetry, $Q_4$, is manifest. We package $3n$ bosonic coordinates, $x^{Aa}$, and $4n$ fermionic superpartners, $\psi^{Aa}$ and $\lambda^A$, as

$$\Phi^{Aa} = x^{Aa} - i\theta \psi^{Aa}, \quad \Lambda^A = i\lambda^A + i\theta b^A,$$

with $n$ auxiliary field $b^A$’s. The supertranslation generator and the supercovariant derivatives are then,

$$Q = \partial_\theta + i\theta \partial_t, \quad D = \partial_\theta - i\theta \partial_t.$$

2
2.1 Two Centers

The general structure of two dyon dynamics can be inferred from the results in Ref. [21]. The latter actually derived the effective action of a single dynamical dyon in the background of an infinitely heavy core BPS state. When the core state consists of a single dyon, the effective action derived there can also be regarded as the interacting “relative” part $L^{rel}$ of a two-dyon effective action, upon the usual decomposition,

$$L = L^{c.m.} + L^{rel},$$

where the trivial center of mass part was understood to be

$$L^{c.m.} = \int d\theta \frac{i}{2} M_{total} D\Phi^{a}_{c.m.} \partial_{t} \Phi^{a}_{c.m.} - \frac{1}{2} M_{total} \Lambda_{c.m.} D\Lambda_{c.m.},$$

with $M_{total} \to \infty$ understood. Here, let us recall basic structures of $L^{rel}$ as dictated by the supersymmetry.

$L^{rel}$ involves only three bosonic coordinates and four fermionic ones and can be further decomposed as

$$L^{rel} = L^{rel}_{0} + L^{rel}_{1},$$

where

$$L^{rel}_{0} = \int d\theta \left( \frac{i}{2} f(\Phi) D\Phi^{a} \partial_{t} \Phi^{a} - \frac{1}{2} f(\Phi) \Lambda D\Lambda + \frac{1}{4} \epsilon_{abc} \partial_{a} f(\Phi) D\Phi^{b} D\Phi^{c} \Lambda \right),$$

with $a = 1, 2, 3$, and

$$L^{rel}_{1} = \int d\theta \left( i K(\Phi) \Lambda - i W(\Phi)_{a} D\Phi^{a} \right),$$

with the condition

$$\partial_{a} \mathcal{K} = \epsilon_{abc} \partial_{b} \mathcal{W}_{c}$$

imposed. Note that this also implies $\partial_{a} \partial_{a} \mathcal{K} = 0$, which is solved by

$$\mathcal{K} = \mathcal{K}(\infty) - \frac{q}{|\vec{x}|}.$$

We will see shortly how $\mathcal{K}(\infty)$ and $q$ can be read off from the underlying Seiberg-Witten theory.

As was claimed, this Lagrangian is invariant under four supersymmetries,

$$\delta_{\epsilon} x^{a} = i \eta^{a}_{mn} \epsilon^{m} \psi^{n},$$

$$\delta_{\epsilon} \psi_{m} = \eta^{a}_{mn} \epsilon^{m} x^{a} + \epsilon_{m} b,$$

$$\delta_{\epsilon} b = -i \epsilon_{m} \psi^{m},$$

(2.10)
with four Grassman parameters $\epsilon^m$ and with $\psi^A \equiv \lambda$. The $\mathcal{N} = 1$ superspace we employed is related to $\epsilon_4$, so $\mathcal{L}_0$ and $\mathcal{L}_1$ are manifestly and individually invariant under these supersymmetry transformation rules. A less obvious fact, which is nevertheless true, is that the two are also individually invariant under all four
\[
\delta_\epsilon \int dt \mathcal{L}^{rel}_0 = 0 = \delta_\epsilon \int dt \mathcal{L}^{rel}_1,
\]
if the auxiliary field $b$ is kept off-shell. This is the feature that allows an easy generalization to $n$ dynamical centers. The auxiliary field $b$ takes the on-shell value,
\[
b = b_{\text{onshell}} \equiv \frac{1}{f} \left( \mathcal{K} + \frac{i}{4} \eta^{pq}_a \partial_a f \psi^p \psi^q \right),
\]
which generates bosonic potential terms of type $K^2/2f$ and mixes up terms in $\mathcal{L}_{0,1}$. Nevertheless, $\mathcal{N} = 4$ supersymmetry of $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ still holds, now in far more complicated on-shell form.

### 2.2 Seiberg-Witten

Before we extend this to $n$ dynamical dyons, we need to understand the role of the core-probe approximation and how it computes $f$, $K$ and $W$ [21] in terms of the quantities that appear in the Seiberg-Witten theory.

Let us consider a collection of charges $\gamma_A$, and represent it as a semiclassical state. The basic information about the semiclassical dyon state comes from the BPS equations of the Seiberg-Witten theory [29, 30, 31, 32]
\[
\vec{F}_i - i\zeta^{-1} \vec{\nabla} \phi_i = 0, \quad \vec{F}^i_D - i\zeta^{-1} \vec{\nabla} \phi^i_D = 0,
\]
where $F = B + iE$ with magnetic field $B$’s and electric field $E$’s, $\phi$’s are unbroken part of the complex adjoint scalars, each of which are labeled by the Cartan index $i = 1, 2, \ldots, r$. $F_D$’s are defined through the low energy $U(1)$ coupling matrix as
\[
\vec{F}^i_D \equiv \tau^{ij} \vec{F}_j, \quad \tau^{ij} = \frac{\partial \phi^j_D}{\partial \phi^i_D}.
\]
The pure phase factor $\zeta$ is determined by the supersymmetry left unbroken by the charge $\gamma$ in a given vacuum, and equals the phase factor of the central charge $Z_\gamma$ of the configuration.

In a core-probe approximation, we split $\gamma_T = \gamma_h + \sum_{A'} \gamma_{A'}$ and treat the latter $n - 1$ as a fixed background of total charge $\gamma_c = \sum_{A'} \gamma_{A'}$. As we saw in the previous
section, the Lagrangian for the dynamical dyon (of charge $\gamma_h$) is characterized by three objects.

The first is the mass function $f = |Z_{\gamma_h}|$ as in

$$L = \frac{1}{2} f \left( \frac{d\vec{x}}{dt} \right)^2 + \cdots ,$$

(2.15)

where

$$Z_{\gamma_h} = \gamma_h^e \cdot \phi + \gamma_h^m \cdot \phi_D ,$$

(2.16)

with the electric part $\gamma_h^e$ and the magnetic part $\gamma_h^m$ of the charge vector $\gamma_{\gamma_h}$. The scalar fields here solve the above BPS equation with the other $n - 1$ charges $\gamma_{A'}$'s as the background point-like sources. The fact we treat such dyons as point-like objects is justified by going very near a marginal stability wall, since this tends to separate charge centers far apart from one another. As we will see shortly, this proximity to marginal stability wall plays a central role in allowing us to construct nonrelativistic low energy dynamics of dyons.

Clearly $|Z_{\gamma_h}|$ acts as the inertia of the probe dyon, which is position-dependent because of the background: this sort of identification is in accordance with general spirit of how one describe well-separated charged objects [33], which has been tested and used successfully for many soliton systems and even lead to exact moduli space metric in some cases [34, 12]. We also use the notation $Z_\gamma$ for the central charge of the charge $\gamma$ so that $Z_\gamma = Z_\gamma(\infty)$.

The other two, more important for the discussion of BPS bound states, are the potential $K^2/2f$ and the vector potential $W$, so that

$$L = \frac{1}{2} f \left( \frac{d\vec{x}}{dt} \right)^2 - \frac{K^2}{2f} \cdot \vec{W} + \cdots ,$$

(2.17)

where these two are determined entirely by the charge distribution of $\gamma_{A'}$'s as [21]

$$dW = *dK , \quad K = \text{Im}[\zeta^{-1} Z_{\gamma_h}] = \text{Im}[\zeta^{-1} Z_{\gamma_h}] - \sum_{A'} \frac{q_{hA'}}{|\vec{x} - \vec{x}_{A'}|} ,$$

(2.18)

with\#5

$$q_{hA'} = \langle \gamma_h, \gamma_{A'} \rangle / 2$$

\#5This convention for the Schwinger product here follows the one used by Denef in Ref. [14, 15]. The original derivation of dyon dynamics from Seiberg-Witten theory in Ref. [21] used a different convention, such that

$$\langle \gamma, \gamma' \rangle = \langle \gamma, \gamma' \rangle_{\text{Denef}} = 2 \langle \gamma', \tilde{\gamma} \rangle_{\text{Lee--Yi}}$$

The tilde emphasizes the fact that the latter also used half-integral electric charges as opposed to integral ones, which is natural when we compute Coulomb energy. Magnetic charges are integral in either convention.
for the Schwinger product.

These are direct consequences of the equations (2.13), combined with the extra assumption of being near the marginal stability wall. Generically, the bosonic potential would have been

$$|Z_h| - \text{Re}[\zeta^{-1} Z_h],$$  \hspace{1cm} (2.19)

but this reduces to

$$\mathcal{K}^2 / 2 |Z_h| = (\text{Im}[\zeta^{-1} Z_h])^2 / 2 |Z_h|, \hspace{1cm} (2.20)$$

as we move near the marginal stability wall defined by alignment of $Z_h$ and $Z_c$ [21]. The reason why we need this proximity to the marginal stability wall is clearly not because of inherent properties of the system, but rather because of the non-relativistic quantum mechanics approximation we employed. Far away from the wall, the potential energy would be not small compared to rest mass of the particles involved, which will bring dynamics to a relativistic one. However, we do not know how to handle interacting and relativistic particles at mechanical level. Nevertheless, this approximation is good enough since we already know that BPS states are stable far away from marginal stability walls.

An important subtlety we wish to point out here is the choice of $\zeta$. In the core-probe limit, it appears that $\zeta = \zeta_c = Z_{\gamma_c} / |Z_{\gamma_c}|$ is the right choice, since we are treating $\gamma_h$ as an external particle in the background given by $\gamma_c = \sum A' \gamma_{A'}$. However, $\zeta$ is tied to the supersymmetry left unbroken by the configuration and furthermore we are interested in the supersymmetric bound states of $\gamma_c$ and $\gamma_h$. Around such a state, the low energy dynamics should have supersymmetries associated with $\gamma_T = \gamma_c + \gamma_h$ rather than those associated with $\gamma_c$.

One can understand this as capturing the backreaction of the background due to the probe. Failing to do so clearly will give us nonsensical answers since, otherwise, the supersymmetry of the bound state in question would not be aligned with the supersymmetry of the moduli dynamics. In the core-probe approximation, the two happen to be the same, $\zeta_T = Z_{\gamma_T} / |Z_{\gamma_T}| = \zeta_c$, simply because the total central charge is dominated by that of the infinitely heavy core state. As we give up the core-probe dichotomy, this accidental identity will no longer hold, and the preceding discussion tells us that one must always use $\zeta_T$.

As we give up the core-probe approximation and treat all charge centers on equal footing, the moduli dynamics will become quite complicated. The part of the above action that remains least affected by this extension is the Lorentz force, coming from $-\vec{x} \cdot \vec{W}$ type couplings. The coefficient $q$ in $\mathcal{W}$ keeps track of how one particle’s quantized electric (magnetic) charges see the other particle’s quantized magnetic

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#6 Importance of the wall in the derivation of low energy dynamics of dyons was also recognized by others [35].
(electric one) charges. \( \mathcal{W} \) is Dirac-quantized and topological, and furthermore can arise only from sum of two-body interactions. Therefore, this part of the interaction can be reliably computed by adding up all pair-wise Lorentz forces, giving us

\[
- \frac{d\vec{x}}{dt} \cdot \vec{W} \rightarrow - \frac{d\vec{x}_A}{dt} \cdot \hat{W}_A
\]

with

\[
\mathcal{W}_{Aa} = \sum_{B \neq A} q_{AB} \mathcal{W}_{a}^{Dirac}(\vec{x}_A - \vec{x}_B),
\]

where \( q_{AB} = \langle \gamma_A, \gamma_B \rangle / 2 \) and \( \mathcal{W}_{a}^{Dirac} \) is the Wu-Yang vector potential [36] of a 4\( \pi \) flux Dirac monopole. Note that the 4\( \pi \) flux of \( \mathcal{W}_{a}^{Dirac} \) dovetails nicely with half-integer-quantized \( q_{AB} \), as demanded by the Dirac quantization.

For general \( n \) also, \( \mathcal{N} = 4 \) supersymmetry constrains the Lagrangian greatly and, as we will see shortly, the potential energy is tied to such minimal couplings. Knowing the latter will allow us to fix, almost completely, the analog of \( K^2 / 2f \) as well. We will presently see how this works in \( n \) center case. A more difficult question is how the kinetic terms would generalize, to which we will only give a general statement rather than precise solution. In this note, our primary interest is in the supersymmetric index for non-threshold bound states, which is independent of details of kinetic term.

### 2.3 Many Centers

For many centers, it is more convenient not to separate out the center of mass coordinate. Let us label the centers by \( A = 1, 2, \ldots, n \) and denote their \( R^3 \) position as \( x^{Aa} \) and the charge \( \gamma_A \). The \( \mathcal{N} = 1 \) superfield content is

\[
\Phi^{Aa} = x^{Aa} - i\theta \psi^{Aa}, \quad \Lambda^A = i\lambda^A + i\theta b^A,
\]

with \( A = 1, 2, \ldots, n \) and \( a = 1, 2, 3 \). \( \mathcal{N} = 4 \) transformation rules are,

\[
\delta_\epsilon x^A = i\eta^{mn}_a \epsilon^m \psi^{Aa}, \quad \delta_\epsilon \psi^A_m = \eta^{a}_{mn} \epsilon^n \dot{x}^A + \epsilon_m b^A, \quad \delta_\epsilon b^A = -i\epsilon_m \dot{\psi}^{Am},
\]

where as before \( \psi^{A4} \equiv \lambda^A \). We again split the Lagrangian into the kinetic part and the potential part,

\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1,
\]

and look for \( \mathcal{L}_{0,1} \) separately, with off-shell \( b^A \)'s.
The $n$-center version of $\mathcal{L}_1$ is, given (2.22), quite obvious,

$$\mathcal{L}_1 = \int d\theta \left( iK_A(\Phi)\Lambda^A - i\mathcal{W}_{Aa}(\Phi)D\Phi^{Aa} \right) ,$$

(2.26)
since the second term gives precisely the Lorentz force among dyons and while the first is induced from the second by $\mathcal{N} = 4$ supersymmetry; One can check easily that

$$\delta_c \int dt \mathcal{L}_1 = 0$$

(2.27)
under all four supersymmetries, provided that

$$\partial_{Aa}\mathcal{K}_B = \frac{1}{2} \epsilon_{abc} (\partial_{Ab}\mathcal{W}_{Bc} - \partial_{Bc}\mathcal{W}_{Ab})$$

(2.28)
and

$$\epsilon_{abc}\partial_{Ab}\partial_{Bc}\mathcal{K}_C = 0 , \quad \partial_{Aa}\partial_{Ba}\mathcal{K}_C = 0 ,$$

(2.29)
for any $A, B, C$. We already learned that

$$\mathcal{W}_{Aa} = \sum_B \frac{\langle \gamma_A, \gamma_B \rangle}{2} \mathcal{W}_{a}^{\text{Dirac}}(\vec{x}^A - \vec{x}^B) ,$$

so $\mathcal{K}$’s also follow immediately via the $\mathcal{N} = 4$ constraints as

$$\mathcal{K}_A = \mathcal{K}_A(\infty) - \frac{1}{2} \sum_B \frac{\langle \gamma_A, \gamma_B \rangle}{|\vec{x}^A - \vec{x}^B|} ,$$

(2.30)
Note that this obeys the constraints except at the submanifold, say $\Delta \equiv \{x^{Aa} : \vec{x}_A = \vec{x}_B, \langle \gamma_A, \gamma_B \rangle \neq 0 \}$. The quantum mechanics can be very singular at such places also, meaning that we should excise $\Delta$ from $R^{3n}$ and impose the regular boundary condition instead.

It remains for us to determine $\mathcal{K}_A(\infty)$’s. These $\mathcal{K}$’s and $\mathcal{W}$’s can be traced back to the original BPS equations (2.13), and found by keeping track of how motion of each center is affected by the presence of the other $n - 1$ centers. After solving the BPS equations, similarly as in the core-probe limit, we learn that

$$\mathcal{K}_A = \text{Im} [\zeta^{-1} \mathcal{Z}_A] = \text{Im} [\zeta^{-1} Z_A] - \frac{1}{2} \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle}{|\vec{x}_A - \vec{x}_B|} ,$$

(2.31)
where $\mathcal{Z}_A$ is computed from the solution to (2.13) with the other $n - 1$ charge centers taken as the background but, nevertheless, with the phase of the total charge, $\zeta = $
\[ \sum_A Z_A / | \sum_A Z_A| \], used in the equations. As we noted above, this is because we must make sure to use the supersymmetries that are preserved by the bound state of all centers. This can be also seen from \( \mathcal{K}_A(\infty) = \text{Im}[\zeta^{-1} Z_A] \), which allows \( \sum_A \mathcal{K}_A(\infty) = 0 \) as demanded by the antisymmetric Schwinger product. Note that this consistency condition would have been violated if we had used different \( \zeta \)'s for different \( \mathcal{K}_A \)'s.

The other piece \( \mathcal{L}_0 \), containing kinetic terms, is a little more involved. The simplest way to find the most general \( \mathcal{L}_0 \) is via an \( \mathcal{N} = 4 \) superspace. For this, note that the collection \( \{ \Phi^a, \Lambda \} \) can be thought of as dimensional reduction of a \( D = 4 \) \( \mathcal{N} = 1 \) vector superfield \([37, 38]\).\(^\#7\) In this map, \( x^a \)'s come from the spatial part of the vector field, the fermions from the gaugino, and the auxiliary field \( b \) from that of the \( D = 4 \) \( \mathcal{N} = 1 \) vector superfield. See appendix A for more detail. Here, we are mainly interested in \( \mathcal{N} = 1 \) form of such a general \( \mathcal{L}_0 \), which is available in Maloney et.al. [28],

\[
\mathcal{L}_0 = \int d\theta i g_{AaBb} D\Phi^A \partial_t \Phi^B - \frac{1}{2} h_{AB} \Lambda^A D \Lambda^B - ik_{AaB} \Phi^A \Lambda^B + \cdots \tag{2.32}
\]

where the ellipsis denotes four cubic terms that we omit here for the sake of simplicity. This \( \mathcal{L}_0 \) is also invariant under the four supersymmetries we listed above,

\[
\delta_\epsilon \int dt \, \mathcal{L}_0 = 0 \tag{2.33}
\]

on its own with \( b^A \)'s off-shell, provided that various coefficient functions derive from a single real function \( L(x) \) of \( 3n \) variables as

\[
g_{AaBb}(\Phi) = \left( \delta_a^e \delta_b^f + \epsilon_a^e \epsilon_b^f \right) \partial_{Ae} \partial_{Bf} L(\Phi),
\]

\[
h_{AB}(\Phi) = \delta_{ab} \partial_{Aa} \partial_{Bb} L(\Phi),
\]

\[
k_{AaB}(\Phi) = \epsilon_{ae} \partial_{Ae} \partial_{Bf} L(\Phi),
\]

\vdots \tag{2.34}

The \( \mathcal{N} = 4 \) supersymmetry requires all those terms as well. See Appendix A for the complete form of \( \mathcal{L}_0 \).

Figuring out the precise form of \( L \) for \( n \) charge centers requires further work. For a single dynamical dyon in the core-probe limit, we know that it is related to the central charge function as \( \partial^2 L = |Z| \). We expect that there exists a similarly intuitive generalization for \( n \) particles case as well. In this note, we are primarily

\(^\#7\)In this version of \( \mathcal{N} = 4 \) superspace, \( \mathcal{L}_1 \) is not obvious. On the other hand, a more extended harmonic superspace form has been found to accommodate both kinetic terms and potential terms [39].
interested in counting nonthreshold bound state, for which details of \( L \) does not enter. Determination of \( L \) can become an important issue, when we begin to consider non-primitive charge states. See next subsection for related comments.

Again, the main point here is that \( L_0 \) and \( L_1 \) are invariant under the four supersymmetries separately when we keep the auxiliary fields \( b^A \)'s off-shell. Combining the two, it follows that the full Lagrangian

\[
L_0 + L_1
\]

is also invariant under all four supersymmetries. Integrating out \( b^A \)'s generates potentials of type \( \sim K^2 \) and mixes up terms in \( L_0 \) and \( L_1 \), but \( N = 4 \) supersymmetries of the entire Lagrangian remain intact.

### 2.4 Kinetic Function \( L \): BPS Dyons vs BPS Black Holes

Note that the potential part \( L_1 \) of the Lagrangian looks identical to the similar expression previously found by Denef [14], which has been later used extensively for counting BPS black holes bound states [25, 22]. The latter relied on \( N = 4 \) quantum mechanical supersymmetry. Although we started with Seiberg-Witten theory for the derivation of \( L_1 \), this part of Lagrangian is entirely determined by \( N = 4 \) supersymmetry combined with long-distance Lorentz forces among charge centers. Thus appearance of the same \( L_1 \) is hardly surprising. In fact, when we apply \( L_1 \) to BPS black holes, it is even more trustworthy, since the Abelian approximation that would underlie such an interaction form is valid all the way to horizon. One cannot say the same for field theory dyons, since at short distance non-Abelian nature must be taken into account. Nevertheless, as long as we are near a marginal stability wall and only long-distance physics matters, it is clear that \( L_1 \) is capable of describing both dyons and black holes.

This does not mean that the moduli dynamics of BPS dyons and those of BPS black holes are identical. The difference resides in the kinetic part \( L_0 \) of the Lagrangian. As demanded by \( N = 4 \) supersymmetry, \( L_0 \) is determined by a single scalar function \( L \) of the \( n \) position vectors \( \vec{x}_A \). For instance, \( L \) for many BPS black holes of an identical charge was found by Maloney et. al. [28]

\[
L(\vec{x}_1, \vec{x}_2, \ldots) = -\frac{1}{16\pi} \int dx^3 \psi^4 \tag{2.36}
\]

where \( \psi = 1 + \sum_A (m/|\vec{x}_A - \vec{x}|) \) with the mass \( m \). On the other hand, for two-center dyon case, we expect smooth behavior near \( \vec{r} = 0 \) [21] since, when the mutual distance is small, non-Abelian cores cannot be ignored and will smooth out Coulombic
singularities. Even if we use the naive Abelian results, $\partial^2 L \sim 1/r$ at most. Comparing this to the two-body case of the supergravity result shows a substantial difference when the two objects begin to overlap.

Indeed, there are situations when the two theories are expected to give different answers. There is no known example of $N = 2$ field theory dyon which is a bound state of two or more identical dyons. For black holes, however, no such restriction seems to exist. If a BPS black hole of charge $\gamma$ exist, we expect BPS black holes of charge $N\gamma$ also to exist, in fact with large entropy. In the present context of moduli quantum mechanics, the latter corresponds to a collection of many charge centers with many flat directions extending to spatial infinities and may be realized as threshold bound states thereof. In such cases, the kinetic term of the effective action at both short distances and long distances could be important. This problem is an important outstanding issue in wall-crossing phenomena in general, for it provided much-needed input data on what dyons or black holes are available, to begin with, to form bound states.

Explicit forms of $L$ for $n$ BPS dyons and for $n$ BPS black holes, respectively, will be studied in a separate work.

3 R-Symmetry, Chirality Operators, and Indices

We wish to compute index of the preceding quantum mechanics

$$\text{Tr} \left( (-1)^F e^{-sH} \right) .$$

Since the quantum mechanics is gapped, of which much discussion will follow in next section, this quantity is truly independent of the parameter $s$. Thus, following the standard arguments, we will compute this in small $s$ limit. Before proceeding, however, it is important to clarify what we mean by the operator $(-1)^F$. In order for the index to make sense, this operator needs to anticommute with supercharge(s),

$$\{(-1)^F, Q\} = 0 ,$$

which is the condition needed for 1-1 matching and thus cancelation between bosonic and fermionic states for nonzero energy eigenvalues. Clearly this is not enough to fix the overall sign of $(-1)^F$ on the Hilbert space, and an index is also plagued by this ambiguity. When we compute an index of standard Dirac operator or de Rham operators, there is usually a canonical choice that is used widely. We will come back to this, later in next section, but the choice is a matter of convenience only and, a priori, has no physical significance.
At field theory level, however, we have an unambiguous and useful definition of such an index, say, the second helicity trace,

$$\Omega = -\frac{1}{2} \text{Tr} \left( (-1)^{2j_3} (2J_3)^2 \right),$$

(3.2)

where the trace $\text{Tr}$ is over a single particle sector of a given charge. We wish to fix the sign of the quantum mechanical index, in accordance with this. Irreducible BPS multiplets, tensor products of half-hyper-multiplet and a spin $j$ multiplet, have the index

$$\Omega \left( [j] \otimes ([1/2] \oplus 2[0]) \right) = (-1)^{2j} (2j + 1),$$

(3.3)

so often we also write,

$$\Omega = \text{Tr} \left( (-1)^{2j_3} \right),$$

(3.4)

with the factored-out half-hypermultiplet understood. This naturally reduces to the low energy dynamics of dyons, which then must correspond to an index defined with a chirality operator that acts exactly like $(-1)^{2j_3}$

$$\Omega \leftrightarrow \text{Tr} \left( (-1)^{2j_3} \right),$$

(3.5)

but of course we need to ask here how such an operator is realized in the quantum mechanics.

As can be inferred from discussions in Ref. [21], the quantum mechanics of previous section are equipped with $SO(4) = SU(2)_L \times SU(2)_R$ R-symmetry. This is easiest to see in how the fermion bilinear couplings to $dK$ and $dW$ combine to give,

$$-\frac{i}{2} \eta^{a}_{mn} \partial_{Aa} K_B \psi^{Am} \psi^{Bn}$$

(3.6)

in the component form, where, as before, $\psi^{Am=1,2,3} = \psi^{Aa=1,2,3}$, $\psi^{A4} \equiv \lambda^A$, and $\eta$ is the 't Hooft self-dual symbol. The above form is precise when the metric is flat, but appropriately modified preserving $SO(4)$ symmetry when it is not. For each particle indexed by $A$, bosonic coordinates are in $(3,1)$ representations while the fermions are in $(2,2)$. Since spatial rotations rotate $\vec{x}^A$ as 3-vectors, $SU(2)_L$ should be interpreted as the rotation group, while $SU(2)_R$ must be descendant of $SU(2)_R$ R-symmetry of the underlying Seiberg-Witten theory. The latter rotates only fermions and leaves the position coordinate intact.#8

In particular, the four supersymmetries are labeled by the $SO(4)$ vector index, and thus are in $(2,2)$ representations. Denoting generators of these two $SU(2)$'s by $J$ and $I$, respectively, we thus find

$$\{(-1)^{2j_3}, Q\} = 0 = \{(-1)^{2j_3}, Q\}.$$  

(3.7)

#8In the core-probe approximation of Ref. [21], only $SU(2)_R$ were generically there, but this was an artefact of treating some of dyon centers as fixed background.
The quantum mechanics have two unambiguous and physically meaningful chirality operators that can be used for index computation. The desired \((-1)^{2J_3}\) is one of them, therefore, we have an unambiguous way of computing the field theory index from the low energy quantum mechanics.

On the other hand, there is an interesting and universal relationship between these pair of chiral operators in the quantum mechanics. Restricting our attention to the relative part of the low energy dynamics again, we have

\[ (-1)^{2J_3} = (-1)^{\sum_{A<B} \langle \gamma_A, \gamma_B \rangle} + n - 1 \times (-1)^{2I_3} \tag{3.8} \]

where \(n\) is the number of dyons in the low energy dynamics. This is easy to see by considering how the two \(SU(2)\) generators are constructed in the quantum mechanics. For \(SU(2)_R\), which rotate only fermions, we have

\[ I_a = \sum_A \left( -\frac{i}{8} \epsilon_{abc} [\hat{\psi}^{Ab}, \hat{\psi}^{Ac}] + \frac{i}{4} [\hat{\psi}^{Aa}, \hat{\lambda}^A] \right) \tag{3.9} \]

where the hat signifies the unit normalized fermion. The spatial rotation generators

\[ J_a = L_a + \sum_A \left( -\frac{i}{8} \epsilon_{abc} [\hat{\psi}^{Ab}, \hat{\psi}^{Ac}] - \frac{i}{4} [\hat{\psi}^{Aa}, \hat{\lambda}^A] \right) \tag{3.10} \]

are similar but differ in two aspects: first, since \(SU(2)_L\) rotates \(\vec{x}_A\)’s, the generators include the orbital angular momentum \(L\); secondly the fermions rotate differently, as reflected in the sign of the last term. This latter difference generates a relative sign between the two chiral operators for each \((2,2)\) representation of fermions, thus explaining \((-1)^{n-1}\). The other sign is equally simple, and come from well-known piece of charge-monopole physics, where the orbital angular momenta is schematically something like

\[ \vec{L} \sim \sum_A (\vec{x}_A \times \vec{\pi}_A) + \sum_{A>B} \frac{\langle \gamma_A, \gamma_B \rangle}{2} \frac{\vec{x}_A - \vec{x}_B}{|\vec{x}_A - \vec{x}_B|} \tag{3.11} \]

with the covariantized momenta \(\vec{\pi}_A\). The orbital angular momentum is constructed from tensor product of spin \(\langle \gamma_A, \gamma_B \rangle/2\) representations times usual integral angular momentum. Then regardless of which particular \(SU(2)_L\) multiplet the state is, integrality vs half-integrality of the orbital angular momentum is unambiguously determined as

\[ (-1)^{2L_3} = (-1)^{\sum_{A>B} \langle \gamma_A, \gamma_B \rangle} \tag{3.12} \]

Note that this does not require \(\vec{L}\) being symmetry operators.

Thus, we have the second helicity trace of \(N = 2\) dyons which can be computed via the low energy quantum mechanics as

\[ \Omega = \text{Tr} \left( (-1)^{2J_3} e^{-sH} \right) = (-1)^{\sum_{A<B} \langle \gamma_A, \gamma_B \rangle} + n - 1 \times \text{Tr} \left( (-1)^{2I_3} e^{-sH} \right) \tag{3.13} \]
In the subsequent computation, with this relation in mind, we will eventually identify \((-1)^{2I_3}\) as the canonical chirality operator \((-1)^F\). For this, there is another sign issue to settle, later when we begin to quote index formula from literature, since the latter come with a canonical choice of \((-1)^F\), which may or may not equal to our choice, \((-1)^{2I_3}\), but we postpone this to end of next section.

Another reason why \((-1)^{2I_3}\) is useful, even though we ultimately want \((-1)^{2J_3}\), can be found in the observation [41] that all explicitly constructed field theory BPS states, to date, are in $SU(2)_R$ singlets times the universal half-hypermultiplet (from the center of mass part in quantum mechanics viewpoint). If this is generally true, we can see that the index with \((-1)^{2I_3}\) is always positive and truly counts the degeneracy. An interesting question, therefore, is whether in the low energy quantum mechanics we derived all supersymmetric bound states are $SU(2)_R$ singlets.

An interesting variant of the second helicity trace is the protected spin character [41],

$$\text{Tr} \left((-1)^{2I_3} y^{2J_3+2I_3}\right),$$

where again we took out the universal half-hypermultiplet from the trace for simplicity. This clearly reduces to, in quantum mechanics,

$$\text{Tr} \left((-1)^{2J_3} y^{2J_3+2I_3}\right).$$

Later we will also see how this quantity is naturally computed, after we reduce the index problem to the more familiar one that relies only the classical moduli space $\mathcal{K} = 0$, by the equivariant index that counts “angular momentum” representations. As we will see, this reduction process cannot carry the entire $\mathcal{N} = 4$ supersymmetry, and, of $SO(4)$ R-symmetry, only a diagonal $SU(2)$ subgroup generated by $J + I$ survives as global symmetry. The equivariant index on $\mathcal{K} = 0$ space does not count representations under spatial rotations but under simultaneous rotation of spatial $SU(2)_L$ and $N = 2$ R-symmetry $SU(2)_R$.\footnote{Of course, if the $SU(2)_R$ singlet hypothesis actually holds for the ground state sector, the end result would not know about $I_3$, anyway. In fact, on the basis of this hypothesis, this equivalence was anticipated previously [22]. Our argument in section 4.4 will prove the identity without such an assumption.}

\section{4 Index Theorem for Distinguishable Centers}

Now we turn to the problem of counting ground states of the above quantum mechanics, or equivalently counting BPS bound states of $n$ dyons. Since the quantum
mechanics has a potential, \( \sim K^2 \), one may expect that the problem can be reduced naturally to another problem on the classical moduli space of \( 2(n - 1) \) dimensions, say,

\[
\mathcal{M}_n = \{ x^{Aa} \mid K_A = 0, \ A = 1, 2, \ldots, n \} / R^3 ,
\]

where the division by \( R^3 \) is to remove the flat center of mass part. This classical moduli space is generically a little more complicated since some of the centers could be associated with identical particles, which we will deal with in the next section.

This reduction is not as straightforward as one might think, however. Ref. [25], for example, suggested that one can ignore the (then unknown) kinetic part of the Lagrangian. Effectively, in our notation, this would involve a geometric quantization of \( \mathcal{L}_1 \),

\[
\mathcal{L}_{\text{geometric}} = \mathcal{L}_1 = -b^A K_A - \mathcal{W}_{Aa} x^{Aa} + \frac{i}{2} \partial_{Aa} \mathcal{K}_{B} \eta_{mn} \psi^{Am} \psi^{Bn} ,
\]

which is obtained by truncating higher-derivative parts in \( \mathcal{L}_0 \). The auxiliary fields, \( b^A \)'s, are now Lagrange multipliers, imposing \( K_A = 0 \) as constraints and leaving a lowest Landau level problem on \( \mathcal{M}_n \) with the magnetic fields \( \sum A d \mathcal{W}_A \). However, computation of the resulting index, if we take \( \mathcal{L}_{\text{geometric}} \) verbatim, generates wrong results relative to other known spectrum; The geometric quantization of \( \mathcal{L}_{\text{geometric}} \) would lead to index formula that is known to generate empirically incorrect answers.

For two body case, for example, the degeneracy \( 2|q| \) has been known to be the correct answer for many explicit constructions. See, for example, Ref. [9] and Ref. [21] for explicit two-dyon bound state construction in the weakly coupled and in the strongly coupled regions of Seiberg-Witten theory, respectively. On the other hand, the naive lowest level Landau problem (or equivalently the geometric quantization problem) gives \( 2|q| + 1 \). One would hope that the effect of fermions in \( \mathcal{L}_{\text{geometric}} \) will fix this, but this apparently does not happen.

The truncation of the kinetic terms in the presence of fermions is quite subtle, since while bosons acquire a symplectic structure thanks to the magnetic field, there is no such analog for fermions. Setting the kinetic term of fermions to zero will cause the canonical commutator ill-defined, making the whole reduction process ambiguous. One can try to reinstate kinetic terms on \( \mathcal{M}_n \) as a regulator, but then, the main issue is that the number of fermions in \( \mathcal{L}_1 \) is \( 4(n - 1) \) real while the number of bosons is \( 2(n - 1) \) real, and these lead to de Rham cohomology problem on \( \mathcal{M}_n \). For not too small \( q \) and when \( \mathcal{M}_n \) is Kähler, for example, the index of such a quantum mechanics coincides precisely with the state counting of the bosonic geometric quantization problem,\(^{11}\) again giving us wrong result for the index.

\(^{11}\)See for example Ref. [40], where in effect a regularized version of these problems were considered with kinetic terms on \( \mathcal{M}_n \) and for its fermionic partners present.
Really at the heart of the problem is, however, the fact that the classical massive directions are in fact no more massive than the classically massless directions. Because the classical moduli space $\mathcal{M}_n$ is of finite size\[12, it comes with various gaps at quantum level, and it so happens that these quantum gaps are one-to-one matched and identical to the gaps associated with the classically massive directions: the dynamics cannot be really split into two distinct sectors of heavy and light modes, at all, and contrary to initial expectation, the reduction to $\mathcal{M}_n$ cannot be justified.

In fact, this lack of separation of scales is easiest to see in how fermions enter the Hamiltonian. Half of fermions get mass from $dK$ while the other hand get mass from $dW$. However, $\mathcal{N} = 4$ supersymmetry of the quantum mechanics tells us that the two are one and the same object, and fermions coupling to $dK$ are no more heavier than those coupling to $dW$.

Fortunately, we can still decouple these classically massive directions in the computation of the index problem. This involves a deformation that breaks all but one supersymmetry, yet because the quantum mechanics is gapped and the surviving supercharge is effectively a Fredholm operator, it can be done while preserving the index. Later in the section and in Appendix B, we explicitly show that, as far as computation of the index goes, we may reduce the moduli dynamics to an effective $\mathcal{N} = 1$ supersymmetric quantum mechanics with target $\mathcal{M}_n$,

$$L_{\text{for index only}}^{\mathcal{N}=1}(\mathcal{M}_n) = \frac{1}{2} G_{\mu\nu} \dot{z}^\mu \dot{z}^\nu + i \frac{1}{2} G_{\mu\nu} \psi^\mu \dot{\psi}^\nu + \cdots - A_\mu \dot{z}^\mu + \cdots, \quad (4.3)$$

where $A$ is a gauge field on $\mathcal{M}_n$ such that

$$dA = F \equiv d\left( \sum_A W_A dx^A \right) \bigg|_{\mathcal{M}_n}. \quad (4.4)$$

and $G$ is the induced metric on $\mathcal{M}_n$. This Lagrangian must be used only for the purpose of computing index.

The key point here is that the number of fermions is exactly half of that in $L_{\text{geometric}}$. Since these fermions live on the tangent bundle of $\mathcal{M}_n$, we have a nonlinear sigma model with real fermions. The relevant wavefunctions are spinors on $\mathcal{M}_n$ and the index in question becomes a Dirac index,

$$\mathcal{I}_n(\{\gamma_A\}) = \int_{\mathcal{M}_n} \text{Ch}(\mathcal{F}) \hat{A}(\mathcal{M}_n) = \int_{\mathcal{M}_n} \text{Ch}(\mathcal{F}) \quad (4.5)$$

with the Chern character $\text{Ch}$ of $\mathcal{F}$. $\hat{A}$ is the $A$-roof genus of the tangent bundle, which will be shown to be trivial for all $\mathcal{M}_n$’s. This formula counts the index when

\[12\]There are also some exotic cases corresponding to the scaling solutions. In these cases, the moduli space is non-compact, from short distance side, but its volume in the naive flat metric is still finite.
we view individual charge centers as distinguishable; in section 5, we will extend the formula appropriately when identical particles are involved and along the way see why the rational invariants of the form $\sim \Omega/p^2$ with integer $p > 1$ appears in various wall-crossing formulae.

The Dirac index found here is consistent with de Boer et. al.’s observation [25] that empirically correct answers emerge for $n = 2$ and $n = 3$ if one assumes that the relevant quantum mechanics admit spinors on $\mathcal{M}_n$ as the wavefunction. This can be then generalized to the refined index (or equivariant index) and make contact with a series of recent works by Manschot et.al [22, 23].

4.1 Two Centers: Reduction to $S^2$

Supersymmetric ground states were found and counted for $n = 2$ case in Ref. [21], which gave the correct answer of $2q$ at the end of the day. As expected, the wavefunctions are all maximized near the classical “true” moduli space $\mathcal{K} = 0$, which was nothing but a two-sphere threaded by a flux of $4\pi q$. However, the wavefunctions can also be seen to be very diffuse, too much so to let us call it “localized” there.

Here, we will illustrate why a naive reduction to $\mathcal{M}_2 = S^2$ by throwing away entire kinetic term is wrong. After the latter procedure, one ends up with $\mathcal{L}_{\text{geometric}}$ for which we need to either geometrically quantize over $S^2$ or regularize the dynamics by reinserting kinetic term on $S^2$ and concentrate on the lowest Landau level. If we follow the second viewpoint, we end up with a two dimensional nonlinear sigma model with four real fermions, so effectively we will have thrown away only the bosonic radial coordinate from the original moduli dynamics.

Let us consider the zero point energy of the relative part of the two-center mechanics. Three bosons can be split into “radial” directions, on which $\mathcal{K}$ and the mass function $f$ depend, and flat “angular” directions. With $\mathcal{K} = a - q/r$ and positive $a$ and $q$, the ground state is at $r = r_0 = q/a$, and the radial direction becomes a harmonic oscillator of frequency $w = a^2/f(r_0)q$, so

$$E_{\text{radial}} \simeq \left( m_b^{\text{radial}} + \frac{1}{2} \right) w \geq \frac{a^2}{2f(r_0)q} .$$

The angular part, although classical flat, also comes with a gap due to the finite volume, and the energy quantization there goes as

$$E_{\text{sphere}} \simeq \frac{\tilde{\mathcal{L}}^2 - q^2}{2f(r_0)r_0^2} \geq \frac{a^2}{2f(r_0)q} ,$$

since the angular momentum is bounded below, in the presence of the flux, by $q$. The four real fermions are paired up into two fermionic oscillators of the same frequency
as above, so we get contribution from the fermion sector as

\[ E_{\text{fermion}} \simeq (m_f + m'_f - 1) w \geq -\frac{a^2}{f(r_0)^q}, \]  

(4.8)

where we again see that there is only one scale in the fermion sector also. Of course, the behavior of fermionic degrees of freedom must be the same as the bosonic ones, since we have supersymmetry.

This shows that, without further deformation, the gap of the classically massive radial direction is exactly the same as the rest of the degrees of freedom. If we wish to localize the problem to \( M_2 = S^2 \), by removing the radial mode, we must do something else so that the gap along the radial direction and the gap along \( M_2 \) are different, but this seems impossible under the \( \mathcal{N} = 4 \) supersymmetry of the quantum mechanics.

Let us remember here that, for the evaluation of index, one needs only two things: a Dirac operator of some kind and a chirality operator that anticommutes with it. One would like to compute the index

\[ \text{Tr}(\mathcal{F}) e^{-sH}, \]  

(4.9)

for interacting part of the theory. Let us, for the sake of definiteness, take \( H = Q^2_4 \), and evaluate

\[ \text{Tr}(\mathcal{F}) e^{-sQ^2_4}. \]  

(4.10)

\( \mathcal{N} = 4 \) supersymmetry is useful since it constrains dynamics but all of them are not really necessary to define an index. It is clear that, as long as we preserve this quantity, we can even break \( \mathcal{N} = 4 \) supersymmetry.

Of course, \( \mathcal{N} = 4 \) supersymmetry is important when it comes to generating correct supermultiplet structure to the bound state, but that only concerns the free center of mass part. The index must be computed from relative interacting part of the dynamics, only for which we will break \( \mathcal{N} = 4 \) supersymmetry.

Thus we are motivated to give up \( d\mathcal{K} = *d\mathcal{W} \) condition, thereby keeping only \( Q = Q_4 \), unbroken. Let us replace

\[ \mathcal{K} \to \xi \mathcal{K}, \]  

(4.11)

with some arbitrarily large number \( \xi \) while keeping \( \mathcal{W} \) as it is. The ground state energy counting is now

\[ E_{\text{radial}} + E_{\text{sphere}} + E_{\text{fermion}} \geq \frac{\xi w}{2} + \frac{w}{2} - \frac{\xi w + w}{2}, \]  

(4.12)

since the half of the fermions (\( \lambda \) and \( \psi^r \)) get the mass from \( d(\xi \mathcal{K}) \) and the other half from \( d\mathcal{W} \). The angular momentum sector mass-gap, \( w/2 = q/2 f(r_0) r_0^2 \), is unchanged
since the classical vacuum, $K = 0$ and thus the radius $r_0$, and $W$ are intact under this rescaling.

It is not difficult to see that the reduced dynamics, after integrating out heavy modes, is a $N = 1$ nonlinear sigma model onto $M_2 = S^2$ coupled to an external vector field $W$. See Appendix B for complete detail of the reduction process. We note that since $M_2 = S^2$ happens to be Kähler, the unbroken supersymmetry gets accidentally extended to $N = 2$, although this is not important for our purpose.

4.2 Many Centers: Reduction to $M_n$

Similarly, we wish to deform the theory by rescaling $K_A \to \xi K_A$, when we have many dynamical charge centers, as well

\[
\begin{align*}
\mathcal{L}_{\text{deformed}} &= \int d\theta \left( \frac{i}{2} g_{AaBb} D\Phi^A \partial_t \Phi^B - \frac{1}{2} h_{AB} \Lambda^A \Lambda^B - i k_{AaBb} \Phi^A \Lambda^B + \cdots \right. \\
&\quad \left. + i \xi K_A(\Phi) \Lambda^A - i W_{Aa}(\Phi) D\Phi^A + \cdots \right) , \tag{4.13}
\end{align*}
\]

where $\xi$ is an arbitrarily large number. As in the two-center case, the bosonic potentials are quadratic in $K_A$'s and there are $n - 1$ “radial” directions that are of mass $\sim \xi$. There are also $2(n - 1)$ fermions that couple to $d(\xi K_A)$'s, so they are also of mass $\sim \xi$. The two sets can be decoupled together, thereby reducing the index problem to $M_n$ with real fermions. It leaves behind a $N = 1$ supersymmetric quantum mechanics onto $M_n$ with $2(n - 1)$ bosons and $2(n - 1)$ real fermions. The process does not affect the free center of mass part, so the latter still comes with 3 bosonic coordinates and 4 fermionic ones.

We may further deform the kinetic part, $\mathcal{L}_0$, by taking the simplest form of the kinetic function,

\[
L = \frac{1}{2} \sum_A m_A \bar{x}^A \cdot \bar{x}^A , \tag{4.14}
\]

which amounts to

\[
g_{AaBb} = \delta_{AB} \delta_{ab} m_A , \quad h_{AB} = \delta_{AB} m_A , \quad k_{AaBb} = 0 , \tag{4.15}
\]

and setting cubic terms to zero as well. The simplest way to justify this deformation is that the kinetic function approaches this flat metric when distances between charge centers approach infinity. This asymptotic form is more than good enough since we can always tune the field theory vacuum, so that we stay arbitrarily near the marginal stability wall. There, $\text{Im}[\xi^{-1} Z_A]$ approaches zero, and the submanifold $M_n$
is arbitrarily large. Since the index cannot change under the continuous and sign-preserving deformation of $\text{Im}[\zeta^{-1}Z_A]$, and since the ambient metric is effectively flat for large $M_n$, the index will be unaffected by this choice of metric.

This leaves us with a very simple $N = 1$ quantum mechanics

$$L_{\text{deformed}} = \int d\theta \left( \frac{i}{2} m_A D\Phi^A a \partial_t \Phi^A a \right) - \frac{1}{2} m_A \Lambda^A D\Lambda^A$$

$$+ i \xi K_A(\Phi) \Lambda^A - i \mathcal{W}_{Aa}(\Phi) D\Phi^A a ,$$

(4.16)

with target $R^{3n}$ modulo submanifolds given by $K_A = \pm \infty$. Of this, the free center of mass positions $R^3$ and the accompanying four real fermions decouples, leaving behind the interacting part of the moduli dynamics onto $R^{3(n-1)}$. This free part is also essential since it generates the basic BPS multiplet structure (whose content equals half of a hypermultiplet) to the bound state. Then, by taking $\xi \to \infty$, we decouple $n - 1$ "radial" directions and $2(n - 1)$ accompanying heavy fermions, and end up with a nonlinear sigma model onto $M_n$ with real $2(n - 1)$ fermionic partners. See appendix for detailed derivation of this fact.

Thus, we arrive at the effective Lagrangian, which can be used for the purpose of computing the index of the original $n$ center problem,

$$L_{\text{for index only}} = \frac{1}{2} G_{\mu\nu} \dot{z}^\mu \dot{z}^\nu - A^A_{\mu} \dot{z}^\mu + \frac{i}{2} G_{\mu\nu} \psi^\mu \dot{\psi}^\nu + \frac{i}{2} G_{\mu\nu} \psi^\mu \dot{z}^\lambda \Gamma^\nu_{\lambda\beta} \psi^{\beta} + \frac{i}{2} F_{\mu\nu} \psi^\mu \psi^\nu$$

(4.17)

again with the induced metric $G$ on $M_n$ and, as we already noted,

$$dA = F \equiv d \left( \sum_A \mathcal{W}_{Aa} dx^A a \right) \bigg|_{M_n}.$$  

(4.18)

Since each $\mathcal{W}_A$ is a sum of Dirac monopoles at $\vec{x} = \vec{x}_B$’s, we find

$$F = d \left( \sum_{A \neq B} \frac{\langle \gamma_A, \gamma_B \rangle}{2} \mathcal{W}_a^{\text{Dirac}}(\vec{x}_A - \vec{x}_B) dx^A a \right) \bigg|_{M_n}$$

$$= d \left( \sum_{A > B} \frac{\langle \gamma_A, \gamma_B \rangle}{2} \mathcal{W}_a^{\text{Dirac}}(\vec{x}_A - \vec{x}_B) dx^A a - dx^B a \right) \bigg|_{M_n}$$

$$= \sum_{A > B} \frac{\langle \gamma_A, \gamma_B \rangle}{2} F^{\text{Dirac}}(\vec{x}_A - \vec{x}_B) ,$$

(4.19)

where $F^{\text{Dirac}}$ is the Dirac monopole of flux $4\pi$. Of four supercharges, $Q_4$ survives the deformation process above, which is then further reduced to $Q_{M_n}$ as heavy modes are integrated out.

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4.3 Index for $n$ Distinguishable Centers

Since this is the plain old nonlinear sigma model twisted by the minimal coupling to $\mathcal{A}$, the reduced supercharge is represented geometrically as the Dirac operator with a $U(1)$ gauge field

$$Q_A \rightarrow Q_{\mathcal{M}_n} = \gamma^\mu (i\nabla_\mu + A_\mu) ,$$

(4.20)

whose index, according to Atiyah-Singer index theorem, is given by

$$I_n(\{\gamma_A\}) = \text{Tr} \left( (-1)^{F_{\mathcal{M}_n}} e^{-sQ^2} \right) = \int_{\mathcal{M}_n} Ch(\mathcal{F}) \hat{A}(\mathcal{M}_n) ,$$

as promised, where we must assume a canonical choice of the chirality operator. This is

$$(-1)^{F_{\mathcal{M}_n}} = (2i)^{n-1} \hat{\psi}^1 \ldots \hat{\psi}^{2(n-1)} ,$$

in terms of properly normalized and ordered fermions. See next subsection for how this choice squares off with physically motivated chirality operators $(-1)^{2J_3}$ and $(-1)^{2I_3}$ of section 3 and how the latter chirality operators reduce on $\mathcal{M}_n$.

Curiously enough, the A-roof genus $\hat{A}$ does not contribute to the index, thanks to the simple topology of $\mathcal{M}_n$. To see this, let us first note that the ambient space, in which $\mathcal{M}_n$ is embedded is essentially $R^{3n}$. For instance, take $\vec{x}_1 = 0$ to remove the translation invariance and make the ambient space $R^{3(n-1)}$, and then impose $\mathcal{K}_A = 0$, of which $n - 1$ are linearly independent. Therefore, $\mathcal{M}_n$ is a complete intersection in $R^{3(n-1)}$. Since A-roof genus is a multiplicative class, we have an identity,

$$\hat{A}(T\mathcal{M}_n) \hat{A}(N\mathcal{M}_n) = \hat{A} \left( TR^{3(n-1)}|_{\mathcal{M}_n} \right) = 1$$

(4.21)

among the tangent and the normal bundles. However, $d\mathcal{K}_A$’s are nowhere vanishing normal vectors on $\mathcal{M}_n$, and thus the normal bundle $N\mathcal{M}_n$ is also topologically trivial\#13, and

$$\hat{A}(T\mathcal{M}_n) = 1.$$  

(4.22)

It is important to note that this decoupling depends only on the topology of the ambient space, namely the original $3n$ dimensional moduli space, near the surface $\mathcal{K}_A = 0$.\#14 This triviality is also implicit in the explicit evaluation of these Dirac

\#13We are indebted to Bumsig Kim for pointing this out to us.

\#14Note that similar argument will not lead to triviality of other multiplicative class since typically they require complex bundles in order to be defined. For instance, $Td(\mathcal{M}_{2k})$ or $c(\mathcal{M}_{2k})$ cannot be argued to be trivial in this manner, for instance, since the normal bundle of $\mathcal{M}_{2k}$ inside the relative position space $R^{3(2k-1)}$ is of odd dimension and, if irreducible, cannot be complex. In particular $\mathcal{M}_2 = S^2$ has a real line as the fibre when embedded into $R^3$, which is consistent with the nontrivial $c_1$. 

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indices in Ref. [23]. Although it turned out there that $\hat{A}(T\mathcal{M}_n)$ factor did make an important difference for evaluation of the equivariant index, the non-equivariant limit is consistent with trivial $\hat{A}$.

4.4 Reduced Symmetry, Index, and Internal Degeneracy

Since we arrived at the nonlinear sigma-model on $\mathcal{M}_n$ only after the deformation of the dynamics, which in particular removes the extended supersymmetry, we must first ask whether various operators survive this procedure of deformation and the subsequent reduction process $\xi \to \infty$. Of the four original supersymmetries, $Q_4$ survives the deformation. It’s on-shell form will be smoothly deformed as well, which goes like

$$Q_4 = \cdots + \lambda^A \mathcal{K}^A \quad \rightarrow \quad Q_4 = \cdots + \xi \lambda^A \mathcal{K}^A .$$

The ellipsis denotes parts unaffected by the deformation. We emphasize again that this supersymmetry is explicitly preserved since the deformed Lagrangian (4.13) is written in the superspace associated with $Q_4$. Then, given that $Q_4$ is a gapped elliptic operator, at $\xi = 1$, this deformation preserves the index as we increase $\xi$ [11]. This $Q_4$ reduces to $Q_{\mathcal{M}_n}$ of the nonlinear sigma model on $\mathcal{M}_n$, and obviously the Hamiltonian, $Q_4^2/2$, gets similarly deformed and eventually reduced to the natural one on $\mathcal{M}_n$.

This leaves the global symmetry operators and the chirality operators. With the $\mathcal{N} = 4$ supersymmetry partially broken, the $SO(4)$ R-symmetry can be easily seen to be broken. On the other hand, the deformation commutes with rotation of $\bar{F}_A$’s, so we expect to see some $SU(2)$ symmetry does survive the process. The question is which $SU(2)$ in $SO(4) = SU(2)_L \times SU(2)_R$ remains unbroken. The answer is the diagonal subgroup, $SU(2)_J$, generated by

$$J_a = J_a + I_a .$$

One can see this in several different ways.

Firstly, both $J$ ($SU(2)_L$) and $I$ ($SU(2)_R$) are broken by themselves, since they both act nontrivially on heavy fermion sector. The diagonal generators $J$’s, on the other hand does not involve $\lambda$ fermions and leave the heavy sector ground state untouched. Secondly, after deformation and reduction to $\mathcal{M}_n$, the dynamics is a nonlinear sigma model, where fermions transform identically to bosons. Recall that bosons and fermions used to belong to $(3,1)$ and $(2,2)$ of $SU(2)_L \times SU(2)_R$. In the reduced dynamics, symmetry properties of the bosons and fermions cannot be different, and indeed under the diagonal subgroup, bosons and fermions transform identically. Finally, after the deformation, the dynamics has only one real supersymmetry $Q_4$ so no R-symmetry is expected. However this supercharge originates from
a \((2, 2)\) multiplet under \(SU(2)_L \times SU(2)_R\), so has to transform nontrivially under either of the two individually. On the other hand, because \(\mathcal{J}\) does not rotate \(\lambda\)'s, \(\mathcal{J}\) commutes with \(Q_4\) and also with its reduced version \(Q_{\mathcal{M}_n}\). At the level of reduced dynamics on \(\mathcal{M}_n\), this \(SU(2)_\mathcal{J}\) is not an R-symmetry but a global symmetry that arises from the universal isometry of \(\mathcal{M}_n\).

While we are on the question of symmetry, let us digress a little and consider the equivariant index or refined index one encounters in literature on wall-crossing, of the generic form

\[
\text{Tr} \left( (-1)^F y^{2j_3} \right)
\]

with a “rotation” generator \(j_3\) along \(z\)-axis. Most such computations are based on some version of low energy quantum mechanics on the classical moduli spaces, our \(\mathcal{M}_n\)'s, but as we saw above, the “rotational symmetry” of \(\mathcal{M}_n\) is in fact not the purely spatial rotation but a diagonal subgroup of spatial rotation \(SU(2)_L\) and the field theory R-symmetry \(SU(2)_R\). Therefore, the refined indices that have been computed are in fact

\[
\text{Tr} \left( (-1)^F y^{2j_3} \right) = \text{Tr} \left( (-1)^F y^{2j_3 + 2I_3} \right)
\]

so actually would equal the protected spin character

\[
\text{Tr} \left( (-1)^{2J_3} y^{2J_3 + 2I_3} \right)
\]

of the field theory, if we are allowed to choose the chirality operator \((-1)^F\) of the quantum mechanics to be \((-1)^{2J_3}\).

So this brings the question of what happens to the two natural chirality operators, \((-1)^{2J_3}\) and \((-1)^{2I_3}\), when we deform and reduce the dynamics in favor of a \(\mathcal{M}_n\) nonlinear sigma-model. As we saw, the two \(SU(2)\) symmetries are lost individually, so operators like \(J_3\) and \(I_3\) can no longer be used to classify eigenstates. Nevertheless, \((-1)^{2J_3}\) and \((-1)^{2I_3}\) are still sensible chirality operators. Even after the deformation, one can show directly \((-1)^{2I_3}\) as a product of all fermions while \((-1)^{2J_3}\) is again the same product of all fermions times \((-1)\sum_{A>B} \langle \gamma_A, \gamma_B \rangle + n - 1\). Both anticommute with the surviving supercharge \(Q_4\), so still defines chirality operators. This is not much of surprise since they simply measure the most rudimentary information about the states, i.e., whether, before deformation, the state was in a integral or in a half-integral representations.

When we reduced the dynamics to \(\mathcal{M}_n\), however, we must properly redefine these chirality operators by evaluating them on vacuum of the heavy oscillators. For instance, consider \((-1)^{2I_3}\) for the simplest \(n = 2\) case. The canonical chirality operator on \(\mathcal{M}_n = S^2\) is, as noted before,

\[
(-1)^{F_{S^2}} = 2i\hat{\psi}^1\hat{\psi}^2,
\]
with the natural orientation arising from embedding of $S^2$ to $R^3$. To relate this to $(-1)^{2I_3}$, we remember to set the heavy fermions, $\psi^3$ and $\lambda$, to their ground state, which gives precisely

$$\langle 0 | (-1)^{2I_3} | 0 \rangle_{\text{heavy}} = (-1)^{F_{S^2}} ,$$

it turns out.\(^{15}\) Clearly, we may repeat this for each sector of 4 fermions labeled by $A$, and find

$$\langle 0 | (-1)^{2I_3} | 0 \rangle_{\text{heavy}} = \prod_A 2i\hat{\psi}_A^1\hat{\psi}_A^2 = (-1)^{F_{M_n}} . \tag{4.28}$$

Therefore, the chirality operator $(-1)^{2I_3}$ prior to the deformation, smoothly descend to the canonical chirality operator on $(-1)^{F_{M_n}}$, upon deformation and subsequent reduction of dynamics, and leads to the standard Dirac index $I_n$.

Since the desired index $\Omega$ needs $(-1)^{2I_3}$ as the chirality operator, we then use (3.13) to relate $(-1)^{2I_3}$ to $(-1)^{2I_3}$, and find an unambiguous answer,

$$\Omega_{\text{distinct}} = (-1)^{\sum_{A>B} \langle \gamma_A, \gamma_B \rangle + n-1} \times I_n(\{\gamma_A\}) . \tag{4.29}$$

On the left hand side, we emphasized the fact we are yet to incorporate the statistics issue. We will see in next section how this generalizes when we impose statistics to the index computation. Before asking the question of statistics, however, there is still one more ambiguity to the expression above, since so far we did not take into account of the internal degeneracy and quantum numbers of the individual charge centers. The left hand side is still defined with respect to $(-1)^{2I_3}$, so adding internal degeneracy factor can be accommodated by writing

$$\Omega_{\text{distinct}} = (-1)^{\sum_{A>B} \langle \gamma_A, \gamma_B \rangle + n-1} \times I_n(\{\gamma_A\}) \times \prod_{A=1}^n \Omega_A \tag{4.30}$$

where individual $\Omega_A$’s are also computed as the trace of $(-1)^{2I_3}$ (as usual modulo the universal half-hypermultiplet part). As usual, we assume that there is no significant coupling of these internal degeneracy to the quantum mechanical degrees of freedom. Sometimes, we will also write this as

$$(-\Omega_{\text{distinct}}) \times (-1)^{\sum_{A>B} \langle \gamma_A, \gamma_B \rangle} = I_n(\{\gamma_A\}) \times \prod_A (-\Omega_A) \tag{4.31}$$

which is more convenient when keeping track of statistics, since, for $SU(2)_R$ singlets, the Bose/Fermi statistics are naturally correlated with the sign of $-\Omega_A$’s.

\(^{15}\)This can be seen most easily when we choose the ordering of $\gamma_A$’s such that $\langle \gamma_A, \gamma_B \rangle$ are all nonnegative for $A > B$, which is also the convention chosen in Ref. [22] for non-scaling cases.
5 Index with Bose/Fermi Statistics and Rational Invariants $\bar{\Omega}$

So far, we pretended that dyons involved are all distinct, and studied the supersymmetric bound states thereof. In reality, this is not quite good enough since we often need to understand bound states of many identical dyons, obeying either fermionic or bosonic statistics. The effective true moduli space, for example, has to be an orbifold of type

$$\mathcal{M}_n/\Gamma$$

where $\Gamma$ is a union of permutation groups that mix up labels for identical particles, with proper action on wavefunctions. Equivalently, the index should be computed with appropriate projection operator inserted,

$$\Omega = \text{Tr} \left( (-1)^{J_3} e^{-sH} \mathcal{P}_\Gamma \right)$$

where $\mathcal{P}_\Gamma$ projects to wavefunctions obeying either Bose or Fermi statistics under the exchange of identical particles.

The orbifolding reduces the volume of the moduli space, so given the index formula which is an integral over the manifold, we should expect to see factors like $1/d!$ as a result of having $d$ identical centers. However, action of $\Gamma$ is not everywhere free, since when identical particles are on top of one another, the action is trivial. There are complicated fixed submanifolds under $\Gamma$, making the problem very involved, and in particular there should be additional contributions from the fixed manifolds under the orbifolding action.

5.1 The MPS Formula

Before we carry out such a computation directly, it is instructive to recall a recent result by Manschot, Pioline, Sen (MPS), who evaded this complication algebraically, and replaced it by a sum of many index problems with distinct charge centers [22]. They argued that one can recover the correct index, by adding indices for a series of artificial problems with a smaller number of charge centers. In this set of effective index problems, the trick requires the following rules: When the reduced problem has $d$ particles of the same kind, MPS divides the index by $1/d!$. When one has a particle of nonprimitive charge as a part of such a reduced problem, one must also use, in place of the true intrinsic degeneracies $\Omega$ of the particles, a mathematical one $\bar{\Omega}$,

$$\bar{\Omega}(\gamma) \equiv \sum_{p|\gamma} \frac{\Omega^+(\gamma/p)}{p^2},$$

(5.3)
where the sum is over the positive integer \( p \) such that \( \gamma/p \) belongs to the quantized charge lattice of the theory. Note that \( \Omega(\gamma) = \bar{\Omega}(\gamma) \) whenever \( \gamma \) is primitive and \( \bar{\Omega}(p\gamma) = \Omega(\gamma)/p^2 \) if no non-primitive charge state exists.

For illustration, let us take two primitive charges \( \beta_1 \) and \( \beta_2 \). Suppose that, among all possible linear combinations of the two, only these two states exist on one side of the marginal stability wall. Labeling the degeneracy by \( \pm \) depending on which side of the wall we are considering, we thus assume that

\[
\Omega^+(m\beta_1 + k\beta_2) = 0, \quad \text{unless } (m,k) = \pm(1,0) \text{ or } (m,k) = \pm(0,1). \quad (5.4)
\]

The sign of \( \Omega_{1,2} \equiv \Omega^+(\beta_{1,2}) \) are correlated with the statistics assignment of the particle; a hypermultiplet has \( \Omega = 1 \) and must be treated as fermions while a vector multiplet has \( \Omega = -2 \) and must be treated as bosons. Under this assumption, we have \( \bar{\Omega}(p\beta_{1,2}) = \Omega(\beta_{1,2})/p^2 \). Manschot et.al.’s formula then simplifies to,

\[
-\Omega^- (m\beta_1 + k\beta_2) \times (-1)^{\sum_{A>B} (\gamma_A, \gamma_B)} \sum_s \frac{1}{m!k!} \mathcal{I}_{m+k}(\beta_1, \beta_1, \ldots, \beta_2, \beta_2, \ldots) (-\Omega_1)^m (-\Omega_2)^k \\
+ \frac{1}{(m-2)!k!} \mathcal{I}_{m-1+k}(2\beta_1, \beta_1, \beta_1, \ldots, \beta_2, \beta_2, \ldots) -\frac{\Omega_1}{2} \left( -\Omega_1 \right)^{m-2} (-\Omega_2)^k \\
+ \frac{1}{(m-3)!k!} \mathcal{I}_{m-2+k}(3\beta_1, \beta_1, \beta_1, \ldots, \beta_2, \beta_2, \ldots) -\frac{\Omega_1}{3} \left( -\Omega_1 \right)^{m-3} (-\Omega_2)^k \\
+ \frac{1}{2!(m-4)!k!} \mathcal{I}_{m-2+k}(2\beta_1, 2\beta_1, \beta_1, \beta_1, \ldots, \beta_2, \beta_2, \ldots) \left( -\frac{\Omega_1}{2^2} \right)^2 (-\Omega_1)^{m-4} (-\Omega_2)^k \\
+ \cdots, \quad (5.5)
\]

where the sum is over all unordered partitions of \( m\beta_1 + k\beta_2 \) respectively, although we listed above only part of the partitions of \( m \). For the overall sign, we re-labeled the individual charges \( \beta_1, \beta_1, \ldots, \beta_2, \beta_2, \ldots \) and called them \( \gamma_A \)'s. This sum and each term in it can be characterized by the following set of rules:

(i) The sum is over all unordered partition of \( m\beta_1 + k\beta_2 = \sum_s d_s \beta_s \) where \( \beta_s = (p_{s1}\beta_1 + p_{s2}\beta_2) \). For each \( \beta_s \), we will have a factor \( \bar{\Omega}(\beta_s) \), so we can, with the current assumption on \( \Omega^+ \)'s, consider only a subset where only one of \( p_{s1} \) and \( p_{s2} \) is nonzero for each \( s \).

(ii) The index \( \mathcal{I}_{n'} \) with \( n' \equiv \sum_s d_s \) effective charge centers. For \( \mathcal{I}_{n'} \), we treat all charge centers as distinguishable, so it is computed by the index theorem of the previous section with \( n' \leq n \) distinguishable centers.
(iii) The combinatoric factor of \(1/d_s!\) for each \(s\). This takes into account of the reduced volume of the moduli space due to the orbifolding by the permutation subgroup \(S(d_s)\) acting on the reduced \(n'\)-center quantum mechanics, but does not address the contribution from the submanifolds fixed by \(S(d_s)\).

(iv) For each effective particle of charge \(p\beta\), with primitive \(\beta\) and \(p > 1\), that shows up in computation of \(I_{n'}\), one further assigns an effective internal degeneracy factor \(-\bar{\Omega}(p\beta) = -\Omega(\beta)/p^2\), in addition to \((-\Omega_1)^{m'}(-\Omega_2)^{k'}\), which reflects the fact that \(m'\) number of \(\beta_1\) centers and \(k'\) number of \(\beta_2\) centers are left as individual.

The last \(-\bar{\Omega}(p\beta) = -\Omega(\beta)/p^2\) should be compared to the naive \((-\Omega(\beta))^p\) degeneracy factor that would be the correct factor if we were considering \(p\) separable particles of charge \(\beta\) instead of one particle of charge \(p\beta\). Finally, the appearance of \(-\Omega's\) instead of \(\Omega's\) is natural, since for example a half-hypermultiplet with \(\Omega = 1\) acts like fermions, while a vector multiplet with \(\Omega = -2\) acts like bosons.

From the quantum mechanics viewpoint, the decomposition (i) clearly has something to do with the orbifolding action \(\Gamma\). Each term in (5.5) arises from a submanifold which is fixed by the product of permutation groups of order \(p_s, \prod_s S(p_s) \subset \Gamma\). For each sector, origins of (ii) and (iii) are also evident as coming from a reduced problem of \(n'\) charge centers and the subsequent volume-reducing action of \(\prod_s S(d_s) = \Gamma/\prod_s S(p_s)\). The only part of this formula which is not evident, so far, from the moduli quantum mechanics viewpoint is the rational degeneracy factor of (iv). Here, we would like to isolate where this comes from, and later derive it directly from the moduli dynamics.

After some careful thinking, it becomes evident that this rational degeneracy factor should come from quantum mechanical degrees of freedom normal to the submanifold fixed by \(S(p')\)'s. Let us consider only \(p > 1\) cases and label them \(p_{s'}\), since otherwise the internal degeneracy factor is \(\Omega(\beta)\) as expected. Subgroup \(S(p_{s'})\)'s permuting these \(p_{s'} > 1\) charges fixes a submanifold \(\mathcal{M}_{n'-\sum(p_{s'}-1)}\) inside \(\mathcal{M}_n\). This fixed submanifold has a codimension \(2\sum(p_{s'}-1)\) in \(\mathcal{M}_n\), since it is spanned by coincidence of \(p_{s'}\) centers, each of which span two directions in \(\mathcal{M}_n\).

Consider the reduced dynamics on the intersection, \(\mathcal{M}_{n'=n-\sum(p_{s'}-1)}\), for computation of \(I_{n'}\) with all such \(p_{s'}\beta_{s'}\) center treated as single particle, respectively. If we start with this reduced index problem, impose the statistics, and pretend that the centers associated with \(p_{s'}\beta_{s'}\) comes with a unit degeneracy we will find a contribution of type

\[
\frac{1}{\prod_s d_s!} \mathcal{I}_{n'} \times (-\Omega_1)^{m'}(-\Omega_2)^{k'},
\]

where \(m'\beta_1 + k'\beta_2 = m\beta_1 + k\beta_2 - \sum_{s'} p_{s'}\beta_{s'}\) and \(n' = m' + k' + \sum_{s'} d_{s'}\). Note that
we took care to include the volume-reducing effect of \( \prod_s S(d_s) = \Gamma / \prod S(p_{s'}) \) via the denominator \( \prod d_s! = m'!k'!(\prod d_{s'}!) \).

This expression is obtained after ignoring the quantum degrees of freedom that are normal to the fixed manifold \( \mathcal{M}_{n-\Sigma(p_{s'-1})} \)'s, and does not agree with MPS formula. The latter is

\[
\frac{1}{\prod_s d_s!} I_{n'} \times (-\Omega_1)^{m'} (-\Omega_2)^{k'} \times \prod_{s'} \frac{-\Omega(\beta_{s'})}{p_{s'}^2}
\]

so the difference is precisely the rational degeneracy factor of \((iv)\). Clearly it comes from quantizing the normal bundles of \( \mathcal{M}_{n-\Sigma(p_{s'-1})} \)'s inside \( \mathcal{M}_n \). On the other hand, for all intent and purpose, this part of quantum mechanics is free, since they have something to do with many identical particles and has no interaction of type \( L_1 \), except for the statistics issue.\(^{#16}\)

This leads us to conclude that the factor, \( \Omega(\beta)/p^2 \), should arise from an index of \( p \) noninteracting identical particles of charge \( \beta \), modulo the center of mass part which already contributed to \( I_{n'=n-p+1} \). The relative dynamics of such identical particles carry \( 2(p-1) \) bosonic degrees of freedom, \( 2(p-1) \) fermionic degrees of freedom, and additionally internal degeneracy of \( |\Omega(\beta)| \) for all \( p \) particles. In next subsection, we will show that precisely such a factor arises from the dynamics of non-interacting and identical \( p \) particles with the internal degeneracy \( \Omega(\beta) \).

The full MPS formula follows the same set of rules, except that one must in general consider an arbitrary set of charges on the + side, and all the partitions of the total charge \( \gamma_T \) in terms of charges of states that exist on + side of the wall. Since the + side of spectrum may then include states of charges \( h\beta_1 + j\beta_2 \) with \( h + j > 1 \), more diverse charge centers will appear for the individual index problems on the right hand side. As we will discuss later, this can be incorporated by treating all such particles on the + side as independent. The only subtlety is when non-primitively charged states exist on the + side; this can be remedied by employing the fully general form, \( \bar{\Omega}(\gamma) \equiv \sum_{p|\gamma} \Omega^+(\gamma/p)/p^2 \) as the effective degeneracy factor. We will also see this most general \( \bar{\Omega} \) emerging from our index computations.

\(^{#16}\)There is a subtlety, again related to whether threshold bound state of identical charges can form. Since we started with the assumption that such nonprimitive state do not exist, it is safe to assume this issue does not complicate our problem. Whether or not we can extend this to theories with threshold bound states, i.e. supergravity, is an open problem.
5.2 Physical Origin of $\Omega(\beta)/p^2$ from $p$ Non-Interacting Identical Particles

Let us first restrict ourselves to bound states involving several identical dyons of charge $\beta$ with $-\Omega^+ (\beta) = \pm 1$. As in the previous discussion, let us consider the bound state of $n$ charge centers, $\gamma_T = \sum A \gamma_A$, $m$ of which are $\beta$’s. The identical nature of the $\beta$ dyons means that the orbifolding group includes the $m$-th order permutation group $S(m)$. We start with the assumption, for simplicity, that $k\beta$ state exists only for $|k| = 1$ on the $+$ side, and then come back for the fully general case in next subsection.

Consider the index reduced on the true moduli space $\mathcal{M}_n$ as described in the previous section, with proper account taken of the bosonic or the fermionic statistics,

$$\Omega^- \left( \sum_A \gamma_A \right) - \Omega^+ \left( \sum_A \gamma_A \right) = \int_{\mathcal{M}_n} \text{tr} \left( \langle X | (-1)^{2J_3} e^{-sQ^2} \mathcal{P}_\Gamma | X \rangle \right) dX , \quad (5.8)$$

via the orbifolding projection operator $\mathcal{P}_\Gamma$. Here $\text{tr}$ means the trace over the fermionic variables as well as other internal discrete degrees of freedom, and we integrate over the bosonic variables $X$ with an appropriate measure. Matching the sign of $-\Omega$ with $(-1)^F$ value of the component dyon states, as we noted in the case of bound state counting in distinguishable centers, this index naturally computes the degeneracy $\Omega^-$’s as

$$\text{Ind}(\{\gamma_A\}; \Gamma) = \int_{\mathcal{M}_n} \text{tr} \left( \langle X | (-1)^{2J_3} e^{-sQ^2} \mathcal{P}_\Gamma | X \rangle \right) dX , \quad (5.9)$$

so we would like to ask whether this reproduces (5.5) and rediscover the rational invariant $\bar{\Omega}$. For this, let us concentrate on the permutation group $S(m)$ part of $\Gamma$ and see how it generates a series of terms, similar to MPS’s wall-crossing formula.

Inside $\mathcal{M}_n$ there are various fixed submanifolds, $\mathcal{M}_{n'}$, of dimension $2(n' - 1)$. The simplest are $\mathcal{M}_{n-p+1}$, fixed under $S(p)$ subgroup of $S(m)$. Note that we use the same notation $\mathcal{M}$ for the fixed submanifolds as the full classical moduli manifold $\mathcal{M}$. This is because all of them are of exactly the same type. For example, the manifold $\mathcal{M}_{n-m+1}$ would also emerge if we started with a different low energy dynamics involving a single center of charge $m\beta$ in place of $m$ centers of charge $\beta$. Ignoring contributions from these fixed manifolds would simply give

$$(-1)^{\sum_{A>B} (\gamma_A \cdot \gamma_B) + n - 1} \left( \frac{T_n}{|\Gamma|} \right) \times \prod_A \Omega_A \quad (5.10)$$

\#17 Because of spin-statistics theorem, there is no irreducible BPS multiplet in $D = 4$ $N = 2$ theories with $\Omega = -1$. The half-hypermultiplet has 1 and vector multiplet has -2. The assumption here is strictly for the illustrative purpose only.
due to the volume-reducing action of $\Gamma$ when it is acting freely. This is the very first term in the MPS formula. Since there are many fixed submanifolds, however, each of them will contribute additively on top of this.

Without loss of generality, let us consider the fixed manifold $\mathcal{M}_{n-p+1}$ associated with the partition $m\beta = p\beta + \beta + \beta + \cdots + \beta$, and label the coordinates along the fixed manifold $\mathcal{M}_{n-p+1}$ by $X'$ and those normal to it by $Y$. Note that among $X'$ are the two coincident (or center of mass) coordinates for the $p\beta$ charge center, so we can think of $Y$'s as the relative position coordinates among these $p$ charge centers; therefore there are $2(p-1)$ $Y$'s. We then formally write the additive contribution from the fixed submanifold $\mathcal{M}_{n-p+1}$ as

$$\Delta_p \times \text{Ind}(\{\gamma_{A'}\} = \{p\beta, \beta, \ldots\}; \Gamma')$$

$$= \Delta_p \times \int_{\mathcal{M}_{n-p+1}} \text{tr}' \left( \langle Y = 0, X'| (-1)^{2\sum_{A'<B'} e^{-sH'} P_{T'} | Y = 0, X'} \rangle \right) dX' \; , \quad (5.11)$$

where $\Gamma' = \Gamma/S(p)$ is the remaining orbifolding group that acts nontrivially on $\mathcal{M}_{n-p+1}$. Here $\text{tr}'$ denotes trace over fermionic and other internal degrees of freedom, except those associated with the $p$ identical $\beta$'s that are held together at $\mathcal{M}_{n-p+1}$.

We factored out the contribution $\Delta_p$ from the normal directions, $Y$, and the superpartners thereof. On the other hand, the second factor is the index of a reduced $n-p+1$ center problem, modulo the internal degeneracy factor of $p\beta$ charge center. Other than this, the computation of this latter factor proceeds on equal footing as (5.8),

$$\int_{\mathcal{M}_{n-p+1}} \text{tr}' \left( \langle Y = 0, X'| (-1)^{2\sum_{A'<B'} e^{-sH'} P_{T'} | Y = 0, X'} \rangle \right) dX'$$

$$\sim (-1)^{\sum_{A'<B'}\langle \gamma_{A'} : \gamma_{B'} \rangle + n-p} \left( \frac{\mathcal{I}_{n-p+1}(\{\gamma_{A'}\})}{|\Gamma'|} \right) \times \prod_{A'=2}^{n-p+1} \Omega_{A'} \; + \; \cdots \quad (5.12)$$

so we may compute the full index recursively. $\Delta_p$ plays the role of the missing internal degeneracy factor $\Omega_{T'}$ here, as it computes the effective contribution from these $p$ coincident $\beta$'s. The ellipsis denotes terms from other fixed submanifold inside $\mathcal{M}_{n-p+1}$ etc.

We will show that $\Delta_p = \pm 1/p^2$, regardless of precise nature of the $\beta$ particles, which also reproduces MPS formula for $\Omega(\beta) = \pm 1$ entirely from the dynamics. Schematically, this factor can be written as

$$\sim \int \text{tr} \left( \langle Y | (-1)^{2\sum_{I=1} e^{-sH} P | Y \rangle \right) dY \; , \quad (5.13)$$
with \( F_\perp \) and \( H_\perp \) defined on \( Y \)'s and the superpartners, again with suitable measure for the bosonic integral. The projection operator

\[
P = \frac{1}{p!} \sum_{\pi \in S(p)} (\mp 1)^{\sigma(\pi)} M_\pi \tag{5.14}
\]

ensures that we isolate wavefunctions of correct Bose/Fermi statistics. Note that the sign in the projection operator is the same as that of \(-\Omega(\beta)\). \( M_\pi \) is the \((p - 1)\) dimensional representation of \( \pi \in S(p) \), common for \((p - 1)\) coordinate doublets \( Y \)'s and for their fermionic partners, \( \psi \)'s. Naturally \( \sigma(\pi) \) is odd or even when \( \pi \) is odd or even.

Since the embedding of \( \mathcal{M}_{n-p+1} \)'s in \( \mathcal{M}_n \) could be very complicated, the exact nature of the decomposition is not entirely clear. On the other hand, the initial index problem is gapped and allows us to take \( s \to 0 \). At least in this limit, the decomposition makes sense intuitively, and, as we will see shortly it suffices to consider an arbitrarily small tubular neighborhood around the fixed manifold \( \mathcal{M}_{n-p+1} \). We take the \( Y \) directions as a ball \( B_{2(p-1)} \) inside a flat \( \mathbb{R}^{2(p-1)} \). Therefore, we have

\[
\Delta_p = \lim_{s \to 0} \int_{B_{2(p-1)}} \text{tr} \left( \langle Y | (-1)^{J_3^+} e^{(s/2)\nabla^2} P | Y \rangle \right) dY . \tag{5.15}
\]

Precisely how we cut-off this neighborhood will not matter, as we will see shortly that a Gaussian integral of squared width \( s \) emerges along \( Y \) directions.

Interestingly, exactly the same kind of object was studied when solving for the famous D0 bound state problems in the 1990’s. The first such computation appeared in Ref. [26] on two-body problem and was later expanded to many body case in Ref. [27]. Here we will adapt and expand the computations in these works. Since there are \( 2(p - 1) \) real fermions, we will choose a polarization of type \( \{ \psi_i, \psi_i^\dagger \} \), so that a general wave function \( | \Psi \rangle \) can be expanded as

\[
| \Psi \rangle = \left( \Psi(Y) + \Psi^{(i)}(Y) \psi_i^\dagger + \frac{1}{2} \Psi^{(i1i2)}(Y) \psi_i^\dagger \psi_i^\dagger + \ldots \right) | 0 \rangle , \tag{5.16}
\]

where \( \Psi^{(i_1 \ldots i_m)}(Y) = \sum_k \lambda_k^{[i_1 \ldots i_m]} \Psi_k(Y) \) and \( \{ \Psi_k \} \) are complete basis of \( Y \)-space wave functions. Since the Hamiltonian is free and does not mix sectors of different fermion numbers, we may evaluate the bosonic and fermionic trace independently. The fermionic trace, for each of \( M_\pi \), is given by

\[
(\pm 1)^p \text{tr}_\psi \left( (-1)^{p-1} (-1)^{N_{\psi^\dagger}} M_\pi \right) , \tag{5.17}
\]

where \( (\pm 1)^p \) arises from the value of \((-1)^{2J_3} \) on \( p \) individual \( \beta \) states. Also the orbital part of \( J_3^\perp \) is always integral in the absence of the minimal coupling contribution.
so \((-1)^{2J^3} \) acting on the quantum mechanical degrees of freedom becomes purely fermionic expression \((-1)^{p-1}(-1)^{N\psi^\dagger}\). The sign in front of the latter comes from converting the chirality operator to a form involving the fermion number operator that counts the creation operators \(\psi^\dagger\).

Then the contribution from \(Y\) direction reads

\[
\Delta_p = \frac{(\pm 1)^p}{p!} \sum_{\pi \in S(p)} \langle \mp 1 \rangle^{\sigma_{\pi}} (-1)^{p-1} \text{tr}_\psi \left( (-1)^{N\psi^\dagger} M_{\pi} \right) \]

\[
\times \int_{B_2(p-1)} \langle Y | e^{s\nabla^2/2} M_{\pi} | Y \rangle dY ,
\]

in the limit of \(s \to 0\). A crucial observation that allows us to proceed systematically is

\[
\text{tr}_\psi \left( (-1)^{F_{\pi}} M_{\pi} \right) = \langle 0 | - \langle 0 | \psi^a M_{\pi}^\alpha_{\pi}^a \psi_{\pi}^a | 0 \rangle + \frac{1}{2} \langle 0 | \psi_{\pi}^a \psi_{\pi}^b M_{\pi}^\alpha_{\pi}^a M_{\pi}^\beta_{\pi}^b \psi_{\pi}^a \psi_{\pi}^b | 0 \rangle + \cdots
\]

\[
= \det(1 - M_{\pi}) ,
\]

and furthermore

\[
\det(1 - M_{\pi}) = \begin{cases} p , & \pi \text{ is a cyclic permutation of order } p \\ 0 , & \text{otherwise} \end{cases}
\]

for which it is important to remember that \(M_{\pi}\) is a \(p - 1\) (rather than \(p\)) dimensional representation of \(S(p)\). Since \((-1)^{\sigma_\pi} = (-1)^{p-1}\) for any cyclic permutation of order \(p\), we find

\[
\Delta_p = \frac{(\pm 1)^p}{p!} \sum_{\pi'} (-1)^{p-1} (-1)^{p-1} \det(1 - M_{\pi'}) \int \langle Y | e^{s\nabla^2/2} M_{\pi'} | Y \rangle dY
\]

\[
= \frac{\pm 1}{p!} \sum_{\pi'} \det(1 - M_{\pi'}) \times \frac{1}{(2\pi s)^{p-1}} \int e^{-(Y-M_{\pi'} Y)^2/2s} dY
\]

\[
= \frac{\pm 1}{p!} \sum_{\pi'} \frac{1}{\det(1 - M_{\pi'})} ,
\]

where the sum is now only over the cyclic permutations of order \(p\). There are precisely \((p - 1)!\) such permutations and they each contribute \(1/p\), via the determinant, so the result is

\[
\Delta_p = \frac{\pm 1}{p^2} ,
\]
as promised. Clearly, we can repeat this when there are several such factors simultaneously, to give,
\[
\Delta_{\{p,\pi\}} = \prod_{s'} \frac{\pm 1}{p_{s'}},
\]  
(5.23)
reproducing \(\Omega(\beta)/p^2\) of the MPS formula with \(\Omega(\beta) = \pm 1\).

The more general case of \(\Omega(\beta) = \pm d\) can be derived similarly. Let us write one particle state as
\[
|\hat{\Psi}\rangle = \hat{\Psi}^{\eta}(x, \psi)|0; \eta\rangle
\]
(5.24)
so that \(\eta = 1, 2, \ldots, d\) labels the internal degeneracy, and \(p\)-particles wave function (without center of mass degree of freedom) can be written as a sum of terms like
\[
\Psi^{\{ij\cdots k\}}_{\{\eta\}}(Y)\psi^\dagger_{(i)}\psi^\dagger_{(j)}\cdots \psi^\dagger_{(k)}|0; \eta_1, \eta_2, \ldots, \eta_p\rangle,
\]
(5.25)
none of which mixes under the free Hamiltonian. Thanks to this, just as the fermionic part and the bosonic part separately contributed, this internal part also factorizes under each \(\pi\). Expressing \(\Delta_p\) as a sum over the elements of permutation group again, we now have an extra factor
\[
\langle \eta_p, \ldots, \eta_1|\eta_{\pi(1)}, \ldots, \eta_{\pi(p)}\rangle,
\]
for each permutation \(\pi\), and the trace over these internal indices. Thus, we arrive at a similarly simple form,
\[
\Delta_p = \frac{(\pm 1)}{p!} \sum_{\eta_1, \ldots, \eta_p} \frac{\langle \eta_p, \ldots, \eta_1|\eta_{\pi(1)}, \ldots, \eta_{\pi(p)}\rangle}{\det(1 - M_{\pi'})}.
\]
(5.26)
As before, the sum is over the permutations of cyclic order \(p\). For such \(\pi'\), the inner product vanishes identically unless all \(\eta_i\)'s are equal to one another, and gives unit if all are equal. The sum over \(\eta\)'s thus collapses to a single sum over \(\eta = \eta_1 = \eta_2 = \cdots = \eta_p\), and
\[
\Delta_p = (\pm 1) \sum_{\eta} \frac{1}{p!} \sum_{\pi'} \frac{1}{\det(1 - M_{\pi'})} = \frac{\pm d}{p^2} = \frac{\Omega(\beta)}{p^2}.
\]
(5.27)
This gives us the only essential ingredient in confirming (5.8), and in fact the fully general version thereof.

For a complete derivation, a recursive argument is needed and we need to consider possibility of low energy dynamics with charge centers both primitive and non-primitive charges simultaneously. This naturally brings us to the most general wall-crossing formula, next.
5.3 General Wall-Crossing Formula

Most of what we derived generalizes to cases with arbitrary spectrum on + side, without much modification, but here we need to point out one subtlety. Suppose the + side of spectrum contains not only a pair of primitively charge states $\gamma$ and $\gamma'$ but also states like $h\gamma + j\gamma'$ a little more involved, and includes states of composite charges such as $m\gamma$ or linear combinations with other charges. (One can also have states with charges completely unrelated to these but those will not participate in the wall-crossing, and therefore irrelevant.) Such a state cannot be considered as a bound state of $h\gamma$'s and $j\gamma'$'s, since the two are mutually repulsive on the + side. Rather, it should be regarded as a completely independent particle of different origin. In fact, for $SU(2)$ theory with a single flavor, a monopole $\gamma$, a quark $\gamma'$ and a dyon $\gamma + \gamma'$ are known to coexist in the central part of the moduli space.

Let us denote charges of these independent particles as $\beta_v$. Since one can form bound states of a given total charge $\gamma_T$ on the “−” side from different combinations of these + side states. We will label each of these physically distinct combination by the upper index inside a parenthesis such that

$$\gamma_T = \sum_{A} m^{(1)}_v \beta_v = \sum_{v} m^{(2)}_v \beta_v = \sum_{v} m^{(3)}_v \beta_v = \cdots . \quad (5.28)$$

In such circumstances, the total degeneracy for $\gamma_T$ has to be computed for each of such bound state problems and summed over,

$$\Omega^- (\gamma_T) - \Omega^+ (\gamma_T) = \sum_{q} \Omega (\{m^{(q)}_v\}) , \quad (5.29)$$

where each term on the right hand side is computed from the $n^{(q)} = \sum m^{(q)}_A$ center quantum mechanics. For each of $\Omega(\{m^{(q)}_v\})$’s, computation of the previous subsection goes through without modification, and will be computed as

$$\Omega (\{m^{(q)}_v\}) = \Omega \left( \sum_{A} \gamma_A = \sum_{v} m^{(q)}_v \beta_v \right) . \quad (5.30)$$

One important detail to remember is that, even if some of charge $\beta_v$’s might be a linear combination of other $\beta_v$’s, each of them are physically unrelated independent particles: The permutation group is simply $\Gamma = \prod_{v} S(m^{(q)}_v)$ for each of these index problems.

Combining this with the results of previous subsection, we reproduce MPS formula in its most general form. Note that, when we reorganize this formula in terms of the index of distinct particles of unit individual degeneracy,

$$I_n(\ldots, \gamma, \ldots) = \int_{\mathcal{M}_n} \text{Ch}(\mathcal{F}) \hat{A}(\mathcal{M}_n) \quad (5.31)$$
the rational invariants $\bar{\Omega}(\gamma)$, multiplying them, will accumulate additive contributions of the form, including $p = 1$ case,

$$\frac{\Omega^+(\gamma/p)}{p^2}$$

for each $\gamma/p = \beta_v$ that appears in one of the expansion $\gamma_T = \sum \nu m^{(q)} \beta_v$. Since we are summing over all possible such expansions, it implies that

$$\bar{\Omega}(\gamma) = \sum_{p|\gamma} \frac{\Omega^+(\gamma/p)}{p^2}$$

will appear as the effective degeneracy factors that multiply $I_n$'s. $p = 1$ terms arises only when $\gamma$ is one of the $\beta_v$'s, while $p > 1$ terms arise from the orbifold fixed sector as in the previous section when $\gamma/p$ is one of $\beta_v$'s. The final expression is

$$\Omega^-(\gamma_T) - \Omega^+(\gamma_T) =$$

$$\ldots$$

$$+ (-1)^{-n'+1+\sum_{A'>B'}|\gamma'_{A'}\gamma'_{B'}|} \times \frac{I_{n'}(\{\gamma'_{1'}, \gamma'_{2'}, \ldots\})}{|\Gamma'|} \times \prod_{A'} \bar{\Omega}(\gamma'_{A'})$$

$$+ \ldots$$

(5.34)

where we wrote a representative form for the partition $\gamma_T = \sum_{A'=1}^{n'} \gamma'_{A'}$ into $n'$ centers and the associated orbifolding group $\Gamma'$ permuting among identical elements in $\{\gamma'_{A'}\}$.

6 Alternative Derivations for Arbitrary Number of Centers

We have so far seen how Dirac index computes wall-crossing formulae with the statistics imposed as a projection operator in the index problem. In particular by decomposing the computation into various fixed submanifold, we effectively reduce the problem to a collection of index problems for certain sets of distinguishable particles. Some of these distinguishable particles originate from real BPS particles, whereas some are mathematical construct. In the latter case, say associated with any fixed submanifolds under the permutation group $S(p)$, an effective internal degeneracy $\Omega(\beta)/p^2$ emerges universally.
In this section, we repeat this exercise for general cases, starting with a string theoretical method of computing orbifold index. While the derivation of previous section is very intuitive and compelling, we wish to further support it with this more rigorous formulation. To handle this problem, we have to figure out how to define the index problem on orbifold singularities. This is a subtle issue in mathematics since most index problems are defined on smooth manifolds. On the other hand, in string theory the orbifold singularities are common and the string theory on such space is well defined. We will use the index for the Dirac-type operator defined in the string theory setting and apply it to the problem of our interest.

In fact in the original paper on the orbifold [45], there appeared the formula of the Euler characteristic on orbifolds. The special feature of this formula is that it has the non-zero contribution from the twisted sectors and it gives the right Euler-characteristic for the blown-up manifold out of the underlying orbifolds when the way of the blown-up of the orbifolds is known. This consequently motivates many mathematical literatures which try to justify the formulae in a rigorous way [46]. For our purpose, the natural object is the Dirac-Ramond index actively studied on 1980’s and we try to read off the index of interest. We are interested in the $U(1)$ equivariant index since we will follow closely the strategy developed by [22]. It turns out that the twisted sectors would not contribute and the entire problem effectively reduces to the usual Atiyah-Bott-Lefshetz theorem [47], with string theory serving as a computational tool.

### 6.1 Dirac-Ramond Operator on Orbifolds

Now let us turn into the Dirac-Ramond index. Dirac-Ramond index is so called the $U(1)$ character-valued index [42]

$$I = \lim_{\tau_2 \to 0} \Tr (-1)^{F_R} \exp (2\pi i \tau_1 P) \exp (-2\pi \tau_2 H),$$

where $H$ is the Hamiltonian for the string worldsheet and $P$ is the momentum of the string. Here $P$ plays the role of the $U(1)$ generator, which is the translation generator along the worldsheet. With $\tau = \tau_1 + i\tau_2$ and $q = e^{2\pi i \tau}$, this index has the contribution from supersymmetric states of world-sheet only and\(^{18}\)

$$I = \Tr (-1)^{F_R} q^{L_0 + \epsilon} \bar{q}^{\bar{L}_0 + \bar{\epsilon}} = \Tr (-1)^{F_R} q^{-P},$$

so that we have the contributions only from left-moving modes. We define $H = L_0 + \bar{L}_0, P = L_0 - \bar{L}_0$ and $\epsilon, \bar{\epsilon}$ is the zero point contributions from the right and the

\(^{18}\)We change the role of $q, \bar{q}$ in Eq. (6.2) for later notational convenience.
left movers respectively. Thus $I$ has the form

$$I = \sum_{\lambda} I_\lambda q^{-\lambda},$$

(6.3)

where $I_\lambda$ is the index on the subspace with momentum $\lambda$, which is essentially the level of the massive string state denoted by $m$. If we take the lowest $\lambda$ this will give the usual index of the Dirac operator. $I$ has the form

$$I = \hat{A}(R) \sum_{(g,t,m)} q^m Ch(F, g) Ch(R, t),$$

(6.4)

with $g$ representation of the gauge group and $t$ tensor representation of the rotation group. The sum is taken over all fields in representation $(g, t)$ at each mass level $m$ of the left mover [44, 48].

Now $I$ is closely related to the string partition function. For example, if we evaluate the Dirac-Ramond index for the NS fermions with anti-periodic boundary conditions along worldsheet time direction in the heterotic string theory, we have [43],

$$I = q^{-k - d/24} \int_M dx_0^\mu d\psi_{-0}^\mu \hat{A}(M) \prod_{n=1}^\infty \frac{\text{det}(1 + q^n e^{iF/2\pi})}{\text{det}(1 - q^n e^{iR/2\pi})},$$

(6.5)

where $x_0^\mu, \psi_{-0}^\mu$ denote the bosonic and fermionic zero modes. When we set $F = R = 0$ we have the typical form of the string partition function. In the above we defined

$$R_{\mu\nu} = \frac{1}{2} R_{\mu\nu\lambda\sigma}(x_0) \psi_{-0}^\lambda \psi_{-0}^\sigma,$$

$$F_{AB} = -\frac{i}{2} F^{M}_\mu T^M_{AB} \psi_{-0}^\mu \psi_{-0}^\nu,$$

(6.6)

and regard the fermion zero modes as the differential forms in the usual way. For our purpose, the gauge bundle is Abelian, so we simply replace $F$ here by $F$ at the end of the computation.

If we take the lowest $q$ this picks up the basic representation and $I$ gives

$$\int_M dx_0^\mu d\psi_{-0}^\mu \hat{A}(M) Ch(F),$$

(6.7)

where we use [44]

$$\sum x^k Ch(F, [k]) = \text{det}(1 + x e^{iF/2\pi}),$$

$$\sum x^k Ch(F, (k)) = \text{det}(1 - x e^{iF/2\pi})^{-1},$$

(6.8)
with \([k],[k]\) being the antisymmetric and the symmetric products of the basic representation. Thus in this way we obtain the formula for the Dirac index from the Dirac-Ramond index.

Now this formalism can be easily generalized to the orbifold cases. The twisted index of the Dirac-Ramond operator is given by \([49]\)

\[
I(g_1,g_2;\tau) = \text{Tr}_{g_2} [q^{L_0 + \epsilon} q^{L_0 + \epsilon} g_1 (-1)^{F_R}] 
\]

where the \(g_1, g_2\) are the twists along \(\sigma_2, \sigma_1\) directions, respectively. Note that we define \(g_1\) twists along temporal direction. Here we label the points of the world sheet torus by the complex quantity \(\sigma_1 + \tau \sigma_2\) with \(\sigma_1, \sigma_2\) having the periodicity \(2\pi\). Consistency of the boundary condition requires that \(g_1, g_2\) commute. In this case the bosonic and the fermionic zero modes exist only along \(M_{g_1g_2}\), which is the space inert under \(g_1\) and \(g_2\). If we decompose the curvature along and the normal directions of dimension \(d_n\) to \(M_{g_1g_2}\) whose dimension is \(d\),

\[
\frac{1}{2\pi} R_{kl} = \frac{i}{4\pi} R_{kl i j} \psi_{-0}^i \psi_{0}^j = \begin{pmatrix} 0 & \omega_{\mu} \\ -\omega_{\mu} & 0 \end{pmatrix}, \quad p = 1 \cdots \frac{d}{2},
\]

\[
\frac{1}{2\pi} R_{ab} = \frac{i}{4\pi} R_{ab i j} \psi_{-0}^i \psi_{0}^j = \begin{pmatrix} 0 & \omega_{p} \\ -\omega_{p} & 0 \end{pmatrix}, \quad p = 1 \cdots \frac{d_n}{2},
\]

while gauge bundle of rank \(d^E\) has the field strength components

\[
\frac{1}{2\pi} F_{AB} = \frac{i}{4\pi} F_{AB i j} \psi_{-0}^i \psi_{0}^j = \begin{pmatrix} 0 & \rho_{r} \\ -\rho_{r} & 0 \end{pmatrix}, \quad r = 1 \cdots \frac{d^E}{2},
\]

where \(i, j, k, l\) denotes the indices along \(M_{g_1g_2}\) directions.

Let’s also define the action of \(g_1, g_2\) on a real bundle \(V\). \(V\) could be either tangent bundle \(TM\) or the gauge bundle \(E\). Under the action of \(g_1, g_2\), \(V\) restricted to \(M_{g_1g_2}\) formally decomposes as

\[
V|_{M_{g_1g_2}} = V^{(0)} \oplus (\oplus_r V^{(r)}).
\]

The second term represents a formal sum of 2-dimensional real bundles on which \(g_1, g_2\) act as rotation by the angles \(2\pi \alpha_r, 2\pi \beta_r\) respectively. Then,

\[
I(g_1,g_2;\tau) = \int_{M_{g_1g_2}} dx_0^i dx_0^i 1 \sqrt{\text{det}'(\hat{\partial} + i \frac{F_{kl}}{2\pi})}
\]

\[
\prod_{r=1}^{d^E} \text{det}_{\alpha_r + \frac{1}{2} \beta_r + \frac{1}{2}}(\hat{\partial} + i \rho_{r}) \text{det}_{-\alpha_r - \frac{1}{2} - \beta_r - \frac{1}{2}}(\hat{\partial} - i \rho_{r})
\]

\[
\prod_{r=1}^{d_n} \text{det}_{\alpha_p, \beta_p}(\hat{\partial} + i \omega_{p}) \text{det}_{-\alpha_p, -\beta_p}(\hat{\partial} - i \omega_{p}),
\]

(6.12)
where \( \text{det}' \) denotes the determinants without the zero modes. Here we denote \( \text{det}_{\alpha,\beta}(\bar{\partial} + \nu) \) denotes the determinants of \( \bar{\partial} + \nu \) of complex modes \( \Psi(\sigma_1, \sigma_2) \) with the boundary conditions,

\[
\Psi(\sigma_1 + 2\pi, \sigma_2) = e^{2\pi i \beta} \Psi(\sigma_1, \sigma_2),
\]

\[
\Psi(\sigma_1, \sigma_2 + 2\pi) = e^{2\pi i \alpha} \Psi(\sigma_1, \sigma_2).
\]

(6.13)

And

\[
\text{det}_{\alpha,\beta}(\bar{\partial}) \equiv d\begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0|\tau)
= e^{i\pi(\alpha\beta - \beta)} q^{\frac{\beta^2 - \beta + \frac{1}{2}}{2}} \prod_{n=1}^{\infty} (1 - q^n e^{2\pi i \alpha})(1 - q^n e^{-2\pi i \alpha}),
\]

(6.14)

with \( 0 \leq \alpha, \beta < 1 \). Finally

\[
\text{det}_{\alpha,\beta}(\bar{\partial} + \nu) = e^{-i\pi \nu} d\begin{bmatrix} \alpha + \nu \\ \beta \end{bmatrix} (0|\tau).
\]

(6.15)

If we evaluate one such component

\[
\sqrt{\text{det}_{\alpha,\beta}(\bar{\partial} + i\rho_r)\text{det}_{-\alpha,\beta}(\bar{\partial} - i\rho_r)}
= e^{\frac{i\pi}{2}(2\alpha\beta - \alpha - \beta)} q^{\frac{(\beta - \frac{1}{2})^2 + \frac{1}{4}}{2}} \prod_{n=1}^{\infty} (1 - q^n e^{2\pi i \alpha} e^{-2\pi \nu})(1 - q^n e^{-2\pi i \alpha} e^{2\pi \nu}),
\]

for \( \beta \neq 0 \) while for \( \beta = 0 \)

\[
\sqrt{\text{det}_{\alpha,0}(\bar{\partial} + i\rho_r)\text{det}_{-\alpha,0}(\bar{\partial} - i\rho_r)}
= e^{-\frac{i\pi}{4} \alpha} q^{\frac{1}{4}} \prod_{n=1}^{\infty} (1 - q^n e^{2\pi i \alpha} e^{-2\pi \nu})(1 - q^n e^{-2\pi i \alpha} e^{2\pi \nu})
\times \sqrt{(1 - e^{2\pi \alpha} e^{-2\pi \nu})(1 - e^{-2\pi \alpha} e^{2\pi \nu})}.
\]

(6.16)

Note that the zero mode contribution is given by

\[
\sqrt{(1 - e^{2\pi i \alpha} e^{-2\pi \nu})(1 - e^{-2\pi i \alpha} e^{2\pi \nu})} = \pm 2i \sinh \pi (\rho_r - i\alpha),
\]

(6.17)

where \( \pm \) sign can be chosen so that it reduces to \( |2\sin \pi \alpha| \) with \( \rho_r = 0 \). Given this
machinery, one can evaluate the twisted index of the orbifold

\[ I(g_1, 0; \tau) = \int_{M_{g_1}} dx_0^i d\psi^i_0 \hat{A}_{M_{g_1}} \prod_{r=1}^{d} \prod_{n=1}^{\infty} \frac{1}{(1 - q^r e^{-2\pi i \omega_r})(1 - q^r e^{2\pi i \omega_r})} \]

\[ \prod_{r=1}^{d} \prod_{n=1}^{\infty} (1 + q^{n+\frac{1}{2} e^{2\pi i \omega_r} e^{-2\pi i \omega_r}})(1 + q^{n-\frac{1}{2} e^{-2\pi i \omega_r} e^{2\pi i \omega_r}}) \times \frac{1}{\pm 2i \sinh \pi (\omega_p - i\alpha_p)} \]

where \( \frac{iR}{2\pi}, \frac{iR'}{2\pi}, \frac{iF'}{2\pi} \) have the matrices with skew eigenvalues are \( \omega_{\mu}, \omega_p - i\alpha_p, \rho_r - i\alpha_r \), respectively. For \( V^{(0)} \) directions, \( \alpha_r = 0 \) is understood. In particular we are interested in the basic representation and we will read off the term of the lowest order in \( q \), which is \( q^{\frac{1}{2}} \) in this case,

\[ \int_{M_{g_1}} dx_0^i d\psi^i_0 \hat{A}(M_{g_1})e^{\frac{iF'}{2\pi}} \prod_{r=1}^{d} \frac{1}{\pm 2i \sinh \pi (\omega_p - i\alpha_p)} = \text{Tr}(-1)^F g_1 , \] (6.19)

which is nothing but the Atiyah-Bott fixed point formula. In writing down the action we discarded the phase factor depending on \( \alpha \) and the zero point energy contribution to the fractional factor of \( q \) to match the usual index result. Note that \( \sinh \) factor gives the \( g_1 \) action on the normal component of the curvature.

Now for general twisted sectors

\[ I(g_1, g_2; \tau) = \int_{M_{g_1 g_2}} \hat{A}(M_{g_1 g_2}) \prod_{n=1}^{\infty} \frac{\det(1 + q^{n-\frac{1}{2} e^{iF'}})}{\det(1 - q^{n} e^{iF'/2\pi})} \]

\[ \times \prod_{r=1}^{d} \prod_{n=1}^{\infty} (1 + q^{n+\frac{1}{2} e^{2\pi i \omega_r} e^{-2\pi i \omega_r}})(1 + q^{n-\frac{1}{2} e^{-2\pi i \omega_r} e^{2\pi i \omega_r}}) \times \frac{1}{\pm 2i \sinh \pi (\omega_p - i\alpha_p)} \]. (6.20)

Here \( F \) is the field strength along \( V^{(0)} \) while \( d^E_n \) denotes the dimension of the vector bundle excluding \( V^{(0)} \). The peculiarity of this expression is that there will be contributions only from the states invariant under the orbifold action. One can easily check that the projection operators \( \frac{1+g_1+g_2^2+\cdots+g_m^{m-1}}{m} \) for an element \( g_1 \) of order \( m \) acting on the twisted sector by \( g_2 \) gives the vanishing contribution to the twisted index for any state not invariant under the orbifold action. Thus one of the nontrivial contribution comes from

\[ \int_{M_{g_1 g_2}} \hat{A}(M_{g_1 g_2}) q^{\frac{1}{2}} e^{\frac{iF'}{2\pi}} , \] (6.21)

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while more complicated combinations of the tensor representations of the normal bundle of the tangent bundle and $V^{(v)}$ are possible. The above term Eq. (6.21) comes from the twisted states localized along $M_{g_1 g_2}$ and singlet under the rotation group of the normal directions to $M_{g_1 g_2}$. If we consider more general states transforming nontrivially under the transverse rotation group, they have more complicated expression, which can be read off from Eq. (6.20).

6.2 Rules for Twisted Sectors

Since we could in principle have the nontrivial contributions from the twisted states, we have to decide if we have to worry about contributions from the lowest twisted states when we are trying to read off the index for the field theory limit. We already know that for Euler characteristic we have to include the contribution from the twisted sectors. We also expect this is the case for the signature of a four manifold which counts the difference in the number of self-dual and the anti-self-dual 2-forms since two forms could arise in the twisted sectors.

To facilitate the discussion, let us consider a specific compactification of string theory and figure out the plausible rules for each index problem. Consider Type IIB compactified on K3. This is a chiral N=(2,0) theory and the matter contents are completely fixed by anomaly constraints. Thus for whole moduli space of K3, the spectrum is the same and this applies for K3 orbifold such as $T^4/Z_2, T^4/Z_3$. The matter content consists of 1 supergravity multiplets and 21 tensor multiplets. Supergravity multiplet consists of graviton and 5 self-dual two-index tensors and gravitinos. One tensor multiplet consists of 5 scalars and one anti-self-dual tensor and chiral fermions. The tensors arise from two massless states $B^i_{\mu\nu}$ with $i = 1, 2$ of Type IIB string theory, and compactify self-dual 4-form $B^+_{\mu\nu\lambda\rho}$ on two forms on K3. On K3 we have 3 self-dual 2-forms and 19 anti-self-dual 2-forms so that we get 3 self-dual tensors and 19 anti-self-dual tensors out of $B^+_{\mu\nu\lambda\rho}$. If we realize K3 as $T^4/Z_2$ orbifold where $Z_2$ acts as $z_1 \rightarrow -z_1, z_2 \rightarrow -z_2$, for a complex coordinate $z_i$ of the torus, we have 16 fixed points and, upon the blowup, each of the fixed point is replaced by Eguchi-Hanson space, which supplies one anti-self-dual 2-form. Hence on each of the twisted sector we obtain one tensor multiplet. Now from this consideration, it’s obvious that we have to include the twisted sector contribution for the computation of the signature of K3. This is given by difference of the number of self-dual tensors and anti-self-dual tensors.

Now consider a Dirac spinor of positive chirality spinor $\eta \in S^+_{K3}$ on K3. If we have such spinors, one obtains the 6-d gravitino from 10-d gravitino

$$\Psi^M \rightarrow \psi_\mu \otimes \eta$$ (6.22)
Thus if we have such spinors on K3, one can obtain gravitinos on 6-dimensions as many as those. One can ask if such spinors could arise from the twisted sectors. But we already saw that on the twisted sectors, only tensor multiplets arise so that we do not have gravitinos from the twisted sectors. Had we obtain the gravitinos from the twisted sectors, we will also have multiple gravitons by supersymmetry, which is quite weird! Thus spinors of K3 can come only from untwisted sectors from the $Z_2$ invariant projection of spinors on $T^4$. These spinors are coming from Ramond sectors of the right moving sectors and we can see why the spinors of K3 could not arise from the twisted sectors. In the untwisted sector $8_s$ of 10-d postive chiral spinor can be represented as Ramond vacuum

$$|\pm, \pm, \pm, \pm>,$$  (6.23)

in the light cone gauge with total number of + sign being even. Here four ±’s denote the eigenvalue of $\exp(\pi J_i)$ where $J_i$ are 4 Cartans of $SO(8)$. Under $SO(4)_1 \times SO(4)_2 = SU(2)_{L1} \times SU(2)_{R1} \times SU(2)_{L2} \times SU(2)_{R2}$ with $SO(4)_2$ action on the transverse K3 directions

$$8^s \rightarrow (2, 1, 2, 1) \oplus (1, 2, 1, 2),$$  (6.24)

and under the $Z_2$ orbifold action only $(2, 1, 2, 1)$ is invariant. Combined with $8_v$ of the right moving modes transforming as a vector under $SO(8)$,

$$8_v \rightarrow (2, 2, 1, 1) \oplus (1, 1, 2, 2),$$  (6.25)

where $(2, 2, 1, 1)$ is invariant under the $Z_2$ action. The combination

$$(2, 1, 2, 1) \otimes (2, 2, 1, 1) \oplus (1, 2, 1, 2) \otimes (1, 1, 2, 2)$$  (6.26)

give rise to the gravitinos of 6-d after $Z_2$ projection (plus fermions of spin $\frac{1}{2}$). Now consider the twisted sectors. The fermions arise from Ramond vacuum of right-moving sectors possibly combining with the NS sector states. Important thing is that Ramond vacuum of the twisted sectors is given by

$$|\pm, \pm>$$  (6.27)

with the total number of + signs even. Note that this state transforms nontrivially only under $SO(4)_1$ and is a singlet under $SO(4)_2$. This could not be a spinor on K3, which transforms nontrivially under $SO(4)_2$. This argument goes through exactly for the other K3 orbifolds.

---

#19 Spinors could arise from the left moving sectors for Type II string theory but the arguments goes parallel to the right-moving modes.

#20 $Z_2$ action on Ramond vacuum can be written as $\exp(\pi(J_3 - J_4))$ where $J_3, J_4$ are the Cartan elements on $SO(4)_2$. 

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For the situation we are interested in, we have to deal with a spinor in $S^+ \otimes E$ with a suitable vector bundle $E$. The associated Dirac index is naturally defined in the heterotic theory. Since the orbifold action acting on the right-movers cannot generate $S^+$ one cannot obtain the desired spinors from the twisted sectors.

This argument goes through Calabi-Yau 3-fold case as well since again the Ramond vacuum of the twisted sector is of the form

$$|\pm>$$  \hspace{1cm} (6.28)

and this is a singlet under $SO(6)$ of the transverse direction of Calabi-Yau 3-fold while spinors on Calabi-Yau 3-fold transform nontrivially under $SO(6)$. Hence in the spacetime supersymmetric theory, we do not have to consider the contribution from the twisted sectors for the Dirac index. This is also true for nonsupersymmetric orbifold, since these states in the right-moving modes are the excitation with the tensor representations of the transverse rotation group around the Ramond vacuum of the type Eq. (6.28). This could not match the spinor representation of the spinors.

Now let’s apply this formalism and carry out the Dirac index computation for the simplest orbifold $T^4/Z_2$. One can use the formula Eq. (6.19). From the projection

$$\frac{1}{2} \text{Tr} \left( (-1)^F (1 + \alpha) \right)$$  \hspace{1cm} (6.29)

with $\alpha$ being $Z_2$ action on $T^4$ we find that the first term $\frac{1}{2} \text{Tr}(-1)^F$ is vanishing since this counts the Dirac index on $T^4$. From the $\frac{1}{2} \text{Tr}(-1)^F g$ one obtains

$$\frac{1}{2} \cdot 16 \cdot \frac{1}{(2\sin\frac{\pi}{2})^2} = 2,$$  \hspace{1cm} (6.30)

where 16 comes from the total number of the fixed points and $2\sin\frac{\pi}{2}$ factor comes from the $Z_2$ action on the normal bundle over the fixed point. Indeed we obtain the right Dirac index on K3. Similar computation can be done for the other toroidal orbifold of $K3$ and gives rise to the same result.

### 6.3 Orbifold Index and the Equivariant Generalization

Let us come back to our problems of counting BPS bound states of charge centers and see how the Bose/Fermi orbifolding is reflected on the equivariant index. Consider the simplest case where we started with a pair of identical particles and other distinguishable ones. The orbifolding group would be then $S_2 = Z_2$. Here let us assume that $-\Omega^+ = \pm 1$, and then the expected answer, from either MPS formula or the derivation in the previous section, is

$$\mp (-1)^{2\gamma_1 \gamma_2} \Omega (2\gamma_1 + \gamma_2) = \frac{1}{2!} I(\gamma_1, \gamma_1, \gamma_2) \pm I(2\gamma_1, \gamma_2) \frac{1}{2^2}.$$  \hspace{1cm} (6.31)
On the other hand, we may also read off the index from the $U(1)$ character-valued index, with the projection operator
\[
\frac{1 \pm \alpha}{2},
\]
with $\alpha^2 = 1$. A new ingredient here is the sign in front of $\alpha$, which chooses whether one retains bosonic or fermionic wavefunctions.

When we apply the formula derived earlier, we have to know how the bundle of interest would transform under $S_n$ actions in general. Note that we are dealing with rank one bundle given by
\[
\mathcal{F} = -\frac{1}{2} \sum_{A \neq B} \epsilon^{abc} \frac{\bar{x}^A - \bar{x}^B}{|\bar{x}^A - \bar{x}^B|^3} q_{AB} \, dx^A \land dx^B.
\]
(6.33)

One can check that the vector bundle is invariant under the exchange of any two $\bar{x}_A, \bar{x}_B$ and hence is invariant under the action of $S_n$. Reading off individual terms from the formula in the last subsection, the result for $S_2 = Z_2$ is
\[
\frac{1}{2} \int_M dx_0^\mu d\psi_0^\mu \hat{A}(M) Ch(F) \pm \frac{1}{2} \int_{M_2} dx_0^\mu d\psi_0^\mu \hat{A}(M_2) Ch(F) \frac{1}{2i \sinh \pi \omega_1'}.
\]
(6.34)

where $M_2$ is the manifold fixed by $Z_2$ action. The first term produces the first term in Eq. (6.31), so let us take a look at the second term. $\omega_1' = \omega_1 - i\alpha_1$ with $\omega_1$ is the eigenvalue of the curvature tensor along the normal bundle and $2\pi\alpha_1$ is the rotation angle of the $Z_2$ action, which is $\pi$. Here we assume that the internal degeneracies are taken into account as explained in the main text. For the 3 body case with 2 identical particles $\omega_1$ does not contribute for the dimensional reason since $2i \sinh \pi \omega_1' = 2 \cosh \pi \omega_1$ gives only even powers of $\omega_1$ and we have the 2-d fixed manifold with 2-d normal bundle. $\sinh$ factor is due to the $Z_2$ action along the normal bundle, which gives rise to $\frac{1}{2\sin \frac{\pi}{2}} = 1/2$. Combined with $1/2$ factor in the projection operator this gives the wanted $\pm 1/2^2$ factor at the second term of Eq. (6.31).

For $S_3$ orbifold and beyond, it is more convenient to consider the $U(1)$ equivariant index and use the fixed point theorem where $U(1)$ is generated by the third component of angular momentum operator $J_3$. For general action of $g$,
\[
\text{Tr}(-1)^F g y^{2\pi J_3} = \int_{M_g} dx_0^\nu d\psi_0^\nu \hat{A}(M, \nu) Ch(F, \nu) \prod_{i=1}^{d_n} \frac{1}{\pm 2i \sinh (\nu L + \pi \omega_p')},
\]
(6.35)

#21Here we have $y^{2\pi J_3}$ while in the previous section we use $y^{2J_3}$ in the index definition. This facilitates the comparison with the corresponding formula appearing in [23].
where \( \omega_p' = \omega_p - i\alpha_p \) with \( \omega_p \) is the eigenvalue of the curvature tensor along the normal bundle and \( 2\pi\alpha_p \) is the rotation angle and \( e^\nu = y \). Following [23], we change the expressions into equivariant characteristic classes accordingly and \( L \) denotes the action of \( J_3 \) on \( T^{(1,0)}(M_n) \). Now consider \( S_2 = Z_2 \) case. This equivariant index can be evaluated by evaluating the fixed points under \( J_3 \). The fixed points are simply north and south poles of the sphere, \( L \) has the eigenvalue \( \pm 1 \) and \( x_r = 0 \) assuming we are dealing with isolated points. Thus,

\[
\frac{1}{i2\sinh(\nu L + \pi \omega_1')}, \frac{1}{2i\sinh(\nu - \frac{\pi i}{2})} = \frac{1}{2\cosh\nu} = \frac{y - y^{-1}}{y^2 - y^{-2}}. \tag{6.36}
\]

Note that the expression is invariant under \( \nu \to -\nu \) so that we don’t have to worry about the sign of \( L = \pm 1 \). This persists for all \( S_n \) case as well.

Now consider \( S_3 \) case. \( S_3 \) projection has the form

\[
\frac{1}{3!}(1 + g + g^2)(1 \pm \alpha), \tag{6.37}
\]

where \( g = (123) \) meaning \( x_1 \to x_2, x_2 \to x_3, x_3 \to x_1 \). Again, we inserted a sign \( \pm \) in front of the order 2 permutation \( \alpha \) to pick out either bosonic or fermionic wavefunctions. Then \( g^2 = (132) \) and \( \alpha = (12) \), meaning \( x_1 \to x_2, x_2 \to x_1 \). One can check that \( g\alpha = (13), g^2\alpha = (23) \). For the bulk contribution without \( g \) or \( \alpha \), the projection operator simply gives \( 1/3! \). Now consider

\[
\pm \frac{1}{3!}(\alpha + g\alpha + g^2\alpha). \tag{6.38}
\]

Each of the three elements give rise to \( Z_2 \) fixed manifold which we already worked out. We have the contribution

\[
\pm \frac{1}{3!} \cdot 3 \cdot \frac{y - y^{-1}}{y^2 - y^{-2}} = \pm \frac{1}{2} \frac{y - y^{-1}}{y^2 - y^{-2}}. \tag{6.39}
\]

Now turn to \( \frac{1}{3!}(g + g^2) \). These two elements are order 3. For each element we evaluate the equivariant index again by going to a fixed submanifold. The \( \hat{A} \) genus combined with Chern characters give rise to the usual angular momentum factor. The relevant factor for the orbifolding effect is

\[
\frac{1}{2\sinh(\nu + \frac{\pi i}{3})}, \frac{1}{2\sinh(\nu - \frac{\pi i}{3})} = \frac{y - y^{-1}}{y^3 - y^{-3}}. \tag{6.40}
\]

Thus, for the refined case

\[
-(-1)^{3(\gamma_1, \gamma_2)} \Omega_{\text{ref}}(3\gamma_1 + \gamma_2) = \frac{1}{3!} I_{\text{ref}}(\gamma_1, \gamma_1, \gamma_1, \gamma_2)
\]

\[
\pm I_{\text{ref}}(2\gamma_1, \gamma_1, \gamma_2) \frac{1}{2} \frac{y - y^{-1}}{y^2 - y^{-2}} + I_{\text{ref}}(3\gamma_1, \gamma_2) \frac{1}{3} \frac{y - y^{-1}}{y^3 - y^{-3}}, \tag{6.41}
\]

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including the factors of the internal degeneracies. Here we are considering the moduli space of three particle configurations invariant under $S_3$. Normal directions mean the deformations away from the fixed configurations in $Z_3$ invariant way. The action is rotation by $\pm 2\pi/3$. Taking $y = 1$ limit, we find

$$-(-1)^{3(\gamma_1, \gamma_2)}\Omega(3\gamma_1 + \gamma_2) = \frac{1}{3!} I(\gamma_1, \gamma_1, \gamma_1, \gamma_2) \pm I(2\gamma_1, \gamma_1, \gamma_2) \frac{1}{2^2} + I(3\gamma_1, \gamma_2) \frac{1}{3^2}$$

(6.42)

reproducing the ordinary index formula of the previous section.

It is now obvious how to generalize for general permutation group. Again we are assuming $-\Omega^+ = \pm 1$. If we consider the various manifolds fixed under the elements of order $m$ the contribution from the normal directions in the index formula gives

$$\frac{1}{2\text{sinh}(\nu + \frac{2\pi i}{m})2\text{sinh}(\nu - \frac{\pi i}{m})2\text{sinh}(\nu + \frac{2\pi i}{m})2\text{sinh}(\nu - \frac{2\pi i}{m})\cdots} = \frac{y - y^{-1}}{y^m - y^{-m}},$$

(6.43)

which becomes $1/m$ in the $y \to 1$ limit. Consider $m\beta_1 + \beta_2$ for general $m$. One has to figure out the combinatoric factors appearing in the projection of $S_m$ group elements. Suppose that $m$ can be written as

$$m = n_1m_1 + \cdots + n_lm_l,$$

(6.44)

The number of ways of the partition of $m$ elements into $n_i$ of $m_i$ elements is given by

$$m! \prod_{i=1}^l \frac{1}{n_i!m_i!}$$

(6.45)

For example if we partition 4 into 2 of 2 elements, we have 3 possibilities, i.e.,

$$12|34, 13|24, 14|23.$$  

(6.46)

For each particular partition of $m$ elements, one has $\prod_{i=1}^l (m_i - 1)!$ permutation elements. For example if we consider the particular partition of 6 into 123|456, from 123 we have two permutation elements (123), (132) and from 456 we have (456), (465) so that one can generate 4 permutation elements out of this particular partition. Hence the total number of the permutation elements arising from the partition eq.(6.44) is

$$m! \prod_{i=1}^l \frac{1}{n_i!m_i!}.$$  

(6.47)
Combined with $\frac{1}{m!}$ in the projection operator of $S_m$ group, and the contribution from
the normal directions in the index theorem, we obtain the factor

$$\prod_{i=1}^{l} \frac{1}{n_i!} \prod_{i=1}^{l} \left( \frac{1}{m_i y^{m_i} - y^{-m_i}} \right)^{n_i}$$  \hspace{1cm} (6.48)

Thus for simple case of $-\Omega^+ = \pm 1$ we have

$$-\Omega_{ref}^-(m\beta_1 + \beta_2)(-1)^{m<\gamma_1,\gamma_2>}$$

$$= \sum_{m=\sum_{i=1}^{l} n_i m_i} \prod_{i=1}^{l} \frac{1}{n_i!} I_{ref}(m_1\beta_1, m_1\beta_1, \ldots, m_l\beta_l, m_l\beta_l) \prod_{i=1}^{l} \left( \frac{1}{m_i y^{m_i} - y^{-m_i}} \cdot -\Omega_i \right)^{n_i} (-\Omega_2)$$

where $I_{ref}$ has $n_i$ of $m_i\beta_i$ factors. The sign of $-\Omega_i$ can be taken account by the
judicious choice of the signs in the projection operator of $S_m$ as was done for $S_2$ and
$S_3$ cases. Thus we obtain the expected answer.

7 Summary and Comments

In this note we showed how $n$ generic BPS dyons of Seiberg-Witten theory interact
with one another, and how the relevant low energy dynamics with $N = 4$ supersym-
metry can be derived in the vicinity of a wall of marginal stability. The resulting
quantum mechanics is specified by three classes of quantities: kinetic term, poten-
tials, and minimal couplings. The latter two turn out to be constrained to each other
by supersymmetry and can be derived exactly, and are universal, in that the general
structure is applicable to BPS black holes as well. The kinetic term may differ, but
for counting non-threshold bound states via index theorem, we only need the asympto-
tic form of the kinetic terms, which fixes effectively the entire Lagrangian. Thanks
to the universal form, this Lagrangian can also be used to compute non-threshold
bound states of BPS black holes as well as those of Seiberg-Witten dyons.

We showed how the usual truncation (in the previous BPS black hole studies)
down to zero locus, $\mathcal{M}_n$, of potentials is misleading because the massgaps along the
classically massive direction are always the same as the quantum massgaps along
$\mathcal{M}_n$, due to the latter’s finite size. Instead, one must sacrifice $N = 4$ supersymmetry,
in favor of an index-preserving $N = 1$ deformation, in order to reduce the problem
to a nonlinear sigma model on $\mathcal{M}_n$.

This gives a definite prescription, hitherto unknown, on how to handle the fermionic
superpartners, and the final form of the index is that of a Dirac operator on $\mathcal{M}_n$ with
an Abelian gauge field $F$ determined unambiguously by the minimal couplings among
dyons/black holes. Along with \( n - 1 \) radial, classical massive directions, \( 2(n - 1) \) fermionic partners become decoupled from the problem, leaving behind a supersymmetric quantum mechanics on \( \mathcal{M}_n \) with real supersymmetry. (Three bosonic and four fermionic variables decouple also, playing the role of the center of mass degrees of freedom.) This shows rigorously why the Dirac index is the relevant one, as was anticipated by de Boer et al. [25].

Since typical wall-crossing problem involves only two linearly independent charge vectors, and thus bound states of many identical BPS states, statistics is of major importance. We address this directly for the index problem by inserting the relevant projection operator \( \mathcal{P}_\Gamma \), and expanding the index to a series involving various fixed submanifolds. Each such contribution consists of two multiplicative factors: one is usual Dirac index on the fixed submanifold and the other is contribution from the normal direction. The latter turns out to be universal and generates a numerical factor \( \sim 1/p^2 \) for each \( p \) coincident and identical particles, times the intrinsic degeneracy of the particle in question. This eventually lead to the rational invariants, \( \Omega(\gamma) = \sum_{\gamma/p} \Omega(\gamma/p)/p^2 \), as the effective degeneracy factor, as was also noted by Manschot et al. [22]. In the end, we have derived the general wall-crossing formula, from the viewpoint of spatially loose BPS bound states by starting from Seiberg-Witten theory, ab initio.\(^\#22\)

Another unexpected bonus, which also shows why the low energy quantum mechanics is never really \( 2(n - 1) \) dimensional but must be addressed with all \( 3n \) position coordinates, was a clarification of various index quantities in literature. The field theory index is most generally computed by the protected spin character,

\[
\text{Tr} \left( (-1)^{2J_3} y^{2J_3+2I_3} \right)
\]

We showed how this quantity maps to the quantum mechanical index

\[
\text{Tr} \left( (-1)^{2J_3} y^{2J_3+2I_3} \right)
\]

with the latter’s \( SO(4) = SU(2)_L \times SU(2)_R \) invariance. This is then further reduced, mathematically, to the usual equivariant index defined over \( \mathcal{M}_n \) as

\[
\text{Tr} \left( \langle 0 | (-1)^{2J_3} | 0 \rangle_{\text{heavy}} y^{2J_3} \right)
\]

where \( J_3 = J_3 + I_3 \) is naturally picked out because in the reduction process, which is a purely mathematical operation, we lose part of the \( SO(4) = SU(2)_L \times SU(2)_R \)

\(^\#22\) The only unresolved issue here is the angular momentum content of the bound states, which is in particular needed when one translates the quantum mechanics index to the second helicity trace of \( D = 4 \) and \( N = 2 \) field theory. For this, we followed the well-known assignment, which has been tested in many explicit examples and widely believed to be correct.
symmetry. At the end of the reduction process, only $SU(2)_J$ survives and rotates bosons and fermions equally, unlike the physical rotation symmetry $SU(2)_L$ which sees that fermions arise from $D = 4$ spinors. Also we saw how the chirality operator $(-1)^{2J_3}$ is related to the canonical one $(-1)^{F_{Mn}}$ as

$$\text{Tr} \left( \langle 0 | (-1)^{2J_3} | 0 \rangle_{\text{heavy}} y^{2J_3} \right) = (-1)^{\sum_{A<B} (\gamma_A \gamma_B) + n - 1} \times \text{Tr} \left( (-1)^{F_{Mn}} y^{2J_3} \right),$$

giving us by-now familiar sign in wall-crossing formulae. This formula is valid before statistics imposed. Imposing Bose/Fermi statistics introduces a further complication, which are addressed in much detail in section 5.

An important question that remains unaddressed satisfactorily here, or in any other existing literature, is whether and under what circumstances threshold bound state appear. Primary examples would be states that are bound state of more than two of the same BPS states. In all of $D = 4$ $N = 2$ field theory examples where explicit states were constructed, no such states ever appeared as far as we know, yet BPS black holes can be easily found to have an integer multiple of a given charge. This tells us that, for addressing such problems, details of the kinetic term in the low energy dynamics are important and perhaps one should deal with $3n$ dimensional low energy dynamics directly, instead of reducing the problem to $\mathcal{M}_n$; $\mathcal{M}_n$ would be no longer compact and thus less useful, since classical ground states would include configuration with arbitrary small separations and those with arbitrary large separations among some charge centers. We further expect that the kinetic terms for dyons and for black holes, respectively, will be substantially different at small mutual separations, accounting the difference. For dyons, this poses additional difficulties since we must face the fact that dyons in field theory are actually non-Abelian objects. For black holes, horizon effectively acts as short-distance cut-off and the configuration remain Abelian at any separation, and the question of kinetic terms at small separation is better posed. We hope to return to this problem in near future.

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Appendix

A $\mathcal{N} = 4$ Superfield Formalism

According to [38] and [37], one can introduce $\mathcal{N} = 4$ superfield by

$$\Phi_{\alpha\beta} = (D_\alpha \bar{D}_\beta + D_\beta \bar{D}_\alpha)V \quad (A.1)$$

with $V$ being the $D = 4$ $N = 1$ vector supermultiplet. Following [38], we define

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - \frac{i}{2} \bar{\theta}_\alpha \frac{\partial}{\partial t}, \quad \bar{D}_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha} - \frac{i}{2} \theta_\alpha \frac{\partial}{\partial t} \quad (A.2)$$

so that

$$\{D_\alpha, \bar{D}_\beta\} = -i \delta_\alpha^\beta \frac{\partial}{\partial t} \quad (A.3)$$

and the indices are raised and lowered by $\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta}$. Alternatively, $\Phi_{\alpha\beta}$ is uniquely determined by the conditions

$$\Phi_{\alpha\beta} = \Phi_{\beta\alpha}, \quad \bar{\Phi}_{\alpha\beta} = \Phi^{\alpha\beta}, \quad D_\alpha \Phi_{\beta\gamma} + D_\beta \Phi_{\gamma\alpha} + D_\gamma \Phi_{\beta\alpha} = 0 \quad (A.4)$$

It is convenient to define real superfields $\hat{\Phi}$ by

$$\hat{\Phi}_a \equiv \frac{1}{2} \epsilon^{\beta\gamma}(\sigma_a)_{\gamma}^\alpha \Phi_{\alpha\beta}$$

$$= \ x_a + \frac{i}{2} \theta \sigma_a \bar{\chi} - \frac{i}{2} \chi \sigma_a \bar{\theta} + \frac{1}{4} \theta \sigma_a \bar{\sigma} \bar{\theta} + \frac{1}{2} \epsilon_{abc} \dot{X}_b \theta \sigma_c \bar{\theta} + \frac{1}{8} \theta \theta \theta \sigma_a \dot{\bar{\chi}} + \frac{1}{8} \theta \theta \theta \sigma_a \dot{\chi} + \frac{1}{16} \theta \theta \theta \theta \dot{\dot{X}}_a \quad (A.5)$$

This defines $\mathcal{N} = 4$ superfield for 3 bosonic and 4 fermionic degrees of freedom. The supersymmetric transformation is given by

$$\delta x_a = \ \frac{i}{2} \epsilon \sigma_a \bar{\chi} - \frac{i}{2} \chi \sigma_a \epsilon$$

$$\delta \chi = -\epsilon \sigma_a \dot{x}_a + \frac{i}{2} \dot{F}$$

$$\delta F = \epsilon \dot{\chi} - \bar{\epsilon} \dot{\bar{\chi}} \quad (A.6)$$

This is a different convention from $N = 1$ so we may have to change the normalization of the fields accordingly to conform to $N = 1$ convention used in the text.
The kinetic term is given by

\[ L_0 = \frac{1}{2} \int d^4\theta L(\Phi_a). \] (A.7)

With \( f \equiv \partial_a \partial_b L(x_b) \) this can be written componentwise\#24

\[
\begin{align*}
L_0 &= f(X)(\frac{1}{2}\dot{x}_a^2 - \frac{i}{4}\dot{\chi}\bar{\chi} + \frac{i}{4}\chi\dot{\bar{\chi}} + \frac{1}{8}F^2) \\
&\quad - \frac{1}{8}F\chi\sigma_a\bar{\chi}\partial_a f + \frac{1}{4}\epsilon_{abc}\dot{\bar{\chi}}(\chi\sigma_c\bar{\chi})\partial_a f - \frac{1}{32}\chi\chi\bar{\chi}\partial_a^2 f.
\end{align*}
\] (A.8)

Now the generalization to \( n \) superfield case is straightforward. One introduces \( n \) superfield \( \Phi^A_a \), \( A = 1, \ldots n \). The kinetic part is given by

\[ L_0 = \frac{1}{2} \int d^4\theta L(\Phi^A_a) \] (A.9)

This action is rather complicated in component form.

Note that in this simple \( \mathcal{N} = 4 \) superspace formulation, potential terms are not obvious,\#25 which is reasonable since the superfields \( \hat{\Phi} \)'s came from the gauge vector multiplet of \( D = 4 \) by dimensional reduction. Furthermore, as we emphasized in the main text, \( \mathcal{N} = 1 \) superspace description is more convenient since the evaluation of the index actually relies only on \( \mathcal{N} = 1 \) supersymmetry. For this, we split the superfields \( \hat{\Phi} \) to 3\( n \) bosonic superfields and \( n \) fermionic superfields as

\[ \Phi^A_a = x^a_A - i\theta\psi^a_A, \quad \Lambda^A = i\lambda^A + i\theta b^A, \quad A = 1, \cdots n, \quad a = 1, 2, 3. \] (A.10)

where the complex fermions \( \chi \) and \( \bar{\chi} \) are decomposed into 3+1 real fermions \( \psi^a \)'s and \( \lambda \)'s, and the auxiliary field \( F \) is now redefined as \( b = F/2 \). With these superfields, the general form of the kinetic Lagrangian \( L_0 \) was worked out in Ref. [28]

\[
\begin{align*}
\mathcal{L}_0 &= \int d\theta \left( \frac{i}{2} g_{AB}^{cb} D\Phi^A_b \dot{\Phi}^B_c - \frac{1}{2} h_{AB}^{\Lambda} D\Lambda^A D\Psi^B - i f_{AB} \dot{\Phi}^A \Lambda^B + \frac{1}{2!} \epsilon^{abc} D\Phi^A_b D\Phi^B_c D\Phi^C_d \\
&\quad + \frac{1}{2!} m_{ABC}^{ab} D\Phi^A_b D\Phi^B_c \Lambda^C + \frac{1}{2!} m_A^{a} D\Phi^A_b \Lambda^B \Lambda^C + \frac{1}{3!} L_{ABC} \Lambda^A \Lambda^B \Lambda^C \right)
\end{align*}
\] (A.11)

\#24 A simple version of this with constant \( f \) would be the dimensional reduction of supersymmetric QED [50].

\#25 Nevertheless, Ref. [51] offers a superconformal example with potential terms, written in an \( \mathcal{N} = 4 \) off-shell form.
with \( D = \frac{d}{d \theta} - i \theta \frac{d}{dt} \), where

\[
\begin{align*}
    g_{AB}^{ab} &= (\delta_d^a \delta_e^b + \epsilon^{fde} \epsilon_{fe}^b) \partial_A \partial_B \psi^L \\
    h_{AB} &= \delta_{ab} \partial_A^a \partial_B^b \\
    f_{AB}^a &= \epsilon_{bc}^a \partial_A^b \partial_B^c \\
    C_{ABC}^{abc} &= \frac{1}{2} \epsilon^{pqh} \epsilon_{pl}^a \epsilon_{qm}^b \epsilon_{hn}^c \partial_A^a \partial_B^b \partial_C^c \psi^L \\
    n_{ABC}^{ab} &= \frac{1}{2} (\epsilon^{pqn} \epsilon_{pl}^a \epsilon_{qm}^b - \epsilon^{l} \delta_{mb} - \epsilon_{bn} \delta_{la}) \partial_A^a \partial_B^b \partial_C^c \psi^L \\
    m_{ABC}^a &= \frac{1}{2} \epsilon^{j} \epsilon_{jm} \epsilon_{jl}^a \partial_A^a \partial_B^b \partial_C^c \psi^L \\
    l_{ABC} &= \frac{1}{2} \epsilon_{abc} \partial_A^a \partial_B^b \partial_C^c \psi^L.
\end{align*}
\]

Note that, for the asymptotic form \( L(x) = \sum_A m_A (\bar{x}_A)^2 / 2 \), all but the first two, \( g \) and \( h \), vanish identically.

## B Reduction to Nonlinear Sigma Model on \( \mathcal{M}_n \)

With \( n \) centers, one starts with \( 3(n - 1) \) bosonic coordinates and \( 4(n - 1) \) fermionic ones, after the free center of mass part is removed from the dynamics. It is convenient to work with a coordinate system where \( n - 1 \) of them equal to independent linear combinations of \( K_A \)'s. In a slight abuse of notation we will denote these again by \( K_A \), now with \( A = 1, \ldots, n - 1 \); although there are \( n \) \( K \)'s, only \( n - 1 \) of them are linearly independent. Thus, we split the relative part of \( r^{AA} \) and \( \psi^{AA} \) as \( Z^M = (K^A, y^\mu) \) and \( \psi^M = (\psi^A, \psi^\mu) \), with \( M = 1, \ldots, 3(n - 1) \) and \( \mu = n, \ldots, 3(n - 1) \). Along the same spirit, we also denote by \( \lambda^A \), \( n - 1 \) linearly independent combinations of \( \lambda \)'s that belong to the relative part of the low energy dynamics. What do we mean by \( \psi^M \)?

We wish to preserve at least one supersymmetry, say \( Q_4 \), and naturally \( \psi^M \) is the superpartner of \( Z^M \),

\[
\psi^M = \frac{\partial Z^M}{\partial r^{AA}} \psi^{AA},
\]

and the kinetic term of \( \psi^M \) includes two factors of \( \partial r^{AA} / \partial Z^M \).

As argued in section 4, it suffices to consider the dynamics with flat metric, which after taking out the center of mass part becomes

\[
g_{AB} = m_{AB} \delta_{ab},
\]

where \( m_{AB} \) is the \( (n - 1) \times (n - 1) \) reduced mass matrix. Expressing this in the
curved coordinate system, $Z^M$, we find that partial derivatives of metric coefficients $g_{MN}$ are nontrivial. In contrast, nothing much happens to $\lambda$’s, other than one of them being taken out as the center of mass part, so their metric is the same reduced mass matrix, $h_{AB} = m_{AB}$, and is constant. Thus, no coordinate-dependent transformations are needed for $\lambda$’s. The deformed Lagrangian with flat kinetic term reads in this coordinate,

$$L = \frac{1}{2} g_{MN} (Z) \dot{Z}^M \dot{Z}^N - \frac{1}{2} \xi^2 (m^{-1})^{AB} K_A (Z) K_B (Z) - W (Z)_M \dot{Z}^M$$

where the crucial middle term in the second line follows from Eq. (B.1) and the anticommuting nature of fermions. We also used $\partial^\mu K_A = 0$.

Since we anticipate that $K$ directions will decouple as $\xi \to \infty$, we split the metric as

$$[g_{MN}] = \begin{pmatrix} H_{AB} & C_A \\ C^T_A & G_{\mu\nu} \end{pmatrix}$$

and the likewise for its inverse

$$[g^{MN}] = \begin{pmatrix} (H - CG^{-1}C^T)^{-1} & -(H - CG^{-1}C^T)^{-1}CG^{-1} \\ -G^{-1}C^T(H - CG^{-1}C^T)^{-1}CG^{-1} & G^{-1} + G^{-1}C^T(H - CG^{-1}C^T)^{-1}C^TG^{-1} \end{pmatrix}$$

Ignoring $W$ and fermion contributions to the conjugate momentum for now for simplicity, the bosonic part of Hamiltonian will then looks something like

$$\mathcal{H} \approx \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + g^{AB} p_A p_B + \frac{1}{2} g^{AB} P_A P_B + \cdots$$

where $P_A = p_A + (H - CG^{-1}C^T)_{AC} g^{C\mu} p_\mu = p_A - (CG^{-1})^\mu_A p_\mu$ $P_A$’s have the standard canonical commutator with $K$’s, so it is clear that, together with $\sim \xi^2 K^2$ terms, they form very heavy harmonic oscillators of frequency $\sim \xi$, settle
down to its ground state sector, and decouple from ground state counting. This leaves behind

\[ H \simeq \frac{1}{2}(G^{-1})^{\mu\nu} p_\mu p_\nu + \cdots \]  

(B.4)

Denoting the canonical conjugate of \( p_\mu \) in this reduced dynamics again by \( y_\mu \), the corresponding Lagrangian would be

\[ L \simeq \frac{1}{2} G_{\mu\nu} \left. \dot{y}^\mu \dot{y}^\nu + \cdots \right|_{\kappa=0} \]  

(B.5)

This makes clear that we could have done the same more simply by imposing \( \kappa = 0 \) at the level of Lagrangian.

Procedure leading up to (B.4) can be repeated in the presence of \( \mathcal{W} \)'s, which simply shift the conjugate momenta in the Hamiltonian, and it is clear that only \( \mathcal{W}_\mu \)'s will survive. We should ask whether this is consistent, since after all \( d\mathcal{W} \)'s are Dirac quantized magnetic fields, and removing some part of the gauge connection could make the remainder ill-defined. However, we have

\[ d\mathcal{W} = \partial_B \mathcal{W}_A dK^B dK^A + (\partial_\mu \mathcal{W}_A - \partial_A \mathcal{W}_\mu) dy^\mu dK^A + \partial_\nu \mathcal{W}_\mu dy^\nu dy^\mu \]

and the pull-back onto \( \mathcal{M}_n \) is simply

\[ \mathcal{M}_n^*(d\mathcal{W}) = \partial_\nu \mathcal{W}_\mu dy^\nu dy^\mu \]  

(B.6)

The pull-back of a well-defined bundle to a smoothly embedded submanifold is still a well-defined bundle, so the reduced gauge connection \( \mathcal{W}_\mu(\kappa = 0) \) is consistent. Thus, the bosonic part of the action reduces to

\[ L \simeq \frac{1}{2} G_{\mu\nu} \left. \dot{y}^\mu \dot{y}^\nu - \mathcal{W}_\mu \right|_{\kappa=0} \dot{y}^\mu + \cdots \]  

(B.7)

leaving us with the question of how to reduce fermion sector.

The fermions enter the Hamiltonian in two places. One is as bilinear connection term added to the conjugate momenta, and the other is an additive contribution of the form

\[ -i \xi \partial_B \mathcal{K}_A \psi^B \chi^A - i \partial_A \mathcal{W}_B \psi^A \psi^B - i(\partial_\mu \mathcal{W}_A - \partial_A \mathcal{W}_\mu) \psi^A \psi^\mu - i \partial_\mu \mathcal{W}_\nu \psi^\mu \psi^\nu \]

#26 Generally \( \kappa \) will mix in the definition of this new \( y_\mu \) coordinates, to reflect the shift of the conjugate momenta, but this becomes irrelevant because dynamics forces \( \kappa = 0 \). Therefore, the same old \( y \) coordinates can be used here.
with the canonical anticommutator among $\psi_M$'s equal to $g^{MN}$. To disentangle heavy $\psi^A$ from light $\psi^\mu$, we shift the light fermions as

$$\tilde{\psi}^\mu \equiv \psi^\mu + \psi^A(H - CG^{-1}C^T)_{AC}g^C_{\mu} = \psi^\mu - \psi^A(CG^{-1})_{A}^\mu,$$

such that

$$\{\psi^A, \tilde{\psi}^\mu\} = 0, \quad \{\tilde{\psi}^\mu, \tilde{\psi}^\nu\} = (G^{-1})^{\mu\nu}.$$

Let us categorize these fermion bilinears into three different pieces,

$$-i\partial_\mu W_\nu \psi^\mu \tilde{\psi}^\nu - iE_{A\mu} \psi^A \tilde{\psi}^\mu + [-i\xi\partial_B K_A \psi^B \lambda^A + \cdots].$$

Terms in the last bracket involve only $\psi^A$ and $\lambda^A$'s with eigenvalues $\sim \xi$, so these will decouple from the low energy spectrum. The potential mixing between heavy and light modes are in

$$E_{A\mu} = \partial_A W_\mu - \partial_\mu W_A + (CG^{-1})^{\nu}_A (\partial_\nu W_\mu - \partial_\mu W_\nu).$$

For heavy sector, this is of course a minor perturbation and ignorable as $\xi \to 0$. For light sector, things look less innocent since the size of this operator is itself not negligible. However, the heavy fermion enters this operator linearly, and always will connect excited states and ground states of heavy fermion sector. This forces the energy eigenvalue differences $(E_n - E_k)$ in the denominator of the perturbation series to be of order $\sim \xi$, such that the perturbation is suppressed by powers of $\sim \xi/E$. In the end, again, the net effect is to turn off the heavy modes $\psi^A$ and $\lambda^A$ completely, leaving behind

$$-i\partial_\mu W_\nu \psi^\mu \psi^\nu$$

only, where we will call this light fermion again as $\psi^\mu$'s. The simplest way to understand this is to recall that any operator linear in heavy fermions will vanish when sandwiched between heavy sector vacuum.

Combining the reduction processes of the bosonic and the fermionic sectors, it is clear that the connection term can be equally reduced to

$$\frac{i}{2} \partial_L g_{MN} \dot{Z}^N \psi^L \psi^M \rightarrow \frac{i}{2} \partial_B G_{\alpha\beta} \dot{y}^\beta \psi^\alpha.$$

Now we can revert from the Hamiltonian to the Lagrangian, after putting all the heavy modes to their ground states, and arrive at the following reduced Lagrangian,

$$\mathcal{L}^{\mathcal{N}=1}_{\text{for index only}}$$

$$= \frac{1}{2} G_{\mu\nu} (\dot{y}^\mu \dot{y}^\nu + i\psi^\mu \tilde{\psi}^\nu) - A_\mu \dot{y}^\mu + \frac{i}{2} F_{\mu\nu} \psi^\mu \psi^\nu - \frac{i}{2} \partial_\mu G_{\alpha\beta} \dot{y}^\beta \psi^\alpha$$

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\[ \frac{1}{2} G_{\mu\nu} \dot{y}^{\mu} \dot{y}^{\nu} + \frac{i}{2} \psi^\mu G_{\mu\nu} (\dot{\psi}^\nu + \Gamma^\nu_{\gamma\delta} \dot{y}^\gamma \psi^\delta) - A_{\mu} \dot{y}^\mu + \frac{i}{2} F_{\mu\nu} \psi^\mu \psi^\nu, \tag{B.8} \]

where we introduced the notation, also used in the main text, \( F = \mathcal{M}_n^*(d\mathcal{W}) \) and its gauge field \( \mathcal{A} \). We already defined \( G \) as the appropriate block of \( g \), but now valued at \( \mathcal{M}_n \). In other words, \( G = \mathcal{M}_n^*(g) \). Remaining fermions live in the co-tangent bundle of \( \mathcal{M}_n \), so the resulting Lagrangian is \( \mathcal{N} = 1 \) non-linear sigma model on \( 2(n - 1) \)-dimensional manifold \( \mathcal{M}_n \), coupled to an Abelian gauge field \( \mathcal{W} \). Supercharge of this dynamics is a Dirac operator on \( \mathcal{M}_n \) coupled to Abelian gauge field \( \mathcal{A} \), and therefore the index is, under the canonical choice of the chirality operator,

\[ \int_{\mathcal{M}_n} \text{Ch}(\mathcal{F}) \hat{A}(\mathcal{M}_n). \]

References

[1] M.K. Prasad and C.M. Sommerfield, “An Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon,” Phys. Rev. Lett. 35 (1975) 760.

[2] E.B. Bogomolny, “Stability of Classical Solutions,” Sov. J. Nucl. Phys. 24 (1976) 449 [Yad. Fiz. 24 (1976) 861].

[3] N. Seiberg and E. Witten, “Monopole Condensation, And Confinement In N=2 Supersymmetric Yang-Mills Theory,” Nucl. Phys. B 426 (1994) 19 [Erratum-ibid. B 430 (1994) 485] [arXiv:hep-th/9407087].

[4] N. Seiberg and E. Witten, “Monopoles, Duality and Chiral Symmetry Breaking in N=2 Supersymmetric QCD,” Nucl. Phys. B 431 (1994) 484 [arXiv:hep-th/9408099].

[5] F. Ferrari and A. Bilal, “The Strong-Coupling Spectrum of the Seiberg-Witten Theory,” Nucl. Phys. B 469, 387 (1996) [arXiv:hep-th/9602082].

[6] K.M. Lee and P. Yi, “Dyons in N=4 Supersymmetric Theories and Three Pronged Strings,” Phys. Rev. D58 (1998) 066005. [hep-th/9804174].

[7] D. Bak, C.K. Lee, K.M. Lee, and P. Yi “Low-energy Dynamics for 1/4 BPS Dyons,” Phys. Rev. D61 (2000) 025001. [hep-th/9906119].

[8] D. Bak, K.M. Lee and P. Yi, “Quantum 1/4 BPS Dyons,” Phys. Rev. D61 (2000) 045003. [hep-th/9907090].
[9] J.P. Gauntlett, N. Kim, J. Park and P. Yi “Monopole Dynamics and BPS Dyons N=2 Super Yang-Mills Theories,” Phys. Rev. D61 (2000) 125012. [hep-th/9912082].

[10] J.P. Gauntlett, C.J. Kim, K.M. Lee and P.Yi “General Low-energy Dynamics of Supersymmetric Monopoles,” Phys. Rev. D63 (2001) 065020. [hep-th/0008031].

[11] M. Stern and P. Yi, “Counting Yang-Mills Dyons with Index Theorems,” Phys. Rev. D62 (2000) 125006. [hep-th/0005275].

[12] E. J. Weinberg and P. Yi, “Magnetic Monopole Dynamics, Supersymmetry, and Duality,” Phys. Rept. 438, 65 (2007) [arXiv:hep-th/0609055].

[13] F. Denef, “Supergravity Flows and D-brane Stability,” JHEP 0008 (2000) 050 [arXiv:hep-th/0005049].

[14] F. Denef, “Quantum Quivers and Hall/Hole Halos,” JHEP 0210 (2002) 023 [arXiv:hep-th/0206072].

[15] F. Denef and G.W. Moore, “Split States, Entropy enigmas, Holes and Halos,” [arXiv:hep-th/0702146].

[16] M. Kontsevich and Y. Soibelman, “Stability Structures, Motivic Donaldson-Thomas Invariants and Cluster Transformations,” [arXiv:0811.2435]

[17] D. Gaiotto, G.W. Moore and A. Neitzke, “Four-dimensional Wall-crossing via Three-dimensional Field Theory,” Commun. Math. Phys. 299 (2010) 163 [arXiv:0807.4723 [hep-th]].

[18] D. Gaiotto, G. W. Moore, A. Neitzke, “Wall-crossing, Hitchin Systems, and the WKB Approximation,” [arXiv:0907.3987 [hep-th]].

[19] H. -Y. Chen, N. Dorey, K. Petunin, “Wall Crossing and Instantons in Compactified Gauge Theory,” JHEP 1006 (2010) 024. [arXiv:1004.0703 [hep-th]].

[20] E. Andriyash, F. Denef, D. L. Jafferis, G. W. Moore, “Wall-crossing from supersymmetric galaxies,” [arXiv:1008.0030 [hep-th]].

[21] S. Lee and P. Yi, “Framed BPS States, Moduli Dynamics, and Wall-Crossing,” JHEP 1104 (2011) 098 [arXiv:1102.1729 [hep-th]].

[22] J. Manschot, B. Pioline and A. Sen, “Wall Crossing from Boltzmann Black Hole Halos,” [arXiv:1011.1258 [hep-th]].
[23] J. Manschot, B. Pioline and A. Sen, “A Fixed point formula for the index of multi-centered N=2 black holes,” JHEP 1105, 057 (2011) [arXiv:1103.1887 [hep-th]].

[24] B. Pioline, “Four ways across the wall,” [arXiv:1103.0261 [hep-th]].

[25] J. de Boer, S. El-Showk, I. Messamah and D. Van den Bleeken, “Quantizing N=2 Multicenter Solutions,” JHEP 0905 (2009) 002 [arXiv:0807.4556 [hep-th]].

[26] P. Yi, “Witten index and threshold bound states of D-branes,” Nucl. Phys. B 505, 307 (1997) [arXiv:hep-th/9704098].

[27] M. B. Green and M. Gutperle, “D Particle bound states and the D instanton measure,” JHEP 9801, 005 (1998) [arXiv:hep-th/9711107].

[28] A. Maloney, M. Spradlin and A. Strominger, “Superconformal Multi-Black Hole Moduli Spaces in Four Dimensions,” JHEP 0204 (2002) 003 [arXiv:hep-th/9911001].

[29] G. Chalmers, M. Rocek and R. von Unge, “Monopoles in quantum corrected N=2 superYang-Mills theory,” arXiv:hep-th/9612195.

[30] A. Mikhailov, N. Nekrasov and S. Sethi, “Geometric realizations of BPS states in N = 2 theories,” Nucl. Phys. B 531 (1998) 345 [arXiv:hep-th/9803142].

[31] A. Ritz, M.A. Shifman, A.I. Vainshtein and M.B. Voloshin, “Marginal Stability and the Metamorphosis of BPS States,” Phys. Rev. D 63 (2001) 065018 [arXiv:hep-th/0006028].

[32] P.C. Argyres and K. Narayan, “String Webs from Field Theory,” JHEP 0103 (2001) 047 [arXiv:hep-th/0101114].

[33] G.W. Gibbons and N.S. Manton, “The Moduli Space Metric for Well Separated BPS Monopoles,” Phys. Lett. B 356 (1995) 32 [arXiv:hep-th/9506052].

[34] K. Lee, E.J. Weinberg and P. Yi, “The Moduli Space of ManyBPS Monopoles for Arbitrary Gauge Groups,” Phys. Rev. D 54 (1996) 1633 [arXiv:hep-th/9602167].

[35] A. Ritz and A. Vainshtein, “Dyon dynamics near marginal stability and non-BPS states,” Phys. Lett. B 668, 148 (2008) [arXiv:0807.2419 [hep-th]].

[36] T.T. Wu and C.N. Yang, “Dirac Monopole without Strings: Monopole Harmonics,” Nucl. Phys. B 107 (1976) 365.
[37] E. A. Ivanov and A. V. Smilga, “Supersymmetric gauge quantum mechanics: Superfield description,” Phys. Lett. B 257 (1991) 79.

[38] V. P. Berezovoj and A. I. Pashnev, “Three-dimensional N=4 extended supersymmetrical quantum mechanics,” Class. Quant. Grav. 8 (1991) 2141.

[39] E. Ivanov and O. Lechtenfeld, “N=4 supersymmetric mechanics in harmonic superspace,” JHEP 0309 (2003) 073 [arXiv:hep-th/0307111].

[40] M. R. Douglas and S. Klevtsov, “Bergman Kernel from Path Integral,” Commun. Math. Phys. 293, 205 (2010) [arXiv:0808.2451 [hep-th]].

[41] D. Gaiotto, G.W. Moore and A. Neitzke, “Framed BPS States,” [arXiv:1006.0146 [hep-th]].

[42] O. Alvarez, T. P. Killingback, M. Mangano, "String Theory and Loop space index theorems,” Comm. Math. Phys. 111(1987) 1.

[43] K. Li, "Character valued index theorems in supersymmetric string theories,” Class. Quantum. Grav. 5 (1988) 95.

[44] Schellekens, Warner, "Anomalies, Characters and strings,” Nucl. Phys. B 287 (1987) 317.

[45] L. Dixon, J. Harvey, C. Vafa and E. Witten, "Strings on orbifolds,” Nucl. Phys. B 261 (1985) 678.

[46] C. Farsi and C. Seaton, "Generalized orbifold Euler characteristics for general orbifolds and wreath products,” [arXiv:0902.1198 [math.DG]] and references therein.

[47] M. F. Atiyah and R. Bott, A Lefshetz fixed point formula for elliptic complexes: I, Annal. Math.,2nd Ser. 86 (1967) 374.; M. F. Atiyah and R. Bott, A lefshetz fixed point formula for elliptic complexes: II, Annal. Math.,2nd Ser. 88 (1968) 451.

[48] E. Witten, "Elliptic Genera And Quantum Field Theory,” Commun.Math.Phys. 109 (1987) 525.

[49] K. Pilch and N. Warner, "String Sturctures and the Index of the Dirac-Ramond Operator on Orbifolds,” Comm. Math. Phys. 115 (1988) 191.

[50] A. V. Smilga, “Pertubative cirrections to effective zero mode hamiltobia in supersymmetric QED,” Nucl. Phys. B 291, 241 (1987).
[51] E. Ivanov, S. Krivonos and O. Lechtenfeld, “N=4, d = 1 supermultiplets from nonlinear realizations of D(2,1: alpha),” Class. Quant. Grav. 21 (2004) 1031 [arXiv:hep-th/0310299].