REPULSION EFFECTS ON BOUNDEDNESS IN A QUASILINEAR ATTRACTION-REPUSSION CHEMOTAXIS MODEL IN HIGHER DIMENSIONS

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Abstract. We consider the following attraction-repulsion Keller-Segel system:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w), \quad x \in \Omega, \quad t > 0, \\
\frac{\partial v}{\partial t} &= \Delta v + \alpha u - \beta v, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial w}{\partial t} &= \Delta w + \gamma u - \delta w, \quad x \in \Omega, \quad t > 0, \\
\end{align*}
\]

with homogeneous Neumann boundary conditions in a bounded domain \( \Omega \subset \mathbb{R}^n \) with smooth boundary. Here all the parameters \( \chi, \xi, \alpha, \beta, \gamma \) and \( \delta \) are positive. The smooth diffusion \( D(u) \) satisfies \( D(u) \geq du^\theta, u > 0 \) for some \( d > 0, \theta \in \mathbb{R} \). It is recently known from [25] that boundedness of solutions is ensured whenever \( \theta > 1 - \frac{2}{n} \). Here, it is shown, if repulsion dominates or cancels attraction in the sense either \{\xi \gamma > \chi \alpha \} or \{\xi \gamma = \chi \alpha, \beta \geq \delta \}, the corresponding initial-boundary value problem possesses a unique global classical solution which is uniformly-in-time bounded for large initial data provided \( \theta > 1 - \frac{2}{n+2} \). In this way, the range of \( \theta > 1 - \frac{2}{n+2} \) of boundedness is enlarged and thus the repulsion effect on boundedness is exhibited.

1. Introduction. To describe the aggregation of Microglia in the central nervous system in Alzheimer’s disease due to the interaction of chemoattractant and chemorepellent, the following attraction-repulsion chemotaxis system was proposed in [29]...
\[
\begin{aligned}
&u_t = \nabla \cdot (D(u) \nabla u) - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w), \quad x \in \Omega, \ t > 0, \\
&\tau_1 v_t = \Delta v + \alpha u - \beta v, \quad x \in \Omega, \ t > 0, \\
&\tau_2 w_t = \Delta w + \gamma u - \delta w, \quad x \in \Omega, \ t > 0, \\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \\
&u(x, 0) = u_0(x), \ \tau_1 v(x, 0) = \tau_1 v_0(x), \tau_2 w(x, 0) = \tau_2 w_0(x), \quad x \in \Omega,
\end{aligned}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^n (n \geq 2)\) with smooth boundary \(\partial \Omega\), the parameters \(\chi, \xi, \alpha, \beta, \gamma\) and \(\delta\) are positive, \(\tau_1, \tau_2 \geq 0\) and \(\nu\) denotes the outward normal vector of \(\partial \Omega\). The unknown \(u(x, t)\) denotes the concentration of cell, \(v(x, t)\) and \(w(x, t)\) describe the concentrations of chemoattractant and chemorepellent, respectively. The model (1.1) was also introduced in [34] to model the quorum sensing effect in the chemotactic movement. Throughout this text, the \(C^2\)-smooth nonlinear diffusion \(D(u)\) is assumed to satisfy
\[
D(u) > 0 \quad \text{for } u \geq 0 \quad \text{and} \quad D(u) \geq d u^\theta \quad \text{for some } d > 0, \theta \in \mathbb{R} \quad \text{and for all } u > 0.
\]

Intuitively, large values of \(\theta\) seem to enhance boundedness of solutions. Neglect of the repulsion effect, the model (1.1) reduces to the widely known (attractive) Keller-Segel chemotaxis model with nonlinear diffusion:
\[
\begin{aligned}
&u_t = \nabla \cdot (D(u) \nabla u) - \chi \nabla \cdot (u \nabla v), \quad x \in \Omega, \ t > 0, \\
&\tau_1 v_t = \Delta v + \alpha u - \beta v, \quad x \in \Omega, \ t > 0,
\end{aligned}
\]

whose solution behavior has been extensively studied in the past four decades in various perspectives (see the survey articles [12, 4] and references therein). More often than not, when \(D(u)\) is a constant, the system (1.3) is known as the classical Keller-Segel model. A striking feature of the classical model (1.3) is the blow-up of solutions in two or higher dimensions [31, 41, 42]. While, when \(D(u)\) is a nonlinear function, there is a critical exponent \(\theta_* = 1 - \frac{2}{n}\) which distinguishes between occurrence and impossibility of blow-up. More precisely, the solution of system (1.3) will globally exist [14, 15, 36, 39] if \(\theta > \theta_*\) and blow up in finite time [5, 7, 8, 13, 30, 32, 16] if \(\theta < \theta_*\) with \(n \geq 3\); when \(\theta = \theta_*\), based on the availability of Lyapunov functional, it has been proved there exists radially symmetric initial data such that the solution of (1.3) will blow up in finite time [17, 22].

Another important subsystem of (1.1) is the following repulsive Keller-Segel model:
\[
\begin{aligned}
&u_t = \nabla \cdot (D(u) \nabla u) + \xi \nabla \cdot (u \nabla w), \quad x \in \Omega, \ t > 0, \\
&\tau_2 w_t = \Delta w + \gamma u - \delta w, \quad x \in \Omega, \ t > 0,
\end{aligned}
\]

Based on a Lyapunov functional different from that of the attractive Keller-Segel model (1.3), the global existence of classical solutions in two dimensions and weak solutions in three and four dimensions were established in [6] with \(D(u) = 1\). Further results on a repulsive Keller-Segel model with nonlinear chemosensitivity can be found in [37].

The attraction-repulsion model (1.1) is a mixed combination of the attractive KS model (1.3) and the repulsive one (1.4). Hence, the mathematical analysis on the boundedness and blow-up of solutions offers great challenges due to the complex interactions between the three components \(u, v\) and \(w\), and the difficulty of constructing a Lyapunov functional. To motivate our study, we summarize the known results on boundedness versus blow-up for (1.1) in the literature as follows:
• $\tau_1 = \tau_2 = \tau$ and $D(u) = 1$: In one dimensional space, the global existence of classical solutions, non-trivial stationary state, asymptotic behavior and pattern formation of the system (1.1) have been studied in [20, 28, 27] with $\tau = 1$. In higher dimensions ($n \geq 2$), Tao and Wang [38] studied the global solvability, boundedness, blow-up, existence of steady states by introducing the transformation $s = \xi w - \chi v$. It has been proved that when repulsion dominates or cancels attraction in the sense of $\xi \gamma \geq \chi \alpha$, the global classical solution will exists for both $\tau = 0$ [38] and $\tau = 1$ [18, 26]. Whereas, if attraction dominates in the sense of $\xi \gamma < \chi \alpha$, then the solution of system (1.1) with $\tau = 0$ will blow up in finite time for large initial mass and exist globally with small initial mass in two dimensional spaces [9, 23, 43]. The large time behavior of solution with small initial data was established in [24]. Part of the above-mentioned results have been recently carried over to the whole space in [19, 35]. However, whether or not the boundedness and blow-up of solution hold for higher dimensions was left as an open problem.

• $\tau_1 = 1, \tau_2 = 0$: When $D(u) = 1$, Jin and Wang [21] firstly detected a Lyapunov functional, and then they established the global existence of uniformly-in-time bounded classical solutions in two dimensional bounded domain with large initial data if the repulsion dominates or cancels attraction (i.e., $\xi \gamma \geq \alpha \chi$). If the attraction dominates (i.e. $\xi \gamma < \alpha \chi$), a critical mass blow-up phenomenon was found. With a nonlinear diffusion $D(u)$, the similar results in higher dimensions were available in [25]; more precisely, therein the authors showed that, for the prototypical choice $D(u) = du^\theta$ and $\theta > 1 - \frac{2}{n}$, the corresponding initial-boundary problem possesses a nonnegative globally bounded solution. On the other hand, if $\theta < 1 - \frac{2}{n}$ and $\xi \gamma < \chi \alpha$, there exist some symmetric initial data such that the corresponding solution blow-up in finite time in the case of $\Omega = B_R(0) \subset \mathbb{R}^n (n \geq 3)$. For the borderline case $\theta = 1 - \frac{2}{n}$ with $n = 3$, there exist radially symmetric solutions which may blow up in finite time in $\Omega = B_R(0)$ with $R > 0$.

As summarized above, when $D(u)$ satisfies (1.2) and $\theta > 1 - \frac{2}{n}$, then boundedness of solutions to (1.1) with $(\tau_1, \tau_2) = (1, 0)$ is guaranteed irrespective of repulsion dominant case $\xi \gamma > \chi \alpha$ or attraction dominant case $\chi \alpha \geq \xi \gamma$. While, the competition between repulsion and attraction plays a significant role in boundedness and other qualitative behaviors as summarized above; besides, the results for (1.4) also convey us that chemo-repulsion mechanism has an effective role in enhancing boundedness and stabilizations [6, 37]. Therefore, when repulsion dominates or cancels attraction (i.e., $\xi \gamma \geq \chi \alpha$), the range of $\theta$ is expected to be enlarged, as observed in [25, Remark 1.2] and it was left as open problem therein. In this note, we aim to exhibit the effect of chemo-repulsion mechanism by resolving the posed problem for the case $(\tau_1, \tau_2) = (1, 0)$ and henceforth we investigate the following attraction-repulsion system

\[
\begin{align*}
\begin{cases}
  u_t = \nabla \cdot (D(u) \nabla u) - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w), & x \in \Omega, \ t > 0, \\
  v_t = \Delta v + \alpha u - \beta v, & x \in \Omega, \ t > 0, \\
  0 = \Delta w + \gamma u - \delta w, & x \in \Omega, \ t > 0, \\
  \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), & x \in \Omega.
\end{cases}
\end{align*}
\]
Employing the same Lyapunov functional as in [25] in a close way, when the repulsion dominates or cancels attraction in the sense either \( \{ \xi \gamma > \chi \alpha \} \) or \( \{ \xi \gamma = \chi \alpha, \beta \geq \delta \} \), (1.6) we establish the refined \( L^{1+\theta} \)-boundedness of \( u \) compared to the \( L^{1} \)-boundedness of \( u \) as used in [25] as the starting point toward our further bootstrap type argument and then the remaining procedure leading to its \( L^{\infty} \)-boundedness is standard and developed since [39]. Namely, we first use the gained \( L^{1+\theta} \)-bound for \( u \) and the \( v \)-equation to improve the regularity of \( \nabla v \) and then, with an elliptic Agmon-Douglis-Nirenberg \( L^{p} \)-estimate applied to the \( u \)-equation, the restriction \( \theta > 1 - \frac{4}{n+2} \) enables us to derive a Grownwall type inequality for the coupled quantity \( \int_{\Omega} u^p + \int_{\Omega} |\nabla v|^{2q} \) for all large \( p, q > 2 \) and, finally, we utilize the well-known Moser-Alikakos type iteration technique (cf. [3] or [39, Lemma A.1]) to conclude the \( L^{\infty} \)-boundedness of \( u \) and then \( W^{1,\infty} \)-boundedness of \( v \) and \( w \). The mathematical formulation of our main result is stated in the next theorem.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^n (n \geq 3) \) be a bounded domain with smooth boundary, \( 0 \leq (u_{0}, v_{0}) \in (C^{0}(\Omega) \times W^{1,\infty}(\Omega)) \) and \( \chi, \xi, \alpha, \beta, \gamma, \delta > 0 \). Suppose that (1.6) holds and that \( D(u) \) satisfies (1.2) with \( \theta > 1 - \frac{4}{n+2} \). Then there exists a unique triple \((u, v, w)\) of nonnegative functions in \( C^{0}(\Omega \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty)) \) which solves (1.5) classically and there exists a constant \( C > 0 \) independent of \( t \) such that
\[
\|u(\cdot, t)\|_{L^{\infty}(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \quad \text{for all} \quad t \in (0, \infty).
\]

**Remark 1.** In [25, Theorem 1.1], the boundedness of classical solutions to (1.5) is obtained under the condition \( \theta > 1 \frac{4}{n} \). This together with Theorem 1.1 ensures the boundedness and global existence for the nonlinear diffusion-repulsion Keller-Segel model (1.5) in \( \geq 3 \)-D under either one of the following cases:
\[
\begin{cases}
\text{Case 1:} & \theta > 1 - \frac{2}{n} , \\
\text{Case 2:} & 1 - \frac{4}{n+2} < \theta \leq 1 - \frac{2}{n} \quad \text{and} \quad \{ \xi \gamma > \chi \alpha \} \quad \text{or} \quad \{ \xi \gamma = \chi \alpha, \beta \geq \delta \} .
\end{cases}
\]

Accordingly, our boundedness especially enlarges the range of \( \theta \) in [25, Theorem 1.1] provided that the repulsion dominates or cancels attraction in the sense of (1.6). Hence, it exhibits the repulsion effect on boundedness in attraction-repulsion models.

**Remark 2.** In the case that \( \Omega = B_{R}(0) \subset \mathbb{R}^3 \) and \( D(u) \) satisfies (1.2) with \( \theta = 1 - \frac{2}{n} \). The authors in [25, Remark 1.5] remarked that the set of initial data \((u_{0}, v_{0}, w_{0})\) enforcing finite-time blow-up in [25, Theorem 1.2] can be proved to be dense with respect to an appropriate topology. While, under (1.6), such blow-up set of initial data is empty, as ensured by Theorem 1.1.

2. **Local existence and preliminaries.** The local existence theorem of (1.1) can be proved by the fixed point theorem and maximum principle along the same line as shown in [38]. We omit the details for convenience.

**Lemma 2.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n (n \geq 2) \). Suppose that \( 0 \leq (u_{0}, v_{0}) \in (C^{0}(\Omega) \times W^{1,\infty}(\Omega)) \) and assume that the diffusion function \( D(s) \in C^{1}(0, \infty) \) satisfies \( D(s) > 0 \) with \( s \in (0, \infty) \). Then there exist \( T_{\max} \in (0, \infty) \) and a unique triple \((u, v, w)\) of nonnegative functions from \( C(\Omega \times [0, T_{\max})) \cap C^{2,1}(\Omega \times (0, T_{\max})) \) solving (1.1) classically in \( \Omega \times (0, T_{\max}) \). Moreover, \( u > 0 \) in \( \Omega \times (0, T_{\max}) \) and
\[
\text{if} \quad T_{\max} < \infty, \quad \text{then} \quad \|u(\cdot, t)\|_{L^{\infty}} \rightarrow \infty \quad \text{as} \quad t \nearrow T_{\max}. \quad (2.1)
\]
By the blowup criterion (2.1) of Lemma 2.1, it suffices to derive \( \|u(\cdot,t)\|_{L^\infty} < \infty \) for all \( t > 0 \) to obtain the global-in-time solutions. We first notice that \( L^1 \)-norm of the solution triple of (1.5) is bounded by integrating equations in (1.5) over \( \Omega \).

**Lemma 2.2.** The solution \((u,v,w)\) of system (1.1) satisfies the following properties

\[
\|u(\cdot,t)\|_{L^1} = \|u_0\|_{L^1},
\]

\[
\|v(\cdot,t)\|_{L^1} = \frac{\alpha}{\beta} \|u_0\|_{L^1} - \left(\frac{\alpha}{\beta} \|u_0\|_{L^1} - \|v_0\|_{L^1}\right) e^{-\beta t},
\]

\[
\|w(\cdot,t)\|_{L^1} = \frac{\gamma}{\delta} \|u_0\|_{L^1}.
\]

The following version of Gagliardo-Nirenberg inequality will be used in several places in our upcoming discussions.

**Lemma 2.3** (Gagliardo-Nirenberg inequality [10]). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with smooth boundary. Let \( l \) and \( k \) be any integers satisfying \( 0 \leq l < k \), and let \( 1 \leq q, r \leq \infty \), and \( p \in \mathbb{R}^+ \), \( \frac{l}{k} \leq a \leq 1 \) such that

\[
\frac{1}{p} - \frac{l}{n} = a \left(\frac{1}{q} - \frac{k}{n}\right) + (1-a) \frac{1}{r}.
\]

Then, for any \( f \in W^{k,q}(\Omega) \cap L^r(\Omega) \), there exist a constant \( c \) depending only on \( \Omega, q, k, r \) and \( n \) such that:

\[
\|D^l f\|_{L^p} \leq c(\|D^k f\|_{L^q}^a \|f\|_{L^r}^{1-a} + \|f\|_{L^r}),
\]

with the following exception: if \( 1 < q < \infty \) and \( k - l - \frac{n}{q} \) is a nonnegative integer, then (2.5) holds only for a satisfying \( \frac{1}{k} \leq a < 1 \).

We should mention that the original Gagliardo-Nirenberg inequality (e.g. see [33]) is stated only for \( r \geq 1 \), but this condition can be easily relaxed to \( r \in (0,p) \) by means of the Hölder’s inequality (cf. [40, Lemma 3.2]).

The existence of Lyapunov function for system (1.5) was firstly found in [21] with \( D(u) = 1 \), which is further developed in [25] for nonlinear diffusion. We here once again will employ this functional in a close way to obtain the key \( L^{1+d} \)-boundedness of \( u \).

**Lemma 2.4.** Let \((u,v,w)\) be the classical solution of (1.5) obtained in Lemma 2.1. Then

\[
\frac{d}{dt} F(u,v,w) + E(u,v,w) = 0,
\]

where

\[
F(u,v,w) = \int_\Omega G(u) + \frac{\chi}{2\alpha} \int_\Omega \beta v^2 + |\nabla u|^2 + \frac{\xi}{2\gamma} \int_\Omega (\delta w^2 + |\nabla w|^2) - \chi \int_\Omega uv,
\]

with \( G(z) := \int_{z_0}^z \int_{\tau_0}^\tau \frac{D(\tau)}{\tau} d\tau d\sigma \) and

\[
E(u,v,w) = \frac{\chi}{\alpha} \int_\Omega v^2 + \int_\Omega u \frac{D(u)}{u} \nabla u - \chi \nabla v + \xi \nabla w.
\]

**Proof.** Multiplying the first equation of (1.5) by \( G'(u) - \chi v + \xi w \) and integrating the result with respect to \( x \) over \( \Omega \), we obtain
Then using (2.11) and the Cauchy-Schwarz inequality, one can readily derive that

\[
\int_{\Omega} u_t(G'(u) - \chi v + \xi w) = \int_{\Omega} \nabla \cdot (D(u)\nabla u - \chi u\nabla v + \xi u\nabla w)(G'(u) - \chi v + \xi w)
\]

\[
= -\int_{\Omega} (D(u)\nabla u - \chi u\nabla v + \xi u\nabla w) \left( \frac{D(u)}{u} \nabla u - \chi \nabla v + \xi \nabla w \right)
\]

\[
= -\int_{\Omega} \left[ \frac{D(u)}{u} \nabla u - \chi \nabla v + \xi \nabla w \right]^2.
\]

(2.8)

A use of integration by part yields

\[
\int_{\Omega} u_t(G'(u) - \chi v + \xi w) = \frac{d}{dt} \int_{\Omega} G(u) - \chi \frac{d}{dt} \int_{\Omega} u v + \chi \int_{\Omega} u v_t + \xi \int_{\Omega} u w. \tag{2.9}
\]

It follows from the second equation of (1.5) that

\[
u = \frac{1}{\alpha} v_n - \frac{1}{\alpha} \Delta v + \frac{\beta}{\alpha} v,
\]

which gives

\[
\int_{\Omega} u v_t = \frac{1}{\alpha} \int_{\Omega} v_t^2 + \frac{1}{2\alpha} \frac{d}{dt} \int_{\Omega} \nabla v^2 + \frac{\beta}{2\alpha} \frac{d}{dt} \int_{\Omega} v^2. \tag{2.10}
\]

Similarly, a substitution of the third equation of (1.1) shows

\[
u = \frac{\delta}{\gamma} w - \frac{1}{\gamma} \Delta w, \tag{2.11}
\]

which gives rise to

\[
\int_{\Omega} u w = \frac{\delta}{2\gamma} \frac{d}{dt} \int_{\Omega} w^2 + \frac{1}{2\gamma} \frac{d}{dt} \int_{\Omega} |\nabla w|^2. \tag{2.12}
\]

The combination of (2.9), (2.10) and (2.12) leads to

\[
\int_{\Omega} u_t(G'(u) - \chi v + \xi w) = \frac{d}{dt} \int_{\Omega} G(u) - \chi \frac{d}{dt} \int_{\Omega} u v + \frac{\beta}{2\alpha} \frac{d}{dt} \int_{\Omega} \nabla v^2 + \frac{\xi}{2\alpha} \frac{d}{dt} \int_{\Omega} |\nabla w|^2 + \frac{\xi}{2\gamma} \frac{d}{dt} \int_{\Omega} |\nabla w|^2,
\]

which together with (2.8) implies (2.7). The proof of Lemma 2.4 is completed. \(\square\)

Employing the energy identity obtained in Lemma 2.4, we next derive \(L^{1+\theta}\) boundedness of the \(u\)-component provided the repulsion dominates or cancels attraction as specified in (1.6), which serves a key starting point towards our further analysis.

**Lemma 2.5.** Let \(\Omega \subset \mathbb{R}^n (n \geq 3)\) and \(D(u)\) satisfy (1.2). If (1.6) holds, then there exists a constant \(C > 0\) independent of \(t\) such that the solution of (1.5) satisfies

\[
\|u(\cdot, t)\|_{L^{1+\theta}} \leq C. \tag{2.13}
\]

**Proof.** Thanks to the mass conservation of \(u\) in (2.2), we shall proceed with \(\theta > 0\). Then using (2.11) and the Cauchy-Schwarz inequality, one can readily derive that

\[
\chi \int_{\Omega} u w = \frac{\chi \delta}{\gamma} \int_{\Omega} w^2 + \frac{\chi}{\gamma} \int_{\Omega} \nabla w \cdot \nabla v
\]

\[
\leq \frac{\chi \delta}{\gamma} \left( \frac{\xi}{2\gamma} \int_{\Omega} w_t^2 + \frac{\chi}{2\xi} \int_{\Omega} v^2 \right) + \frac{\chi}{\gamma} \left( \frac{\xi}{2\chi} \int_{\Omega} |\nabla w|^2 + \frac{\chi}{2\xi} \int_{\Omega} |\nabla v|^2 \right) \tag{2.14}
\]

\[
= \frac{\xi \delta}{2\gamma} \int_{\Omega} w_t^2 + \frac{\chi^2 \delta}{2\xi^2 \gamma} \int_{\Omega} v_t^2 + \frac{\xi}{2\gamma} \int_{\Omega} |\nabla w|^2 + \frac{\chi^2}{2\xi^2 \gamma} \int_{\Omega} |\nabla v|^2.
\]
Substituting (2.14) into (2.7), we obtain

\[ F(u, v, w) \geq \int_{\Omega} G(u) + \left( \frac{\beta \chi}{2 \alpha} - \frac{\chi^2 \delta}{2 \xi \gamma} \right) \int_{\Omega} v^2 + \left( \frac{\chi}{2 \alpha} - \frac{\chi^2}{2 \xi \gamma} \right) \int_{\Omega} |\nabla v|^2 \]

\[ - \int_{\Omega} G(u) + \frac{\chi (\xi \gamma - \chi \alpha)}{2 \xi \gamma \alpha} \int_{\Omega} v^2 + \frac{\chi (\xi \gamma - \chi \alpha)}{2 \xi \gamma} \int_{\Omega} |\nabla v|^2. \]  

(2.15)

Integrating (2.6) with respect to \( t \) and using (2.15), we have

\[ \int_{\Omega} G(u) + \frac{\chi}{\alpha} \int_{0}^{t} \int_{\Omega} v^2 + \int_{0}^{t} \int_{\Omega} u \frac{D(u)}{u} \nabla u - \chi \nabla v + \xi \nabla w \]^2 \]

\[ \leq F(u_0, v_0) - \frac{\chi (\xi \gamma - \chi \alpha)}{2 \xi \gamma \alpha} \int_{\Omega} |\nabla v|^2 + \frac{\chi (\xi \gamma - \chi \alpha)}{2 \xi \gamma} \int_{\Omega} v^2. \]  

(2.16)

In the first case of (1.6), i.e., \( \xi \gamma > \chi \alpha \), using the Gagliardo-Nirenberg inequality (2.3) and the boundedness of \( \|v(t, \cdot)\|_{L^1} \) in (2.3), we have

\[ - \frac{\chi (\xi \gamma - \chi \alpha)}{2 \alpha \xi \gamma} \|v\|^2_{L^2} \leq c_1 \|\nabla v\|_{L^1} \|v\|^2_{L^1} + c_2 \|v\|^2_{L^1} \]

\[ \leq \frac{\chi (\xi \gamma - \chi \alpha)}{2 \xi \gamma} \|\nabla v\|^2_{L^2} + c_3. \]

(2.17)

While, in the second case of (1.6), i.e., \( \xi \gamma = \chi \alpha \) and \( \beta \geq \delta \), one trivially has

\[ - \frac{\chi (\xi \gamma - \chi \alpha)}{2 \xi \gamma \alpha} \int_{\Omega} |\nabla v|^2 - \frac{\chi (\xi \gamma - \chi \alpha)}{2 \alpha \xi \gamma} \int_{\Omega} v^2 = \frac{(\delta - \beta)\chi}{2 \alpha} \int_{\Omega} v^2 \leq 0. \]  

(2.18)

In summary, under (1.6), the combination of (2.16), (2.17) and (2.18) implies that

\[ \int_{\Omega} G(u) \leq F(u_0, v_0) + c_3. \]  

(2.19)

Since \( D(u) \) fulfills (1.2) and \( G(z) := \int_{s_0}^{s} \frac{D(r)}{r} dr ds \), one can easily infer that \( G(u) \geq c_4 u^{\theta+1} - c_5 \) for some \( c_4, c_5 > 0 \). Then the estimate (2.13) follows directly from (2.19).

Once we gain the refined \( L^{1+\theta} \)-boundedness of \( u \) as in (2.13), the remaining procedure leading to its \( L^\infty \)-boundedness is standard and developed since [39]. To make our presentation self-contained, we would like to offer all the necessary details.

**Lemma 2.6.** Let \((u, v, w)\) be a solution of (1.5) defined on its maximal existence interval \([0, T_{max})\). Let \( \theta > 0 \) and

\[ s \in [1, \frac{n(\theta+1)}{n-1-\theta}], \quad \text{if} \quad \theta + 1 \leq n, \]

\[ s \in [1, \infty], \quad \text{if} \quad \theta + 1 > n. \]  

(2.20)

Then the \( v \)-component of the solution of (1.5) satisfies

\[ \|\nabla v(t, \cdot)\|_{L^s} \leq C \quad \text{for all} \quad t \in (0, T_{max}). \]  

(2.21)

**Proof.** The variation-of-constants formula applied to the \( v \)-equation in (1.5) shows

\[ v(t, \cdot) = e^{t(\Delta - \beta)} v_0 + \alpha \int_{0}^{t} e^{(t-\tau)(\Delta - \beta)} u(\cdot, \tau) d\tau \quad \text{for all} \quad t \in (0, T_{max}). \]
The desired estimate is nothing but (2.25).

where the following well-known smoothing holds.

Let Lemma 2.7.

Then we deduce from the boundedness of \( \|u\|_{L^p} \) to guarantee that \( \|u\|_{L^p} \leq c_1 \|u\|_{L^p} \). Hence, it follows from (2.22) that

\[
\|\nabla v(\cdot, t)\|_{L^p} \leq c_1 \|v_0\|_{W^{1,\infty}} + c_2 c_3 \|\alpha\|_{L^\infty}, \quad \forall t > 0, \quad f \in L^n.
\]

Here, \( \lambda_1 \) is the first positive eigenvalue of \(-\Delta\) under homogeneous boundary condition.

Thanks to the restriction of \( s \) in (2.20), we know that \( c_4 \) such that \( c_4 := \int_0^\infty e^{-\beta \sigma} \cdot (1 + \sigma^{-\frac{1}{2}}(\frac{1}{\tau} - \frac{1}{\theta})) d\sigma < \infty. \) Hence, it follows from (2.22) that

\[
\|\nabla v(\cdot, t)\|_{L^p} \leq c_1 \|v_0\|_{W^{1,\infty}} + c_2 c_3 \|\alpha\|_{L^\infty} \quad \text{for all} \quad t \in (0, T_{\text{max}}),
\]

and thereby proves (2.21). The proof of this lemma is thus accomplished.

\[\Box\]

Lemma 2.7. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with smooth boundary. For any \( \varepsilon > 0 \), \( p > 1 \), there exists a constant \( C_\varepsilon > 0 \) such that for each \( u \in L^1(\Omega) \), the solution of the elliptic Neumann problem

\[
-\Delta w + \delta w = \gamma u \quad \text{in} \quad \Omega, \quad \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega (2.23)
\]

satisfies

\[
\int_\Omega w^p \leq \varepsilon \int_\Omega u^p + C_\varepsilon.
\]

Proof. First, we apply the Agmon-Douglis-Nirenberg \( L^p \)-estimates \( p > 1 \) \cite{1, 2} to the linear elliptic problem (2.23) to find a constant \( c_1 > 0 \) such that

\[
\|u\|_{W^{2,p}} \leq c_1 \|u\|_{L^p}. \tag{2.24}
\]

Then we deduce from the boundedness of \( \|u\|_{L^p} \) in (2.4) due to the fact that \( u \in L^1(\Omega) \), the elliptic estimate (2.24), Gagliardo-Nirenberg inequality stated in Lemma 2.3 and Young’s inequality with \( \varepsilon \) that

\[
\int_\Omega w^p = \|w\|^p_{L^p} \leq c_2 \|D^2 w\|^p_{L^p} \|w\|^{p(1-\kappa)}_{L^1} + c_2 \|w\|^p_{L^1} \leq c_3 \|u\|_{L^p}^p + c_3 \leq \varepsilon \|w\|_{L^p}^p + C_\varepsilon, \tag{2.25}
\]

where we used the fact \( p > 1 \) to guarantee

\[
\kappa := \frac{1 - \frac{1}{2}}{1 + \frac{2}{n} - \frac{1}{p}} \in (0, 1).
\]

The desired estimate is nothing but (2.25). \[\Box\]
Remark 3. The larger the exponent \( \theta \) is, the easier the boundedness of \( u \) is. Indeed, if \( \theta + 1 > n \), then Lemma 2.6 tells us that \( \| \nabla v(\cdot, t) \|_{L^\infty} \) is bounded. Then (3.1) of Lemma 3.1 quickly shows that \( \| u(\cdot, t) \|_{L^p} \) is bounded for any \( p > 2 \). Thus, we directly jump to the proof of Theorem 1.1 on P. 13 to obtain that \( \| u(\cdot, t) \|_{L^\infty} \) is bounded. Accordingly, we shall proceed henceforth with \( \theta \leq n - 1 \).

The condition \( \theta > 1 - \frac{4}{n+2} \) (along with \( \theta \leq n - 1 \) as noted in Remark 3) indeed enables us to select certain parameters in an appropriate way such that we have the license to apply Gagliardo-Nirenberg inequality in Lemma 3.2.

Lemma 2.8. Let \( n \geq 2 \), \( \theta > 1 - \frac{4}{n+2} \), \( \bar{p} \geq 1 + 2\theta \) and \( \bar{q} \geq \frac{n^2}{4} \). Then there exist numbers \( p \geq \bar{p}, q \geq \bar{q} \), \( s \in [1, \frac{n(\theta+1)}{n-2\theta}) \), \( \lambda \in (1, \frac{n}{2}) \) and \( \mu > \max\{1, \frac{n}{2}\} \) such that

\[
 p > 1 + 2\theta, \quad (2.26)
\]

\[
 \max \left\{ 1 - \frac{2}{s}, \frac{n-2}{n} \right\} \theta < \frac{1}{\lambda} < 1 - \frac{2}{n} \cdot \frac{1}{q}, \quad (2.27)
\]

\[
 \max \left\{ 1 - \frac{2(q-1)}{s}, \frac{n-2}{n} \right\} \theta < \frac{1}{\mu} < \min \left\{ \frac{2}{1+\theta}, \frac{2}{n} + \frac{n-2}{n} \cdot \frac{1}{q} \right\}, \quad (2.28)
\]

\[
 \frac{p-\theta}{2(1+\theta)} - \frac{1}{2} < \frac{1}{\mu} - \frac{1}{2} < 1, \quad (2.29)
\]

and

\[
 \frac{1}{\mu} - \frac{1}{2} < \frac{q-1}{s} + \frac{1}{\mu} - \frac{1}{2} < 1. \quad (2.30)
\]

Proof. Using the similar argument as in [39, Lemma 2.1], after various honest computations, we obtain the respective equalities directly. \( \square \)

3. Proof of Theorem 1.1. In this section, we show the proof of Theorem 1.1; the idea for the proof starts with the following coupled energy estimates.

Lemma 3.1. Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary. Suppose that \( D(u) \) satisfies (1.2). Assume that \( p > 2 \) and \( q > 2 \). Then there exist two constants \( C_1 > 0 \) and \( C_2 > 0 \) such that for all \( t \in (0, T_{\text{max}}) \), the solution of (1.5) satisfies

\[
 \frac{d}{dt} \left( \int_\Omega u^p + \int_\Omega |\nabla v|^2 q \right) + \frac{2p(p-1)d}{(p+\theta)^2} \int_\Omega |\nabla u^{\frac{p+\theta}{p+\gamma}}|^2 
+ \frac{2(q-1)}{q} \int_\Omega |\nabla \nabla v|^2 + 2q\beta \int_\Omega |\nabla v|^2 + \frac{(p-1)\xi}{2} \int_\Omega u^{p+1} \right) \leq \frac{p(p-1)\lambda^2}{2d} \int_\Omega u^{p-\theta} |\nabla v|^2 + C_1 \int_\Omega u^2 |\nabla v|^{2q-2} + C_2. 
\]

Proof. We multiply the first equation of (1.5) by \( pu^{p-1} \), and integrate the equation with respect to \( x \) over \( \Omega \) to obtain

\[
 \frac{d}{dt} \int_\Omega u^p + p(p-1)d \int_\Omega u^{p-2+\theta} |\nabla u|^2 
\leq p(p-1)\chi \int_\Omega u^{p-1} \nabla u \cdot \nabla v - p(p-1)\xi \int_\Omega u^{p-1} \nabla u \cdot \nabla w
\]
Proof. For \( p, q > 2 \), \( \theta > 1 - \frac{4}{p+2} \), and \( C > 0 \) constant which yields
\[
\frac{d}{dt} \int_{\Omega} u^p + \int_{\Omega} |\nabla v|^{2q} + \frac{2(p-1)d}{p} \int_{\Omega} |\nabla u|^{\frac{p+q}{2}}^2 \\
+ \frac{2(q-1)}{q} \int_{\Omega} |\nabla v|^{\frac{q}{2}} + \frac{(p-1)\xi}{2} \int_{\Omega} u^{p+1} + 2q^2 \int_{\Omega} |\nabla v|^{2q} \leq c_1 \int_{\Omega} u^p + c_2 \int_{\Omega} u^2 |\nabla v|^{2q-2} + c_2.
\]
which yields
\[
\frac{d}{dt} \int_{\Omega} u^p + \frac{p(p-1)d}{2} \int_{\Omega} u^{p-2+\theta} |\nabla u|^{2} + \frac{p(p-1)\chi}{2} \int_{\Omega} u^{p-\theta} |\nabla v|^{2} \\
+ (p-1)\xi \delta \int_{\Omega} u^p w - (p-1)\xi \gamma \int_{\Omega} u^{p+1},
\]
Now, we use Young’s inequality and Lemma 2.7 to estimate the last term in (3.2) as
\[
(p-1)\xi \delta \int_{\Omega} u^p w \leq \frac{(p-1)\xi}{4} \int_{\Omega} u^{p+1} + c_1 \int_{\Omega} w^{p+1} \\
\leq \frac{(p-1)\xi}{2} \int_{\Omega} u^{p+1} + c_2.
\]
(3.3)
Substituting (3.3) into (3.2), one has
\[
\frac{d}{dt} \int_{\Omega} u^p + \frac{2p(p-1)d}{(p+\theta)^2} \int_{\Omega} |\nabla u|^{\frac{p+\theta}{2}}^2 + \frac{p(p-1)\chi}{2} \int_{\Omega} u^{p+1} \\
\leq \frac{p(p-1)\chi^2}{2d} \int_{\Omega} u^{p-\theta} |\nabla v|^{2} + c_2.
\]
(3.4)
Even with the lower regularity of \( \|\nabla v\|_{L^s} (s \in [1, \frac{n}{n+1}]) \) compared to that of Lemma 2.6 and the standard technique to control the resulting boundary integral [14], the following estimate associated with the \( v \)-equation in (1.5) is quite known, cf. [25, Lemma 3.4]:
\[
\frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} + 2q^2 \int_{\Omega} |\nabla v|^{2q} + \frac{2(q-1)}{q} \int_{\Omega} |\nabla v|^{\frac{q}{2}} \leq c_3 \int_{\Omega} u^2 |\nabla v|^{2q-2} + c_4,
\]
(3.5)
Then a combination of (3.4) and (3.5) gives rise to (3.1).

**Lemma 3.2.** Besides the conditions in Lemmas 2.5 and 3.1, assume further \( \theta > 1 - \frac{4}{p+2} \). Then, for \( p > 2 \) and \( q > 2 \) as provided by Lemma 2.8, there exists a constant \( C > 0 \) independent of \( t \) such that the solution of (1.5) satisfies
\[
\|u(\cdot, t)\|_{L^p} \leq C, \text{ for all } t \in (0, T_{\text{max}})
\]
(3.6)
and
\[
\|\nabla v(\cdot, t)\|_{L^{2q}} \leq C, \text{ for all } t \in (0, T_{\text{max}}).
\]
(3.7)

Proof. For \( p, q > 2 \), Lemma 3.1 provides two constants \( c_1 > 0 \) and \( c_2 > 0 \) such that
\[
\frac{d}{dt} \left( \int_{\Omega} u^p + \int_{\Omega} |\nabla v|^{2q} \right) + \frac{2(q-1)}{q} \int_{\Omega} |\nabla v|^{\frac{q}{2}} + \frac{(p-1)\xi}{2} \int_{\Omega} u^{p+1} + 2q^2 \int_{\Omega} |\nabla v|^{2q} \leq c_1 \int_{\Omega} u^p + c_2 \int_{\Omega} u^2 |\nabla v|^{2q-2} + c_2.
\]
(3.8)
In the sequel, we shall utilize the parameters chosen in Lemma 2.8 to bound the two coupling integrals on the right-hand side in terms of the dissipation terms on its left-hand side. To start off, the Hölder inequality gives that

$$
\int_{\Omega} u^{p-\theta} |\nabla v|^2 \leq \left( \int_{\Omega} u^{(p-\theta)\lambda} \right)^{\frac{2}{p-\theta}} \left( \int_{\Omega} |\nabla v|^{2\lambda'} \right)^{\frac{1}{\lambda'}} \tag{3.9}
$$

and

$$
\int_{\Omega} u^2 |\nabla v|^{2q-2} \leq \left( \int_{\Omega} u^{2\mu} \right)^{\frac{1}{\mu}} \left( \int_{\Omega} |\nabla v|^{2(q-1)\mu'} \right)^{\frac{1}{\mu'}} \tag{3.10}
$$

where $\lambda \in \left(1, \frac{n}{n-2}\right)$ and $\mu > \max\{1, \frac{n}{2}\}$ such that

$$
\frac{1}{\lambda} + \frac{1}{\lambda'} = 1, \quad \frac{1}{\mu} + \frac{1}{\mu'} = 1 \iff \lambda' = \frac{\lambda}{\lambda - 1}, \quad \mu' = \frac{\mu}{\mu - 1}. \tag{3.11}
$$

The facts $\lambda > 1$ and $p > 1 + 2\theta$ as in (2.26) entail that

$$
\frac{p + \theta}{2(1 + \theta)} > \frac{p + \theta}{2(p - \theta)\lambda}.
$$

On the other hand, from the first inequality of (2.27), we have

$$
\frac{p + \theta}{2(p - \theta)\lambda} > \frac{n-2}{2n}.
$$

These two inequalities enable us to apply the Gagliardo-Nirenberg inequality to bound

$$
\left( \int_{\Omega} u^{(p-\theta)\lambda} \right)^{\frac{1}{\lambda}} \leq c_3 \|u\|_{L^{\frac{2\mu p + \theta}{p + \theta}}}^{\frac{2\mu}{p + \theta}} \|\nabla u\|_{L^{\frac{2}{p + \theta}}} \tag{3.12}
$$

with

$$
k_1 = \frac{p+\theta}{2(1+\theta)-2(p-\theta)\lambda} + \frac{1}{2} - \frac{1}{q} = \frac{p+\theta}{2(1+\theta)-2(p-\theta)\lambda} - \frac{n-2}{2n} \in (0, 1).
$$

By noting the $L^{1+\theta}$-boundedness of $u$ by (2.13) and setting

$$
a_1 := \frac{(p-\theta)}{p+\theta} \cdot k_1 = \frac{\frac{p-\theta}{2(1+\theta)} - \frac{1}{q} + 1}{\frac{p-\theta}{2(1+\theta)} - \frac{n-2}{2n}} \in (0, 1) \tag{3.13}
$$

due to $\lambda \in \left(1, \frac{n}{n-2}\right)$, we then get from (3.12) that

$$
\left( \int_{\Omega} u^{(p-\theta)\lambda} \right)^{\frac{1}{\lambda}} \leq c_4 \|\nabla u\|_{L^{\frac{2\mu p + \theta}{p + \theta}}}^{\frac{2\mu}{p + \theta}} + c_4. \tag{3.14}
$$

Fixing $s \in \left[1, \frac{n(\theta+1)}{(n-1-\theta)\lambda}\right)$, using the second inequality in (2.27), (3.11) and the Gagliardo-Nirenberg inequality again, we find a $k_2 = \frac{\frac{2}{\lambda'} - \frac{2\lambda'}{\lambda}}{\frac{2}{\lambda'} - \frac{2\lambda'}{\lambda} + \frac{1}{q}} \in (0, 1)$ such that

$$
\left( \int_{\Omega} |\nabla v|^{2\lambda'} \right)^{\frac{1}{\lambda'}} = \|\nabla v\|_{L^{\frac{2\lambda'}{\lambda}}}^{\frac{2\lambda'}{\lambda}} \leq c_6 \|\nabla|\nabla v|^q\|_{L^{\frac{2\lambda}{\lambda}}}^{\frac{2\lambda}{\lambda}} \cdot \|\nabla v\|_{L^{\frac{2\lambda}{\lambda}}}^{\frac{2(1-\lambda)}{\lambda}} + c_6 \|\nabla v\|_{L^\lambda}^{\frac{2\lambda}{\lambda}}. \tag{3.15}
$$

This along with the boundedness of $\|\nabla v\|_{L^s}$ as shown in (2.20) of Lemma 2.6 entails

$$
\left( \int_{\Omega} |\nabla v|^{2\lambda'} \right)^{\frac{1}{\lambda'}} \leq c_7 \|\nabla v\|_{L^s}^{\frac{2b_1}{\lambda}} + c_7 \tag{3.15}
$$
with
\[ b_1 := \frac{1}{q} \cdot \frac{q - \frac{q}{p - \frac{q}{p} + \frac{\theta}{p + \theta}}}{\frac{q}{p} + \frac{\theta}{p + \theta} - \frac{q}{p - \frac{q}{p} + \frac{\theta}{p + \theta}}} = \frac{1}{q} - \frac{1}{p + \theta} \in (0, 1) \quad (3.16) \]

implied by (2.29). Similarly, in light of (2.28) and the boundedness of \( \|u^{\frac{p + \theta}{p}}\|_{L^{2(1 + \theta)/p + \theta}} \), one can find \( k_3 = \frac{p + \theta}{p + \theta} - \frac{\theta}{p + \theta} - \frac{q}{p - \frac{q}{p} + \frac{\theta}{p + \theta}} \in (0, 1) \) and then use G-N inequality to estimate
\[
\left( \int_{\Omega} u^{2\mu} \right)^{\frac{1}{2}} = \|u^{\frac{p + \theta}{p}}\|_{L^{\frac{2}{p + \theta}}} \leq c_8 \|\nabla u^{\frac{p + \theta}{p}}\|_{L^2} ||u^{\frac{p + \theta}{p}}\|_{L^{\frac{2(1 + \theta)}{p + \theta}}} + c_8 \|u^{\frac{p + \theta}{p}}\|_{L^2}^{\frac{2(1 + \theta)}{p + \theta}}
\leq c_9 \|\nabla u^{\frac{p + \theta}{p}}\|_{L^2} + c_9
\]
\[
= c_9 \|\nabla u^{\frac{p + \theta}{p}}\|_{L^2} + c_9, \quad (3.17)
\]

where
\[
a_2 := \frac{2k_3}{p + \theta} = \frac{2}{p + \theta} = \frac{1}{(1 + \theta)p} - \frac{1}{p + \theta} + \frac{1}{\frac{p + \theta}{p} + \frac{\theta}{p + \theta} - \frac{q}{p} + \frac{\theta}{p + \theta}} \in (0, 1). \quad (3.18)
\]

By (2.28), (2.30) and (3.11), it follows that \( k_4 = \frac{q - \frac{q}{p} - \frac{\theta}{p + \theta} - \frac{q}{p} - \frac{\theta}{p + \theta}}{2} \in (0, 1) \). Then the Gagliardo-Nirenberg inequality together with the boundedness of \( \|\nabla v\|_{L^2} \) implies
\[
\left( \int_{\Omega} |\nabla v|^{2(q - 1)\mu'} \right)^{\frac{1}{2(q - 1)\mu'}} = \|\nabla v\|_{L^2}^{2(q - 1)\mu'} \leq c_{10} \|\nabla v\|_{L^2}^{2(q - 1)k_4} \cdot \|\nabla v\|_{L^2}^{2(q - 1)k_4} + c_{10} \|\nabla v\|_{L^2}^{2(q - 1)k_4}
\leq c_{11} \|\nabla v\|_{L^2}^{2k_2} + c_{11}, \quad (3.19)
\]

where
\[
b_2 := \frac{q - 1}{q} \cdot k_4 = \frac{q - 1}{p + \theta} < (0, 1). \quad (3.20)
\]

A collection of (3.9), (3.10), (3.14), (3.15), (3.17) and (3.19) shows that
\[
c_1 \int_{\Omega} \|\nabla v\|^2 + c_1 \int_{\Omega} u^2 |\nabla v|^{q - 2}
\leq c_{12} \|\nabla u^{\frac{p + \theta}{p}}\|_{L^2}^{2a_1} + c_{12} \|\nabla v\|^{q - 2} + c_{12} \|\nabla u^{\frac{p + \theta}{p}}\|_{L^2}^{2a_1} \cdot \|\nabla v\|^{q - 2}
+ c_{12} \|\nabla u^{\frac{p + \theta}{p}}\|_{L^2}^{2a_1} + c_{12} \|\nabla v\|^{q - 2} + c_{12} \|\nabla u^{\frac{p + \theta}{p}}\|_{L^2}^{2a_1} \cdot \|\nabla v\|^{q - 2} + c_{12}, \quad (3.21)
\]

where, due to (2.29) and (2.30), the parameters \( a_1, b_1, a_2, b_2 \) defined respectively by (3.13), (3.16), (3.18) and (3.20) satisfy
\[
a_1 + b_1 = \frac{p - \theta}{2(1 + \theta)} - \frac{1}{p + \theta} \cdot \frac{1}{\frac{p + \theta}{p} + \frac{\theta}{p + \theta} - \frac{q}{p} + \frac{\theta}{p + \theta}} + \frac{1}{p + \theta} < 1, \quad (3.22)
\]
and
\[
a_2 + b_2 = \frac{1}{(1 + \theta)\mu} - \frac{1}{p + \theta} \cdot \frac{1}{\frac{p + \theta}{p} + \frac{\theta}{p + \theta} - \frac{q}{p} + \frac{\theta}{p + \theta}} + \frac{1}{p + \theta} < 1. \quad (3.23)
\]
For $a > 0$ and $b > 0$ with $a + b < 1$, a couple of simple uses of Young’s inequality with epsilon, for any $\varepsilon > 0$ and $X, Y \geq 0$, there exists a constant $C_\varepsilon > 0$ such that

$$X^a + Y^b + X^a Y^b \leq \varepsilon (X + Y) + C_\varepsilon. \tag{3.24}$$

The facts (3.22) and (3.23) enable us to apply (3.24) to (3.21) to deduce

$$c_1 \int_\Omega u^p |\nabla v|^2 + c_1 \int_\Omega u^2 |\nabla v|^{2q} - 2 \leq \varepsilon \left( \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v|^q \right) + c_{13}. \tag{3.25}$$

Substituting (3.25) into (3.8) and choosing $\varepsilon = \min \left\{ \frac{q-1}{q}, \frac{(p-1)d}{p} \right\}$, one has

$$\frac{d}{dt} \left( \int_\Omega u^p + \int_\Omega |\nabla v|^{2q} \right) \leq -\frac{(p-1)\xi \gamma}{2} \int_\Omega u^{p+1} - 2q\beta \int_\Omega |\nabla v|^{2q} + c_{14} \tag{3.26}$$

where we have used the Young’s inequality to get

$$-\frac{(p-1)\xi \gamma}{2} \int_\Omega u^{p+1} \leq -2q\beta \int_\Omega u^p + c_{16}.$$

Solving the Grownwall’s inequality (3.26) gives the existence of a constant $c_{17}$ such that

$$\int_\Omega u^p + \int_\Omega |\nabla v|^{2q} \leq c_{17},$$

which yields the desired estimates (3.6) and (3.7). This completes the proof. \qed

**Proof of Theorem 1.1.** Lemma 3.2 provides a constant $c_1 > 0$ such that, for all $p > n$,

$$\|u(\cdot, t)\|_{L^p} \leq c_1 \text{ for all } t \in (0, T_{\text{max}}),$$

which together with the parabolic and elliptic regularity applied to the second and third equations in (1.5) shows

$$\|\nabla u(\cdot, t)\|_{L^\infty} + \|\nabla v(\cdot, t)\|_{L^\infty} \leq c_2 \text{ for all } t \in (0, T_{\text{max}}). \tag{3.27}$$

Then using (3.27) and Alikakos-Moser type iteration (cf. [3] or [39, Lemma A.1]), one can easily obtain that

$$\|u(\cdot, t)\|_{L^\infty} \leq c_3 \text{ for all } t \in (0, T_{\text{max}}). \tag{3.28}$$

Finally, Theorem 1.1 is an immediate consequence of (3.28) and Lemma 2.1. \qed

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