Manipulation of the Dirac cones and the anomaly in
the graphene related quantum Hall effect

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Abstract. The quantum Hall effect in graphene is regarded to be involving half-integer
topological numbers associated with the massless Dirac particle, this is usually not apparent due
to the doubling of the Dirac cones. Here we theoretically consider two classes of lattice models
in which we manipulate the Dirac cones with either (a) two Dirac points that have mutually
different energies, or (b) multiple Dirac cones having different Fermi velocities. We have shown,
with an explicit calculation of the topological (Chern) number for case (a) and with an adiabatic
argument for case (b) that the results are consistent with the picture that a single Dirac fermion
contributes the half-odd integer series (-3/2, -1/2, 1/2, 3/2, ...) to the Hall conductivity
when the Fermi energy traverses the Landau levels.

1. Introduction

In the graphene quantum Hall effect (QHE)[1, 2, 3], a most striking point is that the Hall
conductivity $\sigma_{xy} = 2n + 1(n = 0, \pm 1, ...)$ in units of $-e^2/h$ with the spin degrees of freedom
dropped here, becomes $\sigma_{xy} = n + 1/2$ when we divide by two to have the contribution from each
Dirac cone, since there are two of them at K and K’ in graphene. The appearance of doubled
Dirac cones is consistent with the Nielsen-Ninomiya theorem[4], which dictates that Dirac cones
should appear in pairs as far as the chiral symmetry is present. However, a natural question is:
can we do better than just dividing by two to convince ourselves on the half-integer topological
numbers.

The quantization into half odd integers is of fundamental interest, but as far as lattice models
are concerned, we always have a periodicity in the Brillouin zone, and the TKNN formula[5]
dictates that integer topological numbers are imperative. For instance, we can conceive lattice
models that is outside the applicability of Nielsen-Ninomiya, but we still end up with integer
Hall conductivity. Here we take another approach to explore this problem: We theoretically
consider two classes of lattice models in which we manipulate the Dirac cones with either (a)
the two Dirac points that have mutually different energies[6], or (b) multiple Dirac cones with
different Fermi velocities. With explicit calculations of the topological (Chern) number, we shall
show that the results for systematically manipulated Dirac cones are consistent with the picture
that a single Dirac fermion contributes the half odd integer series (-3/2, -1/2, 1/2, 3/2, ...) to the Hall conductivity when the Fermi energy traverses the Landau levels.
2. Shifted Dirac cones

A simplest way to realize a model in which two Dirac cones are preserved but have shifted energies is to add a term that is proportional to $\sigma_0$ (unit matrix) with a $k$-dependent coefficient in the space spanned by Pauli matrices[6]. So we introduce a lattice model with a Hamiltonian,

$$\mathcal{H} = \sum_k \hat{c}_k^\dagger \left[ h_k^{gr} + 2t_1(\sin k_1)\sigma_0\right]_{\alpha,\beta} \hat{c}_k \beta,$$

where $\sigma_i$'s are Pauli matrices and $\alpha, \beta$ denote their components. Namely, we have added, on top of the nearest-neighbor hopping $t_0$, an extra hopping involving $\sigma_0$ times a $k$-dependent function. We then lift the degeneracy of the energies of $K$ and $K'$ if the $k$-dependent term has different values between them. A simplest choice is $\propto \sin k_1$. If we go back to the real space, the tight-binding model is as depicted in Fig.1(a), which has extra second-neighbor hoppings on top of the nearest-neighbor ones. The added hoppings has to be only between A-A and B-B for the Dirac cone to be preserved, and they have to be complex for the degeneracy between $K$ and $K'$, mutually time-reversal pairs, to be broken. The model with a complex hopping is fictitious, but we do accomplish shifted Dirac cones as depicted in Fig.1(b). In this model the chiral symmetry is broken, since $\sigma_0$-term in $\mathcal{H}$ invalidates $\{\mathcal{H}, \sigma_z\} = 0$. Nevertheless, the addition of $\sigma_0$ preserves the shape of Dirac cones, along with the species doubling. If we expand the Hamiltonian (1) around $k_0$ (K or K'), we have $h_k \simeq \chi\Delta \sigma_0 - \hbar v_F \left[ (-1)^{\frac{k_0}{2}} \delta k_x \sigma_1 + \delta k_y \sigma_2 \right]$, where $\chi = +1(-1)$ correspond to K (K'), $\Delta = \sqrt{3}t_1$ is (half) the shift, $v_F = \sqrt{3}a t_0/2\hbar$ the Fermi velocity, and $\delta k = k - k_0$.

3. Chern numbers

The calculation of the Hall conductivity, which is a topological Chern number, has to be calculated carefully, since we have to sum up over the contribution from the “Dirac sea”. We can overcome the difficulty with a method employing non-commutative Berry’s connection[7] and its integration over the Brillouin zone to estimate the Chern number with a technique developed...
in the lattice gauge theory.[8] The result for the Chern number in the present model is shown in Fig. 2(b), while Fig. 2(a) is for the ordinary graphene for comparison.

![Figure 2](image-url)

**Figure 2.** Numerically calculated Chern numbers against $E_F$ (upper parts), along with the Landau levels for each of the two Dirac cones with the Chern numbers in each gap displayed (lower parts) for the ordinary graphene ($t_1 = 0$) (a) and for the present model with shifted Dirac cones ($t_1 \neq 0$) (b). The result is for the magnetic flux $\phi = 1/100$.

Figure 2(b) shows that the result is exactly what we expect when we sum the two half-odd-integer series ($..., -3/2, -1/2, 1/2, 3/2, \ldots$) with a shift in energy as $E_F$ traverses shifted sets of Landau levels (as shown in the lower part in the figure). In other words, in a striking contrast to the ordinary graphene where each QHE step has a jump of 2 in the Chern number, the present model exhibits a jump of 1 at each step.[6] The agreement is rather surprising (since there is no apriori reason why the superposition of effective field theory for the vicinities of K and K’ and the lattice model should have the same Chern numbers). Thus, although we have still no half integers for the total Hall conductivity (since a sum of two half-odd integers is an integer), we have indirectly confirmed the half integers. Since no half integers should appear according to TKNN, this is indeed as best as we can confirm the half-integer property of each Dirac cone.

4. Multiple Dirac cones as higher-pseudospin representations of SU(2)

The effective theory for the Dirac cones in graphene is expressed in terms of Pauli matrices, which is a representation of SU(2) with pseudospin $1/2$. Now we pose a question: can we extend this to examine whether the generalized model has half-integer contributions to the Chern number as well? For the two-band case (which may be regarded as having a pseudospin $1/2$) the each matrix is $2 \times 2$, which corresponds to two (A, B) sublattices. Here we show that a realization of higher pseudospin $S$ Dirac cones is possible, where each matrix is $(2S + 1) \times (2S + 1)$ corresponding to $(2S + 1)$ sublattices. By a pseudospin $S$ Dirac cone we mean an effective Hamiltonian around the Dirac point,

$$\mathcal{H} = k_1 \Sigma_1 + k_2 \Sigma_2 + O(k^2),$$

where $\Sigma_i$’s are $(2S + 1)$-dimensional representations of SU(2).

One such model actually appears in the “flat-band model”, where one band is flat in a multiband model.[9] Namely, in a Lieb’s model for the flat band, $S = 1$ SU(2) is realized for a three-band case, where two bands form a Dirac cone, while the other is flat. We can generalize
this to realize for general $S$ with $(2S + 1)$ sites in a unit cell, although the models can again be unrealistic, but does serve as examining half-integer Chern numbers.

![Figure 3.](image)

Although general values of $S$ are permissible, here we focus on half-integer $S = 1/2, 3/2, 5/2, \cdots$ with $2S + 1$ even (while the case of an integer $S$ contains a flat band in the band structure). We can construct the lattice models that realize these pseudospins by inserting extra atoms on edges of each hexagon with real hoppings between them. In this case $(2S + 1)/2$ cones appear around each of K and K' points as displayed in Fig. 3. Let us first look at the Landau level structure ($E_n$ against $n$) in Fig. 4. This can be obtained by quantizing eqn.(2) with $k$ replaced with $k + eA$. The Landau levels have $(2S + 1)$ sequences that correspond to electron and hole Landau levels originating from the $(S + 1/2)$ Dirac cones with different Fermi velocities, and they asymptotically approach $\hbar \omega_c \sqrt{nS_z}$, $(S_z = -S, -S + 1, \cdots, S)$ for large $n$.

The question is the Hall conductivity contributed by the multiple Dirac cones around each of K and K'. Figure 5 displays the result for $S = 5/2$, which is obtained from an effective field theory as follows. We first add a mass term $m \Sigma_3$ to calculate the Chern number in zero magnetic field, following the idea of Haldane[10], and then apply a magnetic field to have Landau levels and let $m \to 0$. Topological numbers should be invariant in such an adiabatic process. In Fig.5 for the contribution to the Chern numbers of each single $S = 5/2$ valley from the three sets of Landau levels associated with the three Dirac cones with different Fermi velocities, we do have half-odd integer series, $\cdots, -5/2, -3/2, 3/2, 5/2, \cdots$ as another generalization of the half-integer contribution from each Dirac cone. For the original lattice model we have two Dirac points, so we can apply the shift between the two Dirac points to resolve them as discussed in the first half of this paper, which is a future problem.

Recently it is recognized[11] that massless Dirac fermions can be realized as surface states of the three-dimensional topological (quantum spin Hall) systems as one manifestation of the bulk-edge correspondence[13, 12]. There the doubling partner exists at the other side of the system, so that the decomposition of the topological numbers into contributions from each Dirac cone...
may become a realistic as well as intriguing question. The work was supported in part by grants-in-aid for scientific research No. 20340098 (YH and HA) from JSPS and No. 22014002(YH) on priority areas from MEXT.

![Figure 4](image-url) **Figure 4.** Landau levels $E_n$ against $n$ for the pseudospin $S = 1/2$ (ordinary graphene) (a), $S = 3/2$ (b), and $S = 5/2$ (c).

![Figure 5](image-url) **Figure 5.** The Hall conductivity from each single valley for $S = 5/2$ (left column) and its decomposition into the contributions from three Dirac cones (right).

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