Benford’s law is a famous law in statistics which states that the leading digits of random variables in diverse data sets appear not uniformly from 1 to 9; the probability that $d$ ($d = 1, \ldots, 9$) appears as a leading digit is given by $\log_{10}(1 + 1/d)$. This paper shows the existence of a random variable with a smooth probability density on $(0, \infty)$ whose leading digit distribution follows Benford’s law exactly. To construct such a distribution the error theory of the trapezoidal rule is used.

Key words: exponential type, leading digits, scale invariance, trapezoidal rule, uniform distribution

1 Introduction

Benford’s law, also called the logarithmic law, is a statistical law which asserts that the leading digits from diverse and different data sets are not uniformly distributed, as might be expected, but follow some peculiar distribution described by a logarithmic function. According to the law, in the sets which obey the law, the number 1 appears as the leading digit about 30% of the time, while 9 appears less than 5% of the time [1, 10]. Benford’s law arises ubiquitously in the fields of chemistry, physics, geology, astronomy, engineering, econometrics, etc, and has a wide range of applications including image processing, network analysis and fraud detection in accounting data [2, 8].

It is shown that the distribution of the leading digits of the product $\prod_{i=1}^{n} X_i$ converges to Benford’s law, as $n \to \infty$, under the assumption that $X_i$ are independent, identically distributed and not purely atomic random variables [3]. It is also shown that if the distribution of a random variable $X > 0$ is the log-normal then the distribution of the leading digits of $X$ approaches to the law as $\sigma^2 \to \infty$, where $\sigma^2$ is the variance of $Y = \ln X$ [3].

The purpose of this note is to construct a continuous distribution on $(0, \infty)$ such that the leading digits of the variable with this distribution follow Benford’s law exactly, without any limiting operations such as the above.

The rest of the paper is organized as follows. Section 2 is a brief introduction of the law. Section 3 investigates the property of continuous distributions on $(0, \infty)$ that satisfy the law exactly. In Section 4, such distribution is proposed by using an error theory of the trapezoidal rule in numerical integrations. Section 5 is the conclusion.
2 Benford’s law

Let \( X \) be a real random variable and its scientific notation or decimal floating-point representation be
\[
X = \pm S(X) \cdot 10^M, \quad 1 \leq S(X) < 10, \quad M \in \mathbb{Z},
\]
and the decimal notation of the significand \( S(X) \) be
\[
S(X) = d_0.d_1d_2 \cdots,
\]
where
\[
d_0 \in \{1, 2, \ldots, 9\}, \quad d_i \in \{0, 1, 2, \ldots, 9\}, \quad i = 1, 2, \ldots.
\]
Benford’s law \([1, 10]\) states that if the data \( X \) are distributed over several orders of magnitude, then the leading digit \( d_0 \) will approximately follow the probability distribution
\[
P (d_0 = d) = \log_{10} (d + 1) - \log_{10} d, \quad d = 1, 2, \ldots, 9,
\]
where \( P (A) \) denotes the probability of an event \( A \). The distribution of the leading digit \( d_0 \) depends, of course, on that of \( X \). Although there are many continuous classical distributions, to the author’s knowledge, none of them gives the Benford’s law exactly. The purpose of this note is to construct a continuous distribution of \( X \) on \((0, \infty)\) such that \( S(X) \) follows Benford’s law exactly.

Hereafter we consider only the case \( X > 0 \), and assume that the base (radix), which will be denoted by \( b \), is not necessarily 10, but instead any integer \( b \geq 2 \). Like the decimal case, we define “scientific notation” for the base \( b \) as
\[
X = b^M \cdot S(X).
\]
In this case, the significand \( S(X) \) and the exponent \( M \) are given by
\[
S(X) = b^F, \quad M = \lfloor \log_b X \rfloor, \quad F = \{ \log_b X \},
\]
where the symbols \( \lfloor \rfloor \) and \( \{ \} \) are the integer and the fractional parts of the argument, respectively.

For the base \( b \), we extend Benford’s law as
\[
P (1 \leq S(X) \leq s) = \log_b s, \quad 1 \leq s < b,
\]
which is often called the strong Benford’s law in base \( b \). We will simply refer to the strong law as Benford’s law. Of course, this definition implies (2) when \( b = 10 \). Since \( F = \log_b S \), we can easily see that the condition (4) is equivalent to
\[
P (0 \leq F \leq \sigma) = \sigma, \quad 0 \leq \sigma < 1,
\]
where \( \sigma = \log_b s \). This means that the fractional part of \( \log_b X \) is uniformly distributed on \([0, 1)\). The uniformity of \( F \) is a key concept in Benford analysis.
3 Piecwise continuous and continuous distributions

Let \( P_S(s) \) be the probability density function (in short pdf) of the significand \( S \). To derive \( P_S(s) \) we differentiate (4) with respect to \( s \). Then we have

\[
P_S(s) = \frac{1}{s \ln b}, \quad s \in [1, b).
\] (6)

If the range of \( X \) is \([1, b)\) then this is also the pdf of \( X \), since in this case \( X = S(X) \). Next we extend the range of \( X \) to a wider range.

Let the range of \( X \) be \([b^{m_0}, b^{m_1}] \) \((m_0 < m_1)\), and \((p_m)\) be the sequence satisfying

\[
p_m \geq 0, \quad m = m_0, m_0 + 1, \ldots, m_1, \quad \sum_{m=m_0}^{m_1} p_m = 1. \tag{7}
\]

Using the sequence, we define the pdf of \( X \) as

\[
P_X(x) = \sum_{m=m_0}^{m_1} \frac{p_m}{x \ln b} \chi_m(x), \quad 0 < x < \infty, \tag{8}
\]

where

\[
\chi_m(x) = \begin{cases} 1, & x \in [b^m, b^{m+1}), \\ 0, & \text{otherwise.} \end{cases}
\]

This function clearly satisfies the requirement for pdf’s

\[
P_X(x) \geq 0, \quad \int_{-\infty}^{\infty} P_X(x) \, dx = 1.
\]

The random variable \( X \) with the density (8) is clearly Benford, since for some \( s \) \((1 \leq s < b)\)

\[
\mathbb{P}(1 \leq S(X) \leq s) = \sum_{m=m_0}^{m_1} \int_{b^m}^{b^{m+1}} P_X(x) \, dx = \sum_{m=m_0}^{m_1} \int_{b^m}^{b^{m+1}} \frac{p_m}{x \ln b} \, dx = \log_b s.
\] (9)

Thus we have easily constructed a distribution of \( X \) on the wider range that satisfies (4).

Next we consider a distribution on \((0, \infty)\). In order to extend the domain of \( P_X(x) \) to \( \mathbb{R}^+ \), we must set \( m_0 = -\infty, m_1 = +\infty \). Moreover, if such a distribution to be continuous then

\[
\cdots = p_{-2} = p_{-1} = p_0 = p_1 = p_2 = \cdots, \tag{10}
\]

which clearly contradicts (7). By the way, since \( p_m \) means

\[
p_m = \mathbb{P}(b^m \leq X < b^{m+1}), \quad m = 0, \pm 1, \pm 2, \ldots, \tag{11}
\]

condition (10) means

\[
\cdots = \mathbb{P}(b^{-1} \leq X < 1) = \mathbb{P}(1 \leq X < b) = \mathbb{P}(b \leq X < b^2) = \cdots.
\]
This is just the hypothesis of *scale invariance* for a scale factor $b$. The hypothesis is often appeared in numerous articles on Benford’s law, since Pinkham [11] derived Benford’s law from this hypothesis. However, Knuth [7] showed that this attractive hypothesis is not possible for all bases $b \geq 2$. Conversely, it was shown by Hamming [6] that if $X$ is Benford, that is, $S(X)$ has the distribution satisfying (4) then for any constants $c > 0$ the significand $S(cX)$ also has that distribution.

We have failed to construct a continuous density on $(0, \infty)$ that gives the law (4). Here we propose another way of constructing such density. Let us introduce the new variable

$$ Y = \ln X, \tag{12} $$

and the pdf of $Y$ be $P_Y(y)$. From the well-known formula of probability theory, we have

$$ P_Y(y) = \left| \frac{dx}{dy} \right| P_X(x) = e^y P_X(e^y) \quad \text{or} \quad P_X(x) = x^{-1} P_Y(\ln x), \tag{13} $$

where $y = \ln x$ [4]. Using $P_Y(y)$, we have for $\sigma = \log_b s$

$$ \mathbb{P}(0 \leq F \leq \sigma) = \mathbb{P}(1 \leq S(X) \leq s) $$

$$ = \sum_{m=-\infty}^{\infty} \mathbb{P}(b^m \leq X \leq b^m s) $$

$$ = \sum_{m=-\infty}^{\infty} \mathbb{P}(m \ln b \leq Y \leq (m + \sigma) \ln b) $$

$$ = \sum_{m=-\infty}^{\infty} \int_{m \ln b}^{(m+\sigma) \ln b} P_Y(y) \, dy. \tag{14} $$

To obtain $P_F(\sigma)$, the pdf of $F$, we differentiate (14) with respect to $\sigma$ under the integral sign to obtain

$$ P_F(\sigma) = \frac{d}{d\sigma} \mathbb{P}(0 \leq F \leq \sigma) $$

$$ = \sum_{m=-\infty}^{\infty} \frac{d}{d\sigma} \left( \int_{m \ln b}^{(m+\sigma) \ln b} P_Y(y) \, dy \right), \tag{15} $$

$$ = \ln b \sum_{m=-\infty}^{\infty} P_Y((m + \sigma) \ln b). $$

This is just the trapezoidal approximation to the infinite integral

$$ \int_{-\infty}^{\infty} P_Y(y) \, dy = 1 \tag{16} $$

with the step size $h = \ln b$. Therefore, if the trapezoidal rule (15) gives the exact value of the integral (16), then the distribution of the fractional part $F = \{\log_b X\}$ is uniformly distributed on $[0, 1)$, and as a result the distribution of $S(X)$ is Benford.
Here we consider the class of pdf functions which satisfies
\[
\int_{-\infty}^{\infty} P_Y(y) \, dy = \ln b \sum_{m=-\infty}^{\infty} P_Y((m + \sigma) \ln b).
\]  
(17)

4 Trapezoidal rule and Benford distribution

Let us consider the integral
\[
I = \int_{a}^{b} f(x) \, dx.
\]  
(18)

To approximate the integral numerically many formulas were developed. Among the formulas, the trapezoidal rule is the most elementary formula. As is well known, the formula cannot be expected to give an accurate result for an integral over a large (finite) interval; this formula gives the exact result only for the case that the integrand \( f(x) \) is a linear or piecewise linear function. Therefore, the formula is seldom used in practice [5]. It is shown, however, that if the interval is infinite or semi-infinite and \( f(x) \) is an analytic function on \( \mathbb{R} \) or \( \mathbb{R}^+ \), then the convergence is tremendously fast [14]. Moreover, Sugihara [13] proved that there exists a class of functions for which the trapezoidal rule gives the exact value of \( I \), when \( a = -\infty \), \( b = +\infty \). Here we show the result by Sugihara:

**Theorem (Sugihara [13])** Let \( f(z) \) be a function that satisfies the following three conditions:

\( f(z) \) is entire, that is, holomorphic for all \( z \in \mathbb{C} \).

\( f(z) \) is exponential type \( A \).

Then for all \( 0 < h < 2\pi / A \)
\[
\int_{-\infty}^{\infty} f(x) \, dx = h \sum_{m=-\infty}^{\infty} f(mh).
\]  
(20)

As an example of the functions belonging to this class with \( A = 1 \), we show \( f(z) = \text{sinc} (z)(= \sin z / z) \). For this function we have
\[
h \sum_{m=-\infty}^{\infty} \text{sinc} (mh) = \int_{-\infty}^{\infty} \text{sinc} (x) \, dx (= \pi)
\]  
(21)

for all \( 0 < h < 2\pi \).
In the present situation, since the integrand is a pdf, we must modify the above conditions as follows:

(a) \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \)

(b) \( f(z) \) is entire, and \( f(x) \geq 0 \) for \( x \in \mathbb{R} \).

(c) \( f(z) \) is exponential type \( A \).

We now return to the topic of Benford’s law. From now on the integrand is not \( f(z) \) but \( P_Y(y) \), and the argument is not \( x \) but \( y \). As an example of the functions that satisfy the three conditions in (22), we can show the following function:

\[
P_Y(y) = \left( \frac{a}{\pi} \right) \text{sinc}^2(a \, y), \quad a > 0.
\]

It can easily be shown that this function satisfies the conditions (a) and (b). It is also shown that the function is exponential type \( 2a \) \cite{12}, since on the imaginary axis

\[
\lim_{v \to \pm\infty} \frac{\log |P_Y(iv)|}{|v|} = 2a.
\]

Therefore, for the \( P_Y(y) \) given by (23) and for the \( b \) in the range

\[
0 < \ln b < \frac{\pi}{a},
\]

eq. (17) is valid. Thus the random variable \( X \) with the density

\[
P_X(x) = x^{-1}P_Y(\ln x) = \left( \frac{a}{\pi x} \right) \text{sinc}^2(a \ln x), \quad x > 0
\]

is Benford, that is, \( S(X) \) satisfy the condition (4) for all \( b \) satisfying

\[
2 \leq b < e^{\pi/a}.
\]

In particular, in order that the variable \( X \) is Benford for \( b = 10 \), we must keep \( a \) in the range

\[
0 < a < \frac{\pi}{\ln 10} = 1.364 \cdots.
\]

Here we consider the mean and variance of the random variable \( X \) with the distribution (25). Let \( \lambda \geq 1 \) be an integer, then \( \lambda \)th moment is

\[
\mathbb{E} [X^\lambda] = \int_0^{\infty} x^\lambda P_X(x) \, dx = \frac{a}{\pi} \int_{-\infty}^{\infty} \text{sinc}^2(ay) e^{\lambda y} \, dy.
\]

where \( \mathbb{E} [\cdot] \) denotes the expected value of the argument. Since in this equation the integrand diverges as \( y \to +\infty \), then the mean and variance of \( X \) do not exist.
5 Conclusion

We have developed a continuous distribution on \((0, \infty)\) of the random variable whose significand obeys Benford’s law exactly by using Sugihara’s theory on the trapezoidal rule [13]. It has been shown that the random variable with the distribution does not have the mean and the variance.

References

[1] F. Benford, The Law of Anomalous Numbers, Proceedings of the American Philosophical Society 78 (1938), 551–572.

[2] Benford Online Bibliography: http://www.benfordonline.net/

[3] A. Berger, T.P. Hill, An Introduction to Benford’s Law, Princeton Univ. Press, 2015.

[4] M. Capiński and E. Kopp, Measure, Integral and Probability, 2nd ed., Springer Undergraduate Mathematics Series, 2003.

[5] G. Evans, Practical Numerical Integration, John Wiley & Sons, 1993.

[6] R.W. Hamming, On the distribution of numbers, Bell Syst. Tech. J. 49 (1970), 1609–1625.

[7] D.E. Knuth, The Art of Computer Programming Volume 2, Seminumerical Algorithms Third Edition, Addison Wesley, 1998.

[8] A.E. Kossovsky, Benford’s Law, Theory, the General Law of Relative Quantities, and Forensic Fraud Detection Applications, World Scientific, 2014.

[9] S.J. Miller ed., Benford’s Law: Theory and applications, Princeton Univ. Press, 2015.

[10] S. Newcomb, Note on the Frequency of Use of the Different Digits in Natural Numbers, Amer. J. Math. 4 (1881), 39–40.

[11] R.S. Pinkham, On the distribution of first significant digits, Ann. Math. Statist. 32 (1961), 1223–1230.

[12] Q.I. Rahman, Functions of Exponential type, Trans. Amer. Math. Soc. 135 (1969), 295–309.

[13] M. Sugihara, A Class of Functions for Which the Trapezoidal Rule Gives the Exact Value of Integral over the Infinite Interval, J. Comp. Appl. Math. 20 (1987), 387–392.

[14] L.N. Trefethen, J.A.C. Weideman, Exponentially Convergent Trapezoidal Rule, SIAM Review, 56 (2014), 385–458.