A NOTE ON DEL PEZZO FIBRATIONS OF DEGREE 1

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ABSTRACT

The purpose of this paper is to extend the results in [11] in the case of del Pezzo fibrations of degree 1. To this end we investigate the anticanonical linear systems of del Pezzo surfaces of degree 1. We then classify all possible effective anticanonical divisors on Gorenstein del Pezzo surfaces of degree 1 with canonical singularities.

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1. Introduction

In [11], it has been proven that smooth del Pezzo fibrations of degree at most 4 over a discrete valuation ring cannot be birationally transformed. Taking $\mathbb{P}^1$-bundle over a discrete valuation ring into consideration, we find the result quite interesting. Because the study of birational maps of del Pezzo fibrations is quite important in birational geometry, we need to pay attention to the result in [11].

In the present paper, we extend the result in [11]. The result in [11] requires the special fibers to be smooth. It is natural to expect the similar result when mild singularities are allowed on the special fibers.

Let $\mathcal{O}$ be a discrete valuation ring with residue field $k$ of characteristic 0. We denote by $K$ the quotient field of $\mathcal{O}$. A model of a variety $X_K$ defined over $K$ is a flat scheme $X$ defined over $\text{Spec} \, \mathcal{O}$ whose generic fiber is isomorphic to $X_K$. Fano fibrations are models of smooth Fano varieties defined over $K$. In particular, del Pezzo fibrations of degree $d$ are models of smooth del Pezzo surfaces of degree $d$ defined over $K$. We let $T = \text{Spec} \, \mathcal{O}$. For a scheme $\pi : Z \rightarrow T$, the scheme-theoretic fiber $\pi^*(o)$ is denoted by $S_Z$, where $o$ is the closed point of $T$.

Let $X/T$ be a Gorenstein del Pezzo fibration. In the present paper, we always assume the following:

- The special fiber $S_X$ is reduced and irreducible.
- Any birational map of $X$ into another del Pezzo fibration over $T$ is biregular on generic fiber.
Such del Pezzo fibrations are studied in [2] and [7]. It should be remarked that the second condition automatically holds if the generic fiber is a del Pezzo surface of degree 1 with Picard number 1 ([6]). The author ([11]) studied birational maps between them with one more condition:

- The log pair \((X, S_X)\) is purely log terminal.

We will also assume this condition throughout the paper. The first and the last conditions force the special fiber \(S_X\) to be a normal Gorenstein del Pezzo surface with canonical singularities.

**Remark.** In [11] we assume that del Pezzo fibrations are \(\mathbb{Q}\)-factorial. But it turns out that we do not have to assume \(\mathbb{Q}\)-factoriality. Indeed, it is enough to assume only Gorenstein singularities.

In this paper, we are mainly interested in del Pezzo fibrations of degree 1. The main task is to extend the results in [11].

**Theorem A.** Let \(X/T\) and \(Y/T\) be del Pezzo fibrations of degree 1. We suppose that the special fiber \(S_X\) of \(X\) has only singularities of type \(A_n\), \(n \geq 3\). Suppose that there is a birational map \(f : X \to Y\). Then exactly one of the following holds:

- The birational map \(f\) is biregular.

- The special fiber \(S_Y\) of \(Y\) has a singularity of type \(E_8\). Moreover, it is a unique singular point of \(S_Y\).

Furthermore, if the anticanonical linear system \(|-K_{S_X}|\) of \(S_X\) has no cuspidal rational curve, then only the first can occur.

**Theorem B.** Notation is as above. Suppose that the special fiber \(S_X\) has only singularities of type \(A_n\), \(n \geq 1\). We assume that \(S_X\) has a singular point \(p\) of type either \(A_1\) or \(A_2\) and that \(f\) is not biregular.

1. There is a cuspidal rational curve \(D\) in \(|-K_{S_X}|\).

2. If every cuspidal rational curve in \(|-K_{S_X}|\) has a cusp outside of singular points of type \(A_2\) and \(A_1\), then \(S_Y\) has only one singular point and the singularity type is \(E_8\).

Suppose that there is an element in \(|-K_{S_X}|\) which has a cusp at the point \(p\).

3. If the point \(p\) has singularity type \(A_1\) and every element of \(|-K_{S_X}|\) has no cusp at singular points different from \(A_1\), then \(S_Y\) has a singular point of type either \(E_8\) or \(E_7\).

4. If the point \(p\) has singularity type \(A_2\), then \(S_Y\) has a singular point of type \(E_8\), \(E_7\), or \(E_6\).

**Theorem C.** Notation is as above. Suppose that the special fiber \(S_X\) has only singularities of type \(A_n\), \(n \geq 1\).
1. If $S_Y$ has a singularity of type $E_6$, then $S_X$ has a cuspidal rational curve in $|−K_{S_X}|$ the cusp of which is an $A_2$-singularity of $S_X$.

2. If $S_Y$ has a singularity of type $E_7$, then $S_X$ has a cuspidal rational curve in $|−K_{S_X}|$ the cusp of which is a singular point of $S_X$ of type either $A_1$ or $A_2$.

Before we proceed, we provide an example which illustrates Theorem A. Let $X$ and $Y$ be subschemes of $P^3_O(1,1,2,3)$ defined by equations $w^2 + z^3 + xy^5 + x^5y = 0$ and $w^2 + z^3 + xy^5 + t^{24}x^5y = 0$, respectively, where $z$ and $w$ are of weight 2 and 3, respectively, and $t$ is a local parameter of $O$. The special fiber of $X$ is smooth. On the other hand, that of $Y$ has a single singular point which is of type $E_8$. There is a birational map $f : X → Y$ defined by $f(x,y,z,w) = (x, t^6 y, t^{10} z, t^{15} w)$.

2. Definitions

In the present section, we briefly explain the essential definitions for this paper. Basically, we use the definitions in birational geometry “textbooks”. For the definitions of log canonical singularities, pure log canonical singularities, centers of log canonicity, 1-complements and so forth (see [8] and [13]).

The most important tool in this paper is the concept of log canonical threshold which was introduced by V. V. Shokurov.

Definition 2.1. Let $(X, B)$ be a log canonical pair. Let $D$ be a $\mathbb{Q}$-Cartier divisor on $X$. The log canonical threshold of $D$ on $(X, B)$ is the number

$$\text{lct}(D; X, B) = \max\{c : K_X + B + cD \text{ is log canonical}\}.$$ 

It is easy to check that $0 ≤ \text{lct}(D; X, B) ≤ 1$.

We need to investigate anticanonical linear systems to obtain our main results. To this end we will use the following number to measure how bad an effective anticanonical divisor can be.

Definition 2.2. Let $X$ be a normal variety with nonempty anticanonical linear system. We suppose that the log pair $(X, 0)$ is log canonical. The total log canonical threshold of $X$ is the number

$$\text{tlct}(X) = \max\{c : K_X + cD \text{ is log canonical for any } D ∈ |−K_X|\}.$$ 

Note that $0 ≤ \text{tlct}(X) ≤ 1$.

In the study of smooth weak del Pezzo surfaces, fundamental cycles, introduced by M. Artin ([3]), give us a great deal of information. In [4], they are called numerical cycles.

Proposition-Definition 2.1. Let $π : Y → X$ be a resolution of a point $p$ on a normal surface $X$. Let $E = \sum E_i$ be the divisor of the $π$-exceptional locus. Then there exists a unique effective exceptional divisor $Γ = \sum a_i E_i$ such that $Γ > 0$, $Γ ⋅ E_i ≤ 0$ for every $E_i$, and $Γ$ is minimal with respect to this property. The divisor $Γ$ is called the fundamental cycle of the bunch $\{E_i\}$.
Proof. See [12]. Q.E.D.

For a minimal resolution of a Du Val singularity, we can easily find the corresponding fundamental cycle. Since the fundamental cycles related to Du Val singularities are essential in this work we list all of them in the Appendix. We will see Kodaira’s classification of degenerations of elliptic curves in the Appendix.

3. Normal Gorenstein del Pezzo surfaces of degree 1 with canonical singularities

Let $S$ be a normal Gorenstein del Pezzo surface of degree 1 with canonical singularities. And let $\pi : \tilde{S} \to S$ be the minimal resolution of $S$. Then the smooth surface $\tilde{S}$ is usually called a weak del Pezzo surface because $-K_{\tilde{S}}$ is nef and big. In the present section we study normal Gorenstein del Pezzo surfaces of degree 1 with canonical singularities via smooth weak del Pezzo surfaces. These surfaces were investigated in [3], [4], [5], [11], [14], and [15]. In our study we pay more attention to their effective anti canonical divisors.

Lemma 3.1. Any element of $| - K_S |$ is reduced and irreducible.

Proof. Let $D = \sum a_i D_i$, where each $D_i$ is a prime divisor. Since the surface $S$ is Gorenstein, each $a_i$ is a nonnegative integer. Then

$$1 = D \cdot (-K_S) = \pi^*(D) \cdot \pi^*(-K_S) = \sum a_i \tilde{D}_i \cdot (-K_{\tilde{S}}) \geq \sum a_i,$$

where $\tilde{D}_i$ is the strict transform of $D_i$ via $\pi$. Note that no $\tilde{D}_i$ is a $-2$-curve. Therefore, $D$ is reduced and irreducible. Q.E.D.

Lemma 3.2. Let $H$ be an element of $| - K_{\tilde{S}} |$. And let $\Gamma$ be a fundamental cycle of the minimal resolution $\pi : \tilde{S} \to S$. If $H$ contains a point of $\Gamma$, then $H = \tilde{D} + \Gamma$, where $\tilde{D}$ is a $-1$-curve.

Proof. See [3, p.53, Corollaire 2]. Q.E.D.

Let $D$ be an element of $| - K_S |$. We consider the pull-back $\pi^*(D)$ of $D$ via $\pi$. Then we may write

$$\pi^*(D) = \tilde{D} + E,$$

where $\tilde{D}$ is the strict transform of $D$ and $E$ consists of $-2$-curves.

Theorem 3.3. Let $p$ be a singular point of $S$. Suppose that $D$ passes through the point $p$. Then the point $p$ is of type $A_n, n \geq 3 \ (D_3, E_6, E_7, E_8, \text{ resp.})$ if and only if the log canonical threshold of $D$ with respect to $K_S$ is 1 (\(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \text{ resp.}\)). Moreover, the divisor $D$ has a node at $p$ if and only if $K_S + D$ is log canonical.

Proof. Since the pull-back $\pi^*(D)$ of $D$ belongs to $| - K_{\tilde{S}} |$, Lemma 3.2 implies

$$\pi^*(K_S + cD) = K_{\tilde{S}} + c\tilde{D} + c\Gamma,$$
where \( c \) is a constant and \( \Gamma \) is the fundamental cycle on \( \tilde{S} \) associated to the point \( p \). Therefore, it is enough to consider the log canonical threshold of \( \tilde{D} + \Gamma \) with respect to \( K_{\tilde{S}} \).

**Claim.** \( \tilde{D} + \text{supp}(\Gamma) \) is a simple normal crossing divisor.

Let \( \Gamma = \sum_{i=1}^{n} a_i E_i \), where each \( E_i \) is a \(-2\)-curve. Observe the equations

\[
0 = \pi^*(D) \cdot E_j = \tilde{D} \cdot E_j + \sum_{i=1}^{n} a_i E_i \cdot E_j, \quad j = 1, \ldots, n.
\]

Then we obtain

\[
M \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix} = - \begin{pmatrix} \tilde{D} \cdot E_1 \\ \tilde{D} \cdot E_2 \\ \vdots \\ \tilde{D} \cdot E_{n-1} \\ \tilde{D} \cdot E_n \end{pmatrix},
\]

where \( M = (E_i \cdot E_j) \). Looking at the fundamental cycles case by case (see the Appendix at the end of the paper), we then see that the claim is true. Moreover the configuration of the effective anticanonical divisor \( \pi^*(D) \) has a perfect match to one of Kodaira’s singular elliptic fibers, \( \tilde{E}_8(II^*) \), \( \tilde{E}_7(III^*) \), \( \tilde{E}_6(IV^*) \), \( \tilde{D}_n(\text{I}^*_{m-4}) \), and \( \tilde{A}_n(\text{I}_{n+1}) \), \( n \geq 3 \).

Since \( \tilde{D} \) is a smooth curve, the first statement immediately follows from the claim.

The Appendix shows that the effective anticanonical divisor \( \pi^*(D) \) is a wheel if and only if the point \( p \) is a singularity of type \( A_n \). This implies the second statement. \( \text{Q.E.D.} \)

**Theorem 3.4.** *Notation as in Theorem 3.3.*

1. The point \( p \) is of type \( A_1 \) (\( A_2 \) resp.) and \( D \) has a cusp at \( p \) if and only if the log canonical threshold of \( D \) is \( \frac{2}{3} \left( \frac{3}{5} \right) \) resp.).

2. If the point \( p \) is of type either \( A_1 \) or \( A_2 \) and \( K_S + D \) is log canonical, then \( D \) has a node at the point \( p \).

**Proof.** The proof is similar to that of Theorem 3.3. We prove the case corresponding to \( A_1 \). Note that \( \tilde{D} \) can meet \( \Gamma \) transversally or tangentially with intersection number 2. If \( \tilde{D} \) meets \( \Gamma \) transversally, then \( K_S + D \) is log canonical and \( D \) has a node at the point \( p \). If \( \tilde{D} \) meets \( \Gamma \) tangentially, then we can easily check that the log canonical threshold of \( \tilde{D} \) with respect to \( K_{\tilde{S}} \) is \( \frac{2}{3} \). And \( D \) has a cusp at the point \( p \).

In the case of \( A_2 \), it follows from the Appendix that \( \tilde{D} \) meets \( \Gamma = E_1 + E_2 \) at two distinct points with \( E_1 \cdot \tilde{D} = E_2 \cdot \tilde{D} = 1 \) or at single point with \( E_1 \cdot \tilde{D} = E_2 \cdot \tilde{D} = 1 \). The first gives us the log canonical threshold 1 and the latter provides the log canonical threshold with \( \frac{2}{3} \). \( \text{Q.E.D.} \)

**Remark.** In the proof of Theorem 3.4, we can also find perfect matches to Kodaira’s singular elliptic fibers. Namely, the singularity type \( A_1 \) corresponds to \( *\tilde{A}_1(\text{III}) \) or \( \tilde{A}_1(\text{I}_2) \) and the singularity type \( A_2 \) corresponds to \( *\tilde{A}_2(\text{IV}) \) or \( \tilde{A}_2(\text{I}_3) \).
Summarizing these results, we obtain perfect information on total log canonical thresholds of normal Gorenstein del Pezzo surfaces of degree 1 with canonical singularities. We can see a beautiful numerical connection between log canonical thresholds and Kodaira’s classification of degenerations of elliptic curves.

**Corollary 3.5.** Let $S$ be a normal Gorenstein del Pezzo surface of degree 1 with canonical singularity as before.

1. $\tilde{E}_8(\text{II}^*)$.
   \[ \text{tlct}(S) = \frac{1}{6} \] if and only if $S$ has an $E_8$ singularity.

2. $\tilde{E}_7(\text{III}^*)$.
   \[ \text{tlct}(S) = \frac{1}{4} \] if and only if $S$ has an $E_7$ singularity and no $E_8$ singularity.

3. $\tilde{E}_6(\text{IV}^*)$.
   \[ \text{tlct}(S) = \frac{1}{3} \] if and only if $S$ has an $E_6$ but neither $E_7$ nor $E_8$.

4. $D_n(\text{I}^n - 4)$, $4 \leq n \leq 8$.
   \[ \text{tlct}(S) = \frac{1}{2} \] if and only if $S$ has a $D_n$ singularity and no exceptional type singularity.

5. $\tilde{A}_2(\text{IV})$.
   \[ \text{tlct}(S) = \frac{2}{3} \] if and only if $S$ has only $A_n$ type singularities and there is an element in $\mid -K_S \mid$ which has a cusp at an $A_2$ singularity of $S$.

6. $\tilde{A}_1(\text{III})$.
   \[ \text{tlct}(S) = \frac{2}{3} \] if and only if $S$ has only $A_n$ type singularities and $\mid -K_S \mid$ contains an element having a cusp at an $A_1$ singularity of $S$ but no element having a cusp at an $A_2$ singularity of $S$.

7. $\tilde{A}_0(\text{II})$.
   \[ \text{tlct}(S) = \frac{5}{6} \] if and only if $S$ has only $A_n$ type singularities and $\mid -K_S \mid$ contains an element having a cusp but no element having a cusp at a singular point of $S$.

8. $\tilde{A}_n(\text{I}_n + 1)$, $n \leq 8$.
   \[ \text{tlct}(S) = 1 \] if and only if $S$ has only $A_n$ type singularities and there is no cuspidal rational curve in $\mid -K_S \mid$.

**Proof.** Let $D$ be an effective anticanonical divisor of $S$. By Lemma 3.1, $D$ is irreducible and reduced. If $D$ passes through a singular point of $S$, then the log canonical threshold of $D$ should obey the rules in Theorems 3.3 and 3.4.

Suppose that $D$ does not pass through any singular point of $S$. We can easily check that the log canonical threshold of $D$ is either 1 or $\frac{5}{6}$. Moreover, $D$ has a cusp if and only if the log canonical threshold of $D$ is $\frac{5}{6}$. Q.E.D.

**Corollary 3.6.** Every effective anticanonical divisor of $S$ passing through a singular point of type $D_n$, $E_6$, $E_7$, or $E_8$ has a cusp at the singular point. If it passes through a singular point of type $A_n$, $n \geq 1$, then it has either a node or a cusp at the singular point.
**Proof.** Let \( D \) be an effective anticanonical divisor of \( S \) passing through a singular point. Note that \( D \) is a Cartier divisor. It then follows from Inversion of adjunction \( [8] \) that \( D \) cannot be smooth. The arithmetic genus of \( D \) is 1 while that of the strict transform of \( D \) via \( \pi \) is 0. The configuration of \( \pi^*(D) \) then implies the results. Q.E.D.

Slightly changing the focus, we look at the number of singularities on the surface \( S \) when it has an exceptional singularity. This observation will later give some remarks on del Pezzo fibrations. The following proposition can be easily derived from \([9]\).

**Proposition 3.7.** Let \( p \) be a singular point of \( S \).

1. The point \( p \) is neither \( A_n \) nor \( D_n \), \( n \geq 9 \).
2. If \( p \) is of type \( A_8 \), \( D_8 \), or \( E_8 \), then it is a unique singular point on \( S \).
3. If \( p \) is of type \( A_7 \), \( D_7 \), or \( E_7 \), then the surface \( S \) has at most two singular points. Moreover, the possible extra singularity is of type \( A_1 \).
4. If \( p \) is of type \( E_6 \), then the surface \( S \) has at most two singular points. The possible extra singularity is of type either \( A_1 \) or \( A_2 \).

**Proof.** Since the minimal resolution \( \tilde{S} \) of the surface \( S \) has degree 1, the rank of \( \tilde{S} \) is 9. Therefore, the number of \(-2\)-curves is at most 8. This fact implies the first three statements. For the last statement, we have to show that the surface \( S \) cannot have the three singularities \( E_6 \), \( A_1 \) and \( A_1 \) at the same time. Because the rank of \( S \) is 1, this can be verified by the classification of singularities on normal Gorenstein del Pezzo surfaces of rank 1 in \([9]\). Q.E.D.

### 4. Proofs of Theorems A, B, and C

For the readers’ convenience we state the result in \([11]\) which is the main method for our proofs.

**Theorem 4.1.** Notation is as in Theorem A. If the del Pezzo fibrations \( X/T \) and \( Y/T \) satisfy the conditions below, then \( f \) is biregular.

- **(Special fiber condition)**
  The special fiber \( S_X \) (\( S_Y \) resp.) is reduced and irreducible. And the log pair \((X, S_X)\) (\( S_Y \) resp.) is purely log terminal.

- **(1-complement condition)**
  For any \( C \in \{-K_{S_X}\} (\{-K_{S_Y}\}, \text{resp.}) \), there exists 1-complement \( K_{S_X} + C_X \) (\( K_{S_Y} + C_Y \) resp.) of \( K_{S_X} \) (\( K_{S_Y} \), resp.) such that \( C_X \) (\( C_Y \), resp.) does not contain any center of log canonicity of \( K_{S_X} + C \) (\( K_{S_Y} + C \) resp.).

- **(Surjectivity condition)**
  Any 1-complement of \( K_{S_X} \) (\( K_{S_Y} \), resp.) can be extended to a 1-complement of \( K_X + S_X \) (\( K_Y + S_Y \), resp.).

- **(Total lc threshold condition)**
  The inequality \( t\text{lct}(S_X) + t\text{lct}(S_Y) > 1 \) holds.
Proof. See [11] Q.E.D.

In our situation, the first three conditions are satisfied (see [11]). The main recipe for the proofs of Theorems A, B, and C is the Total lc threshold condition.

Proof of Theorem A. Suppose that the birational map \( f \) is not biregular. Then \( \text{tlct}(S_X) + \text{tlct}(S_Y) \leq 1 \) by the Total lc threshold condition. We also obtain \( \text{tlct}(S_X) \geq \frac{1}{6} \) from Corollary 3.3. Hence, applying Corollary 3.5 and Proposition 3.7 to the special fiber \( S_Y \) completes the proof of the first assertion.

In order to prove the second assertion, it is enough to see that \( \text{tlct}(S_X) = 1 \) and \( \text{tlct}(S_Y) \geq \frac{1}{6} \) under the conditions. Q.E.D.

Proof of Theorem B. For the first statement suppose there is no cuspidal rational curve in \( | - K_{S_X} | \). We then obtain \( \text{tlct}(S_X) = 1 \). Since \( \text{tlct}(S_Y) \) is always positive, this is a contradiction.

In the second case, Corollary 3.3 implies \( \text{tlct}(S_X) = \frac{2}{3} \) because we assumed that \( f \) is not biregular. Therefore, \( \text{tlct}(S_Y) \) must be \( \frac{1}{6} \). The result then follows from Corollary 3.3.

As for the third statement, we see that the conditions force \( \text{tlct}(S_X) \) to be \( \frac{3}{4} \). Therefore \( \text{tlct}(S_Y) \leq \frac{1}{4} \). Then Corollary 3.3 shows the result.

In the last statement, the conditions give us \( \text{tlct}(S_X) = \frac{2}{3} \). Thus the statement follows from Corollary 3.5. Q.E.D.

Remark. In the statements 3 and 4, we see more than the existence of a certain type of singularity on the special fiber \( S_Y \). To be precise, Proposition 3.7 states the number and types of singularities on \( S_Y \) as in the case of \( E_8 \).

Proof of Theorem C. Noting Proposition 3.7, we see that the conditions imply \( \text{tlct}(S_Y) = \frac{1}{3} \) (resp.). Therefore \( \text{tlct}(S_X) \leq \frac{2}{3} \) (resp.). Since \( S_X \) has only singularities of type \( A_n \), the result follows from Corollary 3.3. Q.E.D.

Appendix

Each diagram represents the fundamental cycle associated to the given singularity type. These fundamental cycles can be easily derived by simple computation. We may refer to [12] or [1]. In each matrix equation the column vector of the left hand side represents the multiplicities of the −2-curves of the fundamental cycle in suitable order.

1. \( A_n, n \geq 1 \)

\[
\begin{pmatrix}
-2 & 1 & 0 & \cdots & 0 & 0 \\
1 & -2 & 1 & 0 & \cdots & 0 \\
0 & 1 & -2 & 1 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 1 & -2 & 1 \\
0 & 0 & \cdots & 0 & 1 & -2 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
0 \\
\vdots \\
1 \\
1 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
1 \\
0 \\
\vdots \\
1 \\
1 \\
\end{pmatrix}
\]
where $n \geq 2$ for the matrix equation.

2. $D_n$, $n \geq 4$

3. $E_6$

4. $E_7$

5. $E_8$
\[
\begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\times
\begin{pmatrix}
2 \\
3 \\
4 \\
5 \\
6 \\
4 \\
2 \\
3 \\
\end{pmatrix}
=
\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]

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