A DENSITY PROBLEM FOR SOBOLEV SPACES ON PLANAR DOMAINS

PEKKA KOSKELA AND YI RU-YA ZHANG

Abstract. We prove that for a bounded simply connected domain \( \Omega \subset \mathbb{R}^2 \), the Sobolev space \( W^{1,\infty}(\Omega) \) is dense in \( W^{1,p}(\Omega) \) for any \( 1 \leq p < \infty \). Moreover, we show that if \( \Omega \) is Jordan, then \( C^\infty(\mathbb{R}^2) \) is dense in \( W^{1,p}(\Omega) \) for \( 1 \leq p < \infty \).

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1. Introduction

Let \( \Omega \subset \mathbb{R}^2 \) be a domain. We define the first order Sobolev space \( W^{1,p}(\Omega) \), \( 1 \leq p \leq \infty \), as the set
\[
\{ u \in L^p(\Omega) \mid \nabla u \in L^p(\Omega; \mathbb{R}^2) \}.
\]
Here \( \nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}) \) is the weak (or distributional) gradient of a locally integrable function \( u \). We equip \( W^{1,p}(\Omega) \) with the non-homogeneous norm:
\[
\| u \|_{W^{1,p}(\Omega)}^p = \int_\Omega |u(x)|^p \, dx + \int_\Omega |\nabla u(x)|^p \, dx
\]
for \( 1 \leq p < \infty \), and
\[
\| u \|_{W^{1,\infty}(\Omega)} = \operatorname{esssup}_{x \in \Omega} |u(x)| + \operatorname{esssup}_{x \in \Omega} |\nabla u(x)|.
\]

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For $1 \leq p < \infty$, it is well-known that smooth functions are dense in $W^{1,p}(\Omega)$ for any domain $\Omega \subset \mathbb{R}^2$. Consequently, if $\Omega$ is a $W^{1,p}$-extension domain, that is, there exists an extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$:

$$Eu(x) = u(x) \text{ for } x \in \Omega \quad \text{and} \quad Eu \in W^{1,p}(\mathbb{R}^2),$$

then we can use global smooth functions to approximate functions in $W^{1,p}(\Omega)$ with respect to $W^{1,p}(\Omega)$-norm. Indeed one extends $u \in W^{1,p}(\Omega)$ to $Eu \in W^{1,p}(\mathbb{R}^2)$, picks a sequence $v_j \in C^\infty(\mathbb{R}^2)$ approximating $Eu$ in $W^{1,p}(\mathbb{R}^2)$-norm and then restricts these $v_j$ to $\Omega$. Notice that Lipschitz domains are extension domains.

If one only wishes to approximate by functions that are smooth up to the boundary, then the Lipschitz condition can be relaxed. Indeed, if $\Omega$ satisfies a cone condition or the weaker segment condition, then $C^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)$. However, it is easy to construct domains $\Omega$ for which $C^\infty(\overline{\Omega})$ fails to be dense. For example, take $\Omega$ to be a slit disk: the unit disk minus a radius. For all this see e.g. [1].

A very different sufficient condition for the density of global smooth functions was given by J.L. Lewis in [9]. He proved that $C^\infty(\mathbb{R}^2)$ is dense in $W^{1,p}(\Omega)$ for every $1 < p < \infty$ when $\Omega$ is a Jordan domain: the bounded component of $\mathbb{R}^2 \setminus \gamma$, where $\gamma$ is a Jordan curve. Lewis’s approximation procedure is based on extending the restriction of the function in question, from a suitable level set, smoothly along the normal vector field of a fixed weak solution of the $q$–Laplace equation

$$\nabla \cdot (|\nabla u|^{q-2} \nabla u) = 0$$

in $\Omega'$, where $\frac{1}{q} + \frac{1}{p} = 1$ and $\Omega \subset \subset \Omega'$. In order to obtain the required estimates, he uses properties of solutions of these equations. This technique leaves the case $p = 1$ open.

Subsequently, W. Smith, A. Stanoyevitch and D.A. Stegenga showed in [11] that domains which satisfy their interior segment property allow approximation of functions in $W^{1,p}(\Omega)$ for $1 \leq p < \infty$ by bounded smooth functions with bounded derivatives and with global smooth functions if the boundary of $\Omega$ satisfies a suitable additional exterior density condition. Their interior segment property is weaker than the usual segment property that actually implies that the boundary is locally the graph of a continuous function. In [11] it was also inquired if the measure density together with lack of two-sided boundary points would suffice for the density of $C^\infty(\overline{\Omega})$, but C.J. Bishop [4] gave a counterexample to this statement.

More recently, A. Giacomini and P. Trebeschi established in [6] density results that especially yield the density of $W^{1,2}(\Omega)$ in $W^{1,p}(\Omega)$ for all $1 \leq p < 2$ when $\Omega$ is bounded and simply connected. They use the Helmholtz decomposition of $L^2(\Omega, \mathbb{R}^2)$ to characterize the orthonormal subspaces of certain Sobolev spaces. Thus only the density of $W^{1,2}(\Omega)$ can be obtained by this technique.

Based on the results above, it is natural to inquire if $W^{1,q}(\Omega)$ is always dense in $W^{1,p}(\Omega)$ for some $q > p$ when $\Omega$ is bounded and simply connected and if even global smooth functions are dense in $W^{1,1}(\Omega)$ when $\Omega$ is Jordan. Our first result gives an even stronger conclusion for the the first problem.

**Theorem 1.1.** If $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain, then $W^{1,\infty}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for any $1 \leq p < \infty$. 

There are plenty of bounded simply connected non-Jordan domains that fail the interior segment condition and hence Theorem 1.1 is not covered by the results discussed above. Our proof of Theorem 1.1 is rather flexible. Especially, it allows us to solve the second problem posed above and to give a new proof for the aforementioned density result [9, Theorem 1] by J.L. Lewis; other consequences of our approach will be recorded in a subsequent paper.

**Corollary 1.2.** If $\Omega$ is a planar Jordan domain, then $C^\infty(\mathbb{R}^2)$ is dense in $W^{1,p}(\Omega)$ for any $1 \leq p < \infty$.

Theorem 1.1 and Corollary 1.2 also give consequences for $BV(\Omega)$, the collection of functions in $L^1(\Omega)$ with bounded variation. Indeed, given $u \in BV(\Omega)$ one always has a sequence of functions $u_j \in W^{1,1}(\Omega)$ that converges to $u$ in $L^1(\Omega)$ and so that the $BV$-energy of $u$, $||Du||(\Omega)$, satisfies

$$||Du||(\Omega) = \lim_{j} ||\nabla u||_{L^1(\Omega)}.$$ 

Based on Theorem 1.1 and Corollary 1.2 we may further assume that $u_j \in W^{1,\infty}(\Omega)$ when $\Omega$ is bounded and simply connected and even that each $u_j$ is the restriction of a global smooth function when $\Omega$ is Jordan. For the theory of $BV$-functions we refer the reader to [2].

The research of this paper has been partially motivated by our attempts to give geometric characterizations for bounded simply connected $W^{1,p}$-extension domains. Indeed, our solution for the case $1 < p < 2$ in [7] uses Lewis's result (Corollary 1.2 for $p > 1$). For the case $p = 1$ we need both Theorem 1.1 and Corollary 1.2, see [8].

The paper is organized as follows. Section 2 contains some preliminaries. We give a decomposition of a bounded simply connected planar domain $\Omega$ and the corresponding partition of unity in Section 3. Section 4 contains the proofs of Theorem 1.1 and Corollary 1.2.

The notation in this paper is quite standard. For example, $C(\cdot)$ refers to a constant that may depend on the given parameters. As usual, the value $C(\cdot)$ may vary between appearances, even within a chain of inequalities. By $a \sim b$ we mean that $b/C \leq a \leq Cb$ for some constant $C \geq 1$. If we need to make the dependence of this constant on the parameters $\cdot$ explicit, we write $a \sim_{(\cdot)} b$. Also $a \lesssim b$ means $a \leq Cb$ with $C \geq 1$, and $a \gtrsim b$ has the analogous meaning.

The Euclidean distance between the sets $A, B \subset \mathbb{R}^n$ is denoted by dist $(A, B)$. We denote by $\ell(\gamma)$ the length of a curve $\gamma$. The Euclidean disk centered at $x$ and with radius $r$ is referred to by $B(x, r)$, and $S^1(x, r)$ is the circle of radius $r$, centered at $x$. For a set $A \subset \mathbb{R}^n$, we refer to its interior by $A^0$, to the boundary by $\partial A$, and to the closure by $\overline{A}$. As usual, $A \subset \subset B$ means that $A$ is compactly contained in $B$.

**2. Preliminaries**

In this section, we introduce some necessary definitions and facts. To begin with, we recall the definition of a Whitney-type set.

**Definition 2.1.** A connected set $A \subset \Omega \subset \mathbb{R}^2$ is called a $\lambda$-Whitney-type set in $\Omega$ with some constant $\lambda \geq 1$ if the following holds.

(i) There exists a disk of radius $\frac{1}{\lambda} \text{diam} (A)$ contained in $A$;

(ii) $\frac{1}{\lambda} \text{diam} (A) \leq \text{dist} (A, \partial \Omega) \leq \lambda \text{diam} (A)$. 


We define the inner distance with respect to $\Omega$ between $x, y \in \Omega$ by
\[
dist_\Omega(x, y) = \inf_{\gamma \subset \Omega} \ell(\gamma),
\]
where the infimum runs over all curves joining $x$ and $y$ in $\Omega$. The inner diameter $\text{diam}_\Omega(E)$ of a set $E \subset \Omega$ is then defined in the usual way.

Let us recall some facts from complex analysis. First of all, recall that conformal maps preserve conformal capacities. More precisely, given a pair $E, F \subset \Omega \subset \mathbb{R}^2$ of continua, define the conformal capacity between $E$ and $F$ in $\Omega$ as
\[
\text{Cap}(E, F, \Omega) = \inf\{\|u\|^2_{W^{1,2}(\Omega)} \mid u \in \Delta(E, F)\},
\]
where $\Delta(E, F)$ denotes the class of all functions $u \in W^{1,2}_{\text{loc}}(\Omega)$, continuous in $E \cup F \cup \Omega$, such that $0 \leq u \leq 1$ on $\Omega$, $u = 1$ on $E$, and $u = 0$ on $F$. By definition, it is clear that if $\tilde{E}, \tilde{F} \subset \tilde{\Omega}$, $E \subset \tilde{E}$, $F \subset \tilde{F}$ and $\Omega \subset \tilde{\Omega}$, then
\[
\text{Cap}(E, F, \Omega) \leq \text{Cap}(\tilde{E}, \tilde{F}, \tilde{\Omega}).
\]
Furthermore if $E, F \subset \mathbb{D}$ are two continua, then
\[
\frac{\min\{\text{diam}(E), \text{diam}(F)\}}{\dist(E, F)} \geq \delta > 0 \implies \text{Cap}(E, F, \mathbb{D}) \geq C(\delta) > 0. \tag{2.1}
\]
Moreover, when $\overline{B(x, r)} \subset B(x, R) \subset \subset \Omega$, we have
\[
\text{Cap}(\overline{B(x, r)}, S^1(x, R), \Omega) \sim \log \left(\frac{R}{r}\right)^{-1}. \tag{2.2}
\]
See [12] for more details. Actually, [12] states these results for “modulus”, but “modulus” is equivalent to conformal capacity (see e.g. [10, Proposition 10.2, Page 54]).

We need the following lemma.

**Lemma 2.2.** Let $E, F \subset \Omega$ be a pair of continua. Then if $\text{Cap}(E, F, \Omega) \geq c_0$, we have
\[
\min\{\text{diam}_\Omega(E), \text{diam}_\Omega(F)\} \gtrsim \dist_\Omega(E, F),
\]
where the constant only depends on $c_0$.

**Proof.** We may assume that $\text{diam}_\Omega(E) \leq \text{diam}_\Omega(F)$ and $2\text{diam}_\Omega(E) \leq \dist_\Omega(E, F)$. Let $z \in E$, and $\frac{\dist_\Omega(E, F)}{\text{diam}_\Omega(E)} = \delta$. We define
\[
f(x) = \begin{cases} 
1, & \text{if } \dist_\Omega(x, z) \leq \text{diam}_\Omega(E) \\
0, & \text{if } \text{dist}_\Omega(x, z) \geq \text{dist}_\Omega(E, F) \\
\frac{\log(\dist_\Omega(E, F) - \log(\text{diam}_\Omega(E, z))}{\log(\dist_\Omega(E, F) - \log(\text{diam}_\Omega(E))}, & \text{otherwise}
\end{cases}
\]

Then a direct calculation via a dyadic annular decomposition with respect to the inner distance gives
\[
c_0 \leq \int_\Omega |\nabla f|^2 \, dx \lesssim \log(\delta)^{-1}.
\]
Hence $\delta \lesssim e^{c_0}$, which means that $\dist_\Omega(E, F) \lesssim \text{diam}_\Omega(E)$. \qed
Recall that hyperbolic geodesics in \( \mathbb{D} \) are arcs of (generalized) circles that intersect the unit circle orthogonally. Moreover, both the hyperbolic metric and hyperbolic geodesics are preserved under conformal maps; see [3, Page 37] for instance. We refer to the hyperbolic distance between a pair of points \( x, y \) in a simply connected planar domain by \( \text{dist}_h(x, y) \).

The following lemma states a distortion property of conformal maps.

**Lemma 2.3** ([3], Theorem 2.10.8). Suppose \( \varphi \) is conformal in the unit disk \( \mathbb{D} \) and \( z, w \in \mathbb{D} \). Then

\[
\exp (-3 \text{dist}_h(z, w))|\varphi'(w)| \leq |\varphi'(z)| \leq \exp (3 \text{dist}_h(z, w))|\varphi'(w)|.
\]

Given a \( \lambda \)-Whitney type set \( A \subset \mathbb{D} \), one has that \( \text{dist}_h(z, w) \leq C(\lambda) \) for all \( z, w \in A \), Hence \( |\varphi'(w)| \sim |\varphi'(z)| \) with a constant only depending on \( \lambda \).

By Lemma 2.3, condition (2.1) and the capacity estimate (2.2), one can verify the following well-known result.

**Lemma 2.4.** Suppose \( \varphi: \Omega \to \Omega' \) is conformal, where \( \Omega = \mathbb{D} \) or \( \Omega' = \mathbb{D} \), and \( Q \subset \Omega \) is a \( \lambda_1 \)-Whitney-type set. Then \( \varphi(Q) \subset \Omega' \) is a \( \lambda_2 \)-Whitney-type set with \( \lambda_2 = \lambda_2(\lambda_1) \).

In the sequel, we often omit the constant \( \lambda \) when we are dealing with a fixed \( \lambda \).

Hyperbolic geodesics have the following important property, often called the Gehring-Hayman inequality.

**Lemma 2.5** ([5]). Let \( \varphi: \mathbb{D} \to \Omega \) be a conformal map. Then for any two points \( x, y \in \mathbb{D} \), denoting the corresponding hyperbolic geodesic in \( \Omega \) by \( \gamma_{x,y} \), and by \( \omega_{x,y} \) any Jordan curve connecting \( x \) and \( y \) in \( \mathbb{D} \), we have

\[
\ell(\varphi(\gamma_{x,y})) \leq C \ell(\varphi(\omega_{x,y})),
\]

where \( C \) is an absolute constant.

Finally, let us recall that bounded smooth functions are dense in \( W^{1,p}(\Omega) \).

**Lemma 2.6.** For any \( 1 \leq p < \infty \), it holds that \( L^\infty(\Omega) \cap C^\infty(\Omega) \) is dense in \( W^{1,p}(\Omega) \) for any domain \( \Omega \subset \mathbb{R}^2 \).

**Proof.** Fix \( v \in W^{1,p}(\Omega) \). Let

\[
v_m = \max\{\min\{v(x), m\}, -m\}.
\]

One can easily check by the absolute continuity of integral that this sequence converges to \( v \) in the Sobolev norm. The claim follows by a standard partition of unity and mollification procedure applied to the functions \( v_m \).

\[ \Box \]

3. Decomposition and partition of unity

3.1. Decomposition of the core of \( \Omega \). Fix a bounded simply connected domain \( \Omega \subset \mathbb{R}^2 \), and consider a conformal map \( \varphi: \mathbb{D} \to \Omega \). For \( l \geq 1 \), let

\[
A_l = \overline{B}(0, 1 - 2^{-l-1}) \setminus B(0, 1 - 2^{-l}).
\]

We define the radial ray \( r_\theta \) as the line segment between the origin and the point \( e^{i\theta} \). For each \( l \in \mathbb{N}, 0 \leq j \leq 2^{l+1} - 1 \) and \( \theta_{l,j} = j2^{-l}\pi \), the collection of radial rays \( r_{\theta_{l,j}} \) cut \( A_l \) into \( 2^{l+1} \)
sets $Q_{l,j}$ labeled counterclockwise respect to $j$ starting from the positive real axis. Moreover, define $A_0 = B(0, 1)$, $Q_{0,0} = \{(x_1, x_2) \in A_0 \mid x_2 \geq 0\}$, and let $Q_{0,1} = \{(x_1, x_2) \in A_0 \mid x_2 \leq 0\}$. By abuse of notation, we sometimes refer also to the closures of the sets $Q_{l,j}$ by $Q_{l,j}$. Notice that all these sets are of Whitney-type.

For $m \geq 2$, set
$$
\Omega_m = \varphi(B(0, 1 - 2^{-m-1})),
$$
and
$$
D_m = \varphi(A_m) \subset \Omega_m.
$$
For $0 \leq j \leq 2^{m+1} - 1$, by Lemma 2.4 the induced set $R_j = \varphi(Q_{m,j}) \subset \Omega$ is also a Whitney type set for $\Omega$. These sets form a decomposition of $D_m$. Apparently the set $R_j$ depends on $m$, but for notational convenience we suppress this.

3.2. Decomposition of the boundary layer of $\Omega$. Now let us decompose $J_m = \Omega \setminus \Omega_m$.

Our aim is to decompose $J_m$ into connected sets such that, for each of them the length of its boundary inside $\Omega$ is controlled, and the distance between any two sets is relatively far if they have no intersection.

To be specific, for each $0 \leq j \leq 2^{m+1} - 1$, define $\beta_j$ to be the shorter arc of
$$
S^1(0, 1 - 2^{-m-1}) \setminus (r_{(2j+1)2^{-m-1}} \cup r_{(j+1)2^{-m}}),
$$
so that
$$
2^{-m-2}\pi \leq \text{dist}(\beta_j, \beta_{j+1}) \leq 2^{-m+1}\pi.
$$
We claim that there exists a hyperbolic geodesic $\gamma^n_j$ connecting $\varphi(\beta_j) \subset \partial R_j$ and $\varphi(\delta^n_j)$ such that $\ell(\gamma^n_j) \lesssim \text{diam}(R_j)$, where each $\delta^n_j$ is the shorter arc of
$$
S^1(0, 1 - 2^{-m-n}) \setminus (r_{(2j+1)2^{-m-1}} \cup r_{(j+1)2^{-m}})
$$
for $n \geq 3$. Observe that $\text{diam}(\varphi(\beta_j)) \sim \text{diam}(R_j) \sim \text{diam}_\Omega(R_j)$ by Lemma 2.3.

Notice that $\text{Cap}(\beta_j, \delta^n_j, \mathbb{D})$ is bounded away from zero by an absolute constant according to (2.1), and hence
$$
\text{Cap}(\varphi(\beta_j), \varphi(\delta^n_j), \Omega) \gtrsim 1.
$$
By Lemma 2.2, we conclude that $\text{dist}_\Omega(\varphi(\beta_j), \varphi(\delta^n_j)) \lesssim \text{diam}_\Omega(R_j)$. The existence of a suitable $\gamma_j^n$ follows by Lemma 2.5.

Parameterize each $\gamma^n_j$ by arclength. Notice that the lengths of $\gamma^n_j$ are uniformly bounded from above by a multiplicative constant times $\text{diam}(R_j)$. Letting $n \to \infty$, by Arzelá-Ascoli lemma, we obtain a curve $\gamma_j$ connecting $R_j$ and the boundary $\partial \Omega$ with $\ell(\gamma_j) \lesssim \text{diam}(R_j)$.

Moreover by Lemma 2.3 for any $0 \leq j \leq 2^{m+1} - 1$, $\text{diam}_\Omega(R_j) \sim \text{diam}_\Omega(R_{j+1})$, where we define $R_{2^{m+1}} = R_0$. Thus by the triangle inequality and Lemma 2.3

$$\text{dist}_\Omega(\gamma^n_j, \gamma^n_{j+1}) \lesssim \text{diam}(R_j),$$

(3.2)

Additionally, we claim that

$$\text{dist}_\Omega(\gamma^n_j, \gamma^n_{j+1}) \gtrsim \text{diam}(R_j).$$

(3.3)

Indeed, first of all, by construction and (3.1) we know that $\gamma^n_j \cap \gamma^n_{j+1} = \emptyset$. Consider a curve $\alpha \subset \Omega$ of length at most $2 \text{dist}_\Omega(\gamma^n_j, \gamma^n_{j+1})$, joining $\gamma^n_j, \gamma^n_{j+1}$ in $\Omega$.

If $16 \text{dist}(\varphi^{-1}(\alpha), Q_{m,j}) \leq \text{diam}(Q_{m,j})$, then there is a subarc $\alpha' \subset \alpha$ such that $\ell(\varphi^{-1}(\alpha')) \sim \text{diam}(Q_{m,j})$ and $\text{dist}(\varphi^{-1}(\alpha'), Q_{m,j}) \leq \frac{1}{8} \text{diam}(Q_{m,j})$. Then by Lemma 2.3 and (3.1) one concludes that

$$\text{diam}(R_j) \lesssim \text{diam}(\alpha') \leq \ell(\alpha') \leq \ell(\alpha) \sim \text{dist}_\Omega(\gamma^n_j, \gamma^n_{j+1}).$$

For the other case where $16 \text{dist}(\varphi^{-1}(\alpha), Q_{m,j}) \geq \text{diam}(Q_{m,j})$, observe that

$$\text{Cap}(\varphi^{-1}(\alpha), Q_{m,j}, \Omega) \gtrsim 1,$$

and by Lemma 2.3 $\text{dist}(\alpha, R_j) \gtrsim \text{diam}(R_j)$. Hence by Lemma 2.2 we conclude that

$$\text{diam}(R_j) \lesssim \text{diam}(\alpha) \leq \ell(\alpha) \sim \text{dist}_\Omega(\gamma^n_j, \gamma^n_{j+1}).$$

Consequently we obtain the claim. Combining (3.2) and (3.3) results in

$$\text{dist}_\Omega(\gamma^n_j, \gamma^n_{j+1}) \sim \text{diam}(R_j),$$

and finally in

$$\text{dist}_\Omega(\gamma_j, \gamma_{j+1}) \sim \text{diam}(R_j),$$

(3.4)

by letting $n \to \infty$.

Denote by $S_j$ the relatively closed subset of $\Omega$ enclosed by $\partial \Omega$, $\partial \Omega_m$, $\gamma_j$ and $\gamma_{j+1}$. Then $J_m = \cup_j S_j$ and $|S_i \cap S_j| = 0$ for any $i \neq j$, where $|A|$ refers to the Lebesgue measure of a set $A$. Thus the sets $S_j$, modulo sets of measure zero, give us a decomposition of $J_m$.

Furthermore, based on (3.4), we claim that

$$\text{dist}_\Omega(S_i, S_j) \gtrsim \max\{\text{diam}(R_i), \text{diam}(R_j)\} \text{ if } R_i \cap R_j = \emptyset,$$

(3.5)

with a constant independent of $\Omega$ and $m$. Indeed any curve $\gamma \subset \Omega$ joining $S_i$ and $S_j$ must pass through the neighbors of $S_i$, namely $R_i \cup R_{i+1} \cup R_{i+2} \cup S_{i-1} \cup S_{i+1}$. A similar conclusion holds also for $S_j$ and its neighbors. Then the desired claim is given by (3.4), the definition of the sets $R_i$, $R_{i-1}$ and $R_{i+1}$ and Lemma 2.3.
3.3. A partition of unity associated to the decomposition. Next we construct a partition of unity related to the decomposition above. Recall that
\[ \Omega = \Omega_m \cup J_m, \]
\[ \Omega_m = \Omega_{m-1} \cup D_m, \]
and for \( D_m \subset \Omega_m \) and \( J_m \), we have \( D_m = \cup_j R_j \) and \( J_m = \cup_j S_j \) respectively.

For \( \Omega_m \), we define a Lipschitz function \( \psi \) in \( \Omega \) such that \( \psi \) is compactly supported in \( \Omega_m \), \( 0 \leq \psi \leq 1 \), \( \psi(x) = 1 \) if \( x \in \Omega_{m-1} \), and \( |\nabla \psi(x)| \lesssim (\text{diam}(R_j))^{-1} \) if \( x \in R_j \), with a constant independent of \( m, j \). This function can be given via the distance function by letting
\[ \psi(x) = \min \left\{ 1, \frac{c_1 \text{dist}(x, J_m)}{\text{dist}(x, \partial \Omega)} \right\}, \]
where the value of \( c_1 \) will be fixed momentarily.

Indeed, \( \psi \) is Lipschitz and, by Leibniz’s rule, for \( x \in R_j \)
\[ |\nabla \psi| \lesssim \frac{1}{\text{dist}(x, \partial \Omega)} + \frac{\text{dist}(x, J_m)}{(\text{dist}(x, \partial \Omega))^2} \lesssim \frac{1}{\text{diam}(R_j)}, \]
where we applied the fact that \( \text{diam}(R_j) \sim \text{dist}(R_j, \partial \Omega) \) for any \( j \). Since \( \psi \) vanishes in \( J_m \), we are left to obtain the correct boundary value on \( \Omega_{m-1} \). For this, notice that each \( x \in \partial \Omega_{m-1} \) belongs to \( R_j \) for some \( j \). Since \( R_j \) is a Whitney-type set,
\[ \text{dist}(x, \partial \Omega) \lesssim \text{diam}(R_j). \]

On the other hand, Lemma 2.3 guarantees that
\[ \text{diam}(R_j) \lesssim \text{dist}(x, J_m). \]
Hence there is a constant \( C_1 \), independent of \( m, j \) so that \( \psi = 1 \) on \( \Omega_{m-1} \) provided \( c_1 \geq C_1 \).

For each \( S_j \), we choose a locally Lipschitz continuous function \( \phi_j \) defined in \( \Omega \) such that the support of \( \phi_j \) is relatively closed in \( \Omega \) and contained in \( c_2 S_j \), \( 0 \leq \phi_j \leq 1 \), \( \phi_j(x) = 1 \) if \( x \in S_j \), and \( |\nabla \phi_j| \lesssim (\text{diam}(R_j))^{-1} \). Here the set \( c_2 S_j \) is defined as
\[ c_2 S_j = \{ x \in \Omega \mid \text{dist}_\Omega(x, S_j) \leq (c_2 - 1) \text{diam}(R_j) \}, \]
for some constant \( c_2 > 1 \) to be determined later. Indeed, we can simply set
\[ \phi_j(x) = \max\{1 - 2[(c_2 - 1) \text{diam}(R_j)]^{-1} \text{dist}_\Omega(x, S_j), 0\} \]
for \( x \in \Omega \).

Let us now choose \( c_2 \) small enough, so that (3.5) and Lemma 2.4 guarantee that
\[ c_2 S_i \cap c_2 S_j = \emptyset \quad \text{if} \quad S_i \cap S_j = \emptyset, \quad c_2 S_i \cap R_j = \emptyset \quad \text{if} \quad R_{i+1} \cap R_j = \emptyset \quad (3.6) \]
and
\[ c_2 S_i \cap \Omega_{m-1} = \emptyset \quad (3.7) \]
for each \( i \).

Towards obtaining a partition of unity, we wish now to choose \( c_1 \) large enough so that \( \psi(x) + \phi_j(x) \geq 1 \) for each \( x \in R_j \). Notice that Lemma 2.3 gives us a constant \( C_2 \) and the fact that \( R_j \) is a Whitney type set a constant \( C_3 \) so that
\[ \text{dist}(x, S_{j-1} \cup S_j) \leq C_2 \text{dist}(x, J_m) \]
and
\[ \text{dist} (x, \partial \Omega) \leq C_3 \text{diam} (R_j) \]
when \( x \in R_j \). Choosing
\[ c_1 = 2 \max \left\{ C_1, \frac{C_2 C_3}{c_2 - 1} \right\} \]
does the job; then \( \phi_{j-1}(x) + \phi_j(x) \geq \frac{1}{4} \) if \( x \in R_j \) and \( \psi(x) \leq \frac{1}{2} \).

We conclude that \( \Phi(x) := \psi(x) + \sum_j \phi_j(x) \geq 1/4 \) for each \( x \in \Omega \) and hence we obtain the desired partition of unity by dividing \( \varphi \) and each \( \phi_j \) by \( \Phi \) in \( \Omega \). By our construction, the new functions that we still denote \( \psi \) and \( \phi_i \) for convenience satisfy the same gradient bounds as the original ones, up to a multiplicative constant.

4. Proof of Theorem 1.1 and Corollary 1.2

The proof of Theorem 1.1 is based on approximating our given function \( u \) via a weighted sum of the functions in the partition of unity from the previous section. Towards this end, for \( m \) to be fixed later and the associated indices \( j \), define
\[ a_j = \int_{\varphi^{-1}(R_j)} u \circ \varphi \, dx. \]
Then \( a_j \) is the average of \( u \circ \varphi \) over \( Q_{m,j} = \varphi^{-1}(R_j) \). Recall here our notation from Section 2.

We need the following technical result.

**Lemma 4.1.** For \( u \in W^{1,p}(\Omega) \) and \( R_i, R_{i+1} \subset \Omega \) defined in Section 3, we have
\[ |a_j - a_{j+1}| \lesssim (\text{diam} (R_j))^{p-2} \int_{R_j \cup R_{j+1}} |\nabla u|^p \, dx \]
and
\[ \int_{R_j} |u - a_j|^p \, dx \lesssim (\text{diam} (R_j))^p \int_{R_j} |\nabla u|^p \, dx, \]
where the constant only depends on \( p \).

**Proof.** First of all by Lemma 2.3 we know that \( u \circ \varphi \in W^{1,p}_{\text{loc}}(\mathbb{D}) \). We apply the usual Poincaré inequality on the nice domain \( Q_{m,j} = \varphi^{-1}(R_j) \) to get
\[ \int_{Q_{m,j}} |u \circ \varphi - a_j|^p \, dx \lesssim (\text{diam} (Q_{m,j}))^p \int_{Q_{m,j}} |\nabla (u \circ \varphi)|^p \, dx. \]
Notice that \( J_\varphi(z) = |\varphi'(z)|^2 \) by conformality of \( \varphi \). Hence our second estimate follows via a change of variable by using chain rule and Lemma 2.3 according to which \( \varphi' \) is essentially constant on \( Q_{m,i} \).

The first inequality follows analogously, using now the Poincaré inequality over \( Q_{m,j} \cup Q_{m,j+1} \) and by adding and subtracting the average over \( Q_{m,j} \cup Q_{m,j+1} \). \( \square \)
Proof of Theorem 1.1. Fix $\epsilon > 0$. Also fix $u \in W^{1,p}(\Omega)$ for given $1 \leq p < \infty$. We may assume that $u$ is smooth and bounded because of Lemma 2.6. We may also require that $\|u\|_{L^\infty(\Omega)} = 1$.

For $m \in \mathbb{N}$ large enough
\[
\|u\|_{W^{1,p}(J_m \cup D_m)}^p \leq \epsilon \quad \text{and} \quad |J_m \cup D_m| \leq \epsilon. \quad (4.1)
\]

Notice that $u|_{\Omega_m} \in W^{1,\infty}(\Omega_m)$ since $\Omega_m$ is compact and $u$ is smooth.

We define a function $u_m$ on $\Omega$ by setting
\[
u_m(x) = u(x)\psi(x) + \sum_j a_j \phi_j(x),
\]
where $\psi(x)$ and $\phi_j(x)$ are the corresponding functions in the partition of unity from the previous section and $a_j$ is as in the beginning of this section.

It is obvious that $u_m \in W^{1,\infty}(\Omega)$ is locally Lipschitz by our construction, since we only have finitely many $R_j$ and the definition of our partition of unity gives the right estimates on the derivatives of of the functions in our partition of unity. Moreover we have $\|u_m\|_{L^\infty(\Omega)} \leq 1$ since $\|u\|_{L^\infty(\Omega)} = 1$, and hence
\[
\|u_m\|_{L^p(J_m \cup D_m)} \leq \epsilon.
\]

Consequently, since $c_2 S_j \cap \Omega_{m-1} = \emptyset$ for any $j$, we only need to check that
\[
\int_{J_m \cup D_m} |\nabla u_m|^p \, dx \lesssim \epsilon.
\]

This actually follows via the Poincaré inequality, Lemma 4.1. Indeed for any $R_i \subset D_m$ with the associated constant $a_i$, Lemma 4.1 and (3.6) give
\[
\int_{R_i} |\nabla u_m|^p \, dx \lesssim \int_{R_i} |\nabla (u_m - a_i)|^p \, dx
\]
\[
\lesssim \int_{R_i} |\nabla [(u(x) - a_i)\psi(x)]|^p \, dx + \sum_{S_j \subset J_m} \int_{R_i} |\nabla [(a_j - a_i)\phi_j(x)]|^p \, dx
\]
\[
\lesssim \int_{R_i} |\nabla u|^p + |u(x) - a_i|^p \text{diam} \,(R_i)^{-p} \, dx + \sum_{S_j \subset J_m} |a_i - a_j|^p \text{diam} \,(R_i)^{2-p}
\]
\[
\lesssim \int_{R_i} |\nabla u|^p \, dx + \sum_{S_j \subset J_m} \int_{R_i \cup R_j} |\nabla u|^p \, dx,
\]
where $R_j$ and $R_i$ are the corresponding Whitney-type sets contained in $D_m$ for $S_j$ and $S_i$, respectively.
Next for each $S_i$, by letting its associated constant to be $a_i$, by Lemma 4.1 \eqref{lemma4.1} and the definition of $\phi_j$, we get

$$\int_{S_i} |\nabla u_m|^p \, dx \lesssim \int_{S_i} |\nabla (u_m - a_i)|^p \, dx \lesssim \sum_{S_j \subset J_m \atop c_2 S_i \cap c_2 S_j \neq \emptyset} \int_{S_i} |\nabla [(a_i - a_j) \phi_j(x)]|^p \, dx \lesssim \sum_{S_j \subset J_m \atop c_2 S_i \cap c_2 S_j \neq \emptyset} |a_j - a_i|^p \text{diam} (R_i)^{2-p} \lesssim \sum_{S_j \subset J_m \atop c_2 S_i \cap c_2 S_j \neq \emptyset} \int_{R_i \cup R_j} |\nabla u|^p \, dx,$$

where $R_j$ and $R_i$ are still the corresponding Whitney-type sets contained in $D_m$ for $S_j$ and $S_i$, respectively.

Since all the sets $R_j$ and $c_2 S_j$ have uniformly finitely many overlaps, the desired estimate follows by summing over $i$. \hfill \square

Let $X$ and $Y$ be two non-empty subsets of $\mathbb{R}^n$. Define the Hausdorff distance $\text{dist}_H(X, Y)$ between them as

$$\text{dist}_H(X, Y) = \max \{ \sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y) \}.$$

We are ready to prove Corollary \ref{corollary1.2}.

**Proof of Corollary \ref{corollary1.2}**. For a given Jordan domain $\Omega \subset \mathbb{R}^2$ we can construct a sequence of Lipschitz domains $\{G_s\}_{s=1}^\infty$ approaching it in Hausdorff distance such that $\Omega \subset \subset G_{s+1} \subset \subset G_s$ for each $s \in \mathbb{N}$. For example, define $G_s$ by subtracting from $\mathbb{R}^2$ all the closed Whitney squares of the complementary domain of $\Omega$ whose sidelength is larger than $2^{-s}$.

Let us recall the proof of Theorem \ref{theorem1.1}. For a function $u \in L^\infty(\Omega) \cap C^\infty(\Omega) \cap W^{1,p}(\Omega)$, we first restricted it on $\Omega_m$ so that \eqref{condition4.1} is satisfied, where the corresponding sets $J_m$ and $D_m$ are defined in Section \ref{section3}. Then we extended the restricted function $u_m$ to each set $S_j$ as the integral average of $u$ on the corresponding set $R_j$. Next we ”glued” these pieces together by our partition of unity, such that the non-zero gradient of $u - u_m$ can only appear in the neighborhoods (with respect to the topology of $\Omega$) of $\partial \Omega_m$ and of the curves $\gamma_j$. We remind that \eqref{equality3.5} was crucial here.

Now let us return to Corollary \ref{corollary1.2}. Fix $u \in C^\infty(\Omega) \cap W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $\epsilon > 0$. When $m$ is large enough, we still truncate $u$ on $\Omega_m$ so that \eqref{condition4.1} holds.

First observe that, when $s$ is large enough, a Whitney-type set contained in $\Omega_m$ is still a Whitney-type set in $G_s$ up to a multiplicative constant, as the domains $G_s$ converge to $\Omega$ in Hausdorff distance. Especially, all the sets $R_j$ are still of Whitney-type.

We furthermore require that $\text{dist}_H(G_s, \Omega)$ is much smaller than the smallest value among $\{ \text{diam} (\gamma_j) \}_{j}$; notice that this is a finite collection. Then, if we extend the end point $z_j \in \partial \Omega$ of each $\gamma_j$ to one of the nearest points on $\partial G_s$, formulas similar to \eqref{equality3.4} and \eqref{equality3.5} still hold for the new curves and sets. Consequently a decomposition of $G_s$ with a corresponding partition of unity can also be constructed via the essence of Section \ref{section3}.
Thus an argument similar to the proof of Theorem 1.1 can be employed for $G_s$. Since $G_s$ is a Lipschitz domain, we can extend functions in $W^{1,\infty}(G_s)$ to global Lipschitz functions. Hence we get a sequence of global Lipschitz functions approximating $u$ in $W^{1,p}(\Omega)$-norm. Applying suitable mollifiers and following a standard diagonal argument, we obtain a sequence of global smooth functions as desired.

□

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Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland

E-mail address: pekka.j.koskela@jyu.fi
E-mail address: yi.y.zhang@jyu.fi