On moments of integrals with respect to Markov additive processes and of Markov modulated generalized Ornstein-Uhlenbeck processes

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Abstract

We establish sufficient conditions for the existence, and derive explicit formulas for the \( \kappa \)th moments, \( \kappa \geq 1 \), of Markov modulated generalized Ornstein-Uhlenbeck processes as well as their stationary distributions. In particular, the running mean, the autocovariance function, and integer moments of the stationary distribution are derived in terms of the characteristics of the driving Markov additive process.

Our derivations rely on new general results on moments of Markov additive processes and (multidimensional) integrals with respect to Markov additive processes.

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1 Introduction

Given a bivariate Markov additive process (MAP) \( ([\xi, \eta], J) = ([\xi_t, \eta_t], J_t)_{t \geq 0} \), the *Markov modulated generalized Ornstein-Uhlenbeck (MMGOU)* process driven by \( ([\xi, \eta], J) \) has been defined in [7] as the process \( (V_t)_{t \geq 0} \) given by

\[
V_t = e^{-\xi t} \left( V_0 + \int_{[0,t]} e^{\xi s} \, d\eta_s \right), \quad t \geq 0,
\]

(1.1)

where the starting random variable \( V_0 \) is conditionally independent of \( ([\xi_t, \eta_t], J_t)_{t \geq 0} \) given \( J_0 \). As shown in [7, Prop. 2.7] the MMGOU process is the unique solution of the stochastic differential equation

\[
dV_t = V_{t-} dU_t + dL_t, \quad t \geq 0,
\]

(1.2)

for another bivariate MAP \( ([U, L], J) = ([U_t, L_t], J_t)_{t \geq 0} \) with \( \Delta U > -1 \) which is uniquely determined by \( ([\xi, \eta], J) \). Hereby, for any càdlàg process \( Z \) we denote by \( Z_{t-} \) the left-hand limit of \( Z \) at time \( t \in (0, \infty) \), set \( Z_{0-} := Z_0 \), and write \( \Delta Z_t = Z_t - Z_{t-} \) for its jumps.

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Assuming that the driving Markov chain $J$ is defined on a finite state space $S$ and is ergodic with stationary distribution $\pi$, it has been shown in [7, Thm. 3.3(a) & Rem. 3.4.2] that the MMGOU process admits a non-trivial stationary distribution if and only if the integral $\int_{(0,t]} e^{s\xi_j} \cdot dL^*_s$ converges $\mathbb{P}_\pi$-almost surely as $t \to \infty$ to some finite-valued random variable $V_\infty$. Hereby, $((\xi^*, L^*), J^*)$ denotes the time-reversal of the Markov additive process $((\xi, L), J)$. In this case, the stationary distribution of the MMGOU process under $\mathbb{P}_\pi$ is uniquely determined as the distribution of
\begin{equation}
V_\infty = - \int_{(0,\infty)} e^{s\xi_j} \cdot dL^*_s. \tag{1.3}
\end{equation}

Necessary and sufficient conditions for $\mathbb{P}_\pi$-a.s. convergence of integrals of the form (1.3) in terms of the characteristics of the appearing processes have been given in [3].

Besides, by [2, Thm. 3.3(b)], the MMGOU process can admit trivial stationary distributions whenever $V$ degenerates to a continuous-time Markov chain that is piecewise constant with discrete stationary distribution. This behaviour will typically be excluded in this paper.

In the special case that $(\xi, \eta)$ (equivalently $(U, L)$) is a bivariate Lévy process, the MMGOU process reduces to the generalized Ornstein-Uhlenbeck (GOU) process, originally introduced in [13] and subsequently studied and applied by various authors, such as [3, 5, 20, 21, 24, 26, 31], to name just a few. Second-order properties of GOU processes like moments and tail distributions are specifically studied in [4], as well as in [3, 21, 24] for special cases.

Another special case of the MMGOU process that has gained attention in the last few years is the Markov modulated Ornstein-Uhlenbeck (MMOU) process. This can be obtained by specifying $U_t = -\gamma_j t$ is a piecewise linear MAP, where $L_t = \sigma_j B_t + \alpha_j t$ is a Markov modulated Brownian motion with drift, where $(\gamma_j, \alpha_j, \sigma_j)$ are constants with $\sigma_j > 0$, $\gamma_j \geq 0$, and $\gamma_j > 0$ for at least one $j \in S$. The MMOU process has been introduced and studied first in [18], where the authors also derive the mean and (autoco-)variance function of the MMOU process using methods from stochastic analysis, while a recursion for higher moments is derived using Laplace transforms. Other articles like e.g. [25] and [40] focused on stationarity properties of MMOU processes. More precisely, in [25], assuming $\alpha_j \equiv 0$, conditions for stationarity of MMOU processes and some expressions for the stationary distribution as scale mixtures are derived. In [40], again assuming $\alpha_j \equiv 0$, expressions for the Fourier transform of the density of an MMOU process with two-state Markov switching (i.e. $|S| = 2$) are provided.

As the stationary distribution of the MMGOU process is the distribution of an exponential functional, at this point we also mention [34], where moments of exponential functionals with deterministic integrator and with a general process with independent increments in the exponential are studied.

In this paper we derive new existence results and formulas for the running mean and the autocovariance function of the MMGOU process, thus generalizing the results for the MMOU process given in [18]. Under suitable conditions we prove that the autocovariance function is exponentially decreasing. This coincides with well-known analogue results for the (G)OU process as well as for the related discrete-time autoregressive AR(1) time series. Further, we derive existence results and formulas for integer moments of the stationary distribution of the MMGOU process. Here the first two moments will be stated explicitly, while we provide a recursion formula for higher order moments. All formulas are given in terms of the characteristics of the driving MAP. In order to prove these results, after recalling various preliminaries on MAPs in Section 2 in Section 3.1 we collect existence results for (exponential) moments of MAPs. Additionally, in this section we prove a general existence result for moments of stochastic integrals with respect to MAPs (Theorem 3.3). This theorem relies on the one hand on a new result on the existence of moments of Lévy driven stochastic integrals up to a stopping time, and on the other hand on a
2 Markov additive processes

Throughout, we assume \( S = \{1, 2, \ldots, |S|\} \) to be a finite set and denote its power set by \( \mathcal{S} \). Let \((X, J) = (X_t, J_t)_{t \geq 0}\) be a Markov process on \( \mathbb{R}^d \times S, d \geq 1\), defined on the filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \( \mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\) denotes a filtration satisfying the usual conditions of completeness and right-continuity (see e.g. \cite{32}), and such that \((X, J)\) is adapted. For any \( j \in S\) we write \( \mathbb{P}_j(\cdot) := \mathbb{P}(\cdot | J_0 = j) \) and \( \mathbb{E}_j[\cdot] \) for the expectation with respect to \( \mathbb{P}_j\), where we always assume \( \mathbb{P}(J_0 = j) > 0\), for every \( j \in S\).

The Markov process \((X, J)\) is called a \((d\text{-dimensional\, Markov\, additive\, process\, with\, respect\, to\, } \mathcal{F} (\mathcal{F}\text{-MAP}), if for all \( s, t \geq 0\) and for all bounded and measurable functions \( f : \mathbb{R}^d \to \mathbb{R}, g : S \to \mathbb{R} \)

\[
\mathbb{E}[f(X_{s+t} - X_s)g(J_{s+t}) | \mathcal{F}_s] = \mathbb{E}_t[f(X_t - X_0)g(J_t)] .
\]

Given a MAP \((X, J)\) the marginal process \(X\) is called the additive component. The process \(J\), which itself is a Markov process with respect to \( \mathcal{F}\), is typically called Markovian component, driving Markov chain/process or modulator.

For any MAP, we assume throughout that \( X_0 = 0 \).

We refer to \cite{11,12} for the first extensive studies on MAPs, to \cite{1} for a short textbook treatment of MAPs, and to the appendix of \cite{13} for numerous results on MAPs. Basic properties of MAPs that will be needed in this article will also be recalled in the upcoming subsections.

2.1 The additive component

From the definition of the MAP \((X, J)\) one can show, cf. \cite{1}, that there exists a sequence of independent \( \mathbb{R}^d\)-valued Lévy processes \( \{X^{(j)}, j \in S\} \) with respect to \( \mathcal{F}\), each with characteristic triplet \( \{(\gamma_X^{(j)}, \Sigma_X^{(j)}, \nu_X^{(j)}), j \in S\}\) with respect to the standard truncation function \( 1_{\{x \leq 1\}}\), and such that, whenever \( J_t = j \) on some time interval \((t_1, t_2)\), the additive component \(\{X_t\}_{t_1 < t < t_2}\) behaves in law as \(X^{(j)}\).

In this paper, we will often consider MAPs \((X, J)\) with a bivariate additive component \(X = (\zeta, \chi)\). Thus the corresponding Lévy processes \( (\zeta^{(j)}, \chi^{(j)}) \) are two-dimensional and their characteristics \( (\gamma_X^{(j)}, \Sigma_X^{(j)}, \nu_X^{(j)}) \) consist of a non-random vector \( \gamma_X^{(j)} = (\gamma_{\zeta^{(j)}}, \gamma_{\chi^{(j)}}) = (\gamma_{\zeta}(j), \gamma_{\chi}(j)) \in \mathbb{R}^2\), a symmetric, non-negative definite 2 × 2 matrix

\[
\Sigma_X^{(j)} = \Sigma_X^{(j)} = \begin{pmatrix}
\sigma_{\zeta \zeta}^{(j)} & \sigma_{\zeta \chi}^{(j)} \\
\sigma_{\chi \zeta}^{(j)} & \sigma_{\chi \chi}^{(j)}
\end{pmatrix} = \begin{pmatrix}
\sigma_{\zeta}^2(j) & \sigma_{\zeta \chi}^{(j)} \\
\sigma_{\chi \zeta}^{(j)} & \sigma_{\chi}^2(j)
\end{pmatrix}
\]

and a Lévy measure \( \nu_X^{(j)} \) on \( \mathbb{R}^2 \setminus \{(0, 0)\} \).

Moreover, whenever the driving chain \( J \) jumps for the \( n \)-th time at a time \( T_n\), say from state \( i \) to state \( j \), it induces an additional jump \( Z_{X,n}^{(j)} \) for \( X\), whose distribution \( F_{X}^{(j)} \) depends only on...
Clearly, return times and exit times are stopping times with respect to $(X, J)$. In a similar way, we define the additive component to be càdlàg from now on. Moreover, the above yields the following path decomposition of the additive component

$$X_t = X_{1,t} + X_{2,t} := \int_{(0,t]} dX^{(J)}_s + \sum_{n\geq 1} \sum_{i,j \in S, i \neq j} Z^{ij}_{X,n} \mathbb{I}_{\{J_{T_{n-1}} = i, J_{T_n} = j, T_n \leq t\}}, \quad t \geq 0, \quad (2.2)$$

where $X_1 = (X_{1,t})_{t \geq 0}$ describes the Lévy process type behaviour of $X$, while $X_2 = (X_{2,t})_{t \geq 0}$ encodes the additional jumps. Conversely, every process $(X, J)$ such that $(J_t)_{t \geq 0}$ is a continuous-time Markov chain with state space $S$ and $X$ has a representation as in (2.2), is a MAP.

We notice that the process $X_1$ in (2.2) is a semimartingale whose characteristics depend on $J$, cf. [16]. This holds also for $X_2$, being an adapted process of finite variation. Therefore, $X$ is a semimartingale, too, and we can define stochastic integrals with respect to $X$.

Furthermore, according to [14] Prop. 2 and Eq. (41), we have

$$X_t \overset{d}{=} \sum_{j \in S} X_{v_j(t)} + \sum_{j \in S} \sum_{i,j \neq j} N^{i \rightarrow j}_t Z^{ij}_{X,t}, \quad t \geq 0, \quad (2.3)$$

where $v_j(t)$ denotes the time $J$ spends in state $j$ up to time $t$, and $N^{i \rightarrow j}_t$ is the number of jumps of $J$ from $i$ to $j$ over the time interval $[0, t]$. Hereby, $Y \overset{d}{=} Z$ reads $Y$ and $Z$ have the same distribution.

### 2.2 The Markovian component

We assume throughout, that the Markovian component $J$ is irreducible, hence ergodic. We denote the intensity matrix of $J$ by $Q = (q_{ij})_{i,j \in S}$, and recall that its entries satisfy $q_{ij} > 0$ for all $i, j \in S$, $i \neq j$, and $q_{ii} = -\sum_{j \in S \setminus \{i\}} q_{ij}$. As $J$ is assumed to be ergodic, it admits a unique stationary distribution that we denote by $\pi = (\pi_j)_{j \in S}$ and we write

$$\mathbb{P}_\pi(\cdot) = \sum_{j \in S} \pi_j \mathbb{P}_j(\cdot). \quad (2.4)$$

For a fixed state $j \in S$ we define the return times to $j$ iteratively by

$$\tau^r_n(j) := \inf\{t > 0 : J_{t-} \neq j, J_t = j\}, \quad \tau^r(j) := \inf\{t > \tau^r_{n-1}(j) : J_{t-} \neq j, J_t = j\}, \quad n \geq 2.$$ 

In a similar way, we define the exit times from the state $j$ by

$$\tau^e_n(j) := \inf\{t > 0 : J_{t-} = j, J_t \neq j\}, \quad \tau^e(j) := \inf\{t > \tau^e_{n-1}(j) : J_{t-} = j, J_t \neq j\}, \quad n \geq 2.$$ 

Clearly, return times and exit times are stopping times with respect to $\mathbb{F}$. The sequences $(\tau^r_n(j) - \tau^r_{n-1}(j))_{n \geq 1}$ and $(\tau^e_n(j) - \tau^e_{n-1}(j))_{n \geq 1}$ are i.i.d. under any $\mathbb{P}_t$. Moreover, under $\mathbb{P}_j$, we may set $\tau^r_0(j) = 0$, which yields an i.i.d. sequence $(\tau^r_{n+1}(j) - \tau^r_n(j))_{n \geq 0}$.
Throughout, let \((N_t)_{t \geq 0}\) be the counting process that describes the number of jumps of \(J\) up to time \(t\). Further, let \(\{T_n, n \geq 1\}\) be the sequence of jump times of \(J\) and set \(T_0 = 0\), such that in particular \(N_t = \sum_{n=1}^{\infty} 1_{\{T_n \leq t\}}\). We also define the counting processes

\[
N_t^{re}(j) := \sum_{\ell=1}^{\infty} 1_{\{\tau_{\ell}^{re}(j) \leq t\}}, \quad N_t^{ex}(j) := \sum_{\ell=1}^{\infty} 1_{\{\tau_{\ell}^{ex}(j) \leq t\}}, \quad j \in S, \quad t \geq 0. \tag{2.5}
\]

The ergodicity of \(J\) and the finiteness of the state space \(S\) imply that \(J\) is positive recurrent, that is \(\mathbb{E}_j[\tau_1^{re}(j)] < \infty\) for all \(j \in S\), cf. [29, Chapter 3.5]. In our setting the return times have finite moments of all orders, i.e. \(\mathbb{E}_j[\tau_1^{re}(j)^k] < \infty\) for all \(k \in \mathbb{N}\). As we were unable to find a suitable reference for this fact, we state it here as lemma and provide a short proof.

**Lemma 2.1.** Let \(J\) be an ergodic Markov chain on \(S\) with \(|S| < \infty\). Then \(\mathbb{E}_j[\tau_1^{re}(j)^k] < \infty\) for all \(j \in S\), \(k \in \mathbb{N}\).

**Proof.** Without loss of generality, we consider \(j = 1\). Then, under \(\mathbb{P}_1\),

\[
\tau_1^{re}(1) = \tau_1^{ex}(1) + \tau,
\]

where \(\tau\) is independent of \(\tau_1^{ex}(1)\) and phase-type distributed. More precisely, \(\tau \sim \text{PH}_{|S|-1}(\alpha, \mathbf{T})\) with

\[
\alpha := \left( \frac{q_{12}}{|q_{11}|}, \ldots, \frac{q_{1|S|}}{|q_{11}|} \right)^\top, \quad \text{and} \quad \mathbf{T} := (q_{ij})_{i,j=2,\ldots,|S|}.
\]

As under \(\mathbb{P}_1\) the variable \(\tau_1^{ex}(1) \sim \text{Exp}(|q_{11}|)\) has finite moments of all orders and the same holds true for \(\tau\) (see e.g. [10, Cor. 3.1.18]), this immediately implies the statement by an application of the binomial theorem. \(\square\)

We introduce the localization process \(\mathbf{A} = (\mathbf{A}_t)_{t \geq 0}\) corresponding to the Markovian component \(J\) of the MAP \((X, J)\) given by

\[
\mathbf{A}_t := \mathbf{e}_J|_{J \rightarrow j} = (1_{\{J_n = j\}})_{j \in S}, \quad t \geq 0,
\]

where \(\mathbf{e}_j\) denotes the \(j\)-th unit vector, such that

\[
\mathbb{E}_j[\mathbf{A}_t] = e^{Q^T t}\mathbf{e}_j. \tag{2.6}
\]

For any real-valued process \(Y\) we write \(\hat{Y}_t := Y_t|_{\mathbf{A}}\) and note that this implies \(1^\top \hat{Y}_t = Y_t\), with \(1\) denoting the column vector of 1’s. As \(\mathbf{A}\) is a regular jump Markov process, by (see [15, Appendix B, Lemma 1.1]) there exists an \(|S|\)-dimensional square integrable \(\mathbb{P}\)-martingale \(\mathbf{M}\) with respect to \(\mathbb{P}\), such that

\[
\mathbf{A}_t = Q^T \int_{(0,t]} \mathbf{A}_s ds + \mathbf{M}_t, \tag{2.7}
\]

where we make the following important notational convention for multivariate stochastic integrals:

**Convention 2.2.** Throughout, the integral \(\int_{(0,t]} H_s dY_s\) has to be understood componentwise in the following cases:

(i) If \(Y\) is a real-valued semimartingale and \(H\) an \(\mathbb{R}^d\)-valued process such that its components \(H^i, i = 1, \ldots, d\), are locally bounded adapted (if \(Y\) is of finite variation) or predictable processes.

(ii) If \(Y\) is an \(\mathbb{R}^d\)-valued semimartingale and \(H\) a real-valued locally bounded predictable process.
For the integral in \( (2.7) \) this implies
\[
\int_{(0,t]} \Lambda_s ds = \left( \int_{(0,t]} \mathbb{1}_{\{J_s = j\}} ds \right)_{j \in S}.
\]

Moreover, again according to Convention \( \[\text{2.7}\] \) integrating (componentwise) by parts, we obtain from \( (2.7) \) for any real-valued semimartingale \( Y \)
\[
\hat{Y}_t = Y_t A_t = \hat{Y}_0 + Q^{T} \int_{(0,t]} \hat{Y}_s ds + \int_{(0,t]} Y_s dM_s + \int_{(0,t]} \Lambda_s dY_s + [Y, A]_t,
\]
where \([Y, A] := \{[Y, \mathbb{1}_{\{J_t = j\}}]\}_{j \in S}\) and \(\mathbb{1}_{\{J_t = j\}}\) is the indicator function of \(J_t = j\).

We notice that, from \( (2.7) \), \(M\) is a martingale which is bounded on compact time intervals, since, for every \( t > 0 \), we have \( \|A_t\|_{\infty} := \sup_{j \in [S]} |A_t e_j^T| = 1 \) and \( \|Q\|_{\infty} := \sup_{i,j \in [S]} |q_{ij}| < \infty \). In particular, by \( \[\text{2.7}\] \) Eq. (14), for every predictable \( H \) such that \( \sup_{t \geq 0} |H_t| \in L^1(\mathbb{P}) \), we have \( \mathbb{E} |\sup_{0 \leq s \leq t} |\int_{(0,s]} H_u dM_u| | < \infty \) for all \( t \geq 0 \). Therefore, we have shown the following.

**Lemma 2.3.** For every predictable process \( H \) with \( \sup_{t \geq 0} |H_t| \in L^1(\mathbb{P}) \) the integral process \( (\int_{(0,t]} H_s dM_s)_{t \geq 0} \) is a centered martingale.

Clearly, \( M \) is a martingale also with respect to \( \mathbb{P}_j \), for any \( j \in S \), and hence with respect to \( \mathbb{P}_\pi \).

### 2.3 The matrix exponent

The **matrix exponent** \( \Psi_X \) of a MAP \((X, J)\) with univariate additive component \(X\) is the matrix in \( \mathbb{C}^{[S] \times [S]} \) defined as
\[
\Psi_X(w) = \text{diag} (\psi_j(w)) + Q^T \circ (E [e^{wZ_{X,1}}])^T_{j,k \in S},
\]
for all \( w \in \mathbb{C} \) such that the right hand side exists, cf. [1] Prop. XI.2.2 or [14]. Hereby and in the following, \( \text{diag}(a_j) \) denotes a diagonal matrix with entries \( a_j, j = 1, \ldots, n \), and \( \circ \) means elementwise multiplication (of matrices) and \( \psi_j(w) = \log E[e^{wX^j_1}] \) is the Laplace exponent corresponding to the \( \text{Lévy process} \ X^j \).

The matrix exponent determines the characteristic function of the additive component \(X\) since, cf. [14] Sec. A.1,
\[
E_\pi [e^{wX_t \mathbb{1}_{\{J_t = j\}}}] = e_j^T e^{i\Psi_X(w)} e_j.
\]

### 2.4 Related processes

As explained in \( \[\text{7}\] \), for any bivariate \( \mathbb{F}\)-MAP \((\zeta, \chi, J)\) the processes \((\zeta + \chi, J) = ((\zeta^t + \chi^t), J_t)_{t \geq 0}\) and \(([\zeta, \chi], J) = ([\zeta^t, \chi^t], J_t)_{t \geq 0}\) are again \( \mathbb{F}\)-MAPs, where \([\zeta, \chi]\) denotes the co-variation of the two semimartingales \(\zeta\) and \(\chi\). In particular, the additive component of \(([\zeta, \chi], J)\) is given by, see \( \[\text{7}\] \) Eq. (2.4)
\[
[\zeta, \chi]_t = \int_{(0,t]} d[\zeta^J(s), \chi^J(s)] + \sum_{n \geq 1} \sum_{i,j \in S} Z_{i,n}^{\zeta,j} Z_{i,n}^{\chi,j} \mathbb{1}_{\{J_{T_n - 1} = i, J_{T_n} = j, T_n \leq t\}}, \quad t \geq 0.
\]
We also recall the dual MAP $(X^*, J^*)$ of $(X, J)$, sometimes also called its time-reversal, which is defined as the process satisfying
\[(X_{(t-s)^-} - X_t, J_{(t-s)^-})_{0 \leq s \leq t} \text{ under } \mathbb{P}_\pi \] is equal in law to \((X^*_s, J^*_s)_{0 \leq s \leq t} \text{ under } \mathbb{P}_\pi^*), \tag{2.12}\]
where \(\pi^* = \pi\) is the unique stationary distribution of \(J^*\), \(\mathbb{P}^*_j(\cdot) = \mathbb{P}(\cdot|J^*_0 = j)\) and \(\mathbb{P}_\pi^*\) defined accordingly; see \cite{14} Lem. 21 for details. Moreover, note that by \cite{14} Sec. A.2, the matrix exponent \(\Psi_{X^*}\) of the dual process \((X^*, J^*)\) fulfills the relation
\[\Psi_{X^*}(w) = (\text{diag } \pi)^{-1} \cdot \Psi_X(-w)^\top \cdot (\text{diag } \pi), \tag{2.13}\]
for all \(w\) where the matrix exponents are defined.

We denote the return and exit times of the dual process by \(\tau^e_n(j)^*\) and \(\tau^r_n(j)^*\), \(n \in \mathbb{N}, j \in S\).

3 Moments of (integrals with respect to) MAPs

Throughout this section let \((X, J)\) denote a MAP on \(\mathbb{R} \times S\) with \(|S| < \infty\), an ergodic Markovian component \(J\), and corresponding localization process \((A_t)_{t \geq 0}\). Using Perron-Frobenius eigenvalue theory it is possible to determine the moments of the additive component process via the matrix exponent \(\Psi_X\). Indeed, cf. \cite{11} Sec. 1.6, the matrix exponent \(\Psi_X(w)\) has a real eigenvalue with maximal real part that we denote by \(\lambda_{\text{max}}(w)\). As shown in \cite{22} Prop.’s 3.2 and 3.4, the eigenvalue \(\lambda_{\text{max}}(w)\) is differentiable and convex. Then, e.g., assuming \(\mathbb{E}_\pi[|X_1|] < \infty\), the first moment of \(X_1\) follows from \cite{11} Cor. XI.2.5] as
\[\mathbb{E}_\pi[X_1] = \frac{d}{dw} \lambda_{\text{max}}(w) \bigg|_{w=0}.\]
In this paper, we aim to circumvent the eigenvalue theory and use the Lévy system of a MAP instead, to derive moment formulas of MAPs as well as stochastic integrals with respect to MAPs that directly depend on the semimartingale characteristics of the processes. Such formulas for the mean and variance of the additive component of \((X, J)\) will be presented in Theorems 3.8 and 3.10 below, after establishing conditions for finiteness of moments in the upcoming subsection.

3.1 Existence of moments

In \cite{14} Thm. 34] conditions for the existence of the mean of the additive component of a MAP are collected. The following lemma generalizes parts of these results to general \(\kappa\)th moments, \(\kappa \geq 1\). Its proof is postponed to Section 5.

**Lemma 3.1.** Let \((X, J)\) be an \(\mathbb{F}\)-MAP and fix \(\kappa \geq 1, t > 0\). Then the following are equivalent:

(i) \(\mathbb{E}_\pi[\sup_{0 < s \leq t} |X_s|^{\kappa}] < \infty\).
(ii) \(\mathbb{E}_\pi[|X_t|^{\kappa}] < \infty\).
(iii) \(\mathbb{E}_{\pi_j}[|X_t|^{\kappa}] < \infty \quad \forall j \in S\).
(iv) \(\mathbb{E}[|X_t^{(j)}|^{\kappa}] < \infty \text{ for all } j \in S, \text{ and } \mathbb{E}[|Z_{X,t}^{(ij)}|^{\kappa}] < \infty \text{ for all } (i, j) \in S^2 \text{ such that } q_{ij} > 0\).

The next lemma provides conditions for the existence of exponential moments of the additive component of \((X, J)\), that will later be useful. Again its proof is given in Section 5.
Lemma 3.2. Let \((X, J)\) be a MAP on \(\mathbb{R} \times S\) and fix \(\kappa > 0\). Assume that

\[
\int_{|x| \geq 1} e^{\kappa|x|} \nu_{X(j)}(dx) < \infty \text{ for all } j \in S, \text{ and } \mathbb{E}\left[e^{\kappa|Z_{X,1}^j|}\right] < \infty \text{ for all } (i, j) \in S^2 \text{ s.t. } q_{ij} > 0.
\]

Then \(\mathbb{E}_\pi \left[\sup_{k \in [0,1]} e^{\kappa|X_k|}\right] < \infty\) and equivalently \(\mathbb{E}_\pi \left[\sup_{s \in [0,t]} e^{\kappa|X_s|}\right] < \infty\) for all \(0 < t < \infty\).

Further insight into the exponential moments of a MAP can also be obtained from the matrix exponent and its eigenvalue with maximal real part. This is illustrated by the following proposition whose proof is to be found in Section 5.

Proposition 3.3. Let \((X, J)\) be a MAP on \(\mathbb{R} \times S\) and choose \(\kappa \in \mathbb{R}\) such that \(\psi_{X(j)}(\kappa) < |q_{jj}|\) for all \(j \in S\), and \(\mathbb{E}[\exp(\kappa Z_{X,1}^j)] < \infty\) for all \((i, j) \in S^2\) with \(q_{ij} \neq 0\).

(i) If for some \(j \in S\)

\[
\max_{i \in S, i \neq j} \left(\frac{1}{|q_{ii}| - \psi_{X(i)}(\kappa)} \sum_{\ell \in S, \ell \neq i,j} q_{i\ell} \mathbb{E}\left[e^{\kappa Z_{X,1}^\ell}\right]\right) < 1,
\]

then \(\mathbb{E}_j \left[e^{\kappa X_{i\tau}(\kappa)}\right] < \infty\).

(ii) If \(\mathbb{E}_j \left[e^{\kappa X_{i\tau}(\kappa)}\right] < \infty\) for some \(j \in S\), and \(\lambda_\text{max}(\kappa) < 0\), then \(\mathbb{E}_j \left[e^{\kappa X_{i\tau}(\kappa)}\right] < 1\).

For our study of moments of the MMGOU processes in Section 4 below, it will be crucial to study moments of stochastic integrals with respect to MAPs up to a certain stopping time. To this aim we continue by showing finiteness of moments of the stochastic integral \(\int_{(0,\tau]} H_s^{-} dX_s\) for an \(\mathbb{R} \times S\)-valued \(\mathcal{F}\)-MAP \((X, J)\), a càdlàg \(\mathcal{F}\)-adapted process \(H\), and a stopping time \(\tau\). We make the following assumption.

Assumption 3.4. The stopping time \(\tau\) has finite moments of every order, that is, \(\mathbb{E}[\tau^n] < \infty\) for every \(n \in \mathbb{N}\), and the càdlàg \(\mathcal{F}\)-adapted process \(H\) satisfies

\[
\mathbb{E}\left[\sup_{0 \leq t \leq \tau} |H_t|^\kappa + \varepsilon\right] < \infty, \quad \kappa \geq 1,
\]

for \(\varepsilon = 0\) if \(\tau\) is bounded, and for some \(\varepsilon > 0\) otherwise.

If \(H\) and \(\tau\) satisfy Assumption 3.4 but \(\tau\) is not bounded, by Hölder’s inequality with exponents \(p = (\kappa + \varepsilon)/\kappa\) and exponent \(q = p/(p - 1) = (\kappa + \varepsilon)/\varepsilon\), we immediately derive the estimate

\[
\mathbb{E}\left[\tau^\eta \sup_{0 \leq t \leq \tau} |H_t|^\kappa\right] \leq \mathbb{E}\left[\sup_{0 \leq t \leq \tau} |H_t|^\kappa + \varepsilon\right]^{1/p} \mathbb{E}\left[\tau^\eta\right]^{1/q} < \infty, \quad \eta > 0.
\]

The next theorem is the main result of this section:

Theorem 3.5. Let \(\kappa \geq 1\). If \((X, J)\) is an \(\mathcal{F}\)-MAP on \(\mathbb{R} \times S\) such that \(\mathbb{E}_\pi[|X_1|^\kappa] < \infty\) and \(\tau\) and \(H\) satisfy Assumption 3.4, then for all \(j \in S\)

\[
\mathbb{E}_j \left[\sup_{0 \leq t \leq \tau} \left|\int_{(0,\tau]} H_s^{-} dX_s\right|^\kappa\right] < \infty.
\]
To prove Theorem 3.5, we use the decomposition $X = X_1 + X_2$ of the MAP $X$ given in 2.2 and consider the integrals with respect to $X_1$ and $X_2$ separately. As $X_1$ is a concatenation of Lévy processes, our studies of this part rely on a generalization of [4] Lem. 6.1 where moments of stochastic integrals with respect to Lévy processes up to a fixed time have been considered. The proof of [4] Lem. 6.1 could not be generalized to integration up to a general stopping time and in Section 5 below we therefore provide an alternative proof.

**Lemma 3.6.** Let $\kappa \geq 1$. If $X$ is a Lévy process such that $\mathbb{E}[|X_1|^\kappa] < \infty$ and $\tau$ and $H$ satisfy Assumption 3.4, then

$$
\mathbb{E}\left[ \sup_{0 \leq t \leq \tau} \left| \int_{(0,t]} H_s \, dX_s \right|^\kappa \right] < \infty.
$$

In order to deal with the integral with respect to the additional jumps of $X$, following [30], observe that we have the representation

$$
X_{2,t} = \sum_{i,j \in S} \Psi_{ij}^t, \quad \Psi_{ij}^t = \int_{(0,t]} \int_{\mathbb{R}} x \, \mu_{ij}^t(dx,ds)
$$

with the integer-valued random measures $\mu_{ij}^t$, $i, j \in S$, $i \neq j$, defined by

$$
\mu_{ij}^t(\omega, dt, dx) = \sum_{n \geq 1} \mathbb{I}\{J_{T_n}^t(\omega) = i, J_{T_n}^t(\omega) = j\} \delta_{(T_n^t(\omega), Z_{X,n}^t(\omega))}(dt, dx) \mathbb{I}\{T_n^t(\omega) < \infty\}.
$$

We stress that the definition of $\mu_{ij}^t$ in [30] (II$^2$ in the notation used in [30]) is flawed: In [30] Eq. (6) the Dirac measure $\delta_{Z_n^t}(dx)$ is replaced by $\mathbb{P}_{Z_n}(dx)$. This however is inconsistent with (3.3).

In the proof of Theorem 3.5 we will apply [27] Thm. 1] to the integral with respect to $X_2$. Therefore, we need the explicit form of the $\mathbb{F}$-dual predictable projection $\nu_{ij}^t$ of $\mu_{ij}^t$, which we are now going to compute. Recall that $F_{X}^{ij}$ denotes the cdf of the law of $Z_{X,1}^t$, and that $N_{i \rightarrow j}^{t} := \mu_{ij}^t([0,t] \times \mathbb{R})$ denotes the number of jumps of $J$ from state $i$ to state $j$ up to $t \geq 0$. By [15] Appendix B, p. 361] (see also [23] p. 290) the process $N_{i \rightarrow j}^{t} - \int_{(0,t]} \mathbb{I}\{J_{s} = i\} q_{ij} \, ds$ is a martingale, hence the $\mathbb{F}$-dual predictable projection $\phi_{ij}^t$ of $N_{i \rightarrow j}^{t}$ is given by

$$
\phi_{ij}^t = \int_{(0,t]} \mathbb{I}\{J_{s} = i\} q_{ij} \, ds, \quad t \geq 0.
$$

From this we derive the following Lemma, see also [23] Sec. 2] and [8] for related results and special cases. Its proof is given in Section 5.

**Lemma 3.7.** For any $i, j \in S$, $i \neq j$, the $\mathbb{F}$-dual predictable projection $\nu_{ij}^t$ of $\mu_{ij}^t$ is given by

$$
\nu_{ij}^t(\omega, dt, dx) = F_{X}^{ij}(dx) \mathbb{I}\{J_{s} = i\} q_{ij} \, dt.
$$

We are now ready to prove Theorem 3.5. For this recall that by Jensen’s inequality for any non-negative numbers $a_1, \ldots, a_n$ and $\kappa \geq 1$, we have

$$
\left( \sum_{i=1}^{n} a_i \right)^\kappa \leq n^{\kappa-1} \sum_{i=1}^{n} a_i^\kappa. \quad (3.5)
$$
Proof of Theorem 3.5. By (2.2) and Minkowski’s inequality we get

\[
\mathbb{E}_j \left[ \sup_{0 \leq t \leq \tau} \left| \frac{\int_0^t H_s \, dX_s}{\kappa} \right| \right]^{\frac{1}{\kappa}} \leq \mathbb{E}_j \left[ \sup_{0 \leq t \leq \tau} \left| \sum_{i \in S} \int_0^t \frac{H_s - \mathbb{1}_{\{J_s = i\}} \, dX_s^i}{\kappa} \right| \right]^{\frac{1}{\kappa}} + \mathbb{E}_j \left[ \sup_{0 \leq t \leq \tau} \left| \int_0^t H_s \, dX_{2,s} \right| \right]^{\frac{1}{\kappa}},
\]

where for the first summand an application of (3.3) yields

\[
\mathbb{E}_j \left[ \sup_{0 \leq t \leq \tau} \left| \sum_{i \in S} \int_0^t \frac{H_s - \mathbb{1}_{\{J_s = i\}} \, dX_s^i}{\kappa} \right| \right] \leq |S|^{\kappa-1} \sum_{i \in S} \mathbb{E}_j \left[ \sup_{0 \leq t \leq \tau} \left| \int_0^t H_s - \mathbb{1}_{\{J_s = i\}} \, dX_s^i \right| \right].
\]

As, by the assumptions of the theorem, we have \(\mathbb{E}[|X_1^{(j)}|] < \infty\) for all \(j \in S\) by Lemma 3.1 and the process \(H \mathbb{1}_{\{J = i\}}\) satisfies Assumption 3.4, this is finite by Lemma 3.6.

We now consider the second summand in (3.3) and note that by Lemma 3.1 from our assumptions, we have \(\mathbb{E}[|Z_{ik}^{(j)}|] < \infty\) for all \((i, j) \in S^2\) such that \(q_{ij} > 0\). From (3.3) and (3.5) it follows

\[
\mathbb{E}_j \left[ \sup_{0 \leq t \leq \tau} \left| \int_0^t H_s \, dX_{2,s} \right| \right] = \mathbb{E}_j \left[ \sup_{0 \leq t \leq \tau} \left| \sum_{i \in S, i \neq k} \int_0^t H_s - d\Psi_s^i \right| \right]
\]

\[
\leq (|S|^2 - |S|)^{\kappa-1} \sum_{i, k \in S, i \neq k} \mathbb{E}_j \left[ \sup_{0 \leq t \leq \tau} \left| \int_0^t H_s - d\Psi_s^i \right| \right]
\]

\[
\leq (2(|S|^2 - |S|))^{\kappa-1} \sum_{i, k \in S, i \neq k} \left( \mathbb{E}_j \left[ \sup_{0 \leq t \leq \tau} \left| \int_0^t H_s - d\phi_s^{ik} \right| \right] \right.
\]

\[
\left. + \mathbb{E}_j \left[ \sup_{0 \leq t \leq \tau} \left| \int_0^t H_s - d\nu_s^{ik} \right| \right] \right) \],
\]

where we define \(\phi_s^{ik} := \mu_s^{ik} - \nu_s^{ik}\) with \(\nu_s^{ik}\) as in Lemma 3.7. Hereby, due to the specific form of \(\nu_s^{ik}\), we get

\[
\mathbb{E}_j \left[ \sup_{0 \leq t \leq \tau} \left| \int_0^t H_s - d\phi_s^{ik} \right| \right] \leq \left( q_{ik} \mathbb{E}[|Z_{ik,X,1,1}|] \right)^{\kappa} \mathbb{E}_j \left[ \sup_{0 \leq t \leq \tau} \left| \int_0^t H_s - \mathbb{1}_{\{J_s = i\}} \, d\phi_s^{ik} \right| \right]
\]

\[
\leq \left( q_{ik} \mathbb{E}[|Z_{ik,X,1,1}|] \right)^{\kappa} \mathbb{E}_j \left[ \tau^{\kappa} \sup_{0 \leq t \leq \tau} |H_t|^\kappa \right] < \infty,
\]

where the last estimate follows by (3.2) with \(\eta = \kappa\), and with \(\varepsilon > 0\) if \(\tau\) is not bounded. This shows that the second expectation in (3.3) is finite for all \(i, k \in S, i \neq k\). Concerning the first expectation in (3.8), using (2.7) Thm. 1, we get

\[
\mathbb{E}_j \left[ \sup_{0 \leq t \leq \tau} \left| \int_0^t H_s - d\phi_s^{ik} \right| \right] = \mathbb{E}_j \left[ \sup_{t \geq 0} \left| \int_0^t H_s \, d\phi_s^{ik} \right| \right]
\]

\[
\leq \begin{cases} C_1^1 \mathbb{E}_j \left[ \int_{(0, \tau]} \left| H_s - x \nu_s^{ik} \right| (ds, dx) \right], & \kappa \in [1, 2) \\ C_2^1 \mathbb{E}_j \left[ \left( \int_{(0, \tau]} \left| H_s - x \nu_s^{ik} \right|^2 (ds, dx) \right)^{\frac{2}{\kappa}} \right] + \mathbb{E}_j \left[ \int_{(0, \tau]} \left| H_s - x \nu_s^{ik} \right|^\kappa (ds, dx) \right], & \kappa \in [2, \infty) \end{cases}
\]

(3.10)
for some constants $C^1_\kappa, C^2_\kappa \in (0, \infty)$. Hence, using (3.9) with $\eta = 1$ or $\eta = \kappa/2$, and with $\varepsilon > 0$ if $\tau$ is not bounded, we get as in (3.10),

$$
\mathbb{E}_j \left[ \sup_{0 \leq t \leq \tau} \left| \int_{(0,t]} \int_\mathbb{R} H_{s-} x \mu_{Z,t}^j (ds, dx) \right|^\kappa \right] < \infty.
$$

Therefore, the right-hand side of (3.8) and hence of (3.6) is finite, and the proof of the theorem is complete. \hfill \Box

3.2 Explicit expressions for the moments

We can now provide an explicit formula for the mean of the additive component of a MAP. The proof of this theorem is given in Section 5.

**Theorem 3.8.** Let $(X, J)$ be an $\mathbb{F}$-MAP on $\mathbb{R} \times S$. Suppose $\mathbb{E}_j[|X_1|] < \infty$ for all $j \in S$, then for all $j \in S$

$$
\mathbb{E}_j[X_1] = \int_0^t e^{Q^T(t-s)} e[X] e^{Q^T_s} ds \cdot e_j,
$$

(3.11)

and in particular

$$
\mathbb{E}_j[X_1] = 1^T e[X] \int_0^t e^{Q^T_s} ds \cdot e_j, \quad \text{and} \quad \mathbb{E}_\pi[X_1] = 1^T e[X] \int_0^t e^{Q^T_s} ds \cdot \pi,
$$

(3.12)

with the expectation matrix

$$
\epsilon[X] := \text{diag} \left( \mathbb{E}[X(j)] \right) + Q^T \circ \left( \mathbb{E}[Z_{1,j}] \right)^\top.
$$

(3.13)

Moreover, $X^\# := \hat{X} - e[X] \int_{(0,]} \Lambda_s ds$ is an $\mathbb{F}$-martingale under $\mathbb{P}$.

The following representation of the expectation of a MAP could also have been derived from [1, Cor. XI.2.5]. We mention it here, as the proof in that source is not fully given. It is an immediate consequence of (3.12) upon noticing that $e^{Q^T_s} \cdot \pi = \pi$ for all $s \geq 0$, and due to [3] Eq. (25.7).

**Corollary 3.9.** Let $(X, J)$ be an $\mathbb{F}$-MAP on $\mathbb{R} \times S$ and suppose that $\mathbb{E}_j[|X_1|] < \infty$ for all $j \in S$, then

$$
\mathbb{E}_\pi[X_1] = \sum_{j \in S} \pi_j \left( \gamma_{X(j)} + \int_{|x| > 1} x \nu_{X(j)}(dx) + \sum_{i \in S, i \neq j} q_{ij} \int_\mathbb{R} x F_{X}^{ij}(dx) \right).
$$

We continue by providing an explicit expression of the variance of the additive component of a MAP in the next theorem.

**Theorem 3.10.** Let $(X, J)$ be an $\mathbb{F}$-MAP on $\mathbb{R} \times S$. Suppose that $\mathbb{E}_j[|X_1|^2] < \infty$ for all $j \in S$, then for all $j \in S, t > 0$

$$
\mathbb{E}_j[X_1^2 A_t] = \left( 2 \int_0^t e^{Q^T(t-s)} e[X] \int_0^s e^{Q^T(s-u)} e[X] e^{Q^T_u} du ds + \int_0^t e^{Q^T(t-s)} e[[X, X]] e^{Q^T_s} ds \right) \cdot e_j,
$$

(3.14)
with \( \epsilon[[X, X]] = \text{diag} \left( \text{Var}(X_{ij}) \right) + Q^\top \circ \left( \mathbb{E} \left[ (Z_{ij})^2 \right] \right)^\top \). In particular this implies
\[
\text{Var}_{j}[X_t] = 1^\top \epsilon[X] \left( 2 \int_0^t \int_0^s e^{Q^\top (s-u)} \epsilon[X] e^{Q^\top u} du ds - \int_0^t e^{Q^\top u} du \cdot 1^\top \epsilon[X] \int_0^t e^{Q^\top s} ds \right) e_j + 1^\top \epsilon[[X, X]] \int_0^t e^{Q^\top s} ds \cdot e_j.
\]

The proof of Theorem 3.10 as given in Section 5 relies on an application of integration by parts and the upcoming Lemma 3.11 which will also be useful for our computations in Section 4. Its proof can also be found in Section 5.

**Lemma 3.11.** Let \((X, J)\) be an \(\mathbb{F}\)-MAP on \(\mathbb{R} \times S\) and \(H\) an \(\mathbb{F}\)-adapted càdlàg process. Assume that \(\mathbb{E}_x[|X_1|] < \infty\) and fix \(t > 0\) such that \(\mathbb{E}[\sup_{0 \leq s \leq t}|H_s|] < \infty\). Then for all \(j \in S\) it holds
\[
\mathbb{E}_j \left[ \int_{(0, t]} H_{s-} dX_s \right] = 1^\top \epsilon[X] \int_{(0, t]} \mathbb{E}_j[\hat{H}_s] ds,
\]
and
\[
\mathbb{E}_j \left[ \int_{(0, t]} \hat{H}_{s-} dX_s \right] + \mathbb{E}_j \left[ \int_{(0, t]} H_{s-} d[X, \Lambda]_s \right] = \epsilon[X] \int_{(0, t]} \mathbb{E}_j[\hat{H}_s] ds.
\]
with the expectation matrix \(\epsilon[X]\) defined in (3.13).

**Remark 3.12.** Via Itô’s formula and similar computations as in the proof of Theorem 3.10 one can also derive closed form expressions for higher moments of the additive component of a MAP, which are then expressed in terms of powers of the expectation matrix, the correlation matrix \(\text{diag}(\text{Var}(X_{ij}))\), and multiple integrals with integrand \(e^{Q^\top t}\). In order to avoid overly lengthy computations we restrain from giving any details, but instead also refer to the recent work [2] where recursive expressions of moments of MAPs similar to the above are presented in Prop. 3.

## 4 Application to Markov modulated GOU processes

In this final section we consider the MMGOU process \((V_t)_{t \geq 0}\) solving the SDE (1.12) for some bivariate MAP \(((U, L), J)\). In Section 4.1 we compute the running mean and the autocovariance function of the MMGOU process under suitable conditions that imply their existence. Afterwards, Section 4.2 focuses on stationary MMGOU processes: We derive suitable conditions for the existence of the moments of the stationary distribution, as well as explicit moment formulas. Similar results as in this section in the special case of the MMOU process have been obtained in [13] and [25]. However, in the latter, no expressions in terms of the driving processes are provided. For the special case of the GOU process, moments of the stationary distribution have been considered in [21] (existence) and [4] (formulas). Related results can also be found in [37], where moments of the stationary distribution of reflected Markov modulated OU processes are computed, and in [34] where moments of exponential functionals of additive processes are studied.

Throughout, we assume \(\Delta U > -1\). This allows us to use the explicit representation (1.11) of the MMGOU process with the MAP \(((\xi, \eta), J)\) that is one-to-one related with \(((U, L), J)\) by
\[
\begin{pmatrix}
U_t \\
L_t
\end{pmatrix} = \begin{pmatrix}
-\xi_t + \frac{1}{2} \int_{(0, t]} \sigma^2_s(J_s) ds + \sum_{0 < s \leq t} (\Delta \xi_s + e^{-\Delta \xi_s} - 1) \\
\eta_t - \int_{(0, t]} \sigma \xi_{t-} \eta_s(J_s) ds + \sum_{0 < s \leq t} (e^{-\Delta \xi_s} - 1) \Delta \eta_s
\end{pmatrix}, \quad t \geq 0,
\]
as shown in [7, Prop. 2.11]. Note that in absence of additional jumps, that is if \( Z^{ij}_{t+n} \equiv Z^{ij}_{t} \equiv 0 \), the conditional independence of \( \xi \) and \( \eta \) (or analogously of \( U \) and \( L \)) given \( J \) implies \( L_t = \eta_t \) for all \( t \geq 0 \). Furthermore, the relation between \( U \) and \( \xi \) in (4.1) is equivalent to

\[
e^{-\xi_t} = \mathcal{E}(U)_t, \quad t \geq 0,
\]

where \( \mathcal{E}(U)_t \) is the (Doléans-Dade) stochastic exponential of \( U \), i.e. the unique solution of the SDE \( dZ_t = Z_{t-}dU_t, \quad t \geq 0 \) with \( Z_0 = \mathcal{E}(U)_0 = 1 \). This intimate relationship between \( U \) and \( \xi \) implies that their moments are also closely related. For \( |S| = 1 \), i.e. for \( U \) and \( \xi \) being Lévy processes, this has been elaborated in [4, Prop. 3.1]. The upcoming two results consider the general case \( |S| \geq 1 \).

**Lemma 4.1.** Let \((\xi, J)\) be a MAP and define the MAP \((U, J)\) via (1.2). Then, for any \( \kappa \geq 1 \), we have

\[
E_{\pi}[(U_1)^{\kappa}] < \infty \quad \text{if and only if} \quad E_{\pi}[e^{-\kappa\xi_1}] < \infty.
\]

**Proof.** The fact that \((\xi, J)\) is a MAP if and only if \((U, J)\) is a MAP with \( \Delta U > -1 \) has been shown in [7]. Further, by Lemma 3.1 we know that \( E_{\pi}[(U_1)^{\kappa}] < \infty \) if and only if \( E_{\pi}[\langle U_1^{\kappa} \rangle] < \infty \) for all \( j \in S \) and \( E_{\pi}[Z^{ij}_{1,1}] < \infty \) for all \( (i, j) \in S^2 \) such that \( q_{ij} > 0 \). The Lévy processes \( U^{(j)}, j \in S \), have finite \( \kappa \)'s moment if and only if \( \int_{|x|>1} x^{\kappa} \nu_{U^{(j)}}(dx) < \infty, \quad j \in S \), cf. [35, Thm. 25.17]. Via the path decomposition (2.2) we observe that (4.1) implies

\[
U^{(j)}_t = -\xi_t^{(j)} + \frac{1}{2} \int_{(0,t]} \sigma^2(j) ds + \sum_{0<s\leq t} \left( e^{-\Delta \xi^{(j)}_s} - 1 + \Delta \xi^{(j)}_s \right), \quad \text{and} \quad Z^{ij}_{1,1} = e^{-Z^{ij}_{1,1}},
\]

and hence \( E_{\pi}[U^{(j)}_1^{\kappa}] < \infty \) is equivalent to \( \int_{|x|>1} e^{-\kappa x} \nu_{U^{(j)}}(dx) < \infty \), and hence to \( E[e^{-\kappa \xi^{(j)}_1}] < \infty \), again by [35, Thm. 25.17]. Summing up, we observe that \( E_{\pi}[(U_1)^{\kappa}] < \infty \) if and only if \( E[e^{-\kappa \xi_1}] < \infty \) for all \( j \in S \) and \( E[e^{-\kappa Z^{ij}_{1,1}}] < \infty \) for all \( (i, j) \in S^2 \) such that \( q_{ij} > 0 \).

Finally, by an analogue computation as in (5.17), we observe that this is in turn equivalent to \( E_{\pi}[e^{-\kappa \xi_1}] < \infty \), which finishes the proof. \( \Box \)

**Proposition 4.2.** Let \((\xi, J)\) be a MAP and define the MAP \((U, J)\) via (1.2). Then for any \( k \in \mathbb{N} \) such that \( E_{\pi}[\sup_{0<s\leq t} \mathcal{E}(U)_s^{(j)}] = E_{\pi}[\sup_{0<s\leq t} e^{-k\xi_1}] < \infty \) for some \( t \geq 0 \) we have the identity

\[
\Psi_{\xi}(-k) = Q^T + \epsilon \left[ kU + \frac{k(k-1)}{2} \int_{(0,t]} \sigma^2(J_s) ds + \sum_{0<s\leq t} ((1 + \Delta U_s)^k - 1 - k \Delta U_s) \right].
\]

In particular it holds

\[
\Psi_{\xi}(-1) = Q^T + \epsilon[U], \quad \text{and} \quad \Psi_{\xi}(-2) = Q^T + 2\epsilon[U] + \epsilon[U, U].
\]

**Proof.** On the one hand, as the matrix exponent \( \Psi_{\xi} \) fulfills (2.10), relation (4.2) immediately implies

\[
E_{\pi}[\mathcal{E}(U)^k_t] = E_{\pi}[e^{-k\xi_1}] = \sum_{i,j \in S} \pi_i E_{\pi}[e^{-k\xi_{i,j}} 1_{\{J_t=j\}}] = 1^T \exp (t \Psi_{\xi}(-k)) \pi.
\]

On the other hand, from the SDE for the Doléans-Dade stochastic exponential, we have \( \Delta \mathcal{E}(U) = \mathcal{E}(U) \Delta U \). Thus we compute via the binomial theorem

\[
\Delta(\mathcal{E}(U)^k_t) = (\mathcal{E}(U)\Delta U)^k - \mathcal{E}(U)^k_{t-} = \sum_{\ell=0}^{k-1} \binom{k}{\ell} \mathcal{E}(U)^{k-\ell}_{t-} (\Delta \mathcal{E}(U))_{t+}^{\ell}
\]
\[ = E(U)^k_{t-} \cdot \sum_{\ell=0}^{k-1} \binom{k}{\ell} (\Delta U)^{k-\ell} = E(U)^k_{t-} \left( (1 + \Delta U)^k - 1 \right). \]

Moreover, since \( dE(U)^k_t = E(U)_{t-} dU_t^c \), we have
\[
\langle E(U)^c, E(U)^c \rangle_t = [E(U), E(U)]_t^c = \int_{(0,t]} E(U)^c_s d\langle U^c, U^c \rangle_s.
\]

Thus Itô’s formula (cf. \([32, \text{Thm. II.32}]\)) yields
\[
E(U)^k_t = 1 + \int_{(0,t]} E(U)^k_s dY_s^{(k)},
\]

where we define
\[
Y_t^{(k)} := kU_t + \frac{k(k - 1)}{2}(U^c, U^c)_t + \sum_{0 < s \leq t} ((1 + \Delta U)^k - 1 - k\Delta U_s)
\]
\[
= k\xi_t + \frac{k^2}{2} \int_{(0,t]} \sigma^2(J_s) t_s + \sum_{0 < s \leq t} (e^{-k\Delta \xi_s} - 1 - k\Delta \xi_s), \quad t \geq 0.
\]

Notice that \((Y^{(k)}, J)\) is an \(F\)-MAP for any \(k\) as can easily be seen by the path description \([22]\). Moreover, due to the given form of \(Y^{(k)}\), we conclude by Lemmas \([4, 5]\) and \([3]\) that our assumptions imply \(E_\pi[Y^{(k)}] < \infty\). Thus, via partial integration as in \((2.8)\), applying \((3.16)\) and arguing via Lemma \([23, 24]\) we obtain
\[
E_\pi[E(U^k_{t}) A_t] = E_\pi[A_0] + (Q^T + \epsilon [Y^{(k)}]) \int_{(0,t]} E_\pi[E(U^k_{s}) A_s] ds.
\]

As \(E(U)_0 = 1\), this ODE is solved uniquely by
\[
E_\pi[E(U^k_{t}) A_t] = \exp \left( t(Q^T + \epsilon [Y^{(k)}]) \right) \pi.
\]

Multiplying with \(1^T\) proves \((1.3)\) due to \((1.3)\). Finally we get \(\epsilon [Y^{(1)}] = \epsilon [U]\), while \(\epsilon [Y^{(2)}] = \epsilon [2U + [U, U]] = 2\epsilon [U] + \epsilon [[U, U]],\) as \(\epsilon [\cdot]\) is additive. This proves \((1.4)\).

**4.1 Mean and autocovariance structure of the MMGOU process**

We start with a technical lemma which provides sufficient conditions for the existence of moments of the MMGOU process.

**Lemma 4.3.** Consider the MMGOU process \((V_t)_{t \geq 0}\) driven by the bivariate MAP \(((\xi, \eta), J)\). Assume that for some \(\kappa \geq 1\) and all \(j \in S\)
\[
E_j \left[ \sup_{0 < s \leq 1} e^{\kappa|\xi_s|} \right] < \infty, \quad E_j \left[ |V_0|^\kappa \right] < \infty, \quad \text{and} \quad E_j \left[ |\eta_1|^\kappa \right] < \infty,
\]
then \(E_j \left[ \sup_{0 < s \leq t} |V_s|^\kappa \right] < \infty\) for all \(j \in S\) and \(t \geq 0\). If \(\kappa = 2\), then in particular \(E_j \left[ \sup_{0 < s \leq t} |V_s|^2 \right] < \infty\) for all \(t \geq 0\).
Proof. Using (1.1) and the inequality (3.5) we have

\[ E_j\left[ \sup_{0<s\leq t} |V_s|^\kappa \right] = E_j\left[ \sup_{0<s\leq t} e^{-\xi_s} \int_{(0,s]} e^{\xi_u - d\eta_u} + e^{-\xi_s} V_0 \right]^\kappa \leq 2^{\kappa-1} \left( E_j\left[ \sup_{0<s\leq t} e^{-\xi_s} \int_{(0,s]} e^{\xi_u - d\eta_u} \right]^\kappa \right) + E_j\left[ \sup_{0<s\leq t} e^{-\xi_s} \right] E_j\left[ |V_0|^\kappa \right], \]

where we used the fact that \( V_0 \) is chosen independently of \((\xi_t, \eta_t), J_t)_{t\geq 0}\). While the second summand is finite by assumption and Lemma 3.2, for the first summand we note that by [7, Lem. 3.1(ii)]

\[ e^{-\xi_s} \int_{(0,s]} e^{\xi_u - d\eta_u} \text{ under } P_\pi \text{ is equal to } - \int_{(0,s]} e^{\xi_u - dL_u^s} \text{ under } P_\pi. \]

As the dual processes \( \xi^* \) and \( L^* \) inherit the moments of \( \xi \) and \( L \), we may conclude via (1.1) and the results in Section 3 that under our conditions \( E_*\left[ \sup_{0<s\leq t} e^{\xi^*_s} \right] < \infty \) and \( E_*\left[ |L|^\kappa \right] < \infty \). Hence finiteness of the first summand follows from Theorem 3.5 and this implies the statement. Finally, if the assumptions hold for \( \kappa = 2 \), then

\[ E_j\left[ \sup_{0<s\leq t} |V_s|^2 \right] \leq E_j\left[ \sup_{0<s\leq t} |V_s|^2 \right]^{\frac{2}{1}} E_j\left[ |V|^2 \right]^{\frac{1}{1}}, \]

is also finite by the above.

**Theorem 4.4.** Consider the MMGOU process \((V_t)_{t\geq 0}\) driven by the bivariate MAP \((\xi, \eta), J\) and solving the SDE (1.2) for the bivariate MAP \((U, L), J\) in (1.1). Assume that for all \( j \in S \)

\[ E_j\left[ \sup_{0<s\leq t} e^{\xi^*_s} \right] < \infty, \quad E_j\left[ |V_0| \right] < \infty, \quad \text{and} \quad E_j\left[ |\eta_1| \right] < \infty. \]

Then for all \( t \geq 0 \) and all \( j \in S \)

\[ E_j\left[ V_t \right] = 1^T e^{(Q^T + e(U)t)e} E_j\left[ \tilde{V}_t \right] + 1^T \int_0^t e^{(Q^T + e(U)(t-s))e} E_j\left[ \tilde{V}_s \right] e^{Q^T} e ds. \quad (4.6) \]

Proof. Recall that \( 1^T e[\tilde{V}_t] = E[V_t] \) for \( \tilde{V}_t := V_t \Lambda_t \). Using (2.8) and the SDE (1.2), and observing that due to Lemma 1.3 we can apply Lemma 2.3, we obtain for any \( j \in S \)

\[ E_j\left[ \tilde{V}_t \right] = E_j\left[ \tilde{V}_0 \right] + Q^T E_j\left[ \int_{(0,t]} \tilde{V}_s ds \right] + E_j\left[ \int_{(0,t]} \tilde{V}_s dU_s \right] + E_j\left[ \int_{(0,t]} \Lambda_s dL_s \right] + E_j\left[ [V, \Lambda]_t \right]. \]

Further, using a Fubini argument and (3.16), this implies

\[ E_j\left[ \tilde{V}_t \right] = E_j\left[ \tilde{V}_0 \right] + Q^T \int_{(0,t]} E_j\left[ \tilde{V}_s \right] ds + e(U) \int_{(0,t]} E_j\left[ \tilde{V}_s \right] ds - E_j\left[ \int_{(0,t]} V_s d[U, \Lambda]_s \right] + e[L] \int_{(0,t]} E_j\left[ \Lambda_s ds \right] - E_j\left[ \int_{(0,t]} d[L, \Lambda]_s \right] + E_j\left[ [V, \Lambda]_t \right] \]

\[ = E_j\left[ \tilde{V}_0 \right] + Q^T \int_{(0,t]} E_j\left[ \tilde{V}_s \right] ds + e(U) \int_{(0,t]} E_j\left[ \tilde{V}_s \right] ds + e[L] \int_{(0,t]} E_j\left[ \Lambda_s ds \right], \quad (4.7) \]

since

\[ \int_{(0,t]} V_s d[U, \Lambda]_s + \int_{(0,t]} d[L, \Lambda]_s = \left[ \int_{(0,t]} V_s dU_s + L, \Lambda \right]_t = [V, \Lambda]_t. \]
Inserting (2.6) yields the ODE
\[ \frac{d}{dt} E_j[\hat{V}_t] = \left( Q^\top + \epsilon [U] \right) E_j[\hat{V}_t] + \epsilon [L] e^{Q^\top} e_j, \]
which is solved uniquely by
\[ E_j[\hat{V}_t] = e^{(Q^\top + \epsilon [U])t} E_j[\hat{V}_0] + \int_0^t e^{(Q^\top + \epsilon [U])(t-s)} \epsilon [L] e^{Q^\top} e_j ds. \]
Multiplying this expression by \( 1^\top \) yields (1.6).

The next theorem proves that the autocovariance function of the MMGOU decreases exponentially if the leading eigenvalue of the matrix exponent \( \Psi_\xi(-1) \) is negative.

**Theorem 4.5.** Consider the MMGOU process \((V_t)_{t \geq 0}\) driven by the bivariate MAP \((\xi, \eta), J)\) and solving the SDE (1.2) for the bivariate MAP \((U, L), J)\) in (1.1). Assume that for all \(j \in S\)
\[ E_j \left[ \sup_{0 < s \leq 1} e^{2[\xi_s]} \right] < \infty, \quad E_j[|V_0|^2] < \infty, \quad \text{and} \quad E_j[|\eta|^2] < \infty. \]

Then the autocovariance structure of \((V_t)_{t \geq 0}\) can be described via
\[ \left( \text{Cov}_j(A_t, V_s) \right) = e^{K(t-s)} \left( \text{Cov}_j(A_s, V_s) \right) \quad \text{with} \quad K := \begin{pmatrix} Q^\top & 0 \\ \epsilon [L] & Q^\top + \epsilon [U] \end{pmatrix}. \tag{4.8} \]
for all \(0 \leq s \leq t\) with starting values \( \text{Cov}_j(A_0, V_0) = 0 \) and \( \text{Cov}_j(\hat{V}_0, V_0) = e_j \text{Var}(V_0) \).
In particular, \( \text{Cov}_j(V_t, V_s) = 1^\top \text{Cov}_j(\hat{V}_t, V_s) \) decreases exponentially if \( \lambda_{\text{max}}(-1) < 0 \).

**Proof.** We start to consider \( \text{Cov}_j(\hat{V}_t, V_s) = E_j[\hat{V}_t V_s] - E_j[\hat{V}_t] E_j[V_s] \) and derive via partial integration over \((s, t)\)
\[ \text{Cov}_j(\hat{V}_t, V_s) = \text{Cov}_j(\hat{V}_s, V_s) + \text{Cov}_j \left( \int_{(s, t]} V_u - dA_u, V_s \right) + \text{Cov}_j \left( \int_{(s, t]} A_u - dV_u, V_s \right) \tag{4.9} \]
\[ + \text{Cov}_j \left( [A, V]_t - [A, V]_s, V_s \right). \]

Hereby, via (1.2) we have
\[ \text{Cov}_j \left( \int_{(s, t]} A_u - dV_u, V_s \right) = \text{Cov}_j \left( \int_{(s, t]} \hat{V}_u - dU_u, V_s \right) + \text{Cov}_j \left( \int_{(s, t]} A_u - dL_u, V_s \right), \quad \text{and} \]
\[ \text{Cov}_j \left( [A, V]_t - [A, V]_s, V_s \right) = \text{Cov}_j \left( \int_{(s, t]} V_u - d[A, U]_u, V_s \right) + \text{Cov}_j \left( \int_{(s, t]} d[A, L]_u, V_s \right), \]
while due to (2.7)
\[ \text{Cov}_j \left( \int_{(s, t]} V_u - dA_u, V_s \right) = \text{Cov}_j \left( Q^\top \int_{(s, t]} \hat{V}_u du, V_s \right) + \text{Cov}_j \left( \int_{(s, t]} V_u - dM_u, V_s \right). \]
As \( E_j[\sup_{0 < s \leq t} |V_s V_t|] < \infty \) by Lemma 1.3, Lemma 2.3 yields that
\[ \text{Cov}_j \left( \int_{(s, t]} V_u - dM_u, V_s \right) = E_j \left[ \int_{(s, t]} V_s V_u - dM_u \right] - E_j[V_s] E_j \left[ \int_{(s, t]} V_u - dM_u \right] = 0. \]
Inserting the above in (4.9) and applying (3.16) we obtain by a straightforward computation

$$\text{Cov}_{j}(\hat{V}_t, V_s)$$

$$= \text{Cov}_{j}(\hat{V}_s, V_s) + (Q^T + \epsilon[U]) \int_{(s,t]} \text{Cov}_{j}(\hat{V}_u, V_s) du + \epsilon[L] \int_{(s,t]} \text{Cov}_{j}(A_u, V_s) du. \quad (4.10)$$

Moreover, again using (2.7) and Lemma 2.3, we derive

$$\text{Cov}_{j}(A_t, V_s) = \text{Cov}_{j}(A_x, V_s) + \text{Cov}_{j}(Q^T \int_{(s,t]} A_u du, V_s) + \text{Cov}_{j}(\int_{(s,t]} dM_u, V_s)$$

$$= \text{Cov}_{j}(A_x, V_s) + Q^T \int_{(s,t]} \text{Cov}_{j}(A_u, V_s) du. \quad (4.11)$$

Equations (4.10) and (4.11) together can now be reformulated in matrix and differential form as

$$\frac{d}{dt} \begin{pmatrix} \text{Cov}(A_t, V_s) \\ \text{Cov}(\hat{V}_t, V_s) \end{pmatrix} = \begin{pmatrix} Q^T & 0 \\ \epsilon[L] & Q^T + \epsilon[U] \end{pmatrix} \begin{pmatrix} \text{Cov}(A_s, V_s) \\ \text{Cov}(\hat{V}_s, V_s) \end{pmatrix} =: K \begin{pmatrix} \text{Cov}(A_s, V_s) \\ \text{Cov}(\hat{V}_s, V_s) \end{pmatrix}, \quad 0 \leq s \leq t,$$

with unique solution as given in (4.8).

The autocovariance function is thus decreasing exponentially if and only if all eigenvalues of $K$ have non-positive real part. Denoting $I = \text{diag}(1)$, we see that

$$\det(K - \lambda I) = \det(Q^T - \lambda I) \det(Q^T + \epsilon[U] - \lambda I).$$

As, in our setting, all eigenvalues of the intensity matrix $Q$ have non-positive real part, cf. [29], it is sufficient to consider the eigenvalues of $Q^T + \epsilon[U]$, i.e. the eigenvalues of $\Psi_{\xi}(-1)$ by (4.3). This finishes the proof.

### 4.2 Stationary MMGOU processes

As already mentioned in the introduction, under our standing assumption that $J$ is defined on a finite state space $S$ and is ergodic with stationary distribution $\pi$, the MMGOU process (1.1) admits a non-trivial stationary distribution if and only if the integral $\int_{(0,t]} e^{\xi_t} - dL_s$ converges $P_\pi-$almost surely as $t \to \infty$ to some finite-valued random variable $V_\infty$. In this case the stationary distribution of the MMGOU process under $P_\pi$ is uniquely determined as the distribution of $V_\infty$ given in (1.3). In this section we analyse the moments of $V_\infty$. We start by providing conditions for their existence before presenting explicit formulas for some integer moments.

**Theorem 4.6.** Consider the MMGOU process $(V_t)_{t \geq 0}$ driven by the bivariate MAP $((\xi, \eta), J)$ and solving the SDE (1.2) for the bivariate MAP $((U, \hat{L}), J)$ in (4.11). Assume that $\lim_{t \to \infty} \xi_t = \infty \ P_\pi$-a.s. and that there exists $\kappa \geq 1$ such that for all $j \in S$

$$E_j^\pi \left[ \int_{(0,\tau_{j}^*]} e^{\xi_{t}} - dL^*_t \right] < \infty \quad \text{and} \quad E_j^\pi \left[ e^{\kappa \xi_{\tau_{j}^*}} \right] < 1. \quad (4.12)$$

Then $(V_t)_{t \geq 0}$ has a stationary solution with distribution $V_t \overset{d}{=} V_\infty$, $t \geq 0$, and in particular

$$E_\pi [V_\infty^\kappa] < \infty.$$

**Proof.** As shown in [7, Thm. 3.3], there exists a finite random variable $V_\infty$ such that $(V_t)_{t \geq 0}$ started with $V_0 \overset{d}{=} V_\infty$ is strictly stationary, if the exponential functional $\int_{(0,t]} e^{\xi_t} - dL_s$ converges in $P_\pi$-probability as $t \to \infty$. This however follows via [6, Prop’s 5.2 and 5.7] from our...
assumptions $\lim_{t \to \infty} \xi_t = \infty \mathbb{P}_\pi$-a.s. and $\mathbb{E}_j[|\int_{(0,\tau^*_{\pi^*}(j))} e^{\xi^*_t - dL^*_s}|^\kappa] < \infty$, since the latter implies $\mathbb{E}_j[\log^+ (\int_{(0,\tau^*_{\pi^*}(j))} e^{\xi^*_t - dL^*_s})] < \infty$.

It thus remains to prove the finiteness of the $\kappa$th moment of $V_\infty$.

By [7, Thm. 3.3] and (2.4) it holds

$$\mathbb{P}^\pi_j (V_\infty \in \cdot) = \mathbb{P}^\pi_j \left( - \int_{(0,\infty)} e^{\xi^*_t - dL^*_s} \in \cdot \right) = \sum_{j \in S} \pi^*_j \cdot \mathbb{P}^\pi_j \left( - \int_{(0,\infty)} e^{\xi^*_t - dL^*_s} \in \cdot \right).$$

Fix $j \in S$ and let $N^*_{r,\pi^*,r^*}$ denote the number of returns of $J^*$ to state $j$ up to time $t > 0$. Then under $\mathbb{P}^\pi_j$ for any $t \geq 0$ we can rewrite the exponential integral as

$$- \int_{(0,t]} e^{\xi^*_t - dL^*_s} = - \int_{(0,\tau^*_{N^*_{r,\pi^*,r^*}}(j)^*)} e^{\xi^*_t - dL^*_s} - \int_{(\tau^*_{N^*_{r,\pi^*,r^*}}(j)^*,t]} e^{\xi^*_t - \xi^*_{r,\pi^*,\tau}} dL^*_s.$$  

Using the same arguments as in the proof of [5] Prop. 4.12], the last factor $\int_{(\tau^*_{N^*_{r,\pi^*,r^*}}(j)^*,t]} \exp(\xi^*_t - \xi^*_{r,\pi^*,\tau}) dL^*_s$ converges in distribution as $t \to \infty$, while $e^{\xi^*_{r,\pi^*,\tau}}$ converges to zero a.s. by assumption. Hence, Slutsky’s theorem yields

$$\lim_{t \to \infty} \left( - \int_{(0,t]} e^{\xi^*_t - dL^*_s} \right) = \lim_{t \to \infty} \left( - \int_{(0,\tau^*_{N^*_{r,\pi^*,r^*}}(j)^*)} e^{\xi^*_t - dL^*_s} \right)$$

under $\mathbb{P}^\pi_j$.

Thus also,

$$\mathbb{E}^\pi_j \left[ \lim_{t \to \infty} \left( - \int_{(0,t]} e^{\xi^*_t - dL^*_s} \right) \right] = \mathbb{E}^\pi_j \left[ \lim_{t \to \infty} \left( - \int_{(0,\tau^*_{N^*_{r,\pi^*,r^*}}(j)^*)} e^{\xi^*_t - dL^*_s} \right) \right]$$

and again by Slutsky’s theorem

$$\mathbb{E}^\pi_j \left[ \lim_{n \to \infty} \left( - \int_{(0,\tau^*_{n}(j)^*)} e^{\xi^*_t - dL^*_s} \right) \right] = \mathbb{E}^\pi_j \left[ \int_{(\tau^*_{n}(j)^*,\tau^*_{n+1}(j)^*)} e^{\xi^*_t - \xi^*_{r,\pi^*,\tau}} dL^*_s \right].$$

It remains to show finiteness of the latter to prove the statement. To this aim we define

$$B^*_{m,j} := \int_{(\tau^*_{m-1}(j)^*,\tau^*_{m}(j)^*)} e^{\xi^*_t - \xi^*_{r,\pi^*,\tau}} dL^*_s, \quad m \in \mathbb{N}.$$  

Then under $\mathbb{P}^\pi_j$ the sequence $(B^*_{m,j})_{m \geq 1}$ is i.i.d. with $\mathbb{E}^\pi_j[|B^*_{1,j}|^\kappa] < \infty$ and for every $m \geq 1$ the variable $B^*_{m,j}$ is independent of $\xi^*_{r,\pi^*,\tau_{m-1}}(j)^*$. We can thus follow the lines of the proof of [24] Prop. 4.1 and derive via Hölder’s inequality that

$$\mathbb{E}^\pi_j \left[ \left( \int_{(0,\tau^*_{n}(j)^*)} e^{\xi^*_t - dL^*_s} \right)^\kappa \right] \leq \mathbb{E}^\pi_j \left[ |B^*_{1,j}|^\kappa \right] \left( \sum_{\ell=1}^n \mathbb{E}^\pi_j \left[ \left( e^{\xi^*_{r,\pi^*,\tau_{\ell-1}(j)^*}} \right)^\kappa \right] \right) \left( \sum_{\ell=1}^n \mathbb{E}^\pi_j \left[ e^{\xi^*_{r,\pi^*,\tau_{\ell-1}(j)^*}} \right] \right)^{\frac{\kappa-1}{\kappa}},$$

where for $\kappa = |\kappa|$ the second factor can be omitted. To show the absolute convergence of the two sums, note that for any $\ell \geq 1$ it holds

$$\xi^*_{r,\pi^*,\tau_{\ell-1}(j)^*} = \sum_{m=1}^\ell \left( \xi^*_{r,\pi^*,\tau_{m}(j)^*} - \xi^*_{r,\pi^*,\tau_{m-1}(j)^*} \right),$$

(4.14)
which is a sum of i.i.d. random variables \((\xi_{\tau_{m}^1(j)^*}, -\xi_{\tau_{m-1}^1(j)^*})_{m \geq 1}\) with \(\xi_{\tau_{m}^1(j)^*}, -\xi_{\tau_{m-1}^1(j)^*} \overset{d}{=} \xi_{\tau_{m}^1(j)^*}\).

Hence
\[
E_j^* \left[ e^{\kappa \xi_{\tau_{m}^1(j)^*}} \right] = E_j^* \left[ e^{\kappa \xi_{\tau_{m-1}^1(j)^*}} \right], \quad \ell \geq 1,
\]
and inserting this in (4.14) we get
\[
E_j^* \left[ \int_{(0, \tau_{m}^1(j)^*]} e^{\kappa \xi_{\tau_{m}^1(j)^*}} dL_s^\kappa \right] \leq E_j^* \left[ |B_{1,j}^\kappa| \right] \left( \sum_{\ell = 1}^n E_j^* \left[ e^{\kappa \xi_{\tau_{m}^1(j)^*}} \right] \right)^{\frac{1}{n}} \left( \sum_{\ell = 1}^n E_j^* \left[ e^{\kappa \xi_{\tau_{m-1}^1(j)^*}} \right] \right)^{\frac{1}{n}}.
\]

Passing to the limit as \(n \to \infty\) this converges absolutely due to assumption (4.12). Since we already proved the existence of the limit in (4.13) this implies the statement.

**Corollary 4.7.** Consider the MMGOU process \((V_t)_{t \geq 0}\) driven by the bivariate MAP \(((\xi, \eta), J)\) and solving the SDE (1.2) for the bivariate MAP \(((\bar{U}, L), J)\) in (1.1). Assume that there exists \(\kappa \geq 1\) such that

(i) \(\psi_{\xi}(\kappa) \in (q_{ij})\) for all \(j \in S\), and \(E[\exp(-\kappa|Z_{\xi,1}^j|)] < \infty\) for all \((i, j) \in S^2\) with \(q_{ij} \neq 0\),

(ii) \(\max_{i \in S, i \neq j} \left( \frac{1}{q_{ij}} - \psi_{\xi}(\kappa) \sum_{\ell \in S, \ell \neq i, j} q_{\ell i} E[e^{-\kappa Z_{\ell,1}^j}] \right) < 1, \quad j \in S,
\)

(iii) the leading eigenvalue \(\lambda_{\kappa}^\xi(\kappa)\) of \(\Psi_{\xi}(\kappa)\) is negative, i.e. \(\Psi_{\xi}(\kappa)\) is negative definite,

(iv) \(E[|L_{1,j}^j|^\kappa] < \infty\) for all \(j \in S\) and \(E[|Z_{L,1}^{ij}|^\kappa] < \infty\) for all \((i, j) \in S^2\) such that \(q_{ij} > 0\).

Then \((V_t)_{t \geq 0}\) has a stationary solution with distribution \(V_t \overset{d}{=} V_\infty, \ t \geq 0\), and in particular \(E_{\infty}[|V_\infty|^\kappa] < \infty\).

**Proof.** We will check the conditions of Theorem 4.6 to prove the statement. First of all, since the function \(w \mapsto \lambda_\kappa(w)\) is strictly convex and smooth, cf. [22] Prop. 3.4, and since \(\lambda_\kappa(0) = 0\), assumption (iii) implies that \(\lambda_\kappa(0) > 0\). As moreover \(\lambda_\kappa(0+) = E_{\infty}[\xi_1]\), we conclude that \(\lim_{t \to \infty} \xi_t = \infty\) a.s. as \(t \to \infty\), cf. [31] Cor. XI.2.9. To prove the conditions in (4.12), observe first that due to assumptions (i) and (ii) by Proposition (3.3) (i) we have
\[
E_j^* \left[ e^{\kappa \xi_{\tau_{m}^1(j)^*}} \right] < \infty
\]
for all \(j \in S\). Further note that relation (2.13) entails that if \(\Psi_{\xi}(\kappa)\) is negative definite the same holds true for \(\Psi_{\xi}(\kappa)\). Thus by Proposition (3.3) (ii) we even have \(E_j^*[e^{\kappa \xi_{\tau_{m}^1(j)^*}}] < 1\) for all \(j \in S\).

Lastly, \(E_j^*[\int_{(0, \tau_{m}^1(j)^*]} e^{\kappa \xi_{\tau_{m}^1(j)^*}} dL_s^\kappa] < \infty\) follows by Theorem 3.5 if we can guarantee that \(E_j^*[|L_{1,j}^j|^\kappa] < \infty\) and \(E_j^*[\sup_{0 < s \leq \tau_{m}^1(j)^*} e^{\kappa \xi_{\tau_{m}^1(j)^*}}] < \infty\). Hereby, \(E_j^*[|L_{1,j}^j|^\kappa] < \infty\) follows from assumption (iv) via Lemma 3.1 since \(L_{1,j}^j \overset{d}{=} -L_{1,j}^j\) and \(Z_{L,1}^{ij} \overset{d}{=} -Z_{L,1}^{ij}\). Moreover, \(E_j^*[\sup_{0 < s \leq \tau_{m}^1(j)^*} e^{\kappa \xi_{\tau_{m}^1(j)^*}}] < \infty\) is equivalent to \(E_j^*[e^{\kappa \xi_{\tau_{m}^1(j)^*}}] < \infty\) due to [35] Thm. 25.18 and a similar computation as in (5.13) conditioning on \(\tau_{m}^1(j)^*\).

Under the above derived conditions for the existence of the \(\kappa\)-th moment of the stationary distribution, the following theorem provides explicit formulae for the first and second moment.
Theorem 4.8. Consider the MMGOU process \((V_t)_{t\geq 0}\) driven by the bivariate MAP \(((\xi, \eta), J)\) and solving the SDE (1.2) for the bivariate MAP \(((U, L), J)\) in (1.1). Assume the conditions of Corollary 4.7 are fulfilled for \(\kappa = 1\), or \(\kappa = 2\), respectively. Then the first two moments of the stationary distribution \(V_\infty\) of the MMGOU process are given by

\[
E_\pi [V_\infty] = -1^T \cdot (Q^T + \epsilon(U))^{-1} \cdot \epsilon[L] \cdot \pi, \quad \text{and}
\]

\[
E_\pi [V_\infty^2] = 1^T (Q^T + 2\epsilon[U] + \epsilon[[U, U]])^{-1} \cdot (2 (\epsilon[L] + \epsilon[[U, U]]) (Q^T + \epsilon[U])^{-1} \epsilon[L] - \epsilon[[L, L]]) \pi.
\]

Proof. To compute the mean, we follow the lines of the proof of Theorem 4.3 where we note that all needed conditions are met due to Lemma 3.1 and Lemma 4.1, due to the relation (4.1), and due to Corollary 4.7. In particular, by Lemma 4.3, it holds \(E_\pi [\sup_{0\leq s \leq t} |V_s|] < \infty\). As we are in the stationary regime here, we clearly have that \(E_\pi [\hat{V}_t] = E_\pi [\hat{V}_0]\) and \(E_\pi [\Lambda_t] = \pi\) for all \(t \geq 0\). Thus (4.14) simplifies to

\[
0 = Q^T \int (0,t] E_\pi [\hat{V}_s] ds + \epsilon[U] \int (0,t] E_\pi [\hat{V}_s] ds + \epsilon[L] \int (0,t] E_\pi [\Lambda_s] ds
\]

\[
\Rightarrow 0 = (Q^T + \epsilon[U]) E_\pi [\hat{V}_0] + \epsilon[L] \cdot \pi.
\]

This immediately implies

\[
E_\pi [V_\infty] = 1^T E_\pi [\hat{V}_0] = -1^T (Q^T + \epsilon(U))^{-1} \cdot \epsilon[L] \cdot \pi \quad (4.15)
\]

as stated, where we note that by (4.3) \(Q^T + \epsilon[U] = \Psi_\xi(-1)\), which is invertible as by assumption \(\lambda^\xi_{\max}(-1) < 0\).

For the second moment, following the first lines of the proof of Theorem 3.10 we derive

\[
E_\pi [V_t^2 \Lambda_t] = E_\pi [V_0^2 \Lambda_0] + Q^T \int (0,t] E_j [V_s^2 \Lambda_s] ds + 2E_j \left[ \int (0,t] \hat{V}_s dV_s \right]
\]

\[
+ E_j \left[ \int (0,t] \Lambda_s - d[V, V]_s \right] + E_j \left[ V^2, \Lambda_t \right]. \quad (4.16)
\]

Hereby notice that Lemma 2.3 was applicable since \(E_j [\sup_{0 \leq s \leq t} |V_s|^2] < \infty\) follows from our assumptions due to Lemma 4.3.

By stationarity it holds \(E_\pi [V_t^2 \Lambda_t] = E_\pi [V_0^2 \Lambda_0]\). Inserting this, the SDE (1.2) and

\[
[V, V]_t = \int (0,t] V_s^2 d[U, U]_t + 2 \int (0,t] V_s - d[U, L]_s + [L, L]_t, \quad t \geq 0,
\]

in (4.15) then yields by a straightforward computation using (3.10)

\[
0 = Q^T \int (0,t] E_\pi [V_s^2 \Lambda_s] ds + 2\epsilon[U] \int (0,t] E_\pi [V_s^2 \Lambda_s] ds + 2\epsilon[L] \int (0,t] E_\pi [\hat{V}_s] ds
\]

\[
+ \epsilon[[U, U]] \int (0,t] E_\pi [V_s^2 \Lambda_s] ds + 2\epsilon[[U, U]] \int (0,t] E_\pi [\hat{V}_s] ds + \epsilon[[L, L]] \int (0,t] E_\pi [\Lambda_s] ds
\]

\[
= (Q^T + 2\epsilon[U] + \epsilon[[U, U]]) E_\pi [V_0^2 \Lambda_0] t + 2 (\epsilon[L] + \epsilon[[U, L]]) E_\pi [\hat{V}_0] t + \epsilon[[L, L]] \pi t.
\]

Inserting the formula (4.15) for the first moment this proves

\[
(Q^T + 2\epsilon[U] + \epsilon[[U, U]]) E_\pi [V_0^2 \Lambda_0] = 2 (\epsilon[L] + \epsilon[[U, L]]) (Q^T - \epsilon[L] \pi - \epsilon[[L, L]] \pi).
\]
Then implying that stationary distribution, given they exist. For \( k > 1 \) as in the proof of Proposition 4.2 one can derive a recursion formula for higher moments of the moment. This agrees with the recursion for the moments of the MMOU process derived in [18, Sec. 3.4].

Remark 4.9. With a similar procedure as used in the proof of Theorem 1.8 and using ideas as in the proof of Proposition 4.2 one can derive a recursion formula for higher moments of the stationary distribution, given they exist. For \( k > 2 \) this yields the recursion

\[
\mathbb{E}_π[V_{k}^X] = 1^T \mathbb{E}_π[V_{k}^X A_0]
\]

\[
= -1^T \left( Q^T + kε[U^c] + \frac{k(k-1)}{2} ε[(U^c,U^c)] + ε \left[ \sum_{0 < s \leq j} ((1 + ΔU_s)^k - 1) \right] \right)^{-1} \cdot \left[ kε[L^c] + k(k-1)ε[(U^c,L^c)] + \left[ \sum_{0 < s \leq j} k ΔL_s (1 + ΔU_s)^{k-1} \right] \right] \mathbb{E}_π[V_{k-1}^X A_0]
\]

\[
+ \left[ \frac{k(k-1)}{2} ε[(L^c)] \right] \mathbb{E}_π[V_{k-2}^X A_0] + \sum_{n=2}^k ε \left[ \sum_{0 < s \leq j} \left( \frac{k}{n} (1 + ΔU_s)^{k-n} ΔL_s \right) \mathbb{E}_π[V_{k-n}^X A_0] \right],
\]

assuming that all needed moment conditions are fulfilled. Note that by Proposition 4.2 the inverted matrix in the recursion equals \( \Psi_ξ(−k) \) and hence the inverse exists, whenever \( \lambda_{max}(−k) < 0 \). Further, we observe that for continuous processes \( U \) and \( L \) the recursion stops with the \( k − 2 \)nd moment. This agrees with the recursion for the moments of the MMOU process derived in [18, Sec. 3.4].

5 Additional proofs for the results in Section 3

5.1 Proofs for the results in Section 3.1

*Proof of Lemma 3.7.* Throughout the proof we set \( t = 1 \). The proof for general \( t > 0 \) works alike. Obviously (i) implies (ii), while the equivalence of (ii) and (iii) is immediate since \( \mathbb{E}_π[X_1^{|t|}] = \sum_{j=1}^{S_1} π_j \mathbb{E}_j[X_1^{|t|}] \). To prove the remaining conclusions we follow ideas from the proof of [13, Thm. 34]: Assume first (iii). Fix any \( j \in S \) and consider the event \( O \), that \( J \) has no transition in \([0, 1]\). Then

\[
\infty > \mathbb{E}_j [X_1^{|t|}] = \mathbb{P}_j(O) \cdot \mathbb{E}_j [X_1^{|t|} | O] + (1 - \mathbb{P}_j(O)) \mathbb{E}_j [X_1^{|t|} | O^c] ≥ e^{-\theta_j k} \mathbb{E}[|X_1^{|t|}|],
\]

implying that \( \mathbb{E}[|X_1^{|t|}|] < \infty \) for all \( j \in S \). Now consider the event \( I \), that the first transition of \( J \) happens before \( t = 1 \) and the second after \( t = 1 \). Then

\[
\infty > \mathbb{E}_j [X_1^{|t|}] ≥ \mathbb{P}_j(I) \cdot \mathbb{E}_j [X_1^{|t|} | I]
\]

\[
= \int_0^1 |q_{jj}| e^{-\theta_j k} \sum_{i \neq j} \frac{q_{ij}}{|q_{jj}|} e^{-\theta_i(1-s)} \mathbb{E}_j [X_s^{|t|} + Z_{X,1}^{ji} + X_{1-s}^{(i)|}] ds.
\]

Hence, especially, \( \mathbb{E}_j [X_s^{|t|} + Z_{X,1}^{ji} + X_{1-s}^{(i)|}] < \infty \) Lebesgue a.e. in \([0, 1]\) for every \( (i, j) \in S^2 \), and we can conclude with (4.3) that for any such \( s \in [0, 1] \)

\[
\mathbb{E}_j [Z_{X,1}^{ji}] = \mathbb{E}_j [(X_s^{|t|} + Z_{X,1}^{ji} + X_{1-s}^{(i)|})^n]
\]
Finally, we show that (iv) implies (i). So, assume (iv) and note that by [35, Thm 25.18] our
This implies (iv).

Via Minkowski’s inequality and (3.5) this yields

\[ \mathbb{E}_j \left[ \sup_{0 < s \leq 1} |X_s|^\kappa \right] \leq \mathbb{E}_j \left[ \left( \sum_{j \in S} \sup_{0 < s \leq 1} |X_s(j)| \right)^\kappa \right] \]

\[ \leq |S|^{\frac{n-1}{\kappa}} \left( \sum_{j \in S} \mathbb{E}\left[ \sup_{0 < s \leq t} |X_s(j)|^\kappa \right] \right)^{\frac{1}{\kappa}} + \left( |S|^2 - |S| \right) \frac{n-1}{\kappa} \mathbb{E}_j[|N_1|^\kappa]\left( \sum_{i, \ell} \mathbb{E}_j[|Z_{X,n,i,\ell}|^{\kappa}] \right)^{\frac{1}{\kappa}} < \infty, \]

where, for the second summand, we used the independence of \( N_1 \) and \( Z_{X,n,i,\ell} \), that \( Z_{X,n,i,\ell} \sim Z_{X,1,i,\ell} \), and that \( N_1 \) has finite moments of all orders.

**Proof of Lemma 3.2.** W.l.o.g. we consider \( \sup_{s \in [0,1]} \). The proof for general \( t < \infty \) is similar.

As earlier on, let \( N_{i \rightarrow j} \) denote the number of transitions of \( J \) from the state \( i \) to the state \( j \) up to time \( t > 0 \). The Lévy processes \( X(j), j \in S \), are independent and also independent of the variables \( Z_{i,\ell} \), and of the counting processes \( N_{i \rightarrow j} \) for all \( i, j \in S \). Thus by monotonicity of the exponential function and using (2.3), we get

\[ \mathbb{E}_\pi \left[ \sup_{s \in [0,1]} e^{\kappa |X_s|} \right] = \mathbb{E}_\pi \left[ \exp \left( \kappa \cdot \sup_{s \in [0,1]} |X_s| \right) \right] \]

\[ \leq \mathbb{E}_\pi \left[ \exp \left( \kappa \left( \sum_{j \in S} \sup_{s \in [0,1]} |X_s(j)| + \sum_{i, j \in S} \sum_{\ell = 1}^{N_{i \rightarrow j}} |Z_{i,\ell,j}| \right) \right) \right] \]

\[ = \left( \prod_{j \in S} \mathbb{E}_\pi \left[ \sup_{s \in [0,1]} e^{\kappa |X_s(j)|} \right] \right) \cdot \mathbb{E}_\pi \left[ \exp \left( \kappa \sum_{i, j \in S} \sum_{\ell = 1}^{N_{i \rightarrow j}} |Z_{i,\ell,j}| \right) \right]. \quad (5.17) \]

The first factor does not involve the initial distribution of \( J \), and by [35 Thms. 25.18 and 25.3] for all \( j \in S \)

\[ \mathbb{E} \left[ \sup_{s \in [0,1]} e^{\kappa |X_s(j)|} \right] < \infty \iff \mathbb{E} \left[ e^{\kappa |X(j)|} \right] < \infty \iff \int_{|x| \geq 1} e^{\kappa |x|} \nu_{X,j}(dx) < \infty. \]

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It thus remains to prove that the second factor in (5.17) is finite if and only if \( E[e^{\kappa Z_{X,1}^{ij}}] < \infty \) for all \( i, j \in S \). Since \((N_{1}^{i-j})_{i,j \in S}\) is independent of the i.i.d. sequences \( \{Z_{X,\ell}^{ij}, \ell \in \mathbb{N}\} \), conditioning on \((N_{1}^{i-j})_{i,j \in S} = (n_{ij})_{i,j \in S} =: \mathbf{n}\), yields

\[
\mathbb{E}_{\pi}\left[ \exp\left( \kappa \sum_{i,j \in S} \sum_{\ell=1}^{N_{1}^{i-j}} |Z_{X,\ell}^{ij}| \right) \right] = \sum_{\mathbf{n} \in \mathbb{N}^{S} \times |S|} \mathbb{P}_{\pi}\left( (N_{1}^{i-j})_{i,j \in S} = \mathbf{n} \right) \prod_{i,j \in S} \left( \mathbb{E}_{\pi}\left[ e^{\kappa Z_{X,1}^{ij}} \right] \right)^{n_{ij}},
\]

which is finite if and only if \( \mathbb{E}[e^{\kappa Z_{X,1}^{ij}}] < \infty \) for all \((i,j) \in S^{2}\) such that \( q_{ij} > 0 \).

**Proof of Proposition 3.3.** Recall first that the transition probabilities of the embedded discrete time Markov chain of \( J \) are given by \( p_{ij} = \frac{q_{ij}}{|\mathcal{Q}|} \) for any \( i, j \in S \). Further, the corresponding Lévy processes \( X^{(j)} \) and the additional jumps \( Z^{ij} \) are all independent, and independent of the holding times of \( J \) in any state. Moreover, for an exponentially distributed random variable \( T \) with rate \( q \), independent of a Lévy process \( (Y_{t})_{t \geq 0} \), the Laplace transform of \( Y_{T} \), whenever it exists, is given by,

\[
\mathbb{E}[e^{\kappa Y_{T}}] = \frac{q}{q - \psi_{Y}(\kappa)},
\]
cf. [36] Eq. (I.3.9)], where \( \psi_{Y} \) is the Laplace exponent of \( Y \) and supposed to exist.

Now we return to our setting. W.l.o.g. we consider \( \mathbb{E}_{1}[e^{\kappa X_{1}^{(i)}}] \), and define the matrices

\[
\mathbf{R} \in \mathbb{R}^{S \times S} \text{ with } R_{ij} := \begin{cases} 0, & i = j, \\ \frac{q_{ij}}{q_{ii} - \psi_{X^{(i)}}(\kappa)} \mathbb{E}[\exp(\kappa Z_{X,1}^{ij})], & i \neq j, \end{cases}
\]

and \( \mathbf{L} \in \mathbb{R}^{S \times S} \) with \( L_{ij} := \begin{cases} 0, & i = 1 \text{ or } j = 1, \\ R_{ij}, & \text{else}. \end{cases} \)

Clearly, by our assumptions, \( \mathbf{R} \) (and hence \( \mathbf{L} \)) is well-defined for the chosen value of \( \kappa \).

Using the notations as introduced in Section 2.2 we note that \( T_{N_{1}^{i^{*}(1)}} = \tau_{i^{*}(1)}^{+} \) and hence, using the defined matrices, we have

\[
\mathbb{E}_{1}[e^{\kappa X_{1}^{(i)}}] = \mathbb{E}_{1}\left[ e^{\kappa (X_{1}^{(i)} - X_{\tau_{i^{*}(1)}^{+} - 1}^{(i)})} \cdots e^{\kappa (X_{2}^{(i)} - X_{1}^{(i)})} e^{\kappa X_{1}^{(i)}} \right] = \sum_{n \geq 2} \mathbb{E}_{1}\left[ e^{\kappa (X_{n}^{(i)} - X_{n-1}^{(i)})} \cdots e^{\kappa (X_{2}^{(i)} - X_{1}^{(i)})} e^{\kappa X_{1}^{(i)}} \right] \mathbb{P}(N_{1}^{i^{*}(1)} = n) = e_{1}^{\top} \cdot \mathbf{R}^{\top} \cdot \left( \sum_{k \geq 0} (\mathbf{L}^{\top})^{k} \right) \mathbf{R} \cdot e_{1}.
\]

We thus observe that \( \mathbb{E}_{1}[e^{\kappa X_{1}^{(i)}}] < \infty \) if and only if the geometric series in (5.18) converges. By [9] Prop. 9.4.13 and Cor. 9.4.10 this holds in particular if \( \|\mathbf{L}\|_{\text{row}} = \|\mathbf{L}\|_{\text{col},\infty} < 1 \). This in turn is equivalent to (5.17) by the definition of \( \mathbf{L} \), thus finishing the proof of (i).

For (ii) assume that \( \mathbb{E}_{1}[e^{\kappa X_{1}^{(i)}}] < \infty \) such that the series in (5.18) converges to

\[
\sum_{k \geq 0} (\mathbf{L}^{\top})^{k} = (\mathbf{I} - \mathbf{L}^{\top})^{-1}.
\]
Recall that for any square matrix $A \in \mathbb{R}^{d \times d}$ the adjugate matrix $A^\dagger = (A^\dagger)_{i,j=1,...,d}$ is defined via

$$A^\dagger_{i,j} := (-1)^{i+j} \det(A_{(j,i)}),$$

where $A_{(i,j)} \in \mathbb{R}^{(d-1) \times (d-1)}$ is the matrix that results from deleting the $i$th row and $j$th column of $A$. Since for an invertible matrix $A$ it holds $A^{-1} = (\det A)^{-1} A^\dagger$, and since $R_{11} = 0$, (5.18) yields

$$\begin{align*}
E_1[e^{X_{i}^{(1)}}] &= e_1^\top \cdot R^\top (I - L)^{-1} \cdot e_1 = \frac{1}{\det(I - L)} e_1^\top \cdot R^\top (I - L)^\dagger \cdot R^\top \cdot e_1 \\
&= \frac{1}{\det(I - L)} \sum_{i=2}^{[S]} \sum_{j=2}^{[S]} (-1)^{i+j} R_{i1} \det((I - L)^\top_{(j,i)}) R_{1j}
\end{align*}$$

Further, a Laplace expansion of $\det(I - R)$ leads to

$$\begin{align*}
\det(I - R) &= \sum_{i=1}^{[S]} (-1)^{1+i} (I - R)_{i1} \det((I - R)_{(i,1)}) \\
&= (1 - R_{11}) \det((I - R)_{(1,1)}) + \sum_{i=2}^{[S]} (-1)^{1+i} (-R_{i1}) \det((I - R)_{(i,1)}) \\
&= \det(I - L) + \sum_{i=2}^{[S]} (-1)^{1+i} (-R_{i1}) \sum_{j=2}^{[S]} (-1)^j (-R_{1j}) \det((I - R)_{(i,1)}(1,j-1)) \\
&= \det(I - L) - \sum_{i=2}^{[S]} \sum_{j=2}^{[S]} (-1)^{i+j} R_{i1} R_{1j} \det((I - L)_{(i,j)}).
\end{align*}$$

Thus, combining (5.19) and (5.20) we obtain

$$E_1[e^{X_{i}^{(1)}}] = \frac{1}{\det(I - L)} (\det(I - L) - \det(I - R)) = 1 - \frac{\det(I - R)}{\det(I - L)}. \quad (5.21)$$

To show the claim, it remains to verify that if $\lambda_{\max}^X(\kappa) < 0$, then the ratio on the right hand side of (5.21) is always positive. To this end we rewrite both determinants in terms of $\det(\Psi_X(\kappa))$ which gives via (2.3)

$$\det(I - R) = (-1)^{[S]} \prod_{i=1}^{[S]} \frac{1}{|q_{ii} - \psi_X(\kappa)|} \det(\Psi_X(\kappa)), \quad (5.22)$$

and

$$\det(I - L) = (-1)^{[S]-1} \prod_{i=2}^{[S]} \frac{1}{|q_{ii} - \psi_X(\kappa)|} \det(\Psi_X(\kappa)_{(1,1)}). \quad (5.23)$$

As $\Psi_X(\kappa)$ is assumed to be negative definite, by Silvester’s criterion (see e.g. [17, Thm. 7.2.5]) it holds $(-1)^{[S]} \det(\Psi_X(\kappa)) > 0$, while $(-1)^{[S]-1} \det(\Psi_X(\kappa)_{(1,1)}) > 0$. Thus (5.22) and (5.23) are positive and the statement follows from (5.21).
Proof of Lemma 3.6. Let \((\gamma, \sigma^2, \nu), W^\sigma\), and \(\mu\) denote the characteristic triplet, the Gaussian part, and the jump measure of the Lévy process \(X\), respectively. Furthermore, we denote by \(\overline{\mu}(dt, dx) := \mu(dt, dx) - d\nu(dx)\) the compensated jump measure of \(X\). We notice that in the given setting \(\int_{(0, \cdot)} \int_{\mathbb{R}} \mu(ds, dx) - \nu(dx)ds\) is a martingale and, for \(\kappa \geq 2\), it is a square integrable martingale.

Because of the Lévy-Itô decomposition of \(X\), since \(\mathbb{E}[|X_t|^\kappa] < \infty\) and \(\kappa \geq 1\), we can write

\[
X_t = \tilde{\gamma} t + W_t^\sigma + \int_{(0,\cdot]} \int_{\mathbb{R}} x \overline{\mu}(ds, dx), \quad t \geq 0,
\]

where \(\tilde{\gamma} := \gamma + \int_{\{|x| > 1\}} x \nu(dx)\). Since \(H_\cdot\) is a locally bounded predictable process, this implies

\[
\int_{(0,\cdot]} H_{s-}dX_s = \tilde{\gamma} \int_{(0,\cdot]} H_{s-}ds + \int_{(0,\cdot]} H_{s-}dW_s^\sigma + \int_{(0,\cdot]} \int_{\mathbb{R}} H_{s-}x \overline{\mu}(ds, dx).
\]

So, by Minkowski’s inequality, we deduce

\[
\mathbb{E}\left[\sup_{0 \leq t \leq \tau} \left| \int_{(0,t]} H_{s-}dX_s \right|^\kappa \right] = \mathbb{E}\left[\sup_{t \geq 0} \left| \int_{(0,t\wedge \tau]} H_{s-}dX_s \right|^\kappa \right]^{\frac{1}{\kappa}} \leq \left|\tilde{\gamma}\right| \mathbb{E}\left[\sup_{t \geq 0} \left| \int_{(0,t\wedge \tau]} H_{s-}ds \right|^\kappa \right]^{\frac{1}{\kappa}} + \mathbb{E}\left[\sup_{t \geq 0} \left| \int_{(0,t\wedge \tau]} H_{s-}dW_s^\sigma \right|^\kappa \right]^{\frac{1}{\kappa}} + \mathbb{E}\left[\sup_{t \geq 0} \left| \int_{(0,t\wedge \tau]} \int_{\mathbb{R}} H_{s-}x \overline{\mu}(ds, dx) \right|^\kappa \right]^{\frac{1}{\kappa}}, \quad (5.24)
\]

Concerning the first summand on the right-hand side in (5.24), we have

\[
\mathbb{E}\left[\sup_{0 \leq t \leq \tau} \left| \int_{(0,t\wedge \tau]} H_{s-}ds \right|^\kappa \right] \leq \mathbb{E}\left[\tau^\kappa \sup_{0 \leq t \leq \tau} |H_t|\right] < \infty,
\]

where the last estimate follows by (3.2) with \(\eta = \kappa\), and \(\varepsilon > 0\) if \(\tau\) is not bounded.

For the Brownian integral in (5.24), by the Burkholder-Davis-Gundy inequality, cf. [33, Cor. IV.(4.2)], we get for some constant \(C_\kappa \in (0, \infty)\)

\[
\mathbb{E}\left[\sup_{t \geq 0} \left| \int_{(0,t\wedge \tau]} H_{s-}dW_s^\sigma \right|^\kappa \right] \leq \sigma^2 C_\kappa \mathbb{E}\left[\left( \int_{(0,\tau]} H_{s-}^2ds \right)^{\kappa/2} \right] \leq \sigma^2 C_\kappa \mathbb{E}\left[\tau^{\kappa/2} \sup_{0 \leq t \leq \tau} |H_t|\right] < \infty,
\]

where the last estimate follows from (3.2) with \(\eta = \kappa/2\) and \(\varepsilon > 0\), if \(\tau\) is not bounded.

It remains to consider the integral with respect to the compensated jump measure in (5.24). By [27, Thm. 1] there exist constants \(C_1^\kappa, C_2^\kappa \in (0, \infty)\) such that

\[
\mathbb{E}\left[\sup_{t \geq 0} \left| \int_{(0,t\wedge \tau]} \int_{\mathbb{R}} H_{s-}x \overline{\mu}(ds, dx) \right|^\kappa \right] \leq \begin{cases} C_1^\kappa \mathbb{E}\left[\int_{(0,\tau]} \int_{\mathbb{R}} |H_{s-}x|^\kappa \nu(dx)ds \right], & \kappa \in [1, 2), \\ C_2^\kappa \mathbb{E}\left[\left( \int_{(0,\tau]} \int_{\mathbb{R}} |H_{s-}x|^2 \nu(dx)ds \right)^{\kappa/2} \right] + \mathbb{E}\left[\int_{(0,\tau]} \int_{\mathbb{R}} |H_{s-}x|^\kappa \nu(dx)ds \right], & \kappa \in [2, \infty), \\ \tilde{C}_1^\kappa \mathbb{E}\left[\tau \sup_{0 \leq t \leq \tau} |H_t|\right], & \kappa \in [1, 2), \\ \tilde{C}_2^\kappa \mathbb{E}\left[\tau^{\kappa/2} \sup_{0 \leq t \leq \tau} |H_t|\right] + C \mathbb{E}\left[\tau \sup_{0 \leq t \leq \tau} |H_t|\right], & \kappa \in [2, \infty), \end{cases}
\]

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where \( \tilde{C}_n^1 := C_n^1 \int_{\mathbb{R}} |x|^\eta \nu(dx) \), \( \tilde{C}_n^2 := C_n^2(\int_{\mathbb{R}} x^2 \nu(dx))^{\eta/2} \) and \( C := \frac{\tilde{C}_n^1}{\tilde{C}_n^2} \tilde{C}_n^2 \) are finite by the assumptions on \( X \). By (3.2) with \( \eta = 1 \) or \( \eta = \kappa/2 \), and with \( \varepsilon > 0 \) if \( \tau \) is not bounded, we see that the right-hand side in the previous estimate is finite. The proof is complete.

\[ \text{Proof of Lemma 3.7.} \] Following [19, Thm. II.1.8], the measure \( \nu_{ij}^2 \) is the \( \mathcal{F} \)-dual predictable projection of \( \mu_{ij}^2 \) if and only if
\[
E \left[ \int_{(0,\infty)} \int_{\mathbb{R}} W(s,x) \mu_{ij}^2(ds, dx) \right] = E \left[ \int_{(0,\infty)} \int_{\mathbb{R}} W(s,x) \nu_{ij}^2(ds, dx) \right] \tag{5.25}
\]
holds for every \( \tilde{\mathcal{P}}(\mathcal{F}) = \mathcal{P}(\mathcal{F}) \otimes \mathcal{B}(\mathbb{R}) \)-measurable non-negative function \( (\omega, t, x) \mapsto W(\omega, t, x) \). Hereby \( \mathcal{P}(\mathcal{F}) \) denotes the \( \sigma \)-algebra of \( \mathcal{F} \)-predictable sets.

Assume that \( W(\omega, t, x) = B_t(\omega)f(x) \), where \( (B_t)_{t \geq 0}, B_t \geq 0 \), is a bounded predictable process and \( f \geq 0 \) a bounded Borel function, then
\[
E \left[ \int_{(0,\infty)} \int_{\mathbb{R}} W(s,x) \mu_{ij}^2(ds, dx) \right] = E \left[ \sum_{n \geq 1} f(Z_{ij}^n) \mathbf{1}_{\{J_{T_n} = i, J_{T_n} = j\}} B_{T_n} \right].
\]
As \( B_{T_n} \) is \( \mathcal{F}_{T_n} \)-measurable, \( B \) being predictable, and \( Z_{ij}^{X,n} \) is independent of \( \sigma(J_{T_n}) \vee \mathcal{F}_{T_n} \) with \( Z_{ij}^{X,n} \) \( \overset{d}{=} \) \( Z_{ij,1} \), we obtain
\[
E \left[ \int_{(0,\infty)} \int_{\mathbb{R}} W(s,x) \mu_{ij}^2(ds, dx) \right] = E \left[ f(Z_{ij}^1) \right] E \left[ \int_{(0,\infty)} B_s dN_s^{-ij} \right] = E \left[ f(Z_{ij}^1) \right] E \left[ \int_{(0,\infty)} B_s d\phi_s^{ij} \right],
\]
where in the last identity we used the properties of the dual predictable projection \( \phi_{ij} \) given in (3.1). Hence, we obtain (5.25) for this special choice of \( W \). Since the class of predictable functions \( W \) of the chosen form generates \( \tilde{\mathcal{P}}(\mathcal{F}) \), the monotone class theorem (cf. [32, Thm. I.1.8]) now yields (5.25) for every non-negative predictable bounded function \( W \). Finally, consider an arbitrary predictable function \( W \geq 0 \). Then \( W^n := W \land n \) is bounded and by the previous step (5.25) holds for \( W^n \). By monotone convergence this implies (5.25) for \( W \) and hence the statement.

\[ \text{5.2 Proofs for the results in Section 3.2} \]

\[ \text{Proof of Theorem 3.8.} \] To prove (3.11) we use the integration by parts formula (2.8) for \( \tilde{X} \), where we first show that the \( \mathbb{R}^{SL} \)-valued local martingale \( \int_{(0,t]} X_s \cdot dM_s \) in (2.8) is an \( \mathbb{R}^{SL} \)-valued centered martingale with respect to \( \mathbb{P}_j \), for every \( j \). Indeed, we have \( E_j[ \sup_{t \in [0,T]} |X_t| ] = E_j[ \sup_{t \geq 0} |X_t| ] < \infty \), for every fixed \( T > 0 \), because of Lemma 3.1. Thus, the claim follows by Lemma 2.3 and we obtain that \( E_j[ \int_{(0,t]} X_s \cdot dM_s ] = 0 \) with \( O \) denoting the column vector of zeros.

So, from (2.8), we obtain
\[
E_j[ \tilde{X}_t ] = E_j[ \tilde{X}_0 ] + \mathbf{Q}^T E_j \left[ \int_{(0,t]} \tilde{X}_s ds \right] + E_j \left[ \int_{(0,t]} \Lambda_s ds \right] + E_j \left[ \left| \Lambda| \tilde{X} \right| \right].
\]
Using that $X_0 = 0$ and $\Delta \Lambda^j = \pm 1$, and recalling the notation introduced in \eqref{eq:27}, this yields

$$
\mathbb{E}_j[\hat{X}_t] = Q^T \mathbb{E}_j \left[ \int_{(0,t]} \hat{X}_s - ds \right] + \mathbb{E}_j \left[ \left( \int_{(0,t]} \mathbb{1}_{\{J_s = i\}} dX_s^{(i)} + \sum_{\ell = 1}^{N_s^{(i)}} \Delta X^{\nu}_{t_{\ell}^{(i)}} \right)_{i \in S} \right] \\
+ \mathbb{E}_j \left[ \left( \sum_{\ell = 1}^{N_s^{(i)}} \Delta X^{(i)}_{t_{\ell}^{(i)}} - \sum_{\ell = 1}^{N_s^{(i)}} \Delta X^{\nu}_{t_{\ell}^{(i)}} \right)_{i \in S} \right] \\
= Q^T \mathbb{E}_j \left[ \int_{(0,t]} \hat{X}_s - ds \right] + \mathbb{E}_j \left[ \left( \int_{(0,t]} \mathbb{1}_{\{J_s = i\}} dX_s^{(i)} \right)_{i \in S} \right] + \mathbb{E}_j \left[ \left( \sum_{\ell = 1}^{N_s^{(i)}} \Delta X^{(i)}_{t_{\ell}^{(i)}} \right)_{i \in S} \right].
$$

(5.26)

As by assumption $\mathbb{E}_j[|X_1|] < \infty$, Lemma \ref{lem:3.1} implies that the processes $X^{(i)}$, $i \in S$, are Lévy processes with $\mathbb{E}[|X^{(i)}_1|] < \infty$ for every $i \in S$. Thus \cite[Lemma 6.1]{4} implies

$$
\mathbb{E}_j \left[ \left( \int_{(0,t]} \mathbb{1}_{\{J_s = i\}} dX_s^{(i)} \right)_{i \in S} \right] = \left( \mathbb{E}\left[X_1^{(i)}\right] \int_{(0,t]} \mathbb{E}_j \left[ \mathbb{1}_{\{J_s = i\}} ds \right] \right)_{i \in S} \\
= \text{diag} \left( \mathbb{E}\left[X_1^{(i)}\right] \right) \left( \int_{(0,t]} \mathbb{P}_j(J_s = i) ds \right)_{i \in S} \\
= \text{diag} \left( \mathbb{E}\left[X_1^{(i)}\right] \right) \int_0^t e^{Q^T s} e_j ds.
$$

(5.27)

via \eqref{eq:20}. It remains to consider the last summand in \eqref{eq:5.26}. Let $N_{k \rightarrow i}^t$ denote the number of transitions of $J$ from the state $k$ to the state $i$ up to time $t > 0$. Then rewriting the sum gives

$$
\mathbb{E}_j \left[ \left( \sum_{\ell = 1}^{N_s^{(i)}} \Delta X^{(i)}_{t_{\ell}^{(i)}} \right)_{i \in S} \right] = \left( \sum_{k \in S} \mathbb{E}_j \left[ \sum_{\ell = 1}^{N_{k \rightarrow i}^t} Z_{X,1}^{k \rightarrow i} \right] \right)_{i \in S} = \left( \sum_{k \in S} \mathbb{E}_j \left[ N_{k \rightarrow i}^t \right] \mathbb{E}\left[Z_{X,1}^{k \rightarrow i}\right] \right)_{i \in S},
$$

by Wald’s equality, since $\{Z_{X,1}^{k \rightarrow i}, \ell \in \mathbb{N}\}$ is an i.i.d. sequence independent of $(N_{k \rightarrow i}^t)_{t \geq 0}$. As e.g. by \cite[Thm. 2]{35}, we have $\mathbb{E}_j[N_{k \rightarrow i}^t] = \int_0^t q_{ki} \mathbb{P}_j(J_s = k) ds$, we further get

$$
\left( \sum_{k \in S} \mathbb{E}_j \left[ N_{k \rightarrow i}^t \right] \mathbb{E}\left[Z_{X,1}^{k \rightarrow i}\right] \right)_{i \in S} = \left( \sum_{k \in S} q_{ki} \mathbb{E}\left[Z_{X,1}^{k \rightarrow i}\right] \int_0^t e_k^T e^{Q^T s} e_j ds \right)_{i \in S} \\
= \left( Q \circ \left( \mathbb{E}\left[Z_{X,1}^{k \rightarrow i}\right] \right)_{i,k \in S} \right)^T \int_0^t e^{Q^T s} e_j ds.
$$

Inserting this and \eqref{eq:5.27} in \eqref{eq:5.26}, and applying a Fubini argument we obtain the following inhomogeneous ODE for $\mathbb{E}_j[\hat{X}_t]$ as function of $t$

$$
\mathbb{E}_j[\hat{X}_t] = Q^T \int_0^t \mathbb{E}_j[\hat{X}_s] ds + \mathbb{E}[X] \int_0^t e^{Q^T s} e_j ds.
$$

This is solved uniquely by

$$
\mathbb{E}_j[\hat{X}_t] = e^{Q^T t} \mathbb{E}_j[\hat{X}_0] + \int_0^t e^{Q^T (t-s)} e[X] e^{Q^T s} e_j ds.
$$

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By our standard assumption $X_0 = 0$ a.s. the first term vanishes and we obtain (3.11). Moreover, multiplying (3.11) with $1^T$ from the left yields (3.12), since $1^T Q^T r = 1^T$.

To finish the proof of the lemma, note that since $E_{\Omega} [|X_t|] < \infty$, we have $E_{\Omega} [|X_t^\#|] < \infty$ for all $j \in S$. We may thus compute

$$E [X_s^\# | F_s] = E [\hat{X}_{t+s} - \hat{X}_s + \hat{X}_s - \epsilon[X] \int_{(0,s]} \Lambda_r \, dr - \epsilon[X] \int_{(s,s+t]} \Lambda_r \, dr | F_s]$$

$$= E [\hat{X}_{t+s} - \hat{X}_s - \epsilon[X] \int_{(s,s+t]} \Lambda_r \, dr | F_s] + \hat{X}_s - \epsilon[X] \int_{(0,s]} \Lambda_r \, dr$$

$$= E_J [\hat{X}_t - \epsilon[X] \int_{(0,t]} \Lambda_r \, dr] + X^\#$$

$$= X^\#,$$

where in the second to last step we have used the MAP property (2.1) in an extended version as shown e.g. in [7, Lem. 2.2].

**Proof of Lemma 3.11** Recall that the process $X^\# := \hat{X} - \epsilon[X] \int_{(0,\cdot]} \Lambda_s \, ds$ defined in Theorem 3.8 is a martingale. Hence, $X^\# := 1^T X^\#$ is a martingale, too, and the stochastic integral $\int_{(0,t]} H_s \cdot dX_s^\#$ is a local martingale. Moreover, it is a true centered martingale, since by the triangle inequality, by the definition of $X^\#$, and by Theorem 3.5 we have

$$E_J \left[ \sup_{0 < u \leq t} \left| \int_{(0,u]} H_s \cdot dX_s^\# \right| \right] \leq E_J \left[ \sup_{0 < u \leq t} \left| \int_{(0,u]} H_s \cdot dX_s \right| \right] + |1^T \epsilon[X]| E_J \left[ \sup_{0 < u \leq t} |H_u| \right] t < \infty.$$  

Thus,

$$E_J \left[ \int_{(0,t]} H_s \cdot dX_s \right] = E_J \left[ \int_{(0,t]} H_s \cdot dX_s^\# \right] + E_J \left[ \int_{(0,t]} H_s \cdot d(1^T \epsilon[X] \int_{(0,s]} \Lambda_v \, dv) \right]$$

$$= 1^T \epsilon[X] E_J \left[ \int_{(0,t]} \hat{H}_s \cdot ds \right],$$

which yields (3.15) by a Fubini argument.

To prove (3.16), for $t \geq 0$ and any $i \in S$, observe that with the notation introduced in (2.5), we get

$$[X, \Lambda]_t = \left( \sum_{\ell=1}^{N_{r,s}^{(i)}} \Delta X_{r,s}^{(i)} - \sum_{\ell=1}^{N_{r,s}^{(i)}} \Delta X_{r,s}^{(i)} \right)_{i \in S},$$

as all components of $\Lambda$ only have jumps of size $\pm 1$. Therefore, by the same arguments as in (3.26),

$$\int_{(0,t]} \hat{H}_s \cdot dX_s + \int_{(0,t]} H_s \cdot d[X, \Lambda]_s$$

$$= \left( \int_{(0,t]} H_s \cdot 1\{J_s = i\} dX_s^{(i)} \right)_{i \in S} + \left( \sum_{\ell=1}^{N_{r,s}^{(i)}} H_{r,s}^{(i)} - \Delta X_{r,s}^{(i)} \right)_{i \in S}. \quad (5.28)$$

Taking the expectation of the components in the first summand in (5.28), [4, Lem. 6.1] gives

$$E_J \left[ \int_{(0,t]} H_s \cdot 1\{J_s = i\} dX_s^{(i)} \right] = E[X_0^{(i)}] \int_{(0,t]} E_J[H_s] 1\{J_s = i\} \, ds,$$
for all $i, j \in S$, which implies

$$
\mathbb{E}_j \left[ \left( \int_{(0,t]} H_{s-} \mathbf{1}_{\{ J_s = i \}} \, dX_s^{(i)} \right)_{i \in S} \right] = \text{diag} \left[ \mathbb{E}[X_1^{(i)}] \right] \int_{(0,t]} \mathbb{E}_j[\mathbf{H}_s] \, ds.
$$

(5.29)

The second summand in (5.28) can be rewritten as

$$
\left( \sum_{\ell=1}^{N^{(i)}} H_{\ell}^{(i)} \Delta X_{\ell}^{(i)} \right)_{i \in S} = \left( \int_{(0,t]} H_{s-} \, dY_{s}^{(i)} \right)_{i \in S}
$$

for the MAPs $(Y^{(i)}, J)$, $i \in S$, defined via $Y_{t}^{(i)} = \sum_{\ell=1}^{N_{t}^{(i)}} \Delta X_{\ell}^{(i)}, \ t \geq 0$. These have the expectation matrices

$$
\mathbb{E}[Y^{(i)}] = Q^T \circ (E[Z_{Y(i),1}^{k,j}])_{k,j \in S} = Q^T \circ \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix},
$$

as follows readily from (2.2). Thus we conclude via (3.15)

$$
\mathbb{E}_j \left[ \left( \sum_{\ell=1}^{N^{(i)}} H_{\ell}^{(i)} \Delta X_{\ell}^{(i)} \right)_{i \in S} \right] = \left( \mathbb{E}_j \left[ \int_{(0,t]} H_{s-} \, dY_{s}^{(i)} \right] \right)_{i \in S}
$$

$$
= \left( 1^T \epsilon Y^{(i)} \int_{(0,t]} \mathbb{E}_j[\mathbf{H}_s] \, ds \right)_{i \in S} = Q^T \circ \left( E[Z_{X,1}^{k,i}] \right)_{k,i \in S} \int_{(0,t]} \mathbb{E}_j[\mathbf{H}_s] \, ds.
$$

(5.30)

Combining (5.29) and (5.30) we get the claim. 

**Proof of Theorem 3.10.** Integration by parts yields $X_t^2 = 2 \int_{(0,t]} X_s - dX_s + [X,X]_s, \ t \geq 0$. Thus, applying (2.8) we observe that

$$
X_t^2 \Lambda_t = Q^T \int_{(0,t]} X_s^2 \Lambda_s - d\Lambda_s + \int_{(0,t]} X_s^2 \, dM_s + 2 \int_{(0,t]} \Lambda_s - dX_s + \int_{(0,t]} \Lambda_s - d[X,X]_s + [X^2, \Lambda]_t.
$$

Taking expectations and applying Lemmas 3.11 and 2.8 this implies

$$
\mathbb{E}_j [X_t^2 \Lambda_t] = Q^T \int_{(0,t]} \mathbb{E}_j[X_s^2 \Lambda_s] \, ds + 2 \mathbb{E}_j \left[ \int_{(0,t]} \tilde{X}_s \, dX_s \right] + \mathbb{E}_j \left[ \int_{(0,t]} \Lambda_s - d[X,X]_s \right] + \mathbb{E}_j \left[ X^2, \Lambda \right]_t.
$$

Recall from Section 2 that $([X,X]_t, J)_{t \geq 0}$ is again an F-MAP and therefore by Theorem 3.8

$$
\mathbb{E}_j \left[ [X,X]_t \right] = 1^T \epsilon [X,X] \int_0^t e^{Q^T s} \, ds \cdot e_j.
$$

Thus via (3.10) we obtain

$$
\mathbb{E}_j [X_t^2 \Lambda_t] = Q^T \int_{(0,t]} \mathbb{E}_j[X_s^2 \Lambda_s] \, ds + 2 \left( \epsilon [X] \int_{(0,t]} \mathbb{E}_j[\tilde{X}_s] \, ds - \mathbb{E}_j \left[ \int_{(0,t]} X_s - d[X, \Lambda]_s \right] \right)
$$

$$
+ \epsilon [X,X] \int_{(0,t]} \mathbb{E}_j[\Lambda_s] \, ds - \mathbb{E}_j \left[ [X,X, \Lambda]_t \right] + \mathbb{E}_j \left[ X^2, \Lambda \right]_t
$$

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\[ Q^\top \int_{(0,t]} \mathbb{E}_j[X_s^2 \Lambda_s] ds + 2 \varepsilon[X] \int_{(0,t]} \mathbb{E}_j[\dot{X}_s] ds + \varepsilon[[X, X]] \int_{(0,t]} \mathbb{E}_j[\Lambda_s] ds. \]

Inserting (2.6) and (5.11) this yields a first order ODE for \( \mathbb{E}_j[X_t^2 \Lambda_t] \) which is uniquely solved by \( \mathbb{E}_j[X_t^2 \Lambda_t] \) as stated in (3.14).

Moreover,

\[
\varepsilon[[X, X]] = \text{diag} \left( \mathbb{E} \left[ [X^{(j)}, X^{(j)}]_1 \right] + Q^\top \circ (\mathbb{E} \left[ (Z^{ij})^2 \right])^T_{i,j \in S} \right)
\]

\[
= \text{diag} \left( \text{Var}(X^{(j)}) + Q^\top \circ (\mathbb{E} \left[ (Z^{ij})^2 \right])^T_{i,j \in S} \right)
\]

as stated. Finally, the stated formula for the variance is easily derived from (3.14) via

\[
\text{Var}_j(X_t) = 1^T \mathbb{E}_j[X_t^2 \Lambda_t] - \mathbb{E}_j[X_t]^2.
\]

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