Complexity of Self-similar Hierarchical Ensembles

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Abstract

Within the framework of generalized combinatorial approach, the complexity is determined for infinite set of self-similar hierarchical ensembles. This complexity is shown to increase with strengthening of the hierarchy coupling to the value, which decreases with growth of both scattering of this coupling and non-extensivity parameter.

Key words: Self-similar hierarchical ensemble; complexity; q-multinomial coefficient.

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Hierarchical structure is one of the most universal peculiarities of complex systems in physics, biology, economics and so on [1]–[5]. Their evolution is shown [6] to reduce to anomalous diffusion process in ultrametric space of the self-similar hierarchical system, whose steady-state distribution over hierarchical levels is given by the Tsallis power-law, being inherent in non-extensive systems [7]. Well-known feature of hierarchical systems consists in that every of statistical ensembles of given level breaks with passage into lower one to subensembles, then every of these breaks to more small subensembles of the following level, and so on (see, for example, Figure 1). From the statistical point of view, the set of above (sub)ensembles is characterized by the complexity, whose value determines the scattering of the hierarchical coupling – in analogy with the entropy in usual statistical systems. This article aims to determine the complexity for self-similar hierarchical ensembles.

According to Ref.[7] the non-extensive statistics is based on the definitions of

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both logarithmic and exponential functions by the equations

\[
\ln_q(x) := \frac{x^{1-q} - 1}{1 - q}, \quad \exp_q(x) := [1 + (1 - q)x]^{\frac{1}{1-q}}; \quad [y]_+ := \max(0, y), \quad q \leq 1,
\]

that are reduced to the usual functions in the limit \( q \to 1 \). Introducing \( q \)-deformed production and ratio of positive values \( x, y \) as follows:

\[
x \otimes_q y = \left[x^{1-q} + y^{1-q} - 1\right]^{\frac{1}{1-q}}, \quad x \odot_q y = \left[x^{1-q} - y^{1-q} + 1\right]^{\frac{1}{1-q}}; \quad x, y > 0,
\]

it is easily to convince that these satisfy to usual properties \( \ln_q(x \otimes_q x) = \ln_q x + \ln_q y, \ln_q(x \odot_q x) = \ln_q x - \ln_q y; \exp_q(x) \otimes_q \exp_q(y) = \exp_q(x + y), \exp_q(x) \odot_q \exp_q(y) = \exp_q(x - y) \).

Within the framework of combinatorial approach [8], the \( q \)-deformed statistics is reduced to consideration of the generalized factorial \( N!_q := 1 \otimes_q \cdots \otimes_q N \),
whose logarithm is as follows:

\[ \ln_q(N!_q) = \frac{\sum_{i=1}^{N} i^{1-q} - N}{1-q}. \]  

(3)

In thermodynamic limit \( N \to \infty \), above sum is estimated by related integral, whose calculation gives

\[ \ln_q(N!_q) = \begin{cases} \frac{N}{2-q} \ln_q N - \frac{N}{2-q} + O(\ln_q N), & q \neq 2, \\ N - \ln N + O(1), & q = 2. \end{cases} \]  

(4)

Defining \( q \)-deformed multinomial coefficient by the equation

\[ \binom{N}{N_1 \ldots N_k}_q := (N!_q) \otimes_q [(N_1!_q) \otimes_q \ldots \otimes_q (N_k!_q)], \]  

(5)

where the set of integers \( N_i \) satisfies to the condition \( N = \sum_{i=1}^{n} N_i \), we find

\[ \binom{N}{N_1 \ldots N_k}_q = \left[ \sum_{i=1}^{N} i^{1-q} - \sum_{i_1=1}^{N_1} i_1^{1-q} - \ldots - \sum_{i_k=1}^{N_k} i_k^{1-q} + 1 \right]^{1/(1-q)}. \]  

(6)

From here, similarly to Eq.(4), one obtains the expression

\[ \ln_q \left( \binom{N}{N_1 \ldots N_k}_q \right) \simeq \begin{cases} \frac{N^{2-q}}{2-q} C_{2-q} \left( \frac{N_1}{N}, \ldots, \frac{N}{N} \right), & q > 0, q \neq 2, \\ -C_1(N) + \sum_{i=1}^{k} C_1(N_i), & q = 2 \end{cases} \]  

(7)

for the Tsallis entropy

\[ C_q(p_1, \ldots, p_N) := -\sum_{i=1}^{N} p_i \ln_q p_i = -\frac{\sum_{i=1}^{N} p_i^q - 1}{1-q}. \]  

(8)

Above formalism is generalized easily for consideration of the hierarchical systems [9]. Assume that the \( N \) states of upper level are distributed over the ensembles \( i = 1, \ldots, n \), every of which contains \( N_i \) states. In turn, \( N_i \) states are bunched in \( m_i \) subensembles \( ij \), every of which contains \( N_{ij} \) states, where the relations \( \sum_{j=1}^{m_i} N_{ij} = N_i, \sum_{i=1}^{n} N_i = N \) are fulfilled. Then, instead of the multinomial coefficient (5) it is necessary to use the expression

\[ \binom{N}{N_{11} \ldots N_{nmn}}_q = \binom{N}{N_1 \ldots N_n}_q \otimes_q \binom{N_1}{N_{11} \ldots N_{1m_1}}_q \otimes_q \cdots \otimes_q \binom{N_n}{N_{n1} \ldots N_{nmn}}_q, \]  

(9)
whose $q$-logarithm is
\[
\ln_q \left( \frac{N}{N_{11} \ldots N_{n_{nm}}} \right)_q = \ln_q \left( \frac{N}{N_{1} \ldots N_{n}} \right)_q + \sum_{i=1}^{n} \ln_q \left( \frac{N_{i_{i1}} \ldots N_{i_{im_i}}}{N_{i_{1}} \ldots N_{i_{im_i}}} \right)_q.
\] (10)

As a result, using estimation (7) arrives at the connection between the complexities of the nearest hierarchical levels
\[
C_Q(p_{11}, \ldots, p_{n_{nm}}) = C_Q(p_{1}, \ldots, p_{n}) + \sum_{i=1}^{n} p_i^Q C_Q \left( \frac{p_{i_{i1}}}{p_i}, \ldots, \frac{p_{i_{im_i}}}{p_i} \right),
\] (11)

whose distributions over states are given by the equations
\[
p_{ij} = \frac{N_{ij}}{N}, \quad p_i = \frac{N_i}{N} \quad (here \, we \, introduce \, 'physical' \, non-extensivity \, parameter \, Q = 2 - q, \, 1 \leq Q \leq 2).
\]

Using definition (8) and connections $p_i = \sum_{j=1}^{m_i} p_{ij}$, at condition $p_i - p_{ij} \ll p_i$ gives estimation
\[
C_Q \left( \frac{p_{i_{i1}}}{p_i}, \ldots, \frac{p_{i_{im_i}}}{p_i} \right) \approx \frac{Q}{2} \sum_{j_i=1}^{m_i} \left( \frac{p_i - p_{ij}}{p_i} \right)^2,
\] (12)

with whose accounting one finds
\[
C_Q(p_{11}, \ldots, p_{n_{nm}}) - C_Q(p_{1}, \ldots, p_{n}) \approx \frac{Q}{2} \sum_{i=1}^{n} p_i^{Q-2} \sum_{j_i=1}^{m_i} (p_i - p_{ij})^2.
\] (13)

If statistical states are distributed within microcanonical ensembles, then both probabilities and related complexities are determined by the level number $n$: $\{p_i\}_1^n \Rightarrow p_n, \{p_{ij}\}_1^{m_i} \Rightarrow p_{n+1}; C_Q(p_1, \ldots, p_n) \Rightarrow C(n), \quad C_Q(p_{11}, \ldots, p_{n_{nm}}) \Rightarrow C(n + 1). \quad As \, a \, result, \, Eq.(13) \, takes \, the \, simplest \, form:
\[
C(n + 1) - C(n) \approx \frac{Q(n + 1)}{2} p_n^{Q-2} (p_n - p_{n+1})^2.
\] (14)

At fixed level number $n \gg 1$, relation obtained is reduced to the differential equation
\[
\frac{\partial^2 C(n)}{\partial p_n^2} = Q(n + 1) \, p_n^{-(2-Q)},
\] (15)

whose integration gives the dependence
\[
C(n) = p_n^Q - \frac{Q}{Q-1} p_n^{Q-1} p_n + \frac{n + 1}{Q-1} p_n^Q
\] (16)

at the boundary conditions
\[
C(n = 0) = 0, \quad \frac{\partial C(n)}{\partial p_n} \bigg|_{n=0} = 0.
\] (17)
Then, using self-similar distribution [6]

\[ p_n = A \left[ \Delta + (Q - 1)(n + 1) \right]^{-(Q-1)/Q}, \quad A \equiv (2 - Q) \left\{ (Q - 1) + \Delta \right\}^{\frac{2-Q}{Q-1}}, \quad (18) \]

where parameter \( \Delta \) determines the scattering over hierarchical levels, we obtain the complexity distribution shown in Figure 2. As it should be, on the upper level \( n = 0 \), where only one statistical ensemble takes place, the complexity is \( C = 0 \). With passage to lower levels the value \( C \) increases, taking maximum value at \( n_0 = (Q - 1) + \Delta \), after which characteristic magnitude \( C_\infty = p_0^Q \) is reached. However, it follows to take into account that above consideration is applicable for level numbers \( n \gg 1 \) only. Therefore, the pointed out maximum is displayed at condition \( \Delta \gg 1 \), when distribution (18) differs very weakly from the exponential one, being inherent in usual additive systems. At moderate scattering of the hierarchical coupling \( \Delta \sim 1 \), the system complexity increases fast from zero to the limit value

\[ C_\infty = \left[ \frac{2 - Q}{(Q - 1) + \Delta} \right]^Q, \quad (19) \]

which decreases with growth of both scattering \( \Delta \) and non-extensivity parameter \( 1 \leq Q \leq 2 \).

Evolution process of the hierarchical ensemble is determined by both diffusion time \( \tau_d = (\Delta^{2-Q}/D)n^Q\tau_0 \) and scale \( \tau = n^2\tau_0 \) of the steady-state reaching, where \( D \) is diffusion coefficient, \( \tau_0 \) is time of passage between nearest levels [6]. At \( t \ll \tau_d \), the diffusion over the hierarchical tree has anomalous character.
to be determined by the law \( n(t) \approx Q^{1/Q}(t/\tau_0)^{1/Q} \). With time growth within the interval \( \tau_d \sim t \ll \tau \), when contributions of anomalous drift and diffusion are comparable, the passage into the normal regime \( n(t) \approx \sqrt{2(t/\tau_0)} \) happens. At \( t \ll \tau \), the distribution over hierarchical levels tends to the steady-state law (18) according to the time dependence

\[
p_n(t) = p_n \left[ 1 - (t/\tau_n)^{-1} \right],
\]

which is reduced to the value \( p_n(t) = 0 \) at \( t \leq \tau_n \), \( \tau_n \equiv (\Delta/QD)n\tau_0, n \neq 0 \) (here, initial distribution \( p_n(t = 0) = \delta_{n0} \) is taken in form of the Kronecker \( \delta \)-symbol).

Non-stationary complexity is determined by Eq.(15), where one should take the boundary conditions

\[
C(n, t = 0) = 0, \quad C(n, t = \infty) = C(n), \quad n \neq 0
\]

instead of Eqs.(17). As a result, we derive the time-dependent complexity in the form

\[
C(n, t) = \left( \frac{p_0^Q}{p_n} - \frac{Q}{Q-1} p_0^{Q-1} \right) p_n(t) + \frac{n + 1}{Q-1} p_n^Q(t), \quad n \gg 1,
\]

being generalization of the steady-state distribution (16). Inserting Eqs.(18), (20) into Eq.(22), we obtain the time dependence of the complexity shown in Figure 3. During the ballistic interval \( t \leq \tau_n \), the complexity keeps the initial value \( C = 0 \), whereas in the course of the time \( t > \tau_n \) it increases fast before the steady-state value (16), being bounded by the maximum value (19).

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Fig. 3. Time dependencies of the complexity on different hierarchy levels (their numbers are noticed near curves) at parameters $\Delta = 1.0, Q = 1.5, D = 1$

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