In this paper we develop the theory of quasispaces (for a Grothendieck topology) and of concrete quasitopoi, over a suitable base category. We introduce the notion of \textit{f-regular category} and of \textit{f-regular functor}. The \textit{f-regular categories} are regular categories in which every family with a common codomain can be factorized into a strict epimorphic family followed by a (single) monomorphism. The \textit{f-regular functors} are (essentially) functors that preserve finite strict monomorphic and arbitrary strict epimorphic families. These two concepts furnish the context to develop the constructions of the theory of concrete quasitopoi over a suitable base category, which is a theory of \textit{pointed quasitopoi}. Our results on quasispaces and quasitopoi, or closely related ones, were already established by Penon in [9], but we prove them here with different assumptions, and under a completely different light.

\section*{INTRODUCTION}

In this paper we develop the theory of quasispaces (for a Grothendieck topology) and of concrete quasitopoi, over a suitable base category. We use the systematic theory of families of arrows in a category (including a characterization of strict epimorphic families as final surjective families in a general context) that we set in [6].

We introduce the notion of \textit{f-regular category} and of \textit{f-regular functor}. The \textit{f-regular categories} are regular categories which have \textit{f-factorizations}, a non-elementary cocompleteness condition: A category has \textit{f-factorizations} if every family \( g_\alpha : X_\alpha \to X \) factorizes in the form \( g_\alpha = m \circ h_\alpha \), where \( m : H \to X \) is a monomorphism and \( h_\alpha : X_\alpha \to H \) is an strict epimorphic family. The \textit{f-regular functors} are (essentially) functors that preserve finite strict monomorphic and arbitrary strict epimorphic families. These two concepts furnish the context to develop the constructions of the theory of concrete quasitopoi.

Our results on quasispaces and quasitopoi, or closely related ones, were already established by Penon in [9], but we prove them here with different assumptions, and under a completely different light. The notion of bounded quasitopos [9] leaves out the paradigmatic examples of quasitopoi, namely, the \textit{legitimate} categories of separated sheaves over a \textit{large} site. In this case, the categories of sheaves are not topoi, and they are not even legitimate categories since the class of morphisms between two sheaves is not in general a set. However, separate sheaves form legitimate categories (insofar the hom-sets are small) which are elementary quasitopoi (in the sense of Penon). They bear to elementary quasitopoi a relation which should be considered as corresponding (in the context of quasitopoi) to the relation that Grothendieck topoi bear to elementary topoi. In this context, the size condition (\textit{bounded}) is unnecessary, and probably misleading. We introduce the abstract notion of \textit{f-quasitopos}, which describes this situation, and generalize the concrete quasitopoi of [5]. Our notion is intermediate:

\[ \text{bounded quasitopos} \Rightarrow \text{f-quasitopos} \Rightarrow \text{elementary quasitopos} \]

The morphisms between \textit{f-quasitopoi} are the \textit{f-regular functors}, which correspond to inverse images of geometric morphisms. The category of \textit{f-quasitopoi} over a suitable base category is a theory of \textit{pointed} quasitopoi.
1. Families of arrows and topological functors

In this section we recall briefly some notions and results from [6] that we shall explicitly need, and in this way fix notation and terminology. For comments and proofs we refer to [6].

Given a category \( T \) and an object \( X \) in \( T \), we shall work with families \( (X_\alpha \to X)_{\alpha \in \Gamma} \) of arrows of \( T \) with codomain \( X \).

1.1. Notation. Given a family \( (X_\alpha \to X)_{\alpha \in \Gamma} \), we shall simply write \( X_\alpha \to X \), omitting as well a label for the index set (the context will always tell whether we are considering a single \( \alpha \) or the whole family).

The diagrammatic notation always denotes a commutative diagram, unless otherwise explicitly indicated.

It is important to point out that we allow the families to be large, that is, not indexed by a set.

1.2. Definition. We say that a family \( Y_\lambda \to X \) refines (is a refinement of) a family \( X_\alpha \to X \) if there is a function between the indices \( \lambda \mapsto \alpha \lambda \) together with arrows \( Y_\lambda \to X_\alpha \) such that

\[
\begin{array}{ccc}
Y_\lambda & \to & X_{\alpha\lambda} \\
\downarrow & & \downarrow \\
X & & \\
\end{array}
\]

1.3. Definition. Given an arrow \( Y \to X \), we say than a family \( Y_\lambda \to X \) is a \( r \)-pull-back of a family \( X_\alpha \to X \) if there is a function between the indices \( \lambda \mapsto \alpha \lambda \) together with arrows \( Y_\lambda \to X_{\alpha\lambda} \) such that

\[
\begin{array}{ccc}
Y_\lambda & \to & X_{\alpha\lambda} \\
\downarrow & & \downarrow \\
Y & \to & X \\
\end{array}
\]

We consider collections \( \mathcal{A} \) of classes of families of arrows with common codomain, one class \( \mathcal{A}_X \) (eventually empty) for each object \( X \) in \( T \). We say that a family in \( \mathcal{A}_X \) is a \( \mathcal{A} \)-family over \( X \).

1.4. Definition (operations on collections).

(1) We denote by \( \textbf{Is}o \) the collection whose only arrows are the isomorphisms.

(2) Given two collections \( \mathcal{A}, \mathcal{B} \) we define the composite \( \mathcal{C} = \mathcal{A} \circ \mathcal{B} \) by means of the following implication:

\[ X_\alpha \to X \in \mathcal{A}_X \quad \text{and} \quad \forall \alpha \; X_{\alpha,\beta} \to X_\alpha \in \mathcal{B}_{X_\alpha} \implies X_{\alpha,\beta} \to X_\alpha \to X \in \mathcal{C}_X \]
Given $\mathcal{A}$ we define a new collection, denoted $\pi \mathcal{A}$, by:

\[ Y_\alpha \to Y \in \pi \mathcal{A} \iff \text{there is } X_\alpha \to X \in \mathcal{A} \text{ and } Y \to X \text{ such that:} \]

\[
\begin{array}{ccc}
Y_\alpha & \to & X_
_{\alpha} \\
\downarrow & & \downarrow \\
Y & \to & X
\end{array}
\]

are pullbacks for all $\alpha$.

Given $\mathcal{A}$ we define a new collection, denoted $s \mathcal{A}$, by:

\[ X_\alpha \to X \in s \mathcal{A} \iff \text{there is a refinement by a family } Y_\lambda \to X \in \mathcal{A}. \]

Notice that $\mathcal{A} \subseteq \pi \mathcal{A}$, and $\mathcal{A} \subseteq s \mathcal{A}$. We set now some properties of collections $\mathcal{A}$ defined by means of these operations:

1.5. **Definition** (properties of collections).

(I) Isomorphisms: $\text{Iso} \subseteq \mathcal{A}$.

(C) Closed under composition: $\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A}$.

(S) Saturated: $s \mathcal{A} \subseteq \mathcal{A}$ (hence $\mathcal{A} = s \mathcal{A}$).

(U) Universal: Given $X_\alpha \to X \in \mathcal{A}$ and $Y \to X$, there exists an $r$-pull-back $Y_\lambda \to Y \in \mathcal{A}$:

\[
\begin{array}{ccc}
Y_\lambda & \to & X_{\alpha \lambda} \\
\downarrow & & \downarrow \\
Y & \to & X
\end{array}
\]

If a collection $\mathcal{A}$ satisfies (S), and the category has finite limits, then (U) is equivalent to:

(U) Stable under pullback: $\pi \mathcal{A} \subseteq \mathcal{A}$ (hence $\mathcal{A} = \pi \mathcal{A}$).

(F) Filtered: Given $X_\alpha \to X \in \mathcal{A}_X$, $Y_\beta \to X \in \mathcal{A}_X$, there exists a common refinement $Z_\lambda \to X \in \mathcal{A}_X$:

\[
\begin{array}{ccc}
Z_\lambda & \to & X_{\alpha \lambda} \\
\downarrow & & \downarrow \\
Y_{\beta \lambda} & \to & X
\end{array}
\]

1.6. **Facts** (about collections). The following statements about given collections hold:

(1) If a collection satisfies the properties (U) and (C) then it satisfies (F).

(2) If $\mathcal{A}$ and $\mathcal{B}$ both satisfy (I), (resp. (C)), (resp. (S)), then so does $\mathcal{A} \cap \mathcal{B}$.

(3) If $\mathcal{A}$ and $\mathcal{B}$ both satisfy (S) and (U), (resp. (S) and (F)), then so does $\mathcal{A} \cap \mathcal{B}$.

An important collection of families are the strict epimorphic families. We recall now this notion from SGA4 [2, I, 10.3, p. 180]:

1.7. **Definition.** Given two families of arrows $f_\alpha : X_\alpha \to X$, $g_\alpha : X_\alpha \to Y$, with the same indexes and domains, we say that $g_\alpha$ is compatible with $f_\alpha$ if for any pair of arrows $(x_\alpha : Z \to X_\alpha$, $x_\beta : Z \to X_\beta)$ with the same domain the following condition holds: $f_\alpha \circ x_\alpha = f_\beta \circ x_\beta$ implies $g_\alpha \circ x_\alpha = g_\beta \circ x_\beta$.

A family $f_\alpha : X_\alpha \to X$ is strict epimorphic if for any family $g_\alpha : X_\alpha \to Y$ which is compatible with $f_\alpha$, there exists a unique $g : X \to Y$ such that $g \circ f_\alpha = g_\alpha$ for all $\alpha$. 

The situation is described in the following diagram, where the family \( g_\alpha \) is compatible with the family \( f_\alpha \):

\[
\begin{array}{c}
X_\alpha \\
\downarrow f_\alpha \\
\downarrow \exists g
\end{array}
\]

\[
\begin{array}{c}
X
\downarrow \exists g \\
\downarrow \exists g
\end{array}
\]

1.8. Definition. Given a functor \( u : T \to S \):

1. An object \( X \) in \( T \) sits over an object \( S \) in \( S \) when \( u(X) = S \). We say also that \( X \) is an object over \( S \). An arrow \( f : X \to Y \) in \( T \) sits over an arrow \( \varphi : S \to T \) in \( S \) when \( u(f) = \varphi \), so that \( X \) (resp. \( Y \)) sits over \( S \) (resp. \( T \)). We say also that \( \varphi \) lifts to an arrow in \( T \) when there exists \( f \) over \( \varphi \). A family \( f_\alpha : X \to Y \) in \( T \) sits over a family \( \varphi_\alpha : S \to T \) in \( S \) when \( u(f_\alpha) = \varphi_\alpha \), for any \( \alpha \). We say also that the family \( \varphi_\alpha \) lifts to a family in \( T \) when there exists \( f_\alpha \) over \( \varphi_\alpha \).

2. We say that two families \( f_\alpha : X_\alpha \to X, \ g_\alpha : X_\alpha \to Y \) in \( T \) which sit over the same family \( \varphi_\alpha : S_\alpha \to S \) in \( S \) are \( u \)-isomorphic if there exists an isomorphism \( \theta : X \to Y \) over \( \id : S \to S \) such that \( \theta \circ f_\alpha = g_\alpha \) for all \( \alpha \).

3. Let \( \mathcal{A} \) be a collection of classes of families in \( T \). We say that \( \mathcal{A} \)-families are unique up to isomorphisms if given any two \( \mathcal{A} \)-families \( f_\alpha : X_\alpha \to X, \ g_\alpha : X_\alpha \to Y \) which sit over the same family \( \varphi_\alpha : S_\alpha \to S \) in \( S \), they are \( u \)-isomorphic by a unique isomorphism. (Notice that when \( u \) is faithful, strict epimorphic families are unique up to isomorphisms.)

4. Consider collections \( \mathcal{A} \) in \( T \) and \( \mathcal{B} \) in \( S \). We say that \( u \) creates \( \mathcal{A} \)-families over \( \mathcal{B} \)-families if given any \( \mathcal{B} \)-family \( \phi_\alpha : S_\alpha \to S \) and an object \( X_\alpha \) over \( S_\alpha \) for every \( \alpha \), there exists an \( \mathcal{A} \)-family \( f_\alpha : X_\alpha \to X \) over \( \phi_\alpha : S_\alpha \to S \).

When the class \( \mathcal{B} \) in \( S \) is the “same” class than the class \( \mathcal{A} \) in \( T \) (that is, if they are denoted by the same letter), we simply say that \( u \) creates \( \mathcal{A} \)-families.

All collections considered in this paper are assumed to be closed under \( u \)-isomorphisms without need to say so explicitly.

1.9. Definition. We consider notions for families in \( T \) relative to the functor \( u \):

1. Given a functor \( u : T \to S \), we say that a family \( f_\alpha : X_\alpha \to X \) in \( T \) is \( u \)-surjective when the family \( u(f_\alpha) \) is strict epimorphic in \( S \).

2. Given a functor \( u : T \to S \), let \( f_\alpha : X_\alpha \to X \) be a family in \( T \) over \( \varphi_\alpha : S_\alpha \to S \) in \( S \). The family \( f_\alpha \) is \( u \)-final if for any family \( g_\alpha : X_\alpha \to Y \) in \( T \) and arrow \( \phi : S \to T \) in \( S \) such that \( g_\alpha \) sits over \( \phi \circ \varphi_\alpha \), there exists a unique \( g : X \to Y \) over \( \phi \) such that \( g \circ f_\alpha = g_\alpha \). (Final families are unique up to isomorphisms in the sense of definition 1.8)

3. By \( FS = F \cap S \) we shall denote the collection of all final and surjective families.

We shall often omit the \( u \) when we write “\( u \)-surjective” or “\( u \)-final”. The situation for final families is described in the following double diagram, where the top diagram sits over the bottom diagram.

\[
\begin{array}{c}
X_\alpha \\
\downarrow f_\alpha \\
\downarrow \exists g
\end{array}
\]

\[
\begin{array}{c}
X
\downarrow \exists g \\
\downarrow \exists g
\end{array}
\]

\[
\begin{array}{c}
S_\alpha \\
\downarrow \varphi_\alpha \\
\downarrow \phi
\end{array}
\]

\[
\begin{array}{c}
S
\downarrow \phi
\end{array}
\]

Let \( sE_X \) be the class of all strict epimorphic families with codomain \( X \). The collection \( sE \) satisfies (I) and (S), but in general it fails to satisfy (C), (U) and (F).
Let $S_X$ be the class of all surjective families with codomain $X$. The collection $S$ satisfies conditions (I) and (S), but fails in general to satisfy (C), (U) and (F).

Let $F_X$ be the class of all final families with codomain $X$. The collection $F$ satisfies conditions (I), (C) and (S), but fail in general to satisfy (U) and (F).

Finally, the collection $FS$ satisfies conditions (I) and (S) by 1.6 (3).

We are interested in the equivalence
\[ \text{Strict epimorphic} \iff \text{Final surjective}. \]

Concerning this we have:

1.11. **Fact.** If the functor $u$ is faithful, and has left and right adjoints (notice that this hypothesis is self-dual), then:

A family is strict epimorphic if and only if is final and surjective, that is, $sE = FS$. And the dual statement: A family is strict monomorphic if and only if is initial and injective.

1.12. **Definition.** A functor $T \xrightarrow{u} S$ is an $E$-functor (resp. $M$-functor) if it is faithful and creates and preserves strict epimorphic families (resp. strict monomorphic families). If we consider only finite families, we have the notions of $E_{\text{fin}}$-functor and $M_{\text{fin}}$-functor.

1.13. **Fact** (about $E$-functors). Given a $E$-functor (resp. $E_{\text{fin}}$-functor) $T \xrightarrow{u} S$, a family (resp. finite family) in $T$ is strict epimorphic if and only if is final and surjective.

1.14. **Fact** (about $M$-functors). Given a $M$-functor (resp. $M_{\text{fin}}$-functor) $T \xrightarrow{u} S$, a family (resp. finite family) in $T$ is strict monomorphic if and only if is initial and injective.

In [6] we have extensively developed a notion of topological functor. We recall here a couple of results we shall need in this paper.

1.15. **Facts** (about topological functors). The following holds:

1. A functor $u : T \rightarrow S$ is topological if and only if it creates initial families.

2. A functor $u : T \rightarrow S$ is topological if and only if considered as a functor $u : T^{\text{op}} \rightarrow S^{\text{op}}$ is topological.

3. A functor $u : T \rightarrow S$ is topological if and only if:
   a) It is faithful and preserves and creates strict epimorphic families.
   b) It has a full and faithful left adjoint $(-)_{\perp} \dashv u$, $u(-)_{\perp} = \text{id}$. □

Topological functors are, in particular, $E$-functors and $M$-functors.

2. $E$-functors

The notions of $E$-functor and $M_{\text{fin}}$-functor, $T \xrightarrow{u} S$, determine a framework of the right generality for several constructions found in many particular situations. Recall that in this case, by 1.13 and 1.14, in the category $T$ strict epimorphic families are the same that final surjective families, and finite strict monomorphic families are the same that finite initial injective families.

An $E$-functor does not necessarily reflect the property of being a strict epimorphic family (example, the forgetful functor of the category of all topological spaces). We establish now some technical results for future use.

$E$-functors (resp $M_{\text{fin}}$-functors) create any colimit (resp. finite limit) that may exists in $S$, and preserve any colimit (resp. finite limit) that may exists in $T$, provided the colimit (resp. finite limit) of the underlying diagram already exists in $S$. 


Proposition. Let $T \xrightarrow{u} S$ be a $\mathcal{E}$-functor (resp. $\mathcal{M}_{\text{fin}}$-functor). Consider a diagram (resp. finite diagram) $\Gamma \xrightarrow{X} T$, $\alpha \mapsto X_\alpha$ in $T$, and the underlying diagram $S_\alpha = uX_\alpha$ in $S$. Suppose a colimit cone $S_\alpha \rightarrow L$ (resp. a limit cone $L \rightarrow S_\alpha$) exists in $S$. Then:

i) A colimit cone $X_\alpha \rightarrow Y$ over $S_\alpha \rightarrow L$ (resp. a limit cone $Y \rightarrow X_\alpha$ over $L \rightarrow S_\alpha$) exists in $T$.

ii) If a colimit cone $X_\alpha \rightarrow Y$ (resp. a limit cone $Y \rightarrow X_\alpha$) exists in $T$, then $u(X_\alpha \rightarrow Y) = S_\alpha \rightarrow L$, (resp. $u(Y \rightarrow X_\alpha) = L \rightarrow S_\alpha$).

Proof. The reader should be able to carefully check the validity of the two statements. The faithfulness of $u$ as well as facts 1.13 (resp. 1.14) are needed.

The proof of the following proposition is immediate.

Proposition. Given functors $T \xrightarrow{u} S \xrightarrow{v} R$, if $u$ and $v$ preserve $s\mathcal{E}$-families (resp. $s\mathcal{M}_{\text{fin}}$-families), and the composite $v \circ u$ creates $s\mathcal{E}$-families (resp. $s\mathcal{M}_{\text{fin}}$-families), then $u$ is a $\mathcal{E}$-functor (resp. $\mathcal{M}_{\text{fin}}$-functor).

Proposition. Consider $u : T \rightarrow S$, $X$ an object in $T$ and the induced functor $u_* : T/X \rightarrow S/G$, where $S = u(X)$. Then, if $u$ is an $\mathcal{E}$-functor so it is $u_*$.

Proof. It is clear that $u_*$ is faithful and that it preserves $s\mathcal{E}$-families. Now we prove that $u_*$ creates $s\mathcal{E}$-families. Let $f_\alpha : X_\alpha \rightarrow X$ be a family in $T/X$ over $s_\alpha : S_\alpha \rightarrow S$ in $S/G$, and an $s\mathcal{E}$-family $\varphi_\alpha$ in $S/G$ as in the diagram below:

\[
\begin{array}{ccc}
S_\alpha & \xrightarrow{\varphi_\alpha} & R \\
\downarrow{\sigma_\alpha} & & \downarrow{\rho} \\
S & & \\
\end{array} \quad \quad \begin{array}{ccc}
X_\alpha & \xrightarrow{g_\alpha} & Y \\
\downarrow{f_\alpha} & & \downarrow{g} \\
X & & \\
\end{array}
\]

The canonical functor $S/G \rightarrow S$ preserves $s\mathcal{E}$-families, so that $\varphi_\alpha$ is an $s\mathcal{E}$-family in $S$. Since $u$ creates $s\mathcal{E}$-families, we have an $s\mathcal{E}$-family $g_\alpha : X_\alpha \rightarrow Y$ in $T$ over $\varphi_\alpha : S_\alpha \rightarrow R$. Moreover, since the family $g_\alpha$ is final surjective in $T$, there exists a unique $g : Y \rightarrow X$ such that $g \circ g_\alpha = f_\alpha$ for all $\alpha$. It is immediate to check that the family $g_\alpha$ in $T/X$ over $\varphi_\alpha$ in $S/G$ is an $s\mathcal{E}$-family in $T/X$.

Definition. A category $\mathcal{T}$ has $f$-factorizations if every family $g_\alpha : X_\alpha \rightarrow X$ factorizes in the form $g_\alpha = m \circ h_\alpha$, where $m : H \rightarrow X$ is a monomorphism and $h_\alpha : X_\alpha \rightarrow H$ is an strict epimorphic family.

Remark. If a family $g_\alpha : X_\alpha \rightarrow X$ factorizes in the form $g_\alpha = m \circ h_\alpha$, where $m : H \rightarrow X$ is a monomorphism, then a family $f_\alpha : X_\alpha \rightarrow Y$ is compatible with $g_\alpha$ if and only if it is compatible with $h_\alpha$. Moreover, if $g_\alpha = m \circ h_\alpha$ is a $f$-factorization then $g_\alpha$ is a strict epimorphic family if and only if $m$ is an isomorphism.

Proposition. If a category $\mathcal{T}$ has $f$-factorizations, then strict epimorphic families compose, that is, the collection $\mathcal{E}$ has property (C).

Proof. We consider a family $X_\alpha \xrightarrow{g_\alpha} X$ and, for any $\alpha$, a family $X_{\alpha, \beta} \xrightarrow{g_{\alpha, \beta}} X_\alpha$, so that we have the composite family $I_{\alpha, \beta} = g_\alpha \circ g_{\alpha, \beta} : X_{\alpha, \beta} \rightarrow X$. Let us suppose that $g_\alpha$ and $g_{\alpha, \beta}$ are strict epimorphic families. To prove that so is $I_{\alpha, \beta}$ we take the $f$-factorization $I_{\alpha, \beta} = m \circ h_{\alpha, \beta}$, with $H \rightarrow X$ mono and $h_{\alpha, \beta}$ an strict epimorphic family. Fixing $\alpha$, it is clear that the family $h_{\alpha, \beta}$ is compatible with $g_{\alpha, \beta}$, hence there exists an arrow $X_\alpha \xrightarrow{h_\alpha} H$ unique such that $h_\alpha \circ g_{\alpha, \beta} = h_{\alpha, \beta}$. Moreover (compose with the epimorphic family $g_{\alpha, \beta}$) we have $m \circ h_\alpha = g_\alpha$ for any $\alpha$. Now we prove that the family $h_\alpha$ is strict epimorphic family. In fact: if $f_\alpha$ is compatible with $h_\alpha$ then the family $f_{\alpha, \beta} = f_\alpha \circ g_{\alpha, \beta}$ is compatible with $I_{\alpha, \beta}$, so that there exists a unique arrow $j_H$ such that $j_H \circ h_{\alpha, \beta} = f_{\alpha, \beta}$, hence (compose with the
epimorphic family \( g_{\alpha,\beta} \) \( f_H \circ h_{\alpha} = g_{\alpha} \). \( f_H \) is unique with this condition because \( h_{\alpha} \) is an epimorphic family (notice that so is \( h_{\alpha,\beta} \)). Finally, remark 2.5 implies that \( m \) is an isomorphism, hence \( l_{\alpha,\beta} \) is an strict epimorphic family. \( \square \)

2.7. Proposition. If \( u \) is an \( E \)-functor and \( S \) has \( f \)-factorizations (definition 2.4), then \( T \) has \( f \)-factorizations.

Proof. Given any family \( g_{\alpha} : X_{\alpha} \to X \) in \( T \), we can take the \( f \)-factorization \( u(g_{\alpha}) = m \circ h_{\alpha} : u(X_{\alpha}) \to H \to u(X) \) in \( S \), and a strict epimorphic family \( f_{\alpha} : X_{\alpha} \to Y \) in \( T \) over \( h_{\alpha} \). But \( h_{\alpha} \) is final surjective, so that there exists \( f \) over \( m \) such that \( g_{\alpha} = f \circ f_{\alpha} \), with \( f \) mono because \( u \) is faithful. \( \square \)

It is well known and easy to prove that if a functor has a right adjoint then it preserves strict epimorphic families. Next we establish that under certain hypotheses this condition is also sufficient for the existence of a right adjoint. Moreover, we give an explicit construction of this adjoint. It is convenient to consider before some particular cases.

2.8. Proposition. If \( u \) is an \( E \)-functor and \( S \) has \( f \)-factorizations (definition 2.4), then \( u \) has a right adjoint. \( \square \)

We indicate the idea of the proof. The construction of the right adjoint is as follows: Given an object \( Y \) in \( S \), consider in \( S \) all arrows of the form \( u(X) \xrightarrow{\varphi} S \). Factor this family \( u(X) \xrightarrow{\psi} H \xrightarrow{m} S \), with a monomorphism \( m \) and the family \( \psi \) strict epimorphic. Let \( R(S) \) and \( X \xrightarrow{\tilde{\varphi}} R(S) \) be a strict epimorphic family over the family \( \psi \). Then, \( R(S) \) is the right adjoint to \( u \) on \( S \).

2.9. Theorem. Consider a diagram of categories and functors where \( u \) and \( u' \) are \( E \)-functors, \( u' \circ F = u \).

\[
\begin{array}{ccc}
T & \xrightarrow{F} & T' \\
\downarrow u & & \downarrow u' \\
S & \xleftarrow{R} & S'
\end{array}
\]

Assume that \( S \) has \( f \)-factorizations (definition 2.4). Then \( F \) has a right adjoint if and only if \( F \) preserves \( sE \)-families. \( \square \)

We indicate the idea of the proof. The construction of the right adjoint is as follows: Given an object \( Y \) in \( T' \) consider in \( S \) all arrows of the form \( u(X) \xrightarrow{\varphi} u'(Y) \) such that \( \varphi = u'(g) \) for some \( F(X) \xrightarrow{g} Y \). Factor this family \( u(X) \xrightarrow{\psi} H \xrightarrow{m} u'(Y) \), for a monomorphism \( m \) and an strict epimorphic family \( \psi \). Let \( G(Y) \) and \( X \xrightarrow{\tilde{\varphi}} G(Y) \) be an strict epimorphic family over the family \( \psi \). Then, \( G(Y) \) is the right adjoint to \( F \) on \( Y \).

Now we set and prove in detail the general theorem:

2.10. Theorem. Consider a diagram of categories and functors where \( u \) and \( u' \) are \( E \)-functors, \( u' \circ F = L \circ u \), and \( R \vdash L \).

\[
\begin{array}{ccc}
T & \xrightarrow{F} & T' \\
\downarrow u & & \downarrow u' \\
S & \xleftarrow{R} & S'
\end{array}
\]

Assume that \( S \) has \( f \)-factorizations (definition 2.4). Then \( F \) has a right adjoint if and only if \( F \) preserves \( sE \)-families.
Proof. Suppose that $F$ preserves strict epimorphic families. For every object $Y$ in $T'$ we construct an object $G(Y)$ in $T$ and an arrow $\epsilon_Y : F(G(Y)) \to Y$ universal from $F$ to $Y$. First, consider arrows $\varphi : u(X) \to R(u'(Y))$ in $S$. To every $\varphi$ it corresponds under the adjunction $R \dashv L$ an arrow $\hat{\varphi} : L(u(X)) \to u'(Y)$. Now, we take in $S$ the family of all arrows $\varphi$ as above such that $\hat{\varphi} = u'(g)$ for some $g : F(X) \to Y$ in $T'$. Notice that we need the equation $u' \circ F = L \circ u$. Since $S$ has $f$-factorizations, there exists a monomorphism $m : H \to R(u'(Y))$ such that $\varphi = m \circ \psi : u(X) \xrightarrow{\psi} H \xrightarrow{m} R(u'(Y))$ for all $\varphi$ in our family, and the family $\psi$ is strict epimorphic. But $u$ creates $s\mathcal{E}$-families, hence there exists an object $G(Y)$ in $T$ and for any $\varphi$ in the family an arrow $\bar{\varphi} : X \to G(Y)$ such that $\bar{\varphi}$ is an strict epimorphic family over the family $\psi$ (in particular $u(G(Y)) = H$). By hypothesis, $F(\bar{\varphi}) : F(X) \to F(G(Y))$ is a $s\mathcal{E}$-family in $T'$, and it sits over the family $L(\psi)$. Moreover it is final and surjective because $u'$ is an $\mathcal{E}$-functor. Recall that the family $g : F(X) \to Y$ considered above sits over the family $\bar{\varphi}$ which factorizes trough $L(\psi)$, so that there exists a unique $\epsilon_Y : F(G(Y)) \to Y$ such that it sits over $m$ and $\epsilon_Y \circ F(\bar{\varphi}) = g$. It remains to prove that this arrow is universal. The above construction shows that given $g$ we have $\bar{\varphi}$ such that $\epsilon_Y \circ F(\bar{\varphi}) = g$. Consider any other arrow $f$ such that $\epsilon_Y \circ F(f) = g$. Then $\bar{m} \circ L(u(f)) = u'(g)$, and applying the adjunction it follows $m \circ u(f) = \varphi = m \circ u(\bar{\varphi})$. Thus, $f = \bar{\varphi}$ because $m$ is a monomorphism and $u$ is faithful. \hfill \square

A functor $F$ as above which has a right adjoint in particular preserves strict epimorphic families, thus it will preserve final surjective families (which are the same). However, it will not preserve arbitrary final families in general.

2.11. Corollary. Let $S, T$ be categories with finite products and $u : T \to S$ an $\mathcal{E}$-functor which preserves finite products. Assume that $S$ is cartesian closed and that it has $f$-factorizations. Then, $T$ is cartesian closed if and only if the cartesian product in $T$ preserves $s\mathcal{E}$-families.

Proof. Apply Theorem 2.10 with the following diagram of functors:

$$
\begin{array}{ccc}
T & \xrightarrow{(-) \times X} & T \\
\downarrow u & & \downarrow u \\
S & \xrightarrow{(-) \circ u} & S
\end{array}
$$

(where $u(X) = S$).

\hfill \square

It is interesting to give some further details of the construction of the exponential in $T$ that follow from the proof of theorem 2.10. Given any two objects $X$ and $Y$ in $T$, the underline object in $S$ of the exponential $Y^X$ is a subobject of the exponential between the underline objects, $u(Y^X) \hookrightarrow u(Y)^{u(X)}$. The family of all arrows $Z \xrightarrow{\bar{\varphi}} Y^X$ such that the arrow $u(Z) \times u(X) \to u(Y)$ (which corresponds by adjointness to the arrow $u(Z) \xrightarrow{u(\bar{\varphi})} u(Y^X) \hookrightarrow u(Y)^{u(X)}$) lifts into $Z \times X \to Y$, is a final surjective family.

We now generalize this corollary to localized exponentials. Recall that if $S$ has finite products, then the usual functor $S/S \to S$ has a right adjoint, and that $S$ has finite limits if and only if each $S/S$ does. Recall also that by definition a category $S$ is locally cartesian closed if $S/S$ is cartesian closed for any object $S$ in $S$. Given an arrow $R \to S$ in a locally cartesian closed category $S$, the pulling-back functor $S/S \to S/R$ has a right adjoint.
2.12. **Theorem.** Let \( S, T \) be categories with finite limits and \( u : T \to S \) an \( \mathcal{E} \)-functor which preserves finite limits. Assume that \( S \) is locally cartesian closed and has \( f \)-factorizations. Then, \( T \) is locally cartesian closed if and only if \( s\mathcal{E} \)-families are universal in \( T \).

*Proof.* Notice that if \( S \) has \( f \)-factorizations then so does each localized category \( S/S \). Then apply corollary 2.11 and proposition 2.3. □

Our last result in this section concerns subobject classifiers.

2.13. **Proposition.** Let \( S, T \) be categories with finite limits and \( u : T \to S \) a \( \mathcal{M}_{fin} \)-functor with a right adjoint \( R \). Assume that \( S \) has \( f \)-factorizations and a strict subobject classifier \( \Omega \). Then, \( R\Omega \) is an strict subobject classifier in \( T \).

*Proof.* Let \( 1 \xrightarrow{t} \Omega \) be the generic strict subobject. Clearly \( 1 = R1 \), and there is a map \( 1 \xrightarrow{at} R\Omega \) in \( T \) corresponding by adjointness to the map \( 1 \xrightarrow{t} \Omega \) in \( S \) (notice that since \( u \) preserves finite limits (proposition 2.1), \( u1 = 1 \)). Given a strict subobject \( M \hookrightarrow X \) in \( T \), the strict subobject \( uM \hookrightarrow uX \) in \( S \) determines the pullback square on the left below, which in turn determines by adjointness the commutative square on the right.

\[
\begin{array}{ccc}
M & \to & X \\
\downarrow \phi & & \downarrow a\phi \\
1 & \xrightarrow{at} & R\Omega
\end{array}
\]

It remains to see that this square is a pullback. This follows by the fact that the strict subobject \( M \hookrightarrow X \) is initial injective (1.14). □

### 3. \( f \)-regular categories and \( f \)-quasitopoi

Recall that a **regular category** is a category with finite limits and such that any arrow can be factorized into a monomorphism composed with an strict epimorphism, and in addition strict epimorphisms are universal. In a regular category strict epimorphisms are the same that regular epimorphisms, and strict epimorphisms compose \([3, 7]\). We introduce now a notion which corresponds to the notion of regular category, but utilizing strict epimorphic families instead of single strict epimorphisms. This notion is not elementary and it means a (co) completeness requirement. We call this notion **\( f \)-regular**, \( f \) for family.

3.1. **Definition.** A category is **\( f \)-regular** when it satisfies:

- \( R1 \) It has all finite limits.
- \( R2 \) Strict epimorphic families are universal (\( U \) in definition 1.5).
- \( R3 \) It has \( f \)-factorizations (definition 2.4).

Notice that (see definition 1.5) the second condition in the definition of \( f \)-regular category means that the class \( s\mathcal{E} \) of strict epimorphic families is stable under pulling-back. In the following remark we set an essential property of \( f \)-regular categories. It follows from 1.6 and 2.6:

3.2. **Remark.** In a \( f \)-regular category the collection \( s\mathcal{E} \) of strict epimorphic families satisfies all five properties in definition 1.5. □

From proposition 2.7 we have:

3.3. **Proposition.** Given a \( \mathcal{E} \)-functor \( Q \xrightarrow{u} S \), with \( S \) a \( f \)-regular category, then \( Q \) is also a \( f \)-regular category if and only if the families created by \( u \) are universal. □
Clearly, $f$-regular categories are regular. Regular categories sufficiently cocomplete (for example if the lattice of subobjects have arbitrary suprema) are $f$-regular. However, we consider $f$-regularity to be the primitive notion since when working with families this is the notion which arises naturally.

3.4. **Proposition.** A category is $f$-regular if and only if it satisfies:

- **R1**) It has all finite limits.
- **R2**) Strict epimorphic families are universal ($U$ in definition 1.5).
- **R4**) It has coequalizers of kernel pairs.
- **R5**) The lattice of subobjects of any object has arbitrary suprema.

**Proof.** Under the presence of R1) and R2) we have:

- **R3**) $\Rightarrow$ **R4):** Given an arrow $S \xrightarrow{f} T$, we can factorize it as $S \xrightarrow{g} H \xhookrightarrow{} T$, with $g$ a strict epimorphism. Then, it readily follows that $S \xrightarrow{g} H$ is a coequalizer of the kernel pair of $f$.

- **R3**) $\Rightarrow$ **R5):** Given a family of subobjects $m_\alpha : H_\alpha \xhookrightarrow{} S$, we can factorize it as $H_\alpha \xrightarrow{i_\alpha} H \xhookrightarrow{},$ with $i_\alpha$ an strict epimorphic family (note that each $i_\alpha$ is mono). Then, it readily follows that $H = \bigsqcup_\alpha H_\alpha$.

- **R4), R5) $\Rightarrow$ R3).** Given a family $X_\alpha \xrightarrow{g_\alpha} X$, take for any $\alpha$ the coequalizer of the kernel pair of $g_\alpha$. We have a factorization $X_\alpha \xrightarrow{h_\alpha} H_\alpha \xrightarrow{m_\alpha} X$, with $h_\alpha$ a strict epimorphism. It is known that from R2) it follows that $m_\alpha$ is a mono (this is an argument due originally to M. Tierney and independently M. Kelly [8], see 2.1.2 and 2.1.3 in [4]). Let $H = \bigsqcup_\alpha H_\alpha$, abuse notation, and write $X_\alpha \xrightarrow{h_\alpha} H$. Then, it can be proved that $h_\alpha$ is a strict epimorphic family.

□

Given a pointed (small) site, the category of separated sheaves is a *bounded quasitopos* in the sense of Penon [9]. In practice the most conspicuous quasitopoi are not bounded. They are categories of separated sheaves for pointed *large* sites. In this case, the categories of sheaves are not topoi, and they are not even legitimate categories since the class of morphisms between two sheaves is not in general a set. However, separate sheaves form legitimate categories (insofar the hom-sets are small) which are elementary quasitopoi in the sense of Penon. They bear to elementary quasitopoi a relation which should be considered as corresponding (in the context of quasitopoi) to the relation that Grothendieck topoi bear to elementary topoi. In this context, the size condition (“bounded”) is unnecessary, and probably misleading. Furthermore, it is not satisfied by many important examples. We introduce now the abstract notion which describes this situation.

3.5. **Definition.** A category $\mathbb{Q}$ is an $f$-*quasitopos* if it satisfies:

- **(QT1)** It has all finite limits.
- **(QT2)** It has all small colimits.
- **(QT3)** It is locally cartesian closed.
- **(QT4)** It has a strict subobject classifier $1 \xrightarrow{t} \Omega$.
- **(QT5)** It has $f$-factorizations.

Since (QT3) implies that strict epimorphic families are universal, $f$-quasitopoi are, in particular, $f$-regular categories.

Recall that a Penon’s (or elementary) quasitopos [9], is a category which satisfies (QT1), (QT3), (QT4), and a elementary (and weaker) form of (QT2), namely: it has all finite colimits. A $f$-quasitopos is, in particular, an elementary quasitopos.

For a *bounded* quasitopos Penon requires QT1 to QT4, and a different (and stronger) form of (QT5), namely: for any object $S$, the lattice of subobjects $P(S)$
should be small. This condition implies that \( P(S) \) is a complete lattice by the existence of small colimits. From proposition 3.4 it follows:

3.6. **Proposition.** In definition 3.5, condition (QT5) is equivalent to: For any object \( S \), the lattice \( P(S) \) of all subobjects of \( S \) is a complete lattice. □

Remark that the existence of an initial object by (QT2) furnishes the factorization of the empty family in (QT5), and vice-versa.

Between the three notions of quasitopoi, we have the (strict) implications:

\[
\text{bounded quasitopos } \Rightarrow \ f\text{-quasitopos } \Rightarrow \ \text{elementary quasitopos}.
\]

Now we consider a kind of functor \( u : Q \to S \) that correspond to inverse images of geometric morphisms of topoi. These functors are relevant when the categories are \( f \)-regular.

3.7. **Definition.** A \( \mathcal{M}_{\text{fin}} \) and \( E \)-functor \( u : Q \to S \) between \( f \)-regular categories is called a \( f \)-regular functor.

The \( f \)-regular functors should be considered as the morphisms of \( f \)-regular categories and \( f \)-quasitopoi. The variance in these large categories should be opposite to the variance in the category of Grothendieck topoi, since \( f \)-quasitopoi are categories of “generalized spaces”, rather than being “generalized spaces” themselves.

Some times it is convenient to consider \( \mathcal{M}_{\text{fin}} \) and \( E \)-functors between categories which are not necessarily \( f \)-regular.

3.8. **Definition.** A \( f \)-regular functor \( u : Q \to S \) between arbitrary categories is a functor such that:

- \( RF1) \) It has a right adjoint.
- \( RF2) \) It is faithful.
- \( RF3) \) It creates and preserves finite strict monomorphic families.
- \( RF4) \) It creates and preserves strict epimorphic families.
- \( RF5) \) It creates and preserves universal strict epimorphic families.

Clearly, when the categories are \( f \)-regular, both definitions coincide (recall proposition 2.8).

From proposition 3.3 it immediately follows:

3.9. **Proposition.** Given a \( f \)-regular functor \( u : Q \to S \), if \( S \) is a \( f \)-regular category, then so is \( Q \). □

3.10. **Theorem.** Given a \( f \)-regular functor \( u : Q \to S \), if \( S \) is is a \( f \)-quasitopos then so is \( Q \).

*Proof.* Follows by proposition 2.1, theorem 2.12 and proposition 2.13. □

The following are basic properties of \( f \)-regular functors:

3.11. **Proposition.** Given a \( f \)-regular functor \( Q \to S \) between arbitrary categories,

1) A finite family in \( Q \) is strict monomorphic if and only if it is initial injective.
2) Any family in \( Q \) is strict epimorphic if and only if it is final surjective.
3) Creates any finite limit that may exist in \( S \), and preserves any finite limit that may exist in \( Q \), provided it already exists in \( S \).
4) Creates any colimit that may exist in \( S \), and preserves any colimit that may exist in \( Q \). When the colimit is universal in \( S \), the created colimit in \( Q \) is also universal.
5) It has a right adjoint \( u \dashv R \), but not necessarily \( uR \cong \text{id} \). It does not have in general a left adjoint.
6) It preserves and creates $f$-factorizations.
7) The usual construction of exponentials in $Q$ out of exponentials in $S$ holds.
8) If $\Omega$ is an strict subobject classifier in $S$, then $R\Omega$ is an strict subobject classifier in $Q$.

Topological and $f$-regular functors are a different kind of $E$-functors. A topological functor may fail to satisfy condition $RF5)$. A $f$-regular functor may not create initial families which are not finite or injective (that is, over non finite or strict monomorphic families), or final families which are not surjective (that is, over non strict epimorphic families). However, often in practice there are functors which are topological and $f$-regular simultaneously (for example, the categories of quasispaces in section 4, or of strict quasispaces in the context described in 5.19. The following results clarify this situation.

3.12. Theorem. A topological functor $u : T \to S$ satisfies conditions $RF1)$ to $RF4)$ in definition 3.8. Thus, it is $f$-regular if (and only if) it satisfies $FR5)$.

Proof. Consider 1.15. By item (3 i) and its dual (which holds by item (2)), a topological functor satisfies $RF2)$, $RF3)$ and $RF4)$. By the dual of item (3 ii), it satisfies $RF1)$. □

3.13. Corollary. If strict epimorphic families in $S$ are universal (in particular, if $S$ is $f$-regular), then:

A topological functor $u : T \to S$ is $f$-regular if (and only if) strict epimorphic families in $T$ are universal. □

3.14. Theorem. A $f$-regular functor $u : Q \to S$ is topological if (and only if) it has a left adjoint $(-)_{\bot} \dashv u$, with $u(-)_{\bot} = id$.

Proof. Follows immediately by 1.15 (3). □

4. Quasispaces over a category $S$

In this section we develop the constructions in the theory of quasitopoi which correspond to the construction of categories of sheaves in the theory of topoi. These constructions, or closely related ones, were considered by Antoine over the category of Sets [1], and by Penon over general categories [9]. The paradigmatic example behind all this are Spanier’s quasitopologies [10].

Notation The hom-sets for any category will be denoted with square brackets. Thus, given any two objects $X$, $Y$ in any category $X$, $[X, Y] \in Set$ denotes the set of arrows in $X$ from $X$ to $Y$.

We recall that a Grothendieck pretopology $J$ on a category $C$ consists of a family of covers $J_C$ for each object $C \in C$ satisfying properties (I), (C) and (U) in definition 1.5 (thus it also satisfies (F), see 1.6 (1)). A Grothendieck topology is a pretopology which in addition satisfies property (S). By adding all families which are refined by a cover any pretopology generates a topology with no other additional families.

From now on we suppose that the following data are given:

(4.1) A category $C$ with a Grothendieck topology $J$, and a functor $u : C \to S$.

For any object $S$ in $S$, there is a presheaf $[u(-), S] : C^{op} \to Set$, with the usual action on morphisms. For any $\varphi : S \to T$ in $S$, composing with $\varphi$ is a natural transformation $[u(-), S] \xrightarrow{\varphi^*} [u(-), T]$.

4.2. Definition. A quasispace is a subpresheaf $X \subseteq [u(-), S]$ satisfying a covering condition. Given $C \in C$, the maps $u(C) \xrightarrow{\varphi} S$ in $X(C)$ are called admissible maps. We also say that $X$ is a quasispace structure on the set $S$. 
Covering condition: Given \( u(C) \xrightarrow{\sigma} S \) and a cover \( C_\alpha \xrightarrow{f_\alpha} C \) in \( J_C \), if the composite \( \sigma \circ u(f_\alpha) \) is admissible for all \( \alpha \), then so is \( \sigma \).

It is convenient to lay down this definition explicitly:

4.3. Definition (explicit). A quasispace is a pair \((S, X)\) where \( S \) is an object of \( \mathcal{S} \) and \( X \) assigns to each object \( C \in \mathcal{C} \) a subset \( XC \subset [u(C), S] \) subject to the following conditions:

Presheaf condition:

\[
(C \xrightarrow{f} D) \in \mathcal{C}, (uD \xrightarrow{\sigma} S) \in XD \Rightarrow (uC \xrightarrow{uf} uD \xrightarrow{\sigma} S) \in XC.
\]

Covering condition:

\[
(C_\alpha \xrightarrow{f_\alpha} C) \in J_C, \forall \alpha (uC_\alpha \xrightarrow{uf_\alpha} uC \xrightarrow{\sigma} S) \in XC_\alpha \Rightarrow (uC \xrightarrow{\sigma} S) \in XC.
\]

Notice that if the covering condition is satisfied over a pretopology, it will be satisfied also over the generated topology.

Notice that the covering condition is not the sheaf condition on the presheaf \( X : \mathcal{C}^{op} \to \mathcal{S} \). Using Grothendieck’s abuse of notation \( C \xrightarrow{\sigma} X \) for \( (uC \xrightarrow{\sigma} S) \in XC \), the later takes the form:

\[
(C_\alpha \xrightarrow{f_\alpha} C) \in J_C, (C_\alpha \xrightarrow{\sigma_\alpha} X) \text{ compatible} \Rightarrow \exists !(C \xrightarrow{\sigma} X) | \sigma \circ f_\alpha = \sigma_\alpha.
\]

We see then that the sheaf condition will be satisfied by a quasispace precisely when the family \( uC_\alpha \xrightarrow{f_\alpha} uC \) is a strict epimorphic family of \( \mathcal{S} \), that is, \( C_\alpha \xrightarrow{f_\alpha} C \) is a surjective family of \( \mathcal{C} \). We have:

4.4. Remark. If the covers are surjective families, then the sheaf condition is equivalent to the covering condition. Thus, in this case, a subpresheaf of \([u(-), S]\) is a sheaf if and only if it is a quasispace.

\(\square\)

Morphisms of quasispaces are arrows \( \varphi \) in \( \mathcal{S} \) such that the natural transformation \( \varphi^* \) sends admissible maps to admissible maps. Explicitly:

4.5. Definition. A morphism of quasispaces \((S, X) \xrightarrow{\varphi} (T, Y)\) is an arrow \( S \xrightarrow{\varphi} T \) in \( \mathcal{S} \) such that:

\[
(uC \xrightarrow{\sigma} S) \in XC \Rightarrow (uC \xrightarrow{\varphi^* \sigma} S \xrightarrow{\varphi} T) \in YC.
\]

We shall denote \( Q \) the category of quasispaces. By definition there is a faithful forgetful functor that we denote \( \varphi : Q \to \mathcal{S} \), \( q(S, X) = S \), \( q(f) = f \).

On any object \( S \in \mathcal{S} \) there is a maximal and a minimal quasispace structure:

4.6. Example. \( S = (S, S_\perp) \), where \( S_\perp C = [uC, S] \).

\[ S_\perp = (S, S_\perp), \text{ where } S_\perp C = \begin{cases} [uC, S] & \text{if the empty family is in } J_C \\ \emptyset & \text{otherwise} \end{cases} \]

\(\square\)

4.7. Example. If \( 1 \) is a terminal object of \( \mathcal{S} \), then \( 1_\perp \) is a terminal object of \( Q \). Observe that \( 1_\perp \neq 1_\perp \). In general, a quasispace structure \( X \) on \( 1 \in \mathcal{S} \), \( 1_\perp \subseteq X \subset 1_\perp \), is determined by a sieve \( \mathcal{K} \subset \mathcal{C} \in \mathcal{C} \Leftrightarrow XC = 1 \) "closed under covers" (condition which is vacuous when the topology is trivial). We see that in general there is a proper class of different quasispace structures on \( 1 \).

\(\square\)

We warn the reader that \( 1_\perp \neq 1_\perp \) even in the case where \( \mathcal{S} = \text{Set} \) is the category of sets and \( 1 \) is the singleton set. In this case, \( 1_\perp \) represents the forgetful functor, but it is not the terminal object \( 1_\perp \) of \( Q \). Classically a further condition is imposed on a quasispace to have the forgetful functor represented by the terminal object (see definition 5.2).
4.8. **Example.** If 0 is an initial object of $S$, then $0_\perp$ is an initial object of $Q$. When 0 is empty (that is, $[S, 0] \neq 0 \iff S = 0$), we shall denote $0 = \emptyset$. In this case, if in addition, the empty family covers $C$ if and only if $uC = \emptyset$, then, there is only one quasispace structure on $\emptyset$, and $\emptyset_\perp = \emptyset^\top$ is also an empty initial object of $Q$. □

4.9. **Yoneda.**

(1) Given any object $C \in C$, $uC$ carries a canonical structure of quasispace, that we denote $\varepsilon C$, defined by stipulating the equivalence:

$uK \to^\sigma uC \in \varepsilon C(K)$

$\exists K \xrightarrow{\kappa} K \in J_K$ and $K_i \xrightarrow{f_i} C$ such that $\sigma \circ uK_i = uf_i$

(in particular, any $uK \to uC$ that lifts to $C$ is admissible for $\varepsilon C$).

The assignment $C \mapsto (uC, \varepsilon C)$ define a functor (which acts as the equality on arrows) $C \xrightarrow{\varepsilon} Q$. We abuse notation and write $\varepsilon C = (uC, \varepsilon C)$.

Clearly $q\varepsilon = u :$

\[
\begin{array}{ccc}
C & \xrightarrow{\varepsilon} & Q \\
\downarrow u & & \downarrow q \\
S
\end{array}
\]

(2) If covers are final families in $C$, then

$uK \to^\sigma uC \in \varepsilon C(K)$

$\exists K \xrightarrow{f} C$ such that $\sigma = uf$

(3) Given any quasispace $(S, X)$, there is an equivalence (natural in $K$)

$uK \to^\sigma S \in XK$

$(uK, \varepsilon K) \xrightarrow{\sigma} (S, X)$ is a morphism of quasispaces

That is, there is an equality of sets $XK = [\varepsilon K, (S, X)]$.

(4) If covers are final families in $C$, then $\varepsilon$ is full.

If $u$ is faithful, then $\varepsilon$ is faithful.

(5) The family $\varepsilon C \to^\sigma (S, X), \sigma \in XC$, is a final family in $Q$.

(6) Given any cover $K_i \to K \in J_K$, the family $\varepsilon K_i \to \varepsilon K$ is a final family in $Q$. If the cover is surjective, it is a final surjective (thus strict epimorphic) family in $Q$.

**proof:** To check items (1) and (3) is sharp but straightforward, and it is left to the reader. Item (2) is immediate from item (1), and item (5) from item (3). Item (4) follows easily from items (2) and (3). Finally, to check item (6) is again sharp but straightforward, or, if the reader prefers, it follows immediately from proposition 4.12 below. □

Initial families in $Q$ are easily characterized. The proof of the following proposition is immediate:

4.10. **Proposition.** A family $(S, X) \to^\varepsilon (S_\alpha, X_\alpha)$ in $Q$ is initial if and only if given any $C$ and $uC \to S$, the following equivalence holds:

$\forall \alpha uC \to S \in XC$

$\forall \alpha uC \to S \xrightarrow{\varepsilon C} S_\alpha \in X_\alpha C$
4.11. **Theorem.** The functor $\mathbb{Q} \xrightarrow{\varphi} \mathbb{S}$ is a topological functor.

**Proof.** We have to see that $q$ creates initial families (see 1.15 (1)). Given quasispaces $(S_\alpha, X_\alpha)$ and a family $S \xrightarrow{\varphi_\alpha} S_\alpha$ in $\mathbb{S}$, define a quasispace structure $X$ on $S$ stipulating that $uC \xrightarrow{\alpha} S \in XC$ by the equivalence in proposition 4.10. It is immediate to check that the pair $(S, X)$ so defined is a quasispace and that the $\varphi_\alpha$ become morphisms of quasispaces $(S, X) \longrightarrow (S_\alpha, X_\alpha)$. □

It follows from 4.10 and 1.15 (2) that $u$ creates final families. However, we shall prove this directly because the proof yields an essential characterization of final families reminiscent of the characterization of epimorphic families of sheaves.

4.12. **Proposition.** A family $(S_\alpha, X_\alpha) \xrightarrow{\varphi_\alpha} (S, X)$ in $\mathbb{Q}$ is final if and only if given any $C$ and $uC \xrightarrow{\sigma} S$, the following equivalence holds:

$$\sigma \in XC \iff \exists C_i \rightarrow C \in J_C, \sigma_i \in X_{\alpha_i}, \text{ such that } uC_i \xrightarrow{\sigma_i} S_{\alpha_i}$$

**Proof.** Given quasispaces $(S_\alpha, X_\alpha)$ and a family $S_\alpha \xrightarrow{\varphi_\alpha} S$ in $\mathbb{S}$, define a quasispace structure $X$ on $S$ stipulating that $uC \xrightarrow{\alpha} S \in XC$ by the equivalence above. To prove the statement we have to check that $(S, X)$ is a quasispace, that the $\varphi_\alpha$ become morphisms of quasispaces $(S_\alpha, X_\alpha) \longrightarrow (S, X)$, and that the resulting family is final.

1) $(X, S)$ is a quasispace:

**presheaf condition:** Given $C \xrightarrow{f} D$ and $uD \xrightarrow{\sigma} S \in XD$, take $C_j \rightarrow C \in T_C$ and $C_j \xrightarrow{f_j} D$ such that $\sigma_j \circ f_j \in X_{\alpha_{ij}} C_j$. This shows that $\sigma \circ uf \in XC$.

**covering condition:** Let $C_i \rightarrow C \in T_C$ and $uC \xrightarrow{\sigma} S$ be such that the composites $(uC_i \rightarrow uC \xrightarrow{\sigma} S) \in XC_i$. Take (for each $i$) a cover $C_{i,j} \rightarrow C_i \in T_{C_i}$ and maps $uC_{i,j} \xrightarrow{\alpha_{i,j}} S_{\alpha_{i,j}} \in X_{\alpha_{i,j}} C_{i,j}$ such that $uC_{i,j} \xrightarrow{\alpha_{i,j}} S_{\alpha_{i,j}}$.

This shows that $\sigma \in XC$ (we use now property (C) of $T$).

2) $(S_\alpha, X_\alpha) \xrightarrow{\varphi_\alpha} (S, X)$ is a quasispace morphism: Given $\alpha$ and a map $uC \xrightarrow{\sigma} S_\alpha \in X_C \alpha C$, consider the diagram:

\[\begin{array}{ccc}
\alpha \circ \sigma & \xrightarrow{\varphi_\alpha} & S \\
\downarrow & & \downarrow \\
\sigma & \xrightarrow{\varphi_\alpha \circ \sigma} & S
\end{array}\]

This shows that $\varphi_\alpha \circ \sigma \in XC$ (we use now property (I) of $T$).

3) Finally, to check that the resulting family is a final family is straightforward. □

An important consequence of this characterization is that final families in $\mathbb{Q}$ are universal. We have:
4.13. **Proposition.** Final families for the functor $Q\to S$ are universal (property (U) in definition 1.5).

**Proof.** Let $(S_\alpha, X_\alpha)\xrightarrow{\varphi} (S, X)$ be a final family and $(T, Y)\xrightarrow{\phi} (S, X)$ a morphism of quasispaces. Consider the $C$-crible on $T$ defined by:

$$uC \xrightarrow{\varphi} T \in P \iff uC \xrightarrow{\theta} S_\alpha, \theta \in X_\alpha C,$$

such that $uC \xrightarrow{\varphi} T \xrightarrow{\phi} S$.

This defines an $r$-pullback in $Q$ by furnishing $uC$ with the yoneda $\varepsilon C$ quasispace structure (4.9). We shall show that the family $(uC, \varepsilon C)\xrightarrow{\varphi} (T, Y),\varphi \in P$, is a final family in $Q$.

Given $uK \xrightarrow{\eta} T \in YK$, the composite $\phi \circ \eta \in XK$. Using proposition 4.12 take $K_i \xrightarrow{s_i} K \in J_K$ and $uK_i \xrightarrow{\eta} S_{\alpha_i} \in X_{\alpha_i} K_i$ such that in the following diagram the exterior commutes:

$$
\begin{array}{ccc}
\eta_i & \downarrow \phi_i & \downarrow \varphi_{\alpha_i} \\
\downarrow uK_i & & \downarrow \varphi \\
\downarrow uK & \phi & \downarrow S \\
\phi \circ \eta & & 
\end{array}
$$

Fill in the middle vertical arrow, with $\varphi_i = \eta \circ us_i$. The square on the right shows that $uK_i \xrightarrow{\varphi_i} T \in P$. Then, the square on the left finishes the proof by another application of proposition 4.12.

Actually we are interested in final surjective families. Concerning surjective families we have:

4.14. **Proposition.** If strict epimorphic families in $S$ are universal, then surjective families for the functor $Q\to S$ are universal (property (U) in definition 1.5).

**Proof.** Just observe that the initial structure determined by the two arrows out of the upper left corner of an $r$-pullback (definition 1.3) taken in $S$ yields a $r$-pull-back in $Q$.

Since surjective and final families have property (S), it follows from 1.6 (3) that when strict epimorphic families in $S$ are universal, final surjective families in $Q$ are universal. Thus, from theorem 4.11 and corollary 3.13 we have:

4.15. **Theorem.** If strict epimorphic families in $S$ are universal (in particular, if $S$ is $f$-regular), then so they are in $Q$, and the functor $Q\to S$ is $f$-regular.

Then, from proposition 3.9 and theorem 3.10 we have:

4.16. **Theorem** (compare [9], 5.8). A category $Q\to S$ of quasispaces over a $f$-regular category $S$ is $f$-regular, and if $S$ is a quasitopos, then so it is $Q$.

Let $\mathbb{P}$ be the category of quasispaces for the trivial (generated by the isomorphisms in $C$) Grothendieck topology. Clearly, the inclusion determines full and faithful functor $Q\xleftarrow{c} \mathbb{P}$, where $Q$ is the category of quasispaces determined by any other topology on $C$. We shall construct a left adjoint $\mathbb{P}\xrightarrow{#} Q$ to the functor $c$, and study its basic properties. We consider the more general situation determined by an inclusion of Grothendieck topologies.
4.17. **Associate Quasispace Functor.** Consider a category \( C \), two Grothendieck topologies \( T \subset J \), and a functor \( C \rightarrow S \). Let \( P \rightarrow S \), \( Q \rightarrow S \) be the respective categories of quasispaces, and \( C \rightarrow h \), \( C \rightarrow c \) the respective yoneda functors (4.9). Let \( Q \rightarrow P \) be the inclusion functor (notice that \( p \circ c = q \), but in general \( c \circ e \neq h \)).

We have:

4.18. **Proposition.** The inclusion functor \( Q \rightarrow P \) has a left adjoint, \( \# \dashv c \), \( P \rightarrow Q \), \( id \rightarrow c \circ \# \), \( \# \circ h = e \).

**Proof.** The proof is sharp but straightforward and it is left to the reader. We mention that the properties (I), (C) and (U) (see definition 1.5) of the covers are essential.

4.19. **Proposition.** The functor \( P \rightarrow Q \) preserves finite strict monomorphic families.

**Proof.** Since \( p \) and \( q \) are, in particular, \( M_{\text{fin}} \)-functors, it is equivalent to work with initial injective families. We shall see that, in fact, \( \# \) preserves any (not necessarily injective) finite initial family. Clearly \( \# \) preserves the empty initial family \( S \) since this family is a quasispace for any topology (example 4.6). Let now \( (S, X) \rightarrow (S_i, X_i) \) be a finite initial family. The following chain of equivalences proves the proposition:

\[
\forall i uK \xrightarrow{\sigma} S \rightarrow S_i \in \#X(K)
\]

\[
\exists K_i \xrightarrow{\alpha} K \in J_K \mid uK_i \rightarrow uK \xrightarrow{\sigma} S \rightarrow X_i(K)
\]

Moreover, \( q \circ \# = p \) and \( \# \circ h = e \).

4.20. **Theorem.** If strict epimorphic families in \( S \) are universal (in particular, if \( S \) is \( f \)-regular), then the functor \( P \rightarrow Q \) is \( f \)-regular.

**Proof.** Recall that \( q \circ \# = p \). We refer to definition 3.8. Conditions RF1) and RF2) are clear. By RF1) we know that \( \# \) preserves \( sE \)-families, and in the previous proposition we established that it preserves \( sM_{\text{fin}} \)-families. By the assumption made, \( p \) and \( q \) are \( f \)-regular functors (theorem 4.15), so conditions RF3) and RF4) follow from proposition 2.2 (and RF5) is equivalent to RF4)).
5. Strict quasispaces over a \( f \)-regular category \( S \)

In the classical examples a third condition is required in the definition of quasispace. In this general setting it means a density condition for the admissible maps. Nothing much can be proved if we do not put some restrictions on the category \( S \). Although not always necessary, the sensible generality is to assume that \( S \) is \( f \)-regular.

5.1. Assumption. The category \( S \) is a \( f \)-regular category (in particular strict epimorphic families in \( S \) have all five properties in definition 1.5).

5.2. Definition. A strict quasispace is a quasispace which in addition satisfies the following condition:

Density condition:

The family \( UC \rightarrow S, \sigma \in XC, \) all \( C \in C \), is strict epimorphic in \( S \).

The category \( sQ \) is defined as the full subcategory of the category \( Q \) of quasispaces whose objects are the strict quasispaces. We denote by \( q_s \) the restriction of the functor \( q \). There is a commutative diagram of faithful functors:

\[
\begin{array}{ccc}
S & \xrightarrow{q} & Q \\
\downarrow i & & \downarrow q \\
sQ & \xleftarrow{i} & Q \\
\end{array}
\]

5.3. Yoneda (continued).

(7) For any object \( C \in C \), the quasispace \( \varepsilon C \) is strict. Thus there is a factorization (that we also denote \( \varepsilon \))

\[
\begin{array}{ccc}
C & \xrightarrow{\varepsilon} & sQ \\
\downarrow & & \downarrow i \\
& & Q \\
\end{array}
\]

(8) Given any strict quasispace \((S, X)\), the family \( \varepsilon C \rightarrow (S, X), \sigma \in XC, \) is a final surjective (thus strict epimorphic) family in \( Q \) (thus also in \( sQ \))

proof: Item (7) is clear since \( \text{id}_{uC} \in \varepsilon C \). Item (8) holds by definition of strict quasispace and item (5) in 4.9.

Any \( f \)-regular functor \( C \rightarrow S \) into a \( f \)-regular category \( S \) is the forgetful functor of a category of strict quasispaces. We have:

5.4. Theorem (compare [9], 5.13). Given a \( f \)-regular functor \( C \rightarrow S \), with \( S \) \( f \)-regular. Consider the Grothendieck topology in \( C \) whose covers are the strict epimorphic families. Then, in the commutative triangle \( C \rightarrow sQ \), the functor \( \varepsilon \) is an equivalence.

proof. From proposition 3.9 we know that \( C \) is \( f \)-regular, so that the strict epimorphic families are a Grothendieck topology. By yoneda 4.9 (4) \( \varepsilon \) is full and faithful. Let now \((S, X)\) be any strict quasispace, and consider a strict epimorphic family \( K \xrightarrow{f} C \) in \( C \) over the strict epimorphic family \( uK \xrightarrow{\varepsilon} S \), all \( K \in C \) and \( \sigma \in XK \).

Since \( f_\# \) is a cover, by yoneda 4.9 (6) the family \( \varepsilon K \xrightarrow{\varepsilon_{f_\#}} \varepsilon C \) is strict epimorphic, and by yoneda 5.3 (8) the family \( \varepsilon K \xrightarrow{\varepsilon} (S, X) \) is also strict epimorphic and sits over \( uK \xrightarrow{\varepsilon} S \). It follows that \( \varepsilon C \) is isomorphic to \((S, X)\) over \( \text{id}_S \).
We see in particular that in this case the functor $q_s$ is $f$-regular. We study now the $f$-regularity condition for the functor $q_s$ in general.

Since strict epimorphic families compose in $\mathcal{S}$, it easily follows:

5.5. **Proposition.** Given a surjective family of quasispaces $(S_\alpha, X_\alpha) \xrightarrow{i_{\alpha}} (S, X)$, if each $(S_\alpha, X_\alpha)$ is strict, then so it is $(S, X)$. $\square$

5.6. **Proposition.** Given a strict epimorphic family of quasispaces $(S_\alpha, X_\alpha) \xrightarrow{q_{\alpha}} (S, X) \in \mathcal{Q}$, if each $(S_\alpha, X_\alpha)$ is strict, then so it is $(S, X)$. **Proof.** Since $\mathcal{Q} \xrightarrow{q} \mathcal{S}$ is topological (thus an $\mathcal{E}$-functor), the given family is final surjective (1.13). Then, by the previous proposition $(S, X)$ is strict. $\square$

Thus we have the following:

5.7. **Corollary.** $s\mathcal{Q}$ is closed under strict epimorphic families in $\mathcal{Q}$, and the functors $q_s$ and $i$ are $\mathcal{E}$-functors. $\square$

5.8. **Proposition.** The functor $s\mathcal{Q} \xrightarrow{i} \mathcal{Q}$ has a right adjoint $q_s \dashv r$, $\mathcal{Q} \xrightarrow{r} s\mathcal{Q}$, and if for any $S \in \mathcal{S}$ the family $uC \xrightarrow{\sigma} S$, all $C \in \mathcal{C}$, is strict epimorphic, then the right adjoint is full and faithful, $r = (-)\top$, $q_s(-)\top = id$. $\square$

**Proof.** Follows by proposition 2.8. Given $S \in \mathcal{S}$, take the family $uC \xrightarrow{\sigma} S$, all $C \in \mathcal{C}$, factor this family $uC \xrightarrow{\psi} H \xrightarrow{m} S$, with a monomorphism $m$ and the family $\psi$ strict epimorphic. Let $rS = (H, rS)$ be a strict epimorphic family over the family $\psi$. Then, $rS$ is the right adjoint to $q_s$ on $S$. The second claim is clear since in this case $H = S$. $\square$

Notice that the condition in the proposition says that the quasispace $S\top$ is strict. When this in not the case, there can not be any strict quasispace structure on $S$, that is, the fiber $s\mathcal{Q}S$ is empty.

5.9. **Proposition.** The inclusion $s\mathcal{Q} \xrightarrow{i} \mathcal{Q}$ has a right adjoint $\mathcal{Q} \xrightarrow{s} s\mathcal{Q}$, $i \dashv s$ (notice that $r = s(-)\top$ and $q_s s \neq q$). **Proof.** Follows by proposition 2.8. Given a quasispace $(S, X)$, we abuse notation and denote $s(S, X) = (sS, X)$, where $sS \subset S$ is given by the factorization of the family of all admissible maps $uC \xrightarrow{\sigma} S$, $C \in \mathcal{C}$ into a strict epimorphic family followed by a monomorphism $uC \xrightarrow{\rightarrow} sS \subset S$. $\square$

We see from proposition 5.5 that a surjective family of strict quasispaces $(S_\alpha, X_\alpha) \xrightarrow{i_{\alpha}} (S, X)$ is final in $s\mathcal{Q}$ if and only if it is characterized by the equivalence in proposition 4.12. This is not the case for non surjective final families in $s\mathcal{Q}$. On spite of having a right adjoint, the inclusion functor $i$ will not preserve final families unless they are surjective. It is clear that the quasispace $S\bot$ (the empty final family, see example 4.6) is not a strict quasispace. More generally, at least when the covers are surjective, we have:

5.10. **Remark.** Any family $(S_\alpha, X_\alpha) \xrightarrow{i_{\alpha}} (S, X)$ between strict quasispaces which is final in $\mathcal{Q}$ (thus characterized by the equivalence in proposition 4.12) is necessarily surjective (assuming the covers to be surjective). **Proof.** For each $UC \xrightarrow{\sigma} S$, $\sigma \in XC$, $C \in \mathcal{C}$, take a cover $C_{\sigma,i} \rightarrow C \in \mathcal{T}_C$ and maps $uC_{\sigma,i} \rightarrow S_{\alpha(\sigma,i)} \in X_{\alpha(\sigma,i)} C_{\sigma,i}$ such that $uC_{\sigma,i} \rightarrow S_{\alpha(\sigma,i)} \xrightarrow{\rightarrow} S_{\alpha(\sigma,i)}$. By property (C) the composite by the lower left corner is strict epimorphic. Then the statement follows by property (S). $\square$
5.11. Proposition. If covers are surjective families in $\mathbb{C}$, final surjective families for the functor $sQ \rightarrow S$ are universal (property (U) in definition 1.5).

Proof. As remarked above, by proposition 5.5 a final surjective family in $sQ$ is characterized by proposition 4.12. Then it follows from 5.3 (7) that the r-pullback constructed in the proof of proposition 4.13 is an r-pullback of strict quasispaces. The resulting family is a final family in $Q$, then by remark 5.10 it is also surjective.

The fact that final surjective families in $sQ$ are universal holds independently of whether the covers are surjective families in $\mathbb{C}$, or not.

However, some other assumptions have to be made. These assumptions are forced upon us by the need that a finite initial injective family of strict quasispaces be in fact an strict quasispace. Given strict quasispaces $(S_\alpha, X_\alpha)$ and a family $S \rightarrow S_\alpha$ in $S$, the initial quasispace structure defined on $S$ in theorem 4.11 will not be strict, unless we make further assumptions. The failure or not of this structure to be strict is related to the fact of whether the functor $q_s$ is $f$-regular (that implies, in particular, that final surjective families in $sQ$ are universal). When any initial injective family of strict quasispaces is a strict quasispace, the functor $q_s$ will be topological.

5.12. Assumption. Given a finite initial injective family of quasispaces $(S, X) \rightarrow (S_\alpha, X_\alpha)$, if each $(S_\alpha, X_\alpha)$ is strict, then so it is $(S, X)$.

5.13. Proposition (Under assumption 5.12). Given a finite strict monomorphic family of quasispaces $(S, X) \rightarrow (S_\alpha, X_\alpha) \in Q$, if each $(S_\alpha, X_\alpha)$ is strict, then so it is $(S, X)$.

Proof. Since $Q \rightarrow S$ is topological (thus a $M_{fin}$-functor), the given family is initial injective (1.14). Then, by the assumption $(S, X)$ is strict.

Thus we have the following:

5.14. Corollary. $sQ$ is closed under finite strict monomorphic families in $Q$, and the functors $q_s$ and $i$ are $M_{fin}$-functors.

In particular, $sQ$ is closed under pullbacks taken in $Q$ (proposition 2.1).

5.15. Theorem (Under assumption 5.12). The functors $sQ \rightarrow S$ and $sQ \rightarrow Q$ are $f$-regular.

Proof. We already know that they have a right adjoint (propositions 5.8 and 5.9) and that they are $E$ and $M_{fin}$-functors (propositions 5.7 and 5.14). To prove that they are $f$-regular it remains to see that strict epimorphic families in $sQ$ are universal. But $Q$ is a $f$-regular category (theorem 4.16), and the inclusion $sQ \rightarrow Q$ is closed under pullbacks and strict epimorphic families. The claim follows.

Then, from proposition 3.9 and theorem 3.10 we have:

5.16. Theorem (Under assumption 5.12), (compare [9], 5.10). A category $sQ \rightarrow S$ of strict quasispaces over a $f$-regular category $S$ is $f$-regular, and if $S$ is a quasitopos, then so it is $sQ$.

From theorems 4.20 and 5.15 it follows
5.17. **Associate Quasispace Functor** (continued). Consider the situation described in 4.17. It is clear that if $(S, X) \in P$ is a strict quasispace, then so it is $\#(S, X) = (S, \#X) \in Q$, and we have a functor $sP \xrightarrow{\#} sQ$ left adjoint to the inclusion $sQ \xrightarrow{i} sP$. There is the following square of pairs of adjoint functors:

\[ \begin{array}{ccc}
\# & \dashv & c,
\mu & \dashv & s,
Q & \xrightarrow{i} & P
\end{array} \]

The inclusion functors commute, and the adjunctions in the full subcategories are the restrictions of the adjunctions in the larger categories

\[ i \circ c = c \circ i, \quad i \circ \# = \# \circ i, \quad s \circ c = c \circ s. \]

(Under assumption 5.12). The functors $i$, $\#$, are $f$-regular functors (the $f$-regularity for the functor $\#$ between strict quasispaces follows because the inclusion functors $i$ are $f$-regular).

The following conditions were considered by Penon ([9], 5.10) to prove that the category $sQ$ is a quasitopos. We use them to insure the validity of assumption 5.12.

5.18. **Proposition.** Let $C \xrightarrow{u} S$ ($S$ f-regular) be such that:

i) The quasispace $1_{\top}$ is strict.

ii) Given any $C \in C$ and a strict subobject $S \hookrightarrow uC = q_{\varepsilon C}$, the initial quasispace structure induced on $S$ is strict.

iii) Given any two objects $C, D \in C$, the product $\varepsilon C \times \varepsilon D$ taken in $Q$ is an strict quasispace.

Then, finite initial injective families of strict quasispaces are strict, that is, the assumption 5.12 holds.

**Proof.** The empty initial injective family is given by $1_{\top}$. We claim that given a strict quasispace $(S, X)$ and a strict monomorphism $T \hookrightarrow S$, the initial quasispace $(T, Y) \hookrightarrow (S, X)$ is strict: We argue over the following diagram:

\[ \begin{array}{ccc}
\varepsilon K & \xrightarrow{\mu} & (P_{\tau}, Z_{\tau}) & \xrightarrow{\sigma_{\mu, \tau}} & \varepsilon C \\
\downarrow{\sigma_{\mu, \tau}} & \downarrow{\tau} & \downarrow{\tau} & \downarrow{\tau} & \downarrow{\tau} \\
(T, Y) & \xrightarrow{\tau} & (S, X)
\end{array} \]

The family $\{\tau, \text{ all } C \in C, \tau \in X C\}$ is strict epimorphic (yoneda 5.3 (8)). Pulling back this family in $Q$ we have the family $\pi_{\tau}$, which is strict epimorphic since $Q$ is $f$-regular (theorem 4.16). Since $Z_{\tau}$ is the initial structure induced by $P_{\tau} \hookrightarrow uC$ (the reader can check this), for each $\tau$, by ii), the family $\{\mu, \text{ all } K \in C, \mu \in Z_{\tau} K\}$ is strict epimorphic. The composite family $\sigma_{\mu, \tau} = \pi_{\tau} \circ \mu$ is strict epimorphic. But $\sigma_{\mu, \tau} \in Y K$, the claim follows.

We let the reader verify that from iii) it easily follows that the product of two (hence any finite product) of strict quasispaces is strict. To finish the proof recall that a initial injective family induces a initial injective map into a product.

In general the functor $sQ \xrightarrow{\top} S$ will not be topological. It is clear that the quasispace $S_{\bot}$ is not strict, but still it may exist a smallest strict quasispace structure on $S$. This is the case in many classical examples (see [5]), and it is equivalent to the fact that the functor $sQ \xrightarrow{\top} S$ is topological. We consider a general situation that include all these examples:
5.19. **Assumption.** There is a class of objects $\mathbb{I} \subset \mathbb{C}$ such that:

a) For each $I$ in $\mathbb{I}$ and $C$ in $\mathbb{C}$, $u$ establish a bijection $I \rightarrow C$ in $\mathbb{C}$.

b) For any $S$ in $\mathbb{S}$, the family of all $uI \rightarrow S$, $I \in \mathbb{I}$, is strict epimorphic.

c) Given any strict epimorphic family $S_\alpha \rightarrow S$ in $\mathbb{S}$, any $I \in \mathbb{I}$ and $uI \rightarrow S$,

there exists $I_i \rightarrow I \in \mathcal{T}_I$, $uI_i \rightarrow S_\alpha$, such that

\[
\begin{array}{ccc}
S_\alpha & \xrightarrow{\varphi_\alpha} & S \\
C_i & \xrightarrow{\sigma} & uI_i \\
C & \xrightarrow{\sigma} & uI
\end{array}
\]

In [5] it is developed the case in which $\mathbb{S}$ is the category of Sets, $\mathbb{I} = \{1\}$, and $uI = 1$ = the singleton set.

5.20. **Proposition** (Under assumption 5.19). Given a quasispace $(S, X)$, the following conditions are equivalent:

i) $(S, X)$ is a strict quasispace.

ii) $uI \rightarrow S \in XI$ for all $I \in \mathbb{I}$ and $\sigma \in [uI, S]$.

**Proof.** Clearly condition b) implies ii) $\Rightarrow$ i). The other implication follows by a straightforward application of conditions c) and a) on the family of all admissible maps. □

Given any $S \in \mathbb{S}$, we can consider the quasispace structure generated by all maps $uI \rightarrow S$, $I \in \mathbb{I}$. This yields:

5.21. **Corollary** (Under assumption 5.19). Given any $S \in \mathbb{S}$, there exists a smallest strict quasispace structure on $S$, that we denote $S_\perp$, and which is defined as follows:

$S_\perp = (S, S_\perp)$, where $uC \rightarrow S \in S_\perp$ if and only if there exists

\[
\begin{array}{ccc}
S_\alpha & \xrightarrow{\varphi_\alpha} & S \\
C_i & \xrightarrow{\sigma} & uI_i \\
C & \xrightarrow{\sigma} & uI
\end{array}
\]

It follows then from 1.15 (3) and corollary 5.7 that the functor $sQ \rightarrow S$ is topological. This can be seen directly as follows:

Using proposition 5.20 it is also immediate to check that given strict quasispaces $(S_\alpha, X_\alpha)$ and any family $S \rightarrow S_\alpha$ in $\mathbb{S}$, the initial quasispace structure defined on $S$ in theorem 4.11 by stipulating the equivalence in proposition 4.10 is strict. It follows:

5.22. **Corollary** (Under assumption 5.19). Given an initial family of quasispaces $(S, X) \rightarrow S_\alpha$ $(S_\alpha, X_\alpha)$, if each $(S_\alpha, X_\alpha)$ is strict, then so it is $(S, X)$. □

5.23. **Corollary** (Under assumption 5.19). The functor $sQ \rightarrow S$ is topological and $sQ$ is closed under initial families in $Q$. □

**Warning:** It does not follow that the functor $i$ is topological (which is not).

From 5.22 we have, in particular:

5.24. **Proposition** (Under assumption 5.19). Assumption 5.12 holds. □

Thus:
5.25. **Corollary** (Under assumption 5.19). The functors \( sQ^{-} \to S \) and \( sQ^{-} \to Q \) are \( f \)-regular, a category \( sQ^{-} \to S \) of strict quasispaces over a \( f \)-regular category \( S \) is \( f \)-regular, and if \( S \) is a quasitopos, then so it is \( sQ^{-} \). Furthermore, the associate quasispace functor situation described in 5.17 holds.

The construction of the strict quasispace \( S_{\perp \ell} \) in corollary 5.21 can be generalized and provides a left adjoint for the inclusion \( sQ^{-} \to Q \).

5.26. **Proposition** (Under assumption 5.19). The inclusion \( sQ^{-} \to Q \) has a left adjoint \( \ell^{-} \to sQ, \ell \dashv i \), such that \( q_s \ell = q \) (notice that \( (-)_{\perp \ell} = \ell((-)_{\perp}) \)).

**Proof.** We give an indication of the proof. The functor \( \ell \) is easily understood: given a quasispace \((S, X)\), we abuse notation and denote \( \ell(S, X) = (S, \ell X) \), \( X \subset \ell X \), where \( \ell X \) is the quasispace generated by \( X \) and all the arrows in \([uI, S]\), all \( I \in I \). Thus, \( \ell(S, X) = (S, X \vee S_{\perp \ell}) \). \( \square \)

**References**

[1] Antoine, P., *Étude élémentaire des catégories d’ensembles structurés*, Bull. Soc. Math. Belgique 18, 142-166 and 387-414, (1966).
[2] Artin M., Grothendieck A., Verdier J. L., *Théorie des topos et cohomologie étale des schémas* (2 vols.), Séminaire de Géometrie Algébrique du Bois-Marie (SGA 4), 1963-64, Lecture Notes in Math. 269 (vol. 1), 270 (vol. 2), Springer-Verlag, (1972).
[3] Barr, M., *Exact categories*, in *Exact categories and categories of sheaves*, Lecture Notes in Math. vol 236, 1-120, Springer-Verlag (1971).
[4] Borceux, F., *Handbook of categorical algebra 2*, Encyclopedia of Mathematics Vol 51, Cambridge Univ. Press, (1994).
[5] Dubuc, E. J., *Concrete quasitopoi*, in *Applications of sheaves* (Proceedings, Durham 1977), Lecture Notes in Math. vol 753, 239-254, Springer-Verlag, (1979).
[6] Dubuc, E. J., Español, L., *Topological functors as familiarly-fibrations*, arXiv:math.CT/0611701 v1, (2006).
[7] Grillet, P.A., *Regular categories*, in *Exact categories and categories of sheaves*, Lecture Notes in Math. vol 236 121-222, Springer-Verlag, (1971).
[8] Kelly, G. M. *Monomorphisms, epimorphisms and pullbacks*, J. of the Austr. Math. Soc. 9, (1969).
[9] Penon, J., *Sur les quasi-topos*, Cahiers Top. Gom. Diff. 18-2, 181-218, (1977).
[10] Spanier, E., *Quasitopologies*, Duke Mathematical Journal 30, 1-14, (1963).