Abstract. The paper is concerned with a zero-sum Stackelberg stochastic linear-quadratic (LQ, for short) differential game over finite horizons. Under a fairly weak condition, the Stackelberg equilibrium is explicitly obtained by first solving a forward stochastic LQ optimal control problem (SLQ problem, for short) and then a backward SLQ problem. Two Riccati equations are derived for constructing the Stackelberg equilibrium. An interesting finding is that the difference of these two Riccati equations coincides with the Riccati equation associated with the zero-sum Nash stochastic LQ differential game, which implies that the Stackelberg equilibrium and the Nash equilibrium are actually identical. Consequently, the Stackelberg equilibrium admits a linear state feedback representation, and the Nash game can be solved in a leader-follower manner.

Keywords. Stochastic differential game, Stackelberg equilibrium, linear-quadratic, two-person, zero-sum, Nash equilibrium, Riccati equation, closed-loop representation.

AMS subject classifications. 91A15, 93E20, 49N10, 49N70.

1 Introduction

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, \(W\) a one-dimensional standard Brownian motion, and \(\mathcal{F} \equiv \{\mathcal{F}_t\}_{t \geq 0}\) the usual augmentation of the natural filtration generated by \(W\). For a given initial state \(x \in \mathbb{R}^n\), consider the following controlled linear stochastic differential equation (SDE, for short) on a finite horizon \([0, T]\):

\[
\begin{aligned}
  dX(s) &= [A(s)X(s) + B_1(s)u_1(s) + B_2(s)u_2(s)]ds \\
  &\quad + [C(s)X(s) + D_1(s)u_1(s) + D_2(s)u_2(s)]dW(s), \\
  X(0) &= x,
\end{aligned}
\]

(1.1)

where \(A, C : [0, T] \to \mathbb{R}^{n \times n}\) and \(B_i, D_i : [0, T] \to \mathbb{R}^{n \times m_i}\) (\(i = 1, 2\)), called the coefficients of the state equation (1.1), are given deterministic functions. The problem involves two players with opposing aims. Each player can affect the evolution of the system (1.1) by selecting his/her own control. In the above, the process \(u_i (i = 1, 2)\) represents the control of Player \(i\), which belongs to the following space:

\[
U_i = \left\{ \varphi : [0, T] \times \Omega \to \mathbb{R}^{m_i} \mid \text{\(\varphi\) is \(\mathcal{F}\)-progressively measurable, and} \quad \mathbb{E} \int_0^T |\varphi(s)|^2ds < \infty \right\}.
\]

The solution \(X(\cdot) \equiv X(\cdot; x, u_1, u_2)\) of (1.1) is called the state process corresponding to \(x\) and \((u_1, u_2)\). The criterion for the performance of \(u_1\) and \(u_2\) is given by the following quadratic functional:

\[
J(x; u_1, u_2) = \mathbb{E} \left\{ \langle GX(T), X(T) \rangle + \int_0^T \left[ \langle Q(s)X(s), X(s) \rangle + \sum_{i=1}^2 \langle R_i(s)u_i(s), u_i(s) \rangle \right] ds \right\},
\]

(1.2)
where $G$ is an $n \times n$ symmetric matrix; $Q : [0, T] \rightarrow \mathbb{R}^{n \times n}$ and $R_i : [0, T] \rightarrow \mathbb{R}^{m_i \times m_i} \ (i = 1, 2)$ are deterministic, symmetric matrix-valued functions.

In our problem, Player 2 is the leader, who announces his/her control $u_2$ first, and Player 1 is the follower, who chooses his/her control accordingly. The criterion functional $J(x; u_1, u_2)$ is regarded as the loss of Player 1 and the gain of Player 2. So whatever the leader announces, the follower will play optimally; that is, Player 1 will select a control $\bar{u}_1(\cdot; u_2, x)$ (depending on the control $u_2$ announced by the leader as well as the initial state $x$) such that $J(x; u_1, u_2)$ is minimized. Knowing this the leader will choose $u_2$ a priori so that $J(x; \bar{u}_1(\cdot; u_2, x), u_2)$ is maximized. Such a game is referred to as a two-person Stackelberg stochastic linear-quadratic (LQ, for short) differential game (denoted by Problem (SG)), in memory of Stackelberg’s pioneering contribution in this field. The main objective of the two players is to find the Stackelberg equilibrium, mathematically defined as follows.

**Definition 1.1.** A control pair $(\bar{u}_1, \bar{u}_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ is called a Stackelberg equilibrium for the initial state $x$ if

$$\inf_{u_1 \in \mathcal{U}_1} J(x; u_1, \bar{u}_2) = J(x; \bar{u}_1, \bar{u}_2) = \sup_{u_2 \in \mathcal{U}_2} \inf_{u_1 \in \mathcal{U}_1} J(x; u_1, u_2).$$

As mentioned earlier, when playing the game, the control selected by the follower depends on the initial state and the control announced by the leader. This means that the optimal control of the follower is a mapping from $\mathcal{U}_2 \times \mathbb{R}^n$ into $\mathcal{U}_1$, which is usually referred as to an Elliot–Kalton strategy. Thus, the players may use the optimal control-strategy pair as an alternative solution to the game. More precisely, we have the following definition.

**Definition 1.2.** Let $\Gamma_1$ be the set of all Elliot–Kalton strategies for Player 1. A control-strategy pair $(\bar{\alpha}_1, u_2) \in \Gamma_1 \times \mathcal{U}_2$ is said to be optimal for the initial state $x$ if

$$J(x; \bar{\alpha}_1(u_2, x), u_2) = \inf_{u_1 \in \mathcal{U}_1} J(x; u_1, u_2), \quad \forall u_2 \in \mathcal{U}_2,$$

$$J(x; \bar{\alpha}_1(\bar{u}_2, x), \bar{u}_2) = \sup_{u_2 \in \mathcal{U}_2} J(x; \bar{\alpha}_1(u_2, x), u_2).$$

Comparing **Definition 1.2** with **Definition 1.1**, it is not hard to see that the outcome $(\bar{u}_1, \bar{u}_2) \equiv (\bar{\alpha}_1(\bar{u}_2, x), \bar{u}_2)$ of an optimal control-strategy pair is a Stackelberg equilibrium.

Since the pioneering work [15] by Stackelberg, the theory of Stackelberg games has been widely used in economics, finance, and engineering; such as the famous principal–agent model (see, for example, [2, 5]). The Stackelberg stochastic LQ differential game was initially studied by Bagchi and Basar [1]. In 2002, a general framework was formulated by Yong [25], in which the leader’s problem was described as an LQ optimal control problem for forward-backward SDEs. By a decoupling method, Yong showed that the open-loop solution can be represented as a state feedback form, provided the associated stochastic Riccati equation is solvable. From then on, there has been extensive research on Stackelberg stochastic LQ game problems. For example, in [4] Bensoussan, Chen, and Sethi established the maximum principle for Stackelberg games; Shi, Wang, and Xiong [14] investigated a Stackelberg stochastic LQ differential game with asymmetric information; Moon [11] studied the case with jump-diffusion systems; Bensoussan et al. [3] considered a mean-field problem with state and control delays; Li and Yu [8] characterized the unique equilibrium of a nonzero-sum Stackelberg LQ game with multilevel hierarchy in a closed form; and Moon and Yang [12] discussed the time-consistent open-loop solutions for time-inconsistent Stackelberg LQ games.

In the literature, it is often assumed that the associated Riccati equations are solvable so that the Stackelberg equilibrium can be constructed explicitly. However, such an assumption seems too strong in certain situations because the solvability of the Riccati equations is merely sufficient, but not necessary for the existence of a Stackelberg equilibrium. Since the solvability of the Riccati equations itself is very difficult, solving Problem (SG) in a general framework is more challenging. The first goal of our paper is to overcome this difficulty in the zero-sum case and then establish a general approach for finding the Stackelberg equilibrium of Problem (SG) by generalizing the recent works [17, 20] on indefinite stochastic LQ optimal control problems to the nonhomogenous case and making some new observations. This can be regarded as one of the main contributions in this paper.
Another important kind of zero-sum stochastic LQ differential games is the so-called Nash game, in which both players announce their decisions simultaneously (see, for example, [13, 21, 23]). In a Nash game, the objective of the players is to find a saddle point \((u_1^\ast, u_2^\ast)\) (also called a Nash equilibrium), defined by

\[
J(x; u_1^\ast, u_2^\ast) \leq J(x; u_1', u_2') \leq J(x; u_1', u_2^\ast), \quad \forall (u_1, u_2) \in U_1 \times U_2.
\]

(1.3)

Such a pair (if exists) is the best choice for both players in the sense that no player can benefit by changing their own control. For simplicity, we shall denote the Nash game by Problem (NG).

Note that though the players in Problem (SG) have opposite objects, they still agree to make some cooperations, because there is a hierarchical structure of decision making between the players. However, the players in Problem (NG) are pure competitors as they are treated on an equal basis. Thus in most of the literature, if not all, Problems (NG) and (SG) are regarded as two different games. In this paper, we shall compare Problem (SG) and Problem (NG) more carefully, and reveal an interesting fact: the Stackelberg equilibrium and the Nash equilibrium coincide under a uniform convexity-concavity condition ((UCC) condition, for short). This is another important contribution of the paper and is completely new in the literature.

1.1 The main results

As mentioned, the purpose of this paper is to develop a general approach for solving Problem (SG) and to establish the connection between Problem (SG) and Problem (NG). We now briefly list our ideas and main results as follows.

(i) We consider first, for a fixed \(u_2 \in U_2\), the follower’s problem, which we denote by Problem (FLQ). By a result form Sun, Li, and Yong [17], we know that Problem (FLQ) admits a unique open-loop optimal control \(\bar{u}_1\) of the form \(\bar{u}_1 = \tilde{\alpha}_1(u_2, x) \equiv \Theta_1 \tilde{X} + v_1 \in U_1\), where the function \(\Theta\) and the process \(v_1\) are determined by the associated Riccati equation and the associated backward SDE (BSDE, for short), respectively. Note that the state process \(\tilde{X}\) depends on the initial state \(x\), and both \(\tilde{X}\) and \(v_1\) depend on the given control \(u_2\). Thus, \(\bar{u}_1\) is a functional of \(u_2\) and \(x\).

(ii) Knowing the follower will use his/her best response \(\bar{u}_1 = \tilde{\alpha}_1(u_2, x)\), the leader’s problem (denoted by Problem (LLQ)) is then to choose a \(\bar{u}_2 \in U_2\) to maximize the utility functional

\[
J(x; \tilde{\alpha}_1(u_2, x), u_2) \equiv J(x; \Theta_1 \tilde{X}(u_2) + v_1(u_2), u_2).
\]

A remarkable feature of the above functional is that it has an explicit representation independent of the forward state process \(\tilde{X}\). Using this crucial observation, we convert the leader’s problem into a backward stochastic LQ optimal control problem.

(iii) We develop some results on backward stochastic LQ optimal control problems with nonhomogeneous terms (see Proposition 2.3), solve the backward control problem derived from the Stackelberg game (see Proposition 3.3), and then verify the resulting control pair is a Stackelberg equilibrium of the game (see Theorem 3.4).

The following is concerned with the connections between Problems (SG) and (NG).

(iv) Under the (UCC) condition (i.e., (H3) and (H5)), we study the leader’s problem by a careful convexity analysis of the criterion functional (see Proposition 4.2) and a closer investigation of backward stochastic LQ optimal control problems (see Proposition 4.7). We obtain the unique optimal control of Problem (LLQ) by solving a new Riccati equation (see Proposition 4.8), in which the auxiliary function introduced in [20] is removed. Then we further show that the Stackelberg equilibrium of Problem (SG) admits a closed-loop representation (see Theorem 4.9).

(v) We find an interesting fact: the solutions to the Riccati equations associated with Problems (FLQ) and (LLQ) can be used to solve the Riccati equation derived in Sun [16] for finding the saddle point of Problem (NG) (see Theorem 5.2). A key point of the proof is to build a bridge between the singular terms of these Riccati equations (see Lemma 5.4), which can be regarded as the most
Theorem 5.2 generalizes the results of [16] at least in two aspects:

• The well-posedness of the Riccati equation associated Problem (NG) is established under a weaker assumption and with a new constructive method (see Remark 5.3).

• An explicit relationship between the Riccati equations associated with Problems (SG), (FLQ) and (LLQ) is established, which is interesting in its own right and new in the literature.

(vi) We observe that the closed-loop systems of Problems (SG) and (NG) coincide (see Theorem 5.9), from which we conclude that the Stackelberg equilibrium obtained in Theorem 3.4 and the unique open-loop saddle point of Problem (NG) are identical (also see Theorem 5.7 for a direct proof). This means that we can solve the Nash game in a leader-follower manner.

The remainder of this paper is structured as follows. In Subsection 1.2, we give a literature review on some closely related topics. Section 2 collects some preliminary results that will be frequently used in the sequel. In Section 3, the Stackelberg equilibrium of Problem (SG) is obtained by solving a forward-backward stochastic LQ optimal control problem. Section 4 is devoted to the closed-loop representation of the Stackelberg equilibrium by some further analysis of backward stochastic LQ optimal control problems. The connection between Problems (SG) and (NG) is established in Section 5, and Section 6 concludes the paper. Some technical details are sketched in Appendix.

1.2 Literature review on the related topics

The LQ control/game theory has occupied the center stage for research in control theory for a long history. Since the purpose of the paper is not to make a lengthy survey on the literature, we only list some closely related works here. In Problem (SG), the follower’s problem is a (forward) stochastic LQ optimal control problem. We refer the reader to the books [26, Chapter 6] and [22] for a detailed study of this subject. In Problem (SG), the leader’s problem is a backward stochastic LQ optimal control problem, which was initially investigated by Lim and Zhou [10], and then generalized by [9, 18, 20] to various cases. The results obtained in Section 3 benefit from the recent work of Sun, Wu, and Xiong [20] a lot. However, to explore the connection between Problems (SG) and (NG) in Sections 4 and 5, we still need to overcome some mathematical difficulties (see, for example, Theorem 5.2) and to make some more accurate observations (see, for example, Proposition 4.2 and Theorem 5.7). For more information and references on Problem (NG), we send the interested reader to the works [28, 13, 6, 7, 21, 27, 16, 19] and the recent book [23] by Sun and Yong.

2 Preliminaries

Throughout the paper, $\mathbb{R}^{n \times m}$ denotes the Euclidean space consisting of $n \times m$ real matrices, endowed with the Frobenius inner product $\langle M, N \rangle = \text{tr}[M^T N]$, where $M^T$ is the transpose of $M$ and $\text{tr}(M)$ is the trace of $M$. The norm induced by $\langle \cdot, \cdot \rangle$ is denoted by $| \cdot |$. The identity matrix of size $n$ is denoted by $I_n$, which is often simply written as $I$ when there is no confusion. When $m = 1$, we simply write $\mathbb{R}^{n \times 1}$ as $\mathbb{R}^n$. Let $\mathbb{S}^n$ be the subspace of $\mathbb{R}^{n \times n}$ consisting of symmetric matrices and $\mathbb{S}_+^n$ (resp., $\mathbb{S}_-^n$) be the subset of $\mathbb{S}^n$ consisting of positive (resp., negative) semidefinite matrices. For $M, N \in \mathbb{S}^n$, we write $M \succeq N$ (resp., $M \succ N$) if $M - N$ is positive semidefinite (resp., positive definite). For an $\mathbb{S}^n$-valued measurable function $F$ on $[0, T]$, we write

$$
\begin{cases}
F \geq 0 & \text{if} \ F(s) \geq 0, \ a.e. \ s \in [0, T], \\
F > 0 & \text{if} \ F(s) > 0, \ a.e. \ s \in [0, T], \\
F \geq \delta I_n & \text{if} \ F(s) \geq \delta I_n, \ a.e. \ s \in [0, T], \ \text{for some} \ \delta > 0.
\end{cases}
$$

Moreover, we use $F \leq 0$, $F < 0$ and $F \ll 0$ to indicate that $-F \geq 0$, $-F > 0$ and $-F \gg 0$, respectively. If $F \gg 0$ (resp., $F \ll 0$), we say that $F$ is uniformly positive (resp., negative) definite. For any Euclidean space $\mathbb{H}$ (which could be $\mathbb{R}^n$, $\mathbb{R}^{n \times m}$, $\mathbb{S}^n$, etc.), we introduce the following spaces:

$$
L^\infty(0, T; \mathbb{H}) = \{ \varphi : [0, T] \to \mathbb{H} \mid \varphi \text{ is essentially bounded} \};
$$
\[
L^2_F,(Ω; H) = \{ ξ : Ω → H \mid ξ \text{ is } F_T\text{-measurable and } E[|ξ|^2] < ∞ \};
\]
\[
L^2(0, T; H) = \{ φ : [0, T] × Ω → H \mid φ \text{ is } F\text{-progressively measurable, and } E \int_0^T |φ(s)|^2 ds < ∞ \};
\]
\[
L^2(Ω; C([0, T]; H)) = \{ φ : [0, T] × Ω → H \mid φ \in F \text{ is continuous, } F\text{-adapted, and } E \sup_{0 ≤ s ≤ T} |φ(s)|^2 < ∞ \}.
\]

To guarantee that Problem (SG) is well-posed, we assume that the coefficients of state equation (1.1) and the weighting matrices in quadratic functional (1.2) satisfy the following conditions.

**(H1).** The coefficients of state equation (1.1) satisfy

\[
A, C ∈ L^∞(0, T; R^{n×n}), \quad B_i, D_i ∈ L^∞(0, T; R^{n×m_i}); \quad i = 1, 2.
\]

**(H2).** The weighting matrices in quadratic functional (1.2) satisfy

\[
G ∈ S^n, \quad Q ∈ L^∞(0, T; S^n), \quad R_i ∈ L^∞(0, T; S^{m_i}); \quad i = 1, 2.
\]

Let (H1) hold. For any \( x \in R^n \) and \( (u_1, u_2) ∈ U_1 × U_2 \), by the standard results of SDEs, state equation (1.1) admits a unique solution \( X ∈ L^2(Ω; C([0, T]; R^n)) \). Then under assumption (H2), the quadratic functional (1.2) is well-defined and thus Problem (SG) is well-posed.

As mentioned in the introduction section, the game with state equation (1.1) and functional (1.2) can be formulated as two different problems (i.e., Problem (SG) and Problem (NG)). Recall Definition 1.1 and Definition 1.2, in which the notions of Stackelberg equilibria and Elliot–Kalton strategies associated with Problem (SG) are introduced. Now let us present an important notion of Problem (NG).

**Definition 2.1.** A control pair \((u_1^*, u_2^*) \in U_1 × U_2\) is called an open-loop saddle point (or a Nash equilibrium) of Problem (NG) for the initial state \( x \in R^n \) if

\[
J(x; u_1^*, u_2^*) ≤ J(x; u_1^*, u_2^*) ≤ J(x; u_1, u_2^*), \quad ∀(u_1, u_2) ∈ U_1 × U_2.
\]

(2.1)

For any \( x \in R^n \), we call \( V(x) \) a value of Problem (NG) at \( x \) if

\[
V(x) = \inf_{u_1 \in U_1} \sup_{u_2 \in U_2} J(x; u_1, u_2) = \sup_{u_2 \in U_2} \inf_{u_1 \in U_1} J(x; u_1, u_2).
\]

(2.2)

**Remark 2.2.** The value function \( V \) of Problem (NG) is well-defined at \( x \in R^n \) only when the following inequality holds:

\[
\inf_{u_1 \in U_1} \sup_{u_2 \in U_2} J(x; u_1, u_2) ≤ \sup_{u_2 \in U_2} \inf_{u_1 \in U_1} J(x; u_1, u_2).
\]

2.1 Backward stochastic LQ optimal control problems with nonhomogeneous terms

In this subsection, we shall generalize the results of indefinite backward stochastic LQ optimal control problems obtained by Sun, Wu, and Xiong [20] to the case with nonhomogeneous terms.

For any given terminal state \( ξ ∈ L^2_F,(Ω; R^n) \), consider the following controlled linear BSDE:

\[
\begin{align*}
\begin{cases}
dY(s) = [A(s)Y(s) + B(s)u_2(s) + C(s)Z(s) + σ(s)] ds \\
+ Z(s) dW(s), & s ∈ [0, T],
\end{cases}
\end{align*}
\]

(2.3)

and the utility functional:

\[
U(ξ; u_2) = E \int_0^T \{ ⟨R u_2, u_2⟩ + ⟨Q Y, Y⟩ + ⟨N Z, Z⟩ + 2 ⟨S_1 Y, u_2⟩ + 2 ⟨S_2 Z, u_2⟩ + 2 ⟨S_3 Y, Z⟩ \} ds
\]
The associated backward stochastic LQ optimal control problem can be stated as follows.

**Problem (BLQ).** For any given terminal state \( \xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \), find a control \( \hat{u}_2 \in \mathcal{U}_2 \) such that
\[
U(\xi; \hat{u}_2) = \sup_{u_2 \in \mathcal{U}_2} U(\xi; u_2) \equiv \bar{U}(\xi). \tag{2.5}
\]

In the following, we are going to find an optimal control of Problem (BLQ), by similar arguments to those employed in [20, Theorem 6.3].

**B1.** The coefficients of state equation (2.3) and the weighting matrices in functional (2.4) satisfy
\[
A, C \in L^\infty(0, T; \mathbb{R}^{n \times n}), \quad B \in L^\infty(0, T; \mathbb{R}^{n \times m_2}), \quad \mathcal{R} \in L^\infty(0, T; \mathbb{S}^{m_2}), \quad Q, N \in L^\infty(0, T; \mathbb{S}^n), \quad S_i \in L^\infty(0, T; \mathbb{R}^{m_2 \times n}); \quad i = 1, 2, 3 \in L^\infty(0, T; \mathbb{R}^{n \times n}), \quad g \in \mathbb{S}^n, \quad g \in \mathbb{R}^n. \tag{2.6}
\]

Moreover, there exists a constant \( \lambda > 0 \) such that
\[
U_0(0; u_2) \leq -\lambda \mathbb{E} \int_0^T |u_2(s)|^2 ds, \quad \forall u_2 \in \mathcal{U}_2, \tag{2.7}
\]
where \( U_0(\xi; u) \) denotes the utility functional \( U(\xi; u) \) with \( \sigma \equiv 0 \) and \( g = 0 \).

It is noteworthy that the condition (2.7) implies \( \mathcal{R} \ll 0 \), whose proof can be found in [20, Corollary 5.3]. Under (B1), by [20, Theorem 6.2] the following Riccati equation admits a unique negative semidefinite solution \( \Sigma^H \in \mathcal{C}(0, T; \mathbb{S}^n) \):
\[
\begin{align*}
\dot{\Sigma}^H - \Sigma^H A^\top - A \Sigma^H + \left[ B + \Sigma^H (S_1^H)\right] \mathcal{R}^{-1} \left[ B^\top + S_1^H \Sigma^H \right] \\
+ \left[ L + \Sigma^H (S_3^H)\right] \left[ I + \Sigma^H N^H \right]^{-1} \Sigma^H \left[ L^\top + S_3^H \Sigma^H \right] = 0, \quad t \in [0, T], \\
\Sigma^H(T) = 0,
\end{align*}
\]
where
\[
\begin{align*}
\mathcal{L} = C - BR^{-1} S_2, \quad N^H = N - S_2^\top R^{-1} S_2 + H, \\
S_1^H = S_1 + B^\top H, \quad S_3^H = S_3 - S_2^\top R^{-1} S_1 + L^\top H,
\end{align*}
\]
with the auxiliary function \( H \) uniquely determined by the following ordinary differential equation (ODE, for short):
\[
\begin{align*}
\dot{H} + HA + A^\top H + Q = 0, \quad t \in [0, T], \\
H(0) = -\mathcal{G}.
\end{align*}
\]
Moreover, the function \( \hat{\Sigma}^H \), defined by
\[
\hat{\Sigma}^H \equiv I + \Sigma^H N^H, \tag{2.11}
\]
is invertible with \( (\hat{\Sigma}^H)^{-1} \in L^\infty(0, T; \mathbb{R}^{n \times n}) \). With the unique solution \( \Sigma^H \) of (2.8), we introduce the following BSDE:
\[
\begin{align*}
d\varphi(s) &= \left\{ [A - \Sigma^H (S_1^H)\right]^\top R^{-1} S_1^H - BR^{-1} S_1^H \varphi \right. \\
&\quad - [L + \Sigma^H (S_3^H)\right]^\top \left[ (\hat{\Sigma}^H)^{-1} \Sigma^H S_3^H \varphi - \beta \right] - \sigma \right\} ds + \beta dW(s), \quad s \in [0, T], \\
\varphi(T) &= -\xi,
\end{align*}
\]
and SDE:
\[
\begin{align*}
dx(s) &= \left\{ \left[ (S_1^H)\right]^\top R^{-1} B^\top + \left( S_1^H \right)^\top R^{-1} S_1^H \Sigma^H + \left( S_1^H \right)^\top \hat{\Sigma}^H\right]^{-1} \left( \Sigma^H \mathcal{L}^\top + \Sigma^H S_3^H \Sigma^H \right) \\
&\quad - A^\top \right\} X + \left( S_1^H \right)^\top \left[ (\hat{\Sigma}^H)^{-1} \Sigma^H S_3^H \varphi - \beta \right] + \left( S_1^H \right)^\top R^{-1} S_1^H \varphi \right\} ds \\
&\quad + \left\{ [A^H (\hat{\Sigma}^H)^{-1} \Sigma^H \mathcal{L}^\top + \Sigma^H S_3^H \Sigma^H] - S_3^H \Sigma^H - L^\top \right\} X \right. \\
&\quad + N^H (\hat{\Sigma}^H)^{-1} \left[ \Sigma^H S_3^H \varphi - \beta \right] - S_3^H \varphi \right\} dW(s), \quad s \in [0, T], \\
X(0) &= g.
\end{align*}
\]
Proposition 2.3. Let (B1) hold. Then for any \( \xi \in L^2_T(\Omega; \mathbb{R}^n) \), the unique optimal control of Problem (BLQ) is given by
\[
\bar{u}_2 = \mathcal{R}^{-1}\left\{ [\mathcal{B}^\top + \mathcal{S}^H\Sigma^H]X + \mathcal{S}^H\varphi \right\} - \mathcal{R}^{-1}\mathcal{S}_2(\Sigma^H)^{-1}\left\{ \mathcal{S}^H\mathcal{L}^\top X + \mathcal{S}^H\Sigma_3^H\Sigma^H X + \Sigma^H\Sigma_5^H - \beta \right\},
\]
where \( \Sigma^H \in C(0,T;\mathbb{S}^n) \), \((\varphi, \beta) \in L^2_T(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_T(0,T;\mathbb{R}^n) \) and \( X \in L^2_T(\Omega; C([0,T];\mathbb{R}^n)) \) are the unique solutions to Riccati equation (2.8), BSDE (2.12) and SDE (2.13), respectively. Moreover, the value function \( \bar{U} \) of Problem (BLQ) is given explicitly by
\[
\bar{U}(\xi) = -\langle \Sigma^H(0)g, g \rangle - 2\langle g, \varphi(0) \rangle - \mathbb{E}\langle H(T)\xi, \xi \rangle + \mathbb{E}\int_0^T \left\{ \langle \mathcal{N}^H(\hat{\Sigma}^H)^{-1}\beta, \beta \rangle + 2\langle (\mathcal{S}_3^H)^\top(\hat{\Sigma}^H)^{-1}\beta, \varphi \rangle - \langle (\mathcal{S}_3^H)^\top(\hat{\Sigma}^H)^{-1}\Sigma\Sigma^H + (\mathcal{S}_4^H)^\top\mathcal{R}^{-1}\mathcal{S}^H\varphi, \varphi \rangle \right\} ds.
\]

By Proposition 2.3, we generalize the results obtained in Sun, Wu, and Xiong [20] to the case with nonhomogeneous terms. Since the proof of Proposition 2.3 is similar to that of [20, Theorem 6.3], we omit it here. Even though, this extension will serve as a foundation for finding a Stackelberg equilibrium of Problem (SG) (see Theorem 3.4). A key point in [20] is that by some transformation techniques, the assumptions \( Q \equiv 0 \) and \( G \equiv 0 \) can be imposed without loss of generality. However, the power of this approach is very limited for our problem, because as a trade off, the associated Riccati equation depends additionally on an auxiliary function \( H \) and the auxiliary function \( H \) will cause some technical difficulties in exploring the connection between Problems (SG) and (NG). In Subsection 4.2, a new representation for the optimal control of Problem (BLQ) will be presented and the auxiliary function \( H \) will be removed.

3 Stackelberg games

In this section, we shall establish a general approach for finding the Stackelberg equilibrium of Problem (SG). The procedure will be divided into two steps.

3.1 The follower’s problem

First, we are going to solve the follower’s problem. For any fixed control \( u_2 \in \mathcal{U}_2 \), the follower’s problem (denoted by Problem (FLQ)) can be stated as follows: Consider the state equation
\[
\begin{aligned}
&dX(s) = \left\{ A(s)X(s) + B_1(s)u_1(s) + B_2(s)u_2(s) \right\} ds \\
&\quad + \left\{ C(s)X(s) + D_1(s)u_1(s) + D_2(s)u_2(s) \right\} dW(s), \quad s \in [0,T],
\end{aligned}
\]
and the cost functional
\[
\mathcal{J}_{u_2}(x; u_1) \equiv J(x; u_1, u_2) = \mathbb{E}\left\{ \langle GX(T), X(T) \rangle + \int_0^T \left[ \langle QX, X \rangle + \langle R_1u_1, u_1 \rangle + \langle R_2u_2, u_2 \rangle \right] ds \right\},
\]
where \( \mathcal{J}_{u_2}(x; u_1) \) denotes the follower’s cost function. The follower (Player 1) wishes to find a control \( \bar{u}_1 \in \mathcal{U}_1 \), depending on \( u_2 \) and \( x \), such that
\[
\mathcal{J}_{u_2}(x; \bar{u}_1) = \inf_{u_1 \in \mathcal{U}_1} \mathcal{J}_{u_2}(x; u_1).
\]

To find an optimal control of Problem (FLQ), we introduce the following assumption.

(H3). There exists a constant \( \lambda > 0 \) such that
\[
J(0; u_1, 0) \geq \lambda \mathbb{E}\int_0^T |u_1(s)|^2 ds, \quad \forall u_1 \in \mathcal{U}_1.
\]

Then by [17, Corollary 4.7], we have the following results.
Proposition 3.1. Let (H1)–(H3) hold. Then for any \( u_2 \in U_2 \) and \( x \in \mathbb{R}^n \), Problem (FLQ) admits a unique optimal control \( \bar{u}_1 \in U_1 \), which admits the following closed-loop representation:

\[
\bar{u}_1(s) = \bar{\alpha}_1(s; u_2, x) \equiv \Theta(s)\bar{X}(s) + v(s) \equiv \Theta(s)\bar{X}(s; x, u_2) + v(s; u_2), \quad s \in [0, T],
\]

where

\[
\Theta = -(R_1 + D_1^TP_1D_1)^{-1}(B_1^TP_1 + D_1^TP_1C),
\]

\[
v = -(R_1 + D_1^TP_1D_1)^{-1}(B_1^TY + D_1^TZ + D_1^TP_1D_2u_2),
\]

with \( P_1 \in C([0, T]; \mathbb{S}^n) \) being the unique strongly regular solution of the Riccati equation:

\[
\begin{aligned}
\dot{P}_1 + P_1A + A^TP_1 + C^TP_1C + Q - (P_1B_1 + C^TP_1D_1)
\times (R_1 + D_1^TP_1D_1)^{-1}(B_1^TP_1 + D_1^TP_1C) = 0,
\end{aligned}
\]

\( P_1(T) = G, \)

(3.5)

\((Y, Z) \equiv (Y(\cdot; u_2), Z(\cdot; u_2))\) solving the BSDE:

\[
\begin{cases}
\frac{dY(s)}{} = -[(A + B_1\Theta)^T Y + (C + D_1\Theta)^T Z + (C + D_1\Theta)^TP_1D_2u_2]ds + ZdW(s), \quad s \in [0, T], \\
Y(T) = 0,
\end{cases}
\]

(3.6)

and \( \bar{X} \equiv \bar{X}(\cdot; x, u_2) \) satisfying the closed-loop system:

\[
\begin{cases}
\frac{d\bar{X}(s)}{} = [(A + B_1\Theta)\bar{X} + B_1v + B_2u_2]ds \\
\quad + [(C + D_1\Theta)\bar{X} + D_1v + D_2u_2]dW(s), \quad s \in [0, T],
\end{cases}
\]

(3.7)

Remark 3.2. Recall from [17, Theorem 4.3] that the unique strongly regular solution \( P_1 \) of Riccati equation (3.5) satisfies

\[
R_1 + D_1^TP_1D_1 \succ 0.
\]

Since the optimal control \( \bar{u}_1 \) admits the closed-loop representation (3.4), we have

\[
J(x; \bar{u}_1(\bar{u}_2), u_2) = J(x; \bar{\alpha}_1(u_2, x), u_2) = \inf_{u_1 \in \mathcal{U}_1} J(x; u_1, u_2), \quad \forall u_2 \in U_2, \ x \in \mathbb{R}^n.
\]

(3.9)

3.2 The leader’s problem and Stackelberg equilibrium

For any \( x \in \mathbb{R}^n \) and \( u_2 \in U_2 \), the follower’s unique optimal control \( \bar{u}_1 \) can be given by (3.4). Knowing this, the leader’s problem (denoted by Problem (LLQ)) becomes: Find a control \( \bar{u}_2 \in U_2 \) such that

\[
J(x; \bar{\alpha}_1(u_2, x), \bar{u}_2) = \sup_{u_2 \in \mathcal{U}_2} J(x; \bar{\alpha}_1(u_2, x), u_2).
\]

(3.10)

From the facts that \( \bar{\alpha}_1(u_2, x) = \Theta \bar{X}(x, u_2) + v(u_2) \), \( \bar{X}(x, u_2) \) is the solution of (3.7) and \( v(u_2) \) is determined by BSDE (3.6), we see that Problem (LLQ) is an optimal control problem for forward-backward SDEs.

By some straightforward calculations, \( J(x; \bar{\alpha}_1(u_2, x), u_2) \) can be rewritten as

\[
J(x; \bar{\alpha}_1(u_2, x), u_2) = J(x; \Theta \bar{X} + v, u_2)
\]

\[
= \langle P_1(0)x, x \rangle + 2\langle Y(0), x \rangle + \mathbb{E} \left\{ \int_0^T \left[ \langle P_1D_2u_2, D_2u_2 \rangle + 2\langle Y, B_2u_2 \rangle + 2\langle Z, D_2u_2 \rangle - \langle (R_1 + D_1^TP_1D_1)^{-1}(B_1^TP_1 + D_1^TP_1C) \right]
\times (R_1 + D_1^TP_1D_1)^{-1}(B_1^TP_1 + D_1^TP_1C) \rangle \right. ds \right\}
\]

(3.11)

It shows that \( J(x; \bar{\alpha}_1(u_2, x), u_2) \) is independent of the state process \( \bar{X} \). Noticing this key point, Problem (LLQ) is converted into an LQ optimal control problem for BSDEs (with nonhomogeneous terms), which is a precondition for our subsequent analysis.
For simplicity, we denote
\[
\begin{cases}
\tilde{R}_1 = D_1^TP_1D_1 + R_1,
\mathcal{A} = (P_1B_1 + C^TP_1D_1)\tilde{R}_1^{-1}B_1^T - A^T, \\
\mathcal{B} = [(P_1B_1 + C^TP_1D_1)\tilde{R}_1^{-1}D_1^T - C^TP_1D_2 - P_1B_2], \\
\mathcal{C} = (P_1B_1 + C^TP_1D_1)\tilde{R}_1^{-1}D_1^T - C^T,
\end{cases}
\]
and
\[
\begin{cases}
\mathcal{R} = D_2^TP_1D_2 - D_1^TP_1D_1\tilde{R}_1^{-1}D_1^TP_1D_2 + R_2, \\
\mathcal{Q} = -B_1\tilde{R}_1^{-1}B_1^T, \\
\mathcal{N} = -D_1\tilde{R}_1^{-1}D_1^T, \\
\mathcal{S}_3 = -D_1\tilde{R}_1^{-1}B_1^T, \\
\mathcal{S}_2 = D_2^T - D_1^TP_1D_1\tilde{R}_1^{-1}D_1^T, \\
\mathcal{S}_1 = B_2^T - D_1^TP_1D_1\tilde{R}_1^{-1}B_1^T.
\end{cases}
\]
With the above notations, BSDE (3.6) and functional (3.11) can be rewritten as
\[
\begin{cases}
dY(s) = [\mathcal{A}(s)Y(s) + \mathcal{B}(s)u_2(s) + \mathcal{C}(s)Z(s)]ds + Z(s)dW(s), \\
Y(T) = 0,
\end{cases}
\]
and
\[
J(x; \bar{\alpha}_1(u_2, x), u_2) = \langle P_1(0)x, x \rangle + \mathbb{E}\left[\int_0^T \left(\langle \mathcal{R}u_2, u_2 \rangle + \langle \mathcal{Q}Y, Y \rangle + \langle \mathcal{N}Z, Z \rangle + 2\langle \mathcal{S}_1Y, u_2 \rangle \right) \right. \\
\left. + 2\langle \mathcal{S}_2Z, u_2 \rangle + 2\langle \mathcal{S}_3Y, Z \rangle \right] ds + 2\langle Y(0), x \rangle,
\]
respectively. Let \( g = x, \mathcal{G} = 0 \) and \( \sigma = 0 \) in (2.4) and (2.3), respectively. Comparing (3.15) with (2.4) yields that
\[
U(0; u_2) + \langle P_1(0)x, x \rangle = J(x; \bar{\alpha}_1(u_2, x), u_2),
\]
where \( U \) is defined by (2.4). Recall the definition (2.9) of \( \mathcal{N}^{\mathcal{H}}, \mathcal{S}_1^{\mathcal{H}}, \mathcal{S}_3^{\mathcal{H}} \) and \( \mathcal{L} \). By Proposition 2.3, we have the following result.

(H4). There exists a constant \( \lambda > 0 \) such that
\[
J(0; \bar{\alpha}_1(u_2, 0), u_2) \leq -\lambda \mathbb{E}\int_0^T |u_2(s)|^2 ds, \quad \forall u_2 \in U_2.
\]

Proposition 3.3. Let (H1)–(H4) hold. Then Problem (LLQ) admits a unique optimal control:
\[
u_2 = \mathcal{R}^{-1}\left\{ [\mathcal{B}^T + \mathcal{S}_1^{\mathcal{H}}\Sigma^{\mathcal{H}}] - \mathcal{S}_2(\mathcal{S}_3^{\mathcal{H}})\Sigma^{\mathcal{H}}\Sigma^{\mathcal{H}} \right\} \hat{x},
\]
where \( \Sigma^{\mathcal{H}} \) is the unique solution of Riccati equation (2.8) and \( \hat{x} \) is uniquely determined by the following SDE:
\[
\begin{cases}
d\hat{x}(s) = \left\{ (\mathcal{S}_3^{\mathcal{H}})^T\mathcal{R}^{-1}\mathcal{B}^T + (\mathcal{S}_3^{\mathcal{H}})^T\mathcal{R}^{-1}\mathcal{S}_1^{\mathcal{H}}\Sigma^{\mathcal{H}} + (\mathcal{S}_3^{\mathcal{H}})^T(\hat{x}^{\mathcal{H}})^{-1}[\Sigma^{\mathcal{H}}\mathcal{L}^T + \Sigma^{\mathcal{H}}\mathcal{S}_3^{\mathcal{H}}\Sigma^{\mathcal{H}}] \\
- A^T \right\} \hat{x} ds + \left\{ \mathcal{N}^{\mathcal{H}}(\hat{x}^{\mathcal{H}})^{-1}[\Sigma^{\mathcal{H}}\mathcal{L}^T + \Sigma^{\mathcal{H}}\mathcal{S}_3^{\mathcal{H}}\Sigma^{\mathcal{H}}] - \mathcal{S}_3^{\mathcal{H}}\Sigma^{\mathcal{H}} - \mathcal{L}^T \right\} \hat{x} dW(s),
\end{cases}
\]
Proof. Note from (3.16) that
\[
U_0(0; u_2) = J(0; \bar{\alpha}_1(u_2, 0), u_2), \quad \forall u_2 \in U_2,
\]
where \( U_0 \) is the utility functional \( U \), defined by (2.4), with \( \sigma = 0 \) and \( g = 0 \). Then assumption (H4) implies that (2.7) holds. Moreover, from the fact \( \tilde{R}_1 = D_1^TP_1D_1 + R_1 \gg 0 \), we get that the coefficients \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) and the weighting matrices \( \mathcal{G}, \mathcal{R}, \mathcal{Q}, \mathcal{N}, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \) satisfy the condition (2.6). Thus under (H1)–(H4), the assumption (B1) holds. Then by Proposition 2.3, Problem (LLQ) admits a unique optimal control
\[
u_2 = \mathcal{R}^{-1}\left\{ [\mathcal{B}^T + \mathcal{S}_1^{\mathcal{H}}\Sigma^{\mathcal{H}}]X + \mathcal{S}_3^{\mathcal{H}}\varphi \right\} \\
- \mathcal{R}^{-1}\mathcal{S}_2(\mathcal{S}_3^{\mathcal{H}})\Sigma^{\mathcal{H}}X + \Sigma^{\mathcal{H}}\mathcal{S}_3^{\mathcal{H}}\Sigma^{\mathcal{H}}X + \Sigma^{\mathcal{H}}\mathcal{S}_3^{\mathcal{H}}\varphi - \beta,
\]
where $X$ and $(\varphi, \beta)$ are the unique solutions of (2.13) and (2.12), with $\sigma \equiv 0$, $g = x$ and $\xi = 0$, respectively. Note that when $\sigma \equiv 0$ and $\xi = 0$, the unique solution of BSDE (2.12) is explicitly given by $(\varphi, \beta) \equiv (0, 0)$. Using the facts $(\varphi, \beta) \equiv (0, 0)$ and $g = x$, SDE (2.13) can be rewritten as
\[
dX(s) = \left\{ (S_t^H)^\top R^{-1} B^\top + (S_t^H)^\top R^{-1} S_t^H \Sigma^H + (S_t^H)^\top (\Sigma^H)^{-1} \Sigma^H \Sigma^T + \Sigma^H S_t^H \Sigma^H \right\} ds - A^\top \{ X(s) \} \, ds + \{ X(s) (\Sigma^H)^{-1} \Sigma^H \Sigma^T + \Sigma^H S_t^H \Sigma^H - S_t^H \Sigma^H - L^\top \} \, dW(s),
\]
where $X(0) = x$.

It follows that $X = \bar{X}$. Substituting $(\varphi, \beta) \equiv (0, 0)$ and $X = \bar{X}$ into (3.19), we get (3.17), which completes the proof.

We conclude this section with the following result.

**Theorem 3.4.** Let (H1)–(H4) hold. Then for any initial state $x \in \mathbb{R}^n$, the control pair $(\bar{u}_1, \bar{u}_2) \equiv (\bar{\alpha}_1(u_2, x), \bar{u}_2)$, obtained in Proposition 3.1 and Proposition 3.3, is a Stackelberg equilibrium of Problem (SG).

**Proof.** From Proposition 3.1 and Proposition 3.3, we see that the control pair $(\bar{u}_1, \bar{u}_2) = (\bar{\alpha}_1(u_2, x), \bar{u}_2)$ satisfies
\[
J(x; \bar{\alpha}_1(u_2, x), u_2) = \inf_{u_1 \in U_1} J(x; u_1, u_2), \quad \forall u_2 \in U_2, \quad (3.21)
\]
\[
J(x; \bar{\alpha}_1(u_2, x), u_2) = \sup_{u_2 \in U_2} J(x; \bar{\alpha}_1(u_2, x), u_2). \quad (3.22)
\]

It follows from Definition 1.1 and Definition 1.2 that $(\bar{u}_1, \bar{u}_2) = (\bar{\alpha}_1(u_2, x), \bar{u}_2)$ is a Stackelberg equilibrium of Problem (SG).

**Remark 3.5.** By [20, Theorem 3.1], the following condition is necessary for open-loop solvability of Problem (LLQ):
\[
U_0(0; u_2) = J(0; \bar{\alpha}_1(u_2, 0), u_2) \leq 0, \quad \forall u_2 \in U_2. \quad (3.23)
\]

Then assumption (H4) is almost necessary for the existence of an optimal control of Problem (LLQ). When Problem (SG) only satisfies (H1)–(H3) and (3.23), one can apply the perturbation approach, developed in [17, 24, 19], to find the Stackelberg equilibrium (if it exists).

## 4 Further analysis of the Stackelberg games

In Theorem 3.4, it has been shown that under (H1)–(H4), Problem (SG) admits a Stackelberg equilibrium $(\bar{u}_1, \bar{u}_2) = (\bar{\alpha}_1(u_2, x), \bar{u}_2)$. However, the assumption (H4) is usually difficult to verify, because it is involved with the optimal strategy $\bar{\alpha}_1(\cdot, \cdot)$ of the follower. In this section, we shall provide a new condition, independent of $\bar{\alpha}_1(\cdot, \cdot)$, to ensure that (H4) holds. Furthermore, under this condition, a closed-loop representation for the Stackelberg equilibrium of Problem (SG) is obtained by a closer investigation of backward stochastic LQ optimal control problems.

### 4.1 Uniform concavity of the functional

For any $t \in [0, T]$, we first introduce the following game problem over $[t, T]$: Consider the state equation
\[
\begin{aligned}
dX(s) &= \left\{ A(s)X(s) + B_1(s)u_1(s) + B_2(s)u_2(s) \right\} ds \\
&\quad + \left\{ C(s)X(s) + D_1(s)u_1(s) + D_2(s)u_2(s) \right\} dW(s), \quad s \in [t, T],
\end{aligned}
\]
and the criterion functional
\[
J(t, x; u_1, u_2) = \mathbb{E}\left\{ J(X(T), T) + \int_t^T \left[ \langle QX, X \rangle + \langle R_1 u_1, u_1 \rangle + \langle R_2 u_2, u_2 \rangle \right] ds \right\}, \quad (4.2)
\]
where $u_i \in \mathcal{U}_i[t, T] \equiv L^2_\mathbb{F}([t, T]; \mathbb{R}^{m_i}); i = 1, 2$. Then,

$$J(0, x; u_1, u_2) = J(x; u_1, u_2), \quad \forall x \in \mathbb{R}^n, u_i \in \mathcal{U}_i; \ i = 1, 2, \quad (4.3)$$

where $J(x; u_1, u_2)$, defined by (1.2), is the criterion functional of Problem (SG). The following result shows that under (H3), the mapping $u_1 \mapsto J(t, 0; u_1, 0)$ is uniformly convex for any $t \in [0, T)$.

**Lemma 4.1.** Let (H1)–(H3) hold. Then for any $t \in [0, T)$,

$$J(t, 0; u_1, 0) \geq \lambda E \int_t^T |u_1(s)|^2 ds, \quad \forall u_1 \in \mathcal{U}_1[t, T], \quad (4.4)$$

where $\lambda > 0$ is the same as that in (H3).

**Proof.** For any $t \in [0, T)$ and $u_1 \in \mathcal{U}_1[t, T]$, define

$$[u_1 \oplus_t 0](s) \triangleq \begin{cases} u_1(s), & s \in [t, T], \\ 0, & s \in [0, t). \end{cases} \quad (4.5)$$

It is clearly seen that $u_1 \oplus_t 0 \in \mathcal{U}_1[0, T] \equiv \mathcal{U}_1$ and

$$J(t, 0; u_1, 0) = J(0, 0; u_1 \oplus_t 0, 0) = J(0; u_1 \oplus_t 0, 0). \quad (4.6)$$

From (H3), we have

$$J(0; u_1 \oplus_t 0, 0) \geq \lambda E \int_0^T |[u_1 \oplus_t 0](s)|^2 ds = \lambda E \int_t^T |u_1(s)|^2 ds. \quad (4.7)$$

Combining (4.6) and (4.7) together, we get (4.4) immediately. \qed

For any $(t, x) \in [0, T) \times \mathbb{R}^n$, by Lemma 4.1 and Proposition 3.1 (with the initial time 0 replaced by $t$), we have

$$J(t, x; \bar{\alpha}_1(u_2, t, x), u_2) \leq J(t, x; u_1, u_2), \quad \forall u_1 \in \mathcal{U}_1[t, T], \ u_2 \in \mathcal{U}_2[t, T], \quad (4.8)$$

where $\bar{\alpha}_1(u_2, t, x) \equiv \bar{\alpha}_1(\cdot; u_2, t, x)$ is defined by (3.4) with the initial time of (3.7) replaced by $t$. Moreover, similar to (3.15), we have

$$J(t, x; \bar{\alpha}_1(u_2, t, x), u_2) = E \left\{ \int_t^T \left[ \langle R u_2, u_2 \rangle + \langle Q Y, Y \rangle + \langle N Z, Z \rangle \\
+ 2 \langle S_1 Y, u_2 \rangle + 2 \langle S_2 Z, u_2 \rangle + 2 \langle S_3 Y, Z \rangle \right] ds + 2 \langle Y(t), x \rangle \right\} + \langle P_1(t) x, x \rangle, \quad (4.9)$$

where $P_1$ is the unique solution to Riccati equation (3.5). $(Y, Z)$ is uniquely determined by

$$\begin{cases} dY(s) = \left[ A(s) Y(s) + B(s) u_2(s) + C(s) Z(s) \right] ds + Z(s) dW(s), & s \in [t, T], \\
Y(T) = 0, \end{cases} \quad (4.10)$$

and the coefficients are defined by (3.12)–(3.13). The optimal control problem with state equation (4.10) and utility (4.9) is a backward LQ problem over the time horizon $[t, T]$. Next, we show that the following condition is sufficient for the uniform concavity of the mapping $u_2 \mapsto J(t, 0; \bar{\alpha}_1(u_2, t, 0), u_2)$.

(H5). There exists a constant $\lambda > 0$ such that

$$J(0, 0; u_2) \leq -\lambda E \int_0^T |u_2(s)|^2 ds, \quad \forall u_2 \in \mathcal{U}_2, \quad (4.11)$$

where $J$ is defined by (1.2).

**Proposition 4.2.** Let (H1)–(H3) and (H5) hold. Then

$$J(t, 0; \bar{\alpha}_1(u_2, t, 0), u_2) \leq -\lambda E \int_t^T |u_2(s)|^2 ds, \quad \forall u_2 \in \mathcal{U}_2[t, T], \ t \in [0, T), \quad (4.11)$$

where $J(t, 0; \bar{\alpha}_1(u_2, t, 0), u_2)$ is defined by (4.9). In particular, assumption (H5) implies that (H4) holds.
Proof. Recall from (4.8) that for any \((t, x) \in [0, T) \times \mathbb{R}^n\) and \(u_2 \in U_2[t, T]\), we have
\[
J(t, x; \bar{\alpha}_1(u_2, t, x), u_2) \leq J(t, x; u_1, u_2), \quad \forall u_1 \in U_1[t, T].
\tag{4.12}
\]
In particular, taking \(x = 0\) and \(u_1 = 0\), the above implies
\[
J(t, 0; \bar{\alpha}_1(u_2, t, 0), u_2) \leq J(t, 0; 0, u_2), \quad \forall u_2 \in U_2[t, T].
\tag{4.13}
\]
Moreover, by the similar arguments to those employed in Lemma 4.1, we get
\[
J(t, 0; 0, u_2) \leq -\lambda E \int_t^T |u_2(s)|^2 ds, \quad \forall u_2 \in U_2[t, T].
\tag{4.14}
\]
Combining (4.13) and (4.14) together, we obtain (4.11) immediately. \(\square\)

The following examples are devoted to comparing the assumptions (H4) and (H5).

**Example 4.3.** For any \(x \in \mathbb{R}\), consider the one-dimensional state equation
\[
\begin{aligned}
\dot{X}(s) &= u_2(s), \quad s \in [0, 1], \\
X(0) &= x,
\end{aligned}
\tag{4.15}
\]
and the quadratic functional
\[
J(x; u_1, u_2) = \int_0^1 \left[ |u_1(s)|^2 - |u_2(s)|^2 \right] ds.
\tag{4.16}
\]
It is directly checked that
\[
\bar{\alpha}_1(s; u_2, x) = 0, \quad s \in [0, 1].
\]
Then
\[
J(0; \bar{\alpha}_1(u_2, 0), u_2) = J(0; 0, u_2), \quad \forall u_2 \in U_2.
\tag{4.17}
\]
Thus, in the example, the assumptions (H4) and (H5) are equivalent.

**Example 4.4.** For any initial pair \((t, x) \in [0, 4) \times \mathbb{R}\), consider the one-dimensional state equation
\[
\begin{aligned}
\dot{X}(s) &= u_1(s) - u_2(s), \quad s \in [t, 4], \\
X(t) &= x,
\end{aligned}
\tag{4.18}
\]
and the quadratic functional
\[
J(t, x; u_1, u_2) = \int_t^4 \left[ |X(s)|^2 + |u_1(s)|^2 - 2|u_2(s)|^2 \right] ds.
\tag{4.19}
\]
It is direct to see that
\[
J(0, 0; u_1, 0) \geq \int_0^4 |u_1(s)|^2 ds, \quad \forall u_1 \in U_1[0, 4],
\]
which implies that (H3) holds. By Proposition 3.1 and Lemma 4.1, we know that for any initial pair \((t, x) \in [0, 4) \times \mathbb{R}\) and \(u_2 \in U_2[t, 4]\), the follower (Player 1) admits a unique optimal control \(\bar{u}_1 \equiv \bar{\alpha}_1(u_2, t, x)\). Note that
\[
J(0, 0; 0, \lambda) = \int_0^4 |\lambda s|^2 - 2\lambda^2 ds = \frac{40}{3}\lambda^2 \to \infty, \quad \text{as} \quad \lambda \to \infty.
\]
Then the following condition does not hold:
\[
J(0, 0; 0, u_2) \leq 0, \quad \forall u_2 \in U_2[0, 4],
\tag{4.20}
\]
due to which the example does not satisfy assumption (H5). Even so, we still have
\[
J(t, 0; \bar{\alpha}_1(u_2, t, 0), u_2) \leq J(t, 0; u_2, u_2) = -\int_t^4 |u_2(s)|^2 ds, \quad \forall u_2 \in U_2[t, 4], \quad t \in [0, 4),
\tag{4.21}
\]
which implies that (H4) still holds. It then follows from Theorem 3.4 that the game admits a Stackelberg equilibrium at any initial pair \((t, x) \in [0, 4) \times \mathbb{R}\). We point out that condition (4.20) is necessary for the existence of an open-loop saddle point (see [16, Theorem 3.3]). Since the criterion functional (4.19) does not satisfy (4.20), the game does not have an open-loop saddle point.
Remark 4.5. The combination of (H3) and (H5) is referred to as a uniform convexity-concavity condition ((UCC) condition, for short) by Sun [16]. In Example 4.4, it has been shown that assumption (H4) is strictly weaker than (H5), due to which we would like to call the assumptions (H3)–(H4) a weak uniform convexity-concavity condition.

4.2 Further results of backward stochastic LQ optimal control problems

For any \((t, x) \in [0, T) \times \mathbb{R}^n\), we begin with this subsection by introducing the following backward stochastic LQ optimal control problem over \([t, T]\) (denoted by Problem (BLQ\(_{[t,T]}\))): Consider the state equation

\[
\begin{aligned}
dY(s) &= [A(s)Y(s) + B(s)u_2(s) + C(s)Z(s)] \, ds + Z(s) \, dW(s), \quad s \in [t, T], \\
Y(T) &= \xi,
\end{aligned}
\]  

(4.22)

and the utility functional

\[
U(t, \xi; u_2) = \mathbb{E}\{ \int_t^T \left[ \langle R u_2, u_2 \rangle + \langle QY, Y \rangle + \langle NZ, Z \rangle + 2\langle S_1 Y, u_2 \rangle \\
+ 2\langle S_2 Z, u_2 \rangle + 2\langle S_1 Y, Z \rangle \right] \, ds + 2\langle Y(t), x \rangle \},
\]

(4.23)

where the coefficients are defined by (3.12)–(3.13). Denote the utility functional \(U(t, \xi; u_2)\) with \(x = 0\) by \(U_0(t, \xi; u_2)\). The following results show that Problem (BLQ\(_{[t,T]}\)) can be solved by introducing a new Riccati equation.

Proposition 4.6. Let (H1)–(H3) and (H5) hold. Then the following Riccati equation admits a unique negative semi-definite solution \(\Sigma \in C(0, T; \mathbb{S}^n)\):

\[
\begin{aligned}
\dot{\Sigma} - \Sigma A^\top - A \Sigma + \Sigma S_1^\top R^{-1} B^\top + B R^{-1} S_1 \Sigma + \Sigma S_1^\top R^{-1} S_1 \Sigma - \Sigma Q \Sigma + B R^{-1} B^\top \\
+ [C - B R^{-1} S_1 - \Sigma S_1^\top R^{-1} S_2 + \Sigma S_2^\top \Sigma] [t + \Sigma N - \Sigma S_2^\top R^{-1} S_2]^{-1} \\
\times [C^\top - S_1^\top R^{-1} B^\top - S_2^\top R^{-1} S_1 \Sigma + S_2 \Sigma] = 0, \\
\Sigma(T) = 0.
\end{aligned}
\]

(4.24)

Moreover, the function \(\hat{\Sigma}\), defined by

\[
\hat{\Sigma} \equiv I + \Sigma N - \Sigma S_2^\top R^{-1} S_2,
\]

(4.25)

is invertible with \(\hat{\Sigma}^{-1} \in L^\infty(0, T; \mathbb{R}^{n \times n})\) and \(\hat{\Sigma}^{-1} \Sigma \in L^\infty(0, T; \mathbb{S}^n)\).

Proof. Comparing (4.23) with (4.9) yields

\[
U_0(t, 0; u_2) = J(t, 0; \alpha_1(u_2, t, 0), u_2), \quad \forall u_2 \in \mathcal{U}_2[t, T],
\]

(4.26)

where \(J(t, x; u_1, u_2)\) is defined by (4.2). Under (H5), by Proposition 4.2 we have

\[
U_0(t, 0; u_2) = J(t, 0; \alpha_1(u_2, t, 0), u_2) \leq -\lambda \mathbb{E} \int_t^T |u_2(s)|^2 \, ds, \quad \forall u_2 \in \mathcal{U}_2[t, T], t \in [0, T).
\]

(4.27)

Denote

\[
\hat{C} = C - B R^{-1} S_2, \quad \hat{S}_3 = S_3 - S_2^\top R^{-1} S_1, \quad \hat{N} = N - S_2^\top R^{-1} S_2, \quad \hat{v}_2 = u_2 + R^{-1} S_2 Z.
\]

(4.28)

Then state equation (4.22) and utility functional (4.23) can be rewritten as:

\[
\begin{aligned}
dY(s) &= [A(s)Y(s) + B(s)u_2(s) + \hat{C}(s)Z(s)] \, ds + Z(s) \, dW(s), \quad s \in [t, T], \\
Y(T) &= \xi,
\end{aligned}
\]

(4.29)

and

\[
\hat{U}(t, \xi; \hat{v}_2) \triangleq \mathbb{E}\{ \int_t^T \langle \hat{R} \hat{v}_2, \hat{v}_2 \rangle + \langle \hat{Q} Y, Y \rangle + \langle \hat{N} Z, Z \rangle \}
\]

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Similarly, we denote the utility functional \( \hat{U}(t, \xi; v_2) \) with \( x = 0 \) by \( \hat{U}_0(t, \xi; v_2) \). By the standard results of BSDEs, we get

\[
\mathbb{E} \int_t^T |v_2(s)|^2 \leq K \mathbb{E} \int_t^T [u_2(s)]^2 + |Z(s)|^2] ds \leq K \mathbb{E} \left[ \int_t^T |u_2(s)|^2 ds + |\xi|^2 \right].
\]

Here, \( K > 0 \) stands for a generic constant which could be different from line to line and is independent of \( t \). Then by (27) and (31) (with \( \xi = 0 \)), we get

\[
\hat{U}_0(t, 0; v_2) = U_0(t, 0; u_2) \leq -\lambda \mathbb{E} \int_t^T |u_2(s)|^2 ds
\]

\[
\leq -\frac{\lambda}{K} \int_t^T |v_2(s)|^2 ds, \quad \forall v_2 \in U_2[t, T], \ t \in [0, T).
\]

Thus by [20, Theorem 5.1 and Corollary 5.3], there exists a constant \( k_0 > 0 \) such that for any \( k > k_0 \) the following Riccati equation

\[
\begin{align*}
\dot{P}_k + P_k A + A^T P_k - (C^T P_k + \tilde{S}_3) & = \left( \tilde{N} + P_k, 0 \right) \left( \tilde{C}^T P_k + \tilde{S}_3 \right), \ t \in [0, T], \\
P_k(T) & = -k I,
\end{align*}
\]

admits a unique solution \( P_k \in C([0, T]; \mathbb{S}^n) \). Then applying the arguments employed in the proof of [20, Theorem 6.2], we get that \( \Sigma = \lim_{k \to \infty} P_k^{-1} \) is the unique solution to the following Riccati equation

\[
\begin{align*}
\dot{\Sigma} & = -\Sigma A^T - A \Sigma + \Sigma S_1^T R^{-1} S_1 - B R^{-1} S_1 \Sigma + \Sigma S_1^T R^{-1} \Sigma S_1 - \Sigma Q \Sigma + BR^{-1} B^T \\
& + \{ \tilde{C} + \Sigma S_3 \} [I + \Sigma \tilde{N} - 1] \Sigma [\tilde{C} + \tilde{S}_3] = 0, \ t \in [0, T], \\
\Sigma(T) & = 0.
\end{align*}
\]

Moreover, \( I + \Sigma \tilde{N} \) is invertible with \( [I + \Sigma \tilde{N}]^{-1} \in L^\infty(0, T; \mathbb{R}^{n \times n}) \) and \( [I + \Sigma \tilde{N}]^{-1} \Sigma \in L^\infty(0, T; \mathbb{S}^n) \). Then by the definition (4.28) of \( \tilde{C}, \tilde{S}_3 \) and \( \tilde{N} \), we get the desired results immediately.

Compared with (2.8), Riccati equation (4.24) does not depend on the auxiliary function \( H \). This new feature will play a crucial role in the proof of Theorem 5.2. A challenging problem is to establish the well-posedness of Riccati equation (4.24) under an assumption like (H4). We hope to come back in our future publications. With the unique solution \( \Sigma \) of (2.8), we introduce the following BSDE:

\[
\begin{align*}
d\varphi(s) & = \{ (A + Q \Sigma - \Sigma S_1^T R^{-1} S_1 - B R^{-1} S_1 \varphi + [B R^{-1} S_1 + \Sigma S_2^T R^{-1} S_2 \\
& - C - \Sigma S_3^T \Sigma^{-1} [\Sigma S_1 - \Sigma S_2^T R^{-1} S_1] \varphi - \beta] \} ds + \beta dW(s), \ s \in [0, T], \\
\varphi(T) & = -\xi,
\end{align*}
\]

and SDE:

\[
\begin{align*}
dX(s) & = \{ -A^T X + Q \Sigma - \Sigma S_1^T R^{-1} B^T - S_1^T R^{-1} S_1 \Sigma - (S_3^T - S_2^T R^{-1} S_2) \\
& \times \dot{\Sigma}^{-1}(\Sigma \Sigma^T - \Sigma S_1^T R^{-1} B^T - \Sigma S_2^T R^{-1} S_1 \Sigma + \Sigma S_3 \Sigma) \} X \\
& + \{ (S_1^T - S_2^T R^{-1} S_2) \Sigma^{-1} (\Sigma S_1 \varphi - \Sigma S_2^T R^{-1} S_1 \varphi - \beta) - Q \varphi + S_1^T R^{-1} S_1 \varphi \} ds \\
& + \{ -C^T S_2^T - C^T S_2^T R^{-1} B^T + (S_3^T - S_2^T R^{-1} S_2) \Sigma - (N - S_2^T R^{-1} S_2) \\
& \times \dot{\Sigma}^{-1}(\Sigma \Sigma^T - \Sigma S_1^T R^{-1} B^T - \Sigma S_2^T R^{-1} S_1 \Sigma + \Sigma S_3 \Sigma) \} X \\
& + (N - S_2^T R^{-1} S_2) \dot{\Sigma}^{-1}(\Sigma S_1 - \Sigma S_2^T R^{-1} S_1) \varphi - \beta \\
& - (S_3^T - S_2^T R^{-1} S_1) \varphi \} dW(s), \ s \in [0, T],
\end{align*}
\]

\[X(0) = x.\]

Then by the standard argument employed in backward stochastic LQ optimal control problems (see [20], for example), we can obtain the unique optimal control of Problem (BLQ\([0,T]\)). The uniqueness of optimal controls of Problem (BLQ\([0,T]\)) comes from the uniform concavity of the utility functional (see Proposition 4.2).
Proposition 4.7. Let (H1)–(H3) and (H5) hold. Then for any \( \xi \in L^2_T(\Omega; \mathbb{R}^n) \), the unique optimal control of Problem (BLQ)\(_{[0,T]}\) is given by

\[
\hat{u}_2 = R^{-1}[B^T + S_1 \Sigma - S_2 \hat{\Sigma}^{-1}(\Sigma C^T - \Sigma S_2^T R^{-1} B^T - \Sigma S_2^T R^{-1} S_1 \Sigma + \Sigma \Sigma \Sigma)]X
+ R^{-1} S_1 \varphi - R^{-1} S_2 \hat{\Sigma}^{-1}((\Sigma S_3 - \Sigma S_2^T R^{-1} S_1)\varphi - \beta),
\]  

(4.37)

where \( \Sigma \in C(0, T; \mathbb{S}^n) \), \((\varphi, \beta) \in L^2_T(\Omega; C([0, T]; \mathbb{R}^n)) \times L^2_T(0, T; \mathbb{R}^n) \) and \( X \in L^2_T(\Omega; C([0, T]; \mathbb{R}^n)) \) are the unique solutions of Riccati equation (4.24), BSDE (4.35) and SDE (4.36), respectively. Moreover, the value function \( U \) of Problem (BLQ)\(_{[0,T]}\) can be represented as

\[
U(0, \xi; \hat{u}_2) = -((\Sigma(0)x, x) - 2\langle x, \varphi(0) \rangle) + \mathbb{E} \int_0^T \left\{ \langle [Q - (S_3^T - S_1^T R^{-1} S_2)\hat{\Sigma}^{-1}(S_3 - S_2^T R^{-1} S_1)]X, \alpha \rangle + \langle (N - S_2^T R^{-1} S_1)\hat{\Sigma}^{-1} \beta, \beta \rangle \right\} ds.
\]  

(4.38)

By Proposition 4.7, we can rewrite Proposition 3.3 as follows.

Proposition 4.8. Let (H1)–(H3) and (H5) hold. Then Problem (LLQ) admits a unique optimal control:

\[
\hat{u}_2 = R^{-1}[B^T + S_1 \Sigma - S_2 \hat{\Sigma}^{-1}(\Sigma C^T - \Sigma S_2^T R^{-1} B^T - \Sigma S_2^T R^{-1} S_1 \Sigma + \Sigma \Sigma \Sigma)]\hat{X},
\]  

(4.39)

where \( \Sigma \) is the unique solution to Riccati equation (4.24) and \( \hat{X} \) is uniquely determined by the following SDE:

\[
d\hat{X}(s) = -[A^T + Q \Sigma - S_1^T R^{-1} B^T - S_2^T R^{-1} S_1 \Sigma - (S_3^T - S_1^T R^{-1} S_2)]\hat{X}ds
- [C^T - S_2^T R^{-1} B^T + (S_3 - S_2^T R^{-1} S_1) \Sigma - (N - S_2^T R^{-1} S_2)]\hat{X}dW(s), \quad s \in [0, T],
\]  

(4.40)

4.3 Closed-loop representation for the Stackelberg equilibrium

In this subsection, we shall show that the Stackelberg equilibrium \((\hat{u}_1, \hat{u}_2)\) obtained in Theorem 3.4 admits a closed-loop representation.

Theorem 4.9. Let (H1)–(H3) and (H5) hold. Let \( P_1 \in C([0, T]; \mathbb{S}^n) \) and \( \Sigma \in C([0, T]; \mathbb{S}^n) \) be the unique solutions to Riccati equations (3.5) and (4.24), respectively. Then Problem (SG) has a Stackelberg equilibrium \((\hat{u}_1, \hat{u}_2) \in U_1 \times U_2\), which admits the following closed-loop representation:

\[
\hat{u}_1 = \bar{\Theta}_1 \hat{X} \equiv \hat{R}_1^{-1}\{B_1 \Sigma - B_1 P_1 - D_1 P_1 C - D_1 \hat{\Sigma}^{-1}(\Sigma C^T - \Sigma S_2^T R^{-1} B^T - \Sigma S_2^T R^{-1} S_1 \Sigma + \Sigma \Sigma \Sigma)]\hat{X}
- \Sigma S_2^T R^{-1} S_1 \Sigma + \Sigma \Sigma \Sigma)]\hat{X} + D_1 P_1 D_2 R^{-1}[B^T + S_1 \Sigma - S_2 \hat{\Sigma}^{-1}(\Sigma C^T - \Sigma S_2^T R^{-1} S_1 \Sigma + \Sigma \Sigma \Sigma)]\hat{X},
\]  

(4.41)

\[
\hat{u}_2 = \bar{\Theta}_2 \hat{X} \equiv R^{-1}[B^T + S_1 \Sigma - S_2 \hat{\Sigma}^{-1}(\Sigma C^T - \Sigma S_2^T R^{-1} B^T - \Sigma S_2^T R^{-1} S_1 \Sigma + \Sigma \Sigma \Sigma)]\hat{X},
\]  

(4.42)

with \( \hat{X} \) being the unique solution of the closed-loop system:

\[
d\hat{X}(s) = \{A(s)\hat{X}(s) + B_1(s)\bar{\Theta}_1(s)\hat{X}(s) + B_2(s)\bar{\Theta}_2(s)\hat{X}(s)\}ds
+ \{C(s)\hat{X}(s) + D_1(s)\bar{\Theta}_1(s)\hat{X}(s) + D_2(s)\bar{\Theta}_2(s)\hat{X}(s)\}dW(s), \quad s \in [0, T],
\]  

(4.43)

\[
\hat{X}(0) = x.
\]

Moreover,

\[
J(x; \hat{u}_1, \hat{u}_2) = (\langle P_1(0) - \Sigma(0) \rangle x, x), \quad \forall x \in \mathbb{R}^n.
\]  

(4.44)
Proof. Taking $u_2 = \tilde{u}_2$ in (3.6), then the unique solution $(Y, Z)$ of (3.6) can be given by

$$Y = -\Sigma \hat{X}, \quad Z = \hat{\Sigma}^{-1}(\Sigma C^T - \Sigma S_2^T R^{-1} B^T - \Sigma S_2^T R^{-1} S_1 \Sigma + \Sigma S_2 \Sigma) \hat{Y},$$

(4.45)

where $\tilde{u}_2$ and $\hat{X}$ are determined by (4.39) and (4.40), respectively. Substituting the above into (3.4) yields that

$$\tilde{u}_1 = \tilde{u}_1(\tilde{u}_2, x) = -\tilde{R}_1 (B_1^T P_1 + D_1^T P_1 C) \hat{X} + \tilde{R}_1 (B_1^T \Sigma \hat{X} - \tilde{R}_1 D_1 \hat{\Sigma}^T \Sigma (\Sigma C^T - \Sigma S_2^T R^{-1} B^T - \Sigma S_2^T R^{-1} S_1 \Sigma + \Sigma S_2 \Sigma) \hat{Y},$$

(4.46)

where $\hat{X}$ is the unique solution of (3.7) with $u_2 = \tilde{u}_2$; that is

$$\begin{cases}
  d\hat{X}(s) = \{A \hat{X} + B_1 \tilde{R}_1 (B_1^T P_1 + D_1^T P_1 C)[\hat{X} - X] + B_1 \tilde{\Theta}_1 \hat{X} + B_2 \tilde{\Theta}_2 \hat{X}\} ds \\
  \hat{X}(0) = x.
\end{cases}$$

(4.47)

To prove that $(\tilde{u}_1, \tilde{u}_2) = (\hat{u}_1, \hat{u}_2)$, by comparing (4.46) and (4.39) with (4.41) and (4.42), it suffices to show that $X = \hat{X} = \hat{X}$. If equation (4.40) can be rewritten as (4.43), then $\hat{X} = \hat{X}$, which implies that $X$ satisfies (4.47). By the uniqueness of the solution to (4.47), we get $X = \hat{X}$ immediately.

Now let us show that (4.40) can be really rewritten as (4.43). Indeed, by the definitions (3.12)–(3.13) of $\mathcal{A}$, $\mathcal{Q}$, $S_1$ and $S_3$, we get

$$\begin{aligned}
  S_1^T R^{-1} B^T + S_1^T R^{-1} S_1 \Sigma - A^T = & Q \Sigma + (S_1^T - S_1^T R^{-1} S_2) \\
  \times \hat{\Sigma}^{-1}(\Sigma C^T - \Sigma S_2^T R^{-1} B^T - \Sigma S_2^T R^{-1} S_1 \Sigma + \Sigma S_2 \Sigma) \\
  = & A - B_1 \tilde{R}_1 (B_1^T P_1 + D_1^T P_1 C + B_1 \tilde{R}_1^T B_1^T \Sigma + [B_2 - B_1 \tilde{R}_1^T D_1] P_1 D_2] \\
  \times [R^{-1} B^T + R^{-1} S_1 \Sigma] - [B_1 \tilde{R}_1^T D_1 + (B_2 - B_1 \tilde{R}_1^T D_1 P_1 D_2) R^{-1} S_2] \\
  \times \hat{\Sigma}^T (\Sigma C^T - \Sigma S_2^T R^{-1} B^T - \Sigma S_2^T R^{-1} S_1 \Sigma + \Sigma S_2 \Sigma) \\
  = & A + B_1 \tilde{\Theta}_1 + B_2 \tilde{\Theta}_2,
\end{aligned}$$

(4.48)

where $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ are defined by (4.41)–(4.42). In a similar way, we also have

$$\begin{aligned}
  S_3^T R^{-1} B^T - C^T = & (S_3 - S_3^T R^{-1} S_1) \Sigma + (N - S_2^T R^{-1} S_2) \\
  \times \hat{\Sigma}^{-1}(\Sigma C^T - \Sigma S_2^T R^{-1} B^T - \Sigma S_2^T R^{-1} S_1 \Sigma + \Sigma S_2 \Sigma) \\
  = & C + D_1 \tilde{\Theta}_1 + D_2 \tilde{\Theta}_2.
\end{aligned}$$

(4.49)

By (4.48) and (4.49), we see that equation (4.40) can be rewritten as (4.43).

When $\xi = 0$ and $\sigma = 0$, the unique solution of BSDE (4.35) is given by $(\varphi, \beta) \equiv (0, 0)$. From the representation (4.38) of the value function $U$, we get

$$U(0, 0; u_2) = -\langle \Sigma(0), x, x \rangle.$$

Substituting the above into (3.16) yields (noting $U(0; u_2) = U(0, 0; u_2)$)

$$J(x; \tilde{u}_1, \tilde{u}_2) = J(x; \tilde{u}_1(\tilde{u}_2, x), \tilde{u}_2) = \langle (P_1(0) - \Sigma(0)) x, x \rangle.$$

This completes the proof. \qed

Remark 4.10. It is noteworthy that the results obtained in Subsections 4.2 and 4.3 still hold true if the assumption (H5) is replaced by (4.11), because (4.11) is sufficient for the well-posedness of Riccati equation (4.24).
5 Connections between Problems (SG) and (NG)

Recall from [16] that the (UCC) condition (H3) and (H5) is sufficient and almost necessary for the solvability of Problem (NG). In this section, under (H3) and (H5), we shall establish some interesting connections between Problem (SG) and Problem (NG).

5.1 Relationship between the Riccati equations

The Riccati equation associated with Problem (NG) reads

\[
\begin{align*}
\dot{P} + PA + A^TP + C^TPC + Q \\
- (PB + C^TPD)(R + D^TPD)^{-1}(B^TP + D^TPC) = 0,
\end{align*}
\]

where

\[
B = (B_1, B_2), \quad D = (D_1, D_2), \quad R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}.
\]

**Definition 5.1.** An absolutely continuous function \(P: [0,T] \to \mathbb{S}^n\) is called a solution of Riccati equation (5.1) if

(i) \(P\) satisfies (5.1) almost everywhere on \([0,T]\), and

(ii) \(R + D^TPD\) is invertible with \((R + D^TPD)^{-1} \in L^\infty(0,T;\mathbb{S}^n)\).

In [16, Definition 4.2], the solution \(P\) of (5.1) is called a strongly regular solution if it also satisfies:

\[(−1)^i+1(R_i + D_i^TPD_i) \gg 0, \quad i = 1, 2.\]  

However, the uniformly positive definiteness (5.3) does not imply the open-loop solvability of Problem (NG), which is different from the situation in control problems (see [16, Example 4.5]). Thus, the condition (5.3) is only used to ensure the invertibility of the singular term \(R + D^TPD\) (i.e., the property (ii) in Definition 5.1).

The following result establishes a connection between Riccati equations (5.1), (3.5) and (4.24), which are introduced for solving Problems (NG), (FLQ) and (LLQ), respectively.

**Theorem 5.2.** Let (H1)–(H3) and (H5) hold. Then Riccati equation (5.1) admits a unique solution

\[P = P_1 - \Sigma,\]

where \(P_1 \in C([0,T];\mathbb{S}^n)\) and \(\Sigma \in C([0,T];\mathbb{S}^n)\) are the unique solutions to Riccati equations (3.5) and (4.24), respectively.

**Remark 5.3.** We emphasize that Theorem 5.2 still holds true if (H5) is replaced by (4.11), because Riccati equation (4.24) is still solvable under (4.11). From Proposition 4.2, we see that the conditions (H3) and (4.11) are strictly weaker than the assumptions (H3) and (H5), which were imposed in [16, Theorem 4.3]. Thus, by Theorem 5.2, first, we establish a connection between the Riccati equations (5.1), (3.5) and (4.24); second, we prove the well-posedness of Riccati equation (5.1) with a new constructive method; third, the assumptions imposed in [16, Theorem 4.3] are relaxed.

To prove Theorem 5.2, we need to make some preparations. The difficulty mainly comes from the singularity of Riccati equations (5.1), (3.5) and (4.24). Recall from Proposition 3.1 that under (H3), Riccati equation (3.5) admits a unique solution \(P_1 \in C([0,T];\mathbb{S}^n)\) satisfying

\[R_1 + D_1^TP_1D_1 \gg 0.\]

Recall from Proposition 4.6 that under (H5), Riccati equation (4.24) admits a unique solution \(\Sigma \in C([0,T];\mathbb{S}^n)\) such that \(\hat{\Sigma} = I + \Sigma \mathcal{N} - \Sigma \mathcal{S}_2^T \mathcal{R}^{-1} \mathcal{S}_2\) is invertible with

\[
\hat{\Sigma}^{-1} = (I + \Sigma \mathcal{N} - \Sigma \mathcal{S}_2^T \mathcal{R}^{-1} \mathcal{S}_2)^{-1} \in L^\infty(0,T;\mathbb{R}^n) \quad \text{and} \quad \hat{\Sigma}^{-1} \Sigma \in L^\infty(0,T;\mathbb{S}^n).
\]

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We also recall the definitions \((3.12)-(3.13)\) of \(A, B, C, Q, N, R, S_i (i = 1, 2, 3)\). Combining \((5.5)\) with the fact \(\Sigma \leq 0\), we have
\[
R_1 + D_1^\top [P_1 - \Sigma]D_1 \gg 0.
\] (5.7)

Denote
\[
\Phi = R_2 + D_2^\top (P_1 - \Sigma)D_2 - D_2^\top (P_1 - \Sigma)D_1[R_1 + D_1^\top (P_1 - \Sigma)D_1]^{-1}D_1^\top (P_1 - \Sigma)D_2, \\
\hat{\Phi} = R^{-1} + R^{-1}S_2 \tilde{\Sigma}^{-1} \Sigma S_1^\top R^{-1}.
\] (5.8) (5.9)

**Lemma 5.4.** The matrix-valued function \(\Phi\) is invertible with its inverse given by \(\Phi^{-1} = \hat{\Phi}\).

By Lemma 5.4, it is straightforward to see that the matrix
\[
R + D^\top (P_1 - \Sigma)D = \begin{pmatrix} [R_1 + D_1^\top (P_1 - \Sigma)D_1 & D_1^\top (P_1 - \Sigma)D_2 \\ D_2^\top (P_1 - \Sigma)D_1 & R_2 + D_2^\top (P_1 - \Sigma)D_2 \end{pmatrix},
\]
is invertible with its inverse given by
\[
[R + D^\top (P_1 - \Sigma)D]^{-1} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix},
\] (5.10)

where
\[
\mathcal{M}_{11} = [R_1 + D_1^\top (P_1 - \Sigma)D_1]^{-1} + [R_1 + D_1^\top (P_1 - \Sigma)D_1]^{-1}D_1^\top (P_1 - \Sigma)D_2 \times \hat{\Phi}D_2^\top (P_1 - \Sigma)D_1[R_1 + D_1^\top (P_1 - \Sigma)D_1]^{-1}, \\
\mathcal{M}_{12} = \mathcal{M}_{21} = -[R_1 + D_1^\top (P_1 - \Sigma)D_1]^{-1}D_1^\top (P_1 - \Sigma)D_2 \hat{\Phi}, \\
\mathcal{M}_{22} = \hat{\Phi}.
\] (5.11)

**Remark 5.5.** Lemma 5.4 serves as a crucial bridge between the singular terms of Riccati equations \((5.1), (3.5)\) and \((4.24)\). The construction of this bridge is technical. Once the explicit form of \(\Phi^{-1}\) is derived, the result can be proved by a lengthy verification. We sketch the proof in Appendix.

### 5.2 Proof of Theorem 5.2

**Uniqueness:** Suppose that \(P, \hat{P} \in C([0, T]; S^n)\) are two solutions of \((5.1)\). Then by Definition 5.1, both \(R + D^\top PD\) and \((R + D^\top \hat{P}D)^{-1}\) are invertible with their inverses belonging to \(L^\infty(0, T; S^n)\). Denote \(\Delta P = P - \hat{P}\). Then \(\Delta P\) satisfies the following linear ordinary differential equation:
\[
\begin{align*}
\Delta \hat{P} + \Delta P A + A^\top \Delta P + C^\top \Delta PC - (\Delta PB + C^\top \Delta PD)(R + D^\top PD)^{-1} \\
\times (B^\top P + D^\top PC) + (PB^\top + C^\top \hat{P}D)(R + D^\top PD)^{-1}D^\top \Delta P D \\
\times (R + D^\top \hat{P}D)^{-1}(B^\top \Delta P + D^\top \Delta PC) = 0, \\
\Delta P(T) = 0.
\end{align*}
\] (5.12)

Note that \(P, \hat{P}, (R + D^\top PD)^{-1}\) and \((R + D^\top \hat{P}D)^{-1}\) are all bounded. Then by a standard argument using the Grönwall’s inequality, we get \(\Delta P \equiv 0\). It follows that Riccati equation \((5.1)\) admits at most one solution.

**Existence:** Note that \(P_1(T) - \Sigma(T) = G\) and
\[
\hat{P}_1 - \hat{\Sigma} = -(P_1 - \Sigma)A - A^\top (P_1 - \Sigma) - C^\top (P_1 - \Sigma)C - Q \\
+ (P_1 B_1 + C^\top P_1 D_1) \hat{R}^{-1} \Sigma (C^\top \Sigma^{-1} \Sigma) - \Sigma B_1 \hat{R}^{-1} \Sigma (D_1^\top P_1 C + B_1^\top P_1) \\
- (P_1 B_1 + C^\top P_1 D_1) \hat{R}^{-1} B_1^\top \Sigma - C^\top \Sigma C - \Sigma Q \Sigma \\
+ [C + \Sigma S_1^\top \hat{\Sigma}^{-1} \Sigma] - S_2^\top \hat{R}^{-1} B^\top - S_2^\top \hat{R}^{-1} \Sigma (S_1^\top \Sigma + S_2) \\
= |B^\top \hat{R}^{-1} S_2 + \Sigma S_1^\top \hat{R}^{-1} S_2 | \hat{\Sigma}^{-1} \Sigma (C^\top + S_2) + \Sigma \hat{B}^\top B + \Sigma S_1^\top \hat{B}^\top B
\]
\[ F \triangleq (I) - [(P_1 - \Sigma)B_1 + C^T(P_1 - \Sigma)D_1]M_{11}[B_1^T(P_1 - \Sigma) + D_1^T(P_1 - \Sigma)C] \]
\[ - [(P_1 - \Sigma)B_1 + C^T(P_1 - \Sigma)D_1]M_{12}[B_2^T(P_1 - \Sigma) + D_2^T(P_1 - \Sigma)C] \]
\[ - [(P_1 - \Sigma)B_2 + C^T(P_1 - \Sigma)D_2]M_{21}[B_1^T(P_1 - \Sigma) + D_1^T(P_1 - \Sigma)C] \]
\[ - [(P_1 - \Sigma)B_2 + C^T(P_1 - \Sigma)D_2]M_{22}[B_2^T(P_1 - \Sigma) + D_2^T(P_1 - \Sigma)C] \]
\[ = 0, \]  
(5.14)

Comparing the above with (5.1), to prove that \( P_1 - \Sigma \) satisfies Riccati equation (5.1), it suffices to show

\[ F \triangleq (I) - [(P_1 - \Sigma)B_1 + C^T(P_1 - \Sigma)D_1]M_{11}[B_1^T(P_1 - \Sigma) + D_1^T(P_1 - \Sigma)C] \]
\[ - [(P_1 - \Sigma)B_1 + C^T(P_1 - \Sigma)D_1]M_{12}[B_2^T(P_1 - \Sigma) + D_2^T(P_1 - \Sigma)C] \]
\[ - [(P_1 - \Sigma)B_2 + C^T(P_1 - \Sigma)D_2]M_{21}[B_1^T(P_1 - \Sigma) + D_1^T(P_1 - \Sigma)C] \]
\[ - [(P_1 - \Sigma)B_2 + C^T(P_1 - \Sigma)D_2]M_{22}[B_2^T(P_1 - \Sigma) + D_2^T(P_1 - \Sigma)C] \]
\[ = 0. \]  
(5.15)

where \( M_{i,j} \) \((i, j = 1, 2)\) is defined by (5.11). By the definitions (3.12) and (3.13), the function \( F \) can be rewritten as

\[ F = P_1B_1(f_1) + \Sigma B_1(f_2) + P_1B_2(f_3) + \Sigma B_2(f_4) + C^TP_1D_1(f_5) + C^TP_1D_2(f_6) + (f_7), \]  
(5.16)

where

\[ (f_2) = -(f_1), \quad (f_4) = -(f_3), \quad (f_5) = (f_1), \quad (f_6) = (f_3), \]  
(5.17)

and

\[ (f_1) = \hat{R}_1^{-1}(D_1P_1C + B_1^TP_1) - \hat{R}_1^{-1}B_1^T\Sigma + \hat{R}_1^{-1}D_1^T\hat{S}^{-1}\Sigma[C^T - S_1^T\Sigma + S_2^T\Sigma^{-1}\Sigma] \]
\[ + S_2\Sigma - \hat{R}_1^{-1}D_1^TP_1D_2\hat{S}^{-1}\Sigma[C^T + S_2\Sigma] + \hat{R}_1^{-1}D_1^TP_1D_2\hat{\Phi}[B^T + S_1\Sigma] \]
\[ - M_{11}[B_1^T(P_1 - \Sigma) + D_1^T(P_1 - \Sigma)C] - M_{12}[B_2^T(P_1 - \Sigma) + D_2^T(P_1 - \Sigma)C], \]  
(5.18)

\[ (f_3) = \hat{R}_1^{-1}S_2\hat{S}^{-1}\Sigma[C^T + S_2\Sigma] - \hat{\Phi}B^T - \hat{\Phi}S_1\Sigma - M_{21}[B_1^T(P_1 - \Sigma) + D_1^T(P_1 - \Sigma)C] \]
\[ - M_{22}[B_2^T(P_1 - \Sigma) + D_2^T(P_1 - \Sigma)C], \]  
(5.19)

\[ (f_7) = C^T\Sigma D_1M_{11}[B_1^T(P_1 - \Sigma) + D_1^T(P_1 - \Sigma)C] + C^T\Sigma D_1M_{12}[B_2^T(P_1 - \Sigma) \]
\[ + D_2^T(P_1 - \Sigma)C] + C^T\Sigma D_2M_{21}[B_1^T(P_1 - \Sigma) + D_1^T(P_1 - \Sigma)C] \]
\[ + C^T\Sigma D_2M_{22}[B_2^T(P_1 - \Sigma) + D_2^T(P_1 - \Sigma)C] - C^T\Sigma\Sigma - C^T\hat{S}^{-1}\Sigma C^T \]
\[ + C^T\hat{S}^{-1}\Sigma S_2\Sigma^{-1}B^T + C^T\hat{S}^{-1}\Sigma S_2\Sigma^{-1}B^T - C^T\hat{S}^{-1}\Sigma S_2\Sigma^{-1}S_1\Sigma. \]  
(5.20)

Thus to prove \( F = 0 \), we only need to show \( (f_i) = 0; i = 1, 3, 7 \). In the following, we shall prove them separately.

1. **Proof of** \((f_3) = 0\). By the definitions of \( \hat{\Phi}, M_{21} \) and \( M_{22} \), \((f_3) \) can be rewritten as

\[ -(f_3) = \hat{R}_1^{-1}S_2\hat{S}^{-1}\Sigma D_1(R_1 + D_1^TP_1D_1)^{-1}B_1^T\Sigma - \hat{\Phi}D_2^T P_1D_1(R_1 + D_1^TP_1D_1)^{-1}B_1^T\Sigma \]
\[ - \hat{\Phi}D_2^TP_1D_1(R_1 + D_1^TP_1D_1)^{-1}[B_1^T(P_1 - \Sigma) + D_1^T(P_1 - \Sigma)C] \]
\[ + \hat{\Phi}D_2^TP_1D_1(R_1 + D_1^TP_1D_1)^{-1}(D_1^TP_1C + B_1^TP_1) - \hat{\Phi}D_2^T \Sigma C \]
\[ - R_1^{-1}S_2\hat{S}^{-1}\Sigma[D_1(R_1 + D_1^TP_1D_1)^{-1}(D_1^TP_1C + B_1^TP_1) - C] \]
\[ = -(f_3)B_1^T(P_1 - \Sigma) - (f_3)C, \]  
(5.21)

where

\[ (f_3) = \hat{R}_1^{-1}S_2\hat{S}^{-1}\Sigma D_1(R_1 + D_1^TP_1D_1)^{-1}B_1^T\Sigma - \hat{\Phi}D_2^T P_1D_1(R_1 + D_1^TP_1D_1)^{-1} \]
\[ + \hat{R}_1^{-1}S_2\hat{S}^{-1}\Sigma D_1(R_1 + D_1^TP_1D_1)^{-1}, \]  
(5.22)

Then from \((f_3) = 0 \) and \((f_3) = 0 \) (see Appendix for the proof), we get \((f_3) = 0 \).
(2) **Proof of** \((f_1) = 0\). We can rewrite \((f_1)\) as

\[
(f_1) = -(f_1)_1[B_2^T (P_1 - \Sigma) + D_2^T P_1 C] + (f_1)_2[B_1^T (P_1 - \Sigma) + D_1^T P_1 C] - (f_1)_3, \quad (5.23)
\]

where \((f_1)_1 = -(f_3)_1 = 0\) and

\[
(f_1)_2 = [R_1 + D_1^T (P_1 - \Sigma) D_1]^{-1} [R_1 + D_1^T (P_1 - \Sigma) D_1]^{-1} D_1^T (P_1 - \Sigma) D_1^T \hat{D} \hat{D}_2 \times \{P_1 - \Sigma\} D_1 R_1^T - R_1^T D_1^T \hat{D} \hat{D}_2 P_1 D_1 \hat{R}_1^T - \hat{R}_1 \hat{R}_1^T D_1^T \hat{D} \hat{D}_2 P_1 D_1 \hat{R}_1^T - \hat{R}_1 \hat{R}_1^T D_1^T \hat{D} \hat{D}_2 P_1 D_1 \hat{R}_1^T, \quad (f_1)_3 = -N_1 D_1^T \Sigma C - N_1 D_1^T \Sigma C - \hat{R}_1 \hat{R}_1^T D_1^T P_1 D_1 \hat{R}_1^T - R_1^T D_1^T \hat{D} \hat{D}_2 P_1 D_1 \hat{R}_1^T - \hat{R}_1 \hat{R}_1^T D_1^T \hat{D} \hat{D}_2 P_1 D_1 \hat{R}_1^T - \hat{R}_1 \hat{R}_1^T D_1^T \hat{D} \hat{D}_2 P_1 D_1 \hat{R}_1^T, \quad (5.24)
\]

Then from \((f_1)_2 = 0\) and \((f_1)_3 = 0\) (see Appendix for the proof), we get \((f_1) = 0\).

(3) **Proof of** \((f_r) = 0\). We can rewrite \((f_r)\) as

\[
(f_r) = (f_r)_1 B_1 P_1 + (f_r)_2 B_1 \Sigma + (f_r)_3 B_2 P_1 + (f_r)_4 B_2 \Sigma + (f_r)_5, \quad (5.26)
\]

where

\[
(f_r)_1 = -(f_r)_2 = -(f_r)_3 = 0, \quad (f_r)_4 = C^T (f_r)_2 = 0, \quad (f_r)_5 = C^T \Sigma D_1 M_{11} D_1^T (P_1 - \Sigma) C + C^T \Sigma D_1 M_{12} D_2^T (P_1 - \Sigma) C + C^T \Sigma D_2 M_{21} D_1^T (P_1 - \Sigma) C + C^T \Sigma D_2 M_{22} D_2^T (P_1 - \Sigma) C - C^T \Sigma C - C^T \Sigma C - M_{11} D_1^T \Sigma C - M_{12} D_1^T \Sigma C - \hat{R}_1 \hat{R}_1^T D_1^T P_1 D_1 \hat{R}_1^T - \hat{R}_1 \hat{R}_1^T D_1^T P_1 D_1 \hat{R}_1^T - \hat{R}_1 \hat{R}_1^T D_1^T P_1 D_1 \hat{R}_1^T - \hat{R}_1 \hat{R}_1^T D_1^T P_1 D_1 \hat{R}_1^T. \quad (5.27)
\]

Then from \((f_r)_5 = 0\) (see Appendix for the proof), we get \((f_r) = 0\).

**Remark 5.6.** From the above proof, we see that **Theorem 5.2** can be proved by comparing \((5.13)\) with \((5.1)\). Although the bridge between the singular terms of Riccati equations \((5.1)\), \((3.5)\) and \((4.24)\) has been established by **Lemma 5.4**, the verification is still technical and lengthy. For more details of the proof, please see Appendix.

### 5.3 Equivalence between Stackelberg equilibria and open-loop saddle points

In **Theorem 5.2**, a connection between the Riccati equations associated with Problems (SG) and (NG) has been established. In this subsection, we shall show that the Stackelberg equilibrium, obtained in **Theorem 4.9**, exactly is the unique open-loop saddle point of Problem (NG).

**Theorem 5.7.** Suppose that \((H1)-(H3)\) and \((H5)\) hold. Then the following results hold.

(i) The Stackelberg equilibrium \((\hat{u}_1, \hat{u}_2) \in U_1 \times U_2\) of Problem (SG), obtained in **Theorem 4.9**, is the unique open-loop saddle point of Problem (NG).

(ii) The value function of Problem (NG) is given by

\[
V(x) = \langle P(0) (0) - \Sigma(0)x, x \rangle = \langle P(0) x, x \rangle, \quad \forall x \in \mathbb{R}^n. \quad (5.28)
\]

**Proof.** (i) Under \((H5)\), by [16, Theorem 4.4] we get that Problem (NG) admits a unique open-loop saddle point \((u_1^*, u_2^*) \in U_1 \times U_2\). By the definition of open-loop saddle points, we have

\[
\inf_{u_1 \in U_1} J_{u_2}(x; u_1) = J_{u_2}(x; u_1^*) \quad \text{and} \quad J(x; u_1^*, u_2^*) = \sup_{u_2 \in U_2} J(x; u_1^*, u_2^*), \quad (5.29)
\]

where \(J_{u_2}\) is defined by \((3.2)\). Then by **Proposition 3.1**, we get

\[
u_1^*(s) = \bar{u}_1(s; u_2^*, x), \quad s \in [0, T], \quad (5.30)
\]
where $\bar{\alpha}_1$ is defined by (3.4). Thus
\[ J(x; u_1^*, u_2^*) = J(x; \bar{\alpha}_1(u_2^*, x), u_2^*). \] (5.31)

Moreover, recall from Proposition 3.1 that
\[ J(x; \tilde{\alpha}_1(u_2, x), u_2) = \inf_{u_1 \in u_1} J(x; u_1, u_2), \quad \forall u_2 \in U_2. \] (5.32)

Then by the second equality in (5.29), we get
\[ J(x; u_1^*, u_2^*) \geq J(x; u_1^*, u_2) \geq J(x; \tilde{\alpha}_1(u_2, x), u_2), \quad \forall u_2 \in U_2. \]

Combining the above with (5.31) yields that
\[ J(x; \tilde{\alpha}_1(u_2^*, x), u_2^*) \geq J(x; \tilde{\alpha}_1(u_2, x), u_2), \quad \forall u_2 \in U_2. \] (5.33)

In other words, $u_2^*$ is an optimal control of Problem (LLQ), which is the leader’s problem.

On the other hand, by Proposition 4.8, Problem (LLQ) admits a unique optimal control $\bar{u}_2 = \tilde{u}_2$ under (H1)–(H3) and (H5). Thus, we must have $u_2^* = \bar{u}_2$. Combining this with the facts $u_1^* = \bar{\alpha}_1(u_2^*, x)$ and $\tilde{\alpha}_1 = \bar{\alpha}_1(\bar{u}_2, x)$, we get $u_1^* = \tilde{\alpha}_1$. It follows that $(\tilde{u}_1, \tilde{u}_2) \in U_1 \times U_2$ is the unique open-loop saddle of Problem (NG).

(ii) By the definition of the value function of Problem (NG) and the fact $(u_1^*, u_2^*) = (\tilde{u}_1, \tilde{u}_2)$, we get
\[ V(x) = J(x; u_1^*, u_2^*) = J(x; \tilde{u}_1, \tilde{u}_2). \]

Then from (4.44) and (5.4), we obtain (5.28).

**Remark 5.8.** We emphasize again that the weak (UCC) condition and the (UCC) condition are almost necessary for the existence of a Stackelberg equilibrium and the existence of an open-loop saddle point, respectively. Then from Theorem 3.4, Example 4.4 and Theorem 5.7, we conclude that the gap between the weak (UCC) condition (i.e., (H3)–(H4)) and the (UCC) condition (i.e., (H3) and (H5)) is the main reason causing the different performances between Problems (SG) and (NG).

Denote
\[ (\Theta_1^\top, \Theta_2^\top)^\top = -(R + D^\top PD)^{-1}(B^\top P + D^\top PC). \] (5.34)

**Theorem 5.9.** Let (H1)–(H3) and (4.11) hold. Then the Stackelberg equilibrium $(\tilde{u}_1, \tilde{u}_2)$ of Problem (SG) can be represented as
\[ \tilde{u}_1 = u_1^* \equiv \Theta_1 X^* \quad \text{and} \quad \tilde{u}_2 = u_2^* \equiv \Theta_2 X^*, \] (5.35)

where $X^*$ is the unique solution of the closed-loop system:
\[
\begin{cases} 
    dX^*(s) = \{A(s)X^*(s) + B_1(s)\Theta_1^\top(s)X^*(s) + B_2(s)\Theta_2^\top(s)X^*(s)\}ds \\
    + \{C(s)X^*(s) + D_1(s)\Theta_1^\top(s)X^*(s) + D_2(s)\Theta_2^\top(s)X^*(s)\}dW(s), 
\end{cases}
\] (5.36)

If (H5) also holds, then $(u_1^*, u_2^*)$ is the unique open-loop saddle point of Problem (NG).

**Proof.** Using the similar argument to that employed in Theorem 5.2, we can get
\[
\hat{\Theta}_1 = -\mathcal{M}_{11}[B_1^\top(P_1 - \Sigma) + D_1^\top(P_1 - \Sigma)C] - \mathcal{M}_{12}[B_2^\top(P_1 - \Sigma) + D_2^\top(P_1 - \Sigma)C], \\
\hat{\Theta}_2 = -\mathcal{M}_{21}[B_1^\top(P_1 - \Sigma) + D_1^\top(P_1 - \Sigma)C] - \mathcal{M}_{22}[B_2^\top(P_1 - \Sigma) + D_2^\top(P_1 - \Sigma)C],
\] (5.37)

where $\hat{\Theta}_i$ and $\mathcal{M}_{i,j}$ $(i, j = 1, 2)$ are defined by (4.41)–(4.42) and (5.11), respectively. Then by the fact $P = P_1 - \Sigma$ obtained in Theorem 5.2, we can rewrite (5.37) as
\[
(\hat{\Theta}_1^\top, \hat{\Theta}_2^\top)^\top = (\Theta_1^\top, \Theta_2^\top)^\top = -(R + D^\top PD)^{-1}(B^\top P + D^\top PC). \] (5.38)

Thus, the Stackelberg equilibrium $(\tilde{u}_1, \tilde{u}_2)$ obtained in Theorem 4.9 can be rewritten as (5.35). If the additional assumption (H5) holds, by Theorem 5.7, the control pair $(u_1^*, u_2^*)$ is the unique open-loop saddle point of Problem (NG).
Remark 5.10. By Theorem 5.9, we show that under (H1)–(H3) and (4.11), the Stackelberg equilibrium of Problem (SG) admits another closed-loop representation (5.35), in terms of the solution to Riccati equation (5.1). When (H5) also holds, (5.36) coincides with the closed-loop system of Problem (NG), which was given in [16, Theorem 4.4].

6 Conclusion

In conclusion, we show that under the weak (UCC) condition (i.e., (H3)–(H4)), a Stackelberg equilibrium of Problem (SG) can be explicitly obtained by solving a forward-backward stochastic LQ optimal control problem (see Theorem 3.4). Interestingly, under the stronger (UCC) condition (i.e., (H3) and (H5)), the Stackelberg equilibrium of Problem (SG) exactly is the unique open-loop saddle point of Problem (NG) (see Theorem 5.7 and Theorem 5.9). It follows that the open-loop saddle point of Problem (NG) can be obtained by considering the game in a leader-follower manner, which is a little surprising. These results are achieved by a careful investigation of backward stochastic LQ optimal control problems (see Proposition 4.2 and Proposition 4.7). Moreover, an explicit relationship between the Riccati equations associated with Problem (NG) (i.e., (5.1)) and Problem (SG) (i.e., (3.5) and (4.24)) is established (see Theorem 5.2). Indeed, we show that (4.24) serves as a bridge between the Riccati equations associated with stochastic LQ optimal controls and two-person zero-sum stochastic LQ Nash games (i.e., Problems (FLQ) and (NG)). As a byproduct, the well-posedness of Riccati equation (5.1) is reestablished by a completely new method, which can help to relax the assumptions imposed by Sun [16].

7 Appendix

7.1 Proof of Lemma 5.4

By the definition of $\mathcal{R}$, it is straightforward to see that

$$
\Phi = \mathcal{R} - D_2^\top \Sigma D_1 + D_2^\top (P_1 - \Sigma) D_1^{-1} D_1^\top P_1 D_2 - D_1^\top P_1 D_1 (P_1 - \Sigma) D_1^{-1} D_1^\top P_1 D_2
$$

Moreover, an explicit relationship between the Riccati equations associated with Problem (NG) (i.e., (5.1)) and Problem (SG) (i.e., (3.5) and (4.24)) is established (see Theorem 5.2). Indeed, we show that (4.24) serves as a bridge between the Riccati equations associated with stochastic LQ optimal controls and two-person zero-sum stochastic LQ Nash games (i.e., Problems (FLQ) and (NG)). As a byproduct, the well-posedness of Riccati equation (5.1) is reestablished by a completely new method, which can help to relax the assumptions imposed by Sun [16].

Then by the definition of $\mathcal{S}_2$, we get

$$
\Phi \hat{\Phi} = I + [D_2^\top - D_2^\top P_1 D_1 \hat{R}_1^{-1} D_1^\top ] \hat{\Sigma}^{-1} [D_2 - D_1 \hat{R}_1^{-1} D_1^\top P_1 D_2] \mathcal{R}^{-1}
$$

Thus to prove Lemma 5.4, noting that $\Phi$ and $\hat{\Phi}$ are symmetric, it suffices to show that $(a) = 0$. By the definition (4.25) of $\hat{\Sigma}$, the function $(a)$ can be simplified as follows:

$$
(a) = \{ D_2^\top - D_2^\top P_1 D_1 \hat{R}_1^{-1} D_1^\top + D_2 D_1 (P_1 - \Sigma) D_1^{-1} D_1^\top - D_2^\top P_1 D_1 \}
$$

Then by the definition of $\mathcal{S}_2$, we get

$$
\Phi \hat{\Phi} = I + [D_2^\top - D_2^\top P_1 D_1 \hat{R}_1^{-1} D_1^\top ] \hat{\Sigma}^{-1} [D_2 - D_1 \hat{R}_1^{-1} D_1^\top P_1 D_2] \mathcal{R}^{-1}
$$

Thus to prove Lemma 5.4, noting that $\Phi$ and $\hat{\Phi}$ are symmetric, it suffices to show that $(a) = 0$. By the definition (4.25) of $\hat{\Sigma}$, the function $(a)$ can be simplified as follows:

$$
(a) = \{ D_2^\top - D_2^\top P_1 D_1 \hat{R}_1^{-1} D_1^\top + D_2 D_1 (P_1 - \Sigma) D_1^{-1} D_1^\top - D_2^\top P_1 D_1
$$

$$
\times [R_1 + D_1^\top (P_1 - \Sigma) D_1^{-1} D_1^\top \Sigma D_1 \hat{R}_1^{-1} D_1^\top ] \hat{\Sigma}^{-1} [D_2 - D_1 \hat{R}_1^{-1} D_1^\top P_1 D_2]
$$

$$
+ \{ D_2 \Sigma D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1^{-1} D_1^\top P_1 D_2] - D_2^\top \Sigma D_2
$$

$$
\times D_1^\top \Sigma D_2 - D_2^\top \Sigma D_2 \} \mathcal{R}^{-1} [D_2^\top - D_2 P_1 D_1 D_1^\top P_1 \hat{R}_1^{-1} D_1^\top \hat{\Sigma}^{-1} [D_2 - D_1 \hat{R}_1^{-1} D_1^\top P_1 D_2].
$$

(7.3)
Further, using the fact
\begin{align*}
D_2^\top \{ I - \Sigma D_1 \hat{R}_1^{-1} D_1^\top - \Sigma |D_2 - D_1 \hat{R}_1^{-1} D_1^\top P_1 D_2| \} R^{-1} \\
\times [D_2^\top - D_2^\top P_1 D_1 \hat{R}_1^{-1} D_1^\top] \hat{\Sigma}^{-1} \Sigma |D_2 - D_1 \hat{R}_1^{-1} D_1^\top P_1 D_2| \\
= D_2^\top \Sigma [D_2 - D_1 \hat{R}_1^{-1} D_1^\top P_1 D_2],
\end{align*}
equality (7.3) can be simplified as follows:
\begin{align*}
(a) = \{ - D_2^2 P_1 D_1 \hat{R}_1^{-1} D_1^\top + D_2^\top P_1 D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} D_1^\top \\
- D_2^\top P_1 D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} D_1^\top \Sigma D_1 \hat{R}_1^{-1} D_1^\top \\
+ D_2^\top \Sigma D_1 \hat{R}_1^{-1} D_1^\top \} \hat{\Sigma}^{-1} \Sigma [D_2 - D_1 \hat{R}_1^{-1} D_1^\top P_1 D_2] \\
- D_2^\top \Sigma D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} \Sigma |D_2 - D_1 \hat{R}_1^{-1} D_1^\top P_1 D_2| \\
+ D_2^\top \Sigma D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} \Sigma |D_2 - D_1 \hat{R}_1^{-1} D_1^\top P_1 D_2| \\
\times R^{-1} [D_2^\top - D_2^\top P_1 D_1 \hat{R}_1^{-1} D_1^\top] \hat{\Sigma}^{-1} \Sigma [D_2 - D_1 \hat{R}_1^{-1} D_1^\top P_1 D_2].
\end{align*}

Then by substituting
\begin{align*}
& D_2^\top \Sigma D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} \Sigma |D_2 - D_1 \hat{R}_1^{-1} D_1^\top P_1 D_2| \\
\times R^{-1} [D_2^\top - D_2^\top P_1 D_1 \hat{R}_1^{-1} D_1^\top] \hat{\Sigma}^{-1} \Sigma [D_2 - D_1 \hat{R}_1^{-1} D_1^\top P_1 D_2] \\
& = D_2^\top \Sigma D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} \Sigma [D_2 - D_1 \hat{R}_1^{-1} D_1^\top P_1 D_2]
\end{align*}
into (7.4), we get
\begin{align*}
(a) = \{ D_2^\top P_1 D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} D_1^\top - D_2^\top P_1 D_1 \hat{R}_1^{-1} D_1^\top \\
- D_2^\top P_1 D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} D_1^\top \Sigma D_1 \hat{R}_1^{-1} D_1^\top + D_2^\top \Sigma D_1 \hat{R}_1^{-1} D_1^\top \\
- D_2^\top \Sigma D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} D_1^\top + D_2^\top \Sigma D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} \\
\times D_2^\top \Sigma D_1 \hat{R}_1^{-1} D_1^\top \} \hat{\Sigma}^{-1} \Sigma [D_2 - D_1 \hat{R}_1^{-1} D_1^\top P_1 D_2] = 0.
\end{align*}

The proof is complete. \(\square\)

### 7.2 Details in the proof of Theorem 5.2

**Verification of \((f_3)_1 = 0.** By the definition (5.9) of \(\hat{\Phi}\), we have
\begin{align*}
\mathcal{R}(f_3)_1 = [I + S_2 \hat{\Sigma}^{-1} \Sigma S_2^\top \hat{\Sigma}^{-1}] D_2^\top (P_1 - \Sigma) D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} \\
- [I + S_2 \hat{\Sigma}^{-1} \Sigma S_2^\top \hat{\Sigma}^{-1}] D_2^\top P_1 D_1 \hat{R}_1^{-1} + S_2 \hat{\Sigma}^{-1} \Sigma D_1 \hat{R}_1^{-1}.
\end{align*}

It follows that
\begin{align*}
\mathcal{R}(f_3)_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1] \\
= D_2^\top P_1 D_1 \hat{R}_1^{-1} D_1^\top \Sigma D_1 - D_2^\top \Sigma D_1 - S_2 \hat{\Sigma}^{-1} \Sigma S_2^\top \hat{\Sigma}^{-1} D_2^\top \Sigma D_1 + S_2 \hat{\Sigma}^{-1} \Sigma D_1 \\
+ S_2 \hat{\Sigma}^{-1} \Sigma S_2^\top \hat{\Sigma}^{-1} D_2^\top P_1 D_1 \hat{R}_1^{-1} D_1^\top \Sigma D_1 - S_2 \hat{\Sigma}^{-1} \Sigma D_1 \hat{R}_1^{-1} D_1^\top \Sigma D_1 \\
= D_2^\top P_1 D_1 \hat{R}_1^{-1} D_1^\top \Sigma D_1 - D_2^\top \Sigma D_1 + S_2 \hat{\Sigma}^{-1} \Sigma D_1 = 0.
\end{align*}

Since \(\mathcal{R} \ll 0\) and \(R_1 + D_1^\top (P_1 - \Sigma) D_1 \gg 0\), the above implies that \((f_3)_1 = 0.\) \(\square\)

**Verification of \((f_3)_2 = 0.** By the fact \((f_3)_1 = 0\), we can simplify \((f_3)_2\) as follows:
\begin{align*}
(f_3)_2 = \{ \hat{\Phi} D_2^\top - \hat{\Phi} D_2^\top (P_1 - \Sigma) D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} D_1^\top - \hat{\Sigma}^{-1} S_2 \hat{\Sigma}^{-1} \Sigma
\end{align*}

Then by the definition (5.9) of \(\hat{\Phi}\) and (7.5), we have
\begin{align*}
\mathcal{R}(f_3)_2 = [I + S_2 \hat{\Sigma}^{-1} \Sigma S_2^\top \hat{\Sigma}^{-1}] D_2^\top \Sigma - S_2 \hat{\Sigma}^{-1} \Sigma.
\end{align*}
$$- [I + S_2 \hat{\Sigma}^{-1} \Sigma S_2^\top R^{-1}] D_2^\top (P_1 - \Sigma) D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} D_1^\top \Sigma$$
$$= D_2^\top P_1 D_1 \hat{R}_1^{-1} D_1^\top \Sigma - S_2 \hat{\Sigma}^{-1} \Sigma D_1 \hat{R}_1^{-1} D_1^\top \Sigma + S_2 \hat{\Sigma}^{-1} \Sigma S_2^\top R^{-1} D_2^\top P_1 D_1 \hat{R}_1^{-1} D_1^\top \Sigma$$
$$- [I + S_2 \hat{\Sigma}^{-1} \Sigma S_2^\top R^{-1}] D_2^\top (P_1 - \Sigma) D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} D_1^\top \Sigma$$
$$= - R(f_3)_1 D_1^\top \Sigma = 0,$$

(7.8)

which implies $(f_3)_2 = 0$. 

**Verification of $(f_1)_2 = 0$.** By the fact $(f_3)_1 = 0$, we get

$$\hat{R}_1(f_1)_2 [R_1 + D_1^\top (P_1 - \Sigma) D_1]$$
$$= D_1^\top \Sigma D_1 + D_1^\top P_1 D_2 R^{-1} S_2 \hat{\Sigma}^{-1} \Sigma D_1 - D_1^\top P_1 D_2 R^{-1} S_2 \hat{\Sigma}^{-1} \Sigma D_1 \hat{R}_1^{-1} D_1^\top \Sigma D_1$$

Then by the definitions of $\hat{\Phi}$, $\hat{\Sigma}$ and $S_2$, the above can be simplified as

$$\hat{R}_1(f_1)_2 [R_1 + D_1^\top (P_1 - \Sigma) D_1]$$
$$= D_1^\top P_1 D_2 \hat{\Sigma}^{-1} \Sigma D_1 - D_1^\top P_1 D_2 R^{-1} S_2 \hat{\Sigma}^{-1} \Sigma D_1 \hat{R}_1^{-1} D_1^\top \Sigma D_1$$

It follows that $(f_1)_2 = 0$. 

**Verification of $(f_1)_3 = 0$.** By the definition of $\hat{\Phi}$ and (7.7), we get

$$[R_1 + D_1^\top (P_1 - \Sigma) D_1] (f_1)_3$$
$$= D_1^\top (P_1 - \Sigma) D_2 \{ \hat{\Phi} D_2 - \hat{\Phi} D_2^\top (P_1 - \Sigma) D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} D_1^\top - R^{-1} S_2 \hat{\Sigma}^{-1} \Sigma C \}$$

The result then follows from $(f_3)_2 = 0$. 

**Verification of $(f_7)_5 = 0$.** Note that

$$(f_7)_5 \triangleq C^\top \Sigma D_1 M_{11} D_1^\top (P_1 - \Sigma) C + C^\top \Sigma D_1 M_{12} D_2^\top (P_1 - \Sigma) C + C^\top \Sigma D_2 M_{21} D_1^\top (P_1 - \Sigma) C + C^\top \Sigma D_2 M_{22} D_2^\top (P_1 - \Sigma) C$$

Thus, to prove $(f_7)_5 = 0$, it is sufficient to show that

$$(f_7)_{51} \triangleq \Sigma D_1 M_{11} D_1^\top + \Sigma D_1 M_{12} D_2^\top + \Sigma D_2 M_{21} D_1^\top + \Sigma D_2 M_{22} D_2^\top + I - \hat{\Sigma}^{-1} = 0.$$ 

(7.10)

By the definition of $M_{ij}, i, j = 1, 2$, we get

$$\hat{\Sigma}(f_7)_{51} \triangleq \Sigma \left\{ - S_2^\top R^{-1} S_2^\top - S_2^\top R^{-1} S_2^\top \Sigma D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} D_1^\top\right.$$ 

$$+ \{ D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} D_1^\top (P_1 - \Sigma) - I \} D_2 \hat{\Phi} D_2^\top \left\{ (P_1 - \Sigma) D_1 \right.$$ 

$$\times \left\{ [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} D_1^\top - I \right\} + \{ N - S_2^\top R^{-1} S_2^\top \} \Sigma \right.$$ 

$$\times \left\{ [D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} D_1^\top (P_1 - \Sigma) - I] D_2 \hat{\Phi} D_2^\top \right.$$ 

$$\times \left\{ (P_1 - \Sigma) D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} D_1^\top - I \right\} \right\}$$

(7.12)

By the definition of $N$, we get

$$(f_7)_{52} \triangleq \{ S_2^\top R^{-1} S_2^\top \Sigma D_1 D_2 \hat{\Phi} D_2^\top - S_2^\top R^{-1} S_2^\top \Sigma D_1 [R_1 + D_1^\top (P_1 - \Sigma) D_1]^{-1} D_1^\top (P_1 - \Sigma) D_2 \hat{\Phi} D_2^\top$$

$$- R(f_3)_1 D_1^\top \Sigma = 0,$$
\[ - D_2^2 \Phi D_2^2 - NP_1 D_1 \Phi D_1^2 \} \{(P_1 - \Sigma)D_1[R_1 + D_1^T (P_1 - \Sigma)D_1]^{-1} D_1^T - I \} \\
- S_2^T R^{-1} S_2 - S_2^T R^{-1} S_2 \Sigma D_1[R_1 + D_1^T (P_1 - \Sigma)D_1]^{-1} D_1^T, \tag{7.13} \]

which yields

\[ (f_T)_{52} = S_2^T R^{-1} S_2 \{ \hat{\Sigma}^{-1} \Sigma S_2 R^{-1} + \Sigma D_1[R_1 + D_1^T (P_1 - \Sigma)D_1]^{-1} D_1^T (P_1 - \Sigma)D_2 \hat{\Phi} \]
\[ - \Sigma D_2 \hat{\Phi} D_2^T + S_2^T R^{-1} D_2^T P_1 D_1 \hat{R}_1^{-1} D_1^T - S_2^T R^{-1} S_2 \Sigma D_1[R_1 + D_1^T (P_1 - \Sigma)D_1]^{-1} D_1^T \]
\[ - S_2^T \{ \hat{\Phi} + R^{-1} S_2 \Sigma D_1[R_1 + D_1^T (P_1 - \Sigma)D_1]^{-1} D_1^T (P_1 - \Sigma)D_2 \hat{\Phi} - R^{-1} S_2 \Sigma D_2 \hat{\Phi} \} \]
\[ \times D_1^T (P_1 - \Sigma)D_1[R_1 + D_1^T (P_1 - \Sigma)D_1]^{-1} D_1^T \]
\[ \triangleq S_2^T R^{-1} S_2 (f_T)_{53} D_2^T + (f_T)_{54} D_1^T. \tag{7.14} \]

By (7.7) and the fact \( \Sigma - \hat{\Sigma} \Sigma (\hat{\Sigma}^{-1})^T = 0 \), we get \((f_T)_{53} = 0\). Moreover,

\[ (f_T)_{54} = S_2^T R^{-1} D_2^T P_1 D_1 \hat{R}_1^{-1} - S_2^T R^{-1} S_2 \Sigma D_1[R_1 + D_1^T (P_1 - \Sigma)D_1]^{-1} \]
\[ - S_2^T R^{-1} D_2^T (P_1 - \Sigma)D_1[R_1 + D_1^T (P_1 - \Sigma)D_1]^{-1} = 0. \tag{7.15} \]

Substituting \((f_T)_{53} = 0\) and \((f_T)_{54} = 0\) into (7.14) yields \((f_T)_{52} = 0\), which then implies \((f_T)_5 = 0\).

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