Expansion Exponents for Nonequilibrium Systems

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Abstract

Local expansion exponents for nonequilibrium dynamical systems, described by partial differential equations, are introduced. These exponents show whether the system phase volume expands, contracts, or is conserved in time. The ways of calculating the exponents are discussed. The principle of minimal expansion provides the basis for treating the problem of pattern selection. The exponents are also defined for stochastic dynamical systems. The analysis of the expansion-exponent behaviour for quasi-isolated systems results in the formulation of two other principles: The principle of asymptotic expansion tells that the phase volumes of quasi-isolated systems expand at asymptotically large times. The principle of time irreversibility follows from the asymptotic phase expansion, since the direction of time arrow can be defined by the asymptotic expansion of phase volume.

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1 Introduction

Macroscopic properties of nonequilibrium dynamical systems are usually characterized by the Gibbs entropy defined for the appropriate probability measure [1,2]. The usage of this entropy has two shortcomings: First, it is defined only for steady states, and an effective definition of entropy far from equilibrium is not known [2]. Second, there appears the problem of singularity resulting from the following [1,2]. A steady state of a dynamical system corresponds to the motion on an attractor. The related probability measure is, typically, singular, because of which the entropy becomes equal to $-\infty$. To avoid this problem, one considers not the entropy itself but the entropy production or entropy production rate, such as the Kolmogorov-Sinai entropy rate [3]. The latter are finite, but again, they are defined only for stationary states.

Employing the Gibbs entropy in nonequilibrium statistical mechanics confronts as well another paradox. This entropy, for an isolated system, as is well known, does not change in time. While the second law of thermodynamics requires that the entropy of an isolated nonequilibrium system would be an increasing function of time. To overcome this contradiction, one introduces modified variants of entropy, among which the most popular is a coarse-grained entropy [4]. Other variants are the von Neumann entropy [5] and the average entropy [6], which change with time for an isolated system not in equilibrium. Lebowitz [7] argues that the relevant entropy for understanding the time evolution of macroscopic systems is not the Gibbs entropy $S_G \equiv -\text{Tr} \rho \ln \rho$ but the Boltzmann entropy $S_B \equiv \ln |\Gamma|$, where $\Gamma$ is the system phase volume. Contrary to the Gibbs entropy, which is constant for an isolated macroscopic system, the Boltzmann entropy should typically increase with time. A weak point in resorting to the Boltzmann entropy $S_B$ is that it is not always clear how to determine the phase volume for a strongly nonequilibrium system.

Instead of trying to construct a special form of entropy that would be valid for nonequilibrium dynamical systems, it is possible to employ another characteristic. The aim of the present paper is to demonstrate that the local expansion exponent can be such a convenient quantity characterizing the temporal evolution of dynamical systems. In Section 2, the local expansion exponent is introduced for an infinite-dimensional dynamical system and its properties are described. Section 3 shows the usefulness of the expansion exponents for treating the problem of pattern selection. The generalization of the expansion exponents to stochastic dynamical systems is given in Section 4. Several examples of Section 5 illustrate how the exponents can be calculated. The notion of quasi-isolated systems is formulated in Section 6 and some particular cases are considered. Section 7 contains conclusions and discussion.

2 Local expansion exponents

Let $x \in \mathbb{D} \subset \mathbb{R}^d$ be a $d$-dimensional set of continuous variables given on a domain $\mathbb{D}$ and let $t \in \mathbb{R}_+ \equiv [0, \infty)$ denote time. Suppose we are interested in the evolution of a family of complex-valued functions $y_i(x,t)$, with $i \in \mathbb{N}_+ \equiv \{1, 2, \ldots\}$, describing a nonequilibrium physical process. It is convenient to employ the matrix notation [8], where the pair $\{i, x\}$
is treated on equal footing as a point in the label space \( \mathbb{N}_+ \times \mathbb{D} \). Then the dynamic state \( y(t) = [y_i(x, t)] \) is presented as a vector-column with respect to the multi-index \( \{i, x\} \). The components \( y_i(x, t) \) may correspond to some physical quantities. In particular, these could be statistical or quantum averages of operators for local observables. This could also be a wave function or a density matrix.

Dynamic state \( y(t) \in \mathcal{F} \) pertains to a phase space \( \mathcal{F} \). Let the velocity field \( v(y, t) = [v_i(x, y, t)] \) be given as a column with respect to the label pair \( \{i, x\} \) and defined in a space tangent to \( \mathcal{F} \). The set of evolution equations can be compactly written as

\[
\frac{d}{dt} y(t) = v(y, t) .
\]

(1)

For the dynamical components \( y_i(x, t) \in \mathcal{F}_i(x) \) pertaining to the phase subspaces \( \mathcal{F}_i(x) \), the structure of the total phase space \( \mathcal{F} \) is that of the tensor product

\[
\mathcal{F} = \otimes_i \otimes_x \mathcal{F}_i(x) .
\]

The continuous product of spaces \( \mathcal{F}_i(x) \), with respect to the continuous variable \( x \), is a direct generalization [9] of the discrete product over the discrete index \( i = 1, 2, \ldots \). The related continuous product of functions \( f(x) \) is naturally defined [9] as

\[
\prod_x f(x) \equiv \exp \int \ln f(x) \, dx ,
\]

with the integration over \( \mathbb{D} \), where any branch of the logarithm may be taken. The elementary phase volume at time \( t \) is interpreted [10,11] as

\[
\delta \Gamma(t) \equiv \prod_i \prod_x \delta y_i(x, t) .
\]

(2)

Introduce the local expansion exponent

\[
X(t) \equiv \ln \left| \frac{\delta \Gamma(t)}{\delta \Gamma(0)} \right| ,
\]

(3)

which shows whether the phase volume, at time \( t \), expands, contracts, or is conserved. Rewriting the definition (3) in the form

\[
|\delta \Gamma(t)| = |\delta \Gamma(0)| \, e^{X(t)} ,
\]

(4)

one sees that positive \( X(t) \) implies expansion, negative \( X(t) \) means contraction, while zero \( X(t) \) tells that the phase volume at time \( t \) does not change. Note that \( X(t) \) could equally be called the local contraction exponent. However, to term it the expansion exponent looks more logical as far as \( X(t) > 0 \) implies expansion.

One may notice that the quantity (3) resembles the entropy production \( S(t) - S(0) \), with the entropy \( S(t) = \ln |\delta \Gamma(t)| \) defining à la Boltzmann. But, clearly, this is just a resemblance since the expansion exponent (3) has sense for any dynamical system and does not require to be ascribed to a statistical system. For the latter, one usually defines
the entropy production being based on the Gibbs entropy, which allows for the description of only steady states [1,2]. But for strongly nonequilibrium systems the Boltzmann form is more appropriate [7].

The expansion exponent (3) is directly related to the multiplier matrix \( \hat{M}(t) = [M_{ij}(x,x',t)] \) with the elements

\[ M_{ij}(x,x',t) \equiv \frac{\delta y_i(x,t)}{\delta y_j(x',0)} \]

and with the initial condition

\[ M_{ij}(x,x',0) = \delta_{ij} \delta(x - x') . \] (6)

Invoking equations (2) and (5), we have

\[ X(t) = \sum_i \int \ln |M_{ii}(x,x,t)| \, dx . \] (7)

This expression can be rewritten in a general form not depending on a particular representation. For this purpose, let us recall that for a matrix \( \hat{A} \) the equality

\[ \text{Tr} \ln \hat{A} = \ln \text{det} \hat{A} \]

is valid. The proof of this equality is based on the fact that the trace of a matrix does not depend on a matrix representation. Then, employing the diagonal representation for \( \hat{A} = [\delta_{mn} A_n] \), one has

\[ \text{Tr} \ln \hat{A} = \sum_n \ln A_n = \ln \prod_n A_n = \ln \text{det} \hat{A} , \]

which proves equality (8). Defining the matrix

\[ \ln |\hat{M}(t)| \equiv [\ln |M_{ij}(x,x',t)|] , \]

equation (7) may be presented as

\[ X(t) = \text{Tr} \ln |\hat{M}(t)| . \] (9)

And using equality (8), we obtain

\[ X(t) = \ln |\text{det} \hat{M}(t)| . \] (10)

Thus, the expansion exponent (3) is expressed through the multiplier matrix (5).

Another expression for the expansion exponent can be found by invoking the evolution equation (1). Define the Jacobian matrix \( \hat{J}(t) = [J_{ij}(x,x',t)] \) with the elements

\[ J_{ij}(x,x',t) \equiv \frac{\delta v_i(x,y,t)}{\delta y_j(x',t)} . \] (11)
The variation of equation (1) over the initial conditions results in the equation

$$\frac{d}{dt} \hat{M}(t) = \hat{J}(t)\hat{M}(t)$$

(12)

for the multiplier matrix. The initial condition for equation (12), according to equality (6), is

$$\hat{M}(0) = \hat{1} \equiv [\delta_{ij} \delta(x - x')] .$$

(13)

With the usage of equation (12), we have

$$\frac{d}{dt} \text{Tr ln } \hat{M}(t) = \text{Tr } \hat{M}^{-1}(t) \frac{d}{dt} \hat{M}(t) = \text{Tr } \hat{M}^{-1}(t)\hat{J}(t)\hat{M}(t) = \text{Tr }\hat{J}(t) ,$$

from where it follows that

$$\frac{d}{dt} \text{Tr ln } \hat{M}(t) = \text{Tr }\hat{J}(t) .$$

Integrating this equation gives

$$\text{Tr ln } \hat{M}(t) = \int_0^t \text{Tr }\hat{J}(t') \, dt' .$$

And equality (8) yields

$$\text{Tr ln } \hat{M}(t) = \ln \text{det } \hat{M}(t) .$$

Comparing the latter two equations results in the equality

$$\text{det } \hat{M}(t) = \exp \left\{ \int_0^t \text{Tr }\hat{J}(t') \, dt' \right\} .$$

(14)

Substituting expressions (14) into equation (10) provides us with the local expansion exponent

$$X(t) = \text{Re } \int_0^t \text{Tr }\hat{J}(t') \, dt' .$$

(15)

Equations (10) and (15) are the main forms for calculating the expansion exponent.

For practical calculational purposes, it is possible to use different representations connected with a chosen basis \( \{ \varphi_n(t) \} \) of the vector-columns \( \varphi_n(t) = [\varphi_{ni}(x, t)] \). In what follows, such a basis is assumed to be orthonormal and complete,

$$\varphi_m^+(t)\varphi_n(t) = \delta_{mn} , \quad \sum_n \varphi_n(t)\varphi_n^+(t) = \hat{1} .$$

From here, we get a useful equality

$$\frac{d\varphi_m^+(t)}{dt} \varphi_n(t) + \varphi_m^+ \frac{d\varphi_n(t)}{dt} = 0$$

(16)

for any indices \( m \) and \( n \). The \( n \)-representations for the multiplier and Jacobian matrices are defined by the elements

$$M_{mn}(t) \equiv \varphi_m^+(t)\hat{M}(t)\varphi_n(t) , \quad J_{mn}(t) \equiv \varphi_m^+(t)\hat{J}(t)\varphi_n(t) .$$

(17)
These elements are connected with each other through equation (12) yielding
\[
\frac{d}{dt} M_{mn}(t) = \sum_{k} \left[ J_{mk}(t) M_{kn}(t) + M_{mk}(t) \varphi_k^+(t) \frac{d\varphi_n(t)}{dt} - \varphi_m^+(t) \frac{d\varphi_k(t)}{dt} M_{kn}(t) \right],
\] (18)
where equality (16) has been invoked. The initial condition for equation (18) is \( M_{mn}(0) = \delta_{mn} \). The \( n \)-representation can be exploited for deriving several important properties.

**Proposition 1.** If the multiplier matrix \( \hat{M}(t) \) possesses the eigenvalues \( \mu_n(t) \), given by the eigenproblem
\[
\hat{M}(t) \varphi_n(t) = \mu_n(t) \varphi_n(t),
\] (19)
with the eigenvectors composing a complete orthonormal basis, then the eigenvalues
\[
\mu_n(t) = \exp \left\{ \int_0^t J_{nn}(t') dt' \right\}
\] (20)
are expressed through the diagonal elements \( J_{nn}(t) \) of the Jacobian matrix.

Proof is given in Appendix A. Note that the Jacobian matrix is, in general, nondiagonal.

**Proposition 2.** Suppose a complete orthonormal basis \( \{ \varphi_n(t) \} \) is such that
\[
\varphi_m^+(t) \frac{d\varphi_n(t)}{dt} = 0 \quad (m \neq n).
\] (21)
Then \( \varphi_n(t) \) are the eigenvectors of the multiplier matrix \( \hat{M}(t) \) if and only if they are also eigenvectors of the Jacobian matrix \( \hat{J}(t) \), with the corresponding eigenvalues \( \mu_n(t) \) and \( J_n(t) \) connected by the relation
\[
\mu_n(t) = \exp \left\{ \int_0^t J_n(t') dt' \right\}.
\] (22)

Proof is in Appendix B. The difference between equations (20) and (22) is in the form of the Jacobian matrix. In equation (20), \( J_{nn}(t) \) is a diagonal element of a generally nondiagonal matrix. While in equation (22), \( J_n(t) \) is an eigenvalue of the Jacobian matrix, which hence is diagonal. In other words, theorem 2 tells that, if condition (21) is satisfied, then the multiplier matrix is diagonal if and only if the Jacobian matrix is diagonal.

**Proposition 3.** Let the Jacobian matrix have the limit
\[
\lim_{t \to \infty} \hat{J}(t) = \hat{J}
\] (23)
and let \( J_n \) be the eigenvalues of \( \hat{J} \). Then the limit
\[
\lim_{t \to \infty} \frac{d}{dt} X(t) = \sum_n \lambda_n
\] (24)
is the sum of the Lyapunov exponents

\[ \lambda_n = \text{Re} J_n = \lim_{t \to \infty} \frac{1}{t} \ln |\mu_n(t)|. \quad (25) \]

Proof is in Appendix C. The sum of the Lyapunov exponents gives the entropy production rate for stationary states \([1,2]\). Hence, the left-hand side of equation (24) has the meaning of the entropy production rate. Recall that summing up only positive Lyapunov exponents leads to the Kolmogorov-Sinai entropy rate \([3]\). Contrary to the entropy production, defined on the basis of the Gibbs entropy \([1,2]\), the local expansion exponent (3) does not meet the problem of singular measure and it has sense not only for stationary states but for any strongly nonequilibrium states.

3 Problem of pattern selection

The local expansion exponent turns out to be a crucial notion for solving the problem of pattern selection. This problem arises when the evolution equations for a physical process possess a multiplicity of solutions corresponding to different spatio-temporal structures \([12]\). When all these solutions are stable, it is not clear how one could distinguish between them and, respectively, what spatio-temporal structures should be treated as more preferable. At the same time, nature does prefer some structures over the others, since some of them arise more often for the given experimental protocol. This problem of pattern selection can be solved as follows.

Let the multiplicity of solutions be parametrized by a label \(\beta\) pertaining to a label manifold \(\mathbb{B} = \{ \beta \}\). The submanifol of \(\mathbb{B}\) related to stable solutions is called \([12]\) stability balloon. Each dynamic state \(y(\beta,t)\), labelled by \(\beta\), corresponds to a particular spatio-temporal pattern. Because of the parametrization of dynamic states by the label \(\beta\), the dependence on this label also comes through the multiplier matrix \(\hat{M}(\beta,t)\), Jacobian matrix \(\hat{J}(\beta,t)\), and the local expansion exponent \(X(\beta,t)\). Since nature distinguishes among possible patterns differentiating them on more or less preferable, there should exist a probability distribution classifying these patterns as more or less preferable. Such a probability distribution of patterns \(p(\beta,t)\) can be obtained by minimizing an information functional

\[ I_p(t) = \int I_p(\beta,t) \, d\beta , \quad \int p(\beta,t) \, d\beta = 1 , \]

under the normalization condition on the probability distribution, where the integration is over the label manifold \(\mathbb{B}\).

The information density \(I_p(\beta,t)\) can be found from the following requirements: The existence of an invariant probability measure

\[ p(\beta,t) |\delta \Gamma(\beta,t)| = p(\beta,0) |\delta \Gamma(\beta,0)| \quad (26) \]

and the occurrence of an invariant information measure

\[ I_p(\beta,t) |\delta \Gamma(\beta,t)| = I_p(\beta,0) |\delta \Gamma(\beta,0)| . \quad (27) \]
These equalities tell us that the probability of a pattern in an elementary phase volume and the amount of information in such a phase volume are temporal invariants. We also require that at the initial time \( t = 0 \) the information density be given by the standard Shannon form

\[
I_p(\beta, 0) = p(\beta, 0) \ln p(\beta, 0) .
\]  

(28)

Then, from the invariance of the probability measure (26), we have

\[
p(\beta, 0) = p(\beta, t) \left| \frac{\delta \Gamma(\beta, t)}{\delta \Gamma(\beta, 0)} \right| .
\]

Substituting this into the Shannon information density (28) and using the invariance of the information measure (27), we obtain the information density of patterns

\[
I_p(\beta, t) = p(\beta, t) \left[ \ln p(\beta, t) + X(\beta, t) \right] .
\]

(29)

Thus, the information functional to be minimized is the pattern information

\[
I_p(t) = \int p(\beta, t) \ln p(\beta, t) \, d\beta + \int p(\beta, t) X(\beta, t) \, d\beta .
\]

(30)

The minimization of the pattern information (30), under the normalization condition on \( p(\beta, t) \), implies the minimization of the conditional information

\[
\tilde{I}_p(t) = I_p(t) + l(t) \left[ \int p(\beta, t) \, d\beta - 1 \right] ,
\]

(31)

in which

\[
l(t) \equiv \ln Z(t) - 1
\]

is a Lagrange multiplier. Minimizing information (31) results in the pattern distribution

\[
p(\beta, t) = \frac{1}{Z(t)} \exp \{ -X(\beta, t) \} ,
\]

(32)

with the normalization factor

\[
Z(t) = \int \exp \{ -X(\beta, t) \} \, d\beta .
\]

The expansion exponent, according to equations (10) and (15), can be determined by one of the forms

\[
X(\beta, t) = \ln | \text{det} \dot{M}(\beta, t) | , \quad X(\beta, t) = \text{Re} \int_0^t \text{Tr} \dot{J}(\beta, t') \, dt' .
\]

(33)

With the first of these forms, the pattern distribution (32) becomes

\[
\frac{p(\beta, t)}{| \text{det} \dot{M}(\beta, t) |} = \frac{Z^{-1}(t)}{| \text{det} \dot{M}(\beta, t) |} ,
\]

(34)
where the normalization factor is

\[ Z(t) = \int \frac{d\beta}{|\det M(\beta, t)|}. \]

The expression for the pattern distribution (32) shows that those patterns are more preferable whose expansion exponents are smaller. The most probable pattern corresponds to the minimal expansion exponent,

\[ \max_\beta p(\beta, t) \iff \min_\beta X(\beta, t), \quad (35) \]

which constitutes the principle of minimal expansion.

The probabilistic approach to pattern selection was advanced in Refs. [10,11], where the pattern distribution was rather postulated by invoking analogies with statistical mechanics. Here we have shown how this pattern distribution can be derived from the minimization of the pattern information. The probabilistic approach to pattern selection was applied to the problem of turbulent photon filamentation [11,13], with theoretical description being in a very good agreement with all experimental observations. This approach makes also the basis for the method of self-similar prediction [14–16] applied for analysing and forecasting possible scenarios in the behaviour of nonequilibrium complex systems, such as markets. In the latter case, the scenario distribution of form (34) was employed.

4 Stochastic dynamical systems

The notion of expansion exponents can be generalized to stochastic dynamical systems. Suppose the evolution equations include a stochastic field \( \xi(t) = [\xi_i(x, t)] \). Then the dependence on this field enters the dynamic state \( y(\xi, t) = [y_i(x, \xi, t)] \) as well as the velocity field \( v(y, \xi, t) = [v_i(x, y, \xi, t)] \). All these quantities are again treated as columns with respect to the multi-index \( \{i, x\} \) spanning the label space \( \mathbb{N}_+ \times \mathbb{D} \). The set of evolution equations acquires the form

\[ \frac{d}{dt} y(\xi, t) = v(y, \xi, t), \quad (36) \]

with the initial condition

\[ y(\xi, 0) = y(0). \quad (37) \]

The observable quantities are obtained after averaging over the stochastic fields. Denoting the stochastic averaging through the double angle brackets \( \ll \ldots \gg \), we have

\[ y(t) \equiv \ll y(\xi, t) \gg. \quad (38) \]

In the elementary phase volume (2), we now need to set

\[ \delta y_i(x, t) = \ll \delta y_i(x, \xi, t) \gg. \quad (39) \]

Then the same definition (3) for the local expansion exponent holds true.
Introducing the stochastic multiplier matrix \( \hat{M}(\xi, t) = [M_{ij}(x, x', \xi, t)] \), with the elements
\[
M_{ij}(x, x', \xi, t) \equiv \frac{\delta y_i(x, \xi, t)}{\delta y_j(x', 0)}
\] (40)
and the initial condition
\[
M_{ij}(x, x', \xi, 0) = \delta_{ij} \delta(x - x') ,
\] (41)
we may define the average multiplier matrix
\[
\hat{M}(t) \equiv \langle \hat{M}(\xi, t) \rangle .
\] (42)
Then expressions (7) and (10) for the expansion exponent retain their sense, with the multiplier matrix (42).

The stochastic Jacobian matrix is defined as \( \hat{J}(\xi, t) = [J_{ij}(x, x', \xi, t)] \), whose elements are
\[
J_{ij}(x, x', \xi, t) \equiv \frac{\delta v_i(x, y, \xi, t)}{\delta y_j(x', \xi, t)}
\] (43)
The variation of the evolution equation (36) yields the equation
\[
\frac{d}{dt} \hat{M}(\xi, t) = \hat{J}(\xi, t)\hat{M}(\xi, t)
\] (44)
for the stochastic multiplier matrix (40), the initial condition being
\[
\hat{M}(\xi, 0) = \hat{1} .
\] (45)
The averaging of equation (44) gives
\[
\frac{d}{dt} \hat{M}(t) = \langle \hat{J}(\xi, t)\hat{M}(\xi, t) \rangle .
\]
Since the right-hand side here, in general, cannot be factorized, there is no presentation for the expansion exponent similar to equation (15). Therefore, for stochastic dynamical systems, the form (10) of the expansion exponent remains the main, where the multiplier matrix is given by equation (42).

With the help of an orthonormal complete basis \( \{\varphi_n(t)\} \), we may pass to the \( n \)-representation
\[
M_{mn}(\xi, t) \equiv \varphi_m^+\hat{M}(\xi, t)\varphi_n(t) , \quad J_{mn}(\xi, t) \equiv \varphi_m^+\hat{J}(\xi, t)\varphi_n(t) ,
\] (46)
which is analogous to definition (17). The elements (46) are connected by the same equation (18). If the eigenproblem
\[
\hat{M}(\xi, t)\varphi_n(t) = \mu_n(\xi, t)\varphi_n(t)
\] (47)
holds true, then the average multiplier matrix (42) possesses the same eigenvectors \( \varphi_n(t) \), with the eigenvalues
\[
\mu_n(t) = \langle \mu_n(\xi, t) \rangle .
\] (48)
It is straightforward to reformulate the theorems 1 and 2 for the stochastic multiplier matrix $\hat{M}(\xi,t)$:

**Proposition 1.** If the stochastic multiplier matrix $\hat{M}(\xi,t)$ possesses the eigenvalues $\mu_n(\xi,t)$, given by the eigenproblem (47), with the eigenvectors forming a complete orthonormal basis, then the eigenvalues

$$\mu_n(\xi,t) = \exp \left\{ \int_0^t J_{nn}(\xi,t')\,dt' \right\} \quad \text{(49)}$$

are expressed through the diagonal elements $J_{nn}(\xi,t)$ of the stochastic Jacobian matrix.

**Proposition 2.** Assume a complete orthonormal basis $\{\varphi_n(t)\}$ is such that condition (21) is valid. Then the stochastic multiplier matrix $\hat{M}(\xi,t)$ is diagonal in the $n$-representation if and only if the stochastic Jacobian matrix is diagonal,

$$M_{mn}(\xi,t) = \mu_n(\xi,t) \delta_{mn} , \quad J_{mn}(\xi,t) = J_n(\xi,t) \delta_{mn} ,$$

their eigenvalues being connected by the relation (49).

But theorem 3 is not valid for stochastic dynamical systems, since $\hat{J}(\xi,t)$, in general, is not defined for $t \to \infty$. Thus, to find the expansion exponent, we have to resort to formula (10). When the stochastic multiplier matrix satisfies the eigenproblem (47), the expansion exponent (10) reduces to the expression

$$X(t) = \sum_n \ln | \ll \mu_n(\xi,t) \gg | . \quad \text{(50)}$$

This is the principal form of the expansion exponent for stochastic dynamical systems.

## 5 Exponents for stochastic systems

Now we shall illustrate how the expansion exponents can be calculated for several examples of stochastic dynamical systems. We shall consider the stochastic field $\xi(t)$ as a real-valued random Gaussian variable, which is centered at zero,

$$\ll \xi(t) \gg = 0 .$$

For the case of coloured noise, the correlation function

$$\ll \xi(t)\xi(t') \gg = C(t-t')$$

can be written [17] as

$$C(t) = C_0 \exp \left( - \frac{|t|}{\tau} \right) .$$

Setting $C_0 = (\gamma_1/\tau + 2\gamma_2^2)$, we may consider two opposite limits, that of a very short correlation time $\tau$ and of a long correlation time. In the short correlation time limit $\tau \to 0$, taking into account that

$$\frac{1}{2\tau} \exp \left( - \frac{|t|}{\tau} \right) \simeq \delta(t) \quad (\tau \to 0) ,$$
we have
\[ \lim_{\tau \to 0} C(t) = 2\gamma_1 \delta(t) . \]
Such a delta-correlated noise is called white because of the uniformity of its spectral function. In the opposite limit of a long correlation time \( \tau \to \infty \), we get
\[ \lim_{\tau \to \infty} C(t) = 2\gamma_2^2 . \]
This kind of noise can be called infrared, since its spectral function is centered at zero energy. In the general case, one should deal with the coloured noise with a finite correlation time. To simplify calculations, at the same time keeping tracks of two admissible limits of short and long correlation times, one can accept the correlation function of the form
\[ C(t) = 2\gamma_1 \delta(t) + 2\gamma_2^2 , \]
representing a mixture of white and infrared noises. In what follows, we shall consider the Gaussian stochastic fields with the correlation function
\[ \langle \xi(t)\xi(t') \rangle = 2\gamma_1 \delta(t-t') + 2\gamma_2^2 , \]
which includes both limiting cases. Setting here \( \gamma_1 \to 0 \), we get the infrared noise, while for \( \gamma_2 \to 0 \), we come to the white noise.

Stochastic differential equations, as is known, can be interpreted either in the sense of Ito or in the sense of Stratonovich [17,18]. We prefer to deal with the Stratonovich interpretation, which is better motivated physically. Another possibility could be to employ the stochastic expansion technique [19,20], by presenting stochastic fields as expansions over smooth functions with random coefficients. This method allows for the usage of the standard differential and integration analysis. The final results of the expansion technique coincide with the corresponding expressions derived by means of the Stratonovich method.

### 5.1 Oscillator with random attenuation

Let us start with a case of an one-dimensional dynamical system described by the evolution equation
\[ \frac{dy}{dt} = (i\omega - \Gamma + \xi) y , \]
in which \( y = y(\xi,t), \xi = \xi(t), \omega \) and \( \Gamma \) are positive parameters. This equation corresponds to an oscillator with random attenuation. The Jacobian related to equation (52) is
\[ J(\xi,t) = i\omega - \Gamma + \xi(t) , \]
because of which the multiplier is
\[ \mu(\xi, t) = \exp \left\{ (i\omega - \Gamma) t + \int_0^t \xi(t') dt' \right\} . \]
For the Gaussian random variables, one has the averaging property

$$\ll \exp \left\{ \alpha \int_0^t \xi(t') \, dt' \right\} \gg = \exp \left\{ \frac{\alpha^2}{2} \int_0^t \ll \xi(t') \xi(t'') \gg \, dt' \, dt'' \right\},$$

where $\alpha$ is a complex number. Then for the local expansion exponent (50), we find

$$X(t) = -\Gamma t + \gamma_1 t + \gamma_2^2 t^2.$$  \hspace{1cm} (55)

If there is no noise, that is $\gamma_1 = \gamma_2 = 0$, the motion is contracting. But in the presence of noise such that either $\gamma_1 > \Gamma$ and $\gamma_2 = 0$ or $\gamma_2 \neq 0$ for any $\gamma_1$, the dynamics is expanding. As is seen, the existence of noise can drastically change the system dynamics.

### 5.2 Stochastic diffusion equation

Consider an infinite-dimensional dynamical system presented by the diffusion equation with a random diffusion coefficient,

$$\frac{\partial y}{\partial t} = (D + \xi) \frac{\partial^2 y}{\partial x^2},$$

where $y = y(x, \xi, t)$, $\xi = \xi(t)$, and $D > 0$. For the spatial variable $x$ given on a finite interval, it is always possible, by an appropriate scaling, to reduce this interval to the unit one, so that $x \in [0, 1]$. The initial condition is

$$y(x, \xi, 0) = y(x, 0).$$  \hspace{1cm} (57)

And let the boundary conditions be

$$y(0, \xi, t) = b_0, \quad y(1, \xi, t) = b_1.$$  \hspace{1cm} (58)

From here, the boundary conditions for the multiplier matrix are

$$M(0, x', \xi, t) = M(1, x', \xi, t) = 0.$$  \hspace{1cm} (59)

Hence, if the multiplier matrix possesses the eigenvectors $\varphi_n = [\varphi_n(x)]$, their components $\varphi_n(x)$ should satisfy the boundary conditions

$$\varphi_n(0) = \varphi_n(1) = 0.$$  \hspace{1cm} (60)

For the Jacobian matrix, associated with equation (56), we have

$$J(x, x', \xi) = (D + \xi) \frac{\partial^2}{\partial x^2} \delta(x - x').$$

The eigenproblem for the Jacobian matrix reads

$$\int_0^1 J(x, x', \xi) \varphi_n(x') \, dx' = J_n(\xi) \varphi_n(x).$$
The eigenfunctions, satisfying the boundary conditions (60), are
\[ \varphi_n(x) = \sqrt{2} \sin(\pi nx) \quad (n = 1, 2, \ldots) . \]

And the eigenvalues of the Jacobian matrix are
\[ J_n(\xi) = -(D + \xi)\pi^2 n^2 . \]  
(62)

The eigenvectors \( \varphi_n \), not depending on time, satisfy condition (21). Then, by theorem 2, the multiplier matrix possesses the same eigenvectors, with the eigenvalues (49), which yields
\[ \mu_n(\xi, t) = \exp \left\{ -\pi^2 n^2Dt - \pi^2 n^2 \int_0^t \xi(t') \, dt' \right\} . \]  
(63)

In this way, the expansion exponent (50) becomes
\[ X(t) = \sum_n \left[ -\pi^2 n^2 Dt + \pi^4 n^4 \left( \gamma_1 t + \gamma_2 t^2 \right) \right] . \]  
(64)

Using the sums
\[ \sum_{n=1}^N n^2 = \frac{N}{6} \left( 2N^2 + 3N + 1 \right), \quad \sum_{n=1}^N n^4 = \frac{N}{30} \left( 6N^4 + 15N^3 + 10N^2 - 1 \right), \]
we obtain
\[ X(t) \simeq -\frac{\pi^2}{3} N^3 Dt + \frac{\pi^4}{5} N^5 \left( \gamma_1 t + \gamma_2 t^2 \right) , \]  
(65)

where \( N \to \infty \). In the absence of noise, when \( \gamma_1 = \gamma_2 = 0 \), the system is contracting. But if either \( \gamma_1 \) or \( \gamma_2 \) are not zero, the phase volume expands.

### 5.3 Stochastic Schrödinger equation

The Schrödinger equation, generated by a Hamiltonian \( H \), with added stochastic background, can be written in the form
\[ \frac{\partial \psi}{\partial t} = (-iH + \xi)\psi , \]  
(66)

which is called normal for dynamical systems. Here \( \psi = \psi(r, \xi, t), \xi = \xi(t), H = H(r) \), and \( \hbar = 1 \).

For the Jacobian matrix, we have
\[ J(r, r', \xi) = (-iH + \xi) \delta(r - r') . \]  
(67)

Let the stationary Schrödinger equation be
\[ H \psi_n = E_n \psi_n \quad \left( \sum_n 1 \equiv N \right) . \]
The Jacobian matrix \( \hat{J}(\xi) \), with the elements (67), possesses the eigenvectors \( \psi = [\psi_n(x)] \)
and the eigenvalues
\[
J_n(\xi) = -iE_n + \xi .
\] (68)

Since the eigenvectors \( \psi \) do not depend on time, condition (21) is valid. Then, by theorem 2, we find the eigenvalues
\[
\mu_n(\xi, t) = \exp \left\{ -iE_n t + \int_0^t \xi(t') \, dt' \right\}
\] (69)
of the multiplier matrix. And for the expansion exponent (50), we find
\[
X(t) = \left( \gamma_1 t + \gamma_2 t^2 \right) N .
\] (70)

When the noise is absent, the phase volume is conserved. But as soon as either \( \gamma_1 \) or \( \gamma_2 \) are not zero, the dynamical system expands.

6 Exponents for quasi-isolated systems

All real physical systems are never completely isolated, but are subject to weak uncontrollable random perturbations from surrounding. This fact has been repeatedly emphasized and discussed in literature [6,9,22–25]. Even more, as has been stressed [26,27], the notion of an isolated system as such is logically self-contradictory. This is because to practically realize the isolation one has to use technical devices acting on the system, and to ensure that the latter is kept isolated, one must apply measuring instruments perturbing the system. The preparation and registration processes may essentially influence the system evolution [5,28]. The impossibility of isolating macroscopic systems from their environments is often considered as the cause of the increase of entropy required by the second law of thermodynamics and the related irreversibility of time arrow [28].

To analyze the influence of weak external noise on the system dynamics, it is useful to introduce the notion of quasi-isolated systems [8]. A physical system is called quasi-isolated if it is subject to the action of very small stochastic perturbations modelling the random influence of surrounding. In order to explicitly show that stochastic perturbations are weak, they can be included in the evolution equations with a small factor \( \alpha \ll 1 \), so that such equations take the form
\[
\frac{d}{dt} y(\alpha \xi, t) = v(y, \alpha \xi, t) .
\] (71)

As a result, the multiplier matrix \( \hat{M}(\alpha \xi, t) \) and the Jacobian matrix \( \hat{J}(\alpha \xi, t) \) also contain this small parameter. And the local expansion exponent becomes
\[
X(\alpha, t) = \ln |\det \ll \hat{M}(\alpha \xi, t) \gg | .
\] (72)

An isolated system could be treated as the asymptotic limit of the related quasi-isolated system, as \( \alpha \to 0 \). However, there is a very gentle point in taking such an
asymptotic limit. A system, being considered for long time, corresponds to the temporal limit $t \to \infty$. The limits $\alpha \to 0$ and $t \to \infty$ may turn to be noncommuting! If these limits for the expansion exponent (72) do not commute, so that

$$[\lim_{t \to \infty}, \lim_{\alpha \to 0}] X(\alpha, t) \neq 0,$$  \hspace{1cm} (73)

then the notion of an isolated system has not much sense, since infinitesimally small random perturbations principally change the system behaviour. Condition (73), when it holds true, can be accepted as a mathematical formulation for the principle of nonexistence of isolated systems.

For illustration, we may employ the examples of Section 5. Thus, for the stochastic oscillator of subsection 5.1 we have

$$X(\alpha, t) = -\Gamma t + \alpha^2 \left( \gamma_1 t + \gamma_2^2 t^2 \right).$$  \hspace{1cm} (74)

This gives us the limits

$$\lim_{t \to \infty} \lim_{\alpha \to 0} X(\alpha, t) = -\infty, \quad \lim_{\alpha \to 0} \lim_{t \to \infty} X(\alpha, t) = +\infty,$$  \hspace{1cm} (75)

where it is assumed that $\gamma_2 \neq 0$. For the stochastic diffusion equation of subsection 5.2, we get

$$X(\alpha, t) \simeq -\frac{\pi^2}{3} N^3 D t + \alpha^2 \frac{\pi^4}{5} N^5 \left( \gamma_1 t + \gamma_2^2 t^2 \right),$$  \hspace{1cm} (76)

from where it follows that

$$\lim_{t \to \infty} \lim_{\alpha \to 0} X(\alpha, t) = -\infty, \quad \lim_{\alpha \to 0} \lim_{t \to \infty} X(\alpha, t) = +\infty.$$  \hspace{1cm} (77)

And for the stochastic Schrödinger equation of subsection 5.3, we find

$$X(\alpha, t) = \alpha^2 \left( \gamma_1 t + \gamma_2^2 t^2 \right) N,$$  \hspace{1cm} (78)

which yields the limits

$$\lim_{t \to \infty} \lim_{\alpha \to 0} X(\alpha, t) = 0, \quad \lim_{\alpha \to 0} \lim_{t \to \infty} X(\alpha, t) = +\infty.$$  \hspace{1cm} (79)

As we see, irrespectively of whether the system without random perturbation has been phase conserving or contracting, it turns, in the long run, to an expanding system as soon as it is influenced by a noise, no matter how weak the noise is.

In this way, at finite times the local expansion exponent can be negative or zero; for some systems it may, probably, fluctuate, similarly to entropy fluctuations in macroscopic kinetics [29].

But there always exist such small random perturbations that at sufficiently long times the expansion exponent becomes positive. This suggests to formulate the principle of asymptotic phase expansion,

$$X(t) > 0 \quad (t \to \infty),$$  \hspace{1cm} (80)
telling that the expansion exponents of quasi-isolated systems become positive at asymptotically large times. Thus, any real physical system can be treated as approximately isolated only for a finite time interval. But sooner or later, the influence of its uncontrollable stochastic environment will prevail and the system phase volume will start expanding.

The principle of asymptotic expansion (80) is explicitly related to the direction of time. Therefore, the irreversibility of time arrow can be treated as a consequence of this principle. One often connects the irreversibility of time with the increase of entropy postulated for isolated systems by the second law of thermodynamics. The increase of entropy could be due to the internal chaotic nature of the microscopic dynamics [7]. However, as is recently reviewed by Zaslavsky [30], chaotic dynamics in real systems does not provide finite relaxation time to equilibrium or fast decay of fluctuations, and chaotic systems are not completely random in the sense originally postulated for statistical systems. Chaotic systems possess a property different from the regular understanding of randomness, a property called persistence of nonequilibrium [30]. Thus, chaotic dynamics of isolated systems cannot provide an explanation for the increase of entropy. Moreover, there is no effective definition of entropy for systems far from equilibrium [2]. Contrary to entropy, the local expansion exponent is defined not only for statistical systems but for arbitrary dynamical systems, strongly as well as weakly nonequilibrium. An important fact is that completely isolated systems do not exist in nature. Any real system can be only quasi-isolated, being subject to uncontrollable random influence of its environment. Although the internal chaotic dynamics, if any, may play the role, but the main cause for the property of asymptotic expansion (80) is the influence of stochastic surrounding. It is this asymptotic phase expansion that indicates the direction of time and makes the time arrow irreversible. The definition of the expansion exponent does not invoke any thermodynamic or statistical notions, but is valid for arbitrary dynamical systems. This is why the irreversibility of time arrow is not a privilege of only statistical and thermodynamic systems, but it is a common property of all evolution processes.

7 Conclusion

The notion of local expansion exponents is introduced being valid for arbitrary dynamical systems, including stochastic dynamical systems. This notion plays a fundamental role in the problem of pattern selection. The probabilistic approach to pattern selection yields the principle of minimal expansion exponent, according to which the most probable pattern at a given time corresponds to the minimal local expansion exponent.

Considering the limits of large times and of weak stochastic sources, as applied to the expansion exponent, makes it possible to give a mathematical formulation for the principle of nonexistence of isolated systems, when these two limits do not commute with each other.

Several examples of quasi-isolated systems suggest, as a plausible generalization, the principle of asymptotic phase expansion, stating that the expansion exponents of quasi-isolated systems become positive at sufficiently long times.

The positive definiteness of the expansion exponent at asymptotically large times
implies the expansion of the system phase volume. The relation between the asymptotic phase expansion and time defines the direction of time and provides the foundation for the principle of *irreversibility of time arrow*, valid for all evolution processes.

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Appendix A: Proof of Proposition 1

When the multiplier matrix satisfies the eigenproblem (19), its elements, defined in equation (17), are diagonal,

\[ M_{mn}(t) = \delta_{mn} \mu_n(t) . \]

Substituting this into equation (18) gives

\[ \delta_{mn} \frac{d}{dt} \mu_n(t) = J_{mn}(t) \mu_n(t) + [\mu_m(t) - \mu_n(t)] \varphi_m^+(t) \frac{d\varphi_n(t)}{dt} , \]

where the property (16) has been used. For \( m = n \), we get the equation

\[ \frac{d}{dt} \mu_n(t) = J_{nn}(t) \mu_n(t) , \]

with the initial condition

\[ \mu_n(0) = 1 , \]

following from condition (13). The solution of the latter equation results in the eigenvalues (20) containing the diagonal elements of the Jacobian matrix. In general, the latter matrix is not diagonal, with the nondiagonal elements being

\[ J_{mn}(t) = \left[ 1 - \frac{\mu_m(t)}{\mu_n(t)} \right] \varphi_m^+(t) \frac{d\varphi_n(t)}{dt} , \]

where \( m \neq n \).

Appendix B: Proof of Proposition 2

If condition (21) is valid, then equation (18) becomes

\[ \frac{d}{dt} M_{nn}(t) = \sum_k J_{mk}(t) M_{kn}(t) + M_{nn}(t) \left[ \varphi_n^+(t) \frac{d\varphi_n(t)}{dt} - \varphi_m^+(t) \frac{d\varphi_m(t)}{dt} \right] . \]

Suppose that \( \varphi_n(t) \) are the eigenvectors of the multiplier matrix \( \dot{M}(t) \), so that the eigenproblem (19) holds. Then the above equation yields

\[ \delta_{mn}(t) \frac{d}{dt} \mu_n(t) = J_{mn}(t) \mu_n(t) , \]

from where it follows that the Jacobian matrix is diagonal

\[ J_{nn}(t) = \delta_{nn} \mu_n(t) , \]

that is, \( \varphi_n(t) \) are also the eigenvectors of \( \dot{J}(t) \). The eigenvalues \( \mu_n(t) \) and \( J_n(t) \) are related by expression (22).

Let now \( \varphi_n(t) \) be the eigenvectors of the Jacobian matrix \( \dot{J}(t) \). Then equation (18) can be solved as

\[ M_{nn}(t) = M_{nn}(0) \exp \left\{ \int_0^t \left[ J_m(t') + \varphi_n^+(t') \frac{d\varphi_n(t')}{dt'} - \varphi_m^+(t') \frac{d\varphi_m(t')}{dt'} \right] dt' \right\} . \]
From here, with the initial condition $M_{mn}(0) = \delta_{mn}$, we have

$$M_{mn}(t) = \delta_{mn} \exp \left\{ \int_0^t J_n(t') \, dt' \right\},$$

which shows that the multiplier matrix is diagonal. Hence, $\varphi_n(t)$ are also the eigenvectors of $\hat{M}(t)$, with the eigenvalues (22).

**Appendix C**: Proof of Proposition 3

Differentiating the local expansion exponent (15) gives

$$\frac{d}{dt} X(t) = \text{Re} \, \text{Tr} \hat{J}(t)$$

for all $t \geq 0$. Due to the existence of the limit (23),

$$\lim_{t \to \infty} \frac{d}{dt} X(t) = \text{Re} \, \text{Tr} \hat{J}.$$

With $J_n$ being the eigenvalues of $\hat{J}$, one has

$$\text{Re} \, \text{Tr} \hat{J} = \sum_n \text{Re} J_n = \sum_n \lambda_n,$$

where $\lambda_n \equiv \text{Re} J_n$ are the Lyapunov exponents. The eigenvectors of $\hat{J}$ do not depend on time, hence they satisfy condition (21). Then, by theorem 2, the limit of $\hat{M}(t)$ at $t \to \infty$ possesses the same eigenvectors as $\hat{J}$. At asymptotically large $t$, because of representation (10) one has

$$X(t) \approx \sum_n \ln |\mu_n(t)| \quad (t \to \infty).$$

And relation (22) gives

$$\ln |\mu_n(t)| \approx \text{Re} \left\{ \int_0^t J_n(t') \, dt' \right\} \quad (t \to \infty).$$

Therefore

$$\lim_{t \to \infty} \frac{1}{t} \ln |\mu_n(t)| = \text{Re} J_n = \lambda_n.$$

Thus, we prove equations (24) and (25).
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