IDENTIFICATION PROBLEMS OF RETARDED DIFFERENTIAL SYSTEMS IN HILBERT SPACES

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Abstract. This paper deals with the identification problem for the \(L^1\)-valued retarded functional differential equation. The unknowns are parameters and operators appearing in the given systems. In order to identify the parameters, we introduce the solution semigroup and the structural operators in the initial data space, and provide the representations of spectral projections and the completeness of generalized eigenspaces. The sufficient condition for the identification problem is given as the so called rank condition in terms of the initial values and eigenvectors of adjoint operator.

1. Introduction. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). Let \( \mathcal{A}(x, D_x) \) be an elliptic differential operator of second order in \( L^1(\Omega) \):

\[
\mathcal{A}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} (a_{i,j}(x) \frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x).
\]

In this paper, we consider the inverse problem for the following retarded functional differential equation defined as \( A_0 u = -\mathcal{A}(x, D_x) u \):

\[
\begin{cases}
    u'(t) = A_0 u(t) + \gamma A_0 u(t - h) + \int_{-h}^{0} a(s)A_0 u(t + s)\, ds, \\
    u(0) = g^0, \quad u(s) = g^1(s), \quad s \in [-h, 0).
\end{cases}
\]  \hspace{1cm} (1.1)

Here \( A_0, \gamma, \) and \( a(\cdot) \) are unknown quantities to be identified and the initial condition \( g = (g^0, g^1) \) is known.

In the field of control engineering, identifiability as a kind of inverse problems, or the parameter estimations of systems has attracted much interest and has been investigated in many references, for example, as for one dimensional heat equation with an unknown spatially-varying conductivity in [1-4], an abstract linear first order evolution equation within the framework of operator theory in [15], and linear retarded functional differential systems in reflexive Banach spaces in [6-8]. In [8, 9]

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the author discussed the control problem for the following retarded system with \( L^1(\Omega) \)-valued controller:

\[
 u'(t) = A_0 u(t) + A_1 u(t-h) + \int_{-h}^{0} a(s) A_2 u(t+s) \, ds + \Phi_0 w(t),
\]

where \( A_i (i = 1, 2) \) are second order linear differential operators with real coefficients, and the controller \( \Phi_0 \) is a bounded linear operator from a control Banach space to \( L^1(\Omega) \). In [6], they established some results concerning identification problems for (1.1) of specific form by taking the observation. Furthermore, Yamamoto and Nakagiri [26] studied the identifiability problem for evolution equations in Banach spaces with unknown operators and initial values by means of spectral theory for linear operators.

In view of Sobolev’s embedding theorem we may also consider \( L^1(\Omega) \subset W^{-1,p}(\Omega) \) if \( 1 \leq p < n/(n-1) \) as is seen in [8]. Hence, we can investigate the system (1.1) in the space \( W^{-1,p}(\Omega) \) considering \( \Phi_0 \) as an operator into \( W^{-1,p}(\Omega) \). Here, we note that the space \( W^{-1,p}(\Omega) \) is \( \zeta \)-convex (as for the definition and fundamental facts of a \( \zeta \)-convex see [2, 3]). Consequently, in view of Dore and Venni [7] the maximal regularity for the linear initial value problem:

\[
 u'(t) = A_0 u(t) + f(t), \quad u(0) = u_0
\]

in the space \( W^{-1,p}(\Omega) \) holds true.

Furthermore, with the aid of a result by Seeley [21] and [8], we can obtain the maximal regularity for solutions of the retarded linear initial value problem (1.1) in the space \( W^{-1,p}(\Omega) \). In view of these results, we deal with an identification problem of (1.1) in \( W^{-1,p}(\Omega) \).

The paper is organized as follows. Section 2 presents some notations.

In Section 3 from the definitions of operator \( A_0 \) and the interpolation theory as in Theorem 3.5.3 of Butzer and Berens [4], we can apply Theorem 3.2 of Dore and Venni [7] to general linear Cauchy problem in the space \( W^{-1,p}(\Omega) \). Thereafter, by using the method of Di Blasio et al. [5] to the system (1.2) with the forcing term \( f \) in place of the control term \( \Phi_0 w \), Section 4 is devoted to studying the wellposedness and regularity for solutions of (1.1) by using a solution semigroup \( S(t) \) in the initial data space \( Z_{p,q} = H^p \times L^q(-h, 0, W^{-1,p}(\Omega)) \), where \( H^p = (W^{1,p}_0(\Omega), W^{-1,p})_{1/q,q}(\Omega) \) for \( 1 < q < \infty \).

In Section 5, in order to identify the parameters, we investigate the spectrum of the infinitesimal generator \( \Lambda \) of \( S(t) \). We will give that the spectrum of \( \Lambda \) is composed of two parts of cluster points and discrete eigenvalues. Moreover, we are concerned with the representations of spectral projections and the problem of completeness of generalized eigenspaces. Based on this result, we establish a sufficient condition for the identification problem is given as the so called rank condition in terms of the initial values and eigenvectors of adjoint operator.

Finally we give a simple example to which our main result can be applied.

2. Notations. Let \( \Omega \) be a region in an \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) and closure \( \bar{\Omega} \). \( C^m(\Omega) \) is the set of all \( m \)-times continuously differential functions on \( \Omega \). \( C^m_0(\Omega) \) will denote the subspace of \( C^m(\Omega) \) consisting of these functions which have compact support in \( \Omega \). \( W^{m,p}(\Omega) \) is the set of all functions \( f = f(x) \) whose derivative \( D^m f \) up to degree \( m \) in distribution sense belong to \( L^p(\Omega) \) . As usual, the norm is then given by
is the formal adjoint of boundary condition by $A$. Let $A$ be a closed linear operator in a Banach space. Then $D(A)$ denotes the domain of $(A)$ and $R(A)$ the range of $A$. $\rho(A)$ denotes the resolvent set of $A$, $\sigma(A)$ the spectrum of $A$, and $\sigma_p(A)$ the point spectrum of $A$. The kernel or null space $\{x \in D(A) : Ax = 0\}$ of $A$ is denoted by $Ker(A)$.

3. Cauchy problems on $\zeta$-convex spaces. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. Consider an elliptic differential operator of second order

$$A(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x) \frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x)$$

where $(a_{i,j}(x) : i, j = 1, \ldots, n)$ is a positive definite symmetric matrix for each $x \in \Omega$, $a_{i,j} \in C^1(\Omega)$, $b_i \in C^1(\Omega)$ and $c \in L^\infty(\Omega)$. The operator

$$A'(x, D_x) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x) \frac{\partial}{\partial x_i}) - \sum_{i=1}^{n} a_{i,j}(b_i(x) \cdot) + c(x)$$

is the formal adjoint of $A$.

For $1 < p < \infty$ we denote the realization of $A$ in $L^p(\Omega)$ under the Dirichlet boundary condition by $A_p$:

$$D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$

$$A_p u = A u \quad \text{for} \quad u \in D(A_p).$$

For $p' = p/(p-1)$, we can also define the realization $A'$ in $L^{p'}(\Omega)$ under Dirichlet boundary condition by $A'_p$:

$$D(A'_p) = W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega),$$

$$A'_p u = A' u \quad \text{for} \quad u \in D(A'_p).$$
It is known that \(-A_p\) and \(-A_p'\) generate analytic semigroups in \(L^p(\Omega)\) and \(L^{p'}(\Omega)\), respectively, and \(A_p^* = A_p'\). For brevity, we assume that \(0 \in \rho(A_p)\). From the result of Seeley \[22\] (see also Triebel \[25, p. 321\]) we obtain that
\[
[D(A_p), L^p(\Omega)]_{\frac{1}{2}} = W_{0}^{1,p}(\Omega),
\]
and hence, may consider that
\[
D(A_p) \subset W_{0}^{1,p}(\Omega) \subset L^p(\Omega) \subset W^{-1,p}(\Omega) \subset D(A_p')^*.\]

Let \((A_p')'\) be the adjoint of \(A_p'\) considered as a bounded linear operator from \(D(A_p')\) to \(L^{p'}(\Omega)\). Let \(\tilde{A}\) be the restriction of \((A_p')'\) to \(W_{0}^{0,1,p}(\Omega)\). Then by the interpolation theory, the operator \(\tilde{A}\) is an isomorphism from \(W_{0}^{1,p}(\Omega)\) to \(W^{-1,p}(\Omega)\). Similarly, we consider that the restriction \((\tilde{A}')'\) of \((A_p')'\) to \(W_{0}^{0,1,p}(\Omega)\) is an isomorphism from \(W_{0}^{1,p}(\Omega)\) to \(W^{-1,p'}(\Omega)\). Furthermore, as seen in Proposition 3.1 in Jeong \[8\], we obtain the following result.

**Proposition 1.** The operators \(\tilde{A}\) and \((\tilde{A}')'\) generate analytic semigroups in \(W^{-1,p}(\Omega)\) and \(W^{-1,p'}(\Omega)\), respectively, and the inequality
\[
\|((\tilde{A})')_{s}\|_{B(W^{-1,p}(\Omega))} \leq C e^{c|s|}, \quad -\infty < s < \infty,
\]
holds for some constants \(C > 0\) and \(\gamma \in (0, \pi/2)\).

We set
\[
H_{p,q} = (W_{0}^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1}{2},q}, \quad q \in (1, \infty). \tag{3.1}
\]

Since \(\tilde{A}\) is an isomorphism from \(W_{0}^{1,p}(\Omega)\) onto \(W^{-1,p}(\Omega)\) and \(W_{0}^{1,q}(\Omega)\) and \(W^{-1,q}(\Omega)\) are \(\zeta\)-convex spaces, it is easily seen that \(H_{p,q}\) is also \(\zeta\)-convex. From the definitions of operator \(\tilde{A}\) and the interpolation space \(H_{p,q}\) as in Theorem 3.5.3 of Butzer and Berens \[4\], we can apply Theorem 3.2 of Dore and Venni \[7\] to general linear Cauchy problem as the following result.

**Proposition 2.** Let \((u_0, f) \in H_{p,q} \times L^q(0, T; W^{-1,p}(\Omega))(1 < q < \infty)\). Then the Cauchy problem
\[
u'(t) = \tilde{A}u(t) + f(t), \quad u(0) = u_0
\]
has a unique solution
\[
u \in L^q(0, T; W_{0}^{1,p}(\Omega)) \cap W^{1,q}(0, T; W^{-1,p}(\Omega)) \hookrightarrow C([0,T]; H_{p,q}).
\]

The last inclusion relation is well known and is an easy consequence of the definition of real interpolation spaces by the trace method.

4. **Retarded equations and lemmas.** In this section, we apply Propositions 3.1 and 3.2 to the retarded functional differential equation in the space \(W^{-1,p}(\Omega)\). Consider the following retarded equation in \(W^{-1,p}(\Omega)\):
\[
\begin{align*}
u'(t) &= A_0u(t) + A_1u(t-h) + \int_{t-h}^{t} a(s)A_2u(t+s)ds + f(t), \quad t \in (0, T] \tag{4.1} \\
u(0) &= g_0, \quad u(s) = g_1(s) \quad s \in [-h, 0).
\end{align*}
\]
Here, \( A_0 = -\bar{A} \), and \( A_t u (t = 1, 2) \) are the restrictions \( W_{0}^{1,p}(\Omega) \) of the linear differential operators \( \bar{A}_t (t = 1, 2) \) with real coefficients:
\[
\bar{A}_t (x, D_x) = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} (\alpha_{i,j}^t(x) \frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} b_i^t(x) + c^t(x),
\]
where
\[
\alpha_{i,j}^t, a_i^t \in C^1(\bar{\Omega}), \quad b_i^t \in C^1(\bar{\Omega}), \quad c^t \in L^\infty(\Omega),
\]
and \((\alpha_{i,j}^t), t = 1, 2\) are positive definite. The kernel \( a(\cdot) \) belongs to \( L^1 (-h, 0) \). For \( q \in (1, \infty) \) we set
\[
Z_{p,q} \equiv H_{p,q} \times L^q(-h, 0; W_{0}^{1,p}(\Omega)).
\]
Using Proposition 2 we can follow the argument as in [5] term by term to deduce the following result (see Proposition 4.1 of [8]).

**Lemma 4.1.** Given \( g = (g^0, g^1) \in Z_{p,q} \) and \( f \in L^q(0, T; W^{-1,p}(\Omega)) \). Then the problem (4.1) has a unique solution
\[
u \in L^q(0, T; W_{0}^{1,p}(\Omega))) \cap W^{1,q}(0, T; W^{-1,p}(\Omega))) \subset C([0, T]; H_{p,q}).
\]
Moreover, we have
\[
\|u\|_{L^q(0, T; W_{0}^{1,p}(\Omega)) \cap W^{1,q}(0, T; W^{-1,p}(\Omega))} \leq c(\|g^0\|_{H_{p,q}})
\]
\[
+ \|g^1\|_{L^q(-h, 0; W_{0}^{1,p}(\Omega))} + \|f\|_{L^q(0, T; W^{-1,p}(\Omega))},
\]
where \( c \) is a constant.

Let \( F \equiv 0 \) in (4.1) and consider the equation on whole \([0, \infty)\). Then by virtue of Lemma 4.1, we can define the solution semigroup \( S(t) (t \geq 0) \) for the system (4.1) as follows [5, Theorem 4.1] or [15, 18]:
\[
S(t) = (u(t; g), u_t(\cdot; g))
\]
where \( g = (g^0, g^1) \in Z_{p,q}, u(t; g) \) is a solution of (4.1) and \( u_t(\cdot; g) \) is the function \( u_t(s; g) = u(t + s; g) \) defined in \([-h, 0] \). It is also known that \( S(t) \) is a \( C_0 \)-semigroup on \( Z_{p,q} \). As in Theorem 4.2 of [5], the infinitesimal generator is characterized as follows.

**Lemma 4.2.** (i) The operator \( S(t) \) is a \( C_0 \)-semigroup on \( Z_{p,q} \).
(ii) The infinitesimal generator \( \Lambda \) of \( S(t) \) is characterized by
\[
D(\Lambda) = \{ g = (g^0, g^1) : g^0 \in W_{0}^{1,p}(\Omega), \; g^1 \in W^{1,q}(-h, 0; W_{0}^{1,p}(\Omega)), \; g^1(0) = g^0, \; A_0 g^0 + A_1 g^1(-h) + \int_{-h}^{0} a(s) A_2 g^1(s) \, ds \in H_{p,q}\},
\]
\[
\Lambda g = (A_0 g^0 + A_1 g^1(-h) + \int_{-h}^{0} a(s) A_2 g^1(s) \, ds, g^1).
\]

The equation (4.1) can be transformed into an abstract equation in \( Z_{p,q} \) as follows.
\[
z'(t) = Az(t) + G(t), \quad z(0) = g,
\]
(4.2)
where $G(t) = (f(t), 0)$, $z(t) = (u(t; g), u_t(\cdot; g)) \in Z_{p,q}$ and $g = (g^0, g^1) \in Z_{p,q}$. The mild solution of initial value problem (4.2) is the following form:

$$z(t; g) = S(t)g + \int_0^t S(t-s)G(s)ds.$$  

We introduce the transposed problem of (4.1):

$$\begin{aligned}
\begin{cases}
y'(t) = A_0^*y(t) + A_1^*y(t-h) + \int_{-h}^0 a(s)A_2^*y(t+s)ds, & t \in (0, T], \\
y(0) = \phi^0, \quad y(s) = \phi^1(s), & s \in [-h, 0).
\end{cases}
\end{aligned}$$  

(4.3)

Here, we remark that $A_0^*, A_1^*, A_2^* \in B(W_0^{1,p'}(\Omega), W^{-1,p'}(\Omega))$. We can also define the solution semigroup $S_T(t)$ of (4.3) by

$$S_T(t)\phi = (y(t; \phi), y_t(\cdot; \phi)) \quad \forall \phi = (\phi^0, \phi^1) \in Z_{p', q'},$$

where $y(t; \phi)$ is the solution of (4.3). Let $A_T$ be the infinitesimal generator of $S_T(t)$ associated with the system (4.3).

For $\lambda \in \mathbb{C}$ we define a densely defined closed linear operator by

$$\Delta(\lambda) = \lambda - A_0 - e^{-\lambda h}A_1 - \int_{-h}^0 e^{\lambda s}a(s)A_2ds,$$

$$\Delta_T(\lambda) = \lambda - A_0^* - e^{-\lambda h}A_1^* - \int_{-h}^0 e^{\lambda s}a(s)A_2^*ds.$$  

The operators $\Delta(\lambda)$ and $\Delta_T(\lambda)$ are bounded in $B(W_0^{1,p'}(\Omega), W^{-1,p'}(\Omega))$ and $B(W_0^{1,p'}(\Omega), W^{-1,p'}(\Omega))$, respectively. Noting that if $\lambda \in \rho(A_0)$

$$\Delta(\lambda) = \{I - (e^{-\lambda h}A_1 + \int_{-h}^0 e^{\lambda s}A_2ds)(\lambda - A_0)^{-1}\}(\lambda - A_0).$$

The structural operator $F$ is defined by

$$\begin{aligned}
[FG]^0 &= g^0, \\
[FG]^1(s) &= A_1g^1(-h-s) + \int_{-h}^s a(\tau)A_2g^1(\tau-s)d\tau, & s \in [-h, 0)
\end{aligned}$$  

(4.4)

for $g = (g^0, g^1) \in Z_{p,q}$. It is easy to see that $F \in B(Z_{p,q}, Z_{p', q'})$, $F^* \in B(Z_{p', q'}, Z_{p,q})$, where

$$\begin{aligned}
[F^* \phi]^0 &= \phi^0, \\
[F^* \phi]^1(s) &= A_1^*\phi^1(-h-s) + \int_{-h}^s a(\tau)A_2^*\phi^1(\tau-s)d\tau, & s \in [-h, 0)
\end{aligned}$$

for $\phi \in Z_{p', q'}$. As in [8, 18] we have that

$$FS_S(t) = S_T^*(t)F^* \quad F^*S_T(t) = S^*(t)F*.$$  

(4.5)

Let $\lambda$ be a pole of $(\lambda - A)^{-1}$ whose order we denote by $k_\lambda$ and $P_\lambda$ be the spectral projection associated with $\lambda$:

$$P_\lambda = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (\mu - A)^{-1}d\mu,$$
where $\Gamma_\lambda$ is a small circle centered at $\lambda$ such that it surrounds no point of $\sigma(A)$ except $\lambda$. And we know that $\bar{\lambda} \in \sigma(A_T)$ is a pole of $(\lambda - A_T)^{-1}$ and the spectral projection is given by

$$P_T^\lambda = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (\mu - A_T)^{-1}d\mu.$$  

As is well known $\lambda$ is an eigenvalue of $A$ and the generalized eigenspace corresponding to $\lambda$ is given by

$$P_\lambda Z_{p,q} = \{P_\lambda u : u \in Z_{p,q}\} = \text{Ker}(\lambda I - A)^k,$$

Let us set

$$Q_\lambda = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (\lambda - \lambda)(\lambda - A)^{-1}d\lambda.$$

Then we remark that

$$Q_{\lambda_j}^k = \frac{1}{2\pi i} \int_{\Gamma_{\lambda_j}} (\lambda - \lambda_j)^i(\lambda - A)^{-1}d\lambda.$$  

It is also well known that $Q_{\lambda_j}^k = 0$ (nilpotent) and $(A - \lambda)P_{\lambda_j} = Q_{\lambda_j}$(cf. [18, 27]).

The following subset of $\sigma(\Lambda)$ are especially of use:

$$\sigma_p(\Lambda) = \text{the point spectrum of } \Lambda,$$

$$\sigma_g(\Lambda) = \{\lambda \in \sigma(\Lambda) : \lambda \text{ is isolated and dim}(P_\lambda Z_{p,q}) = d_{\lambda} < \infty\}.$$  

**Lemma 4.3.** Let $\lambda \in \sigma_p(\Lambda) = \sigma_p(\Delta)$, where $\sigma_p(\Delta) = \{\lambda : \Delta(\lambda) \text{ is not invertival}\}$. Then

1) For any $k = 1, 2, \cdots,$

$$\text{Ker}(\lambda - A)^k = \left\{ (\phi_0^0, e^{\lambda s} \sum_{i=0}^{k-1} (-s)^i \phi_i^0/i!) : \sum_{i=j-1}^{k-1} (-1)^{i-j} \Delta^{(i-j+1)(\lambda)}(\lambda)\phi_i^0/(i-j+1)! = 0, \; j = 1, \cdots, k \right\}.$$  

2) $\lambda \in \rho(A) = \rho(A_T^\ast)$,

$$F(\lambda - A)^{-1} = (\lambda - A_T^\ast)^{-1}F.$$  

In particular, if $\lambda \in \sigma_p(A)$ then

$$FP_\Lambda = (P_T^\lambda)^sF.$$  

The proof of 1) and 2) is from Proposition 7.2 and Theorem 6.1 of Nakagiri [18, 19], respectively.

**Definition 4.4.** The system of generalized eigenspaces of $A$ is complete if

$$\text{Cl}(\text{span}\{ \bigcup_{k=1}^{\infty} \text{Ker}(\lambda - A)^k : \lambda \in \sigma_p(A) \}) = Z_{p,q},$$

where Cl is denotes the closure in $Z_{p,q}$.
We know that \( \lambda \in \sigma_d(A_T) \) if and only if \( \bar{\lambda} \in \sigma_d(A_T) \) and that \( P_\lambda Z_{p,q} = d_\lambda = P_\lambda^T Z_{p',q'} = d_\lambda \). Let \( \{\phi_{\lambda_1}, \ldots, \phi_{\lambda_d}\} \) and \( \{\psi_{\lambda_1}, \ldots, \psi_{\lambda_d}\} \) be the bases of \( P_\lambda Z_{p,q} \) and \( P_\lambda^T Z_{p',q'} \), respectively. As is shown by the same method as Proposition 7.4 and Theorem 8.1 of [18], noting that \( F^* \) is an isomorphism from \( P_\lambda^T Z_{p',q'} \) to \( (P_\lambda)^* Z_{p,q} \), we can suppose that
\[
(F^* \psi_{\lambda_i}, \phi_{\lambda_j}) = \delta_{ij}, \quad i,j = 1, \ldots, d_\lambda. \tag{4.6}
\]

Here, \( (\cdot, \cdot) \) denotes the duality between \( Z^*_{p,q} \) and \( Z_{p,q} \). The duality between \( Z^*_{p,q} \) and \( Z_{p',q'} \) is also denoted by \( (\cdot, \cdot) \).

**Lemma 4.5.** (1) Let \( \lambda \in \sigma_d(A_T) \). Then for any \( g \in Z_{p',q'} \), the spectral projection has the following representation
\[
P_\lambda^T g = \sum_{i=1}^{d_\lambda} (F^* g, \phi_{\lambda_i}) \psi_{\lambda_i}.
\]

(2) Let \( \lambda \in \sigma_d(A_T) \). Then the spectral projection has the following representation
\[
P_\lambda g = \sum_{i=1}^{d_\lambda} (F g, \psi_{\lambda_i}) \phi_{\lambda_i}
\]
for any \( g \in Z_{p,q} \).

**Proof.** We prove only (1) since the proof of (2) is similar. For any \( g \in Z_{p',q'} \), \( P_\lambda^T g \) is written as \( \sum_{i=1}^{m_\lambda} c_i \psi_{\lambda_i} \) for \( c_i \in \mathbb{C} \) and then by (4.6)
\[
(F^* P_\lambda^T g, \phi_{\lambda_j}) = \sum_{i=1}^{m_\lambda} c_i (F^* \psi_{\lambda_i}, \phi_{\lambda_j}) = c_j.
\]
From the Laplace transform of the second equality in (4.5) we have
\[
F^*(\mu - A_T)^{-1} = (\mu - A^*)^{-1} F^*
\]
and
\[
F^* P_\lambda^T \mu \frac{1}{2\pi i} \int_{\Gamma_\lambda} (\mu - A_T)^{-1} d\mu = \frac{1}{2\pi i} \int_{\Gamma_\lambda} F^*(\mu - A_T)^{-1} d\mu = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (\mu - A^*)^{-1} F^* d\mu = (P_\lambda)^* F^*.
\]
Therefore, we have
\[
c_j = (F^* P_\lambda^T g, \phi_{\lambda_j}) = ((P_\lambda)^* F^* g, \phi_{\lambda_j}) = (F^* g, P_\lambda \phi_{\lambda_j}) = (F^* g, \phi_{\lambda_j}).
\]
The proof of (1) is completed. \( \square \)

5. **Identification problem in case** \( A_1 = \gamma A_0 \) \& \( A_2 = A_0 \). In this section we deal with the identification problem in the case where \( A_1 = \gamma A_0 \) with some constant \( \gamma \), \( A_2 = A_0 \) as follows.

\[
\begin{aligned}
&\begin{cases}
u'(t) = A_0 u(t) + \gamma A_0 u(t - h) + \int_0^h a(s) A_0 u(t + s) ds, \\
u(0) = g^0, \quad u(s) = g^1(s), \quad s \in [-h,0].
\end{cases} \tag{5.1}
\end{aligned}
\]

Here \( A_0, \gamma, \) and \( a(\cdot) \) are unknown quantities to be identified and the initial conditions \( g_i = (g_i^0, g_i^1) \in Z_{p,q}, i = 1, \ldots, l \) are known.
We denote by the model system \((5.1)^m\) by the equation \((5.1)\) with \(A_0\), \(\gamma\), \(a\) replaced by \(A_0^m\), \(\gamma^m\), \(a^m\) respectively. The solutions of \((5.1)\) and the model system \((5.1)^m\) are denoted by \(u(t; g)\) and \(u^m(t; g)\), respectively, and the solution semigroup for model system by \(S^m(t)\). We assume that \(A_0^m\) and \(a^m\) satisfy the same type of assumptions as \(A_0\) and \(a\).

The identifiability for \((5.1)\) is to find conditions such that if
\[
u(t; g_i) = u^m(t; g_i), \quad i = 1, \ldots, l,
\]
for \(g_i = (g_{i0}^m, g_{i1}^m) \in Z_{p,q}, i = 1, \ldots, l,\) is a finite set of initial values, then
\[
A_0 = A_0^m, \quad \gamma = \gamma^m, \quad a(s) = a^m(s)
\]
follows.

At first we investigate the spectral properties of the infinitesimal generator \(A^m\) of solution semigroup \(S^m(t)\) for the equation \((5.1)^m\). Since \(\Omega\) is bounded, the imbedding of \(W_0^{1,p}(\Omega)\) to \(H_{p,q}\) is compact. From [1, Theorem 3.4], it follows that the system of generalized eigenspaces of \(A_0\) is complete in \(H_{p,q}\). According to Riesz-Schauder theorem \(A_0^m\) has discrete spectrum
\[
\sigma(A_0^m) = \{\mu_j : j = 1, \ldots\}
\]
which has no point of accumulation except possibly \(\lambda = \infty\).

For \(\lambda \in \mathbb{C}\) we have
\[
\Delta^m(\lambda) = \lambda - m(\lambda)A_0^m
\]
where
\[
m(\lambda) = 1 + \gamma^m e^{-\lambda h} + \int_0^h e^{\lambda s} a^m(s) ds. \tag{5.2}
\]
It is clear that \(m\) is an entire function and
\[
m(\lambda) \to 1 \text{ as } \Re \lambda \to \infty.
\]

Just as Theorems 1 and 2 of [10] for \(A_0^m\) we can prove the following two Lemmas.

**Lemma 5.1.** (1) Let \(\rho(A^m)\) be the resolvent set of the infinitesimal generator \(A^m\) of \(S^m(t)\). Then
\[
\rho(A^m) = \{\lambda : m(\lambda) \neq 0, \quad \frac{\lambda}{m(\lambda)} \notin \rho(A_0^m)\}
\]
\[
= \{\lambda : \Delta(\lambda) \text{ is an isomorphism from } W_0^{1,p}(\Omega) \text{ onto } W^{-1,p}(\Omega)\}.
\]
(2) Let \(\sigma(A^m)\) be the spectrum of \(A^m\). Then
\[
\sigma(A^m) = \sigma_e(A^m) \cup \sigma_p(A^m),
\]
where \(\sigma_e(A^m) = \{\lambda : m(\lambda) = 0\}\) and \(\sigma_p(A^m) = \{\lambda : m(\lambda) \neq 0, \lambda/m(\lambda) \in \sigma(A_0^m)\}\).
Each nonzero point of \(\sigma_e(A^m)\) is not an eigenvalue of \(A^m\) but a cluster point of \(\sigma(A^m)\). \(\sigma_p(A^m)\) consists only of discrete eigenvalues.

(3) Suppose \(m(0) = 0\). Then there exists an analytic function \(g\) on neighborhood at 0 such that \(g(0) \neq 0\) and \(m(\lambda) = \lambda^k g(\lambda)\), and
\[
0 \in \begin{cases} \sigma_p(A^m), & \text{if } k = 1, \\ \sigma_e(A^m), & \text{if } k > 1. \end{cases}
\]

**Lemma 5.2.** Suppose that \(m(0) \neq 0, \gamma^m \neq 0\). Then the system of generalized eigenspaces of \(A^m\) is complete in \(Z_{p,q}\).
The structural operator \( F \) defined by (4.4) is written as
\[
F g = ([F g]^0, [F g]^1),
\]
\[
[F g]^0 = g^0,
\]
\[
[F g]^1(s) = \gamma A_0 g^1(-h - s) + \int_{-h}^{s} a(\tau) A_0 g^1(\tau - s) d\tau, \quad s \in [-h, 0).
\]
for \( g = (g^1, g^\parallel) \in Z_{p,q} \). The \( m^m \) and \( F^m \) are the structural operators of the model system (5.1)m in place of \( m \) in (5.2) and \( F \), respectively.

Let \( \lambda \in \sigma_p(A^m) \), and \( \{ \phi_{\lambda_k} : k = 1, \ldots, d_\lambda \} \) denote the basis of \( P^m_{\lambda}Z_{p,q} \). Let \( A^m_{\mu} \) be the infinitesimal generator of transposed solution semigroup associated with (5.1). Then \( \overline{\sigma} \in \sigma_p(A^m_{\mu}) \). Let \( \{ \psi_{\overline{\sigma}}_k : k = 1, \ldots, d_\lambda \} \) be a basis of \( (P^m)^T_{\lambda}Z_{p,q} \), where \( (P^m)^T_{\lambda} \) denotes the projection of \( A^m_{\mu} \) at \( \mu \). As shown in [21, Theorem 8.1] the projection \( (P^m)^T_{\lambda} \) has the following equivalent representation
\[
P^m_{\lambda} g = \sum_{k=1}^{d_\lambda} (F^m g, \psi_{\overline{\sigma}}_k) \phi_{\lambda_k}, \quad \forall g \in Z_{p,q}.
\]
Throughout this section we shall assume following:

- **RANK CONDITION**: For set of the initial values \( \{ g_1, \ldots, g_l \} \) is said to be satisfy the Rank condition for the model system (5.1)m if and only if
\[
\text{rank}([F^m g_i, \psi_{\overline{\sigma}}_i]) : i \rightarrow 1, \ldots, l, \quad k \downarrow 1, \ldots, d_\lambda \) = \( d_\lambda \), \quad \forall \lambda \in \sigma_p(A^m) \quad (5.3)
\]
for \( n = 1, 2, \ldots \) and \( j = 1, 2, \ldots \).

The assumption of Rank condition is satisfied if and only if
\[
\text{Span}\{P^m_{\lambda} g_1, \ldots, P^m_{\lambda} g_l\} = P^m_{\lambda}Z_{p,q}, \quad \forall \lambda \in \sigma_p(A^m). \quad (5.4)
\]

**Proposition 3.** Assume that \( u(t; g_i) \equiv v^m(t; g_i) \), \( i = 1, \ldots, l \) and the rank condition (5.3) for \( \{ g_1, \ldots, g_l \} \) be satisfied. Further assume that \( m(0) \neq 0 \). Then
\[
\sigma_p(A^m) \subset \sigma_p(A), \quad \sigma_e(A^m) \subset \sigma_e(A), \quad (5.5)
\]
and
\[
A = A^m \text{ on } P^m_{\mu}Z_{p,q}, \quad \forall \lambda \in \sigma_p(A^m). \quad (5.6)
\]

**Proof.** By the definition of semigroups \( S(t) \) and \( S(t)^m \), we have from the assumption that
\[
S^m(t)g_i = S(t)g_i, \quad \forall t > 0, \quad i = 1, \ldots, l. \quad (5.7)
\]
By taking the Laplace transform of (5.7) and using the analytic continuation of the resolvent operators, we have
\[
(\lambda - A^m)^{-1} g_i = (\lambda - A)^{-1} g_i, \quad \forall \lambda \in \sigma_p(A^m) \cap \sigma_p(A), \quad i = 1, \ldots, l. \quad (5.8)
\]
Let \( \lambda_0 \in \sigma_p(A^m) \). First we note that \( \lambda_0 \neq 0 \). Because, if \( \lambda_0 = 0 \), then \( m(\lambda_0) \neq 0 \) and hence \( 0 = \lambda_0/m(\lambda_0) \in \sigma(A^m_{\mu}) \) by Lemma 5.1, which contradicts the fact that \( A^m_{\mu} : W^m_{1,p'}(\Omega) \rightarrow W^{1,1,m}'(\Omega) \) is an isomorphism. We shall show \( \lambda_0 \in \sigma(A) \). Assume contrarily that \( \lambda_0 \in \rho(A) \). Then from Lemma 5.1 there exists a sufficiently small number \( \epsilon > 0 \) such that
\[
\{ \lambda : 0 < |\lambda - \lambda_0| \leq \epsilon \} \subset \rho(A^m), \quad \{ \lambda : |\lambda - \lambda_0| \leq \epsilon \} \subset \rho(A).
\]
Thus, by (5.8), we have
\[ P_{\lambda_0}^m g_i = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \epsilon} (\lambda - A^m)^{-1} g_i d\lambda \]
\[ = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \epsilon} (\lambda - A)^{-1} g_i d\lambda = 0, \quad i = 1, \ldots, l. \]

This implies by the span condition (5.4) for \( \lambda = \lambda_0 \) that \( P_{\lambda_0}^m Z_{p,q} = \{0\} \), which yields the contradiction. Thus, \( \lambda_0 \in \sigma(A) \). Suppose \( \lambda_0 \in \sigma_\epsilon(A) \). Since \( \lambda_0 \neq 0 \) by \( m(\lambda_0) = 0 \) in Lemma 5.1 and \( m(0) \neq 0 \), there exists a sequence \( \{\lambda_n\} \subseteq \sigma_p(A) \) such that \( \lambda_n(\neq \lambda_0) \) converges to \( \lambda_0 \) as \( n \to \infty \). Then we can choose a sufficiently small \( \epsilon > 0 \) and natural number \( N \geq 1 \) such that \( \{\lambda : |\lambda - \lambda_0| = \epsilon\} \subset \rho(A^m) \) and
\[ \{\lambda_n : n \geq N\} \subseteq \{\lambda : 0 < |\lambda - \lambda_0| \leq \epsilon\} \subset \rho(A^m), \]
\[ \lambda_n : 1 \leq n \leq N - 1\} \cap \{\lambda : |\lambda - \lambda_0| \leq \epsilon\} = \emptyset. \]

Since \( \lambda_n \)'s are discrete, we can also choose a positive sequence \( \{\epsilon_n : n \geq N\} \) such that \( \{\lambda : |\lambda - \lambda_0| \leq \epsilon_n\} \subset \rho(A^m) \) for all \( n \geq N \). Therefore, by the residue theorem, we have
\[ P_{\lambda_0}^m g_i = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \epsilon} (\lambda - A^m)^{-1} g_i d\lambda \]
\[ = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \epsilon} (\lambda - A)^{-1} g_i d\lambda = \sum_{n \geq N} \frac{1}{2\pi i} \int_{|\lambda - \lambda_n| = \epsilon_n} (\lambda - A)^{-1} g_i d\lambda \]
\[ = \sum_{n \geq N} \frac{1}{2\pi i} \int_{|\lambda - \lambda_n| = \epsilon_n} (\lambda - A^m)^{-1} g_i d\lambda = \sum_{n \geq N} 0 = 0, \quad i = 1, \ldots, l, \]
which also contradicts the rank condition for \( \lambda = \lambda_0 \). This shows \( \lambda_0 \in \sigma_p(A) \). Since \( \lambda_0 \) is a discrete eigenvalue of \( A \) and \( A^m \), we have for sufficiently small \( \epsilon > 0 \) that
\[ P_{\lambda_0}^m g_i = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \epsilon} (\lambda - A^m)^{-1} g_i d\lambda \]
\[ = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \epsilon} (\lambda - A)^{-1} g_i d\lambda = P_{\lambda_0} g_i, \quad i = 1, \ldots, l. \]

Further, again by (5.8) and (5.9), for all \( i = 1, \ldots, l \) we have
\[ A^m P_{\lambda_0}^m g_i = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \epsilon} A^m(\lambda - A^m)^{-1} g_i d\lambda \]
\[ = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \epsilon} \{\lambda - (\lambda - A^m)\}(\lambda - A^m)^{-1} g_i d\lambda \]
\[ = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \epsilon} \lambda(\lambda - A^m)^{-1} g_i d\lambda - \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \epsilon} g_i d\lambda \]
\[ = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \epsilon} \lambda(\lambda - A^m)^{-1} g_i d\lambda \]
\[ = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \epsilon} \lambda(\lambda - A)^{-1} g_i d\lambda = AP_{\lambda_0} g_i = AP_{\lambda_0}^m g_i. \]

By the span condition (5.4) for \( \lambda = \lambda_0 \), this implies that
\[ A = A^m \quad \text{in} \quad P_{\lambda_0}^m Z_{p,q}. \]
which proves (5.6). Next, let \( \lambda_0 \in \sigma_\varepsilon(A^m) \), then \( m(0) = 0 \), so that \( \lambda_0 \neq 0 \) by the assumption \( m(0) \neq 0 \). Therefore, there exists a sequence \( \{\lambda_n\} \subset \sigma_p(A^m) \) such that \( \lambda_n \) converges to \( \lambda_0 \). Hence from \( \{\lambda_n\} \subset \sigma_p(A^m) \subset \sigma_p(A) \) in (5.5) it follows that \( \lambda_0 \) is a cluster point of \( \sigma_p(A) \) and hence \( \lambda_0 \in \sigma_\varepsilon(A) \).

**Theorem 5.3.** Suppose that \( m(0) \neq 0 \) and \( \gamma^m \neq 0 \). Let the set of initial values \( \{g_1, \ldots, g_l\} \) satisfy the rank condition (5.3) be satisfied. Then

\[
u(t; g_i) \equiv u^m(t; g_i), \quad g_i = (g_i^0, g_i^1) \in \mathbb{Z}_{p,q}, \quad i = 1, \ldots, l \tag{5.12}
\]

implies

\[
A_0 = A^m_0, \quad \gamma = \gamma^m, \quad a(s) \equiv a^m(s). \tag{5.13}
\]

*Proof.* By Proposition 5.1, it follows from (5.12) and the rank condition (5.3) that

\[
A = A^m \text{ in } P^m_0 \mathbb{Z}_{p,q}, \quad \lambda \in \sigma_p(A). \tag{5.14}
\]

Since \( m(0) \neq 0 \) and \( \gamma^m \neq 0 \), by Lemma 5.2 the system of generalized eigenspaces of \( A^m \) is projectively complete, i.e.,

\[
\text{Cl}(\text{span}\{P^m_0 \mathbb{Z}_{p,q} : \lambda \in \sigma_p(A)\}) = \mathbb{Z}_{p,q}. \tag{5.15}
\]

Then by the same argument as in the proof of Theorem 3 in Yamamoto and Nakagiri [26], we can verify by (5.14) and (5.15) that \( D(A^m) = D(A) \) and \( A^m g = Ag \) for any \( g \in D(A^m) \). By Lemma 4.2, this implies

\[
A_0 g^0 + \gamma A_0 g^1(-h) + \int_{-h}^0 a(s) A_0 g^1(s) ds = A^m_0 g^0 + \gamma^m A^m_0 g^1(-h) + \int_{-h}^0 a^m(s) A^m_0 g^1(s) ds \tag{5.16}
\]

for all \( g = (g^0, g^1) \in D(A^m) \). For any \( g^0 \in W_0^{1,p} \) and \( \epsilon \in (0, h) \), let \( g_\epsilon(s) \) be a function in \( W^{1,q}(-h, 0; W_0^{1,p} \Omega) \) such that

\[
g_\epsilon(0) = g^0, \quad g_\epsilon(s) = 0 \quad \text{if } s \in [-h, -\epsilon], \quad \text{and } \int_{-h}^0 \|g_\epsilon(s)\|_{1,p}^q ds \leq \epsilon^q. \tag{5.17}
\]

Then \( g_\epsilon(s) \in D(A^m) \), and we apply this \( g_\epsilon \) to (5.16) to have

\[
(A^m_0 - A_0)g^0 = \int_{-h}^0 (a(s) A_0 - a^m(s) A^m_0) g_\epsilon(s) ds. \tag{5.18}
\]

By using Hölder inequality, we have from (5.17) and (5.18) that

\[
\|A^m_0 - A_0\|_{-1,p} \leq \left( \int_{-h}^0 \|a(s) A_0 - a^m(s) A^m_0\|_{W_0^{1,p} \Omega}^q ds \right)^{1/q'} \left( \int_{-h}^0 \|g_\epsilon(s)\|_{1,p}^q ds \right)^{1/q} \leq \epsilon \|a(\cdot) - a^m(\cdot)\|_{L^{q'}(-h, 0; W_0^{1,p} \Omega)} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0,
\]

so that \( A^m_0 g^0 = A_0 g^0 \) in \( W^{-1,p} \Omega \) for any \( g^0 = W_0^{1,p} \Omega \). Hence \( A^m_0 = A_0 \) follows.

It follows from this and (5.16) that

\[
(\gamma^m - \gamma) A^m_0 g^1(-h) = \int_{-h}^0 (a(s) - a^m(s)) A^m_0 g^1(s) ds, \quad \forall g = (g^0, g^1) \in D(A^m). \tag{5.19}
\]
For any \( f^0 \in W_0^{1,p}(\Omega) \) and \( \epsilon \in (0,h) \), let \( f_\epsilon \) be a function in \( W^{1,q}(-h,0;W_0^{1,p}(\Omega)) \) such that

\[
    f_\epsilon(-h) = f^0, \quad f_\epsilon(s) = 0 \text{ if } s \in [-h+\epsilon,0], \quad \int_{-h}^{0} ||f_\epsilon(s)||_{1,p}^q ds \leq \epsilon^q. \tag{5.20}
\]

Then \((0,f_\epsilon) \in D(A^m)\), and applying this to (5.19) and repeating similar argument as above, we have

\[
    (\gamma^m - \gamma)A_0^m f^0 = 0, \quad \forall f^0 \in W_0^{1,p}(\Omega). \tag{5.21}
\]

Applying \((A_0^m)^{-1}\) to (5.21), we obtain \( \gamma = \gamma^m \). Finally, from (5.19), applying \((A_0^m)^{-1}\) and using the density argument, we have

\[
    \int_{-h}^{0} (g(s) - a^m(s))A_0^m g_1(s) ds = 0 \text{ in } W_0^{1,p}(\Omega), \quad \forall g^1 \in L^q(-h,0;W_0^{1,p}(\Omega)). \tag{5.22}
\]

This implies \( a(s) = a^m(s) \) a.e. \( s \in [-h,0] \).

\[\square\]

**Remark 1.** The rank condition (5.3) can be replaced by

\[
    \text{rank } \{ F^m g_i, \psi_\lambda^0 \} : i = 1, \ldots, l, \quad k \downarrow 1, \ldots, d_\lambda^0 = d_{\lambda}^0, \quad \forall \lambda \in \sigma_p(A^m),
\]

\[
    \{ \psi_\lambda^0 : k = 1, \ldots, d_{\lambda}^0 \} \text{ is a basis of Ker}(\overline{A}^m_0 - \Lambda^m_0) \text{ and dim Ker}(\overline{A}^m_0 - \Lambda^m_0) = d_{\lambda}^0 \text{ (cf. Corollary 1 in [26]).}
\]

6. **Example.** We consider the following retarded functional differential equation of parabolic type:

\[
    \begin{cases}
    \frac{\partial u(t,x)}{\partial t} = \alpha \frac{\partial^2 u(t,x)}{\partial x^2} + \beta \frac{\partial^2 u(t-h,x)}{\partial x^2} + \int_{-h}^{0} a_1(s) \frac{\partial^2 u(t+s,x)}{\partial x^2} ds, & (t,x) \in \mathbb{R}^+ \times (0,\pi), \\
    u(t,0) = u(t,\pi) = 0, & t > 0 \\
    u(0,x) = g_0(x), \quad u(s,x) = g_1(s,x) \quad \text{a.e. } (s,x) \in [-h,0] \times [0,\pi].
    \end{cases}
\]

The initial data \((g_0^1, g_1^1) \in H_{p,q} \times L^q(-h,0;W_0^{1,p}(0,\pi))\), \( p,q \neq 2 \) are known, where \( H_{p,q} \) is defined on the domain \( \Omega = (0,\pi) \) Here, \( \alpha \neq 0, \beta \) and \( a(s) \) are unknown except that \( a_1 \in L^2(-h,0;\mathbb{C}) \). Let \( A_0 \) be the realization in \( H_{p,q} \) of the operator \( \alpha \frac{\partial^2}{\partial x^2} \) with Dirichlet boundary condition, that is,

\[
    A_0 = \alpha \frac{\partial^2}{\partial x^2}, \quad D(A_0) = \{ u \in H_{p,q} : u(t,0) = u(t,\pi) = 0 \}.
\]

Then the eigenvalues and eigenfunctions of \( A_0 \) are \( \mu_n = an^2 \) and \( e_n(x) = \sin(nx) \), \( n = 1, \ldots \), respectively. Let us define as

\[
    \gamma = \beta/\alpha, \quad a(s) = a_1(s)/\alpha \quad s \in [-h,0].
\]

Then the system (6.1) can be written in the same form as of (5.1) on the space \( H_{p,q} \). It is well known that \( \{ e_n : n = 1, \ldots \} \) is an orthogonal base for \( H_{p,q} \), and so \( \{ \sin(nx) : n = 1, \ldots \} \) is complete in \( H_{p,q} \). Thus, we can solve the identification problem of the system (6.1) for parameters \( \alpha, \beta \), and the function \( a_1(\cdot) \) in the terminology of Theorem 5.3.

As an additional result in this case, we consider the system of generalized eigenvalues of \( A \) as defined Lemma 4.2. The spectrum \( \sigma(A) \) of \( A \) is given by

\[
    \sigma(A) = \bigcup_{n=1}^{\infty} \sigma_n,
\]
where

\[ \sigma_n = \{ \lambda \in \mathbb{C} : \Delta_n(\lambda) = \lambda - n^2(\alpha + \beta e^{\lambda h} + \int_{-h}^{0} e^{\lambda s} a_1(s) ds) = 0 \} \]

as seen in [16, 13]. Hence, \( \sigma(A) \) is a countable set consisting entirely of eigenvalues. Let \( \{ \lambda_{nj} \}_{j=1}^{\infty} \) be the set of roots of \( \Delta_n(\lambda) = 0 \) \( (n = 1, 2, \cdots) \) and let \( k_{nj} \) (in many cases \( k_{nj} = 1 \)) be the multiplicity of \( \lambda_{nj} \). The generalized eigenspaces \( P_{\lambda_{nj}} H_{p,q} \) corresponding to \( \lambda_{nj} \in \sigma(A) \) is given by

\[ \text{Span}\{ \exp(\lambda_{nj}s) \sin(nx), \cdots, s^{k_{nj}-1} \exp(\lambda_{nj}s) \sin(nx) \}. \tag{6.2} \]

Since \( \{ \sin(nx), n = 1, \cdots \} \) is complete in \( H_{p,q} \), from (6.2) and [14, Theorem 5.4] it follows the system of generalized eigenspaces of \( A \) is complete. In the special case of the finite dimensional space, \( \sigma(A) \) is a countable set consisting entirely of eigenvalues. Noting that \( \gamma \neq 0 \) and \( 0 \neq \sigma(A) \), the completeness of the system of generalized eigenspaces of \( A \) is equivalent to \( \text{Ker} F^* = \{0\} \) (see Manitius [14]). If \( h \) and \( a_1 \neq 0 \) are known and the multiplicity \( d_{nj} = 1 \) for all \( nj \) and \( q^1 = (g^1_0, 0) \) satisfies \( (g^1_0, \sin(nx)) \neq 0 \), then \( \alpha, \beta \), and \( a_1 \) in (6.1) are identifiable in terminology of Section 5.

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