Stringy corrections to a time-dependent background solution of string and M-Theory

Gustavo Niz$^{1,2,\dagger}$ and Neil Turok$^{2,\ddagger}$

$^1$School of Physics and Astronomy, University of Nottingham, NG7 2RD, UK

$^2$DAMTP, CMS, Wilberforce Road, Cambridge, CB3 0WA, UK

Abstract

We consider one of the simplest time-dependent backgrounds in M-theory, describing the shrinking away of the M-theory dimension with the other spatial dimensions static. As the M-theory dimension becomes small, the situation becomes well-described by string theory in a singular cosmological background where the string coupling tends to zero but the $\alpha'$-corrections become large, near the cosmic singularity. We compute these $\alpha'$-corrections, both for the background and for linearized perturbations, in heterotic string theory, and show they may be reproduced by a map from eleven-dimensional M-theory.
I. INTRODUCTION

One of the most important tests for string and M-theory is to provide a successful account of the cosmic singularity. While the singularity almost certainly cannot be described in classical terms, there seem to be two fundamentally different possibilities for its role in a quantum theory of gravity. Either the singularity marks the first emergence of classical time, or it does not. In the former case, one must explain why the classical universe emerged from the singularity in a dense, rapidly expanding state. In the latter case, one has to understand how a pre-big bang universe can propagate across the singularity and into a hot, expanding phase.

One of the simplest time-dependent backgrounds one can consider in M-theory is compactified Milne spacetime, with metric

\[ ds_{11}^2 = -dt^2 + t^2 d\theta^2 + \sum_{i=1}^{9} (dx^i)^2, \]

where \( \theta \) parameterizes a circle of circumference \( 2\theta_0 \). In heterotic M-theory, which is of special interest for the ekpyrotic and cyclic universe models \cite{1,2} and shall be our main focus here, the circle is modded out by \( Z_2 \), so that \( 0 \leq \theta \leq \theta_0 \).

For all \( t \neq 0 \), the metric is non-degenerate and the spacetime (1) is flat, \( i.e. \) the Riemann curvature vanishes. Also, the four form field strength is zero. Therefore, if M-theory is describable in terms of local covariant field equations for the background bosonic fields then (1) automatically solves those equations.

As suggested in Refs. \cite{3} and \cite{4}, analyticity in \( t \) suggests the possibility of extending the spacetime to the entire range \( -\infty < t < \infty \) so that (1) describes the collapse and re-expansion of the M-theory dimension, or in heterotic M-theory, the collision and separation of two orbifold planes. The conjecture that such a transition through \( t = 0 \) is possible underlies the cyclic and ekpyrotic universe models \cite{1,2}.

When \( t \) is small, the M-theory dimension is small and one should be able to describe the situation in terms of weakly coupled string theory. Under the map from eleven dimensional M-theory to ten-dimensional string theory, at lowest order in \( \alpha' \), (1) yields a cosmological solution in the string frame,

\[ ds_{\text{string}}^2 = |t|(-dt^2 + \sum_{i=1}^{9} (dx^i)^2), \quad g_s = e^\phi = |t|^{3/2}, \]

where \( g_s \) is the string coupling and \( \phi \) is the dilaton. Near \( t = 0 \) stringy interactions are small and string loop effects may be neglected. However, the string frame metric becomes singular so higher order corrections in \( \alpha' \) become increasingly significant. Also, as noted in
The background (1) is not supersymmetric, hence there is no protection against large quantum effects.

The fact that the $\alpha'$ expansion fails certainly does not in itself mean that the background is not a good background for the quantization of strings. In [4] it was shown that $M2$-branes wrapped across the M-theory dimension, which reduce to fundamental strings as $t$ tends to zero, generically obey regular classical evolution through $t = 0$. It was argued that in the small $t$ regime the theory should be described by an expansion in $1/\alpha'$ (i.e. in the string tension), which unfortunately is not yet well understood. In [5] this picture was further elaborated: classically, at least, near $t = 0$ the string evolves as a set of weakly coupled “string bits”, each of which propagates smoothly across $t = 0$. Assuming for now that this picture of the quantum transition makes sense, we can anticipate many of its main features. The production of excited string states may be calculated using semiclassical methods obtaining a sensible finite result [4]. The small parameter in the calculation turns out not to be $\alpha'$ but $\theta_0$, the rapidity of the orbifold plane collision: if $\theta_0$ is small a low density of excited strings is produced. In [5] classical solutions describing strings passing through $t = 0$ have been studied in detail, showing how incoming strings undergo transmutation into a variety of excited massive states (see also [6]).

One cannot expect the transition across $t = 0$ to be describable by any effective theory including only massless fields. If effects involving the string coupling are neglected, the excited states do not decay and they must appear in the final state. Neither the $\alpha'$ expansion nor any resummation of it involving the massless fields alone, can possibly describe such a situation. Instead, a full string field theory approach involving the excited string states will likely be needed.

Having made these necessary qualifications, we nevertheless want to study the stringy $\alpha'$-corrections as the cosmic singularity is approached. In the regime where it is valid, the $\alpha'$ expansion gives some qualitative indications of the effects of virtual massive string states as $t = 0$ approaches, but before the production of real excited string states becomes significant. In particular, it is interesting to check whether the correspondence between the M-theory background (1) and the string theory background (2) survives once $\alpha'$-corrections are included, in the regime where the description of both theories in terms of the massless bosonic fields should still be valid. Here, the flatness of the M-theory background plays a critical role. As long as any higher order corrections can be expressed as powers of the Riemann curvature and the four-form field strength (recall there is no dilaton in eleven dimensions), none of them will have any affect on the linearized perturbations. Any correction they introduce in the perturbation equations necessarily involves at least one power of the background curvature or field strength, both of which are zero. Therefore, the linearized perturbations
continue to be described by Einstein gravity on the M-theory side of the correspondence. All the $\alpha'$-corrections in string theory must therefore correspond to corrections in the map from M to string theory. We shall explicitly construct such a covariant map, order by order in $\alpha'$, and show that it consistently describes our results.

While the technology for computing the $\alpha'$-corrections to the bosonic background fields in string theory has been much studied [7]-[14], several important ambiguities remain. The effective action for the massless fields can be computed either from the tree-level string S-matrix [7] or from the nonlinear $\sigma$-model representing a string propagating in the relevant nontrivial background [8]. In the second method, conformal (Weyl) symmetry of the quantum string requires the vanishing of the $\beta$-functions in the $\sigma$-model. The latter are expressed as equations of motion for the massless bosonic fields, from which one can reconstruct an effective action as a series of geometrical quantities at each order in $\alpha'$.

To lowest (zeroth) order in $\alpha'$, the effective action is just that for general relativity plus the usual terms for the dilaton and the antisymmetric tensor field. Different string theories lead to different $\alpha'$-corrections, but generally, to order $n$ in $\alpha'$, the corrections involve products of the form $R^{m_1}(\nabla \phi)^2 \nabla^2 \phi^{m_3} H^{2m_4}$ with $m_1 + m_2 + m_3 + m_4 = n + 1$, where $R$ is the Riemann tensor, $\phi$ is the dilaton and $H$ is the field strength of the antisymmetric tensor. For type II theories, the $O(\alpha'^1)$ and $O(\alpha'^2)$ terms are zero, and the first correction comes at order $\alpha'^3$ [10]. In the case of the bosonic or heterotic theories, however, there is a nontrivial correction at first order in $\alpha'$ [11, 12]. In this paper, we shall focus on the heterotic string since it is the natural limit of heterotic M-theory when the M-theory dimension becomes small, i.e. at weak string coupling.

Generically, the $\alpha'$ corrections to Einstein’s theory give field equations with higher order time derivatives, possessing spurious solutions with bad physical behavior. As long as one is only interested in perturbation theory in $\alpha'$, such spurious solutions can be safely ignored: the higher order corrections are used only to correct the lower order solutions order by order and no such problems are encountered. Nevertheless, it is important to emphasize that to a given order in $\alpha'$, the action itself is ambiguous.

In fact, for the first nontrivial correction, one can remove the higher order time derivatives by adding certain terms which vanish using the zeroth order equations of motion, as was shown by Zwiebach [13]. The point was generalized by Hull and Townsend [14] who considered arbitrary covariant local field redefinitions such as

$$g'_{\mu \nu} = g_{\mu \nu} + \alpha'(\mu_1 R_{\mu \nu} + \mu_2 \nabla_\mu \phi \nabla_\nu \phi + \mu_3 \nabla_\mu \nabla_\nu \phi + g_{\mu \nu}(\mu_4 R + \mu_5 (\nabla \phi)^2 + \mu_6 \nabla^2 \phi)), \quad \phi' = \phi + \alpha'(\mu_7 R + \mu_8 (\nabla \phi)^2 + \mu_9 \nabla^2 \phi). \quad (3)$$

We have omitted terms involving the antisymmetric tensor field for simplicity. Hull and
Townsend showed that certain special linear combinations of these \( \mu \)-terms do not alter the solutions to the equations of motion. Such field redefinitions lead to extra terms in the effective action which can be written as squares of the lower order \( \beta \)-functions. If any other combination of the \( \mu \)'s is chosen, then the resulting solutions to the equations of motion will be different, but still physically equivalent because they correspond to different definitions of the physical metric and dilaton field. One can try to apply additional arguments like the absence of higher time derivatives, duality symmetry, and so on, in favor of certain choices, but these are likely to remain inconclusive until the full result, including all orders in \( \alpha' \), is known.

The \( \alpha' \)-corrected field equations for the massless string modes have been applied to a variety of different problems, from black hole thermodynamics (see e.g. [16]) to cosmology [17]-[19], and a number of interesting results have been obtained. Here, our intention is to use them to study the approach to a cosmological singularity, before the representation in terms of the massless fields alone fails as we have argued it must.

The paper is divided as follows: in Section II we present the effective action of the heterotic model. The next section describes the cosmological background and its stringy corrections to second order in \( \alpha' \). Section IV shows how these stringy corrections affect cosmological perturbations. We present the tensor modes first and then the scalar perturbations where some care must be taken with the gauge choice. Section V is devoted to checking the proposed correspondence between M-theory and the heterotic string for this particular background. In Section VI, we investigate the qualitative behavior of the \( \alpha' \)-corrected solutions as they approach the singularity. In the last section, we present some conclusions.

II. HETEROISTIC STRING EFFECTIVE ACTION

Our conventions follow the Landau-Lifshitz notation for the curvature tensors, and we use a \((-,-,\ldots,+\)) signature for the metric. Under these assumptions, the form of the heterotic string effective action, in string frame, including the first nontrivial \( \alpha' \)-correction but to zeroth order in the string coupling, is

\[
S = \frac{1}{16\pi G} \int d^{10}x \sqrt{-g} e^{-2\phi} \left( R + 4(\nabla \phi)^2 + \frac{\alpha'}{8} R_{abcd} R^{abcd} + O(\alpha'^2) \right),
\]

where \( G \) is Newton’s constant, and \( \phi \) is the dilaton. We shall not study the antisymmetric tensor contribution because it is zero for the background we are interested in.

As discussed previously, the action (4) presents higher derivative terms at the order \( \alpha' \). However, after a particular field redefinition of the kind \([3]\), with \( \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \ldots \)
\( \mu_0 = 0, \mu_1 = 1, \mu_7 = 1/8 \) and \( \mu_8 = -1/2 \), this effective action takes the form [12, 13, 14]

\[
S = \frac{1}{16\pi G} \int d^{10}x \sqrt{-g} e^{-2\phi} \left( R + 4(\nabla\phi)^2 + \frac{\alpha'}{8} \left( R_{GB}^2 + 16G^{ab}\nabla_a\phi\nabla_b\phi - 16\nabla^2\phi(\nabla\phi)^2 \\
+ 16(\nabla\phi)^4 \right) + O(\alpha'^2) \right),
\]

(5)

which includes the Einstein’s tensor \( G_{ab} = R_{ab} - g_{ab}R/2 \) and the Gauss-Bonnet combination \( R_{GB} = (R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2) \). Although not obvious, the present action yields second order field equations. However, the solutions to the equations of motion remain the same as for the action (4) if one constructs the solutions as a series in \( \alpha' \) about the lowest order solution.

The action (5) has nice properties like the absence of physical ghosts, unitarity and an \( O(d, d) \) symmetry which is related to T-duality [20]. By analyzing string theory four point amplitudes in flat spacetime, Gross and Sloan have worked out the quartic terms in the heterotic string action up to third order in \( \alpha' \) [12]. They did not find any correction at \( \alpha'^2 \). They did find a correction at \( \alpha'^3 \), but while this is sufficient for determining the \( R^4 \) term, for example, it does not include the coefficient of \( (\nabla\phi)^8 \) since that would require an eight point function. Unfortunately, the Gross-Sloan calculation is therefore incomplete for our purposes, since the dilaton is time-dependent in our background. It would be interesting to complete their calculation, to further check the conjectured correspondence between the string and M-theory description of this background. In particular, as we shall argue below, there is an important relation which is satisfied at lowest order by (2), namely \( a^2 \propto e^{2\phi/3} \) where \( a \) is the string frame scale factor. This relation survives the first order correction in \( \alpha' \). We have checked that it does not survive when Gross and Sloan’s third order correction is included, but that result remains inconclusive, as explained, until all the relevant contributions are included.

In order to study solutions of the corrected action (5) we proceed as follows. The equations of motion are expressed as a series of ascending powers of \( \alpha' \). The solution is then expressed as a series in \( \alpha' \), in which the lower order terms enter the field equations as sources for the higher order terms. We first find the \( \alpha' \)-corrected background solution, and then study the tensor and scalar perturbations, using the \( \alpha' \)-corrected background. Since the tensor modes are gauge invariant, it is easier to start the discussion with them, and afterwards we will focus on how to fix the gauge for the scalar modes to get sensible results.
III. COSMOLOGICAL BACKGROUND

In this paper, we study the simplest case, where the string frame background is homogeneous, isotropic and flat in all nine space dimensions:

\[ ds^2 = a^2(t) \left( -N^2(t) dt^2 + \sum_{i=1}^{9} (dx^i)^2 \right), \]

where \( N(t) \) is the lapse function. We are also including a time-dependent dilaton \( \phi \), but we shall set the antisymmetric tensor field to zero, as mentioned above. Now, we insert the background metric (6) into the action (5) and obtain the equations of motion by taking the variation with respect to the scale factor, the dilaton and the lapse function to get the Hamiltonian constraint. The variation of (5) with respect to the scale factor is

\[
0 \equiv 288 a^6 [(a^2)'^2 - 576 a^8 (a^2)' \phi' + 160 a^{10} (\phi')^2 + 288 a^8 (a^2)'' - 144 a^{10} \phi'']
\]

\[
+ \alpha' \left( -189 [(a^2)']^4 - 504 a^2 [(a^2)']^3 \phi' + 1152 a^4 [(a^2)']^2 (\phi'(t))^2 - 576 a^6 (a^2)' (\phi')^3 
\]

\[
+ 80 a^8 (\phi')^4 + 756 a^2 [(a^2)']^2 (a^2)'' - 1008 a^4 (a^2)' \phi' (a^2)'' + 288 a^6 (\phi')^2 (a^2)'' 
\]

\[
- 504 a^4 [(a^2)']^2 \phi'' + 576 a^6 (a^2)' \phi' \phi'' - 144 a^8 (\phi')^2 \phi'' \right),
\]

where \( \phi' = \frac{d\phi}{dt} \). The dilaton variation yields to

\[
0 \equiv -72 a^6 [(a^2)'^2 + 128 a^8 (a^2)' \phi' - 32 a^{10} (\phi')^2 - 72 a^8 (a^2)'' + 32 a^{10} \phi'']
\]

\[
+ \alpha' \left( 63 [(a^2)']^4 + 144 a^2 [(a^2)']^3 \phi' - 288 a^4 [(a^2)']^2 (\phi'(t))^2 + 128 a^6 (a^2)' (\phi')^3 
\]

\[
- 16 a^8 (\phi')^4 - 252 a^2 [(a^2)']^2 (a^2)'' + 288 a^4 (a^2)' \phi' (a^2)'' - 72 a^6 (\phi')^2 (a^2)'' 
\]

\[
+ 144 a^4 [(a^2)']^2 \phi'' - 144 a^6 (a^2)' \phi' \phi'' + 32 a^8 (\phi')^2 \phi'' \right),
\]

and the lapse function constraint (after setting \( N = 1 \)) is

\[
0 = \left( 3 (a^2)' - 2 a^2 \phi' \right) \left[ 48 a^6 (a^2)'^2 - 16 a^8 \phi' + \alpha' \left( 63 [(a^2)']^3 - 126 a [(a^2)']^2 \phi' 
\right.
\]

\[
+ 60 a^4 (a^2)' (\phi')^2 - 8 a^6 (\phi')^3 \right].
\]

To zeroth order in \( \alpha' \), we obtain the Einstein-dilaton equations,

\[
0 = 18 (a^2)' + a^2 \left[ 18 (a^2)'' + a^2 \left( 10 \phi'^2 - 9 \phi'' \right) \right] - 36 a^2 (a^2)' \phi',
\]

\[
0 = 9 (a^2)' + a^2 \left[ 9 (a^2)'' + 4 a^2 \left( \phi'^2 - \phi'' \right) \right] - 16 a^2 (a^2)' \phi',
\]

\[
0 = \left( 3 (a^2)' - 2 a^2 \phi' \right) \left( 3 (a^2)' - a^2 \phi' \right).
\]

which have the following solution:

\[
a^2(t) = t, \quad \phi(t) = \frac{3}{2} \ln(t).
\]
The integration constants \( c, a, \) solution may be computed using only the first order in \( \alpha \) expects a phase transition which cannot be described in general relativity. One does not expect the universe to “bounce” in the field theory description. Rather, one conclusions, this is actually consistent with expectations based on the string bits picture.

In order to find the stringy corrections to the cosmological background we express them as a series in \( \alpha' \), namely

\[
a^2(t) = a_0^2(t) + \alpha' a_I^2(t) + \alpha'^2 a_{I1}^2(t) + \cdots \quad \phi(t) = \phi_0(t) + \alpha' \phi_I(t) + \alpha'^2 \phi_{I1}(t) + \cdots
\]

(13)

where \( a_0^2(t) \) and \( \phi_0(t) \) are given by the zeroth order solutions (11). To first order, the equations for \( a_I \) and \( \phi_I \) are then sourced by the \( \alpha' \)-correction terms evaluated using the zeroth order solutions. Then, to first order in \( \alpha' \) equations (17)-(19) reduce to

\[
0 = 9 + 48 t^2 a_I^2 - 48 t^3 (a_I^2)' - 16 t^4 \phi_I' + 48 t^4 (a_I^2)'' - 24 t^5 \phi_I'',
\]

\[
0 = 9 + 24 t^2 (a_I^2) - 24 t^3 (a_I^2)' - 16 t^4 \phi_I' + 36 t^4 (a_I^2)'' - 16 t^5 \phi_I'',
\]

\[
0 = \frac{3 a_I^2}{t} - 3 (a_I^2)' + 2 t \phi_I',
\]

(14)

which are solved by

\[
a_I^2(t) = -\frac{1}{8 t^2} + c_1 t + c_3, \quad \phi_I(t) = -\frac{3}{16 t^3} + c_2 - \frac{3 c_3}{2 t}.
\]

(15)

The integration constants \( c_1, c_2 \) and \( c_3 \) can be removed by rescaling and shifting time, and by a shift of the dilaton, respectively. Thus, the only nontrivial corrections come from the first term of each field. The negative sign of these terms is interesting: it means that the \( \alpha' \)-corrections act to strengthen the onset of the singularity. As we briefly discuss in the conclusions, this is actually consistent with expectations based on the string bits picture. One does not expect the universe to “bounce” in the field theory description. Rather, one expects a phase transition which cannot be described in general relativity.

As mentioned, the calculations of (12) found no corrections to the effective action at order \( \alpha'^2 \). Assuming all such terms are ruled out, the second order correction to the background solution may be computed using only the first order \( \alpha' \)-correction in the action. At second order in \( \alpha' \), equations (17)-(19) read:

\[
0 = 153 + 768 t^7 (a_I^2)' - 256 t^7 \phi_{I1}' + 768 t^8 (a_I^2)'' - 384 t^8 \phi_{I1}'',
\]

\[
0 = 27 + 96 t^7 (a_I^2)' - 32 t^7 \phi_{I1}' + 72 t^8 (a_I^2)'' - 32 t^8 \phi_{I1}'',
\]

\[
0 = 9 + 192 t^7 (a_I^2)' - 128 t^7 \phi_{I1}',
\]

(16)
and their solution is

\[ a_{II}^2(t) = -\frac{9}{160} t^5 + c_1 t + c_3, \quad \phi_{II}(t) = -\frac{123}{1280} t^6 + c_2 - \frac{3c_3}{2t}. \]  \hspace{1cm} (17)

Again the integration constants \(c_1, c_2\) and \(c_3\) can be removed by coordinate transformations. One can similarly calculate the \(\alpha'\) corrections coming from the Gross-Sloan action, even though this is subject to the caveat made in the introduction. In fact, the coefficient of \(\alpha'^3\) in Gross and Sloan’s correction term is small, and the leading correction to the solution at order \(\alpha'^3\) actually comes from the the first order \(\alpha'\)-correction in the action. The latter correction is

\[ a_{III}^2(t) = -\frac{63}{1280} t^8, \quad \phi_{III}(t) = -\frac{437}{5120} t^9, \]  \hspace{1cm} (18)

where we have removed the integration constants.

Notice that the corrections only become significant for \(t < (\alpha')^{1/3}\), the string time, consistent with naive expectations (see Figure II). Furthermore, the corrections act to strengthen the singularity in the scale factor and the divergence of the dilaton to \(-\infty\). As shall be explained in the conclusions, this is actually consistent with the string bits picture.
IV. COSMOLOGICAL PERTURBATIONS

We now turn to calculating the behaviour of linearized perturbations about this cosmological background (6). The perturbations can, as usual, be decomposed into a transpose-traceless tensor component, a vector piece and a scalar degree of freedom, which each evolve independently. Here, we shall ignore the vector perturbations since there are no vector sources at the linearized level.

A. Tensor Perturbations

We begin with the tensor perturbations, which are gauge invariant and hence technically the easiest to study. The general tensor perturbation may be expressed as a linear combination of gravitational plane waves and, without loss of generality, we can focus on just one wave propagating in the $\hat{k}$ direction, i.e. $h_{ij} \propto e^{i \hat{k} \cdot \vec{x}}$, where

$$ds^2 = a^2(t)\left(-dt^2 + \sum_{i=1}^{9} \sum_{j=1}^{9} (\delta_{ij} + h_{ij})dx^idx^j\right),$$

(19)

and for tensor perturbations, $\sum_j h_{ij}k_j = \sum_j h_{ij}k_j = h_{ii} = 0$, with $|h_{ij}| \ll 1$. To zeroth order in $\alpha'$, the variation of action (5) with respect to the tensor perturbation gives

$$h_{ij}'' + \frac{1}{t}h_{ij}' + k^2h_{ij} = 0,$$

(20)

where $' = d/dt$ as before. The general solution is given in terms of the Bessel functions:

$$h_{ij} = A_{ij}J_0(kt) + \frac{\pi}{2}B_{ij}K_0(kt),$$

(21)

where $A_{ij}$ and $B_{ij}$ are constant matrices, and the factor $\frac{\pi}{2}$ is convenient when expanding for small $k \equiv |\vec{k}|$. The $J_0$ function is regular at $t = 0$ whereas $K_0$ is logarithmically divergent. We shall mainly be interested in the behavior of the perturbations at long wavelengths, $k \to 0$, for which the solution reduces to

$$h_{ij} = A_{ij} + B_{ij} \ln \left(\frac{kt e^\gamma}{2}\right),$$

(22)

where Euler’s constant $\gamma \simeq 0.5772$. Note that $h_{ij}$ has a trivial regular piece proportional to $A_{ij}$, and an irregular component given by the $B_{ij}$ term, which diverges as $t \to 0$. Furthermore, note that the solution tends to minus infinity logarithmically as $k \to 0$.

Through a similar analysis to that explained above for the background, we can compute the $\alpha'$-corrections to the tensor perturbations. First we express the $h_{ij}$ as a series in ascending
where the LHS is just the Bessel differential operator acting on \( h^I_{ij} \), and the RHS is the source term given at small \( k \) in terms of the zeroth order solution \( h^0_{ij} \) \(^{22}\). Solving at small \( k \), we obtain

\[
h^I_{ij} = -\frac{B_{ij}}{6t^3} + (d_1)_{ij} + (d_2)_{ij} \ln(t), \tag{24}
\]

where the integration constants \((d_1)_{ij}\) and \((d_2)_{ij}\) can be absorbed into \( A_{ij} \) and \( B_{ij} \). Thus the only non-trivial correction comes from the term proportional to \( t^{-3} \). This term has the same negative sign we found for the background, and as before, its effect is to strengthen the singular behavior of the solution. The coefficient is again small, so that the corrections only become important at around \( t \sim 1 \) in units of \( \alpha' = 1 \) (see Figure 2).

To second order in \( \alpha' \) we obtain a similar equation for \( h^{II}_{ij}(t) \), sourced by the zeroth and first order solutions, namely

\[
(h^{II}_{ij})'' + \frac{1}{t}(h^{II}_{ij})' + k^2 h^{II}_{ij} = \frac{k^2 h^0_{ij}}{32t^6} + \frac{k^2 h^I_{ij}}{4t^4} - \frac{169 (h^0_{ij})'}{160t^7} - \frac{(h^I_{ij})'}{2t^4} + \frac{7 (h^0_{ij})''}{32t^6} + \frac{(h^I_{ij})''}{t^3}. \tag{25}
\]

It is again simple to solve this in the limit of small \( k \), obtaining

\[
h^{II}_{ij} = -\frac{47B_{ij}}{480t^6} + (d_1)_{ij} + (d_2)_{ij} \ln(t), \tag{26}
\]

where the integration constants can again be absorbed in the zeroth order solution.

One can straightforwardly repeat the process for nonzero \( k \), obtaining the solution as a series in \( kt \):

\[
h_{ij} = h^0_{ij} \left[ 1 - \frac{(kt)^2}{4} - \frac{(kt)^4}{64} + ... - \frac{\alpha'}{t^3} \left( \frac{13(kt)^4}{32} + ... \right) - \frac{\alpha'^2}{t^6} \left( \frac{9(kt)^2}{320} + \frac{251(kt)^4}{2560} + ... \right) \right] + B_{ij} \left[ \frac{(kt)^2}{4} - \frac{3(kt)^4}{128} + ... - \frac{\alpha'}{6t^3} \left( 1 + \frac{29(kt)^2}{4} + \frac{653(kt)^4}{64} + ... \right) \right] + \frac{\alpha'^2}{96t^6} \left( \frac{47}{5} + \frac{547(kt)^2}{20} + \frac{4481(kt)^4}{64} + ... \right), \tag{27}
\]

where \( h^0_{ij} \) is the zeroth order solution \(^{22}\) and the dots represents terms of order \( O(k^6) \). At finite \( k \) we see that even the regular \( A_{ij} \) mode eventually suffers divergent corrections near \( t = 0 \). This is just as one expects. When the physical wavelength of a mode, \( L = t^2/k \), becomes smaller than the string scale, \( L_s \equiv \sqrt{\alpha'} \), the \( \alpha' \)-corrections become large. All of the
FIG. 2: The plot shows a single component of the tensor perturbation $h_{ij}$ in the limit of small $k$ as a function of time, focusing on its behavior near $t = 0$. The zeroth order solution is the solid line and as we move to the right we get the first, second and third order corrections respectively. We have chosen $A_{ij} = -3$ and $B_{ij} = 1$, which corresponds to $k \sim 0.02$ in the solution with non-zero $\vec{k}$, given by equation (22), with $A_{ij} = B_{ij} = 1$.

corrections may be expressed as positive powers of $L/T$ and $L_s/T$ where $T$ is the proper time in string frame, $T = \frac{2}{3}t^{\frac{4}{3}}$.

We have also computed the correction at order $\alpha'^3$, coming from the $\alpha'$ term in the effective action and also from Gross and Sloan’s correction at $\alpha'^3$. For the former we find

$$h^{III}_{ij} = -\frac{2183}{23040} \frac{B_{ij}}{t^9}. \quad (28)$$

We have checked that the Gross-Sloan correction is much smaller than this. But since that term is in any case incomplete, we shall not bother to state the correction here.

The behavior of the corrected solution up to $\alpha'^3$ is shown in Figure 2. Now, let us turn to a consideration of the scalar perturbations.

**B. Scalar Perturbations**

The most general scalar perturbation about our background solution can be written as

$$ds^2_{10} = a^2(t)[-(1 + 2\Upsilon_{10})dt^2 - 2\partial_i\Omega_{10}dx^i dt + [(1 - 2\Psi_{10})\delta_{ij} - 2\partial_i\partial_j\chi_{10}]dx^i dx^j]. \quad (29)$$
It will be useful to keep in mind the Kaluza-Klein map to eleven dimensions, which encodes the geometrical role of the dilaton:

\[
\text{ds}_{11}^2 = e^{\frac{4}{3}(\phi + \delta \phi)} \, d\theta^2 + e^{-\frac{2}{3}(\phi + \delta \phi)} \, ds_{10}^2.
\]  

(30)

The equations of motion for the perturbations are calculated by expanding the action (5) to second order in the perturbations and calculating the resulting Euler-Lagrange equations. In order to calculate the perturbations we have to fix a gauge, but it is useful to identify gauge invariant quantities in terms of which we can express the results. Since all perturbations have the same origin, as gravitational waves in flat eleven-dimensional spacetime, we expect to be able to choose coordinates so that they all obey the same equations of motion. We shall show that by choosing the gauge appropriately in the scalar sector, all the scalar perturbation variables end up obeying exactly the same equations as the tensor perturbations, up to the order in \( \alpha' \) which we work.

Under the change of coordinates \( x^\mu \rightarrow x^\mu + \xi^\mu \), with \( \xi^\mu \) small, it is straightforward to check that the scalar perturbations transform as

\[
\begin{align*}
\Upsilon_{10} &\rightarrow \Upsilon_{10} - \partial_0 \xi^0 - \frac{a'}{a} \xi^0, \\
\delta \phi &\rightarrow \delta \phi - \frac{\phi'}{3} \xi^0, \\
\chi_{10} &\rightarrow \chi_{10} + \xi^s, \\
\Psi_{10} &\rightarrow \Psi_{10} + \frac{a'}{a} \xi^0, \\
\Omega_{10} &\rightarrow \Omega_{10} - \xi^0 + \partial_0 \xi^s,
\end{align*}
\]

(31, 32, 33, 34, 35)

where \( \xi^i \equiv \partial_i \xi^s \). In particular, we notice that

\[
\rho = \Psi_{10} + \frac{1}{3} \delta \phi
\]

(36)

is a gauge-invariant quantity, because \( a'/a = \phi'/3 \). In fact, \( \rho \) is just the eleven-dimensional isotropic perturbation \( \Psi_{11} \), which is gauge invariant just because the spatial metric is static in eleven dimensions. As we shall discuss later, we expect the condition \( a^2 = e^{2\phi'/3} \), necessary for this correspondence to hold, to be enforcible by suitable field redefinitions to all orders in \( \alpha' \).

Now, to fix the gauge, it is convenient to make the choice that the spatial metric perturbation to the eleven-dimensional metric be traceless. This condition ensures that in the long wavelength limit, the solutions are linearized versions of the well-known Kasner solutions. In terms of our ten dimensional variables, this condition reads:

\[
\chi_{10} = \frac{9}{k^2} \left( \Psi_{10} + \frac{1}{3} \delta \phi \right),
\]

(37)
and it fixes the $\xi^a$ gauge freedom completely. One remaining gauge choice is needed to fix $\xi^0$.

A second relation is then found from the field equations. By adjusting the gauge condition order by order in $\alpha'$, we find that the conditions

$$\Upsilon_{10} = -4 \left( 1 - \frac{3 \alpha'}{2 t^3} - \frac{3(5 \alpha')^2}{2 t^6} \right) \Psi_{10},$$

$$\Psi_{10} = -3 \left( 1 + \frac{\alpha'}{2 t^3} + \frac{13 \alpha'^2}{16 t^6} \right) \rho,$$  \hspace{1cm} (38)

result in an equation for $\rho$ which is exactly the same equation as that found earlier for the tensor perturbation $h_{ij}$, for all $k$.

It is helpful to re-interpret the result in eleven dimensions. The general scalar perturbation of (11) involving the lowest Kaluza-Klein modes, and no gauge fields (these are projected out by the $Z_2$ orbifolding in the heterotic theory) is

$$ds_{11}^2 = -(1+2\Upsilon_{11})dt^2 + t^2(1-2\Gamma_{11})d\theta^2 - 2\partial_i \Omega_{11} dx^i dt + [(1-2\Psi_{11})\delta_{ij} - 2\partial_i \partial_j \chi_{11}] dx^i dx^j. \hspace{1cm} (39)$$

Comparing with (29) and (30), we see that

$$\Gamma_{11} = -\frac{2}{3} \delta \phi,$$ \hspace{1cm} (40)

$$\Psi_{11} = \Psi_{10} + \frac{1}{3} \delta \phi = \rho,$$ \hspace{1cm} (41)

$$\Upsilon_{11} = \Upsilon_{10} - \frac{1}{3} \delta \phi.$$  \hspace{1cm} (42)

To lowest order in $\alpha'$, the $(t, \theta)$ part of the eleven-dimensional metric is only conformally perturbed in this gauge (i.e. $\Gamma_{11} = -\Upsilon_{11}$). This is a nice feature in providing a geometrical interpretation of the matching conditions across the bounce, as explained in [21]. This property is spoiled by the higher order $\alpha'$-corrections, if one adopts the naive map (30). However, as we shall see later, once the map is suitably adjusted, this feature of the eleven dimensional metric is retained.

As mentioned, the field $\rho$ obeys the same equation as the tensor modes, for all $k$, confirming the idea of a common higher dimensional origin. In the limit of small $k$, the solution is thus

$$\rho = A + B \ln(t) - \frac{\alpha' B}{6 t^3} - \frac{47 \alpha'^2 B}{480 t^6} + \ldots$$ \hspace{1cm} (43)

For nonzero $k$ we get (27), but with $A$ and $B$ instead of $A_{ij}$ and $B_{ij}$. We can calculate the dilaton perturbation using (38) and the solution for $\rho$. In the limit of small $k$, this is

$$\delta \phi = (A + B \ln(t)) \left( 12 + \frac{9 \alpha'}{2 t^3} + \frac{117 \alpha'^2}{16 t^6} \right) - \frac{B \alpha'}{t^3} \left( 2 + \frac{77 \alpha'}{40 t^3} \right).$$  \hspace{1cm} (44)

If the eleven dimensional picture is correct, we expect it is possible to choose a gauge in which scalars and tensors obey the same equation, to all orders in $\alpha'$. 

14
V. ELEVEN DIMENSIONAL CONNECTION

At lowest order in $\alpha'$, we have the relation $a^2 = e^{2\phi/3}$, and as we have seen, this is preserved at first and second order in $\alpha'$. There are reasons to believe the relation will continue to hold, perhaps after a suitable field redefinition, to all orders. The reason is that in the eleven dimensional picture of M-theory, the gauge fields live on the two orbifold planes which, in the solution (1) neither expand nor contract. This is consistent with the gauge field Lagrangian in heterotic string theory, which takes the form $\int d^{10}x \sqrt{-g} e^{\lambda\phi} F^2$, with $\lambda$ such that the dilaton’s time dependence precisely cancels that of the scale factor, if the relation $a^2 = e^{2\phi/3}$ is satisfied. Therefore, we can expect this relation to hold to all orders, possibly after a field redefinition in string frame, if the metric (1) is a solution of M-theory.

In this section, we want to go further and construct a map from eleven to ten dimensions. Since the eleven dimensional metric should receive no corrections, all of the $\alpha'$-corrections have to arise from the map. We will consider, in order, the background, perturbations at long wavelengths, and then the leading nontrivial $k$-dependence. The most general covariant expression for the ten dimensional string frame metric in terms of the eleven dimensional M-theory metric is

$$g_{(10)}^{\mu \nu} dx_{10}^\mu dx_{10}^\nu = e^\gamma \left( 1 + m_0 \alpha' e^{-\gamma} (\nabla \gamma)^2 + \ldots \right) dx_{11}^\mu dx_{11}^\nu \left( g_{(11)}^{\mu \nu} + m_1 \alpha' e^{-\gamma} \nabla_{\mu} \gamma \nabla_{\nu} \gamma ight) + m_2 \alpha' e^{-2\gamma} \nabla_{\mu} (\nabla_{\nu} e^\gamma) + m_3 \alpha' e^{-\gamma} R_{\mu \nu}^{(11)} + m_4 \alpha' e^{-\gamma} g_{\mu \nu}^{(11)} R^{(11)} + \ldots, \quad (45)$$

where $\mu, \nu$ run over the ten string-theory dimensions and $\gamma = \frac{1}{2} \ln(g_{\theta \theta}^{(11)})$, and the ellipses indicate higher order $\alpha'$-corrections. Since we are working at long wavelengths, we restrict ourselves to terms with only two derivatives. The form of each term and, in particular, the powers of $e^\gamma$ are determined by dimensional analysis.

We compute the RHS of (45) using the eleven dimensional metric (1) with tensor or scalar perturbations. We then compare it with the LHS using the ten dimensional results from the $\alpha'$ expansion, and attempt to fix the coefficients $m_a (a = 0, \ldots, 4)$. If we insist on the relation $a^2 = e^{2\phi/3}$, we have

$$e^{2\phi/3} = e^\gamma \left( 1 + m_0 \alpha' e^{-\gamma} (\nabla \gamma)^2 + \ldots \right). \quad (46)$$

In the background (1), then $\gamma = \ln(t_{11})$ and the spatial components of the map (45) give

$$a_{(10)}^2 \delta_{ij} \quad (47)$$

for the LHS, and

$$e^\gamma \left( 1 + m_0 \alpha' e^{-\gamma} (\nabla \gamma)^2 + \ldots \right) \delta_{ij} \quad (48)$$
for the RHS, consistent by construction with \( a^2 = e^{2\phi/3} \). However, the time-time component of (45) leads to

\[-a^2 = (dt_{11}/dt)^2 t_{11}(1 + m_0\alpha'/t_{11}^3)(-1 + m_1/t_{11}^3),\]

where \( t = t_{10} \). After expanding to first order in \( \alpha' \) and integrating both sides of the map, we find that the eleven dimensional time is related to ten dimensional time by

\[t_{11} = t \left(1 - m_1 \frac{\alpha'}{4t_3^3} + \mathcal{O}(\alpha'^2)\right),\]

(50)

since the term involving \( m_0 \) cancels. Therefore, the time in eleven dimensions is not the same as in ten and, in particular, the singularity at \( t_{11} = 0 \) gets mapped to a positive time in string frame for \( m_1 > 0 \).

To complete the background analysis, we need to compare the dilaton expansion (46) with its stringy corrections (15). After rewriting the stringy corrections using the eleven dimensional time \( t_{11} \), we get a relation for the unknown coefficients \( m_0 \) and \( m_1 \), given by

\[m_0 + \frac{1}{4} m_1 = \frac{1}{8}.\]

(51)

Turning to the tensor perturbations, the only piece of the map that changes is the tensor component in the metric \( g_{ij}^{(11)} \): the LHS of the map (45) reads

\[a^2(\delta_{ij} + h_{ij}) = a^2 \left[\delta_{ij} + A_{ij} + B_{ij} \left(\ln(t) - \frac{\alpha'}{6t^3}\right)\right],\]

(52)

and the RHS reads

\[e^{2\phi/3} \left(\delta_{ij} + h_{ij}^{(11)} + \frac{m_2\alpha'}{2t_{11}^2}\Gamma^0_{ij}\right) = e^{2\phi/3} \left[\delta_{ij} + A_{ij} + B_{ij} \left(\ln(t_{11}) - \frac{m_2\alpha'}{t_{11}^3}\right)\right].\]

(53)

By comparing both sides and using the time transformation (50), we get another constraint for the unknown coefficients \( m_1 \) and \( m_2 \),

\[m_2 + \frac{m_1}{2} = \frac{1}{3}.\]

(54)

In the case of the scalar perturbations the comparison between the LHS and RHS of the map (45) is not as simple as for the tensor modes, due to the fact that we have to take into account the perturbation in \( \gamma \), which is given by \( \gamma \rightarrow \gamma + \delta \gamma = \ln(t)(1 - \Gamma_{11}) \), and, more importantly, there is a remaining gauge freedom. Thus one has to be careful to choose a gauge invariant variable to compare on both sides of the map. As we have seen in the last section, the quantity \( \rho \) defined in (36) is one such quantity, which is furthermore local in time. This means we do not need to use any of the time components of the map (45), which
would have involved an integration, just as occurred for the background. Under the gauge choice (38), the quantity \( \rho \) simplifies to

\[
\rho = A + B \left( \ln(t) - \frac{\alpha'}{6 t^3} \right) + \mathcal{O}(\alpha'^2).
\]

(55) Using the RHS of the map (45) and the dilaton expansion (46), we get the following expressions for \( \Psi_{10} \) and \( \delta\phi \) in terms of 11d quantities

\[
\Psi_{10} = -3 \left( 1 + \alpha' \frac{4 m_0}{t_{11}^3} \right) \Psi_{11} + \alpha' \frac{16 m_0 - m_2}{2 t_{11}^2} \partial_0 \Psi_{11},
\]

\[
\delta\phi = 12 \left( 1 + \alpha' \frac{3 m_0}{t_{11}^3} \right) \Psi_{11} - \alpha' \frac{24 m_0}{t_{11}^2} \partial_0 \Psi_{11},
\]

(56) which results in

\[
\rho \equiv \Psi_{10} + \frac{\delta\phi}{3} = \Psi_{11} - \alpha' \frac{m_2}{2 t_{11}^2} \partial_0 \Psi_{11} = A + B \left( \ln(t_{11}) - \frac{\alpha' m_2}{2 t_{11}^2} \right).
\]

(57) By comparing both equations (55) and (57), we get again condition (54). Therefore, the map is consistent. The terms involving the Ricci tensor and Ricci scalar do not contribute at all to this background, leaving \( m_3 \) and \( m_4 \) unfixed; and the two conditions (51) and (54) can only fix two of the three remaining coefficients. In particular, we can take \( m_0 = 0 \) and then fix the other two parameters to be: \( m_1 = 1/2 \) and \( m_2 = 1/12 \). This choice represents a simple and physical picture: if \( m_0 = 0 \) then \( e^{2\phi/3} = t_{11} \), which implies that the dilaton really measures the eleven-dimensional distance between the two orbifold planes.

Finally, let us see how the map can be extended to describe perturbations at finite \( k \). We focus on tensor perturbations because of their gauge invariance. We include terms which only contribute to a desired order in \( k \) and which do not change the mapping for the background or at lower orders in \( k \). At order \( k^2 \), we have found two terms which provide two unknown parameters needed to recover the 10d result (27). The map with these new extra terms is

\[
g^{(10)}_{\mu\nu} dx_{10}^\mu dx_{10}^\nu = e^\gamma \left( 1 + m_0 \alpha' e^{-\gamma} (\nabla \gamma)^2 + \ldots \right) dx_{11}^\mu dx_{11}^\nu \left( g^{(11)}_{\mu\nu} + m_1 \alpha' e^{-\gamma} \nabla_\mu \gamma \nabla_\nu \gamma \right)
\]

\[
+ m_2 \alpha' e^{-2\gamma} \nabla_\mu (\nabla_\nu e^\gamma) + m_3 \alpha' e^{-\gamma} R^{(11)}_{\mu\nu} + m_4 \alpha' e^{-\gamma} g^{(11)}_{\mu\nu} R^{(11)}_{\mu\nu}
\]

\[
+ m_5 \alpha' (\nabla^\lambda e^{-\gamma}) \nabla_\lambda \left( e^{4\gamma} R^{\alpha\beta}_{\mu\nu} \alpha^\beta \nabla_\alpha \gamma \nabla_\beta \gamma \right) + m_6 \alpha' (\nabla^\lambda e^{-\gamma}) \nabla_\lambda \left( e^\gamma \nabla_\mu \nabla_\nu e^\gamma \right) + \ldots,
\]

(58) where \( R^{\alpha\beta}_{\mu\nu} \) is the 11d Riemann tensor. We calculate the RHS of this map using the tensor-perturbed metric (19), and then we Taylor expand the solution to order \( k^2 \). Only the spatial part contributes to the tensor equation, and by comparing it to the non-zero \( k \) solution (27), we obtain two more constraints for the parameters \( m_i \), given by

\[
m_1 + 2m_2 + 3m_5 + 4m_6 = 0, \quad 9m_1 + 18m_2 + 24m_6 = 58.
\]

(59)
These two additional equations allow us to fix the new constants,

\[
m_1 = \frac{1}{2} - 4m_0, \quad m_2 = \frac{1}{12} + 2m_0 \quad m_5 = -\frac{7}{3}, \quad m_6 = \frac{13}{6},
\]

with \(m_0\) still undetermined, as before. One can extend this map to higher order \(k\) by using the same trick: the operator \((\nabla^\gamma e^{-\gamma})\nabla_\lambda e^{m_5}\) applied on the previous \(k\)-order term, with the correct \(m\)-number for a given \(k\). To conclude this section, we have shown that the first order \(\alpha'\)-corrections, in the ten-dimensional string frame, are all accounted for by a non-trivial mapping from eleven-dimensional Einstein gravity.

VI. DIVERGENCE OF THE \(\alpha'\) SERIES NEAR THE SINGULARITY

In this final section, we want to understand in detail how the \(\alpha'\) expansion fails near the singularity. Because the \((\nabla \phi)^8\) term at order \(\alpha'^3\) has not, to our knowledge, yet been computed, we shall only consider the effect of the order \(\alpha'\) term in the effective action. However, even with only this term, there are some interesting features. Using the fact that the dilaton and the scale factor are related by \(a^2 = e^{2\phi/3}\), the equations of motion for the background (7)-(9) can be reduced to to a single equation

\[
3\alpha'[(a')^4 - a(a')^2 a''] + a^4[(a')^2 + a a''] = 0,
\]

where \(\dot{} = \frac{d}{dt}\). This equation may be simplified by expressing it in terms of the physical Hubble parameter \(H = a'/a^2\) and the proper time \(T = \int a(t)dt\); it becomes

\[
\dot{H}(1 - 3 \alpha' H^2) + 3 H^2 (1 - \alpha' H^2) = 0,
\]

which is easily solved. For small \(H\), i.e. in the regime where the \(\alpha'\) expansion is good, we have \(H \sim 1/3T\), implying \(a \sim T^{1/3} \sim t^{1/2}\), as expected. For large \(H\), the solution goes like \(H \sim 1/T\), implying \(a \sim T \sim e^t\). However, the equation fails at \(T \sim 1\), because \(dH/dT\) becomes infinite, so one cannot get to the large \(H\) regime.

Nevertheless, we can certainly use equation (62) to compute \(H\) as a series in \(\alpha'\); we obtain

\[
H = \frac{1}{3T} + \frac{2 \alpha'}{27 T^3} + \frac{26 \alpha'^2}{729 T^5} + \frac{242 \alpha'^3}{10935 T^7},
\]

which one may check is equivalent to our earlier expressions for \(a(t)\) in conformal time.

Figure 3 shows the series solution (63), order by order in \(\alpha'\), plotted against the solution of equation (62). Serious discrepancies set in around \(T \sim 1\), which of course comes as no surprise.

Clearly, one cannot trust the calculation of the Hubble constant \(H\) for times \(T\) smaller than unity, in string units. If one only has the series (63), how would one go about checking
FIG. 3: Solutions to the Hubble equation (62). The series solution (63) is presented order by order, and also its resummed expression using the Shank’s transformation. The full solution to the Hubble equation (62) is also shown, being badly behaved when $\dot{H} = \infty$.

One way is to attempt to resum the series; we have chosen the Shanks transformation [25], which is simple to apply in this case. If a series has transient such that (in some region of the complex plane) the partial series $I_n$ is equal to $I + \lambda q^n$, where $\lambda q^n$ is the transient and $|q| < 1$, and $I_n \to I$ as $n \to \infty$, then we can use $I_{n-1}$, $I_n$ and $I_{n+1}$ to calculate $I$, which is given by

$$I = \frac{I_{n+1}I_{n-1} - I_n^2}{I_{n+1} + I_{n-1} - 2I_n}. \quad (64)$$

Note, for example that this resummation is exact for the series $1 + x + x^2 + \ldots = 1/(1 - x)$. Applying this to the series (63), one finds a simple pole at $T \sim 0.7$. More important, however, the resummed curve is not very close to the series up to $\alpha'^3$ (see Figure 4). Hence the resummation is unlikely to be correct.

However, some other quantities may be better behaved under resummation. Let us return to consider the background scale factor $a(t)$ and the metric perturbations. As we have seen, the singularity is reached earlier if the $\alpha'$-corrections are included (see Figure 4). We can extrapolate the metric perturbation $a^2 h_{\mu\nu}$ to this point in time to check whether it is finite. To lowest order in $\alpha'$, it is zero, since $t \ln t$ tends to zero. This means one cannot match the
amplitude of this term across \( t = 0 \). However, as shown in Figure 4, this quantity seems to remain finite as the singularity is approached. We can check these conclusions by applying the same simple Shanks resummation, finding in this case that the resummed value is rather close to the series up to \( \alpha'^3 \). This is no more than suggestive, but it may indicate that when all orders in \( \alpha' \) are included these properties will persist.

VII. CONCLUSIONS

In this paper, we have performed a detailed study of \( \alpha' \)-corrections on the simplest ten-dimensional cosmological background solution in string theory, corresponding to eleven-dimensional compactified Milne spacetime in M-theory. We have computed the effect of these corrections on both the background and on linearized scalar and tensor perturbations, to first order in \( \alpha' \). From the M-theory viewpoint, away from the singularity, our background is Riemann-flat in eleven dimensions and hence it should provide an exact background to M-theory. Similarly, one can argue that the linearized perturbations should be exactly described by Einstein gravity in eleven dimensions, since any higher powers of curvature invariants would give no contribution. As a check of this idea, we have verified that, to the order in \( \alpha' \) we compute, all the \( \alpha' \)-corrections in the string frame calculations are indeed possible to generate by field redefinitions. Several “miracles” are necessary in order for this
to occur - for example, the dilaton and the scale factor get corrections in just such a way that the right combination, representing the “transverse” scale factor in eleven dimensions, remains unchanged.

From the string theory viewpoint, we have shown that the $\alpha'$-corrections modify the usual zeroth order solutions so that the singularity occurs sooner, in background conformal time, for both the metric and dilaton fields. Thus the $\alpha'$-corrections do not seem to resist the formation of a cosmic singularity. We have attempted a simple resummation, obtaining a simple pole as the asymptotic behaviour near $t = 0$, and a finite metric perturbation when the scale factor reaches zero. However, such results are inconclusive, because we expect $\alpha'$ expansion to break down well before the singularity, which may be describeable using an expansion in $1/\alpha'$, as discussed in [4, 5].

Clearly, we cannot use the $\alpha'$ expansion to study the singularity itself, nor the transition across it. It will be necessary to develop a more powerful calculational approach, incorporating both the $\alpha'$ expansion at large times, and a new expansion in $1/\alpha'$ near the singularity. Nevertheless, it may be hoped that the results we have obtained will be useful in comparing with this future, more complete treatment.

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