Calabi-Yau generalized complete intersections and aspects of cohomology of sheaves

Qiuye Jia$^{1,2}$, Hai Lin$^1$

$^1$Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, P. R. China
$^2$Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P. R. China

Abstract

We consider generalized complete intersection manifolds in the product space of projective spaces, and work out useful aspects pertaining to the cohomology of sheaves over them. First, we present and prove a vanishing theorem on the cohomology groups of sheaves for subvarieties of the ambient product space of projective spaces. We then prove an equivalence between configuration matrices of complete intersection Calabi-Yau manifolds. We also present a formula of the genus of curves in generalized complete intersection manifolds. Some of these curves arise as the fixed point locus of certain symmetry group action on the generalized complete intersection Calabi-Yau manifolds. We also make a blowing-up along the curves, by which one can generate new Calabi-Yau manifolds. Moreover, an approach on spectral sequences is used to compute Hodge numbers of generalized complete intersection Calabi-Yau manifolds and the genus of curves therein.
1 Introduction

The complete intersection Calabi-Yau manifolds \([1, 2, 3]\) might be a competitive candidate for spacetime model in string theory. In this method of constructing a Calabi-Yau manifold, the complicated Calabi-Yau geometry is embedded into a relatively simple ambient space, such as a product of projective spaces. Many geometric quantities of the Calabi-Yau manifolds can be deduced by their relations to those of the ambient space. In this paper, we explore a generalization of complete intersection Calabi-Yau manifolds \([4]\). These complete intersection Calabi-Yau (CICY) manifolds, including three-folds \([1, 2, 3]\) and four-folds \([5, 6, 7, 8, 9]\), have been constructed and investigated extensively.

Such intersections can be described in the language of line bundles. The complete intersection Calabi-Yau manifolds are defined by the common zero locus of polynomials, which are global sections of line bundles with non-negative degrees. This can be generalized to the construction of new Calabi-Yau manifolds, through bundles with no global sections on the ambient space \([4]\). In this new construction, negative degrees are allowed, that is, one can replace polynomials of non-negative degrees by rational functions. These manifolds are solutions to systems of algebraic equations in a product of projective spaces where the functions in the defining equations may have negative degrees. This is to allow the existence of poles. The Laurent polynomials in these equations have poles, but these poles avoid the common intersection locus, and this largely expands the construction of complete intersection Calabi-Yau varieties \([4, 10, 11]\).

However, we do not require all the line bundles to have global sections at the first place, and we need to take intersections step by step \([4]\), to construct these generalized complete intersection Calabi-Yau (gCICY) varieties. The idea is to take a submanifold \(M\) in the ambient manifold \(A\) and to consider submanifolds \(X\) in \(M\). These submanifolds \(X\) need not be complete intersections in the ambient manifold \(A\). To be more specific, the domain that we require these sections to be regular is decreasing everytime when we take intersections. Hence, although the line bundle does not have a global section on the entire product of projective spaces, it has regular section when restricted to appropriate subvarieties. One then constructs the generalized complete intersection Calabi-Yau in these subvarieties.

In this paper we work on aspects of cohomology of sheaves over generalized complete intersection Calabi-Yau manifolds. We will develop some tools and approaches, in order to understand Calabi-Yau manifolds better. First, we introduce and prove a generalized version of vanishing theorem on the cohomology groups of sheaves that will be used in our computation of Hodge numbers of the gCICYs. This vanishing theorem is a generalization of the original theorem \([12, 13]\) from the case of a single projective space to the case of a product of several projective spaces. In the process of proving this generalized vanishing theorem, we used Poincare residue exact sequences and the method of induction. This cohomology group in the vanishing theorem will appear in the double complexes and the long exact sequences of the cohomology groups that we are interested in computing.

When we consider certain symmetry group actions on gCICYs, some curves would be the
fixed point locus. We make a blow up of the generalized complete intersection Calabi-Yau manifolds along these curves which we identified as the fixed point locus of some involutions. We compute the genus of curves which themselves can be viewed as generalized complete intersection manifolds in a product of projective spaces, and work out a general formula of their genus. This genus formula is needed for the Hodge numbers of the blow ups.

We devote a spectral sequence approach to the computation of Hodge numbers of gCICYs. These Hodge numbers not only encode topological information, but also give the dimensions of the moduli spaces. We show that an approach on spectral sequences can be used to efficiently compute the topological data of the generalized complete intersection manifolds, including the one dimensional case of curves. We build spectral sequences of double complexes of cohomology groups of sheaves and compute their dimensions. Moreover, one of the methods that we obtain the data of an appropriately twisted sheaf is to tensor known exact sequences by locally free sheaves. We also used this method when we prove the generalized vanishing theorem.

We also prove an identity between configuration matrices of complete intersection Calabi-Yau manifolds. This identity was initially put forward by [1] but was not proven before. This identity is useful since it was used in [3] in the classification of generalized complete intersection Calabi-Yau manifolds.

The gCICYs would give us manifolds that didn’t arise in the exhaustive list of CICYs. In addition, an important phenomenon in the deformation of CICYs is that in general, the moduli space of CICYs has dimension less than the moduli space of total deformation class. This is easy to understand: the manifolds that can be defined by polynomials are only a part among all those CY manifolds in this deformation class. It is conceivable that, after generalizing to gCICYs, we would increase the dimension of the moduli spaces. This fact coincides with the result that some of gCICYs are not previously constructed in the list of CICYs. In understanding their properties, we can make use of modern tools in sheaf theory and cohomology theory. Moreover, gCICYs is a promising candidate of important physics model [4, 10, 11]. In conjunction with the characterization of moduli spaces, we are able to understand CY manifolds better.

The organization of this paper is as follows. In Section 2, we briefly review the idea behind the generalization of CICY, and introduce the necessary background. In Section 3, we propose a vanishing theorem on the cohomology groups of sheaves for subvarieties of the ambient product space of projective spaces, with our identified condition. And then in Section 4, we prove an identity between configuration matrices of complete intersection Calabi-Yau manifolds. Afterwards in Section 5, we consider involution of gCICY and the fixed point locus which are curves. We present a general formula of the genus of the curves in gCICY. We also make a blow up along the curves. In Section 6, we present a spectral sequence approach of the cohomology groups of sheaves of gCICY, which are used to compute Hodge numbers and genus of curves, among other things. Finally, we discuss our results and draw some conclusions in Section 7. In Appendix A, we include some details pertaining to a special case of the genus formula. In Appendix B, additional computational details of Hodge
numbers and other topological data are included. In this paper, a variety is assumed to be a scheme of finite type over a field $k$, where $k$ is the complex number field in all the sections but it can be any algebraically closed field in Section 3.

2 Generalized Complete Intersection Manifolds and Calabi-Yau Manifolds

We devote this section to emphasize the idea behind the generalization of CICY [4]. We will explore the construction of generalized complete intersection Calabi-Yau manifolds given in [4]. These manifolds are constructed by line bundles on an ambient space. Here the ambient space $A$ is a product of projective spaces,

$$ A = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t}. \quad (2.1) $$

Let us consider a line bundle $L_a$ on $A$, with multi-degree $(q_1^a, \cdots, q_t^a)$, that is

$$ L_a = O_A (q_1^a, \cdots, q_t^a) = \otimes_i \pi_i^* O_{\mathbb{P}^{n_i}} (q_i^a). \quad (2.2) $$

Here $\pi_i$ is the projection of $A$ onto each projective space factor, $O_{\mathbb{P}^{n_i}} (q_i^a)$ is a line bundle on each projective space, and the $a$ labels each line bundle.

Let $M$ be a submanifold of $A$ that is defined by the common zero locus of several polynomial sections of line bundles on $A$. The degrees of these line bundles can be described by a matrix, which can be called the configuration matrix. Now we can add more columns in the configuration matrix. $X$ is the submanifold of $M$ that is defined by those sections of bundles on $M$ that correspond to those additional columns of the configuration matrix which contain negative entries. In other words, we can write them using configuration matrices as follows:

$$ M = \begin{bmatrix} \mathbb{P}^{n_1} & q_1^1 & \cdots & q_1^P \\ \mathbb{P}^{n_2} & q_2^1 & \cdots & q_2^P \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{P}^{n_t} & q_t^1 & \cdots & q_t^P \end{bmatrix} \subset A. \quad (2.3) $$

$$ X = \begin{bmatrix} \mathbb{P}^{n_1} & q_1^1 & \cdots & q_1^P & q_1^1 \\ \mathbb{P}^{n_2} & q_2^1 & \cdots & q_2^P & q_2^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{P}^{n_t} & q_t^1 & \cdots & q_t^P & q_t^1 \end{bmatrix} \subset M. \quad (2.4) $$

In the above, each column $(q_1^1, \cdots, q_t^1)^T$ of a configuration matrix corresponds to a line bundle $L_a$ with multi-degree $(q_1^1, \cdots, q_t^1)$, as defined in Eq. (2.2). The $a$ labels each column, and equivalently, each line bundle. The integers themselves denote the degrees of the defining polynomials in the homogeneous coordinates of the projective space factors.
The matrix elements of the configuration matrix (2.3) of the ordinary complete intersection \( M \) are non-negative integers. That is, \( q_a^i \geq 0 \), for \( a = 1, \ldots, P \). Each column of the configuration matrix of \( M \), defines a codimension-one hypersurface in the ambient space \( A \). If there are \( P \) columns, then \( M \) is the complete intersection of \( P \) hypersurfaces in \( A \). Each constraint is defined by a polynomial equation, in which the polynomial is a section \( s_a \in H^0(A, \mathcal{L}_a) \neq 0, \forall a = 1, \ldots, P \). Each polynomial is a homogenous polynomial with multi-degree \( (q_a^1, \ldots, q_a^P) \) in the homogenous coordinates of the \( t \) projective space factors. \( M \) is a subvariety in \( A \) with codimension \( P \), because all the degrees of the line bundles, defining \( M \), as a complete intersection, are non-negative.

The original construction of complete intersection Calabi-Yau manifolds did not allow negative integers in the configuration matrix, or in order words, negative degrees for the line bundles [1, 2, 3]. By taking into account non-polynomial deformations in the moduli space of Calabi-Yau varieties, construction of generalized complete intersection Calabi-Yau have been proposed [4]. The generalization is that one allows negative degrees for the line bundles, or in other words, negative integers in the configuration matrix. The additional columns, which are the last \( K - P \) columns in the configuration matrix (2.4) of \( X \), each contain negative integers. We have that for \( \forall a > P, \exists q_a^i < 0 \) in the line bundles defined in Eq. (2.2), in which case these line bundles do not have global sections on \( A \). These additional columns with negative integers correspond to the algebraic equations in products of projective spaces involving rational functions that have negative degrees. The pole locus of these rational functions avoid the common intersection locus \( M \). Denoting the pole locus of these rational functions on \( A \) to be \( \Delta \subset A \), we require that \( M \cap \Delta = \emptyset \).

Below, we illustrate the reason to consider these line bundles with no global sections on the ambient space and sketch the process to take intersections step by step [4]. Because there exists certain negative degree in the line bundle \( \mathcal{L}_a \), for \( a > P \), associated to the additional column in the configuration matrix of \( X \), we have that \( H^0(A, \mathcal{L}_a) = 0 \) and there is no global section of this line bundle over \( A \). As a consequence, \( X \) is not a submanifold in \( A \) defined by the global sections of line bundles on \( A \). Let’s consider a set of subvarieties \( M_a, a = P + 1, \ldots, K \), inside \( M \), and \( M_a \subset M_{a-1} \). And let us denote \( M_P = M \). The \( M_a \) is a subvariety in \( M_{a-1} \) defined by the section \( s_a \in H^0(M_{a-1}, \mathcal{L}_a|_{M_{a-1}}) \neq 0 \) for \( a = P + 1, \ldots, K \).

We take intersections step by step. The domain that we require the section to be regular is decreasing everytime we take intersection. Finally, \( X = M_K \), which is a submanifold in \( M \). So, although the line bundle does not have a global section on the entire product of projective spaces, it has sections that is regular on the subvarieties described above. If we consider those restrictive conditions one by one, say, from the left to the right in the configuration matrix (2.4), rather than simultaneously, we would obtain global sections. Hence \( X \) has codimension \( K - P \) in \( M \). In sum, we have \( X \stackrel{i_X}{\rightarrow} M \stackrel{i_M}{\rightarrow} A \), where \( i_X \) is the inclusion of \( X \) in \( M \), and \( i_M \) is the inclusion of \( M \) in \( A \), and we denote \( i = i_M \circ i_X \).

We now turn to the condition for the above configuration matrix to represent a Calabi-Yau manifold. Since \( X \) is a submanifold of \( M \), and \( M \) is in turn a submanifold of \( A \), we use the adjunction formulas iteratively, see the versions in Appendix B. We hence use the
adjunction formulas of the tangent sheaves in the sequence form iteratively to obtain the Calabi-Yau condition for $X$. Denote the dual of the ideal sheaf of $X$ to be $\mathcal{E}$. We use the fact that $\mathcal{E} = \oplus_a \mathcal{E}_a, \mathcal{E}_a = i^*O_A(q_a^1, \cdots, q_a^K)$, where $q_a^i$ are the matrix elements in the configuration matrix, $a = 1, 2, \cdots, K$. In our process of computing the Chern classes, we use the fact that since $\mathcal{E}$ is line bundle, its Chern class $c_k$ will vanish as long as $k \geq 2$, hence we have

$$c(\mathcal{E}) = \prod_a c(\mathcal{E}_a) = \prod_a (1 + \sum_i q_a^i J_i),$$

where $J_i = c_1(i^*\pi^*O_{\mathbb{P}^{n_i}}(1))$. The dimension of the $i$-th projective space factor is $n_i$. Hence using the adjunction formulas of the tangent sheaves iteratively, we have $c_1(X) = \sum_i (n_i + 1 - \sum_a q_a^i) J_i$. Hence the condition on the matrix elements of the configuration matrix for a Calabi-Yau manifold is

$$\sum_a q_a^i = n_i + 1,$$

where $q_a^i$ here can be either non-negative or negative. The above Calabi-Yau condition is that the sum of the matrix elements of each row equals the dimension of the projective space of the corresponding row plus 1. In the case of ordinary CICY, $q_a^i$ are non-negative only.

We can determine the Hodge numbers of the manifolds and then use them as topological invariants to classify those manifolds. Among other things, the Hodge numbers will give us the complex structure deformation space of the family of manifolds. To be more precise, configuration matrices describe a family of manifolds and a manifold is only determined after choosing its complex structure within that family. We are particularly interested in three-folds and four-folds for the manifolds $X$ due to their usefulness in compactification of string theory.

3 Generalization of a Vanishing Theorem

The original vanishing theorem on the vanishing of the cohomology group $H^0(Y, T_Y)$ for a hypersurface $Y$ in a single projective space, was based on the works of Bott, Deligne, Kodaira-Spencer, Matsumura-Monsky and overviewed in [13]. We are going to generalize it to the case of the product of projective spaces. When we consider complete intersection or generalized complete intersection manifolds, we want to consider the product of projective spaces to be the ambient space. The vanishing theorem that we generalized is useful for computing the Hodge numbers of the generalized complete intersection manifolds. This cohomology group would appear in the double complexes and the long exact sequences of the cohomology groups that we are interested in computing. We will denote the product by

$$A = \prod_{i=1}^t \mathbb{P}^{n_i+1}.$$
Although the results we have in other sections in this paper will only be the analytic case, which specializes the field to be the complex number field, our result in this section also holds for schemes over other algebraically closed fields.

### 3.1 Preparation Using Sheaves

Before presenting the generalized theorem and diving into the details, we develop some general results on divisors and sheaves. The ambient space $A = \prod_{i=1}^{t} P^{n_i+1}$, as the product space of projective spaces, have multi-dimension $n + 1 = (n_1 + 1, n_2 + 1, \ldots, n_t + 1)$. Note that here $n$ is a vector $(n_1, n_2, \ldots, n_t)$ with $t$ components. We also consider the ambient space $A$ to be a smooth scheme over an algebraically closed field $k$ and denote it as $A/k$. The specific case we are going to treat is a subvariety $Y$ of multi-degree $d = (d_1, d_2, \ldots, d_t)$, which is defined by the zero of a homogenous polynomial of degrees $d_1, d_2, \ldots, d_t$ respectively in the homogenous coordinates of the $t$ projective space factors of $A$.

For brevity, for $a = (a_1, a_2, \ldots, a_t)$, we denote the line bundle as follows, $O(a) = \otimes_i \pi_i^* O_{P^{n_i+1}}(a_i)$. We define the sheaves

$$E(a) = E(a_1, \ldots, a_t) = E \otimes O(a) = E \otimes \pi_1^* O_{P^{n_1+1}}(a_1) \otimes \cdots \otimes \pi_t^* O_{P^{n_t+1}}(a_t),$$

when tensoring a given sheaf $E$ with $O(a)$ on $A$, where $\pi_i$ denotes the projection onto the $i$-th factor of $A$. The similar notation will also be used when we restrict the domains to be subschemes of $A$. For $m = (m_1, m_2, \ldots, m_t)$, we denote the sheaves

$$\Omega^m(a) = \pi_1^* \Omega_{P^{n_1+1}}^{m_1} \otimes \cdots \otimes \pi_t^* \Omega_{P^{n_t+1}}^{m_t} \otimes O(a),$$

and in the case when all the $m_i$ are the same, we simply denote the vector $m$ as $m = (m, m, \ldots, m)$.

The canonical bundle of $A$ is $K_A = O_A(-(n + 2))$. This is a short-hand notation for $O_A(-(n_1 + 2), -(n_2 + 2), \ldots, -(n_t + 2))$. The canonical bundle of $Y$ is

$$K_Y = (K_A \otimes I_Y)|_{\bar{Y}},$$

where $I_Y$ is the ideal sheaf of $Y$. We have that $I_Y = O_A(-d)$, hence the canonical bundle of $Y$ is $K_Y = O_Y(d - n - 2)$. The tangent bundle is $T_{Y/k} = \Omega^{n-1}_{\bar{Y}/k} \otimes K_{Y/k}^{-1}$, and hence

$$T_{Y/k} = \Omega^{n-1}_{\bar{Y}/k}(n + 2 - d).$$

We are interested in computing $H^p(Y, T_{Y/k})$, which is useful for and facilitates the determination of the Hodge numbers of the generalized complete intersection manifolds.

Let’s discuss some general results on divisors. Consider the ambient space $A/k$ as a smooth scheme, and $Y$ is a divisor which is smooth as a scheme over $k$. We denote $Der_Y(A/k)$ to be a subsheaf of $T_{A/k}$ that is characterized by the property that it sends the ideal sheaf $I_Y$ to itself. Now $Der_Y(A/k)$ is a locally free $O_A$-module. We denote the twisted
sheaf of forms, defined as the dual of $\text{Der}_Y(A/k)$, by $\Omega^1_{A/k}(\log Y)$, which is also locally free. There is a perfect pairing induced by the contraction of forms along vector fields

$$\Omega^1_{A/k}(\log Y) \times \text{Der}_Y(A/k) \to O_A,$$

hence we have that

$$\Omega^1_{A/k}(\log Y) = \text{Hom}_{O_A}(\text{Der}_Y(A/k), O_A).$$

(3.6)

Since we have $\text{Der}_Y(A/k) \subset T_{A/k}$, there is a reversed inclusion of their duals $\Omega^1_{A/k} \subset \Omega^1_{A/k}(\log Y)$. For $q \geq 0$, we define $\Omega^q_{A/k}(\log Y) = \wedge^q(\Omega^1_{A/k}(\log Y))$.

(3.7)

3.2 Statement of the Theorem and Proof by Induction

Now we present and prove a vanishing theorem on the cohomology groups of sheaves for subvarieties of the ambient product space of projective spaces.

**Theorem 3.1.** Suppose $n_i \geq 0$, $\sum_{i=1}^t d_i \geq 2t + 1 + s$, $d_i \geq 1$, where $s$ is the number of $n_i$ which are 1. The ambient space is the product space of projective spaces $A = \prod_{i=1}^t P_{n_i+1}^{n_i}$. Then for an algebraically closed field $k$, and a smooth hypersurface $Y/k$ of multi-degree $d = (d_1, d_2, \cdots, d_t)$ in $A$, we have $H^0(Y, T_{Y/k}) = 0$.

**Proof.** The idea to prove the above main result is to consider an even more general form of this equation and prove it by descending induction. $A$ is over an algebraically closed field $k$. Since $T_{Y/k} = \Omega^{n-1}_{Y/k}(n + 2 - d)$, it is the case $p = 0$ of the following proposition $C(p)$:

$$C(p) : \quad H^p(Y, \Omega^{n-1-p}_{Y}(n + 2 - (p + 1)d)) = 0.$$  

(3.11)

We aim to prove this equality. When we add $p$ to, or subtract $p$ from a vector, we mean $p$ is a vector $(p, p, \cdots, p)$, and when we do it with a number, we view $p$ as a number.
The technique we use is to twist the short exact sequences. Next we consider the long
exact sequences of the cohomology groups of sheaves associated with them and prove the
vanishing theorems on those related sheaves altogether. The ideal sheaf \( I_Y \) of \( Y \) in \( A \) is
\( O_A(-d) \). We will repeatedly use the following two exact sequences. Tensoring the restriction
exact sequences (3.10) with \( O_A(n + 2 - (p + 1)d) \), we have:
\[
0 \longrightarrow \Omega_A^{n-1-p}(\log Y)(n + 2 - (p + 2)d) \longrightarrow \Omega_A^{n-1-p}(n + 2 - (p + 1)d) \\
\longrightarrow \Omega_Y^{n-1-p}(n + 2 - (p + 1)d) \longrightarrow 0.
\] (3.12)

Tensoring the Poincare residue exact sequences (3.9) with \( O_A(n + 2 - (p + 1)d) \), we have:
\[
0 \longrightarrow \Omega_A^{n-p}(n + 2 - (p + 1)d) \longrightarrow \Omega_A^{n-p}(\log Y)(n + 2 - (p + 1)d) \\
\longrightarrow \Omega_Y^{n-1-p}(n + 2 - (p + 1)d) \longrightarrow 0.
\] (3.13)

We will prove another three equalities at the same time. We list them below:
\[
A(p) : \quad H^p(A, \Omega_A^{n-p}(\log Y)(n + 2 - (p + 1)d)) = 0.
\] (3.14)
And two equalities on the ambient space:
\[
B(p) : \quad H^p(A, \Omega_A^{n-1-p}(n + 2 - (p + 1)d)) = 0.
\] (3.15)
\[
D(p) : \quad H^p(A, \Omega_A^{n-p}(n + 2 - (p + 1)d)) = 0.
\] (3.16)

After taking the long exact sequences of the cohomology groups of sheaves in the above
short exact sequences, the groups in these four equalities above are related. After taking
the long exact sequence of the cohomology groups associated to the short exact sequence
(3.12), this will let us know: If \( B(p) \) and \( A(p + 1) \) both hold, \( C(p) \) shall hold. The long exact
sequence of the cohomology groups associated to the short exact sequence (3.13) will let us
know another implication: If \( C(p) \) and \( D(p) \) both hold, \( A(p) \) shall hold.

Now let us explain the strategy to prove these four equalities. The \( A(p) \) will hold for big
\( p \) for \( p > n_i \), by definition. If \( B(p) \) and \( D(p) \) hold for all \( p \), then by the above derivation,
we have that, once \( A(p + 1) \) holds then this implies that \( C(p) \) and then \( A(p) \) would hold as
well. We are then on the stage of descending induction from \( p + 1 \) to \( p \), which will prove
that \( A(p) \) and \( C(p) \) hold for all \( p \).

We only need to prove \( B(p) \) and \( D(p) \) now. We prove \( D(p) \) then: By Künneth formula, we know
\[
H^p(A, \Omega_A^{n-p}(n + 2 - (p + 1)d)) = \bigoplus_{i=1}^{q} H^q(P^{n+1}, \Omega^{n_i-p}(n_i + 2 - (p + 1)d_i)).
\] (3.17)

We reformulate the Bott formula as follows. \( H^\alpha(P^{n+1}, \Omega_{\mathbb{P}^{n+1}}^\beta(\gamma)) \) will be nonzero only
when:
(1) \( \alpha = 0 \), and either \( \gamma > \beta \geq 0 \) or \( \gamma = \beta = 0 \);
(2) $1 \leq \alpha \leq n$, $\alpha = \beta, \gamma = 0$;

(3) $\alpha = n + 1, \beta \geq 0$ and either $\beta - \gamma > n + 1$ or $\beta = n + 1, \gamma = 0$.

We are going to show that every tensor product that arises in the sum will vanish. Otherwise, suppose there is a nonzero $\otimes_i H^\alpha(\mathbb{P}^{n_i+1}, \Omega^\gamma(n_i + 2 - (p + 1)d_i))$. Consider the index $\beta$ of $\Omega^\beta(\gamma)$, it needs to be non-negative, hence $p \leq n_i, \forall i$. Notice that $\sum_i q_i = p$, we know that $q_i \leq n_i$, hence the last case (3) of non-vanishing will never happen.

If the first case (1) happens, we have $q_i = 0$. If the second condition is $\gamma > \beta \geq 0$, we know that $n_i - (p + 1)d_i > n_i - p$, or $p > (p + 1)d_i$, which is impossible. If the second condition reads $\gamma = \beta = 0$, we know that $n_i = p, n_i + 2 = (p + 1)d_i$, enforcing $p + 2 = (p + 1)d_i$. This can happen only when $p = 0, d_i = 2$, which makes $n_i$ to be 0, leading to contradiction.

Hence every factor in the tensor product has to be in the second case (2), giving us:

\begin{align*}
1 \leq q_i & \leq n_i. \\
n_i - p & = q_i. \\
n_i + 2 & = (p + 1)d_i. 
\end{align*}

Summing these equations over $i$, we get $\sum_i [(p + 1)d_i - p - 2] = p$. So with the condition

$$\sum_i d_i \geq 2t + 1 + s,$$

we have that

$$p \geq (1 + t + s)p + 1 + s.$$

This leads to contradiction. Hence we have proved $D(p)$.

Then we prove $B(p)$: By Künneth formula, we know

$$H^p(A, \Omega^n_{\mathbb{A}}(n + 2 - (p + 1)d)) = \oplus_{\sum_i q_i = p} \otimes_i H^\alpha_i(\mathbb{P}^{n_i+1}, \Omega^{\beta_i(n_i + 2 - (p + 1))}).$$

For the same reason as the $D(p)$ case, the third non-vanishing case will never happen. And we have:

$$p + 1 \leq n_i \quad \text{and} \quad q_i \leq p.$$  

If the first case arises, we will have $q_i = 0$. Moreover, once the second condition specializes to be $\gamma > \beta \geq 0$, we have $n_i + 2 - (p + 1)d_i > n_i - p - 1$, or $(p + 1)(d_i - 1) < 2$, so this enforces $d_i = 1, \forall i$. This then implies that $t = \sum_i d_i \geq 2t + 1 + s$, leading to contradiction. Hence the second condition must be $\gamma = \beta = 0$. This means $n_i = p + 1, n_i + 2 = (p + 1)d_i$. Hence $p + 3 = (p + 1)d_i$. All $d_i$ will be the same. Since $\sum_i d_i > 2t$, we know $d_i \geq 3$. This once again enforces $p = 0, d_i = 3, n_i = 1$. This shows $s = t$, hence $3t = \sum_i d_i \geq 2t + 1 + s$, which is contradictory again.

Now we are left with the second case once again. The equations are

$$n_i - p - 1 = q_i, \quad n_i + 2 = (p + 1)d_i.$$
Summing them over $i$ and comparing these two equations, we have

$$\sum_i n_i = pt + p + t = (p + 1)(\sum_i d_i) - 2t.$$  \hspace{1cm} (3.26)

With the condition (3.21), we have that

$$0 \geq (p - 1)t + 1 + s(p + 1).$$  \hspace{1cm} (3.27)

From this we know $p = 0$, since there would be contradiction if $p \geq 1$. Thus by the first of Eq. (3.26) we know that all $n_i$ have to be 1 and $s = t$. Then Eq. (3.27) leads to contradiction. With condition (3.21), we have showed that the non-vanishing is not possible. Hence we have proved $B(p)$. Therefore by the descending induction above, the theorem is completely proved.

In the generalized vanishing theorem above, there is an interesting condition $\sum_i d_i \geq 2t + 1 + s$. We will make comments on this in the next subsection. Our proof above is for general $n_i \geq 0$, with the condition stated in the theorem.

This result will be useful in our computation of Hodge numbers of the generalized CICYs, as in our approach in Sections 4 and 6 as well as in Appendix A. Further, it is related to the properness of morphisms of schemes.

Our result is also useful for many other models. The theorem we have proved is valid not only for the examples we scrutinize in Sections 5 and 6 but also for many other models, such as the family $X_m$ constructed in [4, 10, 11], which we recall the definition here:

$$X_m = \mathbb{P}^4_{\mathbb{P}^1} \left[ \begin{array}{cc} 1 & 4 \\ m & 2 - m \end{array} \right], \quad F_m = \mathbb{P}^4_{\mathbb{P}^1} \left[ \begin{array}{c} 1 \\ m \end{array} \right].$$  \hspace{1cm} (3.28)

### 3.3 Version for Single Projective Space and Relation to Proper Morphism and Haar Measure

Our theorem 3.1 above is a generalization of the theorem of [12, 13]. The original version for a single projective space is as follows.

**Theorem 3.2.** Suppose $n \geq 0$, $d \geq 3$, and that $(n, d) \neq (1, 3)$. The ambient space is a single projective space $\mathbb{P}^{n+1}_k$. Then for any field $k$, and a smooth hypersurface $M/k$ of degree $d$ in $\mathbb{P}^{n+1}_k$, we have $H^0(M, T_{M/k}) = 0$.

This original version was proved in [12, 13] for any field $k$, in scheme-theoretic formulations. See also [14] for related discussion.

The theorem 3.2 for an algebraically closed field $k$, is implied by our theorem 3.1 as our $t = 1$ case. In theorem 3.2, when the dimension is $n = 1$, the lower bound of the degree $d$ have to be strengthened to be 4. In our theorem 3.1 for $t = 1$, when $n = 1$ which means
\( s = 1 \), our condition \( \sum d_i \geq 2t + 1 + s \) reduces to the condition \( d \geq 2 + 1 + 1 = 4 \), which is in agreement with the condition in theorem \( 3.2 \). So our generalized theorem characterizes the lower bound of the degree in a more uniform way. The original excluded case \( (n, d) \neq (1, 3) \) would now be described in a more uniform and natural formulation.

The vanishing of \( H^0(M, T_M) \) in theorem \( 3.2 \) is related to the projective automorphism of \( M \). The projective automorphism of a hypersurface \( M \) in \( \mathbb{P}^{n+1} \) is the automorphism that is induced by the automorphism of the ambient space \( \mathbb{P}^{n+1} \). \( M \) is a scheme over a field \( k \), and we denote it by \( M/k \). We have its projective automorphism

\[
\text{ProjAut}(M/k) \subset PGL_{n+2}(k).
\]

(3.29)

Suppose \( H_{n,d} \) is a degree \( d \) hypersurface in \( \mathbb{P}^{n+1} \), which plays the role of aforementioned \( M \), see [15]. The following proposition by [16] is on the properness of the action of the group scheme \( PGL_{n+2} \) on \( M \).

**Proposition 3.1.** Suppose \( n \geq 0, d \geq 3 \). The action of the group scheme \( PGL_{n+2} \) on degree \( d \) hypersurface \( H_{n,d} \) in \( \mathbb{P}^{n+1} \)is proper and finite, i.e., the morphism

\[
PGL_{n+2} \times_{\mathbb{Z}} H_{n,d} \rightarrow H_{n,d} \times_{\mathbb{Z}} H_{n,d},
\]

\((g,h) \mapsto (h,g(h))\) (3.30)

is a proper and finite morphism.

The properness of this action of the group scheme is in turn useful for the Haar measure of the matrix integrals for \( PGL_{n+2}(k) \) for a field \( k \) [13]. Since \( PGL_{n+2} \) is from the action on a single projective space, it would be interesting to see whether there is a connection between the generalized vanishing theorem for a product of projective spaces and properties of the action of the product \( \prod_i PGL_{n_i+2} \).

### 4 Identity between Configuration Matrices and its Proof

In this section, we prove an identity, or equivalence, between configuration matrices for complete intersection Calabi-Yau manifolds. This identity means that the manifolds defined by the configuration matrices in the two sides of the identity are diffeomorphic. For the convenience of discussion below, we introduce the notation \( a = (a_2, \cdots, a_k), b = (b_2, \cdots, b_k) \). The indices start with 2 because of the columns that they start from. P. Candelas et. al. proposed the following identification of CY manifolds without proof [1], and we will give the proof of this identity.

**Theorem 4.1.** The complete intersection Calabi-Yau manifolds, defined by the following two configuration matrices, are diffeomorphic, i.e.,

\[
\begin{align*}
\mathbb{P}^1 & \begin{bmatrix} 1 & a \\ 0 & M \end{bmatrix} = \mathbb{P}^1 \begin{bmatrix} a + b \\ nb \end{bmatrix}, \\
\mathbb{P}^n & \begin{bmatrix} 1 & nb \\ 0 & M \end{bmatrix} = \mathbb{P}^{n-1} \begin{bmatrix} a + b \\ nb \end{bmatrix}.
\end{align*}
\]

(4.1)
Here, \(a\) and \(b\) are multi-columns, \(M\) is a block matrix, and \(0\) is a column of zeros.

Proof. We first show how to use Calabi-Yau condition to reduce the number of possible cases to a reasonable extent. Recall that the Calabi-Yau condition is: the sum of a row corresponds to \(P^n\) is \(n_i + 1\). We have that \(a = (a_2, \ldots, a_k), b = (b_2, \ldots, b_k)\). Set \(F := \sum_{i=2}^{k} a_i, G := \sum_{i=2}^{k} b_i\). The Calabi-Yau condition is

\[
F + 1 = 2, \quad nG + 1 = n + 1. \tag{4.2}
\]

So we have \(F = 1, G = 1\). Since all \(a_i, b_i\) are non-negative integers, this means that there is one and only one non-zero term in the \(a\)-row and \(b\)-row respectively. So, the part of the matrix that involves \(P^1 \times P^n\) non-trivially would only have at most 3 columns, depending on whether the non-zero terms are at the same column or not.

Hence we are left with two general cases:

\[
P^1 \begin{bmatrix} 1 & 0 & 1 \\ 1 & n & 0 \\ 0 & M_1 & M_2 \end{bmatrix} = P^n \begin{bmatrix} 1 & 1 \\ n & 0 \\ M_1 & M_2 \end{bmatrix}. \tag{4.3}
\]

\[
P^1 \begin{bmatrix} 1 & 1 \\ 1 & n \\ 0 & M \end{bmatrix} = P^n \begin{bmatrix} 2 \\ n \\ M \end{bmatrix}. \tag{4.4}
\]

Here, \(M_i\) and \(M\) are column vectors.

The observation we need to make is that

\[
P^1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = P^1 \times P^n \times X. \tag{4.5}
\]

Both sides are diffeomorphic to each other.

In the first case, what we need to prove is Eq. (4.3). The reason is as follows: As Eq. (4.5) states, the first column of the left hand side represent a family of \(P^n \times X\) parametrized by \(P^1\). Hence in conjunction with the second column, they should be hypersurfaces in \(P^1 \times P^n \times X\). After embedding them into \(P^1 \times P^n \times X\), by Chow’s theorem, they should be algebraic, hence they should be able to be represented respectively by a configuration matrix. A similar argument applies for the third column as well, but another fact we need is that it is a family of hypersurfaces in \(P^1 \times X\) involving \(P^1\) non-trivially and do not involve \(P^n\). That is to say, when we write them in a configuration matrix of the form

\[
P^1 \begin{bmatrix} 1 & 0 & 1 \\ 1 & n & 0 \\ 0 & M_1 & M_2 \end{bmatrix} = P^n \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ M_1 & M_2 \end{bmatrix}, \tag{4.6}
\]

12
$v_2$ should be zero, and $v_1$ should be non-zero. On the other hand, $u_1$ should not be zero since the first column of the left hand side tells that the parametrization of the family of the hypersurfaces involves $\mathbb{P}^1$ non-trivially. Moreover, being Calabi-Yau is intrinsic, hence new configuration matrix shall satisfy Calabi-Yau condition as well. Hence we have

$$u_1 + v_1 = 2,$$
$$u_2 + v_2 = (n - 1) + 1.$$  \hspace{1cm} (4.7)

Consequently, above equations give us every term we desire, that is $u_1 = 1, u_2 = n, v_1 = 1, v_2 = 0$.

In the second case, what we need to prove is Eq. (4.4). Being similar to the aforementioned case, the left hand side of Eq. (4.4) should be a family of submanifolds of $\mathbb{P}^1 \times \mathbb{P}^{n-1} \times X$. After embedding them into $\mathbb{P}^1 \times \mathbb{P}^{n-1} \times X$, by Chow’s theorem, they should be algebraic, hence they should be able to be represented respectively by a configuration matrix. Moreover, as a manifold, being Calabi-Yau is intrinsic, hence the new configuration matrix will also satisfy Calabi-Yau conditions on the row sums. The right hand side should be a configuration matrix like:

$$\begin{bmatrix}
\mathbb{P}^1 \\
\mathbb{P}^{n-1} \\
X
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
M
\end{bmatrix}.$$  \hspace{1cm} (4.8)

Hence Calabi-Yau condition completely determines the terms exactly as in equation (4.4), that is $q_1 = 2, q_2 = n$.

Hence, the manifolds defined by two different configuration matrices in (4.4), in the sense of Wall’s classification theorems [17], are diffeomorphic.

One can also use the above identities for a part of bigger configuration matrices, where the above matrices on the two sides appear as sub-matrices. Note that the two sides are equivalent at the level of diffeomorphism, but not necessarily have the same complex structure. The two sides can still have a different choice of complex structures and they are in the same complex structure moduli space. This matrix identity is also useful in the class of the manifolds $X_m$ and $F_m$, see [3, 28], constructed and elucidated by [4, 10, 11].

## 5 Genus Formula of Curves in Generalized Complete Intersections and Blow Up

### 5.1 The Configuration Matrix and Fixed Point Locus

We are interested in the involutions of the generalized complete intersection Calabi-Yau manifolds $X$, whose fixed point locus are curves. For convenience, we denote the generalized complete intersection manifold by $X$ and the fixed point locus by $Z$. The configuration
matrix of $X$ is:

$$X = \mathbb{P}^m \begin{bmatrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_p \end{bmatrix}. \tag{5.1}$$

The $a_i, b_i$ here can be either non-negative or negative.

What we want is the data of certain involutions and their fixed point locus. As a first step, we discuss the information about the action of these involutions. Each involution generates a $\mathbb{Z}/2\mathbb{Z}$ subgroup of $G_0 = \text{Aut}(X)$. Moreover, if we have several involutions which are pairwise commutative, then they generate a subgroup of $G_0$, isomorphic to a product of several $\mathbb{Z}/2\mathbb{Z}$ groups. As a subgroup of $G_0$, we denote the product of these $\mathbb{Z}/2\mathbb{Z}$ groups by $G$. We have the involution

$$\iota : X \longrightarrow X, \quad \iota \in G. \tag{5.2}$$

It acts as follows: We consider the involution $\iota$ which can be written in the form $\iota = ((e_0, e_1, e_2, \cdots, e_m), (f_0, f_1, f_2, \cdots, f_n))$, with the fixed point locus being curves. If the component $e_i$ is 1, it changes the sign of its corresponding component in $\mathbb{P}^m$. If $e_i$ is 0, it keeps the corresponding component unchanged. The $f_j$'s are treated in the same manner. For instance, the element $((0, 0, \cdots, 0), (1, 1, \cdots, 0))$ sends $([z_0, z_1, \cdots, z_m], [y_0, y_1, \cdots, y_n])$ to $([z_0, z_1, \cdots, z_m], [-y_0, -y_1, \cdots, y_n])$. These involutions preserve the holomorphic form of $X$. There are two possibilities that a point is fixed under the action of $G$: (i) Every component got acted on is 0; (ii) Every component that is unchanged is 0. The fixed point locus of the involution is the disjoint union of two curves. This observation can be deduced as follows (this argument also applies to more general group actions): If case (i) holds, they remain unchanged under the involution. If not, suppose there is an element $\iota$ in $G$ which acts on $(y_0, y_1)$ non-trivially and at least one of them does not vanish, then $([z_0, \cdots, z_m], [y_0, y_1, \cdots, y_n])$ is sent to $([z_0, \cdots, z_m], [-y_0, -y_1, \cdots, y_n])$ by this element. By the definition of projective spaces, these two points can coincide only when one of them is the other one multiplied by $-1$. In this case, those components that remain unchanged have to be 0, which is case (ii). These are curves inside $X$. The first component $\Gamma_1$ of the curves is defined by $y_0 = y_1 = 0$, and this curve can be viewed as a curve in $\mathbb{P}^m \times \mathbb{P}^{m-2}$ which has coordinates $([z_0, z_1, \cdots, z_m], [y_2, \cdots, y_n])$. The second component $\Gamma_2$ of the curves is defined by $y_2 = y_3 = \cdots = y_n = 0$, and this curve can be viewed as a curve in $\mathbb{P}^m \times \mathbb{P}^1$ which has coordinates $([z_0, z_1, \cdots, z_m], [y_0, y_1])$. Due to that the condition $y_0 = y_1 = y_2 = \cdots = y_n = 0$ is not possible in $\mathbb{P}^n$, these two components will not intersect. The fixed point locus of the involution is

$$Z = \Gamma_1 \cup \Gamma_2. \tag{5.3}$$

It is a disjoint union, since condition (i) and (ii) do not occur at the same time.
5.2 Genus Formula of Curves as Generalized Complete Intersections in Product of Projective Spaces

In this section, we derive a general genus formula for generalized complete intersection curves in product of projective spaces. The curve itself is a generalized complete intersection manifold, in that the configuration matrix for the curve allows negative integer matrix elements. $A$ is the ambient space, which is the product of projective spaces. $\Gamma$ will denote the curve. The entries of the configuration matrix of curves in this section allow negative integers.

Denote the configuration matrix of this family of curves by

$$\Gamma = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_N \\ b_1 & b_2 & b_3 & \cdots & b_N \end{bmatrix} \quad (5.4)$$

where $N = m + n - 1$, with $m, n \in \mathbb{Z}_{>0}$, and the matrix elements $a_i, b_i$ here can be either non-negative or negative integers.

**Theorem 5.1.** The genus $g$ of the above family of curves $\Gamma$ is given by:

$$g = \frac{\left(\sum_{i_1}^{m+n-1} a_i\right) - 1}{\left(\sum_{i_1}^{m+n-1} b_i\right) - 1} + \left(\sum_{i_1 < i_2 < \cdots < i_{m-1}} \left(-\sum_{j=1}^{n-1} a_{i_j}\right)\cdot m\right)\left(\sum_{i_1 < i_2 < \cdots < i_{n-1}} \left(-\sum_{j=1}^{m-1} b_{i_j}\right)\cdot n\right) \quad (5.5)$$

Here the brackets () mean the binomial coefficients.

**Proof.** We will use Koszul complex and spectral sequence to derive the result. The Koszul complex, where the last term is the $m + n - 1$-wedge, is:

$$0 \longrightarrow \wedge^{n+m-1} \mathcal{E}^* \longrightarrow \cdots \longrightarrow \wedge^2 \mathcal{E}^* \longrightarrow \mathcal{E}^* \longrightarrow O_A \longrightarrow O_\Gamma \quad (5.6)$$

We consider the double complex formed by the cohomology groups of these sheaves. The ideal sheaf is $\mathcal{E}^* = \oplus O_A(-a_i, -b_i)$. Moreover, $\wedge^k \mathcal{E}^*$ are the anti-symmetric operations on $(\mathcal{E}^*)^k$, where the power stands for tensor product. So we have $\wedge^k \mathcal{E}^* = \oplus_{i_1 < i_2 < \cdots < i_k} \bigotimes_j O_A(-a_{i_j}, -b_{i_j}) = \oplus_{i_1 < i_2 < \cdots < i_k} O_A(-\sum_j a_{i_j}, -\sum_j b_{i_j})$.

By Künneth formula, we expand

$$H^q(A, O_A(-\sum_j a_{i_j}, -\sum_j b_{i_j})) = \sum_{\alpha + \beta = q} H^\alpha(\mathbb{P}^m, O_{\mathbb{P}^m}(-\sum_j a_{i_j})) \otimes H^\beta(\mathbb{P}^n, O_{\mathbb{P}^n}(-\sum_j a_{i_j})) \quad (5.7)$$

Recall the Bott formula for computation of these cohomology. The only possibly nontrivial terms would be the 0-th or $m$-th cohomology groups for $\mathbb{P}^m$, and the 0-th or $n$-th cohomology groups for $\mathbb{P}^n$, depending on the sum of degrees to be positive or not.
Since in the above spectral sequence only $H^k(A, \wedge^{k-1} \mathcal{E}^*)$ are mapped to $H^1(\Gamma, O_\Gamma)$, the terms we need to consider are as follows:

\[ T_1 := \bigoplus_{i_1 < i_2 < \cdots < i_{m-1}} H^m(\mathbb{P}^m, O_{\mathbb{P}^m}(- \sum_{j=1}^{m-1} a_{i_j})) \otimes H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(- \sum_{j=1}^{m-1} b_{i_j})). \tag{5.8} \]

\[ T_2 := \bigoplus_{i_1 < i_2 < \cdots < i_{n-1}} H^0(\mathbb{P}^m, O_{\mathbb{P}^m}(- \sum_{j=1}^{n-1} a_{i_j})) \otimes H^n(\mathbb{P}^n, O_{\mathbb{P}^n}(- \sum_{j=1}^{n-1} b_{i_j})). \tag{5.9} \]

\[ T_3 := H^m(\mathbb{P}^m, O_{\mathbb{P}^m}(- \sum_{i=1}^{m+n-1} a_i)) \otimes H^n(\mathbb{P}^n, O_{\mathbb{P}^n}(- \sum_{i=1}^{m+n-1} b_i)). \tag{5.10} \]

Next we consider $T_1$, and $T_2$ can be treated in the same manner. There are restrictions on the indices. The 0-th cohomology groups would be non-zero only when $-\sum_{j=1}^{m-1} b_{i_j} \geq 0$. They have dimensions \((-\sum_{j=1}^{m-1} b_{i_j})_{m} + n\) in this situation. Here the brackets \(\binom{\cdot}{\beta}\) mean the binomial coefficients \(\frac{\alpha!}{\beta!((\alpha - \beta)!}\)). We denote the set of such $m - 1$-indices $(i_1, i_2, \cdots, i_{m-1})$ as $I_1$.

As for the $m$-th cohomology groups, they will be non-zero only when $-\sum_{j=1}^{m-1} a_{i_j} \leq -(m + 1)$, being of dimensions \((-\sum_{j=1}^{m-1} a_{i_j} - 1)_{m} - (\sum_{j=1}^{m-1} a_{i_j})_{m-1}\).

We add all those terms with non-vanishing cohomology groups. We set \(\binom{\alpha}{\beta}\) to be 0 as long as $\alpha < \beta$ or $\beta < 0$. We need to consider all possible tensor products. So the total dimension of $T_1$ is:

\[ \dim T_1 = \sum_{i_1 < i_2 < \cdots < i_{m-1}} \left( \binom{\sum_{j=1}^{m-1} a_{i_j} - 1}{\sum_{j=1}^{m-1} a_{i_j} - m - 1} \right) \sum_{i_1 < i_2 < \cdots < i_{m-1}} \binom{n}{\binom{\sum_{j=1}^{m-1} b_{i_j}}{n}}. \tag{5.11} \]

In the above, the summation is over the index set $I_1$.

We denote $I_2$ the set of $n - 1$-indices $(i_1, i_2, \cdots, i_{n-1})$ such that $-\sum_{j=1}^{n-1} a_{i_j} \geq 0$. We have

\[ \dim T_2 = \sum_{i_1 < i_2 < \cdots < i_{n-1}} \left( \binom{\sum_{j=1}^{n-1} a_{i_j} + m}{\sum_{j=1}^{n-1} a_{i_j}} \right) \sum_{i_1 < i_2 < \cdots < i_{n-1}} \binom{n}{\binom{\sum_{j=1}^{n-1} b_{i_j} - 1}{\sum_{j=1}^{n-1} b_{i_j} - n - 1}}. \tag{5.12} \]

Here, the summation is over the index set $I_2$.

The only left term is $T_3$, which has only one component. By Bott formula, and since $-\sum_{i=1}^{m+n-1} a_i \leq -(m + 1)$, $-\sum_{i=1}^{m+n-1} b_i \leq -(n + 1)$, we find that its dimension is

\[ \dim T_3 = \left( \binom{\sum_{i=1}^{m+n-1} a_i - 1}{\sum_{i=1}^{m+n-1} a_i - m - 1} \right) \left( \binom{\sum_{i=1}^{m+n-1} b_i - 1}{\sum_{i=1}^{m+n-1} b_i - n - 1} \right). \tag{5.13} \]
This is a general formula for the genus of smooth curves as generalized complete intersections, as a function of the data of the configuration matrix. This formula is useful when we calculate the Hodge numbers of the blow up of the generalized complete intersection manifolds along curves.

We also mention a special case as follows. For the simplest example of the product space of projective spaces, we consider the special case $A = \mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $[z_0, z_1], [y_0, y_1]$ respectively. For example, a polynomial $F$ of bi-degree $(d_1, d_2)$ defines a curve $C$. Below, we will also add singular points $P$ on the Riemann surface, for a more generality. We have the following:

**Proposition 5.1.** For $C$ as above, its genus $g$ is given by:

$$
g = (d_1 - 1)(d_2 - 1) - \sum_P r_P(r_P - 1).$$

(5.14)

where $r_P$ is the multiplicity of $P$ in $C$.

The smooth part of this formula (5.14) is a special case of our above general formula (5.5), when we plug in $m = 1$, $n = 1$, $a_1 = d_1$ and $b_1 = d_2$. We prove genus formula in this form in Appendix A. Among other things, the main tool we are going to use is the Riemann-Roch theorem.

### 5.3 Hodge Numbers of Blow Up

We have considered involutions of $gCICY$, and then identified some fixed point locus of the involutions. We can then also perform a blow-up on the manifold. Since the involutions preserve the holomorphic form, the blow up along the fixed point locus, is still Calabi-Yau. Finally, we can derive the Hodge numbers of the blow ups from the cohomology of the exceptional divisors, for example [18]. Moreover, we relate the genus of the curves to the Hodge numbers. In conjunction with the genus formula (5.5) in section 5.2 we will determine the Hodge numbers of the blow up of the manifold. We consider a special case of threefolds (5.1), whose configuration matrix is

$$X = \mathbb{P}^1 \oplus \mathbb{P}^4 \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix},$$

(5.15)

where $a_i$ and $b_i$ can be either non-negative or negative integers, and the Calabi-Yau condition is $a_2 = 2 - a_1$ and $b_2 = 5 - b_1$. We denote the coordinates of $\mathbb{P}^1$ and $\mathbb{P}^4$ by $[z_0, z_1], [y_0, y_1, y_2, y_3, y_4]$ respectively. The group that we will consider to act on $A = \mathbb{P}^1 \times \mathbb{P}^4$ would be $G = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. In $G$, the action $((0,0),(1,1,0,0,0))$ represents the involution:

$$\iota : X \longrightarrow X. \quad ([z_0, z_1], [y_0, y_1, y_2, y_3, y_4]) \mapsto ([z_0, z_1], [-y_0, -y_1, y_2, y_3, y_4]).$$

(5.16)
By the consideration in Section 5.1, \( \tau \) fixes a point if one of these two situations happens: 
\[ y_0 = y_1 = 0 \] or \[ y_2 = y_3 = y_4 = 0. \] The second situation arises since in the projective space, we can multiply all coordinates by \(-1\) without changing the point. Denote these two parts of fixed locus by \( \Gamma_1, \Gamma_2 \) respectively.

Now we consider the example with \( a_1 = 3, b_1 = 2 \), which is the threefold considered in [4, 19], which gives \( X = P_1 \times P_2 \left[ \begin{array}{cc} 3 & -1 \\ 2 & 3 \end{array} \right] \). On \( \Gamma_1 \), we have \( y_0 = y_1 = 0 \), which reduces \( A \) to be \( \tilde{A}_1 = P_1 \times P_2 \), whose coordinates are \( ([z_0, z_1], [y_2, y_3, y_4]) \). In this case, \( \Gamma_1 \) would be
\[
\Gamma_1 = P_1 \times P_2 \left[ \begin{array}{cc} 3 & -1 \\ 2 & 3 \end{array} \right].
\]
(5.17)

This curve is a generalized complete intersection manifold in \( P_1 \times P_2 \). Using our general formula (5.5) in Section 5.2 yields that this curve has genus \( g(\Gamma_1) = 8 \).

Next we consider \( \Gamma_2 \). On \( \Gamma_2 \), we would have \( y_2 = y_3 = y_4 = 0 \), and this reduces \( A \) to be \( \tilde{A}_2 = P_1 \times P_1 \) which has coordinates \( ([z_0, z_1], [y_0, y_1]) \). Consequently, this part of fixed locus is a curve in \( P_1 \times P_1 \), which is
\[
\Gamma_2 = P_1 \times P_1 \left[ \begin{array}{cc} 3 \\ 2 \end{array} \right].
\]
(5.18)

Using our general formula (5.5) in Section 5.2 yields that its genus is \( g(\Gamma_2) = 2 \).

The genus gives the data of Hodge numbers. We have that \( \dim H^1(\Gamma_j, O_{\Gamma_j}) = g(\Gamma_j) \), \( j = 1, 2 \), and hence \( \dim H^1(\Gamma_j, Z) = 2g(\Gamma_j) \) and \( \dim H^0(\Gamma_j, Z) = 1 \). Now we make a blow-up of the original gCICY along the curves, that is in the form (5.3). The involution preserves the holomorphic form, and hence the blow-up along the fixed point locus of the involution is still a Calabi-Yau.

The Hodge structure of the blow up is described by the following theorem [18].

**Theorem 5.2.** Denote \( X_Z \) to be the blow up of \( X \) along \( Z \). We have the following isomorphism of Hodge structures:
\[
H^k(X, Z) \oplus (\oplus_{i=0}^{r-2} H^{k-2i-2}(Z, Z)) \simeq H^k(X_Z, Z),
\]
where \( r \) is the codimension of \( Z \) in \( X \), and \( r - 1 = \text{rank}(E) \), with \( E := \tau^{-1}(Z) \) being the exceptional divisor viewed as a bundle over \( Z \).

Here, the exceptional divisor \( E := \tau^{-1}(Z) \) is a \( P^1 \) bundle over the curves. When we take \( k = 0, 1 \), we know that \( h^0(X) \) and \( h^1(X) \) would stay unchanged when we take the blowing-up. The change happens in the case where \( k = 2, 3 \). The only non-trivial summand involving \( Z \) would be those ones corresponding to \( i = 0 \). Consider the case \( k = 2 \). The term from \( Z \) is \( H^0(Z, Z) \). These two components will not intersect since the coordinates of them can not all be zero. Hence every component will increase \( h^2(X_Z) \) by 1. Hence totally it will increase by \( \dim H^0(Z, Z) = 2 \). We have \( h^2(X_Z) = h^2(X) + 2 = 4 \) and \( h^{1,1}(X_Z) = 4 \). In the case \( k = 3 \),
the summand involving $Z$ would be $H^1(Z, \mathbb{Z})$, offering additional dimension $\dim H^1(Z, \mathbb{Z}) = 2g(\Gamma_1) + 2g(\Gamma_2)$, which, in this example, is 20. Hence $h^3(X_Z) = h^3(X) + 20 = 94 + 20 = 114$. Since CY condition gives us $h^{3,0}(X_Z) = 1$, we know that $h^{2,1}(X_Z) = \frac{1}{2}(114 - 2) = 56$, in this case.

Our genus formula of the curves (5.3) is needed when we consider Hodge numbers of blow-up of gCICYs. These blow-ups can provide new Calabi-Yau manifolds and their variants in the moduli space of Calabi-Yaus. They are also useful for string compactification and non-perturbative superpotentials in lower dimensional field theory after the compactification.

6 Spectral Sequence Approach

The previous method requires much effort in computation, whereas the spectral sequence approach needs much less elaboration since it can treat all sheaves in the Koszul sequences of four or more terms at one time. This is related to the codimension of the submanifolds as follows: the Koszul sequence of a submanifold of codimension $d$ is of length $d + 2$. So if the codimension is bigger, we need to treat Koszul sequences of more terms.

6.1 Spectral Sequences

When the sequence is no longer of three terms, spectral sequence truly demonstrates its power comparing with iterative usage of short exact sequences. We denote $O_X(b_1, \cdots, b_t) := O_A(b_1, \cdots, b_t)|_X$, and we use spectral sequences to compute

$$ h^*(X, O_X(b_1, \cdots, b_t)). \quad (6.1) $$

We use $h^*(X, F)$ to denote the list of dimensions of cohomology groups ($h^0, h^1, h^2, \cdots$) where $h^i(X, F) = \dim H^i(X, F)$.

As we have explained, the manifold $X$ can be constructed by line bundles. Suppose we have line bundles $E_i$ respectively. The number of columns in the configuration matrix is $K$, and hence there are $K$ line bundles. Then when we consider the restriction map $r : O_A \longrightarrow O_X$, the kernel would be the dual of $E := \oplus_{i=1}^K E_i$. And the map from $E^*$ to $O_A$ can be obtained by contracting with the dual $E$. Analyzing kernels of these maps iteratively, we find a sequence:

$$ 0 \longrightarrow \wedge^K E^* \longrightarrow \cdots \longrightarrow \wedge^2 E^* \longrightarrow E^* \longrightarrow O_A \longrightarrow O_X \longrightarrow 0. \quad (6.2) $$

For brevity, in the above notation, we have used the brief notation $O_X$ to mean $i_*O_X$, where $i : X \to A$ is the inclusion. When we want to derive data on cohomology groups of other sheaves, we can tensor this sequence by locally free sheaves or bundles on $A$.

For example, in our case in Section 5.3, $E = E_1 \oplus E_2$. For instance, for $a_1 = 3, b_1 = 2$, we have $E_1 = O_A(3, 2), E_2 = O_A(-1, 3)$. Hence $E^* = O_A(-3, -2) \oplus O_A(1, -3), \wedge^2 E^* =$
$O_A(-2, -5)$. We notice that $\wedge^2 E^* = O_A(-2, -5) = K_A$ is the canonical bundle of $A$. This is not a coincidence. As a matter of fact, this is a criterion of Calabi-Yau condition.

The compact manifold $X$ is a Calabi-Yau threefold if the structure sheaf $O_X$ fits into a resolution sequence:

$$0 \longrightarrow F^*_K \longrightarrow \cdots \longrightarrow F^*_2 \longrightarrow F^*_1 \longrightarrow F^*_0 \longrightarrow O_X \longrightarrow 0.$$  \hfill (6.3)

Here, we require $F^*_0 = O_A$, $F^*_K = K_A$, the canonical bundle of $A$. We can tensor the above sequence with a locally free sheaf $F$ of interest, and make an double complex $W^{p,q}$ of their sheaf cohomology $H^q(A, F^*_p \otimes F)$.

We will then turn to the example we have considered in Section 5.3. The goal here is to use the powerful tool spectral sequence to determine Hodge numbers. Our approach using multiple short exact sequences.

For the example in Section 5.3, the sequence (6.2) becomes

$$0 \longrightarrow \wedge^2 E^* \longrightarrow E^* \longrightarrow O_A \longrightarrow O_X \longrightarrow 0.$$ \hfill (6.4)

More specifically, in this example, the above sequence is

$$0 \longrightarrow O_A(-2, -5) \longrightarrow O_A(-3, -2) \oplus O_A(1, -3) \longrightarrow O_A \longrightarrow O_X \longrightarrow 0.$$ \hfill (6.5)

To obtain $h^*(X, O_X(-1, 3))$, we tensor this sequence by the sheaf $F = O_A(-1, 3)$:

$$0 \longrightarrow O_A(-3, -2) \longrightarrow O_A(-4, 1) \oplus O_A \longrightarrow O_A(-1, 3) \longrightarrow O_X(-1, 3) \longrightarrow 0.$$ \hfill (6.6)

Other ingredients we need would be the cohomology groups of sheaves involved in this sequence. We build the following double complex $W^{p,q}$ consisting of their cohomology groups:

$$
\begin{array}{c}
H^0(A, O_A(-3, -2)) \longrightarrow H^0(A, O_A(-4, 1) \oplus O_A) \longrightarrow H^0(A, O_A(-1, 3)) \longrightarrow H^0(X, O_X(-1, 3)) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
H^1(A, O_A(-3, -2)) \longrightarrow H^1(A, O_A(-4, 1) \oplus O_A) \longrightarrow H^1(A, O_A(-1, 3)) \longrightarrow H^1(X, O_X(-1, 3)) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots
\end{array}
$$ \hfill (6.7)

In the double complex above, the vertical direction is labeled by $q$, and the horizontal direction is labeled by $p$. We have used the fact that $H^i(X, O_X(b_1, \cdots, b_l)) = H^i(A, i_*O_X(b_1, \cdots, b_l))$, for each $i$, which can be easily derived from Lemma 2.10 of Ch. III in [20].

Those cohomology groups involving $A$ can be computed through Bott formula and Künneth formula. We have that

$$h^*(A, O_A(-1, 3)) = (0, 0, 0, 0, 0, 0).$$ \hfill (6.8)
Similarly,
\[ h^*(A, O_A(-3, -2)) = (0, 0, 0, 0, 0). \] (6.9)
The next sheaf we consider is \( O_A(-4, 1) \):
\[ h^*(A, O_A(-4, 1)) = (0, 15, 0, 0, 0). \] (6.10)
And finally, a familiar one:
\[ h^*(A, O_A) = (1, 0, 0, 0, 0). \] (6.11)

The dimensions of the cohomology groups of sheaves in the double complex \( W^{p,q} \) are in the following table:

| \( O_A(-3, -2) \) | \( O_A(-4, 1) \oplus O_A \) | \( O_A(-1, 3) \) | \( O_X(-1, 3) \) | \( h^0(X, O_X(-1, 3)) = 14 \) | \( h^1(X, O_X(-1, 3)) = 0 \) |
|----------------|-----------------|----------------|----------------|-----------------|----------------|
| 0              | 1               | 0              | \( h^0(X, O_X(-1, 3)) \) | \( h^1(X, O_X(-1, 3)) \) |
| 0              | 15              | 0              | \( h^0(X, O_X(-1, 3)) \) | \( h^1(X, O_X(-1, 3)) \) |
| \( \vdots \)  | \( \vdots \)   | \( \vdots \)  | \( \vdots \)      | \( \vdots \)      |

In the double complex context, since the \( r \)-th differential operators of a spectral sequence would send an element \( r \) steps rightward and \( r - 1 \) steps upward, the only \( H^k(X, O_X(-1, 3)) \) that possibly has a source would be \( H^0(X, O_X(-1, 3)) \). Its dimension can be calculated from counting the Euler characteristic. The Euler characteristic is counted with respect to the total complex \( M^k := \bigoplus_{p+q=k} W^{p,q} \). The Euler characteristic number of the complex of sheaves equals the Euler characteristic number of the total complex \( M^k \). This is used when we calculate the dimensions of the abelian groups involved. We have then
\[ h^*(X, O_X(-1, 3)) = (14, 0, 0, 0). \] (6.13)

### 6.2 Genus of Curves

We can also use spectral sequences of double complexes to compute the genus of curves. Recall that when we want to determine the Hodge numbers of the blow up of \( X \), we have determined the genus of the curves along which it is blown-up. The genus of them can be determined by our genus formula (5.5) in Section 5.2. For example, one of the curves is characterized as:
\[ \Gamma_1 = \mathbb{P}^1 \left[ \begin{array}{cc} 3 & -1 \\ 2 & 3 \end{array} \right]. \] (6.14)

We can also compute similarly for the other curve \( \Gamma_2 \). By the relation \( g(\Gamma_1) = \dim H^1(\Gamma_1, O_{\Gamma_1}) \), we turn this into a sheaf theory computation. Now we take \( \bar{A} = \mathbb{P}^1 \times \mathbb{P}^2 \). We have a similar four-term sequence:
\[ 0 \rightarrow O_{\bar{A}}(-2, -5) \rightarrow O_{\bar{A}}(-3, -2) \oplus O_{\bar{A}}(1, -3) \rightarrow O_{\bar{A}} \rightarrow O_{\Gamma_1} \rightarrow 0. \] (6.15)
We build the following double complex $W^{p,q}$ consisting of their cohomology groups:

$$
\begin{array}{c}
\begin{array}{c}
H^0(\tilde{A}, O_{\tilde{A}}(-2, -5)) \\
\downarrow \\
H^1(\tilde{A}, O_{\tilde{A}}(-2, -5))
\end{array}
\end{array} \rightarrow 
\begin{array}{c}
\begin{array}{c}
H^0(\tilde{A}, O_{\tilde{A}}(-3, -2) \oplus O_{\tilde{A}}(1, -3)) \\
\downarrow \\
H^1(\tilde{A}, O_{\tilde{A}}(-3, -2) \oplus O_{\tilde{A}}(1, -3))
\end{array}
\end{array} \rightarrow 
\begin{array}{c}
\begin{array}{c}
H^0(\tilde{A}, O_{\tilde{A}}) \\
\downarrow \\
H^1(\tilde{A}, O_{\tilde{A}})
\end{array}
\end{array} \rightarrow 
\begin{array}{c}
\begin{array}{c}
H^0(\Gamma_1, O_{\Gamma_1}) \\
\downarrow \\
H^1(\Gamma_1, O_{\Gamma_1})
\end{array}
\end{array}
$$

The dimensions of the cohomology groups we need are listed below:

\begin{align*}
h^*(\tilde{A}, O_{\tilde{A}}(-2, -5)) &= (0, 0, 0, 6), \\
h^*(\tilde{A}, O_{\tilde{A}}(-3, -2)) &= (0, 0, 0, 0), \\
h^*(\tilde{A}, O_{\tilde{A}}(1, -3)) &= (0, 0, 2, 0), \\
h^*(\tilde{A}, O_{\tilde{A}}) &= (1, 0, 0, 0).
\end{align*}

The dimensions of their cohomology groups in the double complex $W^{p,q}$ are:

$$
\begin{array}{cccc}
O_{\tilde{A}}(-2, -5) & O_{\tilde{A}}(-3, -2) \oplus O_{\tilde{A}}(1, -3) & O_{\tilde{A}} & O_{\Gamma_1} \\
0 & 0 & 1 & h^0(\Gamma_1, O_{\Gamma_1}) = 1 \\
0 & 0 & 0 & h^1(\Gamma_1, O_{\Gamma_1}) = 8 \\
0 & 2 & 0 & h^2(\Gamma_1, O_{\Gamma_1}) = 0 \\
6 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots
\end{array}
$$

In the table above, the three-level differential operator will map the cohomology group where the 6 seats, to $H^1(\Gamma_1, O_{\Gamma_1})$, and the two-level operator would send the 2 to $H^2(\Gamma_1, O_{\Gamma_1})$. Except these two places, there are no other cohomology groups that can be mapped to any one of them, forming images to reduce their dimensions when we take cohomology. And, lower level operators would map it to the cohomology groups where the 0 seats, which means that they would keep to be kernels, remaining unchanged when we take cohomology. Hence they would be stable until they touch $H^1(\Gamma_1, O_{\Gamma_1})$. We can also count the Euler characteristic as before. In conclusion, we have $g = h^1(\Gamma_1, O_{\Gamma_1}) = 2 + 6 = 8$, which is exactly the genus of this curve. This agrees with our general genus formula (5.5) in Section 5.2.

7 Discussion

We worked out aspects of cohomology of sheaves on the generalized complete Calabi-Yau manifolds, and developed some tools and approaches to understand them better. These manifolds are constructed from line bundles in ambient product spaces. For the generalized
case, one can make use of line bundles which do not have global sections on the ambient space, but have sections when restricted to appropriate subvarieties. One then constructs the generalized complete intersection Calabi-Yau in these subvarieties. Hence approaches and methods of sheaves are very necessary for these generalized constructions. The main tools we used include cohomology of sheaves, Bott formula, Künneth formula, and spectral sequences.

We have presented and proved a vanishing theorem of cohomology groups of sheaves which naturally generalized the original one for a single projective space to the case for a product of several projective spaces. The technique we used is to twist the Poincare residue short exact sequences. Next we considered the long exact sequences of cohomology groups associated with them and proved the vanishing theorem. This is useful for computing the Hodge numbers of complete intersection and generalized complete intersection manifolds in a product of projective spaces.

In the version of the vanishing theorem for a single projective space, the vanishing is related to the fact that the action of the projective linear group $PGL_{n+2}(k)$ on degree $d$ hypersurface in $\mathbb{P}^{n+1}$ induces a morphism which is proper and finite. This in turn is useful for the Haar measure of the matrix integrals for $PGL_{n+2}(k)$ \cite{13}. It would be good to understand whether there are generalizations of these relations in the cases of a product of projective spaces.

Moreover, we proved an identity between configuration matrices for complete intersection Calabi-Yau manifolds. The manifolds from the configuration matrices of the two sides of this identity are equivalent at the level of diffeomorphism. There are also other types of identities between configuration matrices \cite{4,1,10}. Besides identities, there are other types of relationships between configuration matrices pertaining to deformations and geometric transitions of the varieties \cite{4,10}. These identities and relationships are important and useful in the classification of gCICY \cite{4}.

Further, we have identified some involutions of gCICY. We identified some fixed point locus of involutions of gCICY, which are curves. We considered blow-up of gCICY along the curves, which can produce new Calabi-Yau manifolds. The blow-ups along the fixed point locus of the involutions provided new Calabi-Yaus and their variants, which can be useful for string compactification and non-perturbative superpotentials after the compactification. This enlarged the range of the construction of Calabi-Yau manifolds, and is important for the moduli space of Calabi-Yau manifolds, in which topologically distinct Calabi-Yau manifolds may be connected to each other by geometric transitions \cite{21,22}, such as via blowing-downs and blowing-ups. And this process of geometric transitions could also involve \cite{23} non-Kähler Calabi-Yau manifolds, which include complex non-Kähler manifolds with a trivial canonical bundle.

We presented a genus formula for curves in generalized complete intersection manifolds, which is useful for computing the Hodge numbers of blow-ups of the generalized complete intersection manifolds along the curves. These curves themselves can also be viewed as generalized complete intersection manifolds. Since these curves are the fixed point locus of
involutions which preserve the holomorphic forms, it would be interesting to identify the
fixed point locus of more general quotient symmetries in the automorphism groups.

Many of the construction of gCICYs have K3-fibrations and elliptic fibrations [10,11,4],
which are widely useful for string dualities, because of the fiber structures, such as the
heterotic/IIA duality and heterotic/F-theory duality. These fibrations can widely appear in
the context of heterotic string theory, for example [24,25,26,23,27] and references therein.
In the context of heterotic theory, it would be very interesting to consider, in addition, vector
bundles on these generalized complete intersection Calabi-Yau manifolds.

Furthermore, we used a spectral sequence approach that facilitates the computations
of the cohomology group of sheaves of the generalized complete intersection manifolds. It
is particularly useful for subvarieties with codimension one or higher. This approach is
useful for computing the Hodge numbers and the genus of curves in generalized complete
intersection manifolds. These Hodge numbers play an important role when we consider the
defformation classes.

Constructing examples of Calabi-Yau manifolds is an interesting theme due to mirror
symmetry and the search for mirror dual manifolds. Some of the generalized complete
intersection manifolds would be new to previous classifications. Some of the generalized
models may be related to weighted projective spaces by nontrivial blowing-downs [11]. One
can describe mirror dual manifolds by transposition of the degree matrices [28, 29, 30, 31]
in weighted projective spaces. It may be interesting to find mirror dual manifolds of these
new models and to understand the mirror duality from the point of view of the worldsheet
theory better.

These new types of Calabi-Yau manifolds open up new possibilities for analysis on the
worldsheet theory of strings on these spaces as target spaces, and may give new insights in
the context of gauged linear sigma models [32, 11, 33]. The defining equations with negative
degrees for the line bundles involve Laurent polynomials, which have their poles. However,
their poles are carefully avoided through beforehand intersection. The superpotential terms
on the worldsheet would involve Laurent polynomials, which would be a new phenomenon.

The CICYs in products of projective spaces have also enabled the computation of in-
stanton corrections in string theory compactifications, see for example [34, 35]. There are
new divisors in gCICY that are not from the divisors of the ambient product space of
projective spaces, unlike the ordinary CICY case. One can wrap branes inside these new
divisors in gCICY. Euclidean branes wrapping nontrivial new divisors contribute to the non-
perturbative superpotentials in lower dimensional field theory after the compactification,
under specific conditions [36, 35] on the cohomology groups of sheaves. This provides new
types of instanton corrections in the non-perturbative superpotentials, when compactified to
lower dimensions.
Acknowledgments

We would like to thank X. Gao, S.-J. Lee, B. Wu, and S.-T. Yau for communications or discussions. The work was supported in part by Yau Mathematical Sciences Center and Tsinghua University.

A Another Genus Formula

In this appendix, we prove proposition 5.1 in Section 5. The main tool we are going to use is the Riemann-Roch theorem. The expression (5.14) is similar to the case of the genus of curves of a single degree $n$ in $\mathbb{P}^2$, which can be denoted as $\mathbb{P}^2[n]$, see for example [37]. Our cases are different from theirs in the single projective space.

We denote the curves to be

$$X = \mathbb{P}^1 \left[ \begin{array}{l} d_1 \\ d_2 \end{array} \right],$$

which are bi-degree $(d_1, d_2)$ curves in $A = \mathbb{P}^1 \times \mathbb{P}^1$. We will pick appropriate lines to intersect $X$ to construct appropriate divisor for the calculation of the genus. Here we also introduce a divisor to estimate the irregularity of $X$:

$$E = \sum_P (r_P - 1)P. \quad (A.1)$$

For a generic point, $r_P - 1 = 0$, it would not arise in the above sum. Another concept we will need is adjointness: A form $G$ is said to be adjoint to $X$, if and only if $G \geq E$.

We can choose linear forms which are degree-one homogeneous polynomials $H_1, H_2$ on the variables of the first and second factor spaces respectively. The $H_i$ will pinpoint the location of the $i$-th factor, determining the ratio of its coordinates. Then, we are left with an equation on the variables of the other space factor. Hence a generic choice of $H_i$ would generate $d_j$ points, $j \neq i$. We denote the equation and its divisor with the same letter. Set

$$H_1 \cap X = Q_1 + \cdots + Q_{d_2} := Q, \quad H_2 \cap X = S_1 + \cdots + S_{d_1} := S. \quad (A.2)$$

We consider

$$E_m = m_2S + m_1Q - E, \quad (A.3)$$

where $m = (m_1, m_2)$ is a bi-index. For convenience, we introduce $M := \frac{1}{2} \deg E = \sum_P \frac{r_P(r_P - 1)}{2}$. We have:

$$\deg E_m = m_1d_2 + m_2d_1 - 2M. \quad (A.4)$$

We make the following definitions:

$$N_m := \{ \text{Forms of bi-degree } m = (m_1, m_2) \text{ which are adjoint to } X \}. \quad (A.5)$$
\( F_k := \{ \text{All forms of bi-degree } k = (k_1, k_2) \}. \) \hspace{1cm} (A.6)

We also have that
\[
L(D) := \{ f | \text{div } f + D \geq 0 \} \quad \text{and} \quad l(D) = \dim L(D).
\] \hspace{1cm} (A.7)

In order to use Riemann-Roch, we need to calculate \( l(E_m) \). To this end, we will construct an exact sequence involving forms of certain bi-degree with respect to the homogeneous coordinates.

To calculate \( l(E_m) \), we construct following map:
\[
\varphi : N_m \longrightarrow L(E_m), \quad \varphi(G) = \frac{G}{H_{1}^{m_1}H_{2}^{m_2}}. \] \hspace{1cm} (A.8)

Next we consider the kernel of this map, since it sends a form to the function field of the curve defined by \( F = 0 \), it will be zero if and only if it is divisible by \( F \).

The next result is that \( \varphi \) is onto. For any \( f = \frac{B}{C} \in L(E_m) \) where \( B, C \) are forms of the same degree. By definition we have \( \text{div}(B) - \text{div}(C) + \text{div}(E_m) \geq 0 \). This gives us forms \( \Psi, \Phi \) such that \( BH_{1}^{m_1}H_{2}^{m_2} = \Psi C + \Phi F \). After taking restriction to the quotient \( F = 0 \), this gives us \( f = \frac{\Psi}{H_{1}^{m_1}H_{2}^{m_2}} \). Notice that \( \text{div}(\Psi) = \text{div}(BH_{1}^{m_1}H_{2}^{m_2}) - \text{div}(C) \geq E \), hence we see that \( \Psi \in N_m \) and \( f = \varphi(\Psi) \). Combining above characterization of ker \( \varphi \) and the surjectivity, we have the following exact sequence:
\[
0 \longrightarrow F_{m-d} \longrightarrow N_m \longrightarrow L(E_m) \longrightarrow 0, \tag{A.9}
\]
where \( m = (m_1, m_2) \) and \( m - d = (m_1 - d_1, m_2 - d_2) \).

In smooth situation, \( F_k \) and \( N_k \) would coincide, both of them have dimension \((k_1+1)(k_2+1)\). The exact sequence (A.9) gives us
\[
l(E_m) = (m_1 + 1)(m_2 + 1) - (m_1 - d_1 + 1)(m_2 - d_2 + 1). \tag{A.10}
\]

Then, using Riemann-Roch we have:
\[
g = \deg E_m - l(E_m) + 1 = (d_1 - 1)(d_2 - 1). \tag{A.11}
\]

Now we consider a more general case, by adding the correction term of singular points \( P \). To this end, we introduce the following notation: \( N = N(k; r_1J_1, r_2J_2, \ldots, r_aJ_a) \) is the space of forms of bi-degree \( k = (k_1, k_2) \), whose corresponding curves have multiplicity at least \( r_i \) at point \( J_i \). Next we will count the dimension of \( N(m; r_1J_1, r_2J_2, \ldots, r_aJ_a) \). Here by a linear form, we mean a form of bi-degree (1,1). We have the following proposition.

**Proposition A.1.**

\[
\dim N(m; r_1J_1, r_2J_2, \ldots, r_aJ_a) \geq (m_1 + 1)(m_2 + 1) - \sum_j \frac{r_j(r_j + 1)}{2} \tag{A.12}
\]
with equality when \( m_i \geq (\sum_j r_j) - 1, i = 1, 2. \)

26
Proof. Without loss of generality, we consider a point \( T = ([0, 1], [0, 1]) \). A form can be written as \( F(z_0, z_1, y_0, y_1) = \sum_{t_0+i_1=m_1, j_0+j_1=m_2} a_{I} z_{i_1}^{j_0} y_{j_1}^{i_0} \) with \( I = (i_1, i_2), J = (j_1, j_2) \) are multi-indices. We call the curve cut out by \( F \) as \( F \) as well. It has multiplicity at least \( r \) at \( T \) if and only if those coefficients with \( i_0 + j_0 < r \) are zero. There are \( 1 + 2 + \cdots + r = \frac{r(r+1)}{2} \) such coefficients, so such condition will decrease the dimension at most \( \frac{r(r+1)}{2} \). With equality if it is the first condition to impose, and it can be less if several such conditions are all assumed. Hence we have claimed inequality.

For the equality when \( m_i \) is large, we put an induction on \( t := (\sum_i r_i) - 1 \). If, say \( m_1 = 1 \), then this enforces \( t = 0 \) or \( 1 \). If \( t = 0 \), there would only be one \( r_i \), and the dimension count above will give the result since the redundancy in dimension decrease will not arise. If \( t = 1 \), for the same reason as before, we just need to consider \( a = 2, r_1 = r_2 = 1 \), so we are considering a curve passing through 2 distinct points. This gives us 2 linearly independent linear equations on the coefficients, which decreases the dimension of \( N(m; r_1 J_1, r_2 J_2, \ldots, r_a J_a) \) by 2.

Next we will consider cases where \( m_i > 1, t > 1 \). Suppose, firstly, each \( r_i \) is 1. Every term \( \frac{r_1(r+1)}{2} \) would be 1. By induction, we only need to show that, every time we add a \( J_i \), the dimension goes down by 1. Set \( N_s \) to be \( N(m; r_1 J_1, r_2 J_2, \ldots, J_s) \). All we need to do is to prove is that \( N_s \neq N_{s-1} \). We choose linear forms \( H_i \) passing through \( J_i \) but not other \( J_j \), another form \( L_0 \) not passing all \( J_i \) is used to complete the degree, we set \( L_0 \) to have bi-degree \( (m_1 - 1, m_2 - 1) \). Set \( F = L_1 L_2 \cdots L_{s-1} L_0 \), and it will be in \( N_{s-1} \) but not \( N_s \), so \( N_s \not\subset N_{s-1} \).

Secondly, if \( r = r_1 > 1 \). Once again we set \( T = J_1 = ([0, 1], [0, 1]) \). In order to decrease \( t \), we consider:

\[
N_0 = N(m; (r - 1) J_1, r_2 J_2, \ldots, r_a J_a). \tag{A.13}
\]

We want to show that the dimension goes down at least \( r \) when we come from \( N(m; (r - 1) J_1, r_2 J_2, \ldots, r_a J_a) \) to \( N(m; r J_1, r_2 J_2, \ldots, r_a J_a) \). This fact, the proved inequality together with the induction hypothesis which gives the dimension of \( N(m; (r - 1) J_1, r_2 J_2, \ldots, r_a J_a) \) will complete the induction. To this end, we consider write \( F \in N_0 \) in the form \( F = \sum_{i=1}^{r-1} a_i z_i^{i_0} y_1^{r-1-i} F_i + F' \), where \( F' \) contains those terms with greater sum of the degrees of \( z_0 \) and \( y_0 \). The multiplicity of \( J_1 \) is the smallest sum of the degrees of \( z_0 \) and \( y_0 \). Define \( N_i, i \geq 1 \) to be the subspace of \( N_0 \) satisfying \( a_0 = \cdots = a_{i-1} = 0 \). And notice that \( N_r \) is the subspace of \( N_0 \) that all \( a_i \) mentioned vanishes, enforcing the multiplicity of \( J \) to be at least \( r \). Hence we have a decreasing sequence of spaces \( N_i \supseteq N_{i+1}, i = 0, 1, \ldots, r - 1 \), and \( N_r = N(m; r J_1, r_2 J_2, \ldots, r_a J_a) \). Consequently, similar to the first case, all we need to show is that \( N_i \neq N_{i+1}, i = 0, 1, \ldots, r - 1 \). And we will take this strengthened version of dimension count as a part of the induction process.

We make an induction now. Define \( W_0 = N(m - (1, 0); (r - 2) J_1, r_2 J_2, \ldots, r_a J_a) \). For \( F \in W_0 \), write \( F = \sum_{i=0}^{r-1} b_i z_0^{i_0} y_0^{r-2-i} F_i + F' \) and define \( W_k \) to be the subspace of \( W_0 \) consisting of forms with \( b_j = 0, j < k \). By induction, we have \( W_i \supseteq W_{i+1}, i = 0, 1, \ldots, r - 2 \), and \( W_{r-1} = N(m - (1, 0); (r - 1) J_1, r_2 J_2, \ldots, r_a J_a) \). So we can pick \( G_i \) such that \( G_i \in W_i, G_i \notin W_{i+1} \).
Then \( y_0 G_i \in N_i \) and \( y_0 G_i \notin N_{i+1} \). The case where the power of \( y_0 \) can not be increased can be treated by multiplying \( z_0 \): pick \( G_i \) as above, we know \( z_0 G_{r-2} \in N_{r-1}, z_0 G_{r-2} \notin N_r \). Hence \( N_i \neq N_{i+1} \), and this completes the proof. \( \square \)

With proposition \[ \text{A.1} \] on dimension, we return to the proof of \[ \text{5.14} \], now the \( N_m \) should be

\[
N_m = N(m; (r_P - 1)P).
\] (A.14)

\( P \) runs over those with \( r_P \geq 2 \). Using the exact sequence \[ \text{A.9} \] we have

\[
l(E_m) = (m_1 + 1)(m_2 + 1) - M - (m_1 - d_1 + 1)(m_2 - d_2 + 1).
\] (A.15)

Then, using Riemann-Roch we hence have

\[
g = \deg E_m - l(E_m) + 1
= m_1d_2 + m_2d_1 - 2M - ((m_1 + 1)(m_2 + 1) - M - (m_1 - d_1 + 1)(m_2 - d_2 + 1)) + 1
= (d_1 - 1)(d_2 - 1) - M.
\] (A.16)

Hence we proved proposition \[ \text{5.1} \] in Section \[ 5 \].

**B Topological Data of Subvarieties and Generalized Complete Intersections**

In this Appendix, we include supplementary details on determining the Hodge numbers of the manifolds and using them as topological invariants to classify those manifolds. We will explore the construction in \[ 3 \]. In this construction, \( A \) is the ambient space, \( M \) is the polynomial intersection, and \( X \) is the gCICY. In sum, we have \( X \subset M \subset A \). The main tools we will use are Hodge numbers and we compute them by cohomology groups of sheaves. The methods in this Appendix are alternative and complimentary to the approach in Section \[ 6 \] using spectral sequences of double complexes.

**B.1 Adjunction Formula, Koszul Sequence and Euler Sequence**

To compute the cohomology groups of sheaves, we need the following short exact sequences:

**Lemma B.1.** *(Adjunction Formula)* For a divisor \( D \) in the manifold \( M \), we have

\[
0 \longrightarrow TD \longrightarrow TM|_D \longrightarrow O_M(D)|_D \longrightarrow 0.
\] (B.1)

**Lemma B.2.** *(Koszul sequence)* Suppose \( I_D \) is the ideal sheaf of the divisor \( D \subset M \), we have

\[
0 \longrightarrow I_D \longrightarrow O_M \longrightarrow O_M|_D \longrightarrow 0.
\] (B.2)
Since the Euler sequence respects the direct sum, we have the following:

**Lemma B.3. (Euler sequence)** For \( A = A_1 \times A_2 \), \( A_1 = \mathbb{P}^{n_1} \), \( A_2 = \mathbb{P}^{n_2} \), we have

\[
0 \longrightarrow O_{A_1} \oplus O_{A_2} \longrightarrow O_A(1,0)^{\oplus(n_1+1)} \oplus O_A(0,1)^{\oplus(n_2+1)} \longrightarrow TA \longrightarrow 0. \tag{B.3}
\]

Tensoring the Koszul exact sequence with \( TM \) and \( O_M(D) \) respectively, we obtain the following exact sequences:

\[
0 \longrightarrow O_M(-D) \otimes TM \longrightarrow TM \longrightarrow TM|_D \longrightarrow 0. \tag{B.4}
\]

\[
0 \longrightarrow O_M \longrightarrow O_M(D) \longrightarrow O_M(D)|_D \longrightarrow 0. \tag{B.5}
\]

### B.2 Data of Subvariety \( M \)

Now we choose the example \( A = \mathbb{P}^1 \times \mathbb{P}^4 \),

\[
M = \begin{bmatrix} 3 \\ \mathbb{P}^4 \\ 2 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}. \tag{B.6}
\]

The reason we choose this example is that this is one of the simplest cases with multiple columns and negative degrees, and it is a good demonstration of computations of cohomology groups of sheaves and Hodge numbers. Also, it is good for comparing with other computational methods for this example \([4, 19]\). Since \( X \) is a submanifold of \( M \), and \( M \) is in turn a submanifold of \( A \), we can use the adjunction formulas in the short exact sequence form, iteratively. In order to derive the exact sequence involving the data we need, we tensor appropriate sheaf with the exact sequence we have. In particular, we can tensor it by tangent sheaves.

The Koszul sequence and adjunction formula for \( M \) are:

\[
0 \longrightarrow O_A(-M) \longrightarrow O_A \longrightarrow O_A|_M \longrightarrow 0. \tag{B.7}
\]

\[
0 \longrightarrow TM \longrightarrow TA|_M \longrightarrow O_A(M)|_M \longrightarrow 0. \tag{B.8}
\]

Our first goal is to compute the Hodge numbers of \((M, TM)\). In this case, \( I_M = O_A(-M) = O_A(-3, -2) \). To this end, we compute the cohomology groups of \( TA \) from the first sequence through the long exact sequence of cohomology groups associated with this short exact sequence. Then, we tensor the Koszul sequence with \( TA \) to obtain the following exact sequence

\[
0 \longrightarrow TA \otimes O_A(-3, -2) \longrightarrow TA \longrightarrow TA|_M \longrightarrow 0. \tag{B.9}
\]

Once we can compute the cohomology of \( TA \otimes O_A(-3, -2) \), we would be able to compute the cohomology of \( TA|_M \) from this sequence. In order to use the last sequence to compute the data of \( TM \), we would need the data of \( O_A(3, 2)|_M \). This can be obtained by tensoring Koszul sequence with \( O_A(3, 2) \):

\[
0 \longrightarrow O_A \longrightarrow O_A(3, 2) \longrightarrow O_A(3, 2)|_M \longrightarrow 0. \tag{B.10}
\]
The cohomology of $TA \otimes O_A(-3, -2)$ can be computed by Künneth formula. We consider the Bott vanishing theorem, as a special case of Borel-Weil-Bott theorem.

**Theorem B.1. (Bott Vanishing Theorem) [38]**

$$H^q(\mathbb{P}^n, O_{\mathbb{P}^n}(k)) = 0, \text{ if } \begin{cases} q \neq 0, 0 \leq k \\ \forall q, -n \leq k < 0 \\ q \neq n, k \leq -(n+1) \end{cases}. \quad (B.11)$$

Another fact that we shall need is, when $k \geq 0$, $H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(k))$ is the space of degree-$k$ homogeneous polynomials, and there is a perfect pairing:

$$H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(k)) \times H^n(\mathbb{P}^n, O_{\mathbb{P}^n}(-k-n-1)) \rightarrow \mathbb{C}. \quad (B.12)$$

So the dimensions of the two factors would be the same. So when $j \leq -(n+1)$, we have a nonnegative $k$ such that $k = -j - (n+1)$. Using the above duality, we have

$$h^0(\mathbb{P}^n, O_{\mathbb{P}^n}(k)) = h^n(\mathbb{P}^n, O_{\mathbb{P}^n}(j)) = \binom{n+k}{n} = \binom{-j-1}{-j-n-1}. \quad (B.13)$$

In sum, we have:

**Theorem B.2. (Bott Formula) [39]**

$$h^q(\mathbb{P}^n, \Omega^p(k)) = \begin{cases} \binom{k+n-p}{k} \binom{k-1}{p} & q = 0, 0 \leq p \leq n, k > p \\ 1 & k = 0, 0 \leq p = q \leq n \\ \binom{-k+p}{-k-1} \binom{-k-1}{-n-p} & q = n, 0 \leq p \leq n, k < p - n \\ 0 & \text{otherwise} \end{cases}. \quad (B.14)$$

Taking $p = 0$, we will recover the formulas for $h^q(\mathbb{P}^n, O_{\mathbb{P}^n}(k))$.

To use this for $A$, which is the product of projective spaces, we need:

**Theorem B.3. Künneth’s Formula**

$$H^q(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_m}, O(q_1, q_2, \ldots, q_m)) = \oplus_{k_1+k_2+\cdots+k_m=q} (H^{k_1}(\mathbb{P}^{n_1}, O_{\mathbb{P}^{n_1}}(q_1)) \otimes H^{k_2}(\mathbb{P}^{n_2}, O_{\mathbb{P}^{n_2}}(q_2)) \otimes \cdots \otimes H^{k_m}(\mathbb{P}^{n_m}, O_{\mathbb{P}^{n_m}}(q_m))). \quad (B.15)$$

We also need the characterization of the tangent bundle using the hyperplane bundle,

$$TP^n = \Omega^{n-1} \otimes O(n+1). \quad (B.16)$$

Now we proceed to compute the cohomology groups of $O_A(i, j)$. Notice that those cohomology groups on the right hand side of Künneth formula will be nonzero only when
Writing out the long exact sequence associated to the Euler sequence and since the cohomology respects direct sum, we have

\begin{align*}
  h^k(A, TA) &= 0, k \geq 1, \\
  h^0(A, TA) &= 2 \times 2 + 5 \times 5 - 2 = 27, \\
  h^*(A, TA) &= (27, 0, 0, 0, 0, 0). \tag{B.17}
\end{align*}

The second step shall be the calculation of \( h^k(A, TA \otimes O_A(-3, -2)) \). Denote the projection of \( A \) onto the first and the second factor by \( \pi_1, \pi_2 \) respectively. We have:

\[
  TA \otimes O_A(-3, -2) = [\pi_1^*(T\mathbb{P}^1 \otimes O_{\mathbb{P}^1}(-3)) \otimes \pi_2^*O_{\mathbb{P}^4}(-2)] \oplus [\pi_1^*O_{\mathbb{P}^1}(-3) \otimes \pi_2^*(T\mathbb{P}^4 \otimes O_{\mathbb{P}^4}(-2))] \tag{B.18}
\]

Now we substitute above formulas into the expression of \( TA \otimes O_A(-3, -2) \), to get \( E_1 = \pi_1^*O_{\mathbb{P}^1}(1) \otimes \pi_2^*O_{\mathbb{P}^4}(-2) \). By K"unneth formula we know \( h^q(\mathbb{P}^4, O_{\mathbb{P}^4}(-2)) \) will always be zero due to the Bott formula. Hence \( h^0(A, E_1) = 0 \). Then we turn to \( E_2 = \pi_1^*O_{\mathbb{P}^1}(-3) \otimes \pi_2^*(\Omega^3 \otimes O_{\mathbb{P}^4}(3)) \). The \( h^q(\mathbb{P}^1, O(-3)) \) is nonzero only when \( q = n = 1 \), being 3. Then we consider \( h^q(\mathbb{P}^4, \Omega^3 \otimes O_{\mathbb{P}^4}(3)) \). In this case we have \( k = 3, p = 3, n = 4 \), they will never satisfy the conditions for nonzero ones in Bott formula, hence those Hodge numbers will always vanish. So \( h^q(A, E_2) = 0 \). In conjunction with \( \text{(B.9)} \), the Hodge numbers of \( TA|_M \) would be the same as that of \( TA \), that is:

\[
  h^*(M, TA|_M) = (27, 0, 0, 0, 0). \tag{B.19}
\]

To use the adjunction formula for \( M \), we will need to calculate the Hodge numbers of \( O_A(3, 2)|_M \). This can be obtained from the twisted Koszul sequence \( \text{(B.10)} \): The only nonzero Hodge number for the cohomology groups of \( O_A \) would locate at the 0-th cohomology group, which is 1. The only nonzero Hodge number of \( O_A(3, 2) \) would locate at the 0-th cohomology group, which is 60. So the only nonzero Hodge number of \( O_A(3, 2)|_M \) would be at the 0-th cohomology group, being 60 - 1 = 59. Summing up, we have

\[
  h^*(A, O_A(3, 2)) = (60, 0, 0, 0, 0, 0), \quad h^*(A, O_A(3, 2)|_M) = (59, 0, 0, 0, 0, 0). \tag{B.20}
\]

So by the long exact sequence associated to \( \text{(B.8)} \), we see that

\[
  h^*(M, TM) = (0, 32, 0, 0, 0). \tag{B.21}
\]

### B.3 Data of Generalized Complete Intersection \( X \)

Our essential goal would be the data of \( X \). The adjunction formula of \( X \) reads

\[
  0 \rightarrow TX \rightarrow TM|_X \rightarrow O_M(-1, 3)|_X \rightarrow 0. \tag{B.22}
\]
As long as the Hodge number of the latter two sheaves are both determined, we would be able to determine the Hodge numbers of \((X, TX)\). First, for \(O_M(-1, 3)|_X\), we need the Koszul sequence for \(X\):

\[
0 \rightarrow O_M(1, -3) \rightarrow O_M \rightarrow O_M|_X \rightarrow 0.
\]  
(B.23)

Tensoring with \(O_M(-1, 3)\) we have:

\[
0 \rightarrow O_M \rightarrow O_M(-1, 3) \rightarrow O_M(-1, 3)|_X \rightarrow 0.
\]  
(B.24)

Recall the information in (B.7), we need the Hodge numbers of \(O_A(-3, -2)\). All of these Hodge numbers will vanish due to Künneth formula and the fact that \(h^q(\mathbb{P}^4, O_{\mathbb{P}^4}(-2)) = 0, \forall q\). So the Hodge numbers of \(O_M\) would be the same as that of \(O_A\):

\[
h^*(M, O_M) = (1, 0, 0, 0, 0).
\]  
(B.25)

For the Hodge numbers of \(O_M(-1, 3)\), we tensor (B.7) with \(O_A(-1, 3)\) to obtain:

\[
0 \rightarrow O_A(-4, 1) \rightarrow O_A(-1, 3) \rightarrow O_A(-1, 3)|_M \rightarrow 0.
\]  
(B.26)

The Hodge numbers of the first two sheaves can be computed through the Bott formula. The cohomology groups of \(O_A(-4, 1)\) would be nonzero only for the 1-st cohomology group for the first factor and the 0-th cohomology group for the second factor. And its \(h^1\) is 15,

\[
h^*(A, O_A(-4, 1)) = (0, 15, 0, 0, 0, 0).
\]  
(B.27)

For the second sheaf, once again, we can observe that the first factor has trivial cohomology groups, that is \(h^q(\mathbb{P}^1, O_{\mathbb{P}^1}(-1)) = 0, \forall q\), due to Bott formula, giving

\[
h^*(A, O_A(-1, 3)) = (0, 0, 0, 0, 0, 0).
\]  
(B.28)

So, combining the long exact sequence for (B.26), we have

\[
h^*(M, O_M(-1, 3)) = (15, 0, 0, 0, 0).
\]  
(B.29)

This result, (B.24) and (B.25) together, gives

\[
h^*(X, O_M(-1, 3)|_X) = (14, 0, 0, 0, 0).
\]  
(B.30)

In spite of \(h^0(A, O_A(-1, 3)) = 0\), we see that \(h^0(M, O_M(-1, 3)) \neq 0\). This means that \(X\) is an algebraic submanifold in \(M\), although \(X\) is not a submanifold in \(A\) defined by the global sections of line bundles on \(A\).

So our focus now should be the cohomology groups of \(TM|_X\). Tensoring the sequence (B.23) with \(TM\), we have short exact sequence involving \(TM|_X\):

\[
0 \rightarrow TM \otimes O_M(1, -3) \rightarrow TM \rightarrow TM|_X \rightarrow 0.
\]  
(B.31)

32
The Hodge numbers of $TM$ have been computed before, hence all we need to do is to compute the Hodge numbers of $TM \otimes O_M(1, -3)$. The above process used for $TM$ can be used here again after tensoring with $O(1, -3)$. The Euler sequence will become:

$$0 \longrightarrow O_A(1, -3)^{\oplus 2} \longrightarrow O_A(2, -3)^{\oplus 2} \oplus O_A(1, -2)^{\oplus 5} \longrightarrow TA \otimes O_A(1, -3) \longrightarrow 0. \quad (B.32)$$

The anjunction formula will become:

$$0 \longrightarrow TM \otimes O_M(1, -3) \longrightarrow (TA \otimes O_A(1, -3))|_M \longrightarrow O_M(4, -1) \longrightarrow 0. \quad (B.33)$$

Firstly we compute the Hodge numbers of $O_M(4, -1)$. We tensor the Koszul sequence (B.7) with $O_A(4, -1)$ to obtain:

$$0 \longrightarrow O_A(1, -3) \longrightarrow O_A(4, -1) \longrightarrow O_M(4, -1) \longrightarrow 0. \quad (B.34)$$

By the Bott formula applied to the first two sheaves, the second factor which arise when we involve the Künneth formula would always vanish, hence they have identically trivial cohomology groups. Thus the long exact sequence with respect to this short exact sequence enforces $O_M(4, -1)$ to have trivial cohomology groups,

$$h^*(M, O_M(4, -1)) = (0, 0, 0, 0, 0). \quad (B.35)$$

What is left to be done is the cohomology of $TA \otimes O_A(1, -3)|_M$. Tensor (B.7) with $TA \otimes O_A(1, -3)$, we know the following sequence is exact:

$$0 \longrightarrow TA \otimes O_A(-2, -5) \longrightarrow TA \otimes O_A(1, -3) \longrightarrow (TA \otimes O_A(1, -3))|_M \longrightarrow 0. \quad (B.36)$$

The cohomology of the first two sheaves can be computed with the help of (B.16). Similar to the computation on $TA \otimes O_A(-3, -2)$, this sheaf will be:

$$TA \otimes O_A(-2, -5) = [\pi^*_1(TP^4 \otimes O_{P^1}(-2)) \otimes \pi^*_2 O_{P^4}(-5)] \oplus [\pi^*_1 O_{P^1}(-2) \otimes \pi^*_2(TP^4 \otimes O_{P^4}(-5))] \quad (B.37)$$

\[ := E_3 \oplus E_4. \]

The first term is $E_3 = \pi^*_1 O_{P^1} \otimes \pi^*_2 O_{P^4}(-5)$. The Hodge numbers of the first factor would be $(1, 0)$. That of the second sheaf would be obtained by Bott formula, $h^4(P^4, O_{P^4}(-5)) = 1$. So the Hodge numbers of $E_3$ is $h^*(A, E_3) = (0, 0, 0, 0, 1, 0)$. The second part is $E_4 = \pi^*_1 O_{P^1}(-2) \otimes \pi^*_2(\Omega^3 \otimes O_{P^4})$. The Hodge numbers of the first factor are $h^*(P^4, O_{P^1}(-2)) = (0, 1)$. As for $\Omega^3 \otimes O_{P^4}$, we have $k = 0, p = 3, n = 4$, so the only nontrivial Hodge number would be at $q = p = 3$, being 1. Subsequently, $h^*(P^4, \Omega^3 \otimes O_{P^4}) = (0, 0, 0, 1, 0)$. These two sets of Hodge numbers give us $h^*(A, E_4) = (0, 0, 0, 0, 1, 0)$. Since the cohomology is compatible with direct sum, we see that

$$h^*(A, TA \otimes O_A(-2, -5)) = (0, 0, 0, 0, 2, 0). \quad (B.38)$$
Likewise we consider
\[ TA \otimes O_A(1, -3) = [\pi_1^*(T\mathbb{P}^1 \otimes O_{\mathbb{P}^1}(1)) \otimes \pi_2^*O_{\mathbb{P}^4}(-3)] \oplus [\pi_1^*O_{\mathbb{P}^1}(1) \otimes \pi_2^*(T\mathbb{P}^4 \otimes O_{\mathbb{P}^4}(-3))] \] (B.39)
\[ := E_5 \oplus E_6. \]

We can simplify above expressions to be \( E_5 = \pi_1^*O_{\mathbb{P}^1}(3) \otimes \pi_2^*O_{\mathbb{P}^4}(-3) \) and \( E_6 = \pi_1^*O_{\mathbb{P}^1}(1) \otimes \pi_2^*(\Omega^3 \otimes O_{\mathbb{P}^4}(2)) \). The second factor of \( E_6 \) has trivial cohomology groups, hence \( h^*(A, E_6) = (0, 0, 0, 0, 0, 0) \). For \( E_5 \), we know the only nonzero Hodge number for \( O_{\mathbb{P}^1}(1) \) would be \( h^0(A, O_{\mathbb{P}^1}(1)) = 2 \). For the second factor, we have \( k = 2, p = 3, n = 4 \) in the Bott formula, which gives trivial cohomology groups. Thus \( h^*(A, E_6) = (0, 0, 0, 0, 0, 0) \). Hence the second sheaf in sequence (B.36) has trivial cohomology groups:
\[ h^*(A, TA \otimes O_A(1, -3)) = (0, 0, 0, 0, 0, 0). \] (B.40)

By this and (B.33), the long exact sequence associated with (B.36) gives us:
\[ h^*(M, (TA \otimes O_A(1, -3))|_M) = (0, 0, 0, 2, 0). \] (B.41)

Together with (B.33), it is now possible to determine the cohomology of \( TM \otimes O_M(1, -3) \).

Since the third sheaf has trivial cohomology groups, the first two sheaves would have the same cohomology groups:
\[ h^*(M, TM \otimes O_M(1, -3)) = (0, 0, 0, 2, 0). \] (B.42)

In addition to this, with (B.21), we are now ready to determine \( h^*(X, TM|_X) \). We have a six-term long exact sequence for (B.31),
\[ 0 \rightarrow H^1(M, TM)_{32} \rightarrow H^1(X, TM|_X) \rightarrow 0 \]
\[ \rightarrow 0 \rightarrow H^2(X, TM|_X) \rightarrow H^3(M, TM \otimes O_M(1, -3))_{2} \rightarrow 0. \] (B.43)

Here, the subscripts of the cohomology groups denote their dimensions. So we have
\[ h^*(X, TM|_X) = (0, 32, 2, 0). \] (B.44)

Finally, we can compute the data of \( TX \). A six-term long exact sequence for (B.32) will be
\[ 0 \rightarrow H^0(X, O_M(-1, 3))|_X 14 \rightarrow H^1(X, TX) \rightarrow H^1(X, TM|_X)_{32} \]
\[ \rightarrow 0 \rightarrow H^2(X, TX) \rightarrow H^2(X, TM|_X)_{2} \rightarrow 0. \] (B.45)

So we have
\[ h^*(X, TX) = (0, 46, 2, 0). \] (B.46)

Hence \( h^{2,1}(X) = 46, h^{1,1}(X) = 2, h^{3,0}(X) = 1 \) and \( h^{3}(X) = 94 \).

In the above, we have used Serre duality and CY condition, from which we know that
\[ H^1(X, TX) = H^2(X, TX^*) = H^{2,1}(X^*). \] (B.47)

and
\[ H^2(X, TX) = H^1(X, TX^*) = H^{1,1}(X^*). \] (B.48)
References

[1] P. Candelas, A. M. Dale, C. A. Lutken and R. Schimmrigk, “Complete Intersection Calabi-Yau Manifolds,” Nucl. Phys. B 298 (1988) 493.

[2] P. Green and T. Hubsch, “Calabi-Yau Manifolds as Complete Intersections in Products of Complex Projective Spaces,” Commun. Math. Phys. 109 (1987) 99.

[3] S.-T. Yau, “Compact three-dimensional Kähler manifolds with zero Ricci curvature,” Proc. Symp. on Anomalies, Geometry, Topology, World Scientific Publishing, Singapore (1985) 395–406.

[4] L. B. Anderson, F. Apruzzi, X. Gao, J. Gray and S. J. Lee, “A new construction of Calabi-Yau manifolds: Generalized CI CYs,” Nucl. Phys. B 906 (2016) 441 [arXiv:1507.03235 [hep-th]].

[5] I. Brunner, M. Lynker and R. Schimmrigk, “Unification of M-theory and F-theory Calabi-Yau fourfold vacua,” Nucl. Phys. B 498 (1997) 156 [hep-th/9610195].

[6] J. Gray, A. S. Haupt and A. Lukas, “All Complete Intersection Calabi-Yau Four-Folds,” JHEP 1307 (2013) 070 [arXiv:1303.1832 [hep-th]].

[7] J. Gray, A. Haupt and A. Lukas, “Calabi-Yau Fourfolds in Products of Projective Space,” Proc. Symp. Pure Math. 88 (2014) 281.

[8] J. Gray, A. S. Haupt and A. Lukas, “Topological Invariants and Fibration Structure of Complete Intersection Calabi-Yau Four-Folds,” JHEP 1409 (2014) 093 [arXiv:1405.2073 [hep-th]].

[9] L. B. Anderson, X. Gao, J. Gray and S. J. Lee, “Multiple Fibrations in Calabi-Yau Geometry and String Dualities,” JHEP 1610 (2016) 105 [arXiv:1608.07555 [hep-th]].

[10] P. Berglund and T. Hubsch, “On Calabi-Yau generalized complete intersections from Hirzebruch varieties and novel K3-fibrations,” arXiv:1606.07420 [hep-th].

[11] P. Berglund and T. Hubsch, “A Generalized Construction of Calabi-Yau Models and Mirror Symmetry,” SciPost Phys. 4 (2018) 009 [arXiv:1611.10300 [hep-th]].

[12] P. Deligne, “Cohomologie des intersections complètes”, Lect. Notes in Math. 340 (1973) 39–61.

[13] N. M. Katz and P. Sarnak, “Random matrices, Frobenius eigenvalues, and monodromy”, American Mathematical Society Colloquium 45, AMS, Providence, RI, 1999.

[14] P. Deligne, “Les intersections complètes de niveau de Hodge un.”, Invent. Math. 15 (1972) 237–250.
[15] H. Matsumura and P. Monsky, “On the automorphisms of hypersurfaces”, J. Math. Kyoto Univ. 3 (1963) 347–361.

[16] D. Mumford and J. Fogarty, “Geometric Invariant Theory”, Springer-Verlag, Berlin, 1982.

[17] C. T. C. Wall, “Classification problems in differential topology. V”, Invent. Math. 1 (1966) 355–374.

[18] C. Voisin, “Hodge theory and complex algebraic geometry I”, Cambridge Studies in Advanced Mathematics 76, Cambridge University Press, 2002.

[19] A. Garbagnati and B. van Geemen, “A remark on generalized complete intersections,” Nucl. Phys. B 925 (2017) 135 [arXiv:1708.00517 [math.AG]].

[20] R. Hartshorne, “Algebraic Geometry”, Springer-Verlag, Berlin, 1977.

[21] M. Reid, “The moduli space of 3-folds with $K=0$ may nevertheless be irreducible,” Math. Ann. 278 (1987) 329–334.

[22] M. Gross, “Primitive Calabi-Yau threefolds,” J. Differential Geom. 45 (1997) no. 2, 288–318.

[23] H. Lin, B. Wu and S.-T. Yau, “Heterotic String Compactification and New Vector Bundles,” Commun. Math. Phys. 345 (2016) 457–475 [arXiv:1412.8000 [hep-th]].

[24] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, “Vacuum Configurations for Superstrings”, Nucl. Phys. B 258 (1985) 46.

[25] J. J. Heckman, H. Lin and S.-T. Yau, “Building Blocks for Generalized Heterotic/F-theory Duality,” Adv. Theor. Math. Phys. 18 (2014) 1463–1503 [arXiv:1311.6477 [hep-th]].

[26] I. V. Melnikov, R. Minasian and S. Sethi, “Heterotic fluxes and supersymmetry,” JHEP 1406 (2014) 174 [arXiv:1403.4298 [hep-th]].

[27] H. Lin and T. Zheng, “Higher dimensional generalizations of twistor spaces,” J. Geom. Phys. 114 (2017) 492–505 [arXiv:1609.09438 [math.DG]].

[28] P. Berglund and T. Hubsch, “A Generalized construction of mirror manifolds,” Nucl. Phys. B 393 (1993) 377 [Stud. Adv. Math. 9 (1998) 327] [hep-th/9201014].

[29] V. V. Batyrev, “Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties,” J. Alg. Geom. 3 (1994) 493 [alg-geom/9310003].
S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, “Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces,” Nucl. Phys. B 433 (1995) 501 [hep-th/9406055].

P. Berglund and S. H. Katz, “Mirror symmetry constructions: A review,” Stud. Adv. Math. 1 (1996) 87 [hep-th/9406008].

E. Witten, “Phases of N=2 theories in two-dimensions,” Nucl. Phys. B 403 (1993) 159 [hep-th/9301042].

E. Sharpe, “A few recent developments in 2d (2,2) and (0,2) theories,” Proc. Symp. Pure Math. 93 (2015) 67 [arXiv:1501.01628 [hep-th]].

C. Beasley and E. Witten, “New instanton effects in string theory,” JHEP 0602 (2006) 060 [hep-th/0512039].

E. Witten, “Nonperturbative superpotentials in string theory,” Nucl. Phys. B 474 (1996) 343 [hep-th/9604030].

L. B. Anderson, F. Apruzzi, X. Gao, J. Gray and S. J. Lee, “Instanton superpotentials, Calabi-Yau geometry, and fibrations,” Phys. Rev. D 93 (2016) 086001 [arXiv:1511.05188 [hep-th]].

W. Fulton, “Algebraic Curves”, Addison-Wesley, 1989.

R. Bott, “Homogeneous vector bundles”, Ann. of Math. (2) 66 (1957) 203–248.

C. Okonek, M. Schneider, H. Spindler, “Vector bundles on complex projective spaces”, Progress in Mathematics 3, Birkhauser, 1980.