Sampling From the Wasserstein Barycenter

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Abstract. This work presents an algorithm to sample from the Wasserstein barycenter of absolutely continuous measures. Our method is based on the gradient flow of the multimarginal formulation of the Wasserstein barycenter, with an additive penalization to account for the marginal constraints. We prove that the minimum of this penalized multimarginal formulation is achieved for a coupling that is close to the Wasserstein barycenter. The performances of the algorithm are showcased in several settings.

1. INTRODUCTION

The barycenter in the space of probability measures equipped with the Wasserstein distance, first introduced by Agueh and Carlier (2011) gained a lot of popularity in recent years. The most striking property of the Wasserstein space is probably how its geodesics can be interpreted as a displacement of particles in $\mathbb{R}^d$, allowing the Wasserstein barycenter of measures to take into account the geometry of the underlying space $\mathbb{R}^d$. Thanks to its geometric interpretation, it proved useful in a variety of applications. For instance, it has been applied in image processing: Julien et al. (2011) developed an algorithm for texture synthesis built around this measure, Barré et al. (2020) used the Wasserstein barycenter of images to improve the precision of gas emission sourcing, and it played a central role in Gramfort, Peyré, and Cuturi (2015) to compare brain image data. In the field of fairness, Gordaliza et al. (2019) and Le Gouic, Loubes, and Rigollet (2020) leveraged the distribution to tackle the problem of fairness in regression and classification problems. Barycenters also found applications in Bayesian inference; Srivastava, Cevher, et al. (2015) and Srivastava, Li, and Dunson (2018) proposed a method to accelerate the computation of posterior distributions for Bayesian inference using the Wasserstein barycenter.

More formally, let us denote by $(P_2(\mathbb{R}^d), W_2)$ the set of all measures defined on $\mathbb{R}^d$ with finite second order moment, endowed with the Wasserstein distance

$$W_2 : (\mu, \nu) \mapsto \sqrt{\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\gamma(x, y)},$$

where the infimum is taken over the set of couplings of $\mu, \nu$. A Wasserstein barycenter $b$ of $\mu_1, \ldots, \mu_n$ with weights $\lambda_1, \ldots, \lambda_n$ is a minimizer of the map $\nu \mapsto \sum \lambda_i W_2^2(\mu_i, \nu)$, i.e. the measure $b$ corresponds to a Fréchet mean of $\mu_1, \ldots, \mu_n$ in the Wasserstein space $(P_2(\mathbb{R}^d), W_2)$. Equivalently, the multimarginal formulation of the barycenter problem asserts that $b$ is obtained by pushing forward the minimizer $\gamma^*$ of a functional $G$ (see equation (2)) defined over the set of couplings of the $\mu_i$’s. Under mild conditions on the $\mu_i$’s, both problems admit unique solutions, and yield the same measure. Section 2 provides more details on the Wasserstein barycenter.

Most methods in the literature propose to estimate the Wasserstein barycenter with a discrete measure e.g. Cuturi and Doucet (2014), Solomon et al. (2015), and Benamou et al. (2015).
However, this approach does not scale well with the dimension as noted by Altschuler and Boix-Adserà (2021). Indeed, they showed that computing the infimum of \( \nu \mapsto \sum \lambda_i W_2^2(\mu_i, \nu) \) when the \( \mu_i \) are discrete measures is a NP-hard problem in the dimension, even when only approximate solutions are acceptable. We can, however, avoid estimating the density altogether and focus on generating samples distributed according to the barycenter of known measures. Given the broad applicability of the Wasserstein barycenter and of sampling techniques in general, we believe that such sampling procedures should be part of the statistician’s toolbox.

We are motivated by the success of sampling methods in high dimensional settings, which is due to the possibility of integrating functions against the target measure without resorting to discretization on a large grid. Markov chain methods have become popular tools to this end. The most well known among them are probably Hamiltonian Monte Carlo methods, variants of the Metropolis-Hastings algorithm relying on simulations to generate diverse samples from a distribution (Bishop 2006, Chapter 11), and Langevin diffusion (Pavliotis 2014, Chapter 4), which transports points along random trajectories to redistribute them according to the target measure. Such transportation methods are computationally attractive since one only needs to store minimal information about how to move the particles at each iteration.

In their celebrated work, Jordan, Kinderlehrer, and Otto (1998) showed that the marginals of Langevin diffusion are distributed according to gradient flows in the Wasserstein space of the Kullback-Leibler divergence with respect to the stationary measure. This insight, studied rigorously in Ambrosio, Gigli, and Savaré (2008), brought to light a connection between sampling and optimization which added a new perspective to the study of Monte Carlo algorithms (e.g. Vempala and Wibisono 2019 analyze the Unadjusted Langevin Algorithm and Chewi, Le Gouic, Lu, Maunu, Rigollet, and Stromme 2020 analyze the Mirror Langevin diffusion in this framework). Inspired by this insight, we aim to minimize the multimarginal formulation \( G \) with gradient descent in the Wasserstein space. This procedure, like Langevin diffusion, iteratively redistributes randomly initialized points to produce a sample from the barycenter. We therefore recover an approximation of the optimal coupling through the samples, which can be desirable in applications.

However since the barycenter is defined by a constrained optimization problem, one cannot expect that the constraints will remain satisfied along the gradient flow. This issue of constraints has been tackled in a variety of ways in the literature. A natural solution consists in restricting the domain of possible directions at each iteration, which hints to the Frank-Wolfe algorithm, i.e. to choose the steepest descent direction in the admissible set. This approach was studied in Luise et al. (2019) within the context of regularized optimal transport, where the regularized Wasserstein barycenter (also known as the Sinkhorn barycenter) minimizes a sum of Sinkhorn divergences. When the admissible set of directions forms a Reproducing Kernel Hilbert Space (RKHS) with suitable kernel, Shen et al. (2020) propose to generate samples distributed according to the Sinkhorn barycenter by iterative pushforward of an initial measure with the map \( \text{id}_{\mathbb{R}^d} - h \cdot d \), where \( h \) is a step size and \( d \) is the direction of steepest descent in the RKHS. Both methods operate on discrete measures and the authors prove that the continuous measure is recovered as a (weak) limit when the number of samples increases to ensure consistency. We refer to Peyré and Cuturi (2020) for details about regularized optimal transport. We note that in some settings the issue of constraints can be addressed more easily, for example in the Bures-Wasserstein manifold (i.e. the subspace of Gaussian measures in the Wasserstein space) where the barycenter problem reduces to a finite dimensional optimization problem. Chewi, Maunu, et al. (2020) propose a gradient descent algorithm for the original formulation of the barycenter problem. However, we
obviously cannot hope that similar properties hold for arbitrary continuous measures.

In this paper, we perform gradient descent on a penalized functional \( F^\alpha : P_2((\mathbb{R}^d)^n) \to \mathbb{R}^+ \) obtained from \( G \) by adding a penalization term to control the distance between the coupling marginals and the \( \mu_i \)'s. We weigh the penalization with a coefficient \( \alpha \) and control the induced error with \( \alpha \). Taking advantage of the differential calculus on the Wasserstein space, we can define a gradient flow for \( F^\alpha \). To implement this procedure, we focus on the popular SVGD algorithm introduced in Liu and Wang (2016) to approximate the gradient of the penalization; technical details are given in Section 4.

Contributions. Inspired by the work of Jordan, Kinderlehrer and Otto cited above, we introduce a new sampling algorithm, called BARYGD, that performs kernelized gradient descent on a well chosen functional \( F^\alpha \) built as a penalized version of the now classical multimarginal formulation of the barycenter problem introduced by Agueh and Carlier (2011). We show that our method is consistent: with fixed penalization strength \( \alpha \), if gradient descent yields a coupling \( \gamma_{\varepsilon,\alpha} \) such that \( F^\alpha(\gamma_{\varepsilon,\alpha}) - \inf F^\alpha \leq \varepsilon \), then we get a quantitative bound on the Wasserstein distance between the approximate barycenter obtained from \( \gamma_{\varepsilon,\alpha} \) and the true barycenter obtained from the optimal coupling \( \gamma^* \). This bound vanishes when \( \varepsilon \to 0 \) and \( \alpha \to \infty \). As a consequence, we show that the minimizers of \( F^\alpha \) converge to \( \gamma^* \) when \( \alpha \to \infty \) and quantify the rate of convergence. Furthermore, we perform numerical experiments with the algorithm in several settings (see Figures 1 and 2).

Organization. In Section 2 we describe the Wasserstein barycenter in more details. We present and analyze our method in Section 3. Section 4 is devoted to implementation details. In Section 5, we discuss some open questions.

Notation. All probability measures are assumed to be absolutely continuous with respect to the Lebesgue measure and, with a slight abuse of notation, we will identify measures with their densities. We write \( X \sim \nu \) when random variable \( X \) has distribution \( \nu \). For \( \mu, \nu \in P_2(\mathbb{R}^d) \) and a measurable map \( f : \mathbb{R}^d \to \mathbb{R}^d \), we write \( \nu = f_* \mu \) if \( f(X) \sim \nu \) when \( X \sim \mu \). The set of couplings of \( \mu \) and \( \nu \), i.e. probability measures on \( \mathbb{R}^d \times \mathbb{R}^d \) with marginals \( \mu \) and \( \nu \), is denoted by \( \Pi(\mu, \nu) \). We denote marginals by indices so \( \gamma_1 = \mu \) and \( \gamma_2 = \nu \) for \( \gamma \in \Pi(\mu, \nu) \); the notation extends naturally to couplings of \( n \) measures. We denote Euclidean gradients by \( \nabla \) and write \( \nabla_W \) for gradients in the Wasserstein space. Finally, we write indifferently \( c_t = c(t) \) and \( c' = \dot{c} = \partial_t c \) when \( c : [a, b] \to X \) is a curve in some space \( X \).

2. WASSERSTEIN BARYCENTERS

This section briefly presents background notions on the Wasserstein space and its barycenters that were first introduced in the seminal paper of Agueh and Carlier (2011). We refer to Le Gouic and Loubes (2010) for questions of existence and stability of the barycenter.

The Wasserstein space over \( \mathbb{R}^d \) is defined as the set \( P_2(\mathbb{R}^d) \) of probability measures over \( \mathbb{R}^d \) with finite second order moment, endowed with the distance \( W_2 \) defined by

\[
W_2^2(\mu_0, \mu_1) = \inf_{\gamma \in \Pi(\mu_0, \mu_1)} \int |x - y|^2 \, d\gamma(x, y), \quad \forall \mu_0, \mu_1 \in P_2(\mathbb{R}^d)
\]

where the infimum is taken over the set \( \Pi(\mu_0, \mu_1) \) of all couplings \( \gamma \) of \( \mu_0 \) and \( \mu_1 \). The Wasserstein
space is a geodesic space: for every $\mu_0$ and $\mu_1$ in $P_2(\mathbb{R}^d)$ there exists a path $(\mu_t)_{t \in [0,1]}$ such that
\[
W_2(\mu_s, \mu_t) = |s - t| W_2(\mu_0, \mu_1), \quad \forall s, t \in [0,1].
\]
Such a path $(\mu_t)_{t \in [0,1]}$ is called a constant-speed geodesic between its end points $\mu_0$ and $\mu_1$ — we will often shorten it to geodesic. A functional $G$ defined over $P_2(\mathbb{R}^d)$ is said to be geodesically convex if it is convex along every geodesic.

Let $\mu_1, \ldots, \mu_n \in P_2(\mathbb{R}^d)$ and let $\lambda_1, \ldots, \lambda_n$ be strictly positive weights such that $\sum \lambda_i = 1$. A Wasserstein barycenter of $(\mu_i)$ with weights $(\lambda_i)$ is defined as any measure
\[
b \in \arg\min_{\nu \in P_2(\mathbb{R}^d)} \sum \lambda_i W_2^2(\mu_i, \nu),
\]
i.e. a barycenter corresponds to a Fréchet mean of the $\mu_i$ in the metric space $(P_2(\mathbb{R}^d), W_2)$.

The minimization problem $[1]$ is equivalent to the following multimarginal problem. Let $T(x) = \sum \lambda_i x_i$ for any $x \in (\mathbb{R}^d)^n$ and recall that $\Pi(\mu_1, \ldots, \mu_n)$ denotes the set of all couplings of the $\mu_i$. The infimum of the functional
\[
G : \gamma \mapsto \int \sum \lambda_i |x_i - T(x)|^2 \, d\gamma(x)
\]
over $\Pi(\mu_1, \ldots, \mu_n)$ is achieved for a coupling $\gamma^*$ such that
\[
b = T_{\#} \gamma^*,
\]
(3a)
\[
G(\gamma^*) = \sum \lambda_i W_2^2(\mu_i, T_{\#} \gamma^*).
\]
(3b)
Moreover, if one of the $\mu_i$’s is absolutely continuous with respect to the Lebesgue measure, then $\gamma^*$ is unique. Note that by introducing the probability measure $P = \sum \lambda_i \delta_{\mu_i}$ on $P_2(\mathbb{R}^d)$, the right hand side of (3b) corresponds to the variance of $P$. With this notations, $b$ minimizes the variance functional $\nu \mapsto \sum \lambda_i W_2^2(\mu_i, \nu)$ of $P$ and $b$ corresponds to the barycenter of $P$.

In the next section, we leverage the smooth structure of the Wasserstein space in order to develop an optimization scheme for $G$ that will be the base of our sampling algorithm.

## 3. SAMPLING WITH GRADIENT FLOWS

This section defines the flow gradient of a functional on the Wasserstein space. Some useful details of this technique introduced in Jordan, Kinderlehrer, and Otto (1998) are gathered in Appendix $\text{B}$. 

### 3.1. Definition of the problem

We aim to optimize the functional $G$ defined in Section 2 with a gradient flow on the Wasserstein space. For simplicity, we first define the cost function $c$ by
\[
c : x \mapsto \sum \lambda_i |x_i - T(x)|^2
\]
then, for any \( \gamma \in P_2((\mathbb{R}^d)^n) \), we have

\[
G(\gamma) = \int c \, d\gamma.
\]

The set of marginal constraints \( \Pi(\mu_1, \ldots, \mu_n) \) is convex in \( L^2((\mathbb{R}^d)^n) \) but is not geodesically convex in the Wasserstein space, which makes the multimarginal problem a difficult non-convex optimization problem on \( P_2(\mathbb{R}^d) \). Since a gradient descent scheme on \( G \) will inevitably leave the constraint set \( \Pi(\mu_1, \ldots, \mu_n) \), we modify the problem by enforcing the constraints with a penalization term that ensures that the marginals are close to the constraint set \( \Pi(\mu_1, \ldots, \mu_n) \). We also assume that \( \nabla F \) is defined Lebesgue-almost everywhere. Such inequalities were introduced by Sturm (2003) to describe curvature properties of geodesic spaces and have played a central role in understanding the behavior of the empirical Wasserstein barycenter (see Ahidar-Coutrix, Le Gouic, and Paris (2020) and Le Gouic et al. (2019)). We refer to Appendix C for details on variance inequalities. We can now state the following result.

**Proposition 1.** Suppose at least one of the \( \mu_i \)'s is absolutely continuous. Then, for any \( \alpha > 0 \), the functional \( F^\alpha \) admits at least one minimizer in \( P_2((\mathbb{R}^d)^n) \). Moreover, this minimizer is absolutely continuous with respect to the Lebesgue measure.

Given a minimizer \( \gamma^\alpha \) of \( F^\alpha \), the measure \( T_\# \gamma^\alpha \), dubbed the associated barycenter to \( \gamma^\alpha \), should be close to the barycenter \( b \) of \( P \) thanks to (3a) and (3b). Our next result quantifies this proximity with respect to \( \alpha \) under extra assumptions on \( \mu_1, \ldots, \mu_n \).

We first assume that \( \mu_1, \ldots, \mu_n \) satisfy the Poincaré inequality with a strictly positive constant \( C_P \), i.e. that for any \( i \in [1, n] \) and any Lipschitz \( f \in L^2(\mu_i) \), we have

\[
|f|_{L^2(\mu_i)}^2 \leq C_P |\nabla f|_{L^2(\mu_i)}^2 \tag{4}
\]

where \( \nabla f \) is defined Lebesgue-almost everywhere. Such inequalities are very common in the sampling setting. We also assume that \( P = \sum \lambda_i \delta_{\mu_i} \) satisfies a variance inequality with constant \( k \), i.e. that there exists a strictly positive constant \( k \) such that for any \( \nu \in P_2(\mathbb{R}^d) \), it holds

\[
kW_2^2(\nu, b_P) \leq B(\nu) - B(b_P),
\]

where \( b \) is the barycenter of \( \mu_1, \ldots, \mu_n \), and \( B : \nu \mapsto \sum \lambda_i W_2^2(\mu_i, \nu) \) is the functional appearing in the original formulation of the barycenter problem.

Such inequalities were introduced by Sturm (2003) to describe curvature properties of geodesic spaces and have played a central role in understanding the behavior of the empirical Wasserstein barycenter (see Ahidar-Coutrix, Le Gouic, and Paris (2020) and Le Gouic et al. (2019)). We refer to Appendix C for details on variance inequalities. We can now state the following result.

**Proposition 2.** Let \( \alpha \geq 1 \) and \( \varepsilon \leq 1 \). Suppose \( \mu_1, \ldots, \mu_n \) satisfy a Poincaré inequality with constant \( C_P \) and that \( \sum \lambda_i \delta_{\mu_i} \) satisfies a variance inequality with constant \( k \). Then there exists a
strictly positive constant $C$, only depending on the variance $\sigma^2 = \int c \, d\gamma^\star$ and $C_\mathcal{P}$, such that for any $\gamma^{\varepsilon, \alpha} \in \mathcal{P}_2((\mathbb{R}^d)^n)$ satisfying $F_\alpha(\gamma^{\varepsilon, \alpha}) - \inf F_\alpha \leq \varepsilon$, it holds

$$\frac{k}{4} W_2^2(T_# \gamma^{\varepsilon, \alpha}, T_# \gamma^\star) \leq \varepsilon + \frac{C}{\sqrt{\alpha}}. \quad (5)$$

Proposition 2 shows that any approximate minimizer of $F_\alpha$ produces a barycenter which distance to the desired barycenter $b$ is of order $O(1/\sqrt{\alpha})$ and can therefore be controlled to arbitrary precision with $\alpha$.

The fact that the minimization of $F_\alpha$ is carried out without constraints on the marginals of $\gamma$ allows to use classical optimization techniques. The next section is devoted to the introduction of the gradient flow on the Wasserstein space of $F_\alpha$ that will be the backbone of our algorithm.

3.2. Minimizing scheme

We first briefly recall that the Wasserstein gradient of a functional $F : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ at a point $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ is a vector field on $\mathbb{R}^d$ given by the Euclidean gradient of its first variation $f$, i.e.

$$\nabla_W F(\nu) := x \mapsto \nabla f(x) \in L^2(\mathbb{R}^d, \mathbb{R}^d; \nu) \quad (6)$$

where $f$ is such that for any signed measure $\delta$ with $\int d\delta = 0$, we have

$$F(\nu + \varepsilon \delta) = F(\nu) + \varepsilon \int f d\delta + o(\varepsilon).$$

For instance, the Wasserstein gradient of $\mu \mapsto \chi^2_\nu(\mu)$ is given by

$$\nabla_W \chi^2_\nu(\mu) = 2\nabla_\nu.\mu.$$

Intuitively, when a probability measure is seen as a large number of particles, the Wasserstein gradient of a function defines a vector field along which each particle should be moved in order to locally maximize the function. More details about the differential calculus on the Wasserstein space are provided in Appendix 3.

To sample from the barycenter of $P = \sum \lambda_i \delta_{\mu_i}$, we consider the flow described by

$$\begin{cases}
\partial_t X_t = -\nabla_W F_\alpha(\gamma(t))(X_t) \\
X(0) = X_0 \sim \gamma(0),
\end{cases} \quad (7)$$

where $\gamma(t)$ denotes the distribution of $X_t$ at time $t > 0$. Recall that for a measure $\gamma \in \mathcal{P}_2((\mathbb{R}^d)^n)$, we write $\gamma_i$ for its $i$-th marginal. Using (6), the Wasserstein gradient of $F_\alpha$ is given by

$$\nabla_W F_\alpha(\gamma)(x) = \nabla c(x) + 2\alpha \sum \lambda_i \nabla \frac{\gamma_i}{\mu_i}(x),$$

for any $\gamma \in \mathcal{P}_2((\mathbb{R}^d)^n)$ and $x \in \mathbb{R}^d$.

The explicit Euler scheme for (7) provides an approximation of $(X_{m})_m$ for $t_{m+1} - t_m = h_m$,

$$X_{m+1} = X_m - h_m \left[ \nabla c(X_m) + 2\alpha \sum \lambda_i \nabla \frac{\gamma_i(t_m)}{\mu_i}(X_m) \right], \quad (8)$$

where $\gamma(t_m)$ is the distribution of $X_m$ and $h_m$ is a given step size.
However, note that since \( \gamma(t) \) is unknown, this discretization scheme is not implementable as it is. In order to implement it, we use a kernel to approximate the Wasserstein gradient of the penalization term in \( F_\alpha \) as is done in Liu and Wang (2016) and Chewi, Le Gouic, Lu, Maunu, and Rigollet (2020). The next section provides more details.

4. IMPLEMENTATION

As noted in Section 3, Algorithm (8) cannot be implemented directly since there is no canonical way to recover the unknown distribution \( \gamma(t_m) \) of the current state \( X_{t_m} \) from the mere knowledge of \( X_m \). In this section, we present a concrete implementation of the scheme (8).

The SVGD algorithm. We consider the popular SVGD algorithm introduced in Liu and Wang (2016) which uses an explicit kernel \( K \) independent of the target measure. The kernel \( K \) can be chosen to have a convenient analytical form (e.g. a Gaussian kernel) amenable to direct evaluation. Let \( \pi \propto e^{-V} \), with \( V \in C^1(\mathbb{R}^d) \), be the target measure. SVGD aims to minimize the \( \chi^2 \) divergence by transporting randomly initialized points along the flow described by

\[
\partial_t X_t = -K_\pi \nabla_\pi \chi^2_\pi(\mu_t)(X_t),
\]

where \( X_t \sim \mu_t \) and \( K_\pi : f \mapsto \int K(\cdot, x)f(x) d\pi(x) \) is the integral operator associated with \( K \).

Note that integration by parts yields

\[
K_\pi \nabla_\pi \chi^2_\pi(\mu_t)(X_t) = \int \{K(X_t, x) \nabla V(x) - \nabla_2 K(X_t, x)\} d\mu_t(x).
\]

Approximating the integrals by averaging with samples \( (X^i_t)_{i=1,...,N} \) distributed according to \( \mu_t \), one gets the SVGD algorithm

\[
X_{i+1}^j = X_i^j + \frac{h_t}{N} \sum_{j=1}^{N} \left\{ -K(X_i^j, X^j_\ell) \nabla\log \mu_j(X^j_\ell) + \nabla_2 K(X_i^j, X^j_\ell) \right\}
\]

where the initial points \( X^1_0, \ldots, X^N_0 \) are chosen at random.

Proposed algorithm. To minimize \( F_\alpha \), we implement (9), replacing each gradient of the penalization sum, \( \sum_{i=1}^{n} \lambda_i \chi^2_{\mu_i} \), with the SVGD iteration for each marginal. Thus, we couple \( n \) batches of \( N \) points and generate \( N \) samples approximately distributed according to the barycenter \( T_\gamma \) of a coupling \( \gamma \) minimizing \( F_\alpha \). Our iteration takes the form

\[
X_{t+1}^{ij} = X_t^{ij} - h_t \nabla j c(X_t) + \frac{\alpha h_t \lambda_j}{N} \sum_{\ell=1}^{N} \left\{ K(X_t^{ij}, X_\ell^{ij}) \nabla \log \mu_j(X_\ell^{ij}) + \nabla_2 K(X_t^{ij}, X_\ell^{ij}) \right\}
\]

where \( j \in [1,n] \) is the index of marginal \( \mu_j \) and \( X_t^{ij}, i \in [1,N] \), stands for the \( i \)-th sample associated with the \( j \)-th marginal.

Numerical details. All the simulations are carried out with a Gaussian kernel

\[
K(x, y) = \exp\left(-|x - y|^2\right) \quad \forall x, y \in \mathbb{R}.
\]
The step size $h_t$ as well as $\alpha = \alpha_t$ vary over iterations, starting from $h_0 = 0.1$ and $\alpha_0 = 1000$. We first keep the parameters fixed and let gradient descent concentrate the particles on the graph of an increasing function before doubling the penalization strength while keeping $h_t \cdot \alpha_t$ constant. Thus, we enforce the marginal constraints once the coupling corresponds to an optimal transport. In Figure 2 we the step-size evolves according to AdaGrad (see Duchi, Hazan, and Singer (2011)). We give further experimental results in Appendix A.

5. CONCLUSION

We have presented a new algorithm to sample from the Wasserstein barycenter of a finite set of measures. Our numerical experiments illustrate its convergence in several examples. We also proved a theoretical bound for samples drawn from almost minimizers of $F^\alpha$, the gradient flow of which inspired our algorithm. It remains to formally prove that this algorithm, a kernelized gradient descent of $F^\alpha$, converges to a minimum of $F^\alpha$ and bound the rate of convergence. On another note, it would be interesting to import other common tools of optimization to the Wasserstein setting in order to derive a sampling algorithm by optimizing $F^\alpha$.

ACKNOWLEDGMENTS

We thank Philippe Rigollet for useful discussions. Thibaut Le Gouic was supported by NSF award IIS-1838071.
Figure 1: Top plot: 200 samples drawn from $T_{\# N(0,1)}$, $\text{arctan}_{\# N(0,1)}$ using BARYGD with a Gaussian kernel and initial step size $h = 10^{-3}$, and the resulting kernel density estimate. Bottom plot: Trajectories of the samples evolving over 220 iterations, from green to red samples. We can see the arctan transport map is approximately reconstructed.
Figure 2: Top figure: Top plot: 150 samples drawn from the barycenter of two unit variance normal distributions (shown in light gray) using our algorithm with Gaussian kernel with $\alpha = 10^3$ and $h_{\text{opt}} = 10^{-4}$, and the resulting kernel density estimator. Bottom plot: Trajectories of the samples evolving over 1000 iterations, starting from green to red sample. The transport map is almost perfectly reconstructed. Bottom figure: the Wasserstein distance to the true barycenter over iterations plotted in log – log scale. From the graph, we read that BARYGD performs as $O(1/\sqrt{m})$ in the number of iterations $m$. 
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A. ADDITIONAL SIMULATIONS

In this section, we illustrate our algorithm by other experiments. We also test the LAWGD algorithm introduced by Chewi, Le Gouic, Lu, Maunu, and Rigollet (2020) to approximate the gradient of the penalization in the Euler scheme (8).

The LAWGD algorithm. LAWGD generates a batch of samples from the target measure $\pi \propto e^{-V}$ by transporting randomly initialized points $X^1_0, \ldots, X^N_0 \in \mathbb{R}^d$ along the flow described by

$$\partial_t \mu_t = \text{div}(\mu_t \nabla L^{-1}(\mu_t))$$

where $L = \Delta - \nabla V \cdot \nabla$ is the infinitesimal generator of a Markov diffusion process having stationary measure $\pi$. It is assumed to have discrete spectrum and that its eigenvectors $\psi_k$ (associated with strictly positive eigenvalues $\lambda_k$) form a basis of $L^2(\pi)$. An eigendecomposition then yields

$$L^{-1} = \sum_k \frac{\psi_k \otimes \psi_k}{\lambda_k},$$

which leads to the LAWGD algorithms upon discretization:

$$X^i_{m+1} = X^i_m - \frac{h_m}{N} \sum_{j=1}^N \nabla L^{-1}(X^i_m, X^j_m)$$

where $h_m$ is the step size at iteration $m$. In contrast to SVGD where one uses $\nabla V$ directly, here the gradient intervenes only indirectly to compute the $\psi_k$ while the actual computations are carried out with $\nabla L^{-1}$. To compute eigendecomposition (12), we implement the Schrödinger scheme described in Chewi, Le Gouic, Lu, Maunu, and Rigollet (2020, Section 5).

Barycenter of Gaussians. Let $b$ denote the Wasserstein barycenter of $\mu_i = \mathcal{N}(m_i, S_i)$, $i = 1, \ldots, n$. We have $b = \mathcal{N}(m^*, S^*)$ where $m^* = \sum \lambda_i m_i$ and $S^*$ is the fixed point of $S \mapsto \sum \lambda_i (S^{1/2} S_i S^{1/2})^{1/2}$ on positive semidefinite matrices. Moreover, the Wasserstein distance between two Gaussians is explicit:

$$W_2^2(\mu_i, \mu_j) = |m_i - m_j|^2 + \text{tr} \left( S_i^2 + S_j^2 - 2(S_i S_j)^{1/2} \right)$$

for any $i, j \in [1, n]$. See Malagò and Pistone (2018) for details on the geometry of Gaussians and Agueh and Carlier (2011, Section 6) for the characterization of the barycenter of Gaussians. These results allow us to compare our approximate barycenters to the true barycenters for Gaussian marginals.

Experimental setup. In our experiments, we first consider testing BARYGD in one dimension with measures obtained by pushing forward the standard normal distribution by an increasing map. With three marginals, to choose the parameters $\alpha_t$ and $h_t$, we start with $h_0 = 0.1$ and $\alpha_0 = 1000$ and double $\alpha_t$ and divide $h_t$ by 2—keeping $\alpha_t h_t = \text{cte}$—whenever all marginal samples are increasing functions of each other. Reference samples from the barycenter against which to compare our results. Figures 1 and 2 illustrate our algorithm with two marginals. Figures 4 and 5 shows results with three marginals. The golden line represents the Gangbo-Święch map $T = (T^1, T^2, T^3)$ where $T^i$ are the coordinate transformations, e.g. $T^1 = \text{id}, T^2 = \arctan, T^3 =$
(an affine transformation) when the distributions are $\mathcal{N}(0, 1)$, $\arctan \mathcal{N}(0, 1)$ and $\mathcal{N}(\mu, \sigma^2)$.

Then we test the algorithm in two dimensions by sampling from the barycenter of three Gaussian distributions. In this case, we choose $\alpha = 1000$ and $h = 10^{-4}$. This is shown in Figure 6. The colored regions show the kernel estimates of the marginals. The contour lines represent the true marginals and barycenter.

Choosing an appropriate sequence of step sizes that ensures gradient descent converges is a difficult problem. We focused on testing BARYGD with SVGD as our implementation of this variant proved easier to parametrize. However, as illustrated on Figure 7, we have observed that when BARYGD with LAWGD kernel is well parametrized, it can yield qualitatively better results than when SVGD is used.

**Effect of the penalization strength.** We illustrate the effect of the penalization coefficient, we generate samples distributed according to the barycenter of two Gaussians in two dimensions (see Figure 3). In this experiment, the step size and the penalization strength are kept fixed. We observe that the algorithm the quality of the approximate barycenter increases with the penalization.
Figure 3: 100 samples (red) drawn from the barycenter of two almost orthogonal normal distributions (gray patches) $\mathcal{N}(0, \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix})$ and $\mathcal{N}(0, \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix})$ with $\varepsilon = 10^{-2}$ using BARYGD. We used a Gaussian kernel and fixed step size $h = 10^{-2}\alpha^{-1}$. The resulting density estimates are shown as red patches. Initial samples are shown in light green. The density of the true barycenter is represented by the black contour lines and the densities of the marginals by the white contour lines. The algorithm ran for 2000 iterations with penalization strengths $\alpha = 1, 10^2, 10^3$ (top left to top right to bottom plots). The barycentric weights are $\lambda_1 = \lambda_2 = 0.5$. We see that the marginals and the barycenter are increasingly better approximated as the value of $\alpha$ increases.
Figure 4: Top plot: 300 samples drawn from the barycenter of three normal distributions using BARYGD with a Gaussian kernel, and the density estimator. Initial samples are shown in light green. Bottom plot: Evolution of the samples over 600 iterations. We see the linear transport map is well reconstructed. The "front" is formed when the algorithm starts imposing the marginal constraints.
Figure 5: Top plot: 300 samples drawn from the barycenter of two normal distributions and $\arctan \#\mathcal{N}(0,1)$ using BARYGD with a Gaussian kernel, and the density estimator. Initial samples are shown in light green. Bottom plot: Evolution of the samples over 600 iterations. We see the samples coil around the transport map, approximately reconstructing it. The "front" is formed when the algorithm starts imposing the marginal constraints.
Figure 6: 150 samples (red) drawn from the barycenter of three Gaussian distributions in the plane using BARYGD with an adaptive Gaussian kernel and adaptive step size (AdaGrad) over 1000 iterations. Initial points are shown in green. The colored regions represent kernel density estimates of the marginals computed based on the generated samples. Contour lines represent the reference measures.
Figure 7: Left: 100 samples drawn from the barycenter of two Gaussian distributions with BARYGD with LAWGD kernel over 300 iterations. We see that the density is well approximated. Right: Coupling at the end of the algorithm. The points are aligned on a line, as expected when transporting Gaussians.
B. FIRST ORDER DIFFERENTIAL STRUCTURE IN WASSERSTEIN SPACES

The fundamental elements upon which a Riemannian structure is constructed are firstly the curves, from which the tangent spaces are constructed, then the metric to give the tangent space an Euclidean structure. To do so on the Wasserstein space, it is helpful to think of a probability measure as a vast collection of appropriately distributed particles in $\mathbb{R}^d$. Then, properties of the measure can be phrased in terms of properties of the particles and vice versa.

Let $\mu_0 \in P_2(\mathbb{R}^d)$ and $X_0 \sim \mu_0$ be a point on an integral curve $(X_t)_t$ of a vector field $(V_t)_t$, i.e. $\partial_t X_t = V_t(X_t)$, and let $\mu_t$ be the law of $X_t$. Differentiating $\mu_t$ by duality with smooth, compactly supported functions $f \in C_\infty(\mathbb{R}^d)$ yields

$$\langle \partial_t \mu_t, f \rangle = \mathbb{E} \partial_t f(X_t) = \mathbb{E} \langle \nabla f(X_t), V_t(X_t) \rangle = \int_{\mathbb{R}^d} \nabla f \cdot V_t \, d\mu_t \quad (= \langle \partial_t \mu_t, f \rangle)$$

(14)

$$= - \int_{\mathbb{R}^d} f \, \text{div}(\mu_t V_t)).$$

In particular $(\mu_t)_t$ satisfies the conservation of mass equation

$$\partial_t \mu_t = -\text{div}(\mu_t V_t)$$

(15)

in a weak sense. In comparison with the Riemannian analogue of (14), equality (14) suggests to interpret the tangent vector to the curve at time $t$ as the vector field $V_t$. However, equation (15) does not uniquely define such a vector as replacing $V_t$ with $V_t + \hat{V}_t$ where $\text{div}(\mu_t \hat{V}_t) = 0$, gives the same curve $\mu_t$ for a different "tangent vector".

Fortunately, optimal transport theory provides a natural choice for $V_t$ when one wants to transport $\mu_t$ to a nearby $\mu_{t+h}$ at minimal quadratic cost: gradients $\nabla(\psi_t - \tfrac{1}{2} \| \cdot \|^2)$ of Kantorovich potentials $\psi_t$ — i.e. convex functions such that $\mu_{t+h} = \nabla \psi_t \# \mu_t$. Indeed, the optimal trajectory between $X_{t+h}$ and $X_t$ is a geodesic: $X_{t+h} = (1-h)X_t + hX_{t+h}$ for $0 \leq h \leq 1$, thus

$$\langle \partial_h X_{t+h} \rangle_{|h=0} = \nabla(-\tfrac{1}{2} |X_t|^2 + \psi_t(X_t)).$$

Motivated by this case, one then defines the tangent space at $\mu_t$ as

$$T_{\mu_t}P_2(\mathbb{R}^d) = \left\{ \psi \in C_\infty(\mathbb{R}^d) \mid \nabla \psi \in L^2(\mu_t) \right\},$$

equipped with the Hilbert structure inherited from $L^2(\mathbb{R}^d, \mathbb{R}^d; \mu_t)$. We denote by $\langle \cdot, \cdot \rangle_\mu$ such a scalar product given by $\langle f, g \rangle_\mu = \int (f, g) \, d\mu$ for $f, g : \mathbb{R}^d \rightarrow \mathbb{R}^d$. This Hilbert structure is also consistent with the Benamou-Brenier formula for the Wasserstein metric

$$W_2^2(\mu_0, \mu_1) = \inf_{\mu \in \Gamma(\mu_0, \mu_1)} \int_0^1 |V_t|^2_{\mu_t} \, dt$$

(16)

where $\Gamma(\mu_0, \mu_1) = \{(\mu_t)_{|t} \mid \mu(0) = \mu_0$ and $\mu(1) = \mu_1, \partial_t \mu_t = -\text{div}(\mu_t V_t)\}$. Equation (16) is a perfect analogue of the length formula in Riemannian manifolds. Moreover, it verifies that optimal transport curves are indeed the geodesics in the Wasserstein space.

Now let $F : P_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ be a functional on the Wasserstein space and let $f$ denote its first
variation \( f \), i.e. the function satisfying

\[
(\partial_t F(\mu + t\rho))|_{t=0} = \int_{\mathbb{R}^d} f \, d\rho
\]

for signed measures \( \rho \) with total mass \( \int_{\mathbb{R}^d} d\rho = 0 \). To define the gradient of \( F \) at a point \( \mu \in P_2(\mathbb{R}^d) \), let \( (\mu_t)_t \) be a curve such that \( \mu_0 = \mu \). Then formally \( \mu_t = \mu_0 - t \text{div}(\mu_0 V_0) + o(t) \) and

\[
F(\mu_t) = F(\mu_0) - t \int_{\mathbb{R}^d} f \text{div}(\mu_0 V_0) + o(t)
\]

so

\[
(\partial_t F(\mu_t))|_{t=0} = \langle \nabla_W F(\mu_0), V_0 \rangle_{\mu_0} = \langle \nabla f, V_0 \rangle_{\mu_0}.
\]

This resembles the behavior of Riemannian gradients and suggests that we define the Wasserstein gradient of \( F \) as the gradient of its first variation, i.e. that we set

\[
\nabla_W F(\mu) := \nabla f : \mathbb{R}^d \to \mathbb{R}^d \subset L^2(\mathbb{R}^d, \mathbb{R}^d, \mu)
\]

With this machinery, functionals over the Wasserstein space can be optimized by following Wasserstein gradient flows. This was the main result of the seminal paper of Jordan, Kinderlehrer, and Otto [1998], in which they proved that the marginal distributions of Langevin diffusion forms a path in the Wasserstein space that is the gradient flow of the Kullback-Leibler divergence.

Let \( \mu \in P_{2}^{ac}(\mathbb{R}^d) \). For completeness, we compute the gradient of \( \chi^2_\mu \) at \( \nu \in P_{2}^{ac}(\mathbb{R}^d) \). Let \( \delta \) be a signed measure such that \( \int d\delta = 0 \). Then, for small \( t \), we have

\[
\chi^2_\mu(\nu + t\delta) = \chi^2_\mu(\nu) + 2t \int \left( \frac{\nu}{\mu} - 1 \right) \delta + o(t) = \chi^2_\mu(\nu) + 2t \int \frac{\nu}{\mu} \delta + o(t)
\]

hence the first variation of \( \chi^2_\mu \) at \( \nu \) is \( 2\nu/\mu \), and

\[
\nabla_W \chi^2_\mu(\nu) = 2\nabla_{\nu/\mu}.
\]

C. PROOFS

In the next two sections, we recall some notions that will appear in the proofs. We recall useful notation in the third section and then the different proofs are gathered.

C.1. The \( \chi^2 \) transportation inequality

We recall a transportation inequality which connects the \( \chi^2 \) penalization with the Wasserstein distance under the Poincaré inequality. We say that a measure \( \mu \in P_2(\mathbb{R}^d) \) satisfies a \( \chi^2 \)-transport inequality with positive constant \( c \), denoted by \( T^{\chi^2}_2(c) \), if we have

\[
W^2_2(\mu, \nu) \leq 2c \chi^2_\mu(\nu)
\]

for all \( \nu \in P_2(\mathbb{R}^d) \). The following important result is due to Ding [2015] (see also the note by Ledoux [2018]) and makes the connection between the \( \chi^2 \) divergence and the Wasserstein distance precise.

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Proposition 3 (Ding). Let $\mu \in P_2(\mathbb{R}^d)$ and $c \in \mathbb{R}_+^*$. If $\mu$ satisfies a Poincaré inequality with constant $c$ then it satisfies $T^2_2(2c)$. Conversely, if $\mu$ satisfies $T^2_2(c)$ then it satisfies a Poincaré inequality with constant $2c$.

C.2. Variance inequalities

Variance inequalities are crucial to the proof of Proposition 2 in Subsection C.3. Moreover, they naturally ensure uniqueness of the barycenter. These inequalities were introduced by Sturm (2003) in his investigation of the curvature of general metric spaces. They have since played a central role in the study of convergence rates of empirical barycenters in Ahidar-Coutrix, Le Gouic, and Paris (2020) and Le Gouic et al. (2019). Recently, Chewi, Maunu, et al. (2020) gave a simple condition on the optimal transport from the barycenter to any other measure implying a variance inequality.

Recall that a probability measure $P \in P_2(P_2(\mathbb{R}^d))$ with barycenter $b_P \in P_2(\mathbb{R}^d)$ satisfies a variance inequality with constant $k \in [0, 1]$ if

$$kW^2_2(\nu, b_P) \leq \int [W^2_2(\mu, \nu) - W^2_2(\mu, b_P)] P(d\mu)$$

(17)

for any $\nu \in P_2(\mathbb{R}^d)$. Note that such an inequality always holds with $k = 0$ and that if $k > 0$ then the barycenter is unique. Variance inequalities express that the variance functional $\nu \mapsto \int W^2_2(\mu, \nu) P(d\mu)$ behaves quadratically around the barycenter $b_P$.

From Sturm (2003), it is known that variance inequalities with constant 1 are satisfied for probability measures defined over spaces of non positive Alexandrov curvature. The situation is more contrasted for measures defined over spaces of non negative curvature, such as $P_2(\mathbb{R}^d)$. On such spaces, variance inequalities may not hold for $k > 0$. Let us recall a recent result in Chewi, Maunu, et al. (2020, Theorem 6) in that direction.

Let $P \in P_2(P_2(\mathbb{R}^d))$ and let $b_P \in P_2(\mathbb{R}^d)$ be a barycenter of $P$. For any $\nu \in P_2(\mathbb{R}^d)$, denote by $\varphi_\nu$ the Kantorovich potential from $b_P$ to $\nu$. If there exists $k : P_2(\mathbb{R}^d) \to \mathbb{R}_+^*$ such that $\varphi_\nu$ is $k(\nu)$-strongly convex for $P$-almost any $\nu \in P_2(\mathbb{R}^d)$, and $\int \varphi_\nu \, dP(\nu) = 1$, then $P$ satisfies a variance inequality with constant $\int k(\nu) P(d\nu)$.

C.3. Notation

We recall that $T$ denotes for the map $x \mapsto \sum \lambda_i x_i$, where the $\lambda_i$ are the barycentric weights, and that, for any $\alpha \geq 0$, we minimize the functional

$$F^\alpha : \gamma \mapsto \int c \, d\gamma + \alpha \sum \lambda_i \chi_{\mu_i}^2(\gamma) \text{ where } c(x) = \sum \lambda_i |x_i - T(x)|^2 \quad \forall x \in (\mathbb{R}^d)^n.$$

We write $\gamma^*$ for the barycenter coupling for the $\mu_i$ with weights $\lambda_i$, i.e., it satisfies

$$T_{\#\gamma^*} = \arg\min_{\nu \in P_2(\mathbb{R}^d)} \sum \lambda_i W^2_2(\mu_i, \nu).$$

(18)

We assume throughout that the measures $\mu_1, \ldots, \mu_n$ satisfy the Poincaré inequality with constant $C_P$. 

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C.4. Proof of Proposition 1

Let $\alpha > 0$. We begin by showing that $F^\alpha$ is lower semi-continuous, i.e. that for every convergent sequence $(\gamma^k)$, with limit $\gamma$, we have

$$F^\alpha(\gamma) \leq \liminf_{k \to \infty} F^\alpha(\gamma^k).$$

Let $R = \sum \lambda_i \chi_{\partial_i}$ and $F : \gamma \mapsto \int c \, d\gamma$ so that $F^\alpha = F + \alpha R$. By Polyanskiy and Wu (2019, Theorem 3.6), $\alpha R$ is lower semi-continuous with respect to the weak topology on $P_2((\mathbb{R}^d)^n)$ and it remains to show that this property also holds for $F$. Let $m > 0$ and $c_m = c \wedge m$. Since $c$ is continuous, it follows that $c_m \in \mathcal{C}_b^0((\mathbb{R}^d)^n)$, and one sees that $c_m \geq 0$ is increasing in $m$ (i.e. $c_m < c_m'$ when $m < m'$). Moreover, it is clear that $c_m \underset{m \to \infty}{\longrightarrow} c$ pointwise. For any $k \geq 0$ we have

$$\int c_m \, d\gamma^k \leq \int c \, d\gamma^k$$

and taking the limit inferior in $k$ on both sides we obtain

$$\liminf_{k \to \infty} \int c_m \, d\gamma^k = \int c_m \, d\gamma \leq \liminf_{k \to \infty} \int c \, d\gamma^k$$

where the equality follows by weak convergence. We now use the monotone convergence theorem to obtain

$$\int c \, d\gamma \leq \liminf_{k \to \infty} \int c \, d\gamma^k$$

which shows that $F$ is lower semi-continuous with respect to the weak topology on $P_2((\mathbb{R}^d)^n)$. Hence, $F^\alpha$ is lower semi-continuous.

Next, we show that there exists a minimizer for $F^\alpha$. It is clear that $\inf F^\alpha \geq 0$. Let $(\gamma^k)$ be a minimizing sequence of probability measures on $P_2((\mathbb{R}^d)^n)$, i.e. it satisfies $F^\alpha(\gamma^k) \underset{k \to \infty}{\longrightarrow} \inf F^\alpha$. Then, from a certain rank $k_0$, for any $k \geq k_0$, we have $F^\alpha(\gamma^k) \leq \inf F^\alpha + 1$. Because for any coupling $\gamma$, we have

$$F^\alpha(\gamma) = m_2(\gamma) + \sum \lambda_i \int (|T(x)|^2 - 2 \langle x_i, T(x) \rangle) \gamma(\, dx) + \alpha \sum \lambda_i R_{\mu_i}(\gamma), =: m_2(\gamma) + r(\gamma),$$

and since the assumption implies that $r(\gamma^k) \leq \inf F^\alpha + 1$, it holds

$$m_2(\gamma^k) \leq 2(\inf F^\alpha + 1) < \infty.$$  \hspace{1cm} (19)

This guarantees that the sequence $(\gamma^k)$ is tight. Indeed, if by contradiction $(\gamma^k)$ were not tight, then at least for some $i \in [1, n]$ the sequence $(\gamma^k_i)$ is not tight either (if all the $\gamma_i$’s were tight, then, for any $\varepsilon > 0$ there is a compact set $K_{\varepsilon}^i$ such that for all $k$ we have $\gamma^k_i(\mathbb{R}^d \setminus K_{\varepsilon}^i) \leq \varepsilon/n$, and then $\gamma^k((\mathbb{R}^d)^n \setminus K_{\varepsilon}^i) \leq \varepsilon$, which would make $(\gamma^k)$ tight as well). Now if the sequence $(\gamma^k)$ is not tight, then for some $m > 0$, for any compact set $K \subset \mathbb{R}^d$ we have $\gamma^k_i(\mathbb{R}^d \setminus K) > m$, for a subsequence still denoted by $(\gamma^k)$, and then $\sum \int |x_i|^2 \, d\gamma^k_i(x_i) = m_2(\gamma^k) = \infty$, which contradicts (19). Thus $(\gamma^k)$ is tight. Then, by Villani (2009, Lemma 6.14), at least a sub-sequence of $(\gamma^k)$ converges to some absolutely continuous $\gamma \in P_2((\mathbb{R}^d)^n)$ (if $\gamma$ were to be singular, we would have $F^\alpha(\gamma) = \infty$ due to the penalization term, which cannot be). The lower semi-continuity of $F^\alpha$ implies that $\gamma$ is a minimizer, which concludes the proof.
C.5. Proof of Proposition 2

Notation. We define the set of \( \varepsilon \)-approximate minimizers of \( F^\alpha \)
\[
M^{\varepsilon,\alpha} = \left\{ \gamma \in P_2((R^d)^n) \mid 0 \leq F^\alpha(\gamma) - \inf F^\alpha \leq \varepsilon \right\}.
\]
where \( \alpha \) and \( \varepsilon \) are strictly positive.

C.5.1 Main result

The main result of this subsection is Proposition 2. It states that, assuming the \( \mu_i \) satisfy a variance inequality and a Poincaré inequality, for any \( \alpha \geq 1 \) and a small \( \varepsilon \), the distance between barycenters of couplings in \( M^{\varepsilon,\alpha} \) and the barycenter \( T^\# \gamma^* \) is controlled with \( \varepsilon \) and \( 1/\sqrt{\alpha} \). The implication of this result is that, in the regime \( \varepsilon \ll 1 \ll \alpha \) we are interested in, this distance is bounded by \( \varepsilon + 1/\sqrt{\alpha} \), up to a constant. We recall the precise statement.

Proposition 2. Let \( \alpha \geq 1 \) and \( \varepsilon \leq 1 \). Suppose \( \mu_1, \ldots, \mu_n \) satisfy the Poincaré inequality with constant \( C_P \) and that \( \sum \lambda_i \delta_{\mu_i} \) satisfies a variance inequality with constant \( k \). Then there exists a constant \( C_1 \), only depending on the variance \( \sigma^2 = \int c \, d\gamma^* \) and \( C_P \), such that for any \( \gamma^{\varepsilon,\alpha} \in M^{\varepsilon,\alpha} \),
\[
\frac{k}{4} W^2_2 (T^\# \gamma^{\varepsilon,\alpha}, T^\# \gamma^*) \leq \varepsilon + \frac{C_1}{\sqrt{\alpha}}.
\]

Let us now sketch the proof. For any \( \gamma^{\varepsilon,\alpha} \in M^{\varepsilon,\alpha} \), we have
\[
\frac{1}{2} W^2_2 (T^\# \gamma^{\varepsilon,\alpha}, T^\# \gamma^*) \leq W^2_2 (T^\# \gamma^{\varepsilon,\alpha}, T^\# \tilde{\gamma}^{\varepsilon,\alpha}) + W^2_2 (T^\# \tilde{\gamma}^{\varepsilon,\alpha}, T^\# \gamma^*),
\]
where \( \tilde{\gamma}^{\varepsilon,\alpha} \) denotes the coupling associated to the multimarginal problem for the marginals of \( \gamma^{\varepsilon,\alpha} \). Our first step is to show that under the Poincaré inequality couplings in \( M^{\varepsilon,\alpha} \) are close to \( \Pi(\mu_1, \ldots, \mu_n) \), i.e. their marginals are, up to a constant, \( \sqrt{\varepsilon/\alpha} \)-close to the desired marginals; this is Lemma 4. Assuming that \( \sum \lambda_i \delta_{\mu_i} \) satisfies a variance inequality, we then use Lemmas 4 and 5 to derive a bound on the distance between the barycenters of \( \tilde{\gamma}^{\varepsilon,\alpha} \) and \( \gamma^* \).

Lemma 4. Suppose \( \mu_1, \ldots, \mu_n \) satisfy the Poincaré inequality with constant \( C_P \). Recall \( \sigma^2 = \int c \, d\gamma^* \). Let \( \alpha > 0 \), \( \varepsilon \leq \sigma^2 \) and let \( \gamma^{\varepsilon,\alpha} \in M^{\varepsilon,\alpha} \). Then
\[
\sum \lambda_i W^2_2 (\gamma_i^{\varepsilon,\alpha}, \mu_i) \leq \frac{8 C_P \sigma^2}{\alpha}.
\]

Lemma 5. Suppose the \( \mu_i \) satisfy the \( C_P \)-Poincaré inequality. Let \( \alpha \geq \sigma^2 \), \( \varepsilon \leq \sigma^2 \) and let \( \gamma^{\varepsilon,\alpha} \in M^{\varepsilon,\alpha} \). Further, let \( \tilde{\gamma}^{\varepsilon,\alpha} \) be the barycenter coupling with weights \( \lambda_1, \ldots, \lambda_n \) for \( \gamma_1^{\varepsilon,\alpha}, \ldots, \gamma_n^{\varepsilon,\alpha} \), and define
\[
C_2 = 4 \sigma \left( 2 C_P + \sqrt{2 C_P} \right).
\]
Then
\[
\int c \, d\gamma^{\varepsilon,\alpha} - \int c \, d\gamma^* \leq \varepsilon,
\]
\[
\int c \, d\tilde{\gamma}^{\varepsilon,\alpha} - \int c \, d\gamma^* \leq \frac{C_2}{\sqrt{\alpha}},
\]
and
\[
\int c \, d\gamma^{\varepsilon,\alpha} - \int c \, d\tilde{\gamma}^{\varepsilon,\alpha} \leq \varepsilon + \frac{3 C_2}{\sqrt{\alpha}}.
\]
Our second step is to show that a perturbed variance inequality holds for $\sum \lambda_i \delta_{\gamma_i}$ for any $\gamma \in M^{x,\alpha}$; this is Lemma 6. We exploit this result to derive a bound on the distance between the barycenters of $\gamma^{x,\alpha}$ and $\bar{\gamma}^{x,\alpha}$. However, it introduces a supplementary term (the second one in the right hand side of (23)), which we need to control with $\varepsilon$ and $1/\sqrt{\alpha}$. We carry this out with Lemmas 4 and 5. We conclude the proof by injecting the two distance bounds back in (20).

**Lemma 6.** Let $P, Q \in P_2(P_2(\mathbb{R}^d))$. Assume that $P$ satisfies a $k$-variance inequality and that $P$ and $Q$ admit barycenters $b_P$ and $b_Q \in P_2(\mathbb{R}^d)$. Let

$$\sigma^2(P) = \int W_2^2(\nu, b_P) P(d\nu) \quad \text{and} \quad \sigma^2(Q) = \int W_2^2(\nu, b_Q) Q(d\nu).$$

Then, for any $\mu \in P_2(\mathbb{R}^d),$

$$\frac{k}{2} W_2^2(\mu, b_Q) \leq \int (W_2^2(\nu, \mu) - W_2^2(\nu, b_Q)) Q(d\nu) + C(\mu) W_2(Q, P) \quad (23)$$

where

$$C(\mu) = 2\sqrt{8[\sigma^2(Q) + \sigma^2(P) + W_2^2(b_P, b_Q)] + 2W_2^2(\mu, b_Q)}$$

**C.5.2 Proofs**

**Proof of Lemma 4** Let $\gamma^\alpha$ be a minimizer of $F^\alpha$. Let $R = \sum \lambda_i \chi_{\mu_i}^2$. The definition of $M^{x,\alpha}$ gives

$$0 \leq \int c d\gamma^x\alpha - \int c d\gamma^\alpha + \alpha R(\gamma^x\alpha) - \alpha R(\gamma^\alpha) \leq \varepsilon,$$

whence

$$\alpha R(\gamma^x\alpha) \leq \varepsilon - \int c d\gamma^x\alpha + \int c d\gamma^\alpha + \alpha R(\gamma^\alpha).$$

Since $\gamma^\alpha$ is minimizing, we have $F^\alpha(\gamma^\alpha) \leq F^\alpha(\gamma^\star)$ and, subtracting $\int c d\gamma^x\alpha$ from both sides, we find

$$- \int c d\gamma^x\alpha + \int c d\gamma^\alpha + \alpha R(\gamma^\alpha) \leq \int c d\gamma^\star - \int c d\gamma^\alpha \leq \int c d\gamma^\star,$$

hence $\alpha R(\gamma^x\alpha) \leq \varepsilon + \int c d\gamma^\star$. Now, by Proposition 5, the Poincaré inequality implies $W_2^2(\mu_j, \cdot) \leq 4C_P R_{\mu_j}$ for any $j \in [1, n]$ and the assumption that $\varepsilon \leq \int c d\gamma^\star = \sigma^2$, thus

$$\sum \lambda_i W_2^2(\gamma^x\alpha_i, \mu_i) \leq 4C_P R(\gamma^x\alpha) \leq 4C_P \frac{\varepsilon}{\alpha} \left( \varepsilon + \int c d\gamma^\star \right) \leq \frac{8C_P \sigma^2}{\alpha},$$

which completes the proof. 

**Proof of Lemma 5**

(First inequality) By definition,

$$\int c d\gamma^x\alpha - \int c d\gamma^\star = F^\alpha(\gamma^x\alpha) - \alpha R(\gamma^x\alpha) - F^\alpha(\gamma^\star) \leq F^\alpha(\gamma^x\alpha) - F^\alpha(\gamma^\star)$$

and adding and subtracting $F^\alpha(\gamma^\alpha)$ yields

$$F^\alpha(\gamma^x\alpha) - F^\alpha(\gamma^\star) = \left\{ F^\alpha(\gamma^x\alpha) - F^\alpha(\gamma^\star) \right\} + \left\{ F^\alpha(\gamma^\alpha) - F^\alpha(\gamma^\star) \right\} \leq \varepsilon,$$

$$\leq \varepsilon \text{ since } \gamma^x\alpha \in M^{x,\alpha} \quad \text{and} \quad \leq 0 \text{ since } \gamma^\alpha \text{ minimizes } F^\alpha.$$
Proceeding similarly as for (24), we obtain using Lemma 4 that

\[ \int c \, d\tilde{\gamma}^{\varepsilon,\alpha} - \int c \, d\gamma^* = \sum \lambda_i \left( W_2^2(\tilde{\gamma}_i^{\varepsilon,\alpha}, T\#\tilde{\gamma}_i^{\varepsilon,\alpha}) - W_2^2(\mu_i, T\#\gamma^*) \right) \leq \sum \lambda_i \left( W_2^2(\tilde{\gamma}_i^{\varepsilon,\alpha}, T\#\gamma^*) - W_2^2(\mu_i, T\#\gamma^*) \right) \]

where we used the fact that \( \gamma^* \) is not optimal for the \( \tilde{\gamma}_i^{\varepsilon,\alpha} \) to assert that \( \sum \lambda_i W_2^2(\tilde{\gamma}_i^{\varepsilon,\alpha}, T\#\tilde{\gamma}_i^{\varepsilon,\alpha}) \leq \sum \lambda_i W_2^2(\gamma_i^{\varepsilon,\alpha}, T\#\gamma^*) \). Combining the triangle inequality with the identity \( a^2 - b^2 = (a - b)(a + b) \) for all \( a, b \in \mathbb{R} \) yields

\[ \int c \, d\tilde{\gamma}^{\varepsilon,\alpha} - \int c \, d\gamma^* \leq \sum \lambda_i W_2^2(\gamma_i^{\varepsilon,\alpha}, T\#\gamma^*) + 2 \sqrt{\sum \lambda_i W_2^2(\gamma_i^{\varepsilon,\alpha}, \mu_i) \sum \lambda_i W_2^2(\mu_i, T\#\gamma^*)}. \]

We conclude with Lemma 4 which yields

\[ \int c \, d\tilde{\gamma}^{\varepsilon,\alpha} - \int c \, d\gamma^* \leq \frac{8C_p \sigma^2}{\alpha} + \frac{\sigma \sqrt{8C_p}}{\sqrt{\alpha}} \leq C \frac{2}{\sqrt{\alpha}} \]

since we assume that \( \sigma \geq \varepsilon \).

**Second inequality** By equality of the original and multimarginal formulations of the barycenter problem for \( \gamma_i^{\varepsilon,\alpha} \) and \( \gamma^* \), we have

\[ \int c \, d\gamma_i^{\varepsilon,\alpha} - \int c \, d\gamma_i^{\varepsilon,\alpha} \leq \sum \lambda_i W_2^2(\gamma_i^{\varepsilon,\alpha}, T\#\gamma_i^{\varepsilon,\alpha}) \]

Adding and subtracting \( \int c \, d\gamma_i^{\varepsilon,\alpha} \) in (24), we get

\[ \left\{ \int c \, d\gamma_i^{\varepsilon,\alpha} - \int c \, d\gamma_i^{\varepsilon,\alpha} \right\} + \left\{ \int c \, d\tilde{\gamma}^{\varepsilon,\alpha} - \int c \, d\gamma^* \right\} \leq \varepsilon \]

and by moving the second term of the left hand side to the right hand side, we have

\[ \int c \, d\gamma_i^{\varepsilon,\alpha} - \int c \, d\gamma_i^{\varepsilon,\alpha} \leq \varepsilon + \int c \, d\gamma^* - \int c \, d\tilde{\gamma}^{\varepsilon,\alpha}. \]

(Third inequality) Adding and subtracting \( \int c \, d\tilde{\gamma}^{\varepsilon,\alpha} \) in (24), we obtain using Lemma 4 that

\[ \int c \, d\gamma^* - \int c \, d\tilde{\gamma}^{\varepsilon,\alpha} \leq \sum \lambda_i W_2^2(\gamma_i^{\varepsilon,\alpha}, \mu_i) + 2 \sqrt{\sum \lambda_i W_2^2(\gamma_i^{\varepsilon,\alpha}, \mu_i) \sum \lambda_i W_2^2(\gamma_i^{\varepsilon,\alpha}, T\#\tilde{\gamma}^{\varepsilon,\alpha})} \]

However, by optimality of \( \tilde{\gamma}^{\varepsilon,\alpha} \) and using Inequality (24) and the assumption that \( \alpha \geq \sigma^2 \),

\[ \sum \lambda_i W_2^2(\gamma_i^{\varepsilon,\alpha}, T\#\tilde{\gamma}^{\varepsilon,\alpha}) = \int c \, d\gamma^* \leq \int c \, d\tilde{\gamma}^{\varepsilon,\alpha} \leq \sigma^2 + \frac{8C_p \sigma^2}{\alpha} + \frac{\sigma \sqrt{8C_p}}{\sqrt{\alpha}} \leq \left( \sigma + \sqrt{8C_p} \right)^2. \]
Therefore, we proved
\[
\int c \, d\gamma^* - \int c \, d\gamma^{z,\alpha} \leq \frac{8C_p \sigma^2}{\alpha} + \frac{\sigma \sqrt{8C_p}}{\sqrt{\alpha}} \left( \sigma + \sqrt{8C_p} \right) \\
\leq \frac{1}{\sqrt{\alpha}} \left( 4\sigma (2C_p + \sqrt{2C_p}) + 16\sigma C_p \right) = \frac{4\sigma}{\sqrt{\alpha}} \left( 6C_p + \sqrt{2C_p} \right) \leq \frac{3C^2}{\sqrt{\alpha}}
\]
which concludes the proof together with [25].

**Proof of Lemma [8]** Let \( \mu \in P_2(\mathbb{R}^d) \). From the variance inequality for \( P \), we have
\[
kW_2^2(\mu, b_P) \leq \int W_2^2(\mu, \nu) \, P(d\nu) - \int W_2^2(\nu, b_P) \, P(d\nu) = \int W_2^2(\mu, \nu)(P - Q)(d\nu) + \int W_2^2(\nu, b_Q) \, Q(d\nu) \\
- \int W_2^2(\nu, b_P) \, P(d\nu) + \int (W_2^2(\mu, \nu) - W_2^2(b_Q, \nu)) \, Q(d\nu) \\
\leq \int W_2^2(\mu, \nu)(P - Q)(d\nu) + \int W_2^2(\nu, b_P)(Q - P)(d\nu) \\
+ \int (W_2^2(\mu, \nu) - W_2^2(\nu, b_Q)) \, Q(d\nu),
\]
where we used the optimality of \( b_Q \) as a barycenter for \( Q \) in the last line. We start by bounding the first term in [26]. Let \( \Gamma \) be an optimal coupling between \( P \) and \( Q \). Then
\[
\int W_2^2(\mu, \nu)(P - Q)(d\nu) = \int (W_2^2(\mu, \nu) - W_2^2(\mu, \zeta)) \, \Gamma(d\nu \, d\zeta) \\
= \int (W_2(\mu, \nu) - W_2(\mu, \zeta))W_2(\mu, \nu) + W_2(\mu, \zeta) \, \Gamma(d\nu \, d\zeta),
\]
and, from the triangle inequality, we have \( W_2(\mu, \nu) \leq W_2(\mu, \zeta) + W_2(\zeta, \nu) \), hence
\[
\int W_2^2(\mu, \nu)(P - Q)(d\nu) \leq \int W_2(\nu, \zeta)(W_2(\mu, \nu) + W_2(\mu, \zeta)) \, \Gamma(d\nu \, d\zeta).
\]
Recall the notation
\[
\sigma^2(\mu; P) = \int W_2^2(\nu, \mu) \, P(d\nu) \quad \text{and} \quad \sigma^2(\mu; Q) = \int W_2^2(\nu, \mu) \, Q(d\nu).
\]
Applying Cauchy-Schwarz and Young inequalities,
\[
\int W_2^2(\mu, \nu)(P - Q)(d\nu) \leq W_2(Q, P) \sqrt{2(\sigma^2(\mu; P) + \sigma^2(\mu; Q))}.
\]
Now, since variance inequality holds in the reverse direction for \( k = 1 \) in nonnegatively curved space (see Ahidar-Coutrix, Le Gouic, and Paris [2020] Theorem 3.2))
\[
\sigma^2(\mu; P) - \sigma^2(b_P; P) \leq W_2^2(\mu, b_P) \quad \text{and} \quad \sigma^2(\mu; Q) - \sigma^2(b_Q; P) \leq W_2^2(\mu, b_Q),
\]
whence
\[
\int W_2^2(\nu, \mu)(P - Q)(d\nu) \leq W_2(Q, P) \sqrt{2(\sigma^2(P) + \sigma^2(Q) + W_2^2(\mu, b_P) + W_2^2(\mu, b_Q))}.
\]
We follow similar steps to bound the second term of (26) and obtain that
\[
\int W_2^2(\nu, b_P)(Q - P)(d\nu) \leq W_2(Q, P)\sqrt{2(\sigma^2(Q) + \sigma^2(P) + W_2^2(b_Q, b_P))}
\]

Injecting these bounds into (26) and recalling the definition of \(A\), we have
\[
kW_2^2(\mu, b_P) \leq \int (W_2^2(\mu, \nu) - W_2^2(b_Q, \nu))Q(d\nu) + A(\mu; Q, P)W_2(Q, P).
\]

We conclude by applying (27) twice, with \(\mu\) and \(b_Q\), and the triangle inequality to get
\[
kW_2^2(\mu, b_Q) \leq kW_2^2(b_Q, b_P) + kW_2^2(\mu, b_P)
\]
\[
\leq kW_2^2(b_Q, b_P) + A(\mu; Q, P)W_2(Q, P) + \int (W_2^2(\nu, \mu) - W_2^2(\nu, b_Q))Q(d\nu)
\]
\[
\leq \int (W_2^2(\mu, \nu) - W_2^2(b_Q, \nu))Q(d\nu) + W_2(Q, P)(A(b_Q; Q, P) + A(\mu; Q, P)).
\]

\[\blacksquare\]

**Proof of Proposition 2** The proof follows the outline described above. Let \(\gamma^{\varepsilon, \alpha} \in M^{\varepsilon, \alpha}\) and let \(\bar{\gamma}^{\varepsilon, \alpha}\) be the barycentric coupling for \(\gamma_1^{\varepsilon, \alpha}, \ldots, \gamma_n^{\varepsilon, \alpha}\) with weights \(\lambda_1, \ldots, \lambda_n\). Recall that \(C_2 = 4\sigma (4C_P + \sqrt{2C_P})\). From the triangle and Young’s inequalities, we have
\[
\frac{1}{2}W_2^2(T\#\gamma^*, T\#\bar{\gamma}^{\varepsilon, \alpha}) \leq W_2^2(T\#\gamma^*, T\#\bar{\gamma}^{\varepsilon, \alpha}) + W_2^2(T\#\bar{\gamma}^{\varepsilon, \alpha}, T\#\gamma^*).. \tag{28}
\]

**(Step 1. Bound on** \(\text{Step 1. Bound on } W_2(T\#\gamma^*, T\#\bar{\gamma}^{\varepsilon, \alpha})** We start by bounding the distance between the barycenters of \(\bar{\gamma}^{\varepsilon, \alpha}\) and \(\gamma^*\). Using the variance inequality for \(P\), we have
\[
kW_2^2(T\#\gamma^*, T\#\bar{\gamma}^{\varepsilon, \alpha}) \leq \sum \lambda_i(W_2^2(\mu_i, T\#\bar{\gamma}^{\varepsilon, \alpha}) - W_2^2(\mu_i, T\#\gamma^*))
\]
\[
= \sum \lambda_i(W_2(\mu_i, T\#\bar{\gamma}^{\varepsilon, \alpha}) - W_2(\mu_i, T\#\gamma^*))(W_2(\mu_i, T\#\bar{\gamma}^{\varepsilon, \alpha}) + W_2(\mu_i, T\#\gamma^*))
\]
and, by the triangle applied to \(W_2(\mu_i, T\#\bar{\gamma}^{\varepsilon, \alpha})\) in both factors, we get
\[
kW_2^2(T\#\bar{\gamma}^{\varepsilon, \alpha}, T\#\gamma^*)
\]
\[
\leq \sum \lambda_i(W_2(\mu_i, \gamma_i^{\varepsilon, \alpha}) + W_2(\gamma_i^{\varepsilon, \alpha}, T\#\bar{\gamma}^{\varepsilon, \alpha}) - W_2(\mu_i, T\#\gamma^*))
\]
\[
\cdot (W_2(\mu_i, \gamma_i^{\varepsilon, \alpha}) + W_2(\gamma_i^{\varepsilon, \alpha}, T\#\bar{\gamma}^{\varepsilon, \alpha}) + W_2(\mu_i, T\#\gamma^*))
\]
\[
= \sum \lambda_iW_2^2(\mu_i, \gamma_i^{\varepsilon, \alpha}) + 2\sum \lambda_iW_2(\mu_i, \gamma_i^{\varepsilon, \alpha})W_2(\gamma_i^{\varepsilon, \alpha}, T\#\bar{\gamma}^{\varepsilon, \alpha})
\]
\[
+ \sum \lambda_i(W_2^2(\gamma_i^{\varepsilon, \alpha}, T\#\bar{\gamma}^{\varepsilon, \alpha}) - W_2^2(\mu_i, T\#\gamma^*)). \tag{29}
\]

We now bound the second term of (29). Using Cauchy-Schwarz the second term is bounded by
\[
\sum \lambda_iW_2(\mu_i, \gamma_i^{\varepsilon, \alpha})W_2(\gamma_i^{\varepsilon, \alpha}, T\#\bar{\gamma}^{\varepsilon, \alpha}) \leq \sqrt{\sum \lambda_iW_2^2(\mu_i, \gamma_i^{\varepsilon, \alpha})}\sqrt{\sum \lambda_iW_2^2(\gamma_i^{\varepsilon, \alpha}, T\#\bar{\gamma}^{\varepsilon, \alpha})}. \tag{30}
\]
Recall that $\sigma^2(P) = \sum \lambda_i W_2^2(\mu_i, T_{\#}\gamma^*)$ denotes the variance of $P$. By optimality of $T_{\#}\tilde{\gamma}^{\varepsilon,\alpha}$, using the triangle and Young inequalities yields

$$
\sum \lambda_i W_2^2(\gamma_i, T_{\#}\tilde{\gamma}^{\varepsilon,\alpha}) = \sum \lambda_i W_2^2(\gamma_i, T_{\#}\gamma^*) \leq 2 \sum \lambda_i (W_2^2(\gamma_i, \mu_i) + W_2^2(\mu_i, T_{\#}\gamma^*)) \leq 2 \left( \frac{8C_P\sigma^2}{\alpha} + \sigma^2 \right). 
$$

Therefore, using Lemma 4 again, and the assumption $\alpha \geq \sigma^2$,

$$
\sum \lambda_i W_2(\mu_i, \gamma_i^{\varepsilon,\alpha}) W_2(\gamma_i^{\varepsilon,\alpha}, T_{\#}\tilde{\gamma}^{\varepsilon,\alpha}) \leq \frac{\sigma \sqrt{16C_P}}{\sqrt{\alpha}} \sqrt{8C_P + \sigma^2}
$$

For the last term of (29), the multimarginal formulation for $\gamma^{\varepsilon,\alpha}$ and $\gamma^*$ shows that the last sum equals $\int c \, d\gamma^{\varepsilon,\alpha} - \int c \, d\gamma^*$, which we control by Lemma 5. The first term of (29) is bounded by a direct application of Lemma 4. Therefore, we get from (29):

$$
k W_2^2(T_{\#}\tilde{\gamma}^{\varepsilon,\alpha}, T_{\#}\gamma^*) \leq \frac{\sigma \sqrt{8C_P}}{\sqrt{\alpha}} + \frac{2\sigma \sqrt{16C_P}}{\sqrt{\alpha}} \sqrt{8C_P + \sigma^2} + \frac{C_2}{\sqrt{\alpha}} \approx \frac{C_P\sigma^2}{\sqrt{\alpha}}
$$

(Step 2. Bound on $W_2(T_{\#}\gamma^{\varepsilon,\alpha}, T_{\#}\tilde{\gamma}^{\varepsilon,\alpha})$) We first bound $W_2(T_{\#}\gamma^{\varepsilon,\alpha}, T_{\#}\tilde{\gamma}^{\varepsilon,\alpha})$ by applying Lemma 6 with $\mu = T_{\#}\gamma^{\varepsilon,\alpha}$ and

$$
P := \sum \lambda_i \delta_{\mu_i} \quad \text{and} \quad Q := P^{\varepsilon,\alpha} := \sum \lambda_i \delta_{\gamma_i^{\varepsilon,\alpha}}.
$$

Since $P$ satisfies a $k$-variance inequality by assumption, this gives us

$$
\frac{k}{2} W_2^2(T_{\#}\gamma^{\varepsilon,\alpha}, T_{\#}\tilde{\gamma}^{\varepsilon,\alpha}) \leq \sum \lambda_i \left( W_2^2(\gamma_i, T_{\#}\gamma^*) - W_2^2(\gamma_i^{\varepsilon,\alpha}, T_{\#}\gamma^*) \right) + C(T_{\#}\gamma^{\varepsilon,\alpha}) W_2(P^{\varepsilon,\alpha}, P)
$$

where

$$
C(T_{\#}\gamma^{\varepsilon,\alpha}) = 2\sqrt{\delta \sigma^2(P^{\varepsilon,\alpha}) + \sigma^2(P) + W_2^2(T_{\#}\gamma^*, T_{\#}\gamma^{\varepsilon,\alpha})} + 2W_2^2(T_{\#}\gamma^{\varepsilon,\alpha}, T_{\#}\tilde{\gamma}^{\varepsilon,\alpha}).
$$

First, we bound $C(T_{\#}\gamma^{\varepsilon,\alpha})$. Using the multimarginal problem for $\gamma^{\varepsilon,\alpha}$, the variance of $P^{\varepsilon,\alpha}$ is easily bounded with Lemma 5. Indeed, we have

$$
\sigma^2(P^{\varepsilon,\alpha}) = \sum \lambda_i W_2^2(\gamma_i, T_{\#}\gamma^{\varepsilon,\alpha}) = \int c \, d\gamma^{\varepsilon,\alpha} \leq \sigma^2 + \frac{C_2}{\sqrt{\alpha}}.
$$

Then we bound $W_2^2(T_{\#}\gamma^{\varepsilon,\alpha}, T_{\#}\tilde{\gamma}^{\varepsilon,\alpha})$. Using again triangle and Young inequalities, we get the following bound by Lemma 5

$$
W_2^2(T_{\#}\gamma^{\varepsilon,\alpha}, T_{\#}\tilde{\gamma}^{\varepsilon,\alpha}) \leq 2 \sum \lambda_i W_2(T_{\#}\gamma^{\varepsilon,\alpha}, \gamma_i^{\varepsilon,\alpha}) + 2 \sum \lambda_i W_2(\gamma_i^{\varepsilon,\alpha}, T_{\#}\tilde{\gamma}^{\varepsilon,\alpha})
\leq 2 \int c \, d\gamma^{\varepsilon,\alpha} + 2 \int c \, d\tilde{\gamma}^{\varepsilon,\alpha}
\leq 2\varepsilon + \frac{6C_2}{\sqrt{\alpha}} + 4\sigma^2(P^{\varepsilon,\alpha})
\leq 6\frac{C_2}{\sqrt{\alpha}} + 6\sigma^2 + \frac{4C_2}{\sqrt{\alpha}}.
$$
Since, $\sigma^2 = \sigma^2(P) = \int c \, d\gamma^*$, it follows from gathering our bounds and using (33) that

$$C(T\#\gamma^\varepsilon,\alpha)$$



$$\leq 2 \sqrt{8 \left[ 2\sigma^2 + \frac{\sigma \sqrt{8C_P}}{\sqrt{\alpha}} + \frac{2\sigma \sqrt{16C_P}}{\sqrt{\alpha}} \sqrt{8C_P + \sigma^2} + \frac{C_2}{\sqrt{\alpha}} \right] + 12 \frac{C_2}{\sqrt{\alpha}} + 12\sigma^2 + 8 \frac{C_2}{\sqrt{\alpha}} + \frac{\sigma^2}{C_2} + \frac{2\sigma \sqrt{16C_P}}{\sqrt{\alpha}} \frac{C_2}{\sqrt{\alpha}} + \frac{\sigma^2}{C_2}} + 2 \sqrt{8\sigma^2 + 21 \frac{C_2}{\sqrt{\alpha}} + 2\sigma \sqrt{16C_P} \sqrt{8C_P + \sigma^2} =: A^b \approx \sqrt{\frac{C_P}{\sqrt{\alpha}} + \sigma^2}. \quad (35)$$

Remark that $W^2_2(P,P^\varepsilon,\alpha) \leq \sum \lambda_i W^2_2(\mu_i,\gamma^\varepsilon,\alpha)$. Collecting our bounds, we can now apply (34) to bound $W^2_2(T\#\gamma^\varepsilon,\alpha, T\#\hat{\gamma}^\varepsilon,\alpha)$ by using Lemma 5

$$\frac{k}{2} W^2_2(T\#\gamma^\varepsilon,\alpha, T\#\hat{\gamma}^\varepsilon,\alpha)$$



$$\leq \sum \lambda_i \left( W^2_2(\gamma_i^\varepsilon,\alpha, T\#\gamma^\varepsilon,\alpha) - W^2_2(\gamma_i^\varepsilon,\alpha, T\#\hat{\gamma}^\varepsilon,\alpha) \right) + A^b W^2_2(P,P^\varepsilon,\alpha)$$



$$\leq \int c \, d\gamma^\varepsilon,\alpha - \int c \, d\hat{\gamma}^\varepsilon,\alpha + A^b \sqrt{\sum \lambda_i W^2_2(\gamma_i^\varepsilon,\alpha, \mu_i)}$$



$$\leq \varepsilon + \frac{1}{\sqrt{\alpha}} \left( 3C_2 + \sigma A^b \sqrt{8C_P} \right). \quad (36)$$

(Step 3. Conclusion) We conclude by applying the bounds (36) and (33) to (28) to obtain

$$\frac{k}{2} W^2_2(T\#\gamma^*, T\#\hat{\gamma}^\varepsilon,\alpha) \leq 2\varepsilon + \frac{\sigma \sqrt{8C_P}}{\sqrt{\alpha}} + \frac{2\sigma \sqrt{16C_P}}{\sqrt{\alpha}} \sqrt{8C_P + \sigma^2} + \frac{C_2}{\sqrt{\alpha}} + \frac{2}{\sqrt{\alpha}} \left( 3C_2 + \sigma A^b \sqrt{8C_P} \right).$$

The proof is complete setting

$$2C_1 = \sigma \sqrt{8C_P} + 2\sigma \sqrt{16C_P} (8C_P + \sigma^2) + C_2 + 2 \left( 3C_2 + \sigma A^b \sqrt{8C_P} \right). \quad (37)$$