An explicitly solvable model of the spontaneous \textit{PT}-symmetry breaking

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Abstract

We contemplate the pair of the purely imaginary delta-function potentials on a finite interval with Dirichlet boundary conditions. The two parameter model exhibits nicely the expected quantitative features of the unavoided level crossing and of a ”phase-transition” complexification of the energies. Combining analytic and numerical techniques we investigate strength- and position-dependence of its spectrum.

1 The Model

Systems with point interactions play an important role in quantum mechanics since they keep their exact solvability while describing real systems \cite{1}. Conception of point interactions in pseudo-Hermitian quantum mechanics appeared in various works \cite{2}, \cite{3}, \cite{4}, \cite{5}.

The aim of present article is to describe spectral properties of one particle confined in infinite square well with two point interactions of imaginary strength. The corresponding time-less Schrödinger equation reads

\[ H\psi = (-\triangle - i\xi \delta(x + a) + i\xi \delta(x - a))\psi = E\psi. \] (1)

where $H$ acts on continuous functions from $L_2((-1, 1), dx)$ vanishing in $\pm 1$ and whose derivatives have an imaginary step $\psi'(\pm a_R) - \psi'(\pm a_L) = \pm i\xi \psi(\pm a)^3$.

Considering energy $E$ as a general complex number, following the standard procedure we arrive at the secular equation

\[ \sinh 2u + \frac{\xi^2}{4u^2} (\sinh(4ua - 2u) + \sinh 2u - 2 \sinh 2ua) = 0, \] (2)

where $u$ is defined as $E = -u^2$. Invariance of the equation with respect to complex conjugation of its root was rather expected due to the $P$-pseudo

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\textsuperscript{3}Indexes $L$, $R$ denote direction in which we approach the point.
Hermiticity of the Hamiltonian \((H^\dagger = PHP)\). Due to the lack of physically relevant interpretation for non-real eigenvalues \(E\), we will be interested in real energies only. In that case, (2) acquires the following simplified form,

\[
\sin 2k + \frac{\xi^2}{k^2} \sin 2ka \sin^2 k(1 - a) = 0, \quad k = Im(u). \tag{3}
\]

Neither (2) nor (3) can be solved without numerical methods. Before presentation of numerical solution, we will focus on qualitative properties of (3) that can be derived analytically.

2 Spectral lines - robust vs. fragile

Performing a formal manipulation, the equation (3) can be rewritten,

\[
\xi^2 = -\frac{\sin 2k}{\sin 2ka \sin^2 k(1 - a)} k^2. \tag{4}
\]

In (1), we required \(\xi\) to be non-negative real number. It restricts the domain of \(\xi = \xi(k)\) since the right-hand side of (4) must be positive.

Let us take the following set

\[
\mathcal{J} = \left( J_a^- \cap I^- \right) \cup \left( J_a^+ \cap I^+ \right) \cup M, \tag{5}
\]

where

\[
k \in I^\pm \Leftrightarrow \pm \sin 2k > 0, \quad k \in J_a^\pm \Leftrightarrow \pm \sin 2ka > 0, \quad k \in M \Leftrightarrow \sin 2k = \sin 2ka = 0.
\]

Assuming that \(k \in \mathcal{J}\), we define \(\xi\) to be equal to its limit in the points where \(\sin k(1 - a) = 0\). Then the function \(\xi = \xi(k)\) is continuous on the open interval \(J_a^\pm \cap I^\mp\). If \(k \in M\), equation (3) is fulfilled for any real \(\xi\).

Let us discuss now the behavior of \(\xi\) at the boundaries of the subintervals of \(\mathcal{J}\). The numerator \(\sin 2k\) equals zero in one boundary point at least. It is due to \(a < 1\) and \(\xi = 0\) in this case. We will say that the boundary point is of the first type. On the other hand when \(\sin 2ka\) vanishes, \(\xi\) tends to infinity and the boundary is of the second type.

Thus, we can observe three kinds of spectral lines. The first one is supported by an interval from \(\mathcal{J}\) with both boundaries of the first type. It reaches its maximum \(\xi_{max}\) on the interval so it is bounded. The spectral lines of the second type tend to infinity and are supported by subintervals of \(J_a^\pm \cap I^\mp\) with mixed types of boundaries. The third kind is unbounded as well and is supported by \(k \in M\). That are semi-lines perpendicular to
Figure 1: The function $\xi = \xi(k)$ for $a = 2/3$. All three types of spectral lines are present, $\xi = 5t$, $k = j\pi$.

$k$-axes. In terminology used in [6], the energy levels of the first type are called \textbf{fragile} while the remaining ones are \textbf{robust} (see Fig. 1).

The third class of spectral lines corresponds to wave functions having nodes at the interaction points and it appears only for rational $a$. On two intervals $j_\pm \subset \mathcal{J}_\pm \cap \mathcal{I}$ that are connected by $k \in M$, spectral lines of the first and the third kind are stuck together. We speak about \textbf{unavoided level crossing} in this case. Due to simple form of (4), the level crossing can be described quantitatively. Let us take $a = \frac{P}{Q}$ where $P$ and $Q$ are coprime integers. The level crossing appears if $2k = lQ\pi$ and $l(Q - P)$ is odd number, $l$ is integer. In that case, the crossing of spectral lines occurs for

$$\xi_{\text{cross}} = \frac{l\pi Q^2}{2P^2 \sin \frac{\pi(Q - P)}{2}}.$$ 

The above analysis provided us a qualitative knowledge about behavior of $k = k(\xi)$ implicitly given by (2). The fragile spectral lines cease to be real when $\xi$ exceeds a critical value $\xi_{\text{max}}$. Thus for any $a$, we can find a sequence of the critical couplings $\xi_{\text{crit},n}$ where $n$ is the excitation of the higher energy from the couple of merging energy levels. The sequences of critical couplings have their minimum $\xi_{\text{min}} \sim 2.4931$ for $a \sim 0.3335$. It is an interesting feature of our model that the sequence is not monotonic. We call it \textbf{an overstepping complexification} of energy levels (see Fig 2.). The numerical solution $k = k(\xi)$ of (2) in the supercritical regime is in Fig.3.

3 Conclusion and Outlook

The presented model of a particle captured in an infinite square well with two point $PT$-symmetric interactions \cite{11} offers an interesting spectral behav-
We studied a parametric dependence of energy levels on a position and strength of the point impurities. The analysis provided us a good qualitative control over various phenomena appearing in the spectrum. We described presence of fragile and robust spectral lines as well as appearance of unavoidable level crossing of energy levels. These analytical observations were supported by numerical results. In the numerically obtained data, we observed a new phenomenon of ”overstepping” complexification where merging of energy levels is not sequent with respect to the excitation but proceeds at random.

The article is a continuation of research started in [5]. In this sense, it is straightforward to find perturbation series for energy in the case of generic $a$. It has been shown that for $a = 1/2$, the effect of non-Hermitian interaction
was negligible for high excitations. This fact enables approximate separability of positive-definite metric performed originally in [7] and consequent computation of physically relevant quantities like position expectation value and its time development.

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