A h-Version of the Least Squares Collocation Method for the Biharmonic Equation in Irregular Domains

Vasily Belyaev, Luka Bryndin and Vasily Shapeev
Khristianovich Institute of Theoretical and Applied Mechanics, Russian Academy of Sciences, 4/1 Institutskaya street, Novosibirsk, 630090 Russia
E-mail: belyaevasily@mail.ru

Abstract. The numerical solution of boundary value problems for the biharmonic equation in irregular domains was obtained by new h-versions of the high-order accuracy least squares collocation method. In particular, we consider discrete boundary and multiply connected domains. It is shown that our approximate solution is high order accurate. We represent some suitable numerical examples. The numerical results are compared with those found by other authors who used a high order finite difference method. The biharmonic equation was applied to model the stress-strain state of isotropic thin irregular plates in the theory of thin plates.

1. Introduction
Biharmonic equations have significant applications in many areas of science and technology. Classical examples can be found in fluid, solid mechanics and many other areas. For example, in fluid dynamics, the stream function satisfies the biharmonic equation. In solid mechanics, the solution of the biharmonic equation can be used to represent the Airy stress function. In the theory of thin plates, the solution of the biharmonic equation can be used for simulating the stress-strain state of an isotropic plate under the action of a transverse load [1]. We present some suitable non-trivial numerical examples in the latter case. Biharmonic equations are difficult to solve due to the fourth order derivatives in the differential equation. Generally, problems involving high-order PDEs and complex geometries are more difficult to solve than those with second-order PDEs and regular geometries, respectively.

At the present time, numerical solutions of the boundary value problems for the biharmonic equation in irregular domains are usually obtained through the use of the finite difference method (FDM) [2, 3], because of the ease of grid generation and fast solving. On top of that, the biharmonic equation was decoupled to two Poisson equations in [2, 3] and some extra computational effort for iteration was needed in this kind of method. The finite element method (FEM) is another popular method for solving this problem [4–8] using unstructured grids. Although the solution techniques for fourth order equations by FDMs and FEMs are well developed, there are not many results available for dealing with arbitrary shapes and complex boundary conditions [9]. Spectral methods and methods of spectral elements have become increasingly popular in the computation of continuum mechanics problems. The main advantage of these methods is the exponential rate of convergence if the unknown solution is sufficiently smooth. However, in real physical problems, the smoothness of the solution is
generally weak, the spectral method is less effective or flexible than FEMs. In particular, it is worth highlighting the works [9–12]. The biharmonic equation was solved in irregular domains in these papers. There are other numerical approaches to the biharmonic equation (see, for example, [13] and the references therein). However, most numerical methods are limited to rectangular domain.

In the present study the boundary value problems for the nonhomogeneous biharmonic equation in irregular domains are solved by the least squares collocation (LSC) method [14,15]. In the first case the irregular domain is embedded in a rectangle covered by the regular grid with rectangular cells. The second case is based on the triangulation of the original domain. In the LSC method the original differential problem is projected into a finite-dimensional linear functional space serving for the construction of approximate solutions. The approximate solution is most often obtained as a linear combination with indeterminate coefficients of the elements of a basis defined in this space. The overdetermined system of linear algebraic equations (SLAE) for unknown coefficients is constructed of collocation equations, boundary conditions and matching conditions in each cell of the grid. Krylov subspaces [16], multi-grid complexes [17] and diagonal preconditioner [18] are used in the new h-versions.

In the present paper, we first present numerical examples for solving the Dirichlet problem for an inhomogeneous biharmonic equation in irregular domains. We compare our results with those in [2, 3]. Finally, we apply the LSC method for calculating the stress-strain state of isotropic thin irregular plates under the action of transverse loads.

2. Description of the LSC Method

Let us consider the Dirichlet problem for the biharmonic equation for \( u(x_1, x_2) \)

\[
\Delta^2 u = f(x_1, x_2), \quad (x_1, x_2) \in \Omega, \quad u|_{\partial\Omega} = g_1(x_1, x_2), \quad u_n|_{\partial\Omega} = g_2(x_1, x_2) \quad (2.1)
\]

in the irregular domain \( \Omega \subset \mathbb{R}^2 \) with the boundary \( \partial\Omega \), where \( \Delta^2 = \frac{\partial^4}{\partial x_1^4} + 2 \frac{\partial^4}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4}{\partial x_2^4} \), \( u_n = \frac{\partial u}{\partial n} \), \( \vec{n} \) is the unit normal pointing outward, \( u(x_1, x_2) \) is the unknown function, \( f(x_1, x_2) \), \( g_1(x_1, x_2) \), and \( g_2(x_1, x_2) \) are given functions. The boundary conditions also are set on hole boundaries if the domain is multiply connected. Given the limits on the length of this paper, the description of the LSC method for simply connected irregular domain is given here. However, we represent some examples for multiply connected domains.

In this paper \( \Omega \) is the domain with discrete boundary (figure 1 [14,15]) or polygonal domain (figure 2). The program builds a continuous double spline \((x_1(t), x_2(t))\) if boundary is smooth or a piece-wise spline if boundary has breakpoints in the first case for describing \( \partial\Omega \). The irregular domain is embedded in a rectangle covered by the regular grid \((N_1 \times N_2)\) with rectangular cells. The elongated irregular cells (i-cells) are attached to the neighboring ones (figure 1). This h-version of the LSC method is called “h-version 1”. The readers are referred to [14] for more information. The polygonal domain is divided into “big” triangles, which are divided in turn into “small” triangles (figure 1). The “small” triangles equal each other in a particular “big” triangle and are similar to it. This h-version of the LSC method is called “h-version 2”. Let \( N_{\text{cells}} \) be the number of cells in each version.

For convenience we introduce the local coordinate system in each \( j \)-th cell

\[
y_1 = \frac{(x_1 - x_{1j})}{h_1}, \quad y_2 = \frac{(x_2 - x_{2j})}{h_2}, \quad (2.2)
\]

where \((x_{1j}, x_{2j})\) is the cell center, \( h_1 \) and \( h_2 \) are the characteristic sizes of the \( j \)-th cell, \( j = 1, ..., N_{\text{cells}} \), \( v(y_1, y_2) = u(x_1(y_1), x_2(y_2)) \). In the first case [14], for example, \( 2h_1 \) and \( 2h_2 \) are the sizes of the rectangular cells in the \( x_1 \) and \( x_2 \) directions, respectively. The approximate
solution in each \( j \)-th cell is sought in the form of a linear combination of the basis functions

\[ \phi_i(y_1, y_2) \in \{y_1^{\alpha_1}y_2^{\alpha_2} | 0 \leq \alpha_1 \leq K, \ 0 \leq \alpha_2 \leq K - \alpha_1\}, \ i = 1, ..., N_{nu}, \ N_{nu} = (K + 1)(K + 2)/2: \]

\[ v_{hj}(y_1, y_2) = \sum_{i=1}^{N_{nu}} b_{ij} \phi_i. \]

The unknown coefficients \( b_{ij} \) in the LSC method are determined from an overdetermined “local” system of equations in each cell consisting of the collocation equations, the matching conditions and boundary conditions.

The collocation equations multiplied by \( h_1^2 h_2^2 \) are written down at the collocation points \((y_{1c}, y_{2c})\) (figure 1, 2) in each \( j \)-th cell as:

\[ k_c \left( h_2^2 \frac{\partial^4 v_{hj}}{\partial y_1^4} + 2 \frac{\partial^4 v_{hj}}{\partial y_1^2 \partial y_2^2} + h_1^2 \frac{\partial^4 v_{hj}}{\partial y_2^4} \right) = k_c h_1^2 h_2^2 f(x_1(y_{1c}), x_2(y_{1c})), \]

where \( c = 1, ..., 16 \ (K = 4) \) and \( k_c \) is controlling weight.

The matching conditions (multiplied by \( h_1 h_2 \) in (2.5)) are written down at the matching points (figure 1) in each \( j \)-th cell as:

\[ k_{m0} v_{hj} + k_{m1} \frac{\partial v_{hj}}{\partial n_j} = k_{m0} \hat{v}_h + k_{m1} \frac{\partial \hat{v}_h}{\partial n_j}, \]

\[ h_1 h_2 \left( k_{m2} \frac{\partial^2 v_{hj}}{\partial n_j^2} + k_{m3} \frac{\partial^3 v_{hj}}{\partial n_j^3} \right) = h_1 h_2 \left( k_{m2} \frac{\partial^2 \hat{v}_h}{\partial n_j^2} + k_{m3} \frac{\partial^3 \hat{v}_h}{\partial n_j^3} \right), \]

where \( n_j \) denotes the unit outer normal to the boundary of the \( j \)-th cell, \( v_{hj} \) and \( \hat{v}_h \) are the limits of the function \( v_{hj} \) as its arguments tend to the cell side from within and outside the cell, \( k_{m0}, k_{m1}, k_{m2}, k_{m3} \) are controlling weights.
If a side of a cell adjoins the boundary of the domain $\Omega$, then the boundary conditions are written down at several points $\in \delta \Omega$ (Fig. 1) as

$$k_{b_0} v_{hj} = k_{b_0} g_1(x_1, x_2), \quad (2.6)$$

$$k_{b_1} \left( \frac{n_1}{h_1} \frac{\partial v}{\partial y_1} + \frac{n_2}{h_2} \frac{\partial v}{\partial y_2} \right) = k_{b_1} g_2(x_1, x_2), \quad (2.7)$$

where $k_{b_0}, k_{b_1}$ are controlling weights.

The SLAE obtained by combining the equations (2.3)–(2.7) in all cells of the computational domain (global system) is solved in the iterative process of Gauss-Seidel. In this process one “global iteration” consists of sequential solution of the local SLAE in all cells of the domain. A system matrix of equations is reduced to the upper triangular form by the orthogonal method when constructing the solution in each cell. The iterative process continues while the condition is true

$$\max_{rj} |b_{rj}^{n+1} - b_{rj}^n| > \epsilon, \quad (2.8)$$

where $b_{rj}^n$ — $r$-th ($r = 1, \ldots, N_{nu}$) coefficient of the polynomial which approximates the solution in the cell with the number $j$ on the $n$-th iteration. The value $\epsilon$ is a given small constant [14].

3. Numerical examples

3.1. Dirichlet problem

Example 1 (using h-version 1). We try to recover the values of the function $u(x_1, x_2) = x_1^3 \ln(1 + x_2) + x_2^2 \left( \frac{1}{1 + x_1} \right)$ in the domain defined by skinny ellipse $\frac{x_1^2}{0.5^2} + \frac{x_2^2}{0.15^2} \leq 1$ from the knowledge $\Delta^2 u$ inside domain and $u$, $u_n$ on the domain boundary. The table 1 shows the absolute errors $L_\infty$ for various grids (maximal difference between the computed solution and the exact one). The following parameters were used in the calculations: $k_c = k_{m_2} = k_{m_3} = k_{b_0} = k_{b_1} = 1, k_{m_1} = 0.01, k_{m_3} = 0.015, \epsilon = 10^{-14}$. We compare our results to numerical examples given in [2, 3]. In our previous paper [14] the absolute error is slightly bigger than in [3]. The FDM proposed in [3] has the convergence rate bigger than the LSC method when $K = 4$. However, the biharmonic equation of fourth order was splitted into two coupled Poisson equations in the other papers [2, 3]. On top of that, the domain boundary is given by the double splines in this paper (also in [14]), and not analytically, as was done in the other papers. Using the LSC method for solving two Poisson equations substantially increases the accuracy of the calculations. This fact is shown in the next experiment. The new h-version of the LSC method was proposed and implemented in this paper with arbitrary $K$ (table 1, $K = 6, 8, 10$). In this case the absolute error is substantially smaller than in the other papers [2, 3, 14]. Roundoff errors play role when $K = 10$ and grid size is $32 \times 16$.

3.2. Calculation of the Stress-Strain State

In the theory of thin plates, the biharmonic equation can be used to represent a plate deflection $w$, where $f = q/D$, $q$ is the intensity of the load distributed over the upper plate surface, $D = Eh^3/12(1 - \nu^2)$ is the flexural rigidity of the plate, $h$ is the thickness of the plate, $E$ is Young’s modulus, $\nu$ is Poisson’s ratio. For example, one of the following conditions can be set on each piece of the plate boundary:

$$w = 0, \quad \frac{\partial w}{\partial n} = 0 \quad \text{— a clamped edge}, \quad (3.1)$$

$$w = 0, \quad M_n w = 0 \quad \text{— a simple support}, \quad (3.2)$$
The exact values of the inner boundary are following: 

\[(5 \pi x + 5)\sin(x/2) + 5), (5 \pi x + 5)sin(x/2) + 5)\).

Example 2 (using h-version 2). Let the projection of the middle plane of the plate on the plane \((x_1, x_2)\) has polygonal biconnected form (figure 3). The vertices of the exterior boundary are following: 

\(5 \pi x + 5)sin(x/2) + 5)\), (5 \pi x + 5)sin(x/2) + 5)\).

The exact values of the \(w\) and the bending moment \(M_n\) are set on the boundary \(\partial \Omega\) in the numerical experiment in order to use the analytical solution (3.4) for the test. The solution of these equations is greatly simplified in this case due to the polygonal boundary. The biharmonic equation can be decoupled as two Poisson equations with the Dirichlet boundary condition [1].

\[\begin{align*}
M_n,w = 0, & V_nw = 0 \quad \text{a free edge,} \\
\end{align*}\]

(3.3)

Table 1. Results of numerical experiments for the example 1

| \(N_1 \times N_2\) | \(L_\infty\) | Rate | \(N_1 \times N_2\) | \(L_\infty\) | Rate | \(L_\infty\) | Rate |
|-----------------|--------|------|-----------------|--------|------|--------|------|
| 64 \times 32    | 3.65e-4 | —    | 8 \times 4      | 1.97e-5| —    | —      | —    |
| 128 \times 64   | 9.54e-5 | 1.9  | 16 \times 8     | 1.56e-6| 3.65 | 3.0e-6 | —    |
| 256 \times 128  | 2.08e-5 | 2.2  | 32 \times 16    | 1.52e-7| 3.35 | 1.1e-7 | 4.8  |
| 512 \times 256  | 4.98e-6 | 2.1  | 64 \times 32    | 1.24e-8| 3.61 | 1.5e-9 | 6.2  |

Table 2. The results of the numerical experiments for the example 2.

| \(N_{cells}\) | \(E_{N_1,N_2}\) | Rate | \(N_{iter}\) | \(E_{N_1,N_2}\) | Rate | \(N_{iter}\) |
|---------------|---------------|------|---------------|---------------|------|---------------|
| 48            | 3.97e-4       | —    | 45            | 1.29e-4       | —    | 81 (81)       |
| 192           | 3.74e-5       | 3.40 | 43            | 4.43e-6       | 4.86 | 81 (81)       |
| 768           | 6.24e-6       | 2.58 | 81            | 1.37e-7       | 5.01 | 86 (81)       |
| 3072          | 1.15e-6       | 2.43 | 241           | 4.63e-9       | 4.88 | 97 (85)       |
where \( x \) on the plane \([x_1, x_2]\) is the controlling weight of the number of iteration, CPU time in seconds spent in solving the global system is also shown in the table 2. The following parameters were used in the first case: \( h = 0.1 \text{ m}, E = 200 \text{ GPa}, \nu = 0.28, d_1 = d_2 = 9.65 \text{ m}, \epsilon = 10^{-10}, k_e = k_{m1} = k_{m2} = k_{b0} = k_{b2} = 1, k_{m1} = 0.12, k_{m3} = 0.125, \) where \( k_{b2} \) is the controlling weight before the second equation in (3.2). The equations of the discrete problem and parameters before them were taken from [19] in the second case. The number of iterations is indicated in parentheses in the column \( N_{iter} \) for solving the first Poisson’s equation when \( \epsilon = 10^{-14}. \)

**Example 3 (using h-version 1).** Let the projection of the middle plane of the plate on the plane \((x_1, x_2)\) has multiply connected form. The outer boundary is circular curve \((x_1 - 5)^2 + (x_2 - 5)^2 = 5^2\) with 5 holes: \((x_1 - 3)^2 + (x_2 - 3)^2 \leq 0.7^2, (x_1 - 3)^2 + (x_2 - 7)^2 \leq 0.6^2, (x_1 - 7)^2 + (x_2 - 3)^2 \leq 0.5^2, (x_1 - 7)^2 + (x_2 - 7)^2 \leq 0.4^2, (x_1 - 5)^2 + (x_2 - 5)^2 \leq 0.3^2.\) The table 3 shows the relative errors \( E_{N1, N2} \) for exact solution from the example 2.

| \(N_1 \times N_2\) | \(E_{N1, N2}\) | \(N_{iter}\) | Rate | CPU time (s) |
|---------------------|----------------|-------------|------|-------------|
| 8 \times 8          | 8.63(-4)       | 81          | —    | 0.18        |
| 16 \times 16        | 1.26(-4)       | 81          | 2.77 | 0.593       |
| 32 \times 32        | 1.19(-5)       | 121         | 3.40 | 3.15        |
| 64 \times 64        | 2.37(-6)       | 201         | 2.32 | 20.51       |

**Example 4 (using h-version 2).** Let us consider a problem with an unknown analytical solution. Suppose the isotropic plate from previous example is under the action of the uniform transverse load \( q = \text{const} \). The inner edges are simple supported (3.2). Four exterior edges are clamped (3.1) and two edges are free (3.3). We use the behavior of the value stress intensity in the problem solution domain to illustrate the stress state of the plate

\[
I = \frac{1}{\sqrt{2}} \sqrt{(\sigma_x - \sigma_y)^2 + \sigma_x^2 + \sigma_y^2 + 6\sigma_{xy}^2},
\]

where \( \sigma_x = -E(w_{xx} + \nu w_{yy}), \sigma_y = -E(w_{yy} + \nu w_{xx}), \sigma_{xy} = -E(w_{xy}), E = \frac{E x_3}{1-\nu^2}, x_3 \in [-h/2, h/2]. \)
The following parameters were used in the calculations: $h = 0.1$ m, $E = 200$ GPa, $\nu = 0.28$, $q = 1$ MPa, $x_3 = h/2$. The figure 3 shows the deflection form $w$ and the figure 4 shows the stress intensity $I$ (3.5).

**Example 5 (using h-version 1).** Let the projection of the middle plane of the plate on the plane $(x_1, x_2)$ has segment form with clamped edges. The following parameters were used in the calculations: $h = 0.1$ m, $E = 200$ GPa, $\nu = 0.28$, $q = 500$ KPa, $x_3 = h/2$. The figure 5 shows the deflection form $w$ and the figure 6 shows the stress intensity $I$ (3.5).

![Figure 5. The deflection value $w$ of the strain plate (example 5)](image1)

![Figure 6. The stress intensity $I$ (example 5)](image2)

4. Conclusions

The new h-versions of the LSC method of high-order accuracy are proposed and implemented for the numerical solution of the nonhomogeneous biharmonic equation in the irregular domains. The convergence of the approximate solutions to the exact ones was verified on a sequence of grids in numerical experiments with analytical solutions. It is shown that the LSC method in many cases is better than FDMs.

Acknowledgments

The research was carried out within the framework of the Program of Fundamental Scientific Research of the state academies of sciences in 2013-2020 (project No. AAAA-A17-117030610136-3).

References

[1] Timoshenko S P and Woinowsky-Krieger S 1959 *Theory of Plates and Shells, 2dn edn* (New York: McGraw-Hill Book Company) p 580

[2] Chen G, Li Z and Lin P 2007 *Adv. Comput. Math.* 29 (2) 113–33

[3] Ben-Artzi M, Chorev I, Croisille J P and Fishelov D 2009 *SIAM J. Numer. Anal.* 47 (4) 3087–108

[4] Brenner S C 1989 *SIAM J. Numer. Anal.* 26 (5) 1124–38

[5] Mayo A and Greenbaum A 1992 *SIAM J. Sci. Comput.* 13 (1) 101–18

[6] Hanisch M R 1993 *SIAM J. Numer. Anal.* 30 (1) 184–214

[7] Davini C and Pitacco I 2000 *SIAM J. Numer. Anal.* 38 (3) 820–36

[8] Eymard R, Galloné T, Herbin R and Linke A 2012 *Math. Comp.* 812 (280) 2019–48

[9] Shao W, Wu X and Chen S 2012 *Engin. Anal. Bound. Elem.* 36 (12) 1787–98

[10] Mai-Duy N, See H and Tran-Cong T 2009 *Appl. Math. Modell.* 33 (1) 284–99

[11] Shao W and Wu X 2015 *Appl. Math. Modell.* 39 (9) 2554–69

[12] Zhang Q and Chen L 2017 *Comp. Math. Appl.* 75 (12) 2958–68

[13] Jiang Y, Wang B and Yuesheng X 2014 *SIAM J. Numer. Anal.* 52 (5) 2530–2554

[14] Shapeev V P and Belyaev V A 2018 *Computat. Technolog.* 23 (3) 15–30 (in Russian)

[15] Belyaev V A and Shapeev V P 2018 *AIP Conf. Proceedings* 2027 030094–1–030094–9

[16] Saad Y 1992 *Numerical Methods for Large Eigenvalue Problems* (Manchester: Manchester University Press) p 346

[17] Fedorenko R P 1964 *USSR Comput. Math. Math. Phys.* 4 (3) 559–64

[18] Ramšák M and Škerget L A 2004 *Int. J. Numer. Meth. Fluids.* 46 (8) 815–47

[19] Belyaev V A and Shapeev V P 2018 *Computat. Technolog.* 23 (3) 15–30 (in Russian)