Singular polynomials for the symmetric group
and Krawtchouk polynomials

Charles F. Dunkl∗
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Abstract
A singular polynomial is one which is annihilated by all Dunkl ope-
rators for a certain parameter value. These polynomials were fi rst studied
by Dunkl, de Jeu and Opdam, (Trans. Amer. Math. Soc. 346 (1994),
237-256). This paper constructs a family of such polynomials associated
with the irreducible representation \((N - 2, 1, 1)\) of the symmetric group
\(S_N\) for odd \(N\) and parameter values \(-\frac{1}{2}, -\frac{3}{2}, -\frac{4}{2}, \ldots\). The method depends on
the use of Krawtchouk polynomials to carry out a change of var iables in
a generating function involved in the construction of nonsymmetric Jack
polynomials labeled by \((m, n, 0, \ldots, 0)\), \(m \geq n\).

1 Introduction
We will study polynomials on \(\mathbb{R}^N\) with certain properties relating to the action
of the symmetric group \(S_N\) acting as a finite reflection group (of type \(A_{N-1}\)).
Let \(\mathbb{N}\) denote \(\{1, 2, 3, \ldots\}\) and \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\); for \(\alpha \in \mathbb{N}_0^N\) let \(|\alpha| = \sum_{i=1}^N \alpha_i\) and
define the monomial \(x^\alpha\) to be \(\prod_{i=1}^N x_i^{\alpha_i}\); its degree is \(|\alpha|\). Consider elements of
\(S_N\) as functions on \(\{1, 2, \ldots, N\}\) then for \(x \in \mathbb{R}^N\) and \(w \in S_N\) let \((xw)_i = x_{w(i)}\)
for \(1 \leq i \leq N\); and extend this action to polynomials by \(wf(x) = f(xw)\). This
has the effect that monomials transform to monomials, \(w(x^\alpha) = x^{\alpha_{w}}\) where
\((w\alpha)_i = \alpha_{w^{-1}(i)}\) for \(\alpha \in \mathbb{N}_0^N\). (Consider \(x\) as a row vector, \(\alpha\) as a column
vector, and \(w\) as a permutation matrix, with 1’s at the \((w(j), j)\) entries.) The
reflections in \(S_N\) are the transpositions, denoted by \((i, j)\) for \(i \neq j\), interchanging
\(x_i\) and \(x_j\).

In [3] the author constructed for each finite reflection group a parametrized
commutative algebra of differential-difference operators; for the symmetric group
there is one parameter \(\kappa \in \mathbb{C}\) and the definition is as follows:

Definition 1 For any polynomial \(f\) on \(\mathbb{R}^N\) and \(1 \leq i \leq N\) let
\[
\mathcal{D}_i f(x) = \frac{\partial}{\partial x_i} f(x) + \kappa \sum_{j \neq i} \frac{f(x) - (i, j) f(x)}{x_i - x_j}.
\]

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It was shown in [3] that $D_i D_j = D_j D_i$ for $1 \leq i, j \leq N$ and each $D_i$ maps homogeneous polynomials to homogeneous polynomials. A specific parameter value $\kappa$ is said to be a singular value (associated with $S_N$) if there exists a nonzero polynomial $p$ such that $D_j p = 0$ for $1 \leq i \leq N$: and $p$ is called a singular polynomial. It was shown by Dunkl, de Jeu and Opdam [3, p.248] that the singular values for $S_N$ are the numbers $-\frac{1}{n}$ where $n = 2, \ldots, N, j \in N$ and $n \nmid j$ (n does not divide $j$). In this paper we construct singular polynomials for the values $\frac{1}{N}$ $(j \in N$ and $N - 1 \nmid j)$ with a new result for the case of $N$ being odd and $-\frac{1}{N} = -l - \frac{1}{2}, l \in N_0$.

There are conjectures in [5, p.255] regarding some general properties of the singular polynomials for $S_N$ but these are not as yet established. Here is an easy example of singular polynomials: let $a_N (x) = \prod_{1 \leq i < j \leq N} (x_i - x_j)$, the alternating polynomial. Then $(i, j) a_N (x) = -a_N (x)$ for any transposition $(i \neq j)$; further $\frac{\partial}{\partial x_i} a_N (x) = a_N (x) \sum_{j \neq i} \frac{1}{x_i - x_j}$. Thus for any $l \in N_0$ we have $D_i \{a_N (x)^{2l+1}\} = (2l + 1 + 2\kappa) a_N (x)^{2l+1} \sum_{j \neq i} \frac{1}{x_i - x_j}$ (for each $i$), which shows that $a_N (x)^{2l+1}$ is singular for $\kappa = -l - \frac{1}{2}$. Irreducible representations of $S_N$ are labeled by partitions of $N$ (see, for example, Macdonald [7, p.114]); the polynomial $a_N (x)^{2l+1}$ is associated with the representation $(1, l, \ldots, 1)$ (more precisely, the span $\mathbb{R} a_N (x)^{2l+1}$ an $S_N$-module of isotype $(1, 1, \ldots, 1)$).

Our construction is in terms of nonsymmetric Jack polynomials which are defined to be the simultaneous eigenfunctions of the pairwise commuting operators $D_i x_i - \kappa \sum_{j < i} (i, j), 1 \leq i \leq N$ (details about these may be found in the book by Dunkl and Xu [3, Ch.8]). When $\kappa > 0$ these operators are self-adjoint with respect to the inner product on polynomials defined by

$$\langle f, g \rangle_T = c_\kappa \int_{T^N} f (x) g (x) \prod_{1 \leq i < j \leq N} |x_i - x_j|^{2\kappa} dm (x),$$

where $T^N$ is the $N$-fold complex torus $\{z \in \mathbb{C} : |z| = 1\}^N$, $x_j = e^{i \theta_j}$ for $-\pi < \theta_j \leq \pi$ (and $1 \leq j \leq N$) and the standard measure is $dm (x) = \prod_{j=1}^N d \theta_j$. The constant $c_\kappa$ is chosen so that $(1, 1, \ldots, 1) = 1$ (computed by means of the Macdonald-Mehta-Selberg integral). The nonsymmetric Jack polynomials are labeled by $N_0^N$; in this paper only the labels $(m, n, 0, \ldots, 0)$ will occur.

### 2 The Basic Polynomials

These are the relevant results from Dunkl [4]. All the polynomials considered here have coefficients in $\mathbb{Q} (\kappa)$ (rational functions of $\kappa$ with rational coefficients). The polynomials $p_{mn} (x)$ are defined by the generating function

$$\sum_{m, n=0}^\infty p_{mn} (x) s^m t^n = (1 - sx_1)^{-1} (1 - tx_2)^{-1} \prod_{i=1}^N ((1 - sx_i) (1 - tx_i))^{-\kappa} \quad (1)$$
the product rule:

\begin{align*}
\text{Definition 2} & \quad \text{For } m \geq n \text{ let} \\
\omega_{mn} &= p_{mn} + \sum_{j=1}^{n} (-\kappa)_{j} \frac{(m-n+1)_{j-1}}{(\kappa + m - n + 1)_{j} j!} ((m-n+j) p_{m+j,n-j} + j p_{n-j,m+j}), \\
\omega_{nn} &= (1, 2) \omega_{mn}. \\

\text{Observe that } (1, 2) p_{mn} = p_{nm}, \text{ and the coefficients of } \omega_{mn} \text{ in the formula are independent of } N. \text{ Also there is the symmetry property } \omega_{ij} = \omega_{ji} \text{ for } 2 < i < j \leq N. \text{ It can be shown that } D_{1} x_{i} \omega_{mn} = ((N-1) \kappa + m + 1) \omega_{mn} \text{ and } (D_{2} x_{i} - \kappa (1, 2)) \omega_{mn} = ((N-2) \kappa + n + 1) \omega_{mn} \text{ for } m \geq n. \text{ By means of the product rule:}

D_{1} (fg) &= f D_{1} g + \frac{\partial f}{\partial x_{i}} g + \kappa \sum_{j \neq i} \frac{f - (i,j)}{x_{i} - x_{j}} (i,j) g, \\

\text{we can show } D_{1} x_{i} \omega_{mn} &= (1 + (N-3) \kappa) \omega_{mn} + \kappa ((1, i) + (2, i)) \omega_{mn} \text{ and } \\
(D_{1} x_{i} - \kappa \sum_{j<i} (i,j)) \omega_{mn} &= ((N-i) \kappa + 1) \omega_{mn} \text{ for each } i > 2. \text{ Thus } \omega_{mn} \text{ is the nonsymmetric Jack polynomial labeled by } (m,n,0,\ldots,0). \text{ Note that } \omega_{mn} \text{ is defined whenever } \kappa \notin -\mathbb{N}, \text{ although } \omega_{mn} = 0 \text{ for certain values of } N,m,n \text{ and } \kappa. \text{ In fact this is the key ingredient of our construction.}

\textbf{Theorem 3} \quad \text{The following hold for all } \kappa \notin -\mathbb{N}:

1. for } m > n, \quad D_{1} (\omega_{mn}) = (N \kappa + m) \omega_{m-1,n} \\
\quad + \frac{((N-1) \kappa + n) \kappa}{\kappa + m - n} (\omega_{n-1,m} - \frac{\kappa}{\kappa + m - n + 1} \omega_{m,n-1}); \\
2. for } m \geq n, \quad D_{2} \omega_{mn} = ((N-1) \kappa + n) \left( \omega_{m,n-1} - \frac{\kappa}{\kappa + m - n + 1} \omega_{n-1,m} \right); \\

\text{3. for } m > n, \quad D_{2} \omega_{mn} = ((N-1) \kappa + n) \left( \omega_{m,n-1} - \frac{\kappa}{\kappa + m - n + 1} \omega_{n-1,m} \right);
3. for $m = n$, $D_1 \omega_{nn} = ((N - 1) \kappa + n) \left( \omega_{n-1,n} - \frac{\kappa}{\kappa + 1} \omega_{n,n-1} \right)$.

**Proof.** It was shown in [4, p.192] that both $D_1 \omega_{mn}$ and $D_2 \omega_{mn}$ are in the span of $\{\omega_{m-1,n}, \omega_{n,m-1}, \omega_{m,n-1}, \omega_{n-1,m}\}$. This implies that only the coefficients of $p_{m-1,n}, p_{n,m-1}, p_{m,n-1}, p_{n-1,m}$ need to be calculated. Let $g_1, g_2, ...$ denote polynomials of the form $\sum_{j=0}^{n-2} (c_j p_{m+n-1-j,j} + c'_j p_{j,m+n-1-j})$ with coefficients $c_j, c'_j \in \mathbb{Q}(\kappa)$. Both formulae (2), (3) are used.

Suppose $m > n$, then

$$D_1 \omega_{mn} = (N \kappa + m) p_{m-1,n} + \kappa p_{m,n-1} - \kappa p_{n-1,m} - \frac{\kappa (m - n + 1)}{\kappa + m - n + 1} (N \kappa + m + 1) p_{m,n-1} + g_1,$$

then since $\omega_{m-1,n} = p_{m-1,n} - \frac{\kappa}{\kappa + m - n} \left( (m - n) p_{m,n-1} + p_{n-1,m} \right) + g_2$ it follows that $D_1 \omega_{mn} - (N \kappa + m) \omega_{m-1,n} = \frac{(N-1) \kappa + n}{\kappa + m - n} \left( p_{m-1,n} - \frac{\kappa}{\kappa + m - n + 1} p_{n-1,m} \right) + g_3$. This proves part (1).

Next suppose $m \geq n$ then

$$D_2 \omega_{mn} = (N \kappa + n) p_{m,n-1} + \kappa p_{n-1,m}$$

$$- \frac{\kappa}{\kappa + m - n + 1} (N \kappa + m + 1) p_{n-1,m} + g_4$$

$$= (N - 1) \kappa + n \left( p_{m,n-1} - \frac{\kappa}{\kappa + m - n + 1} p_{n-1,m} \right) + g_4,$$

and this proves part (2). Part (3) follows from (2) by setting $m = n$ and applying the transposition (1, 2).

The following evaluation formula for $\omega_{mn} (1^N)$ (where $1^N = (1, 1, \ldots, 1) \in \mathbb{R}^N$) is a special case of a general result for nonsymmetric Jack polynomials (see [6, p.310]). Here is a self-contained proof.

**Proposition 4** For $m \geq n$,

$$\omega_{mn} (1^N) = \frac{(N \kappa + 1)_m ((N - 1) \kappa + 1)_n}{(m - n)! n! (\kappa + m - n + 1)_n}.$$

**Proof.** By the negative binomial theorem $p_{ij} (1^N) = \frac{(N \kappa + 1)_j (N \kappa + 1)_n}{n^j}$. Substituting this in Definition 2 yields

$$\omega_{mn} (1^N) = \frac{(N \kappa + 1)_m (N \kappa + 1)_n}{m! n!}$$

$$+ \sum_{j=1}^{n} \frac{(-\kappa)_j (m - n + 1)_j}{(\kappa + m - n + 1)_j j!} (m - n + 2j) \frac{(N \kappa + 1)_{m+j} (N \kappa + 1)_{n-j}}{(m+j)! (n-j)!}.$$
For now, assume \( m > n \) then \((m - n + 1)_j (m - n + 2j) = (m - n)_j (\frac{m-n}{2} + 1)_j / (\frac{m-n}{2})_j \) and

\[
\omega_{mn} (1^N) = \frac{(N\kappa + 1)_m (N\kappa + 1)_n}{m! n!} \times \sum_{j=0}^{n} \frac{(-n}_j (\kappa)_j (N\kappa + 1 + m)_j (m - n)_j (\frac{m-n}{2} + 1)_j}{(m + 1)_j (\kappa + m - n + 1)_j (-N\kappa - n)_j (\frac{m-n}{2})_j j!}.
\]

The sum is a terminating well-poised \(_5F_4\) whose value is

\[
\frac{(m - n + 1)_n (-N\kappa + \kappa - n)_n}{(\kappa + m - n + 1)_n (-N\kappa - n)_n},
\]

a formula of Dougall (see Bailey [1, p.25]). The stated formula follows by using the reversal \((a - n)_n = (-1)^n (1 - a)_n\). The formula is also valid when \( m = n \) (consider \( z = m - n \) as a variable, then the limit of \((z)_j (\frac{1}{2} + 1)_j / (\frac{1}{2})_j\) as \( z \to 0 \) is 1 for \( j = 0 \) and \( 2 \times j! \) for \( j \geq 1 \).

### 3 Restriction to \( N = 2 \)

The main results of this paper revolve around the vanishing of \( \omega_{mn} \) for certain values of \( N, m, n \) and \( \kappa \), but it is also necessary to show certain \( \omega_{mn} \neq 0 \). This will be accomplished by restricting to \( N = 2 \) (setting \( x_i = 0 \) for \( i > 2 \)) and finding an explicit formula for \( \omega_{mn} \).

**Definition 5** For \( m \geq n \) let

\[
f_{mn} (x_1, x_2) = (x_1 x_2)^n \sum_{j=0}^{m-n} \frac{\kappa + j}_j (m - n - j)_j}{(m - n - j)!} x_1^{m-n-j} x_2^j.
\]

**Proposition 6** For \( m \geq n \), \( D_1 x_1 f_{mn} = (\kappa + m + 1) f_{mn} \) and 
\( (D_2 x_2 - \kappa (1, 2)) f_{mn} = (n + 1) f_{mn} \).

**Proof.** It is clear from the generating function [1] that \( \omega_{m,0} = p_{m,0} = f_{m,0} \) for \( m \geq 0 \). This implies \( D_1 x_1 f_{m,0} = (\kappa + m + 1) f_{m,0} \) and \( D_2 x_2 f_{m,0} = (1 + \kappa (1, 2)) f_{m,0} \) (it can be shown directly from the generating function that \( D_1 x_1 p_{m,0} = (\kappa + m + 1) p_{m,0} \) for \( N = 2 \)). By the product rule [1]

\[
D_1 x_1 f_{mn} = (x_1 x_2)^n D_1 x_1 f_{m-n,0} + x_1 f_{m-n,0} \frac{\partial}{\partial x_1} (x_1 x_2)^n
= (\kappa + m - n + 1 + n) (x_1 x_2)^n f_{m-n,0},
\]

and

\[
(D_2 x_2 - \kappa (1, 2)) f_{mn} = (x_1 x_2)^n (D_2 x_2 - \kappa (1, 2)) f_{m-n,0} + x_2 f_{m-n,0} \frac{\partial}{\partial x_2} (x_1 x_2)^n
= (n + 1) (x_1 x_2)^n f_{m-n,0}.
\]
as claimed. ■

Since the joint eigenfunctions of the commuting operators \( D_1 x_1 \) and \( (D_2 x_2 - \kappa (1, 2)) \) are uniquely determined for generic \( \kappa \) (including \( \kappa > 0 \)) we see that \( f_{mn} \) is a scalar multiple of \( \omega_{mn} \). Evaluation at \( x = (1, 1) \) determines the constant.

**Proposition 7** For \( N = 2, m \geq n \),

\[
\omega_{mn} = \frac{(2\kappa + m - n + 1)n_n}{(\kappa + m - n + 1)n_n} f_{mn}.
\]

**Proof.** By Proposition 4 \( \omega_{mn}(1^2) = \frac{(2\kappa + 1)m_n}{(\kappa + m - n + 1)n_n} \) while \( f_{mn}(1^2) = \frac{(2\kappa + 1)m_n}{(m - n)!} \). Since \( (2\kappa + 1)m_n = (2\kappa + 1)m_n \) this completes the proof. ■

We observe that \( f_{mn} \neq 0 \) provided (as assumed throughout) that \( \kappa \notin -\mathbb{N} \). This leads to the following nontriviality result.

**Corollary 8** For \( N \geq 2, m \geq n \) the polynomial \( \omega_{mn} \neq 0 \) provided that \( 2\kappa \neq -j \) where \( j = m - n + 1, m - n + 2, \ldots, m \).

### 4 Some singular polynomials

These results already appeared in [4], and serve as illustration.

**Proposition 9** For \( N \geq 2 \) and \( n \in \mathbb{N} \) such that \( N \nmid n \) (equivalently, \( \gcd(N, n) < N \)), \( \omega_{n,0} \) is a singular polynomial for \( \kappa = -\frac{n}{N} \).

**Proof.** Note that \( \omega_{n,0} = p_{n,0} \). By formula (2) \( D_1 \omega_{n,0} = (N\kappa + n) \omega_{n-1,0} \) and \( D_i \omega_{n,0} = 0 \) for each \( i > 1 \). Further \( p_{n,0}(1, 0, \ldots) = \frac{(\kappa + 1)n_n}{n!} \) which is not zero provided \( \kappa \notin -\mathbb{N} \). ■

These polynomials have been studied by Chmutova and Etingof [2] in the context of representations of the rational Cherednik algebra.

**Proposition 10** For \( N \geq 4 \) and \( n \in \mathbb{N} \) such that \( \gcd(N - 1, n) < \frac{N-1}{2} \), \( \omega_{nn} \) is a singular polynomial for \( \kappa = -\frac{n}{N-1} \).

**Proof.** The condition \( \gcd(N - 1, n) < \frac{N-1}{2} \) is equivalent to excluding the values \( \kappa = -j, -j + \frac{1}{2} \) for \( j \in \mathbb{N} \). By Theorem 3 we have \( D_1 \omega_{nn} = \left( (N - 1)\kappa + n \right) \left( \omega_{n-1,n} - \frac{\kappa}{\kappa + 1} \omega_{n,n-1} \right) \) which is zero for \( \kappa = -\frac{n}{N-1} \), similarly \( D_2 \omega_{nn} = 0 \), and \( D_i \omega_{nn} = 0 \) for all \( i > 2 \) (for all \( \kappa \)). Corollary 8 shows that \( \omega_{nn} \neq 0 \) since \( 2\kappa \notin -\mathbb{N} \). ■

Suppose that \( M \) is an irreducible \( S_N \)-module of homogeneous polynomials (that is, \( M \) is a linear subspace of the space of polynomials, and is invariant under the action of each \( w \in S_N \), and has no proper nontrivial invariant subspaces)
then $M$ is of some isotype (corresponding to an irreducible representation of $S_N$) labeled by a partition $\tau$ of $N$.

**Proposition 11** Suppose $\tau$ is a partition of $N$ and the homogeneous polynomial $f$ is of degree $n$ and of isotype $\tau$, that is, span $\{wf : w \in S_N\}$ is an $S_N$-module on which $S_N$ acts by the irreducible representation corresponding to $\tau$, then

$$\sum_{i=1}^{N} x_i D_i f = (n + \kappa \mu(\tau)) f,$$

where $\mu(\tau) = \binom{N}{2} - \frac{1}{2} \sum_{j=1}^{N} \tau_j (\tau_j + 1 - 2j)$.

**Proof.** It is easy to show that $\sum_{i=1}^{N} x_i D_i f = \sum_{i=1}^{N} x_i \frac{\partial f}{\partial x_i} + \kappa \sum_{1 \leq i < j \leq N} (1 - (i, j)) f$

for any polynomial $f$. The operator $\sum_{1 \leq i < j \leq N} (1 - (i, j))$ is constant on span $\{wf : w \in S_N\}$, and its value is given by Young’s formula,

$$\binom{N}{2} - \frac{1}{2} \sum_{j=1}^{N} \tau_j (\tau_j + 1 - 2j) \quad \text{(see [3, p.177]).}$$

The Euler operator $\sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i}$ gives the degree of $f$. \(\blacksquare\)

Thus a necessary condition for a homogeneous polynomial $f$ of isotype $\tau$ to be singular is that $\kappa = -\frac{\text{deg } f}{\mu(\tau)}$. The isotype for span $\{w \omega_{n,0} : w \in S_N\}$ when $\kappa = -\frac{N}{2}$ is $(N - 1, 1)$ and $\mu((N - 1, 1)) = N$. For the values $\kappa = -\frac{N}{2}$ with $2\kappa \notin \mathbb{N}$ the isotype of span $\{w \omega_{nn} : w \in S_N\}$ is $(N - 2, 2)$ and $\mu((N - 2, 2)) = 2N - 2$. It is exactly the filling of the gap at $\gcd(N - 1, n) = \frac{N-1}{2}$ for $N$ being odd that we consider in the sequel. Before we leave this section we point out that the singular polynomials described so far do not depend on $N$ (with the exception just noted), that is, the singularity property holds for all $N$. This no longer holds once we consider isotypes corresponding to partitions with more than two parts.

## 5 Singular polynomials for half-integer parameter values

In this section we show that the polynomials $\omega_{(2l+1)(m+1),(2l+1)m}$ are singular for $N = 2m + 1$ and $\kappa = -l - \frac{1}{2}$ for $l \in \mathbb{N}_0$ and $m \in \mathbb{N}$. By Theorem [3] we already know $D_l \omega_{(2l+1)(m+1),(2l+1)m} = 0$ for $i \geq 2$. Thus we need to show that $\omega_{(2l+1)(m+1)-1,(2l+1)m} = 0$ for these choices of $N, \kappa$. This will be done by introducing a new basis of polynomials related to the $\{p_{mn}\}$ basis by a linear relation involving Krawtchouk polynomials.

**Definition 12** The homogeneous polynomials $q_{mn}$ (for $m, n \in \mathbb{N}_0$) are defined by

$$\sum_{m, n=0}^{\infty} q_{mn} (x) u^m v^n = \sum_{i,j=0}^{\infty} p_{ij} (x) (u + v)^i (u - v)^j,$$

the generating function converges for $|u|, |v| < (\max_i |x_i|)/2$. 

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We state the basic properties of the symmetric Krawtchouk polynomials (see Szegő, [8, p.36]). They are orthogonal for the binomial distribution with parameter \( \frac{1}{2} \). Fix \( n \in \mathbb{N} \) then the Krawtchouk polynomial of degree \( m \) (parameters \( n, \frac{1}{2} \)) with \( 0 \leq m \leq n \) is given by

\[
K_m(t; n) = \frac{1}{(m)!} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} \binom{t-j}{n-j}.
\]

Then the following hold for \( 0 \leq m, l \leq n \):

1. \( K_m(0; n) = 1 \), normalization;
2. \( (1-s)^t (1+s)^{n-t} = \sum_{m=0}^{n} s^m \binom{n}{m} K_m(l; n) \), generating function;
3. \( 2^{-n} \sum_{l=0}^{n} \binom{n}{l} K_m(t; n) K_l(t; n) = \delta_{ml} \binom{n}{m}^{-1} \), orthogonality;
4. \( K_m(l; n) = \frac{\Gamma(n)}{\Gamma(l+1)} \binom{n}{l} \), hypergeometric polynomial;
5. \( K_m(l; n) = K_l(m; n) \), symmetry;
6. \( K_m(n-t; n) = (-1)^m K_m(t; n) \), parity.

Any expansion in \( \{ p_{n-i,i} : 1 \leq i \leq n \} \) can be transformed to one in \( \{ q_{n-i,i} : 1 \leq i \leq n \} \) \( (n \in \mathbb{N}) \) by means of Krawtchouk polynomials.

**Lemma 13** Suppose \( n \in \mathbb{N} \) and \( f = \sum_{i=0}^{n} c_i p_{n-i,i} \) with coefficients \( c_i \in \mathbb{Q}(\kappa) \), then

\[
f = \frac{1}{2^n} \sum_{i=0}^{n} q_{n-i,i} \sum_{j=0}^{n} \binom{n}{j} c_j K_j(j; n).
\]

**Proof.** In Definition 12 replace \( u,v \) by \( \frac{s+t}{2}, \frac{s-t}{2} \) respectively then \( p_{n-j,i} \) equals the coefficient of \( s^{n-j} t^i \) in \( 2^{-n} \sum_{l=0}^{n} q_{n-i,i} (s+t)^n (s-t)^l = 2^{-n} \sum_{i=0}^{n} q_{n-i,i} \sum_{j=0}^{n} \binom{n}{j} K_j(i; n) s^{n-j} t^i \). The lemma now follows from the symmetry relation.

We use the lemma to show that for special values of \( n, i, \kappa \) the coefficients of \( \omega_{n-i,i} \) with respect to \( \{ q_{n-j,i} : 0 \leq j \leq n \} \) have a vanishing property: \( \omega_{n-i,i} = \sum_{j=0}^{n} c_j q_{n-j,i} \) and \( c_j = 0 \) for \( j > n - 2i \). We will also find a similar result for \( \omega_{n-i,i} + \omega_{i,n-i} \).

**Proposition 14** Suppose \( n \) is even, \( 1 \leq i \leq \frac{n}{2} \) and \( \kappa = -\frac{1}{2} (n - 2i + 1) \), then

\[
\omega_{n-i,i} = \sum_{j=0}^{n-2i} c_j q_{n-j,i} \text{ with coefficients } c_j \in \mathbb{Q}.
\]

**Proof.** Substitute \( \kappa = -\frac{1}{2} (n - 2i + 1) \) in Definition 12 for \( \omega_{n-i,i} \) to obtain

\[
\omega_{n-i,i} = p_{n-i,i} + \sum_{l=1}^{i} \frac{(n-2i+1)_l}{l!} p_{n-i+l,i-l} + \frac{(n-2i+1)_{l-1}}{(l-1)!} p_{l-i,n-i+l}.
\]
Extract the coefficients of $\omega_{n-i,i}$ with respect to $p_{n-j,j}$ as follows: for $0 \leq j \leq i-1$ replace $l$ by $i-j$ in the first part of the sum, the value is

$$\frac{(n-2i+1)_{i-j}}{(i-j)!} = \frac{(n-2i+i-j)!}{(n-2i)! (i-j)!} = \frac{(i-j+1)_{n-2i}}{(n-2i)!}.$$ 

This is also valid for $j = i$; the coefficient is zero for $i+1 \leq j \leq n-i$; for $n-i < j \leq n$ replace $l$ by $j-n + i$ then

$$\frac{(n-2i+1)_{j-1}}{(j-1)!} = \frac{(j-n+i)_{n-2i}}{(n-2i)!} = (-1)^n \frac{(i-j+1)_{n-2i}}{(n-2i)!},$$

(reversal of the Pochhammer symbol) but $n$ is even, so the value is $\frac{(i-j+1)(n-2i)}{(n-2i)!}$. As before, $\omega_{n-i,i} + \omega_{i,n-i} = \sum_{j=0}^{n-2i-1} c_j q_{n-j,j}$ with coefficients $c_j \in \mathbb{Q}$.

**Proposition 15** Suppose $n$ is odd, $0 \leq i < \frac{n}{2}$ and $\kappa = -\frac{1}{2} (n-2i)$, then $\omega_{n-i,i} + \omega_{i,n-i} = \sum_{j=0}^{n-2i-1} c_j q_{n-j,j}$ with coefficients $c_j \in \mathbb{Q}$.

**Proof.** It follows from Definition 2 that

$$\omega_{n-i,i} + \omega_{i,n-i} = \sum_{l=0}^{i} \frac{(-\kappa) (n-2i)_l}{(\kappa + n - 2i + 1)_l} \frac{n-2i + 2l}{n-2i} (p_{n-i+l,l-i} + p_{l-i,n-i+l}).$$

When $\kappa = -\frac{1}{2} (n-2i)$ we obtain $\frac{(-\kappa) (n-2i)_l}{(\kappa + n - 2i + 1)_l} = \frac{n-2i}{n-2i+2l}$. As before, $\frac{(n-2i)}{n} = \frac{(l+1)(n-2i-1)}{(n-2i-1)!}$, and replace $l$ by $i-j$ and $j-n+i$ respectively for the ranges $0 \leq j \leq i$ and $n-i \leq j \leq n$ respectively. Also $(j-n+i+1)_{n-2i-1} = (-1)^{n-1} (i-j+1)_{n-2i-1}$ and $n$ is odd. The polynomial $(i-j+1)_{n-2i-1}$ in $j$ vanishes at $i+1, \ldots, n-i-1$. Similarly to the previous proposition we find that

$$\omega_{n-i,i} + \omega_{i,n-i} = \frac{1}{2n} \sum_{l=0}^{i} \sum_{j=0}^{n-l} q_{n-l,t} \frac{(n-j+1)_{n-2i-1} K_l(j;n)}{(n-2i-1)!},$$

and the coefficient of $q_{n-l,t}$ vanishes for $l > n-2i-1$. Because $(i-j+1)_{n-2i-1} = (i-(n-j)+1)_{n-2i-1}$ the coefficients of $q_{n-l,t}$ also vanish when $l$ is odd.

We finish the construction of singular polynomials by showing for certain values of $\kappa, N, n, i$ that the polynomials $q_{n-l,t}$ vanish for $0 \leq l \leq n-2i$. First we consider some partial products in the generating function.
Definition 16 The power series \( A_n(u; \kappa), B_n(u; \kappa) \) (arbitrary \( \kappa \in \mathbb{C}, \ |u|, |v| < \frac{1}{2} \) and \( n \in \mathbb{N}_0 \)) are given by

\[
(1 - (u + v))^{-\kappa} (1 - (u - v))^{-\kappa} = \sum_{n=0}^{\infty} A_n(u; \kappa) v^n,
\]

\[
(1 - (u + v))^{-\kappa - 1} (1 - (u - v))^{-\kappa} = \sum_{n=0}^{\infty} B_n(u; \kappa) v^n.
\]

When \( \kappa = -l - \frac{1}{2}, l \in \mathbb{N}_0 \) some of the series \( A_n, B_n \) are actually polynomials.

Lemma 17 For \( n \leq 2l \), the functions \( A_n(u; -l - \frac{1}{2}), B_n(u; -l - \frac{1}{2}) \) are polynomials in \( u \) of degree \( 2l + 1 - n, 2l - n \) respectively.

Proof. For the first part

\[
\sum_{n=0}^{\infty} A_n(u; \kappa) v^n = (1 - u)^{-2\kappa} \left( 1 - \left( \frac{v}{1-u} \right)^2 \right)^{-\kappa} = \sum_{j=0}^{\infty} \frac{(\kappa)_j}{j!} v^j (1 - u)^{-2\kappa - 2j}.
\]

Thus \( A_n = 0 \) if \( n \) is odd and \( A_{2j}(u; -l - \frac{1}{2}) = \frac{(-l+\frac{1}{2})_j}{j!} (1 - u)^{2l+1-2j} \), which is a polynomial of degree \( 2l + 1 - 2j \) provided \( 2j \leq 2l \). For the second part

\[
\sum_{n=0}^{\infty} B_n(u; \kappa) v^n = (1 - u)^{-2\kappa - 1} \left( 1 + \frac{v}{1-u} \right) \left( 1 - \left( \frac{v}{1-u} \right)^2 \right)^{-\kappa - 1} = \sum_{j=0}^{\infty} \frac{(\kappa + 1)_j}{j!} \left( v^j (1 - u)^{-2\kappa - 2j} + v^{j+1} (1 - u)^{-2\kappa - 2j - 2} \right).
\]

Thus \( B_n(u; -l - \frac{1}{2}) = \frac{(-l+\frac{1}{2})_j}{j!} (1 - u)^{2l-n} \) (with \( j = \lfloor \frac{n}{2} \rfloor \)), a polynomial of degree \( 2l - n \) provided \( n \leq 2l \).

To be precise the polynomials \( A_n(u; -l - \frac{1}{2}) \) are of degree \( \leq 2l + 1 - n \) (being 0 when \( n \) is odd).

Proposition 18 Let \( \kappa = -l - \frac{1}{2}, l \in \mathbb{N}_0 \) and suppose that \( n \leq 2l \) then \( m + n \geq N(2l + 1) - 1 \) implies \( q_{mn} = 0 \).

Proof. By combining the generating function \( \mathcal{H} \) for \( \{p_{mn}\} \) and Definition 11 we obtain

\[
\sum_{m,n=0}^{\infty} q_{mn} u^m v^n = \sum_{\alpha \in \mathbb{N}_0^N} B_{\alpha_1}(u x_1; \kappa) (-1)^{\alpha_2} B_{\alpha_2}(u x_2; \kappa) \times \prod_{s=3}^{N} A_{\alpha_s}(u x_s; \kappa) x^{\alpha} v^{|\alpha|}.
\]
For a fixed $n \leq 2l$ the coefficient of $v^n$ is the sum over $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = n$. For each $\alpha$ with $|\alpha| = n$ (implying that each $\alpha_i \leq 2l$ and the Lemma applies) the corresponding term is a product of polynomials in $u$ of degree $(2l - \alpha_1) + (2l - \alpha_2) + \sum_{s=3}^{N} (2l + 1 - \alpha_s) = N(2l + 1) - 2 - n$. This shows that the coefficient of $u^n v^n$ vanishes if $m \geq N(2l + 1) - 1 - n$. 

**Theorem 19** Let $\kappa = -l - \frac{1}{2}$ and $N = 2m + 1$, with $l \in \mathbb{N}_0, m \in \mathbb{N}$ then $\omega_{(2l+1)(m+1),(2l+1)m}$ is a singular polynomial, of isotype $(N - 2, 1, 1)$, and $(1 + (1, 2)) \omega_{(2l+1)(m+1),(2l+1)m} = 0$.

**Proof.** Let $a = (2l + 1)(m + 1), b = (2l + 1)m$. Since $-2\kappa = 2l + 1 < a - b + 1$ Corollary [5] shows that $\omega_{ab} \neq 0$. By Theorem [5] $D_2 \omega_{ab} = (2m\kappa + b) \left( \omega_{a,b-1} - \frac{\kappa}{\kappa + a - b - 1} \omega_{b-1,a} \right) = 0$ for $\kappa = -\frac{b}{2m} = -l - \frac{1}{2}$. Similarly $D_1 \omega_{ab} = (N\kappa + a) \omega_{a-1,b}$. By Proposition [14] $\omega_{a-1,b} = \sum_{j=0}^{a-1-b} c_j q_{a+b-1-j,j}$ (since $-\frac{1}{2} (a - b) = \kappa$) with some coefficients $c_j \in \mathbb{Q}$. Since $a - 1 - b = 2l$ and $a + b - 1 = N(2l + 1) - 1$, Proposition [15] shows that each $q_{a+b-1-j,j} = 0$ (for $j \leq 2l$).

Let $M = \text{span} \{ w \omega_{(2l+1)(m+1),(2l+1)m} : w \in S_N \}$. By the invariance properties of $\{ D_i : 1 \leq i \leq N \}$ any nonzero element of $M$ is also singular. In general for $m > n$ the $S_N$-module span $\{ w \omega_{mn} : w \in S_N \}$, which realizes the representation of $S_N$ induced up from the trivial representation of $S_1 \times S_1 \times S_{N-2}$, decomposes into the isotypes $(N - 2, 1, 1), (N - 2, 2), (N - 1, 1), (N)$ (see [7], p.115)). The eigenvalues $\mu (\tau)$ are $2N, 2N - 2, N, 0$ respectively (see Proposition [11]); but the singularity condition implies $N(2l + 1) + k\mu (\tau) = 0$ thus the latter three can not contain singular polynomials for $\kappa = -l - \frac{1}{2}$ (note the degree of the polynomial $\omega_{ab}$ is $N(2l + 1)$). This implies that $M$ is of isotype $(N - 2, 1, 1)$ and hence is of dimension $\binom{N-1}{2}$.

By Proposition [15] $\omega_{ab} + \omega_{ba} = \sum_{j=0}^{a-b-1} c_j q_{a+b-1-j,j}$ (since $-\frac{1}{2} (a - b) = \kappa$ and $a + b$ is odd) with $c_j \in \mathbb{Q}$. Since $a - b - 1 = 2l$ and $a + b = N(2l + 1)$, Proposition [15] shows that each $q_{a+b-1-j,j} = 0$. It is interesting that the parameters of the singular polynomials just barely satisfy the various inequalities appearing in the preparatory results.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA
CHARLOTTESVILLE, VA 22904-4137, U.S.
cfd5z@virginia.edu