Lepton Masses in a Minimal Model with Triplet Higgs Bosons and $S_3$ Flavor Symmetry

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Abstract

Viable neutrino and charged lepton masses and mixings are obtained by imposing a $S_3 \times Z_4 \times Z_3$ flavor symmetry in a model with a few additional Higgs. We use two $SU(2)_L$ triplet Higgs which are arranged as a doublet of $S_3$, and standard model singlet Higgs which are also put as doublets of $S_3$. We break the $S_3$ symmetry in this minimal model by giving vacuum expectation values (VEV) to the additional Higgs fields. Dictated by the minimum condition for the scalar potential, we obtain certain VEV alignments which allow us to maintain $\mu - \tau$ symmetry in the neutrino sector, while breaking it maximally for the charged leptons. This helps us to simultaneously explain the hierarchical charged lepton masses, and the neutrino masses and mixings. In particular, we obtain maximal $\theta_{23}$ and zero $\theta_{13}$. We allow for a mild breaking of the $\mu - \tau$ symmetry for the neutrinos and study the phenomenology. We give predictions for $\theta_{13}$ and the CP violating Jarlskog invariant $J_{CP}$, as a function of the $\mu - \tau$ symmetry breaking parameter. We also discuss possible collider signatures and phenomenology associated with lepton flavor violating processes.

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1 Introduction

Proof of neutrino masses and mixing from a series of outstanding experimental efforts, spanning many decades and using neutrinos from myriad types of natural as well as man-made sources, have opened a window to physics beyond the standard model of particle physics. Though there still remains a lot to be learnt about neutrino properties, a lot is already known \[1\]. The two mass squared differences \(\Delta m_{21}^2\) and \(\Delta m_{31}^2\), and the two mixing angles \(\theta_{12}\) and \(\theta_{23}\) are now fairly well determined. The third mixing angle \(\theta_{13}\), though still not determined, is known to be small. How small, is the question which will be answered in the next generation oscillation experiments. The current 3\(\sigma\) allowed ranges of the oscillation parameters are given as \[2\]

\[
7.1 \times 10^{-5} eV^2 < \Delta m_{21}^2 < 8.3 \times 10^{-5} eV^2, \quad 2.0 \times 10^{-3} eV^2 < \Delta m_{31}^2 < 2.8 \times 10^{-3} eV^2, \quad (1)
\]

\[
0.26 < \sin^2 \theta_{12} < 0.42, \quad \sin^2 2\theta_{23} > 0.9, \quad \sin^2 \theta_{13} < 0.05. \quad (2)
\]

The best-fit of the global neutrino oscillation data corresponds to maximal \(\theta_{23}\) and zero \(\theta_{13}\). This has prompted the speculation that a \(\mu - \tau\) permutation symmetry \[3\] might exist in the neutrino sector.\(^2\) This symmetry should however be broken for the charged leptons, for which we know that a hierarchy exists between the masses of the \(\mu\) and \(\tau\). It cannot exist in the quark sector either, where the hierarchy between the masses is larger and mixing angles are known to be very small. In addition, the fact that the best-fit solar mixing angle is close to \(\sin^2 \theta_{12} = 1/3\) suggests that the lepton mixing matrix has the tribimaximal (TBM) mixing form \[5\]. Challenge for model builders lies in constructing an aesthetically simple and phenomenologically viable model, which could explain all aspects of the fermion masses and mixing. One way of generating the observed pattern of fermion masses is to impose certain flavor symmetries. The \(A_4\) group has received a lot of attention recently \[6-8\] as a way of generating TBM mixing for the neutrinos. However, there are some drawbacks of these models. The basic version of some of these models might need some fine tuning \[8\]. They are also not able to give a consistent explanation of the quark mass hierarchies and the CKM mixing. Various extensions of the models with \(A_4\) family symmetry have been proposed \[9\] in order to address these issues. However, most of these require additional discrete symmetries and many more scalar particles.

Another discrete group which has been extensively discussed in the literature as a family symmetry group is the \(S_3\) permutation group \[10-16\]. Many of these models predict TBM mixing. However, the main challenge still is to predict TBM mixing for neutrinos and at the same time reproduce an almost diagonal charged lepton mass matrix with the correct mass hierarchies. The quark sector also needs to be explained. Most models considered so far use right-handed neutrinos and type I seesaw for generating the neutrino masses. Only in \[13\] the authors consider a model with triplet Higgs, and hence employ a type I+II seesaw mechanism to generate the correct neutrino masses and mixing. We will present a model

\(^1\) We define \(m_{ij}^2 = m_i^2 - m_j^2\).

\(^2\) A \(L_\mu - L_\tau\) family symmetry could also very naturally give maximal \(\theta_{23}\) and zero \(\theta_{13}\) \[4\].

\[1\] We define \(m_{ij}^2 = m_i^2 - m_j^2\).

\[2\] A \(L_\mu - L_\tau\) family symmetry could also very naturally give maximal \(\theta_{23}\) and zero \(\theta_{13}\) \[4\].
which does not have any right-handed neutrinos and uses $SU(2)_L$ triplet Higgs to generate Majorana neutrino masses. This has never been considered before. We show how small neutrino masses can be explained in this model without invoking the seesaw mechanism. We also show how the neutrino mixing can be explained naturally by imposing the discrete flavor symmetry $S_3$.

The $S_3$ group has the $S_2$ permutation group as its subgroup. If one identifies this subgroup with the $\mu - \tau$ exchange symmetry, then it is straightforward to get vanishing $\theta_{13}$ and maximal $\theta_{23}$ for the neutrinos. However, since the same group acts on the charged leptons as well, this would lead to $\mu$ and $\tau$ masses of the same order. In addition this would lead to a highly non-diagonal mass matrix for the charged leptons, which is undesirable in this case. Therefore, the $S_3$ group should be broken in such a way that $\mu - \tau$ permutation symmetry remains intact for the neutrinos but gets badly broken for the charged leptons.

In a recent paper [16], the authors have used this idea to generate a viable scenario which explains almost all aspects of the lepton masses within a framework of a $S_3 \times Z_3$ family symmetry. They also extend the model with additional $Z_3$ symmetries and Higgses to explain also the mass and mixing pattern of the quarks. In order to get the desired mass matrices, it is mandatory to have certain alignment for the vacuum expectation values (VEV) of the extra standard model singlet Higgs particles, which transform non-trivially under $S_3$. The authors of [16] use a supersymmetric version of their model with a few extra driving fields to explain the required VEV alignments.

In this paper, we propose a model with $S_3 \times Z_4 \times Z_3$ family symmetry. The additional $Z_3$ symmetry is required for obtaining the correct form of the charged lepton mass matrix. As in [16], we preserve the $\mu - \tau$ symmetry in the neutrino sector while breaking it almost maximally for the charged leptons. Note that the field content and hence the mass generation mechanism of our model is completely different. In particular, we introduce two $SU(2)_L$ triplet Higgs in our model for generating the neutrino masses. The charged lepton masses are generated by the standard Higgs doublet. We postulate two additional $S_3$ doublets of Higgs which are $SU(2)_L \times U(1)_Y$ singlets, to generate the desired lepton mass matrices. The $S_3$ group is broken spontaneously when the singlet Higgs acquire VEVs. The VEVs are aligned in such a way that the residual $\mu - \tau$ symmetry is intact for the neutrinos but broken maximally for the charged leptons. We justify the VEV alignment by explicitly minimizing our scalar potential. We do not need to impose supersymmetry. We show that under the most general case, the minimization condition of our scalar potential predicts a very mild breaking of the $\mu - \tau$ symmetry for the neutrinos. We study the phenomenological viability and testability of our model both in the exact as well as approximate $\mu - \tau$ symmetric cases. We give predictions for the mass squared differences, mixing angles, absolute neutrino mass scale, beta decay and neutrino-less double beta decay. It should be possible to extend our model to reproduce the quark mass hierarchy and CKM mixing such that the complete model is anomaly free [17].

The paper is organized as follows: We give an overview of the $S_3$ group in the appendix. In section 2 we introduce the particle content of our model and write down the

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3Work in progress.
Table 1: Transformation properties of matter and flavon fields under the flavor groups.

| Field | $H$ | $l_1$ | $D_l$ | $\tau_R$ | $\mu_R$ | $\nu_R$ | $\Delta$ | $\phi_e$ | $\xi$ |
|-------|-----|-------|-------|---------|---------|---------|---------|---------|-------|
| $S_3$ | 1   | 1     | 2     | 1       | 1       | 1       | 2       | 2       | 2     |
| $Z_4$ | $i$ | $-1$  | 1     | 1       | $-i$    | 1       | $-1$    | $i$     | $-1$  |
| $Z_3$ | 1   | 1     | 1     | $\omega$| $\omega^2$| 1       | $\omega$| 1       |       |

mass matrices for the neutrinos and charged leptons. In section 3 we present the phenomenological implications of our model in the exact and approximate $\mu - \tau$ symmetric case. We discuss in detail the possible collider phenomenology and lepton flavor violating channels which could be used to provide smoking gun evidence for our model. Section 4 is devoted to justifying the alignment needed for the Higgs VEVs. We end in section 5 with our conclusions.

2 The Model

We present in Table 1 the particle content of our model and their transformation properties under the discrete groups $S_3, Z_4$ and $Z_3$. The Higgs $H$ is the usual $SU(2)_L$ doublet,

$$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix},$$

which transforms as singlet under $S_3$. The Higgs $\Delta_1$ and $\Delta_2$ are $SU(2)_L$ triplets,

$$\Delta_i = \begin{pmatrix} \Delta_i^+ / \sqrt{2} \\ \Delta_i^0 \\ -\Delta_i^0 / \sqrt{2} \end{pmatrix},$$

which transform as a doublet

$$\Delta = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix},$$

under $S_3$. We introduce two additional $S_3$ scalar doublets $\phi_e$ and $\xi$,

$$\phi_e = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

which are singlets under $SU(2)_L \times U_Y(1)$ and are hence our new flavon fields. The $SU(2)_L \times U_Y(1)$ lepton doublets are distributed in the $S_3$ multiplets as follows:

$$D_l = \begin{pmatrix} l_2 \\ l_3 \end{pmatrix},$$

transforms as a doublet under $S_3$, where $l_2 = (\nu_{\mu L}, \mu_L)^T$ and $l_3 = (\nu_{\tau L}, \tau_L)^T$, while

$$l_1 = \begin{pmatrix} \nu_{e L} \\ e_L \end{pmatrix},$$
transforms as a singlet. The right-handed fields $e_R$, $\mu_R$ and $\tau_R$ transform as 1 under $S_3$. The corresponding charges of the particles under $Z_4$ and $Z_3$ has been summarized in Table 1.

### 2.1 Neutrino Masses and Mixing

Given the field content of our model and their charge assignments presented in Table 1, the most general $S_3 \times Z_4 \times Z_3$ invariant Yukawa part of the Lagrangian (leading order) giving the neutrino mass can be written as

$$ - L^\nu_y = \frac{y_2}{\Lambda} (D_l D_l) (\xi \Delta) + \frac{y_1}{\Lambda} (D_l D_l) (\xi \Delta)^2 + 2 y_3 l_1 D_l \Delta + \frac{y_4}{\Lambda} l_1 l_1 \xi \Delta + h.c. + ... $$

where $\Lambda$ is the cut-off scale of the theory and the underline sign in the superscript represents the particular $S_3$ representation from the tensor product of the two $S_3$ doublets. Since $(D_l D_l)$ and $\xi \Delta$ are $2 \times 2$ products which could give either 1 or 2, and since we can obtain 1 either by $1 \times 1$ or $2 \times 2$, we have two terms coming from $(D_l D_l) (\xi \Delta)$. The $(D_l D_l) (\xi \Delta)$ as $1' \times 1'$ term does not contribute to the neutrino mass matrix. Note that the presence of the $Z_4$ symmetry prevents the appearance of the usual 5 dimensional $D_l D_l HH$ and $l_1 l_1 HH$ Majorana mass term for the neutrinos. In fact, the neutrino mass matrix is completely independent of $H$ due to the $Z_4$ symmetry. In addition, there are no Yukawa couplings involving the neutrinos and the flavon $\phi_e$ due to $Z_4$ or/and $Z_3$ symmetry. The $S_3$ symmetry is broken spontaneously when the flavon $\xi$ acquires a vacuum expectation value (VEV):

$$ \langle \xi \rangle = \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right). $$

Finally, the $SU(2)_L \times U_Y(1)$ breaks at the electroweak scale giving VEVs to the triplets as

$$ \langle \Delta \rangle = \left( \begin{array}{c} \langle \Delta_1 \rangle \\ \langle \Delta_2 \rangle \end{array} \right), \quad \text{where} \quad \langle \Delta_i \rangle = \left( \begin{array}{cc} 0 & 0 \\ v_i & 0 \end{array} \right), $$

neutrinos get massive and their mass matrix is given as

$$ m_\nu = \begin{pmatrix} 2 y_4 \frac{w}{\Lambda} & 2 y_3 v_2 & 2 y_3 v_1 \\ 2 y_3 v_2 & 2 y_1 \frac{u_1 v_1}{\Lambda} & 2 y_2 \frac{w}{\Lambda} \\ 2 y_3 v_1 & 2 y_2 \frac{w}{\Lambda} & 2 y_1 \frac{u_1 v_1}{\Lambda} \end{pmatrix}, $$

where $w = u_1 v_2 + u_2 v_1$. For the VEV alignments

$$ v_1 = v_2, \quad \text{and} \quad u_1 = u_2, $$

the neutrino mass matrix reduces to the form

$$ m_\nu = \begin{pmatrix} 2 y_4 \frac{2 u_1 v_1}{\Lambda} & 2 y_3 v_1 & 2 y_3 v_1 \\ 2 y_3 v_1 & 2 y_1 \frac{u_1 v_1}{\Lambda} & 2 y_2 \frac{2 u_1 v_1}{\Lambda} \\ 2 y_3 v_1 & 2 y_2 \frac{2 u_1 v_1}{\Lambda} & 2 y_1 \frac{u_1 v_1}{\Lambda} \end{pmatrix}. $$

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4The term $(l_i D_l \Delta)$ denotes $(l_i^T C \tau_2 D_l \Delta)$, where $C$ is the charge conjugation operator.
We will motivate our choice of the VEV alignments in section 4 where we will show that one can expect this from the minimization condition of the scalar potential. Denoting $\frac{u_1}{\Lambda}$ as $u_1'$ the mass matrix becomes

$$m_\nu = 2v_1 \begin{pmatrix} 2y_4 u_1' & y_3 & y_3 \\ y_3 & y_1 u_1' & 2y_2 u_1' \\ y_3 & 2y_2 u_1' & y_1 u_1' \end{pmatrix},$$

(15)

where $u_1' = \frac{u_1}{\Lambda}$ and it is less than 1. Redefining $2y_4 u_1'$ as $y_4$, $y_1 u_1'$ as $y_1$ and $2y_2 u_1'$ as $y_2$, the final form of the mass matrix is

$$m_\nu = 2v_1 \begin{pmatrix} y_4 & y_3 & y_3 \\ y_3 & y_1 & y_2 \\ y_3 & y_2 & y_1 \end{pmatrix}. $$

(16)

Note that if the VEV $u_1 = 0$, we would obtain the matrix

$$m_\nu = 2v_1 y_3 \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. $$

(17)

This is a very well known form of the neutrino mass matrix. It returns inverted neutrino mass spectrum with eigenvalues $\{-2\sqrt{2}v_1 y_3, 2\sqrt{2}v_1 y_3, 0\}$, and bimaximal mixing with $\theta_{23} = \theta_{12} = \pi/4$ and $\theta_{13} = 0$. The only family symmetry considered in the literature for obtaining the form of the mass matrix given by Eq. (17) is $L_e - L_\mu - L_\tau$ [18]. We have here obtained this form of $m_\nu$ from a completely new kind of flavor symmetry. Of course exact bimaximal mixing is ruled out by the solar neutrino and KamLAND data [2]. Besides, as one can see from the eigenvalues of this neutrino mass matrix, that $\Delta m^2_{21} = 0$. This is untenable in the light of the experimental data. In order to generate the correct $\Delta m^2_{21}$ and deviation of $\theta_{12}$ from maximal that is consistent with the data, one has to suitably perturb $m_\nu$ (for instance, see [19] as one example of such a model in the framework of the Zee-Wolfenstein ansatz). In the $S_3$ model that we consider here, this is very naturally obtained if we allow non-zero VEV for $\xi$. The strength of the additional terms is linearly proportional to $u_1/\Lambda$ and could be naturally small.

In what follows, we will consider all values of $u_1/\Lambda$ from very small to $\sim 1$. The eigenvalues of the most general matrix given by Eq. (16) are

$$m_i = v_1 \left(y_1 + y_2 + y_4 - \sqrt{y_1^2 + y_2^2 + y_4^2 + 8y_3^2 + 2y_1 y_2 - 2y_1 y_4 - 2y_2 y_4}\right),$$

(18)

$$m_j = v_1 \left(y_1 + y_2 + y_4 + \sqrt{y_1^2 + y_2^2 + y_4^2 + 8y_3^2 + 2y_1 y_2 - 2y_1 y_4 - 2y_2 y_4}\right),$$

(19)

$$m_3 = 2v_1 \left(y_1 - y_2\right).$$

(20)

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5For all analytical results given in this section we have assumed the model parameters to be real for simplicity. We check the phenomenological viability and testability of our model in the next section for complex Yukawas and VEVs.
Note that the only difference between \( m_i \) and \( m_j \) comes in the sign of the quantity within square root. We know that the solar neutrino data provides evidence for \( \Delta m^2_{21} > 0 \) at more than 6\( \sigma \) [2]. Therefore, the choice of \( m_1 \) and \( m_2 \) in Eqs. (18) and (19) is determined by the condition \( m_2 > m_1 \) viz., the larger eigenvalue corresponds to \( m_2 \). The eigenvectors are given as

\[
U_i = \begin{pmatrix} \frac{y_1 + y_2 - y_4 + \sqrt{a}}{2y_3} \\ \frac{1}{b} \\ \frac{1}{c} \end{pmatrix}, \quad U_j = \begin{pmatrix} \frac{y_1 + y_2 - y_4 - \sqrt{a}}{2y_3} \\ \frac{1}{b} \\ \frac{1}{c} \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix},
\]

where \( U_i \) corresponds to the eigenvalue given in Eq. (18) and \( U_j \) to that in Eq. (19). Whether \( U_1 \equiv U_i \) or \( U_j \) depends on whether \( m_i \) is smaller or larger than \( m_j \). The quantities \( b \) and \( c \) are the normalization constants given by

\[
b^2 = 2 + \frac{(y_1 + y_2 - y_4 + \sqrt{a})^2}{(2y_3)^2},
\]

and

\[
c^2 = 2 + \frac{(y_1 + y_2 - y_4 - \sqrt{a})^2}{(2y_3)^2},
\]

and \( a \) is given as

\[
a = y_1^2 + y_2^2 + y_4^2 + 2y_1y_2 + 2y_1y_4 - 2y_2y_4.
\]

From Eqs. (18), (19) and (20) we obtain

\[
\Delta m^2_{21} = 4v_1^2(y_1 + y_2 + y_4)\sqrt{a},
\]

\[
\Delta m^2_{31} = v_1^2(3y_1 - y_2 + y_4 - \sqrt{a})(y_1 - 3y_2 - y_4 + \sqrt{a}).
\]

The mixing angles can be seen from Eq. (21) to be

\[
\theta'_{13} = 0, \\
\tan \theta'_{23} = 1, \\
\tan \theta'_ {12} = \frac{(y_1 + y_2 - y_4 - \sqrt{a})b}{(y_1 + y_2 - y_4 + \sqrt{a})c}.
\]

Note that neither the ratio of the two mass squared differences \( \Delta m^2_{21}/\Delta m^2_{31} \), nor the mixing angles depend on the value of the triplet VEV \( v_1 \). They only depend on the Yukawas couplings. Only the absolute mass square differences \( \Delta m^2_{21} \) and \( \Delta m^2_{31} \) individually depend on the triplet VEV. The effective neutrino mass predicted for neutrino-less double beta decay is given as

\[
|m_{\nu ee}| = |2v_1y_4|,
\]

while the effective mass squared observable in beta decay \( m^2_\beta \) and the total neutrino mass crucial for cosmology \( m_t \) are given as

\[
m^2_\beta = \sum_i |m_i|^2 |U_{ei}|^2, \quad \text{and} \quad m_t = \sum_i |m_i|,
\]

respectively.
2.2 Charged Lepton Masses and Mixing

The Yukawa Lagrangian up to order $1/\Lambda^3$ giving the charged lepton mass is

\[- \mathcal{L}_e^y = \frac{\gamma^y}{\Lambda} \tau R H^\dagger (D_t \phi_e) + \frac{\gamma^b}{\Lambda^3} \tau R H^\dagger (D_t \phi_e) \frac{1}{2} (\xi \xi)^2 + \frac{\gamma'}{\Lambda^2} \tau R H^\dagger (D_t \phi_e) (\phi_e \phi_e)^2 \]
\[+ \frac{\gamma''}{\Lambda^3} \tau R H^\dagger (D_t \phi_e) (\phi_e \phi_e)^2 + \frac{\gamma'}{\Lambda^2} \tau R H^\dagger l_1 (\phi_e \xi) \]
\[+ \frac{\beta'}{\Lambda} \bar{\mu} R H^\dagger (D_t \phi_e) + \frac{\beta''}{\Lambda^2} \bar{\mu} R H^\dagger l_1 (\phi_e \phi_e \xi) + \frac{\alpha''}{\Lambda^3} \bar{\nu} R H^\dagger l_1 (\phi_e \phi_e \phi_e) \]
\[+ \frac{\alpha'}{\Lambda} \bar{\nu} R H^\dagger (D_t \phi'_e) (\phi'_e \phi'_e)^2 + \frac{\alpha'}{\Lambda} \bar{\nu} R H^\dagger (D_t \phi'_e) (\phi'_e \phi'_e)^2 + h.c + \ldots \]

While $Z_4$ symmetry was sufficient to get the desired $m_\nu$, the extra $Z_3$ symmetry had to be introduced in order to obtain the correct form for the charged lepton mass matrix. We reiterate that the presence of the $Z_4$ symmetry ensures that the flavon doublet $\phi_e$ couples to charged leptons only. This is a prerequisite since we wish to break $S_3$ such that the $\mu - \tau$ symmetry remains intact for the neutrinos while it gets maximally broken for the charged leptons. For neutrinos the $\mu - \tau$ symmetry was kept intact by the choice of the vacuum alignments given in Eq. (13). To break it maximally for the charged leptons we choose the VEV alignment

\[\langle \phi_e \rangle = \begin{pmatrix} v_e \\ 0 \end{pmatrix} . \]

Once $S_3$ is spontaneously broken by the VEVs of the flavons and $SU(2)_L \times U(1)_Y$ by the VEVs of the standard model doublet Higgs, we obtain the charged lepton mass matrix (leading terms only)\(^\text{6}\)

\[m_l = \begin{pmatrix} \alpha'' \lambda^2 & 0 & 0 \\ \beta'' \lambda u'_1 & \beta' \lambda & 0 \\ \gamma'' u'_2 & \gamma' u'_2 & \gamma \end{pmatrix} v_{SM} \lambda , \]

where $v_{SM} = \langle H \rangle$ is the VEV of the standard Higgs, $\lambda = v_e / \Lambda$, $u'_1 = u_1 / \Lambda$ and $u'_2 = u_2 / \Lambda$. The charged lepton masses and mixing matrix are obtained from

\[m_{l_{\text{diag}}}^2 = U_1 m_l U_1^\dagger , \]

giving the masses as

\[m_\tau \simeq \gamma \lambda v_{SM}, \quad m_\mu \simeq \beta' \lambda^2 v_{SM}, \quad m_e \simeq \alpha'' \lambda^3 v_{SM} . \]

For $\lambda \simeq 2 \times 10^{-2} \simeq \frac{\lambda_c}{2}$ where $\lambda_c$ is the Cabibbo angle, the correct mass hierarchy of $\tau$ to $\mu$ to $e$ as well as their exact numerical values can be obtained by choosing $\gamma = 0.36$, $\beta' = 1.01$

\(^6\)While this form of charged lepton mass matrix has been obtained using $Z_4 \times Z_3$ symmetry, similar viable forms can be obtained using other $Z_n$ symmetries. For example, we have explicitly checked that $Z_6 \times Z_2$ and $Z_8 \times Z_2$ symmetries also give viable structure for $m_l$. 

8
and $\alpha'' = 0.25$ and $v_{SM} = 246$ GeV. Note that the masses of the charged leptons do not depend on the VEV of $\xi$, however the mixing angles involved depend on this parameter. For $u'_1 = u'_2 = u' \simeq 10^{-1}$ and $\gamma', \gamma''$ and $\beta''$ of the order unity, we get the charged lepton mixing angles as:

$$
\sin \theta_{12} \simeq \lambda^2, \quad \sin \theta_{23} \simeq 0.1\lambda, \quad \sin \theta_{13} \simeq 0.1\lambda^2.
$$

(34)

Since $U = U_l^\dagger U_\nu$, where $U$ is the observed lepton mixing matrix and $U_\nu$ is the matrix which diagonalizes $m_\nu$ given by Eq. (14), the contribution of charged lepton mixing matrix would be very tiny. For $\sin \theta_{13}$ the maximum contribution from the charged lepton is $\mathcal{O}(10^{-4})$. In any case, in what follows we show all results for $U = U_l^\dagger U_\nu$.

3 Phenomenology

3.1 Exact $\mu - \tau$ Symmetry Limit

We have already presented in Eqs. (25) and (33) the expressions for the neutrino mass squared differences and charged lepton masses, while in Eqs. (26) and (34) we have given the mixing angles in the lepton sector in terms of model parameters. We argued that for $\lambda \simeq 2 \times 10^{-2} \simeq \frac{\lambda^2}{2}$, we obtain the charged lepton mass hierarchy in the right ballpark.

Since neutrino masses are directly proportional to $v_1$, it is phenomenologically demanded that the magnitude of this VEV should be small. In fact, since $\Delta m^2_{31} \propto v^2_1$, we take $v^2_1 \sim 10^{-4} - 10^{-3}$ eV$^2$, and find that all experimentally observed neutrino masses and mixing constraints are satisfied. It is not unnatural to expect such a small value for $v^2_1$. For instance, in the most generic left-right symmetric models,

$$
v_1 \equiv v_L \sim v_{SM}^2/v_R,
$$

(35)

where $v_{SM}$ is the electroweak scale and $v_R$ is the VEV of the $SU(2)_R$ Higgs triplet. It is natural to take $v_R \sim 10^{13} - 10^{15}$ GeV for which we get $v_1 \sim 1 - 10^{-2}$ eV.

Allowing $v^2_1$ to take any random value between $10^{-4} - 10^{-3}$ eV$^2$ we have checked that the neutrino mass spectrum obtained is hierarchical. For larger values of $v_1$ of course one would get larger values for the absolute neutrino mass scale and for $v_1 \sim 1$ eV, we expect a quasi-degenerate neutrino mass spectrum. In all our plots we keep $v^2_1$ between $10^{-4} - 10^{-3}$, eV$^2$. In Figure [we show the prediction for $\sin^2 \theta_{12}$ as a function of the model parameters $y_i$’s. In each panel we show the dependence of $\sin^2 \theta_{12}$ on a given $y_i$, allowing all the others to vary randomly. Here we have assumed normal mass hierarchy for the neutrinos. For the charged lepton sector we have assumed a fixed set of model parameters which give viable charged lepton masses and we took $\lambda \simeq 2 \times 10^{-2}$. We note that for normal hierarchy ($\Delta m^2_{31} > 0$):

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<sup>7</sup>Our choice of $u'_1 = u'_2 \simeq \mathcal{O}(10^{-1})$ will be justified from large $y_1$ and $y_2$ in Fig[2] and large $y_4$ in Fig[3] which we will show in the next section.
Figure 1: Scatter plots showing the range of the solar mixing angle $\sin^2 \theta_{12}$ as a function of the model parameters in $m_\nu$ in the exact $\mu - \tau$ symmetry limit and for normal hierarchy. In each of the panels all other parameters except the one appearing in the $x$-axis are allowed to vary freely.

- $y_1 = 0$ and $y_2 = 0$ are not allowed.
- There is almost negligible dependence of $\sin^2 \theta_{12}$ on $y_1$ and $y_2$ for $|y_1| > 1$ and $|y_2| > 1$ respectively.
- The range of $\sin^2 \theta_{12}$ decreases with $|y_3|$ and $|y_4|$.

Figure 2 gives the scatter plots showing allowed ranges for the model parameters in two-dimensional parameter spaces, taking two parameters at a time and allowing the rest to vary freely. We have considered normal hierarchy in this figure. We note from the figure that for normal hierarchy:

- $y_1 = 0$ and $y_2 = 0$ are not allowed as we had seen before. With $y_1 = 0$ we would have obtained neutrino mass matrix Eq. (10) with two texture zeros in $\mu - \mu$ and $\tau - \tau$ elements and with $e - \mu$ and $e - \tau$ entries same in the mass matrix it will not be possible to get a normal-hierarchy [20]. This could also be explained from the explicit analytical form of the eigenvalues. However, as we will see from Figure 3.
inverted hierarchy can occur in this case. With \( y_2 = 0 \) one gets neutrino mass matrix with one texture zero in \( \mu - \tau \) element which is not viable for normal ordering [21]. The allowed values of \( y_1 \) for normal hierarchy are highly correlated with the allowed values of \( y_2 \) and they are necessarily of opposite signs. One obtains a rough linear dependence between the allowed values of \( y_1 \) and \( y_2 \).

- \( y_4 = 0 \) is allowed and there is very little correlation of allowed values of \( y_4 \) with \( y_1 \) and \( y_2 \). For \( y_4 = 0 \), one gets a neutrino mass matrix Eq.(16) with one texture zero in \( e - e \) element which can produce normal hierarchy only [21]. This predicts \( |m_{\nu_{ee}}| = 0 \).

- \( y_3 \) and \( y_4 \) are strongly correlated.

In Figure 3 we show the corresponding allowed ranges for the model parameters for inverted hierarchy. In each of the panels, the parameters that do not appear on the \( x \) and \( y \)-axes are allowed vary randomly. From a comparison of Figures 2 and 3 we can observe that the allowed areas in the parameter space is almost complementary.\(^8\) We find that for the inverted hierarchy:

\(^8\)Of course the same set of model parameter values would never give both normal and inverted hierarchy.
• $y_1 = 0$ and $y_2 = 0$ simultaneously are still not allowed, though now we can have $y_1 = 0$ or $y_2 = 0$ separately when the other parameter is within a certain favorable (non-zero) range. As for normal hierarchy, allowed values of $y_1$ and $y_2$ are highly correlated. As before there is a linear dependence between them.

• $y_3 = 0$ and $y_4 = 0$ are not allowed here.

In the left panel of Figure 4 we show the variation of the effective neutrino mass $m_{\nu_{ee}}$ with the model parameter $y_4$. The effective mass predicted for neutrino-less double beta decay in our model is $|m_{\nu_{ee}}| = |2v_1y_4|$. We have allowed $v_1$ to vary freely in the range $10^{-1} - 10^{-2}$. From Figure 4 one can clearly see that our model predicts $m_{\nu_{ee}} \lesssim 0.07$ eV. The next generation of neutrino-less double experiments are expected to probe down to $m_{\nu_{ee}} = 0.01 - 0.05$ eV [22]. The middle panel of this figure shows the total predicted neutrino mass $m_t$ and right panel shows $m_\beta^2$. We find that the total neutrino mass $m_t$ (in eV) varies within the range $0.05 < m_t < 0.28$, while the effective mass squared observable in beta decay $m_\beta^2 \simeq \mathcal{O}(10^{-4} - 10^{-2})$ eV$^2$. The KATRIN experiment will be sensitive to $m_\beta > 0.3$ eV [23].
3.2 Mildly Broken $\mu - \tau$ Symmetry Limit

So far we have assumed that the $S_3$ breaking in the neutrino sector is such that the residual $\mu - \tau$ symmetry is exact. This was motivated by the fact that $S_2$ is a subgroup of $S_3$ and we took a particular VEV alignment given in Eq. (13). We will try to justify this choice of VEV alignment from minimization of the scalar potential. In this subsection we will assume that the $\mu - \tau$ symmetry is mildly broken. This could come from explicit $\mu - \tau$ breaking terms in the Lagrangian. In the next section we will see that in our model this comes naturally after the minimization of the scalar potential due to the deviation of the VEV alignments from that given in Eq. (13). Small breaking of the VEV alignments could also come from radiative corrections and/or higher order terms in the scalar part of the Lagrangian. Any breaking of $\mu - \tau$ symmetry will allow $\theta_{23}$ to deviate from maximal and $\theta_{13}$ from zero. Any non-zero $\theta_{13}$ will open up the possibility of low energy CP violation in the lepton sector. For the sake of illustration we consider a particular $\mu - \tau$ symmetry breaking for $m_\nu$, which results from the deviation of the VEV alignment from Eq. (13). We will see in the next section that this deviation is small and could come from $v_1 \neq v_2$ and/or $u_1 \neq u_2$. For the sake of illustration we consider only the breaking due to $v_1 \neq v_2$. We will see that from the minimization of the scalar potential one can take $v_1 = v_2(1 + \epsilon)$. As a result the neutrino mass matrix (12) becomes

$$m_\nu = 2v_2 \begin{pmatrix} y_1u'(2 + \epsilon) & y_3 & y_3(1 + \epsilon) \\ y_3 & y_1u'(2 + \epsilon) & y_2u'(1 + \epsilon) \\ y_3(1 + \epsilon) & y_2u'(2 + \epsilon) & y_1u'(1 + \epsilon) \end{pmatrix}. \tag{36}$$

We show the values of $|U_{e3}| \equiv \sin \theta_{13}$ predicted by the above $m_\nu$ as a function of the symmetry breaking parameter $|\epsilon|$ in the right panel of Figure 5. The left panel panel of

![Figure 4: Scatter plots showing variation of $|m_{\nu_{ee}}|$, $m_\ell$ and $m_\beta^2$ with the model parameter $y_4$.](image-url)
Figure 5: The Jarlskog invariant $J_{CP}$ (left panel) and $\sin \theta_{13}$ as a function of the $\mu - \tau$ symmetry breaking parameter $|\epsilon|$.

The Jarlskog invariant $J_{CP}$ is given by

$$J_{CP} = \text{Im}\left\{U_{e1} U_{\mu2} U_{e2}^* U_{\mu1}^*\right\},$$

as a function of $|\epsilon|$. We note that the model predicts values of $\sin \theta_{13} \lesssim 10^{-1}$ and $J_{CP} \lesssim 10^{-2}$ with the exact value determined by the extent of symmetry breaking. This could give $\sin^2 2\theta_{13} \lesssim 0.04$, which is just within the sensitivity reach of the forthcoming reactor experiments like Double Chooz [24] and long baseline accelerator experiments like T2K [25] and NOνA [26]. These values of $\theta_{13}$ and $J_{CP}$ could give a large positive signal in the next generation high performance long baseline experiments using neutrino beams from Neutrino Factories, Superbeams and Beta-beams [27].
3.3 Collider Signature and Lepton Flavor Violation

Recent discussion on the analysis of the scalar potential and the Higgs mass spectrum for models with one triplet Higgs can be found in [28]. In our model with two Higgs triplets we get mixing between the two doubly charged Higgs $\Delta^{++}_1$ and $\Delta^{++}_2$. The physical Higgs fields can be obtained from the scalar potential and are given by

$$H^{++}_1 = \Delta^{++}_1 \cos \theta + \Delta^{++}_2 \sin \theta,$$

(38)

$$H^{++}_2 = -\Delta^{++}_1 \sin \theta + \Delta^{++}_2 \cos \theta,$$

where the mixing angle $\theta$ is

$$\tan 2\theta = \frac{c_2(u_1^2 + u_2^2) + c_2' u_1 u_2}{h'_6(u_2^2 - u_1^2) - h_6|v_c|^2}.$$  

(39)

The parameters $c_2$, $c_2'$, $h_6$ and $h'_6$ are defined in Eq. (50). In the exact $\mu - \tau$ limit, which can be realized by setting $h_6 = 0$ and $h'_6 = 0$, the mixing angle $\theta$ is of course $\pi/4$. Even when $h_6$ and $h'_6$ are $\neq 0$, since the couplings involved in Eq. (39) are expected to be of comparable strengths and since $v^2 <\approx u_1^2$ (from the observed masses and mixing) and $u_2^2 - u_1^2$ is small (see next section) we obtain nearly maximal mixing. In the approximate limit where we take $u_1 \approx u_2 = u$ and neglect $|v_1|^2$, $|v_c|^2$ and $v^{2}_{SM}$ in comparison to $u^2$, the square of the masses of the doubly charged Higgs are given by

$$M^{++}_1 \approx -a + u^2 \left[(4e_1 + 2e'_1) - (2e_2 + e'_2)\right],$$

(40)

$$M^{++}_2 \approx -a + u^2 \left[(4e_1 + 2e'_1) + (2e_2 + e'_2)\right].$$

(41)

The quantities $e_1$, $e'_1$, $e_2$ and $e'_2$ are dimensionless coefficients in the scalar potential and will be explained in the next section. We will see in the next section that $a$ in Eqs. (40) and (41) has a mass dimension 2 and comes as the co-efficient of the $\Delta^{++}_1 \Delta^{--}_1$ term in the scalar potential. We note that the masses are modified due the mixing between $\Delta^{++}_1$ and $\Delta^{++}_2$, and depend on the VEV $u$. The difference between the square of the masses of the two doubly charged Higgs depends only on $u$ and the coefficients $e_2$ and $e'_2$. Measuring this mass squared difference at a collider experiment will provide a handle on the VEV $u$, which could then be used in conjunction with the lepton mass and mixing data to constrain the new scale $\Lambda$. In the most natural limit where we take all coupling constants $e_1$, $e'_1$, $e_2$ and $e'_2$ to be of the same order, then $M^{++}_1 \approx -a$ and $M^{++}_2 \approx -a + 9e_1u^2$.

The most distinctive signature of the existence of triplet Higgs can be obtained in collider experiments, through the production and subsequent decay of the doubly charged Higgs particle(s) [29–31]. The doubly charged Higgs, if produced, would decay through the following possible channels:

$$H^{++} \rightarrow H^+H^+,$$

$$H^{++} \rightarrow H^+W^+, $$

$$H^{++} \rightarrow l^+l^+, $$

$$H^{++} \rightarrow W^+W^+.$$  

(42)

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9We will see this in the next section.
Note that our model has two doubly charged and two singly charged Higgs, however we have suppressed the corresponding indices in Eq. (12). Likewise, we have suppressed the flavor indices of the leptons. The first two decay modes depend on the mass difference between the singly and doubly charged Higgs and hence might be kinematically suppressed compared to the last two channels. We therefore do not consider them any further. The decay rate \( H_{1,2}^{++} \rightarrow W^+W^+ \) is proportional to the square of triplet Higgs VEVs \( v_{1,2} \), while the decay rate to dileptons is inversely proportional to them. As a result the ratio of the decay rates for the two channels is proportional to \( v_{1,2}^{-4} \) and is given as [30]

\[
\frac{\Gamma(H_{1,2}^{++} \rightarrow l_a^+ l_b^+)}{\Gamma(H_{1,2}^{++} \rightarrow W^+W^+)} \approx \left( \frac{m_\nu}{M_{H_{1,2}^{++}}} \right)^2 \left( \frac{v_{SM}}{v_{1,2}} \right)^4 ,
\]

where \( M_{H_{1,2}^{++}} \) is the mass of the doubly charged Higgs and \( m_\nu \) is the scale of neutrino mass. It has been shown [30] from a detailed calculation that for \( M_{H_{1,2}^{++}} \approx 300 \) GeV and \( v_{1,2} \lesssim 10^{-4} \) GeV, decay to dileptons will dominate. For our model \( v_1^2 \approx v_2^2 \approx 10^{-3} - 10^{-4} \) eV\(^2\) and hence we can safely neglect decays to \( W^+W^+ \). The decay rate to dileptons is given as [29–31]

\[
\Gamma(H_{1,2}^{++} \rightarrow l_a^+ l_b^+) = \frac{1}{4\pi(1 + \delta_{ab})} |F_{ab}|^2 M_{H_{1,2}^{++}} ,
\]

while the branching ratio for this decay mode is

\[
BR_{ab} = BR(H_{1,2}^{++} \rightarrow l_a^+ l_b^+) = \frac{2}{(1 + \delta_{ab}) \Sigma_{ab}|F_{ab}|^2} ,
\]

where \( F_{ab} \) are the relevant vertex factors which directly depend on the form of the neutrino mass matrix. Using Eq. (3), we have tabulated in Table 2 the vertex factors for all possible interaction channels in our model. We see that apart from \( e-\mu \) or \( e-\tau \) combinations given in the table, all vertices have extra suppression factor of \( \frac{u_{ij}^2}{\Lambda} \). All other vertices arising from Eqs. (9) and (29) will involve the flavon fields \( \xi \) and/or \( \phi \) and will be suppressed by higher orders in \( \Lambda \). We therefore do not give them here. We had argued from Eq. (39) that in the exact \( \mu-\tau \) symmetric limit \( \theta = \pi/4 \). One can then immediately see from Table 2 that \( H_2^{++}ee \) and \( H_2^{++}\mu\tau \) couplings are zero. Therefore, in the exact \( \mu-\tau \) symmetric limit, the decay of \( H_2^{++} \) to \( ee \) and \( \mu\tau \) is strictly forbidden. We have noted above that even when we do not impose exact \( \mu-\tau \) symmetry, \( \theta \approx \pi/4 \) and hence these decay channels will be suppressed. The branching ratio of all the other decay modes are determined by the corresponding Yukawa couplings. Generally speaking, since all the vertices other than \( H_{1,2}^{++} e\mu \) and \( H_{1,2}^{++} e\tau \) are \( \frac{u_{ij}^2}{\Lambda} \) suppressed, branching ratio of these channels will be larger than all others, assuming equal values of \( y_1,y_2,y_3 \) and \( y_4 \). However, \( y_3 \) and \( y_4 \) could be small for normal hierarchy while inverted hierarchy could be produced for very small \( y_1 \) and \( y_2 \). This will give a handle on determining the neutrino parameters in general and the neutrino mass hierarchy in particular [31]. For instance, if the decay modes of doubly charged Higgs to \( e\mu \) and \( e\tau \) are not observed at a collider experiment, then it would imply small \( y_3 \), which would disfavor the inverted hierarchy.
Signature of doubly charged Higgs could in principle also be seen in lepton flavor violating processes. However, in the framework of our model all additional contribution to \( l_i \rightarrow l_j \gamma \) are smaller than what is expected in the standard model. One can check from Table 2 that the only additional diagram which does not have any \( \Lambda \) (or \( u_{1,2}/\Lambda \)) suppression contributes to \( \tau \rightarrow \mu \gamma \). However, even this diagram will be suppressed due to \( M_{H_{1,2}^{++}} \gg M_W \) [32]. The presence of \( H_{1,2}^{++} \) will allow the decay modes of the form \( l_i \rightarrow l_j l_k \) at the tree level, where \( l_i, l_j \) and \( l_k \) are leptons of any flavor. The branching ratios for \( \mu \rightarrow eee \) and \( \tau \rightarrow eee \) in our model for exact \( \mu - \tau \) symmetry is given by [33]

\[
BR(\mu \rightarrow eee) \simeq \frac{1}{16 G_F^2 \Lambda^2 M_{H_{1,2}^{++}}^4} |y^3_3|^2 |y^3_1|^4.
\] (46)

Thus we see that even this process is suppressed by \( u_{21}^2/\Lambda^2 \) compared to other models with triplet Higgs. Branching ratio for all other lepton flavor violating decay modes such as \( \tau \rightarrow \mu \mu \mu \) are further suppressed. The only decay mode which comes unsuppressed is \( \tau \rightarrow e e \mu \), for which the branching ratio is given by

\[
BR(\tau \rightarrow e e \mu) \simeq \frac{1}{4 G_F^2 M_{H_{1,2}^{++}}^4} |y_{31}|^4.
\] (47)

The current experimental constraint on this decay mode is \( BR(\tau \rightarrow e e \mu) < 2 \times 10^{-7} \) [34], which constrains our model parameter \( y_3 \) as (assuming \( M_{H_{1,2}^{++}} \sim 300 \text{ GeV} \))

\[
|y_3| \lesssim 10^{-1}.
\] (48)

Recall that \( y_3 \) is predicted to be large for the inverted hierarchy while it could be tiny for the normal hierarchy. On the face of it then it appears that the bound given by Eq. (48) disfavors the inverted hierarchy for our model. However, recall that the allowed values of \( y_3 \) shown in Figs. 2 and 3 were presented assuming \( v_1^2 \) to lie between \( 10^{-3} - 10^{-4} \text{ eV}^2 \). However, \( v_1^2 \) could be higher and since what determines the mass squared differences \( \Delta m_{21}^2 \) and \( \Delta m_{31}^2 \) is the product of \( v_1^2 \) and the Yukawas, higher \( v_1^2 \) would imply smaller values of the latter. For instance, we could have taken \( v_1^2 \sim 10^{-1} - 10^{-2} \text{ eV}^2 \) and in that case inverted hierarchy would still be allowed. We reiterate that the bound given by Eq. (48) has been obtained assuming \( M_{H_{1,2}^{++}} \sim 300 \text{ GeV} \). For more massive doubly charged Higgs the branching ratio would go down. On the other hand, if one uses bounds from lepton flavor violating decays to constrain the Yukawas, then one would obtain corresponding limits on the value of \( v_1 \). We conclude that with improved bounds on lepton flavor violating decay modes, one could test our model and/or the neutrino mass hierarchy predicted by our model.
Up to terms of dimension four, the flavor basis of the charged leptons are the same.

### Table 2: Doubly charged Higgs triplet and lepton vertices and the corresponding vertex factors $F_{ab}$

| Vertices | Vertex factors $F_{ab}$ |
|----------|-------------------------|
| $e\mu H^+_{1+}$ | $2y_3\sin\theta CP_L$ |
| $e\mu H^+_{2+}$ | $2y_3\cos\theta CP_L$ |
| $e\tau H^+_{1+}$ | $2y_3\cos\theta CP_L$ |
| $e\tau H^+_{2+}$ | $2y_3\sin\theta CP_L$ |
| $eeH^+_{1+}$ | $y_1\frac{(\sin\theta_{u1} + \sin\theta_{u2})}{\Lambda} CP_L$ |
| $eeH^+_{2+}$ | $y_1\frac{(\cos\theta_{u1} - \sin\theta_{u2})}{\Lambda} CP_L$ |
| $\mu\tau H^+_{1+}$ | $y_2\frac{(\sin\theta_{v1} + \sin\theta_{v2})}{\Lambda} CP_L$ |
| $\mu\tau H^+_{2+}$ | $y_2\frac{(\cos\theta_{v1} - \sin\theta_{v2})}{\Lambda} CP_L$ |
| $\tau\tau H^+_{1+}$ | $y_3\frac{(\cos\theta_{v1})}{\Lambda} CP_L$ |
| $\tau\tau H^+_{2+}$ | $y_3\frac{(\sin\theta_{v2})}{\Lambda} CP_L$ |
| $\mu\mu H^+_{1+}$ | $y_4\frac{(\cos\theta_{v1})}{\Lambda} CP_L$ |
| $\mu\mu H^+_{2+}$ | $y_4\frac{(\sin\theta_{v2})}{\Lambda} CP_L$ |

Table 2: Doubly charged Higgs triplet and lepton vertices and the corresponding vertex factors $F_{ab}$ where a and b are generation indices. The charged lepton mass matrix is almost diagonal in our model. In this analysis we have considered that mass basis and flavor basis of the charged leptons are the same.

## 4 The Vacuum Expectation Values

Up to terms of dimension four, the $S_3 \times Z_4 \times Z_3$ invariant scalar potential (cf. Table 1) is given by

$$V = \sum_i V_i$$

where

$$V_1 = -a Tr[\Delta^\dagger \Delta] + b (Tr[\Delta^\dagger \Delta])^2$$
$$V_2^a = [-c(\xi \xi) + h.c] + c'(\xi' \xi)$$
$$V_2^b = [d(\xi \xi)^{\dagger} + h.c] + d'(\xi' \xi')^\dagger (\xi \xi)^2 + [d''(\xi' \xi')^\dagger (\xi \xi) + h.c]$$
$$V_3^a = [e_1 Tr[(\Delta^\dagger \Delta)^{\dagger}(\xi \xi) + h.c] + e'_1 Tr[(\Delta^\dagger \Delta)^{\dagger}(\xi \xi)]$$
$$V_3^b = [e_2 Tr[(\Delta^\dagger \Delta)^{\dagger}](\xi \xi)^2 + h.c] + e'_2 Tr[(\Delta^\dagger \Delta)^{\dagger}](\xi \xi)^2$$
$$V_3^c = h'_{1} Tr[\Delta^\dagger \Delta^2] (\xi' \xi')^\dagger + h''_{1} (\xi' \xi')^\dagger (\phi_1^\dagger \phi_2)$$
$$V_4 = f_1 Tr[(\Delta^\dagger \Delta)^{\dagger} (\Delta^\dagger \Delta)^{\dagger}] + f_2 Tr[(\Delta^\dagger \Delta)^{\dagger} (\Delta^\dagger \Delta)^{\dagger}] + f_3 Tr[(\Delta^\dagger \Delta)^{\dagger} (\Delta^\dagger \Delta)^{\dagger}]$$
$$V_5 = -h_1 (\phi_1^\dagger \phi_2) + h_2 (\phi_1^\dagger \phi_2) + h_3 (\phi_1^\dagger \phi_2) + h_4 (\phi_1^\dagger \phi_2) + h_5 (\phi_1^\dagger \phi_2) + h_6 Tr[\Delta^\dagger \Delta^2] (\phi_1^\dagger \phi_2)$$
$$V_6 = h_4 Tr[\Delta^\dagger \Delta^2] (\phi_1^\dagger \phi_2) + h_5 Tr[\Delta^\dagger \Delta^2] (\phi_1^\dagger \phi_2) + h_6 Tr[\Delta^\dagger \Delta^2] (\phi_1^\dagger \phi_2)$$
$$V_7^a = [l_1 (\xi \xi)^{\dagger} (\phi_1^\dagger \phi_2) + h.c] + l''_{1} (\xi \xi)^{\dagger} (\phi_1^\dagger \phi_2)$$
$$V_7^b = [l_2 (\xi \xi)^{\dagger} (\phi_1^\dagger \phi_2) + h.c] + l''_{1} (\xi \xi)^{\dagger} (\phi_1^\dagger \phi_2) + l_1 (H^\dagger H) (\phi_1^\dagger \phi_2)$$
$$V_8 = a_1 Tr[\Delta^\dagger \Delta] (H^\dagger H) + [a_2 (H^\dagger H)(\xi \xi) + h.c] - \mu^2 (H^\dagger H) + \lambda (H^\dagger H)^2$$
$$+ r(H^\dagger \tau_i H) Tr[\Delta^\dagger \tau_i \Delta] + a''_{2} (H^\dagger H)(\xi \xi)$$

(50)
The underline sign in the superscript represents the particular $S_3$ representation from the tensor product of the two $S_3$ doublets. The superscripts “2” without the underline represent the square of the term. The quantities with primes are obtained following Eq. (84)

$$\xi' = \sigma_1(\xi)^\dagger = \left(\begin{array}{c} \xi_1^2 \\ \xi_2^2 \end{array}\right), \quad \phi'_e = \sigma_1(\phi_e)^\dagger = \left(\begin{array}{c} \phi_2^2 \\ \phi_1^2 \end{array}\right), \quad \Delta' = \sigma_1(\Delta)^\dagger = \left(\begin{array}{c} \Delta_2^\dagger \\ \Delta_1^\dagger \end{array}\right). \quad (51)$$

The potential given by Eqs. (49) and (50) has to be minimized. The singlets $\xi$ and $\phi_e$ pick up VEVs which spontaneously breaks the $S_3$ symmetry at some high scale, while $\Delta$ picks up a VEV when $SU(2)_L \times U(1)_Y$ is broken at the electroweak scale. The VEVs have already been given in Eqs. (10), (11), and (30). Though we begin by taking the VEV alignment for $\langle \phi_e \rangle$ in order to obtain the VEV alignments for $\langle \Delta \rangle$ and $\langle \xi \rangle$, we stress that this does not pose any serious threat to our model and it can be shown to be perfectly self-consistent. For the sake of keeping the algebra simple we take the VEVs of $\Delta_1$ and $\Delta_2$ to be complex but the VEVs of $\xi$ to be real. We have checked that a complex VEV for $\xi$ does not bring any qualitative change to our calculations.

We denote $v_1 = |v_1|e^{i\alpha_1}, v_2 = |v_2|e^{i\alpha_2}$, where $v_1$ and $v_2$ are the VEVs of $\Delta_1$ and $\Delta_2$. Substituting this in Eqs. (49) and (50) we obtain

$$V = (-a + 4c_1u_1u_2 + e'_1(u_1^2 + u_2^2) + h_4|v_c|^2 + a_1v_{SM}^2)(|v_2|^2 + |v_1|^2) + (b + f_1 + f_2)(|v_2|^2 + |v_1|^2)^2$$

$$-4cu_1u_2 + c'(u_1^2 + u_2^2) + 8du_1^2u_2^2 + d'(u_1^4 + u_2^4) + 4d''u_1u_2(u_1^2 + u_2^2) + 2(f_3 - 2f_2)|v_1|^2|v_2|^2$$

$$+2|v_1||v_2|e_2(u_1^2 + u_2^2) + e'_2u_1u_2| \cos(\alpha_2 - \alpha_1) + (-h_1 + 4l_1u_1u_2)|v_c|^2 + l''_1|v_c|^2(u_1^2 + u_2^2)$$

$$+h_3|v_c|^4 + 4a_2v_{SM}^2u_1u_2 + [-h_6|v_c|^2 + h'_6(u_2^2 - u_1^2)](|v_2|^2 - |v_1|^2) - h''_6(u_2^2 - u_1^2)|v_c|^2$$

$$+a''_2v_{SM}^2(u_1^2 + u_2^2) + l_4v_{SM}^2|v_c|^2 - \mu^2v_{SM}^2 + \lambda v_{SM}^4 \quad (52)$$

where we have absorbed $r$ in the redefine $a_1$. The minimization conditions are:

$$\frac{\partial V}{\partial (\alpha_2 - \alpha_1)} = 0, \quad (53)$$

$$\frac{\partial V}{\partial (|v_1|)} = 0, \quad (54)$$

$$\frac{\partial V}{\partial (|v_2|)} = 0, \quad (55)$$

$$\frac{\partial V}{\partial u_1} = 0, \quad (56)$$

$$\frac{\partial V}{\partial u_2} = 0, \quad (57)$$
\[ \frac{\partial V}{\partial |v_c|} = 0 . \]  

From Eq. (53) we obtain the condition,
\[ 2|v_1||v_2|[e_2(u_1^2 + u_2^2) + e'_2u_1u_2] \sin(\alpha_2 - \alpha_1) = 0 . \]  

Hence
\[ \alpha_2 = \alpha_1 , \]  

as long as \(|v_1|, |v_2| \) and \([e_2(u_1^2 + u_2^2) + e'_2u_1u_2] \neq 0 \). Eq. (54) leads to the condition,
\[ -2a|v_1| + 4B(|v_2|^2 + |v_1|^2)|v_1| + 2|v_1|[4e_1u_1u_2 + e'_1(u_1^2 + u_2^2)] + 2|v_2|[e_2(u_2^2 + u_1^2) + e'_2u_1u_2] \\
+ 4F|v_1||v_2|^2 + 2h_4|v_1||v_c|^2 + 2a_1u_1|v^2_S| + 2h_6|v_c|^2|v_1| - 2h'_6|v_1|(u_2^2 - u_1^2) = 0 , \]  

where we have defined \( B = (b + f_1 + f_2), \ F = f_3 - 2f_2 \) and we have used \( \alpha_2 = \alpha_1 \). Using Eq. (54) we obtain,
\[ -2a|v_2| + 4B(|v_2|^2 + |v_1|^2)|v_2| + 2|v_2|[4e_1u_1u_2 + e'_1(u_1^2 + u_2^2)] + 2|v_1|[e_2(u_2^2 + u_1^2) + e'_2u_1u_2] \\
+ 4F|v_2||v_1|^2 + 2h_4|v_2||v_c|^2 + 2a_1u_1|v^2_S| - 2h_6|v_c|^2|v_2| + 2h'_6|v_1|(u_2^2 - u_1^2) = 0 . \]  

Multiplying Eq. (61) by \(|v_2| \) and Eq. (62) by \(|v_1| \) and subtracting one from the other we obtain,
\[ (2e_2(u_1^2 + u_2^2) + 2e'_2u_1u_2 + 4F|v_1||v_2|)(|v_1|^2 - |v_2|^2) = 4|v_1||v_2|[h_6|v_c|^2 - h'_6(u_2^2 - u_1^2)] . \]  

In the limit \( h_6 = 0 \) and \( h'_6 = 0 \) we get \(|v_2| = |v_1| \) (if \( e_2, e'_2 \) and \( F \neq 0 \) simultaneously), which is required for exact \( \mu-\tau \) symmetry in the neutrino sector. However, there is no \textit{a priori} reason to assume that \( h_6, \) and \( h'_6 \) are zero. In the most general case keeping non-zero \( h_6 \) and \( h'_6, \) we obtain
\[ |v_1|^2 = |v_2|^2 + \frac{4|v_1||v_2|[h_6|v_c|^2 - h'_6(u_2^2 - u_1^2)]}{2e_2(u_1^2 + u_2^2) + 2e'_2u_1u_2 + 4F|v_1||v_2|} . \]  

Since \(|v_1||v_2| \ll u_1u_2 \) and \((u_1^2 + u_2^2) \) we neglect the \( 4F|v_1||v_2| \) term from the denominator. For \( u_1 \simeq u_2 = u \) and \( e_2 \simeq e'_2, \) one obtains
\[ |v_1|^2 = |v_2|^2 + \frac{2|v_1||v_2|h_6|v_c|^2}{3e_2u^2} . \]  

For a fixed \( v_2, \) this is a quadratic equation in \( v_1 \) which allows the solution \( v_1 \simeq v_2(1 + \epsilon) \) where \( \epsilon = \frac{h_6|v_c|^2}{3e_2u^2} . \) For \( h_6 \) and \( e_2 \) of the same order and \( \frac{u}{X} = 10^{-1}, \frac{v}{X} = 10^{-2} \) we obtain \( \epsilon \simeq 10^{-2} \ll 1 \). This would give rise to a very mild breaking of the \( \mu-\tau \) symmetry. We have discussed this case in section 3.2.
Using Eqs. (50) and (57) and repeating the same exercise we get the deviation from $u_1 = u_2$ as

$$u_1^2 = u_2^2 + \frac{A}{B}$$

(66)

where $A$ and $B$ are

$$A = 4u_1u_2[h_6''|v_c|^2 - h_6'(|v_2|^2 - |v_1|^2)]$$

(67)

and

$$B = (-4c + 16du_1u_2 - 4d'u_1u_2 + 4d''(u_1^2 + u_2^2) + 4e_1(|v_1|^2 + |v_2|^2) + 2e_2|v_1||v_2| + 4l_1|v_c|^2 + 4a_2v_{SM}^2),$$

(68)

and using the same arguments as above, it is not hard to see that the deviation from $u_1 = u_2$ is also mild. Again, $u_1 = u_2$ is satisfied when $h_6' = 0$ and $h_6'' = 0$. Since $h_6 = 0$ is also required for $|v_1| = |v_2|$ to be satisfied, we conclude that exact $\mu - \tau$ symmetry for neutrinos demands that $h_6 = 0$, $h_6' = 0$ and $h_6'' = 0$ simultaneously.

Finally, from the last minimization condition (58) we get the solution,

$$|v_c|^2 = \frac{1}{4h_3} \left[ 2h_6(|v_2|^2 - |v_1|^2) + 2h_6''(u_2^2 - u_1^2) - 2l_1''(u_1^2 + u_2^2) + 2h_1 - 2h_4(|v_1|^2 + |v_2|^2) - 8u_1u_2l_1 - 2l_4v_{SM}^2 \right].$$

(69)

We next use the condition (69) to estimate the cut-off scale $\Lambda$. Since $h_1$ defines in Eq. (50) gives the square of the mass of the $\phi_2$ fields, it could be large. The other couplings $h_3, l_1, l_4, h_4, h_6', l_1'$ and $h_6$ are dimensionless and can be assumed to have roughly the same order of magnitude which should be much much smaller than $h_1$. Dividing both sides of Eq. (69) by $\Lambda^2$ and using $|v_1|^2 \simeq |v_2|^2 = 10^{-3}$ eV$^2$, $\frac{v_{\mu}}{\Lambda} \simeq 2 \times 10^{-2}$, $\frac{v_{\tau}}{\Lambda} \simeq 10^{-1}$ and hence $(\frac{v_{\mu}}{\Lambda})^2 < (\frac{v_{\tau}}{\Lambda})^2 < (\frac{v_{\mu}}{\Lambda})^2$, we get

$$\Lambda^2 \simeq \frac{h_1}{4l_1 + 2l_1''} \times 10^2 \text{ GeV}^2.$$  

(70)

Note that $h_1$ has mass dimension 2 and in principle could be large. If we take $h_1$ in TeV range, for example if we take $\sqrt{h_1} = 10$ TeV, then the cut-off scale of the theory is fixed as $10^2$ TeV, where we have taken $l_1$ and $l_1'' \simeq O(1)$. From $\frac{v_{\mu}}{\Lambda} = \frac{v_{\tau}}{\Lambda} \sim 10^{-1}$ and $\frac{v_{\mu}}{\Lambda} \sim 10^{-2}$, we then obtain $u_{1,2} = 10$ TeV and $v_c = 1$ TeV. We reiterate that the constraints from the lepton masses themselves do not impose any restriction on the cut-off scale and the VEVs. One can obtain estimates on them only through limits on the masses of the Higgs. For instance, from Eqs. (40) and (41) one could in principle estimate $u$ by measuring the difference between doubly charged Higgs masses. This could then be combined with the neutrino data to get $\Lambda$, and finally use the charged lepton masses to get $v_c$. 

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Since we consider a model with triplet Higgs to generate Majorana neutrino masses, it is pertinent to make some comments regarding breaking of lepton number and possible creation of a massless goldstone called Majoron [35]. If we wish to conserve lepton number in our effective Lagrangian giving neutrino masses (cf. Eq. (9)), we would have to assign lepton number $-2$ to our Higgs fields $\Delta$, and lepton number 0 to the flavor fields $\xi$. One could then fear that Majorons would be created when $\Delta$ get VEVs. However, lepton number is only an accidental symmetry of the standard model. It is possible that this symmetry was broken in the theory at the high scale. In any case, we do not consider lepton number to be a good symmetry of our theory. For instance, we could break it explicitly by giving a lepton number to the fields $\xi$. In that case one would not break lepton number spontaneously and there would be no Majoron.

These VEV alignments have been obtained by assuming no effect of renormalization group running. However, it is understood that the running from the high scale where $S_3$ is broken to the electroweak scale where the masses are generated, will modify the VEV alignments. Another way the VEV alignments could get modified is through higher dimensional terms in the scalar potential. Due to the $Z_4$ as well as $Z_3$ symmetry that we have imposed, one cannot get terms of dimension five in the scalar potential. The possible next order terms in $V$ would therefore be terms of dimension six. These terms would be suppressed by $\Lambda^2$ and are therefore expected to be much less important in $V$.

5 Conclusions

We have attempted to provide a viable model for the lepton masses and mixing by imposing a $S_3 \times Z_4 \times Z_3$ family symmetry. Our model requires two $SU(2)_L$ Higgs triplets arranged in the doublet representation of $S_3$. In addition we need 2 sets of flavon $S_3$ doublets which are singlets with respect to the standard model. By suitably arranging our fermions in the different representations of $S_3$, we constructed the Yukawa part of the neutrino and charged lepton Lagrangian. Desired structure for the mass matrices were obtained by giving suitable $Z_4$ and $Z_3$ charges to the particles. In particular, the most common dimension five operator $LLHH$, which gives Majorana neutrino masses is strictly forbidden in our model by the flavor symmetry. This term arises in seesaw models. Type-I seesaw is forbidden by $Z_4$ symmetry and since it would have required right-handed standard model singlet neutrinos (at a high scale), these are therefore naturally absent in our model. Type-II seesaw would require the coupling $\Delta HH$ which is forbidden by the $S_3$ flavor symmetry. Due to the presence of the Higgs triplets, we have Majorana neutrino masses from dimension four operator $l_i D_l \Delta$ itself. In addition, we have masses generated by other dimension five operators and together they provide a phenomenological correct form for the neutrino mass matrix. The observed charged lepton masses can be obtained very naturally from our model.

Neutrino data demands $\theta_{23}$ to be maximal and $\theta_{13}$ to be zero. This hints towards the presence of $\mu - \tau$ symmetry in the neutrino sector. On the other hand the wide disparity

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10Here $L$ represents the lepton doublet and $H$ the standard model Higgs doublet.
between the $\mu$ and $\tau$ masses demands that this exchange symmetry does not exist for the charged leptons. The $\mu - \tau$ reflection symmetry is a subgroup of $S_3$ and we break $S_3$ in such a way that this exchange symmetry is preserved for the neutrinos, while it is maximally violated for the charged lepton. This allowed for simultaneous explanation of the peculiar mixing pattern of the neutrinos and the strong hierarchical mass pattern for the charged lepton. One requires a certain alignment for the VEVs for the Higgses for achieving this. The VEV alignment in our model comes from the minimum condition of the scalar potential without any need for imposing supersymmetry and/or additional driving fields. We performed an explicit minimization of the scalar potential to justify the VEV alignment required. We showed that exact $\mu - \tau$ symmetry could be obtained if certain conditions are satisfied. In the most general case, we obtained a deviation from exact $\mu - \tau$ symmetry. However, we showed that these deviations are very extremely small in our model.

We studied the phenomenological viability and predictions of our model in the exact $\mu - \tau$ symmetric limit and produced plots showing correlations between the different model parameters. We gave predictions for $\sin^2 \theta_{12}$, $\Delta m^2_{21}$, $\Delta m^2_{31}$, effective mass in neutrino-less double beta decay, the observed mass squared in direct beta decay and total mass of the neutrinos relevant to cosmological data. We also allowed for mild breaking of the $\mu - \tau$ symmetry and calculated $\theta_{13}$ and the strength of CP violation in the lepton sector. We showed results for one illustrative symmetry breaking scenario and concluded that one would be able to observe such small $\theta_{13}$ and CP violation in next-generation long baseline experiments involving powerful beams.

Our model predicts lepton flavor violating processes such at $\tau \rightarrow eee\mu$ at the tree level. This and other lepton flavor violating processes could therefore be used to constrain the model as well as the neutrino mass hierarchy. Production and subsequent decay of the doubly charged Higgs at particle colliders is a smoking gun signal for the existence of triplet Higgs. We showed that in our model since the triplet VEV is required to be very small, decay to dileptons would predominate. The lepton flavors involved in the final lepton pair could be used to distinguish this model from the other models with triplet Higgs. These signatures could also be used to distinguish the inverted and normal hierarchy.

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## Appendix: The $S_3$ Permutation Symmetry Group

The group $S_3$ is the permutation group of three distinct objects, and is the smallest non-abelian symmetry group. It consists of a set of rotations which leave an equilateral triangle invariant in three dimensions. The group has six elements divided into three conjugacy classes. The generators of the group are $S$ and $T$ which satisfy

$$S^2 = T^3 = (ST)^2 = 1.$$  \hspace{1cm} (71)

The elements are given by the permutations

$$G \equiv \left\{ e, (1 2), (1 3), (2 3), (1 2 3), (3 2 1) \right\},$$  \hspace{1cm} (72)

which can be written in terms of the generators as

$$G \equiv \left\{ e, ST, S, TS, T^2, T \right\}.$$  \hspace{1cm} (73)

One can see that the $S_3$ group contains two kinds of subgroups. It can be easily checked that the subgroup of elements

$$G_{Z_3} \equiv \left\{ e, T, T^2 \right\},$$  \hspace{1cm} (74)

form a group under $Z_3$. In addition, there are three $S_2$ permutation subgroups\footnote{The group $S_2$ is isomorphic to $Z_2$.}

$$G_{S_{12}} \equiv \left\{ e, (1 2) \right\}, \quad G_{S_{13}} \equiv \left\{ e, (1 3) \right\}, \quad G_{S_{23}} \equiv \left\{ e, (2 3) \right\}.$$  \hspace{1cm} (75)
In this paper we are mainly interested in the permutation subgroup which corresponds to \( \mu - \tau \) exchange symmetry. We will break the \( S_3 \) group for neutrinos in such a way that \( \mu - \tau \) symmetry remains intact once neutrino mass terms are generated after electroweak symmetry breaking. On the other hand, for the charged leptons we will break it maximally in order to generate the desired hierarchy between the \( \mu \) and \( \tau \) masses.

The group contains two one-dimensional and one two-dimensional irreducible representations. The one-dimensional representations are given by

\[
1 : \quad S = 1, \quad T = 1 \quad \quad (76) \\
1' : \quad S = -1, \quad T = 1 . \quad \quad (77)
\]

The two-dimensional representation is given by

\[
2 : \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} . \quad \quad (78)
\]

The character table is given in Table 3. Using the Table we can write down the rules for the tensor products. For the one-dimensional irreducible representations we have

\[
1 \times 1 = 1, \quad 1 \times 1' = 1', \quad 1' \times 1' = 1 . \quad \quad (79)
\]

Tensor products between two doublets \( \psi = (\psi_1, \psi_2)^T \) and \( \phi = (\phi_1, \phi_2)^T \) are given as

\[
2 \times 2 = 1 + 1' + 2 , \quad \quad (80)
\]

where

\[
1 \equiv \psi_1 \phi_2 + \psi_2 \phi_1 , \quad \quad (81) \\
1' \equiv \psi_1 \phi_2 - \psi_2 \phi_1 , \quad \quad (82) \\
2 \equiv \begin{pmatrix} \psi_2 \phi_2 \\ \psi_1 \phi_1 \end{pmatrix} . \quad \quad (83)
\]

The complex conjugate doublet \( \psi^* \) is given as \( 2^* \) for which the generators are \( S^* \) and \( T^* \). One can easily check that \( \psi^* \) does not transform as doublet (2) of \( S_3 \) and therefore for this case a meaningful way of writing the tensor products for the conjugate fields is by defining

\[
\psi' \equiv \sigma_1 \psi^* = \begin{pmatrix} \psi_2^* \\ \psi_1^* \end{pmatrix} . \quad \quad (84)
\]

Using the relations \( \sigma_1 S^* \sigma_1 = S \) and \( \sigma_1 T^* \sigma_1 = T \) one can show that \( \psi' \) transforms as a doublet. Then the tensor products \( \psi' \times \phi \) are given by Eq. (80) where

\[
1 \equiv \psi_1^* \phi_1 + \psi_2^* \phi_2 , \quad \quad (85) \\
1' \equiv \psi_1^* \phi_2 - \psi_2^* \phi_1 , \quad \quad (86) \\
2 \equiv \begin{pmatrix} \psi_1^* \phi_2 \\ \psi_2^* \phi_1 \end{pmatrix} . \quad \quad (87)
\]
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