Property \((T_{L^\Phi([0,1])})\) for Orlicz spaces \(L^\Phi([0,1])\)

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Abstract

We show that for a uniformly convex uniformly smooth real-valued Orlicz space \(L^\Phi([0,1])\), a locally compact second countable group has Kazhdan’s property \((T)\) if and only if it has property \((T_{L^\Phi([0,1])})\).

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1 Introduction

In \([\text{BFGM07}]\), they generalize Kazhdan’s property \((T)\) for linear isometric representations on Banach spaces: Let \(G\) be a topological group and \(B\) a Banach space. A linear isometric \(G\)-representation on \(B\) is a continuous homomorphism \(\rho : G \to O(B)\), where \(O(B)\) denotes the group of all invertible linear isometries \(B \to B\), and continuous means the action map \(G \times B \to B\) is jointly continuous. We say that such a representation almost has invariant vectors if for all compact subsets \(K \subset G\)

\[
\inf_{\|v\|=1} \max_{g \in K} \|\rho(g)v - v\| = 0.
\]

Denote by \(B^{\rho(G)}\) the closed subspace of \(G\)-fixed vectors in \(B\). Then the \(G\)-representation \(\rho\) descends to a linear isometric \(G\)-representation \(\hat{\rho}\) on \(B/B^{\rho(G)}\).

Definition 1 \([\text{BFGM07}]\). Let \(B\) be a Banach space. A topological group \(G\) is said to be have property \((T_B)\) if for any linear isometric \(G\)-representation \(\rho : G \to O(B)\), the quotient \(G\)-representation \(\hat{\rho} : G \to O(B/B^{\rho(G)})\) does not almost have invariant vectors.

Kazhdan’s property \((T)\) is equivalent to property \((T_H)\) for a Hilbert space \(\mathcal{H}\). For \(L^p\) spaces, Bader, Furman, Gelander and Monod proved the following:

Theorem 2 \([\text{BFGM07}]\). Let \(G\) be a locally compact second countable group, and \((\Omega, \mu)\) a standard Borel space.

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If $G$ has property $(T_{L^p([0,1])})$ for some $1 < p < \infty$, then $G$ has Kazhdan’s property $(T)$. 

(2) If $G$ has Kazhdan’s property $(T)$, then $G$ has property $(T_{L^p(\mu)})$ for any $\sigma$-finite measure $\mu$ and any $1 \leq p < \infty$.

A Banach space $B$ is said to be uniformly convex if for every $\epsilon > 0$ there is $\delta(\epsilon) > 0$ such that $\|u + v\| \leq 2(1 - \delta(\epsilon))$ whenever $\|u\| = \|v\| = 1$ and $\|u - v\| \geq \epsilon$. A Banach space $B$ is said to be uniformly smooth if for each $\epsilon' > 0$ there is $\delta'(\epsilon') > 0$ such that $\|u + v\| \geq 2 - \epsilon'\|u - v\|$ whenever $\|u\| = \|v\| = 1$ and $\|u - v\| \leq \delta'(\epsilon')$. A Banach space $B$ is said to be $ucus$ if it is uniformly convex and uniformly smooth. For example, $L^p$ spaces with $1 < p < \infty$ are $ucus$ Banach space.

T. Yokota asked us whether results about group action on $L^p$-spaces can be generalized to Orlicz spaces, which are generalized function spaces of $L^p$-spaces. We prove

**Theorem 3.** Let $G$ be a locally compact second countable group, $\Phi$ an $N$-function, and $L^\Phi([0,1])$ a $ucus$ real-valued Orlicz space with the gauge norm. Then $G$ has Kazhdan’s property $(T)$ if and only if it has property $(T_{L^\Phi([0,1])})$.

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## 2 Orlicz spaces

This section refers to [RR91].

**Definition 4.** A function $\Phi : [0, +\infty) \to [0, +\infty]$ is said to be a Young function if it is

1. convex, i.e., $\Phi(st_1 + (1-s)t_2) \leq s\Phi(t_1) + (1-s)\Phi(t_2)$ for all $t_1, t_2 \in \mathbb{R}$ and $s \in [0, 1]$;
2. $\Phi(0) = 0$;
3. $\lim_{t \to \infty} \Phi(t) = +\infty$.

The function $\Phi^* : [0, +\infty) \to [0, +\infty]$ defined by

$$\Phi^*(s) := \sup\{st - \Phi(t) : t \geq 0\}$$

is called the complementary function of the Young function $\Phi$. The function $\Phi^*$ is also a Young function.

A Young function $\Phi$ is called an $N$-function if it is a Young function satisfying $0 < \Phi(t) < \infty$ for all $t \in (0, \infty)$, and $\lim_{t \to 0} \Phi(t)/t = 0$, $\lim_{t \to \infty} \Phi(t)/t = +\infty$. A Young function $\Phi$ is said to satisfy the $\Delta_2$-condition if there are $K > 0$ and $t_0 \geq 0$ such that

$$\Phi(2t) \leq K\Phi(t) \quad \text{for all } t \geq t_0.$$
A Young function $\Phi$ is said to satisfy the $\nabla_2$-condition if there are $\ell > 1$ and $t_0 \geq 0$ such that
\[ 2\ell \Phi(t) \leq \Phi(\ell t) \quad \text{for all } t \geq t_0. \]

An N-function $\Phi$ satisfies $\Delta_2$-condition if and only if $\Phi^*$ satisfies $\nabla_2$-condition. If an N-function $\Phi$ satisfies $\Delta_2$-condition and $\nabla_2$-condition, then there are $t_0 \geq 0$, $q > p > 1$, $C > 0$ and $D > 0$ such that for all $t \geq t_0$
\[ D|t|^p \leq \Phi(t) \leq C|t|^q. \]

An N-function $\Phi$ is said to satisfy the Milnes condition, if for each $\epsilon > 0$ there exist $K(\epsilon) > 1$ and $t_0(\epsilon) \geq 0$ such that
\[ K(\epsilon)\Phi'(t) \leq \Phi'((1 + \epsilon)t) \quad \text{for all } t \geq t_0(\epsilon), \]
where $\Phi'$ is the left derivative of $\Phi$. If an N-function $\Phi$ satisfies the Milnes condition, then it satisfies $\nabla_2$-condition. For example, the function $\Phi_p(t) := |t|^p$ ($1 < p < \infty$) is an N-function satisfying the $\Delta_2$-condition and the Milnes condition. The function $\Phi_1(t) := |t|$ and the function $\Phi_\infty(t) := 0$ if $t \in [-1, 1]$; $\Phi_\infty(t) := +\infty$ otherwise, are not N-function.

Let $\Omega$ be a $\sigma$-finite measure space with a positive measure $\mu$. Set
\[ I_\Phi(f) := \int_\Omega \Phi(|f|)d\mu \]
for a measurable function $f$.

**Definition 5.** For a Young function $\Phi$, the space
\[ L_\Phi = L_\Phi(\Omega, \mu) := \{ f : \Omega \to \mathbb{R}, \text{ measurable : } I_\Phi(af) < \infty \text{ for some } a > 0 \} / \sim \]
is called an Orlicz space, where $f \sim g$ means $f = g$ $\mu$-a.e..

If $\Phi$ satisfies $\Delta_2$-condition, the simple functions are dense in $L_\Phi(\Omega, \mu)$.

**Definition 6.** Let $\Phi$ be a Young function. For any measurable function $f$ on $\Omega$, we define
\[ N_\Phi(f) := \inf \left\{ b > 0 : I_\Phi \left( \frac{f}{b} \right) \leq 1 \right\} \in [0, \infty], \]
where it is understood that $\inf(\emptyset) = +\infty$. The $N_\Phi$ is a norm on $L_\Phi(\Omega, \mu)$, which is called the gauge norm (or the Luxemburg-Nakano norm). For a measurable function $f$ on $\Omega$, we define
\[ \| f \|_\Phi := \sup \left\{ \int_\Omega |f\psi|d\mu : I_{\Phi^*}(\psi) \leq 1 \right\}. \]
The $\| \cdot \|_\Phi$ is a norm on $L_\Phi(\Omega, \mu)$, which is called the Orlicz norm. The norm spaces $(L_\Phi, N_\Phi)$ and $(L_\Phi, \| \cdot \|_\Phi)$ are Banach spaces.
Since \((L^\Phi_p, N_\Phi_p) = (L^p, \|\cdot\|_L^p)\) for \(1 \leq p \leq \infty\), Orlicz spaces with the gauge norms are generalization of \(L^p\) spaces.

**Theorem 7.** If \(\Phi\) is a Young function satisfying \(\Delta_2\)-condition, then \((L^\Phi)^* = L^{\Phi^*}\), and for each \(v^* \in (L^\Phi)^*\), there is a unique \(\psi_{v^*} \in L^{\Phi^*}\) such that
\[
v^*(f) = \int f \psi_{v^*} d\mu, \quad f \in L^\Phi
\]
\[
\|v^*\| = \sup\{|v^*(f)| : N_\Phi(f) \leq 1, f \in L^\Phi\} = \|\psi_{v^*}\|_\Phi.
\]
\[
\|v^*\|^* := \sup\{|v^*(f)| : \|f\|_\Phi \leq 1, f \in L^\Phi\} = N_{\Phi^*}(\psi_{v^*}).
\]

**Theorem 8** ([Mil57], [Aki72]). Let \(\Phi\) be a N-function. Assume \((\Omega, \mu)\) is nonatomic. Then the following are equivalent:

1. \((L^\Phi(\Omega, \mu), N_\Phi)\) is ucu;
2. \((L^\Phi(\Omega, \mu), \|\cdot\|_\Phi)\) is ucu;
3. \(\Phi\) and \(\Phi^*\) are strictly convex and satisfy the Milnes condition.

3. \((T_{L^\Phi([0,1])}) \Rightarrow (T)\)

If \(B\) is a ucu Banach space, for any linear isometric \(G\)-representation \(\rho : G \to O(B)\), we can define an associated dual linear isometric \(G\)-representation \(\rho^* : G \to O(B^*)\) defined by
\[
(\rho^*(g)v^*)u := v^*(\rho(g)u) \quad \text{for } g \in G, u \in B, v^* \in B^*.
\]

By Proposition 2.6 in [BFGM07], \(B = B^{\rho(G)} \oplus B'\), where
\[
B' := \{u \in B \mid v^*(u) = 0 \text{ for all } v^* \in (B^*)^{\rho(G)}\}.
\]

Furthermore, property \((T_B)\) can be rephrased as follows: For any linear isometric \(G\)-representation \(\rho : G \to O(B)\), the restriction \(\rho' : G \to O(B')\) of \(\rho\) to \(B'\) does not almost have invariant vectors.

**Theorem 9.** Let \(G\) be a locally compact second countable group, and \(\Phi\) an N-function such that \((L^\Phi([0,1]), N_\Phi)\) is ucu. For the Orlicz space \(L^\Phi([0,1])\) with the gauge norm \(N_\Phi\), if \(G\) has property \((T_{L^\Phi([0,1])})\), then it has Kazhdan’s property \((T)\).

**Proof.** We follow the arguments in [BFGM07]. Assume that \(G\) does not have Kazhdan’s property \((T)\). Connes and Weiss [CW80] construct a measure-preserving, ergodic \(G\)-action on a standard nonatomic probability space \((\Omega, \mu)\) which admits an asymptotically invariant measurable subsets \(E_n\) for all \(g \in G\), namely,
\[
\mu(E_n) = \frac{1}{2} \quad \text{and} \quad \mu(gE_n \Delta E_n) \to 0 \quad \text{for all } g \in G.
\]
Consider the linear isometric $G$-representation $\rho$ on $B = L^\Phi(\Omega, \mu)$ defined by $\rho(g)f(x) = f(g^{-1}x)$. Since $B^* = L^{\Phi^*}(\Omega, \mu)$, the dual linear isometric $G$-representation $\rho^* : G \to O(B^*)$ is written as $\rho^*(g)\psi(x) = \psi(gx)$ for $\psi \in B^*$. As the action of $G$ on $(\mathcal{X}, \mu)$ is ergodic, $(B^*)^\rho(G) = (L^\Phi(\Omega, \mu))^\rho(G) = \mathbb{R}\chi_\Omega$, the constant functions on $\Omega$. Hence the canonical complement $B'$ of $B^\rho(G)$ is

$$B' = L^\Phi_0(\mu) := \{f \in L^\Phi(\mu) : \int_\Omega f d\mu = 0\}.$$ 

Let $f_n := \Phi^{-1}(1)(2\chi_{E_n} - \chi_\Omega)$. Then the sequence $\{f_n\}_{n=1}^\infty$ lies in $L^\Phi_0(\mu)$, and

$$N_\Phi(f_n) = \inf \left\{ b > 0 : \int_\Omega \Phi \left( \frac{\Phi^{-1}(1)}{b} \right) d\mu(x) \leq 1 \right\}$$

is

$$= \inf \left\{ b > 0 : \Phi \left( \frac{\Phi^{-1}(1)}{b} \right) \leq 1 \right\}$$

$$= 1.$$

It still satisfies

$$N_\Phi(\rho(g)f_n - f_n) = N_\Phi(2\Phi^{-1}(1)\chi_{E_n} \triangle E_n)$$

$$= 2\Phi^{-1}(1) \inf \left\{ b > 0 : \int_{gE_n \triangle E_n} \Phi \left( \frac{1}{b} \right) d\mu(x) \leq 1 \right\}$$

$$= 2\Phi^{-1}(1) \inf \left\{ b > 0 : \mu(gE_n \triangle E_n) \Phi \left( \frac{1}{b} \right) \leq 1 \right\}$$

$$\to 0$$

for all $g \in G$. This means $B'$ almost has the invariant vectors $\{f_n\}_{n=1}^\infty$. Since $B$ is ucus, $G$ does not have property $(T_{L^\Phi([0,1])})$. \hfill \square

4 Linear isometries on $L^\Phi([0,1])$

Let $\Phi$ be an N-function with $\Phi(1) = 1$, $\mu$ the Lebesgue measure on $[0,1]$. By Proposition 4 at 3.2 in [KR94] the Orlicz space $L^\Phi([0,1])$ with the gauge norm $N_\Phi$ has Fatou’s property. Hence it is a rearrangement-invariant function space. Assume $(L^\Phi([0,1]), N_\Phi)$ is not isometric to $L^2([0,1])$. By [KR94], for a surjective linear isometry $U$ on $L^\Phi([0,1])$, there exist a Borel function $h : [0,1] \to \mathbb{R}\setminus\{0\}$, and an invertible Borel map $T : [0,1] \to [0,1]$ such that

(i) for any Borel set $A \subset [0,1]$, $\mu(T^{-1}A) = 0$ if and only if $\mu(A) = 0$, and

(ii) for all $f \in L^\Phi([0,1])$

$$Uf(x) = h(x)f(T(x)) \quad \text{a.e. } x \in [0,1].$$

(1)

Since $\mu \circ T^{-1}$ is a measure which is absolutely continuous with respect to $\mu$, it has a Radon-Nykodym derivative $r$, which satisfies

$$\mu(T^{-1}A) = \int_A r(x) d\mu, \quad \int_A f(x) d\mu(x) = \int_{T^{-1}A} f(x) r(x) d\mu(x)$$

$$5$$
for any Borel set $A \subset [0,1]$ and any $f \in L^\Phi([0,1])$. Under this situation, by the same proof of Theorem 5.4.10 in [FJ03], the equation (1) implies

$$\Phi(|h(x)|\alpha) = r(x)\Phi(\alpha)$$

for almost all $x \in [0,1]$ and all $\alpha > 0$.

### 5 $(T) \Rightarrow (T_{L^\Phi([0,1])})$

Let $\Phi, \Psi$ be two N-functions. The map

$$\phi_{\Phi, \Psi} : L^\Phi([0,1]) \to L^\Psi([0,1]); f \mapsto \phi_{\Phi, \Psi}(f) := \Psi^{-1} \circ \Phi(|f|) \text{sign}(f)$$

is called the generalised Mazur map, where $\text{sign}(f)(x) := f(x)/|f(x)|$.

**Lemma 10 ([Del05]).** Suppose there exist $t_0 \geq 0$, $q_i > p_i \geq 1$, $C_i > 0$ and $D_i > 0$ such that for all $t \geq t_0$

$$D_1|t|^{p_1} \leq \Phi(t) \leq C_1|t|^{q_1} \quad \text{and} \quad D_2|t|^{p_2} \leq \Psi(t) \leq C_2|t|^{q_2}.$$

Then $\phi_{\Phi, \Psi} : S(L^\Phi) \to S(L^\Psi)$ is a uniform homeomorphism, where $S(L^\Phi), S(L^\Psi)$ are the unit spheres of $L^\Phi([0,1])$ and $L^\Psi([0,1])$ with the gauge norms.

Using this, we give a homomorphism $O(L^\Phi([0,1]))$ to $O(L^2([0,1]))$.

**Lemma 11.** Let $\Phi$ be an N-function such that $(L^\Phi([0,1]), N_\Phi)$ is not isometric to $L^2([0,1])$. Then the conjugation

$$U \mapsto \phi_{\Phi, l^2} \circ U \circ \phi_{l^2, \Phi}$$

is a homomorphism from $O(L^\Phi([0,1]))$ to $O(L^2([0,1]))$.

**Proof.** For a simple function $f \in L^2([0,1])$, using the equations (1) and (2) we have

$$\phi_{\Phi, l^2} \circ U \circ \phi_{l^2, \Phi}(f)(x)$$

$$= \phi_{\Phi, l^2} \circ U \left( \Phi^{-1}(|f(x)|^2) \text{sign}(f(x)) \right)$$

$$= \phi_{\Phi, l^2} \left( h(x)\Phi^{-1}(|f(Tx)|^2) \text{sign}(f(Tx)) \right)$$

$$= \left( \Phi \left((h(x)\Phi^{-1}(|f(Tx)|^2) \text{sign}(f(Tx)) \right) \right)^{\frac{1}{2}} \text{sign}(h(x) \text{sign}(f(Tx)))$$

$$= \left( r(x)\Phi \left((|f(Tx)|^2) \right) \right)^{\frac{1}{2}} \text{sign}(h(x) \text{sign}(f(Tx)))$$

$$= r(x)^{\frac{1}{2}} f(Tx) \text{sign}(h(x))$$
Hence the map $U \mapsto \phi_{\Phi,t^2} \circ U \circ \phi_{t^2,\Phi}$ is linear. Furthermore, we have
\[
\|\phi_{\Phi,t^2} \circ U \circ \phi_{t^2,\Phi}(f)\|_{L^2}^2 = \int_{[0,1]} |\phi_{\Phi,t^2} \circ U \circ \phi_{t^2,\Phi}(f)(x)|^2 d\mu(x)
\]
\[
= \int_{[0,1]} |r(x)\frac{4}{T} f(Tx) \text{sign}(h(x))|^2 d\mu(x)
\]
\[
= \int_{[0,1]} |r(x)|^2 |f(Tx)|^2 d\mu(x)
\]
\[
= \int_{[0,1]} |f(x)|^2 d\mu(x)
\]
\[
= \|f\|_{L^2}^2.
\]

If we define $(\phi_{\Phi,t^2} \circ U \circ \phi_{t^2,\Phi})^{-1} := \phi_{\Phi,t^2} \circ U^{-1} \circ \phi_{t^2,\Phi}$, then
\[
(\phi_{\Phi,t^2} \circ U \circ \phi_{t^2,\Phi})^{-1} \phi_{\Phi,t^2} \circ U \circ \phi_{t^2,\Phi} = \phi_{\Phi,t^2} \circ U^{-1} \circ \phi_{t^2,\Phi} \circ \phi_{\Phi,t^2} \circ U \circ \phi_{t^2,\Phi} = \text{id}.
\]

Hence the map $U \mapsto \phi_{\Phi,t^2} \circ U \circ \phi_{t^2,\Phi}$ is an invertible linear isometry on the subspace $\{ \text{simple functions} \} \subset L^2([0,1])$. Hence by extending the map $U \mapsto \phi_{\Phi,t^2} \circ U \circ \phi_{t^2,\Phi}$ to the linear isometry on $L^2([0,1])$, we have the homomorphism from $O(L^\Phi([0,1]))$ to $O(L^2([0,1]))$. \hfill \Box

**Theorem 12.** Let $G$ be a locally compact second countable group, and $\Phi$ an $N$-function such that $(L^\Phi([0,1]), N_\Phi)$ is ucs. For the Orlicz space $L^\Phi([0,1])$ with the gauge norm $N_\Phi$, if $G$ has Kazhdan’s property (T), then it has property $(T_{L^\Phi([0,1])})$.

**Proof.** We follow the arguments in [BFGM07]. We may assume $\Phi(1) = 1$. Assuming that $G$ does not have property $(T_{L^\Phi([0,1])})$. Write $B = L^\Phi([0,1])$ and $H = L^2([0,1])$. Since $B$ is ucs, there is a representation
\[
\rho : G \to O(B)
\]
so that the restriction $\rho' : G \to O(B')$ almost has invariant vectors, i.e. for any compact subset $K \subset G$ and $n \in \mathbb{N}$, there exist unit vectors $f_n \in B' \subset B$ so that
\[
\max_{g \in K} N_\Phi(\rho'(g)f_n - f_n) < \frac{1}{n}.
\]
Let us then define $\pi : G \to O(H)$ by $\pi(g) = \phi_{\Phi,t^2} \circ \rho(g) \circ \phi_{t^2,\Phi}$. Then $\phi_{\Phi,t^2}$ maps $B^\rho(G)$ onto $H^{\pi(G)}$. As the unit sphere $S(B')$ of $B'$ is uniformly separated from $B^\rho(G)$ by uniform convexity of $B$, the uniform continuity of the generalized Mazur map implies that $v_n := \phi_{\Phi,t^2}(f_n)$ is a sequence in the unit sphere $S(H)$ of $H$ such that there exists $\delta > 0$ satisfying $\text{dist}(v_n, H^{\pi(G)}) \geq \delta$ and
\[
\varphi_n(g) := \|\pi(g)v_n - v_n\|_H \to 0 \quad \text{as } n \to \infty.
\]
Let $w_n$ denote the projection of $v_n$ to $\mathcal{H}' = (\mathcal{H}^{\pi(G)})^\perp$. Then $\|w_n\|_{\mathcal{H}} \geq \delta > 0$ for all $n$ and

$$\|\pi(g)w_n - w_n\|_{\mathcal{H}} \leq \varphi_n(g) \to 0 \quad \text{as} \quad n \to \infty.$$ 

Thus the restriction $\pi'$ of $\pi$ to $\mathcal{H}'$ does not $G$-invariant vectors, but almost does. Hence $G$ does not have Kazhdan’s property $(T)$. \hfill \Box

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