Fixing Improper Colorings of Graphs

Konstanty Junosza-Szaniawski\textsuperscript{1}, Mathieu Liedloff\textsuperscript{2}, and Paweł Rzążewski\textsuperscript{1}

\textsuperscript{1} Warsaw University of Technology, Faculty of Mathematics and Information Science, Koszykowa 75, 00-662 Warszawa, Poland
\{k.szaniawski,p.rzazewski\}@mini.pw.edu.pl

\textsuperscript{2} Université d’Orléans, INSA Centre Val de Loire, LIFO, 45067 Orléans, France
mathieu.liedloff@univ-orleans.fr

Abstract. In this paper we consider a variation of a recoloring problem, called the $r$-Color-Fixing. Let us have some non-proper $r$-coloring $\phi$ of a graph $G$. We investigate the problem of finding a proper $r$-coloring of $G$, which is “the most similar” to $\phi$, i.e. the number $k$ of vertices that have to be recolored is minimum possible. We observe that the problem is NP-complete for any $r \geq 3$, but is Fixed Parameter Tractable (FPT), when parametrized by the number of allowed transformations $k$. We provide an $O^*(2^n)$ algorithm for the problem (for any fixed $r$) and a linear algorithm for graphs with bounded treewidth. Finally, we investigate the fixing number of a graph $G$. It is the maximum possible distance (in the number of transformations) between some non-proper coloring of $G$ and a proper one.

1 Introduction

Many problems in real-life applications have a dynamic nature. When the constraints change, the previously found solution may no longer be optimal or even feasible. Therefore often there is needed to recompute the solution (preferably using the old one). This variant is called a reoptimization and has been studied for many combinatorial problems, e.g. TSP (see Ausiello et al. \cite{1}), Shortest Common Superstring (see Bilò et al. \cite{2}) or Minimum Steiner Tree (see Zych and Bilò \cite{17}). We also refer the reader to the paper of Shachnai et al. \cite{16}, where the authors describe a general model for combinatorial reoptimization.

Another family of problems, in which we deal with transforming one solution to another, is reconfiguration. Here we are given two feasible solutions and want to transform one into another by a series of simple transformations in such a way that every intermediate solution is feasible. When we consider a reconfiguration version of the graph coloring problem, we want to transform one proper coloring into another one in such a way that at every step we can recolor just one vertex and the coloring obtained after this change is still proper.

A special attention has been paid to determining if a given graph $G$ is $r$-mixing, i.e. if for any two proper $r$-colorings of $G$ you can transform one into
another (maintaining a proper \(r\)-coloring at each step). Cereceda \textit{et al.} \cite{8,10} characterize graphs, which are 3-mixing and they provide a polynomial algorithm for recognizing them. Determining if a graph is \(r\)-mixing for any \(r \geq 4\) is \textsc{PSPACE}-complete \cite{7}. There are also some results showing that a graph \(G\) is \(r\)-mixing, where \(r\) is some function of \(G\). For example, Jerrum \cite{14} showed that every graph \(G\) is \((\Delta(G) + 2)\)-mixing. This bound has been recently refined by Bonamy and Bousquet \cite{6}, who proved that every graph is \((\chi_g(G) + 1)\)-mixing, where \(\chi_g(G)\) denotes the Grundy number of \(G\), i.e. the highest possible number of colors used by a greedy coloring of \(G\). Clearly \(\chi_g(G) \leq \Delta(G) + 1\).

Another direction of research in \(r\)-mixing graphs is the maximum number of transformations necessary to obtain one \(r\)-coloring from another one (i.e. the distance between those colorings). Bonamy and Bousquet \cite{6} show that if \(r \geq \text{tw}(G) + 2\) (where \(\text{tw}(G)\) denotes the \textit{treewidth} of \(G\)), then any two \(r\)-colorings of \(G\) are in distance of at most \(2(n^2 + n)\), while for \(r \geq \chi_g(G) + 1\), any two \(r\)-colorings are in distance of at most \(4 \cdot \chi_g(G) \cdot n\).

A slightly different problem has been considered by Felsner \textit{et al.} \cite{13}. They also transformed one \(r\)-coloring to another one using some local changes, but did not require the initial coloring to be proper (the final one still has to be proper). Also, a vertex could be recolored to color \(x\) if it did not have any neighbor colored with \(x\) (strictly speaking, any out-neighbor, as the authors were considering directed graphs). They showed that if \(G\) is a 2-orientation (i.e. every out-degree is equal to 2) of some maximal bipartite planar graph (i.e. a plane quadrangulation), then every proper 3-coloring of \(G\) could be reached in \(O(n^2)\) steps from any initial (even non-proper) 3-coloring of \(G\). Similar results hold for 4-colorings and 3-orientations of maximal planar graphs (i.e. triangulations).

In this paper we consider a slightly different problem. We start with some (possibly non-proper) \(r\)-coloring and ask for the minimum number of transformations needed to obtain a proper \(r\)-coloring (any proper \(r\)-coloring, not the specific one). We are allowed to change colors of vertices arbitrarily, provided that we recolor just one vertex in each step. We mainly focus on the computational aspects of determining if, starting with some given \(r\)-coloring of \(G\), we can reach a proper \(r\)-coloring in at most \(k\) steps.

The paper is organized as follows. In Section 3 we show that our problem is \textsc{NP}-complete for any \(r \geq 3\) and polynomial otherwise (here \(k\) is a part of the input). In Section 4 we provide an \(O^*(2^n)\) algorithm for the problem. In the next two Sections we focus on the parametrized complexity (we refer the reader to the book by Downey and Fellows \cite{12} for an introduction to the parametrized complexity theory). Namely, we show that our problem is in \textsc{FPT}, when parametrized by \(k\) (Section 5) and provide a linear algorithm solving the problem for graphs with bounded treewidth (Section 6). In Section 7 we investigate the \textit{fixing number} of \(G\), i.e. the maximum (over all initial colorings \(\varphi\)) distance from \(\varphi\) to a proper coloring of \(G\).

\[\[
1\text{ In } O^* \text{ notation we suppress factors, which are polynomial in the input size.}\]