A unified approach to realize universal quantum gates in a coupled two-qubit system with fixed always-on coupling

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Abstract

We demonstrate that in a coupled two-qubit system any single-qubit gate can be decomposed into two \textit{conditional} two-qubit gates and that any \textit{conditional} two-qubit gate can be implemented by a manipulation analogous to that used for a controlled two-qubit gate. Based on this we present a unified approach to implement universal single-qubit and two-qubit gates in a coupled two-qubit system with fixed always-on coupling. This approach requires neither supplementary circuit or additional physical qubits to control the coupling nor extra hardware to adjust the energy level structure. The feasibility of this approach is demonstrated by numerical simulation of single-qubit gates and creation of two-qubit Bell states in rf-driven inductively coupled two SQUID flux qubits with realistic device parameters and constant always-on coupling.

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During the past decade, a variety of physical qubits have been explored for possible implementation of quantum gates. Of those solid-state qubits based on superconducting devices have attracted much attention because of their advantages of large-scale integration and easy connection to conventional electronic circuits.\cite{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15} Superconducting single-qubit gates \cite{16, 17} and two-qubit gates \cite{12, 13, 14, 15} have been demonstrated recently.

However, building a practical quantum computer requires to operate a large number of multi-qubit gates simultaneously in a coupled multi-qubit system. It has been demonstrated that any type of multi-qubit gate can be decomposed into a set of universal single-qubit gates and a two-qubit gate, such as the controlled-NOT (CNOT) gate \cite{16, 17}. Thus it is imperative to implement the universal single-qubit and two-qubit gates in a multi-qubit system with the minimum resource and maximum efficiency \cite{18}.

Implementing universal single-qubit gates and two-qubit gates in coupled multi-qubit systems can be achieved by turning off and on the coupling between qubits \cite{18, 19, 20}. In these schemes, supplementary circuits were required to control inter-qubit coupling. However, rapid switching of the coupling results in two serious problems. The first one is gate errors caused by population propagation between qubits. Because the computational states of the single-qubit gates are not a subset of the eigenstates of the two-qubit gates the populations of the computational states propagate from one qubit to another when the coupling is changed, resulting in additional gate errors. The second one is additional decoherence introduced by the supplementary circuits \cite{21, 22}. This is one of the biggest obstacles for quantum computing with solid-state qubits, particularly in coupled multi-qubit systems \cite{2}. In addition, the use of supplementary circuits also significantly increase the complexity of fabrication and manipulation of the coupled qubits.

To circumvent these problems, a couple of alternative schemes, such as those with untunable coupling \cite{21} and always-on interaction \cite{22}, have been proposed. In the first scheme, each logic qubit is encoded by extra physical qubits and coupling between the encoded qubits is constant but can be turned off and on effectively by putting the qubits in and driving them out of the interaction free subspace. In the second scheme, the coupling is always on but the transition energies of the qubits are tuned individually or collectively. These schemes can overcome the problem of undesired population propagation but still suffer from those caused by the supplementary circuits needed to move the encoded qubits and tune the transition energies. Moreover, the use of encoded qubits also requires a significant number of additional physical qubits.

In this Letter, we present a unified approach to implement universal single-qubit and two-qubit gates in a coupled two-qubit system with fixed always-on coupling. In this approach, each single-qubit gate is realized via two \textit{conditional} two-qubit gates and each \textit{conditional} two-qubit gate is implemented with a manipulation analogous to that used for a controlled two-qubit gate (e.g., CNOT) in the same subspace of the coupled two-qubit system without additional circuits or physical qubits. Since the computational states of the single-qubit gates are a subset of those of the two-qubit gates the gate errors due to population propagation are completely eliminated. The effectiveness of the approach is demonstrated by numerically simulating the single-qubit gates and creating the Bell states in a unit of inductively coupled two superconducting quantum interference device (SQUID) flux qubits with realistic device parameters and constant always on coupling.

Consider a basic unit consisting of two coupled qubits which we call a control qubit ($Cq$) and a target qubit ($Tq$) for convenience. An eigenstate of the coupled qubits is
denoted by \( |n\rangle = |ij\rangle \), which can be well approximated by the product of an eigenstate of \( \mathcal{C}_q \), \( |i\rangle \), and that of \( \mathcal{T}_q \), \( |j\rangle \), \( |n\rangle \equiv |ij\rangle = |i\rangle |j\rangle \), for weak inter-qubit coupling. The computational states of the coupled qubits are \( |1\rangle = |00\rangle \), \( |2\rangle = |01\rangle \), \( |3\rangle = |10\rangle \), and \( |4\rangle = |11\rangle \), which correspond to the eigenstates for \( i, j = 0, 1 \) and \( n = 1, 2, 3, 4 \), respectively. In general, the result of an operation on \( \mathcal{C}_q \) depends on the state of \( \mathcal{T}_q \) since \( \mathcal{C}_q \) is coupled to and hence influenced by \( \mathcal{T}_q \). Suppose \( U \) is a unitary operator acting on a single qubit in the four-dimensional (4D) Hilbert space spanned by \( |n\rangle \). In order to perform \( U \) on \( \mathcal{C}_q \) one wants the state of \( \mathcal{C}_q \) evolves according to \( U \) independent of the state of \( \mathcal{T}_q \) which is left unchanged. This operation is denoted by \( U^{(2)} = U^{(1)} \otimes I^{(1)}_T \), where, the \( 4 \times 4 \) unitary matrix \( U^{(2)} \) and the \( 2 \times 2 \) unitary matrix \( U^{(1)}_T \) are the representations of \( U \) in the Hilbert space of the coupled qubits and in the subspace of \( \mathcal{C}_q \) while the \( 2 \times 2 \) unitary matrix \( I^{(1)}_T \) is the representation of the identity operation in the subspace of \( \mathcal{T}_q \). If the matrix elements of \( U^{(1)}_T \) are \( u_{ij} \) \((i, j = 1, 2)\), the operation can be decomposed into two operations as

\[
U^{(2)} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & u_{11} & 0 & u_{12} \\
0 & 0 & 1 & 0 \\
0 & u_{21} & 0 & u_{22}
\end{pmatrix} \begin{pmatrix}
 u_{11} & 0 & u_{12} & 0 \\
0 & 1 & 0 & 0 \\
u_{21} & 0 & u_{22} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = U^{(2)}_0 U^{(2)}_1, \tag{1}
\]

where, the \( 4 \times 4 \) unitary matrices \( U^{(2)}_0 \) and \( U^{(2)}_1 \) are also in the 4D Hilbert space of the coupled qubits. Since \( U^{(2)}_0 \) only involves the states \( |00\rangle \) and \( |10\rangle \) and \( U^{(2)}_1 \) only involves \( |01\rangle \) and \( |11\rangle \) they represent operations on \( \mathcal{C}_q \) when \( \mathcal{T}_q \) is in the state \( |0\rangle \) and \( |1\rangle \), respectively. If the state of \( \mathcal{T}_q \) is \( |0\rangle \) \((|1\rangle)\), \( U^{(2)}_0 \) \((U^{(2)}_1)\) implements the desired operation \( U \) while \( U^{(2)}_1 \) \((U^{(2)}_0)\) does nothing. Thus \( U^{(2)}_0 \) and \( U^{(2)}_1 \) represent two conditional two-qubit gates of the coupled qubits. Eq. 1 shows that for the coupled qubits any single-qubit gate can be realized via two conditional two-qubit gates. This property is universal and is shown schematically in Fig. 1, where the single-qubit operation \( U^{(2)} \) and its equivalent operation via two conditional two-qubit gates \( U^{(2)}_0 \) and \( U^{(2)}_1 \) are illustrated. The conditional gates are denoted by the open and closed circles for the state of \( \mathcal{T}_q \) being \( |0\rangle \) and \( |1\rangle \), respectively.

The most obvious advantage of implementing single-qubit gates this way is that the conditional two-qubit gates can be realized using essentially the same set of operations used for controlled two-qubit gates. To show this, let us consider a rf-driven coupled two-qubit system with sufficiently different energy-level spacings \( \Delta E_{13}, \Delta E_{24}, \Delta E_{12}, \) and \( \Delta E_{34} \), where \( \Delta E_{nm} \) is the level spacing between the states \( |n\rangle \) and \( |n\rangle \). To implement, for example, a single-qubit NOT gate, we apply two \( \pi \) pulses to the \( \mathcal{C}_q \): the first one is resonant with \( \Delta E_{13} \) and the second with \( \Delta E_{24} \). Because both pulses are largely detuned from \( \Delta E_{12} \) and \( \Delta E_{34} \), they do not induce unintended population transfer when the fields are sufficiently weak. If the state of \( \mathcal{T}_q \) is \( |0\rangle \) \((|1\rangle)\), the first \( (\text{second}) \) pulse accomplishes the desired transformation. Therefore, when the two microwave pulses are applied to \( \mathcal{C}_q \), either sequentially or simultaneously, the NOT gate is accomplished and the gate time is essentially the same as that of a stand-alone \( \mathcal{C}_q \) if the both pulses are applied simultaneously.

To implement a two-qubit gate such as CNOT in coupled qubits, we apply a \( \pi \) pulse resonant with \( \Delta E_{34} \) to \( \mathcal{T}_q \). In this case, the state of \( \mathcal{T}_q \) flips if and only if the state of \( \mathcal{C}_q \) is \( |1\rangle \). Since the energy level structure for the conditional two-qubit gates and the CNOT gate could be the same the universal single-qubit gates and CNOT gate can be implemented in the same fashion without adjusting inter-qubit coupling as long as \( \mathcal{C}_q \) and \( \mathcal{T}_q \) can be addressed individually by microwave pulses, which is rather straightforward to realize with solid-state qubits in general and with rf SQUID flux qubits in particular.

For concreteness, we demonstrate how to realize the single-qubit and two-qubit gates in rf-driven two SQUID flux qubits with constant always-on coupling. The coupled flux qubits comprise two SQUIDs coupled inductively through their mutual inductance \( M \). Each SQUID consists of a superconducting loop of inductance \( L \) interrupted by a Josephson tunnel junction characterized by its critical current \( I_c \) and shunt capacitance \( C \). A flux-biased SQUID with total magnetic flux \( \Phi \) enclosed in the loop is analogous to a “flux” particle of mass \( m = C \Phi_0^2 \) moving in a one-dimensional potential, where \( \Phi_0 = h/2e \) is the flux quantum. For simplicity, we assume that the two SQUIDs are identical. In this case, \( C_i = C \), \( L_i = L \), and \( I_{ci} = I_c \) for \( i = 1, 2 \).

![FIG. 1: Single-qubit gate](image)

The Hamiltonian of the coupled qubits is\(^\_\)\(^\_\)\(^\_\)\(^\_\)

\[ H(x_1, x_2) = H_0(x_1) + H_0(x_2) + H_{12}(x_1, x_2), \]

where \( H_0(x_i) \) is Hamiltonian of the ith single qubit given by \( H_0(x_i) = p_i^2/2m + \omega_{LC}^2 (x_i - x_{ci})^2 / 2 - E_f \cos(2\pi x_i) \) and \( H_{12} \) is the interaction between the SQUIDs given by \( H_{12} = m \omega_{LC}^2 \kappa (x_1 - x_{ci}) (x_2 - x_{2i}) / 2 \). Here, \( x_i = \Phi_i / \Phi_0 \) is the canonical coordinate of the ith “flux” particle and \( p_i = -i \hbar \partial / \partial x_i \) is the canonical momentum conjugate to \( x_i \), \( x_{ci} = \Phi_{ci} / \Phi_0 \) is the normalized external
flux bias of the $i$th qubit, $E_J = \frac{\Delta E_{J,i}^2}{\beta_L} / 4\pi^2$ is the Josephson coupling energy. $\beta_L = 2 \pi L \Phi_0 / \Phi_0$ is the potential shape parameter, $\omega_{LC} = 1 / \sqrt{LC}$ is the characteristic frequency of the SQUID, and $\kappa = 2 \pi / L$ is the coupling strength. The coupled SQUID qubits are a multi-level system. The eigenenergies $E_n$ and eigenstates $|n\rangle$ are obtained by numerically solving the eigenvalue equation of $H(x_1, x_2)$ using the two-dimensional Fourier-grid Hamiltonian method [27].

Most of the single-qubit gates, such as the NOT gate $(|0\rangle \rightarrow |1\rangle)$, can be described by rotations of an angle $\theta$ about an axis perpendicular to the $z$ axis, $R_z(\theta) = \cos(\theta/2)|0\rangle\langle 0| + e^{i\theta/2}|1\rangle\langle 1|$. The maximally entangled two-qubit states are realized by applying two resonant pulses to the two coupled SQUID qubits with the following CNOT gate $|ij\rangle \rightarrow |i\rangle |J(\theta) |j\rangle$ after the two conditional two-qubit rotations. (b) The rotation angle $\theta$ vs. pulse width. (c) Populations of the states $|00\rangle$ and $|10\rangle$ vs. pulse width.

Entanglement is one of the most profound characteristics of quantum systems which plays a crucial role in quantum information processing and communication [10]. The maximally entangled two-qubit states are referred to as the Bell states. To create the Bell states from a state $|ij\rangle$, a Hadamard gate on $C_q$, $H^{(2)}$, which is decomposed into two conditional two-qubit Hadamard gates $H_1^{(2)}$ and $H_2^{(2)}$, and a CNOT gate are commonly used [see FIG. 3(a)]. The two conditional two-qubit Hadamard gates are implemented by applying two $\pi/2$ pulses $x_{C1}$ and $x_{C2}$ to $C_q$ and the following CNOT gate is implemented by applying a $\pi$ pulse $x_T$ to $T_q$, as shown

![FIG. 2: Arbitrary single-qubit rotation about an axis perpendicular to $z$ axis in the coupled SQUID flux qubits. (a) Quantum circuit: the state $|ij\rangle$ evolves to the state $R^{(2)}(\theta) |ij\rangle = [R^{(2)}(\theta)|i\rangle |j\rangle$ after the two conditional two-qubit rotations. (b) The rotation angle $\theta$ vs. pulse width. (c) Populations of the states $|00\rangle$ and $|10\rangle$ vs. pulse width.](image-url)
in FIG. 3(b). The amplitudes and frequencies of $x_{C1}$ and $x_{C2}$ are the same as those used in FIG. 2 and those of $x_T$ are $x_{T0} = 5 \times 10^{-5}$ and $\omega_T = 0.0592\omega_{LC} = 2\pi \times 4.7$ GHz. In FIG. 3(c), we plot the population evolution of the computational states when the initial state is $|00\rangle$. It is shown clearly that the coupled qubits evolve first into a product state $\langle 00 | + | 10 \rangle / \sqrt{2}$ from the initial state $|00\rangle$ after the first two $\pi/2$ pulses and then into a Bell state $\langle 00 | + | 11 \rangle / \sqrt{2}$ from the product state after the second $\pi$ pulse.

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