Explicit Methods for Hilbert Modular Forms of Weight 1

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Abstract

In this article we present an algorithm that uses the graded algebra structure of Hilbert modular forms to compute the adelic $q$-expansion of a Hilbert modular form of (partial) weight as the quotient of modular forms of higher weights. Moreover, our algorithm is designed in such a way that it computes in all characteristics simultaneously. The main improvement to existing methods is that our algorithm can be applied in (partial) weight 1, which fills a gap left by standard computational methods.

1 Introduction

Established methods to compute Hilbert modular forms (HMFs) are restricted to weight at least 2. As with classical modular forms, the weight 1 case is more intricate. However, one can use the graded algebra structure of HMFs to compute the adelic $q$-expansion of a HMF of (partial) weight 1 as the quotient of HMFs of higher weights.

If $f$ and $E$ are HMFs of level $\mathfrak{N}$, weights $k$ and $k'$ and characters $\mathcal{E}$ and $\mathcal{E}'$ respectively, then their product $f \cdot E$ is a HMF of weight $k + k'$ and character $\mathcal{E} \cdot \mathcal{E}'$. So, if the adelic $q$-expansion of $E$ is invertible, then any HMF of weight $k$ is the quotient of a HMF of weight $k + k'$ by the form $E$, i.e.

$$\mathcal{M}_k(\mathfrak{N}, \mathcal{E}) \subset \frac{1}{E} \mathcal{M}_{k+k'}(\mathfrak{N}, \mathcal{E} \cdot \mathcal{E}') .$$

Moreover, the left hand side is stable under the action of the Hecke algebra of weight $k$, so we can shrink the right hand side by taking the largest subspace of $\frac{1}{E} \mathcal{M}_{k+k'}(\mathfrak{N}, \mathcal{E} \cdot \mathcal{E}')$ that is stable under the action of the Hecke algebra. The underlying philosophy is that we can reduce this candidate space until it only contains HMFs of weight $k$. This approach was used for the first time for classical modular forms in 1978 by Joe Buhler in [2] and later by Kevin Buzzard in [3] and George Schaeffer in [10], and for classical HMFs over quadratic fields with narrow class number 1 by Richard Moy and Joel Specter in [8].

The key observation that enables these results to extend to HMFs is that we can verify whether or not a candidate fraction of HMFs is indeed a HMF by using its truncated adelic $q$-expansion. That is, we prove that the square of the truncated adelic $q$-expansion of such a fraction $g/E$ coincides with the adelic $q$-expansion of a HMF (of higher weight) if and only if the quotient of the truncated adelic $q$-expansions of $g$ and $E$ coincides with the adelic $q$-expansion of a HMF (of lower weight).

We apply this approach to develop an algorithm that computes HMFs with coefficients in $\mathbb{C}$ and any weight as well as HMFs over finite fields with parallel weight. In particular, we prove that our algorithm computes in almost all characteristics simultaneously, in the sense that given weight $k$, level $\mathfrak{N}$ and character $\mathcal{E}$, the output of our algorithm to compute HMFs with coefficients in a number field with trivial class group includes a finite set of primes $\mathcal{L}$.
such that for all primes $p$ not contained in $\mathcal{L}$ satisfying certain conditions in higher weight, all HMFs of weight $k$, level $\mathfrak{N}$ and character $\mathcal{E}$ over $\overline{\mathbb{F}}_p$ lift to characteristic zero. Finally, we use our algorithm to find explicit examples of non-liftable HMFs of parallel weight 1 by running the algorithms for primes contained in $\mathcal{L}$.

**Notation**

Throughout this article, $K$ will denote a totally real number field of degree $n > 1$ and $\mathcal{O}_K$ its ring of integers. If $a$ is an element of $K$ we will denote the image of $a$ under the $n$ distinct embeddings of $K$ into $\mathbb{R}$ by $a^{(1)}, \ldots, a^{(n)}$. An element $a$ of $K$ is said to be totally positive, denoted $a \gg 0$, if $a^{(i)} > 0$ for all embeddings of $K$ into $\mathbb{R}$. If $a$ is a subset of $K$, we will denote $a^+$ the subset of totally positive elements of $a$. For example $\mathcal{O}_K^{\times,+}$ is the set of totally positive units in $\mathcal{O}_K$. We denote the set of all integral ideals of $K$ by $I_K$, the narrow class group of $K$ by $\text{Cl}^+$ and the narrow class number of $K$ by $h^+$.

For $n$-tuples $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ we write

$$z^k = \prod_{i=1}^{n} z_i^{k_i} \quad \text{and} \quad \text{tr}(z) = \sum_{i=1}^{n} z_i.$$ 

We extend this notation to $K$ by identifying $\xi \in K$ with the $n$-tuple $(\xi^{(1)}, \ldots, \xi^{(n)})$ in $\mathbb{R}^n$, i.e.

$$\xi^k = \prod_{i=1}^{n} (\xi^{(i)})^{k_i} \quad \text{and} \quad \text{tr}(\xi) = \sum_{i=1}^{n} \xi^{(i)}.$$ 

Moreover, we write $k_0 = \max_i \{k_i\}$. If $\ell$ is an integer we write $\ell = (\ell, \ldots, \ell)$ to distinguish the integer $\ell$ from the parallel vector $\underline{\ell}$. If $\mathfrak{N}$ is an ideal of $\mathcal{O}_K$, $\mathcal{E}$ a (Dirichlet) character mod $\mathfrak{N}$ and $R$ a $\mathbb{Z}[\frac{1}{\mathfrak{N}}]$-algebra, then we denote the $R$-module of HMFs and cuspidal HMFs of weight $k$, level $\mathfrak{N}$ and character $\mathcal{E}$ over $R$ by $\mathcal{M}_k(\mathfrak{N}, \mathcal{E}; R)$ and $S_k(\mathfrak{N}, \mathcal{E}; R)$ respectively. If $W$ is a subset of $\mathbb{Z}^n$ that contains $0 = (0, 0, \ldots, 0)$ and that is closed under addition, then $\mathcal{M}_W(\Gamma_1(\mathfrak{N}); R)$ denotes the graded $R$-algebra of HMFs of weights $k$ in $W$ and congruence subgroup $\Gamma_1(\mathfrak{N})$. We will write $\mathcal{T}_k(\mathfrak{N}, \mathcal{E}; R)$ for the $R$-algebra of Hecke operators acting on $\mathcal{M}_k(\mathfrak{N}, \mathcal{E}; R)$. For a detailed approach of HMFs and Hecke operators see for example [1], [6], [11] or [12].

## 2 Adelic $q$-expansion

In this section we construct the graded $R$-algebra of adelic power series (of weights in $W$) and formulate conditions on the ring $R$ such that this construction is well defined. Finally, we will show that a suitable $q$-expansion principle from $\mathcal{M}_W(\Gamma_1(\mathfrak{N}); R)$ to this graded $R$-algebra of adelic power series exists.

Let $\mathfrak{N}$ be an integral ideal of $K$ and $W$ a subset of $\mathbb{Z}^n$ that contains $0 = (0, 0, \ldots, 0)$ and is closed under addition. Let $R$ be a $\mathbb{Z}[\frac{1}{\mathfrak{N}}]$-algebra satisfying:

1. The element $\varepsilon^{k/2}$ is a unit in $R$ for all $\varepsilon$ in $\mathcal{O}_K^{\times,+}$ and all $k$ in $W$; \hfill (1)
2. The element $\xi^{(\mathfrak{N}_{k})/2}$ is a unit in $R$ for all $\xi$ in $\mathcal{O}_K^{\times,+}$ and all $k$ in $W$. \hfill (2)

**Remark 2.1.** 1. The fields $\overline{\mathbb{Q}}$ and $\mathbb{C}$ both satisfy conditions (1) and (2) for any $W$. 

2. If $\mathfrak{N}$ is an integral ideal of $K$ and $W$ contains $0 = (0, 0, \ldots, 0)$ and is closed under addition, then $\mathcal{M}_W(\Gamma_1(\mathfrak{N}); R)$ is a graded $R$-algebra of HMFs of weights in $W$ and congruence subgroup $\Gamma_1(\mathfrak{N})$. We will write $\mathcal{T}_k(\mathfrak{N}, \mathcal{E}; R)$ for the $R$-algebra of Hecke operators acting on $\mathcal{M}_k(\mathfrak{N}, \mathcal{E}; R)$. For a detailed approach of HMFs and Hecke operators see for example [1], [6], [11] or [12].
2. If \( W \) is the subset of parallel weights, then any \( \mathbb{Z}[\frac{1}{N(\mathfrak{m})}] \)-algebra satisfies both (1) and (2).

3. If \( W \) is not contained in the set of parallel weights and \( \text{Char}(R) = p > 0 \) then \( R \) does not satisfy (2) since \( p^{(\lambda_n - k)/2} \) is not contained in \( R^\times \) for any non-parallel weight vector \( k \). However, this does not necessarily imply that the adelic \( q \)-expansion principle is not valid in this setting, only that our specific construction is not applicable.

Let \( \{ t_\lambda \}_{\mathcal{C}^+} \) be a full set of representatives of the narrow class group of \( K \). We define the \( R \)-module of adelic power series over \( R \), denoted \( R_\mathcal{A}[[q^{I_K}]] \), as the \( R \)-module whose elements consist of an \( h^+ \)-tuple \( a(0) \) in \( R_{\mathcal{C}^+} \) together with a rule associating to every non trivial ideal \( m \) of \( \mathcal{O}_K \) an element \( a_m \) of \( R \). To emphasise that these adelic power series will be the adelic \( q \)-expansions of HMFs, we write

\[
R_\mathcal{A}[[q^{I_K}]] := R_{\mathcal{C}^+} \oplus \bigoplus_{0 \neq b \in \mathcal{O}_K} R \cdot q^b
\]

\[
= \left\{ (a(0), t_\lambda)_{t_\lambda} \in R_{\mathcal{C}^+} + \sum_{0 \neq b \in \mathcal{O}_K} a_b q^b \mid \text{with all } a_* \in R \right\}.
\]

For any weight vector \( k \) in \( W \) we define the \( R \)-module of geometric power series of weight \( k \) over \( R \) as the \( R \)-module of \( h^+ \)-tuples of formal power series where the coefficients of the power series at \( \lambda \) are indexed by the totally positive elements of \( t_\lambda \) and satisfy \( a_{\lambda, \varepsilon} \xi = \varepsilon^{k/2} a_{\lambda, \xi} \) for all \( \varepsilon \) in \( \mathcal{O}_K^{x^+, +} \), i.e.

\[
R_{k, \{ t_\lambda \}}[[q^{O^+_K}]] := \left\{ (a_{\lambda, 0} + \sum_{\xi \in t_\lambda^+} a_{\lambda, \xi} q^{\xi})_{t_\lambda} \middle| a_{\lambda, \varepsilon} = \varepsilon^{k/2} a_{\lambda, \xi} \text{ for all } \varepsilon \in \mathcal{O}_K^{x^+, +} \right\},
\]

where all \( a_{\xi, t_\lambda} \) lie in \( R \). Note that the module of adelic power series does not depend on the choice of representatives \( \{ t_\lambda \}_{\mathcal{C}^+} \). The module of geometric power series over \( R \) does depend on the choice of representatives \( \{ t_\lambda \}_{\mathcal{C}^+} \). However, we will show that the modules obtained by different choices of representatives are isomorphic. Moreover, the \( R \)-module \( \bigoplus_{k \in W} R_{k, \{ t_\lambda \}}[[q^{O^+_K}]] \) has a natural structure of a graded \( R \)-algebra by componentwise multiplication. The following proposition will allow us to view the adelic power series as a graded \( R \)-algebra.

**Proposition 2.2.** Let \( R \) be a \( \mathbb{Z}[\frac{1}{N(\mathfrak{m})}] \)-algebra satisfying conditions (1) and (2). The choice of representatives \( \{ t_\lambda \}_{\mathcal{C}^+} \) induces an isomorphism of \( R \)-modules \( \Psi_{k, \{ t_\lambda \}} \) between geometric power series and adelic power series.

**Proof.** We construct the map \( \Psi_{k, \{ t_\lambda \}} \) by

\[
\Psi_{k, \{ t_\lambda \}} : R_{k, \{ t_\lambda \}}[[q^{O^+_K}]] \to R_\mathcal{A}[[q^{I_K}]]:
\]

\[
(a_{\lambda, 0} + \sum_{\xi \in t_\lambda^+} a_{\lambda, \xi} q^{\xi})_{t_\lambda} \mapsto (a_{\lambda, 0})_{t_\lambda} + \sum_{0 \neq b \in \mathcal{O}_K} a_b q^b,
\]

where \( a_b = a_{\lambda, \xi} \xi^{(\lambda_n - k)/2} \) with \( \xi \) and \( \lambda \) such that \( b = \xi \lambda^{-1} t_\lambda \). One checks that \( \Psi_{k, \{ t_\lambda \}} \) is a well defined morphism of \( R \)-modules and that its inverse is given by

\[
\Phi_{k, \{ t_\lambda \}} : R_\mathcal{A}[[q^{I_K}]] \to R_{k, \{ t_\lambda \}}[[q^{O^+_K}]]:
\]

\[
(a_{(0), \{ t_\lambda \}})_{t_\lambda} + \sum_{0 \neq b \in \mathcal{O}_K} a_b q^b \mapsto (a_{(0), \{ t_\lambda \}} + \sum_{\xi \in t_\lambda^+} a_{\lambda, \xi} q^{\xi})_{t_\lambda^+}.
\]
with \( a_{\lambda,\xi} = a_{\xi t_{\lambda}^{-1}} \cdot \xi^{(k-\mathfrak{h}_0)/2} \). Note that \( \xi t_{\lambda}^{-1} \) is an integral ideal of \( K \) since \( \xi \) is an element of \( t_{\lambda} \), hence \( a_{\xi t_{\lambda}^{-1}} \) is well defined. \( \square \)

**Definition 2.3.** Let \( R \) be a \( \mathbb{Z}[\frac{1}{N(\mathfrak{N})}] \)-algebra satisfying conditions (1) and (2). The graded \( R \)-algebra of adelic power series (of weights \( W \)) is defined as the \( R \)-module

\[
R_W[q^{f_k}] := \bigoplus_{k \in W} R_k[q^{f_k}]
\]

equipped with the graded \( R \)-algebra structure induced by the isomorphisms \( \Psi_{k,\{t_{\lambda}\}} \) and \( \Phi_{k,\{t_{\lambda}\}} \), i.e. if \( f \) and \( g \) are adelic power series of weight \( k \) and \( k' \) respectively then

\[
f \cdot g = \Psi_{k+k',\{t_{\lambda}\}}(\Phi_{k,\{t_{\lambda}\}}(f) \cdot \Phi_{k',\{t_{\lambda}\}}(g))
\]

by definition.

**Theorem 2.4.** The graded \( R \)-algebra \( R_W[q^{f_k}] \) is independent of the choice of representatives \( \{t_{\lambda}\}_{\mathfrak{C}^1} \).

**Proof.** Let \( \{t_{\lambda}\}_{\mathfrak{C}^1} \) and \( \{t_{\lambda}'\}_{\mathfrak{C}^1} \) be two choices of representatives of the narrow class group of \( K \) and let \( \{\xi_{\lambda}\}_{\mathfrak{C}^1} \) be totally positive elements of \( K \) such that

\[
\xi_{\lambda} t_{\lambda} = t_{\lambda}' \text{ for all } [t_{\lambda}] \text{ in } C_{1^+}.
\]

We define an isomorphism of \( R \)-modules as follows

\[
\phi_{k,\{\xi_{\lambda}\}} : R_{k,\{t_{\lambda}\}}[q^{\mathfrak{O}_{\lambda}^k}] \to R_{k,\{t_{\lambda}'\}}[q^{\mathfrak{O}_{\lambda}^k}] : \left( \sum_{\mathfrak{e}' \in t_{\lambda}^+} a_{\lambda,\mathfrak{e}' q^{\mathfrak{e}'}} \right)_{\mathfrak{C}^1} \mapsto \left( \sum_{\mathfrak{e}' \in t_{\lambda}^+} a_{\lambda,\mathfrak{e}' q^{\mathfrak{e}'}} \xi_{\lambda}^{(k-\mathfrak{h}_0)/2} q^{\mathfrak{e}' / q^{\mathfrak{e}'}} \right)_{\mathfrak{C}^1}.
\]

By construction of \( \phi_{k,\{\xi_{\lambda}\}} \) the following diagram commutes.

\[
\begin{array}{ccc}
R_{k,\{t_{\lambda}\}}[q^{\mathfrak{O}_{\lambda}^k}] & \xrightarrow{\phi_{k,\{\xi_{\lambda}\}}} & R_{k,\{t_{\lambda}'\}}[q^{\mathfrak{O}_{\lambda}^k}] \\
\downarrow{\Psi_{k,\{t_{\lambda}\}}} & & \downarrow{\Psi_{k,\{t_{\lambda}'\}}} \\
R_W[q^{f_k}] & \xrightarrow{\phi_{k,\{\xi_{\lambda}\}}} & R_{k,\{t_{\lambda}'\}}[q^{\mathfrak{O}_{\lambda}^k}]
\end{array}
\]

Finally, one checks that the isomorphism \( \phi_{k,\{\xi_{\lambda}\}} \) induces an isomorphism of graded \( R \)-algebras. \( \square \)

**Theorem 2.5** (The geometric \( q \)-expansion principle). Let \( R \) be a \( \mathbb{Z}[\frac{1}{N(\mathfrak{N})}] \)-algebra satisfying condition (1). Then there exists a natural injective morphism of graded \( R \)-algebras

\[
\mathcal{M}_W(\Gamma_1(\mathfrak{N}); R) \to \bigoplus_{k \in W} R_{k,\{t_{\lambda}\}}[q^{\mathfrak{O}_{\lambda}^k}]
\]

called the geometric \( q \)-expansion principle. If \( R' \) is a \( \mathbb{Z}[\frac{1}{N(\mathfrak{N})}] \)-algebra containing \( R \), then

\[
\mathcal{M}_W(\Gamma_1(\mathfrak{N}); R) = \left\{ f \in \mathcal{M}_W(\Gamma_1(\mathfrak{N}); R') \mid a_{\lambda,\xi}(f) \in R \text{ for all } \xi \in t_{\lambda}^+ \right\}.
\]
Proof. See [9, Theorem 6.7].

Corollary 2.6 (The adelic $q$-expansion principle). Let $R$ be a $\mathbb{Z}[[N(\mathfrak{m})]]$-algebra satisfying conditions (1) and (2).

1. The geometric $q$-expansion principle composed with the isomorphisms $\{\Phi_{k,\phi}(t_{\lambda})\}$ induces an injective morphism of graded $R$-algebras from $\mathcal{M}_W(\Gamma_1(N)); R)$ to $R_W[[q^{1/K}]]$ called the adelic $q$-expansion map.

2. The image of $\mathcal{M}_W(\Gamma_1(N)); R)$ in $R_W[[q^{1/K}]]$ is independent of the choice of representatives $\{t_{\lambda}\}$.

3. Let $R'$ be a $\mathbb{Z}[[N(\mathfrak{m})]]$-algebra containing $R$. Then

$$\mathcal{M}_W(\Gamma_1(N)); R) = \left\{ f \in \mathcal{M}_W(\Gamma_1(N)); R') \mid a_{(0)}(f) \in R^{Cl^+} \text{ and } a_{b} \in R \right\}.$$ 

Proof. This follows from Proposition 2.2, Theorem 2.4 and Theorem 2.5.

Proposition 2.7. Let $R$ be a $\mathbb{Z}[[N(\mathfrak{m})]]$-algebra satisfying conditions (1) and (2) and $\mathcal{E}$ an $R$-valued character mod $N$. The action of the Hecke operators on the adelic $q$-expansion of $\mathcal{M}_k(\mathfrak{H}, \mathcal{E}; R)$ is given by

$$a_{(0)}[t_{\lambda}](T_a(f)) = \sum_{a \subseteq b} \mathcal{E}(b)N(b)^{k_0-1}a_{(0)}[t_{\lambda}, a/b^2](f),$$

$$a_m(T_a(f)) = \sum_{m \cdot a \subseteq b} \mathcal{E}(b)N(b)^{k_0-1}a_{ma/b^2}(f).$$

Proof. See [11, Section 2].

Corollary 2.8. Let $R'$ be a ring that contains $R$ and let $B$ be a positive integer such that the Hecke algebra $\mathcal{T}_k(\mathfrak{H}, \mathcal{E}, R')$ is generated by the Hecke operators $T_{b'}$ with $N(b') \leq B$ as an $R$-module. Then

$$\mathcal{M}_k(\mathfrak{H}, \mathcal{E}; R) = \left\{ f \in \mathcal{M}_k(\mathfrak{H}, \mathcal{E}, R') \mid a_{(0)}(f) \in R^{Cl^+} \text{ and } a_{b} \in R \text{ for all } 0 \neq b < \mathcal{O}_K \text{ with } N(m) \leq B \right\}.$$ 

Proof. Let $f$ be a HMF in $\mathcal{M}_k(\mathfrak{H}, \mathcal{E}, R')$ such that $a_{(0)}(f) \in R^{Cl^+}$ and $a_{b}(f) \in R$ for all non-trivial ideals $b$ with $N(b) \leq B$. By Corollary 2.6 it suffices to show that $a_{b}(f) \in R$ for all non-trivial ideals $b$. Let $m$ be a non-trivial ideal of $K$. Then $T_m$ is an $R$-linear combination of Hecke operators $T_{b_i}$ with $N(b_i) \leq B$, hence

$$a_{m}(f) = a_{O_K}(T_m f) = a_{O_K} \left( \sum_i r_i T_{b_i} f \right) = \sum_i r_i a_{b_i}(f).$$

Since all $a_{b_i}(f)$ and $r_i$ are elements of $R$, so is $a_{m}(f).$
Truncated power series

Let $R$ be a $\mathbb{Z}[[1/\mathfrak{N}]]$-algebra satisfying condition (1) and (2) and let $B$ be a positive number, then by multiplicativity of the norm and by the definition of multiplication of adelic power series, Definition 2.3, the following subgroup is an ideal of $R_W[[q^{1/k}]]$

$$(q^B) = \left\{ \sum_{m \in \mathcal{O}_K} a_m q^m \mid a_m = 0 \text{ for all } N(m) < B \right\}.$$  

We call the quotient, $R_W[[q^{1/k}]]/(q^B)$, the ring of adelic power series mod $q^B$. We denote $\pi_B$ for the projection map onto $R_W[[q^{1/k}]]/(q^B)$.

Definition 2.9. The Sturm bound of weight $k$, level $\mathfrak{N}$ and character $\mathcal{E}$ over $R$ is the smallest positive integer $B$ such that the adelic $q$-expansion map followed by the natural projection of $q$-expansions

$$\mathcal{M}_k(\mathfrak{N}, \mathcal{E}; R) \rightarrow R_W[[q^{1/k}]]/(q^B)$$

is an injective morphism of $R$-modules.

We define the $R$-module of fractional Hilbert modular forms of weight $k$, level $\mathfrak{N}$, character $\mathcal{E}$ over $R$ as

$$\mathcal{M}_k^f(\mathfrak{N}, \mathcal{E}; R) := \left\{ \frac{f}{g} \mid k_1 - k_2 = k, \mathcal{E}_1/\mathcal{E}_2 = \mathcal{E}, f \in \mathcal{M}_{k_1}(\mathfrak{N}, \mathcal{E}_1; R), g \in \mathcal{M}_{k_2}(\mathfrak{N}, \mathcal{E}_2; R) \text{ and } g \text{ in } R_W[q^{1/k}] \right\}.$$  

Since the HMFs $f$ and $g$ are elements of respectively $H^0(X_1(\mathfrak{N}) \times \text{Spec}(R), \omega_{\mathcal{E}, k}^{\otimes 1})$ and $H^0(X_1(\mathfrak{N}) \times \text{Spec}(R), \omega_{\mathcal{E}, k}^{\otimes 2})$ with $X_1(\mathfrak{N})$ the Hilbert modular variety of level $\Gamma_1(\mathfrak{N})$, $\omega_{\mathcal{E}, k}$ the modular line bundle of weight $k$ and character $\mathcal{E}$, their quotient $f/g$ is an element of $H^0(X_1(\mathfrak{N}) \times \text{Spec}(R), \omega_{\mathcal{E}, k}^{\otimes 1} \otimes \mathcal{K})$ with $\mathcal{K}$ the sheaf of meromorphic functions on $X_1(\mathfrak{N}) \times \text{Spec}(R)$. The condition $g/f$ in $R_W[q^{1/k}]$ means precisely that $f/g$ has a well-defined adelic $q$-expansion in $R_W[[q^{1/k}]]$. In particular, the $q$-expansion is over integral ideals of $K$ rather than fractional ideals. The following lemma and theorem will give necessary and sufficient conditions on a fractional HMF to be a HMF.

Lemma 2.10. Let $f$ be a fractional HMF over $R$. Then $f^2$ is a HMF if and only if $f$ is a HMF.

Proof. This follows from the fact that the $R$-module of HMF is precisely the $R$-submodule of the fractional HMFs that do not admit any poles.  

The following theorem is the key observation that enables us to develop our algorithm. It shows that we only need a limited amount of precision in order to conclude that a fractional HMF is a HMF.

Theorem 2.11. Let $k$ and $k'$ be weight vectors in $W$, let $\mathcal{E}$ and $\mathcal{E}'$ be $R$-valued characters mod $\mathfrak{N}$, let $E$ be a HMF in $\mathcal{M}_{k'}(\mathfrak{N}, \mathcal{E}'; R)$, let $B$ be the Sturm bound of weight $2k + 2k'$, level $\mathfrak{N}$ and character $\mathcal{E}^2\mathcal{E}'^2$ and let $\tilde{f}$ be the truncated adelic $q$-expansion mod $q^B$ of a fractional HMF in $E^{-1}\mathcal{M}_{k+k'}(\mathfrak{N}, \mathcal{E}\mathcal{E}'; R)$.

Then $\tilde{f}^2$ agrees with the adelic $q$-expansion mod $q^B$ of a HMF in $\mathcal{M}_{2k}(\mathfrak{N}, \mathcal{E}^2; R)$ if and only if $\tilde{f}$ agrees with the adelic $q$-expansion mod $q^B$ of a HMF in $\mathcal{M}_k(\mathfrak{N}, \mathcal{E}; R)$. 


Algorithm 1

Given an invertible adelic \( q \)-expansion \( f \) of a HMF \( E \) in \( \mathcal{M}_{k+k'}(\mathfrak{M}, \mathcal{E}''; R) \) and the adelic \( q \)-expansion \( h \) of a finite set of generators of \( \mathcal{M}_{k+k'}(\mathfrak{M}, \mathcal{E}''; R) \), computes the largest Hecke stable submodule of the image of \( E^{-1} \mathcal{M}_{k+k'}(\mathfrak{M}, \mathcal{E}''; R) \) in \( R_W[[q^k]]/(q^B) \).

\[
V_0 \leftarrow \text{the image of the adelic } q \text{-expansion map of } E^{-1} \mathcal{M}_{k+k'}(\mathfrak{M}, \mathcal{E}''; R) \text{ mod } q^B
\]

\( i \leftarrow 1 \)

\textbf{while } \( V_{i-1} \) is not \( T_m \)-stable for some \( T_m \) \textbf{do}

\[
V_i \leftarrow \left( T_m|V_{i-1} \right)^{-1} \left( \pi_B/\mathfrak{M}(m) \right)(V_{i-1})
\]

\( i \leftarrow i + 1 \)

\textbf{end while}

\textbf{return } V_i
Algorithm 2 Given an invertible adelic \( q \)-expansion mod \( q^B \) of a HMF \( E \) in \( \mathcal{M}_{k'}(\mathfrak{N}, \mathcal{E}'; R) \) and the adelic \( q \)-expansion mod \( q^B \) of a basis of \( \mathcal{M}_{k+k'}(\mathfrak{N}, \mathcal{E}'; R) \), computes the adelic \( q \)-expansion mod \( q^B \) of all normalised eigenforms in \( \mathcal{M}_k(\mathfrak{N}, \mathcal{E}; R) \).

\[
\begin{aligned}
V_0 & \leftarrow \text{the image of the adelic } q \text{-expansion map of } E^{-1}\mathcal{M}_{k+k'}(\mathfrak{N}, \mathcal{E}'; R) \text{ mod } q^B \\
V & \leftarrow \text{the largest Hecke stable submodule of } V_0 \text{ using Algorithm 1} \\
< \beta_1, \ldots, < \beta_\ell > & \leftarrow \text{simultaneous eigenspaces of } V \\
\text{return } \{ \frac{1}{a_1(\beta_i)} \beta_i \mid \beta_i^2 \in \mathcal{M}_{2k+2k'}(\mathfrak{N}, \mathcal{E}^2; R) \} 
\end{aligned}
\]

Theorem 3.1. Let \( R \) be a \( \mathbb{Z}[\frac{1}{N(\mathfrak{m})}] \)-algebra satisfying conditions (1) and (2). Let \( k \) and \( k' \) be weight vectors in \( \mathcal{W} \), let \( \mathcal{E} \) and \( \mathcal{E}' \) be \( R \)-valued characters and let \( E \) be a HMF in \( \mathcal{M}_{k'}(\mathfrak{N}, \mathcal{E}; R) \) such that \( a_{1(0)}(E) \) is invertible in \( R^\times \). Let \( V \) be the largest Hecke stable submodule of the image of the adelic \( q \)-expansion map of \( E^{-1}\mathcal{M}_{k+k'}(\mathfrak{N}, \mathcal{E}'; R) \) in \( R_W[q^B]/(q^B) \).

1. The submodule \( V \) contains the image of the adelic \( q \)-expansion map of \( \mathcal{M}_k(\mathfrak{N}, \mathcal{E}; R) \).

2. If the bound \( B \) is larger than the Sturm bound for \( \mathcal{M}_{2k+2k'}(\mathfrak{N}, \mathcal{E}^2; R) \), then the \( R \)-module of adelic \( q \)-expansions mod \( q^B \) of \( \mathcal{M}_k(\mathfrak{N}, \mathcal{E}; R) \) is precisely the \( R \)-module of \( q^B \)-adic power series \( v \) in \( V \) such that \( v^2 \) agrees with the adelic \( q \)-expansion of a HMF in \( \mathcal{M}_{2k}(\mathfrak{N}, \mathcal{E}^2; R) \) mod \( q^B \).

3. If either \( R \) is a field or an \( S \)-algebra for some PID \( S \) such that \( \mathcal{M}_{k+k'}(\mathfrak{N}, \mathcal{E}'; R) \) is free of finite rank as an \( S \)-module and the bound \( B \) is such that the Hecke algebra \( \mathcal{T}_k(\mathfrak{N}, \mathcal{E}; \text{Frac}(S) \otimes_S R) \) is generated as an \( R \)-module by the Hecke operators \( T_\mathfrak{m} \) with \( N(\mathfrak{m})^2 \leq B \), then Algorithm 1 computes the largest Hecke stable submodule \( V \) of the image of the adelic \( q \)-expansion map of \( E^{-1}\mathcal{M}_{k+k'}(\mathfrak{N}, \mathcal{E}'; R) \) in \( R_W[q^B]/(q^B) \) in finite time.

4. If \( R \) is an algebraically closed field, the bound \( B \) is larger than the Sturm bound for \( \mathcal{M}_{k+k'}(\mathfrak{N}, \mathcal{E}'; R) \) and the Hecke algebra \( \mathcal{T}_k(\mathfrak{N}, \mathcal{E}; R) \) is generated by all Hecke operators \( T_\mathfrak{m} \) with \( N(\mathfrak{m}) \leq B \), then Algorithm 2 computes the adelic \( q \)-expansion mod \( q^B \) of all normalised eigenforms of \( \mathcal{M}_k(\mathfrak{N}, \mathcal{E}; R) \).

Finally, the analogous statements hold for \( R \)-modules of cuspidal HMFs.

Proof. 1. The \( R \)-module of adelic \( q \)-expansions mod \( q^B \) of \( \mathcal{M}_k(\mathfrak{N}, \mathcal{E}; R) \) is a Hecke stable submodule of the image of the adelic \( q \)-expansion map of \( E^{-1}\mathcal{M}_{k+k'}(\mathfrak{N}, \mathcal{E}'; R) \) in \( R_W[q^B]/(q^B) \), hence it is contained in the largest Hecke stable submodule.

2. This follows from Theorem 2.11.

3. As in Algorithm 1, we take \( V_0 \) to be the \( R \)-module of adelic \( q \)-expansions mod \( q^B \) of \( E^{-1}\mathcal{M}_{k+k'}(\mathfrak{N}, \mathcal{E}'; R) \) and \( \{ \mathfrak{m}_i \} \) the ideals of \( \mathcal{O}_R \) such that

\[
V_i := (T_\mathfrak{m}_i|_{V_{i-1}})^{-1}(\pi_B/\mathfrak{m}_i)(V_{i-1}) \subset V_{i-1}.
\]

Clearly, the largest Hecke-stable submodule \( V \) is contained in each of the spaces \( V_i \). Hence, if the algorithm terminates, the result is the largest Hecke-stable subspace of \( V \). It remains to show that the algorithm does terminate after a finite number of iterations.

Note that by construction, each of the inclusion \( V_i \subset V_{i-1} \) is strict, so we obtain a descending chain of \( R \)-modules

\[
V_0 \supset V_1 \supset \ldots \supset V_{i-1} \supset V_i \supset \ldots \supset V.
\]
If $R$ is a field we obtain a strictly decreasing sequence of positive integers

$$\dim_R(V_0) > \dim_R(V_1) > \cdots > \dim_R(V_{i-1}) > \dim_R(V_i) > \ldots > \dim_R(V).$$

Since, $V_0$ is finite dimensional, the descending chain $V_i$ terminates after at most $\dim_R(V_0)$ iterations.

Let $S$ be the PID such that $M_{k+k'}(\mathcal{U}, \mathcal{E}'; R)$ is free of finite rank as an $S$-module, by the above argument, it suffices to show that

$$\text{rank}_S(V_{i-1}) = \text{rank}_S(V_i) \text{ if and only if } V_{i-1} = V_i.$$ 

We will show that the following two statements hold for any bound $B$ such that the Hecke algebra $T_k(\mathcal{U}, \mathcal{E}; \text{Frac}(S) \otimes R)$ is generated as an $R$-module by the Hecke operators $T_m$ with $N(m)^2 \leq B$.

(i) For all $v_0$ in $\text{Frac}(S) \otimes S R_W[\sqrt[k]{B}]/(B)$ and all $s \in S \setminus \{0\}$,

$$s \cdot v_0 \in V_0 \text{ and } \pi_{\sqrt[q]{B}}(v_0) \in R_W[\sqrt[k]{B}]/(\sqrt[q]{B}) \text{ if and only if } v_0 \in V_0.$$

(ii) For all $v$ in $R_W[\sqrt[k]{B}]/(B)$, all $s \in S \setminus \{0\}$ and all integers $i \geq 0$,

$$s \cdot v \in V_i \text{ if and only if } v \in V_i.$$

The ‘only if’ part of (i) is immediate. Conversely, let $v_0$ be an element of $\text{Frac}(S) \otimes S R_W[\sqrt[k]{B}]/(B)$ and $s$ a non-zero element of $S$ such that $s \cdot v_0 \in V_0$ and $\pi_{\sqrt[q]{B}}(v_0) \in R_W[\sqrt[k]{B}]/(\sqrt[q]{B})$. By definition of $V_0$, there exists a HMF $f$ in $M_{k+k'}(\mathcal{U}, \mathcal{E}'; R)$ such that

$$s \cdot v_0 \cdot E \equiv f \mod B.$$

Now $f/s$ is a HMF in $M_{k+k'}(\mathcal{U}, \mathcal{E}'; \text{Frac}(S) \otimes R)$ whose coefficients mod $q^B$ agree with those of $v_0 \cdot E$. The coefficients of $v_0$ and $E \mod \sqrt[q]{B}$ are elements of $R$, hence the coefficients of $f/s \mod \sqrt[q]{B}$ are contained in $R$. Since the $R$-module $T_{k+k'}(\mathcal{U}, \mathcal{E}'; \text{Frac}(S) \otimes R)$ is generated by the Hecke operators $T_b$ with $N(b) \leq \sqrt[q]{B}$, Corollary 2.8 implies that $f/s$ is a HMF in $M_{k+k'}(\mathcal{U}, \mathcal{E}'; R)$. So $f/(s \cdot E)$ is a fractional HMF in $E^{-1}M_{k+k'}(\mathcal{U}, \mathcal{E}'; R)$ whose adelic $q$-expansion mod $q^B$ agrees with $v_0$, i.e. $v_0$ is an element of $V_0$.

Next, we prove statement (ii). Again, the ‘only if’ part is immediate. We prove the converse by induction on $i \geq 0$. Conversely, the case $i = 0$ is a special case of statement (i). Let $i > 0$ be an integer, $v \in R_W[\sqrt[k]{B}]/(B)$ and $s \in S \setminus \{0\}$ such that $s \cdot v \in V_i$. By construction of $V_i$ this means that $s \cdot v \in V_{i-1}$ and $T_m(s \cdot v) = \pi_{B/N(m)}(V_{i-1})$, i.e. $s \cdot v \in V_{i-1}$ and there exists an element $v'$ in $V_{i-1}$ such that

$$T_m(s \cdot v) = \pi_{B/N(m)}(v').$$

The element $v'/s$ is an element of $\text{Frac}(S) \otimes S V_{i-1} \subset \text{Frac}(S) \otimes S V_0$ such that

$$\pi_{\sqrt[q]{B}}(v'/s) = \pi_{\sqrt[q]{B}} \circ T_m(v) \in R_W[\sqrt[k]{B}]/(\sqrt[q]{B}).$$

Note that the projection mod $\sqrt[q]{B}$ of $T_m(v)$ is well defined since $B > N(m)^2$. So by statement (i), $v'/s$ is an element of $V_0$. In particular, $v'/s$ is an element of $V_0$. In particular, $v'/s$ is an element of $V_0$. In particular, $v'/s$ is an element of $V_0$. In particular, $v'/s$ is an element of $V_0$. In particular, $v'/s$ is an element of $V_0$. In particular, $v'/s$ is an element of $V_0$. In particular, $v'/s$ is an element of $V_0$. In particular, $v'/s$ is an element of $V_0$. In particular, $v'/s$ is an element of $V_0$. In particular, $v'/s$ is an element of $V_0$. In particular, $v'/s$ is an element of $V_0$. In particular, $v'/s$ is an element of $V_0$. In particular, $v'/s$ is an element of $V_0$. In particular, $v'/s$ is an element of $V_0$. In particular, $v'/s$ is an element of $V_0$.
$R_W[q^{1k}]/(q^B)$ and $s \cdot v'/s = v' \in V_{i-1}$, the induction hypothesis implies that $v'/s$ is an element of $V_{i-1}$. So

$$T_m(v) = \pi_B/(N_m)(v'/s) \in \pi_B/(N_m)(V_{i-1})$$

and $v \in V_{i-1}$, hence $v$ is an element of $V_i$. This completes the proof by induction of part (ii).

Finally, we show that (ii) implies that

$$\text{rank}_S(V_{i-1}) = \text{rank}_S(V_i) \text{ if and only if } V_{i-1} = V_i$$

for all $i \geq 0$. Suppose that $\text{rank}_S(V_{i-1}) = \text{rank}_S(V_i)$, then $\text{Frac}(S) \otimes_S V_{i-1} = \text{Frac}(S) \otimes_S V_i$. Let $v$ be an element of $V_{i-1}$, then there exists an element $s$ in $S$ such that $s \cdot v \in V_i$. By statement (ii), this implies that $v \in V_i$.

4. To show the algorithm is well defined, it suffices to prove that all simultaneous eigenspaces of $V$ are 1-dimensional. By Proposition 2.7 any normalised eigenvector $f$ in $R_W[q^{1k}]/(q^B)$ satisfies

$$T_m(f) = a_m(f) \cdot f.$$ 

So if $\beta$ and $\beta'$ are normalised simultaneous eigenvectors in the same simultaneous eigenspace, then $a_m(\beta) = a_m(\beta')$ for all ideals $m$ with $N(m) \leq B$, so $\beta = \beta' \mod q^B$. Since $\beta^2 \in \mathcal{M}_{2k+2k'}(\mathfrak{N}, \mathcal{E}\mathcal{E}'^2; R)$ for all $\beta$ in the output of the algorithm and since $B$ is larger than the Sturm bound for $\mathcal{M}_{2k+2k'}(\mathfrak{N}, \mathcal{E}\mathcal{E}'^2; R)$, we can conclude by Theorem 2.11 that each of the adelic power series mod $q^B$ in the output of the algorithm agrees with the adelic $q$-expansion of some normalised HMF $f$ in $\mathcal{M}_k(\mathfrak{N}, \mathcal{E}; R)$, with $f$ a normalised eigenvector for all Hecke operators $T_m$ with $N(B) \leq m$. Since these Hecke operators generate the full Hecke algebra, the form $f$ is a normalised eigenform.

One application of our algorithm is to compute examples of non-liftable HMFs of parallel weight 1. The following corollary to Theorem 3.1 will allow us to compute the $q$-expansion of HMFs with coefficients in $\mathbb{F}_p$, for almost all primes $p$ simultaneously, under certain conditions. That is, it will compute a space of HMFs and a finite list of primes such that the projection morphism is surjective for all primes $p$ not in the list. This enables us to find explicit non-liftable HMFs by rerunning the algorithm in characteristic $p$ for primes in the finite list.

**Corollary 3.2.** Let $\mathcal{O}$ be a ring of integers with trivial class group, $k$ and $k'$ be parallel weight vectors, $\mathfrak{N}$ an integral $\mathcal{E}$ and $\mathcal{E}'$ characters mod $\mathfrak{N}$ with values in $\mathcal{O}$ and $E$ a HMF in $\mathcal{M}_{k'}(\mathfrak{N}, \mathcal{E}; \mathcal{O}[\mathfrak{N}(0)])$ whose constant coefficient $a_{(0)}(E)$ has no component equal to 0. Let $\tilde{R}$ be the smallest $\mathbb{Z}[\mathfrak{N}(0)]^{-}$-algebra containing $\mathcal{O}[\mathfrak{N}(0)]^{-}$ and such that all components of $a_{(0)}(E)$ are invertible in $\tilde{R}$, i.e.

$$\tilde{R} := \mathcal{O}[\frac{1}{\mathfrak{N}(0)}, \frac{1}{a_{(0)}(E)}].$$

If Algorithm 1 over the ring $\tilde{R}$ yields a candidate submodule $V$ of $\tilde{R}_W[q^{1k}]$ such that the adelic $q$-expansion map followed by the natural projection mod $q^B$ is an isomorphism of $\tilde{R}$-modules from $\mathcal{M}_k(\mathfrak{N}, \mathcal{E}; \tilde{R})$ to $V$, then it also yields a finite set of prime ideals $\mathcal{L}$ such that for all primes $p$ not contained in $\mathcal{L}$ the conditions

1. The projection morphism $\mathcal{M}_{k+k'}(\mathfrak{N}, \mathcal{E}E'; \tilde{R}) \to \mathcal{M}_{k+k'}(\mathfrak{N}, \mathcal{E}E'; \tilde{R}/\mathfrak{p})$ is surjective and
2. The Sturm bound for $\mathcal{M}_k(\mathfrak{M}, \mathcal{E}; \tilde{R}/p)$ is smaller than the precision $B$,

imply that the projection $\mathcal{M}_k(\mathfrak{M}, \mathcal{E}; \tilde{R}) \to \mathcal{M}_k(\mathfrak{M}, \mathcal{E}; \tilde{R}/p)$ is surjective. The analogous statement holds for $R$-modules of cuspidal HMFs.

Proof. Note that the ring $\tilde{R}$ is a PID since it is the localisation of a PID. Moreover, $a_0(E)$ is a unit in $\tilde{R}^{\mathfrak{M}}$ and both $\mathcal{E}$ and $\mathcal{E}'$ are $\tilde{R}$-valued characters. In particular we can apply Theorem 3.1 and Algorithm 1 both over $\tilde{R}$ and any quotient $\tilde{R}/p$ with $p$ a prime ideal of $\tilde{R}$.

Let $V(\tilde{R})$ be the output of Algorithm 1. Note that by construction $V(\tilde{R})$ is the solution of a system of linear equations defined over $\tilde{R}$. Let us denote $V(\tilde{R}/p)$ for the $\tilde{R}/p$-vectorspace of solutions of the mod $p$-reduced linear system. Then for almost all primes $p$ we have

$$V(\tilde{R}/p) = V(\tilde{R})/p,$$

i.e. for almost all primes $p$ the solution of the reduced system is the reduction of the solution of the system. We define $\mathcal{L}$ to be the set of primes for which this equality does not hold. Note that since $\tilde{R}$ is a PID, we can compute the Smith normal form of the matrix representation of the linear system of equations defining $V$. Moreover the set $\mathcal{L}$ is contained in the set of primes dividing the pivot elements of the Smith normal form of the matrix representation of the linear system of equations defining $V$, hence we can explicitly determine all primes $p$ such that $V(\tilde{R}/p) \neq V(\tilde{R})/p$.

Let $p$ be a prime not contained in $\mathcal{L}$ and such that the projection morphism

$$\mathcal{M}_{k+k'}(\mathfrak{M}, \mathcal{E}; \tilde{R}) \to \mathcal{M}_{k+k'}(\mathfrak{M}, \mathcal{E}; \tilde{R}/p)$$

is surjective, then the first part of Theorem 3.1 implies that the candidate space $V(\tilde{R}/p)$ contains the adelic $q$-expansions mod $q^B$ of $\mathcal{M}_k(\mathfrak{M}, \mathcal{E}, \tilde{R}/p)$.

Let us write $q^B(-)$ for the image of the adelic $q$-expansion map followed by the natural projection mod $q^B$. Then, the first assumption of the corollary says that

$$q^B(\mathcal{M}_k(\mathfrak{M}, \mathcal{E}; \tilde{R}))/p \to q^B(\mathcal{M}_k(\mathfrak{M}, \mathcal{E}; \tilde{R}/p)) \to V(\tilde{R}/p)$$

is an isomorphims of $\tilde{R}$-modules. Hence, we obtain the following commutative diagram

$$\begin{array}{ccc}
q^B(\mathcal{M}_k(\mathfrak{M}, \mathcal{E}; \tilde{R}))/p & \to & q^B(\mathcal{M}_k(\mathfrak{M}, \mathcal{E}; \tilde{R}/p)) \\
\downarrow & & \downarrow \\
V(\tilde{R}/p) & \to & V(\tilde{R}/p)
\end{array}$$

where all injections are inclusions. In particular, we obtain

$$q^B(\mathcal{M}_k(\mathfrak{M}, \mathcal{E}; \tilde{R}))/p = q^B(\mathcal{M}_k(\mathfrak{M}, \mathcal{E}; \tilde{R}/p)).$$

Finally, if $p$ is a prime such that the Sturm bound for weight $k$, level $\mathfrak{M}$ and character $\mathcal{E}$ over $\tilde{R}/p$ is less than the precision $B$, then this implies that the projection

$$\mathcal{M}_k(\mathfrak{M}, \mathcal{E}; \tilde{R}) \to \mathcal{M}_k(\mathfrak{M}, \mathcal{E}; \tilde{R}/p)$$

is surjective. □

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Remark 3.3. 1. If the $\mathbb{Z} \left[ \frac{1}{N(\mathfrak{n})} \right]$-algebra $R$ in Theorem 3.1.3 is itself free of finite rank as an $S$-module for some PID $S$, then $\mathcal{M}_{k+k'}(\mathfrak{n}, \mathcal{E}, \mathcal{E}'; R)$ is free of finite rank as an $S$-module. In particular we can apply Algorithm 1 over the ring of integers of any number field since any such ring is free of finite rank as a $\mathbb{Z}$-module.

2. There are 46 cyclotomic fields with class number one, see for example [13, Theorem 11.1]. This classification gives an indication of the orders of the characters $\mathcal{E}$ and $\mathcal{E}'$ in Corollary 3.2. In particular, $\mathbb{Q}(\zeta_n)$ has class number one for $n$ up to 22.

3. If $F$ is a number field of class number greater than one, we cannot apply Corollary 3.2 to compute in all characteristics simultaneously. However, we can compute HMFs over any quotient $\mathcal{O}_F/p$ using Algorithm 1.

4 Numerical Examples

In this section, we discuss explicit results obtained by our implementation of the above algorithms in Magma.

Limitations

In order to apply Theorem 3.1 and Corollary 3.2 to prove that an adelic power series obtained from Algorithms 1 or 2 corresponds to the adelic $q$-expansion of a HMF in $\mathcal{M}_k(\mathfrak{n}, \mathcal{E}, \mathbb{F}_p)$ the precision $B$ must be larger than the Sturm bound for $\mathcal{M}_{2k+2k'}(\mathfrak{n}, \mathcal{E'}, \mathbb{F}_p)$. However, the best known bound on the Sturm bound, $2(k+k')N(\mathfrak{n})^3$, is beyond what we can compute in a reasonable time with the available computing power.

This remark holds in all weights, levels and characters. More precisely, our computations in characteristic 0 yield a candidate space that is an upper bound, i.e. the output of the algorithm contains the space of adelic $q$-expansions of HMFs, but this inclusion could be strict if the precision is lacking.

A second obstruction to proving that the computed space of adelic power series agrees with the adelic $q$-expansions of HMFs arises from the first condition in Corollary 3.2 which requires the projection morphism

$$\mathcal{M}_{k+k'}(\mathfrak{n}, \mathcal{E}, \mathbb{R}) \to \mathcal{M}_{k+k'}(\mathfrak{n}, \mathcal{E}^2, \mathbb{R}/p)$$

to be surjective. For parallel weight at least 3 this is precisely the main result of [7]. However, no such result is proven for parallel weight 2. If the projection morphism is not surjective for a given prime $p$, the computed candidate space is only a lower bound, i.e. there could, a priori, exist more HMFs in $\mathcal{M}_k(\mathfrak{n}, \mathcal{E}, \mathbb{R}/p)$.

Given enough time and computational power, one could use our algorithms to compute a bound on the Sturm bound in individual cases. One could also use Corollary 3.2 to verify that the projection morphism in parallel weight 2 is indeed surjective for all primes $p$ since all HMFs in parallel weight larger than 2 are liftable. These computations would then yield proven adelic $q$-expansions. However, the time required is beyond reasonable.

Instead, we run our algorithm with increasing precision in steps of 500. If the number of linearly independent eigenforms remains the same after increasing the bound we are led to believe that we have computed with sufficient precision. In the example that follows, using coefficients with ideals up to norm at most 2000 sufficed.
A Non-liftable Example

The real quadratic field $\mathbb{Q}(\sqrt{6})$ has class number 1 and narrow class number 2. Let $\omega$ denote $\sqrt{6}$, let $\mathfrak{N}_{331} = (25 + 7\omega)$ be a prime ideal above 331 and let $\mathcal{E}$ be the unique quadratic character mod $\mathfrak{N}_{331}$.

The Eisenstein series $E_4(\mathcal{E}, 1)$, see [4, Proposition 2.1] is cuspidal. However, the Eisenstein series $E_4(\mathcal{E}', 1)$ with $\mathcal{E}'$ the primitive character inducing $\mathcal{E}$ has constant term $a_{\mathcal{O}} = \left[\frac{1}{12}, \frac{1}{12}\right]$. In particular, $12 \cdot E_4(\mathcal{E}', 1)$ is an (old) HMF in $M_4(\mathfrak{N}_{331}, \mathcal{E}, \mathbb{Z}[\frac{1}{N(\mathfrak{N}_{331})}])$ whose adelic $q$-expansion is invertible in $(\mathbb{F}_p)^{12}[[q^{1/3}]]$ for all primes $p$ and $W$ the set of parallel weight vectors.

So we can apply algorithms 1 and 2 to obtain

$$\dim(S_4(\mathfrak{N}_{331}, \mathcal{E}, \mathbb{C})) = 0$$

$$\dim(S_4(\mathfrak{N}_{331}, \mathcal{E}, \mathbb{F}_p)) = \begin{cases} 2 & \text{if } p = 3 \\ 0 & \text{else.} \end{cases}$$

Moreover, we can compute normalised eigenforms $f$ and $f^\sigma$ with coefficients in $\mathbb{F}_9 = \mathbb{F}_3(\zeta)$ where $\zeta^4 = 1$. The eigenform $f$ and its Galois conjugate $f^\sigma$ span the space $S_4(\mathfrak{N}_{331}, \mathcal{E}, \mathbb{F}_3)$. For primes $p$ with norm up to 25 we list the norm of the prime, a generator $\alpha_p$ of the prime and the coefficient, hence Hecke eigenvalue, of the normalised eigenforms $a_p(f)$ and $a_p(f^\sigma)$ in Table 1.

The absolute and relative frequencies of elements of $\mathbb{F}_9$ occurring as a Hecke eigenvalue for primes up to norm 2000 are given in Table 2. These frequencies suggest that the image of the associated Galois representation in $\text{GL}_2(\mathbb{F}_9)$ is one of the following subgroups $H_1 \cong \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $H_2 = \text{SmallGroup}(48, 33)$ or $H_3 = \text{SmallGroup}(144, 130)$. The images of these subgroups in $\text{PGL}_2(\mathbb{F}_9)$ are respectively isomorphic to $\mathbb{Z}_4$, $A_4$ and $\text{SmallGroup}(36, 9)$.

| $N(p)$ | $\alpha_p$ | 2 | 3 | 5 | 5 | 19 | 19 | 23 | 23 |
|--------|-------------|---|---|---|---|----|----|----|----|
|        |             | $-\zeta$ | 1 | $\zeta$ | 0 | 2 | 0 | $\zeta$ | 0 |
|        | $a_p(f)$    | $\zeta$ | 1 | $-\zeta$ | 0 | 2 | 0 | $-\zeta$ | 0 |

Table 1: The Hecke eigenvalues for primes $p = (\alpha_p)$ with norm less than 25 for the normalised eigenforms $f$ and $f^\sigma$ spanning the space $S_4(\mathfrak{N}_{331}, \mathcal{E}, \mathbb{F}_3)$.

| $x$ | $a_p(f)$ abs. freq. | $a_p(f)$ rel. freq. | $a_p(f^\sigma)$ abs. freq. | $a_p(f^\sigma)$ abs. freq. |
|-----|----------------------|---------------------|-----------------------------|-----------------------------|
| 0   | 71                   | 0.24                | 71                          | 0.24                        |
| 1   | 56                   | 0.19                | 56                          | 0.19                        |
| -1  | 52                   | 0.18                | 52                          | 0.18                        |
| $\zeta$ | 63   | 0.21                | 53                          | 0.18                        |
| $-\zeta$ | 53   | 0.18                | 63                          | 0.21                        |

Table 2: The absolute and relative frequencies of elements $x \in \mathbb{F}_9 = \mathbb{F}_3(\zeta)$ occurring as the Hecke eigenvalues for primes $p$ with norm less than 2000 for the normalised eigenforms $f$ and $f^\sigma$ spanning the space $S_4(\mathfrak{N}_{331}, \mathcal{E}, \mathbb{F}_3)$.
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