Gravitational fluctuations in de Sitter cosmology

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Abstract. De Sitter spacetime plays an important role in cosmology: the geometry of most inflationary models is close to de Sitter spacetime and so will be the late-time behavior of the present universe with accelerated expansion. In linearized perturbation theory the metric fluctuations in a de Sitter cosmology describe very well the anisotropies of the microwave background and the observed large scale structure. Recently there has been some interest in the need to go beyond the linear approximation to include the effect of matter loops. This will allow testing perturbation theory in a de Sitter background, checking possible large back-reaction effects on the de Sitter geometry, and also eventually to discriminate between inflationary models that lead to similar results at tree level. Working in the framework of stochastic gravity, or equivalently in the large $N$ expansion, one may derive the two-point correlations for the gravitational fluctuations incorporating the effects of matter loops. One may characterize the quantum gravitational fluctuations in a gauge invariant way by the two-point functions of the linearized Riemann tensor. This can be given in terms of the two-point linearized Einstein and Weyl tensors. Assuming minimally coupled scalar fields in a de Sitter invariant state, the two-point functions of the linearized Einstein tensor over the de Sitter background have been computed in terms of de Sitter invariant bi-tensors.

1. Introduction
De Sitter spacetime plays a key role in our understanding of the universe. On the one hand, the anisotropies of the cosmic microwave background and the observed large structure of the universe are successfully explained by a short early inflationary phase [1, 2]. On the other hand, the present universe is undergoing also a phase of accelerated expansion [3] which might be driven by a small non-vanishing cosmological constant. The geometry of most inflationary models as well as the late time behavior of our expanding universe are well approximated by the de Sitter geometry.

For this reason quantum gravitational perturbations around de Sitter and near-de Sitter spacetimes have been extensively studied in the context of inflationary models. The standard analysis is based on the linearized computation of the two-point function of the metric perturbations [4, 5]. But more recently the need for matter loop corrections has been emphasized by Weinberg [6], see also [7], who pointed out the importance of testing perturbation theory in cosmology given the great success of the tree level approximation. Another reason to include the effect of matter loops is to discriminate between different inflationary models that give very similar results at the tree level [8, 9]. A series of works including matter loop corrections in different models have been performed using different approaches and models [10, 11, 12] including stochastic gravity [13]. However, certain aspects such as the significance of possible
large infrared effects, have not been completely settled yet. The standard theoretical framework
assumes that the background dynamics of de Sitter and near-de Sitter spacetimes is not affected
by large quantum corrections. However, there is a number of studies [14, 15, 16, 17, 18] that
claim that one-loop corrections could give a significant back-reaction in inflationary models. It
has also been argued that higher-order radiative corrections from graviton two-loops could give
rise to a secular screening of the cosmological constant for pure gravity in de Sitter spacetime
[19, 20]. Such nonlinear effects are dominated by infrared modes which are copiously generated
by the exponential expansion and should be fairly insensitive to the short-distance behavior of
quantum gravity. However, whether these large effects really take place is still under debate,
mainly due to the gauge dependence of the results [21, 22, 23, 24, 25, 26, 27].

It is also worth noting that interacting field theories in de Sitter spacetime are usually plagued
with infrared divergencies from loop diagrams in a perturbative expansion in the interaction
parameter that leads to the breakdown of perturbation theory. An approach due to Starobinsky
[28, 29] has been able to go beyond by non-perturbatively accounting for the infrared effects.
It is based on the observation that the infrared part of the scalar field may be considered as a
classical stochastic field with a probability distribution described by a Fokker-Planck equation,
this is because the infrared modes outside the horizon undergo fast decoherence. In this way
Starobinski and Yokoyama [30] were able to show that there is a de Sitter invariant state for a low-
mass scalar field (that is, a mass smaller than the Hubble scale) with a quartic self-interaction.
Also Riotto and Sloth [31] where able to evaluate the infrared finite correlation functions
by resumming higher loop-diagrams non-perturbatively for $O(N)$-invariant scalar theory with
quartic self-interactions; they use the Fokker-Planck equation and also a two-particle-irreducible
formulation. These results are in agreement with a recent result by Marolf and Morrison [32]
who prove by analytical continuation of the interacting Euclidean vacuum that for cubic and
quartic interactions there is an interacting de Sitter invariant vacuum which is well behaved in
the infrared and stable. Although for low masses the different terms of the perturbative series
do not decrease as already noted in Ref. [33], where the similarities of the large fluctuations
on superhorizon scales in de Sitter with some examples in condensed matter physics, near the
vicinity of a critical point, are pointed out. Other authors, however, like Polyakov [34, 35] argue
that interacting scalar fields in de Sitter are unstable and that this can lead to the decay, or
screening, of the cosmological constant. This result is based on Feynman diagrams which are
infrared divergent in the “in-out” formalism due to the exponential growth of the de Sitter
volume element. The use of the “in-out” perturbation theory, however, has been known to lead
to unstabilities which are not present in the “in-in” formalism [36, 37]; see also [38, 39] in relation
to Polyakov’s proposal.

Here, instead of a self-interacting scalar field we consider a free massive scalar field interacting
with metric perturbations on a de Sitter background. To gain insight into the possibility
of important quantum effects in de Sitter, it seems convenient to study quantum metric
perturbations around de Sitter spacetime keeping de Sitter invariance always manifest. This may
be achieved with the formalism of de Sitter-invariant bi-tensors developed by Allen and Jacobson
[40]. The tree-level graviton two-point function has been computed using this formalism by
several authors [41, 42, 43]. A conclusion that can be drawn from their results is that the
two-point function of the linearized Riemann tensor $R^{ab}_{\phantom{ab}cd}$, which is gauge-invariant around de
Sitter spacetime, decays like the fourth power of the physical distance between the two points
[44]. Here, we summarize the main results obtained in Ref. [45] on the effects of matter loops
on quantum gravitational fluctuations. A key ingredient for this calculation is the connected
symmetrized two-point function of the stress tensor operator of the matter fields. Obtaining
this object in de Sitter spacetime, in a de Sitter-invariant vacuum, and with the formalism of
de Sitter-invariant bitensors, is one of the main results reported here.

We work in the framework of stochastic gravity. Stochastic gravity, which is based in
the Einstein-Langevin equations, is a perturbative extension of semiclassical gravity which incorporates the stress tensor fluctuations [46, 47, 48, 49, 50], as a gravitational source. Moreover, in analogy with the results in some simple open quantum systems [51] and the results of stochastic gravity in Minkowski spacetime [52], it has been shown that the correlation functions of the metric fluctuations obtained in stochastic gravity reproduce the two-point metric fluctuations in the quantum theory of gravity interacting with \( N \) quantum fields to leading order in an \( 1/N \) expansion [53, 54]. Thus, stochastic gravity is a useful framework to study quantum metric fluctuations, and can be applied to the problem of cosmological perturbations to some advantage. It exactly reproduces the standard tree level results for cosmological perturbations when both the gravitational and matter field perturbations are linear and quantized [56, 57]. But since it incorporates the full effect of the matter fields it goes beyond the tree level by incorporating matter loop corrections [13]. Here we are concerned with the gravitational fluctuations in de Sitter spacetime including these corrections. More specifically, we consider fluctuations which are directly induced by the two-point functions of the stress tensor such as the linearized Einstein tensor fluctuations.

2. Stochastic gravity

General relativity is known to be a non-renormalizable quantum field theory, however, it makes perfect sense when it is regarded as an effective field theory [58, 59] which has a range of validity below a certain energy scale, often taken as the Planck mass. This low energy effective theory results from the separation of the low energy gravitational quantum effects from the high energy contributions. The action of the gravitational effective field theory is organized as a sum of all possible interactions which are consistent with general covariance and local Lorentz invariance in a series which has higher and higher powers of derivatives of the metric, as an energy expansion, with coefficients that must be determined by experiments. The lowest order terms are the cosmological constant and the Einstein-Hilbert terms.

Regarding quantum gravity as an effective field theory results in a perfectly well behaved quantum field theory. The semiclassical limit can be formally derived as the leading-order \( 1/N \) approximation of quantum gravity [60], interacting with \( N \) free quantum fields when Newton’s gravitational constant \( G \) is rescaled to \( \bar{G} = GN \). In this limit, in which \( \bar{G} \) is kept fixed, one arrives after path integration at a theory where the gravitational field can be treated as a configuration and the quantum fields are fully quantized. If we call \( g_{ab} \) the metric tensor and \( \hat{\phi} \) a scalar field (for notational simplicity we write just one scalar field) one arrives at the semiclassical Einstein equation for the metric \( g_{ab} \):

\[
G_{ab}[g] = \bar{\kappa}\langle T_{ab}[\bar{g}] \rangle_{\text{ren}}, \tag{1}
\]

where \( \bar{\kappa} = \kappa N \) with \( \kappa = 8\pi G = 8\pi/m_P^2 = 8\pi l_P^2 \), where \( m_P \) and \( l_P \) are the Planck mass and Planck length, respectively, and \( T_{ab} = T_{ab}[\bar{\phi}^2] \) is the stress tensor operator which is quadratic in the field operator \( \bar{\phi} \). This operator, being the product of distribution valued operators, is ill defined and needs to be regularized and renormalized; the subscript \( \text{ren} \) means that the operator has been renormalized. The angle brackets on the right hand side mean that the expectation value of the stress tensor operator is computed in some quantum state, say \( |\psi\rangle \), compatible with the geometry described by the metric \( g_{ab} \). On the left hand side of equation (1), \( G_{ab}[g] \) stands for the Einstein tensor for the metric \( g_{ab} \). We also implicitly assume that the local geometric terms that are needed to renormalize the stress tensor operator are included on the right hand side; these are basically a term with a cosmological constant and two higher derivative terms quadratic in the curvature that can be derived from an action with the square of the Ricci scalar and another term with the square of the Weyl tensor, these terms come with two arbitrary renormalization parameters. These higher derivative terms are already present in
the gravitational effective field theory. Thus, the effect of the high energy modes of the matter fields is to produce shifts in the coefficients of the effective field action up to terms quadratic in the curvature, which involve up to four derivatives of the metric tensor.

The purpose of stochastic gravity is to extend perturbatively the semiclassical theory in order to include the quantum fluctuations of the stress tensor. The physical observable that measures these fluctuations is the noise kernel: a four index bi-tensor defined as the expectation value of the anticommutator of the stress tensor operator $T_{ab}$,

$$N_{abc'd'}(x, x') = \frac{1}{2} \langle \{T_{ab}(x), T_{c'd'}(x')\} \rangle - \langle T_{ab}(x) \rangle \langle T_{c'd'}(x') \rangle.$$  

This expectation value is taken in the background metric $g_{ab}$ which we assume to be a solution of the semiclassical equation (1). An important property of the symmetric bi-tensor, $N_{abc'd'}(x, x') = N_{bc'd'a}(x', x)$, is that it is finite because the ultraviolet divergences of $T_{ab}$ are cancelled by the subtraction of $\langle T_{ab} \rangle$. Since the operator $T_{ab}$ is selfadjoint $N_{abc'd'}(x, x')$, which is the expectation value of an anticommutator, is real and positive semi-definite. This last property allows for the introduction of an auxiliary classical Gaussian stochastic tensor $\xi_{ab}$ defined by

$$\langle \xi_{ab}(x) \rangle_s = 0, \quad \langle \xi_{ab}(x) \xi_{c'd'}(x') \rangle_s = N_{abc'd'}(x, x').$$

This stochastic tensor is symmetric $\xi_{ab} = \xi_{ba}$ and divergenceless, $\nabla^a \xi_{ab} = 0$, as a consequence of the fact that the stress tensor operator is divergenceless. The subscript $s$ means that the expectation value is just a classical stochastic average.

The idea now is to modify the semiclassical Einstein equation (1) by introducing a linear correction to the metric tensor $g_{ab}$, such as $g_{ab} + h_{ab}$, which accounts consistently for the fluctuations of the stress tensor. The simplest equation is obtained by adding to the right hand side of equation (1), but evaluated for the perturbed metric, the stochastic tensor just introduced. Subtraction of the semiclassical equation (1) leads to the following equation,

$$G^{(1)}_{ab}[g + h] = \kappa \langle T^{(1)}_{ab} [g + h] \rangle_{\text{ren}} + \kappa \xi_{ab}[g],$$

where we recall that the background metric $g_{ab}$ is assumed to be a solution of equation (1). As indicated by the superscript (1) this stochastic equation must be thought of as a linear equation for the metric perturbation $h_{ab}$ which will behave consequently as a stochastic field tensor. This equation is gauge invariant with respect to diffeomorphisms defined by any field on the background spacetime [47].

The stochastic equation (4) is known as the Einstein-Langevin equation, and predicts that the gravitational field has stochastic fluctuations over the background $g_{ab}$. It is linear in $h_{ab}$, thus its solutions can be written in terms of a solution of the homogeneous equation containing information on the initial conditions, $h^0_{ab}(x)$, and the auxiliary field $\xi^{cd}(x')$ with the retarded propagator of equation (4) with vanishing initial conditions, $G^{\text{ret}}_{abcd}(x, x')$. Form this and equations (3), the two-point correlations for the metric perturbations are:

$$\langle h_{ab}(x) h_{cd}(y) \rangle_s = \langle h^0_{ab}(x) h^0_{cd}(y) \rangle_s + \kappa^2 \int d^4 x' d^4 y' \sqrt{g(x') g(y')} G^{\text{ret}}_{abef}(x, x') N^{efgh}(x', y') G^{\text{ret}}_{cdgh}(y, y').$$

These two-point stochastic correlations, which are independent of the auxiliary stochastic tensor introduced above, are the most relevant physical observables.

It turns out that these stochastic correlations are connected to the correlations of the quantized metric fluctuations in the large $N$ expansion. In fact, it can be shown that they correspond exactly to the symmetrized two-point quantum metric correlations obtained in the large $N$ expansion: $\langle \{h_{ab}(x), h_{cd}(y)\} \rangle = 2 \langle h_{ab}(x) h_{cd}(y) \rangle_s$, where $h_{ab}(x)$ mean the quantum
Another relevant isometry is the antipodal isometry $-X$ under Lorentz transformations is the position vector in Minkowski spacetime of the point $H + 1)$-dimensional Minkowski spacetime, whose points satisfy the equation $X^A(x)X^B(x) = H^{-2}$, where $H$ is a parameter with dimensions of mass, $\eta_{AB}$ the Minkowski metric, and $X^A(x)$ is the position vector in Minkowski spacetime of the point $x$. The hyperboloid is invariant under Lorentz transformations $X^A = \Lambda^A_B X^B$ and these define the isometries $\sigma$ of $dS_D$.

3. Stress tensor correlation in de Sitter spacetime

3.1. De Sitter geometry and de Sitter invariant bi-tensors

The $D$-dimensional de Sitter spacetime, $dS_D$, may be defined as a submanifold embedded in $(D + 1)$-dimensional Minkowski spacetime, whose points satisfy the equation $\eta_{AB}X^A(x)X^B(x) = H^{-2}$, where $H$ is a parameter with dimensions of mass, $\eta_{AB}$ the Minkowski metric, and $X^A(x)$ is the position vector in Minkowski spacetime of the point $x$. The hyperboloid is invariant under Lorentz transformations $X^A = \Lambda^A_B X^B$ and these define the isometries $\sigma$ of $dS_D$.

One may write the de Sitter metric in coordinates $(\tilde{\eta}, \chi, \theta, \varphi)$ which span a finite range $-\pi < \tilde{\eta} < 0$, $0 \leq \chi < \pi$, with $(\theta, \varphi)$ the standard spherical coordinates. It is given by

$$ds^2 = \frac{1}{H^2 \sin^2 \tilde{\eta}} \left( -d\tilde{\eta}^2 + d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right),$$

where the sections $\tilde{\eta} = \text{const}$ are three spheres. These coordinates cover the entire de Sitter spacetime. They represent a closed universe which contracts for $-\pi < \tilde{\eta} < -\pi/2$ and expand for $-\pi/2 < \tilde{\eta} < 0$. The de Sitter conformal diagram, for the conformal metric $(H^2 \sin^2 \tilde{\eta}) ds^2$, may be represented in terms of the coordinates $(\chi, \tilde{\eta})$, each point of the conformal diagram represents a $S^2$ sphere of radius $\sin \chi$.

To introduce the formalism of the de Sitter invariant bi-tensors we first define the basic bi-scalar

$$Z(x, x') = \eta_{AB}X^A(x)X^B(x'),$$

where $x, x' \in dS_D$, and where we assume $H = 1$ for simplicity. The value of $H$ will be restored in the final expressions. Note that $Z$ is invariant under the de Sitter group, if $\sigma$ is an isometry, then $Z(\sigma(x), \sigma(x')) = Z(x, x')$ and under the antipodal transformation $Z(\alpha(x), x') = -Z(x, x')$. This bi-scalar is related to the Minkowskian distance, defined by $d^2(x, x') = \eta_{AB}[X^A(x) - X^A(x')] [X^B(x) - X^B(x')]$, which is the length of the straight line in the embedding $(D + 1)$-Minkowski spacetime that joins $x$ and $x'$. It follows that $Z$ and $d$ are related through

$$d^2 = 2(1 - Z).$$

Therefore, $Z = 1$ when $x'$ is on the light cone of $x$, $Z > 1$ when the points are timelike separated, and $Z < 1$ when the points are spacelike separated. It should also be noted that $Z \in (-\infty, \infty)$, since the Minkowskian distance between two points on the hyperboloid can take any value.

Without loss of generality we can take the point $x$ in the conformal diagram $(\chi = 0, \tilde{\eta} = -\pi/2)$ and consider the spacelike straight curve with $\tilde{\eta} = \text{const}$ connecting $x$ and its antipodal $\alpha(x)$ which in that diagram is given by $(\chi = \pi, \tilde{\eta} = -\pi/2)$. This curve is a geodesic, because it is
a straight line on a cylinder tangent to the hyperboloid. The Minkowskian distance between \( x \) and any other point \( x' \) in this geodesic satisfies \( 0 \leq d(x, x') \leq 2 \). This implies

\[
-1 \leq Z(x, x') \leq 1.
\] (9)

Any other spacelike geodesic passing through \( x \) can be obtained from this one by an isometry that leaves \( x \) invariant (it also leaves \( \sigma(x) \) invariant). This is analogous to the case of the sphere, where any geodesic (meridian) passing through the north pole is obtained from a given one by a rotation that leaves the north pole (and south pole) invariant. This means that any point \( y' \) connected to \( x \) by a spacelike geodesic satisfies \( y' = \sigma(x') \) for some \( x' \) in the above geodesic, and where \( \sigma \) is an isometry such that \( \sigma(x) = x \). De Sitter invariance of \( Z \) then implies \( Z(x, y') = Z(\sigma(x), \sigma(x')) = Z(x, x') \). Consequently, equation (9) holds for any point \( x' \) connected with \( x \) by a spacelike geodesic. In other words, spacelike separated points \( x, x' \) such that \( Z(x, x') < -1 \) cannot be joined by a geodesic. For timelike geodesics, however, there are theorems that guarantee the existence of geodesics between any two timelike separated points when the spacetime is globally hyperbolic [55].

A bi-tensor is said to be maximally symmetric if it is invariant under all the isometries of the manifold. The following bi-tensors are maximally symmetric [40] in \( dS_D \): the distance, \( \mu(x, x') \), along the shortest geodesic joining \( x \) and \( x' \), also called the geodesic distance; the unit vectors, \( n_a(x, x') \) and \( n_{a'}(x, x') \), tangent to the geodesic at the points \( x \) and \( x' \) respectively, pointing outward from it; the parallel propagator, \( g_{ab}(x, x') \), which parallel-transports a vector from \( x' \) to \( x \) along the geodesic; and the metric tensors, \( g_{ab}(x, x') \) and \( g_{a'\nu}(x, x') \), at the points \( x \) and \( x' \) respectively. A theorem by Allen and Jacobson [40] states that any maximally symmetric bi-tensor is a linear combination of products of \( n_a, n_{a'}, g_{ab}, g_{a'\nu} \) and \( g_{ab\nu} \), with coefficients that depend only on \( \mu \).

If \( x(\lambda) \) is the shortest geodesic joining \( x \) and \( x' \), and its tangent vector is \( v^a(\lambda) \), and we take \( x(0) = x \) and \( x(1) = x' \), the geodesic distance between \( x \) and \( x' \) is

\[
\mu(x, x') = \int_0^1 d\lambda [g_{ab}(x(\lambda))v^a(\lambda)v^b(\lambda)]^{1/2}.
\] (10)

Note that \( \mu(x, x') \) is a real number for spacelike separated points, and an imaginary number for timelike separated points. The sign prescription for the square root is chosen in such a way that \( \mu(x, x') > 0 \) for spacelike separated points, and \( \mu(x, x') = i\xi(x, x') \), with \( \xi(x, x') > 0 \), for timelike separated points. The bi-tensors \( n_a \) and \( n_{a'} \) are defined by the equations \( n_a = \nabla_a \mu \), \( n_{a'} = \nabla_{a'} \mu \). Allen and Jacobson’s theorem can be applied to the computation of the covariant derivatives of \( n_a, n_{a'} \) and \( g_{ab\nu} \), which are also maximally symmetric bi-tensors. The corresponding \( \mu \)-dependent coefficients were computed in Ref. [40].

For spacelike geodesics, \(-1 \leq Z < 1\), the geodesic distance \( \mu \) of \( dS_D \) is given by the equation

\[
\mu = \cos^{-1} Z,
\] (11)

where \( \cos^{-1} \) is the principal value of the inverse cosine, which is single-valued. For example, \( \cos^{-1} 1 = 0 \), and \( \cos^{-1}(-1) = \pi \). This is easily seen for the geodesic with \( \tilde{\eta} = \text{const} \) joining \( (\chi = 0, \tilde{\eta} = -\pi/2) \) and \( (\chi = \pi, \tilde{\eta} = -\pi/2) \). Other spacelike geodesics are obtained from this one by an isometry, and therefore, by de Sitter invariance of \( \mu \) and \( Z \), Eq. (11) is true for all spacelike geodesics. For spacelike separated points not joined by a geodesic one may still define \( \mu \) by analytical continuation [40, 45]. The definition of \( n_a, n_{a'} \) and of the parallel propagator can then be extended. A geometric interpretation of this analytic extension for spacelike separated points not joined by geodesics is given in Ref. [45].
3.2. Noise kernel in de Sitter spacetime

Consider a quantum field in $dS_D$, whose stress tensor operator is $T_{ab}$, in a vacuum state $|0\rangle$ that we assume to be de Sitter invariant. We want to compute the noise kernel $N_{abc'd'}$ defined by Eq. (2). Since $|0\rangle$ is a de Sitter-invariant state, the expectation value of any observable must be de Sitter-invariant. Therefore, $N_{abc'd'}$ is a de Sitter-invariant bi-tensor and Allen and Jacobson’s theorem may be used. On the other hand, $N_{abc'd'}(x, x') = N_{c'd'ab}(x', x)$, and, since the stress tensor is symmetric, we also have that $N_{abc'd'} = N_{bac'd'}$. The most general bi-tensor that is a linear combination of products of $n_a$, $n_c$, $g_{ab}$, $g_{av'}$ and $g_{a'v}$, with coefficients that depend only on $\mu$, and that satisfies the above constraints is

$$N_{abc'd'} = P(\mu)n_an_bn_cn_dn_d + Q(\mu)(n_anbg_{c'd'} + n_cn_dg_{ab}) + R(\mu)(n_an_cg_{bd'} + n_bn_dg_{ac} + n_an_dg_{bc} + n_bn_cg_{ad'}) + S(\mu)(g_{ac}g_{bd'} + g_{bc}g_{ad'}) + T(\mu)g_{ab}g_{c'd'}.$$  

(12)

Moreover, the stress tensor conservation law implies $\nabla^a N_{abc'd'} = 0$ which imposes some differential equations for the coefficients $P, Q, R, S, T$. Also there are further restrictions on these coefficients imposed by the condition that the noise kernel should be unambiguous when $x$ and $x'$ are antipodal, see Ref. [45] for the details.

Now let us assume a free massive minimally coupled scalar field, $\xi = 0$, whose stress tensor is given by the coincidence limit, $T_{ab}(x) = \lim_{x' \to -x} D_{ab}(x, x')\phi(x)\phi(x')$, where $D_{ab}$ is a second order differential operator [45]. The stress tensor two-point function of the free scalar field is obtained by substituting the previous expression in (2). Since the field $\phi$ is free, and $|0\rangle$ is a vacuum state, the four-point function appearing in this equation can be written in terms of the Wightman function, $G^+(x, x') = \langle 0 | \phi(x)\phi(x') | 0 \rangle$,

$$\langle 0 | \phi(x)\phi(x')\phi(x')\phi(x''') | 0 \rangle = G^+(x, x'')G^+(x', x''')+G^+(x, x')G^+(x'', x''')+G^+(x, x'')G^+(x'', x').$$

This equation is obtained by expanding the field $\phi$ in terms of creation and annihilation operators. Note that, in the particular case where the product of fields on the left-hand side is time-ordered, this equation also follows from Wick’s theorem. Substituting into (2) using the previous expression for the stress tensor, and taking the limit $x'' \to x$ and $x''' \to x'$, one finds $N_{abc'd'}$ in terms of $G^+$ [45].

We now use that the vacuum $|0\rangle$ is de Sitter-invariant, and impose that the points $x$ and $x'$ are spacelike separated. For spacelike separated points, the Wightman function is the expectation value of an observable, so that it must be a de Sitter-invariant bi-scalar. Thus it must have the form $G^+ = G(\mu)$. On the other hand, the Wightman function satisfies the Klein-Gordon equation. Imposing as boundary conditions that $G(\mu)$ diverges only at the coincidence limit, $\mu \to 0$, and that it does so in the same way as the standard Wightman function in Minkowski spacetime, the only solution [45] is $G(\mu) = c_{m,D}F\left(h_+, h_-, \frac{D}{2} \frac{1+Z}{Z} \right)$, where $F$ denotes the hypergeometric function, $h_\pm$ and $c_{m,D}$ are constants, $2h_\pm = (D - 1) \pm \sqrt{(D - 1)^2 - 4m^2}$ and $c_{m,D}(4\pi)^{D/2}\Gamma\left(\frac{D}{2}\right) = \Gamma(h_+)\Gamma(h_-)$, and we are using that $Z = \cos \mu$, as follows from (11). One can see that there is a vacuum whose Wightman function for spacelike separated points is just that expression, which is the Bunch-Davies vacuum. Substituting $G(\mu)$ into the noise kernel, one finds that the noise kernel in the Bunch-Davies vacuum has the form (12), and the functions $P, Q, R, S, T$ are explicitly given in terms of $G(\mu)$; see [45] for the details.

4. Quantum gravitational fluctuations

To characterize the gravitational fluctuations we may compute the two-point function of the linearized Riemann tensor. The Riemann tensor of the background de Sitter spacetime with suitably raised indices is given by $R_{abcd} = (R/6)\delta^c_{[a}\delta^d_{b]}$. Its linear perturbation is gauge invariant
because its Lie derivative with respect to an arbitrary vector field vanishes. The Riemann tensor is entirely determined by the Ricci tensor (or equivalently the Einstein tensor) and the Weyl tensor. Both the linearized Einstein and the Weyl tensors over a de Sitter background are separately gauge invariant, and the Weyl tensor also vanishes in the background.

The previous results for the noise kernel can be directly used to get information on the quantum gravitational fluctuations in de Sitter spacetime. It turns out that the lowest-order contribution to the two-point function for the linearized Einstein tensor is of order $l_P^5$ and comes directly from the one-loop contributions of the matter fields. Both from the stochastic gravity formalism [48, 49, 54, 50] or from the Einstein equation to linear order in the metric perturbations (including the matter fields as well as the metric perturbations). Using order reduction [61] one can see that the two higher derivative quadratic curvature tensors, which are implicit in the Einstein-Langevin equation (4), and which appear as counterterms in the renormalization of the stress tensor, both vanish in this case because they become zero for a de Sitter background. Thus only the linearized Einstein tensor survives, and one obtains the following result for the (connected) symmetrized two-point function of the Einstein tensor at that order:

$$\langle G^{(1)}_{ab} (x) G^{(1)}_{cd} (x') \rangle_c = 64 \pi^2 \frac{1}{l_P^4} N^{-1} N^{a'}_{b'} N^{c'}_{d'} (x, x'),$$

where $G^{(1)}_{ab}$ is the linearized Einstein tensor, and $N^{a'}_{b'} N^{c'}_{d'}$ is the noise kernel. Therefore, the noise kernel in de Sitter gives the leading order contribution to the symmetrized two-point function of the linearized Einstein tensor. From the results for the noise kernel [45], for massless fields, $m = 0$, and long distances $d$, we have

$$\langle G^{(1)}_{ab} (x) G^{(1)}_{cd} (x') \rangle_c \sim \frac{l_P^2 H^4}{N d^4},$$

and for low mass fields, $m \ll H$, we have

$$\langle G^{(1)}_{ab} (x) G^{(1)}_{cd} (x') \rangle_c \sim \frac{l_P^2 H^8}{N} (H d)^{-\frac{4m^2}{3m^2}},$$

which implies very long range correlations.

The two-point function for the linearized Weyl tensor which is also gauge invariant gets contributions of order $l_P^4$ and higher from matter loops, but also a non-vanishing tree-level contribution of order $l_P^5$ even in the absence of matter loops, which was calculated by Kouris [44]. For spacelike separated points and long distances one has

$$\langle W^{(1)}_{abcd} (x) W^{(1)}_{a'b'c'd'} (x') \rangle_c \sim \frac{l_P^2 H^2}{N d^4} + O \left( \frac{l_P^2 H^4}{N} \right).$$

The matter loop contributions of the order $l_P^4$ have not been computed yet. When combined with the tree-level result for the correlations of the Weyl tensor in Ref. [44], our results for the correlations of the Einstein tensor completely characterizes the quantum fluctuations of the full Riemann tensor. It should be pointed out that the two-point function corresponding to the cross-correlation of the Einstein and Weyl tensors is of order $l_P^6$ (rather than $l_P^5$). This can be interpreted as the absence of correlations between the tree-level fluctuations of the Weyl tensor, with an amplitude of order $l_P$, and the fluctuations of the Ricci tensor, with an amplitude of order $l_P^5$. The two-point function for the linearized Weyl tensor given in the previous expression may be compared to the two-point function computed over a Minkowski background, given by

$$\langle W^{(1)}_{abcd} (x) W^{(1)}_{a'b'c'd'} (x') \rangle_c \sim \frac{l_P^2}{N d^4}. $$
This result agrees also with the the two-point function for the linearized Weyl tensor in a de Sitter background at short distances.

Note that in contrast to the existing calculations of loop corrections to the spectrum of cosmological perturbations in inflationary models, where the self-interaction of the matter fields often plays an important role [10, 11], the only interaction vertices associated with the free fields that we have considered here are those corresponding to their gravitational interaction. Moreover, we do not consider a non-vanishing homogeneous background configuration for the scalar fields (which would give also a tree-level contribution to the stress-tensor two-point function [56]) and the usual treatment for the correlations of the scalar-type metric perturbations [6, 11, 12] cannot be directly applied. We believe that considering the correlations of the Riemann tensor and expressing them in terms of maximally invariant tensors is very useful in order to analyze whether de Sitter invariance is broken by loop corrections.

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