Topological quiver matrix models and quantum foam

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Abstract

We study the matrix models that describe the BPS bound states of branes arising from the quiver picture of the derived category. These theories have a topological partition function that localizes to the Euler character of the anti-ghost bundle over the classical BPS moduli space. We examine the effective internal geometry of D6/D2 bound states in the local vertex geometry, using BPS 0-brane probes. The Kahler blowups of the Calabi-Yau that we find utilizing these quiver theories are a realization of A-model quantum foam in the full IIA theory.

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1 Introduction

The idea that the topological A-model involves quantized fluctuations of the Kahler geometry of a Calabi-Yau was introduced in [23]. They argued that the topological string partition function could be reproduced by summing over non-Calabi-Yau blow ups along collections of curves and points. The equivalent Donaldson-Thomas theory, given by a topologically twisted $\mathcal{N} = 2$ $U(1)$ gauge theory on the Calabi-Yau, involves a sum over singular instantons, which can be blown up to obtain the fluctuations of the geometry.

The connection between this theory of BPS bound states of D2 and D0 branes to a 6-brane was further explained in [13] by lifting to M-theory. The D6-brane lifts to a Taub-NUT geometry in 11 dimensions, and the Donaldson-Thomas theory results in precisely that repackaging of the Gopakumar-Vafa invariants counting bound states of D2/D0 at the center of the Taub-NUT which is required to reproduce the A-model.

In this work, I will try to elucidate the role of Kahler quantum geometry when the topological A-model is embedded in the full IIA theory. We will show that the effective internal geometry experienced by BPS 0-brane probes of bound states of 2-branes and a D6 is exactly the blow up of the Calabi-Yau along the wrapped curves! This will be done in the context of the quiver matrix models that describe the low energy dynamics, so we will see the topological string theory emerge from a matrix model, with quite a different flavor than [12].

Therefore, if such a bound state in the mixed ensemble were constructing in $\mathbb{R}^{3,1}$, then in the probe approximation, the effective geometry of the Calabi-Yau would be literally the “foam” envisioning in [23]. Moreover, the work of [11] relating D4/D2/D0 bound states with classical black hole horizons to multi-centered D6/D6 configurations means this might have implications for BPS black holes with non-vanishing classical entropy.

More subtle probes of the geometry using $N$ 0-branes are able to discern the presence of a line bundle associated to the exceptional divisors of the blow up. Thus we see that the 2-branes are blown up to 4-branes, which can be dissolved into $U(1)$ flux on the D6. Moreover, only a small number of 0-branes can “fit” comfortably on the exceptional divisors, which might indicate that their Kahler size is of order $g_s$, as predicted in [23]. We will exactly reproduce the know topological vertex amplitudes from the quivers we discover, by counting fixed points of the equivariant $T^3$ action.

The topological quiver matrix models we shall discuss are quite interesting in their own right. Consider the BPS bound states of D6/D4/D2/D0 system in IIA theory compactified on a Calabi-Yau 3-fold. For large Kahler moduli, the D6 brane worldvolume theory is described by the 6+1 dimensional nonabelian Born-Infeld action. At low energies, we can integrate out the Kaluza-Klein modes in the internal manifold, presumably giving rise to a 0+1 dimensional superconformal quantum mechanics, which is the gauge theory dual to the $AdS_5$ near horizon geometry. The full superconformal theory remains mysterious, but there have been numerous attempts to understand it, see for instance [19].

Our interest is in the index of BPS ground states, which is captured by the topological twisted version of the physical theory. In the case of D6 branes, this is the twisted $\mathcal{N} = 2$ Yang Mills in six dimensions. The low energy description is now a 0-dimensional topological theory, that is a matrix model with a BRST trivial action. This topological version of the dual theory is far simpler, and can be written down exactly in non-trivial situations.

The physical interpretation of the quiver matrix model changes dramatically as one moves from the Gepner point to the large volume limit in the Kahler structure moduli space. It is very satisfying to see that the quiver description, which is most at home at the Gepner point, gives the correct result for the index of BPS states even in the opposite noncompact limit we will consider. At large volume, the quiver should be thought of as the collective coordinates of instantons in the worldvolume theory on the D6 brane.

Note that the moduli on which the theory depends are really the background moduli, defined at infinity in spacetime. In the case of 6-brane bound state we are examining, there are no BPS states at the naive attractor value of the moduli, so the attractor flow is always a split flow. Amazingly, this doesn’t impair the ability of the quiver theory to capture the index of bound
states, and furthermore we are able to read off certain aspects of the attractor flow tree \[11\] from the shape of the matrix model potential.

One of the most powerful worldsheet techniques to explore string theory compactified on a Calabi-Yau given by a complete intersection in projective space are gauged linear sigma models with superpotential. These two dimensional worldsheet theories can be constructed explicitly and exactly, and flow in the IR to the conformal non-linear sigma on the Calabi-Yau. For the purpose of understanding BPS saturated questions, the linear model itself already gives the exact answer, just as in the context of the topological string; finding the IR fixed point is unnecessary.

The holomorphically wrapped D-branes we are interested in are described by boundary linear sigma models, which have been studied extensively in the context of the B-model. The branes of the topological B-model can be encoded in the boundary conditions for the open B-model worldsheet, which are naively associated to vector bundles on the Calabi-Yau. In fact, a much richer structure of boundary data has been discovered, associated to D6 and anti-D6 branes with tachyons condensed. Mathematically speaking, the holomorphic branes are objects in the derived category of coherent sheaves \[14\], with the tachyons becoming maps between the sheaves associated to the 6-branes. The BPS branes of IIA theory require a stability condition (for generic Kahler moduli, \(\pi\)-stability of the triangulated category) not present for branes of the topological B-model.

At the Gepner point in the Calabi-Yau moduli space, this description simplifies, and one can generate all of the configurations of branes by condensing tachyons between a complete collection of fractional branes. The chain maps of the derived category are then encoded in the linear maps between a quiver of vector spaces, as shown in \[17\]. In general there will be a nontrivial superpotential, which must be minimized to obtain the supersymmetric ground states \[5\]. The effective topological theory of BPS branes is therefore the topological quiver matrix model, whose partition function localizes to the Euler characteristic of the anti-ghost bundle over the Kahler quotient of the quiver variety.

As we move in the Calabi-Yau moduli space out to the large volume, certain degrees of freedom will become frozen in the local limit. We find the quiver for the remaining dynamical degrees of freedom, and discover that there are a few extra terms in the effective action. In the local limit, branes with different dimensions, or those wrapping cycles with different asymptotics in the local Calabi-Yau, become sharply distinguished. This corresponds to the breaking of some gauge groups \(U(N + M) \to U(N) \times U(M)\) of the Gepner point quiver. There are residual terms in the action from the off-diagonal components of the original D-term, which need to retained even in the local limit. Crucially, however, the local limit is universal, and the action we find is independent of the specific global geometry into which is was embedded, so our analysis is consistent.

The index of BPS states always looks, from the large volume perspective, like the Euler character of some bundle over the resolution of the instanton moduli space. The classical instanton moduli are described by the zero modes of the instanton solutions in the topologically twisted gauge theory living on the 6-branes. We will find this moduli space more directly, using the quiver description of the derived category of coherent sheaves, and taking the appropriate large volume limit.

The resulting holomorphic moduli space literally gives the moduli space of B-branes as the solution to some matrix equations \[7\], moded out by the gauge group. There are exactly the pieces of the open string worldsheet gauged linear sigma model which survive the topological string reduction to the finite dimensional \(Q\)-cohomology. This moduli space is explicitly independent of the Kahler moduli.

It is crucial that much of this moduli space does not correspond to BPS bound states of holomorphically wrapped branes in IIA theory. This is already clear because the stability of bound states depends on the background Kahler moduli. There is a clear realization of this fact in the gauge theory living on the branes, whose instantons satisfy the Hermitian Yang-Mills...
The holomorphic moduli space only imposes the F-term conditions, namely

\[ F^{2,0} = 0, \quad (1) \]

and gauges the complexified gauge group, \( GL(N, \mathbb{C}) \), while the physical instanton moduli space is the Kahler quotient, obtained from the D-term condition,

\[ F^{1,1} \wedge \omega^{d-1} = r\omega^d, \quad (2) \]

which plays the role of the moment map, together with the compact gauge group \( U(N) \).

In the case of D4/D2/D0 bound states on local Calabi-Yau, whose BPS sector is described by the Vafa-Witten twist of \( N = 4 \) Yang Mills theory on a 4-manifold, the partition function reduces to the Euler character of the instanton moduli space. There the quiver realization is exactly the ADHM construction of this finite dimensional space. This can be promoted to a matrix model with the same partition function, moreover the structure of the fermionic terms will be associated to the tangent bundle over the classical moduli space. Satisfyingly, this theory has exactly the 10 expected scalars and 16 fermions of the dimensional reduction of the maximally supersymmetric gauge theories in higher dimensions. Moreover, the action looks like a topologically twisted version of the usual D0 worldvolume theory. In flat space, there are no dynamical degrees of freedom associated to the motion of the noncompact D2 and D4 branes, but there are near massless bifundamentals associated to the 4-0 and 2-0 strings.

Our matrix models are the analogous construction for 6-branes. There is no reason for the antighost bundle of the matrix model to be the tangent bundle, although their dimensions are equal when the virtual dimension of the moduli space is zero. In fact, we will find that this obstruction bundle is indeed different in the case of D6/D0 bound states.

The quivers we will find provide are sensitive probes of the effective geometry, enabled one to go beyond calculations of the Euler characteristic. An interesting structure emerges, in which different effective quivers for the dynamical 0-brane fields are related by flops in the blow up geometry of the Calabi-Yau they probe. This leads to a more detailed understanding of the quantum foam picture of the topological A-model.

2 Review of quivers and topological matrix models

2.1 A quiver description of the classical D6-D0 moduli space

We will begin by constructing the classical moduli space of \( N \) D0 branes in the vertex geometry. The gauged linear model can be obtained by dimensional reduction of the instanton equations in higher dimensions. Consider bound states of holomorphically wrapped branes which can be expressed as flux dissolved into the worldvolume of the top dimensional brane. Then the classical moduli space of these instantons is determined by the solutions of the Hermitian Yang-Mills equations,

\[ F^{(2,0)} = 0, \quad F^{(1,1)} \wedge \omega^{d-1} = r\omega^d. \quad (3) \]

The reduction of the 6d gauge field to zero dimensions results in 6 Hermitian scalar fields, which can be conveniently combined into complex \( Z_i \), for \( i = 1, 2, 3 \). In terms of these matrices, the F-term condition is the zero dimensional reduction of \( F^{(2,0)} = 0 \),

\[ [Z_i, Z_j] = 0. \quad (4) \]

We find the D-term constraint as the analog of \( F^{(1,1)} = r\omega \), where \( \omega \) is the Kahler form,

\[ \sum_{i=1}^{3} [Z_i, Z_i^\dagger] = 0, \quad (5) \]
which one can think of as the moment map associated to the $U(N)$ action on adjoint fields. Here it is impossible to find solutions with nonzero $r$, since the left hand side is traceless. This is equivalent to the fact that one cannot add a Fayet-Iliopoulos term when there are only adjoint fields. The F-term condition (4) comes from the superpotential

$$W = \text{Tr} Z_1[Z_2, Z_3],$$

by the usual BPS condition that $\partial W = 0$.

The gauged linear sigma model description of the topologically twisted version of the world-volume theory of $N$ D0 branes in $\mathbb{C}^3$ can be obtained by computing the appropriate $\text{Ext}$ groups in the derived category description. The fields in the quiver theory are exactly the 0-0 strings that in this case give three complex adjoint fields, $Z_i$, of $U(N)$, which can be naturally interpreted as the collective coordinates of $N$ points in $\mathbb{C}^3$.

Adding the background D6 brane extends this quiver by a new $U(1)$ node, with a single field, $q$, coming from the 0-6 strings, transforming in the bifundamental, as shown in [35]. As described by Witten, the new quiver data which determines the zero dimensional analog of the instanton equations (3) is

$$[Z_i, Z_j] = 0$$

$$\sum_{i=1}^{3} [Z_i, Z_i^\dagger] + q^\dagger q = rI_N$$

The Fayet-Iliopoulos parameter, $r$, will turn out to be the mass of the 0-6 fields, as we will see more clearly in the matrix model. This mass is determined by the asymptotic B-field needed to preserve supersymmetry in the D6/D0 bound states [35]. In the absence of a B-field, $r < 0$, and $q$ is a massive field, hence there would be a stable nonsupersymmetric vacuum. For $r > 0$, which is the case of interest for us, the now tachyonic $q$ condenses to give a supersymmetric bound state. Note that there are no new possible terms in the superpotential (6) because of the shape of the quiver, hence the bifundamental can only appear as we have written it in the D-term.

The holomorphic quotient space that is naturally determined by the F-term is $\mathcal{M}_{\text{hol}} = X/GL(N, \mathbb{C})$, where $X$ is the space of commuting complex matrices. This is exactly the moduli space of branes in the B-model, which depends purely on holomorphic data. It turns out to be a larger space, with less structure, than the Kahler quotient $\mathcal{M}_{\text{Kahler}} = X//U(N)$ that describes the physical BPS bound states in II theory. If an additional stability condition, which in this case is the existence of a cyclic vector for the representation of the quiver, is imposed on $\mathcal{M}_{\text{hol}}$ then it can be shown to be identical to the Kahler quotient, following the reasoning of [27].

### 2.2 Quiver matrix model of D0 branes

The index of BPS states of 0-branes can be found using the topologically twisted reduction of the D0 matrix model of [7], where the time directions drops out for the obvious reasons when considering the structure of ground states. This results in a zero dimensional theory, i.e. a matrix model, with a BRST like supersymmetry.

Following [28] and [29], we will first consider the theory describing only D0 branes in $\mathbb{C}^3$, by utilizing the formalism developed by [34] to study Euler characteristics by means of path integrals. The action of the topological supersymmetry is given by

$$\delta Z_i = \psi_i, \quad \delta \psi_i = [\phi, Z_i], \quad \delta \phi = 0$$
$$\delta \tilde{\phi} = \eta, \quad \delta \eta = [\phi, \tilde{\phi}]$$
$$\delta \varphi = \zeta, \quad \delta \zeta = [\phi, \varphi]$$
$$\delta \chi_i = H_i, \quad \delta H_i = [\phi, \chi_i]$$
$$\delta \chi = H, \quad \delta H = [\phi, \chi],$$
where all fields are complex $U(N)$ adjoints, except for $\chi$ and $H$ which are Hermitian. This corresponds to the $d=10$ case of (29) in a slightly different notation that is more convenient in the context of compactification on a Calabi-Yau 3-fold. The $Z_i$ are the D0 collective coordinates in the internal manifold, while $\varphi$ morally lives in the bundle of $(3,0)+(0,3)$ forms over the Calabi-Yau, which practically means that it is a scalar.

This is the reduction to zero dimensions of the twisted $\mathcal{N}=4$ Vafa-Witten theory, and equivalently, of the twisted $\mathcal{N}=2$ gauge theory in six dimensions. We will write a $Q$-trivial action for this topological matrix model, which can be interpreted as the topological twisted version of the $0+1$ dimensional superconformal quiver quantum mechanics [19] describing these bound states. The topological property results from the fact we are only interested in computed the index of BPS states, which can be described in this zero dimensional theory.

Choose the sections, in the language of (34), to be
\[
s_i = \Omega_{ijk} Z_j Z_k + [\varphi, Z_k^\dagger]
\]
\[
s = \sum_i [Z_i, Z_i^\dagger] + [\varphi^\dagger, \varphi],
\]
where $\Omega_{ijk}$ is the holomorphic 3-form, which we can assume to be given by the antisymmetric $\epsilon_{ijk}$ for the $\mathbb{C}^3$ vertex geometry.

The action is given by $S = t\{Q, V\}$, where
\[
V = \text{Tr} \left( \chi_i^\dagger (H_i - s_i) \right) + \text{Tr} (\chi (H - s)) + \text{Tr} \left( \psi_i [\bar{\phi}, Z_i^\dagger] \right) + \text{Tr} \left( \zeta^\dagger [\bar{\phi}, \varphi] \right) + \text{Tr} \left( \eta [\varphi, \bar{\phi}] \right).
\]
Following (34), we write the bosonic terms after integrating out $H_i$ and $H$, which appear quadratically in the action, to obtain
\[
S_{\text{bosonic}} = \text{Tr} \left( s_i^\dagger s_i + s^2 + [\phi, Z_i][\bar{\phi}, Z_i]^\dagger + [\varphi, \varphi^\dagger][\bar{\phi}, \bar{\phi}]^\dagger + [\bar{\phi}, \bar{\phi}]^2 \right),
\]
where the $\text{Tr}[\phi, Z_i][\bar{\phi}, Z_i]^\dagger$ terms arise from the twisted superpotential, thus coupling the vector multiplet, $\phi$, with the chiral matter fields charged under it.

This theory has a total of 9 real scalars, coming from the $Z_i$, $\varphi$, and $\phi$, transforming the adjoint of $U(N)$, corresponding to the 9 collective coordinates of a point-like brane. It is clear, following the reasoning of (34), that the partition function will count, with signs, the Euler character of the moduli space of gauge equivalence classes of solutions to the equations, $s_i = 0$, $s = 0$. Moreover, the field $\phi$ can be easily eliminated from the theory, since it simple acts to enforce the $U(N)$ gauge symmetry on the moduli space. To relate this to the ADHM type data we discussed in the previous section, we need to understand what happens to the extra collective coordinate, $\varphi$. In fact, we will prove a vanishing theorem, and find that it does not contribute to the instanton moduli space. Its main function in the matrix model is to ensure that all of the solutions are counted with the same sign, so that the Euler character is correctly obtained, in the same spirit as (34).

Note that there are 4 dynamical complex adjoint fields in the matrix model, the $Z_i$ and $\varphi$, which satisfy 3 holomorphic equations (8). The Kahler quotient by $U(N)$ gives 1 real D-term equation from the moment map, and quotients by $U(N)$. Hence the expected dimension of the moduli space is $8-6-1-1 = 0$, however it may not consist of isolated points in practice unless the sections are deformed in a sufficiently generic manner.

In order to fully make sense of this theory, we need to regulate the noncompactness of $\mathbb{C}^3$ in some way. For local analysis on toric Calabi-Yau, the natural choice is to add torus-twisted mass terms, which will further enable one to localize the path integral to the fixed points of the induced action of $U(1)^3$. It is possible to understand the toric localization in the matrix model language. We want to turn on the zero dimensional analog of the $\Omega$-background explained in (23). Exactly as in that case, we twist the $U(N)$ gauge group by the torus $U(1)^3$ symmetry. That is, we gauge a nontrivial $U(N) = SU(N) \times U(1)$ subgroup of $U(N) \times U(1)^3$. 
The effect of this on the topological matrix model is to change the BRST operator so that $Q^2$ generates one of the new gauge transformations. That is, the fields transform as

$$
\delta Z_i = \psi_i
$$

$$
\delta \psi_i = [\phi, Z_i] - \epsilon_i Z_i,
$$

(10)

where the $\epsilon_i$ determine the embedding of the gauged $U(1)$ inside $U(1) \times U(1)^3$.

This changes the terms involving $\phi$ in the bosonic action to become $\sum \text{Tr}([\phi, Z_i] - \epsilon_i Z_i)([\bar{\phi} - \epsilon Z_i]^t)$, which forces the adjoint fields $Z_i$ to be localized with respect to the torus action. We will soon see that the partition function is independent of the choice of weights $\epsilon_i$ as long as the superpotential itself is invariant under the chosen group action, so that

$$
\epsilon_1 + \epsilon_2 + \epsilon_3 = 0.
$$

This is natural, since only these torus actions are subgroups of the $SU(3)$ holonomy, and thus the equivariant twisted theory continues to preserve supersymmetry.

There is an induced action of $U(1)^3$ on the other fields in the theory, $\chi_i \to e^{i\alpha_i - i\epsilon_i} \chi_i$ and $\varphi \to e^{\alpha \varphi}$, where $\alpha = \sum \epsilon_i$, which changes the action of $Q$ in the obvious way.

To understand the measure factor at the solutions that contribute to the partition function we need to examine the fermionic piece of the action,

$$
S_f = \text{Tr} \left( \phi [\chi_i, \chi_i^t] + \bar{\phi} ([\psi_i, \psi_i^t] - [\zeta, \zeta^t]) + \Omega_{ijk} \psi_i [Z_j, \chi_k^t] + \zeta [Z_i, \chi_i^t] - \psi_i [\chi_i, \varphi] \right)
$$

$$
+ \text{Tr} \left( (\chi + i\eta)([\psi_i, Z_i^t] - [\zeta, \varphi^t]) \right).
$$

(11)

The anti-ghost, $\chi$, which enforces the moment map condition, naturally pairs with $\eta$, the fermionic partner of the $U(N)$ multiplet, to produce the Kahler quotient of the moduli space. We see that $\eta$ appears linearly in the action, thus when it is integrated out, the delta function constraint $[\psi_i, Z_i^t] - [\zeta, \varphi^t] = 0$ is enforced. This forces the fermions $\psi_i$ and $\zeta$ to lie in the sub-bundle of the tangent bundle normal to the D-term condition, $s = 0$. Therefore the Euler character of the Kahler quotient will be obtained, as in [34].

The topological nature of the partition function means that we are free to change the coupling, $t$, which results in a BRST trivial change of the theory, leaving the partition function invariant. Taking the limit $t \to \infty$, the matrix model localizes to the classical moduli space defined by

$$
\Omega_{ijk} Z_j Z_k + [\varphi, Z_i^t] = 0
$$

$$
\sum_i [Z_i, Z_i^t] + [\varphi^t, \varphi] = 0,
$$

(12)

where we must mod out by the gauge group, $U(N)$.

Near the solutions to the BPS equations the potential can be approximated as a Gaussian, and the partition function will pick up a determinant factor. From the bosonic part of the action [29] expanded near a solution, there are quadratic terms of the form $\text{Tr}(\delta \phi (\delta^* s_i))$, where here $\delta$ is the variation. Terms such as $\text{Tr} (s_i^t (s_i^* (\delta s_i)))$ cannot arise since $s_i = 0$ for the BPS solutions. For each such contribution, there is an analogous fermionic term of the form $\text{Tr}(\chi_i^t (\delta s_i))$. After integrating out the anti-ghosts, the quadratic terms involving the ghosts $\psi_i$ will exactly match those of the fields $Z_i$. Call the resulting quadratic form $A$.

In addition there are the twisted superpotential terms of the form $[\phi, Z_i] + \epsilon_i Z_i]^2$, which give a total contribution to the one loop determinant

$$
\frac{1}{\det(\text{ad} \phi + \alpha - \epsilon_i)} \det(\text{ad} \phi + \alpha - \epsilon_i)
$$

$$
\frac{1}{\det(\text{ad} \phi + \alpha)} \det(\text{ad} \phi + \epsilon_i)
$$

(13)

which exactly cancels, up to sign when we include the measure factor for diagonalizing $\phi$, if $\alpha = \sum \epsilon_i = 0$, as in [29]. This is all there is for the bound state of $N$ 0-branes, since there are
no nontrivial fixed points, hence \( A = 0 \). For future reference, the answer in general would be

\[
\frac{\det(A + T)}{\det(A + T')},
\]

where \( T \) and \( T' \) are the \( \epsilon_i \) dependent pieces in the quadratic approximation. Thus for \( \sum \epsilon_i = 0 \), we still have exact cancellation, even at the nontrivial fixed points. This is no surprise, since we designed the theory for precisely this effect.

Naively one would expect that the anti-ghost bundle whose Euler character the matrix model will compute is simply spanned by \( \chi_i \) and \( \chi \) fibered trivially over the moduli space. However, by looking at the fermion kinetic terms involving \( \psi_i \) and \( \zeta \), one finds that the bundle is obstructed when restricted to the manifold obtained after imposing the F-flatness condition. In particular, there are terms in the action,

\[
\text{Tr} \left( \psi_i \left( \Omega_{ijk} [\chi_j^\dagger, Z_k] - [\chi, Z_i^\dagger] \right) + \zeta \left( [\chi_i^\dagger, Z_i^\dagger] + [\chi, \varphi^\dagger] \right) \right),
\]

which require that

\[
\Omega_{ijk} [Z_i^\dagger, \chi_j] = [Z_k, \chi],
\]

\[
[Z_i, \chi_i] = [\chi, \varphi].
\]

This defines the anti-ghost bundle, or rather complex of bundles, whose Euler character is the index of BPS states computed by the matrix integral. It is obviously distinct from the tangent bundle, although it has the same rank, and their Euler characters may thus differ.

Now we proceed to find the vanishing theorem that shows the field \( \varphi \) does not contribute to the instanton equations. First, notice that using the F-term condition,

\[
\sum_k ||[\varphi^\dagger, Z_k]||^2 = \text{Tr}[\varphi^\dagger, Z_k] \Omega_{ijk} Z_j Z_k = \frac{1}{2} \text{Tr} \varphi^\dagger \Omega_{ijk} [Z_k, [Z_j, Z_k]] = 0,
\]

by the Jacobi identity, and the antisymmetry of \( \Omega_{ijk} \). This means that

\[
[[\varphi^\dagger, Z_k]] = 0,
\]

and moreover \( [Z_i, Z_j] = 0 \), as we had hoped.

What can we learn from the D-term? Rewriting it as

\[
0 = \sum_k ||[\varphi^\dagger, Z_k]||^2 + ||[\varphi^\dagger, \varphi]||^2 + 2 \sum \text{Tr}[\varphi^\dagger, \varphi][Z_i, Z_i^\dagger],
\]

we see that all the terms are positive, since

\[
\text{Tr} \varphi^\dagger \left( \varphi, [Z_i, Z_i^\dagger] \right) = - \text{Tr}[\varphi^\dagger, Z_i][Z_i^\dagger, \varphi] - \text{Tr}[\varphi^\dagger, Z_i^\dagger][\varphi, Z_i] = ||[Z_i, \varphi]||^2,
\]

by the Jacobi identity and equation (13). Therefore we see that \( \varphi \) commutes with both the \( Z_i \) and their conjugates, and can be trivially factored out of the theory.

At this point, all of the fields are on the same footing, since we can use the same reasoning to show that \( [Z_i, Z_j^\dagger] = 0 \), and the theory localizes to the trivial branch of moduli space. This is exactly the marginally supersymmetric state of \( N \) independent D0 branes. Note that if we rewrite the adjoint fields in terms of 9 Hermitian collective coordinates, and expand all of the terms appearing in the action, we will see the full \( SO(9) \) rotational symmetry of non-relativistic D0 branes in flat space. This will soon be broken by the addition of a D6 brane, and the presence of a B-field.
3 Solving the matrix model for D6/D0 in the vertex

Now we will proceed to find the matrix model description of the quantum foam theory of bound states of 1 D6 brane and \( N \) D0 branes in the vertex. The only new field comes from the 0-6 strings, which results in a chiral scalar in the low energy description. This appears in the matrix model as a new topological multiplet,

\[
\delta q = \rho \\
\delta \rho = \phi q,
\]

living in the fundamental of \( U(N) \). See the left side of Figure 1 for the quiver diagram of the internal Calabi-Yau degrees of freedom.

It is clear that moving the D0 branes off the D6 brane in the normal directions will give a mass to the 0-6 strings. Thus there should be a term in the quiver action of the form \( q^\dagger \phi \varphi \dagger q \). The similar mass term, \( q^\dagger \bar{\phi} \phi q \), is automatically included in the twisted superpotential, as we shall see below. Moreover, the BPS bound states must therefore have \( \varphi^\dagger q = 0 \), which, however cannot obviously be implemented by any superpotential. This requires the addition of new anti-ghosts,

\[
\delta \xi = h \\
\delta h = \phi \xi,
\]

living in the fundamental of \( U(N) \). The number of new fields is thus cancelled by the new equations, and we are still left with a moduli space of expected dimension 0.

As we will see later in similar examples, this extra equation should be understood as a symptom of working in a noncompact geometry. Quiver theories normally arise by considering the theory of fractional branes at a Gepner point where the central charges are all aligned. The exact nature of that quiver depends upon the compactification, but in the local limit, will be reduced to those found here. The general pattern is that a larger gauge group gets broken to \( U(N) \times U(M) \) where the \( N \) and \( M \) are the numbers of branes that are distinguished (for example as D0 and D6) in the large volume limit we are interested in, far from the Gepner point. There will typically be some off-diagonal D-term which persists even in the large volume limit, and \( \varphi^\dagger q = 0 \) is just such an example.

The \( U(N) \) D-term now includes the contribution of the bifundamental, replacing the section \( s \) by

\[
s = \sum_i [Z_i, Z_i^\dagger] + [\varphi^\dagger, \varphi] + qq^\dagger - r I,
\]

where \( r \) is the mass of the field \( q \). The anti-ghost \( \xi \) appears in the action in the form

\[
\{ Q, \xi^\dagger (h - \varphi q) \}.
\]

Putting everything together, we find that the bosonic part of the action is

\[
S_b = \left| \Omega_{ijk} Z_i Z_j + [\varphi, Z_k^\dagger] \right|^2 + \left| \sum_i [Z_i, Z_i^\dagger] + [\varphi^\dagger, \varphi] + qq^\dagger - r I \right|^2 + q^\dagger \varphi \varphi^\dagger q + ||\phi, Z_i||^2 + ||\phi, \varphi^\dagger||^2 + Tr[\phi, \bar{\phi}]^2.
\]

It is clear that the only effect of the Fayet-Iliopoulos parameter, \( r \), is to make the 0-6 strings tachyonic by adding the term \(-r \ qq^\dagger\) to the action. As explained in [35] the mass of these bifundamentals is determined by the asymptotic B-field. In particular, they are massive, \( r < 0 \), in the absence of a magnetic field, so that the minimum of the action has nonzero energy and there are no BPS states. For a sufficiently strong B-field, \( q \) will become tachyonic and can condense into a zero energy BPS configuration.
The argument we used before implies that the theory localizes on solutions to the equations,
\[ [Z_i, Z_j] = 0 \]
\[ [Z_i, \varphi^\dagger] = 0 \]
\[ \sum_i [Z_i, Z_i^\dagger] + [\varphi^\dagger, \varphi] + q^\dagger q = rI \]
\[ q^\dagger \varphi = 0. \]  \hfill (26)

Note that equation (18) still holds, since the superpotential it was derived from is unchanged.

We find a vanishing theorem from the D-term as before, computing
\[ 0 = \left| \sum_i [Z_i, Z_i^\dagger] + [\varphi^\dagger, \varphi] + q^\dagger q - rI \right|^2 = \left| [Z_i, Z_i^\dagger] + q^\dagger q - rI \right|^2 + \text{Tr}[\varphi^\dagger, \varphi]^2 + 2 \sum \text{Tr}[\varphi^\dagger, \varphi][Z_i, Z_i^\dagger] + 2 q^\dagger [\varphi^\dagger, \varphi] q. \]  \hfill (27)

Only the last term is different, and using the fact that \( q^\dagger \varphi = 0 \), as required by the potential for moving the D0 branes off the D6 in the spacetime direction, we see that all terms on the right hand side of (27) are positive squares. Hence the moduli space consists of solutions to the quiver equations (7), with the field \( \varphi \) totally decoupling from the classical moduli space. It will, however, give a crucial contribution to the 1-loop determinant at the fixed points.

### 3.1 Computing the Euler character of the classical moduli space

The moduli space of BPS states is given by the solutions to (7) up to \( U(N) \) gauge equivalence. We want to determine the Euler characteristic of this quiver moduli space using equivariant techniques. There is a natural action of the \( U(1)^3 \) symmetry group of the vertex geometry on this classical instanton moduli space, given by \( Z_i \to \lambda_i Z_i, \quad q \to \lambda q \). The Euler character is localized to the fix points, which are characterized by the ability to undo the toric rotation by a gauge transformation, that is,
\[ [\phi, Z_i] = \epsilon_i Z_i \]
\[ q^\dagger \phi - \alpha q = \epsilon q. \]  \hfill (28)
for some $\phi \in su(N)$, and where $\alpha \in u(1)$ implements the $U(1)$ gauge transformation, which can obviously be re-absorbed into the $U(N)$ transformation parameterized by $\phi$.

This is implemented in the matrix model as before, by changing the $Q$ action so that it squares to the new $U(N)$ subgroup of $U(N) \times U(1)^3$ that is being gauged. Consistency requires that $\varphi$ and $\xi$ have weights $\epsilon_1 + \epsilon_2 + \epsilon_3$ and $\epsilon - \epsilon_1 + \epsilon_2 + \epsilon_3$, respectively, under the torus action. The twisted superpotential terms in the path integral are modified in the obvious way, schematically, from $\text{Tr}[\varphi, \xi]$, exactly after including the Vandermonde determinant from diagonalizing $\phi$. The weight at the fixed point can be determined as follows, assuming that $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ to preserve the $SU(3)$ holonomy. It is easy to check as before that contributions to the quadratic terms in the variation of $Z_i$ away from the fixed point exactly match those of $\psi_i$. The twisted superpotential gives rise to additional contributions

$$\text{Tr}([\delta Z_i, \phi] + \epsilon_i \delta Z_i) \left( [\bar{\phi}, \delta Z_i ] + \epsilon_i \delta Z_i \right) + ||\varphi, \phi||^2 + q^1(\bar{\phi} + \epsilon)(\phi + \epsilon)q,$$

(29)

to the bosonic fields, and for the anti-ghosts,

$$||\chi_i, \phi| - \epsilon_i \chi_i||^2 + \xi^1(\bar{\phi} + \epsilon)(\phi + \epsilon)\xi,$$

(30)

where we used the fact that our torus action lives in $SU(3)$. Therefore, everything cancels exactly after including the Vandermonde determinant from diagonalizing $\phi$.

To begin finding the fixed points, chose the gauge by requiring that $\phi$ of (28) is diagonal, denoting the eigenvalues by $\phi_a$. This is a far more judicious choice then trying to diagonalize the $Z_i$, since they are not Hermitian, and all of the interesting bound states in fact require them to be non-diagonalizable. The noncommutativity of $Z_i$ and $Z_i^\dagger$ thus implied can be understood physically as resulting from the background B-field we have turned on to produce supersymmetric bound states in the D6/D0 system. This field is indeed proportional to the Kahler form, $\omega = \sum_i dz_i^* \wedge dz_i$.

In this gauge, the equivariant condition gives the strong constraint that

$$(Z_i)_{ab}(\phi_a - \phi_b - \epsilon_i) = 0, \quad g_a(\phi_a - \alpha - \epsilon) = 0.$$

(31)

For generic $\epsilon_i$, this forces most of the components of $Z_i$ to vanish, and it is useful to think of the nonzero elements as directed lines connecting a pair of the $N$ points indexed by $\phi_a$.

It is convenient to represent the action of the $Z_i$ as translation operators on a finite collection, $\eta$, of $N$ points in $\mathbb{Z}^3$ associated to the eigenvectors of $\phi$, with adjacency determined by the condition

$$\phi_a - \phi_b = \epsilon_i,$$

(32)

which must hold whenever $(Z_i)_{ab} \neq 0$. More precisely, associating each eigenvector of $\phi$ to a point $x^a$ in $\mathbb{Z}^3$, whenever (32) is satisfied, the points are related spatially by

$$x_i^a = x_i^b + 1.$$

In relating the quiver description to the six dimensional Donaldson-Thomas theory, recall that the fixed points of the induced $(\mathbb{C}^\times)^3$ action on equivariant ideals of the algebra of polynomials on $\mathbb{C}^3$ can be encoded as three dimensional partitions in the positive octant in the lattice $\mathbb{Z}^3$ [23]. These are equivalent in the standard way to coherent, torus invariant sheaves with point-like support at the origin in $\mathbb{C}^3$.

What does the F-flatness condition, $[Z_i, Z_j] = 0$, mean for the torus localized configurations? Clearly this implies the fact that if $p, q \in \eta$ can be connected by a particular sequence of positive translations, $Z_i$, remaining within $\eta$ at each stage, then all such paths generated by other orders of the same $Z_i$ must also lie in $\eta$. It is easy to see that this means that $\eta = \pi - \pi'$ is the difference between two three dimensional partitions, $\pi' \subseteq \pi$. Moreover, the values of the nonzero $(Z_i)_{ab}$ can be chosen such that the matrices indeed commute.

This has a nice interpretation in the language of the derived category of coherent sheaves used to describe all equivariant B-model boundary states in the vertex geometry. Let $E_\pi$ be the
equivariant sheaf associated to the partition $\pi$ following [23]. Then the configuration for general $\eta$ is associated to the complex

$$\mathcal{E}_\pi \rightarrow \mathcal{E}_{\pi'},$$

which, although perfectly fine as a B-model brane, is always an unstable object in the full theory, for any value of the B-field. It has D0 charge of $N$, and vanishing D6 charge, since we have condensed the D6/\(\overline{\text{D}}\)6 tachyon. At the level of modules, the worldvolume of this B-brane is described by $M = \mathcal{I}_{\pi'}/\mathcal{I}_\pi$. This is exactly what we should have expected, since we haven’t yet imposed the D-term constraint, which gives the stability condition in the quiver language.

The dimensional reduction of the (1,1) part of the field strength that appears in (6) is naturally diagonal in the basis we have chosen. For most points in $\eta$ it is possible to find $Z_i$ satisfying all the conditions, such that $[Z_i, Z_i^\dagger] = Z_i Z_i^\dagger - Z_i^\dagger Z_i$ has a positive eigenvalue at that point. The exception are the points $p \in \eta$ on the interior boundary, $\pi'$, since they are killed by $Z_i^\dagger$ and have only the negative contribution $-Z_i^\dagger Z_i$.

Therefore to satisfy the D-term condition, these negative eigenvalues must be cancelled by the contribution of the D6/D0 bifundamental, $q^j q$. The single D6 brane only gives us one vector $q^j \in \mathbb{C}^N$ to work with, hence only one negative eigenvalue of $[Z_i, Z_i^\dagger]$ can be cancelled. Hence there are only solutions when the interior has a single corner, namely when $\pi'$ is trivial, and $\eta$ is a three dimensional partition. Therefore we have reproduced the crystals first related to the A-model in [31].

In this case, the $Z_i$ act as multiplication operators in the algebra $\mathcal{A} = \mathbb{C}[x, y, z]/\mathcal{I}_\pi$, which is the algebra of polynomials on the nonreduced subscheme found in the nonabelian branch of $N$ points in three dimensions. The holomorphic version of the stability condition is the existence of a cyclic vector, $q \in \mathbb{C}^N$, such that polynomials in the $Z_i$, acting on $q$, generate the entire vector space. This is obvious for $Z_i$ given by translation operators in the dimension $N$ unital algebra, $\mathcal{A}$, with $q = 1 \in \mathbb{C}[x, y, z]$. Moreover, it can be shown that the D-term constraint is equivalent to this algebraic stability requirement.

### 3.2 Generalization to the $U(M)$ Donaldson-Thomas theory

The $U(M)$ Donaldson-Thomas theory describes the bound states of $M$ D6 branes with D2 and D0. The instanton moduli space is described by the same quiver equations (7) as before, where $q$ is now a $(N, M)$ bifundamental.

Applying the analysis of section 2, we find that the $Z_i$ act as translations on a nested partition, $\eta = \pi - \pi'$, with $N$ boxes. In the same way as before the moment map constraint can be saturated only when $q^j q$ can cancel the negative contributions to $\sum[Z_i, Z_i^\dagger]$ on the interior corners. Therefore $\eta$ can have at most $M$ interior corners. Equivalently, choose $M$ points in $\mathbb{Z}^3$, and construct overlapping three dimensional partitions based on each point. This will define a permissible $\eta$.

Consider the example of $U(2)$ Donaldson-Thomas invariants. Then it is clear from the quiver description of the moduli space that the equivariant bound states are associated to partitions in an “L-shaped” background, as shown. This background is the most general with two corners, moreover configurations consisting of decoupled partitions resting independently on the corners are obviously not included. Therefore we can explicitly determine the partition function using the results of [31] and [24] about the statistical mechanics of melting crystals, obtaining

$$Z_{U(2)} = \sum_{n,m,k>0} (M(q) C_{[nm][kn]} q^{(nm^2+n^2k)/2 - S(n) S(m, k)}) + \sum_{n,m>0} (M(q) C_{[nm]} q^{nm^2/2 - S(n) S(m)}),$$

(33)

where $[nm]$ is the rectangular $n \times m$ Young diagram.

The first term is the generating function of crystals on the L-shape given by the asymptotic rectangular Young diagrams, where the power of $q$ is present because of the framing factor in the
topological vertex relative to the crystal partition function. This term leads to an over-counting of contributions to the $U(2)$ theory because of the inclusion of decoupled partitions supported independently in the two corners, hence these are subtracted off by the second term. It is also possible for the two interior corners to lay in the same plane, which is captured by the final two terms in (33). The partition functions $S^{(n, m)}$ are defined in [24].

Although the result (34) is not very transparent, it is notable as the first calculation of the $U(2)$ Donaldson-Thomas theory, even in the vertex. It would be very interesting to try to confirm this formula mathematically, as well as the more implicit (although still fully calculable order by order) answer for $U(M)$.

4 The full vertex $C_{\mu\nu\eta}$

The topological A-model partition function has been shown to be given by a dual description in terms of the bound states of a single 6-brane with chemical potentials turned on for D2 and D0 branes at large values of the background B-field [23]. In toric Calabi-Yau this has been checked explicitly, since both the A-model and the Donaldson-Thomas theory localize onto equivariant contributions in the vertex glued along the legs of the toric diagram. The general vertex, which thus determines the entire partition function of the A-model on any toric Calabi-Yau, can also be described by a quiver matrix model.

We will construct this quiver, and see that it reproduces the known answer for the topological partition function. In addition, this model gives much additional interesting information about the moduli space of these bound states then merely the Euler character. We will find that in general, the effective geometry seen by the dynamical D0 branes depends in an intriguing way on the background Kahler moduli, and undergoes flop transitions as one crosses walls where the attractor trees change shape, as in [11]. In the next section, we will see how the quantum foam picture of fluctuating Kahler geometry arises naturally in the quiver description, and further surprises of the effective geometry will become apparent.

The effective geometry we explore with 0-branes is, of course, the geometry of the moduli space of BPS states. This can receive various corrections both in $\alpha'$ and $g_s$. In the local case, however, the situation is more under control, since the Calabi-Yau moduli are always at large volume. Moreover, the torus action extends to the effective geometry, and we see no reason that this $T^3$ symmetry should be violated by corrections, at least in the local limit. Hence the corrections to the Kahler structure on the effective moduli appear likely to be simple renormalizations of the already present Kahler parameters, i.e. the dependence of the FI terms on the background moduli receives $\alpha'$ corrections, which is no surprise.

4.1 One nontrivial asymptotic of bound D2 branes

We want to determine the low energy effective quantum mechanics of BPS D6/D2/D0 bound states. Note that the D6/D2 system is T-dual to a 4-brane with bound 0-branes, and is described by the same ADHM quiver [6]. As one dials the background values of the Kahler moduli, the D6/D2/D0 configurations are more naturally regarded as either point-like instantons in a D6 brane with D2 (singularly supported) “flux” turned on, a collection of 2-branes bound to a D6 with D0 charge, or a D2/D0 bound state attached to the D6. Only the first interpretation will be relevant for us, as we will soon see.

The low energy spectrum of 2-0 strings in flat space consists of a single tachyonic bifundamental multiplet, denoted here by $B$, which remains tachyonic for all values of the background B-field. The BPS states are obtained by condensing this field to cause the D0 branes to dissolve into flux on the D2. In our situation, however, if we turn on a B-field, the 6-2 strings can become far more tachyonic. As we will see quantitatively below, this field condenses first, melting the D2 branes into $U(1)$ flux on the D6, and giving a large positive contribution to the mass of the usual D2/D0 and D6/D0 bifundamentals. They become irrelevant to the moduli space of BPS states, but, surprisingly, an initially massive multiplet in the 2-0 spectrum receives an opposite
A particularly simple way of determining the relevant spectrum of low energy 2-0 fields motivated by string theory is to look at the large volume limit of a D2-D0 bound state wrapping a compact 2-cycle. Consider \( k \) D2 branes on the \( S^2 \) of the resolved conifold, which are described by the quiver shown in Figure 2 with total D0 charge \( N + k/2 \) (including induced charge). The superpotential implies that

\[
C_1 D_a C_2 = C_2 D_a C_1, \quad D_1 C_a D_2 = D_2 C_a D_1,
\]

for the supersymmetric vacua.

The motion of the 0-branes is governed by the collective coordinates on the conifold, \( c_1 d_2 = c_2 d_1 \), where \((d_1 : d_2)\) are projective coordinates on \( P^1 \). At first, we want to focus on the case where there are \( N \) point-like bound D0 branes near one vertex. I will work holomorphically to avoid considerations of stability, which will be strongly affected by the D6 brane we will introduce shortly. This can be done by fixing the gauge,

\[
D_2 = \begin{pmatrix} I_N & 0_{N \times k} \end{pmatrix},
\]

which implicitly assumes that all of the D0 branes are in the patch \( d_2 \neq 0 \), consistent with the desired noncompact limit we will take to obtain the vertex geometry.

For this gauge choice, we have broken the \( U(N) \times U(N+k) \) symmetry down to \( U(N) \times U(k) \), and obtained the “localized” quiver shown. This gives the quiver for D2-D0 branes in flat space, including the \( A_a \) fields which are massive without the presence of the D6 and background B-field. They will play a crucial role for us, as we will soon see.

The remaining quiver fields for the conifold decompose under the breaking of the gauge group as

\[
C_a = \begin{pmatrix} Z_a \\ A_a \end{pmatrix}, \quad D_1 = \begin{pmatrix} Z_3 & B \end{pmatrix}.
\]

The F-flatness equation (34) implies that

\[
[Z_1, Z_2] = 0, \quad [Z_2, Z_3] = BA_1, \quad [Z_3, Z_1] = BA_2, \quad A_1 Z_2 = A_2 Z_1.
\]

To obtain the full quiver for 2-0 in \( C^3 \), one needs to modify these equations in the natural way to include the motion of the D2 branes, which are rigid in the conifold example. These equations naturally result from extremizing the superpotential (40) below.

In order to obtain the correct moduli space for \( N > 1 \) and \( k > 1 \) these is one extra ingredient we have so far neglected. This can be motivated in two different ways. First, in the above derivation of the quiver by taking a limit of the resolved conifold geometry, the D-term of the original \( U(N+k) \) symmetry has an off-diagonal component in the decomposition to \( U(N) \times U(k) \). This would imply the existence of an additional constraint,

\[
A_a Y^\dagger_a + Z^\dagger_a A_a = 0,
\]

where we have included the generalization to a D2 brane bound state in a nontrivial configuration encoded in the \( Y_a \) adjoints of \( U(k) \).

Moreover, this term is needed to describe the ordinary contribution to the mass of the 2-0 strings when the D2 and D0 branes are separated in transverse directions. That is, there must be a term in the low energy effective theory of the form

\[
\text{Tr} \left( A_a Y^\dagger_a Y_a A^\dagger_a + Z^\dagger_a A_a A^\dagger_a Z_a \right).
\]
We will soon see that the F-term, $|\partial W|^2$, only gives an appropriate mass to the bifundamentals $A_1$ when there is a distance $(Y_2)_i - (Z_2)_j \neq 0$ between a pair of eigenvalues, however the $A_2$ mode must also receive a mass. Later, it will become clear the condition $Z^\dagger_a A_a = 0$ is redundant, and hence the associated terms in the action can be absorbed into the already existing ones, but the constraint

$$A_a Y^\dagger_a = 0$$

is new, and must be imposed in addition to the ordinary quiver conditions. This is similar to the mass term needed for the 6-0 strings generated by separation of the branes in the direction transverse to the Calabi-Yau. Note that there is no off-diagonal residual gauge symmetry, since the form of $B_2$ breaks it to exactly $U(N) \times U(k)$; this would be an issue for $GL(N + k, \mathbb{C})$, however.

The correct bundle over the classical moduli space can be found by including the fields that are relevant for motion normal to the Calabi-Yau, playing an analogous role to $\phi$ and $\phi$ in the D0 theory. By the transversal $SO(7)$ rotational symmetry broken from $SO(9, 1)$ by the presence of the D2 brane, there must exist 3 additional low energy 0-2 modes, a complex chiral field, $\tilde{A}$, and a real vector, analogous to $\phi$. The vector multiplet is associated to the unbroken off-diagonal component of $GL(N + k, \mathbb{C})$, and only its associated D-term survives in the Kahler description, as we have seen. The presence of $\tilde{A}$ is important for finding the correct 1-loop determinant at the fixed, although it is vanishing for all BPS solutions.

The dynamical topological multiplets, describing the motion (adjoint fields) and tachyons (bifundamentals) of the D0 branes are

$$\delta Z_i = \psi_i, \quad \delta \psi_i = [\phi, Z_i]$$
$$\delta \phi = \zeta, \quad \delta \phi = [\phi, \phi]$$
$$\delta q = \rho, \quad \delta \rho = \phi q$$
$$\delta B = \beta, \quad \delta \beta = B \phi - \phi' B$$
$$\delta A_a = \alpha_a, \quad \delta \alpha_a = A_a \phi' - \phi A_a$$
$$\delta \tilde{A} = \tilde{\alpha}$$
$$\delta \tilde{\alpha} = \tilde{A} \phi' - \phi \tilde{A}$$

where $i = 1, 2, 3$ and $a = 1, 2$, and $B, A_a$ are the lowest lying 2-0 string modes, with masses $m_B = -m_A < 0$ at all values of the background Kahler moduli. There are auxiliary multiplets

$$\delta \phi = 0$$
$$\delta \tilde{\phi} = \eta$$
$$\delta \tilde{\eta} = [\phi, \phi'],$$

as before. We regard the D2 brane moduli as frozen due to the noncompactness of their world-volume, but they can still be derived from the T-dual D4/D0 system as:

$$\delta Y_a = \xi, \quad \delta \xi = [\phi', Y_a]$$
$$\delta J = \nu, \quad \delta \nu = \phi' J$$
$$\delta K = \kappa, \quad \delta \kappa = K \phi'$$
$$\delta \phi' = 0,$$

where the completion to the full ten dimensional theory is ignored, as we are not concerned with with anti-ghost bundle over this moduli space, having already chosen a particular point due to the noncompactness of the wrapped 2-cycles.

The topological nature of the partition function means that the Euler character of the obstruction bundle is a deformation invariant when it satisfies the proper convergence properties. In particular, the cubic terms in the superpotential are sufficient to determine the exact result, as higher order corrections, even if they exist, will not affect the answer, although they may
correct the geometry of moduli space. The superpotential can be read off from the quiver diagram, including the usual D0 Chern-Simons like term as well as the natural superpotential of the D2/D0 system, giving

\[ W = \text{Tr} (\Omega_{ijk} Z_i Z_j Z_k + \epsilon_{ab} Z_a A_b + \epsilon_{ab} A_a Y_b + q KB), \]  

which implies the F-flatness conditions,

\[ [Z_1, Z_2] = 0, \quad [Z_3, Z_a] = A_a B, \quad B Z_a = Y_a B, \]

\[ Z_1 A_2 + A_1 Y_2 = Z_2 A_1 + A_2 Y_1 + q K, \quad KB = 0, \quad Bq = 0. \]  

(41)

The moment maps are

\[ \sum [Z_i, Z_d] + \sum A_a A_a^\dagger - B^\dagger B + qq^\dagger = r I_N, \]

\[ \sum [Y_a, Y_d^\dagger] - \sum A_a A_a^\dagger + BB^\dagger + K^\dagger K - JJ^\dagger = r I_M, \]  

(42)

which square to give the D-term contribution to the topological matrix model. Note in particular that \( r' > r \) so that the usual 2-0 string is classically tachyonic, as it must be.

The collective coordinates, \( Y_a \), of the D2 brane will encode their configuration as bound flux in the D6, which we will regard as a fixed asymptotic condition, due to the noncompactness of the D2 worldvolume. There is also the F-term constraint for the D6/D2 system,

\[ [Y_1, Y_2] = JK. \]  

(43)

First let us understand why these are physically the correct conditions. The FI parameters serve to give masses to the bifundamentals via terms of the form \(-2 r' \text{Tr} A_a A_a^\dagger\), and we find that the 0-2 tachyon has bare mass \(-2(r' - r) < 0\) for all values of the background B-field. There is a pair of massive fields, \( A_a \), in the 0-2 spectrum with mass \(2(r' - r)\) that allow us to write the usual superpotential. The 6-2 string \( K \), which is also tachyonic with mass \(-2r' < -2(r' - r)\), will be the first to condense, dissolving the D2 branes into flux on the D6.

Recall that the D6/D2 system is T-dual to D4/D0, and the bound D0 flux is described by the ADHM construction [3]. Therefore the combination \( \sum [Y_a, Y_a^\dagger] + K^\dagger K - JJ^\dagger \) is an \( M \times M \) matrix with positive eigenvalues, which results in a large positive contribution, from the term \( \text{Tr} B^\dagger (\sum [Y_a, Y_a^\dagger] + K^\dagger K - JJ^\dagger) B \) in the action, to the effective mass of \( B \) in the vacuum with nonzero \( K \). Therefore we find that \( B = 0 \) in the supersymmetric vacuum, which also agrees with the F-term condition. This can be thought of as a quantum corrected mass for \( B \) after integrating out the heavy field, \( K \).

As can easily be seen from the structure of the D-terms, the effective masses are \( m_a = -m_B \), hence these bifundamentals now condense, binding the D0 branes to the D2. Putting everything together, we see that the background D2 configuration is described by a Young diagram, \( \lambda \), with \( M \) boxes, encoded in the matrices \( Y_a \), as expected. The D0 collective coordinates obey \( [Z_i, Z_j] = 0 \), giving a difference, \( \eta \), of three dimensional partitions, and the D-term can only be satisfied when the interior corners are exactly the image of the map \( A_a \) from \( \mathbb{C}^M \to \mathbb{C}^N \).

Moreover, there is a holomorphic equation which implies that

\[ Z_1 A_2 + A_1 Y_2 = Z_2 A_1 + A_2 Y_1. \]  

(44)

It is also clear from the discussion of the effective mass of the bifundamentals that \( Y_a A_a^\dagger = 0 \) for \( a = 1, 2 \). This means that only one of the terms on each side of (44) can be nonvanishing. The geometric interpretation in terms of crystal configurations is that the \( z_3 = 0 \) plane of \( \eta \) together with \( \lambda \) forms a Young diagram, and by the commutativity of the \( Z_i \), this implies that \( \eta \) is exactly a three dimensional partition in the background of \( \lambda \times z_3 \) axis, as expected from [31].

To further check that (44) is the correct F-flatness condition even for non-torus invariant points in the BPS moduli space, note that shifting the positions of the D0 and D2 brane together
by sending $Z_i \mapsto Z_i + x_i I_N$ and $Y_a \mapsto Y_a + x_a I_k$ doesn’t change the solutions for the $A_\alpha$, as expected. It is easy to confirm that in the generic branch of moduli space, specifying $3$ independent eigenvalues of mutually diagonalizable $Z_i$ and likewise $2k$ distinct eigenvalues of $Y_a$ totally determines the $A_\alpha$ up to gauge equivalence. This is because (44) essentially allows one to solve for $A_2$ in terms of $A_1$, and the condition (49) together with the D-terms fix $A_1$ up to the $U(1)^{N+k-1}$ symmetry left unbroken by the choice of $Z_i$ and $Y_a$.

The path integral can be calculated as before by introducing a toric regulator. The fields transform as before under the $U(1)^3$ action on $\mathbb{C}^3$, with

$$A_\alpha \rightarrow e^{i\epsilon_\alpha} A_\alpha, \quad B \rightarrow e^{i\epsilon_3} B, \quad \bar{A} \rightarrow e^{i\sum \epsilon_i} \bar{A},$$

and consistently for the anti-ghosts. Therefore the full result for the determinant is given by

$$\frac{(\text{ad} \phi + \epsilon_1 + \epsilon_2)(\text{ad} \phi + \epsilon_1 + \epsilon_3)(\text{ad} \phi + \epsilon_2 + \epsilon_3)(\phi - \phi' + \epsilon_1 + \epsilon_2)(\phi' - \phi + \epsilon_3 + \epsilon_1)}{(\text{ad} \phi + \epsilon_3)(\text{ad} \phi + \epsilon_2)(\text{ad} \phi + \epsilon_1)(\phi - \phi' + \epsilon_1 + \epsilon_3)(\phi' - \phi + \epsilon_2 + \epsilon_3)} \times \frac{(\phi' - \phi + \epsilon_3 + \epsilon_2)}{(\phi' - \phi + \epsilon_1)} \frac{(\phi - \phi' + \epsilon_1 + \epsilon_2 + \epsilon_3)(\phi + \epsilon_3 + \epsilon_1)}{(\phi - \phi' + \epsilon_1 + \epsilon_2 + \epsilon_3)(ad \phi + \epsilon_1 + \epsilon_2 + \epsilon_3)},$$

which precisely cancel when the toric weights sum to zero, and we include the Vandermonde from the $\phi$ integral. The measures from the fields governing motion of noncompact objects have not been included since they are frozen, not integrated.

### 4.2 Multiple asymptotics and a puzzle

Now we would like to understand the quiver matrix model that describes the bound states of D0 branes to our D6 when D2 charge is turned on in more than one of the three toric 2-cycles. The index of BPS states on this theory, with a fixed configuration of the nondynamical fields associated to motion of noncompact objects, should give exactly the topological vertex of [2]. Generalizing the quiver, it is again clear that the frozen D2 adjoint fields should give a stable representation of the constraint $[Y_1, Y_2] = 0$: these are the asymptotic Young diagrams appearing in the topological vertex, $C_{\mu \nu \rho}$. See Figure 1 for the complete quiver diagram including all fields.

The quiver is given as before, with a node, $U(k_i)$, for each of the three stacks of frozen D2 branes, and bifundamentals 2-2’ strings, which have the same low energy field content as a T-dual pair of 4-0 bifundamentals. Only the cubic terms in the superpotential are relevant, and the possible terms allowed by the quiver (which also respect the obvious rotational symmetries in $\mathbb{C}^3$) are

$$W_0 = \epsilon_{ijk} \text{Tr} (Z_i Z_j Z_k) + \epsilon_{ijk} \text{Tr} (B^i (Z_j A^i_k + A^i_j Y_k)) + \epsilon_{ijk} \text{Tr} (J^i_j B^i A^i_k) + K_i B_i q,$$

for the D0 degrees of freedom and

$$W_2 = \text{Tr} (J^1_2 J^2_3 J^3_1 - J^1_3 J^3_2 J^2_1) + \epsilon_{ijk} \text{Tr} (J^i_j Y^i_k J^i_j) + K_i J^i_j \bar{K}_j,$$

where we have only included couplings of the dynamical fields. The frozen 6-2 modes must satisfy the equations of motion obtained from $\partial W_{\text{frozen}} = 0$, where

$$W_{\text{frozen}} = K_i \bar{Y}^i \bar{K}_i + \epsilon_{ijk} \text{Tr} (\bar{Y}^i Y^j Y^k),$$

where the $\bar{Y}^i$ are exactly 0 in the vacuum, being associated to motion transverse to the D6 brane, and have been introduced simply to be able to write the superpotential. Note that it is possible to rescale the fields, while preserving rotational invariance in $\mathbb{C}^3$, in such a way to set all of the relative coefficients in the superpotential to unity.
The physical moduli space is the Kahler quotient of the resulting algebraic variety by $U(N) \times U(k_1) \times U(k_2) \times U(k_3) \times U(1)$, where the overall $U(1)$ acts trivially on all of the fields. This means that there is a D-term in the action given by the square of the equations

$$
\sum_{i=1}^{3} [Z_i, Z_i^\dagger] + \sum_{i \neq a=1}^{3} (A_a^i) (A_a^i)^\dagger + q q^\dagger = r_N I_N
$$

$$
\sum_{a \neq i} \left( [Y_a^i, Y_a^i] - A_a^i A_a^i + J_a^i J_a^i - J_a^i J_a^i \right) + \bar{I} I^\dagger - \bar{I} I^\dagger = r_i I_{k_i},
$$

where the FI parameters, $r$, are determined by the background values of the Kahler moduli in a complicated way. We will later graph the loci where various combinations of these Kahler parameters of the moduli space vanish, which can be found by looking for walls of marginal stability where the central charges of some of the constituent branes align.

The fact that we are working in a local geometry obtained as a noncompact limit far from the Gepner point of the global Calabi-Yau again means that there will exist residual off-diagonal D-terms. Collecting all of the relevant equations, one has that

$$
A_j^i Y_j^{i\dagger} + Z_j^i A_j^i + A_j^i J_j^{i\dagger} = 0
$$

$$
J_j^i Y_{j\dagger}^i + Y_{j\dagger}^i J_j^i = 0
$$

The 2-2’ bifundamental strings are localized in $\mathbb{C}^3$, stretched along the minimum distance between the orthogonal 2-branes (and thus living at any non-generic intersection), so they are also dynamical fields. Looking at the form of $W_2$, it is possible to see that the 2-6 system without D0 branes has no unlifted moduli after fixing the frozen degrees of freedom, that is, the semi-classical BPS moduli space is a point. This can be easily seen, since if the D2 branes are separated by a large distance in $\mathbb{C}^3$, all of the $J_j^i$ become heavy. Thus the moduli space we are interested in is indeed the effective geometry as seen purely by the 0-brane probes.

There is a new feature in this quiver, since it is not immediately obvious which of the 6-2 and 2-2’ strings should given background VEVs. We will find that depending on the values of the background Kahler moduli and B-field, the BPS ground states will live in different branches of the moduli space of these fields, which are frozen from the point of view of local D0 dynamics. The resulting effective quivers for the D0 degrees of freedom will be different, although the final computation of the Euler character turns out to be independent. In the next section, it will become clear what this means in terms of quantum foam. The alternative quiver realizations will exactly correspond to the resolutions, related by flops, of the blown up geometry experienced by the D0 branes, viewed as probes. It is thus very natural that the effective Kahler structure depends on the background Kahler moduli, via the Fayet-Iliopoulos parameters.

To get a feel for the way this works, consider the simplest example with two nontrivial asymptotics, $C_{\mathbb{R}^3}$. There are two equivariant configurations of the D6-D2 system that are consistent with the F-term condition, $\partial W_{\text{frozen}} = 0$, namely $Y_j^i = 0$ and either $J_j^1 = 0, J_j^2 = 1$ or $J_j^1 = 0, J_j^2 = 1$. Only one of these solutions can satisfy the D-term constraint for the D2 $U(1)$ gauge groups, for any given value of the FI parameters. For definiteness, let us focus on the latter solution. Then $N$ D0 branes probing this background will be described by the quiver as before, with the new condition that

$$
Z_3 A_2^1 + A_3^1 J_2^1 = Z_2 A_3^1,
$$

whose equivariant fixed points can be interpreted as a skew three dimensional partition generated by the commuting $Z_i$ in the usual way. The vectors $A_3^1$ and $A_2^1$ form the interior corners of the partition, saturating the moment map condition, while $A_1^1 = 0$ by (37), and $A_0^3$, interpreted
as a vector in $\mathbb{C}^N$, is simply given be $Z_2 A_3^3$ by equation (51). Using the further holomorphic equation that

$$Z_1 Z_2 A_3^3 = Z_1 A_2^2 J_2^1 = Z_3 A_1^2 J_2^1,$$  

(52)

we see that this means the skew partition exactly fits into the empty room crystal configuration associated to $C_{\square\square\square}$.

The essential idea extends to arbitrary background D2 brane configurations. In particular, there are usually many choices in determining the frozen fields, which depend of the FI parameters, and thus on the background Kahler moduli. One way to solve the constraints is to start with one asymptotic, and condense the tachyonic 2-6 field, being sure to tune the FI parameters so that it is along the direction of steepest descent in the potential. This procedure will give new contributions to the effective masses of the 2-2' strings, and there will again be some region in the background moduli space where they become unstable and condense. Finally, some of the 2-2 and 2'-2 fields condense. The D-term condition for each first in the background moduli space where they become unstable and condense. Finally, some of the new contributions to the effective masses of the 2-2' strings, and there will again be some region so that it is along the direction of steepest descent in the potential. This procedure will give

$$\text{Effective geometry, blowups, and marginal stability}$$

Consider the quiver describing a single D0 brane in the vertex with one asymptotic 2-brane bound state encoded in the Young diagram, $|\mu| = k$. Then the relevant dynamical fields are the D0 coordinates and the 2-0 tachyons, which satisfy the F-flatness relation

$$A_1 (Y_2 - z_2) = A_2 (Y_1 - z_1),$$  

(53)

where $Y_\alpha (\mu)$ are fixed matrices. It is easy to check that the projectivization of this holomorphic moduli space obtained by moding out by the residual nontrivial $\mathbb{C}^k$ gauge symmetry left unbroken by the $Y_\alpha$ is exactly the blow up of $\mathbb{C}^3$ along the ideal defined by $\mu$. Suppose that all of the parallel 2-branes are separated in the transverse $z_1-z_2$ plane. Then, working in the holomorphic language, we can use $GL(k, \mathbb{C})$ to simultaneously diagonalize the $Y_\alpha$, and in that basis we therefore find

$$(A_1)_m ((y_2)_m - z_2) = (A_2)_m ((y_1)_m - z_1), \quad m = 1, \ldots, k$$

These $k$ equations define the blow up of $\mathbb{C}^2$ at $k$ distinct points; the $z_2$ direction is a simple product with the resulting smooth, non Calabi-Yau 4-manifold. As we imagine bringing the D2 branes on top of each other, exploring other branch of the Hilbert scheme of $k$ points in $\mathbb{C}^2$, the effective geometry experiencing by the 0-brane will be the blow up along that more complicated
ideal sheaf. The smoothness of the Hilbert scheme in two complex dimensions (which is the moduli space of the 2-branes bound to a D6) assures one that no subtleties can emerge in this procedure, as can be confirmed in specific examples.

More generally, using $N$ 0-branes as a probe, we would see a geometry related to, but more complicated than, the Hilbert scheme of $N$ points in this blow up variety. In particular, the 0-branes are affected by residual flux associated to the blown up 2-branes, so that globally they live in sections of the canonical line bundle associated to the exceptional divisors, rather than the trivial bundle. More abstractly, the associated sheaves fit into an exact sequence

$$0 \to \mathcal{I} \to \mathcal{L} \to i_* \mathcal{O}_Z \to 0,$$

where $\mathcal{L}$ is the nontrivial line bundle, and $Z$ encodes the bound point-like subschemes. This doesn’t change the nature of the Euler character of the moduli, which is determined by the local singularities. Intriguingly, there is a further refinement which naturally associates a size of order $g_s$ to the exceptional divisors. It is impossible for more than a small number (typically two in our examples) of D0 branes to “fit” on the exceptional divisor. It would be interesting to see if this feature also emerged in the study of the exact BPS moduli space of $N$ 0-branes on a Calabi-Yau in the small volume limit.

In many examples with two or three nontrivial asymptotics, there exist multiple resolutions of the blown up geometry, as shown in the simple example of $C_{\Box \Box}$ (see Figure 3). We can see this as well in trying to write down the quiver description of these moduli spaces: there are different choices of background D2 moduli which, however, all have equal Euler characteristic. In fact, there is a clear physical reason for this, to be found in a more detailed analysis of the lines of marginal stability. Depending on the values of the background Kahler moduli, different bifundamental tachyons will condense first. The index of BPS states is independent of which attractor tree we follow, as long as no lines of marginal stability are crossed.

The partition function for the general vertex reduces to the Euler characteristic of the anti-ghost bundle over the classical moduli space, because of the topological nature of the matrix integral. As we saw in the previous section, this Euler character can be localized to the fixed points of a $T^3$ action induced on the moduli space, and the weights at the fixed points are all equal when the torus twist preserves the $SU(2)$ holonomy, as shown by computing the 1-loop determinant near the fixed point.

We now investigate the properties of the moduli space itself, which it the Kahler quotient of a holomorphic variety defined by quadratic equations, resulting from the cubic superpotential. The three F-term equations,

$$Z_i A_j^k + A_i^k Y_j^k + A_j^i J_i^k = Z_j A_i^k + A_j^k Y_i^k + A_i^j J_j^k,$$  \hspace{1cm} (54)

where $i, j, k$ are distinct and not summed over, together with $[Z_i, Z_j] = 0$ serve to define the holomorphic moduli space of the dynamical low energy degrees of freedom.
In our noncompact setup, we have not specified the boundary conditions for the 2-branes, which are frozen in the local analysis. If they are free to move in the transverse directions, generically they will not intersect, and the effective geometry probed by a BPS 0-brane will be the blowup along the wrapped 2-cycles, as can be seen by going to a complex basis in which $Z_i$ and $A^j_i$ are all diagonal. The 2-2’ strings will become massive, and exit the low energy spectrum, when the branes are separated, and (54) becomes $k_1 + k_2 + k_3$ equations describing the local geometry $\mathbb{C} \times \mathcal{O}(-1) \rightarrow \mathbb{P}^1$ near each of the blown up 2-cycles.

It is also very interesting to consider compactifications in which the D2 branes wrap rigid or obstructed 2-cycles. In that case, they will have a nontrivial intersection, and a more sophisticated analysis is needed. As we will show in the following examples, the effective geometry is again the blow up along the wrapped subschemes. It is here that multiple smooth resolutions of the geometry may exist, corresponding to different attractor trees in which the order in which the tachyonic fields condense changes. No walls of marginal stability are crossed, hence the index of BPS remains constant.

The supersymmetric bound states we are discussing only exist when the asymptotic B-field is sufficiently large. From the perspective of Minkowski space, as the B-field flow down to the attractor value, various walls of marginal stability will be crossed. Thus the full geometry is multi-centered. Certain features of the attractor flow tree are visible in the quiver matrix model, so we first examine them from supergravity.

The central charge of a D6/D4/D2/D0 bound state described by a complex of sheaves $\mathcal{E}^*$ over the Calabi-Yau is, at large radius,

$$Z(\mathcal{E}^*) = \int e^{-\omega} ch(\mathcal{E}^*) \sqrt{Td(X)},$$

where $\omega = B + iJ$ is the Kahler $(1,1)$-form on $X$ [4]. At small volumes, this would receive worldsheet instanton corrections, hence the following analysis should only be expected to indicate the qualitative behavior of the attractor flow. This is sufficient for our purposes, since the local limit automatically assumes large volume.

In our case of $k_i$ 2-branes and $N$ 0-branes bound to a D6 brane in the equivariant vertex, the central charge (55) is computed as

$$\int e^{\sum t_i \omega_i} (1 + k_1 \omega_2 \wedge \omega_3 + k_2 \omega_1 \wedge \omega_3 + k_3 \omega_1 \wedge \omega_2 + N \omega_1 \wedge \omega_2 \wedge \omega_3) = t_1 t_2 t_3 + \sum t_i k_i + N, \quad (56)$$

where the $t_i$ are Kahler parameters associated to toric legs. They may be independent or obey some relations, depending on the embedding of the vertex into a compact 3-fold.

From this one can check the well known fact that the 6-0 BPS bound state only exists when $\arg(t^3) > 0$, that is $|B_{zz}| > (1/\sqrt{3}) |g_{zz}|$. Similar calculations have been done in [11] when there are also D2 branes. We are most interested in the behavior of the 2-branes as the moduli approach the attractor values. Note that in a local analysis, the attractor flow tree itself cannot be determined without knowing the compactification; this is obvious because when the $t_i$ become small, the vertex is no longer a good approximation.

All of these details of the global geometry are hidden in the FI parameters and the VEVs of the frozen fields appearing in the effective action for the 0-brane degrees of freedom. This is consistent with the expectation that the theory is defined at the asymptotic values of the moduli, where the 2-brane fields are heavy and their fluctuations about the frozen values can be integrated out. The topological nature of the matrix models guarantees that the only effect will be a ratio 1-loop determinants, which in fact cancel up to a sign.

As the moduli flow according to the attractor equations from the large B-field asymptotic limit needed to support the 6-brane BPS states, there are various possible decays. To find the walls along which they occur, one finds the condition for the central charges of the potential fragments to align. We find that in our examples, the 0-branes first split off.

The remaining D6/D2 bound state will itself decay at another wall of marginal stability. Depending on the asymptotic value of the moduli, the 2-branes wrapped one of the three toric
Figure 4: Walls of marginal stability for the decays $\Gamma \rightarrow \Gamma_0 + \Gamma_{6/2}$ (dashed line), $\Gamma_{6/2} \rightarrow \Gamma_1 + \Gamma'$ (dotted), and $\Gamma_{6/2} \rightarrow \Gamma_2 + \Gamma''$ (solid).

legs will fragment off. Because the relevant Kahler moduli space is six real dimensional even in the local context, the full stability diagram is difficult to depict on paper, so we will content ourselves with the representative cross-section shown in Figure 4. The charges in this example are denoted by

$$\Gamma = \omega_1 \wedge \omega_2 \wedge \omega_3 + \omega_1 + \omega_2 + 1$$
$$\Gamma_{6/2} = \omega_1 \wedge \omega_2 \wedge \omega_3 + \omega_1 + \omega_2$$
$$\Gamma_1 = \omega_1$$
$$\Gamma_2 = \omega_2.$$  

We choose the background Kahler moduli to be

$$t_1 = 16i + x, \quad t_2 = 14i + y, \quad t_3 = 4i + 4,$$

and plot the $x$, $y$ plane which parameterizes the asymptotic value of the B-field along the curves wrapped by the D2 branes.

It is immediately apparent that as the B-field in the $z_1$ and $z_2$ planes decrease along the attractor flow, the 0-branes separate along the first curve shown, and then the D6/D2 fragment intersects one of the two 2-brane curves. We shall see below how this phenomenon is imprinted on the matrix model. Therefore, in our local analysis, the 2-brane degrees of freedom are frozen in a pattern which depends on the attractor flow tree. It is natural that the indices calculated for the different effective 0-brane geometries are equal, since we see that they arise from the same theory in the global Calabi-Yau geometry.

5.1 Probing the effective geometry of a single 2-brane with $N$ 0-branes

Let us first understand a very simple example from this point of view, namely the effective geometry induced by a single D2 brane, corresponding to the vertex $C_{\square}$. Consider probing the induced geometry wrapped by the D6 brane using $N$ 0-branes. The F-term equation (44) tells us that $Z_1 A_2 = Z_2 A_1$, where we set the D2 position to 0 without loss of generality. Ignoring the $z_3$ direction for now, as it simply goes along for the ride, we see that this can be related to the quiver for a D4/D2/D0 bound state in $O(-1) \rightarrow \mathbb{P}^1$, which lifts to D6/D4/D0 in the three dimensional blow up. The D4 brane is equivalent to the line bundle mentioned before, since it can be dissolved into smooth flux on the 6-brane worldvolume.

To see this, consider the two node quiver for $O(-1) \rightarrow \mathbb{P}^1$ obtained by dropping one dimension from the quiver of resolved conifold. Setting the field $D = (I0)$ for quiver charges $N$ and
$N + 1$ splits $U(N + 1) \to U(N) \times U(1)$, and allows one to define

$$Z_a = C_a D,$$

and $A_a$ to be the $U(1)$ piece of $C_a$. Then the relation $C_1 BC_2 = C_2 BC_1$ implies that $Z_1 A_2 = Z_2 A_1$ and $[Z_1, Z_2] = 0$, as desired.

The two moduli spaces are not quite identical, however, because it is not always possible to choose a gauge such that $D = (I \ 0)$, and conversely, given such a form for $D$, it is impossible to satisfy the D-term condition when $Z_3 = 0$. Therefore the structure of the moduli spaces near the zero section of $O(-1)$ differ. In particular, it is easy to see that if $Z_1 = Z_2 = 0$ in some $M$ dimensional subspace of $\mathbb{C}^N$ then there is a $\mathbb{P}^1$ of $A_1, A_2$ if $M = 1$, a single point if $M = 2$, and a violation of the D-term if $M > 2$. This is the sense in which the resolution is small - only a single 0-brane can fit conformably.

Lifting to the full $\mathbb{C}^3$ adds one complex dimension to the 2-brane, hence the even far from the exceptional divisor the 0-branes will see a $U(1)$ flux turned on in the D6 gauge theory. This has no effect on the moduli space of a single BPS D0, but will twist the global structure of the $N$ D0 moduli space as follows.

Ordinarily, the Hilbert scheme of points is a kind of non-smooth “resolution” of the classical space of $N$ points in a 3-fold, $\text{Sym}^N(X)$. For simplicity, consider the case of $\text{Sym}^2(X)$, which looks like the bundle $T/\mathbb{Z}_2 \to \Delta$ near the diagonal embedding, $\Delta : X \leftrightarrow \text{Sym}^2 X$, where $T$ is the tangent bundle, with the natural $\mathbb{Z}_2$ action $z_i \to -z_i$ in local coordinates on the fiber. The $z_i$ should be thought of as the small relative separation of the two points. In this special case, the Hilbert scheme $\text{Hilb}^2(X)$, is locally the smooth resolution of the $\mathbb{Z}_2$ orbifold, which replaces the fiber by $O(-2) \to \mathbb{P}^2$.

Therefore near the diagonal, the Hilbert scheme of two points on $X$ is given by

$$O(-2) \otimes K \longrightarrow \mathbb{P}T \downarrow \bigg\downarrow \bigg\downarrow \bigg\downarrow \bigg\downarrow X,$$

where the $O(-2)$ bundle is fibered trivially over the base, and $K$ is the canonical bundle over $X$. When a $U(1)$ flux on the D6 worldvolume is turned on, carrying D4 charge, the $O(-2)$ of the Hilbert scheme becomes fibered over the diagonal in the sense of $\mathcal{L}$, the $U(1)$ bundle over $X$. That is, we should write $\mathcal{L}_X \otimes \mathcal{O} \otimes O(-2)_{\mathbb{P}^2}$ in the above diagram of the geometry where the points are close together. This is compactified in a natural way to the full Hilbert scheme with a background line bundle, and a similar situation holds for $N \geq 2$ D0 branes.

Intriguingly, the effective canonical class of $X$, in the sense of the what line bundle appears in the above description of the Hilbert scheme, turns out to be the trivial bundle in our case, since $\mathcal{L}$ exactly cancels the fibration of $K$ by the nature of the blow up construction. It would be interesting to understand if there is any more robust way in which the blow ups are effectively Calabi-Yau.

### 5.2 Example 2: the geometry of $C_{\Box\Box}$

Let us work out the example of $C_{\Box\Box}$ in detail, as it already possesses most of the new features. The classical moduli of the D2 branes are the coordinates $y_2^1, y_3^1$ and $y_1^2, y_3^2$ respectively. Depending on the details of the compactification, which determine the superpotential for these fields, it may be possible to separate the branes in the $z_3$ direction. In that case, the 2-2' strings develop a mass and exit the low energy spectrum. Hence $N$ probe D0 branes will have a BPS moduli space given by the $U(1)^2 \times SU(N)$ Kahler quotient of

$$\begin{align*}
(Z_2 - y_2^1) A_1^1 &= (Z_3 - y_3^1) A_1^2 \\
(Z_1 - y_1^2) A_2^2 &= (Z_3 - y_3^2) A_2^3 \\
[Z_i, Z_j] &= 0.
\end{align*}$$

(58)
For $g_1^1 \neq g_2^3$, this looks locally, near each of the D2 branes, like the geometry of $N$ D0 branes in the “small” blowup discussed before. In particular, a single D0 probe sees exactly the blow up geometry along the two disjoint curves.

More interesting behavior occurs when the 2-branes cannot be separated in this way, for example in the closed vertex geometry in which the wrapped $\mathbb{P}^1$’s are rigid. This is more closely connected with the equivariant result, which requires the wrapped 2-cycles to have such a non-generic intersection by torus invariance. Here we must use the fact that

$$J_j^i (A_j^i)^\dagger = 0,$$

which results from the term [(50)] in the effective action.

The effective geometry of the $C_{\square\square\square}$ vertex, as probed by an individual BPS D0 brane is therefore determined by the Kahler quotient of the variety,

$$z_1 a_3^1 = a_3^1 j_2^1$$
$$z_1 a_3^2 = z_3 a_1^2,$$

by $U(1) \times U(1)$ under which the fields have charges

$$z_1, j_2^1, a_1^2, a_3^2,$$
$$0, 1, 1, 0, 0,$$
$$0, 1, -1, 1, 1.$$

The first equation implies that $(z_2, a_3^2)$ lives in the $\mathcal{O}(-1)$ bundle over the $\mathbb{P}^1$ with homogeneous coordinates $(a_3^1 : j_2^1)$. Moreover the second equation fixes $(z_1, z_3)$ to be fibered in $\mathcal{O}(-1)$ on the compactification of the $a_3^2$ direction. This is shown in the toric diagram (see Figure 3), which is exactly the resolution of the blow up of $\mathbb{C}^3$ along the non-generically intersecting $z_1$ and $z_2$ lines!

Reversing the roles of the two D2 branes will result in a flopped geometry (which in this case is coincidentally geometrically equivalent to the first). It seems reasonable to conjecture that as one dials the background moduli, and thus implicitly the FI parameters, such that the roles of the two branes are exchanged, the attractor tree pattern changes without crossing a wall of marginal stability. This behavior was observed in a related system in [11]. We are using the intuitive identification between the attractor flow and the line of quickest descent in the potential of the matrix model.

### 5.3 An example with three 2-branes

The blowup of the vertex geometry along all three torus invariant legs has a total of four smooth resolutions, related by conifold flops. This is the right description when the legs are expected to become rigid spheres in the global geometry, so they cannot be holomorphically deformed away from the triple intersection. Let us see explicitly how one of these blowups arises from the quiver description of $C_{\square\square\square}$.

Suppose that the 2-brane wrapping the $z_1$ plane has melted first into the background D6, that is, $I^1 \neq 0$. Without any 0-brane probes, the dynamical fields are the $J_j^i$, which satisfy F-term conditions derived before requiring that $J_j^i J_j^i = 0$, $J_j^i J_j^k = 0$, and $J_j^i I^i = 0$ for all distinct $i, j, k$. It is easy to satisfy the D-term constrain, for appropriate values of the FI parameters, by taking $J_1^1 J_1^1 \neq 0$ and all others vanishing. These can be gauged away using the $U(1)$’s associated to the D2 worldvolumes, as expected, therefore the BPS configuration is completely determined by the frozen fields, and the moduli space is a point.

Probing this system with a D0 brane, we find that $\partial W = 0$ implies

$$z_1 a_3^1 = a_3^1 j_2^1$$
$$z_2 a_3^2 = z_3 a_1^2$$
$$z_1 a_2^3 = a_2^3 j_1^1.$$

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The Kahler structure is determined by the D-terms,

\[ |J_1|^2 + |a_1|^2 - |a_2|^2 = r_1 \]
\[ |J_2|^2 + |a_3|^2 = r_2 \]
\[ |J_3|^2 + |a_3|^2 = r_3, \]

where we have absorbed the contributions of the frozen degrees of freedom associated to non-compactness into the FI parameters. Completing the Kahler quotient by \( U(1)^3 \), one obtains the toric web diagram shown in Figure 5.

As we have seen before, the fact that the Euler character is 4 instead of the expected 3 is not a contradiction, since if \( r_1 > 0 \), than one of the \( J_a \) ≠ 0, and the central \( \mathbb{P}^1 \) gets shrunken to a point. It is reasonable to expect that this is not a wall of marginal stability, but rather a change in the attractor tree pattern, since the \( J_a = 0 \) branch of the moduli space for \( r_1 < 0 \) is connected to the configurations in which the 2-branes are separated. If that was the case, then a full analysis of the fermionic terms in the matrix model would show that the anti-ghost bundle is not the tangent bundle in this case, and only contributes 1 from the resolved \( \mathbb{P}^1 \). This seems more natural than a true jump in the index, because the \( r_1 > 0 \) classical moduli space is singular, and one would expect an Euler character of 2 from the conifold singularity if we were using the tangent bundle.

The other resolutions of the blow up are also related by changing the pattern of the attractor flow, condensing first one of the other 2-6 tachyons. These are related geometrically via flops through the symmetric resolution shown in Figure 5.

6 Conclusions and further directions

We constructed matrix models possessing a topological supersymmetry from the quiver descriptions of holomorphic branes in a Calabi-Yau by adding the associated fermions, introducing the multiplets resulting from motion in the Minkowski directions, and imposing the constraints using the anti-ghost field method of [34]. These matrix models are the topologically twisted version of the D-brane theory in the extreme IR limit; that is, they are theories of the BPS sector of the (unknown) dual superconformal quantum mechanics.

The partition function was shown to localize to the Euler character of an obstruction bundle over the classical moduli, by proving the appropriate vanishing theorems. We evaluated the partition function by regularizing the path integral by gauging part of the \( U(1)^3 \) torus symmetry, and using the localization of this equivariant version to the fixed points of the torus action, giving exact agreement with known results.

The toric vertex geometry we studied can be obtained as the local limit of various compact Calabi-Yau manifolds. The \( T^3 \) symmetry guarantees that the information of the global geometry
only affects the local degrees of freedom that remain dynamical in the limit through the values of the FI parameters and the background VEVs of certain frozen fields. Quite generally, some gauge groups of global quiver will be broken by the frozen VEVs, and the associated residual off-diagonal components of the D-term can appear in the effective matrix model.

We saw that the geometry of the BPS moduli space of a single 0-brane in a D6/D2 background in the vertex is exactly the blowup along the curves wrapped by the 2-branes. The structure of the moduli space of $N$ probe 0-branes was also investigated, revealing interesting behavior. This can be viewed as a first step in embedding the quantum foam picture of the A-model into the full IIA theory. In particular we found that effective internal geometry seen by 0-brane probes is indeed the blow ups of the Calabi-Yau predicted in $^{23}$. It might be interesting to relate these ideas to the open/closed duality discussed in $^{20}$. Although the discussion there is in the context of finding the closed string dual of Lagrangian branes in the A-model, they find a sum over Calabi-Yau geometries that depend on which term one one looks at in the Young diagram expansion of the open string holonomy. From the quantum foam point of view, these are related to the 2-brane bound states, so there could be some connection to the effective geometries we have investigated.

In this paper we have focused on the quiver for Donaldson-Thomas theory in the vertex geometry, but the extension of these ideas to many toric geometries should be straightforward. Likewise, even the bound states of branes on a compact Calabi-Yau such as the quintic, which have a quiver description, may be computed using similar topological matrix models. In those cases, the technical difficulty of finding all of the fixed points of a toric action on the classical BPS moduli space increases proportionally. Therefore the hope would be to find a way around performing the direct evaluation used here.

It would be very interesting to try to apply the techniques of matrix models at large $N$ to the quiver theories constructed here. It is natural to expect that an intriguing structure should emerge in the expansion about that large charge limit. This would make sharp the idea that these matrix models give the “CFT” dual of the square of the topological string, at least in the OSV regime, since one would identify the perturbative topological string expansion with the $1/N$ corrections in the quiver matrix model.

Closer to the topic of the 6-brane theories studied in this paper, one would also hope to find a large charge expansion describing the analog of the limit shapes in $^{31}$. The quiver theories would contain further information about the large charge moduli space away from the torus fixed points. Such a method of finding the asymptotic expansion of the partition function could have implications for the entropy enigma found in $^{11}$. On the other hand, because our matrix models are topological, the naive perturbative expansion is trivial, cancelling between bosons and fermions, so novel techniques would be required to obtain the large $N$ limit.

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