A UNIFIED SCHEME OF APPROACH TO RAMANUJAN CONJECTURES

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Abstract. The Ramanujan conjecture for modular forms of holomorphic type was proved by Deligne [2] almost half a century ago: the proof, based on his proof of Weil’s conjectures, was an achievement of algebraic geometry. Quite recently [10], we proved the conjecture in the case of Maass forms of the full unimodular group: we show here that the proof given in this case works in the holomorphic case as well.

1. Introduction

This is an analyst’s proof of the classical Ramanujan conjecture for modular forms of the group $\Gamma = SL(2, \mathbb{Z})$ of holomorphic type. As such, it bypasses the deep proof by Deligne of the Weil conjectures. Our point is not to give a (much) shorter proof of the conjecture, but to give substance to our present belief that the scheme of proof experienced first in the Maass case could constitute an efficient approach to Ramanujan-related conjectures in general. The proof to follow may also be regarded as a belated answer to a question raised by Chandrasekharan [1, p.140] or implicitly raised by Manin [6, p.99].

The method originated from developments in pseudodifferential analysis [9], to be briefly alluded to in Section 6, for the main purpose of explaining terminology. The basic step consists in replacing the hyperbolic half-plane, the classical domain in such questions, by the plane: however, while the group $SL(2, \mathbb{R})$ acts on tempered distributions, in the case of Hecke distributions [10], by linear changes of coordinates, the analogue here will be a representation of $SL(2, \mathbb{R})$ in $\mathcal{S}'(\mathbb{R}^2)$ containing all terms from a certain realization of the holomorphic discrete series.

The proof of the Ramanujan-Petersson conjecture in the Maass case [10] and the present proof of the Ramanujan-Deligne theorem [2] follow totally parallel ways: the relation between the two is close to the relation
between the principal and holomorphic discrete series of $SL(2, \mathbb{R})$. The algebraic part (the present Section 3, culminating in Proposition 3.3), is strictly equivalent to Section 5 in [10], but more spectral-theoretic developments are needed in the Maass case. An axiomatic version containing the two cases would probably be possible, up to a point, but would not make easy reading. This has to wait until more general groups than $SL(2, \mathbb{R})$ are considered, a task better left to mathematicians younger than the present author.

Since our emphasis is on the method rather than the result, we limit ourselves, here, to modular forms of even weight (i.e., $m + 1 = 12, 14, \ldots$) for the full unimodular group, though it would be easy to drop the first assumption, or to introduce characters, while not so easy to consider congruence groups in place of $SL(2, \mathbb{Z})$. Recall that, given a holomorphic cusp-form $f$ of even weight $m + 1$ and of Hecke type, with a Fourier expansion

$$f(z) = \sum_{N \geq 1} b_n e^{2\pi i nz},$$

the Ramanujan conjecture or, rather, the Ramanujan-Deligne theorem, is the inequality $|b_p| \leq 2p^m$ for $p$ prime. The function $f$ is normalized in Hecke’s way if $b_1 = 1$, in which case one has simply $T_pf = b_pf$ [3, p.101], $T_p$ denoting the usual Hecke operator.

One first builds a representation $A_{\text{na}}$ of $SL(2, \mathbb{R})$ in $S'(\mathbb{R}^2)$ preserving for every $m = 1, 3, \ldots$ the part of $S'(\mathbb{R}^2)$ consisting of distributions that transform under $A_{\text{na}}(\gamma)$ with $\gamma \in SO(2)$ like the function $(x_1 - ix_2)^m$; next, for $m = 1, 3, \ldots$, an operator $\theta_m$ from the subspace of $S'(\mathbb{R}^2)$ just defined to the space of holomorphic functions in the hyperbolic half-plane, with the property that, under $\theta_m$, the corresponding restriction of the representation $A_{\text{na}}$ transfers to the representation commonly denoted as $g \mapsto D_{m+1}(g)$ from the holomorphic discrete series of $SL(2, \mathbb{R})$.

Any cusp-form $f$ of even weight $m + 1$ and of Hecke type can be obtained as a (non-explicit) linear combination of quite explicit so-called Poincaré series. Next, given any such Poincaré series $P_M$, one can build an object (almost a distribution) $\Sigma_M$ in the plane the image of which under $\theta_m$ coincides with $P_M$. Besides, given a prime $p$, one constructs a linear endomorphism $T_p^{\text{plane}}$ of the subspace of $S'(\mathbb{R}^2)$ consisting of distributions invariant under the multiplication by $e^{i\pi|x|^2}$, which has the property that, under $\theta_m$, $T_p^{\text{plane}}$ transfers to $T_p$. One is thus left with the problem of finding an appropriate bound, in the above-defined space of tempered distributions, for the operator $T_p^{\text{plane}}$. Taking advantage of the simple algebraic structure
of powers of this operator, one finally obtains the required bound, using standard methods of analysis.

The scheme of proof is the same in the Maass case and the holomorphic case, and the whole Section 3 could be reduced to [10, section 5]. Most analytic developments, though not equivalent in the two cases, are totally similar. At the end, the Maass case is more difficult, as Hecke eigenforms are not linear combinations of Poincaré-type series: instead, some spectral theory is required.

2. A representation of $SL(2, \mathbb{R})$

In this section, we introduce the object (almost a distribution) $\Xi_M$, the image under $\theta_m$ of which is a standard Poincaré series (5.1). Decomposing $\Xi_M$ as $\sum_{(a,c)=1} I_{a,c}$ (2.16), the aim of this admittedly computational section is to prove the estimate (2.33) in Proposition 2.5. The method is strikingly similar to the one in [10, section 7].

We denote as $\mathcal{A}_n$ the representation of $SL(2, \mathbb{R})$ in any of the spaces $\mathcal{S}(\mathbb{R}^2)$, $L^2(\mathbb{R}^2)$, $\mathcal{S}'(\mathbb{R}^2)$, unitary in the second case, defined on generators by the equations, in which the Euclidean Fourier transform is defined as
\[
(F_{\text{euc}} h)(x) = \int_{\mathbb{R}^2} h(y) e^{-2\pi i \langle x, y \rangle} dx,
\]

\[
(i) \quad \mathcal{A}_n \left( \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \right) h(x) = h(x) e^{ixc|x|^2}, \quad x \in \mathbb{R}^2;
\]

\[
(ii) \quad \mathcal{A}_n \left( \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \right) h = -i F_{\text{euc}} h;
\]

\[
(iii) \quad (\mathcal{A}_n \left( \left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) \right) h)(x) = a^{-1} h(a^{-1} x), \quad x \in \mathbb{R}^2, \ a \neq 0. \quad (2.1)
\]

We first show that the representation $\mathcal{A}_n$ contains all representations $D_{m+1}$ from the holomorphic discrete series of representations of $SL(2, \mathbb{R})$.

**Proposition 2.1.** Given $m = 1, 2, \ldots$ and $h \in \mathcal{S}(\mathbb{R}^2)$, set for $z$ in the hyperbolic half-plane $\mathbb{H} = \{ z : \text{Im} \ z > 0 \}$
\[
(\theta_m h)(z) = \int_{\mathbb{R}^2} (x_1 + i x_2)^m e^{i\pi z|x|^2} h(x) dx. \quad (2.2)
\]

For every $g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{R})$, one has $\theta_m(\mathcal{A}_n(g) h) = D_{m+1}(g) \theta_m h$, with
\[
(D_{m+1} \left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \right) f)(z) = (bz + d)^{-m-1} f \left( \frac{az + c}{bz + d} \right). \quad (2.3)
\]
Proof. One may assume that $g$ is one of the generators of $SL(2,\mathbb{R})$ listed in (2.1): the only non-immediate case is (ii). Writing

\[
(\theta_m (-i \mathcal{F}^{\text{euc}} h))(z) = -i \int_{\mathbb{R}^2} (x_1 + i x_2)^m e^{i \pi z |x|^2} \mathcal{F}^{\text{euc}}(x) \, dx
\]

\[
= -i \langle \mathcal{F}^{\text{euc}} ((x_1 + i x_2)^m e^{i \pi z |x|^2}), h \rangle, \quad (2.4)
\]

using the classical formula for the Fourier transform of products of radial functions by “spherical harmonics” (e.g. [7])

\[
\mathcal{F}^{\text{euc}} ((x_1 + i x_2)^m f(|x|)) = (x_1 + i x_2)^m \times 2\pi i^{-m} |x|^{-m} \int_0^\infty f(t) t^{m+1} J_m(2\pi t |x|) \, dt
\]

and the equation [5, p.93]

\[
\int_0^\infty e^{i\pi z t^2} t^{m+1} J_m(2\pi t |x|) \, dt = \frac{1}{2\pi} |x|^m (-iz)^{-m-1} \exp \left( \frac{i\pi |x|^2}{z} \right). \quad (2.6)
\]

one obtains

\[
(\theta_m (-i \mathcal{F}^{\text{euc}} h))(z) = z^{-m-1} (\theta_m h) \left( -\frac{1}{z} \right), \quad (2.7)
\]

which is the desired case of (2.3).

□

Analysis in the plane $\mathbb{R}^2$ will be based on the use of the commuting operators

\[
2i\pi \mathcal{A} = -i \left( x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right), \quad 2i\pi \mathcal{A}^\flat = 1 + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \quad (2.8)
\]

the first of which commutes with all transformations $\mathcal{A} g$. The second one will make integrations by parts of a usual type possible, while appropriate polynomials in the first one will improve convergence by a totally different means, to wit by killing distributions $\mathcal{S}$ solutions of $(2i\pi \mathcal{A}) \mathcal{S} = m \mathcal{S}$ for small “uninteresting” integral values of $m$.

The following calculation introduces the objects $\psi_M$ the transforms of which under the operators $\mathcal{A} g$ with $g \in \Gamma$ constitute the individual terms of the Poincaré-type series $\mathcal{T}_M$ mentioned in the introduction.
Lemma 2.2. With $M = 1, 2, \ldots$, let $\phi_M(x_1, x_2) = e^{2i\pi x_1 \sqrt{2M}}$ and

$$
\psi_M(x) = (\text{Ana} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \phi_M)(x) = -i \left( F_{\text{euc}} \phi_M \right)(x) = -i \delta(x_1 - \sqrt{2M}) \delta(x_2).
$$

If $\left( \begin{smallmatrix} a & \varepsilon \\ c & -\varepsilon \end{smallmatrix} \right) \in \text{SL}(2, \mathbb{Z})$ and $a \neq 0$, one has

$$
\left( \text{Ana} \left( \begin{pmatrix} -a & \varepsilon \\ -c & -\varepsilon \end{pmatrix} \right) \psi_M \right)(x) = a^{-1} \exp \left( \frac{2i\pi M \tau}{a} \right)
\times \exp \left( \frac{2i\pi x_1 \sqrt{2M}}{a} \right) \exp \left( \frac{i\pi c |x|^2}{a} \right),
$$

where $\tau$ is defined by the congruence $c \tau \equiv 1 \mod a$. Also, with $\varepsilon = \pm 1$,

$$
\left( \text{Ana} \left( \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix} \right) \psi_M \right)(x) = \varepsilon \psi_M(\varepsilon x).
$$

Proof. Assuming $ac \neq 0$, let us compute $\text{Ana} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \phi_M = \text{Ana} \left( \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} \right) \psi_M$. One has

$$
\left( \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} \right) = \left( \begin{pmatrix} -c^{-1} & 0 \\ 0 & -c \end{pmatrix} \right) \left( \begin{pmatrix} 1 & ac \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} \right).
$$

Since $\text{Ana} \left( \begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} \right) \psi_M = \exp(-\frac{2i\pi Md}{c}) \psi_M$, one has

$$
\text{Ana} \left( \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} \right) \psi_M = \exp \left( -\frac{2i\pi Md}{c} \right) \text{Ana} \left( \begin{pmatrix} -c^{-1} & 0 \\ 0 & -c \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \psi_M.
$$

Now, $\left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right)$: from this decomposition and (2.9), one obtains

$$
\left( \text{Ana} \left( \begin{pmatrix} 1 & ac \\ 0 & 1 \end{pmatrix} \right) \psi_M \right)(x) = -i F_{\text{euc}} \left[ e^{-i\pi ac |x|^2} e^{2i\pi x_1 \sqrt{2M}} \right]
= -i F_{\text{euc}} \left( y \mapsto e^{-i\pi ac |y|^2} \right)(x_1 \sqrt{2M}, x_2)
= -\frac{1}{ac} \exp \left( \frac{2i\pi M}{ac} \right) \exp \left( -\frac{2i\pi x_1 \sqrt{2M}}{ac} \right) \exp \left( \frac{i\pi |x|^2}{ac} \right).
$$
From (2.13) and (2.14), one has
\[
\begin{align*}
\left[ \mathcal{A}_n \left( \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} \right) \psi_M \right] (x) \\
= \exp \left( -\frac{2i\pi Md}{c} \right) \left[ \mathcal{A}_n \left( \begin{pmatrix} -c & 0 \\ 0 & -c \end{pmatrix} \right) \mathcal{A}_n \left( \begin{pmatrix} 1 & ac \\ 0 & 1 \end{pmatrix} \right) \psi_M \right] (x) \\
= \exp \left( -\frac{2i\pi Md}{c} \right) (-c) \left( \mathcal{A}_n \left( \begin{pmatrix} 1 & ac \\ 0 & 1 \end{pmatrix} \right) \psi \right) (-cx)
\end{align*}
\]
\[
= \frac{1}{a} \exp \left( -\frac{2i\pi Md}{c} \right) \exp \left( \frac{2i\pi M}{ac} \right) \exp \left( -2i\pi \frac{x_1 \sqrt{2M}}{a} \right) \exp \left( i\pi \frac{c |x|^2}{a} \right)
\]
\]
Finally, \( \exp \left( -\frac{2i\pi Md}{c} \right) \exp \left( \frac{2i\pi M}{ac} \right) = \exp \left( -\frac{2i\pi Mb}{a} \right) = \exp \left( \frac{2i\pi Mc}{a} \right) \), which gives (2.10).

Note that (2.10) remains valid if \( a \neq 0 \) and \( c = 0 \), though the proof is not. Also, with \( \varepsilon = \pm 1 \), one has \( \left[ \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \right] = \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \), so that \( \mathcal{A}_n \left( \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \right) \psi_M = \mathcal{A}_n \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \phi_M \), which gives (2.11) if \( \varepsilon = 1 \), if \( \varepsilon = -1 \) as well since \( \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \).

□

Setting \( \Gamma_{\infty} = \left\{ \left( \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right) \right\} \) and \( \Gamma_{\infty}^* = \left\{ \left( \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} : n \in \mathbb{Z} \right) \right\} \), one can define formally the series
\[
\mathcal{T}_M = \sum_{g \in \Gamma \setminus \Gamma_{\infty}} \mathcal{A}_n(g) \phi_M = \sum_{g \in \Gamma \setminus \Gamma_{\infty}^*} \mathcal{A}_n(g) \psi_M
\]
\[
= \sum_{(a,c) = 1} \mathcal{A}_n \left( \begin{pmatrix} -a \\ -c \end{pmatrix} \right) \psi_M = \sum_{(a,c) = 1} I_{a,c}, \quad (2.16)
\]
with
\[
I_{a,c}(x) = a^{-1} \exp \left( -\frac{2i\pi Mc}{a} \right) \exp \left( \frac{2i\pi x_1 \sqrt{2M}}{a} \right) \exp \left( i\pi \frac{c |x|^2}{a} \right). \quad (2.17)
\]
However, the series does not converge weakly in \( S'(\mathbb{R}^2) \); to recover this, it will be necessary to consider in place of this would-be distribution its image under an appropriate polynomial in the operator \( 2i\pi \mathcal{A} \). Also, the core of this proof of the Ramanujan-Deligne theorem will consist in \( q \)-dependent estimates (the cases when \( q \to \infty \) and \( q \to 0 \) must both be considered) for the series
\[
\sum_{(a,c) = 1} \langle I_{a,c}, q^{1-2i\pi A^2} h \rangle = \sum_{(a,c) = 1} \langle q^{1+2i\pi A^2} I_{a,c}, h \rangle. \quad (2.18)
\]
We examine now the individual terms of this series. No uniformity with respect to \( M \) (the integer present in the definition of \( I_{a,c} \)) is needed, or claimed.

**Lemma 2.3.** One has if \( ac \neq 0 \) and \( h \in S(\mathbb{R}^2) \) the estimate

\[
\left| \langle I_{a,c}, q^{-2i\pi A} h \rangle \right| \leq \left( \frac{a^2}{q^2} + c^2 q^2 \right)^{-\frac{1}{2}} |h|, \tag{2.19}
\]

where \( |.| \) is a continuous norm on \( S(\mathbb{R}^2) \) which it is not necessary to make explicit. On the other hand,

\[
\left| \langle I_{1,0}, q^{-2i\pi A} h \rangle \right| + \left| \langle I_{0,1}, q^{-2i\pi A} h \rangle \right| \leq (q + q^{-1})^{-1} |h|, \tag{2.20}
\]

which can be improved to the fact that \( \langle I_{1,0}, q^{-2i\pi A} h \rangle \) is a rapidly decreasing function of \( q \) as \( q \to \infty \), while \( \langle I_{0,1}, q^{-2i\pi A} h \rangle \) is a rapidly decreasing function of \( \frac{1}{q} \) as \( q \to 0 \).

**Proof.** The special cases when \( ac = 0 \) are taken care of by the equations

\[
I_{1,0}(x) = \exp \left( 2i\pi x_1 \sqrt{2M} \right), \quad I_{0,1}(x) = i \delta(x_1 - \sqrt{2M})\delta(x_2), \tag{2.21}
\]

so that

\[
\langle I_{1,0}, q^{-2i\pi A} h \rangle = q \langle \mathcal{F}_{\text{euc}} h \rangle (-q\sqrt{2M}, 0), \quad \langle I_{0,1}, q^{-2i\pi A} h \rangle = -iq^{-1}h(\sqrt{2M}q, 0). \tag{2.22}
\]

We assume from now on that \( ac \neq 0 \). One has

\[
(-i\mathcal{F}_{\text{euc}} I_{a,c})(x) = \left[ (-i\mathcal{F}_{\text{euc}}) \mathcal{A}_a \left( \begin{pmatrix} 1 & -2 \\ 0 & c \end{pmatrix} \right) \psi_{M} \right](x)
\]

\[
eq c^{-1} \exp \left( -2i\pi M \frac{a}{c} \right) \exp \left( \frac{2i\pi \sqrt{2M} x_1}{c} \right) \exp \left( -i\pi a |x|^2 c \right), \tag{2.23}
\]

a consequence of (2.10), (2.17). Since \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left( \begin{pmatrix} a & -c \\ -a & c \end{pmatrix} \right) = \left( \begin{pmatrix} -a & c \\ c & a \end{pmatrix} \right) \), it follows that

\[-i\mathcal{F}_{\text{euc}} I_{a,c} = I_{c,-a}. \]

Also, \( \mathcal{F}_{\text{euc}} q^{2i\pi A^2} = q^{-2i\pi A^2} \mathcal{F}_{\text{euc}} \), hence

\[-i\mathcal{F}_{\text{euc}} \left( q^{2i\pi A^2} I_{a,c} \right) = q^{-2i\pi A^2} I_{c,-a}. \tag{2.24}
\]

The right-hand side of (2.19) is invariant if one changes \( a, c, q \) to \( c, -a, q^{-1} \), and one may thus reduce the proof to the case when \( q \leq 1 \).
One has

\[
\langle I_{a,c}, q^{-2i\pi A^5} h \rangle = \exp \left( \frac{2i\pi Mc}{a} \right) \times \\
\frac{q}{a} \int_{\mathbb{R}^2} \exp \left( \frac{2i\pi qx_1\sqrt{2M}}{a} \right) \exp \left( \frac{i\pi cq^2(x_1^2 + x_2^2)}{a} \right) h(x_1, x_2) \, dx_1 \, dx_2,
\]

(2.25)

and the integral can be rewritten as the product of a constant of modulus 1 by

\[
\frac{q}{a} \int_{\mathbb{R}^2} \exp \left( \frac{i\pi cq^2}{a} (x_1^2 + x_2^2) \right) h \left( x_1 - \frac{\sqrt{2M}}{cq}, x_2 \right) \, dx_1 \, dx_2.
\]

(2.26)

If \( cq^2 \leq 1 \), one has \( \left( \frac{a^2}{q^2} + c^2q^2 \right)^{-\frac{1}{2}} > 2^{-\frac{1}{2}} \frac{q}{a} \) and the desired estimate is immediate. Assume now that \( cq^2 > 1 \), in which case \( \left( \frac{a^2}{q^2} + c^2q^2 \right)^{-\frac{1}{2}} \) is of the order of \( \frac{1}{cq} \), and the estimate is immediate if \( cq \leq 1 \) since, in this case, \( \frac{q}{a} \leq q \leq 1 \leq \frac{1}{cq} \). Finally, if \( cq^2 > 1 \) and \( cq > 1 \), the stationary phase method gives for the integral the bound \( O \left( \frac{a}{cq^2} \right) \), to be completed by the remark that \( \frac{q}{a} \times \frac{a}{cq^2} = \frac{1}{cq} \).

\( \square \)

Integrations by parts will make it possible to improve the estimate, by powers of \( \frac{q}{a} \) or \( cq \).

**Lemma 2.4.** Set, with \( j = 1, 2 \),

\[
B_j = 2\pi \sqrt{2M} \frac{q}{a} x_j, \quad D_j = 2\pi \sqrt{2M} \frac{1}{cq} \left( -\frac{1}{2i\pi} \frac{\partial}{\partial x_j} \right).
\]

(2.27)

One has

\[
\langle I_{a,c}, q^{-2i\pi A^5} (2i\pi A) h \rangle = \langle I_{a,c}, q^{-2i\pi A^5} B_2 h \rangle = \langle I_{a,c}, q^{-2i\pi A^5} D_2 h \rangle.
\]

(2.28)
Proof. One has

\[ (-2i\pi A) \exp \left( \frac{2i\pi qx_1 \sqrt{2M}}{a} \right) = \exp \left( \frac{2i\pi qx_1 \sqrt{2M}}{a} \right) \times \frac{2\pi qx_2 \sqrt{2M}}{a}. \]  

(2.29)

We compute \( \langle I_{a,c}, q^{1-2i\pi A^i} (2i\pi A) h \rangle \) with the help of (2.25). Since the transpose of \( A \) is \( -A \), one can, instead of replacing \( h \) by \( 2i\pi A h \), keep \( h \) as it is and replace the product of the two exponentials on the right-hand side of (2.25) by its image under \(-2i\pi A\), an operator that acts trivially on the second exponential. Using (2.29), one obtains the first equation (2.28), the main benefit of which lies in the presence of the factor \( q^a \) in the operator \( B^2 \).

One can gain, instead, a factor \( \frac{1}{cq} \), noting that one has also

\( \langle I_{a,c}, q^{-2i\pi A^i} (2i\pi A) h \rangle = \langle I_{a,c}, q^{-2i\pi A^i} D_2 h \rangle \)  

(2.30)

with \( D_2 = 2\pi \sqrt{2M} \frac{1}{cq} \left( -\frac{1}{2\pi} \frac{\partial}{\partial x_j} \right) \). To see this, it suffices to note that the operator of multiplication by \( \frac{qx_2}{a} \) has the same effect as the operator \(-D_2\) on the exponential \( \exp \left( -i\pi cq^2 (x_1^2 + x_2^2) \right) \), while the other exponential present in the integrand of (2.25) depends only on \( x_1 \). This time, do not forget to change sign when transposing \( D_2 \).

\[ \square \]

Proposition 2.5. Define by induction the polynomials \( P_j(A) \) such that

\[ P_0(A) = 1 \]  

and

\[ P_{j+1}(A) = \prod_{|r| \leq j} (A + r) \times P_j(A). \]  

(2.31)

There is a sequence of polynomials \( Q_j(\alpha, \beta) \) in the variables \( \alpha, \beta \), of the same degree as \( P_j \) and without any term of total degree \( <j \), such that one has for every \( h \in S(\mathbb{R}^2) \) the identity

\[ \langle I_{a,c}, q^{-2i\pi A^i} P_j(2i\pi A) h \rangle = \langle I_{a,c}, q^{-2i\pi A^i} Q_j(B_1, B_2) h \rangle. \]  

(2.32)

One has for some continuous norm \( \| \cdot \| \) on \( S(\mathbb{R}^2) \) depending only on \( j \) the estimate, valid for every \( h \in S(\mathbb{R}^2) \),

\[ \| \langle I_{a,c}, q^{-2i\pi A^i} P_j(2i\pi A) h \rangle \| \leq \left( \frac{a^2}{q^2} + c^2 q^2 \right)^{-\frac{1}{2}} \left( 1 + \frac{a^2}{q^2} + c^2 q^2 \right)^{\frac{1}{2}} \| h \|. \]  

(2.33)
Proof. The first equation (2.28) means that, with $P_1(A) = A$, (2.32) will hold if taking $Q_1(B_1, B_2) = B_2$. For simplicity of notation, set $A = 2i\pi \mathcal{A}$ and $L = q^{1-2i\pi \mathcal{A}^2}$, thus writing the identity just recalled as $I_{a,c}(LAh) = I_{a,c}(LB_2h)$. It is convenient to introduce the combinations $X = -iB_1 + B_2$ and $Y = iB_1 + B_2$, so that

$$[A, X] = X, \quad [A, Y] = -Y, \quad B_2 = \frac{X + Y}{2}$$

and, more generally,

$$X^r A = AX^r - r R^r, \quad Y^r A = AY^r + r Y^r,$$ (2.35)

and to search in place of $Q_j(B_1, B_2)$ for a polynomial $R_j(X, Y)$ with the same degree requirements. Given two polynomials $P(A)$ and $R(X, Y)$, write $P(A) \sim R(X, Y)$ if, for every $h \in S(R^2)$, one has $I_{a,c}(LP(A)h) = I_{a,c}(LR(X, Y)h)$. More generally, one can extend this notion of equivalence to the case when “polynomials” $S(A, X, Y)$ in $A, X, Y$ (i.e., sums of ordered monomials) are considered in place of polynomials $R(X, Y)$. Note that an equivalence $P(A) \sim S(A, X, Y)$ remains one if both “polynomials” are multiplied on the right by the same polynomial in $A$.

Using (2.35), one can push $A$ to the left, or let is disappear, until, starting from a monomial $X^{r_1}Y^{r_2}$, one obtains, using at the end that $A \sim B_2 = \frac{X + Y}{2}$,

$$X^{r_1}Y^{r_2} A = X^{r_1} [AY^{r_2} + r_2 Y^{r_2}] = [AX^{r_1} - r_1 X^{r_1}] Y^{r_2} + r_2 X^{r_1} Y^{r_2}
\sim \frac{1}{2} [X^{r_1+1}Y^{r_2} + X^{r_1}Y^{r_2+1}] - (r_1 - r_2) X^{r_1} Y^{r_2}.$$ (2.36)

Then, one has

$$X^{r_1}Y^{r_2}(A + r_1 - r_2) \sim \frac{1}{2} [X^{r_1+1}Y^{r_2} + X^{r_1}Y^{r_2+1}]$$ (2.37)

There were already no terms of total degree $< j$ in $R_j(X, Y)$, and there are none in $R_{j+1}(X, Y)$ in view of (2.36). Multiplying $R_j(X, Y)$ on the right by the polynomial $\prod_{|r| \leq k} (A + r)$ has the effect of killing all monomials of total degree $j$.

For the second part, we note, in view of the coefficient $\frac{2\pi \sqrt{2M}}{qa}$ apparent in the definition of $B_2$, that the application of $B_2^j$ makes it possible to obtain an extra coefficient $(\frac{2}{\pi})^{-j}$, at the price, of course, of changing the
norm \( | \cdot | \). Now, one can use the operator \( D_2 \) in place of \( B_2 \), changing the coefficient \( (\frac{a}{b})^{-j} \) to \( (qc)^{-j} \).

\[ \square \]

3. The Hecke operator and its powers

The calculations in the present section are an exact replica of those made in [10, Section 5] in the Maass case. The algebra is the same, a consequence on one hand of the fact that the relation \( R \tau[\beta] = \tau\left(\frac{A}{p}\right)R \), (cf. (3.9)) on which the whole section 3 depends, will still hold if, instead of what has been introduced in Lemma 3.2, we define \( R \) by the equation

\[ (Rh)(x, \xi) = p^{-\frac{1}{2}}h \left(p^{\frac{1}{2}}x, p^{-\frac{1}{2}}\xi\right) \]

and \( \tau[\beta] = \exp(\beta\xi \frac{\partial}{\partial x}) \). On the other hand, the analogue of (3.13), replacing \( R \) in the way just indicated and replacing \( T_{\text{plane}} \) by the operator with the same role in the Maas case, holds too.

One could thus, forgetting self-containedness, dispense with the present section or with Section 5 in [10]. This coincidence does not extend to the analytic developments making up the rest of either paper, though most are still similar in the two cases.

The Poincaré series \((\theta_m \Sigma_M)(z)\) is a modular form of weight \(2k = m+1\): it will be convenient, if redundant, to keep both \(k\) and \(m\). On the class of holomorphic functions \(f\) invariant under the change \(z \mapsto z + 1\), one defines for \(p\) prime [3, (6.13)] the operator \(T_p\) such that

\[ (T_pf)(z) = p^m f(pz) + \frac{1}{p} \sum_{s \mod p} f\left(\frac{z + s}{p}\right) . \quad (3.1) \]

Recall [3, (6.39)] that a cusp-form with the Fourier expansion \(f(z) = \sum_{n \geq 1} b_n e^{2\pi i nz}\) is of Hecke type if, for any prime \(p\), \(T_p f\) is a multiple of \(f\), of necessity by the factor \(\frac{b_p}{b_1}\).

**Lemma 3.1.** Define on distributions \(\Sigma\) invariant under \(\text{Ana}(\Gamma^\infty)\) the operator \(T_p^{\text{plane}}\) such that

\[ \left(T_p^{\text{plane}}\Sigma\right)(x) = p^{\frac{m}{2}} - 1 \Sigma\left(\frac{x}{\sqrt{p}}\right) + p^{\frac{m}{2}} \sum_{s \mod p} \Sigma(x \sqrt{p}) \exp\left(\frac{i\pi s |x|^2}{p}\right) , \quad (3.2) \]
in other words
\[ p^{-\frac{1}{2}-i\pi A^\natural} T_p^{\text{plane}} = p^{-\frac{1}{2}-i\pi A^\natural} + \sum_{s \mod p} \exp \left( \frac{i\pi s |x|^2}{p} \right) p^{-\frac{1}{2}+i\pi A^\natural}. \] (3.3)

If \( \mathfrak{T} \in S'(\mathbb{R}^2) \) is invariant under \( \mathfrak{A}^\natural(\Gamma^\bullet) \), one has for \( m = 3, 5, \ldots \) the identity
\[ T_p \left( \theta_m \mathfrak{T} \right) = \theta_m \left( T_p^{\text{plane}} \mathfrak{T} \right). \] (3.4)

**Proof.** One has
\[ (T_p \theta_m \mathfrak{T})(z) = \int_{\mathbb{R}^2} (x_1 + ix_2)^m \mathfrak{T}(x) \left[ p^m e^{i\pi \frac{p^2}{2} |x|^2} + \frac{1}{p} \sum_{s \mod p} \exp \left( \frac{i\pi \left( \frac{z + s}{p} \right) |x|^2}{p} \right) \right] dx \]
\[ = \int_{\mathbb{R}^2} (x_1 + ix_2)^m e^{i\pi x \frac{|x|^2}{2}} \left[ p^{\frac{m}{2}-1} \mathfrak{T} \left( \frac{x}{\sqrt{p}} \right) + p^{\frac{m}{2}} \mathfrak{T} (x \sqrt{p}) \exp \left( \frac{i\pi s |x|^2}{p} \right) \right] dx. \] (3.5)

**Lemma 3.2.** Denote as \( R \) the operator \( p^{-\frac{1}{2}-i\pi A^\natural} \). Define, for \( \beta \in \mathbb{R} \), the operator \( \tau[\beta] \) as the operator of multiplication by the function \( \exp \left( i\pi \beta |x|^2 \right) \).

If one introduces for every \( j \in \mathbb{Z} \) the space \( \text{Inv}(p^j) \) consisting of tempered distributions invariant under \( \tau[p^j] \), the operator \( R^\ell \) acts for every \( \ell \in \mathbb{Z} \) from \( \text{Inv}(p^j) \) to \( \text{Inv}(p^{j-\ell}) \). Given two linear endomorphisms \( A_1 \) and \( A_2 \) of \( \text{Inv}(1) \), write \( A_1 \sim A_2 \) if the two operators (which may well extend to a larger space) coincide there. On the other hand, with \( \tau[.] \) as just defined, introduce the operators
\[ \sigma_r = \frac{1}{p^r} \sum_{s \mod p^r} \tau \left[ \frac{s}{p^r} \right], \quad \sigma_r^{(\ell)} = \frac{1}{p^r} \sum_{s \mod p^r} \tau \left[ \frac{sp^{\ell-r}}{p^r} \right], \] (3.6)

the first (the case \( \ell = 0 \) of \( \sigma_r^{(\ell)} \)) on \( \text{Inv}(1) \), the second on \( \text{Inv}(p^\ell) \). One has
\[ R = p^{-\frac{1}{2}-i\pi A^\natural}, \quad R^{-1} = p^{\frac{1}{2}+i\pi A^\natural}, \quad \text{so that} \quad p^{-\frac{1}{2}} T_p^{\text{plane}} = R + \sigma_1^{(1)} R^{-1}. \] (3.7)

One has for every pair \( r, \ell \) of non-negative integers
\[ R^\ell \sigma_r \sim \sigma_{r+\ell}, \quad R^{-\ell} \sigma_r \sim \sigma_r^{(\ell)} R^{-\ell}, \quad \sigma_r R^{-1} \sigma_1 \sim R^{-1} \sigma_{r+1}. \] (3.8)
Proof. A distribution $\mathcal{S}$ lies in $\text{Inv}(p^\ell)$ if it is invariant under the multiplication by $\exp(i\pi p^\ell |x|^2)$: in particular, $\psi_M \in \text{Inv}(1)$ for $M = 1, 2, \ldots$.

That $R^\ell$ sends $\text{Inv}(p^j)$ to $\text{Inv}(p^{j-\ell})$ is immediate. So is the fact that $R\tau[\beta] = \tau[p^{\ell+1} R].$ Write then

$$R \sigma_r = \frac{1}{p^r} \sum_{s \mod p^r} \tau \left[ \frac{s}{p^{r+1}} R \right], \quad \sigma_{r+1} R = \frac{1}{p^{r+1}} \sum_{s_1 \mod p^{r+1}} \tau \left[ \frac{s_1}{p^{r+1}} R \right].$$

(3.9)

The first operator makes sense on $\text{Inv}(1)$ because if $S$ lies in this space, $R S \in \text{Inv}(1)$, and the knowledge of $b \mod p^r$ implies that of $b p^{r+1}$ up to a multiple of $1/p$. That $R \sigma_r$ and $\sigma_{r+1} R$ agree on $\text{Inv}(1)$ follows. By induction on $\ell$, $R^\ell \sigma_r \sim \sigma_{r+1} R^\ell$. Next,

$$\left( R^{-\ell} \sigma_r \mathcal{S} \right)(x) = R^{-\ell} \left[ x \mapsto p^{-r} \sum_{s \mod p} \mathcal{S}(x) \exp \left( i\pi s |x|^2 \right) \right]$$

$$= p^{\ell-r} \sum_{s \mod p^r} \mathcal{S}(p^{\frac{1}{2}} x) \exp \left( i\pi s p^{\ell-r} |x|^2 \right)$$

$$= p^{-r} \sum_{s \mod p^r} \exp \left( i\pi s p^{\ell-r} |x|^2 \right) \left( p^{\ell(\frac{1}{2} + i\pi A^2)} \mathcal{S} \right)(x),$$

(3.10)

so that $R^{-\ell} \sigma_r \sim \sigma_{r}^{(\ell)} R^{-\ell}$.

Finally, if $\mathcal{S} \in \text{Inv}(1)$, $\sigma_1 \mathcal{S} \in \text{Inv}(p^{-1})$, so that $R^{-1} \sigma_1 \mathcal{S} \in \text{Inv}(1)$, and the operator $\sigma_r R^{-1} \sigma_1$ is well-defined on this space. One has

$$(\sigma_r R^{-1} \sigma_1 \mathcal{S})(x_1, x_2) = \sigma_r \left[ (x_1, x_2) \mapsto p \left( \sigma_1 \mathcal{S} \right) \left( p^{\frac{1}{2}} x \right) \right]$$

$$= p^{-r+1} \sum_{s \mod p^r} \left( \sigma_1 \mathcal{S} \right) \left( p^{\frac{1}{2}} x \right) \exp \left( i\pi s |x|^2 \right) = p \mathcal{S}(p^{\frac{1}{2}} x) = (R^{-1} \mathcal{S})(x)$$

(3.11)
with
\[
\mathcal{T}(x) = p^{-r} \sum_{s \mod p^r} \sigma_1 \mathcal{G}(x) \exp \left( \frac{i\pi s |x|^2}{p^{r+1}} \right) = p^{-r-1} \sum_{s \mod p^r} \mathcal{G}(x) \exp \left( \frac{i\pi s |x|^2}{p^{r+1}} + \frac{i\pi s' |x|^2}{p} \right). \tag{3.12}
\]

As \(s\) and \(s'\) run through the classes indicated as a subscript, \(s + p^r s'\) describes a full class modulo \(p^{r+1}\), so that the right-hand side of (3.12) is the same as \((R^{-1} \sigma_{r+1} \mathcal{G})(x_1, x_2)\). In other words, \(\sigma_r R^{-1} \sigma_1 \sim R^{-1} \sigma_{r+1}\).

\[\boxdot\]

**Proposition 3.3.** Set
\[
\tilde{T} = p^{-\frac{m}{2}} T^\text{plane}_p = R + \sigma_1^{(1)} R^{-1} = R + R^{-1} \sigma_1. \tag{3.13}
\]

Given \(k = 1, 2, \ldots\) and \(\ell\) such that \(0 \leq \ell \leq k\), there are non-negative integers \(\alpha_{k, \ell}^{(0)}, \alpha_{k, \ell}^{(1)}, \ldots, \alpha_{k, \ell}^{(\ell)}\), satisfying the conditions:

(i) \(\alpha_{k, \ell}^{(0)} + \alpha_{k, \ell}^{(1)} + \cdots + \alpha_{k, \ell}^{(\ell)} = \binom{k}{\ell}\) for all \(k, \ell\),

(ii) \(2\ell - k - r \leq 0\) whenever \(\alpha_{k, \ell}^{(r)} \neq 0\),

such that one has the identity (between two operators on the space of \(\Gamma^\infty\)-invariant distributions \(\mathcal{G}\))
\[
\tilde{T}^k = \sum_{\ell=0}^{k} R^{k-2\ell} \left( \alpha_{k, \ell}^{(0)} I + \alpha_{k, \ell}^{(1)} \sigma_1 + \cdots + \alpha_{k, \ell}^{(\ell)} \sigma_\ell \right). \tag{3.14}
\]

**Proof.** By induction. Assuming that the given formula holds, we write \(\tilde{T}^{k+1} = \tilde{T}^k (R + R^{-1} \sigma_1)\), using the equations \(\sigma_r R \sim R \sigma_{r-1}\) \((r \geq 1)\) and \(\sigma_r R^{-1} \sigma_1 \sim R^{-1} \sigma_{r+1}\). We obtain

\[
\tilde{T}^{k+1} = \sum_{\ell=0}^{k} R^{k+1-2\ell} \left( \alpha_{k, \ell}^{(0)} I + \alpha_{k, \ell}^{(1)} I + \alpha_{k, \ell}^{(2)} \sigma_1 + \cdots + \alpha_{k, \ell}^{(\ell)} \sigma_{\ell-1} \right)
+ \sum_{\ell=0}^{k} R^{k-1-2\ell} \left( \alpha_{k, \ell}^{(0)} \sigma_1 + \alpha_{k, \ell}^{(1)} \sigma_2 + \cdots + \alpha_{k, \ell}^{(\ell)} \sigma_{\ell+1} \right), \tag{3.15}
\]
or

\[
\tilde{T}^{k+1} = \sum_{\ell=0}^{k} R^{k+1-2\ell} \left( \alpha_{k,\ell}^{(0)} I + \alpha_{k,\ell}^{(1)} I + \alpha_{k,\ell}^{(2)} \sigma_1 + \cdots + \alpha_{k,\ell}^{(\ell)} \sigma_{\ell-1} \right) \\
+ \sum_{\ell=1}^{k+1} R^{k+1-2\ell} \left( \alpha_{k,\ell-1}^{(0)} \sigma_1 + \alpha_{k,\ell-1}^{(1)} \sigma_2 + \cdots + \alpha_{k,\ell-1}^{(\ell-1)} \sigma_{\ell-1} \right).
\] (3.16)

The point (i) follows, using \((k+1)\ell + k = (k+1)^2\). Next, looking again at (3.15), one observes that, in the expansion of \(\tilde{T}^{k+1}\), the term \(R^{k+1-2\ell}\) is the sum of two terms originating (in the process of obtaining \(\tilde{T}^{k+1}\) from \(\tilde{T}^k\)) from the terms \(R^{k-2\ell}\sigma_{r+1}\) and \(R^{k-2\ell+2}\sigma_{r-1}\). The condition \(2\ell - (k+1) - r \leq 0\) is certainly true if either \(2\ell - k - (r+1) \leq 0\) or \((2\ell - 2) - k - (r-1) \leq 0\), which proves the point (ii) by induction.

\(\square\)

4. The main estimate

Again the method is similar to the one developed in [10, section 8].

From Proposition 2.5, it follows that, for \(j \geq 2\), the series

\[
\mathcal{G}_M^{(j)} : = \sum_{(a,c) = 1} P_j(-2i\pi A) I_{a,c},
\] (4.1)

with \(I_{a,c}\) as defined in (2.17), is a well-defined tempered distribution. The main point of our proof will consist in establishing a satisfactory bound, in \(S'(\mathbb{R}^2)\), for the image of \(\mathcal{G}_M^{(j)}\) under the operator \((T_{p}^{\text{plane}})^{2N}\), with a large \(N\). This operator was made explicit in Proposition 3.3 (take \(k = 2N\) there), and we shall analyze individually the terms \(R^{2N-2\ell}\sigma_r\) (with \(r \geq 0\) and \(r \geq 2\ell - 2N\)) of the decomposition (3.14). We set \(q = p^{\ell} - N\) so that \(R^{2N-2\ell} = q^{1+2i\pi A^2}\), an operator the transpose of which is the operator \(q^{-1} - 2i\pi A^2\): one has \(q \leq 1\) if \(\ell \leq N\), \(q > 1\) if \(\ell > N\). Note that the operator \(A\) commutes with all operators present below, to wit \(A^2\), the \(\tau[\gamma]'s\), and \(\sigma_r\).

**Proposition 4.1.** Given a positive integer \(M\) and \(j \geq 2\), let \(\mathcal{G}_M^{(j)}\) be the distribution defined in (4.1). Given a prime \(p\), a number \(\varepsilon > 0\) and \(h \in S(\mathbb{R}^2)\), the expression \((q + q^{-1})^{-\varepsilon} \langle q^{1+2i\pi A^2} \sigma_r (\mathcal{G}_M^{(j)}), h \rangle = (q + q^{-1})^{-\varepsilon} \langle q^{1+2i\pi A^2} \sigma_r (\mathcal{G}_M^{(j)}), h \rangle\) is bounded in a way independent of \(r =
0, 1, . . . and of q > 0 such that p′ ≥ q^2.

Proof. The operator σ_r is an arithmetic average of operators τ[\frac{s}{p'}] with 0 ≤ s < p', and the estimate to be proven would be so if the corresponding estimate, with σ_r replaced by τ[\frac{s}{p'}] for any fixed s, were. As τ[\frac{s}{p'}] is the operator of multiplication by exp\left(\frac{i\pi s |x|^2}{p'}\right), one has

\[ \tau\left[\frac{s}{p'}\right] q^{1-2i\pi A^3} h = q^{1-2i\pi A^3} \tilde{h} \quad \text{with} \quad \tilde{h}(x) = h(x) \exp\left(\frac{i\pi s q^2 |x|^2}{p'}\right). \] (4.2)

Denote as \overset{\circ}{I}_{a,c} the distribution obtained from I_{a,c}, as made explicit in (2.17), by dropping the constant factor exp\left(\frac{2i\pi M_p}{a}\right), of absolute value 1: then, \overset{\circ}{I}_{a,c} makes sense for nonzero values of a, c, no longer assuming that a, c are integers. Inserting the factor exp\left(\frac{i\pi s q^2 |x|^2}{p'}\right) and grouping it with the factor exp\left(\frac{i\pi c q^2 |x|^2}{a}\right) already present in \overset{\circ}{I}_{a,c}(qx), one obtains

\[ \langle \overset{\circ}{I}_{a,c}, \tau\left[\frac{s}{p'}\right] q^{1-2i\pi A^3} h \rangle = \langle \overset{\circ}{I}_{a,c+\frac{a}{p'}}, q^{1-2i\pi A^3} h \rangle. \] (4.3)

This is identical to \langle \overset{\circ}{I}_{a,c}, h \rangle, save for the replacement of c by c_1 = c + \frac{a}{p'}.

When a = 0, one has c_1 = c = ±1 and the desired estimate has been obtained in Lemma 2.3. We assume from now on that a ≠ 0.

Let us first consider, for any fixed s, the sum of terms for which c_1 ≥ \frac{1}{2} or c_1 < -\frac{1}{2}. We use the bound (Proposition 2.5)

\[ \left| \langle \overset{\circ}{I}_{a,c}, \tau\left[\frac{s}{p'}\right] P_j (2i\pi A) q^{1-2i\pi A^3} h \rangle \right| \leq q \times \left(1 + \frac{a^2}{q^2} + c_1^2 q^2\right)^{-\frac{1}{2}} \left|h\right|^2. \] (4.4)

With

\[ \gamma = \frac{s}{p'} \in [0, 1[, \quad c_1 = c + \gamma a, \quad a' = \frac{a}{q}, \quad c' = cq, \] (4.5)
one has
\[
\frac{a^2}{q^2} + c_1^2 q^2 = a'^2 + (c' + \gamma a' q^2)^2 = (a' c') \left( \frac{1 + \gamma^2 q^4}{\gamma q^2} \right) \left( a' c' \right)
\]
\[
\geq (2 + \gamma^2 q^4)^{-1}(a'^2 + c'^2) = (2 + \gamma^2 q^4)^{-1} \left( \frac{a^2}{q^2} + c^2 q^2 \right). \quad (4.6)
\]

If \( q \leq 1 \), it suffices to note that, if \( j \geq 3 \),
\[
q \int_{-\infty}^{\infty} \int_{1}^{\infty} \left( 1 + \frac{x^2}{q^2} + y^2 q^2 \right)^{-\frac{j-1}{2}} dx dy = O(q), \quad (4.7)
\]
to obtain the desired result (even the more so). The same kind of estimate would not do for \( q > 1 \) because of the bad factor \( (2 + \gamma^2 q^4)^{\frac{j+1}{2}} \).

However, in this case, the sum of the expressions (4.4) extended to the set of pairs \( a, c \) such that \( a \geq 1 \) and \( c_1 = c + \frac{sa}{p^r} > \frac{1}{2} \) or \( c_1 \leq -\frac{1}{2} \) would still be satisfactory. The more difficult remaining part of the series is the sum extended to the set of pairs \( a, c \) such that \( a \geq 1 \), \(-\frac{1}{2} < c + \frac{sa}{p^r} \leq \frac{1}{2} \), under the assumption that \( q > 1 \).

With \( g = P_j(2i\pi A) h \) and \( a \geq 1 \), set
\[
f^{(j)}_{a,c}(t) = I_{a,c+ta} \left( q^{-2i\pi A} P_j(2i\pi A) h \right) \times \text{char}(-\frac{1}{2} < c + ta \leq \frac{1}{2}) \quad (4.8)
\]
\[
= \int_{-\infty}^{\infty} g \left( q(c+ta)x - \frac{q}{a} \right) e^{2i\pi x} dx \times \text{char} \left( -\frac{1}{2} < \frac{c}{a} < t < \frac{1}{2} - \frac{c}{a} \right). \quad (4.9)
\]

There is another tacit constraint on \( c \), to wit \( (a, c) = 1 \). We must estimate now the sum
\[
\sum_{(a, c) = 1} \frac{1}{p^r} \sum_{0 < s \leq p^r} f^{(j)}_{a,c} \left( \frac{s}{p^r} \right), \quad (4.10)
\]
truly a sum over \( a \geq 1 \) only since, the term corresponding to a given value of \( a \) and \( s \) can be nonzero for only one value of \( c \).
One has
\[
\frac{1}{p^r} \sum_{0 < a \leq p^r} \sum_{a \geq 1} f_{a,c}^{(j)} \left( \frac{b}{p^r} \right) = \int_0^1 \sum_{a \geq 1} f_{a,c}^{(j)}(t) \, dt
\]
\[
+ \frac{1}{2p^r} \sum_{a \geq 1} \left[ f_{n,m}^{(j)}(1) - f_{a,c}^{(k)}(0) \right] - \frac{1}{p^r} \int_0^1 B_1(p^r t) \frac{d}{dt} \sum_{a \geq 1} f_{a,c}^{(j)}(t) \, dt,
\]
(4.11)
and we shall treat one-by-one the various terms of this Euler-Maclaurin expansion, recalling that \(|B_1| \leq \frac{1}{2} \). In the last term, one must not forget the discontinuities of the function \(f_{a,c}^{(j)}\), implied by the presence of the characteristic function as a factor of \(f_{a,c}^{(j)}\).

Considering the first integral on the right-hand side, we use Proposition 2.5 to write
\[
|f_{a,c}^{(j)}(t)| \leq C \left( \frac{a^2}{q^2} + q^2(c + ta)^2 \right)^{-\frac{1}{2}} \left[ 1 + \frac{a^2}{q^2} + q^2(c + ta)^2 \right]^{-\frac{1}{2}}
\]
\[
\times \text{char}(-\frac{1}{2} < c + ta \leq \frac{1}{2}),
\]
(4.12)
so that, setting \(s = \frac{t}{q}\),
\[
\int_0^1 \sum_{a \not= 0} \left| f_{a,c}^{(j)}(t) \right| \, dt \leq \sum_{a \geq 1} \frac{1}{qa} \int_0^\infty \left( \frac{a^2}{q^2} + t \right)^{-\frac{1}{2}} \left( 1 + \frac{a^2}{q^2} + t \right)^{-\frac{1}{2}} \, ds
\]
\[
\leq \sum_{a \geq 1} \frac{1}{qa} \int_0^\infty \left[ \left( \frac{a^2}{q^2} \right)^\varepsilon \right]^\frac{1}{2} (t^{1-\varepsilon})^{\frac{1}{2}} (1 + t)^{-\frac{1}{2}} \, dt
\]
\[
= C a^{-1-\varepsilon} q^{-1+\varepsilon}
\]
for some constant \(C > 0\) if \(0 < \varepsilon < 1 \leq j\). The first factor ensures summability, and the factor \(q^{-1+\varepsilon}\) is just the one we hoped for.
Let us consider the other terms of the Euler-Maclaurin decomposition (4.11). From (4.12), one has if $c \neq 0$, $j \geq 1$ and $0 < \varepsilon \leq 1$

$$\left| f^{(j)}_{a,c}(0) \right| \leq C \left( \frac{a^2}{q^2} + q^2 c^2 \right)^{-\frac{1}{2}} \left[ 1 + \frac{a^2}{q^2} + q^2 c^2 \right]^{-\frac{1}{2}} \leq C \left( \frac{a^2}{q^2} + q^2 \right)^{-1}$$

$$\leq C \left[ \left( \frac{a^2}{q^2} \right)^{\frac{1+\varepsilon}{2}} (q^2)^{1-\frac{1}{2}} \right]^{-1} = C a^{-1-\varepsilon} q^{2\varepsilon}.$$  \hspace{1cm} (4.13)

The $m$-summability is ensured. The (bad) factor $q^{2\varepsilon}$ is taken care of by the coefficient $p^{-r}$ apparent in (4.11), since we have assumed that $2\ell - 2N - r \leq 0$, hence $p^r \geq q^2$, so that a factor $q^{-2+2\varepsilon}$ remains. The term $\frac{1}{2p^r} f^{(j)}_{a,c}(1)$ is bounded in exactly the same way.

Finally, one remarks that $\frac{d}{dt} f^{(j)}_{a,c}(t)$ is the sum of three terms: the major one is obtained by the equation (4.8) if one replaces there the function $h$ by the function $h_1(x_1, x_2) = q^2 x_2 \frac{\partial h}{\partial x_1}$, and $g$ by $g_1 = P_j(2i\pi A) h_1$. We have just recalled the assumption $p^{-r} q^2 \leq 1$. The other two terms are Dirac masses at $t = a^{-1} (\pm \frac{1}{2} - c)$ and are to be treated just as (4.13), only replacing $\frac{a^2}{q^2} + q^2 c^2$ by $\frac{a^2}{q^2} + \frac{q^2}{4}$. Given $a$, the number of available $c$, in view of the condition $t \in [0, 1]$, is at most $|a|$, and a comparison with an integral shows that $\sum |a| \left( \frac{a^2}{q^2} + \frac{q^2}{4} \right)^{-j-1} = O(q^{1-j})$ if $\geq 2$. This concludes the proof of Proposition 4.1.

$\square$

5. The Ramanujan-Deligne theorem

This section is radically different from [10, section 9], which demands more spectral-theoretic developments.

Though the series $\mathfrak{S}_M = \sum_{g \in \Gamma \backslash \Gamma^*} \text{Ana}(g) \psi_M = \sum_{(a,c)=1} I_{a,c}$ considered in (2.16) does not converge in $S'(R^2)$, the series of its $\theta_m$-transforms does converge as a holomorphic function in $\mathbb{H}$ and its sum is given, with $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma$, as

$$(\theta_m \mathfrak{S}_M)(z) = -i (2M)^{\frac{m}{2}} \sum_{(a,c)=1} (-dz + b)^{-m-1} \exp \left( 2i\pi M \frac{-cz + a}{dz + b} \right),$$ \hspace{1cm} (5.1)

a standard Poincaré series. This expansion is a consequence of the equation

$$(\theta_m \psi_M)(z) = -i (2M)^{\frac{m}{2}} e^{2i\pi Mz}$$ \hspace{1cm} (5.2)
Theorem 5.1. (Deligne) The coefficients $b_p$ of the Fourier series decomposition of a holomorphic cusp-form $f$ of even weight $2k = m + 1$ and Hecke type, normalized by the condition $b_1 = 1$, satisfy for $p$ prime the estimate $|b_p| \leq 2p^{k - \frac{1}{2}}$.

Proof. Let $M_0$ be the dimension of the linear space of cusp-forms of weight $2k$: then $[3, p.54]$, $f$ is of necessity a linear combination of the Poincaré series $\theta_M$ (cf. (5.1)) with $M \leq M_0$. For each such $M$, and large $N$, we consider the image of $\theta_M$ under $(p - \frac{1}{2} - i\pi A^5 + p - \frac{1}{2} + i\pi A^5 \sigma_1)^{2N}$, i.e., using Proposition 3.3, under the operator

$$
\left(p^{-\frac{1}{2} - i\pi A^5} + p^{-\frac{1}{2} + i\pi A^5} \sigma_1\right)^{2N} = \sum_{\ell=0}^{2N} \sum_{0 \leq r \leq \ell} \alpha_{2N,\ell}^{(r)} b^{(\ell-N)(1+2i\pi A^5)} \sigma_r.  \quad (5.3)
$$

The $(2N)$-trick has been borrowed from Langlands's suggestion in matters related to the Grand Ramanujan Conjecture $[4, p.716]$.

Recall that $\mathcal{S}_M^{(j)} = P_j(-2i\pi A) \mathcal{T}_M$ and that, with $q = p^\ell - N$, so that $\mathcal{S}_M^{(j)} = q^{1+2i\pi A^5}$, Proposition 4.1 gives for the expression $\langle q^{1+2i\pi A^5} \sigma_r \mathcal{S}_M^{(j)}, h \rangle$ the bound $(q + q^{-1})^\epsilon \| h \|$, for some continuous norm $\| . \|$ on $\mathcal{S}(\mathbb{R}^2)$ independent of $\ell$ and $r$. It follows that

$$
\left| \langle \left(p^{-\frac{1}{2} - i\pi A^5} + p^{-\frac{1}{2} + i\pi A^5} \sigma_1\right)^{2N} \mathcal{S}_M^{(j)}, h \rangle \right| \leq \| h \| \times (2p)^{N\epsilon} \sum_{\ell=0}^{2N} \sum_{0 \leq r \leq \ell} \alpha_{2N,\ell}^{(r)}.
$$

Using Proposition 3.3 again, one has

$$
\sum_{\ell=0}^{2N} \sum_{0 \leq r \leq \ell} \alpha_{2N,\ell}^{(r)} = \sum_{\ell=0}^{2N} \binom{2N}{\ell} = 2^{2N}. \quad (5.5)
$$

The product $(2p)^{-N\epsilon} 2^{-2N} \left(p^{-\frac{m}{2} T_p^\text{plane}}\right)^{2N} \mathcal{S}_M^{(j)}$ thus remains, as $N \to \infty$, in a bounded subset of the space of tempered distributions. To say it differently, $(2p)^{-N\epsilon} 2^{-2N} \left(p^{-\frac{1}{2}} T_p^\text{plane}\right)^{2N} \mathcal{T}_M$ remains in a bounded subset of the weak dual of the space $P_j(-2i\pi E) \mathcal{S}(\mathbb{R}^2)$. 

and of Proposition 2.1.
If \( f \), admitting the expansion
\[
f(z) = \sum_{n \geq 1} b_n e^{2i\pi nz}
\]
with \( b_1 = 1 \), coincides with a linear combination
\[
\sum_j \beta_j \theta_m \mathcal{T}_{M_j},
\]
one has for \( p \) prime and \( N = 1, 2, \ldots \), according to (3.4),
\[
b_p^{2N} f = \sum_j \beta_j T_p^{2N} \theta_m \mathcal{T}_{M_j}.
\]
On the other hand, \((2i\pi A)(x_1 + ix_2)^{-m} = m (x_1 + ix_2)^{-m} \) and \( \theta_m \left( P_j(-2i\pi A) \mathcal{T}_{M_j} \right) = P_j(-m) \theta_m \mathcal{T}_{M_j} \).

Now, \( P_j(-m) \neq 0 \) if one chooses, say, \( j = 3 \), since \( m + 1 \geq 12 \) and the zeros of the polynomial \( P_j \) are \(-j, -j + 1, \ldots, j \). It then follows from the estimate obtained that \(|b_p| \leq 2p^{\frac{N}{2}}\).

6. A CASE OF NON-GEOMETRIC QUANTIZATION: ANAPLECTIC VS METAPLECTIC

This short section addresses itself to people interested in pseudodifferential analysis or in geometric quantization theory. Also, it will explain notation.

The Weyl symbolic calculus \( \Psi \) connects to the metaplectic representation \( \text{Met} \) by the general covariance identity \( \text{Met}(g) \Psi(S) \text{Met}(g^{-1}) = \Psi(S \circ g^{-1}) \). This connection between the metaplectic representation and the natural geometric action of \( SL(2, \mathbb{R}) \) in \( \mathbb{R}^2 \) may be regarded as an essential feature of geometric quantization, though this term is more often taken in the more specialized sense of building the representation on the left-hand side (by means of polarizations of appropriate type) from the one on the right-hand side. Changing \( SL(2, \mathbb{R}) \) to other groups, this scheme, including the symbolic calculus, has a large number of variants.

So as to obtain more general possibilities, let us denote as \( \text{Met}(g) \) the representation such that \( \text{Met}(g) \mathcal{S} = \mathcal{S} \circ g^{-1} \). The space \( L^2(\mathbb{R}^2) \) decomposes as the continuous sum of spaces of distributions of given degrees of homogeneity in \(-1 + i\mathbb{R}\) and parity, and Maass forms can be realized as automorphic objects of the representation \( \text{Met} \). This led to our proof of the Ramanujan-Petersson conjecture [10] in the Maass case. The notation \( \text{Met}(\_\_) \) prepares the way for a more general concept. In the present paper, the representation \( \text{Met} \) is replaced by \( \text{Ana} \) so that, on the symbol side, the holomorphic discrete series, as opposed to the principal series of \( SL(2, \mathbb{R}) \)
associated to the decomposition of functions in $\mathbb{R}^2$ into homogeneous components, shows.

How does this relate to the covariance question? There is a uniquely defined symbolic calculus $\text{Alt}$, dubbed the alternative pseudodifferential calculus in [9], together with a representation $\text{Ana}$, called the anaplectic representation (it exists also in the $n$-dimensional case [8], but the generalization is not straightforward), the pair of which enjoys the covariance identity, with $\mathfrak{g}$ in an appropriate space,

$$\text{Met}(g) \text{Alt}(\mathfrak{g}) \text{Ana}(g^{-1}) = \text{Alt} (\text{Ana}(g) \mathfrak{g}). \quad (6.1)$$

So far as this covariance identity is concerned, the pair $\text{Ana}$, $\text{Ana}$ is fully similar to the pair $\text{Met}$, $\text{Met}$. The analogy goes further since, just as in the Weyl calculus, the anaplectic representation combines with the Heisenberg representation in a just as satisfactory way. We refer to [8, 9] for more developments.

The anaplectic representation is not unitary, but pseudo-unitary for a certain non-degenerate scalar product, making it appear as the sum of the “central” representation from the complementary series and of a signed (non-unitarizable) version of the same. Let us just mention that the spectrum in this theory of the harmonic oscillator, defined in the usual way in terms of the infinitesimal operators of the Heisenberg representation, is no longer the set $\left\{\frac{1}{2}, \frac{3}{2}, \ldots \right\}$ but $\mathbb{Z}$. All the objects we are usually fond of playing with, such as Gaussian functions, creation and annihilation operators, have full, though completely exotic, analogues in the anaplectic theory.
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