Higher order first integrals and quantum gravitational anomalies

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Abstract. Hidden symmetries on curved space-times are investigated in connection with higher rank Killing tensors. It is shown that at the quantum level the conformal Killing vectors and tensors do not in general produce operators that commute with the Klein-Gordon operator.

1. Introduction
The concept of symmetries is one of the key concepts in physics, Noether’s theorem giving a correspondence between symmetries and conservation laws.

The evolution of a dynamical system is described in the entire phase-space and from this point of view it is natural to go in search of conserved quantities to genuine symmetries of the complete phase-space, not just the configuration one. Such symmetries are associated with higher rank symmetric Stackel-Killing (SK) tensors which generalize the Killing vectors. These higher order symmetries are known as hidden symmetries and the corresponding conserved quantities are quadratic, or, more general, polynomial in momenta. Also SK tensors play a pivotal role in Hamilton-Jacobi theory of separation of variables and the integrability of finite-dimensional Hamiltonian systems [1]. Another natural generalization of the Killing vectors is represented by the antisymmetric Killing-Yano (KY) tensors which in many respects are more fundamental than the SK tensors.

The conformal extension of the SK tensor equation determines the conformal Stackel-Killing (CSK) tensors which define first integrals of motion of the null geodesics. Investigations of the hidden symmetries of the higher dimensional space-times have pointed out the role of the conformal Killing-Yano (CKY) tensors to generate background metrics with black hole solutions.

Passing from the classical motions to the hidden symmetries of a quantized system it is necessary to investigate the corresponding quantum conserved quantities and separability of the equations of motion. Especially in the case of hidden symmetries it can appear anomalies representing discrepancies between the conservation laws at the classical level and the corresponding ones at the quantum level.

The plan of the paper is as follows. In Section 2 the generalized Killing equations are presented in connection with higher order conserved quantities. In Section 3 we discuss the special role of the KY tensors in generating higher order symmetries. In the next Section we describe the relationship between conformal symmetries and the corresponding quantum operators and analyze the existence of quantum gravitational anomalies. Finally, the last Section is devoted to conclusions.
2. Symmetries and conserved quantities
Let us consider the Hamilton function describing the geodesic motion in a curved \( n \)-dimensional (pseudo-)Riemmanian space \( \mathcal{M} \) with the metric \( g \):

\[
H = \frac{1}{2} g^{\mu \nu} p_\mu p_\nu .
\]  

A conserved quantity of motion can be expanded as a power series in momenta

\[
K = K_0 + \sum_{k=1}^{p} \frac{1}{k!} K^{(\mu_1 \cdots \mu_k)}(x) p_{\mu_1} \cdots p_{\mu_k},
\]  

having a vanishing Poisson bracket with the Hamiltonian.

For the conservation of \( K \) its terms must satisfy

\[
K^{(\mu_1 \cdots \mu_k ; \mu)} = 0,
\]

where a semicolon denotes the covariant differentiation corresponding to the Levi-Civita connection and round brackets denote full symmetrization over the indices enclosed. A symmetric tensor \( K^{(\mu_1 \cdots \mu_k)} \) satisfying (3) is called a SK tensor of rank \( k \). Let us note that for any geodesic \( \gamma \) with tangent vector \( \dot{x}^\mu = p^\mu \)

\[
Q_K = K^{(\mu_1 \cdots \mu_k)} \dot{x}^{\mu_1} \cdots \dot{x}^{\mu_k},
\]

is constant along \( \gamma \).

3. Killing-Yano tensors
KY tensors are a different generalization of Killing vectors which can be studied on a manifold. They were introduced by Yano [2] from a purely mathematical perspective and later on it turned out they have many interesting properties relevant to physics. In recent years the KY tensors are related to a multitude of different topics such as supersymmetries, index theorems, supergravity theories, and so on [3, 4].

A KY tensor \( Y_{\mu_1 \cdots \mu_p} \) is antisymmetric satisfying the following equation:

\[
Y_{\mu_1 \cdots \mu_{p-1} ; \mu_p ; \nu} = 0.
\]

The first connection with the symmetry properties of the geodesic motion is the observation that along every geodesic \( \gamma \) in \( \mathcal{M} \), \( Y_{\mu_1 \cdots \mu_{p-1} ; \nu} \dot{x}^\nu \) is parallel.

These two generalizations SK and KY of the Killing vectors could be related. Let \( Y_{\mu_1 \cdots \mu_p} \) be a KY tensor, then the symmetric tensor field

\[
K_{\mu \nu} = Y_{\mu \mu_2 \cdots \mu_p} Y_{\nu ; \mu_2 \cdots \mu_p},
\]

is a SK tensor and it sometimes refers to this SK tensor as the associated tensor with \( Y_{\mu_1 \cdots \mu_p} \).

Having in mind the special role of null geodesic for the motion of massless particles, it is convenient to look for conformal generalization of KY tensor. Let us mention also the remarkable role of the conformal Killing-Yano (CKY) tensors in the study of the properties of higher dimensional black holes (see e. g. [5, 6] and the cites contained therein). In what follows we limit ourselves to CKY tensors of rank 2 which satisfy [7]

\[
Y_{\mu \nu ; \lambda} + Y_{\lambda \nu ; \mu} = \frac{2}{n-1} \left( g_{\lambda \mu} Y_{\nu ; \sigma} + g_{\nu ; (\mu} Y_{\lambda ) ; \sigma} \right).
\]
There is also a conformal generalization of the SK tensors, namely a symmetric tensor $K_{\mu_1 \cdots \mu_p} = K_{(\mu_1 \cdots \mu_p)}$ is called a CSK tensor if it obeys the equation

$$K_{(\mu_1 \cdots \mu_p; \nu)} = g_{\nu (\mu_1} \tilde{K}_{\mu_2 \cdots \mu_p)},$$

where the tensor $\tilde{K}$ is determined by tracing the both sides of equation (8). Let us note that in the case of CSK tensors, the quantity (4) is constant only for null geodesics $\gamma$. There is also a similar relation between CKY and CSK tensors as in equation (6) [7].

4. Quantum gravitational anomalies

In order to find the necessary conditions for the existence of constants of motion in a first-quantized system we replace momenta by derivatives and look for operators commuting with the Hamiltonian:

$$H = \square = \nabla_\mu g^{\mu\nu} \nabla_\nu = \nabla_\mu \nabla^\mu,$$

(9)
corresponding to a free scalar particle and the covariant Laplacian is acting on scalars.

Many times the classical conserved quantities associated with SK tensors do not generally transfer to the quantized systems producing so-called quantum anomalies [8]. In what follows we shall analyze the quantum anomalies in the case of SK and CSK tensors and, to make things more specific, we confine ourselves to the case of tensors of rank 1 and 2.

For the beginning let us consider the conserved operator corresponding to a conformal Killing vector $K^\mu$ in the quantized system

$$Q_V = K^\mu \nabla_\mu,$$

(10)
in order to identify a quantum gravitational anomaly we shall evaluate the commutator $[\square, Q_V] \Phi$ for $\Phi \in C^\infty(M)$, solutions of the Klein-Gordon equation with the Klein-Gordon operator (9). A straightforward calculation gives

$$[H, Q_V] = \frac{2 - n}{n} K_\lambda^{\lambda \mu} \nabla_\mu + K_\lambda^{\lambda \mu} \nabla_\mu.$$

(11)

As it is expected, in the case of ordinary Killing vectors the r. h. s. of this commutator vanishes and there are no quantum gravitational anomalies. But for conformal Killing vectors, we confront with a quite different situation. Even if we evaluate the r. h. s. of (11) on solutions of the massless Klein-Gordon equation, $\square \Phi(x) = 0$, the term $K_\lambda^{\lambda \mu} \nabla_\mu$ survives. It is possible to find some particular spaces in which the r. h. s. of (11) vanishes, but in general the system is affected by quantum gravitational anomalies.

In the case of SK and CSK tensors of rank 2 the quantum analog of the classical conserved quantity (4) is

$$Q_T = \nabla_\mu K^{\mu\nu} \nabla_\nu,$$

(12)

where $K^{\mu\nu}$ is a SK (3) or a CSK (8) tensor.

Now the evaluation of the commutator between $H$ and the quantum operator (12) is more involved than in case of Killing vectors and the result of this tedious evaluation is [9, 10]:

$$[\square, Q_T] = 2 \left( \nabla^{(\lambda} K^{\mu\nu)} \right) \nabla_\lambda \nabla_\mu \nabla_\nu + 3 \nabla_\tau \left( \nabla^{(\lambda} K^{\tau\nu)} \right) \nabla_\lambda \nabla_\nu + \left\{ - \frac{4}{3} \nabla_\lambda \left( R_{\nu [\lambda} K^{\mu]} \right) \right\} \nabla_\nu + \nabla_\lambda \left( \frac{1}{2} g_{\tau\sigma} \left( \nabla^{(\tau} K^{\sigma\nu)} - \nabla^{\sigma} K^{(\tau\nu)} + \nabla_\nu \nabla^{(\lambda} K^{\mu)} \right) \right\} \nabla_\nu.$$

(13)

We mention that the last term is missing in the corresponding equation in [8]. Note also that the terms are arranged into groups with three, two and just one derivatives and consequently it is impossible to have compensations between them.
However in the case of SK tensors all the symmetrized derivatives vanish and we end up with a simpler result

\[ [\Box, Q_T] = -\frac{4}{3} \nabla_{\mu} (R_{\sigma}^{\ [\mu} K_{\nu]}^{\sigma}) \nabla_{\nu}. \] (14)

Since the r. h. s. does not vanish identically, SK tensors exhibit quantum anomalies, i.e. the classical conservation law does not transfer to the quantum level.

Although in general SK tensors do not give quantum mechanical symmetries, there are a few notable conditions for which the r. h. s. of (14) vanishes. The simplest case in which this happens is obviously when the space is Ricci flat. A slightly more general case is when the space is Einstein, i.e. \( R_{\mu\nu} \propto g_{\mu\nu} \) (that is, if the vacuum Einstein equations are fulfilled). In this case we get that the r. h. s. of the commutator involves \( K^{[\mu\nu]} \) and consequently vanishes since SK tensors are symmetric. A more interesting and quite unexpected case is represented by SK tensors associated with KY tensors of rank 2 as in (6) [8], a situation which occurs for some spaces.

For CSK tensors the existence of some favorable conditions which could prevent the appearance of quantum anomalies is quite impossible. In the case of CSK tensors we could not simplify any more the commutator (13) since for them the symmetrized derivatives do not vanish. Even if we evaluate the commutator for CSK tensors associated with CKY tensors we do not obtain a cancellation of anomalies [10]. Therefore we are not able to find any favorable circumstances on the CSK tensors in order to achieve a conserved quantum operator.

5. Concluding comments

The (C)SK and (C)KY tensors are related to a multitude of different topics such as classical integrability of systems together with their quantization, supergravity, string theories, hidden symmetries in higher dimensional black-holes space-times, etc.

To conclude let us discuss shortly some problems that deserve a further attention. More elaborate examples working in a \( N \)-dimensional curved space-time and involving higher ranks (\( k > 2 \)) SK tensors will be presented elsewhere. The concept of generalized (C)KY symmetry in the presence of a skew-symmetric torsion is more widely applicable and may become very powerful [11].

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