Normal form decomposition for Gaussian-to-Gaussian superoperators

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In this paper we explore the set of linear maps sending the set of quantum Gaussian states into itself. These maps are in general not positive, a feature which can be exploited as a test to check whether a given quantum state belongs to the convex hull of Gaussian states (if one of the considered maps sends it into a non positive operator, the above state is certified not to belong to the set). Generalizing a result known to be valid under the assumption of complete positivity, we provide a characterization of these Gaussian-to-Gaussian (not necessarily positive) superoperators in terms of their action on the characteristic function of the inputs. For the special case of one-mode mappings we also show that any Gaussian-to-Gaussian superoperator can be expressed as a concatenation of a phase-space dilatation, followed by the action of a completely positive Gaussian channel, possibly composed with a transposition. While a similar decomposition is shown to fail in the multi-mode scenario, we prove that it still holds at least under the further hypothesis of homogeneous action on the covariance matrix.

I. INTRODUCTION

Gaussian Bosonic States (GBSs) play a fundamental role in the study of continuous-variable (CV) quantum information processing [113, 114], i.e. super-operators mapping the set of GBSs into itself. In the last two decades, a great deal of attention has been devoted in characterizing these objects. In particular the community focused on Gaussian Bosonic Channels (GBCs) [5], i.e. Gaussian transformations which are completely positive (CP) and provide hence the proper mathematical representation of data-processing and quantum communication procedures which are physically implementable [11]. On the contrary, less attention has been devoted to the study of Gaussian superoperators which are not CP or even non-positive. A typical example of such mappings is provided by the phase-space dilatation lines (specifically the phase-invariant Gaussian Bosonic maps) [5–10]. GBSs are characterized by the property of having Gaussian Wigner quasi-distribution and describe Gibbs states of Hamiltonians which are quadratic in the field operators of the system. Further, in quantum optics they include coherent, thermal and squeezed states of light and can be easily created via linear amplification and loss.

Directly related to the definition of GBSs is the notion of Gaussian transformations [1, 3, 4], i.e. super-operators mapping the set of Gaussian states into itself. In particular the community focused on Gaussian Bosonic Channels (GBCs) [5], i.e. Gaussian transformations which are completely positive (CP) and provide hence the proper mathematical representation of data-processing and quantum communication procedures which are physically implementable [11]. On the contrary, less attention has been devoted to the study of Gaussian superoperators which are not CP or even non-positive. A typical example of such mappings is provided by the phase-space dilatation lines (specifically the phase-invariant Gaussian Bosonic maps) [5–10]. GBSs are characterized by the property of having Gaussian Wigner quasi-distribution and describe Gibbs states of Hamiltonians which are quadratic in the field operators of the system. Further, in quantum optics they include coherent, thermal and squeezed states of light and can be easily created via linear amplification and loss.

Given \( \hat{\rho} \in \mathcal{G} \) one can identify another Gaussian density operator \( \hat{\rho}' \) which admits the function \( W_{\hat{\rho}}^{(\lambda)}(\mathbf{r}) \) as Wigner distribution, i.e. \( W_{\hat{\rho}}^{(\lambda)}(\mathbf{r}) = W_{\hat{\rho}'}^{(\lambda)}(\mathbf{r}) \). On the other hand, there exist inputs \( \hat{\rho} \) for which \( W_{\hat{\rho}}^{(\lambda)}(\mathbf{r}) \) is no longer interpretable as the Wigner quasi-distribution of any quantum state: in this case in fact \( W_{\hat{\rho}}^{(\lambda)}(\mathbf{r}) \) results to be the Wigner quasi-distribution \( W_{\hat{\rho}}^{(r)}(\mathbf{r}) \) of an operator \( \hat{\theta} \) which is not positive [12] (for example, any pure non-Gaussian state has this property for any \( \lambda \neq \pm 1 \) [13]). Accordingly phase-space dilatations [1] should be considered as “un-physical” transformations, i.e. mappings which do not admit implementations in the laboratory. Still dilatations and similar exotic Gaussian-to-Gaussian mappings turn out to be useful mathematical tools that can be em-
ployed to characterize the set of states of CV systems in a way which is not dissimilar to what happens for positive (but not completely positive) transformations in the analysis of entanglement [13]. In particular Bröcker and Werner [12] used [1] to study the convex hull $\mathcal{C}$ of Gaussian states (i.e. the set of density operators $\hat{\rho}$ which can be expressed as a convex combination of elements of $\mathcal{G}$). The rational of this analysis is that the set $\mathcal{F}$ of density operators which are mapped into proper output states by this transformation includes $\mathcal{C}$ as a proper subset, see Fig. 2. Accordingly if a certain input $\hat{\rho}$ yields a $W^{(\lambda)}(\mathbf{r})$ which is not the Wigner distribution of a state, we can conclude that $\hat{\rho}$ is not an element of $\mathcal{C}$. Finding mathematical and experimental criteria which help in identifying the boundaries of $\mathcal{C}$ is indeed a timely and important issue which is ultimately related with the characterization of non-classical behavior in CV systems, see e.g. Refs. [15–25] and citations therein.

In this context a classification of non-positive Gaussian-to-Gaussian operations is mandatory, and the aim of the present work is to initiate such an analysis. We stress that the general form of a linear map sending Gaussian states into Gaussian states is known only assuming complete positivity from the beginning. One goal of this paper is finding this form without this hypothesis: we prove that the action of such transformations on the covariance matrix and on the first moment must be linear, and we write explicitly the transformation properties of the characteristic function (Theorem III.1). In the classical case, any probability measure can be written as a convex superposition of Dirac deltas, so the convex hull of the Gaussian measures coincides with the whole set of measures. A simple consequence of this fact is that a linear transformation sending Gaussian measures into Gaussian (and then positive) measures is always positive. Nothing of this holds in the more interesting quantum case, so we focus on it, and use Theorem III.1 to get a decomposition which, for single-mode operations, shows that any linear quantum Gaussian-to-Gaussian transformation can always be decomposed as a proper combination of a dilatation (1) followed by a CP Gaussian mapping plus possibly a transposition. We also show that our decomposition theorem applies to the multi-mode case, as long as we restrict the analysis to Gaussian transformations which are homogeneous at the level of covariance matrix. For completeness we finally discuss the case of contractions. In Sec. IV we present the main result of the manuscript, i.e. the decomposition theorem for single-mode Gaussian-to-Gaussian transformations. The multi-mode case is then analyzed in Sec. V. The paper ends hence with Sec. VI where we present a brief summary and discuss some possible future developments. In Appendix A we prove the unboundedness of phase-space dilatations with respect to the trace norm.

II. NOTATION

In this section we introduce the notation and some basic definitions which are useful to set the problem.

A. Symplectic structure

Consider a CV system constituted by $n$ independent modes $\hat{Q}^j, \hat{P}^k$ with quadratures satisfying the canonical commutation relations

$$[\hat{Q}^j, \hat{P}^k] = i\delta^{jk} \mathbb{1}, \quad j, k = 1, \ldots, n. \quad (2)$$

Organizing these operators in the column vector $\hat{\mathbf{R}}$ of elements

$$\hat{\mathbf{R}} \equiv \left(\hat{Q}^1, \hat{P}^1, \ldots, \hat{Q}^n, \hat{P}^n\right)^T, \quad (3)$$

equation (2) can be casted in the equivalent form

$$[\hat{R}^i, \hat{R}^j] = i\Delta^{ij} \mathbb{1}, \quad (4)$$

FIG. 2. (Color online) Pictorial representation of the structure of the set of states $\mathcal{G}$ of a CV system. $\Psi$ is the subset of density operators $\hat{\rho}$ which have non-negative Wigner distribution [10]. $\mathcal{F}$ is set of states of states which instead are mapped into proper density operators by an arbitrary dilatation (1). $\mathcal{G}$ is the set of Gaussian states and $\mathcal{C}$ its convex closure. $\mathcal{G}, \mathcal{F}, \mathcal{C}$ and $\mathcal{S}$ are closed under convex convolution, $\mathcal{G}$ is not. For a detailed study of the relations among these sets see Ref. [12].
where $\Delta$ is the skew-symmetric matrix
\[
\Delta = \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
which defines the symplectic form of the problem. Accordingly a real matrix $S$ is said to be symplectic if it preserves $\Delta$, i.e. if the following identity holds
\[
S\Delta S^T = \Delta.
\]

B. Characteristic and Wigner functions

For any density operator $\hat{\rho} \in \mathcal{F}$ we define its symmetrically ordered characteristic function as
\[
\chi(k) \equiv \text{Tr} \left( \hat{\rho} \exp \left( i k^T \hat{R} \right) \right), \quad k \in \mathbb{R}^{2n}.
\]
This formula makes sense for any trace-class operator $\hat{\rho}$, the associated $\chi(k)$ being a bounded continuous function. By using the Parceval-type formula \[1\]
\[
\text{Tr} \left( \hat{\rho}_1 \hat{\rho}_2 \right) = \int \frac{\chi_{\hat{\rho}_1}(k) \chi_{\hat{\rho}_2}(k)}{(2\pi)^n} dk,
\]
one can extend the correspondence \[7\] to Hilbert-Schmidt operators, the associated $\chi(k)$ being a square-integrable function. The correspondence is the isomorphism between the Hilbert space $\mathcal{F}$ of Hilbert-Schmidt operators with the inner product given by the left hand side of \[8\], and that of square-integrable functions of $k$. The operator $\hat{\rho}$ can be expressed as
\[
\hat{\rho} = \int \chi_{\hat{\rho}}(k) e^{-ik^T \hat{R}} \frac{dk}{(2\pi)^n}.
\]
We also define its Wigner function as the Fourier transform of the characteristic function, square integrable for any $\hat{\rho} \in \mathcal{F}$:
\[
W_{\hat{\rho}}(r) = \int \chi_{\hat{\rho}}(k) e^{-ik^T r} \frac{dk}{(2\pi)^n}.
\]

C. Heisenberg uncertainty

The covariance matrix of $\hat{\rho}$ is defined as
\[
\sigma^{ij} \equiv \text{Tr} \left( \hat{\rho} \left( \hat{R}^j - \langle \hat{R}^j \rangle, \hat{R}^i - \langle \hat{R}^i \rangle \right) \right),
\]
with $\langle \hat{R}^i \rangle \equiv \text{Tr} \left[ \hat{\rho} \hat{R}^i \right]$, provided the second moments of $\hat{\rho}$ are finite. The positivity of $\hat{\rho}$ together with the commutation relations \[4\] imply the Robertson-Heisenberg uncertainty relation
\[
\sigma \geq \pm i \Delta.
\]

We also remind that the Williamson theorem \[26\] ensures that given a symmetric positive definite matrix $\sigma$, there exists a symplectic matrix $S$ such that
\[
S\sigma S^T = \bigoplus_{i=1}^{n} \nu_i 1_2, \quad \nu_i > 0.
\]
The $\nu_i$ are called symplectic eigenvalues of $\sigma$ and can be computed as the positive ordinary eigenvalues of the matrix $i\Delta \sigma$ (they come in couples of opposite sign). The Heisenberg uncertainty principle \[12\] is hence satisfied iff they are all greater than one:
\[
\nu_j \geq 1, \quad j = 1, \ldots, n.
\]
The symplectic condition \[6\] simplifies in the case of one mode. Indeed for any $2 \times 2$-matrix $M$ we have
\[
M \Delta M^T = \Delta \det M,
\]
therefore a $2 \times 2$-matrix $S$ is symplectic iff
\[
\det S = 1.
\]
We remind also that a symmetric positive definite $2 \times 2$-matrix $\sigma$ has a single symplectic eigenvalue, given by
\[
\nu^2 = \det \sigma;
\]
then from \[14\] $\sigma \geq 0$ is the covariance matrix of a quantum state iff
\[
\det \sigma \geq 1.
\]

D. Gaussian states

States with positive Wigner function \[10\] form a convex subset $\Psi$ in the space of the density operators $\mathcal{F}$ of the system. The set $\mathcal{F}$ of Gaussian states is a proper subset of $\Psi$. A Gaussian state $\hat{\rho}_x, \sigma \in \mathcal{F}$ with covariance matrix $\sigma$ and first moments $x$ is defined by the property of having Gaussian characteristic function
\[
\chi(k) = e^{-\frac{1}{2}k^T \sigma k + i k^T x},
\]
i.e. Gaussian Wigner function
\[
W(x) = \frac{1}{\sqrt{\det (\pi \sigma)}} e^{-(r-x)^T \sigma^{-1}(r-x)}.
\]
For $\sigma = 1_{2n}$ we obtain the family of coherent states $\hat{\rho}_x, 1_{2n}, x \in \mathbb{R}^{2n}$. The Husimi function $\text{Tr}(\hat{\rho}_x 1_{2n})$ of any bounded operator $\hat{\rho}$ uniquely defines $\hat{\rho}$. It follows that the linear span of the set of coherent states, and hence of all Gaussian states, is dense in the Hilbert space of Hilbert-Schmidt operators $\mathcal{F}$. Similarly, these linear spans are dense in the Banach space of trace-class operators $\mathcal{S}$. 
E. The Wigner-positive states and the convex hull of Gaussian states

Starting from the vacuum, devices as simple as beam-splitters combined with one-mode squeezers permit (at least in principle) to realize all the elements of $\mathcal{G}$. Then, choosing randomly according to a certain probability distribution which Gaussian state to produce, it is in principle possible to realize all the states in the convex hull $\mathcal{C}$ of the Gaussian ones, i.e. all the states $\hat{\rho}$ that can be written as

$$\hat{\rho} = \int \hat{\rho}_{x,\sigma} \, d\mu(x,\sigma) \ ,$$

(21)

where $\mu$ is the associated probability measure of the process.

It is easy to verify that $\mathcal{C}$ is strictly larger than $\mathcal{G}$, i.e. there exist states $\hat{\rho}$ which are not Gaussian. On the other hand, one can observe that (21) implies

$$W_{\hat{\rho}}(r) = \int \frac{1}{\sqrt{\det(\pi \sigma)}} e^{-(r-x)^T \sigma^{-1}(r-x)} \, d\mu(x,\sigma) > 0 \ ,$$

(22)

so also $\mathcal{C}$ is included into $\mathcal{G}$, see Fig. 2. There are however elements of $\mathcal{G}$ which are not contained in $\mathcal{C}$; for example, any finite mixture of Fock states

$$\hat{\rho} = \sum_{n=0}^{N} p_n |n\rangle \langle n| \quad N < \infty \quad p_n \geq 0 \quad \sum_{n=0}^{N} p_n = 1$$

(23)

is not even contained in the weak closure of $\mathcal{C}$, even if some of them have positive Wigner function $\mathcal{G}$.

III. CHARACTERIZATION OF GAUSSIAN-TO-GAUSSIAN MAPS

Determining whether a given state $\hat{\rho}$ belongs to the convex hull $\mathcal{C}$ of the Gaussian set is a difficult problem $\mathcal{G}$13. Then, there comes the need to find criteria to certify that $\hat{\rho}$ cannot be written in the form (21). A possible idea is to consider a non-positive superoperator $\Phi$ sending any Gaussian state into a state $\mathcal{G}$12. By linearity $\Phi$ will also send any state of $\mathcal{C}$ into a state, therefore if $\Phi(\hat{\rho})$ is not a state, $\hat{\rho}$ cannot be an element of $\mathcal{C}$; in other words, the transformation $\Phi$ acts as a mathematical probe for $\mathcal{C}$. In what follows we will focus on those probes which are also Gaussian transformations, i.e. which not only send $\mathcal{G}$ into states, but which ensure that the output states $\Phi(\hat{\rho})$ are again elements of $\mathcal{G}$. Then the following characterization theorem holds

Theorem III.1. Let $\Phi$ be a linear bounded map of the space $\mathcal{H}$ of Hilbert-Schmidt operators, sending the set of Gaussian states $\mathcal{G}$ into itself. Then its action in terms of the characteristic function $\chi$, the first moments and the covariance matrix $\alpha$ is of the form

$$\phi (\mathbf{k}) \rightarrow \chi (K^T \mathbf{k}) e^{-\frac{1}{2} \mathbf{k}^T \alpha \mathbf{k} + \mathbf{k}^T \mathbf{y}_0} \ ,$$

(24)

$$\phi (\mathbf{x}) \rightarrow K \mathbf{x} + \mathbf{y}_0 \ ,$$

(25)

$$\phi (\sigma) \rightarrow K \sigma K^T + \alpha \ ,$$

(26)

where $\mathbf{y}_0$ is an $\mathbb{R}^n$ vector, and $K$ and $\alpha$ are $2n \times 2n$ real matrices such that $\alpha$ is symmetric, and for any $\sigma \geq \pm i \Delta$

$$K \sigma K^T + \alpha \geq \pm \Delta \ .$$

(27)

The condition (27) imposes that $\Phi(\hat{\rho})$ is a Gaussian state for any Gaussian $\hat{\rho}$. As already mentioned it is weaker than (15), which also ensures the mapping of Gaussian states into Gaussian states but further enforces complete positivity. An example of mappings fulfilling (27) but not (15) is provided the dilatations defined in Eq. (1). Such mappings in fact, while explicitly not CP (12), correspond to the transformations (24) where we set $\mathbf{y}_0 = 0$ and take

$$K = \lambda \mathbf{l}_{2n} \ , \quad \alpha = 0 \ ,$$

(28)

with $|\lambda| > 1$. At the level of the covariance matrices (26), this implies $\sigma' = \lambda^2 \sigma$ which clearly still preserve the Heisenberg inequality (12) (indeed $\lambda^2 \sigma \geq \sigma \geq \pm i \Delta$), ensuring hence the condition (27). Dilatations are not bounded with respect to the trace norm (see Theorem 4.1 of Appendix A). This explains why Theorem III.1 is formulated on the space of Hilbert-Schmidt operators. Indeed, via the Parceval formula we can prove that dilatations are bounded in this space:

$$\|\phi(\hat{\rho})\|^2 = \int |\chi_k(\lambda \mathbf{k})|^2 \left(\frac{d\mathbf{k}}{2\pi}\right)^n =$$

$$\int |\chi_k(\mathbf{k})|^2 \left(\frac{d\mathbf{k}}{2\pi \lambda^2}\right)^n = \frac{1}{\lambda^{2n}} \|\hat{\rho}\|^2 \ .$$

For $\lambda = \frac{1}{\mu}$ with $|\mu| > 1$ the transformation (28) yields a contraction of the output Wigner quasi-distribution. In the Hilbert space $\mathcal{H}$, the contraction by $\lambda$ is $\lambda^{2n}$ times the adjoint of the dilatation by $\lambda = \frac{1}{\mu}$, as follows from the Parceval formula (12). As different from the dilatations, these mappings no longer ensure that all Gaussian states will be transformed into proper density operators. For instance, the vacuum state is mapped into a non-positive operator (this shows in particular that the contractions and hence the adjoint dilatations are non-positive maps).

Another example of transformation not fulfilling the CP requirement (15) but respecting (27) is the (complete) transposition

$$K = T_{2n} \quad \alpha = 0 \ ,$$

(29)

that is well-known not to be CP. Unfortunately, being positive it cannot be used to certify that a given state is not contained in the convex hull $\mathcal{C}$ of the Gaussian ones. Is there anything else? We will prove that for one mode, any channel satisfying (27) can be written as a dilatation
composed with a completely positive channel, possibly composed with the transposition \( (29) \), see Fig. 3. We will also show that in the multi-mode case this simple decomposition does not hold in general; however, it still holds if we restrict to the channels that do not add noise, i.e. with \( \alpha = 0 \).

**Proof.** Let the Gaussian state \( \hat{\rho}_{\mathbf{x}, \sigma} \) be sent into the Gaussian state \( \hat{\rho}_{\mathbf{y}, \tau} \) with covariance matrix \( \tau(\mathbf{x}, \sigma) \) and first moment \( y(\mathbf{x}, \sigma) \), with the characteristic function

\[
\chi_{\hat{\rho}_{\mathbf{x}, \sigma}}(k) \equiv \chi_{y, \tau}(k) = e^{-\frac{1}{4}k^T \tau k + ik^T y} .
\]  

(30)

We first remark that the functions \( \tau(\mathbf{x}, \sigma) \) and \( y(\mathbf{x}, \sigma) \) are continuous. The map \( \Phi \) is bounded and hence continuous in the Hilbert-Schmidt norm. The required continuity follows from

**Lemma III.2.** The bijection \( (\mathbf{x}, \sigma) \rightarrow \hat{\rho}_{\mathbf{x}, \sigma} \) is bicontinuous in the Hilbert-Schmidt norm.

The proof of the lemma follows from the Parceval formula by direct computation of the Gaussian integral

\[
\int \left| \chi_{\hat{\rho}_{\mathbf{x}, \sigma}}(k) - \chi_{\hat{\rho}_{\mathbf{x}', \sigma'}}(k) \right|^2 \frac{dk}{(2\pi)^n}.
\]

Next, we have the identity

\[
\int \hat{\rho}_{\mathbf{x}', \sigma'} \mu_{\mathbf{x}, \sigma}(d\mathbf{x}') = \hat{\rho}_{\mathbf{x}, \sigma} ,
\]

(31)

where \( \mu_{\mathbf{x}, \sigma} \) is Gaussian probability measure with the first moments \( \mathbf{x} \) and covariance matrix \( \sigma \), which is verified by comparing the quantum characteristic functions of both sides.

Applying to both sides of this identity the continuous map \( \Phi \) we obtain

\[
\int \hat{\rho}_{\mathbf{y}(\mathbf{x}', \sigma'), \tau(\mathbf{x}', \sigma')} \mu_{\mathbf{x}, \sigma}(d\mathbf{x}') = \hat{\rho}_{\mathbf{y}(\mathbf{x}, \sigma'), \tau(\mathbf{x}, \sigma')} .
\]

By taking the quantum characteristic functions of both sides, we obtain

\[
\int \chi_{\mathbf{y}(\mathbf{x}', \sigma'), \tau(\mathbf{x}', \sigma')}(k) \mu_{\mathbf{x}, \sigma}(d\mathbf{x}') = \chi_{\mathbf{y}(\mathbf{x}, \sigma'), \tau(\mathbf{x}, \sigma')}(k) , \quad k \in \mathbb{R}^n .
\]  

(32)

Now notice that \( \mu_{\mathbf{x}, \sigma} \) is the fundamental solution of the diffusion equation:

\[
du = \frac{1}{4} \partial_i d\sigma^{ij} \partial_j u ,
\]

(33)

where \( d \) is the differential with respect to \( \sigma \), i.e.

\[
d = \sum_{i,j=1}^{m} d\sigma^{ij} \frac{\partial}{\partial \sigma^{ij}}
\]

(34)

and

\[
\partial_i = \frac{\partial}{\partial x^i} ,
\]

(35)

with the sum over the repeated indices. Relation \( (32) \) means that for any fixed \( k \), the function \( u(\mathbf{x}, \sigma) = \chi_{\mathbf{y}(\mathbf{x}, \sigma'), \tau(\mathbf{x}, \sigma')}(k) \) is the solution of the Cauchy problem for the equation \( (32) \) with the initial condition \( u(\mathbf{0}, 0) = \chi_{\mathbf{y}(\mathbf{x}, \sigma'), \tau(\mathbf{x}, \sigma')}(k) \). Since the last function is bounded and continuous, the solution of the Cauchy problem is infinitely differentiable in \( (\mathbf{x}, \sigma) \) for \( \sigma > 0 \). Substituting

\[
u(\mathbf{x}, \sigma) = \exp \left[ -\frac{1}{4} k^T \tau(\mathbf{x}, \sigma') k + i k^T y(\mathbf{x}, \sigma') \right] .
\]

(36)

into \( (33) \) and differentiating the exponent, we obtain the identity

\[
-\frac{1}{4} k^T d\tau k + i k^T dy =
\]

\[
= \frac{1}{4} \left( \frac{1}{4} k^T \partial_i \tau k - i k^T \partial_i y \right) d\sigma^{ij} \left( \frac{1}{4} k^T \partial_j \tau k - i k^T \partial_j y \right) - \frac{1}{16} k^T \left( \partial_i d\sigma^{ij} \partial_j \tau k + \frac{i}{4} k^T \partial_i d\sigma^{ij} \partial_j y \right) .
\]

(37)

We can now compare the two expressions. Since the left hand side contains only terms at most quadratic in \( k \), we get

\[
\partial_i \tau = 0 ,
\]

i.e. \( \tau \) does not depend on \( \mathbf{x} \). Then, the right hand side simplifies into

\[
-\frac{1}{4} k^T \left( \partial_i y d\sigma^{ij} \partial_j y^T \right) k + \frac{i}{4} k^T \partial_i d\sigma^{ij} \partial_j y .
\]

(38)

Comparing again with the left hand side, we get

\[
d\tau(\sigma) = \partial_i y d\sigma^{ij} \partial_j y^T
\]

(38)

\[
dy(\mathbf{x}, \sigma) = \frac{1}{4} \partial_i d\sigma^{ij} \partial_j y .
\]

(39)

Since \( d\tau(\sigma) \) does not depend on \( \mathbf{x} \), also \( \partial_i y \) cannot, i.e. \( y \) is a linear function of \( \mathbf{x} \):

\[
y(\mathbf{x}, \sigma) = K(\sigma) \mathbf{x} + y_0(\sigma) ,
\]

(40)

where \( K(\sigma) \) and \( y_0(\sigma) \) are still arbitrary functions. But now \( (39) \) becomes

\[
dy(\mathbf{x}, \sigma) = 0 ,
\]

i.e. \( y \) does not depend on \( \sigma \), i.e.

\[
y = K\mathbf{x} + y_0
\]

(42)

with \( K \) and \( y_0 \) constant. Finally, \( (38) \) becomes

\[
d\tau(\sigma) = K d\sigma K^T ,
\]

(43)

that can be integrated into

\[
\tau(\sigma) = K\sigma K^T + \alpha .
\]

(44)
Thus we get that the transformation rules for the first and second moments are given by Eqs. (25) and (26). The positivity condition for quantum Gaussian states implies (27). The map defined by (24) correctly reproduces (25) and (26), so it coincides with $\Phi$ on the Gaussian states. Since it is linear and continuous, and the linear span of Gaussian states is dense in $\mathcal{F}$, it coincides with $\Phi$ on the whole $\mathcal{F}$.

Remark III.3. A similar argument can be used to prove that any linear positive map $\Phi$ of the Banach space $\mathcal{S}$ of trace-class operators, leaving the set of Gaussian states globally invariant, has the form (24). By Lemma 2.2.1 of [27] any such map is bounded, and the proof of Theorem III.1 can be repeated, with $\mathcal{F}$ replaced by $\mathcal{S}$. In addition, since the trace of operator is continuous on $\mathcal{S}$, the formula (24) implies preservation of trace. However, the positivity condition is difficult to express in terms of the map parameters $y_0, K, \alpha$.

On the other hand, if $\Phi$ is completely positive then the necessary and sufficient condition is

$$\alpha \geq \pm i(\Delta - \Delta_K) , \quad (45)$$

where

$$\Delta_K \equiv K\Delta K^T . \quad (46)$$

Thus $\Phi$ is a quantum Gaussian channel [5]. The condition Eq. (27) is subdue by this more stringent (see below) constraint.

For automorphisms of the $C^*$-algebra of the Canonical Commutation Relations a similar characterization, based on a different proof using partial ordering of Gaussian states, was first given in [28, 29].

Remark III.4. There is a counterpart of Theorem III.1 in probability theory:

Theorem III.5. Let $\Phi$ be an endomorphism (linear bounded transformation) of the Banach space $\mathcal{M}(\mathbb{R}^n)$ of finite signed Borel measures on $\mathbb{R}^n$ (equipped with the total variation norm) having the Feller property (the dual $\Phi^*$ leaves invariant the space of bounded continuous functions on $\mathbb{R}^n$). Then, if $\Phi$ sends the set of Gaussian probability measures into itself, $\Phi$ is a Markov operator whose action in terms of characteristic functions is of the form (24), with the condition (27) replaced by $\alpha \geq 0$.

Proof. The proof is parallel to the proof of Theorem III.1 with replacement of (51) by the corresponding identity for Gaussian probability measures. As a result, we obtain that the action of $\Phi$ in terms of characteristic functions is given by (24) for any measure $\mu$ which is a linear combination of Gaussian probability measures. For arbitrary measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ the characteristic function of $\Phi(\mu)$ is

$$\chi_{\Phi(\mu)}(k) = \int e^{ik^T \cdot x} \Phi(\mu)(dx) = \int \Phi^*\left(e^{ik^T \cdot x}\right) \mu(dx) ,$$

where $\Phi^*\left(e^{ik^T \cdot x}\right)$ is continuous bounded function by the Feller property. Since the linear span of Gaussian probability measures is dense in $\mathcal{M}(\mathbb{R}^n)$ in the weak topology defined by continuous bounded functions (it suffices to take Dirac’s deltas, i.e. probability measures degenerated at the points of $\mathbb{R}^n$ , the formula (24) extends to characteristic function of arbitrary finite signed Borel measure on $\mathbb{R}^n$. The action of $\Phi$ on the moments is given by (25) and (26). The positivity of the output covariance matrix when the input is a Dirac delta implies $\alpha \geq 0$.

A. Contractions

A contraction by $\lambda = \frac{1}{\mu}$ behaves properly on the restricted subset $\mathcal{G}^{(\cdot)}_{\mu^2}$ of $\mathcal{G}$ formed by the Gaussian states whose covariance matrix admits symplectic eigenvalues larger than $\mu^2$. Indeed all elements of $\mathcal{G}^{(\cdot)}_{\mu^2}$ will be mapped into proper Gaussian output states by the contraction (and by linearity also the convex hull of $\mathcal{G}^{(\cdot)}_{\mu^2}$ will be mapped into proper output density operators). We will prove that any transformation with this property can be written as a contraction of $1/\mu$, followed by a transformation of the kind of Theorem III.1. Let us first notice that:

Lemma III.6. A set $(K, \alpha)$ satisfies (27) for any $\sigma$ if $\sigma$ has all the symplectic eigenvalues greater than $\mu^2$ if $\sigma' = \sigma/\mu^2$ is a state. Then (27) is satisfied for any $\sigma \geq \pm i\mu^2 \Delta$ if

$$\mu^2 K\sigma' K^T + \alpha \geq \pm i\Delta \quad \forall \sigma' \geq \pm i\Delta , \quad (47)$$

i.e. iff $(\mu K, \alpha)$ satisfies (27) for any $\sigma \geq \pm i\Delta$.

Proof. $\sigma$ has all the symplectic eigenvalues greater than $\mu^2$ iff $\sigma \geq \pm i\mu^2 \Delta$, i.e. iff $\sigma' = \sigma/\mu^2$ is a state. Then (27) is satisfied for any $\sigma \geq \pm i\mu^2 \Delta$ if

$$\mu^2 K\sigma' K^T + \alpha \geq \pm i\Delta \quad \forall \sigma' \geq \pm i\Delta , \quad (47)$$

Then we can state the following result:

Corollary III.7. Any transformation associated with $(K, \alpha)$ satisfying (27) for any state in $\mathcal{G}^{(\cdot)}_{\mu^2}$ (i.e. for any $\sigma \geq \pm i\mu^2 \Delta$) can be written as a contraction of $1/\mu$, followed by a transformation satisfying (27) for any state in $\mathcal{G}$ (i.e. for any $\sigma \geq \pm i\Delta$).

IV. ONE MODE

Here we will give a complete classification of all the one-mode maps (24) satisfying (27).

We will need the following

Lemma IV.1. A set $(K, \alpha)$ satisfies (27) iff

$$\sqrt{\det \alpha} \geq 1 - |\det K| . \quad (48)$$
Proof. For one mode, $\sigma \geq 0$ satisfies $\sigma \geq \pm i \Delta$ iff $\det \sigma \geq 1$, and condition (27) can be rewritten as

$$\det (K \sigma K^T + \alpha) \geq 1, \quad \forall \sigma \geq 0, \det \sigma \geq 1.$$  \hspace{1cm} (49)

To prove (49) $\Rightarrow$ (48) consider first the case $\det K \neq 0$. Choosing $\sigma$ such that $K \sigma K^T = \frac{\sqrt{\det K}}{\sqrt{\det \alpha}} \alpha$, we have $\sigma \geq 0, \det \sigma \geq 1$. Inserting this into (49), we obtain

$$\left(1 + \frac{|\det K|}{\sqrt{\det \alpha}}\right)^2 \det \alpha \geq 1$$

or, taking square root,

$$\left(1 + \frac{|\det K|}{\sqrt{\det \alpha}}\right) \sqrt{\det \alpha} \geq 1.$$  \hspace{1cm} (50)

hence (48) follows.

If $\det K = 0$, then there is a unit vector $e$ such that $Ke = 0$. Choose $\sigma = -\epsilon e e^T + e e^T$, where $\epsilon > 0$, and $e_1$ is a unit vector orthogonal to $e$. Then $\sigma \geq 0, \det \sigma = 1$, and $K \sigma K^T = \epsilon A$, where $A = K e_1 e_1^T K^T \geq 0$. Inserting this into (49), we obtain

$$\det (\epsilon A + \alpha) \geq 1, \quad \forall \epsilon \geq 0,$$

hence (48) follows.

To prove (48) $\Rightarrow$ (49), we use Minkowski’s determinant inequality

$$\sqrt{\det (A + B)} \geq \sqrt{\det A} + \sqrt{\det B} \quad \forall \ A, B \geq 0.$$  \hspace{1cm} (51)

We have for all $\sigma \geq 0, \det \sigma \geq 1$,

$$\sqrt{\det (K \sigma K^T + \alpha)} \geq$$

$$\geq |\det K| \sqrt{\det \sigma} + \sqrt{\det \alpha} \geq$$

$$\geq |\det K| + \sqrt{\det \alpha} \geq 1,$$  \hspace{1cm} (52)

where in the last step we have used (48).

To compare transformations satisfying (48) with CP ones, we need also

Lemma IV.2. A set $(K, \alpha)$ characterizes a completely positive transformation (i.e. satisfies (45)) iff

$$\sqrt{\det \alpha} \geq |1 - \det K|.$$  \hspace{1cm} (53)

Proof. For one mode, using (15),

$$\Delta K = K \Delta K^T = \det K \Delta,$$  \hspace{1cm} (54)

and (45) becomes

$$\alpha \geq \pm i(1 - \det K) \Delta.$$  \hspace{1cm} (55)

Recalling (18), for linearity (54) becomes exactly

$$\det \alpha \geq (1 - \det K)^2.$$  \hspace{1cm} (56)

We recall here that a complete classification of single mode CP maps has been provided in Refs. [30, 31].

We are now ready to prove the main result of this section.

Theorem IV.3. Any map $\Phi$ satisfying (27) can be written as a dilatation possibly composed with the transposition, followed by a completely positive map. In more detail, given a set $(K, \alpha)$ satisfying (27),

a1: If

$$0 \leq \det K \leq 1,$$  \hspace{1cm} (57)

$\Phi$ is completely positive.

a2: If

$$\det K > 1,$$  \hspace{1cm} (58)

$\Phi$ can be written as a transposition composed with a completely positive map.

b1: If

$$-1 \leq \det K < 0,$$  \hspace{1cm} (59)

$\Phi$ can be written as a transposition composed with the addition of Gaussian noise given by $\alpha$.

b2: If

$$\det K < -1,$$  \hspace{1cm} (60)

$\Phi$ can be written as a dilatation of $\sqrt{|\det K|}$ composed with the transposition, followed by the symplectic transformation given by

$$S = \frac{K}{\sqrt{|\det K|}},$$  \hspace{1cm} (61)

composed with the addition of Gaussian noise given by $\alpha$. 

Proof. a: Let us start from the case
\[ \det K \geq 0 . \] (62)
a1: If
\[ 0 \leq \det K \leq 1 , \] (63)
\[ \text{[48] and [52] coincide, so } \Phi \text{ is completely positive.} \]
a2: If
\[ \det K > 1 , \] (64)
we can write \( K \) as
\[ K = S \sqrt{\det K} I_2 , \] (65)
where
\[ S = \frac{K}{\sqrt{\det K}} \] (66)
is symplectic since \( \det S = 1 \). Then \( \Phi \) can be written as a dilatation of \( \sqrt{\det K} > 1 \), followed by the symplectic transformation given by \( S \), composed with the addition of the Gaussian noise given by \( \alpha \).

b: If
\[ \det K < 0 , \] (67)
we can write \( K \) as
\[ K = K' T , \] (68)
where \( T \) is the one-mode transposition
\[ T = \begin{pmatrix} 1 \\ -1 \end{pmatrix} , \] (69)
and
\[ \det K' = -\det K > 0 . \] (70)

From (48) we can see that also \( K' \) satisfies
\[ \sqrt{\det \alpha} \geq 1 - |\det K'| , \] (71)
and we can exploit the classification with positive determinant, ending with the same decomposition with the addition of the transposition after (or before, since they commute) the eventual dilatation.

\[ \Box \]

V. MULTI-MODE CASE

In the multi-mode case, a classification as simple as the one of theorem IV.3 does not exist. However, we will prove that if \( \Phi \) does not add any noise, i.e \( \alpha = 0 \), the only solution to (27) is a dilatation possibly composed with a (total) transposition, followed by a symplectic transformation. We will also provide examples that do not fall in any classification like IV.3, i.e. that are not composition of a dilatation, possibly followed by a (total) transposition, and a completely positive map.

We will need the following lemma:

Lemma V.1.
\[ \inf_{\sigma \geq \pm i} w^\dagger \sigma w = |w^\dagger \Delta w| \quad \forall \ w \in \mathbb{C}^{2n} . \] (72)

Proof.
a. Lower bound The lower bound for the LHS is straightforward: for any \( \sigma \geq \pm i \Delta \) and \( w \in \mathbb{C}^{2n} \) we have
\[ w^\dagger \sigma w \geq \pm i w^\dagger \Delta w , \] (73)
and then
\[ \inf_{\sigma \geq \pm i} w^\dagger \sigma w \geq |w^\dagger \Delta w| . \] (74)

b. Upper bound To prove the converse, let
\[ w = w_1 + i w_2 , \quad w_j \in \mathbb{R}^{2n} , \]
where without lost of generality we assume \( w_1 \neq 0 \). Then
\[ w^\dagger \sigma w = w_1^\dagger \sigma w_1 + w_2^\dagger \sigma w_2 , \quad |w^\dagger \Delta w| = 2 |w_1^\dagger \Delta w_2| . \]
Assume first \( w_1^\dagger \Delta w_2 = \epsilon \neq 0 \). Then we can introduce the symplectic basis \( \{ e_j, h_j \}_{j=1,\ldots,n} \), where
\[ e_1 = \frac{w_1}{\sqrt{\epsilon}} , \quad h_1 = \frac{\text{sign}(\epsilon)}{\sqrt{\epsilon}} w_2 . \]

Expressed in this basis the question (72) reduces to the first mode, and the infimum is attained by the matrix of the form
\[ \sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \sigma_{n-1} , \]
where \( \sigma_{n-1} \) is any quantum correlation matrix in the rest \( n-1 \) modes.

Consider next the case where \( w_1^\dagger \Delta w_2 = 0 \) and \( w_2 \) is not proportional to \( w_1 \). In this context we introduce the symplectic basis \( \{ e_j, h_j \}_{j=1,\ldots,n} \), where
\[ e_1 = w_1 , \quad e_2 = w_2 . \]

Accordingly the identity (72) reduces to the first two modes, and the infimum is attained by the matrices of the form
\[ \sigma(\epsilon) = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \oplus \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \oplus \sigma_{n-2} , \]
where \( \sigma_{n-2} \) is any quantum correlation matrix in the rest \( n-2 \) modes, and \( \epsilon \to 0 \).
Finally, if \( w_2 = c w_1, \ c \in \mathbb{R}, \) we introduce the symplectic basis \( \{ e_j, h_j \}_{j=1,...,n}, \) where \( e_i = w_1. \) The question [72] reduces to the first mode, and the infimum is attained by the matrices of the form

\[
\sigma(\epsilon) = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \oplus \sigma_{n-1},
\]

where \( \sigma_{n-1} \) is any quantum correlation matrix in the rest \( n - 1 \) modes, and \( \epsilon \to 0. \)

A simple consequence of lemma V.1 is

**Lemma V.2.** Any \( \alpha \) satisfying [27] for some \( K \) is positive semidefinite.

**Proof.** The constraint [27] implies

\[
(K^T k)^T \sigma (K^T k) + k^T \alpha k \geq 0
\]

for any \( \sigma \geq \pm i \Delta \) and \( k \in \mathbb{R}^{2n}. \) Taking the inf over \( \sigma \geq \pm i \Delta, \) and exploiting lemma V.1 with \( w = K^T k, \) we get

\[
k^T \alpha k \geq 0,
\]

i.e. \( \alpha \) is positive semidefinite. \( \square \)

Lemma V.1 lets us rephrase our problem: the constraint [27] can be written as

\[
(K^T w)^T \sigma (K^T w) + w^T \alpha w \geq |w^T \Delta w|,
\]

\( \forall \sigma \geq \pm i \Delta, \forall w \in \mathbb{C}^{2n}. \) Taking the inf over \( \sigma \) in the LHS we hence get

\[
|w^T \Delta_K w| + w^T \alpha w \geq |w^T \Delta w|, \quad \forall w \in \mathbb{C}^{2n},
\]

(78)

with \( \Delta_K \) as in Eq. (46). Notice that, as for the complete positivity constraint (45), since \( K \) enters in (78) only through \( |w^T \Delta_K w|, \) whether given \( K \) and \( \alpha \) satisfy [27] depends not on the entire \( K \) but only on \( \Delta_K. \)

The easiest way to give a general classification of the channels satisfying (78) (and then [27]) would seem choosing a basis in which \( \Delta \) is as in [42], and then try to put the antisymmetric matrix \( \Delta_K \) in some canonical form using symplectic transformations preserving \( \Delta. \) However, the complete classification of antisymmetric matrices under symplectic transformations is very involved [32], and in the multi-mode case the problem simplifies only if we consider maps \( \Phi \) that do not add noise, since in this case the constraint (78) rules out almost all the equivalence classes. In the general case, we will provide examples showing the other possibilities.

### A. No noise

The main result of this section is the classification of the maps \( \Phi \) that do not add noise (\( \alpha = 0 \)) and satisfy [27].

**Theorem V.3.** A map \( \Phi \) with \( \alpha = 0 \) satisfying [27] can always be decomposed as a dilatation [27] composed with the transposition, followed by a symplectic transformation: i.e.

\[
K = S \kappa \mathds{1}_{2n} \quad \text{or} \quad K = ST \kappa \mathds{1}_{2n},
\]

(79)

with \( \kappa \geq 1 \).

**Proof.** With \( \alpha = 0 \) and

\[
w = w_1 + i w_2, \quad w_i \in \mathbb{R}^{2n},
\]

(80)

(78) becomes

\[
|w_1^T \Delta_K w_2| \geq |w_1^T \Delta w_2|,
\]

(81)

i.e. all the matrix elements of \( \Delta_K \) are in modulus bigger than the corresponding ones of \( \Delta \) in any basis. In particular, if some matrix element \( \Delta_{ij}^K \) vanishes, also \( \Delta_{ij} \) must vanish. Let us choose a basis in which \( \Delta_K \) has the canonical form

\[
\Delta_K = \bigoplus_{i=1}^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus 0_{2n-r},
\]

(82)

where

\[
r \equiv \text{rank} \Delta_K.
\]

For (81), in this basis \( \Delta \) must be of the form

\[
\Delta = \bigoplus_{i=1}^n \begin{pmatrix} -\lambda_i \\ \lambda_i \end{pmatrix} \oplus 0_{2n-r}, \quad |\lambda_i| \leq 1.
\]

(83)

Since \( \Delta \) has full rank, there cannot be zeroes in its decomposition, so \( r \) must be \( 2n. \)

We will prove that all the eigenvalues \( \lambda_i \) must be equal. Let us take two eigenvalues \( \lambda \) and \( \mu, \) and consider the restriction of \( \Delta \) and \( \Delta_K \) to the subspace associated to them:

\[
\Delta_K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Delta = \begin{pmatrix} \lambda & 0 \\ -\lambda & \mu \end{pmatrix}.
\]

(84)

If we change basis with the rotation matrix

\[
R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]

(85)

\[
\Delta \mapsto R \Delta R^T, \quad \Delta_K \mapsto R \Delta_K R^T,
\]

(86)

\( \Delta_K \) remains of the same form, while \( \Delta \) acquires off-diagonal elements proportional to \( \lambda - \mu. \) Since for (81) the off-diagonal elements of \( \Delta \) must vanish also in the new basis, the only possibility is \( \lambda = \mu. \) Then all the \( \lambda_i \) must be equal, and \( \Delta_K \) must then be proportional to \( \Delta:

\[
\Delta_K = \frac{1}{\lambda} \Delta, \quad 0 < |\lambda| \leq 1,
\]

(87)
where we have put all the $\lambda_i$ equal to $\lambda \neq 0$ (since $\Delta$ is nonsingular they cannot vanish). Relation (87) means
\[ K\Delta K^T = \frac{1}{\lambda} \Delta, \] (88)
i.e.
\[ \left(\sqrt{|\lambda|} K\right) \Delta \left(\sqrt{|\lambda|} K\right)^T = \text{sign}(\lambda) \Delta. \] (89)
If $0 < \lambda \leq 1$, we can write $K$ as a dilatation of
\[ \kappa = \frac{1}{\sqrt{\lambda}}, \] (90)
composed with a symplectic transformation given by
\[ S = \sqrt{\lambda} K, \] (91)
i.e.
\[ K = S \kappa I_{2n}, \quad S\Delta S^T = \Delta. \] (92)
If $-1 \leq \lambda < 0$, since the total transposition $T$ changes the sign of $\Delta$:
\[ T\Delta T^T = -\Delta, \] (93)
we can write $K$ as a dilatation of
\[ \kappa = \frac{1}{\sqrt{|\lambda|}}, \] (94)
composed with $T$ followed by a symplectic transformation:
\[ K = S T \kappa I_{2n}, \quad S\Delta S^T = \Delta. \] (95)

\section*{B. Examples with nontrivial decomposition}

If $\alpha \neq 0$, a decomposition as simple as the one of theorem IV.3 does no more exist: here we will provide some examples in which the canonical form of $\Delta_K$ is less trivial, and that do not fall in any classification like the precedent one. Essentially, they are all based on this observation:

**Proposition V.4.** If $\alpha$ is the covariance matrix of a quantum state, i.e. $\alpha \geq \pm i \Delta$, the constraint (27) is satisfied by any $K$.

Since for one mode the decomposition of theorem IV.3 holds, we will provide examples with two-mode systems. We will always consider bases in which
\[ \Delta = \begin{pmatrix} \frac{1}{\lambda} & -1 \\ -1 & \frac{1}{\lambda} \end{pmatrix}, \] (96)

\subsection*{1. Partial transpose}

The first example is the partial transpose of the second subsystem, composed with a dilatation of $\sqrt{\nu}$ and the addition of the covariance matrix of the vacuum as noise:
\[ K = \sqrt{\nu} \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}, \quad \nu > 0, \quad \alpha = I_4. \] (97)
In this case we have
\[ \Delta_K = \begin{pmatrix} -\nu & \sqrt{\nu} \\ \sqrt{\nu} & -\nu \end{pmatrix}, \quad \nu > 0, \] (98)
and $i(\Delta - \Delta_K)$ has eigenvalues
\[ \pm (1 + \nu), \quad \pm (1 - \nu), \] (99)
so that one of them is $|1 + |\nu|| > 1$, and the complete positivity requirement (45)
\[ I_4 \geq \pm i(\Delta - \Delta_K) \] (100)
cannot be fulfilled by any $\nu \neq 0$.

We will prove that this map cannot be written as a dilatation, possibly composed with the transposition, followed by a completely positive map. Indeed, suppose we can write $K$ as
\[ K = K' \lambda I_4 \] (99)
\[ \text{or} \quad K = K' T_4 \lambda I_4, \quad \lambda \geq 1. \] (101)
Then
\[ \Delta_{K'} = \pm \frac{1}{\lambda^2} \Delta_K \] (102)
is always of the form (98) with
\[ \nu' = \pm \frac{\nu}{\lambda^2}, \] (103)
and also the transformation with $K'$ cannot be completely positive.

\subsection*{2. Q exchange}

As second example, we take for the added noise $\alpha$ still the covariance matrix of the vacuum, and for the matrix $K$ the partial transpose of the first mode composed with the exchange of $Q^1$ and $Q^2$ followed by a dilatation of $\sqrt{\nu}$:
\[ \alpha = I_4 \geq \pm i \Delta, \quad K = \sqrt{\nu} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \nu > 0. \] (104)
With this choice,

\[
\Delta_K = \begin{pmatrix}
0 & \nu \\
-\nu & 0
\end{pmatrix} . \quad (105)
\]

The transformation is completely positive iff

\[
\mathbb{I}_4 \geq \pm i(\Delta - \Delta_K) , \quad (106)
\]

and since the eigenvalues of \( i(\Delta - \Delta_K) \) are

\[
\pm \sqrt{1 + \nu^2} , \quad (107)
\]

(106) is never fulfilled for any \( \nu \neq 0 \).

As before, we will prove that this map cannot be written as a dilatation, possibly composed with the transposition, followed by a completely positive map. Indeed, suppose we can write \( K \) as

\[
K = K' \lambda \mathbb{I}_4 \quad \text{or} \quad K = K' T_4 \lambda \mathbb{I}_4 , \quad \lambda \geq 1 . \quad (108)
\]

Then

\[
\Delta_{K'} = \pm \frac{1}{\lambda^2} \Delta_K
\]

is always of the form (105) with

\[
\nu' = \pm \frac{\nu}{\lambda^2} , \quad (110)
\]

and also the transformation with \( K' \) cannot be completely positive.

VI. CONCLUSIONS

In this paper we have explored both at the classical and quantum level the set of linear transformations sending the set of Gaussian states into itself without imposing any further requirement, such as positivity. We have proved that the action on the covariance matrix and on the first moment must be linear, and we have found the form of the action on the characteristic function. Focusing on the quantum case, for one mode we have obtained a complete classification, stating that the only not CP transformations in the set are actually the total transposition and the dilatations (and their compositions with CP maps). The same result holds also in the multi-mode scenario, but it needs the further hypothesis of homogeneous action on the covariance matrix, since we have shown the existence of non-homogeneous transformations belonging to the set but not falling into our classification.

The dilatations are then confirmed to be (at least in the single mode or in the homogeneous action cases) the only transformation in the class (24) that can act as a probe for the convex hull of the Gaussian states \( \mathcal{G} \).

VII. ACKNOWLEDGEMENTS

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Appendix A: Unboundedness of dilatations

**Theorem A.1.** For any \( \lambda \neq \pm 1 \) the phase-space dilatation by \( \lambda \) is not bounded in the Banach space \( \mathcal{T} \) of trace-class operators.

**Proof.** Fix \( \lambda \neq \pm 1 \), and let \( \Theta \) be the phase-space dilatation by \( \lambda \). Suppose \( \Theta \) to be bounded, i.e.

\[
\left\| \Theta (\hat{X}) \right\|_1 \leq \|\Theta\|\left\|\hat{X}\right\|_1 \quad \forall \hat{X} \in \mathcal{T} . \quad (A1)
\]

Let also

\[
p_n^{(m)} := \langle m | \Theta (m) | m \rangle | n \rangle . \quad (A2)
\]

Eq. (A1) implies

\[
\sum_{n=0}^{\infty} \left| p_n^{(m)} \right| \leq \|\Theta\| \quad \forall m \in \mathbb{N} . \quad (A3)
\]

The moment generating function of \( p^{(m)} \) is [12]

\[
g_m(q) := \sum_{n=0}^{\infty} p_n^{(m)} e^{-i n q} = \frac{1 - \tau}{1 - e^{-i q}} \left( \frac{1 - \tau e^{i q}}{e^{i q} - \tau} \right)^m , \quad (A4)
\]

where \( q \in \mathbb{R} \) and

\[
\tau := \frac{\lambda^2 - 1}{\lambda^2 + 1} . \quad (A5)
\]

Define

\[
a_m := \frac{1 - \tau}{\sqrt{m} \tau (1 + \tau)} . \quad (A6)
\]

Let \( \phi \in C_c^\infty (\mathbb{R}) \) be an infinitely differentiable test function with compact support. We must then have

\[
\sum_{n=0}^{\infty} \phi (a_m (n - \lambda^2 m)) \left| p_n^{(m)} \right| \leq ||\phi||_\infty \|\Theta\| . \quad (A7)
\]

Expressed in terms of the Fourier transform of \( \phi \)

\[
\tilde{\phi}(k) = \int_{-\infty}^{\infty} \phi(x) e^{i k x} \, dx , \quad (A8)
\]

(A7) becomes

\[
\sum_{n=0}^{\infty} \left( \int_{-\infty}^{\infty} \tilde{\phi}(k) e^{i \lambda^2 a_m k} e^{-i k a_m n} \frac{dk}{2\pi} \right) | p_n^{(m)} | \leq ||\phi||_\infty \|\Theta\| . \quad (A9)
\]
Since the sum of the integrands is dominated by the integrable function
\[ \frac{\left| \Theta \right|}{2\pi} \left| \tilde{\phi}(k) \right|, \]
we can bring the sum inside the integral, getting
\[ \int_{-\infty}^{\infty} \tilde{\phi}(k) g_m(a_m k) e^{i \lambda^2 m a_m k} \frac{dk}{2\pi} \leq \left\| \phi \right\| \left\| \Theta \right\|. \quad (A10) \]

Since for any \( k \)
\[ \lim_{m \to \infty} \left( g_m(a_m k) e^{i \lambda^2 m a_m k} \right) = e^{\frac{i k^3}{3}}, \quad (A11) \]
(see subsection [A1]), by the dominated convergence theorem
\[ \lim_{m \to \infty} \int_{-\infty}^{\infty} \tilde{\phi}(k) g_m(a_m k) e^{i \lambda^2 m a_m k} \frac{dk}{2\pi} = \int_{-\infty}^{\infty} \tilde{\phi}(k) e^{\frac{i k^3}{3}} \frac{dk}{2\pi} = \int_{-\infty}^{\infty} \phi(x) \text{Ai}(x) dx, \quad (A12) \]
where \( \text{Ai}(x) \) is the Airy function. Now we get
\[ \int_{-\infty}^{\infty} \text{Ai}(x) \phi(x) dx \leq \left\| \Theta \right\| \left\| \phi \right\| \quad \forall \phi \in C_c^\infty(\mathbb{R}). \quad (A13) \]

Since the Airy function is continuous and the set of its zeroes has no accumulation points (except \(-\infty\), there exists a sequence of test functions \( \phi_r \in C_c^\infty(\mathbb{R}), r \in \mathbb{N} \) with \( \left\| \phi_r \right\|_\infty = 1 \) approximating \( \text{sgn}(\text{Ai}(x)) \), i.e. such that
\[ \lim_{r \to \infty} \int_{-\infty}^{\infty} \text{Ai}(x) \phi_r(x) dx = \int_{-\infty}^{\infty} |\text{Ai}(x)| dx = \infty, \quad (A14) \]

implying
\[ \left\| \Theta \right\| = \infty. \quad (A15) \]

\[ 1. \text{ Computation of the limit in } (A11) \]

Here we compute explicitly the limit in (A11). It is better to rephrase it in terms of
\[ q := a_m k, \quad q \to 0 \quad (A16) \]
(remember that \( a_m \sim 1/\sqrt{m} \)). Putting together (A11), (A4), (A5) and (A6), we have to compute
\[ \lim_{q \to 0} \left( \frac{1 - \tau}{1 - \tau e^{i q}} \right) \left( \frac{1 - \tau e^{i q}}{e^{i q} \tau} \right) \left( \frac{1 - \tau e^{i \frac{1}{3} q}}{e^{i \frac{1}{3} q} \tau} \right) \left( \frac{k^3(1 - \tau^3)}{\pi^3(1 + \tau^3)} \right) = e^{\frac{ik^3}{3}}. \quad (A17) \]

The first term on the left-hand-side tends to one. The second term on the left-hand-side inside can be treated via Taylor expansion, i.e.
\[ \frac{1 - \tau e^{i q}}{e^{i q} \tau} = 1 + i q^3 \frac{\tau(1 + \tau)}{3(1 - \tau)^3} + \mathcal{O}(q^5) \quad (A18) \]
for \( q \to 0 \). This gives
\[ \lim_{q \to 0} \left( \frac{1 - \tau e^{i q}}{e^{i q} \tau} \right) \left( \frac{1 - \tau e^{i \frac{1}{3} q}}{e^{i \frac{1}{3} q} \tau} \right) \left( \frac{k^3(1 - \tau^3)}{\pi^3(1 + \tau^3)} \right) = \]
\[ \lim_{q \to 0} \left( 1 + i q^3 \frac{\tau(1 + \tau)}{3(1 - \tau)^3} + \mathcal{O}(q^5) \right) \left( \frac{k^3(1 - \tau^3)}{\pi^3(1 + \tau^3)} \right) = \]
\[ = e^{\frac{ik^3}{3}}, \quad (A19) \]
which proves the identity of (A17).

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[1] A. S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory (Edizioni della Normale, Pisa 2011).
[2] A. Ferraro, S. Olivares, and M. G. A. Paris, Gaussian states in continuous variable quantum informan (Napoli, Bibliopolis 2005).
[3] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. 77, 513 (2005).
[4] C. Weedbrook, et al., Rev. Mod. Phys. 84, 621 (2012).
[5] A. S. Holevo and R. F. Werner, Phys. Rev. A 63, 032312 (2001).
[6] V. Giovannetti, S. Lloyd, L. Maccone and P. W. Shor, Phys. Rev. Lett. 91, 047901 (2003).
[7] V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, J. H. Shapiro, and H. P. Yuen, Phys. Rev. Lett. 92, 027902 (2004).
[8] M. M. Wolf, D. Perez-Garcia, and G. Giedke, Phys. Rev. Lett. 98, 130501 (2007).
[9] V. Giovannetti, A. S. Holevo, and R. García-Patrón, Comm. Math. Phys. 334:3, 1553-1571 (2015).
[10] V. Giovannetti, R. García-Patrón, N. J. Cerf, and A. S. Holevo, Nat. Phot. 8, 796 (2014).
[11] C. M. Caves, and P. D. Drummond, Rev. Mod. Phys. 66, 481 (1994).
[12] T. Broecker and R. F. Werner, J. Math. Phys. 36, 62 (1995).
[13] N. C. Dias and J. N. Prata, Rep. Mat. Phys. 63, 43-54 (2009).
[14] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[15] R. Filip and L. Mišta Jr., Phys. Rev. Lett. 106, 200401 (2011).
[16] M. G. Genoni, M. L. Palma, T. Tufarelli, S. Olivares, M. S. Kim, and Matteo G. A. Paris, Phys. Rev. A 87, 062104 (2013).
[17] M. L. Palma, J. Stanners, M. G. Genoni, T. Tufarelli, S. Olivares, M. S. Kim, and M. G. A. Paris, Phys. Scr. T160, 014035 (2014).
[18] C. Hughes, M. G. Genoni, T. Tufarelli, M. G. A. Paris, and M. S. Kim, Phys. Rev. A 90, 013810 (2014).
[19] A. Mari, K. Kieling, B. Melholt Nielsen, E. S. Polzik, and J. Eisert, Phys. Rev. Lett. 106, 010403 (2011)
[20] T. Kiesel, W. Vogel, M. Bellini, and A. Zavatta, Phys. Rev. A 83, 032116 (2011)
[21] Th. Richter and W. Vogel, Phys. Rev. Lett. 89, 283601 (2002)
[22] M. Ježek, I. Straka, M. Mičuda, M. Dušek, J. Fiurášek, and R. Filip, Phys. Rev. Lett. 107, 213602 (2011)
[23] M. G. Genoni, M. G. A. Paris, and K. Banaszek, Phys. Rev. A 76, 042327 (2007).
[24] M. G. Genoni, M. G. A. Paris, and K. Banaszek, Phys. Rev. A 78, 060303 (2008).
[25] M. G. Genoni and M. G. A. Paris, Phys. Rev. A 82, 052341 (2010).
[26] J. Williamson, Am. J. of Math. 58, 141 (1936).
[27] E. B. Davies: Quantum theory of open systems, Academic Press, London 1976.
[28] B. Demoen, P. Vanheuverzwijn and A. Verbeure, Lett. Math. Phys. 2, 161-166 (1977).
[29] M. Fannes, Comm. Math. Phys 51, 55-66 (1976).
[30] A. S. Holevo, Problems of Information Transmission, 43, 1 (2007).
[31] F. Caruso, V. Giovannetti and A. S. Holevo, New J. Phys. 8, 310 (2006).
[32] P. Lancaster and L. Rodman, Linear Algebra and its Applications 406, 1-76 (2005)