A new Solution of the Fractional-order Singular Linear Time-Invariant System

Zaiyong Feng1,2, Ning Chen1,* , Lingling Wu2, Yi Zhang2

1College of Mechanical and Electronic Engineering, Nanjing Forestry University, Nanjing 210037, Jiangsu, China.
2Department of Social Science, Nanjing institute of Railway Technology, Nanjing 210031, Jiangsu, China.

*Corresponding author e-mail: chenning@njfu.edu.cn

Abstract A new solution, the distributional solution of the fractional-order singular linear time-invariant control system (FOSLTIS) is studied in the paper. On the basis of proposing the definition of the Caputo fractional derivatives of Dirac function \( \delta(t) \), the Laplace transform \( \mathcal{L}\{\zeta D_0^\alpha \delta(t)\} = s^\alpha (\alpha \in \mathbb{R}^+) \) was obtained, then \( \mathcal{L}\{\zeta D_0^\alpha \delta(t)\} (\alpha \in \mathbb{R}^+) \) was derived. With regard to FOSLTIS, the system was decomposed into fast subsystem and slow subsystem by restricted equivalent transformation. Using Laplace transformation and taking \( \mathcal{L}\{\zeta D_0^\alpha \delta(t)\} (\alpha \in \mathbb{R}^+) \) into account, distributional solution for the fast subsystem was obtained. Two examples illustrated the correctness of the distributional solution of the FOSLTIS.

1. Introduction

The fractional-order system has been successfully used in many fields with characteristics such as "process memory" and "historical heredity" [1-5]. On the other hand, the singular system is suitable for describing complex system in which there is a coupling relationship between variables, like differential constraints or algebraic constraints [6-7]. The fractional-order singular system inherits the characteristics and advantages of both the fractional-order system and the singular system, thus it’s getting more and more attentions from researchers.

Studies on the fractional-order singular system focus on: (1) Properties of the fractional-order singular system. After N’Doye I proposed the fractional-order singular system model, studies on the fractional-order singular system’s stability and stabilization, admissibility and so on were carried on by one after another [8-12]; (2) The solution of the fractional-order singular linear system. Kaczorek T, Feng Z Y, Chen N, etc. in [13-15] studied the equivalent standard form of the fractional-order singular linear system, and discussed the solution of the system; (3) The application of the fractional-order singular system in electrical engineering and control science, such as fractional-order circuits, design...
of observers, etc. in [16, 17]. Furthermore, researches on solving Fractional-Order Singular Linear Time-Invariant System (FOSLTI S) mainly include three methods: The Shuffle method [18], Drazin inverses method [19] and the Weierstrass-Kronecker standard form method [16].

In this paper, we studied a new solution, namely, the distributional solution of the FOSLTI S. We firstly studied the Laplace transform of $\mathcal{L}\left(\mathcal{C}_0t^\alpha\delta(t)\right), \alpha \in R^+$. Secondly we derived the distributional solution of the FOSLTI S on the basis of the Laplace transform of $\mathcal{L}\left(\mathcal{C}_0t^\alpha\delta(t)\right), \alpha \in R^+$. Two illustrative examples were proposed in the end.

2. Description of the Problem

Consider the following FOSLTI S (1) [15, 16]:

\[
\begin{align*}
E_0^\alpha dx(t) &= Ax(t) + Bu(t), \quad x(0) = x_0; \\
y(t) &= Cx(t) + Du(t); \\
\end{align*}
\]

(1)

where $x(t), y(t), u(t)$ are the state variable, output variable and control input variable of the system respectively and $x(t) \in R^n, y(t) \in R^n, u(t) \in R^r$.

The coefficient matrices satisfy $E, A \in R^{nxn}, B \in R^{nxr}, C \in R^{nxn}$ and $D \in R^{nxr}$. The fractional derivative $\mathcal{C}_0t^\alpha x(t)$ adopts the Caputo derivative, and the derivative order $\alpha$ satisfies $0 < \alpha < 1$. Considering the existence and uniqueness of the solution of the system (1) [14], it is assumed that the matrix pair $(E, A)$ is regular, i.e., there exists some $s \in C$ making $\det\left(s^dE - A\right) \neq 0$.

We mainly study the distributional solution of FOSLTI S in the following sections.

3. Basic Definitions and Preliminaries

As non-classical function (generalized function), the Dirac function $\delta(t)$ and its derivative $\delta^{(i)}(t), (i \in N)$ are defined as follows [20]:

**Definition 1** $\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}, \int_{-\infty}^{\infty} \delta(t)dt = 1.$

**Definition 2** Denoting $\delta^{(i)}(t)$ as the $i$th-order derivative $(i \in N)$ of $\delta(t)$, we have $\delta^{(i)}(t) = 0 , t \neq 0$. If the function $f(t) \in C[a, b], 0 \in [a, b]$ and it is $i$th-order differentiable at $t = 0$, then we have $\int_{a}^{b} f(t)\delta^{(i)}(t)dt = (-1)^i f^{(i)}(0)$.

The full definition of the function $f(t)$’s Caputo fractional derivative $\mathcal{C}_0t^\alpha f(t)$ is as follows:

**Definition 3** [1] If $f(t)$ has continuous derivatives up to the $m$-order, then:

\[
\mathcal{C}_0t^\alpha f(t) = \mathcal{C}_0t^{\alpha - m} f^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{t} \left(t-\tau\right)^{m-\alpha-1} f^{(m)}(\tau)d\tau, (0 \leq m-1 < \alpha < m, \ m \in N^+).
\]

Particularly, when $0 < \alpha < 1$, $\mathcal{C}_0t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} f'(\tau) (t-\tau)^{\alpha-1} d\tau .

The following properties on the Laplace transformation are commonly used in fractional theory.
Property 1[1] \( \mathcal{L}\left\{ C_0^p f(t) \right\} = s^p F(s) - \sum_{k=0}^{m-1} s^{p-k-1} f^{(k)}(0), \quad 0 \leq m-1 < p < m. \)

4. \( C_0^a D_0^\alpha \delta(t) \) And Its (Inverse) Laplace Transformation

There has not been research on the definition of Caputo fractional derivative of \( \delta(t) \) till now. Through usage of the above definitions and property, we investigate the Laplace transformation of \( C_0^a D_0^\alpha \delta(t) \), i.e. Theorem 1 and Theorem 2.

**Theorem 1** \( \mathcal{L}\left[ C_0^a D_0^\alpha \delta(t) : s \right] = s^\alpha, \quad \alpha \in \mathbb{R}^+. \)

**Proof:** When \( 0 \leq m-1 < \alpha < m, \quad m \in \mathbb{N}^+ \),

\[ \mathcal{L}\left[ C_0^a D_0^\alpha \delta(t) : s \right] = \mathcal{L}\left[ \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \delta^{(m)}(\tau) d\tau : s \right] \]

\[ = \frac{1}{\Gamma(m-\alpha)} \mathcal{L}\left[ t^{m-\alpha-1} \delta^{(m)}(t) \right] = \frac{1}{\Gamma(m-\alpha)} \mathcal{L}\left[ t^{m-\alpha-1} \right] \cdot \mathcal{L}\left[ \delta^{(m)}(t) \right] \]

Because \( m-1 < \alpha < m \), \(-1 < m-\alpha-1 < 0\) and \( \mathcal{L}\left[ \delta^{(i)}(t) \right] = s^i, \quad i \in \mathbb{N} \) [20], then we obtain

\[ \mathcal{L}\left[ C_0^a D_0^\alpha \delta(t) : s \right] = \frac{1}{\Gamma(m-\alpha)} \mathcal{L}\left[ t^{m-\alpha-1} \right] \cdot \mathcal{L}\left[ \delta^{(m)}(t) \right] = \frac{1}{\Gamma(m-\alpha)} \cdot \frac{\Gamma(m-\alpha)}{s^{m-\alpha}} \cdot s^\alpha = s^\alpha. \]

Implementing the inverse Laplace transformation to Theorem 1, Theorem 2 is derived.

**Theorem 2** \( \mathcal{L}^{-1}\left[ s^\alpha \right] = C_0^a D_0^\alpha \delta(t), \quad \alpha \in \mathbb{R}^+. \)

5. Distributional Solution of the FOSLTIS

Since the coefficient matrix pair \((E,A)\) in system (1.1) is regular, Left multiplying \( P_1 \) on the system (1.1), letting \( x= P_2 \begin{bmatrix} x_1 \end{bmatrix} \begin{bmatrix} x_2 \end{bmatrix} \end{bmatrix} \), then the system may be equivalently transformed into the following system (2) \((0 < \alpha \leq 1)\):

\[
\begin{cases}
C_0^a D_0^\alpha x_1(t) = A_0 x_1(t) + B_1 u(t), \quad x_1(0) = x_{10}; \quad (2.1) \\
C_0^a D_0^\alpha x_2(t) = A_2 x_2(t) + B_2 u(t); \quad x_2(0) = x_{20}; \quad (2.2)
\end{cases}
\]

\( N \) is a nilpotent matrix, and if its index is \( h \), then \( N^{h+1} \neq 0 \), \( N^i = 0, i = 1,2,\ldots,h-1 \).

We refer to the subsystem (2.1) of system (2) as the slow subsystem, the subsystem (2.2) as the fast subsystem. Next, we discuss the solutions of the two subsystems of system (2) respectively.

Obviously, the slow subsystem (2.1) is a common fractional linear system. For the solution of the slow subsystem (2.1), Kaczorek T proposed Lemma 1 as below [13]:

**Lemma 1** The solution of slow subsystem (2.1) has the following form:

\[
x_1(t) = x_{10}(t,x_{10}) + x_{10}(t,u) = \Phi_0(t) x_{10} + \int_0^t \Phi(t-\tau) B_1 u(\tau) d\tau \quad (3)
\]
Where, $\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A_k}{(k\alpha+1)} t^{(k+1)\alpha}$, $\Phi(t) = \sum_{k=0}^{\infty} \frac{A_k t^{(k+1)\alpha}}{(k+1)\alpha}$. The $E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{(\alpha k + 1)\Gamma(\alpha k + 1)}$ is the Mittag-Leffler function of single parameter.

When the order $\alpha = 1$, $\Phi_0(t) = \Phi(t) = e^{\alpha t}$, solution (3) degenerates to the classical solution of common linear systems. For the solution of the fast subsystem (2.2), we give Theorem 3 as below:

**Theorem 3** Distributional solution of the fast subsystem (2.2) is as below (4):

$$X_2(t,u,x_{20}) = x_2(t,x_{20}) + x_{2u}(t,u)$$

Where, $X_2(t,x_{20}) = -\sum_{k=0}^{h-1} N^{k} B_2 \left[ c D_t^{k\alpha} u(t) + \sum_{i=0}^{l-1} D_t^{k\alpha-i} \delta(t) u^i(0) \right]$ (4.1)

$$x_{2u}(t,u) = \sum_{k=0}^{h-1} N^{k} B_2 \left[ c D_t^{k\alpha} u(t) + \sum_{i=0}^{l-1} D_t^{k\alpha-i} \delta(t) u^i(0) \right]$$ (4.2)

$h$ is the index of the nilpotent matrix $N$, $l_k = \left[ k\alpha \right]$, $k = 0, 1, 2, \cdots, h - 1$.

**Proof:** Applying the Laplace transformation to the fast subsystem (2.2), we obtain:

$$N \left( s^\alpha X_2(s) - s^{\alpha-1} x_2(0) \right) = lX_2(s) + B_2 U(s)$$

After transposition and sorting out: $\left( Ns^\alpha - I \right) X_2(s) = Ns^{\alpha-1} x_2(0) + B_2 U(s)$

Thus: $X_2(s) = \left( Ns^\alpha - I \right)^{-1} \left[ Ns^{\alpha-1} x_2(0) + B_2 U(s) \right]$ (4.3)

Expanding $\left( Ns^\alpha - I \right)^{-1}$ by the Neumann series, namely $\left( Ns^\alpha - I \right)^{-1} = -\sum_{k=0}^{h-1} N^{k} s^{\alpha k}$, taking $h$ is the index of the nilpotent matrix $N$ into account, the above expansion may be simplified to the sum of the finite terms, i.e. $\left( Ns^\alpha - I \right)^{-1} = -\sum_{k=0}^{h-1} N^{k} s^{\alpha k}$. After substituting it into Equation(4.3), we have:

$$X_2(s) = -\sum_{k=0}^{h-1} N^{k} s^{\alpha k} \left[ Ns^{\alpha-1} x_2(0) + B_2 U(s) \right] = -\sum_{k=0}^{h-1} N^{k+1} s^{(k+1)\alpha-1} x_2(0) + N^{k} s^{\alpha k} B_2 U(s)$$

$$= -\sum_{k=0}^{h-1} N^{k+1} s^{(k+1)\alpha-1} x_2(0) - \sum_{k=0}^{h-1} N^{k} s^{\alpha k} B_2 U(s) = -\sum_{k=0}^{h-1} N^{k} s^{\alpha k} x_2(0) - \sum_{k=0}^{h-1} N^{k} s^{\alpha k} B_2 U(s)$$

Implementing the inverse Laplace transformation to the above equation with Theorem 2, the following is obtained:

$$X_2(t) = \mathcal{L}^{-1} \left[ -\sum_{k=0}^{h-1} N^{k} s^{\alpha k-1} x_2(0) - \sum_{k=0}^{h-1} N^{k} s^{\alpha k} B_2 U(s) \right] = -\sum_{k=0}^{h-1} N^{k} \mathcal{L}^{-1} \left[ s^{\alpha k-1} \right] x_2(0) - \sum_{k=0}^{h-1} N^{k} s^{\alpha k} B_2 U(s)$$

$$= -\sum_{k=0}^{h-1} N^{k} \mathcal{L}^{-1} \left[ c D_t^{k\alpha-1} \delta(t) x_2(0) \right] - \sum_{k=0}^{h-1} N^{k} s^{\alpha k} B_2 U(s)$$ (4.4)

To calculate $\mathcal{L}^{-1} \left[ \sum_{k=0}^{h-1} N^{k} s^{\alpha k} B_2 U(s) \right]$, consider the following:...
\[ \mathcal{D}_t^{\alpha} D_t^{k\alpha} u(t) = s^{k\alpha} U(s) - \sum_{i=0}^{l_k-1} s^{k\alpha-1-i} u^{(i)}(0), \quad (l_k - 1 < k\alpha \leq l_k, l_k \in N^+) \]

After transposition, \( s^{k\alpha} U(s) = \mathcal{D}_t^{\alpha} D_t^{k\alpha} u(t) + \sum_{i=0}^{l_k-1} s^{k\alpha-1-i} u^{(i)}(0) \), then, \[
- s^{k\alpha} U(s) = \mathcal{D}_t^{\alpha} D_t^{k\alpha} u(t) + \sum_{i=0}^{l_k-1} C_D^{k\alpha} \delta(t) u^{(i)}(0)
\]

Hence, \[
- \left[ \sum_{k=0}^{h-1} N^k s^{k\alpha} B_2 U(s) \right] = \sum_{k=0}^{h-1} N^k B_2 \left[ \mathcal{D}_t^{\alpha} D_t^{k\alpha} u(t) + \sum_{i=0}^{l_k-1} C_D^{k\alpha} \delta(t) u^{(i)}(0) \right].
\]

Substitute the above equation into (4.4), we obtain the solution of the fast subsystem

\[
x_2(t) = - \sum_{k=1}^{h-1} N^k C_D^{k\alpha} \delta(t) x_{20} - \sum_{k=0}^{h-1} N^k B_2 \left[ \mathcal{D}_t^{\alpha} D_t^{k\alpha} u(t) + \sum_{i=0}^{l_k-1} C_D^{k\alpha} \delta(t) u^{(i)}(0) \right]
\]

Obviously, \( x_2(t, u, x_{20}) = x_{2a}(t, x_{20}) + x_{2a}(t, u) \), where

\[
x_2(t, x_{20}) = - \sum_{k=1}^{h-1} N^k C_D^{k\alpha} \delta(t) x_{20}, \quad x_{2a}(t, u) = - \sum_{k=0}^{h-1} N^k B_2 \left[ \mathcal{D}_t^{\alpha} D_t^{k\alpha} u(t) + \sum_{i=0}^{l_k-1} C_D^{k\alpha} \delta(t) u^{(i)}(0) \right]
\]

Proof completed.

It can be seen that the solution (4) contains linear combinations of the Caputo fractional derivative of distributional function \( \delta(t) \) and the control input \( u(t) \). Therefore, (4) is distributional solution. Obviously, \( x_2(t, x_{20}) = - \sum_{k=1}^{h-1} N^k C_D^{k\alpha} \delta(t) x_{20} \) is the response of the fast subsystem to the initial value \( x_{20} \). \( x_{2a}(t, u) = - \sum_{k=0}^{h-1} N^k B_2 \left[ \mathcal{D}_t^{\alpha} D_t^{k\alpha} u(t) + \sum_{i=0}^{l_k-1} C_D^{k\alpha} \delta(t) u^{(i)}(0) \right] \) is the response of the fast subsystem to the input \( u(t) \). The index of matrix \( N \) is \( h \), \( l_k = \left\lceil k\alpha \right\rceil \), \( k = 0, 1, 2, \cdots, h - 1 \).

6. Verification with Examples

Example 1 Distributional Solution of the Fast Subsystem

Consider the system (5):

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} C_D^{0.5} x(t) = x(t) + \begin{bmatrix}
-1 \\
1
\end{bmatrix} u(t), \quad x_0 = \begin{bmatrix}
x_{10} \\
x_{20}
\end{bmatrix}
\]

Obviously, the nilpotent matrix \( N = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \), its index \( h = 2 \), \( B = \begin{bmatrix}
-1 \\
1
\end{bmatrix} \). From theorem 3, its distributional solution is \( x(t, u, x_0) = x_i(t, x_0) + x_{2a}(t, u) \).
\[
x(t, x_0) = -\sum_{k=1}^{N} C D_t^{k-1} \delta(t)x_0 = -N C D_t^{-0.5} \delta(t)x_0 = \left[ -\frac{C}{0}D_t^{-0.5} \delta(t)x_{20} \right]
\]

\[
x_a(t, u) = -\sum_{k=0}^{N} B C D_t^{k-1} u(t) + \sum_{i=0}^{k-1} C D_t^{k-1-i} \delta(t)u(0) = -Bu(t) - NB\left[ \frac{C}{0}D_t^{-0.5} u(t) + \frac{C}{0}D_t^{-0.5} \delta(t)u(0) \right]
\]

Hence, \[x(t, u, x_0) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{C}{0}D_t^{-0.5} \delta(t)x_{20} + u(t) - \frac{C}{0}D_t^{-0.5} u(t) - \frac{C}{0}D_t^{-0.5} \delta(t)u(0) \\ -u(t) \end{bmatrix}.
\]

For the correctness of the solution, the system (5) is equivalent to:
\[
\begin{cases}
\frac{C}{0}D_t^{0.5} x_1(t) = x_1(t) - u(t) \\
0 = x_2(t) + u(t)
\end{cases}
\]

Thus \(x_1(t)\) is obviously correct from (5.2). Substituting \(x_1(t)\) in the solution into equation (5.1), performing 0.5-order integration on both sides of equation (5.1), we have:
\[
x_2(t) - x_{20} = x_2(t) + \frac{C}{0}D_t^{-0.5} u(t) - \left( u(t) - u(0) \right) - u(0) - \frac{C}{0}D_t^{-0.5} u(t) = -u(t) - x_{20}
\]

Considering \(x_3(t) = -u(t)\), the correctness of \(x_1(t)\) is validated.

**Example 2 Distributional Solution of the Fractional Singular Linear System**

Consider the system (6):
\[
E C D_t^{0.8} x(t) = Ax(t) + Bu(t), \quad x(0) = x_0.
\]

where \(E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 5 & 2 & 4 & 2 \\ 3 & 2 & 3 & 3 \\ 2 & 0 & 1 & 0 \end{bmatrix} \), \(A = \begin{bmatrix} 8 & -1 & 3 & 0 \\ 10 & -7 & 0 & 4 \\ 15 & -7 & 2 & 3 \\ 1 & 1 & 0 & 1 \end{bmatrix} \), \(B = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \).

Taking \(P_1 = \begin{bmatrix} -2 & -3 & 3 & 4 \\ 4 & 7 & -6 & -9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \), \(P_2 = \begin{bmatrix} 2 & -1 & -2 & -1 \\ 3 & -1 & -3 & -2 \\ -4 & 2 & 4 & 3 \\ 1 & 0 & 0 & -1 \end{bmatrix} \), then: \((E, A) \sim P_1(E, A)P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \).

Thus (6) is transformed into (7):
\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} C D_t^{0.8} x_1(t) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} x_1(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(t) \tag{7.1}
\]
\[
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} C D_t^{0.8} x_2(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_2(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \tag{7.2}
\]

Noting that \(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}^{T2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \), the solution of the slow subsystem (7.1) is:
\[
x_1(t) = x_{10}(t, x_{10}) + x_{10}(t, u) = \Phi_0(t) x_{10} + \int_0^t \Phi(t - \tau) B u(\tau) d\tau
\]
The solution of the fast subsystem (7.2) is:

$$x(t) = x_2(t) + x_2(u(t)),$$

where

$$x_2(t) = (t-\tau)^{\alpha+0.2} I + \frac{(t-\tau)^{0.6}}{\Gamma(0.8)} I + \frac{(t-\tau)^{1-1}}{\Gamma(1.6)} I \int_{0}^{t} (t-\tau) u(\tau) d\tau.$$

The solution of the fast subsystem (7.2) is:

$$x(t) = x(t) + x(u(t)),$$

where

$$x(t) = 0.2 0.6 0.8$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$

$$x(t) = 0 1 1 1 1.$$
[5] Ortigueira, Manuel Duarte, and J. A. T. Machado. "Fractional signal processing and applications." Signal Processing, 107(2015)11:197-197.

[6] G. R. Duan, Analysis and Design of Descriptor Linear Systems, Springer Press, New York, 2010.

[7] Anna Filasová, and Dušan Krokavec. "Enhanced approach to PD control design for linear time-invariant descriptor systems." Journal of Physics Conference Series 783 (2017) 1:012-037.

[8] N’Doye I, Darouach M, Zasadzinski M, Radhy, N.E. "Observers design for singular fractional-order systems." 50th IEEE Conference on Decision and Control and European Control conference, 2011.

[9] Ji, Yude, and J. Qiu. "Stabilization of fractional-order singular uncertain systems." ISA Transactions, 56(2015):53-64.

[10] S. Liu, X.-F. Zhou, X. Li, W. Jiang. "Asymptotical stability of Riemann–liouville fractional descriptor systems with multiple time-varying delays." Applied Mathematics Letters, 65(2017)1:32-39.

[11] Yao, Y. U., Z. Jiao, and C. Y. Sun. "Sufficient and Necessary Condition of Admissibility for Fractional-order Singular System." Acta Automatica Sinica, 39(2013)12:2160-2164.

[12] Marir, Saliha, M. Chadli, and D. Bouagada. "A novel approach of admissibility for singular linear continuous-time fractional-order systems." International Journal of Control Automation & Systems, 15(2017)2:959-964.

[13] T. Kaczorek."Descriptor fractional linear systems and electrical circuits." International Journal of Applied Mathematics & Computer Science, 21(2011)2:379-384.

[14] Z.-Y. Feng, N. Chen. "On the Existence and Uniqueness of the Solution of Linear Fractional Differential-Algebraic System." Mathematical Problems in Engineering, 2016(2016):1-9.

[15] Batiba, Iqbal, et al. "The General Solution of Singular Fractional-Order Linear Time-Invariant Continuous Systems with Regular Pencils." Entropy, 20(2018)6:400

[16] Kaczorek, T. "Singular fractional linear systems and electrical circuits." International Journal of Applied Mathematics and Computer Science, 21 (2011) 2:379-384.

[17] Lin, Chong. "Necessary and sufficient conditions of observer-based stabilization for a class of fractional-order descriptor systems." Systems & Control Letters, 112(2018):31-35.

[18] Kaczorek, T. "Reduction and decomposition of singular fractional discrete-time linear systems." Acta Mechanica Automatica, 5(2011)4:1–5.

[19] Kaczorek, T. "Drazin inverse matrix method for fractional descriptor continuous-time linear systems." Bull. Pol. Acad. Sci. Tech. Sci, 62(2014)3:409–412.

[20] S. L. Campbell, Descriptor systems of differential equations, Pitman Publishing, London, 1980.