The ramification filtration in certain $p$-extensions

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Abstract. We show that the recent result of Castañoeda and Wu about the ramification filtration in certain $p$-extensions of function fields of prime characteristic $p$ is equally valid over local fields of mixed characteristic $(0, p)$. Apart from being applicable to both equicharacteristic and mixed characteristic cases, our method has the advantage of being purely local, purely conceptual, more natural, and much shorter.

Il faut le faire.
— Samuel Eilenberg

1. Introduction. — Let $p$ be a prime number and $K$ a local field with perfect residue field of characteristic $p$. Let $G$ be a $p$-group which has the property that every subgroup of $G$ is the intersection of the family of index-$p$ subgroups of $G$ containing it, and let $L$ be a totally ramified $G$-extension of $K$. When $K$ has characteristic $p$, Castañeda and Wu [2, Theorem 4.4] have recently established a relationship between the possible exponents of the differentials of intermediate extensions $L|E|K$ which have degree $p$ over $K$ and the lower ramification breaks of $L|K$. We show that the same relationship holds even when $K$ has characteristic 0, and it is more easily derived by using the ramification filtration in the upper numbering and Herbrand’s theorem. We also remark that the hypothesis on the group $G$ forces it to be commutative of exponent $p$.

2. Certain $p$-groups. — Let us first show that the $p$-groups which were introduced in [2, 2-5] are the same as $F_p$-spaces:

Lemma 2.1. — If $G$ is a $p$-group (of order $> 1$) in which every subgroup $G' \subset G$ is the intersection of the family of index-$p$ subgroups of $G$ containing $G'$, then $G$ is commutative of exponent $p$.

Proof: Taking $G'$ to be the trivial subgroup of $G$, we conclude that the intersection of the family of all index-$p$ subgroups of $G$ is trivial. Let

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P be the maximal quotient of G which is commutative of exponent p and let N be the kernel of the projection \( G \rightarrow P \); we have to show that N is trivial, for which it is enough to show that \( N \subset H \) for every index-p subgroup \( H \subset G \), as we have seen. Recall that every index-p subgroup in a p-group is normal. As \( G/H \) is commutative of exponent p, there is a unique morphism \( P \rightarrow G/H \) such that the projection \( G \rightarrow G/H \) factors as \( G \rightarrow P \rightarrow G/H \) (by the maximality of P), and hence \( N \subset H \). Therefore N is trivial, \( G = P \), and the lemma is proved.

3. The upper ramification breaks. — It follows that G-extensions \( L|K \) for which the group G satisfies the hypothesis of Lemma 2.1 are the same as abelian extensions of exponent p. We next determine the upper ramification breaks of such extensions in terms of the degree-p extensions of K contained in \( L \), necessarily cyclic over K.

**Proposition 3.1.** — For every abelian extension \( L \) of \( K \) of exponent p, the upper ramification breaks \( t \) of \( L|K \) are precisely the same as the possible ramification breaks of intermediate extensions \( L|E|K \) which are cyclic and of degree p over \( K \). In particular, \( t \in \mathbb{Z} \).

**Proof:** We will use the compatibility of the ramification filtration in the upper numbering \( (G^u)_{u \in \mathbb{R}} \) on \( G = \text{Gal}(L|K) \) with the passage to the quotient (Herbrand’s theorem [6, Chapter IV, Proposition 14]).

Explicitly, denoting by \( G^{u+} \) the union of \( G^{u+\varepsilon} \) (\( \varepsilon > 0 \)), if there is an intermediate extension \( L|E|K \) which is (cyclic) of degree p over \( K \), and if the unique ramification break of \( E|K \) occurs at \( t \) (necessarily an integer), then \( C^t = C \) and \( C^{t+} = \{1\} \), where \( C = \text{Gal}(E|K) \). But by Herbrand’s theorem, \( C^t = \overline{G^t} \) and \( C^{t+} = \overline{G^{t+}} \), where \( \overline{H} \) denotes the image in \( C \) of a subgroup \( H \subset G \) under the projection \( G \rightarrow C \). Hence \( G^{t+} \neq G^t \), and an upper ramification break occurs for \( L|K \) at \( t \). (The hypothesis that G be an \( F_p \)-space has not been used yet.)

For the converse, suppose that an upper ramification break occurs at \( t \) for \( L|K \), so that \( G^{t+} \neq G^t \). Since G is an \( F_p \)-space, there is an index-p subgroup \( H \subset G \) such that \( G^{t+} \subset H \) but \( G^t \not\subset H \), so that \( L^H \) is a degree-p (cyclic) extension of \( K \). For \( C = \text{Gal}(L^H|K) = G/H \), by Herbrand’s theorem we have \( C^t = \overline{G^t} = C \) because \( G^t \not\subset H \), and \( C^{t+} = \overline{G^{t+}} = \{1\} \) because \( G^{t+} \subset H \), so the unique ramification break for \( L^H|K \) occurs at \( t \) (which therefore has to be an integer). This completes the proof.

**Remark 3.2.** It is known that the upper ramification breaks of any abelian extension of \( K \) are integers (Hasse-Arf, [6, Chapter V, Theorem 1]). Proposition 3.1 implies and sharpens Hasse-Arf for abelian extensions of exponent p by specifying which integers can occur.
Remark 3.3. In view of Proposition 3.1, one would like to determine the possibilities for the unique ramification break $t$ of a ramified degree-$p$ cyclic extension $E|K$. For this purpose, denote by

$$b^{(i)} = i + [(i-1)/(p-1)] \quad (i > 0)$$

the sequence of positive integers which are prime to $p$. If $K$ has characteristic 0, let $e$ be the ramification index of $K|\mathbb{Q}_p$. If moreover $K$ does not contain a primitive $p$-th root $\zeta$ of 1, the possibilities for $t$ are $b^{(1)}, \ldots, b^{(e)}$ and each of them does occur [3, Proposition 63]. If $K$ contains $\zeta$, there is one further possibility, namely $pe_1 = b^{(e)} + 1$, where $e_1$ is the ramification index of $K|\mathbb{Q}_p(\zeta)$ (so that $e = (p-1)e_1$), and it does occur for $E = K(\sqrt[p]{\pi})$, where $\pi$ is a uniformiser of $K$ [3, Corollary 62]. If $K$ has characteristic $p$, each $b^{(i)}$ is a possibility, and each of them does occur [4, Proposition 14].

Remark 3.4. The special case of Proposition 3.1 when $K$ is a finite extension of $\mathbb{Q}_p$ can be found in the recent preprint of Capuano and Del Corso [1, Proposition 7]. Incidentally, [4] and [5] derive the results of [1] without using class field theory and also treat the equicharacteristic case.

Remark 3.5. If the unique ramification break of a ramified degree-$p$ cyclic extension $E$ of $K$ occurs at $t$, then the exponent of the different of $E|K$ is $d = (1 + t)(p - 1)$ [6, Chapter IV, Proposition 4], so $t$ can be recovered from $d$ and Proposition 3.1 could have been stated in terms of the possible $d$ instead of the possible $t$ for such $E \subset L$.

4. The lower ramification breaks. — Let us finally show how to recover the main result of Castañeda and Wu [2, Theorem 4.4] (and extend it to the mixed characteristic case) from the foregoing.

So let $L|K$ be a totally ramified abelian extension of exponent $p$, and assume that the degree $[L : K]$ is finite, so that the filtration in the lower numbering is defined on $G = \text{Gal}(L|K)$. Let

$$d_1 < d_2 < \cdots < d_n$$

be the exponents of the different of $E|K$, for the various degree-$p$ extensions $E$ of $K$ in $L$. We have to compute the lower ramification breaks of $L|K$ in terms of these $d_i$ and the indices $(G^t : G^{t+})$ for the various upper ramification breaks $t$ of $L|K$.

We have seen in Remark 3.5 that the unique ramification breaks of intermediate extensions $L|E|K$ which are cyclic and of degree $p$ over $K$ occur precisely at

$$t_1 < t_2 < \cdots < t_n,$$
where \( d_i = (1 + t_i)(p - 1) \) for every \( i \in [1, n] \). Note that \( t_1 > 0 \) because \( L/K \) is a totally ramified \( p \)-extension (of degree \( > 1 \)).

It follows from Proposition 3.1 that the upper ramification breaks of \( L/K \) occur precisely at these \( t_i \). For each \( i \in [1, n] \), let \( f_i \) be the codimension of \( G^{t_i} \) in \( G^{t_i} \), or equivalently \( p^{f_i} = (G^{t_i} : G^{t_i+}) \) (so that \( f_i > 0 \)). Note that \( G^{t_i+} = G^{t_i+} \) for every \( i \in [1, n] \). All this information is neatly summarised by the diagram

\[
\{1\} = G^{t_1} \subset_f G^{t_2} \subset G^{t_3} \subset \cdots \subset G^{t_n-1} \subset G^{t_n-1} \subset G^{t_n-1} \subset \cdots \subset G^{t_1} = G
\]

where the notation \( V \subset_c W \) means that \( V \) is a codimension-\( c \) subspace of the \( \mathbb{F}_p \)-space \( W \).

**Proposition 4.1.** — The lower ramification breaks of the extension \( L/K \) occur precisely at \( l_1 = t_1 \) and at \( l_i = l_{i-1} + (t_i - t_{i-1})p^{f_1+f_2+\cdots+f_{i-1}} \) for \( i \in [2, n] \).

**Proof:** We use the formula \( l_i = \psi_{|L/K}(t_i) \) for the passage from the upper numbering to the lower numbering [6, Chapter IV, §3]. Recall that

\[
\psi_{|L/K}(v) = \int_0^v (G^0 : G^w) \, dw.
\]

The result follows upon tabulating the indices \( (G^0 : G^w) \) for every real \( w \in [0, +\infty[ \):

| \( w \) | \( [0, t_1) \) | \( [t_1, t_2) \) | \( [t_2, t_3) \) | \( \cdots \) | \( [t_{n-1}, t_n) \) | \( [t_n, +\infty[ \) |
|---|---|---|---|---|---|
| \( (G^0 : G^w) \) | 1 | \( p^{f_1} \) | \( p^{f_1+f_2} \) | \( \cdots \) | \( p^{f_1+f_2+\cdots+f_{n-1}} \) | \( p^{f_1+f_2+\cdots+f_n} \) |

**Remark 4.2.** More examples of such computations can be found in [4, Section 4] and a comprehensive summary in [5].

**5. Summary.** — Let \( p \) be a prime number, and let \( G \) be a \( p \)-group in which every subgroup is the intersection of the index-\( p \) subgroups of \( G \) containing it. We show that \( G \) has to be commutative of exponent \( p \).

Let \( K \) be a local field with perfect residue field of characteristic \( p \), and let \( L/K \) be a totally ramified \( G \)-extension. If \( d_1 < d_2 < \cdots < d_n \) is the set of exponents of the different of the various degree-\( p \) extensions of \( K \) in \( L \), then the upper ramification breaks of \( L/K \) occur precisely at the integers \( t_1 < t_2 < \cdots < t_n \), where \( d_i = (1 + t_i)(p - 1) \), and the lower ramification breaks precisely at \( l_1 = t_1 \) and at \( l_i = l_{i-1} + (t_i - t_{i-1})p^{f_1+f_2+\cdots+f_{i-1}} \) for every \( i \in [2, n] \), where \( f_j \) is the codimension of \( G^{t_j} \) in the \( \mathbb{F}_p \)-space \( G^{t_j} \) for every \( j \in [1, n] \).
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