Attraction and repulsion in conformal gravity

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ABSTRACT
We use numerical integration to solve the field equations of conformal gravity, assuming a metric that is static and spherically symmetric. Our solution is an extension of that found by Mannheim and Kazanas; it indicates, as expected, that gravitation in this model should be attractive on small scales and repulsive on large ones.

Key words: gravitation — cosmology: theory.

1 INTRODUCTION
Conformal gravity (CG), along with other alternate theories of gravity, has been described in a review by Mannheim (2006); this paper will be referred to as PM. There have been important developments since this review was written (Mannheim 2012), but they will not concern us here.

CG is based on the conjecture that at the deepest level the laws of nature should be conformally invariant. Gravitation theory is therefore based on an action principle derived from the Weyl tensor, $W^\mu\nu$ (PM, sections 5.1 and 8.7). An early achievement in the theory of CG was the discovery by Mannheim and Kazanas of the analytic solution of the field equations $W^\mu\nu = 0$ (1) for a static, spherically symmetric source, analogous to the Schwarzschild solution in conventional gravitation (Mannheim & Kazanas 1989; this paper will be referred to as MK. See also PM, section 9.2). The line element has the form

$$ds^2 = -B(r) \, dt^2 + \frac{dr^2}{B(r)} + r^2 \, d\Omega_2$$

(2)

in the notation of Weinberg (1972).

MK showed that outside a static, spherically symmetric source the function $B(r)$ is given by

$$B(r) = 1 - \frac{\beta(2 - 3\beta r)}{r} - 3\beta r + \gamma r - kr^2.$$ 

(3)

The final $kr^2$ term is important only at cosmological distances, and serves to embed the metric in a de Sitter universe. The term proportional to $1/r$ is analogous to the Schwarzschild solution. The $\gamma r$ term is new, and has been used by Mannheim and collaborators (Mannheim & O’Brien 2012) to explain galactic rotation curves without recourse to dark matter.

Mannheim has also constructed a conformally invariant cosmological model (PM, sections 3.5, 8.7, and 10). In addition to the usual fermion fields, he introduces a scalar field, $S$, that undergoes a symmetry breaking transition and acquires a constant non-zero vacuum expectation value, $S_0$. The resulting field equations are (PM, equation 188):

$$4\alpha g W^\mu\nu = T^\mu\nu,$$

(4)

where $\alpha_g$ is a dimensionless coupling constant, and the energy–momentum tensor, $T^\mu\nu$, is given in PM, equation (66):

$$T^\mu\nu = \frac{i}{2} \bar{\psi} \gamma^\mu(x) \left[ \gamma^\nu + \Gamma^\nu(x) \right] \psi - \frac{1}{4} \delta^{\mu\nu} h S_0 \bar{\psi} \psi$$

$$- \frac{1}{6} S_0^2 \left( R^\mu\nu - \frac{1}{4} g^\mu\nu R \right).$$

(5)

The fermion contribution will not concern us here, because we will deal only with the vacuum equations. But we note that even if the fermion fields are zero, $T^\mu\nu$ is not zero, but depends on $S_0$ and $R^\mu\nu$:

$$T^\mu\nu = -\frac{1}{6} S_0^2 \left( R^\mu\nu - \frac{1}{4} g^\mu\nu R \right).$$

(6)

In developing this cosmological model, Mannheim points out that for the Friedmann–Robertson–Walker (FRW) space normally assumed, $W^\mu\nu$ is identically zero, so the field equations are simply

$$T^\mu\nu = 0,$$

(7)

identical to the equations of conventional cosmology but with an effective gravitational constant that is negative (PM, equation 224):

$$G_{\text{eff}} = -\frac{3c^3}{4\pi S_0^2}.$$ 

(8)

2 STRUCTURE OF THE FIELD EQUATIONS FOR A STATIC, SPHERICALLY SYMMETRIC SOURCE

We start with the observation that the solution obtained by MK for a static, spherically symmetric source did not use the field equation (4) but the simpler equation (1).
Combining equation (4) with (5), the complete field equations can be written as

\[ W^{\mu\nu} + \eta \left( R^{\mu\nu} - \frac{1}{4} g^{\mu\nu} R \right) = 0, \tag{9} \]

where \( \eta = S_0 / (24\Omega_0) \) is a constant of dimension length \( -2 \). We shall assume the magnitude and sign of \( \eta \) are completely unknown, and for the purposes of this paper can be chosen as we wish. We note, however, that some numerical work on the matching of interior and exterior solutions (Wood & Moreau 2001) suggests that \( \eta \) is positive. We will call the first term in equation (9) the ‘main term’, and the rest the ‘eta terms’. Since \( \eta \) is not dimensionless, we can expect that the main term will dominate for small \( r \), and the eta terms for large \( r \).

If the eta terms alone are set to zero, we get the equation

\[ R^{\mu\nu} - \frac{1}{4} g^{\mu\nu} R = 0, \tag{10} \]

which leads to the same line element as before, equation (2), with a metric function of the form \( B(r) = 1 + a/r + b r^2 \), without the linear term of MK.

This leads to the conjecture that at small scales the solution of MK is appropriate, but that at some distance of the order of \( \eta^{-1/2} \) the solution goes over to one of Schwarzschild form. The main purpose of this paper is to demonstrate, by numerical integration, that a solution of that kind does, indeed, exist.

### 3 THE INTEGRATION: ASSUMPTIONS AND PARAMETERS

As shown in Table 1, we use parameters approximately equal to those given by Mannheim for NGC 3198 (Mannheim & O’Brien 2012). Integration is carried out over six orders of magnitude, from \( r_{\text{surface}} \) to a cosmological distance, \( r_{\text{max}} \).

The line element has the form (2) at the beginning, when the main terms of the field equation dominate, and at the end, when the eta terms dominate. In the transition region we cannot be so sure, and a more exact treatment might use the more general line element with two unknown functions:

\[ ds^2 = -B(r) \, dr^2 + C(r) \, dr^2 + r^2 \, d\Omega^2. \tag{11} \]

For simplicity, however, we will assume that, for weak fields, the form (2) is adequate throughout.

The tensor equation (4) gives three different equations, but because of the relations between them it is only necessary to integrate one. The \( rr \) component yields the extension of MK equation (14); primes denote differentiation with respect to \( r \):

\[
B^{-1} \left[ W^{rr} + \eta \left( R^{rr} - \frac{1}{4} g^{rr} R \right) \right]
\]

\[
= \frac{1}{6} B' B'' - \frac{1}{12} (B')^2 - \frac{1}{3r} (B B'' - B' B') \]

\[
- \frac{1}{3r^2} \left[ B B'' + (B')^2 \right] + \frac{2}{3r^3} B B' - B^2 + \frac{1}{3r^4}
\]

\[
+ \frac{\eta}{4} \left( B'' + \frac{2}{r^2} - \frac{2B}{r^4} \right). \tag{12} \]

Setting the right-hand side equal to zero we get a third-order equation that can be integrated straightforwardly.

The linear term in the solution of MK is sufficiently small in practice that \( B(r) \) never deviates far from unity. We therefore integrate the equation for the function \( A(r) = B(r) - 1 \):

\[
A'' + \left\{ -\frac{1}{12} (A')^2 + \frac{1}{3r} (A' A') \right\}
\]

\[
- \frac{1}{3r^2} \left[ (A + 1) A'' + (A')^2 \right] + \frac{2}{3r^3} (A + 1) A' \]

\[
- \frac{A(A + 2)}{3r^4} + \frac{\eta}{4} \left( A'' - \frac{2A}{r^2} \right) \right\} \left/ \left( \frac{A + 1}{3r} - \frac{A'}{6} \right) \right. \tag{13} \]

Some details of the integration are given in the appendix. Here we simply present Fig. 1. In the fitting process, the parameter \( k \) has been varied to make the curve reach \( -1.0 \times 10^{-6} \) at \( r_{\text{max}} \). We have arranged for this small \( r^2 \) component at the end to match a de Sitter space; at the right of the graph, the curve is headed for a horizon at \( r = 1.0 \times 10^{30} \), where \( A = -1 \) and \( B = 0 \). We note that \( |k| \ll |\eta| \), a necessary condition if the cosmological term is only to become noticeable for \( r > r_{\text{centre}} \).

We see, first, that for suitable choices of \( \eta \) and \( k \), a solution of the field equations does exist that duplicates the solution of MK for small \( r \), and gives a Schwarzschild-like solution for large \( r \). Further, we find that \( \eta \) should be chosen negative. If we change the sign of \( \eta \), we find that even for values of \( |k| \) as large as \( 1/r^2_{\text{centre}} \), the final value of \( A \) is always large and negative.

| Parameter | Value      | Units |
|-----------|------------|-------|
| \( r_{\text{surface}} \) | 1.0E20     | m     |
| \( r_{\text{max}} \)     | 1.0E26     | m     |
| \( r_{\text{centre}} \)  | 1.0E23     | m     |
| \( \eta \)            | -1.0E-46   | m⁻²   |
| \( \beta \)           | 5.39E13    | m     |
| \( \gamma \)          | 1.97E-28   | m⁻¹   |
| \( k \)               | (-1.42E-51)| m⁻²   |

**Figure 1.** \( B - 1 \) versus \( r \), using outward integration with \( \log(r/r_{\text{centre}}) \) as the independent variable. This graph uses parameters for NGC 3198, as listed in Table 1.
4 COMMENTS

Our graph indicates an attractive force for \( r < r_{\text{centre}} \), and a repulsive force for \( r > r_{\text{centre}} \), in accordance with the well-known feature of Mannheim’s model, that gravitation is apparently attractive on small scales but repulsive on large ones.

Checking Fig. 1 against observations may prove difficult, since so much of the figure corresponds to distances greater than \( 1.0 \times 10^{22} \) m. We emphasize that in this paper, for simplicity, we have assumed space to be free of matter for \( r > r_{\text{surface}} \). Mannheim & O’Brien (2012) have pointed out, however, that distant matter can have an observable effect on galactic rotation curves, and our analysis would have to be extended to take this into account.

As a first estimate of how distant matter might affect galactic rotation curves if Fig. 1 is appropriate, let us consider a simplified model in Minkowski space. Suppose sources within a distance of \( r_{\text{centre}} \) produce a linear gravitational potential, \( kr \), with \( k > 0 \). Summing over all these sources, and approximating the sum by an integral, we can show straightforwardly that they produce a potential near the origin that goes like \( c_1 r^2 \), with \( c_1 > 0 \). In other words, a test particle released a short distance from the origin will be drawn back towards it.

Suppose in addition that sources more distant than \( r_{\text{centre}} \) generate a potential more suitable for the right half of Fig. 1, namely a Newtonian potential, \( M/r \), added to a cosmological component, \( \kappa r^2 \), with \( \kappa < 0 \). Imagine that these sources are distributed uniformly out to some horizon, \( r_{\text{max}} \). The integrated effect of these sources near the origin can be shown to be a potential \( c_2 r^2 \), with \( c_2 < 0 \).

We see that local and distant sources have opposite effects, and determining which will prevail is a delicate matter, requiring a full relativistic treatment. This will not be attempted here. We observe, however, that if, after the \( S \to S_0 \) transition, there is a time interval during which the horizon distance is smaller than \( r_{\text{centre}} \), then all sources will be attractive, and we may regain the type of cosmology familiar from Einsteinian relativity.

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APPENDIX A: DETAILS OF THE INTEGRATION

We use the integrator \textsc{radau5}, as described in Hairer, Norsett & Wanner (1987).

For initial conditions we use equation (3), the solution of MK, evaluated at \( r_{\text{surface}} \).

Rather than use equation (13) directly, we transform it in two ways. First, we change the independent variable to \( z = \log(r/r_{\text{centre}}) \). Using dots to denote differentiations with respect to \( z \), the equation becomes

\[
\ddot{A} = -2\dot{A} + 3\dot{A} + \left[ \frac{1}{4(A + 1) - 2A} \right] \times \left\{ (-\dot{A} + \dot{A})^2 + 4\dot{A}(-\dot{A} + \dot{A}) \\
- 4[(A + 1)(-\dot{A} + \dot{A}) + \dot{A}^2] \\
+ 8(A + 1)\dot{A} - 4A(A + 2) \\
+ 3\eta r_0^2 \exp(2z)(-\dot{A} + \dot{A} - 2A) \right\}.
\]

(A1)

The second transformation is simply a scale change. \( A(r) \) is always much less than unity, and its derivatives are smaller still. We can get more manageable magnitudes by multiplying all variables by a constant, \( P \), chosen so that \( PA(r_{\text{surface}}) = 1 \). The scale factor is removed after the integration is complete. Writing \( \dot{A}_P = P\dot{A} \), \( \ddot{A}_P = P\ddot{A} \), etc.,

\[
\ddot{A}_P = -2\dot{A}_P + 3\dot{A}_P + \left[ \frac{1}{4(A_P + P) - 2A_P} \right] \times \left\{ (-\dot{A}_P + \dot{A}_P)^2 + 4\dot{A}_P(-\dot{A}_P + \dot{A}_P) \\
- 4[(A_P + P)(-\dot{A}_P + \dot{A}_P) + \dot{A}_P^2] \\
+ 8(A_P + P)\dot{A}_P - 4A_P(A_P + 2P) \\
+ 3P\eta r_0^2 \exp(2z)(-\dot{A}_P + \dot{A}_P - 2A_P) \right\}.
\]

(A2)

(Because \( A(r) \) is always small, we can use either this complete equation or its linearized form; the resulting graphs are very similar.)

Before beginning the main integration, we find two values of \( k \), one of which drives \( A(r) \) to a large positive value at \( r_{\text{max}} \), while the other drives \( A(r) \) to a large negative value. These two values are used as input to a root-finding program, such as \textsc{rtbis} from Numerical Recipes (Press et al. 2007). Each pass through the root finder involves a complete integration, with the output being the value of \( A(r_{\text{max}}) \). \( k \) is automatically adjusted in this way to bring \( A(r_{\text{max}}) \) to the chosen final value.

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