Research Article

Existence of Solutions for Superlinear Second-Order System with Noninstantaneous Impulses

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Variational methods are used in order to establish the existence of nontrivial weak solution for superlinear second-order system with noninstantaneous impulses. The main result is obtained when a kind of definition of the weak solution for this system is introduced. Meanwhile, an example is presented to illustrate the main result.

1. Introduction

In this paper, we consider the following superlinear system with noninstantaneous impulsive effects:

\[
\begin{align*}
-\ddot{u} &= \lambda b_i(t)\nabla F_i(u - u(t_{i+1})), & t \in [s_i, t_{i+1}], i \in M, \\
\dot{u}(t) &= \alpha_i, & t \in [t_i, s_i], i \in M \setminus \{0\}, \\
\dot{u}(s_i) &= \dot{u}(s_i^-), & i \in M \setminus \{0\}, \\
u(0) &= u(T), & \dot{u}(0) = \alpha_0,
\end{align*}
\]

(1)

where \(u(t) = (u^1(t), u^2(t), \ldots, u^N(t))\), \(\lambda > 0\), \(b_i \in F_i(s_i, t_{i+1})\), \(F_i \in \mathcal{F}\), \(\alpha_i \in \mathbb{R}^N\), \(0 = s_0 < t_1 < s_1 < t_2 < s_2 < \cdots < t_m < s_m < t_{m+1} = T\), the impulses start abruptly at points \(t_i\) and keep the derivative constant on a finite time interval \([t_i, s_i]\), \(\dot{u}(s_i^+) = \lim_{s \to s_i^-} \dot{u}(s)(i \in M \setminus \{0\})\). Here, \(m\) is a given positive integer and \(T > 0\). Denote

\[M = \{0, 1, 2, \ldots, m\}\]

\[\mathcal{F} = \{F \in C^1(\mathbb{R}^N, \mathbb{R}): F(0) = 0\}\]

(2)

\[\mathcal{L}[s, t] = \{b \in L^1([s, t] \setminus \{0\}): b \geq 0 \text{ a.e. in } [s, t]\}\]

where \(0 \leq s < t \leq T\).

This paper is mainly motivated by two facts. For the past thirty years, impulsive differential equations have been studied extensively in the literature. There are many approaches to investigate solutions for impulsive differential equations, such as the method of lower and upper solutions, fixed point theory, coincidence degree theory, and geometric approach (see for example [1–8]). In addition, variational methods have been also dealt with impulsive differential equations since papers [9, 10] appeared. Meanwhile, many existence and multiplicity results of solutions for various impulsive differential equations have been obtained. We refer the reader to a large quantity of the references for papers [9, 10]. It is worth noting that in all these references, most of impulsive effects are instantaneous. However, in [11, 12], the authors considered a kind of noninstantaneous impulsive effects, which is motivated by paper [13]. On the other hand, in recent years, some new critical point theorems are used to study nonlinear differential equations with no impulsive effects (see for example, [14–21]). Based on these two facts, we investigate the existence of nontrivial solution for (1).

This paper is organized as follows. In Section 2, we recall a critical point theorem in [16] and introduce a definition of the weak solution for (1), which generalizes Definition 2.3 in [12] to an \(N\)-dimensional case. In Section 3, we present our result and complete its proof via variational methods. The result is new in the field of studying noninstantaneous...
impulsive problem. In Section 4, we present an example in order to illustrate our result.

2. Preliminaries

The following critical point theorem is crucial in our arguments.

**Theorem 1** (see [16], Theorem 3.2). Let $X$ be a real Banach space and let $\Phi, \Psi : X \rightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functions such that $\Phi$ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{u \in \Phi^{-1}(\gamma, \infty)} \Psi(u)$ is finite and assume that for each $\lambda \in [0, r/(\sup_{u \in \Phi^{-1}(\gamma, \infty)} \Psi(u))]$, the functional $I_\lambda = \Phi - \lambda \Psi$ satisfies the (PS)-condition and it is unbounded from below. Then, for each $\lambda \in [0, r/(\sup_{u \in \Phi^{-1}(\gamma, \infty)} \Psi(u))]$, the functional $I_\lambda$ admits at least two distinct critical points.

Consider the Hilbert space:

$$H_0^1 := \{ u \in H^1(0, T; \mathbb{R}^N) \mid u(0) = u(T) \} ,$$

with the inner product

$$ \langle u, v \rangle_{H_0^1} = \int_0^T (\dot{u}(t), \dot{v}(t)) dt + \int_0^T (u(t), v(t)) dt, \forall u, v \in H_0^1,$$

where $(\cdot, \cdot)$ is the inner product in $\mathbb{R}^N$. The corresponding norm is defined by

$$\| u \|_{H_0^1} = \left( \int_0^T |\dot{u}(t)|^2 dt + \int_0^T |u(t)|^2 dt \right)^{1/2}, \forall u \in H_0^1.$$

Meanwhile, for every $u, v \in H_0^1$, we define

$$ \langle u, v \rangle_1 = \int_0^T (\dot{u}(t), \dot{v}(t)) dt,$$

which is also an inner product in $H_0^1$, whose corresponding norm is that

$$\| u \|_1 := \langle u, u \rangle_1^{1/2}.$$

Poincare’s inequality [22] implies that

$$\int_0^T |u(t)|^2 dt \leq \frac{1}{\lambda_1} \int_0^T |\dot{u}(t)|^2 dt,$$

where $\lambda_1 = \pi^2/T^2$ is the first eigenvalue of the Dirichlet problem

$$-\ddot{u}(t) = \lambda u(t), t \in [0, T] ; u(0) = u(T).$$

Hence,

$$\sqrt{\frac{\lambda_1}{\lambda_1 + 1}} \| u \|_{H_0^1} \leq \| u \| \leq \| u \|_{H_0^1}.$$  \hspace{1cm} (10)

That is, $\| \cdot \|$ is equivalent to $\| \cdot \|_{H_0^1}$. It is well known that $(H_0^1, \| \cdot \|_{H_0^1})$ is compactly embedded in $C([0, T], \mathbb{R}^N); \| \cdot \|_\infty)$, where $\| u \|_\infty = \max_{t \in [0, T]} |u(t)|.$ Since the norms $\| \cdot \|_{H_0^1}$ and $\| \cdot \|$ are equivalent, there exists a positive number $\gamma$ such that

$$\| u \|_\infty \leq \gamma \| u \|,$$

where $\gamma := \sqrt{2(1 + (1/\lambda_1))} \max \{ \sqrt{T}, 1/\sqrt{T} \}$.

Following the idea of the variational approach for impulsive differential equation of [11, 12] and combining with (1), one has that for every $v \in H_0^1$,

$$\int_0^T (\ddot{u}(t), \dot{v}(t)) dt = -\sum_{i=1}^{m} \int_{t_i}^{t_{i+1}} (-\ddot{u}(t), \dot{v}(t)) dt + \sum_{i=1}^{m} \int_{t_i}^{t_{i+1}} (-\ddot{u}(t), \dot{v}(t)) dt$$

$$= \int_0^T (\ddot{u}(t), \dot{v}(t)) dt - \sum_{i=1}^{m} (\ddot{u}(t_i) - \ddot{u}(t_i^+), \dot{v}(t_i))$$

$$+ \sum_{i=1}^{m} (\ddot{u}(t_i^-) - \ddot{u}(t_i), \dot{v}(t_i))$$

$$= \int_0^T (\ddot{u}(t), \dot{v}(t)) dt + \sum_{i=1}^{m} (-\ddot{u}(t_i^+) + \alpha_i, \dot{v}(t_i)).$$

(12)

In addition, combining with (1) again, one has

$$\sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \lambda(b_i(t)F_i(u - u(t_{i+1})), v(t_{i+1})) dt$$

$$= \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} (\ddot{u}(t), \dot{v}(t)) dt$$

$$+ \sum_{i=1}^{m} (\ddot{u}(t_i^+) + \alpha_i, \dot{v}(t_i)).$$

(13)

(12) and (13) show that

$$\int_0^T (\ddot{u}(t), \dot{v}(t)) dt = \int_0^T (\ddot{u}(t), \dot{v}(t)) dt + \sum_{i=1}^{m} (\ddot{u}(t_i^+) - \ddot{u}(t_i), \dot{v}(t_i))$$

$$+ \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \lambda(b_i(t)F_i(u - u(t_{i+1})), v(t_{i+1})) dt.$$  \hspace{1cm} (14)
On the other hand,
\[
\int_0^T (\tilde{u}(t), v(t)) dt = \sum_{i=1}^m \int_{t_i}^{t_{i+1}} \lambda (b_i(t) F_i(u - u(t_i)), v(t)) dt \\
+ \sum_{i=1}^m \int_{t_i}^{t_{i+1}} \left( \frac{d}{dt}(a_i) + \lambda v(t) \right) dt \\
= \lambda \sum_{i=1}^m \int_{t_i}^{t_{i+1}} (b_i(t) F_i(u - u(t_i)), v(t)) dt.
\]
(15)

Hence, (14) and (15), together with \( v(t_{m+1}) = v(T) = 0 \), imply that
\[
\int_0^T (\tilde{u}(t), \dot{v}(t)) dt = \sum_{i=1}^m (a_{i-1} - a_i, v(t_i)) \\
+ \lambda \sum_{i=1}^m \int_{t_i}^{t_{i+1}} (b_i(t) \nabla F_i(u(t) - u(t_i)), v(t) - v(t_i)) dt.
\]
(16)

Similar to Definition 2.3 in [12], we can introduce the following concept of a weak solution for problem (1).

**Definition 2.** Say that \( u \in H_0^1 \) is a weak solution for problem (1) if (16) holds for any \( v \in H_0^1 \).

Now, we consider the functionals \( I_A, \Phi, \Psi : H_0^1 \rightarrow \mathbb{R} \) defined by putting
\[
\Phi(u) = \frac{1}{2} \|u\|^2,
\]
(17)
\[
\Psi(u) = \frac{1}{\lambda} \sum_{i=1}^m (a_{i-1} - a_i, u(t_i)) + \sum_{i=1}^m \int_{t_i}^{t_{i+1}} b_i(t) F_i(u(t) - u(t_i)) dt,
\]
(18)
\[
I_A(u) = \Phi(u) - \lambda \Psi(u).
\]
(19)

Since \( b_i \in L^2[\xi_i, \xi_{i+1}], F_i \in \mathcal{F} \) and \( a_i \in \mathbb{R}^N \), \( \Phi \) and \( \Psi \) are two continuously Gâteaux differentiable with
\[
\Phi'(u)v = \int_0^T (\tilde{u}(t), \dot{v}(t)) dt,
\]
(20)
\[
\Psi'(u)v = \frac{1}{\lambda} \sum_{i=1}^m (a_{i-1} - a_i, v(t_i)) \\
+ \sum_{i=1}^m \int_{t_i}^{t_{i+1}} (b_i(t) \nabla F_i(u(t) - u(t_i)), v(t) - v(t_i)) dt,
\]
(21)
for every \( u, v \in H_0^1 \). Hence,
\[
I_A'(u)v = \Phi'(u)v - \lambda \Psi'(u)v.
\]
(22)

Namely, \( I_A \in C^1(H_0^1, \mathbb{R}) \). Definition 2, together with (20)–(22), shows that a critical point of \( I_A \) corresponds to a weak solution for (1).

**3. Main Result and Its Proof**

From now on, we refer to \( M \) as the range of \( i \), unless specifically stated. Let \( \|b_i\|_1 \) be the usual norm in \( L^1[\xi_i, \xi_{i+1}] \). The following theorem is our main result.

**Theorem 3.** Suppose that
\[
(F) \text{ There exist } r_i > 0 \text{ and } \mu_i > 2 \text{ such that for } |\xi| \geq r_i,
\]
\[
0 < \mu_i F_i(\xi) \leq (\nabla F_i(\xi), \xi),
\]
(23)

hold. Then, for every \( b_i \in L^2[\xi_i, \xi_{i+1}] \setminus \{0\} \) and for every
\[
\lambda \in \left[ 0, \sup_{c \geq \bar{c}} \frac{(c^2/2k^2) - c \sum_{i=1}^m (a_{i-1} - a_i)}{\sum_{i=1}^m \max_{|\xi| \leq c} F_i(\xi)} \right],
\]
(24)

(1) admits at least one nontrivial weak solution.

**Proof.** We complete the proof in four steps.

Step 1. Clearly, \( H_0^1 \) is a real Banach space. \( \Phi, \Psi \) are two continuously Gâteaux differentiable functional. In addition, (17), (18), and \( F_i \in \mathcal{F} \) imply that \( \Phi \) is bounded from below and \( \Phi(0) = \Psi(0) = 0 \).

Step 2. \( I_A \) satisfies the (PS)-condition. That is, every \( \{u_n\} \) such that \( I_A(u_n) \) is bounded and \( I_A'|u_n| \rightarrow 0 \) as \( n \rightarrow +\infty \) contains a convergent subsequence.

By (11), we have
\[
\sum_{i=1}^m (a_{i-1} - a_i, u(t_i)) \leq \sum_{i=1}^m |a_{i-1} - a_i| \|u\|_{\infty} \leq \sum_{i=1}^m |a_{i-1} - a_i| \|u\|,
\]
(25)

Let \( \mu = \min \{ \mu_i : i \in M \} \), then \( \mu > 2 \). From (F) and (25), we deduce that
\[
\mu I_A(u_n) - I_A'(u_n)u_n = \left( \frac{\mu}{2} - 1 \right) \|u_n\|^2 - (\mu - 1) \sum_{i=1}^m (a_{i-1} - a_i, u_n(t_i)) \\
- \lambda \sum_{i=1}^m \int_{t_i}^{t_{i+1}} b_i(t)(\mu F_i(u_n(t) - u_n(t_i))) \\
- (\nabla F_i(u_n(t) - u_n(t_i)), u_n(t) - u_n(t_i)) dt \\
\geq \left( \frac{\mu}{2} - 1 \right) \|u_n\|^2 - (\mu - 1) \sum_{i=1}^m |a_{i-1} - a_i| \|u_n\|,
\]
(26)

which implies that \( u_n \) is bounded in \( H_0^1 \). Since \( H_0^1 \) is a reflexive Banach space, passing to a subsequence if necessary, we may assume that there is \( u_0 \in H_0^1 \) such that \( u_n \rightharpoonup u_0 \) in \( H_0^1 \). Then, \( \{u_n\} \) converges uniformly to \( u_0 \) on \([0, T] \) and \( u_n \rightharpoonup u_0 \) in \( L^2([0, T] ; \mathbb{R}^N) \). In addition,
for every $\xi \in \mathbb{R}^N$. Hence, by (17)–(19) and (32)–(34), we have

$$ I_1(kv) = \frac{1}{2} \|kv_0\|^2 - \sum_{i=1}^{m} (a_{i,1} - a_{i,1} kv_0(t_i)) dt + \sum_{i=0}^{m} b_i(t_i) (kv_0(t_i) - kv_0(t_{i+1})) dt $$

$$ \leq \frac{1}{2} k^2 + k y \sum_{i=0}^{m} |a_{i,1} - a_{i,1} - a_i| (2k) a+b + \beta T $$

$$ = -\alpha \sum_{i=0}^{m} \|b_i\|_{(t_i-s_i)(2a+b)} + \frac{1}{2} k^2 + \gamma \sum_{i=0}^{m} |a_{i,1} - a_{i,1}| k + \beta T, $$

which shows that $I_1(kv_0) \longrightarrow -\infty$ as $k \longrightarrow \infty$ because the coefficient of $k^2$ is negative.

**Step 4.** $\lambda \in [0, \sup((c^2/2k^2) c^2/2k^2 - c \sum_{i=0}^{m} |a_{i,1} - a_{i,1}|)/\sum_{i=0}^{m} \|b_i\|_{\max_{\xi \in \mathbb{R}^N} F_i(\xi)}, c \in \mathbb{R}^+ \longrightarrow \infty$. Let $c$ be such that

$$ 0 < \lambda < \frac{c^2/2k^2 - c \sum_{i=0}^{m} |a_{i,1} - a_{i,1}|}{\sum_{i=0}^{m} \|b_i\|_{\max_{\xi \in \mathbb{R}^N} F_i(\xi)}}. $$

and put

$$ r = \frac{c^2}{2k^2}. $$

By (18), one has

$$ \sup_{a \in \Phi^{-1}(\{0,1\})} \Psi(u) \leq \sum_{i=0}^{m} |a_{i,1} - a_{i,1}| + \sum_{i=0}^{m} \|b_i\|_{\max_{\xi \in \mathbb{R}^N} F_i(\xi)} < +\infty, $$

$$ 0 < \lambda < \frac{r}{c \sum_{i=0}^{m} \|b_i\|_{\max_{\xi \in \mathbb{R}^N} F_i(\xi)}} \leq \sup_{a \in \Phi^{-1}(\{0,1\})} \Psi(u). $$

Now, we apply Theorem 1 to conclude that (1) admits at least one nontrivial weak solution.

**4. Example**

Let us consider the following problem:

$$ -\ddot{u} = (t+1)(u(t_{i+1}) - u(t_i))^2, \quad t \in (s_i, t_{i+1}], i = 0, 1, $$

$$ \dot{u}(t) = (0.1, 0.1, 0.1, 0.12), \quad t \in (t_i, s_{i+1}], i = 0, 1, $$

$$ u(s) = \dot{u}(s), \quad i = 0, 1, $$

$$ u(0) = 0 = \dot{u}(0), $$

where $0 = s_0 < t_1 = 0.1 < 0.9 = s_1 < t_2 = 1.$

In this case, $N = 3, T = \lambda = 1, b_i(t) = t + 0.1, t \in (s_i, t_{i+1}], F_i(u) = (1/4) u_i^4, i = 0, 1, k = \sqrt{2(1+1/10^2)}, c = 0.5,$ one can verify that all the conditions in Theorem 3 are satisfied and (39) admits at least one nontrivial weak solution.
Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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References

[1] S. Hu and V. Lakshmikantham, “Periodic boundary value problems for second order impulsive differential systems,” Nonlinear Analysis, vol. 13, no. 1, pp. 75–85, 1989.
[2] Y. Li and Q. Zhou, “Periodic solutions to ordinary differential equations with impulses,” Science in China, vol. 36, pp. 778–790, 1993.
[3] X. Lin and D. Jiang, “Multiple positive solutions of Dirichlet boundary value problems for second order impulsive differential equations,” Journal of Mathematical Analysis and Applications, vol. 321, no. 2, pp. 501–514, 2006.
[4] B. Liu and J. Yu, “Existence of solution for m-point boundary value problems of second-order differential systems with impulses,” Applied Mathematics and Computation, vol. 125, no. 2-3, pp. 155–175, 2002.
[5] D. Qian and X. Li, “Periodic solutions for ordinary differential equations with sublinear impulsive effects,” Journal of Mathematical Analysis and Applications, vol. 303, no. 1, pp. 288–303, 2005.
[6] D. Qian, L. Chen, and X. Sun, “Periodic solutions of superlinear impulsive differential equations: a geometric approach,” Journal of Differential Equations, vol. 258, no. 9, pp. 3088–3106, 2015.
[7] S. H. Saker and J. O. Alzabut, “Existence of periodic solutions, global attractivity and oscillation of impulsive delay population model,” Nonlinear Analysis: Real World Applications, vol. 8, no. 4, pp. 1029–1039, 2007.
[8] S. H. Saker and J. O. Alzabut, “Periodic solutions, global attractivity and oscillation of an impulsive delay host-macroparasite model,” Mathematical and Computer Modelling, vol. 45, no. 5-6, pp. 531–543, 2007.
[9] Y. Tian and W. Ge, “Applications of variational methods to boundary-value problem for impulsive differential equations,” Proceedings of the Edinburgh Mathematical Society, vol. 51, no. 2, pp. 509–527, 2008.
[10] N. J. Juan and D. O’Regan, “Variational approach to impulsive differential equations,” Nonlinear Analysis: Real World Applications, vol. 10, pp. 680–690, 2009.
[11] L. Bai and J. J. Nieto, “Variational approach to differential equations with not instantaneous impulses,” Applied Mathematics Letters, vol. 73, pp. 44–48, 2017.
[12] L. Bai, J. J. Nieto, and X. Wang, “Variational approach to non-instantaneous impulsive nonlinear differential equations,” The Journal of Nonlinear Sciences and Applications, vol. 10, no. 5, pp. 2440–2448, 2017.

[13] E. Hernández and D. O’Regan, “On a new class of abstract impulsive differential equations,” Proceedings of the American Mathematical Society, vol. 141, pp. 1641–1649, 2013.
[14] M. Scehchter, Minimax Systems and Critical Point Theory, Birkhauser, Boston, MA, USA, 2009.
[15] G. Bonanno, R. Livrea, and M. Scehchter, “Some notes on a superlinear second order Hamiltonian system,” Manuscripta Mathematica, vol. 154, no. 1-2, pp. 59–77, 2017.
[16] G. Bonanno, “Relations between the mountain pass theorem and local minima,” Advances in Nonlinear Analysis, vol. 1, pp. 205–220, 2012.
[17] G. Bonanno and G. D’Agui, “Two non-zero solutions for elliptic Dirichlet problems,” Zeitschrift für Analysis und ihre Anwendungen, vol. 35, no. 4, pp. 449–464, 2016.
[18] B. Ricceri, “A further refinement of a three critical points theorem,” Nonlinear Analysis, vol. 74, no. 18, pp. 7446–7454, 2011.
[19] B. Ricceri, “Addendum to ”A further refinement of a three critical points theorem” [Nonlinear Anal. 74 (2011) 7446-7454],” Nonlinear Analysis, vol. 75, no. 5, pp. 2957–2958, 2012.
[20] B. Ricceri, “A three critical points theorem revisited,” Nonlinear Analysis: Theory, Methods & Applications, vol. 70, no. 9, pp. 3084–3089, 2009.
[21] B. Ricceri, “A further three critical points theorem,” Nonlinear Analysis: Theory, Methods & Applications, vol. 71, no. 9, pp. 4151–4157, 2009.
[22] K. C. Chang and Y. Q. Lin, Functional Analysis, Peking University Press, Beijing, China, 1987.