The Origin of the area law of the entropy of a quantum field in a black hole

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It is shown that that the area law for the entropy of a quantum field in the Schwarzschild black hole is due to the quantum statistics. The entropies for one particle, a Boltzmann gas, a quantum mechanical gas obeying Bose-Einstein or Fermi-Dirac statistics, and a quantum field in the Schwarzschild black hole are calculated using the microcanonical ensemble approach and the brick wall method. The area law holds only when the effect of quantum statistics is dominated.

Since the discovery of the black hole entropy $S^{BH} = A_H/(4l_p^2)$ by Bekenstein and Hawking, Euclidean path-integral approach, microcanonical functional integral approach were used to derive black hole thermodynamical properties.

The statistical-mechanical foundation of the entropy of a black hole was discussed in connection with the information approach and with the entanglement entropy. The entanglement entropy is proportional to the area of the horizon but diverges due to the presence of arbitrary high modes near the horizon. To resolve this problem, many renormalization or regularization methods were introduced and these methods were discussed and compared in Ref. 20.

It is generally believed that the entropy of a quantum field in a black hole is proportional to the area of the black hole horizon. The presence of the event horizon makes the one-particle number of states divergent and singles out only the area dependence of the number of states. But it does not determine the functional form of the entropy on area and if we use the result of Padmanabhan the entropy of one-particle system is not proportional to the area but dependent on the logarithm of the area. Therefore the following question arises: What is the origin of the area law for the entropy of the quantum field? There are many different properties between the classical one-particle system and quantum field system. The purpose of the present Letter is to examine which aspect of quantum fields gives the area law.

For this purpose, we adopt the micro-canonical approach and the brick-wall method. First we reproduce the Padmanabhan’s result to calculate the entropy of the classical one-particle system. This result can be applied to the quantum one-particle system. Secondly, we consider the many particle systems with the energy and the particle number fixed. Here we examine the differences of statistical behavior among the Boltzmann statistics, Bose-Einstein statistics and Fermi-Dirac statistics. Finally we remove the constraint on the particle number and we take ensemble sum over all particle number states for the quantum-field statistics.

The line element of Schwarzschild black hole is described by

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2d^2\Omega_2,$$

where $d\Omega_2$ is the metric of the unit 2-sphere, and $g_{tt} = -1/g_{rr} = -(1 - 2M/r)$. Since $\xi^0 = (1, 0, 0, 0)$ is a timelike Killing vector, we can define a covariant, conserved energy of a point particle to be $H(p, x) = \xi^\alpha p_\alpha$.

Before turning on to a system of many particles, we first consider a particle with the mass $\mu$ in a static spherical box around a Schwarzschild black hole. Here the inner and the outer radii of the box are $2M + h$ and $L$, respectively. The phase volume of the classical particle for a fixed energy $E$ is the volume of a hypersurface satisfying $H(p, x) = E$, or

$$\frac{p_t^2}{g_{rr}} + \frac{p_0^2}{g_{\theta\theta}} + \frac{p_{\phi}^2}{g_{\phi\phi}} = \frac{E^2}{-g_{tt}} - \mu^2.$$

On the other hand, for a quantum particle which satisfies

$$[\nabla_\mu \nabla^\mu - \mu^2] \Psi = 0,$$

using the WKB solution

$$\Psi = e^{-iEt + im\theta + iS(r, \theta)},$$

we have the same constraint equation with $p_r = \partial S/\partial r, p_\theta = \partial S/\partial \theta$. In a box normalization with an appropriate boundary condition like as in the brick wall method, a discrete momentum eigenvalue corresponds to one quantum state per unit volume. Then the sum over the quantum states can be rewritten as the integral over the space and the momentum.

Thus both in the classical and the quantum systems, the volume of the hypersurface $g_1(E) = \int d^3p d^2x \delta(E - H(p, x))$ gives the number of states for
a given energy and can be obtained by \( \partial \Gamma / \partial E \), where
\[
\Gamma(E) = \frac{4\pi}{3} \int_{\text{box}} d^3 x \sqrt{\gamma} \left( \frac{E^2}{-\mu^2} - \mu^2 \right)^{3/2},
\]
where \( \gamma \) is the determinant of the spatial part of the metric. When the inner wall of the box approach to the horizon \((h \to 0)\), in the leading order we have
\[
\Gamma(E) \sim \frac{2\pi}{3} \frac{A}{\kappa^2 \epsilon^2} E^3,
\]
(6)
\[
g_1(E) \sim \frac{2\pi}{\kappa^2 \epsilon^2} E^2,
\]
(7)
where \( A \) is the area of the inner wall of the box, \( \kappa = 1/(4M) \) is the surface gravity, at the horizon and \( \epsilon \) is the proper distance from the horizon to the box
\[
\epsilon = \int_{2M}^{2M+h} dr \sqrt{g_{rr}} = 2\sqrt{2M} h.
\]
(8)
In fact, the one-particle phase volume of the system can be consulted in Ref. \[22\]. Here we rederived it to introduce our notations.

From the expression \[1\] and the entropy formula \( S(E) = \ln g_1(E) \) we get the leading behavior of the entropy of the particle
\[
S \sim \ln \left( \frac{2\pi A}{\kappa^3 \epsilon^2} E^2 \right).
\]
(9)
As seen from this equation the entropy of the one-particle system is proportional to the logarithm of the area of the black hole.

Now let us extend our result to many particle systems. The number of accessible states with total energy \( \mathcal{E} \) and total number \( N \) is \[23\]
\[
g_N(\mathcal{E}) = \sum_{\{n_i\}} W\{n_i\},
\]
(10)
where the sum extends over all sets of integers \( \{n_i\} \) satisfying the conditions
\[
\mathcal{E} = \sum_i E_i n_i, \quad N = \sum_i n_i.
\]
(11)
Here \( W\{n_i\} \) is the number of states of the system corresponding to the set of occupation numbers \( \{n_i\} \), is given by
\[
W\{n_i\} = \begin{cases} 
\prod_i \frac{g_1(n_i)}{n_i!} & \text{(Boltzmann)}, \\
\prod_i \frac{(n_i + g_1(n_i) - 1)!}{n_i!(g_1(n_i) - 1)!} & \text{(Bose)}, \\
\prod_i \frac{g_1(n_i)!}{n_i!(g_1(n_i) - n_i)!} & \text{(Fermi)}
\end{cases}
\]
(12)
for Boltzmann statistics, Bose-Einstein statistics, and Fermi-Dirac statistics, respectively.

In fact, \( g_N(\mathcal{E}) \) is quite well approximated by \( W\{\bar{n}_i\} \) where \( \{\bar{n}_i\} \) is a set of occupation numbers that maximize \( W\{n_i\} \) subject to \[11\]
\[
\bar{n}_i = \begin{cases} 
\frac{g_1(E_i)}{z^{\frac{3}{E_i}}} & \text{(Boltzmann)}, \\
\frac{g_1(E_i)}{z^{-\frac{3}{E_i}} + 1} & \text{(Bose and Fermi}).
\end{cases}
\]
(13)
Here the Lagrangian multipliers \( z \) and \( \beta \) are determined by the two conditions of Eq. \[11\]. Then the entropy \( S = \ln W\{\bar{n}_i\} \) is given by using the Stirling’s formula:
\[
S = \sum_i \left[ \bar{n}_i \ln \left( 1 + \frac{g_1(n_i)}{n_i} \right) + g_1(n_i) \ln \left( 1 + \frac{\bar{n}_i}{g_1(n_i)} \right) \right] \quad \text{(Bose)},
\]
\[
S = \sum_i \left[ \bar{n}_i \ln \left( \frac{g_1(E_i) - 1}{n_i} \right) - g_1(E_i) \ln \left( 1 - \frac{\bar{n}_i}{g_1(n_i)} \right) \right] \quad \text{(Fermi)}.
\]
(15)
The explicit calculation using \[11\] and \[13\] gives the entropy
\[
S = \left\{ \begin{array}{ll}
\frac{N}{2} \left[ 3 + 2 \ln \left( \frac{2\alpha \mathcal{E}^3}{27N^2} \right) \right] & \text{(Boltzmann)}, \\
\frac{[4f_4(z) - f_3(z)] \ln z}{2\alpha \beta^3} & \text{(Bose and Fermi)}.
\end{array} \right.
\]
(16)
where \( \alpha = \frac{2\pi A}{\kappa^2 \epsilon^2} \) and \( f_n(z) = \frac{1}{(n-1)!} \int_0^\infty dx x^{n-1}/(e^x + 1) \) (Bose and Fermi) can be written as
\[
f_n(z) = \begin{cases} 
\sum_{l=1}^{[n/2]} C_{2l} \ln(n_z - 2l) + O \left( \frac{1}{z} \right) & \text{for } \beta \gg 1, \text{(Fermi)}.
\end{cases}
\]
(16)
using the Sommerfeld expansion for large \( z \). Here \([x]\) denotes the greatest integer which is not greater than \( x \), and \( I_n = (n - 1)!(2n)(1 - 2^{-n}) \zeta(n) \).

Let us discuss the result of Bose statistics, to see the change of the entropy behavior from the logarithm dependent one to the linear one of the area of the black hole. Note that Eq. \[2\] gives restriction that the ground state energy must be greater than \( E_0 = \mu c/(4M) \). In fact this is extremely small quantity so we can set it effectively to zero. With this choice the constraint equations \[11\] give the following equation for \( \beta, z \).
\[
\int dE \frac{g_1(E)}{e^{\beta E} / z - 1} + n_0 = N, \quad \int dE \frac{E g_1(E)}{e^{\beta E} / z - 1} = \mathcal{E},
\]
(17)
where we isolated the average number of particles in the zero energy state \( n_0 = z/(1 - z) \) which is the zero energy divergence in Eq. \[13\] at \( z = 1 \). These constraint equations become
\[
\frac{f_3(z)}{\beta^3} + \frac{n_0}{2\alpha} = \frac{N}{2\alpha^3} \frac{f_4(z)}{\beta^4} = \frac{\mathcal{E}}{6\alpha}, \quad (18)
\]

For a small \( z \), we can approximate \( f_n(z) = z + z^2/2^n \) and we have
\[
z \approx \frac{27N^4}{2\alpha e^3}, \quad (19)
\]
\[
\beta \approx \left(1 - \frac{27N^4}{32\alpha e^3}\right) \frac{3N}{\mathcal{E}}. \quad (20)
\]

These equations hold only for a relatively small number of particles compared to the phase volume of one-particle system with energy \( \mathcal{E}/N \) (Note that the \( z \ll 1 \) means \( N/[\alpha(\mathcal{E}/N)^3] \ll 1 \)). Then the entropy is given by
\[
S \approx N \ln \frac{4\pi A e^3}{27\kappa e^2 N^4}, \quad (21)
\]
which is quite similar to the Boltzmann gas in [13]. If we consider the on-shell entropy by using Eq. (20) and \( \beta\kappa = 2\pi \), we get
\[
S \approx N \ln \frac{A (1 - 27N^4/32\alpha e^3)}{2\pi^2 N\kappa e^2}, \quad (22)
\]
which is proportional to the logarithm of the area.

\[
N/(2a)
\]

![Plot for \( a = 100 \)](image)

**FIG. 1.** \( \mathcal{E}-N \) diagram. The line is depicted for \( \beta = 0.75 \) with \( \alpha = 100 \). The number of particles in the ground states varies from 0 to 1000.

For \( z \sim 1(n_0 \gg 1) \), from (18) we have
\[
\mathcal{E} = \frac{6\alpha \zeta(4)}{\beta^4}, \quad N = n_0 + 2\alpha \zeta(3)/\beta^3, \quad (23)
\]
where we used the approximation \( f_i(z \sim 1) \approx \zeta(i) \). Here the energy value in (23) is the maximum energy the system can have for a fixed \( \beta \). On the other hand the number of particles can grow indefinitely due to the presence of the zero energy particles as can be seen in Fig. 1. This phenomenon that the zero energy particles pile up in the ground state is the Bose-Einstein condensation. Now the entropy is given by
\[
S = \frac{16\pi \zeta(4)}{\beta^3 \kappa e^2} A. \quad (24)
\]

Here we note that the black hole entropy in this limit obeys the area law.

Now let us consider the Fermi statistics. The constraint equations (14) can be rewritten as
\[
\int dE \frac{g(E)}{e^{\beta E}/z + 1} = N, \quad \int dE \frac{Eg(E)}{e^{\beta E}/z + 1} = \mathcal{E}, \quad (25)
\]
and after some algebra we get
\[
\frac{f_3(z)}{\beta^3} = \frac{N}{2\alpha} \frac{f_4(z)}{\beta^4} = \frac{\mathcal{E}}{6\alpha}. \quad (26)
\]
For small \( z \) the solution of these equations are exactly the same as those of Bose statistics (21). On the other hand, for large \( z \) we get
\[
\mathcal{E} = \frac{\alpha}{4} \left[ \left( \frac{\ln z}{\beta} \right)^4 + 6 \left( \frac{\ln z}{\beta} \right)^2 + \frac{I_4}{\beta^4} \right],
\]
\[
N = \frac{\alpha}{3} \left[ \left( \frac{\ln z}{\beta} \right)^3 + 3 \left( \frac{\ln z}{\beta} \right)^2 \frac{I_2}{\beta^2} + \frac{I_4}{\beta^4} \right]. \quad (27)
\]

Then the entropy obeys the area law again:
\[
S = \frac{\alpha}{\beta} \left( I_2 \left( \frac{\ln z}{\beta} \right)^2 + \frac{I_4}{3} \right). \quad (28)
\]

All the previous results are obtained under the assumption that the total number of particles, \( N \), is constant. However, these can be regarded as quantum mechanical limits because the Fock space of quantum field theory is spanned by a complete set of particle number states. Therefore when we count the accessible states in the microcanonical ensemble for a quantum field we should consider all possible number states, with the energy being fixed. Then it is natural to vary the particle numbers and we fix the energy only:
\[
\mathcal{E} = \sum_i n_i E_i = 6\alpha \frac{f_4(z)}{\beta^4} \quad (29)
\]
both for Bose and Fermi statistics. The number of states \( g(\mathcal{E}) \) for this ensemble can be obtained by summing \( g_N(\mathcal{E}) \) over all possible \( N \):
\[
g(\mathcal{E}) = \sum_{N=0}^{\infty} g_N(\mathcal{E}). \quad (30)
\]
satisfies (29) becomes solution of (32). Then the distribution of particles which
constraint (29). The entropy now is given by
entropy principle directly to Eq. (12) with the con-
This result can be obtained by applying the maximal
entropy principle automatically selects the
states whose particle numbers give the area law [see (33)].
The role of large particle number is to expose the ef-
fact of quantum statistics. In the case of quantum fields,
the area law for the entropy of a quantum field in the
Schwarzschild black hole is due to the quantum statis-
tics. The entropy obeys Boltzmann statistics [see (9) for one-particle,
or quantum field case [see (34)], the main contribution
of quantum particles], the entropy is dependent on the
logarithm of the area. Therefore we can conclude that
the entropy of Bose gas in equilibrium with the black hole
is \(7\) \(\frac{f_4(1)}{\beta}\frac{\pi f_4(z)}{f_4(z)} A\),
where we used \(f_n'(z) = f_{n-1}(z)/z\). Further, from the variation of \(g_N(E)\) we have
\[
\delta \ln g_N(E) = \frac{\alpha}{\beta z} \delta z F(z) \ln z = 0, \tag{32}
\]
where \(F(z) = 3f_3(z)/4f_4(z) - f_2(z)\). Since \(F(z)\) is always
negative for \(z > 0\) and \(F(z \to 0) \sim z, z = 1\) is the only
solution of (32). Then the distribution of particles which
satisfies (29) becomes
\[
\bar{n}_\pm = \frac{g_1(E)}{e^{\beta E} - 1} \text{ (Bose and Fermi).} \tag{33}
\]
This result can be obtained by applying the maximal
entropy principle directly to Eq. (22) with the con-
straint (29). The entropy now is given by
\[
S_E = \frac{16\pi f_4(1)}{\beta^3 \kappa^2 e^2} A, \tag{34}
\]
where \(f_4(1) = \zeta(4)\) for Bose statistics and \(f_4(1) = (7/8) \cdot \zeta(4)\) for the Fermi statistics and the suffix \(E\)
was introduced to stress that we have fixed only the energy
by the constraint (29). Further it can be verified that
\(\partial S_E/\partial E = \beta\) using (18) and (26). If we consider the
equilibrium with the black hole of the inverse tempera-
ture \(\beta = \beta_{BH} = 2\pi/\kappa\), the entropy is given by
\[
S_E = \frac{2f_4(1)}{\pi^2 e^2} A. \tag{35}
\]
The relative ratio between the entropy of Fermi gas and
the entropy of Bose gas in equilibrium with the black hole is
\(7/8\).

Now we can answer to our main question about area
law of the entropy. As can be seen in the cases of
large particle number limit of quantum mechanical sys-
tems [see (24) (Bose-Einstein) and (28) (Fermi-Dirac)]
or quantum field case [see (4)], the main contribution
of the entropy is proportional to the area. On the other
hand, if the number of particles is small or if the sys-
tem obeys Boltzmann statistics [see (1) for one-particle,
(13) for Boltzmann gas, and (2) for small number limit
of quantum particles], the entropy is dependent on the
logarithm of the area. Therefore we can conclude that
the area law for the entropy of a quantum field in the
Schwarzschild black hole is due to the quantum statis-
tics. The role of large particle number is to expose the ef-
fact of quantum statistics. In the case of quantum fields,
the maximal entropy principle automatically selects the
states whose particle numbers give the area law [see (23)].

We would like to thank Prof. Yoonbai Kim and Prof.
Philliar Oh for stimulating discussions and Prof. R.
Ruffini for his encouragement. H.C.K. and J.Y.J. are
supported by the Korean Science and Engineering Foundation
for the post-doctoral fellowships and the Center
for Theoretical Physics (SNU). M.H.L. is supported by the
Center for Molecular Sciences (KAIST).

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