We study mixing and diffusion properties of passive scalars driven by generic rough shear flows. Genericity is here understood in the sense of prevalence and (ir)regularity is measured in the Besov–Nikolskii scale $B^{\alpha}_{1,\infty}$, $\alpha \in (0,1)$. We provide upper and lower bounds, showing that in general inviscid mixing in $H^{1/2}$ holds sharply with rate $r(t) \sim t^{1/(2\alpha)}$, while enhanced dissipation holds with rate $r(\nu) \sim \nu^{\alpha/(\alpha+2)}$. Our results in the inviscid mixing case rely on the concept of $\rho$-irregularity, first introduced by Catellier and Gubinelli (Stoc. Proc. Appl. 126, 2016) and provide some new insights compared to the behavior predicted by Colombo, Coti Zelati and Widmayer (Ars Inven. Anal., 2021).

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CONTENTS

1. Introduction 1
2. Preliminaries 8
   2.1. Prevalence 8
   2.2. A useful class of Gaussian transverse measures 9
3. Inviscid mixing 10
   3.1. Lower bounds in terms of regularity 10
   3.2. Upper bounds in terms of $\rho$-irregularity 11
   3.3. Prevalence statements and proof of Theorem 1.4 14
4. Enhanced dissipation 16
   4.1. Lower bounds in terms of regularity 16
   4.2. Wei’s irregularity condition 18
   4.3. Sufficient conditions for stochastic processes 22
   4.4. Prevalence statements and proof of Theorems 1.5, 1.1 24
5. Further comments and future directions 26
Appendix A. Besov spaces 28
Appendix B. A simple extension of a result by Wei 29
References 31

1. INTRODUCTION

We are interested in the long time behavior of solutions $f$ to

$$\begin{cases}
\partial_t f + u \partial_x f = \nu \Delta f \\
|f|_{t=0} = f_0, \quad T \int f_0(x,y) dx = 0
\end{cases}$$

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Note: This document has been written using GNU TeXmacs [28].
on the 2-dimensional flat torus $\mathbb{T}^2$. The PDE (1.1) is an advection-diffusion equation associated to a shear flow $u = u(y) : T \to \mathbb{R}$, $f : \mathbb{R}^2 \times T^2 \to \mathbb{R}$ with initial condition $f_0 \in L^2(\mathbb{T}^2)$ and where $\nu \in [0, 1]$ is the diffusion coefficient. Defining $\tilde{u} : T^2 \to \mathbb{R}^2$ as $\tilde{u}(x, y) := (u(y), 0)^T$, equation (1.1) may be written as
\[
\partial_t f + \tilde{u} \cdot \nabla f = \nu \Delta f
\]
which is the equation for a passive scalar $f$ advected by the velocity field $\tilde{u}$. Note that $\tilde{u}$ is a divergence-free vector field and a stationary solution to 2D Euler equations.

Exactly for this reason, shear flows have received a lot of attention in the literature, in connection to the problem of understanding the interaction between mixing and diffusion in fluid mechanics and the transfer of energy from large to small scales for the scalar $f$. In particular, shear flows are sufficiently simple to allow explicit calculations, while presenting a highly non-trivial behavior, as already observed by Kelvin in [32] in the case of the Couette flow $\bar{u}(y) = y$.

Observe that for continuous $u$, eq. (1.1) can be solved explicitly by Feynman–Kac formula, giving
\[
f_t(x, y) = \mathbb{E}\left[ f_0 \left( x - \int_0^t u \left( y + \sqrt{2\nu} B^1_s \right) ds + \sqrt{2\nu} B^1_t, y + \sqrt{2\nu} B^2_t \right) \right]
\]
where $B = (B^1, B^2)$ is a standard 2D Brownian motion (Bm). In the case $\nu = 0$ we obtain
\[
f_t(x, y) = f_0(x - tu(y), y).
\]
Both formulas (1.3) and (1.4) can then be extended to the case $u \in L^1(\mathbb{T})$, \footnote{For $u \in L^1(\mathbb{T})$, the formal expression $\int_0^t u \left( y + \sqrt{2\nu} B^1_s \right) ds$ in (1.3) can be made rigorous using the local time of $B^2$; alternatively, equation (1.1) can be solved by applying the Fourier transform in the $x$-variable and solving the family of equations for $\hat{f}^k = P_k f$, see the beginning of Appendix B for more details.} in which case eq. (1.1) must be understood in the weak sense, and generate continuous semigroups $e^{t(-u\partial_x + \nu\Delta)}$ on $L^2(\mathbb{T}^2)$. Yet, they do not provide any immediate insight on the long time behavior of the solution $f$, in particular on the decay in time of quantities like $\|f_t\|_{H^{-s}}$ and $\|f_t\|_{L^2}$.

Following the line of research initiated in [43], [10], we consider rough shear flows, in the sense of requiring $u \in B^\alpha_{1,\infty}(\mathbb{T})$ for some $\alpha \in (0, 1)$. Here $B^\alpha_{1,\infty}(\mathbb{T})$ denote the Besov–Nikolskii spaces, see Appendix A for their definition.

We are interested in understanding the behavior of generic $u \in B^\alpha_{1,\infty}(\mathbb{T})$, a problem explicitly left open in [10]. For this purpose we adopt the measure-theoretic notion of genericity given by the theory of prevalence, developed by Hunt, Sauer and Yorke [29] to provide an analogous of “Lebesgue almost every” on infinite dimensional spaces, see Section 2.1 for more details. In what follows the expression “for almost every $\phi \in E^r$, where $E$ is a function space, is understood in the sense of prevalence.

The next statement summarizes our main findings.

\textbf{Theorem 1.1.} Let $\alpha \in (0, 1)$. The following hold:
\begin{enumerate}
\item For almost every $u \in B^\alpha_{1,\infty}(\mathbb{T})$ we have inviscid mixing in the scale $H^{1/2}(\mathbb{T}^2)$, in the following sense: for any $\tilde{\alpha} > \alpha$, there exists $C = C(\alpha, \tilde{\alpha}, u)$ such that, for any $f_0 \in H^{1/2}(\mathbb{T})$ satisfying $\int_{\mathbb{T}} f(x, \cdot) dx \equiv 0$, it holds
\[
\|e^{-t\nu \partial_x} f_0\|_{H^{-1/2}} \leq Ct^{-\frac{\tilde{\alpha}}{2}} \|f_0\|_{H^{1/2}} \quad \forall t \geq 0.
\]
\end{enumerate}
ii. For almost every \( u \in B^0_{1,\infty}(T) \) we have enhanced dissipation in the following sense that: for any \( \alpha > 1 \) there exist \( C_\alpha = C(\alpha, \dot{\alpha}, \nu) \) such that, for any \( f_0 \in L^2(T) \) satisfying \( \int_T f(x, \cdot)dx = 0 \), it holds

\[
\|e^{t(-u \partial_x + \Delta)}f_0\|_{L^2} \leq C_1 \exp \left(-C_2 t^\nu \frac{\dot{\alpha}}{2}\right) \|f_0\|_{L^2} \quad \forall t \geq 0, \nu \in [0,1].
\]

In the above statement, the condition \( \int_T f(x, \cdot)dx = 0 \) is necessary, as it naturally ensures that \( f \) witnesses the effect of the transport operator \( u \partial_x \); indeed \( g_t(y) := \int_T f_t(x,y)dx \) must solve the standard heat equation \( \partial_t g = \nu \partial^2_y g \) and thus cannot exhibit any mixing/enhanced dissipation effect.

There is no obvious a priori reason to work with the spaces \( B^0_{1,\infty}(T) \) (e.g. in \cite{10} the authors deal with \( C^\alpha(T) = B^0_{\alpha,\infty}(T) \)), rather they arise naturally in our analysis. One of the main intuitions of the present paper is the identification of such spaces as the correct one for studying generic inviscid mixing and enhanced dissipation properties of shear flows. At the same time, let us mention that the only truly relevant parameter is \( \alpha \in (0,1) \): indeed statements similar to those of Theorem 1.1 can be given for the (smaller) spaces \( B^\alpha_{p,q}(T) \) for any choice of \( p, q \in [1, \infty] \), see Remark 1.6 below.

Before moving further, let us heuristically motivate the connection between Points i. and ii. of Theorem 1.1 and why it is natural to expect \( \nu^{\alpha/\alpha+2} \) to appear, given the decay \( \|f_t\|_{H^{-1/2}} \lesssim t^{-1/(2\alpha)} \). In fact, the argument can be given in a much more general framework: let \( f'' \) be a solution to (1.2) with \( \nu > 0 \), \( \int_T f_0(z)dz = 0 \) and \( \bar{u} : \mathbb{T}^d \to \mathbb{R}^d \) be a divergence free vector field; then \( f'' \) satisfies the energy balance

\[
\frac{d}{dt}\|f''\|_{L^2}^2 = -2\nu \|\nabla f''\|_{L^2}^2.
\]

Now assume the solution \( f \) to the transport equation \( \partial_t f + \bar{u} \cdot \nabla f = 0 \) to satisfy the decay \( \|f_t\|_{H^{-s}} \lesssim t^{-s/\alpha} \) for suitable parameters \( \alpha > 0, s \in (0,1) \) (for \( s > 1 \), one may reduce to \( s = 1 \) by Riesz–Thorin interpolation theorem). For \( \nu \ll 1 \) and sufficiently short times, we expect \( f'' \) and \( f \) to stay close and therefore \( f'' \) to exhibit the same decay as \( f \). By the interpolation inequality

\[
\|f\|_{L^2} \lesssim \|f\|_{H^{-s}}^{1/2} \|\nabla f\|_{L^2}^{1/2},
\]

we deduce that

\[
\frac{d}{dt}\|f''\|_{L^2}^2 \sim \nu \|f''\|_{L^2}^{2(\alpha+1)/\alpha} \|\nabla f''\|_{L^2}^2 \gtrsim \nu \|f''\|_{H^{-s}}^2 \gtrsim \nu t^\frac{\alpha}{\alpha+2}.
\]

Assume for simplicity \( \|f_0\|_{L^2} = 1 \) and define \( \tau > 0 \) to be the first time such that \( \|f''\|_{L^2} = 1/2 \). Integrating (1.5) over \([0, \tau]\) we obtain

\[
1 \sim 2^\frac{\alpha}{\alpha+2} - 1 \gtrsim \nu \int_0^\tau t^\frac{\alpha}{\alpha+2} dt \sim \nu \frac{\tau^{1+\frac{\alpha}{\alpha+2}}}{\alpha+2} = \left(\nu^{\frac{\alpha}{\alpha+2}}\right)^{\frac{\alpha}{\alpha+2}}
\]

Namely, in order for the energy \( \|f''\|_{L^2} \) to be reduced by half by the dynamics, we need to wait for at most \( \tau \lesssim \nu^{-\alpha/(\alpha+2)} \). Iterating the argument on intervals \([n\tau, n(\tau + 1)]\) would then produce an asymptotic decay at least of the form \( \exp(-Ct^\nu\frac{\dot{\alpha}}{2}) \).

While the argument is clearly heuristic, it predicts the correct exponent \( \frac{\alpha}{\alpha+2} \) and works for any choice of the parameter \( s > 0 \) (in particular for \( s = 1/2 \)) as in Theorem 1.1 and not only for \( s = 1 \), which is the case receiving the most attention in the literature.
Unfortunately, there are only few rigorous quantitative results connecting explicitly inviscid mixing and enhanced dissipation properties (see [13] and the references therein) and they appear not to be optimal. For instance for \( s \in (0,1] \), an application of Corollary 2.3 from [13] would only predict a decay

\[
\|f_t\|_{L^2} \leq \exp(-C\nu^{q_s} t)\|f_0\|_{L^2}, \quad q_s := \frac{\alpha(1 + s)}{\alpha + s + \alpha s};
\]

in particular \( q_1 = \frac{2\alpha}{\alpha + 1} \) while \( q_{1/2} = \frac{3\alpha}{3\alpha + 1} \).

**Relation with existing literature.** Understanding the interaction between mixing and diffusion is one of the most fundamental problems in fluid mechanics, dating back to the works of Kelvin [32] and Reynolds [39].

In the pioneering work [11], such relation has been formalized mathematically by introducing the concept of relaxation enhancing flows; the result has been recently revisited in a more quantitative fashion in the works [13, 18]. The use of weak norms \( H^{-s} \) in order to quantify mixing of passive scalars was first introduced in [34].

Shear flows and circular flows in particular have been recently studied by several authors, employing a variety of technique, including stationary phase methods and hypocoercivity schemes [2, 12, 14], spectral methods [43] and stochastic analysis [15]. Roughly speaking, the main known results for (1.1) are the following:

- If \( u \in C^{n+1} \) has a finite number of critical points with maximal order \( n \), then enhanced dissipation holds with \( r(\nu) \sim \nu^{\frac{n}{n+2}}(1 + \log \nu^{-1})^{-1} \), see Theorem 1.1 in [2].
- There exist \( u \in C^\alpha, \alpha \in (0,1) \), for which enhanced dissipation holds with \( r(\nu) \sim \nu^{\frac{\alpha}{\alpha+1}} \), see Theorem 5.1 from [43].
- The above results are sharp, up to logarithmic corrections, in the sense that for \( u \in C^{n+1} \) (resp. \( u \in C^\alpha \)) the best possible rate is \( r(\nu) \sim \nu^{\frac{n}{n+2}} \) (resp. \( r(\nu) \sim \nu^{\frac{\alpha}{\alpha+1}} \)), see Theorem 4 in [15]; the proof is based on the Lagrangian Fluctuation Dissipation relation introduced in [16], [17].

Let us also mention the remarkable stable mixing estimate obtained in [12] for \( u \) satisfying Assumption (H) therein. Motivated by the above results, the authors of [10] explore the mixing and enhanced dissipation properties of rough shear flows, namely \( u \) sharply \( \alpha \)-Hölder for \( \alpha \in (0,1) \). In particular, they construct a Weierstrass-type flow \( u \) such that the following hold (see Theorem 1.1 in [10]):

1. enhanced dissipation holds with rate \( r(\nu) \sim \nu^{\frac{\alpha}{\alpha+1}} \), confirming the results from [43];
2. along suitable sequences \( t_n \to \infty \), inviscid mixing holds on \( H^1 \) with rate \( r(t) \sim t^{1/\alpha} \):
   \[
   \|e^{-t_n u\partial_x} f_0\|_{H^{-1}} \lesssim t_n^{-\frac{1}{\alpha}}\|f_0\|_{H^1}.
   \]
3. however, to the authors’ surprise, there exist other sequences \( \tilde{t}_n \to \infty \) on which inviscid mixing only holds with rate \( r(t) \sim t \), in the sense that
   \[
   \|e^{-\tilde{t}_n u\partial_x} f_0\|_{H^{-1}} \gtrsim \tilde{t}_n^{-1}\|f_0\|_{H^1}.
   \]

In particular, the inviscid mixing rate \( r(t) \sim t \) is the same attained by suitable Lipschitz functions; the authors wonder whether such a discrepancy between Points 2. and 3. is to be expected for generic flows \( u \in C^\alpha \), see the paragraph “Perspectives”, p.3 in [10].

The main aim of the present work is to give a negative answer to the above question, while letting a more natural picture emerge in the context of generic rough shear flows. Theorem 1.1
shows that for generic \( u \in B_{1,\infty}^\alpha \) (similarly for \( u \in C^\alpha \), see Remark 1.6) inviscid mixing holds on \( H^{1/2} \) with rate \( r(t) \sim t^{1/2\alpha} \), uniformly over all \( t \geq 0 \). Such a decay is also the best possible, see Theorem 1.4 below. On the other hand, Theorem 1.1 confirms the enhanced dissipation rate \( r(\nu) \sim \nu^{\alpha/(\alpha+2)} \), already identified in \([43, 10]\), as a property of generic shear flows.

We believe that the use of less standard spaces \( B_{1,\infty}^\alpha \) and mixing norms \( H^{-s} \) with \( s \neq 1 \) to be some of the main contributions of this work, compared to previous literature; they arise naturally in computations, rather than being a mathematical artifact. A complete picture is however still missing; for instance, the question whether generic \( u \in B_{1,\infty}^\alpha \) satisfy inviscid mixing on \( H^1 \) with rate \( r(t) \sim t^{1/\alpha} \) is still open and goes beyond our current methods.

**Structure of the proof.** As done frequently in the literature, in order to prove Theorem 1.1 for the PDE (1.1), we will pass to study its hypoelliptic counterpart

\[
\partial_t f + u \partial_x f = \nu \partial_y^2 f
\]

again under the assumption \( \int_T f_0(x, y) dx = 0 \) for all \( y \in \mathbb{T} \).

For \( k \in \mathbb{Z}_0 := \mathbb{Z} \setminus \{0\} \), define the Fourier transform in the \( x \)-variable as

\[
(P_k f)(y) := \int_T f(x, y) e^{-ikx} dx
\]

so that any \( f : \mathbb{T}^2 \to \mathbb{R} \) has a decomposition \( f(x, y) = \sum_k (P_k f)(y)e^{ikx} \). If \( f \) solves (1.6), then for each \( k \in \mathbb{Z}_0 \) the function \( f^k_i := P_k f_i \) solves the one dimensional complex valued PDE (harmonic oscillator)

\[
\partial_t f^k + ikuf^k = \nu \partial_y^2 f^k.
\]  

For \( k \in \mathbb{Z}_0, \nu \geq 0 \) and \( u \in L^1(\mathbb{T}) \), the PDE (1.7) has an associated semigroup on \( L^2(\mathbb{T}; \mathbb{C}) \), which we denote by \( e^{t(-iku + \nu \partial_y^2)} \); observe that the parameter \( k \), up to its sign, may be removed by the rescaling \( \tilde{t} = t|k|, \tilde{\nu} = \nu/|k| \). In this way the study of asymptotic behavior of \( f^k \) may be reduced to that of \( f^\pm \), which motivates the following definitions.

Note that whenever we refer to a rate \( r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \), we always assume it to be a continuous, increasing function.

**Definition 1.2.** A velocity field \( u \in L^1(\mathbb{T}) \) is said to be mixing on the scale \( H^s(\mathbb{T}; \mathbb{C}) \), \( s \geq 0 \), with rate \( r_{s, \text{mix}} \), if there exist a constant \( C > 0 \) such that

\[
\|e^{-iku}\|_{H^s \to H^{-s}} \leq C \frac{1}{r_{s, \text{mix}}(t|k|)} \quad \forall \ k \in \mathbb{Z}_0, \ t \geq 1.
\]  

**Definition 1.3.** A velocity field \( u \in L^1(\mathbb{T}) \) is said to be diffusion enhancing on \( L^2(\mathbb{T}; \mathbb{C}) \) with rate \( r_{\text{diff}} \) if there exists a constant \( C > 0 \) such that

\[
\|e^{(-iku + \nu \partial_y^2)}\|_{L^2 \to L^2} \leq C \exp \left( -r_{\text{diff}} \left( \frac{\nu}{|k|} \right) |k| t \right) \quad \forall \ k \in \mathbb{Z}_0, \nu \in (0, 1], \ t \geq 1.
\]  

The following theorems, which are the main results of the paper, provide sharp inviscid mixing and enhanced diffusion statements for generic shear flows. In particular, they describe precisely the behavior of solutions to (1.1) at each Fourier level \( P_k \).

**Theorem 1.4** (Inviscid case \( \nu = 0 \)). Let \( \alpha \in (0, 1) \).

a) **Lower bound.** Suppose that \( u \in B_{1,\infty}^\alpha(\mathbb{T}) \) is mixing on the scale \( H^{1/2}(\mathbb{T}; \mathbb{C}) \) with rate \( r_{1/2, \text{mix}} \), in the sense of Definition 1.2; then necessarily \( r_{1/2, \text{mix}}(t) \lesssim t^{1/\alpha} \).

b) Upper bound. Almost every \( u \in B^\alpha_{1,\infty}(\mathbb{T}) \) satisfying the following property: for any \( \tilde{\alpha} > \alpha \), \( u \) is mixing on the scale \( H^{1/2}(\mathbb{T}; \mathbb{C}) \) with rate \( r_{1/2, \text{mix}}(t) \gtrsim t^{\tilde{\alpha}}/t^{\tilde{\alpha}/2} \).

**Theorem 1.5** (Dissipative case \( \nu > 0 \)). Let \( \alpha \in (0, 1) \).

a) Lower bound. Suppose that \( u \in B^\alpha_{1,\infty}(\mathbb{T}) \) is diffusion enhancing with rate \( r_{\text{diff}} \), in the sense of Definition 1.3; then necessarily \( r_{\text{diff}}(\nu) \lesssim \nu^{\alpha/2} \).

b) Upper bound. Almost every \( u \in B^\alpha_{1,\infty}(\mathbb{T}) \) satisfies the following property: for any \( \tilde{\alpha} > \alpha \), \( u \) is diffusion enhancing with rate \( r_{\text{diff}}(\nu) \gtrsim \nu^{\tilde{\alpha}/(\tilde{\alpha}+2)} \).

Theorems 1.4 and 1.5 will be proven respectively in Sections 3 and 4, which are structured in a very similar way. Roughly speaking, the strategy we adopt in proving upper and lower bounds may be summarized in three main steps:

1. In both cases, the lower bound follows from estimates which explicitly employ the regularity assumption \( u \in B^\alpha_{1,\infty} \); in the case \( \nu > 0 \), we need to preliminary establish a Lagrangian Fluctuation-Dissipation relation for the PDE (1.7) (see Proposition 4.2) similarly in spirit to what was done in [15].

2. The upper bound is satisfied by any \( u \) enjoying a suitable analytic property, which encodes its irregularity. It turns out that the right properties are given respectively by \( \rho \)-irregularity (see Definition 3.4) for \( \nu = 0 \) and by Wei’s irregularity condition (see Definition 4.4) for \( \nu > 0 \). A shear flow \( u \) satisfying any of such properties necessarily enjoys only limited regularity in the scales \( B^\alpha_{p,q} \) (see Proposition 3.8 and Lemma 4.8), confirming that these are the correct spaces to work with.

3. Finally, we show that a.e. \( u \in B^\alpha_{1,\infty} \) is \( \rho \)-irregular (resp. satisfies Wei’s condition), see Section 3.3 (resp. Section 4.4). This is achieved by probabilistic methods, using the law of fractional Brownian motions (see Section 2.2 for details) to construct a measure witnessing the prevalence of such properties.

**Remark 1.6.** Let us stress that points a) of Theorems 1.4-1.5 hold for all \( u \in B^\alpha_{1,\infty} \), not only generic elements. Since \( \mathbb{T} \) is finite, we have the embeddings \( B^\alpha_{p,q} \hookrightarrow B^\alpha_{1,\infty} \) for any \( p,q \in [1, \infty] \), thus the lower bound is true for all \( u \in B^\alpha_{p,q} \) as well. On the other hand, the proofs of points b) of Theorems 1.4-1.5 can be easily readapted to provide the same statements for almost every \( u \in B^\alpha_{p,q} \) for any choice of \( p,q \in [1, \infty] \).

In particular, one could always work with the spaces \( C^\alpha = B^\alpha_{\infty,\infty} \) if desired. There are however several reasons for working with \( B^\alpha_{1,\infty} \) or more generally \( B^\alpha_{p,q} \) instead of \( C^\alpha \).

Mathematically, such spaces include genuinely discontinuous functions, as well as (possibly continuous) functions of finite \( p \)-variation for any \( p \in [1, \infty] \); it holds

\[
B^{1/p}_{p,1} \hookrightarrow V^p_e \hookrightarrow V^p \hookrightarrow B^{1/p}_{p,\infty},
\]

see Proposition 4.3 from [35], Proposition 2.3 from [23] for more details.

Physically, a simple way to explain singularities in fully developed turbulence is by means of structure functions (see e.g. [21]), which are closely related to the finite difference characterization of Besov spaces \( B^\alpha_{p,\infty} \). Turbulence is also believed to be closely connected to multifractality (again we refer to the appendix of [21]), a feature which is absent from generic \( u \in C^\alpha \) (which are monofractal) but instead manifested by almost every \( u \in B^\alpha_{p,q} \), see [31, 20, 19].

Our results show that the only relevant parameter in understanding mixing and enhanced dissipation rates for a.e. \( u \in B^\alpha_{p,q} \) is \( \alpha \in (0, 1) \), regardless of the values of \( p,q \); thus there is no apparent connection between mixing and multifractal features of \( u \), at least in the setting of shear flows.
Structure of the paper. In Section 2 we shortly recall some of the main tools we will be working with, specifically the theory of prevalence and a relevant class of Gaussian processes, which includes fractional Brownian motion.

Sections 3 and 4 contain the proofs of Theorems 1.4 and 1.5 and are designed in a similar manner: in both cases we will first prove the lower bound, then introduce the concept of \( \rho \)-irregularity (resp. Wei’s condition) and explain its connection to the upper bound, as well as to the irregularity of \( u \); finally, we show by probabilistic means that a.e. \( u \in B^\rho_{\infty, \infty} \) satisfies such property. The end of Section 4 also contains the proof of Theorem 1.1.

In Appendix A we collect some well known results on Besov spaces, while Appendix B contains a technical extension of the results from [43] needed to work in our setting.

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Notations and conventions. We will use the notation \( a \lesssim b \) to mean that there exists a constant \( c > 0 \) such that \( a \leq cb \); \( a \lesssim_x b \) highlights the dependence \( c = c(x) \). The notation \( a \sim b \) stands for \( a \lesssim b \) and \( b \lesssim a \), similarly for \( a \sim_x b \).

Whenever needed, we will identify the \( d \)-dimensional torus \( \mathbb{T}^d \) with either \( [0, 2\pi]^d \) or \( [-\pi, \pi]^d \) with periodic boundary condition, and functions \( \varphi : \mathbb{T}^d \to \mathbb{R} \) with \( 2\pi \)-periodic functions defined on \( \mathbb{R}^d \). We will use \( d_{T^d}(x, y) \) to denote the canonical distance on the flat torus \( \mathbb{T}^d \), namely \( d_{T^d}(x, y) = \inf_{k \in \mathbb{Z}^d} |x + 2\pi k - y| \), where \( | \cdot | \) denotes the Euclidean distance on \( \mathbb{R}^d \). With a slight abuse, we will keep writing \( | x | \) for \( x \in \mathbb{T}^d \) to denote \( d_{T^d}(x, 0) \).

\( L^p(\mathbb{T}^d) \) denotes classical Lebesgue spaces, \( C^\alpha(\mathbb{T}^d) \) Hölder spaces and \( H^s(\mathbb{T}^d) = W^{s, 2}(\mathbb{T}^d) \) fractional Sobolev spaces. \( B^\alpha_{p,q}(\mathbb{T}^d) \) denotes Besov spaces on \( \mathbb{T}^d \); we refer to Appendix A for a detailed discussion of their definition and main properties. Here let us shortly recall, that for \( \alpha \in (0, 1) \) and \( p \in [1, \infty) \), \( f \in B^\alpha_{p,q}(\mathbb{T}^d) \) if and only if \( f \in L^p(\mathbb{T}^d) \) and it has finite Gagliardo-Niremberd type seminorm

\[
\left\| f \right\|_{B^\alpha_{p,q}(\mathbb{T}^d)} := \sup_{x \neq y \in \mathbb{T}^d} \frac{\| f(\cdot + x) - f(\cdot + y) \|_{L^p}}{d_{T^d}(x, y)^s}, \tag{1.10}
\]

see equations (A.1)-(A.2) for more details. Similarly, \( B^\alpha_{p,q}(0, \pi) \) denotes Besov spaces on \( [0, \pi] \).

Given \( p \in [1, \infty) \) and a compact interval \( I \subset \mathbb{R} \), we denote by \( V^p = V^p(I) \) the Banach space of functions \( f : I \to \mathbb{R} \) of finite \( p \)-variation, with norm

\[
\| f \|_{V^p} = \| f(0) \| + \sup_{\pi \in \Pi(I)} \left( \sum_{[t_i, t_{i+1}] \in \pi} | f(t_{i+1}) - f(t_i) |^p \right)^{\frac{1}{p}},
\]

where the supremum is taken over the set \( \Pi(I) \) of all finite partition of \( I \), identified with sequences \( \{ t_i \}_{i=0}^n \) such that \( \min I = t_0 < t_1 < \cdots < t_n = \max I \). \( V^p_I \) stands for the closed subspace of \( V^p \) of continuous functions. \( V^p(T) \) is defined by identifying \( T \) with the interval \( [-\pi, \pi] \).

Whenever a stochastic process \( X = (X_t)_{t \geq 0} \) is considered, if not specified we tacitly assume the existence of an abstract underlying filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \), such that the \( \sigma \)-algebra \( \mathcal{F} \) and the filtration \( (\mathcal{F}_t)_{t \geq 0} \) satisfy the usual assumptions and \( (X_t)_{t \geq 0} \) is adapted to \( (\mathcal{F}_t)_{t \geq 0} \). Whenever we say that \( (\mathcal{F}_t)_{t \geq 0} \) is the natural filtration generated by \( X \), then it is
tacitly implied that it is actually its right continuous, normal augmentation wrt. \( P \). We denote by \( E \) integration (equiv. expectation) wrt. the probability \( \mathbb{P} \).

2. Preliminaries

2.1. Prevalence. The theory of prevalence has been developed by Hunt, Sauer and Yorke in [29] in order to provide a measure theoretic notion of genericity in infinite dimensional spaces. It is a natural generalization of the concept of “full Lebesgue measure sets” from the finite dimensional setting. We follow here the exposition given in [29], although for our purposes it will be enough to work with Banach spaces \( E \).

Definition 2.1. Let \( E \) be a complete metric vector space. A Borel set \( A \subset E \) is said to be shy if there exists a measure \( \mu \) such that:

i. There exists a compact set \( K \subset E \) such that \( 0 < \mu(K) < \infty \).

ii. For every \( v \in E \), \( \mu(v + A) = 0 \).

In this case, the measure \( \mu \) is said to be transverse to \( A \). More generally, a subset of \( E \) is shy if it is contained in a shy Borel set. The complement of a shy set is called a prevalent set.

Sometimes it is said informally that the measure \( \mu \) “witnesses” the prevalence of \( A^c \).

It follows immediately from Point i. of Definition 2.1 that, if such a measure \( \mu \) exists, then it can be assumed to be a compactly supported probability measure on \( E \). On the other hand, in order to exhibit the existence of \( \mu \) satisfying Points. i.-ii., it suffices to find another tight probability measure \( \tilde{\mu} \) only satisfying requirement ii. If \( E \) is separable, then any probability measure on \( E \) is tight and therefore Point i. is automatically satisfied.

The following properties hold for prevalence (all proofs can be found in [29]):

1. If \( E \) is finite dimensional, then a set \( A \) is shy if and only if it has zero Lebesgue measure.
2. If \( A \) is shy, then so is \( v + A \) for any \( v \in E \).
3. Prevalent sets are dense.
4. If \( \dim(E) = +\infty \), then compact subsets of \( E \) are shy.
5. Countable union of shy sets is shy; conversely, countable intersection of prevalent sets is prevalent.

From now, whenever we say that a statement holds for a.e. \( v \in E \), we mean that the set of elements of \( E \) for which the statement holds is a prevalent set. Property 1. states that this convention is consistent with the finite dimensional case.

In the context of a function space \( E \), it is natural to consider as probability measure the law induced by an \( E \)-valued random variable. Namely, given stochastic process \( W \) defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) taking values in a separable Banach space \( E \), in order to show that a property \( P \) holds for a.e. \( f \in E \), it suffices to show that

\[
\mathbb{P} (f + W \text{ satisfies property } P) = 1, \quad \forall f \in E.
\]

(2.1)

Clearly, we are assuming that the set \( A = \{ w \in E : w \text{ satisfies property } P \} \) is Borel measurable; if \( E \) is not separable, we need to additionally require that the law of \( W \) is tight, so as to satisfy Point i. of Definition 2.1.

As a consequence of properties 4. and 5., the set of all possible realizations of a probability measure \( \mu \) on a separable infinite dimensional Banach space is a shy set, as it is contained in a countable union of compact sets (this is true more in general for any tight measure on a Banach space). This fact highlights the difference between a statement of the form “Property \( P \) holds for a.e. \( f \) (in the sense of prevalence)” and “Property \( P \) holds for \( \mu \text{-a.e. } f \)”}; indeed, the second
statement doesn’t provide any information regarding whether the property might be prevalent or not. Intuitively, the elements satisfying a prevalence statement are “many more” than just the realizations of a given measure $\mu$.

2.2. A useful class of Gaussian transverse measures. From now on, given an interval $[0,T]$ and a probability measure $\mu$ on $C([0,T])$, we will denote by $(X_t)_{t \in [0,T]}$ the associated canonical process, which is given by $X_t(\omega) = \omega(t)$ for $\omega \in C([0,T])$, and by $\mathcal{F}_t = \sigma(\{X_s, s \leq t\})$ the associated natural filtration.

A key point of the present work is to verify that suitable properties $\mathcal{P}$ are satisfied by a.e. $f \in E$ for suitable $E = B_{T,\infty}$. The discussion from Section 2.1, in particular equation (2.1), suggests to look for classes of processes which are stable under deterministic additive perturbations and in [24] we identified the local nondeterministic (LND) Gaussian processes as a useful class in the study of prevalence in function spaces. We recall in the next definition that a real valued process $X$ is Gaussian if for any $s,t \in [0,T]$, $(X_t,\ldots,X_n) \in \mathbb{R}^n$-valued Gaussian variable.

**Definition 2.2.** Given $\beta > 0$, a real valued Gaussian process $X$ is strongly locally non-deterministic with parameter $\beta$, $\beta$-SLND for short, if there exists a constant $C_X$ such that

$$\text{Var}(X_t|\mathcal{F}_s) \geq C_X|t-s|^{2\beta}$$

(2.2)

uniformly over $s,t \in [0,T]$ with $s < t$.

In (2.2) above, $\text{Var}(\cdot|\mathcal{F}_s)$ denotes the conditional variance; equivalently, Definition 2.2 amount to the condition that, for any $s < t$, there is a decomposition $X_t = X_{s,t}^{(1)} + X_{s,t}^{(2)}$ where $X_{s,t}^{(1)}$ is Gaussian and adapted to $\mathcal{F}_s$ while $X_{s,t}^{(2)}$ is Gaussian, independent of $\mathcal{F}_s$, with variance $\text{Var}(X_{s,t}^{(2)}) \geq C_X|t-s|^{2\beta}$. The increments of the process $X$ are therefore “intrinsically chaotic” in a way that can be quantified precisely by the parameter $\beta$. Let us shortly mention that Definition 2.2 is not the only notion of LND in the literature and there are several non-equivalent ones; see [44] for a review.

The importance of the $\beta$-SLND property comes from the following elementary fact, which can be readily checked from the definition (see also Remark 26 from [24]); in the statement, $f : [0,T] \rightarrow \mathbb{R}$ can be naturally unbounded.

**Lemma 2.3.** Let $(X_t)_{t \in [0,T]}$ be a $\beta$-SLND Gaussian process and $f : [0,T] \rightarrow \mathbb{R}$ be a measurable function; then $X + f$ is also a $\beta$-SLND Gaussian process.

Lemma 2.3 will be our main leverage to establish prevalence statements, as it reduces the difficulty to that of verifying that any $\beta$-SLND Gaussian process satisfies $\mu$-a.s. the property $\mathcal{P}$ of interest; this will indeed be the strategy implemented in Sections 3.3 and 4.4 respectively.

In this sense, we could work with any possible Gaussian law $\mu$ whose associated canonical process is $\beta$-SLND, without further specification. To keep things less abstract, we will however use a well-known one-parameter family from this class, which are the laws $\{\mu^H, H \in (0,1)\}$ of fractional Brownian motion (fBm) of parameter $H \in (0,1)$. The material recalled next is mostly classical and can be found in the monograph [37].

The law of fBm of Hurst parameter $H \in (0,1)$ is defined as the unique Gaussian measure $\mu^H$ on $\Omega = C([0,T])$ such that

$$\int_\Omega X_t(\omega)\mu^H(d\omega) = 0,$$

$$\int_\Omega X_t(\omega)X_s(\omega)\mu^H(d\omega) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$
For $H = 1/2$, the law of fBm corresponds to the classical Wiener measure; instead for $H \neq 1/2$, the associated canonical process $X$ is not a semimartingale nor a Markov process.

The support of $\mu^H$ in terms of Besov spaces is well understood, with sharp results going back to [9] (see also [42] for a modern proof which extends to the vector valued case): it holds
\[
\mu^H(C^{H-\varepsilon}) = 1 \quad \forall \varepsilon > 0, \quad \mu^H(B^H_{p,\infty}) = 1 \quad \forall p \in [1, \infty),
\]
while
\[
\mu^H(C^H) = 0, \quad \mu^H(B^H_{p,q}) = 0 \quad \forall p, q \in [1, \infty).
\]
In particular fBm trajectories are sharply not $H$-Hölder continuous, but by Ascoli–Arzelà $\mu^H$ is a tight probability measure on $B^H_{p,\infty}$ for any $\varepsilon > 0$ and any $p \in [1, \infty]$. As promised, this class of Gaussian measures does satisfy the LND property.

**Lemma 2.4.** Let $X$ be the canonical process associated to $\mu^H$, $H \in (0, 1)$. Then $X$ is $H$-SLND; moreover, the Gaussian process $Y_t := \int_0^T X_s ds$ is $(1 + H)$-SLND.

**Proof.** The first claim is classical and can be found in the review [44] and the references therein; alternative, a self-contained proof, based on the Mandelbrot–Van Ness representation of fBm, is given in Section 2.4 from [24]; the same representation can be used to establish the second half of the claim involving the process $Y$, see Example iv. from Section 4.2 in [24]. \qed

Among the reasons for using $\mu^H$, instead of just any Gaussian measure satisfying a suitable LND condition, let us finally mention that this process can be simulated numerically in a very efficient way.

3. **Inviscid mixing**

This section contains the proof of Theorem 1.4, which we split in several steps.

Recall the setting: in order to study the transport equation $\partial_t f + u \partial_x f = 0$, we pass to Fourier modes $f^k(y) = (P_k f_t)(y)$, solving $\partial_t f^k + iku f^k = 0$; namely $f^k_t(y) = e^{-ikty}f^k_0(y)$.

It is then natural to take a slightly more general perspective and study maps of the form $y \mapsto e^{i\xi u(y)}g(y)$ with $\xi \in \mathbb{R}$, $g \in H^s(\mathbb{T})$.

3.1. **Lower bounds in terms of regularity.** We show here that the regularity of $u$, measured in the Besov–Nikolskii scale $B^{\alpha}_{1,\infty}$, necessarily implies a lower bound on the decay of solutions in the $H^{-1/2}$-norm. The proof is partly inspired by that of Proposition 3.2 from [10].

**Lemma 3.1.** Let $u \in B^{\alpha}_{1,\infty}(\mathbb{T})$ for some $\alpha \in (0, 1)$. Then for any $g \in H^1(\mathbb{T})$ there exists a constant $C = C(\alpha, g)$ such that
\[
\|e^{i\xi u}g\|_{H^{-1/2}} \geq C(1 + \|u\|_{B^{\alpha}_{1,\infty}})^{-\frac{1}{4\alpha}} |\xi|^{-\frac{1}{2\alpha}} \quad \forall |\xi| \geq 1. \quad (3.1)
\]

**Proof.** Fix $\xi$ with $|\xi| \geq 1$ and set $\tilde{g} := e^{i\xi u}g$; we claim that $\tilde{g} \in B^{\alpha/2}_{2,\infty}$. By Sobolev and Besov embeddings, $g \in L^\infty \cap B^\alpha_{2,\infty}$; $e^{i\xi u} \in L^\infty$, so it’s enough to show that $e^{i\xi u} \in B^{\alpha/2}_{2,\infty}$. By the basic estimate $|e^a - e^b| \leq \sqrt{2}|a - b|^{1/2}$, it holds
\[
\left\| e^{i\xi u(\cdot + y)} - e^{i\xi u(\cdot + \tilde{y})} \right\|_{L^2} \lesssim |\xi|^{1/2}\|u(\cdot + y) - u(\cdot + \tilde{y})\|_{L^1}^{1/2} \lesssim |\xi|^{1/2}\|u\|_{B^{\alpha}_{1,\infty}}^{1/2} d_T(y, \tilde{y})^{\alpha/2}.
\]
By the equivalent characterization of Besov–Nikolskii spaces, this implies
\[ ||e^{i\xi u}||_{B^{s/2}_{2,\infty}} \lesssim 1 + |\xi|^{1/2}||u||_{B^{s/2}_{1,\infty}}^{1/2} \lesssim (1 + ||u||_{B^{s}_{1,\infty}})^{1/2}|\xi|^{1/2} \]
and so by Proposition A.4 in Appendix A we conclude that \( \bar{g} \in B^{s/2}_{2,\infty} \) with
\[ ||\bar{g}||_{B^{s/2}_{2,\infty}} \lesssim ||g||_{H^1} (1 + ||u||_{B^{s}_{1,\infty}})^{1/2} |\xi|^{1/2} \]  
(3.2)
Clearly \( ||\bar{g}||_{L^2} = ||g||_{L^2} \). Using the interpolation inequality from Corollary A.6 in Appendix A (for the choice \( s_1 = 1/2, s_2 = \alpha/2 \)) we obtain
\[ ||g||_{L^2} \leq ||\bar{g}||_{L^2} \lesssim ||\bar{g}||_{H^{-1/2}} ||\bar{g}||_{B^{s/2}_{2,\infty}} \]  
(3.3)
Rearranging now the terms in (3.3) and applying the estimate (3.2) we find
\[ ||g||_{H^{-1/2}} \gtrsim ||\bar{g}||_{B^{s/2}_{2,\infty}} ||g||_{L^2}^{1+\alpha} \gtrsim ||g||_{L^2}^{1+\alpha/2} ||\bar{g}||_{H^{1/2}} (1 + ||u||_{B^{s}_{1,\infty}})^{-\frac{1}{2\alpha}} |\xi|^{-\frac{1}{2\alpha}} \]  
(3.4)
where the hidden constant in (3.4) only depends on \( \alpha \). Using the definition of \( \bar{g} \) and relabelling the constant to include the \( g \)-dependent terms yields the conclusion. □

**Corollary 3.2.** Let \( u \in B^{s}_{1,\infty}(\mathbb{T}) \) be mixing on \( H^{1/2}(\mathbb{T}) \) with rate \( r_{1/2-mix} \), in the sense of Definition 1.2. Then there exists a constant \( C = C(\alpha, u) \) such that
\[ r_{1/2-mix}(t) \leq Ct \frac{1}{\pi} . \]

**Proof.** Consider \( g(y) = e^{iy} \), so that \( ||g||_{H^{1/2}} \sim ||g||_{H^1} \sim 1 \); then by Definition 1.2 applied for the choice \( k = 1 \) and Lemma 3.1 for \( \xi = -t \), it holds
\[ \frac{1}{r(t)} \geq ||e^{-it\varphi}||_{H^{1/2}} \sim ||g||_{H^{-1/2}} \gtrsim ||e^{-it\varphi}||_{H^{-1/2}} (1 + ||u||_{B^{s}_{1,\infty}})^{-\frac{1}{2\alpha}} t^{-\frac{1}{2\alpha}} ; \]
up to relabelling constants, this yields the conclusion. □

**Remark 3.3.** In fact, the statement of Lemma 3.1 can be generalized as follows. For \( \alpha \in (0, 1) \), \( u \in B^{s}_{1,\infty}(\mathbb{T}) \), \( g \in H^{1}(\mathbb{T}) \) and any \( s > 0 \) there exists a constant \( C(\alpha, g, s) \) such that
\[ ||e^{i\xi u}||_{H^{-s}} \gtrsim C(1 + ||u||_{B^{s}_{1,\infty}})^{-\frac{s}{2\alpha}} |\xi|^{-\frac{s}{2\alpha}} \forall |\xi| \geq 1. \]
Then arguing as in Corollary 3.2 by choosing \( g(y) = e^{iy} \), one can conclude that the best possible rate for inviscid mixing on the scale \( H^{s}(\mathbb{T}) \) is \( r_{s-mix}(t) \sim t^{s/\alpha} \). Taking \( s = 1 \) provides the rate \( t^{1/\alpha} \), which is in line with Proposition 3.2 from [10].

### 3.2. Upper bounds in terms of \( \rho \)-irregularity

The concept of \( \rho \)-irregularity was first introduced in [6] in the study of regularization by noise phenomena. Its applications to PDEs have been subsequently explored in [7, 8, 24, 5].

**Definition 3.4.** Let \( \gamma \in [0, 1) \), \( \rho > 0 \); a measurable map \( u : [0, \pi] \to \mathbb{R} \) is said to be \((\gamma, \rho)\)-irregular if there exists a constant \( C > 0 \) such that
\[ \left| \int_{I} e^{i\xi u(z)} dz \right| \leq C|I|^\gamma |\xi|^{-\rho} \forall \xi \in \mathbb{R}, I \subset [0, \pi] \]  
(3.5)
where \( I \) stands for a subinterval of \([0, \pi]\) and \( |I| \) denotes its length. A similar definition holds for \( u : \mathbb{R} \to \mathbb{R} \); a map \( u : \mathbb{T} \to \mathbb{R} \) is said to be \((\gamma, \rho)\)-irregular if its \( 2\pi \)-periodic extension \( u : \mathbb{R} \to \mathbb{R} \) has this property. We say that \( u \) is \( \rho \)-irregular for short if there exists \( \gamma > 1/2 \) such that it is \((\gamma, \rho)\)-irregular.
Proof. For \(\gamma, \rho\) by periodicity it can be identified with a function on \([-\pi, \pi]\). Hence for \(u : [0, \pi] \to \mathbb{R}\), by (3.5) it holds

\[
\|\Phi_{\gamma}^u\|_{\gamma, \rho} = \sup_{\xi \in \mathbb{R}, 0 \leq s \leq t \leq \pi} \frac{|\Phi_{\gamma}^u(\xi) - \Phi_{\rho}^u(\xi)|}{|t - s|^{\gamma} |\xi|^\rho}.
\]

The property of \(\rho\)-irregularity may be rephrased in the following form, more suited for our purposes.

**Lemma 3.5.** Let \(u : \mathbb{T} \to \mathbb{R}\) be \((\gamma, \rho)\)-irregular, then

\[
\|e^{i\xi u}\|_{B_{\infty, \infty}^{1,-1}} \lesssim \|\Phi_{\gamma}^u\|_{\gamma, \rho}^{-\rho} \quad \forall \xi \in \mathbb{R}.
\]

**Proof.** For \(\tilde{y} \in [-\pi, \pi]\) and \(\xi \in \mathbb{R}\), define the function

\[
v_\xi(\tilde{y}) = \int_{-\pi}^{\tilde{y}} e^{i\xi u(y)}dy - \left(\frac{\tilde{y} + \pi}{2\pi}\right) \int_{-\pi}^{\pi} e^{i\xi u(y)}dy,
\]

by periodicity it can be identified with a function on \(\mathbb{T}\). Then by definition of \((\gamma, \rho)\)-irregularity it holds \(\|v_\xi\|_{C^\gamma} \lesssim \|\Phi_{\gamma}^u\|_{\gamma, \rho}^{-\rho}\) and so by Proposition A.2 we deduce that

\[
\|e^{i\xi u}\|_{B_{\infty, \infty}^{1,-1}} = \left\|v_\xi' + \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi u(y)}dy\right\|_{B_{\infty, \infty}^{1,-1}} \lesssim \|\Phi_{\gamma}^u\|_{\gamma, \rho}^{-\rho}.
\]

The relation between \(\rho\)-irregularity and inviscid mixing comes from the next result.

**Lemma 3.6.** Let \(u : \mathbb{T} \to \mathbb{R}\) be \((\gamma, \rho)\)-irregular for some \(\gamma > 1/2\). Then there exists a constant \(C = C(\gamma)\) such that

\[
\|e^{i\xi g}\|_{H_{-1/2}} \leq C \|\Phi_{\gamma}^u\|_{\gamma, \rho}^{-\rho} \|g\|_{H_{1/2}} \quad \forall \xi \neq 0, g \in H_{1/2}.
\]

As a consequence, \(u\) is mixing on the scale \(H_{1/2}\) with rate \(r_{1/2, \text{mix}}(t) = t^\rho\), in the sense of Definition 1.2.

**Proof.** The proof of the estimate (3.6) relies on several properties of Besov spaces, for which we refer the reader to Appendix A. By assumption \(\gamma + 1/2 > 1\), thus we can apply Proposition A.3 (for the choice \(s_1 = \gamma - 1, s_2 = 1/2, p_1 = q = \infty, p_2 = p = 2\)) and Lemma 3.5 to obtain

\[
\|e^{i\xi u}\|_{B_{2,\infty}^{1,-1}} \lesssim \|\Phi_{\gamma}^u\|_{\gamma, \rho}^{-\rho} \|g\|_{B_{2,\infty}^{1/2}} \lesssim \|\Phi_{\gamma}^u\|_{\gamma, \rho}^{-\rho} \|g\|_{H_{1/2}}.
\]

Again by the hypothesis \(\gamma - 1 > -1/2\) and so by Besov embeddings \(B_{2,\infty}^{1/2} \hookrightarrow H^{-1/2}\), yielding the first claim. Applying estimate (3.6) for \(k \in \mathbb{Z}_0\), \(\xi = -tk\) gives

\[
\|e^{-itku}\|_{H_{1/2} \hookrightarrow H^{-1/2}} \leq C \frac{\|\Phi_{\gamma}^u\|_{\gamma, \rho}}{(t|k|)^\rho}
\]

and thus the conclusion. \(\square\)
The property of $\rho$-irregularity implies roughness of $u$, as the name suggests. To quantify this precisely, we recall the concept of Hölder roughness, as presented in [22].

**Definition 3.7.** A measurable map $u : \mathbb{T} \to \mathbb{R}$ is said to be $\alpha$-Hölder rough if there exists $L = L_\alpha(u)$ such that: for any $\delta > 0$ and any $\bar{y} \in \mathbb{T}$, there exists $z \in \mathbb{T}$ satisfying
\[
  d_T(\bar{y}, z) \leq \delta \quad \text{and} \quad |u(\bar{y}) - u(z)| \geq L_\alpha(u)\delta^\alpha.
\]

The optimal constant $L_\alpha(u)$ is called the modulus of $\alpha$-Hölder roughness of $u$.

Definition 3.7 is equivalent to requiring
\[
  L_\alpha(u) = \inf_{\bar{y} \in \mathbb{T}, \delta > 0} \sup_{z \in B_\delta(\bar{y})} \frac{|u(z) - u(\bar{y})|}{\delta^\alpha} > 0. \tag{3.7}
\]

A detailed study of analytic properties of $\rho$-irregular paths was carried out in Section 5 of [24]; in particular, there exists a critical parameter $\alpha^*$, associated to the pair $(\gamma, \rho)$, linked to the (ir)regularity of $u$ in Hölder and Besov–Nikolskii scales.

**Proposition 3.8.** Let $u : \mathbb{T} \to \mathbb{R}$ be $(\gamma, \rho)$-irregular and define $\alpha^* := (1 - \gamma)/\rho$. Then:

a) $u$ is $\alpha$-Hölder rough for any $\alpha > \alpha^*$ with $L_\alpha(u) = +\infty$.

b) $u$ has infinite $p$-variation on any subinterval $I \subset \mathbb{T}$ and for any $p > 1/\alpha^*$.

c) $u$ does not belong to $B^\alpha_{1, \infty}$ for any $\alpha > \alpha^*$.

**Proof.** For functions $u : [0, T] \to \mathbb{R}$, points a) and b) are proved in [24], cf. Corollary 65 and Corollary 68 therein; we recall here shortly the idea of proof.

Going through the proof of Theorem 63 from [24], one can establish the (much stronger) fact that, if $u$ is $(\gamma, \rho)$-irregular, then for any $\tilde{\alpha} > \alpha^*$ it holds
\[
  \lim_{\varepsilon \to 0^+} \inf_{y \in (0, T)} \varepsilon^{-1} \mathcal{L}(h \in (0, \varepsilon) : |u(y+h) - u(y)| \geq \varepsilon^{\tilde{\alpha}}) = 1, \tag{3.8}
\]

where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{R}$. In particular, there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, it must hold
\[
  \mathcal{L}(h \in (0, \varepsilon) : |u(y+h) - u(y)| \geq \varepsilon^{\tilde{\alpha}}) \geq \varepsilon/2 > 0 \quad \forall y \in (0, T);
\]

therefore for any $y \in (0, T)$ we can find infinitely many, arbitrarily small $h$ such that $|u(y+h) - u(y)| \geq h^{\tilde{\alpha}}$; playing with the arbitrariness of $\tilde{\alpha}$, one can then easily establish both properties of Hölder roughness and infinite $p$-variation.

Up to identifying $u : \mathbb{T} \to \mathbb{R}$ with a $2\pi$-periodic function, it’s easy to check that property (3.8) carries over to this setting as well, as it is only related to the local behaviour or $u$ around any fixed $y$; same goes for the proofs of points a) and b).

We now focus on establishing claim c), which is instead an original contribution of this work. Fix $\alpha > \alpha^*$ and choose $\tilde{\alpha} \in (\alpha^*, \alpha)$; by estimate (3.8) (with the infimum taken over $y \in \mathbb{T}$ instead of $(0, T)$), for all $\varepsilon > 0$ sufficiently small, it must hold
\[
  \pi \leq \int_T \varepsilon^{-1} \mathcal{L}(h \in (0, \varepsilon) : |u(y+h) - u(y)| \geq \varepsilon^{\tilde{\alpha}}) dy
\]
\[
  \leq \int_T \varepsilon^{-1-\tilde{\alpha}} \int_0^\varepsilon |u(y+h) - u(y)| dh dy
\]
\[
  = \varepsilon^{-1-\tilde{\alpha}} \int_0^\varepsilon \|u(\cdot + h) - u(\cdot)\|_{L^1} dh
\]
\[
  \leq \varepsilon^{\alpha-\tilde{\alpha}} \|u\|_{B^\alpha_{1, \infty}}.
\]
where in the second passage we used Markov’s inequality. Since $\alpha > \tilde{\alpha}$, letting $\varepsilon \to 0^+$ we can conclude that $\|u\|_{B^\alpha_{1,\infty}} = +\infty$.

\[ \square \]

**Remark 3.9.** If $u$ is $\rho$-irregular, then Proposition 3.8-c) implies that $u$ does not belong to $B^\alpha_{1,\infty}$ for any $\alpha > (2\rho)^{-1}$. Conversely, if $u \in B^\alpha_{1,\infty}$, then it can only be $\rho$-irregular for parameters $\rho$ satisfying $\rho \leq (2\alpha)^{-1}$.

### 3.3. Prevalence statements and proof of Theorem 1.4.

Given the results of Sections 3.1–3.2, it is natural to wonder whether generic elements of $B^\alpha_{1,\infty}$ are “almost as irregular as possible”, in the sense of being $\rho$-irregular for any $\rho < (2\alpha)^{-1}$; we provide here a positive answer.

In order to do so, we will first prove the statement for elements of $B^\alpha_{1,\infty}(0, \pi)$, see Theorem 3.11, and only later deduce the same property for $B^\alpha_{1,\infty}(T)$ by a “deperiodization” procedure (cf. Corollary 3.13 below).

Differently from Section 2.2, whenever dealing with a measure $\mu$ supported on $C([0, \pi])$, it will be useful to denote by $u = \{u_y\}_{y \in [0, \pi]}$ the associated canonical process; we will instead employ the letter $\varphi$ to denote deterministic functions, either defined on $[0, \pi]$ or on $T$.

Before proceeding further, we need to recall the following key result established in [24], cf. Theorem 29 therein.

**Proposition 3.10.** Let $\mu$ be a Gaussian measure on $C([0, T])$ whose canonical process $u$ is $\beta$-SLND for some $\beta > 0$. Then for any $\rho < (2\beta)^{-1}$ it holds

$$\mu^H(u \text{ is } \rho\text{-irregular}) = 1.$$

We can combine Proposition 3.10 with the invariance of the $\beta$-SLND property from Lemma 2.3 to deduce a first prevalence statement.

**Theorem 3.11.** Let $\alpha \in (0, 1)$; then a.e. $\varphi \in B^\alpha_{1,\infty}(0, \pi)$ is $\rho$-irregular for every $\rho < (2\alpha)^{-1}$.

**Proof.** Given $\rho > 0$, define the set

$$A_\rho = \{ \varphi \in B^\alpha_{1,\infty}(0, \pi) : \varphi \text{ is } \rho\text{-irregular} \};$$

it holds

$$A_\rho = \bigcup_{n,m=3}^{\infty} A_{\rho, n, m},$$

with

$$A_{\rho, n, m} := \{ \varphi \in B^\alpha_{1,\infty}(0, \pi) : \varphi \text{ is } (\gamma, \rho)\text{-irr., for } \gamma = \frac{1}{2} + \frac{1}{m}, \|\Phi^\varphi\|_{\gamma, \rho} \leq m \}.$$

The sets $A_{\rho, n, m}$ are closed in the topology of $B^\alpha_{1,\infty}(0, \pi)$ (the map $\varphi \mapsto \|\Phi^\varphi\|_{\gamma, \rho}$ is lower semicontinuous in the topology of $L^1(0, \pi)$), thus $A_\rho$ is Borel measurable. If we show that $A_\rho$ is prevalent in $B^\alpha_{1,\infty}(0, \pi)$ for any $\rho < (2\alpha)^{-1}$, then the same holds for

$$A = \left\{ \varphi \in B^\alpha_{1,\infty}(0, \pi) : \varphi \text{ is } \rho\text{-irregular for every } \rho < \frac{1}{2\alpha} \right\} = \bigcap_{n=1}^{\infty} A_{\frac{1}{2n} - \frac{1}{\pi}},$$

providing the conclusion.

Now fix $\rho < (2\alpha)^{-1}$ and choose $H \in (0, 1)$ such that $H > \alpha$, $\rho < (2H)^{-1}$; denote by $\mu^H$ the law of fractional Brownian motion on $C([0, \pi])$ and by $u = \{u_y, y \in [0, \pi]\}$ the associated canonical process. Since $\mu^H$ is supported on $C^{H-\varepsilon}([0, \pi])$ for any $\varepsilon > 0$ and $H > \alpha$, it is also a tight probability measure on $B^\alpha_{1,\infty}(0, \pi)$; thus we only need to verify Property ii. from Definition 2.1, equivalently property (2.1) for $E = B^\alpha_{1,\infty}$. 


Fix \( \varphi \in B^{1,\infty}_1(0, \pi) \); by Proposition 2.4, \( u \) is a \( H \)-SLND process and so by Lemma 2.3 the same holds for \( u + \varphi \). In turn, by our choice of the parameters and Proposition 3.10, this implies that \( \varphi + u \) is \( \mu^H \)-a.s. \( \rho \)-irregular; as the argument holds for any \( \varphi \in B^{1,\infty}_1(0, \pi) \), we have shown that

\[
\mu^H(\varphi + \mathcal{A}_\rho) = 1 \quad \forall \varphi \in B^{1,\infty}_1(0, \pi),
\]

namely that \( \mu^H \) witnesses the prevalence of \( \mathcal{A}_\rho \) in \( B^{1,\infty}_1(0, \pi) \).

We pass to show how to exploit Theorem 3.11 to establish similar statement for functions defined on the torus.

We identify the torus \( T \) with the interval \( [-\pi, \pi] \), up to \( -\pi \sim \pi \); thus any measurable function \( \varphi : T \to \mathbb{R} \) can be identified with \( \varphi : [-\pi, \pi] \to \mathbb{R} \) such that \( \varphi(-\pi) = \varphi(\pi) \). Any such \( \varphi \) is in a 1-1 correspondence with a pair \((\varphi_1, \varphi_2)\) of measurable functions defined on \([0, \pi]\), given by \( \varphi_1(y) := \varphi(y), \varphi_2(y) := \varphi(-y) \); they satisfy the constraint \( \varphi_1(\pi) = \varphi_2(\pi) \). The \( \rho \)-irregularity property of the periodic function \( \varphi \) is actually equivalent to that of the aperiodic functions \( \varphi_i \).

**Lemma 3.12.** A measurable function \( \varphi : \mathbb{T} \to \mathbb{R} \) is \( (\gamma, \rho) \)-irregular if and only if the functions \( \varphi_1, \varphi_2 : [0, \pi] \to \mathbb{R} \) are so.

**Proof.** The proof is elementary. Given \( I \subset [-\pi, \pi] \), setting \( I_1 = I \cap [0, \pi], I_2 = I \cap [-\pi, 0] \) it holds \( \max\{|I_1|, |I_2|\} \leq |I| \leq 2 \max\{|I_1|, |I_2|\} \), so that

\[
\max\{\|\Phi^{\varphi_1}\|_{\gamma, \rho}, \|\Phi^{\varphi_2}\|_{\gamma, \rho}\} \leq \|\Phi^{\varphi}\|_{\gamma, \rho} \leq 2 \max\{\|\Phi^{\varphi_1}\|_{\gamma, \rho}, \|\Phi^{\varphi_2}\|_{\gamma, \rho}\}. \tag*{\square}
\]

Conversely, given a measurable \( \tilde{\varphi} : [0, \pi] \to \mathbb{R} \), we can associate it another function \( \varphi = T\tilde{\varphi} : \mathbb{T} \to \mathbb{R} \) by setting \( T\tilde{\varphi}(y) = \tilde{\varphi}(|y|) \), which corresponds to \((T\varphi_1) = (T\varphi_2) = \tilde{\varphi} \). It immediately follows from Lemma 3.12 that \( T\tilde{\varphi} \) is \( (\gamma, \rho) \)-irregular if and only if \( \tilde{\varphi} \) is so; it is also easy to check that, if \( \varphi \in B^{1,\infty}_1(0, \pi) \cap L^\infty(0, \pi) \), then \( T\varphi \in B^{1,\infty}_1(\mathbb{T}) \).

We are finally ready to prove a prevalence statement in \( B^{1,\infty}_1(\mathbb{T}) \).

**Corollary 3.13.** Let \( \alpha \in (0, 1) \), then a.e. \( \varphi \in B^{1,\infty}_1(\mathbb{T}) \) is \( \rho \)-irregular for any \( \rho < (2\alpha)^{-1} \).

**Proof.** The proof that the set

\[
\mathcal{A} := \left\{ \varphi \in B^{1,\infty}_1(\mathbb{T}) : \varphi \text{ is } \rho \text{-irregular for any } \rho < \frac{1}{2\alpha} \right\}
\]

is Borel in the topology of \( B^{1,\infty}_1(\mathbb{T}) \) is identical to that of Theorem 3.11 and thus omitted; as therein, we can introduce the sets \( \mathcal{A}_\rho \) and reduce the task to establish the prevalence of the set \( \mathcal{A}_\rho \) for any fixed \( \rho < (2\alpha)^{-1} \).

Choose \( H \in (0, 1) \) such that \( H > \alpha, \rho < (2H)^{-1} \) and denote by \( \mu^H \) the associated law of fBM; since it is supported on \( B^{1,\infty}_1(0, \pi) \cap L^\infty(0, \pi) \), we can define a new measure on \( B^{1,\infty}_1(\mathbb{T}) \) by \( \nu^H := T_2 \mu^H \), where \( (T\varphi)(y) = \varphi(|y|) \) for \( y \in [0, \pi] \) and \( T \) denotes the pushforward measure.

Recall the notation \( \varphi_1, \varphi_2 \) from Lemma 3.12: for any \( \varphi \in B^{1,\infty}_1(\mathbb{T}) \) it holds

\[
\nu^H(\varphi + \mathcal{A}) = \mu^H\left( \big\{ u \in B^{1,\infty}_1(0, \pi) : Tu + \varphi \text{ is } \rho \text{-irregular} \big\} \right)
\]

\[
= \mu^H\left( \bigcap_{i=1}^{2} \big\{ u \in B^{1,\infty}_1(0, \pi) : u + \varphi_i \text{ is } \rho \text{-irregular} \big\} \right) = 1;
\]

in the last passage we used the already established properties of the measure \( \mu^H \) from the proof of Theorem 3.11, as well as the fact that the intersection of sets of full measure is still of full measure. Overall, this shows that \( \nu^H \) witnesses the prevalence of the set \( \mathcal{A}_\rho \); the conclusion follows using the fact that countable intersection of prevalent sets is prevalent. \( \square \)
We are now ready to complete the proof of Theorem 1.4. The lower bound comes from Corollary 3.2, while the upper bound from a combination of Lemma 3.6 and Corollary 3.13.

4. Enhanced dissipation

This section contains the proof of Theorem 1.5 split in several steps. Recall the setting: we want to study the asymptotic behavior of the family of complex-valued PDEs (1.7), equivalently obtain upper and lower bounds on

$$\|e^{tL_{k,\nu}}\|_{L^2(\mathbb{T}; C) \to L^2(\mathbb{T}; C)} \quad \text{as } t \to \infty,$$

where $L_{k,\nu} := -iku + \nu \partial^2_y$.

4.1. Lower bounds in terms of regularity. We show here that if $u$ has regularity of degree $\alpha \in (0,1)$, as measured in a suitable Besov–Nikolskii scale, then the its best possible diffusion enhancing rate is $r_{\text{dif}}(\nu) \sim \nu^{\alpha/(2+\alpha)}$. The precise statement goes as follows.

**Proposition 4.1.** Let $u \in B^{\alpha,1}_{1,\infty}(\mathbb{T})$ be diffusion enhancing with rate $r_{\text{dif}}$, in the sense of Definition 1.3; then there exists a constant $C > 0$ such that

$$r_{\text{dif}}(\nu) \leq C\nu^{\alpha/(2+\alpha)}$$

for all $\nu \in (0,1]$.

In order to provide estimates for $e^{tL_{k,\nu}}$ it is convenient to study more generally the properties of solutions $g: \mathbb{T} \to \mathbb{C}$ to

$$\partial_t g + i\xi u g = \nu \partial^2_y g$$

in function of the parameters $\xi \in \mathbb{R}$, $\nu \in (0,1)$ and the shear flow $u$.

The proof of Proposition 4.1 follows a similar strategy to [15] and is based on deriving a Lagrangian Fluctuation-Dissipation relation (FDR) for the PDE (4.1), which is a result of independent interest.

**Proposition 4.2.** Let $u \in L^1(\mathbb{T})$, $g$ be a solution to (4.1) with initial data $g_0 \in L^2(\mathbb{T}; \mathbb{C})$; for any $(t,y) \in \mathbb{R}_{\geq 0} \times \mathbb{T}$, define the complex random variable

$$Z_t^y = \exp \left( -i\xi \int_0^t u(y + \sqrt{2\nu}B_s) \, ds \right) g_0 \left( y + \sqrt{2\nu}B_t \right)$$

where $B$ is a standard real-valued BM. Then we have the following Lagrangian FDR:

$$\|g_0\|_{L^2}^2 - \|g_t\|_{L^2}^2 = \int_T \text{Var}(Z_t^y) dy.$$

**Proof.** Without loss of generality, we can assume $u$ and $g_0$ to be smooth, as identity (4.2) in the general case will follow from an approximation argument (the definition of $Z_t^y$ is meaningful for any $u \in L^1(\mathbb{T})$, thanks to the properties of the local time of a Brownian motion). Let us however first show that the r.h.s. of (4.2) is a well-defined quantity, which can be estimated independently of the smoothness of $u$, $g_0$. Indeed, for any $t \geq 0$ it holds

$$\int_T \mathbb{E}[|Z_t^y|^2] dy = \int_T \mathbb{E}[|g_0|^2(y + \sqrt{2\nu}B_t)] dy = \mathbb{E}\left[ \int_T |g_0|^2(y + \sqrt{2\nu}B_t) dy \right] = \|g_0\|_{L^2}^2,$$

where in the last step we used the invariance of the $L^2$-norm of $g_0$ under (random) translations; the pointwise bound $\text{Var}(Z_t^y) \leq \mathbb{E}[|Z_t^y|^2]$ then readily yields an estimate for the r.h.s. of (4.2).
In our setting, we take $C = \nu \partial_y^2 |g|^2 - 2\nu |\partial_y g|^2$. Let $\tilde{h}$ where $\tilde{h} \mid_{\partial (T, \tilde{B})}$

**Proof.**

Recall the elementary identity $2 \text{Var}(X) = E[|X - \tilde{X}|^2]$ for $\tilde{X}$ being an i.i.d. copy of $X$. In our setting, we take

$$\tilde{Z}_t^\nu = \exp \left( -i \xi \int_0^t u \left( y + \sqrt{\nu} \tilde{B}_s \right) \, ds \right) g_0 \left( y + \sqrt{\nu} \tilde{B}_t \right)$$

where $\tilde{B}$ is another BM independent of $B$. Therefore

$$\|g_0\|^2_{L^2} - \|g_t\|^2_{L^2} \leq \frac{1}{2} \int_T E[|Z_t^\nu - \tilde{Z}_t^\nu|^2] \, dy$$

and more generally, the map $(t, y) \mapsto |g|^2(t, y)$ satisfies

$$\frac{d}{dt} \int_T [h \mid_{\partial (T, \tilde{B})}]^2 = -2\nu |\partial_y g|^2.$$ 

Now let $h$ to be a solution of $\partial_t h = \nu \partial_y^2 h$ with initial data $h_0 = |g_0|^2$. It holds

$$\frac{d}{dt} \int_T [h \mid_{\partial (T, \tilde{B})}]^2 = -2\nu |\partial_y g|^2$$

which implies that

$$\|g_0\|^2_{L^2} - \|g_t\|^2_{L^2} = 2\nu \int_0^t \|\partial_y g\|^2_{L^2} = \int_T [h_t(y) - |g_t(y)|^2] \, dy.$$

Finally, since by Feynman–Kac, $h(t, y) = E \left[ |g_0|^2(y + \sqrt{\nu} B_t) \right]$, we obtain

$$\|g_0\|^2_{L^2} - \|g_t\|^2_{L^2} = \int_T \left( E \left[ |g_0|^2(y + \sqrt{\nu} B_t) \right] - E[|Z_t^\nu|^2] \right) \, dy$$

which gives the conclusion. \hfill $\Box$

**Lemma 4.3.** Let $g_0 \in H^1(T; \mathbb{C})$, $u \in B^\alpha_{1,\infty}(T)$ for some $\alpha \in (0, 1)$ and $\xi \in \mathbb{R}$. Then there exists $C = C(\alpha) > 0$ such that the solution $g$ to (4.1) satisfies

$$\|g_0\|^2_{L^2} - \|g_t\|^2_{L^2} \leq C \|g_0\|^2_{H^1} \left( \nu t + \|u\|_{B^\alpha_{1,\infty}} |\xi| \nu^{1+\frac{1}{2}} \right) \forall \nu, t > 0.$$

**Proof.** Recall the elementary identity $2 \text{Var}(X) = E[|X - \tilde{X}|^2]$ for $\tilde{X}$ being an i.i.d. copy of $X$. In our setting, we take

$$\tilde{Z}_t^\nu = \exp \left( -i \xi \int_0^t u \left( y + \sqrt{\nu} \tilde{B}_s \right) \, ds \right) g_0 \left( y + \sqrt{\nu} \tilde{B}_t \right)$$

where $\tilde{B}$ is another BM independent of $B$. Therefore

$$\|g_0\|^2_{L^2} - \|g_t\|^2_{L^2} = \frac{1}{2} \int_T E[|Z_t^\nu - \tilde{Z}_t^\nu|^2] \, dy$$

and more generally, the map $(t, y) \mapsto |g|^2(t, y)$ satisfies

$$\frac{d}{dt} \int_T [h \mid_{\partial (T, \tilde{B})}]^2 = -2\nu |\partial_y g|^2.$$ 

Now let $h$ to be a solution of $\partial_t h = \nu \partial_y^2 h$ with initial data $h_0 = |g_0|^2$. It holds

$$\frac{d}{dt} \int_T [h \mid_{\partial (T, \tilde{B})}]^2 = -2\nu |\partial_y g|^2$$

which implies that

$$\|g_0\|^2_{L^2} - \|g_t\|^2_{L^2} = 2\nu \int_0^t \|\partial_y g\|^2_{L^2} = \int_T [h_t(y) - |g_t(y)|^2] \, dy.$$

Finally, since by Feynman–Kac, $h(t, y) = E \left[ |g_0|^2(y + \sqrt{\nu} B_t) \right]$, we obtain

$$\|g_0\|^2_{L^2} - \|g_t\|^2_{L^2} = \int_T \left( E \left[ |g_0|^2(y + \sqrt{\nu} B_t) \right] - E[|Z_t^\nu|^2] \right) \, dy$$

which gives the conclusion. \hfill $\Box$
Using the inequality $|e^{i\xi a} - e^{i\xi b}| \leq \sqrt{2} |\xi|^{1/2} |b - a|^{1/2}$ and the characterization of Besov spaces in terms of finite differences (see Appendix A), we deduce
\[
\|g_0\|_{L^2}^2 - \|g_t\|_{L^2}^2 \lesssim \mathbb{E} \left[ \left\| g_0 \left( t + \sqrt{\nu} B_t \right) - g_0 \left( t + \sqrt{\nu} \tilde{B}_t \right) \right\|_{L^2}^2 \right] + \|g_0\|_{L^\infty}^2 |\xi| \mathbb{E} \left[ \left\| \int_0^t u \left( t + \sqrt{\nu} B_s \right) - u \left( t + \sqrt{\nu} \tilde{B}_s \right) \right\|_{L^1} \right] ds \]
\[
\lesssim \|g_0\|_{H^1}^2 \left( \nu \mathbb{E} \|B_t - \tilde{B}_t\|^2 + |\xi| \int_0^t \mathbb{E} \left[ \left\| u \left( t + \sqrt{\nu} B_s \right) - u \left( t + \sqrt{\nu} \tilde{B}_s \right) \right\|_{L^1} \right] ds \right) \]
\[
\lesssim \|g_0\|_{L^2}^2 \left( \nu t + \|u\|_{B_{1,\infty}^\alpha} \|\nu\|^\frac{3}{2} t^{1+\frac{\alpha}{2}} \right) ;
\]
computing the last expectation yields the conclusion. □

We are now ready to complete the

**Proof of Proposition 4.1.** The proof goes along the same lines as Lemma 2 from [15]. We argue by contradiction. Assume there exists no such constant $C$, then it must hold
\[
\liminf_{\nu \to 0^+} \nu^{-\frac{\alpha}{2\alpha+2}} r_{\text{dif}}(\nu) = +\infty.
\] (4.3)

Now take $g_0(y) = (2\pi)^{-1/2} e^{iy}$, so that $\|g_0\|_{L^2} = 1 \sim \|g_0\|_{H^1}$; by Definition 1.3 and Lemma 4.3 applied to $\xi = 1$ we deduce that there exist constants $C_1, C_2 > 0$ such that, for any $\nu \leq 1$ and $t \geq 1$, it holds
\[
1 - C_1 e^{-r_{\text{dif}}(\nu)t} \leq 1 - \|e^{tL_{1,\nu}}\|_{L^2}^2 \leq 1 - \|g_t\|_{L^2}^2 \leq C_2 \|g_0\|_{H^1}^2 \left( \nu t + \|u\|_{B_{1,\infty}^\alpha} \nu^{\frac{3}{2}} t^{1+\frac{\alpha}{2}} \right) \]
\[
\leq C_2 (1 + \|u\|_{B_{1,\infty}^\alpha}) \nu^{\frac{3}{2}} t^{1+\frac{\alpha}{2}}.
\]

Let $\nu_n \downarrow 0$ be a sequence realizing the liminf in (4.3) and choose
\[
t_n = \left( r_{\text{dif}}(\nu_n) \nu_n^{-\alpha/(\alpha+2)} \right)^{-1/2};
\]
then we obtain
\[
1 - C_1 \exp \left( - \left( \nu_n^{-\frac{\alpha}{2\alpha+2}} r_{\text{dif}}(\nu) \right)^{1/2} \right) \lesssim \nu_n^{-\frac{\alpha}{2\alpha+2}} \left( \nu_n^{-\frac{\alpha}{2\alpha+2}} r_{\text{dif}}(\nu) \right)^{\frac{\alpha+2}{2}}.
\]
Taking the limit as $n \to \infty$ on both sides we find $1 \leq 0$ which is absurd. □

4.2. Wei’s irregularity condition. A major role in the analysis of dissipation enhancement by rough shear flows is played by the following condition, first introduced in [43].

**Definition 4.4.** We say that $u \in L^1(0,T)$ satisfies Wei’s condition with parameter $\alpha > 0$ if, setting $\psi(y) = \int_0^y u(z) dz$, it holds
\[
\Gamma_\alpha(u) := \left[ \inf_{\delta \in (0,1)} \inf_{\gamma \in [0,T-\delta]} \int_{\gamma}^{\gamma+\delta} \left| \psi(y) - c_1 - c_2 y \right|^2 dy \right]^{1/2} > 0.
\] (4.4)

A similar definition holds for $u \in L^1_{\text{loc}}(\mathbb{R})$; $u \in L^1(\mathbb{R})$ is said to satisfy Wei’s condition once it is identified with a $2\pi$-periodic map on $\mathbb{R}$. 
Remark 4.5. Denoting by $P_1$ the set of all polynomials of degree at most one, for $u \in L^1_{\text{loc}}(\mathbb{R})$ the definition is equivalent to
\[
\Gamma_\alpha(u) = \left( \inf_{I \subset \mathbb{R}, |I| < 1} |I|^{-2\alpha - 3} \inf_{P \in P_1} \int_I |\psi(y) - P(y)|^2 \, dy \right)^{1/2} > 0;
\]
this highlights its "complementarity" to the seminorm $\|\cdot\|_{L^1_1}^{2,2\alpha+3}$ associated to the higher order Campanato space $L^2_1$, as defined in [4]. Observe that $\Gamma_\alpha$ is homogeneous, i.e. $\Gamma_\alpha(\lambda u) = \lambda \Gamma_\alpha(u)$ for all $\lambda \geq 0$.

The importance of condition (4.4) comes from the following result.

Theorem 4.6. Let $u \in L^1(\mathbb{T})$ be such that $\Gamma_\alpha(u) > 0$ for some $\alpha > 0$. Then there exist positive constants $C_1, C_2$, depending on $\alpha$ and $\Gamma_\alpha(u)$, such that
\[
\|e^{L_{x,v}}\|_{L^2 \to L^2} \leq C_1 \exp \left( -C_2 \nu^{\alpha+2} |k|^{\frac{2}{\alpha+2}} t \right) \quad \forall \nu \in (0,1), k \in \mathbb{Z}_0, t \geq 0. \tag{4.5}
\]
Namely, $u$ is diffusion enhancing with rate $r_{\text{diff}}(x) \sim x^{\alpha/(\alpha+2)}$, in the sense of Definition 1.3.

The statement comes from Theorem 5.1 from [43]; therein $u$ is required to be continuous, but this restriction is not necessary, see Appendix B for the proof.

Following the same approach as in Section 3, we proceed to show that the condition $\Gamma_\alpha(u)$ implies irregularity of $u$; we start by relating it to the property of $\alpha$-Hölder roughness, in the sense of Definition 3.7.

Lemma 4.7. Let $u \in L^1(\mathbb{T})$ be such that $\Gamma_\alpha(u) > 0$ for some $\alpha > 0$. Then $u$ is $\alpha$-Hölder rough and it holds $L_\alpha(u) \geq \Gamma_\alpha(u)$.

Proof. Fix $\delta > 0$, $\bar{y} \in [-\pi, \pi]$; it holds
\[
\inf_{c_1, c_2 \in \mathbb{R}} \int_{\bar{y}}^{\bar{y} + \delta} |\psi(y) - c_1 - c_2 y|^2 \, dy \leq \int_{\bar{y}}^{\bar{y} + \delta} |\psi(y) - \psi(\bar{y}) - \psi'(\bar{y})(y - \bar{y})|^2 \, dy
\]
\[
\leq \int_{\bar{y}}^{\bar{y} + \delta} \left( \int_{\bar{y}}^{y} |u(z) - u(\bar{y})| \, dz \right)^2 \, dy
\]
\[
\leq \delta^{2\alpha+3} \left( \sup_{z \in B_\delta(\bar{y})} \frac{|u(z) - u(\bar{y})|}{\delta^\alpha} \right)^2.
\]
As the inequality holds for all $\delta$ and $\bar{y}$, we obtain $\Gamma_\alpha(u)^2 \leq L_\alpha(u)^2$ and the conclusion. \qed

We can also relate Wei’s condition to regularity in the Besov–Nikolskii scales $B^\alpha_{1,\infty}$.

Lemma 4.8. Let $u \in L^1(\mathbb{T})$ be such that $\Gamma_\alpha(u) > 0$ for some $\alpha \in (0,1)$. Then $u$ does not belong to $B^\alpha_{1,\infty}$ for any $\tilde{\alpha} > \alpha$ and does not belong to $B^q_{1,q}$ for any $q < \infty$.

Proof. For any $\bar{y} \in [-\pi, \pi]$ and $\delta > 0$ it holds
\[
\delta^{2\alpha+3} \Gamma_\alpha(u)^2 \leq \int_{\bar{y}}^{\bar{y} + \delta} \left( \int_{\bar{y}}^{y} |u(z) - u(\bar{y})| \, dz \right)^2 \, dy
\]
\[
\leq \int_{\bar{y}}^{\bar{y} + \delta} \left( \int_{\bar{y}}^{\bar{y} + \delta} |u(z) - u(\bar{y})| \, dz \right)^2 \, dy
\]
thus implying that
\[
\inf_{y \in \mathbb{T}} \int_0^\delta |u(\bar{y} + h) - u(\bar{y})| \, dh \geq \delta^{1 + \alpha} \Gamma_\alpha(u) \quad \forall \delta \in (0, 1).
\] (4.6)

Now fix $\bar{\alpha} > \alpha$; starting from (4.6) and arguing as in the proof of Proposition 3.8 (with $\varepsilon$ replaced by $\delta$), one obtains
\[
2\pi \Gamma_\alpha(u) \leq \delta^{\bar{\alpha} - \alpha} \|u\|_{B^\bar{\alpha}_1^{\infty}},
\]
which implies the first claim by letting $\delta \to 0^+$. Integrating (4.6) over $\bar{y} \in \mathbb{T}$ yields
\[
\int_0^\delta \|u(\cdot + h) - u(\cdot)\|_{L^1} \, dh \geq \delta^{1 + \alpha} \Gamma_\alpha(u) \quad \forall \delta \in (0, 1); \tag{4.7}
\]
now assume by contradiction that $u \in B^q_{1,q}$ for some $q < \infty$, then by its equivalent characterization (see Appendix A) and the uniform integrability of $h \mapsto h^{-1 - \alpha q} \|u(\cdot + h) - u(\cdot)\|_{L^1}^q$ it must hold
\[
\lim_{\delta \to 0^+} \int_0^\delta \frac{\|u(\cdot + h) - u(\cdot)\|_{L^1}^q}{|h|^{1 + \alpha q}} \, dh = 0. \tag{4.8}
\]
On the other hand, by estimate (4.7) and Jensen’s inequality, it holds
\[
\int_0^\delta \frac{\|u(\cdot + h) - u(\cdot)\|_{L^1}^q}{|h|^{1 + \alpha q}} \, dh \geq \delta^{1 + \alpha} \int_0^\delta \|u(\cdot + h) - u(\cdot)\|_{L^1}^q \, dh \\
\geq \delta^{-q(1 + \alpha)} \left( \int_0^\delta \|u(\cdot + h) - u(\cdot)\|_{L^1} \, dh \right)^q \\
\geq \Gamma_\alpha(u)^q > 0
\]
uniformly in $\delta \in (0, 1)$, contradicting (4.8).

\section*{Remark 4.9.} It follows from Lemma 4.8 and the construction presented Section 2 from [10] that, for any $\alpha \in \mathbb{Q}$ as in Lemma 2.1 therein, there exists a Weierstrass-type function which belongs to $C^\alpha(\mathbb{T})$, satisfies Wei’s condition with parameter $\alpha$ and does not belong to $B^\bar{\alpha}_p$ for any $p \in [1, \infty], q \in [1, \infty)$, nor to any $B^{\bar{\alpha}}_{p,q}$ with $\bar{\alpha} > \alpha$.

In light of Theorem 4.6, in order to show that almost every shear flow $u$ enhances dissipation, it will suffice to show that almost every $u$ satisfies Wei’s condition. We therefore need to find sufficient conditions in order for $\Gamma_\alpha(u) > 0$ to hold. We start with the following simple fact, whose proof is almost identical to that of Lemma 3.12, which simplifies the problem by allowing us to work with not necessarily periodic functions.

\section*{Lemma 4.10.} A map $u : \mathbb{T} \to \mathbb{R}$ satisfies $\Gamma_\alpha(u) > 0$ if and only if the maps $u_i : [0, \pi] \to \mathbb{R}$ defined by $u_1(y) = u(y), u_2(y) = u(-y)$ do so.

In this way, we can reduce the task to identifying sufficient conditions for functions defined on a standard interval $[0, \pi]$. For any $\delta > 0$, we denote by $\Delta^\delta_\alpha f(y) = f(y + 2\delta) - 2f(y + \delta) + f(y)$.

\section*{Lemma 4.11.} For any $\alpha > 0$ and any $(\bar{y}, \delta)$ it holds
\[
\delta^{-2\alpha - 3} \inf_{c_1, c_2} \int_{\bar{y}}^{\bar{y} + \delta} |\psi(y) - c_1 - c_2 y|^2 \, dy \geq \frac{1}{12} \left( \int_{\bar{y}}^{\bar{y} + \delta} \frac{1}{1 + \varepsilon} \, dy \right)^{-2(1 + \alpha)} \tag{4.9}
\]
Proof. First observe that \( \Delta_2^2(c_1 + c_2y) \equiv 0 \) for any \( c_1, c_2 \) and that for any \( f \) it holds
\[
\int_{\bar{y}}^{\bar{y} + \delta} |f(y)|^2 dy \geq \frac{1}{12} \int_{\bar{y}}^{\bar{y} + \delta} |\Delta_2^2 f(y)|^2 dy.
\]
Next, applying Jensen inequality for \( g(x) = x - \frac{n}{2(1+n)} \), which is convex on \((0, \infty)\), it holds
\[
\left( \frac{1}{\delta} \int_{\bar{y}}^{\bar{y} + \delta} |\Delta_2^2 f(y)|^2 dy \right)^{-\frac{n}{2(1+n)}} \leq \frac{1}{\delta} \int_{\bar{y}}^{\bar{y} + \delta} |\Delta_2^2 f(y)|^{-\frac{n}{2(1+n)}} dy.
\]
Algebraic manipulations of the second inequality and the choice \( f(y) = \psi(y) - c_1 - c_2y \) yield (4.9).

In view of Lemma 4.11, given \( \alpha > 0 \) and an integrable \( u : [0, \pi] \to \mathbb{R} \), we define
\[
G_\alpha(\bar{y}, \delta) := \int_{\bar{y}}^{\bar{y} + \delta} |\Delta_2^2 \psi(y)|^{-\frac{1}{2(1+n)}} dy,
\] (4.10)
where \( \psi \) is defined as usual by \( \psi(y) = \int_0^y u(z) dz \).

Lemma 4.12. For any \( \alpha \in (0, 1) \) and \( \varepsilon > 0 \), define \( \beta := \alpha + \varepsilon(1 + \alpha) \) and
\[
K_{\alpha, \varepsilon}(u) := \sup_{n \in \mathbb{N}, 1 \leq k \leq 2^{n-1}} 2^{-n\varepsilon} G_\alpha(\pi k 2^{-n}, \pi 2^{-n-1}).
\]
Then there exists a constant \( C = C(\alpha, \varepsilon) \) such that
\[
\Gamma_\beta(u) \geq C(K_{\alpha, \varepsilon}(u))^{-1 - \alpha}.
\]
Proof. First observe that, for any \( \beta \in (0, 1) \),
\[
|\Gamma_\beta(u)|^2 \sim_{\delta} \inf_{\delta \in (0, 1/3), \bar{y} \in [0, 1-3\delta]} \delta^{-2\beta - 3} \inf_{c_1, c_2 \in \mathbb{R}} \int_{\bar{y}}^{\bar{y} + 3\delta} |\psi(y) - c_1 - c_2y|^2 dy
\]
so to conclude it suffices to provide a lower bound on the latter for our choice of \( \beta \). Fix \((\bar{y}, \delta)\) and choose \( n \in \mathbb{N} \) and \( k \in \{1, \ldots, 2^n - 1\} \) such that
\[
\delta \in (\pi 2^{-n}, \pi 2^{-n+1}], \quad \bar{y} \in [\pi (k-1) 2^{-n}, \pi k 2^{-n}]
\]
so that \([\bar{y}, \bar{y} + 3\delta] \supseteq [\bar{y}, \bar{y} + 3\bar{\delta}]\) for the choice \( \bar{y} = \pi k 2^{-n}, \bar{\delta} = \pi 2^{-n-1} \). As a consequence,
\[
\delta^{-2\beta - 3} \inf_{c_1, c_2 \in \mathbb{R}} \int_{\bar{y}}^{\bar{y} + 3\delta} |\psi(y) - c_1 - c_2y|^2 dy
\geq 2^{\beta} \delta^{-2\beta - 3} \inf_{c_1, c_2 \in \mathbb{R}} \int_{\bar{y}}^{\bar{y} + 3\delta} |\psi(y) - c_1 - c_2y|^2 dy
\geq 2^{\beta} \delta^{-2(\beta - \alpha)} \left( \int_{\bar{y}}^{\bar{y} + \delta} |\Delta_2^2 \psi(y)|^{-\frac{1}{2(1+n)}} dy \right)^{-2(1+\alpha)}
\]
where in the second passage we employed inequality (4.9) and then the definition of \( \beta \). Overall we deduce by the definition of \( K \) and the choice of \((\bar{y}, \delta)\) that
\[
\delta^{-2\beta - 3} \inf_{c_1, c_2 \in \mathbb{R}} \int_{\bar{y}}^{\bar{y} + 3\delta} |\psi(z) - c_1 - c_2z|^2 dz \geq_{\beta} K_{\alpha, \varepsilon}(u)^{-2(1+\alpha)};
\]
taking the infimum over \((\delta, y)\) then yields the conclusion. \( \square \)
4.3. **Sufficient conditions for stochastic processes.** In order to establish prevalence statements, we want to run the same programme as in Section 3.3, exploiting the properties of LND Gaussian processes and their fundamental translation invariance from Lemma 2.3. In order for this strategy to work, we need an equivalent of Proposition 3.10; this is precisely the aim of this section, cf. Corollary 4.17 below. Its proof requires a few preparations; we start with the following intermediate, general result.

**Proposition 4.13.** Let \( u : [0, \pi] \to \mathbb{R} \) be an integrable stochastic process, \( \psi = \int_0^\pi u_s ds \) and suppose that there exist \( \lambda, \kappa > 0, \alpha \in (0, 1) \) such that

\[
\sup_{\delta \in (0, 1), \bar{y} \in [0, \pi - \delta]} \mathbb{E}[\exp(\lambda G_\alpha(\bar{y}, \delta))] \leq \kappa
\]

for \( G \) as defined in (4.10). Then for any \( \beta > \alpha \) it holds \( \mathbb{P}(\Gamma_\beta(u) > 0) = 1 \).

**Proof.** By virtue of Lemma 4.12, for \( \beta = \alpha + \varepsilon (1 + \alpha) \) it holds

\[
\mathbb{P}(\Gamma_\beta(u) > 0) \geq \mathbb{P}(K_{\alpha, \varepsilon}(u) < \infty),
\]

so to conclude it suffices to show that \( \mathbb{P}(K_{\alpha, \varepsilon}(u) < \infty) = 1 \) for all \( \varepsilon > 0 \). Given \( \lambda \) as in the hypothesis, define the random variable

\[
J := \sum_{n \in \mathbb{N}} 2^{-2n} \sum_{k=1}^{2^{n-1}} \exp(\lambda G_\alpha(\pi k 2^{-n}, \pi 2^{-n-1})).
\]

By assumption \( \mathbb{E}[J] < \infty \), so that \( \mathbb{P}(J < \infty) = 1 \). For any \( n, k \) it holds

\[
G_\alpha(\pi k 2^{-n}, \pi 2^{-n-1}) \leq \frac{1}{\lambda} \log(2^{2n}J) \lesssim \frac{n}{\lambda} (1 + \log J)
\]

which implies that

\[
Y := \sup_{n \in \mathbb{N}, 1 \leq k \leq 2^{-n-1}} \frac{1}{n} G_\alpha(\pi k 2^{-n}, \pi 2^{-n-1}) \lesssim \frac{1}{\lambda} (1 + \log J) < \infty \quad \mathbb{P}\text{-a.s.}
\]

Finally, for any \( \varepsilon > 0 \) it holds \( K_{\alpha, \varepsilon}(u) \lesssim \varepsilon Y \), which yields the conclusion. \( \square \)

In order to apply Proposition 4.13 to suitable LND Gaussian processes, we will need the three Lemmas 4.14-4.16 below.

The next elementary lemma often appears in the probabilistic literature in connection to so-called Krylov or Khasminskii type of estimates, see Lemma 1.1 from [38] for a slightly more general statement. For the sake of completeness, we give the proof.

**Lemma 4.14.** Let \( X \) be a real valued, nonnegative stochastic process, defined on an interval \( [t_1, t_2] \), adapted to a filtration \( \{F_s\}_{s \in [t_1, t_2]} \); suppose there exists a deterministic \( C > 0 \) such that

\[
\text{ess sup}_{\omega \in \Omega} \mathbb{E} \left[ \int_{s}^{t^\prime} X_r | F_s \right] \leq C \quad \forall s \in [t_1, t_2].
\]

Then for any \( \lambda \in (0, 1) \) it holds

\[
\mathbb{E} \left[ \exp \left( \frac{\lambda}{C} \int_{t_1}^{t^\prime} X_r dr \right) \right] \leq (1 - \lambda)^{-1}.
\]

**Proof.** Up to rescaling \( X \), we may assume \( C = 1 \). It holds

\[
\mathbb{E} \left[ \exp \left( \lambda \int_{t_1}^{t^\prime} X_r dr \right) \right] = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E} \left[ \left( \int_{t_1}^{t^\prime} X_r dr \right)^n \right] = \sum_{n=0}^{\infty} \lambda^n I_n
\]
where
\[ I_n = \mathbb{E} \left[ \int_{t_1 < r_1 < \ldots < r_n < t_2} X_{r_1} \ldots X_{r_n} \, dr_1 \ldots dr_n \right]. \]

By the assumptions and the non-negativity of \( X \), it holds
\[ I_n = \int_{t_1 < r_1 < \ldots < r_{n-1} < t_2} \mathbb{E} \left[ X_{r_1} \ldots X_{r_{n-1}} \int_{r_{n-1}}^t X_{r_n} \, dr_n \right] \, dr_1 \ldots dr_{n-1} \]
\[ = \int_{t_1 < r_1 < \ldots < r_{n-1} < t_2} \mathbb{E} \left[ X_{r_1} \ldots X_{r_{n-1}} \mathbb{E} \left[ \int_{r_{n-1}}^t X_{r_n} \, dr_n \big| \mathcal{F}_{r_{n-1}} \right] \right] \, dr_1 \ldots dr_{n-1} \]
\[ \leq \int_{t_1 < r_1 < \ldots < r_{n-1} < t_2} \mathbb{E} [X_{r_1} \ldots X_{r_{n-1}}] \, dr_1 \ldots dr_{n-1} = I_{n-1} \]
which iteratively implies \( I_n \leq 1 \). Therefore we obtain
\[ \mathbb{E} \left[ \exp \left( \lambda \int_0^t X_u \, du \right) \right] \leq \sum_{n=0}^{\infty} \lambda^n = (1 - \lambda)^{-1}. \]

**Lemma 4.15.** Let \( Z \sim \mathcal{N}(m, \sigma^2) \) be a real valued Gaussian variable. Then for any \( \theta \in (0, 1) \) there exists \( c_\theta > 0 \) such that
\[ \mathbb{E} \left[ |Z|^{-\theta} \right] \leq c_\theta \sigma^{-\theta}. \]

**Proof.** Set \( Z = \sigma N + m \), then \( \mathbb{E} \left[ |Z|^{-\theta} \right] = \sigma^{-\theta} \mathbb{E} \left[ |N - x|^{-\theta} \right] \) for \( x = -m/\sigma \); therefore is sufficed to show that
\[ \sup_{x \in \mathbb{R}} \mathbb{E} \left[ |N - x|^{-\theta} \right] = \sup_{x \in \mathbb{R}} \int_{x \in \mathbb{R}} |x - y|^{-\theta} p(y) \, dy = \| | \cdot |^{-\theta} * p \|_{L^\infty} < \infty \]
where \( p \) stands for the Gaussian density \( p(x) = (2\pi)^{-1/2} \exp(-|x|^2/2) \). By Young’s inequality it holds
\[ \| | \cdot |^{-\theta} * p \|_{L^\infty} \leq \| | \cdot |^{-\theta} 1_{|\cdot|<1} * p \|_{L^\infty} + \| | \cdot |^{-\theta} 1_{|\cdot|\geq1} * p \|_{L^\infty} \]
\[ \leq \| | \cdot |^{-\theta} 1_{|\cdot|<1} \|_{L^1} \| p \|_{L^\infty} + \| | \cdot |^{-\theta} 1_{|\cdot|\geq1} \|_{L^\infty} \| p \|_{L^1} \]
\[ \leq (2\pi)^{-1/2} \| | \cdot |^{-\theta} 1_{|\cdot|<1} \|_{L^1} + 1 < \infty \]
which gives the conclusion. \( \square \)

**Lemma 4.16.** Let \( Y : [0, \pi] \rightarrow \mathbb{R} \) be a \((1 + H)\)-SLND Gaussian process with constant \( C_Y \), in the sense of Definition 2.2. Then for any \( \alpha > H \) there exists \( \lambda = \lambda(\alpha, H, C_Y) > 0 \) s.t.
\[ \mathbb{E} \left[ \exp \left( \lambda \int_{\bar{y}}^{\bar{y} + \delta} |\Delta_\delta^H Y_y|^{-\frac{1}{1+\alpha}} \, dy \right) \right] \leq 2 \quad \forall \delta \in (0, 1), \bar{y} \in [0, \pi - \delta]. \]

**Proof.** The result follows Lemmas 4.14 and 4.15 applied to the process \( X_y = |\Delta_\delta^H \psi_y|^{-\frac{1}{1+\alpha}}. \) Indeed, denote by \( \mathcal{F}_y \) the natural filtration generated by \( \psi \) and let \( \mathcal{G}_y := \mathcal{F}_{y+2\delta} \). It is clear that \( \Delta_\delta^H \psi_y = Y_{y+2\delta} - 2Y_{y+\delta} + Y_y \) is \( \mathcal{G}_y \)-adapted; for any \( [z, y] \subset [\bar{y}, \bar{y} + \delta] \) it holds
\[ \text{Var}(\Delta_\delta^H Y_y | \mathcal{G}_z) = \text{Var}(Y_{y+2\delta} | \mathcal{F}_{z+2\delta}) \geq C_Y |y - z|^{2(1+H)}. \]
As a consequence, we have a decomposition $\Delta^2_\delta Y_y = Z^{(1)}_{z,y} + Z^{(2)}_{z,y}$ with $Z^{(1)}_{z,y}$ adapted to $G_z$ and $Z^{(2)}_{z,y}$ Gaussian and independent of $G_z$: therefore

$$\mathbb{E} \left[ \int_y^\vartheta \Delta^2_\delta Y_y| \frac{1}{1+\alpha} dy \right| G_z] = \int_y^\vartheta \mathbb{E} \left[ Z^{(2)}_{z,y} + \frac{1}{1+\alpha} \right] (Z^{(1)}_{u,y}) dy.$$  

By Lemma 4.15, since $\text{Var}(Z^{(2)}_{z,y}) \geq C_Y|y - z|^{2(1+H)}$ and $\theta = (1 + \alpha)^{-1} \in (0, 1)$, it holds

$$\sup_{x \in \mathbb{R}} \mathbb{E} \left[ |Z^{(2)}_{z,y} + x| \frac{1}{1+\alpha} \right] \leq C \text{Var}(Z^{(2)}_{z,y})^{-\frac{1}{2(1+\alpha)}} \lesssim C_{\alpha,H,C_Y} |y - z|^{-\frac{1+H}{1+\alpha}}$$

and thus

$$\mathbb{E} \left[ \int_z^{\vartheta + \delta} |\Delta^2_\delta Y_y| \frac{1}{1+\alpha} dy \right| G_z] \lesssim \int_y^\vartheta |y - z|^{-\frac{1+H}{1+\alpha}} dy$$

$$\lesssim \int_0^1 |r|^{-\frac{1+H}{1+\alpha}} dr \sim C(\alpha, H, C_Y)$$

where the estimate is uniform over $z \in [\vartheta, \vartheta + \delta]$, $\vartheta \in \mathbb{T}$ and $\delta \in (0, 1)$. Choosing

$$\lambda = \frac{1}{2C(\alpha, H, C_Y)},$$

we obtain the conclusion by applying Lemma 4.14. \qed

With Lemmas 4.14-4.16 at hand, we can finally verify that suitable Gaussian processes verify Wei's condition with probability 1; we give the statement in full generality, but we stress that the most relevant example verifying the hypothesis below is the canonical process $X$ associated to $\mu^H$, as granted by Lemma 2.4.

**Corollary 4.17.** Let $X : [0, \pi] \to \mathbb{R}$ be a Gaussian process such that

$$Y_y = \int_0^y X_z dz$$

is $(1 + H)$-SLND for some $H \in (0, 1)$. Then

$$\mathbb{P}(\Gamma_\alpha(X) > 0) = 1$$

for any $\alpha > H$.

**Proof.** It follows immediately combining Lemma 4.16 and Proposition 4.13. \qed

### 4.4. Prevalence statements and proof of Theorems 1.5, 1.1.

Similarly to Section 3.3, in order to prove prevalence statements in $B^\alpha_{1,\infty}(\mathbb{T})$, we will actually start by establishing their analogues on $B^\alpha_{1,\infty}(0, \pi)$.

**Theorem 4.18.** Let $\alpha \in (0, 1)$; then a.e. $\varphi \in B^\alpha_{1,\infty}(0, \pi)$ satisfies $\Gamma_\beta(\varphi) > 0$ for all $\beta > \alpha$.

**Proof.** Fix $\alpha \in (0, 1)$ and define $A := \{ \varphi \in B^\alpha_{1,\infty}(0, \pi) : \Gamma_\beta(\varphi) > 0 \text{ for all } \beta > \alpha \}$; it holds

$$A = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_{n,m} := \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \left\{ \varphi \in B^\alpha_{1,\infty}(0, \pi) : \Gamma_\beta(\varphi) \geq \frac{1}{m} \text{ for } \beta = \alpha + \frac{1}{n} \right\}.$$  

The sets $A_{n,m}$ are closed in the topology of $B^\alpha_{1,\infty}(0, \pi)$ (the map $\varphi \mapsto \Gamma_\beta(\varphi)$ is upper semicontinuous in the topology of $L^1(0, \pi)$), thus $A$ is Borel measurable. In order to conclude, it is enough to show that for any fixed $\beta > \alpha$, the set $A_\beta := \{ \varphi \in B^\alpha_{1,\infty}(0, \pi) : \Gamma_\beta(\varphi) > 0 \}$ (which is Borel by the same line of argument) is prevalent.
Now fix $\beta > \alpha$ and choose $H \in (\alpha, \beta)$; denote by $\mu^H$ the law of fBm of parameter $H$ on $C([0, \pi])$ and by $u = \{u_y\}_{y \in [0, \pi]}$ the associated canonical process. Since $\mu^H$ is supported on $C^{H-\varepsilon}([0, \pi])$ for any $\varepsilon > 0$ and $H > \alpha$, it is also a tight probability measure on $B^a_1(0, \pi)$. By Lemma 2.4, the associated process $\psi = \int_0^\pi u(y)dy$ is $(1 + H)$-SLND and so by Lemma 2.3 the same holds for $f + \psi$, for any measurable $f : [0, \pi] \to \mathbb{R}$.

In particular, for a given $\varphi \in B^a_1(0, \pi)$, taking $f = \int_0^\pi \varphi(y)dy$, it follows from Corollary 4.17 and the choice $\beta > H$ that

$$\mu^H(\varphi + A_\beta) = \mu^H\left(\{u \in B^a_1(0, \pi) : \Gamma(u + \varphi > 0)\}\right) = 1.$$ 

As the reasoning holds for any $\varphi \in B^a_1(0, \pi)$, we deduce that $\mu^H$ witnesses the prevalence of $A_\beta$ and we obtain the conclusion. \(\square\)

As in Section 3.3, we define for $\tilde{\varphi} : [0, \pi] \to \mathbb{R}$ the map $(T\tilde{\varphi})(y) = \tilde{\varphi}(|y|)$; conversely for $\varphi : \mathbb{T} \to \mathbb{R}$, $\varphi_1(y) := \varphi(y)$, $\varphi_2(y) := \varphi(-y)$. Recall that if $\tilde{\varphi} \in B^a_1(0, \pi)$, then $T\tilde{\varphi} \in B^a_1(0, \pi)$.

**Corollary 4.19.** Almost every $\varphi \in B^a_1(0, \pi)$ satisfies $\Gamma_\beta(\varphi) > 0$ for all $\beta > \alpha$.

**Proof.** The proof is very similar to that of Corollary 3.13, again employing measures of the form $\nu^H = T\mu^H$ for suitable $H \in (0, 1)$; specifically, once we fix $\beta > \alpha$ and we define a subset $A_\beta$ of $B^a_1(0, \pi)$ as in the proof of Theorem 4.18, it suffices to choose $H \in (\alpha, \beta)$. In this way $\mu^H$ is tight on $B^a_1(0, \pi)$ and $\nu^H$ is tight on $B^a_1(0, \pi)$; the verification that

$$\nu^H(\varphi + A_\beta) = 1 \ \forall \varphi \in B^a_1(0, \pi)$$

is almost identical to that of Corollary 3.13, only this time invoking Lemma 4.10 and Theorem 4.18. \(\square\)

At this point we have all the ingredient to close the dissipative case.

**Proof of Theorem 1.5.** The lower bound comes from Proposition 4.1, while the upper bound from a combination of Theorem 4.6 and Corollary 4.19. \(\square\)

The main result of the paper, Theorem 1.1, is now a direct consequence of Theorems 1.4 and 1.5. In fact, let us record here a slightly sharper estimate. Given $f \in L^2(\mathbb{T}^2)$, for any $s \in \mathbb{R}$ define

$$\|f\|_{L^2_H^s}^2 := \sum_{k \in \mathbb{Z}} \|P_k f\|_{H^s(\mathbb{T}, C)}^2 = \sum_{(k, \eta) \in \mathbb{Z}^2} (1 + |\eta|^2)^s |\hat{f}(k, \eta)|^2;$$

it’s clear that, for $s \geq 0$, $\|f\|_{L^2_H^s} \leq \|f\|_{H^s(\mathbb{T}^2)}$ and $\|f\|_{H^{-s}(\mathbb{T}^2)} \leq \|f\|_{L^2_H^{-s}}$.

**Theorem 4.20.** Almost every $u \in B^a_1(0, \pi)$ satisfies the following property: for any $\tilde{\alpha} > \alpha$, there exists $C = C(\alpha, \tilde{\alpha}, u)$ such that, for any $f_0 \in H^{1/2}(\mathbb{T}^2)$ with $P_0 f_0 \equiv 0$, it holds

$$\|e^{t\tilde{\alpha}x} f_0\|_{L^2_H^{-1/2}} \leq C t^{-\frac{\tilde{\alpha}}{4\pi}} \|f_0\|_{L^2_H^{1/2}}. \quad (4.11)$$

Almost every $u \in B^a_1(0, \pi)$ satisfies the following property: for any $\tilde{\alpha} > \alpha$ there exist $C_i = C(\alpha, \tilde{\alpha}, u)$ such that, for any $f_0 \in L^2(\mathbb{T}^2)$ with $P_0 f_0 \equiv 0$, it holds

$$\|e^{-t(\tilde{\alpha} - \nu^\Delta)} f\|_{L^2(\mathbb{T}^2)} \leq C_1 \exp \left(-C_2 t\nu^\frac{\tilde{\alpha}}{2}\right) \|f_0\|_{L^2(\mathbb{T}^2)}. \quad (4.12)$$
Proof. By Theorem 1.4 b), for almost every \( u \in B^α_{1,∞}(T) \) and any \( \tilde{α} > α \) it holds
\[
\|e^{-τu∂_x}f_0\|_{L^2_tH^{−1/2}_x}^2 = \sum_{k ∈ \mathbb{Z}_0} \|P_k(e^{-τu∂_x}f_0)\|_{H^{−1/2}}^2 \leq \sum_{k ∈ \mathbb{Z}_0} (t|k|)^{−1/2} \|P_kf_0\|_{H^{−1/2}}^2 \lesssim t^{−1/2} \|f_0\|_{L^2_tH^{−1/2}_x}^2
\]
proving (4.11). Denote \( L_ν = −u∂_x + ν∂^2_ν \), so that \( −u∂_x + νΔ = L_ν + ν∂^2_x \), where the operators \( L_ν \) and \( ν∂^2_x \) commute; also observe that \( P_k(e^{τ\partial_ν^2}f) = e^{−t\lambda^2}P_kf \).

Combining these properties with Theorem 1.5, for almost every \( u \in B^α_{1,∞}(T) \) and any \( \tilde{α} > α \) it holds
\[
\|e^{(−u∂_x+νΔ)}f\|_{L^2}^2 = \sum_{k ∈ \mathbb{Z}_0} \|P_k(e^{τ\partial_ν^2}e^{τL_ν}f)\|_{L^2}^2 = \sum_{k ∈ \mathbb{Z}_0} e^{−2k^2} \|P_k(e^{τL_ν}f)\|_{L^2}^2 \lesssim \sum_{k ∈ \mathbb{Z}_0} \exp \left( −2t|k|^2 − Cτν^{α/2}|k|^2 \right) \|P_kf\|_{L^2}^2 \lesssim \exp \left( −Cτν^{α/2} \right) \sum_{k ∈ \mathbb{Z}_0} \|P_kf\|_{L^2}^2
\]
which yields (4.12).

5. Further comments and future directions

We have shown in this paper that generic rough shear flows satisfy both inviscid mixing and enhanced dissipation properties, with rates sharply determined by the regularity parameter \( α \in (0, 1) \) in the Besov scale \( B^α_{1,∞} \). In the enhanced dissipation case, this confirms the intuition from [10]; instead in the inviscid mixing one, it shows that the behavior presented by Weierstrass-type functions constructed therein is not generic in the sense of prevalence. Our results provide a connection to the property of \( ρ \)-irregularity, which was never observed in this context, and highlight the importance of working with mixing scales \( H^s \) with \( s \neq 1 \).

We conclude by presenting a few additional remarks and open problems arising from this work.

1. We are currently unable to determine whether there is a clear connection between the properties of \( ρ \)-irregularity and Wei’s condition. Lemma 3.6, together with the trivial estimate \( \|f_1\|_{H^{−1}} \leq \|f_1\|_{H^{−1/2}} \), imply that for \( α \in (0, 1/2) \) the shear flows \( u \in C^α \) constructed in [10] satisfy \( Γ_α(u) > 0 \) but are not \( ρ \)-irregular with \( ρ \sim (2α)^{−1} \). Heuristically, this fact is similar to the existence of flows with small dissipation time which are not mixing, like the cellular flows presented in [30].

2. The above argument also implies the existence of Weierstrass type functions which are not \( ρ \)-irregular, for suitable values \( ρ \). We believe this problem was open in the probabilistic community, although never been explicitly addressed in the literature.

3. Even without establishing a direct connection to Wei’s condition, it would be interesting to show that functions \( u \) which are \( ρ \)-irregular for \( ρ \sim (2α)^{−1} \) are diffusion enhancing with rate \( r_{diff}(ν) \sim ν^{α/(α+2)} \), in line with the heuristic argument presenting in the introduction. Since such \( u \) are mixing, they are indeed diffusion enhancing with a suitable rate by [11]; however the quantitative results from [13] only imply the worsened rate \( ν^{3α/(1+3α)} \) and it is rather unclear how to “bridge the gap”.

4. Going through the same proof as in Lemma 3.6, one can show that if \( u \) is \( (γ, ρ) \)-irregular with \( γ > 1 − s \), then \( u \) is mixing on the scale \( H^s \) with rate \( r_{s-mix}(t) = t^γ \). In the case \( γ = 0 \)
an even simpler proof based on duality and integration by parts provides mixing on the scale $H^1$ with the same rate. In fact, since $H^1(\mathbb{T}; \mathbb{C})$ is an algebra, by integration by parts it holds

$$
|\langle e^{i\xi u} f, g \rangle| = \left| \int_{-\pi}^{\pi} e^{i\xi u(y)} f(y)g(y)dy \right| \\
\leq \left| (fg)(-\pi) \right| \left| \int_{-\pi}^{\pi} e^{i\xi u(y)}dy \right| \\
+ \int_{-\pi}^{\pi} \left| (fg)'(y) \right| \left| \int_{-\pi}^{\pi} e^{i\xi u(z)}dz \right| dy \\
\lesssim \left( \|fg\|_{L^\infty} + \| (fg)' \|_{L^1} \right) \|\Phi^u\|_{\mathcal{O},\rho} |\xi|^{-\rho} \\
\lesssim \|f\|_{H^1} \|g\|_{H^1} \|\Phi^u\|_{\mathcal{O},\rho} |\xi|^{-\rho}
$$

which by duality implies $\|e^{i\xi u} f\|_{H^{-1}} \lesssim \|f\|_{H^1} \|\Phi^u\|_{\mathcal{O},\rho} |\xi|^{-\rho}$ and so the claim.

5. It is however an open problem to provide examples of $(0, \rho)$-irregular functions $u$, for any $\rho < 1$. See Remark 69 in [24] for a deeper discussion. There are several examples of $(0, 1)$-irregular, including the choice $u(y) = y$; by Proposition 1.4 from [7], it is enough to require the existence of $\delta > 0$ such that

$$
\frac{1}{\delta} \leq |u'(y)|, \quad \frac{|u''(y)|}{|u'(y)|} \leq \delta \quad \forall y \in [0, \pi]; 
$$

(5.1)

observe the similarity of condition (5.1) with Assumption (H) from [12].

6. The property of $(\gamma, \rho)$-irregularity can be reformulated in terms of the (Fourier transform of) occupation measure of $u$, namely the family $\{\mu^u_I, I \leq \pi \}$ given by $\mu^u_I = u_I \mathcal{L}_I$ where $I$ are subintervals of $\mathbb{T}$ and $\mathcal{L}_I$ stands for the Lebesgue measure on $I$; see Section 2.3 from [24] for more details. Closely related to the occupation measure of $u$ is its local time, namely the Radon–Nikodym derivative $d\mu^u_I/d\mathcal{L}_T$, which has been intensively studied for stochastic processes, see the review [25]. The following question arises naturally: is it possible to link the mixing properties of $u$ to the regularity of its local time?

7. In the paper we have always focused on the scales $B^\alpha_{1,\infty}(\mathbb{T})$ with $\alpha \in (0, 1)$. If one is instead interested in the mixing properties of generic $u \in C(\mathbb{T})$, much faster rates are available. Indeed for any $\beta > 1$ it’s possible to construct $\tilde{u}^\beta \in C([0, \pi])$ satisfying

$$
\left| \int_{y_1}^{y_2} e^{i\xi u^\beta(z)}dz \right| \lesssim_{\gamma, \beta} |y_2 - y_1|^\gamma \exp \left( -C_{\gamma, \beta} |\xi|^{\frac{2}{1+\beta}} \right) \quad \forall \xi \in \mathbb{R}, \quad 0 \leq y_1 \leq y_2 \leq \pi 
$$

(5.2)

and so by symmetrization the same holds for $w^\beta := T\tilde{u}^\beta$. Such $\tilde{u}^\beta$ are given by typical realization of the so called $(2\beta)$-log Brownian motion, see [27] for its definition and Propositions 48 and 49 from [24] for the proof of (5.2). In fact, one could use the law of such process to prove that a.e. $u \in C(\mathbb{T})$ satisfies (5.2) for any $\beta > 1$ (the value $\beta = 1$ can only be attained by Caratheodory functions, which are naturally discontinuous). Arguing as in the proof of Lemma 3.6 one can deduce that such $u$ are exponentially mixing, in the sense that they satisfy the estimate

$$
\|e^{i\xi u} g\|_{H^{-1}} \lesssim \exp \left( -C_{\gamma, \beta} |\xi|^{\frac{2}{1+\beta}} \right) \|g\|_{H^1} \quad \forall g \in H^1(\mathbb{T}; \mathbb{C}) 
$$

(5.3)

and so that

$$
\|e^{-tu\partial_t} f\|_{L^2_T H^{-1}} \lesssim \exp \left( -C_{\gamma, \beta} t^{\frac{2}{1+\beta}} \right) \|f\|_{L^2_T H^{-1}} 
$$

(5.4)

for all $f \in H^1(\mathbb{T}^2)$ satisfying $P_0 f \equiv 0$.

8. Finally, let us point out that the property of $\rho$-irregularity also holds for generic vector valued functions $u : [0, 1] \to \mathbb{R}^d$ (resp. $u : \mathbb{T} \to \mathbb{R}^d$), for any $d \in \mathbb{N}$, see [24]. In particular, similar
statements to part i. of Theorem 1.1 can be established for “higher dimensional” shear flows of the form

$$\partial_t f + \bar{u} \cdot \nabla f = \nu \Delta f$$

for \( f : \mathbb{T}^{d+1} \to \mathbb{R} \), \( \bar{u}(x_1, \ldots, x_{d+1}) := (u(x_{d+1}), 0)^T \); observe that for \( d = 2 \), \( \bar{u} \) is a stationary solution to 3D Euler equations. In light of [11], the vector field \( \bar{u} \) constructed by a \( \rho \)-irregular \( u \) is diffusion enhancing; thus can be applied in the study of suppression of blow-up by mixing phenomena similarly to what was done in [33, 3, 30].

**Appendix A. Besov spaces**

In this appendix we record fundamentals on Besov spaces \( B^s_{p,q} \) on the torus \( \mathbb{T}^d \), although in the paper we only need the case \( d = 1 \). For a gentle introduction on spaces on \( \mathbb{R}^d \) we refer to the monograph [1]; see also the classical paper [41] for spaces on an interval \( I \subset \mathbb{R} \). All their properties transfer to the analogous spaces on \( \mathbb{T}^d \) by a clever use of Poisson summation formula, see [26], [36]. Alternatively, periodic Besov spaces have been treated in Chapter 3 of [40].

Given a dyadic partition of the unity \( (\chi, \varphi) \) and the associated Littlewood–Paley blocks \( \{ \Delta_j \}_{j \geq -1} \), the (inhomogeneous) Besov spaces \( B^s_{p,q}(\mathbb{T}^d) \) with \( s \in \mathbb{R}, p, q \) are defined as the set of tempered distributions \( f \in \mathcal{S}'(\mathbb{T}^d) \) such that

$$\| f \|_{B^s_{p,q}} := \left( \sum_{j=-1}^{\infty} 2^{jsq} \| \Delta_j f \|_{L^p}^q \right)^{1/q} < \infty$$

with the usual conventions when \( q = \infty \). \( (B^s_{p,q}(\mathbb{T}^d), \| \cdot \|_{B^s_{p,q}}) \) are Banach spaces and the definition is independent of the choice of the partition of unity \( (\chi, \varphi) \). Besov spaces are handy to use due to their many properties, including functional embeddings and behavior under derivation and multiplication; we recall them briefly.

**Proposition A.1** ([1], Prop. 2.71). Let \( 1 \leq p_1 \leq p_2 \leq \infty \) and \( 1 \leq q_1 \leq q_2 \leq \infty \). Then for any \( s \in \mathbb{R} \), the space \( B^s_{p_1,q_1} \) continuously embeds in \( B^s_{p_2,q_2} \).

Also recall the following basic facts: for any \( \varepsilon > 0 \) and any \( p, q \in [1, \infty] \), the space \( B^s_{p,q} \) continuously embeds in \( B^{s-\varepsilon}_{p,1} \), as can be checked using the definition; for any \( p \in [1, \infty] \), we have the embeddings

$$B^0_{p,1} \hookrightarrow L^p \hookrightarrow B^0_{p,\infty}$$

see for instance Remark A.3 from [36] for the second statement.

**Proposition A.2** ([36], Prop. A.5). Let \( s \in \mathbb{R}, p, q \in [1, \infty], i \in \{1, \ldots, n\} \). Then the map \( f \mapsto \partial_i f \) is a continuous linear operator from \( B^s_{p,q} \) to \( B^{s-1}_{p,q} \).

**Proposition A.3** ([36], Prop. A.7). Let \( s_1, s_2 \in \mathbb{R} \) and \( p, p_1, p_2, q \in [1, \infty] \) be such that

$$s_1 < 0 < s_2, \quad s_1 + s_2 > 0, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2};$$

then \( (f, g) \mapsto fg \) is a well-defined continuous bilinear map from \( B^{s_1}_{p_1,q} \times B^{s_2}_{p_2,q} \) to \( B^{s_1}_{p,q} \). 

**Proposition A.4** ([1], Cor. 2.86). For any \( s > 0 \) and \( p, q \in [1, \infty] \), the space \( B^s_{p,q} \cap L^\infty \) is an algebra and there exists a constant \( C = C(s) \) such that

$$\| fg \|_{B^s_{p,q}} \leq C(\| f \|_{L^\infty} \| g \|_{B^s_{p,q}} + \| f \|_{B^s_{p,q}} \| g \|_{L^\infty}) \quad \forall f, g \in B^s_{p,q} \cap L^\infty.$$
Choosing

The result is well known, see Theorem 2.80 from [1] for the statement on $C$.

For any $u$ such a restriction is not necessary and in fact the result holds for any $u$.

Applying Lemma A.5 for the choice $\theta = 1/2$ we find

$$\|f\|_{L^p} \ll \|f\|_{B^{s_1}_p} \ll \|f\|_{B^{s_2}_2} \ll \|f\|_{H^{-1}}.$$

We may assume $\|f\|_{B^{s_1}_p} = 1$; for any $N \geq 0$ it holds

$$\|f\|_{B^{s_1}_p} = \sum_{j < N} 2^j \|f\|_{L^p} + \sum_{j \geq N} 2^j \|\Delta f\|_{L^p}$$

$$\|f\|_{B^{s_2}_2} \ll \sum_{j < N} 2^{-j(s_2-s_1)} + \sum_{j \geq N} 2^{-j(1-s_2)}$$

Choosing $N$ such that $\|f\|_{B^{s_1}_p} \sim 2^{-N(s_2-s_1)}$ the conclusion then follows.

We conclude this appendix by proving some interpolation inequalities, which played a fundamental role in the proofs in Section 3.1.

**Lemma A.5.** Let $p \in [1, \infty]$, $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$ and $\theta \in (0, 1)$. Then there exists a constant $C = C(p, s_2 - s_1, \theta)$ such that

$$\|f\|_{B^{s_1}_p + (1-\theta)s_2} \ll \|f\|_{B^{s_1}_p} \ll \|f\|_{B^{s_2}_2} \quad \forall f \in B^{s_2}_2.$$  

**Proof.** The result is well known, see Theorem 2.80 from [1] for the statement on $\mathbb{R}^d$; let us provide a self-contained proof.

We may assume $\|f\|_{B^{s_1}_p} = 1$; for any $N \geq 0$ it holds

$$\|f\|_{B^{s_1}_p} = \sum_{j < N} 2^j \|f\|_{L^p} + \sum_{j \geq N} 2^j \|\Delta f\|_{L^p}$$

$$\|f\|_{B^{s_2}_2} \ll \sum_{j < N} 2^{-j(s_2-s_1)} + \sum_{j \geq N} 2^{-j(1-s_2)}$$

Choosing $N$ such that $\|f\|_{B^{s_1}_p} \sim 2^{-N(s_2-s_1)}$ the conclusion then follows.

**Corollary A.6.** For any $s_1, s_2 > 0$ there exists a constant $C(s_1, s_2)$ such that

$$\|f\|_{L^2} \ll \|f\|_{B^{s_1}_2} \ll \|f\|_{H^{-s_2}} \quad \forall f \in B^{s_2}_2.$$  

**Proof.** Applying Lemma A.5 for the choice $p = 2, \theta = s_2/(s_1 - s_2)$ and using Besov embeddings we find

$$\|f\|_{L^2} \ll \|f\|_{B^{s_1}_2} \ll \|f\|_{B^{s_2}_2} \ll \|f\|_{H^{-s_2}},$$

APPENDIX A. A SIMPLE EXTENSION OF A RESULT BY WEI

Theorem 5.1 from [43] requires the restriction to work with $u \in C(\mathbb{T})$, but we show here that such a restriction is not necessary and in fact the result holds for any $u \in L^1(\mathbb{T})$, as stated in Theorem 4.6. Let us recall the setting: we are interested in the decay of solutions to complex valued PDEs of the form

$$\partial_t f + iuf = \nu \partial_y^2 f.$$  

(1.1)
Equation B.1 is well-posed (in the weak sense) for any \( u \in L^1(\mathbb{T}) \) and \( f_0 \in L^2(\mathbb{T}) \). Indeed, for smooth \( u \), any solution \( f \) to (B.1) satisfies
\[
\partial_t \|f\|_{L^2}^2 + 2\nu \|\partial_y f\|_{L^2}^2 = 0,
\]
thus implying that it belongs to \( L^2(0, T; H^1(\mathbb{T})) \) \( \mapsto L^2(0, T; C(\mathbb{T})) \); therefore we have uniform estimates for \( iu \tilde{f} \in L^2(0, T; \mathbb{L}^1(\mathbb{T})) \) only depending on \( \|u\|_{L^1} \). Arguing by weak compactness one can then easily construct weak solutions to (B.1) for any \( u \in L^1(\mathbb{T}) \), establish their uniqueness, and show that they are the strong limit in \( C([0, T]; \mathbb{L}^2(\mathbb{T})) \) of those to smooth \( u \). Overall, this defines the semigroup \( t \mapsto e^{t(\nu \partial_y^2 - iu)} \) on \( L^2(\mathbb{T}) \) for any \( u \in L^1(\mathbb{T}) \) and \( \nu > 0 \).

Identifying \( u \in L^1(\mathbb{T}) \) with a 2\( \pi \)-periodic function, its primitive \( \psi \) is a (non periodic) element of \( C(\mathbb{R}) \), well defined up to additive constant; for given \( \delta \in (0, 1) \), define
\[
\omega_1(\delta, u) := \inf_{x, c_1, c_2 \in \mathbb{R}} \int_{x - \delta}^{x + \delta} |\psi(y) - c_1 - c_2 \delta|^2 dy.
\]
Denote by \( F : \mathbb{R} \to [0, \pi/2] \) the inverse of \( x \mapsto 36x \tan x \), which is a one-to-one increasing function. The next statement summarizes some of the main findings from [43].

**Lemma B.1.** Let \( u \in C(\mathbb{T}) \) and \( \nu > 0 \) be fixed; then for all \( \delta \in (0, 1) \) and \( t \geq 0 \) it holds
\[
\|e^{t(\delta \nu^2 - iu)}\|_{L^2 \rightarrow L^2} \leq \exp\left( \frac{\pi}{2} - t\nu\delta^2 F(\delta\nu^{-2}(\omega_1(\delta, u)))^2 \right). \tag{B.2}
\]

**Proof.** By time rescaling, the solution \( f \) to (B.1) is given by \( f(t, y) = f^\nu(t\nu, y) \) where \( f^\nu \) solves \( \partial_t f^\nu + iu' f^\nu = \partial_y^2 f^\nu \) where \( u' = u/\nu \); applying the Gearhart–Prüss theorem (Theorem 1.3 from [43]) to \( f^\nu \), it holds
\[
\|f_t\|_{L^2} \leq \exp\left( \frac{\pi}{2} - t\nu\psi_1(u') \right)
\]
where \( \psi_1(u) \) is defined as in Section 4 from [43]. By Lemma 4.3 therein and the 2-homogeneity of \( u \mapsto \omega(\delta, u) \), for any \( \delta \in (0, 1) \) it holds
\[
\psi_1(u') \geq \delta^{-2} F(\delta(\omega_1(\delta, u')))^2 = \delta^{-2} F(\delta\nu^{-2}(\omega_1(\delta, u)))
\]
which gives the conclusion. \( \Box \)

We can now give the

**Proof of Theorem 4.6.** By time rescaling, we can restrict to the case \( k = 1 \). Now let \( u \in L^1(\mathbb{T}) \) be a function satisfying \( \Gamma_\alpha(u) > 0 \) for some \( \alpha \in (0, 1) \) and consider a family \( \{u', \varepsilon > 0\} \) of continuous functions satisfying \( \|u'' - u\|_{L^1} \leq \varepsilon \). Denote by \( \psi^\varepsilon \) the primitive of \( u'' \); by the basic inequality \( a^2 \geq b^2/2 - (a - b)^2 \), for any \( \delta \in (0, 1) \) it holds
\[
\omega_1(\delta, u') \geq \frac{1}{2} \omega_1(\delta, u) - \sup_{x \in \mathbb{R}} \left| \int_{x - \delta}^{x + \delta} \psi'^\varepsilon(y) - \psi(y) \right|^2 dy
\geq \frac{1}{2} \omega_1(\delta, u) - 2\delta \|u - u''\|^2_{L^1(\mathbb{T})}.
\]
Combined with the fact that by definition \( \omega_1(\delta, u) \geq 2^{2\alpha+3}\delta^{2\alpha+3}\Gamma_\alpha(u)^2 \), we deduce
\[
\omega_1(\delta, u') \geq 2^{2\alpha+2}\delta^{2\alpha+3}\Gamma_\alpha(u)^2 - 2\delta \varepsilon^2 \quad \forall \delta \in (0, 1), \varepsilon > 0. \tag{B.3}
\]
Now fix \( \nu > 0 \) and define \( C_1 = e^{\pi/2}, C_2 = 2^{2\alpha+2}\Gamma_\alpha(u)^2 \); applying Lemma B.1 to \( u'' \), exploiting the fact that \( F \) is increasing, and choosing \( \delta = \nu^{1/(\alpha+2)} \), we obtain
\[
\|e^{t(\delta \nu^2 - iu''')}\|_{L^2 \rightarrow L^2} \leq C_1 \exp\left( -t\nu^{\alpha+2} F\left( C_2 - 2\nu^{-2}\delta^{2\alpha+3}\varepsilon^2 \right)^2 \right) \quad \forall t \geq 0. \tag{B.4}
\]
where the estimate holds for all $\varepsilon > 0$ small enough such that $C_2 - 2\nu^{-2(\alpha+1)/(\alpha+2)}\varepsilon^2 > 0$.

Since the semigroup $e^{\nu(\partial^2_y - iu^2)}$ pointwise converges to $e^{\nu(\partial^2_y - iu^2)}$ as $\varepsilon \to 0^+$, passing to the limit on both sides of (B.4) gives the conclusion. \qed

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