Super-Calogero model with $OSp(2|2)$ supersymmetry: is the construction unique?

Pijush K. Ghosh

Theory Division,
Saha Institute of Nuclear Physics,
Kolkata 700 064, India.

Abstract

We show that the construction of super-Calogero model with $OSp(2|2)$ supersymmetry is not unique. In particular, we find a new co-ordinate representation of the generators of the $OSp(2|2)$ superalgebra that appears as the dynamical supersymmetry of the rational super-Calogero model. Both the quadratic and the cubic Casimir operators of the $OSp(2|2)$ are necessarily zero in this new representation, while they are, in general, nonzero for the super-Calogero model that is currently studied in the literature. The Casimir operator that exists in the new co-ordinate representation is not present in the case of the existing super-Calogero model. We also discuss the case of $N$ free superoscillators and superconformal quantum mechanics for which the same conclusions are valid.

*Electronic address: pijush@theory.saha.ernet.in
I. INTRODUCTION

Ever since the supersymmetric extension of the rational Calogero model was introduced in [8], various aspects of this class of supermodels have been studied [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. These supermodels are exactly solvable [8, 9, 10, 11] and play an important role in our understanding of the integrable structure of many-particle quantum mechanics [12, 13, 14]. Further, they are relevant in the study of the superstrings [15], black-holes [16], many-particle superconformal quantum mechanics [17, 18, 19], superpolynomials [20], spin chains [21, 22] etc.

The rational super-Calogero model has a dynamical $OSp(2|2)$ supersymmetry [8]. The super-Hamiltonian and the other generators satisfying the structure equations of $OSp(2|2)$ are represented in terms of a set of bosonic and fermionic variables. In all subsequent discussions of the super-Calogero model introduced in [8], the same co-ordinate representation of the $OSp(2|2)$ generators have been used. Although the construction of [8] is a very standard one, it is pertinent to ask whether the $OSp(2|2)$ superextension of the rational Calogero model is unique or not? In other words, is it possible to have a different co-ordinate representation of the generators of the $OSp(2|2)$ than the one originally introduced in [8]? We indeed find that such a possibility exists and the $OSp(2|2)$ superextension of the Calogero model is not unique.

In particular, we show that the generators of the $OSp(2|2)$ algebra can be represented in terms of the co-ordinates of the super-Calogero model in two different ways. One of them can be shown to be equivalent to the construction of Ref. [8] and we refer to this particular co-ordinate representation of the generators as ‘standard superextension’ in all subsequent references. The corresponding super-Hamiltonian will be referred as standard super-Calogero model. The other representation of the generators, although appeared in the study of the supersymmetric magnetic monopoles [25], has never been discussed in the literature in the context of general many-particle supersymmetric quantum mechanics or Calogero models. Both the quadratic and the cubic Casimir operators in this case are
necessarily zero. Thus, the representation of the group $OSp(2|2)$ is necessarily ‘atypical’\cite{23, 24}, i.e., the Casimir operators do not specify the spectrum completely. We refer to this case as ‘atypical superextension’ for all future purposes. We refer the corresponding super-Hamiltonian as the ‘chiral super-Calogero Hamiltonian’, since it is known from the study of supersymmetric magnetic monopoles\cite{25} that the eigenstates for such a representation have definite chirality.

The differences between these two co-ordinate representations are the following. First of all, the Hamiltonian of the standard super-Calogero model and the chiral super-Calogero model are different. Further, in contrast to the ‘atypical superextension’, the quadratic and the cubic Casimir operators of the ‘standard superextension’ in terms of the co-ordinate representation are not necessarily zero. The representation of the algebra can be either ‘typical’ or ‘atypical’ depending on the particular condition used on the wavefunction. The eigenvalues of the Casimir’s can be zero only for the special case corresponding to the ‘atypical’ representation of the group $OSp(2|2)$. Moreover, for the ‘atypical superextension’, there exists a Scasimir operator\cite{26}. This is an even operator which commutes with all the bosonic generators and anti-commutes with all the fermionic generators of the $OSp(2|2)$. This even operator is called Scasimir, because it squares to the Casimir of the $OSp(1|1)$, a subgroup of $OSp(2|2)$. The Scasimir does not exists for the $OSp(2|2)$ in the case of the standard superextension.

We also show that the angular part of the super-Calogero model, in the N-dimensional hyper-spherical coordinate, can be factorized in terms of first order differential operators that commute with the total Hamiltonian. Thus, eigen spectrum of these first order operators can be used to find the eigen spectrum of the angular part of the super-Calogero model. The result is valid for both the standard and the atypical superextension with the difference that the factorization can be done in two independent ways for the former case, while only one factorization is possible for the later case. To the best of our knowledge, this has never been realized before for the standard superextension. The reformulation of the problem in terms of two real supercharges made us to see this subtle connection. We also show that super-
Calogero model, for both the standard and the atypical superextension, can be deformed suitably to admit $\mathcal{N} = 0$ supersymmetry [27].

Our plan of presenting the results is the following. We briefly review about the $\text{OSp}(2|2)$ supersymmetry, the Casimir and the Scasimir operators in the next section. In section III, we explicitly write down the coordinate representations of the $\text{OSp}(2|2)$ generators and present our results. The standard and the atypical superextension are discussed in sections III.A and III.B, respectively. The results presented till the end of the section III.B are quite general. We consider the specific cases of superoscillators, superconformal quantum mechanics [28, 29] and super-Calogero models in section IV. Finally, we conclude with discussions in section V.

II. SUSY ALGEBRA

The $\text{OSp}(2|2)$ algebra is described in terms of the set of fermionic generators $f \equiv \{Q_1, Q_2, S_1, S_2\}$ and the set of bosonic generators $b \equiv \{H, D, K, Y\}$. The structure-equations of the $\text{OSp}(2|2)$ are [23, 24],

\[
\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta}H, \quad \{S_\alpha, S_\beta\} = 2\delta_{\alpha\beta}K, \quad \{Q_\alpha, S_\beta\} = -2\delta_{\alpha\beta}D + 2\epsilon_{\alpha\beta}Y,
\]

\[
[H, Q_\alpha] = 0, \quad [H, S_\alpha] = -iQ_\alpha, \quad [K, Q_\alpha] = iS_\alpha, \quad [K, S_\alpha] = 0,
\]

\[
[D, Q_\alpha] = -\frac{i}{2}Q_\alpha, \quad [D, S_\alpha] = \frac{i}{2}S_\alpha, \quad [Y, Q_\alpha] = \frac{i}{2}\epsilon_{\alpha\beta}Q_\beta, \quad [Y, S_\alpha] = \frac{i}{2}\epsilon_{\alpha\beta}S_\beta,
\]

\[
[Y, H] = [Y, D] = [Y, K] = 0, \quad \alpha, \beta = 1, 2,
\]

along with the algebra of its $O(2, 1)$ subgroup,

\[
[H, D] = iH, \quad [H, K] = 2iD, \quad [D, K] = iK.
\]

The Casimir of the $O(2, 1)$ is $C = \frac{1}{2}(HK + KH) - D^2$. The quadratic and the cubic Casimir operators of $\text{OSp}(2|2)$ are given by [23, 24],

\[
C_2 = C + \frac{i}{4}[Q_1, S_1] + \frac{i}{4}[Q_2, S_2] - Y^2,
\]

\[
C_3 = C_2Y - \frac{Y}{2} + \frac{i}{8}([Q_1, S_1]Y + [Q_2, S_2]Y + [S_1, Q_2]D
\]

\[
- [S_2, Q_1]D + [Q_1, Q_2]K + [S_1, S_2]H).
\]
The particular form of the cubic Casimir $C_3$ has been derived from the general expression given in Ref. [24]. We will show that both $C_2$ and $C_3$ are zero for the ‘atypical’ superextension, while non-zero for the ‘standard’ superextension.

The subgroup $OSp(1|1)$ of $OSp(2|2)$ is described either by the set of generators $A_1 \equiv \{H, D, K, Q_1, S_1\}$ or $A_2 \equiv \{H, D, K, Q_2, S_2\}$. Let us first consider the set $A_1$. The Casimir of the $OSp(1|1)$ is given by,

$$C_1 = C + \frac{i}{4}[Q_1, S_1] + \frac{1}{16}. \tag{4}$$

One can define an even operator $C_s$,

$$C_s = i[Q_1, S_1] - \frac{1}{2}, \tag{5}$$

which has the property that it commutes with all the bosonic generators and anticommutes with all the fermionic generators of the set $A_1$. Moreover, it satisfies, $C_1 = \frac{1}{4}C_s^2$ and consequently, $C = \frac{1}{4}C_s(C_s - 1) - \frac{3}{16}$. The operator $C_s$ is known as Scasimir and has important applications in the description of the nonrelativistic dynamics of a spin-$\frac{1}{2}$ particle in the background of monopoles, Dirac-Coulomb problem etc. [25, 30]. The Casimir and the Scasimir for the set $A_2$ are given by,

$$\tilde{C}_1 = C + \frac{i}{4}[Q_2, S_2] + \frac{1}{16}, \quad \tilde{C}_s = i[Q_2, S_2] - \frac{1}{2}. \tag{6}$$

with the properties $\tilde{C}_1 = \frac{1}{4}\tilde{C}_s^2$ and $C = \frac{1}{4}\tilde{C}_s(C_s - 1) - \frac{3}{16}$. This implies that, in general, the Casimir $C$ can be factorized in two different ways, either in terms of $C_s$ or $\tilde{C}_s$.

We will show that the Scasimir $C_s$ and $\tilde{C}_s$ are identical for the ‘atypical’ superextension. Consequently, the Casimir $C$ can be factorized only in one way. Moreover, the Scasimir commutes with all the even operators $b$ and anticommutes with all the odd operators $f$ of the $OSp(2|2)$. Thus, the Scasimir of the $OSp(1|1)$ can be promoted to be the Scasimir of $OSp(2|2)$. On the other hand, for the standard superextension, $C_s$ and $\tilde{C}_s$ are different. Neither $C_s$ nor $\tilde{C}_s$ or any linear combination of them can be promoted to be the Scasimir of $OSp(2|2)$.

The generator of the compact rotation $R$ and the creation-annihilation operator $B_\pm$ of the
\( O(2, 1) \) are given by,

\[
R = \frac{1}{2}(H + K), \quad B_{\pm} = \frac{1}{2}K - \frac{1}{2}H \pm iD,
\]

and satisfy the algebra,

\[
[R, B_{\pm}] = \pm B_{\pm}, \quad [B_{-}, B_{+}] = 2R.
\]

The structure-equations (1) can be written in the Cartan basis where \( R \) and \( Y \) are simultaneously diagonal. In the Cartan basis, define the operators,

\[
\mathcal{H}_+ = R + Y, \quad \mathcal{H}_- = R - Y.
\]

We will show that both \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) have different co-ordinate representations corresponding to ‘atypical’ and ‘standard’ superextension. \( \mathcal{H}_+ \) is related to the \( \mathcal{H}_- \) and the vice versa through an automorphism of the \( OSp(2|2) \) algebra. So, it is suffice to consider either of them as the defining super-Hamiltonian.

### III. CO-ORDINATE REPRESENTATION

We will now show that it is possible to choose two different co-ordinate representations of the generators of \( OSp(2|2) \). Our choice is such that the generators of the subgroup \( OSp(1|1) \), namely \( A_1 \equiv \{H, D, K, Q_1, S_1\} \), have the same co-ordinate representation for both the atypical and standard superextension. It is only the generators that enlarge the \( OSp(1|1) \) to \( OSp(2|2) \), namely \( \{Q_2, S_2, Y\} \), that are chosen to be different for these two cases.

We follow the convention of considering \( \{Q_2, S_2, Y\} \) as part of the generators of the ‘standard’ superextension, while \( \{\hat{Q}_2, \hat{S}_2, \hat{Y}\} \) as that of the ‘atypical’ superextension. Consequently, any new operator/generator that is defined using any one of the operators \( \{\hat{Q}_2, \hat{S}_2, \hat{Y}\} \) will be denoted by \( \hat{O} \) to distinguish it from \( O \) that is constructed out of \( \{Q_2, S_2, Y\} \).

The operator \( R \) is defined solely in terms of \( H, D \) and \( K \) in Eq. (7). So, it has the same co-ordinate representation for both the ‘standard’ and the ‘atypical’ superextension. On the other hand, \( Y = \frac{1}{2}\{Q_1, S_2\} \) and \( \hat{Y} = \frac{1}{2}\{Q_1, \hat{S}_2\} \) are chosen to be different for these two cases. It is now obvious from Eq. (9) that both the Hamiltonian \( \mathcal{H}_{\pm} = R \pm Y \) and
\( \mathcal{H}_\pm = R \pm \hat{Y} \) have different co-ordinate representations, corresponding to ‘atypical’ and ‘standard’ superextension.

We introduce \( 2N \) entities \( \xi_i \) satisfying the following real Clifford algebra,
\[
\{ \xi_i, \xi_j \} = 2g_{ij}, \quad g_{ij} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix},
\]
where \( I \) is an \( N \times N \) identity matrix. The signature of the metric \( g_{ij} \) is such that the square of the \( \xi_i \)'s is equal to 1 or \( -1 \) depending on whether \( i > N \) or \( i \leq N \), respectively. In particular,
\[
\xi_{N+i}^2 = -\xi_i^2 = 1, \quad i = 1, 2, \ldots, N.
\]
(11)

It is known that one can introduce a new operator \( \gamma_5 \),
\[
\gamma_5 = \xi_1 \xi_2 \ldots \xi_{2N-1} \xi_{2N},
\]
(12)
which anticommutes with all the \( \xi_i \)'s. Further, we can introduce two idempotent operators
\[
\gamma_5^\pm = \frac{1}{2}(1 \pm \gamma_5) \quad \text{with} \quad (\gamma_5^\pm)^2 = \gamma_5^\pm \quad \text{and} \quad \gamma_5^+ \gamma_5^- = \gamma_5^- \gamma_5^+ = 0, \quad \text{since} \quad \gamma_5^2 = 1.
\]
The operators \( \gamma_5 \) and \( \gamma_5^\pm \) play an important role in constructing independent real supercharges.

Most of our discussion will be based on the above real Clifford algebra. However, we also need the complexified version of the above algebra in order to make connection with the known results in the literature. We define a set of fermionic variables \( \psi_i \) and their conjugates \( \psi_i^\dagger \) as,
\[
\psi_i = \frac{i}{2}(\xi_i - \xi_{N+i}), \quad \psi_i^\dagger = \frac{i}{2}(\xi_i + \xi_{N+i}).
\]
(13)
These fermionic variables satisfy the Clifford algebra,
\[
\{ \psi_i, \psi_j \} = \{ \psi_i^\dagger, \psi_j^\dagger \} = 0, \quad \{ \psi_i, \psi_j^\dagger \} = \delta_{ij}.
\]
(14)
The fermionic vacuum \( |0\rangle \) and its conjugate \( |\bar{0}\rangle \) in the \( 2^N \) dimensional fermionic Fock space are defined as, \( \psi_i|0\rangle = 0, \psi_i^\dagger|\bar{0}\rangle = 0 \). The fermion number corresponding to each particle is defined as, \( n_i = \psi_i^\dagger \psi_i \) and the total fermion number \( n = \sum_i n_i \). The total fermion number \( n = 0 \) for the fermionic vacuum and \( n = N \) for the conjugate vacuum. An equivalent expression for \( \gamma_5 \) in terms of \( n_i \) can be written as,
\[
\gamma_5 = (-1)^N \prod_{i=1}^N (2n_i - 1).
\]
(15)
The action of $\gamma_5$ on a state $|n\rangle$ with fermion number $n$ is, $\gamma_5|n\rangle = (-1)^n|n\rangle$, where $0 \leq n \leq N$. Note that $\gamma_5$ leaves the fermionic vacuum invariant, i.e., $\gamma_5|0\rangle = |0\rangle$. On the other hand, $\gamma_5|\bar{0}\rangle = (-1)^N|\bar{0}\rangle$, implying that the conjugate vacuum is invariant for even $N$ and changes sign for odd $N$.

Consider the following Hamiltonian,

$$H = \frac{1}{2N} \sum_{i=1}^{N} (p_i^2 + W_i^2) - \frac{1}{2} \sum_{i,j=1}^{N} \xi_i \xi_{N+j} W_{ij}, \quad W_i = \frac{\partial W}{\partial x_i}, \quad W_{ij} = \frac{\partial^2 W}{\partial x_i \partial x_j}. \quad (16)$$

The superpotential $W = \ln G$ with $G$ being a homogeneous function of degree $d$. This ensures that the bosonic potential in $H$ scales inverse-squarely. Subsequently, we define the Dilatation operator $D$ and the conformal operator $K$ as,

$$D = -\frac{1}{4} \sum_{i=1}^{N} (x_i p_i + p_i x_i), \quad K = \frac{1}{2} \sum_i x_i^2. \quad (17)$$

The operators $H$, $D$ and $K$ together satisfy an $O(2,1)$ algebra. We now introduce the supercharges $Q_1$ and $S_1$,

$$Q_1 = \frac{1}{\sqrt{2}} \sum_{i=1}^{N} [-i \xi_i p_i + \xi_{N+i} W_i], \quad S_1 = -\frac{i}{\sqrt{2}} \sum_{i=1}^{N} \xi_i x_i. \quad (18)$$

The supercharges $Q_1$ and $S_1$ squares to the Hamiltonian $H$ and the conformal generator $K$, respectively.

The Casimir $C$ of the $O(2,1)$ algebra can be shown to be equivalent to the angular part of the $H$ in the $N$-dimensional hyper-spherical co-ordinates, up to an additional constant shift. In particular,

$$C = \frac{1}{4} \left[ \sum_{i<j} L_{ij}^2 + r^2 \left( \sum_{i=1}^{N} W_i^2 - \sum_{i,j=1}^{N} \xi_i \xi_{N+j} W_{ij} \right) + \frac{1}{4} N(N-4) \right], \quad (19)$$

where $L_{ij} = x_i p_j - x_j p_i$ and $r^2 = \sum_i x_i^2$. Note that both $W_i^2$ and $W_{ij}$ scale inverse-squarely. So, in the $N$ dimensional hyperspherical co-ordinates $r^2(\sum_i W_i^2 - \sum_{i,j} \xi_i \xi_{N+j} W_{ij})$ contains only angular variables. In Ref. [32], such a relation between the Casimir operator and the angular part of a Hamiltonian with dynamical $O(2,1)$ symmetry and involving only bosonic co-ordinates was derived. Eq. (19) generalizes the work of [32] by including both bosonic and fermionic degrees of freedom. Note that the relation between the Casimir and the angular
part of the Hamiltonian is valid as long as there is an underlying $O(2,1)$ symmetry and does not depend on whether the Hamiltonian is supersymmetric or not.

**A. Standard superextension**

We consider the case of ‘standard’ superextension in this subsection. We first present our results in terms of real supercharges and at the end of this subsection, we use complex supercharges to make correspondence with the current literature. To the best of our knowledge, the many-particle supersymmetric quantum mechanics with dynamical $OSp(2|2)$ supersymmetry has never been discussed in terms of real supercharges. Although both the formulation give the same result, the formulation in terms of real supercharges brings out certain subtle features in connection with the Casimir’s of the model. We believe such features have never been discussed previously.

The generators $Q_2$, $S_2$ and $Y$ are given by,

$$Q_2 = -\frac{1}{\sqrt{2}} \sum_i [\xi_{N+i} p_i + i \xi_i W_i],$$

$$S_2 = -\frac{1}{\sqrt{2}} \sum_i \xi_{N+i} x_i,$$

$$Y = -\frac{1}{4} \left( \sum_i \xi_{N+i} \xi_i + 2d \right).$$

The structure-equations (11) of $OSp(2|2)$ are satisfied with the above co-ordinate representation of the generators. In order to see that the Casimir operators $C_2$ and $C_3$ are non-zero, we first find,

$$i[Q_1, S_1] = \frac{N}{2} + \frac{i}{2} \sum_{i,j=1}^N \xi_i \xi_j L_{ij} + \sum_{i,j=1}^N \xi_{N+i} \xi_j x_j W_i,$$

$$i[Q_2, S_2] = \frac{N}{2} - \frac{i}{2} \sum_{i,j=1}^N \xi_{N+i} \xi_{N+j} L_{ij} - \sum_{i,j=1}^N \xi_{N+i} \xi_j x_i W_j,$$

$$Y^2 = \frac{1}{16} \left( 4d^2 + N + 4d \sum_i \xi_{N+i} \xi_i - \frac{1}{2} \sum_{i \neq j} [\xi_{N+i}, \xi_{N+j}] \xi_i \xi_j \right).$$

Note that $C$ in Eq. (11) contains a term proportional to $L_{ij}^2$. On the other hand, $i[Q_1, S_1] + i[Q_2, S_2]$ does not contain any term proportional to $L_{ij}^2$. It only contains a term proportion
to $L_{ij}$ and $Y^2$ is independent of bosonic coordinates. This implies that $C_2$ in (3) can not be zero. Similarly, the first term of $C_3$ in (3) contains a term of the form $L_{ij}^2 Y$. However, such a term can not be generated from rest of the terms in the definition of $C_3$. So, the cubic Casimir $C_3$ is also non-zero. It should be noted here that although the Casimir’s $C_2$ and $C_3$ are nonzero in terms of the coordinate representation, their eigenvalues can be zero for the special case corresponding to the atypical representation of the $OSp(2|2)$ algebra.

The coordinate representation of the operators $C_s = i[Q_1, S_1] - \frac{1}{2}$ and $\bar{C}_s = i[Q_2, S_2] - \frac{1}{2}$ can be simply found from the first two equations of (21). Recall that $C_s$ is the Scasimir of the $OSp(1|1)$ corresponding to the set of generators $A_1$. Similarly, $\bar{C}_s$ is the Scasimir of the $OSp(1|1)$ corresponding to the set of generators $A_2$. We find that $C_s$ neither commutes with $Y$ nor anticommutes with $Q_2$ and $S_2$. So, $C_s$ can not be considered as the Scasimir of the full $OSp(2|2)$. Similarly, $[\bar{C}_s, Y] \neq 0$ and $\{Q_1, \bar{C}_s\} \neq 0 \neq \{S_1, \bar{C}_s\}$. The operator $\bar{C}_s$ either can not be considered as the Scasimir of the $OSp(2|2)$. An explicit calculation shows that $[C_s + \bar{C}_s, Y] = 0$. However, the anticommutators between $C_s + \bar{C}_s$ and the supercharges $Q_1, Q_2, S_1, S_2$ are nonzero. Thus, we conclude that the Scasimir’s of the subgroup $OSp(1|1)$ or any linear combination of them can not be promoted to be the Scasimir of the full $OSp(2|2)$.

A few comments are in order at this point. The Casimir $C$ of the $O(2, 1)$ is equivalent to the angular part of the Hamiltonian $H$ in $N$ dimensional hyper-spherical coordinates. We have seen that $C$ can be factorized either in terms of $C_s$ or $\bar{C}_s$. Both $C_s$ and $\bar{C}_s$ commute with $H$ and between themselves. So, $C_s$, $\bar{C}_s$ and $C$ can be diagonalized simultaneously. Both $C_s$ and $\bar{C}_s$ being first order differential operator, it may be much easier to solve them compared to $C$. Once the eigenvalues of $C_s, \bar{C}_s$ are known, the eigenvalues of $C$ can be computed easily. Secondly, the same method can also be used for a Hamiltonian,

$$H' = H + V(r),$$  \hspace{1cm} (22)

for which the supersymmetry is explicitly broken by the introduction of an arbitrary potential $V(r)$. Note that the angular part of $H'$ and $H$ are still the same. Further, both $C_s$ and $\bar{C}_s$ commutes with $H'$. So, $C_s$ and $\bar{C}_s$ can be used to solve the non-supersymmetric Hamiltonian.
This is the first example of $\mathcal{N} = 0$ supersymmetry in the context of many-particle quantum mechanics.

In order to see that the coordinate representation given by eqs. (16,17,18,19,20) indeed corresponds to the standard superextension, we write the super-Hamiltonian $H^\pm$ in terms of $\psi_i$ and $\psi_i^\dagger$ as,

$$H^\pm = \frac{1}{4} \sum_i \left( p_i^2 + W_i^2 + x_i^2 \right) + \frac{1}{4} \sum_{i,j} \left[ \psi_i, \psi_j^\dagger \right] W_{ij} \pm \frac{1}{2} \left( n - \frac{N}{2} - d \right).$$

This form matches with the super-Hamiltonian with dynamical $OSp(2|2)$ supersymmetry that was constructed in Ref. [10]. The supercharges are,

$$F^L_{\pm} = \pm \frac{i}{2} \sum \psi_i \left( p_i - iW_i \pm ix_i \right), \quad F^R_{\pm} = \pm \frac{i}{2} \sum \psi_i^\dagger \left( p_i + iW_i \pm ix_i \right), \quad \left( F^L_{\pm} \right)^\dagger = F^R_{\mp},$$

with $H_{\pm} = \{ F^L_{\pm}, F^R_{\mp} \}$. The total fermion number $n$ commutes with $H_{\pm}$. Projecting $H_{\pm}$ in the zero fermion sector and $H_{-}$ in the $N$-fermion sector, we get the purely bosonic Hamiltonian $H^b_{\pm}$. In particular,

$$H^b_{\pm} = \frac{1}{4} \sum_i \left( p_i^2 + W_i^2 \pm W_{ii} + x_i^2 \right) - \frac{1}{2} \left( \frac{N}{2} \pm d \right).$$

The bosonic Hamiltonian $H^b_{\pm}$ belongs to the general class of Hamiltonian producing Calogero models [10]. The ground state wavefunction of $H^b_{\pm}$ is, $\phi_+ = e^{W - \frac{i}{4} \sum x_i^2}$, while that of $H^b_{-}$ is, $\phi_- = e^{-W - \frac{i}{4} \sum x_i^2}$. The ground-state energy is zero for both the cases. The parameters of $H^b_{\pm}$ are contained in the superpotential $W$. Note that $\phi^b_+$ and $\phi^b_-$ can not be square integrable for the same range of parameters, since $W = \ln G$ with $G$ a homogeneous function. Thus, the admissible range of parameters due to the square integrability of wave-functions, is different for $H^b_+$ and $H^b_-$.

**B. Atypical Superextension**

We introduce the supercharges $\hat{Q}_2$, $\hat{S}_2$ and the bosonic generator $\hat{Y} = \frac{1}{2} \{ Q_1, S_2 \}$ for the ‘atypical’ superextension as [25],

$$\hat{Q}_2 = -i\gamma_5 Q_1, \quad \hat{S}_2 = -i\gamma_5 S_1,$$
\[ \dot{Y} = \frac{\gamma_5}{2} \left[ \frac{i}{2} \sum_{i,j} \xi_i \xi_j L_{ij} + \sum_{i,j} \xi_{N+i} \xi_j W_{ij} x_j + \frac{N}{2} \right]. \] (26)

The above co-ordinate representation of the generators is not identical to the one given in Eq. (20), only if the total number of particles \( N \) is greater than or at least equal to two, i.e. \( N \geq 2 \). So, our results emerge as the many-particle or higher dimensional features of the usual one dimensional supersymmetric quantum mechanics. The structure-equations (1) are satisfied by the bosonic generators \( \{H, D, K, \dot{Y}\} \) and the fermionic generators \( \{Q_1, \dot{Q}_2, S_1, \dot{S}_2\} \). The cubic Casimir now takes a simple form [25],

\[ C_3 = (\dot{Y} - \frac{\gamma_5}{4}) C_2. \] (27)

Using the defining relation for \( \dot{Y} \), we find \( \dot{Y} = \frac{\gamma_5}{2} (C_s + \frac{1}{2}) \) and consequently, \( \dot{Y}^2 = \frac{1}{4} (C_s^2 + C_s + \frac{1}{4}) \). Expressing all other terms in \( C_2 \) in terms of \( C_s \), it is now easy to show that,

\[ C_2 = 0, \quad C_3 = (\dot{Y} - \frac{\gamma_5}{4}) C_2 = 0. \] (27)

The spectrum is not completely specified by the eigenvalues of the Casimir’s and the representation is necessarily ‘atypical’. Further, note that the operators \( C_s = \dot{C}_s \). Using the structure-equations of \( OSp(2|2) \), we find that \( C_s(\dot{C}_s) \) commutes with \( H, D, K, \dot{Y} \) and anti-commutes with \( Q_1, \dot{Q}_2, S_1, \dot{S}_2 \). So, the operator \( C_s \) is also the Casimir of the \( OSp(2|2) \) supergroup and can be used to determine the eigenspectrum.

As in the case of standard superextension, \( \mathcal{N} = 0 \) supersymmetry is also present for the ‘atypical’ superextension. The only difference being that we now have only one independent operator \( C_s \) that factorized the Casimir \( C \). The co-ordinate representation of \( C_s \) can still be determined from the first equation of Eq. (21) and it can be used to find the spectrum of non-supersymmetric Hamiltonian \( H' \).

The super-Hamiltonian \( \hat{\mathcal{H}}_\pm \) can be written in terms of \( \psi_i \) and \( \psi_i^\dagger \) as,

\[ \hat{\mathcal{H}}_\pm = \frac{1}{4} \sum_i \left( \hat{p}_i^2 + W_i^2 + x_i^2 \right) + \frac{1}{4} \sum_{i,j} \left[ \psi_i \psi_j^\dagger \right] W_{ij} \pm \frac{\gamma_5}{4} \left[ N + 2d \right. \]

\[ - i \sum_{i,j} \left( \psi_i^\dagger \psi_j^\dagger + \psi_i^\dagger \psi_j \right) e^W L_{ij} e^{-W} - i \sum_{i,j} \left( \psi_i \psi_j - \psi_j^\dagger \psi_i \right) e^{-W} L_{ij} e^W \]

\[ - \sum_{i,j} \left( \psi_i^\dagger \psi_j + \psi_j^\dagger \psi_i \right) \left( x_i W_j + x_j W_i \right) \]. \] (28)

The supercharges are,

\[ \hat{F}_L^\pm = \gamma_5^- \left( F_{L+}^R + F_{L-}^R \right), \quad \hat{F}_R^\pm = \gamma_5^+ \left( F_{R-}^L + F_{R+}^L \right), \quad \left( \hat{F}_L^\pm \right)^\dagger = \hat{F}_R^\pm, \] (29)
with \( \hat{\mathcal{H}}_\pm = \{ \hat{F}^L, \hat{F}^R \} \). Comparing (23) and (28), it is obvious that \( \mathcal{H}_\pm \) is different from \( \hat{\mathcal{H}}_\pm \). However, the total fermion number \( n \) is not a conserved quantity for \( \hat{\mathcal{H}}_\pm \). Thus, the states cannot be labeled in terms of \( n \). Note that the fermion-number violating terms appear only in the expression for \( \hat{Y} \). The operator \( \hat{Y} \) is simultaneously diagonalized with \( R \) and, in general, eigenvalue of \( Y \) can be any complex number \([23, 24] \). If we choose the eigenvalue of \( \hat{Y} \) to be \( Y^+ = \frac{1}{4}(N + 2d) \) and project \( R \) to the fermionic vacuum \( |0\rangle \), we see that \( \hat{\mathcal{H}}_- \) contains the purely bosonic Hamiltonian \( \mathcal{H}_b^- \). Similarly, if we choose the eigenvalue of \( \hat{Y} \) to be \( Y^- = -\frac{1}{4}(N - 2d) \) and project \( R \) to the conjugate fermionic vacuum \( |\bar{0}\rangle \), the purely bosonic Hamiltonian \( \mathcal{H}_b^- \) appears from \( \hat{\mathcal{H}}_+ \). The eigenstates of \( \hat{Y} \) corresponding to the eigenvalues \( Y^\pm \) are \( \phi^+|0\rangle \) and \( \phi^-|\bar{0}\rangle \), respectively. Note that \( \phi^+|0\rangle \) and \( \phi^-|\bar{0}\rangle \) are simultaneous eigenstates of \( R \). We conclude that the purely bosonic Hamiltonian \( \mathcal{H}_b^\pm \) can be obtained from both \( \mathcal{H}_\pm \) and \( \hat{\mathcal{H}}_\pm \) through appropriate projections.

Finally, a few comments are in order. We have followed the convention of having the same coordinate representation for the generators \( H, D, K, Q_1, S_1 \) for both standard and the atypical superextension. The coordinate representation of \( Q_2, S_2 \) and \( Y \) are different for these two cases. We mention here about two other possibilities. Let us define,

\[
\hat{Q}_1 = -i\gamma_5Q_2, \quad \hat{S}_1 = -i\gamma_5S_2, \quad \hat{Y} = \frac{1}{2}\{\hat{Q}_1, S_2\}.
\]  

(30)

The operators \( \{\hat{Q}_1, Q_2, \hat{S}_1, S_2, H, D, K, \hat{Y}\} \) constitute a second `atypical' superextension of the bosonic Hamiltonian \( \mathcal{H}^b_\pm \). Note that \( \hat{Y} = -\frac{i\gamma_5}{2}[Q_2, S_2] \) and \( \hat{Y} = \frac{i\gamma_5}{2}[Q_1, S_1] \), where the commutators \([Q_1, S_1]\) and \([Q_2, S_2]\) are determined from Eq. (21). So, the first chiral super-model \( \hat{\mathcal{H}}_\pm \) and the second chiral super-model \( \hat{\mathcal{H}} = R \pm \hat{Y} \) have different co-ordinate representations. Similarly, \( \{\hat{Q}_1, \hat{Q}_2, \hat{S}_1, \hat{S}_2, H, D, K, -Y\} \) constitute another `standard' superextension of the bosonic Hamiltonian \( \mathcal{H}^b_\pm \). However, in this case, \( \mathcal{H}_\pm \) of Eq. (23) simply changes to \( \mathcal{H}_\pm \).

One may ask at this point whether the supercharges \( \{Q_1, Q_2, \hat{Q}_1, \hat{Q}_2\} \) can be used to construct an extended supersymmetry involving the Hamiltonian \( H \). Although each of these supercharges squares to \( H \), some of the anti-commutators involving two different super-
charges are non-zero. In particular, we find the following relations,

\[ H = Q_1^2 = Q_2^2 = \hat{Q}_1^2 = \hat{Q}_2^2, \]
\[ \{Q_1, Q_2\} = \{\hat{Q}_1, \hat{Q}_2\} = \{Q_1, \hat{Q}_2\} = \{Q_2, \hat{Q}_1\} = 0, \]
\[ \{Q_1, \hat{Q}_1\} = i\gamma_5 [Q_1, Q_2], \quad \{Q_2, \hat{Q}_2\} = -i\gamma_5 [Q_1, Q_2], \quad \]

where the last two anti-commutators are nonzero. So, an extended supersymmetry for \( H \) with the supercharges \( \{Q_1, Q_2, \hat{Q}_1, \hat{Q}_2\} \) can not be constructed.

**IV. EXAMPLES**

Our discussions so far have been confined to the general superpotential \( W \). Specific choices of \( W \) lead to different interesting physical models. We consider here three examples, (i) free superoscillators, (ii) superconformal quantum mechanics and (iii) super-Calogero models. The ‘standard’ super-models for these three cases are exactly solvable. Although the representation of the \( OSp(2|2) \) is necessarily ‘atypical’ for chiral super-models, the spectrum can be obtained exactly following [25]. However, the spectrum generating algebra may be bigger than \( OSp(2|2) \) for some cases. This is definitely the case for chiral super-Calogero model. The study of the complete spectrum of the chiral super-models thus needs a detail investigation, which is beyond the scope of this paper. We write down below only the standard and the chiral super-Hamiltonian and their ground states for the three examples considered in this paper. The excited states belonging to the \( OSp(2|2) \) multiplet may be obtained [25] by acting the strings of creation operators \( B_+, \hat{F}^L_+, \hat{F}^R_+ \) on the groundstate. Note that \( \hat{F}^{L,R}_+ \) is linear in both \( \psi_i \) and \( \psi_i^\dagger \), which is consistent with the fact that fermion number \( n \) is not a conserved quantity for the chiral super-models.

**A. Superoscillators**

Consider the superpotential \( W = 0 \). Consequently, we have to take \( d = 0 \), which is the degree of homogeneity of \( G \). The Hamiltonian \( H \) is that of \( N \) free superparticles. Both
$\mathcal{H}_\pm$ and $\hat{\mathcal{H}}_\pm$ describe the Hamiltonian of $N$ free superoscillators in different co-ordinate representation. In particular,

$$
\mathcal{H}_\pm = \frac{1}{4} \sum_i \left( p_i^2 + x_i^2 \right) \pm \frac{1}{2} \left( n - \frac{N}{2} \right),
$$

$$
\hat{\mathcal{H}}_\pm = \frac{1}{4} \sum_i \left( p_i^2 + x_i^2 \right) \pm \frac{\gamma_5}{4} \left[ N - i \sum_{i,j} \left( \psi_j^\dagger \psi_j + \psi_i^\dagger \psi_i + \psi_j^\dagger \psi_i - \psi_i^\dagger \psi_j \right) L_{ij} \right].
$$

(32)

Note that for $N = 1$, $\mathcal{H}_\pm$ and $\hat{\mathcal{H}}_\pm$ are identical. The differences show up only for $N \geq 2$. The groundstates of $\mathcal{H}_\pm$ are $e^{-\frac{1}{2}r^2}|0\rangle$ and $e^{-\frac{1}{2}r^2}|\bar{0}\rangle$, respectively. The ground-state energy is zero for both the cases. On the other hand, the zero-energy groundstates of $\hat{\mathcal{H}}_\pm$ are $e^{-\frac{1}{2}r^2}|\text{odd}\rangle$ and $e^{-\frac{1}{2}r^2}|\text{even}\rangle$, respectively. The symbol $|\text{odd}\rangle(|\text{even}\rangle)$ denotes a fermionic state with odd(even) fermion number $n$. It is a very special feature of the chiral superoscillators that the groundstate can be constructed using any odd (even) fermion number $n$. For $W \neq 0$, it turns out that the even state is necessarily the fermionic vacuum $|0\rangle$ and the odd state is the conjugate vacuum $|\bar{0}\rangle$ for odd $N$. The total fermion number is not a conserved quantity for $\hat{\mathcal{H}}_\pm$. So, linear combinations of states with odd(even) fermion numbers are also exact eigen states of superoscillators. However, linear combination of states with with odd and even fermion numbers are not valid eigenstates, since mixing of different chiralities are not allowed.

B. Super-conformal quantum mechanics

We choose $W = \eta \ln r$ and consequently, $d = \eta$. The Hamiltonian $H$ for this case is an example of superconformal quantum mechanics. The Hamiltonian $H$ has no normalizable ground-states and the prescription [29] is to study the evolution of the operator $\mathcal{H}_\pm$. The new representation suggests that an alternative prescription may be to study the evolution of the chiral Hamiltonian $\hat{\mathcal{H}}_\pm$ instead of $\mathcal{H}_\pm$. In particular, these Hamiltonians are given by,

$$
\mathcal{H}_\pm = \frac{1}{4} \left( \sum_i p_i^2 + \eta \left( \eta - \frac{N + 2}{r^2} \right) + r^2 \right) - \frac{\eta}{2r^2} \sum_{i,j} \psi_j^\dagger \psi_j \left( \delta_{ij} - 2 \frac{x_i x_j}{r^2} \right) \pm \frac{1}{2} \left( n - \frac{N}{2} - \eta \right),
$$

$$
\hat{\mathcal{H}}_\pm = \frac{1}{4} \left( \sum_i p_i^2 + \eta \left( \eta - \frac{N + 2}{r^2} \right) + r^2 \right) - \frac{\eta}{2r^2} \sum_{i,j} \psi_j^\dagger \psi_j \left( \delta_{ij} - 2 \frac{x_i x_j}{r^2} \right) \pm \frac{\gamma_5}{4} \left[ N + 2\eta \right].
$$
Note that for \( N = 1 \), \( \mathcal{H}_\pm \) and \( \hat{\mathcal{H}}_\pm \) are identical. The differences show up only for \( N \geq 2 \). For \( \eta = 0 \), the Hamiltonian of superoscillators is reproduced. The zero-energy groundstate of \( \mathcal{H}_+ \) and \( \hat{\mathcal{H}}_+ \) is, \( r_0^\eta e^{-\frac{1}{2}r^2} |0\rangle, \eta \geq 0 \). The restriction on \( \eta \) comes from the self-adjointness of the super-Hamiltonian. The zero-energy groundstate of \( \mathcal{H}_- \) is \( \chi = r^{-\eta} e^{-\frac{1}{2}r^2} |0\rangle, \eta < 0 \). The same wave-function \( \chi \) is the zero-energy ground-state of \( \hat{\mathcal{H}}_+ \) for odd \( N \) only. For even \( N \), \( \chi \) is an exact eigenstate of \( \hat{\mathcal{H}}_+ \) with non-zero energy.

C. Calogero Models

The superpotential for the \( A_{N+1} \)-type supersymmetric rational Calogero model is given by,

\[
G_{A_{N+1}} = \prod_{i<j}(x_i - x_j)^\eta, \quad d = \frac{g}{2} N(N-1).
\]

The Hamiltonian \( \mathcal{H}_\pm \) and \( \hat{\mathcal{H}}_\pm \) are,

\[
\mathcal{H}_\pm = \frac{1}{4} \sum_i \left( p_i^2 + x_i^2 + \sum_{j \neq i} \frac{g(g - 1)}{(x_i - x_j)^2} \right) + \frac{g}{2} \sum_{i \neq j} \frac{\psi_i^\dagger \psi_i - \psi_j^\dagger \psi_j}{(x_i - x_j)^2} \pm \frac{1}{2} \left( n - \frac{N}{2} - \frac{g}{2} N(N-1) \right),
\]

\[
\hat{\mathcal{H}}_\pm = \frac{1}{4} \sum_i \left( p_i^2 + x_i^2 + \sum_{j \neq i} \frac{g(g - 1)}{(x_i - x_j)^2} \right) + \frac{g}{2} \sum_{i \neq j} \frac{\psi_i^\dagger \psi_i - \psi_j^\dagger \psi_j}{(x_i - x_j)^2} \pm \frac{\gamma_5}{4} \left[ n + gN(N-1) \right]
- i \sum_{i,j} \left( \psi_i^\dagger \psi_j + \psi_j^\dagger \psi_i \right) \left( G_{A_{N+1}} L_{ij} G_{A_{N+1}}^{-1} \right)
- i \sum_{i,j} \left( \psi_i \psi_j - \psi_j \psi_i \right) \left( G_{A_{N+1}}^{-1} L_{ij} G_{A_{N+1}} \right)
- g \sum_{i,j} \left( \psi_i^\dagger \psi_j + \psi_j^\dagger \psi_i \right) \left( \sum_{k \neq j} x_i(x_j - x_k)^{-1} + \sum_{k \neq i} x_j(x_i - x_k)^{-1} \right).
\]

After separating out the centre of mass, the standard super-Calogero model for \( N = 2 \) is shown to be identical to the Hamiltonian of \( N = 1 \) superconformal quantum mechanics of Ref. [28]. For the chiral super-Calogero model with \( N = 2 \), the centre of mass can not be separated out from the residual degree of freedom. In the limit of vanishing \( g \) and arbitrary \( N \), we recover the case of free superoscillators. The zero-energy groundstate of \( \mathcal{H}_+ \) and \( \hat{\mathcal{H}}_+ \) is \( G e^{-\frac{1}{2}r^2} |0\rangle, \eta > 0 \). Similarly, the zero-energy groundstate of \( \mathcal{H}_- \) is \( \chi_1 = G^{-1} e^{-\frac{1}{2}r^2} |0\rangle, \eta < 0 \). As in the case of superconformal quantum mechanics, \( \chi_1 \) is the zero energy eigenstate of \( \hat{\mathcal{H}}_+ \) for odd \( N \) only. For even \( N \), \( \chi_1 \) is an exact eigenstate of \( \hat{\mathcal{H}}_+ \) with nonzero energy.
The superpotential for the $BC_N$-type supersymmetric rational Calogero model is given by,

$$G_{BC_N}(\lambda, \lambda_1, \lambda_2) = \prod_{i<j} (x_i^2 - x_j^2)^\lambda \prod_k x_k^{\lambda_1} \prod_l (2x_l)^{\lambda_2}, \quad d = \lambda N(N - 1) + N\lambda_1 + N\lambda_2,$$

where $\lambda$, $\lambda_1$ and $\lambda_2$ are arbitrary parameters. The $D_N$-type model is described by $\lambda_1 = \lambda_2 = 0$, while $\lambda_1 = 0 (\lambda_2 = 0)$ describes $C_{N+1}(B_{N+1})$-type Hamiltonian. Any particular result for $BC_N$ type model can be obtained using this form of superpotential from the equations in Sec. III, which we do not repeat here.

V. SUMMARY & DISCUSSIONS

We have constructed the supersymmetric extension of the rational Calogero model with $OSp(2|2)$ dynamical supersymmetry that is different from the one in Ref. 8 in many respects. First of all, the super-Hamiltonian have different co-ordinate representations in these two constructions. In the new construction, the quadratic and the cubic Casimir operators are necessarily zero, thereby having only ‘atypical’ representation. Further, the Scasimir of the subgroup $OSp(1|1)$ can be promoted to be the Scasimir of the full $OSp(2|2)$. On the other hand, the cubic and the quadratic Casimir’s are, in general, non-zero for the standard construction. The Scasimir of the $OSp(1|1)$ can not be promoted to be the Scasimir of the full $OSp(2|2)$ in this case.

We have also shown that $C$, which is equivalent to the angular part of the super-Hamiltonian $H$ in the N-dimensional hyper-spherical coordinate, can be factorized in terms of first order differential operators that commute with the Hamiltonian. Thus, eigen spectrum of these first order operators can be used to find the eigenspectrum of $C$. The result is valid for both standard and the atypical superextension. Surprisingly, this was never realized before for the standard superextension. The reformulation of the problem in terms of two real supercharges made us to see this subtle connection. We have also shown that super-Calogero model, for both the standard and the atypical superextension, can be deformed suitably to admit $\mathcal{N} = 0$ supersymmetry.
The supercharge $-\sqrt{2}Q_2$ can be identified with a $2N$ dimensional Euclidean Dirac operator which is independent of the bosonic coordinates $x_{N+1}, x_{N+2}, \ldots, x_{2N}$. The supercharge $Q_2$ being part of the $OSp(2|2)$ algebra, its spectrum can be readily derived algebraically. For the special case of rational super-Calogero model of $A_{N+1}$-type,

$$W_i = g \sum_{j(\neq i)} \frac{1}{x_i - x_j}, \quad i, j = 1, 2, \ldots, N,$$

which describes a many-particle Coulomb interaction. The mathematical as well as physical significance of this many-particle Dirac-Coulomb operator needs to be well understood.

The standard super-Calogero model is exactly solvable, although the $OSp(2|2)$ symmetry alone can not determine the complete spectrum. The spectrum generating algebra is higher than the $OSp(2|2)$. One would like to know at this point whether the chiral super-Calogero model is also exactly solvable or not? Part of the spectrum corresponding to the $OSp(2|2)$ symmetry can of course be obtained analytically. However, it is desirable to study different aspects related to integrability and exact solvability of the chiral super-Calogero model.

**Acknowledgments**

I would like to thank Ryu Sasaki for useful comments on the manuscript. This work is supported (DO No. SR/FTP/PS-06/2001) by SERC, DST, Govt. of India through the Fast Track Scheme for Young Scientists:2001-2002.

[1] F. Calogero, J. Math. Phys. (N.Y.) 10 (1969) 2191; 10 (1969) 2197.
[2] B. Sutherland, J. Math. Phys.(N.Y.) 12 (1971) 246; 12 (1971) 251; Phys. Rev. A 4 (1971) 2019.
[3] M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. 71 (1981) 314; 94 (1983) 6.
[4] A. Polychronakos, Les Houches Lectures 1998, [hep-th/9902157](http://arxiv.org/abs/hep-th/9902157).
[5] E. D’Hoker and D. H. Phong, [hep-th/9912271](http://arxiv.org/abs/hep-th/9912271); A. Gorsky and A. Mironov, [hep-th/0011197](http://arxiv.org/abs/hep-th/0011197).
[6] A. J. Bordner, E. Corrigan and R. Sasaki, Prog. Theor. Phys. 102 (1999) 499, hep-th/9905011
    S. P. Khastgir, A. J. Pocklington and R. Sasaki, J. Phys. A: Math. Gen. 33 (2000) 9033, hep-th/0005277.

[7] B. Basu-Mallick, P. K. Ghosh and Kumar S. Gupta, Phys. Lett A311 (2003) 87, hep-th/0208132.
    B. Basu-Mallick, P. K. Ghosh and Kumar S. Gupta, Nucl. Phys. B659 (2003) 437, hep-th/0207040.
    B. Basu-Mallick and Kumar S. Gupta, Phys. Lett. A292 (2001) 36, hep-th/0109022.

[8] D. Z. Freedman and P. F. Mende, Nucl. Phys. B344 (1990) 317.

[9] L. Brink, T. H. Hansson, S. Konstein and M. A. Vasiliev, Nucl. Phys. B401 (1993) 591, hep-th/9302023.

[10] P. K. Ghosh, Nucl. Phys. B595 (2001) 519, hep-th/0007208.

[11] L. Brink, A. Turbiner and N. Wyllard, J. Math. Phys. 39 (1998) 1285, hep-th/9705219.

[12] B. S. Shastry and B. Sutherland, Phys. Rev. Lett. 70 (1993) 4029.

[13] P. K. Ghosh, A. Khare and M. Sivakumar, Phys. Rev. A58 (1998) 821, cond-mat/9710206.
    C. Efthimiou and D. Spector, Phys. Rev. A56 (1997) 208, quant-ph/9702017.

[14] A. J. Bordner, N. S. Manton and R. Sasaki, Prog. Theor. Phys. 103 (2000) 463, hep-th/9910033.

[15] A. Dabholkar, Nucl. Phys. B368 (1992) 293; J. P. Rodrigues and A. J. van Tonder, Int. J. Mod. Phys. A8 (1993) 2517, hep-th/9204061.

[16] G. W. Gibbons and P. K. Townsend, Phys. Lett. B 454, 187 (1999), hep-th/9812034.

[17] P. K. Ghosh, J. Phys. A34 (2001) 5583, hep-th/0009055.

[18] N. Wyllard, J. Math. Phys. 41 (2000) 2826, hep-th/9910160.

[19] S. Bellucci, A. Galajinsky and Sergey Krivonos, hep-th/0304087.
    A. V. Galajinsky, hep-th/0302156.

[20] P. Desrosiers, L. Lapointe and P. Mathieu, Nucl. Phys. B606 (2001) 547, hep-th/0103178.
    hep-th/0305038.

[21] T. Deguchi and P. K. Ghosh, J. Phys. Soc. Jap. 70 (2001) 3225, hep-th/0012058.
[22] M. V. Ioffe and A. I. Neelov, J. Phys. A33 (2000) 1581, quant-ph/0001063.

[23] W. Nahm and M. Scheunert, J. Math. Phys. 17 (1976) 868; M. Scheunert, W. Nahm and V. Rittenberg, J. Math. Phys. 18 (1977) 146; 18 (1977) 155.

[24] L. Frappat, P. Sorba and A. Sciarrino, hep-th/9607161.

[25] E. D’Hoker and L. Vinet, Comm. Math. Phys. 97 (1985) 391.

[26] D. Arnaudon and M. Bauer, Lett. Math. Phys. 40 (1997) 307, hep-th/9605020; D. Arnaudon, M. Bauer and L. Frappat, Comm. Math. Phys. 187 (1997) 429, hep-th/9605021.

[27] D. Spector, Phys. Lett. B474 (2000) 331, hep-th/0001008.

[28] V. de Alfaro, S. Fubini and G. Furlan, Nuovo Cimento A34 (1976) 569.

[29] S. Fubini and E. Rabinovici, Nucl. Phys. B245 (1984) 17.

[30] F. De Jonghe et. al., Phys. Lett. B359 (1995) 114, hep-th/9507046; B. A. Horvath, A. J. Macfarlane and J.-W. Van Holten, Phys. Lett. B486 (2000) 346, hep-th/0006118; M. Plyushchay, Phys. Lett. B485 (2000) 187, hep-th/0005122.

[31] R. Coquereaux, Phys. Lett. B115 (1982) 389.

[32] P. J. Gambardella, J. Math. Phys. (N.Y.) 16 (1975) 1172.