UNIFORMIZATION OF SIMPLY CONNECTED FINITE TYPE LOG-RIEMANN SURFACES

KINGSHOOK BISWAS AND RICARDO PEREZ-MARCO

ABSTRACT. We consider simply connected log-Riemann surfaces with a finite number of infinite order ramification points. We prove that these surfaces are parabolic with uniformizations given by entire functions of the form $F(z) = \int Q(z)e^{P(z)} \, dz$ where $P, Q$ are polynomials of degrees equal to the number of infinite and finite order ramification points respectively.

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1. Introduction

In [BPM10a] we defined the notion of log-Riemann surface, as a Riemann surface $S$ equipped with a local diffeomorphism $\pi : S \rightarrow \mathbb{C}$ such that the set of points $\mathcal{R}$ added in the completion $S^\times = S \sqcup \mathcal{R}$ of $S$ with respect to the flat metric on $S$ induced by $\pi$ is discrete. The mapping $\pi$ extends to the points $p \in \mathcal{R}$, and is a covering of a punctured neighbourhood of $p$ onto a punctured disk in $\mathbb{C}$; the point $p$ is called a ramification point of $S$ of order equal to the degree of the covering $\pi$ near $p$. The finite order ramification points may be added to $S$ to give a Riemann surface $S^\times$, called the finite completion of $S$. In this article we are interested in log-Riemann surfaces of finite type, i.e. those with finitely many ramification points and finitely generated fundamental group, in particular simply connected log-Riemann surfaces of finite type. We prove the following:

**Theorem 1.1.** Let $S$ be a log-Riemann surface with $d_1 < +\infty$ infinite order ramification points and $d_2 < +\infty$ finite order ramification points (counted with multiplicity), such that the finite completion $S^\times$ is simply connected. Then $S$ is biholomorphic to $\mathbb{C}$ and the uniformization $\tilde{F} : \mathbb{C} \rightarrow S^\times$ is given by an entire function.
$F = \pi \circ \tilde{F}$ of the form $F(z) = \int Q(z)e^{P(z)}dz$ where $P, Q$ are polynomials of degrees $d_1, d_2$ respectively.

Conversely we have:

**Theorem 1.2.** Let $P, Q \in \mathbb{C}[z]$ be polynomials of degrees $d_1, d_2$ and $F$ an entire function of the form $F(z) = \int Q(z)e^{P(z)}dz$. Then there exists a log-Riemann surface $S$ with $d_1$ infinite order ramification points and $d_2$ finite order ramification points (counted with multiplicity) such that $F$ lifts to a biholomorphism $\tilde{F} : \mathbb{C} \to S^\times$.

The entire functions of the above form were first studied by Nevanlinna [Nev32], who essentially proved Theorem 1.1, although his proof is in the classical language. The uniformization theorem was also rediscovered by M. Taniguchi [Tan01] in the form of a representation theorem for a class of entire functions defined by him called "structurally finite entire functions". The techniques we use are very different and adapted to the more general context of log-Riemann surfaces. In a forthcoming article [BPM10b] we use these techniques to generalize the above theorems to a correspondence between higher genus finite type log-Riemann surfaces and holomorphic differentials on punctured Riemann surfaces with isolated singularities of "exponential type" at the punctures (locally of the form $ge^h dz$ where $g, h$ are germs meromorphic at the puncture).

The proof of Theorem 1.1 proceeds in outline as follows: we approximate $S$ by simply connected log-Riemann surfaces $S_n^\times$ with finitely many ramification points of finite orders such that $d_1$ ramification points of $S_n^\times$ converge to infinite order ramification points. The surfaces $S_n$ converge to $S$ in the sense of Caratheodory (as defined in [BPM10a]) and by the Caratheodory convergence theorem proved in [BPM10a], the uniformizations $\tilde{F}_n$ of $S_n$ converge to the uniformization $\tilde{F}$ of $S$. The uniformizations $\tilde{F}_n$ are the lifts of polynomials $F_n = \pi_n \circ \tilde{F}_n$, such that the nonlinearities $G_n = F_n''/F_n'$ are rational functions of uniformly bounded degree with simple poles at the critical points of $F_n$. As these critical points go to infinity as $n \to \infty$, the nonlinearity of the function $F = \pi \circ \tilde{F}$ is a polynomial, from which it follows that $F$ is of the form $\int Q(z)e^{P(z)}dz$.

To prove Theorem 1.2 we use the converse of Caratheodory convergence theorem: we approximate $F = \int Q(z)e^{P(z)}dz$ by polynomials $F_n = \int Q(z)(1 + \frac{P(z)}{n})^n dz$. The polynomials $F_n$ define log-Riemann surfaces $S_n$ which then converge in the sense of Caratheodory to a log-Riemann surface $S$ defined by $F$, and a study of the log-Riemann surfaces $S_n$ shows that the log-Riemann surface $S$ has $d_1$ infinite order ramification points and $d_2$ finite order ramification points (counted with multiplicity).

We develop the tools necessary for the proofs in the following sections. We first describe a "cell decomposition" for log-Riemann surfaces, which allows one to approximate finite type log-Riemann surfaces by log-Riemann surfaces with finitely many ramification points of finite order. The cell decomposition allows us to read the fundamental group of a log-Riemann surface from an associated graph, and to prove a parabolicity criterion for simply connected log-Riemann surfaces which in particular implies that the log-Riemann surfaces $S$ and $S_n$ considered in the proof of Theorem 1.1 are parabolic.
2. Cell decompositions of log-Riemann surfaces

We recall that a log-Riemann surface $(\mathcal{S}, \pi)$ comes equipped with a path metric $d$ induced by the flat metric $|d\pi|$. Any simple arc $(\gamma(t))_{t \in I}$ in $\mathcal{S}$ which is the lift of a straight line segment in $\mathbb{C}$ is a geodesic segment in $\mathcal{S}$; we call such an arc unbroken geodesic segments. Note that an unbroken geodesic segment is maximal if and only if, as $t$ tends to an endpoint of $I$ not in $I$, either $\gamma(t)$ tends to infinity, or $\gamma(t) \to p \in \mathcal{R}$.

2.1. Decomposition into stars. Let $w_0 \in \mathcal{S}$. Given an angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, for some $0 < \rho(w_0, \theta) \leq +\infty$, there is a unique maximal unbroken geodesic segment $\gamma(w_0, \theta) : [0, \rho(w_0, \theta)) \to \mathcal{S}$ starting at $w_0$ which is the lift of the line segment $\{\pi(w_0) + te^{i\theta} : 0 \leq t < \rho(w_0, \theta)\}$, such that $\gamma(w_0, \theta)(t) \to w^* \in \mathcal{R}$ if $\rho(w_0, \theta) < +\infty$.

**Definition 2.1.** The star of $w_0 \in \mathcal{S}$ is the union of all maximal unbroken geodesics starting at $w_0$,

$$V(w_0) := \bigcup_{\theta \in \mathbb{R}/2\pi\mathbb{Z}} \gamma(w_0, \theta).$$

Similarly we also define for a ramification point $w^*$ of order $n \leq +\infty$ the star $V(w^*)$ as the union of all maximal unbroken geodesics $\gamma(w^*, \theta)$ starting from $w^*$, where the angle $\theta \in [-n\pi, n\pi]$:

$$V(w^*) := \{\gamma(w^*, \theta)(t) : 0 \leq t < \rho(w^*, \theta), -n\pi \leq \theta \leq n\pi\}.$$

**Proposition 2.2.** For $w_0 \in \mathcal{S}$ the star $V(w_0)$ is a simply connected open subset of $\mathcal{S}$. The boundary $\partial V(w_0) \subset \mathcal{S}$ is a disjoint union of maximal unbroken geodesic segments in $\mathcal{S}$.

**Proof:** Since $\mathcal{R}$ is closed, the function $\rho(w_0, \theta)$ is upper semi-continuous in $\theta$, from which it follows easily that $V(w_0)$ is open. Moreover $\pi$ is injective on each $\gamma(w_0, \theta)$, hence is a diffeomorphism from $V(w_0)$ onto its image $\mathbb{C} - F$, where $F$ is the disjoint union of closed line segments $\{\pi(w_0) + te^{i\theta} : \rho(w_0, \theta) < +\infty, t \geq \rho(w_0, \theta)\}$; clearly $\mathbb{C} - F$ is simply connected. By continuity of $\pi$, each component $C$ of $\partial V(w_0)$ is contained in $\pi^{-1}(\gamma)$ for some segment $\gamma$ in $F$, hence is an unbroken geodesic segment $(\alpha(t))_{t \in I}$. Since $C$ is closed in $\mathcal{S}$, $C$ must be maximal. \(\Box\)

The set of ramification points $\mathcal{R}$ is discrete, hence countable. Let $L \supset \pi(\mathcal{R})$ be the union in $\mathbb{C}$ of all straight lines joining points of $\pi(\mathcal{R})$. Then $\mathbb{C} - L$ is dense in $\mathbb{C}$. By a generic fiber we mean a fiber $\pi^{-1}(z_0) = \{w_i\}$ of $\pi$ such that $z_0 \in \mathbb{C} - L$.

**Proposition 2.3.** Let $\{w_i\}$ be a generic fiber. Then:

1. The stars $\{V(w_i)\}$ are disjoint.
2. The connected components of the stars $\partial V(w_i)$ are geodesic rays $\gamma : (0, +\infty) \to \mathcal{S}$ such that $\gamma(t) \to w^* \in \mathcal{R}$ as $t \to 0$, $\gamma(t) \to \infty$ as $t \to \infty$.
3. The union of the stars is dense in $\mathcal{S}$:

$$\mathcal{S} = \bigcup_i V(w_i) = \bigcup_i V(w_i).$$
Proof: (1): If \( w \in V(w_i) \cap V(w_j) \) then the geodesic segments from \( w \) to \( w_i, w_j \) are lifts of \( \{w, z_0\} \), so by uniqueness of lifts (\( \pi \) is a local diffeomorphism) \( w_i = w_j \).

(2): By the previous Proposition, each component of \( \partial V(w_i) \) is a maximal unbroken geodesic segment \( \gamma : (0, r) \to S \) with \( \lim_{t \to 0} \gamma(t) = w^* \in \mathcal{R} \) where \( w^* \) is a ramification point such that \( \pi(\gamma) \) is a straight line segment contained in the straight line through \( \pi(w_i) \) and \( \pi(w^*) \). If \( r < +\infty \) then \( \gamma(t) \to w^*_i \in \mathcal{R} \) as \( t \to r \), so \( \pi(w_i) \) must lie on the straight line through \( \pi(w^*_i), \pi(w^*_i) \), contradicting the fact that \( \{w_i\} \) is a generic fiber. Hence \( r = +\infty \).

(3): Given \( p \in S \), if \( \pi(p) \neq z_0 \), take a path \( (p(t))_{0 < t < \epsilon} \subset S \) converging to \( p \) as \( t \to 0 \) such that the line segments \( [\pi(p(t)), z_0] \) make distinct angles at \( z_0 \), then the discreteness of \( \mathcal{R} \) implies that for \( t \) small enough these line segments admit lifts; again by discreteness of \( \mathcal{R} \) for some \( i \) we have \( p(t) \in V(w_i) \) for all \( t \) small, and \( p \in V(w_i) \).

It is easy to see that for \( w_i \neq w_j \), the components of \( \partial V(w_i), \partial V(w_j) \) are either disjoint or equal, and each component can belong to at most two such stars. The above Propositions hence give a cell decomposition of \( S \) into cells \( V(w_i) \) glued along boundary arcs \( \gamma \subset \partial V(w_i), \partial V(w_j) \).

2.2. The skeleton and fundamental group. Let \( \pi^{-1}(z_0) = \{w_i\} \) be a generic fiber. The 1-skeleton of the cell decomposition into stars gives an associated graph:

**Definition 2.4.** The skeleton \( \Gamma(S, z_0) \) is the graph with vertices given by the stars \( V(w_i) \), and an edge between \( V(w_i) \) and \( V(w_j) \) for each connected component \( \gamma \) of \( \partial V(w_i) \cap \partial V(w_j) \). Each edge corresponds to a geodesic ray \( \gamma : (0, +\infty) \to S \) starting at a ramification point. This gives us a map from edges to ramification points, \( \text{foot} : \gamma \mapsto \text{foot}(\gamma) := \lim_{t \to 0} \gamma(t) \in \mathcal{V}(w_i) \cap \mathcal{V}(w_j) \).

For \( w^* \in \mathcal{R} \) we let \( C(w^*) = \{\gamma : \text{foot}(\gamma) = w^*\} \).

We omit the proof of the following proposition which is straightforward:

**Proposition 2.5.** If \( w^* \) is of finite order \( n \) then \( C(w^*) = (\gamma_i)_{1 \leq i \leq n} \) is a cycle of edges in \( \Gamma \) of length \( n \). If \( w^* \) is of infinite order then \( C(w^*) = (\gamma_i)_{i \in \mathbb{Z}} \) is a bi-infinite path of edges in \( \Gamma \).

We can compute the fundamental group of a log-Riemann surface from its skeleton:

**Proposition 2.6.** The log-Riemann surface \( S \) deformation retracts onto \( \Gamma(S, z_0) \). In particular \( \pi_1(S) = \pi_1(\Gamma(S, z_0)) \).

**Proof:** Let \( \partial V(w_i) = \bigcup_{j \in J_i} \gamma_{ik} \) be the decomposition of \( \partial V(w_i) \) into its connected components. Choose points \( v_{ik} \in \gamma_{ik} \), satisfying \( v_{ik} = v_{jk} \) if \( \gamma_{ik} = \gamma_{kj} \). Choose simple arcs \( \alpha_{ik}, k \in J_i \), joining \( w_i \) to \( v_{ik} \) within \( V(w_i) \), with \( \alpha_{ik} \cap \alpha_{ik'} = \{w_i\} \). Then \( V(w_i) \) deformation retracts onto the union of the arcs \( \alpha_{ik} \); moreover for \( i, j \in I \) we can choose the retractions compatibly on arcs \( \gamma \subset \partial V(w_i) \cap \partial V(w_j) \), giving a retraction of \( S \) onto the union of all arcs \( \alpha_{ik}, i \in I, k \in J_k \), which is homeomorphic to \( \Gamma(S, z_0) \). \( \diamond \)
The relation of \( \Gamma(S, z_0) \) to the finitely completed log-Riemann surface \( S^\times \) is as follows:

**Definition 2.7.** The finitely completed skeleton \( \Gamma^\times(S, z_0) \) is the graph obtained from \( \Gamma(S, z_0) \) as follows: for each finite order ramification point \( w^* \), add a vertex \( v = v(w^*) \) to \( \Gamma(S, z_0) \), remove all edges in the cycle \( C(w^*) \) and add an edge from \( v_i \) to \( v \) for each vertex \( v_i \) in the cycle \( C(w^*) \).

Then as above we have:

**Proposition 2.8.** The finitely completed log-Riemann surface \( S^\times \) deformation retracts onto the finitely completed skeleton \( \Gamma^\times(S, z_0) \).

**Proof:** Let \( w^* \) be a finite order ramification point. Observe that in the proof of the previous Proposition, for \( \gamma = \gamma_{ik} \) an edge in \( C(w^*) \), in the finitely completed log-Riemann surface the arc \( \alpha_{ik} \) can be be homotoped to an arc \( \tilde{\alpha}_{ik} \) from \( w_i \) to \( w^* \). Then \( S^\times \) deformation retracts onto the union of the arcs \( \alpha_{ik}, \tilde{\alpha}_{ik} \) which is homeomorphic to \( \Gamma^\times(S, z_0) \).

Given a graph \( \Gamma \) satisfying certain compatibility conditions along with the information of the locations of the ramification points, we can also construct an associated log-Riemann surface \( S \) with skeleton \( \Gamma \):

**Proposition 2.9.** Let \( \Gamma = (V, E) \) be a connected graph with countable vertex and edge sets and a map \( \text{foot}: E \to \mathbb{C} \). For each vertex \( v \) let \( E_v \) be the set of edges with a vertex at \( v \) and let \( R_v = \text{foot}(E_v) \). Assume that the following hold:

1. The image \( \text{foot}(E) \subset \mathbb{C} \) is discrete.
2. For all vertices \( v \) and points \( z \in R_v \), the intersection \( \text{foot}^{-1}(z) \cap E_v \) has exactly two edges, labelled \( \{e_z(v, +), e_z(v, -)\} \).
3. For an edge \( e \) between vertices \( v, v' \) with \( \text{foot}(e) = z \), either \( e = e_z(v, +) = e_z(v', -) \) or \( e = e_z(v, -) = e_z(v', +) \).

Then there exists a log-Riemann surface \( S \) with skeleton \( \Gamma(S, z_0) = \Gamma \) for some \( z_0 \in \mathbb{C} \).

**Proof:** Let \( L \subset \mathbb{C} \) be the union of all straight lines through pairs of points in \( \text{foot}(E) \), and let \( z_0 \in \mathbb{C} - L \). For each vertex \( v \) of \( \Gamma \), let \( L_v \) be the union of the half-lines \( l_z \) starting at points \( z \in R_v \) with direction \( z - z_0 \). By assumption (1) this collection of half-lines is locally finite. Let \( U_v \) be the domain \( \mathbb{C} - L_v \). Equip \( U_v \) with the path metric \( d(a, b) = \inf_\beta \int_a^b |dz| \) (infimum taken over all rectifiable paths \( \beta \) joining \( a \) and \( b \)). Then the metric completion \( U_v^* \) of \( U_v \) is given by adjoining for each \( z \in R_v \) two copies of \( l_z \) (the two 'sides' of the slit \( l_z \)) intersecting at a point \( z_v \), which we denote by

\[ U_v^* = U_v \bigcup_{z \in R_v} (l_z(v, +) \cup l_z(v, -)) \]

where we take \( l_z(v, +) \) to be the 'upper side' and \( l_z(v, -) \) the 'lower' side (so \( z \to l_z(v, +) \) if \( z \to l_z \) in \( U_v \) with \( \arg(z - z_0) \) increasing and \( z \to l_z(v, -) \) if \( z \to l_z \) in \( U_v \) with \( \arg(z - z_0) \) decreasing). The inclusion of \( U_v \) in \( \mathbb{C} \) extends to a local isometry \( \pi_v: U_v^* \to \mathbb{C} \) with \( \pi_v(l_z(v, +)) = \pi_v(l_z(v, -)) = l_z \).
Let $S^*$ be

$$S^* = \bigsqcup_{v \in V} U^*_v / \sim$$

with the following identifications: for each edge $e$ with vertices $v, v'$ and foot($\gamma$) = $z$, if $e = e_z(v, +) = e_z(v', -)$ we paste isometrically the half-lines $l_z(v, +), l_z(v', -)$, otherwise we paste isometrically $l_z(v, -), l_z(v', +)$. The identifications are compatible with the maps $\pi_v$, giving a a map $\pi : S^* \to \mathbb{C}$. We let $R \subset S^*$ be the subset corresponding to the points \{z_0\} and $\bar{S} = S^* - R$.

Since $\pi(R) = \text{foot}(E)$ is discrete, the set $R$ is discrete. Moreover $\pi$ restricted to $S$ is a local isometry, and the completion of $S$ with respect to the induced path metric is precisely $S^*$, hence $S$ is a log-Riemann surface. The fiber $\pi^{-1}(z_0)$ is generic since $z_0 \in \mathbb{C} - L$. The stars with respect to this fiber are precisely the open subsets $U_v \subset S$. For any star $U_v$ its closure in $S^*$ is the image of $U_v^* \subset S$.

For vertices $v, v'$, according to the above identifications between $U^*_v, U^*_v$ in $S^*$, each component of $\partial U_v \cap \partial U_{v'}$ (if non-empty) is a half-line $l$ arising from an edge $e$ between $v_1, v_2$, of either the form $l = l_z(v, +) = l_z(v', -)$ or $l = l_z(v, -) = l_z(v', +)$.

It follows that $\Gamma(S, z_0) = \Gamma$. \(\diamond\)

2.3. Truncation and approximation by finite sheeted surfaces. We can use the decomposition into stars to approximate any log-Riemann surface by finite sheeted log-Riemann surfaces by “truncating” infinite order ramification points to finite order ramification points. More precisely we have:

**Theorem 2.10.** Let $(S, p)$ be a pointed log-Riemann surface. Then:

1. There exists a sequence of pointed log-Riemann surfaces $(S_n, p_n)$ converging to $(S, p)$ in the Caratheodory topology such that each $S_n$ has only finitely many ramification points all of finite order.

2. If $S^*$ is simply connected then all the surfaces $S^*_n$ are simply connected.

We recall the definition of convergence of log-Riemann surfaces in the Caratheodory topology from [BPM10]: $(S_n, p_n) \to (S, p)$ if for any compact $K \subset S$ containing $p$ there exists $N = N(K) \geq 1$ such that for all $n \geq N$ there is an isometric embedding $\iota_{n,K}$ of $K$ into $S_n$, mapping $p$ to $p_n$ which is a translation in the charts $\pi, \pi_n$ on $S, S_n$.

**Proof of Theorem 2.10** (1): Since the generic fibers are dense in $S$ we may assume without loss of generality that $p = w_0$ lies in a generic fiber $\{w_i\} = \pi^{-1}(z_0)$. Let $V_i = V(w_i)$ be the corresponding stars and $\Gamma = \Gamma(S, z_0)$ the associated skeleton, equipped with the graph metric $d_\Gamma$ (where each edge has length 1). For any star $V_i$ and $R > 0$, the set $V_i \cap B(w_i, R)$ is compact, so it contains at most finitely many ramification points. It follows that the collection of edges

$$E(V_i, R) := \{ \gamma : \gamma \text{ is an edge with a vertex at } V_i, \text{foot}(\gamma) \in \overline{B(w_i, R)} \}$$

is finite, and hence so is the corresponding collection of vertices

$$V(V_i, R) := \{ V_j : \gamma \in E(V_i, R) \text{ is an edge between } V_i, V_j \}.$$

For $n \geq 1$ we define collections of edges and vertices $(E_{n,k})_{1 \leq k \leq n}, (V_{n,k})_{1 \leq k \leq n}$ as follows:
We let \( E_{n,1} = E(V_0, n), V_{n,1} = V(V_0, n) \) and for \( 1 < k \leq n, \)
\[
E_{n,k} := \bigcup_{V_i \in V_{n,k-1}} E(V_i, n) \\
V_{n,k} := \bigcup_{V_i \in V_{n,k-1}} V(V_i, n)
\]

This gives us finite connected subgraphs \( \Gamma_n = (V_{n,n}, E_{n,n}) \) of \( \Gamma \). Let
\[
\tilde{S}_n = \bigcup_{V \in V_{n,n}} V \subset S^+ 
\]
be the corresponding union of stars in \( S^+ \). It is a Riemann surface with boundary, each boundary component being an edge \( \gamma \) of \( \Gamma_n \). We paste appropriate boundary components isometrically to obtain a Riemann surface without boundary \( S_n = S_n/\sim \) as follows:

We let \( R_n \) be the set of ramification points \( \{ \text{foot}(\gamma) : \gamma \in E_{n,n} \} \). For \( w^* \in R_n \) we let \( \Gamma_n(w^*) \) be the subgraph of \( \Gamma_n \) consisting of vertices \( V_i \) and edges \( \gamma \) such that \( w^* = \text{foot}(\gamma) \in V_i \). Two cases arise:

(i) The ramification point \( w^* \) is of finite order: Then there are finitely many stars \( V_i \) such that \( w^* \in V_i \). If \( \Gamma_n(w^*) \) does not contain all of them, then the union of stars \( \bigcup_{V_i \in \Gamma_n(w^*)} V_i \) has two boundary components, both of which are lifts of a half-line in \( \mathbb{C} \) starting at \( \pi(w^*) \); in this case we can paste the two components by an isometry which is the identity in charts.

(ii) The ramification point \( w^* \) is of infinite order: Then the union of stars \( \bigcup_{V_i \in \Gamma_n(w^*)} V_i \) always has two boundary components, both of which are lifts of a half-line in \( \mathbb{C} \) starting at \( \pi(w^*) \); we paste the two components by an isometry which is the identity in charts.

Let \( q_n : \tilde{S}_n \to S_n/\sim \) denote the quotient of \( \tilde{S}_n \) under the identifications made in (i), (ii). The subset \( S_n := (\tilde{S}_n/\sim) - q_n(R_n) \) is a Riemann surface without boundary. Since the identifications are compatible with the map \( \pi, \pi \) induces a map \( \pi_n : S_n \to \mathbb{C} \) which is a local diffeomorphism. The completion of \( S_n \) with respect to the flat metric induced by \( \pi_n \) is isometric to \( \tilde{S}_n/\sim \), so that \( S_n \) is a log-Riemann surface with finite ramification set \( q_n(R_n) \); it is clear from the construction in (i), (ii) above that these ramification points are all of finite order. We let \( p_n = q_n(p) \).

Any compact \( K \subset S \) containing \( p \) can only intersect finitely many stars \( V_i \) and hence \( K \subset \tilde{S}_n \) for \( n \) large enough. Moreover for \( n \) large \( K \) does not intersect the boundary of \( \tilde{S}_n \) (which is contained in stars going to infinity in \( \Gamma \) as \( n \) goes to infinity), hence the quotient map \( q_n \) isometrically embeds \( K \) in \( S_n \). Thus \( (S_n, p_n) \) converges to \( (S, p) \) as required.

(2): The graph \( \Gamma(S_n, z_0) \) can be obtained by adding edges to the finite graph \( \Gamma_n \) between certain vertices corresponding to edges in the sets \( C(w^*), w^* \in R_n \), to give cycles \( C(q_n(w^*)) \) in \( \Gamma(S_n, z_0) \). If \( S^+ \) is simply connected then by Proposition 2.8 the graph \( \Gamma^+(S, z_0) \) is a tree. It follows from the construction of \( \Gamma^+(S, z_0) \) that \( \pi_1(\Gamma(S, z_0)) \) is generated by cycles corresponding to finite order ramification points.
and hence \( \pi_1(\Gamma(S_n, z_0)) \) is generated by the cycles \( C(q_n(w^*)) \). In constructing \( \Gamma^\times(S_n, z_0) \) from \( \Gamma(S_n, z_0) \) these cycles become trivial so \( \pi_1(\Gamma^\times(S_n, z_0)) \) is trivial. \( \diamond \)

2.4. Compactness for uniformly finite type log-Riemann surfaces. The family of finite type log-Riemann surfaces with a given uniform bound on the number of ramification points is compact, in the following sense:

**Theorem 2.11.** Let \( (S_n, p_n) \) be a sequence of pointed log-Riemann surfaces with ramification sets \( \mathcal{R}_n \). If for some \( M, \epsilon > 0 \) we have \( \# \mathcal{R}_n \leq M, d(p_n, \mathcal{R}_n) > \epsilon \) for all \( n \) then there is a pointed log-Riemann surface \( (S, p) \) with ramification set \( \mathcal{R} \) such that \( \# \mathcal{R} \leq M \) and \( (S_n, p_n) \) converges to \( (S, p) \) along a subsequence.

**Proof:** Composing \( \pi_n \) with a translation if necessary we may assume \( \pi_n(p_n) = 0 \) for all \( n \). Since \( d(p_n, \mathcal{R}_n) > \epsilon \) we can change \( p_n \) slightly (within the ball \( B(p_n, \epsilon) \)) to assume without loss of generality that the fiber \( \pi_n^{-1}(0) \) containing \( p_n \) is generic. Let \( \Gamma_n \) be the corresponding skeleton and \( v_{n,0} \) the vertex containing \( p_n \). Passing to a subsequence we may assume the projections \( \pi_n(\mathcal{R}_n) \) converge (in the Hausdorff topology) to a finite set \( \{ w_1^*, \ldots, w_N^* \} \cup \{ \infty \} \subset C - B(0, \epsilon) \) (where \( N \leq M \)), and for all \( n \) lie in small disjoint neighbourhoods \( B_{1, \ldots, B_N} \) and \( B \) of the points of \( R = \{ w_1^*, \ldots, w_N^* \} \) and \( \infty \) respectively.

Let \( \gamma_1, \ldots, \gamma_N \) be generators for the group \( G = \pi_1(C - R) \) where each \( \gamma_i \) is a simple closed curve in \( C - (B \cup_i B_i) \) starting at the origin with winding number one around \( B_i \) and zero around \( B_j, j \neq i \). There is a natural action of \( G \) on the vertices of \( \Gamma_n \): given a vertex \( v \), let \( w \) be the point of the fiber \( \pi_n^{-1}(0) \) in \( v \). Then any \( g \in G \) has a unique lift \( \tilde{g} \) to \( S_n \) starting at \( w \). Let \( g \cdot v \) be the vertex of \( \Gamma_n \) containing the endpoint of \( \tilde{g} \).

We define a graph \( \Gamma'_n = (V_n, E_n) \) as follows: the vertex set \( V_n \) is the orbit of \( v_{n,0} \) under \( G \). We put an edge \( e \) between distinct vertices \( v, v' \) of \( \Gamma'_n \) for each generator \( \gamma \in \{ \gamma_1^\pm, i = 1, \ldots, N \} \) such that \( v' = \gamma \cdot v \). We define foot \( n(e) = w_i^* \) if the edge \( e \) corresponds to either of the generators \( \gamma_i, \gamma_i^{-1} \). This defines a map \( \text{foot}_n : E_n \to R \subset C \).

For \( v \in V_n \) let \( E_v \) be the set of edges with a vertex at \( v \) and \( R_v = \text{foot}_n(E_v) \subset R \). Since \( \gamma_i \cdot v = v \) if and only if \( \gamma_i^{-1} \cdot v = v \), it follows that for \( z = w_i^* \in R_v \), the intersection \( \text{foot}_n^{-1}(z) \cap E_v \) consists of precisely the two edges corresponding to the generators \( \gamma_i, \gamma_i^{-1} \); we label these edges as \( e_z(v, +), e_z(v, -) \).

It is easy to see that the graphs \( \Gamma'_n \) satisfy the hypotheses of Proposition 2.8. Since each vertex has valence at most \( 2N \), the balls \( B(v_n,0,k) \) are finite, so we can pass to a subsequence such that the pointed graphs \( \Gamma'_n, v_{n,0} \) converge to a limit pointed graph \( \Gamma = (V, E, v_0) \), in the sense that for any \( k \geq 1 \), for all \( n \) large enough there is an isomorphism \( i_n \) of the ball \( B(v_0, k) \) with \( B(v_{n,0}, k) \) taking \( v_0 \) to \( v_{n,0} \). We may also assume that the isomorphisms \( i_n \) for different \( n \) are compatible with the mappings \( \text{foot}_n \) and the labeled edges \( e_z(v, +), e_z(v, -) \), thus inducing a corresponding mapping foot : \( E \to R \subset C \) and a labeling of the edges of \( \Gamma \). Then the limit graph \( \Gamma \) satisfies the hypotheses of Proposition 2.8 and we obtain a corresponding pointed log-Riemann surface \( (S, p) \) ramified over the points of \( R \) such that \( \Gamma(S,0) = \Gamma \), with \( p \) in a generic fiber \( \pi^{-1}(0) \), and the star containing \( p \).
corresponding to the vertex \( v_0 \) of \( \Gamma \). Moreover \( S \) has at most \( N \) ramification points. It is easy to see that any compact \( K \subset S \) containing \( p \) embeds isometrically in all the log-Riemann surfaces \( S_n \) via an isometry \( \iota_n \) such that \( \iota_n(p) = p_n, \iota'_n(p) = 1 \), hence \((S_n, p_n)\) converges to \((S, p)\). ♣

2.5. Decomposition into Kobayashi-Nevanlinna cells. Let \( S \) be a log-Riemann surface with \( R \neq \emptyset \). We define a cellular decomposition of \( S \) due to Kobayashi \([\text{Kob35}]\) and Nevanlinna \([\text{Nev53}]\) which is useful in determining the type (parabolic or hyperbolic) of simply connected log-Riemann surfaces.

**Definition 2.12.** Let \( w^* \in R \). The Kobayashi-Nevanlinna cell of \( w^* \) is defined to be the set
\[
W(w^*) := \{ w \in S^* | d(w, w^*) < d(w, R - \{w^*\}) \}
\]

**Proposition 2.13.** The Kobayashi-Nevanlinna cells satisfy:
1. Any \( w \in W(w^*) \) lies on an unbroken geodesic \([w^*, w] \subset W(w^*)\). In particular \( W(w^*) \subset V(w^*) \) is open and path-connected.
2. The boundary of \( W(w^*) \) is a locally finite union of geodesic segments.
3. \( S = \bigcup_{w^* \in R} W(w^*) \)

**Proof:** (1): For any \( w \in W(w^*), w \neq w^* \), since \( R \neq \emptyset \) there is a maximal unbroken geodesic \( \gamma(w, \theta) \) converging to a point of \( R \) at one end, and since \( w^* \) is the point in \( R \) closest to \( w \), there must be such a geodesic \([w, w^*] \) converging to \( w^* \). Moreover for any \( w' \in [w, w^*], w'_i \in R - \{w^*\} \), we have
\[
d(w^*, w') = d(w^*, w) - d(w, w') < d(w_i^*, w) - d(w, w') \leq d(w_i^*, w')
\]
hence \([w, w^*] \subset W(w^*)\).

(2): Let \( w \in \partial W(w^*) \). By discreteness of \( R \) there are finitely many ramification points \( w^* = w^*_1, \ldots, w^*_n \) at minimal distance \( r > 0 \) from \( w \), and \( n \geq 2 \). The disc \( B(w, r) \) is a euclidean disk, with the points \( w^*_i \) lying on its boundary; the angular bisectors of the sectors formed by \([w, w^*_i], [w, w^*_{i+1}]\) then are equidistant from \( w^*_i, w^*_{i+1} \) and lie in \( \partial W(w^*_i) \cap \partial W(w^*_{i+1}) \), while all other points in the disk lie in \( W(w^*_i) \) for some \( i \). Hence a neighbourhood of \( w \) in \( \partial W(w^*) \) is given either by a geodesic segment passing through \( w \) (if \( n = 2 \)) or by two geodesic segments meeting at \( w \) (if \( n > 2 \)).

(3): Any \( w \in S \) belongs to \( \overline{W(w^*)} \) for any ramification point \( w^* \) at minimal distance from \( w \). ♣

2.6. Kobayashi-Nevanlinna parabolicity criterion. We consider a log-Riemann surface \( S \) such that the finite completion \( S^\infty \) is simply connected. We will use the following theorem of Nevanlinna \([\text{Nev53}]\) p. 317:
Theorem 2.14. Let $F \subset S^\times$ be a discrete set and $U : S^\times - F \to [0, +\infty)$ be a continuous function such that:

1. $U$ is $C^1$ except on at most a family of locally finite piecewise smooth curves.
2. $U$ has isolated critical points.
3. $U \to +\infty$ as $z \to F$ or as $z \to \infty$.

For $\rho > 0$ let $\Gamma_\rho$ be the union of the curves where $U = \rho$, and let

$$L(\rho) = \int_{\Gamma_\rho} |\text{grad}_z U||dz|.$$

where $|\text{grad}_z U||dz|$ is the conformally invariant differential given by $\sqrt{(\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2} |dz|$ for a local coordinate $z = x + iy$. If the integral

$$\int_0^\infty \frac{d\rho}{L(\rho)}$$

is divergent then the surface $S^\times$ is parabolic.

We now define a function $U$ on $S$ as follows:

Let $\omega$ be the continuous differential $\omega := |d\arg(w - w^*)|$, where for each $w \in S$, $w^*$ is a ramification point such that $w \in \bar{W}(w^*)$. Fix a base point $w_0 \in S$ and define $\tau : S \to [0, +\infty)$ by

$$\tau(w) := \inf \int_{w_0}^{w} \omega$$

where the infimum is taken over all paths from $w_0$ to $w$. We define another non-negative continuous function $\sigma : S \to [0, +\infty)$ by

$$\sigma(w) := |\log|w - w^*||$$

where as before for each $w \in S$ the point $w^*$ is a ramification point such that $w \in \bar{W}(w^*)$.

Then the sum $U = \tau + \sigma : S \to \mathbb{R}$ is a function satisfying the conditions (1)-(3) of the above theorem. The map $t = \sigma + i\tau$ gives a local holomorphic coordinate away from the boundaries of the Kobayashi-Nevanlinna cells, for which we have $|\text{grad}_z U||dt| = \sqrt{2} |dt|$. On a level set $\Gamma_\rho = \{U = \rho\}$ we have $0 \leq \tau \leq \rho, t = (\rho - \tau) + i\tau$, so $|\text{grad}_z U||dt| = \sqrt{2} |dt| = 2 |d\tau|$. For a given $\theta > 0$, the connected components of the level set $\{\tau(w) = \theta\}$ are Euclidean line segments which are half-lines or intervals; let $0 \leq n(\theta) \leq \infty$ denote the number of such line segments. Each such segment intersects $\Gamma_\rho$ in at most one point; hence we obtain

$$L(\rho) = \int_{\Gamma_\rho} |\text{grad}_z U||dt| = 2 \int_{\Gamma_\rho} |d\tau| \leq \int_0^\rho n(\theta)d\theta$$

Using Theorem 2.14 above, we obtain the following:

Theorem 2.15. Let $S$ be a log-Riemann surface such that $S^\times$ is simply connected. For $\theta > 0$ let $0 \leq n(\theta) \leq \infty$ denote the number of connected components of the level set $\{\tau(w) = \theta\}$. If the integral

$$\int_0^\infty \frac{d\rho}{\int_0^\rho n(\theta)d\theta}$$

is divergent then $S^\times$ is biholomorphic to $\mathbb{C}$. 
This implies:

**Corollary 2.16.** Let $S$ be a log-Riemann surface with a finite number of ramification points such that $S^\times$ is simply connected. Then $S$ is biholomorphic to $\mathbb{C}$.

**Proof:** In this case the function $n(\theta)$ is bounded above by twice the number of ramification points of $S$, so $\int_0^\rho n(\theta)d\theta \leq C\rho$ and hence the integral in Theorem 2.15 diverges. $\diamond$

### 3. Uniformization theorems

We can now prove Theorem 1.1 as follows:

**Proof of Theorem 1.1** Let $p \in S$. Let $D_1, D_2$ be the numbers of infinite and finite order ramification points respectively of $S$. By Corollary 2.16 the log-Riemann surface $S^\times$ is biholomorphic to $\mathbb{C}$. The approximating finitely completed log-Riemann surfaces $S_n^\times$ given by Theorem 2.10 are also biholomorphic to $\mathbb{C}$ and for $n$ large all have $D_1 + D_2$ ramification points. Let $\tilde{F} : \mathbb{C} \to S^\times$ and $\tilde{F}_n : \mathbb{C} \to S_n^\times$ be corresponding normalized uniformizations such that $\tilde{F}(0) = p, \tilde{F}_n(0) = p_n, \tilde{F}'_n(0) = 1$, with inverses $G = \tilde{F}^{-1}, G_n = \tilde{F}_n^{-1}$. By Theorem 1.2 of [BPM10a] the entire functions $F_n = \pi_n \circ \tilde{F}_n$ converge uniformly on compacts to the entire function $F = \pi \circ \tilde{F}$. Since $\pi_n : S_n^\times \to \mathbb{C}$ is finite to one, the entire function $F_n$ has a pole at $\infty$ of order equal to the degree of $\pi_n$, and is hence a polynomial. The nonlinearities $R_n = F''_n/F'_n$ are rational functions whose poles are simple poles with integer residues at the critical points of $F_n$, which are images of the ramification points of $S_n$ under $G_n$. Thus the rational functions $R_n$ are all of degree $D_1 + D_2$, converging normally to $F''/F'$, so $R = F''/F'$ is a rational function of degree at most $D$.

Each ramification point $w^*$ of $S$ corresponds to a ramification point $w^*_n$ of $S_n$ of order converging to that of $w^*$. We note that for $n$ large any compact $K \subset S^\times$ containing $p$ embeds into the approximating surfaces $S_n^\times$. Since the maps $G_n$ converge to $G$ uniformly on compacts of $S^\times$ by Theorem 1.1 [BPM10a], the images under $G_n$ of ramification points in $S_n^\times$ corresponding to finite ramification points in $S$ converge to their images under $G$, giving in the limit $D_2$ simple poles of $R$, with residue at each equal to the order of the corresponding finite ramification point of $S$ minus one.

On the other hand the infinite order ramification points of $S$ are not contained in $S^\times$, so the images of the corresponding ramification points in $S_n^\times$ under $G_n$ cannot be contained in any compact in $\mathbb{C}$ and hence converge to infinity. The rational functions $R_n$ have a simple zero at infinity, and have $D_1$ simple poles converging to infinity. Applying the Argument Principle to a small circle around infinity it follows that $R$ has a pole of order $D_1 - 1$ at infinity.

Thus $R$ is of the form

$$
\frac{F''}{F'} = \sum_{i=1}^{D_2} \frac{m_i - 1}{z - z_i} + P'(z)
$$
where $m_1, \ldots, m_{D_1}$ are the orders of the finite ramification points of $S$ and $P$ is a polynomial of degree $D_1$. Integrating the above equation gives

$$F(z) = \pi(p) + \int_0^z (t - z_1)^{m_1-1} \cdots (t - z_{D_2})^{m_{D_2}-1} e^{P(t)} dt$$

as required. ∘

We can prove the converse using the above Theorem and the compactness Theorem. We need a lemma:

**Lemma 3.1.** Let $(S_n, p_n)$ converge to $(S, p)$. If all the surfaces $S_n^\times$ are simply connected then $S^\times$ is simply connected.

**Proof:** We may assume the points $p_n, p$ belong to generic fibers. Let $\Gamma_n, \Gamma$ denote the corresponding skeletons. Let $\gamma$ be a loop in $S^\times$ based at $p$. We may homotope $\gamma$ away from the finite ramification points to assume that $\gamma \subset S$. By Proposition 2.4 $\gamma$ corresponds to a path of edges $\alpha = \{e_1, \ldots, e_n\}$. By induction on the number of edges we may assume that $\alpha$ is simple. If $\text{foot}(\alpha) = \{w^*\}$ is a singleton then $w^*$ is a finite ramification point and $\gamma$ is trivial in $S^\times$. Otherwise there are distinct ramification points $w_1^*, w_2^* \in \text{foot}(\alpha)$. Considering the isometric embedding of $\gamma$ in $S_n$ for $n$ large gives a path $\gamma_n$ and a corresponding path of edges $\alpha_n$; for $n$ large, it follows that there are distinct ramification points in $\text{foot}_n(\alpha_n)$, hence $\gamma_n$ is non-trivial in $S_n^\times$, a contradiction. ∘

**Proof of Theorem 1.2** Given an entire function $F$ with $F''(z) = Q(z)e^{P(z)}$ we can approximate it by polynomials $F_n$ such that $F_n'(z) = Q(z)(1 + P(z)/n)^n$. Let $Z_n = \{P = -n\} \cup \{Q = 0\} \cup \subset \subset \mathbb{C}$ be the zeroes of $F_n'$. The pair $(S_n = \mathbb{C} - Z_n, \pi_n = F_n : \mathbb{C} - Z_n \to \mathbb{C})$ is a log-Riemann surface with finite ramification set $\mathcal{R}_n$ which can be naturally identified with $Z_n$, the order of a ramification point being the local degree of $F_n$ at the corresponding point of $Z_n$.

For $n$ large the surfaces $S_n$ all have the same number of ramification points $D = D_1 + D_2$ where $D_1$ is the degree of $P$ and $D_2$ the number of distinct zeroes of $Q$. Moreover since $F_n'$ converge uniformly on compacts, choosing a point $z_0$ such that $Q(z_0) \neq 0$, for all $n$ large $|F_n'|$ is uniformly bounded away from 0 on a fixed neighbourhood of $z_0$, so $d(z_0, \mathcal{R}_n)$ is uniformly bounded away from 0. It follows from Theorem 2.11 that $(S_n, p_n = z_0)$ converge along a subsequence to a limit log-Riemann surface $(S, p)$ with finitely many ramification points such that $\pi(p) = z_0$. Since $S_n^\times$ is simply connected for all $n$, by the previous Lemma $S^\times$ is simply connected. By Theorem 2.10 $S^\times$ is biholomorphic to $\mathbb{C}$. Let $\tilde{F} : \mathbb{C} \to S^\times$ be a normalized uniformization such that $\tilde{F}(z_0) = p, \tilde{F}'(z_0) = F'(z_0)$. It follows from Theorem 1.2 of [BPM10a] that the maps $F_n$ converge normally to $\pi \circ \tilde{F}$, so $F = \pi \circ \tilde{F}$. Thus $F$ defines the uniformization of a simply connected log-Riemann surface with finitely many ramification points. The degrees of $Q, P$ relate to the numbers of finite poles and poles at infinity respectively of the nonlinearity $F''/F'$; the relations between the degrees of $Q, P$ and the numbers of finite and infinite order ramification points of $S$ then follow from the previous Theorem. ∘

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RKM Vivekananda University, Belur Math, WB-711 202, India

CNRS, LAGA, UMR 7539, Université Paris 13, Villetaneuse, France