PROFITE PRO-C*-ALGEBRAS AND PRO-C*-ALGEBRAS OF
PROFITE GROUPS

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Abstract. We define the profinite completion of a C*-algebra, which is a
pro-C*-algebra, as well as the pro-C*-algebra of a profinite group. We show
that the continuous representations of the pro-C*-algebra of a profinite group
correspond to the unitary representations of the group which factor through
a finite group. We define natural homomorphisms from the C*-algebra of a
locally compact group and its profinite completion to the pro-C*-algebra of
the profinite completion of the group. We give some conditions for injectivity
or surjectivity of these homomorphisms, but an important question remains
open.

The C*-algebra $C^*(G)$ of a locally compact group $G$ is a well known construc-
tion. One of the main motivations for it is the correspondence between unitary
representations of $G$ and C*-algebra representations of $C^*(G)$.

A profinite group is an inverse limit $\lim_{\lambda \in \Lambda} G_\lambda$ of finite groups
having the discrete topology for each $\lambda$. Such a group is compact, and therefore
has a conventional C*-algebra. However, we can also form the inverse limit of the
C*-algebras $C^*(G_\lambda)$. The result is a pro-C*-algebra, which we introduce and study
in this paper. Its representation theory is related to a part of the representation
theory of $G$, namely the unitary representations which factor through finite groups.
This is the part of the representation theory of $G$ which should be thought of as
being compatible with the structure of $G$ as a profinite group.

In Section 1 we present some relevant basic facts about pro-C*-algebras. The
pro-C*-algebra of a profinite group will be an example of a profinite pro-C*-algebra,
so we develop the theory of these, and of the profinite completion of a C*-algebra.
Generally this completion is far too large to be of any interest, but it seems to be
useful in connection with the C*-algebras of profinite groups.

Section 2 contains the definition and basic properties of the pro-C*-algebra of a
profinite group. We give here the relation between the representations of the group
and its pro-C*-algebra.

In Section 3 for a locally compact group $G$, we use a homomorphism from $C^*(G)$
to the pro-C*-algebra of the profinite completion $\overline{G}$ of $G$ to study the relation
between $C^*(G)$ and its profinite completion on the one hand, and the pro-C*-algebra
of $\overline{G}$ on the other hand. One can’t expect a tight relationship in general. We

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do prove that the map from $C^*(G)$ is injective for a countable amenable residually finite group (Theorem 3.9). However, there are residually finite groups whose C*-algebras are not residually finite dimensional; for these, the map from $C^*(G)$ is never injective. We do not know what happens when the group is residually finite but not amenable and its C*-algebra is residually finite dimensional. The map from the profinite completion is always surjective (Corollary 3.14). For a compact group, it is an isomorphism if and only if the group is profinite (Proposition 3.16), but can be an isomorphism for noncompact groups, sometimes for trivial reasons (Example 3.17 and Example 3.18).

1. Profinite pro-C*-algebras

In this section, we define pro-C*-algebras, profinite pro-C*-algebras, and the profinite completion of a (residually finite dimensional) C*-algebra. As will become clear, the profinite completion is usually very large. (See Example 1.15.) In the next section, we will see situations in which either the profinite completion is not quite so large, or there is a completion which is smaller and more appropriate for the situation.

We approximately follow the definition of a pro-C*-algebra of 1.2 of [16]. The terminology in the literature is inconsistent; in [10] and elsewhere, the term “pro-C*-algebra” is used for what we call here the completion of a pro-C*-algebra.

**Definition 1.1.** Let $A$ be a complex *-algebra. A C* seminorm on $A$ is an algebra seminorm $p$ on $A$ which satisfies $p(a^*a) = p(a)^2$ for all $a \in A$. If $p$ is a C* seminorm, we denote by $\ker(p)$ the set $\ker(p) = \{a \in A : p(a) = 0\}$.

**Lemma 1.2.** Let $A$ be a complex *-algebra and let $p$ be a C* seminorm on $A$. Then:

1. $\ker(p)$ is a *-ideal in $A$, closed if $A$ has a topology and $p$ is continuous.
2. $p$ induces a norm on $A/\ker(p)$, with respect to which $A/\ker(p)$ satisfies all the axioms for a C*-algebra with the possible exception of completeness.
3. If $A$ is a C*-algebra, then $p$ is automatically continuous; in fact, $p(a) \leq \|a\|$ for all $a \in A$. Moreover, $A/\ker(p)$ is already complete.

**Proof.** The first two parts are immediate. For the third, set $q(a) = \max(p(a), \|a\|)$ for $a \in A$. Then $q$ is a C* norm on $A$. The obvious map $A \to A/\ker(q)$ is algebraically a *-homomorphism, hence contractive. Therefore $p(a) \leq \|a\|$ for all $a \in A$. Completeness of $A/\ker(p)$ now follows from the fact that it is the range of a homomorphism of C*-algebras. □

**Notation 1.3.** Let $A$ be a complex *-algebra. If $p$ is a C* seminorm on $A$, we denote by $A_p$ or $A/\ker(p)$ the completion of $A/\ker(p)$.

**Definition 1.4.** A pro-C*-algebra is a pair $(A, (p_\lambda)_{\lambda \in \Lambda})$ consisting of a C*-algebra $A$ and a family $(p_\lambda)_{\lambda \in \Lambda}$ of C* seminorms on $A$, indexed by a directed set $\Lambda$, such that $\lambda \leq \mu$ implies $p_\lambda \leq p_\mu$. We also refer to $(p_\lambda)_{\lambda \in \Lambda}$ as a pro-C*-algebra structure on $A$. 
We have omitted two of the three conditions in 1.2 of [16], because they are not satisfied in many of our examples. We give these conditions in the next definition.

**Definition 1.5.** Let $A$ be a C*-algebra, and let $(p_\lambda)_{\lambda \in \Lambda}$ be a pro-C*-algebra structure on $A$.

1. We say that $(p_\lambda)_{\lambda \in \Lambda}$ is full if the following condition is satisfied. Whenever $(a_\lambda)_{\lambda \in \Lambda}$ is a family of elements $a_\lambda \in A_\lambda$ which is consistent in the sense that for $\lambda \geq \mu$, the image of $a_\lambda$ in $A_\mu$ is $a_\mu$, and such that we also have $\|a_\lambda\| \leq 1$ for all $\lambda \in \Lambda$, then there exists $a \in A$ such that $\|a\| \leq 1$ and such that $a + \ker(p_\lambda) = a_\lambda$ for all $\lambda \in \Lambda$.

2. We say that $(p_\lambda)_{\lambda \in \Lambda}$ is faithful if $\|a\| = \sup_{\lambda \in \Lambda} p_\lambda(a)$ for all $a \in A$.

The condition of Definition 1.5(1) asserts that the inverse limit of the closed unit balls of the quotients $A_\lambda$ is the closed unit ball of $A$.

Lemma 1.2(3) implies that one inequality in Definition 1.5(2) is automatic: we always have $\|a\| \geq \sup_{\lambda \in \Lambda} p_\lambda(a)$.

**Remark 1.6.** Let $A$ be a C*-algebra, and let $(p_\lambda)_{\lambda \in \Lambda}$ be a pro-C*-algebra structure on $A$. Then $(p_\lambda)_{\lambda \in \Lambda}$ defines a topology on $A$, in which a net $(a_\alpha)_{\alpha \in I}$ in $A$ converges to $a \in A$ if and only if $p_\lambda(a_\alpha - a) \to 0$ for all $\lambda \in \Lambda$. When $(p_\lambda)_{\lambda \in \Lambda}$ is understood, we write $\overline{A}$ for the completion of $A$. This algebra is an inverse limit of C*-algebras (namely, the C*-algebras $A/\ker(p_\lambda)$), or a pro-C*-algebra in the sense of Definition 1.1 of [10].

The pro-C*-algebra structure $(p_\lambda)_{\lambda \in \Lambda}$ is faithful if and only if the map $A \to \overline{A}$ is injective, equivalent, if and only if $\bigcap_{\lambda \in \Lambda} \ker(p_\lambda) = \{0\}$. The pro-C*-algebra structure $(p_\lambda)_{\lambda \in \Lambda}$ is full if and only if the map $A \to \overline{A}$ has range equal to the C*-algebra of bounded elements of $\overline{A}$ in the sense of Definition 1.10 of [10].

See [10] and the references there for more on the theory of inverse limits of C*-algebras.

**Definition 1.7.** Let $A$ be a C*-algebra. Two pro-C*-algebra structures on $A$ are said to be equivalent if the topologies they define, as in Remark 1.6 are equal.

**Definition 1.8.** A C*-algebra $A$ is called residually finite dimensional if it has a faithful family of finite dimensional representations.

**Definition 1.9.** A pro-C*-algebra $(A, (p_\lambda)_{\lambda \in \Lambda})$ is profinite if $A/\ker(p_\lambda)$ is finite dimensional for all $\lambda \in \Lambda$. If $A$ is any C*-algebra, we define its profinite pro-C*-algebra structure to be the collection of all C* seminorms $p$ on $A$ such that $A/\ker(p)$ is finite dimensional (justification in Lemma 1.10 below), and we define the profinite completion of $A$ to be $A$ equipped with this pro-C*-algebra structure.

The C* seminorms in the second part of Definition 1.9 are continuous, by Lemma 1.2(3).

**Lemma 1.10.** Let $A$ be a C*-algebra. Then:

1. The collection of all C* seminorms $p$ on $A$ such that $A/\ker(p)$ is finite dimensional is a pro-C*-algebra structure.
2. The profinite pro-C*-algebra structure of Definition 1.9 is profinite.
3. The profinite pro-C*-algebra structure of Definition 1.9 is faithful if and only if $A$ is residually finite dimensional.
Proof. All parts of the lemma are easy. □

We will not use this fact, but we point out that Theorem 6.1 of [14] shows that a pro-C*-algebra \((A,(p_\lambda)_{\lambda \in \Lambda})\) is profinite if and only if \(A\) is semireflexive as a topological vector space, that is (see Sections 5.3 and 5.4 in Chapter 4 of [13]) the map from \(A\) to its strong second dual (as a topological vector space) is bijective (but not necessarily a homeomorphism).

Commutative C*-algebras are residually finite dimensional, as is \(C_0(X,M_n)\) for any locally compact Hausdorff space \(X\) and any \(n \in \mathbb{Z}_{>0}\). Theorem 7 of [3] shows that the full C*-algebra of the free group on two generators is residually finite dimensional.

**Proposition 1.11.** Let \((A,(p_\lambda)_{\lambda \in \Lambda})\) be a profinite pro-C*-algebra. Set \(I = \bigcap_{\lambda \in \Lambda} \ker(p_\lambda)\). Then \(A/I\) is residually finite dimensional.

**Proof.** Let \(a \in A \setminus I\). It suffices to find a finite dimensional representation \(\pi\) of \(A\) such that \(\pi(a) \neq 0\). Choose \(\lambda\) such that \(p_\lambda(a) \neq 0\), let \(\sigma\) be an injective finite dimensional representation of the finite dimensional C*-algebra \(A/\ker(p_\lambda)\), and take \(\pi\) to be the composition of \(\sigma\) with the quotient map \(A \to A/\ker(p_\lambda)\). □

**Corollary 1.12.** Let \(A\) be a C*-algebra, and let \((p_\lambda)_{\lambda \in \Lambda}\) be the collection of C* seminorms of its profinite completion. Set \(I = \bigcap_{\lambda \in \Lambda} \ker(p_\lambda)\). Then \(A/I\) is residually finite dimensional.

**Proof.** The profinite completion is profinite. □

The profinite completion of \(A\) is universal for finite dimensional representations of \(A\).

**Proposition 1.13.** Let \(A\) be a C*-algebra, and let \(\pi: A \to L(H)\) be a representation of \(A\) on a Hilbert space \(H\). Then the following are equivalent:

1. \(\pi\) is continuous in the profinite pro-C*-algebra structure of \(A\).
2. \(\pi(A)\) is finite dimensional.
3. There is a finite set \(F\) of finite dimensional representations of \(A\) such that \(\pi\) is a direct sum of copies of representations in \(F\).

Moreover, the restriction map \(\sigma \mapsto \sigma|_A\), from continuous representations of \(A\) to representations of \(A\) which are continuous in the profinite pro-C*-algebra structure, is bijective.

In particular, the irreducible continuous representations of \(A\) are exactly the irreducible finite dimensional representations of \(A\).

**Proof of Proposition 1.13.** Let \((p_\lambda)_{\lambda \in \Lambda}\) be the profinite pro-C*-algebra structure of \(A\).

We prove the first part. That (1) implies (2) follows from the fact that any continuous homomorphism from \(A\) with any pro-C*-algebra structure must factor through the quotient by the kernel of one of the seminorms in the pro-C*-algebra structure. To see that (2) implies (3), we observe that a finite dimensional C*-algebra has only finitely many unitary equivalence classes of irreducible representations, that they are all finite dimensional, and that every representation is a direct sum of irreducible representations. That (3) implies (1) is obvious.
The second part is clear from the following observation. Let $\pi$ be a representation of $A$ which is continuous for the pro-$C^*$-algebra structure $(p_\lambda)_{\lambda \in \Lambda}$. Then $\pi$ extends uniquely to the completion $\overline{A}$.

\begin{proof}
Let $\alpha_0 : A \to \prod_{\pi \in R} L(H_{\pi})$ be the homomorphism given by $\alpha_0(a) = (\pi(a))_{\pi \in R}$ for $a \in A$. By Proposition 1.13, a net $(a_\lambda)_{\lambda \in \Lambda}$ in $A$ converges to $a \in A$ in the topology from the profinite pro-$C^*$-algebra structure if and only if $\alpha_0(a_\lambda) \to \alpha_0(a)$. Therefore $\alpha_0$ induces a unique continuous homomorphism $\alpha : \overline{A} \to \prod_{\pi \in R} L(H_{\pi})$. Moreover, if $\alpha$ is injective, it will follow that $\alpha$ is a homeomorphism onto its image.

It therefore remains to prove that $\alpha$ is bijective. We first consider injectivity. The kernel of the map $A \to \overline{A}$ is the intersection of the kernels of all homomorphisms to finite dimensional $C^*$-algebras, and $\ker(\alpha_0)$ is the intersection of the kernels of all finite dimensional irreducible representations of $A$. These are clearly equal, and injectivity follows.

Since $\overline{A}$ is complete, surjectivity will follow from density of $\alpha_0(A)$ in $\prod_{\pi \in R} L(H_{\pi})$. For this, it is enough to prove that if $F \subset R$ is finite, $\pi_0 \in R \setminus F$, and $c \in L(H_{\pi_0})$, then there exists $b \in A$ such that $\pi_0(b) = c$ and $\pi(b) = 0$ for all $\pi \in F$. Set $I = \ker(\pi_0)$ and let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for $I$. Let $\pi \in F$. Since $\pi$ and $\pi_0$ are distinct, finite dimensional, and irreducible, we have $\ker(\pi_0) \nsubseteq \ker(\pi)$. Therefore $\pi(I) \neq 0$, whence $\pi(I) = L(H_{\pi_0})$. So $(\pi(e_\lambda))_{\lambda \in \Lambda}$ is an approximate identity for $L(H_{\pi_0})$. Thus $\pi(e_\lambda) \to 1$ in norm. Since $F$ is finite, there is $\lambda_0 \in \Lambda$ such that $||1 - \pi(e_\lambda)|| < \frac{1}{3}$ for all $\pi \in F$. Let $f : [0,1] \to [0,1]$ be a continuous function such that $f(0) = 0$ and $f(t) = 1$ for all $t \in \left[\frac{1}{2},1\right]$. Then $f(\pi(e_{\lambda_0})) = 1$ for all $\pi \in F$. Set $a = 1 - f(e_{\lambda_0})$, getting $\pi_0(a) = 1$ and $\pi(a) = 0$ for all $\pi \in F$. Now choose $b_0 \in A$ such that $\pi(b_0) = c$, and set $b = ab_0$. This is the required element.

\end{proof}

**Example 1.15.** Let $X$ be a locally compact Hausdorff space. Then $C_0(X)$ is residually finite dimensional. It follows from Proposition 1.14 that the topology determined by the profinite pro-$C^*$-algebra structure on $C_0(X)$ is the topology of pointwise convergence, and the completion $\overline{C_0(X)}$ consists of all functions (not necessarily continuous or bounded) from $X$ to $\mathbb{C}$.

In particular, the profinite pro-$C^*$-algebra structure on $C_0(X)$ is usually not full (Definition 1.5(1)), not even if $X$ is compact.

Since we make extensive use of the multiplier algebra $M(A)$ of a $C^*$-algebra $A$, we summarize some of its properties for convenient reference.

**Theorem 1.16.** Let $A$ be a $C^*$-algebra.

1. Let $B$ be a $C^*$-algebra and let $\varphi : A \to B$ be a surjective homomorphism. Then there exists a unique homomorphism $\tilde{\varphi} : M(A) \to M(B)$ such that $\tilde{\varphi}(a) = \varphi(a)$ for all $a \in A$. Moreover, $\tilde{\varphi}$ is unital.

2. Let $H$ be a Hilbert space, and let $\pi : A \to L(H)$ be a nondegenerate representation. Then there exists a unique representation $\tilde{\pi} : M(A) \to L(H)$
whose restriction to $A$ is $\pi$. Moreover, $\bar{\pi}$ is unital, and if $\pi$ is injective then so is $\bar{\pi}$.

(3) If $A$ is residually finite dimensional, then so is $M(A)$.

Proof. Part (1) follows from Theorem 3.1.8 of [8], with $A/\ker(\varphi) \cong B$ in place of $I$ and $M(A)/\ker(\varphi)$ in place of $A$.

When $\pi$ is injective, existence and injectivity in part (2) follow from Proposition 3.12.3 of [9]. We get the general case from this case by combining this case with part (1). Uniqueness follows from the fact that a representation is uniquely determined by its restriction to any ideal for which the restriction is nondegenerate. It is easy to see that $\bar{\pi}$ is unital.

To prove part (3), in part (2) choose a faithful family $(\pi_\lambda)_{\lambda \in \Lambda}$ of finite dimensional representations of $A$. Then $\bar{\pi} = \bigoplus_{\lambda \in \Lambda} \pi_\lambda$. So $M(A)$ is residually finite dimensional. □

Remark 1.17. Let $(A, (p_\lambda)_{\lambda \in \Lambda})$ be a pro-$C^*$-algebra. Then there is an induced pro-$C^*$-algebra structure on $M(A)$, given by the family $(q_\lambda)_{\lambda \in \Lambda}$ of $C^*$ seminorms defined as follows. For $\lambda \in \Lambda$, let $\kappa_\lambda: A \to A/\ker(p_\lambda)$ be the quotient map. Then $q_\lambda(a) = \|\kappa_\lambda(a)\|$.

We really get a $C^*$-algebra structure this way, since if $\lambda_1 \leq \lambda_2$ then the homomorphism $\kappa_{\lambda_1, \lambda_2} : A/\ker(p_{\lambda_1}) \to A/\ker(p_{\lambda_2})$ is surjective, so extends to a homomorphism $\bar{\kappa}_{\lambda_1, \lambda_2} : M(A/\ker(p_{\lambda_1})) \to M(A/\ker(p_{\lambda_2}))$, and $\bar{\kappa}_{\lambda_1, \lambda_2} \circ \kappa_{\lambda_1} = \kappa_{\lambda_2}$ by uniqueness in Proposition 1.18.

Proposition 1.18. Let $A$ be a $C^*$-algebra. Then the identity map of $A$ extends uniquely to a continuous homomorphism from the multiplier algebra $M(A)$ to the completion $\overline{A}$ of $A$ in the topology from the profinite completion pro-$C^*$-algebra structure on $A$. Moreover, the resulting homomorphism is injective if and only if $A$ is residually finite dimensional.

Proof. Let $(p_\lambda)_{\lambda \in \Lambda}$ be the collection of $C^*$ seminorms defining the pro-$C^*$-algebra structure of the profinite completion of $A$. For $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$, further let $\kappa_\lambda: A \to A/\ker(p_{\lambda})$ and $\kappa_{\lambda, \mu}: A/\ker(p_{\lambda}) \to A/\ker(p_{\mu})$ be the quotient maps.

It follows from Theorem 1.16(1) that there is a unique family of homomorphisms $\varphi_\lambda : M(A) \to A/\ker(p_\lambda)$ such that $\varphi_\lambda$ extends the quotient map $\kappa_\lambda : A \to A/\ker(p_{\lambda})$. Uniqueness further implies that whenever $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$, we have $\kappa_{\lambda, \mu} \circ \varphi_\mu = \varphi_\lambda$. It follows that there is a unique homomorphism $\varphi : M(A) \to \overline{A}$ such that $\kappa_\lambda \circ \varphi = \varphi_\lambda$ for all $\lambda \in \Lambda$. It is clear that $\varphi$ is the unique continuous homomorphism from $M(A)$ to $\overline{A}$ which extends the identity map of $A$.

Now assume $A$ is residually finite dimensional; we prove that $\varphi$ is injective. Let $x \in M(A)$ be nonzero. Choose $a \in A$ such that $xa \neq 0$. The element $xa$ is in $A$, so there is $\lambda \in \Lambda$ such that $p_\lambda(xa) \neq 0$. Then $\varphi_\lambda(x)\kappa_\lambda(a) = \kappa_\lambda(xa) \neq 0$, so $\varphi_\lambda(x) \neq 0$. Thus $\varphi(x) \neq 0$.

Finally, assume that $\varphi$ is injective; we prove that $A$ is residually finite dimensional. By construction, $\overline{A}$ has a faithful family of continuous finite dimensional representations on Hilbert spaces. Therefore any $C^*$-algebra contained in $\overline{A}$ is residually finite dimensional. Since $\varphi$ is injective, the conclusion follows. □
Proposition 1.19. Let $A$ and $B$ be $\mathcal{C}^*$-algebras, and let $\varphi: A \rightarrow B$ be a homomorphism. Then $\varphi$ is continuous with respect to the topologies on $A$ and $B$ coming from their profinite completions.

Proof. This follows immediately from the fact that if $\pi$ is a finite dimensional representation of $B$, then $\pi \circ \varphi$ is a finite dimensional representation of $A$. \Halmos

As is clear from Example 1.15, the profinite completion of a $\mathcal{C}^*$-algebra is in general extremely large. Here is a situation in which it is not so bad. We will apply the result to the $\mathcal{C}^*$-algebras of compact groups.

Proposition 1.20. Let $S$ be a set, and for $s \in S$ let $n(s) \in \mathbb{Z}_{>0}$. Set $A = \bigoplus_{s \in S} M_{n(s)}$, and give it the profinite pro-$\mathcal{C}^*$-algebra structure (Definition 1.9). Further give $M(A)$ the pro-$\mathcal{C}^*$-algebra structure coming from Remark 1.17. Then:

1. We have $M(A) = \left\{ a \in \prod_{s \in S} M_{n(s)} : \sup_{s \in S} \|a_s\| < \infty \right\}$.

2. The pro-$\mathcal{C}^*$-algebra topologies (Definition 1.6) on $A$ and $M(A)$ both come from the obvious identifications of $A$ and $M(A)$ with subsets of $\prod_{s \in S} M_{n(s)}$ by restricting the product topology.

3. Both $A$ and $M(A)$ are equal to $\prod_{s \in S} M_{n(s)}$.

4. The algebras of bounded elements (in the sense of Definition 1.10 of [10]) in both $A$ and $M(A)$ are equal to $M(A)$.

5. The pro-$\mathcal{C}^*$-algebra structure on $A$ is faithful (Definition 1.5(2)).

6. The pro-$\mathcal{C}^*$-algebra structure on $M(A)$ is faithful and full (Definition 1.5(1)).

Proof. Part (1) is clear.

Part (2) for $A$ follows from Proposition 1.14. For $M(A)$, by Proposition 1.14 and the definition of the pro-$\mathcal{C}^*$-algebra structure on $M(A)$, the $\mathcal{C}^*$ seminorms in this pro-$\mathcal{C}^*$-algebra structure are exactly the seminorms $p_F \left( (a_s)_{s \in S} \right) = \sup_{s \in F} \|a_s\|$ for finite sets $F \subset S$. These define the product topology on $\prod_{s \in S} M_{n(s)}$, proving the statement about $M(A)$.

The remaining parts of the proposition are now clear. \Halmos

2. PRO-$\mathcal{C}^*$-ALGEBRAS OF PROFINITE GROUPS

In this section, we define the pro-$\mathcal{C}^*$-algebra of a profinite group $G$, and connect it to the representation theory of $G$. The construction makes sense for any locally compact group $G$, although in general it reflects the behavior of the profinite completion of $G$ rather than of $G$. We give the construction in this generality, since we use it in Section 3 to compare the profinite pro-$\mathcal{C}^*$-algebra structure on $\mathcal{C}^*(G)$ with the pro-$\mathcal{C}^*$-algebra of the profinite completion of $G$.

We recall the following definition. (See the beginning of Section 2.1 of [12].)

Definition 2.1. A topological group $G$ is said to be profinite if $G$ is topologically isomorphic to an inverse limit $\varprojlim \lambda G_\lambda$ of an inverse system of finite groups $G_\lambda$, in which $G_\lambda$ is given the discrete topology for each $\lambda \in \Lambda$.

In particular, a profinite group is compact, Hausdorff, and totally disconnected. Conversely, every compact Hausdorff totally disconnected group is profinite. See Theorem 2.1.3 of [12]. See [12] for much more on profinite groups.
The following notation is based on Section 3.1 of [12].

**Notation 2.2.** For any group $G$ and any subgroup $H$, we denote by $[G : H]$ the index of $H$ in $G$. For any topological group $G$ (assuming the discrete topology if no other topology is obvious or specified), we let $\mathcal{N}_G$ be the set of closed normal subgroups $N \subset G$ such that the index $[G : N]$ is finite. We order $\mathcal{N}_G$ by reverse inclusion.

**Remark 2.3.** Let $G$ be a topological group. If $M, N \in \mathcal{N}_G$ then also $M \cap N \in \mathcal{N}_G$. (In fact, $[G : M \cap N] \leq [G : M] \cdot [G : N]$.) Therefore $\mathcal{N}_G$ is a directed set.

The following definition is at the beginning of Section 3.2 of [12].

**Definition 2.4.** Let $G$ be a topological group. We define its *profinite completion* $\overline{G}$ to be the inverse limit of the quotients $G/N$ as $N$ runs through $\mathcal{N}_G$.

The profinite completion $\overline{G}$ is obviously a profinite group.

**Notation 2.5.** Let $G$ be a topological group. There is an obvious continuous homomorphism from $G$ to $\overline{G}$, which we denote by $\gamma_G$. For $N \in \mathcal{N}_G$, we write $\overline{N}$ for the closure $\gamma_G(N)$ of the image of $N$ in $\overline{G}$.

**Lemma 2.6.** Let $G$ be a topological group. Then $N \mapsto \overline{N}$ defines a bijection from $\mathcal{N}_G$ to $\mathcal{N}_{\overline{G}}$. Moreover, for every $N \in \mathcal{N}_G$, the map $\gamma_G$ induces an isomorphism $G/N \rightarrow \overline{G}/\overline{N}$.

**Proof.** See parts (a), (b), and (d) of Proposition 3.2.2 of [12].

We recall that a topological group $G$ is called *residually finite* if the intersection of the closed normal subgroups of finite index in $G$ is $\{1\}$. Clearly, then, the map $G \rightarrow \overline{G}$ is injective if and only if $G$ is residually finite.

**Remark 2.7.** We recall that if $G$ is a locally compact group, then there is a standard continuous unital homomorphism from the measure algebra $M(G)$ to the multiplier algebra $M(C^*(G))$ which extends the homomorphism $L^1(G) \rightarrow C^*(G)$ coming from the definition of $C^*(G)$ as the universal enveloping $C^*$-algebra of $L^1(G)$. (See 7.1.5 of [9].) For each $g \in G$, there is then a unitary $u_g \in M(C^*(G))$ obtained as the image of the point mass measure at $g$, regarded as an element of $M(G)$.

We further recall the correspondence between unitary representations of $G$ and nondegenerate representations of $C^*(G)$. (See Proposition 7.1.4 of [9] and its proof.) Let $\mu$ be a left Haar measure on $G$. If $v \mapsto v_g : G \rightarrow U(H)$ is a unitary representation of $G$ on a Hilbert space $H$, then the corresponding representation $\pi : C^*(G) \rightarrow L(H)$ is defined on $L^1(G)$ by the formula, for $\xi, \eta \in H$,

$$\langle \pi(f)\xi, \eta \rangle = \int_G f(g)\langle v_g\xi, \eta \rangle \, d\mu(g).$$

Given a nondegenerate representation $\pi$, we recover $v$ as follows. First let $\overline{\pi}$ be the unique extension to a unital homomorphism from $M(C^*(G))$ to $L(H)$, as in Theorem [1.16][2]. Then, with $u_g$ as in the previous paragraph, we have $v_g = \overline{\pi}(u_g)$.

**Lemma 2.8.** Let $G$ be a locally compact group, let $F$ be a finite group, and let $\rho : G \rightarrow F$ be a surjective continuous homomorphism. Then there is a unique surjective homomorphism $\pi : C^*(G) \rightarrow C^*(F)$ whose extension to a homomorphism $\overline{\pi} : M(C^*(G)) \rightarrow C^*(F)$ satisfies, in the notation of Remark 2.7, the relation...
\[ \pi(u_g) = u_{\mu(g)} \] for all \( g \in G \). For \( f \in C_c(G) \), and with \( \mu \) being a left Haar measure on \( G \), it is given by the formula

\[ \pi(f) = \sum_{x \in F} \left( \int_{\mu^{-1}(x)} f \, d\mu \right) u_x. \] (2.1)

**Proof.** We may assume that \( C^*(F) \) is a unital subalgebra of \( L(H) \) for some finite dimensional Hilbert space \( H \). Then everything follows from Remark 2.7 except \( \pi(C^*(G)) = C^*(F) \) and (2.1). For (2.1), let \( \varphi(f) \) denote the right hand side; then a calculation shows that for all \( \xi, \eta \in L^2(F) \), we have \( \langle \varphi(f)\xi, \eta \rangle = \langle \pi(f)\xi, \eta \rangle \). It is now clear that \( \pi(f) \in C^*(F) \). Therefore \( \pi(a) \in C^*(F) \) for all \( a \in C^*(G) \).

It remains only to prove that \( \pi \) is surjective. It is enough to show that for every \( x \in F \), the unitary \( u_x \in C^*(F) \) is in the range of \( \pi \). Set \( W = \rho^{-1}(x) \), which is a nonempty open and closed subset of \( G \). Then there is \( f \in C_c(G) \) such that \( \text{supp}(f) \subset W \) and \( \int_G f \, d\mu = 1 \). The formula for \( \pi(f) \) implies that \( \pi(f) = u_x \). □

**Lemma 2.9.** Let \( G \) be a locally compact group, let \( F_1 \) and \( F_2 \) be finite groups, and let

\[ \rho_1 : G \to F_1, \quad \rho_2 : G \to F_2, \quad \text{and} \quad \gamma : F_1 \to F_2 \]

be surjective continuous homomorphisms such that \( \rho_2 = \gamma \circ \rho_1 \). Let

\[ \pi_1 : C^*(G) \to C^*(F_1), \quad \pi_2 : C^*(G) \to C^*(F_2) \quad \text{and} \quad \varphi : C^*(F_1) \to C^*(F_2) \]

be the homomorphisms of Lemma 2.8 and let \( \tilde{\pi}_1 \) and \( \tilde{\pi}_2 \) be the extensions of \( \pi_1 \) and \( \pi_2 \) as in Theorem 1.16(2). Then \( \pi_2 = \varphi \circ \pi_1 \) and \( \tilde{\pi}_2 = \varphi \circ \tilde{\pi}_1 \).

**Proof.** Clearly \( \pi_2 \) and \( \varphi \circ \pi_1 \) are both surjective. Using the fact that \( M(C^*(F_1)) = C^*(F_1) \) (since \( C^*(F_1) \) is unital), we check that \( \pi_2(u_g) = (\varphi \circ \pi_1)(u_g) \) for all \( g \in G \). So \( \tilde{\pi}_2 = \varphi \circ \pi_1 \) by the uniqueness statement in Lemma 2.8. That \( \tilde{\pi}_2 = \varphi \circ \tilde{\pi}_1 \) now follows from the uniqueness statement in Theorem 1.16(2). □

**Proposition 2.10.** Let \( G \) be any compact group. Let \( \overline{C^*(G)} \) be the completion of \( C^*(G) \) in its profinite pro-\( C^* \)-algebra structure. Then the \( C^* \)-algebra \( B \) of bounded elements of \( \overline{C^*(G)} \) (in the sense of Definition 1.10 of [10]) is unital and contains \( C^*(G) \) as an ideal, and the induced map \( M(C^*(G)) \to B \) is an isomorphism of \( C^* \)-algebras. Moreover, \( M(C^*(G)) \) is residually finite dimensional.

In particular, the profinite pro-\( C^* \)-algebra structure of \( M(C^*(G)) \) is full in the sense of Definition 1.11.

**Proof of Proposition 2.10.** Since all irreducible representations of \( G \) are finite dimensional, there are an index set \( S \) and numbers \( n(s) \in \mathbb{Z}_{\geq 0} \) such that we can write \( C^*(G) \) as a direct sum \( C^*(G) = \bigoplus_{s \in S} M_{n(s)} \). The result now follows from Proposition 1.20. □

**Definition 2.11.** Let \( G \) be a locally compact group. For each \( N \in \mathcal{N}_G \) (Notation 2.2), we define a \( C^* \) seminorm \( \| \cdot \|_N \) on \( C^*(G) \) by letting \( \kappa_N : C^*(G) \to C^*(G/N) \) be the homomorphism obtained from Lemma 2.8 using the quotient map, and setting \( \| a \|_N = \| \kappa_N(a) \| \). We define the pro-\( C^* \)-algebra of \( G \) to be \( C^*(G) \) equipped with the pro-\( C^* \)-algebra structure consisting of the seminorms \( \| \cdot \|_N \) as \( N \) runs through \( \mathcal{N}_G \). Letting \( \overline{\kappa}_N : M(C^*(G)) \to C^*(G/N) \) be the extension as in Theorem 1.16(2), we further define the multiplier pro-\( C^* \)-algebra of \( G \) to be
$M(C^*(G))$ equipped with the pro-$C^*$-algebra structure consisting of the seminorms $a \mapsto \|\kappa_N(a)\|$ as $N$ runs through $\mathcal{N}_G$.

Before proving elementary facts about these pro-$C^*$-algebra structures, we give a lemma which will be used again later.

**Lemma 2.12.** Let $G$ be a profinite group. Let $v$ be a unitary representation of $G$ on a finite dimensional Hilbert space $H$. Then there is $N \in \mathcal{N}_G$ and a unitary representation $w$ of $G/N$ on $H$ such that $v_g = w_{gN}$ for all $g \in G$.

**Proof.** Choose an open set $W$ of the unitary group $U(H)$ such that $W$ contains no subgroups of $U(H)$ other than $\{1\}$. Let $V = \{g \in G : v_g \in W\}$.

Then $V$ is an open subset of $G$. Since $G$ is profinite, there is $N \in \mathcal{N}_G$ such that $N \subset V$. Since $W$ contains no nontrivial subgroups, it follows that $v_g = 1$ for all $g \in N$. Therefore $v$ induces a representation $w$ of $G/N$ on $H$. □

**Proposition 2.13.** Let $G$ be a locally compact group.

1. The collection of seminorms $\| \cdot \|_N$ of Definition 2.11 is a pro-$C^*$-algebra structure on $C^*(G)$, and the collection of seminorms $a \mapsto \|\kappa_N(a)\|$ of Definition 2.11 is a pro-$C^*$-algebra structure on $M(C^*(G))$.

2. Suppose, in addition, that $G$ is profinite. Then the pro-$C^*$-algebra structure on $C^*(G)$ is faithful and is equivalent (Definition 1.17) to the profinite pro-$C^*$-algebra structure (Definition 1.13) of $C^*(G)$. The pro-$C^*$-algebra structure on $M(C^*(G))$ is faithful and full.

**Proof.** We prove (1). Lemma 2.9 implies that if $N_1, N_2 \in \mathcal{N}_G$ satisfy $N_2 \subset N_1$, then $\|a\|_{N_1} \leq \|a\|_{N_2}$ for all $a \in C^*(G)$. Moreover, $\mathcal{N}_G$ is directed by Remark 2.8. Thus, we have pro-$C^*$-algebra structures on $C^*(G)$ and $M(C^*(G))$.

Now assume that $G$ is profinite.

We first prove that the pro-$C^*$-algebra structure on $C^*(G)$ is equivalent to the one defining the profinite completion of $C^*(G)$. Since the $C^*$-algebras of finite groups are finite dimensional, it suffices to prove that if $p$ is a $C^*$ seminorm on $C^*(G)$ such that $C^*(G)/\ker(p)$ is finite dimensional, then there is closed normal subgroup $N$ of finite index in $G$ such that $\| \cdot \|_N \geq p$.

Represent $C^*(G)/\ker(p)$ unitarily and faithfully on a finite dimensional Hilbert space $H$. Thus, we have a homomorphism $\pi : C^*(G) \to L(H)$ whose range contains 1 and such that $p(a) = \|\pi(a)\|$ for all $a \in C^*(G)$. Then $\pi$ comes from a unitary representation $v \mapsto v_g$ of $G$ on $H$. Let $N$ and $w : G/N \to L(H)$ be as in Lemma 2.12. Let $\psi : C^*(G/N) \to L(H)$ be the corresponding representation of $C^*(G/N)$. Then $\psi \circ \kappa_N$ and $\pi$ are both nondegenerate homomorphisms from $C^*(G)$ to $L(H)$ whose extensions to homomorphisms $M(C^*(G)) \to L(H)$ send $u_g$ to $v_g$ for all $g \in G$. Therefore $\psi \circ \kappa_N = \pi$. For all $a \in A$, we thus have $\|\kappa_N(a)\| \geq \|\pi(a)\| = p(a)$. This proves the equivalence of pro-$C^*$-algebra structures.

The rest of (2) now follows from Proposition 2.10. □

The following is the analog for profinite groups of Proposition 1.13.

**Proposition 2.14.** Let $G$ be a profinite group. Then a nondegenerate representation $\pi$ of the pro-$C^*$-algebra of $G$ (in the sense of Definition 2.11) is continuous if and only if the image of $G$ under the corresponding unitary representation of $G$
is finite. In particular, there is a bijective correspondence between continuous non-degenerate representations of the pro-C*-algebra of $G$ and unitary representations of $G$ with finite range.

**Proof.** The first part is just the statement that $\pi$ is continuous if and only if it factors through the C*-algebra of a finite quotient of $G$.

The second part follows from the first and from Remark 2.7. $\square$

### 3. The relation between the profinite completion of $C^*(G)$ and the C*-algebra of the profinite completion of $G$

In this section, we construct a homomorphism $\varphi_G$ from the C*-algebra of a locally compact group $G$ to the completion of the pro-C*-algebra of its profinite completion $\overline{G}$, and we study its properties. Two questions seem to be interesting: when is $\varphi_G$ injective, and when is the extension of $\varphi_G$ to the profinite completion of $C^*(G)$ surjective or an isomorphism? The first question is really about injectivity of a homomorphism $C^*(G) \to M(C^*(\overline{G}))$. It thus does not involve pro-C*-algebras, although they provide the motivation. We give a positive answer when $G$ is residually finite and either discrete amenable or abelian. Residual finiteness is necessary for fairly trivial reasons, but we show by example that it is not sufficient. For the second question, we prove that the extension is always surjective. If $G$ is profinite, then the extension is an isomorphism. If $G$ is compact, the converse is true, but there are noncompact groups for which the extension is an isomorphism.

**Proposition 3.1.** Let $G$ be a locally compact group, with profinite completion $\overline{G}$, and let $\gamma_G: G \to \overline{G}$ be the canonical map, as in Notation 2.3. For $N \in \mathcal{N}_G$, let $\rho_N: G \to G/N$ be the quotient map, and let $\kappa_N: C^*(G) \to C^*(G/N)$ be the map of Lemma 2.8. Further, following Proposition 2.10, as applied to $\overline{G}$, let $\sigma_N: M(C^*(\overline{G})) \to C^*(G/N)$ be the restriction to $M(C^*(\overline{G}))$ of the map $C^*(\overline{G}) \to C^*(G/N)$ which comes via Definition 2.11 from the expression of $\overline{G}$ as the inverse limit of the finite quotients $G/N$. Then there exists a unique homomorphism $\varphi_G: C^*(G) \to M(C^*(\overline{G}))$ such that $\sigma_N \circ \varphi_G = \kappa_N$ for all $N$. Moreover:

1. $\varphi_G$ extends to a homomorphism $\tilde{\varphi}_G$ from $M(C^*(G))$ to $M(C^*(\overline{G}))$ such that, in the notation of Remark 2.7, we have $\tilde{\varphi}_G(u_g) = u_{\gamma_N(g)}$ for all $g \in G$.
2. $\varphi_G$ is nondegenerate, that is, $\varphi_G(C^*(G)) \cdot C^*(\overline{G}) = C^*(\overline{G})$.

**Proof.** For each $M \in \mathcal{N}_G$, let $\tau_M: \lim_{\leftarrow N \in \mathcal{N}_G} C^*(G/N) \to C^*(G/M)$ be the standard map from the inverse limit. Using Lemma 2.9 to check compatibility of the maps, we see that there is a unique homomorphism $\psi: C^*(G) \to \lim_{\leftarrow N \in \mathcal{N}_G} C^*(G/N)$ such that $\tau_N \circ \psi = \kappa_N$ for all $N$. Since $C^*(G)$ is a C*-algebra, the range of $\psi$ lies in the C*-algebra of bounded elements of the inverse limit. In the notation of Proposition 2.10 applied to the compact group $\overline{G}$, this inverse limit is $C^*(\overline{G})$. It follows from Proposition 2.10 that the bounded elements can be identified with $M(C^*(\overline{G}))$. Thus we can take $\varphi_G$ to be the corestriction of $\psi$ to $M(C^*(\overline{G}))$. Uniqueness of $\varphi_G$ is obvious from uniqueness of $\psi$.

We prove (1). Let $\tilde{\kappa}_N$ be the extension of $\kappa_N$ to a map $M(C^*(G)) \to C^*(G/N)$. It follows from Lemma 2.8 that $\tilde{\kappa}_N(u_g) = u_{gN}$ for all $g \in G$. The maps $\tilde{\kappa}_N$ are also compatible with the inverse system defining $C^*(\overline{G})$, and therefore give a map...
\( \varphi_G : M(C^*(G)) \rightarrow C^*(\overline{G}) \) such that \( \varphi_G(u_g) = u_{\gamma_G(g)} \) for all \( g \in G \). For the same reasons as for \( \varphi_G \), the range of \( \varphi_G \) is contained in \( M(C^*(\overline{G})) \).

We now prove nondegeneracy (part 2). Let \( \mu \) be a left Haar measure on \( G \). Let \( \mu \) be normalized left Haar measure on \( G \).

We claim that if \( M, N \in \mathcal{N}_G \) with \( M \subset N \), if \( f \in C_c(G) \) is supported in \( N \), and if \( a \in C(\overline{G}) \) is constant on cosets of \( N \), then

\[
\kappa_M(f)\sigma_M(a) = \left( \int_G f \, d\mu \right) \sigma_N(a).
\]

To prove this, let \( S \) be a set of coset representatives for \( M \) in \( G \). Then \( \gamma_G(S) \) is a set of coset representatives for \( M \) in \( \overline{G} \), by Lemma 2.6. Also, for \( x \in G/M \) denote the corresponding unitary in \( C^*(G/M) \) by \( u_x \). We use the formula of Lemma 2.8 at the first step and \( \text{supp}(f) \subset N \) at the second step to write

\[
\kappa_M(f) = \sum_{g \in S} \left( \int_{gM} f \, d\mu \right) u_{gM} = \sum_{g \in S \cap N} \left( \int_{gM} f \, d\mu \right) u_{gM}.
\]

Using the uniqueness part of Lemma 2.8 we see that

\[
\sigma_M|_{C^*(\overline{G})} : C^*(\overline{G}) \rightarrow C^*(G/M)
\]

is also a homomorphism of the form of Lemma 2.8. Using the formula of Lemma 2.8 at the first step, and the fact that \( a \) is constant on cosets of \( N \) and \( M \subset N \) at the second step, we thus get

\[
\sigma_M(a) = \sum_{h \in S} \left( \int_{\gamma_G(h)M} a \, d\mu \right) u_{hM} = [G : M]^{-1} \sum_{h \in S} a(\gamma_G(h))u_{hM}.
\]

Now multiply:

\[
\kappa_M(f)\sigma_M(a) = [G : M]^{-1} \sum_{g \in S \cap N} \left( \int_{gM} f \, d\mu \right) \sum_{h \in S} a(\gamma_G(h))u_{ghM}.
\]

Since \( a \) is constant on cosets of \( N \), we can replace \( a(\gamma_G(h)) \) by \( a(\gamma_G(gh)) \). Because \( gS \) is also a system of coset representatives of \( M \) in \( G \), we have

\[
[G : M]^{-1} \sum_{h \in S} a(\gamma_G(gh))u_{ghM} = \sigma_M(a).
\]

Since \( f \) vanishes off \( N \), we therefore get

\[
\kappa_M(f)\sigma_M(a) = \sum_{g \in S \cap N} \left( \int_{gM} f \, d\mu \right) \sigma_M(a) = \left( \int_G f \, d\mu \right) \sigma_M(a).
\]

This proves the claim.

We next claim that if \( f \) and \( a \) are as in the claim, then \( \varphi_G(f)a = (\int_G f \, d\mu) a \). This follows from the fact that the intersection of the kernels \( \ker(\sigma_M) \), as \( M \) runs through all \( M \in \mathcal{N}_G \) such that \( M \subset N \), is zero. We can now choose \( f \in C_c(G) \) with \( \text{supp}(f) \subset N \) such that \( \int_G f \, d\mu = 1 \), getting \( \varphi_G(f)a = a \).

To complete the proof of nondegeneracy, it is enough to prove that the set

\[
T = \{ a \in C(\overline{G}) : \text{there is } N \in \mathcal{N}_G \text{ such that } a \text{ is constant on cosets of } N \}
\]
is dense in $C^* (\overline{G})$. It follows from uniform continuity of elements of $C(\overline{G})$ that $T$ is dense in $C(\overline{G})$. The proof is completed by observing that $C(\overline{G})$ is dense in $C^* (\overline{G})$. \hfill $\square$

The main business of this section is the study of the maps $\varphi_G$ and $\tilde{\varphi}_G$.

The first question we consider is when $\varphi_G$ is injective. Formally, this question doesn’t involve pro-$C^*$-algebras at all; they merely provide the motivation.

**Proposition 3.2.** Let $G$ be a locally compact group, with profinite completion $\overline{G}$, with canonical map $\gamma_G: G \to \overline{G}$, and with $\varphi_G: C^*(G) \to M (C^* (\overline{G}))$ as in Proposition 3.1. Let $H$ be a Hilbert space, let $g \mapsto v_g$ be a unitary representation of $G$ on $H$, and let $\pi: C^*(G) \to L(H)$ be the corresponding nondegenerate representation of $C^*(G)$. Then the following are equivalent:

1. The representation $g \mapsto v_g$ factors through $\overline{G}$: there exists a unitary representation $x \mapsto w_x$ of $\overline{G}$ on $H$ such that $v_g = w_{\gamma_G (g)}$ for all $g \in G$.

2. The representation $\pi$ factors through $M (C^* (\overline{G}))$: there exists a nondegenerate representation $\mu: C^* (\overline{G}) \to L(H)$ whose extension $\tilde{\mu}$ to $M (C^* (\overline{G}))$ satisfies $\pi = \tilde{\mu} \circ \varphi_G$.

**Proof.** Assume (1). Since $\overline{G}$ is compact, $w$ is a direct sum of finite dimensional representations. It therefore suffices to prove (2) under the assumption that $w$ is finite dimensional.

Apply Lemma 2.12 to $w$, with $\overline{G}$ in place of $G$, and use Lemma 2.6 to find $N \in \mathcal{N}_G$ and a unitary representation $w(0)$ of $\overline{G}/N$ on $H$ such that $w_x = w(0)_x$ for all $x \in \overline{G}$. Let $\mu: C^* (\overline{G}) \to L(H)$ and $\mu(0): C^* (\overline{G}/N) \to L(H)$ be the nondegenerate representations of $C^*$-algebras corresponding to $w$ and $w(0)$. Let $\nu: C^* (\overline{G}) \to C^* (\overline{G}/N)$ be the map of Lemma 2.8. Thus $\mu(0) \circ \nu = \mu$. Let

$$\tilde{\mu}: M (C^* (\overline{G})) \to L(H) \quad \text{and} \quad \tilde{\nu}: M (C^* (\overline{G}/N)) \to C^* (\overline{G}/N)$$

be the extensions of $\mu$ and $\nu$. Following Lemma 2.6, identify $\overline{G}/N$ with $G/N$, and thus identify $\tilde{\nu}$ with the map $\sigma_N$ in the statement of Proposition 3.1. Also let $\kappa_N: C^*(G) \to C^*(G/N)$ be as there. Then, using Proposition 3.1 at the second step,

$$\tilde{\mu} \circ \varphi_G = \mu(0) \circ \tilde{\nu} \circ \varphi_G = \mu(0) \circ \kappa_N.$$ 

It follows from Proposition 3.1.2 that $\tilde{\mu} \circ \varphi_G$ is nondegenerate. Let

$$\tilde{\varphi}_G: M(C^*(G)) \to M (C^* (\overline{G})) \quad \text{and} \quad \kappa_N: M(C^*(G)) \to C^*(G/N)$$

be the extensions of $\varphi_G$ and $\kappa_N$ (the first coming from Proposition 3.1.1). Then

$$\tilde{\mu} \circ \tilde{\varphi}_G = \mu(0) \circ \tilde{\kappa}_N.$$ 

For $g \in G$, we thus have

$$(\tilde{\mu} \circ \tilde{\varphi}_G)(u_g) = (\mu(0) \circ \tilde{\kappa}_N)(u_g) = w(0)_{\gamma_G (g)} N = w_{\gamma_G (g)} = v_g = \pi(u_g).$$

Since $\tilde{\mu} \circ \tilde{\varphi}_G$ is nondegenerate, it follows that $\tilde{\mu} \circ \varphi_G = \pi$. This is the required factorization of $\pi$.

Now assume (2). Let $\tilde{\pi}$ be the extension of $\pi$ to $M(C^*(G))$, and let $\tilde{\varphi}_G$ be as in Proposition 3.1.1. Then $\tilde{\mu} \circ \tilde{\varphi}_G$ also extends $\pi$, so $\tilde{\mu} \circ \tilde{\varphi}_G = \tilde{\pi}$. For $g \in G$, we then have, using Proposition 3.1.1 at the third step,

$$v_g = \tilde{\pi}(u_g) = (\tilde{\mu} \circ \tilde{\varphi}_G)(u_g) = \tilde{\mu}(u_{\gamma_G (g)}) = w_{\gamma_G (g)}.$$
This proves (1).

**Corollary 3.3.** Let $G$ be a locally compact group, with profinite completion $\overline{G}$, and let $\varphi_G: C^*(G) \to M(\overline{C^*(G)})$ be as in Proposition 3.1. Let $I \subset C^*(G)$ be the intersection of the kernels of the representations of $C^*(G)$ associated as in Remark 2.7 to representations of $G$ with finite range. Then $\ker(\varphi_G) = I$.

**Proof.** Let $a \in \ker(\varphi_G)$. Let $v: G \to L(H)$ be an arbitrary representation of $G$ with finite range, with associated representation $\pi: C^*(G) \to L(H)$. Then $v$ factors through $\overline{G}$, so Proposition 3.2 implies that $\pi$ factors through $M(\overline{C^*(G)})$. Therefore $\pi(a) = 0$. This shows that $a \in I$.

Conversely, let $a \in I$. For $N \in N_G$, let $\kappa_N$ and $\sigma_N$ be as in the statement of Proposition 3.1. Then $\kappa_N(a) = 0$ for all $N \in N_G$. Therefore $\sigma_N(\varphi_G(a)) = 0$ for all $N \in N_G$. The last sentence of Proposition 2.13(2), applied with $\overline{G}$ in place of $G$, now implies that $\varphi_G(a) = 0$.

**Corollary 3.4.** Let $G$ be a locally compact group, let $\varphi_G: C^*(G) \to M(\overline{C^*(G)})$ be as in Proposition 3.1 and let $I$ be as in Corollary 3.3. Then $\varphi_G$ is injective if and only if $I = 0$.

**Proof.** This is immediate from Corollary 3.3.

**Corollary 3.5.** Let $G$ be a locally compact group, and suppose that the map $\varphi_G: C^*(G) \to M(\overline{C^*(G)})$ of Proposition 3.1 is injective. Then $G$ is residually finite.

**Proof.** Every unitary representation of $G$ with finite range is a direct sum of finite dimensional representations. Corollary 3.4 therefore implies that there is a family $\{v^{(\lambda)}\}_{\lambda \in \Lambda}$ of finite dimensional unitary representations of $G$, each with finite range, such that the representation $\pi: C^*(G) \to L(H)$, associated to $v = \bigoplus_{\lambda \in \Lambda} v^{(\lambda)}$, is injective. Theorem 4.16(2) implies that its extension $\tilde{\pi}$ to $M(C^*(G))$ is also injective. Since $\tilde{\pi}(u_g) = v_g$ for all $g \in G$, we conclude that $v$ is injective. Since $v$ is a direct sum of representations with finite range, it follows that $G$ is residually finite.

**Example 3.6.** The converse to Corollary 3.5 is false. Indeed, the group $G = \text{SL}_3(\mathbb{Z})$ is residually finite, but its full group $C^*$-algebra is not residually finite dimensional (by the main theorem of [11]). Since $M(\overline{C^*(G)})$ is residually finite dimensional (by Proposition 2.10), it follows that $\varphi_G$ is not injective.

The method of Example 3.6 suggests the following question, to which we do not know the answer.

**Question 3.7.** Is there a residually finite locally compact group $G$ such that $C^*(G)$ is residually finite dimensional but $\varphi_G$ is not injective?

The free group $F_2$ on two generators is a candidate. The algebra $C^*(F_2)$ is residually finite dimensional by Theorem 7 of [13]. Thus, the direct sum of all representations of $F_2$ on finite dimensional Hilbert spaces yields a faithful representation of $C^*(G)$. However, it is not in general possible to approximate a representation of $G$ on a finite dimensional Hilbert space pointwise by representations on the same Hilbert space with finite range. This follows from two facts. The first is Jordan’s Theorem (see page 91 of [7]), according to which for every $n \in \mathbb{Z}_{>0}$ there
is \( l(n) \in \mathbb{Z}_{>0} \) that every finite subgroup of the unitary group of \( M_n \) contains an abelian normal subgroup of index at most \( l(n) \). (For recent proofs and strengthenings, see Section 2 of [2] and Proposition 2.3 of [15].) The second is the existence of injective representations of \( F_2 \) in the unitary matrix groups \( U(n) \). For example, define
\[
 u = \begin{pmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{pmatrix}.
\]
Then, by Section 4 of [11], the elements \((uv)^2\) and \((uv)^2\) generate a copy of \( F_2 \) inside \( U(3) \).

By itself, the situation described in the previous paragraph does not show that the direct sum of all unitary representations of \( F_2 \) with finite range is not faithful on \( C^*(F_2) \).

We can give two positive results.

**Proposition 3.8.** Let \( G \) be a residually finite locally compact abelian group. Then the map \( \varphi_G: C^*(G) \to M(C^*(\overline{G})) \) of Proposition 3.1 is injective.

Not all locally compact abelian groups are residually finite. The group \( S^1 \) with its usual topology, and \( \mathbb{Q} \) with the discrete topology, are counterexamples. For both of these, \( M(C^*(\overline{G})) = \mathbb{C} \).

**Proof of Proposition 3.8.** To make the notation less awkward, set \( P = \overline{G} \), and abbreviate \( \gamma_G: G \to P \) to \( \gamma \). We let \( \hat{G} \) and \( \hat{P} \) denote the Pontryagin duals of \( G \) and \( P \), and we let \( \hat{\gamma}: \hat{P} \to \hat{G} \) be the map induced by \( \gamma \). We can identify \( C^*(G) \) with \( C_0(\hat{G}) \) and \( M(C^*(\overline{G})) \) with the algebra \( C_0(\hat{P}) \) of bounded continuous functions on \( \hat{P} \). Then \( \varphi_G \) is given by \( \varphi_G(f) = f \circ \hat{\gamma} \).

Since \( G \) is residually finite, \( \gamma \) is injective. Therefore \( \hat{\gamma} \) has dense range. (See 24.41(b) of [6].) So \( f \mapsto f \circ \hat{\gamma} \) is injective. \( \square \)

**Theorem 3.9.** Let \( G \) be an amenable residually finite discrete group. Then the map \( \varphi_G: C^*(G) \to M(C^*(\overline{G})) \) of Proposition 3.1 is injective.

**Proof.** The following argument is taken from the proof of Proposition 3.3 of [17], and was pointed out to us by David Kerr.

Since \( G \) is amenable, we can identify \( G \) with a subalgebra of \( L(l^2(G)) \) via the regular representation. Let \( a \in C^*(G) \) be nonzero. We show that \( a \) is not in the ideal \( J \) of Corollary 3.3. This will give the result.

Without loss of generality \( \|a\| = 1 \). Choose \( b \in C^*(G) \) such that \( \|b - a\| < \frac{1}{4} \) and \( b \) is a finite linear combination of the standard unitaries \( u_g \). That is, there are a finite set \( S \subset G \) and numbers \( \beta_g \in \mathbb{C} \) for \( g \in S \) such that \( b = \sum_{g \in S} \beta_g u_g \). We have \( \|b\| > \frac{3}{4} \), so there is \( \xi \in l^2(G) \) with finite support such that \( \|\xi\| = 1 \) and \( \|b\xi\| > \frac{1}{2} \). Let \( \delta_g \in l^2(G) \) be the standard basis element corresponding to \( g \in G \), and use similar notation for other groups. Then there are a finite set \( T \subset G \) and numbers \( \alpha_g \in \mathbb{C} \) for \( g \in T \) such that \( \xi = \sum_{g \in T} \alpha_g \delta_g \).

We let
\[
 ST = \{ gh: g \in S \text{ and } h \in T \} \quad \text{and} \quad T^{-1} = \{ g^{-1}: g \in T \}.
\]
Since \( G \) is residually finite, there is \( N \in \mathcal{N}_G \) such that the restriction to \( ST \) of the quotient map \( G \to G/N \) is injective. Let \( v: G \to L(l^2(G/N)) \) be the composition
of this quotient map with the regular representation of $G/N$. Let $\pi: C^*(G) \to L(l^2(G/N))$ be the corresponding homomorphism. Set $\eta = \sum_{g \in ST} \alpha_g \delta_g N$. Then $\|\eta\| = 1$ since the vectors $\delta_g N$ are orthonormal. For $g \in ST$, define

$$\lambda_g = \sum_{h \in g^{-1}T} \beta_h \alpha_{h^{-1}g}.$$ 

Then

$$b \xi = \sum_{g \in ST} \lambda_g \delta_g \quad \text{and} \quad \pi(b) \eta = \sum_{g \in ST} \lambda_g \delta_g N.$$

As $g$ runs through $ST$, the elements $\delta_g$ and $\delta_g N$ form orthonormal systems in $l^2(G)$ and in $l^2(G/N)$. Therefore

$$\|\pi(b)\eta\|^2 = \sum_{g \in ST} |\lambda_g|^2 = \|b \xi\|^2.$$

So

$$\|\pi(a)\| > \|\pi(b)\| - \frac{1}{4} \geq \|\pi(b)\eta\| - \frac{1}{4} = \|b \xi\| - \frac{1}{4} > \frac{1}{4} - \frac{1}{4} = \frac{1}{4}.$$

Therefore $\pi(a) \neq 0$. Since $\pi$ comes from a representation of $G$ which factors through the finite group $G/N$, this shows that $a \notin I$. \hfill \Box

We give an explicit example of a nonabelian group covered by Theorem 3.9, with a direct proof that the map $\varphi_G: C^*(G) \to M(C^*(G))$ of Proposition 3.4 is injective.

**Example 3.10.** Let $G$ be the discrete Heisenberg group, that is,

$$G = \left\{ \begin{pmatrix} 1 & n & l \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} : l, m, n \in \mathbb{Z} \right\}.$$

We can identify $G$ with the group generated by elements $g$, $h$, and $z$, subject to the relations

$$gh = zhg, \quad zg = zg, \quad \text{and} \quad zh = hz.$$

Here,

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ (See Section VII.5 of [4], where these elements are called $u$, $v$, and $w$.)

By Theorem VII.5.1 of [4], a complete set of representatives of the unitary equivalence classes of irreducible representations of $C^*(G)$ is given by the representations $\pi^{(n,k,\alpha,\beta)}$ defined as follows. We require that $n \in \mathbb{Z}_{>0}$, that $k \in \{1, 2, \ldots, n\}$ and be relatively prime to $n$, and that $\alpha$ and $\beta$ both be in the arc

$$J_n = \{ \exp(2\pi it) : 0 \leq t < \frac{1}{n} \}.$$ 

Let $s_n \in M_n$ be the cyclic shift, defined on the standard basis vectors $\delta_j \in \mathbb{C}^n$ by $s_n \delta_n = \delta_1$ and $s_n \delta_j = \delta_{j+1}$ for $j = 1, 2, \ldots, n$. Then $\pi^{(n,k,\alpha,\beta)}$ is determined by

$$\pi^{(n,k,\alpha,\beta)}(g) = \alpha \cdot \text{diag}(1, e^{2\pi ik/n}, e^{2\pi i 2k/n}, \ldots, e^{2\pi i (n-1)k/n}),$$

$$\pi^{(n,k,\alpha,\beta)}(h) = \beta s_n, \quad \text{and} \quad \pi^{(n,k,\alpha,\beta)}(z) = e^{2\pi ik/n} \cdot 1.$$ 

(It is not stated in [4], but one easily checks that these representations actually all exist.)
Proposition 3.11. Let $G$ be a locally compact group. Give $M(C^*_G)$ the pro-$C^*$-algebra structure of Proposition 2.13. Give $C^*_G$ its profinite pro-$C^*$-algebra structure (Definition 1.9). Then the map $\varphi_G : C^*_G \to M(C^*_G)$ of Proposition 3.1 is continuous for the topologies induced by these pro-$C^*$-algebra structures.

Moreover, $\varphi_G$ extends to a continuous homomorphism

$$\varphi_G : C^*_G \to M(C^*_G).$$

Proof: The second statement follows from the first.

To prove the first statement, it is enough to prove the following. Let $p$ be any $C^*$ seminorm in the pro-$C^*$-algebra structure on $M(C^*_G)$. Then the $C^*$ seminorm $p \circ \varphi_G$ is in the pro-$C^*$-algebra structure on $C^*_G$.

So let $p$ be such a $C^*$ seminorm. By definition, there is $N \in \mathcal{N}_{\overline{G}}$, with corresponding homomorphism $\kappa_N : C^*_G \to C^*_G/N$ and extension $\tilde{\kappa}_N : M(C^*_G) \to C^*_G/N$, such that $p(a) = \|\tilde{\kappa}_N(a)\|$ for all $a \in M(C^*_G)$. Then $C^*_G/\ker(p \circ \varphi_G)$ is isomorphic to a subalgebra of $C^*_G/N$, and is hence finite dimensional. Therefore $p \circ \varphi_G$ is in the pro-$C^*$-algebra structure on $C^*_G$.

Theorem 3.12. Let $G$ be a locally compact group. Let $R$ be a set consisting of one representative $v : G \to L(H_v)$ of each unitary equivalence class of finite dimensional irreducible representations of $G$. Let

$$F = \{ v \in R : v \text{ has finite range} \}.$$

Then there is a commutative diagram

$$\begin{array}{ccc}
C^*_G & \overset{\varphi_G}{\longrightarrow} & M(C^*_G) \\
\alpha \downarrow & & \beta \downarrow \\
\prod_{v \in R} L(H_v) & \overset{\rho}{\longrightarrow} & \prod_{v \in F} L(H_v),
\end{array}$$

in which the vertical maps $\alpha$ and $\beta$ are isomorphisms of topological algebras, the top horizontal map is as in Proposition 3.11 and the bottom horizontal map $\rho$ is the obvious projection map.
Proof. For \( v \in R \) let \( \pi^v : C^*(G) \to L(H_v) \) be the corresponding representation of \( C^*(G) \). Let \( S \) be the set of \( v \in R \) which extend continuously to representations of \( G \).

The map \( \alpha \) is obtained from Proposition 1.14 using the set of representations \( \{ \pi^v : v \in R \} \). To obtain \( \beta \), we begin by applying Proposition 2.13(2) and parts (2) and (3) of Proposition 1.20 with \( A = C^* (\hat{G}) \). This identifies \( M( \hat{C}^* (\hat{G}) ) \) and \( M( \hat{C}^*(\hat{G})) \) as

\[
M( \hat{C}^* (\hat{G}) ) = \left\{ b \in \prod_{v \in S} L(H_v) : \sup_{v \in S} \| b_v \| < \infty \right\}
\]

and we take \( \beta \) to be induced by the obvious inclusion.

We next identify \( \rho \). Let \( a \in C^*(G) \), and let \( \overline{\alpha} \) be its image in \( \hat{C}^*(G) \). Then \( \alpha(\overline{\alpha}) = (\pi^v(a))_{v \in R} \) by Proposition 1.14. Using Proposition 3.11, we get \( (\beta \circ \overline{\alpha})(\overline{\pi})(\overline{\pi}) = (\pi^v(a))_{v \in S} \). It is now obvious that the diagram commutes.

It remains to identify \( S \) with \( F \). Since \( \hat{G} \) is profinite, this identification is immediate from Lemma 2.12.

Example 3.13. Take \( G = \mathbb{Z} \). Referring to Theorem 3.12, we can identify \( R \) with the dual group \( \hat{G} \), that is, with the unit circle \( S^1 \). We thus get the identification \( C^*(G) = C(S^1) \). We can identify \( F \) as

\[
F = \{ \zeta \in S^1 : \text{there is } n \in \mathbb{Z}_{>0} \text{ such that } \zeta^n = 1 \}.
\]

The topology from the profinite pro-\( C^* \)-algebra structure on \( C(S^1) \) is the topology of pointwise convergence on \( S^1 \), and, by Theorem 3.12, the completion is the set of all functions from \( S^1 \) to \( \mathbb{C} \). (See Example 1.15.) The topology from the \( C^* \)-algebra structure on \( C(S^1) = C^*(G) \) obtained via \( \varphi_G \) is the topology of pointwise convergence on \( F \). It is still faithful, but its completion is the set of all functions from \( F \) to \( \mathbb{C} \).

Corollary 3.14. Let \( G \) be a locally compact group. Then the map \( \overline{\varphi}_G : \hat{C}^*(\hat{G}) \to M( \hat{C}^*(\hat{G}) ) \) of Proposition 3.11 is surjective.

Proof. The map \( \rho \) of Theorem 3.12 is always surjective.

Corollary 3.15. Let \( G \) be a locally compact group. Then the following are equivalent.

1. The map \( \overline{\varphi}_G : \hat{C}^*(\hat{G}) \to M( \hat{C}^*(\hat{G}) ) \) of Proposition 3.11 is bijective.
2. Every finite dimensional irreducible representation of \( G \) has finite range.
3. Every finite dimensional representation of \( G \) has finite range.

Moreover, when these conditions are satisfied, \( \overline{\varphi}_G \) is a homeomorphism.

Proof. The equivalence of (1) and (2) is clear from Theorem 3.12. The equivalence of (2) and (3) follows from the fact that every finite dimensional representation of \( G \) is a finite direct sum of finite dimensional irreducible representations.

It is tempting to hope that the map \( \overline{\varphi}_G \) of Proposition 3.11 is a homeomorphism if and only if \( G \) is profinite. This is true when \( G \) is compact, but fails in general.

Proposition 3.16. Let \( G \) be a compact group. Then the map \( \overline{\varphi}_G : \hat{C}^*(\hat{G}) \to M( \hat{C}^*(\hat{G}) ) \) of Proposition 3.11 is a homeomorphism if and only if \( G \) is profinite.
Proof. By Corollary 3.15 we must check that $G$ is profinite if and only if every finite dimensional representation of $G$ has finite range.

If $G$ is profinite, then every finite dimensional representation of $G$ has finite range by Lemma 2.12.

Suppose $G$ is compact and not profinite. Let $\overline{G}$ be the profinite completion of $G$ and let $\gamma_G : G \to \overline{G}$ the canonical map, as in Notation 2.5. Then $\gamma_G(G)$ is dense in $\overline{G}$ by construction, and its range is compact, so $\gamma_G$ is surjective. Let $H = \ker(\gamma_G)$. Then $H$ is compact and nontrivial, and hence has a nontrivial finite dimensional irreducible representation $\sigma_0$. The range of $\sigma_0$ is infinite because $H$ contains no nontrivial subgroups with finite index. Induce $\sigma_0$ to a representation $\sigma$ of $G$. The restriction of $\sigma$ to $H$ contains $\sigma_0$ as a summand (by Frobenius reciprocity; see for example Theorem 7.4.1 in [5]). Therefore, when $\sigma$ is written as a direct sum of finite dimensional irreducible representations of $G$, at least one of them, say $\pi$, has the property that $\sigma_0$ is a summand in $\pi|_H$. This representation $\pi$ is a finite dimensional representation of $G$ whose range is not finite. □

Example 3.17. Let $G$ be the abelian group $G = \bigoplus_{n=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$, with the discrete topology. Then $G$ is not profinite because it is not compact. (In fact, its profinite completion is $\prod_{n=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$.) Nevertheless, the map $\varphi_G : C^*(G) \to M\left(C^*(\overline{G})\right)$ of Proposition 3.11 is a homeomorphism.

We verify the criterion of Corollary 3.15(2). The irreducible representations of $G$ are all one dimensional. Since every element of $G$ has order 1 or 2, and $S^1$ has only two such elements, the range of every irreducible representation of $G$ has at most two elements.

We give a second example, an infinite countable amenable group for which both $C^*(G)$ and $M(C^*(G))$ are just $\mathbb{C}$.

Example 3.18. Let $G$ be the group of finitely supported even permutations of a countable infinite set, with the discrete topology. This group is countable and simple. It is amenable, since it is the increasing union of finite subgroups. It is not profinite because it is not compact. Nevertheless, we claim that the map $\varphi_G : C^*(G) \to M(C^*(G))$ of Proposition 3.11 is a homeomorphism.

We verify the criterion of Corollary 3.15(3), by showing that $G$ has no nontrivial finite dimensional representations. Since $G$ is simple, it suffices to show that $G$ has no faithful finite dimensional representations.

Suppose $u : G \to L(\mathbb{C}^n)$ is a faithful representation. Jordan’s Theorem (see page 91 of [2], or Theorem 2.1 of [2]) provides $l \in \mathbb{Z}_{>0}$ such that every finite subgroup of $G$ contains an abelian normal subgroup of index at most $l$. Since $G$ contains a copy of every finite group, this is obviously impossible.

References
[1] M. B. Bekka, On the full $C^*$-algebras of arithmetic groups and the congruence subgroup problem, Forum Math. 11(1999), 705–715.
[2] E. Breuillard and B. Green, Approximate groups, III: the unitary case, preprint (arXiv:1006.5160v2 [math.GR]).
[3] M.-D. Choi, The full $C^*$-algebra of the free group on two generators, Pacific J. Math. 87(1980), 41–48.
[4] K. R. Davidson, $C^*$-Algebras by Example, Fields Institute Monographs no. 6, Amer. Math. Soc., Providence RI, 1996.
[5] A. Deitmar and S. Echterhoff, Principles of Harmonic Analysis, Springer Science+Business Media, New York, 2008.
[6] E. Hewitt and K. A. Ross, Abstract Harmonic Analysis, Vol. I: Structure of Topological Groups, Integration Theory, Group Representations, 2nd ed., Grundlehren der Mathematischen Wissenschaften no. 115, Springer-Verlag, New York, etc., 1979.
[7] C. Jordan, Mémoire sur les équations différentielles linéaires à intégrale algébrique, J. reine angew. Math. 84 (1878), 89–215.
[8] G. J. Murphy, C*-Algebras and Operator Theory, Academic Press, Boston, San Diego, New York, London, Sydney, Tokyo, Toronto, 1990.
[9] G. K. Pedersen, C*-Algebras and their Automorphism Groups, Academic Press, London, New York, San Francisco, 1979.
[10] N. C. Phillips, Inverse limits of C*-algebras, J. Operator Theory 19 (1988), 159–195.
[11] C. Radin and L. Sadun, On 2-generator subgroups of SO(3), Trans. Amer. Math. Soc. 351 (1999), 4469–4480.
[12] L. Ribes and P. Zalesskii, Profinite Groups, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3rd Series, vol. 40, Springer-Verlag, Berlin, etc., 2000.
[13] H. H. Schaefer, Topological Vector Spaces, Third printing corrected, Graduate Texts in Mathematics, Vol. 3, Springer-Verlag, New York, Heidelberg, Berlin, 1971.
[14] K. Schmüdgen, Über LMC*-Algebren, Math. Nachr. 68 (1975), 167–182.
[15] A. B. Thom, Convergent sequences in discrete groups, Canad. Math. Bull., to appear; doi:10.4153/CMB-2011-155-3. arXiv:1003.4093v2 [math.GR].
[16] D. Voiculescu, Dual algebraic structures on operator algebras related to free products, J. Operator Theory 17 (1987), 85–98.
[17] S. Wassermann, Exact C*-Algebras and Related Topics, Lecture Notes Series No. 19, Research Institute of Mathematics Global Analysis Research Center, Seoul National University, Seoul, Korea, 1994.

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