Quantization of classical integrable systems
Part IV: systems of resonant oscillators

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Abstract
By applying methods already discussed in a previous series of papers by the same authors, we construct here classes of integrable quantum systems which correspond to \( n \) fully resonant oscillators with nonlinear couplings. The same methods are also applied to a series of nontrivial integral sets of functions, which can be constructed when additional symmetries are present due to the equality of some of the frequencies. Besides, for \( n = 3 \) and resonance 1:1:2, an exceptional integrable system is obtained, in which integrability is not explicitly connected with this type of symmetry. In this exceptional case, quantum integrability can be realized by means of a modification of the symmetrization procedure.

1 Introduction
In [3] we have already given examples of applications to concrete systems of a general procedure [2], with which a classical integrable system can generally be transformed into a quasi-integrable quantum system [1] (see also references therein). In this paper we consider systems describing an arbitrary number \( n \) of oscillators with a fully resonant set of frequencies. More exactly, in the classical case we describe a class of integrable sets of functions \( F \) such that
\[
F_1 = l_1 I_1 + \cdots + l_n I_n,
\]
where
\[
I_i = \frac{1}{2}(p_i^2 + x_i^2)
\]
for \( i = 1, \ldots, n \), and \( l_1, \ldots, l_n \) are nonzero integer values, i.e., \( l \in (\mathbb{Z} \setminus \{0\})^n \) (one can assume that they have no common divisors other than 1). Hence the set \( F \) contains the hamiltonian \( H = F_1 \) of a set of fully resonant linear oscillators. We are able to construct a whole class of integrable sets of functions by exploiting the condition of resonance. Furthermore, when some of the frequencies are equal to each other, it is possible to construct a wider class of nontrivial integrable systems, by exploiting the symmetry provided by these equalities.

These integrable sets contain in general \( 2n - k \) elements, where \( k \) is equal to the number of elements in the central subset [4]. This number, for the various integral sets here considered, can take all possible values from 1 to \( n \). Each integrable set can be applied to all systems whose hamiltonian is an arbitrary function of the central elements, and so to a whole class of systems which describe nonlinear oscillations with completely resonant frequencies of small oscillations. It means that if the amplitude of oscillations tends to zero,
then in this limit the oscillations tend to the linear ones with a completely resonant set of frequencies. The situation \( n = k = 3, |h_1| = |h_2| \neq 0 \), is studied in particular detail. For the special case \( |h_1| = |h_2| = 1, |h_3| = 2 \), we are able to construct an exceptional integrable system which is not explicitly based on the symmetry connected with the two equal frequencies.

Using the results of \([2]\), we then construct the quantum analogues of all these classical integrable systems. For the exceptional system, a modification of the symmetrization procedure is required in order to obtain a quasi-integrable quantum system. This modification consist in the addition of a lower order term to the direct symmetrization of one of the classical functions, and thus is of the same type as the modification that was required in \([3]\) for the integration of the free quantum rigid body in 6-dimensional space.

The investigations of the integrable systems considered in \([3]\) and in the present paper illustrate in concrete situations the general concepts presented in \([1]\) and \([2]\). Note however that the systems considered in \([3]\), describing the motion of particles in a central force field and the free rotation of a rigid body, have application in physics only when the dimension \( n \) of configuration space is equal to 3. On the contrary, the systems of oscillators which are considered here are interesting for applications for any not too large value of \( n \). For example, spectral lattices which are constructed from quantum integrable systems are useful for studying data of spectrography in physics and molecular chemistry. The points of such lattices are vectors \( \lambda = (\lambda_1, \ldots, \lambda_k) \), such that each vector is made of eigenvalues \( \lambda_i, i = 1, \ldots, k \), of the central operators \( F_1, \ldots, F_k \) of an integrable set \( F = (F_1, \ldots, F_k; F_{k+1}, \ldots, F_{2n-k}) \) of linear differential operators, and these eigenvalues correspond to common eigenfunctions \( \psi \) of these operators: \( F_j \psi = \lambda_i \psi, i = 1, \ldots, k \). More details can be found for example in \([5, 6, 7]\). Sets of nonlinear fully resonant oscillators also play a central role in the study of important infinite-dimensional systems \([8, 9]\).

### 2 Classical oscillators with completely resonant set of frequencies

It is easy to construct a linear basis of the infinite-dimensional linear space of all real polynomial functions in \((x, p)\) which are in involution with \( F_1 = l_1 I_1 + \cdots + l_n I_n \). To this purpose, it is useful to introduce the functions

\[
  z_j = \frac{x_j + i p_j}{\sqrt{2}}, \quad \bar{z}_j = \frac{x_j - i p_j}{\sqrt{2}},
\]

so that \( \bar{z}_j \) is the complex conjugate of \( z_j \). A linear basis in the space of all monomials in \((x, p)\) is obviously given by the real and imaginary parts of all monomials in \((z, \bar{z})\), i.e., by the functions of the form \( \text{Re} \, P_{a,b} \) and \( \text{Im} \, P_{a,b} \), where \( P_{a,b} := z_1^{a_1} \bar{z}_1^{b_1} \cdots z_n^{a_n} \bar{z}_n^{b_n} \) and \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_+^n \), \( b \in Z_n^+, (b_1, b_2, \ldots, b_n) \in \mathbb{Z}_+^n \). We have

\[
  \{z_i, z_j\} = \{\bar{z}_i, \bar{z}_j\} = 0, \quad \{z_i, \bar{z}_j\} = i \delta_{ij}
\]

for \( i, j = 1, \ldots, n \), and \( I_i = z_i \bar{z}_i \). It follows that

\[
  [F_1, P_{a,b}] = -i \text{Im} \cdot (a - b) P_{a,b},
\]
where \( l \cdot (a - b) = \sum_{j=1}^{n} l_j(a_j - b_j) \). We have \((P_{a,b})^* = P_{b,a}\), where * denotes complex conjugation. From (2.3) one thus obtains

\[
\begin{align*}
\{F_1, \Re P_{a,b}\} &= l \cdot (b - a) \Im P_{a,b}, \\
\{F_1, \Im P_{a,b}\} &= -l \cdot (b - a) \Re P_{a,b}.
\end{align*}
\]  

We see therefore that a basis of the linear space of all polynomials in involution with \( F_1 \) is given by the set \( \mathcal{P}_l := \{ \Re P_{a,b}, \Im P_{a,b}, l \cdot (b - a) = 0 \} \).

### 2.1 General symmetries

It is easy to find \( 2n - 1 \) functionally independent elements of \( \mathcal{P}_l \), for any set of frequencies \( l = (l_1, \ldots, l_n) \in (\mathbb{Z} \setminus \{0\})^n \). Hence, the system with hamiltonian function \( F_1 \) is integrable with number of central integrals \( k = 1 \). For example, if \( l_i \geq 1 \) for \( i = 1, \ldots, n \), then it is easy to check that the function \( F_1 \) is involutive with the functions of the set \( F = (F_1, I_2, I_3, \ldots, I_n, R_{12}, R_{13}, \ldots, R_{1n}) \), where

\[
R_{ij} := \Im(z_i^{l_i} \bar{z}_j^{l_j}) \quad \text{for} \quad 1 \leq i < j \leq n.
\]

If \( l_i \) and \( l_j \) have a common divisor \( d \), then one can divide both numbers by \( d \) and consider instead of \( R_{ij} \) the polynomial

\[
R'_{ij} := \Im(z_i^{l_i'} \bar{z}_j^{l_j'}), \quad \text{where} \quad l_i' = l_i/d, \quad l_j' = l_j/d.
\]

Moreover, it is easy to see that the functions of the set \( F \) are functionally independent, so that this set is integrable with one central integral \( F_1 \). An integrable set \( F \) of similar form can also be constructed when the hamiltonian function \( F_1 \) has frequencies \( l_1, \ldots, l_n \) of different signs. In this case, one takes

\[
R_{ij} := \begin{cases} 
\Im(z_i^{l_i} \bar{z}_j^{l_j}) & \text{if} \; l_i l_j > 0, \\
\Im(z_i^{l_i} \bar{z}_j^{-l_j}) & \text{if} \; l_i l_j < 0.
\end{cases}
\]  

It is also possible to find other integrable sets of functions, with number \( k \) of central integrals ranging from 2 up to the maximum possible value \( n \). As we already explained in [3] about a similar situation for one-particle systems with \( SO(n) \) symmetry (see there proposition 2.4 and related comments), the existence of such sets with higher \( k \) allows one to construct a wider class of integrable systems, by taking as hamiltonian any arbitrary function of the central elements of an integrable set. In the present case, functions of this type can often be represented as polynomials in \((x, p)\) of degree \( > 2 \). In such cases, they can be considered as perturbations to the hamiltonian \( F_1 \) of resonant oscillators, which become negligible in the limit of small amplitude of oscillations.

We are now going to describe a general procedure for the construction of a remarkable class of integral sets of functions of various \( k \). For any \( m \in \mathbb{Z}^n \), consider the function

\[
R_m = \zeta_1^{m_1} \zeta_2^{m_2} \cdots \zeta_n^{m_n},
\]

where

\[
\zeta_j^{m_j} := \begin{cases} 
z_j^{m_j} & \text{if} \; m_j \geq 0, \\
z_j^{-m_j} & \text{if} \; m_j < 0.
\end{cases}
\]  

Note that \((\zeta_j^{m_j})^* = \zeta_j^{-m_j}\), so that \( R_m = R_{-m} \). For \( r \in \mathbb{Z}^n \), let us define

\[
J_r = \sum_{j=1}^{n} r_j I_j,
\]  

3
so that $F_i = J_i$. Similarly to (2.4) we have
\[
\{ J_r, \text{Re } R_m \} = r \cdot m \text{ Im } R_m, \quad \{ J_r, \text{Im } R_m \} = -r \cdot m \text{ Re } R_m. \tag{2.9}
\]
Consider the sets of functions $A_l = (\text{Re } R_m, \ m \cdot l = 0)$, $B_l = (\text{Im } R_m, \ m \cdot l = 0)$, and $I = (I_1, I_2, \ldots, I_n)$. It is easy to see that $\mathcal{U}_l := (I, A_l, B_l)$ is a set of functions in involution with $F_1$, which is closed with respect to Poisson brackets. This means that the Poisson bracket of two elements of $\mathcal{U}_l$ can always be expressed as a function of other elements of $\mathcal{U}_l$. Note also that
\[
(\text{Re } R_m)^2 + (\text{Im } R_m)^2 = |R_m|^2 = I_1^{[m]} I_2^{[m_2]} \cdots I_n^{[m_n]}. \tag{2.10}
\]
Put
\[
F_i = J_{r(i)} \quad \text{for } i = 1, \ldots, n, \tag{2.11}
\]
where $r^{(1)}, \ldots, r^{(n)}$ are $n$ linearly independent elements of $\mathbb{Z}^n$, with $r^{(1)} = l$. Obviously $F^{(n)} = (F_1, \ldots, F_n)$ is an integrable set with $k = n$. It is however possible to obtain other integrable sets with $k$ central elements, where $k$ is any number such that $1 \leq k < n$, by adding to $F^{(n)}$ suitable (functions of) elements of $\mathcal{U}_l$. A simple possible choice is to put
\[
F_{n+k} = \text{Im } R_{m(i)} \quad \text{for } i = 1, \ldots, n-k, \tag{2.12}
\]
where $m^{(1)}, \ldots, m^{(n-k)}$ are $n-k$ linearly independent elements of $\mathbb{Z}^n$, such that $r^{(i)} \cdot m^{(j)} = 0 \ \forall \ i = 1, \ldots, k$ and $\forall \ j = 1, \ldots, n-k$. Using the second of (2.9) it is immediate to see that $\{ F_i, F_j \} = 0 \ \forall \ i = 1, \ldots, k$ and $\forall \ j = 1, \ldots, 2n - k$. Furthermore, it is not difficult to show that the set $F^{(k)} = (F_1, \ldots, F_{2n-k})$ is functionally independent. It follows that $F^{(k)}$ is an integrable set with $k$ central elements $(F_1, \ldots, F_k) = (J_{r^{(1)}}, \ldots, J_{r^{(k)}})$.

It is also possible to construct integrable sets of functions whose central subset contains elements of $A_l$ or $B_l$. For example, one can obtain a set with $k = n$ by taking $F_l = J_{r^{(i)}}$ for $i = 1, \ldots, n-1$, $F_n = \text{Im } R_m \in B_l$, where $r^{(1)} = l$ and $m \cdot r^{(i)} = 0 \ \forall \ i = 1, \ldots, n-1$. More generally, let us consider the sets of functions $R_1 := (\text{Im } R_{m(l)})$, $\ldots$, $\text{Im } R_{m(n-1)})$ and $R_2 := (R_{m(l)+1}), \ldots, R_{m(n)})$, where $m^{(1)}, \ldots, m^{(l+1)}, \ldots, m^{(h)} \in \mathbb{Z}^n$ are linearly independent vectors such that $m^{(i)} \cdot l = 0$ and $m^{(i)} \cdot m^{(j)} = 0 \ \forall \ i = 1, \ldots, h$ and $j = 1, \ldots, k'$, and $\forall \ p = 1, \ldots, n$. The latter condition obviously ensures that $\{ \text{Im } R_{m(l)}, \text{Im } R_{m(n)} \} = 0$. For example, for $n \geq 6$ oscillators of equal frequencies, i.e., $l = 1 \ \forall \ i = 1, \ldots, n$, a pair of sets of such type, with $k' = 3$ and $h = n - 4$, is given by $R_1 = (R_{12}, R_{34}, R_{56})$, $R_2 = (R_{78}, R_{79}, \ldots, R_{7n})$, where the functions $R_{ij}$ are defined by formula (2.5). Clearly one must have in general $k' \leq n/2$. Obviously the case $h = k'$ corresponds to $R_2 = \emptyset$. It is also easy to see that, when $h > k'$, then one has necessarily $h < n - k'$. Furthermore, let us take $n-k'$ linearly independent vectors $r^{(1)}, \ldots, r^{(n-k')}$ in $\mathbb{Z}^n$, with $r^{(1)} = l$, $r^{(i)} \cdot m^{(j)} = 0 \ \forall \ i = 1, \ldots, n-k'$ and $j = 1, \ldots, k'$, in such a way that the first $n-h$ vectors of this set satisfy the additional relations $r^{(i)} \cdot m^{(j)} = 0 \ \forall \ i = 1, \ldots, n-h$ and $j = k' + 1, \ldots, h$. It is then easy to see that the set
\[
(F_1, \ldots, F_{2n-k}) = (J_{1}, R_{1}; J_{2}, R_{2}), \tag{2.13}
\]
where $k = n - h + k'$, $J_{1} = (J_{r^{(1)}}, \ldots, J_{r^{(n-h)}})$, $J_{2} = (J_{r^{(n-h+1)}}, \ldots, J_{r^{(n-k')}})$, is an integrable set with $k$ central elements $(F_1, \ldots, F_k) = (J_{1}, R_{1})$. 

4
2.2 Additional symmetries for equal frequencies

It often occurs in physics that all frequencies, or part of them, are equal to one another or have the same absolute value. In these cases there exist also other types of integrable sets. For example, let us suppose that $l_i = l_j \forall i, j = 1, \ldots, n$. In this case the system with Hamiltonian $F_1$ is invariant with respect to the action of the group $SO(n)$ on configuration space, and can be identified with the system describing a point particle moving in the central potential $U(r) = r^2/2$ in $n$-dimensional space. It is then possible to construct additional integrable sets of functions by making use of proposition 2.4 of [3]. In the present case, the procedure can also be generalized to the case in which only the absolute values of the frequencies are equal to one another. More precisely, let us suppose that $l_i = \epsilon_i$ for $i = 1, \ldots, n$, where $\epsilon_i = \pm 1 \forall i = 1, \ldots, n$. Consider the functions

$$w_i := \begin{cases} z_i & \text{if } \epsilon_i = +1, \\ \bar{z}_i & \text{if } \epsilon_i = -1, \end{cases}$$

for $i = 1, \ldots, n$, and let $\bar{w}_i$ denote the complex conjugate of $w_i$. We have

$$\{w_i, w_j\} = \{\bar{w}_i, \bar{w}_j\} = 0,$$ and $I_i = z_i\bar{z}_i = w_i\bar{w}_i \forall i, j = 1, \ldots, n$. It follows that $\{I_i, w_j\} = -i\delta_{ij}\epsilon_j w_i$, and

$$\{F_1, w_j\} = -iw_j, \quad \{F_1, \bar{w}_j\} = iw_j,$$

where $F_1 = \sum_{i=1}^n \epsilon_i I_i$. Let us consider the momenta

$$P_{ij} = i(w_i\bar{w}_j - w_j\bar{w}_i) = -P_{ji}.$$ From (2.15) it follows that

$$\{F_1, P_{ij}\} = 0 \forall i, j = 1, \ldots, n.$$ Note that

$$P_{ij} = \begin{cases} \pm i(z_i\bar{z}_j - z_j\bar{z}_i) = \pm(x_ip_j - x_jp_i) & \text{if } \epsilon_i = \epsilon_j = \pm 1, \\ \pm i(z_i\bar{z}_j - \bar{z}_i\bar{z}_j) = \mp(x_ip_j + x_jp_i) & \text{if } \epsilon_i = -\epsilon_j = \pm 1. \end{cases}$$

Hence, for $\epsilon_i = \epsilon_j = 1$, $P_{ij}$ is formally identical to the momentum introduced in [3] for a particle in a central force field (although the physical meaning of variables $x$ and $p$ is here different). We have

$$\{P_{ij}, P_{hk}\} = -\epsilon_i\delta_{ih}P_{jk} - \epsilon_j\delta_{jk}P_{ih} + \epsilon_i\delta_{ih}P_{jh} + \epsilon_j\delta_{jh}P_{ik},$$

which has to be compared with the analogous Poisson bracket in [3]. It is then easy to see that

$$\{P^2, P_{ij}\} = 0 \forall i, j = 1, \ldots, n,$$

where

$$P^2 := \sum_{i<j} \epsilon_i\epsilon_j P_{ij}^2.$$ Since (2.19) is formally identical with the analogous poisson bracket for a particle in a central force field, it is easy to see that proposition 2.4 of [3] can be generalized in the following way.

5
Proposition 2.1. For any $n \geq 2$, for any sequence $(\epsilon_1, \ldots, \epsilon_n)$, with $\epsilon_i = \pm 1 \ \forall \ i = 1, \ldots, n$, and for any $z = 1, \ldots, n - 1$, it is possible to construct in a recursive manner sets $Z_{n,z}$ and $L_{n,z}$ of polynomial functions of degree $\leq 2$ in the variables $P_n := (P_{ij}, 1 \leq i < j \leq n)$, with the following properties:

1. $Z_{n,z}$ contains $z$ elements,
2. $L_{n,z}$ contains $2(n - z - 1)$ elements,
3. the set $\Pi_{n,z} := (Z_{n,z}, L_{n,z})$ is functionally independent,
4. $\{Z_{n,z}, \Pi_{n,z}\} = 0$.

In this proposition, momenta $P_{ij}$ are obviously defined according to formula (2.16). From (2.17) it follows that $\{F_1, \Pi_{n,z}\} = 0$. Hence the sets of functions $F_{n,z} := (F_1, Z_{n,z}; L_{n,z})$ are integrable sets, with subset of central elements $(F_1, Z_{n,z})$. We have $\sharp F_{n,z} = 2n - k$ and $k = z + 1$. Hence the number $k$ of central elements of these sets can take all values from $k = 2$ to $k = n$.

We are now able to describe a general class of integrable sets, which includes those presented above as particular cases, and which can be applied to any arbitrary set of frequencies $l \in (\mathbb{Z} \setminus \{0\})^n$. Let us divide the frequencies into $u$ groups, with $1 \leq u \leq n$, so that each group contains one or more frequencies whose absolute values are equal to one another. Namely, without loss of generality, we suppose that

$$l_i = \begin{cases} 
\epsilon_is_1 & \text{for } i = 1, \ldots, p_1, \\
\epsilon_is_2 & \text{for } i = p_1 + 1, \ldots, p_2, \\
\vdots & \vdots \\
\epsilon_is_u & \text{for } i = p_{u-1} + 1, \ldots, n,
\end{cases}$$

(2.20)

where $s_1, \ldots, s_u \in \mathbb{N}$, $\epsilon_i = \pm 1 \ \forall \ i = 1, \ldots, n$, $1 \leq p_1 < p_2 < \ldots < p_{u-1} < n$. The number of frequencies contained in the various groups are $q_h = p_h - p_{h-1} \geq 1$ for $h = 1, \ldots, u$, where $p_0 := 0$, $p_u := n$. Obviously $\sum_{h=1}^u q_h = n$. If the absolute values of all frequencies of the system are pairwise different, one can only take $u = n$, $q_h = 1 \ \forall \ h = 1, \ldots, n$. Note however that, at variance with the analogous partition of the generalized moments of inertia $\lambda_1, \ldots, \lambda_n$ of a rigid body, given in [3], here we do not require that the absolute values of the frequencies of different groups be pairwise different. Hence, it is possible in general that $s_h = s_j$ for some $h \neq j$. This means that, if the system includes any set of more than one oscillators, whose frequencies have a common absolute value, one is free to divide them into subgroups in any arbitrary possible way. Each choice will lead to different realizations of integrable sets of functions, according to the procedure which we are going to describe.

Let $w_i$ be defined by formula (2.14) for $i = 1, \ldots, n$. For each $h = 1, \ldots, u$, let us consider the sets of momenta

$$P_{(h)} = \begin{cases} 
\emptyset & \text{if } q_h = 1, \\
(P_{ij}, p_{h-1} < i < j \leq p_h) & \text{if } q_h > 1,
\end{cases}$$

(2.21)

where $P_{ij}$ is defined by formula (2.16). Note that $\{P_{(h)}, P_{(j)}\} = 0$ if $h \neq j$. For each $h$ such that $q_h > 1$, let us construct sets of functions $Z_{q_h,z_h}(P_{(h)})$ and
\( L_{q_h, z_h}(P_h) \) by means of proposition 2.1, where \( z_h \) can be arbitrarily chosen among the possible values 1, 2, \ldots, \( q_h - 1 \). For those \( h \) such that \( q_h = 1 \) we put instead \( z_h = 0 \) and \( Z_{q_h, z_h}(P_h) = L_{q_h, z_h}(P_h) = \emptyset \). We have

\[
z Z_{q_h, z_h}(P_h) = z_h, \\
z L_{q_h, z_h}(P_h) = 2(q_h - z_h - 1),
\]

and

\[
\{ Z_{q_h, z_h}(P_h), \Pi_{q_j, z_j}(P_j) \} = 0 \tag{2.22}
\]

\( \forall h, j = 1, \ldots, u \), where \( \Pi_{q_j, z_j}(P_j) = (Z_{q_j, z_j}, L_{q_j, z_j}) \).

We can write

\[
F_1 = \sum_{h=1}^u s_h K_h, \quad K_h := \sum_{i=p_h-1+1}^{p_h} \epsilon_i I_i.
\]

In analogy with (2.17) we have

\[
\{ K_h, P_j \} = 0, \tag{2.23}
\]

so that

\[
\{ K_h, \Pi_{q_j, z_j}(P_j) \} = 0 \quad \forall h, j = 1, \ldots, u. \tag{2.24}
\]

This implies, in particular, that \( \{ F_1, \Pi_{q_j, z_j}(P_j) \} = 0 \). Let us introduce the functions

\[
W_h := \sum_{i=p_h-1+1}^{p_h} \epsilon_i w_i^2, \quad h = 1, \ldots, u.
\]

It is easy to check that

\[
\{ W_h, W_j \} = 0, \quad \{ W_h, \bar{W}_j \} = 4i\delta_{jh} K_h, \quad \{ W_h, P_j \} = 0, \quad \{ W_h, K_j \} = 2i\delta_{jh} W_h \tag{2.25}
\]

\( \forall j, h = 1, \ldots, u \), where \( W_h \) denotes as usual the complex conjugate of \( W_h \). Note also the relation

\[
W_h W_h = K_h^2 - P_h^2, \tag{2.27}
\]

where

\[
P_h^2 := \sum_{p_h-1 < i < j \leq p_h} \epsilon_i \epsilon_j P_{ij}^2.
\]

We are now going to construct integrable sets of functions which contain the set \( \tilde{F} = (F_1, \Pi_{q_1, z_1}(P_1), \ldots, \Pi_{q_u, z_u}(P_u)) \). For the choice of the additional elements, we shall follow a procedure which is similar to the one we used before in section 2.1. Here the sets of functions \( (K_1, \ldots, K_u) \) and \( (W_1, \ldots, W_u) \) will play the role that was formerly played by the sets \( (I_1, \ldots, I_n) \) and \( (z_1, \ldots, z_n) \) respectively. We shall in fact replace formula (2.6) by

\[
R_m = \Omega_1^m \Omega_2^m \cdots \Omega_u^m, \tag{2.28}
\]

where \( m \in \mathbb{Z}^u \) and

\[
\Omega_h^m := \begin{cases} W_h^m & \text{if } m_h \geq 0, \\
W_h^{-m} & \text{if } m_h < 0. \end{cases} \tag{2.29}
\]
Moreover, we shall replace (2.8) by

\[ J_r = \sum_{h=1}^{u} r_h K_h, \tag{2.30} \]

where \( r \in \mathbb{Z}^u \). Note that \( F_1 = J_s \). Using the second of (2.26) we easily obtain the Poisson brackets

\[ \{ J_r, \text{Re } R_m \} = 2r \cdot m \text{ Im } R_m, \quad \{ J_r, \text{Im } R_m \} = -2r \cdot m \text{ Re } R_m, \tag{2.31} \]

which have to be compared with (2.9).

Consider the sets of functions

\[ A_s = (\text{Re } R_m, s \cdot m = 0), \quad B_s = (\text{Im } R_m, s \cdot m = 0), \quad K = (K_1, \ldots, K_u) \quad \text{and} \quad P^2 = (P^2_1, \ldots, P^2_u). \]

Using relations (2.24)–(2.27), it is easy to verify that \( U_s := (K, P^2, A_s, B_s) \) is a set of functions in involution with \( \tilde{F} \), which is closed with respect to Poisson brackets. We can obtain an integrable set analogous to the one defined by formulas (2.11)–(2.12), by adding to the set \( \tilde{F} \) suitable combinations of elements of \( U_s \). Let \( r^{(1)}, \ldots, r^{(u)} \) be \( u \) linearly independent elements of \( \mathbb{Z}^u \), with \( r^{(1)} = s \). Choose \( k' \in \mathbb{N} \) such that \( 1 \leq k' \leq u \), and construct the set of \( u - k' \) functions

\[ R = (\text{Im } R_{m(1)}, \ldots, \text{Im } R_{m(u-k')}) \]

where \( m^{(1)}, \ldots, m^{(u-k')} \) are \( u - k' \) linearly independent elements of \( \mathbb{Z}^u \), such that \( r^{(h)} \cdot m^{(j)} = 0 \ \forall \ h = 1, \ldots, k' \) and \( \forall \ j = 1, \ldots, u - k' \). Obviously \( R = \emptyset \) if \( k' = u \). Consider then the set of functions

\[ F = (J_1, Z; J_2, R, L), \tag{2.32} \]

where

\[ J_1 = (J_{r^{(1)}}, \ldots, J_{r^{(k')}}), \]
\[ J_2 = (J_{r^{(k'+1)}}, \ldots, J_{r^{(u)}}), \]
\[ Z = (Z_{q_1, z_1(P^{(1)}_1)}, \ldots, Z_{q_u, z_u(P^{(u)}_u)}), \]
\[ L = (L_{q_1, z_1(P^{(1)}_1)}, \ldots, L_{q_u, z_u(P^{(u)}_u)}). \]

Note that \( J_{r^{(1)}} = J_s = F_1 \). We have

\[ \sharp J_1 = k', \quad \sharp J_2 = u - k', \quad \sharp R = u - k', \]
\[ \sharp Z = z := \sum_{h=1}^{u} z_h, \quad \sharp L = 2 \sum_{h=1}^{u} (q_h - z_h - 1) = 2(n - z - u). \]

Hence

\[ \sharp (J_1, Z) = k := k' + z, \]
\[ \sharp F = 2n - z - k' = 2n - k. \]

We have obviously

\[ \{ J_1, J_1 \} = \{ J_1, J_2 \} = 0. \]

From (2.24) it follows that

\[ \{ J_1, Z \} = \{ J_1, L \} = \{ J_2, Z \} = 0. \]
From the second of (2.31) it follows that
\[ \{ J_1, R \} = 0. \]
From (2.22) it follows that
\[ \{ Z, Z \} = \{ Z, L \} = 0. \]
Finally, from the first of (2.26) it follows that
\[ \{ Z, R \} = 0. \]

It is also possible to show that the set \( F \) is functionally independent. We are therefore able to formulate the following

**Proposition 2.2.** The set \( F \) defined by (2.32) is an integrable set of functions with \( k \) central elements \((J_1, Z)\), where \( k = k' + z \).

From this proposition it obviously follows that any system with hamiltonian
\[ H = f(J_1, Z), \]
where \( f \) is any function of \( k \) variables, is integrable with the same integrable set \( F \).

It is also possible to obtain an integrable set of functions \( F \supset \tilde{F} \), whose central subset includes elements of \((A_s, B_s)\). To this purpose, we shall follow a procedure similar to the one which led us before to the set (2.13). If \( 1 \leq k' \leq u/2 \), let \( m^{(1)}, \ldots, m^{(k')}, m^{(k'+1)}, \ldots, m^{(b)} \in \mathbb{Z}^u \) be linearly independent vectors, such that \( m^{(i)} \cdot s = 0 \) and \( m^{(i)} \cdot m^{(j)} = 0 \) \( \forall i = 1, \ldots, h \) and \( j = 1, \ldots, k' \), and \( \forall p = 1, \ldots, n \). It is easy to see that one must have either \( h = k' \) or \( k' < h < u - k' \). Furthermore, let us take \( u - k' \) linearly independent vectors \( r^{(1)}, \ldots, r^{(u-k')} \in \mathbb{Z}^u \), with \( r^{(1)} = s \), \( r^{(i)} \cdot m^{(j)} = 0 \) \( \forall i = 1, \ldots, u - k' \) and \( j = 1, \ldots, k' \), in such a way that the first \( u - h \) vectors of this set satisfy the additional relations \( r^{(i)} \cdot m^{(j)} = 0 \) \( \forall i = 1, \ldots, u - h \) and \( j = k' + 1, \ldots, h \). Let us then consider the set
\[ F = (J_1, R_1, Z; J_2, R_2, L, \mathcal{L}), \] (2.33)
where
\[ J_1 = (J_{r^{(1)}}, \ldots, J_{r^{(u-k')}}), \]
\[ J_2 = (J_{r^{(u-k'+1)}}, \ldots, J_{r^{(u)}}), \]
\[ R_1 = (\text{Im} R_{m^{(1)}}, \ldots, \text{Im} R_{m^{(u')}}), \]
\[ R_2 = (\text{Im} R_{m^{(u'+1)}}, \ldots, \text{Im} R_{m^{(h)}}). \]

It is easy to see that \( F \) contains \( 2n - k \) elements, where \( k = u - h + k' + z \).

**Proposition 2.3.** The set \( F \) defined by (2.33) is an integrable set of functions with \( k \) central elements \((F_1, \ldots, F_k) = (J_1, R_1, Z)\).

Hence any system with hamiltonian \( H = f(J_1, R_1, Z) \), where \( f \) is any function of \( k \) variables, is integrable with the same integrable set \( F \).
3 Quantum oscillators with completely resonant set of frequencies

In order to maintain the correspondence between our notation and the one usually employed in physics, in this section we shall associate with the classical impulses \( p \) the complex operators \( \hat{p} = -i\partial/\partial x \), where \( i = \sqrt{-1} \). We have with this convention \( \hat{p} = \hat{p}^* \), where \( \hat{p}^* \) denotes the hermitian conjugate of the operator \( \hat{p} \). It is obvious that this modification does not substantially affect the general results on quantization which have been obtained in the preceding papers of this series, although some formulas have to be corrected by the introduction of one or more factors \( i \). The standard canonical commutation relations become

\[
[x_i, x_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{p}_i, x_j] = -i\delta_{ij}
\]  

for \( i, j = 1, \ldots, n \), where \( \delta_{ij} \) is the Kronecker symbol. The standard quantization of the functions \( z_j, \bar{z}_j \) is given by

\[
\hat{z}_j = \frac{x_j + i\hat{p}_j}{\sqrt{2}}, \quad \hat{z}_j^* = \frac{x_j - i\hat{p}_j}{\sqrt{2}},
\]

where \( \hat{z}_j^* \) denotes the hermitian conjugate of the operator \( \hat{z}_j \). These operators satisfy the commutation relations

\[
[\hat{z}_i, \hat{z}_j] = [\hat{z}_i^*, \hat{z}_j^*] = 0, \quad [\hat{z}_i, \hat{z}_j^*] = \delta_{ij}
\]  

for \( i, j = 1, \ldots, n \). Note that, with the conventions presently adopted for the quantization of impulses \( p \), the Poisson bracket of two functions is now replaced by the commutator of the corresponding operators multiplied by \( i \).

Let as above \( \tilde{F}_1 = l_1 l_1 + \cdots + l_n I_n \) be the hamiltonian function of the system describing linear oscillations with a completely resonant set of frequencies. The standard quantization \( \hat{\tilde{F}}_1 \) of \( \tilde{F}_1 \) clearly coincides with its symmetrization \( \tilde{F}_1^{\text{sym}} \) with respect to \( (x, \hat{p}) \). On the other hand, since the operators \( \hat{\tilde{z}}, \hat{\tilde{z}}^* \) are linear combinations of the operators \( x, \hat{p} \), it is easy to see that the symmetrization with respect to \( (x, \hat{p}) \) is equivalent to the symmetrization with respect to \( (\hat{\tilde{z}}, \hat{\tilde{z}}^*) \). We thus have

\[
\hat{F}_1 = l_1 \hat{I}_1 + \cdots + l_n \hat{I}_n,
\]

where

\[
\hat{I}_j = \tilde{I}_j^{\text{sym}} = \frac{x_j^2 + \hat{p}_j^2}{2} = \frac{\hat{\tilde{z}}_j \hat{\tilde{z}}_j^* + \hat{\tilde{z}}_j^* \hat{\tilde{z}}_j}{2}.
\]

In the general situation described by relations (2.20), we can write

\[
\tilde{K}_h := \sum_{i=p_{h-1}+1}^{p_h} \epsilon_i \hat{I}_i, \quad h = 1, \ldots, u.
\]

Let us introduce the operators

\[
\hat{w}_i := \begin{cases} \hat{z}_i & \text{if } \epsilon_i = +1, \\ \hat{z}_i^* & \text{if } \epsilon_i = -1, \end{cases}
\]

which satisfy the relations

\[
[\hat{w}_i, \hat{w}_j] = [\hat{w}_i^*, \hat{w}_j^*] = 0, \quad [\hat{w}_i, \hat{w}_j^*] = \epsilon_i \delta_{ij},
\]  

for \( i, j = 1, \ldots, u \). The standard canonical commutation relations become

\[
[x_i, x_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{p}_i, x_j] = -i\delta_{ij}
\]  

for \( i, j = 1, \ldots, n \), where \( \delta_{ij} \) is the Kronecker symbol. The standard quantization of the functions \( z_j, \bar{z}_j \) is given by

\[
\hat{z}_j = \frac{x_j + i\hat{p}_j}{\sqrt{2}}, \quad \hat{z}_j^* = \frac{x_j - i\hat{p}_j}{\sqrt{2}},
\]

where \( \hat{z}_j^* \) denotes the hermitian conjugate of the operator \( \hat{z}_j \). These operators satisfy the commutation relations

\[
[\hat{z}_i, \hat{z}_j] = [\hat{z}_i^*, \hat{z}_j^*] = 0, \quad [\hat{z}_i, \hat{z}_j^*] = \delta_{ij}
\]  

for \( i, j = 1, \ldots, n \). Note that, with the conventions presently adopted for the quantization of impulses \( p \), the Poisson bracket of two functions is now replaced by the commutator of the corresponding operators multiplied by \( i \).
\[
\hat{J}_j = \frac{\hat{w}_j^* \hat{w}_j + \hat{w}_j \hat{w}_j^*}{2},
\]
for \(i, j = 1, \ldots, n\). We then define
\[
\hat{P}_{ij} = i(\hat{w}_i \hat{w}_j^* - \hat{w}_j \hat{w}_i^*) = -\hat{P}_{ji}, \quad i, j = 1, \ldots, n,
\]
and
\[
\hat{W}_h = \sum_{i=p_{h-1}+1}^{p_h} \epsilon_i \hat{w}_i^2, \quad h = 1, \ldots, u.
\]
Clearly all operators of the set \((\hat{K}, \hat{P}, \hat{W}, \hat{W}^*)\) coincide with the symmetrization with respect to \((\hat{w}, \hat{w}^*)\) of the corresponding classical functions. It then follows from proposition 3.1 (case 1) of [2] that the commutators (multiplied by \(i\)) between these operators have the same form as the Poisson brackets between the corresponding classical functions. In particular, from (2.23), (2.25) and (2.26) we get
\[
[\hat{K}_h, \hat{P}_{(j)}] = 0, \quad [\hat{W}_h, \hat{P}_{(j)}] = 0, \quad (3.4)
\]
\[
[\hat{W}_h, \hat{W}_j] = 0, \quad [\hat{W}_h, \hat{W}_j^*] = 4\delta_{jh}\hat{K}_h, \quad [\hat{W}_h, \hat{K}_j] = 2\delta_{jh}\hat{W}_h \quad (3.5)
\]
\forall j, h = 1, \ldots, u, where \(\hat{P}_{(j)}\) is the standard quantization of the set \(P_{(j)}\) defined by formula (2.21). We also introduce the operators
\[
\hat{R}_m = \hat{\Omega}_1^m \hat{\Omega}_2^m \cdots \hat{\Omega}_u^m,
\]
where \(m \in \mathbb{Z}^u\) and
\[
\hat{\Omega}_h^m := \begin{cases} 
\hat{W}_h^m & \text{if } m_h \geq 0, \\
(\hat{W}_h^* )^{-m_h} & \text{if } m_h < 0.
\end{cases} \quad (3.7)
\]
Note that one need not specify the ordering of the operators on the right-hand side of (3.6), since all these operators commute according to (3.5).

Let us consider the integrable set (2.32) for the classical system. We put
\[
\hat{J}_1 = (\hat{J}_{r(1)}, \ldots, \hat{J}_{r(u')}), \\
\hat{J}_2 = (\hat{J}_{u'+1}, \ldots, \hat{J}_{(u)})
\]
where
\[
\hat{J}_{r(i)} = \sum_{h=1}^{u} \epsilon_i^{(h)} \hat{K}_h, \quad i = 1, \ldots, u.
\]
We put also
\[
\hat{Z} = (\hat{Z}_{q_1, z_1}(\hat{P}_{(1)}), \ldots, \hat{Z}_{q_u, z_u}(\hat{P}_{(u)}) ), \\
\hat{L} = (\hat{L}_{q_1, z_1}(\hat{P}_{(1)}), \ldots, \hat{L}_{q_u, z_u}(\hat{P}_{(u)}) ),
\]
where polynomials \(\hat{Z}_{n,z}\) and \(\hat{L}_{n,z}\) have the same form as the classical ones \(Z_{n,z}\) and \(L_{n,z}\) (symmetrization is here unnecessary, since in these polynomials all monomials of degree 2 are squares of components of \(\hat{P}\)). We finally put
\[
\hat{R} = (\text{Im } \hat{R}_{m^{(1)}}, \ldots, \text{Im } \hat{R}_{m^{(u'-1)}}),
\]
11
The operators of the set $\hat{R}$ obviously coincide with the symmetrization with respect to $(\hat{W}, \hat{W}^*)$ of the functions of the set $R$. From the isomorphism between the two Lie algebras respectively generated by the functions $(K, P, W, \bar{W})$ and by the operators $(\hat{K}, \hat{P}, \hat{W}, \hat{W}^*)$, it follows that one can deduce the commutation relations

$$[\hat{J}_1, \hat{J}_2] = [\hat{J}_1, \hat{J}_2] = 0,$$

$$[\hat{J}_1, \hat{Z}] = [\hat{J}_1, \hat{L}] = [\hat{J}_2, \hat{Z}] = [\hat{J}_2, \hat{L}] = 0$$

from the corresponding Poisson bracket relations, by making use of proposition 4.2, case a, of [2]. In order to prove that

$$[\hat{Z}, \hat{Z}] = [\hat{Z}, \hat{L}] = 0,$$

we note that relations

$$[\hat{Z}_{q_1}, \hat{Z}_{q_2} (\hat{P}_{h_1}), \hat{Z}_{q_3}, \hat{Z}_{q_4} (\hat{P}_{h_2})] = [\hat{Z}_{q_1}, \hat{Z}_{q_2} (\hat{P}_{h_1}), \hat{L}_{q_3}, \hat{z}_j (\hat{P}_{h_2})] = 0$$

for $h \neq j$ follow from $[\hat{P}_{h_1}, \hat{P}_{h_2}] = 0$. For $h = j$ they can instead be deduced from proposition [22] by repeating the same arguments that were used in [3] to deduce proposition 2.7 from proposition 2.4 in the analogous situation of the central force field. Finally, from the second of (3.4) it follows that

$$[\hat{Z}, \hat{R}] = 0.$$

**Proposition 3.1.** The set $\hat{F} = (\hat{J}_1, \hat{Z}; \hat{J}_2, \hat{R}, \hat{L})$ is a quasi-integrable set of operators with $k$ central elements $(\hat{J}_1, \hat{Z})$, where $k = k' + z$.

**Proof.** We have just shown that the elements of the set $\hat{F}$ satisfy the required commutation relations. It is not too difficult to complete the proof of this proposition, by showing that the set is also quasi-independent. \qed

From this proposition it follows that any quantum system with hamiltonian operator $\hat{H} = f(\hat{J}_1, \hat{Z})$, where $f$ is an arbitrary polynomial of $k$ variables, is integrable with the same integrable set of operators $\hat{F}$.

In a similar way one can quantize the integrable set (2.33). Let us define in this case

$$\hat{J}_1 = (\hat{J}_{1(1)}, \ldots, \hat{J}_{1(u-h)}),$$

$$\hat{J}_2 = (\hat{J}_{1(u-h+1)}, \ldots, \hat{J}_{1(u-k')}),$$

$$\hat{R}_1 = (\text{Im} \hat{R}_{m(1)}, \ldots, \text{Im} \hat{R}_{m(u')}),$$

$$\hat{R}_2 = (\text{Im} \hat{R}_{m(u'+1)}, \ldots, \text{Im} \hat{R}_{m(h)}).$$

**Proposition 3.2.** The set $\hat{F} = (\hat{J}_1, \hat{R}_1, \hat{Z}; \hat{J}_2, \hat{R}_2, \hat{L})$ is a quasi-integrable set of operators with $k$ central elements $(\hat{J}_1, \hat{R}_1, \hat{Z})$, where $k = u - h + k' + z$.

Hence any quantum system with hamiltonian operator $\hat{H} = f(\hat{J}_1, \hat{R}_1, \hat{Z})$, where $f$ is an arbitrary polynomial of $k$ variables, is integrable with the same integrable set of operators $\hat{F}$. 

12
4 The case $n = 3$

In section 2 we have constructed a remarkable class of integrable sets of functions for an arbitrary system of resonant oscillators. However there may exist other integrable sets which do not belong to the class we have considered. In the present section we shall make an attempt to systematically classify all integrable sets of polynomial functions in the canonical variables $(x, p)$, or equivalently $(z, \bar{z})$, for $n = 3$. Since our goal is to obtain the largest possible class of integrable systems, we shall restrict our attention to integrable sets of functions having the largest number $k$ of central elements, i.e., $k = 3$. Hence, we shall aim at characterizing all sets $F = (F_1, F_2, F_3)$ of functionally independent polynomial functions, such that $F_1 = J_1 = l_1 I_1 + l_2 I_2 + l_3 I_3$ and $\{F_i, F_j\} = 0$ for $i, j = 1, 2, 3$. To each of these sets there corresponds a class of integrable systems with hamiltonian $H = f(F)$, where $f$ is an arbitrary function of three variables. This class is obviously the same for two integrable sets which are functionally equivalent to each other, i.e., such that the elements of one set are locally functions of the elements of the other set. Therefore in the following, whenever we shall mention an integrable set, we shall implicitly refer to the whole equivalence class to which it belongs.

If the three frequencies are pairwise different, i.e., $l_i \neq l_j$ for $i \neq j$, according to the discussion of section 2.1 there exist integrable sets with $k = 3$ of the form

$$F = (F_1, J_r, f(I_1, I_2, I_3, \text{Im } R_m)),$$

where $f$ is an arbitrary function of four variables such that the set $F$ is functionally independent. Here $r \in \mathbb{Z}^3$ and $m \in \mathbb{Z}^3$ are two nonvanishing vectors such that $r$ is linearly independent of $l$, and $l \cdot m = r \cdot m = 0$. For example, the trivial integrable set $F = (F_1, I_2, I_3)$ corresponds to taking $r = (0, 1, 0)$ and $f(x_1, x_2, x_3, x_4) = x_3$ in (4.1). Note that there exist only 4 functionally independent functions in involution with both $F_1$ and $F_2 = J_r$. Since 4 such functions are $(I_1, I_2, I_3, \text{Im } R_m)$, for any integrable set such that $F_2 = J_r$ we can locally write $F_3 = f(I_1, I_2, I_3, \text{Im } R_m)$ as in (4.1). For example, owing to (2.11), we obtain $F_3 = \text{Re } R_m$ by taking $f(x_1, x_2, x_3, x_4) = \pm \sqrt{x_1^{\text{Re}} x_2^{\text{Re}} x_3^{\text{Im}}} - x_4^{\text{Im}}$.

For the classification of integrable sets it is useful to introduce, besides the concept of functional equivalence given in [3], also that of “canonical equivalence” between integrable sets.

**Definition 4.1.** We say that two integrable sets $F$ and $F'$ are **canonically equivalent** if there exists a symplectic (i.e., linear and canonical) transformation which transforms one set into the other. In such a case, we say that also the two classes of functional equivalence, to which the two sets respectively belong, are canonically equivalent. This means that, if $G$ is functionally equivalent to $F$, and $G'$ is functionally equivalent to $F'$, then we say that $G$ is canonically equivalent to $G'$. In particular, two functionally equivalent sets are also canonically equivalent.

Let $G_l$ be the group of all the symplectic transformations $g : (z, \bar{z}) \mapsto (Z, \bar{Z})$ which leave $F_1$ invariant, i.e., such that $F_1(z, \bar{z}) = F_1(Z, \bar{Z})$, or explicitly

$$l_1 z_1 \bar{z}_1 + l_2 z_2 \bar{z}_2 + l_3 z_3 \bar{z}_3 = l_1 Z_1 \bar{Z}_1 + l_2 Z_2 \bar{Z}_2 + l_3 Z_3 \bar{Z}_3.$$

(4.2)
Given an integrable set \( F = (F_1(z, \bar{z}), F_2(z, \bar{z}), F_3(z, \bar{z})) \), we can for any \( g \in G \) construct a set \( F' = (F_1(Z, \bar{Z}), F_2(Z, \bar{Z}), F_3(Z, \bar{Z})) \) which is canonically equivalent to \( F \)

Let us consider the Lie algebra \( L \) of the second degree real polynomials which are in involution with \( F_1 \). The group \( G \) contains all the one-parameter subgroups of canonical transformations generated by the elements of \( L \). For any \( G \in L \) we write the associated subgroup in the form \( z \mapsto Z(\tau) \), with

\[
\frac{dZ(\tau)}{d\tau} = \{Z(\tau), G\}, \quad Z(0) = z.
\]  

(4.3)

If the three frequencies are pairwise different, a linear basis of this algebra is given by the set \((I_1, I_2, I_3)\). Let us consider the subgroup generated by \( G = I_1 \). According to (4.3) we have

\[
Z_1(\tau) = e^{i\tau} z_1, \quad Z_2(\tau) = z_2, \quad Z_3(\tau) = z_3.
\]

Similar formulas hold for \( G = I_2 \) and \( G = I_3 \). Therefore, all transformations of the group \( G \) have the form

\[
Z_1 = \exp(i\phi_1) z_1, \quad Z_2 = \exp(i\phi_2) z_2, \quad Z_3 = \exp(i\phi_3) z_3,
\]

with \( \phi_i \in \mathbb{R} \) for \( i = 1, 2, 3 \). This means that \( G \) is an abelian group isomorphic to \( U(1) \times U(1) \times U(1) \).

It can be useful to classify integrable sets \( F = (F_1, F_2, F_3) \) of polynomial functions according to the degree of polynomials \( F_2 \) and \( F_3 \). We shall always suppose that these polynomials have been chosen in such a way that \( \deg F_2 \leq \deg F_3 \), and that there do not exist functionally equivalent sets of lower degree. More precisely, we suppose that there does not exist another functionally equivalent integrable set of polynomials \( (F_1', F_2', F_3') \) such that \( \deg F_2' < \deg F_2 \) or \( \deg F_2' = \deg F_3 \) and \( \deg F_3' < \deg F_3 \).

**Definition 4.2.** Let \( F = (F_1, F_2, F_3) \) be an integrable set, where \( F_1 = l_1 I_1 + l_2 I_2 + l_3 I_3 \) and \( \deg F_2 \leq \deg F_3 \). We say that \( F \) is a simple integrable set if \( \deg F_3 = 2 \). We say that a simple integrable set \( F \) has degree \( d \) if \( \deg F_3 = d \).

Note that all integrable sets of the form (4.1), which were obtained using the general methods of sections 2.1 and 2.2 are simple. Since a symplectic transformation does not alter the degree of a polynomial function, an integrable set which is canonically equivalent to a simple integrable set is also simple, and two canonically equivalent simple integrable sets have the same degree.

4.1 The case \(|l_1| = |l_2| \neq |l_3|

If \( l_1 = l_2 \), by applying the results of section 2.2 we obtain another type of simple integrable sets, in addition to that given by formula (4.1). It has the form

\[
F = (F_1, P_{12}, f(I_1 + I_2, I_3, \text{Im } R, \text{Re } R)),
\]  

(4.4)

where \( f \) is an arbitrary function of 4 variables such that the set \( F \) is functionally independent, \( P_{12} = 2\text{Im} (\bar{z}_1 z_2) \), and

\[
R = \begin{cases} 
    (z_1^2 + z_2^2)|l_3| z_3^{2|l_1|} & \text{if } l_1 l_3 > 0, \\
    (z_1^2 + z_2^2)|l_3|^{2|l_1|} z_3 & \text{if } l_1 l_3 < 0.
\end{cases}
\]  

(4.5)
In a similar way, one can see that there exist also simple integrable sets of the form
\[ F = (F_1, Q_{12}, f(I_1 + I_2, I_3, \text{Im} S, \text{Re} S)), \]
where \( Q_{12} = 2\text{Re}(\bar{z}_1 z_2) \), and
\[
S = \begin{cases} 
(z_1^2 - 2) |z_3| z_3^{2|z_1|} & \text{if } l_1 l_3 > 0, \\
(z_1^2 - 2) |z_3| z_3^{2|z_1|} & \text{if } l_1 l_3 < 0.
\end{cases}
\]
(4.7)

Analogous integrable sets also exist for \( l_1 = -l_2 \). For the sake of simplicity, in the following we shall write down explicitly only the formulas which are valid for \( l_1 = l_2 \).

**Proposition 4.1.** For \( l_1 = l_2 \neq l_3 \), the Lie algebra \( \mathcal{L}_l \) of the second degree real polynomials in involution with \( F_1 \) has linear dimension five. A basis of this algebra is given by the set \( L = (L_1, \ldots, L_5) \), where
\[
L_1 = -\text{Re}\{\bar{z}_1 z_2\} = -\frac{\bar{z}_1 z_2 + z_1 \bar{z}_2}{2} = \frac{x_1 x_2 + p_1 p_2}{2} = -\frac{1}{2} Q_{12},
\]
\[
L_2 = -\text{Im}\{\bar{z}_1 z_2\} = i \frac{\bar{z}_1 z_2 - z_1 \bar{z}_2}{2} = \frac{i p_1 x_2 - x_1 p_2}{2} = -\frac{1}{2} P_{12},
\]
\[
L_3 = -\frac{I_1 - I_2}{2} = -\frac{\bar{z}_1 z_3 + \bar{z}_3 z_2}{2} = \frac{-x_1^2 - p_1^2 + x_2^2 + p_2^2}{4},
\]
\[
L_4 = -\frac{I_1 + I_2}{2} = -\frac{\bar{z}_1 z_3 + \bar{z}_3 z_2}{2} = \frac{-x_1^2 + p_1^2 + x_2^2 + p_2^2}{4},
\]
\[
L_5 = I_3 = \bar{z}_3 z_3 = \frac{x_3^2 + p_3^2}{2}.
\]
(4.8)

The first four of these functions do not depend on \((x_3, p_3)\). They satisfy the algebraic relation
\[
L_1^2 + L_2^2 = |\bar{z}_1 z_2|^2 = I_1 I_2 = L_4^2 - L_3^2,
\]
(4.9)

or
\[
\sum_{i=1}^{3} L_i^2 = L_4^2.
\]
(4.10)

The Poisson brackets between the elements of \( L \) are
\[
\{L_i, L_j\} = \sum_{k=1}^{3} \varepsilon_{ijk} L_k \quad \text{for } i, j, k = 1, 2, 3,
\]
(4.11)
\[
\{L_\mu, L_4\} = \{L_\mu, L_5\} = 0 \quad \text{for } \mu = 1, \ldots, 5,
\]
(4.12)

where \( \varepsilon_{ijk} \) is the completely antisymmetric tensor such that \( \varepsilon_{123} = 1 \).

**Proof.** By using (4.3) it is easy to check that all elements of \( \mathcal{L}_l \) can be expressed as linear combinations of elements of \( L = (L_1, \ldots, L_5) \). Since the set \( L \) is obviously linearly independent, it forms a basis of \( \mathcal{L}_l \). Introducing the Pauli matrices
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
\[
15
\]
and defining
\[ \sigma_4 = E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]
we can write
\[ L_\mu = -\frac{1}{2} y^* \sigma_\mu y, \quad \mu = 1, 2, 3, 4, \quad (4.13) \]
where
\[ y = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \]
From (4.13), using relations (2.2) one easily obtains
\[ \{ L_\mu, L_\nu \} = \frac{i}{4} y^*[\sigma_\mu, \sigma_\nu]y. \]
Then relations (4.11)–(4.12) can be immediately verified using the well-known commutation relations between matrices \( \sigma \).

Let us consider the subgroups of \( G_l \) generated by \( L_i \), with \( i = 1, 2, 3 \). According to (4.3) we have
\[ dY(\tau) d\tau = \{ Y(\tau), L_i \} = -i \sigma_i Y(\tau), \]
\[ dZ_3(\tau) d\tau = \{ Z_3(\tau), L_i \} = 0, \]
where
\[ Y(\tau) = \begin{pmatrix} Z_1(\tau) \\ Z_2(\tau) \end{pmatrix}. \]
Hence
\[ Y(\tau) = \exp\left( -i \frac{\tau}{2} \sigma_i \right) y, \]
\[ Z_3(\tau) = z_3. \]
Similarly, for the two subgroups generated by \( L_4 \) and \( L_5 \) we have respectively
\[ Y(\tau) = \exp\left( -i \frac{\tau}{2} \right) y, \]
\[ Z_3(\tau) = z_3, \]
and
\[ Y(\tau) = y, \]
\[ Z_3(\tau) = e^{i\phi} z_3. \]

From these formulas one can easily derive the general form of the elements of \( G_l \).

**Proposition 4.2.** For \( l_1 = l_2 \neq l_3 \) the group \( G_l \) is made by all the transformations of the form
\[ Y = U y, \]
\[ Z_3 = e^{i\phi} z_3, \quad (4.14) \]
where \( U \) is a unitary \( 2 \times 2 \) matrix and \( \phi \in \mathbb{R} \). Hence the group \( G_l \) is isomorphic to \( U(2) \times U(1) \).
Using (4.13), we find that under a transformation of the form (4.14) the functions (4.8) behave as

\[
L_i \mapsto -\frac{1}{2} Y^* \sigma_i Y = -\frac{1}{2} y^* U^* \sigma_i U y = \sum_{j=1}^{3} R_{ij}(U) L_j \quad \text{for } i = 1, 2, 3,
\]

\[
L_4 \mapsto -\frac{1}{2} Y^* \sigma_4 Y = -\frac{1}{2} y^* U^* \sigma_4 U y = L_4,
\]

\[
L_5 \mapsto \bar{Z}_3 Z_3 = \bar{z}_3 z_3 = L_5,
\]

where \( R_{ij}(U) \) are the elements of the 3 \( \times \) 3 matrix \( R(U) \in SO(3) \) which is associated with \( U \) by the standard three-dimensional real representation \( R : U(2) \rightarrow SO(3) \). The kernel of this representation is made of all the matrices \( U \in U(2) \) of the form \( U = e^{i \psi} E \), with \( \psi \in \mathbb{R} \).

Using (2.3) it is easy to see that any real polynomial in involution with \( F_1 \) can be obtained as an algebraic combination of the elements of a finite set. For \( l_1 = p > 0 \), \( l_3 = q > 0 \), this set includes, besides the five second degree polynomials \( L_\mu \), the polynomials \( \operatorname{Re}\{z_1^{q-s} z_2^2 z_3^p\} \) and \( \operatorname{Im}\{z_1^{q-s} z_2^2 z_3^p\} \) for \( 0 \leq s \leq q \). If \( l_1 = p < 0 \), one can write analogous formulas with \( z_3^p \) in place of \( z_3^p \).

Defining (for \( p > 0 \))

\[
A_{q,s} = \frac{z_1^{q-s} z_2^2 z_3^p}{\sqrt{(q-s)!s!}},
\]

we can write for any \( a_s, b_s \in \mathbb{R} \)

\[
a_s \operatorname{Re}\{z_1^{q-s} z_2^2 z_3^p\} + b_s \operatorname{Im}\{z_1^{q-s} z_2^2 z_3^p\} = \operatorname{Re}\{c_s A_{q,s}\},
\]

with

\[
c_s = \sqrt{(q-s)!s!}(a_s + ib_s) \in \mathbb{C}.
\]

Note that

\[
\sum_{s=0}^{q} A_{q,s}^2 = \frac{1}{q!} (z_1^2 + z_2^2)^q z_3^p = \frac{R}{q!},
\]

where \( R \) is the function defined in (4.15).

The transformation properties of the \( A_{q,s} \) under the action of the group \( G_t \) are determined by their Poisson brackets with the functions \( L_\mu \). A direct calculation provides

\[
\{i L_\mu, A_{q,s}\} = \sum_{h=0}^{q} (J_{q,h})_{hs} A_{q,h},
\]

with

\[
(J_{q,1})_{hs} = \frac{1}{2} \left[ \delta_{h-1,s} \sqrt{(q-s) h} + \delta_{s-1,h} \sqrt{(q-h)s} \right],
\]

\[
(J_{q,2})_{hs} = \frac{1}{2i} \left[ \delta_{h-1,s} \sqrt{(q-s) h} - \delta_{s-1,h} \sqrt{(q-h)s} \right],
\]

\[
(J_{q,3})_{hs} = \left( \frac{q}{2} - h \right) \delta_{hs},
\]

\[
(J_{q,4})_{hs} = \frac{q}{2} \delta_{hs},
\]

\[
(J_{q,5})_{hs} = p \delta_{hs}.
\]
It follows that
\[ \{ iL_\mu, \sum_{s=0}^{q} c_s A_{q,s} \} = \sum_{s=0}^{q} c'_s A_{q,s}, \]
where
\[ c'_s = \sum_{r=0}^{q} (J_{q,\mu})_{sr} c_r, \]
or in compact notation \( c' = J_{q,\mu} \cdot c \). The numerical coefficients on the right-hand side of (4.16) have been chosen in such a way that the matrices \( J_{q,\mu} \) are hermitian. Other properties of these matrices, which can be directly verified using formulas (4.18)–(4.22), follow from simple general considerations. Since \( F_1 = qL_5 - 2pL_4 \), the relation \( \{ F_1, A_{q,s} \} = 0 \) implies \( qJ_{q,5} = 2pJ_{q,4} \). Furthermore, as a consequence of the Jacobi identity we have for any function \( G \)
\[ \{ L_\mu, \{ L_\nu, G \} \} - \{ L_\nu, \{ L_\mu, G \} \} = \{ \{ L_\mu, L_\nu \}, G \}. \]
By applying the previous equation with \( G = \sum_{s=0}^{q} c_s A_{q,s} \), one can see that the commutation relations between the matrices \( J_{q,\mu} \) are determined by the Poisson brackets (4.11)–(4.12) between the functions \( L_\mu \). We have accordingly
\[ [J_{q,i}, J_{q,j}] = i \sum_{k=1}^{3} \varepsilon_{ijk} J_{q,k} \] (4.23)
for \( i, j = 1, 2, 3 \), and
\[ [J_{q,\mu}, J_{q,4}] = [J_{q,\mu}, J_{q,5}] = 0 \] (4.24)
for \( \mu = 1, \ldots, 5 \). Looking at formulas (4.18)–(4.20) one can actually recognize the well-known \((q + 1)\)-dimensional irreducible representation of \( su(2) \), which satisfies the relation
\[ \sum_{i=1}^{3} J_{q,i}^2 = \frac{q(q+2)}{4} E, \]
\( E \) being the identity matrix of rank \( q + 1 \). It follows that the functions \( A_{q,s} \) belong to the corresponding irreducible representation of \( U(2) \).

In the following of the present section, we shall always suppose that \( l_1 = l_2 = p > 0, l_3 = q > 0, q \neq p \). However, using the methods which we already applied in section 2, the results can be easily generalized to all cases in which \( |l_1| = |l_2| \neq |l_3| \).

**Theorem 4.3.** Any simple integrable set is canonically equivalent to a set \( F = (F_1, F_2, F_3) \) with
\[ F_2 = d_1I_1 + d_2I_2, \] (4.25)
where \( d_1 \in \mathbb{N} \) and \( d_2 \in \mathbb{Z} \) do not have common divisors, and \( |d_2| \leq d_1 \).

**Proof.** If \( \deg F_2 = 2 \), according to proposition 4.1 we can write \( F_2 = \sum_{\mu=1}^{5} \alpha_{\mu} L_\mu \), with \( \alpha_{\mu} \in \mathbb{R} \). By replacing \( F_2 \) with \( F_2 - (\alpha_5/q)F_1 \), we obtain a functionally equivalent set such that \( F_2 \) does not contain \( L_5 = I_3 \). Furthermore, by performing an appropriate transformation of \( G_l \), according to the first of (4.15) we can operate a rotation on the \( L_i, i = 1, 2, 3 \), in such a way to eliminate from
$F_2$ the terms proportional to $L_1$ and $L_2$. After these operations we obtain a canonically equivalent set with

$$F_2 = \beta_3 L_3 + \beta_4 L_4 = \gamma_1 I_1 + \gamma_2 I_2,$$

where $\beta_3 = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$, $\beta_4 = \alpha_4 + 2p\alpha_5/q$, $\gamma_1 = -(\beta_3 + \beta_4)/2$, $\gamma_2 = (\beta_3 - \beta_4)/2$. We further note that, applying when necessary the canonical transformation that interchanges $z_1$ and $z_2$, we can always ensure that $|\gamma_1| \geq |\gamma_2|$.

In general $F_3$ can be written in the form

$$F_3 = \sum_{(a,b) \in K} \text{Re}\{c_{a,b} P_{a,b}\},$$

where $P_{a,b} := z_1^{a_1} z_2^{a_2} z_3^{a_3} z_4^{a_4}$, $c_{a,b} \in \mathbb{C}$ $\forall (a,b) \in K$, and $K \subset \mathbb{Z}_1^3 \times \mathbb{Z}_1^3$ is a finite set such that

$$p(a_1 - b_1 + a_2 - b_2) + q(a_3 - b_3) = 0 \quad \forall (a,b) \in K.$$  \hspace{1cm} (4.27)

From (4.26) we have in general

$$\{F_2, P_{a,b}\} = -i[\gamma_1(a_1 - b_1) + \gamma_2(a_2 - b_2)] P_{a,b},$$

so that the condition $\{F_2, F_3\} = 0$ implies

$$\gamma_1(a_1 - b_1) + \gamma_2(a_2 - b_2) = 0 \quad \forall (a,b) \in K.$$  \hspace{1cm} (4.28)

If $a_1 - b_1 = a_2 - b_2 = 0$, then from (4.27) we obtain $a_3 - b_3 = 0$ and $\text{Re}\{c_{a,b} P_{a,b}\} = (\text{Re} \ c_{a,b}) I_1^{a_1} I_2^{a_2} I_3^{a_3}$. Hence, if $a_1 - b_1 = a_2 - b_2 = 0 \forall (a,b) \in K$, we have clearly $(F_1, F_2, F_3) \approx (I_1, I_2, I_3) \approx (F_1, I_1, I_3)$, where $\approx$ denotes functional equivalence. This means that the thesis of the theorem holds with $d_1 = 1, d_2 = 0$ in formula (4.25). On the other hand, if there exists $(a,b) \in K$ such that, say, $a_2 - b_2 \neq 0$, it follows from (4.28) that $\gamma_2/\gamma_1 = -(a_1 - b_1)/(a_2 - b_2) \in \mathbb{Q}$. Since one can always redefine $F_2$ by multiplying it by a constant, the thesis follows immediately from (4.26). \hfill $\Box$

It follows from Theorem 4.4 that any simple integrable set is canonically equivalent to a set of the form (4.1), with $r = (d_1, d_2, 0)$. In particular, any set of the form (4.1) or (4.8) can be converted into the form (4.1) via a symplectic transformation. This can be checked directly, by observing that under a transformation of the form (4.13), with

$$U = \exp \left( i \frac{\pi}{4} \sigma_1 \right) = \frac{1 + i\sigma_1}{\sqrt{2}}, \quad \phi = 0,$$

we have according to (4.15) $P_{12} \rightarrow I_1 - I_2$. Similarly, for

$$U = \exp \left( -i \frac{\pi}{4} \sigma_2 \right) = \frac{1 - i\sigma_2}{\sqrt{2}}, \quad \phi = 0,$$

we have $Q_{12} \rightarrow I_1 - I_2$.

When $r = (d_1, d_2, 0)$, the vector $m$ in (4.1) must satisfy the conditions

$$p(m_1 + m_2) + q m_3 = 0,$$

$$d_1 m_1 + d_2 m_2 = 0.$$
We see that, if \( d_1 = d_2 \), we can take \( m = (-1,1,0) \), which means that \( \text{Im} R_m = 2P_{32} \). For \(-d_1 \leq d_2 < d_1 \) we can take instead \( m = (-d_2 h, d_1 h, -pk) \), where \( h \) and \( k \) are two positive integers without common divisors such that \( q/(d_1 - d_2) = h/k \).

We list in Table 1 some examples of simple integrable sets, which are obtained in this way for \( l = (1,1,2) \).

| \( F_2 \) | \( F_3 \) | degree |
|---|---|---|
| \( I_1 \) | \( \text{Re}\{z_2^2z_3\} \) | 3 |
| \( I_1 - I_2 \) | \( \text{Re}\{z_1z_2z_3\} \) | 3 |
| \( 3I_1 + I_2 \) | \( \text{Re}\{z_1z_3^2z_3\} \) | 5 |
| \( 3I_1 - I_2 \) | \( \text{Re}\{z_1z_3^2z_2\} \) | 6 |
| \( 2I_1 + I_2 \) | \( \text{Re}\{z_1^2z_2^2z_3\} \) | 7 |
| \( 5I_1 + I_2 \) | \( \text{Re}\{z_1z_2^2z_3^2\} \) | 8 |
| \( 5I_1 + 3I_2 \) | \( \text{Re}\{z_1^2z_2^3z_3\} \) | 9 |
| \( 5I_1 - I_2 \) | \( \text{Re}\{z_1z_2z_3^2\} \) | 9 |

Table 1: Examples of simple integrable sets for \( l = (1,1,2) \).

### 4.2 A non-simple integrable algebra

In the preceding section we have shown how one can construct simple integrable sets of arbitrarily high degree. On the other hand, we do not know of any general method to obtain non-simple integrable sets, that is integrable sets of arbitrarily high degree. On the other hand, we do not know of any general method to obtain non-simple integrable sets, that is integrable sets \( F \), with \( 2 < \deg F_2 \leq \deg F_3 \), such that there exists no functionally equivalent set \( F' \) with \( \deg F'_2 = 2 \). Nevertheless, we are able to exhibit a concrete example of non-simple integrable set for \( l = (1,1,2) \). Let us define

\[
D_0 = \text{Re}\{z_1^2z_3\} = \sqrt{2}\text{Re}\{A_{2,0}\}, \quad C_0 = \text{Im}\{z_1^2z_3\} = \sqrt{2}\text{Im}\{A_{2,0}\},
\]
\[
D_1 = \text{Re}\{z_1z_2z_3\} = \text{Re}\{A_{2,1}\}, \quad C_1 = \text{Im}\{z_1z_2z_3\} = \text{Im}\{A_{2,1}\},
\]
\[
D_2 = \text{Re}\{z_1z_2^3\} = \sqrt{2}\text{Re}\{A_{2,2}\}, \quad C_2 = \text{Im}\{z_1z_2^3\} = \sqrt{2}\text{Im}\{A_{2,2}\},
\]

and consider the functions

\[
F_1 = I_1 + I_2 + 2I_3, \quad F_2 = C_0 + 2C_2, \quad F_3 = 2C_0^2 + I_1M_3^2, \tag{4.29}
\]

where \( M_3 := P_{12} = 2\text{Im}\{z_1z_2\} \). Clearly \( \{F_1, F_2\} = \{F_1, F_3\} = 0 \). Furthermore, by exploiting formulas (4.17)–(4.22), or by direct computation, one easily finds

\[
\begin{align*}
\{M_3, C_0\} & = -2C_1, \\
\{M_3, C_2\} & = 2C_1, \\
\{I_1, C_0\} & = 2D_0, \\
\{I_1, C_2\} & = 0.
\end{align*}
\]

(4.30) (4.31)

We also have

\[
\{C_0, C_2\} = \frac{i}{4}(z_1^2z_2^2 - z_1^2z_2^2) = -\frac{M_3N_3}{4},
\]

where \( N_3 := Q_{12} = 2\text{Re}\{z_1z_2\} \). Using these relations one finds

\[
\{F_2, F_3\} = 2M_3(C_0N_3 - D_0M_3 - 2I_1C_1) = 0 \tag{4.32}
\]

20
where the last equality follows from
\[
C_0 N_3 - D_0 M_3 = 2\text{Im}\{\bar{z}_1^2 z_3\}\text{Re}\{z_1 \bar{z}_2\} + 2\text{Re}\{\bar{z}_1^2 z_3\}\text{Im}\{z_1 \bar{z}_2\} = 2\text{Im}\{z_1 \bar{z}_2^2 z_3\} = 2I_1 C_1.
\]

It is easy to see that the set \( F = (F_1, F_2, F_3) \) is functionally independent. Therefore \( F \) is an integrable set with \( \deg F_2 = 3, \deg F_3 = 6 \). This implies that any system with hamiltonian \( \hat{H} = f(\hat{F}) \), where \( f \) is an arbitrary function of three variables, is also integrable. It is also easy to obtain a similar integrable set for any other \( l \) such that \(|l_1| = |l_2| = 1, |l_3| = 2\). By means of symplectic transformations of the form (4.14), we can then obtain a whole class of non-simple integrable sets which are canonically equivalent to \( F \). However, we do not know at present of any other class of non-simple integrable sets, either for \( l = (\pm 1, \pm 1, \pm 2) \) or for any other value of \( l \). Attempts to discover other examples, also with the aid of specially made computer programs, have proven unsuccessful.

### 4.3 Quantization

Let us now consider the problem of the quantization of the integrable sets that we have obtained in sections 4.1 and 4.2. According to proposition 3.1 (case 1) of [2], all simple integrable sets can be straightforwardly quantized by symmetrization with respect to the operators (\( \hat{z}, \hat{z}^* \)) which were defined in section 3. Therefore any quantum system with hamiltonian operator \( \hat{H} = f(\hat{F}) \), where \( \hat{F} = F^\text{sym} \) is the symmetrization of a simple integrable set and \( f \) is an arbitrary function of three variables, is quasi-integrable.

As regards the non-simple algebra (4.29), let \( F_i^\text{sym}, i = 1, 2, 3 \), be the operators which are obtained by symmetrization of the functions \( F_i \) with respect to the operators (\( \hat{z}, \hat{z}^* \)). Using proposition 4.1 (case a) of [2] we obtain that
\[
[F_1^\text{sym}, F_2^\text{sym}] = [F_1^\text{sym}, F_3^\text{sym}] = 0.
\]
However this proposition does not allow us to evaluate \([F_2^\text{sym}, F_3^\text{sym}]\), since both the involved operators have degree higher than 2. Let us then make use of the general formula of Moyal brackets. With the conventions adopted in this section, this formula for two generic polynomials \( H(z, \bar{z}) \) and \( F(z, \bar{z}) \) can be written as \([H^\text{sym}, F^\text{sym}] = G^\text{sym}\), where
\[
G = \sum_{k \in \mathbb{N}} \sum_{|\alpha + \beta| = 2k + 1} (-1)^{|\beta|} \frac{\partial^{|\alpha + \beta|} H}{\partial z^\alpha \partial \bar{z}^\beta} \frac{\partial^{\alpha + \beta} F}{\partial z^\beta \partial \bar{z}^\alpha}.
\]

By applying this formula we obtain
\[
[F_2^\text{sym}, F_3^\text{sym}] = \frac{5}{2} i \hat{D}_0,
\]
where \( \hat{D}_0 \) is the standard quantization of the function \( D_0 \) (symmetrization is in this case unnecessary). On the other hand, we have from (4.31) and from proposition 3.1 of [2] that
\[
[F_2^\text{sym}, \hat{I}_1] = -i [F_2, I_1]^\text{sym} = 2i \hat{D}_0.
\]
Since $[F_{sym}^1, \hat{I}_1] = 0$, it follows from the two above equalities that the three operators
\[
\hat{F}_1 = F_{sym}^1, \quad \hat{F}_2 = F_{sym}^2, \quad \hat{F}_3 = F_{sym}^3 - \frac{5}{4}\hat{I}_1
\]
are pairwise in involution. Since these operators are also quasi-independent, one concludes that the set $\hat{F} = (\hat{F}_1, \hat{F}_2, \hat{F}_3)$ is a quasi-integrable set of operators. Hence, any quantum system with Hamiltonian operator $\hat{H} = f(\hat{F})$, where $f$ is an arbitrary function of three variables, is quasi-integrable. We have thus shown that it is possible to quantize our example (4.29) of non-simple integrable set. However, to this purpose the general procedure of quantization by symmetrization needs to be modified. The modification is represented by the introduction of the term $-(5/4)\hat{I}_1$ in the expression of the operator $\hat{F}_3$. This fact presents an analogy with the situation for a free quantum rigid body in 6-dimensional space [3], and also with some results that have already been obtained for other types of integrable quantum systems [10][11].

References

[1] M. Marino and N. N. Nekhoroshev, Quantization of classical integrable systems. Part I: quasi-integrable quantum systems, arXiv:1001.4685 [math-ph] (2010).

[2] M. Marino and N. N. Nekhoroshev, Quantization of classical integrable systems. Part II: quantization of functions on Poisson manifolds, arXiv:1001.4701 [math-ph] (2010).

[3] M. Marino and N. N. Nekhoroshev, Quantization of classical integrable systems. Part III: systems in $n$-dimensional Euclidean space, arXiv:1001.4885 [math-ph] (2010).

[4] N. N. Nekhoroshev, Action-angle variables and their generalizations, Trans. Moscow Math. Soc. 26, 180–198 (1972).

[5] N. N. Nekhoroshev, D. A. Sadovskii and B. I. Zhilinskii, Fractional monodromy of resonant classical and quantum oscillators, C. R. Acad. Sci. Paris, Sér. I, 335, 985–988 (2002).

[6] N. N. Nekhoroshev, D. A. Sadovskii and B. I. Zhilinskii, Fractional Hamiltonian monodromy, Ann. Henri Poincaré 7, 1099–1211 (2006).

[7] N. N. Nekhoroshev, Fractional monodromy in the case of arbitrary resonances, Sbornik: Mathematics 198, 383–424 (2007).

[8] D. Bambusi and N. N. Nekhoroshev, A property of exponential stability in nonlinear wave equations near the fundamental linear mode, Physica D 122, 73–104 (1998).

[9] N. N. Nekhoroshev, Strong stability of the approximate fundamental mode of the nonlinear string equation, Trans. Moscow Math. Soc. 63, 151–217 (2002).

[10] J. Hietarinta, Classical versus quantum integrability, J. Math. Phys. 25, 1833–1840 (1984).
[11] J. Hietarinta and B. Grammaticos, On the $\hbar^2$ correction terms in quantum integrability, *J. Phys. A, Math. Gen.* **22**, 1315–1322 (1989).