Potentials and Jacobian algebras for tensor algebras of bimodules

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Abstract
We introduce and study potentials, mutations and Jacobian algebras in the framework of tensor algebras associated with symmetrizable dualizing pairs of bimodules on a symmetric algebra over any commutative ground ring. The graded context is also considered by starting from graded bimodules, and the classical non simply-laced context of modulated quivers with potentials is a particular case. The study of potentials in this framework is related to symmetrically separable algebras, and we have two kinds of potentials: the symmetric and the non symmetric ones. When the Casimir ideal of the symmetric algebra coincides with its center, all potentials appear as symmetric potentials and their manipulation mimics the simply laced study of quivers with potentials. This useful information suggests that, for applications to cluster algebras theory and related fields, one may restrict a further study of modulated quivers with potentials to the setting where the ground symmetric algebra is separable over a field. Associated with this work is a generalized construction of Ginzburg dg-algebras and cluster categories associated with graded modulated quivers with potentials.

Keywords: potential, modulated quiver, mutation, Jacobian algebra, cluster tilted algebra.

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Contents
1 Introduction 2
2 Trace maps and symmetrizable Dualizing pairs of bimodules 5
3 Potentials and Jacobian algebras 10
  3.1 Tensor path algebras of modulated quivers 10
  3.2 Casimir morphisms and projective bases for tensor path algebras 11
  3.3 Non simply-laced generalization of potentials and their Jacobian ideals 12
  3.4 Intrinsic description of potentials, modulated quivers with potentials 15
4 Reduction of modulated quivers with potentials 17
  4.1 The cyclic Leibniz rule and the chain-rule 17
  4.2 The reduction process 21
5 Symmetric potentials 28
6 Some examples in the inseparable context 34
7 Mutations of modulated quivers with potentials 36
8 Examples of mutations in the mutation class of Dynkin type $F_4$ 39

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9 Graded modulated quiver with potentials

10 The cluster category of a graded modulated quiver with potential

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References

1. Introduction

The main purpose in this paper is to extend to a suitable general framework some recent aspects of the theory of quivers with potentials and corresponding Jacobian algebras started in [1]. The first motivation of this work is a result of [3] relating the mutation of cluster tilting objects in 2-Calabi-Yau categories to the mutation of quivers with potentials. In the simply laced case, the theory of quivers with potentials was motivated by several sources: superpotentials in physics [4, 5, 6], Calabi-Yau algebras [7, 8, 9, 10], cluster algebras. The original motivation for the study of quivers with potentials comes from the theory of cluster algebras introduced and studied in a series of papers [11, 12, 13, 14] by S. Fomin and A. Zelevinsky. The underlying combinatorics of the theory of cluster algebras is embodied in skew-symmetric integer matrices and their mutations, or equivalently, in valued quivers without loops and their mutations. However, most of the time, recent categorifications of cluster theory restrict to the simply-laced case, that is the one corresponding to skew-symmetric matrices or equivalently to 2-acyclic quivers without loops.

The present framework and the method

In this introductory discussion, we do not provide explicit definitions for some notions announced here and kindly refer the reader to the text for full detailed definitions. The general framework considered here is based on the existence of the so-called trace maps on simple algebras [15, §22]. We let \( k \) be any commutative ring and \((K, t)\) a symmetric \( k \)-algebra, finitely generated projective as \( k \)-module; here \( t \in \text{Hom}_k(K, k) \) is a strongly non-degenerate trace map for \( K \), that is, \( t \) induces an isomorphism of \( K \)-bimodules \( \text{Hom}_k(K, k) \) taking each \( a \in K \) to the \( k \)-linear map \( t(a -) : b \mapsto t(ab) \). Let \( B \) be a \( k \)-bimodule, finitely generated projective as left \( K \)-module and right \( K \)-module. Then \( B \) appears as part of a data \( \{ B, B^*, b \} \) which we call a symmetrizable dualizing pair of \( K \)-bimodules, here \( B \otimes B^* \otimes B^* \otimes B \mathrel{\overset{b}{\longrightarrow}} K \) is a strongly non-degenerate bilinear form and, \( t \) is a symmetrizing map for \( b \), that is, \( tb(x \otimes \xi) = tb(\xi \otimes x) \) for all \( x \in B \) and \( \xi \in B^* \), see Definition 2.1. The data \( Q = (B, K, t) \) is called a k-modulated quiver having \( B \) as arrow bimodule. The path algebra \( kQ \) of \( Q \) (or the path algebra of \( B \)) is the tensor algebra of \( B \) over \( k \); thus \( kQ = T_k(B) = \bigoplus_{i \geq 0} B^i \) where \( B^0 = kQ_i \) is the \( i \)-fold tensor product of \( B \) over \( k \) (referred to as the bimodule generated by all length-\( i \) paths in \( Q \)), with \( B^0 = K \). The complete path algebra of \( Q \) is given by \( \overline{kQ} = \prod_{i \geq 0} B^i \). Write \( \overline{kQ}_{(d)} = \prod_{i \geq 0} B^i \) for all natural number \( d \geq 1 \) and let \( J_{\overline{kQ}} = \overline{kQ}_1 \). Then \( kQ \) is a topological algebra with \( \overline{kQ} \)-adic topology and \( J_{\overline{kQ}} \) is referred as the closed arrow ideal of \( Q \). Observe that the classical non simply-laced context is recovered when \( k \) is a direct product \( \prod_{i \in [1, n]} k_i \) of division algebras over a field \( k \), here \( n \geq 1 \) is a natural number, \( [1, n] = \{1, \ldots, n\} \) and each \( k_i \) is viewed as subfield in \( k \) with unit \( 1_i \). On the other hand, the simply-laced context is obtained when \( k \) occurs as elementary semisimple algebra \( k^n = \prod_{i \in [1, n]} k_i \) over a field \( k \), here \( k_i = k \) for all \( i \in [1, n] \); in this case \( B \) is a central \( k^n \)-bimodule and the data \( Q = (B, k^n) \) may be referred to as a \( k \)-quiver, the arrows of \( Q \) correspond bijectively to the union of \( k \)-bases of \( 1_i B 1_j \), with \( i, j \in [1, n] \).

For a \( k \)-quiver \( Q = (B, k^n) \), a potential \( W \) on \( Q \) was defined as a possibly infinite sum of cyclic elements in \( kQ_{(2)} \); to \( W \) is associated a closed ideal \( J_W \), called the Jacobian ideal of \( W \) and generated by the cyclic derivatives of \( W \) with respect to the arrows of \( Q \), the quotient algebra \( J(Q, W) := \overline{kQ}J_W \) is called the Jacobian algebra [1]. Next, we enrich the framework just described by starting with \( G \)-graded \( K \)-bimodules \( B \) for an abelian group \( G \) and considering potentials of homogeneous degree with respect to \( G \)-grading.
In the present framework, in order to get an appropriate notion of potential with respect to cyclic derivatives we must lift ordinary permutations of arrows from simply laced path algebras to a kind of skew permutations for tensor algebras \( kQ \). This can be achieved in two complementary ways. Let us describe the most general and intrinsic method of our study. Given a symmetrizable dualizing pair of \( k \)-bimodules \( \{ M, M', \beta \} \), we observe that the induced non-degenerate bilinear forms \( M \otimes M' \rightarrow K \) and \( M' \otimes M \rightarrow K \) are dual morphisms and their dual morphisms give rise the following Casimir morphisms \( \hat{\beta} : M \otimes M' \rightarrow M' \otimes M \) (see subsection 2). These Casimir morphisms enjoy surprisingly nice properties and are fundamental for a notion of skew permutation inside tensor path algebras: the left permutation and the right permutation of \( \hat{\beta} \) coincide with \( \hat{\beta} \) and reciprocally, the left permutation and the right permutation of \( \hat{\beta} \) coincide with \( \hat{\beta} \), so that the complete cyclic permutation of each of the above Casimir morphisms stays invariant. Referring to the last property we say that each Casimir morphism \( \hat{\beta} \in \{ \hat{\beta}, \hat{\beta} \} \) is cyclically stable. Thanks to some crucial properties of Casimir morphisms, potentials for modulated quivers we easily defined as morphisms of \( k \)-bimodules \( K \rightarrow kQ(2) \), equivalently potentials correspond to \( K \)-central elements in \( kQ(2) \). For the second but complementary approach of our study, we restrict to symmetric potentials: they can be obtained from elements of the central \( Z(K) \)-bimodule \( kQ \otimes K \). Indeed, the ordinary cyclic permutation of tensor elements from simply laced path algebras appears to be well-defined on \( kQ \otimes K \), and the manipulation of symmetric potentials becomes less technical. In particular, if the \( k \)-algebra \( K \) is separable over a ground field \( k \), then by a result of Donald G. Higman [25], the Casimir ideal \( Z(K) \) of \( K \) coincides with the center \( Z(K) \) of \( K \) and potentials on \( Q \) coincides with symmetric potentials, the latter also holds when \( K \) is a symmetrically (or strongly) separable algebra over any commutative ring. The special treatment of symmetric potentials in this work is motivated by a recent work of B. Keller on deformations of Calabi-Yau differential graded categories and on Ginzburg differential graded categories, in which the author considers potentials in a path category. We must draw the attention of the anonymous reader that the obstruction to the reduction process is of the same nature as the obstruction to the generalization of the well-known Gabriel’s theorem for presentation of finite dimensional algebras. Gabriel’s theorem states that any finite dimensional algebra over an algebraically closed field admits a presentation by a quiver with relations, whereas the non simply-laced analogue of this result states that any finite dimensional algebra \( A \) over a field, with Jacobson radical \( J_A \), admits a presentation by a modulated quiver with relations provided \( A \) can be given

\[ \text{On the obstruction to the reduction.} \] We must draw the attention of the anonymous reader that the obstruction which arises when trying to reduce a modulated quiver with potential is of the same nature as the obstruction to the generalization of the well-known Gabriel’s theorem for presentation of finite dimensional algebras. Gabriel’s theorem states that any finite dimensional algebra over an algebraically closed field admits a presentation by a quiver with relations; whereas the non simply-laced analogue of this result states that any finite dimensional algebra \( A \) over a field, with Jacobson radical \( J_A \), admits a presentation by a modulated quiver with relations provided \( A \) can be given
The latter splitting condition is satisfied if the ground field is perfect. For an arbitrary ring thus each element \( x \) written either as enveloping \( k \)-algebra \( K^e \) is not semisimple. Thus, if \( K \) is separable over a field \( k \), then, as a tensor product over a field of two separable \( k \)-algebras, \( K^e \) is a separable \( k \)-algebra and hence semisimple (see [29, Cor 11.6.8]), in this case the obstruction to the reduction of modulated quivers with potentials disappears exactly as in the case of presentation of finite dimensional algebras by modulated quivers with relations.

**Description of main results and organization of the paper**

The first main result of this work is the reduction Theorem 4.6; it establishes the reduction up to weak right-equivalences. Focusing on symmetric potentials, the reduction process is refined in Theorem 5.4 under some natural splitting conditions, it can be obtained up to right-equivalences as in the simply-laced case. Now, whenever the reduction is possible, for each central idempotent \( e \in K \) satisfying some natural condition, we define the mutation of a modulated quiver with potential at "e" up to weak right-equivalences (or right-equivalences if the Casimir ideal \( \Delta \{K\} \) coincides with \( Z \{K\} \)), and Theorem 7.4 states that the mutation at \( e \) is a well-defined involution on the set of (weak) right-equivalence classes of modulated quivers with potentials. Of a special interest, we deduce (in Corollary 5.5) that in the setting of a separable algebra \( K \) over a field, all potentials are symmetric ones and the study of modulated quivers with potentials in such a context mimics the simply-laced study: cyclic (left or right) permutations are images under a Casimir operator of corresponding ordinary permutations. This is indeed a useful information: for applications to cluster algebras theory, one may restrict a further non simply-laced study of modulated quivers with potentials to the setting of a perfect ground field where things behave smoothly. Next, considering graded modulated quiver with homogeneous potentials, we extend our mains results to the graded context.

The paper is organized as follows. Section 2 is dedicated to the main technical tools about symmetrizable dualizing pairs, in Section 3 we begin the discussion of the general approach to potentials and cyclic derivatives. Section 4 deals with the reduction problem in full generality and in section 5 we focus on symmetric potentials and sharpen the main result from section 4. Examples and illustrations of reduction are postponed to Section 6. Then, after discussing on mutations of modulated quivers with potentials in Section 7, more examples of sequences of mutations and reductions in the Dynkin type \( F_4 \) appear in Section 8. In section 9 we consider graded modulated quiver with potentials of homogeneous degree and provide arguments showing that the results of preceding sections hold in the graded context. In guise of application, in the last section we introduce non simply-laced Ginzburg dg-algebras and cluster categories associated with graded modulated quivers with potentials, generalizing the construction of cluster categories associated with graded quivers with potentials from [16, Def 3.5] and [20, § 4].

**Some perspectives**

In the present work, we have not investigated rigid modulated quivers with potentials and non-degeneracy of mutation as done in [1, §6.7.8]; also we have not studied decorated representations of modulated quivers with potentials. However, at least in the presence of separability over a base field or more specially in the setting of a perfect ground field, we believe a general study of mutations of decorated representations of modulated quivers with potentials should be affordable. In order to understand the cluster categories associated with modulated quiver with (nonzero) potentials, it is natural to prove the following about non simply-laced Ginzburg dg-algebras.

**Conjecture:** Keller’s result on the 3-Calabi-Yau property of simply-laced Ginzburg dg-categories holds in the general framework, at least when separability over a field \( k \) is assumed.

**Conventions, matrix mutation and valued quiver mutation**

We let \( k \) be a commutative ring and \( K \) a \( k \)-algebra assumed to be finitely generated projective as \( k \)-module. Tensor (path) algebras occur as tensor algebras of \( K \)-bimodules \( B \), with \( B \) assumed to be finitely generated and projective as left \( K \)-module and right \( K \)-module. The tensor product \( B \otimes_k B' \) of two \( K \)-bimodules \( B \) and \( B' \) is denoted by \( B \otimes B' \) or simply by \( B B' \).

The composition of any two morphisms \( f : X \longrightarrow Y \) and \( g : Y \longrightarrow Z \) in a given category is written either as \( g f \), \( g f \) or as \( g f \). We shall sometimes deal with infinite linear combinations which naturally occur: thus each element \( x = (x_\lambda)_{\lambda \in \Sigma} \) of a direct product \( \prod_{\lambda \in \Sigma} B_\lambda \) of left or right \( K \)-modules appears naturally as an infinite sum \( x = \sum_{\lambda \in \Sigma} x_\lambda \).
Matrix mutation. Let $n \in \mathbb{N}$ be a nonzero integer, we write $[1, n] = \{1, \ldots, n\}$. Let $B = (b_{i,j})_{1 \leq i,j \leq n}$ be a matrix with integer entries, $B$ is assumed skew-symmetrizable, that is, there exists a diagonal $n \times n$-matrix $E = (1, \ldots, n_0)$ of nonzero positive integers such that $b_{i,j}n_j = -b_{j,i}n_i$ for all $i, j \in [1, n]$. The mutation of $B$ at direction $k \in [1, n]$ is the skew-symmetrizable matrix $B' = \mu_k(B)$ described as follows: define the common sign of each pair $a, b \in \mathbb{Z}$ by $\text{sign}(a, b) = \text{sign}(\text{sign}(a) + \text{sign}(b))$ where $\text{sign}(0) = 0$; then $B' = (b'_{i,j})$ is given by this following mutation rule: if $k \in \{i, j\}$ then $b'_{i,j} = -b_{i,j}$, otherwise we have $b'_{i,j} = b_{i,j} + \text{sign}(b_{i,k}, b_{k,j})b_{i,k}b_{k,j}$.

Valued quiver mutation. An arbitrary (locally finite) valued quiver $Q$ with valuation $d$ consists of a set of points $I = Q_0$, and disjoint finite sets $Q_1(i, j)$ of valued arrows from $i$ to $j$, where the valuation of each $\alpha \in Q_1(i, j)$ is a pair of natural numbers $d(\alpha) = (d_i, d_j)$ and $\alpha$ may be pictured as $\xrightarrow{i, j} \alpha$ or as $\alpha : i \to j$; the valuation $d$ is required to be right (or left) symmetric, where the (minimal right) symmetrizing map $I \xrightarrow{\mu_0} \mathbb{N}$ for $d$ prescribes for each $i \in I$ a non-zero integer $n_i \in \mathbb{N}$ such that $d_i, n_j = d_j, n_i$ for all $\alpha \in Q_1(i, j)$. Arrows with valuation $(0, 0)$ are referred to as $0$-valued (or trivially valued) arrows, they are normally not drawn in pictures, it is understood that a 0-valued arrow is not counted among the arrows of the valued quiver. For an integer $m \geq 2$, the valued quiver is $m$-acyclic if it contains no $m$-cycle, that is, a path of length $m$ of the form $i_1 \xrightarrow{i_2, i_3} \cdots \xrightarrow{i_{m-1}, i_m} \xrightarrow{i_m, i_1}$. The composite of two paths $\omega \in Q(i, j)$ and $\omega' \in Q(j, t)$ is their concatenation denoted by $\omega \omega'$ or $\omega' \omega$.

For a valued quiver $Q$ over a set points $I$ with valuation $d$, define its normal form as the valued quiver $Q_i$ over without parallel arrows, with valuation still denoted by $d$ and obtained from $Q$ by replacing each finite set $Q_1(i, j) = \{a_1, \ldots, a_m\}$ by a one-element set consisting of a single valued arrow $\alpha : \xrightarrow{i, j} \sum_{s=1}^m d_{s,i}d_{s,j}$ with $(d_{1,i}, d_{1,j}) = \sum_{s=1}^m (d_{s,i}, d_{s,j})$. Thus a valued quiver is normalized whenever it coincides with its normal form. Let $Q$ be a normalized valued quiver over $I$ with valuation $d$, then $Q$ is completely defined by its set of points and its valuation. Let $\mu$ be a point not lying on a $2$-cycle in $Q$. The mutation of $Q$ at $\mu$ is the normalized valued quiver $Q' = \mu_k(Q)$ over $I$ with valuation $d'$ as follows:

(a) For any valued arrows $\alpha : x \overset{a}{\to} b$ and $\beta : k \overset{d}{\to} y$ starting or ending at $\mu$ in $Q$, there are corresponding valued arrows $\alpha^* : x \overset{a}{\to} b$ and $\beta^* : k \overset{d}{\to} y$ in $Q'$.  
(b) For each pair $i, j \in I \setminus \{\mu\}$ we have: $d'_j = \max(d_j, d_j' - d_i, 0)$; equivalently, $\alpha' = \max(d_\alpha, d_\alpha' - d_i, 0)$.

The above description of mutation is canonical: we never add superfluous 2-cycles. This contrasts with ordinary quiver mutation where superfluous 2-cycles are added and then, some of them are "simplified" in a non canonical way. By a little abuse of language, if $Q$ and $Q'$ are any valued quivers over a set $I$, we still write $Q' = \mu_k(Q)$ if the normal form of $Q'$ is the mutation at $\mu$ of the normal form of $Q$. Below is an illustration of valued quiver mutations, where the first two are respectively the normal forms of the last two ones:

![Mutation Illustration](image)

We observe that 2-acyclic normalized valued quivers without loops over a set $I$ correspond bijectively to skew-symmetrizable matrices with integer coefficients indexed by $I$, in such a way that valued quiver mutation and matrix mutation agree: let $Q$ be a normalized 2-acyclic valued quiver without loops over $I$ with valuation $d$, and $(b_{i,j})_{i,j \in I}$ the corresponding skew-symmetrizable matrix, then $(b_{i,j}, b_{j,i}) = (\alpha d_j - d'_{1,j}, d_i - d'_{i,j})$.

2. Trace maps and symmetrizable Dualizing pairs of bimodules

We write $Z(K)$ for the center of $K$; the $K$-center $Z_K(B)$ of a $K$-bimodule $B$ is the $Z(K)$-submodule of $B$ consisting of all elements $x$ with $ax = xa$ for all $a \in K$. Recall that the left dual $B' = \text{Hom}_K(KB, K)$, the $k$-dual
Hom_{k}(K B_k, k) and the right dual \( B^o = \text{Hom}_K(B_K, K) \) of \( B \) consist respectively of left \( K \)-linear maps, \( k \)-linear maps and right \( K \)-linear maps on \( B \), with actions defined as follows: for \( a, b \in K, v \in B, \xi \in \text{Hom}_K(K B_k, k) \) and \( v \in B^o \), we have \( (a \cdot a) \cdot b)(x) = u(x-a)b, (a \cdot \xi)(b)(x) = \xi(b-x) \) and \( (a \cdot v) \cdot b)(x) = b \cdot \xi(x) \) for every \( x \in B \). The bimodule \( B \) is dualizing if the left dual and the right dual of \( B \) are isomorphic. Recall that \( K \) is Frobenius if there is an isomorphism \( \phi : \text{Hom}_K(K k) \rightarrow \text{Hom}_K(k K) \) of left or right \( K \)-modules; if additionally \( \phi \) is a \( K \)-bimodule morphism then \( K \) is called a symmetric Frobenius algebra or simply a symmetric algebra and may be denoted by \((K, t)\) with \( t = \phi(1) \). Symmetric algebras and traces are related as in the following definition.

**Definition 2.1.**

(i) A \( k \)-linear trace form (or simply a trace) on \( K \) is any element \( t \) in the \( K \)-center of \( \text{Hom}_K(K k) \): thus \( t(a b) = t(b a) \) for all \( a, b \in K \). The radical of \( t \) is the ideal \( R_t := \{ a \in K : \forall b \in K, t(ab) = 0 \} \), and \( t \) is non-degenerate if its radical is zero. When the induced \( K \)-bimodule morphism \( K \rightarrow \text{Hom}_K(K k) \), \( a \mapsto (b \mapsto t(ab)) \) is an isomorphism, \( t \) is called strongly non-degenerate and in this case \((K, t)\) is symmetric.

(ii) The Casimir morphism \( \delta_{K,0,k} : k \rightarrow k \otimes_k \text{Hom}_K(K, k) \cong K \otimes_k K \) associated with any symmetric algebra \((K, t)\) takes the unit of \( k \) to a Casimir element \( \sum_{e \in A} e_s \circ e_s^* \in Z_K(K \otimes_k K) \) characterized by the identities:

\[
\text{for all } a \in K, \quad \sum_{e \in A} e_s t(e_s^* a) = a = \sum_{e \in A} t(e_s a)e_s^*. \tag{2.1}
\]

In part (ii) above, \( \{ e_s : s \in A \} \) is a finite generating set for \( K \) over \( K \), and since \( K \) is a projective \( k \)-module we choose a right inverse \( K^\otimes \rightarrow k \otimes K \) for \( p \) yielding a generating set \( \{ e_s : s \in A \} \) for the dual \( \text{Hom}_K(K, k) \) to which corresponds a \( "\text{dual generating set} \" \{ e_s^* : s \in A \} \subset K \) with \( t(e_s^* e) = \delta_k : a \mapsto t(e_s^* a) = e_s(a) \). Identities (2.1) yields the following observation.

**Remark 2.2.** Any \( K \)-bimodule \( B \) over symmetric algebra \((K, t)\) is dualizing: the canonical maps \( t = t_{\rightarrow} : B \rightarrow \text{Hom}_K(B, k) \) and \( t_{\leftarrow} = t_{\leftarrow} : B^o \rightarrow \text{Hom}_K(B, k) \) are bimodule isomorphisms, and for all \( v \in \text{Hom}_K(B, k) \) we have: \( t_{\leftarrow}^{-1}(v) : x \mapsto \sum_{e \in A} e_s v(e_s^* x) \) and \( t_{\rightarrow}^{-1}(v) : (x \mapsto \sum_{e \in A} e_s x(e_s^*)) \).

We then introduce the first main tool for the study of potentials in a general framework.

**Definition 2.3.** Let \( B, B' \) be \( K \)-bimodules together with a bimodule morphism \( b : B \otimes B' \otimes B' \rightarrow B \rightarrow K \) referred to as the bilinear form.

(a) The data \( \{ B, B', b \} \) is a symmetrizable pairing over \((K, t)\) if properties (i) and (ii) below hold.

(i) \((K, t)\) is a symmetric algebra, and \( t \) is a symmetrizing trace for \( b \), that is, \( t(b(x \otimes x')) = t(b(x' \otimes x)) \) for all \( x \in B, x' \in B' \).

(ii) \( b \) is non-degenerate, that is, the adjoint morphisms \( b_{1,1} : B^o \rightarrow B^o : x' \mapsto b(- \otimes x') \) and \( b_{1,2} : B^o \rightarrow B^o : x \mapsto b(x \cdot \otimes -) \) (or equivalently the adjoint morphisms \( b_{2,1} : B^o \rightarrow B^o : x' \mapsto b(- \otimes x') \) and \( b_{2,2} : B^o \rightarrow B^o : x \mapsto b(x \cdot \otimes -) \) are injective.

(b) The ordered data \( \{ B, B', b \} \) is a symmetrizable weakly dualizing pair over \((K, t)\) if \( B \) is projective as left and right \( K \)-module, conditions (i)-(ii) hold and the adjoint morphism \( b_{1,1} \) (or equivalently the adjoint morphism \( b_{2,2} \)) is bijective. If in addition, \( B \) (and thus \( B' \)) is finitely generated as left and right \( K \)-module, then the ordered data \( \{ B', B; b \} \) is also a symmetrizable weakly dualizing pair over \((K, t)\) and we call the (non ordered) data \( \{ B, B'; b \} \) a symmetrizable dualizing pair of bimodules, \( b \) strongly non-degenerate, \( B \) and \( B' \) are called mutually dual and we write: \( B' = B^* \) and \( B = B^{\circ o} \).

Often in a weakly dualizing pair \( \{ B, B'; b \} \) we shall omit to specify the bilinear form \( b \), in this case we write:

\( (x \otimes \xi) = b(x \otimes \xi) \) and \( \xi \otimes x = b(\xi \otimes x) \) for all \( x \in B, \xi \in B' \).

Note that each symmetric algebra \((K, t)\) gives rise to a natural symmetrizable dualizing pair \( \{ K, K \} \) with the bilinear form given by the multiplication of \( K \). We need the following lemma which gives a large class of symmetric algebras as well as the existence of nonzero traces for finite-dimensional local algebras.
Lemma 2.1.  (a) Let $K$ be any $k$-algebra. If $K$ has a non-degenerate $k$-linear trace $t$, then the $Z(K)$-module $Z_K(\text{Hom}_K(K,k))$ is free of dimension one and each non-degenerate trace on $K$ is given by $ct$ where $c \in Z(K)$ is a central unit.

(b) Suppose $K$ is finite-dimensional over a field $k$, with Jacobson radical $J_k$. Then there is a nonzero trace $t \in \text{soc}(K\text{Hom}_K(K,k)) \cap \text{soc}(\text{Hom}_K(K,k)_K)$. The $k$-algebra $K = K/J_K$ is symmetric and each $K$-bimodule $M$, finite-dimensional over $k$, is part of a symmetrizable dualizing pair $(M, M^*; b)$.

Proof. (a) Suppose $t$ is a non-degenerate trace on $K$ and let $\tau$ be any trace on $K$, since clearly the dual $\text{Hom}_K(K,k)$ is a free left $K$-module of dimension one; there exists $c \in K$ such that $\tau = ct$. We must show that $c \in Z(K)$, thus let $a, b \in K$: we have $t(caab) = \tau(ab) = \tau(ba) = t(ba) = t(ab)$, thus $t(ca - ac)b = 0$ for all $b \in K$, so that $ca - ac \in R_t = 0$, hence $c \in Z(K)$. Now suppose $t = ct$ is also non-degenerate, then we must also have $t = c't$ for some $c' \in Z(K)$, so that $t = c'ct$ and $\tau = cc't$, yielding that $cc' = 1 = cc'$, this proves part (a) of the lemma.

(b) It is a standard result that finite dimensional simple algebras over a field and hence semisimple algebras are symmetric Frobenius algebras. This can be done by invoking the existence of the so-called reduced trace for simple algebras which are finite-dimensional over their centers. Hence the semisimple $k$-algebra $K$ is symmetric for some trace $\hat{t}$, and if $\pi : K \longrightarrow K$ is the natural projection, then we get a $k$-linear trace $t = \hat{t} \circ \pi$ for $K$ with radical $R_t = J_K$ and with $t \in \{ \text{soc}(K\text{Hom}_K(K,k)) \cap \text{soc}(\text{Hom}_K(K,k)_K) \}$. The rest of the proof follows from Remark 2.2 together with the observation that the bilinear form associated with a symmetrizable dualizing pair $(M, M^*; b)$ is induced by the corresponding non-degenerate trace form: indeed assume $B^*$ is a $K$-bimodule isomorphic to one of (and thus to all) the standard duals of a $K$-bimodule $B$, finite-dimensional over $k$, choose an isomorphism $\phi : B^* \longrightarrow \text{Hom}_K(B,k)$; then $\phi$ yields a symmetrizable dualizing pair $(B, B^*; b)$ over $(K, t)$ with $b$ given as follows: let $x \in B, u \in B^*$, by Remark 2.2 write $t \circ u_1 = \phi(u) = t \circ u_2$ with $u_1 \in B$ and $u_2 \in B^*$, then $b(x \circ u) = u_1(x)$ and $b(u \circ x) = u_2(x)$.

Mutually dual projective bases and Casimir elements

Assume $B$ is part of a symmetrizable weakly dualizing pair $(B, B^*; b)$ over $(K, t)$. Choose a split sequence $K \longrightarrow B\longrightarrow K^{(p)}\longrightarrow K$ for the left $K$-module $B$, where $\pi$ is a split epimorphism with right inverse $\pi'$, $K^{(p)}$ is a direct sum of copies of $K$ indexed by a (possibly infinite) cardinal $p$. We get a left projective basis $\{ (x_s : s \in p), \{ \hat{x}_s, s \in p \} \}$ for $B$ characterized by the following property: $x = \sum_{s \in p} \hat{x}_s(x)x_s$ for all $x \in B$, and since moreover for each $u \in B^*$, the map $x \mapsto \sum_{s \in p} \langle \hat{x}_s(x), u \rangle(x) = \sum_{s \in p} \langle \hat{x}_s(x), u(x_s) \rangle = u(x)$ is a well-defined element of $B, u$ naturally occurs as (possibly infinite) sum: $u = \sum_{s \in p} \hat{x}_s(x)$. We refer to the (possible infinite) sum $\sum_{s \in p} \hat{x}_s(x)$ as the Casimir element associated with the left projective $K$-module $B$ and its left dual. Thus, if $p$ is a finite cardinal, then under the natural isomorphism $\phi : B \otimes \text{Hom}_K(KB_K; B) : \phi(u \otimes x)(z) = u(z)x$ (induced by the adjunction of tensor product), the pre-image of the identity map is given by the Casimir element. Similarly, the Casimir element $\sum_{q \in q} y_q \otimes \hat{y}_q$ and the right projective basis $\{ (y_s : s \in q), \{ \hat{y}_s : s \in q \} \}$ associated with the right projective $K$-module $B$ and its right dual have the following characterizing property: $x = \sum_{s \in p} y_s \hat{y}_s(x)$ and $u = \sum_{s \in p} u(y_s)\hat{y}_s$ for all $x \in B$ and $u \in B^*$, and when $q$ is a finite cardinal, the Casimir element associated with $B_{\text{q}}$ is the pre-image of the identity map under the natural isomorphisms $\psi : B \otimes B^* \longrightarrow \text{Hom}_K(B_{\text{q}}, B_{\text{q}}^*) : \psi(y \otimes v)(z) = yv(z)$. Using the adjoint isomorphisms $b_{1,1} : B^* \longrightarrow B$ and $b_{2,1} : B^* \longrightarrow B^*$, we get two pairs $\{ (x_s : s \in p), \{ x_s^* : s \in p \} \}$ and $\{ (y_s : s \in q), \{ y_s^* : s \in q \} \}$ of left projective bases and a right projective basis associated with $B$ and its weak dual $B^*$, having the following characterizing identities where $x \in B, \xi \in B^*$ and the formula expressing each $\xi$ may (naturally) appears as an infinite sums:

$$\sum_{s \in p} b(x \otimes x_s^*)x_s = x = \sum_{r \in q} y_r b(y_r^* \otimes x) \quad \text{and} \quad \sum_{s \in p} x_s^* b(x_s \otimes \xi) = \xi = \sum_{s \in q} b(\xi \otimes y_s) y_s^*. \quad (2.2)$$
The "elements" $\mathfrak{z}_{M^* \otimes \mathfrak{n}} = \sum_{\ell \in \mathfrak{p}} x_\ell \otimes x_\ell$ and $\mathfrak{z}_{B \otimes \mathfrak{n}^*} = \sum_{\ell \in \mathfrak{p}} y_\ell \otimes y_\ell^*$ are again referred to as Casimir elements associated with $\mathfrak{b}$ (or with the pair $(B, B^*; \mathfrak{b})$), note in view of equations (2.2) that these Casimir elements are K-central.

Now suppose we are given two symmetrizable pairing $(M, M^*; \beta)$ and $(M', M'^*; \beta')$ respectively over $(K, \tau)$ and $(K, \tau')$. Then for a $k$-linear map $f : M \rightarrow M'$, its left dual $f'$ and its right dual $f''$ (if they exist) are the unique $k$-linear maps $f', f'' : M'^* \rightarrow M^*$ defined by the condition: $\beta'(f(-) \otimes -) = \beta(- \otimes f(-))$ and $\beta''(f(-) \otimes -) = \beta(- \otimes f(-))$. We say that $f$ is left dualizing (respectively, right dualizing) when $f$ (respectively, $f''$) exists. Note that, when they exist, $f'$ and $f''$ need not coincide if the symmetrizable requirement on our pairing of bimodules is omitted.

$f$ is dualizing if $f$ and $f''$ exist and coincide, in this case their common value $f^*$ is called the dual of $f$.

**Lemma 2.2.** Let $(B, B^*; \mathfrak{b})$ and $(B', B'^*; \mathfrak{b}')$ be symmetrizable pairing over $(K, \mathfrak{t})$, and $f \in \text{Hom}_K(B, B)$,

1. If $f$ is left dualizing then it is left K-linear and $f'$ is right K-linear; if $f$ is right dualizing then it is right K-linear and $f''$ is left K-linear. If $f$ is a K-bimodule morphism, then it is dualizing whenever $f$ admits a left or right dual, in this case the dual of $f$ is the unique K-bimodule morphism $f^* : B^* \rightarrow B$ with the following property:

$$b'(f(-) \otimes -) = b(- \otimes f''(-)) \quad \text{or equivalently} \quad b'(- \otimes f(-)) = b(f'(-) \otimes -). \quad (2.3)$$

2. If the data $(B, B^*; \mathfrak{b})$ is weakly dualizing, then any morphism $f : B \rightarrow B'$ of left K-modules (respectively, right K-modules, K-bimodules) is left dualizing (respectively, right dualizing, dualizing).

**Proof.** For part (1), simply apply the fact that the bilinear forms $b$ and $b'$ are non-degenerate and symmetrizable via the same non-degenerate trace map $\mathfrak{t}$. In part (2), the one ordered data $(B, B^*; \mathfrak{b})$ is weakly dualizing over $(K, \mathfrak{t})$, so that we have adjoint bimodule isomorphisms $b_{1, \mathfrak{t}} : B \rightarrow B$ and $b_{2, \mathfrak{t}} : B' \rightarrow B'$. Thus, when $f : B \rightarrow B'$ is left K-linear, the composition map along the sequence

$$B \xrightarrow{f^*} B^* \xrightarrow{b_{1, \mathfrak{t}}} B' \xrightarrow{b_{2, \mathfrak{t}}} B'',$

is clearly a left dual for $f$. Similarly, if $f$ is right K-linear then its right dual exists. When $f$ is a bimodule morphism, part (1) and the previous arguments prove that $f$ is dualizing. \hfill \square

**Lemma 2.3.** Let $(B, B^*; \mathfrak{b})$ and $(B', B'^*; \mathfrak{b}')$ be symmetrizable dualizing pairs over $(K, \mathfrak{t})$. Then the left dual of any left K-linear isomorphism $f : B \rightarrow B'$ yields: $(f^{-1} \otimes f)(\mathfrak{z}_{B^* \otimes \mathfrak{n}}) = \mathfrak{z}_{B'^* \otimes \mathfrak{n}}$. Dually, the right dual of any right K-linear isomorphism $f : B \rightarrow B'$ yields: $(f' \otimes (f')^{-1})(\mathfrak{z}_{B \otimes \mathfrak{n}^*}) = \mathfrak{z}_{B' \otimes \mathfrak{n}^*}$.

**Proof.** Let $f : B \rightarrow B'$ be an isomorphism of left K-modules, then in view the last part of Lemma 2.2, $f$ and $f^{-1}$ are left dualizing and clearly $f^{-1} = (f')^{-1}$. Write $\mathfrak{z}_{B^* \otimes \mathfrak{n}} = \sum_{n=1}^N x_n \otimes x_n$ for the Casimir element in $B^* \otimes \mathfrak{b}$. Now let $x' \in B, \xi' \in B'^*$, applying the characterizing properties (2.2) for the Casimir element $\mathfrak{z}_{B^* \otimes \mathfrak{n}}$ and the definition of the left dual $f^{-1}$ we have:

$$x' = f f^{-1}(x') = f \left( \sum_{n=1}^N b(x' \otimes x_n) x_n \right) = \sum_{n=1}^N b(f^{-1}(x') \otimes x_n) f(x_n) = \sum_{n=1}^N b'(x' \otimes f^{-1}(x_n)) f(x_n);$$

$$\xi' = (f^{-1} f)(\xi') = (f^{-1} \left( \sum_{n=1}^N x_n \otimes f(\xi') \right)) = \sum_{n=1}^N f^{-1}(x_n) b(x_n \otimes f(\xi')) = \sum_{n=1}^N f^{-1}(x_n) b(f(x_n) \otimes \xi'),$$

showing in virtue of (2.2) that the element $(f^{-1} \otimes f)(\mathfrak{z}_{B^* \otimes \mathfrak{n}}) = \sum_{n=1}^N f^{-1}(x_n) \otimes f(x_n)$ is as claimed the Casimir element in $B'^* \otimes B'$. The dual statement is proved in the same way. \hfill \square

**Products of symmetrizable dualizing pairing**

First, note that if $B$ and $B'$ are K-bimodules, projective and finitely generated as left and right K-modules, then their tensor product $B \otimes B'$ (over K) is still finitely generated projective as left and right K-module. Suppose $(B, B^*; \mathfrak{b})$ and $(B', B'^*; \mathfrak{b}')$ are symmetrizable dualizing pairs over $(K, \mathfrak{t})$. We can form the product $(B, B^*; \mathfrak{b}) \otimes (B', B'^*; \mathfrak{b}'; B \otimes B') = (B \otimes B', B'^* \otimes B'^*; \mathfrak{b} \otimes \mathfrak{b}' + \mathfrak{b} \otimes \mathfrak{b}')$ with the induced bilinear form $\mathfrak{b} \otimes \mathfrak{b}' : x \otimes x' \rightarrow \mathfrak{b}(x \otimes x')$. This is well-defined in virtue of (2.2). For $x' \in B', u \in B^*$ and $u' \in B'^*$ we have $(\mathfrak{b} \otimes \mathfrak{b}')(x \otimes x' \otimes u \otimes u') = \mathfrak{b}(\mathfrak{b}'(x \otimes x') \otimes u)$ and thus $(\mathfrak{b} \otimes \mathfrak{b}')(u' \otimes u \otimes x \otimes x') = \mathfrak{b}'(\mathfrak{b}(u' \otimes x \otimes x'))$. \hfill 8
One defines in the same way the product of any finite number of dualizing pairs of bimodules. Observe that the pair \( \{B, B^*; b\} \) also induces two symmetrizable dualizing pairs \( \{B \otimes B^*, B \otimes B^*\} \) and \( \{B^* \otimes B, B^* \otimes B\} \) in which \( B \otimes B^* \) and \( B^* \otimes B \) are self-dual bimodules. The next lemma gives a simple but crucial observation.

**Lemma 2.4.** (1) For a symmetrizable dualizing pair \( \{B, B^*; b\} \) over \( (K, t) \), the dual morphisms of the bilinear forms \( b_1 : B \otimes B^* \rightarrow K \) and \( b_2 : B^* \otimes B \rightarrow K \), with \( b_1(x \otimes u) = b(x \otimes u) \) and \( b_2(u \otimes x) = b(u \otimes x) \) for all \( x \in B, u \in B^* \), coincide with the Casimir morphisms \( \epsilon_{B \otimes B^*} : K \rightarrow B \otimes B^* \) and \( \epsilon_{B^* \otimes B} : K \rightarrow B^* \otimes B \), taking the unit element of \( K \) to the corresponding Casimir elements.

(2) Suppose \( \{B, B^*\} \otimes \{B', B'^*\} = \{B \otimes B', B^* \otimes B'^*\} \) is the product of symmetrizable dualizing pairs over \( (K, t) \). Then the corresponding Casimir elements are given by \( \epsilon_{B \otimes B'^*} = \sum_{i=1}^q \sum_{j=1}^q (y_i \otimes y'_j) \otimes (y^*_j \otimes y^*_i) \) and \( \epsilon_{B^* \otimes B'^*} = \sum_{i=1}^p \sum_{j=1}^p (x'_i \otimes x^*_j) \otimes (x_j \otimes x^*_i) \), where \( \epsilon_{B \otimes B'^*} = \sum_{i=1}^q y_i \otimes y^*_i, \epsilon_{B^* \otimes B'} = \sum_{i=1}^p x^*_i \otimes x_i, \epsilon_{B \otimes B'^*} = \sum_{j=1}^p y'_j \otimes y^*_j \) and \( \epsilon_{B^* \otimes B'} = \sum_{j=1}^p x'^*_j \otimes x'^*_i \).

**Proof.** The proof is a direct application of the definition of Casimir elements and the dual of a morphism. □

**Derivative operators.** For a symmetrizable dualizing pair \( \{M, M^*; b\} \) over \( (K, t) \), let \( A := T_K(M) = K \otimes M \otimes (M \otimes M) \oplus (M \otimes M) \oplus \ldots \) be the tensor algebra of the \( K \)-bimodule \( M \), then write \( \partial = \partial_M := b \otimes 1 : M^* \otimes M \otimes A \rightarrow A \) and \( \partial' = \partial_{M'} := 1 \otimes b : A \otimes M \otimes M^* \rightarrow A \), respectively referred to as **left derivative operator** and **right derivative operator**. We now conclude this subsection with a property of a cyclical stability.

**Lemma 2.5.** Let \( \{M, M^*; b\} \) be a symmetrizable dualizing pair over \( (K, t) \), let \( \delta = \delta_{M \otimes M^*} \) and \( \delta' = \delta_{M^* \otimes M} \).

(1) For every bimodule morphism \( K \rightarrow M \), we have: \( b(m(1) \otimes \cdot) = m = b(\cdot \otimes m(1)) \), and \( m \) is cyclically stable, that is, the bimodule morphisms \( \epsilon_{M} := (\text{Id}_M \otimes b) \circ (1 \otimes m \otimes \text{Id}_M) \circ \delta_{M \otimes M^*} \) and \( \epsilon_{M^*} := (b \otimes \text{Id}_M) \circ (\text{Id}_M \otimes m \otimes \text{Id}_M) \circ \delta_{M^* \otimes M} \) coincide with \( m \).

(2) Consider the following morphisms referred to as left or right permutations of \( \delta \) and \( \delta' \):

\[
\begin{align*}
\epsilon_\delta &= \partial_M (1 \otimes M^* \otimes M) \otimes \delta' \\
\epsilon_{\delta'} &= \partial_{M^*} (1 \otimes M \otimes M^*) \otimes \delta
\end{align*}
\]

(2.4)

Then \( \delta \) and \( \delta' \) are cyclically equivalent: \( \epsilon_{\delta'} = \delta = \epsilon_\delta \) and \( \epsilon_{\delta'} = \delta' = \epsilon_\delta \). In particular \( \delta \) and \( \delta' \) are cyclically stable: \( \epsilon_{\delta'}^2 := \epsilon_{\delta'}(\epsilon_\delta(\delta')) = \delta = \epsilon_{\delta'}^2 := \epsilon_{\delta'}(\epsilon_\delta(\delta')) = \delta' = \epsilon_{\delta'}^2 := \epsilon_{\delta'}(\epsilon_\delta(\delta')) = \delta' \).

**Proof.** For part (1), let \( E \rightarrow \otimes K \) be a morphism of bimodules, in respect to the data \( \{M, M^*; b\} \) and \( (K, K) \), and in view of Lemma 2.2 the dual \( M^* \rightarrow K \) of \( m \) exists and satisfies the following relation: \( b(m(1) \otimes \cdot) = K(1 \otimes \text{Id}(\cdot)) \otimes K = m = K(m(\cdot) \otimes 1) \otimes K = b(\cdot \otimes m(1)) \). In view of Lemma 2.4, the Casimir morphism \( \delta \), sending the unit of \( K \) to the Casimir element \( \delta(1) = \sum_{i=1}^q y_i \otimes y^*_i \), is the dual of the bilinear form \( B \otimes B^* \rightarrow K \). Thus, using the relation \( b(m(1) \otimes \cdot) = b(\cdot \otimes m(1)) \), we get: \( \epsilon_{\delta}(1) = (\text{Id}_M \otimes b) \circ (1 \otimes m \otimes \text{Id}_M) \circ \delta_{M \otimes M^*} = \sum_{i=1}^q y_i \otimes y^*_i \).

Part (2) is a direct application of the definition of Casimir morphisms and identities (2.2). Indeed write \( \delta'(1) = \sum_{i=1}^q y_i \otimes x_i \in M^* \otimes M \). We then have \( (\epsilon_\delta'\delta')(1) = (\partial(1 \otimes \delta' \otimes \text{Id}) \circ \delta')(1) = (\partial(1 \otimes \sum_{i=1}^q y_i \otimes x_i \otimes y^*_i) = \sum_{i=1}^q (b(y_i \otimes x_i) \cdot x_i) \otimes y^*_i = \sum_{i=1}^q y_i \otimes y^*_i \).

9
we have \( \varepsilon_n' \cdot 1 = \left( \partial(1 \otimes \mathcal{J} \otimes 1) \circ \mathcal{J} \right) \cdot 1 = \partial \left( \sum_{i=1}^{n} y_i \otimes x_i^* \otimes x_k \otimes y_i^* \right) = \sum_{i=1}^{n} y_i \left( \sum_{k=1}^{n} x_k \otimes b(x_k \otimes y_i^*) \right) = \sum_{i=1}^{n} y_i \otimes y_i^* = \mathcal{J}(1) \), hence \( \varepsilon_n' = \mathcal{J} \). In the same way one checks that \( \varepsilon_n \mathcal{J} = \varepsilon_n' \mathcal{J} \).

\[ \square \]

3. Potentials and Jacobian algebras

3.1. Tensor path algebras of modulated quivers

**Definition 3.1.** Given a symmetrizable dualizing pair \( \{ B, B^*; b \} \) over a symmetric algebra \( \{ K, t \} \), we refer to the data \( Q = (B, K, t) \) as \( k \)-modulated quiver, \( B \) is therefore referred to as the arrow bimodule of \( Q \). The dual of \( Q \) is the modulated quiver \( Q^* = (K^*, B, t) \).

Note that we may decompose \( K \) as finite direct product \( \prod_{i \in I} k_i \) of indecomposable \( k \)-algebras \( k_i \), each \( k_i \) is viewed as subalgebra in \( K \) and the unity of \( K \) is \( 1 = \sum_{i \in I} 1_i \), where \( 1_i \in k_i \) is the unity of \( k_i \), the set \( \{ 1_i : i \in I \} \) is then a system of central primitive orthogonal idempotents for \( K \). Thus, \( \{ K, t \} \) occurs as direct product of symmetric algebras \( (k_i, t_i) \) with \( t_i = t_{k_i} \), \( i \in I \), and the pair \( \{ B, B^*; b \} \) occurs as direct sum of symmetrizable dualizing pairs \( \{ i_j, B_j, B_j^*; i_j b \} \) over \( k_i \) \( \times \) \( k_j \) (\( i_j, t_j, t_{k_j} \)) with \( B_j = 1_j \cdot B \cdot 1_j \). When \( B_j \) is nonzero, we have an arrow from \( i \) to \( j \) in \( Q \) pictured as \( \alpha(i,j) : i_j B_j \otimes 1_j \) or simply as \( \alpha(i,j) : i_j B_j \). We say that \( Q \) has no loop if \( B_j \) is zero for all \( i \), we also write \( Q_m(i,j) \) for the set of all length-\( m \) paths from \( i \) to \( j \), while \( Q(i,j) \) denotes the set of all paths from \( i \) to \( j \). Observe that, if moreover each \( k \)-algebra \( k_i \) is a division algebra then the case of classical modulated quivers is recovered and the underlying (normalized) valued quiver of \( Q \) over \( I \) has valued arrows \( (i_i,j_i) \) with \( d_j = \dim k_i (B_j) \) and \( i d_j = \dim k_i (B_j) \) for all \( i, j \in I \).

In the sequel we assume that \( Q := (B, K, t) \) is a \( k \)-modulated quiver. The tensor path algebra \( kQ \) of \( Q \) is by definition the tensor algebra of the \( k \)-bimodule \( B \), thus \( kQ := T_K(B) = \bigoplus_{m \geq 0} kQ_m \), where \( kQ_m = B^m \) is the \( m \)-fold tensor product of \( B \), referred to as the bimodule of degree-\( m \) homogeneous elements (or the bimodule of all length-\( m \) paths), with \( B^0 = K, B^1 = B \) and \( B^{m+1} = B^m \otimes B \) for all \( m \geq 1 \). We let \( kQ_\{1\} = \bigoplus_{m \geq 1} kQ_m \), the ideal \( kQ_{\{1\}} \) is referred to as the arrow ideal of \( kQ \) and we have \( kQ_kQ_{\{1\}} = K \). In general the arrow ideal of \( kQ \) needs not coincide with the Jacobson radical of \( kQ \), unless \( K \) is semisimple and \( Q \) is acyclic (that is, there exists some \( m \geq \) with \( B^m = 0 \)). Next, the complete tensor path algebra of \( Q \) is the direct product \( k\widehat{Q} = \prod_{m \geq 0} B^m \), and the closed arrow ideal of \( k\widehat{Q} \) is given by \( k\widehat{Q}_\{1\} = \bigcap_{m \geq 1} B^m \), the latter coincides with the Jacobson radical of \( k\widehat{Q} \) whenever \( K \) is semisimple. For all \( i, j \in I \), each \( k_i \)-bimodule \( 1_j k\widehat{Q} 1_j \) is referred to as the bimodule of all elements \( \xi \) with source \( s(\xi) = i \) and target \( t(\xi) = j \).

The \( J_{k\widehat{Q}} \)-adic topology. The \( J_{k\widehat{Q}} \)-adic topology on \( k\widehat{Q} \) admits as system of open neighbourhoods of \( 0 \) the family \( \{ J_{k\widehat{Q}}^l \}_{l \geq 0} \), with \( J_{k\widehat{Q}}^l = \bigcap_{m \geq l} B^m \) for each \( l \geq 0 \). The closure of each subset \( S \subset k\widehat{Q} \) is given by \( \overline{S} = \bigcap_{l \geq 0} (S + J_{k\widehat{Q}}^l) \).

**Remark 3.2.** (a) \( k\widehat{Q} \) coincides with the projective limit \( \bigcup_{l \geq 0} k\widehat{Q}/J_{k\widehat{Q}}^l \), thus the \( J_{k\widehat{Q}} \)-adic topology on \( k\widehat{Q} \) is complete and separate. Next, let \( F = \sum_{\lambda=(\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m} a_{\lambda_1} \cdots a_{\lambda_m} \) be any power series over \( K \) for some natural number \( m \), then for all \( u = (u_1, \ldots, u_m) \) with \( u_1, \ldots, u_m \in J_{k\widehat{Q}}^\{1\} \), the infinite sum \( F(u) = \sum_{\lambda=(\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m} a_{\lambda_1} u_{\lambda_1} \cdots a_{\lambda_m} u_{\lambda_m} \) defines a unique element in \( k\widehat{Q} \) given as the limit \( \lim_{\lambda \to \infty} F_\lambda(u) \) of the series of partial sums \( F_\lambda(u) := \sum_{\theta \leq \lambda} h_\theta(u) \), where \( h_\theta(u) = a_{\theta_1} u_{\theta_1} \cdots a_{\theta_m} u_{\theta_m} \) for each \( \theta = (\theta_1, \ldots, \theta_m) \in \mathbb{N}^m \).

(b) Let \( S \subset k\widehat{Q} \) be any \( k \)-submodule, then \( S = \left\{ \sum x_l : x_l \in S \cap F_{k\widehat{Q}}^l \right\} \).

\[ \square \]
3.2. Casimir morphisms and projective bases for tensor path algebras

Let \( l \geq 0 \) be a fixed natural number, in view of Lemma 2.4 and the discussion preceding it, we have an induced symmetrizable dualizing pair \( \{ B(l), B^*(l); b^l \} \) over \( (K, t) \), where \( \{ B(0), B^*(0); y^0 \} \) coincides the natural dualizing pair \( \{ K, K \} \), \( b^0 \) being the multiplication of \( K \). We also have the following Casimir morphisms:

\[
\begin{align*}
\delta_{(l)} & : K \longrightarrow B(l) \otimes B^*(l) : 1 \mapsto \sum_{y \in \mathcal{Q}_o^l} y \otimes y^*, \quad \text{with } \mathcal{Q}_o = \{ 1 \} = \mathcal{Q}_o^*, \\
\delta'_{(l)} & : K \longrightarrow B^*(l) \otimes B(l) : 1 \mapsto \sum_{y \in \mathcal{Q}_o} x^* \otimes x, \quad \text{with } \mathcal{Q}_o = \{ 1 \} = \mathcal{Q}_o^*.
\end{align*}
\]

(3.2)

Here the pair \( (\mathcal{Q}, \mathcal{Q}^*) \), with \( \mathcal{Q}^* = \{ x^* : x \in \mathcal{Q} \} \), is a left projective basis for the left \( K \)-module \( B(l) \) and its dual, and \( (\mathcal{Q}^*, \mathcal{Q}) \), with \( \mathcal{Q} = \{ y^* : y \in \mathcal{Q}^* \} \), is a right projective basis for the right \( K \)-module \( B \) and its dual. We get two symmetrizable weakly dualizing pairs \( \{ kQ^*, \tilde{kQ}, b \} \) and \( \{ kQ, \tilde{kQ}^*, b \} \) with induced bilinear forms:

\[
\tilde{b} : kQ \otimes kQ^* \otimes kQ \otimes \tilde{kQ} \longrightarrow K, \quad \tilde{b}(x \otimes \xi) = \sum_{l \geq 0} b^l(x_l \otimes \xi_l) \quad \text{for all } \xi = (\xi_l)_{l \geq 0} \in kQ^* \text{ and } x = (x_l)_{l \geq 0} \in \tilde{kQ}.
\]

(3.3)

\[
\tilde{b} : kQ \otimes \tilde{kQ}^* \otimes kQ \otimes \tilde{kQ} \longrightarrow K, \quad \tilde{b}(\chi \otimes \zeta) = \sum_{l \geq 0} b^l(\chi_l \otimes \zeta_l) \quad \text{for all } \zeta = (\zeta_l)_{l \geq 0} \in \tilde{kQ}^* \text{ and } \chi = (\chi_l)_{l \geq 0} \in kQ.
\]

(3.4)

For all natural numbers \( n, m \) with \( m \neq 0 \) and each symbol \( s \in \{ 1, r \} \) we put:

\[
\mathcal{Q}_o^m = \mathcal{Q}_o^1 \otimes \cdots \otimes \mathcal{Q}_o^1, \quad \text{and } \mathcal{Q}_o^{m^*} = \mathcal{Q}_o^1 \otimes \cdots \otimes \mathcal{Q}_o^1.
\]

(3.5)

\[
\mathcal{Q}_o = \bigcup_{l \geq n} \mathcal{Q}_o^l, \quad \text{with dual projective basis: } \mathcal{Q}_o^\prime = \bigcup_{l \geq n} \mathcal{Q}_o^{l^*}.
\]

Hence, \( (\mathcal{Q}_o^l, \mathcal{Q}_o^{l^*}) \) and \( (\mathcal{Q}_o^l, \tilde{kQ}) \) are two pairs of projective bases associated with the symmetrizable weakly dualizing pair \( \{ kQ^*, \tilde{kQ}; b \} \) and we have the following characterizing identities:

\[
\sum_{\chi \in \mathcal{Q}_o^l} \chi^* \tilde{b}(\chi \otimes -) = 1_{kQ^*} = \sum_{\omega \in \mathcal{Q}_o^l} \tilde{b}(\cdot \otimes \omega) \omega^* \text{ and } \sum_{\chi \in \mathcal{Q}_o^l} \tilde{b}(\cdot \otimes \chi^*) \chi = 1_{\tilde{kQ}} = \sum_{\omega \in \mathcal{Q}_o^l} \omega \tilde{b}(\omega^* \otimes -).
\]

(3.6)

One can derive similar conclusions for the symmetrizable weakly dualizing pair \( \{ kQ, \tilde{kQ}^*; b \} \).

Continuous morphisms of path algebras.

**Definition 3.3.** Let \( M \) be a \( K \)-bimodule, then an algebra morphism \( f : T_K(M) \longrightarrow kQ \) is called a **morphism of path algebras** if \( f|_{kK} = 1_{kK} \) and \( f(M) \subset kQ^\prime \). In this case we let \( f_l : M \longrightarrow B(l) \), \( l \geq 1 \), be the family of \( K \)-bimodule morphisms such that \( f|_{M} = (f_l)_{l \geq 1} \).

In the classical case of a semisimple algebra \( K \), one checks that an algebra morphism \( f \) as above is a path algebra morphism if \( f|_{kK} = 1_{kK} \). Recall that if \( A \) is a \( k \)-algebra with a \( K \)-bimodule structure such that the unity of \( A \) is \( K \)-central, then any \( K \)-bimodule morphism \( f_1 : B \longrightarrow A \) uniquely extends to a morphism \( f : kQ \longrightarrow A \) of \( k \)-algebras.

**Proposition 3.1.** Given any \( k \)-modulated quiver \( Q' = (B', K, t) \), the two following statements are true.

(a) Any family \( (\phi_l)_{l \geq 1} \) of \( K \)-bimodule morphisms \( \phi_l : B \longrightarrow B(l) \) defines a unique continuous morphism \( \phi : kQ \longrightarrow kQ' \) of topological path \( k \)-algebras. Furthermore, \( \phi \) is an isomorphism if and only if \( \phi_1 : B \longrightarrow B' \) is.
(b) Any path algebra morphism \( \phi : \widehat{kQ} \rightarrow kQ' \) is continuous and, if \( \phi \) is also surjective then for every subset \( I \subset kQ \) such that \( \text{Ker}(\phi) \cap I = \varnothing \) we have \( \phi(I) = \widehat{\phi(I)} \). Consequently any path algebra isomorphism \( \phi : \widehat{kQ} \rightarrow kQ' \) is a homeomorphism of topological path \( k \)-algebras.

Proof.

**Statement (a).** For the first part of (a), the existence an extension \( \phi \) follows by the universal property of \( kQ \) and by Remark 3.2, the continuity and hence the uniqueness of \( \phi \) follow by statement (b). For the second part of (a), if \( \phi \) is an isomorphism of algebras, then \( \phi : B \rightarrow B' \) is clearly an isomorphism of \( kQ \)-bimodules. Conversely, assume that \( \phi \) is an isomorphism of \( kQ \)-bimodules, thus without lost of generality we can also assume that \( B' = B \) and \( \phi_1 \equiv 1_B \). With notations of (2.1) and (3.5), we take a left projective basis \( (\mathcal{Q}_b, \mathcal{Q}_o) \) for \( kQ \) and its weak dual \( \widehat{kQ} \), and a projective basis \( \{ c_s, e_s^* : s \in \Lambda \} \) of the symmetric algebra \( kQ \) over \( k \). The system \( S = \{ c_s\chi : s \in \Lambda, \chi \in \mathcal{Q}_b \} \) is a "projective basis" of \( kQ \) over \( k \) with the corresponding dual "projective basis" \( S^* = \{ \chi^*e_s^* : s \in \Lambda, \chi \in \mathcal{Q}_b \} \). The elements of \( S \) being ordered in an increasing order of their degree, in view of identities (3.6), each element \( x \in \widehat{kQ} \) is written as an infinite k-linear combination \( x = \sum_{\chi \in \mathcal{S}} c_s\chi \), and the infinite matrix representing the map \( \phi \) relatively to the projective basis \( S \) is a triangular matrix with the "1's" on its diagonal, and hence is invertible, consequently \( \phi \) is bijective.

**Statement (b).** Let \( J = J_{kQ}^1 \), \( J' = J_{kQ}^2 \), and \( \phi : \widehat{kQ} \rightarrow kQ' \) a morphism of path algebras. Thus \( \phi|_k = \mathbb{1}_k \) and \( \phi(B) \subset J' \). The definition of the \( J \)-adic topology shows each subset \( U \subset \widehat{kQ} \) containing a power of \( J \) is open. By assumption, \( \phi(B) \subset J' \), implying that \( \phi(J') \subset J'' \) for all \( l \geq 1 \), so that \( \phi^{-1}(J'') \supset J' \), showing that each pre-image \( \phi^{-1}(J'') \) is an open set. Hence \( \phi \) is continuous.

Now assume that \( \phi \) is surjective. The previous discussion shows that \( \phi(J') = J'' \) for all \( l \geq 1 \), in particular \( \phi^{-1}(J'') = \phi^{-1}(\phi(J')) = J' + \text{Ker}(\phi) \). Let \( J \subset kQ \) with \( \text{Ker}(\phi) \subset J \). Using the fact that \( \phi^{-1}(v \cap v') = \phi^{-1}(v) \cap \phi^{-1}(v') \) and \( \phi^{-1}(\phi(U)) = U + \text{Ker}(\phi) \) for all subsets \( V, V' \subset kQ, U \subset \widehat{kQ} \), we have: \( \phi^{-1}(\phi(I)) = \phi^{-1}(\bigcap_{i=1}^n (\phi(I) + J')) = \bigcap_{i=1}^n \phi^{-1}(\phi(I) + J') = \bigcap_{i=1}^n (I + \text{Ker}(\phi) + J') \subset \bigcap_{i=1}^n (I + J') = \widehat{T} = T \); but also \( T = \bigcap_{i=1}^n (I + J') \subset \bigcap_{i=1}^n (I + \text{Ker}(\phi) + J') \). Consequently, \( \phi^{-1}(\phi(T)) = T \), so that \( \phi(T) = \phi(T) \). \( \square \)

**Definition 3.4** ([1, 2.5]). Let \( \phi \) be a path algebra automorphism of \( \widehat{kQ} \). Then \( \phi \) is a change of arrows if \( \phi(2) \equiv (\phi(1))_{g \geq 2} = 0 \). If \( \phi_1 = \mathbb{1}_B \) then \( \phi \) is a unital triangular automorphism. We say that \( \phi \) has depth \( d \geq 1 \) if \( \phi_k = 0 \) for all \( 1 \leq k \leq d \), in this case \( \phi(u) - u \in J_{kQ}^{d+2} \) for all \( u \in J_{kQ}^d \).

### 3.3. Non simply-laced generalization of potentials and their Jacobian ideals

**Definition 3.5.** A potential on \( Q \) is any \( kQ \)-bimodule morphism \( K \rightarrow \Omega kQ(2) \); thus the bimodule of potentials on \( Q \) identifies with the \( Z(kQ) \)-bimodule \( Z_k(\widehat{kQ}(2)) \) of all \( K \)-central elements in \( J_{kQ}^1 kQ = \Omega kQ(2) \).

By Lemma 2.5, \( m \in Z_k(\widehat{kQ}(2)) \) is cyclically stable, that is, homogeneous components of \( m \) are cyclically stable.

**Example 3.6.** Suppose \( K = E \times F \times L \) as product of indecomposable symmetric \( k \)-algebras. Below, the second modulated quiver is obtained from the first by a transformation named latter on as mutation.

\[
\begin{array}{c}
E & \xrightarrow{1B_2, 1B_2^*} & F & \xrightarrow{2B_3, 2B_3^*} & L \\
\end{array}
\]

Then a potential on the second modulated quiver is given by the Casimir \( \chi(1B_2^* \otimes 2B_3^* \otimes 1B_2) \).
For the symmetrizable weakly dualizing pair \( \{k^Q, \bar{k}^Q, \bar{b}\} \), the associated left derivative operator and right derivative operator are respectively \( \partial^l := \bar{b} \otimes 1 : k^Q \otimes \bar{k}^Q \longrightarrow \bar{k}^Q \) and \( \partial^r := 1 \otimes \bar{b} : k^Q \otimes k^Q \longrightarrow k^Q \), they are explicitly described as follows: for all \( \xi \in B^{(s)}, v \in B^{(d)} \) with \( d < l, x \in B^{(l)} \) and \( u \in \bar{k}^Q \), we have
\[
\partial^l (\xi \otimes v) = 0 = \partial^r (v \otimes \xi), \quad \partial^l (\xi \otimes xu) = \partial^r (\xi \otimes x)u = \bar{b}(\xi \otimes x)u = b^l(\xi \otimes x)u \quad \text{and} \quad \partial^r (ux \otimes \xi) = u\partial^r (x \otimes \xi) = u\bar{b}(x \otimes \xi) = ub^l(x \otimes \xi) .
\] (3.7)

We observe that the left derivative operator is a morphism of \( \bar{k}^Q\)-\( k^Q \)-bimodules, while the right derivative operator is morphism of \( \bar{k}^Q\)-\( k^Q \)-bimodules. The following observations are direct generalizations of identities (2.2) and (3.6).

**Remark 3.7.** For all natural number \( l \in \mathbb{N} \),
\[
\sum_{\chi \in \dot{Q}_l} \chi^* \partial^l (\chi \otimes -) = I_{kQ} = \sum_{\omega \in \dot{Q}_l} \partial^r (- \otimes \omega) \omega^* \quad \text{and} \quad \sum_{\chi \in \dot{Q}_l} \partial^r (- \otimes \chi^*) \chi = I_{kQ_{(l)}} = \sum_{\omega \in \dot{Q}_l} \omega \partial^r (\omega^* \otimes -) .
\] (3.8)

Moreover, taking a componentwise composition, on \( k^Q \otimes \bar{k}^Q \otimes k^Q \) we have \( \partial \partial^r = \partial^r \partial^l = \partial^r \partial^l \) .

Now let \( k^m \longrightarrow \bar{k}^Q_{(2)} \) be a potential on \( Q \). The action of the left derivative operator on \( m \) yields the bimodule morphism \( \partial^l m = \partial^l \circ (1 \otimes m) : k^Q \longrightarrow \bar{k}^Q \) and the action the right derivative operator yields the bimodule morphism \( \partial^r m = \partial^r \circ (m \otimes 1) : k^Q \longrightarrow \bar{k}^Q \). Thus, when \( m \) is identified with \( m(1) \), for each \( \xi \in k^Q \) we have:
\[
\partial^l m := (\partial^l m)(\xi) = \partial^l (\xi \otimes m) \quad \text{and} \quad \partial^r m := (\partial^r m)(\xi) = \partial^r (m \otimes \xi) .
\]

Note that we have Casimir morphisms: \( \dot{g}_{(1)} : K \longrightarrow B^{(l)} \otimes B^{(l)} \) and \( \dot{g}_{(1)} : K \longrightarrow B^{(l)} \otimes B^{(l)} \) described by (3.2).

**Definition 3.8 (Skew permutations).** Let \( l \in \mathbb{N} \). The left and the right permutation operators of order \( l \) are morphisms \( \varepsilon_l^l, \varepsilon_l^r \) \( : Z_K(\bar{k}^Q_{(2)}) \longrightarrow Z_K(\bar{k}^Q_{(2)}) \) acting on potentials \( K \longrightarrow B^{(l)} \cdot B^{(d)} \) by:
\[
\varepsilon_l^l m := \partial^l (1 \otimes m \otimes 1) \circ \dot{g}_{(1)}^l : K \longrightarrow B^{(l)} \quad \text{and} \quad \varepsilon_l^r m := \partial^r (1 \otimes m \otimes 1) \circ \dot{g}_{(1)}^r : K \longrightarrow B^{(l)} .
\]

Thus \( \varepsilon_l^0 = I_{Z_K(\bar{k}^Q_{(2)})} = \varepsilon_l^0 \) and \( \varepsilon_l^l \) and \( \varepsilon_l^r \) act as identity maps on homogeneous potentials of degree \( d \).

**Proposition 3.2.** (1) For all potential \( m \) on \( \bar{k}^Q \), the action \( \partial^r (\varepsilon_l^r(m)) : B^{(l)} \longrightarrow \bar{k}^Q \) of the right derivative operator on the left permutation of order \( l \) of \( m \) is equal to the action \( \partial^r (m) : B^{(l)} \longrightarrow \bar{k}^Q \) of the left derivative operator on \( m \). Likewise, the action \( \partial^r (\varepsilon_l^l(m)) : B^{(l)} \longrightarrow \bar{k}^Q \) of the right derivative operator on the right permutation of order \( l \) of \( m \) is equal to the action \( \partial^r (m) : B^{(l)} \longrightarrow \bar{k}^Q \) of the right derivative operator on \( m \).

(2) For every \( l \in \mathbb{N} \), we have \( \varepsilon_l^l \circ \varepsilon_l^r = 1_{Z_K(\bar{k}^Q_{(2)})} = \varepsilon_l^r \circ \varepsilon_l^l \).

(3) We have a cyclic permutation operator \( \varepsilon_{(l)} : Z_K(\bar{k}^Q_{(2)}) \longrightarrow Z_K(\bar{k}^Q_{(2)}) \) defined on homogeneous potentials \( m \) of degree \( d + 1 \) by:
\[
\varepsilon_{(l)} m = \sum_{\ell=0}^d \varepsilon_l^l m = \sum_{\ell=0}^d \varepsilon_l^r m .
\]
Consequently, there is a cyclic derivative operator \( \partial : k^Q \otimes_{Z(K)} Z_K(\bar{k}^Q_{(2)}) \otimes Z_K(\bar{k}^Q_{(2)}) \otimes_{Z(K)} k^Q \longrightarrow \bar{k}^Q \) such that \( \partial (\xi \otimes m) = \partial^l (\xi \otimes \varepsilon_{(l)} m) = \partial^r (\varepsilon_{(l)} m \otimes \xi) \) for all \( \xi \in k^Q \) and \( m \in Z_K(\bar{k}^Q_{(2)}) \). Hence the action of the cyclic derivative on each potential \( m \) is the bimodule morphism \( \partial m : k^Q \longrightarrow \bar{k}^Q \) with \( \partial m = \partial \varepsilon_{(l)} m = \partial^r \varepsilon_{(l)} m \).

(4) Let \( \xi \in B^{(s)} \) and \( \zeta \in B^{(t)} \) with \( 1 \leq s, t \in \mathbb{N} \). On the \( Z(K) \)-module of potentials we have:
\[
\partial((\xi \otimes \zeta) \circ -) - \partial^r ((\xi \otimes \zeta) \circ \varepsilon_{(l)} (-)) = \partial^r (\varepsilon_{(l)} (\zeta \otimes -) \circ \xi) = \partial^r (\varepsilon_{(l)} (\xi \otimes -) \circ \zeta) = \partial (\varepsilon_{(l)} (\xi \otimes -) \circ (\zeta \otimes -)) = \partial (\varepsilon_{(l)} (\xi \otimes -) \circ (\zeta \otimes -)) = \partial (\varepsilon_{(l)} (\xi \otimes -) \circ (\zeta \otimes -)) = \partial (\varepsilon_{(l)} (\xi \otimes -) \circ (\zeta \otimes -)) .
\]

13
Proof. Statement (1). Let $m$ be a potential on $Q$ and $l \in N$. For Casimir morphisms $\zeta_{i_0} : K \rightarrow B(l) \otimes B^*(l)$ and $\zeta'_{i_0} : K \rightarrow B^*(l) \otimes B(l)$, as in (3.2) we have $\zeta_i(1) = \sum_{y \in \epsilon_i(1)} y \otimes y^*$ and $\zeta'_i(1) = \sum_{x \in \epsilon_i(1)} x^* \otimes x$, where \((\epsilon_i, \epsilon'_i)\) is left and right projective bases for the bimodule $B(l)$ and its dual. By definition $\partial^\epsilon(\epsilon_i^l(m)) = \partial^\epsilon \circ (\epsilon_i^l(m) \otimes 1)$. Definition 3.8 shows that $\epsilon_i^l(m) = \partial^\epsilon \circ (1 \otimes m \otimes 1) \circ \gamma_i$. Using (3.7) and identities (3.8) from Remark 3.7, for all $\xi \in B^*(l)$ we have:
\[
(\partial^\epsilon(\epsilon_i^l(m))) = \partial^\epsilon \left( \sum_{x \in \xi} x^* \otimes \epsilon_i^l(m) \otimes x \right) = \partial^\epsilon \left( \sum_{x \in \xi} x^* \otimes \epsilon_i^l(m) \right) = \partial^\epsilon(\xi \otimes \epsilon_i^l(m)) = (\partial^\epsilon m)(\xi).
\]
Thus $\partial^\epsilon(\epsilon_i^l(m)) = \partial^\epsilon m$ on $B^*(l)$. Similarly, one proves using $\gamma_i$ that $\partial^\epsilon(\epsilon_i^l(m)) = \partial^\epsilon m$ on $B^*(l)$.

Statement (2). Let $m$ be any potential on $Q$. By statement (1) and identities (3.8) we have:
\[
epsilon_i^l(m) = \partial^\epsilon \circ (1 \otimes \epsilon_i^l(m) \otimes 1) \circ \gamma_i = (1 \otimes \partial^\epsilon \circ (\epsilon_i^l(m) \otimes 1)) \circ \gamma_i = (1 \otimes \partial^\epsilon \circ \epsilon_i^l(m)) \circ \zeta_i = (1 \otimes \partial^\epsilon \circ \epsilon_i^l(m)) \circ \zeta_i = \partial^\epsilon \left( \sum_{x \in \xi} x^* \otimes \epsilon_i^l(m) \right) = \partial^\epsilon(\xi \otimes \epsilon_i^l(m)) = (\partial^\epsilon m)(\xi).
\]

Statement (3). To show that the cyclic permutation operator $\epsilon_c$ is properly defined, it suffices to consider the case of an homogeneous potential $m$ of degree $d + 1$ with $d \geq 1$. Statement (2) and the cyclical stability of $m$ show that $\epsilon_i^d(m) = m$ and $\sum_{l=0}^d \epsilon_i^l(m) = \sum_{l=0}^d \epsilon_i^d \epsilon_i^{d-1} \epsilon_i^{d-2} \cdots \epsilon_i^1 \epsilon_i^0 = m$, thus $\epsilon_c$ is properly defined. For the existence of the cyclic derivative operator, consider an arbitrary potential $m$. Observe that the cyclically stability also shows that $\epsilon_i^l \epsilon_i^c = \epsilon_c = \epsilon_i^c \epsilon_i$ for all natural number $l$. Let $\xi = \sum_{l=0}^n \xi_l \in kQ^*$ and $x \in kQ$ such that $\partial^\epsilon(\xi \otimes \epsilon_c m) = \sum_{l=0}^n \partial^\epsilon(\xi_l \otimes \epsilon_i^l(m)) = \sum_{l=0}^n \partial^\epsilon(\epsilon_i^l(m) \otimes \xi_l) = \partial^\epsilon(\epsilon_c m \otimes \xi)$, establishing the existence and the definition of the cyclic derivative operator.

Statement (4). We apply the definition of the cyclic derivative and the fact the left derivative and the right derivative pointwise commute. Let $\zeta \in B^*(kQ)$, $\epsilon' \in B^*(l)$, on the $Z(K)$-module of potentials we have:
\[
\partial^\epsilon((\zeta \otimes \epsilon_c(i))) = \partial^\epsilon(\zeta \otimes \epsilon_c(i)) = \partial^\epsilon(\zeta \otimes (\epsilon_i^l(i))) = \partial^\epsilon(\zeta \otimes \epsilon_i^l(m) \otimes \epsilon_i^l(i)) = \partial^\epsilon(\zeta \otimes \epsilon_i^l(m)) = \partial^\epsilon(\zeta \otimes \epsilon_i^l(m)) = \partial^\epsilon(\zeta \otimes \epsilon_i^l(m)) = \partial^\epsilon(\zeta \otimes \epsilon_i^l(m)) = \partial^\epsilon(\zeta \otimes \epsilon_i^l(m)).
\]

This establishes the identities of statement (4) and completes the proof of the proposition. \(\square\)

In the sequel, for $\xi \in kQ^*$ and $x \in kQ$ we put:
\[
\partial^\epsilon x = \partial^\epsilon(\xi \otimes x), \quad \partial^\epsilon x = \partial^\epsilon(\xi \otimes x) \quad \text{and} \quad \partial^\epsilon x = \partial^\epsilon(\xi \otimes x) = \partial^\epsilon(\xi \otimes x).
\]

Definition 3.9. For a potential $m \in Z_K(kQ_2)$, the closure $J_m := (\partial^\epsilon m)(B^*)$ of the ideal $\langle \partial^\epsilon m \rangle(B^*)$ in $kQ$ is the Jacobi ideal of $m$, the corresponding Jacobian algebra is $J_m = \mathcal{J}(Q, m) = kQ_2 J_m$.

Definition 3.10. The $Z(K)$-module skew $[Z_K(kQ_2), Z_K(kQ_2)]$ of skew commutators consists of finite sums of potentials of the form $m \circ \epsilon_c m$ (or equivalently, of the form $m \circ \epsilon_c m$). The closed $Z(K)$-module of skew commutators is the closure skew $[Z_K(kQ_2), Z_K(kQ_2)]$ of skew $[Z_K(kQ_2), Z_K(kQ_2)]$. Two potentials $m$ and $m'$ are cyclically equivalent whenever $m \sim m'$ lies in skew $[Z_K(kQ_2), Z_K(kQ_2)]$, in this case we have $\partial^\epsilon m = \partial^\epsilon m'$ for all $\xi \in B^*$, $J_m = J_{m'}$, and $\mathcal{J}(Q, m) = \mathcal{J}(Q, m')$. \(\square\)
3.4. Intrinsic description of potentials, modulated quivers with potentials

Lemma 3.3. Let \( \{M, M^*; b\} \) be a symmetrizable weakly dualizing pair, \( \{(U, U^*; \rho) \) and \( \{\overline{U}, \overline{U}^*; \overline{\rho}\} \) two symmetrizable pairings over the symmetric algebra \((K, t)\). Let \((\epsilon) : 0 \rightarrow U \xrightarrow{f} M \xrightarrow{g} \overline{U} \rightarrow 0 \) and \((\epsilon^*) : 0 \rightarrow \overline{U} \xrightarrow{g^*} M^* \xrightarrow{f^*} U^* \rightarrow 0 \) be mutually dual exact sequences of K-bimodule morphisms such that \((\epsilon)\) splits as sequence of left K-linear maps. Then, the dual sequence \((\epsilon^*)\) splits as sequence of right K-linear maps and the following holds. Let \(U \xrightarrow{f} M \xrightarrow{g} \overline{U} \) be left K-linear morphisms such that the map \(h := [f, g^*] : U \oplus \overline{U} \rightarrow M\) is a left K-linear isomorphism with inverse \(h^{-1} = \left( f^*, g \right) : M \rightarrow U \oplus \overline{U}\), then \(f^*\) and \(g^*\) are left dualizing, \(h = \left( f^*, g \right) : M^* \rightarrow U^* \oplus \overline{U}^*\) and \(h^{-1} = \left[ f^*, g \right]\). Moreover, if \((\epsilon)\) also splits as sequence of right K-linear maps then \(\{U, U^*; \rho\}\) and \(\{\overline{U}, \overline{U}^*; \overline{\rho}\}\) are weakly dualizing pairs.

Proof. By assumption on \((\epsilon)\), let \(U \xrightarrow{f} M \xrightarrow{g} \overline{U} \) be left K-linear morphisms with \(f^* f = 1_U\) and \(g g^* = 1_{\overline{U}}\) and \(f^* g = 0\). Part (2) of Lemma 2.1 states that \(f^*\) is left dualizing, and clearly \(f^* \circ f = 1_U\), implying (using basic module theory) that the dual sequence \((\epsilon^*)\) splits as sequence of right K-linear morphisms and there is a unique right K-linear morphism \(g^* : M^* \rightarrow \overline{U}^*\) such \(1_M = f^* g^* + g^* g\), \(g^* g = 1_{\overline{U}^*}\) and \(g^* f = 0\). Next, we must check that \(g^* : M^* \rightarrow \overline{U}^*\) also serves as left dual for \(g^* : U \rightarrow M^*\). Let \(\xi \in M^*\) and \(\overline{\xi} = \varphi(x) \in \overline{U}\) with \(x \in M\). Then, we have \(\xi M^* \rightarrow (\xi^*)^* = \overline{\xi}\) is left dualizing, and clearly \(f^* \circ f = 1_U\), thereby \(f^*\) is a bimodule morphism while \(g^*\) is right K-linear. We want to compute the composition: \(\theta := \tilde{\varphi} \circ \varphi^{-1} : U^* \oplus \overline{U}^* \rightarrow U \oplus \overline{U}\). Writing each element in \(U^* \oplus \overline{U}^*\) as formal sum \(v + \zeta\) with \(v \in U^*\) and \(\zeta \in \overline{U}^*\), we get:

\[
\theta(v + \zeta) = \left( \Phi \rightarrow v \right) \left( \left( \varphi(-) \circ (f^*(v) + g^*(\zeta)) \right) \right) = \tilde{\varphi}(\varphi(-) \circ (f^*(v) + g^*(\zeta)))
\]

Thus \(\theta\) is the direct sum of \(\rho_{\varphi} : U^* \rightarrow U^*\) and \(\rho_{\tilde{\varphi}} : \overline{U}^* \rightarrow \overline{U}^*\), implying that these adjoint maps are isomorphisms. Since \(M\) is projective as left and right \(K\)-module, so are \(U\) and \(\overline{U}\). Thus, for the last statement of the lemma, if \((\epsilon)\) also splits as sequence of right K-linear maps, then \(U\) and \(\overline{U}\) are also projective as right K-modules, and we deduce that \(\{U, U^*; \rho\}\) and \(\{\overline{U}, \overline{U}^*; \overline{\rho}\}\) are weakly dualizing.

The following result shows that potentials appear as Casimir elements provided some splitting conditions hold.

Proposition 3.4. (1) Let \(\{M, M^*; \beta\}\) and \(\{M', M'^*; \beta'\}\) be symmetrizable dualizing pairs over \((K, t)\), and \(m : K \rightarrow M \otimes M'\) an homogeneous potential. Then the right derivative \(\partial m : M'^* \rightarrow M\) and the left derivative \(\partial m : M^* \rightarrow M'\) are mutually dual morphisms; they induce three symmetrizable pairings \(\{U, V; \gamma\}, \{\overline{U}, \overline{U}^*; \overline{\beta}\} \) and \(\{\overline{V}, \overline{V}^*; \overline{\beta}\}\) with \(U = \text{Im}(\partial m), V = \text{Im}(\partial m), U^* = \text{Ker}(\partial m)\), \(\overline{U} = M U, \overline{V} = \text{Ker}(\partial m)\) and \(\overline{V} = M' V, \overline{V}^* = \text{Ker}(\partial m)\), together with the following pairs of mutually dual exact sequences of canonical injections and projections:

\[
(\epsilon) : \overline{U} \xrightarrow{f} M \xrightarrow{g} \overline{U} \quad \text{and} \quad (\epsilon') : \overline{V} \xrightarrow{f} M' \xrightarrow{g} \overline{V}.
\]
Moreover, if \((e^*)\) and \((e')\) split as sequences of left and right K-linear maps, then so does \((e)\) and \((e')\) and, \(\{U, V; \gamma\}, \{\ol{U}, \ol{U} \, ; \ol{\beta}\}\) and \(\{\ol{V}, \ol{V} \, ; \ol{\beta}\}\) are dualizing. In this case, let \(h \, := \, [1, \ol{\beta}] : \, U \oplus \ol{U} \to M\) be a right K-linear isomorphism with inverse \(h^{-1} \, = \, [\beta]\), and \(h' \, := \, [1, \ol{\beta}] : \, V \oplus \ol{V} \to M'\) a left K-linear isomorphism with inverse \(h'^{-1} \, = \, [\beta']\), then \((h^{-1})^* \, = \, [\rho', 1]\), \((h'^{-1})^* \, = \, [\rho, 1]\) and we have:

\[
\begin{align*}
\mathcal{J}_{M^* \otimes M'} &= (h \otimes (h^{-1})^*)(\mathcal{J}_{U \otimes V} \otimes (V \otimes \ol{V})) = (1 \otimes \rho')(\mathcal{J}_{U \otimes V}) + (\ol{\beta} \otimes 1)(\mathcal{J}_{V \otimes \ol{V}}), \\
\mathcal{J}_{M^* \otimes M'} &= (h'^{-1} \otimes h')(\mathcal{J}_{U \otimes V} \otimes (V \otimes \ol{V})) = (\rho' \otimes 1)(\mathcal{J}_{U \otimes V}) + (\ol{\beta} \otimes 1)(\mathcal{J}_{V \otimes \ol{V}}) \quad \text{and} \quad m = \mathcal{J}_{U \otimes V}.
\end{align*}
\]

(2) Let \(W \in B^{(2)}\) be a potential. Then the cyclic derivative \(B^* \xrightarrow{\partial W} B\) is a self-dual morphism inducing two symmetric pairings \(\{B_0, B_0\}\) and \(\{\ol{B}, \ol{B}\}\) with \(B_0 = \text{Im}(\partial W), \, \ol{B} = B B_0\) and \(B' = \text{Ker}(\partial W)\), together with mutually dual exact sequences of canonical injections and projections: \((\partial) : \, \ol{B} \xrightarrow{\ol{\beta}} B \xrightarrow{\beta} B_0, \, (\partial^*) : \, B_0^* \xrightarrow{\rho} B \xrightarrow{\rho} \ol{B}\). Moreover, if \(B_0\) is a direct summand in \(B\) then \(Q = Q_{\text{triv}} \oplus \ol{Q}\) as direct sum of two modulated quivers, with \(Q = (B, K, t), \, Q_{\text{triv}} = (B_0, K, t)\) and we have \(e_i W = \mathcal{J}_{q_0 \otimes q_0} \in B^{(2)}\).

_Proof._ Since part (2) is a direct application of part (1), we only need to prove part (1). But then, in view of Lemma 2.3 and Lemma 3.3, it suffices to establish the first part of (1) and the identity \(m = \mathcal{J}_{U \otimes V}\) in the second part of (1). To start, we must show that the left derivative \(f : = \partial m : M^* \to M\) and the right derivative \(f' : = \partial m : M^* \otimes M\) are mutually dual. Once again by Lemma 2.2, \(f\) and \(f'\) are already dualizing morphisms. Let \(\xi \in M^*, \, \xi' \in M'^*, \) we have: \(\beta(\xi \otimes f'(\xi')) = \beta(\xi) f(\xi) = \xi \partial y \partial \xi m = \partial y \partial \xi m = \beta(\xi, \partial y \partial \xi, m), \) showing in view of Lemma 2.2 that \(f' = \partial y \partial \xi = f\). Next, the pair \((f, f')\) clearly induces a well-defined bilinear form \(\gamma : \, U \otimes V \otimes V \to K\) such that: \(\gamma(f(\xi) \otimes f'(\xi')) = : \beta(\xi \otimes f'(\xi')) = \beta(\xi' \otimes f(\xi))\) and \(\gamma(f(\xi')) \circ f(\xi)\) for all \(\xi, \xi' \in M^*\), \(\xi' \in M'^*\), and \(\gamma\) is also non-degenerate, since \(\beta\) and \(\beta'\) are (strongly) non-degenerate. We get that the data \(\{V, U, \gamma\}\) is a symmetric pairing over \((K, t)\). In the same way, that there are canonically induced symmetrized pairing \(\{U, \ol{U}, \xi \partial y \partial \xi\} \) and \(\{V, \ol{V}, \xi \partial y \partial \xi\} \) over \((K, t)\).

Now, we want to show that the sequence \((e)\) is a self-dual morphism in \((K, t)\). Since the projection defined by \(f\) is dualizing and \((e')\) is the short exact sequence \(U^* \to M^* \to M^* \otimes M\) defined by the sub submodule \(U \subset M\). For all \(r \in U \subset M\) and \(\xi \in M^*\), we have: \(\gamma(u \otimes p(\xi)) = \gamma(u \otimes f(\xi)) = \beta(u, \xi)\), and we have: \(B(\xi, x)\). showing that the inclusion \(\ol{U} \to M\) is the right dual of \(p\) and hence the dual of \(p\) in view of Lemma 2.2. For all \(q_0 \in \ol{U}\) and \(x \in M\), writing \(\ol{\beta}\) for the coset of \(x \) in \(\ol{U} = M U\), by definition we have: \(\ol{\beta}(q_0, x) = \beta(q_0, x)\), showing that the inclusion \(\ol{U} \to M\) is dualizing and its dual is the canonical projection \(\partial : \, M \to M U\). In the same way, on can check that there are mutually dual exact sequences \((e')\) and \((e')\) over \((K, t)\).

As said before, the proof of the result of (1), except for the relation \(m = \mathcal{J}_{U \otimes V}\) is given by Lemmas 2.3 and 3.3. But, writing \(\mathcal{J}_{M \otimes M} = \sum_{r=1}^n y_r \otimes y_r\) and \(\mathcal{J}_{M^* \otimes M'} = \sum_{r=1}^n y_r' \otimes y_r'\) for the Casimir elements in \(M \otimes M^*\) and \(M^* \otimes M'\), by (3.8) we have \(m = \sum_{r=1}^n y_r \otimes \partial y_r m = (1 \otimes p)(\mathcal{J}_{M \otimes M})\) and \(m = \sum_{r=1}^n (\partial y_r \partial y_r) = (1 \otimes p')(\mathcal{J}_{M^* \otimes M'})\). Hence, the equality \(m = \mathcal{J}_{U \otimes V}\) follows from the relations \(\mathcal{J}_{M \otimes M} = (1 \otimes p')(\mathcal{J}_{U \otimes V}) + (\ol{\beta} \otimes 1)(\mathcal{J}_{V \otimes \ol{V}})\) and \(\mathcal{J}_{M^* \otimes M'} = (p' \otimes 1)(\mathcal{J}_{U \otimes V}) + (\ol{\beta} \otimes 1)(\mathcal{J}_{V \otimes \ol{V}})\). 

For a potential \(m\) on \(Q\) with degree-2 component \(m_2 \in B \otimes B\), let \(B_{\text{triv}} = \partial (B^* \otimes m_2)\). Then we have an induced symmetrized pairing \(\{U, V\}\) with \(U = (\partial m_2)(B^*), \) \(V = (\partial m_2)(B^*)\) and \(B_{\text{triv}} \subseteq U \oplus V\). Thus, if \(U\) and \(V\) are also projective as left and right K-modules then the pair \(\{U, V\}\) is dualizing.

**Definition 3.11.** The potential \(m\) is 2-loop free if as left and right K-module \(B_{\text{triv}}\) is a direct summand in \(B\) and \(U \cap V = 0\). In this case, the paring \(\{U, V\}\) is dualizing, \(m_2 = \mathcal{J}_{U \otimes V} \in U \otimes V\) is a Casimir element and \(B_{\text{triv}} = U \oplus V\); the data \((Q, m)\) is called a modulated quiver with potential.
4. Reduction of modulated quivers with potentials

The main results in section require some preparation about Jacobian ideals.

4.1. The cyclic Leibniz rule and the chain-rule.

In the study of quivers with potentials, the cyclic Leibniz rule is an easy consequence of the fact that any simply-laced path algebra has a "symmetric" path k-basis and the computation of cyclic derivatives only requires the ordinary cyclic permutation of arrows in the quiver. However, such a symmetry is generally absent in the present framework. Thanks to properties of symmetrizable dualizing pairs, the following result controls skew permutations of potentials along morphisms of K-bimodules.

**Proposition 4.1.** Let \( f : \mathbb{K}_q(U) \to \mathbb{K}_q(V) \) and \( h : \mathbb{K}_q(V) \to \mathbb{K}_q(W) \) be path algebra morphisms, \( \{U, U^*; \beta\} \) and \( \{V, V^*; \mu\} \) symmetrizable dualizing pairs over \((K, t)\). Let \( W = \sum_{k=1}^q y_k \otimes v_k \) and \( S = \sum_{k=1}^q u_k \otimes x_k \) be potentials with \( y_k \in U, x_k \in V, u_k \in \mathbb{K}_q(U), v_k \in \mathbb{K}_q(V) \). Then, for all \( l \geq 1 \) and potentials \( (f_1 \otimes h)(W) = \sum_{k=1}^q f_1(y_k) \otimes h(v_k), \ (f \otimes h)(S) = \sum_{k=1}^q f(y_k) \otimes h(x_k), \ (f \otimes h)(W) \) and \( (f \otimes h)(S) \) we have:

\[
(h \otimes f)(\varepsilon, W) = \sum_{k=1}^q \sum_{\ell=1}^N \frac{b(x^* \otimes f(y_k))}{y} h(v_k) \otimes x = \sum_{\ell=1}^N \varepsilon^l \sum_{k=1}^q f_1(y_k) \otimes h(v_k) = \sum_{\ell=1}^N \varepsilon^l (f_1 \otimes h)(W)
\]

\[
(h \otimes f)(\varepsilon, S) = \sum_{y \in \mathcal{Q}_0} y \otimes \sum_{k=1}^q f(y_k) b(h(x_k) \otimes y^*) = \sum_{l \in \mathcal{N}_*} \varepsilon^l \sum_{k=1}^q f(y_k) \otimes h(x_k) = \sum_{l \in \mathcal{N}_*} \varepsilon^l (f \otimes h)(S)
\]

Moreover, any morphism \( \phi : \mathbb{K}_q(U) \to \mathbb{K}_q(W) \) of path algebras over the same symmetric algebra \((K, t)\) sends cyclically equivalent potentials to cyclically equivalent ones.

**Proof.** Fix a natural number \( l \geq 1 \), in view of (3.2) we have Casimir morphisms \( \beta_{(j)} : K \to B^{(l)} \otimes B^{(l)}, \beta_{(j)}^l : K \to B^{(l)} \otimes B^{(l)} \) with \( \beta_{(j)} = \sum_{y \in \mathcal{Q}_0} y \otimes y^* \) and \( \beta_{(j)}^l = \sum_{y \in \mathcal{Q}_0} x^* \otimes x \). We also consider the Casimir elements \( \beta_{u*} \equiv \sum_{u \in \mathcal{S}} u \otimes u^* \) and \( \beta_{u*v} \equiv \sum_{v \in \mathcal{S}'} v^* \otimes v \) associated with the dualizing pairs \( \{U, U^*; \beta\} \) and \( \{V, V^*; \mu\} \). Let us prove that (4.1) holds. By Lemma 2.2, each bimodule morphism \( f_1 : U \to B^{(l)} \) is dualizing and its dual \( f_1^* : B^{(l)} \to U^* \) is characterized by the relations: \( \beta(u \otimes f^*(\xi)) = b'((f(u) \otimes \xi) \otimes f(u)) \) for all \( u \in U \) and \( \xi \in B^{(l)} \). Let \( m_t = \sum_{k=1}^q f_1(y_k) \otimes h(v_k) \) be K-central as image of \( m_t \).

In the following computation of \( \varepsilon^l m_t \), use (2.2) for \( f_1^*(x^*) \) in the second row and \( f_1(u) \) in the height row:
3.1, \( b \) is dense in \( 0 \). In the next step we develop a differential calculus on potentials. Consider the topological

For the last part of the proposition, let \( \phi : k\mathcal{Q} \to k\mathcal{Q} \) be a morphism of path algebras. As in Proposition 3.1, \( \phi \) is continuous and induced by a family of \( K \)-bimodule morphisms \( \phi_t : B \to B(t) \), \( t \geq 1 \). For all potential \( m = \sum_{k=1}^{q} y_k \otimes v_k \in B(d) \) with \( y_k \in B \), letting \( m_t := \sum_{k=1}^{q} \phi(y_k) \otimes \phi(v_k) \) yields that:

\[
\phi(m - \varepsilon, m) = \phi(m) - \phi(\varepsilon, m) = \sum_{t \in \mathbb{N}^*} m_t - \sum_{t \in \mathbb{N}^*} \varepsilon_t m_t = \sum_{t \in \mathbb{N}^*} (m_t - \varepsilon_t m_t).
\]

Therefore, \( \phi \) sends any skew commutator to an element of the closed \( Z(K) \)-module of skew commutators in \( k\mathcal{Q} \), and since \( \phi \) is continuous we conclude that \( \phi \) sends the closed \( Z(K) \)-module of skew commutators in \( k\mathcal{Q} \) to the closed \( Z(K) \)-module of skew commutators in \( k\mathcal{Q} \). Hence \( \phi \) sends cyclically equivalent potentials to cyclically equivalent ones.

In the next step we develop a differential calculus on potentials. Consider the topological \( K \)-bimodule 

\[
k\mathcal{Q} \otimes k\mathcal{Q} = \prod_{d \geq 0} (B(d) \otimes_k B(e)),
\]

having as system of open neighborhoods of 0 the subbimodules \( \prod_{d+e \geq m} (B(d) \otimes_k B(e)), m \geq 0 \). Thus \( k\mathcal{Q} \otimes k\mathcal{Q} \) is dense in \( k\mathcal{Q} \otimes k\mathcal{Q} \). When we fix projective bases \((k\mathcal{Q}^*, k\mathcal{Q}^*)\) and \((k\mathcal{Q}^+, k\mathcal{Q}^+)\), we equally lift the corresponding Casimir morphisms to the following: \( \tilde{f}_{(l)} : k \to B(l) \otimes B(l) \) and \( \tilde{f}_{(l)} : k \to B^*(l) \otimes B(l) \), with \( \tilde{f}_{(l)} = \sum_{y \in k\mathcal{Q}} y \otimes y^* \) and \( \tilde{f}_{(l)} = \sum_{x \in k\mathcal{Q}} x^* \otimes x \). Left and right derivative operator on \( k\mathcal{Q} \) are naturally extended to derivative operators \( \partial^L, \partial^R : k\mathcal{Q}^* \otimes (k\mathcal{Q} \otimes k\mathcal{Q}) \to k\mathcal{Q} \otimes k\mathcal{Q} \) as follows: for all \( \xi \in k\mathcal{Q}^* \) and \( v_1 \otimes v_2 \in k\mathcal{Q} \otimes k\mathcal{Q} \) we have

\[
\tilde{f}_{(l)} := \tilde{f}_{(l)}(1_{k\mathcal{Q}}) \otimes 1_{k\mathcal{Q}})
\]


\[ \partial_{v_1}(v_1 \otimes v_2) = \partial'((v_1 \otimes v_2) \otimes \xi) := v_1 \otimes (\partial_{v_2}) \]

Assume that the homogeneous potential. The cyclic permutation of \( k, v \in B(l), l \in \mathbb{N}, \) we have:

\[ \Delta^k_0(v) = \sum_{l \in \mathbb{N}} \Delta^k_0(v_1) = \partial^k_1 \sum_{l \in \mathbb{N}} \delta_{v}(y_{\partial^k_1}(v_1) \otimes y_{\partial^k_1}(v_2)), \]

\[ (u_{\partial^k_1} v) \square w = u \cdot w \cdot v. \]

**Notations.** Let \( d, k \geq 1, \) for a product \( u_{k,0} \cdot u_{k,1} \cdots u_{k,d} \) of elements of \( \mathbb{kQ}, \) we put: \( u_{k,0} := 1 = u_{k,d}, \)

\( u_{k,r} := u_{k,0} \cdots u_{k,r-1} \) and \( u_{k,r} := u_{k,r+1} \cdots u_{k,d} \) for \( 0 < r < d. \) Recall that we have a weakly dualizing pair \( \{(\mathbb{kQ}^*), \mathbb{kQ}, \mathbb{b}\}, \) we also put \( \mathbb{b} := \mathbb{b}^0 : K \otimes K \rightarrow K \) is the multiplication of \( K. \)

**Lemma 4.2** (cyclic Leibniz rule). Let \( m = \sum_{k=1}^{n} u_{k,0} \cdot u_{k,1} \cdots u_{k,d} \) be a potential on \( Q \) with \( d \geq 1 \) and \( u_{k,r} \in \mathbb{kQ}. \) Then for all \( \xi \in B^* \) the following cyclic Leibniz rule holds:

\[ \partial_{x} m = \sum_{r=0}^{d} \sum_{x \in \mathbb{kQ}} \sum_{k=1}^{n} \Delta^k_0(\mathbb{b}_r(x^* \otimes u_{k,r})u_{k,r}) \square (u_{k,r} \cdot x) \]

\[ = \sum_{r=0}^{d} \sum_{x \in \mathbb{kQ}} \sum_{k=1}^{n} \Delta^k_0(\mathbb{b}_r(x^* \otimes u_{k,r} \otimes y^*) \square (y_{u_{k,r} \otimes y^*}). \]

**Proof.** For \( k \in \{1, \ldots, n\}, \) let \( u_{k} = \sum_{l \in N_{\mathbb{kQ}}} u_{k,r,l} \) with \( u_{k,r,l} \in B(l), l \geq 1. \) Recall that \( \mathbb{rQ} = \bigcup_{l \geq 1} \mathbb{rQ}_{l} \) with \( \mathbb{rQ}_{l} = \{1\} \in \mathbb{rQ}^* \) and in view (3.3), for all \( x \in \mathbb{rQ}_{l} \) and \( u \in \mathbb{kQ} \) we have \( \mathbb{b}^0(x^* \otimes u) = 0. \) We will use induction on \( d \geq 1 \) to establish (L) \( : \partial_{x} m = \sum_{r=0}^{d} \sum_{x \in \mathbb{kQ}} \sum_{k=1}^{n} \Delta^k_0(\mathbb{b}_r(x^* \otimes u_{k,r})u_{k,r} \square (u_{k,r} \cdot x). \]

Assume that \( d = 1. \) Then \( m = \sum_{l \in N_{\mathbb{kQ}} \cdots} \sum_{k=1}^{n} u_{k,0,1} \cdot u_{k,1,1} \cdot x = \sum_{p \geq 2} m_p \) where \( m_p := \sum_{l \in N_{\mathbb{kQ}}} \sum_{k=1}^{n} u_{k,0,1} \cdot u_{k,1,1} \cdot x \) is a degree-\( p \) homogeneous potential. The cyclic permutation of \( m \) is given by

\[ \varepsilon_{p} m = \sum_{p \geq 2} \varepsilon_{p} m_p = \sum_{p \geq 2} \sum_{l \in N_{\mathbb{kQ}}} \varepsilon_{l} m_p = \sum_{p \geq 2} \sum_{l \in N_{\mathbb{kQ}}} \sum_{k=1}^{n} \partial(x^* \otimes u_{k,0,1} \cdot u_{k,1,1} \cdot x) \]

\[ = \sum_{p \geq 2} \sum_{l \in N_{\mathbb{kQ}}} \sum_{k=1}^{n} \partial(x^* \otimes u_{k,0,1} \cdot u_{k,1,1} \cdot x) + \sum_{p \geq 2} \sum_{l \in N_{\mathbb{kQ}}} \sum_{k=1}^{n} \partial(x^* \otimes u_{k,0,1} \cdot u_{k,1,1} \cdot x). \]

Let \( \varepsilon_{p} m = S_1 + S_2 \) where \( S_1 \) is the first term in the last line above and \( S_2 \) the second. In view (4.3) defining operators \( \Delta^k_0 \) and \( \square, \) we compute the left derivative \( \partial_{x} S_1 \) as follows.

\[ \partial_{x} S_1 := \partial_{x} \sum_{p \geq 2} \sum_{l \in N_{\mathbb{kQ}}} \sum_{k=1}^{n} \partial(x^* \otimes u_{k,0,1} \cdot u_{k,1,1} \cdot x) \]

\[ = \partial_{x} \sum_{l \in N_{\mathbb{kQ}}} \sum_{k=1}^{n} \partial(x^* \otimes u_{k,0,1} \cdot u_{k,1,1} \cdot x) \]

\[ = \sum_{l \in N_{\mathbb{kQ}}} \sum_{k=1}^{n} \partial(x^* \otimes u_{k,0,1} \cdot u_{k,1,1} \cdot x) = \sum_{l \in N_{\mathbb{kQ}}} \sum_{k=1}^{n} \partial(x^* \otimes u_{k,0,1} \cdot u_{k,1,1} \cdot x) \]

\[ = \sum_{k=1}^{n} \Delta^k_0(u_{k,0,1}) \square u_{k,1,1} = \sum_{x \in \mathbb{kQ}} \sum_{k=1}^{n} \Delta^k_0(\mathbb{b}_0(x^* \otimes u_{k,0,1})u_{k,0,1}) \square (u_{k,0,1} \cdot x). \left( * \right) \]

And, using (3.5) describing projective bases associated with \( Q, \) for all \( t = l + s \geq l \) with \( 0 \leq s < t', \) we have \( \mathbb{rQ}_{l} = \mathbb{rQ}_{l} \otimes \mathbb{rQ}_{s} := \{x \otimes z : x \in \mathbb{rQ}_{l}, z \in \mathbb{rQ}_{s}\}; \) thus the left derivative \( \partial_{x} S_2 \) is computed as follows.
\[ \partial_\xi S_2 := \partial_\xi \sum_{p \geq 1} \sum_{l' = p}^{l - 1} \sum_{x \in \mathfrak{Q}_l} \sum_{z \in \mathfrak{Q}_l'} \mathfrak{P}(x^* \otimes u_{k,0,l} u_{k,1,l'}) x \]

\[ = \partial_\xi \sum_{l' = 0}^{l - 1} \sum_{x \in \mathfrak{Q}_l} \sum_{z \in \mathfrak{Q}_l'} \mathfrak{P}(z^* \otimes x^* \otimes u_{k,0,l} u_{k,1,l'}) x z \]

\[ = \partial_\xi \sum_{l' = 0}^{l - 1} \sum_{x \in \mathfrak{Q}_l} \sum_{z \in \mathfrak{Q}_l'} \mathfrak{P}(z^* \otimes \tilde{b}(x^* \otimes u_{k,0,l}) u_{k,1,l'}) x z \]

\[ = \partial_\xi \sum_{l' = 0}^{l - 1} \sum_{x \in \mathfrak{Q}_l} \sum_{z \in \mathfrak{Q}_l'} \mathfrak{P}(z^* \otimes (\sum_{x \in \mathfrak{Q}_0} \tilde{b}(x^* \otimes u_{k,0,l})) u_{k,1,l'}) x z \]

Hence, combining (**) and (*) above, (L) is proved for \( d = 1 \). For the induction step, assume \( d > 1 \) and the result true for \( d - 1 \). We write: \( \mathfrak{m} = \sum_{k = 1}^{n} (u_{k,0} u_{k,1} u_{k,2} \cdots u_{k,d}) \), by a direct application of the induction assumption and of the proof of the case \( d = 1 \) above we get:

\[ \partial_\xi \mathfrak{m} = \partial_\xi \sum_{k = 1}^{n} (u_{k,0} u_{k,1} u_{k,2} \cdots u_{k,d}) \]

\[ = \sum_{x \in \mathfrak{Q}_0} \sum_{k = 1}^{n} \Delta_\xi(\tilde{b}(x^* \otimes u_{k,0,l}) u_{k,1,l}) \triangledown x = \sum_{x \in \mathfrak{Q}_0} \sum_{k = 1}^{n} \Delta_\xi(\tilde{b}(x^* \otimes u_{k,0,l}) u_{k,1,l}) \triangledown x \]

Hence, (L) is proved. Dually, the Leibniz rule involving only the operator \( \Delta_\xi \) holds.

**Lemma 4.3** (cyclic chain-rule). Let \( \phi : \mathfrak{Q} \longrightarrow \mathfrak{Q}' \) be a morphism of path algebras for a given modulated quiver \( \mathfrak{Q}' = (B', K, t) \). Then for all potential \( \mathfrak{m} \) on \( \mathfrak{Q} \) and all \( \xi \in B'^* \) we have:

\[ \partial_\xi \phi(\mathfrak{m}) = \sum_{y \in \mathfrak{Q}_1} (\Delta_\xi \phi(y)) \triangledown \phi(\partial_\xi \mathfrak{m}) = \sum_{x \in \mathfrak{Q}_1} (\Delta_\xi \phi(x)) \triangledown \phi(\partial_\xi \mathfrak{m}). \tag{4.5} \]

**Proof.** Since each potential \( W \) decomposes as sum of homogeneous potentials, it suffices to prove the chain-rule for homogeneous potentials. Thus we may assume that \( \mathfrak{m} \) is homogeneous and write

\[ \mathfrak{m} = \sum_{k = 1}^{n} u_{k,0} u_{k,1} u_{k,2} \cdots u_{k,d} \in B^{d+1} \] for some \( d \geq 1 \) and \( u_{k,r} \in B \), \( r = 0, 1, \ldots, d \). We have

\[ \phi(\mathfrak{m}) = \sum_{k = 1}^{n} \phi(u_{k,0} u_{k,1} \cdots u_{k,d}) \] and in view of (3.5) we have projective bases \((B'_t, \mathfrak{Q}'_t^*)\), \((\mathfrak{Q}'_t, \mathfrak{Q}'^*_t)\), \((\mathfrak{Q}'_s, \mathfrak{Q}'^*_s)\) and \((\mathfrak{Q}'_r, \mathfrak{Q}'^*_r)\) with \( t, s, r \in \mathbb{N} \) and \( \mathfrak{Q}'_t = \{1\} = Q'_s^* \) and \( \mathfrak{Q}'_r = \{1\} = Q'^*_r \). Now, let \( \xi \in B'^* \), we will establish the chain-rule (4.5) in terms of operator \( \Delta_\xi \). The cyclic Leibniz rule (4.2) yields that:
\[ \partial_v \phi(m) = \sum_{r=0}^{d} \sum_{x \in \mathcal{Q}_0} \sum_{k=1}^{n} \Delta^\dagger \left( \partial^* \left( x^{\ast \ast} \otimes \phi(u_{k,<r}) \right) \right) \phi(u_{k,r}) \square \left( \phi(u_{k,r}) x' \right) \]

\[ = \sum_{y \in \mathcal{Q}_1} \sum_{x' \in \mathcal{Q}_0} \sum_{r=0}^{d} \sum_{k=1}^{n} \Delta^\dagger \left( \phi(y) \right) \square \left( \partial^* \left( x^{\ast \ast} \otimes \phi(u_{k,<r}) \right) \right) \phi(u_{k,r}) \phi(u_{k,r}) x'. \]

\[ \text{For each } r > 0, \text{ we have dualizing pairs } \{ B(r), B^{(r)} \} \text{ and } \{ B \otimes B^{(d-r)}, B^{(d-r)} \otimes B^* \}, \text{ and for K-bimodule morphisms } f_r := \phi_{|_{B^r}} : B^r \rightarrow k\mathcal{Q} \text{ and } h_r := 1_B \otimes (\phi_{|_{B^{(d-r)}}}) : B \otimes B^{(d-r)} \rightarrow k\mathcal{Q}', \text{ we observe} \]

\[ (f_r \otimes h_r)(m) = \sum_{y \in \mathcal{Q}_1} \phi(u_{k,<r}) u_{k,r} \phi(u_{k,<r}). \]

whence, invoking relation (4.1) of Proposition 4.1 to control the left permutation of \( m \) with respect the pair \( \{ B(r), B^{(r)} \} \), we get:

\[ (h_r \otimes f_r)(\varepsilon^*_m) = \sum_{y \in \mathcal{Q}_1} \sum_{x' \in \mathcal{Q}_0} \sum_{k=1}^{n} \bar{b}^*_y \left( x^{\ast \ast} \otimes \phi(u_{k,<r}) \right) u_{k,r} \phi(u_{k,<r}) x'. \]

For each \( y \in \mathcal{Q}_1 \), let us compute the term \( S_{y,r} := \partial^* \left( \sum_{x' \in \mathcal{Q}_0} \sum_{k=1}^{n} \bar{b}^*_y \left( x^{\ast \ast} \otimes \phi(u_{k,<r}) \right) u_{k,r} \phi(u_{k,<r}) x' \right): \)

\[ S_{y,r} = \partial^* \left( \sum_{x' \in \mathcal{Q}_0} \sum_{k=1}^{n} \bar{b}^*_y \left( x^{\ast \ast} \otimes \phi(u_{k,<r}) \right) u_{k,r} \phi(u_{k,<r}) x' \right) = \partial^* \left( (h_r \otimes f_r)(\varepsilon^*_m) \right). \]

From (**) and (*) we get:

\[ \partial_v \phi(m) = \sum_{y \in \mathcal{Q}_1} \Delta^\dagger \left( \phi(y) \right) \square \left( \phi(\partial^*_y, \sum_{r=0}^{d} \varepsilon^*_m) \right) = \sum_{y \in \mathcal{Q}_1} \Delta^\dagger \left( \phi(y) \right) \square \left( \phi(\partial^*_y, m) \right) = \sum_{y \in \mathcal{Q}_1} \Delta^\dagger \left( \phi(y) \right) \square \left( \phi(\partial^*_y, m) \right). \]

Dually, the chain-rule in terms of operator \( \Delta^\dagger \) also holds. 

\[ 4.2. \text{The reduction process} \]

Throughout this subsection, let \( (\mathcal{Q}, m) \) be a modulated quiver with potential where \( \mathcal{Q} = (B, K, t) \). We have \( m = (m_l)_{l \geq 2} \) with \( m_l \in B(l) \); put \( m = m_2 + m_3 \) with \( m_3 = (m_l)_{l \geq 3} \). Let \( U = (\partial^* m_2)(B^*), V = (\partial^* m_2)(B^*) \),
\( B_{\text{triv}} = (\partial m_2)(B^*) \), \( \mathcal{B} = B/B_{\text{triv}} \) and \( \mathcal{B}' = \ker(\partial m_2) \subset B^* \). Applying Proposition 3.4 and the fact that \( m \) is 2-loop free (Definition 3.11), we get the following.

Remark 4.1. \( U \cap V = 0 \), \( B_{\text{triv}} = U \oplus V \) and over \((K,t)\) we have induced symmetrizable dualizing pairs \( \{U,V\} \), \( \{B_{\text{triv}}, B_{\text{triv}}^t\} \) and \( \{\mathcal{B}, \mathcal{B}'; \hat{\mathcal{B}}\} \), together with mutually dual canonical exact sequences

\[
\begin{align*}
&\vartheta^t : \mathcal{B}' \to B^* \to B_{\text{triv}} \\
&\vartheta : \mathcal{B} \to B \to B_{\text{triv}}
\end{align*}
\]

which split as sequences of left and right \( K \)-modules. For all \( \xi \in \mathcal{B}' \), \( x \in B_{\text{triv}} \) we have:

\[
\vartheta(\xi \otimes x) = \mathcal{B}(\xi \otimes \varrho(x)) = 0 = \mathcal{B}(\varrho(x) \otimes \xi) = \varrho(x \otimes \xi).
\]

Definition 4.2. The bimodule \( B_{\text{triv}} \) is the trivial part of \( B \), \( Q_{\text{triv}} := \{B_{\text{triv}}, K, t\} \) the trivial part of \( Q \) and \( (Q, m)^{\text{triv}} := (Q_{\text{triv}}, \rho) \) the trivial part of \( (Q, m) \). The bimodule \( \mathcal{B} := B/B_{\text{triv}} \) is the reduced part of \( B \) and \( Q_{\text{red}} = \mathcal{Q} := (\mathcal{B}, K, t) \) the reduced part of \( Q \). The modulated quiver with potential \((Q, m)\) is reduced if \( m \) belongs to \( J_{kQ}^2 \), that is, if \( m_2 = 0 \). \((Q, m)\) is trivial if the reduced part of \( B \) is zero. The trivial part of \((Q, m)\) splits if \( B_{\text{triv}} \) is a direct summand in \( B \).

A note on presentations of Jacobian algebras. The first obstruction to reduce a modulated quiver with potential is of the same nature as the obstruction to the presentation of finite-dimensional algebras over non algebraically closed fields by modulated quivers with relations (see [23]). Indeed, let \( A := J'(Q, \rho) \) and \( J_A := J_{kQ}^2 J_{m} \); if \( A \) admits a presentation by a modulated quiver with relations, then \( A \) is an \((A/J_A)\)-bimodule and \( J_A^2 \) is a direct summand in \( J_A \). Now let \( \hat{J} := J_{kQ}^2 \) and observe that \( B_{\text{triv}} = B \cap (J_A^2 + J_m) \) and we have:

\[
\mathcal{J}J_A^2 = (\hat{J}/J_m)((J^2 + J_m)/J_m) \cong \hat{J}(J^2 + J_m) = (B + J^2)(J^2 + J_m) = \mathcal{B}(B \cap (J^2 + J_m)) = \mathcal{B}B_{\text{triv}} = \mathcal{B},
\]

illustrating (1.ii) below.

Definition 4.3. A reduction on \((Q, m)\) is a path algebra epimorphism \( \phi : \mathcal{Q} \to \mathcal{Q}' \) (with \( \mathcal{Q}' = (B', K, t) \)) such that \( \phi(m) \) is reduced and the following conditions hold.

\[\begin{align*}
(1.\text{i}) & \text{ Ker}(\phi) \text{ is the closed ideal generated by the image of a } K\text{-bimodule morphism } f = \left[ \begin{smallmatrix} f \end{smallmatrix} \right] : B_{\text{triv}} \to B_{\text{triv}} \oplus J_{kQ}^2 \text{ with } \text{Im}(f) \subset (\partial m)(B^*). \\
(1.\text{ii}) & \text{ Let } \pi : \mathcal{Q} \to \mathcal{Q}/\text{Ker}(\phi) \text{ be the natural projection and } \varrho : (B + \text{Ker}(\phi)) \to \text{Ker}(\phi) \text{ the } K\text{-bimodule epimorphism with } \rho = \varrho \circ \pi_{|A}. \text{ Then } \varrho \text{ has a right inverse } \varrho' : \mathcal{B} \to (B + \text{Ker}(\phi))/\text{Ker}(\phi) \text{ which lifts to a left (respectively, right) } K\text{-linear map } \rho' : \mathcal{B} \to B \text{ such that } \phi \circ \rho' = \phi \circ \rho'.
\end{align*}\]

A reduction \( \phi \) splits whenever \( \rho' : \mathcal{B} \to B \) is a bimodule morphism; in this case \((Q, m)^{\text{triv}} \) splits.

Lemma 4.4. \( a \) Let \( J_0 \) be any closed ideal in \( \mathcal{Q} \) satisfying (1.ii) above. Then \( (\partial m)(B^*) \subset f(B_{\text{triv}}) + (\partial m)(B') \) and consequently, \( J_m = J_0 + (\partial m)(B') \).

\[\begin{align*}
(1.\text{b}) & \text{ Let } I \\&I' \text{ be two } k\text{-modules in a path algebra } A \text{ with } J\text{-adic topology, where } J \text{ is the complete arrow ideal in } A. \text{ Then the closed module } \mathcal{I} \text{ coincides with } I + I'. \text{ If } I \text{ and } I' \text{ are ideals in } A \text{ then } \mathcal{I} \subset \mathcal{I}', \text{ consequently, if } I' \subset I + J^I + J'I \text{ then } I' \subset \mathcal{I}.
\end{align*}\]

Proof. Since part (a) is a direct consequence of the assumptions and the definition of \( \mathcal{B}' \) and \( J_m \), we turn to part (b). Under \( J\)-adic topology, the closure of a subset \( S \subset A \) is \( \overline{S} = \bigcap_{l \in \mathbb{N}} (S + J^l) = \bigcap_{l \in \mathbb{N}} (S + J^l) \), with \( l \in \mathbb{N} \). Thus each subset \( S + J^l \) is closed, and for two \( k\)-modules \( I, I' \subset A \) we have:

\[
\overline{I + I'} = \bigcap_{l \in \mathbb{N}} (I + J^l) + \bigcap_{l \in \mathbb{N}} (I' + J^l) = \bigcap_{l \in \mathbb{N}} (I + J^l + I' + J^l) = \bigcap_{l \in \mathbb{N}} (I + J^l + J^l) = \overline{I + I'}.
\]

Thus \( \overline{I + I'} = \overline{I + I'} = \overline{I + I'} \), showing that \( \mathcal{I} \subset \mathcal{I}' \). Assume that \( I \) and \( I' \) are ideals in \( A \). For all \( l \in \mathbb{N} \), note that \( \mathcal{I} \subset (I + J^l) \subset \mathcal{I}' = I + J^l \), implying that \( \mathcal{I} \subset \mathcal{I}' \). Now, suppose that \( I' \subset I + J^I + J'I \), we must show that \( I' \subset \mathcal{I} \). Applying the relations just proved, we have:

\[
\begin{align*}
I' \subset (I + J^I + J'I) & \subset I + J(I + J^I + J'I) + (I + J^I + J'I)J \subset I + J(I + J^I + J'I + J'I)J \subset \overline{I + J^I + J'I + J'I}J \subset I + J^I + J'I + J'I + J'I + J'I + J'I + J'I = \overline{I + I'}.
\end{align*}
\]

22
thus repeating the previous procedure, for each \( l \geq 2 \) we get \( I' \subset I + \sum_{r=0}^{l-1} (J^{l-r}J^{r}) \subset I + J^{l} = I + J' \), implying that \( I' \subset I \) as claimed. This completes the proof of the lemma. \( \square \)

Let \( L := \overline{kQ^*B_{\text{triv}}kQ} \) be the closed ideal generated by \( B_{\text{triv}} \). We get the following facts on Jacobian ideals.

**Theorem 4.5.** Let \( \phi : \overline{kQ} \longrightarrow \overline{kQ'} \) be any path \( k \)-algebra epimorphism with \( Q' = (B', K, 1) \). Then the following statements hold.

1. If \( \phi \) is an isomorphism, then \( \phi(J_m) = J_{\phi(m)} \) and there is an induced \( k \)-morphism \( J(Q, m) \longrightarrow J(Q', \phi(m)) \).

2. If \( \phi \) is a reduction on \( (Q, m) \) then \( \phi(J_m) = J_{\phi(m)} \) and \( J(Q, m) \cong J(Q', \phi(m)) \). Moreover there is a left (respectively, right) \( K \)-linear isomorphism \( [I, \rho'] : L \otimes \overline{kQ'} \longrightarrow \overline{kQ} \) with \( \rho'(B') \subset B \) and \( \rho'(u)v = \rho(u)\rho'(v) \) for all \( u, v \in J_{\overline{kQ'}} \). Thus \( \phi \) is an isomorphism and \( \phi_l : B_{\text{triv}} \longrightarrow B' \) of \( \phi \) is a bimodule isomorphism with dual \( \phi^*_l : B^* \longrightarrow B'^* \). Write \( m \) as sum \( \sum m_l \) of homogeneous potentials \( m_l \in B(l) \). We have \( \phi_l(J_l) = \overline{J_l} \), \( \phi_l(m) = \sum m_l \phi_l(m_l) \) and each \( \phi_l(m_l) \) belongs to \( \{0\} \cup (\overline{J_l} \setminus \overline{J^l+1}) \). Let \( \xi' \in B'^* \).

**Definition 4.4.** Let \( \phi : \overline{kQ} \longrightarrow \overline{kQ'} \) be a reduction on \( (Q, m) \). The data \( (Q', \phi(m)) \) is a reduced modulated quiver with potential associated with \( (Q, m) \) and \( \phi \) is referred to as reduction from \( (Q, m) \) to \( (Q', \phi(m)) \). We also refer to \( \ker(\phi) \) as trivial part in \( J_m \).

**Proof of Theorem 4.5.** We let \( \tilde{J} = J_{\overline{kQ}} \) and \( \tilde{J} = J_{\overline{kQ'}} \). We have dualizing pairs \( B, B^*; b, \{\overline{B}, \overline{B}^*; \overline{b}\} \) and \( B', B'^*; b' \). Let us prove part (1). The chain-rule (4.5) shows that, for all \( \xi' \in B'^* \) we have \( \partial_{\xi'}(\phi(m)) = \sum \{ \Delta_{\xi'}(\phi(y)) \otimes \phi(\partial_{\xi'}m) \} \), implying by the surjectivity of \( \phi \) that \( J_{\phi(m)} \subset \phi(J_m) \). Next, assume that \( \phi \) is an isomorphism. Applying the previous observations to \( \phi^{-1} \) and \( \phi(m) \) shows that \( J_m \subset \phi^{-1}J_{\phi(m)} \) and then \( \phi(J_m) \subset \phi(\phi^{-1}(J_{\phi(m)})) \). Hence, \( \phi(J_m) = J_{\phi(m)} \). The degree-1 component \( \phi_1 : B_{\text{triv}} \longrightarrow B' \) of \( \phi \) is a bimodule isomorphism with dual \( \phi^*_1 : B^* \longrightarrow B'^* \). Write \( m \) as sum \( \sum m_l \) of homogeneous potentials \( m_l \in B(0) \). We have \( \phi_l(J_l) = \overline{J_l} \), \( \phi_l(m) = \sum m_l \phi_l(m_l) \) and each \( \phi_l(m_l) \) belongs to \( \{0\} \cup (\overline{J_l} \setminus \overline{J^l+1}) \). Let \( \xi' \in B'^* \).

Note that: \( \partial_{\xi'}(\phi(m)) = 0 \) if and only if \( \partial_{\xi'}(\phi_l(m_l)) = 0 \) for all \( l \geq 2 \).

For each \( l \geq 2 \), by the chain-rule we have \( \partial_{\xi'}(\phi_l(m_l)) = \sum \{ \Delta_{\xi'}(\phi(y)) \otimes \phi(\partial_{\xi'}m_l) \} \) with each \( \phi(\partial_{\xi'}m_l) \) lying in \( \{0\} \cup (\overline{J_l} \setminus \overline{J^l+1}) \). For each \( l \geq 2 \), we have \( \phi(x) = \sum \phi_l(x) \) with \( \phi_l(x) \in B(l) \), thus \( \Delta_{\xi'}(\phi_l(x)) = \partial_{\xi'}(\phi_l(\phi_l(x)\otimes x)) = (b' \otimes \phi(x)) \otimes x \). We get that, if \( \partial_{\xi'}(\phi_l(m_l)) = 0 \) then the term \( \Delta_{\xi'}(\phi_l(x)) \) is zero.

Theorem 4.3 implies that \( B \cap (\ker(\phi) + J_{\overline{kQ}}) = B_{\text{triv}} \phi_1 : B_{\text{triv}} \longrightarrow B' \) yields a bimodule isomorphism \( \phi_1 : B \longrightarrow B' \) with \( \phi_1 = \overline{\phi_1} \), the projection \( \rho : B \longrightarrow \overline{B} \) induces a morphism \( \overline{\rho} : (B + \ker(\phi)) \longrightarrow \overline{B} \) with \( \overline{\rho}(x + \ker(\phi)) = \rho(x) \) for all \( x \in B \). Still by assumption, \( \overline{\rho} \) has a right inverse \( \overline{\rho}' : \overline{B} \longrightarrow B + \ker(\phi) \) which lifts to a left (respectively, right) \( K \)-linear map \( \rho' : B \longrightarrow B \) such that \( \rho \circ \rho' = \phi_1 \circ \rho' \). Hence, without loss of generality, we may assume that \( B' = \overline{B} \), \( \overline{\phi_1} = \overline{\phi} \) and \( \rho' \) is right \( K \)-linear. Thus \( \phi_1 = \rho, \phi \circ \rho' = \phi_1 \circ \rho' = 1_{\overline{B}} \), we have a right \( K \)-linear isomorphism \( h' : [\mathbb{I}, \rho'] : B_{\text{triv}} \longrightarrow B \) by duality a left \( K \)-linear isomorphism
\[(h^{r-1})^* = [j,1] : B_{\text{triv}} \oplus B^* \longrightarrow B^*\] where \(j\) is a left \(K\)-linear right inverse for \(p = \partial m_2 : B^* \longrightarrow B_{\text{triv}}\). For some \(p, q \in \mathbb{N}\), we have Casimir elements \(\mathfrak{z}_{\text{triv}} \in \mathfrak{z}_{\text{triv}}\) and \(\mathfrak{z}_{p \otimes m} = \sum_{k=1}^{p+q} y_k y_k + \sum_{k=1}^{p} y_{p+k} \otimes y_{p+k}.\) For \(k \in [1, p]\) and \(s \in [p+1, p+q]\), put \(y^*_k = j(c_k) \in B^*\) and \(y_s = r^*(\mathfrak{z}_s)\). Part (1) of Proposition 3.4 states that
\[
\mathfrak{z}_{p \otimes m} = (1 \otimes J)(\mathfrak{z}_{p \otimes m}) + (p' \otimes 1)(\mathfrak{z}_{p \otimes m}) = \sum_{k=1}^{p+q} y_k y^*_k. \quad (\ast)
\]
By Remark 4.1 we have: \([b(\xi \otimes x) = \mathfrak{f}(\xi \otimes p(x)) = 0 = \mathfrak{f}(p(x) \otimes \xi) = b(x \otimes \xi)\) for all \(\xi \in \mathfrak{f}^*, x \in B_{\text{triv}}\). Let \(\xi \in \mathfrak{f}^*\). Using the chain-rule, identities (3.8) as well as previous observations, we have:
\[
\phi(\partial m) = \phi(\partial \xi) = \sum_{k=1}^{p+q} b(y_k \otimes y_k) \phi(\partial y_k) + \sum_{k=1}^{p} b(y_k \otimes y_k) \phi(\partial y_k) = \sum_{k=1}^{p+q} \mathfrak{f}(y_k \otimes y_k) \phi(\partial y_k) = \sum_{k=1}^{p+q} \mathfrak{f}(y_k \otimes y_k) \phi(\partial y_k) = \sum_{k=1}^{p+q} \mathfrak{f}(y_k \otimes y_k) \phi(\partial y_k) = \sum_{k=1}^{p+q} \mathfrak{f}(y_k \otimes y_k) \phi(\partial y_k).
\]
Hence, \(\phi((\partial m)^*) \subset J_{\phi(m)} + (\mathfrak{f}(J_{\phi(m)} + \phi(J_{\mathfrak{f}(m)}))), \Phi\). But Lemma 4.4-(b) shows that \(J_m\) coincides with the closure of \(\Phi(\phi) + ((\partial m)^*)\), implying that \(\phi(J_m)\) is contained in the closure of \(J_{\phi(m)} + (\mathfrak{f}(J_m) + \phi(J_{\mathfrak{f}(m)})), \Phi\) and, applying part (b) of Lemma 4.4 we get \(J_{\phi(m)} \subset J_{\phi(m)}\). By the chain-rule we also have \(J_{\phi(m)} \subset J_{\phi(m)}\). Thus \(J_{\phi(m)} = J_{\phi(m)}\) and the latter also shows that \(\phi\) induces an isomorphism of Jacobian algebras from \(\mathcal{J}(\mathfrak{f}, m)\) to \(\mathcal{J}(\mathfrak{f}'(m), \phi(m))\). To complete the proof of (2), we will extend the right \(K\)-linear map \(\rho' : \mathfrak{f} \longrightarrow \mathfrak{f}\) to a continuous right \(K\)-linear morphism again denoted by \(\rho' : k\mathfrak{Q}^* \longrightarrow k\mathfrak{Q}, \rho'_{\mid \mathfrak{Q}} = 1_{\mathfrak{Q}}\). Recall that \(L\) is the closed ideal in \(k\mathfrak{Q}\) generated by \(B_{\text{triv}}\). With notations of \((\ast)\) above, let \(a_{\mathfrak{Q}} = \{y_k : k \in [1, p+q]\}\) and \(b_{\mathfrak{Q}^*} = \{y_k^* : k \in [1, p+q]\}\). Then \((a_{\mathfrak{Q}}, b_{\mathfrak{Q}^*})\) is a right projective basis for the pair \(\{B, B^*; b\}\), while \((\{y_k : k \in [1, p]\}, \{x_k : k \in [1, p]\})\) and \((\{y_{p+k} : k \in [1, q]\}, \{y_{p+k} : k \in [1, q]\})\) are right projective bases for the pairs \(\{B_{\text{triv}}, B_{\text{triv}}\}\) and \(\{\mathfrak{f}, \mathfrak{f}^*; b\}\), respectively. And by definition we have: \(\rho'((\mathfrak{f}_i) = y_k\) for all \(k \in [p+1, p+q]\). In view of subsection 3.2, we form corresponding right projective bases \((a_{\mathfrak{Q}}^*, a_{\mathfrak{Q}^*})\) and \((b_{\mathfrak{Q}^*}, b_{\mathfrak{Q}})\) for the weakly dualizing pairs \(\{kQ^*, k\mathfrak{Q}; b\}\) and \(\{k\mathfrak{Q}^*, k\mathfrak{Q}; b\}\) respectively. Here \(a_{\mathfrak{Q}^*} = \{1\} \cup a_{\mathfrak{Q}}\), with \(1^* = 1 \in K\); each \(y \in a_{\mathfrak{Q}}\) expresses as \(y = y_{i_1} \otimes \cdots \otimes y_{i_l}\) with \(l \geq 1\) and \(i_1, \ldots, i_l \in [1, q + p]\), the corresponding dual is \(y^* = y_{i_l} \otimes \cdots \otimes y_{i_1}\). A similar description is given for \(a_{\mathfrak{Q}}\). Next, put \(Y = a_{\mathfrak{Q}} \cap L\); it consists of basis elements \(y_{i_1} \otimes \cdots \otimes y_{i_l}\) such that at least one of the integers \(i_1, \ldots, i_l\) belongs to \([1, p]\). Also put \(Y^* = a_{\mathfrak{Q}^*} \cap L\). Therefore, \(\rho'\) is defined on each basis element \(y = y_{i_1} \otimes \cdots \otimes y_{i_l} \in a_{\mathfrak{Q}}\) by: \(\rho'(y) := y_{i_1} \otimes \cdots \otimes y_{i_l}\). Thus, in virtue of identities (3.6) from subsection 3.2, for each \(x \in k\mathfrak{Q}\) and \(\mathfrak{f} = \sum_{\mathfrak{f} \in a_{\mathfrak{Q}^*}} \mathfrak{f} \mathfrak{b}(y^* \otimes x)\) we have \(\rho'(\mathfrak{f}) = \sum_{\mathfrak{f} \in a_{\mathfrak{Q}^*}} \rho'(\mathfrak{f}) \mathfrak{b}(y^* \otimes x)\). By construction, \(\rho' : k\mathfrak{Q}^* \longrightarrow k\mathfrak{Q}\) has the desired properties.

Different presentations of Jacobian algebras by reduced modulated quivers with potentials can be compared using the following concept.

**Definition 4.5.** Let \((\mathfrak{Q}, m')\) be another modulated quiver with potential \(\mathfrak{Q}' = (B', K, t)\). A weak right-equivalence between \((\mathfrak{Q}, m)\) and \((\mathfrak{Q}', m')\) is a path algebra isomorphism \(\phi : k\mathfrak{Q} \longrightarrow k\mathfrak{Q}'\) such that \(J_{\phi(m)} = J_{\phi(m')}.\) If moreover \(\phi(m)\) is cyclically equivalent to \(m'\) then \(\phi\) is a right-equivalence.
Under the assumption that the trivial part of \((Q, m)\) splits, the first main result of this work gives the existence and uniqueness of split reductions up to weak right-equivalences. As before, \(\rho : k\overline{Q} \longrightarrow k\overline{Q}\) is the natural projection.

**Theorem 4.6** (reduction theorem). Assume the trivial part of \((Q, m)\) splits and write \(Q = Q_{\text{triv}} \oplus Q_{\text{red}}\). Then there is a right-equivalence \(\phi\) from \((Q, m)\) to a direct sum \((Q_{\text{triv}}, m_{\text{triv}}) \oplus (Q_{\text{red}}, m_{\text{red}})\), yielding a split reduction \(\pi_m = \rho \circ \phi\) from \((Q, m)\) into a reduced modulated quiver with potential \(\text{red}(Q, m) = (Q_{\text{red}}, m_{\text{red}})\), with kernel \(I_{\text{red}}^m\) such that \(m - m_{\text{red}}\) is cyclically equivalent to an element in \((I_{\text{red}}^m)2\). Furthermore, the split reduction process \(\text{red} : (Q, m) \longrightarrow \text{red}(Q, m)\) is a well-defined operation on weak-right equivalence classes of modulated quivers with potentials.

The proof of the first part of Theorem 4.6 is the object of discussion in Lemma 4.7 to Lemma 4.8.

**Lemma 4.7.** Let \(N\) be a direct summand in \(B\) and \(S\) a potential lying in the closed ideal \((\overline{N})\) generated by \(N\). Then \(S\) is cyclically equivalent to a potential lying in \(NJ_{k\overline{Q}}\) and to a potential lying in \(J_{k\overline{Q}}\).

**Proof.** Elements of \((\overline{N})\) can be written as possibly infinite sums of elements \(u_l, l \geq 1\), with \(u_l \in (N) \cap B^{(l)} = \sum_{l=0}^{l-1} B^{(s)} NB^{(l-s-1)}\). Thus \(S = \sum_{l=0}^{l=1} S_l\) where \(S_l\) is a potential lying in \((N) \cap B^{(l+1)}\). By assumption, \(B = N \oplus N'\) for some submodule \(N' \subset B\). There is a corresponding decomposition \(B^* = N^* \oplus N'^*\) such that for all \((\xi, \xi') \in N^* \times N'^*\) and \((x, x') \in N \times N'\) we have: \(b(\xi \otimes x') = 0 = b(x' \otimes \xi)\) and \(b(\xi' \otimes x) = 0 = b(x \otimes \xi')\).

For each \(l \geq 1\) we have \(B^{(l+1)} = (N \oplus N')B^{(l)} = (\xi \oplus N'^*)(N \cdot B^{(l-s)}) \oplus N'^{(l)}(N' \cdot B^{(l-s)})\), and each \(S_l \in (N) \cap B^{(l+1)}\) expresses as sum \(\sum_{l=0}^{l=1} S_l\) of potentials with \(S_l \in N'^{(s)}N \cdot B^{(l-s)}\). Hence the left permutation \(e_{l,s}^lS_{l,s}\) of order \(s\) of each \(S_l\) belongs to \(NB^{(l)}\) while the right permutation \(e_{l,s}^{-1}S_{l,s}\) of order \(l-s\) of each \(S_l\) belongs to \(B^{(l)}N\). Thus \(S\) is cyclically equivalent to \(N \cdot J_{k\overline{Q}}\) and to a potential lying in \(J_{k\overline{Q}}\). □

Denoting as before the reduced part of \(B\) by \(\overline{B}\) and using the assumption in Theorem 4.6, we simply write \(B = B_{\text{triv}} \oplus \overline{B}\). Part (2) of Proposition 3.4 shows that the pair \(\{B, B^*; b\}\) occurs as direct sum of naturally induced dualizing pairs \(\{B_{\text{triv}}, B_{\text{triv}}^*\} \oplus \{\overline{B}, \overline{B}^*\}\) with \(B^* = B_{\text{triv}}^* \oplus \overline{B}^*\) and we have a right inverse \(B_{\text{triv}}^{\text{red}} \longrightarrow B^*\) for the cyclic derivative \(\partial_{B_{\text{triv}}} : B^* \longrightarrow B_{\text{triv}}\) such that \(\text{Im}(\partial_{B_{\text{triv}}}) = B_{\text{triv}}^*\). In view of Remark 4.1, letting \(U = \partial_{B_{\text{triv}}} B^*\) and \(V = \partial_{B_{\text{triv}}} B^*\), we have \(U \cap V = 0, B_{\text{triv}} = U \oplus V, \{B_{\text{triv}}, B_{\text{triv}}^*\} = \{U, U^*\} \oplus \{V, V^*\}\) as direct sum of naturally induced dualizing pairs with \(U^* = j_{\text{triv}}(V)\) and \(V^* = j_{\text{triv}}(U)\) and \(\partial_{B_{\text{triv}}} : B_{\text{triv}}^* \longrightarrow B_{\text{triv}}\) occurs as direct sum of the partial derivatives \(\partial_{B_{\text{triv}}} : U^* \longrightarrow U\) and \(\partial_{B_{\text{triv}}} : U^* \longrightarrow V\).

Let us summarise previous observations:

\begin{equation}
B_{\text{triv}} = U \oplus V, B = B_{\text{triv}} \oplus \overline{B},\text{ and }\{B, B^*\} = \{U, U^*\} \oplus \{V, V^*\} \oplus \{\overline{B}, \overline{B}^*\}\end{equation}

**Lemma 4.8.** With the assumption that the trivial part of \((Q, m)\) splits, there exists a unital/matrix automorphism \(\phi : k\overline{Q} \longrightarrow k\overline{Q}\) such that \(\phi(m)\) is cyclically equivalent to a potential \(\overline{m}\) in the form (4.9) with
$u_k = 0 = v_k$ for all $k \in [1, p]$ and such that: $\phi_{l-k} = 1_{k-Q}$. Moreover, the map $\pi_m = \rho \phi$ is a reduction on $(Q, m)$ with $\text{Ker}(\pi_m) = J_{m}^{\text{triv}}$ and $m - \pi_m(m) \equiv_{k-Q} (m_2) \in (J_{m}^{\text{triv}})^2$.

**Proof.**

**Claim.** Suppose $S$ is a $d$-split potential written in the form (4.9). Then there exists a unital triangular automorphism $\varphi : k-Q \rightarrow k-Q$ having depth $d$, with $\varphi_{|_{k-Q}} = 1_{k-Q}$, such that $\varphi(S)$ is cyclically equivalent to a 2$d$-split potential $S'$, with $S' - S \in J_{k-Q}^{2d+2}$.

We write $S = \delta_{u \in V} + S_1 + S_2 + S_1$, with $S_1 \in U \otimes J_{k-Q}^{d+1}$, $S_2 \in J_{k-Q}^{d+1} \otimes V$, $S_1 \in J_{k-Q}^{d+1}$, keeping the notations of (4.9) for $S$. Then we have a unital triangular automorphism $\varphi : k-Q \rightarrow k-Q$ having depth $d$, defined by letting:

$$\varphi_{|_{k-Q}} = 1_{k-Q}, \quad \varphi_{|_{U}} = 1_{U} - (\partial^d S_2) \circ \text{triv} : U \rightarrow U \otimes J_{k-Q}^{d+1}$$

and $\varphi_{|_{V}} = 1_{V} - (\partial S_1) \circ \text{triv} : V \rightarrow V \otimes J_{k-Q}^{d+1}$.

Thus, for all $k \in [1, p]$ we have: $\varphi(y_k) = y_k - u_k$ and $\varphi(x_k) = x_k - v_k$ with $v_k \in J_{k-Q}^{d+1}$ and $u_k \in J_{k-Q}^{d+1}$.

Since $\varphi$ has depth $d$, we have $\varphi(u_k) = u_k + u'_k$ and $\varphi(v_k) = v_k + v'_k$ for some $u'_k, v'_k \in J_{k-Q}^{d+1}$. We get $\varphi(S) = \sum_{k=1}^{p} (y_k - u_k)(x_k - v_k) + (y_k - u_k)(v_k + v'_k) + (u_k + u'_k)(x_k - v_k) + S_1 = \sum_{k=1}^{p} y_k \otimes x_k + W + S_1$, where $W = \sum_{k=1}^{p} (y_k \otimes v'_k + u'_k \otimes x_k - u_k \otimes v'_k - u'_k \otimes v_k - u_k \otimes v_k) \in J_{k-Q}^{2d+2}$ is a potential. Since $k-Q = L \otimes k-Q$ with $L = k-Q B_{\text{triv}} k-Q$, we can write $W = W' + W$ for two potentials $W' \in L \otimes J_{k-Q}^{2d+2}$ and $W \in J_{k-Q}^{2d+2}$. But using again Lemma 4.7 and the fact that $B_{\text{triv}} = U \otimes V$, we get that $W'$ is cyclically equivalent to a sum $W'' = W'_1 + W'_2$ of two potentials $W'_1 = \sum_{k=1}^{p} y_k \otimes v'_k + u'_k \otimes x_k \in U \otimes J_{k-Q}^{2d+2}$ and $W'_2 = \sum_{k=1}^{p} u'_k \otimes x_k \in J_{k-Q}^{2d+2} \otimes V$ with $u'_k = \partial^d S'_2 \in J_{k-Q}^{d+1}$ and $v'_k = \partial S'_2 \in J_{k-Q}^{d+1}$. Hence, $W - (W'' + W)$ lies in $J_{k-Q}^{2d+2}$ and in the closed module $\text{skew}\{Z_{K}(k-Q(2)), Z_{K}(k-Q(2))\}$ of skew commutators in $k-Q$.

Thus, the desired potential $S'$ is given by:

$$S' = \sum_{k=1}^{p} y_k \otimes x_k + S'_1 + S'_2 + (W + S_1)$$

with $S'_1 = \sum_{k=1}^{p} y_k \otimes v'_k$ and $S'_2 = \sum_{k=1}^{p} u'_k \otimes x_k$, with $u'_k = \partial^d S'_2 \in J_{k-Q}^{d+1}$ and $v'_k = \partial S'_2 \in J_{k-Q}^{d+1}$. This completes the proof of our claim.

Next, starting with a 1-split potential $S_1$ in the form (4.9) and using successively the above claim, one constructs a sequence of potentials $S_1, S_2, \ldots$, and a sequence of unital triangular automorphisms $\phi_1, \phi_2, \ldots$, having the following properties:

- (p0): $m \equiv_{cyclic} S_1$
- (p1): $S_d$ is 2$d$-split.
- (p2): $\phi_d$ has depth 2$d$.
- (p3): $\phi_d(S_d) \equiv_{cyclic} S_{d+1}$ and each element $C_d := \phi_d(S_d) - S_{d+1}$ lies in $J_{k-Q}^{d+2} \cap \text{skew}\{Z_{K}(k-Q(2)), Z_{K}(k-Q(2))\}$.

Using (p2) we set $\phi = \lim_{l \to \infty} \phi_1 \phi_{l-1} \cdots \phi_1$. By part (a) of Remark 3.2 and Proposition 3.1, is a well-defined unital triangular automorphism of $k-Q$ such that $\phi_{l-k} = 1_{k-Q}$. And by (p3), each element $C_d := \phi_d(S_d) - S_{d+1}$ lies in $J_{k-Q}^{d+2} \cap \text{skew}\{Z_{K}(k-Q(2)), Z_{K}(k-Q(2))\}$. But by Proposition 4.1, any unital triangular automorphism sends skew $\text{skew}\{Z_{K}(k-Q(2)), Z_{K}(k-Q(2))\}$ to itself. Thus, using again (p0), we get that $\phi_1 \phi_{l-1} \cdots \phi_1(m) \equiv_{cyclic} \phi_1 \phi_{l-1} \cdots \phi_1(S_1) = S_{l+1} + \sum_{d=1}^{l} \phi_1 \phi_{d-1} \cdots \phi_1(C_d)$ for all $l \geq 1$; and passing to the limit as $l$ tends to $\infty$, we have $\phi(m) \equiv_{cyclic} \phi(S_1) = \lim_{l \to \infty} S_1 + \phi(\sum_{l=1}^{\infty} (\phi \phi_{l-1} \cdots \phi_1)^{-1}(C_d))$. Letting $\bar{m} = \lim_{l \to \infty} S_1$, we get that $\phi(m)$ is cyclically equivalent to $\bar{m}$ and, in view of (p1), $\bar{m}$ is in the form (4.9) with $u_k = 0 = v_k$. To complete the proof, we now let $\pi_m := \rho \phi$. Then $\pi_m$ is clearly a split reduction with $\text{Ker}(\pi_m) = \varphi^{-1}(L)$. Let us check that $\varphi^{-1}(L)$ coincides with the closed ideal $J_{m}^{\text{triv}} = (J_B(\text{triv}))$, where we recall that $f := (\rho \phi) \circ \text{triv} : B_{\text{triv}} \rightarrow B_{\text{triv}} \otimes J_{k-Q}^{2}$, with $\rho \phi \circ \text{triv} = 1_{B_{\text{triv}}}$. We have $\bar{m} = \delta_{u \in V} + W = \sum_{k=1}^{p} y_k \otimes x_k + W$ for some potential $W \in J_{k-Q}^{2}$. In view of
(4.6) and (4.7) above, \( B = B_{\text{triv}} \oplus B \) with \( B_{\text{triv}} = U \oplus V \), \( B^* = B_{\text{triv}}^* \oplus B^* \) with \( B_{\text{triv}}^* = U^* \oplus V^* \), and (using again Proposition 3.4-(1)) the Casimir element in \( B \otimes B^* \) expresses as \( \delta_{B \otimes B^*} = \delta_{B_{\text{triv}} \otimes B_{\text{triv}}^*} + \delta_{B \otimes B^*} \). Write \( \delta_{B_{\text{triv}} \otimes B_{\text{triv}}^*} = \sum_{k=1}^r z_k \otimes z_k^* \). Since \( \phi(B_{\text{triv}}) = I_{\mathbb{P}} \), we deduce that \( \Delta_k^\phi(z \otimes z) = \Delta_k^\phi(z) = 0 \) for all \( z \in B \) and \( \xi \in B_{\text{triv}}^* \).

Hence, the chain-rule (4.5) and the fact that \( \phi(m) \equiv \text{cyc} \m \) give the following conclusion: for all \( \xi \in B_{\text{triv}}^* \) we have \( \partial \m = \partial \phi(m) = \sum_{k=1}^r \Delta_k^\phi(z_k) \otimes \phi(\partial \m) \otimes \phi(\partial \m) \in \phi(J_m) \). Hence \( L \subseteq \phi(J_m) \), and applying the inverse uniiangular automorphism \( \phi^{-1} \) to \( m \), we also have \( J_m \subseteq \phi^{-1}(L) \), thus \( \phi(J_m) \subseteq \phi^{-1}(L) \). Hence \( \phi(J_m) \) is already a path algebra morphism, that is, \( \phi(J_m) \) is already a path algebra morphism, that is, \( \phi(J_m) = J_m \). Consequently, we get: \( \phi(m - \pi_m(m)) = \phi(m) - \phi(m) \equiv \text{cyc} \m - \rho(m) = m_{Q \oplus V} \in L^2 \), so that \( m - \pi_m(m) \equiv \text{cyc} \phi^{-1}(m) \). This completes the proof of Lemma 4.8 and the first part in Theorem 4.6.

The rest of this section is consecrated to the proof of the second part of Theorem 4.6. Keeping the same assumptions \( (Q, m) \), to each direct sum decomposition \( B = B_{\text{triv}} \oplus B \) corresponds a split reduction \( \pi_m : kQ \rightarrow kQ \) with kernel denoted by \( J_m \), such that \( \phi(J_m) = I_{kQ} \). The second part of Theorem 4.6 is given by the following lemma.

**Lemma 4.9.** Let \( \varphi : kQ \rightarrow kQ' \) be a weak right-equivalence between \( (Q, m) \) and a modulated quiver with potential \( (Q', m') \) with \( Q' = (B', K, \ell) \). Then \( (Q', m') \) sits and, for any split reduction \( \varphi' := \pi'_{m'} : kQ' \rightarrow kQ' \) corresponding to a direct sum decomposition \( B' = B'_{\text{triv}} \oplus B'_{\text{t}} \) of \( K \)-bimodules, we have a weak right-equivalence \( \psi := (\varphi' \circ \phi) \) as \( kQ \rightarrow kQ \) between \( \text{red}(Q, m) \) and \( \text{red}(Q', m') \).

**Proof.** We write \( \varphi = \pi_m : kQ \rightarrow kQ \) for the reduction defined by \( m \) with respect to a direct sum decomposition \( B = B_{\text{triv}} \oplus B \). Let us agree with the following abbreviations: \( J = J_m, J' = J_{m'}, J = \varphi(J), J' = \varphi(J') \). Recall that the trivial parts \( J_m \) and \( B_{\text{triv}} \) are such that: \( B_{\text{triv}} = (J + J') \cap B = (J^2 + J') \cap B \). Similarly, \( B_{\text{triv}} = (J' + J^2) \cap B' = (\text{Ker}(\varphi') + \text{Ker}(\varphi')) \cap B' \).

In view of Proposition 3.1-(a), the degree-1 homogeneous component of isomorphism \( \phi \) is a \( K \)-bimodule isomorphism \( \phi : B = B_{\text{triv}} \oplus B \rightarrow B_{\text{triv}} \), and \( B' = \phi_1(B_{\text{triv}}) \oplus \phi_1(B) \). But using Theorem 4.5 and the assumption on \( \phi \), we get \( J_{\phi'(m)} = \phi(J_m) = J_m \). As a path algebra isomorphism, \( \phi(J') = J' \) for all \( l \in \mathbb{N} \). We then have: \( \phi(B_{\text{triv}}) = \phi(J + J^2) \cap B \) and \( \phi(B_{\text{triv}}) = (\phi(J) + \phi(J^2)) \cap \phi(B) = (J' + J^2) \cap \phi(B) \), so that \( \phi_1(B_{\text{triv}}) = (J' + J^2) \cap \phi_1(B) = (J' + J^2) \cap B' = B_{\text{triv}} \). Thus, applying \( B_{\text{triv}} \) splits \( B' \).

For the rest of the proof, fix a direct sum decomposition \( B' = B_{\text{triv}} \oplus B' \) and let \( \overline{Q} \) be the reduced modulated quiver associated with \( B' \). Then \( \varphi' := \pi_{m'} : kQ' \rightarrow kQ' \) be the corresponding split reduction. Recall that \( \psi_k \in kQ \) and \( \varphi' \) are \( kQ \rightarrow kQ \), hence, letting \( \overline{m} = m \) and \( \overline{m'} = m' \), we derive the following conclusions:

\[ \begin{align*}
J & = J_{\overline{m}} = \varphi(J_m) \subseteq J \quad J' = J_{\overline{m'}} = \varphi(J_{m'}) \subseteq J' \quad \text{(4.7)}
\end{align*} \]

The algebra morphism \( \psi := (\varphi' \circ \phi) \) is already a path algebra morphism, that is, \( \psi_k \in K_{kQ} \). In the previous paragraph, we proved that \( \phi_1(B_{\text{triv}}) = B'_{\text{triv}} \). Write the bimodule isomorphism \( \phi_1 : B_{\text{triv}} \rightarrow B' \) in a matrix form: \( \phi_1 = [\phi_{11}, \phi_{12}] : B \oplus B_{\text{triv}} \rightarrow B \oplus B'_{\text{triv}} \), with \( \phi_{11} : B \rightarrow B'_{\text{triv}} \), \( \phi_{12} : B \rightarrow B_{\text{triv}} \), and \( \phi_{21} : B_{\text{triv}} \rightarrow B'_{\text{triv}} \). Hence, \( \phi_{11} \) is also an isomorphism. If we put \( \phi_{12} = \phi_{12} = [\phi_{12}] : B \rightarrow B' \) and \( \phi_{21} = \phi_{21} : B_{\text{triv}} \rightarrow B_{\text{triv}} \), then the \( v = \phi_{21}(z) + \phi_{21}(z) \) belonging to \( B_{\text{triv}} \), thus \( \psi(z) = \phi' \phi(z) = \phi' \phi_{11}(z) + \phi'(v) = 
\)
with \( \varphi'(v) \in J^2_{\overline{B}} \). Then the degree-1 component of \( \psi \) coincides with the isomorphism \( \phi_{1,1} : B \rightarrow \overline{B}' \), implying by Proposition 3.1-(a) that \( \psi \) is a path algebra isomorphism. It now remains to check that \( \psi(J) = \overline{J}' \).

In view of (**) above, we have: \( \psi(J) = \varphi'(\varphi(J)) \subseteq \varphi'(\varphi(J)) = \varphi'(J)' = \overline{J}' \), so that \( \psi(J) \subseteq \overline{J}' \). Reciprocally, let \( z' \in \overline{J}' \), then let \( z' = \psi(z) \) for some \( z \in k\overline{Q} \); we have to check that \( z \) belongs to \( J \). We have \( z' = \varphi' \varphi(J) \), so that \( z' = \varphi'(\varphi(x)) \) for some \( x \in J \) and \( \varphi'(z) = \psi(z) = z' = \varphi'(\varphi(x)) \). Thus \( \varphi(z - x) = \phi(z) - \phi(x) \in \text{Ker}(\varphi') \subset J' = \varphi(J) \), showing that \( z - x \in J \).

But then, \( x \) being already an element in \( J \) we get \( z \in J \cap k\overline{Q} \), so that \( z = \varphi(z) \in J \). We conclude that \( \psi(J) = J' \) and \( \psi \) is a weak right-equivalence between \( \text{red}(Q', m') \) and \( \text{red}(Q', m) \).

When the trivial part of (\( Q, m \)) does not split, reductions as described in Definition 4.3 may not exist. However, examples from section 6 illustrate the fact that reduction or a notion a skew reduction still survive is some cases; but weak right-equivalence is still too restrictive to be a comparison tool between non-split reductions.

5. Symmetric potentials

The main result of this section is that the study of modulated quivers with symmetric potentials mimics the simply-laced study of quivers with potentials; in particular the sophisticated issue of skew permutations of general potentials is made easy for symmetric potentials.

As before, \( Q = (B, K, t) \) is a fixed modulated quiver over \( (K, t) \); the data \( (k\overline{Q}, \overline{Q}, \overline{Q}) \) and \( (k\overline{Q}, \overline{Q}) \) are respectively right projective and left projective bases associated with the dualizing pair \( \{B, B^*; b\} \).

Let \( \sum_{s \in \Lambda} e_s \otimes e_s^* \in Z_K(k\overline{Q}) \) be the Casimir element of the symmetric \( k \)-algebra \((K, t)\), then the set \( \{e_s, e_s^* : s \in \Lambda\} \) is a projective \( k \)-basis of \( K \) characterized by identities (2.1) which we recall: for all \( a \in K \),
\[
\sum_{s \in \Lambda} e_s(a) = \sum_{s \in \Lambda} \text{tr}(a e_s) e_s^*.
\]
The enveloping algebra \( K^e = K\overline{Q}K^e \) is endowed with the involution sending each \( (a \otimes b) \) to \( (a \otimes b)^* := b \otimes a \). Each \( K \)-bimodule \( M \) is naturally viewed as right and left \( K^e \)-module; we have: \( x(a \otimes b) = bxa = (a \otimes b)^* x \) for all \( x \in M \) and \( a, b \in K \).

Consider the \( K \)-module \( k\overline{Q}\otimes_{K^e} K \) which is identified with the \( K \)-module \( k\overline{Q}[K, k\overline{Q}] \), while \([K, k\overline{Q}] \) is the \( K \)-module generated by commutators \([a, v] := av - va \), with \( a \in K \) and \( v \in k\overline{Q} \). Indeed, writing \( \pi \) for the coset \( v + [K, k\overline{Q}] \) with \( v \in k\overline{Q} \), the map \( k\overline{Q}\otimes_{K^e} K \rightarrow k\overline{Q}[K, k\overline{Q}] : v \otimes a \mapsto \pi v \) yields a natural isomorphism with inverse \( k\overline{Q}[K, k\overline{Q}] \rightarrow k\overline{Q}\otimes_{K^e} K : \pi \mapsto v \otimes 1 \). In the sequel, each \( v \otimes 1 \in k\overline{Q}\otimes_{K^e} K \) will be simply denoted by \( v_{\otimes e} \).

Lemma 5.1. (a) We have a map \( \varphi_k = k\overline{Q}Z_k(k\overline{Q}) : v \mapsto \sum_{s \in \Lambda} e_s v e_s^* = \sum_{s \in \Lambda} e_s^* v e_s \), referred to as Casimir operator for \( k\overline{Q} \). It induces a \( Z(K) \)-linear map \( \varphi_k : k\overline{Q}_{\otimes e} K \rightarrow k\overline{Q}_{\otimes e} k \otimes k\overline{Q} : v_{\otimes e} \mapsto \varphi_k(v_{\otimes e}) = \varphi_k(v) \).

(b) The \( Z(K) \)-module \( k\overline{Q}_{\otimes e} K \) enjoys an ordinary cyclic permutation operator \( k\overline{Q}_{\otimes e} K \rightarrow k\overline{Q}_{\otimes e} K \) such that for every \( v = v_0 \cdots v_n \in B^{(n+1)} \) indexed over the cyclic group \( \mathbb{Z}_{n+1} := \{0, \ldots, n\} \), with \( x_i \in B \) and with corresponding ordinary cyclic permutation \( \varepsilon^n_{v} v = \sum_{i=0}^{n} x_{i} \cdots x_{i+n} \), we have: \( e_s(v_{\otimes e}) = \varepsilon^n_{v} v_{\otimes e} \).

Proof. Part (a) readily follows by the characterizing identities (2.1) of the Casimir element \( \sum_{s \in \Lambda} e_s \otimes e_s^* \in Z_K(K^e) \). And part (b) follows by the fact that \( au_{\otimes e} = ua_{\otimes e} \) for all \( u \in k\overline{Q} \) and \( a \in K \).

The module \( \varphi_k(k\overline{Q}) \) is called the Casimir ideal of \( Z(K) \) ([24, §3.2], [28, §2]) and does not depends on the choice of a projective \( k \)-basis \( \{e_s, e_s^* : s \in \Lambda\} \) and, in view of part (a) of Lemma 2.1, \( \varphi_k \) (K) does not depend on the tracing \( t \) chosen for \( K \). We shall refer to the elements of \( \varphi_k(k\overline{Q}) \) as symmetric potentials, and in view of part (b) of Lemma 5.1 we will also refer to the elements of \( k\overline{Q}_{\otimes e} K \) as symmetric potentials.

Next we consider the \( K \)-module \( B^e := \text{Hom}_{\text{mod}-K}(B, K^e) \) where \( B \) is regarded as right \( K^e \)-module, thus the natural left \( K^e \)-module structure of \( B^e \) is such that for all \( \xi \in B^e \) and \( a, b \in K \) we have \((a \otimes b)\xi(x) = (a \otimes b)\xi(x), \)
with $x \in B$. We refer to $B^\circ$ as the dual of $B$ as bimodule. For all $\alpha \in \mathbb{Z}(K)$, $x \in B$ and $\alpha \in B^\circ$, we note that $\alpha(x) = \alpha(xa)$. Hence, we naturally define a partial (left) derivatorivative $\partial : B^\circ \otimes \mathbb{Z}(K) \rightarrow B^\circ$ such that for all $\alpha \in B^\circ$, $x \in B$ and $v \in kQ$ we have: $\partial_v(xv,1) = \partial(\alpha \otimes (xv,1)) = \alpha(x)v$. Now, as in the simply-laced case, the cyclic derivatorivative $\partial : B^\circ \otimes \mathbb{Z}(K) \rightarrow B^\circ$ acts on symmetric potentials as follows: let $v = x_0 \cdots x_n \in B^{(n+1)}$ be any homogeneous tensor indexed over $\mathbb{Z}_{n+1} = \{0, \ldots, n\}$, with $x_i \in B$ and with corresponding ordinary cyclic permuation $c^\text{ord}_v = \sum_{i=0}^n x_i \cdots x_{i+n}$, then

$$\partial_v(v,1) := \partial(\alpha \otimes (v,1)) = \partial_{\alpha}(c^\text{ord}_v,1) = \sum_{i=0}^n \alpha(x_i)(x_{i+1} \cdots x_{i+n})$$

(5.1)

As for general potentials, to a symmetric potential $S \in kQ_{\otimes e}K$ is associated a Jacobian ideal $J_S := (\text{Im}(\partial_S))$.

The next result shows that the class of Jacobian ideals obtained from symmetric potentials in $kQ_{\otimes e}K$ and the corresponding ordinary cyclic derivaritive is exactly the class of Jacobian ideals obtained from symmetric potentials in $\mathfrak{h}(kQ_{(2)})$ and cyclic skew permutation and cyclic skew derivaritive.

**Proposition 5.5.** (1) The trace of the symmetric algebra $(K, t)$ yields a bimodule isomorphism $\tilde{t} : B^* \rightarrow B^\circ$ such that, for all $\alpha \in B^\circ$, we have: $\tilde{t}(\alpha(\alpha) \otimes -) = (\mathbb{I} \otimes \mathbb{I}) \alpha$ or equivalently, $b(- \otimes \tilde{t}(\alpha)) = (t \otimes \mathbb{I}) \alpha$. And for all $\xi \in B^\circ$ and $x \in B$ we have: $\tilde{t}^{-1}(\xi)(x) := \sum_{s \in \Lambda} b(\xi \otimes e_s)(x) = \sum_{s \in \Lambda} \epsilon_s \otimes b(xe^*_s \otimes x)$. Thus $\{B, B^\circ\}$ is a dualizing pair naturally isomorphic to the pair $\{B, B^*; b\}$.

(2) Let $S \in kQ_{\otimes e}K$ be a symmetric potential and $\mathfrak{h}(S)$ its image in $\mathfrak{h}(kQ_{(2)})$. Then $\epsilon_{\alpha}(\mathfrak{h}(S)) = \tilde{\epsilon}(\epsilon_S)$; and for every $\alpha \in B^\circ$ and $\xi = \tilde{t}(\alpha) \in B^\circ$ we have $\partial_{\alpha}S = \partial_{\tilde{\epsilon}}(\mathfrak{h}(S))$.

**Proof.** The proof of part (1) follows by a direct application of the characterizing identities (2.1) for the Casimir element $\mathfrak{h} \epsilon = \sum_{s \in \Lambda} \epsilon_s \otimes e^*_s$. Let us prove part (2). Let $S = v_0,1 \in kQ_{\otimes e}K$. To show that $\epsilon_{\alpha}(\mathfrak{h}(S)) = \tilde{\epsilon}(\epsilon_S)$, we may assumed without lost of generality that $v = x_0 \cdots x_n \in B^{(n+1)}$ is an homogeneous element indexed over the cyclic group $\mathbb{Z}_{n+1} = \{0, \ldots, n\}$, with $x_i \in B$. We have $\epsilon_{\alpha}(v_0,1) = (\sum_{i=0}^n x_i \cdots x_{i+n})\otimes 1$.

Writing $j_{B^\circ \otimes e} = \sum_{x \in \mathbb{Q}_1} x^* \otimes x$, we compute the skew left permutation of $\mathfrak{h}(S)$ as follows:

$$\epsilon_{\alpha}(\mathfrak{h}(S)) = \xi \left(\sum_{s \in \Lambda} \sum_{x \in \mathbb{Q}_1} b(x^* \otimes e_s, x)\right) = \sum_{s \in \Lambda} \sum_{x \in \mathbb{Q}_1} \sum_{t \in \mathbb{Q}_1} b(t \otimes e_s, x)\right) x \cdots x_0 e^*_s \otimes x = \sum_{s \in \Lambda} b(\xi \otimes e_s)(x) = \sum_{r, s \in \Lambda} b(\xi \otimes e_s)(x)$$

(5.1)

We deduce that: $\epsilon_{\alpha}(\mathfrak{h}(S)) = \sum_{i=0}^n \epsilon_{s_i}(\mathfrak{h}(S)) = \tilde{\epsilon}(\sum_{i=0}^n x_i \cdots x_{i+n})\otimes 1 = \tilde{\epsilon}(\epsilon_S)$. Next, $S$ being assumed to be any general symmetric potential, write $\epsilon_S = (\sum_{k=1}^p x_k v_k)\otimes 1$ with $x_k \in B$ and $v_k \in kQ$ for each $k \in [1, p]$.

For all $\xi \in B^\circ$ and $\alpha = \tilde{t}(\xi) \in B^\circ$, we have:

$$\partial_{\alpha}S = \sum_{k=1}^p \alpha(x_k)v_k = \sum_{k=1}^p \sum_{s \in \Lambda} b(\xi \otimes e_s x_k)\otimes e^*_s v_k = \sum_{k=1}^p \sum_{s \in \Lambda} b(\xi \otimes e_s x_k)\otimes e^*_s v_k = \sum_{k=1}^p \sum_{s \in \Lambda} b(\xi \otimes e_s x_k)\otimes e^*_s v_k$$

completing the proof of part (2).
Remark 5.1. The natural surjective map of topological bimodules $\pi_c : k\overline{Q}_k \to k\overline{Q}_{\mathcal{S}c} K : x \mapsto \pi(x) \otimes 1$ preserves permutations of tensors elements and cyclical equivalence, hence any property of a simply laced potential $w \in k\overline{Q}_k$ with respect to cyclical equivalence is transferred to the symmetric potential $w_{\mathcal{S}c} 1 \in k\overline{Q}_{\mathcal{S}c} K$. In particular, when $k$ is a field, the study of quivers with potentials with respect to cyclical equivalence applies to potentials in $k\overline{Q}_{\mathcal{S}c} K$.

The above connection been made, we can derive the following useful result on symmetric potentials.

**Lemma 5.3.** Suppose $k$ is a field. Let $I$ be a closed ideal in $J_{k\overline{Q}}$ and $J$ the closure of an ideal generated by finitely many elements $m_1, \ldots, m_p \in J_{k\overline{Q}_k}$. Then any symmetric potential $S$ belonging to $(I-J)_{\mathcal{S}c} 1$ is cyclically equivalent to a symmetric potential lying in $(I m_1 + \cdots + I m_p)_{\mathcal{S}c} 1$, thus $\tilde{\Delta}(S)$ is cyclically equivalent to a symmetric potential $W$ lying in $(I m_1 + \cdots + I m_p)$.

**Proof.** Let $S = v_{\mathcal{S}c} 1$ with $v \in IJ$. Then in view of Remark 5.1, the fact that $S$ is cyclically equivalent to a symmetric potential lying in $(I m_1 + \cdots + I m_p)_{\mathcal{S}c} 1$ is given by the corresponding simply-laced result in $k\overline{Q}_k$ (see [1, Lem. 13.8]). By part (2) of Proposition 5.2, $\varepsilon_c (\tilde{\Delta}(S)) = \tilde{\Delta} (\varepsilon_c S)$, implying that the symmetric potential $\tilde{\Delta}(S)$ is cyclically equivalent to a symmetric potential $W$ lying in $(I m_1 + \cdots + I m_p)$. □

Let us mention that a direct proof of Lemma 5.3 above, though a little bit technical, is still possible. Indeed, the notions of “$C$-space and $D$-space” used in [1, §13] to prove the simply laced analogue of Lemma 5.3 are easily seen to be special cases of symmetrizable weakly dualizing pairs of bimodules $(M, M^*)$ where $M$ arises as a union $\bigcup M_n$ of $K$-bimodules $M_n$ which are finite-dimensional as free and semisimple $k$-modules, with $0 = M_0 \subset M_1 \subset M_2 \subset \cdots$.

The assumption that $k$ is a field is directly required only by Lemma 5.3 above, it enables us to quickly establish the second main result of this work as follows.

**Theorem 5.4.** Under the Casimir operator $\tilde{\Delta} : k\overline{Q}_{\mathcal{S}c} K \to (k\overline{Q})$ and the natural isomorphism $B^\circ \cong B^*$, ordinary permutations and cyclic derivatives of symmetric potentials from $k\overline{Q}_{\mathcal{S}c} K$ agree with skew permutations and cyclic derivatives of their images in $\tilde{\Delta} (k\overline{Q})$, and when the Casimir ideal $\tilde{\Delta}(K)$ coincides with the center of $K$, all potentials on $Q$ are symmetric. Moreover, over a field $k$, the split reduction of modulated quivers with symmetric potentials $(Q, m)$ such that the cyclic derivative $B^* \xrightarrow{\partial m} \text{In}(\partial m)$ also splits can be defined up to right-equivalences.

Before proving Theorem 5.4, the following question retains our attention.

**Question 1.** When does the Casimir ideal $\tilde{\Delta}(K)$ of a symmetric algebras $K$ coincide with the center of $K$?

Recall the following definition to be compared with [26, Defn 2.1], [27, thm.3.1] presenting nine equivalent characterizations of symmetrically separable algebras.

**Definition 5.2.** A $k$-algebra $A$ is symmetrically separable (or strongly separable) if there exists a $k$-linear trace $\tau$ on $A$ such that $(A, \tau)$ is symmetric and the associated Casimir element $\sum_{s=1}^r \varepsilon_s \otimes \varepsilon_s^*$ is a (symmetric) separability idempotent for $A$: in particular $\sum_{s=1}^r \varepsilon_s \varepsilon_s^* = 1 = \sum_{s=1}^r \varepsilon_s^* \varepsilon_s$.  

30
If $\hat{\mathbb{A}}(K) = Z(K)$, then for all $K$-bimodule $M$ we also have $\hat{\mathbb{A}}(M) = Z_K(M)$.

Part (a) of Lemma 2.1 shows that if $K$ is symmetrically separable algebra then $\hat{\mathbb{A}}(K) = Z(K)$.

Thanks to a well-known Higman’s Theorem, Question 1 is completely solved when the ground ring is a field.

**Remark 5.3.**

- By Higman’s Theorem [25, thm 10] (or [28, thm 1]), separable algebras over a field are exactly those symmetric algebras $K$ such that $\hat{\mathbb{A}}(K) = Z(K)$.
- By a well-known result (see P.M. Cohn [29, Cor 11.6.8]), the tensor product over a field of two separable algebras is again separable and hence semisimple.

**Corollary 5.5.** If $K$ is separable over a field $k$, then so is the enveloping algebra $K^e$, $\hat{\mathbb{A}}(K) = Z(K)$, all potentials on $Q$ are symmetric and can treated symmetrically using ordinary cyclic permutation and ordinary cyclic derivative from $\hat{\mathbb{A}}_{Q^e\otimes \mathbb{A}} K$, and the reduction of every $k$-modulated quiver with potential is well-defined up to right-equivalences. This is in particular the case when $k$ is a perfect field.

**Proof of Theorem 5.4.**

The first part of Theorem 5.4 is a direct consequence of Proposition 5.2. We dedicate the rest of this section to establish the last part of Theorem 5.4. Thus $(K, t)$ is symmetric over a field $k$, and $(Q, m)$ is a modulated quiver with symmetric potential having a split trivial part such that $\partial m : B^* \rightarrow \text{Im}(\partial m)$ also splits. We need to construct and manipulate unitriangular automorphisms. In the following the assumption that $k$ is a field is not used.

**Lemma 5.6.** Let $f : B^* \rightarrow M$ be a split bimodule epimorphism with right inverse $f'$, $W \in M^eM$ a potential with $M' \subset J^2_{kK}$ and $M \subset J^1_{kK}$, and put $W' = (1_{M'} \otimes f')(W)$.

Then the following assertions hold.

1. There is a bimodule morphism $\alpha : B \rightarrow M'$ such that $W = \sum_{y \in Q_1} b_y \otimes f(y^*)$ with $b_y := \alpha(y)$ for all $y \in Q_1$. There is a unitriangular automorphism $\phi : \hat{\mathbb{A}}_{Q^e \otimes \mathbb{A}} K$ with $\phi_{1_{B}} = \mathbb{I}_B + \alpha$.

2. Let $S$ be a reduced potential on $Q$ and the unitriangular automorphism above. Then $\phi(S) - S = \sum_{y \in Q_1} b_y \partial_{*,y} S$ is cyclically equivalent to a potential $S'$ in $I_{Q^e} J^2_{kK}$ where $I$ is the closed ideal given by:

$$I = \{(b_y : y \in Q_1) \} = \{(b_x : x \in Q_1) \}$$

with $b_x = \partial^*(W' \otimes x)$ for each $x \in Q_1$. Moreover, if $S$ is symmetric, then so is $S'$.

**Proof.** Let us prove (1). The element $W' = (1_{M'} \otimes f')(W) \in J^2_{kK} B^*$ is $K$-central since $W$ is, hence $W'$ is a potential on $Q \otimes Q^*$ and we get a bimodule morphism $\alpha := \partial W' : B \rightarrow M'$. In view of (3.8) we have $W' = \sum_{y \in Q_1} \partial^* (W' \otimes y) \otimes y^*$, and since $f \circ f' = 1_{M'}$ we get $W = (1_{M'} \otimes f)(W') = \sum_{y \in Q_1} b_y \otimes f(y^*)$ with $b_y = \partial^* (W' \otimes y) = \alpha(y)$ for each $y \in Q_1$. By Proposition 3.1, the bimodule morphism $1_{B} + \alpha : B \rightarrow B \oplus M' \subset B \oplus J^2_{kK}$ induces a unitriangular automorphism $\phi$ of $\hat{\mathbb{A}}_{Q^e \otimes \mathbb{A}} K$ with $\phi_{1_{B}} = \mathbb{I}_B + \alpha$.

We now turn to the proof of part (2). We have $\{b_y : y \in Q_1\} \rightarrow K = \mathbb{I}(\partial^* W') = K \cdot \{b_x : x \in Q_1\}$ with $b_x = \partial^* (W' \otimes x)$ and $b_x = \partial^*(W' \otimes x)$ for each $(x, y) \in Q_1 \times Q_1$. We start with the case of an homogeneous potential $S = \sum_{i=1}^{n} u_{i,0} u_{i,1} \cdots u_{i,d}$ with $d \geq 2$ and $u_{i,r} \in B$ for all $(i, r) \in [1, n] \times [0, d]$. As in the statement of the cyclic Leibniz rule (4.4), for each $(i, r) \in [1, n] \times [0, d]$ we write: $u_{i, < r} = \prod_{k=0}^{r-1} u_{i,k} \prod_{k=r+1}^{d} u_{i,k}$ and $u_{i, > r} = \prod_{k=r+1}^{d} u_{i,k}$, where the empty products coincide with 1 in $K$. Then, expanding $\phi(S)$ we write: $\phi(S) = \sum_{i=1}^{n} (u_{i,0} + \alpha(u_{i,0}))(u_{i,1} + \alpha(u_{i,1})) \cdots (u_{i,d} + \alpha(u_{i,d})) = S + S_1 + S_{2(1)}$, where $S_1 = \sum_{r=0}^{d-1} \sum_{i=1}^{n} u_{i, < r} - \alpha(u_{i,r}) u_{i, > r}$.
For each \( r \in [0, d] \), the term \( S_{1,r} := \sum_{i=1}^{n} u_{i,<r} \alpha(u_{i,r}) u_{i,r} \) is an homogeneous potential, which is then cyclically equivalent to the left permutation \( \varepsilon_{r} S_{1,r} \) and we have:

\[
\varepsilon_{r} S_{1,r} = \sum_{x \in \mathcal{Q}_{1}} \sum_{i=1}^{n} b\left(x^{*} \otimes u_{i,<r}\right)\alpha(u_{i,r}) u_{i,r} = \sum_{x \in \mathcal{Q}_{1}} \sum_{i=1}^{n} \alpha\left(\hat{b}(x^{*} \otimes u_{i,<r}) u_{i,r}\right) u_{i,r}
\]

\[
= \sum_{x \in \mathcal{Q}_{1}} \sum_{i=1}^{n} \alpha(\gamma) \sum_{y \in \mathcal{Q}_{1}} b\left(y^{*} \otimes \hat{b}(x^{*} \otimes u_{i,<r}) u_{i,r}\right) u_{i,r}
\]

\[
= \sum_{x \in \mathcal{Q}_{1}} \sum_{i=1}^{n} \alpha(\gamma) \sum_{y \in \mathcal{Q}_{1}} \hat{b}\left(x^{*} \otimes u_{i,<r}\right) u_{i,r}
\]

\[
= \sum_{x \in \mathcal{Q}_{1}} \sum_{i=1}^{n} \alpha(\gamma) \hat{b}\left(x^{*} \otimes u_{i,<r}\right) u_{i,r}
\]

\[
= \sum_{x \in \mathcal{Q}_{1}} \sum_{i=1}^{n} \alpha(\gamma) \hat{b}\left(x^{*} \otimes u_{i,<r}\right) u_{i,r}
\]

Thus, \( S_{1} = \sum_{r=0}^{d} \varepsilon_{r} S_{1,r} = \sum_{r=0}^{d} \sum_{x \in \mathcal{Q}_{1}} \sum_{i=1}^{n} b\left(x^{*} \otimes u_{i,<r}\right) \alpha(u_{i,r}) u_{i,r} \) is cyclically equivalent to a potential \( \varepsilon_{r} S \) for all \( r \in [1, d] \), and \( v_{r} \in N_{k}^{n} \bigcap I \).

As above, we get that the right permutation \( \varepsilon_{r} S_{2,r} \) belongs to \( \sum_{x \in \mathcal{Q}_{1}} J_{k}(J_{k}^{d} \bigcap I)b_{x} \), implying that \( S_{2} \) is cyclically equivalent to an element of \( \sum_{x \in \mathcal{Q}_{1}} J_{k}(J_{k}^{d} \bigcap I)b_{x} \).

Now, for a general potential \( S \in J_{k}^{d} \), written as sum of homogeneous potentials, the previous discussion shows that \( \phi(S) - S - \sum y_{x} \partial_{y} S \) is cyclically equivalent an element of the form \( \sum c_{x} b_{x} \), where \( c_{x} = \sum_{y \in \mathcal{Q}_{1}} y_{x} c_{x,y} \), with \( c_{x,y} \in J_{k}^{d-1} \bigcap I \). Since \( I \) is closed, each \( c_{x} \) is a well-defined element of \( J_{k}^{d} \). Thus \( \phi(S) - S - \sum y_{x} \partial_{y} S \) is cyclically equivalent to a potential \( S' \) in \( J_{k}^{d} \). Finally, if \( S \) is symmetric, then part (2) of Proposition 5.2 implies that \( S' \) is also symmetric.

As in the simply-laced case, the fact that the reduction of \((\mathcal{Q}, m)\) is defined up to right-equivalences will be derived as consequence of the following result whose proof relies on Lemma 5.6 and Lemma 5.3.

**Proposition 5.7.** Let \( S \) and \( S' \) be reduced potentials on \( \mathcal{Q} \) such that \( S' - S \) is cyclically equivalent to a potential \( S'' \) in \((J_{S})^{2}\). Then the following statements hold.

1. \( J_{S} = J_{S} \).
2. Suppose \( S \) and \( S'' \) are symmetric and the cyclic derivative \( B^{*} \overset{\partial S}{\longrightarrow} \text{Im}(\partial S) \) splits. Then \((\mathcal{Q}, S)\) is right-equivalent to \((\mathcal{Q}, S')\); more precisely there exists a unipotential automorphism \( \phi \) of \( k\mathcal{Q} \) such that \( \phi(S) \) is cyclically equivalent to \( S' \) and \( \phi(u) = u \in J_{S} \) for all \( u \in k\mathcal{Q} \).

**Proof.** Put \( \tilde{J} = J_{k}^{d} \). We have \( J_{S} \subset \tilde{J}^{2} \) because \((\mathcal{Q}, S)\) is reduced. Part (1) is an easy consequence of the cyclic-Leibniz rule (4.4): indeed, for all \( \xi \in B^{*} \) we have \( \partial_{\xi} S' - \partial_{\xi} S = \partial_{\xi} S'' = \tilde{J}_{S} + J_{S}\tilde{J}, \) implying in view of part (4) of Lemma 4.4 that \( J_{S} = J_{S} \).

For part (2), by assumption we have a split epimorphism \( f : B^{*} \longrightarrow M \) with \( M = \partial(B^{*} \otimes S) \) and \( f(\xi) = \partial_{\xi} S \) for all \( \xi \in B^{*} \). Let \( f' \) be a right inverse for \( f \). Using induction on \( n \), we will construct a sequence of unitriangular automorphisms \( \phi_{n} : k\mathcal{Q} \longrightarrow k\mathcal{Q}, \) with \( n \geq 0 \) and \( \phi_{0} = 1_{k\mathcal{Q}} \), taking each generator \( y \in n\mathcal{Q}_{1} \) to \( y + b_{y},n \) and having the following properties:

1. \( b_{y,n} \in \tilde{J}^{n+1} \bigcap J_{S} \) for all \( y \in n\mathcal{Q}_{1} \) and all \( n \in \mathbb{N} \).
2. The sum \( \sum_{y \in n\mathcal{Q}_{1}} y \partial_{y} S \) is a symmetric potential and \( S' \) is cyclically equivalent to the symmetric potential \( \phi_{0}\phi_{1} \cdots \phi_{n-1}(S + \sum_{y \in n\mathcal{Q}_{1}} y \partial_{y} S) \) for all \( n \geq 1 \).
The existence of $\phi_1$ with the desired properties follows by part (1) of Lemma 5.6 and by Lemma 5.3 in which we take $I = J_S = J$ and $m_0 = f(y^\ast) = \partial_S S$.

Now assume that, for some $n \geq 1$, we have already defined $\phi_1, \ldots, \phi_n$ having the desired properties. We then want to construct a unitriangular automorphism $\phi_{n+1}$ such that properties (i)-(ii) are satisfied with $n$ replaced by $n + 1$. By part (2) of Lemma 5.6, $(S + \sum_{y \in \mathbb{Q}_1} b_{y,n}\partial_y S) - \phi_n(S)$ is cyclically equivalent to a symmetric potential $W_1$ belonging to $\hat{J}(\hat{J}_n \cap J_S)^2$. In particular, observing that $\hat{J}(\hat{J}_n \cap J_S)^2 \subseteq (\hat{J}_{n+2} \cap J_S)J_S$, we deduce that $S - \phi_n(S)$ is cyclically equivalent to a symmetric potential in $(J_S)^2$. Thus combining part (1) of Theorem 4.5 together with the already proved part (1) of Proposition 5.7, we conclude that $\phi_n(J_S) = J_{\phi_n(S)} = J_S$. It follows that the symmetric potential $(S + \sum_{y \in \mathbb{Q}_1} b_{y,n}\partial_y S) - \phi_n(S)$ is in fact cyclically equivalent to a symmetric potential $\phi_n(W_2)$ belonging to $\phi_n((\hat{J}_{n+2} \cap J_S)J_S)$, where $W_2 = \phi_n^{-1}(W_1)$ is a symmetric potential in $(\hat{J}_{n+2} \cap J_S)J_S$. But then applying Lemma 5.3 to $I = \hat{J}_{n+2} \cap J_S$ and $J = J_S$, we get that $W_2$ is cyclically equivalent to a symmetric potential $W$ lying in $(\hat{J}_{n+2} \cap J_S)M$, and part (1) of Lemma 5.6 yields a unitriangular automorphism $\phi_{n+1} : k\mathcal{Q} \to k\mathcal{Q}$ taking each $y \in \mathbb{Q}_1$ to an element $y + b_{y,n+1}$ with $b_{y,n+1} \in \hat{J}_{n+2} \cap J_S$ and such that $W = \sum_{y \in \mathbb{Q}_1} b_{y,n+1}\partial_y S$

Now, the fact that $(S + \sum_{y \in \mathbb{Q}_1} b_{y,n}\partial_y S) - \phi_n(S)$ is cyclically equivalent to $\phi_n(W) = \phi_n\left( \sum_{y \in \mathbb{Q}_1} b_{y,n+1}\partial_y S \right)$ shows that $(S + \sum_{y \in \mathbb{Q}_1} b_{y,n}\partial_y S)$ is cyclically equivalent to $\phi_n(S + \sum_{y \in \mathbb{Q}_1} b_{y,n+1}\partial_y S)$, thus the assumption that $S'$ is cyclically equivalent to $\phi_0\phi_1 \cdots \phi_{n-1}(S + \sum_{y \in \mathbb{Q}_1} b_{y,n+1}\partial_y S)$ shows that $S'$ is cyclically equivalent to $\phi_0\phi_1 \cdots \phi_{n-1} \phi_n(S + \sum_{y \in \mathbb{Q}_1} b_{y,n+1}\partial_y S)$. We have therefore constructed a unitriangular automorphism $\phi_{n+1}$ such that properties (i)-(ii) are satisfied with $n$ replaced by $n + 1$, completing the induction step.

Now, in view of property (i), letting $\phi = \lim_{n \to \infty} \phi \ast \cdots \ast \phi$, we get a well-defined unitriangular automorphism of $k\mathcal{Q}$ such that $\phi(u) - u \in J_S$ for all $u \in k\mathcal{Q}$. And letting $n$ tends to $\infty$ in property (ii), we conclude that $S$ is cyclically equivalent to $\phi(S)$, completing the proof of part (2) of Proposition 5.7.

Now, using Proposition 5.7, we will show that the reduction of $(\mathcal{Q}, m)$ can be defined up to right-equivalences. Thus, let $\phi : k\mathcal{Q} \to k\mathcal{Q}'$ be a right-equivalence between $(\mathcal{Q}, m)$ and a modulated quiver with potential $(\mathcal{Q}', m')$ where $\mathcal{Q}' = (B', K, t)$. Since $\phi$ is obviously a weak right-equivalence and the reduction is defined up to weak-right-equivalences by the reduction Theorem 4.6, we derive the following conclusions: $(\mathcal{Q}', m')$ has a split trivial part, and keeping the notations of (4.6) and (4.7) we write: $\mathcal{Q} = \mathcal{Q}_{\text{triv}} \oplus \mathcal{Q}', \mathcal{Q}' = \mathcal{Q}'_{\text{triv}} \oplus \mathcal{Q}$ as direct sums of naturally induced modulated quivers where $\mathcal{Q}_{\text{triv}} = (B_{\text{triv}}, K, t)$ with $B_{\text{triv}} = \partial(B^\ast \oplus m_2)$ and $m_2$ denotes the degree-2 component of $m$, $B = B_{\text{triv}} \oplus \mathcal{B}$, $\mathcal{B} = (B, K, t)$; and similarly $\mathcal{Q}'_{\text{triv}} = (B'_{\text{triv}}, K, t)$ with $B'_{\text{triv}} = \partial(B'^\ast \oplus m'_2)$, the degree-one component $\phi_1 : B_{\text{triv}} \to B'$ of $\phi$ is an isomorphism with $\phi_1(B_{\text{triv}}) = B'_{\text{triv}}$ and $B' = B'_{\text{triv}} \oplus \mathcal{B}$, $\mathcal{B}' = (B', K, t)$.

Still by Theorem 4.6, consider the reduction $\pi_m : k\mathcal{Q} \to k\mathcal{Q}$ from $(\mathcal{Q}, m)$ into $\text{red}(\mathcal{Q}, m) = (\mathcal{Q}', \mathcal{M})$ and reduction $\pi_{m'} : k\mathcal{Q} \to k\mathcal{Q}'$ from $(\mathcal{Q}', m')$ into $\text{red}(\mathcal{Q}, m) = (\mathcal{Q}', \mathcal{M}')$. Recall that

\[ \mathcal{M} = \pi_m(m), \mathcal{M}' = \pi_m(m'), \pi_m|_{\mathcal{M}} = \text{id} \text{ and } \pi_m|_{\mathcal{M}'} = \text{id}. \]

Next, $\phi$ induces a weak right-equivalence $\psi := \pi_{m'}\phi_{k\mathcal{Q}} : k\mathcal{Q} \to k\mathcal{Q}'$ between reduced modulated quivers with potentials $(\mathcal{Q}, \mathcal{M})$ and $(\mathcal{Q}', \mathcal{M}')$, and we have:

\[ \phi(J_{\mathcal{M}}) = J_{\mathcal{M}'}; \quad \pi_m(J_{\mathcal{M}}) = J_{\mathcal{M}} \text{ and } J_{\psi(\mathcal{M})} = \psi(J_{\mathcal{M}}) = J_{\mathcal{M}'} = \pi_m(J_{\mathcal{M}'}). \]

To show that $(\mathcal{Q}, \mathcal{M})$ and $(\mathcal{Q}', \mathcal{M}')$ are right-equivalent, by Proposition 5.7 it suffices to prove that:

(\(\alpha\)) The potential $S := \psi(\mathcal{M})$ is symmetric and $\mathcal{M}' - S$ is cyclically equivalent to a symmetric potential in $(J_S)^2$. 33
(β) The cyclic derivative $\partial S : \overline{B}^* \to \text{Im}(\partial S)$ splits.

For (α), we have $S = \psi(\mathfrak{m}) = \pi_m^\prime \phi(\pi_m^\prime(m))$, showing that $S$ is symmetric since $m$ is. The last part of Lemma 4.8 shows that $m - \pi_m^\prime(m)$ is cyclically equivalent to a symmetric potential $W$ lying in $(\text{Ker}(\pi_m^\prime))^2$. Since reductions and weak right-equivalences send cyclically equivalent potentials to cyclically equivalent ones, we deduce that:

$$\overline{m}^\prime - S = \pi_m^\prime(m) - \pi_m^\prime(\phi(\pi_m^\prime(m))) = \pi_m^\prime(m - \phi(\pi_m^\prime(m)))$$

By the definition of a right-equivalence, $m^\prime \equiv \pi_m^\prime(\phi(m))$.

But, using (α) above we get $\pi_m^\prime(\phi(J_m)) = \pi_m^\prime(J_m) = \mathfrak{j}_m = \psi(J_m) = J_S$, so that $\pi_m^\prime(\phi(J_m)^2) = (J_S)^2$. Hence, $\overline{m}^\prime - S$ is cyclically equivalent to a symmetric potential in $(J_S)^2$, completing the proof of (α).

We now turn to the proof of (β). Observe that the map $\overline{B}^* \to \text{Im}(\partial S)$ splits whenever its kernel is a direct summand in $\overline{B}^*$, but since $S = \psi(\mathfrak{m})$ and $\psi$ is a path algebra isomorphism, applying part (1) of Theorem 4.5 we get that, $\text{Ker}(\partial S(\mathfrak{m}))$ is a direct summand in $\overline{B}^*$ if and only if $\text{Ker}(\partial(\mathfrak{m}))$ is a direct summand in $\overline{B}$, hence, we have to show that the cyclic derivative $\overline{B}^* \to \text{Im}(\partial(\mathfrak{m}))$ splits. By the reduction Theorem 4.6 (or Lemma 4.8), there is a unithomorphim $\varphi : (Q/kQ) \to kQ$ such that $\varphi(m)$ is cyclically equivalent to $\mathfrak{j}_{U \otimes V} + \mathfrak{m}$ with $B_{\text{triv}} = U \oplus V$ for an induced dualizing pair $\{U, V\}$. And By assumption the cyclic derivative $B^* \to \text{Im}(\partial(\mathfrak{m}))$ splits, implying that the cyclic derivative $B^* \to \text{Im}(\partial(\varphi(m)))$ splits, so that:

the cyclic derivative $\partial(\mathfrak{j}_{U \otimes V} + \mathfrak{m}) : B^* \to \text{Im}(\partial(\mathfrak{j}_{U \otimes V} + \mathfrak{m}))$ splits.  

(***)

Recall that $B = B_{\text{triv}} \oplus \overline{B}$, $B_{\text{triv}} = U \oplus V = \text{Im}(\partial(\mathfrak{j}_{U \otimes V}))$, $\mathfrak{m} \in J_{Q/kQ}^2$, $B^* = B_{\text{triv}} \oplus \overline{B}^*$ where $\overline{B}^*$ is the kernel of the cyclic derivative $\partial(\mathfrak{j}_{U \otimes V}) : B^* \to \text{Im}(\partial(\mathfrak{j}_{U \otimes V}))$. Therefore, $\partial(\mathfrak{j}_{U \otimes V} + \mathfrak{m}) = \partial(\mathfrak{j}_{U \otimes V}) \cap \mathfrak{m}$ and $B^* \to \text{Im}(\partial(\mathfrak{m}))$ also splits and completing the proof of (β). Hence, the proof of the last part of Theorem 5.4 is finished.

6. Some examples in the inseparable context

Here we illustrate the fact that the reduction of a modulated quiver with potential $(Q, m)$ may still be carried even if the trivial part of $(Q, m)$ does not split.

Let $k = F_2(u,v,u^{-1})$ be the non perfect function field of one variable over the prime field $F_2$ of characteristic 2; $E = F_2(u^2)$ and $\mathfrak{f} = F_2(u^\frac{1}{2}) = k\{1, u^\frac{1}{2}, u, u^2 \}$ = $E(1, u^2)$. Then $E$ and $\mathfrak{f}$ are finite-dimensional inseparable extensions of $k$. Let $K = k \times k \times k$ with $k_1 = F = k_2$ and $k_3 = k$; each $k_i$ is viewed as subfield in $K$ with unity $1_i$, thus the unity of $K$ is $1_1 + 1_2 + 1_3$. For $\lambda \in \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}\}$ we write $u^{\lambda} = 1_i, u \in k_i$. We have a symmetric k-algebra $(K, t)$ where $t$ is the natural k-linear trace induced by its restriction on $F$ as follows: $t(1) = 1$ and $t(u^\frac{1}{2}) = t(u) = t(u^2) = 0$. We have Casimir elements $\mathfrak{j}_{\mathfrak{g}, t} = 1 \otimes 1 \otimes u^{1/2} \otimes u^{-1} + u^\frac{1}{2} \otimes u^{-1} \otimes u^\frac{1}{2} + u^2 \otimes u^{-2}$ and $\mathfrak{j}_{K} = \sum_{s=0}^{9} e_s \otimes e_s^* \in \mathcal{S} \{(e_s)_{1 \leq s \leq 9} = (1, 1, 1, 1, 1, 1, 1, 1, 1)\}$. The canonical element $\sum_{s=0}^{8} e_s e_s^*$ is equal to $1_3$ and the Casimir ideal $\mathfrak{I}(K)$ coincides with $k_3$, showing that $\mathfrak{I}(K) = \mathbb{Z}(K) = K$. Let us pose some useful notations: for $M \in \{F, F_{\mathfrak{g}}\}$ and $i, j \in \{1, 2, 3\}$, let $M_{ij}$ have the natural $k_i, k_j$-bimodule structure on $M$; when viewed as element in $M_{ij}$ each $x \in M$ may still be written as $x$ or be subscripted as $x_{ij}$ more precision is needed. In particular put, $z_j := z_{ij} = 1_{ij} + u^{\frac{1}{2}} \otimes u^{-1} \in F_{\mathfrak{g}} F_j$ with $(i, j) = (1, 2), (2, 1)$.

Consider the modulated quiver $Q = (B, K, t) : k_1 \to k_2$ with $B = F_{\mathfrak{g}} F_2 \oplus F_1 \oplus F_2 \oplus F_3$, and $B^* = F_{\mathfrak{g}} F_1 \oplus F_2 \oplus F_3 \oplus F_3$. The bilinear form in the symmetrizable dualizing pair $\{B, B^*; b\}$ is induced by
we have $b((x \otimes y) \otimes (x' \otimes y')) = b((y \otimes x') \otimes (x \otimes y')) = x' b((x' \otimes y') \otimes (x \otimes y'))$; for $i = 1, 2$, $x \in F_i$ and $x' \in F'_i$, we have $b(x \otimes x') = x x'$ and $b(x' \otimes x) = x' x$. Note that $b(1 \otimes x) = 1 + 1 = 0$. The Casimir operator $\delta : kQ \to kQ, x \mapsto \sum_{y \in Q_1} y \otimes y^*$, vanishes on $(F \otimes F_2) \otimes F_1 \oplus (F_2 \otimes F)$. For Casimir element $\delta_{\alpha \otimes \beta} = \sum_{x \in Q_1} x^* \otimes x$, we compute the cyclic derivatives $\partial_{1z_2} \alpha \otimes \partial_{2z_1} \beta$, for all $\alpha, \beta \in \mathcal{B}$.

**Example 6.1:** A nonsymmetric potential of degree 2. $W := 1 \otimes 1$, is a nonsymmetric potential on $Q$, the subbundle $U = F_1 \{1, z_2\} F_2$ is one-dimensional on both sides and is not a direct summand in $F \otimes F_2$. We compute: $\partial_{1z_2}^2 W = 11$, and $\partial_{(z_1 \otimes 1)}^2 W = 1z_2$. Thus for the modulated quiver with potential $(Q, W)$ we have $B_{\text{triv}} = U \oplus 2F_1$ and $B := B_{\text{red}} = U \oplus 2F_3 \oplus 3F_1$ where $U = (F \otimes F_2) / W$. Let $\mathcal{Q} = (B, k, t)$ be the corresponding reduced modulated quiver. We get that $J_W$ coincides with the closed ideal $L = (B_{\text{triv}})$ and the natural projection $\rho : kQ \to k\mathcal{Q}$ is a non-split reduction from $(Q, W)$ to $(\mathcal{Q}, 0)$.

**Example 6.2:** A family $(m_n)_{n \in \mathbb{N}}$ of nonsymmetric potentials. Keep the notations from Example 6.1. For each $n \in \mathbb{N}$, we consider the nonsymmetric potential with the convention that $W^0$ is the unity of $K$:

$$m := W + SW^n \text{ where } S := (1 \otimes 1) \otimes (\alpha u^1) + (u^4 \otimes 1) \otimes \alpha \in \mathcal{B}$$

Thus $S$ is a symmetric potential, we get $S = z_2 \otimes u^2 \alpha + (1 \otimes 1) \otimes (\alpha u^1 + u^1 \alpha) - (1 \otimes u^4) = z_2 u^2 + (1 \otimes 1) u^2 \alpha$. The degree-2 component of $m$ is $m_2 = W = z_2 \otimes 1_1, 1 \in F \otimes F_2 \otimes 2F_1$ and, as in Example 6.1 above, $(Q, m_n)$ is a modulated quiver with potential such that $B_{\text{triv}} = U \oplus 2F_1$ and $B = U \oplus 2F_3 \oplus 3F_1$.

We will compute the cyclic derivatives $\partial_{1z_2} m_n$ and $\partial_{2z_1} m_n$. We need the following permutations: $SW^n, z_2^2 (SW^n), \ldots, z_2^n (SW^n)$. We have $\partial_{1z_2} (S) = u^2 \alpha$ and by definition, $\partial_{1z_2} (SW^n) = z_2 \otimes z_1 \otimes S 2^n (SW^n) = z_2 \otimes 1_1 \otimes S + 0 = W$. Thus $\partial_{1z_2} (SW^n) = u^2 \alpha W^n + z_2 (SW^n - 1) W + SW^n + \ldots + W^{n-1} S$ for all $n \geq 1$. Apply a similar argument to compute $\partial_{2z_1} (SW^n)$. We deduce that

$$\partial_{1z_2} m_n = z_2 + (1 + \sum_{r=0}^{n-1} W^r W^{n-r})_1, (2)$$

$$\partial_{2z_1} m_n = (1 + \sum_{r=0}^{n-1} W^r W^{n-r})_1, (2)$$

Let $J_0$ be the closed ideal in $kQ$ generated by $\partial_{1z_2} m_n$ and $\partial_{2z_1} m_n$. Let $\rho : kQ \to k\mathcal{Q}$ be the natural projection. The right $K$-linear map $\mathcal{U} \to F \otimes F_2$, taking the coset $1 \otimes 1$ to $1 \otimes 1$, yields a right $K$-linear map $\rho' : B \to B$ such that $\rho_{\alpha \beta} = 1_{(\mathcal{U} \otimes 1, 1)} \otimes 1_1 \otimes 1_1$, inducing a bimodule morphism $\rho' : \pi : kQ \to k\mathcal{Q}$ is the natural projection. Thus $J_0 = \text{Ker} (\pi)$ satisfies condition (1)(ii) of Definition 4.3 for trivial parts of Jacobian ideals (that is, kernels of reductions). Now we have the two following cases.

(a) The case $n \geq 1$. By (6.2) above we get that $1 \otimes 1$ is the element $(1 + \sum_{r=0}^{n-1} W^r W^{n-r})_1, (2)$. Therefore the natural projection $\rho : kQ \to k\mathcal{Q}$ is a non-split reduction from $(Q, m_n)$ to $(\mathcal{Q}, 0)$.
(b) The case of potential \( m_0 = W + S \). Here, \( \partial_{z_1}, m_0 = z_2 \in J_0 \) and \( \partial_{z_2}, m_0 = z_1 + \frac{1}{2} \alpha \in J_0 \).

However, \( z_1, z_2 \in F_1 \) is \( F \)-central while the \( E \)-central element \( 2 \alpha \) is not \( F \)-central, and we note that the \( K \)-bimodule generated by the set \( \{ \partial_{z_1}, m_0, \partial_{z_2}, m_0 \} \) contains an element in \( \mathcal{J}^{2}_{k \mathcal{Q}} \), namely the element \( \gamma = (\alpha u_1 + \alpha u_2) = (z_1 + \frac{1}{2} \alpha) + \frac{1}{2} (z_1 + \frac{1}{2} \alpha)u_1^2 \). Thus \( J_0 \) does not satisfy condition (1.i) in Definition 4.3. Indeed, one can check that there is no \( K \)-bimodule morphism \( f = \left[ \begin{array}{c|c} I & \hat{1}_{n_{triv}} \end{array} \right] : B_{triv} \rightarrow B_{triv} \oplus J^{2}_{k \mathcal{Q}} \) such that \( f(B_{triv}) \subset (\partial m_0)(B^*) \), showing that there is no reduction on \( (\mathcal{Q}, m_0) \) as described by Definition 4.3.

However, (as in the proof of (2) of Theorem 4.5) the right \( K \)-linear map \( \rho': B \rightarrow B \) extends to a right \( K \)-linear morphism \( \rho': k \mathcal{Q} \rightarrow k \mathcal{Q} \) such that the map \( \mathcal{P} = \pi \circ \rho' : k \mathcal{Q} \rightarrow k \mathcal{Q} / I_0 \) is a surjective morphism of topological algebras; one checks that \( \text{Ker}(\mathcal{P}) \) is the closed ideal \( I_0 \) generated by the element \( \gamma \) above. We have an epimorphism of path algebras \( \phi : k \mathcal{Q} \rightarrow k \mathcal{Q} / I_0 \) defined on the bimodule \( B = \mathcal{F} \otimes k \mathcal{Q} / I_0 \) as follows:

\( \phi \) is induced by \( \rho \) over \( \mathcal{F} \otimes k \mathcal{Q} / I_0 \), and over \( k \mathcal{Q} / I_0 \), we have: \( \phi(z_1) = -u_2^2 \alpha + I_0 = u_2^2 \alpha + I_0 \).

Thus \( \text{Ker}(\phi) = J_0 \) (and we note that \( \phi(m_0) = 0 \)). Let \( m_0 = \rho(m_0) \in k \mathcal{Q} / I_0 \), then \( m_0 = 1 \otimes 1_2 \otimes (\alpha u_1 + \alpha u_2) \in \mathcal{F}_1 \otimes k \mathcal{Q} / I_0 \). For the reduced quiver with potential \( (\mathcal{Q}, m_0) \), the morphism of topological algebras \( \phi \) above yields an isomorphism of Jacobian algebras \( \mathcal{J}(\mathcal{Q}, m_0) \cong \mathcal{J}(\mathcal{Q}, m_0) \).

**Skew reductions.** In point (b) above, \( \phi \) is an instance of what we may refer to as **skew reduction**.

With previous observations, it is not difficult to derive the following consequence of Theorem 4.5.

**Corollary 6.1.** Let \( (\mathcal{Q}, m) \) be a modulated quiver with potential, with \( B_{triv} = U \oplus V \) and \( m_2 = \gamma_{U \otimes V} \) and \( \mathcal{Q} \) the corresponding reduced modulated quiver. Suppose that \( B = V \oplus B_1 \) for some subbimodule \( B_1 \) containing \( U \) such that \( m - m_2 \in k \mathcal{Q} \) where \( \mathcal{Q} = (B_1, K, t) \). Then there is a reduction or a skew reduction from \( (\mathcal{Q}, m) \) to a reduced modulated quiver with potential \( (\mathcal{Q}, \rho(m)) \).

**7. Mutations of modulated quivers with potentials**

Let us mention that a motivation for lifting mutations of quivers with potentials to mutations of modulated quivers with potentials comes from a successful non simply-laced generalization of cluster structures on 2-Calabi-Yau categories over arbitrary fields. As before, we fix a modulated quiver with potential \( (\mathcal{Q}, m) \) over a symmetric algebra \( (k, t) \), with \( \mathcal{Q} = (B, K, t) \); where \( B \) is part of a symmetrizable dualizing pair \( \{ B, B^* \} \). Also, \( (k\mathcal{Q}, k\mathcal{Q}^*) \) and \( (k\mathcal{Q}^1, k\mathcal{Q}^2) \) stand respectively for a chosen right projective basis and left projective basis for \( B \) over \( K \).

Note that we can write \( K = \prod_{i \in I} k_i \) as direct product of indecomposable \( k \)-algebras \( k_i \), each \( k_i \) viewed as subalgebra in \( K \) with \( 1_i \). The unity of \( K \) is \( 1 = \sum_i 1_i \), the set \( \{ 1_i : i \in I \} \) is a system of central primitive orthogonal idempotents for \( K \), referred to as **set of points of \( \mathcal{Q} \)**. We fix a point \( e \) of \( \mathcal{Q} \) and write \( t = 1 - e \), such that:

\[
\text{The idempotent } e \text{ is loop-free and 2-loop free, that is, } eB e = 0 \text{ and } (B e) \cap (eB) = 0.
\]

Replacing if necessary \( m \) by a cyclically equivalent potential, we have: \( em = 0 = me \).

(7.1)

In view of the first part of (7.1), we derive the following relations:

\[
B = B e \oplus B^* \oplus e \tau B e^* = B^* \oplus B^* e \oplus \tau B^* e \tau, \quad \text{and}
\]

\[
\{ B, B^* \} = \{ B e, B^* e \} \oplus \{ e B, B^* e \} \oplus \{ \tau B e, \tau B^* e \} \quad \text{as naturally induced pairing.}
\]

(7.2)

**Definition 7.1.** Whenever (7.1) is satisfied, the **semi-mutation** of \( (\mathcal{Q}, m) \) at point \( e \) is the modulated quiver with potential \( \tilde{\mu}_e (\mathcal{Q}, m) = (\tilde{\mu}_e (\mathcal{Q}), \tilde{m}) \) described as follow:
(i) $\tilde{\mu}_e(Q) = (\tilde{B}, \tilde{K}, t)$ with $\tilde{B} = [BeB] \oplus eB^* \oplus B^*e \oplus \tilde{B} \tau\tilde{B}$, where $[BeB]$ is still the bimodule $BeB$ regarded as being part of the arrow bimodule of $\tilde{\mu}_e(Q)$, and letting $\tilde{B}^* = eB^* \oplus B^*e \oplus \tilde{B} \tau\tilde{B}^*$, the data \{ $\tilde{B}, \tilde{B}^*$ \} is a symmetrizable dualizing pair canonically induced by the pair \{ $B, B^*$ \}. Each tensor element in $[BeB]$ may be written as $[xy]$ or $[x \otimes y]$ with $x \in B$ and $y \in eB$.

(ii) $\tilde{m} = [m] + \tilde{\lambda}_e$ with $\tilde{\lambda}_e = \lambda_{B \oplus eB^* \oplus B^*e \oplus \tilde{B} \tau\tilde{B}} = \sum_{y \in e \mathcal{Q}_1} \sum_{z \in \mathcal{Q}_1} [m_{yz}] \otimes \phi^* \otimes e \phi^*$, where $[m]$ coincides with $m$ but is regarded as an element in the complete path algebra of $\tilde{\mu}_e(Q)$.

Observe (using (7.1)) that $\tilde{m}$ is necessarily 2-loop free, so that $(\tilde{\mu}_e(Q), \tilde{m})$ is indeed a modulated quiver with potential. Using part (2) of Lemma 2.5 we obtain:

Remark 7.2. $\epsilon_1(\tilde{\lambda}_e) = \sum_{x \in \mathcal{Q}_1} \sum_{y \in \mathcal{Q}_1} \epsilon(x \otimes [y \mathrm{e} z] \otimes \phi^* \otimes e \phi^*) = [\lambda_{B \oplus eB^* \oplus B^*e \oplus \tilde{B} \tau\tilde{B}}, \tilde{\lambda}_e]$, thus $\tilde{m} = [m] + \lambda_{B \oplus eB^* \oplus B^*e \oplus \tilde{B} \tau\tilde{B}}$.

Remark 7.3. Let $(Q', m')$ be another modulated quiver with potential with $(Q' = (B', K, t))$ such that $eB^* = 0 = B^*e$. Then $\tilde{\mu}_e(Q \oplus Q', m + m') = \tilde{\mu}_e(Q, m) \oplus (Q', m')$.

Theorem 7.1. For each modulated quiver with potential $(Q, m)$ satisfying condition (7.1), the right-equivalence class of the semi-mutation $\tilde{\mu}_e(Q, m) = (\tilde{\mu}_e(Q), \tilde{m})$ is determined by that of $(Q, m)$.

Proof. Let $Q = (B, K, t)$ with $\tilde{B} = B \oplus eB^* \oplus B^*e$. Clearly, $\tilde{B}^*$ is a naturally induced symmetrizable dualizing pair \{ $\tilde{B}, \tilde{B}^*$ \} with $\tilde{B}^* = B^* \oplus B \oplus eB$. Then, the natural embedding $\tilde{B} \longrightarrow \tilde{B}^*$ identifies $\overline{kQ}$ with a closed subalgebra in $k\overline{Q}$. We also have a natural embedding $\tilde{B} \longrightarrow \overline{kQ}$ sending each degree-1 element $[xy]$ of $\tilde{B}$ to the tensor element $xy$ in $k\overline{Q}$, allowing us to identify $\overline{kQ}$ with a closed subalgebra in $k\overline{Q}$. Under this identification, $\tilde{m}$ now viewed as as element in $k\overline{Q}$.

Lemma 7.2. Every automorphism $\phi$ of $k\overline{Q}$ extends to an automorphism $\Phi$ of $k\overline{Q}$ such that: for all $\xi \in B^*$ we have $\Phi(\epsilon^* \otimes [y \mathrm{e} z] \otimes \phi^* \otimes e \phi^*) = [\lambda_{B \oplus eB^* \oplus B^*e \oplus \tilde{B} \tau\tilde{B}}, \xi]$, and $\Phi(\overline{kQ}) = k\overline{Q}$.

Proof. We first check that $\phi^*$ is indeed a morphism of $k$-bimodules on $eB^*$. Let $\xi \in B^*$, and $a, b \in K$; we have: $\Phi(\epsilon^* \otimes [y \mathrm{e} z] \otimes \phi^* \otimes e \phi^*) = [\lambda_{B \oplus eB^* \oplus B^*e \oplus \tilde{B} \tau\tilde{B}}, \xi]$. Using identities (2.2) in the sequel we write $xa = \sum_{z \in \mathcal{Q}_1} \epsilon(xa \otimes z^\ast)z$, thus $\phi^*(\epsilon^* \otimes [y \mathrm{e} z] \otimes \phi^* \otimes e \phi^*) = [\lambda_{B \oplus eB^* \oplus B^*e \oplus \tilde{B} \tau\tilde{B}}, \xi]$. Similarly, it is easily checked that $\phi^*(\epsilon \otimes [y \mathrm{e} z] \otimes \phi^* \otimes e \phi^*) = [\lambda_{B \oplus eB^* \oplus B^*e \oplus \tilde{B} \tau\tilde{B}}, \xi]$. Let us show that the degree-1 component $\Phi^1 : B \oplus eB^* \oplus B^*e \longrightarrow B \oplus eB^* \oplus B^*e$ of $\Phi$ is an automorphism. $\Phi^1|_B = \phi_1 : B \longrightarrow B$, and by part (a) of Proposition 3.1, $\phi_1$ is an automorphism of $B$. For all $\xi \in B^*$, we have: $\Psi^1(\epsilon^* \otimes [y \mathrm{e} z] \otimes \phi^* \otimes e \phi^*) = [\lambda_{B \oplus eB^* \oplus B^*e \oplus \tilde{B} \tau\tilde{B}}, \xi]$. Thus, $\phi^1|_{B^*e} : B^*e \longrightarrow B^*e$. Similarly, $\Phi^1|_{B^*e} : B^*e \longrightarrow B^*e$. Therefore, $\phi^1$ is an automorphism of $B^*$ and part (a) of Proposition 3.1 yields that $\Phi$ is an automorphism of $k\overline{Q}$ extending $\Phi$. We now prove the first identity in (7.3). In view of identities (3.6), for each $u \in B$ we know that $\phi^{-1}(u) = \sum_{x \in \mathcal{Q}_1} \phi^{-1}(u \otimes \phi^{-1}(u)) \otimes \phi(x)$. Then, we compute:
\[ j_{(eB^\ast)\otimes (e\ast)} = \sum_{z \in Q_1} \sum_{x \in Q_1} \phi(x) \phi^{-1}(ze \otimes x^\ast) \otimes \phi(x^\ast) \phi(x) = \sum_{x \in Q_1} \phi(x) \phi(x^\ast) \phi(x) = \phi(j_{(eB^\ast)\otimes (e\ast)}). \]

The second identity of (7.3) is established in the same way and the last identity follows by the definition of \( \phi \).

The reduction Theorem 4.6 together with Theorem 7.1 above yield the following result.

**Corollary 7.3.** Suppose (7.1) holds and the trivial part \((\tilde{\mu}_e(Q), \tilde{m})_{triv}\) splits. Then the weak right-equivalence class of \( \text{red}(\tilde{\mu}_e(Q, m)) \) is determined by that of \((Q, m)\).

**Definition 7.4.** With the assumptions of Corollary 7.3, the mutation of \((Q, m)\) at point \( e \) is the reduced modulated quiver with potential \( \text{red}(\tilde{\mu}_e(Q), \tilde{m}) \), unique up to weak right-equivalence: we write \( \mu_e(Q, m) = \text{red}(\tilde{\mu}_e(Q), \tilde{m}) \).

Another important result of this work establishes that, every mutation is an involution.

**Theorem 7.4.** The mutation \( \mu_e \) is an involution over the set of weak right-equivalence classes of the modulated quivers with potentials \((Q, m)\) satisfying (7.1) and whose semi-mutations have a split trivial part. If moreover \( m \) is a symmetric and \( \partial m : B^* \longrightarrow \partial(B^* \otimes m) \) also splits, then \( \mu_e \) is an involution up to right-equivalences.

**Proof.** Suppose that \((Q, m)\) is a reduced modulated quiver with potential satisfying the assumptions of the theorem. Then write: \( \tilde{\mu}_e^2(Q, m) = \tilde{\mu}_e(Q, m) = (\tilde{Q}, \tilde{m}) \). In view of reduction Theorem 4.6 and Theorem 5.4 (for the symmetric case), it suffices to show that \((\tilde{Q}, \tilde{m})\) is right-equivalent to \((Q, m) \oplus (\tilde{Q}_{triv}, W)\) where \( W \) is cyclically equivalent to the degree-2 component \( m_2 \) of \( m \). By definition, \( \tilde{B} = [BeB] \oplus [eB^* \otimes B^* e] \oplus W^e B^e \). We have (7.4) and (7.5) below where (7.4) uses the assumption that (7.1) holds.

\[ \tilde{B} = [eB^* B^e] \oplus Be \oplus eB \oplus [BeB] \oplus \partial B = [eB^* B^e] \oplus [BeB]. \]  

(7.4)

\[ \tilde{m} = [m] + \sum_{y \in Q_1} \sum_{z \in Q_1} yez[z^* e \otimes ey^*] + \sum_{y \in Q_1} \sum_{z \in Q_1} yez[z^* e \otimes ey^*]. \]  

(7.5)

But using part (2) of Lemma 2.5 we know that: \( \varepsilon_e(j_{[a^* B^e] \otimes [eB^e]}) = j_{[a^* B^e] \otimes [eB^e]} \sum_{y \in Q_1} yez [z^* e \otimes ey^*]. \)

Hence,

\[ \tilde{m} \equiv_{\mu_e} m' := [m] + \sum_{y \in Q_1} \sum_{z \in Q_1} yez[z^* e \otimes ey^*]. \]  

(7.6)

We then consider the trivial modulated quiver with potential \( \tilde{Q}_{triv}^e = [BeB] \oplus [eB^* B^e] \oplus [BeB]^e \) and \( W = \tilde{\mu}_{\partial} = [BeB] \oplus [eB^* B^e] \oplus [BeB]^e \) and \( W = j_{[a^* B^e] \otimes [eB^e]} \sum_{y \in Q_1} yez[z^* e \otimes ey^*]; \) (note that \( W \) is of course cyclically equivalent to \( m_2 \)). Now, to prove Theorem 7.4 it suffices to show that the modulated quiver with potential \( (\tilde{Q}, m') \) is right-equivalent to \((Q, m) \oplus (\tilde{Q}_{triv}, W)\), here \( m' \) is given by (7.6) above. The term \( S = j_{[a^* B^e] \otimes [eB^e]} \sum_{y \in Q_1} yez[z^* e \otimes ey^*] \) of \( m' \) is a potential and the right derivative morphism \( [BeB] \rightarrow B \otimes eB \) is a bimodule isomorphism taking each \([yez] \) to \( y \otimes ez \) for all \( y, z \in \mathbb{Q}_1 \). Whence the following unitriangular automorphism \( \varphi : kQ \rightarrow kQ \) whose restriction on the bimodule \( \tilde{B} = B \oplus ((eB^* B^e) \oplus [BeB]) \) is given by:

\[ \varphi|_{B \oplus [eB^* B^e]} = 1_{B \oplus [eB^* B^e]} \] and \( \varphi|_{[BeB]} = 1_{[BeB]} - \partial S : [BeB] \rightarrow [BeB] \otimes Be \oplus eB, \)

thus \( \varphi([yez]) = [yez] - y \otimes ez \) for all \( y, z \in \mathbb{Q}_1 \).
Thus we have a\algebra with Casimir element $8$. Examples of mutations in the mutation class of Dynkin type $K$ are $\text{Casimir operator is } s$.

Next, we deduce the unitriangular automorphism $\varphi'$ of $\widehat{kQ}$ with 

$$\varphi'|_{B\oplus B} = 1_{B\oplus B} \text{ and } \varphi'|_{B e e B'} - f : [B^* e e B'] \rightarrow [B^* e e B'] \oplus \widehat{J^2},$$

thus $\varphi'([z^* e \otimes e y^*]) = [z^* e \otimes e y^*] - f([z^* e \otimes e y^*])$ for all $y, z \in \mathbb{Q}_i$.

We get that $\varphi'(m') = m + W$. Hence, letting $\phi = \varphi' \circ \varphi$, we obtain a right-equivalence from $(\widehat{Q}, \widehat{m})$ to $(Q, m) \oplus (\widehat{Q}_{\text{triv}}, W)$, completing the proof of Theorem 7.4.

8. Examples of mutations in the mutation class of Dynkin type $F_4$

Consider the $\mathbb{R}$-algebra $K = k_1 \times k_2 \times k_3 \times k_4$, with $k_1 = \mathbb{R} = k_2 = \mathbb{R}, k_3 = \mathbb{C}, k_4 = \mathbb{C}$, regarded as $\mathbb{R}$-subalgebras of $K$ with respective unities $1_1, 1_2, 1_3, 1_4$. An $\mathbb{R}$-basis of $K$ set $S := \{1_1, 1_2, 1_3, 1_4\}$ where for $s = 3, 4$, the element $i_s \in \mathbb{C}$ corresponds to the complex number $i \in \mathbb{C}$. We have the canonical trace $t : K \rightarrow \mathbb{R}$ with $t(1_s) = 1$ for each $s \in \{1, 4\}$ and $t(1_3) = t(1_4) = 0$. Then $(K, t)$ is a symmetric and separable $\mathbb{R}$-algebra with Casimir element $\sum_{s=1}^{4} 1_s \otimes 1_s^* + i_3 \otimes i_3^* + i_4 \otimes i_4^* = \sum_{s=1}^{4} 1_s \otimes 1_s - i_3 \otimes i_3 - i_4 \otimes i_4$. Let $M$ be a $K$-bimodule. Recall that $Z(M)$ is the central $Z(K)$-bimodule consisting of $K$-central elements in $M$, the associated Casimir operator is $\tilde{3}_M : M \rightarrow Z(M); x \mapsto \tilde{3}_M(x) = \sum_{s=1}^{4} 1_{x^s} 1_{x^s}^* + i_3 x^s i_3^* + i_4 x^s i_4^* = \sum\limits_{s=1}^{4} 1_s x^s 1_s - i_3 x^s i_3 - i_4 x^s i_4$.

Also recall by Corollary 5.5 that, since $\mathbb{R}$ is a perfect field, any potential on an $\mathbb{R}$-modulated quiver is symmetric, cyclic (left or right) permutation mimics the simply laced case: it is obtained as the image under the Casimir operator of the corresponding ordinary cyclic (left or right) permutation.

As in the illustrative section 6, we fix some notations for some useful $K$-bimodules here. For all $(M, i, j) \in \mathcal{X}(K) := \{ \mathbb{R} \times \mathbb{R}, \mathbb{R} \times \mathbb{R}, \mathbb{C} \times \mathbb{C}, \mathbb{C} \times \mathbb{C}, \mathbb{C} \times \mathbb{C}, \mathbb{C} \times \mathbb{C}, \mathbb{C} \times \mathbb{C}, \mathbb{C} \times \mathbb{C}, \mathbb{C} \times \mathbb{C}, \mathbb{C} \times \mathbb{C})$, we write $i_{M_j}$ for the natural $k_i$-$k_j$-bimodule structure on $M$; and when viewed as element in $i_{M_j}$ each $x \in M$ is still written as $x$ (if this can be easily inferred from the context), otherwise $x$ is subscripted as $x_{ij}$. For each $(\mathbb{C}, s, t) \in \mathcal{X}(K)$, the conjugate natural bimodule $\mathcal{C}_t$ is obtained by conjugating the right module structure of the natural bimodule $\mathcal{C}_s$: thus we have $z_{ij} z_{ij} := z_{ij} z_{ij}$ for all $z \in k_s, z' \in \mathbb{C}_t$ and $x \in \mathcal{C}_t$. Also put $i_1 = 1$ and $i_2 = i$ as elements of $\mathcal{C}_t$, and observe for example that $x_{11} i_1 = - x_{12} i_2 = - x_{13} i_3 = - x_{14} i_4$.

Below, each symmetrizable dualizing pair over $(K, t)$ is (naturally isomorphic) to one of the following.

- The self-dual pairs \{\mathbb{R}_{ij}, \mathbb{R}_{ij}\} or \{\mathbb{C}_s, \mathbb{C}_s\}, with $(i, j) = (1, 2)$, $(2, 1)$ and $(s, t) = (2, 3)$, $(3, 2)$, $(3, 4)$, $(4, 3)$; here associated bilinear forms are given by the ordinary multiplication.

- The pairs \{\mathbb{C}_s, \mathbb{C}_t\} with $s = 3, 4$ and with associated bilinear forms given by the ordinary multiplication $\mathcal{C}_s \otimes \mathcal{C}_t$ and the map $\mathcal{C}_s \otimes \mathcal{C}_t \rightarrow \mathbb{R}_1 : (z \otimes z') \mapsto (z \otimes z') = t(zz')$. 

39
The conjugate pairs \( \{ \tilde{\mathcal{C}}, \mathcal{C}_\tau \} \) with \((s, t) = (3, 4), (4, 3)\) and with associated bilinear forms induced by conjugating the second argument of the ordinary multiplication: \( \tilde{\mathcal{C}} \otimes \mathcal{C}_\tau \rightarrow \mathcal{C}_\tau \otimes \tilde{\mathcal{C}} : (z \otimes z') \mapsto z z' \).

Now start with the modulated quiver with zero potential \( Q^{(0)} : R_1 \xrightarrow{a^{R_2}} R_2 \xrightarrow{a^{C_3}} C_3 \xrightarrow{C_4} C_4 \). First observe the following picture of successive mutations of the underlying valued quivers \( \Phi_4 \) of \( Q^{(0)} \):

\[
\Phi_4 : 1 \xrightarrow{\mu_2} 2 \xrightarrow{1,2,3} 4 \xrightarrow{\mu_3} 3 \xrightarrow{1,2,3} 4 \xrightarrow{\mu_4} 2
\]

The first mutation below is clear from the definition of mutation, where as usual a tensor element \( x \otimes y \) is also written as \( xy \) or \( yx \) and where the bimodule \( [\mathcal{R}_2 \otimes \tilde{\mathcal{C}}_3] \) is naturally identified with \( \mathcal{C}_3 \):

\[
(Q^{(0)}, 0) \xrightarrow{\mu_2} (Q^{(1)}) : \tilde{\mathcal{C}}_3 \rightarrow \mathcal{C}_4, \quad W_1 := \tilde{\mathcal{J}}_1 = \tilde{\mathcal{J}}_{2,3,4,1} \quad \xrightarrow{\mu_3} (Q^{(1)}) : \mathcal{C}_4 \rightarrow \mathcal{C}_3, \quad W_1.
\]

For the semi-mutation \( \mu_3 \) above, we naturally identify \( [i \mathcal{C}_2 \otimes \tilde{\mathcal{C}}_3 \mathcal{C}_2] \) and \([\mathcal{C}_2 \otimes \tilde{\mathcal{C}}_3 \mathcal{C}_2] \) with \( \mathcal{C}_2 \) and \( \tilde{\mathcal{C}}_3 \) respectively. We have \( \mathcal{W}_1 = [W_1] + \tilde{\mathcal{J}}_3 \). Here the Casimir element \( \tilde{\mathcal{J}}_3 \) is the sum of two Casimir elements: \( \tilde{\mathcal{J}}_3 = \tilde{\mathcal{J}}_{c_2 \otimes (2c_3, c_4)} + \tilde{\mathcal{J}}_{c_2 \otimes (c_3, c_2)} \), with \( \tilde{\mathcal{J}}_{c_2 \otimes (2c_3, c_4)} = i_{c_2} j_{c_3} j_{c_4} + i_{c_2} (-j_{c_4}) \) and \( \tilde{\mathcal{J}}_{c_2 \otimes (c_3, c_2)} = j_{c_2} j_{c_3} j_{c_4} \). Thus \( \mathcal{W}_1 = i_{c_2} j_{c_3} j_{c_4} \) with 2-cyclic component \( \mathcal{W}_{1,2} = i_{c_2} j_{c_3} j_{c_4} \). In view of the Casimir element \( \tilde{\mathcal{J}}_{c_2 \otimes (2c_3, c_4)} = i_{c_2} j_{c_3} j_{c_4} + i_{c_2} (-j_{c_4}) \), we have \( \partial(\tilde{\mathcal{J}}_3), (\mathcal{W}_1) = j_{c_2} j_{c_3} j_{c_4} \) and \( \partial(\tilde{\mathcal{J}}_3), (\mathcal{W}_1) = j_{c_2} j_{c_3} j_{c_4} \). Thus the trivial bimodule for \( (Q^{(1)}, \mathcal{W}_1) \) is \( \mathcal{R}_1 \oplus \mathcal{R}_2 \mathcal{C}_3 \) (and is of course a direct summand of the arrow bimodule of \( \mathcal{Q}^{(1)} \)), the corresponding reduced bimodule is \( \mathcal{R}_1 \mathcal{I}_1 \oplus \mathcal{C}_4 \mathcal{R}_2 \mathcal{I}_4 \mathcal{C}_4 \); the closed ideal \( J^{\text{red}} \) of \( \mathcal{W}_3 \) is the kernel of a reduction \( \pi : \mathcal{Q}(\mathcal{Q}^{(1)}) \rightarrow \mathcal{W}_3 \) which fixes the reduced arrow bimodule and such that: \( \pi_1(j_1) = 0, \pi_3(j_1 + j_2 j_3 j_4) = 0 \) so that \( \pi_1(j_1) = -j_1 j_2 j_3 j_4 \). Thus the reduced potential is given by \( \pi(\mathcal{W}_1) = -i_{c_2} j_{c_3} j_{c_4} + j_{c_2} j_{c_3} j_{c_4} \). Naturally identifying \( \mathcal{R}_1 \mathcal{I}_1 \) with \( \mathcal{R}_2 \), the previous details are summarized in the following picture:

\[
(Q^{(1)}) : 1 \xrightarrow{\mu_2} 2 \xrightarrow{1,2,3} 4 \xrightarrow{\mu_3} 3 \xrightarrow{\mu_4} 2 \quad \text{reduction} \quad (Q^{(2)}) : 1 \xrightarrow{\mu_2} 2 \xrightarrow{1,2,3} 4 \xrightarrow{\mu_3} 3 \xrightarrow{\mu_4} 2 \quad \text{with} \quad W_2 := -i_{c_2} j_{c_3} j_{c_4} + j_{c_2} j_{c_3} j_{c_4}.
\]

We can perform more mutations as shown is the following picture, where one should notice the presence of the conjugate natural bimodule \( \tilde{\mathcal{C}}_\tau \) in the last modulated quiver and \( W_4 := \frac{1}{2}(j_{c_1} j_{c_3} + j_{c_1} j_{c_3}) j_{c_\tau} \).
Remark 8.1. If instead of a perfect field we consider a non-perfect field, then all the sequences of mutations and reductions above can still be performed, provided, in view of Corollary 6.1, skew reductions are also allowed.
9. Graded modulated quiver with potentials

This section is motivated by [34, § 6.2] about graded quivers with potentials. We will quickly explain why the results of preceding sections holds in the graded context. We fix an abelian group $G$ which should be $\mathbb{Z}$, $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}^n$ for some $n, p \in \mathbb{N}$. Let $\mathcal{C}$ be an additive category. A $G$-graded object in $\mathcal{C}$ is just a family $X = (X^p)_{p \in \mathbb{Z}}$ of objects of $\mathcal{C}$; the degree-$p$ component of $X$ is $X^p$. A graded morphism $f : X \rightarrow Y$ of degree $n \in G$ between two graded objects consists of a family of morphisms $f^p : X^p \rightarrow Y^p + n$, $p \in \mathbb{Z}$. Graded morphisms of degree 0 are simply referred to as graded morphisms. A complex (or a dg (differential graded) object) in $\mathcal{C}$ consists of a graded object $X = (X^p)_{p \in \mathbb{Z}}$ together with a differential $d = d_X$, the latter is a graded morphism of degree 1 such that $d \circ d = 0$ (that is, $d^p \circ d^{p-1} = 0$ for all $p \in G$). When the category $\mathcal{C}$ has all direct sums, we identify each graded object $X$ with the direct sum $\bigoplus X^p$. Giving a $k$-algebra $\Lambda$, denote by $\text{Gr}(\Lambda)$ the category of $G$-graded (right) $\Lambda$-modules and graded morphisms (of degree 0). Let $M = \bigoplus M^p$ in $\text{Gr}(\Lambda)$. The $G$-graded left $\Lambda$-module $M' = \text{Hom}_A(M, A)$ has components $M'^p = \text{Hom}_A(M^-p, A), p \in G$; with this $G$-grading, $\text{Hom}_A(M, A)$ is called the dual of the $G$-graded $\Lambda$-module $M$. For $n \in G$, the $n$-shift of $M$ is the graded module $M[n]$ with components $(M[n])^p = M^{p+n}$ for all $p \in G$. The tensor product of a $G$-graded left $\Lambda$-module $L$ by $M$ is the $G$-graded $k$-module $M \otimes_\Lambda M$ with components: $(M \otimes_\Lambda M)^n = \bigoplus_{p+q=n} L^p \otimes_\Lambda M^q, n \in G$.

**Graded modulated quivers and their complete path algebras.** A modulated quiver $Q = (B, K, t)$ is $G$-graded if the (finitely generated) arrow $k$-bimodule $B$ is $G$-graded (thus, $B = \bigoplus B^p$ and only finitely many components $B^p$ are nonzero). Assume that $Q$ is $G$-graded. The path algebra $kQ$ is a topological $G$-graded algebra with respect to $kQ$-adic topology on $kQ$, with grading induced by that of $Q$ and with $k$ lying in degree 0. For $p \in G$, the degree-$p$ component $B^p$ of $B$ should not be confused with the notation $B^l$ for $l \in \mathbb{N}$, the latter being the $l$-fold tensor product of $B$ over $k$; in particular $(B^l)^p$ is not the degree-$p$ component of $B^l$. The complete path algebra $\widehat{kQ}$ of $Q$ is the completion of $kQ$ with respect to $kQ$-adic topology in the category $\text{Gr}(k)$ of $G$-graded $k$-modules; thus $\widehat{kQ}$ coincides with the projective limit (in $\text{Gr}(k)$) of the natural inverse system

$$K = kQ/J_{kQ} \llongrightarrow \bigoplus_{a \leq d \leq t} B^{(d)} \cong \bigoplus_{a \leq d \leq t+1} B^{(d)} \cong kQ/J_{kQ}^1 \llongleftarrow \bigoplus_{a \leq d \leq t+1} B^{(d)} \cong kQ/J_{kQ}^1 \llongrightarrow \cdots$$

As $G$-graded $k$-module, we get: $\widehat{kQ} = \bigoplus_{p \in G} \widehat{kQ}^p$ with degree-$p$ component $\widehat{kQ}^p = \bigoplus_{l \geq 0} (B^{(l)})^p$.

Recall that $B$ is part of a symmetrizable dualizing pair $(B, B^*; b)$. Since the dual $B^*$ is canonically isomorphic to the $k$-dual $\text{Hom}_k(B, k)$, we get the following observation.

**Remark 9.1.** Endowing $B^*$ with the dual $G$-grading induced by that of $B$, the dualizing pair $(B, B^*; b)$ arises as direct sum of induced dualizing pairs $(B^p, (B^p)^*) = \{B^p, (B^p)^* \}_{p \in G}$, the bilinear form $b$ is $G$-graded of degree 0 and vanishes on $B^p \otimes (B^p)^\ast$ and $(B^p)^\ast \otimes B^p$ for all $p, q \in G$ with $q \neq -p$.

In the sequel, let $n \in G$ and $(Q, m)$ be a $G$-graded modulated quiver with potential homogeneous of degree $n$, $(m$ needs not be homogeneous with respect to path-grading). Each component $m_{ij} \in B^{(d)}$ of $m$ (with respect to path grading) is therefore homogeneous of degree $n \in G$ and the following lemma is an easy observation.

**Lemma 9.1.** For each $0 < d \in \mathbb{N}$, the derivative morphisms $\partial^d m, \partial^d m : (B^*)^{(d)} \rightarrow \widehat{kQ}$ and $\partial^d m : B^* \rightarrow \widehat{kQ}$ are $G$-graded morphisms of degree $n \in G$. In particular the trivial part $Q_{\text{triv}} = (B_{\text{triv}}, K, t)$ and the reduced part $Q = \overline{Q}_{\text{red}} = (\overline{B}, K, t)$ are naturally $G$-graded with $G$-gradings induced by that of $B$, the natural projection $\rho : \widehat{kQ} \rightarrow \overline{kQ}$ is a $G$-graded morphism, left and right $K$-linear right inverses to the projection $\partial^d m : B^* \rightarrow B_{\text{triv}}$ can be chosen as $G$-graded morphisms of degree $-n$.

Note that two cyclically equivalent potentials (of homogeneous degree with respect to $G$-grading) have the same degree. We adapt the notion of (weak) right-equivalence and reductions to the graded context. Below, $Q' = (B', K, t)$ is another graded modulated quiver.
Definition 9.2. Let $\phi : \widehat{kQ} \to \widehat{kQ}$ be a graded algebra morphism. Then $\phi$ is a path algebra morphism if $\phi|_K = 1_K$ and $\phi(B) \subset J_{kQ}$, in this case $\phi|_B = (\phi)_{\geq 1} : B \to J_{kQ}$ with $\phi : B \to B^{(l)}$, $l \geq 1$. We call $\phi$ a reduction on $(Q,m)$ if $\phi$ is a path algebra epimorphism satisfying the graded version of properties (1.i) and (1.ii) from Definition 4.3, namely:

(1.i) $\text{Ker}(\phi)$ is the closed graded ideal in $\widehat{kQ}$ generated by the image of a graded $K$-bimodule morphism $f = [\tilde{f}] : B_{\text{triv}} \to B_{\text{triv}} \oplus J^2_{kQ}$ with $\text{Im}(f) \subset (\partial m)(B^*)$.

(1.ii) Let $\pi : \widehat{kQ}/\text{Ker}(\phi)$ be the natural projection, $\rho : B \to B = B/B_{\text{triv}}$ the natural projection, and $\tilde{\pi} : (B + \text{Ker}(\phi))/\text{Ker}(\phi) \to \tilde{B}$ the $K$-bimodule epimorphism with $\rho = \tilde{\pi} \circ \pi$. Then $\rho$ has a right inverse $\rho^{-1} : \tilde{B} \to B$ such that $\rho \circ \rho^{-1} = \phi_1 \circ \rho^{-1}$.

Similarly, (weak) right equivalences between G-graded modulated quivers with potentials are defined.

As direct consequence of Lemma 9.1 above, we have the following.

Corollary 9.2. Unitriangular automorphisms appearing in Lemma 4.8 and the proof of Proposition 5.6 can be constructed as G-graded algebra morphisms.

Applying the previous discussion and Definition 9.2 above, we get that the results from the first section to the fifth (as well as the setting of [1, §2-6]) generalize to the graded context. We therefore state the following.

Theorem 9.3. The reduction Theorem 4.6 and its symmetric version Theorem 5.4 holds for G-graded modulated quivers with potentials of degree $n \in G$. In particular, the reduced potential is also of degree $n$.

Mutation of graded modulated quiver with potentials.

Let $e \in K$ be a point in $Q$ ($e$ belongs to a system of central primitive orthogonal idempotents for $K$) satisfying (7.1). We then adapt Definition 7.1 to the graded context as follows (compare with [34, §6.2]).

Definition 9.3. The left semi-mutation of $(Q,m)$ at point $e$ is the graded modulated quiver with potential $\tilde{\mu}_e(Q,m)$ whose underlying non $G$-graded modulated quiver with potential is the semi-mutation $(\tilde{\mu}_e(Q), \tilde{m})$ with arrow bimodule $\tilde{B} = [BeB] \oplus eB^* \oplus B^*e \oplus \pi B \pi$, and the G-graded arrow bimodule of $\tilde{\mu}_e(Q,m)$ is $\tilde{\mu}_e(B) = [BeB] \oplus eB^* \oplus B^*e \oplus \pi B \pi$ where the $G$-grading of $[BeB]$ is induced by that of the tensor product $B \otimes B$. Similarly, the right semi-mutation $\tilde{\mu}_e^r(Q,m)$ is defined by letting $\tilde{\mu}_e^r(B) = [BeB] \oplus eB^* \oplus B^*e \oplus \pi B \pi$.

In the above definition, the potential $\tilde{m}$ is homogeneous of the same degree $n \in G$ as $m$. We obtain the following graded version of Theorem 7.1 and Corollary 7.3, where Theorem 9.3 above is also used.

Theorem 9.4. The right-equivalence classes of the left semi-mutation $\tilde{\mu}_e(Q,m)$ and the right semi-mutation $\tilde{\mu}_e^r(Q,m)$ are determined by that of $(Q,m)$. Thus, if the trivial part $(\tilde{\mu}_e(Q), \tilde{m})_{\text{triv}}$ splits, then the weak right-equivalence classes of $\text{red}(\tilde{\mu}_e(Q,m))$ are determined by that of $(Q,m)$.

Let $s \in \{\ldots, e, e\}$, in the situation of previous theorem, the reduced G-graded modulated quiver with potential $\tilde{\mu}_e^s(Q,m)_{\text{red}}$ is the left (or right) mutation of $(Q,m)$ at point $e$, it is unique up to weak right-equivalence (or right equivalence if $k$ is a perfect field). We also deduce the following result.

Theorem 9.5. In the graded context, left and right mutation are again involutive on the set of weak right-equivalence classes of G-graded modulated quivers with potentials of homogeneous degree in $G$.

10. The cluster category of a graded modulated quiver with potential

Here the abelian group $G$ is $\mathbb{Z}$ and we keep the notions of graded objects as defined in the previous section. We let $n \in G$ and $(Q,m)$ be a graded modulated quiver with potential homogeneous of degree $n - 3 \in G$, where as before $Q = (B,K,t)$ and $(K,t)$ is a symmetric k-algebra. $(Q,m)$ is Jacobian-finite whenever $m \in kQ$ and the Jacobian algebra $\mathcal{J}(Q,m)$ is finitely generated as k-module.
10.1. Complete Ginzburg dg-algebra and the generalized cluster category

Refer to [30] for concepts about differential graded categories and differential graded algebras (in short, dg-categories, dg-algebras). Simply-laced Ginzburg dg-algebra appears in [8, sec 4.2] (for Q concentrated in degree 0 and n = 3), see also [18, § 2.5] and [31, § 6.2]. To Q we associate a graded modulated quiver $\hat{Q} = Q \oplus Q^*[n-2] \oplus K K_K[n-1]$ where the G-graded modulated quiver $Q^* = (B^*, K, t)$ is the dual of Q; thus the $G$-graded arrow bimodule of $Q$ is $B = B \oplus B^* \oplus K K_K$ with $K K_K$ concentrated in degree 0.

**Definition 10.1.** The complete Ginzburg dg-algebra $\hat{\Gamma}_n = \hat{\Gamma}_n(Q, m)$ is the complete path algebra $k \hat{Q}$ of the graded modulated quiver $\hat{Q}$. $\hat{\Gamma}_n$ is endowed with the unique continuous differential $k \hat{Q} \xrightarrow{\partial} k \hat{Q}$, satisfying the Leibniz rule: $\partial(uv) = \partial(u)v + (-1)^d u \partial(v)$ for all $u \in \hat{\Gamma}_n$, given on $B$ as follows:

- $\partial$ vanishes on $B$, and $\partial|_{B^*} = \partial m : B^* \xrightarrow{\partial} k \hat{Q}$, $\xi \mapsto \partial(\xi) = \partial m$.
- The restriction of $\partial$ on the self-dual natural bimodule $k K_K$ is the Casimir morphism

$$\partial_{B^* \otimes B} : k K_K \xrightarrow{\partial} (B \otimes B^*) \oplus (B^* \otimes B),$$

thus for all $a \in K$ we have: $\partial(a) = a(\sum_{y \in Q_1} y y^* - \sum_{x \in Q_1} x^* x) = (\sum_{y \in Q_1} y y^* - \sum_{x \in Q_1} x^* x)a$, where $(a_Q, a_Q^*)$ and $(l_Q, l_Q^*)$ are respectively right and left projective bases for $B$ defined by the Casimir morphisms $\partial_{B^* \otimes B}$ and $\partial_{B \otimes B^*}$.

In case $m$ lies in $k Q$, the non-complete Ginzburg dg-algebra $\Gamma_n$ is the path algebra $k \hat{Q}$ endowed with the differential defined above.

**Remark 10.2** ([20, Lem 2.8] for the simply laced case). If $Q$ is concentrated in degree 0 and $n = 3$, then $\mathcal{J}(Q, m)$ coincides with the 0-homology $H^0(\hat{\Gamma}_3)$ of the differential graded algebra $\hat{\Gamma}_3$.

Let $D \hat{\Gamma}_n$ be the derived category of $\hat{\Gamma}_n$ and view $\hat{\Gamma}_n$ as object of $D \hat{\Gamma}_n$. The perfect derived category of $\hat{\Gamma}_n$ is the smallest full triangulated subcategory $\operatorname{per} \hat{\Gamma}_n$ of $D \hat{\Gamma}_n$ generated by $\hat{\Gamma}_n$ and closed under taking direct summands. Denote by $D_{\mathfrak{sl}} \hat{\Gamma}_n$ the subcategory of $D \hat{\Gamma}_n$ consisting of dg modules $M$ with finite-length total homology, that is, the homology $H(M) = \bigoplus_{p \in \mathbb{Z}} H^p(M)$ has finite length over $k$. For $n = 3$, it is shown in the simply-laced framework that $D_{\mathfrak{sl}} \hat{\Gamma}_n$ is a triangulated subcategory of $\operatorname{per} \hat{\Gamma}_n$ [20, §2.15.2.18.4], and $D_{\mathfrak{sl}} \hat{\Gamma}_n$ enjoys a relative $n$-Calabi-Yau property in $D \hat{\Gamma}_n$ [19, Lem 4.1] and [31, Thm 6.3].

**Definition 10.3.** When $Q$ is concentrated in degree 0 and $n = 3$, the cluster category $C_{(Q, m)}$ associated with $(Q, m)$ is the idempotent completion of the triangulated quotient $\operatorname{per} \mathcal{J} D^{\mathbb{B}} \hat{\Gamma}_3$.

The following questions arise naturally since Calabi-Yau property is fundamental in cluster theory.

(a) Does the relative Calabi-Yau property of $D_{\mathfrak{sl}} \hat{\Gamma}_n$ in $D \hat{\Gamma}_n$ survive in the non-simply-laced framework?

(b) Suppose $k$ is a field and $(Q, m)$ Jacobian-finite. Is $C_{(Q, m)}$ Hom-finite, $(n-1)$-Calabi-Yau? Is $\hat{\Gamma}_n$ a cluster-tilting object in $C_{(Q, m)}$?

A dg-algebra $A$ is homologically smooth if $A \in \operatorname{per}(A^e)$ where $A^e = A^e \otimes A$ is the enveloping dg $k$-algebra of $A$. And $A$ is $n$-Calabi-Yau as bimodule if in $D(A^e)$ there is a bimodule isomorphism $R \operatorname{Hom}_{A^e}(A, A^e) \xrightarrow{\sim} A[-n]$. A notion of topological and homological smoothness is defined for bilaterally pseudocompact dg-algebras [20, §7.11].

We expect the following result due to Bernhard Keller to hold in the general framework.

**Theorem 10.1** ([31, Thm 6.3], [20, Thm 7.17]). The non-complete (resp. complete) Ginzburg dg algebra (or dg category) of a quiver with potential is (topologically) homologically smooth and 3-Calabi-Yau as bimodule.

**Conjecture 2.** Generalized Ginzburg dg-algebras (dg-categories) For $n = 3$, $\Gamma_n$ and $\hat{\Gamma}_n$ are (topologically) homologically smooth and $n$-Calabi-Yau as bimodules, at least when the symmetric algebra $K$ is separable over a field.
In the sequel, suppose $Q$ is concentrated in degree 0. With exactly the same argument as in [16, Thm 3.6] and [20, § 7.20], we derive the following.

**Theorem 10.2** ([16, Thm 3.6], [20, § 7.20] for simply-laced case). Suppose Conjecture 2 holds and $k$ is a field. Then the generalized cluster category $C_{Q,m}$ of a Jacobian-finite modulated quiver with symmetric potential is $\text{Hom}$-finite 2-Calabi-Yau and the image $T$ of the free module $\Gamma$ into $C_{Q,m}$ is a cluster tilting object such that $\text{End}_{C_{Q,m}}(T)$ coincides with the Jacobian algebra $J(Q)$.

Recall the following interesting characterization of cluster categories inside the context of 2-Calabi-Yau categories.

**Theorem 10.3** (Keller-Reiten). Assume $k$ is a perfect field. Let $C$ be the stable category of a Frobenius category such that $C$ is 2-Calabi-Yau; let $T \subset C$ be a cluster tilting subcategory. Then, if the category $\text{mod}T$ of finite presented modules over $T$ is hereditary then $C$ is exactly equivalent to the cluster category $C_T = D^b(\text{mod}T)[(\tau^{-1}1)]^Z$.

**Corollary 10.4**. If Conjecture 2 holds and $k$ is a perfect field, then for an acyclic $Q$ the category $C_{Q,0}$ is exactly equivalent to the cluster category $C_Q$ of [21].

**Proof.** The argument of the proof is the same as in the simply-laced case. When $k$ is a perfect field and Conjecture 2 holds, it follows by Theorem 10.2 that $C_{Q,0}$ is 2-Calabi-Yau, admitting a cluster tilting object $T$ such that $\text{End}_C(T) = kQ$, so that we have the expected result in view of Keller-Reiten Theorem 10.3. \hfill \Box

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