Extended quasi-cyclic constructions of quantum codes and entanglement-assisted quantum codes

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Received: date / Accepted: date

Abstract Construction of quantum codes and entanglement-assisted quantum codes with good parameters via classical codes is an important task for quantum computing and quantum information. In this paper, by a family of one-generator quasi-cyclic codes, we provide quasi-cyclic extended constructions that preserve the self-orthogonality to obtain stabilizer quantum codes. As for the computational results, some binary and ternary stabilizer codes with good parameters are constructed. Moreover, we present methods to construct maximal-entanglement entanglement-assisted quantum codes by means of the class of quasi-cyclic codes and their extended codes. As an application, some good maximal-entanglement entanglement-assisted quantum codes are obtained and their parameters are compared.

Keywords Quasi-cyclic codes · Extended constructions · Quantum codes · Entanglement-assisted quantum codes

1 Introduction

Quantum error-correcting codes (QECCs, for short) were introduced to reduce the effects of environmental and operational noise (decoherence). Reducing
the decoherence or controlling the decoherence to an acceptable level is a key challenge for researchers. After the pioneering work of Shor and Steane in [31, 33], the theory of QECCs has been developed rapidly. In [6], it has been proven that binary QECCs can be constructed from classical codes over $\mathbb{F}_2$ or $\mathbb{F}_4$ with certain self-orthogonal properties. Then, Ashikhmin et al. [2] generalized these results to non-binary case. Afterwards, many good QECCs have been constructed by using classical linear codes over finite fields [10, 12, 16, 18, 25, 27, 34, 38]. Another important discovery in the quantum error-correcting area was the entanglement-assisted quantum codes (EAQECCs, for short). The concept of an EAQECC was introduced by Brun et al. [5], which overcame the barrier of the self-orthogonal condition. They proved that if shared entanglement is available between the sender and receiver in advance, non-self-orthogonal classical codes can be used to construct EAQECCs. There are many researchers presented some constructions of good EAQECCs [4, 14, 19, 20, 22, 23, 24, 29, 35, 37].

Quasi-cyclic (QC) codes form an important class of linear codes with a rich algebraic structure, which is the generalization of cyclic codes. Many QC codes have improved the earlier known minimum distances [7, 32]. Moreover, QC codes meet a modified version of the Gilbert-Vashamov (GV) bound [17, 21]. Naturally, QC codes can also be applied to construct QECCs. In [15], Hagiwara et al. studied constructions of QECCs from QC LDPC codes with a probabilistic method. In 2018, Galindo et al. [11] used two-generators QC codes that were dual-containing to construct QECCs. In 2019, Ezerman et al. [9] employed QC codes with large Hermitian hulls to provide QECCs and gained a record-breaking binary $[[31, 9, 7]]$ QECC. In [25, 26], we have obtained some good QECCs from one-generator and two-generators QC codes that are symplectic self-orthogonal, respectively. It is well-known that extended constructions of linear error-correcting codes are extremely effective methods to obtain new codes with good performance, such as famous generalized Reed-Muller codes, generalized Reed-Solomon codes and so on. In fact, extended constructions can also be utilized to construct QECCs and EAQECCs. In [34], Tonchev presented doubling extended constructions and obtained a new optimal binary $[[28, 12, 6]]$ QECC, which improved the corresponding lower bounds on minimum distance at that time. In 2018, Guenda et al. [14] applied extended constructions to design families of EAQECCs with good error-correcting performance requiring desirable amounts of entanglement. However, until now, there are no related extended constructions of QECCs and EAQECCs from QC codes.

Inspired by the above work, we provide QC extended constructions to obtain QECCs and maximal-entanglement EAQECCs. This paper is organized as follows. In Sect. 2, we discuss some preliminary concepts and propose a family of one-generator QC codes, which are self-orthogonal with respect to the Hermitian inner product. Sect. 3 provides QC extended constructions that preserve the self-orthogonality to construct good QECCs. In Sect. 4, we present methods of constructing maximal-entanglement EAQECCs based on these QC codes and their extended codes. Sect. 5 concludes this paper.
2 Preliminaries

let \( \mathbb{F}_q \) be the finite field with \( q^2 \) elements where \( q \) is a power of prime \( p \). It is obvious that the characteristic of \( \mathbb{F}_q \) is \( p \). Given two vectors \( u = (u_0, u_1, \ldots, u_{n-1}) \) and \( v = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{F}_q^n \), their Hermitian inner product is defined as \( \langle u, v \rangle_h = \sum_{i=0}^{n-1} u_i^q v_i \). Recall that an \([n, k]_{q^2}\) linear code \( \mathcal{C} \) is a linear subspace of \( \mathbb{F}_q^n \) with dimension \( k \). For any codeword \( c \in \mathcal{C} \), the Hamming weight of \( c \) is the number of nonzero coordinates in \( c \). The minimum distance \( d \) of a linear code \( \mathcal{C} \) equals to the smallest weight of its nonzero codewords. A generator matrix is a \( k \times n \) matrix whose rows form a basis for \( \mathcal{C} \). Given a linear code \( \mathcal{C} \subset \mathbb{F}_q^n \), the Hermitian dual code of \( \mathcal{C} \) is \( \mathcal{C}^\perp = \{ v \in \mathbb{F}_q^n | \langle u, v \rangle_h = 0, \forall u \in \mathcal{C} \} \). If \( \mathcal{C} \subset \mathcal{C}^\perp \), then the code \( \mathcal{C} \) is Hermitian self-orthogonal. The Hermitian hull of \( \mathcal{C} \) is the intersection \( \Hull_h(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}^\perp \). For a \( k \times n \) matrix \( G = (g_{ij})_{k \times n} \) and a vector \( v = (v_0, v_1, \ldots, v_n) \) over \( \mathbb{F}_q^2 \) (viewed as a \( 1 \times n \) matrix), we define \( G^v = (g_{ij}^v)_{k \times n} \) and \( v^\dagger = (v_0^\dagger, v_1^\dagger, \ldots, v_n^\dagger) \). Denote by \( A^\dagger \) and \( v^\dagger \) the conjugate transpose matrices of \( A \) and \( v \). It is well-known that a linear code \( \mathcal{C} \) is Hermitian self-orthogonal if and only if \( GG^\dagger = 0 \), where \( G \) is a generator matrix of \( \mathcal{C} \) and \( 0 \) denotes the zero matrix.

Let \( \mathbb{R}_n = \mathbb{F}_q[x]/(x^n - 1) \) and \( \mathcal{C} \) be a \( q^2 \)-ary linear code of length \( 2n \). For any codeword \( c = (c_0, c_1, \ldots, c_{n-1}, c_n, c_{n+1}, \ldots, c_{2n-1}) \in \mathcal{C} \), define \( \psi(c) = (c_{n-1}, c_0, c_1, c_2, \ldots, c_{2n-2}) \). If \( \mathcal{C} = \psi(\mathcal{C}) \), then we call the code \( \mathcal{C} \) a quasi-cyclic (QC) code of length \( 2n \) and index 2 over \( \mathbb{F}_q^2 \). Note that a vector \( c = (a_0, a_1, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}) \in \mathbb{F}_q^{2n} \) can be identified with \( (a(x), b(x)) \in \mathbb{R}_n \), where \( a(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \) and \( b(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1} \). Further, \( \psi(c) \) corresponds to \( (xa(x), xb(x)) \) in \( \mathbb{R}_n^2 \). Therefore, a QC code \( \mathcal{C} \) of index 2 can be viewed algebraically as an \( \mathbb{R}_n \)-submodule of \( \mathbb{R}_n^2 \). If \( \mathcal{C} \) is generated by \( G(x) = (g_1(x), g_2(x)) \in \mathbb{R}_n^2 \), then \( \mathcal{C} \) is a one-generator QC code of index 2. Define \( g(x) = \gcd(g_1(x), g_2(x), x^n - 1) \) and \( h(x) = (x^n - 1)/g(x) \). Polynomials \( g(x) \) and \( h(x) \) are the generator polynomial and the parity-check polynomial of \( \mathcal{C} \), respectively. Further, the dimension of \( \mathcal{C} \) is \( \deg(h(x)) \). One can refer to \([11]\) for more details.

Attached to polynomial \( f(x) = f_0 + f_1 x + \cdots + f_{n-1} x^{n-1} \in \mathbb{R}_n \), we define \( \hat{f}(x) = f_0 + f_{n-1} x + f_{n-2} x^2 + \cdots + f_1 x^{n-1} \) and \( f^q(x) = f_0^q + f_{1}^q x + \cdots + f_{n-1}^q x^{n-1} \). In addition, if \( f(x) \cdot h(x) = x^n - 1 \), then \( \hat{f}(x) = x^{\deg(h(x))} h(\frac{x}{b}) \). In the following, a class of one-generator QC codes that are Hermitian self-orthogonal is introduced, whose some properties can be obtained from \([11]\).

Definition 1 Let \( \mathcal{C}_{q^2}(f, g) \) be a QC code over \( \mathbb{F}_q^2 \) of length \( 2n \) generated by \( (g(x), f(x)g(x)) \), where \( f(x) \) and \( g(x) \) are polynomials in \( \mathbb{R}_n \) such that \( g(x) \) divides \( x^n - 1 \).

Lemma 1 ([11], Proposition 14) The Hermitian dual code \( \mathcal{C}^{\perp}_{q^2}(f, g) \) of the QC code \( \mathcal{C}_{q^2}(f, g) \) over \( \mathbb{F}_q^2 \) is generated by the pairs \( (g^q(x), 0) \) and \( (-f^q(x), 1) \).

Lemma 2 ([11], Proposition 15) A sufficient condition for \( \mathcal{C}_{q^2}(f, g) \) to be contained in its Hermitian dual code \( \mathcal{C}^{\perp}_{q^2}(f, g) \) is \( g^{\perp q}(x) \mid g(x) \).
Note that if a linear code $C$ is self-orthogonal, then the dual dimension is larger than or equal to that of $C$. Since $C^\perp_h = (C^\perp)^q$, we conclude that $C^\perp_h$ and $C^\perp$ have the same weight distribution, where $C^\perp$ denotes the usual Euclidean dual code of $C$. In order to obtain the exact Hermitian dual distance of $C$, for simplicity, we can firstly calculate the weight distribution of $C$, and then apply the following well-known MacWilliams equation.

**Theorem 1** \([28]\) If $C$ is an $[n, k, d]$ linear code over $\mathbb{F}_q$ with weight enumerator

$$W_C(x, y) = \sum_{i=0}^{n} A_i x^{n-i} y^i,$$

where $A_i$ denotes the number of codewords in $C$ with Hamming weight $i$. Then, the weight enumerator of the Euclidean dual code $C^\perp$ is given by

$$W_{C^\perp}(x, y) = q^{-k} W_C(x + (q - 1)y, x - y).$$

### 3 Extended quasi-cyclic constructions of quantum codes

Recall that a $q$-ary quantum error-correcting code (QECC) of length $n$ is a $K$-dimensional subspace of the $q^n$-dimensional Hilbert space $(\mathbb{C}^q)^\otimes n$, where $\mathbb{C}$ denotes the complex field. If $K = q^k$, then the QECC is represented by $[[n, k, d]]_q$, where $d$ is the minimum distance. Just as the classical case, one of the main problems in quantum error correction is to construct QECCs with good parameters. When fixing the code length $n$ and dimension $k$, we expect to gain a big minimum distance $d$. Conversely, when the minimum distance $d$ is equal, we want the code rate $\frac{k}{n}$ to be greater. Available in \([13]\), there is a database of best known binary QECCs. For ternary QECCs, code tables \([8]\) are kept online by Edel according to their explicit constructions. In order to evaluate the superiority of QECCs, Feng et al. \([10]\) presented a quantum Gilbert-Vashamov (GV) bound as follows, which is closely related to the size of the finite field.

**Theorem 2** \([10]\, quantum Gilbert-Vashamov bound) Let $n > k \geq 2$ with $n \equiv k \pmod{2}$ and $d \geq 2$. Then there exists a pure stabilizer QECC with parameters $[[n, k, d]]_q$ if the inequality

$$\frac{q^{n-k+2}-1}{q^2-1} > \sum_{i=1}^{d-1} (q^2-1)^{i-1} \binom{n}{i}$$

is satisfied.

One can check that almost all the QECCs meet this bound. If not, these codes usually have particularly good parameters. It is generally known that there exists an important connection between QECCs and classical Hermitian self-orthogonal linear codes from the following theorem.
Theorem 3 A Hermitian self-orthogonal \([n,k]_{q^2}\) linear code \(C\) such that there are no vectors of weight less than \(d\) in \(C^\perp\) yields a QECC with parameters \([n,n-2k,d]_q\).

Via the Hermitian self-orthogonal quasi-cyclic codes introduced in Definition 1, next, we will provide our QC extended constructions that preserve the self-orthogonality.

Proposition 1 Assume that \(g(x)\) and \(f(x)\) are polynomials in \(\mathbb{R}_n\) satisfying \(g(x) \mid g(q^2)\), then the QC code \(C_{q^2}(f,g) = [2n, n - \deg(g(x)), d]_{q^2}\) with generator matrix \(G = (G_1, G_2)\) is Hermitian self-orthogonal. Let \(C_i\) (i = 1, 2) be a linear code generated by \(G_i\). If there exists a codeword \(x(i) \in C_i^\perp\) such that \((x(i), x(i))_h = p - 1\), where \(p\) is the characteristic of \(F_{q^2}\). Then

(i) The code \(C'\) with generator matrix

\[
G' = \begin{pmatrix}
G_1 & G_2 & 0 \\
\vdots & \vdots & \vdots \\
x^{(1)} & 0 & 0 \\
0 & \cdots & 0 & 1
\end{pmatrix}
\]

is a Hermitian self-orthogonal \([2n + 1, n - \deg(g(x)) + 1]\) linear code with Hermitian dual distance

\[
d^\perp_h \leq d(C'^\perp_h) \leq d^\perp_h + 1,
\]

where \(d^\perp_h\) denotes the Hermitian dual distance of \(C_{q^2}(f,g)\).

(ii) The code \(C''\) with generator matrix

\[
G'' = \begin{pmatrix}
G_1 & G_2 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
x^{(1)} & 0 & 0 & 1 \\
0 & \cdots & 0 & 0 & 1
\end{pmatrix}
\]

is a Hermitian self-orthogonal \([2n + 2, n - \deg(g(x)) + 2]\) linear code with Hermitian dual distance

\[
d^\perp_h \leq d(C''^\perp_h) \leq d^\perp_h + 2,
\]

where \(d^\perp_h\) denotes the Hermitian dual distance of \(C_{q^2}(f,g)\).

Proof Obviously, the linear codes \(C'\) and \(C''\) have parameters \([2n + 1, n - \deg(g(x)) + 1]\) and \([2n + 2, n - \deg(g(x)) + 2]\), respectively. A simple computation shows that

\[
G' G'^\dagger = \begin{pmatrix}
G_1 & G_2 & & & \vdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x^{(1)} & 0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
G_1 G_1^\dagger + G_2 G_2^\dagger & G_1 x^{(1)} \\
x^{(1)} G_1^\dagger & x^{(1)} x^{(1)^\dagger} + 1
\end{pmatrix}
\]
and

\[
G''G''^\dagger = \begin{pmatrix}
G_1 & G_2 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ddots & 0 \\
x^{(1)} & 0 & \cdots & 1 \\
x^{(2)} & 0 & \cdots & 0 \\
\end{pmatrix}
= \begin{pmatrix}
G_1G_1^\dagger + G_2G_2^\dagger & G_1x^{(1)} & G_2x^{(2)} \\
x^{(1)}G_1^\dagger & x^{(1)}x^{(1)} + 1 & 0 \\
x^{(2)}G_2^\dagger & x^{(2)}x^{(2)} + 1 & 0 \\
\end{pmatrix}
\]

Since the QC code \(C_{q^2}(f, g)\) is Hermitian self-orthogonal, then \(GG^\dagger = G_1G_1^\dagger + G_2G_2^\dagger = 0\), where 0 is the zero matrix. If there exists a codeword \(x^{(i)} \in C_{q^i}(i = 1, 2)\) such that \((x^{(i)}, x^{(i)})_h = p - 1\), then it is easy to see that \(G'G'^\dagger\) and \(G''G''^\dagger\) are both zero matrices. It is equivalent to say that linear codes \(C'\) and \(C''\) are both Hermitian self-orthogonal. Further, as every \(d_{\perp h} - 1\) columns of \(C\) are linearly independent, then every \(d_{\perp h} - 1\) columns of \(G'\) and \(G''\) are obviously linearly independent. It follows that \(d_{\perp h} \leq d(C'_{q^2}) \leq d_{\perp h} + 1\) and \(d_{\perp h} \leq d(C''_{q^2}) \leq d_{\perp h} + 2\), where \(d_{\perp h}\) denotes the Hermitian dual distance of \(C_{q^2}(f, g)\).

By Theorem 3 and Proposition 1, we have the following result directly.

**Theorem 4** With the above notations, let \(C_{q^2}(f, g)\) be a self-orthogonal QC code \(C_{q^2}(f, g)\) with respect to the Hermitian inner product. Then it provides two QECCs with parameters \([2n + 1, 2\deg(g(x)) - 1, d(C'_{q^1})]_q\) and \([2n + 2, 2\deg(g(x)) - 2, d(C''_{q^1})]_q\), respectively. Moreover, \(d_{\perp h} \leq d(C'_{q^2}) \leq d_{\perp h} + 1\) and \(d_{\perp h} \leq d(C''_{q^2}) \leq d_{\perp h} + 2\), where \(d_{\perp h}\) denotes the Hermitian dual distance of \(C_{q^2}(f, g)\).

In the following, we will construct some good QECCs over small finite fields \(F_2\) and \(F_3\) according to Theorem 4. We compute it by the algebra system Magma. Let \(\omega\) and \(\xi\) be primitive elements of \(F_4\) and \(F_9\). For simplicity, elements \(0, 1, \omega, \omega^2\) in \(F_4\) and \(0, 1, \xi, \xi^2, \xi^3, \xi^4, \xi^5, \xi^6, \xi^7\) in \(F_9\) are represented by \(0, 1, 2, 3, 4, 5, 6, 7, 8\), respectively.

**Example 1** Assume that \(q = 2\) and \(n = 15\). Consider the following polynomials in \(\mathbb{F}_4[x]/(x^{15} - 1)\),

\[
g(x) = x^9 + 3x^8 + x^7 + x^5 + 3x^4 + 2x^2 + 2x + 1,
\]

\[
f(x) = 2x^3 + 2x^2 + 2x + 1.
\]

Since \(g(x) \mid x^{15} - 1\) and \(g^{\perp}(x) \mid g(x)\), then by Lemma 2, the QC code \(C_{q^2}(f, g)\) is Hermitian self-orthogonal. Select a codeword \(x^{(1)} = (1, 3, 2, 1, 3, 2, 1, 3, 2, 1, 3, 2, 1, 3, 2)\),
2, 1, 3, 2) ∈ C⊥. According to Proposition 1 (i), we can construct a Hermitian self-orthogonal linear code C′ with generator matrix

\[
G = \begin{pmatrix}
 1 & 2 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 1 & 3 & 2 & 2 & 2 & 0 & 0 & 0 & 1 \\
 0 & 1 & 2 & 2 & 0 & 1 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 3 & 2 & 2 & 2 & 1 & 3 & 2 & 0 \\
 0 & 0 & 1 & 2 & 2 & 0 & 3 & 1 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 3 & 2 & 3 & 2 & 1 & 3 & 2 & 0 \\
 0 & 0 & 0 & 1 & 2 & 2 & 0 & 3 & 1 & 1 & 3 & 1 & 0 & 0 & 1 & 3 & 2 & 3 & 2 & 1 & 3 & 2 & 0 \\
 1 & 3 & 2 & 1 & 3 & 2 & 1 & 3 & 2 & 1 & 3 & 2 & 1 & 3 & 2 & 1 & 3 & 2 & 1 & 3 & 2 & 0 \\
\end{pmatrix}
\]

Using algebraic software Magma [3], we have that C′ is a [31, 7, 16]4 linear code and its weight enumerator is

\[
0^1 16^3 18^6 30^5 20^2 350^2 24^4 800^2 26^3 150^2 28^5 160^0 20^0 2520^0 22^0 3900^0 24^0 5400^0 26^0 3150^0 28^0 780^0.
\]

By the MacWilliams equation, we can gain that the weight enumerator of C⊥ is

\[
0^1 5^5 10^5 30^3 50^3 120^3 210^3 165^3 150^3 120^3 90^3 60^3 45^3 30^3 20^3 15^3 12^3 10^3 7^3 6^3 5^3 4^3 3^3 2^3 1^3 0.
\]

Hence, the Hermitian dual distance of C′ is 5. By Theorem 4, an optimal QECC with parameters [[31, 17, 5]]2 will be provided. Moreover, according to the propagation rule in [6], there will also exists a QECC with parameters [[32, 17, 5]]2. By comparison with Grassl’s code tables [13], we note that our codes are better than the best-known [[31, 17, 4]]2 and [[32, 17, 4]]2 QECCs, respectively. Hence, our QECCs break the current records.

**Example 2** Write q = 3 and n = 10. Choose the following polynomials in \(F_9[x]/\langle x^{10} - 1 \rangle\),

\[
g(x) = x^6 + 7x^5 + 5x^4 + x^2 + 3x + 5, \quad f(x) = x^3 + 2x^2 + 5x + 1.
\]

By Lemma 2, the QC code \(C_9(f, g)\) is Hermitian self-orthogonal. Select codewords \(x^{(1)} = (1, 1, 8, 2, 1, 2, 6, 0, 1) \in C_{12}^⊥\) and \(x^{(2)} = (1, 7, 3, 8, 5, 7, 7, 0, 3, 2) \in C_{12}^⊥\). By Proposition 1 (ii), a Hermitian self-orthogonal linear code C′ can be constructed, whose generator matrix is given as follows

\[
G'' = \begin{pmatrix}
 5 & 5 & 1 & 0 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 5 & 8 & 2 & 7 & 6 & 4 & 5 & 2 & 5 & 1 & 0 \\
 0 & 5 & 3 & 1 & 0 & 5 & 1 & 0 & 0 & 1 & 5 & 8 & 2 & 7 & 6 & 4 & 5 & 2 & 5 & 0 & 0 \\
 0 & 0 & 5 & 3 & 1 & 0 & 5 & 1 & 0 & 1 & 5 & 8 & 2 & 7 & 6 & 4 & 5 & 2 & 0 & 0 \\
 0 & 0 & 0 & 5 & 3 & 1 & 0 & 5 & 1 & 2 & 5 & 1 & 5 & 8 & 2 & 7 & 6 & 4 & 5 & 0 & 0 \\
 1 & 1 & 8 & 2 & 1 & 2 & 6 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 3 & 8 & 5 & 7 & 0 & 3 & 2 & 0 & 1
\end{pmatrix}
\]

Calculate that C′ is a [22, 6, 10]9 linear code and its weight enumerator is

\[
0^1 10^1 16^1 23^1 38^1 44^2 45^2 15^1 37^1 16^1 110^2 17^1 328^2 51^1 87^1 520^1 19^1 183^1 36^2 20^1 412^1 28^2 21^1 1266^1 42^2 2386^0.
\]
By the MacWilliams equation, we obtain that $\mathcal{C}'$ is a $[22, 16, 5]_9$ linear code. One can check that it is the best-known classical code according to Grassl’s code tables [13]. By Theorem 4, there exists a QECC with parameters $[[22, 10, 5]]$. To testify the superiority of the code, we find that our code exceeds the quantum GV bounds. In [9], the authors gave a QECC with parameters $[[22, 8, 5]]$. Obviously, the code rate of our QECC is higher.

**Example 3** Now set $q = 2$ and $n = 51$. Define polynomials

$$g(x) = x^{35} + 2x^{34} + 3x^{33} + 3x^{32} + 3x^{31} + 2x^{26} + x^{25} + 2x^{24} + x^{23} + 2x^{22} + 2x^{20} + 3x^{18} + 2x^{17} + 3x^{15} + 2x^{13} + 2x^{12} + x^{11} + x^{10} + 3x^9 + 2x^5 + x^4 + 2x^2 + 2x + 2,$$

$$f(x) = 2x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + 2x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

in $\mathbb{F}_4[x]/(x^{51} - 1)$. By Lemma 2, the QC code $\mathcal{C}_0(f, g)$ is Hermitian self-orthogonal. Choose a codeword $x^{(1)} = (1, 2, 0, 0, 0, 3, 2, 0, 2, 3, 3, 0, 2, 3, 1, 3, 0, 2, 2, 3, 2, 0, 3, 1, 3, 2, 1, 2, 2, 3, 0, 1, 0, 2, 0, 3, 0, 1, 2, 2, 0, 3, 0, 0) \in \mathcal{C}_1 \perp$. By Proposition 4 (i), we obtain a Hermitian self-orthogonal $[103, 17, 38]_4$ linear code, whose weight enumerator is given as follows

$$0^{1}, 38^{2}, 48^{3}, 56^{4}, 95^{5}, 110^{6}, 122^{7}, 58^{8}, 51^{9}, 97^{10}, 54^{11}, 148^{12}, 256^{13}, 691^{14}, 227^{15}, 58^{16}, 51^{17}, 22^{18}, 60^{19}, 278^{20}, 441^{21}, 62^{22}, 119^{23}, 59^{24}, 407^{25}, 64^{26}, 139^{27}, 55^{28}, 487^{29}, 66^{30}, 136^{31}, 538^{32}, 139^{33}, 68^{34}, 507^{35}, 97^{36}, 71^{37}, 148^{38}, 101^{39}, 77^{40}, 229^{41}, 33^{42}, 293^{43}, 51^{44}, 183^{45}, 76^{46}, 294^{47}, 109^{48}, 623^{49}, 393^{50}, 60^{51}, 245^{52}, 249^{53}, 80^{54}, 264^{55}, 268^{56}, 533^{57}, 82^{58}, 181^{59}, 506^{60}, 84^{61}, 982^{62}, 79^{63}, 961^{64}, 86^{65}, 413^{66}, 285^{67}, 65^{68}, 88^{69}, 322^{70}, 360^{71}, 40^{72}, 90^{73}, 118^{74}, 253^{75}, 79^{76}, 52^{77}, 286^{78}, 61^{79}, 94^{80}, 597^{81}, 61^{82}, 96^{83}, 43^{84}, 32^{85}, 63^{86}, 98^{87}, 162^{88}, 27^{89}$$

By the MacWilliams equation, the Hermitian dual distance of $\mathcal{C}'$ is 7. By Theorem 3, a QECC with parameters $[[103, 69, 7]]$ can be constructed, which surpasses the best-known $[[103, 69, 6]]$ QECC at now [13].

In the following, four tables will be given to illustrate that many good QECCs can be obtained by our extended QC constructions. Here we just give some good QECCs over small finite fields $\mathbb{F}_2$ and $\mathbb{F}_3$ via the extended construction provided in Proposition 1 (i). Tables 1 and 3 contain some Hermitian self-orthogonal linear codes $\mathcal{C}'$ over $\mathbb{F}_2$ and $\mathbb{F}_9$. These codes are used to construct good binary and ternary QECCs in Tables 2 and 4, respectively. In Table 2, some the best-known or optimal binary QECCs with length less than or equal to 127 are provided, most of which have different weight distributions with the best-known QECCs in Grassl’s code tables [13]. In Table 4, we construct some good ternary QECCs, which all exceed the quantum GV bounds and have higher code rate than QECCs available in [8]. For simplicity, we write coefficients of polynomials in ascending order to denote polynomials. The exponents of the elements indicate the number of the consecutive same elements. For example, the polynomial $1 + \xi^4 x^2 + x^3 + x^4$ over $\mathbb{F}_9$ is represented by 1052.
Table 1  Hermitian self-orthogonal extended QC codes $\mathcal{C}'$ over $F_4$.

| n   | $f(x)$, $g(x)$ and $x^{(1)}$ over $F_4$ | Codes $\mathcal{C}'$ |
|-----|---------------------------------|-------------------|
| 7   | 12, $101^2$, $(13)^23^1$        | [15, 4, 8]_4      |
| 17  | $3^{11}$, $132^20^22^23^1$     | [35, 9, 14]_4     |
| 23  | $1^921$, $10(100)^23^1$, $10232^20^331302^032^03^0$ | [47, 12, 20]_4   |
| 29  | $1^9212$, $(133)^2(133)^221$, $1021^20^1302^030^12^13^2$ | [59, 15, 24]_4   |
| 31  | $1^73$, $1^702(01)2^010^310^3$, $10^4132301^2013101^21^270^1302^032^03^21^2$ | [63, 11, 24]_4   |
| 31  | $1^7212$, $10^13^01^03^21^2$, $1^02^02^23^30^310^330^13^01^20^212^10^23^0$ | [63, 16, 22]_4   |
| 37  | $1^42013$, $12^20213^23102^02^1$, $(10^213^23^1)^21^213^10^32^030^102120^102$ | [75, 19, 26]_4   |
| 39  | $1^432013$, $121^3021^21^2310^213^3$, $1^03^23^23203^131313^1(23)^20^23^312^10^3$ | [79, 19, 32]_4   |
| 41  | $1^72^1$, $131210(31)^2101^23^22^2101^23^301213^1$, $130^323201^22^212^10^23^032^012^10^13^10^2^0$ | [83, 11, 30]_4   |
| 55  | $1^{10}2$, $130132^23020^32^131^102^6^3^31^01^2102^013(31)^42^01$, | [111, 13, 46]_4   |
| 55  | $(13)^22^10^130^22^210^22^103^20^23^03^02^12^02^03^210^220^32^03^210^2$ |        |
| 63  | $1^7212$, $3123^12^302^23^22^213^31^22^210^23^02^22^01^20^02^23^31^210^212^1$, $1^202^23^2320^310^221^211^20^130^21^22^10^13^20^13^2$ | [127, 13, 52]_4 |
| 63  | $1^7212$, $31^23^20^22^32^230^23^31^210^22^02^01^20^22^22^212^23^210^23^31^210^22^2120^13^210^2$ | [127, 16, 50]_4 |
| 63  | $10^230^3(20)^212^1^212^1^20^2^23^212^31^213^23^13^20^2^0^23^0^2^212^31^2$ |        |

Table 2  The best-known binary QECCs from extended QC codes $\mathcal{C}'$.

| $\mathcal{C}_{1,1}$ $f(x)$, $g(x)$ | $\mathcal{C}_{1,2}$ $f(x)$, $g(x)$ | Our QECCs |
|-----------------|-----------------|----------|
| [15, 4, 8]_4   | [15, 11, 3]_4   | [15, 7, 3]_2 |
| [35, 9, 14]_4  | [35, 26, 5]_4   | [35, 17, 5]_2 |
| [47, 12, 20]_4 | [47, 35, 6]_4   | [47, 23, 6]_2 |
| [59, 15, 24]_4 | [59, 44, 7]_4   | [59, 29, 7]_2 |
| [63, 11, 24]_4 | [63, 32, 5]_4   | [63, 41, 5]_2 |
| [63, 16, 22]_4 | [63, 47, 7]_4   | [63, 31, 7]_2 |
| [75, 19, 26]_4 | [75, 56, 8]_4   | [75, 37, 8]_2 |
| [79, 19, 32]_4 | [79, 60, 8]_4   | [79, 41, 8]_2 |
| [83, 11, 30]_4 | [83, 72, 5]_4   | [83, 61, 5]_2 |
| [111, 13, 46]_4| [111, 98, 5]_4  | [111, 85, 5]_2 |
| [127, 13, 52]_4| [127, 114, 5]_4 | [127, 101, 5]_2 |
| [127, 16, 50]_4| [127, 111, 6]_4 | [127, 95, 6]_2 |
Table 3 Hermitian self-orthogonal extended QC codes $C'$ over $\mathbb{F}_9$.

| $n$ | $f(x), g(x), x^{(1)}$ over $\mathbb{F}_9$ | Codes $C'$ |
|-----|-------------------------------------|------------|
| 11  | 12486, 15^{103}, 129245487^3        | [23, 6, 12]_9 |
| 17  | $1^4_{5121}, 5215371561, 1^6_{3680}1720823472$ | [35, 9, 16]_9 |
| 23  | $1^8_{212}, 150_{511}(10)^201, 18452373054381^6383157^2$ | [47, 12, 23]_9 |
| 35  | $1^8_{21}, 5208270^7(75)^540276513148^2731, 1050^22676308^2116^202384^273487^280^2$ | [71, 9, 26]_9 |
| 41  | $1^7_{506}, 58354913507452^26126526^218730175081741, 1743516718^230141^2786273^281(28)^2245^28631^242$ | [83, 5, 39]_9 |
| 65  | $1^7_{51}, 173681^1057206^22847641684587643^2746825^280^21340275868531, 17361^225412708058626127^2805(26)^21287080(08)^2128642857381682^264214$ | [131, 12, 59]_9 |

4 Quasi-cyclic constructions of entanglement-assistant quantum codes

Entanglement-assistant quantum error-correcting codes (EAQECCs) can be regarded as generalized QECCs, which can break the self-orthogonal conditions. An $[[n, k, d; c]]_q$ EAQECC encodes $k$ logical qubits into $n$ physical qubits with the help of $c$ copies of entangled ebits. In particular, if $c = 0$, then the EAQECC is a standard stabilizer QECC. Similar to the QECCs, EAQECCs can also be constructed by classical linear codes in the following theorem.

**Theorem 5** ([36]) If $C$ is an $[[n, k, d; c]]_q$ classical code over $\mathbb{F}_q$ with parity check matrix $H$, then $C^{\perp_h}$ stabilizes an EAQECC with parameters $[[n, 2k - n + c, d; c]]_q$, where $c = \text{Rank}(HH^\dagger)$ is the number of entangled ebits required.
In 2018, Guenda et al. [14] established a relation between the required number of entangled ebits and the dimension of the Hermitian hull of a classical linear code.

**Theorem 6** ([14]) Let $\mathcal{C}$ be a classical $[n, k, d]_q$ code with parity check matrix $H$ and generator matrix $G$. Then $\text{Rank}(HH^\dagger)$ and $\text{Rank}(GG^\dagger)$ are independent of $H$ and $G$ so that

$$\text{Rank}(HH^\dagger) = n - k - \dim(\text{Hull}_h(\mathcal{C})) = n - k - \dim(\text{Hull}_h(\mathcal{C}^\perp_h)),$$

and

$$\text{Rank}(GG^\dagger) = k - \dim(\text{Hull}_h(\mathcal{C})) = k - \dim(\text{Hull}_h(\mathcal{C}^\perp_h)).$$

where $\text{Hull}_h(\mathcal{C}) = \text{Hull}_h(\mathcal{C}^\perp_h) = \mathcal{C} \cap \mathcal{C}^\perp_h$. Obviously, $c = \text{Rank}(HH^\dagger) = \text{Rank}(GG^\dagger) + n - 2k$.

For an $[[n, k, d; c]]_q$ EAQECC, it is called maximal-entanglement when $c = n - k$. Refs. [4, 19, 20, 24, 37] have revealed that maximal-entanglement EAQECCs can both reach the EA-quantum capacity and EA-hashing bound asymptotically, which can provide higher code rate and lower SNR (signal to noise ratio). Therefore, it is worthwhile to exploit how to construct maximal-entanglement EAQECCs with good performances.

In the following, using the QC code $\mathcal{C}_q^2(f, g)$, we present construction methods to obtain maximal-entanglement EAQECCs. By Lemma 1, we know that the Hermitian dual code of $\mathcal{C}_q^2(f, g)$ is generated by the pairs $(g^\perp q(x), 0)$ and $(-f^\perp q(x), 1)$. Hence, code $\mathcal{C}_q^2(f, g)$ has a parity check matrix as follows

$$H = \begin{pmatrix} H_1 & 0 \\ H_2 & I_n \end{pmatrix},$$

where $\deg(g(x)) \times n$ matrix $H_1$ and $n \times n$ matrix $H_2$ are respectively circulant matrices determined by $g^\perp q(x)$ and $-f^\perp q(x)$. $I_n$ denotes the $n \times n$ identity matrix. Let matrix $M$ be the conjugate transpose of the circulant matrix defined by $f(x)$. One can see that $H_2 + M = 0$. In the rest of the paper, we suppose that gcd($f(x), x^n - 1) = 1$. It is easily deduced that matrices $M$ and $H_2$ are invertible.

**Theorem 7** With the previous notions, suppose that $\mathcal{C}_q^2(f, g)$ is a $[2n, n - \deg(g(x)), d]_q$ QC code with parity check matrix

$$H = \begin{pmatrix} H_1 & 0 \\ H_2 & I_n \end{pmatrix},$$

where $H_1, H_2$ be a nonsingular matrix. Define $P = H_1^\dagger (H_1 H_1^\dagger)^{-1} H_1 - (H_1 H_2)^{-1}$. If the number 1 isn’t in the eigenvalue set of $P$, then there exist two maximal-entanglement EAQECCs with parameters $[[2n, n - \deg(g(x)), d; n + \deg(g(x))]]_q$ and $[[2n, n + \deg(g(x)), d^\perp_h; n - \deg(g(x))]]_q$, respectively, where $d^\perp_h$ denotes the Hermitian dual distance of $\mathcal{C}_q^2(f, g)$. 
Proof Since $\mathcal{C}_{q^2}(f, g)$ is a $[2n, n - \deg(g(x)), d_1, q]$ linear code, then its Hermitian dual $\mathcal{C}_{q^2}^\perp(f, g)$ has parameters $[2n, n + \deg(g(x)), d_2, q]$. Applying Theorem 5, it provides two $[[2n, -2\deg(g(x)) + c_1, d; c_1]]_q$ and $[[2n, 2\deg(g(x)) + c_2, d_2, c_2]]_q$ EAQECCs, where $d^\perp$ denotes the Hermitian dual distance of $\mathcal{C}_{q^2}(f, g)$. Next we compute the number of entangled ebits $c_1$ and $c_2$. Note that

$$HH^\dagger = \begin{pmatrix} H_1 & 0 \\ H_2 & I_n \end{pmatrix} \begin{pmatrix} H_1^\dagger & H_1^\dagger H_2 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} H_1 H_1^\dagger & H_1 H_2^\dagger + I_n \\ H_2 H_1^\dagger & H_2 H_2^\dagger + I_n \end{pmatrix}.$$

By the hypothesis, $H_1 H_1^\dagger$ is a nonsingular matrix, then we define the following matrices

$$A = \begin{pmatrix} (H_1 H_1^\dagger)^{-1} & 0 \\ 0 & I_n \end{pmatrix}, \quad B = \begin{pmatrix} I_{\deg(g(x))} & 0 \\ -H_2 H_1^\dagger & I_n \end{pmatrix},$$

$$C = \begin{pmatrix} I_{\deg(g(x))} & 0 \\ 0 & H_2^{-1} \end{pmatrix}, \quad D = \begin{pmatrix} I_{\deg(g(x))} & 0 \\ 0 & (H_2^{-1})^{-1} \end{pmatrix}.$$

Then

$$AH H^\dagger = \begin{pmatrix} I_{\deg(g(x))} & (H_1 H_1^\dagger)^{-1} H_1 H_1^\dagger H_2^\dagger \\ H_2 H_1^\dagger & H_2 H_2^\dagger + I_n \end{pmatrix}$$

and

$$BA H H^\dagger = \begin{pmatrix} I_{\deg(g(x))} & (H_1 H_1^\dagger)^{-1} H_1 H_1^\dagger H_2^\dagger \\ 0 & -H_2 H_1^\dagger (H_1 H_1^\dagger)^{-1} H_1 H_2^\dagger + H_2 H_2^\dagger + I_n \end{pmatrix}.$$

If the number 1 isn’t an eigenvalue of matrix $P = (H_1 H_1^\dagger)^{-1} H_1 (H_2 H_2^\dagger)^{-1}$, then $-P + I_n$ and $CB AH H^\dagger D$ are both full rank matrices. Note that matrices $A$, $B$, $C$ and $D$ are all invertible, then $c_1 = \text{Rank}(HH^\dagger) = \text{Rank}(AH H^\dagger) = \text{Rank}(BA H H^\dagger D) = n + \deg(g(x))$. According to Theorem 6, $c_2 = \text{Rank}(CC^\dagger) = \text{Rank}(HH^\dagger) + 2n - 2(n + \deg(g(x))) = n - \deg(g(x))$. As a consequence, it provides two maximal-entanglement EAQECCs with parameters $[[2n, n - \deg(g(x)), d; n + \deg(g(x))]]_q$ and $[[2n, n + \deg(g(x)), d^\perp; n - \deg(g(x))]]_q$, respectively.

Next we will construct some binary maximal-entanglement EAQECCs with good parameters according to Theorem 7. Similarly, let $\omega$ be the primitive element, and elements $0, 1, \omega, \omega^2 \in F_4$ are represented by $0, 1, 2, 3$.

**Example 4** Set $q = 2$ and $n = 7$. Select the following polynomials in the quotient ring $F_4[x]/(x^7 - 1)$,

$$g(x) = x + 1, \quad f(x) = x^5 + 2x^4 + 3x^3 + 2x^2 + 3x.$$
Then $g^q(x) = x^6 + x^3 + x^4 + x^2 + x + 1$, $f^q(x) = 2x^6 + 3x^5 + 2x^4 + 3x^3 + x^2$,

$$H_1 = (111111), \quad H_2 = \begin{pmatrix} 0013232 \\ 200132 \\ 3200132 \\ 2320013 \\ 1323200 \\ 0132320 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0023230 \\ 0002323 \\ 3000232 \\ 2300023 \\ 3230002 \\ 0232300 \end{pmatrix}.$$ 

By calculation, $\mathcal{C}_4(f,g)$ is a linear code with parameters $[14, 6, 7]_4$. Since the characteristic polynomial of matrix $P$ is $x(x^3 + \omega)(x^3 + \omega^2)$, it is easy to see that the number 1 isn’t in its eigenvalue set. Applying Theorem 7, it can provide a maximal-entanglement EAQECC with parameters $[[14, 6, 7: 8]]_2$. According to Lu’s code tables of maximal-entanglement EAQECs in [22], our code is optimal and better than the best-known $[[14, 6: 8]]_2$ EAQECs. So it breaks the current records.

**Example 5** Let $q = 2$, $n = 11$ and define the following polynomials in the quotient ring $\mathbb{F}_4[x]/(x^{11} - 1)$,

$$g(x) = x^6 + 3x^5 + 3x^4 + 2x^2 + 2x + 1, \quad f(x) = x^4 + 2x^3 + 3x^2 + x.$$ 

Then $g^q(x) = x^6 + 3x^4 + x^3 + x^2 + 2x + 1$, $f^q(x) = x^{10} + 2x^9 + 3x^8 + x^7$. We calculate that $\mathcal{C}_4(f, g)$ is a $[22, 5, 13]_4$ linear code and its weight enumerator is

$$0^1 1^{56} 2^{66} 3^{1598} 4^{16^6} 5^{17^6^6} 6^0 7^{18^3} 8^{19^{132}} 9^{20^{33}} 10^{31} 11^{32} 12^{33} 13^{34} 14^{35} 15^{49} 16^{43} 17^{52} 18^{64} 19^{79} 20^{89} 21^{99}.$$ 

Using the MacWilliams equation, the weight enumerator of $\mathcal{C}_4^{[2, 4]}(f, g)$ is given as follows,

$$0^{14} 1^{46275} 2^{65676} 3^{52437} 4^{1364056} 5^{82050290} 6^{9562740} 7^{1037269804} 8^{11122099016} 9^{122435494302} 10^{13774526170} 11^{14393685534} 12^{1538956696} 13^{163136710621} 14^{17332193204} 15^{183767434720} 16^{19174803664} 17^{20786523551} 18^{212247456015} 19^{22306444469}.$$ 

Hence, $\mathcal{C}_4^{[2, 4]}(f, g)$ is an optimal linear code with parameters $[22, 17, 4]_4$ and meets requirements of Theorem 7. Hence, a maximal-entanglement EAQEC with parameters $[[22, 17, 4: 5]]_2$ can be constructed. It has better parameters than the $[[23, 17, 2: 6]]_2$ maximal-entanglement EAQEC appearing in [22], whose minimum distance does not increase when the entangled states are added.

We provide some binary maximal-entanglement EAQECs in Tables 5 and 6, which are derived from quaternary QC codes $\mathcal{C}_4(f, g)$ and $\mathcal{C}_4^{[2, 4]}(f, g)$, respectively. Compared to the parameters of maximal-entanglement EAQECs available in [22], our EAQECs have better performances.

In [19, 20], authors have showed that almost $[[n, k, d; c]]$ EAQECs were not equivalent to any standard $[[n + c, k, d]]$ QECCs and had better performances than all $[[n + c, k, d]]$ QECCs. Even if a maximal-entanglement $[[n, k, d; c]]$
is a \([2n + 1, n - \deg(g(x)) + 1]\) linear code with Hermitian dual distance
\[d^{+s} \leq d(C^{+s}) \leq d^+ + 1,\]
where $d^{⊥h}$ denotes the Hermitian dual distance of $C_{q^h}(f, g)$. Moreover,
\[
\text{Rank}(G' G'^{-1}) = n - \deg(g(x)) + 1.
\]
(ii) The code $C''$ with generator matrix
\[
G'' = \begin{pmatrix}
G_1 & G_2 & \vdots & \vdots \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \ddots & 0 \\
\end{pmatrix}
\]
\[
is a \ [2n + 2, n - \deg(g(x)) + 2] \ linear \ code \ with \ Hermitian \ dual \ distance
\[
d^{⊥h} \leq d(C''^{⊥h}) \leq d^{⊥h} + 2,
\]
where $d^{⊥h}$ denotes the Hermitian dual distance of $C_{q^h}(f, g)$. Moreover,
\[
\text{Rank}(G' G''^{-1}) = n - \deg(g(x)) + 2.
\]

The process of proof is similar to Proposition 1, we omit it here. From Theorem 5 and Proposition 2, the following result can be concluded directly.

**Theorem 8** Let $q > 2$ be a prime power and $C_{q^h}(f, g)$ a QC code with generator $G = (G_1, G_2)$ satisfying the conditions of Theorem 7. Then, there exist two maximal-entanglement EAQECCs with parameters $[[2n + 1, n - \deg(g(x)) + 1]]$ and $[[2n + 2, n - \deg(g(x)) + 2]]$, respectively. Moreover, $d^{⊥h} \leq d(C''^{⊥h}) \leq d^{⊥h} + 1$ and $d^{⊥h} \leq d(C''^{⊥h}) \leq d^{⊥h} + 2$, where $d^{⊥h}$ denotes the Hermitian dual distance of $C_{q^h}(f, g)$.

**Example 6** Assume that $q = 9$ and $n = 10$. Let $\zeta$ be a primitive element of $\mathbb{F}_{81}$. Consider the following polynomials in $\mathbb{F}_{81}[x]/(x^{10} - 1)$,
\[
g(x) = x^7 + \zeta^{44} x^6 + \zeta^{58} x^5 + \zeta^{52} x^4 + \zeta^{36} x^3 + \zeta^{10} x^2 + \zeta^{44} x + \zeta^{48},
\]
\[
f(x) = \zeta^{14} x^2 + \zeta^2 x + 1.
\]
Then $g^{⊥q}(x) = \zeta^8 x^3 + \zeta^{12} x^2 + \zeta^{36} x + 1$, $f^q(x) = \zeta^{18} x^9 + \zeta^{46} x^8 + 1$,
\[
H_1 = \begin{pmatrix}
\zeta^{2000000000} & \zeta^{36} & \zeta^{12} & 0000000000 \\
\zeta^{12} & \zeta^{58} & \zeta^{52} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{12} & \zeta^{58} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{12} & \zeta^{58} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{12} & \zeta^{58} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{12} & \zeta^{58} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{12} & \zeta^{58} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{12} & \zeta^{58} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{12} & \zeta^{58} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{12} & \zeta^{58} & \zeta^{0000000000} \\
\end{pmatrix},
\]
\[
H_2 = \begin{pmatrix}
\zeta^{60} & \zeta^{19} & \zeta^{12} & 0000000000 \\
\zeta^{12} & \zeta^{11} & \zeta^{12} & 0000000000 \\
\zeta^{0000000000} & \zeta^{11} & \zeta^{12} & 0000000000 \\
\zeta^{0000000000} & \zeta^{11} & \zeta^{12} & 0000000000 \\
\zeta^{0000000000} & \zeta^{11} & \zeta^{12} & 0000000000 \\
\zeta^{0000000000} & \zeta^{11} & \zeta^{12} & 0000000000 \\
\zeta^{0000000000} & \zeta^{11} & \zeta^{12} & 0000000000 \\
\zeta^{0000000000} & \zeta^{11} & \zeta^{12} & 0000000000 \\
\zeta^{0000000000} & \zeta^{11} & \zeta^{12} & 0000000000 \\
\zeta^{0000000000} & \zeta^{11} & \zeta^{12} & 0000000000 \\
\end{pmatrix},
\]
\[
P = \begin{pmatrix}
\zeta^{48} & \zeta^{14} & \zeta^{10} & 0000000000 \\
\zeta^{10} & \zeta^{36} & \zeta^{52} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{36} & \zeta^{52} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{36} & \zeta^{52} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{36} & \zeta^{52} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{36} & \zeta^{52} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{36} & \zeta^{52} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{36} & \zeta^{52} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{36} & \zeta^{52} & \zeta^{0000000000} \\
\zeta^{0000000000} & \zeta^{36} & \zeta^{52} & \zeta^{0000000000} \\
\end{pmatrix}.
\]

Since the characteristic polynomial of matrix $P$ is $x(x + \zeta^{10})(x + \zeta^{30})(x + \zeta^{10})(x + \zeta^{30})(x + \zeta^{60})(x + \zeta^{20})^3$, then the number 1 isn’t an eigenvalue of the matrix $P$. Moreover, $H_1 H_1^T$ is nonsingular. Therefore, the QC code $C_{q^h}(f, g)$ with a generator matrix
\[
G = \begin{pmatrix}
\zeta^{48} & \zeta^{44} & \zeta^{10} & \zeta^{36} & \zeta^{52} & \zeta^{58} & \zeta^{44} & 0000000000 \\
\zeta^{44} & \zeta^{48} & \zeta^{10} & \zeta^{36} & \zeta^{52} & \zeta^{58} & \zeta^{44} & 0000000000 \\
\zeta^{10} & \zeta^{36} & \zeta^{52} & \zeta^{58} & \zeta^{44} & 0000000000 \\
\zeta^{20} & \zeta^{36} & \zeta^{52} & \zeta^{58} & \zeta^{44} & 0000000000 \\
\zeta^{36} & \zeta^{52} & \zeta^{58} & \zeta^{44} & 0000000000 \\
\zeta^{52} & \zeta^{58} & \zeta^{44} & 0000000000 \\
\zeta^{58} & \zeta^{44} & 0000000000 \\
\end{pmatrix}
\]
satisfies the conditions of Theorem 7.

We choose \( x^{(1)} = (\zeta^{44}, \zeta^{71}, \zeta^{56}, \zeta^{22}, \zeta^{52}, \zeta^{73}, \zeta^{58}, \zeta^{33}) \) and \( x^{(2)} = (\zeta^{18}, \zeta^{41}, 2, \zeta^{10}, \zeta^{17}, \zeta^{31}, \zeta^{61}, \zeta^{75}, \zeta^{58}) \). From \( x^{(1)} x^{(1)\dagger} = \zeta^{60} \) and \( x^{(2)} x^{(2)\dagger} = \zeta^{50} \), we may take \( \alpha_1 \) and \( \alpha_2 \) both equal to 1. Hence, the extended QC code \( C'' \) has a generator matrix as follows

\[
G'' = \begin{bmatrix}
\zeta^{48} & \zeta^{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\zeta^{44} & \zeta^{71} & \zeta^{56} & \zeta^{33} & \zeta^{58} & \zeta^{33} & 0 & 0 & 0 & 0 & 1 \\
\zeta^{71} & \zeta^{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\zeta^{44} & \zeta^{71} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\zeta^{71} & \zeta^{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Applying the MacWilliams equation, we calculate that \( C'' \perp_h \) is a \([22, 17, 5; 5]_{81} \) linear code. According to Theorem 8, a new maximal-entanglement EAQECC with parameters \([22, 17, 5; 5]_{9}\) can be derived, which is superior to the codes with parameters \([23, 17, 3; 6]_{9}\) appeared in [22]. Note that a standard pure \([22, 10, 5]_{9}\) QECC is the best code meeting the quantum GV bounds with code length \( n = 22 \) and minimum distance \( d = 5 \). Compared with this standard QECC, our constructed EAQECC has \( 9^7 = 4782969 \) more codewords for the same code length and minimum distance although we add 5 entanglement ebits indeed.

5 Conclusions

In this paper, by a class of one-generator QC codes, we presented QC extended constructions that preserved the self-orthogonality. As an application, some good stabilizer QECCs over small finite fields \( \mathbb{F}_2 \) and \( \mathbb{F}_3 \) were obtained. In the binary case, some of our quantum codes broken or matched the current records. In the ternary case, our codes filled some gaps or had better performances than the current results.

It is well-known that the most common way of constructing QECCs now is from cyclic and constacyclic codes [1,16,27,38]. But in most cases, in order to gain good QECCs, we need the code length \( n \) to divide \( q^s - 1 \) for some positive integer \( s \). From our extended QC constructions, one can see that our method can break through the restriction partly, which produces QECCs with more flexible code lengths. Further, we have constructed maximal-entanglement EAQECCs from QC codes and their extended codes as well. Some good maximal-entanglement EAQECCs were derived and their parameters were compared. To the best of our knowledge, this is the first attempt to construct maximal-entanglement EAQECCs from QC codes and their extended codes.

However, one can find that our construction only can provide QECCs and EAQECCs with a relatively small distance. As the dimension increase, calculating the exact Hermitian dual distance will be computationally intractable (NP-hard) even if we used the MacWilliams equation. So in future study, a lower bound for our QC extended construction is extremely valuable.
Acknowledgments

This work is supported by National Natural Science Foundation of China (Nos.11471011, 11801564, 11901579).

References

1. Aly, S.A., Klappenecker, A., Sarvepalli, P.K.: On quantum and classical BCH codes. IEEE Trans. Inf. Theory 53(3), 1183-1188 (2007)
2. Ashikhmin, A., Knill, E.: Nonbinary quantum stabilizer codes. IEEE Trans. Inf. Theory 47(7), 3065-3072 (2001)
3. Bosma, W., Cannon, J., Playoust, C.: The MAGMA algebra system I: the user language. J. Symb. Comput. 24, 235-265 (1997)
4. Bowen, G.: Entanglement required in achieving entanglement-assisted channel capacities. Phys. Rev. A 66, 052313 (2002)
5. Brun, T., Devetak, I., Hsieh, M.: Correcting quantum errors with entanglement. Science. 314, 436-439 (2006)
6. Calderbank, A.R., Rains, E.M., Shor, P.W., Sloane, N.J.A.: Quantum error correction via GF(4). IEEE Trans. Inf. Theory 44(4), 1369-1387 (1998)
7. Daskalov, R., Hristov, P.: New binary one-generator quasi-cyclic codes. IEEE Trans. Inf. Theory 49, 3001-3005 (2003)
8. Edel, Y.: Table of quantum twisted codes. electronic address: www.mathi.uni-heidelberg.de/~yves/matritzen/QTBC/QTBCindex.html (Accessed 5 Sep. 2020)
9. Ezerman, M.F., Ling, S., Özkaya, B., Solé, P.: Good stabilizer codes from quasi-cyclic codes over $F_4$ and $F_9$. [arXiv:1906.03864v1] (2019)
10. Feng, K., Ma, Z.: A finite Gilbert-Varshamov bound for pure stabilizer quantum codes, IEEE Trans. Inf. Theory 50, 3323-3325 (2004).
11. Grassl, M., Codetables. electronic address: http://www.codetables.de/ (Accessed 5 Sep. 2020)
12. Guenda, K., Jitman, S., Gulliver, T.A.: Constructions of good entanglement assisted quantum error correcting codes. Des. Codes. Cryptogr. 86, 121-136 (2018)
13. Hagiwara, M., Kasai, K., Imai, H., Sakaniwa, K.: Spatially-coupled quasi-cyclic quantum LDPC codes. in Proc. 2011 IEEE ISIT. Nice. France 638-642 (2007)
14. Kai, X., Zhu, S., Tang, Y.: Quantum negacyclic codes. Phys. Rev. A 88(1), 012326(1-5) (2013)
15. Kasami, T.: A Gilbert-Varshamov bound for quasi-cyclic codes of rate $\frac{1}{2}$. IEEE Trans. Inf. Theory 20, 679 (2008)
16. Ketkar, A., Klappenecker, A., Kumar, S.: Nonbinary stabilizer codes over finite fields. IEEE Trans. Inf. Theory 52, 4892-4914 (2006)
17. Lai, C.Y., Brun, T.A.: Entanglement-assisted quantum error correcting codes with imperfect ebits. Phys. Rev. A 86, 032319 (2012)
18. Lai, C.Y., Brun, T.A.: Entanglement increases the error-correcting ability of quantum error-correcting codes. Phys. Rev. A 88, 012320 (2013)
19. Li, R., Ma, W., Ma, Y., Liu, Y., Cao, H.: Entanglement-assisted quantum MDS codes from constacyclic codes with large minimum distance. Finite Fields Appl. 53, 309-325 (2018)
20. Lu, L., Li, R., Guo, L., Fu, Q.: Maximal entanglement-assisted quantum codes constructed from linear codes. Quantum Inf. Process. 14(1), 165-182 (2014)
25. Lv, J., Li, R., Wang, J.: New binary quantum codes derived from one-generator quasi-cyclic codes. IEEE Access 7, 85782-85785 (2019)
26. Lv, J., Li, R., Wang, J.: An explicit construction of quantum quantum stabilizer codes from quasi-cyclic codes. IEEE Communication Letters. 1-1 (2020)
27. Ma, Z., Lu, X., Peng, K., Feng, D.: On non-binary quantum BCH codes. LNCS 3959, 675-683 (2006)
28. MacWilliams, F.J., Sloane, N.J.A.: The Theory of Error Correcting Codes. Amsterdam, The Netherlands: North-Holland Math. Library 16, (1996)
29. Qian, J., Zhang, L.: On MDS linear complementary dual codes and entanglement-assisted quantum codes. Des. Codes Cryptogr. 86(7) 1565-1572 (2018)
30. Séguin, G.E., Drolet, G.: The theory of 1-generator quasi-cyclic codes. Technical Reports, Department of Electrical and Computer Engineering, Royal Military College, Kingston, ON, Canada (1990)
31. Shor, P.W.: Scheme for reducing decoherence in quantum memory. Phys. Rev. A 52(4), 2493-2496 (1995)
32. Siap, I., Aydın, N., Ray-Chaudhuri, D.K.: New ternary quasi-cyclic codes with better minimum distances. IEEE Trans. Inf. Theory 46, 1554-1558 (2000)
33. Steane, A.: Multiple-particle interference and quantum error correction. Proc. R. Soc. A Math. Phys. Eng. Sci. 452 (1954), 2551-2577 (1996)
34. Tonchev, V.D.: The existence of optimal quaternary [28, 20, 6] and quantum [[28, 12, 6]]. J. Algebra Comb. Discrete Appl. 1(1), 13-17 (2013).
35. Wang, J., Li, R., Lv, J., Guo, G., Liu, Y.: Entanglement-assisted quantum error correction codes with length n = q^2 + 1. Quantum Inf. Process. 18, 1-21 (2019)
36. Wilde, M., Brun, T.: Optimal entanglement formulas for entanglement-assisted quantum coding. Phy. Rev. A 77, 064302 (2008)
37. Wilde, M.M., Hsieh, M.H., Babar, Z.: Entanglement-assisted quantum turbo codes. IEEE Trans. Inf. Theory 60, 1203-1222 (2014)
38. Zhu, S., Sun, Z., Li, P.: A class of negacyclic BCH codes and its application to quantum codes. Des. Codes Cryptogr. 86(10), 2139-2165 (2018)