Metastable massive gravitons from an infinite extra dimension*

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July 18, 2018

Abstract

Motivated by stringy considerations, Randall & Sundrum have proposed a model where all the fields and particles of physics, save gravity, are confined on a 4-dimensional brane embedded in 5-dimensional anti-deSitter space. Their scenario features a stable bound state of bulk gravity waves and the brane that reproduces standard general relativity. We demonstrate that in addition to this zero-mode, there is also a discrete set of metastable bound states that behave like massive 4-dimensional gravitons which decay by tunneling into the bulk. These are resonances of the bulk-brane system akin to black hole quasinormal modes—as such, they give rise to the dominant corrections to 4-dimensional gravity. The phenomenology of braneworld perturbations is greatly simplified when these resonant modes are taken into account, which is illustrated by considering gravitational radiation emitted from nearby sources and early universe physics.

*This essay received an honourable mention in the 2005 Gravity Research Foundation essay competition
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String theory, one of the leading candidates for the ‘theory of everything,’ tells us that the universe has more than four dimensions, but everyday experience seems to suggest otherwise. Over the years, there have been several attempts to explain why extra dimensions are hidden, including the conventional assumption that they have a compact topology with radii on the order of the Planck length. But recently, certain developments in non-perturbative string theory have provided alternatives to this ‘Kaluza-Klein’ compactification. The key ingredient is so-called $d$-branes, which are $(d + 1)$-dimensional hypersurfaces on which the standard model particles and fields can be consistently confined. This raises the possibility that the observable universe is a 3-brane, i.e., we are living on a ‘braneworld’ [1, review].

But in these scenarios, gravity is free to propagate in the full higher-dimensional ‘bulk’ manifold. This is potentially worrisome, since the force of gravity we are familiar with behaves in an entirely 4-dimensional manner. For example, in a higher-dimensional universe we would expect deviations from Newton’s inverse square law at large distances, and such deviations have never been measured. We could again invoke compactification to rescue the model, but there is a different intriguing strategy: What if we were able to find scenarios where the bulk graviton was, in some sense, *dynamically* bound to the brane? Could we then allow the extra dimensions to be infinite?

In 1999, Randall & Sundrum \[2\] proposed a phenomenological realization of this idea in 5-dimensional anti-deSitter space. Their model has the line element

$$ds^2 = e^{-2|y|/\ell} \eta_{\alpha\beta} dx^\alpha dx^\beta + dy^2, \quad G_{ab}^{(5)} = (6/\ell^2) g_{ab}^{(5)},$$

where $\ell$ is the anti-deSitter length scale. The brane is identified with the intrinsically flat geometric defect at $y = 0$, and the Israel junction conditions imply that it is actually a thin sheet of vacuum energy. Fluctuations of this model can be written as $\eta_{\alpha\beta} \rightarrow \eta_{\alpha\beta} + h_{\alpha\beta}$ with

$$h_{\alpha\beta} = e^{-|y|/2\ell} \psi_k(t, y) e^{-ik \cdot x} \epsilon_{\alpha\beta}, \quad \epsilon_{\alpha\beta} = \text{constant},$$

where $\psi_k$ satisfies a one-dimensional wave equation

$$-\ell^2 \partial_t^2 \psi_k = -\partial_z^2 \psi_k + V_k(z) \psi_k, \quad z = e^{y/\ell} \geq 1.$$

Here, we have restricted attention to one half of the bulk. The potential $V_k$ is shown in Figure[1]. The delta function enforces the boundary condition $\partial_z (z^{3/2} \psi_k) = 0$ at $z = 1,$
Figure 1: The potential governing gravity waves in the Randall-Sundrum braneworld which ensures that the matter content of the brane is unaltered by the perturbation. The
attractive nature of the delta function allows for a normalizable, stable bound state of
the brane and the 5-dimensional graviton
\[
\psi^{(0)}_k = e^{i\omega t} z^{-3/2}, \quad \omega = k. \tag{4}
\]
From the point of view of 4-dimensional brane observers, the fluctuation behaves exactly
like a massless spin-2 field propagating on a flat background. Hence, this so-called ‘zero-
mode’ reproduces 4-dimensional weak field gravity on the brane, and shows how one can
recover standard general relativity with an infinite extra dimension.

However, \(\psi^{(0)}_k\) is not the only solution to (3). There is a whole spectrum of modes
parameterized by a separation constant \(m\):
\[
\psi^{(m)}_k = e^{i\omega t} \sqrt{z} \left[ H^{(1)}_1(m\ell)H^{(2)}_2(m\ell z) - H^{(2)}_1(m\ell)H^{(1)}_2(m\ell z) \right], \quad \omega^2 = k^2 + m^2, \tag{5}
\]
where \(H^{(n)}_\nu\) are Hankel functions. The dispersion relation on the right implies that each
mode looks like a massive spin-2 field to a brane observer. So, in addition to the bound
state solution that reproduces general relativity, there is a continuum of massive graviton
‘Kaluza-Klein’ modes that predict deviations from it. A complete description of brane
gravity must include both.

What is the essential behaviour of the massive modes on the brane, and under what
circumstances are they important relative to the zero-mode? To be sure, a direct attack
on this problem with Green’s functions can be mounted, but we pursue a more physical
approach by asking a different, more subtle question: Does the potential shown in Figure
\(\Box\) treat each value of \(m\) democratically, or do certain masses tend to dominate the gravi-

ity wave spectrum? That is, are any resonant massive modes within the Kaluza-Klein
Resonant phenomena play a prominent role in many branches of physics, and can often be used to simply characterize the most important features of a given system’s dynamics. There are at least two examples of this that are useful to highlight: In black hole perturbation theory, the master equations governing gravity wave propagation support resonant solutions known as quasinormal modes \( [3] \), which tend to dominate the late time behaviour of scattered gravity waves, irrespective of the initial configuration of the perturbation. A second example comes from accelerator physics, where resonances in scattering cross sections allows one to identify bound states of elementary particles \( \textit{without} \) having a complete theoretical grasp on the interactions between them. In both cases, we do not need to actually solve the equations, subject to a given source, to predict the behaviour of the system—the resonant effects are predominant. These model problems suggest that the key to gaining physical intuition about the massive modes in the braneworld is to find the resonant solutions of \( [3] \).

Asymptotically far from the brane, the exact solution \( [5] \) reduces to a superposition of travelling waves, and the ratio of the coefficients of the outgoing and incoming contributions defines the scattering matrix. As in the two example problems cited above, the resonant modes of our system are defined by the poles of this scattering matrix.\(^1\) A simple calculation \( [6] \) yields that these correspond to a discrete set of \( \textit{complex} \) masses

\[
\{ m_j \ell \} = \{ \mu_j \} = \{ 0.419 + 0.577 i, \ 3.832 + 0.355 i, \ 7.016 + 0.350 i \ \ldots \}. \tag{6}
\]

If \( k \) is real, the dispersion relation \( \omega^2 = k^2 + m^2 \) implies that these modes have \( \text{Im} \omega > 0 \), i.e., they are exponentially damped in time. In this sense, they are exactly analogous to the quasinormal modes of perturbed black holes. Because these solutions decay in time, it is sensible to call them the metastable bound states of the brane and the bulk graviton. A useful interpretation comes from recalling Gamov’s classic 1928 treatment of the radioactive alpha decay \( [7] \), where the \( \alpha \)-particle is thought to be trapped in a potential well surrounding a much heavier partner. In that problem, the metastable bound states represent solutions where the \( \alpha \)-particle is mostly localized near the daughter nucleus, but there is a finite probability of it tunnelling through the potential barrier and out to infinity. In this spirit, we see that the resonant masses \( [6] \) represent a spectrum of massive 4-dimensional gravitons mostly localized on the brane, but subject to decay by tunnelling.

\(^1\)For a comprehensive introduction to the theory of one-dimensional scattering and the associated resonant phenomena, see the textbooks by Taylor \( [4] \) or Landau and Lifshitz \( [5] \).
into the bulk.

This opens up a whole new way of looking at perturbative gravity in the Randall-Sundrum scenario. Before, one thought of a zero-mass graviton accompanied by continuum of massive spin-2 fields. Now, we realize that the essential behaviour is that of a stable massless graviton augmented by a discrete family of quasi-bound massive cousins.

To gain a better understanding of the phenomenology of these ‘particles,’ we consider a wavepacket of gravitational radiation on the brane. We assume motion in the $x$-direction, and a momentum space profile $\alpha(k)$. The pulse’s evolution on the brane will dominated by contributions from the zero-mode and resonant masses:

$$
\delta h_{\alpha\beta} \sim \epsilon_{\alpha\beta} \int dk \alpha(k) \left[ \exp \frac{ik(t - x)}{n_j} \right].
$$

(7)

Here, $j$ runs over the resonances, the $c_j$ expansion coefficients are determined from the initial extra-dimensional pulse profile, and $n_j$ is the complex reflective index:

$$
n_j = n_j(k) = \frac{k}{\omega_j(k)} = \frac{k\ell}{\sqrt{(k\ell)^2 + \mu_j^2}}.
$$

(8)

Hence, $h_{\alpha\beta}$ is a superposition of a discrete pulses corresponding to the zero-mode and massive gravitons. Since $n_j$ has a nonzero real and imaginary parts, each of the massive pulses behaves like it is travelling in an absorptive medium, slower than the speed of light. On the other hand, the zero mode acts like it is propagating in a vacuum.

If $\alpha(k)$ is peaked about some $k = k_0$, we can sensibly ask: How fast do the massive pulses travel? And how far do they get before decaying? We define the group velocity $v_j$ and lifetime $\tau_j$ of each of the modes in the usual way:

$$
v_j = \Re \omega'_j(k_0) = \Re n_j(k_0), \quad \tau_j = \frac{1}{\Im \omega_j(k_0)} = \frac{\Im n_j(k_0)}{k_0}.
$$

(9)

Together, these give an attenuation length $d_j$, which is the distance a given massive mode travels before its amplitude decreases by a factor of $e$:

$$
\frac{d_j}{\ell} = \frac{v_j\tau_j}{\ell} = \frac{k_0\ell}{\Re \mu_j \Im \mu_j}.
$$

(10)

The denominator on the right is of order unity or larger, so we see that modes with $k_0\ell \gg 1$ can travel for large distances, while modes with $k_0\ell \ll 1$ have very short streaming lengths on the brane.
Now, tabletop experiments of Newton’s law limit $\ell \lesssim 0.1$ mm, which immediately dashes any hopes of seeing any bulk effects in the gravity waves emitted from nearby sources. The reasoning is as follows: Astrophysical systems have sizes much larger than 0.1 mm, which means that any gravitational radiation will primarily be composed of partial waves with $k\ell \ll 1$. Thus, the signal from the massive modes will only propagate for a minuscule distance $d \ll \ell$ along the brane before decaying away, making their direct detection impossible.\(^2\)

The situation is quite different when we consider cosmological braneworld scenarios, where the motion of the brane in the extra dimension accounts for the expansion of the universe. In the early universe epoch $H\ell \sim 1$ of such a model, all sub-horizon modes will have $k\ell \gtrsim 1$. Hence, the attenuation length of the massive gravitons will be of order the horizon size or larger, implying that they should have an important effect of the gravity wave background. However, our calculations to this point have been for a static brane, and it is unclear how general brane motion affects the quasi-particle excitations. But some (heuristic) progress can be made by considering the small-scale fluctuations, which have

$$k\ell \gg 1 \implies \omega_j(k) \approx k \left[ 1 + \frac{\mu^2}{2(k\ell)^2} \right] \implies |\omega_j(k)| \gg H. \quad (11)$$

The last inequality means that the typical oscillation timescale is much less than the speed of cosmological expansion, so it is safe to say that these modes ‘see’ the brane to be stationary, and are still valid resonant solutions. Hence, in the early universe the high frequency component of the gravity wave background will effectively be governed by a discrete spectrum of spin-2 fields obeying the above dispersion relation. As the universe expands the ‘fingerprint’ of these fields expands with it, eventually decoupling from the massive modes when its size grows larger than $\ell$. Therefore, this ‘back of the envelope’ calculation predicts that the relic gravity wave background carries within it primordial signatures of the extra dimension, encoded in $\omega_j(k)$ and thereby the discrete set of complex numbers $\{\mu_j\}$.

So, what has our knowledge of the resonances between bulk gravity waves and the brane achieved for us? We have seen that the late time behaviour of metric fluctuations is dominated by a discrete set of spin-2 fields with complex masses plus the zero-mode.

\(^2\)A complimentary result can be derived by integrating over real values of $\omega$ in (7) and assuming that $k$ is complex. Then, one finds that the attenuation length becomes long only when the source involves frequencies with $\omega \ell \gg 1$, i.e., with $\omega \gg 10^{12}$ sec\(^{-1}\).
Kinematically, the resonant massive modes behave like gravitons travelling in an absorptive medium—the dissipation is due to the tunnelling of gravitational radiation into the extra dimension. For a given AdS length scale $\ell$ in the bulk, we have calculated the lifetime and streaming-length of these particles on the brane, which are of order $\ell$ and $\ell/c$ respectively. With $\ell \lesssim 0.1$ mm, we see that the massive part of the spectrum cannot play a large role astrophysical processes in the nearby universe, but may be significant in earlier epochs where $H\ell \sim 1$. This highlights the ephemeral nature of bulk effects on the brane in the Randall-Sundrum scenario, and why such a model is a credible description of the real world. By and large, we see that Einstein’s theory of gravitation can—in principle—live peacefully with infinite extra dimensions, and the regions of conflict are neatly parameterized by discrete spectrum of massive decaying gravitons.

I would like to thank Chris Clarkson, Roy Maartens, and David Wands for helpful discussions and encouragement, and NSERC for financial support.

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