The Generalized Inverse Computing Method of a Class of Block Matrix from Models for Flow in Porous Materials

Tangwei Liu¹*, Jingying Zhou², Kangxiu Hu³, Shuo Chen⁴ and Jinfeng Lai⁵

¹,²,³,⁴,⁵School of Science, East China University of Technology, 418 Guanglan Road, Nanchang, 330013, China

*Corresponding author

Abstract. The generalized inverse method of block matrix plays an effective and important role during the computing procedure of certain mathematical models. Some numerical methods for flow models in porous materials are discussed in this paper. Firstly, some partial differential equation models of flow in porous materials are given. Then a class of block matrix from the discrete systems of mathematic models by the multiscale finite element methods are analyzed. Finally, the generalized inverse of the two-by-two block matrix are obtained and the algorithm for computing the minus inverse is presented. The presented methods have an advantage of concise computing procedure for the simulation of models.

Keywords: porous materials; partial difference equation model; inverse computing method; block matrix.

1. Introduction

Flow in porous materials can be described by some partial differential equation (PDE) mathematic models[1,2,4]. Recently, there have been some advances in numerical methods of these models [3,5,6]. In general speaking, multi-scale finite element methods (MsFEM) are regarded as the general numerical methods to solve the PDE models with special boundary conditions[1,2,4]. In previous studies, some scholars have given many expressions for some generalized inverse of partitioned matrix [7,9,10]. However, it is still difficult to calculate the generalized inverse of some special block matrix[8]. In this paper, the generalized inverse of two by two block matrices is used to solve a special class of linear systems derived from the numerical solution of PDEs for flow in porous materials. The brief and efficient computing methods is presented for the procedure simulation of models.

2. Two by Two Block Matrix from the Flow Models

In this section, we first consider the following coupled PDEs model for porous materials consisting of two equations for pressure $P$ and total Darcy velocity $U$ in a spatial domain [4].

\[ \nabla \cdot U(x,t) + q(x) = (\phi \mu C) \frac{\partial P(x,t)}{\partial t}, x \in \Omega, t > 0, \]  \hspace{1cm} (1)

\[ U(x,t) = -\lambda k(x) \nabla P(x,t), \]  \hspace{1cm} (2)

\[ P(x,0)=P_0, \]  \hspace{1cm} (3)

where $\Omega$ is a domain in $\mathbb{R}^d$ ($d=2$), $\phi$, $\mu$, $c$, $q$ and $k(x)$ denote the porosity viscosity, the compressibility, source term and the permeability tensor, respectively. We assume that $U=U(x,t)$, 

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\( P = P(x, t) \) are unknown functions. And here we assume that the equations (1)-(3) are equipped with the following boundary conditions

\[
\mathbf{U} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega,  
\]

where \( \mathbf{n} \) denotes the outward normal of \( \partial \Omega \).

The Laplace transform of the function \( p \) and its inversion formulas are given by

\[
\bar{P}(x, s) = L[P(x, t)] = \int_0^\infty e^{-st} P(x, t) dt ,
\]

\[
P(x, t) = L^{-1}[\bar{P}(x, s)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \bar{P}(x, s) ds ,
\]

where \( s = c + i\omega; c, \omega \in \mathbb{R} \).

By using the mixed MsFEM, we obtain the approximations of the Laplace transform of the pressure \( P \) and total Darcy velocity \( \mathbf{U} \):

\[
\bar{U}_H = \sum_{i=1}^N \bar{U}_i(s) \varphi_i(x) \text{ and } \bar{P}_H = \sum_{i=1}^N \bar{P}_i(s) \cdot \varphi_i(x)
\]

where the coefficients \( \bar{p} \) and \( \bar{U} \) are obtained by solving the following linear equations [4]:

\[
\begin{bmatrix}
H & C^T \\
C & L
\end{bmatrix}
\begin{bmatrix}
\bar{U} \\
-\bar{P}
\end{bmatrix}
= \begin{bmatrix}
0 \\
F
\end{bmatrix}.
\]

We note that

\[
H = \left[ \int_\Omega \varphi_i \cdot (\lambda K^{-1}) \varphi_j \, dx \right],  
\]

\[
C = \left[ \int_\Omega \varphi_i \, \text{div} \varphi_j \, dx \right],  
\]

\[
L = \left[ \int_\Omega \varphi_i \cdot \varphi_j \, dx \right],  
\]

\[
F = \left[ \int_\Omega h \cdot \varphi_j \, dx \right],
\]

where \( \varphi_i, \varphi_j \) represent velocity and pressure basis function \((i, j = 1, 2, \cdots, N)\), respectively.

3. The Generalized Inverse of a Class of Two-by-Two Block Matrix

3.1. The Basic Concept of Inverse Matrices

There are many kinds of inverse matrices for block matrices such as minus inverse, plus inverse, least square generalized inverse, reflexive generalized inverse and minimum norm generalized inverse. In this section, we consider only the minus inverse of four-block matrices.

For the matrix \( A \in F^{m \times n} \), we consider the following formulas:

\( (i) \) \( AXA = A, \forall A \in F^{m \times n} \).

For an \( m \times n \) matrix \( A \), if there exists a matrix \( X \) satisfy (i), then matrix \( X \) is called a minus inverse of \( A \), and denoted by \( A^{-} \). And for any \( A \in F^{m \times n} \), minus inverse \( A^{-} \) exists.
3.2. Minus Inverse of a Two-by-Two Block Matrix

In this section, we consider the generalized inverse of the following two-by-two block matrix

\[
A = \begin{pmatrix}
H & C^T \\
C & L
\end{pmatrix}.
\]  

(7)

We discuss the minus inverse \( A^- \) of matrix \( A \) when \( C = C^T = 0 \).

We partition \( A^- \) into appropriate blocks as follows

\[
A^- = \begin{pmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix}.
\]  

(8)

Since \( AA^- A = A \), we have

\[
AA^- A = \begin{pmatrix}
H & 0 \\
0 & L
\end{pmatrix}
\begin{pmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix}
\begin{pmatrix}
H & 0 \\
0 & L
\end{pmatrix} = \begin{pmatrix}
HB_{11} & HB_{12} \\
LB_{21} & LB_{22}
\end{pmatrix} = A = \begin{pmatrix}
H & 0 \\
0 & L
\end{pmatrix}.
\]

From \( HB_{11} H = H \), \( LB_{22} L = L \), we obtain \( B_{11} = H^- \), \( B_{22} = L^- \).

The general solution of the matrix equation \( AXB = 0 \) is \( X = Y - AA^- YBB^- \), where \( Y, B \) are some arbitrary matrices of appropriate order.

So the general solutions of the matrix equation \( HB_{12} L = 0 \) and \( LB_{21} H = 0 \) are

\[
B_{12} = Y - H^- HYYL^- L, B_{21} = Z - L^- LZHH^- L,
\]

where \( Y \) and \( Z \) are arbitrary matrices of appropriate order.

We have

\[
A^- = \begin{pmatrix}
H & 0 \\
0 & H
\end{pmatrix}^- = \begin{pmatrix}
H^- & Y - H^- HYYL^- L \\
Z - L^- LZHH^- L
\end{pmatrix}.
\]  

(9)

As \( A_{12} \) and \( A_{21} \) are nonzero, we discuss some singular and nonsingular cases about matrix \( A \).

I) When matrix \( A \) is singular, assume that \( \det(A_{11}) \neq 0 \), by the elementary transformation of the partitioned matrix

\[
\begin{pmatrix}
E & 0 \\
-CH^- & E
\end{pmatrix}
\begin{pmatrix}
H & C^T \\
C & L
\end{pmatrix}
\begin{pmatrix}
E & -H^- C^T \\
0 & E
\end{pmatrix} = \begin{pmatrix}
H & 0 \\
0 & L - CH^- C^T
\end{pmatrix}.
\]

Notes

\[
P = \begin{pmatrix}
E & 0 \\
-CH^- & E
\end{pmatrix}, Q = \begin{pmatrix}
E & -H^- C^T \\
0 & E
\end{pmatrix}, D = \begin{pmatrix}
H & 0 \\
0 & L - CH^- C^T
\end{pmatrix},
\]

such that \( PAQ = D \).

According to \( A \) singular, \( \det(A_{22} - A_{21} A_{11}^{-1} A_{12}) \neq 0 \), the following formula can be obtained.
\[ A^{-1} = \begin{pmatrix} H & C^T \\ C & L \end{pmatrix} = QD^*P = \begin{pmatrix} E & -C^TH^- \\ 0 & E \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & L-CH^-C^T \end{pmatrix} \begin{pmatrix} E & 0 \\ -C^TH^- & E \end{pmatrix} \]

\[ = \begin{pmatrix} E & -H^*C^T \\ 0 & E \end{pmatrix} \begin{pmatrix} H^- & 0 \\ 0 & (L-CH^-C^T)^{-} \end{pmatrix} \begin{pmatrix} E & 0 \\ -CH^- & E \end{pmatrix} \]

\[ = \begin{pmatrix} H^- + H^*C^T(L-CH^-C^T)^{-}CH^- & -H^*C^T(L-CH^-C^T)^{-} \\ -(L-CH^-C^T)^{-}CH^- & (L-CH^-C^T)^{-} \end{pmatrix}. \]  

(10)

If \( \det(L) \neq 0 \), then \( \det(H-C^T L \cdot C) \neq 0 \). We have

\[ A^{-1} = \begin{pmatrix} (H-C^T L \cdot C)^{-} & -(H-C^T L \cdot C)^{-} \\ -L^*C(H-C^T L \cdot C)^{-} & L^* + L^*C(H-C^T L \cdot C)^{-} \end{pmatrix}. \]  

(11)

II) When \( A \) is nonsingular, if \( \det(H) \neq 0 \), then \( PAQ = D \Rightarrow A = P^*DQ^* \).

Since

\[ (P^*DQ^*)(P^*DQ^*)^{-} = P^*DQ^*QD^*PP^*DQ^* = P^*DD^*DQ^* = P^*DQ^* , \]

\( D \) is a quasi-diagonal matrix, the following formula can be obtained

\[ D^{-1} = \begin{pmatrix} H & 0 \\ 0 & L-CH^-C^T \end{pmatrix}^{-} = \begin{pmatrix} H^- & 0 \\ 0 & (L-CH^-C^T)^{-} \end{pmatrix} . \]

\[ A^{-1} = \begin{pmatrix} H^- + H^*C^T(L-CH^-C^T)^{-}CH^- & -H^*C^T(L-CH^-C^T)^{-} \\ -(L-CH^-C^T)^{-}CH^- & (L-CH^-C^T)^{-} \end{pmatrix}. \]  

(12)

Similarly, when \( \det(A_2) \neq 0 \), we can get

\[ A^{-1} = \begin{pmatrix} (H-C^T L \cdot C)^{-} & -(H-C^T L \cdot C)^{-} \\ -L^*C(H-C^T L \cdot C)^{-} & L^* + L^*C(H-C^T L \cdot C)^{-} \end{pmatrix}. \]  

(13)

If \( A \) is semi-positive definite Hermite matrix, similar conclusions will be drawn regardless of whether \( H \) or \( L \) is invertible or not.

\( A \) is semi-positive definite Hermite matrix, there exists matrix \( \overline{B} \), such that \( A = \overline{B}^T \overline{B} \).

Note that

\[ \overline{B} = (B_1, B_2), A = \begin{pmatrix} H & C^T \\ C & L \end{pmatrix}, \]

one has the following formula

\[ \begin{pmatrix} H & C^T \\ C & L \end{pmatrix} = \begin{pmatrix} B_1^H B_1 & B_1^H B_2 \\ B_2^H B_1 & B_2^H B_2 \end{pmatrix}. \]

The equations \( A^H A x = 0 \) and \( A x = 0 \) have the same solution.

Since \( A^H A(A^H A)^{-} A^H A - A = A^H A \), the following formulas be obtained.

\[ A^H (A(A^H A)^{-} A^H A - A) = 0, A(A^H A)^{-} A^H A - A = 0, A = A(A^H A)^{-} A^H A, \]
\[
C^T H^{-1} H = B_2^H B_1 (B_1^H B_1)^{-1} B_1^H B_1 = B_2^H B_1 = C^T,
\]
\[
HH^{-1} C^T = B_1^H B_1 (B_1^H B_1)^{-1} B_1^H B_2 = B_1^H B_2 = C^T.
\]

Notes

\[
P = \begin{pmatrix} E & 0 \\ -C^T H^{-1} \end{pmatrix},
Q = \begin{pmatrix} E & -H^{-1} C^T \\ 0 & E \end{pmatrix},
D = \begin{pmatrix} H & 0 \\ 0 & L - CH^{-1} C^T \end{pmatrix},
\]

we obtain the following formulas

\[
D = PAQ = \begin{pmatrix} E & 0 \\ -CH^{-1} \end{pmatrix} \begin{pmatrix} H & C^T \\ C & L \end{pmatrix} \begin{pmatrix} E & -H^{-1} C^T \\ 0 & E \end{pmatrix}
\]
\[
= \begin{pmatrix} E & 0 \\ -CH^{-1} \end{pmatrix} \begin{pmatrix} B_1^H B_1 & B_1^H B_2 \\ B_2^H B_1 & B_2^H B_2 \end{pmatrix} \begin{pmatrix} E & -H^{-1} C^T \\ 0 & E \end{pmatrix}
= \begin{pmatrix} H & 0 \\ 0 & L - CH^{-1} C^T \end{pmatrix}.
\]

Obviously, \( P \) and \( Q \) are invertible, we have

\[
A^{-1} = QD^{-1} P = Q \begin{pmatrix} H & 0 \\ 0 & L - CH^{-1} C^T \end{pmatrix}^{-1} P
= \begin{pmatrix} E & -H^{-1} C^T \\ 0 & E \end{pmatrix} \begin{pmatrix} H^{-1} & 0 \\ 0 & (L - CH^{-1} C^T)^{-1} \end{pmatrix}
\begin{pmatrix} E & 0 \\ -CH^{-1} & E \end{pmatrix}
= \begin{pmatrix} H^{-1} + H^{-1} C^T (L - CH^{-1} C^T)^{-1} CH^{-1} & -H^{-1} C^T (L - CH^{-1} C^T)^{-1} CH^{-1} \\ -(L - CH^{-1} C^T)^{-1} CH^{-1} & (L - CH^{-1} C^T)^{-1} \end{pmatrix}. 
\]

Using the above results, an algorithm for computing matrix \( A^{-1} \) is given as following.

Algorithm 1 Implementing process of computing matrix \( \begin{pmatrix} H & C^T \\ C & L \end{pmatrix} \)

1. Compute \( H^{-1} \);
2. Compute \( (L - CH^{-1} C^T)^{-1} \);
3. Compute \( A^{-1} \) according to (14).

4. Generalized Inverse Solutions of a Class of Linear System

For the linear system (5), we obtain the inverse of the block matrix as following by the algorithm 1.

\[
\begin{pmatrix} H & C^T \\ C & L \end{pmatrix}^{-1} = \begin{pmatrix} H^{-1} + H^{-1} C^T (L - CH^{-1} C^T)^{-1} CH^{-1} & -H^{-1} C^T (L - CH^{-1} C^T)^{-1} CH^{-1} \\ -(L - CH^{-1} C^T)^{-1} CH^{-1} & (L - CH^{-1} C^T)^{-1} \end{pmatrix}. 
\]

We have

\[
\begin{pmatrix} \tilde{U} \\ -\tilde{P} \end{pmatrix} = \begin{pmatrix} H^{-1} + H^{-1} C^T (L - CH^{-1} C^T)^{-1} CH^{-1} & -H^{-1} C^T (L - CH^{-1} C^T)^{-1} CH^{-1} \\ -(L - CH^{-1} C^T)^{-1} CH^{-1} & (L - CH^{-1} C^T)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ F \end{pmatrix},
\]
\[
\tilde{U} = -H^{-1} C^T (L - CH^{-1} C^T)^{-1} F, \quad \tilde{P} = -(L - CH^{-1} C^T)^{-1} F.
\]

At last we can compute the numerical value of \( U(x,t) \), \( P(x,t) \) by \( \tilde{U}(x,s) \), \( \tilde{P}(x,s) \) using the numerical inversion formula of Laplace transform.
5. Conclusion
In this paper, we derive a new brief computing procedure by the generalized inverse of the block two-by-two linear systems, which arise from partial differential equation models of flow in porous materials. Numerical solutions of pressure and Darcy velocity in the mathematic models can be obtained. The results is meaningful for the numerical simulation of flow in porous materials.

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