Application of nonextensive statistics to particle and nuclear physics

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Abstract: We present an overview of possible imprints of non-extensivity in particle and nuclear physics. Special emphasis is put on the intrinsic fluctuations present in the system under consideration as the possible source of nonextensivity. The possible connection of nonextensivity and the self organized criticality apparently being observed in some cosmic rays and hadronic experiments will also be discussed.

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1 Introduction - how we got to it

Our encounter with notion of nonextensivity originated with observation that in some cosmic ray data (like depth distribution of starting points of cascades in Pamir lead chamber [1]) one encounters deviations from the usual exponential distributions of some variables towards the power-like ones:

\[
\frac{dN}{dT} = \text{const} \cdot \exp \left( - \frac{T}{\lambda} \right) \Rightarrow \text{const} \cdot \left[ 1 - (1-q) \frac{T}{\lambda} \right]^\frac{1}{1-q}. \tag{1}
\]

Here \(N\) denotes the number of counts at depth \(T\) (cf. [1] for details). Whereas in [1] we have proposed as explanation a possible fluctuations of the mean free path \(\lambda\) in eq. (1) characterised by relative variance \(\omega = \frac{\langle \sigma^2 \rangle - \langle \sigma \rangle^2}{\langle \sigma \rangle^2} \geq 0.2\), in [2] the same data were fitted by power-like (Lévy type) formula as above keeping \(\lambda\) fixed.
and setting \( q = 1.3 \). In this way we have learned about Tsallis statistics and Tsallis nonextensive entropy and distributions \([3]\). Following this approach we have then demonstrated \([4,5]\) that, indeed,

\[
L = \exp \left( -\frac{x}{\lambda_0} \right) \Rightarrow L_q = \exp_q \left( -\frac{x}{\lambda_0} \right) = \langle \exp \left( -\frac{x}{\lambda} \right) \rangle ,
\]

(2)

with \( q = 1 \pm \omega \) for \( q > 1 \) (+) and \( q < 1 \) (−), i.e., there is connection between the measure of fluctuations \( \omega \) and the measure of nonextensivity \( q \) (it has been confirmed recently in \([6]\)). This will be the first subject discussed here. The other will be connected with the self-organized criticality form of nonextensivity as apparently seen in hadronic processes.

2 Temperature fluctuations?

The most interesting example of the possible existence of such fluctuations is the trace of power like behaviour of the transverse momentum distribution in multiparticle production processes encountered in heavy ion collisions \([7,8]\). Such collisions are of special interest because they are the only place where new state of matter, the Quark Gluon Plasma, can be produced \([9]\). Transverse momentum distributions are believed to provide information on the temperature \( T \) of reaction, which is given by the inverse slope of \( dN/dp_T \), if it is exponential one. If it is not, question arises what we are really measuring. One explanation is the possible flow of the matter, the other, which we shall follow here, is the nonextensivity (or rather fluctuations leading to it). Namely, as was discussed in detail in \([2]\) the extreme conditions of high density and temperature occurring in ultrarelativistic heavy ion collisions can invalidate the usual BG approach and lead to \( q > 1 \), i.e., to

\[
\frac{dN(p_T)}{dp_T} = \text{const} \cdot \left[ 1 - (1 - q) \frac{\sqrt{m^2 + p_T^2}}{T} \right]^{1/q} .
\]

(3)
Here $m$ is the mass of produced particle and $T$ is, for the $q = 1$ case, the \textit{temperature} of the hadronic system produced. Although very small ($|q - 1| \sim 0.015$) this deviation, if interpreted according to eq. (2), leads to quite large relative fluctuations of temperature existing in the nuclear collisions, $\Delta T/T \simeq 0.12$. It is important to stress that these are fluctuations existing in small parts of hadronic system in respect to the whole system rather than of the event-by-event type for which, $\Delta T/T = 0.06/\sqrt{N} \to 0$ for large $N$ (cf. \cite{11} for relevant references). Such fluctuations are potentially very interesting because they provide direct measure of the total heat capacity $C$ of the system:

\[
\frac{\sigma^2(\beta)}{\langle \beta \rangle^2} = \frac{1}{C} = \omega = q - 1
\]

($\beta = \frac{1}{T}$) in terms of $\omega = q - 1$. Therefore, measuring \textit{both} the temperature of reaction $T$ and (via nonextensivity $q \neq 1$) its total heat capacity $C$, one can not only check whether an approximate thermodynamics state is formed in a single collision but also what are its theromdynamical properties (especially in what concerns the existence and type of the possible phase transitions \cite{11}).

To observe such fluctuations an event-by-event analysis of data is needed \cite{11}. Two scenarios must be checked: (a) $T$ is constant in each event but because of different initial conditions it fluctuates from event to event and (b) $T$ fluctuates in each event around some mean value $T_0$. Fig. 1 shows typical event obtained in simulations performed for central $Pb + Pb$ collisions taking place for beam energy equal $E_{\text{beam}} = 3$ A·TeV in which density of particles in central region (defined by rapidity window $-1.5 < y < 1.5$) is equal to $\frac{dN}{dy} = 6000$ (this is the usual value given by commonly used event generators \cite{12}). In case (a) in each event one expects exponential dependence with $T = T_{\text{event}}$ and possible departure from it would occur only after averaging over all events. It would reflect fluctuations originating from different initial conditions for each particular collision. This situation is illustrated in Fig. 1a where $p_T$ distributions for $T = 200$ MeV (black symbols) and
\( T = 250 \text{ MeV} \) (open symbols) are presented. Such values of \( T \) correspond to typical uncertainties in \( T \) expected at LHC accelerator at CERN. Notice that both curves presented here are straight lines. In case (b) one should observe departure from the exponential behaviour already on the single event level and it should be fully given by \( q > 1 \). It reflects situation when, due to some intrinsically dynamical reasons, different parts of a given event can have different temperatures \([4,5]\). In Fig. 1b black symbols represent exponential dependence obtained for \( T = 200 \text{ MeV} \) (the same as in Fig. 1a), open symbols show the power-like dependence as given by \((3)\) with the same \( T \) and with \( q = 1.05 \) (notice that the corresponding curve bends slightly upward here). In this typical event we have \( \sim 18000 \) secondaries, i.e., practically the maximal possible number. Notice that points with highest \( p_T \) correspond already to single particles. As one can see, experimental differentiation between these two scenarios will be very difficult, although not totally impossible. On the other hand, if successful it would be very rewarding - as we have stressed before.

One should mention at this point that to the same category of fluctuating temperature belongs also attempt \([13]\) to fit energy spectra in both the longitudinal and transverse momenta of particles produced in the \( e^+e^- \) annihilation processes at high energies, novel nonextensive formulation of Hagedorn statistical model of hadronization process \([10,6]\) and description of single particle spectra \([14]\).

3 Nonexponential decays

Another hint for intrinsic fluctuations operating in the physical system could be the known phenomenon of nonexponential decays \([15]\). Spontaneous decays of quantum-mechanical unstable systems cannot be described by the pure exponential law (neither for short nor for long times) and survival time probability is \( P(t) \propto t^{-\delta} \) instead of exponential one. It turns out \([13]\) that
by using random matrix approach, such decays can emerge in a natural way from the possible fluctuations of parameter $\gamma = 1/\tau$ in the exponential distribution $P(t) = \exp(-\gamma t)$. Namely, in the case of multichannel decays (with $\nu$ channels of equal widths involved) one gets fluctuating widths distributed according to gamma function

$$P_{\nu}(\gamma) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu}{2 < \gamma >}\right)^{\frac{\nu}{2} - 1} \exp\left(-\frac{\nu \gamma}{2 < \gamma >}\right),$$

and strength of their fluctuations is given by relative variance $\frac{\langle(\gamma - <\gamma>)^2\rangle}{<\gamma>^2} = \frac{2}{\nu}$, which decreases with increasing $\nu$. According to [4], it means therefore that, indeed,

$$L_q(t, \tau_0) = \frac{2 - q}{\tau_0} \left[1 - (1 - q)\frac{t}{\tau_0}\right]^{\frac{1}{1-q}},$$

with the nonextensivity parameter equal to $q = 1 + \frac{2}{\nu}$.

4 Self-organized criticality in cascade processes

For non-equilibrium phenomena the sources of power-law distributions are self-organized criticality (SOC - nonequilibrium systems are continuously driven by their own internal dynamic to a critical state with power-laws omnipresent) [16] and stochastic multiplicative processes (power-law is generated by the presence of underlying replication of events) [17]. All they can be unified in terms of generalized nonextensive statistics [3]. It has been recently argued [18,19] that in systems of quarks and gluons formed in collision processes there exists evidence of SOC behaviour. The picture proposed in [18] is that system of interacting soft gluons can be, and should be, considered as an open, dynamical, complex system with many degrees of freedom, remaining in general far from thermal and/or chemical equilibrium. Applied first to description of formation of colour-singlet gluon clusters in inelas-
tic diffraction scattering this argumentation has been recently extended to cover also quarks [19]. This allowed to describe the existing data on high transverse momentum jet production in a uniform way (jet is, loosely speaking, a bunch of collimated hadrons going in one direction). Namely, the $E_T$ jet cross section ($E_T$ is the energy of such jet measured in direction perpendicular to the collision axis) is given by simple power law:

$$\frac{d^2\sigma}{dE_Td\eta} \propto E_T^{-\alpha},$$

(7)

with $\alpha \sim 5$, 7 and 9 for small, intermediate and large values of $E_T$, respectively, indicating three different possible scenarios operating at different ranges of $E_T$ (according to [19]).

Similar situation has been encountered in cosmic ray physics where energy spectra of particles from atmospheric cascades exhibit power-like behaviour, which can be described by Lévy type distribution [20]. For sufficiently large energy fraction $x_N$ allocated to the cascade with $N$ generations it is given by

$$P(x_N) \propto \left(\frac{x_N}{\langle x_N \rangle}\right)^{-\alpha}, \quad \alpha = \frac{1}{q - 1},$$

(8)

where $q$ is nonextensivity index depending on the number of generations $N$, $q = \frac{3}{2} - \frac{c^{N-1}}{2}$, ($c$ is $N$-independent parameter which should be fitted, here $c = 0.55$). For events without cascading $N = 1$ and $q = 1$, i.e., one gets usual exponential distribution. For large $N$ (in practice for $N \geq 6$) $q \to 3/2$ (and $\alpha \to 2$ in (8)), which is limiting value available here for $q$ (explaining why in such experiments one always observes $\alpha \leq 2$). This result tells us that the exactly power-like behaviour of spectra is achieved only asymptotically, for long enough cascades. This has been clearly demonstrated in [20] on some experimental data.

Coming back to eq. (7) we argue that, as can be seen in Fig. 2, the same data can be described by a suitably modified equivalent
of eq. (8):

$$\frac{d^2\sigma}{dE_Td\eta}|_{\eta=0} = c \cdot E_T^{-\alpha} \cdot [1 - \text{Erf}(a)],$$

(9)

this time with only one exponent for all values of $E_T$: $\alpha = 5.01$ and 4.9 for energies $\sqrt{s} = 546$ and 1800 GeV considered in [19], respectively. The correction factor to the power law comes from accounting for smearing out of the initial conditions of the cascade (not present in (8)) with $a = (1 - \alpha)(\frac{\delta}{\sqrt{2}}) + \frac{\sqrt{2}}{\delta}\ln \Delta_0$ and $\Delta_0 = \frac{E_T}{E_T^{(0)}}$ (ratios of the actual energy of jet $E_T$ to its threshold energy $E_T^{(0)}$). This smearing was assumed in the following log-normal form: $P(\Delta)d(\ln \Delta) = \frac{1}{\sqrt{2\pi}} \exp \left[-\left(\frac{\ln \Delta - \ln \Delta_0}{2\delta}\right)^2\right]$. Parameter $\delta$ is dispersion of the smearing distribution and is set to be equal $\delta = 0.74$ and 0.79 for the above respective energies. The value $\alpha \sim 5$ in (9) emerges in a natural way from that of 2 in (8). The point is that the main interaction in hadronic collisions proceeds between quarks, which are produced in the quark-gluon QCD cascade process from the initial dressed valence quarks in nucleon. Therefore jets are produced by collision of two such quarks with energy fractions $x$ and $y$ and energy $M$ of such collision is $M = \sqrt{z} \cdot s \sim \sqrt{z}$ (where $z = x \cdot y$ and where $s$ is initial invariant energy of reaction squared). Therefore distribution in $z$ is of the form of convolution: $P(z) = P(x) \otimes P(y) \sim z^{-3}$, which results in

$$\frac{d\sigma}{dz} = \frac{1}{M} \frac{d\sigma}{dM} \sim M^{-6} \quad \text{or} \quad \frac{d\sigma}{dM} \sim M^{-5}. \quad \text{(10)}$$

Because $E_T \sim M$ eq. (10) becomes eq. (7). Therefore value of $\alpha = 5$ is what really comes from the pure cascade. All deviations from it needed to fit data come from the initial conditions properly accounted for.
There is steadily growing evidence that some peculiar features observed in particle and nuclear physics (including cosmic rays) can be most consistently explained in terms of the suitable applications of nonextensive statistic of Tsallis. Here we were able to show only some selected examples, more can be found in [5]. However, there is also some resistance towards this idea, the best example of which is provided in [21]. It is shown there that mean multiplicity of neutral mesons produced in $p - \bar{p}$ collisions as a function of their mass (in the range from $m_\eta = 0.55$ GeV to $M_\Upsilon = 9.5$ GeV) and the transverse mass $m_T$ spectra of pions (in the range of $m_T \simeq 1 \div 15$ GeV), both show a remarkable universal behaviour following over 10 orders of magnitude the same power law function $C \cdot x^{-P}$ (with $x = m$ or $x = m_T$) with $P \simeq 10.1$ and $P \simeq 9.6$, respectively. In this work such a form was just postulated whereas it emerges naturally in $q$-statistics with $q = 1 + 1/P \sim 1.1$ (quite close to results of [13]). We regard it as new, strong imprint of nonextensivity present in multiparticle production processes. This interpretation is additionally supported by the fact that in both cases considered in [21] constant $c$ is the same. Apparently there is no such phenomenon in $AA$ collisions which has simple and interesting explanation: in nuclear collisions volume of interaction is much bigger what makes the heat capacity $C$ also bigger. This in turn, cf. eq. (4), makes $q$ smaller. On should then, indeed, expect that $q_{\text{hadronic}} \gg q_{\text{nuclear}}$, as observed.

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References

[1] G.Wilk and Z.Wlodarczyk, *Phys. Rev.* **D50** (1994) 2318.

[2] G.Wilk and Z.Wlodarczyk, *Nucl. Phys.* **B (Proc. Suppl.) A75** (1999) 191.

[3] C.Tsallis, in *Nonextensive Statistical Mechanics and its Applications*, S.Abe and Y.Okamoto (Eds.), Lecture Notes in Physics LPN560, Springer (2000).

[4] G.Wilk and Z.Wlodarczyk, *Phys. Rev. Lett.* **84** (2000) 2770.

[5] G.Wilk and Z.Wlodarczyk, *Chaos, Solitons and Fractals* **13/3** (2001) 581.

[6] See lecture by C.Beck in this issue and in [cond-matt/0105371](cond-matt/0105371) and [cond-mat/0105374](cond-mat/0105374).

[7] W.M.Alberico, A.Lavagno and P.Quarati, *Eur. Phys. J.* **C12** (2000) 499.

[8] O.V.Utyuzh, G.Wilk and Z.Wlodarczyk, *J. Phys.* **G26** (2000) L39.

[9] Cf. proceedings of *Quark Matter’99*, *Nucl. Phys.* **A525** (1999).

[10] C.Beck, *Physica* **A286** (2000) 164.

[11] O.V.Utyuzh, G.Wilk and Z.Wlodarczyk, *How to observe fluctuating temperature?*, [hep-ph/0103273](hep-ph/0103273).

[12] K.J.Escola, *On predictions of the first results from RHIC*, [hep-ph/0104058](hep-ph/0104058), to be published in Quark Matter’01 proceedings, *Nucl. Phys. A* (2001).

[13] I.Bediaga, E.M.F.Curado and J.M.de Miranda, *Physica* **A286** (2000) 156.

[14] F.S.Navarra, O.V.Utyuzh, G.Wilk and Z.Wlodarczyk, *Violation of the Feynman scaling law as a manifestation of nonextensivity*, [hep-ph/0009163](hep-ph/0009163), to be published in *Nuovo Cim.* **24C** (2001), July-September issue.

[15] G.Wilk and Z.Wlodarczyk, *Non-exponential decays and nonextensivity*, [hep-ph/0103114](hep-ph/0103114).

[16] M.Paczuski, S.Maslow and B.Pak, *Phys. Rev.* **E53** (1996) 414.

[17] S.C.Manrubia and D.H.Zanette, *Phys. Rev.* **E59** (1999) 4945.

[18] C.Boros et al., *Phys. Rev.* **D61** (2000) 094010 and references therein.

[19] Fu Jinghua, Meng Ta-chung, R.Rittel and K.Tabelow, *Phys. Rev. Lett.* **86** (2001) 1961.

[20] M.Rybczyński, Z.Wlodarczyk and G.Wilk, *Nucl. Phys.* **B (Proc. Suppl.) 97** (2001) 81.

[21] M.Gaździcki and M.I.Gorenstein, *Power Law in Hadron Production*, [hep-ph/0103010](hep-ph/0103010), to be published in *Phys. Lett. B* (2001).
Fig. 1. (a) Normal exponential $p_T$ distributions (i.e., $q = 1$) for $T = 200$ MeV (black symbols) and $T = 250$ MeV open symbols). (b) Typical event from central $Pb + Pb$ at $E_{beam} = 3$ A·TeV (cf. text for other details) for $= 200$ MeV for $q = 1$ (black symbols) exponential dependence and $q = 1.05$ (open symbols).

Fig. 2. The high energy $\bar{p}p$ collisions data used in [19] (cf. their Fig. 1) fitted by our eq. (9). See text for details.