Two-term, asymptotically sharp estimates for eigenvalue means of the Laplacian

Evans M. Harrell II
School of Mathematics, Georgia Institute of Technology
Atlanta GA 30332-0160, USA
harrell@math.gatech.edu

Joachim Stubbe
EPFL, MATHGEOM-FSB, Station 8, CH-1015 Lausanne, Switzerland
Joachim.Stubbe@epfl.ch

Abstract
We present asymptotically sharp inequalities for the eigenvalues $\mu_k$ of the Laplacian on a domain with Neumann boundary conditions, using the averaged variational principle introduced in [14]. For the Riesz mean $R_1(z)$ of the eigenvalues we improve the known sharp semiclassical bound in terms of the volume of the domain with a second term with the best possible expected power of $z$.

In addition, we obtain two-sided bounds for individual $\mu_k$, which are semiclassically sharp. In a final section, we remark upon the Dirichlet case with the same methods.

Key words: Neumann Laplacian, Dirichlet Laplacian, semiclassical bounds for eigenvalues

2010 Mathematics Subject Classification: 58J50, 47F05,47A75
Two-term, asymptotically sharp estimates for eigenvalue means of the Laplacian

Evans M. Harrell II and Joachim Stubbe

11 June, 2016

Contents

1 Introduction 1

2 Proofs of the main results 6
  2.1 Refinement of Kröger’s inequality: Theorem 1.1 . . . . . . . . . . . 6
  2.2 Two-term spectral bounds: Proof of Theorem 1.2 . . . . . . . . . .7

3 Riesz means of Laplacians on rectangles 10

4 Two-term estimates for Dirichlet Laplacians by averaging 14

A Refinements of Young’s and Hölder’s inequality 15

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary $\partial \Omega$. We mainly consider here the eigenvalue problem for the Laplacian with Neumann boundary conditions,

$$\begin{align*}
-\Delta u &= \mu u \quad \text{on } \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{align*}$$

(1.1)

The spectrum (1.1) consists of an ordered sequence of eigenvalues $\mu_j$ tending to infinity,

$$0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots$$

(1.2)

The corresponding normalized eigenfunctions are denoted $u_j$. Neumann eigenvalues satisfy the same Weyl asymptotic relation as the better studied Dirichlet eigenvalues, viz.,

$$\lim_{j \to \infty} \mu_j j^{-2/d} = C_d |\Omega|^{-2/d},$$

(1.3)

where $|\Omega|$ denotes the volume of $\Omega$ and the “classical constant” $C_d$ is given by

$$C_d = (2\pi)^{2-2/d} B_d,$$

(1.4)
where \( B_d = \frac{\pi^{d/2}}{\Gamma(1 + d/2)} \) is the volume of the \( d \)-dimensional unit ball. In 1961, Pólya showed that
\[
\mu_j \leq C_d |\Omega|^{-2/d} (j - 1)^{-2/d}
\]
for all positive integers \( j \) when \( \Omega \) is any tiling domain of \( \mathbb{R}^d \), and the opposite inequality in the Dirichlet case, for which we denote the eigenvalues \( \lambda_j \):
\[
\lambda_j \geq C_d |\Omega|^{-2/d} j^{-2/d}.
\]
His still unproven conjecture is that these inequalities hold for all bounded domains \( \Omega \subset \mathbb{R}^d \). In other words the Weyl limit (1.3) is approached from below in the Neumann case and above for Dirichlet.

Whereas there are universal relations among eigenvalues of the Dirichlet problem, for the Neumann problem, Colin-de-Verdière showed in 1987 [9] that for any finite nondecreasing \( 0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_k \), there exists a bounded domain having these values as the first \( k \) eigenvalues. Therefore inequalities among Neumann eigenvalues must incorporate geometric properties of \( \Omega \) to be of interest. (See, e.g. [2, 4, 3] for discussions of universal eigenvalue bounds and related references.)

Other convenient ways to study the spectrum rely on the counting function,
\[
N(\mu) := \#\{\mu_j : \mu_j < \mu\},
\]
and, in a tradition going back to Berezin [6], Riesz means, \( R_\sigma(z) := \sum_j (z - \mu_j)^{-\sigma} \), or, resp., \( \sum_j (z - \lambda_j)^{-\sigma} \). Here \( x_+ \) denotes the positive part of \( x \). \( N(z) \) can be interpreted as the limit of \( R_\sigma(z) \) when \( \sigma \to 0 \). For instance, Berezin proved the equivalent of the summed version of (1.6) in the Riesz-mean form,
\[
\sum_j (z - \lambda_j)^+ \leq L^{cl}_{1,d} |\Omega| z^{1+d/2},
\]
where
\[
L^{cl}_{\gamma,d} := \frac{\Gamma(\gamma + 1)}{(4\pi)^{d/2}\Gamma(\gamma + 1 + d/2)}.
\]
In recent years, beginning with a paper by Melas [21], there has arisen an industry to improve (1.8) by including further terms in lower powers of \( z \). An improvement incorporating the best expected succeeding power, \( z^{d-1/2} \), was obtained in the Dirichlet case by Geisinger-Laptev-Weidl [13], and we refer to that paper for further background.

Our main goal here is to achieve analogous improvements in Riesz-means for Neumann eigenvalues in terms of \( z \) to the expected powers. In addition, we obtain two-sided bounds for individual eigenvalues \( \mu_k \), which are semiclassically sharp. For this we rely on the averaged variational introduced in [14] and a series of analytic inequalities. In a final section, we also treat the Dirichlet case with the same methods. An appendix contains a discussion of improvements to Young’s and Hölder’s inequalities.
An important step towards Pólya’s conjecture in the Neumann case was taken in 1991 by Kröger, who by applying a variational estimate for the sum of the first $k$ eigenvalues, obtained the asymptotically sharp inequality

$$
\frac{d+2}{d} \sum_{j=1}^{k} \mu_j \leq C_d |\Omega|^{-2/d} k^{1-2/d}.
$$

(1.10)

Later, using the Fourier transforms of the eigenfunctions $u_j$, Laptev proved the Riesz-mean inequality equivalent to Kröger’s estimate (1.10),

$$
\sum_{j} (z - \mu_j)^+ \geq L_{1,d}^d |\Omega| z^{1+d/2},
$$

(1.11)

for all $z \geq 0$. (See also [19].)

Our first result is an improvement of (1.10) using a refinement of Young’s inequality for real numbers, which not only improves the estimates of Riesz means and sums, but also provides a bound on individual eigenvalues. It will be useful to introduce the following notation.

$$
m_k := C_d \left( \frac{k}{|\Omega|} \right)^{2/d}, \quad S_k := \frac{\frac{d+2}{d} \sum_{j=1}^{k} \mu_j}{m_k}.
$$

(1.12)

In these terms $m_k$ is the Weyl expression, and Kröger’s inequality (1.10) is expressed as $S_k \leq 1$. We shall prove the following refinement of Kröger’s inequality.

**Theorem 1.1.** Let $d \geq 2$. Then for all $k \geq 0$ the Neumann eigenvalue $\mu_{k+1}$ satisfies

$$
m_k^2 (1 - S_k) \geq (\mu_{k+1} - m_k)^2.
$$

(1.13)

I.e.,

$$
m_k \left( 1 - \sqrt{1 - S_k} \right) \leq \mu_{k+1} \leq m_k \left( 1 + \sqrt{1 - S_k} \right).
$$

(1.14)

Kröger’s bound corresponds to replacing the right side of (1.13) by 0. One may further ask whether there is an additional remainder term improving the right side of the universal inequality (1.13), which contains more explicit information on the geometry of $\Omega$. The asymptotic expansion of the counting function suggests that under sufficient regularity conditions the $(d-1)$-dimensional volume of the boundary $\partial \Omega$ (see [15, 22]) may appear:

$$
\mathcal{N}(\mu) \approx C_d^{d/2} |\Omega| \mu^{d/2} + \frac{1}{4} C_{d-1}^{(d-1)/2} |\partial \Omega| \mu^{(d-1)/2},
$$

(1.15)

and therefore, for the Riesz mean,

$$
R_1(z) := \sum_{j=1} \left( z - \mu_j \right)^+ \approx L_{1,d}^d |\Omega| z^{1+d/2} + \frac{1}{4} L_{1,d-1}^{d-1} |\partial \Omega| z^{(d+1)/2}.
$$

(1.16)
In the present paper we present a two-term bound for $R_1(\mu)$, using additional geometrical information on $\Omega$. To this end, for any unit vector $v \in \mathbb{R}^d$ let $\delta_v$ be the width of $\Omega$ in the $v$-direction, that is,

$$
\delta_v(\Omega) := \sup \{ v \cdot (x - y) : x, y \in \Omega \} = \max \{ v \cdot (x - y) : x, y \in \partial \Omega \}. \quad (1.17)
$$

We note that $\delta_v(\Omega)$ always lies between twice the inradius and the diameter of $\Omega$.

We prove the following.

**Theorem 1.2.** Let $d \geq 2$. Then for each unit vector $v \in \mathbb{R}^d$ and for all $z \geq 0$

$$
\sum (z - \mu_j)^+ \geq L_{1,d}^d |\Omega|^d z^d \frac{1}{2} + \frac{1}{4} L_{1,d-1}^d \frac{|\Omega|}{\delta_v(\Omega)} z^d + \frac{1}{96} (2\pi)^{2-d} B_d \frac{|\Omega|}{\delta_v(\Omega)^d} z^d. \quad (1.18)
$$

Together with the with the semiclassical bound (1.11) this implies the improved estimate

$$
\sum (z - \mu_j)^+ \geq L_{1,d}^d |\Omega|^d z^d \frac{1}{2} + \left( \frac{1}{4} L_{1,d-1}^d \frac{|\Omega|}{\delta_v(\Omega)} z^d + \frac{1}{96} (2\pi)^{2-d} B_d \frac{|\Omega|}{\delta_v(\Omega)^d} z^d \right)^+. \quad (1.19)
$$

Both inequalities (1.18) and (1.19) will follow from our proof. Although the bound (1.19) improves (1.18), we work in most cases with (1.18) since we are mainly interested in large $z$. An exception is Corollary 1.4 below, where we use the estimate (1.19). We also remark that while the first term is sharp, the second term in Eq. (1.18) appears too small by a factor $1/2$. Indeed, for the box $\Omega = [0, 1]^{d-1} \times [0, \delta]$ the bound (1.18) differs from the asymptotic formula (1.16) by a factor $1/2$, since with $\delta_v(\Omega) = \delta$, comparing the second term of (1.18) and the asymptotic expansion (1.16) we obtain

$$
\frac{|\Omega|}{\delta_v(\Omega)} = 1, \quad |\partial \Omega| = 2 + (d - 1) \delta,
$$

in which $\delta$ can be chosen arbitrarily small. More generally, this argument applies to any domain of the form $\Omega = \Omega' \times [0, \delta]$ such that $\Omega'$ is bounded in $\mathbb{R}^{d-1}$ with finite boundary, since

$$
\frac{|\Omega|}{\delta_v(\Omega)} = |\Omega'|, \quad |\partial \Omega| = 2|\Omega'| + |\partial \Omega'| \delta.
$$

From our method of proof it will be seen that for these kinds of domains the lower bound (1.18) can be improved to the optimal lower bound consistent with the asymptotic formula (1.16). It is less clear whether the improvement can be obtained in the absence of a product structure.
Corollary 1.3. Let $d \geq 2$ and $\Omega = \Omega' \times [0, \delta]$ be a bounded domain. Then for any unit vector $v \in \mathbb{R}^d$ and for all $\mu \geq 0$,

$$\sum_{j=1}^{n} (z - \mu_j)^+ \geq L_{1,d}^1 |\Omega| z^{\frac{d+1}{2}} + \frac{1}{2} L_{1,d-1}^1 \frac{|\Omega|}{\delta_v(\Omega)} z^{\frac{d+1}{2}} - \frac{1}{24} (2\pi)^{2-d} B_d \frac{|\Omega|}{\delta_v(\Omega)^2} z^{\frac{d}{2}}.$$

(1.20)

Note that by means of the integral transform

$$\int_0^\infty (z - \lambda - t) \gamma^{-1} dt, \quad \gamma > 1,$$

Eq. (1.18) implies further bounds for higher Riesz means, viz.,

$$\sum_{j=1}^{n} (z - \mu_j)^+ \geq L_{\gamma,d}^1 |\Omega| z^{\frac{d+1}{2}} + L_{\gamma,d-1}^1 \frac{|\Omega|}{4\delta_v(\Omega)} z^{\frac{d+1}{2}} - \frac{\pi}{96} L_{\gamma,d-2} \frac{|\Omega|}{\delta_v(\Omega)^2} z^{\frac{d+1}{2}}.$$

(1.21)

for any $\gamma \geq 1$, as well as a strengthened version by means of eq.(1.19). This moreover implies that Pólya's conjecture (1.5) can be proved with an improvement for domains in product form:

Corollary 1.4. Suppose that $\Omega = \Omega_1 \times \Omega_2$, where Pólya’s conjecture (1.5) holds for $\Omega_1$ and $\Omega_2$ is a domain of finite measure. Then

$$N(z) \geq 1 + |\Omega| L_{0,d}^1 z^{\frac{d}{2}} + |\Omega| \left( \frac{L_{0,d+1}^1}{4\sqrt{\pi} \cdot 4\delta_v(\Omega_2)} z^{\frac{d}{2}} - \frac{L_{0,d+2}^1}{384\delta_v(\Omega_2)^2} z^{\frac{d}{2}-1} \right).$$

(1.22)

This implies Pólya's conjecture for $\Omega$, when only the first two terms in this expression are kept.

The proof of the main Theorem 1.2 is based on an averaged variational principle introduced by the authors [14], which was later used in [11] to extend and simplify Kröger's results for certain operators on manifolds. The averaged variational principle uses only basic properties of quadratic forms and an averaging over an orthonormal basis or, more generally, a frame. Quoting from the formulation in [11]:

Lemma 1.5. Consider a self-adjoint operator $H$ on a Hilbert space $\mathcal{H}$, the spectrum of which is discrete at least in its lower portion, so that $-\infty < \mu_0 \leq \mu_1 \leq \ldots$. The corresponding orthonormalized eigenvectors are denoted $\{\psi^{(i)}\}$. The closed quadratic form corresponding to $H$ is denoted $Q(\varphi, \varphi)$ for vectors $\varphi$ in the quadratic-form domain $Q(H) \subset \mathcal{H}$. Let $f_\zeta \in \mathcal{H}$ be a family of vectors indexed by a variable $\zeta$ ranging over a measure space $(\mathcal{M}, \Sigma, \sigma)$. Suppose that $\mathcal{M}_0$ is a subset of $\mathcal{M}$. Then for any $z \in \mathbb{R}$,

$$\sum_{j} (z - \mu_j)^+ \int_{\mathcal{M}_0} \langle \psi^{(j)}, f_{\zeta} \rangle^2 d\sigma \geq \int_{\mathcal{M}_0} \langle z \|f_{\zeta}\|^2 - Q(f_{\zeta}, f_{\zeta}) \rangle d\sigma,$$

(1.23)

provided that the integrals converge.
2 Proofs of the main results

2.1 Refinement of Kröger’s inequality: Theorem 1.1

The quadratic-form domain of the Neumann Laplacian $-\Delta^N$ on a Euclidean domain $\Omega$ is the restriction to $\Omega$ of functions in the Sobolev space $H^1_0(\mathbb{R}^d)$ [10] (which is normally but not always the same as $H^1(\Omega)$), and the quadratic form corresponding to $-\Delta^N$ is

$$Q(f, f) = \int_{\Omega} |\nabla f|^2 \, dx.$$  \hspace{1cm} (2.1)

The trial functions $f(x) = e^{ip \cdot x}$ are admissible, so choosing them as in [17] leads after a calculation to the following bound for the eigenvalues of the Neumann Laplacian. (The set $\mathfrak{M}$ is chosen as $\{p \in \mathbb{R}^d\}$ with Lebesgue measure, and $\mathfrak{M}_0$ is the ball of radius $R$. See [17, 11] for details of the calculation.)

$$\mu_{k+1} R^d - \frac{d}{d+2} R^{d+2} \leq m_k^{d/2} \left( \frac{1}{k} \sum_{i=1}^{k} \mu_i \right)$$ \hspace{1cm} (2.2)

for all $R > 0$, cf. (1.12). Putting $R^d = m_k^{d/2} x^{d/2}$, we get the bound

$$\frac{d+2}{d} \sum_{i=1}^{k} \mu_i \leq m_k \left( \frac{d+2}{d} \frac{\mu_{k+1}}{m_k} - \frac{d+2}{d} \frac{\mu_{k+1}}{m_k} x^2 + x^{d+2} \right).$$

We choose $x = x_k = \frac{\mu_{k+1}}{m_k}$. This yields

$$\frac{d+2}{d} \sum_{i=1}^{k} \mu_i - m_k \leq m_k \frac{2}{d} \left( \frac{d+2}{2} x_k - \frac{d}{2} - x_k \right).$$ \hspace{1cm} (2.3)

We may assume that $d \geq 2$, since when $d = 1$ all eigenvalues are explicitly known. Then $p = \frac{d}{2} \geq 1$, and, therefore, the function $g_p(x)$ defined in (A.6) is $\leq 0$. Hence we obtain:

$$\frac{d+2}{d} \sum_{i=1}^{k} \mu_i - m_k \leq -m_k (x_k - 1)^2,$$ \hspace{1cm} (2.4)

which strengthens Kröger’s estimate

$$\frac{d+2}{d} \sum_{i=1}^{k} \mu_i \leq m_k = C_d \frac{k^{2/d}}{|\Omega|^{2/d}}$$

and yields the bound on $\mu_{k+1}$ claimed in (1.14).
2.2 Two-term spectral bounds: Proof of Theorem 1.2

Proof. Let \( v \in \mathbb{R}^d \) be a unit vector. After a translation we may suppose that \( \Omega \subset \mathbb{R}^d \) is a bounded domain such that \( \Omega \subset \{ x \in \mathbb{R}^d : 0 \leq v \cdot x \leq L \} \), that is, in the \( v \) direction all \( x \in \Omega \) are contained in an interval of length \( L \). We shall choose \( L \) later as \( L = 2\delta_0(\Omega) \). Fixing \( v \), we may choose a coordinate system such that \( v \) is a standard unit vector of the canonical basis of \( \mathbb{R}^d \). We apply the averaged variational principle 1.5 with test functions of the form

\[
 f(x) = (2\pi)^{-\frac{d+1}{2}} e^{i p \cdot x} \phi_n(v \cdot x),
\]

where \( p = p - (p \cdot v) v \) and \( \phi_n \) is an eigenfunction of the Neumann Laplacian on an interval of length \( L \), that is,

\[
 -\phi_n''(y) = \kappa_n \phi_n(y) \quad \text{on } [0,L] \text{ and } \phi_n'(0) = \phi_n'(L) = 0.
\]

Recall that the eigenvalues \( \kappa_n \) are given by \( \kappa_n = \frac{(\pi n)^2}{L^2} \), \( n \in \mathbb{N} \) and the (normalized) eigenfunctions are given by \( \phi_0(y) = L^{-1/2} \) and \( \phi_n(y) = \frac{\sqrt{2}}{L} \cos \left( \frac{\pi ny}{L} \right) \), where \( n \) ranges over the positive integers. With these test functions, the variational principle implies that

\[
 \sum_{j=1}^k (z - \mu_j)^2 |\langle f, u_j \rangle|^2 \geq (2\pi)^{1-d} (z - |p|) \int_\Omega \phi_n(v \cdot x)^2
\]

\[
 - (2\pi)^{1-d} \int_\Omega \phi_n'(v \cdot x)^2
\]

for any \( z \in [\mu_k, \mu_{k+1}] \). When \( n > 0 \) we apply the trigonometric identities \( \cos^2 t = \frac{1}{2} + \frac{1}{2} \cos 2t \) and \( \sin^2 t = \frac{1}{2} - \frac{1}{2} \cos 2t \) to \( \phi_n(v \cdot x)^2 \) and \( \phi_n'(v \cdot x)^2 \), respectively. Then for all \( n \geq 0 \), (2.7) becomes

\[
 \sum_{j=1}^k (z - \mu_j)^2 |\langle f, u_j \rangle|^2 \geq (2\pi)^{1-d} L^{-1} |\Omega| (z - |p|)^2 \left( \frac{(\pi n)^2}{L^2} \right)
\]

\[
 + (2\pi)^{1-d} L^{-1} \left( z - |p| \right)^2 \left( \frac{(\pi n)^2}{L^2} \right) \left( 1 - \delta_{0,n} \right) \int_\Omega \cos \left( \frac{2\pi n v \cdot x}{L} \right),
\]

where \( \delta_{0,n} \) denotes the Kronecker delta. On the right side we integrate over the set \( \Phi_k = \{ (p, n) \in \mathbb{R}^{d-1} \times \mathbb{N} : |p| + \frac{\pi^2 n^2}{L^2} \leq z \} \) while on the left side over the larger set \( \mathbb{R}^{d-1} \times \mathbb{N} \), using Parseval’s identity. We shall prove in Lemma 3.2 below that for all \( R > 0 \),

\[
 \sum_{k \geq 0} (R^2 - k^2)_+ \geq \max \left( \frac{2R^3}{3} + \frac{R^2}{2} - \frac{R}{6}, R^2 \right).
\]
By applying the lower bound (2.9) to the sum over $n$ and then integrating over $p_\perp$ we obtain an explicit lower bound for $\int \sum_{\Phi_k} \left( z - |p_\perp|^2 - \frac{(\pi n)^2}{L^2} \right)$. Since $\int \max(f, g) \geq \max(\int f, \int g)$, this yields
\[
\sum_{j=1}^{k} (z - \mu_j) \geq \frac{2}{d+2} (2\pi)^{-d} B_d|\Omega| z^{\frac{d}{2}+1} + \frac{1}{d+1} (2\pi)^{1-d} B_{d-1} |\Omega| L^{-1} z^{\frac{d+1}{2}} - \frac{1}{24} (2\pi)^{2-d} B_d |\Omega| L^{-2} z^{\frac{d}{2}} + G(z),
\] where
\[
G(z) := \int \sum_{\Phi_k} (2\pi)^{1-d} (z - |p_\perp|^2 + \frac{(\pi n)^2}{L^2})(1 - \delta_{0,n}) \int_\Omega \cos \left( \frac{2\pi n \cdot x}{L} \right).
\]

It remains to control $G(z)$, which could in principle be positive or negative. In fact, by averaging (2.10) in a certain way we shall show that $G$ can be dropped altogether. To this end we choose $L$ large enough that $\Omega$ is also contained in $\{ x \in \mathbb{R}^d : 0 \leq v \cdot x \leq L/2 \}$ when translated by $L/2$. This means nothing else than assuming that $\Omega \subset \{ x \in \mathbb{R}^d : 0 \leq v \cdot x \leq L/2 \}$. In the corresponding Neumann eigenfunctions we have to replace $v \cdot x$ by $v \cdot x + L/2$. We may apply the averaged variational principle on both sets (the eigenvalues $\frac{(\pi n)^2}{L^2}$, $n \in \mathbb{N}$ remain unchanged). Since
\[
\frac{1}{2} \left( \cos \left( \frac{2\pi n v \cdot x}{L} \right) + \cos \left( \frac{2\pi n (v \cdot x + L/2)}{L} \right) \right) = \begin{cases} \cos \left( \frac{2\pi n v \cdot x}{L} \right) & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}, \end{cases}
\]
all odd $n$ may be dropped from $G(z)$, leaving only cosine functions of the form $\cos \left( \frac{4\pi n v \cdot x}{L} \right)$ with $n$ a positive integer. We apply the same averaging procedure with a translation by $L/4$. Since
\[
\frac{1}{2} \left( \cos \left( \frac{4\pi n v \cdot x}{L} \right) + \cos \left( \frac{2\pi n (v \cdot x + L/4)}{L} \right) \right) = \begin{cases} \cos \left( \frac{4\pi n v \cdot x}{L} \right) & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}, \end{cases}
\]
again the terms containing odd integers may be dropped. Since $G(z)$ contains only a finite number of contributions, after a finite sequence of averages with shifts $L/2^n$, every contribution will be eliminated. Hence
\[
\sum_{j=1}^{k} (z - \mu_j) \geq \frac{2}{d+2} (2\pi)^{-d} B_d|\Omega| z^{\frac{d}{2}+1} + \frac{1}{d+1} (2\pi)^{1-d} B_{d-1} |\Omega| L^{-1} z^{\frac{d+1}{2}} - \frac{1}{24} (2\pi)^{2-d} B_d |\Omega| L^{-2} z^{\frac{d}{2}}.
\]
We may now choose \( L = 2\delta_v(\Omega) \), which yields the statement of the theorem.

To prove Corollary 1.3 we note that when \( \Omega = \Omega' \times [0, \delta] \) we may choose \( \mathbf{v} = \mathbf{e}_n \). As a consequence

\[
\int_{\Omega} \phi_n(\mathbf{v} \cdot \mathbf{x})^2 = |\Omega'|, \quad \int_{\Omega} \phi_n'(\mathbf{v} \cdot \mathbf{x})^2 = \frac{(\pi n)^2}{L^2} |\Omega'|,
\]

and no translations are needed. Therefore we may choose \( L = \delta \) which yields the bound (1.20).

From the bound (2.9) it is straightforward to derive the simpler expression

\[
\sum_{k \geq 0} (R^2 - k^2)_+ \geq \frac{2R^3}{3} + \frac{R^2}{3},
\]

containing only two terms. This yields the following spectral bound.

**Corollary 2.1.** Let \( d \geq 2 \). Then for any unit vector \( \mathbf{v} \in \mathbb{R}^d \) and for all \( \mu \geq 0 \)

\[
\sum_{j=1}^{k} (\zeta - \mu_j)_+ \geq L_{1,d}^2 |\Omega| z^\frac{d}{2} + 1 + L_{1,d-1}^2 \frac{|\Omega|}{6\delta_v(hull(\Omega))} z^\frac{d}{2} + \frac{\pi}{2} |\Omega| z^\frac{d}{2} + \frac{\pi}{2} \mu^3/2.
\]

The term containing the width \( \delta_v \) can be estimated by geometric properties of the convex hull of \( \Omega \), since \( \delta_v(\Omega) \) coincides with \( \delta_v(hull(\Omega)) \). For example, in 2 dimensions,

\[
\int_{S^1} \delta_v = 2|\partial hull(\Omega)|.
\]

With Corollary 2.1 by choosing \( \mathbf{v} \) so that \( \delta_v \) equals the mean width \( w \) of \( hull(\Omega) \) (= the average of \( \delta_v \) uniformly over directions \( \mathbf{v} \)), we obtain a correction involving the isoperimetric ratio of \( \Omega \),

\[
\sum_{j=1}^{k} (\mu - \mu_j)_+ \geq L_{1,2}^2 |\Omega| \mu^2 + L_{1,1}^2 \frac{\pi |\Omega|}{6|\partial hull(\Omega)|} \mu^{3/2}.
\]

In arbitrary dimensions, if \( \delta_v \) is chosen equal to \( w \), then, following Bourgain [7], the final term in (2.13) can be bounded from below in terms of the *isotropic constant*,

\[
L_{hull(\Omega)}^2 := \frac{\det(M_{hull(\Omega)})}{\Vol(hull(\Omega))^{1+1/d}},
\]

where the inertia matrix \( M_{ij} = \int_{hull(\Omega)} x_i x_j dx \) has been minimized with respect to the choice of the origin. Finding the optimal upper bound for the ratio \( \frac{w}{L} \) for convex \( \Omega \) is an open problem in analysis. In [23], Milman has, for example, proved an upper bound for \( w \) in the form of a universal constant times \( \sqrt{d \log(d)^2} \).

It has been known since the work of Ball [5] that under various further assumptions convex bodies satisfy reverse isoperimetric inequalities, with which Inequality

\[
\text{Equation (2.13)}
\]
can be connected to additional geometric properties of \( \text{hull}(\Omega) \). See, e.g., [24].

Finally, we prove Corollary 1.4. It was shown by Laptev [18] that if Pólya’s conjecture holds on a domain \( \Omega_1 \), then it holds on arbitrary Cartesian products of the form \( \Omega_1 \times \Omega_2 \). (The Dirichlet case was treated in [18].) In fact, the same argument allows improve bounds on the counting function, benefitting from the improved bounds for sums coming from \( \Omega_2 \), as follows.

**Proof.** Suppose that \( \Omega_1 \subset \mathbb{R}^{d_1} \) \( d_1 \geq 2 \) is a domain for which Pólya’s conjecture holds. Let \( d_1 + d_2 = d \), \( \Omega = \Omega_1 \times \Omega_2 \) with \( \Omega_2 \subset \mathbb{R}^{d_2} \) of finite measure. The Neumann eigenvalues \( \mu_j \) of \( \Omega \) are of the form \( \mu_j = \mu_{j_1} + \mu_{j_2} \) where \( \mu_{j_1}, \mu_{j_2} \) are the Neumann eigenvalues of \( \Omega_1, \Omega_2 \), respectively. Therefore,

\[
\sum_j (z - \mu_j)_+^0 = \sum_{j_2} \sum_{j_1} (z - \mu_{j_2} - \mu_{j_1})_+^0 \geq 1 + L_{0,d}^{cl} |\Omega_1| \sum_{j_2} (z - \mu_{j_2})_{+}^{d_1/2}.
\]

Since \( d_1/2 \geq 1 \), using (1.21) and (1.9) we obtain

\[
N(z) \geq 1 + \frac{L_{0,d}^{cl}}{\Omega_1 |\Omega_2|} \left( L_{\gamma,d}^{cl} z^{\frac{d}{2}} + \frac{L_{\gamma,d}^{cl} - 2}{4\delta_v(\Omega_2)} z^{\frac{d}{2} - \frac{1}{2}} - \frac{\pi L_{\gamma,d}^{cl}}{96 \delta_v(\Omega_2)^2} z^{\frac{d}{2} + \gamma - 1} \right) \geq 1 + |\Omega| \left( L_{0,d}^{cl} z^{\frac{d}{2}} + \frac{L_{0,d}^{cl} - 1}{\sqrt{4\pi} \cdot 4\delta_v(\Omega_2)} z^{\frac{d}{2} - \frac{1}{2}} - \frac{L_{0,d}^{cl}}{384\delta_v(\Omega_2)^2} z^{\frac{d}{2} + \gamma - 1} \right),
\]

as well as

\[
N(z) \geq 1 + |\Omega| L_{0,d}^{cl} z^{\frac{d}{2}}.
\]

Combining both estimates we prove the claim.

3 Riesz means of Laplacians on rectangles

In this section we derive upper and lower bounds for Riesz means of Neumann and Dirichlet Laplacians, respectively, on the rectangle \( R := [0,l_1] \times [0,l_2] \).

**Theorem 3.1.** Let \( \mu_i^R, \lambda_i^R \) denote the eigenvalues of the Neumann Laplacian and the Dirichlet Laplacian on \( R = [0,l_1] \times [0,l_2] \). Suppose that \( l_1 \leq l_2 \). Then the
Finally, for all \( R > 0 \) and for all \( \beta > 0 \), following estimates hold

\[
\frac{3\pi}{128} \left( \frac{l_2}{l_1} + \frac{l_1}{l_2} + \frac{32}{3\pi} \right) \mu + \frac{3\pi}{64} \left( \frac{1}{l_1} + \frac{1}{l_2} \right) \mu^{1/2} + \frac{3\pi^{3/2} \gamma^{1/2}}{64l_2 l_1^{1/2}} \mu^{1/4} \\
\geq \sum_{j=1}^{k} (\mu - \mu_j^R) + - \frac{|R|}{8\pi} \mu^2 - \frac{|\partial R|}{6\pi} \mu^{3/2} \\
\geq -\frac{\pi}{24} \left( \frac{l_2}{l_1} + \frac{l_1}{l_2} - \frac{6}{\pi} \right) \mu - \frac{i}{12} \left( \frac{1}{l_1} + \frac{1}{l_2} \right) \mu^{1/2} - \frac{\pi^{3/2} \gamma^{1/2}}{12l_2 l_1^{1/2}} \mu^{1/4},
\]

and

\[
\frac{3\pi}{128} \left( \frac{l_2}{l_1} + \frac{l_1}{l_2} + \frac{32}{3\pi} \right) \lambda + \frac{\pi}{12} \left( \frac{1}{l_2} - \frac{9}{16l_1} \right) \lambda^{1/2} + \frac{3\pi^{3/2} \gamma^{1/2}}{64l_2 l_1^{1/2}} \lambda^{1/4} \\
\geq \sum_{j=1}^{k} (\lambda - \lambda_j^R) + - \frac{|R|}{8\pi} \lambda^2 + \frac{|\partial R|}{6\pi} \lambda^{3/2} \\
\geq -\frac{\pi}{24} \left( \frac{l_2}{l_1} + \frac{l_1}{l_2} - \frac{6}{\pi} \right) \lambda + \frac{i}{12} \left( \frac{1}{l_1} - \frac{9}{16l_2} \right) \lambda^{1/2} - \frac{\pi^{3/2} \gamma^{1/2}}{12l_2 l_1^{1/2}} \lambda^{1/4}.
\]

Proof. The Riesz mean for the Neumann Laplacian on \( R \) is given by

\[
R_1^N(z) = \sum_{n_1, n_2 \geq 0} \left( z - \frac{(\pi n_1)^2}{l_1^2} - \frac{(\pi n_2)^2}{l_2^2} \right)_+
\]

We need the following polynomial upper and lower bounds for one-dimensional Riesz means \( \sum (R^2 - k^2)_+ \), in particular \( 2.9 \).

Lemma 3.2. For all \( R > 0 \),

\[
\max \left( \frac{2R^3}{3} + \frac{R^2}{2} - \frac{R}{6}, R^2 \right) \leq \sum_{k \geq 0} (R^2 - k^2)_+ \leq \frac{2R^3}{3} + \frac{R^2}{2} + \frac{3R}{32},
\]

and for all \( R > 0, \beta > 0 \),

\[
\max \left( \frac{\sqrt{\pi} \Gamma(\beta + 2)}{2 \Gamma(\beta + 5/2)} R^{2\beta + 3} + \frac{1}{2} R^{2\beta + 2}, \frac{\sqrt{\pi} \Gamma(\beta + 2)}{12 \Gamma(\beta + 3/2)} R^{2\beta + 1}, R^{2\beta + 2} \right) \\
\leq \sum_{k \geq 0} (R^2 - k^2)^{\beta + 1} \\
\leq \frac{\sqrt{\pi} \Gamma(\beta + 2)}{2 \Gamma(\beta + 5/2)} R^{2\beta + 3} + \frac{1}{2} R^{2\beta + 2} + \frac{3\sqrt{\pi} \Gamma(\beta + 2)}{64 \Gamma(\beta + 3/2)} R^{2\beta + 1}.
\]

Finally, for all \( R > 0 \),

\[
\sum_{k \geq 0} \sqrt{(R^2 - k^2)_+} \leq \frac{\pi R^2}{4} + \frac{R}{2} + \frac{\sqrt{2R}}{2}.
\]
The lemma will be proved below. Assuming it for now, we continue the proof of the theorem for the Neumann Laplacian on the rectangle \([0, l_1] \times [0, l_2]\). Since

\[
R_1^N(z) = \frac{\pi^2}{l_2^2} \sum_{n_1,n_2 \geq 0} \left( \frac{l_2^2 z}{\pi^2} - \frac{l_2^2 n_1^2}{l_1^2} - \frac{n_2^2}{l_2^2} \right),
\]

by applying the lower bound (3.3) we get:

\[
R_1^N(z) \geq \frac{2\pi^2 l_2}{3l_1^3} \sum_{n_1 \geq 0} \left( \frac{l_2^2 z}{\pi^2} - n_1^2 \right)^{3/2} + \frac{\pi^2}{2l_1^3} \sum_{n_1 \geq 0} \left( \frac{l_2^2 z}{\pi^2} - n_1^2 \right)^{1/2} - \frac{\pi^2}{6l_1 l_2} \sum_{n_1 \geq 0} \left( \frac{l_2^2 z}{\pi^2} - n_1^2 \right)^{1/2}.
\]

Applying the lower bounds (3.3), (3.4) and the upper bound (3.5) we get

\[
\frac{2\pi^2 l_2}{3l_1^3} \sum_{n_1 \geq 0} \left( \frac{l_2^2 z}{\pi^2} - n_1^2 \right)^{3/2} \geq \frac{l_1 l_2}{8\pi} z^2 + \frac{l_2}{3\pi} z^{3/2} - \frac{\pi}{24} \frac{l_2}{l_1} z,
\]

\[
\frac{\pi^2}{2l_1^3} \sum_{n_1 \geq 0} \left( \frac{l_2^2 z}{\pi^2} - n_1^2 \right)^{1/2} \geq \frac{l_1}{3\pi} z^{3/2} + \frac{z}{4} - \frac{\pi}{12l_1^2} z^{1/2}.
\]

and

\[
-\frac{\pi^2}{6l_1 l_2} \sum_{n_1 \geq 0} \left( \frac{l_2^2 z}{\pi^2} - n_1^2 \right)^{1/2} \geq -\frac{\pi}{24} \frac{l_1}{l_2} z - \frac{\pi}{12l_1^2} z^{1/2} + \frac{\pi^3}{24} \frac{1/2}{12l_2^{1/2}} z^{1/4}.
\]

Summarizing all estimates, we get the lower bound of (3.1). Similarly, we get the upper bound of (3.1) interchanging \(l_1\) and \(l_2\). The Riesz mean for the Dirichlet Laplacian on \(R\) is given by

\[
R_1^D(z) = \frac{\pi^2}{l_2^2} \sum_{n_1,n_2 \geq 1} \left( \frac{l_2^2 z}{\pi^2} - \frac{l_2^2 n_1^2}{l_1^2} - \frac{n_2^2}{l_2^2} \right).
\]

The corresponding one-dimensional bounds are those of Lemma 3.2 subtracting \(R^2\), \(R^{2j+2}\), and respectively \(R\) in (3.3), (3.4) and the upper bound (3.5), leading to a change of the sign of the second term, from which we get the bounds (3.2) of the theorem.

We next prove Lemma 3.2.

Proof. Start from the identity

\[
\sum_{k \geq 0} (R^2 - k^2)_+ = R^2 + R^2[R] - \frac{[R]^3}{3} - \frac{[R]^2}{2} - \frac{[R]}{6},
\]
where \( [R] \) denotes the integer part of \( R \). We substitute the periodic sawtooth function \( \psi(t) = \left( t - [t] - \frac{1}{2} \right) \), in terms of which

\[
\sum_{k \geq 0} (R^2 - k^2) = \frac{2R^3}{3} + \frac{R^2}{2} - \frac{R}{6} + \left( \frac{1}{4} - \psi(R)^2 \right) \left( R - \frac{\psi(R)}{3} \right)
\]

(3.7)

since both factors of the product are nonnegative. This lower bound is exact when \( R \) is an integer. Since \( \sum_{k \geq 0} (R^2 - k^2) = R^2 \) trivially for all \( 0 < R < 1 \), the lower bound follows. For the upper bound, we wish to replace \( \left( \frac{1}{4} - \psi(R)^2 \right) \left( R - \frac{\psi(R)}{3} \right) \) by a linear expression in \( R \) for \( R \geq 0 \), or, equivalently, find an upper bound for

\[
F(R) := \left( \frac{1}{4} - \psi(R)^2 \right) \left( 1 - \frac{\psi(R)}{3R} \right).
\]

Because on each interval \( (n, n+1) \) the function \( \psi(R) \) is antisymmetric about \( n+\frac{1}{2} \) and negative on \( (n, n+\frac{1}{2}) \), the maximum is to be sought in an interval of the form \( (n, n+\frac{1}{2}) \). On these subintervals, the second factor decreases when \( R \) is replaced by \( R + 1 \), while the first factor is positive and unchanged. Hence, the maximum of \( F(R) \) occurs where \( 0 < R < \frac{1}{2} \). In this interval, however, an elementary calculus exercise shows that the maximizing value is \( R = \frac{3}{8} \), and thus \( F(R) \leq F(\frac{3}{8}) = \frac{25}{96} \).

Substituting this into the first line of (3.7) yields the claim. We observe that the upper and lower bounds in (3.3) coincide uniquely when \( R = \frac{3}{8} \). To prove (3.4) we note that for all \( \beta > 0 \),

\[
\int_0^\infty \sum_{k \geq 0} (R^2 - k^2)_+ t^{\beta-1} dt = \frac{1}{\beta(\beta+1)} \sum_{k \geq 0} (R^2 - k^2)_{\beta+1}^+
\]

\[
= 2 \int_0^\infty \sum_{k \geq 0} (s^2 - k^2)_+ s (R^2 - s^2)^{\beta-1} ds,
\]

and then apply the bounds (3.3). It remains to show (3.5). We start from the identity

\[
\sum_{k \geq 0} \sqrt{(R^2 - k^2)_+} = \frac{\pi R^2}{4} + \frac{R}{2} - \int_0^R t(R^2 - t^2)^{-1/2} \left( t - [t] - \frac{1}{2} \right) dt.
\]

For any continuous increasing function \( f : [0, R] \to \mathbb{R} \) and any positive integer \( k \leq R \),

\[
\int_{k-1}^k \psi(t) f(t) dt = \int_0^{\frac{1}{2}} \left( \frac{1}{2} - s \right) \left( f(k - s) - f(k - 1 + s) \right) ds \geq 0.
\]

(3.8)
Consequently,
\[
\sum_{k \geq 0} \sqrt{(R^2 - k^2)}_+ \leq \frac{\pi R^2}{4} + \frac{R}{2} - \int_{[R]}^R t(R^2 - t^2)^{-1/2} \left( t - \left\lfloor t \right\rfloor - \frac{1}{2} \right) dt. \quad (3.9)
\]

The integral between \([R]\) and \(R\) can also be computed explicitly. Define \(\rho = \frac{[R]}{R}\) and \(\kappa = \sqrt{1 - \rho^2}\). Then for all \(R > 0\) we have \(1 - \min \left(1, \frac{1}{R}\right) \leq \rho \leq 1\). Hence \(0 < \kappa < 1\) if \(R < 1\) and \(0 < \kappa < R^{-1} \sqrt{2R - 1}\) otherwise. Then
\[
\int_{[R]}^R \frac{t \psi(t)}{\sqrt{R^2 - t^2}} \, dt = R^2 \int_{\rho}^{1} \frac{s^2 - \rho s - \frac{\pi}{2}}{\sqrt{1 - s^2}} \, ds
\]
\[
= \frac{R^2}{2} \left( \arcsin \kappa - \kappa \sqrt{1 - \kappa^2} \right) - \frac{R \kappa}{2}.
\]

We also note that \(\kappa \mapsto \arcsin \kappa - \kappa \sqrt{1 - \kappa^2} - \frac{2\kappa^3}{3}\) is increasing. It follows that
\[
\int_{[R]}^R \frac{t \psi(t)}{\sqrt{R^2 - t^2}} \, dt \leq - \frac{R^2 \kappa^3}{3} + \frac{R \kappa}{2},
\]
proving the claim. \(\square\)

4 Two-term estimates for Dirichlet Laplacians by averaging

For Dirichlet Laplacians on a bounded domain \(\Omega\) our strategy will be to enclose \(\Omega\) in a box \(B\) and then to use the averaged variational principle to estimate the Riesz means of the Dirichlet Laplacian on \(B\) in terms of expectations with the eigenfunctions of \(-\Delta\Omega\). Thus suppose that \(\Omega \subset B\) where \(B = \prod_{\alpha=1}^{d} [0, L_{\alpha}]\) is a box of volume \(|B| = \prod_{\alpha=1}^{d} L_{\alpha}\). We let \(v_{\Omega}^0\) denote the Dirichlet eigenfunctions on \(\Omega\), and, similarly, for \(B\) we define
\[
v_{k}^B(x) = \prod_{\alpha=1}^{d} \psi_{n_{\alpha}}(x_{\alpha}),
\]
where \(\psi_{n_{\alpha}}(x_{\alpha}) := \sqrt{\frac{2}{L_{\alpha}} \sin \left( \frac{n_{\alpha} \pi x_{\alpha}}{L_{\alpha}} \right)}\), corresponding to eigenvalues \(\lambda_{k}^B = \prod_{\alpha=1}^{d} \frac{\pi^2 n_{\alpha}^2}{L_{\alpha}^2}\) with \(n_{\alpha} \in \mathbb{Z}_+\). By the variational principle,
\[
\sum_{j}(z - \lambda_{j}^B)_+ \left| \langle v_{k}^\Omega, v_{j}^\Omega \rangle_B \right|^2 \geq z \int_{B} |v_{k}^\Omega|^2 \, dx - \int_{B} |\nabla v_{k}^\Omega|^2 \, dx. \quad (4.1)
\]
Since $\nu^\Omega_k \in H^1_0(\Omega)$, all integrals reduce to integrals on $\Omega$. On the right side we take a finite sum in $k$ while on the left we sum over all $k$ and apply the completeness relation, obtaining

$$\sum (z - \lambda^B_j)^+ \geq \sum (z - \lambda^\Omega_j)^+.$$  \hfill (4.2)

To apply the translation argument as above we suppose that $l_\alpha$ is at least twice the width of $\Omega$ in the $\alpha$ direction and note that the average of $\psi^2_{n,\alpha}(x_\alpha)$ and its translate by $\frac{l_\alpha}{2}$ is $\frac{1}{l_\alpha}$. Therefore,

$$\frac{|\Omega|}{|B|} \sum (z - \lambda^B_j)^+ \geq \sum (z - \lambda^\Omega_j)^+,$$  \hfill (4.3)

which improves Berezin-Li-Yau. Consider, for example, the case $d = 2$ where applying the upper bound in (3.2) of theorem 3.1 for the Dirichlet Laplacian on a rectangle $B$ with side lengths $l_1, l_2$ we obtain the explicit upper bound

$$\sum (\lambda - \lambda^B_j)^+ \leq L^2_{1,2} |\Omega| \lambda^2 - \frac{1}{4} L^2_{1,1} \frac{d\Omega}{|B|} \lambda^{3/2} + F(l_1, l_2, \lambda) |\Omega|,$$  \hfill (4.4)

where $F(l_1, l_2, \lambda)$ is shorthand notation for the lower-order terms of the left side in (3.2).

**A  Refinements of Young’s and Hölder’s inequality**

In [2.1], we rely on an improvement of Young’s inequality in order to strengthen Kröger’s inequality with (2.4). Improvements of Young’s inequality that are adequate for this purpose already exist in the literature [1],[10],[12], but we take the opportunity in this appendix to present an efficient approach to deriving improvements to Young’s and Hölder’s inequalities.

To begin, let $p > -1$. For $x \geq 0$ we define the strictly concave function $y_p(x)$ by

$$y_p(x) := (p + 1)x - p - x^{p+1}.$$  \hfill (A.1)

The unique critical point of $y_p(x)$ occurs at $x = 1$. Since $y_p(1) = 0$, Young’s inequality follows in the following formulation:

1. $y_p(x) \leq 0$ for all $x \geq 0$ if $p \geq 0$

2. $y_p(x) \geq 0$ for all $x \geq 0$ if $-1 < p \leq 0$.

Before deriving an improvement, we first note that the case $-1 < p \leq 0$ is equivalent to the case $p \geq 0$ by means of the duality

$$y_p(x) = -(p + 1)y_q(z), \quad (p + 1)(q + 1) = 1, \quad z = x^{q+1},$$
the fixed point of which is the trivial case $p = q = 0$. In the following we therefore only consider the case $p > 0$. Putting $x = a/b^{1/p}$, defining $s = p + 1$, $r = \frac{p + 1}{p}$, such that $\frac{1}{r} + \frac{1}{s} = 1$, and dividing by $p + 1$, we obtain the classical version of Young’s inequality:

$$ab - \frac{b^r}{r} - \frac{a^s}{s} \leq 0, \quad a, b \geq 0. \quad (A.2)$$

There are basically two refinements discussed in [1, 16, 12], which as we shall show follow directly from identities for the functions $y_p(x)$. First, we consider the family of functions $f_p$ defined by

$$f_p(x) := y_p(x) + \left(x^{(p+1)/2} - 1\right)^2 = 2y_{(p-1)/2}(x). \quad (A.3)$$

Clearly

1. $f_p(x) \leq 0$ for all $x > 0$ if $p \geq 1$,
2. $f_1(x) = 0$ for all $x > 0$,
3. $f_p(x) \geq 0$ for all $x > 0$ if $0 < p \leq 1$.

When $p \geq 1$ we have $s = p + 1 \geq 2$, and with $x = a/b^{1/p}$ the refinement of Young’s inequality becomes:

$$ab - \frac{b^r}{r} - \frac{a^s}{s} \leq -\frac{1}{s}\left(a^{s/2} - b^{r/2}\right)^2, \quad a, b \geq 0, \quad s \geq 2 \geq r > 1. \quad (A.4)$$

When $0 \leq p \leq 1$ the inequality is reversed. Exchanging $a$ and $b$ as well as $r$ and $s$, we get:

$$ab - \frac{b^r}{r} - \frac{a^s}{s} \geq -\frac{1}{r}\left(a^{s/2} - b^{r/2}\right)^2, \quad a, b \geq 0, \quad s \geq 2 \geq r > 1. \quad (A.5)$$

Another refinement follows from considering the family of functions $g_p$ defined by

$$g_p(x) = y_p(x) + p(x-1)^2 = px^2 - (p-1)x - x^{p+1} = x y_{p-1}(x). \quad (A.6)$$

We observe that

1. $g_p(x) \leq 0$ for all $x > 0$ if $p \geq 1$,
2. $g_1(x) = 0$ for all $x > 0$,
3. $g_p(x) \geq 0$ for all $x > 0$ if $0 < p \leq 1$.

When $p \geq 1$ we have $s = p + 1 \geq 2$, and with $x = a/b^{1/p}$ the refinement of Young’s inequality becomes:

$$ab - \frac{b^r}{r} - \frac{a^s}{s} \leq -\frac{1}{r}\left(a - b^{r-1}\right)^2 b^{2-r}, \quad a, b \geq 0, \quad s \geq 2 \geq r > 1. \quad (A.7)$$
When $0 \leq p \leq 1$ we find a reversed inequality. Exchanging $a$ and $b$ as well as $r$ and $s$, we obtain:

$$ab - \frac{b^r}{r} - \frac{a^s}{s} \geq -\frac{1}{s} (b - a^{s-1})^2 a^{2-s}, \quad a, b \geq 0, \quad s \geq 2 \geq r > 1.$$ (A.8)

Although we do not use it in this paper, we further note that refinements of Hölder’s inequality, cf. [8], are easily obtained from the inequalities above as follows:

Let $M$ be a measure space and $a \in L^s(M)$, $b \in L^r(M)$ such that $\|a\|_s = \|b\|_r = 1$ where $r^{-1} + s^{-1} = 1$, $s \geq 2 \geq r > 1$ and $\|\cdot\|_p$ denotes the usual norm in $L^p(M)$. Then by integrating the pointwise inequalities (A.4) and (A.5),

$$1 - \frac{1}{r} \int \left( |a|^r/2 - |b|^{r/2} \right)^2 \leq \int |ab| \leq 1 - \frac{1}{s} \int \left( |a|^{s/2} - |b|^{r/2} \right)^2,$$ (A.9)

with equality if and only if $|a|^s = |b|^r$ pointwise almost everywhere. We also may directly make the replacements $a \rightarrow t^{-1}a$, $b \rightarrow tb$ in (A.4) and (A.5) and after integration optimize with respect to $t$. This yields the slightly improved inequalities:

$$\left( 1 - \frac{1}{2} \int \left( |a|^{s/2} - |b|^{r/2} \right)^2 \right)^{2/r} \leq \int |ab| \leq \left( 1 - \frac{1}{2} \int \left( |a|^{s/2} - |b|^{r/2} \right)^2 \right)^{2/s}. $$ (A.10)

When integrating the pointwise inequalities (A.7) and (A.8):

$$1 - \frac{1}{s} \int \left( |b| - |a|^{s-1} \right)^2 |a|^{2-s} \leq \int |ab| \leq 1 - \frac{1}{r} \int \left( |a| - |b|^{r-1} \right)^2 |b|^{2-r},$$ (A.11)

with equality if and only if $|a|^s = |b|^r$ pointwise almost everywhere.

References

[1] J. M. Aldaz, A stability version of Hölder’s inequality, J. Math. Analysis Appl. 343(2008)842–852.

[2] M. S. Ashbaugh, Universal eigenvalue bounds of Payne-Polya-Weinberger, Hile-Protter and H. C. Yang, - Proc. Indian Acad. Sci. Math. Sci. 112, 2002, 3–30.

[3] M. S. Ashbaugh, Universal inequalities for the eigenvalues of the Dirichlet Laplacian, in prep.

[4] M. S. Ashbaugh and L. Hermi, A unified approach to universal inequalities for eigenvalues of elliptic operators. Pacific J. Math. 217 (2004)201–219.

[5] K. Ball, Volume ratios and a reverse isoperimetric inequality, J. London Math. Soc. (1991) s2-44 (2): 351–359. doi: 10.1112/jlms/s2-44.2.351

17
[6] F. Berezin, Convariant and contravariant symbols of operators. *Izv. Akad. Nauk SSSR* 37:1134–1167. [In Russian, English transl. in *Math. USSR-Izv.*, 6:1117-1151 (1973).]

[7] J. Bourgain, On the distribution of polynomials on high-dimensional convex sets. Geometric aspects of functional analysis (1989–90), Lecture Notes in Math., Vol. 1469, Springer, Berlin, (1991), 127–137.

[8] E. A. Carlen, R. L. Frank, and E. H. Lieb, *Stability estimates for the lowest eigenvalue of a Schrödinger operator*. Geometric and Functional Analysis 24, Issue 1, (2014), 63–84.

[9] Colin de Verdière, Y. (1987), Construction de laplaciens dont une partie finie du spectre est donnée, Ann. Sci. École Norm. Sup. (4) 20: 599–615. MR 90d:58156 Zbl 0636.58036

[10] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1987. Oxford Science Publications.

[11] A. El Soufi, E. M. Harrell II, S. Ilias, and J. Stubbe, *On Sums of Eigenvalues of Elliptic Operators on Manifolds*, to appear in J. Spectral Theory. [arXiv:1507.02632](https://arxiv.org/abs/1507.02632).

[12] S. Furuichi, *On refined Young inequalities and reverse inequalities* J. Math. Inequal. 5(2011)21–31.

[13] L. Geisinger, A. Laptev, and T. Weidl, Geometrical versions of improved Berezin–Li–Yau inequalities. *J. Spectr. Theory* 1 (2011), 87109 DOI 10.4171/JST/4

[14] E. M. Harrell II and J. Stubbe, On sums of graph eigenvalues. *Linear Algebra Appl.*, 455(2014)168–186. Corrigendum *Ibid*. 458(2014)699–700.

[15] V. Y.Ivrii, Second term of the spectral asymptotic expansion of the Laplace-Beltrami operator on manifolds with boundary, Funct. Anal. Appl. 14(1980), 98–106.

[16] F. Kittaneh and Y. Manasrah, *Improved Young and Heinz inequalities for matrices* J. Math. Analysis Appl. 361(2010)262–269.

[17] P. Kröger, *Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space* J. Funct. Analysis 106(1992), 353–357. MR1165859 (93d:47091)

[18] A. Laptev, Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces, J. Funct. Anal. 151, pp.531–545, 1997

[19] A. Laptev, Spectral inequalities for PDE’s and their applications, AMS/IP Studies in Advanced Mathematics Volume 51, 2012
[20] S. Larson, On the remainder term of the Berezin inequality on a convex domain, preprint 2015. [arXiv:1509.06705]

[21] A. D. Melas, A lower bound for sums of eigenvalues of the Laplacian, Proc. Amer. Math. Soc. 131 (2003), 631–636.

[22] R. Melrose, Weyl’s conjecture for manifolds with concave boundary, Proc. Sym-pos. Pure Math. 36(1980),257–274.

[23] E. Milman, On the mean-width of isotropic convex bodies and their associated Lp-centroid bodies, Int. Math. Res. Not. 11 (2015)3408–3423.

[24] S.-L. Pan and H. Zhang, A Reverse Isoperimetric Inequality for Convex Plane Curves, Beiträge zur Algebra und Geometrie Contributions to Algebra and Geometry 48 (2007) 303–308.