The Aharonov–Bohm effect in conical space

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Abstract
Conical space emerges inevitably as an outer space of any topological defect of the vortex type. Quantum-mechanical scattering of a nonrelativistic particle by a vortex centred in conical space is considered, and effects of the transverse size of the vortex are taken into account. Paradoxical peculiarities of the short-wavelength limit of scattering are discussed.

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1. Introduction
In 1957, the issue of the cylindrical gravitational waves of Einstein and Rosen [1] was reviewed by Weber and Wheeler [2] who were seeking for arguments in favour of the physical reality of such waves. As a byproduct of their survey, they found that there could be the circumstances in the theory of general relativity, under which a spacetime is locally flat almost everywhere but is not Minkowskian in the region of its flatness. Such a spacetime was named conical, and the date of its discovery is fixed in [2]: in a footnote the authors thank Professor M Fierz for the permission to quote from his letter of 14 May 1957, where he showed that conical spacetime emerges in the asymptotics of the imploding–exploding gravitational wave solution at large distances from the axis of the cylindrical symmetry. A little bit later the properties of conical spacetime were studied in detail by Marder [3, 4] who, in particular, predicted a specific gravitational lensing effect—the doubling of the image of objects located behind the axis of the symmetry (see [4]).

In the same year 1957, Abrikosov [5] discovered that a magnetic vortex can be formed in the type-II superconductors, and later this result was rederived in a more general context in relativistic field theory [6]. Such string-like structures denoted as the Abrikosov–Nielsen–Olesen (ANO) vortices arise as topological defects in the aftermath of phase transitions with spontaneous breakdown of continuous symmetries; the general condition of the existence of
these structures is that the first homotopy group of the group space of the broken symmetry group should be nontrivial.

What is the relation between conical spaces and ANO vortices? At first sight there is nothing, but at a more close look one can note that, since the ANO vortex is a topological defect, it is characterized by nonzero energy distributed along its axis, which in turn, according to general relativity, is a source of gravity. As can be shown (for details see the next section), this source makes the spacetime outside the vortex to be conical. Since the squared Planck length enters as a factor before the stress–energy tensor in the Einstein–Hilbert equation, the deviations from the Minkowskian metric are of the order of the squared quotient of the Planck length to the correlation length, the latter characterizing the size of the topological defect, i.e. the thickness of the vortex. For superconductors this quotient is vanishingly small and effects of the conicity are surely negligible. However, topological defects of the type of ANO vortices may arise in a field which is seemingly rather different from the condensed matter physics—in cosmology. This was realized by Kibble [7, 8] and Vilenkin [9, 10] (see also [11]), and from the beginning of the 1980s, such topological defects in cosmology are known under the name of cosmic strings. Cosmic strings with the thickness of the order of the Planck length are definitely ruled out by astrophysical observations, and there remains a room for cosmic strings with the thickness which is more than 3.5 orders larger than the Planck length (see, e.g., [12]), although the direct evidence for their existence is lacking.

In 1959, Aharonov and Bohm [13] considered the quantum-mechanical scattering of a charged particle on a magnetic vortex and found an effect that does not depend on the depth of penetration of the charged particle into the region of the vortex flux. Thus, it was demonstrated for the first time that the quantum-mechanical motion of charged particles can be affected by the magnetic flux from the impenetrable for the particles region. This effect which is alien to classical physics has a great impact on the development of various fields in quantum physics, ranging from particle physics and cosmology to condensed matter and mesoscopic physics (see, e.g., reviews [14–16]).

In the late 1980s, the quantum-mechanical scattering of a test particle in conical space was considered by t’Hooft [17] and Jackiw et al [18, 19]; later the consideration was extended to the case of a magnetic vortex placed along the axis of conical space, i.e. to the quantum-mechanical scattering on a cosmic string [20]. It should be noted that in the above works, as well as in [13], the effects of the thickness of the strings were neglected. This shortcoming was remedied, and the finite-thickness effects were taken into account both for the vortex in ordinary flat space [14] and for the vortex in conical space [21] (see also [22]). Paradoxical peculiarities of the Aharonov–Bohm effect in conical space, including the issue of the quantum-classical correspondence, are discussed in the present paper.

2. Abrikosov–Nielsen–Olesen vortex and conical space

Let us start with the Lagrangian of the Abelian Higgs model:

\[ L = -\frac{1}{4} F_{\rho\nu} F^{\rho\nu} - \frac{1}{2} \left[ (\partial^\rho - i e A^\rho) \psi_H \right] \left[ (\partial_\rho - i e A_\rho) \psi_H \right] \frac{\lambda}{4} \left( \psi_H^* \psi_H - \frac{\sigma^2}{2} \right)^2, \]

(1)

where units \( c = \hbar = 1 \) are used and the metric signature is chosen as \((-1, 1, 1, 1)\). The ground state of the model is characterized by the nonzero vacuum expectation value of the absolute value of the complex scalar Higgs field:

\[ \langle \text{vac} | | \psi_H | | \text{vac} \rangle = \frac{1}{\sqrt{2}} \sigma, \]

(2)
where the value of $\sigma$ is implied to be real positive. Writing the complex field $\psi_H$ in terms of two real fields $\chi_H$ and $\tilde{\chi}_H$:

$$\psi_H(x) = \frac{1}{\sqrt{2}}[\sigma + \chi_H(x)]e^{i\tilde{\chi}_H(x)},$$  \hspace{1cm}(3)$$
and eliminating the field $\tilde{\chi}_H$ by gauge transformation, $A_\rho \to A_\rho + e_H^{-1}\partial_\rho \tilde{\chi}_H$, one can present $L(1)$ in the form

$$L = -\frac{1}{2}F_{\rho\rho'}F^{\rho\rho'} - \frac{1}{2}m_H^2A^\rho A_\rho - \frac{1}{2}(\partial^\rho\chi_H)(\partial_\rho\chi_H) - \frac{1}{2}m_H^2\chi_H^2 + \cdots,$$  \hspace{1cm}(4)$$
where dots correspond to cubic and quartic self-interaction terms of $\chi_H$ and to a quartic interaction term of $\chi_H$ with $A_\rho$, and

$$m_H^2 = \epsilon_H^2\sigma^2, \quad m_H^2 = \frac{1}{2}\lambda\sigma^2.$$  \hspace{1cm}(5)$$
Thus, the physical content of the model is the vector particle with mass $m_H$ and the scalar particle with mass $m_A$.

A static cylindrically symmetric solution in the model is given by the configuration \[6\]

$$\psi_H = \frac{1}{\sqrt{2}}\sigma r_\nu(r)e^{i\nu_\nu},$$  \hspace{1cm}(6)$$
$$A_\nu = \frac{n}{e_H}[(\nu_A(r))^2], \quad A_0 = A_r = A_3 = 0,$$  \hspace{1cm}(7)$$
where the cylindrical $(r, \varphi, z)$ coordinates are used, $n \in \mathbb{Z}$ ($\mathbb{Z}$ is the set of integers), and $r_H$ and $\nu_A$ satisfy the system of nonlinear differential equations:

$$\begin{cases} r^{-1}\underbrace{\partial_r r_\nu}_H + \frac{1}{2}m_H^2(1 - \nu_A^2)\nu_H - \nu_\nu r^{-2}(1 - \nu_A^2)^2\nu_H = 0, \\
\nu_\nu r^{-1}\partial_r r_\nu^2 + m_H^2\nu_H(1 - \nu_A^2) = 0,
\end{cases} \hspace{1cm}(8)$$
with the boundary conditions

$$\nu_H(0) = \nu_A(0) = 0, \quad \nu_H(\infty) = \nu_A(\infty) = 1.$$  \hspace{1cm}(9)$$
The analytical form of the solution to (8) and (9) is unknown, but the extensive analysis involving the use of rigorous methods (see [23]) and numerical calculations yields that $\nu_H$ and $\nu_A$ tend to zero at the origin as $\nu_H \propto \alpha r^{[m]}$ and $\nu_A = \alpha r$, while approaching exponentially fast to the unity value as $\nu_H = 1 - b_H r^{-1/2} \exp(-m_H r)$ and $\nu_A = 1 - b_A r^{1/2} \exp(-m_A r)$. The only nonvanishing component of the gauge field tensor $F_{\rho\rho'}$ is

$$B^3 \equiv r^{-1}F_{\rho\nu} = r^{-1}\partial_\nu A_\rho = 2n(e_H r)^{-1}\nu_H(\partial_\nu \nu_A).$$  \hspace{1cm}(10)$$
Thus, one notes that at large distances from the symmetry axis (at $r > m_H^{-1}$ and $r > m_A^{-1}$) the solution to (8) and (9) corresponds to the ground state: $\psi_H = \sigma r H e^{i\nu_\nu}$ and $B^3 = 0$.

The solution is characterized by two cores: the one (where the Higgs field differs from its vacuum value) has the transverse size of the order of the correlation length $r_H = m_H^{-1}$, and the other one (where the gauge field strength is nonzero) has the transverse size of the order of the penetration depth $r_A = m_A^{-1}$. The value of quotient $\kappa \equiv r_A/r_H = e_H^3/\sqrt{\lambda/2}$ which is known as the Ginzburg–Landau parameter distinguishes between the type-I ($\kappa < 1/\sqrt{2}$) and the type-II ($\kappa > 1/\sqrt{2}$) superconductors. The solution to (8) and (9) in the case of the type-II superconductors is known as the Abrikosov vortex, while the solution in the general case of either $\kappa > 1/\sqrt{2}$ or $\kappa < 1/\sqrt{2}$ may be denoted as the ANO vortex. The ground state manifold, i.e., the spatial region outside the vortex, is not simply connected; the first homotopy group $\pi_1$ is nontrivial, $\pi_1 = \mathbb{Z}$, and thus the vortices are characterized by a winding number $n \in \mathbb{Z}$, see (6) and (7).
The stress–energy tensor corresponding to the ANO vortex has diagonal nonvanishing components only, and
\[ T_{00} = -T_{33} = \frac{1}{2} (B^2 + r^{-2}) \left( (\partial_\varphi - i e_\mu A_\mu) \psi \right)^2 + \frac{\lambda}{4} \left( |\psi|^2 - \frac{\sigma^2}{2} \right)^2 \]
where
\[ T_{\rho\rho} = \frac{1}{2} \sigma^2 \left( (\partial_\varphi - i e_\mu A_\mu) \psi \right)^2 + \frac{m^2}{n^2} \left( 1 - \frac{r^2}{\bar{r}_A^2} \right)^2 + n^2 r^{-2} \left[ 4 m^2 \bar{r}_A^2 (\partial_\varphi - i e_\mu A_\mu)^2 + \bar{r}_A^2 \left( 1 - \frac{r^2}{\bar{r}_A^2} \right)^2 \right]. \] (11)

As for \( T_{\sigma\sigma} \) and \( r^{-2} T_{\varphi\varphi} \), they are negative with their absolute values being much smaller than \( T_{00} \) (see, e.g., [24]).

Using (8), relation (11) can be recast into the form
\[ T_{00} = -T_{33} = \frac{1}{2} \sigma^2 \left( r^{-1} \partial_\varphi (r \tau_H \partial_\varphi \tau_H) + \frac{m^2}{n^2} \left( 1 - \frac{r^2}{\bar{r}_H^2} \right) + 4 n^2 m^2 \bar{r}_A^2 r^{-2} \tau_A^2 \right). \] (12)

In the square brackets on the right-hand side of (12), the first two terms are nonvanishing in the core of the order of correlation length \( \tau_H \) and the third term is nonvanishing in the core of the order of penetration depth \( r_A \).

The stress–energy tensor is a source of gravity according to the Einstein–Hilbert equation:
\[ R_{\rho\rho} - \frac{1}{2} g_{\rho\rho} R = 8\pi G T_{\rho\rho}, \] (13)
where \( R_{\rho\rho} \) is the Ricci tensor, \( R = g^{\rho\sigma} R_{\rho\sigma} \) is the scalar curvature, \( G = \frac{\hbar}{2\pi} \) is the gravitational constant (\( \hbar \) is the Planck length); we use the notations adopted in [25]. Taking the trace over Lorentz indices in (13), one gets that the spacetime region of the vortex core is characterized by the positive scalar curvature,
\[ \bar{r}_H \]
and the third term is nonvanishing in the core of the order of correlation length \( \tau_H \) and the third term is nonvanishing in the core of the order of penetration depth \( r_A \).

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\[ \mu = \int_0^{2\pi} d\varphi \int_0^\infty dr r T_{00} = \pi \sigma^2 (I_H + I_A), \] (14)
where
\[ I_H = \frac{1}{4} \int_0^\infty du u \left( 1 - \bar{r}_H^2 \right), \quad I_A = \frac{4 n^2}{4} \int_0^\infty du \frac{\bar{r}_A^2 (\partial_\varphi - i e_\mu A_\mu)^2}{\bar{r}_A^2 + r^2}, \] (15)
and the functions of dimensionless variables are introduced: \( \bar{r}_H(m_A r) \equiv \bar{r}_H(r) \) and \( \bar{r}_A(m_A r) \equiv \bar{r}_A(r) \). The integrands in integrals (15) are damped as \( e^{-u} \) at large values of \( u \), and, therefore, the upper limit of integration in (15) is of the order of unity rather than infinity. One can conclude that the dependence of the linear energy density \( \mu \) on the parameter \( \kappa \) is rather weak. In order to solve (13) outside the vortex core, it suffices to use the approximation neglecting the transverse size of the core. Then the stress–energy tensor is expressed in terms of \( \mu \) as
\[ T_{00} = -T_{33} = \frac{\delta(r)}{r} \Delta(\varphi), \quad T_{\sigma\sigma} = T_{\varphi\varphi} = 0, \] (16)
where \( \Delta(\varphi) = (2\pi)^{-1} \sum_{n \in \mathbb{Z}} e^{in\varphi} \) is the delta-function for the compact (angular) variable. Solving (13) with \( T_{\rho\rho} \) in the form of (16), one gets the metric outside the vortex core, which is given by the squared length element
\[ ds^2 = -d\tilde{r}^2 + (1 - 4G\mu)^{-1} d\varphi^2 + (1 - 4G\mu)\tilde{r}^2 d\tilde{\varphi}^2 + (dx^2)^2 \]
where
\[ \tilde{r} = r (1 - 4G\mu)^{-1/2}, \quad 0 < \psi < 2\pi (1 - 4G\mu). \]
This is the metric of conical space: a surface which is transverse to the axis of the vortex is isometric to the surface of a cone with the deficit angle equal to $8\pi G\mu$.

In view of (14), the value of the linear energy density can be estimated as $\mu \approx \pi \sigma^2$. The Abelian Higgs model (1) contains two dimensionless parameters, $\lambda$ and $\epsilon_\text{H}$, and by varying their values, one gets the variety of superconductors of types I and II. One can fix the value of one parameter, say $\lambda$, and then the variety of superconductors is obtained by varying the value of the other parameter, $\epsilon_\text{H}$. By fixing $\lambda = \frac{2}{\pi}$, one gets $\mu \approx m_\text{H}^2$, and then the deviation from the Minkowskian metric is estimated as $G\mu \approx (l_{\text{Pl}} / r_\text{H})^2$, as it has been announced in the introduction.

A more consistent treatment involves the analysis of the full system of coupled equations for the metric and the vortex-forming gauge and Higgs fields [24]; it yields the same results as presented above.

To conclude this section, we list the global (spatial-point-independent) parameters of the ANO vortex: flux of the gauge field strength

$$\Phi = \int_{\text{core}} d\sigma B^3, \quad (18)$$

and the linear energy density (compare with (14) and (15))

$$\mu = \int_{\text{core}} d\sigma T_{00}, \quad (19)$$

where the integration is over the transverse section of the core of the vortex. The flux is directly related to the gauge-Higgs coupling, $\Phi = 2\pi c_\text{H}^2/\hbar$ (see (10)), while the density can be estimated as $\mu \approx m_\text{H}^2 / c_\text{H}$ or $\mu \approx \hbar c / r_\text{H}^2$, where constants $c$ and $\hbar$ are recovered. One more parameter should be introduced, and this is the transverse radius of the vortex core,

$$r_c = \max\{r_\text{H}, r_\Lambda\}, \quad (20)$$
i.e. $r_\text{H}$ for the type-I superconductors and $r_\Lambda$ for the type-II superconductors.

3. Quantum-mechanical scattering in conical space

We shall study the quantum-mechanical scattering of a nonrelativistic test particle by an ANO vortex. Since the motion of the particle along the vortex axis is free, we only need to consider the two-dimensional motion on the surface which is orthogonal to the vortex axis. The Schrödinger equation for the wavefunction describing the stationary scattering state has the form

$$H\psi(r, \varphi) = \hbar^2 k^2 / 2m \psi(r, \varphi), \quad (21)$$

where $m$ is the particle mass and $k$ is the absolute value of the particle wave vector.

The conical nature of space outside the vortex core is characterized by the dimensionless parameter

$$\eta = 4G\mu c^4, \quad (22)$$

where, as well as in (21), constants $c$ and $\hbar$ are recovered. As it has been already mentioned, this parameter is vanishingly small for vortices in superconductors, while for cosmic strings it is restricted to the range $0 < \eta < 4 \times 10^{-7}$ [12]. However, it may appear that a more wide range of $\eta$ is of physical interest: in particular, the topological defects in graphene correspond to carbon monolayer nanocones with deficit angles $2\pi \eta$ being positive and negative integer
multiples of $\pi/3$ [26, 27]. In view of this, we make our consideration as general as possible by extending the range of values of $\eta$ to $1 > \eta > -\infty$.\(^3\)

The behaviour of the Schrödinger Hamiltonian at large distances from the scattering centre is crucial for the construction of scattering theory; therefore, it suffices at first to write down the Hamiltonian outside the vortex core:

$$H = -\frac{\hbar^2}{2m} \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{(1-\eta^2)^2} \left( \frac{\partial^2}{\partial \phi^2} - i \frac{\Phi}{\Phi_0} \right)^2 \right], \quad r > r_c, \quad (23)$$

where $\Phi_0 = 2\pi \hbar c e^{-1}$ is the London flux quantum. The quantum-mechanical particle is coupled to the vortex-forming gauge field with constant $e$ which, in general, differs from $e\Omega$; the case of $e = e\Omega/2$ corresponds to the Bardeen–Cooper–Schrieffer model of superconductivity, when the condensate of Cooper pairs is described phenomenologically by the Higgs field. A vortex in the type-II superconductors has the flux equal to one semifluxon, $\Phi = \frac{1}{2}\Phi_0$, while a vortex in the type-I superconductors can have the flux equal to an integer multiple of a semifluxon, $\Phi = \frac{n}{2}\Phi_0$ ($n \in \mathbb{Z}$).

Hamiltonian (23) should be compared with the Hamiltonian in the absence of the vortex (i.e. at $\Phi = 0$ and $\eta = 0$):

$$H_0 = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right). \quad (24)$$

Thus, the interaction is given by the difference between (23) and (24), which can be written in the form

$$H - H_0 = v(x) + v^j(x) \left( -i \frac{\partial}{\partial x^j} \right) + v^{ij}(x) \left( -\frac{\partial^2}{\partial x^i \partial x^j} \right), \quad (25)$$

where we have introduced notations $x = (x^1, \ldots, x^n), r = r \cos \phi, x^2 = r \sin \phi$ and $j, j' = 1, 2$.

It should be noted that the interaction of the form of (25) is of short range, if the coefficient functions $v, v^j$ and $v^{ij}$ decrease as $O(r^{-1-\varepsilon})$ at $r \to \infty$ ($\varepsilon > 0$), and then scattering theory can be constructed in the usual way (see, e.g., [28]). However, even for particle scattering by a purely magnetic vortex ($\Phi \neq 0$ and $\eta = 0$) interaction (25) is of long range since the coefficient function $v^j$ decreases as $O(r^{-1})$ at $r \to \infty$. Because of the long-range nature of the interaction in this case, it is impossible to choose a plane wave as the incident wave, as it has been noted by Aharonov and Bohm [13]. Nevertheless, it is possible to construct scattering theory in this case and obtain in its framework the Aharonov–Bohm scattering amplitude (see [29]).

Hörmander [30] studied a class of interactions of the form (25) containing both a short-range part and a long-range part characterized by real coefficient functions that decrease in the limit $r \to \infty$ as $O(r^{-\varepsilon})$ ($0 < \varepsilon \leq 1$). He formulated certain additional requirements under which scattering theory can be constructed, and as he notes in his monograph [30], ‘the existence of modified wave operators is proved under the weakest sufficient conditions among all those known at the present time’.

Hörmander’s conditions are satisfied by the interaction in the problem of scattering by a purely magnetic vortex,

$$v \sim O(r^{-2}) \quad \text{and} \quad v^j \sim O(r^{-1}), \quad r \to \infty$$

\(^3\) If one considers a set of noncompact simply connected surfaces imbedded in three-dimensional Euclidean space, then $2 \geq \eta > -\infty$ and asymptotically conical surfaces ($1 > \eta > -\infty$), as well as an asymptotically cylindrical surface ($\eta = 1$), have an infinite area, whereas surfaces with $2 \geq \eta > 1$ have a finite area (e.g. a surface with $\eta = 2$ is a sphere with a puncture). A cylindrically symmetric space, where a surface orthogonal to the symmetry axis is asymptotically conical, can be denoted, in general, as a conical space. Thus, such spaces are characterized by $\eta$ from the range $1 > \eta > -\infty$.\(^4\)
(v and $v^j$ are real, and $v^{jj} = 0$), and, for instance, by the interaction in the problem of scattering by a Coulomb centre,

$$v \sim O(r^{-1}), \quad r \to \infty$$

(v is real and $v^j = v^{jj} = 0$). In contrast, the interaction in the problem of scattering by a vortex in conical space ($\Phi \neq 0$, $\eta \neq 0$) does not satisfy Hörmander’s conditions:

$$v \sim O(r^{-2}), \quad v^j \sim O(r^{-1}) \quad \text{and} \quad v^{jj} \sim O(1), \quad r \to \infty,$$

(26)

where $v^j$, in contrast to $v$ and $v^{jj}$, is a complex function (more precisely, the imaginary part of $v^j$ of order $r^{-1}$ is due to the nondecrease of the real quantity $v^{jj}$ in the limit $r \to \infty$). Nevertheless, even in this last case scattering theory can be constructed, and this has been done in [21], based on earlier works [18–20].

According to this theory, the scattering matrix in the wave vector representation is

$$S(k, \varphi'; \varphi) = \frac{1}{2} \frac{\delta(k - k')}{\sqrt{k'k}} e^{2i\kappa(r_c - \xi_c)} \left\{ \Delta \left( \varphi - \varphi' + \frac{\eta \pi}{1 - \eta} \right) \exp \left[ -\frac{i\Phi}{\Phi_0(1 - \eta)} \right] + \Delta \left( \varphi - \varphi' - \frac{\eta \pi}{1 - \eta} \right) \exp \left[ i\Phi \pi \frac{1}{\Phi_0(1 - \eta)} \right] + \delta(k - k') \frac{e^{i\pi/4}}{\sqrt{2\pi k}} \right\}

\left[ W[k, \varphi - \varphi'] + \delta(k - k') \frac{e^{i\pi/4}}{\sqrt{2\pi k}} \right],

(27)

where the final (k) and initial (k') two-dimensional wave vectors of the particle are written in polar variables, and $\xi_c = \int_0^{r_c} ds$ is the geodesic radius of the vortex core. All the delta-functions of angular variables are enclosed in the figure brackets, whereas the transition matrix (the last term in (27)) is free of such delta-functions. Note that in the case of the short-range interaction one has $2\Delta(\varphi - \varphi')$ instead of the figure brackets in (27). Thus, one can see that, due to the long-range nature of the interaction, even the conventional relation between the scattering matrix and the transition matrix is changed, involving now a distorted unity matrix (the first term in (27)) instead of the usual one, $\delta(k - k')\Delta(\varphi - \varphi')(kk')^{-1/2}$.

The transition matrix contains the scattering amplitude ($f$) which is given by the expression

$$f(k, \varphi - \varphi') = e^{i\kappa(r_c - \xi_c)} f_0(k, \varphi - \varphi') + f_c(k, \varphi - \varphi'),

(28)

where

$$f_0(k, \varphi) = -\frac{e^{i\pi/4}}{\sqrt{2\pi k}} \sum_{n \in \mathbb{Z}} \exp[i\eta(\varphi - \pi)] \sin(\alpha_n \pi),

(29)

$$f_c(k, \varphi) = -\exp[2i\kappa(r_c - \xi_c) - i\pi/4] \sqrt{\frac{2}{\pi k}}

\times \sum_{n \in \mathbb{Z}} \exp[i\eta(\varphi - \pi) - i\alpha_n \pi] \frac{W[\sqrt{\xi_c}j_n(\xi_c)k, \sqrt{\xi_c}j_n(\xi_c)k]}{W[\sqrt{\xi_c}j_n(\xi_c)k, \sqrt{\xi_c}h_n^{(1)}(\xi_c)k]}.

(30)

$$\alpha_n = |n - \Phi/\Phi_0|(1 - \eta)^{-1},

(31)

$J_\nu(u)$ and $H^{(1)}_\nu(u)$ are the Bessel and the first-kind Hankel functions of order $\nu$, $\kappa_n(\int_0^{r_c} ds, k)$ is the partial wave solution which is unique and regular inside the vortex core, and the Wronskians of functions $j^{(1)}(\xi_c)$ and $j^{(2)}(r_c)$ is defined as

$$W[j^{(1)}(\xi_c), j^{(2)}(r_c)] = j^{(1)}(\xi_c)[\partial_r j^{(2)}(r)]|_{r = r_c} - [\partial_r j^{(1)}(r)]|_{r = r_c} j^{(2)}(r_c).

It is also instructive to present the $r \to \infty$ asymptotics of the scattering wave solution to the Schrödinger equation (21):

$$w_\nu(u) \sim \frac{1}{u^{1/2}} \left( 1 + \frac{\alpha^2}{2u} + \frac{1}{2} \frac{\alpha^4}{32u^2} + \ldots \right),

(32)

where $\alpha = \sqrt{\lambda + i\mu}$.
\[ \psi(x, k') = (2\pi)^{-1} \sum_l \exp\left[-ikr(1 - \eta)(\phi - \phi' - \pi + 2l\pi)\right] \]
\[ \times \exp\left[i\Phi_{0}\left(\phi - \phi' - \pi + 2l\pi\right)\right] + f(k, \phi - \phi')[2\pi(1 - \eta)\sqrt{r}]^{-1} \exp(ikr) + O(r^{-1}), \]

(32)

where \( x = (r \cos \phi, r \sin \phi), \ k' = (k \cos \phi', k \sin \phi'), \) and the summation is over integer values of \( l \) that satisfy the condition
\[ \frac{\phi' - \phi}{2\pi} - \frac{1}{2}\frac{\eta}{1 - \eta} < l < \frac{\phi' - \phi}{2\pi} + 1 + \frac{1}{2}\frac{\eta}{1 - \eta}. \]

(33)

The scattering amplitude \( f(k, \phi - \phi') \) enters (32) as the factor before the outgoing wave \( [2\pi(1 - \eta)\sqrt{r}]^{-1} \exp(ikr)r^{-1/2} \). In the case of the short-range interaction, the term of order \( O(1) \) is the plane wave \( [2\pi(1 - \eta)\sqrt{r}]^{-1} \exp(ikr) \) which is interpreted as the incident wave. In the case of scattering by a purely magnetic vortex, the incident wave is distorted, differing from the plane wave and taking the form \( [2\pi(1 - \eta)\sqrt{r}]^{-1} \exp[ikr \cos(\phi - \phi')] \exp[i\Phi_{0}^{-1}(\phi - \phi')] \) [13]. When space is conical, the distortion is much stronger and the incident wave is given by the finite sum which is of the order of \( O(1) \) in (32); the distortion of the incident wave in conical space was first obtained in [17, 18].

The independent part of \( r_{c} \) of the scattering amplitude, \( f_{0}(k, \phi) \) (29), is determined by the sum which can exactly be taken [20] yielding
\[ f_{0}(k, \phi) = -\frac{e^{i\pi/4}}{2\sqrt{2\pi}k} \exp\left[i\left[\left[ \frac{\Phi}{\Phi_{0}} \right] \left( \phi + \frac{\eta\pi}{1 - \eta} \right) - \frac{i\Phi\pi}{\Phi_{0}(1 - \eta)} \right] \right] \]
\[ \times \left[ \cot\left(\frac{1}{2}\left(\phi + \frac{\eta\pi}{1 - \eta}\right)\right) + i\right] - \exp\left[i\left[\left[ \frac{\Phi}{\Phi_{0}} \right] \left( \phi - \frac{\eta\pi}{1 - \eta} \right) + \frac{i\Phi\pi}{\Phi_{0}(1 - \eta)} \right] \right] \]
\[ \times \left[ \cot\left(\frac{1}{2}\left(\phi - \frac{\eta\pi}{1 - \eta}\right)\right) + i\right], \]

(34)

where \([u]\) denotes the integer part of quantity \( u \) (i.e. the integer which is less or equal to \( u \)). In the limit \( r_{c} \rightarrow 0 \), one gets \( \xi_{c} \rightarrow 0 \) and \( f_{c}(k, \phi) \rightarrow 0 \), since function \( \kappa_{c} \) is regular, while \( J_{c} \) is vanishing and \( H_{c}^{(1)} \) is divergent at the origin (to be more specific, \( f_{c}(k, \phi) \) decreases at least as \( O|\ln^{-1}(k_{c})| \) at \( r_{c} \rightarrow 0 \) [21]). Thus, \( f_{0}(k, \phi) \) (34) is the amplitude of scattering by an idealized (singular) vortex of zero thickness. The long-range nature of interaction exhibits itself in the divergence of scattering amplitude (34) in two directions which are symmetric with respect to the forward direction, \( \phi = \pm \eta\pi(1 - \eta)^{-1} \); note that the amplitude of scattering by a purely magnetic vortex \((\eta = 0) \) diverges in the forward direction, \( \phi = 0 \) [13].

Since dimensionless quantity \( \sqrt{k}f_{c}(k, \phi) \) depends on \( r_{c} \) through dimensionless product \( kr_{c} \), the limit \( r_{c} \rightarrow 0 \) is the same as the limit \( k \rightarrow 0 \). Therefore, in the long-wavelength limit \((k \rightarrow 0) \), \( f_{c}(k, \phi) \) is negligible as compared to \( f_{0}(k, \phi) \), and the differential cross section in this limit takes the form
\[ \frac{d\sigma}{d\phi} = |f_{0}(k, \phi)|^{2} = \frac{1}{4\pi k} \left[ \frac{1}{2\sin^{2}\left[\frac{1}{2}\left(\phi + \frac{\eta\pi}{1 - \eta}\right)\right]} + \frac{1}{2\sin^{2}\left[\frac{1}{2}\left(\phi - \frac{\eta\pi}{1 - \eta}\right)\right]} \right] \]
\[ - \frac{\cos \left[\left[\frac{2\Phi_{0}}{\Phi_{0}} - (2\left[\phi + \frac{\eta\pi}{1 - \eta}\right] + 1)\frac{\pi}{1 - \eta}\right] \right]}{\sin \left[\frac{1}{2}\left(\phi + \frac{\eta\pi}{1 - \eta}\right)\right] \sin \left[\frac{1}{2}\left(\phi - \frac{\eta\pi}{1 - \eta}\right)\right]} \] .

(35)

In particular, if the vortex flux is equal to an integer multiple of a semifluxon, \( \Phi = \frac{\pi}{2}\Phi_{0} \), then
\[ \frac{d\sigma}{d\phi} = \frac{1}{8\pi k} \left[ \frac{\sin^{2}\left(\frac{\eta\pi}{1 - \eta}\right)}{\sin^{2}\left(\frac{1}{2}\frac{\eta\pi}{1 - \eta}\right) - \sin^{2}\left(\frac{\phi}{2}\right)} \right]^{2} \]

(36)
for even \( n \), and
\[
\frac{d\sigma}{d\varphi} = \frac{1}{2\pi k} \frac{\sin^2 \left( \frac{\varphi}{2} \right) \cos^2 \left( \frac{1}{2} \frac{n\pi}{1 - \eta} \right)}{\sin^2 \left( \frac{1}{2} \frac{n\pi}{1 - \eta} \right) - \sin^2 \left( \frac{\varphi}{2} \right)}
\]  
(37)

for odd \( n \). Cross section (36) coincides with the cross section in the case of zero flux, which was first obtained in [18]. Note that cross section (37) vanishes in the forward direction, \( \varphi = 0 \).

Due to the divergence of the differential cross section at \( \varphi = \pm \eta \pi (1 - \eta)^{-1} \), the total cross section,
\[
\sigma_{\text{tot}} = \int_0^{2\pi} d\varphi \frac{d\sigma}{d\varphi},
\]

is infinite in the long-wavelength limit.

The effects of the finite thickness of the vortex core become important at shorter wavelengths of the scattered particle.

4. Scattering of a short-wavelength particle

The dependent part on \( r_c \) of the scattering amplitude, \( f_c(k, \varphi) \) (30), is determined by the structure of the vortex core, i.e. by the distribution of the gauge field strength and the Higgs field inside the core; note that, unlike \( f_0(k, \varphi) \), \( f_c(k, \varphi) \) is smooth and infinitely differentiable function of \( \varphi \). Since our topic is the Aharonov–Bohm effect, we would like to make the region of the core inaccessible for the quantum-mechanical particle, and, hence, we impose a boundary condition on the wavefunction at the edge of the core. In the context of the Aharonov–Bohm effect, the Dirichlet boundary condition is mostly used,
\[
\psi \big|_{r=r_c} = 0,
\]

(38)
i.e. it is assumed that quantum-mechanical particles are perfectly reflected from the vortex core.

If condition (38) is imposed, then one gets
\[
f_c(k, \varphi) = -\exp\left[2ik(r_c - \xi_c) - i\pi/4\right] \sqrt{\frac{2}{\pi k}} \sum_{n \in \mathbb{Z}} \exp\left[i(n\varphi - \pi) - i\alpha_n\pi\right] \frac{J_{\alpha_n}(kr_c)}{H_{\alpha_n}(kr_c)}.
\]

(39)

Unlike the case of \( f_0(k, \varphi) \) (29), the infinite sum in (39) cannot be calculated explicitly. However, the summation can be performed in the case \( kr_c \gg 1 \) yielding [21]
\[
f_c(k, \varphi) = -\exp\left[2ik(r_c - \xi_c)(1 - \eta)\sqrt{r_c/2}\right] \sum_{l \in \mathbb{Z}} \cos \left[\frac{1}{2}(1 - \eta)(\varphi - \pi + 2l\pi)\right]
\]
\[
\times \exp \left\{i\Phi^{-1}(\varphi - \pi + 2l\pi) - 2ikr_c \cos \left[\frac{1}{2}(1 - \eta)(\varphi - \pi + 2l\pi)\right]\right\},
\]

(40)

where the finite sum is over integers \( l \) that satisfy the condition (compare with (33))
\[
-\frac{\varphi}{2\pi} - \frac{1}{2} \frac{1 - \eta}{1 - \eta} < l < -\frac{\varphi}{2\pi} + 1 + \frac{1}{2} \frac{\eta}{1 - \eta},
\]

(41)

and terms of the order of \( \sqrt{r_c} O\left[(kr_c)^{-1/6}\right] \) and smaller are neglected.

Since \( f_0(k, \varphi) \) is proportional to \( k^{-1/2} \) (34), the differential cross section in the short-wavelength limit, \( kr_c \gg 1 \), is given by
\[
\frac{d\sigma}{d\varphi} = |f_c(k, \varphi)|^2,
\]

(42)

where \( f_c(k, \varphi) \) is given by (40).
Figure 1. Classical trajectories of scattered particles and scattering angle: $\omega_0 = -\frac{\eta}{2\pi} \pi$ at $-\infty < \eta < 0$ and $\omega_0 = (2n - \frac{\eta}{2\pi}) \pi$ at $\frac{2n\pi}{2n+1} < \eta < \frac{2n\pi}{2n+1} (n \geq 1)$.

Figure 2. Classical trajectories of scattered particles and scattering angle: $\omega_0 = \frac{\eta}{2\pi} \pi$ at $0 < \eta < \frac{1}{2}$ and $\omega_0 = (\frac{\eta}{2\pi} - 2n) \pi$ at $\frac{2n\pi}{2n+2} < \eta < \frac{2n\pi}{2n+2} (n \geq 1)$.

In the case of a purely magnetic vortex ($\eta = 0$) there is only one term ($l = 0$) in the sum in (40), and therefore the dependence on the vortex flux disappears in the cross section
\[ \frac{d\sigma}{d\varphi} = \frac{1}{2} r_c \sin \frac{\varphi}{2} \quad (0 < \varphi < 2\pi), \] (43)
which is the cross section for scattering of a classical point particle by an impenetrable cylindrical shell of radius $r_c$, see p 1381 of [31]. This result is easy to understand, since the short-wavelength limit, $k \to \infty$, can be regarded as the classical limit, $\hbar \to 0$, in view of the relation $k = \text{momentum}/\hbar$.

However, in the case of a vortex in conical space the dependence on the vortex flux survives in the cross section in the short-wavelength limit, if the number of terms in the sum in (40) is more than 1. Before analysing the situation with this number, let us make a digression concerning scattering in classical mechanics.

If the vortex core is impenetrable for a classical point particle, then its scattering does not depend on the vortex flux and is purely kinematic, if the thickness of the vortex core is neglected. There is no scattering in coordinates $\tilde{r}, \tilde{\varphi}$ (see (17)), and, going over to the angular variable $\varphi$, one gets classical trajectories depicted in figures 1 and 2, where the vortex is directed perpendicular to the plane of the figure and its position is indicated by the dot. The scattering angle is independent of the impact parameter and is equal to $\omega_0$ or $-\omega_0$ ($0 \leq \omega_0 \leq \pi$) depending on the side from which the particle approaches the vortex. Depending on the value of $\eta$, the trajectories either do not intersect (figure 1) or do intersect (figure 2); the value of $\omega_0$ itself depends on $\eta$. The region of angles $-\omega_0 < \varphi < \omega_0$ in figure 1 may be denoted as the region of shadow (no objects from this region can be seen by an observer to the left of the vortex). The region of angles $-\omega_0 < \varphi < \omega_0$ in figure 2 may be denoted as the region of double image (every object from this region has double image for an observer to the left of the vortex).

Returning to quantum-mechanical scattering, we note that the number of terms in the distorted incident wave in (32) is equal to the number of finite terms in the scattering amplitude in the short-wavelength limit (40). This number denoted by $n_l$ in the following is even in the
region of classical shadow or classical double image and is odd otherwise. Moreover, the value of \( n_l \) outside the shadow is larger by 1 than that in the shadow, whereas the value of \( n_l \) outside the double-image region is smaller by 1 than that in the double-image region. To be more precise, in the case \(-\infty < \eta < 0\), we have \( n_l = 1 \) outside the shadow and \( n_l = 0 \) in the shadow (the main contribution is decreasing as \( \sqrt{r_c} O[(kr_c)^{-1/3}] \)); in the case \( 0 < \eta < 1/2 \), we have \( n_l = 1 \) outside the double-image region and \( n_l = 2 \) in the double-image region; in the case \( 1/2 < \eta < 2/3 \), we have \( n_l = 3 \) outside the shadow and \( n_l = 2 \) in the shadow; in the case \( 2/3 < \eta < 3/4 \), we have \( n_l = 3 \) outside the double-image region and \( n_l = 4 \) in the double-image region and so on with increasing values of \( n_l \). In particular, in the case \( 0 < \eta < 1/2 \), which is most relevant for the cosmic string phenomenology, we obtain the differential cross section in the short-wavelength limit:

\[
\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi} = \frac{r_c}{2}(1 - \eta)^2 \cos \left( \frac{1}{2}(1 - \eta)(\varphi - \pi) \right), \quad \frac{\eta\pi}{1 - \eta} < \varphi < 2\pi - \frac{\eta\pi}{1 - \eta}, \tag{44}
\]

and

\[
\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi} = r_c(1 - \eta)^2 \left\{ \cos \left( \frac{1}{2}(1 - \eta)\varphi \right) \sin \left( \frac{1}{2}\eta\pi \right) + \sqrt{\sin^2 \left( \frac{1}{2}\eta\pi \right) - \sin^2 \left( \frac{1}{2}(1 - \eta)\varphi \right)} \right\}
\times \cos \left[ 2\pi \Phi \Phi_0^{-1} + 4kr_c \sin \left( \frac{1}{2}(1 - \eta)\varphi \right) \cos \left( \frac{1}{2}\eta\pi \right) \right], \quad \frac{\eta\pi}{1 - \eta} < \varphi < \frac{\eta\pi}{1 - \eta}, \tag{45}
\]

In the strictly forward direction, we get

\[
\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi} = 2r_c(1 - \eta)^2 \sin \left( \frac{1}{2}\eta\pi \right) \cos^2 \left( \Phi \Phi_0^{-1} \pi \right), \quad \varphi = 0. \tag{46}
\]

In particular, if the vortex flux is equal to an integer multiple of a semifluxon, \( \Phi = \frac{n}{2}\Phi_0 \), then (45) takes the form

\[
\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi} = r_c(1 - \eta)^2 \left\{ \cos \left( \frac{1}{2}(1 - \eta)\varphi \right) \sin \left( \frac{1}{2}\eta\pi \right) \pm \sqrt{\sin^2 \left( \frac{1}{2}\eta\pi \right) - \sin^2 \left( \frac{1}{2}(1 - \eta)\varphi \right)} \right\}
\times \cos \left[ 4kr_c \sin \left( \frac{1}{2}(1 - \eta)\varphi \right) \cos \left( \frac{1}{2}\eta\pi \right) \right], \quad \frac{\eta\pi}{1 - \eta} < \varphi < \frac{\eta\pi}{1 - \eta}, \tag{47}
\]

where the upper (lower) sign corresponds to even (odd) \( n \).

It should be noted that the total cross section is independent of the vortex flux and finite in the short-wavelength limit:

\[
\sigma_{\text{tot}} = 2r_c(1 - \eta), \tag{48}
\]

where the last relation is also valid in the general case \( 1 > \eta > -\infty \); the decreasing terms of order \( r_c O[(kr_c)^{-1/3}] \) and smaller are omitted in (48), as well as in (44)–(47).

5. Discussion

Usually, the effects of non-Euclidean geometry are identified with the effects which are due to the curvature of space. This is not the case in general, and there are spaces which are flat almost everywhere but non-Euclidean even in the vast region where they are locally flat; this gives rise to non-Euclidean effects in such a region. Conical space remains to be non-Euclidean in
the whole even if the transverse size of the region of nonzero curvature is shrunk to zero. The non-Euclidean effect in this locally flat space is that the space serves as a lens for propagating beams of light. Two parallel beams after bypassing the axis of spatial symmetry from different sides either converge (and intersect), or diverge, depending on the value of deficit angle $2\pi \eta$. It is evident that conical space serves as a concave lens in the case of negative values of the deficit angle and as a convex lens in the case of some bounded positive values of the deficit angle ($0 < \eta < 1/2$); it is less evident, although it is true, that, as the deficit angle grows further ($1/2 < \eta < 1$), conical space serves in turn as a concave and as a convex lenses, see figures 1 and 2.

Conical space per se is worth of interest, and, moreover, the attention to this subject is augmented by the fact that conical space emerges inevitably as an outer space of any topological defect which is characterized by the nontrivial first homotopy group. Although the ANO vortices yielding a noticeable amount of the deficit angle have yet to be found, all hypothetical possibilities in theory should be elaborated.

Bearing the above in mind, we consider the quantum-mechanical scattering of a nonrelativistic particle by an ANO vortex. This is the most general extension of the scattering Aharonov–Bohm effect to the case when space outside a magnetic vortex is conical. From the aspect of scattering theory, this corresponds to a situation when the interaction with a scattering centre is not of the potential type, see (25), and is even nondecreasing at large distances from the centre, see (26). Such a fairly strong interaction violates the conditions which are needed according to H"ormander [30] for constructing scattering theory in the case of the long-range interaction. Despite this fact, a comprehensive scattering theory has been constructed [21], and we rely mostly on the results of this work.

The long-range nature of the interaction reveals itself in the divergence of the scattering amplitude in the long-wavelength limit in two directions which are symmetric with respect to the forward direction, see (34); this should be compared with the case of scattering by a magnetic vortex in Euclidean space, when the amplitude diverges in one, forward, direction [13]. The long-range nature of the interaction is also revealed in the distortion of the unity matrix in the relation between the S-matrix and the scattering amplitude, see (27), and in the distortion of the incident wave in the asymptotics of the scattering wavefunction, see (32).

Both in the cases of Euclidean and conical spaces, the differential cross section is a periodic function of the vortex flux with the period equal to the London flux quantum. A peculiarity of conical space is that the cross section vanishes in the forward direction, if the vortex flux equals half of the London flux quantum, see (37) at $\varphi = 0$ for the long-wavelength limit and (46) at $\Phi = \frac{1}{4} \Phi_0$ for the short-wavelength limit.

In the present paper we have considered scattering of a nonrelativistic spinless particle. For particles with spin, the appropriate spin connections which are dependent on $\eta$ should be introduced in Hamiltonian (23). Thus, unlike scattering by an impenetrable vortex in Euclidean space, such a scattering in conical space depends on the spin of a scattered nonrelativistic particle. In particular, for a spin-1/2 particle all results of the present paper are modified in the following way: one should change $\Phi \Phi_0^{1/2}$ to $\Phi \Phi_0^{1/2} \mp 1/2 \eta$, where two signs correspond to two spin states which are defined by projections of spin on the vortex axis.

As the wavelength of a scattered particle increases, the effects of the core structure of the vortex die out, and the differential cross section becomes independent of the transverse size of the core, see (35). Evidently, the long-wavelength limit corresponds to the extreme quantum limit, when the wave aspects of matter are exposed to the maximal extent.

As the wavelength of a scattered particle decreases, the effects of the core structure of the vortex become prevailing. The short-wavelength limit corresponds to the classical limit, when the wave aspects of matter are suppressed in favour of the corpuscular ones. Therefore, one
would anticipate that, provided the vortex core is made impenetrable for a scattered particle, the cross section becomes independent of the vortex flux in this limit.

This anticipation is confirmed for the total cross section (48) which corresponds to the classical expression that can simply be interpreted as the quotient of the circumference of the core to the half of the complete angle (note that only half of the core edge is exposed to incident particles). However, this anticipation is overturned for the differential cross section which remains to be dependent on the vortex flux, see (45) for the case \(0 < \eta < 1/2\). For instance, if one considers a cosmic string corresponding to the ANO vortex in the type-I superconductor, then the vortex radius is equal to the correlation length, \(r_c = r_H\); the vortex flux is equal to an integer multiple of a semifluxon, \(\Phi = n_2 \Phi_0\), and the differential cross section for the forward scattering of a short-wavelength spinless particle by such a cosmic string is (see (46) at small \(\eta\))

\[
\frac{d\sigma}{d\phi} = 4\pi l_P^2 r_H^{-1}, \quad \text{even } n. \tag{49}
\]
\[
\frac{d\sigma}{d\phi} = 0, \quad \text{odd } n. \tag{50}
\]

For a spin-1/2 particle the result is the same at even \(n\) only, and differs from zero otherwise:

\[
\frac{d\sigma}{d\phi} = 16\pi^3 l_P^6 r_H^{-5}, \quad \text{odd } n. \tag{51}
\]

This should be compared with the case \(\eta = 0\) when differential cross section (43) vanishes in the forward direction, irrespective of the value of the particle spin and the value of the vortex flux.

The reason of the discrepancy between the short-wavelength and classical limits lies in expression (40) which yields nondecreasing at \(kr_c \gg 1\) terms in the scattering amplitude; each term contains the vortex flux in its phase and only there. The number of these terms is determined by (41) and coincides with the number of terms in the distorted incident wave in (32). This number is 1 at \(\eta = 0\) and at \(\eta < 0\) (out of the region of classical shadow), and, thence, the absolute value of the amplitude is independent of the vortex flux in this case. The number is more than 1 at \(0 < \eta < 1\) (at \(0 < \eta < 1/2\) in the region of classical double image), and, thence, due to the interference of different terms, the periodic dependence on the vortex flux survives in the short-wavelength limit in the absolute value of the amplitude.

Although the given explanation is quite comprehensive, the simple and transparent physical arguments, in our opinion, are lacking. The Aharonov–Bohm effect, i.e. the periodic dependence on the flux of the impenetrable vortex, is due to the nontrivial topology of space with the excluded vortex region; this topology is the same both for Euclidean space and for conical space with either positive or negative deficit angle. Classical and quantum-mechanical motion depends on the spatial geometry, i.e. on the value of the deficit angle, and this is of no surprise. A real puzzle is: why conical space with positive deficit angle distinguishes itself by the discrepancy between the short-wavelength and classical limits?

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