Rigidity of conformal minimal two-spheres in a compact symmetric space

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Abstract: In this paper we studied rigidity of submanifolds, rigidity is an important property of submanifolds in differential geometry. Then we mainly proved that any conformal minimal two-sphere immersed in a compact symmetric space N is determined, up to an isometry of N, by its first and second fundamental forms.

1. Introduction
Theory of submanifolds is one of the most interesting objects in differential geometry. Rigidity of submanifolds is an important classification of submanifolds. A classical rigidity theorem in differential geometry states that a surface in Euclidean 3 space is locally determined up to congruence by its first and second fundamental forms. There are many generalizations and variations of this result. In this paper, we will hypothesize that ambient space is a compact symmetric space N and study rigidity of a conformal minimal sphere immersed in a compact symmetric space.

We know that global and local rigidities of submanifolds in some special symmetric spaces were studied, for example, complex projective spaces CP^n and complex Grassmann manifolds Gk,n (1 < k < n − 1).

For complex projective space, any holomorphic curve is locally (or globally) determined by its first fundamental form, up to a rigid motion (cf. [4][9]); Furthermore, any super-minimal surface (or conformal minimal sphere) in a complex projective space is locally (or globally) determined by its Gauss curvature and kähler angle, up to a rigid motion (cf.[1][6]).

For complex Grassmann manifold Gk,n (1 < k < n − 1), a holomorphic curve does not be determined by its first fundamental form (cf. [7]).

Since any compact symmetric space may be totally geodesically immersed in its group of isometries (Cartan embedding), we can use theory of Lie groups to study submanifolds in compact symmetric space, especially uniqueness and existence questions of submanifolds.

We will prove a global rigidity theorem of a conformal minimal two-sphere in a compact symmetric space as follows.

Theorem. Any conformal minimal two-sphere immersed in a compact symmetric space N is determined, up to an isometry of N, by its first and second fundamental forms.

2. Preliminaries
Let N be a compact Riemann symmetric space, and let G be the isometry group of N with Lie algebra g and bi-invariant metric ⟨·,·⟩. We fix a base point x ∈ N. Let s ∈ G be the involutive isometry with isolated fixed point x. Then the map τ : G → G determined by s as follows:

\[ g \mapsto sgs \]
for all $g \in G$ is an involution of $G$, which is the *involution* at $x$ (cf. [10]).

Let $H$ be the isotropy group of $x$ with Lie algebra $\mathfrak{q}$. Then $(G^c)_0 \subset H \subset G^c$ (cf. [2]) and we have a symmetric decomposition of $g$ as follows:

$$g = p \oplus q$$

into ±1-eigenspaces of $d\tau$ satisfying the following relations

$$[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}, \quad [\mathfrak{q}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{q},$$

where $[\cdot, \cdot]$ is Lie bracket. Moreover the map $g \to T_xN$ given by

$$X \mapsto \frac{d(\exp TX_x)}{dt}\bigg|_{t=0}$$

is an isomorphism on $p$. Inverting this at each point gives a $g$-valued 1-form on $N$, which provides an isomorphism of $TN$ with a subbundle $[\mathfrak{p}]$ of the trivial bundle $N \times g$ (cf. [3]).

Define a map $i : N \to G$ given by

$$i(\xi \cdot x) = gsg^{-1}$$

which is well defined. Then $i$ is a totally geodesic immersion (cf. [5]). Moreover $i(N)$ is a globally Riemannian symmetric space that is totally geodesically embedded in $G$ (cf.[2][10]). Set

$$\tilde{N} = \{gsg^{-1} \mid g \in G \}.$$

Then $i(N) = \tilde{N}$.

Let $M$ be a 2-dimensional Riemannian manifold, and let $\varphi : M \to G$ be a smooth map. Then $\varphi$ is harmonic if and only if $\varphi$ satisfies the following equations:

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0, \quad d^*\alpha = 0, \quad (1)$$

where $\alpha = \varphi^* \theta$ and $\theta$ is the left Maurer-Cartan form of $G$.

The first equation of (1) is called the Maurer-Cartan equation, which shows that the connection $d + \alpha$ on the trivial principal bundle $M \times G$ is flat; The second equation of (1) shows that $\alpha$ is co-closed (cf. [11]).

Introducing a local complex coordinate $z = x + \sqrt{-1}y$ and $\overline{z} = x - \sqrt{-1}y$ on $M$. Set

$$A_z = \frac{1}{2} \varphi^{-1} \overline{\partial} \varphi$$

and $A_{\overline{z}} = \frac{1}{2} \varphi^{-1} \partial \varphi$, where $\partial = \frac{\partial}{\partial x}$ and $\overline{\partial} = \frac{\partial}{\partial \overline{y}}$. Then $\alpha = 2A_z dz + 2A_{\overline{z}} d\overline{z}$, and (1) is equivalent to (cf. [12])

$$\partial A_z = [A_z, A_z], \quad \overline{\partial} A_{\overline{z}} = [A_z, A_{\overline{z}}], \quad (2)$$

where $2A_z dz$ is a $G^C$-valued (1,0) form and $2A_{\overline{z}} d\overline{z} = 2A_z d\overline{z}$.

Extending the metric of $G$ to $G^C$ by usual method, for convenience, we still denote the metric of $G^C$ by $\langle \cdot, \cdot \rangle$. Then the induced metric $ds^2$ by $\varphi$ on $M$ is given by

$$ds^2 = 4 < A_z, A_z > dz d\overline{z} + 8 < A_z, A_{\overline{z}} > dz d\overline{z} + 4 < A_{\overline{z}}, A_{\overline{z}} > d\overline{z} d\overline{z}. \quad (3)$$

From (3) it follows that $\varphi$ is conformal if and only if $< A_z, A_z > = < A_{\overline{z}}, A_{\overline{z}} > = 0$. Note the following facts:

(i) If $M$ is a Riemann sphere and $\varphi$ is harmonic, then $\varphi$ is conformal;

(ii) If $\varphi$ is a conformal immersion, then $\varphi$ is harmonic if and only if $\varphi$ is minimal.
Now we always assume that $\varphi : M \to G$ is a conformal minimal immersion. If we identify $\varphi^{-1}\tau G$ with $M \times g$ via $\alpha$, then the pullback of the Levi-Civita connection on $G$ is given by

$$DY = dY + \frac{1}{2}[\alpha, Y]$$

for all $Y \in \Gamma(M \times g)$. Let $\nabla$ be the induced connection on $M$, and let $\sigma$ be the second fundamental form of $\varphi$. Then

$$\sigma(X, Y) = D_X(\alpha(Y)) - \alpha(\nabla_X Y) \quad (4)$$

for all $X, Y \in TM$.

Set $ds^2 = 2Fdzd\overline{z}$, where $F = 4 < A_x, A_z >$. Then by a simple computation we get

$$\nabla \overline{\beta} = (\overline{\partial} \log F) \overline{\partial}, \quad \nabla \beta = (\partial \log F) \partial, \quad \nabla \beta = \nabla \overline{\beta} = 0.$$

Hence by a straightforward computation the second fundamental form $\sigma$ of conformal minimal immersion $\varphi$ is given by

$$\sigma^C = 2F\overline{\partial}(A_z/F)dzd\overline{z} + 2F\overline{\partial}(A_x/F)d\overline{z}d\overline{z}. \quad (5)$$

where $\sigma^C$ is the complexification of $\sigma$.

### 3. Rigid Theorem for Conformal Minimal Spheres

Let $\psi : M \to \mathcal{N}$ be a smooth map, and let $i \circ \psi = \varphi$. Then $\psi$ is a conformal minimal immersion if and only if $\varphi : M \to \tilde{\mathcal{N}} \subset G$ is a conformal minimal immersion (cf. [8]).

Set $\varphi = gsg^{-1}$, where $g : M \to G$. By a symmetric decomposition of $g$, we have

$$g^{-1}d g = \theta_1 \oplus \theta^{-1},$$

where $\theta_1 : TM \to \mathfrak{q}$ and $\theta^{-1} : TM \to \mathfrak{p}$. Then we have the following proposition:

**Proposition 3.1.** Let $\varphi : M \to \tilde{\mathcal{N}} \subset G$ be a conformal minimal immersion. Then

$$\varphi = gsg^{-1} \quad \text{and} \quad \alpha = -2g\theta^{-1}g^{-1}.$$  

**Proof.** By $\varphi = gsg^{-1}$ we have

$$\alpha = g(s\theta s - \theta)g^{-1}.$$  

The result is obtain from $s\theta s = \theta_1$ and $s\theta^{-1} = -\theta^{-1}$. Q.E.D.

Given another conformal minimal immersion $\tilde{\psi} : M \to \tilde{\mathcal{N}} \subset G$ and $i \circ \tilde{\psi} = \tilde{\varphi}$. Set

$$\tilde{\varphi} = fsf^{-1}, \quad \tilde{\alpha} = \varphi^{-1}d\tilde{\alpha} \quad \text{and} \quad \tilde{\theta} = f^{-1}d f \quad \text{etc.}$$

**Proposition 3.2.** Let $\psi(m_0) = \tilde{\psi}(m_0) = x$ for fixed $m_0 \in M$. Then $\psi$ and $\tilde{\psi}$ are congruent, up to an isometry of $\mathcal{N}$, if and only if there exists a fixed element $\Omega \in H$ such that $\varphi = \Omega \tilde{\varphi} \Omega^{-1}$

**Proof.** By $\varphi = gsg^{-1}$ and $\tilde{\varphi} = fsf^{-1}$ we have

$$\psi(m) = g(m) \cdot x, \quad \tilde{\psi}(m) = f(m) \cdot x$$

for all $m \in M$.

If $\psi$ and $\tilde{\psi}$ are congruent, up to an isometry of $\mathcal{N}$, then there exists a fixed element $\Omega \in G$ such that

$$\psi = \Omega \cdot \tilde{\psi}.$$  

This shows that $g = \Omega f \ (\mod H)$, i.e. $\varphi = \Omega \tilde{\varphi} \Omega^{-1}$. Because of $g(m_0) \cdot x = f(m_0) \cdot x = x$, so $g(m_0), f(m_0) \in H$. Hence $\Omega \in H$.
Conversely, if there exists a fixed element \( \Omega \in G \) such that \( \varphi = \Omega \tilde{\varphi} \Omega^{-1} \). Then we have \( \psi = \Omega \cdot \tilde{\psi} \), i.e. \( \psi \) and \( \tilde{\psi} \) are congruent, up to an isometry of \( N \).

**Remark.** Proposition 3.2 shows that if two conformal minimal surfaces \( \psi \) and \( \tilde{\psi} \) with \( \psi(m_0) = \tilde{\psi}(m_0) = x \) at a base point \( m_0 \) are congruent, up to an isometry of \( N \), then \( \psi^* ds^2 = \tilde{\psi}^* ds^2 \) and \( \sigma_{\psi} = \Omega \sigma_{\tilde{\psi}} \Omega^{-1} \) for some fixed element \( \Omega \in H \), where \( \sigma_{\psi} \) and \( \sigma_{\tilde{\psi}} \) are the second fundamental forms of \( \varphi \) and \( \tilde{\varphi} \) respectively.

Let \( \varphi \) and \( \tilde{\varphi} \) be two conformal minimal surfaces immersed in a symmetric space \( N \) with \( \psi(m_0) = \tilde{\psi}(m_0) = x \) at a base point \( m_0 \). We say that \( \psi \) and \( \tilde{\psi} \) have the same second fundamental form if \( \sigma_{\psi} = \Omega \sigma_{\tilde{\psi}} \Omega^{-1} \) for some fixed element \( \Omega \in H \).

Are they congruent, up to an isometry of \( N \), if \( \psi \) and \( \tilde{\psi} \) have the same first and second fundamental forms? In the following section we will answer this problem.

We need the following lemma.

**Lemma 3.3.** ([9]) Let \( f, \tilde{f} : M \rightarrow G \) be two smooth maps of a connected manifold \( M \) into Lie group \( G \). Then \( f(m) = q \tilde{f}(m) \) for all \( m \in M \) and fixed \( q \in G \) if and only if \( f^* \theta = \tilde{f}^* \theta \), where \( \theta \) is the Maurer-Cartan form on \( G \).

Now we prove the following rigid theorem on conformal minimal spheres in a symmetric space.

**Theorem 3.4.** Two conformal minimal spheres immersed in a compact symmetric space \( N \) with the same first and second fundamental forms are congruent, up to an isometry of \( N \).

Proof. Let \( \psi \) and \( \tilde{\psi} \) be two conformal minimal spheres immersed in \( N \) with the same first and second fundamental forms, and let \( i \circ \psi = \varphi = g s g : M \rightarrow \tilde{N} \subset G \) and \( i \circ \tilde{\psi} = \tilde{\varphi} = f s f : M \rightarrow \tilde{N} \subset G \).

Let \( \psi(m_0) = \tilde{\psi}(m_0) = x \) at fixed \( m_0 \in M \). Since \( \varphi(m_0) = \tilde{\varphi}(m_0) = s, f(m_0), g(m_0) \in H \).

Without loss of generality, we may assume that \( f(m_0) = g(m_0) = I \). Otherwise, if \( g(m_0) \neq I \), then we consider that conformal minimal immersion \( g(m_0)^{-1} \varphi g(m_0) \), which is congruent with \( \varphi \). By Proposition 3.2 we only need to prove that there exists a fixed element \( q \in H \) such that \( \varphi = q \tilde{\varphi} q^{-1} \).

Set \( \alpha = \varphi^* \theta = 2 A z dz + 2 A \bar{z} d\bar{z} \) and \( \tilde{\alpha} = \tilde{\varphi}^* \tilde{\theta} = 2 B z dz + 2 B \bar{z} d\bar{z} \). Since they have the same first and second fundamental forms, \( F = \tilde{F} \) and there exist a fixed element \( q \in H \) such that

\[
F \bar{\partial}(A_z / F) = q F \bar{\partial}(B_z / \tilde{F}) q^{-1}, \quad F \bar{\partial}(A_{\bar{z}} / F) = q \bar{F} \bar{\partial}(B_{\bar{z}} / \tilde{F}) q^{-1}.
\]

Hence we have

\[
A_z = q B_z q^{-1} + C(z) F,
\]

Where \( C : M \rightarrow g^C \) is an anti-holomorphic map in \( z \).

Since \( M \) is a sphere, then \( C(z) = c \) (constant map). Therefore we have \( A_z = q B_z q^{-1} + c F \).

Namely

\[
\alpha = q \tilde{\alpha} q^{-1} + 2 c F dz + 2 \bar{c} F d\bar{z}
\]

In particular, by Proposition 3.1 at \( m_0 \) we have
\[ \alpha(m_0) = -2\partial_\perp(m_0), \quad \tilde{\alpha}(m_0) = -2\tilde{\partial}_\perp(m_0). \]

namely, \( \alpha(m_0) \) and \( \tilde{\alpha}(m_0) \) are \( \mathbf{p}^c \)-valued forms. Hence \( cdz + \overline{c}d\overline{z} \) is a \( \mathbf{p}^c \)-valued form, i.e.
\[ c = (A_z(m_0) - qB_z(m_0)q^{-1})/F(m_0), \quad \overline{c} = (A_z(m_0) - qB_z(m_0)q^{-1})/F(m_0) \in \mathbf{p}^c. \]
Since \( A_z(m_0) \) and \( A_z(m_0) (qB_z(m_0)q^{-1}) \) and \( qB_z(m_0)q^{-1} \) are mutually orthogonal and they have the same length, \( c = \overline{c} = 0 \) or \( c \) and \( \overline{c} \) are linearly independent.

On the other hand, from (2) and (6) it follows that
\[ \overline{c}\overline{\partial}F = \partial A_z - q\overline{\partial}B_zq^{-1}, \quad c\overline{\partial}F = \overline{\partial}A_z - q\overline{\partial}B_zq^{-1}. \]

This shows that \( c\overline{\partial}F + \overline{c}\partial F = 0 \). If \( c \neq 0 \), then \( \partial F = \overline{\partial}F = 0 \), which implies that \( F \) is constant everywhere on \( M \) and curvature of \( M \) equals zero. This contradicts with the Gauss-Bonnet-Chern formula. Therefore \( c = 0 \) and it follows that
\[ \alpha = q\tilde{\alpha}q^{-1}. \]

which implies that \( \varphi = q_0q\tilde{\varphi}q^{-1} \) by Lemma 3.3.

Because \( \varphi(m_0) = \tilde{\varphi}(m_0) = s \) for fixed \( q_0 \in G \), so \( q_0 = I \).

Q.E.D

Remark. Local rigidity theorem of a holomorphic curve in \( G_{n,2n} \) was proved by Griffiths (cf. [9]).

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