Diagonal Form of the Varchenko Matrices

Yibo Gao, YiYu Zhang

Abstract

Varchenko [7] defined the Varchenko matrix associated to any real hyperplane arrangement and computed its determinant. The definition extends straightforwardly to pseudosphere arrangements and thus oriented matroids. Here we will show by explicit construction and proof by contradiction that the Varchenko matrix of a pseudosphere arrangement has a diagonal form if and only if it has no degeneracy.

1 Introduction

Varchenko defined the Varchenko matrix associated with any real hyperplane arrangement in [7] and computed its determinant, which has a very nice factorization. Naturally, one may ask about its Smith normal form or diagonal form over some integer polynomial ring. The Smith normal forms of the $q$-Varchenko matrices for certain types of hyperplane arrangements were first studied by Denham and Hanlon in [4] and more recently by Cai and Mu in [3].

In this paper, we prove that the Varchenko matrix of a real hyperplane arrangement has a diagonal form if and only if the arrangement is semigeneral. Furthermore, both the definition of the Varchenko matrix and the above result extend straightforwardly to pseudosphere arrangements, which also correspond to oriented matroids through the Topological Representation Theorem of Folkman and Lawrence (see [1]).

We define pseudosphere arrangements and the associated Varchenko matrices in section 2. In section 3, we use combinatorial techniques and matrix operations to explicitly construct a diagonal form of the Varchenko matrix associated with any semigeneral pseudosphere arrangement. We prove by contradiction that the Varchenko matrix of any arrangement with degeneracy does not have a diagonal form in section 4.

It follows immediately that while the $q$-Varchenko matrix of any semigeneral arrangement has a Smith normal form, the corresponding Varchenko matrix doesn’t in general. Besides, our construction serves as an alternative proof for a special case, i.e., that of real semigeneral hyperplane arrangements, of Varchenko’s theorem on the determinant of the Varchenko matrix.

2 Preliminaries

In this paper, we will mostly follow the notation in [6].

2.1 Pseudosphere arrangements

Definition 2.1. Let $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : ||x|| = 1\}$ be the unit sphere in $\mathbb{R}^{d+1}$. We say that $S \subseteq \mathbb{S}^d$ is a pseudosphere if there exists a homeomorphism $H : \mathbb{S}^d \to \mathbb{S}^d$ such that $S = H(S^{d-1})$, where $S^{d-1} = \{x \in \mathbb{S}^d : x_{d+1} = 0\}$. The sphere $\mathbb{S}^d$ is divided by $S$ into two sides (closed and connected hemispheres) $S^+$ and $S^-$ with $S^+ \cap S^- = S$.

Let $I = \{1, 2, \ldots, n\}$. A (real finite) pseudosphere arrangement $\mathcal{A} = (S_i)_{i \in I}$ is a set of pseudospheres in $\mathbb{S}^d$ such that for all $B \subseteq I$:

a) The intersection $S_B = \bigcap_{i \in B} S_i$ is either empty or homeomorphic to a sphere of some dimension;

b) If $S_B \neq \emptyset$, then for all $S_i \in \mathcal{A}$ such that $S_B \nsubseteq S_i$, the intersection $S_B \cap S_i$ is a pseudosphere in $S_B$.

Furthermore, $S_B \cap S_i$ divides $S_B$ into two sides $S_B \cap S^+_i$ and $S_B \cap S^-_i$.

In this paper we only work with real finite pseudosphere arrangement.

We say that $\mathcal{A}$ is a signed pseudosphere arrangement if, in addition, we designate the positive and negative sides of each $S_i \in \mathcal{A}$.
Define the dimension of the intersection \( S_B = \bigcap_{i \in B} S_i \) to be the dimension of the sphere homeomorphic to \( S_B \).

If \( B \subseteq E \) implies \( \dim(S_B) = d - |B| \) for all \( S_B \neq \emptyset \), then we say that \( \mathcal{A} \) is a semigeneral arrangement (or \( \mathcal{A} \) is in semigeneral position) in \( \mathbb{S}^d \). In particular, we call \( \mathcal{A} \) a general arrangement (or \( \mathcal{A} \) is in general position) in \( \mathbb{S}^d \) if \( B \subseteq E \), \( |B| \leq d \) implies \( \dim(S_B) = d - |B| \) and \( B \subseteq E \), \( |B| > d \) implies \( S_B = \emptyset \).

**Definition 2.2.** Let \( L(\mathcal{A}) \) be the set of all nonempty intersections of pseudospheres in \( \mathcal{A} \), including \( \mathbb{S}^d \) as the intersection over the empty set. The set \( L(\mathcal{A}) \) comes naturally with a partial order defined by reverse inclusion. We call \( L(\mathcal{A}) \) the intersection poset of \( \mathcal{A} \). In particular, the minimum element in \( L(\mathcal{A}) \) is \( \mathbb{S}^d \).

For all \( S_B \in L(\mathcal{A}) \), define the subarrangement \( \mathcal{A}_{SB} = \{ S_i \in \mathcal{A} : S_B \subseteq S_i \} \) and the arrangement \( \mathcal{A}_{SB} \) of \( \mathcal{A}_{SB} \) in \( S_B \).

### 2.2 The Varchenko Matrix

The Varchenko matrix was initially defined for any real (finite) hyperplane arrangement in \([7]\). The definition can be extended straightforwardly to real (finite) pseudosphere arrangements.

Recall that a region \( R \) of a pseudosphere arrangement \( \mathcal{A} = (S_i)_{i \in I} \) is a connected component of the complement of \( \bigcup_{i \in I} S_i \) in \( \mathbb{S}^d \). We can associate each region \( R \) with a sign vector \( X^R = \{ \pm 1 \}^I \) such that \( X^R_i = +1 \) if \( R \in S_i^+ \) and \( X^R_i = -1 \) if \( R \in S_i^- \) for all \( i \in I \). Note that each region has a unique sign vector. Denote by \( \mathbb{R}(\mathcal{A}) \) the set of regions of \( \mathcal{A} \) and let \( r(\mathcal{A}) = |\mathbb{R}(\mathcal{A})| \).

Now we assign to \( S_i \in \mathcal{A} \) an indeterminate \( x_i \). For any pair of regions \( (R, R') \) of \( \mathcal{A} \), set

\[
\text{sep}(R, R') = \{ S_i \in \mathcal{A} : R \text{ separates } R \text{ and } R' \} = \{ i \in I : X_i^R = X_i^{R'} \}.
\]

The set \( \text{sep}(R, R') \) is well-defined since each region lies in exactly one of \( S_i^+ \) and \( S_i^- \) for all \( i \).

**Definition 2.3.** The Varchenko matrix \( V(\mathcal{A}) = [V_{R'R}] \) of a pseudosphere arrangement \( \mathcal{A} \) is the \( r(\mathcal{A}) \times r(\mathcal{A}) \) matrix with rows and columns indexed by \( \mathbb{R}(\mathcal{A}) \) and entries

\[
V_{R'R} = \prod_{S_i \in \text{sep}(R, R')} x_i.
\]

If \( \mathcal{A} \) is a hyperplane arrangement, then we obtain the original definition of the Varchenko matrix associated with a hyperplane arrangement. For example, the Varchenko matrix of the arrangement in Fig. 1 is

\[
V = \begin{bmatrix}
1 & x_1 & x_1 x_2 & x_1 x_3 & x_2 x_3 & x_1 x_2 x_3 \\
1 & x_1 & x_2 & x_3 & x_1 x_3 & x_1 x_2 x_3 \\
x_1 & x_2 & x_2 & 1 & x_2 x_3 & x_1 x_2 x_3 \\
x_1 x_3 & x_3 & x_2 x_3 & 1 & x_1 & x_1 x_2 \\
x_3 & x_1 x_3 & x_1 x_2 x_3 & x_1 & 1 & x_2 & x_1 x_2 \\
x_2 x_3 & x_1 x_2 x_3 & x_1 x_3 & x_1 x_2 & x_2 & 1 & x_1 \\
x_1 x_2 x_3 & x_2 x_3 & x_3 & x_2 & x_1 x_2 & x_1 & 1
\end{bmatrix}.
\]

It turns out that the determinant of the Varchenko matrix has an elegant factorization. We formulate this result via Möbius functions as in \([6, \text{sec} 6]\).

**Definition 2.4.** The Möbius function of \( L(\mathcal{A}) \) is defined by

\[
\mu(M, M') = \begin{cases}
1 & \text{if } M = M' \text{ in } L(\mathcal{A}) \\
- \sum_{M \leq N < M'} \mu(M, N) & \text{if } M < M' \text{ in } L(\mathcal{A})
\end{cases}.
\]

Furthermore, we set \( \mu(M) = \mu(\mathbb{S}^d, M) \).

Define the characteristic polynomial of \( \mathcal{A} \) to be \( \chi_{\mathcal{A}}(t) = \sum_{M \in L(\mathcal{A})} \mu(M) t^{|\text{dim}(M)|} \).

**Theorem 2.5** (Varchenko \([7]\)). Let \( \mathcal{A} = \{ S_1, \ldots, S_n \} \) be a pseudosphere arrangement. If \( M \in L(\mathcal{A}) \), set

\[
x_M = \prod_{i \in \mathbb{S}_i} x_i, \quad n(M) = r(\mathcal{A}^M) \text{ and } p(M) = \left| \frac{d}{dt} \chi_{\mathcal{A}}(1) \right|.
\]

Then

\[
\det V(\mathcal{A}) = \prod_{M \in L(\mathcal{A}), M \neq 0} (1 - x_M^R)^{\frac{d}{dM} p(M)}.
\]
For instance, the Varchenko matrix associated with the arrangement in Figure 1 has determinant
\[ \det(V) = (1 - x_1^2)^3(1 - x_2^2)^3(1 - x_3^2)^3. \]

**Definition 2.6.** Fix any numbering of the regions of \((A)\). Let \(R_k, R_m, R_n \in \mathcal{R}(\mathcal{A})\), where \(k, m, n\) are not necessarily distinct. The *distance* \(l_k(m, n)\) between \(R_k\) and \(R_m \cup R_n\) is the product of the indeterminates \(x_i\) of all pseudospheres \(S_i\) that separate both \(R_k, R_m\) and \(R_k, R_n\), i.e., \(x_i^{R_m} = X_i^{R_n} = -X_i^R\).

It follows that
\[ l_k(m, n) = \prod_{S_i \in \text{sep}(R_k, R_m) \cap \text{sep}(R_k, R_n)} x_i. \]

Observe that by definition of \(V\), the entry \(V_{mn} = \frac{V_{mk} \cdot V_{kn}}{l_k(m, n)^2}\).

### 2.3 Diagonal Form

Let \(A, B\) be \(n \times n\) square matrices over \(\mathbb{Z}[x_1, x_2, \ldots, x_n]\).

**Definition 2.7.** We say that the square matrix \(A\) is *equivalent* to the square matrix \(B\) over the ring \(R\), denoted by \(A \sim B\), if there exist matrices \(P, Q\) over \(R\) such that \(\det(P), \det(Q)\) are units in \(R\) and \(PAQ = B\).

In other words, the matrix \(A\) is equivalent to \(B\) if and only if we can get from \(A\) to \(B\) by a series of row and column operations (subtracting a multiple of a row/column from another row/column, or multiplying a row/column by a unit in \(R\)). It is easy to check that \(\sim\) is an equivalence relation.

For all \(k \leq n\), let \(\gcd(A, k)\) be the greatest common divisor of all the determinants of \(k \times k\) submatrices of \(A\).

**Lemma 2.8.** If \(A \sim B\), then \(\gcd(A, k) = \gcd(B, k)\) for all \(k = 1, 2, \ldots\), and \(\text{rank}(A) = \text{rank}(B)\).

**Proof.** Follow the same arguments for Theorem 6.5 in [5, 6.1].

**Definition 2.9.** Let \(A\) be an \(n \times n\) square matrix over the ring \(R\). We say that \(A\) has a *diagonal form* over \(R\) if there exists a diagonal matrix \(D = \text{diag}(d_1, d_2, \ldots, d_n)\) in \(R\) such that \(A \sim D\). In particular, if \(d_i \neq d_{i+1}\) for all \(1 \leq i \leq n - 1\), then we call \(D\) the *Smith normal form (SNF)* of \(A\) in \(R\).

It is known that the SNF of a matrix exists and is unique if we are working over a principal ideal domain. But the SNF of a matrix may not exist if we are working over \(R\), the ring of integer polynomials.

For example, the matrix \[
\begin{bmatrix}
    x & 0 \\
    0 & x + 2
\end{bmatrix}
\] does not have an SNF over \(R\).

**Lemma 2.10.** If the SNF of a matrix \(A\) exists, then it is unique up to units.

**Proof.** Let \(D\) be one of the SNFs of \(A\). Suppose that \(A \sim D = \text{diag}(d_1, \ldots, d_n)\) where \(d_k \neq d_{k+1}\) for \(k = 1, \ldots, n - 1\).

It is easy to see that \(\gcd(D, k) = d_1 \cdots d_k\), so \(d_1 \cdots d_k = \gcd(A, k)\) for \(k = 1, \ldots, n - 1\) by Lemma 2.8.

Given a matrix \(A\), the above equations and the condition that \(d_k \neq d_{k+1}\) for \(k = 1, \ldots, n - 1\) are sufficient to solve for \(d_k\). Namely, \(d_k = 0\) if \(\gcd(A, k) = 0\); otherwise \(d_k\) equals a unit times \(\gcd(A, k) / \gcd(A, k - 1)\). Here, \(\gcd(A, 0) = 1\).
The next lemma follows directly from the transitivity of $\sim$ and the uniqueness of SNF.

**Lemma 2.11.** If $A \sim B$ and if one of $A$, $B$ has SNF, then the other also has SNF and $\text{SNF}(A) = \text{SNF}(B)$.

**Definition 2.12.** Let $A$ be a matrix over the ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$. Let $x$ be an indeterminate corresponding to $S$. Define $A_{x_i=f_i(q), x_2=f_2(q), \ldots, x_n=f_n(q)}$ to be the matrix over the ring $\mathbb{Z}[q]$ obtained by replacing each $x_i$ by $f_i(q)$ in $A$.

For example, when $V$ is a Varchenko matrix, the matrix $V_{s=q, \ldots, z=q}$ is called the $q$-Varchenko matrix.

**Lemma 2.13.** Let $A, B$ be matrices over the ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$. If $A \sim B$, then

$$A_{x_i=f_i(q), x_2=f_2(q), \ldots, x_n=f_n(q)} \sim B_{x_i=f_i(q), x_2=f_2(q), \ldots, x_n=f_n(q)}.$$  

### 3 The Main Result

**Theorem 3.1.** Let $I = \{1, 2, \ldots, N\}$ be a finite set and $\mathcal{A} = (S_i)_{i \in I}$ be a pseudosphere arrangement in $\mathbb{S}^d$. Let $x_i$ be an indeterminate corresponding to $S_i$ for all $i \in I$. Then the Varchenko matrix $V$ associated with $\mathcal{A}$ has a diagonal form over $\mathbb{Z}[x_1, \ldots, x_N]$ if and only if $\mathcal{A}$ is in semigeneral position. In that case, the diagonal form of $V$ has diagonal entries $\prod_{i \in B} (1 - x_i^2)^{k}$ for all $B \subseteq I$ such that $S_B \in L(\mathcal{A})$.

**Corollary 3.2.** Let $\mathcal{A}$ be any semigeneral pseudosphere arrangement in $\mathbb{S}^d$. The $q$-Varchenko matrix $V_q$ of $\mathcal{A}$ has an SNF over the ring $\mathbb{Z}[q]$. The diagonal entries of its SNF are of the form $(1 - q^2)^k$, $k = 0, 1, \ldots, d$, and the multiplicity of $(1 - q^2)^k$ equals the number of elements in $L(\mathcal{A})$ with dimension $d - k$.

**Corollary 3.3.** Let $\mathcal{A}$ be a semigeneral pseudosphere arrangement in $\mathbb{S}^d$ and $V$ its Varchenko matrix. Then

$$\det(V) = \prod_{i \in I} (1 - x_i^2)^{m_i},$$

where $m_i = |\{S_B \in L(\mathcal{A}) : S_B \subseteq S_i\}|$.

Thus, our proof also serves as an alternative proof for a special case of Theorem 2.5.

**Remark 3.4.** In fact the definition of the Varchenko matrix can be generalized to oriented matroids via the following corollary of the Topological Representation Theorem of Folkman and Lawrence. Hence the results in this paper also apply directly to oriented matroids.

**Theorem 3.5.** (c.f. [1, Sec 1.4]) There is a one-to-one correspondence between signed essential pseudosphere arrangements in $\mathbb{S}^d$ (up to topological equivalence) and simple rank $d + 1$ oriented matroids (up to isomorphism).

### 4 Construction of the Diagonal Form of the Varchenko Matrices of Semigeneral Arrangements

In this section, we prove the sufficient condition of Theorem 3.1 by explicitly constructing the diagonal form of the Varchenko matrix of a semigeneral arrangement.

Assume as before that we are working in $\mathbb{S}^d$.

**Definition 4.1.** Define $\phi : \mathbb{Z}[x_1, \ldots, x_N] \rightarrow \mathbb{Z}[x_1, \ldots, x_N]$ to be the function satisfying the following properties:

(a) $\phi(p + q) = \phi(p) + \phi(q)$ for all $p, q \in \mathbb{Z}[x_1, \ldots, x_N]$.
(b) $\phi(p \cdot q) = \phi(p) \phi(q)$ for all monomials $p, q \in \mathbb{Z}[x_1, \ldots, x_N]$ with gcd$(p, q) = 1$.
(c) $\phi(x_i^k) = x_i$ if $k \geq 2$ and $\phi(x_i^k) = x_i^k$ if $k = 0, 1$ for all $i = 1, \ldots, N$.
(d) $\phi(0) = 0$.

It is easy to check that $\phi$ is well-defined and unique. In fact, $\phi(p)$ is obtained from $p$ by replacing all exponents $e \geq 3$ by 2.

**Proposition 4.2.** For all $i \in \{1, 2, \ldots, N\}$ and $p \in \mathbb{Z}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N]$,

(a) $\phi(x_i^2 \cdot (1 - x_i^2 \cdot p)) = x_i^2 \cdot \phi(1 - p);$  
(b) $\phi((1 - x_i^2)(1 - x_i^2 \cdot p)) = 1 - x_i^2.$
Lemma 4.5.\) The above identities follow directly from the definition of \(\varphi\). \(\square\) \(\square\)

**Definition 4.3.** A set of regions \(\mathcal{B} \subseteq \mathcal{B}(\mathcal{A})\) *encompasses a point* \(x \in \mathbb{S}^d\) if the interior of the closure of the union of these regions contains \(x\).

A set of regions \(\mathcal{B} \subseteq \mathcal{B}(\mathcal{A})\) *encompasses an element* \(M \in \mathcal{L}(\mathcal{A})\) if there exists a point \(x \in M\) such that \(\mathcal{B}\) encompasses \(x\).

In other words, an element \(M\) is encompassed by a set of regions \(\mathcal{B}\) if a nontrivial part of \(M\) with nonzero relative measure is encompassed by some regions in \(\mathcal{B}\).

Let \(\mathcal{C}(\mathcal{B})\) be the set of elements of the intersection poset that are encompassed by \(\mathcal{B}\). Note that \(\mathcal{C}(\emptyset) = \emptyset\) and \(\mathcal{C}(\{R\}) = \{\mathbb{S}^d\}\) for any \(R \in \mathcal{B}(\mathcal{A})\).

In Figure 2, for example, all points on the segment of \(S_1\) between region \(R_1\) and \(R_4\) are encompassed by the set of regions \(\{R_1, R_2, R_3, R_4, R_5\}\). So \(\mathcal{C}(\{R_1, R_2, R_3, R_4, R_5\}) = \{\mathbb{S}^2, S_1, S_2, S_3, S_5\}\) and \(\mathcal{C}(\{R_1, R_2, R_3, R_4, R_5, R_6\}) = \{\mathbb{S}^2, S_1, S_2, S_3, S_5, S_3 \cap S_5\}\).

![Figure 2: General position of 5 (pseudo)lines in \(\mathbb{S}^2\); region \(R_i\) is labeled as \(i\).](image)

**Definition 4.4.** Fix a numbering of the regions of \(\mathcal{A}\). We say that region \(R_k\) is the first to encompass \(M\) for some \(M \in \mathcal{L}(\mathcal{A})\) if \(M \in \mathcal{C}(\{R_1, R_2, \ldots, R_k\})\) and \(M \notin \mathcal{C}(\{R_1, R_2, \ldots, R_{k-1}\})\).

One can see that \(\gcd(V_{i,k}, x_{i,1}, x_{i,2}, \ldots, x_{i,m}) \neq 1\) for any \(1 \leq j < k\), where \(R_k\) is the first to encompass \(M = S_{i_1} \cap \cdots \cap S_{i_m} \in \mathcal{L}(\mathcal{A})\).

For example, in Figure 2, we see that region \(R_5\) is the first to encompass \(S_5\) and region \(R_6\) is the first to encompass \(S_3 \cap S_5\).

**Lemma 4.5.** There exists a numbering \(\pi\) (and order of coloring) of \(\mathcal{B}(\mathcal{A})\) such that \(\mathcal{C}(\{R_1, R_2, \ldots, R_k\}) = k\) for all \(k = 0, 1, \ldots, r(\mathcal{A})\). Let \(\mathcal{B}^{(k)} = \{R_1, R_2, \ldots, R_k\}\) be the set of the regions with the first \(k\) indices. Set \(M^{(k)} = \mathcal{C}(\mathcal{B}^{(k)}) \setminus \mathcal{C}(\mathcal{B}^{(k-1)})\). Then for any \(k = 1, \ldots, r(\mathcal{A})\), \(\pi\) has the following properties:

(a) The interior of the closure of \(\bigcup \mathcal{B}^{(k)}\) is connected.
(b) For all \(M \in \mathcal{L}(\mathcal{A})\), the subset \(\{x : x \in M, \mathcal{B}^{(k)} \text{ encompasses } x\} \subseteq M\) is connected.
(c) If \(R_k\) is the first to encompass \(M = S_B\) where \(B \subseteq I\), then \(R_k\) is the first colored region in the cone formed by all \(S_{i_1}, i \in B\) that contains \(R_k\).
(d) For all \(M \in \mathcal{L}(\mathcal{A})\), \(M\) is cut into connected closed sections \(M_1, M_2, \ldots\) by all pseudospheres \(S_i \in \mathcal{A}\) that intersect \(M\). Let \(R^{(k)}_{M_i} = \{R_j : 1 \leq j \leq k, M_i \in R^{(k)}_j\}\). Then the interior of the closure of \(\bigcup_{R \in \mathcal{B}^{(k)}_M} R\) is connected.

**Lemma 4.5** is saying that there is a way for us to add (and color) regions of \(\mathcal{B}(\mathcal{A})\) one by one such that whenever we add (and color) a region, we can encompass exactly one new element in \(\mathcal{L}(\mathcal{A})\).

The labeling of the regions in Figure 2 is such a numbering. The interior of the closure of \(\{R_1, R_2, R_3, R_4, R_5\}\) and \(\{R_1, R_2, R_3, R_4, R_5, R_6\}\) are connected. If we add \(R_6\) to the closure of \(\{R_1, R_2, R_3, R_4, R_5\}\), \(R_6\) is the first colored region in \(S_3^+ \cap S_5^+\), i.e., it comes before \(R_7, R_8, \ldots, R_{12}\). Property (d) is saying that the interior of the closure of all colored regions around any intersection point, (pseudo)line segment or (pseudo)ray is connected.
Proof of Lemma 4.5. We will prove Lemma 4.5 by induction on \( k \), the number of regions added and colored. The base case \( k = 1 \) is trivial since we encompass \( S_1 \) after adding the first region \( R_1 \).

Suppose Lemma 4.5 holds after adding the first \( k - 1 \) regions. Let \( M \) be an element of the smallest dimension in the set \( \{ M \in L(\mathcal{A}) : \exists R \in \mathcal{B}(\mathcal{A}) \setminus \mathcal{B}^{(k-1)} \text{ such that } \{ R \} \cup \mathcal{B}^{(k-1)} \text{ encompasses } M \} \). Suppose \( M \) is the intersection of \( s \) pseudospheres \( S_{i_1}, \ldots, S_{i_s} \). Let \( R_k \) be the uncolored region such that \( \{ R_k \} \cup \mathcal{B}^{(k-1)} \) encompasses \( M \).

Claim 4.6. \( |M^{(k)}| = |\mathcal{C}(\{ R_1, R_2, \ldots, R_k \})| - |\mathcal{C}(\{ R_1, R_2, \ldots, R_{k-1} \})| \leq 1. \)

Suppose that \( M^{(k)} = \mathcal{C}(\{ R_1, R_2, \ldots, R_k \}) \setminus \mathcal{C}(\{ R_1, R_2, \ldots, R_{k-1} \}) \neq \emptyset \). Then:

(i) If \( s = 1 \), then \( M \in M^{(k)} \) is some pseudosphere \( S \in \mathcal{A} \);

(ii) If \( s \geq 2 \), then \( M \in M^{(k)} \in L(\mathcal{A}) \) and \( \dim(M) = d - s \).

Furthermore, the four properties in Lemma 4.3 remain true.

In other words, after adding a new region \( R_k \), we encompass at most one new element \( M \in L(\mathcal{A}) \).

Proof of Claim. (i) If \( s = 1 \), then the closure of any uncolored region is connected to at most one colored region by some pseudosphere. Let \( R_k \) be connected by \( S_a \) to some \( R_{m_1}, 1 \leq m_1 \leq k - 1 \). Hence we definitely have \( S_a \in \mathcal{C}(\mathcal{B}^{(k)}) \). Note that \( R_k \) is connected to only one colored region, so \( |M^{(k)}| \leq 1 \). Therefore \( M^{(k)} \subseteq \{ S_a \} \).

We want to show that the four properties still hold after adding \( R_k \). Property (a) remains true by the induction hypothesis on \( k - 1 \) regions. If \( M^{(k)} = \{ S_a \} \), then \( R_k \) has to be the only colored region in \( S_a \). It follows that \( S_a \notin \mathcal{C}(\mathcal{B}^{(k-1)}) \) and \( \{ x : x \in S_a, \mathcal{B}^{(k)} \text{ encompasses } x \} = R_k \cap R_{m_1} \) is connected and nonempty. Therefore (c) holds for \( k \). For any \( x \in \mathcal{C}(\mathcal{B}^{(k-1)}) \), we have \( \{ x : x \in M, \mathcal{B}^{(k)} \text{ encompasses } x \} = \{ x : x \in M, \mathcal{B}^{(k-1)} \text{ encompasses } x \} \), which is connected by the induction hypothesis. Hence (b) still holds. Observe that \( R_{m_1} \) is changed after we add \( R_k \) only if \( M_i \in R_{m_1} \). Since \( R_k \) is the only colored region in \( S_a \), \( M_i \in \mathcal{B}^{(k)} \cap \mathcal{B}^m \). Now \( \bigcup_{R_i \notin M^{(k)}} R_i \) is connected by the induction hypothesis, and \( R_{i_1}, R_{m_1} \) are connected by a nontrivial (i.e. with nonzero relative measure) part of \( S_a \), so \( \bigcup_{R_i \notin M^{(k)}} R_i \) is also connected.

(ii) If \( s \geq 2 \), then by our induction hypothesis (d), \( R_k \) is the only uncolored region of the \( 2^s \) regions whose closure contain \( M = S_{i_1} \cap \cdots \cap S_{i_s} \). For all \( i = 1, 2, \ldots, s \), let \( S_{i_0} \) be the pseudosphere that connects \( R_k, R_{m_1} \), where \( 1 \leq m_1 \neq \cdots \neq m_s \leq k - 1 \). Clearly \( M \in \mathcal{C}(\mathcal{B}^{(k)}) \).

Assume to the contrary that we encompass at least two distinct elements \( M, M' \in L(\mathcal{A}) \) by adding \( R_k \). If \( M' \) is some pseudosphere \( S' \in \mathcal{A} \), then at least one of \( R_{m_1}, \ldots, R_{m_s} \) would be on the same side of \( S' \) as \( R_k \). By the induction hypothesis (c), this implies that \( S' \) has been encompassed before adding \( R_k \), a contradiction to our assumption.

Hence \( M' \) has to be the intersection of some \( S'_{i_1}, \ldots, S'_{i_d} \), \( 1 < d \in \mathcal{A} \), all of which border \( R_k \). Denote the \( t \) regions connected to \( R_k \) by \( M' = R_{i_1} \cup \cdots \cup R_{i_t} \). All \( R_{i_1} \) have to be colored before we add \( R_k \), or we wouldn’t encompass \( M' \) by adding \( R_k \). Let \( K_i' = \bigcap_{j=1}^{i} Y_{j}^{X_i} \), where \( Y_i^{X_i} = + \) if \( R_k \in S_{i_1}^{+} \) and \( Y_i^{X_i} = - \) if \( R_k \in S_{i_1}^{-} \). Hence \( K_i' \) is a polyhedron containing \( R_k \) with \( S_{i_1}, \ldots, S_{i_d} \) as facets and \( M' \) as its “tip”. Similarly, let \( K_i = \bigcap_{j=1}^{i} Y_{j}^{X_i} \), where \( Y_i^{X_i} = + \) if \( R_k \in S_{i_1}^{+} \) and \( Y_i^{X_i} = - \) if \( R_k \in S_{i_1}^{-} \).

Figure 3: Examples of case (ii) in \( S^2 \).

(1) If \( \{ S_{i_1}, 1 < i \leq t \} \cap \{ S_{i_k}, 1 < k \leq s \} = \emptyset \), then a nontrivial part of \( M' \) is inside the interior of \( K_t \) and a nontrivial part of \( M \) is inside the interior of \( K_i' \). Hence, we can find a region \( R_{i_j}, 1 \leq m_j < k \) that is
contained in $K'_i$. Note that $R_{m,i}$ and $R'_i$ are on different sides of $S_{a,i}$ for all $i$. Using the induction hypothesis (a), $R_{m}$ and $\bigcup R'_i$ have to be connected by colored regions on both sides of $S_{a,i}$. Let $R'$ be one of these regions such that $R'$ is bordered by $S_{a,i}$ and $R'$ is on the same side of $S_{a,i}$ as $\bigcup R'_i$. Pick any $v \neq j$ such that $R_{m,i}$ and $R_{m,j}$ are on different sides of $S_{a,j}$. Therefore $R_{m,i}$ and $R'$ are on the same side of $S_{a,j}$. Since $R_{m,i}$ is an uncolored region between $R'$ and $R_{m,j}$, we can find an element $l \subseteq S_{a,j}, l \in L(\mathcal{A})$ with $\dim(l) > 0$ such that $l$ has nontrivial intersection with the boundaries of $R', R_{m,i}$ and $R_{m,j}$. Note that $l$ is encompassed before we add $R_{m,i}$, but $\{x : x \in l, \mathcal{A}(k-1)\text{-encompasses } x\}$ is not connected, which contradicts the induction hypothesis (b).

(2) If there exist some $S_{a,i} \in \{S'_i, 1 \leq i \leq t\} \cap \{S_{a,s}, 1 \leq s \leq s\}$, then we can find a region $R'_i$ such that $R_{m,i}, R'_i$, and $R_{m,j}$ are on the same side of $S_{a,i}$. Note that $S_{a,i}$ borders $R'_i$, and $R_{m,j}$ is an uncolored region between $R_{m,i}$ and $R'_i$. Again we can find an element $l$ such that $l \subseteq S_{a,i}, l \in L(\mathcal{A}), \dim(l) > 0$ and $l$ has nontrivial intersection with the boundaries of $R_{m,i}, R_{m,j}$, and $R'_i$. Now $l$ is encompassed before we add $R_{m,i}$, but $\{x : x \in l, \mathcal{A}(k-1)\text{-encompasses } x\}$ is not connected, which is a contradiction to the induction hypothesis (b).

Hence we conclude that $M(k) \subseteq \{M\}$.

Now we want to show that the four properties remain true after adding $R_{k}$. Suppose $R_{k}$ is the first to encompass $M(k)$. If $R_{k}$ is not the first colored region in $K_{k}$, then we can find $R_{j}, 1 \leq j \leq k - 1$ that is the colored region closest to $M(k)$ in $K_{k}$ before we add $R_{k}$. If $R_{j}$ is strictly inside the interior of $K_{k}$, then we can apply the argument in (1) of (ii) above by replacing $R'$ with $R_{j}$. Else use the argument in (ii).(2) where $R_{j}$ is equivalent to $R'_{i}$. In both cases we can find an encompassed element on the boundary of $K_{k}$ whose encompassed parts are not connected, a contradiction to the induction hypothesis (b). Therefore $R_{k}$ has to be the first colored region in $K_{k}$, so property (c) holds for $k$. Properties (a), (b) and (d) follow immediately.

Hence we’ve completed the induction. \(\square\)

Note that after adding all regions in $\mathcal{A}(\mathcal{A})$, we encompass all elements $M \in L(\mathcal{A})$. Since $\mathcal{A}$ is semigeneral, so $|L(\mathcal{A})| = |\mathcal{A}(\mathcal{A})|$. Therefore we add exactly one new element after adding (and coloring) a new region, i.e., $|\mathcal{A}(\{R_{1}, R_{2}, \ldots, R_{k}\})| = k$ for all $k = 0, 1, \ldots, r(\mathcal{A})$. \(\square\)

Now we are ready to show the diagonal form of the Varchenko matrix of an arrangement $\mathcal{A}$ in semigeneral position. Fix a numbering of $\mathcal{A}(\mathcal{A})$ that satisfy all the properties in Lemma 4.5.

**Definition 4.7.** Let $P = [P_{i,j}]$ be an $N \times N$ symmetric matrix with entries $P_{i,j} \in \mathbb{Z}[x_1, \ldots, x_r]$. If $P_{k,k}$ is a factor of $P_{k,n}$, denoted as $P_{k,k}|P_{k,n}$, for all $n = 1, \ldots, N$, then we can define a matrix operation $T^{(k)}$ as follows:

$(T^{(k)}P)_{m,n} = P_{m,k}P_{n,k} - P_{m,n}P_{k,k}$ for all $m, n = 1, \ldots, N, (m, n) \neq (k, k)$;

$(T^{(k)}P)_{k,k} = P_{k,k}$.

In other words, we can apply $T^{(k)}$ to $P$ only when $P_{k,k}|P_{k,i}$ for all $i$. For each $i \neq k$, we subtract row $i$ by $\frac{P_{i,k}}{P_{k,k}}$ times row $k$ to get $P'$. After operating on all rows, we then subtract column $j$ of $P'$ by $\frac{P_{k,j}}{P_{k,k}}$ times column $k$ of $P'$ to get $T^{(k)}P$. It is easy to check that $T^{(k)}$ is a well-defined operation. The resulting matrix is also symmetric and the entries $(i, k), (k, i)$ are 0 for all $i \neq k$.

Set $V^{(0)} = V$. For all $k = 0, 1, \ldots, r(\mathcal{A})$, let $V^{(k)}$ be the matrix obtained by applying $T^{(1)}, \ldots, T^{(k)}$ in order to $V$. Denote entry $(m, n)$ of $V^{(k)}$ as $V^{(k)}_{m,n}$. Let $M(k) = S_{a_1} \cap S_{a_2} \cap \cdots \cap S_{a_k}$ and $A_k = \{S_{a_1}, S_{a_2}, \ldots, S_{a_k}\}$.

**Lemma 4.8.** a) $V^{(k)}_{k,k} = \prod_{a \in A_k} (1 - x_a^2)$;

b) $V^{(k)}_{m,n} = 0$ for all $m \neq n \leq k$; $V^{(k)}_{m,m} = V^{(m)}_{m,m}$ for all $m \leq k$;

c) $V^{(k)}_{m,n} = V^{(m)}_{m,n} \cdot \varphi(k) \prod_{i=1}^{k} (1 - x_i^2 (m,n))$ if at least one of $m, n$ is greater than $k$.

**Proof.** We will justify the Lemma by induction on $k$. The statements hold for the base case $k = 0$ by the definition of $V^{(0)} = V$.

Suppose that the statements hold for $k - 1$.

It follows from Lemma 4.5.c that $k$ is the first and only colored region in the polyhedron $K_{a_1a_2\ldots a_k}$. For all $1 \leq i \leq s$, there exists a unique $r_i$ such that $1 \leq r_i \leq k - 1$ and $S_{a_i}$ connects $r_i$ and $k$, i.e., $\text{sep}(r_i, k) = \{S_{a_i}\}$. Furthermore, $r_i \neq r_j$ for all $i \neq j$. 

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Remark 4.9. $V_{m,k}^{(k-1)} = V_{m,k} \cdot V_{k,k}^{(k-1)}$ or 0.

Proof. If $m$ is not in $K_{a_1a_2 \cdots a_s}$, then $m$ and $k$ are on different sides of at least one pseudosphere, say $S_{a_1}$. Then $S_{a_1} \notin \text{sep}(r_1,m)$, i.e., $\text{sep}(r_1,m) \cap \text{sep}(r_1,k) = \text{sep}(r_1,m) \cap \{S_{a_1}\} = \emptyset$. Therefore $l_{r_1}(m,k) = 1$. By the induction hypothesis, we have

$$V_{m,k}^{(k-1)} = V_{m,k} \cdot \varphi\left((1 - l_{r_1}^2(m,k)) \cdot \prod_{j=1, j \neq r_1}^{k-1} (1 - l_j^2(m,k))\right) = 0.$$

If $m$ is in $K_{a_1a_2 \cdots a_s}$, then $m$ and $k$ are on the same side of $S_{a_i}$ for all $1 \leq i \leq s$. Therefore, for any region $j = 1, 2, \ldots, k-1$, at least one of $\{S_{a_1}, \ldots, S_{a_s}\}$ separates $j$ and $m \cap k$, say $S_{a_i}, i \in \{1, 2, \ldots, s\}$. Since $S_{a_i} \subseteq \text{sep}(j,m)$, we can deduce that $x_{a_i} | l_j(m,k)$. Note that for all $1 \leq i \leq s, S_{a_i} \subseteq \text{sep}(r_1,m)$, i.e., $\text{sep}(r_1,m) \cap \text{sep}(r,k) = \text{sep}(r,m) \cap \{S_{a_i}\} = \{S_{a_i}\}$. Therefore $l_{r_1}(m,k) = x_{a_i}$. Applying the results of Proposition 4.2.2, we get

$$V_{m,k}^{(k-1)} = V_{m,k} \cdot \varphi\left((1 - l_{r_1}^2(m,k)) \cdot \prod_{j=1, j \neq r_1}^{k-1} (1 - l_j^2(m,k))\right) = V_{m,k} \cdot (1 - x_{a_1}^2)(1 - x_{a_2}^2) \cdots (1 - x_{a_s}^2).$$

On the other hand,

$$V_{k,k}^{(k-1)} = V_{k,k} \cdot \varphi\left(\prod_{j=1}^{k-1} (1 - l_j^2(k,k))\right) = \varphi\left((1 - l_{r_1}^2(k,k)) \cdot \prod_{j=1, j \neq r_1}^{k-1} (1 - l_j^2(k,k))\right) = (1 - x_{a_1}^2)(1 - x_{a_2}^2) \cdots (1 - x_{a_s}^2).$$

Hence we have

$$V_{m,k}^{(k-1)} = V_{m,k} \cdot (1 - x_{a_1}^2) = V_{m,k} \cdot V_{k,k}^{(k-1)}.$$

Since $V_{k,k}^{(k-1)} | V_{m,k}^{(k-1)}$ for all $m = 1, 2, \ldots, r(\omega')$, we can apply the matrix operation $T^{(k)}$ to $V^{(k-1)}$. By definition of $T^{(k)}$, if $m \neq n \leq k$, then $V_{m,n}^{(k)} = 0$; else,

$$V_{m,n}^{(k)} = V_{m,n}^{(k-1)} - \frac{V_{m,k}^{(k-1)} \cdot V_{n,k}^{(k-1)}}{V_{k,k}^{(k-1)}}.$$

(2)

It follows immediately that $V_{m,k}^{(k)} = V_{k,k}^{(k-1)} = (1 - x_{a_1}^2)(1 - x_{a_2}^2) \cdots (1 - x_{a_s}^2)$. Therefore, claim (a) holds for $k$. In addition, we can deduce from Remark 4.9 that if at least one of $m, n$ is not contained in $K_{a_1a_2 \cdots a_s}$, then

$$V_{m,k}^{(k)} = V_{m,k}^{(k-1)} - \frac{V_{m,k}^{(k-1)} \cdot V_{n,k}^{(k-1)}}{V_{k,k}^{(k-1)}} = V_{m,k}^{(k-1)} - 0 = V_{m,n} \cdot \varphi\left(\prod_{i=1}^{k} (1 - l_i^2(m,n))\right) = V_{m,n} \cdot \varphi\left(\prod_{i=1}^{k} (1 - l_i^2(m,n))\right).$$

Note that if $m \neq n \leq k$, i.e., neither of $m, n$ is contained in $K_{a_1a_2 \cdots a_s}$, then by the induction hypothesis $V_{m,n}^{(k)} = V_{m,n}^{(k-1)} = 0$. If $m = n, k$, then $V_{m,m}^{(k)} = V_{m,m}^{(k-1)} = \cdots = V_{m,m}^{(m)}$. Hence (b) also holds for $k$.

In order to prove (c), it suffices to show that if $m, n$ are both contained in $K_{a_1a_2 \cdots a_s}$, then

$$V_{m,n} \cdot \varphi\left(\prod_{i=1}^{k} (1 - l_i^2(m,n))\right) - V_{m,n} \cdot \varphi\left(\prod_{i=1}^{k} (1 - l_i^2(m,n))\right) = V_{m,n}^{(k-1)} - V_{m,n}^{(k)}.$$
Using linearity of $\varphi$, we can combine the two terms on the left hand side of the above equation:

\[
LHS = V_{m,n} \cdot \varphi(l_k^2(m,n) \prod_{i=1}^{k-1} (1 - l_i^2(m,n))).
\]  

So we only need to show that

\[
\varphi(l_k^2(m,n) \prod_{i=1}^{k-1} (1 - l_i^2(m,n))) = l_k^2(m,n) \cdot V_{k,k}^{(k-1)}.
\]  

Applying Proposition 4.2(a) we know that

\[
I_k^2(m,n) \mid \varphi(l_k^2(m,n) \prod_{i=1}^{k-1} (1 - l_i^2(m,n))) \mid I_k^2(m,n) \mid \varphi(l_k^2(m,n) \cdot l_i^2(m,n)).
\]  

Hence we can pull out $I_k^2(m,n)$ on the left hand side:

\[
\varphi(l_k^2(m,n) \cdot \prod_{i=1}^{k-1} (1 - l_i^2(m,n))) = l_k^2(m,n) \cdot \varphi(\prod_{i=1}^{k-1} (1 - l_i^2(m,n))),
\]

where $l_i^2(m,n) = \frac{\varphi(l_k^2(m,n) \cdot l_i^2(m,n))}{l_k^2(m,n)}$ for all $i = 1, 2, \ldots, k - 1$.

Note that $I_i(m,n)$ is exactly the distance between $i$ and $m \cup n$ in $\mathcal{A}_{m,n} = \mathcal{A} \setminus (\text{sep}(k,m) \cap \text{sep}(k,n)) = \mathcal{A} \setminus \text{sep}(k,m \cup n)$. Therefore it suffices to show that in $\mathcal{A}_{m,n}$,

\[
\varphi(\prod_{i=1}^{k-1} (1 - l_i^2(m,n))) = V_{k,k}^{(k-1)}.
\]

Observe that after deleting $\text{sep}(k,m) \cap \text{sep}(k,n)$, either one of $m, n$, say $m$, merges with $k$, or else $m$ and $n$ share a border with $k$ respectively. Applying the results of Remark 4.9, in the first case we have

\[
\varphi(\prod_{j=1}^{k-1} (1 - l_j^2(m,n))) = V_{k,k}^{(k-1)}.
\]

In the second case, $\text{sep}(r_i, m \cup n) = \text{sep}(r_i, k) \cup \text{sep}(k, m \cup n)$. Therefore, $\hat{l}_i(m,n) = l_i(k,k)$ and

\[
\varphi(\prod_{j=1}^{k-1} (1 - \hat{l}_j^2(m,n))) = \varphi((1 - \hat{l}_{r_1}^2(m,n)) \cdots (1 - \hat{l}_{r_s}^2(m,n)) \cdot \prod_{j \neq r_1, \ldots, r_s}^{k-1} (1 - \hat{l}_j^2(m,n)))
\]

\[
= (1 - x_{r_1}^2) \cdots (1 - x_{r_s}^2) = V_{k,k}^{(k-1)}.
\]

Hence we conclude that (c) holds for $k$, and this completes the induction. \hfill \square

An immediate corollary of Lemma 4.8 is that when $k = r(\mathcal{A})$, $V_{m,n}^{r(\mathcal{A})} = 0$ for all $m \neq n$. Thus we have reduced $V$ to a diagonal matrix. We know from Lemma 4.5.c that by adding region $k$ we encompass exactly one new element $M(k) \in L(\mathcal{A})$, so each entry $V_{k,k}^{(k)} = \prod_{a \in A_k} (1 - x_a^2)$ appears exactly once on the diagonal of $V^{r(\mathcal{A})}$. Hence we’ve proven the sufficient condition of Theorem 3.1.
5 Nonexistence of the Diagonal Form of the Varchenko Matrices of Arrangements Not in Semigeneral Positions

In this section, we will prove the necessary condition of Theorem 3.1.

Lemma 5.1. Suppose that $\mathcal{A}$ is a finite pseudosphere arrangement in $\mathbb{S}^d$ and $S \notin \mathcal{A}$ is a $(d-1)$-dimensional pseudosphere. If $V(\mathcal{A} \cup \{S\})$ has a diagonal form over $\mathbb{Z}[x_1, \ldots, x_n]$, then $V(\mathcal{A})$ also has a diagonal form over $\mathbb{Z}[x_1, \ldots, x_n]$.

Proof. Let $x_1, \ldots, x_n$ be the indeterminates for pseudospheres in $\mathcal{A}$ and $x_{n+1}$ for $S$. Set $V = V(\mathcal{A})$ and $V^{(0)} = V(\mathcal{A} \cup \{S\})$.

Let $V^{(1)}$ be the matrix obtained by setting $x_{n+1} = 1$ in $V^{(0)}$. Observe that the $i^{th}$ row (column) and the $j^{th}$ row (column) of $V^{(1)}$ is the same for all $i \neq j$ if $V_i^{(0)} = x_{n+1}$, i.e., region $i$ and $j$ are separated only by $S$. Apply row and column operations to eliminate repeated rows (columns), and we will get $V^{(1)} \sim V \oplus 0$, where $0$ is the all zero matrix of dimension $k \times k$ and $k = r(\mathcal{A} \cup \{S\}) - r(\mathcal{A})$.

If $V^{(0)}$ has a diagonal form over $\mathbb{Z}[x_1, \ldots, x_{n+1}]$, then we can assign an integer value to $x_{n+1}$ and hence $V^{(1)}$ and $V \oplus 0$ have a diagonal form over $\mathbb{Z}[x_1, \ldots, x_n]$.

Let $D$ be the diagonal form of $V \oplus 0$. According to Theorem 2.5, $\det(V) \neq 0$; by Lemma 2.8, $\text{rank}(D) = \text{rank}(V^{(1)})$, which is equal to the dimension $r(\mathcal{A})$ of $V$. Therefore, the number of zeros on $D$'s diagonal is $k$.

Note that there exist matrices $P, Q$ of dimension $r(\mathcal{A} \cup \{S\})$ and unit determinant such that $P(V \oplus 0) = DQ$. We can also write the matrices in the following way, where $D'$ is the diagonal matrix obtained from eliminating the all zero rows and columns in $D$:

$$
\begin{bmatrix}
P_1 & P_3
\end{bmatrix} \begin{bmatrix}
P_2
0
\end{bmatrix} V 0 = D' \begin{bmatrix}
0
0
0
\end{bmatrix} Q_1 Q_2;
$$

$$
\begin{bmatrix}
P_1 V
0
\end{bmatrix} = D' Q_1 \begin{bmatrix}
0
0
\end{bmatrix} Q_2.
$$

It is easy to check that $P_1 V = 0$, $D' Q_2 = 0$, $P_1 V = D' Q_1$.

Since $\det(V) \neq 0$ and $\det(D') \neq 0$, $P_1$ and $Q_2$ have only 0 entries. Therefore, $1 = \det(P_1) = \det(P_1) \det(P_2); 1 = \det(Q) = \det(Q_1) \det(Q_4)$. The only units in $\mathbb{Z}[x_1, \ldots, x_n]$ are 1 and $-1$, so we can assume that $\det(P_1) = \det(Q_1) = 1$. Thus, $D'$ is a diagonal form of $V$.

Now we've arrived at the main theorem of this section, which is also the necessary condition of Theorem 3.1.

Theorem 5.2. Let $\mathcal{A}$ be a pseudosphere arrangement in $\mathbb{S}^d$ that is not semigeneral. Then $V(\mathcal{A})$ does not have a diagonal form over $\mathbb{Z}[x_1, \ldots, x_n]$.

Proof. Using Lemma 5.1, we can delete many pseudospheres in $\mathcal{A}$ as possible so that the resulting arrangement, denoted again as $\mathcal{A}$, satisfies the nonsemigeneral property and the minimum condition, i.e., if we delete any pseudosphere, the remaining arrangement will be semigeneral.

Note that there must exist $S_1, \ldots, S_p \in \mathcal{A}$ with nonempty intersection such that $\dim(S_1 \cap \cdots \cap S_p) \neq d - p$. Hence $\dim(S_1 \cap \cdots \cap S_p) \geq d - p + 1$. If $\dim(S_1 \cap \cdots \cap S_p) \geq d - p + 2$, then $\dim(S_2 \cap \cdots \cap S_p) \geq d - (p - 1)$, contradicting the minimum condition of $\mathcal{A}$. Therefore, $\dim(S_1 \cap \cdots \cap S_p) = d - 1$. Also by the minimum condition, $\mathcal{A} \cup \{S_1, \ldots, S_p\}$.

Without loss of generality, we can assume that $p = d + 1$ (by projecting all pseudospheres to a smaller subspace). Thus we only need to consider the case of $\mathcal{A} = \{S_1, \ldots, S_{d+1}\}$ in $\mathbb{S}^d$, where the intersection $S_1 \cap \cdots \cap S_{d+1}$ is a single point and any pseudosphere arrangement formed by a subset of $\mathcal{A}$ with cardinality $d$ is semigeneral and nonempty.

Now we can deduce the structure of the intersection poset $L(\mathcal{A})$: $L(\mathcal{A})$ consists of the intersection of any $k$ pseudospheres in $\mathcal{A}$ for all $k = 0, 1, \ldots, d - 1$ and the point $S_1 \cap \cdots \cap S_{d+1}$, which is also the intersection of any $d$ pseudospheres in $\mathcal{A}$. It follows that for all $S \in \mathcal{A}$, $\mathcal{A}S$ is the pseudosphere arrangement of $d$ pseudospheres of dimension $d - 1$ intersecting at one point in $\mathbb{R}^{d-1}$, so $r(\mathcal{A}S) = 2^d - 2$, $r(\mathcal{A}) = 2^{d+1} - 2$. 


Let $x_1, \ldots, x_{d+1}$ be the indeterminates of the pseudospheres in $\mathcal{A}$. By Theorem 2.5, $\det V(\mathcal{A}) = (1 - x_i^2)^{d-2} (1 - x_j^2)^{d-2} (1 - x_k^2)^{d-2} (1 - x_{l+1}^2)^{d-2} (1 - x_1^2 \cdots x_{d+1}^2)$.

Set $S_1 \cap \cdots \cap S_{d+1}$ as the origin $(0, 0, \ldots, 0)$. Pick any pseudosphere $S \in \mathcal{A}$. Then $S$ separates the space into two half spaces $S_1, S_2$. Since $\mathcal{A}$ is symmetric about the origin, exactly half of $S(\mathcal{A})$ lies in $S_1$, i.e., $r(S_1) = \frac{1}{2} r(\mathcal{A}) = 2^d - 1$. We also know that $r(S^\delta) = 2^d - 2 = (2^d - 1) - 1$. Thus, for all but one region $R$ in $S_1$, the intersection of its closure and $S$ has dimension $d - 1$. The intersection of the closure of $R$ with $S$ is the point $S_1 \cap \cdots \cap S_{d+1}$.

Here comes an important observation: If we restrict to the Varchenko matrix of regions in $S_1$, denoted by $\mathcal{A}(S_1)$, we obtain the Varchenko matrix of the pseudosphere arrangement of $d$ pseudospheres in $\mathbb{R}^{d-1}$ in general position. This matrix is equivalent to the Varchenko matrix for $\mathcal{A}^\delta \cup R$. Intuitively, we can view $R$ as the inner region of the point $S_1 \cap \cdots \cap S_{d+1}$.

We’ll prove the theorem by contradiction. Suppose that $D$ is a diagonal form of $V(\mathcal{A})$, then $V \sim D$.

First, we want to show that in $D$’s diagonal entries, $1 - x_i$ and $1 + x_i$ must appear in the form of $1 - x_i^2$ for all $i = 1, \ldots, d + 1$.

For all $i = 1, \ldots, d + 1$, consider $V_{x_i=3, x_j=0 \forall j \neq i}$. It can be decomposed into blocks of $[1\ 3\ 1]$ and identity matrices, so its SNF has diagonal entries $1$ and $8$ with multiplicities. Now $D_{x_i=3, x_j=0 \forall j \neq i}$ has the same SNF by Lemma 2.11. Note that $1 - x_i = -2$, $1 + x_i = 4$, and their products are all powers of $2$, so $D_{x_i=3, x_j=0 \forall j \neq i}$ is already in SNF. We have equal numbers of $4$ and $-2$, so they must pair up in the form of $1 - x_i^2$ or we won’t have only $1$ and $8$ on the diagonal. In addition, $1 - x_i^2$ can appear at most once in each diagonal entry of $D$.

We can ignore the terms $1 - x_1 \cdots x_{d+1}$ and $1 + x_1 \cdots x_{d+1}$ for the moment since we will assign at least one of $x_i$ to $0$ in the following steps.

If we set $x_{d+1} = 0$, we will get two blocks of matrices corresponding to a general position using the earlier observation. If we set all other indeterminates equal to the indeterminate $q$, the diagonal form of $V_{x_{d+1}=0}$ has diagonal entries $(1 - x_i^2)$ and $(1 - x_j^2)$ (twice) for all $k = 0, \ldots, d - 1$, $1 \leq i < \cdots < k \leq d$. Thus, the SNF of $V_{x_i=0, x_j=q \forall j \neq i}$ (over $\mathbb{Z}[q]$) has diagonal entries $(1 - q^2)^k (2^d k)$ times for all $k = 0, 1, \ldots, d - 1$.

We call a diagonal entry of $D$ a $k$-entry if after setting $x_1 = \ldots = x_{d+1} = q$, it becomes $(1 - q^2)^k$. All diagonal entries of $D_{x_i=0, x_j=q \forall j \neq i}$ have the form $(1 - q^2)^k$ for some $k$ so it is already in SNF. Since SNF is unique, $D_{x_i=0, x_j=q \forall j \neq i}$ has diagonal entries $(1 - q^2)^k (2^d k)$ times for all $k = 0, 1, \ldots, d - 1$.

We then compute the exact number of $k$-entries in $D$. The number of $k$-entries in $D$ is $0$ if $k > d$; otherwise, there exists some $i \in \{1, \ldots, d+1\}$ such that $D_{x_i=0, x_j=q \forall j \neq i}$ has a diagonal entry $(1 - q^2)^d$, which leads to a contradiction.

**Claim 5.3.** The number of $(d - 2k - 1)$-entries in $D$ is $2(d+1)^{d-2k-2}$ and the number of $(d - 2k - 2)$-entries in $D$ is $0$.

**Proof.** We’ll prove the claim by induction on $k$. It is true when $k = 0$.

If the claim holds for $k$, i.e., the number of $(d - 2k)$-entries in $D$ is $0$. Assume that the number of $(d - 2k - 1)$-entries in $D$ is $m$ and $1 - x_i^2$ appears in $a_i$ of these entries.

Set $x_1 = 0$ and all other indeterminates equal to $q$. Since there are exactly $2(d - 2k - 1)$-entries of $(1 - q^2) d - 2k - 1$‘s and no $(d - 2k)$-entries in $D$, $m - a_i = 2(d - 2k - 1)$. Therefore $a_i$ is a constant with respect to $i$.

By a simple double counting of the total number of $(1 - \square^2)$‘s in those entries, where $\square = x_1, \ldots, x_{d+1}$, we have

$$\sum_{i=1}^{d+1} a_i = \binom{m}{d - 2k - 1} = m \cdot (d - 2k - 1).$$

It is easy to check that $m = 2(d+1)^{d-2k-2}$ and $a_i = 2(d+1)^{d-2k-2} - 2\binom{d - 2k - 1}{d - 2k - 2} = 2\binom{d - 2k - 1}{d - 2k - 2}$ for all $i = 1, \ldots, d + 1$.

Now if we assign $0$ to $x_i$ and $q$ to all other indeterminates, we already have $(1 - q^2)^d d - 2k - 2$ occuring $a_i = 2(d - 2k - 2)$ times so we do not need more. Therefore, all the $(d - 2k - 2)$-entries in $D$ must contain $1 - x_i^2$. It is true for all $i = 1, \ldots, d + 1$ so we have a contradiction unless the number of $(d - 2k - 2)$-entries is $0$.

This completes the induction and proves the claim. □

If $d$ is even, the number of $0$-entries of $\mathcal{A}$ is $0$ in $D$. In other words, if we assign $1$ to all indeterminates, $D$ becomes the all zero matrix with rank $0$, while the rank of $V$ becomes $1$, which is a contradiction by Lemma 2.8.
If $d$ is odd, we have to take into account the terms $1 - x_1 \cdots x_{d+1}$ and $1 + x_1 \cdots x_{d+1}$. As before, we will first show that they must pair up.

Let $x_1 = 3$, $x_i = 1$ for all $i \geq 2$ in $V$. Then in $V$, row $i$ is identical to row $j$ if region $i$ and $j$ are on the same side of $S_1$. Eliminating repeated rows with row and column operations, we get \[
\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \oplus 0
\]
where 0 is the all zero matrix, which has SNF \{1, 8, 0 (multiple times)\}. Note that when $d$ is odd, there are no 1−entries. Thus 1 and 8 must come from a combination of $1 - x_1 \cdots x_{d+1}$ and $1 + x_1 \cdots x_{d+1}$. Hence they must appear in the form of $1 - x_2 \cdots x_{d+1}$.

Furthermore, $1 - x_1^2 \cdots x_{d+1}$ must appear alone in a 0-entry. Otherwise, since there are no 1−entries in $D$, after assigning $x_i = 1$ for $i \geq 2$ we will end up with a matrix with only two 1’s on the diagonal and 0 everywhere else. Since $d$ is odd, the number of 0−entries is 2. One of them is $1 - x_2^2 \cdots x_{d+1}$ and the other one can only be a true 1.

Consider $V_{x_1=\cdots=x_{d+1}=3}$ and $D_{x_1=\cdots=x_{d+1}=3}$. Since $V_{x_1=\cdots=x_{d+1}=3}$ has a submatrix \[
\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix},
\]
so $\gcd(V_{x_1=\cdots=x_{d+1}=3}, 2) \leq 8$.

On the diagonal of $D_{x_1=\cdots=x_{d+1}=3}$, there is one 1, one $1 - 3^{d+2}$ and all other entries are multiples of $(1 - 3^2)^2 = 64$ since there is no 1-entry.

Note that $1 - 3^{d+2} = (1 - 3^2)(1 + 3^2 + 3^4 + \cdots + 3^{2d})$. Since $d$ is odd, so $1 + 3^2 + 3^4 + \cdots + 3^{2d}$ is even and $16 | (1 - 3^{2d+2})$. Therefore, $16 | \gcd(D_{x_1=\cdots=x_{d+1}=3}, 2)$, which leads to a contradiction by Lemma 2.8.

Hence we conclude that $V(a^f)$ does not have a diagonal form.

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