GLOBAL WELL-POSEDNESS FOR THE TWO-DIMENSIONAL EQUATIONS OF NONHOMOGENEOUS INCOMPRESSIBLE LIQUID CRystal FLOWS WITH NONNEGATIVE DENSITY

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Abstract. In this paper, we first establish the global well-posedness of strong solutions of the simplified Ericksen-Leslie model for nonhomogeneous incompressible nematic liquid crystal flows in dimensions two, if the initial data satisfies some smallness condition. It is worth pointing out that the initial density is allowed to contain vacuum states and the initial velocity can be arbitrarily large. Next, we present a Serrin’s type criterion, depending only on $\nabla d$, for the breakdown of local strong solutions. As a byproduct, the global strong solutions with large initial data are obtained, provided the macroscopic molecular orientation of the liquid crystal materials satisfies a natural geometric angle condition (cf. [19]).

1. Introduction. Liquid crystals are substances that exhibit a phase of matter that has properties between those of a conventional liquid and those of a solid crystal (cf. [12]). The hydrodynamic theory of liquid crystals was first developed by Ericksen and Leslie during the period of 1958 through 1968 (see [9, 10, 20, 21]). Since then, many remarkable developments have been made from both theoretical and applied aspects, however, many physically important and mathematically fundamental problems still remain open. In this paper, we consider a two-dimensional
simplified Erckten-Leslie model for the nonhomogeneous incompressible nematic liquid crystals:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P &= \Delta u - \nabla d \cdot \Delta d, \\
\text{div} u &= 0, \\
d_t + u \cdot \nabla d &= \Delta d + |\nabla d|^2 d,
\end{align*}
\]

where \( \rho : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}^+ \) is the density of the fluid, \( u : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}^2 \) is the velocity field, \( P : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R} \) is the pressure, and \( d : \mathbb{R}^2 \times [0, \infty) \to S^2 \) (the unit sphere in \( \mathbb{R}^3 \), i.e. \( |d| = 1 \)) represents the averaged macroscopic/continuum molecular orientations.

Although system (1.1)–(1.4) is a simplified version of the Ericksen-Leslie model, it still retains the most interesting mathematical properties without losing the basic nonlinear structure of the original Ericksen–Leslie model [9, 10, 20, 21]. Roughly speaking, the system (1.1)–(1.4) is a system of the nonhomogeneous Navier-Stokes equations for the incompressible flows coupled with the equation for the heat flow of harmonic maps, and thus, its mathematical analysis is full of challenges. In particular, if \( \rho \equiv \text{Const.} \), then it turns into the following homogeneous system which models the incompressible flows of nematic liquid crystal:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P &= \Delta u - \nabla d \cdot \Delta d, \\
\text{div} u &= 0, \\
ds_t + u \cdot \nabla d &= \Delta d + |\nabla d|^2 d,
\end{align*}
\]

with \( |d| = 1 \). Moreover, if \( u \equiv 0 \) in (1.5)–(1.7), then it reduces to the following equation for the heat flow of harmonic maps:

\[
d_t = \Delta d + |\nabla d|^2 d, \quad |d| = 1.
\]

There has been a lot of literature on the mathematical studies of (1.5)–(1.7) and (1.8), see, for example, [14, 13, 23, 25, 26, 24, 37, 35] and [3, 4, 5, 6, 32], respectively. In the following, we briefly recall some related mathematical results of the liquid crystal flows. In a series of papers, Lin [23] and Lin-Liu [25, 26] initiated the mathematical analysis of (1.5)–(1.7) in 1990s. More precisely, to relax the nonlinear constraint \( |d| = 1 \), they proposed an approximate model of Ericksen-Leslie system with variable length by Ginzburg-Landau functionals, that is, the equation (1.7) with \( |d| = 1 \) is replaced by

\[
d_t = \Delta d + |\nabla d|^2 d, \quad |d| = 1.
\]

In [23, 25], the authors proved the global existence of classical and weak solutions of (1.5), (1.6), (1.9) in dimensions two and three, respectively. The partial regularity of suitable weak solutions was also studied in [26]. In contrast with (1.9), it is much more difficult to deal with the nonlinear term \( |\nabla d|^2 d \) with \( |d| = 1 \) on the right-hand side of (1.4) or (1.7) from the mathematical point of view. In two independent papers, Hong [13] and Lin-Lin-Wang [24] showed the global existence of weak solutions of (1.5)–(1.7) in dimensions two, and proved that the solutions are smooth away from at most finitely many singular times which is analogous to that for the heat flows of harmonic maps (see [3, 32]). The global existence of smooth solution with small initial data of (1.5)–(1.7) was also proved [24, 37] and [35] in dimensions two and three, respectively. The general Ericksen-Leslie system,
in which the viscous (Leslie) stress and/or elastic (Ericksen) stress are included, was also considered by some researchers. In [34], Wang-Zhang-Zhang proved the local well-posedness of the general 3D Ericksen-Leslie system, and obtained the global well-posedness with special Ericksen stress coefficient and small initial data. Recently, Wang-Wang [33] studied the global weak solutions of 2D full Ericksen-Leslie system with general Ericksen stress coefficient.

For the approximate nonhomogeneous equations (1.1)–(1.4), the global existence of weak solutions with generally large initial data was proved in [28, 16], and the global regularity of the solution with strictly positive density was studied when the initial density may contain vacuum.

Assume that the initial data $\rho_0, u_0, d_0$ satisfies (1.1)–(1.4) and $(\rho_0, u_0, d_0)$ is a given unit vector (i.e. $|e| = 1$).

To state our main results, we first introduce the definition of strong solutions of (1.1)–(1.4), (1.10) and (1.11).

**Definition 1.1.** A pair of functions $(\rho, u, P, d)$ is called a strong solution of (1.1)–(1.4), (1.10) and (1.11) on $\mathbb{R}^2 \times [0, T]$, if $\rho \geq 0$ and $|d| = 1$ for all $(x, t) \in \mathbb{R}^2 \times [0, T]$,

\[
\begin{align*}
\rho - \tilde{\rho} &\in C([0, T]; H^2), \quad \rho_t \in L^\infty(0, T; H^1), \\
u &\in C([0, T]; H^2) \cap L^2(0, T; H^3), \quad \rho^{1/2} u_t \in L^\infty(0, T; L^2), \\
u_t &\in L^2(0, T; H^1), \quad \nabla P \in C([0, T]; L^2) \cap L^2(0, T; H^1), \\
d - e &\in C([0, T]; H^3), \quad (\nabla^2 d, d_t) \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2),
\end{align*}
\]

and $(\rho, u, P, d)$ satisfies (1.1)–(1.4) a.e. on $\mathbb{R}^2 \times (0, T]$.

Now, our first result concerning the global strong solutions with small data can be stated in the following theorem.

**Theorem 1.1.** Assume that the initial data $(\rho_0, u_0, d_0)$ satisfies

\[
\begin{align*}
\rho_0 &\geq 0, \quad (\rho_0 - \tilde{\rho}, u_0) \in H^2, \quad d_0 - e \in H^3, \quad \text{div} u_0 = 0, \quad |d_0| = 1, \\
\Delta u_0 - \nabla P_0 - \nabla d_0 \cdot \Delta d_0 &\equiv \rho_0^{1/2} g \text{ for some } (\nabla P_0, g) \in L^2.
\end{align*}
\]
Then there exists a unique global strong solution \((\rho, u, P, d)\) of (1.1)–(1.4), (1.10) and (1.11) on \(\mathbb{R}^2 \times [0, T] \) for any \(T \in (0, \infty)\), provided
\[
\exp\left(2 \left(\|\rho_0^{1/2}u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2\right)\right)\|\nabla d_0\|_{L^2}^2 \leq \frac{1}{16}.
\]

(1.14)

It is worth mentioning that the smallness condition (1.14) stated in Theorem 1.1 implies that \((\rho_0, u_0)\) can be arbitrarily large, if \(\|\nabla d_0\|_{L^2}\) is suitably small. This is analogous to the one in [37]. Moreover, it also shows that the Cauchy problem of nonhomogeneous Navier-Stokes equations (i.e. \(d \equiv \text{Const.}\)) with large initial data containing vacuum has a global strong solution on \(\mathbb{R}^2 \times [0, T] \) for all \(0 < T < \infty\). Thus, Theorem 1.1 also generalizes the result due to Huang-Wang [15].

The proof of Theorem 1.1 is mainly based on a critical Sobolev inequality of logarithmic type (see Lemma 2.4), which was originally due to Brezis-Wainger [2] (see also [29, 31]), and then was developed by Huang-wang [15] in the study of the global well-posedness of nonhomogeneous MHD equations with nonnegative density. Compared with MHD equations, the strong nonlinearities \(\nabla d \cdot \Delta d\) and \(\|\nabla d\|^2 d\) in (1.2) and (1.4) will cause some additional difficulties. Moreover, the arguments in [15] actually depend on the size of the domain and cannot be applied directly to the case of the whole space. We shall utilize a Poincaré’s type inequality (see Lemma 2.2) and the material derivative \(\|\rho^{1/2}u\|_{L^2}\) to overcome this difficulty. It is worth mentioning here that the strictly positive far-field condition \(\tilde{\rho} > 0\) plays an important role in our analysis.

For the generally large initial data, it is still an interesting and open problem whether the strong solution blows up or not in finite time. In [24] and [14], the authors proved respectively that the following blowup criteria for the two-dimensional equations of (1.5)–(1.7):
\[
\lim_{T \to T^*} \int_0^T (\|u\|_{L^4}^4 + \|\nabla d\|_{L^4}^4) \, dt = \infty \quad \text{and} \quad \lim_{T \to T^*} \int_0^T \|\nabla d\|_{L^\infty} \, dt = \infty,
\]
(1.15)

where \(0 < T^* < \infty\) is the maximal time of the existence of strong solutions to (1.5)–(1.7). Motivated by the proofs of Theorem 1.1, we can prove the following mechanism for possible breakdown of strong solutions, which is a natural extension of the ones in [24, 14].

**Theorem 1.2.** Assume that \(0 < T^* < \infty\) is the maximal time of the existence of strong solutions \((\rho, u, P, d)\) to (1.1)–(1.4), (1.10) and (1.11) with generally large initial data \((\rho_0, u_0, d_0)\) satisfying (1.13). Then,
\[
\lim_{T \to T^*} \int_0^T \|\nabla d\|_{L^r}^s \, dt = \infty,
\]
(1.16)

where \(r\) and \(s\) satisfy
\[
\frac{1}{r} + \frac{1}{s} \leq \frac{1}{2}, \quad 2 < r \leq \infty.
\]
(1.17)

Theorem 1.2 implies that for any \(0 < T < \infty\), if the left-hand side of (1.16) is finite, then the problem (1.1)–(1.4), (1.10) and (1.11) has a global strong solution \((\rho, u, P, d)\) on \(\mathbb{R}^2 \times (0, T)\).

Based upon a frequency localization argument combined with the concentration-compactness approach, Lei-Li-Zhang [19] recently proved the following interesting rigidity theorem for the approximate harmonic maps.
Proposition 1.1. ([19, Theorem 1.5]) Given positive constants \( C_0 \in (0, \infty) \) and \( \varepsilon \in (0, 1) \), assume that \( d : \mathbb{R}^2 \rightarrow S^2 \) satisfying \( \nabla d \in H^1 \), \( \| \nabla d \|_{L^p} \leq C_0 \) and \( d_3 \geq \varepsilon \). Then there is a positive constant \( \delta_0 \in (0, 1) \), depending only on \( C_0 \) and \( \varepsilon \), such that
\[
\| \nabla d \|_{L^4}^4 \leq (1 - \delta_0) \| \nabla^2 d \|_{L^2}^2,
\]
which particularly implies
\[
\| \Delta d + |\nabla d|^2 d \|_{L^2}^2 \geq \frac{\delta_0}{2} (\| \Delta d \|_{L^2}^2 + \| \nabla d \|_{L^4}^4).
\]

As an immediate consequence of Theorem 1.2 and Proposition 1.1, we can remove the smallness restriction (1.14) on the initial data and prove the following global existence result of strong solutions with large initial data, provided the macroscopic molecular orientation of the liquid crystal materials satisfies a natural geometric angle condition. This extends the Lei-Li-Zhang’s result (cf. [19]) to the case of nonhomogeneous incompressible liquid crystal flows with initial vacuum.

Theorem 1.3. Let \( e_3 = (0, 0, 1) \in S^2 \) and let \( d_{03} \) be the third component of \( d_0 \). In addition to the condition (1.13) of Theorem 1.1, assume further that
\[
d_{03} \geq \varepsilon \quad \text{and} \quad d_0 - e_3 \in L^2
\]
holds for some uniform positive constant \( \varepsilon \in (0, 1) \). Then for any \( T \in (0, \infty) \), there exists a unique global strong solution \((\rho, u, P, d)\) of (1.1)–(1.4), (1.10) and (1.11) on \( \mathbb{R}^2 \times [0, T] \).

The rest of the paper is organized as follows. In Sect. 2, we state some known inequalities and facts which will be used later. The proof of Theorem 1.1 will be done in Sect. 3, based on the local existence theorem and the global a priori estimates. In Sect. 4, we outline the proofs of Theorems 1.2 and 1.3.

2. Preliminaries. In this section, we list some useful lemmas which will be frequently used in the next sections. We first recall the well-known Ladyzhenskaya and Sobolev inequalities (see, for example, [17, 1]).

Lemma 2.1. For \( f \in H^1 \), it holds for any \( 2 \leq p < \infty \) that
\[
\| f \|_{L^4} \leq \sqrt{2} \| f \|_{L^2} \| \nabla f \|_{L^2},
\]
\[
\| f \|_{L^p} \leq C(p) \| f \|_{L^2} \| \nabla f \|_{L^2}^{1 - 2/p},
\]
where \( C(p) \) is a positive constant depending on \( p \). In addition, if \( f \in W^{1,p} \cap H^2 \) with \( p > 2 \), then there exists a universal positive constant \( C \) such that
\[
\| f \|_{L^\infty} \leq C \| f \|_{W^{1,p}} \leq C \| f \|_{H^2}.
\]

We will also use the following Poincaré type inequality, which implies that the velocity \( u \) actually belongs to \( L^2 \) even that the vacuum states may appear.

Lemma 2.2. For given positive number \( \tilde{\rho} \in (0, \infty) \), assume that \( \rho - \tilde{\rho} \in L^2 \cap L^\infty \) with \( \tilde{\rho}(x) \geq 0 \) and that \((\rho^{1/2} v, \nabla v) \in L^2 \). Then there exists a positive constant \( C \), depending only on \( \tilde{\rho} \), \( \| \rho - \tilde{\rho} \|_{L^2} \) and \( \| \rho - \tilde{\rho} \|_{L^\infty} \), such that
\[
\| v \|_{L^2} \leq C(\tilde{\rho}, \| \rho - \tilde{\rho} \|_{L^2 \cap L^\infty}) \left( \| \rho^{1/2} v \|_{L^2} + \| \nabla v \|_{L^2} \right).
\]
**Lemma 2.3.** Consider the following stationary Stokes equations:

\[ -\Delta U + \nabla P = f, \quad \text{div} U = 0 \quad \text{in} \quad \mathbb{R}^2. \]

Then for any \( f \in W^{m,p} \) with \( m \in \mathbb{Z}^+ \) and \( p > 1 \), there exists a positive constant \( C \), depending only on \( m \) and \( p \), such that

\[
\|\nabla^2 U\|_{W^{m,p}} + \|\nabla P\|_{W^{m,p}} \leq C\|f\|_{W^{m,p}}. \tag{2.5}
\]

To derive the \( L^2 \)-norm of the gradient of the velocity, we shall apply a critical Sobolev inequality of logarithmic type which was proved by Huang-Wang (cf. [15]) and is originally due to Brezis-Wainger [2] (see also [29, 31]). This is the key tool for the proofs of Theorems 1.1–1.3.

**Lemma 2.4.** Assume that \( f \in L^2(s,t;H^1) \cap L^2(s,t;W^{1,q}) \) with \( 0 \leq s < t < \infty \) and \( q > 2 \). Then there exists a positive constant \( C(q) \), independent of \( s \) and \( t \), such that

\[
\|f\|_{L^2(s,t;L^\infty)} \leq C \left[ 1 + \|\nabla f\|_{L^2(s,t;L^2)} \ln^{1/2} (e + \|f\|_{L^2(s,t;W^{1,q})}) \right]. \tag{2.6}
\]

In the case when the density is nonnegative, the local strong solutions to (1.1)–(1.4), (1.10) and (1.11) was obtained in [36]. Indeed, in [36] the authors only considered the case of smooth bounded domains, however, as pointed out in [7], the similar procedure also works for the whole space by means of the standard domain expansion technique. For simplicity, we quote the following local existence theorem of strong solutions without proofs.

**Lemma 2.5.** Assume that the conditions of Theorem 1.1 hold. Then, there exists a positive time \( T_0 \in (0, \infty) \) such that the Cauchy problem (1.1)–(1.4), (1.10) and (1.11) admits a unique strong solution on \( \mathbb{R}^2 \times (0, T_0) \).

3. **Proof of Theorem 1.1.** Assume that the conditions of Theorem 1.1 hold. Let \( 0 < T^* < \infty \) be the first blowup time of a strong solution \( (\rho, u, P, d) \) to the Cauchy problem (1.1)–(1.4), (1.10) and (1.11). In order to prove Theorem 1.1, it suffices to prove that there actually exists a generic positive constant \( 0 < M < \infty \), depending only on the initial data \( (\rho_0, u_0, d_0) \) and \( T^* \), such that

\[
\mathcal{E}(T) := \sup_{0 \leq t \leq T} \left( \|\rho - \tilde{\rho}\|_{H^2}^2 + \|u\|_{H^2}^2 + \|d - e\|_{H^2}^2 + \|\rho^{1/2} u_t\|_{L^2}^2 + \|d_t\|_{H^1}^2 \right)
+ \int_0^T \left( \|u\|_{H^3}^2 + \|\nabla d\|_{H^2}^2 + \|u_t\|_{H^1}^2 + \|d_t\|_{H^2}^2 \right) dt \leq M \tag{3.1}
\]
which, combined with (3.5), immediately leads to (3.4). Integrating by parts, we find

\[ 0 \leq \rho(x, t) \leq \|\rho_0\|_{L^\infty} \quad \text{for all} \quad (x, t) \in \mathbb{R}^2 \times [0, T]. \quad (3.2) \]

Moreover, multiplying (1.1) by \( q|\rho - \bar{\rho}|^{q-2}(\rho - \bar{\rho}) \) with \( q \geq 2 \) and integrating it by parts over \((0, t)\), by (1.3) we find

\[ \| (\rho - \bar{\rho})(t) \|_{L^q} = \| \rho_0 - \bar{\rho} \|_{L^q} \quad \text{for} \quad \forall t \in [0, T]. \quad (3.3) \]

In view of (1.1)–(1.4), we have the following standard energy estimates.

**Lemma 3.1.** For any \( 0 < T < T^* \), one has

\[
\sup_{0 \leq t \leq T} \left( \| \rho^{1/2} u \|_{L^2}^2 + \| \nabla d \|_{L^2}^2 \right) + 2 \int_0^T \left( \| \nabla u \|_{L^2}^2 + \| \Delta d + |\nabla d|^2 d \|_{L^2}^2 \right) dt \\
\leq \int \left( \| \rho^{1/2} u \|_{L^2}^2 + |\nabla d|^2 \right) (x, 0) dx := E_0. \quad (3.4)
\]

**Proof.** Multiplying (1.2) by \( u \) in \( L^2 \) and integrating by parts, by (1.3) we have

\[
\frac{1}{2} \frac{d}{dt} \int \rho |u|^2 dx + \int |\nabla u|^2 dx = - \int (u \cdot \nabla d \cdot \Delta d) dx. \quad (3.5)
\]

In view of the fact that \( |d| = 1 \), multiplying (1.4) by \( (\Delta d + |\nabla d|^2 d) \) in \( L^2 \) and integrating by parts, we find

\[
\frac{1}{2} \frac{d}{dt} \int |\nabla d|^2 dx + \int |\Delta d + |\nabla d|^2 d|^2 dx \\
= \int (u \cdot \nabla d \cdot \Delta d) dx + \int (|\nabla d|^2 d \cdot d + |\nabla d|^2 u \cdot \nabla d \cdot d) dx \\
= \int (u \cdot \nabla d \cdot \Delta d) dx + \frac{1}{2} \int (|\nabla d|^2 \partial_t |d|^2 + |\nabla d|^2 u \cdot |\nabla d|^2) dx \quad (3.6)
\]

which, combined with (3.5), immediately leads to (3.4). \( \square \)

To be continued, we need the following key estimate on \( \| \nabla^2 d \|_{L^2(0, T; L^2)}. \)

**Lemma 3.2.** Assume that the initial data satisfies

\[
\exp \left( 2 \left( \| \rho_0^{1/2} u_0 \|_{L^2}^2 + \| \nabla d_0 \|_{L^2}^2 \right) \right) \| \nabla d_0 \|_{L^2}^2 \leq \frac{1}{16}. \quad (3.7)
\]

then for any \( 0 < T < T^* \), it holds that

\[
\sup_{0 \leq t \leq T} \| \nabla d \|_{L^2}^2 + \int_0^T \| \nabla^2 d \|_{L^2}^2 dt \leq \frac{1}{16}, \quad (3.8)
\]
and moreover,
\[
\sup_{0 \leq t \leq T} \|d - e\|_{L^2}^2 + \int_0^T \|\nabla d\|_{L^2}^2 dt \leq C. \tag{3.9}
\]

**Proof.** On one hand, we easily obtain from the identity |d| = 1 that
\[
\int |\Delta d + |\nabla d|^2|dx = \int (|\Delta d|^2 + |\nabla d|^4) dx + 2 \int |\nabla d|^2(\Delta d)dx = \int (|\Delta d|^2 - |\nabla d|^4) dx. \tag{3.10}
\]

On the other hand, by (1.3) we deduce after integrating by parts that
\[
\int (u \cdot \nabla d \cdot \Delta d) dx = -\int (\partial_j u^i \partial_i d^k \partial_j d^k + \partial_i u^i \partial_j d^k \partial_j d^k) dx = -\int (\partial_j u^i \partial_i d^k \partial_j d^k) dx \leq \|\nabla u\|_{L^2} \|\nabla d\|_{L^2}^2, \tag{3.11}
\]

where and in what follows the repeated indices denotes the summation over the indices. Thus, substituting (3.10), (3.11) into (3.6) and recalling the fact that

and thus, one infers from the local existence theorem and the continuity argument that there exists a $T_1 > 0$ such that

\[
\|(\nabla d)(t)\|_{L^2} \leq \frac{1}{8}, \quad \forall t \in [0, T_1]. \tag{3.13}
\]

Set
\[
\bar{T} := \sup\{T \mid (3.13) \text{ holds for all } t \in (0, T)\}.
\]

Then it follows from (3.12)–(3.13) that for any $t \in [0, \bar{T})$,
\[
\frac{d}{dt}\|\nabla d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 \leq 4\|\nabla u\|_{L^2}^2 \|\nabla d\|_{L^2}^2,
\]

which, together with Gronwall’s inequality and (3.4), gives
\[
\|\nabla d\|_{L^2}^2 + \int_0^t \|\nabla^2 d\|_{L^2}^2 d\tau \leq \exp\left(4 \int_0^t \|\nabla u\|_{L^2}^2 d\tau\right) \|\nabla d_0\|_{L^2}^2 \leq 2e^{2E_0}\|\nabla d_0\|_{L^2}^2 \leq \frac{1}{16} \tag{3.14}
\]

Combining (3.4), (3.14) with the continuity argument immediately shows that (3.13) holds for all $0 < \bar{T} < T^*$. The proof of (3.8) is therefore finished.
As a result, multiplying (1.4) by \(d - e\) in \(L^2\), integrating by parts, using (1.3), (2.1) and the fact that \(|d| = 1\), we find
\[
\frac{1}{2} \frac{d}{dt} \|d - e\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 \leq \int |\nabla d|^2 |d - e| dx \leq \|\nabla d\|_{L^4} \|d - e\|_{L^2} \\
\leq C \|\nabla d\|_{L^2} \|\nabla d\|_{L^2} \|d - e\|_{L^2} \\
\leq \frac{1}{2} \|\nabla d\|_{L^2}^2 + C \|\nabla d\|_{L^2}^2 \|d - e\|_{L^2}^2,
\]
which, together with (3.8) and Gronwall’s inequality, proves (3.9). \(\square\)

By Lemmas 2.4 and 3.2, we can now derive the estimates of \(\|\nabla u\|_{L^2}\) and \(\|\nabla^2 d\|_{L^2}\) which is the most important step among the proofs.

**Lemma 3.3.** For any \(0 < T < T^*\), one has
\[
\sup_{0 \leq t \leq T} \left( \|u\|_{H^1}^2 + \|\nabla d\|_{H^1}^2 + \|d_t\|_{L^2}^2 \right) \\
\quad + \int_0^T \left( \|\nabla^2 u\|_{L^2}^2 + \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|d_t\|_{H^1}^2 + \|\nabla d\|_{L^2}^2 \right) dt \leq C,
\]
which particularly gives
\[
\int_0^T \|\rho^{1/2} u_t\|_{L^2}^2 dt \leq C.
\]

**Proof.** Let \(\dot{f} := f_t + u \cdot \nabla f\) denote the material derivative. Also set
\[
M(d) := \nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 I, \quad (\nabla d \otimes \nabla d)_{ij} := \frac{\partial d}{\partial x_i} \frac{\partial d}{\partial x_j}, \quad 1 \leq i, j \leq 2.
\]
Then it is easily seen that
\[
\nabla d \cdot \Delta d = \text{div}(M(d)).
\]

To prove (3.15), multiplying (1.2) by \(u_t\) and integrating by parts, by (3.2) and Cauchy-Schwarz’s inequality we deduce
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\rho^{1/2} \dot{u}\|_{L^2}^2 \\
= - \int \text{div}(M(d)) \cdot u_t dx + \int \rho u \cdot \nabla u \cdot \dot{u} dx \\
= \frac{d}{dt} \int M(d) : \nabla u dx - \int M(d)_t : \nabla u dx + \int \rho u \cdot \nabla u \cdot \dot{u} dx \\
\leq \frac{d}{dt} \int M(d) : \nabla u dx + \frac{1}{2} \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \frac{1}{4} \|\nabla d_t\|_{L^2}^2 + C(\|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2) \|\nabla u\|_{L^2}^2,
\]
where \(A : B := \sum_{i,j=1}^2 a_{ij} b_{ij}\) for \(A = (a_{ij})_{2 \times 2}\) and \(B = (b_{ij})_{2 \times 2}\). As a result,
\[
\frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\rho^{1/2} \dot{u}\|_{L^2}^2 \\
\leq 2 \frac{d}{dt} \int M(d) : \nabla u dx + \frac{1}{2} \|\nabla d_t\|_{L^2}^2 + C(\|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2) \|\nabla u\|_{L^2}^2.
\]

(3.17)

Next, one easily obtains from (1.4) that
\[
\frac{d}{dt} \|\nabla d\|_{L^2}^2 + (\|d_t\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) \leq C \int (|u|^2 |\nabla d|^2 + |\nabla d|^4) dx \\
\leq C (\|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2) \|\nabla d\|_{L^2}^2.
\]

(3.18)
To deal with the term \( \| \nabla d_t \|_{L^2}^2 \) on the right-hand side of (3.17), we first apply \( \nabla \) to both sides of (1.4) to get that
\[
\nabla d_t - \nabla \Delta d = -\nabla (u \cdot \nabla d) + \nabla (|\nabla d|^2 d),
\]
from which we deduce after direct computations that
\[
\int d \frac{d}{dt} \| \nabla^2 d \|_{L^2}^2 + \left( \| \nabla d_t \|_{L^2} + \| \nabla \Delta d \|_{L^2}^2 \right)
\leq C \int \left( |\nabla d|^6 + |\nabla d|^2 |\nabla^2 d|^2 + |\nabla u|^2 |\nabla d|^2 + |u|^2 |\nabla^2 d|^2 \right) dx
\]
(3.20)
where we have used (2.1) and (3.4) to get that
\[
\| \nabla d \|_{L^8}^6 \leq C \| \nabla d \|_{L^2}^2 \| \nabla d_t \|_{L^2}^2 \| \nabla^2 d \|_{L^2}^2 \leq C \| \nabla d \|_{L^8}^2 \| \nabla^2 d \|_{L^2}^2.
\]
Thanks to (2.1) and (3.4), we have
\[
\int M(d) \cdot \nabla u dx \leq \frac{1}{4} \| \nabla u \|_{L^2}^2 + C \| \nabla d \|_{L^2}^2 \| \nabla^2 d \|_{L^2}^2 \leq \frac{1}{4} \| \nabla u \|_{L^2}^2 + C_1 \| \nabla^2 d \|_{L^2}^2.
\]
Thus, taking this into account, multiplying (3.20) by 2C_1 + 1, and adding the resulting inequality, (3.17) and (3.18) together, we obtain after integrating the resulting inequality over \((s, t)\) with \(0 \leq s < t < T\) that
\[
\left( \| \nabla u \|_{L^2}^2 + \| \nabla d \|_{H^1}^2 \right) (t) + \int_s^t \left( |\rho^{1/2} \dot{u}|_{L^2}^2 + \| d_t \|_{H^1}^2 + \| \nabla^2 d \|_{H^1}^2 \right) d\tau
\leq C \left( \| \nabla u \|_{L^2}^2 + \| \nabla d \|_{H^1}^2 \right) (s) + C \int_s^t \left( \| \nabla u \|_{L^\infty}^2 + \| \nabla d \|_{H^1}^2 \right) d\tau,
\]
and consequently,
\[
\left( \| \nabla u \|_{L^2}^2 + \| \nabla d \|_{H^1}^2 \right) (t) + \int_s^t \left( |\rho^{1/2} \dot{u}|_{L^2}^2 + \| d_t \|_{H^1}^2 + \| \nabla^2 d \|_{H^1}^2 \right) d\tau
\leq C \left( \| \nabla u \|_{L^2}^2 + \| \nabla d \|_{H^1}^2 \right) (s) \exp \left( C \int_s^t \left( |\nabla u \|_{L^\infty}^2 + \| \nabla d \|_{H^1}^2 \right) d\tau \right).
\]
(3.21)
Clearly, it remains to estimate \( \| (u, \nabla d) \|_{L^\infty} \). To this end, let
\[
\Phi(t) := e + \sup_{0 \leq \tau \leq t} \left( \| \nabla u \|_{L^2}^2 + \| \nabla d \|_{H^1}^2 \right) + \int_0^t \left( |\rho^{1/2} \dot{u}|_{L^2}^2 + \| d_t \|_{H^1}^2 + \| \nabla^2 d \|_{H^1}^2 \right) d\tau.
\]
First, in view of (2.4) and (3.2)–(3.4), we have
\[
\| u \|_{L^2}^2 \leq C \left( \| \rho^{1/2} \dot{u} \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 \right) \leq C \left( 1 + \| \nabla u \|_{L^2}^2 \right).
\]
(3.22)
Next, using Lemma 2.3, (2.1), (3.2) and (3.4), we deduce from Hölder’s and Cauchy-Schwarz’s inequalities that
\[
\| \nabla^2 u \|_{L^2} \leq C \left( \| \rho \dot{u} \|_{L^2} + \| \nabla d \cdot \Delta d \|_{L^2} \right)
\leq C \left( \| \rho^{1/2} \dot{u} \|_{L^2} + \| \nabla^2 d \|_{L^2}^2 + \| \nabla^3 d \|_{L^2} \right),
\]
(3.23)
which, combined with (3.8), yields

\[
\int_s^t \| \nabla^2 u \|^2_{L^2} d\tau \leq C \int_s^t \left( \| \rho^{1/2} \dot{u} \|^2_{L^2} + \| \nabla^2 d \|^2_{L^2} + \| \nabla^3 d \|^2_{L^2} \right) d\tau
\leq C \sup_{s \leq \tau \leq t} \| \nabla^2 d \|^2_{L^2} + C \int_s^t \left( \| \rho^{1/2} \dot{u} \|^2_{L^2} + \| \nabla^3 d \|^2_{L^2} \right) d\tau. \tag{3.24}
\]

Recalling the definition of \( \Phi(T) \), using (3.4), (3.22) and (3.24), we infer from Lemma 2.4 that for any 0 \( \leq s < t \leq T < T^* \),

\[
\| u \|^2_{L^2(s,t;L^\infty)} \leq C \left( 1 + \| \nabla u \|^2_{L^2(s,t;L^2)} \ln \left( e + \| u \|_{L^2(s,t;W^{1,4})} \right) \right) \leq C \left( 1 + \| \nabla u \|^2_{L^2(s,t;L^2)} \ln \left( e + \| u \|_{L^2(s,t;H^2)} \right) \right) \leq C \left( 1 + \| \nabla u \|^2_{L^2(s,t;L^2)} \ln \left( C \Phi(t) \right) \right). \tag{3.25}
\]

In a similar manner, by (2.6), (3.8) and (3.9) one has

\[
\| \nabla d \|^2_{L^2(s,t;L^\infty)} \leq C \left( 1 + \| \nabla^2 d \|^2_{L^2(s,t;L^2)} \ln \left( e + \| \nabla d \|_{L^2(s,t;W^{1,4})} \right) \right) \leq C \left( 1 + \| \nabla^2 d \|^2_{L^2(s,t;L^2)} \ln \left( e + \| \nabla d \|_{L^2(s,t;H^2)} \right) \right) \leq C \left( 1 + \| \nabla^2 d \|^2_{L^2(s,t;L^2)} \ln \left( C \Phi(t) \right) \right). \tag{3.26}
\]

So, inserting (3.25) and (3.26) into (3.21) shows that for any 0 \( \leq s < t \leq T < T^* \),

\[
\Phi(t) \leq C \Phi(s) \exp \left\{ C_2 \left( \| \nabla u \|^2_{L^2(s,t;L^2)} + \| \nabla^2 d \|^2_{L^2(s,t;L^2)} \right) \ln \left( C_1 \Phi(t) \right) \right\} \leq C \Phi(s) \left[ C_1 \Phi(t) \right]^{C_2} \left( \| \nabla u \|^2_{L^2(s,t;L^2)} + \| \nabla^2 d \|^2_{L^2(s,t;L^2)} \right). \tag{3.27}
\]

Note that the bounds in (3.4) and (3.8) are independent of \( T \). Thus, it follows from (3.4) and (3.8) that there exists a positive constant \( \delta > 0 \) such that

\[
C_2 \left( \| \nabla u \|^2_{L^2(T-\delta,T;L^2)} + \| \nabla^2 d \|^2_{L^2(T-\delta,T;L^2)} \right) \leq \frac{1}{2},
\]

which, inserted into (3.27) and combined with Cauchy-Schwarz’s inequality, yields

\[
\Phi(T) \leq C \Phi(T-\delta) \left[ C_1 \Phi(T) \right]^{1/2} \leq \frac{1}{2} \Phi(T) + C \Phi^2(T-\delta),
\]

and consequently,

\[
\Phi(T) \leq C(T) \Phi^2(T-\delta). \tag{3.28}
\]

As a result of (3.28), we see that \( \Phi(T) \) is bounded for any \( 0 < T < T^* \), since the local existence theorem indicates \( \Phi(T-\delta) < \infty \) for any \( 0 < T < T^* \). This, together with (3.22) and (3.23), finishes the proof of (3.15). Note that, the boundedness of \( \| u \|_{L^2} \) can be easily obtained from (1.4).

Furthermore, recalling the definition of material derivative (i.e. \( \cdot \cdot \cdot \)), one gets from (2.1), (3.2) and (3.15) that

\[
\int_0^T \| \rho^{1/2} u_t \|^2_{L^2} dt \leq \int_0^T \| \rho^{1/2} u \|^2_{L^2} dt + \int_0^T \int \rho |u|^2 |\nabla u|^2 dxdt.
\]
and Cauchy-Schwarz’s inequality we deduce

\[
\leq C + C \int_0^T \| u \|_{L^2}^2 \| \nabla u \|_{L^2}^2 dt
\]

\[
\leq C + C \int_0^T \| u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} dt
\]

\[
\leq C + C \int_0^T \| \nabla^2 u \|_{L^2}^2 dt \leq C,
\]

which immediately proves (3.16). The proof of Lemma 3.3 is therefore complete.

Next, we proceed to estimate \( \| \rho^{1/2} u_t \|_{L^2} \) and \( \| \nabla d_t \|_{L^2} \).

**Lemma 3.4.** For any \( 0 < T < T^* \), one has

\[
\sup_{0 \leq t \leq T} \left( \| \rho^{1/2} u_t \|_{L^2}^2 + \| d_t \|_{H^1}^2 \right) + \int_0^T \left( \| \nabla u_t \|_{L^2}^2 + \| d_t \|_{L^2}^2 \right) dt \leq C,
\]

and moreover,

\[
\sup_{0 < t \leq T} \left( \| u_t \|_{H^2}^2 + \| \nabla d \|_{H^2}^2 \right) + \int_0^T \left( \| \nabla u \|_{W^{1,4}}^2 + \| \nabla^2 d \|_{H^2}^2 \right) dt \leq C.
\]

**Proof.** Differentiating (1.2) with respect to \( t \) gives

\[
\rho u_{tt} + \rho u \cdot \nabla u_t - \Delta u_t = -\rho_t u_t + u \cdot \nabla u - \rho u \cdot \nabla P_t - \text{div} M_t,
\]

which, multiplied by \( u_t \) in \( L^2 \) and integrated by parts over \( \mathbb{R}^2 \), results in

\[
\frac{1}{2} \frac{d}{dt} \| \rho^{1/2} u_t \|_{L^2}^2 + \| \nabla u_t \|_{L^2}^2
\]

\[
= -\int \rho_t (u_t + u \cdot \nabla u) \cdot u_t dx - \int \rho u_t \cdot \nabla u \cdot u_t dx + \int M_t \cdot \nabla u_t dx \tag{3.31}
\]

\[
:= I_1 + I_2 + I_3.
\]

We are now in a position of estimating the right-hand side of (3.31) term by term. First, using (1.1) and integrating by parts, by Lemmas 2.1–2.2, (3.2), (3.15) and Cauchy-Schwarz’s inequality we deduce

\[
I_1 = \int \left( \rho u \cdot \nabla |u_t|^2 + \rho u \cdot \nabla (u \cdot \nabla u \cdot u_t) \right) dx
\]

\[
\leq C \int \left( \rho |u_t| \| \nabla u_t \| + \rho |u| \| \nabla u \| \| u_t \| + \rho |u| \| \nabla^2 u \| \| u_t \| + \rho |u| \| \nabla u \| \| \nabla u_t \| \right) dx
\]

\[
\leq C \left( \| u \|_{L^\infty} \| \rho^{1/2} u_t \|_{L^2} \| \nabla u_t \|_{L^2} + \| u \|_{L^\infty} \| \nabla u \|_{L^2} \| \rho^{1/2} u_t \|_{L^2} \right)
\]

\[
+ C \left( \| u \|_{L^2}^2 \| \nabla^2 u \|_{L^2} + \| u \|_{L^2}^2 \| \nabla u \|_{L^2} \| \nabla u_t \|_{L^2} \right)
\]

\[
\leq \frac{1}{4} \| \nabla u_t \|_{L^2}^2 + C \left( 1 + \| \rho^{1/2} u_t \|_{L^2}^2 + \| \nabla^2 u \|_{L^2}^2 \right),
\]

where we have used the following estimates:

\[
\| u \|_{L^\infty}^2 \leq C \| u \|_{W^{1,4}} \leq C \left( \| u \|_{H^1}^2 + \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} \right) \leq C \left( 1 + \| \nabla^2 u \|_{L^2} \right), \tag{3.32}
\]

and

\[
\| u_t \|_{L^2} \leq C \left( \| \rho^{1/2} u_t \|_{L^2} + \| \nabla u_t \|_{L^2} \right), \quad \| u_t \|_{L^2}^2 \leq C \| u_t \|_{L^2} \| \nabla u_t \|_{L^2} \tag{3.33}
\]
In view of (2.1), (2.3), (3.15) and (3.33), we can estimate $I_2$ as follows:

$$
|I_2| \leq C \|\nabla u\|_{L^4} \|u_t\|_{L^4} \|\rho^{1/2} u_t\|_{L^2} \\
\leq C \|\nabla u\|_{L^2}^{1/2} \|\nabla^2 u\|_{L^2}^{1/2} \|u_t\|_{L^2}^{1/2} \|\nabla u_t\|_{L^2}^{1/2} \|\rho^{1/2} u_t\|_{L^2} \\
\leq C \left( \|\nabla u_t\|_{L^2}^{1/2} + \|\rho^{1/2} u_t\|_{L^2}^{1/2} \right) \|\nabla u_t\|_{L^2}^{1/2} \|\nabla^2 u\|_{L^2}^{1/2} \|\rho^{1/2} u_t\|_{L^2} \\
\leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + C \left( 1 + \|\nabla^2 u\|_{L^2}^{1/2} + \|\rho^{1/2} u_t\|_{L^2}^{1/2} \right).
$$

Finally, it is easily seen from (2.3) and (3.15) that

$$
|I_3| \leq C \|\nabla d\|_{L^\infty} \|\nabla u_t\|_{L^2} \|\nabla u_t\|_{L^2} \\
\leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + C \left( 1 + \|\nabla^3 d\|_{L^2}^2 \right) \|\nabla u_t\|_{L^2}^2.
$$

Substituting the estimates of $I_1$, $I_2$ and $I_3$ into (3.31), one obtains

$$
\frac{d}{dt} \|\rho^{1/2} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \\
\leq C \left( 1 + \|\nabla^2 u\|_{L^2}^2 + \|\rho^{1/2} u_t\|_{L^2}^2 \right) + C \left( 1 + \|\nabla^3 d\|_{L^2}^2 \right) \|\nabla u_t\|_{L^2}^2.
$$

(3.34)

To estimate $\|\nabla d_t\|_{L^2}$, we differentiate (1.4) with respect to $t$ to get

$$
d_{tt} - \Delta d_t = (\|\nabla d\|^2 d - u \cdot \nabla d)_t,
$$

and hence, using Lemma 2.1, (3.15) and (3.33), we deduce

$$
\frac{d}{dt} \|\nabla d_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2 \\
\leq C \int \left( \|\nabla d\|^2 \|\nabla d_t\|^2 + \|\nabla d_t\|^4 \|d_t\|^2 + \|u_t\|^2 \|\nabla d_t\|^2 + \|u\|^2 \|\nabla d_t\|^2 \right) dx \\
\leq C \left( \|\nabla d\|_{L^\infty}^2 + \|u\|_{L^\infty}^2 \right) \|\nabla d_t\|_{L^2}^2 + C \left( \|\nabla d\|_{L^\infty}^2 \|d_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 \|u_t\|_{L^2}^2 \right) \\
\leq C \left( \|\nabla d\|_{L^2}^2 + \|u_t\|_{L^2}^2 \right) \|\nabla d_t\|_{L^2}^2 + C \left( 1 + \|\nabla d\|_{L^2}^2 + \|\rho^{1/2} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \right),
$$

(3.35)

where we have also used (2.1), (2.3) and (3.15) to get that

$$
\|\nabla d\|_{L^\infty}^2 \leq C \|\nabla d\|_{W^{1,4}}^4 \leq C \left( 1 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) \leq C \left( 1 + \|\nabla^3 d\|_{L^2}^2 \right).
$$

Now, multiplying (3.34) by $2C_1 + 1$ and adding it to (3.35), we see that

$$
\frac{d}{dt} \left( \|\rho^{1/2} u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 \right) \|\nabla u_t\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2 \\
\leq C \left( 1 + \|\nabla d\|_{L^2}^2 + \|u\|_{L^2}^2 \right) \|\nabla d_t\|_{L^2}^2 \|\rho^{1/2} u_t\|_{L^2}^2, \\
$$

which, combined with (3.15), (3.16) and Gronwall’s inequality, leads to (2.29), since the compatibility condition stated in (1.13) implies that $(\rho^{1/2} u_t)(x, 0) \in L^2$ is well defined.

Using (2.1), (2.3), (3.15) and (2.29), we have by (2.5) that

$$
\|\nabla^2 u\|_{L^2} \leq C \left( \|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} + \|\nabla d \cdot \Delta d\|_{L^2} \right) \\
\leq C \left( \|\rho^{1/2} u_t\|_{L^2} + \|u\|_{L^4} \|\nabla u\|_{L^4} + \|\nabla d\|_{L^4} \|\nabla^2 d\|_{L^4} \right) \\
\leq C \left( 1 + \|\nabla^2 u\|_{L^2}^{1/2} + \|\nabla^3 d\|_{L^2}^{1/2} \right).
$$

(3.36)
Similarly, one also infers from (3.19) that
\[
\|\nabla^3 d\|_{L^2} \leq C \left( \|\nabla d\|_{L^2} + \|\nabla (u \cdot \nabla d)\|_{L^2} + \|\nabla (|\nabla d|^2)\|_{L^2} \right)
\]
which, together with (3.36) and Young’s inequality, yields
\[
\sup_{0 \leq t \leq T} (\|u\|_{H^2} + \|\nabla d\|_{H^2}) \leq C. \quad (3.37)
\]
Using (3.2), (3.29), (3.33) and (3.37), we deduce from (2.5) and Lemma 2.1 that
\[
\int_0^T \|\nabla^2 u\|_{L^4}^4 \leq C \int_0^T \left( \|\rho u_t\|_{L^4}^2 + \|\rho u \cdot \nabla u\|_{L^4}^2 + \|\nabla d \cdot \Delta d\|_{L^4}^2 \right) dt
\]
\[
\leq C \int_0^T \left( \|\rho^{1/2} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|u\|_{H^2}^2 + \|\nabla d\|_{H^2}^4 \right) dt
\]
\[
\leq C + C \int_0^T \|\nabla u_t\|_{L^2}^2 dt \leq C.
\]
Moreover, by virtue of Lemma 2.1, (3.15), (3.29) and (3.30) we infer from (1.4) that
\[
\int_0^T \|\nabla d\|_{H^2}^2 dt \leq C \int_0^T \left( \|u_t\|_{H^2}^2 + \|u \cdot \nabla d\|_{H^2}^2 + \|\nabla (|\nabla d|^2)\|_{H^2}^2 \right) dt
\]
\[
\leq C \int_0^T \left( 1 + \|u_t\|_{H^2}^2 + \|u\|_{H^2}^2 \|\nabla d\|_{H^2}^2 + \|\nabla d\|_{H^2}^2 + \|\nabla (|\nabla d|^2)\|_{H^2}^2 \right) dt
\]
\[
\leq C,
\]
where we have used the following Moser’s type calculus inequality (see [30] ) that for \( f, g \in H^s(\mathbb{R}^2) \) with \( s \geq 2 \),
\[
\|fg\|_{H^s} \leq C (\|f\|_{L^\infty} \|g\|_{H^s} + \|f\|_{H^s} \|g\|_{L^\infty}) \leq C \|f\|_{H^s} \|g\|_{H^s}.
\]
This, together with (3.37) and (3.38), leads to (3.30) immediately.

The last step is to estimate the first and second order derivatives of the density.

**Lemma 3.5.** For any \( 0 < T < T^* \), one has
\[
\sup_{0 \leq t \leq T} (\|\nabla \rho\|_{H^1} + \|\rho_t\|_{H^1}) + \int_0^T \|\nabla u\|_{H^2}^2 dt \leq C. \quad (3.38)
\]

*Proof.* Differentiating (1.1) with respect to \( x_i \) \( (i = 1, 2) \), multiplying the resulting equation by \( |\nabla \rho|^{p-2} \partial_i \rho \) with \( p \geq 2 \), and integrating it by parts over \( \mathbb{R}^2 \), we obtain after summing up
\[
\frac{d}{dt} \|\nabla \rho\|_{L^p}^p \leq C \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^p}^p \leq C \|\nabla u\|_{W^{1,4}} \|\nabla \rho\|_{L^p}^p,
\]
which, combined with (3.30) and Gronwall’s inequality, yields
\[
\|\nabla \rho\|_{L^p}^p \leq C \|\nabla \rho_0\|_{L^p}^p \exp \left( C \int_0^T \|\nabla u\|_{W^{1,4}} dt \right) \leq C, \quad \forall \ p \geq 2. \quad (3.39)
\]
Similarly, by (3.39) we also deduce from (1.1) that
\[
\frac{d}{dt} \|\nabla^2 \rho\|_{L^2}^2 \leq C \|\nabla u\|_{L^\infty} \|\nabla^2 u\|_{L^4} \|\nabla \rho\|_{L^4} \|\nabla^2 \rho\|_{L^2} + C \|\nabla \rho\|_{L^4} \|\nabla^2 \rho\|_{L^2} ,
\]
so that
\[
\|\nabla^2 \rho\|_{L^2}^2 \leq C \|\nabla u\|_{W^{1,4}} (1 + \|\nabla^2 \rho\|_{L^2}^2) \exp \left( C \int_0^T \|\nabla u\|_{W^{1,4}} \, dt \right) \leq C. \tag{3.40}
\]
As a result of (3.29), (3.39) and (3.40), one easily gets from (1.1) that
\[
\|\rho_t\|_{H^1} \leq C.
\]
Finally, it follows from (2.5), (3.2), (3.15), (3.29) and (3.38) that
\[
\|\nabla u\|_{H^2} \leq C (\|\rho u\|_{H^1} + \|\rho u \cdot \nabla u\|_{H^1} + \|\nabla d \cdot \Delta d\|_{H^1})
\]
\[
\leq C \left( \|\rho^{1/2} u\|_{L^2} + \|\nabla \rho\|_{L^4} \|u_t\|_{L^4} + \|\nabla u_t\|_{L^2} \right)
\]
\[
+ C \left( \|u\|_{H^2} + \|u\|_{L^\infty} \|\nabla \rho\|_{L^4} \|u\|_{L^4} + \|\nabla d\|_{H^1} \right)
\]
\[
\leq C \left( 1 + \|\nabla u_t\|_{L^2} \right),
\]
where we have also used Lemmas 2.1 and 2.2. Consequently,
\[
\int_0^T \|\nabla u\|_{H^2}^2 \, dt \leq C + C \int_0^T \|\nabla u_t\|_{L^2}^2 \, dt \leq C,
\]
which, together with (3.39) and (3.40), proves (3.38).

Collecting all the estimates in (3.2), (3.3) and Lemmas 3.1–3.5 together, we arrive at (3.1), and hence, the proof of Theorem 1.1 is complete.

4. Proofs of Theorems 1.2 and 1.3. This section is concerned with the proofs of Theorems 1.2 and 1.3. We first prove Theorem 1.2 by using contradiction arguments. So, to do this, we assume otherwise that
\[
\lim_{T \to T^*} \int_0^T \|\nabla d\|_{L^4}^r \, dt \leq M_0 < \infty
\]
with any \((r, s)\) satisfying (1.17).

We begin the proof with the observation from the proof of Theorem 1.1 (see (3.12)) that, to remove the smallness condition (1.14) and to obtain a global strong solution with generally large initial data, it suffices to achieve the estimate of \(\|\nabla^2 d\|_{L^2(0, T; L^2)}\) for any \(0 < T < T^*\). Moreover, it follows from (3.10) and (3.4) that for any \(0 < T < T^*\),
\[
\int_0^T \|\nabla^2 d\|_{L^2}^2 \, dt = \int_0^T \|\Delta d + |\nabla d|^2 d\|_{L^2}^2 \, dt + \int_0^T \|\nabla d\|_{L^4}^4 \, dt \leq \frac{1}{2} E_0 + \int_0^T \|\nabla d\|_{L^4}^4 \, dt.
\]
Therefore, to bound \(\|\nabla^2 d\|_{L^2(0, T; L^2)}\), we only need to deal with \(\|\nabla d\|_{L^4(0, T; L^4)}\). This will be done in the following.

On one hand, assume that \((r, s)\) satisfies
\[
\frac{1}{r} + \frac{1}{s} \leq \frac{1}{2} \quad \text{with} \quad 4 \leq r \leq \infty. \tag{4.3}
\]
Then, using Hölder’s inequality, (2.1) and (3.4), we find
\[
\|\nabla d\|_4^4 \leq C \|\nabla d\|_2^{(r-4)/(r-2)} \|\nabla d\|_L^{r-2} \|\nabla^2 d\|_L^2 \leq C \|\nabla d\|_L^r \|\nabla^2 d\|_L^2 \leq \frac{1}{2} \|\nabla^2 d\|_L^2 + C (1 + \|\nabla d\|_L^s).
\]
(4.4)

Thus, putting (4.4) into (4.2) and using (4.1), we obtain
\[
\int_0^T \|\nabla^2 d\|_L^2 dt \leq E_0 + C \int_0^T (1 + \|\nabla d\|_L^s) dt \leq C,
\]
(4.5)

provided \((r, s)\) satisfies (4.3).

On the other hand, assume that \((r, s)\) satisfies
\[
\frac{1}{r} + \frac{1}{s} \leq \frac{1}{2} \text{ with } 2 < r < 4.
\]
(4.6)

Then, by virtue of (2.2), (3.4) and Hölder’s inequality we find that
\[
\|\nabla d\|_L^4 \leq C \|\nabla d\|_L^{1-\alpha} \|\nabla d\|_L^2 \|\nabla^2 d\|_L^\alpha \|\nabla^2 d\|_L^2 \leq C \|\nabla d\|_L^{1-\alpha} \|\nabla^2 d\|_L^\alpha \|\nabla^2 d\|_L^2
\]
(4.7)

with
\[
\frac{2r}{r-2} < p < \infty, \quad \alpha = \frac{(4-r)p}{4(p-r)} \in \left(0, \frac{1}{2}\right).
\]

As a result of (4.7), we have by Young’s inequality that
\[
\|\nabla d\|_L^4 \leq C \|\nabla d\|_L^{4(1-\alpha)} \|\nabla^2 d\|_L^{4\alpha(p-2)/p}
\leq \frac{1}{2} \|\nabla^2 d\|_L^2 + C \|\nabla d\|_L^{4(1-\alpha)p/(p-2\alpha(p-2))}
\leq \frac{1}{2} \|\nabla^2 d\|_L^2 + C (1 + \|\nabla d\|_L^s),
\]
(4.8)

since direct calculations give
\[
\frac{4(1-\alpha)p}{p-2\alpha(p-2)} = \frac{2r}{r-2} \leq s.
\]

Thus, putting (4.8) into (4.2), by (4.1) we also obtain (4.5), provided \((r, s)\) satisfies (4.6).

**Proof of Theorem 1.2.** We have proved that there exists a positive constant \(C\), depending on the initial data, \(T^*\) and \(M_0\), such that for any \(0 < T < T^*\),
\[
\int_0^T \|\nabla^2 d\|_L^2 dt \leq C;
\]
provided (4.1) holds. Thus, with the help of this and (3.2)–(3.4), following the arguments in the proofs of Lemmas 3.3–3.5, we arrive at (3.1), which, combined with the local existence theorem (see Lemma 2.5), implies the solutions can be extended beyond \(T^*\). This immediately leads to a contradiction of \(T^*\), and hence, the proof of Theorem 1.2 is complete. \(\Box\)
Proof of Theorem 1.3. In fact, by applying the maximum principle to the equation of $d_3$ (i.e. the third component of $d$), we have

$$\inf_{x \in \mathbb{R}^2} d_3(x, t) \geq \inf_{x \in \mathbb{R}^2} d_{03} \geq \varepsilon, \quad \forall t > 0.$$  

So, it follows from (3.4) and (1.19) that for any $0 < T < \infty$,

$$\int_0^T \left( \|\Delta d\|_{L^2}^2 + \|\nabla d\|_{L^4}^4 \right) dt \leq C(E_0, \varepsilon) \int_0^T \|\Delta d + |\nabla d|^2 d\|_{L^2}^2 dt \leq C(E_0, \varepsilon).$$  

(4.9)

This, combined with Theorem 1.2 with $r = s = 4$, implies that the strong solutions of (1.1)–(1.4), (1.10) and (1.11) exist for all $T > 0$. The proof of Theorem 1.3 is thus finished.

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