THE IDEAL STRUCTURE OF THE HECKE C*-ALGEBRA
OF BOST AND CONNES

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Abstract. We compute explicitly the primitive ideal space of the Bost-Connes Hecke C*-algebra by embedding it as a full corner in a transformation group C*-algebra and applying a general theorem of Williams. This requires the computation of the quasi-orbit space for the action of $\mathbb{Q}_+^*$ on the space $\mathcal{A}_f$ of finite adeles. We then carry out a similar computation for the action of $\mathbb{Q}_+^*$ on the space $\mathcal{A} = \mathcal{A}_f \times \mathbb{R}$ of full adeles.

Introduction

In their work on phase transitions in number theory, Bost and Connes introduced a noncommutative Hecke C*-algebra $\mathcal{C}_Q$ and gave a presentation of this algebra. The present authors recognised this presentation as that of a semigroup crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes N^\infty$, and analysed the representations of $\mathcal{C}_Q$ using techniques previously developed for studying Toeplitz algebras [6]. Laca showed in [7] that this approach gives useful insight to many of the constructions of Bost and Connes, and used the universal properties of semigroup crossed products to simplify and extend their arguments.

From a C*-algebraic point of view, however, the Hecke C*-algebra $\mathcal{C}_Q$ remains poorly understood. Here we shall contribute a description of the ideal structure of $\mathcal{C}_Q$. More precisely, we shall describe the primitive ideal space Prin$\mathcal{C}_Q$ and the topology on it: since the lattice of ideals in any C*-algebra $A$ is isomorphic to the lattice of closed subsets of Prin$A$, this gives a complete description of the ideal structure.

To do this, we use the realisation of $\mathcal{C}_Q$ as $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes N^\infty$. Following a standard route (see [6, 8]), we dilate the endomorphisms $\alpha_n$ to automorphisms $\alpha_\infty$ of a direct limit $C^*(\mathbb{Q}/\mathbb{Z})_\infty$, and then realise $C^*(\mathbb{Q}/\mathbb{Z})_\infty \rtimes N^\infty$ as a full corner in the ordinary crossed product $C^*(\mathbb{Q}/\mathbb{Z})_\infty \rtimes N^\infty$. The direct limit $C^*(\mathbb{Q}/\mathbb{Z})_\infty$ is naturally isomorphic to the algebra $C_0(\mathcal{A}_f)$ of continuous functions on the space $\mathcal{A}_f$ of finite adeles, and the action $\alpha_\infty$ is conjugate to the action of the diagonally embedded copy of $\mathbb{Q}_+^*$ by division on $\mathcal{A}_f$. We analyse Prin$(C_0(\mathcal{A}_f) \rtimes \mathbb{Q}_+^*)$ using a theorem of Williams which identifies it as a quotient space of $\mathbb{Q} \times \mathbb{Q}_+^*$, where $\mathbb{Q}$ is the quasi-orbit space for the action (see §1). The main part of our work, therefore, is to compute this quasi-orbit space; the parametrisation we obtain allows us to give an elegant description of the topology (see Theorem 2.8).

The space $\mathcal{A}_f$ of finite adeles tells only part of the story, and one should also consider the “infinite place” — that is, use the space $\mathcal{A} = \mathcal{A}_f \times \mathbb{R}$ of full adeles. (In Connes’ latest analysis [2], for example, he has implicitly discarded $\mathcal{C}_Q$ in favour of $C_0(\mathcal{A}) \rtimes \mathbb{Q}_+^*$.) We have been able to extend our analysis of the quasi-orbits to the action of $\mathbb{Q}_+^*$ on $\mathcal{A}$, and thus obtain a parametrisation of Prin$(C_0(\mathcal{A}) \rtimes \mathbb{Q}_+^*)$ and a description of the topology. The inclusion of the infinite place significantly changes the primitive ideal space, largely

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because the orbits of invertible adeles are now closed rather than dense, giving rise to a copy of the idele class group inside the primitive ideal space (see Corollary 5.8).

1. Williams’ theorem

Consider an action of a second countable locally compact abelian group \( G \) on a second countable locally compact space \( X \). Recall that a quasi-orbit for the action is the set of points with a given orbit closure; the quasi-orbit space \( Q(X/G) \) is the quotient of \( X \) by the relation

\[
x \sim y \iff G \cdot x = G \cdot y.
\]

Notice that, because \( G \) is abelian, the isotropy groups \( G_x \) are constant on quasi-orbits. The Mackey machine, as presented by Green [5], says that every primitive ideal is associated to a unique quasi-orbit (in a sense made precise in [5]) and is induced from the isotropy group of that quasi-orbit. To get specific representations, let \( \varepsilon_x \) denote the representation \( f \mapsto f(x) \) of \( C_0(X) \). Then for every \( \gamma \in \hat{G} \), \( (\varepsilon_x \cdot \gamma) \) is a covariant representation of \( (C_0(X), G_x) \), and the induced representation \( \text{Ind}^G_{G_x}(\varepsilon_x \cdot \gamma) \) is an irreducible representation of \( C_0(X) \times G \) (see [4, Proposition 4.2]). We shall use the following form of Williams’ theorem.

**Theorem 1.1** (Williams). Define an equivalence relation on \( Q(X/G) \times \hat{G} \) by

\[
([x], \gamma) \sim ([y], \chi) \iff [x] = [y] \text{ and } \gamma|_{G_x} = \chi|_{G_x},
\]

where \( G_x \) denotes the isotropy group at \( x \). Then the map

\[
(1.1) \quad ([x], \gamma) \mapsto \ker \text{Ind}^G_{G_x}(\varepsilon_x \cdot \gamma)
\]

induces a homeomorphism of the quotient \( (Q(X/G) \times \hat{G})/\sim \) onto \( \text{Prim}(C_0(X) \times G) \).

**Proof.** An examination of the proof of [4, Lemma 4.10] shows that the map \( (x, \gamma) \mapsto \ker \text{Ind}(\varepsilon_x \cdot \gamma) \) of \( X \times \hat{G} \) into \( \text{Prim}(C_0(X) \times G) \) is the map discussed at the beginning of [4, §5]. Since second countable transformation groups are automatically quasi-regular ([3, §5]; see [5, Corollary 19] for a more general result), it follows from [4, Corollary 3.2] that every second countable transformation group \( (X, G) \) with \( G \) amenable is Effros-Hahn regular in the sense of [4, Definition 4.12]. This applies in particular when \( G \) is abelian, so the present theorem is a restatement of [4, Theorem 5.3] as it applies to second countable abelian transformation groups; that Williams’ quotient \( \Gamma \) of \( X \times \hat{G} \) can be viewed as a quotient of \( Q(X/G) \times \hat{G} \) is observed before [4, Theorem 5.3].

**Remark 1.2.** In our case, the isotropy groups will be either \( G \) or \( \{e\} \). When \( G_x = G \), the induced representation \( \text{Ind}^G_{G_x}(\varepsilon_x \cdot \gamma) \) is just \( \varepsilon_x \cdot \gamma \). When \( G_x = \{e\} \), the representation \( \text{Ind}^G_{G_x}(\varepsilon_x \cdot \gamma) \) is equivalent to \( M_x \times \lambda \) acting on \( L^2(G) \), where

\[
(M_z(f)\xi)(s) = f(s \cdot x)\xi(s) \quad \text{and} \quad \lambda_t(\xi)(s) = \xi(t^{-1}s).
\]

To see this, note that the representation considered in [4] is given by the left action \( (R, V) \) of \( (C_0(X), G) \) on the Hilbert \( C_0(X) \)-module \( C_c(G, X) \) described in the middle of page 340 of [4]. The unitary \( U : C_c(G, X) = C_c(G, X) \otimes \mathbb{C} \to L^2(G) \) of [4, Lemma 2.14] is given by \( U_z(r) = z(r, r \cdot x) \), and the calculations

\[
U(R_x(f)z)(r) = (R_x(f)z)(r, r \cdot x) = f(r \cdot x)z(r, r \cdot x) = (M_x(f)U(z))(r)
\]
Lemma 2.1. The isotropy group at every $a \in \mathcal{A}_f \setminus \{0\}$ is trivial, and the isotropy group at $0 \in \mathcal{A}_f$ is $\mathbb{Q}^*_+.$

Proof. For each $r \in \mathbb{Q}^*_+, we have $ra := (ra_p)_{p \in \mathcal{P}},$ so $ra = a$ implies that $ra_p = a_p$ in $\mathbb{Q}_p$ for every prime $p.$ Clearly $a \neq 0$ implies that $a_p \neq 0$ for at least one prime $p,$ and, for such a prime, $ra_p = a_p$ implies $r = 1,$ because $\mathbb{Q}_p$ is a field. The second assertion is obvious. \qed

Remark 2.2. Because the isotropy group at 0 is all of $\mathbb{Q}^*_+,$ Williams’ theorem says that there must be a copy of $\hat{\mathbb{Q}}^*_+$ in the primitive ideal space of $C_\mathbb{Q}.$ It is easy to identify this copy: if $\epsilon : C^*(\mathbb{Q}/\mathbb{Z}) \to \mathbb{C}$ is the augmentation homomorphism given by $\epsilon(\delta_r) = 1$ for every $r \in \mathbb{Q}/\mathbb{Z},$ then for each $\gamma \in \hat{\mathbb{Q}}^*_+,$ the pair $(\epsilon, \gamma|_{\mathbb{N}^\times})$ is a one-dimensional covariant representation of $(C^*(\mathbb{Q}/\mathbb{Z}), \mathbb{N}^\times, \alpha),$ giving a one-dimensional representation of $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^\times.$

Next we need to compute the closure of an orbit. To simplify things we note that the orbits for the action of $\mathbb{Q}^*_+$ by division are the same as for the action by multiplication, and compute the latter.

Lemma 2.3. For each $a \in \mathcal{A}_f,$ the closure of the corresponding orbit is

\begin{equation}
(2.1) \quad \mathbb{Q}^*_+a = \{ b \in \mathcal{A}_f : a_p = 0 \implies b_p = 0 \}.
\end{equation}
Proof. It is clear that $\subseteq$ holds, so we concentrate on proving $\supseteq$. Let $b$ belong to the right-hand side, and notice that it is enough to show that $mb \in \mathbb{Q}_+^*a$ for some integer $m$. We may choose $m$ such that $mb \in \prod_p \mathbb{Z}_p$, and then we can easily write down a typical basic open neighbourhood of $mb$:

$$V = \prod_{p \in F} V_p \times \prod_{p \notin F} \mathbb{Z}_p$$

for $F$ a finite set of primes and $V_p$ open sets in $\mathbb{Z}_p$. We can further assume that $a \in \prod \mathbb{Z}_p$ without changing its orbit or the set of primes $p$ for which $a_p$ vanishes.

Let $F' := \{ p \in F : a_p \neq 0 \}$, and notice that for every $p \in F'$ we can write $a_p = p^{\nu_p(a_p)} u_p$ for some unit $u_p \in \mathbb{Z}_p^*$. Then $p^{\nu_p(a_p)} a_p^{-1} V_p = u_p^{-1} V_p$ is an open set in $\mathbb{Z}_p$, and hence contains a positive integer $k_p$; we then have that $k_p p^{-\nu_p(a_p)} \in a_p^{-1} V_p$.

We now choose $l_p$ large enough to ensure that the ball $B(k_p p^{-\nu_p(a_p)}; p^{-l_p + \nu_p(a_p)})$ in $\mathbb{Q}_p$ is contained in $a_p^{-1} V_p$. (Notice that because $V_p$ is open in $\mathbb{Z}_p$, and $a_p$ is nonzero in $\mathbb{Q}_p$, we have that $a_p^{-1} V_p$ is open in $\mathbb{Q}_p$.) By the Chinese Remainder Theorem, there is an integer $n$ such that

$$n \equiv k_p \left( \prod_{q \in F' \setminus \{p\}} q^{\nu_q(a_q)} \right) \mod p^l_p \quad \text{for all } p \in F'.$$

But now for each $p \in F'$,

$$\left| \frac{k_p}{p^{\nu_p(a_p)}} - \frac{n}{\prod_{q \in F'} q^{\nu_q(a_q)}} \right|_p = \left| \frac{k_p \prod_{q \in F' \setminus \{p\}} q^{\nu_q(a_q)} - n}{\prod_{q \in F'} q^{\nu_q(a_q)}} \right|_p \leq p^{-l_p + \nu_p(a_p)}.$$

Next define

$$r := \left( \frac{n}{\prod_{q \in F'} q^{\nu_q(a_q)}} \right),$$

and notice that for all $p \in F'$ we have $r \in B(k_p p^{-\nu_p(a_p)}; p^{-l_p + \nu_p(a_p)}) \subset a_p^{-1} V_p$, so that $ra_p \in V_p$. If $p \in F \setminus F'$, in which case $a_p = 0$, then $ra_p = 0 = mb_p$, so clearly $ra_p \in V_p$. If $p \notin F$, then $\prod_{q \in F'} q^{\nu_q(a_q)}$ is coprime to $p$, and $r$ embeds as a $p$-adic integer (that is, $r \in \mathbb{Z}_p$), and hence $ra_p \in \mathbb{Z}_p$. Thus $ra \in \prod_{p \in F'} V_p \times \prod_{p \notin F} \mathbb{Z}_p$, and $mb \in \mathbb{Q}_+^*a$. \hfill $\Box$

The quasi-orbit space $\mathcal{Q}(\mathcal{A}_f/\mathbb{Q}_+^*)$ is the quotient of $\mathcal{A}_f$ by the equivalence relation $\sim$ defined by $a \sim b$ if and only if $\mathbb{Q}_+^* a = \mathbb{Q}_+^* b$. The above characterisation of the orbit closures shows that $a \sim b$ if and only if $\{ p : a_p = 0 \} = \{ p : b_p = 0 \}$, hence the quasi-orbits are in one-to-one correspondence with the subsets of the set of prime numbers. The power set $2^P$ has a natural power-cofinite topology in which the basic open sets are

$$U_G := \{ T \in 2^P : T \cap G = \emptyset \} = 2^{P \setminus G},$$

where $G$ ranges over the finite subsets of $P$. This collection is indeed the basis for a topology because $U_G \cap U_H = U_{G \cup H}$.

**Proposition 2.4.** For each $a \in \mathcal{A}_f$ define $S(a) := \{ p \in P : a_p = 0 \}$. Then the map $a \mapsto S(a)$ induces a homeomorphism of the quasi-orbit space $\mathcal{Q}(\mathcal{A}_f/\mathbb{Q}_+^*)$ onto the space $2^P$ with the power-cofinite topology defined above.

**Proof.** Let $q : a \mapsto q(a)$ be the quasi-orbit map of $\mathcal{A}_f$ to $\mathcal{Q}(\mathcal{A}_f/\mathbb{Q}_+^*)$. It follows easily from the characterisation of orbit closures that the map $a \mapsto S(a)$ factors through $q$, and that $q(a) \mapsto S(a)$ is a bijection.
By the Lemma on page 221 of [3] we know that the map \( q : \mathcal{A}_f \to \mathcal{Q}(\mathcal{A}_f/\mathbb{Q}_+^*) \) is continuous and open. Hence the sets \( q(V) \), as \( V \) runs through a basis for the topology on \( \mathcal{A}_f \), form a basis for the quotient topology on \( \mathcal{Q}(\mathcal{A}_f/\mathbb{Q}_+^*) \). By definition, a typical basic open neighbourhood of \( a \) in \( \mathcal{A}_f = \prod_{p \in \mathcal{P}} (\mathbb{Q}_p : \mathbb{Z}_p) \) is \( W = a + V \) with \( V \) a product of the form

\[
V = \prod_{p \in F} V_p \times \prod_{p \not\in F} \mathbb{Z}_p,
\]

in which \( F \) is a finite subset of \( \mathcal{P} \) and \( V_p \) is an open neighbourhood of \( 0 \) in \( \mathbb{Q}_p \) for every \( p \in F \). After relabelling \( a + \mathbb{Z}_p \) as \( V_p \) for those (finitely many) \( p \not\in F \) for which \( a_p \not\in \mathbb{Z}_p \), and enlarging the set \( F \) accordingly, we see that \( W \) is a product of the same form as \( V \).

Since the bijection \( q(a) \mapsto S(a) \) carries \( q(V) \) to \( S(V) \), we want to prove that the collection \( \{ S(V) \} \) with \( V \) as above is precisely the basis \( \{ U_G \} \) for the topology on \( 2^\mathcal{P} \).

Given \( V \) as above, the set \( \{ p \in F : 0 \not\in V_p \} \) is finite, and every finite subset of \( \mathcal{P} \) arises for some \( V \). Thus it suffices to prove that for every \( V \) as above,

\[
S(V) = U_{\{ p \in F : 0 \not\in V_p \}}.
\]

First we prove \( \supseteq \):

\[
\begin{align*}
  b \in V & \implies b_p \in V_p \quad \text{for all } p \in F \\
  & \implies b_p \neq 0 \quad \text{if } 0 \not\in V_p \\
  & \implies S(b) \cap \{ p \in F : 0 \not\in V_p \} = \emptyset \\
  & \implies S(b) \in U_{\{ p \in F : 0 \not\in V_p \}}.
\end{align*}
\]

To prove \( \subseteq \), let \( G := \{ p \in F : 0 \not\in V_p \} \) and suppose \( T \) is a subset of \( \mathcal{P} \) such that \( T \cap G = \emptyset \). We need to find \( b = (b_p) \in V \) such that \( S(b) = T \). If \( p \in T \) choose \( b_p = 0 \); if \( p \not\in T \) but \( p \in F \), choose \( b_p \in V_p \setminus \{0\} \); finally, if \( p \not\in T \cup F \), simply take \( b_p = 1 \in \mathbb{Z}_p \). It is clear that \( S(b) = T \). To see that \( b \in V \), notice that \( b_p \in V_p \) for \( p \in F \cap T \) because \( T \cap G = \emptyset \), and obviously \( b_p \in V_p \) for \( p \in F \setminus T \).

Now that we have described the quasiorbit space and the isotropy groups, our description of \( \text{Prim}(C_0(\mathcal{A}_f) \times \mathbb{Q}_+^*) \) follows immediately from Theorem [1].

**Proposition 2.5.** Define a relation on \( 2^\mathcal{P} \times \mathbb{Q}_+^* \) by

\[
(S, \gamma) \sim (T, \chi) \iff \begin{cases} S = T & \text{if } S \neq \mathcal{P} \\ S = T \text{ and } \gamma = \chi & \text{if } S = \mathcal{P}. \end{cases}
\]

Then \( \sim \) is an equivalence relation. For each nonzero \( a \in \mathcal{A}_f \) the ideal \( I_S := \ker \text{Ind} \varepsilon_a = \ker(M_a \times \lambda) \) depends only on \( S := \{ p : a_p = 0 \} \), and the maps

\[
S \mapsto I_S \quad \text{and} \quad \gamma \mapsto \ker(\varepsilon_0 \times \gamma)
\]

combine to give a homeomorphism of \( (2^\mathcal{P} \times \mathbb{Q}_+^*)/\sim \) onto \( \text{Prim}(C_0(\mathcal{A}_f) \times \mathbb{Q}_+^*) \).

**Proof.** This follows from a direct application of Theorem [1] using the characterisation of the quasi-orbit space given by Proposition 2.4.

**Remark 2.6.** Since the isotropy groups are either trivial or \( \mathbb{Q}_+^* \), we can also describe \( (2^\mathcal{P} \times \mathbb{Q}_+^*)/\sim \) as the disjoint union

\[
(2^\mathcal{P} \setminus \{ \mathcal{P} \}) \sqcup \mathbb{Q}_+^*.
\]
therefore follows from standard properties of Morita equivalence \cite{10, Corollary 3.33}.\[\pi\] is equivalent to the compression of \[\text{intertwines the natural left action of}\]

But the map \[\text{combine to give a homeomorphism of}\]

The space \[\text{Proof.}\]

The preceding discussion shows that the map of \[\text{equivalence and induces a homeomorphism of}\]

Composition of the homeomorphism of \[\text{Lemma 2.7.}\]

Let \[\text{following standard lemma.}\]

Theorem 2.8. \[\text{For each proper subset}\]

Then, using the notation of Remark 1.2, \[\text{is the primitive ideal of}\]

and is determined by the covariant representation \[\text{is irreducible and its kernel depends only on}\]

The maps \[\text{combine to give a homeomorphism of}\]

Proof. The preceding discussion shows that the map of \[\text{is the composition of the homeomorphism of}\]

This shows that for \[\text{have the same kernel} \] if and only if \[\text{are in the same quasi-orbit. It is natural to ask when the representations}\]

First we need to characterise the zero divisors of \[\mathbb{Z}.\]
Lemma 2.9. Let \( a \in \mathcal{Z} = \prod_p \mathbb{Z}_p \). Then the following are equivalent:

1. \( a \) is not a zero divisor;
2. \( a_p \neq 0 \) for every \( p \in \mathcal{P} \);
3. the set \( \mathbb{Q}_+^* a \cap \mathcal{Z} \) is dense in \( \mathcal{Z} \).

Proof. Since the operations in \( \mathcal{Z} = \prod_p \mathbb{Z}_p \) are componentwise, (1) is trivially equivalent to (2).

Suppose (2) holds, let \( F \) be a finite subset of \( \mathcal{P} \), and let \( V_p \) be an open subset of \( \mathbb{Z}_p \) for each \( p \in F \), so that \( V = \prod_{p \in F} V_p \times \prod_{p \notin F} \mathbb{Z}_p \) is a basic open set in \( \prod_p \mathbb{Z}_p \). Let \( n = \prod_{p \in F} p^{a_p} \), and define \( u = (u_p)_{p \in \mathcal{P}} \in \prod_p \mathbb{Z}_p \) by

\[
u_p = \begin{cases}
\frac{1}{n} a_p & \text{if } p \in F \\
1 & \text{if } p \notin F.
\end{cases}
\]

Then \( u \) is a unit, \( n \) divides \( a \), and \( \frac{1}{n} a_p = u_p \) for \( p \in F \). Since multiplication by a unit is a homeomorphism, \( u^{-1} V \) is an open set, and since the canonical embedding of \( \mathbb{N}^\infty \) is dense in \( \mathcal{Z} \), there exists \( m \in \mathbb{N}^\infty \) such that \( m \in u^{-1} V \). Thus \( \frac{m}{n} a_p = m u_p \in V_p \) for every \( p \in F \), so \( \frac{m}{n} a \in V \), proving (3).

Conversely, suppose \( a_p = 0 \) for some \( p \). Then \( \mathbb{Q}_+^* a \cap \mathcal{Z} \) is contained in the proper closed set \( \{ z : z_p = 0 \} \) and cannot be dense, so (3) implies (2). \( \square \)

Proposition 2.10. Suppose \( a \in \mathcal{Z} \) and \( a \neq 0 \).

1. The representation \( \pi_a \times V \) is faithful if and only if \( a \) is not a zero divisor.
2. Suppose \( b \in \mathcal{Z} \setminus \{0\} \). The representations \( \pi_a \times V \) and \( \pi_b \times V \) are unitarily equivalent if and only if \( a \) and \( b \) are in the same \( \mathbb{Q}_+^* \)-orbit.

Proof. By [3, Proposition 3.7], \( \pi_a \times V \) is faithful if and only if \( \pi_a \) is faithful on \( C^*(\mathbb{Q}/\mathbb{Z}) \cong C(\mathcal{Z}) \). Since the kernel of \( \pi_a \) in \( C(\mathcal{Z}) \) is \( \{ g : g|_{\mathbb{Q}_+^* a} = 0 \} \), the assertion (1) follows from Lemma 2.3.

Next we prove (2). If \( b = ra \) for some \( r \in \mathbb{Q}_+^* \), then \( V_r \) implements the equivalence between \( \pi_a \times V \) and \( \pi_b \times V \). Now suppose that \( \pi_a \times V \) is unitarily equivalent to \( \pi_b \times V \); we have to show that \( a \) and \( b \) lie in the same \( \mathbb{Q}_+^* \)-orbit.

Denote the canonical conditional expectation from \( C_0 \) to \( C^*(\mathbb{Q}/\mathbb{Z}) \) by \( \Phi \); then \( \omega_b : T \mapsto \Phi(T)(b) \) is the vector state \( T \mapsto \langle (\pi_b \times V)(T)e_1, e_1 \rangle \). If \( \pi_b \times V \) is unitarily equivalent to \( \pi_a \times V \), then there is a unit vector \( \xi \) in the Hilbert space of \( \pi_a \times V \) such that \( \langle \pi_a \times V(T)\xi, \xi \rangle = \omega_b(T) \). Write \( \xi = \sum_r c_r e_r \), where the sum is over the set \( \{ r \in \mathbb{Q}_+^* : ra \in \mathcal{Z} \} \), and choose a finite subset \( F \) of \( \mathbb{Q}_+^* \) such that \( \sum_{r \notin F} |c_r|^2 < 1/2 \). If \( b \notin \mathbb{Q}_+^* a \) there exists \( f \in C^*(\mathbb{Q}/\mathbb{Z}) \) with \( \|f\| \leq 1 \) such that \( \hat{f}(b) = 1 \) and \( \hat{f}(ra) = 0 \) for every \( r \in F \). But then

\[
1 = \hat{f}(b) = \omega_b(f) = \langle \pi_a(f)\xi, \xi \rangle = \sum_r \langle \hat{f}(ra)c_re_r, c_re_r \rangle \leq \sum_{r \notin F} |c_r|^2 < 1/2,
\]

which is a contradiction. So \( b \) has to be in \( \mathbb{Q}_+^* a \). \( \square \)

3. The action of \( \mathbb{Q}^* \) on the full adeles

In this section we compute the primitive ideal space of the transformation group \( C^*-algebra C_0(\mathcal{A}) \times \mathbb{Q}^* \) of the multiplicative action of \( \mathbb{Q}^* \) on the full adeles \( \mathcal{A} = \mathcal{A}_f \times \mathbb{R} \). If \( a \in \mathcal{A} \), we write \( a_f \) for the finite part \( (a_p)_{p \in \mathcal{P}} \) and \( a_\infty \) for the value of \( a \) at the infinite prime, so that \( a = (a_f, a_\infty) \in \mathcal{A}_f \times \mathbb{R} \).
For each finite prime $p$, let $v_p : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ be the $p$-adic valuation. (If $x \in \mathbb{Q}_p^*$, then $v_p(x)$ is by definition the unique integer such that $x p^{-v_p(x)}$ is in $\mathbb{Z}_p^*$; if $x = 0$ we let $v_p(x) = \infty$.) By abuse, if $a \in A$, we write $v_p(a) = v_p(a_p)$. If $a$ is invertible, then $v_p(a)$ is finite for every $p$ and vanishes for almost all $p$; it follows that the product $\prod_{p \in \mathcal{P}} p^{-v_p(a)}$ is a rational number, and it is easy to check that $a \operatorname{sign}(a_{\infty}) \prod_{p \in \mathcal{P}} p^{-v_p(a)} \in \prod_{p \in \mathcal{P}} \mathbb{Z}_p^* \times \mathbb{R}_+^*$ for every invertible adele $a$. Thus, every invertible adele has the form $ru$ for some $u \in \prod_p \mathbb{Z}_p^* \times \mathbb{R}_+^*$ and $r \in \mathbb{Q}^*$.

Clearly every such product is invertible, and the factorisation is unique, so we have a bijection

$$
(r, u) \in \mathbb{Q}^* \times \left( \prod_{p \in \mathcal{P}} \mathbb{Z}_p^* \times \mathbb{R}_+^* \right) \mapsto ru \in A^*.
$$

(3.1)

As for the action of $\mathbb{Q}^*$ on $A_f$, 0 is the only adele with nontrivial isotropy, so the quasi-orbit space is the key to understanding the primitive ideal space. We begin by characterising the orbit closures.

**Lemma 3.1.** If $u$ is an invertible adele, then the orbit $\mathbb{Q}^* u$ is closed in $A$.

**Proof.** We may assume that $u = (u_f, u_{\infty}) \in \prod_p \mathbb{Z}_p^* \times \mathbb{R}_+^*$ without changing the orbit. Suppose now $a$ is in the closure of $\mathbb{Q}^* u$, and let $r_n$ be a sequence of nonzero rationals such that $r_n u \to a$ as $n \to \infty$. Fix $\varepsilon < 1/(4u_{\infty})$. Since $\prod_p \mathbb{Z}_p \times (-\varepsilon, \varepsilon)$ is a neighbourhood of 0 in $A$, there exists $N \in \mathbb{N}$ such that for every $n > N$,

$$
|r_n u - a| < \prod_p \mathbb{Z}_p \times (-\varepsilon, \varepsilon).
$$

Then for every $m, n > N$ we have

$$
(r_n - r_m)u = (r_n u - a) - (r_m u - a) \in \left( \prod_p \mathbb{Z}_p \times (-\varepsilon, \varepsilon) \right) - \left( \prod_p \mathbb{Z}_p \times (-\varepsilon, \varepsilon) \right) = \prod_p \mathbb{Z}_p \times (-2\varepsilon, 2\varepsilon).
$$

Since $u_f \in \prod_p \mathbb{Z}_p^*$, $(r_n - r_m)u \in \prod_p \mathbb{Z}_p$ implies $r_n - r_m \in \mathbb{Z}$. On the other hand, $(r_n - r_m)u_{\infty} \in (-2\varepsilon, 2\varepsilon)$ implies

$$
|r_n - r_m| < \frac{4\varepsilon}{u_{\infty}} < 1.
$$

Since $r_n - r_m$ is an integer, the sequence $r_n$ must be eventually constant, and hence $a = r_n u \in \mathbb{Q}^* u$. \hfill \Box

Next we characterise the orbit closure of noninvertible adeles. The result is similar to Lemma 3.1 for the finite adeles, but the proof is slightly more involved because we need to control the size of the archimedean absolute value.

**Lemma 3.2.** If $a \in A$ is not invertible, then

$$
\mathbb{Q}^* a = \{ b : a_p = 0 \implies b_p = 0 \}.
$$

**Proof.** The inclusion $\subset$ is clear, and we only need to show that if $b_p$ is zero whenever $a_p$ is zero, then $b$ can be approximated from within the orbit of $a$. Since $A_f$ is the restricted product of the $\mathbb{Q}_p$ relative to the $\mathbb{Z}_p$, there is an integer $n$ such that $(na)_p \in \mathbb{Z}_p$ for all $p \in \mathcal{P}$, and $na$ has the same orbit and the same zeros as $a$. So we may suppose without loss of generality that $a \in \prod_p \mathbb{Z}_p \times \mathbb{R}$. The same argument shows that we might as well assume that $b \in \prod_p \mathbb{Z}_p \times \mathbb{R}$. 


Suppose then that \( b \in \prod_p \mathbb{Z}_p \times \mathbb{R} \) satisfies \( b_p = 0 \) if \( a_p = 0 \). As a subspace of \( \mathcal{A} \), the set \( \prod_p \mathbb{Z}_p \times \mathbb{R} \) has the product topology, so a typical basic open neighbourhood of \( b \) has the form

\[
V = \prod_{p \in F} V_p \times \prod_{p \notin F} \mathbb{Z}_p \times (x, y),
\]

where \( F \) is a finite subset of primes and \((x, y)\) is an open interval in \( \mathbb{R} \).

Let \( F' := \{ p \in F : a_p \neq 0 \} \); notice that whenever \( p \in F \setminus F' \) we trivially have \( ra_p = 0 = a_p = b_p \in V_p \) for all \( r \in \mathbb{Q}^* \). We want to find \( r \in \mathbb{Q}^* \) such that

1. \( ra_p \in V_p \) for \( p \in F' \),
2. \( ra_p \in \mathbb{Z}_p \) for \( p \in \mathcal{P} \setminus F \),
3. \( ra_\infty \in (x, y) \).

For \( p \in F' \), write \( a_p \) as \( p^{v_p(a_p)}u_p \) for some unit \( u_p \in \mathbb{Z}_p^* \), and choose \( k_p \in \mathbb{Z} \) such that \( k_p \in u_p^{-1}V_p \). Then \( k_pp^{-v_p(a_p)} \in a_p^{-1}V_p \), and since \( a_p^{-1}V_p \) is open in \( \mathbb{Q}_p \), there exists \( \ell_p \in \mathbb{N} \) such that

\[
(3.2) \quad B\left(k_pp^{-v_p(a_p)}, p^{-\ell_p+v_p(a_p)}\right) \subset a_p^{-1}V_p.
\]

We will verify (i) by finding \( r \) in this ball.

We deal with two cases separately.

**Case I:** Suppose there exists a prime \( Q \) such that \( a_Q = 0 \). If \( Q = \infty \), the result follows from Lemma 2.3. So we suppose \( Q \neq \infty \).

Let \( N_{F'} = \prod_{p \in F'} (p^{\ell_p}) \). Choose \( m \in \mathbb{N} \) such that

\[
\frac{N_{F'}}{Q^m} < \frac{y - x}{a_\infty}.
\]

By the Chinese Remainder Theorem, the congruences

\[
n = k_pQ^m \left( \prod_{q \in F' \setminus \{p\}} q^{v_q(a_q)} \right) \mod p^{\ell_p}\quad \text{for } p \in F'
\]

have a solution \( n_0 \), and the set of solutions is then \( n_0 + \left( \prod_{p \in F'} p^{\ell_p} \right) \mathbb{Z} \). We claim that if \( n \) is one of these solutions and we define \( r \) by

\[
(3.3) \quad r := \frac{n}{Q^m \prod_{q \in F'} q^{v_q(a_q)}},
\]

then \( r \) is in the ball of \((3.2)\). To see this, we compute

\[
\left| \frac{k_p}{p^{v_p(a_p)}} - \frac{n}{Q^m \prod_{q \in F'} q^{v_q(a_q)}} \right|_p = \left| k_pQ^m \prod_{q \in F' \setminus \{p\}} q^{v_q(a_q)} - n \right|_p = \left| \frac{cp^{\ell_p}}{Q^m \prod_{q \in F'} q^{v_q(a_q)}} \right|_p \quad \text{for some } c \in \mathbb{Z}
\]

\[
\leq p^{-\ell_p+v_p(a_p)} \quad \text{by definition of } | \cdot |_p.
\]

This says precisely that every \( r \) of the form \((3.3)\) lies in the ball of \((3.2)\), and hence satisfies (i).

To see that such \( r \) also satisfy (ii), note that \( a_Q = 0 \) implies \( ra_Q = 0 \in \mathbb{Z}_Q \). If \( p \notin F' \cup \{Q\} \), then \( r \in \mathbb{Z}_p \) because \( p \) does not divide the denominator. Hence \( ra_p \in \mathbb{Z}_p \) for all \( p \notin F \).
Now recall that we chose \( m \) large enough to ensure that every interval of length \((y - x)/a_\infty\) contains at least one of the elements of
\[
\frac{n_0 + \left( \prod_{p \in F'} p^{f_p} \right) \mathbb{Z}}{Q^m \prod_{q \in F'} q^{v_q(a_q)}} = \frac{n_0}{Q^m \prod_{q \in F'} q^{v_q(a_q)}} + \frac{N_{F'}}{Q^m}.
\]
Therefore there exists \( n \in n_0 + \left( \prod_{p \in F'} p^{f_p} \right) \mathbb{Z} \) such that the corresponding \( r \) satisfies \( ra_\infty \in (x, y) \). This \( r \) satisfies (i), (ii) and (iii).

**Case II:** Suppose \( a_p \neq 0 \) for every \( p \in \mathcal{P} \cup \{ \infty \} \). Then, because \( a \) is not invertible, there must exist infinitely many primes dividing \( a \). Let \( N_F = \prod_{p \in F'} (p^{f_p - v_p(a_p)}) \). We can choose a set \( G \) of primes disjoint from \( F \) such that every \( q \in G \) divides \( a \), and such that
\[
\frac{N_F}{\prod_{q \in G} q} < \frac{y - x}{a_\infty}.
\]
A similar argument to that of Case I, with \( Q^m \) replaced by \( \prod_{q \in G} q \), now yields a rational number \( r \) satisfying (i), (ii) and (iii).

\[\square\]

**Proposition 3.3.** The quasi-orbit map \( q : \mathcal{A} \to \mathcal{Q}(\mathcal{A}^*/\mathbb{Q}^*) \) is given by:
\[
a \in \mathcal{A} \mapsto q(a) = \begin{cases} \mathbb{Q}^*a & \text{if } a \text{ is invertible,} \\ \{ b \in \mathcal{A} \setminus \mathcal{A}^* : b_p = 0 \text{ iff } a_p = 0 \} & \text{if } a \text{ is not invertible.} \end{cases}
\]

**Proof.** The result is a direct application of Lemma 3.1 and Lemma 3.2. \[\square\]

We aim to obtain a better description of this quasi-orbit space and of its topology, along the lines of Proposition 2.4. This will involve parametrising the quasi-orbit space and then describing the topology on the parameter space.

One part of the parameter space will be the power set \( \mathbb{P} = \mathcal{P} \cup \{ \infty \} \); we again endow \( \mathbb{P} \) with its power-cofinite topology, generated by the basic open sets \( U_G = \mathbb{P}^{\mathbb{G}} \), where \( \mathbb{G} \) runs through the finite subsets of \( \mathbb{P} \).

The other part of the parameter space will be \( \mathcal{U} := \prod_{p \in \mathbb{P}} \mathbb{Z}_p^* \times \mathbb{R}_+^* \) with the product topology, which is the topology of \( \mathcal{U} \) as a subspace of the ideles, see for instance [13, Chapter IV.3] or [11, §5.1]. Although the topology of the ideles as a restricted product is strictly stronger than the subspace topology they inherit from \( \mathcal{A} \), the two topologies coincide on \( \mathcal{U} \). To see this, suppose \( u_n \to u \) in the adelic topology of \( \mathcal{U} \). Then for each \( p \), \( (u_n)_p \) converges to \( (u)_p \) in \( \mathbb{Z}_p \), and hence in \( \mathbb{Z}_p^* \), because \( u \) and \( u_n \) all lie in the subset \( \mathcal{U} \) of \( \mathcal{A} \). It follows that \( u_n \to u \) in the product topology of \( \mathcal{U} \), that is, in the idele topology.

**Proposition 3.4.** The map \( \chi : \mathcal{A} \to \mathbb{P} \cup \mathcal{U} \) defined by
\[
a \in \mathcal{A} \mapsto \chi(a) = \begin{cases} a \text{ sign}(a_\infty) \prod_{p \in \mathbb{P}} p^{-v_p(a)} \in \mathcal{U} & \text{if } a \text{ is invertible,} \\ \{ p \in \mathbb{P} : a_p = 0 \} \in \mathbb{P} & \text{if } a \text{ is not invertible.} \end{cases}
\]
factors through the quasi-orbit map, and induces a bijection \( \chi(a) \mapsto q(a) \) of \( \mathbb{P} \cup \mathcal{U} \) onto the quasi-orbit space \( \mathcal{Q}(\mathcal{A}/\mathbb{Q}^*) \).

**Proof.** Every subset of \( \mathbb{P} \) is the vanishing set of a noninvertible adele, and every \( u \in \mathcal{U} \) is equal to \( \chi(u) \), so \( \chi \) maps \( \mathcal{A} \) onto \( \mathbb{P} \cup \mathcal{U} \). It remains to show that \( \chi(a) = \chi(b) \) if and only if \( q(a) = q(b) \) for every \( a, b \) in \( \mathcal{A} \).

Suppose \( \chi(a) = \chi(b) \). This implies that either \( a \) and \( b \) are both invertible, or else they are both not invertible.
• If they are both invertible, we have
\[ a \cdot \text{sign}(a_\infty) \prod_p p^{-v_p(a)} = b \cdot \text{sign}(b_\infty) \prod_p p^{-v_p(b)}, \]
hence \( b = \pm a \prod_p p^{-v_p(a)+v_p(b)} \in \mathbb{Q}^* a \), from which \( q(a) = q(b) \).
• If they are both not invertible, \( \{ p \in \mathbb{P} : a_p = 0 \} = \{ p \in \mathbb{P} : b_p = 0 \} \) and clearly \( q(a) = q(b) \).

Suppose now \( q(a) = q(b) \). This also implies that either \( a \) and \( b \) are both invertible, or else they are both not invertible.
• If they are both invertible, we can write \( b = ra \) for some \( r \in \mathbb{Q}^* \), and then
\[
\chi(b) = (ra) \cdot \text{sign}(ra_\infty) \prod_p p^{-v_p(ra)}
\]
\[
= (r \cdot \text{sign}(r) \prod_p p^{-v_p(r)}) \cdot (a \cdot \text{sign}(a_\infty) \prod_p p^{-v_p(a)})
\]
\[
= a \cdot \text{sign}(a_\infty) \prod_p p^{-v_p(a)} = \chi(a),
\]
because \( r \cdot \text{sign}(r) \prod_p p^{-v_p(r)} = 1 \).
• If they are both not invertible, then
\[
\{ c \in A \setminus A^* : c_p = 0 \iff a_p = 0 \} = \{ c \in A \setminus A^* : c_p = 0 \iff b_p = 0 \},
\]
and from this it follows that \( a_p = 0 \) if and only if \( b_p = 0 \), so \( \chi(a) = \chi(b) \).

This completes the proof. \( \square \)

The set \( U \) is a locally compact group under multiplication, and the restriction of \( \chi \) to the ideles \( A^* \) is a group homomorphism onto \( U \) with kernel \( \mathbb{Q}^* \). Thus \( U \) is isomorphic to the idele class group \( A^*/\mathbb{Q}^* \).

The next step is to topologise \( 2^\mathbb{P} \sqcup U \) so as to make the bijection from Proposition 3.3 a homeomorphism. The right topology comes from the power-cofinite topology on \( 2^\mathbb{P} \) and the product topology on \( U \), but it is necessary to specify how these two parts interact. First, to deal with a subtlety arising from the difference between adelic and idealic topologies, we need to consider the absolute value function on \( A \).

The absolute value \( \|a\| \) of an adele \( a \) is, by definition, the product of the normalized \( p \)-adic absolute values:
\[
\|a\| := |a_\infty| \prod_{p \in \mathcal{P}} p^{-v_p(a)},
\]
where \( v_p \) is the \( p \)-adic valuation and \( |a_\infty| \) is the usual absolute value of the real component of \( a \). The absolute value vanishes at \( a \) if and only if either \( a_p = 0 \) for some \( p \) or \( v_p(a) > 0 \) for infinitely many \( p \); in other words, \( \|a\| = 0 \) if and only if \( a \) is not invertible. Since noninvertibles are dense in \( A \), the absolute value is not a continuous function.

**Lemma 3.5.** The absolute value is an upper semi-continuous function on \( A \).

**Proof.** The maps \( a \mapsto p^{-v_p(a)} \) for \( p \in \mathcal{P} \) and \( a \mapsto |a_\infty| \) are all continuous on \( A \), so for each finite subset \( F \) of \( \mathcal{P} \) the product
\[
\xi_F : a \in A \mapsto |a_\infty| \prod_{p \in F} p^{-v_p(a)}
\]
is continuous, and, the net \((\xi_F)_F\) directed by the finite subsets of \(\mathcal{P}\) under inclusion converges pointwise to the absolute value. For each finite \(G \subset \mathcal{P}\) and each choice of \(k_p \in \mathbb{Z}\) for \(p \in G\), the set

\[
W := \prod_{p \in G} p^{k_p} \mathbb{Z}_p \times \prod_{p \notin G} \mathbb{Z}_p \times \mathbb{R}
\]

is open in \(\mathcal{A}\), and every adele belongs to some \(W\) (given \(a \in \mathcal{A}\), it suffices to choose \(G = \{p : v_p(a) < 0\}\) and \(k_p = v_p(a)\)). Moreover, for every \(b \in W\), if \(v_p(b) < 0\) then \(p \in G\).

It follows that the tail net \(\{\xi_F\}_{F \supset G}\) is nonincreasing when restricted to \(W\), because all the factors greater than 1 have been included in \(\xi_F(b)\) already. Thus, the restriction of \(\| \cdot \| = \lim_F \xi_F\) to \(W\) is the infimum of the restricted tail net and hence is upper semicontinuous on \(W\). Since each \(W\) is open and their union is all of \(\mathcal{A}\), this completes the proof.

From (3.1) we know that the map \((r, u) \mapsto ru\) is a bijection of \(\mathbb{Q}^* \times \mathcal{U}\) onto \(\mathcal{A}^*\). Since \(\mathbb{Q}^*\) has the discrete topology, and the product (idelic) topology on \(\mathcal{U}\) coincides with the adelic topology, it is clear that the map is continuous. However, its inverse is not continuous: let \(p_n\) be the \(n\)th prime and define a sequence \(a_n \in \mathcal{A}^*\) by \((a_n)_q = 1\) for \(q \neq p_n\) and \((a_n)_{p_n} = p_n\); then \(a_n \to 1 \in \mathcal{A}\), but \((u_n)_\infty \to 0\), so \((r_n, u_n)\) cannot converge in \(\mathbb{Q}^* \times \mathcal{U}\). Notice that the absolute value \(\|a_n\|\) tends to zero in this example; the following lemma shows that this is crucial.

**Lemma 3.6.** For each \(\varepsilon > 0\) let \(F_\varepsilon := \{a \in \mathcal{A} : \|a\| \geq \varepsilon\}\) and \(U_\varepsilon := \{u \in \mathcal{U} : u_\infty \geq \varepsilon\}\), considered with their relative topologies as subsets of the adeles and the ideles, respectively. Then \(F_\varepsilon\) is closed, \(\chi(F_\varepsilon) = U_\varepsilon\), and the map \((r, u) \mapsto ru\) is a homeomorphism of \(\mathbb{Q}^* \times U_\varepsilon\) onto \(F_\varepsilon\).

**Proof.** By Lemma 3.3, \(F_\varepsilon\) is closed, and the preceding discussion establishes that \((r, u) \mapsto ru\) is a continuous bijection, so it only remains to show that the inverse is continuous. Suppose \(a_n \to a \in F_\varepsilon\); we need to show that \(u_i := \chi(a_i)\) converges to \(u := \chi(a)\) in \(U_\varepsilon\) and that \(r_i := r(a_i)\) is eventually equal to \(r(a)\). Dividing everything by \(r(a)\), we may assume that \(a \in U_\varepsilon\), and show that \(r_i\) is eventually 1. For each \(n \in \mathbb{N}\), let \(p_n\) be the \(n\)th prime, and let

\[
U_n := \prod_{j \leq n} \mathbb{Z}_{p_j}^* \times \prod_{j > n} \mathbb{Z}_{p_j} \times I
\]

where \(I\) is an interval in \(\mathbb{R}_+\) containing \(a_\infty\). For each \(n\), \(a_i\) is eventually in \(U_n\), and then

\[
v_{p_j}(r_i) = 0 \quad \text{for} \quad j \leq n \quad \text{and} \quad v_{p_j}(r_i) \geq 0 \quad \text{for} \quad j > n.
\]

It follows that \(r_i\) is eventually an integer, and that either \(r_i = 1\) or \(r_i \geq p_i\) (because \(p_i\) is the smallest possible prime factor of \(r(a_i)\)). Since \(r_i u_i = a_i\) converges to \(a \in F_\varepsilon\), we know that \((r_i u_i)_\infty\) converges to \(a_\infty \geq \varepsilon\). But \((u_\lambda)_\infty\) is bounded away from 0, so \(\{r_i\}\) has to remain bounded, and this can only happen if \(r_i\) is eventually 1.

**Theorem 3.7.** Let \(\tau\) be the topology on the parameter space \(2^\mathcal{P} \sqcup \mathcal{U}\) pulled back from the quotient topology on the quasi-orbit space under the bijection of Proposition 3.3 (so that \(\tau\) is the quotient topology induced by the map \(\chi : \mathcal{A} \to 2^\mathcal{P} \sqcup \mathcal{U}\)). Then the \(\tau\)-closure of a set \(B \subset 2^\mathcal{P}\) is

\[
\overline{B} = \begin{cases} 
\text{the power-cofinite closure of } B & \text{if } B \text{ is not power-cofinite dense in } 2^\mathcal{P}, \\
2^\mathcal{P} \sqcup \mathcal{U} & \text{if } B \text{ is power-cofinite dense in } 2^\mathcal{P}.
\end{cases}
\]
and the $\tau$-closure of a set $C \subset U$ is

$$\overline{C^\tau} = \begin{cases} 
\text{the idelic closure of } C & \text{if } \{c_\infty : c \in C\} \text{ is bounded away from } 0, \\
2^p \cup U & \text{if } 0 \in \{c_\infty : c \in C\}. 
\end{cases}$$

**Proof.** First we show that the power-cofinite closure of $B$ is always contained in $\overline{B^\tau}$. Let $T$ be in the power-cofinite closure of $B$; we will show that that every (basic) open set containing $T$ intersects $B$. We can assume that the basic open set is $\chi(W)$ for $W = \prod_p V_p \times \prod_p \Z_p$, where $V_p$ is an open subset of $\Q_p$ for every $p$ in the finite subset $F$ of $\P$, and that $T = \chi(t)$ for $t \in W$. We need to find an adele $w$ such that

- $w_p \in V_p$ for $p \in F$ and $w_p \in \Z_p$ for $p \notin F$ (so that $w \in W$);
- $w$ is not invertible in $A$ (so that $\chi(w) = \{p : w_p = 0\}$);
- $\chi(w) \in B$.

The set $G = \{p \in F : 0 \notin V_p\}$ is finite and disjoint from $T = \{p : t_p = 0\}$, because $t \in W$. Thus $T \in U_G$, and because $T$ is in the power-cofinite closure, there exists $b \in B \cap U_G$ — that is, $b \cap G = \emptyset$. Choose $w$ as follows:

$$w_p = \begin{cases} 
t_p & \text{if } p \in G \\
p & \text{if } p \notin b \cap G \\
0 & \text{if } p \in b. 
\end{cases}$$

Then $w$ is not invertible (even if $b = \emptyset$) and $\chi(w) = b$. This proves that $T \in \overline{B^\tau}$.

Next we show that if $B$ is not power-cofinite dense in $2^p$ then the power-cofinite closure $\overline{B^{pc}}$ of $B$ contains $\overline{B^\tau}$. Because $B$ is not dense, it misses some basic open set $U_G$, and $\overline{B^{pc}} \subset 2^p \setminus U_G$. We claim that $2^p \setminus U_G$ is also $\tau$-closed. To see this, write

$$2^p \setminus U_G = \chi(\chi^{-1}(2^p \setminus U_G)) = \chi(\{a \in A : \chi(a) \cap G \neq \emptyset\}).$$

and then observe that the set

$$\{a \in A : \chi(a) \cap G \neq \emptyset\} = \{a \in A : \{p : a_p = 0\} \cap G \neq \emptyset\} = \{a \in A : \text{for some } p \in G \text{ we have } a_p = 0\}$$

is a finite union of $\Q^*$-invariant closed sets. Thus $2^p \setminus U_G = \chi(\{a \in A : \chi(a) \cap G \neq \emptyset\})$ is $\tau$-closed, as claimed. This implies that the $\tau$-closure of $B$ is contained in $2^p$, and indeed is contained in

$$\overline{B^{pc}} = \cap\{2^p \setminus U_G : B \cap U_G = \emptyset\}.$$

If $B$ is power-cofinite dense in $2^p$, then $\emptyset$ is in the power-cofinite closure of $B$. The initial paragraph of the proof, with $T = \emptyset$, shows that $\emptyset \in \overline{B^\tau}$; we will show that $\{\emptyset\}$ itself is $\tau$-dense. Let $W$ be any open set in $A$. Choose a non-invertible adele $a$ such that $a_p \neq 0$ for every $p$. Lemma 3.2 implies that $W$ contains $ra$ for some $r \in \Q^*$. Then $\emptyset = \chi(a) = \chi(ra) \in \chi(W)$. Thus $\{\emptyset\}$ is $\tau$-dense, and we must have $\overline{B^\tau} = 2^p \cup U$. This completes the description of $\overline{B^\tau}$.

Next let $C \subset U$, and assume first that $0 \in \overline{c_\infty : c \in C}$. We claim that every singleton $\{p\}$ belongs to $\overline{C^\tau}$. There is a strictly increasing sequence $n_k$ such that each interval $(p^{-(n_k+1)}, p^{-n_k})$ contains $c_\infty$ for some $c \in C$, and for each $k$ we choose one such $c = c_k$. Since $n_k \to \infty$, we have $(p^{n_k+1}c_k)_p \to 0$ in $\Z_p$. Moreover, for all $k$ we have $(p^{n_k+1}c_k)_q \in \Z_q$.
for every finite prime \( q \neq p \) and \((p^{n_k+1}c_k)_\infty \in (1, p] \). Since \( \mathbb{Z}_p \times \prod_{q \neq p} \mathbb{Z}_q \times [1, p] \) is compact, we can assume by passing to a subsequence that \((p^{n_k+1}c_k)_q \to a_q \) for every \( q \in \mathbb{P} \); then \( a_q \) vanishes precisely when \( q = p \), and hence \( \chi(a) = \{ p \} \). By continuity of \( \chi \), we know that \( \chi(p^{n_k+1}c_k) \to \chi(a) = \{ p \} \) in \((2^P \cup U, \tau)\), and, since \( \chi(p^{n_k+1}c_k) = \chi(c_k) = c_k \in C \), it follows that \( \{ p \} \in \overline{C}^\tau \), as claimed.

Since every singleton \( \{ p \} \) is in \( \overline{C}^\tau \), \( \overline{C}^\tau \) meets every basic open neighbourhood \( U_G \) of \( \emptyset \) in \( 2^P \), and \( \emptyset \in \overline{C}^\tau \). Since we have already seen that \( \{ \emptyset \} \) is \( \tau \)-dense, it follows that \( \overline{C}^\tau = 2^P \cup U \).

Assume now that \( c_\infty \geq \epsilon \) for every \( c \in C \), and let \( F_\epsilon := \{ a \in A : \| a \| \geq \epsilon \} \). The inclusion map \( U_\epsilon \hookrightarrow F_\epsilon \to 2^P \cup U \) is continuous, so the closure \( \overline{C}_U^{\epsilon} \) of \( C \) in \( U \) is certainly contained in the \( \tau \)-closure of \( C \) in \( 2^P \cup U \). On the other hand, because the inverse of \((r, u) \mapsto ru \) is continuous by Lemma 3.6, the set \( \overline{Q}^* \overline{C}_U^{\epsilon} \) is closed in \( F_\epsilon \). Since it is also \( \overline{Q}^* \)-invariant, its image \( \overline{C}_U^{\epsilon} = \chi(\overline{Q}^* \overline{C}_U^{\epsilon}) \) is closed in the quotient topology; in other words, it is \( \tau \)-closed. Thus \( \overline{C}_U^{\tau} \subset \overline{C}_U^{\epsilon} \), and we conclude that \( \overline{C}_U^{\tau} = \overline{C}_U^{\epsilon} \).

From here it is relatively easy to describe Prim\((C_0(A) \rtimes \overline{Q}^*)\) and its topology. Recall that for each \( a \in A \) there is a covariant representation \((M_a, \lambda)\) of \((C_0(A), \overline{Q}^*)\) on \( L^2(\overline{Q}^*)\) given by:

\[
(M_a f)\xi(r) = f(ra)\xi(r) \quad \text{and} \quad \lambda_s\xi(r) = \xi(s^{-1}r).
\]

The primitive ideals of \( C_0(A) \rtimes \overline{Q}^* \) are the kernels of these representations; there are three types corresponding to the three types of orbit closures, and our description of the topology on Prim\((C_0(A) \rtimes \overline{Q}^*)\) follows from Williams' theorem (see the comments in Remark 2.6).

**Corollary 3.8.**

1. The map \( \gamma \mapsto \ker(\varepsilon_0 \times \gamma) \) is an injection of \( \overline{Q}^* \) into Prim\((C_0(A) \rtimes \overline{Q}^*)\), and its image is a closed subset of Prim\((C_0(A) \rtimes \overline{Q}^*)\).
2. If \( a \neq 0 \) is not invertible, then \( \ker(M_a \times \lambda) \) depends only on \( S(a) := \{ p \in \mathbb{P} : a_p = 0 \} \), and the map \( S(a) \mapsto \ker(M_a \times \lambda) \) is an injection of \( 2^P \setminus \{ \mathbb{P} \} \) into Prim\((C_0(A) \rtimes \overline{Q}^*)\).
3. If \( a \in A \) is invertible, then \( \ker(M_a \times \lambda) \) depends only on the orbit \( Q^*a \), which contains a unique element \( u \in U := \prod_{p \in \mathbb{P}} \mathbb{Z}_p^* \times \mathbb{R}_1^* \); the map \( u \mapsto \ker(M_u \times \lambda) \) is an injection of the set \( U \) into Prim\((C_0(A) \rtimes \overline{Q}^*)\).

Taken together, these maps parametrize the primitive spectrum of \( C_0(A) \rtimes \overline{Q}^* \) in the sense that their images are disjoint and their union is all of Prim\((C_0(A) \rtimes \overline{Q}^*)\). Define \( \overline{A} \) for a nonempty subset \( A \) of \( \overline{Q}^* \sqcup (2^P \setminus \{ \mathbb{P} \}) \cup U \) by

\[
\overline{A} := \begin{cases}
\overline{Q}^* & \text{if } A \subset \overline{Q}^*; \\
\mathbb{Q}^* \sqcup (\overline{A}^{oc} \setminus \{ \mathbb{P} \}) & \text{if } A \subset 2^P \setminus \{ \mathbb{P} \} \text{ and } \overline{A}^{oc} \neq 2^P; \\
\overline{A}^{oc} & \text{if } A \subset U \text{ and } 0 \notin \{ \| a \| : a \in A \}; \\
(\overline{Q}^* \sqcup (2^P \setminus \{ \mathbb{P} \}) \cup U) & \text{if } A \subset 2^P \setminus \{ \mathbb{P} \} \text{ and } \overline{A}^{oc} = 2^P; \\
o & \text{or if } A \subset U \text{ and } 0 \in \{ \| a \| : a \in A \}.
\end{cases}
\]

Then \( A \mapsto \overline{A} \) (has an obvious extension that) satisfies Kuratowski’s closure axioms, and the resulting topology on \( \overline{Q}^* \sqcup (2^P \setminus \{ \mathbb{P} \}) \cup U \) makes the parametrisation a homeomorphism.
Remark 3.9. (1) The injection of $\hat{Q}^*$ is a homeomorphism onto its image in $\text{Prim}(C_0(A) \rtimes Q^*)$. Indeed, the map $\varepsilon_0$ induces an isomorphism of $(C_0(A) \rtimes Q^*)/(\ker \varepsilon_0) \times Q^*)$ onto $C^*(Q^*)$, and $\hat{Q}^*$ is the primitive ideal space of the quotient viewed as a closed subset of $\text{Prim}(C_0(A) \rtimes Q^*)$.

(2) The second injection is also a homeomorphism onto its image: to see this, just note that the power-cofinite closure of $A \subset 2^P$ is equal to $2^P \cap \overline{A}$. Indeed, the image under $\chi$ of the sequence $\{a_n\}$ defined before Lemma 3.8 is closed in the idelic topology on $U$, but is $\tau$-dense in $2^P \cup U$, and hence dense for the relative topology on $U \subset 2^P \cup U$.

(3) The third injection is not a homeomorphism onto its range. Indeed, the image under $\chi$ of the sequence $\{a_n\}$ defined before Lemma 3.6 is closed in the idelic topology on $U$, but is $\tau$-dense in $2^P \cup U$, and hence dense for the relative topology on $U \subset 2^P \cup U$.

(4) The representations associated to elements of $U$ are all CCR representations: the map $r \mapsto ru$ is a homeomorphism of $Q^*$ onto the discrete set $Q^*u$, and hence the image of the representation $M_u \times \lambda$ is isomorphic to $K(\ell^2(Q^*))$ by the Stone-von Neumann theorem.

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