THE GLOBAL INDICES OF LOG CALABI-YAU VARIETIES
–A SUPPLEMENT TO FUJINO’S PAPER: THE INDICES OF LOG CANONICAL SINGULARITIES–

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Abstract. This paper gives the all possible global indices of log Calabi-Yau 3-folds with standard coefficients on the boundaries and having lc, non-klt singularities. This follows easily from the discussion in the paper: The indices of log canonical singularities by Fujino.

1. Introduction

In this paper, we study a log pair \((X, B_X)\) with a normal projective variety \(X\) defined over \(\mathbb{C}\) and a boundary \(B_X\) of standard coefficients (i.e., \(B_X = \sum b_i B_i\), where \(b_i = 1\) or \(1 - 1/m\) for \(m \in \mathbb{N}\)). A pair \((X, B_X)\) is called a log Calabi-Yau variety if it has lc singularities and \(K_X + B_X \equiv 0\). For a log Calabi-Yau variety \((X, B_X)\) assume that there exists \(r \in \mathbb{N}\) such that \(r(K_X + B_X) \sim 0\). (For \(\dim X \leq 3\) this holds true for every log Calabi-Yau variety, by the abundance theorem ([4], 11.1.3) and [5]). We define the global index \(\text{Ind}(X, B_X)\) by the minimum of such \(r\).

It is well known that a non-singular surface \(X\) with \(K_X \equiv 0\) has \(\text{Ind}(X, 0) = 1, 2, 3, 4, 6\). Blache [4] proved that a normal surface \(X\) with \(K_X \equiv 0\) and having lc non-klt singularity has also \(\text{Ind}(X, 0) = 1, 2, 3, 4, 6\). This is generalized into the case that log Calabi-Yau surface \((X, B_X)\) has lc and non-klt singularities in [12, 2.3].

In this paper we prove the following:

**Theorem 1.1.** Let \((X, B_X)\) be a log Calabi-Yau 3-fold with lc non-klt singularities. Then \(r \in \mathbb{N}\) can be the global index \(\text{Ind}(X, B_X)\), if and only if \(\varphi(r) \leq 20\) and \(r \neq 60\), where \(\varphi\) is the Euler function. In particular the global index is bounded.

This theorem is a corollary of the following:

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Theorem 1.2. Assume the Abundance Theorem and $G$-equivariant log Minimal Model Program for dimension $\leq n$, where $G$ is a finite group. Let $(X, B_X)$ be an $n$-dimensional log Calabi-Yau variety with non-klt singularities. If the conjectures $(F_j')$ and $(F_l)$ in [3] hold true for $j = n - 1$, $l \leq n - 2$, then the global index $\text{Ind}(X, B_X)$ is bounded.

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2. The global indices

2.1. Throughout this paper, we use the notation and the terminologies in [3]. We assume the Abundance Theorem and the $G$-equivariant log Minimal Model Program (as is well known, these hold for dimension $\leq 3$ by [1, 11.1.3], [3] and [4, 2.21]).

2.2. Let $(X, B_X)$ be an $n$-dimensional log Calabi-Yau variety. Since we assume the Abundance Theorem, there exists $r \in \mathbb{N}$ such that $r(K_X + B_X) \sim 0$. Let $\pi : (Y, B) \to (X, B_X)$ be the index 1 cover with $K_Y + B = \pi^*(K_X + B_X)$.

Here the index 1 cover is constructed as follows: let $r = \text{Ind}(X, B_X)$, then there exists a rational function $\varphi$ on $X$ such that $r(K_X + B_X) = \text{div}(\varphi)$; take the integral closure $Y$ in $K(X)(\sqrt[r]{\varphi})$. Note that $K_Y + B \sim 0$, that $B = \pi^*(\lfloor B_X \rfloor)$ is a reduced divisor and that $\pi$ ramifies only over the components of $B_X$ whose coefficients are $< 1$, as the coefficients of $B_X$ are standard. Since $K_X + B_X$ is lc (resp. klt) if and only if $K_Y + B$ is lc (resp. klt), $(Y, B)$ is log Calabi-Yau of global index 1. Therefore we obtain that every log Calabi-Yau variety $(X, B_X)$ is the quotient of a log Calabi-Yau variety of global index 1 by the action of a finite cyclic group.

2.3. Let $G$ be the cyclic group acting on a log Calabi-Yau variety $(Y, B)$ of global index 1. Since $G$ acts on $\Gamma(Y, K_Y + B) = \mathbb{C}$, there is a corresponding representation $\rho : G \to GL(\Gamma(Y, K_Y + B)) = \mathbb{C}^*.$

Lemma 2.4. Under the notation above, Let $(X, B_X)$ be the quotient $(Y, B)/G$ by $G$. Then $\text{Ind}(X, B_X) = |\text{Im}\rho|$. 

Proof. For a generator $\theta \in \Gamma(Y, K_Y + B)$, $\theta^{\text{Im} \rho}$ is $G$-invariant, therefore $\Gamma(X, |\text{Im} \rho|(K_X + B_X)) \neq 0$, which yields $\text{Ind}(X, B_X) \leq |\text{Im} \rho|$. Conversely, for a generator $\eta \in \Gamma(X, \text{Ind}(X, B_X)(K_X + B_X))$, $\pi^* \eta \in \Gamma(Y, \text{Ind}(X, B_X)(K_Y + B))$ is $G$-invariant. If we write $\pi^* \eta = a \theta^{\text{Ind}(X, B_X)} (a \in \mathbb{C})$, for a generator $g \in G$, $(a \theta^{\text{Ind}(X, B_X)}) g = a \epsilon^{\text{Ind}(X, B_X)} \theta^{\text{Ind}(X, B_X)} = a \theta^{\text{Ind}(X, B_X)}$, where $\epsilon$ is a primitive $|\text{Im} \rho|$-th root of unity. Hence, $\text{Ind}(X, B_X) \geq |\text{Im} \rho|$.

2.5. Now we are going to study lc and non-klt log Calabi-Yau varieties.

Let $(Y, B)$ be an $n$-dimensional log Calabi-Yau variety of global index 1 with lc and non-klt singularities. Assume that a cyclic group $G$ acts on $(Y, B)$. Then we have a projective $G$-equivariant log resolution $\varphi : \tilde{Y} \to Y$ of $(Y, B)$. Indeed, let $\varphi' : Y' \to Y$ be the canonical resolution of $(Y, B)$ constructed in [3], then $\varphi'$ is projective and $\varphi'^{-1}(B) \cup$ (the exceptional set) is normal crossing divisor. By the blow up at a suitable $G$-invariant center, we obtain the divisor with simple normal crossings. Define the subboundary $F$ on $\tilde{Y}$ by $K_{\tilde{Y}} + F = \varphi^*(K_Y + B)$. Run $G$-equivariant log MMP for $K_{\tilde{Y}} + F_B$ over $Y$ (The notation $F_B$ is in [3, 1.5] and $F_B = F_c$ in our case). Then we obtain $G\mathbb{Q}$-factorial dlt pair $f : (Y', B') \to (Y, B)$ over $(Y, B)$. Since $K_{Y'} + B'$ is $f$-nef and $(Y, B)$ is lc, we obtain that $K_{Y'} + B' = f^*(K_Y + B) \sim 0$. By [3, 2.4], $B'$ has at most two connected components.

Definition 2.6 (for the local version, see [4, 4.12]). Let $(Y, B)$ and $(\tilde{Y}, F)$ be as in 2.5. We define

$$\mu = \mu(Y, B) := \min \{ \dim W \mid W \in CLC(\tilde{Y}, F) \}.$$ 

Note that in case $B'$ is connected, then $0 \leq \mu \leq n - 1$ and in case $B'$ has two connected components, then $\mu = n - 1$.

Case 1 ($B'$ is connected)

There exist a $G$-isomorphism $\Gamma(Y, K_Y + B) \simeq \Gamma(Y', K_{Y'} + B')$ and an exact sequence:

$$0 = \Gamma(Y', K_{Y'}) \to \Gamma(Y', K_{Y'} + B') \to \Gamma(B', (K_{Y'} + B')|_{B'}) = \mathbb{C},$$

where the last term is isomorphic to $\Gamma(B', K_{B'})$, as $K_{Y'} + B'$ is a Cartier divisor. Therefore, we have only to check the action of $G$ on $\Gamma(B', K_{B'})$.

Proposition 2.7 (for the local case, see [4, 4.11]). If there exists a non-zero admissible section in $\Gamma(B', m_0 K_{B'})$, then $G$ acts on $\Gamma(B', m_0 K_{B'})$ trivially.

Proof. The proof is the same as that of [4, 4.11]. We have only to note that $B' = E = E'$ in our case. \[\square\]
Proposition 2.8 (for the local case, see [3, 4.14]). Assume that \( \mu(Y, B) \leq n-2 \). Then there exists a non-zero admissible section \( s \in \Gamma(B', m_0K_{B'}) \) with \( m_0 \in D_\mu \). In particular, \( s \) is \( G \)-invariant. Thus, \( \text{Ind}((Y, B)/G) \in I_\mu \).

Proof. The proof is the same as that of [3, 4.14]. Again \( B' = E = E^c \).

Proposition 2.9. Assume that \( B' \) is connected and \( \mu(Y, B) = n-1 \). Then \( \text{Ind}(Y, B)/G \in I_{n-1} \).

Proof. In this case, \( B' \) is irreducible, therefore \((Y', B')\) is plt. Then, by Adjunction [3, 17.6], \( B' \) is klt and \( K_{B'} \sim 0 \). Now apply [2.4].

Case 2 (\( B' \) has two connected components).

Note that \( B' \) is the disjoint union of two irreducible components, therefore \((Y', B')\) is plt (see [3, 2.4]). Run \( G \)-equivariant log MMP for \( K + B' - \epsilon B' \), then we obtain a \( G \)-equivariant contraction \( p : Y'' \to Z \) of an extremal face for \( K + B'' - \epsilon B'' \) to a lower dimensional variety \( Z \), where \( B'' = B'_1 \amalg B''_2 \) is the divisor on \( Y'' \) corresponding to \( B' \). Here \( \dim Z = n-1 \), because \( h^{n-1}(Y', \mathcal{O}_{Y'}) = h^1(Y', K_{Y'}) \neq 0 \). We also obtain that \( B''_i \)'s are generic sections of \( p \). Since \((Y'', B'')\) is plt and \( K_{Y''} + B'' \sim 0 \), each \( B''_i \) has canonical singularities and \( K_{B''_i} \sim 0 \) by [3, 17.6]. Then the birational image \( Z \) has \( K_Z \sim 0 \), and therefore it has canonical singularities. Since the group \( G = \langle \gamma \rangle \) acts on \( B'' \), the subgroup \( H := \langle \gamma^2 \rangle \) acts on each \( B''_i \) \((i = 1, 2) \). Consider the exact sequence:

\[
0 = \Gamma(Y'', K_{Y''} + B''_2) \to \Gamma(Y'', K_{Y''} + B'') \xrightarrow{\alpha} \Gamma(B''_1, K_{B''_1}),
\]

where \( \alpha \) is an \( H \)-equivariant isomorphism. On the other hand, the homomorphism \( \Gamma(B''_1, K_{B''_1}) \to \Gamma(Z, K_Z) \) induced from \( p|_{B''_1} \) is also an \( H \)-equivariant isomorphism. Hence, for two representations \( \rho : G \to GL(\Gamma(Z, K_Z)) \) and \( \rho' : G \to GL(\Gamma(Y'', K_{Y''} + B'')) \), we obtain the equality \( |\rho(H)| = |\rho'(H)| \). Note that, for any representation \( \lambda : G \to \mathbb{C}^* \), \( \lambda(H) = \lambda(G) \) if and only if \( |\lambda(G)| \) is an odd number. If we denote \( |\rho(G)| \) by \( r \), then \( r \in I_{n-1} \), and either: (1) \( |\rho'(G)| = r \) or (2) \( |\rho'(G)| = 2r \) and \( r \) is odd or (3) \( |\rho'(G)| = r/2 \) and \( r/2 \) is odd. By defining \( I_k'' := I_k' \cup \{2r \mid r \in I_k' \text{ odd} \} \cup \{r/2 \mid r \in I_k', r/2 \text{ odd} \} \), we obtain:

Proposition 2.10. Assume \( B' \) has two connected components. Then \( \text{Ind}((Y, B)/G) \in I''_{n-1} \).

By [2.8, 2.9 and 2.10], we obtain Theorem [1.2]. In particular, for the 3-dimensional case \( G \)-equivariant log MMP, the Abundance Theorem and \((F_j^i), (F_l) \quad (j = 2, l \leq 1) \) hold. Here note that \( I_0 = \{1, 2\}, I_1 = \).
\{1, 2, 3, 4, 6\} and \(I'_2 = \{ r \in \mathbb{N} \mid \varphi(r) \leq 20, r \neq 60 \}\) by \[10\] and \[9\]. By the list of the values of \(I'_2\) in \[9, \text{Table 1}\], we can check that \(I'_2 = I'_2\). Therefore we obtain the necessary condition of the global index \(\text{Ind}(X, B_X)\) in Theorem \[1.1\].

The following shows that it is the sufficient condition of the global index:

**Example 2.11.** Let \(r\) be a positive integer that satisfies \(\varphi(r) \leq 20\) and \(r \neq 60\). Then by \[8\] and \[9\], there exists a \(K3\)-surface \(S\) with an action \(G\) of order \(r\) and \(r = |\text{Im}\rho|\). Let \(Y = S \times \mathbb{P}^1\) and \(B = S \times \{0\} + S \times \{\infty\}\). Let \(G\) act on \(Y\) by trivial action on \(\mathbb{P}^1\) and the action above on \(S\). Let \((X, B_X)\) be the quotient of \((Y, B)\) by \(G\) with \(K_Y + B = \pi^*(K_X + B_X)\). Then \((X, B_X)\) is a log Calabi-Yau 3-fold with global index \(r\).

**Remark 2.12.** We can also prove Theorem \[1.1\] by using \[4\] instead of \[3\]. Indeed, we used \[3\] only for propositions \[2.7\] and \[2.8\]. For the 3-dimensional case, these propositions can be replaced by the discussion on the order of the action of \(G\) on \(H^2(F_B, \mathcal{O}_{F_B})\) for type \((0, 0)\) and \((0, 1)\). Theorems \[4, 4.5\] and \[4, 4.12\] give the same results as in \[2.3\].

**Remark 2.13.** Osamu Fujino informed the author that the boundedness of the indices of log Calabi-Yau 3-folds also follows from \[3, 4.17\] and the proof of \[3, 4.14\]. By this proof we obtain the index in \(I_2\) instead of \(I'_2\).

**Remark 2.14.** If we assume \((F'_n)\), then it is clear that \(n\)-dimensional klt log Calabi-Yau variety has the global index \(r \in I'_n\) by Lemma \[2.4\]. Therefore klt log Calabi-Yau surface has the global index \(r\) such that \(\varphi(r) \leq 20\) and \(r \neq 60\).

For a klt log Calabi-Yau 3-fold with \(B_X = 0\), the global index satisfies the same condition as above \[3, \text{Corollary 5}\].

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