COVERS OF POINT-HYPERPLANE GRAPHS

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Abstract. We construct a cover of the non-incident point-hyperplane graph of projective dimension 3 for fields of characteristic 2. If the cardinality of the field is larger than 2, we obtain an elementary construction of the non-split extension of $SL_4(F)$ by $F^6$.

1. Introduction

The non-incident point-hyperplane graph $H_n(F)$ has as vertex set the non-incident pairs of a point and a hyperplane in the projective geometry of projective dimension $n$ over a field $F$. Two distinct vertices are adjacent if the points and hyperplanes are mutually incident. These graphs have been studied extensively, cf. Gramlich [2]. One of their properties is that $H_{n+1}(F)$ is locally $H_n(F)$ for all $F$ and $n$, and that every connected and locally $H_n(F)$ graph is isomorphic to $H_{n+1}(F)$ whenever $n > 2$.

This property does not necessarily hold if $n \leq 2$. Indeed, if $n \in \{0, 1\}$ then it is easily seen not to hold. In Gramlich [2], a covering graph of $H_3(F_2)$, constructed by means of a computer algebra computation, shows that it does not hold for $n = 2, F = F_2$. In this paper we give a computer-free construction of a covering graph of $H_3(F)$ for char $F = 2$, thus providing counterexamples to the local recognizability of $H_3(F)$ for this wider class. This is the content of our main theorem, Theorem 3.6, which is based on a construction developed in Sections 2 and 3. These sections are based on the second author’s Master’s thesis [3].

In Section 4 we find automorphisms of this covering graph, generating an extension of $SL_4(F)$ by $F^6$, which is non-split if $|F| > 2$. Consequently, as a byproduct we find an elementary construction of the non-split extension discussed by Bell [1] and Griess [3].

1.1. Notation and conventions. We let groups act on the right. Hence we will often consider the set of right cosets of a subgroup $H$ of some group $G$. In order to avoid confusion with the set minus operation, we will denote this set as $H \backslash G$.

All graphs are simple and undirected. The adjacency of vertices $u$ and $v$ is denoted by $u \perp v$. For a graph $\Gamma$, we let $V(\Gamma)$ be the set of its vertices and $D(\Gamma)$ be the set of its darts or oriented edges; that is, the set of ordered pairs of vertices $(u, v)$ for which $u \perp v$.

2. The voltage assignment

In this section, we discuss a general method of constructing covers of a given graph by means of voltage assignments. For a general introduction to voltage assignments, see Malnič et al. [4].

For all vertices $v$ of a graph, we call the induced graph on the neighbourhood of $v$ the local graph of $\Gamma$ at $v$. Let $\Gamma$ and $\Delta$ be two graphs. A map $\alpha: V(\Gamma) \to V(\Delta)$ preserving adjacency such that the local graph at every vertex of $\Gamma$ is mapped isomorphically to the local graph at its image, is called a local isomorphism. If $\Gamma$ and $\Delta$ are connected and a local isomorphism from $\Gamma$ to $\Delta$ exists, we call $\Gamma$ a cover of $\Delta$.

Let $N$ be a group. A map $\ell: D(\Delta) \to N$ such that $\ell(u, v) = \ell(v, u)^{-1}$ is called a voltage assignment of $\Delta$. We will often write $\ell_{u,v}$ for $\ell(u, v)$, or $\ell_{i,j}$ for $\ell_{v_i, v_j}$. The lift of $\Delta$ with respect to $\ell$ is the graph with vertex set $V(\Delta) \times N$, where $(u, m) \perp (v, n)$ if and only if $u \perp v$ and $\ell(u, v) = mn^{-1}$.

Given a path $P = (v_0 \perp v_1 \perp \cdots \perp v_l)$, we call $\ell_{0,1} \ell_{1,2} \cdots \ell_{l-n-1,n}$ the voltage of $P$, denoted by $\ell(P)$. Using induction it is immediate that for any $m \in N$, we find an induced path in the lift from $(v_0, m)$ to $(v_n, \ell(P)^{-1}m)$. 
Let $\Delta$ be a connected graph with voltage assignment $\ell: D(\Delta) \to N$. Let $\Gamma$ be the lift of $\Delta$ with respect to $\ell$. Then it is easy to see that $\Gamma$ is connected if and only if for every $n \in N$ and every $v_0 \in V(\Delta)$, there is an $i \in \mathbb{N}$ and a cycle $(v_0, v_1, \ldots, v_i = v_0)$ such that $\ell_0, \ell_1, \ldots, \ell_{i-1}, \ell_i = n$. It is equally easy to see that for all $n \in N$ and $v \in V(\Delta)$, the local graph at $(v, n)$ in $\Gamma$ is isomorphic to the local graph at $v$ in $\Delta$, if and only if for every triangle $u, v, w$, we have $\ell_{u,v} \ell_{v,w} \ell_{w,u} = 1$. These two observations lead to the following straightforward lemma.

**Lemma 2.1.** Let $\Delta$ be a connected graph with voltage assignment $\ell': D(\Delta) \to N'$. Let $T$ be the normal closure of the subgroup of $\Gamma$ generated by the voltages of all triangles.

Define $N = T \setminus N'$ and $\ell_{u,v} = T\ell'_{u,v}$. Let $M$ be the subgroup of $\Delta$ generated by the voltages (with respect to $\ell$) of all cycles. Let $\Gamma$ be the lift of $\Delta$ with respect to $\ell$.

Then the map $\alpha: V(\Gamma) \to V(\Delta)$, $(v, n) \mapsto v$ is a local isomorphism and every connected component of $\Gamma$ is an $[M]$-fold cover of $\Delta$.

Let $G$ be a group of automorphisms of $\Delta$ with an action on $N$. We will say that $\ell$ is $G$-equivariant if and only if for all $g \in G$ and $v \perp w \in \Delta$, we have that $\ell_{v,w} = (\ell_{v,w})^g$.

Group-equivariant voltage assignments enable the group to lift to a group of automorphisms of the lift. This is the content of the next lemma, the proof of which is again straightforward.

**Lemma 2.2.** Let $G$ be a subgroup of $\text{Aut} \Delta$ such that $\ell$ is $G$-equivariant. Let $\Gamma$ be the lift of $\Delta$ with respect to $\ell$. Then $G \times N$ acts faithfully on $\Gamma$ by the action $(v, n)(g, k) = (v^g, n^k)$.

Now suppose we have the setup of Lemma 2.1. Suppose that $M$ is Abelian and that $\ell$ is $G$-equivariant. Choose a vertex $v \in V(\Delta)$. For all $g \in G$, choose a path $P_g$ from $v^g$ to $v$, and let $\lambda(g)$ be the voltage of $P_g$. Then the following lemma holds.

**Lemma 2.3.** The stabilizer in $G \times N$ of the connected component $\Gamma_0$ of $\Gamma$ containing $(v, 0)$ is $H = \{(g, \lambda(g) + m) \mid g \in G, m \in M\}$, which is an extension of $G$ by $M$.

**Proof.** Since $(v, 0)(g, \lambda(g) + m) = (v^g, \lambda(g) + m)$ and since the path induced on $P_g$ starting at $(v^g, \lambda(g) + m)$ ends at $(v, m)$, we have that $H$ stabilizes $\Gamma_0$. If any element $(g, n)$ stabilizes $\Gamma_0$, it maps $(v, 0)$ to an element $(v^g, n)$ such that there is a path from $(v^g, \lambda(g))$ to $(v^g, n)$. Then the projection of that path down to $\Delta$ is a cycle; hence $\lambda - n \in M$. So $H$ is the full stabilizer of $\Gamma_0$. Therefore it is a group.

The kernel of the projection onto the first coordinate is $\{1\} \times M$, so that is a normal subgroup. The quotient by that subgroup is $G$.

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### 2.1. The reduct.

Let a graph $\tilde{\Delta}$ be given. We define an equivalence relation $\sim$ on the vertices by

$$v \sim w \iff v^\perp = w^\perp,$$

where $v^\perp$ is the set of neighbours of $v$. The adjacency relation carries over in a natural way from the vertices to the equivalence classes of $\sim$ (which are necessarily cocliques). Hence we can mod out $\sim$ and obtain a new graph $\Delta/\sim = \tilde{\Delta}$. We call $\Delta$ the reduct of $\tilde{\Delta}$, and write $\tilde{v}$ for the vertex in $\Delta$ representing the equivalence class containing $v$.

We will need this well-known construction in Section 4 in order to link the projective graph $H_n(\mathbb{F})$ to its affine version, $\tilde{H}_n(\mathbb{F})$ (defined in the same section).

**Lemma 2.4.** Let $\ell: D(\tilde{\Delta}) \to N$ be a voltage assignment on a graph $\tilde{\Delta}$ without isolated vertices. Let $\Gamma$ be the lift of $\tilde{\Delta}$ with respect to $\ell$. Then the following assertions are equivalent:

(i) for all $u, v, w \in V(\tilde{\Delta})$ such that $u \sim v$ and $v \perp w$, we have $\ell_{w,u} = \ell_{v,w},$

(ii) for all $(u, m), (v, n) \in V(\tilde{\Gamma})$, we have that $(u, m) \sim (v, n)$ if and only if $m = n$ and $u \sim v$.

**Proof.**

\cite{[1]}: Let $u, v, w$ be such that $u \sim v \perp w$. Then $(u, 1) \sim (v, 1)$. Now $(u, 1) \perp (w, \ell_{w,u})$ and $(v, 1) \perp (w, \ell_{w,v})$; this implies that $(u, 1) \perp (w, \ell_{w,v})$. But $(u, 1)$ has only one neighbour of shape $(w, -)$. Hence $\ell_{w,v} = \ell_{v,w}$.

\cite{[2]}: Suppose that $u \sim v \perp w$ for $u, v, w \in V(\tilde{\Delta})$. Then for any $n \in N$, both $(u, n)$ and $(v, n)$ are adjacent to $(w, \ell_{w,u} n)$. Similarly for other neighbours of $u$ and $v$. So $(u, n) \sim (v, n)$. 

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Now let us assume \((u, m) \sim (v, n)\). Then \(u \sim v\). Since \(\tilde{\Delta}\) contains no isolated vertices, there exists a \((w, j)\) such that \((v, n) \perp (w, j)\); then \(\ell_{v, w} = nj^{-1}\) and also \(\ell_{u, v} = mj^{-1}\). Since the two are the same, we find that indeed \(m = n\).

If the conditions of Lemma 2.4 hold, we will say that \(\ell\) is reductive. If \(\ell\) is reductive, then by \((\ref{2.4})\) we can write \((\tilde{v}, n)\) for \((v, n)\); and then by \((\ref{2.4})\), \(\ell_{u, \tilde{v}}\) is well defined as \(\ell_{u, v}\).

In the following three corollaries, \(\tilde{\Delta}\) will be a connected graph. \(\ell : D(\Delta) \to N\) will be a reductive voltage assignment. \(\tilde{\Gamma}\) will be the lift of \(\tilde{\Delta}\) with respect to \(\ell\). Furthermore, \(\Delta = \tilde{\Delta}/\sim \) and \(\Gamma = \tilde{\Gamma}/\sim\).

**Corollary 2.5.** Define \(\ell' : D(\Delta) \to N\) by \(\ell'_{u, \tilde{v}} = \ell_{u, v}\). Then \(\Gamma\) is the lift of \(\Delta\) with respect to \(\ell'\).

**Corollary 2.6.** Let \(G\) be a group of automorphisms of \(\tilde{\Delta}\). Suppose \(\ell\) is \(G\)-equivariant. Then \(G \ltimes N\) has an action on \(\Gamma\) defined by \((\tilde{v}, n)g = (\tilde{v}g, n\cdot g)\).

**Corollary 2.7.** Let \(\ell' : D(\Delta) \to N'\) be a reductive voltage assignment on \(\tilde{\Delta}\) that is \(G\)-equivariant for some group \(G\) of automorphisms of \(\tilde{\Delta}\). Let \(T\) be the normal closure of the subgroup of \(N'\) generated by the voltages of cycles.

Define \(N = T \\setminus N'\) and \(\ell_{u, v} = T\ell'_{u, v}\). Let \(M\) be the subgroup of \(N\) generated by the voltages of cycles. Then the map \(\alpha : \Gamma \to \Delta, (\tilde{v}, n) \mapsto \tilde{v}\) is a local isomorphism and every connected component of \(\Gamma\) is an \(|M|\)-fold cover of \(\Delta\).

Choose a vertex \(v \in \Delta\) and for all \(g \in G\), choose a path \(P_g\) from \(v^0\) to \(v\), and set \(\lambda(g) = \ell(P_g)\).

If \(M\) is Abelian, then the stabilizer in \(G \ltimes N\) of the connected component of \(\Gamma\) containing \((v, 0)\) is \(\{(g, \lambda(g) + m) \mid g \in G, m \in M\}\), which is an extension of \(G\) by \(M\).

3. Point-Hyperplane Graphs

Consider the projective geometry \(\mathbb{P}_n(F)\) of (projective) dimension \(n\) over the field \(F\). We denote incidence by \(\subset\) or \(\supset\) and the projective dimension by \(\dim\). Now define the graph \(H_n(F)\) to have vertex set

\[
\{(x, X) \mid x, X \in \mathbb{P}_n(F), \dim x = 0, \dim X = n - 1, x \not\subset X\}
\]

and adjacency defined by

\[
(x, X) \perp (y, Y) \iff x \subset Y \text{ and } y \subset X.
\]

For this graph, with \(n = 3\) and \(F = \mathbb{F}_2\) of characteristic 2, we will build a cover as follows. We first construct an affine version \(\tilde{H}_3(F)\), the reduct of which is \(H_3(F)\). Then we recall some multilinear algebra in order to define a voltage assignment for \(\tilde{H}_3(F)\). This provides an \(|F^6|\)-fold cover of \(H_3(F)\), the reduct of which is an \(|F^6|\)-fold cover of \(H_3(F)\).

\(V\) will be the vector space \(\mathbb{P}^{n+1}\) with basis \(e_1, e_2, \ldots\) and dual basis \(f_1, f_2, \ldots\), so \(\mathbb{P}_n(F) = \mathbb{P}(V)\).

We define \(\tilde{H}_n(F)\) to be the graph with vertex set

\[
\{v \otimes f \mid v \in V, f \in V^*, f(v) \neq 0\},
\]

and adjacency defined by

\[
v \otimes f \perp w \otimes g \iff f(w) = v(g) = 0.
\]

**Lemma 3.1.** Let \(n \geq 2\). Then \(\tilde{H}_n(F)/\sim\) is isomorphic to \(H_n(F)\).

**Proof.** Clearly, for all \(\alpha \in \mathbb{F}\), we have \(v \otimes f \sim \alpha(v \otimes f)\). Also, if \(v \otimes f \sim w \otimes g\), then \(w\) is in the intersection of all \(n\)-dimensional subspaces of \(V\) containing \(v\), and \(g\) is in the intersection of all \(n\)-dimensional subspaces of \(V^*\) containing \(f\). Hence \(w \otimes g = \alpha(v \otimes f)\), for some \(\alpha \in \mathbb{F}^*\). So the vertex sets of \(\tilde{H}_n(F)\) and \(H_n(F)\) are equal. Clearly the adjacency relation is also the same.

We will proceed to recall some basic multilinear algebra which will be needed for constructing the voltage assignment of \(\tilde{H}_n(F)\).

We fix \(n = 3\) so that \(\dim V = 4\) and let

\[
\bigwedge^k V = V^* \otimes k / \langle v_1 \otimes v_2 \otimes \cdots \otimes v_k \mid v_i = v_j \text{ for some } i \neq j \rangle
\]
be the $k$th Grassmannian of $V$. The image of $v_1 \otimes \cdots \otimes v_k$ in $\bigwedge^k V$ is denoted $v_1 \wedge \cdots \wedge v_k$. Let $G$ be a group with a linear action on $V$; this induces a natural action on $\bigwedge^k V$. Now $G \leq \text{SL}(V)$ if and only if $G$ stabilizes every element of $\bigwedge^4 V$. We will mostly be using the case where $k = 2$. We need the following elementary lemmas.

**Lemma 3.2.** There is a canonical isomorphism between $(\bigwedge^2 V)^*$ and $\bigwedge^2 (V^*)$, that preserves the induced action of $\text{GL}(V)$.

**Sketch of proof.** Let $B: \bigwedge^2 (V^*) \times \bigwedge^2 V \to \mathbb{F}$ be defined for $\hat{f} = f_1 \wedge f_2 \in \bigwedge^2 (V^*)$ and $\hat{v} = v_1 \wedge v_2 \in \bigwedge^2 V$ by

$$B(\hat{f}, \hat{v}) = f_1(v_1)f_2(v_2) - f_1(v_2)f_2(v_1),$$

and extended by linearity. We define $b_\beta(\hat{v}) = B(\hat{f}, \hat{v})$. Then $\beta: f \mapsto b_\beta$ is an isomorphism. □

Because of the preceding lemma, we will drop the parentheses in the future and write $\bigwedge^2 V$.

**Lemma 3.3.** Let $V$ be a vector space of dimension 4 over a field $\mathbb{F}$. Fix an isomorphism $\chi$ between $\bigwedge^4 V$ and $\mathbb{F}$. Let $G$ be a subgroup of $\text{SL}(V)$. Then there is a canonical isomorphism between $\bigwedge^2 V^*$ and $\bigwedge^2 V$, which respects the natural induced group actions of $G$ on $\bigwedge^2 V^*$ and $\bigwedge^2 V$.

**Sketch of proof.** We can set up a bilinear mapping $B$ from $\bigwedge^2 V \times \bigwedge^2 V$ to $\mathbb{F}$ as follows:

$$B(v_1 \wedge v_2, w_1 \wedge w_2) = (v_1 \wedge v_2 \wedge w_1 \wedge w_2)\chi,$$

and extended by linearity. Now let $\psi$ map $\hat{w} \in \bigwedge^2 V$ to the linear functional that maps $\hat{v} \in \bigwedge^2 V$ to $B(\hat{v}, \hat{w})$. Then $\psi$ is a vector space isomorphism. □

We will often let $\phi$ denote the inverse of $\psi$, and if we consider it appropriate, leave the map out entirely.

For an arbitrary vector space $V$, we let

$$S_2(V) = (V \otimes V)/\langle v \otimes w - w \otimes v \mid v, w \in V \rangle$$

be the second order symmetric tensor of $V$. Then the natural action of $G$ on $V \otimes V$ induces a natural action on $S_2(V)$. We denote the image of $v \otimes w$ in $S_2(V)$ by $vw$. We will often write $w^2$ for $ww$.

Now let $\text{char} \, \mathbb{F} = 2$, and let $W = \bigwedge^2 V = \bigwedge^2 (V^*)$ of dimension 6. Then $S_2(W)$ has dimension 21.

**Lemma 3.4.** The subspace $W^{(2)}$ of $S_2(W)$, defined as

$$W^{(2)} = \langle \hat{w}^2 \mid \hat{w} \in W \rangle,$$

has dimension 6 and is invariant under the induced action of $\text{GL}(V)$.

**Sketch of proof.** If $(\hat{e}_i^0)_{i=1}^6$ is a basis for $W$, then $(\hat{e}_i^2)_{i=1}^6$ is a basis of $W^{(2)}$. For any $g \in \text{GL}(V)$, $\hat{w} \in W$ we have $(\hat{w}^2)^g = (\hat{w}^g)^2$. □

We now fix an isomorphism between $\bigwedge^4 V$ and $\mathbb{F}$.

**Lemma 3.5.** Let $w, x, y, z \in V$ such that $w \wedge x \wedge y \wedge z = 1$. Then

$$U = (w \wedge x)(y \wedge z) + (w \wedge y)(z \wedge x) + (w \wedge z)(x \wedge y)$$

does not depend on the choice of $w, x, y, z$. Furthermore, it is fixed under the induced action of $\text{SL}(V)$.

**Proof.** The map

$$\Delta: (w, x, y, z) \mapsto (w \wedge x)(y \wedge z) + (w \wedge y)(z \wedge x) + (w \wedge z)(x \wedge y)$$

is 4-linear and alternating. There is only one such map: the determinant. Hence for tuples of vectors such that $\det(w, x, y, z) = w \wedge x \wedge y \wedge z = 1$, we find that $\Delta$ must be constant.

Since the image of $\Delta(w, x, y, z)$ under an element of $\text{SL}(V)$ is $\Delta(w', x', y', z')$ for some tuple satisfying $w' \wedge x' \wedge y' \wedge z' = 1$, the element $U$ is fixed under $\text{SL}(V)$. □
3.1. The voltage assignment $\ell$. We now restrict our attention to $H_3(\mathbb{F})$, for some $\mathbb{F}$ with $\text{char}\mathbb{F} = 2$. Hence $\dim V = 4$ and we retain $W$, $W^{(2)}$ and $U$ as in the previous section. We let $\ell : D(H_3(\mathbb{F})) \to S_2(W)$ assign the voltage

$$h_1(v_1)^{-1}h_2(v_2)^{-1}(v_1 \land v_2)(h_1 \land h_2)$$

(1)

to the dart from $v_1 \otimes h_1$ to $v_2 \otimes h_2$, and let $\ell^U : D(H_3(\mathbb{F})) \to S_2(W)/(U)_{\mathbb{F}_2}$ be the composition of $\ell$ with the natural projection of $S_2(W)$ to $S_2(W)/(U)_{\mathbb{F}_2}$.  

**Theorem 3.6.** Let $\Gamma$ be the lift of $H_3(\mathbb{F})$ with respect to $\ell^U$, and let $\Gamma = \widetilde{\Gamma}/\sim$. Then every connected component of $\Gamma$ is an $[\mathbb{F}^6]$-fold cover of $H_3(\mathbb{F})$.

For proving Theorem 3.6 we need some auxiliary lemmas.

**Lemma 3.7.** The voltage assignment $\ell$, defined in (1), is reductive.

**Proof.** Let $\widetilde{u}_1 = v_1 \otimes h_1, \widetilde{u}_2 = v_2 \otimes h_2, \widetilde{u}_3 = v_3 \otimes h_3$ be vertices of $H_3(\mathbb{F})$ satisfying $\widetilde{u}_1 \sim \widetilde{u}_2 \perp \widetilde{u}_3$. Since $\widetilde{u}_1 \sim \widetilde{u}_2$, we can write $v_2 \otimes h_2$ as $\alpha v_1 \otimes h_1$ for some $\alpha \in \mathbb{F}$. Then

$$\ell_{\widetilde{u}_2, \widetilde{u}_3} = \alpha^{-1}h_1(v_1)^{-1}h_3(v_3)^{-1}(v_1 \land v_3)(h_1 \land h_3) = \ell_{\widetilde{u}_1, \widetilde{u}_3}.$$ 

\[\square\]

**Lemma 3.8.** Each triangle in $H_3(\mathbb{F})$ has voltage $U$.

**Proof.** Let $(v_1 \otimes h_1), (v_2 \otimes h_2), (v_3 \otimes h_3)$ be a triangle in $H_3(\mathbb{F})$. Then $\{v_i\}$ and $\{h_i\}$ are both linearly independent sets. Hence the intersection of the null spaces of $\{h_i\}$ has dimension 1; choose $t$ nonzero in it. Then $\{v_1, v_2, v_3, t\}$ is linearly independent. We define $u = (v_1 \land v_2 \land v_3 \land t)^{-1}t$. Then $\{v_1, v_2, v_3, u\}$ is a basis for $V$.

Since $h_1(v_3), h_2(v_3), h_1(u), h_2(u)$ all vanish, $(h_1 \land h_2)^{\phi} = \alpha(v_3 \land u)$ for some nonzero $\alpha \in \mathbb{F}$. From this we can derive that $1 = v_1 \land v_2 \land v_3 \land u = \alpha^{-1}[h_1(v_1)h_2(v_2) + h_1(v_2)h_2(v_1)] = \alpha^{-1}h_1(v_1)h_2(v_2)$. Hence $(h_1 \land h_2)^{\phi} = h_1(v_1)h_2(v_2)(v_3 \land u);$ similarly we obtain $(h_1 \land h_3)^{\phi} = h_1(v_1)h_3(v_3)(v_2 \land h_2)$ and $(h_2 \land h_3)^{\phi} = h_2(v_2)h_3(v_3)(v_1 \land h_1)$. So if $\{i, j, k\} = \{1, 2, 3\}$, then the voltage of the dart from $v_i \otimes h_i$ to $v_j \otimes h_j$ is $h_i(v_i)^{-1}h_j(v_j)^{-1}(v_i \land v_j)(h_i \land h_j)^{\phi} = (v_1 \land v_j)(v_k \land u)$. The sum of the voltages is then

$$(v_1 \land v_2)(v_3 \land u) + (v_1 \land v_3)(v_2 \land u) + (v_2 \land v_3)(v_1 \land u) = U.$$ 

\[\square\]

For the following four lemmas, note that we regard $S_2(W)$ as a group only here, so the subgroups are the subspaces over $\mathbb{F}_2$ – not necessarily over $\mathbb{F}$.

**Lemma 3.9.** The voltage of a cycle in $H_3(\mathbb{F})$ of length four is in the $\mathbb{F}_2$-space $W^{(2)} \oplus (U)$.

**Proof.** Note that $U \notin W^{(2)}$, so the space is indeed a direct sum.

Consider a cycle consisting of $v_0 \otimes h_0, \ldots, v_3 \otimes h_3$. We define $\alpha_i = h_i(v_i)^{-1}$. We first assume that $v_0 = v_2$. Then the voltage of the quadrangle is equal to $(v_0 \land v_1)(\alpha_1 h_1 \land (\alpha_0 h_0 + \alpha_2 h_2))^{\phi} + (v_0 \land v_3)(\alpha_3 h_3 \land (\alpha_0 h_0 + \alpha_2 h_2))^{\phi}$. Note that $(\alpha_0 h_0 + \alpha_2 h_2)(v_i) = 0$ for all $i$; for $i = 1$ or 3 because of the adjacencies in the graph, and for $i = 0$ or 2 because then $(\alpha_0 h_0 + \alpha_2 h_2)(v_i) = 0$. Hence the null space of $\alpha_0 h_0 + \alpha_2 h_2$ contains all $v_i$.

We will show that we may assume that $(h_i)$ is a linearly independent set. Let us assume that a nontrivial linear combination of $(h_i)$ is zero. We will distinguish cases according to the sets of nonzero coefficients.
If only the coefficients for two adjacent vertices are nonzero, say for $h_0$ and $h_1$, then their null spaces coincide. Hence $v_0$ is in the null space of $h_0$. Contradiction.

If only the coefficients for $h_0$ and $h_2$ are nonzero, then we have $v_0 \otimes h_0 \sim v_2 \otimes h_2$; then the voltage is the same as if we replace $h_2$ by $h_0$, and if we do that we get the sum of two voltages of 2-cycles, which is 0.

If only the coefficients for $h_1$ and $h_3$ are nonzero, then say $h_3 = \gamma h_1$. The voltage is then $(v_0 \wedge (\alpha_1 v_1 + \gamma \alpha_3 v_3))(h_1 \wedge (\alpha_0 h_0 + \alpha_2 h_2)) = 0$. But since both $h_1$ and $\alpha_0 h_0 + \alpha_2 h_2$ vanish on both $v_0$ and $\alpha_1 v_1 + \gamma \alpha_3 v_3$, the elements $(v_0 \wedge (\alpha_1 v_1 + \gamma \alpha_3 v_3))$ and $(h_1 \wedge (\alpha_0 h_0 + \alpha_2 h_2))$ only differ by a scalar factor. Hence the voltage is in $W^2$.

If exactly three of the coefficients are nonzero, say those for $h_0$, $h_1$, and $h_2$, then the common null space of any pair of those is equal to the common null space of the three. In particular, the common null space of $h_0$ and $h_2$ is contained in the null space of $h_1$. So $v_1$ is in the null space of $h_1$. Contradiction.

If all four coefficients are nonzero, the common null space of $h_1$ and $\alpha_0 h_0 + \alpha_2 h_2$ coincides with the common null space of $h_3$ and $\alpha_0 h_0 + \alpha_2 h_2$; hence for some nonzero $\lambda, \mu \in \mathbb{F}$ we have $\alpha_3 h_3 = \lambda \alpha_1 h_1 + \mu (\alpha_0 h_0 + \alpha_2 h_2)$. Then the voltage of the quadrangle is $(v_0 \wedge (v_1 + \lambda v_3))(\alpha_1 h_1 \wedge (\alpha_0 h_0 + \alpha_2 h_2)) = 0$. Again, both $\alpha_1 h_1$ and $\alpha_0 h_0 + \alpha_2 h_2$ are 0 on both $v_0$ and $v_1 + \lambda v_3$. Therefore the voltage is in $W^2$.

So $(h_i)$ is a linearly independent set. Then $(\alpha_i h_i)$ is a basis for $V^*$. Let $(x_i)$ be its dual basis (so, a basis of $V$). We will now determine the isomorphism between $\bigwedge^2 V^*$ and $\bigwedge^2 V$ with respect to these bases. Let $D = (x_0 \wedge \cdots \wedge x_3)^{-1}$. Let $w_0, w_1$ be arbitrary vectors in $V$ and let us write $w_j = \sum_{i=0}^{3} \epsilon_{i,j} x_i$. Now let us see what element of $\bigwedge^2 V^*$, expressed in $(\alpha_i h_i)_{i=0}^3$, corresponds to e.g. $x_{2,3}$. For clarity, we will use $\lfloor$ and $\lceil$ to denote application of a functional in this computation.

\begin{equation}
\begin{aligned}
(x_2 \wedge x_3)[w_0 \wedge w_1] & = (\eta_{0,0} \eta_{1,1} + \eta_{0,1} \eta_{1,0})(x_0 \wedge x_1 \wedge x_2 \wedge x_3) = \\
& = D^{-1}(\alpha_0 h_0[w_0] \alpha_1 h_1[w_1] + \alpha_0 h_0[w_1] \alpha_1 h_1[w_0]) = D^{-1}(\alpha_0 h_0 \wedge \alpha_1 h_1)[w_0 \wedge w_1].
\end{aligned}
\end{equation}

We see that $\alpha_i h_i \wedge \alpha_j h_j$ corresponds to $Dx_{k,\ell}$, where $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$.

Let us write $v_j = \sum_{i=0}^{3} \epsilon_{i,j} x_i$. Then $\epsilon_{i,j} = \alpha_i h_i(v_j) = h_i(v_j)^{-1} h_i(v_j)$; in particular, $\epsilon_{i,i+1} = 0$. Using the fact that $v_0 = v_2$, we can find the values of $\epsilon_{i,j}$ to be as in Table 1, where $\beta$ and $\gamma$ are arbitrary scalars.

If we now define $x_4 = x_0 + x_2$, then the voltage of the quadrangle is equal to

\begin{equation}
(x_4 \wedge (x_1 + \gamma x_3))(x_{2,3} \wedge x_{0,3}) + (x_4 \wedge (\beta x_1 + x_3))(x_{1,2} \wedge x_{0,1}),
\end{equation}

which evaluates to $\gamma (x_{3,4})^2 + \beta (x_{1,4})^2 \in W^2$. This proves the lemma for quadrangles with $v_0 = v_2$.

Duality gives us that the same holds if $h_0 = h_2$. Let us call quadrangles for which two opposite vertices have the vector or the functional in common, special.

Now let us assume that we have a non-special quadrangle $v_0 \otimes h_0, \ldots, v_3 \otimes h_3$. If $v_0 \otimes h_2$ is a vertex (i.e. $h_2(v_0) \neq 0$), then we can split the quadrangle into two special quadrangles. See Figure 1.

This leaves us with the case where for all $i$, we have $h_{i+2}(v_i) = 0$. Then $\alpha_i h_i(v_j) = \delta_{ij}$, where $\delta_{ij}$ is Kronecker’s symbol. In particular, the $v_i$ are linearly independent. Hence they form a basis.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\textbf{j} & 0 & 1 & 2 & 3 \\
\hline
0,2 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & $\gamma$ \\
3 & 0 & $\beta$ & 0 & 1 \\
\hline
\end{tabular}
\caption{The values of $\epsilon_{i,j}$. The values of $\beta$ and $\gamma$ are arbitrary.}
\end{table}
Lemma 3.10. The voltage of a cycle in $\widetilde{H}_3(\mathbb{F})$ of length 5 is in the $\mathbb{F}_2$-span $W^{(2)} \oplus \langle U \rangle$.

Proof. Let us take a pentagon $v_0 \otimes h_0, \ldots , v_4 \otimes h_4$. Since for all $i$, we have $v_i \otimes h_i \sim \alpha_i v_i \otimes h_i$, we can replace $v_i \otimes h_i$ by $\alpha_i v_i \otimes h_i$. This has the effect of subdividing the pentagon into a quadrangle (with voltage in $W^{(2)} \oplus \langle U \rangle$) and a new pentagon. Now if we prove that the new pentagon has a voltage in $W^{(2)} \oplus \langle U \rangle$, then the old one also has that property. The process is shown in Figure 2. In this way we ensure that $h_i(v_i) = 1$ for all $i$.

Choose an index $i$ and consider the index set $\{i - 2, i, i + 2\}$ (modulo 5). Now suppose that the common null space $\mathcal{N}_i$ of $h_{i-2}$, $h_i$, and $h_{i+2}$ is not contained in $\mathcal{V}_i = \langle v_{i-2}, v_i, v_{i+2} \rangle$. Then take some $v \in \mathcal{N}_i \setminus \mathcal{V}_i$, and some $h \in V^*$ such that $h$ is zero on $\mathcal{V}_i$, but not on $v$. Then the vertex $v \otimes h$ is adjacent to $v_{i-2} \otimes h_{i-2}$, $v_i \otimes h_i$ and $v_{i+2} \otimes h_{i+2}$. Hence we can subdivide the pentagon into two quadrangles and a triangle, as in Figure 3. Therefore its voltage is in $W^{(2)} \oplus \langle U \rangle$.

Now suppose that for all indices $i$, we have that $\mathcal{N}_i \subseteq \mathcal{V}_i$. Since $\dim \mathcal{N}_i \geq 1$, there is a nonzero vector in $\mathcal{V}_i$ on which $h_{i-2}$, $h_i$ and $h_{i+2}$ are all zero. Say this vector is

$$v = \lambda_{i-2}v_{i-2} + \lambda_i v_i + \lambda_{i+2}v_{i+2}.$$  

If $\lambda_i = 0$, then also

$$0 = h_{i \pm 2}(\lambda_{i \pm 2}v_{i \pm 2} + \lambda_{i \mp 2}v_{i \mp 2}) = \lambda_{i \pm 2},$$

contradicting $v \neq 0$. So we may assume $\lambda_i \neq 0$ and, multiplying $v$ by a suitable scalar, $\lambda_i = 1$. Then

$$0 = h_{i \pm 2}(v) = \lambda_{i-2}h_{i-2}(v_{i-2}) + 1 + \lambda_{i+2}h_i(v_{i+2});$$

$$0 = h_{i-2}(v) = \lambda_i - 2 + h_{i-2}(v_i);$$

$$0 = h_{i+2}(v) = \lambda_{i+2} + h_{i+2}(v_i).$$
Figure 2. Replacing a vertex in a pentagon by a scalar multiple in order to have \( h_i(v_i) = 1 \).

Figure 3. Splitting a pentagon into a triangle and two quadrangles.

So we find \( \lambda_{i \pm 2} = h_{i \pm 2}(v_i) \) and hence
\[
(3) \quad h_{i-2}(v_i)h_i(v_{i-2}) + h_{i+2}(v_i)h_i(v_{i+2}) = 1.
\]
If we sum Equation (3) over all \( i \), the right hand side is 1. But every term on the left hand side occurs twice, so the left hand side is 0. Contradiction. \( \square \)

Corollary 3.11. The voltage of a cycle in \( \tilde{H}_3(F) \) is in the \( F_2 \)-space \( W^{(2)} \oplus \langle U \rangle \).

Proof. By Lemma 1.3.5 of Gramlich \[2\], the diameter of \( H_3(F) \) is two. It follows that the diameter of \( \tilde{H}_3(F) \) is also two.

Lemmas 3.8, 3.9 and 3.10 tell us that the lemma holds for all cycles of length at most 5. Let \( c = (v_0 \perp v_1 \perp \cdots \perp v_n = v_0) \) be a shortest cycle with a voltage not in \( W^{(2)} \oplus \langle U \rangle \); so \( n \geq 6 \). There is a path of length at most 2 from \( v_0 \) to \( v_3 \). Let us call this path \( P \). This gives us two new
Proof of Theorem 4.1. We apply Corollary 2.7. Lemma 3.8 gives us $H_3$, an extension of $SL_\lambda$. By permuting the basis vectors and by choosing different values for $\lambda$, we obtain an $F_2$-basis for $W(2)$. Hence every connected component of $\Gamma$ is a $|W(2)|$-fold cover of $\Delta$. Since $W(2) \cong F_6$, we have finished the proof.

4. A GROUP OF AUTOMORPHISMS

Let $\Gamma$ be the graph of Theorem 4.1 and set $N = S_2(W)/\langle U \rangle_{F_2}$ and $M = W(2)$. The group $SL_4(F)$ acts on $H_3(F)$, so, by Corollary 2.6, the group $SL_4(F) \rtimes N$ acts on $\Gamma$. By the results of Section 3, an extension of $SL_4(F)$ by $M$ acts on a connected component of $\Gamma$.

Theorem 4.1. Let $charF = 2$ and $|F| > 2$. Then the group $H$ of automorphisms of $\Gamma$ obtained as the stabilizer in $SL_4(F) \rtimes N$ of a connected component of $\Gamma$, is a non-split extension of $SL_4(F)$ by $F^6$.

Proof. Let $\alpha \in F$ be an element outside the ground field. Let $F$ denote the additive subgroup $(1, \alpha)$ of $F$ of order 4. For $x \in F$, let $A_x$ be the element of $L_4(F)$ fixing $e_2$ and $e_4$, and mapping $e_1$ and $e_3$ to $e_1 + xe_2$ and $e_3 + xe_4$, respectively. We will show that the subgroup $A = \{A_x \mid x \in F\}$ does not lift to a subgroup of $H$.

Let $v_x = (e_1 + xe_2, f_1) \in V(\Delta)$ and let $v_x$ be the corresponding vertex $(e_1 + xe_2, (f_1)) \in V(\Delta)$; similarly, let $\bar{u} = (e_3, f_3) \in V(\bar{\Delta})$ and let $u$ be the corresponding vertex $(e_3, (f_3)) \in V(\bar{\Delta})$. Then $v_x \perp u$ for all $x \in F$.

We define a basis for $W$, and write down the images under $A_x$.

$$
\begin{align*}
    w_1 &= e_1 \land e_2 = (f_3 \land f_4) \to \cdots = w_1; \\
    w_2 &= e_1 \land e_3 = (f_2 \land f_4) \to \cdots = w_2 + xw_3 + xw_4 + x^2w_5; \\
    w_3 &= e_1 \land e_4 = (f_2 \land f_3) \to \cdots = w_3 + xw_5; \\
    w_4 &= e_2 \land e_3 = (f_1 \land f_4) \to \cdots = w_4 + xw_5; \\
    w_5 &= e_2 \land e_4 = (f_1 \land f_3) \to \cdots = w_5; \\
    w_6 &= e_3 \land e_4 = (f_1 \land f_2) \to \cdots = w_6.
\end{align*}
$$

The voltage along the dart between $u$ and $v_x$ is

$$(e_1 + xe_2) \land e_3) (f_1 \land f_3) = w_2w_5 + xw_4w_5.$$
Now $[x,m]^2 = [0, x^2 w_5^2 + m A x + m]$, so $[x,m]$ has order two if and only if $m A x + m = x^2 w_5^2$. By elementary linear algebra we find that this is true for $x \neq 0$ if and only if $m \in w_3 + S$, where $S = \langle w_7^2, w_5^2 + w_2^2, w_5^2, w_6^2 \rangle_Y$. Note that $S$ is $A$-invariant.

Let $s(x) = w_7^2 + c(x)$. Then $s(x) \in S$. Since $[1, c(1)][\alpha, c(\alpha)] = [\alpha + 1, c(\alpha + 1)]$, we have $s(\alpha + 1) = w_7^2 + \alpha w_2^2 + c(1) A \alpha + c(\alpha) = w_7^2 + (\alpha + \alpha^2) w_5^2 + s(1) + s(\alpha) \not\in S$, a contradiction.

Since $A$ does not lift to a subgroup of $H$, neither does $\text{SL}_4(\mathbb{F})$. In other words, the extension of $\text{SL}_4(\mathbb{F})$ by $M$ is non-split.

\[ \square \]

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