Transformable calculation schemes in geometrically nonlinear problems of mechanics of sandwich plates with the contour reinforcing beams

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Abstract. The transformable calculation schemes for geometrically nonlinear problems of mechanics of sandwich plates with contour reinforcing beams are constructed. To solve the boundary value problem, it was used by the finite sum method. The solving the nonlinear problem arising in this case using a two-layer iterative process with the lowering of nonlinearity on the lower layer was carried out. A software package implemented in the Matlab environment has been developed. Numerical experiments were carried out and their results were analyzed.

1. Introduction
Sandwich structures are widely used in aerospace engineering, shipbuilding, transport engineering, as well as in construction, when increased rigidity and minimum weight are required [1-9]. Sandwich structural elements on the contour almost always have reinforcing beams that provide load transfer to the carrier layers when interacting with other structural elements (Figure 1). In [1], an analysis of the variants of their design; for small deformations and medium displacements, a refined geometrically nonlinear theory was constructed, which makes it possible to describe the process of their subcritical deformation and to reveal all possible forms of buckling of the supporting layers and supporting elements. It is based on introducing into consideration, as unknown contact forces, the interaction of the outer layers with a core in the form of transverse tangential stresses, constant in thickness, and the outer layers and the core with reinforcing beams at all points of their interface surfaces, taking into account the boundary conditions.
We assume that within the framework of such a model, when the tangential stress $\tau_1$ reaches the destructive stress value $\tau^*$ (at $\tau_1 = \tau^*$), separation occurs between the core and the reinforcing rod, which, in turn, leads to transformation of the originally formulated boundary conditions in the core to the conditions of the form $\tau_1 = 0$, i.e., to the necessity of consistently statement of the two problems. To solve the problem, the finite sum method was used. The solving of the nonlinear problem arising in this case was carried out using a two-layer iterative process with the lowering of nonlinearity on the lower layer. A software package implemented in the Matlab environment has been developed. Numerical experiments were carried out, and their results were analyzed.

2. Statement of the problem.

Applied to a sandwich plate experiencing a cylindrical bend under the action of a longitudinal force $T_{\tau}^- = T_{\tau}^+$ (see Figure 1), the equilibrium equations of the outer layers derived in [10] and their kinematic conditions of conjugation with the core will have the form (here and below, $k = 1, 2$ is the layer number, $\delta_{(1)} = 1$, $\delta_{(2)} = -1$ and subscript after the comma indicates the derivative with respect to the longitudinal coordinate)

$$
N_{(1)}^{(k)} + \delta_{(k)}(E_{(3)}w_{(2)}^{(2)} - w_{(1)}^{(1)})/2h + X_{(3)}^{(k)} = 0, \quad T_{(1)}^{(k)} + \delta_{(k)}q_1 + X_1^{(k)} = 0, \quad u^{(k)} = u_{(2)}^{(2)} - H_{(1)}w_{(1)}^{(2)} - H_{(2)}w_{(1)}^{(2)} + 2h\eta l_{/G_{(13)}} - 2h^2q_{(1,1)}^{(1)}/(3E_{(3)}) = 0,
$$

(1)

Here, tangential forces $T_{(1)}^{(k)}$ and generalized shear forces $N_{(1)}^{(k)}$ through unknown axial displacements $u_{(1)}^{(k)}$, deflections $w_{(k)}^{(2)}$ of the points of the middle surfaces of the carrier layers and tangential stresses $q_1$ in the core are expressed by the formulas

$$
N_{(1)}^{(k)} = Q_{(1)}^{(k)} + H_{(k)}q_1, Q_{(1)}^{(k)} = M_{(1)}^{(k)} + T_{(1)}^{(k)}\omega^{(k)},
$$

(2)

$$
M_{(1)}^{(k)} = -D_{(1)}^{(k)}w_{(1)}^{(2)} + T_{(1)}^{(k)}\omega^{(k)} = B_{(k)}(u_{(1)}^{(k)} + (\omega^{(k)})^2/2)^2,
$$

where $\omega^{(k)} = w_{(1)}^{(2)}$ are rotation angles, $B_{(k)} = 2h_{(k)}(E^{(k)})/(1 - v_{12}^{(k)}v_{21}^{(k)})$ and $D_{(k)} = B_{(k)}h_{(k)}^2/3$ are the bending stiffness and tension-compressive stiffness of the $k$-th layer, having a thickness $2h_{(k)}$ and made of a material with elastic modulus $E^{(k)}$ and Poisson’s coefficients $v_{12}^{(k)}, v_{21}^{(k)}$; $H_{(k)} = h_{(k)} + h$, $2h$ is the thickness of the core having the elastic modulus $E_{(3)}$ in the transverse direction and the transverse shear modulus $G_{(13)}$; $M_{(1)}^{(k)}$ is the internal bending moment in the $k$-th layer; $Q_{(1)}^{(k)}$ are shear forces in the $k$-th carrier layer without taking into account the tangential stresses in the core.

Introducing unknown contact forces $(Q_{(1)}^{(k)}, Q_{(13)}^{(k)})$, moments $(L_{(1)}^{(k)})$ and contact tangential stresses $\tau_1$ of the interaction of the outer layers and the core with reinforcing beams, we find that solutions of equations (1), (2) must satisfy the boundary conditions (for $x_1 = x_1^+$)

$$
M_{(1)}^{(k)} = L_{(1)}^{(k)}, \quad T_{(1)}^{(k)} = Q_{(1)}^{(k)}, \quad q_1 = \tau_1, \quad M_{(1)}^{(k)} + T_{(1)}^{(k)}\omega^{(k)} + H_{(k)}q_1 = Q_{(13)}^{(k)} + h\tau_1,
$$

(3)

We suppose that when a plate is loaded, a stress-strain state that is symmetric with respect to the cross section $x_1 = 0$ is formed in it. Then at $x_1 = 0$ the following boundary conditions are true

$$
\omega^{(k)} = 0, \quad N_1^{(k)} = 0, \quad u_1^{(k)} = 0, \quad q_1 = 0.
$$

(4)

When cylindrical bending of a sandwich plate beams, reinforcing it in the end sections $x_1 = x_1^-$, $x_1 = x_1^+$, are transformed into absolutely solid bodies. Therefore, the equilibrium equations of such a reinforcing rod in the cross section $x_1 = x_1^+$ will have the following form

$$
\begin{align*}
    f_{\tau}^+ &= \left[-(Q_{(1)}^{(1)} + Q_{(13)}^{(2)}) + T_{\tau}^+\right] = 0, \quad f_{\tau}^- = \left[-(Q_{(1)}^{(1)} + Q_{(13)}^{(2)}) + 2h\tau_1\right] + T_{\tau}^- = 0, \\
    f_{\varphi}^+ &= \left[-(L_{(1)}^{(1)} + L_{(1)}^{(2)}) - Q_{(1)}^{(1)}H_{(1)} + Q_{(1)}^{(2)}H_{(2)} - B_{(2)}^+(Q_{(1)}^{(1)} + Q_{(13)}^{(1)}) + 2h\tau_1\right] + m_{\varphi}^+ = 0,
\end{align*}
$$

(5)
where $m_s^+ = T_s^+ z_i - T_s^+ \zeta_i$ is the linear bending moment specified in the end section. The displacements $U$, $W$ and rotation $\phi$ formed in the reinforcing beams must satisfy the following kinematic conditions for the conjugation of the beams with the carrier layers

$$u_1^{(k)} - (U - \delta_{1k})H(k)\phi = 0 \text{ for } \delta_Q^{(1)} \neq 0, \quad w_1^{(k)} - (W + B_2^+ \phi) = 0 \text{ for } \delta_Q^{(1)} \neq 0,$$

$$w_1^{(k)} + \phi = 0 \text{ for } \delta_L^{(k)} \neq 0, \quad w_1^{(1)} + w_2^{(2)} + 2h^2 q_{11} - 2(W + B_2^+ \phi) = 0 \text{ for } \delta_1 \neq 0. \quad (6)$$

We note that in (6) the unknown $U$, $W$ are the displacements of the points of the axial line $O_x^+$ (Figure 1) of the reinforcing beam, and $\phi$ is the angle of rotation of the cross section.

3. Numerical experiments

Numerical experiments were carried out for the case of hinge fixing of the carrier layers with a reinforcing beam when $L_1^{(k)} = 0$. For the formulated boundary value problem in the form of five differential equations (1) with the boundary conditions (3) and ten algebraic equations (5), (6), it is required to determine the vector function of the unknowns

$$X = \{w_1^{(1)}, w_2^{(2)}, u_1^{(1)}, u_2^{(2)}, q_{11}, w_1^{(1)} + w_2^{(2)} + 2h^2 q_{11} - 2(W + B_2^+ \phi), \tau_1, U, W, \phi\}.$$

To do this, in accordance with the finite sum method, we reduce the original problem to a system of Volterra type integro-algebraic equations of the second kind with additional relations for determining the unknown integration constants. As a result, relative to the vector function

$$Y(x) = \{w_1^{(1)}, w_2^{(2)}, u_1^{(1)}, u_2^{(2)}, q_{11}, w_1^{(1)} + w_2^{(2)} + 2h^2 q_{11} - 2(W + B_2^+ \phi), \tau_1, U, \phi\},$$

we arrive at the resolving system of sixteen integro-algebraic equations ($w_2^{(k)} = w_1^{(k)}$).

For approximation of integral equations in [11–13], a method of collocations by Gaussian nodes and a method for constructing the integrating matrices, which are analogous to integral operators generating these equations, are proposed. As a result, a geometrically non-linear problem can be represented in a finite-operator form

$$A_1(Y) + A_2(Y) = F(T_s^+), \quad (7)$$

where $A_1$ is linear, $A_2$ is nonlinear operators.

To find the solution to problem (7), we will use a two-layer iterative process with lowering nonlinearity to the lower layer [14-24] with a preconditioner, which is the linear part $A_1$ of the operator of the scheme (7):

$$A_1\left(\frac{Y^{(n+1)} - Y^{(n)}}{\rho}\right) + (A_1 + A_2)(Y^{(n)}) = F(T_s^+), \quad (8)$$

where $\rho > 0$ is an iterative parameter.

In this work, the central place is occupied by the modeling of the stratification in the cross section of the connection of the filler with the reinforcing rod. To this end, when carrying out calculations by tabulating by end load $T_s^+$, the value of the contact tangential stress $\tau_1$ of interaction was tracked. When equality $\tau_1 = \tau^*$ is achieved, stratification of the core and reinforcing beam occurs.

An algorithm based on the incremental Lagrange method, also called the parameter continuation method [25–30], has been developed for the numerical investigation of the process. With such a value of the load $T_s^*$ at which the tangential stresses reach destructive values, i.e., $\tau_1 = \tau^*$, the new geometrically nonlinear problems are introduced into consideration. As a new loading parameter, tangential stress $\tau_1 = -\tau^*$ is chosen to zero out the stresses formed at the junction points of the plate with the rod. These tasks are already formulated on the basis of (1), (3)–(6), with respect to the
increment functions $\Delta Y$ of the unknowns, and the last equation of the system (6) is replaced by the following $\tau_1 = -\tau^*$. For the numerical implementation of the iterative method (8) of solving problem (7) on the axial compression of a plate by a force $T^*_x$ applied to a reinforcing rod with eccentricity, a software package has been developed, implemented in Matlab. On its basis, numerical calculations were carried out with the following values of the geometric and elastic parameters of the plate and the reinforcing rod (Figure 1): $x_1 = a = 10$ cm, $2h_{(1)} = 2h_{(2)} = 0.1$ cm, $h = 1$ cm, $G_{13} = 25$ MPa, $E_3 = 50$ MPa, $E_1^{(k)} = 133 \cdot 10^3$ MPa, $V_{12}^{(k)} = V_{21}^{(k)} = 0.3$, $k = 1,2$, $z_A = \xi_A = B_2^{(+)} = h + h_{(1)}$. The number of points of a mesh entered in the finite-dimensional approximation of the problem is assumed to be $N = 256$, this ensures complete convergence of the used numerical method for all parameters of the stress-strain state of the structure. Calculations according to (8) were carried out as long as the norm of the residual $\| F(T^*_x) - (A_1 + A_2)(Y^{(n)}) \|$ and the difference between the iterations $\| Y^{(n+1)} - Y^{(n)} \|$ remained greater than the specified accuracy $\varepsilon = 5 \cdot 10^{-8}$. When finding a solution to the problem of the stress-strain state before the moment of stratification, calculations were performed by tabulating on the face load $T^*_x$ until the tangential stress $\tau_1$ reached the value of $\tau^* = 0.288$ MPa.

Figure 2 shows the change in tangential stresses in the core along the plate. It can be seen that, due to the effect of a compressive force with an eccentricity on the beam, they are not equal to zero along the entire length of the plate (at the points of connection of the core with the reinforcing beam, they must take a zero value after separation). It can be seen that they reach the maximum value in the neighborhood of conjugation the core with the reinforcing beam.

Figure 3 shows the graphs of the functions of increments $\Delta q_1$, and Figure 4 shows the graphs of functions $\tilde{q}_1 = q_1 + \Delta q_1$ for a given value of the increment of the tangential stress $\Delta \tau_1 = -\tau^* = 0.288$ MPa. Comparing the obtained results, it can be seen that, after separation, due to the equality $\tilde{\tau}_1 = 0$, there occurs a significant redistribution of transverse tangential stresses with their concentration in the outer layers in the neighborhood of the cross section $x = x_1$. The tangential stresses in the core, formed before the separation, are redistributed in the interface zone with the reinforcing rod, reaching the maximum value of $\tilde{q}_1 \approx 0.4$ MPa (Figure 4). Note that the membrane normal stresses in the carrier layers after delamination are subject to slight changes.

![Figure 2](image2.png)  
Figure 2. Transverse tangential stresses in a placeholder ($q_1$, MPa)

![Figure 3](image3.png)  
Figure 3. Increment of transverse tangential stresses in a core ($\Delta q_1$, MPa)
Figure 4. Transverse tangential stresses in a core after separation ($\bar{q}_1$, MPa)

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