The geometry of polynomial identities

C. Procesi

Abstract. In this paper we stress the role of invariant theory and in particular the role of varieties of semisimple representations in the theory of polynomial identities of an associative algebra. In particular, using this tool, we show that two PI-equivalent finite-dimensional fundamental algebras (see Definition 2.19) have the same semisimple part. Moreover, we carry out some explicit computations of codimensions and cocharacters, extending work of Berele [8] and Kanel-Belov [6], [7].

Keywords: polynomial identities, fundamental algebras, invariant theory.

Dedicated to J.-P. Serre with admiration

§1. Introduction

In the last 50 years, or more, several papers have dealt with connections between non-commutative algebra and algebraic geometry.

Sometimes the term non-commutative algebraic geometry has been used, which I find misleading since there are several quite disjoint instances of those connections. Loosely, the only unifying ideas are some information on various sorts of spectral theory of operators.

In this paper we want to examine a particular theory, that of algebras with polynomial identities, PI-algebras for short, and show how a sequence of varieties of semisimple representations (see §3.3.1) appears to be naturally associated with a PI-theory (see Theorems 3.1 and 3.14).

These varieties are among a series of discrete, geometric and combinatorial invariants of PI-theories. Together with these one should also consider certain moduli spaces which parametrize certain coherent sheaves on these representation varieties. At the moment these objects seem to be hard to treat although some hints will be given in the last part of the paper.

In this paper algebra will always mean associative algebra over some field $F$. We will also assume that $F$ is of characteristic 0, otherwise too many difficult and sometimes unsolved problems arise. When convenient we shall also assume that $F$ is algebraically closed.

AMS 2010 Mathematics Subject Classification. 15A24, 16R10, 16R30.
Let us recall the main definitions. For a comprehensive discussion of the basic material we refer the reader to Rowen’s book [32] or Drensky [14] and Drensky-Formanek [15].

We start from the free associative algebra \( F\langle X \rangle \), in some finite or countable set of variables \( x_i \), with basis the \textit{words} in these variables, and whose elements we think of as \textit{non-commutative polynomials}.

Given any algebra \( A \) we have, for every map \( X \to A, x_i \mapsto a_i \), an associated \textit{evaluation} of polynomials \( f(x_1, \ldots, x_m) \mapsto f(a_1, \ldots, a_m) \in A \).

\textbf{Definition 1.1.} (i) A polynomial \( f(x_1, \ldots, x_m) \in F\langle X \rangle \) is a \textit{polynomial identity} of \( A \), or a PI, if \( f(a_1, \ldots, a_m) = 0 \) for all evaluations in \( A \).

(ii) A PI-algebra is an algebra \( A \) which satisfies a non-zero polynomial identity.

(iii) Two algebras \( A, B \) are PI-equivalent if they satisfy the same polynomial identities.

The set \( \text{Id}(A) \) of PI of \( A \) in the variables \( X \) is an ideal of \( F\langle X \rangle \) closed under all possible endomorphisms of \( F\langle X \rangle \), that is, substitutions of the variables \( x_i \) with polynomials.

Such an ideal \( I \) is called a \textit{T-ideal}. The quotient \( F\langle X \rangle/I \) satisfies all the identities in \( I \).

In fact, \( F\langle X \rangle/\text{Id}(A) \) is a free algebra in the category of all algebras which satisfy the PI of \( A \). When \( X \) is countable and \( I \) is a T-ideal, then \( I \) is the ideal of all polynomial identities of \( F\langle X \rangle/I \). One usually says that \( F\langle X \rangle/I \) is a relatively free algebra.

\textit{Remark 1.2.} There is a small annoying technical point: it is usually necessary to work with algebras without 1 (in the sense that either there is truly not a 1 or we do not consider it as part of the axioms). In this case the free algebra has to be taken without 1. This implies a different notion of T-ideals since in the case of algebras with 1, a T-ideal is also stable under specializing a variable to 1, while this is not allowed in the other case. We leave it to the reader to understand in which setting we are working.

As usual, \textit{polynomials} have a dual aspect of symbolic expressions or functions. The same happens in PI theory where, if \( I \) is the T-ideal of identities of an algebra \( A \), then \( F\langle X \rangle/I \) can be identified with an algebra of polynomial maps \( A^X \to A \) which commute with the action of the automorphism group of \( A \).

Its algebraic combinatorial nature is quite complex and usually impossible to describe in detail even for very special algebras \( A \). In general the algebra of all polynomial maps commuting with the action of the automorphism group of \( A \) is strictly larger. A major example is when \( A = M_n(F) \), the algebra of \( n \times n \) matrices over \( F \), where one quickly finds a connection with classical invariant theory. Even this basic example cannot be described in full except in the trivial case \( n = 1 \) and the non-trivial case \( n = 2 \) (see [23]).

A major difficulty in the theory is the fact that relatively free algebras \( F\langle X \rangle/I \), even when \( X \) is finite, almost never satisfy the Noetherian condition (see Amitsur [4]) that the ideals (right, left or two-sided) have a finite set of generators. This is not just a technical point, but reflects some deep combinatorics appearing.
This is in part overcome by the solution of the Specht problem by Kemer [19] (Theorem 2.6 below), which states that $T$-ideals are finitely generated in the appropriate sense.

§ 2 is devoted to a quick overview of the required Kemer theory. In particular we shall stress the role of fundamental algebras; see Definition 2.19 and Theorem 2.24, which are the building blocks of the PI-equivalence classes of finite-dimensional algebras.

In § 3 we start to obtain some consequences of Kemer theory. The first goal of this paper is to show, Theorem 3.1, that, when $X$ is finite, for a given relatively free algebra $F(x)/I$ there is a canonical finite filtration by $T$-ideals $K_i$ such that the quotients $K_i/K_{i-1}$ have natural structures of finitely generated modules over special finitely generated commutative algebras $T_i$. In fact each piece $K_i/K_{i-1}$ is a two-sided ideal in some special trace algebra which is finitely generated as a module over $T_i$. The algebras $T_i$ are coordinate rings of certain representation varieties (§ 3.3.1) and the word geometry, appearing in the title, refers to the geometric description of the algebraic varieties supporting the various modules $K_i/K_{i-1}$; see Theorem 3.14.

At the next step we show that these varieties are natural quotient varieties parametrizing semisimple representations. As an important consequence, in Corollary 3.15 we show that if two fundamental algebras are PI-equivalent, then they have the same semisimple part.

Then, from some explicit information on these varieties and using some general facts of invariant theory, we shall deduce a number of corollaries. We compute the dimensions of the relatively free algebras, Proposition 3.20, and, in Corollary 3.31, the growth of the cocharacter sequence; see Definition 2.3.

In the final § 4 we shall indicate a method to classify finite-dimensional algebras up to PI-equivalence.

§ 2. Growth and the theory of Kemer

In this section we recall basic results of PI theory which motivate our research and are needed in the rest of this paper.

Growth. Given $n$, we let $V_n$ denote the space of multilinear polynomials of degree $n$ in the variables $x_1, \ldots, x_n$. The space $V_n$ has as a basis the monomials $x_{\sigma(1)} \cdots x_{\sigma(n)}$ as $\sigma$ runs over all permutations of $1, 2, \ldots, n$, so $\dim V_n = n!$.

Identities can always be multilinearized, whence the subset $\text{Id}(A) \cap V_n$ plays a special role and, in characteristic zero, the ideal $\text{Id}(A)$ is completely determined by the sequence of multilinear identities \{Id(A) ∩ $V_n$\}$_{n\geq 1}$. In order to study $\dim(\text{Id}(A) \cap V_n)$, we introduce the quotient space $V_n/(\text{Id}(A) \cap V_n)$ and its dimension

$$c_n(A) := \dim(V_n/(\text{Id}(A) \cap V_n)).$$

The integer $c_n(A)$ is the $n$th codimension of $A$. Clearly, $c_n(A)$ determines $\dim(\text{Id}(A) \cap V_n)$ since $\dim V_n$ is known.
The study of growth for a PI-algebra $A$ is mainly the study of the rate of growth of the sequence $c_n(A)$ of its codimensions as $n$ goes to infinity. For a full survey we refer to [29]. We have the following basic property proved by Regev.

**Theorem 2.1** [28]. The sequence $c_n(A)$ is always exponentially bounded.

We then have the integrality theorem of Giambruno–Zaicev (conjectured by Amitsur).

**Theorem 2.2** [17]. Let $A$ be a PI-algebra over a field $F$ with $\text{char}(F) = 0$. Then the limit

$$\lim_{n \to \infty} c_n(A)^{1/n} \in \mathbb{N}$$

exists and is an integer, called the exponent.

The space $V_n/(\text{Id}(A) \cap V_n)$ is a representation of the symmetric group $S_n$ acting by permutations of the variables.

**Definition 2.3.** The $S_n$-character $\chi_{S_n}(V_n/(\text{Id}(A) \cap V_n))$ of this space is denoted by $\chi_n(A)$:

$$\chi_n(A) = \chi_{S_n}(V_n/(\text{Id}(A) \cap V_n))$$

and is called the $n$th cocharacter of $A$.

Since $c_n(A) = \deg \chi_n(A)$, cocharacters are a refinement of codimensions and an important tool in their study. By theorems of Amitsur–Regev and Kemer, $\chi_n(A)$ is supported on some $(k, \ell)$-hook. Shirshov’s height theorem [33] then implies that the multiplicities of the irreducible characters in the cocharacter sequence are polynomially bounded.

One of the goals of this paper is to discuss (see §§3.3.4 and 3.4) some further information one can gather on these numbers using geometric methods.

**2.1. Three fundamental theorems.** We need to review the results and some of the techniques of Kemer theory presented in the monograph [19]; see also [3] or the forthcoming book [2]. A fundamental theorem of Kemer states the following.

**Theorem 2.4** (Kemer). If $X$ is a finite set, a non-zero $T$-ideal $I$ of $F\langle X \rangle$ is the ideal of polynomial identities in the variables $X$ for a finite-dimensional algebra $A$.

In other words, any finitely generated PI-algebra is PI-equivalent to a finite-dimensional algebra.

This is in fact the first part of a more general statement. Let us consider the Grassmann algebra, thought of as a super-algebra, in countably many odd generators $G := \bigwedge [e_1, e_2, \ldots]$ decomposed as $G = G_0 \oplus G_1$ into its even and odd part.

**Theorem 2.5** (Kemer). Every PI-algebra $R$ is PI-equivalent to the Grassmann envelope $G_0 \otimes A_0 \oplus G_1 \otimes A_1$ of a finite-dimensional super-algebra $A = A_0 \oplus A_1$.

The algebra $A$ is of course not unique. Nevertheless some normalizations in the choice of $A$ can be made, and the purpose of this paper is to show that there is a deep geometric structure of the algebra $F\langle X \rangle/I$ which reflects the structure of these normalized algebras.
A major motivation of Kemer was to solve the Specht problem, that is, to prove the following theorem.

**Theorem 2.6** (Kemer). *All T-ideals are finitely generated as T-ideals.*

This implies that, when working with T-ideals, we can use the standard method of Noetherian induction, that is, every non-empty set of T-ideals contains maximal elements.

2.1.1. **Alternating polynomials.** Kemer theory is based on the existence of some special alternating polynomials which are not identities, that is, they do not belong to a given T-ideal \( \Gamma \). So let us start by reviewing this basic formalism.

Let us fix some positive integers \( \mu, t, s \). We want to construct multilinear polynomials in some variables \( X \) and possibly other variables \( Y \). We want to have \( N = \mu t + s(t + 1) \) variables \( X \) decomposed into \( \mu \) disjoint subsets called *small layers* \( X_1, \ldots, X_\mu \) with \( t \) elements each, and \( s \) *big layers* \( Z_1, \ldots, Z_s \) with \( t + 1 \) elements each.

We consider polynomials alternating in each layer. Such a space of polynomials is obtained, by operations of substitution of variables, from a finite-dimensional space \( M_{\mu, t, s}(X, W) \) constructed as follows.

Take \( N + 1 \) variables \( w_1, w_2, \ldots, w_{N+1} \) and consider the space spanned by the \( N! \) monomials

\[
w_1 x_{\sigma(1)} w_2 x_{\sigma(2)} \cdots w_N x_{\sigma(N)} w_{N+1}, \quad \sigma \in S_N.
\]

In this space the subgroup \( G := S_\mu^t \times S_s^t \) acts by permuting the monomials and thus we have a subspace \( M_{\mu, t, s}(X, W) \) of dimension \( N!/(t!^\mu (t+1)!^s) \) with basis all possible polynomials alternating in these layers.

Since in general we work with algebras without 1, we also need to add all polynomials obtained from these by specializing some of the variables \( w_i \) to 1.

We thus obtain a space \( M_X = M_{\mu, t, s}(X, W) \) of polynomials in \( X \) and some of the \( W \), alternating in the layers of \( X \), so that, if we take any polynomial \( f(X, Y) \) which is multilinear and alternating in the layers of \( X \) and depends on other variables \( Y \), this polynomial is obtained as a linear combination of elements of \( M_X \) after substitution of the variables \( w_i \) by polynomials in the variables \( Y \).

In particular, this shows how from a space \( M_{\mu, t, s}(X, W) \) one may deduce, by variable substitution, larger spaces \( M_{\mu', t', s'}(X, W) \), \( \mu' \geq \mu, \ t' \geq t, \ s' \geq s \).

Of particular importance are the *Capelli polynomials*, introduced by Razmyslov [26]:

\[
C_m(x_1, x_2, \ldots, x_m; w_1, w_2, \ldots, w_{m+1}) := \sum_{\sigma \in S_m} \varepsilon_\sigma w_1 x_{\sigma(1)} w_2 x_{\sigma(2)} \cdots w_m x_{\sigma(m)} w_{m+1}, \tag{1}
\]

where \( \varepsilon_\sigma \) is the sign of a permutation \( \sigma \). In fact this polynomial plays a role analogous to that of the classical Capelli identity, which is instead an identity of differential operators.

Since one often needs to analyze algebras without 1, it is useful to introduce the *Capelli list* \( C_m \) of all polynomials deduced from \( C_m(x_1, x_2, \ldots, x_m; w_1, w_2, \ldots, w_{m+1}) \) by specializing one or more variables \( w_i \) to 1.

Here is another fact of the theory, a consequence of Theorem 2.4.
Proposition 2.7. A PI-algebra is PI-equivalent to a finite-dimensional algebra if and only if it satisfies some Capelli identity.

In this paper we want to concentrate on finite-dimensional algebras, so we shall assume from now on that some Capelli identity is satisfied. In this way we do not need to introduce super-algebras nor to apply Theorem 2.5. Nevertheless most results can be extended to super-algebras in a more or less straightforward way, as will be presented in the forthcoming book [2].

2.1.2. Use of Schur–Weyl duality. Recall that, given a vector space $V$ with dim $V = k$, each tensor power $V^\otimes d$ is acted on by the general linear group $\text{GL}(V)$ and the symmetric group $S_d$ which span two algebras, each the centraliser of the other. In characteristic 0 both algebras are semisimple and thus we have the Schur–Weyl duality decomposition into isotypic components, indexed by partitions $\lambda \vdash d$ of height $\text{ht}(\lambda) \leq \text{dim} V = k$:

$$V^\otimes d = \bigoplus_{\lambda \vdash d, \text{ht}(\lambda) \leq k} S_\lambda(V) \otimes M_\lambda. \quad (2)$$

The $M_\lambda$ are the irreducible representations of the symmetric group $S_d$, constructed from the theory of Young symmetrizers, with characters $\chi_\lambda$. The modules $S_\lambda(V)$ are irreducible representations of $\text{GL}(V)$ which in fact can be thought of as polynomial functors on vector spaces and called the Schur functors (see [25]).

We consider the free algebra $F\langle X \rangle$ as the tensor algebra $T(V)$ over an infinite-dimensional vector space $V$, with basis the variables $X := \{x_1, x_2, \ldots \}$, and take a $T$-ideal $I$.

Remark 2.8. If we want to stress the basis $X$ of $V$, we also write $S_\lambda(V) = S_\lambda(X)$.

Since a $T$-ideal is stable under variable substitutions, it is in particular stable under the action of the linear group of $V$, which we define to be $\text{GL}(V) := \bigcup_m \text{GL}(m, F)$. We can decompose $T(V)/I$ into irreducible representations of this group, which are deduced from formula (2). If we assume that $I$ contains a Capelli identity $C_{m+1}$ (or a Capelli list), we have a restriction, deduced from these Capelli identities, on the height of the partitions appearing in $T(V)/I$:

$$T(V)/I := \bigoplus_d \bigoplus_{\lambda \vdash d, \text{ht}(\lambda) \leq m} n_\lambda S_\lambda(V). \quad (3)$$

Highest weights. One can then apply the theory of highest-weight vectors (which belongs to the theory of Lie groups and algebraic groups). In our case the notion of weight is just the multidegree in the variables $x_i$. An element whose weight is the sequence $k_1, k_2, \ldots$ is a non-commutative polynomial homogeneous of degree $k_i$ in each $x_i$.

Weights are usually equipped with the dominance order (which in Lie theory arises from the theory of roots). In our case, a sequence $k_1, k_2, \ldots$ is greater than $h_1, h_2, \ldots$ in the dominance order if $k_1 \geq h_1$, $k_1 + k_2 \geq h_1 + h_2$, ..., $k_1 + \cdots + k_i \geq h_1 + \cdots + h_i$, ...
Consider in $GL(V)$ the unipotent group $U$ of (strictly upper triangular) linear transformations of the type $x_i \mapsto x_i + \sum_{j<i} a_{i,j} x_i$.

One knows that the subspace $S_\lambda(V)^U$ of $U$-invariants in the space $S_\lambda(V)$ is 1-dimensional and generated by an element $u$ of weight $\lambda$. That is, $u$ is homogeneous of degree $h_i$ in each $x_i$, where $h_i$ is the length of the $i$th row of $\lambda$. In fact $u$ is a highest-weight vector using the dominance order of weights. Thus if the height of $\lambda$ does not exceed $m$, then this element depends only upon the first $m$ variables.

On the other hand, since $S_\lambda(V)$ is an irreducible representation of $GL(V)$, it is generated by this highest-weight vector. One deduces the following assertion from Theorem 2.4.

**Theorem 2.9.** A $T$-ideal $\Gamma \subset F\langle X \rangle$ in countably many variables $X$ is the ideal of identities of a finite-dimensional algebra if and only if it contains a Capelli list.

**Remark 2.10.** This allows us, in order to study the multiplicities $n_\lambda$, to reduce the number of variables (and hence the dimension of $V$) to any chosen finite number with the constraint of being $\geq m$.

2.1.3. *The Kemer index.* A main tool in the theory of Kemer is given by introducing a pair of non-negative integers $\beta(\Gamma)$, $\gamma(\Gamma)$, called the *Kemer index* of $\Gamma$, for every $T$-ideal $\Gamma$ which contains some Capelli identities. These numbers give a first measure of which of the spaces $M_{\mu,t,s}(X,W)$ are not entirely contained in $\Gamma$ (or do not consist of polynomial identities of some given algebra).

**Definition 2.11.** For every $T$-ideal $\Gamma$ which contains some Capelli identities, we let $\beta(\Gamma)$ be the greatest integer $t$ such that for every $\mu \in \mathbb{N}$ there exists a $\mu$-fold $t$-alternating (in the $\mu$ layers $X_i$ with $t$ elements) polynomial not in $\Gamma$:

$$f(X_1, \ldots, X_\mu, Y) \notin \Gamma.$$

We then let $\gamma(\Gamma)$ be the maximum $s \in \mathbb{N}$ for which there exists, for all $\mu$, a polynomial $f(X_1, \ldots, X_\mu, Z_1, \ldots, Z_s, Y) \notin \Gamma$, alternating in $\mu$ small layers $X_i$ with $\beta(\Gamma)$ elements, and $s$ big layers $Z_j$ with $\beta(\Gamma) + 1$ elements.

The pair $(\beta(\Gamma), \gamma(\Gamma))$ is the *Kemer index* of $\Gamma$, denoted by $\text{Ind} \Gamma$.

For an algebra $A$ we define the Kemer index of $A$, denote by $\text{Ind}(A)$, to be the Kemer index of the ideal of polynomial identities of $A$.

We order the Kemer indices lexicographically and then observe the following.

**Remark 2.12.** If $\Gamma_1 \subset \Gamma_2$ are two $T$-ideals, we have

$$\text{Ind} \Gamma_1 \geq \text{Ind} \Gamma_2.$$

If $\Gamma = \bigcap_{i=1}^k \Gamma_i$ is an intersection of $T$-ideals, then

$$\text{Ind} \Gamma = \max \text{Ind} \Gamma_i.$$

If $A = \bigoplus_{i=1}^k A_i$ is a direct sum of algebras, then

$$\text{Ind}(A) = \max \text{Ind}(A_i).$$
Remark 2.13. By definition, writing $s := \gamma(\Gamma)$, there is a minimum $\mu_0 = \mu_0(\Gamma)$ such that there is no polynomial $f(X_1, \ldots, X_\mu, Z_1, \ldots, Z_{s+1}, Y) \notin \Gamma$ alternating in $\mu \geq \mu_0$ layers $X_i$ with $\beta(\Gamma)$ elements and in $s+1$ layers $Z_j$ with $\beta(\Gamma)+1$ elements.

Definition 2.14. A polynomial $f(X_1, \ldots, X_\mu, Z_1, \ldots, Z_{s}, Y) \notin \Gamma$ alternating in $\mu$ layers $X_i$ with $\beta(\Gamma)$ elements and in $s = \gamma(\Gamma)$ layers $Z_j$ with $\beta(\Gamma)+1$ elements with $\mu > \mu_0 + 1$, will be called a $\mu$-Kemer polynomial.

A Kemer polynomial is by definition a $\mu$-Kemer polynomial for some $\mu > \mu_0 + 1$.

Remark 2.15. A Kemer polynomial $f(X_1, \ldots, X_\mu, Z_1, \ldots, Z_{s}, Y)$ which is also linear in a variable $w$, not in the layers $Z_j$, has the following property. Fix one of the layers $X_i$ which does not contain the variable $w$, add to it the variable $w$ and alternate these $t+1$ variables. If $w$ is also in no layer $X_j$, we produce a polynomial alternating in $\mu - 1 \geq \mu_0$ layers with $t$ elements and in $s+1$ layers with $t+1$ elements. Otherwise, if $w \in X_j$, $i \neq j$, we produce a polynomial alternating in $\mu - 2 \geq \mu_0$ layers with $t$ elements and in $s+1$ layers with $t+1$ elements. By the definition of $\mu_0$ and of a Kemer polynomial, this is always an element of $\Gamma$. Hence if $\Gamma = \text{Id}(A)$, it is a polynomial identity of $A$.

Example 2.16. 1) If $A = M_n(F)$, then every $(n^2 + 1)$-alternating polynomial is in $\text{Id}(A)$. Conversely, using the Capelli polynomials (1), the product (taken in an ordered way)

$$C_{\mu,n^2}(X_1, \ldots, X_\mu, Y) := \prod_{i=1}^{\mu} C_{n^2}(x_{i,1}, \ldots, x_{i,n^2}; y_{i,1}, \ldots, y_{i,n^2}, y_{i,n^2+1}),$$

(X_i := \{x_{i,1}, \ldots, x_{i,n^2}\})

(4)

evaluated in suitable $e_{ij}$ can take any value $e_{hk}$ for every $\mu$ and, in particular, it is not an identity. Hence $\beta(M_n(F)) = n^2$ and $\gamma(M_n(F)) = 0$.

2) On the contrary, if $A$ is a finite-dimensional nilpotent algebra with $A^s \neq 0$ and $A^{s+1} = 0$, we see that its Kemer index is $(0, s)$.

In a subtle and not completely understood way, all other cases are a mixture of these two special cases.

Remark 2.17. We could take a slightly different point of view and define $\mu$-Kemer polynomials for a $T$-ideal $\Gamma$ with Kemer index $(t, s)$ to be the elements of $M_{\mu,t,s}(X, W)$ which are not in $\Gamma$, and then deduce some larger $T$-ideal by evaluations of these polynomials.

2.2. Fundamental algebras. We need to recall, without proof, some steps in Kemer theory and fix some notation.

Let $A$ be a finite-dimensional algebra over a field $F$ of characteristic 0, $J := \text{rad} A$ its Jacobson radical and $\overline{A} := A/J$ a semisimple algebra.

By Wedderburn’s principal theorem (see [1]) we can write

$$A = \overline{A} \oplus J = R_1 \oplus \cdots \oplus R_q \oplus J, \quad R_i \text{ simple},$$

(5)

where, if we assume that $F$ is algebraically closed, for every $i$ the simple algebra $R_i$ is isomorphic to $M_{n_i}(F)$ for some $n_i$. By Example 2.16, 2), we can assume from now on that $\overline{A} \neq 0$. We call $\overline{A}$ the semisimple part of $A$. 
Definition 2.18. We set $t_A := \dim_F A/J = \dim_F \overline{A}$ and write $s_A + 1$ for the nilpotency index of $J$, that is, $J^{s_A} \neq 0$ and $J^{s_A+1} = 0$. We call the pair $t_A, s_A$ the $t, s$-index of $A$.

It is easily seen that the $t, s$-index of $A$ is greater than or equal to the Kemer index $\text{Ind}(A)$. So it is important to understand when these two indices coincide.

With the previous notation consider the quotient map $\pi: A \twoheadrightarrow A/J = \bigoplus_{i=1}^{q} R_i$.

For every $i$, $1 \leq i \leq q$, put

$$R^{(i)} := \bigoplus_{1 \leq j \leq q, j \neq i} R_j \quad \text{and} \quad A_i := \pi^{-1}R^{(i)} \subset A.$$  \hspace{1cm} (6)

We have for all $i$ that $A_i$ satisfies all polynomial identities of $A$ and $t_{A_i} < t_A$.

We need to construct a further algebra $A_0$ which satisfies all polynomial identities of $A$ and $t_{A_0} = t_A$ but $s_{A_0} < s_A$.

This algebra is constructed as the free product $\overline{A} \star F\langle X \rangle$ modulo the ideal generated by all polynomial identities of $A$, for a sufficiently large $X$ (in fact we shall see in Lemma 4.8 that $s$ variables suffice), and finally modulo the ideal of elements of degree $\geq s$ in the variables $X$. By definition, $t_{A_0} = t_A$, $s_{A_0} = s_A - 1$.

We have $\text{Id}(A) \subset \bigcap_{i=0}^{q} \text{Id}(A_i) \cap \text{Id}(A_0)$ by construction. If $\text{Id}(A) = \bigcap_{i=1}^{q} \text{Id}(A_i) \cap \text{Id}(A_0)$, then $A$ is PI-equivalent to $\bigoplus_{i=1}^{q} A_i \oplus A_0$ and all the $t, s$-indices of these algebras are strictly less than the $t, s$-index of $A$. This suggests the following definition.

Definition 2.19. We say that $A$ is fundamental if

$$\text{Id}(A) \varsubsetneq \bigcap_{i=0}^{q} \text{Id}(A_i).$$  \hspace{1cm} (7)

A polynomial $f \in \bigcap_{i=0}^{q} \text{Id}(A_i) \setminus \text{Id}(A)$ will be called fundamental.

Example 2.20. An algebra of block-triangular matrices is fundamental.

By induction one has the following assertion.

Proposition 2.21. Every finite-dimensional algebra $A$ is PI-equivalent to a finite direct sum of fundamental algebras.

Let us now study the properties of a fundamental multilinear polynomial $f$. By definition we have an evaluation in $A$ which is different from 0.

Definition 2.22. We may assume that each variable has been substituted by a semisimple (resp. radical) element. We call such an evaluation restricted.

By an abuse of terminology, let us call the corresponding variables semisimple (resp. radical).
Lemma 2.23. Take a non-zero restricted evaluation, in $A$, of a multilinear fundamental polynomial $f$.

1) We then have that there is at least one semisimple variable evaluated in each $R_i$, for all $i = 1, \ldots, q$ (the property of being full).

2) We have exactly $s_A$ radical substitutions (the property $K$).

The next result is usually presented in the literature divided in two parts, called the first and second Kemer lemmas.

Theorem 2.24. A finite-dimensional algebra $A$ is fundamental if and only if its Kemer index equals the $t,s$-index.

In this case, given any fundamental polynomial $f$, we have $\mu$-Kemer polynomials in the $T$-ideal $\langle f \rangle$ generated by $f$, for every $\mu$.

Remark 2.25. Conversely, there is a $\mu$ such that all $\mu$-Kemer polynomials are fundamental.

We now introduce a definition.

Definition 2.26. A $T$-ideal $I$ is called primary if it is the ideal of identities of a fundamental algebra.

A $T$-ideal $I$ is irreducible if it is not the intersection $I = J_1 \cap J_2$ of two $T$-ideals $J_1, J_2$ properly containing $I$.

Proposition 2.27. Every irreducible $T$-ideal containing a Capelli list is primary, and every $T$-ideal containing a Capelli identity is a finite intersection of primary $T$-ideals.

Proof. By Theorem 2.6 we may apply Noetherian reduction on $T$-ideals. So every $T$-ideal is a finite intersection of irreducible $T$-ideals, for otherwise there is a maximal one which does not have this property and we quickly get a contradiction.

If a $T$-ideal $I$ contains a Capelli list, it is the $T$-ideal of PI of a finite-dimensional algebra, which by Proposition 2.21 is PI-equivalent to a direct sum of fundamental algebras. Hence $I$ is the intersection of the ideals of polynomial identities of these algebras. Since it is irreducible, it must coincide with the ideal of polynomial identities of one summand, a fundamental algebra. $\square$

In the theory of primary decomposition we may define an irredundant decomposition $I = J_1 \cap \cdots \cap J_k$ of a $T$-ideal $I$ into primary ideals. In a similar way, every finite-dimensional algebra $B$ is PI-equivalent to a direct sum $A = \bigoplus_i A_i$ of fundamental algebras which is irredundant in the sense that $A$ is not PI-equivalent to any proper algebra $\bigoplus_{i \neq j} A_i$. We call this an irredundant sum of fundamental algebras.

It is NOT true that the $T$-ideal of polynomial identities of a fundamental algebra is irreducible. The following example was suggested to me by Belov.

Consider the two fundamental algebras $A_{1,2}, A_{2,1}$ of block upper-triangular $3 \times 3$ matrices stabilizing a partial flag formed by a subspace of dimension 1, 2 respectively. They have both a semisimple part isomorphic to $M_2(F) \oplus F$. By a theorem of Giambruno–Zaicev [18], the two $T$-ideals of polynomial identities are respectively $I_2I_1$ and $I_1I_2$, where $I_k$ denotes the ideal of identities of $k \times k$ matrices. By a theorem of Bergman–Lewin [9] these two ideals are different.
Now in their direct sum \( A_{1,2} \oplus A_{2,1} \) consider the algebra \( L \) which on the diagonal has equal entries in the two \( 2 \times 2 \)-blocks and in the two \( 1 \times 1 \)-blocks. It is easy to see that \( L \) is PI-equivalent to \( A_{1,2} \oplus A_{2,1} \) and that it is fundamental. Its \( T \)-ideal is not irreducible but it is \( I_2 I_1 \cap I_1 I_2 \).

2.2.1. The role of traces. Let \( A \) be a fundamental algebra over a field \( L \), with index \( t, s \), and \( f \) a \( \mu \)-Kemer polynomial for \( \mu \) so large that \( f \) is also fundamental.

It follows from Lemma 2.23 that the evaluation in the small layers factors through the radical. So let us fix one of the small layers (say, of the variables \( x_1, \ldots, x_t \)) and for simplicity denote the remaining variables by \( Y \). Thus, having fixed some evaluation, which we denote by \( Y \), of the variables \( Y \) outside the chosen small layer of variables, we deduce from \( f \) a multilinear alternating map (still denoted by \( f \)) from \( A/J \) to \( A \) in \( t \) variables. Since \( \dim A = t \), this map must then have the form
\[
f(x_1, x_2, \ldots, x_t, Y) = \det(\pi_1, \ldots, \pi_t)u(Y),
\]
where the \( \pi_i \) are the classes in \( \overline{A} \) of the evaluations of the variables \( x_i \) and we have chosen a trivialization of \( \bigwedge^t \overline{A} \).

Since \( f \) is fundamental and there are \( s \) radical evaluations in the big layers, which are all contained in \( Y \), we also have \( u(Y) \in J^s \).

As a consequence, we deduce the following important identity, understood as an equality of functions on \( A \):
\[
f(zx_1, zx_2, \ldots, zx_t, Y) = \det(\pi) f(x_1, x_2, \ldots, x_t, Y),
\]
where \( \det(\pi) \) means the determinant of the left multiplication by \( \pi \) on \( A/J \).

When we polarize this identity, we have the characteristic polynomial on the right-hand side. In particular, the following identity holds as an equality of functions on \( A \):
\[
\sum_{k=1}^t f(x_1, \ldots, x_{k-1}, zx_k, x_{k+1}, \ldots, x_t, Y) = \text{tr}(\pi) f(x_1, \ldots, x_t, Y),
\]
where \( \text{tr}(\pi) \) is again the trace of the left multiplication by \( \pi \) on \( A/J \).

We now observe that the left-hand side of this formula is still alternating in all the layers in which \( f \) is alternating and not an identity, since for generic \( z \) we have \( \text{tr}(\pi) \neq 0 \). Thus it is again a \( \mu \)-Kemer polynomial.

We can then repeat the argument and arrive at the following result.

**Lemma 2.28.** The function resulting from multiplying a \( \mu \)-Kemer polynomial, evaluated in \( A \), by a product \( \text{tr}(\pi_1) \text{tr}(\pi_2) \cdots \text{tr}(\pi_k) \), where the \( z_i \) are new variables, is the evaluation in \( A \) of a new \( \mu \)-Kemer polynomial (also involving the variables \( z_i \)).

This discussion easily extends to a direct sum \( A = \bigoplus_{i=1}^q A_i \) of fundamental algebras (with radicals \( J_i \)) over some field \( L \), all with the same Kemer index \( (t, s) \), giving the formula
\[
\sum_{i=1}^q t_i(\pi) f_i(x_1, \ldots, x_t, Y) = \sum_{k=1}^t f(x_1, \ldots, x_{k-1}, zx_k, x_{k+1}, \ldots, x_t, Y),
\]
where \( f_i(x_1, \ldots, x_t, Y) \) is the projection to \( A_i \) of the evaluation \( f(x_1, \ldots, x_t, Y) \), while \( t_i(\bar{z}) \) is the trace of the left multiplication by the class of \( z \) on \( A_i \). In this case if \( \mu \) is sufficiently large and \( f \) is a \( \mu \)-Kemer polynomial for one of the algebras \( A_i \), then \( f \) is either a polynomial identity or a \( \mu \)-Kemer polynomial for each of the \( A_j \). It is then convenient to think of \( A \) as a module over the direct sum of \( q \) copies of \( L \). Then we can write \( t(\bar{z}) := (t_1(\bar{z}), \ldots, t_q(\bar{z})) \in L^q \) and we have

\[
t(\bar{z})f(x_1, \ldots, x_t, Y) = \sum_{k=1}^{t} f(x_1, \ldots, x_{k-1}, zx_k, x_{k+1}, \ldots, x_t, Y).
\]

Notice furthermore that if \( \mu \) is sufficiently large and \( f \) is a generic \( \mu \)-Kemer polynomial, it is not a PI for any of the algebras \( A_i \).

### 2.2.2. \( T \)-ideals of finite Kemer index

We have remarked that a \( T \)-ideal in the free algebra in countably many variables is of finite Kemer index if and only if it contains some Capelli list.

The following fact could also be obtained from the theory of Zubrilin [37].

**Proposition 2.29.** If \( \Gamma \) is a \( T \)-ideal with finite Kemer index \((t,s)\), then there is a \( \nu \in \mathbb{N} \) such that every \( \nu \)-Kemer polynomial \( f(X_1, \ldots, X_{\nu}, Z_1, \ldots, Z_s, W) \in R(X) \) has the property that, for two extra variables \( y, z \notin X \) one has, modulo \( \Gamma \),

\[
\forall i = 1, \ldots, \nu: \quad f(zX_1, \ldots, X_{\nu}, Z_1, \ldots, Z_s, W) \equiv_{\Gamma} f(X_1, \ldots, zX_i, \ldots, X_{\nu}, Z_1, \ldots, Z_s, W); \quad (11)
\]

\[
f(zyX_1, X_2, \ldots, X_\nu, Z_1, \ldots, Z_s, W) \equiv_{\Gamma} f(yzX_1, X_2, \ldots, X_\nu, Z_1, \ldots, Z_s, W).
\]

**Proof.** By Theorem 2.9 and Proposition 2.21 we know that \( \Gamma \) is the ideal of identities of a finite direct sum of fundamental algebras \( A = \bigoplus_{i=1}^{m} A_i \). Also by Remark 2.12 the Kemer index of \( \Gamma \) is the maximum of the Kemer indices of the fundamental algebras \( A_i \). Thus we may assume that the first \( k \) algebras in the list \( A_i \) have this maximal Kemer index and decompose \( A = B \oplus C \), where \( B = \bigoplus_{i=1}^{k} A_i \). It follows that there is some \( \nu \in \mathbb{N} \) such that any \( \nu \)-Kemer polynomial for \( \Gamma \) is a \( \nu \)-Kemer polynomial for \( B \) while it is a polynomial identity for \( C \).

Then formula \((11)\) is valid if and only if it is valid modulo \( \Gamma' := \text{Id}(B) \), that is, as an equality of functions on \( B \), and this is ensured by formula \((8)\). \( \square \)

### 2.2.3. Generic elements

Let \( A \) be a finite-dimensional algebra over an algebraically closed field \( F \), and \( \dim_F A = n \). Fix a basis \( a_1, \ldots, a_n \) of \( A \). Given \( m \in \mathbb{N} \) (or \( m = \infty \)), consider \( L \), the rational function field \( F(\Lambda) \), where \( \Lambda = \{ \lambda_{i,j}, i = 1, \ldots, n, j = 1, \ldots, m \} \) are \( mn \) indeterminates.

We can construct \( m \) generic elements \( \xi_j := \sum_{i=1}^{n} \lambda_{i,j} a_i \) for \( A \), and in \( A \otimes L \) we construct the algebra \( F_A(m) = F(\xi_1, \ldots, \xi_m) \) generated, over the field \( F \), by these generic elements. This is clearly isomorphic to the relatively free algebra in the variety of algebras generated by \( A \), the quotient of the free algebra modulo the identities of \( A \) in \( m \) variables.
The proof uses a basic tool of PI-theory, the Shirshov basis: for a finitely generated PI-algebra $R = F\langle \xi_1, \ldots, \xi_\ell \rangle$ there exists a finite number $N = N(\ell, s)$ depending only on the number $\ell$ of generators and the degree $s$ of a polynomial identity satisfied by $R$, such that every monomial in the variables $\xi_j$ is a linear combination with coefficients in $F$ of products of powers $a_1^{n_1}a_2^{n_2} \cdots a_j^{n_j}$, $j \leq N$, where $a_i$ are monomials in the generators $\xi_j$ of degree less than $s$ (the degree of a PI satisfied by $R$) [32].

If $A = \bigoplus_{i=1}^k A_i$ is a direct sum of finite-dimensional algebras (usually assumed to be fundamental), let $J = \bigoplus_{i=1}^k J_i$ be its radical and $A/J = \bigoplus_{i=1}^k \overline{A}_i = \bigoplus_{i=1}^k A_i/J_i$.

We may choose a basis $a_1, \ldots, a_n$ of $A$ over $F$ as the union of bases of the summands $A_i$. We may also choose the basis for each summand $A_i$ (decomposed as $\overline{A}_i \bigoplus J_i$) to be formed of a basis of $J_i$ and one of $\overline{A}_i$. Then when we construct $m$ generic elements $\xi_j := \sum_{i=1}^n \lambda_{i,j} a_i$ for $A$, by the choice of the basis each is the sum $\xi_j = \sum_{i=1}^k (\xi_{j,i} + \eta_{j,i})$, where the elements $\zeta_{j,i} := \xi_{j,i} + \eta_{j,i}$ are generic for $A_j$. The $\eta_{j,i}$ are generic for $\overline{A}_j$, while the $\eta_{j,i}$ are generic for the radical $J_j$, and all involve disjoint variables.

For each $j = 1, \ldots, k$ we also have the relatively free algebra $\mathcal{F}_{A_j}(m) = F(\zeta_{1,j}, \ldots, \zeta_{m,j})$ and an injection

$$\mathcal{F}_A(m) \subset \bigoplus_{j=1}^k \mathcal{F}_{A_j}(m) \subset A \otimes_F L = \bigoplus_{j=1}^k A_j \otimes_F L.$$  

Notice that the radical of $A \otimes_F L$ is $J \otimes_F L$ and modulo the radical this is $\overline{A} \otimes_F L$, which is some direct sum of matrix algebras $\bigoplus_i M_{n_i}(L)$.

The projection $p: A \to \overline{A} = \bigoplus_{i=1}^k \overline{A}_i$ of coordinates $p_1, p_2, \ldots, p_k$ induces a map of algebras generated by the generic elements:

$$p: \mathcal{F}_A \to \bigoplus_i \mathcal{F}_{\overline{A}_i} \subset \bigoplus_i \overline{A}_i \otimes_F L, \quad p: \xi_i \mapsto (\zeta_{i,1}, \ldots, \zeta_{i,k}).$$

Remark 2.30. The kernel of $p$ is a nilpotent ideal while the image is isomorphic to the domain $\mathcal{F}_{M_n(F)}$ of generic $n \times n$ matrices, where $n$ is the maximum of the degrees $n_i$ (where $\overline{A} = \bigoplus_i M_{n_i}(F)$).

We then set

$$a \in \mathcal{F}_A, \quad t(a) := (t_1(a), \ldots, t_k(a)) \in L^{\oplus k} \quad (12)$$

by setting $t_i(a)$ to be the trace of the left multiplication by the image of $a$ under $p_i$ on the summand $\overline{A}_i \otimes_F L$. We let

$$T_A(m) := F[t(a)]|_{a \in \mathcal{F}_A} \subset L^{\oplus k} \quad (13)$$

be the (commutative) algebra generated over $F$ by all the elements $t(a), a \in \mathcal{F}_A(m)$. From now on we assume that $m$ is fixed and drop the symbol $(m)$, writing simply $\mathcal{F}_A(m) = \mathcal{F}_A$, $T_A(m) = T_A$.

Theorem 2.31. $T_A$ is a finitely generated $F$-algebra and $T_A\mathcal{F}_A$ is a finitely generated module over $T_A$.

Proof. The proof uses a basic tool of PI-theory, the Shirshov basis: for a finitely generated PI-algebra $R = F\langle \xi_1, \ldots, \xi_\ell \rangle$ there exists a finite number $N = N(\ell, s)$ depending only on the number $\ell$ of generators and the degree $s$ of a polynomial identity satisfied by $R$, such that every monomial in the variables $\xi_j$ is a linear combination with coefficients in $F$ of products of powers $a_1^{n_1}a_2^{n_2} \cdots a_j^{n_j}$, $j \leq N$, where $a_i$ are monomials in the generators $\xi_j$ of degree less than $s$ (the degree of a PI satisfied by $R$) [32].
A basic example is given by the algebra of matrices \( A \). In fact the coefficients are polynomials in \( t(a^2) \) for \( j \leq \max(\dim \overline{A}_i) \).

For this let \( n_i := \dim \overline{A}_i \). The projection of \( a \) in \( \overline{A}_i \otimes_F L \) satisfies \( H_{n_i}(x) \), where we take for \( H_{n_i}(x) \) the Cayley–Hamilton polynomial induced by the left multiplication on \( \overline{A}_i \otimes_F L \). This is a universal expression in \( x \) and the elements \( t_i(a^j), j \leq n_i \), where \( t_i(a^j) \) is the trace of the left action on \( \overline{A}_i \otimes_F L \) of the projection of \( a^j \).

Thus if we use the formal Cayley–Hamilton polynomial for \( n_i \times n_i \) matrices, but using as trace of \( a \) the \( k \)-tuple \( t(a) := (t_1(a), \ldots, t_k(a)) \) for all \( a \), we see that, if \( \overline{a} \) denotes the image of \( a \) in \( \bigoplus_i \overline{A}_i \otimes_F L \) we have \( H_{n_i}(\overline{a})\overline{a} \in \bigoplus_i \overline{A}_i \otimes_F L \) has 0 in the \( i \)th component, so

\[
\prod_{i=1}^k H_{n_i}(\overline{a}) = 0 \quad \text{in} \quad p(T_A F_A) \subset \bigoplus_i \overline{A}_i \otimes_F L.
\]

Now every element of the kernel of \( p \) is nilpotent of some fixed degree \( s \), and we finally deduce that

\[
\left( \prod_{i=1}^k H_{n_i}(a) a \right)^s = 0 \quad \text{in} \quad T_A F_A.
\]

We have multiplied by \( a \) since we do not assume that the algebra has a 1.

Let \( d := s(\sum_i n_i^2 + k) \) be the degree of \( \left( \prod_{i=1}^k H_{n_i}(x) x \right)^s \).

We take the list \( a_1, \ldots, a_N \) of monomials of degree less than or equal to the degree \( s \) of a PI, generating a Shirshov basis \( a_{i_1}^{n_1} a_{i_2}^{n_2} \cdots a_{i_j}^{n_j} \) for \( F_A \). Let \( T'_A \) be the subalgebra of \( T_A \) generated by the coefficients of the characteristic polynomials of the monomials \( a_i, i = 1, \ldots, N \). By construction, it is finitely generated.

Then using the identity (14), every monomial \( a_{i_1}^{n_1} a_{i_2}^{n_2} \cdots a_{i_j}^{n_j} \) is a linear combination of monomials \( a_{i_1}^{m_1} a_{i_2}^{m_2} \cdots a_{i_j}^{m_j} \), where \( m_i < d \) for all \( i \), with coefficients in \( T'_A \).

Next, since we know that \( t(t(a)b) = t(a)t(b) \), it follows that \( T_A \) is generated by the traces \( t(M) \), where \( M \) is a monomial in the Shirshov basis with exponents less than \( d \). Thus \( T_A \) is a finitely generated algebra and \( F_A T_A \) is spanned over \( T_A \) by this finite number of monomials. \( \square \)

**Example 2.32.** A basic example is given by the algebra of matrices \( A = M_t(F) \). In this case the algebra of generic elements is known as the *generic matrices*. The commutative algebra \( T_A(Y) \) (to be denoted by \( T_t(Y) \)) equals the algebra of invariants of \( m := |Y| \) matrices under conjugation, and the algebra \( T_A F_A \) is the algebra of equivariant maps (under conjugation) between \( m \)-tuples of matrices to matrices.

Assume now that \( A = \bigoplus_{i=1}^m A_i \) is a direct sum of fundamental matrices. Decompose \( A = B \oplus C \), where we may assume that \( B = \bigoplus_{i=1}^k A_i \) is the sum of the \( A_i \) with maximal Kemer index (the same Kemer index as \( A \)) and \( C \) is the direct sum of the remaining algebras. For \( \mu \) sufficiently large, a \( \mu \)-Kemer polynomial for \( A \) is a polynomial identity on \( C \) and either a Kemer polynomial or a polynomial identity for \( A_i, i = 1, \ldots, k \). So in this case we call any polynomial with this property a Kemer polynomial for \( A \). By formula (9), as extended to direct sums, a Kemer
polynomial evaluated in $\mathcal{F}_A \subset A \otimes_F L = \bigoplus_i A_i \otimes_F L$ satisfies formula (9), where $\text{tr}(\overline{z})$ is the element of $\mathcal{T}_A$ in formula (12).

In fact, since such polynomials vanish on $C$, the formulae factor through $\mathcal{F}_B, \mathcal{T}_B$, which are quotients of $\mathcal{F}_A, \mathcal{T}_A$.

We can then interpret Lemma 2.28 as follows.

**Corollary 2.33.** The ideal $K_A$ of $\mathcal{F}_A$ spanned by evaluations of Kemer polynomials is a $\mathcal{T}_A$-submodule and thus a common ideal in $\mathcal{F}_A$ and $\mathcal{F}_A \mathcal{T}_A$.

In fact, under the quotient map $\mathcal{F}_A \to \mathcal{F}_B$ the ideal $K_A$ maps isomorphically to the corresponding ideal $K_B$. In other words, the action of $\mathcal{F}_A \mathcal{T}_A$ on $K_A$ factors through $\mathcal{F}_B \mathcal{T}_B$.

The importance of this corollary lies in the fact that the non-commutative object $K_A$ is, by Theorem 2.31, a finitely generated module over a finitely generated commutative algebra, so we can apply to it all the methods of commutative algebra. This is the goal of the ensuing sections. At this point the ideal $K_A$ depends on $A$ and not only on the $\mathcal{T}$-ideal, but as we shall see one can also remove this dependence and define an intrinsic object which plays the same role.

§ 3. The canonical filtration

3.1. Rationality and a canonical filtration. We want to draw some interesting consequences from the theory developed so far.

Let $R(X) = R := F(X)/I$ be a relatively free algebra in a finite number $k$ of variables $X$.

We have seen that $I$ is the $\mathcal{T}$-ideal of identities in $k$ variables of a finite-dimensional algebra $A = \bigoplus_i A_i$, a direct sum of fundamental algebras which may also be chosen irredundant.

Choosing such an $A$, we write $R = \mathcal{F}_A(X)$ and identify $\mathcal{F}_A(X)$ with the corresponding algebra of generic elements of $A$.

As usual, we decompose this direct sum into two parts, $A = B \oplus C$, where $B$ is the direct sum of the $A_i$ with the same Kemer index as $A$ while $C$ is the sum of those of strictly lower Kemer index.

We have $\text{Id}(A) = \text{Id}(B) \cap \text{Id}(C)$ and thus an embedding $\mathcal{F}_A(X) \subset \mathcal{F}_B(X) \oplus \mathcal{F}_C(X)$ of the corresponding relatively free algebras. Let us drop $X$ for simplicity.

We can apply Theorem 2.31 to $B$ and embed $\mathcal{F}_B \subset \mathcal{F}_B \mathcal{T}_B$, which is a finitely generated module over the finitely generated commutative algebra $\mathcal{T}_B$ (both graded).

Let $K_0 \subset R$ be the $\mathcal{T}$-ideal generated by the Kemer polynomials of $B$ for sufficiently large $\mu$. Since these polynomials are PI of $C$, it follows that under the embedding $\mathcal{F}_A(X) \subset \mathcal{F}_B(X) \oplus \mathcal{F}_C(X)$ the ideal $K_0$ maps isomorphically to the corresponding ideal in $\mathcal{F}_B$ which, by Corollary 2.33, is a finitely generated module over $\mathcal{T}_B$.

Since $K_0 \subset R$ is a $\mathcal{T}$-ideal, the algebra $R_1 := R/K_0$ is also a relatively free algebra, now with strictly lower Kemer index.

So we can repeat the construction and let $K_1 \subset R$ be the $\mathcal{T}$-ideal such that $K_0 \subset K_1$ and $K_1/K_0$ is the $\mathcal{T}$-ideal in $R_1$ generated by the corresponding Kemer polynomials.
If we iterate this construction, we must stop after a finite number of steps since the Kemer index strictly decreases at each step.

**Theorem 3.1.** 1) We have a filtration $0 \subset K_0 \subset K_1 \subset \cdots \subset K_u = R$ of $T$-ideals such that $K_{i+1}/K_i$ is the $T$-ideal in $R_i := R/K_i$ generated by the corresponding Kemer polynomials for $\mu$ suitably large.

2) The Kemer index of $R_i$ is strictly smaller than that of $R_{i-1}$.

3) Each algebra $R_i$ has a quotient $\overline{R}_i$ which can be embedded $\overline{R}_i \subset \overline{R}_i T_{\overline{R}_i}$ in a finitely generated module over a finitely generated commutative algebra $T_{\overline{R}_i}$ and such that $K_{i+1}/K_i$ is mapped injectively to an ideal of $\overline{R}_i T_{\overline{R}_i}$.

4) In particular, $K_{i+1}/K_i$ has the structure of a finitely generated module over the finitely generated commutative algebra $T_{\overline{R}_i}$.

**Corollary 3.2 (Belov [6]).** If $R := F\langle X \rangle/I$ is a relatively free algebra in a finite number of variables $X$, then its Hilbert series

$$H_R(\rho) := \sum_{k=0}^{\infty} \dim(R_i)\rho^i$$

is a rational function of the form

$$\frac{p(\rho)}{\prod_{j=1}^{N}(1 - \rho h_j)}, \quad h_j \in \mathbb{N}, \quad p(\rho) \in \mathbb{Z}[\rho].$$

(15)

**Proof.** Clearly, $H_R(\rho) = \sum_{i=0}^{n-1} H_{K_{i+1}/K_i}(\rho)$. We know that $K_{i+1}/K_i$ is a finitely generated module over a finitely generated graded algebra $T_{\overline{R}_i}$.

If $T_{\overline{R}_i}$ is generated by elements $a_1, \ldots, a_m$ of degrees $h_i$, then it follows from commutative algebra that

$$H_{K_{i+1}/K_i}(\rho) = \frac{p_i(\rho)}{\prod_{j=1}^{m}(1 - \rho h_j)}, \quad p_i(\rho) \in \mathbb{Z}[\rho].$$

Summing these (finitely many) rational functions, we have the result. □

In Theorem 3.21 and Corollary 3.23 we will apply a deeper geometric analysis in order to compute from the Hilbert series the dimensions of the relatively free algebras.

One can generalize these results by considering a relatively free algebra in $k$ variables $X$ as multi-graded by the degrees of the variables $X$, and then we write down its generating series of the multi-grading

$$H_R(x) = \sum_{h_1, \ldots, h_k} \dim(R_{h_1 \ldots h_k})x_1^{h_1} \cdots x_k^{h_k}.$$ (16)

This is of course the graded character of the induced action of the torus of diagonal matrices (which is a standard choice for a maximal torus of the general linear group $GL(k)$) acting linearly on the space of variables. The same argument actually shows that $H_R(x)$ is a *nice rational function* (see Definition 3.25) with denominator a product of factors of type $1 - x_1^{h_1} \cdots x_k^{h_k}$.

Thus the series in formula (16) should be interpreted in terms of the representation theory of $GL(k)$. In each degree $d$ the homogeneous part $R_d$ of the algebra $R$ is a quotient of the representation $V^\otimes d$, $\dim V = k$. 
By the Schur–Weyl duality discussed in §2.1.2 and by formula (3) we have

$$R_d = \bigoplus_{\lambda \vdash d} m_\lambda S_\lambda(V), \quad m_\lambda \leq \chi_\lambda(1).$$

That is, $m_\lambda$ does not exceed the dimension of the corresponding irreducible representation of the symmetric group $S_d$. In fact the following holds by Remark 2.10.

**Proposition 3.3.** If the number of variables is larger than $m$ and $R$ satisfies the Capelli list $C_m$, then $m_\lambda$ equals the multiplicity of $\chi_\lambda$ in the cocharacter of $A$ in Definition 2.3.

The character of $S_\lambda(V)$ is the corresponding Schur function $S_\lambda(x_1, \ldots, x_k)$ (a symmetric function), so that we finally have

$$H_R(x) = \sum_d \sum_{\lambda \vdash d} m_\lambda S_\lambda(x_1, \ldots, x_k). \quad (17)$$

We have seen that the numbers $m_\lambda$ are the multiplicities of the cocharacters. So it will also be interesting to write directly a generating function for these multiplicities:

$$\overline{H}_R(\rho_1, \ldots, \rho_k) = \sum_d \sum_{\lambda \vdash d} m_\lambda \rho^\lambda. \quad (18)$$

This has to be interpreted using the theory of highest weights (see §2.1.2). The weight is then a dominant weight sum of the fundamental weights $\omega_i := \prod_{j=1}^i x_j$, which is the highest weight of $\bigwedge^i F^k$. If $m_i$ is the number of columns of length $i$ of the partition $\lambda$, then the corresponding dominant weight is $\sum_i m_i \omega_i$, that is, the character $\prod_{i=1}^k (\prod_{j=1}^i x_j)^{m_i}$. We identify partitions with dominant weights and thus write

$$\lambda = \sum_i m_i \omega_i, \quad \rho_i := \rho^{\omega_i}, \quad \rho^\lambda = \prod_i \rho_i^{m_i}. \quad$$

A highest-weight vector $v_\lambda$ of weight $\lambda$ generates an irreducible representation $S_\lambda(F^k)$, and $S_\lambda(F^k)^U$ is 1-dimensional and spanned by $v_\lambda$.

In the following paragraphs, we use the theory of highest-weight vectors to show that the function (18) is also rational and of a special type (connected with the theory of partition functions).

Finally, in Theorem 3.39 we will apply this theory to give a precise quantitative result on the growth of the colength of $R$.

### 3.2. A close look at the filtration.

Our next goal is to show that the commutative rings $T_{R_i}$, which we have deduced from some fundamental algebras, can formally be derived using only properties of the $T$-ideals. For this we need to recall the theory of polynomial maps.

#### 3.2.1. Polynomial maps.

The notion of a polynomial map is quite general and we refer to Roby, [30] and [31]. We shall only use polynomial maps $f: M \to N$ between vector spaces over an infinite field (in fact, a field of characteristic $0$). Then the theory simplifies: a polynomial map is just a map given by polynomials in the coordinates.
A polynomial map \( f : M \to N \), homogeneous of degree \( t \), between two vector spaces factors as the map \( m \mapsto m^\otimes t \) to the symmetric tensors \( S^t(M) := (M^\otimes t)^{S_t} \) and a linear map \( S^t(M) \to N \). If \( A \) is an algebra, then \( A^\otimes t \) and \( S^t(A) = (A^\otimes t)^{S_t} \) are also algebras and we have the following general definition and fact.

**Definition 3.4.** A polynomial map, homogeneous of degree \( t \), between two algebras \( A, B \) is said to be **multiplicative** if \( f(ab) = f(a)f(b) \) for all \( a, b \in A \).

**Proposition 3.5** (Roby [31]). A polynomial map, homogeneous of degree \( t \), between two algebras \( A, B \) is multiplicative if and only if the induced map \( S^t(A) \to B \) is an algebra homomorphism.

We can apply this theory to \( A \), equal to a free algebra \( F\langle Y \rangle \), a positive integer \( t \) and the subalgebra of symmetric tensors \( S^t(F\langle Y \rangle) = (F\langle Y \rangle^\otimes t)^{S_t} \). We can treat the map \( z \mapsto z^\otimes t \) as a universal multiplicative polynomial map homogeneous of degree \( t \).

A general theorem of Ziplies [35] interpreting the second fundamental theorem of matrix invariants of Procesi and Razmyslov, that is, the Procesi–Razmyslov theory of trace identities [22], [26], [27], states the following.

**Theorem 3.6.** (Ziplies) The maximal Abelian quotient of \( S^t(F\langle Y \rangle) \) is isomorphic to the algebra \( T_t(Y) \) of invariants of \( t \times t \) matrices in the variables \( Y \).

(Vaccarino) This isomorphism is induced by the explicit multiplicative map, homogeneous of degree \( t \):

\[
\det : F\langle Y \rangle \to T_t(Y), \quad \det : f(y_1, \ldots, y_m) \mapsto \det f(\xi_1, \ldots, \xi_m),
\]

where the \( \xi_i \) are generic \( t \times t \) matrices.

The second part is due to Vaccarino [34]. At the moment the theorem is presented in the literature only in characteristic 0. The map is well defined and surjective in all characteristics by a theorem of Donkin [13]. By a paper of Zubkov [36] it actually follows that it is always an isomorphism, as will be explained elsewhere in a joint paper with De Concini [12].

### 3.2.2. Kemer polynomials.

The proof of Corollary 2.33 is based on the fact that a \( T \)-ideal \( \Gamma \) in finitely many variables can be presented as the ideal of identities of a finite-dimensional algebra \( A \), a direct sum of fundamental algebras.

We now want to show that this structure of the \( T \)-ideal of Kemer polynomials is independent of \( A \) and thus gives some information on the possible algebras \( A \) having \( \Gamma \) as ideal of identities.

Denote by \( R(Y) \) and \( R(Y \cup X) \) the relatively free algebras in the variety associated with \( \Gamma \) in these corresponding variables. Let us first use an auxiliary algebra \( A \) having \( \Gamma \) as ideal of identities and let \( K_A \subset R(Y \cup X) \) be the space of Kemer polynomials previously defined starting from \( A \). Changing the algebra \( A \) to some \( A' \) may change the space \( K_A \) but only for the \( \mu \)-Kemer polynomials up to some \( \mu_1 \). We shall soon see how to free ourselves from this irrelevant constraint. Let \( K_\nu \) be the space of \( \nu \)-Kemer polynomials in the relatively free algebra \( F(X)/\Gamma \) in which the small layers are taken from the variables in \( X \) (and may also depend on the variables \( Y \)).
If \( z \in F(Y) \), we have a linear map, which we shall denote by \( \tilde{z} \), given by (choosing one of the small layers):
\[
\tilde{z} : f(x_1, x_2, \ldots, x_t, X, W) \mapsto f(zx_1, zx_2, \ldots, zx_t, X, W) \mod \Gamma.
\]  
(19)

By Proposition 2.29 there is an intrinsic \( \nu \in \mathbb{N} \) such that, by formula (11),
1) the operators \( \tilde{z} \) do not depend on the small layer chosen and commute;
2) the map \( z \mapsto \tilde{z} \) is a multiplicative polynomial map, homogeneous of degree \( t \), from the free algebra \( F(Y) \) to a commutative algebra of linear operators.

Therefore the Zpies–Vaccarino theorem (Theorem 3.6) tells us the following.

**Corollary 3.7.** Formula (19) defines a module structure on \( K_\nu \) over the algebra \( T_t(Y) \) of invariants of \( t \times t \) matrices in the variables \( Y \).

This module structure does not depend on \( A \) and thus is independent of the embedding of the relatively free algebra in \( A \otimes \mathbb{C} \).

On the other hand, choose \( A = \bigoplus_{i=1}^k A_i \), a direct sum of fundamental algebras, so that \( \Gamma = \text{Id}(A) \). Decompose \( A = B \oplus C \), where \( B = \bigoplus_{i=1}^u A_i \) is the direct sum of the \( A_i \) with maximal Kemer index \( (t, s) \) equal to the Kemer index of \( \Gamma \).

If \( \nu \) is sufficiently large, we know that \( K_\nu \) vanishes on \( C \) and in the previous embedding \( \tilde{z} \) coincides with the multiplication by \( (\det(\tilde{z})) \in \mathbb{C}^u \). When we polarize from \( z \in R(Y) \), we see by formula (9) that multiplication by \( \text{tr}(z) := (\text{tr}(z_1), \ldots, \text{tr}(z_u)) \) maps \( K_\nu \) into \( K_\nu \) and, by definition, lies in \( T_A = T_A(Y) \) (which now depends on \( \nu \)). In fact, by definition, \( T_A \) is generated by these elements.

**Lemma 3.8.** The action of \( T_A \) on \( K_\nu \) is faithful, so that the algebra \( T_A(Y) \) is the quotient of \( T_t(Y) \) modulo the ideal annihilator of the module \( K_\nu \).

This ideal is independent of \( \nu \) for \( \nu \) large.

**Proof.** We have constructed \( T_A \) from formula (13) as contained in the direct sum of the algebras \( T_{A_i}, i = 1, \ldots, u \). By definition, \( T_A \) depends only on the semisimple part of \( A \), so it is the coordinate ring of a variety which depends only upon the algebra \( \bigoplus_i M_{n_i}(F) \), that is, upon the numbers \( n_i \).

So in the end the action on \( K \) of the algebra \( T_t(Y) \) of invariants of \( t \times t \) matrices in the variables \( Y \) factors through the map
\[
\pi : T_t(Y) \to T_A(Y) \subset \bigoplus_{i=1}^u T_{A_i}(Y).
\]  
(20)

We claim that the composite of \( \pi \) with any projection to a summand \( T_{A_i}(Y) \) is surjective. Indeed, the first ring is generated by the traces of the monomials evaluated in all \( t \)-dimensional representations while the second is generated by the same traces but only for those representations which factor through the left action of \( A_i \). The nature of this subvariety will be seen in § 3.3.1.

Let \( K_{\nu,i} \) be the image of \( K_\nu \) in \( F_{A_i}T_{A_i}(Y) \). We have that each \( K_{\nu,i} \) is torsion-free over \( T_{A_i} \), which is a domain, since \( K_{\nu,i} \subset J_i^8 \otimes \mathbb{C} \) is contained in a vector space over the field \( L \).

Since \( K_\nu \subset \bigoplus_{i=1}^u K_{\nu,i} \) and for each \( i \) the restriction to \( K_{\nu,i} \) is non-zero, we finally have that \( T_A \) acts faithfully on \( K_\nu \), so that we have defined the homomorphism in formula (20). \( \square \)
We now want to free ourselves from the auxiliary variables $X$ and evaluate in all possible ways the elements of $K$, in the relatively free algebra $\mathcal{R}(Y)$ of $A$ in the variables $Y$, thus obtaining a $T$-ideal $K_T$ in $\mathcal{R}(Y)$.

**Theorem 3.9.** The module action of $T_t(Y)$ on $K$ induces a unique module action on $K_T$ compatible with substitutions of variables in $Y$.

This action factors through a faithful action of its image $\overline{T}_A(Y)$ in $T_A(Y)$.

*Proof.* If we work inside the non-intrinsic algebra generated by $R(Y)$ and $T_A$, we have that $K_T$ is a $T_A$-submodule and this module structure is by definition compatible with substitutions of variables in $Y$.

On the other hand, since the elements of $K_T$ can be obtained by specializing the elements of $K$, there is a unique way in which the module action of $T_t(Y)$ on $K$ can induce a module action on $K_T$ compatible with substitutions of variables in $Y$. It is a faithful $\overline{T}_A(Y)$-action since $T_A$ acts faithfully on $K_TL$. □

Notice that $K_T \subset B \otimes L$, so all the computations are just for the algebra $B = \bigoplus_i A_i$. Hence from now on we shall just assume that $A = B$ and $C = 0$.

### 3.3. Representation varieties

Our next task is to describe the algebraic varieties whose coordinate rings are the rings $T_A$.

#### 3.3.1. The varieties $W_{n_1,\ldots,n_q}$. For any given $t$ and $m$ consider the space $M_t(F)^m$ of $m$-tuples of $t \times t$ matrices. We think of this as the set of $t$-dimensional representations of the free algebra $F\langle X \rangle$ in $m$ variables $x_1,\ldots,x_m$.

The projective linear group $\mathrm{PGL}(t,F)$ acts on this space by simultaneous conjugation so that its orbits are the isomorphism classes of such representations. It is well known (see [5]) that the closed orbits correspond to semisimple representations. So, by geometric invariant theory, the quotient variety $V_t(m) := M_t(F)^m / \! / \mathrm{PGL}(t)$ parametrizes equivalence classes of semisimple representations of dimension $t$. As soon as $m \geq 2$, its generic points correspond to irreducible representations, which give closed free orbits, so the variety has dimension $(m - 1)t^2 + 1$.

The coordinate ring of this variety is the ring of $\mathrm{PGL}(t,F)$-invariants, which we shall denote by $T_{t,m}$ (or, as above, by $T_t(Y)$ if we denote by $Y$ the $m$ matrix variables).

In characteristic 0 the algebra $T_t(Y)$ is generated by the traces of the monomials in the matrix variables, while in all characteristics by the work of Donkin [13] we need all coefficients of the characteristic polynomials of monomials, which can be taken to be primitive.

Given non-negative integers $h_i, n_i$, $i = 1,\ldots,q$, with $\sum_{i=1}^q h_i n_i = t$, we may consider, inside the variety $M_t(F)^m / \! / \mathrm{PGL}(t)$, the subvariety $W_{h_1,\ldots,h_q;n_1,\ldots,n_q}$ of semisimple representations which can be obtained as direct sums $\bigoplus_i h_i N_i$ of semisimple representations $N_i$ of dimension $n_i$ for each $i = 1,\ldots,q$. Of course, generically each $N_i$ is irreducible.

We are interested in the special case $h_i = n_i$, which we denote by $W_{n_1,\ldots,n_q}$.

The variety $W_{h_1,\ldots,h_q;n_1,\ldots,n_q}$ is the natural image of the product $\prod_{i=1}^q V_{n_i}(m)$, where $V_{n_i}(m)$ is the variety of semisimple representations of $m$-tuples of $n_i \times n_i$ matrices, under the map $j : N_1,\ldots,N_q \mapsto \bigoplus_i h_i N_i$.

We see in fact that this map $j$ can be considered as a restriction.
Thus we have the following remark.

Every representation of the form $\bigoplus_{i=1}^q h_i M_{n_i}(F)$ of block-diagonal matrices, where the block $M_{n_i}(F)$ appears embedded into $h_i$ equal blocks. Of course, this subalgebra is by definition isomorphic to $M_{n_1,\ldots,n_q} := \bigoplus_{i=1}^q M_{n_i}(F)$. We denote this inclusion isomorphism by $j_{h_1,\ldots,h_n}$.

When we restrict the invariants $T_\ell(Y)$ to this subalgebra, we see that when $z$ is some polynomial in $Y$, the restriction of the function $\text{tr}(z)$ to this subalgebra equals $\sum_{i=1}^q h_i \text{tr}_i(z)$, where $\text{tr}_i(z)$ is in the algebra $T_{n_i}(Y)$.

This means that $T_\ell(Y)$ maps to the $G = \prod_{i=1}^q \text{PGL}(n_i,F)$ invariants, that is, the coordinate ring of $\prod_{i=1}^q V_{n_i}(m)$, which is $T_{n_1}(Y) \otimes \cdots \otimes T_{n_n}(Y)$.

We have shown that, as $\prod_{i=1}^q V_{n_i}(m)$ is the quotient of $m$ copies of the space $\bigoplus_{i=1}^q M_{n_i}(F)$ under the group $G = \prod_{i=1}^q \text{PGL}(n_i,F)$ acting by conjugation, we have the following commutative diagram, where $\pi_G$, $\pi_{\text{PGL}(t,F)}$ are quotients under the two groups:

$$
\begin{array}{ccc}
\bigoplus_{i=1}^q M_{n_i}(F) & \xrightarrow{j_{n_1,\ldots,n_q}} & M_{n_1}(F) \otimes \cdots \otimes M_{n_n}(F) \\
\pi_G & & \pi_{\text{PGL}(t,F)} \\
\prod_{i=1}^q V_{n_i}(m) & \xrightarrow{j} & W_{h_1,\ldots,h_q;n_1,\ldots,n_q}
\end{array}
$$

Every representation of the form $\bigoplus_{i=1}^q h_i N_i$ with $N_i$ of dimension $n_i$ can be conjugated into $\bigoplus_{i=1}^q h_i M_{n_i}(F)$, but usually this can be done in several different ways. Thus we have the following remark.

**Remark 3.10.** The map $j : \prod_{i=1}^q V_{n_i}(m) \to W_{n_1,\ldots,n_q}$ is surjective but it is almost never an isomorphism. It is an isomorphism only when $q = 1$.

We claim that the two varieties have the same dimension. For this it is enough to show that the generic fibre is finite. The generic fibre is obtained when all the summands $N_i$ in $\prod_{i=1}^q V_{n_i}(m)$ and in $W_{n_1,\ldots,n_q}$ are irreducible and not just semi-simple. In this case, if we have several indices $i$ with the same $n_i$, then in the expression $\bigoplus_{i=1}^q n_i N_i$ we may permute the indices of irreducible representations of the same dimension so they come from different ways of arranging them in the factors $V_{n_i}(m)$. Thus if we have certain multiplicities $h_1,\ldots,h_q$ of the different indices $n_i$, we see that the generic fibre is formed by $\prod_i h_i!$ points.

In fact even if there are no multiplicities, so the map $j$ is birational, then the same argument may show that in non-generic fibres we may perform some of these permutations and so the map is not usually bijective.

**Example 3.11.** $t = 2 = 1 + 1$. The variety $V_1(m)$ is just an affine space whose coordinate ring is the polynomial ring $F[x_1,\ldots,x_m]$. So $V_1(m) \times V_1(m)$ is the 2$m$-dimensional affine space whose coordinate ring is the polynomial ring $F[x_1,\ldots,x_m,y_1,\ldots,y_m]$.

For the other ring we take monomials in the elements $(x_i,y_i)$, which should be thought of as diagonal $2 \times 2$ matrices. Such a monomial is a pair formed by a monomial in $x_i$ and the same monomial in $y_i$. Its trace is the sum over these two monomials, a symmetric function in the exchange $\tau$ between $x_i$, $y_i$.

The map is generically 2 to 1; the image of the coordinate rings is the ring of $\tau$-invariants.
In fact we have an even stronger statement. Let $U_i$ denote the coordinate ring of the variety $V_{n_i}(m)$, so that $U_1 \otimes U_2 \otimes \cdots \otimes U_q$ is the coordinate ring of the variety $\prod_{i=1}^q V_{n_i}(m)$.

**Lemma 3.12.** $U_1 \otimes U_2 \otimes \cdots \otimes U_q$ is integral over $T_A(m)$, spanned by monomials of degree bounded by some number independent of $m$.

**Proof.** Consider a polynomial
\[
f(\rho) = \rho^t - a_1 \rho^{t-1} + a_2 \rho^{t-2} + (-1)^t a_t
\]
with roots $x_1, \ldots, x_t$. Since the elements $a_i$ are the elementary symmetric functions in the $x_i$, we may even take the $x_i$ as indeterminates.

Given an integer $k \in \mathbb{N}$ and a set $S \subset \{1, \ldots, t\}$ with $h$ elements, we put $X_S^k := \sum_{i \in S} x_i^k$.

Consider the following polynomial of degree $N := \binom{t}{h}$:
\[
\rho^N + \sum_{i=1}^N (-1)^i b_i \rho^{N-i} := \prod_{S \subset \{1, \ldots, t\}, |S| = h} (\rho - X_S^k).
\]
The coefficients $b_i$ of this polynomial are clearly symmetric functions in the variables $x_i$, so they are expressible as polynomials in the elements $a_i$. In fact $b_i$ is a polynomial of degree $ki$ in the variables $x_i$, so it is a polynomial of this weight when we give to $a_i$ the weight $i$.

Each algebra $U_i$ is generated by the traces $\text{tr}(M_i)$ of the monomials $M$ acting on the $i$th summand. Thus the element $\text{tr}(M_i)$ is a sum of $n_i$ eigenvalues taken out of the list of the $t$ eigenvalues of the monomial $M$ acting on the direct sum. We deduce a universal polynomial of degree $\binom{t}{n_i}$ satisfied by $\text{tr}(M_i)$ with coefficients polynomials in the elements $\text{tr}(M^k)$, $k = 1, \ldots, t$. □

3.3.2. The support of Kemer polynomials. We now apply the previous discussion to Kemer polynomials. First let us take a fundamental algebra $D$ with semisimple part $\bigoplus_{i=1}^q M_{n_i}(F)$, $t = \sum_i n_i^2$. We have constructed a map from $T_d(Y)$ to $T_D$.

**Lemma 3.13.** The algebra $T_D$ is the coordinate ring of the irreducible subvariety $W_{n_1, \ldots, n_u}$ of $M_t(F)^m \mathord{/\!/} \text{PGL}(t)$.

**Proof.** By definition, $T_D$ is the algebra generated by the traces of the $m$-tuples of elements of $\bigoplus_{i=1}^q M_{n_i}(F)$, $t = \sum_i n_i^2$, acting on itself by left multiplication. So the lemma follows from the previous discussion as summarized by the diagram (21). □

Let $A = \bigoplus_{i=1}^u A_i$ be a direct sum of fundamental algebras, all with the same Kemer index $(t, s)$. With each $A_i$ we have associated the following irreducible subvariety of $M_t(F)^m \mathord{/\!/} \text{PGL}(t, F)$:
\[
W_i := W_{n_1, \ldots, n_{q_i}}, \quad A_i = \bigoplus_{j=1}^{q_i} M_{n_j}(F), \quad \sum_{j=1}^{q_i} n_j^2 = t.
\]
We have thus a rather interesting fact.
Theorem 3.14. The image of the algebra $T_t(Y)$, acting on the space $K_R$ of $\nu$-Kemer polynomials, is for large $\nu$ the coordinate ring of a subvariety of $M_t(F)^m // \text{PGL}(t)$, possibly reducible and always equal to the union of the subvarieties $W_i$, $i = 1, \ldots, u$.

In other words, for large $\nu$ the $T_t(Y)$-module $K_R$ of $\nu$-Kemer polynomials is supported on this subvariety.

Notice that $T_t(Y)$ is an intrinsically defined object, and then $\bigcup_i W_i$ is also intrinsic, being the support of $K_R$. The subvarieties $W_i = W_{n_1, \ldots, n_q}$ reflect the structure of the semisimple parts of the fundamental algebras $A_i$ which may appear as summands of maximal Kemer index of an algebra $A$ having as identities the given $T$-ideal.

There are some subtle points in this construction. First of all, by $K_R$ we mean the $T$-ideal generated by $K_\nu$ for $\nu$ large, in the sense that this variety stabilizes for $\nu$ large. It is possible that some variety $W_i$ is contained in another $W_j$. This gives an embedded component, which may not be visible just by the structure of the module $K_R$ but depends on the embedding $K_R \subset \bigoplus_i K_i$, which in turn depends on the particular choice of $A$ such that $\Gamma = \text{Id}(A) = \bigcap_{i=1}^u \text{Id}(A_i)$. This appears as some primary decomposition and it is worthy of further investigation.

A specific element of $K_R$ vanishes on one of the varieties $W_i$ if and only if it is a polynomial identity for the corresponding summand $A_i$.

Corollary 3.15. If $R$ is the relatively free algebra associated with a fundamental algebra $A$ with semisimple part $\overline{A} = \bigoplus_{j=1}^q M_{n_j}(F)$, then the module of Kemer polynomials is supported on the irreducible variety $W_{n_1, \ldots, n_q}$.

In particular, if two fundamental algebras are PI-equivalent, then they have the same semisimple part.

In general if we have two equivalent PI-algebras $A = \bigoplus_i A_i$, $B = \bigoplus_j B_j$, each a direct sum of fundamental algebras, we see that the semisimple components which give maximal varieties of representations are uniquely determined.

This answers at least partially the question on how intrinsic are the constructions associated with a particular choice of an algebra $A$ having a chosen $T$-ideal as the ideal of polynomial identities.

We shall thus make use of the following refinement of the Kemer index.

Definition 3.16. Given a $T$-ideal $\Gamma$ of identities of a fundamental algebra, we take as the index of $\Gamma$ the pair $\overline{A}, s$, where $s$ is the second Kemer index (while the first is $\dim \overline{A}$).

3.3.3. Dimension. For an associative algebra $R$ with 1 over a field $F$, Gel’fand and Kirillov [16] defined a dimension as follows. Let $V \subset R$ be a finite-dimensional subspace with 1 $\in V$. Let $V_n$ denote the span of all products of $n$ elements of $V$ and set $d_V(n) := \dim V_n$.

Definition 3.17. The Gel’fand–Kirillov dimension (GK-dimension) is defined by the formula

$$\dim R := \sup_V \limsup_{n \to \infty} \frac{\log d_V(n)}{\log n}.$$
If $R$ is generated by $V$, then

$$\dim R = \limsup_{n \to \infty} \frac{\log d_V(n)}{\log n}.$$ 

In general a finitely generated non-commutative algebra may have infinite dimension or a dimension which is not an integer (see \cite{10}).

A special case is when $R$ is graded and its Hilbert series $H_R(\rho) := \sum_{k=0}^{\infty} \dim_F R_k \rho^k$ is a rational function of the type in formula (15) (this is a rather strong constraint on $R$). Since

$$\frac{1}{1 - \rho^h} = \sum_{i=0}^{\infty} \rho^{ih},$$

one has

$$\frac{\sum_{j=0}^{r} a_j \rho^j}{\prod_{j=1}^{N} (1 - \rho^{h_j})} = \sum_{j=0}^{r} a_j \rho^j \prod_{j=1}^{N} \sum_{i=0}^{\infty} \rho^{ih_j}.$$ 

The function $\prod_{j=1}^{N} (1 - \rho^{h_j})^{-1}$ is the Hilbert series of the polynomial algebra in generators $x_1, \ldots, x_N$ with $x_i$ of degree $h_i$. Writing $\prod_{j=1}^{N} \sum_{i=0}^{\infty} \rho^{ih_j} = \sum_{k=0}^{\infty} c_k \rho^k$, we see that $c_k$ is a non-negative integer which counts in how many ways the integer $k$ can be written in the form $k = \sum_{j=1}^{N} i_j h_j$, $i_j \in \mathbb{N}$.

Such a function $c_k$ is classically known as a partition function. It coincides on the positive integers with a quasi-polynomial of degree $N - 1$. Quasi-polynomial means in this case that when we restrict $c_k$ to each coset of $\mathbb{Z}$ modulo the least common multiple of the $h_j$, this function coincides on the positive integers in this coset with a polynomial of degree $N - 1$. There is an extensive literature on such functions (see \cite{11}).

If we expand the rational function in formula (15) in a power series $\sum_{k=0}^{\infty} d_k \rho^k$, we still have that after a finite number of steps the function $d_k$ coincides with a quasi-polynomial $D(k)$, but its degree may be strictly less than $N - 1$. One has the following theorem.

**Theorem 3.18.** The dimension of $R$ is the order of the pole of $H_R(\rho)$ at $\rho = 1$ and equals $n + 1$, where $n$ is the degree of the quasi-polynomial $D(k)$.

If $R$ is a commutative algebra, finitely generated over a field $F$, then it has a finite dimension which can be defined in several ways, either as the Krull dimension, that is, the length of a maximal chain of prime ideals, or as the Gel’fand–Kirillov dimension or, finally, when $R$ is a domain, as the transcendence degree of the field of fractions of $R$ over $F$. For a finitely generated commutative graded algebra $R$, the Hilbert series is a rational function of the type in formula (15) and the dimension of $R$ equals the dimension of its associated affine variety $V(R)$.

For a module $M$ we have the notion of the support of $M$, that is, the set of points $p \in V(R)$ where $M$ is non-zero in the sense that $M \otimes_R R_p \neq 0$, where $R_p$ is the local ring at $p$. Then the dimension of $M$ equals the dimension of its support. The support is computed as follows.
Remark 3.19. It is well known that a finitely generated module $M$ over a commutative Noetherian ring $R$ has a finite filtration $0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = M$ such that $M_{i+1}/M_i = R/P_i$, where $P_i$ is a prime ideal.

If $R$, $M$ are graded and $R$ is finitely generated, then the $M_i$ can be taken to be graded, the Hilbert series of $M$ is the sum of the Hilbert series of the quotients $R/P_i$, and the dimension of $M$ is the maximum of the dimensions of the quotients $R/P_i$.

In the language of algebraic geometry, $R$ determines the algebraic variety $V(R)$ and the ideals $P_i$ determine some subvarieties with coordinate rings $R/P_i$. One sees by induction that $M$ is supported on the union of these subvarieties and thus the dimension of $M$ is the dimension of its support.

If $M$ is torsion-free over a domain $R$, then it contains a free module $R^k \subset M$ such that $M/R^k$ is supported on a proper subvariety and, therefore, has lower dimension. Then the dimension of $M$ equals that of $R^k$.

We will apply this to non-commutative relatively free algebras.

3.3.4. The dimension of relatively free algebras. We take as the definition of dimension for a relatively free algebra the one given by its Hilbert series, which measures growth and can be seen to be equal to the Gel’fand–Kirillov dimension. It is almost never equal to the Krull dimension which instead equals the dimension of $R/J$, where $J$ is the nilpotent radical. It is well known that when $R$ is a relatively free algebra, $R/J$ is a ring of generic matrices since the only semiprime $T$-ideals are the $T$-ideals of identities of matrices.

This problem was treated by Belov in [7]. There he obtains results similar to our Proposition 3.20 in a different language and by a different method, which can also be applied in positive characteristic.

Let us first analyze a fundamental algebra $D$ with semisimple part $\bigoplus_{i=1}^q M_{n_i}(F)$.

Take the relatively free algebra in $m$ variables $\mathcal{F}_D(m) := F\langle \xi_1, \ldots, \xi_m \rangle$ for $D$. We have an inclusion $\mathcal{F}_D(m) \subset \mathcal{F}_D(m)\mathcal{T}_D(m)$ and also an inclusion $K_D \subset \mathcal{F}_D(m)$ of the ideal generated by Kemer polynomials.

Since both $K_D$ and $\mathcal{F}_D(m)\mathcal{T}_D(m)$ are finitely generated torsion-free modules over $\mathcal{T}_D(m)$, it follows that the dimensions of the Hilbert series of $K_D$, $\mathcal{F}_D(m)$, $\mathcal{F}_D(m)\mathcal{T}_D(m)$, $\mathcal{T}_D(m)$ are all equal to the Gel’fand–Kirillov dimension of $\mathcal{T}_D(m)$. We have already computed this dimension: $\dim \mathcal{T}_D(m) = (m-1)t + q$, where $t = \sum_{i=1}^q n_i^2$ is also the first Kemer index.

Proposition 3.20. 1) The GK-dimension of the relatively free algebra in $m$ variables for a fundamental algebra $D$ is $(m-1)t + q$, where $t$ is the first Kemer index.

2) The GK-dimension of the relatively free algebra in $m$ variables for a direct sum $\bigoplus_i D_i$ of fundamental algebras is the maximum of the GK-dimensions of the relatively free algebras in $m$ variables for the fundamental algebras $D_i$.

3) If $\Gamma$ is a $T$-ideal containing a Capelli list and $\Gamma = \bigcap_i \Gamma_i$ for some irreducible $T$-ideals $\Gamma_i$, then the dimension of $F\langle X \rangle/\Gamma$ is the maximum of the dimensions of the $F\langle X \rangle/\Gamma_i$, each of these being the relatively free algebra for a fundamental algebra.

Proof. We have just proved the first part. As to the second, we remark that the relatively free algebra in $m$ variables for a direct sum $\bigoplus_i D_i$ is contained in the direct
sum of the relatively free algebras in \( m \) variables for the summands \( D_i \), so its dimension is smaller than or equal to the maximum of the dimensions of the algebras relative to the summands. On the other hand, each relatively free algebra in \( m \) variables for the summands \( D_i \) is also a quotient of the relatively free algebra in \( m \) variables for the direct sum, so the claim follows. The third part follows from the second and the fact that an irreducible \( T \)-ideal containing a Capelli list is the ideal of polynomial identities of a fundamental algebra (see Proposition 2.27). \( \square \)

There is also a more intrinsic approach using the filtration of \( R \) given by Theorem 3.1. The dimension of \( R \) is the maximum of the dimensions of the modules \( K_{i+1}/K_i \). By Theorem 3.14 each of these modules is supported on a union of the varieties \( W_{n_1,\ldots,n_k} \). On each of these subvarieties the module is torsion-free, so it has the same dimension as its support. We deduce the following theorem.

**Theorem 3.21.** The dimension of \( R \) is the maximum dimension of the varieties \( W_{n_1,\ldots,n_k} \).

Now the dimension of \( W_{n_1,\ldots,n_k} \) is \( mt - \sum_{i=1}^q (n_i^2 - 1) = (m - 1)t + q \), so we see that as \( m \) grows the maximum is obtained from the factors \( R_i \) for which the first Kemer index equals the Kemer index \( t \) of \( R \), and among these the one with maximal \( q \).

**Definition 3.22.** The integer \( q \) is an invariant of the \( T \)-ideal, called the \( q \)-invariant.

**Corollary 3.23.** The dimension of \( R(m) \) for \( m \) large is \( (m - 1)t + q \), where \( t \) is the first Kemer index and \( q \) is the \( q \)-invariant.

Observe that when \( R \) is the ring of generic \( n \times n \) matrices, we have \( t = n^2 \) and \( q = 1 \) and the formula is valid for all \( m \geq 2 \). By [18], Theorem 4, the formula also holds for all \( m \geq 2 \) in the case of block-triangular matrices with \( q \) equal to the number of blocks and \( t \) equal to the dimension of the semisimple part.

**3.4. Cocharacters.** We now want to extend the work of Berele [8] and Belov [6] in which they show how the cocharacter multiplicities are described by partition functions. This requires some standard preliminaries.

**3.4.1. Partition functions.** The notion of a partition function can be discussed for any sequence of integral vectors \( S := \{a_1,\ldots,a_m\}, \ a_i \in \mathbb{Z}^p \), for which there is a linear function \( f \) with \( f(a_i) > 0 \) for all \( i \).

Then one defines the partition function \( P_S(b) \) on \( \mathbb{Z}^p \) by the formula

\[
 b \in \mathbb{Z}^p, \quad P_S(b) = \# \left\{ (\rho_1,\ldots,\rho_m) \in \mathbb{N}^m \left| \sum_{i=1}^m \rho_i a_i = b \right. \right\}.
\]

Of course, \( P_S(b) = 0 \) unless \( b \) lies in the positive cone

\[
 C(S) := \left\{ \sum_i x_i a_i \left| x_i \in \mathbb{R}^+ \right. \right\}
\]

\(^2\)This restriction is essential. Otherwise the value of the partition function is \( \infty \).
generated by the elements $a_i$. The assumption $f(a_i) > 0$ for all $i$ means that $C(S)$ is pointed, that is, it does not contain any line (only half-lines).

It is customary to express the partition function by a generating function. This is a series in $p$ variables $x_1, \ldots, x_p$. When $b = (b_1, \ldots, b_p) \in \mathbb{Z}^p$ we set $x^b := \prod_{i=1}^p x_i^{b_i}$ and then, setting
\[
P_S = \sum_{b \in \mathbb{Z}^p} P_S(b) x^b,
\]
one sees that
\[
P_S = \frac{1}{\prod_{i=1}^p (1 - x^{a_i})}.
\]
This formal series is also truly convergent on some region of space. We can interpret this in terms of graded algebras (or geometrically as a torus embedding). Let $R_S = F[y_1, \ldots, y_m]$ be the polynomial algebra in $m$ variables $y_i$ to which we give a $\mathbb{Z}^p$-grading by assigning the degree $a_i$ to $y_i$. For every graded vector space $V = \bigoplus_{a \in \mathbb{Z}^p} V_a$ with $\dim V_a < \infty$ for all $a$, we can define its graded Hilbert series
\[
H_V = \sum_{a \in \mathbb{Z}^p} \dim(V_a) x^a.
\]
One has to be careful when manipulating such series since in general the product of two such series makes no sense, so if we have two graded vector spaces $V, W$ with the previous restriction on graded dimensions, in general $V \otimes W$ does not satisfy this restriction. The product makes sense if we restrict to series supported on a given pointed cone. In this case we have $H_{V \otimes W} = H_V H_W$.

In general let us consider a finitely generated graded $R_S$-module $M$. Then the following lemma holds.

**Lemma 3.24.** The partition function of $S$ coincides with the Hilbert series of $R_S$. The Hilbert series of $M$ has the form
\[
H_M = \frac{p(x)}{\prod_{i=1}^m (1 - x^{a_i})}, \quad p(x) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_p^{\pm 1}]. \tag{22}
\]
That is, $p(x)$ is a finite linear combination with integer coefficients of Laurent monomials.

If $S \subset \mathbb{N}^p$ and $M$ is graded in $\mathbb{N}^p$, then the elements $p(x)$ are polynomials.

**Definition 3.25.** In [8], Berele calls a rational function of the type in formula (22) with $p(x)$ a polynomial a nice rational function.

If we set all the variables $x_i$ equal to $\rho$ in a nice rational function $H$, we have a nice rational function of $\rho$.

The order of the pole at $\rho = 1$ of this rational function will be called the dimension of $H$.

This dimension gives information on the growth of the coefficients of the generating function
\[
H(\rho) := \sum_{k=0}^\infty c_k \rho^k = \frac{p(\rho)}{\prod (1 - \rho^{h_i})}.
\]
Proposition 3.26. The function $c_k$ for $k$ sufficiently large is a quasi-polynomial, that is, a polynomial on each coset modulo the least common multiple $m$ of the integers $h_i$.

If $c_k \geq 0$ for $k \gg 0$, then the dimension (which equals the order of the pole of this function at $\rho = 1$) equals the maximum degree of these polynomials plus 1.

Proof. Let $m$ be the least common multiple of the integers $h_i$. Writing $m = h_i k_i$ for each $i$, we have

$$1 - \rho^m = (1 - \rho^{h_i}) \sum_{j=0}^{k_i-1} \rho^{h_i j}.$$ 

It follows that $H(\rho)$ can be written in the form $\frac{P(\rho)}{(1-\rho^m)^i}$ with $P(\rho)$ some polynomial. Since for $i > 0$ we have

$$\frac{\rho^m}{(1-\rho^m)^i} = \frac{1 - (1 - \rho^m)}{(1-\rho^m)^i} = \frac{1}{(1-\rho^m)^i} - \frac{1}{(1-\rho^m)^{i-1}},$$

it follows that $H(\rho)$ has an expansion in partial fractions

$$H(\rho) = \sum_{i=1}^{d} \frac{p_i(\rho)}{(1-\rho^m)^i} + q(\rho),$$

where all the $p_i(\rho)$ are polynomials of degree $< m$ in $\rho$ and $q(\rho)$ is a polynomial.

Then notice that the generating function of

$$\frac{\rho^j}{(1-\rho^m)^i} = \sum_{k=0}^{\infty} \binom{i+k-1}{i-1} \rho^{mk+j}, \quad 0 \leq j < m,$$

gives a polynomial on the coset $\mathbb{Z}m + j$, with positive values, of degree $i - 1$.

It follows that after a finite number of steps (given by the polynomial $q(\rho)$) the function $c_k$ is a polynomial on each coset, and of degree $d - 1$ on some cosets where the coefficients of $p_d(\rho)$ are different from 0. If $c_k$ is definitely positive, it follows that all the coefficients of $p_d(\rho)$ are non-negative. In particular, $p_d(1) \neq 0$ and the order of the pole of $H(\rho)$ at 1 is clearly $d$. □

3.4.2. The theory of Dahmen–Micchelli. We want to recall quickly some features of the theory of partition functions. As before, let $S$ be a list of integral vectors. We want to decompose the cone $C(S)$ into big cells and define its singular and regular points.

The singular points are defined as all linear combinations of some subset of $S$ which does not span $\mathbb{R}^p$. The regular points are the complement of the singular points.

The big cells are defined as the connected components of the open set of regular points in the cone $C(S)$.

One finally needs the notion of a quasi-polynomial on $\mathbb{Z}^p$. This is a function $f$ on $\mathbb{Z}^p$ for which there exists a subgroup $\Lambda \subset \mathbb{Z}^p$ of finite index such that the restriction of $f$ to each coset of $\Lambda$ in $\mathbb{Z}^p$ coincides with some polynomial.
In the theory of partition functions, a major role is played by a finite-dimensional space of quasi-polynomials introduced by Dahmen–Micchelli. Let us recall their theory (see [11] for a complete exposition).

- Given a list of integer vectors $S := \{a_1, \ldots, a_m\}$, $a_i \in \mathbb{Z}^p$, spanning $\mathbb{R}^p$, we call a subset $Y$ of $S$ a cocircuit if it is minimal with the property that $S \setminus Y$ does not span $\mathbb{R}^p$. The set of all cocircuits of $S$ will be denoted by $\mathcal{E}(S)$.
- With $S$ is associated a remarkable convex polytope, the zonotope $Z(S) := \{\sum_{i} \rho_i a_i \mid 0 \leq \rho_i \leq 1\}$.
- The faces of this zonotope come in opposite pairs corresponding to the cocircuits of $S$. Moreover, $Z(S)$ can be paved by parallelepipeds associated with all the bases of $\mathbb{R}^p$ which can be extracted from $S$. The volume of each of these parallelepipeds is a positive integer, the absolute value of the determinant of the corresponding basis elements.
- For a list $Y$ of integral vectors we then define the difference operator $\nabla_Y := \prod_{y \in Y} \nabla_y$, where $\nabla_y f(x) = f(x) - f(x - y)$.
- The Dahmen–Micchelli space $DM(S)$ is the space of integral-valued functions on $\mathbb{Z}^p$ satisfying the system of difference equations $\nabla_Y f = 0$, $Y \in \mathcal{E}(S)$.
- The space $DM(S)$ is a space of quasi-polynomials of degree $m - p$. It is a free Abelian group of dimension $\delta(S)$, the weighted number of bases extracted from $S$, or the volume of $Z(S)$.
- In fact if we take a generic shift $a - Z(S)$, $a \in \mathbb{R}^p$, then the set $a - Z(S) \cap \mathbb{Z}^p$ has exactly $\delta(S)$ elements and the restriction of $DM(S)$ to $a - Z(S) \cap \mathbb{Z}^p$ is an isomorphism with the space of integer-valued functions $a - Z(S) \cap \mathbb{Z}^p$.

In the following main theorem on partition functions, for which we refer to [11], it is assumed that $S$ spans $\mathbb{R}^p$.

**Theorem 3.27.** $P_S$ is supported on the intersection of the lattice $\mathbb{Z}^p$ with the cone $C(S)$.

Given a big cell $\mathfrak{c}$ of $C(S)$ and a point $a \in \mathfrak{c}$ very close to 0, one has that $a - Z(S) \cap \mathbb{Z}^p$ intersects $C(S)$ only in 0.

$P_S$ coincides on each big cell $\mathfrak{c}$ with the quasi-polynomial in the space $DM(S)$ which is 1 at 0 and equals zero at the other points of $a - Z(S) \cap \mathbb{Z}^p$.

Finally the partition function may be interpreted as counting the number of integer points in the variable convex compact polytope

$$V(b) := \left\{(\rho_1, \ldots, \rho_m) \mid \rho_i \in \mathbb{R}^+, \sum_i \rho_i a_i = b\right\}.$$ 

Thus the partition function is asymptotic to the volume of this variable polytope. This volume function, denoted by $T_S(b)$, is a spline, that is, a piecewise-polynomial function on $C(S)$, which is a polynomial of degree $m - p$ on each big cell. Again, there is a remarkable theory behind these functions; see [11].
3.4.3. U-invariants. We want to apply the previous theory to cocharacters and prove Theorem 3.33: the generating function \( \sum_{\lambda} m_{\lambda} x^\lambda \) of the multiplicities of the cocharacters is a nice rational function.

In order to prove this theorem, we need to recall some basic facts on highest-weight vectors and U-invariants, where U is the subgroup of the linear group consisting of strictly upper-triangular matrices with 1 on the diagonal.

Recall that we have seen (Proposition 3.3) that the multiplicity \( m_{\lambda} \) of a cocharacter equals the multiplicity of the space spanned by the highest-weight vectors of weight \( \lambda \) in the relatively free algebra in \( k \) variables provided that we are considering an algebra of dimension \( k \) or, more generally, an algebra which satisfies the Capelli list \( C_{k+1} \) (Remark 2.10).

By Theorem 3.1 the Hilbert series of \( R \) is a sum of the contributions of the finitely many factors \( K_{i+1}/K_i \) which are all modules over finitely generated algebras. Moreover, these are all stable under the action of the linear group \( G = GL(k) \) so that we have a decomposition into irreducible representations

\[ K_{i+1}/K_i = \bigoplus_{\lambda} S_{\lambda}(F^k)^{\oplus m_{i,\lambda}}. \]  

These multiplicities \( m_{i,\lambda} \) can be computed by taking the vector space of U-invariants

\[ (K_{i+1}/K_i)^U = \bigoplus_{\lambda} (S_{\lambda}(F^k)^U)^{m_{i,\lambda}}. \]

Since \( \dim S_{\lambda}(F^k)^U = 1 \) (this space is generated by a single vector of weight \( \lambda \)), we have that the graded Hilbert series of \( (K_{i+1}/K_i)^U \), which is \( \mathbb{N}^k \)-graded by the coordinates \( m_i \) of the dominant weights, is the generating function of the multiplicities \( m_{i,\lambda} \).

We thus have that \( (K_{i+1}/K_i)^U \) is a graded module over a graded polynomial algebra in finitely many variables \( \omega_1, \ldots, \omega_k \), and the number \( m_{i,\lambda} \) is the dimension of its component of degree \( \lambda = \sum_j n_j \omega_j \).

So our aim is to prove that \( (K_{i+1}/K_i)^U \) is finitely generated as a graded module over the polynomial algebra in the variables \( \omega_1, \ldots, \omega_k \). Then we can apply Lemma 3.24 and Theorem 3.27 which, properly interpreted, give the desired result, Theorem 3.33.

This actually is a standard fact on reductive groups. Let us explain its proof.

Let \( G \) be a reductive group in characteristic 0. Its coordinate algebra \( F(G) \) decomposes, under the left and right actions of \( G \), as \( F(G) = \bigoplus V_i \otimes V_i^* \), where \( V_i \) runs over all irreducible rational representations of \( G \).

If we fix a Borel subgroup \( B \) with unipotent radical \( U \), then the algebra \( F(G)^U = \bigoplus V_i \otimes (V_i^*)^U \) \((U \text{ acting on the right})\) is a finitely generated algebra over which \( G \) acts on the left. In fact it is generated by the irreducible representations associated with the fundamental weights.

This algebra is the coordinate ring of an affine variety \( \overline{G/U} \) which contains \( G/U \) as an open orbit.

For every subgroup \( H \) of \( G \) consider the set \( F(G)^H \) of invariants under the right action. The group \( G \) acts by the left action on \( F(G)^H \). Given a representation \( V \) of \( G \), the group \( G \) then acts diagonally on \( V \otimes F(G)^H \).
Lemma 3.28. For every representation \( V \) of \( G \) we have the equality
\[
V^H = (V \otimes F(G)^H)^G.
\]
In fact this is given by restricting to \((V \otimes F(G)^H)^G\) the explicit map
\[
\pi: V \otimes F(G) \to V, \quad \pi: v \otimes f(g) \mapsto vf(1).
\]
If \( V \) is an algebra with a \( G \)-action by automorphisms, then this identification is an isomorphism of algebras.

Proof. The space of polynomial maps from \( G \) to \( V \) is clearly \( V \otimes F(G) \), where \( v \otimes \phi \) corresponds to the map \( g \mapsto \phi(g)v \). We act on such maps by \( G \times G \): the right action is \( f^h(g) := f(gh) \) while for the left action we use \( hf(g) := h(f(h^{-1}g)) \). These two actions commute, and the left action on maps is the tensor product of the action on \( V \) and the left action on \( F(G) \), that is, if \( f = v \otimes \phi \), then we have \( hf = hv \otimes h \phi \).

A map \( f: G \to V \) is \( G \)-equivariant under the left action on \( G \) if \( f(hg) = hf(g) \). This means that \( f \) is invariant under the left action on maps. For such a map \( f \) we have \( f(g) = f(g1) = gf(1) \). Conversely, given any vector \( v \in V \), the map \( g \mapsto gv \) is \( G \)-equivariant. Thus we have a canonical identification
\[
j: (V \otimes F(G))^G \cong V, \quad j(f) := f(1) \quad \iff \quad j: v \otimes \phi \mapsto \phi(1)v.
\]
Moreover, the map \( j: (V \otimes F(G))^G \to V \) is \( G \)-equivariant if we use the right \( G \)-action on \((V \otimes F(G))^G\). Indeed, for an equivariant map \( f \) we have \( f^h(g) := f(gh) \) (induced by the right action), which maps to \( j(f^h) = f^h(1) = f(h) = hf(1) = hj(f) \).

Under this identification, take \( a = \sum_j v_j \otimes \phi_j \in V \otimes F(G)^H \) invariant under the right action of some subgroup \( H \). This means that for the corresponding map \( a: G \to V \), given by \( a(g) = \sum_j \phi_j(g)v_j \), we have \( a(gh) = a(g) \) for all \( h \in H \) and, therefore, \( a(h) = a(1) \) for all \( h \in H \). If \( a \) is \( G \)-equivariant, we see that this is equivalent to the fact that \( j(a) = \sum_j \phi_j(1)v_j \in V^H \).

As to the second statement, it is enough to observe that \( \pi \) (in formula (25)) is a homomorphism of algebras, whence so is \( j \). \( \square \)

This allows us to replace for \( G \)-modules the invariant theory for \( U \), which is not ruled by Hilbert theory since \( U \) is not reductive, by that of the reductive group \( G \) and obtain in this way the desired statements on finite generation. In fact a simple argument as in Hilbert theory shows the following.

Let \( M \) be a finitely generated module over a finitely generated commutative algebra \( A \). Assume that a linearly reductive group \( G \) acts on \( M \) and \( A \) by automorphisms in a compatible way, that is, \( g(am) = g(a)g(m), a \in A, m \in M \).

Lemma 3.29. The space of invariants \( M^G \) is finitely generated as a module over the finitely generated algebra \( A^G \).

Proof. Consider the \( A \)-submodule \( AM^G \) of \( M \). It is finitely generated by some elements \( m_1, \ldots, m_k \in M^G \). Thus, if \( u \in M^G \), we have \( u = \sum_i a_im_i, a_i \in A \). Now the map \( A^k \to M \) given by \( \sum_i a_im_i \) is \( G \)-equivariant, so it commutes with the projection to the invariants (in \( A \) called the Reynolds operator \( R \)), so \( u = \sum_i R(a_i)m_i, R(a_i) \in A^G \). Hence \( M^G \) is generated over \( A^G \) by the elements \( m_i \). \( \square \)
At this point we only have to apply the theory of graded modules over graded algebras (here the grading is by the semigroup of dominant weights, which can be identified with \( \mathbb{N}^k \)) and use the fact that the algebra \( F(G)^G \) is finitely generated by elements which have as weight the fundamental weights. Thus we deduce the following theorem.

**Theorem 3.30.** If \( V \) is a finitely generated module over a finitely generated algebra \( A \) with \( G \)-action, then the space \( V^G = (V \otimes F(G)^G)^G \) is a finitely generated module over the finitely generated algebra \( A^G = (A \otimes F(G)^G)^G \).

**Proof.** We have that \( V \otimes F(G)^U \) is a finitely generated module over the finitely generated algebra \( A \otimes F(G)^U \). The module structure is compatible with the diagonal \( G \)-action. Hence the claim follows from Lemma 3.29. \( \square \)

**Corollary 3.31.** If \( V \) and \( A \) are as before and \( V = \bigoplus_\lambda m_\lambda S_\lambda(F^k) \), then the generating function \( \sum_\lambda m_\lambda \rho^\lambda \) is a rational function in the variables \( \rho_i := \rho^{\omega_i} \), and its denominator is a product of factors of type

\[
1 - \rho^{\mu_i} = 1 - \rho^{\sum_i m_i \omega_i} = 1 - \prod_i \rho_i^{m_i}.
\]

The elements \( \mu_i = \sum_i m_i \omega_i, m_i \in \mathbb{N} \), are dominant weights of some finite set of irreducible representations whose highest weights generate \( A^U \) as algebra.

**Proof.** If \( V = \bigoplus_\lambda m_\lambda S_\lambda(F^k) \), we have \( V^U = \bigoplus_\lambda m_\lambda S_\lambda(F^k)^U \) and \( S_\lambda(F^k)^U \) is one-dimensional and is generated by a vector of weight \( \lambda \). So \( \sum_\lambda m_\lambda \rho^\lambda \) is the Hilbert series of the graded module \( V^U \) (graded by dominant weights). Then compute the generating function of \( V^U \) using its identification with \( (A \otimes F(G)^U)^G \) and then apply Lemma 3.24 and Lemma 3.29. We only need to remark that the torus \( T \) acts on \( (A \otimes F(G)^U)^G \) by acting on \( F(G) \) on the right and that the weight is preserved under this identification. \( \square \)

We will apply the previous theory in the case when \( V, A \) are graded and the action of \( GL(k) \) on \( V_m \) and \( A_m \) is a polynomial action of degree \( m \) (in particular, no inverse of the determinant appears).

Under these hypotheses the action of the torus of diagonal matrices \( a_1, \ldots, a_k \) determines the weight decomposition, hence also the multi-grading and the grading.

The fundamental weight \( \rho_i := \omega_i = a_1 a_2 \cdots a_i \) has degree \( i \). So the ordinary Hilbert series \( \sum_n \dim V_n^U \) of the space \( V^U \) is given by the substitution \( \rho_i := \rho^{\omega_i} \mapsto \rho^i \), so that \( \rho \sum_i m_i \omega_i \mapsto \rho \sum_i m_i \).

Notice that in degree \( n \) the dimension of \( V_n^U \) equals the length or the number of irreducible components into which \( V_n \) decomposes.

**Corollary 3.32.** The length of \( V_n \) is a nice rational function whose numerator is a polynomial in \( \mathbb{Z}[\rho] \) and whose denominator is a product of factors of type \( 1 - \rho^{\mu_i} \).

### 3.4.4. Cocharacters

We want to apply the previous theory to cocharacters. Let \( A \) be a PI-algebra satisfying a Capelli identity (or rather a Capelli list) \( C_m \). By Kemer’s theorem, it is in fact PI-equivalent to a finite-dimensional algebra.

We then have that the cocharacter \( \chi_k = \sum_{|\lambda| = k, \mu(\lambda) < m} n_{k, \lambda} \lambda \chi_\lambda \) is a sum of irreducible characters \( \chi_\lambda \) associated with partitions of height \( < m \).
Consider the generating function \( \sum_k \sum_{\lambda \vdash k} n_{\lambda} \rho^\lambda \). We write
\[
\rho^\lambda = \prod_{i=1}^{m-1} \rho_i^{n_i} := \rho^n, \quad n := (n_1, \ldots, n_{m-1}),
\]
where \( n_i \) equals the number of columns of \( \lambda \) of length \( i \).

**Theorem 3.33.** The generating function \( \sum_{\lambda} n_{\lambda} \rho^\lambda \) of the multiplicities of the cocharacters is a nice rational function (in the sense of Definition 3.25), that is, it has the form
\[
H_{\text{co}} = \frac{p(\rho)}{\prod_{i=1}^{m} (1 - \rho^{n_i})}, \quad p(\rho) \in \mathbb{Z}[\rho_1, \ldots, \rho_{m-1}]. \tag{26}
\]

The generating function of the colengths is also a nice rational function (\( \mathcal{R} \)).

**Proof.** From what we have seen, \( n_{\lambda} \) is the multiplicity of the irreducible representation \( S_{\lambda}(X) \) in the relatively free algebra \( \mathcal{F}_A(X) \) associated with \( A \). We know that this multiplicity equals the multiplicity of the space \( S_{\lambda}(X)^U \) (Remarks 2.8 and 2.10), which equals \( S_{\lambda}(\{x_1, \ldots, x_{m-1}\})^U \). Now we have for the free algebra in \( m - 1 \) variables the canonical filtration, where each \( K_i/K_{i-1} \) is a GL(\( k \))-module, so it has a generating series of cocharacters \( H_{\text{co}}^i \) and \( H_{\text{co}} = \sum_i H_{\text{co}}^i \). Furthermore, \( K_i/K_{i-1} \) satisfies the hypotheses of Theorem 3.30 with \( V = K_i/K_{i-1} \) and \( A = T_i \). So for each \( K_i/K_{i-1} \) we may apply Corollary 3.31, giving a contribution to formula (26) of the same type.

For the colength we apply Corollary 3.32. \( \square \)

**Remark 3.34.** In principle, the rational function describing the generating series of the cocharacters contains all the information on the codimension. The series of codimensions is obtained from the series of cocharacters by the formal linear substitution of each monomial \( \rho^a \) by \( \chi_a(1)\rho^{|a|} \). The properties of this linear map on the space of power series which are expressed by rational functions may be worthy of further investigation.

### 3.5. Invariants of several copies of \( V \).

In §3.5.1 we shall deduce some precise estimates on the dimension (see Definition 3.25) of the rational functions expressing cocharacters and colength for a fundamental algebra. We need some general facts first. Let us ask the following question. Let \( V \) be a vector space of some dimension \( k \) and \( G \) a semisimple group acting on \( V \). The invariants of \( m \) copies \( V^m \) under the action of \( G \) are also a representation of GL(\( m, F \)). In fact from Cauchy’s formula we have
\[
S[(V^*)^m] = \bigoplus_{\lambda} S_{\lambda}(V^*) \otimes S_{\lambda}(F^m) \quad \implies \quad S[(V^*)^m]^G = \bigoplus_{\lambda} S_{\lambda}(V^*)^G \otimes S_{\lambda}(F^m).
\]

We are interested in understanding the multiplicity with which a given representation \( S_{\lambda}(F^m) \) appears. First, it may appear only if the height of \( \lambda \) does not exceed \( \dim V \). We know that if the height of \( \lambda \) does not exceed \( \dim V \), this multiplicity equals \( \dim S_{\lambda}(V)^G \) provided that \( m \) is larger than the height of \( \lambda \).
Thus this multiplicity stabilizes for \( m \geq \dim V = k \) and we are reduced to compute it for \( m = k \). If \( U \) is the unipotent subgroup of strictly upper-triangular matrices in \( SL(k, F) \), we have

\[
S[(V^*)^k]^{G \times U} = \bigoplus_{\lambda} S_{\lambda}(V^*)^G \otimes S_{\lambda}(F^k)^U, \quad \dim S_{\lambda}(F^k)^U = 1.
\]

Thus the generating function of these multiplicities is the generating function of \( S[(V^*)^m]^{G \times U} \).

In particular, for the growth we need to compute the dimension of the algebra \( S[(V^*)^m]^{G \times U} \). We compute it as follows. From §3.4 we have

\[
S[(V^*)^k]^{G \times U} = (S[(V^*)^k] \otimes F(SL(k))^U)^{G \times SL(k)}.
\]  

**Theorem 3.35.** If \( G \) acts faithfully on \( V \) and \( G \times SL(k) \) acts freely on a non-empty open subset of the variety \( V^k \times SL(k)/U \), then the dimension of \( S[(V^*)^k]^{G \times U} \) is

\[
\frac{k^2 + k}{2} - \dim G.
\]

**Proof.** By formula (27) this dimension is the dimension of the quotient variety of \( G \times SL(k) \) acting on the variety \( V^k \times W \), where \( W \) is the variety with coordinate ring \( F(SL(k))^U \). This variety contains \( SL(k)/U \) as an open dense subset. Hence its dimension is \((k^2 - 1) - \frac{k^2 - k}{2} = \frac{k^2 + k - 2}{2}\).

The group \( SL(k, F) \) is semisimple and simply connected. So by a theorem of Popov (see [21], Corollary of Proposition 1) the coordinate ring of \( SL(k, F) \) is factorial. Then, since \( U \) is a connected group, the ring \( F(SL(k))^U \) is factorial and, therefore, the variety \( X := V^k \times W \) is factorial.

For a semisimple group \( H \) acting on an irreducible affine variety \( X \) which is also factorial and with the generic orbit equal to \( H \) (or just with finite stabilizer) one knows by another theorem of Popov [20] that the generic orbit is closed. This implies, by Hilbert’s theory, that the generic orbit equals the generic fibre of the quotient map and hence the quotient variety has dimension \( \dim X - \dim H \).

These hypotheses are satisfied in our case and we have

\[
\dim X = \dim V^k + \dim SL(k)/U = k^2 + \frac{k^2 + k - 2}{2}, \quad \dim H = \dim G + k^2 - 1
\]

since \( H = G \times SL(k) \). The formula follows. \( \square \)

We see from this formula that the strong hypotheses of Theorem 3.35 can hold only for somewhat small \( G \). Hence the following criterion is useful.

**Proposition 3.36.** If \( G \) acts faithfully on a space \( V \) of dimension \( k \) and the generic stabilizer of the action of \( G \) on \( V \) is a torus, then \( G \times SL(k) \) acts freely on a non-empty open subset of \( V^k \times SL(k)/U \) and \( G \times U \) acts freely on a non-empty open subset of \( V^k \).

**Proof.** In fact let us look at the stabilizer of \((v_1, \ldots, v_k) \). It is the subgroup of \( G \times U \) stabilizing \((v_1, \ldots, v_k) \). Then, for a generic choice of \((v_1, \ldots, v_k) \), this is
the stabilizer of a generic orbit of $G \times U$ on $V^k$. So it is enough to show that this is trivial.

The action of $(g, u)$ on a vector $(v_1, \ldots, v_k)$ is the action of $u$ on $(gv_1, \ldots, gv_k)$. Moreover $u = 1 + \Lambda$ is some triangular matrix with $\lambda_{j,i} = 0$ unless $j < i$, so finally

$$(g, u)(v_1, \ldots, v_k) = (w_1, \ldots, w_k), \quad w_i = gv_i + \sum_{j: j < i} \lambda_{j,i}gv_j. \quad (29)$$

Hence if $(g, u)$ stabilizes the vector $(v_1, \ldots, v_k)$, we must have $v_i = w_i$ for all $i$.

In particular, $v_1 = gv_1$, $v_2 = gv_2 + \lambda v_1$. So since $v_1$ is generic, $g$ is a semisimple element, being in a torus. Decompose $V$ into the space of invariants and a stable complement: $V = V^g \oplus V_g$ and, for all $i$, write $v_i = a_i + b_i$, $a_i \in V^g$, $b_i \in V_g$. Notice that $v_1 = a_1 \in V^g$.

Thus consider $a_2 + b_2 = v_2 = gv_2 + \lambda v_1 = a_2 + gb_2 + \lambda a_1$. This implies that $b_2 - gb_2 = \lambda a_1 \in V_g \cap V^g = \{0\}$. Hence $b_2 - gb_2 = 0$, but then $b_2 \in V^g$ and, therefore, $b_2 = 0$.

Since $b_2$ is generic in $V_g$, this implies that $V_g = 0$, that is, $g = 1$. Thus we deduce from formula (29) that $\sum_{j: j < i} \lambda_{j,i}v_j = 0$ for all $i$.

Then, since $(v_1, \ldots, v_k)$ are generic, they are linearly independent and this implies that $\lambda_{j,i} = 0$ and also $u = 1$. □

3.5.1. Colenlength. We want now to investigate the colength of $R = \bigoplus_{n=0}^{\infty} R_n$, where $R$ is the relatively free algebra, in $k$ variables, of some finite-dimensional algebra. By definition the colength is the function $\ell(R_n)$ of $n$ which measures the number of irreducible representations of $GL(k, F)$ decomposing the part $R_n$ of degree $n$. If $k$ is larger than the degree $m$ of a Capelli list satisfied by $A$, we also know that the colength stabilizes.

We need some preliminaries. Let $\overline{A} = \bigoplus_{i=1}^{q} M_{n_i}(F)$ be a semisimple algebra, $t = \dim \overline{A}$, and $G = \prod_{i=1}^{q} G_i = \prod_{i=1}^{q} \text{PSL}(n_i)$ its automorphism group. Denote by

$$S_{\overline{A}} := S[(\overline{A}^t)^t]_G = \bigotimes_{i=1}^{q} S[\left(\left(M_{n_i}(F^*)^t\right)^t\right]^{G_i} = \bigotimes_{i=1}^{q} T_{n_i}(t) \quad (30)$$

the ring of invariants of $t$ copies of $\overline{A}$ under $G$. The action of $\text{SL}(t, F)$ on $S_{\overline{A}}$ is induced by its action on $\overline{A} = \overline{A} \otimes_F F^t$.

If $U$ is the unipotent group of strictly upper-triangular matrices in $\text{SL}(t, F)$, then the algebra $S_{\overline{A}}^U$ comes from formula (27) for $V = \overline{A} = \bigoplus_{i=1}^{q} M_{n_i}(F)$ under the group $G = \prod_{i} \text{PSL}(n_i)$, which acts faithfully on $\overline{A}$, and it is semisimple. Moreover, a generic element $u$ of $\overline{A}$ is a list of matrices, each with distinct eigenvalues, so that the stabilizer of $u$ in $G$ is a product of maximal tori. We have thus verified all the hypotheses of Proposition 3.36. We can thus apply Theorem 3.35 and then the formula for the dimension of $S_{\overline{A}}^U$ is given by formula (28), where $k = \dim \overline{A} = t = \sum_{i} n_i^2$ while $\dim G = \sum_{i=1}^{q} (n_i^2 - 1) = t - q$.

**Proposition 3.37.** If $A = \bigoplus_{i=1}^{q} M_{n_i}(F)$, $G = \prod_{i} \text{PSL}(n_i)$ the dimension of the algebra $S_{\overline{A}}^U$ in formula (27) equals $\frac{t^2 - t}{2} + q$. 
Proof. Here we apply formula (28) with \( k = t \) and \( \dim G = t - q \). □

We now want to apply this to \( F_A \), the relatively free algebra of a fundamental algebra \( A \), and the trace ring \( T_A \).

**Proposition 3.38.** If \( F_A \) is the relatively free algebra of a fundamental algebra \( A \) with \( \overline{A} = \bigoplus_{i=1}^{q} M_{n_i}(F) \), then the dimension of the rational function of colength is

\[
\frac{t^2 - t}{2} + q, \quad t = \sum_{i=1}^{q} n_i^2.
\]

**Proof.** The proposed dimension in formula (31) is the dimension of the colength function of \( S_{\overline{A}} \) (see (30)). By Lemma 3.12 the ring of invariants \( S_{\overline{A}} \) is a finite (torsion-free) module over the trace ring \( T_A \). So, by Theorem 3.30, \( S_{\overline{A}}^U \) is a finite (torsion-free) module over \( T_A^U \) and hence the dimension of \( T_A^U \) equals that of \( S_{\overline{A}}^U \).

Since \( T_A F_A \) is a finite torsion-free module over \( T_A \), Theorem 3.30 yields that \( (T_A F_A)^U \) is a finite torsion-free module over \( T_A^U \), whence they have the same dimension. But also \( F_A \) contains a \( T \)-ideal \( K_A \) which is an ideal for \( T_A F_A \). Hence \( K_A^U \subset F_A^U \) is an ideal for \( (T_A F_A)^U \). It follows that \( K_A^U, F_A^U, (T_A F_A)^U, T_A^U \) all have the same dimension, equal to that of \( S_{\overline{A}} \) and given by formula (31). □

For a general relatively free algebra \( R \), using the standard filtration (Theorem 3.1), we have that the colength of \( R \) is the sum of the colengths of the factors \( K_{i+1}/K_i \) and we have seen that the generating function \( H_{\ell(R)} := \sum_{n=0}^{\infty} \ell(R_n)\rho^n \) is a nice rational function with denominator a product of the factors \( 1 - \rho^{d_i} \). So for large \( n \), the colength \( \ell(R_n) \) is a quasi-polynomial of degree \( d - 1 \), where \( d \) is the order of the pole of \( H_{\ell(R)} \) at \( \rho = 1 \). It is then equal to the maximum of the dimensions of the colength functions for the factors \( K_{i+1}/K_i \).

Each of these factors \( K_{i+1}/K_i \) is a finite module over some finitely generated algebra \( T_{\overline{R}_i} \subset \bigoplus_j T_{A_j} \), where \( \overline{R}_i \) (the quotient of \( R_i \) mentioned above) is the relatively free algebra associated with an algebra \( A = \bigoplus_j A_j \), a direct sum of the fundamental algebras \( A_j \), all with the same Kemer index equal to the Kemer index of \( R_i \). The algebra \( T_{\overline{R}_i} \subset \bigoplus_j T_{A_j} \) is the coordinate ring of a union of the varieties \( W_{\overline{A}_j} \) associated with the fundamental algebras \( A_j \). Moreover, \( K_{i+1}/K_i \subset \bigoplus_j (K_{i+1}/K_i)_j \), and each \( (K_{i+1}/K_i)_j \) is torsion-free over the corresponding \( T_{A_j} \). We thus have from Proposition 3.26 that the number \( d \) is also the maximum order of the pole of the Hilbert series of the colength for \( T_{A_j} \) and, therefore, the maximum of the dimensions of the algebras \( T_{A_j}^U \).

Let us summarize these results for \( R(m) \) a relatively free algebra in \( m \) variables satisfying some Capelli list \( C_{k+1} \). Denote the quotients of the standard filtration by \( \overline{R}_i \).

**Theorem 3.39.** The generating function of the colength of \( R(m) \) is a nice rational function which stabilizes for \( m \geq k \).

When \( m \geq k \), this rational function has dimension \( \max \left( \frac{t_i^2 - t_i}{2} + q_i \right) \), where \( t_i \) is the first Kemer index of \( \overline{R}_i \) while \( q_i \) is its \( q \)-invariant (see Definition 3.22).

**Proof.** The only thing which requires some proof concerns the dimension. The dimension of a nice rational function \( \sum_i c_i \rho^i \) associated with a generating sequence
\(c_i > 0\) is the order of the pole at \(\rho = 1\). Hence one easily has from Proposition 3.26 that the dimension of a sum of several nice rational functions of the form \(\sum c_i \rho^i\), \(c_i > 0\), equals the maximum of their dimensions. In our case one has to compute the maximum arising from the colength in the standard filtration and finally the argument is, using the previous discussion for fundamental algebras, like that in Theorem 3.21. \(\square\)

§ 4. Model algebras

4.1. The canonical model of fundamental algebras. We now want to discuss the problem of choosing a canonical fundamental algebra in each PI-equivalence class. In Corollary 3.15 we saw that PI-equivalent fundamental algebras have isomorphic semisimple parts. Thus it is natural to study fundamental algebras \(A\) with a given fixed semisimple part \(\overline{A} = \bigoplus_{i=1}^{q} M_{n_i}(F)\) so that \(\beta(A) = t = \sum_{i=1}^{q} n_i^2\).

Remark 4.1. From Lemma 2.23 it follows that the second Kemer index \(\gamma(A)\), which for a fundamental algebra coincides with the maximum \(s\) for which \(J^s \neq 0\), must be \(\geq q - 1\). The case \(\gamma(A) = q - 1\) is attained by all possible algebras of upper-triangular matrices with semisimple part \(\overline{A}\).

So our present goal is to analyze fundamental algebras with index \((\overline{A}, s)\) (see Definition 3.16), where \(s \geq q - 1\) (and thus with Kemer index \(t = \dim \overline{A}, s\)). We now use Definition 2.26 and Proposition 2.27.

Consider the \(T\)-ideal \(I\) of identities of a fundamental algebra \(A = \overline{A} \oplus J\). We saw in Corollary 3.15 that the semisimple part \(\overline{A}\) is determined by \(I\).

We want to construct a canonical fundamental algebra having semisimple part \(\overline{A}\) and \(I\) as the \(T\)-ideal of identities. This algebra is constructed as in § 2.2.

First we construct a universal object. Take the free product \(\overline{A} \star F(X)\) for \(X = \{x_1, \ldots, x_m\}\). We assume that \(m \geq s\). We now take this free product modulo the ideal of elements of degree \(\geq s + 1\) in the variables \(X\). Call the resulting algebra \(\mathcal{F}_{\overline{A},s}(X)\). This is a finite-dimensional algebra with semisimple part \(\overline{A}\) and Jacobson radical \(J\) of nilpotency \(s + 1\) generated by the elements \(x_i\). It satisfies a universal property among such algebras.

Remark 4.2 (universal property). Given a finite-dimensional algebra \(A\) with semisimple part \(\overline{A}\) and Jacobson radical \(J\) of nilpotency \(s + 1\), any map \(X \rightarrow J\) extends to a unique homomorphism of \(\mathcal{F}_{\overline{A},s}(X)\) to \(A\) which is the identity on \(\overline{A}\).

4.1.1. Description of \(\mathcal{F}_{\overline{A},s}(X)\). Let \(V\) be the vector space with basis the elements \(x_i\). In degree \(h\) the algebra \(\mathcal{F}_{\overline{A},s}(X) = \mathcal{F}_{\overline{A},s}(V)\) can be described as follows.

Definition 4.3. Let \(M\) be the monoid in two generators \(a, b\) with relation \(a^2 = a\).

The elements of this monoid correspond to words in \(a, b\) in which \(aa\) never appears as a subword. When we multiply two such words and have a factor \(aa\) appearing, we reduce it to \(a\) by the rule \(a^2 = a\).

Now with such a word \(w\) we associate a tensor product \(T_w\) of the factors \(\overline{A}\) whenever we have an \(a\) and \(V\) when we have a \(b\). For example,

\[w = abbab \mapsto T_w = \overline{A} \otimes V \otimes V \otimes \overline{A} \otimes V.\]
The multiplication of two such words $w_1w_2$ according to the previous rule induces a multiplication $T_{w_1}T_{w_2} \subset T_{w_1w_2}$. We take the corresponding tensor product and if we get a factor $\overline{A} \otimes \overline{A}$, then we replace it by $\overline{A}$ by multiplication.

Then we see that $\mathcal{F}_{\overline{A},s}(X)$ is the direct sum $\bigoplus T_w$, where $w$ runs over all words of the previous type with at most $s$ occurrences of $b$. It is a graded algebra over this monoid, truncated at degree $s$ in $b$.

As a representation of $\text{GL}(m,F) \times G = \text{GL}(V) \times G$, the algebra $\mathcal{F}_{\overline{A},s}(V)$ in degree $h$ is a direct sum of $c_{i,j}$ spaces, each isomorphic to $\overline{A}^{\otimes i} \otimes V^{\otimes j}$, where $c_{i,j}$ is the number of words in $M$ with $a$ appearing $i$ times and $b$ appearing $j$ times. The summands correspond to the types of elements in the free product which are monomials in $j$ elements of $V$ and $i$ elements of $\overline{A}$.

**Corollary 4.4.** Having fixed $s$, the $\text{GL}(V)$-invariant subspaces of $\mathcal{F}_{\overline{A},s}(V)$ all intersect $\mathcal{F}_{\overline{A},s}(V_s)$, where $V_s$ is the subspace of dimension $s$ spanned by the first $s$ variables.

**Proof.** This is a property of each $V^\otimes h$, $h \leq s$. A $\text{GL}(V)$-invariant subspace is generated by its highest-weight vectors which in $V^\otimes h$ depend on the first $h$ elements of a chosen basis of $V$ (the variables). $\square$

This corollary gives another justification for restricting our analysis to the algebras $\mathcal{F}_{\overline{A},s}$ (with $X$ a set of $s$ variables).

**Proposition 4.5.** When $s \geq q - 1$, we have that $\mathcal{F}_{\overline{A},s}$ is a fundamental algebra with index $(\overline{A}, s)$ (see Definition 3.16).

**Proof.** This is done by following the proof of Kemer’s lemma and proving the existence of Kemer polynomials. We just sketch the proof. If $\overline{A} = \bigoplus_{i=1}^d M_{n_i}(F)$, take for each $i$ a product $P_i$ of $k+s$ Capelli polynomials in $n_i^2$ variables $X_{i,j}$, $i = 1, \ldots, q$, $j = 1, \ldots, k+s$. When evaluated in suitable matrix units, each $P_i$ is non-zero.

Take the product $M_0P_1M_1P_2\cdots M_qP_qM_q$ of all these polynomials, where the $M_i$ are monomials in $s$ variables $z_1, \ldots, z_s$ so that each variable appears once, and $M_i$ for $i = 1, \ldots, q_1$ has degree at least 1. Here we use the hypothesis $s \geq q - 1$.

Then form $k$ layers $X_j := \bigcup_i X_{i,j}$, $j = 1, \ldots, k$, with $t = \sum_i n_i^2$ elements, and $s$ layers $Z_j := z_j \bigcup_i X_{i,j+k}$, $j = 1, \ldots, s$, with $t + 1$ elements.

Then alternate all these layers independently, getting a polynomial $f$. One may verify, by looking carefully at the description of $\mathcal{F}_{\overline{A},s}$, that by evaluating $z_j$ in the elements $x_j$ and the other variables in matrix units, one can construct non-zero evaluations for $f$. Thus $f$ is a Kemer polynomial and gives the Kemer index $t, s$. $\square$

**Definition 4.6.** The algebra $\mathcal{F}_{\overline{A},s}$ is the universal fundamental algebra for the pair $\overline{A}, s$.

Remark that $\mathcal{F}_{\overline{A},s}$ is defined for all $s$, but only if $s \geq q - 1$ do we have that $\mathcal{F}_{\overline{A},s}(X)$ is fundamental by Proposition 4.5.

4.1.2. The algebra $\mathcal{A}_s(X)$ associated with a fundamental algebra $A$. Given a finite-dimensional algebra $A$ with given index $t, s$, we define $\mathcal{A}_s(X)$ to be $\mathcal{F}_{\overline{A},s}(X)$ modulo
the ideal generated by all polynomial identities of \( A \). All these polynomial identities vanish on \( \overline{A} \) and, therefore, they take values in the ideal generated by the elements \( x_i \).

Therefore the Jacobson radical \( J_s \) of \( A_s(X) \) is the ideal generated by the elements \( x_i \), and its semisimple part is \( \overline{A} \).

By construction and Remark 4.2, given any list of elements \( a_1, \ldots, a_m \in J \), there is a morphism \( \pi : A_s(X) \rightarrow A \) which is the identity on \( \overline{A} \) and maps \( x_i \mapsto a_i \).

For \( m \) sufficiently large this morphism may be chosen so that the radical \( J_s \) maps surjectively to the radical \( J \). Then the map of \( A_s(X) \) to \( A \) is also surjective, whence \( A_s(X) \) satisfies the same polynomial identities as \( A \).

Definition 4.7. We now define \( F_{\overline{A},s} \) and \( A_s \) to be the algebras \( F_{\overline{A},s}(X) \) and \( A_s(X) \), where \( X = \{ x_1, \ldots, x_s \} \) is formed by \( s \) variables.

Lemma 4.8. \( A_s \) satisfies the same identities as \( A \).

Proof. By construction, all identities of \( A \) are satisfied by \( A_s \), so we need to prove the reverse. Take a multilinear polynomial \( f \) which is not a PI of \( A \). We need to show that it does not vanish on \( A_s \).

There is a substitution of \( f \) in \( A \) which is non-zero. We may assume that it is restricted (see Definition 2.22). In this substitution at most \( s \) variables are in the radical, the others being in some matrix units of \( \overline{A} \). We now take the same substitution for matrix units in \( \overline{A} \) and substitute the remaining variables (which we may call \( y_1, \ldots, y_k, k \leq s \)) for \( x_1, \ldots, x_k \in X \subset A_s \).

By the universal property, the evaluation of \( f \) in \( A \) factors through this evaluation in \( A_s \), which is therefore different from 0. □

Lemma 4.9. Let \( B, A = B/I \) be two finite-dimensional algebras, \( J \) the radical of \( B \) and \( I \subset J \), so that \( A, B \) have the same semisimple part \( \overline{A} \).

Assume that \( A, B \) have the same nilpotency index and \( A \) is fundamental. Then \( B \) is fundamental and has the same Kemer index as \( A \).

Proof. The assumption implies that \( A, B \) have the same \( t, s \) index. By hypothesis, the Kemer index of \( A \) equals the \( t, s \) index. Now the Kemer index of \( B \), which is less than or equal to its \( t, s \) index, cannot be less than the Kemer index of \( A \). Hence the statement follows from Theorem 2.24. □

Proposition 4.10. \( F_{\overline{A},s} \) and \( A_s \) are fundamental algebras with index \( (\overline{A}, s) \).

Proof. For \( F_{\overline{A},s} \) this is the content of Proposition 4.5. Then the statement follows from Theorem 2.24 since by construction its Kemer index equals the \( t, s \) index. □

Definition 4.11. We call \( A_s \) the canonical model of \( A \).

Notice a property of this model: if \( f(y_1, \ldots, y_k) \) is a multilinear non-commutative polynomial, then in order to verify whether it is a PI of \( A_s \), one has to choose in all possible ways a subset of the variables \( y \) with \( h \leq s \) elements, evaluate these variables in the canonical elements \( x_1, \ldots, x_h \), and then evaluate all the other variables into matrix units of \( \overline{A} \). If all these finitely many evaluations are 0, then \( f \) is a PI. This follows from the universal property in Remark 4.2.
Notice also that, by construction, the nilpotent subalgebra of $\mathcal{F}_{A,s}(X)$ generated by the $x_i$ is a relatively free nilpotent algebra.

**Question.** When is the $T$-ideal of identities of $\mathcal{F}_{A,s}$ irreducible, and how can it be described?

**Remark 4.12.** The algebra $R$ of block upper-triangular matrices relative to a decomposition of a vector space $V = \bigoplus_{i=1}^q V_i$, $\dim V_i = n_i$, can be presented as a quotient of $\mathcal{F}_{A,q-1}$ as follows. $R$ is described as the direct sum $\bigoplus_{i \leq j} \text{hom}(V_i, V_j)$. Take the elements $E_{i,i+1}$ to be a non-zero matrix in the corresponding block $\text{hom}(V_i, V_{i+1})$. It is easily seen that the homomorphism mapping $\mathcal{F}_{A,q-1}$ to $R$ which is the identity on $A$ and maps $x_i \mapsto E_{i,i+1}$ is surjective.

### 4.2. Some complements.

#### 4.2.1. Generalized identities.** The algebra $\mathcal{F}_{A,s}(X)$ should be thought of as a free algebra in a suitable category, so we give the following definition.

**Definition 4.13.** Given an algebra $R$, an $R$-algebra is any associative algebra $S$ with a bimodule action of $R$ on $S$ satisfying

\[
(r(s_1 s_2) = (rs_1)s_2, \quad (s_1 s_2)r = s_1(s_2r), \quad (s_1 r)s_2 = s_1(rs_2), \quad \forall r \in R, \quad s_1, s_2 \in S.
\] (32)

Given an $R$-algebra $S$, we can give $R \oplus S$ the structure of algebra by setting

\[(r_1, s_1)(r_2, s_2) = (r_1 s_1, r_1 s_2 + s_1 r_2 + s_1 s_2).
\]

The axioms (32) are necessary and sufficient for $R \oplus S$ to be associative.

The canonical model has the following universal property. Consider a nilpotent algebra $R$ with $R^{s+1} = 0$ and equipped with an $\overline{A}$-algebra structure according to Definition 4.13. Then any map $j: X \to R$ extends to a map $\overline{j}: \mathcal{F}_{A,s}(X) \to \overline{A} \oplus R$ equal to the identity on $\overline{A}$.

If $\overline{A} \oplus R$ satisfies the PI of $A$, the map $\overline{j}$ factors through $\mathcal{F}_{\overline{A},s}(X) \to \mathcal{A}_s(X) \to \overline{A} \oplus R$.

In particular, $\mathcal{F}_{\overline{A},s}(X)$ and $\mathcal{A}_s(X)$ behave as relatively free $\overline{A}$-algebras.

The endomorphisms of $\mathcal{F}_{\overline{A},s}(X)$ (resp. $\mathcal{A}_s(X)$) which are the identity on $\overline{A}$ correspond to arbitrary substitutions of the variables $x_i$ by elements of the radical.

This gives rise to the notion of a $T$-ideal in $\mathcal{F}_{\overline{A},s}(X)$, or the ideal of generalized identities.

In particular, the action of the linear group $\text{GL}(m, F)$ on the vector space with basis the elements $x_i$, and also the automorphism group $G$ of $\overline{A}$, extend to give a group $\text{GL}(m, F) \times G$ of automorphisms of $\mathcal{F}_{\overline{A},s}(X)$ and $\mathcal{A}_s(X)$. Thus the kernel of the quotient map $\mathcal{F}_{\overline{A},s}(X) \to \mathcal{A}_s(X)$ is stable under the group $\text{GL}(m, F) \times G$ of automorphisms.

Let us now restrict to the basic case $|X| = s$. 

Given a fundamental algebra $A$ with index $(\overline{A}, s)$, the verbal ideal of $\mathcal{F}_{\overline{A}, s}$ defining the algebra $A$ is clearly a $T$-ideal $J \subset \mathcal{F}_{\overline{A}, s}$ with the property that $J$ does not contain the entire subspace of elements of degree $s$ (this is of course an open condition). But not all $T$-ideals $J \subset \mathcal{F}_{\overline{A}, s}$ with the property that $J$ does not contain the entire subspace of elements of degree $s$ are of this type. In fact $\mathcal{F}_{\overline{A}, s}/J$ need not be fundamental as even the simplest examples show (see Example 4.14).

**Example 4.14.** $\overline{A} = F$, $s = 1$. If $e$ is the identity of $F$, then the algebra $\mathcal{F}_{\overline{A}, s}$ is 5-dimensional with basis $e, x, ex, xe, exe$. This algebra is fundamental with possible Kemer polynomials $[x, y]z_1 \cdots z_\mu, z_1 \cdots z_\mu [x, y]$. It satisfies the identity $\overline{z}[x, y]w$. One can see that the verbal ideal $J$ associated with the commutative law $[x, y]$ is 2-dimensional: $[e, x], [e, ex], [e, xe] = [e, x] - [e, ex]$. Modulo $J$, this is a commutative non-fundamental algebra.

One can now parametrize $T$-ideals $J = \bigoplus_{i=1}^s J_i$ (where $J_i$ is the part of degree $i$ of $\mathcal{F}_{\overline{A}, s}$) as finitely many projective varieties according to the Hilbert series $\sum_{i=1}^s \dim J_i \rho^i$ or even the character as representations of the group $G \times GL(s, F)$, with $G$ the group of automorphisms of $\overline{A}$.

It is easily verified that, given such a graded character, the set of subspaces of $\mathcal{F}_{\overline{A}, s}$ carrying this character is a product of Grassmann varieties, and inside this variety the condition to be an ideal or a $T$-ideal is closed.

This leaves open two questions.

1. Given a $T$-ideal $J$ of $\mathcal{F}_{\overline{A}, s}$, is the condition that $\mathcal{F}_{\overline{A}, s}/J$ is fundamental open, closed or locally closed?

2. Given a $T$-ideal $J$ of $\mathcal{F}_{\overline{A}, s}$ such that the algebra $A := \mathcal{F}_{\overline{A}, s}/J$ is fundamental, we clearly have that $A$ is a quotient of the corresponding relatively free algebra $A_s$. So we ask the same question about the condition that $A = A_s$.

The final problem is thus to understand whether the set of relatively free algebras $A_s$ has some natural structure of an algebraic variety.

**4.2.2. A canonical structure of a Cayley–Hamilton algebra.** We finally make a further connection with a construction from invariant theory. We equip $\mathcal{F}_{\overline{A}, s}$ with the canonical structure of a Cayley–Hamilton algebra (see [24]) as follows.

Given $a \in \mathcal{F}_{\overline{A}, s}$, we set $t(a) := \sigma \tr(\overline{a})$, where $\overline{a} = (a_1, \ldots, a_q) \in \overline{A} = \bigoplus_{i=1}^q M_{n_i}(F)$ and we have $\tr(\overline{a}) = \sum_{i=1}^q \tr(a_i)$.

This trace $t(a)$ is in fact the trace of the semisimple representation of $\overline{A}$ where each irreducible module $F^{n_i}$ relative to the $i$th block of $\overline{A}$ appears with multiplicity $s$. The Cayley–Hamilton identity for this representation is $\text{CH}_N(x)^s$, where $\text{CH}_N(x)$ is the Cayley–Hamilton polynomial with $N = \sum_i n_i$ for the semisimple representation $\bigoplus_{i=1}^q F^{n_i}$. Then $(\text{CH}_N(x)x)^s$ is the Cayley–Hamilton polynomial for the previous representation plus the trivial $(0)$ module $F^s$. Since this polynomial has no constant term, it can be evaluated in the algebra $\mathcal{F}_{\overline{A}, s}$ without $1$.

Arguing as in formula (14), we see that with this definition of trace, the elements of $\mathcal{F}_{\overline{A}, s}$ satisfy the $(N + 1)s$ Cayley–Hamilton identity $(\text{CH}_N(x)x)^s$.

By theorems of the author [24], we then have universal embeddings $\mathcal{F}_{\overline{A}, s} \subset M_{(N+1)s}(U), A_s \subset M_{(N+1)s}(U_A)$, where $U$ and $U_A = U/I$ are commutative rings on which the projective linear group $G := \text{PGL}((N+1)s)$ acts and we have canonical
isomorphisms and a commutative diagram

\[
\begin{array}{ccc}
F_{A,s} & \xrightarrow{\sim} & M_{(N+1)s}(U)^G \\
\downarrow \pi & & \downarrow \pi \\
A_s & \xrightarrow{\sim} & M_{(N+1)s}(U_A)^G \\
\downarrow \pi & & \downarrow \pi \\
A & \xrightarrow{\sim} & M_{(N+1)s}(U_A)^G \\
\end{array}
\]

(33)

with the vertical maps surjective.

We can make a further reduction using the special property that \( U_{\overline{A}} \) is the coordinate ring of the variety of all embeddings of \( \overline{A} \) in \( M_{(N+1)s}(F) \) which are compatible with the given trace. Since \( \overline{A} \) is semisimple, all these embeddings are conjugate. This means that \( U_{\overline{A}} \) is the coordinate ring of a homogeneous space \( G/G' \), where \( G' \) is the group of automorphisms fixing a given embedding \( i: \overline{A} \to M_{(N+1)s}(F) \), so that by the double-centralizer theorem we have that \( i \) is an isomorphism to \( M_{(N+1)s}(F)^{G'} \). Since the mappings \( U \to U_A \to U_{\overline{A}} \) are \( G \)-equivariant, one can replace these rings by the coordinate ring of the fibre of \( i \), and \( G \) by \( G' \). This gives a commutative diagram

\[
\begin{array}{ccc}
F_{\overline{A},s} & \xrightarrow{\sim} & M_{(N+1)s}(U)^{G'} \\
\downarrow \pi & & \downarrow \pi \\
A_s & \xrightarrow{\sim} & M_{(N+1)s}(U_A)^{G'} \\
\downarrow \pi & & \downarrow \pi \\
A & \xrightarrow{\sim} & M_{(N+1)s}(F)^{G'} \\
\end{array}
\]

(34)

with the vertical maps surjective.

The coordinate ring \( U \) of this fibre has an explicit description as the quotient of the polynomial ring in the entries of \( s \) variables \( \xi_i \), which are matrices of size \( (N+1)s \times (N+1)s \), modulo all the relations which imply that \( F_{\overline{A},s} \to M_{(N+1)s}(U) \) is a morphism determined by the values of \( x_i \mapsto \xi_i \), extending the embedding \( i \).

Bibliography

[1] A. A. Albert, *Structure of algebras*, Amer. Math. Soc. Colloq. Publ., vol. 24, Amer. Math. Soc., New York 1939.

[2] E. Aljadeff, A. Giambruno, C. Procesi, and A. Regev, *Rings with polynomial identities and finite dimensional representations of algebras*, book in preparation.

[3] E. Aljadeff, A. Kanel-Belov, and Y. Karasik, “Kemer’s theorem for affine PI algebras over a field of characteristic zero”, *J. Pure Appl. Algebra* 220:8 (2016), 2771–2808.
[4] S. A. Amitsur, “A noncommutative Hilbert basis theorem and subrings of matrices”, *Trans. Amer. Math. Soc.* **149** (1970), 133–142.

[5] M. Artin, “On Azumaya algebras and finite-dimensional representations of rings”, *J. Algebra* **11**:4 (1969), 532–563.

[6] A. Ya. Belov, “On the rationality of Hilbert series of relatively free algebras”, *Uspekhi Mat. Nauk* **52**:2(314) (1997), 153–154; English transl., *Russian Math. Surveys* **52**:2 (1997), 394–395.

[7] A. Ya. Belov, “The Gel’fand–Kirillov dimension of relatively free associative algebras”, *Mat. Sb.* **195**:12 (2004), 3–26; English transl., *Sb. Math.* **195**:12 (2004), 1703–1726.

[8] A. Berele, “Applications of Belov’s theorem to the cocharacter sequence of p.i. algebras”, *J. Algebra* **298**:1 (2006), 208–214.

[9] G. M. Bergman and J. Lewin, “The semigroup of ideals of a fir is (usually) free”, *J. London Math. Soc.* (2) **11**:1 (1975), 21–31.

[10] W. Borho and H. Kraft, “Über die Gelfand–Kirillov-dimension”, *Math. Ann.* **220**:1 (1976), 1–24.

[11] C. De Concini and C. Procesi, *Topics in hyperplane arrangements, polytopes and box-splines*, Universitext, Springer, New York 2010.

[12] C. De Concini and C. Procesi, *Invariant theory of matrices*, book in preparation.

[13] S. Donkin, “Invariants of several matrices”, *Invent. Math.* **110**:2 (1992), 389–401.

[14] V. Drensky, *Free algebras and PI-algebras. Graduate course in algebra*, Springer-Verlag, Singapore 2000.

[15] V. Drensky and E. Formanek, *Polynomial identity rings*, Adv. Courses Math. CRM Barcelona, Birkhäuser Verlag, Basel 2004.

[16] I. M. Gel’fand and A. A. Kirillov, “Fields associated with enveloping algebras of Lie algebras”, *Dokl. Akad. Nauk SSSR* **167**:3 (1966), 503–505; English transl., *Soviet Math. Dokl.* **7** (1966), 407–409.

[17] A. Giambruno and M. Zaicev, “Exponential codimension growth of PI algebras: an exact estimate”, *Adv. Math.* **142**:2 (1999), 221–243.

[18] A. Giambruno and M. Zaicev, “Minimal varieties of exponential growth”, *Adv. Math.* **174**:2 (2003), 310–323.

[19] A. R. Kemer, *Ideals of identities of associative algebras*, Diss. Doct. Phys.-Math. Sci., Barnaul 1988; English transl., Transl. Math. Monogr., vol. 87, Amer. Math. Soc., Providence, RI 1991.

[20] V. L. Popov, “Stability criteria for the action of a semisimple group on a factorial manifold”, *Izv. Akad. Nauk SSSR Ser. Mat.* **34**:3 (1970), 523–531; English transl., *Math. USSR-Izv.* **4**:3 (1970), 527–535.

[21] V. L. Popov, “Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector bundles”, *Izv. Akad. Nauk SSSR Ser. Mat.* **38**:2 (1974), 294–322; English transl., *Math. USSR-Izv.* **8**:2 (1974), 301–327.

[22] C. Procesi, “The invariant theory of $n \times n$-matrices”, *Advances in Math.* **19**:3 (1976), 300–381.

[23] C. Procesi, “Computing with $2 \times 2$ matrices”, *J. Algebra* **87**:2 (1984), 342–359.

[24] C. Procesi, “A formal inverse to the Cayley–Hamilton theorem”, *J. Algebra* **107**:1 (1987), 63–74.

[25] C. Procesi, *Lie groups. An approach through invariants and representations*, Universitext, Springer, New York, NY 2007.

[26] Yu. P. Razmyslov, “Trace identities of full matrix algebras over a field of characteristic zero”, *Izv. Akad. Nauk SSSR Ser. Mat.* **38**:4 (1974), 723–756; English transl., *Math. USSR Izv.* **8**:4 (1974), 724–760.
[27] Yu. P. Razmyslov, *Identities of algebras and their representations*, Nauka, Moscow 1989; English transl., Transl. Math. Monogr., vol. 138, Amer. Math. Soc., Providence, RI 1994.

[28] A. Regev, “Existence of identities in $A \otimes B$”, *Israel J. Math.* 11:2 (1972), 131–152.

[29] A. Regev, *Growth for algebras satisfying polynomial identities*, preprint, 2014.

[30] N. Roby, “Lois polynômes et lois formelles en théorie des modules”, *Ann. Sci. École Norm. Sup.* (3) 80 (1963), 213–348.

[31] N. Roby, “Lois polynômes multiplicatives universelles”, *C. R. Acad. Sci. Paris Sér. A-B* 290:19 (1980), A869–A871.

[32] L. H. Rowen, *Polynomial identities in ring theory*, Pure Appl. Math., vol. 84, Academic Press [Harcourt Brace Jovanovich, Publishers], New York–London 1980.

[33] A. I. Shirshov, “On rings with identities”, *Mat. Sb.* 43(85):2 (1957), 277–283. (Russian)

[34] F. Vaccarino, “Generalized symmetric functions and invariants of matrices”, *Math. Z.* 260:3 (2008), 509–526.

[35] D. Ziplies, “Abelianizing the divided powers algebra of an algebra”, *J. Algebra* 122:2 (1989), 261–274.

[36] A. N. Zubkov, “A generalization of the Razmyslov–Procesi theorem”, *Algebra i Logika* 35:4 (1996), 433–457; English transl., *Algebra and Logic* 35:4 (1996), 241–254.

[37] K. A. Zubrilin, “On the largest nilpotent ideal in algebras that satisfy Capelli identities”, *Mat. Sb.* 188:8 (1997), 93–102; English transl., *Sb. Math.* 188:8 (1997), 1203–1211.

Claudio Procesi
Mathematics Department,
University of Rome ‘La Sapienza’, Italy
E-mail: procesi@mat.uniroma1.it

Received 1/AUG/15
Edited by A. V. DOMRIN