Probability representation of quantum observable and quantum states

V. N. Chernega¹, O. V. Man’ko¹,², V. I. Man’ko¹,³

¹ - Lebedev Physical Institute, Russian Academy of Sciences
Leninskii Prospect 53, Moscow 119991, Russia
² - Bauman Moscow State Technical University
The 2nd Baumanskaya Str. 5, Moscow 105005, Russia
³ - Moscow Institute of Physics and Technology (State University)
Institutskii per. 9, Dolgoprudnyi, Moscow Region 141700, Russia
Corresponding author e-mail: manko@sci.lebedev.ru

Abstract

We introduce the probability distributions describing quantum observables in conventional quantum mechanics and clarify their relations to the tomographic probability distributions describing quantum states. We derive the evolution equation for quantum observables (Heisenberg equation) in the probability representation and give examples of the spin-1/2 (qubit) states and the spin observables. We present quantum channels for qubits in the probability representation.

Keywords: quantum supramatism, probability representation, quantum observables, qubit states, Heisenberg evolution equation.

1 Introduction

In conventional quantum mechanics, quantum states are identified either with wave functions (pure states) [1] or with density matrices (mixed states) [2, 3]. The observables like positions or momenta as well as spin variables are associated with Hermitian operators [4, 5, 6] acting in Hilbert spaces. Other formulations of the quantum-system states associating with the states the functions on the phase space like the Wigner function [7], the Husimi–Kano function [8, 9], and the Glauber–Sudarshan function [10, 11] have been
developed to obtain the formulation of quantum states more similar to the formulation of the states in classical statistical mechanics.

Recently, the tomographic probability representation of quantum states was suggested \[\text{[12]}\]; in this representation, the quantum states are identified with fair probability distributions connected with density matrices in its phase-space representations by integral transforms; e.g., the Radon transform \[\text{[13]}\] of the Wigner function provides the optical tomogram \[\text{[14, 15]}\], which is a standard probability distribution of continuous homodyne quadrature of photon depending on an extra parameter called the local oscillator phase, which can be measured \[\text{[16]}\].

The probability distributions determining the spin states were considered in \[\text{[17, 18, 19, 20, 21, 22, 23, 24]}\], and the tomographic probability representation of quantum states was studied in \[\text{[25, 26, 27, 28, 29, 30, 31, 32, 33, 34]}\].

The tomographic probabilities identified with quantum states can be associated with density operators, in view of the formalism of star-product quantization \[\text{[35, 36, 37, 38, 39]}\] analogous to the procedure where the phase-space quasidistributions of quantum states, like the Wigner function, are presented within the star-product framework in \[\text{[40]}\] (see also recent reviews \[\text{[41, 42]}\]). On the other hand, quantum observables associated with Hermitian operators are presented within the star-product framework by symbols of the operators, which are some functions on the phase space, say, in the Wigner–Weyl representation or the functions of discrete variables in the spin-tomographic description of qudit states.

The aim of this work is to extend the probability representation of quantum states to describe the quantum observables in conventional quantum mechanics by fair probability distributions depending on extra parameters. Formally, we address the problem of constructing the invertible map of Hermitian matrices (not only nonnegative trace-class ones) onto sets of probability distributions depending on random variables and extra parameters. We construct such probability representation of quantum observables for systems with finite-dimensional Hilbert spaces of states which are spin-1/2 systems or systems of qubits.
2 Probability Representation for Qubit Observables

As was shown in [43, 44], the arbitrary qubit density $2 \times 2$ matrix $\rho$ can be presented in the form

$$\rho = \begin{pmatrix} p_3 & p^* - \gamma^* \\ p - \gamma & 1 - p_3 \end{pmatrix}, \quad \gamma = \frac{1 + i}{2}, \quad p = p_1 + ip_2, \quad p_3^* = p_3, \quad (1)$$

where $0 \leq p_1, p_2, p_3 \leq 1$ are the probabilities to have $m = +1/2$ spin projections on directions $x, y, z$, respectively. The three probabilities must satisfy the inequality

$$(p_1 - 1/2)^2 + (p_2 - 1/2)^2 + (p_3 - 1/2)^2 \leq 1/4. \quad (2)$$

In this section, we demonstrate that an arbitrary spin-1/2 observable can be described by probabilities $0 \leq p_1(a), p_2(a), p_3(a), p_1(b), p_2(b), p_3(b) \leq 1$, where $a$ and $b$ are some real nonnegative numbers, and for these numbers the inequalities

$$(p_1(a) - 1/2)^2 + (p_2(a) - 1/2)^2 + (p_3(a) - 1/2)^2 \leq 1/4,$$  

$$(p_1(b) - 1/2)^2 + (p_2(b) - 1/2)^2 + (p_3(b) - 1/2)^2 \leq 1/4 \quad (3)$$

hold. To show this, we construct the following map of an arbitrary Hermitian matrix $H = H^\dagger$ onto a nonnegative Hermitian matrix $\rho(x)$ with unit trace. The matrix elements of the matrix $\rho(x)$ depend on the parameters $-\infty \leq x \leq \infty$.

Now we express the matrix $\rho(x)$ in terms of matrix elements of the matrix $H$ as follows:

$$\rho_{11}(x) = \frac{H_{11} + x}{H_{11} + H_{22} + 2x}, \quad \rho_{12}(x) = \frac{H_{12}}{H_{11} + H_{22} + 2x},$$  

$$\rho_{21}(x) = \frac{H_{21}}{H_{11} + H_{22} + 2x}, \quad \rho_{22}(x) = \frac{H_{22} + x}{H_{11} + H_{22} + 2x}. \quad (4)$$

It is obvious that for $x \geq |x_0|$, where $x_0$ is the smallest of two eigenvalues of the Hermitian matrix $H$, the matrix $\rho(x)$ satisfies the conditions $\rho(x) = \rho^\dagger(x)$, Tr$\rho(x) = 1$, and $\rho(x) \geq 0$. This means that the matrix $\rho(x)$ for such values of the parameter $x$ can be interpreted as the density matrix of the qubit state and, in view of this fact, it can be presented in the form [41].

One can check that, if one takes two different values of the parameter $x$, e.g., $x = a$ and $x = b$, where $a, b \geq |x_0|$, the matrix of observable $H$ can be expressed in terms of the
matrix elements of the two density matrices $\rho(a)$ and $\rho(b)$, namely,

$$H_{11} = \frac{ap_3(b)(1 - 2p_3(a)) - bp_3(a)(1 - 2p_3(b))}{p_3(a) - p_3(b)},$$

$$H_{11} + H_{22} = \frac{a - b + 2(bp_3(b) - ap_3(a))}{p_3(a) - p_3(b)},$$

$$H_{12} = (H_{11} + H_{22} + 2a)p_{12}(a) = (H_{11} + H_{22} + 2b)p_{12}(b),$$

$$H_{21} = H_{12}^*.$$ (5)

These relations provide the matrix elements of the observable $H$ in terms of the probabilities $p_1(a), p_2(a), p_3(a), p_1(b), p_2(b)$, and $p_3(b)$. The observable $H$ can be, e.g., the Hamiltonian; also

$$p_{12}(a) = p_1(a) - ip_2(a) - (1 - i)/2.$$ (6)

If $H = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $\rho(x) = \begin{pmatrix} 2^{-1} + (2x)^{-1} & 0 \\ 0 & 2^{-1} - (2x)^{-1} \end{pmatrix}$ at $x > 1$.

3 Spin-1/2 Tomography

The tomographic probability distribution $w(m, \vec{n})$ describing the qubit state was defined in [17, 18] as diagonal matrix elements of the density matrix in the rotated reference frame, i.e.,

$$w(m, \vec{n}) = \langle m | u^\rho u^\dagger | m \rangle,$$ (7)

where the unitary matrix $u_{jk}$ $(j, k = 1, 2)$ is expressed in terms of the Euler angles as follows:

$$u_{jk} = \begin{pmatrix} \cos \frac{\theta}{2} e^{i(\phi + \psi)/2} & \sin \frac{\theta}{2} e^{i(\phi - \psi)/2} \\ -\sin \frac{\theta}{2} e^{-i(\phi - \psi)/2} & \cos \frac{\theta}{2} e^{-i(\phi + \psi)/2} \end{pmatrix},$$ (8)

and $m = \pm 1/2$ is the spin projection on the direction determined by the unit vector $\vec{n}$,

$$\vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$$

The function $w(m, \vec{n})$, called the tomographic probability distribution, gives the probability to have the spin-projection $m$ on the direction $\vec{n}$. The density matrix $\rho$ is expressed in terms of the tomographic probability distribution $w(m, \vec{n})$ [17, 18]. Since there are only three parameters determining the density matrix $\rho$, information contained in the qubit
Thus, the tomogram \( w(m, \vec{n}) \), where \( \vec{n}_1 = (1, 0, 0) \), \( \vec{n}_2 = (0, 1, 0) \), and \( \vec{n}_3 = (0, 0, 1) \), is sufficient to obtain the density matrix.

This matrix is given by (1), where \( p_1, p_2, \) and \( p_3 \) are the probabilities to have \( m = +1/2 \) on the above directions. An arbitrary tomogram is expressed in terms of the probabilities \( p_1, p_2, \) and \( p_3 \) (\( p = p_1 + ip_2, \gamma = (1 + i)/2 \)) as follows:

\[
w(m, \vec{n}) = \left( \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \right) \left( \begin{pmatrix} p_3 & p^* - \gamma^* \\ p - \gamma & 1 - p_3 \end{pmatrix} \right) \left( \begin{pmatrix} u^*_{11} & u^*_{21} \\ u^*_{12} & u^*_{22} \end{pmatrix} \right). \]

Thus, the tomogram (7) reads

\[
w(+1/2, \vec{n}) = (\vec{p} - \vec{p}_0)\vec{n} + 1/2,
\]

where \( \vec{p}_0 = (1/2, 1/2, 1/2) \) and \( w(-1/2, \vec{n}) = 1 - w(+1/2, \vec{n}) \). Thus, we describe an arbitrary qubit state by means of three probabilities in any rotated reference frame.

Applying the formula obtained to the spin-1/2 tomogram, we can express the density matrix \( \rho_u = u\rho u^\dagger \) in terms of probabilities \( p_1, p_2, \) and \( p_3 \) written as components of the vector \( \vec{p} = (p_1, p_2, p_3) \) and vectors \( \vec{x}', \vec{y}', \) and \( \vec{z}' \) obtained by rotations of three basis vector \( \vec{x}, \vec{y}, \) and \( \vec{z} \) given by the unitary matrix \( u \). This means that we have three orthogonal vectors \( \vec{x}', \vec{y}', \) and \( \vec{z}' \) obtained by the rotation \( O(3) \) from the initial vectors \( \vec{x}, \vec{y}, \) and \( \vec{z}, \) i.e., \( \vec{x}' = O\vec{x}, \vec{y}' = O\vec{y}, \) and \( \vec{z}' = O\vec{z} \). Using these expressions, one can get the probabilities \( p_1', p_2', \) and \( p_3' \) determined by the matrix \( \rho' = u\rho u^\dagger \) in the form

\[
p_1' = L_{11}p_1 + L_{12}p_2 + L_{13}p_3 + C_1,
\]
\[
p_2' = L_{21}p_1 + L_{22}p_2 + L_{23}p_3 + C_2,
\]
\[
p_3' = L_{31}p_1 + L_{32}p_2 + L_{33}p_3 + C_3,
\]

where the matrix \( L \) and vector \( \vec{C} \) are expressed in terms of the unitary matrix as follows:

\[
L_{31} = u_{12}u^*_{11} + u_{11}u^*_{12}, \quad L_{32} = i(u_{12}u^*_{11} - u_{11}u^*_{12}), \quad L_{33} = |u_{11}|^2 - |u_{12}|^2,
\]
\[
L_{13} = \text{Re} (u_{11}u^*_{21}) - \text{Re} (u_{12}u^*_{22}), \quad L_{12} = \text{Re} (i(u_{12}u^*_{21}) - \text{Re} (i(u_{11}u^*_{22}),
\]
\[
L_{11} = \text{Re} (u_{12}u^*_{21}) + \text{Re} (u_{11}u^*_{22}), \quad L_{23} = \text{Im} (u_{12}u^*_{22}) - \text{Im} (u_{11}u^*_{21}),
\]
\[
L_{22} = \text{Im} (i(u_{11}u^*_{22}) - \text{Im} (i(u_{12}u^*_{21}), \quad L_{21} = -\text{Im} (u_{12}u^*_{21}) - \text{Im} (u_{11}u^*_{22}),
\]
\[
C_1 = \text{Re} (-\gamma u_{12}u^*_{21} - \gamma^* u_{11}u^*_{22} + u_{12}u^*_{22} + \gamma*),
\]
\[
C_2 = \text{Im} (\gamma u_{12}u^*_{21} + \gamma^* u_{11}u^*_{22} - u_{12}u^*_{22} - \gamma*),
\]
\[
C_3 = -\gamma u_{12}u^*_{11} - \gamma^* u_{11}u^*_{12} + |u_{12}|^2.
\]
The density matrix $\rho'$ is expressed in terms of probabilities $(p_1, p_2, p_3) = \vec{p}$ as

$$\rho' = (\sigma_0/2) + (\vec{p} - \vec{p}_0) \left( \vec{z}' \sigma_z + \vec{x}' \sigma_x + \vec{y}' \sigma_y \right)$$

with the vector $\vec{p}_0 = (1/2, 1/2, 1/2)$. This form shows that the probabilities $p'_1 = (\vec{p} - \vec{p}_0) \cdot \vec{x}'$, $p'_2 = (\vec{p} - \vec{p}_0) \cdot \vec{y}'$, and $p'_3 = (\vec{p} - \vec{p}_0) \cdot \vec{z}' + 1/2$ are the probabilities to obtain the spin projection $m = +1/2$ along the directions given by vectors $\vec{x}'$, $\vec{y}'$, and $\vec{z}'$, respectively.

For unitary transform of the density matrix $\rho \rightarrow \rho' = \sum_s \mathcal{P}_s u_s \rho u_s^\dagger$, where we have the probability distribution $1 \geq \mathcal{P}_s \geq 0$, $\sum_s \mathcal{P}_s = 1$, and $u_s$ are unitary matrices, the tomogram $w(+1/2, \vec{n})$ converts into

$$w(+1/2, \vec{n}) = \sum_{s,k} \mathcal{P}_s (\vec{p} - \vec{p}_0)_k O_{jk}^s n_k + 1/2.$$ 

Here, $O_{jk}^s$ are real orthogonal $3 \times 3$ matrices ($O^T = O^{-1}$). Then the new density matrix $\rho'$ reads

$$\rho' = \frac{1}{2} \sigma_0 + (\vec{p} - \vec{p}_0) \sum_s \left[ (\sigma_z O^s \mathcal{P}_s) \vec{z} \right] \sigma_z + (\sigma_x \mathcal{P}_s O^s) \vec{x} + (\sigma_y \mathcal{P}_s O^s) \vec{y}. $$

The $3 \times 3$ matrix $\sum_s \mathcal{P}_s O^s$ is the convex sum of orthogonal $3 \times 3$ matrices $O^s$. In the above formula, it acts on the basis vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$.

Thus, the new probabilities obtained due to the unitary transform are linear combinations of the $\vec{p}$ components, i.e.,

$$p'_1 = \sum_s \sum_{k=1}^3 \mathcal{P}_s O_{k1}^s (\vec{p} - \vec{p}_0)_k;$$

$$p'_2 = \sum_s \sum_{k=1}^3 \mathcal{P}_s O_{k2}^s (\vec{p} - \vec{p}_0)_k;$$

$$p'_3 = \sum_s \sum_{k=1}^3 \mathcal{P}_s O_{k3}^s (\vec{p} - \vec{p}_0)_k + 1/2.$$

The expressions obtained describe the quantum channels for qubit states in the probability representation.

### 4 Spin-1/2 Observable Tomograms

To provide the probability description of spin observables, we construct a tomogram of the matrix $\rho(x)$ for an arbitrary parameter $x$. First, we introduce the function

$$w(m, \vec{n}, x) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} \rho_{11}(x) & \rho_{12}(x) \\ \rho_{21}(x) & \rho_{22}(x) \end{bmatrix} \begin{bmatrix} u_{11}^* & u_{12}^* \\ u_{21}^* & u_{22}^* \end{bmatrix}_{mm}, \quad m = \pm 1/2. \quad (10)$$

6
Here, we use the map of the matrix indices

\[ 1 \ 1 \leftrightarrow 1/2 \ 1/2, \quad 12 \leftrightarrow 1/2 \ -1/2, \quad 21 \leftrightarrow -1/2 \ 1/2, \quad 22 \leftrightarrow -1/2 \ -1/2. \]

For \( x > |x_o| \), the function \( w(m, \bar{n}, x) \geq 0 \) satisfies the normalization condition \( \sum_{m=-1/2}^{1/2} w(m, \bar{n}, x) = 1 \).

Following the derivation of probabilities for the given density matrices, we introduce the probabilities \( P_3(a) \) and \( P_3(b) \) given by Eq. (3) as follows:

\[
P_3(a) = \frac{H_{11} + a}{H_{11} + H_{22} + 2a}, \quad P_3(b) = \frac{H_{11} + b}{H_{11} + H_{22} + 2a}. \tag{11}
\]

Also the probabilities \( P_1(a), P_2(a), P_1(b), \) and \( P_2(b) \) are determined by the relations

\[
P_1(a) - iP_2(a) - \gamma^* = \frac{H_{12}}{H_{11} + H_{22} + 2a}, \quad P_1(b) - iP_2(b) - \gamma^* = \frac{H_{12}}{H_{11} + H_{22} + 2b}. \tag{12}
\]

Thus, we obtain the tomograms of observable \( H \) for \( \bar{n} = 0, 0, 1 \); they read

\[
w(+1/2, \bar{n}, a) = P_3(a) = \frac{H_{11} + a}{H_{11} + H_{22} + 2a}, \quad w(+1/2, \bar{n}, b) = P_3(b) = \frac{H_{11} + b}{H_{11} + H_{22} + 2b}. \tag{13}
\]

Also for an arbitrary \( \bar{n} \), we have

\[
w(+1/2, \bar{n}, a) = (\bar{P}(a) - \bar{P}_0) \bar{n} + 1/2, \quad w(+1/2, \bar{n}, b) = (\bar{P}(b) - \bar{P}_0) \bar{n} + 1/2.
\]

5 Triangle Geometry of Tomographic Probabilities of Observables in the Quantum Suprematism Picture

Since we introduce the map of the spin-1/2 observable onto two density matrices \( \rho(a) \) and \( \rho(b) \), we can apply the tomographic description of the density matrices and known geometrical properties of qubit states formulated in terms of the triangle geometry using the Triada of Malevich’s squares [44] in the quantum suprematism picture. The specific feature of the probability representation of the qubit observable is related to the fact that the Hermitian matrix \( H \) is connected with two density matrices; this means that we use the probabilities \( P_1(a), P_2(a), P_1(b), P_2(b) \), and \( P_3(a), P_3(b) \) to describe the observable.

Thus, the probabilities are associated with vertices \( A_1(a), A_2(a), A_3(a) \) and \( A_1(b), A_2(b), A_3(b) \) of the triangles shown in Figs. 1 and 2. These vertices are located on
Figure 1: Triangle $A_1(a)A_2(a)A_3(a)$ corresponding to three probabilities $P_1(a)$, $P_2(a)$, and $P_3(a)$ determining the density matrix $\rho(x = a)$.

Figure 2: Triangle $A_1(b)A_2(b)A_3(b)$ corresponding to three probabilities $P_1(b)$, $P_2(b)$, and $P_3(b)$ determining the density matrix $\rho(x = b)$.

the sides of the equilateral triangle with a side length of $\sqrt{2}$ \cite{44}. The two triangles of Malevich’s squares determined by the probabilities $0 \leq P_1(a)$, $P_2(a)$, $P_3(a)$, $P_1(b)$, $P_2(b)$, $P_3(b) \leq 1$ are shown in Figs. 3 and 4.

The sums of areas of the squares read

\[
S_a = 2[3(1 - p_1(a) - p_2(a) - p_3(a)) + 2p_1^2(a) + 2p_2^2(a) + 2p_3^2(a) \\
+ p_1(a)p_2(a) + p_2(a)p_3(a) + p_3(a)p_1(a)],
\]

\[
S_b = 2[3(1 - p_1(b) - p_2(b) - p_3(b)) + 2p_1^2(b) + 2p_2^2(b) + 2p_3^2(b) \\
+ p_1(b)p_2(b) + p_2(b)p_3(b) + p_3(b)p_1(b)].
\]

There are bounds for the sums of the areas due to hidden quantum correlations in artificial qubit states $\rho(a)$ and $\rho(b)$ associated with the observable $H$. The minimum area is $3/2$. If the observable $H$ is the Pauli matrix $\sigma_z$, i.e., it corresponds to the spin projection on the
Figure 3: Triada of Malevich’s squares containing complete information on the density matrix $\rho(x = a)$.

Figure 4: Triada of Malevich’s squares containing complete information on the density matrix $\rho(x = b)$.
Figure 5: Two triangles corresponding to the spin projection of observable $\sigma_x$ with vertices $A_1(a)$, $A_2(a)$, $A_3(a)$ (left) and $A_1(b)$, $A_2(b)$, $A_3(b)$ (right) in triangle geometrical picture of quantum observables.

$z$ axis, $H = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the triangles with vertices $A_1(a)$, $A_2(a)$, $A_3(a)$ and $A_1(b)$, $A_2(b)$, $A_3(b)$ look as shown in Fig. 5. The length of sides $A_1(a)A_2(a)$ and $A_1(b)A_2(b)$ is equal to $\sqrt{2}/2$. In this case, the maximum area $S(a)$ and $S(b)$ is $5/2$, and the minimum area is $3/2$.

The area $S(a) = 3/2$ corresponds to the matrix $\rho(a) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

6 The Evolution Equation for Qubit Observables in the Probability Representation

Given an observable $A_{jk}$ in the matrix form and a Hamiltonian $H_{jk}$ ($j, k = 1, 2$), the Heisenberg equation for the observable $A(t)$ reads

$$\frac{\partial A(t)}{\partial t} = i[H, A(t)].$$

The corresponding “density matrix” $\rho(x, t)$

$$\rho(x, t) = \frac{1}{\text{Tr} A(t) + 2x}(A(t) + x1_2)$$

satisfies the evolution equation

$$\frac{\partial \rho(x, t)}{\partial t} = i[H, \rho(x, t)].$$

To obtain this equality, we employed the property $\frac{\partial}{\partial t}(\text{Tr} A(t) + 2x) = 0$, which follows from Eq. [14]. Since $\rho(x, t)$ can be expressed in terms of probabilities $p_1(x, t), p_2(x, t)$,
and $p_3(x,t)$,
\[
\rho(x,t) = \begin{pmatrix} p_3(x,t) & p_1(x,t) - ip_2(x,t) - \gamma^* \\ p_1(x,t) + ip_2(x,t) - \gamma & 1 - p_3(x,t) \end{pmatrix},
\] (17)
the evolution equation (16) can be presented as the system of equations for the probability vector \( \vec{p}(x,t) = \begin{pmatrix} p_1(x,t) \\ p_2(x,t) \\ p_3(x,t) \end{pmatrix} \) given by the following expression:
\[
\frac{d\vec{p}(x,t)}{dt} = L\vec{p}(x,t) + \vec{C}.
\] (18)
Here, the $3 \times 3$ matrix $L$ reads
\[
L = \begin{pmatrix} 0 & H_{11} - H_{22} & -2 \text{Im} H_{21} \\ H_{22} - H_{11} & 0 & 2 \text{Re} H_{21} \\ 2 \text{Im} H_{21} & -2 \text{Re} H_{21} & 0 \end{pmatrix},
\]
the three-vector $\vec{C}$ is \( \vec{C} = \begin{pmatrix} \text{Im} H_{21} + (H_{22} - H_{11})/2 \\ -\text{Re} H_{21} + (H_{11} - H_{22})/2 \\ 2 \text{Im} (\gamma H_{12}) \end{pmatrix} \), and the number $H_{jk}$ can be expressed in terms of probabilities $p_1(a)$, $p_2(a)$, $p_3(a)$ and $p_1(b)$, $p_2(b)$, $p_3(b)$ given by (11)–(13). This means that the equation for $\rho(x,t)$ has the form containing only the probabilities $p_1(x,t)$, $p_2(x,t)$, $p_3(x,t)$, $p_1(a)$, $p_2(a)$, $p_3(a)$, and $p_1(b)$, $p_2(b)$, $p_3(b)$.

The unitary evolution preserves the eigenvalues of observable $A$. Due to this, the vector $\vec{p}(x,t)$ for chosen parameters $a$ and $b$ has the nonnegative components varying in the domain $0 \leq p_j(x,t) \leq 1$ and satisfying the inequalities providing the nonnegativity condition for the density matrix $\rho(x,t)$.

7 Conclusions

To conclude, we list the main results of this work.

We presented qubit states by the density matrix with matrix elements expressed in terms of three measurable probabilities of spin-1/2 projections in three perpendicular directions and obtained compact formula (9) for spin tomogram; the matrix of an arbitrary observable is expressed in terms of two probability distributions given by (5) and (6). Also we demonstrate the Heisenberg evolution equation for an arbitrary observable as a
system of linear kinetic evolution equations for probabilities given by Eq. (18); the coefficients of the equations are also expressed in terms of the probabilities describing arbitrary Hamiltonian matrix elements. Thus, we formulated all the ingredients of quantum mechanics — states, observables, quantum evolution equation — in terms of probabilities; with the states and observables being identified in a conventional formulation of quantum mechanics with vectors in Hilbert spaces and operators acting in the Hilbert spaces. Note that the probabilities in classical probability theory are related to the other geometrical structure — simplexes.

We found the possibility to map the quantum states and observables formulated using the Hilbert space characteristics onto the probabilities associated with characteristics of the simplexes. This is done for the spin-1/2 system but we conjecture that the construction of an analogous map can be found for arbitrary quantum systems. We present such construction for qutrits in a future publication.

Acknowledgments

The formulation of the problem of the evolution equation for qubit observables in the probability representation and the results of Sec. 2 are due to V. I. Man’ko, who is supported by the Russian Science Foundation under Project No. 16-11-00084; this work was partially performed at the Moscow Institute of Physics and Technology. The authors thank A. Avanesov for correcting some formulas of [44].

References

[1] E. Schrödinger, Ann. Phys. (Liepzig), 79, 489 (1926).
[2] L. D. Landau, Z. Phys., 45, 430 (1927).
[3] J. von Neumann, Nach. Ges. Wiss. Göttingen, 11, 245 (1927).
[4] P. Dirac, The Principles of Quantum Mechanics, Oxford University Press (1930).
[5] G. Esposito, G. Marmo, G. Miele, and E. C. G. Sudarshan, Advanced Concepts in Quantum Mechanics, Cambridge University Press, UK (2015).
[6] G. Esposito, G. Marmo, and E. C. G. Sudarshan, *From Classical to Quantum Mechanics: An Introduction to the Formalism, Foundations and Applications*, Cambridge University Press (2004).

[7] E. Wigner, *Phys. Rev.*, **40**, 749 (1932).

[8] K. Husimi, *Proc. Phys. Math. Soc. Jpn*, **23**, 264 (1940).

[9] Y. Kano, *J. Math. Phys.*, **6**, 1913 (1986).

[10] R. J. Glauber, *Phys. Rev. Lett.*, **10**, 84 (1963).

[11] E. C. G. Sudarshan, *Phys. Rev. Lett.*, **10**, 84 (1963).

[12] S. Mancini, V. I. Man’ko, and P. Tombesi, *Phys. Lett. A*, **213**, 1 (1996).

[13] J. Radon, *Berichte Sächsische Akademie der Wissenschaften*, **29**, 262, Leipzig (1917).

[14] J. Bertrand and P. Bertrand, *Found. Phys.*, **17**, 397 (1989).

[15] K. Vogel and H. Risken, *Phys. Rev. A*, **40**, 2847 (1989).

[16] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, *Phys. Rev. Lett.*, **70**, 1244 (1993).

[17] V. V. Dodonov and V. I. Man’ko, *Phys. Lett. A*, **239**, 335 (1997).

[18] V. I. Man’ko and O. V. Man’ko, *J. Exp. Theor. Phys.*, **85**, 430 (1997).

[19] O. V. Man’ko, in: B. Gruber and M. Ramek (Eds.), *Proceedings of the International Conference “Symmetries in Science X” (Bregenz, Austria, 1997)*, Plenum Press, New York (1998), p. 207.

[20] G. M. D’Ariano, L. Maccone, and M. Paini, *J. Opt. B: Quantum Semiclass. Opt.*, **5**, 77 (2003).

[21] S. Weigert, *Phys. Rev. Lett.*, **84**, 802 (2000).

[22] J.-P. Amiet and S. Weigert, *J. Opt. B: Quantum Semiclass. Opt.*, **1**, L5 (1999).

[23] S. N. Filippov and V. I. Man’ko, *Phys. Scr.*, **83**, 058101 (2011).
[24] S. N. Filippov and V. I. Man’ko, J. Russ. Laser Res., 32, 56 (2011).

[25] A. Ibort, V. I. Man’ko, G. Marmo, et al., Phys. Scr., 79, 065013 (2009).

[26] O. V. Man’ko, V. I. Man’ko, and G. Marmo, Phys. Scr., 62, 446 (2000).

[27] M. Asorey, P. Facchi, V. I. Man’ko, et al., Phys. Rev. A, 76, 012117 (2007).

[28] M. Asorey, P. Facchi, V. I. Man’ko, et al., Phys. Rev. A, 77, 042115 (2008).

[29] O. V. Man’ko, V. I. Man’ko, G. Marmo, et al., Phys. Lett. A, 357, 255 (2006).

[30] O. V. Man’ko and V. I. Man’ko, J. Russ. Laser Res., 18, 407(1997).

[31] A. B. Klimov, O. V. Man’ko, V. I. Man’ko, et al., J. Phys. A: Math. Gen., 35, 6101 (2002).

[32] O. V. Man’ko, in: Proceedings of the Wigner Centennial Conference (Pecs, Hungary, 2002), The Official Electronic Proceedings, paper 30; Acta Physica Hungarica A, Series Heavy Ion Physics, 19/3-4, 313 (2004).

[33] V. N. Chernega and O. V. Man’ko, Phys. Scr., 30, 074052 (2015).

[34] M. A. Man’ko, V. I. Man’ko, G. Marmo, et al., Nuovo Cimento C, 36, 163 (2013).

[35] O. V. Man’ko, V. I. Man’ko, and G. Marmo, J. Phys. A: Math. Gen., 35, 699 (2002).

[36] O. V. Man’ko, V. I. Man’ko, and G. Marmo, “Tomographic map within the framework of star-product quantization,” in: Proceedings of the Conference “Quantum Theory and Symmetries” (Krakow, 2001), WorldScientific (2002), p. 126.

[37] O. V. Man’ko and V. N. Chernega, JETP Lett., 97, 557(2013).

[38] O. V. Manko, Phys. Scr., T135, 014004 (2009).

[39] S. V. Kuznetsov, O. V. Man’ko, and N. V. Tcherniega, J. Opt. B: Quantum Semiclass. Opt., 5, S503 (2003).

[40] R. L. Stratonovich, J. Exp. Theor. Phys., 5, 1206 (1957).
[41] O. V. Man’ko, V. I. Man’ko, G. Marmo, and P. Vitale, Phys. Lett. A, 360, 522 (2007).

[42] F. Lizzi and P. Vitale, SIGMA, 10, 36 (2014).

[43] V. I. Man’ko, G. Marmo, F. Ventriglia, and P. Vitale, J. Phys. A: Math. Theor., 50, 335402 (2017).

[44] V. N. Chernega, O. V. Man’ko, and V. I. Man’ko, J. Russ. Laser Res., 38, 141 (2017).