Perturbations and Stability of Black Ellipsoids

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Abstract

We study the perturbations of two classes of static black ellipsoid solutions of four dimensional vacuum Einstein equations. Such solutions are described by generic off–diagonal metrics which are generated by anholonomic transforms of diagonal metrics. The analysis is performed in the approximation of small eccentricity deformations of the Schwarzschild solution. We conclude that such anisotropic black hole objects may be stable with respect to the perturbations parametrized by the Schrodinger equations in the framework of the one–dimensional inverse scattering theory.

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1 Introduction

In a series of works new classes of solutions of four dimensional (4D) and 5D vacuum Einstein equations with ellipsoid and toroidal symmetry are constructed [1]. Such solutions are generated by anholonomic deformations of the Schwarzschild metric, of static or stationary configurations, and depend anisotropically on angular coordinates. They are described by generic off–diagonal metric ansatz which are effectively diagonalized with respect to anholonomic frames with associated nonlinear connection structure. A study of horizons and geodesic behaviour [2, 3] concluded that for small deformations of the spherical symmetry, for instance, to a resolution ellipsoid one. Such solutions define black ellipsoid objects (static black holes with ellipsoidal horizons and anisotropic polarizations of constants). With respect to anholonomic frames the new solutions are given by certain metric coefficients which are similar to those from the Reissner–Nordstrom metric, but with an effective, polarized, ”electromagnetic” charge, induced by off–diagonal vacuum gravitational interactions. This differs substantially from the usual static electrovacuum solutions with spherical symmetry which are exact solutions

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of the Einstein–Maxwell equations (see, for instance Ref. [4], as a general reference on 
"the mathematical theory of black holes").

The aim of this paper is to study perturbations of black ellipsoids and to prove 
that there are such static ellipsoid like configurations which are stable with respect 
to perturbations of a fixed type of anisotropy (i. e. for certain imposed anholonomic constraints). The main idea of a such proof is to consider small (ellipsoidal, or another type) deformations of the Schwarzschild metric and than to apply the already developed methods of the theory of perturbations of classical black hole solutions, with a re–
definition of the formalism for adapted anholonomic frames.

The theory of perturbations of the Schwarzschild spacetime black holes was initiated 
in Ref. [5], developed in a series of works, e. g. Refs [6, 7], and related [8] to the theory 
of inverse scattering and its ramifications (see, for instance, Refs. [9]). The results 
on the theory of perturbations and stability of the Schwarzschild, Reissner–Nordstrom 
and Kerr solutions are summarized in a monograph [4]. As alternative treatments of 
the stability of black holes we cite in Ref. [10].

The paper has the following structure: In Sec. 2 we introduce an off–diagonal ansatz 
which parametrizes two classes of anholonomic static deformations of the Schwarzschild 
solutions which describe black ellipsoid like objects (the formulas for the components of 
Ricci and Einstein tensors are outlined in the Appendix). In Sec. 3 we investigate the 
axial metric perturbations governed by some one–dimensional Schroedinger equations 
with nonlinear potential and anisotropic gravitational polarizations. Section 4 is de-
voted to a stability analysis of polar metric perturbations; a procedure of definition of 
formal solutions for polar perturbations is formulated. In Sec. 5 we prove the stability 
of static anholonomically deformed solutions of the Sewarschild metric with respect to 
perturbations treated in the framework of the inverse scattering theory based on the 
one–dimensional Schrodinger equation with paremetric potentials. A discussion and 
conclusions are contained in Sec. 6.

2 Metrics Describing Perturbations of Anisotropic 
    Black Holes

We consider a four dimensional pseudo–Riemannian quadratic linear element

\[
ds^2 = \Omega(r, \varphi) \left[ -\left(1 - \frac{2m}{r} + \frac{\varepsilon}{r^2}\right)^{-1} dr^2 - r^2 q(r) d\theta^2 - \eta_3(r, \theta, \varphi) r^2 \sin^2 \theta d\varphi^2 \right] + \left[1 - \frac{2m}{r} + \frac{\varepsilon}{r^2} \eta(r, \varphi) \right] \delta t^2,
\]

with

\[
\delta \varphi = d \varphi + \varepsilon w_1(r, \varphi) dr, \quad \text{and} \quad \delta t = dt + \varepsilon n_1(r, \varphi) dr,
\]

where the local coordinates are denoted \( u = \{ u^\alpha = (r, \theta, \varphi, t) \} \) (the Greek indices \( \alpha, \beta, ... \) will run the values 1,2,3,4), \( \varepsilon \) is a small parameter satisfying the conditions

\( 0 \leq \varepsilon \ll 1 \) (for instance, an eccentricity for some ellipsoid deformations of the spherical symmetry) and the functions \( \Omega(r, \varphi), q(r), \eta_3(r, \theta, \varphi) \) and \( \eta(\theta, \varphi) \) are of necessary smooth class. The metric [11] is static, off–diagonal and transforms into the usual
Schwarzschild solution if $\varepsilon \to 0$ and $q, \eta_3 \to 1$; it describes at least two classes of static black hole solutions generated as small anholonomic deformations of the Schwarzschild solution [1, 2, 3].

The geodesic and the horizon structure of the first class of solutions was investigated in Ref. [2]. It was proved that for the data

$$\eta_3(r, \varphi) = \eta_3[0](r, \varphi) + \varepsilon \lambda_3(r, \varphi) + \varepsilon^2 \gamma_3(r, \varphi) + ..., \quad \eta_4(r, \varphi) = 1 + \varepsilon \lambda_4(r, \varphi) - \varepsilon^2 \lambda_4(r, \varphi) \left(1 - \frac{2m}{r}\right)^{-1} + ..., \quad \eta(r, \varphi) = \lambda_4(r, \varphi) \left(r^2 - 2mr\right) + 1, \Omega = 1,$$

$$\varepsilon \sqrt{\eta_3} = \eta_0 \partial \sqrt{\eta_4}/\partial \varphi, \eta_0 = \text{const}, w_1 = 0,$$

and

$$n_1(r, \varphi) = n_{1[1]}(r) + n_{1[2]}(r) \int d\varphi \eta_3(r, \varphi) / \left(\sqrt{\eta_4(x, \varphi)}\right)^3, \eta_4^* \neq 0;$$

$$= n_{1[1]}(r) + n_{1[2]}(r) \int d\varphi \eta_3(r, \varphi), \eta_4^* = 0;$$

$$= n_{1[1]}(r) + n_{1[2]}(r) \int d\varphi / \left(\sqrt{\eta_4(r, \varphi)}\right)^3, \eta_3^* = 0;$$

where the limit $\partial \sqrt{\eta_4}/\partial \varphi \to 0$ is considered for $\varepsilon \to 0$ and the functions $\eta_3(r, \varphi)$ and $n_{1[1,2]}(r)$ are stated by some boundary conditions, there is defined a static black hole with ellipsoidal horizon (a black ellipsoid).

Another class of black ellipsoids with anisotropic conformal symmetries, with a nontrivial conformal factor $\Omega$ which induces a nonzero value for the coefficient $w_1$ (see details in Ref. [3]), can be defined by a metric (1) with the data

$$\Omega(r, \varphi) = 1 - \varepsilon r^2 \sin^2 \theta \eta_3(r, \theta, \varphi); \quad \eta(r, \varphi) = r \eta_3(r, \theta, \varphi) \sin^2 \theta \left(r^2 - 2mr\right)^2/2m;$$

$$\epsilon w_1(r, \varphi) = \partial_1 \Omega(\partial \Omega/\partial \varphi)^{-1}, \partial \eta_3/\partial \varphi \neq 0;$$

$$0, \partial \eta_3/\partial \varphi = 0;$$

and

$$n_1(r, \varphi) = n_{1[1]}(r) + n_{1[2]}(r) \int d\varphi \eta_3(r, \varphi),$$

where, for this type of solutions, the function $\eta_3(r, \theta, \varphi)$ is chosen as to have a value $\phi(r, \varphi) = \eta_3(r, \theta, \varphi) \sin^2 \theta$ depending only on variables $r$ and $\varphi$.

The maximal analytic extensions of such locally anisotropic black hole metrics in the framework of general relativity theory with anholonomic frames were constructed in Refs. [2, 3]. It was shown that the condition of vanishing of the coefficient $1 - 2m/r + \varepsilon \eta(r, \varphi)/r^2$ before $\delta t$ defines the equation of a static non–spherical horizon for a so called ”locally anisotropic” black hole (for a corresponding parametrization of $\eta$ we may construct a resolution ellipsoid horizon). There is a similarity of the metrics of type (1) with the Reissner–Norstrom solution: the anisotropic metric is obtained as a linear
approximation on $\varepsilon$ from some exact vacuum solutions of the Einstein equations, but the Reissner–Norstrom one is a static exact solution of the Einstein–Maxwell equations.

We can apply the perturbation theory for the metric (11) (not paying a special attention to some particular parametrization of coefficients for one or another class of anisotropic black hole solutions) and analyze its stability by using the results of Ref. [4] for a fixed anisotropic direction, i.e. by imposing certain anholonomic frame constraints for an angle $\varphi = \varphi_0$ but considering possible perturbations depending on three variables ($u^1 = x^1 = r, u^2 = x^2 = \theta, u^4 = t$). If we prove that there is a stability on perturbations for a value $\varphi_0$, we can analyze in a similar manner another values of $\varphi$. A more general perturbative theory with variable anisotropy on coordinate $\varphi$, i.e. with dynamical anholonomic constraints, connects the approach with a two dimensional inverse problem which makes the analysis more sophisticate.

We note that in a study of perturbations of any spherically symmetric system and, for instance, of small ellipsoid deformations, without any loss of generality, we can restrict our considerations to axisymmetric modes of perturbations. Non–axisymmetric modes of perturbations with an $e^{im\varphi}$ dependence on the azimuthal angle $\varphi$ ($n$ being an integer number) can be deduced from modes of axisymmetric perturbations with $n = 0$ by suitable rotations since there are not preferred axes in a spherically symmetric background. The ellipsoid like deformations may be included into the formalism as some low frequency and constrained perturbations.

For simplicity, in this paper, we restrict our study only to fixed values of the coordinate $\varphi$ assuming that anholonomic deformations are proportional to a small parameter $\varepsilon$; we shall investigate the stability of solutions only by applying the one dimensional inverse methods. The metric (11) to be investigated is with a deformed horizon and transforms into the usual Schwarzschild solution at long radial distances (see Refs. [2, 3] for details on such off–diagonal metrics).

Let us consider a quadratic metric element

$$ds^2 = -e^{2\mu_1}(du^1)^2 - e^{2\mu_2}(du^2)^2 - e^{2\mu_3}(du_3)^2 + e^{2\mu_4}(du_4)^2,$$

$$\delta u^3 = d\varphi - q_1dx^1 - q_2dx^2 - \omega dt,$$

$$\delta u^4 = dt + n_1dr$$

where

$$\mu_\alpha(x^k, t) = \mu_\alpha^{(c)}(x^k, \varphi_0) + \delta\mu_\alpha^{(c)}(x^k, t),$$

$$q_i(x^k, t) = q_i^{(c)}(r, \varphi_0) + \delta q_i^{(c)}(x^k, t),$$

$$\omega(x^k, t) = 0 + \delta\omega^{(c)}(x^k, t)$$

with

$$e^{2\mu_1^{(c)}} = \Omega(r, \varphi_0)(1 - \frac{2m}{r} + \frac{\varepsilon}{r^2})^{-1},$$

$$e^{2\mu_2^{(c)}} = \Omega(r, \varphi_0)r^2,$$

$$e^{2\mu_3^{(c)}} = \Omega(r, \varphi_0)r^2 \sin^2 \theta \eta_3(r, \varphi_0),$$

$$e^{2\mu_4^{(c)}} = 1 - \frac{2m}{r} + \frac{\varepsilon}{r^2}\eta(r, \varphi_0),$$
and some non-trivial values for \( q^{(\varepsilon)}_i \) and \( \varepsilon n_i \),

\[
q^{(\varepsilon)}_1 = \varepsilon w_1(r, \varphi_0),
\]

\[
n_1 = \varepsilon \left( n_{1[1]}(r) + n_{1[2]}(r) \int_{\varphi_0}^{\varphi} \eta_3(r, \varphi) d\varphi \right).
\]

We are distinguishing two types of small deformations from the spherical symmetry. The first type of deformations, labeled with the index \((\varepsilon)\) are generated by some \(\varepsilon\) terms which define a fixed ellipsoid like configuration and the second type ones, labeled with the index \((\varsigma)\), are some small linear fluctuations of the metric coefficients.

The general formul\(\)as for the Ricci and Einstein tensors for metric elements of class \(\mathcal{H}\) with \(n_1 = 0\) are given in \[4\]. We compute similar values with respect to anholonomic frames, when, for a conventional splitting \(u^a = (x^i, y^a)\), the coordinates \(x^i\) and \(y^a\) are treated respectively as holonomic and anholonomic ones. In this case the partial derivatives \(\partial / \partial x^i\) must be changed into certain ‘elongated’ ones

\[
\frac{\partial}{\partial x_1} \rightarrow \frac{\delta}{\partial x_1} = \frac{\partial}{\partial x_1} - w_1 \frac{\partial}{\partial \varphi} - n_1 \frac{\partial}{\partial t},
\]

\[
\frac{\partial}{\partial x_2} \rightarrow \frac{\delta}{\partial x_1} = \frac{\partial}{\partial x_1} - w_2 \frac{\partial}{\partial \varphi} - n_2 \frac{\partial}{\partial t},
\]

see details in Refs \[1, 2, 3\]. In the ansatz \(\mathcal{H}\), the anholonomic contributions of \(w_i\) are included in the coefficients \(q_i(x^k, t)\), there is only one nonzero value \(w_1\) and, in consequence, we have to introduce elongations of partial space derivatives only on the \(t\)-variable. For convenience, in the Appendix we present the necessary formulas for \(R_{\alpha\beta}\) (the Ricci tensor) and \(G_{\alpha\beta}\) (the Einstein tensor) computed for the ansatz \(\mathcal{H}\) with three holonomic coordinates \((r, \theta, \varphi)\) and on anholonomic coordinate \(t\) (in our case, being time like), with the partial derivative operators

\[
\partial_1 \rightarrow \delta_1 = \frac{\partial}{\partial r} - n_1 \frac{\partial}{\partial t}, \partial_2 = \frac{\partial}{\partial \theta}, \partial_3 = \frac{\partial}{\partial \varphi},
\]

and for a fixed value \(\varphi_0\).

A general perturbation of an anisotropic black–hole described by a quadratic line element \(\mathcal{H}\) results in some small quantities of the first order \(\omega\) and \(q_i\), inducing a dragging of frames and imparting rotations, and in some functions \(\mu_\alpha\) with small increments \(\delta \mu_\alpha\), which do not impart rotations. Some coefficients contained in such values are proportional to \(\varepsilon\), another ones are considered only as small quantities. The perturbations of metric are of two generic types: axial and polar one. We investigate them separately in the next two Sections.

### 3 Axial metric perturbations

Axial perturbations are characterized by non–vanishing \(\omega\) and \(q_i\) which satisfy the equations

\[
R_{3i} = 0,
\]
see the explicit formulas for such coefficients of the Ricci tensor in the Appendix. The resulting equations governing axial perturbations, $\delta R_{31} = 0$, $\delta R_{32} = 0$, are respectively

$$
\partial_2 \left( e^{3\mu_i^{(c)} + \mu_i^{(c)} - \mu_1^{(c)} - \mu_2^{(c)}} Q_{12} \right) = -e^{3\mu_i^{(c)} - \mu_1^{(c)} + \mu_2^{(c)}} \partial_4 Q_{14},
$$

$$
\delta_1 \left( e^{3\mu_i^{(c)} + \mu_i^{(c)} - \mu_1^{(c)} - \mu_2^{(c)}} Q_{12} \right) = e^{3\mu_i^{(c)} - \mu_1^{(c)} + \mu_2^{(c)}} \partial_4 Q_{24},
$$

where

$$
Q_{ij} = \delta_i q_j - \delta_j q_i, Q_{14} = \partial_4 q_i - \delta_i \omega
$$

and for $\mu_i$ there are considered unperturbed values $\mu_i^{(c)}$. Introducing the values of coefficients (5) and (6) and assuming that the perturbations have a time dependence of type $\exp(i\sigma t)$ for a real constant $\sigma$, we rewrite the equations (7)

$$
\frac{1 + \varepsilon (\Delta^{-1} + 3r^2 \phi/2)}{r^4 \sin^3 \theta \eta_3^{3/2}} \partial_2 Q^{(n)} = -i\sigma \delta \omega - \sigma^2 q_1,
$$

$$
\frac{\Delta}{r^4 \sin^3 \theta \eta_3^{3/2}} \delta_1 \left( Q^{(n)} \left[ 1 + \varepsilon \left( \frac{\eta - 1}{\Delta} - r^2 \phi \right) \right] \right) = i\sigma \partial_2 \omega + \sigma^2 q_2
$$

for

$$
Q^{(n)}(r, \theta, \varphi_0, t) = \Delta Q_{12} \sin^3 \theta = \Delta \sin^3 \theta (\partial_2 q_1 - \delta_1 q_2), \Delta = r^2 - 2mr,
$$

where $\phi = 0$ for solutions with $\Omega = 1$ and $\phi(r, \varphi) = \eta_3 (r, \theta, \varphi) \sin^2 \theta$, i. e. $\eta_3 (r, \theta, \varphi) \sim \sin^{-2} \theta$ for solutions with $\Omega = 1 + \varepsilon$...

We can exclude the function $\omega$ and define an equation for $Q^{(n)}$ if we take the sum of the (9) subjected by the action of operator $\partial_2$ and of the (10) subjected by the action of operator $\delta_1$. Using the relations (5), we write

$$
r^4 \delta_1 \left\{ \frac{\Delta}{r^4 \eta_3^{3/2}} \left[ \delta_1 \left( Q^{(n)} + \varepsilon \left( \frac{\eta - 1}{\Delta} - r^2 \phi \right) \right) \right] \right\} +
\sin^3 \theta \partial_2 \left[ 1 + \varepsilon (\Delta^{-1} + 3r^2 \phi/2) \right] \partial_2 Q^{(n)} = \frac{\sigma^2 r^4}{\Delta \eta_3^{3/2}} Q^{(n)} = 0.
$$

The solution of this equation is searched in the form $Q^{(n)} = Q + \varepsilon Q^{(1)}$ which results in

$$
r^4 \partial_1 \left( \frac{\Delta}{r^4 \eta_3^{3/2}} \partial_1 Q \right) + \sin^3 \theta \partial_2 \left( \frac{1}{\sin^3 \theta \eta_3^{3/2}} \partial_2 Q \right) + \frac{\sigma^2 r^4}{\Delta \eta_3^{3/2}} Q = \varepsilon A (r, \theta, \varphi_0),
$$

where

$$
A (r, \theta, \varphi_0) = r^4 \partial_1 \left( \frac{\Delta}{r^4 \eta_3^{3/2}} \partial_t \right) - r^4 \partial_1 \left( \frac{\Delta}{r^4 \eta_3^{3/2}} \partial_1 Q^{(1)} \right) - \sin^3 \theta \partial_2 \left[ 1 + \varepsilon (\Delta^{-1} + 3r^2 \phi/2) \partial_2 Q^{(1)} - \frac{\sigma^2 r^4}{\Delta \eta_3^{3/2}} Q^{(1)} \right],
$$

with a time dependence like $\exp[i\sigma t]$. 

6
It is possible to construct different classes of solutions of the equation (11). At the first step we find the solution for \( Q \) when \( \epsilon = 0 \). Then, for a known value of \( Q(r, \theta, \varphi_0) \) from
\[
Q^{(n)} = Q + \epsilon Q^{(1)},
\]
we can define \( Q^{(1)} \) from the equations (9) and (10) by considering the values proportional to \( \epsilon \) which can be written
\[
\begin{align*}
\partial_1 Q^{(1)} &= B_1(r, \theta, \varphi_0), \\
\partial_2 Q^{(1)} &= B_2(r, \theta, \varphi_0),
\end{align*}
\]
where
\[
B_1 = n_1 \frac{\partial Q}{\partial t} - \frac{1}{2} \partial_1 \left[ Q \left( \frac{\eta - 1}{\Delta} - r^2 \phi \right) \right]
\]
and
\[
B_2 = - \left( \Delta^{-1} + 3 r^2 \phi / 2 \right) \partial_2 Q - \sigma^2 r^4 \sin^3 \theta \ w_1.
\]
The integrability condition of the system (12), \( \partial_1 B_2 = \partial_2 B_1 \) imposes a relation between the polarization functions \( \eta_3, \eta, w_1 \) and \( n_1 \) (for a corresponding class of solutions, see formulas (2), or (3)). In order to prove that there are stable anisotropic configurations of anisotropic black hole solutions, we may consider a set of polarization functions when \( A(r, \theta, \varphi_0) = 0 \) and the solution with \( Q^{(1)} = 0 \) is admitted. This holds, for example, if
\[
\Delta n_1 = n_0 r^4 \eta_3^{3/2}, \ n_0 = \text{const}.
\]
In this case the axial perturbations are described by the equation
\[
\eta_3^{3/2} r^4 \partial_1 \left( \frac{\Delta}{r^4 \eta_3^{3/2}} \partial_1 Q \right) + \sin^3 \theta \partial_2 \left( \frac{1}{\sin^3 \theta} \partial_2 Q \right) + \frac{\sigma^2 r^4}{\Delta} Q = 0
\]
which is obtained from (11) for \( \eta_3 = \eta_3(r, \varphi_0) \), or for \( \phi(r, \varphi_0) = \eta_3(r, \theta, \varphi_0) \sin^2 \theta \).

In the limit \( \eta_3 \to 1 \) the solution of equation (13) is investigated in details in Ref. [4]. Here, we prove that in a similar manner we can define exact solutions for non-trivial values of \( \eta_3 \). The variables \( r \) and \( \theta \) can be separated if we substitute
\[
Q(r, \theta, \varphi_0) = Q_0(r, \varphi_0) C_{\nu+1/2}^{-3/2}(\theta),
\]
where \( C^n_{\nu} \) are the Gegenbauer functions generated by the equation
\[
\left[ \frac{d}{d\theta} \sin^{2\nu} \theta \frac{d}{d\theta} + n (n + 2\nu) \sin^{2\nu} \theta \right] C^n_{\nu}(\theta) = 0.
\]
The function \( C_{\nu+1/2}^{-3/2}(\theta) \) is related to the second derivative of the Legendre function \( P_1(\theta) \) by formulas
\[
C_{\nu+1/2}^{-3/2}(\theta) = \sin^3 \theta \frac{d}{d\theta} \left[ \frac{1}{\sin \theta} \frac{dP_1(\theta)}{d\theta} \right].
\]
The separated part of (13) depending on radial variable with a fixed value \( \varphi_0 \) transforms into the equation
\[
\eta_3^{3/2} r^4 \left[ \frac{d}{dr} \left( \frac{\Delta}{r^4 \eta_3^{3/2}} \frac{dQ_0}{dr} \right) + \left( \sigma^2 - \mu^2 \frac{\Delta}{r^4} \right) Q_0 \right] = 0,
\]
\[
(14)
\]
where $\mu^2 = (l - 1)(l + 2)$ for $l = 2, 3, \ldots$. A further simplification is possible for $\eta_3 = \eta_3(r, \varphi_0)$ if we introduce in the equation (14) a new radial coordinate

$$r_# = \int \eta_3^{3/2}(r, \varphi_0)r^2 dr$$

and a new unknown function $Z^{(n)} = r^{-1}Q_0(r)$. The equation for $Z^{(n)}$ is a Schrödinger like one–dimensional wave equation

$$\left(\frac{d^2}{dr_#^2} + \frac{\sigma^2}{\eta_3^{3/2}}\right)Z^{(n)} = V^{(n)}Z^{(n)}$$

(15)

with the potential

$$V^{(n)} = \Delta r^5 \left[\mu^2 - r^4 d\left(\frac{\Delta}{r^4 \eta_3^{3/2}}\right)\right]$$

(16)

and polarized parameter

$$\tilde{\sigma}^2 = \sigma^2 / \eta_3^{3/2}.$$ 

This equation transforms into the so–called Regge–Wheeler equation if $\eta_3 = 1$. For instance, for the Schwarzschild black hole such solutions are investigated and tabulated for different values of $l = 2, 3$ and 4 in Ref. [4].

We note that for static anisotropic black holes with nontrivial anisotropic conformal factor, $\Omega = 1 + \varepsilon\ldots$, even $\eta_3$ may depend on angular variable $\theta$ because condition that $\phi(r, \varphi_0) = \eta_3(r, \theta, \varphi_0) \sin^2 \theta$ the equation (13) transforms directly in (15) with $\mu = 0$ without any separation of variables $r$ and $\theta$. It is not necessary in this case to consider the Gegenbauer functions because $Q_0$ does not depend on $\theta$ which corresponds to a solution with $l = 1$.

We may transform (15) into the usual form,

$$\left(\frac{d^2}{dr_*^2} + \sigma^2\right)Z^{(n)} = \tilde{V}^{(n)}Z^{(n)}$$

if we introduce the variable

$$r_* = \int dr_# \eta_3^{-3/2}(r_#, \varphi_0)$$

for $\tilde{V}^{(n)} = \eta_3^{3/2}V^{(n)}$. So, the polarization function $\eta_3$, describing static anholonomic deformations of the Scharzschild black hole, ”renormalizes” the potential in the one–dimensional Schrödinger wave–equation governing axial perturbations of such objects.

We conclude that small static ”ellipsoid” like deformations and polarizations of constants of spherical black holes (the anisotropic configurations being described by generic vacuum off–diagonal metric ansatz) do not change the type of equations for axial perturbations: one modifies the potential barrier,

$$V^{(-)} = \Delta r^5 \left[(\mu^2 + 2) r - 6m\right] \rightarrow \tilde{V}^{(n)}$$

and re–defines the radial variables

$$r_* = r + 2m \ln (r/2m - 1) \rightarrow r_*(\varphi_0)$$

with a parametric dependence on anisotropic angular coordinate which is caused by the existence of a deformed static horizon.
4 Polar metric perturbations

The polar perturbations are described by non–trivial increments of the diagonal metric coefficients, \( \delta \mu_\alpha = \delta \mu_\alpha^{(e)} + \delta \mu_\alpha^{(c)} \), for

\[
\mu_\alpha^{(e)} = \nu_\alpha + \delta \mu_\alpha^{(e)}
\]

where \( \delta \mu_\alpha^{(c)}(x^k, t) \) parametrize time depending fluctuations which are stated to be the same both for spherical and/or spheroid configurations and \( \delta \mu_\alpha^{(e)} \) is a static deformation from the spherical symmetry. Following notations \( \text{(5)} \) and \( \text{(6)} \) we write

\[
e^{\nu_1} = r/\sqrt{|\Delta|}, \quad e^{\nu_2} = r\sqrt{|q(r)|}, \quad e^{\nu_3} = rh_3 \sin \theta, \quad e^{\epsilon t} = \Delta/r^2
\]

and

\[
\delta \mu_1^{(e)} = -\frac{\epsilon}{2} (\Delta^{-1} + r^2 \phi), \quad \delta \mu_2^{(e)} = \delta \mu_3^{(e)} = -\frac{\epsilon}{2} r^2 \phi, \quad \delta \mu_4^{(e)} = \frac{\epsilon \eta}{2 \Delta}
\]

where \( \phi = 0 \) for the solutions with \( \Omega = 1 \).

Examining the expressions for \( R_{ii} \), \( R_{12}, R_{33} \) and \( G_{11} \) (see the Appendix) we conclude that the values \( Q_{ij} \) appear quadratically which can be ignored in a linear perturbation theory. Thus the equations for the axial and the polar perturbations decouple. Considering only linearized expressions, both for static \( \epsilon \)–terms and fluctuations depending on time about the Schwarzschild values we obtain the equations

\[
\begin{aligned}
\delta_1 (\delta \mu_2 + \delta \mu_3) + (r^{-1} - \delta_1 \mu_4) (\delta \mu_2 + \delta \mu_3) - 2r^{-1} \delta \mu_1 &= 0 \quad (\delta R_{41} = 0), \\
\partial_2 (\delta \mu_1 + \delta \mu_3) + (\delta \mu_2 - \delta \mu_3) \cot \theta &= 0 \quad (\delta R_{42} = 0), \\
\partial_2 \delta_1 (\delta \mu_3 + \delta \mu_4) - \delta_1 (\delta \mu_2 - \delta \mu_3) \cot \theta - (r^{-1} - \delta_1 \mu_4) \partial_2 (\delta \mu_1) - (r^{-1} + \delta_1 \mu_4) \partial_2 (\delta \mu_1) &= 0 \quad (\delta R_{42} = 0), \\
e^{2\mu_4} [2 (r^{-1} + \delta_1 \mu_4) \delta_1 (\delta \mu_3) + r^{-1} \delta_1 (\delta \mu_3 + \delta \mu_4 - \delta \mu_1 + \delta \mu_2) + \delta_1 [\delta (\delta \mu_3)] - 2r^{-1} \delta_1 \mu_1 (r^{-1} + 2 \delta_1 \mu_4)] - 2e^{-2\mu_4} \partial_4 [\partial_4 (\delta \mu_3)] + r^{-2} \{ \partial_2 [\partial_2 (\delta \mu_3)] + \partial_2 (2 \delta \mu_3 + \delta \mu_4 + \delta \mu_1 - \delta \mu_2) \cot \theta + 2 \delta \mu_2 \} &= 0 \quad (\delta R_{33} = 0), \\
e^{-2\mu_4} [r^{-1} \delta_1 (\delta \mu_4) + (r^{-1} + \delta_1 \mu_4) \delta_1 (\delta \mu_2 + \delta \mu_3)] - 2\epsilon^{-2\mu_4} \partial_4 [\partial_4 (\delta \mu_3 + \delta \mu_2)] + r^{-2} \{ \partial_2 [\partial_2 (\delta \mu_3)] + \partial_2 (2 \delta \mu_3 + \delta \mu_4 - \delta \mu_2) \cot \theta + 2 \delta \mu_2 \} &= 0 \quad (\delta G_{11} = 0).
\end{aligned}
\]

The values of type \( \delta \mu_\alpha = \delta \mu_\alpha^{(e)} + \delta \mu_\alpha^{(c)} \) from \( \text{(17)} \) contain two components: the first ones are static, proportional to \( \epsilon \), and the second ones may depend on time coordinate \( t \). We shall assume that the perturbations \( \delta \mu_\alpha^{(c)} \) have a time–dependence \( \exp[\sigma t] \) so that the partial time derivative “\( \partial_4 \)” is replaced by the factor \( i \sigma \). In order to treat both type of increments in a similar fashion we may consider that the values labeled with \( (\epsilon) \) also oscillate in time like \( \exp[\sigma^{(e)} t] \) but with a very small (almost zero) frequency \( \sigma^{(e)} \to 0 \). There are also actions of ”elongated” partial derivative operators like

\[
\delta_1 (\delta \mu_\alpha) = \partial_1 (\delta \mu_\alpha) - \epsilon n_1 \partial_4 (\delta \mu_\alpha).
\]
To avoid a calculus with complex values we associate the terms proportional \( \varepsilon n_1 \partial_4 \) to amplitudes of type \( \varepsilon n_1 \partial_4 \) and write this operator as

\[
\delta_1 (\delta \mu_\alpha) = \partial_1 (\delta \mu_\alpha) + \varepsilon n_1 \sigma (\delta \mu_\alpha).
\]

For the "non-perturbed" Schwarzschild values, which are static, the operator \( \delta_1 \) reduces to \( \partial_1 \), i.e. \( \delta_1 \nu_\alpha = \partial_1 \nu_\alpha \). Hereafter we shall consider that the solution of the system (17) consists of a superposition of two linear solutions, \( \delta \mu_\alpha = \delta \mu_\alpha^{(c)} + \delta \mu_\alpha^{(s)} \); the first class of solutions for increments will be provided with index \( \varepsilon \), corresponding to the frequency \( \sigma^{(c)} \) and the second class will be for the increments with index \( \varsigma \) and correspond to the frequency \( \sigma^{(s)} \). We shall write this as \( \delta \mu_\alpha^{(A)} \) and \( \sigma^{(A)} \) for the labels \( A = \varepsilon \) or \( \varsigma \) and suppress the factors \( \exp[\sigma^{(A)} t] \) in our subsequent considerations. The system of equations (17) will be considered for both type of increments.

We can separate the variables by substitutions (see the method in Refs. [7, 4])

\[
\delta \mu_1^{(A)} = L^{(A)}(r) P_l(\cos \theta), \quad \delta \mu_2^{(A)} = [T^{(A)}(r) P_l(\cos \theta) + V^{(A)}(r) \partial^2 P_l/\partial \theta^2],
\]

\[
\delta \mu_3^{(A)} = \left[ T^{(A)}(r) P_l(\cos \theta) + V^{(A)}(r) \cot \theta \partial P_l/\partial \theta \right], \quad \delta \mu_4^{(A)} = N^{(A)}(r) P_l(\cos \theta)
\]

and reduce the system of equations (17) to

\[
\delta_1 \left( N^{(A)} - L^{(A)} \right) = \left( r^{-1} - \partial_1 \nu_4 \right) N^{(A)} + \left( r^{-1} + \partial_1 \nu_4 \right) L^{(A)},
\]

\[
\delta_1 L^{(A)} + \left( 2r^{-1} - \partial_1 \nu_4 \right) N^{(A)} = - \left[ \delta_1 X^{(A)} + \left( r^{-1} - \partial_1 \nu_4 \right) X^{(A)} \right],
\]

and

\[
2r^{-1} \delta_1 \left( N^{(A)} \right) - l(l+1)r^{-2}e^{-2\nu_1}N^{(A)} - 2r^{-1}(r^{-1} + 2\partial_1 \nu_4)L^{(A)} - 2(r^{-1} + \partial_1 \nu_4)\delta_1 \left[ N^{(A)} + (l-1)(l+2)V^{(A)}/2 \right] - (l-1)(l+2)2r^{-2}e^{-2\nu_1} \left( V^{(A)} - L^{(A)} \right) - 2\sigma^{(A)}e^{-4\nu_1} \left[ L^{(A)} + (l-1)(l+2)V^{(A)}/2 \right] = 0,
\]

where we have introduced new functions

\[
X^{(A)} = \frac{1}{2}(l-1)(l+2)V^{(A)}
\]

and considered the relation

\[
T^{(A)} - V^{(A)} + L^{(A)} = 0 \quad (\delta R_{42} = 0).
\]

We can introduce the functions

\[
\bar{L}^{(A)} = L^{(A)} + \varepsilon \sigma^{(A)} \int n_1 L^{(A)} dr, \quad \bar{N}^{(A)} = N^{(A)} + \varepsilon \sigma^{(A)} \int n_1 N^{(A)} dr,
\]

\[
\bar{T}^{(A)} = N^{(A)} + \varepsilon \sigma^{(A)} \int n_1 N^{(A)} dr, \quad \bar{V}^{(A)} = V^{(A)} + \varepsilon \sigma^{(A)} \int n_1 V^{(A)} dr,
\]

for which

\[
\partial_1 \bar{L}^{(A)} = \delta_1 \left( L^{(A)} \right), \quad \partial_1 \bar{N}^{(A)} = \delta_1 \left( N^{(A)} \right), \quad \partial_1 \bar{T}^{(A)} = \delta_1 \left( T^{(A)} \right), \quad \partial_1 \bar{V}^{(A)} = \delta_1 \left( V^{(A)} \right).
\]
and, this way it is possible to substitute in (19) and (20) the elongated partial derivative \( \delta_1 \) by the usual one acting on "tilded" radial increments.

By straightforward calculations (see details in Ref. [4]) one can check that the functions

\[
Z^{(+)}_{(A)} = r^2 \frac{6mX^{(A)}/r(l-1)(l+2) - L^{(A)}}{r(l-1)(l+2)/2 + 3m}
\]

satisfy one-dimensional wave equations similar to (15) for \( Z^{(\eta)} \) with \( \eta_3 = 1 \), when \( r_* = r_* \),

\[
\left( \frac{d^2}{dr_*^2} + \sigma^2_{(A)} \right) \tilde{Z}^{(+)}_{(A)} = V^{(+)} Z^{(+)}_{(A)},
\]

(22)

\[
\tilde{Z}^{(+)}_{(A)} = Z^{(+)}_{(A)} + \varepsilon \sigma_{(A)} \int n_1 Z^{(+)}_{(A)} dr,
\]

where

\[
V^{(+)} = \frac{2\Delta}{r^5[r(l-1)(l+2)/2 + 3m]^2} \times \left\{ 9m^2 \left[ \frac{r}{2} (l-1)(l+2) + m \right] + \frac{1}{4} (l-1)^2(l+2)^2 r^3 \left[ 1 + \frac{1}{2} (l-1)(l+2) + \frac{3m}{r^2} \right] \right\}.
\]

(23)

For \( \varepsilon \to 0 \), the equation (22) transforms in the usual Zerilli equation [11, 4].

To complete the solution we give the formulas for the "tilded" \( L^{(A)} \), \( X^{(A)} \) and \( N^{(A)} \) factors,

\[
\tilde{L}^{(A)} = \frac{3m}{r^2} \tilde{\Phi}_{(A)} - \frac{(l-1)(l+2)}{2r} \tilde{Z}^{(+)}_{(A)},
\]

(24)

\[
\tilde{X}^{(A)} = \frac{(l-1)(l+2)}{2r} (\tilde{\Phi}_{(A)} + \tilde{Z}^{(+)}_{(A)}),
\]

\[
\tilde{N}^{(A)} = \left( m - \frac{m^2 + r^4 \sigma^2_{(A)}}{r - 2m} \right) \frac{\tilde{\Phi}_{(A)}}{r^2} - \frac{(l-1)(l+2)r}{2(l-1)(l+2) + 12m} \frac{\partial \tilde{Z}^{(+)}_{(A)}}{\partial r_*}
\]

\[
- \frac{(l-1)(l+2)}{[r(l-1)(l+2) + 6m]^2} \times
\]

\[
\left\{ \frac{12m^2}{r} + 3m(l-1)(l+2) + \frac{r}{2} (l-1)(l+2) [(l-1)(l+2) + 2] \right\},
\]

where

\[
\tilde{\Phi}_{(A)} = (l-1)(l+2)e^{\nu_4} \int \frac{e^{-\nu_4} \tilde{Z}^{(+)}_{(A)}}{[l(l-1)(l+2)]r + 6m} dr.
\]

Following the relations (21) we can compute the corresponding "untiled" values and put them in (18) in order to find the increments of fluctuations driven by the system of equations (17). For simplicity, we omit the rather cumbersome final expressions.

The formulas (24) together with a solution of the wave equation (22) complete the procedure of definition of formal solutions for polar perturbations. In Ref. [4] there are tabulated the data for the potential (23) for different values of \( l \) and \( (l-1)(l+2)/2 \). In the anisotropic case the explicit form of solutions is deformed by terms proportional to \( \varepsilon n_1 \sigma \). The static ellipsoidal like deformations can be modeled by the formulas obtained in the limit \( \sigma_\varepsilon \to 0 \).
5 The Stability of Black Ellipsoids

The problem of stability of anholonomically deformed Schwarzschild metrics to external perturbation is very important to be solved in order to understand if such static black ellipsoid like objects may exist in general relativity. In this context we address the question: Let be given any initial values for a static locally anisotropic configuration confined to a finite interval of $r_*$, for axial perturbations, and $r_*$, for polar perturbations, will one remain bounded such perturbations at all times of evolution?

We have proved that even for anisotropic configurations every type of perturbations are governed by one dimensional wave equations of the form

$$\frac{d^2 Z}{d\rho^2} + \sigma^2 Z = V Z$$

(25)

where $\rho$ is a radial type coordinate, $Z$ is a corresponding $Z^{(n)}$ or $Z^{(s)}$ with respective smooth real, independent of $\sigma > 0$ potentials $\tilde{V}(n)$ or $V(-)$ with bounded integrals. For such equations a solution $Z(\rho, \sigma, \varphi_0)$ satisfying the boundary conditions

$$Z \rightarrow e^{i\sigma\rho} + R(\sigma)e^{-i\sigma\rho} \quad (\rho \rightarrow +\infty),$$

$$\rightarrow T(\sigma)e^{i\sigma\rho} \quad (\rho \rightarrow -\infty)$$

(the first expression corresponds to an incident wave of unit amplitude from $+\infty$ giving rise to a reflected wave of amplitude $R(\sigma)$ at $+\infty$ and the second expression is for a transmitted wave of amplitude $T(\sigma)$ at $-\infty$), provides a basic complete set of wave functions which allows to obtain a stable evolution. For any initial perturbation that is smooth and confined to finite interval of $\rho$, we can introduce the integral

$$\psi(\rho, 0) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{\psi}(\sigma, 0)Z(\rho, \sigma)d\sigma$$

and define, at later times, the evolution of perturbations,

$$\psi(\rho, t) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{\psi}(\sigma, 0)e^{i\sigma t}Z(\rho, \sigma)d\sigma.$$ 

The Schrodinger theory guarantees the conditions

$$\int_{-\infty}^{+\infty} \psi(\rho, 0)\psi(\rho, 0)^* d\rho = \int_{-\infty}^{+\infty} \hat{\psi}(\sigma, 0)\hat{\psi}(\sigma, 0)^* d\sigma = \int_{-\infty}^{+\infty} \psi(\rho, 0)\psi(\rho, 0)^* d\rho,$$

from which the boundedness of $\psi(\rho, t)$ follows for all $t > 0$.

In our consideration we have replaced the time partial derivative $\partial / \partial t$ by $i\sigma$, which was represented by the approximation of perturbations to be periodic like $e^{i\sigma t}$. This is connected with a time–depending variant of (25), like

$$\frac{\partial^2 Z}{\partial t^2} = \frac{\partial^2 Z}{\partial \rho^2} - V Z.$$
Multiplying this equation on $\partial Z/\partial t$, where $Z$ denotes the complex conjugation, and integrating on parts, we obtain

$$\int_{-\infty}^{+\infty} \left( \frac{\partial Z}{\partial t} \frac{\partial^2 Z}{\partial \rho^2} + \frac{\partial Z}{\partial \rho} \frac{\partial^2 Z}{\partial t \partial \rho} + VZ \frac{\partial Z}{\partial t} \right) d\rho = 0$$

providing the conditions of convergence of necessary integrals. This equation added to its complex conjugate results in a constant energy integral,

$$\int_{-\infty}^{+\infty} \left( \left| \frac{\partial Z}{\partial t} \right|^2 + \left| \frac{\partial Z}{\partial \rho} \right|^2 + V|Z|^2 \right) d\rho = \text{const},$$

which bounds the expression $|\partial Z/\partial t|^2$ and excludes an exponential growth of any bounded solution of the equation (25). We note that this property holds for every type of "ellipsoidal" like deformation of the potential, $V \rightarrow V + \varepsilon V^{(1)}$, with possible dependences on polarization functions as we considered in (16) and/or (23).

The general properties of the one–dimensional Schrodinger equations related to perturbations of holonomic and anholonomic solutions of the vacuum Einstein equations allow us to conclude that there are locally anisotropic static configurations which are stable under linear deformations.

In a similar manner we may analyze perturbations (axial or polar) governed by a two–dimensional Schrodinger wave equation like

$$\frac{\partial^2 Z}{\partial t^2} = \frac{\partial^2 Z}{\partial \rho^2} + A(\rho, \varphi, t) \frac{\partial^2 Z}{\partial \varphi^2} - V(\rho, \varphi, t)Z$$

for some functions of necessary smooth class. The stability in this case is proven if exists an (energy) integral

$$\int_{0}^{\pi} \int_{-\infty}^{+\infty} \left( \left| \frac{\partial Z}{\partial t} \right|^2 + \left| \frac{\partial Z}{\partial \rho} \right|^2 + \left| A \frac{\partial Z}{\partial \rho} \right|^2 + V|Z|^2 \right) d\rho d\varphi = \text{const}$$

which bounds $|\partial Z/\partial t|^2$ for two–dimensional perturbations. For simplicity, we omitted such calculus in this work.

Finally, we note that this way we can also prove the stability of perturbations along "anisotropic" directions of arbitrary anholonomic deformations of the Schwarzschild solution which have non–spherical horizons and can be covered by a set of finite regions approximated as small, ellipsoid like, deformations of some spherical hypersurfaces. We may analyze the geodesic congruence on every deformed sub-region of necessary smoothly class and proof the stability as we have done for the resolution ellipsoid horizons. In general, we may consider horizons of with non–trivial topology, like vacuum black tori, or higher genus anisotropic configurations. This is not prohibited by the principles of topological censorship [12] if we are dealing with off–diagonal metrics and associated anholonomic frames [1]. The vacuum anholonomy in such cases may be treated as an effective matter which change the conditions of topological theorems.
6 Outlook and Conclusions

It is a remarkable fact that, in spite of appearance complexity, the perturbations of static off–diagonal vacuum gravitational configurations are governed by similar types of equations as for diagonal holonomic solutions. The origin of this mystery is located in the fact that by anholonomic transforms we effectively diagonalized the off–diagonal metrics by "elongating" some partial derivatives. This way the type of equations governing the perturbations is preserved but, for small deformations, the systems of linear equations for fluctuations became "slightly" nondiagonal and with certain tetradic modifications of partial derivatives and differentials. In details, the question of relating of particular integrals of such systems associated with systems of linear differential equations is investigated in Ref. [4]. In our case one holds the same relations between the potentials $\tilde{V}^{(\eta)}$ and $V^{(-)}$ and wave functions $Z^{(\eta)}$ and $Z^{(+)}$ with that difference that the physical values and formulas where polarized by some anisotropy functions $\eta_3(r, \theta, \varphi), \Omega(r, \varphi), q(r), \eta(r, \varphi), w_1(r, \varphi)$ and $n_1(r, \varphi)$ and deformed on a small parameter $\varepsilon$.

We also observe that the "anisotropic" potentials $\tilde{V}^{(\eta)}$ and $V^{(-)}$ may be defined for such polarizations $\eta_3(r, \theta, \varphi)$ as they would be smooth functions, integrable over the range of $r_*, (-\infty, +\infty)$ and positive everywhere. For real $\sigma$, the anisotropic solutions, represent ingoing and outgoing waves of type $e^{\pm i\sigma r_*}(r_* \rightarrow \pm \infty)$. We conclude that the underlying physical problem is one of reflection and transmission of incident waves (from $+$ or $-\infty$) by some anisotropically deformed one–dimensional potential barriers $\tilde{V}^{(\eta)}$ and/or $V^{(-)}$, for every fixed anisotropic angle $\varphi_0$. Because at distances far away from deformed horizons, the metrics of anisotropic black holes transforms into the usual Schwarzschild solution we can prove the equality of the reflection and the transmission coefficients for the axial and the polar perturbations, even the wave equations are anholonomically deformed. Such formulas are tabulated in Ref. [4] and may be used for anisotropic solutions but with some redefined coefficients. The general properties of the one–dimensional potential–scattering are preserved for small anholonomic deformations; they are concerned with some solutions of one dimensional Schrödinger's wave equations.

We emphasize that we proved the stability of black ellipsoids by fixing any anisotropic directions in the off–diagonal metrics, $\varphi = \varphi_0$ i.e. under certain anholonomic constraints imposed on vacuum gravitational configurations. A more general consideration with variable $\varphi$ relates the problem to the two dimensional Schrödinger’s potential–scattering problem, as well to an anholonomic Newman–Penrose formalism which makes the solution of the problem of stability to be more sophisticated. Nevertheless, we are having strong arguments that the stability of anisotropic static solutions proved in the parametric approximation $\varphi = \varphi_0$ will hold at least for such polarizations which model two dimensional scattering effects containing some particular one dimensional stable Schrödinger’s potentials. The one–dimensional perturbation analysis is the first, very important, step in investigating the stability of any physical system; it should be included into the higher dimensional and/or less constrained approaches.

We found that the origin of the results that the functions $Z^{(-)}$ and $Z^{(+)}$ and their anisotropic extensions $Z^{(\eta)}$ and $Z^{(\eta, A)}$ do in fact satisfy very similar one–dimensional
wave equations stems from the fact that some deep properties of static solutions of
the Einstein equations which hold both for diagonal and off–diagonal static–stationary
vacuum and/or electrovacuum metrics.

Finally, we conclude that there are static black ellipsoid vacuum configurations
which are stable with respect to one dimensional perturbations, axial and/or polar ones,
governed by solutions of the corresponding one–dimensional Schrodinger equations.
The problem of stability of such objects with respect to two, or three, dimensional
perturbations, and the possibility of modeling such perturbations in the framework
of a two–, or three–, dimensional inverse scattering problem is a topic of our further
investigations.

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Appendix

We compute the coefficients of the curvature tensor as

$$R^\alpha_{\beta\gamma\tau} = \delta^\alpha_\gamma \Gamma^\beta_\tau - \delta^\alpha_\tau \Gamma^\beta_\gamma + \Gamma^\alpha_\tau \Gamma^\beta_\gamma - \Gamma^\alpha_\gamma \Gamma^\beta_\tau,$$

of the Ricci tensor as

$$R^\alpha_{\beta\gamma\alpha} = R_{\beta\gamma}$$

and of the Einstein tensor as

$$G_{\beta\gamma} = R_{\beta\gamma} - \frac{1}{2} g_{\beta\gamma} R$$

for $R = g^\beta_\gamma R_{\beta\gamma}$ and $\delta_\gamma = \partial_\gamma$ for $\gamma = 2, 3, 4$ and $\delta_1 = \partial_1 - n_1 \partial/\partial t$. Straightforward
computations for the quadratic line element \(\text{(1)}\) give

$$R_{11} = -e^{-2\mu_1} \left[ \delta_{11}^2 (\mu_3 + \mu_4 + \mu_2) + \delta_1 \mu_3 \delta_1 (\mu_3 - \mu_1) + \delta_1 \mu_2 \delta_1 (\mu_2 - \mu_1) + \delta_1 \mu_4 \delta_1 (\mu_4 - \mu_1) - e^{-2\mu_2} \left[ \delta_{22} \mu_1 + \delta_2 \mu_1 \delta_2 (\mu_3 + \mu_4 + \mu_1 - \mu_2) \right] \right] + e^{-2\mu_4} \left[ \delta_{44} \mu_1 + \delta_4 \mu_1 \delta_4 (\mu_3 - \mu_1 + \mu_1 + \mu_2) \right] - \frac{1}{2} e^{2(\mu_3 - \mu_1)} \left[ e^{-2\mu_2} Q_{12}^2 + e^{-2\mu_4} Q_{14}^2 \right];$$

$$R_{12} = -e^{-\mu_1-\mu_2} \left[ \delta_2 \delta_1 (\mu_3 + \mu_2) - \delta_2 \mu_1 \delta_1 (\mu_3 + \mu_1) - \delta_1 \mu_2 \delta_4 (\mu_3 + \mu_1) + \delta_1 \mu_3 \delta_2 \mu_3 + \delta_1 \mu_4 \delta_2 \mu_4 \right] + \frac{1}{2} e^{2\mu_3 - 2\mu_4 - \mu_1 - \mu_2} Q_{14} Q_{24};$$

$$R_{31} = -\frac{1}{2} e^{2\mu_3 - \mu_1 - \mu_2} \left[ \delta_2 \left( e^{3\mu_3 + \mu_4 - \mu_1 - \mu_2} Q_{21} \right) + \delta_4 \left( e^{3\mu_3 - \mu_4 + \mu_2 - \mu_1} Q_{41} \right) \right];$$

$$R_{33} = -e^{-2\mu_1} \left[ \delta_{11}^2 \mu_3 + \delta_1 \mu_3 \delta_1 (\mu_3 + \mu_4 + \mu_2 - \mu_1) \right] - e^{-2\mu_2} \left[ \delta_{22} \mu_3 + \delta_2 \mu_3 \delta_2 (\mu_3 + \mu_4 - \mu_2 + \mu_1) \right] + \frac{1}{2} e^{2(\mu_3 - \mu_1 - \mu_2)} Q_{12}^2 + e^{-2\mu_4} \left[ \delta_{44} \mu_3 + \delta_4 \mu_3 \delta_4 (\mu_3 - \mu_4 + \mu_2 + \mu_1) \right] - \frac{1}{2} e^{2(\mu_3 - \mu_4)} \left[ e^{-2\mu_2} Q_{24}^2 + e^{-2\mu_1} Q_{14}^2 \right];$$
\[ R_{41} = -e^{-\mu_1-\mu_4}[\partial_4 \delta_4(\mu_3 + \mu_2) + \delta_1 \mu_3 \partial_4(\mu_3 - \mu_1) + \delta_1 \mu_2 \partial_4(\mu_2 - \mu_1) \\
-\delta_1 \mu_4 \partial_4(\mu_3 + \mu_2)] + \frac{1}{2} e^{2\mu_3-\mu_4-\mu_1-\mu_2} Q_{12} Q_{34}, \]

\[ R_{43} = -\frac{1}{2} e^{2\mu_3-\mu_1-\mu_2} [\partial_4 e^{3\mu_3-\mu_4-\mu_1+\mu_2} Q_{14}] + \partial_2 (e^{3\mu_4-\mu_4+\mu_1-\mu_2} Q_{21}), \]

\[ R_{44} = -e^{-2\mu_1}[\partial_4^2(\mu_1 + \mu_2 + \mu_3) + \partial_4 \mu_3 \partial_4(\mu_3 - \mu_4) + \partial_4 \mu_3 \partial_4(\mu_1 - \mu_4) + \partial_4 \mu_2 \partial_4(\mu_2 - \mu_4)] + e^{-2\mu_1}[\partial_1^2 \mu_4 + \delta_1 \mu_4 \partial_1(\mu_3 + \mu_2 - \mu_1 + \mu_2)] + \]

\[ e^{-2\mu_2}[\partial_2^2 \mu_4 + \partial_2 \mu_3 \partial_2(\mu_3 + \mu_4 - \mu_1 + \mu_2)] - \frac{1}{2} e^{2\mu_3-\mu_4} [e^{-2\mu_1} Q_{14} + e^{-2\mu_2} Q_{24}], \]

where the rest of the coefficients are defined by similar formulas with a corresponding changings of indices and partial derivative operators, \( R_{22}, R_{42} \) and \( R_{32} \) is like \( R_{11}, R_{41} \) and \( R_{31} \) with with changing the index \( 1 \to 2 \). The values \( Q_{ij} \) and \( Q_{i4} \) are defined respectively

\[ Q_{ij} = \delta_j q_i - \delta_i q_j \] and \( Q_{i4} = \partial_4 q_i - \delta_i \omega. \]

The nontrivial coefficients of the Einstein tensor are

\[ G_{11} = e^{-2\mu_2}[\partial_1^2(\mu_3 + \mu_4) + \partial_2(\mu_3 + \mu_4) \partial_2(\mu_4 - \mu_2) + \partial_2 \mu_3 \partial_2 \mu_3] - \]

\[ e^{-2\mu_4}[\partial_1^2(\mu_3 + \mu_4) + \partial_1(\mu_3 + \mu_2) \partial_1(\mu_3 - \mu_4) + \partial_1 \mu_3 \partial_1 \mu_3] + \]

\[ e^{-2\mu_1}[\partial_1^2 \mu_4 + \delta_1(\mu_3 + \mu_2) + \delta_1 \mu_3 \delta_1 \mu_2] - \]

\[ \frac{1}{4} e^{2\mu_3} [e^{-2(\mu_1+\mu_2)} Q_{12}^2 - e^{-2(\mu_1+\mu_4)} Q_{14}^2 + e^{-2(\mu_2+\mu_3)} Q_{24}^2], \]

\[ G_{33} = e^{-2\mu_1}[\partial_1^2(\mu_4 + \mu_2) + \delta_1 \mu_4 \delta_1(\mu_4 - \mu_1 + \mu_2) + \delta_1 \mu_5 \delta_1(\mu_2 - \mu_1)] + \]

\[ e^{-2\mu_2}[\partial_1^2(\mu_4 + \mu_1) + \partial_2(\mu_4 - \mu_2 + \mu_1) + \partial_2 \mu_1 \partial_2(\mu_1 - \mu_2)] - \]

\[ e^{-2\mu_4}[\partial_1^2(\mu_1 + \mu_2) + \partial_1 \mu_1 \partial_1(\mu_1 - \mu_4) + \partial_1 \mu_2 \partial_1(\mu_2 - \mu_4) + \partial_1 \mu_2 \partial_1 \mu_2] + \]

\[ \frac{3}{4} e^{2\mu_3} [e^{-2(\mu_1+\mu_2)} Q_{12}^2 - e^{-2(\mu_1+\mu_4)} Q_{14}^2 + e^{-2(\mu_2+\mu_3)} Q_{24}^2], \]

\[ G_{44} = e^{-2\mu_1}[\partial_1^2(\mu_3 + \mu_2) + \delta_1 \mu_3 \delta_1(\mu_3 - \mu_1 + \mu_2) + \delta_1 \mu_5 \delta_1(\mu_2 - \mu_1)] - \]

\[ e^{-2\mu_2}[\partial_1^2(\mu_3 + \mu_1) + \partial_2(\mu_3 - \mu_2 + \mu_1) + \partial_2 \mu_1 \partial_2(\mu_1 - \mu_2)] - \frac{1}{4} e^{2(\mu_3-\mu_1-\mu_2)} Q_{12}^2 \]

\[ + e^{-2\mu_4}[\partial_1 \mu_3 \partial_1(\mu_1 + \mu_2) + \partial_1 \mu_1 \partial_1 \mu_2] - \frac{1}{4} e^{2(\mu_3-\mu_4)} [e^{-2\mu_1} Q_{14}^2 - e^{-2\mu_2} Q_{24}^2]. \]

The component \( G_{22} \) is to be found from \( G_{11} \) by changing the index \( 1 \to 2 \).

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