Lateral-Torsional Buckling of C-Beams with Varying Inertia

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Abstract. The present work deals with a numerical study on the flexural-torsional buckling of a thin-walled beam with uniformly varying C-section. Approximate solutions to the examined problem are obtained through the Dirichlet variational method and the Rayleigh-Ritz discretization procedure. The presented results prove to be asymptotically convergent for increasing numbers of test functions. This convergence property highlights the technical utility of the proposed approach, on considering the difficulty of obtaining analytic solutions for buckling problems of thin-walled beams with non-uniform cross-section.

1. Introduction

Thin-walled open-section members with continuous variation of the cross-sectional areas are increasingly employed in structural, mechanical and aeronautical engineering applications, since they allow for optimising strength, stiffness and stability with significant material savings and economic benefits. When the cross sections vary along the beam axis, thin-walled open section beams show relatively low bending and torsional stiffness coefficients. They tend to buckle out of the plane by deflecting laterally and twisting, as a compressive load is applied [1-3]. The solution to this coupled stability problem can be quite complex and difficult to obtain in analytic form. A number of theoretical and numerical approaches have been proposed in the literature to face this problem, including the Initial Parameter Method, Bifurcation an Dynamic Approaches, Energy Methods, Imperfection Sensitivity Analyses and Finite Element Approximations [4-7]. Worth of mention are the works by Elishakoff [8] on the behaviour of variable cross section beams, and by El-Mahdy [9] on the inelastic lateral-torsional buckling of symmetric beams with variable flange ratio.

We develop in the present paper a numerical investigation on the elastic flexural-torsional buckling of a simply supported thin-walled beam with uniformly varying C-cross section, through a numerical approach base on the Dirichlet variational method. The adopted method provides numerical estimates of the buckling load of the examined beam. It relies on the principle of stationary potential energy and employs the Rayleigh-Ritz discretisation procedure, with varying number of test functions. The given numerical result show that such an approach qualifies as a robust and versatile tool that allows to obtain reliable upper bounds of the true critical load, on considering the remarkable complexity of the mathematical problem ruling the lateral-torsional buckling problem of beams with variable cross-section.

2. Mechanical model

Let us consider a simple supported thin-walled beam made of S420 steel. We assume that that such a beam has C-section with height and width that vary along the length according to the following laws:

\[ B'(z) = -0.01z + 28.5 \]
\[ H'(z) = 0.10z + 87.0 \]  

(1)
while the cross-section thickness $\delta$ remains constant ($\delta = 4\text{ mm}$).

The beam material is linear elastic with Young’s modulus $E = 210\text{ GPa}$; it is subjected to an axial compressive load, $F$, at the centroid $G_A$ of the free end cross section (as shown in figure 1); it is defined with respect to a right-handed rectangular coordinate system $Gxyz$ (axes $z$ coincides with the longitudinal centroidal axis of the member, while $x$ and $y$ coincide with the two principal axes of the cross section).

![Diagram](image)

**Figure 1.** Theoretical model: load arrangement and constrain conditions (a); basic notation (b, c).

### 2.1. First order torsional and flexural equilibrium equation of thin-walled beam

According to the Vlasov torsion theory, the differential equations of equilibrium describing the flexural-torsional behaviour of the analysed beam [1] are

\[
\begin{align*}
\left( E I_y \ u'' \right)' + F \ u' - F \ ȳ_G \ \varphi' &= 0 \\
\left( E I_x \ v'' \right)' + F \ v' + F \ ȳ_G \ \varphi' &= 0 \\
C_\omega \ \varphi''' + C_\omega' \ \varphi'' - \left( W_z - \frac{FI_c}{A} \right) \ \varphi'' - \left( W_z - \frac{FI_c}{A} \right)' \ \varphi' - F \ u'' \ ȳ_G + F \ v'' \ ȳ_G &= 0
\end{align*}
\]

in which the apex represents the $z$-derivative operation; $A, I_x$ and $I_y$ are the cross-section area and the second moments of area about the $x$ and $y$ axes, respectively; $I_c$ is the polar moment of area about shear centre; $W_z, K_G$ and $C_\omega$ are St. Venant warping rigidity and warping constant, respectively; $\hat{y}_G$ and $\hat{y}_G$ are the coordinates of the centroid with respect to the coordinate system $G\hat{x}\hat{y}\hat{z}$, which is obtained by a translation of the reference $Gxyz$ coordinate system in the shear centre of the free end. The in-plane displacement of an arbitrary centroid point with coordinates $x$ and $y$ are

\[
\begin{align*}
\{u = u_c - (y - y_c) \ \varphi \\
v = v_c + (x - x_c) \ \varphi
\end{align*}
\]

where $u_c$ and $v_c$ are the displacements of the shear centre, $\varphi$ is the angle of twist of the cross section, and $x_c$ and $y_c$ are the shear centre coordinates. Due to the variability of the cross sections along the beam and their symmetry with respect to the $z$ axis, the location of the shear centre will be determined by the following equations.
where $\gamma_c = 0$}

2.2. Total Potential Energy method and Rayleigh-Ritz discretization procedure

Let $E$ be the total potential energy which characterises the equilibrium condition in the analysed elastic system. The Dirichlet variational principle assumes that the stationarity of the second variation of $E$

$$\partial_2 E = L_2 + W$$

is associated to a critical position of equilibrium. In equation (5), $L_2$ and $W$ are the potential of applied forces and the strain energy, respectively. The use of Total Potential Energy method leads to an approximate values of the critical buckling load as an upper bound for the real value. The accuracy of the solution depends on using approximate deformation shapes that are as close as possible to the exact deformation shapes and also satisfy the boundary conditions of the problem.

According to Rayleigh-Ritz discretisation, a class of kinematically admissible configurations can be describe by the finite series

$$s_j = \sum_{i=1}^{m} q_{j,i} \psi_{j,i} \quad j = 1, 2, 3$$

where $\psi$-terms are arbitrarily chosen coordinate functions (i.e. a system of orthogonal functions) and $q_{i,j} = q_{i,j}(u, v, \theta)$ is a corresponding set of $3m$ Lagrangian multipliers.

Substituting equation (6) into the energy expression (5) the total energy $E$ becomes a quadratic function of the Lagrangian multipliers and the minimum condition provides a system of $3m$ linear homogeneous equations (necessary and sufficient conditions of equilibrium) from which the critical buckling force $F_c$ can be calculated without solving the nonlinear equilibrium equations (2).

$$\frac{\partial E}{\partial q_{i,j}} = 0 \quad j = 1, 2, 3 \quad i = 1, \ldots, m$$

3. Numerical results

Our numerical results assume the following class of kinematically admissible configurations

$$\left\{ \begin{array}{l}
  u_c = \sum_{i=1}^{m} u_i \sin \left( \frac{m \pi \hat{z}}{L} \right) \\
  v_c = \sum_{i=1}^{m} v_i \sin \left( \frac{m \pi \hat{z}}{L} \right) \\
  \theta = \sum_{i=1}^{m} \theta_i \sin \left( \frac{m \pi \hat{z}}{L} \right)
\end{array} \right.$$  

which satisfy the boundary conditions of the analysed problem, namely

$$\left\{ \begin{array}{l}
  u(0) = u''(0) = u(l) = u''(l) = 0 \\
  v(0) = v''(0) = v(l) = v''(l) = 0 \\
  \theta(0) = \theta''(0) = \theta(l) = \theta''(l) = 0
\end{array} \right.$$
Table 1 shows the obtained lower eigenvalues of the characteristic equation (i.e. the critical value) by taking one, two or three terms of the equation series (8). The results given in Table 1 highlight an asymptotical convergence of the employed numerical scheme, with limiting value of the buckling load approximately equal to 15.6 kN. On increasing the number of waves \( m \) up to 3, the numerical solutions exhibit asymptotic convergence, and progressively reducing in amplitude predictions of the buckling load.

| \( \mathcal{F}_c \) [kN] |
|-----------------|
| One wave \( (m = 1) \) | 16.4 |
| Two waves \( (m = 2) \) | 15.7 |
| Three waves \( (m = 3) \) | 15.6 |

4. Conclusions
We have conducted a numerical investigation of nonlinear stability of thin-walled C-section with uniformly varying inertia along the beam axis. First, we have derived the equilibrium equations of the analysed problem from the Vlasov theory [1-3]. The mathematical formulation of this problem leads to a homogeneous system of three linear ordinary differential equations with non-constant coefficients, whose exact solution is challenging to obtain in closed form. Therefore, we have numerically predicted an upper bound of the buckling load by means of the Dirichlet variational approach and the Rayleigh-Ritz algorithm. The numerical results presented in Sect. 3 demonstrate that the critical load corresponding to coupled flexural-torsional buckling asymptotically converges with the increasing order of test functions. This remarkable result leads us to conclude that the proposed numerical approach is a robust tool for obtaining technically reliable predictions of the true buckling load of beams with non-uniform cross-section. It is worth noting that such approach can be usefully generalised to the stability analysis of a variety of mechanical systems, with special focus on mechanical metamaterials, hierarchical metamaterials and structures equipped with curved members [10–36]. An experimental validation of the proposed theory is also addressed to future research, through the fabrication and testing of physical models at a reduced scale, on using, e.g., sustainable additive manufacturing techniques [37-44]. Future extensions of the present research will also include an experimental study of the effects played by stiffening diaphragms on the lateral-torsional behaviour of thin-walled beams with non-uniform cross-sections.

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