Structure of 2-skeletons of higher dimensional regular polytopes

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Abstract

In our paper [2], we have classified simple regular polyhedral BP-complexes, which are polyhedral complexes satisfying certain natural conditions on their vertex structures. As an addendum of this classification, we prove that 2-skeletons of higher dimensional regular polytopes are simple regular polyhedral complexes, but not polyhedral BP-complexes. The proof is done by a detailed investigation of their vertex structures.

1. Introduction

Let $X$ be a connected homogeneous 2-dimensional locally finite cell complex equipped with a piecewise linear metric. By definition, $X$ is homogeneous if each 1-cell of $X$ is a proper face of some 2-cell. We will introduce some definitions according to [2]. We call $X$ a polyhedral complex if it is obtained by gluing isometric edges of convex polygons. More precisely, we assume that $X$ satisfies the following:

1. The boundary of each 2-cell of $X$ has at least 3 vertices.
2. $X$ is equipped with a piecewise linear metric $d$ such that if $e$ is a 2-cell of $X$ with $n$ vertices, then with respect to the intrinsic metric induced by $d$, the closure $\bar{e}$ of $e$ is isometric to some $n$-gon in the Euclidean plane.

A polyhedral complex $X$ is said to be simple if the intersection of any two faces is connected, and regular if every face of $X$ is congruent to a regular polygon and the space of directions at every vertex is isometric to each other.

Recall that the space of directions $\Sigma_v$ at a vertex $v$ on a polyhedral complex $X$ is by definition the set of initial directions of unit speed geodesics emanating from $v$ with the extended interior metric induced from the angle metric between two directions. That is, $\Sigma_v$ is isometric to a distance sphere $S_\ell(v)$ equipped with the extended intrinsic
metric induced from \( d/\varepsilon \) for a sufficiently small \( \varepsilon > 0 \), where \( d \) is a piecewise linear metric on \( X \).

For a regular polyhedral complex \( X \), the metric graph isometric to the space of directions at a vertex of \( X \) is called the vertex structure of \( X \), and denoted by \( \Sigma_X \).

A polyhedral complex \( X \) is called a polyhedral BP-complex if it satisfies the following:

(B) For any vertex \( v \in X \), \( \Sigma_v \) is a Blaschke graph, that is, the diameter and the injectivity radius of \( \Sigma_v \) are the same.

(P) For any vertex \( v \in X \), the diameter of \( \Sigma_v \) is less than \( \pi \).

If \( X \) is regular, the above two conditions imply that \( \Sigma_X \) is a Blaschke graph and the diameter of it is less than \( \pi \).

In our previous paper [2], we have studied simple regular polyhedral BP-complexes, and proved that they are either the surface of a Platonic solid, a \( p \)-dodecahedron, a \( p \)-icosahedron, an \( m \)-covered regular \( n \)-gon for some \( m \geq 2 \) or a complete tripartite polyhedron. We have also remarked that 2-skeletons of higher dimensional regular polytopes are simple regular polyhedral complexes, but not polyhedral BP-complexes without proof. As an addendum to this remark in [2], we prove in this paper the following

**Theorem 1.** The 2-skeleton of a regular polytope of dimension greater than 3 does not satisfy the condition (B), that is, it is not a polyhedral BP-complex.

Note that the 3-dimensional regular polytopes are the Platonic solids. Therefore any 2-skeleton of them is a regular polyhedron, that is, a simple regular polyhedral BP-complex.

To prove this theorem, we note that regular polytopes of dimension greater than 3 are classified as follows (cf. Chapters VII and VIII of [1]):

1. Three ordinary classes of regular polytopes: a regular simplex, a cross polytope and a measure polytope.

2. Three special four-dimensional regular polytopes: an octaplex (24-cell), a dodecaplex (120-cell) and a tetraplex (600-cell).

Then, for each \( n \)-dimensional polytope of these classes with \( n \geq 4 \), we look into their 2-skeletons and determine the diameter and the injectivity radius of the vertex structure. This concludes that they do not satisfy the condition (B).

2. Three ordinary classes of regular polytopes

2.1. Regular \( n \)-simplex

Let \( \alpha_n \) be a regular \( n \)-simplex with vertices \( \{ v_0, v_1, \ldots, v_n \} \). Then the 2-skeleton \( A_n \) of \( \alpha_n \) is formed by the \( n+1 \) vertices \( \{ v_0, v_1, \ldots, v_n \} \), \( n+1C_2 \) edges \( \{ v_i v_j (= v_j v_i) | i < j \} \) and \( n+1C_3 \) faces (regular triangles) \( \{ v_i v_j v_k | i < j < k \} \), where \( \binom{i}{j} \) is the number of combinations choosing \( j \) elements out of \( i \) elements. Then \( A_n \) is a simple regular
polyhedral complex, and the vertex structure $\Sigma_{A_n}$ is a complete graph of $n$ vertices with a metric such that each edge is $\pi/3$ in length. Hence the injectivity radius of $\Sigma_{A_n}$ is $\pi/2$ and the diameter of $\Sigma_{A_n}$ is $2\pi/3$ for $n \geq 4$, that is, $A_n$ is a simple regular polyhedral complex satisfying the condition $(P)$, but not $(B)$ for $n \geq 4$.

In fact, for any point $x \in \Sigma_{A_n}$, there exists a loop of circumference $\pi$ containing $x$ on $\Sigma_{A_n}$, which is the shortest embedded circle on $\Sigma_{A_n}$. So the minimizing property of a geodesic emanating from $x$ is broken at the distance $\pi/2$, and a geodesic segment of length less than $\pi/2$ is minimizing. Therefore the injectivity radius of $\Sigma_{A_n}$ is $\pi/2$.

Next, we consider the diameter of $\Sigma_{A_n}$. For the vertex $v_0 \in \alpha_n$, the 1-skeleton of the link $L_{v_0}$ of $v_0$ with the induced metric from $\alpha_n$ is homothetic to $\Sigma_{v_0} \cong \Sigma_{A_n}$. Let $x \in \Sigma_{A_n}$ be a mid point of the edge corresponding to $v_1v_2$ of $L_{v_0}$, and $y$ a mid point of the edge corresponding to $v_3v_4$. Then the distance from $x$ to $y$ is $2\pi/3$. It is also easily seen that, for any two point of $\Sigma_{A_n}$, the distance of these two points is less than or equal to $2\pi/3$. Hence we have the diameter of $\Sigma_{A_n}$ is $2\pi/3$.

2.2. Cross polytope

An $n$-dimensional cross polytope $\beta_n$ is defined from $\beta_{n-1}$ inductively as follows: $\beta_1$ is a segment with two vertices $\{v_1, w_1\}$ as end points. $\beta_2$ is a dipyramid (a suspension) with new vertices $\{v_2, w_2\}$ based on $\beta_1$, that is, a join $\beta_1 \ast \{v_2, w_2\}$, which is a solid square. $\beta_3$ is a dipyramid with $\{v_3, w_3\}$ based on $\beta_2$, namely a 3-fold join $\beta_1 \ast \{v_2, w_2\} \ast \{v_3, w_3\}$, a solid octahedron. $\beta_n$ is obtained as a dipyramid with $\{v_n, w_n\}$ based on $\beta_{n-1}$, namely a $n$-fold join $\beta_1 \ast \{v_2, w_2\} \ast \cdots \ast \{v_n, w_n\}$. Note here that the interior of $\beta_n$ is considered to consist of exactly one $n$-cell, not two $n$-cells. For example, $\beta_2$ consists of four vertex $\{v_1, w_1, v_2, w_2\}$, four 1-cells $\{v_1v_2, v_2w_1, w_1w_2, w_2v_1\}$ and just one 2-cell $\{v_1v_2w_1w_2\}$. Also note that it has a metric such that every edge of $\beta_n$ has same length.

The 2-skeleton $B_3$ of $\beta_3$ is an octahedron. In general, the 2-skeleton $B_n$ of $\beta_n$ is constructed by $2n$ vertices $\{v_1, w_1, \ldots, v_n, w_n\}$, $2^2 \cdot nC_2$ edges $\{v_iv_j, v_iw_j, w_iw_j \mid i < j\}$ and $2^3 \cdot nC_3$ faces (regular triangles) $\{v_iv_jvk, v_iv_jwk, v_iw_jvk, v_iw_jwk, w_iv_jvk, w_iw_jvk, w_iw_jwk \mid i < j < k\}$.

Then $B_n$ is a simple regular polyhedral complex, and the vertex structure $\Sigma_{B_n}$ is homeomorphic to a 1-skeleton of $\beta_{n-1}$ with a metric such that each edge is $\pi/3$ in length.

For example, $\Sigma_{B_n} \cong \Sigma_{v_1}$ corresponds to the 1-skeleton of $\{v_2, w_2\} \ast \cdots \ast \{v_n, w_n\}$. Hence the injectivity radius of $\Sigma_{B_n}$ is $\pi/2$, and the diameter of $\Sigma_{B_n}$ is $2\pi/3$ for $n \geq 4$. Hence $B_n$ is a simple regular polyhedral complex satisfying the condition $(P)$, but not $(B)$ for $n \geq 4$.

In fact, for any point $x$ on the 1-skeleton of $\{v_2, w_2\} \ast \cdots \ast \{v_n, w_n\}$, let $z_i$ be an edge containing $x$, where $z_i$ is either $v_i$ or $w_i$. From $n \geq 4$, we can take a third number $k$ with $k \neq i$ and $k \neq j$, and then a loop joining these vertices $z_i, z_j, z_k$ is the shortest embedded circle on the 1-skeleton. Therefore, for any point $x \in \Sigma_{B_n}$, we can take a loop containing $x$ on $\Sigma_{B_n}$, which is the shortest embedded circle of circumference $\pi$. So, as in the case $A_n$, the injectivity radius of $\Sigma_{B_n}$ is $\pi/2$.

Next, we consider the diameter of $\Sigma_{B_n} \cong \Sigma_{v_1}$. Let $x \in \Sigma_{v_1}$ be a mid point of the edge corresponding to $v_2v_3$, and $y$ a mid point of the edge corresponding to $w_2w_3$. Then the distance from $x$ to $y$ is $2\pi/3$. It is also easily seen that, for any two points
of Σ, the distance of these two points is less than or equal to 2π/3. Hence we have the diameter of Σ is 2π/3.

2.3. Measure polytope (hypercube)

For a positive number a, let I be a closed interval [0, a]. A n-dimensional measure polytope γn, which is also called a hypercube, is obtained as n Cartesian products of I in Rn. Then the vertices of γn are the set of 2n points V = {x = (x1, x2, · · · , xn) ∈ Rn | xi = 0, a}. Then it is easily seen that, for two vertices v = (v1, · · · , vn), w = (w1, · · · , wn) ∈ V, v is adjacent to w if and only if there exists a number i such that vi ̸= vi and vj = wj for j ̸= i. For any vertex v ∈ V, the number of adjacent vertex to v is n, that is, the number of edges at v is n, and hence the number of the edges of γn is n · 2n/2 = n · 2n−1. A 2-dimensional face of γn is obtained as

\{x_1\} × · · · × I × · · · × I × · · · × \{x_n\}

for 1 ≤ i < j ≤ n, where xk is 0 or a for any k ̸= i, j. So the number of faces is nC2 · 2n−2. These vertices, edges and faces form the 2-skeleton Cn of γn, which is a simple regular polyhedral complex.

For a vertex v = (v1, · · · , vn) ∈ V, the space of directions Σv of Cn is a complete graph of n vertices with a metric such that each edge is π/2 in length. In fact, regarding Σv as Sε(γ) for a sufficiently small ε, each vertex or edge of Σv is corresponding to the edge or face at v on Cn containing it, respectively. Let vx and vy be two distinct edges at v. In order to confirm that Σv is a complete graph, it suffices to find the face containing the vertices v, x and y. Since the vertices x = (x1, · · · , xn) and y = (y1, · · · , yn) are adjacent to v, there are two distinct numbers i, j such that vi ̸= xi and vj ̸= yj and the other components of x and y are equal to that of v. Since the face (square)

\{x_1\} × · · · × I × · · · × I × · · · × \{x_n\}

contains the vertices v, x and y, the two vertices on Σv corresponding to the edges vx and vy are adjacent. Note that x is not adjacent to y on γn. This implies that Σv is a complete graph with a metric such that the length of each edge is π/2 which is the interior angle of a square.

In the same way as the case of An, we have the injectivity radius of Σv is 3π/4 and the diameter is π. Therefore Cn is a simple regular polyhedral complex satisfying neither the condition (B) nor (P).

3. Three special four-dimensional regular polytopes

Next, we discuss the vertex structures of 2-skeletons of three special four-dimensional regular polytopes. In order to investigate them, it will be fine to use the Schläfi symbol. So we recall the Schläfi symbol.

The Schläfi symbol is a notation of the form \{p, q, r, · · · \}. For integer p, the symbol \{p\} denotes a regular p-gon, namely p-sided regular polygon. The symbol \{p, q\} denotes a Platonic solid whose surface is formed by regular p-gons, with q of
them surrounding each vertex. For example, the Schl"afli symbol of a Platonic solid whose surface is a tetrahedron is \{3,3\}, and in the case of an octahedron it is \{3,4\}. A regular 4-dimensional polytope with \(r\) Platonic solids of type \(\{p,q\}\) surrounding each edge is represented by \(\{p,q,r\}\).

Although it is not necessary in this paper, we will introduce the Schl"afli symbol in general cases. In general, the Schl"afli symbol is defined inductively as follows: a regular \(n\)-polytope is of type \(\{p_1,p_2,\cdots,p_{n-2},p_{n-1}\}\) if it has \(p_{n-1}\) numbers of \((n-1)\)-polytopes of type \(\{p_1,p_2,p_3,\cdots,p_{n-2}\}\) around each \((n-3)\)-face. In particular, a regular \(n\)-simplex \(\alpha_n\) is \(\{3,3,\cdots,3\}\), an \(n\)-dimensional cross polytope \(\beta_n\) is \(\{3,3,\cdots,4\}\), and an \(n\)-dimensional measure polytope \(\gamma_n\) is \(\{4,3,\cdots,3\}\).

### 3.1. Octaplex (24-cell)

It is known that an octaplex is consists of 24 solid octahedra, with 24 vertices, 96 edges and 96 faces. It is also known the Schl"afli symbol of an octaplex is \(\{3,4,3\}\), that is, an octaplex consists of solid octahedra, with 3 of them around each edge.

Let \(S_{24}\) be a 2-skeleton of an octaplex, where 24 means the number of constituent solid octahedra. For a vertex \(v\in S_{24}\), the space of directions \(\Sigma_v\) is isometric to the 1-skeleton of a regular cube whose edges are of length \(\pi/3\).

In fact, it is known that the vertex figure of an octaplex \(\{3,4,3\}\) is a solid cube \(\{4,3\}\). More exactly, the number of edges containing \(v\) is \((\text{the number of edges}) \times 2 / (\text{the number of vertices}) = 96 \times 2 / 24 = 8\), which is the number of the vertices of \(\Sigma_v\). Since the regular octahedron is a constituent element, the shortest embedded circle on \(\Sigma_v\) is a 4-cycle, a cycle of length 4. Surrounding 3 octahedra around each edge of octaplex, the 3 faces(squares) gather around a vertex on the vertex figure. Therefore \(\Sigma_v\) is isometric to the 1-skeleton of a regular cube with a metric such that the length of each edge is \(\pi/3\), the interior angle of a regular triangle.

Now it is clear that the injectivity radius and the diameter of \(\Sigma_v\) are \(2\pi/3\) and \(\pi\), respectively. In more detail, since the shortest embedded circle is a 4-cycle of the circumference \(4\pi/3\), the injectivity radius is \(2\pi/3\). The diameter \(\pi\) is attained as the distance between a vertex and the antipodal vertex (as vertices on a cube) of \(\Sigma_v\). Hence \(S_{24}\) is a simple regular polyhedral complex satisfying neither the condition (B) nor (P).

### 3.2. Dodecaplex (120-cell)

A dodecaplex is consists of 120 solid dodecahedra, with 600 vertices, 1200 edges and 720 faces. The Schl"afli symbol of a dodecaplex is \(\{5,3,3\}\), that is, a dodecaplex consists of solid dodecahedra, with 3 of them around each edge.

Let \(S_{120}\) be a 2-skeleton of a dodecaplex. For a vertex \(v\in S_{120}\), the space of directions \(\Sigma_v\) is isometric to the 1-skeleton of a regular tetrahedron whose edges are of length \(3\pi/5\).

In fact, it is known that the vertex figure of a dodecaplex \(\{5,3,3\}\) is a solid regular tetrahedron \(\{3,3\}\). More exactly, the number of edges at \(v\) is \(1200 \times 2 / 600 = 4\), which is the number of the vertices of \(\Sigma_v\). Since the regular dodecahedron is a constituent element, the shortest embedded circle on \(\Sigma_v\) is a 3-cycle. Surrounding 3 dodecahedra around each edge of dodecaplex, the 3 faces(triangles) gather around a vertex on the vertex figure. Therefore \(\Sigma_v\) is isometric to the 1-skeleton of a regular tetrahedron with
a metric such that the length of each edge is $3\pi/5$, the interior angle of a regular pentagon.

Then the injectivity radius and the diameter of $\Sigma_v$ are $9\pi/10$ and $6\pi/5$, respectively. In more detail, since the shortest embedded circle is a $3$-cycle of the circumference $9\pi/5$, the injectivity radius is $9\pi/10$. The diameter $6\pi/5$ is attained as the distance between such two points $x_1$ and $x_2$ that $x_i$ is a mid point of a edge $e_i$ of $\Sigma_v$ with $e_1 \cap e_2 = \emptyset$. Hence $S_{120}$ is a simple regular polyhedral complex satisfying neither the condition (B) nor (P). It is notable that the diameter of $\Sigma_v$ is greater than $\pi$.

3.3. Tetraplex(600-cell)

A tetraplex is consists of 600 solid tetrahedra, with 120 vertices, 720 edges and 1200 faces. The Schl"afli symbol of a tetraplex is $\{3, 3, 5\}$, that is, a tetraplex consists of solid tetrahedra, with 5 of them around each edge.

Let $S_{600}$ be a 2-skeleton of a tetraplex. For a vertex $v \in S_{600}$, the space of directions $\Sigma_v$ is isometric to the 1-skeleton of a regular icosahedron whose edges are of length $\pi/3$.

In fact, it is known that the vertex figure of a tetraplex $\{3, 3, 5\}$ is a solid regular icosahedron $\{3, 5\}$. More exactly, the number of edges at $v$ is $720 \times 2/120 = 12$, which is the number of the vertices of $\Sigma_v$. Since the regular tetrahedron is a constituent element, the shortest embedded circle on $\Sigma_v$ is a 3-cycle. Surrounding 5 tetrahedra around each edge of tetraplex, the 5 faces(triangles) gather around a vertex on the vertex figure. Therefore $\Sigma_v$ is isometric to the 1-skeleton of a regular icosahedron with a metric such that the length of each edge is $\pi/3$, the interior angle of a regular triangle.

Then the injectivity radius of $\Sigma_v$ is $\pi/2$, since the shortest embedded circle is a 3-cycle of the circumference $\pi$. We also see the diameter of $\Sigma_v$ is $\pi$ as follows.

For any vertex $x \in \Sigma_v$, there is the unique antipodal vertex $y \in \Sigma_v$ (as vertices on a icosahedron). Then $d(x, y) = 3 \cdot \pi/3 = \pi$, where $d$ is the distance on $\Sigma_v$. Hence the diameter is not less than $\pi$.

For any two points $p, q \in \Sigma_v$, let $\{v_1, v_2\}$ and $\{w_1, w_2\}$ be two pair of adjacent vertices on $\Sigma_v$ such that $p \in v_1v_2$ and $q \in w_1w_2$. If $v_i$ is the antipodal vertex of $w_j$ of $\Sigma_v$, we may assume $v_1$ is the antipodal vertex of $w_2$ by changing the indices if necessary. Then $v_i$ is not the antipodal vertex of $w_i$ for $i = 1, 2$. Let $e_i$ be the shortest path joining $v_i$ and $w_i$. Since the end points of $e_i$ are not antipodal vertices to each other, the length of $e_i$ is either $0, \pi/3$ or $2\pi/3$. Then $e_1 \cup w_1w_2 \cup e_2 \cup v_2v_1$ is a loop passing through $p, q$, whose length is less than or equal to $2\pi$. Therefore $d(p, q) \leq \pi$. If any pair $\{v_i, w_j\}$ are not antipodal vertices to each other, $v_i$ is not the antipodal vertex of $w_i$ for $i = 1, 2$ naturally. Therefore we have $d(p, q) \leq \pi$ in the same way as above. Since $\pi$ is attained as the distance between a vertex and the antipodal vertex, we have the diameter of $\Sigma_v$ is $\pi$.

Hence $S_{600}$ is a simple regular polyhedral complex satisfying neither the condition (B) nor (P).

Summarizing the above arguments in sections 2 and 3, we complete the proof of Theorem 1.
References

[1] H.S.M. Coxeter, *Regular polytopes*, Dover, New York, 1973.

[2] J. Itoh and F. Ohtsuka, *A natural generalization of regular convex polyhedra*, Topol. Appl. 219 (2017) 43-54.