LANTERN RELATIONS AND RATIONAL BLOWDOWNS

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ABSTRACT. We discuss a connection between the lantern relation in mapping class groups and the rational blowing down process for 4-manifolds. More precisely, if we change a positive relator in Dehn twist generators of the mapping class group by using a lantern relation, the corresponding Lefschetz fibration changes into its rational blowdown along a copy of the configuration $C_2$. We exhibit examples of such rational blowdowns of Lefschetz fibrations whose blowup is homeomorphic but not diffeomorphic to the original fibration.

1. Introduction

Lefschetz fibrations relate the topology of symplectic 4-manifolds to the combinatorics on positive relators in Dehn twist generators of mapping class groups. Fuller introduced a substitution technique for constructing positive relators to obtain an example of non-holomorphic Lefschetz fibrations of genus three [15], [14]. Many constructions of Lefschetz fibrations as positive relators can be interpreted as generalizations of his construction (cf. [5]), while it has been less investigated what such substitutions mean geometrically.

In this paper we study a particular substitution, the lantern substitution (or the $L^±_1$ substitution in short), for positive relators of mapping class groups. The corresponding surgical operation on Lefschetz fibrations turns out to be the rational blowing down process, which was discovered by Fintushel and Stern [6], along a copy of the configuration $C_2$ (i.e. a $-4$-framed unknot in Kirby diagrams). Applying a theorem of Usher [19], we give examples of such rational blowdowns of Lefschetz fibrations whose blowup is homeomorphic but not diffeomorphic to the original fibration.

In Section 2 we review the lantern relation in mapping class groups and define the lantern substitution for positive relators. We discuss a relation between lantern relations and rational blowdowns in Section 3 and state the main theorem in Section 4. We then exhibit some examples in Section 5 and end by observing other relations in Section 6.

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2. Lantern relations and substitutions

Let $\Sigma_g$ be a closed oriented surface of genus $g \geq 2$ and $\mathcal{M}_g$ the mapping class group of $\Sigma_g$. We denote by $\mathcal{F}$ the free group generated by all isotopy classes $\mathcal{S}$ of simple closed curves on $\Sigma_g$. There is a natural epimorphism $\varpi : \mathcal{F} \to \mathcal{M}_g$ which sends (the isotopy class of) a simple closed curve $a$ on $\Sigma_g$ to the right-handed Dehn twist $t_a$ along $a$. We set $\mathcal{R} := \mathrm{Ker} \varpi$ and call each element of $\mathcal{R}$ a relator in the generators $\mathcal{S}$ of $\mathcal{M}_g$. A word in the generators $\mathcal{S}$ is called positive if it includes no negative exponents. We put $w(c) := t_{a_1}^{e_1} \cdots t_{a_r}^{e_r}(c) \in \mathcal{S}$ for $c \in \mathcal{S}$ and $W = a_1^{e_1} \cdots a_r^{e_r} \in \mathcal{F}$ ($a_1, \ldots, a_r \in \mathcal{S}, e_1, \ldots, e_r \in \{ \pm 1 \}$), and put $wV := w(c_1) \cdots w(c_n) \in \mathcal{F}$ for $V = c_1 \cdots c_n \in \mathcal{F}$ ($c_1, \ldots, c_n \in \mathcal{S}$).

We begin with a precise definition of the lantern relation [2], [9].

**Definition 2.1.** Let $a$ and $b$ be simple closed curves on $\Sigma_g$ with geometric intersection number $2$ and algebraic intersection number $0$. We orient $a$ and $b$ locally on a neighborhood of each intersection point $p \in a \cap b$ such that the intersection number $(a \cdot b)_p$ at $p$ is $+1$. Resolving all intersection points according to the local orientations, we obtain a new simple closed curve $c$. A regular neighborhood of $a \cup b$, which can be chosen to include $c$, is a genus $0$ subsurface $\Sigma$ of $\Sigma_g$ with four boundary components. We denote simple closed curves parallel to four boundary components of $\Sigma$ by $d_1, d_2, d_3$, and $d_4$. The relation

$$t_{d_1} t_{d_2} t_{d_3} t_{d_4} = t_a t_b t_c$$

is called the lantern relation. We put $L := L(a, b) = abcd_1^{-1}d_2^{-1}d_3^{-1}d_4^{-1} \in \mathcal{R}$.

Let $\varrho \in \mathcal{R}$ ($\varrho \neq 1$) be a positive relator of $\mathcal{M}_g$. Let $a, b, c, d_1, d_2, d_3$, and $d_4$ be curves as in Definition 2.1. Suppose that $\varrho$ includes $d_1 d_2 d_3 d_4$ as a subword:

$$\varrho = U \cdot d_1 d_2 d_3 d_4 \cdot V \quad (U, V \in \mathcal{F})$$

Since $\varrho$ and $U \cdot L \cdot U^{-1}$ are both relators of $\mathcal{M}_g$, the positive word

$$\varrho' = U \cdot abc \cdot V = U \cdot abcd_1^{-1}d_2^{-1}d_3^{-1}d_4^{-1} \cdot d_1 d_2 d_3 d_4 \cdot V = U \cdot L \cdot U^{-1} \cdot \varrho$$

is also a relator of $\mathcal{M}_g$. The length of the word $\varrho'$ is equal to that of $\varrho$ minus one.

**Definition 2.2.** We say that $\varrho'$ is obtained by applying an $L$-substitution to $\varrho$. Conversely, $\varrho$ is said to be obtained by applying an $L^{-1}$-substitution to $\varrho'$. We also call these two kinds of operations lantern substitutions (cf. [5]).

We next recall a definition of Lefschetz fibrations (cf. [13], [8]).

**Definition 2.3.** Let $M$ be a closed oriented smooth $4$-manifold. A smooth map $f : M \to S^2$ is called a Lefschetz fibration of genus $g$ if it satisfies the following conditions:

(i) $f$ has finitely many critical values $b_1, \ldots, b_n \in S^2$ and $f$ is a smooth fiber bundle over $S^2 - \{ b_1, \ldots, b_n \}$ with fiber $\Sigma_g$;

(ii) for each $i$ ($i = 1, \ldots, n$), there exists a unique critical point $p_i$ in the singular fiber $f^{-1}(b_i)$ such that $f$ is locally written as $f(z_1, z_2) = z_1^2 + z_2^2$ with respect to some local complex coordinates around $p_i$ and $b_i$ which are compatible with orientations of $M$ and $S^2$;

(iii) no fiber contains a $-1$-sphere.

**Remark 2.4.** A more general definition can be found in Chapter 8 of [8]. We treat also Lefschetz fibrations with boundary in the proof of Theorem 3.1.
Suppose that \( g \geq 2 \). According to theorems of Kas and Matsumoto, there exists a one-to-one correspondence between the isomorphism classes of Lefschetz fibrations and the equivalence classes of positive relators modulo simultaneous conjugations
\[
c_1 \cdots c_n \sim w(c_1) \cdots w(c_n),
\]
and elementary transformations
\[
c_1 \cdots c_{i+1} \cdots c_n \sim c_1 \cdots c_{i+1} (c_i) \cdots c_n,
\]
where \( c_1 \cdots c_n \in \mathcal{R} \) is a positive relator in the generator \( \mathcal{S} \) and \( w \in \mathcal{F} \). This correspondence is described by using the holonomy (or monodromy) homomorphism induced by the classifying map of \( f \) restricted on \( S^2 - \{b_1, \ldots, b_n\} \) (cf. \([8],[13]\), and \([3]\)). We denote (the isomorphism class of) a Lefschetz fibration associated to a positive relator \( \varrho \in \mathcal{R} \) by \( M_{\varrho} \to S^2 \).

Let \( \varrho, \varrho' \in \mathcal{R} \) be positive relators of \( \mathcal{M}_g \) and \( M_{\varrho}, M_{\varrho'} \) the corresponding Lefschetz fibrations over \( S^2 \), respectively. Suppose that the relator \( \varrho' \) is obtained by applying an \( L \)-substitution to the relator \( \varrho \). The Euler characteristic and the signature of a Lefschetz fibration \( M_{\varrho'} \to S^2 \) with monodromy \( \varrho' \) are computed as follows:
\[
e(M_{\varrho'}) = e(M_{\varrho}) - 1, \quad \sigma(M_{\varrho'}) = \sigma(M_{\varrho}) + 1
\]
([5], Theorem 4.3 and Proposition 3.12). We investigate relations between \( M_{\varrho} \) and \( M_{\varrho'} \) and several properties of them in the subsequent sections.

3. Rational blowdowns via lantern relations

Let \( \varrho, \varrho' \in \mathcal{R} \) be positive relators of \( \mathcal{M}_g \) and \( M_{\varrho}, M_{\varrho'} \) the corresponding Lefschetz fibrations over \( S^2 \), respectively.

**Theorem 3.1.** If \( \varrho' \) is obtained by applying an \( L \)-substitution to \( \varrho \), then the 4-manifold \( M_{\varrho'} \) is a rational blowdown of \( M_{\varrho} \) along a configuration \( C_2 \subset M_{\varrho} \).

**Proof.** We take a subsurface \( \Sigma \) of \( \Sigma_g \) and curves \( a, b, c, d_1, d_2, d_3, d_4 \) on \( \Sigma \) as in Definition [21]. Let \( N, N' \) be Lefschetz fibrations over \( D^2 \) with fiber \( \Sigma \) corresponding to the positive words \( d_1 d_2 d_3 d_4, abc \), respectively.

![Figure 1](image)

Drawing a Kirby diagram of \( N \), sliding the central \( -1 \)-framed unknot over other three \( -1 \)-framed unknots, and canceling three 1-handle/2-handle pairs, we obtain a \( -4 \)-framed unknot (Figure 1). Thus \( N \) is diffeomorphic to the total space of a \( D^2 \)-bundle over \( S^2 \) with Euler number \( -4 \), which is denoted by \( C_2 \) in [5] (see also [8], Section 8.5).
Drawing a Kirby diagram of $N'$ and sliding and canceling handles as in Figure 2, we obtain a pair of a dotted circle and a $+1$-framed unknot with linking number +2. This means that $N'$ is diffeomorphic to a rational 4-ball with boundary $L(4,1)$, which is denoted by $B_2$ in \[\text{(6)}\] (see also \[\text{8}\], Section 8.5).

From construction, $N$ (resp. $N'$) can be considered a submanifold of $M_\rho$ (resp. $M_\rho'$). It is also easily seen that $M_\rho - \text{int} \ N$ and $M_\rho' - \text{int} \ N'$ are diffeomorphic to each other. Hence we have

$$M_\rho \approx N' \cup_{\partial N} (M_\rho - \text{int} \ N) \approx B_2 \cup_{L(4,1)} (M_\rho - \text{int} \ C_2).$$

This completes the proof of Theorem 3.1. \hfill \Box

4. Smooth structures

Let $\varrho, \varrho' \in \mathcal{R}$ be positive relators of $M_\varrho$ and $M_\varrho, M_\varrho'$ the corresponding Lefschetz fibrations over $S^2$, respectively. Suppose that $\varrho'$ is obtained by applying $k$ times $L$-substitutions ($k \geq 1$), elementary transformations, and simultaneous conjugations to $\varrho$. Suppose also that $e(M_\varrho) + \sigma(M_\varrho) \geq 2$. We choose a positive relator $\zeta \in \mathcal{R}$ ($\zeta \neq 1$) such that $M_\zeta - \nu F$ is simply-connected and either the word $\zeta$ includes at least one separating curve as a factor, or $\sigma(M_\varrho) + \sigma(M_\zeta)$ is not divisible by 16. Here $\nu F$ is an open fibered neighborhood of a regular fiber $F$ of $M_\zeta$. Taking a fiber sum of $M_\varrho$ (resp. $M_\varrho'$) with $M_\zeta$, we obtain a new Lefschetz fibration $M_1 := M_\varrho \#_F M_\zeta$ (resp. $M_2 := M_\varrho' \#_F M_\zeta$) with monodromy $\varrho \cdot W_\zeta$ (resp. $\varrho' \cdot W_\zeta$) for some $W \in \mathcal{F}$ (resp. $W' \in \mathcal{F}$). It is obvious that $\varrho \cdot W_\zeta$ is obtained by applying $k$ times $L$-substitutions, elementary transformations, and simultaneous conjugations to $\varrho \cdot W_\zeta$.

**Theorem 4.1.** The 4-manifold $M_1$ is homeomorphic but not diffeomorphic to a $k$ times blowup $M_2 \# k\mathbb{C}P^2$ of $M_2$. Moreover, both of these 4-manifolds do not dissolve.

**Proof.** Let $j : F_i \hookrightarrow M_i$ be the inclusion map from a general fiber $F_i$ into the total space $M_i$ ($i = 1, 2$). The induced homomorphism $j_\# : \pi_1(F_i) \to \pi_1(M_i)$ is surjective and the kernel of $j_\#$ includes the normal subgroup $N$ of $\pi_1(M_i)$ generated by the vanishing cycles of $M_i$ (cf. \[\text{1}\], Lemma 3.2). Since $M_i - \nu F$ is simply-connected and $j_\#$ is the composition of homomorphisms $\pi_1(F_i) \to \pi_1(M_i - \nu F) \to \pi_1(M_i)$, the group $\pi_1(M_i)$ must be trivial ($i = 1, 2$).

$M_1$ is a non-spin 4-manifold because either it has a component of a separating singular fiber which represents a homology class of square $-1$, or $\sigma(M_1)$ is not divisible by 16. It is easily seen from the observation above that $e(M_2) = e(M_1) - k$ and $\sigma(M_2) = \sigma(M_1) + k$. By virtue of Freedman’s classification theorem, both of $M_1$ and $M_2 \# k\mathbb{C}P^2$ is homeomorphic to $\# b_2^+(M_1)\mathbb{C}P^2 \# b_2^-(M_1)\overline{\mathbb{C}P}^2$ because they are simply-connected, non-spin, and have the same Euler characteristic and the same signature.

$M_1$ is a fiber sum $M_\varrho \#_F M_\zeta$ of non-trivial Lefschetz fibrations $M_\varrho$ and $M_\zeta$. By Gompf’s theorem (\[\text{8}\], Theorem 10.2.18), $M_1$ admits a symplectic structure with
symplectic fibers. It follows from a theorem of Usher \[19\] that $M'$ is a minimal symplectic 4-manifold. Since $b^+_2(M') = b^+_2(M) - b_1(M) + b^+_2(M) + 2g - 1 > 1$, $M'$ does not contain any smooth $-1$-sphere as a consequence of Seiberg-Witten theory ([17], [18], cf. [8], Remark 10.2.4(a)). On the other hand, $M_2\#k\mathbb{CP}^2$ has a natural smooth $-1$-sphere. Hence $M_1$ and $M_2\#k\mathbb{CP}^2$ cannot be diffeomorphic.

Because $M_1$ and $M_2\#k\mathbb{CP}^2$ admit symplectic structure and $b^+_2(M) > 1$, these manifolds can not be diffeomorphic to $\#b^+_2(M)\mathbb{CP}^2 + b^-_2(M)\overline{\mathbb{CP}}^2$ ([16], [10], cf. [8], Theorem 10.1.14). □

**Remark 4.2.** We do not use any explicit property of rational blowdowns to prove Theorem 4.1. The proof above is rather similar to that of Theorem 4.8 of [3]. It is likely that $M_0$ is homeomorphic but not diffeomorphic to $M_0'\#k\mathbb{CP}^2$ (without taking fiber sums with $M_1$) in a general setting. On the other hand, a certain rational blowdown along $C_2$ happens to be diffeomorphic to an honest blowdown of the original 4-manifold: $E(1)_2(\approx E(1)\approx \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2)$ is a rational blowdown of $E(1)\#\overline{\mathbb{CP}}^2\approx \mathbb{CP}^2\#10\overline{\mathbb{CP}}^2$ along $C_2$ ([6], Proposition 3.2, cf. [8], Theorem 8.5.9 and Theorem 8.3.11).

5. **Examples**

We apply theorems in previous sections to explicit examples.

**Example 5.1.** Let $\varrho := \tilde{F}^{\text{even}}_{g-h-1}\tilde{F}^{\text{even}}_{h-1}$ and $\varrho' := V_h(2 \leq h \leq g - 2)$ be the relators of $\mathcal{M}_g(g \geq 4)$ constructed in Section 4 of [5] and $\varsigma := Q$ the positive relator of $\mathcal{M}_g(g \geq 2)$ constructed in Section 4 of [3]. Since $\varrho'$ is obtained by applying an $L$-substitution to $\varrho$, it turns out from Theorem 3.1 that $M_{\varrho'}$ is a rational blowdown of $M_{\varrho}$ along a copy of $C_2$. The Euler characteristic and the signature of $M_{\varrho'}$ are equal to $12g^2 + 6g + 8gh - 8h^2 + 7$ and $-6g^2 - 8g - 4gh + 4h^2 - 3$, respectively. $M_{\varsigma} - \nu F$ is simply-connected and $\varsigma$ includes one separating curve. The Euler characteristic and the signature of $M_{\varrho}$ are $2g^2 + 7$ and $-(g^2 + 2g + 3)$ for even $g$, and $2g^2 + 4g + 7$ and $-(g + 2)^2$ for odd $g$, respectively. We set $M_1 := M_{\varrho}\#F M_{\varsigma}$ and $M_2 := M_{\varrho}'\#F M_{\varsigma}$. It follows from Theorem 4.1 that $M_1$ is homeomorphic but not diffeomorphic to $M_2\#\overline{\mathbb{CP}}^2$ and both of these do not dissolve. If we use $Q^n(n \geq 2)$ instead of $Q$, we obtain infinitely many pairs of homeomorphic but non-diffeomorphic 4-manifolds for a fixed $g (\geq 4)$.

**Example 5.2.** Let $X_3$ and $X_{3,3}$ be the Lefschetz fibrations of genus 3 defined in §4 of [3]. A positive relator $\varrho$ (resp. $\varrho'$) representing the monodromy of $X_3$ (resp. $X_{3,3}$) is given as follows (see Figure 3, Figure 4, and Figure 2 of [3]).

$$
\varrho := (c_1c_2x_1c_3c_6c_4c_3c_2c_5c_6c_7)^3, \quad \varrho' := (\tilde{y}_1x_1ty_5y_2c_8f_1c_8c_2\tilde{x}_3r_3)^3,
$$

where we put $r := f_3^{-1}(c_4)$. We apply elementary transformations and simultaneous conjugations to $\varrho$ as follows.

$$
\varrho = (c_1c_2x_1c_3c_6c_4x_2c_5c_6c_7)^3 = (c_1c_2x_1c_3f_1^{-1}(c_4) \cdot c_8c_2c_4x_2c_5c_6c_7)^3
$$

$$
\sim f_1^{-1}(c_2c_3x_1c_3c_6c_4 \cdot f_1(c_4) \cdot x_2c_5c_6c_7)^3
$$

$$
\sim (c_1c_2) \cdot c_1c_3c_4c_6c_8c_5 \cdot f_1(c_4) \cdot x_2c_5c_6c_7)^3
$$

$$
\sim (c_1c_2) \cdot x_1c_3c_4c_6c_8 \cdot f_1(c_4) \cdot x_2c_5c_6c_7 \cdot c_1)^3
$$
Example 5.3. Let $\varrho$ (resp. $\varrho'$) be a positive relator of $\mathcal{M}_2$ given as follows (see Figure 5, Figure 6, Figure 7, and Figure 4 of \cite{[4]}).

\begin{align*}
\varrho &= (c_5c_4c_3c_2c_1c_0c_4c_3c_2c_1)^2, \\
\varrho' &= c_3(\delta)_{c_5c_4c_3c_2c_1}(x) \cdot \bar{k}h c_5^{-1}(c_5c_4c_3c_2c_1) \cdot k \cdot c_3^{-1}(h)_{c_5c_4c_3c_2c_1} = \tau' (\bar{y}_1 := c_1(c_2))
\end{align*}

We apply elementary transformations to $\varrho'$ as follows.

\begin{align*}
\varrho' &= (\bar{y}_1 x_1 c_5c_4c_3c_2c_1c_0c_4c_3c_2c_1)^3 = (\bar{y}_1 x_1 c_5c_4c_3c_2c_1c_0c_4c_3c_2c_1)^3 \\
&= (\bar{y}_1 x_1 c_5c_4c_3c_2c_1c_0c_4c_3c_2c_1)^3 \\
&= (\bar{y}_1 x_1 c_5c_4c_3c_2c_1c_0c_4c_3c_2c_1)^3
\end{align*}

Thus $\tau'$ is obtained by applying three times $L$-substitutions to $\tau$ by virtue of the lantern relation $f_1tv = c_1c_5c_7$, and $M_2 = X_3$ turns out to be a rational blowdown of $M_0 = X_3$ along three copies of $C_2$ from Theorem 5.1.

We set $c := (c_1c_2c_3c_4c_5c_6c_7c_8c_9c_{10})^2 \in R$ and put $M_1 := M_0 #_p M_{\bar{c}}$ and $M_2 := M_0 #_p M_{\bar{c}}$. Both of $M_1$ and $M_2 # 3\mathbb{CP}^2$ are simply-connected and have the Euler characteristic 56 and signature $-36$. Hence Theorem 4.1 tells us that $M_1$, $M_2 # 3\mathbb{CP}^2$, and $# 9\mathbb{CP}^2 # 45\mathbb{CP}^2$ are homeomorphic but mutually non-diffeomorphic.

We next exhibit an example of lantern substitution for genus 2 fibrations and pose a problem about it.

\begin{align*}
\varrho := (c_5c_4c_3c_2c_1c_0c_4c_3c_2c_1)^2, \\
\varrho' := c_3(\delta)_{c_5c_4c_3c_2c_1}(x) \cdot \bar{k}h c_5^{-1}(c_5c_4c_3c_2c_1) \cdot k \cdot c_3^{-1}(h)_{c_5c_4c_3c_2c_1} = \tau' (\bar{y}_1 := c_1(c_2))
\end{align*}
Let $M_\varrho$ (resp. $M_\varrho'$) be the corresponding Lefschetz fibration of genus 2 over $S^2$. It is well-known that $M_\varrho$ is diffeomorphic to $\mathbb{CP}^2 \# 13 \mathbb{CP}^2$ (cf. [8]). $\varrho'$ is obtained by applying elementary transformations and four times $L$-substitutions to $\varrho$ as follows.

$$\varrho = (c_5 c_4 c_3 e_2 c_1^2 c_2 c_3 c_4 c_5)^2$$

$$\sim c_5 c_4 c_3 e_2 c_1 \cdot c_1 e_2 \cdot c_3 (c_4) \cdot c_3 c_5 \cdot c_2 c_1 \cdot c_1 c_2 c_3 c_4 c_5$$

$$\rightarrow c_5 c_4 c_3 e_2 c_1 \cdot c_1 e_2 \cdot c_3 (c_4) \cdot c_1 c_2 c_3 c_4 c_5$$

$$\sim c_5 c_4 c_3 \cdot c_1^2 \cdot c_3 (c_4) \cdot c_3 c_5 \cdot c_2 c_1 \cdot c_1 c_2 c_3 c_4 c_5$$

$$\rightarrow c_5^2 c_4^2 \cdot c_4 c_3 \cdot e_2 (c_2) \cdot c_2 \cdot c_3 (c_4) \cdot c_1 c_2 c_3 c_4 c_5$$

$$\rightarrow c_3 \delta x \cdot c_4 c_3 \cdot e_2 (c_2) \cdot c_2 \cdot c_3 (c_4) \cdot c_1 c_2 c_3 c_4 c_5$$

$$\sim \delta x c_4 c_3 \cdot e_2^2 (c_2) \cdot c_2 \cdot c_3 (c_4) \cdot c_1 c_2 c_3 c_4 c_5$$

$$\sim \delta x c_4 c_3 \cdot e_2 (c_2) \cdot c_2 \cdot c_3 (c_4) \cdot c_1 c_2 c_3 c_4 c_5$$

$$\sim \delta x c_4 c_3 \cdot e_2 (c_2) \cdot c_2 \cdot c_3 (c_4) \cdot c_1 c_2 c_3 c_4 c_5$$
and has the Euler characteristic 12 and signature $\chi$. Does $M$ generated by all commutativity, all braid, all 2-chain, and all lantern relators. We is a Lefschetz fibration of genus 2 with 6 non-separating, 2 separating singular

$$E$$

where the symbol $c$ and $E$ are simply-connected and have the Euler characteristic 36 and

$$M$$

$$\approx \mathbb{C}P^2 \# 13\mathbb{C}P^2 \# \mathbb{C}P^2$$

along copies of $C_2$ from Theorem 3.1.

We set $\varsigma := \rho \in \mathcal{R}$ and put $M_1 := M_{\rho} \#_F M_{\varsigma}$ and $M_2 := M_{\rho'} \#_F M_{\varsigma}$. Both of $M_1$ and $M_2 \# 4\mathbb{C}P^2$ are simply-connected and have the Euler characteristic 36 and signature $-24$. Hence Theorem 3.1 tells us that $M_1, M_2 \# 4\mathbb{C}P^2$, and $\# 5\mathbb{C}P^2 \# 29\mathbb{C}P^2$ are homeomorphic but mutually non-diffeomorphic.

We denote the manifold $M_{\rho'}$ of Example 4.3 by $E$. Since $E$ is simply-connected and has the Euler characteristic 12 and signature $-8$, $E$ is homeomorphic to $E(1) = \mathbb{C}P^2 \# 9\mathbb{C}P^2$ from Freedman’s classification theorem.

Problem 5.4. Does $E$ decompose into a non-trivial fiber sum of other Lefschetz fibrations? Is $E$ isomorphic to a fiber sum of two copies of Matsumoto's fibration?

If $E$ decomposes into a non-trivial fiber sum, then it is not diffeomorphic to $E(1)$ by virtue of Usicher’s theorem [19]. Matsumoto’s fibration (Example B of [13]) is a Lefschetz fibration of genus 2 with 6 non-separating, 2 separating singular fibers, and its total space is diffeomorphic to $S^2 \times T^2 \# 4\mathbb{C}P^2$. It is easy to see that an appropriately twisted fiber sum of two copies of Matsumoto’s fibration is homeomorphic but not diffeomorphic to $E(1)$. Another possible way to examine the manifold $E$ would be to compute the Seiberg-Witten invariants of $E$ by the formula [6] of Fintushel and Stern.

6. OTHER RELATIONS

We finally observe effects of substitutions for other relations. Luo [11] improved Gervais’ infinite presentation [7] of $\mathcal{M}_g$ to show that the set $\mathcal{R}$ of relators is normally generated by all commutativity, all braid, all 2-chain, and all lantern relators. We briefly review definitions of these relations but lantern relation.
Let $a, b$ be disjoint essential simple closed curves on $\Sigma_g$. The relation

$$t_a t_b = t_b t_a$$

in $\mathcal{M}_g$ is called a **commutativity relation**. A regular neighborhood $\Sigma$ of $a \cup b$ is the disjoint union of two annuli.

Let $a, b$ be simple closed curves on $\Sigma_g$ which intersect transversely at one point. The relation

$$t_a t_b t_a = t_b t_a t_b$$

in $\mathcal{M}_g$ is called a **braid relation**. A regular neighborhood $\Sigma$ of $a \cup b$ is a torus with one boundary component. Let $c$ be a simple closed curve parallel to the boundary of $\Sigma$. The relation

$$(t_a t_b)^6 = t_c$$

in $\mathcal{M}_g$ is called a **chain relation of length 2**, or **2-chain relation** in short.

Both sides of each relation above correspond to Lefschetz fibrations over $D^2$ with fiber $\Sigma$. It is not difficult to draw Kirby diagrams of those manifolds and find out what they are (cf. [8], Chapter 8). We actually obtain the following table.

| relation    | manifold for LHS | manifold for RHS | common boundary |
|-------------|------------------|------------------|-----------------|
| commutativity | $D^2 \amalg D^2$ | $D^2 \amalg D^2$ | $S^1 \amalg S^1$ |
| braid       | $X(S^2, -2)$     | $X(S^2, -2)$     | $\mathbb{RP}^3$ |
| 2-chain     | $\mathcal{M}_c(2, 3, 6)$ | $X(T^2, -1)$ | $\Sigma(2, 3, 6)$ |
| lantern      | $C_2$           | $B_2$           | $L(4, 1)$       |

The symbol $X(B, e)$ stands for the total space of a $D^2$-bundle over $B$ with Euler number $e$. The Milnor fiber $\mathcal{M}_c(2, 3, 6)$ and the Brieskorn manifold $\Sigma(2, 3, 6)$ are defined by

$$\mathcal{M}_c(2, 3, 6) := \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^6 = \varepsilon \} \cap D^6,$$

$$\Sigma(2, 3, 6) := \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^6 = 0 \} \cap S^5$$

(see [8], Figure 8.13 for Kirby diagram). Substitutions for commutativity and braid relations do not change the original manifold (cf. [20], Figure 34 and [3], Appendix A), whereas those for 2-chain and lantern relations do.

It might be interesting to extend the table above to that for various other relations such as chain relations of length $n$ ($\geq 3$), the star relation, and Matsumoto’s relations [12]. No relation seems to be known to correspond to a rational blowing down process along $C_p$ for $p \geq 3$.

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