Construction of Directed Strongly Regular Graphs as Generalized Cayley Graphs

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Abstract
Directed strongly regular graphs were introduced by Duval in 1998 as one of the possible generalization of classical strongly regular graphs to the directed case. Duval also provided several construction methods for directed strongly regular graphs. In this paper, an infinite family of directed strongly regular graphs with parameters $\lambda = \mu = t - 1$ is constructed, as generalized Cayley graphs of cyclic groups.

Keywords: Directed strongly regular graphs; Generalized Cayley graph; Automorphism

1 Introduction
All graphs considered in this paper will be finite simple directed graphs; so the graphs will have no loops or multiple edges. Let $\Gamma$ be a directed graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For any $x, y \in V(\Gamma)$, we say that $x$ is adjacent to $y$, denoted by $x \rightarrow y$, if there is an edge directed from $x$ to $y$, while $x \not\rightarrow y$ means that $x$ is not adjacent to $y$. Let $\Gamma$ be a graph with $V(\Gamma) = \{x_1, x_2, ..., x_n\}$. The adjacency matrix $A = A(\Gamma)$ of $\Gamma$ is the $n \times n$ matrix whose rows and columns are indexed by the vertices such that $A_{ij} = 1$ if $x_i$ is adjacent to $x_j$ and $A_{ij} = 0$ otherwise.

A directed graph is called a directed strongly regular graph with parameters $(n, k, t, \lambda, \mu)$, denoted simply by DSRG$(n, k, t, \lambda, \mu)$, if its adjacency matrix $A$ satisfies

$$AJ = JA = kJ$$

and

$$A^2 = tI + \lambda A + \mu(J - I - A),$$

where $I$ is the identity matrix of order $n$, and $J$ is the $n \times n$ matrix of all 1’s. Thus, each vertex of DSRG$(n, k, t, \lambda, \mu)$ has $k$ out-neighbors and $k$ in-neighbors, including $t$ neighbors counted as both in- and out-neighbors of the vertex. For vertices $x \neq y$, the number of paths of length two from $x$ to $y$ is $\lambda$ if $x \rightarrow y$ and $\mu$ otherwise.

A DSRG with $t = k$ is an undirected strongly regular graph. A DSRG with $t = 0$ is a graph known as doubly-regular tournament [4]. Thus we will only consider DSRGs
with \( 0 < t < k \) in this paper. Duval proved the following conditions for the existence of a DSRG(\( n, k, t, \lambda, \mu \)) with \( 0 < t < k \):

\[
k(k + (\mu - \lambda)) = t + (n - 1)\mu, \quad (2)
\]

\[
(\mu - \lambda)^2 + 4(t - \mu) = d^2, \quad d \mid 2k - (\mu - \lambda)(n - 1),
\]

\[
\frac{2k - (\mu - \lambda)(n - 1)}{d} \equiv n - 1 \pmod{2},
\]

\[
\left| \frac{2k - (\mu - \lambda)(n - 1)}{d} \right| \leq n - 1,
\]

where \( d \) is a positive integer, and

\[
0 \leq \lambda < t < k, 0 < \mu \leq t < k, -2(k - t - 1) \leq \mu - \lambda \leq 2(k - t).
\]

The eigenvalues of a DSRG(\( n, k, t, \lambda, \mu \)) are

\[
k > \rho = \frac{1}{2}(\lambda - \mu + d) > \sigma = \frac{1}{2}(\lambda - \mu - d),
\]

with the multiplicities

\[
1, \quad \frac{-k + \sigma(n - 1)}{\rho - \sigma}, \quad \text{and} \quad \frac{k + \rho(n - 1)}{\rho - \sigma}
\]

respectively.

Parameter sets satisfying these conditions from Duval’s paper are called feasible. Since there are so many limitations on the possible sets of parameters, directed strongly regular graphs are quite rare. Duval provided an initial list of feasible parameter sets in his paper, but a more complete list is available in [3].

There are numerous construction methods for DSRG. In [4], Duval described some methods including constructions using quadratic residue, block construction of permutation matrices and the Kronecker product. In addition, some of known constructions use combinatorial designs [8], coherent algebras [8, 15], finite geometries [7, 8, 9, 16], matrices [4, 6, 9], regular tournaments [6, 11], block matrices [1], finite incidence structures [18, 2] and Cayley graphs [5, 10, 11, 15].

In addition, non-existence of some directed strongly regular graphs were investigated [7, 11, 12, 13, 14]. From [4], an eigenvalue of a DSRG(\( n, k, t, \lambda, \mu \)) is 0 if and only if \( t = \mu \). If 0 is an eigenvalue of the adjacency matrix, then the rank of the adjacency matrix is the sum of multiplicities of non-zero eigenvalues, i.e., \( 1 + \frac{d}{2} \). Thus we define the rank of a feasible parameter set with \( t = \mu \) to be \( 1 + \frac{d}{2} \) (even if no directed strongly regular graph exists with these parameters). In [12, 14], Jorgensen gave the sufficient and necessary conditions on existence of a directed strongly regular graph with parameters \( (n, k, t, \lambda, \mu) \) and with adjacency matrix of rank \( r \leq 5 \) based on an enumeration of \( r \times r \) matrices with entries in \( \{0, 1\} \). Therefore, some feasible parameters are excluded.

In this paper, we construct an infinite family of DSRGs is constructed as generalized Cayley graph of cyclic groups. The organization of the paper is as follows. In section 2, we give the definition of generalized Cayley graphs and a necessary and sufficient condition that makes it easy to check whether a generalized Cayley graph is a DSRG. Section 3 contains some necessary conditions on the existence of DSRGs as generalized Cayley graphs of abelian groups. In section 4, we construct an infinite family of DSRGs with parameters \( \lambda = \mu = t - 1 \) as generalized Cayley graph of cyclic groups. Among the feasible parameters listed in [3], this construction realizes the feasibility of parameters such as \((24, 13, 8, 7, 7), (32, 17, 10, 9, 9), (40, 29, 22, 21, 21)\) and so on.
2 Cayley Graphs and Generalized Cayley Graphs

Hereafter, we will denote the identity element of a group \( G \) by \( e \).

**Definition 2.1** Let \( G \) be a finite group and \( A \subseteq G \setminus e \). The Cayley graph of \( G \) generated by \( A \), denoted by \( \text{Cay}(G, A) \), is the digraph \( \Gamma \) such that \( V(\Gamma) = G \) and \( x \rightarrow y \) if and only if \( x^{-1}y \in A \).

**Definition 2.2** For any finite group \( G \), the group ring \( \mathbb{Z}[G] \) is defined as the set of all formal sums of elements of \( G \), with coefficients from \( \mathbb{Z} \). The operations + and \( \cdot \) on \( \mathbb{Z}[G] \) are given by

\[
\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g)g
\]

and

\[
\left( \sum_{g \in G} a_g g \right) \cdot \left( \sum_{h \in G} b_h h \right) = \sum_{g, h \in G} (a_g \cdot b_h)gh.
\]

The group ring \( \mathbb{Z}[G] \) is a ring with multiplicative identity \( e \). For any subset \( X \) of \( G \), we use \( X \) to denote the element of the group ring \( \mathbb{Z}[G] \) that is the sum of all elements of \( X \), namely \( X = \sum_{x \in X} x \). The lemma below allows us to express a necessary and sufficient condition for a Cayley graph to be directed strongly regular in terms of the group ring.

**Lemma 2.3** The Cayley graph \( \text{Cay}(G, A) \) of \( G \) with respect to \( A \) is a directed strongly regular graph with parameters \((n, k, t, \lambda, \mu)\) if and only if \( |G| = n \), \( |A| = k \), and

\[
A^2 = te + \lambda A + \mu(G - e - A).
\]

**Lemma 2.4** A DSRG \((n, k, t, \lambda, \mu)\) with \( 0 < t < k \) cannot be a Cayley graph of an abelian group.

Generalized Cayley graphs were introduced by Marušić etc. \[17\]. The definition is given below.

**Definition 2.5** Let \( G \) be a group, \( A \) be a subset of \( G \) and \( \alpha \) be an automorphism of \( G \) such that \( \alpha^2 = \epsilon \), the identity automorphism of \( G \). The generalized Cayley graph \( \text{GC}(G, A, \alpha) \) of \( G \) with respect to the ordered pair \((A, \alpha)\) is the directed graph with elements of \( G \) as its vertices, and \( x \rightarrow y \) if and only if \( \alpha(x^{-1})y \in A \).

It is clear that a generalized Cayley graph has the following properties.

**Proposition 2.6** Let \( \text{GC}(G, A, \alpha) \) be the generalized Cayley graph of \( G \) with respect to the ordered pair \((A, \alpha)\), then

(i) \( \text{GC}(G, A, \alpha) \) has no loops if and only if \( \alpha(x^{-1})x \notin A \) for each \( x \in G \),

(ii) \( \text{GC}(G, A, \alpha) \) is undirected if and only if \( \alpha(A) = A^{-1} \),

(iii) \( \text{GC}(G, A, \alpha) \) is the Cayley graph \( C(G, A) \) by choosing \( \alpha = \epsilon \),

(iv) If \( \alpha \) has order 2, then \( \text{GC}(G, A, \alpha) \) is connected if and only if \( A \) is a left generating set for \((G, \ast)\), where \( x \ast y = \alpha(x)y \) for all \( x, y \in G \).
In (i) of above proposition, “$\alpha(x^{-1})x \notin A$ for each $x \in G$” ensures that $GC(G, A, \alpha)$ is a simple digraph. Thus as long as $GC(G, A, \alpha)$ was mentioned in the rest of the paper, it is assumed that $A \subseteq G \setminus \{\alpha(x^{-1})x \mid x \in G\}$ without special instructions. Next, we give a necessary and sufficient condition for a generalized Cayley graph to be directed strongly regular in terms of the group ring.

**Theorem 2.7** The generalized Cayley graph $GC(G, A, \alpha)$ of $G$ with respect to the ordered pair $(A, \alpha)$ is a directed strongly regular graph with parameters $(n, k, t, \lambda, \mu)$ if and only if $|G| = n$, $|A| = k$, and

$$\overline{\alpha(A)} \cdot \overline{A} = t e + \lambda B + \mu(G - e - B),$$

(6)

where $B = \{x^{-1} \alpha(x) a \mid x \in G, a \in A\}$.

**Proof.** For any vertex $z$, it is clear that the sets of out-neighbors and in-neighbors of $z$ are

$$\alpha(z) A = \{\alpha(z) a \mid a \in A\}, \quad \alpha(z A^{-1}) = \{\alpha(z a^{-1}) \mid a \in A\}$$

respectively by the definition of the generalized Cayley graph. Let $x, y \in G$, the number of paths of length 2 from $x$ to $y$ in $GC(G, A, \alpha)$ is $|\alpha(x) A \cap \alpha(y A^{-1})|$. That is the number of ordered pairs $(a_1, a_2) \in A \times A$ such that $\alpha(x) a_1 = \alpha(y a_2^{-1})$, i.e., $x^{-1} y = \alpha(a_1) a_2$. In other words, it is the coefficient of $x^{-1} y$ in $\overline{\alpha(A)} \cdot \overline{A}$.

If $x = y$, then $x^{-1} y = e$. If $x \rightarrow y$, then $y = \alpha(x) a$ and then $x^{-1} y = x^{-1} \alpha(x) a$ for some $a \in A$. If $x \neq y$ and $x \not\rightarrow y$, then $y \in G \setminus \{\alpha(x) A, x\}$ and thus $x^{-1} y \in G \setminus \{x^{-1} \alpha(x) A, e\}$. The result now follows in terms of the group ring $Z[G]$. \hfill \Box

Note that Theorem [2.7] is in accordance with Lemma [2.3] when $\alpha = \epsilon$.

### 3 DSRGs as Generalized Cayley Graphs of Abelian Groups

In this and next section, we consider only the generalized Cayley graph $GC(G, A, \alpha)$ where the order of $\alpha$ is 2. Let $G$ be an abelian group of order $n$ and $A \subseteq G \setminus \{\alpha(x^{-1})x \mid x \in G\}$ with $|A| = k$. It is clear that $H = \{x^{-1} \alpha(x) \mid x \in G\}$ is a subgroup of $G$. For each $x \in G$, let $Hx$ be the coset containing $x$ of $H$ in $G$. Then

$$B = \{x^{-1} \alpha(x) a \mid x \in G, a \in A\} = \cup_{a \in A} H a.$$

From $H \alpha(x) = Hx$ for each $x \in G$, we have again that $B = \cup_{a \in \alpha(A)} H a$.

**Theorem 3.1** Let $G$ be an abelian group, $A \subseteq G \setminus \{\alpha(x^{-1})x \mid x \in G\}$ and $\alpha$ be an automorphism of order 2 of $G$. If the generalized Cayley graph $GC(G, A, \alpha)$ is a DSRG($n, k, t, \lambda, \mu$) with $0 < t < k$, then $\lambda = \mu$.

**Proof.** Let $GC(G, A, \alpha)$ be a DSRG($n, k, t, \lambda, \mu$) with $0 < t < k$, then the equation (6) is satisfied by Theorem [2.7]. Thus, we have

$$k^2 = (t - \mu) + (\lambda - \mu) |B| + \mu n$$

by checking the number of elements in two side of the equation [6]. However, it is follows from [2] that

$$k^2 = (t - \mu) + (\lambda - \mu) k + \mu n.$$
If \( \lambda \neq \mu \), then \(|B| = k\). Thus \( B = A \) and \( B = \alpha(A) \) since \( B = \bigcup_{a \in A} Ha \supseteq A \), \( B \supseteq \alpha(A) \) and \(|A| = |\alpha(A)| = k\). Thus \( \alpha(A) = A \) and the equation \( 3 \) becomes \( 5 \). This means that the Cayley graph \( C(G, A) \) is a DSRG \((n, k, t, \lambda, \mu)\). It contradicts to Lemma 2.4. Therefore, \( \lambda = \mu \).

Similar to Lemma 2.4 we could rewrite above theorem as follows.

**Corollary 3.2** A DSRG \((n, k, t, \lambda, \mu)\) with \( 0 < t < k \) and \( \mu \neq \lambda \) cannot be a generalized Cayley graph of an abelian group.

**Corollary 3.3** Let \( G \) be an abelian group, \( A \subseteq G \setminus \{\alpha(x^{-1})x \mid x \in G\} \) and \( \alpha \) be an automorphism of order 2 of \( G \). Then the generalized Cayley graph \( GC(G, A, \alpha) \) is a DSRG \((n, k, t, \lambda, \mu)\) with \( 0 < t < k \) if and only if \(|G| = n, |A| = k \) and

\[
\overline{\alpha(A)} \cdot A = (t - \mu)e + \mu \overline{G}.
\]

### 4 An Infinite Family of DSRGs as Generalized Cayley Graphs of Cyclic Groups

In this section, we will construct an infinite family of directed strongly regular graphs with parameters \( \lambda = \mu = t - 1 \) as generalized Cayley graphs of cyclic groups. Let \( G = \langle g \rangle \) be the cyclic group of order \( n \). It is known that for a positive integer \( s \), the map

\[
\alpha: G \longrightarrow G \\
g \longmapsto g^s.
\]

is an automorphism of \( G \) if and only if \( n \) and \( s \) are relatively prime. The order of \( \alpha \) is 2 if and only if \( n \mid (s^2 - 1) \) and \( s \neq 1 \).

**Construction 4.1** Let \( k, n \) and \( \mu \) be positive integers such that \( k^2 - 1 = n\mu \), then the automorphism \( \alpha: g \longmapsto g^{-k} \) of \( G \) has order 2. Set \( A = \{g, g^2, g^3, \ldots, g^k\} \). Then the generalized Cayley graph \( GC(G, A, \alpha) \) is a DSRG \((n, k, \mu + 1, \mu, \mu)\).

**Proof.** It is clear that \( H = \{x^{-1}\alpha(x) \mid x \in G\} = \langle g^{k+1} \rangle \) and \( H \cap A = \emptyset \). The following table presents the exponents of the \( k^2 \) elements in the multiset \( \alpha(A) \cdot A = \{\alpha(x) \cdot x \mid x \in G\} \), with rows indexed by elements in \( \alpha(A) \) and columns indexed by elements in \( A \).

|      | 1   | 2   | 3   | \cdots | \( k - 1 \) | \( k \) |
|------|-----|-----|-----|--------|-----------|-------|
| \(-k\) | \(-k + 1\) | \(-k + 2\) | \(-k + 3\) | \cdots | \(-1\) | 0 |
| \(-2k\) | \(-2k + 1\) | \(-2k + 2\) | \(-2k + 3\) | \cdots | \(-k - 1\) | \(-k\) |
| \(-3k\) | \(-3k + 1\) | \(-3k + 2\) | \(-3k + 3\) | \cdots | \(-2k - 1\) | \(-2k\) |
| \cdots | \cdots | \cdots | \cdots | \cdots | \cdots | \cdots |
| \(- (k - 1)k \equiv k - 1\) | k | k + 1 | k + 2 | \cdots | 2k - 2 | 2k - 1 |
| \(-k^2 \equiv -1\) | 0 | 1 | 2 | \cdots | \( k - 2\) | \( k - 1\) |

Obviously,

\[
\overline{\alpha(A)} \cdot A = (\mu + 1)e + \mu \overline{G} - e.
\]

(7)

The result now follows from Theorem 2.7. \(\square\)
In Theorem 1 of [11], Jorgensen defined the same graph under the assumption $\mu \mid (k-1)$. He proved it is directed strongly regular by using the definition of DSRG. Here, we generalize it to the general case and our proof is much easier than him. Jorgensen also mentioned that the graphs he constructed are generalized Cayley graphs.

Set $A_h = A_h$, for some $h \in H$. Then the generalized Cayley graph $GC(G, A_h, \alpha)$ is again a DSRG($n, k, \mu + 1, \mu, \mu$). In addition, for a DSRG($n, k, \mu + 1, \mu, \mu$), we can conclude $n\mu = k^2 - 1$ by the equation (2). Therefore, Construction 4.1 gives a construction of DSRG($n, k, \mu + 1, \mu, \mu$)s for all positive integers $k$ and $\mu$. Since the complement of a DSRG is also a DSRG, among the feasible parameters listed in [3], this construction realizes the feasibility of parameters such as $(24, 13, 8, 7, 7)$, $(24, 10, 5, 3, 5)$, $(32, 17, 10, 9, 9)$, $(32, 14, 7, 5, 7)$, $(40, 29, 22, 21, 21)$, $(40, 10, 3, 1, 3)$ and so on.

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