Abstract

Let \( p \) be a fixed prime. We show that the number of isomorphism classes of finite rings of order \( p^n \) is \( p^\alpha \), where \( \alpha = \frac{4}{27} n^3 + O(n^{5/2}) \). This result was stated (with a weaker error term) by Kruse and Price in 1969; a problem with their proof was pointed out by Knopfmacher in 1973. We also show that the number of isomorphism classes of finite commutative rings of order \( p^n \) is \( p^\beta \), where \( \beta = \frac{2}{27} n^3 + O(n^{5/2}) \). This result was stated (again with a weaker error term) by Poonen in 2008, with a proof that relies on the problematic step in Kruse and Price’s argument.
1 Introduction

For a positive integer $N$, let $f_{\text{rings}}(N)$ be the number of (isomorphism classes of) finite rings of cardinality $N$. (We do not assume that our rings have a multiplicative identity.) What can be said about this function?

Write $N = \prod_{i=1}^{t} p_{i}^{n_{i}}$, where the integers $p_{i}$ are distinct primes. It is not hard to see that a ring $R$ of cardinality $N$ may be uniquely written as a direct sum $R = \bigoplus_{i=1}^{t} R_{i}$, where $R_{i}$ is a ring of order $p_{i}^{n_{i}}$, and hence $f_{\text{rings}}(N) = \prod_{i=1}^{t} f_{\text{rings}}(p_{i}^{n_{i}})$. So we may specialise to the case when $N$ is a prime power. For the rest of this paper, we assume that $N = p^{n}$, where $p$ is prime.

It seems very hard to provide exact values for $f_{\text{rings}}(p^{n})$, so it is natural to ask about asymptotic enumeration. We think of $p$ as being fixed, and $n$ increasing: How fast does this function grow? About fifty years ago, an interesting paper by Kruse and Price [13] addressed this question, taking inspiration from the then-recent enumeration of finite $p$-groups by Graham Higman [9] and C.C. Sims [20]. Kruse and Price [13, Corollary 5.9] (see also [14, Chapter V]) state that the number of (isomorphism classes of) finite rings of cardinality $p^{n}$ is $p^{\alpha}$, where $\alpha = \frac{1}{27}n^{3} + O(n^{8/3})$. The structure of their argument can be summarised as follows. For the lower bound, they construct $p^{\frac{1}{27}n^{3} - O(n^{2})}$ rings of cardinality $N$ by taking quotients of a certain $\mathbb{F}_{p}$-algebra of nilpotency class 2 on $r$ generators, where $r \approx 2n/3$. (Here $\mathbb{F}_{p}$ is the finite field of order $p$.) This $\mathbb{F}_{p}$-algebra has about $p^{\frac{1}{27}n^{3}}$ ideals of index $p^{n}$, and it can be shown that each isomorphism class of rings appears at most $p^{O(n^{2})}$ times as a quotient by an ideal of this form. This approach is inspired by Higman’s lower bound for the number of isomorphism classes of groups of order $p^{n}$. To provide a corresponding upper bound, Kruse and Price divide the problem into three steps:

1. Reduce to the case of $\mathbb{F}_{p}$-algebras.

2. Provide an upper bound on the number of nilpotent $\mathbb{F}_{p}$-algebras, using techniques inspired by Sims’ enumeration of (nilpotent) groups of order $p^{n}$.

3. Show that once the isomorphism class of the (necessarily nilpotent) Jacobson radical $J(R)$ is fixed, there are few possibilities for the algebra $R$ itself. There is an interesting parallel with Pyber’s (much later) results on group enumeration [17] here: Pyber shows that the number
of isomorphism classes of groups whose Sylow subgroups have been chosen (up to isomorphism) is small.

The first step of this approach is ingenious but, sadly, flawed: In 1973, Knopfmacher [12, Page 169] already points out that the reduction does not work. We provide counterexamples to this reduction in the appendix below.

The main aim of our paper is to show how the upper bound in [13] may nevertheless be established, by providing analogues of the last two steps above that work without the initial reduction to algebras. First, in Section 2, we provide an upper bound on the number of nilpotent rings, rather than just nilpotent \( \mathbb{F}_p \)-algebras, of cardinality \( p^n \). This uses techniques similar to those in Sims [20] and in Kruse and Price [13] (but see below). Secondly, in Section 3 we show that once the isomorphism class of the (nilpotent) Jacobson radical \( J(R) \) is fixed, there are few possibilities for the ring \( R \). Our argument is (and needs to be) rather different from the argument in [13].

In fact, our arguments in Section 2 make use of a trick due to Craig Seeley and M.F. Newman [15] (see Blackburn, Neumann and Venkataraman [3, Chapter 5] for details) which was originally applied to improve error terms in group enumeration. Adapting this trick to the situation of rings, we are able to show (Theorem 3.6) that the number of rings of cardinality \( p^n \) is \( p^\alpha \), where

\[
\alpha = \frac{4}{27}n^3 + O(n^{5/2}).
\]

We note (see Theorem 3.9) that the same statement holds when enumerating rings with identity.

More recently, Poonen [16, Section 11] has used the same problematic initial reduction to algebras [16, Lemma 11.1] in order to enumerate commutative rings. In Section 4 we provide a proof of this enumeration also, with an improved error term. We show (Theorem 4.5) that the number of commutative rings of cardinality \( p^n \) is \( p^\beta \), where

\[
\beta = \frac{2}{27}n^3 + O(n^{5/2}).
\]

The reduction to nilpotent rings in Section 3 can be used unchanged in the commutative situation. However, the enumeration of nilpotent commutative rings requires extra work. As before, our results allow us to enumerate commutative rings with identity; see Theorem 4.8.

Finally, in the appendix, we exhibit counterexamples to the proof of Kruse and Price.
Acknowledgement We dedicate this paper to the memory of Peter M. Neumann. Peter was responsible for initiating the authors’ collaboration on this paper, bringing us together after Robin wrote to him with counterexamples to Kruse and Price’s argument and early ideas for avoiding the problematic reduction to $\mathbb{F}_p$-algebras. Simon was one of Peter’s many D.Phil. students, and he would like to acknowledge the lasting influence of Peter’s knowledge, advice and encouragement throughout his career. He will be greatly missed.

2 Finite nilpotent rings

Let $R$ be a nilpotent ring of order $p^n$, and define $\overline{R} = R/pR$. Roughly speaking (using the insight of Sims) either $\overline{R}/\overline{R}^3$ has very restricted structure (so there are few possibilities for it), or there exists a small set of elements of $R$ whose images in $\overline{R}$ generate a sub-algebra containing $\overline{R}^2$. The multiplicative structure of this small set determines most of the structure of $R$, allowing us to provide a tight enumeration of nilpotent rings. The arguments in this section are similar to [13], although we use an idea of Seeley and Newman to improve the error term in our enumeration.

We begin with two structural results (Lemmas 2.1 and 2.2) concerning $\mathbb{F}_p$-algebras $R$, before proving the main theorem of this section (Theorem 2.3). Define the Sims dimension of the $\mathbb{F}_p$-algebra $R$ to be the least dimension of $S/R^2$ as $S$ ranges over the subalgebras $S$ of $R$ such that $S^3 = R^2$. The following result is Lemma 5.5 of [13], although we use an idea of Seeley and Newman to improve the error term in our enumeration.

Lemma 2.1. Let $R$ be a nilpotent $\mathbb{F}_p$-algebra such that $r = \dim(R/R^2)$, $s$ is the Sims dimension of $R$, $t = \dim(R^2)$ and $R^3 = 0$. If $S$ is a subalgebra of $R$ that contains $R^2$, then

$$\dim(S^2) - \dim(S/R^2) \leq t - s + 1.$$ 

Lemma 2.2. Let $r, s, t$ be positive integers and let $\alpha(r, s, t)$ be a real number. Suppose that $p^{\alpha(r, s, t)}$ is the number of (isomorphism classes of) $\mathbb{F}_p$-algebras $R$ such that $r = \dim(R/R^2)$, $s$ is the Sims dimension of $R$, $t = \dim(R^2)$ and $R^3 = 0$, then

$$\alpha(r, s, t) \leq r^2(t - s) + O((r + t)^{8/3}) \quad (1)$$

and

$$\alpha(r, s, t) \leq r^2(t - s) + \frac{1}{2}rst + O((r + t)^{5/2}) \quad (2)$$
Proof. The inequality (1) is [13, Theorem 5.4].

To prove (2), let \( x_1, x_2, \ldots, x_r \) be elements of \( R \) that map onto a basis for \( R/R^2 \). The ring \( R \) is determined up to isomorphism by the \( r^2 \) products \( x_i x_j \) given by \( 1 \leq i, j \leq r \). Now \( R^2 \) contains \( p^t \) elements, so there are at most \( p^t \) ways of choosing all our \( x_i x_j \). Hence \( \alpha(r, s, t) \leq r^2 t \). If \( s \leq 2r^{1/2} + 1 \), then this last inequality for \( \alpha(r, s, t) \) gives (2) as desired.

Now suppose that \( s > 2r^{1/2} + 1 \) and let \( f = \lceil r^{1/2} \rceil \) be the greatest integer not exceeding \( r^{1/2} \) and let \( g = \lceil r/f \rceil \) be the least integer not less than \( r/f \). Writing \( \langle v_1, v_2, \ldots, v_k \rangle \) for the \( \mathbb{F}_p \)-subspace spanned by elements \( v_1, v_2, \ldots, v_k \in R \), we may define \( \mathbb{F}_p \)-subspaces \( V_1, V_2, \ldots, V_g \) of \( R \) by

\[
V_i = \langle x((i-1)f+1, x((i-1)f+2, \ldots, x((i-1)f+f) \quad \text{for} \quad 1 \leq i < g
\]

and

\[
V_g = \langle x((g-1)f+1, x((g-1)f+2, \ldots, x_r \rangle.
\]

Taking \( S = V_i + V_j + R^2 \) in Lemma 2.1 and noting that \( R^3 = 0 \), we get

\[
\dim(V_i + V_j)^2 \leq \dim(V_i) + \dim(V_j) + t - s + 1 \leq 2f + t - s + 1 \leq 2r^{1/2} + t - s + 1. \tag{3}
\]

Let \( d = 2f + t - s + 1 \). From (3) and (4), there is a \( d \)-dimensional subspace \( W_{ij} \) of \( R^2 \) that contains \( (V_i + V_j)^2 \). Now the number of such subspaces is

\[
\frac{(p^f - 1)(p^f - p) \ldots (p^f - p^{d-1})}{(p^d - 1)(p^d - p) \ldots (p^d - p^{d-1})}. \tag{5}
\]

When \( 0 \leq i < d \), we have

\[
p^f - p^i < p^t \leq p^{i+1} - p^i \leq p^{t-d+i+1} - p^{t-d+i} = p^{t-d+1}(p^d - p^i),
\]

so

\[
\frac{p^f - p^i}{p^d - p^i} \leq p^{t-d+1}. \tag{6}
\]

From (5) and (6), the number of ways of choosing each subspace \( W_{ij} \) is at most \( p^{d(t-d+1)} \), so the number of ways of choosing all \( \binom{g}{2} \) subspaces \( W_{ij} \) is at most \( p \) to the power \( \binom{g}{2} d(t - d + 1) \). Once all the \( W_{ij} \) have been chosen,
there are at most \( p^d \) choices for each product \( x_ix_j \) with \( 1 \leq i, j \leq r \). So

\[
\alpha(r, s, t) \leq \left( \frac{g}{2} \right) d(t - d + 1) + r^2 d
\]

\[
\leq \left( \frac{g}{2} \right) (2f + t - s + 1)(s - 2f) + r^2(2f + t - s + 1).
\]

Now \( f = r^\frac{1}{2} + O(1) \) and \( g = r^\frac{1}{2} + O(1) \), so

\[
\alpha(r, s, t) \leq \frac{1}{2}r(t - s)s + r^2(t - s) + O((r + t)^{5/2})
\]

\[
\leq r^2(t - s) + \frac{1}{2}rst - \frac{1}{2}rs^2 + O((r + t)^{5/2})
\]

\[
\leq r^2(t - s) + \frac{1}{2}rst + O((r + t)^{5/2}).
\]

For a representative \( R \) of each of the \( p^\alpha(r, s, t) \) isomorphism classes of \( \mathbb{F}_p \)-algebras of the form above, choose a subalgebra \( S \) such that \( S^2 = R^2 \) and \( \dim S/R^2 = s \). We choose a basis \( x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_t \) of \( R \), which we call the standard basis for \( R \), with the following properties: the elements \( x_1 + R^2, x_2 + R^2, \ldots, x_s + R^2 \) form a basis for \( S/R^2 \); the elements \( x_1 + R^2, x_2 + R^2, \ldots, x_r + R^2 \) form a basis for \( R/R^2 \); the elements \( y_1, y_2, \ldots, y_t \) form a basis for \( R^2 \); each element \( y_i \) may be written in the form \( y_i = x_kx_\ell \) for some \( k, \ell \in \{1, 2, \ldots, s\} \). A standard basis exists, since \( R \) has cube zero and Sims dimension \( s \). For each element \( y_i \), we choose an equality of the form \( y_i = x_kx_\ell \) for some \( k, \ell \in \{1, 2, \ldots, s\} \), and call this the standard monomial representation of \( y_i \).

**Theorem 2.3.** The number of (isomorphism classes of) nilpotent rings of order \( p^n \) is \( p^\alpha \), where

\[
\alpha = \frac{4}{27}n^3 + O(n^{5/2}).
\]

**Proof.** From [13, Theorem 2.2], the number of \( \mathbb{F}_p \)-algebras of cube zero and order \( p^n \) is \( p^{\alpha'} \), where \( \alpha' = 4n^3/27 + O(n^2) \). Hence it is sufficient to prove that \( p^\alpha \) is an upper bound for the number of nilpotent rings of order \( p^n \).

Let \( R \) be a nilpotent ring of order \( p^n \), and let \( \overline{R} = R/pR \), which is an \( \mathbb{F}_p \)-algebra of order \( p^w \) where \( 1 \leq w \leq n \). The additive structure of \( R \) is isomorphic to an abelian group of order \( p^w \) and rank \( w \). We fix some standard representative \( G \) for this isomorphism class of abelian groups. Let \( r = \dim(\overline{R}/\overline{R}^2) \), let \( s \) be the Sims dimension of \( \overline{R} \), let \( t = \dim(\overline{R}^2/\overline{R}^3) \) and
\[ u = \dim(R^d). \] Let \( m \) be the least integer such that \( R^m = 0 \). For \( h \geq 2 \), define \( u_h = \dim(R^h/R^{h+1}). \)

The number of choices for the isomorphism class of an abelian group \( G \) of order \( p^n \) is the number of partitions of \( n \), which is at most \( 2^{n-1} < p^n \). There are at most \( n + 1 \) choices for each of \( w, r, s, t \) and \( m \). The integer \( u \) is then determined by \( u = w - (r + t) \). We see that \( u_h = 0 \) for \( h \geq m \). Now, \((u_2, u_3, \ldots, u_{m-1})\) is a sequence of \( m - 2 \) positive integers that sum to \( u + t \), and so there are at most \( 2^{u+t-1} < p^n \) choices for the integers \( u_h \). So we have made \((n+1)^5 p^{2^n} = p^{O(n)} \) choices in all. From now on, we assume that \( w, r, s, t, m \) and the integers \( u_h \) are fixed.

The quotient \( R/R^3 \) is an \( \mathbb{F}_p \)-algebra of order \( p^{r+t} \) and cube zero, whose square has order \( p^t \) and whose Simp dimension is \( s \). We choose one of the \( p^{3(r,s,t)} \) isomorphism classes of \( R/R^3 \), and from now on we assume this choice is also fixed.

Let \( \bar{x}_1, \ldots, \bar{x}_r \in \bar{R} \) map onto the first \( r \) elements of the standard basis for \( R/R^3 \). Let \( \bar{S} \) be the subring of \( \bar{R} \) generated by the first \( s \) elements \( \bar{x}_1, \ldots, \bar{x}_s \). Because our basis of \( R/R^3 \) is standard, \( \bar{S}^2 + \bar{R}^3 = \bar{R}^3 \). By [13, Theorem 5.2], using the fact that \( \bar{R}^m = 0 \), we see that

\[ \bar{S}^i = \bar{R} \quad \text{for all integers } i \geq 2. \] (7)

For \( i \geq 2 \), let \( d(i) = \dim(R^i) \). So \( d(2) = u + t \), and \( d(i) = \sum_{h=i}^{m-1} u_h \) when \( i \geq 3 \). We choose a basis \( \bar{e}_1, \ldots, \bar{e}_{u+t} \) of \( R^3 \) in such a way that \( \bar{e}_1, \ldots, \bar{e}_{d(i)} \) is a basis of \( R \) for all \( i \geq 2 \). Indeed, we choose the basis in the following way, so that each element of the basis is a monomial in \( \bar{x}_1, \ldots, \bar{x}_s \). We first choose monomials \( \bar{e}_{u+1}, \ldots, \bar{e}_{u+t} \) in \( \bar{x}_1, \ldots, \bar{x}_s \) whose images in \( \bar{R}/\bar{R}^3 \) form the final \( t \) elements of the standard basis for \( \bar{R}/\bar{R}^3 \); we use the standard monomial representations of the standard basis to do this. Since (7) holds, the set of elements \( \{\bar{x}_i \bar{x}_j : 1 \leq i \leq s \text{ and } u + 1 \leq j \leq u + t\} \) span \( \bar{R} \) modulo \( \bar{R}^4 \). We choose a subset \( \bar{e}_{d(4)+1}, \ldots, \bar{e}_{d(3)} \) of this set which is a basis of \( \bar{R} \) modulo \( \bar{R}^4 \). We continue in this way, choosing \( \bar{e}_{d((i+1)+1}, \ldots, \bar{e}_{d(i)} \) to be a basis of \( \bar{R} \) modulo \( \bar{R}^{i+1} \) contained in the set \( \{\bar{x}_i \bar{x}_j : 1 \leq i \leq s \text{ and } d(i) + 1 \leq j \leq d(i-1)\} \), to produce the basis of the form we require.

We have chosen the integers \( w, r, s, t, m \), the integers \( u_h \), and the isomorphism class of \( R/R^3 \). Though we do not need this, we remark that \( R \) is determined up to isomorphism by these choices, together with a knowledge
of each of the \( r^2 + s(w - r) \) products
\[
\overline{x}_i \overline{x}_j \quad \text{for all } i, j \text{ such that } 1 \leq i, j \leq r
\]
and \( \overline{x}_i \overline{e}_j \) for all \( i, j \) such that \( 1 \leq i \leq s \) and \( 1 \leq j \leq u + t = w - r \)
as a linear combination of the \( \overline{e}_k \). We can see this as follows. First note that
the way we have chosen our basis allows us to deduce a representation of
each \( \overline{e}_j \) as a monomial in \( \overline{x}_1, \ldots, \overline{x}_s \) from these \( r^2 + s(w - r) \) products. Then
note that a product of two elements of the form \( \overline{x}_i \) or \( \overline{e}_j \) can be computed
from associativity together with the fact that each \( \overline{e}_j \) is a monomial of at
least second degree in \( \overline{x}_1, \ldots, \overline{x}_s \). (We give a similar argument for \( R \), in a
little more detail, below.)

There are coefficients \( \lambda_{i,j,k} ; \mu_{i,j,k} ; \nu_{i,j,k} \in \mathbb{F}_p \) such that
\[
\overline{x}_i \overline{x}_j = \sum_{k=1}^{u} \lambda_{i,j,k} \overline{e}_k + \sum_{k=u+1}^{u+t} \mu_{i,j,k} \overline{e}_k \text{ where } 1 \leq i, j \leq r,
\]
\[
\overline{x}_i \overline{e}_j = \sum_{k=1}^{d(k+1)} \nu_{i,j,k} \overline{e}_k \text{ where } 1 \leq i \leq s, \overline{e}_j \in \overline{R}^k \setminus \overline{R}^{k+1} \text{ and } 2 \leq h \leq m - 2,
\]
\[
\overline{x}_i \overline{e}_j = 0 \text{ where } 1 \leq i \leq s \text{ and } \overline{e}_j \in \overline{R}^{m-1}.
\]
The values of coefficients \( \mu_{i,j,k} \) are determined by the isomorphism class of
\( \overline{R}/\overline{R}^3 \), since the images of \( \overline{x}_1, \ldots, \overline{x}_r, \overline{e}_{u+1}, \ldots, \overline{e}_{u+t} \) in \( \overline{R}/\overline{R}^3 \) form a standard
basis. There are \( r^2 u \) coefficients \( \lambda_{i,j,k} \), each of which is an element of \( \mathbb{F}_p \), so
the number of possible values of all the \( \lambda_{i,j,k} \) is \( p^{r^2 u} \). For each \( h \) in the range
\( 2 \leq h \leq m - 1 \), there are \( u_h \) values of \( j \) for which \( \overline{e}_j \in \overline{R}^h \setminus \overline{R}^{h+1} \), so the total
number of coefficients \( \nu_{i,j,k} \) is
\[
s \sum_{h=2}^{m-2} u_h d(h + 1) = s \sum_{h=2}^{m-2} u_h (u_{h+1} + u_{h+2} + \ldots + u_{m-1})
\]
\[
= \frac{1}{2} s ((u_2 + u_3 + \ldots + u_{m-1})^2 - u_2^2 - u_3^2 - \ldots - u_{m-1}^2)
\]
\[
\leq \frac{1}{2} s ((u_2 + u_3 + \ldots + u_{m-1})^2 - u_2^2)
\]
\[
\leq \frac{1}{2} s ((u + t)^2 - t^2).
\]

Hence the number of choices for the \( r^2 \) products \( \overline{x}_i \overline{x}_j \) and the \( s(w - r) \)
products \( \overline{x}_i \overline{e}_j \) is at most \( p^\beta \), where
\[
\beta = r^2 u + \frac{1}{2} s ((u + t)^2 - t^2)
\]
\[
= r^2 (w - r - t) + \frac{1}{2} s ((w - r)^2 - t^2),
\]
since \( u = w - r - t \). By [13, Corollary 5.3], \( s \leq t + 1 \), so

\[
\beta \leq r^2(w - r - t) + \frac{1}{2}s\{(w - r)^2 - (s - 1)^2\}. 
\]

We now fix one of the \( p^\beta \) choices for the coefficients \( \lambda_{i,j,k} \) and \( \nu_{i,j,k} \), thus determining the isomorphism class of \( \overline{R} \).

Lift the basis elements \( \overline{x}_1, \ldots, \overline{x}_r \) arbitrarily to elements \( x_1, \ldots, x_r \) of \( R \). Each element \( \overline{x}_i \) is a monomial in the elements \( \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_s \), as a consequence of the products above. We lift \( \overline{e}_i \) to the corresponding monomial \( e_i \in R \) in the elements \( x_1, x_2, \ldots, x_s \). The elements \( e_1, \ldots, e_{u+t}, x_1, \ldots, x_r \) are additive generators of \( R \), since \( R \) is a group of prime power order under addition and the elements additively generate \( R \) modulo \( pR \).

We have already chosen an additive group \( G \) for \( R \). There are at most \( p^n \) ways of representing an element \( x_i \) or \( e_i \) as an element of \( G \). There are at most \( n \) elements to represent, so there are at most \( p^{n^2} \) choices for these representations. After making these choices, addition in the ring (with elements represented as sums of the elements \( x_i \) and \( e_i \) in \( G \)) is determined. We choose the values of the \( r^2 \) products \( x_ix_j \) where \( 1 \leq i, j \leq r \) and the \( s(w - r) \) products \( x_ie_j \) where \( 1 \leq i \leq s \) and \( 1 \leq j \leq u + t = w - r \). Each of these \( r^2 + s(w - r) \) products is known modulo \( pR \) and the number of elements in \( pR \) is \( p^{n-w} \) so the number of possibilities for these products, after the choices we have fixed above, is at most \( p^\gamma \), where

\[
\gamma = \{r^2 + s(w - r)\}(n - w).
\]

Once we have made these choices, we claim that multiplication in \( R \), and hence the isomorphism class of \( R \), is determined. To see this, note that left multiplication by \( x_i \) is determined by the products above for any \( 1 \leq i \leq s \), since \( G \) is generated by the elements \( x_j \) and \( e_j \). So left multiplication by any monomial in \( x_1, x_2, \ldots, x_s \) is determined, by induction on the degree of the monomial. Each element \( e_j \) is a monomial in \( x_1, x_2, \ldots, x_s \), and (by our choice of elements \( e_j \)) a suitable monomial may be deduced from the products above. So left multiplication by each element \( e_j \) is also determined. We now show that left multiplication by \( x_a \), for \( s + 1 \leq a \leq r \), is a consequence of the products above. To show this, it is sufficient to consider each product \( x_ax_b \) for \( 1 \leq b \leq w - r \). We know that \( e_b \) is equal to some monomial \( x_{i_1}x_{i_2} \cdots x_{i_k} \) where \( k \geq 2 \) and \( 1 \leq i_1, i_2, \ldots, i_k \leq s \). So, by associativity, \( x_ax_b = (x_ax_{i_1}) \cdot (x_{i_2} \cdots x_{i_k}) \). The second factor is a monomial in \( x_1, x_2, \ldots, x_s \), and so is determined. The first factor is a sum of elements \( e_j \) since it lies
in $R^2$, and its value is determined by one of the products we have fixed. Since left multiplication by each element $e_j$ is determined, the product $x_a e_b$ is determined. Since addition in the ring is fixed, we see that left multiplication by any sum of the elements $x_i$ and $e_i$, in other words by any element of the ring, is determined using the distributive property of multiplication. So the product of any two elements in our ring is determined, and our claim follows.

To summarise, we have shown that the number of isomorphism classes of rings of order $p^n$ is at most $p^{\delta + O(n^2)}$, where $\delta$ is the maximum value of

$$\alpha(r, s, t) + \{r^2 + s(w-r)\}(n-w) + r^2(w-r-t)$$

$$+ \frac{1}{2} s\{(w-r)^2 - (s-1)^2\} \quad (9)$$

over all possible values of $r, s, t$ and $w$. It suffices to show that $\delta \leq \frac{4}{27} n^3 + O(n^{5/2})$.

If $w = n$ and $u = 0$ then $R = \overline{R}$ and $R^3 = 0$, in which case the desired result follows from \[13, \text{Theorem 2.2}\]. So we may assume that $u > 0$ or $w < n$. From \[13, \text{Corollary 5.3}\], $0 \leq s \leq t + 1$. Hence

$$r + s \leq r + t + 1 \leq r + t + u + 1 = w + 1 \leq n + 1$$

where one of the last two inequalities is strict; thus $r + s \leq n$. Clearly $0 \leq s \leq r$. Now let $x = r/n, y = s/n, z = t/n$ and $v = w/n$. Then

$$0 \leq y \leq x \quad \text{and} \quad x + y \leq 1. \quad (10)$$

Following Newman and Seely's idea, we consider separately the cases $x \leq 3/5$ and $x \geq 3/5$.

When $x \leq 3/5$, we substitute the first inequality from Lemma 2.2 into (9). Allowing the final error term to absorb minor terms, we get

$$\delta \leq \{sw + (r-s)r\}(n-w) + r^2(w-r-s) + \frac{1}{2} s\{(w-r)^2 - s^2\} + O(n^{8/3}).$$

So

$$\frac{\delta}{n^3} \leq (yv + x^2 - xy)(1-v) + x^2(v-x-y) + \frac{1}{2} y((v-x)^2 - y^2) + O(n^{-1/3})$$

$$= x^2(1-x) + \frac{1}{2} y\{2 - (x+1)^2 - (1-v)^2\} - \frac{1}{2} y^3 + O(n^{-1/3})$$

$$\leq x^2(1-x) + \frac{1}{2} y\{2 - (x+1)^2\} - \frac{1}{2} y^3 + O(n^{-1/3})$$

$$\leq 18/125 + O(n^{-1/3}),$$

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where the last inequality follows by applying standard calculus techniques over the region where the constraints $x \leq 3/5$ and (10) hold. Hence, as $\frac{18}{125} < \frac{4}{27}$, we get $\delta \leq \frac{4}{27}n^3 + O(n^{5/2})$ when $x \leq 3/5$, as desired.

When $x \geq 3/5$ we substitute the second inequality from Lemma 2.2 into (9). Allowing the final error term to absorb minor terms, we get

$$\delta \leq \{sw + (r - s)r\}n - w + r^2(w - r - s) + \frac{1}{7}rst + \frac{1}{7}s\{(w - r)^2 - s^2\} + O(n^{5/2}).$$

So we find that

$$\frac{\delta}{n^3} \leq (yv + x^2 - xy)(1 - v) + x^2(v - x - y) + \frac{1}{2}xyz$$
$$+ \frac{1}{2}y((v - x)^2 - y^2) + O(n^{-\frac{1}{2}})$$
$$\leq \frac{4}{27} + O(n^{-\frac{1}{2}}),$$

the last inequality following by applying standard calculus techniques over the region where the constraints $x \geq 3/5$, $z \leq 1$, $v \leq 1$ and (10) hold.

Hence in both cases, whether $x \leq 3/5$ or $x \geq 3/5$, we have

$$\delta \leq \frac{4}{27}n^3 + O(n^{5/2}).$$

3 Finite rings in general

For a ring $R$, we write $J(R)$ for the Jacobson radical of $R$. The following theorem provides some structure theory for finite rings that we require.

**Theorem 3.1.** Let $R$ be a finite non-nilpotent $p$-ring, not necessarily with an identity element. There exists a subring $S$ of $R$ such that:

(i) $S + J(R) = R$,

(ii) $S$ has a multiplicative identity,

(iii) $J(S) = pS$, and

(iv) $S \cap J(R) = pS$.

We say that the subring $S$ is a *coefficient ring* of $R$. (This terminology comes from the situation when $R$ has an identity element, as then every element of $R$ can be written as a polynomial in the generators of $J(R)$ with coefficients in $S$. This is not necessarily true in the more general situation.)
Proof. Let $S$ be a subring of $R$ that is minimal subject to (i) holding. (Clearly a minimal subring exists, since $R$ is finite and since (i) holds when $S = R$.) We will show that properties (ii), (iii) and (iv) all hold.

We establish property (ii) first. The ring $R$ is artinian, as it is finite. Since the Jacobson radical $J(R)$ of $R$ is nilpotent, but $R$ is not nilpotent, $J(R)$ is a proper ideal of $R$. The quotient $R/J(R)$ is semi-simple and non-trivial, and so the Wedderburn–Artin Theorem implies in particular that $R/J(R)$ has a multiplicative identity. By property (i) this identity may be written in the form $s + J(R)$ for some $s \in S$. Since $J(R)$ is nilpotent, it is a nil ideal. So the idempotent $s + J(R)$ may be lifted to an idempotent $e \in R$. Indeed (see, for example, the proof of [10, Proposition III.8.3]), $e$ may be taken to be a polynomial in $s$, and so we may assume that $e \in S$. Now $eSe$ is a subring of $S$ with identity $e$. Moreover, since $e + J(R) = s + J(R)$, we see that $e + J(R)$ is the identity in $R/J(R)$ and so $eSe + J(R) = S + J(R)$. By the minimality of $S$, we see that $eSe = S$, and so property (ii) is established.

We now establish property (iii), using an approach that is inspired by [4, Lemma 1].

Let $R$ have characteristic $p^k$. Then $pR$ is a nil (two-sided) ideal; indeed $pR$ is actually nilpotent, since $(pR)^k = 0$. Since $J(R)$ contains all nil ideals [10, Theorem I.6.2], we see that $pR \subseteq J(R)$.

Now, $J(R/J(R))$ is trivial [10, Theorem I.2.2]. Since (i) holds, the natural map from $S$ to $R/J(R)$ is surjective, and so (see [10, Proposition I.7.1]) the radical $J(S)$ is mapped into the trivial subring $J(R/J(R))$. Hence

$$J(S) \subseteq J(R).$$

In particular, combining with (11), we find that

$$pS \subseteq J(R).$$

The quotient ring $S/pS$ is an $\mathbb{F}_p$-algebra. By (11), we find that $J(S/pS) = J(S)/pS$ (see [5, Proposition 10.4.3]). Now, the Wedderburn–Malcev Theorem for $\mathbb{F}_p$-algebras [6, Theorem 72.19] shows that there exists a subring $T$ of $S$ containing $pS$ with the property that there is an additive decomposition

$$S/pS = T/pS \oplus J(S/pS) = T/pS \oplus J(S)/pS.$$
We see that

\[ R = S + J(R) = (T + J(S) + pS) + J(R) = T + J(R). \]

By the minimality of \( S \), we find that \( T = S \), and so (13) implies that

\[ J(S) \subseteq pS. \]

This, together with (11) implies property (iii).

To establish property (iv), first note that \( pS \subseteq pR \subseteq J(R) \) and so \( pS \subseteq S \cap J(R) \). Secondly, note that \( J(R) \) is a nil ideal of \( R \) (as it is nilpotent), so \( S \cap J(R) \) is a nil ideal of \( S \). Hence

\[ S \cap J(R) \subseteq J(S) = pS, \]

by (iii) and because \( J(S) \) contains all nil ideals in \( S \) (see [10, Corollary to Theorem I.6.2]). Hence (iv) follows and the theorem is proved.

We comment that there is a natural, but less elementary, proof of Theorem 3.1 (iii) using Azumaya’s generalized Wedderburn-Malcev theorem (see [1, Proposition 19] or [2, Theorem 33]). In this approach, we define \( Z \) to be the set of integer multiples of \( e \) (so \( S \) is a \( Z \)-algebra) and use Azumaya’s theorem to deduce that there exists a separable \( Z \)-algebra \( T \) with \( T \subseteq S \) such that \( S = T + J(S) \). We may show that \( J(T) = pT \) by observing that \( pT \) is a nilpotent ideal, and that \( T/pT \) is a separable (and hence semisimple) \( Z/pZ \)-algebra. Since \( J(S) \subseteq J(R) \), we see that \( R = S + J(R) = T + J(R) \) and so \( S = T \) by the minimality of \( S \). Since \( J(T) = pT \) we see that (iii) follows.

We require some extra information about the subring \( S \) in the theorem above. First, a result due to Clark gives much more information about the structure of \( S \). Recall that, for integers \( k \) and \( r \), the Galois ring \( GR(p^k, r) \) may be defined by

\[ GR(p^k, r) = \mathbb{Z}_{p^k}[x]/(f(x)), \]

where \( \mathbb{Z}_{p^k} \) denotes the integers modulo \( p^k \), and where \( f(x) \in \mathbb{Z}_{p^k}[x] \) is a monic polynomial of degree \( r \) which is irreducible modulo \( p \). Note that (see [18, Section 3] for example) the isomorphism class of the ring \( GR(p^k, r) \) is determined by \( k \), \( r \) and \( p \). Note also that \( GR(p, r) \cong \mathbb{F}_{p^r} \).
Lemma 3.2. Let $S$ be a finite $p$-ring with an identity. Then $S$ is a direct sum of full matrix rings over Galois rings if and only if $J(S) = pS$.

Proof. See Clark [4, Lemma 3].

Corollary 3.3. The number of isomorphism classes of rings $S$ that are the coefficient ring of some finite non-nilpotent $p$-ring of cardinality $p^n$ is at most $n \, 2^{3n}$. In particular, there are $p^{O(n)}$ choices for the isomorphism class of $S$.

Proof. We have $|S| = p^s$, where $1 \leq s \leq n$, and so there are at most $n$ choices for $s$. Suppose $s$ is fixed. Theorem 3.1 and Lemma 3.2 together imply that $S$ is a direct sum of full matrix rings over Galois rings:

$$S \cong S_1 \oplus S_2 \oplus \cdots \oplus S_t,$$

for some positive integers $t$, $m_i$, $k_i$ and $r_i$. Clearly the isomorphism class of $S$ is determined by the choice of these integers. Now, $|S_i| = p^{s_i}$, where $s_i = m_i^2 k_i r_i$. Since $S$ is a direct sum of the rings $S_i$, we see that

$$s = \sum_{i=1}^{t} s_i \geq \sum_{i=1}^{t} k_i.$$

In particular, $(k_1, k_2, \ldots, k_t, s + 1 - \sum_{i=1}^{t} k_i)$ is an ordered (positive) integer partition of $s + 1$. There are $2^s$ ordered integer partitions of $s + 1$, and so there are at most $2^s$ choices for the integers $t$ and $k_1, k_2, \ldots, k_t$. Similarly, there are at most $2^s$ choices for the integers $r_1, r_2, \ldots, r_t$ and, since

$$s \geq \sum_{i=1}^{t} m_i^2 \geq \sum_{i=1}^{t} m_i,$$

there are at most $2^s$ choices for the integers $m_1, m_2, \ldots, m_t$. Hence the number of choices for the isomorphism class of $S$ is at most

$$n(2^s)^3 \leq n \, 2^{3n},$$

as required.

Theorem 3.4. Let $S$ be a non-trivial direct sum of full matrix rings over Galois rings. Suppose that $|S| \leq p^n$. The number of isomorphism classes of (not necessarily unital) left $S$-modules of cardinality at most $p^n$ is at most $p^{2n^2 + 3n + 1} = p^{O(n^2)}$. The same statement holds for right $S$-modules.
Proof. We prove the theorem for left-modules. The proof for right-modules is identical.

We begin with considering the special case when \( S = M_{m \times m}(\text{GR}(p^k, r)) \) for positive integers \( m, k \) and \( r \).

We claim that \( S \) is 2-generated (as a ring) in this special case. To see this, first note that \( \text{GR}(p^k, r) \) is 1-generated: any element \( \zeta \in \text{GR}(p^k, r) \) whose image in the natural map onto the finite field \( \text{GR}(p, r) \) is primitive (or, more generally, lies in no proper subfields) will generate \( \text{GR}(p^k, r) \). We then note that, writing \( E_{i,j} \in S \) for the matrix with \((i,j)\) entry 1 and all other entries 0, the ring \( S \) is generated by \( g_1 \) and \( g_2 \) where \( g_1 = \zeta E_{1,1} \) and where \( g_2 \) is the cyclic permutation matrix defined by

\[
g_2 = E_{1,2} + E_{2,3} + \cdots + E_{m-2,m-1} + E_{m-1,m} + E_{m,1}.
\]

This establishes our claim.

A (not necessarily unital) left \( S \)-module \( V \) of cardinality at most \( p^n \) is determined by its structure as an abelian group, together with the pair of maps in \( \text{End}(V) \) induced by the left action of each of \( g_1 \) and \( g_2 \) on \( V \). (Here, \( \text{End}(V) \) is the set of abelian group homomorphisms from \( V \) to itself.) Now, as an abelian group we find that

\[
V \cong \mathbb{Z}_{p^{a_1}} \oplus \mathbb{Z}_{p^{a_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{a_r}},
\]

for some positive integers \( a_1, a_2, \ldots, a_r \) where \( \sum_{i=1}^r a_i \leq n \). In particular, the sequence \((a_1, a_2, \ldots, a_r, n+1-\sum_{i=1}^r a_i)\) is an ordered partition of \( n+1 \) into positive integers, and so there are at most \( 2^n \) choices for these integers. Since any element of \( \text{End}(V) \) is specified by the images of \( r \) generators for \( V \), we see that

\[
|\text{End}(V)| \leq |V|^r \leq (p^n)^n = p^{n^2},
\]

so there are at most \( p^{n^2} \) choices for the action of each of \( g_1 \) and \( g_2 \) on \( V \). Hence the number of left \( S \)-modules is at most \( p^{2n^2+n} \), and the theorem follows in this case.

We now consider the general case, so

\[
S = S_1 \oplus S_2 \oplus \cdots \oplus S_t \text{ where } S_i = M_{m_i \times m_i}(\text{GR}(p^{k_i}, r_i))
\]

(14)

for some positive integers \( t, m_i, k_i \) and \( r_i \). Note that, since \( |S| \leq p^n \), we have \( t \leq n \).
Let $e_i \in S$ be the identity matrix in $S_i$, so 

$$e_i e_j = \begin{cases} 
0 & \text{when } i \neq j, \\
 e_i & \text{when } i = j.
\end{cases}$$

The identity element 1 of $S$ is $\sum_{t=1}^t e_i$. The subring $e_i Se_i$ is the $i$th ring in the sum (14), and so is isomorphic to $S_i$.

Let $V$ be a left $S$-module of cardinality at most $p^n$. Setting $V_0$ to be the kernel of the map $v \mapsto 1v$ on $V$, there is an additive decomposition of $V$ of the form 

$$V = V_0 \oplus 1V = V_0 \oplus e_1V \oplus e_2V \oplus \cdots \oplus e_tV.$$ 

For $i \in \{1, 2, \ldots, t\}$, write $V_i = e_iV$. For $i \in \{0, 1, \ldots, t\}$, define the non-negative integer $v_i$ by $|V_i| = p^{v_i}$. Since $\sum_{i=0}^t v_i \leq n$, and since $t \leq n$, we see that $\sum_{i=0}^t (v_i + 1) \leq 2n + 1$. Hence the sequence $(v_0 + 1, v_1 + 1, \ldots, v_t + 1, 2n + 2 - \sum_{i=0}^t)$ is an ordered integer partition of $2n + 2$, and so there are at most $2^{2n+1}$ possibilities for the integers $v_i$.

Since $S$ acts trivially on $V_0$, the isomorphism class of the module $V_0$ is entirely determined by its abelian group structure. Since $|V| = p^{v_0}$, the argument used in the special case above shows that there are at most $2^{v_0}$ possibilities for $V_0$ once $v_0$ is fixed.

Let $i \in \{1, 2, \ldots, t\}$ be fixed. When $i \neq j$, the subring $e_j Se_j$ of $S$ acts trivially on $V_i$. So $V_i$ is entirely determined as a left $S$-module by its left $e_iSe_i$-module structure. Since $e_iSe_i \cong S_i$, the special case of the theorem we have already established shows that there are at most $p^{2v_i^2 + v_i}$ possibilities for the isomorphism class of $V_i$.

So the number of isomorphism classes of left $S$-modules $V$ is at most 

$$2^{2n+1}2^{v_0} \prod_{i=1}^t p^{2v_i^2 + v_i} \leq p^{2n^2 + 3n + 1},$$

since $\sum_{i=0}^t v_i \leq n$, and since $2 \leq p$. Hence the theorem follows. 

\textbf{Theorem 3.5.} Let $r$ and $n$ be integers such that $0 \leq r \leq n$. Let $J$ be a nilpotent ring of cardinality $p^r$. The number of isomorphism classes of rings $R$ of cardinality $p^n$ with $J(R) \cong J$ is at most $np^{7n^2 + 9n + 2} = p^{O(n^2)}$.

\textit{Proof.} The theorem is clearly true when $r = n$, since $R \cong J$ in this case. So we may assume that $r < n$. 

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Let $R$ be a ring of cardinality $R \cong J$. By replacing $R$ by a suitable isomorphic copy, we may assume that $J(R) = J$. We aim to show that the isomorphism class of the ring $R$ is determined by a certain sextuple of algebraic structures. Indeed, we aim to show that the isomorphism class of $R$ is determined by: the isomorphism class of a coefficient ring $S$; the isomorphism classes of left and right $S$-modules corresponding to left and right multiplication of $S$ on $J$; two abelian group isomorphisms that identify the underlying sets of these $S$-modules with $J$; an abelian group isomorphism that determines how $pS$ embeds in $J$. We will then proceed to count the number of possibilities for these structures.

Let $S$ be a set of representatives for isomorphism classes of coefficient rings for rings of order $p^n$. For each such ring $S$, choose a fixed set $C_S$ of coset representatives for $pS$ in $S$.

Choose a coefficient ring $U$ for $R$, and let $S \in S$ be isomorphic to $U$. So there exists an isomorphism $\theta : S \to U$. By Theorem 3.1(iv), we know that $pU \subseteq J$. So the restriction of $\theta$ to $pS$ is an injective abelian group homomorphism $\psi : pS \to J$.

Theorem 3.1(i) and (iv) implies that every element of $R$ may be uniquely written in the form $\theta(s) + x$ where $s \in C_S$ is one of the coset representatives for $pS$ in $S$ chosen above and where $x \in J$.

Let $V_S$ and $V'_S$ be sets of representatives for the isomorphism classes of, respectively, all left $S$-modules of cardinality $p^t$ and all right $S$-modules of cardinality $p^t$. Since $J$ is a left ideal in $R$, left multiplication by $U$ makes $J$ into a left $U$-module. Using the isomorphism $\theta$ between $S$ and $U$, we see that $J$ is a left $S$-module. Let $V \in V_S$ be isomorphic to the left $S$-module that arises in this way, and let $\phi : V \to J$ be the induced module isomorphism (which, in particular, is an isomorphism of abelian groups). Similarly, right multiplication gives rise to a right $S$-module $V' \in V'_S$ and an isomorphism $\phi' : V' \to J$. For all $s \in S$, $v \in V$ and $v' \in V'$,

$$sv = \theta(s)\phi(v) \quad \text{and} \quad v's = \phi'(v')\theta(s).$$

We have shown that each ring $R$ gives rise to (at least one) sextuple $(S, \psi, V, \phi, V', \phi')$ where $S \in S$, $V \in V_S$ and $V' \in V'_S$, where $\psi : pS \to J$ is an injective group homomorphism, and where the maps $\phi : V \to J$ and $\phi' : V' \to J$ are isomorphisms of abelian groups.

We may form a ring $T$ that is isomorphic to $R$ by taking all formal sums $s + x$ with $s \in C_S$ and $x \in J$ as our underlying set, and defining addition
and multiplication as follows. Let \( s_1, s_2 \in C_S \). Now, \( s_1 + s_2 = s + y \) for some \( s \in C_S \) and \( y \in pS \), and \( s_1s_2 = s' + y' \) for some \( s' \in C_S \) and \( y' \in pS \). We define the sum of elements \( s_1 + x_1 \) and \( s_2 + x_2 \) in \( T \) to be

\[
s + \psi(y) + x_1 + x_2,
\]

and the product of these elements to be

\[
s' + \psi(y') + \phi(s_1 \phi^{-1}(x_2)) + \phi'(\phi'^{-1}(x_1) s_2) + x_1 x_2.
\]

We see that \( T \cong R \), via the isomorphism that maps \( s + x \) to \( \theta(s) + x \) for all \( s \in C_S \) and \( x \in J \). Since \( T \) is defined only using the sextuple \((S, \psi, V, \phi, V', \phi')\), this information is sufficient to determine the isomorphism class of \( R \).

It remains to count the number of possibilities for \((S, \psi, V, \phi, V', \phi')\). There are at most \( n2^{3n} \) possibilities for \( S \), by Corollary 3.3. Once \( S \) is fixed, there are at most \( p^{2n^2+3n+1} \) possibilities for each of the \( S \)-modules \( V \) and \( V' \), by Theorem 3.4. The functions \( \psi \), \( \phi \) and \( \phi' \) are all abelian group homomorphisms between \( p \)-primary groups of order at most \( p^n \). Such groups are generated by at most \( n \) elements, and a homomorphism is determined by the images of a generating set, so the number of choices for each of \( \psi \), \( \phi \) and \( \phi' \) is at most \( (p^n)^n = p^{n^2} \). Hence the number of possibilities for our sextuple is at most

\[
n2^{3n}(p^{2n^2+3n+1})^2(p^{n^2})^3 \leq n p^{7n^2+9n+2} = p^{O(n^2)}.
\]

Each sextuple is associated with at most one isomorphism class of rings \( R \), and every ring \( R \) is associated with at least one sextuple. So the theorem follows.

**Theorem 3.6.** The number of isomorphism classes of rings of cardinality \( p^n \) is \( p^\delta \) where \( \delta = \frac{4}{27}n^3 + O(n^{5/2}) \).

**Proof.** Theorem 2.2 of [13] shows the number of \( \mathbb{F}_p \)-algebras of cube zero and dimension \( n \) is \( p^{\frac{8}{27}n^3 + O(n^2)} \). This provides the lower bound we need. So to prove the theorem it is sufficient to show that there are at most \( p^{\frac{4}{27}n^3 + O(n^{5/2})} \) rings of order \( p^n \).

The Jacobson radical \( J(R) \) of a ring \( R \) of cardinality \( p^n \) is nilpotent (as \( R \) is artinian), and has cardinality \( p^r \) for some integer \( r \) such that \( 0 \leq r \leq n \). By Theorem 2.3 there are at most \( p^{\frac{r^3}{3} + O(r^{5/2})} \) nilpotent rings of cardinality \( p^r \), and so there are at most \( p^{\frac{4}{27}r^3 + O(r^{5/2})} \) choices for the isomorphism class of \( J(R) \). Once the isomorphism class of \( J(R) \) is fixed, Theorem 3.5 shows that
there are at most \( np^{7n^2+9n+2} \) choices for the isomorphism class of \( R \). Hence the number of rings of cardinality \( p^n \) is at most

\[
\sum_{r=0}^{n} np^{7n^2+9n+2} p^{\frac{r^3}{2}+O(r^{5/2})} = p^{\frac{n^3}{2}+O(n^{5/2})},
\]
as required.

The high-level lesson we might take from the theorem above is that the structure of non-nilpotent rings is very restricted: the leading term of the enumeration function is provided by nilpotent rings. Indeed, it may well be the case (though we are far from having a proof) that the proportion of rings of order \( p^n \) that are non-nilpotent tends to 0 as \( n \to \infty \). The following theorem provides a result in this direction.

**Theorem 3.7.** Let \( f_s(n) \) be the number of isomorphism classes of rings \( R \) such that \(|R| = p^n\) and \( R/J(R) \geq p^s \). There exists a positive real number \( \sigma \) such that, when we set \( s = \sigma \sqrt{n} \),

\[
\lim_{n \to \infty} \frac{f_s(n)}{f_{\text{rings}}(n)} = 0.
\]

We note that Kruse and Price [13, Theorem 5.10] provide a weaker version of this result, setting \( s = \varepsilon n \) for some (arbitrary) positive real number \( \varepsilon \).

**Proof.** Theorem 2.3 shows that the number of isomorphism classes of nilpotent rings of cardinality \( p^r \) is at most \( p^{\frac{n^3}{2} + \kappa n^{5/2}} \) for some positive constant \( \kappa \). Let \( \sigma \) be a positive real number so that \( \frac{4}{9} \sigma > \kappa \), and define \( s = \sigma \sqrt{n} \). Following the approach of the proof of Theorem 3.6, we see that

\[
f_s(n) \leq \sum_{r=0}^{n-[s]} np^{7n^2+9n+2} p^{\frac{r^3}{2}+\kappa r^{5/2}} = p^{\alpha},
\]

where

\[
\alpha \leq \frac{4}{27} (n-s)^3 + \kappa n^{5/2} + O(n^2) = \frac{4}{27} n^3 + (\kappa - \frac{4}{9} \sigma) n^{5/2} + O(n^2).
\]

Since \( f_{\text{rings}}(n) \geq p^{\frac{n^3}{2}+O(n^2)} \) by [13, Theorem 2.2], we see that

\[
f_s(n)/f_{\text{rings}}(n) \leq p^{(\kappa - \frac{4}{9} \sigma)n^{5/2} + O(n^2)} \to 0
\]
as \( n \to \infty \), since \( \frac{4}{9} \sigma > \kappa \). \( \square \)
We remark that, even though it is possible that most rings are nilpotent, the number of non-nilpotent rings is nevertheless large:

**Theorem 3.8.** The number of isomorphism classes of non-nilpotent rings of cardinality $p^n$ is $p^{\frac{2}{27}n^3 + O(n^{5/2})}$.

**Proof.** The upper bound is provided by Theorem 3.6. The lower bound may be proved by considering rings that are a direct sum of the form $F_p \oplus N$, where $N$ is a nilpotent ring of order $p^n - 1$. There are $p^{\frac{2}{27}n^3 + O(n^{5/2})}$ choices for the isomorphism class of $N$, by Theorem 2.3, and so the theorem follows. \[\]

Finally we remark that, since the rings constructed for the lower bound all have an identity element, the following theorem holds:

**Theorem 3.9.** The number of isomorphism classes of rings with identity that have cardinality $p^n$ is $p^{\frac{2}{27}n^3 + O(n^{5/2})}$.

### 4 Finite commutative rings

This section contains proofs of our theorems on the enumeration of finite commutative rings. Some preliminary results (Theorem 4.1 and Lemma 4.3) are easily proved by adapting the proofs of their more general non-commutative versions. For example, in a ring $R$ with $r$ generators $x_1, \ldots, x_r$, we usually need to consider $r^2$ possible products $x_i x_j$, whereas when $R$ is commutative, there are at most $\frac{1}{2}r(r + 1)$ such products that are distinct. However, in order to prove Theorem 4.4, a more thorough modification of the proof of Theorem 2.3 is needed.

We use the following theorem for our lower bound.

**Theorem 4.1.** The number of (isomorphism classes of) commutative nilpotent $F_p$-algebras with cube zero and dimension $n \geq 2$ is $p^n$, where $\alpha = \frac{2}{27}n^3 + O(n^2)$.

**Proof.** The lower bound is a consequence of Poonen [16, Lemma 9.1] and [16, Theorem 9.2], so it suffices to establish a corresponding upper bound.

Let $f(n, r)$ be the number of (isomorphism classes of) commutative $F_p$-algebras $A$ such that $\dim A = n$, $\dim(A/A^2) = r$ and $A^3 = 0$. By following the proof of [13, Theorem 2.1], but using the free commutative $F_p$-algebra $F$ of cube zero and rank $r$, and observing that $F^2$ has dimension $\frac{1}{2}(r + 1)$,
we may prove the following. First, if \( \frac{1}{2}r(r + 1) < n - r \), then \( f(n, r) = 0 \). Secondly, if \( \frac{1}{2}r(r + 1) \geq n - r \), then

\[
\frac{1}{2}r(r+1)(n-r)-(n-r)^2-r^2 \leq \log_p f(n, r) \leq \frac{1}{2}r(r+1)(n-r)-(n-r)^2+n-r.
\]

The theorem now follows by observing that the value of \( \frac{1}{2}r^2(n-r) \) takes its maximum value of \( 2n^3/27 \) when \( r = 2n/3 \), and following the proof of [13, Theorem 2.2].

\[\square\]

**Lemma 4.2.** Let \( R \) be a commutative nilpotent \( \mathbb{F}_p \)-algebra such that \( r = \dim(R/R^2) \), \( s \) is the Sims dimension of \( R \), \( t = \dim(R^2) \) and \( R^3 = 0 \). There exist \( x_1, \ldots, x_r \in R \) such that \( x_1 + R^2, x_2 + R^2, \ldots, x_r + R^2 \) form a basis for \( R/R^2 \), such that \( \langle x_1, x_2, \ldots, x_s \rangle^2 = R^2 \) and such that for \( 1 \leq i \leq s - 1 \)

\[x_ix_{i+1} \notin \langle x_1, x_2, \ldots, x_i \rangle^2.\]

In particular,

\[\langle x_1 \rangle^2 \subset \langle x_1, x_2 \rangle^2 \subset \cdots \subset \langle x_1, x_2, \ldots, x_s \rangle^2\]

is a strictly increasing sequence of subalgebras.

**Proof.** By definition of the Sims dimension, there is a subalgebra \( S \) of \( R \) such that \( \dim S/R^2 = s \) and such that \( S^2 = R^2 \). Moreover, we may assume that no proper subalgebra \( T \) of \( S \) has \( T^2 = R^2 \). The existence of \( x_1, x_2, \ldots, x_s \in R \) with the properties we want follows from [16, Proposition 10.1] (where, in the notation of Proposition 10.1, we take \( V = S/R^2, W = R^2 \), and multiplication as our symmetric bilinear map from \( V \times V \) to \( W \)). The elements \( x_1, x_2, \ldots, x_s \) are linearly independent modulo \( R^2 \), so there exist \( x_{s+1}, x_{s+2}, \ldots, x_r \in R \) so that \( x_1 + R^2, x_2 + R^2, \ldots, x_r + R^2 \) form a basis for \( R/R^2 \). So the lemma follows. \[\square\]

Here is a commutative version of our Lemma 2.2.

**Lemma 4.3.** Let \( r, s, t \) be positive integers and let \( \alpha(r, s, t) \) be a real number. Suppose that \( p^{\alpha(r, s, t)} \) is the number of (isomorphism classes of) commutative \( \mathbb{F}_p \)-algebras, \( R \), such that \( r = \dim(R/R^2) \), \( s \) is the Sims dimension of \( R \), \( t = \dim(R^2) \) and \( R^3 = 0 \). Then

\[
\alpha(r, s, t) \leq \frac{1}{2}r^2(t-s) + O((r+t)^{8/3}) \tag{15}
\]

and

\[
\alpha(r, s, t) \leq \frac{1}{2}r^2(t-s) + \frac{1}{2}rst + O((r+t)^{5/2}). \tag{16}
\]
Proof. The inequality (15) is Poonen [16, Proposition 10.4]. The proof of (16) closely follows the proof of (2) in Lemma 2.2 above. The sole change is to observe that we only require knowledge of the \( \frac{1}{2} r(r+1) \) products \( x_i x_j \) where \( 1 \leq i \leq j \leq r \) since we are working in a commutative ring so, in the notation of the proof of Lemma 2.2,

\[
\alpha(r, s, t) \leq \left( \frac{g}{2} \right) d(t-d+1) + \frac{1}{2} r(r+1) d \leq \frac{1}{2} r^2 (t-s) + \frac{1}{2} rs t + O\left((r+t)^{5/2}\right). \quad \Box
\]

As in Section 2, for each representative of the \( \alpha^{(r,s,t)} \) isomorphism classes of commutative \( \mathbb{F}_p \)-algebras of the form above we choose a standard basis \( x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_t \) of \( R \). As before, the elements of the standard basis should have the properties that: the elements \( x_1 + R^2, x_2 + R^2, \ldots, x_r + R^2 \) form a basis for \( R/R^2 \); the elements \( y_1, y_2, \ldots, y_t \) form a basis for \( R^2 \); each element \( y_i \) may be written in the form \( y_i = x_k x_\ell \) for some \( k, \ell \in \{1, 2, \ldots, s\} \).

In addition, we require the following two properties. First, we require that

\[
\langle x_1 \rangle^2 \subset \langle x_1, x_2 \rangle^2 \subset \cdots \subset \langle x_1, x_2, \ldots, x_s \rangle^2
\]

is a strictly increasing sequence of subalgebras. Secondly, writing \( q_i = \dim \langle x_1, x_2, \ldots, x_i \rangle^2 \), we require that \( y_1, y_2, \ldots, y_{q_i} \) is a basis of \( \langle x_1, x_2, \ldots, x_i \rangle^2 \). A standard basis exists, by Lemma 4.2. Note that, since \( q_1 < q_2 < \cdots < q_s = t \), we find that

\[
q_i \leq t - s + i \text{ for } 1 \leq i \leq s. \quad (17)
\]

For each element \( y_j \), we choose a representation of the form \( y_j = x_k x_\ell \) for some \( k, \ell \in \{1, 2, \ldots, s\} \), and call this the standard monomial representation of \( y_j \). Clearly we may insist that \( k \leq \ell \) here. Moreover, when \( q_{i-1} < j \leq q_i \), we may (and we do) choose \( \ell = i \).

**Theorem 4.4.** The number of (isomorphism classes of) commutative nilpotent rings of cardinality \( p^n \) is \( p^n \), where \( \alpha = \frac{2}{27} n^3 + O(n^{5/2}) \).

**Proof.** From Theorem 4.1 the number of commutative \( \mathbb{F}_p \)-algebras with cube zero and order \( p^n \) is \( p^n \), where \( \alpha' = 2n^3/27 + O(n^2) \). Hence it is sufficient to prove that \( p^n \) is an upper bound for the number of commutative nilpotent rings of order \( p^n \).

As in the proof of Theorem 2.3, we choose a concrete abelian group \( G \) of order \( p^n \) and rank \( w \) that will be isomorphic to the additive group of \( R \). We set \( R = R/pR \), so \( R \) is an \( \mathbb{F}_p \)-algebra of order \( p^w \), and choose \( r = \dim(R/R^2) \),

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\[ t = \dim(\mathbb{R}^2/\mathbb{R}^3), \quad u = \dim(\mathbb{R}^3) \]
and the Sims dimension \( s \) of \( \mathbb{R} \) (which is equal to the Sims dimension of \( \mathbb{R}/\mathbb{R}^3 \)). We choose \( m \), the least integer such that \( \mathbb{R}^m = 0 \), and we choose integers \( u_h = \dim(\mathbb{R}^h/\mathbb{R}^{h+1}) \). The argument in the proof of Theorem 2.3 shows we have made \( p^{O(n)} \) choices so far. We choose one of the \( p^{a(r,s,t)} \) isomorphism classes for the \( \mathbb{F}_p \)-algebra \( \mathbb{R}/\mathbb{R}^3 \).

For a nilpotent ring \( R \) of cardinality \( p^n \) that respects the choices we made above, we use a standard basis for \( \mathbb{R}/\mathbb{R}^3 \) to construct a basis

\[ x_1, x_2, \ldots, x_r, e_1, e_2, \ldots, e_{u+t} \]

for \( \mathbb{R} \) where each element \( e_j \) is a monomial in \( x_1, x_2, \ldots, x_s \) just as in the proof of Theorem 2.3. In particular, \( e_{u+u+2}, \ldots, e_u \) form a basis for \( \mathbb{R}^3 \) modulo \( \mathbb{R}^4 \), and each element \( e_{u+u+k} \) has the form \( \bar{x}_a \bar{x}_b \) where \( 1 \leq a \leq s \) and \( u < b \leq u + t \). We claim that we may insist in addition that \( u < b \leq u + q_a \).

To prove the claim, it suffices to show that the set

\[ S = \{ \bar{x}_a \bar{x}_b : 1 \leq a \leq s \text{ and } u < b \leq u + q_a \} \]

spans \( \mathbb{R}^3 \) modulo \( \mathbb{R}^4 \). The argument in the proof of Theorem 2.3 shows that the larger set

\[ \hat{S} = \{ \bar{x}_a \bar{x}_b : 1 \leq a \leq s \text{ and } u < b \leq u + t \} \]

spans \( \mathbb{R}^3 \) modulo \( \mathbb{R}^4 \). An element in \( \hat{S} \setminus S \) may be written in the form \( \bar{x}_a \bar{x}_b \) where \( b > u + q_a \). We show that such an element is a sum of elements of \( S \) modulo \( \mathbb{R}^4 \), which will establish our claim. Since \( b > u + q_a \), we see that \( \bar{x}_b = \bar{x}_f \bar{x}_g \) with \( f \leq g \) and \( g > a \). Using the fact that our ring is commutative,

\[ \bar{x}_a \bar{x}_b = \bar{x}_a \bar{x}_f \bar{x}_g = \bar{x}_g(\bar{x}_f \bar{x}_a). \]

Since \( a < g \) and \( f \leq g \), we see that \( \bar{x}_f \bar{x}_a + \mathbb{R}^3 \) is contained in \( \langle x_1, x_2, \ldots, x_g \rangle^2 + \mathbb{R}^3 \). So, working modulo \( \mathbb{R}^3 \), the product \( \bar{x}_f \bar{x}_a \) is a sum of elements \( \bar{e}_j \) with \( u < j \leq u + q_g \). Hence, working modulo \( \mathbb{R}^4 \), the product \( \bar{x}_g(\bar{x}_f \bar{x}_a) \) is a sum of elements in \( S \), and our claim follows.

We construct an additive generating set \( x_1, x_2, \ldots, x_r, e_1, e_2, \ldots, e_{u+t} \) for \( R \) using the monomial representations of \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{u+t} \) just as before. We identify the elements of this generating set with elements of \( G \): this requires making \( p^{O(n^2)} \) choices.
The argument in Theorem 2.3 shows that the isomorphism class of \( R \) is determined once we know the products \( x_i x_j \) for \( 1 \leq i, j \leq r \) and the products \( x_i e_j \) for \( 1 \leq i \leq s \) and \( 1 \leq j \leq u + t \) (as elements of \( G \)). Since \( R \) is commutative, we only need to know the products \( x_i x_j \) with \( 1 \leq i \leq j \leq r \) and the products \( x_i e_j \) for \( 1 \leq i \leq s \) and \( 1 \leq j \leq u + t \). Unfortunately, if we count the number of possibilities for these equations using the methods in Theorem 2.3 we do not get a sufficiently tight bound: we produce a bound of the form \( p^{cn^3 + O(n^{5/2})} \) with \( c > 2/27 \). We improve this bound by showing that we do not need to know all of the products \( x_i e_j \). Indeed we claim that multiplication in our ring is determined once we know the following products (as elements of \( G \)):

(i) the products \( x_i x_j \) for all \( i, j \) such that \( 1 \leq i \leq j \leq r \),

(ii) the products \( x_i e_j \) for all \( i, j \) such that \( 1 \leq i \leq s \) and \( u + 1 \leq j \leq u + q_t \),

(iii) the products \( x_i e_j \) for \( 1 \leq i \leq s \) and \( 1 \leq j \leq u \).

To prove this claim, we first observe, just as in Theorem 2.3, that these equations allows us to express every element \( e_j \) as a monomial in \( x_1, x_2, \ldots, x_s \). (Our more careful choice of \( e_{u-u_3+1}, e_{u-u_3+2}, \ldots, e_u \) is required at this point.) We show that the products \( x_i e_j \) for \( 1 \leq i \leq s \) and \( 1 \leq j \leq u + t \) are determined by (i) to (iii) above (together with the additive structure \( G \) of \( R \)). This suffices to prove our claim, using the argument in the proof of Theorem 2.3. Indeed, writing \( \text{char}(R) = p^\kappa \), we will show that the products \( p^a x_i e_j \) are determined for \( 0 \leq a \leq \kappa \) by induction on \( \kappa - a \) (the case \( a = 0 \) giving us what we need). Clearly \( p^a x_i e_j = 0 \) when \( a = \kappa \) (and so \( p^a x_i e_j \) is determined). Now suppose, as an inductive hypothesis, that \( 0 < a \leq \kappa \) and the products \( p^a x_i e_j \) are determined by (i) to (iii). We show that \( p^{a-1} x_i e_j \) is determined. When \( j \leq u + q_t \) the product is determined by the equations (ii) and (iii), so we may assume that \( j > u + q_t \). We may write \( e_j = x_{a'} x_{b'} \) where \( a' \leq b' \) and \( b' > i \) (and this expression is determined by (i)). Since \( R \) is commutative, \( p^{a-1} x_i e_j = p^{a-1} x_{b'}(x_i x_{a'}) \). Since \( a' \leq b' \) and \( i < b' \) we find that \( x_i x_{a'} \in \langle a_1, a_2, \ldots, x_{b'} \rangle^2 \), and so

\[
x_i x_{a'} \in \langle e_1, e_2, \ldots, e_{u+q_t}, pe_{u+q_t+1}, \ldots, pe_{u+t} \rangle.
\]

Since multiplication is distributive and the product \( x_i x_{a'} \) is determined by (i), we see that \( p^{a-1} x_i e_j \) is determined once we know \( p^{a-1} x_{b'} x \) where

\[
x \in \{ e_1, e_2, \ldots, e_{u+q_t}, pe_{u+q_t+1}, \ldots, pe_{u+t} \}.
\]
But the products
\[ p^{a-1}x_v e_1, p^{a-1}x_v e_2, \ldots, p^{a-1}x_v e_{u+q_v} \]
are determined by the equations (ii) and (iii), and the products
\[ p^{a-1}(px_1 e_{u+q_v}), p^{a-1}(px_2 e_{u+q_v+2}), \ldots, p^{a-1}(px_1 e_{u+t}) \]
are determined by our inductive hypothesis. Thus, since multiplication is
distributive, \( p^{a-1}x_v e_j \) is determined. So our result follows by induction.

We have now shown that the isomorphism class of \( R \) is fixed once we
have chosen the products (i), (ii) and (iii) above. We now provide an upper
bound for the number of these choices.

Each product \( x_i x_j \) lies in \( R^2 \), and is already determined modulo \( R^3 + pR \)
because we have chosen \( \mathfrak{p} \) as a standard basis. Since \( |pR| = p^{n-w} \) and
\( |R^3| = p^w \) we see that \( |R^3 + pR| = p^{n-w+u} = p^{n-r-t} \) and so there are at
most \( p^{n-r-t} \) choices for the product \( x_i x_j \). There are \( \frac{1}{2}r(r+1) \) products
of the form (i), and so the number of choices for for these products is at most
\( p \) to the power \( \frac{1}{2}r(r+1)(n-r-t) \). Hence the number of choices for the
products (i) is at most \( p \) to the power \( \frac{1}{2}r^2(n-r-t) + O(n^2) \).

For a fixed integer \( i \) with \( 1 \leq i \leq s \), there are \( q_i \) choices of integers \( j \) such
that \( u+1 \leq j \leq u+q_i \). The product \( x_i e_j \) lies in \( R^3 \), and \( |R^3| \leq |R^3 + pR| =
p^{n-r-t} \). Therefore the number of choices for the equations (ii) is at most \( p \) to the power \( \sum_{i=1}^s q_i(n-r-t) \).

Using (17), we see that
\[
\sum_{i=1}^s q_i(n-r-t) \leq \sum_{i=1}^s (t-s+i)(n-r-t) = \left(s(t-s) + \frac{1}{2}s(s+1)\right)(n-r-t).
\]
So the number of choices for the products (ii) is at most \( p \) to the power
\( (st-\frac{1}{2}s^2)(n-r-t) + O(n^2) \).

A product \( x_i e_j \) of the form (iii) lies in \( pR + \langle e_1, e_2, \ldots, e_{j-1} \rangle \), which is
a subgroup of \( G \) of order \( p^{n-w+j-1} \). So the number of choices for the
products (iii) is at most \( p \) to the power \( s \sum_{j=1}^u (n-w+j-1) \). Since
\( w = r+t+u \), the number of choices for the products (iii) is at most \( p \) to the power \( su(n-r-t) - \frac{1}{2}su^2 + O(n^2) \).

To summarise, we have shown that the number of isomorphism classes of
nilpotent commutative rings of order \( p^n \) is at most \( p^{\delta + O(n^2)} \), where \( \delta \) is the
maximum value of
\[
\alpha(r,s,t) + \frac{1}{2}r^2(n-r-t) + (st-\frac{1}{2}s^2)(n-r-t) + su(n-r-t) - \frac{1}{2}su^2
\]

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over all non-negative integers \(r, s, t, u\) such that \(r + t + u \leq n\), \(r \geq 1\) and \(s \leq \min(r, t + 1)\). Since \(\delta\) is an increasing function of \(u\), we may assume that \(r + t + u = n\) in our maximisation.

We bound \(\delta\) using standard techniques from calculus, just as in the proof of Theorem 2.3. Indeed, when \(r \leq (3/5)n\) we may use the upper bound (15) for \(\alpha(r, s, t)\) and show that \(\delta \leq (9/125)n^3 + O(n^{8/3})\). Since \(9/125 < 2/27\), this implies that \(\delta \leq (2/27)n^3 + O(n^{5/2})\) when \(r \leq (3/5)n\). When \(r \geq (3/5)n\) we may use the upper bound (16) for \(\alpha(r, s, t)\) to show that \(\delta \leq (2/27)n^3 + O(n^{5/2})\). So the theorem follows.

Using Theorems 3.5 and 4.4 and mimicking the proof of Theorem 3.6 yields the following general result.

**Theorem 4.5.** The number of isomorphism classes of commutative rings of cardinality \(p^n\) is \(p^n\delta\), where \(\delta = \frac{2}{27}n^3 + O(n^{5/2})\).

Theorems 3.7, 3.8 and 3.9 all have commutative analogues:

**Theorem 4.6.** Let \(g(n)\) be the number of isomorphism classes of commutative rings of order \(p^n\). Let \(g_s(n)\) be the number of isomorphism classes of commutative rings \(R\) such that \(|R| = p^n\) and \(R/J(R) \geq p^s\). There exists a positive real number \(\sigma\) such that, when we set \(s = \sigma\sqrt{n}\),

\[
\lim_{n \to \infty} \frac{g_s(n)}{g(n)} = 0.
\]

**Theorem 4.7.** The number of isomorphism classes of commutative non-nilpotent rings of cardinality \(p^n\) is \(p^{2/27}n^3 + O(n^{5/2})\).

**Theorem 4.8.** The number of isomorphism classes of commutative rings with identity that have cardinality \(p^n\) is \(p^{2/27}n^3 + O(n^{5/2})\).

We omit the proofs of these theorems, since they are straightforward modifications of the proofs in Section 3.

**Appendix: Some counterexamples**

In this appendix we provide counterexamples to Kruse and Price’s proposed reduction to \(\mathbb{F}_p\)-algebras [13, Theorem 3.1], which also apply to the argument of Poonen [16, Lemma 11.1]. After explaining some details of the method used, Examples 1 to 4 below illustrate problems with the proof. The most
serious problem is illustrated by Example 2 below: A process that is claimed to transform a ring $R$ of cardinality $p^n$ into an $n$-dimensional $\mathbb{F}_p$-algebra does not, in fact, always produce an associative object. Since this process produces objects that are not $\mathbb{F}_p$-algebras, we do not see how to use it to reduce the enumeration of rings to the enumeration of $\mathbb{F}_p$-algebras. So we are forced to produce a proof that avoids this reduction.

We also mention that there is a problem with Theorem 4.2 of [13], which bounds the number of $\mathbb{F}_p$-algebras with a fixed Jacobson radical: Example 5 shows that Theorem 4.2 is not true as stated. It is possible to prove a version of Theorem 4.2 with a slightly weaker bound, but we do not provide full details since our corrected proof does not require this result.

The reduction to $\mathbb{F}_p$-algebras

Theorem 3.1 of [13] states that if the number of pairwise non-isomorphic $p$-algebras ($\mathbb{F}_p$-algebras in our terminology) of dimension $n$ is $p^n$, then the number of pairwise non-isomorphic rings of order $p^n$ is less than $p^{\alpha+n^2+n}$. The proof begins as follows. Let $R$ be a ring of order $p^n$. Let $x_1, \ldots, x_m$ be a basis for the additive group of $R$, with $\text{char } x_i = p^{k_i}$ for $1 \leq i \leq m$, so that $\sum_{i=1}^m k_i = n$. Next, for $1 \leq i \leq m$ and $0 \leq j < k_i$, define $y_{ij} = p^j x_i$. (Here, as in [14] and [16], minor misprints in [13] have been corrected.) Rename the $y_{ij}$ in any convenient (predetermined) ordering as $z_1, \ldots, z_n$. Then there are integers $\phi_{ijk}$ for $1 \leq i, j, k \leq n$, such that $0 \leq \phi_{ijk} < p$ and

$$z_i z_j = \sum_{k=1}^n \phi_{ijk} z_k.$$ 

Moreover, the ring $R$ is determined up to isomorphism by the structure constants $\phi_{ijk}$.

Now define an $\mathbb{F}_p$-algebra $A$ with a basis $e_1, \ldots, e_n$ by setting

$$e_i e_j = \sum_{k=1}^n \phi_{ijk} e_k.$$ 

Kruse and Price assert that “By construction the multiplicative semigroups of $A$ and $R$ are isomorphic, so associativity of $A$ is equivalent to associativity of $R$”. This is not true, as the following examples demonstrate.
Example 1. Let $R = \mathbb{Z}/8\mathbb{Z}$, the ring of integers modulo 8, then the group of units of $R$ is the direct sum of two cyclic groups, each of order 2. On the other hand, it is easy to verify that $A \cong F_2[X]/(X^3)$, whose group of units is cyclic of order 4, generated by $(1 + X) + (X^3)$.

Example 2. The previous example showed that the multiplicative semi-groups of $R$ and $A$ need not be isomorphic. Here we construct an example in which $A$ is not even associative. Define a ring $R$ as follows. Let $p = 3$. Take the additive group of $R$ to be $C(3^2) \oplus C(3^2)$, the direct sum of two cyclic groups, each of order 9. Let $x_1, x_2$ be a basis for this additive group and define multiplication by the equations

\[
x_1^2 = 5x_2, \\
x_1x_2 = x_2x_1 = x_1 + x_2, \\
x_2^2 = 2x_1 + 3x_2,
\]

and their consequences under the distributive laws. To check that $R$ itself is associative, note that

\[
x_1^2x_2 = 5x_2 = x_1 + 6x_2 = x_1^2 + x_1x_2 = x_1(x_1x_2), \text{ and} \\
x_1x_2^2 = 2x_1^2 + 3x_1x_2 = 3x_1 + 4x_2 = x_1x_2 + x_2^2 = (x_1x_2)x_2.
\]

The associativity of other products of base elements follows from commutativity, for example $x_1(x_2x_1) = (x_2x_1)x_1 = (x_1x_2)x_1$.

Now take $z_1 = x_1, z_2 = x_2, z_3 = px_1 = 3x_1$ and $z_4 = px_2 = 3x_2$. Then

\[
z_1^2 = 5x_2 = 2x_2 + 3x_2 = 2z_2 + z_4, \\
z_1z_2 = z_2z_1 = x_1x_2 = x_1 + x_2 = z_1 + z_2, \\
z_2^2 = 2x_1 + 3x_2 = 2z_1 + z_4, \\
z_2z_4 = z_4z_2 = 3x_2^2 = 6x_1 = 2z_3,
\]

so exactly similar expressions for $e_1^2$ and so on hold in the algebra $A$. Hence

\[
e_1^2 e_2 = (2e_2 + e_4)e_2 = 2e_2^2 + e_4e_2 = 2(2e_1 + e_4) + 2e_3 = e_1 + 2e_3 + 2e_4.
\]

But

\[
e_1(e_1e_2) = e_1(e_1 + e_2) = e_1^2 + e_1e_2 = (2e_2 + e_4) + (e_1 + e_2) = e_1 + e_4,
\]

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so $A$ is not associative.

In the middle of their proof of Theorem 3.1 Kruse and Price write that “for any ring of order $p^n$ we can construct [an $\mathbb{F}_p$-algebra] of dimension $n$ and an associated basis for the algebra”. This might suggest that the isomorphism class of $A$ (whether associative or not) is independent of the basis $x_1, \ldots, x_m$ of the additive group of $R$. Indeed this interpretation becomes explicit in [16] Lemma 11.1 where it is stated that “the construction . . . defines a map from the set of isomorphism classes of rings of order $p^n$ to the set of pairs $(\overrightarrow{a}, A)$ where $\overrightarrow{a} = (a_1, \ldots, a_m)$ is a sequence of positive integers summing to $n$ and $A$ is a based rank-$n \mathbb{F}_p$-algebra.” Our next example shows that the algebra $A$ does indeed depend on the chosen basis $x_1, \ldots, x_m$ of $R$.

**Example 3.** The ring $R$ of Example 2 can be more simply presented if we observe that the element $7x_1 + x_2 = e$ (say) is a (necessarily unique) identity element, since $(7x_1 + x_2)x_1 = 7x_1^2 + x_1x_2 = 35x_2 + x_1 + x_2 = x_1$ and $(7x_1 + x_2)x_2 = 7x_1x_2 + x_2^2 = 7x_1 + 7x_2 + 2x_1 + 3x_2 = x_2$. Let $x = 4e + x_1$. It is easy to verify that $e, x$ is a basis for $R$ and that multiplication is given by $x^2 = 3e$. With this basis one can show that $A \cong \mathbb{F}_3[X]/(X^4)$. This is very different from the non-associative algebra constructed in our previous example.

**Example 4.** Both Kruse and Price (Theorem 3.1) and Poonen (Lemma 11.1) assert that the construction of $A$ from $R$ can be reversed and that “for any $p$-algebra of dimension $n$ with a given basis we can construct at most one ring of type $p(k_1, \ldots, k_m)$”. A counterexample to this claim is given by defining two rings $R_i$ ($i = 1, 2$) with additive group $C(3^2) \oplus C(3^2)$ and basis $e_i, x_i$. We take $e_i$ as the identity element of $R_i$. Multiplication in $R_1$ is given by $x_1^2 = 3e_1$. In $R_2$ it is given by $x_2^2 = 6e_2$. It is easy to check that $R_1$ contains no element $y$ such that $y^2 = 6e_1$. Hence $R_1$ and $R_2$ are not isomorphic.

**Enumerating $\mathbb{F}_p$-algebras with fixed Jacobson radical**

Theorem 4.2 of [13] states that if $S$ is a finite semisimple $q$-algebra and $N$ is a nilpotent $q$-algebra of dimension $n$, then the number of pairwise non-isomorphic $q$-algebras $A$ with radical $N$ and $A/N \cong S$ is less than $q^{2n^2}$. 

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Example 5. A counterexample to the statement of [13, Theorem 4.2] is given by taking $q = p = 2, S = \mathbb{F}_2, n = 1$ and $N$ to be the null algebra on a cyclic group of order 2. In this case $q^{2n^2} = 4$. Let $x$ be the unique generator of $N$ and let $A$ be any algebra with radical $N$ such that $A/N \cong \mathbb{F}_2$. We can lift the identity element of $\mathbb{F}_2$ to an idempotent $e$ in $A$. Then $e^2 = e, x^2 = 0$ and each of the four possibilities (i) $ex = xe = x$, (ii) $ex = xe = 0$, (iii) $ex = x, xe = 0$ and (iv) $ex = 0, xe = x$ gives a way of defining multiplication in $A$ and these lead to 4 non-isomorphic algebras $A$.

We note that the bound in the proof of [13, Theorem 4.2] misses a term corresponding to the number of choices for the dimensions of various irreducible submodules involved (the integers $n_i, k_i$ and $a_i$, in the notation of [13]). We believe this missing term is the root of the problem with this bound, but we remark that the number of choices for these integers is small.

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