Limits of functions and elliptic operators

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Abstract. We show that a subspace $S$ of the space of real analytical functions on a manifold that satisfies certain regularity properties is contained in the set of solutions of a linear elliptic differential equation. The regularity properties are that $S$ is closed in $L^2(M)$ and that if a sequence of functions $f_n$ in $S$ converges in $L^2(M)$, then so do the partial derivatives of the functions $f_n$.

Keywords. Elliptic regularity; real-analytic manifolds; hypoelliptic.

The limit $f$ of a sequence $f_n$ of complex analytical functions (under uniform convergence on compact sets) is complex analytical. Furthermore all partial derivatives of $f_n$ converge to the corresponding partial derivatives of $f$. This is in contrast to the case of real analytical functions. In fact, by the Weierstrass approximation theorem, every continuous real function on a compact domain is the uniform limit of real analytical functions on this domain.

The reason for the contrast between the complex and the real analytical cases is of course that complex analytical functions satisfy an elliptic differential equation, namely the Cauchy–Riemann equation (or alternatively because they satisfy the Laplace equation), while no such equation is satisfied in the analytical case.

Here we show that this phenomenon is universal, namely, whenever we have a class $S$ of (real analytical) functions on a closed manifold $M$ that have regularity properties similar to those of holomorphic functions, all functions in $f \in S$ satisfy an elliptic differential equation $Pf = 0$.

Our motivation is that in many geometric situations rigidity phenomena are associated with elliptic operators which are often hidden, i.e., not a priori related to the geometry. Two striking instances of this are the Seiberg–Witten equations for smooth four-dimensional manifolds and $J$-holomorphic curves in Symplectic topology. Hence it is of interest to show that there are situations where there must be elliptic operators, even though they are not a priori present.

First we recall the definition of the Sobolev spaces $W^{2,k}$ where $k \geq 0$ is an integer. We will not need the general case when $k \geq R$.

DEFINITION 0.1.

Let $k \geq 0$ be an integer. Suppose $f$ and $g$ are smooth, real valued functions on $\mathbb{R}^n$ with compact support, we define the Sobolev inner product $\langle f, g \rangle_{2,k}$ by

$$\langle f, g \rangle_{2,k} = \sum_{j=0}^{k} \sum_{I} \int_{\mathbb{R}^n} \partial^I f (x) \partial^I g (x) dx;$$

where $I$ is a multi-index and $\partial^I$ denotes the partial derivative with respect to $I$. 
DEFINITION 0.2.
Suppose \( M \) is a manifold, let \( \{ U_i \} \) be a locally finite cover of \( M \) by subsets homeomorphic to \( \mathbb{R}^n \) and let \( \{ \pi_i \} \) be a partition of unity subordinate to this cover with \( \text{supp}(\pi_i) \subseteq U_i \) compact. For smooth compactly supported functions \( f \) and \( g \) on \( M \), define the Sobolev inner product \( \langle f, g \rangle_{L^2_k} \) by
\[
\langle f, g \rangle_{L^2_k} = \sum_i \langle \pi_i f, \pi_i g \rangle_{L^2_k}.
\]
where \( \langle \pi_i f, \pi_i g \rangle_{L^2_k} \) denotes the Sobolev inner product on \( U_i = \mathbb{R}^n \).

The above definition depends on the choice of the cover \( U \), but different covers give equivalent inner products.

DEFINITION 0.3.
The Sobolev space \( W^{2k}(M) \) is the Hilbert space completion of the space \( C^\infty_c(M) \) of smooth functions on \( M \) with compact support with respect to the Sobolev inner product \( \langle \cdot, \cdot \rangle_{L^2_k} \).

When \( k = 0 \) we get the Hilbert space \( L^2(M) \) with its usual inner product. The definitions above coincide with the definitions using Fourier transforms.

We can now state our main result.

Theorem 0.4. Let \( S \) be a subspace of real analytical functions on a compact real analytical manifold \( M \) that is closed under the \( L^2 \)-norm on \( M \). Assume further that if \( f_n \in S \) is a sequence of functions such that \( f_n \to f \) in \( L^2(M) \), then \( f_n \to f \) in all Sobolev spaces \( W^{2k}(M) \), \( k \in \mathbb{N} \). Then there is an analytical elliptic differential operator \( P \) on \( M \) such that \( \forall f \in S ; P f = 0 \).

Remark 0.5. The analogous result for sections of a bundle on \( M \) holds and can be proved in exactly the same way.

A differential operator \( P \) that satisfies elliptic regularity on every open set \( U \) (i.e., if \( u \) is a distribution on \( U \) with \( Pu = f \), \( f \) smooth, then \( u \) is smooth) is called hypoelliptic.

Such operators have been characterised among operators with constant coefficients by Hörmander [2]. What we consider here is a different situation where our class of functions may not be given by a differential equation. What we can conclude is also weaker – we only know that \( S \) is contained in the set of solutions to an elliptic differential equation.

We now outline the proof. By using the hypothesis, we show that on the space \( S \), the \( L^2 \)-norm is equivalent to the \( W^{22} \)-norm. From this we deduce that the space \( S \) is finite dimensional. Next, for each \( x \in M \), the partial derivatives at \( x \) give linear functionals on \( S \). By using the finite-dimensionality of \( S \), we show that at \( x \) we can find an elliptic differential equation satisfied by \( S \). The same method yields elliptic differential equations on certain semi-analytical sub-varieties. Finally, we use the local Noetherian property of real analytic varieties to deduce that we can globally construct an elliptic differential operator \( P \) with \( Pf = 0 \) \( \forall f \in S \).

Only the final step in the above outline uses analyticity. We shall show, however, that the hypothesis of analyticity is essential for our result.

1. Finite dimensionality of \( S \)
In this section we show that \( S \) is finite dimensional. First we make an elementary observation about subspaces of Hilbert spaces.
Let $H_1$ and $H_2$ be Hilbert spaces with norms $k_k_1$ and $k_k_2$ respectively. Assume that as sets $H_2 - H_1$ with $k_k_1$, $8x \leq H_2$. The following result will be applied to the case when $H_1 = L^2(M)$ and $H_2 = W^{2,2}(M)$.

**Proposition 1.1.**

Let $S$ be a subspace of $H_2$ that is closed in $H_1$ and $H_2$ so that the subspace topologies induced by $H_1$ and $H_2$ coincide. Then there is a constant $C > 0$ such that $k_k_2 Ck_k_1; 8x < S$.

**Proof.** By hypothesis, the identity map from $S$ with its topology as a subspace of $H_1$ to $S$ with its topology as a subspace of $H_2$ is continuous. Hence it must be bounded from which the conclusion follows.

Note that by the hypothesis in our main theorem, the above result applies to $S$ with $H_1 = L^2(M)$ and $H_2 = W^{2,2}(M)$. We next show that a space $S$ satisfying the hypothesis of the main theorem is finite dimensional.

**Lemma 1.2.** Let $M$ be a closed manifold and let $S$ be a subspace of $W^{2,2}(M) - L^2(M)$ such that there exists $C > 0$ such that for $f \in S$, $k_k_2 Ck_k_2 < k_k_2$. Then $S$ is finite dimensional.

**Proof.** Suppose $S$ is infinite dimensional, then let $\{x_n\}$ denote an $L^2$-orthonormal sequence in $S$. By hypothesis, for all $j \geq 2, x_n \in W^{2,2}$ and $k_k_2 Ck_k_2 < C$. By the Rellich lemma it follows that the sequence $x_n$ has a convergent sequence in $L^2$. But, as the vectors $x_n$ are $L^2$-orthonormal, this is impossible. Thus, $S$ must be finite dimensional.

**2. Pointwise differential equations**

We can now construct elliptic differential equations satisfied by the functions in $S$ at a single point $x \in M$. Choose a system of local coordinates. Observe that partial derivatives at $x$ give linear functionals on $S$, i.e., elements of the dual $S^0$ of $S$. These generate a subspace $V_x$ of $S^0$. As $S^0$ is finite dimensional, $V_x$ is generated by finitely many partial derivatives, and hence those of order at most $k$ for some $k$. We denote these differential operators by $P_1, \ldots, P_m$.

Now let $E$ be an elliptic operator of order greater than $k$, for instance a power of the Laplacian. Then $f \in E f(x)$ is an element of $S^0$, hence is spanned by $P_i$. Thus, at each $f \in S$ satisfies a relation $\langle E, \Sigma P_i f(x) \rangle = 0$. As this has the same leading term as $E$, this is an elliptic differential equation.

Note that the relations $P_1, \ldots, P_m$ are independent as elements of $S^0$ on an open set (as independence is an open condition). Let $r(x) = \dim V_x$. Let $f_1, \ldots, f_N$ be a basis for $S$. In the special case where $r(x)$ is a constant function (for instance $r(x) = \dim (S)$, the maximum possible value, at all points), we shall see that we have a global elliptic operator even in the absence of analyticity.

**Proposition 2.1.**

Suppose $r(x) = m$ is a constant. Then there is an elliptic differential operator $E$ such that $E f = 0$, for all $f \in S$. 

Proof. We first show that there is a uniform degree \( k \) so that the operators of degree at most \( k \) span \( V \) for all \( x \in M \). For \( x \in M \), let \( P_1, \ldots, P_m \) be operators independent at \( x \) and let \( U_j \) be the set where these operators are independent. This is an open set as linear independence is an open condition (for instance, by considering determinants). By hypothesis, \( M \) is the union of the sets \( U_j \). By compactness we can find finitely many such sets \( U_j \) whose union is \( M \). Let \( k \) be the maximal degree of the differential operators associated to these sets.

Now, let \( E \) be an elliptic operator of order greater than \( k \). For each \( U_j \), we have differential operators \( P_1, \ldots, P_m \) which are independent at each \( x \in U_j \) and hence span \( V \). Hence we have a relation \( Ef(x) = \sum c_i(x)P_i f(x) \) for \( f \in S \).

We next show that each \( c_i(x) \) is smooth as a function of \( x \in U_j \). Let \( x_0 \in U_j \) be an arbitrary point. We shall show that \( c_i(x) \) is smooth at \( x_0 \).

As the operators \( P_1, \ldots, P_m \), are independent at \( x_0 \) as functionals on \( S \), there exist \( i \in \{1, \ldots, m \} \) such that \( P_i g_i(x_0) = \delta_{ij} \). It follows that for \( x \in U_j \), \( P_i g_i(x) = \delta_{ij} + a_{ij}(x) \) with \( a_{ij}(x) \) smooth functions and \( a_{ij}(0) = 0 \).

Let \( A(x) \) denote the matrix with entries \( a_{ij}(x) \) and let \( V(x) \) (respectively \( C(x) \)) denote the (column) vector with entries \( E g_i(x) \) (respectively \( c_i(x) \)). Note that \( V(x) \) is smooth as a function of \( x \). As \( E g_i(x) = \sum_{j=1}^m P_j g_i(x) c_j(x), \) i.e., \( V(x) = (\mathcal{V} + A(x)) C(x) \), we have \( C(x) = (\mathcal{V} + A(x))^{-1} V(x) \).

Now, \( A(x_0) = 0 \) and it is well-known that \( M \uparrow (\mathcal{V} + M)^{-1} \) is smooth as a function of \( M \) at \( M = 0 \) (by using the implicit function theorem or Taylor expansions). Hence \( C(x) \) is smooth at \( x_0 \), i.e., each \( c_i \) is smooth at \( x_0 \), as required. Let \( E_0^0 = E - \sum c_i P_i \). This is an elliptic operator annihilating \( S \).

To construct an elliptic operator globally, we take a partition of unity \( \pi \) subordinate to the cover \( \bigcup_{i=1}^s U_i \) and let \( E_0 = \sum_i \pi_i E_i \). Then each \( f \in S \) is in the kernel of \( E_0 \) and \( E_0 \) is elliptic as, by construction, the leading term of \( E_0 \) is the same as that of \( E \).

Without assuming analyticity, however, our main result fails in general. To see this, we let \( M = S^1 = \mathbb{R}/\mathbb{Z} \), and construct a function \( f \) on \( S^1 \) so that, for \( n > 1 \), \( f^{(n)}(l=n) \neq 0 \) but \( f^{(k)}(l=n) = 0 \) for \( k < n \). Let \( S \) be the (one-dimensional) span of \( f \).

An elliptic differential operator \( E \) on a one-dimensional manifold is a differential operator \( P(D) \) whose leading coefficient is non-zero at all points. The function \( f \) does not satisfy \( Ef = 0 \) for any such operator as if \( d \) is the order of \( E \), by construction \( Ef(l=d) \neq 0 \).

3. Globalisation in the analytical case

In the analytical case, we shall construct sets similar to \( U_j \) above. These are now open in the real analytic topology, i.e., one whose sub-basis is generated by sets of the form \( f(x) \neq 0 \) where \( f \) is an analytical function.

We shall need two basic facts regarding the real analytical topology (see, for instance [2]). Firstly, any closed set is defined by a single equation, as given a closed set \( F = Z(g_1, \ldots, g_p) = \{ x \in M, g_i(x) = 0 \} \) for \( g_i \) then, we have \( F = Z(g_1^2 + \cdots + g_p^2) \). Secondly, as the ring of power series is Noetherian, the real analytical topology is locally Noetherian. As \( M \) is compact, the real analytical topology on \( M \) is Noetherian.

As \( M \) is analytical, by a theorem of Morrey [3] and Grauert [11] there is an analytical Riemannian metric on \( M \). Hence the Laplacian (with respect to an analytical metric) is an analytical elliptic operator on \( M \) and so are its powers. It follows that there are analytical elliptic differential operators on \( M \) of arbitrarily high orders.
In the rest of this section, we make the convention that all differential operators we consider are analytical ones globally defined on $M$. In particular we shall use the notation of the previous sections, but with $V_t$ now being the subspace of $S^0$ generated by global analytical operators acting on $S$ at $x$ and $r (x)$ its dimension.

We shall inductively construct sequences of sets $F_i$ and $V_i$, with $F_i$ a decreasing sequence of closed sets and $V_i$ open, and finitely many elliptic differential operators that span $V_x$ for $x \in F_i \setminus V_i$.

Let $F_1 = M$ and note that this is a closed subset of $M$. On $F_1 = M$, let $m_1 = r (x)$ be the maximum value of $r (x)$ (which is attained as $r (x) \in \mathbb{Z}$, $0 < r (x) \dim S$) for all $x \in M$ and let $P_1^1, \ldots, P_1^m, m = m_1$, be (analytic) differential operators with $f \in \mathcal{P}_1 f (x)$ independent in $S^0$. Then the set $V_1$ where the $P_1^j$'s are independent is an open set in the analytical topology. It follows that for all $x \in V_1$, the functionals $P_1^j (x)$ span $V_x$. Let $q_1$ be the maximum order of differential operators $P_1^j$.

Next, let $F_2$ be the complement of $V_1$. Let $r (x) = m_2 = m_1$, $z \leq F_2$, be the maximum value of $r (x)$ on $F_2$ and let $P_2^1, \ldots, P_1^m, m = m_2$, be different operators with $f \in \mathcal{P}_2 f (x)$ independent functionals in $S^0$. Let $V_2$ be the open set (in the real analytical topology) where $P_2^j$'s are independent as functionals in $S^0$. Then these span $V_x$ for all $x \in F_2 \setminus V_2$. Let $q_2$ be the maximum order of these differential operators.

Inductively, given $F_k$ and $V_k$, we define $F_{k+1} = F_k \cap V_k$ and let $m_{k+1}$ be the maximum rank of $V_x$ on $F_{k+1}$. As above, we construct differential operators $P_{k+1}^j$ and let $V_{k+1}$ be the set on which these are independent. These span $V_x$ for all $x \in F_{k+1} \setminus V_{k+1}$. Let $q_k$ be the maximal order of the differential operators constructed above.

Now, by the local Noetherian property, the above process must stabilise. It follows that for some $n$, $F_n = V_n$. Let $q$ be the maximum of the numbers $q_j$, $1 \leq j \leq n$ and note that on each set $F_i \setminus V_i$, we have differential operators of degree at most $q$ that span at each point the subspace $V_x$. Let $g_n$ be analytical functions such that $F_n = Z (g_n)$.

Let $E$ be an analytic elliptic operator of order greater than $q$. We construct inductively analytic elliptic operators $E_n, \ldots, E_1$ with $E_1$ being the required operator. First, note that on $F_n \setminus V_n$, we can find an operator $G_n$, with analytical coefficients, of order at most $q$ so that $E f (x) = G_n f (x)$ for all $x \in F_n \setminus V_n, f \in S$. Let $E_n = E - G_n$.

Next, observe that on $F_n \setminus V_n$, the function $g_n$ does not vanish, and hence the operator $E_n = g_n$ is well-defined. Hence there is an operator $G_n$, with analytical coefficients, of order at most $k$ so that $(E_n = g_n) f (x) = G_n f (x)$ for all $x \in F_n \setminus V_n, f \in S$. Let $E_n = E_n - g_n G_n$. This annihilates $S$ on $F_n \setminus V_n$ by construction and also on $F_n$ as it coincides with $E_n$ on $F_n$. Thus $E_n$ annihilates $S$ on $F_n$.

Similarly, given an elliptic operator $E_k$ that annihilates $S$ on $F_k$, we can construct an elliptic operator $E_k$ that annihilates $S$ on $F_k$. Proceeding inductively, we obtain an operator that annihilates $S$ on $F_1 = M$.

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References

[1] Grauert H, On Levi’s problem and the imbedding of real-analytic manifolds, *Ann. Math.* 68(2) (1958) 460–472
[2] Hörmander L, The analysis of linear partial differential operators, II, Differential operators with constant coefficients, Grundlehren der Mathematischen Wissenschaften (Berlin: Springer-Verlag) (1983) vol. 257

[3] Morrey C B Jr, The analytic embedding of abstract real-analytic manifolds, Ann. Math. 68(2) (1958) 159–201

[4] Narasimhan R, Introduction to the theory of analytic spaces, Lect. Notes Math. (Berlin-New York: Springer-Verlag) (1966) vol. 25