Threshold analyses and Lorentz violation

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In the context of threshold investigations of Lorentz violation, we discuss the fundamental principle of coordinate invariance, the role of an effective dynamical framework, and the conditions of positivity and causality. Our analysis excludes a variety of previously considered Lorentz-breaking parameters and opens an avenue for viable dispersion-relation investigations of Lorentz violation.

I. INTRODUCTION

For several decades, a sizeable amount of theoretical work has been directed towards a unified quantum description of all fundamental interactions including gravity. Comparable experimental efforts have been inhibited, partly by the expected minuscule size of Planck-scale effects, and partly by the absence of a realistic underlying framework. One approach to overcome this phenomenological obstacle is to identify exact symmetries in present-day physics that may be violated in underlying theories and are amenable to high-precision tests.

Lorentz and CPT invariance satisfy these criteria. They are cornerstones of our present understanding of nature at the fundamental level, and certain Lorentz and CPT tests belong to the most precise null experiments available. Moreover, a mechanism that could cause spontaneous Lorentz and CPT breaking in the framework of string field theory was discovered more than a decade ago [1]. Subsequent investigations have considered the possibility of Lorentz violation in other contexts, such as spacetime foam [2,3], nontrivial spacetime topology [4], loop quantum gravity [5], realistic noncommutative field theories [6], and spacetime-varying couplings [7].

Low-energy effects of Lorentz breaking are described by the Standard-Model Extension (SME) [8–12]. This dynamical framework is constructed to contain all Lorentz- and CPT-violating lagrangian terms consistent with coordinate invariance, a fundamental requirement to be discussed below. The SME has provided the basis for numerous experimental Lorentz- and CPT-violation searches involving hadrons [13–15], protons and neutrons [16–18], electrons [18–22], photons [23–27], muons [28], and neutrinos [8,29,30].

Within the SME, it is straightforward to confirm that spacetime-symmetry breakdown generally modifies one-particle dispersion relations [23,8,9,11]. This feature of Lorentz violation permits the prediction of possible experimental signatures based purely on kinematical arguments. For example, primary ultra-high-energy cosmic rays (UHECR) at energies eight orders of magnitude below the Planck mass have been observed. At these scales, Lorentz-breaking effects might be amplified relative to the ones in low-energy experiments leading to potentially observable threshold modifications for particle reactions. This idea has been adopted in a number of recent investigations [2,29,31–42]. In many of these studies, however, the dispersion relations are constructed phenomenologically without reference to an effective dynamical framework and other physical principles.

The goal of the present work is to investigate how some of the arbitrariness in the construction of Lorentz-violating dispersion relations can be removed. Our investigation is based on the principle of coordinate invariance and on the condition of compatibility with an effective dynamical framework. It is argued that these two features are fundamental enough for being physical requirements, while maintaining relative independence of the details of the Planck-scale theory. Moreover, the implementation of general dynamical properties significantly increases the scope of threshold investigations. We also discuss positivity and causality, properties which further add to the viability of kinematical analyses. Throughout we assume energy-momentum conservation.

In Sec. 2, we comment on the requirement of coordinate independence and its consequences for dispersion relations. Section 3 discusses dispersion relations from the viewpoint of compatibility with the SME. In Sec. 4, issues regarding positivity and causality are addressed. Further results and a discussion of sample dispersion relations often considered in the literature can be found in Sec. 5. A brief summary is contained in Sec. 6.

II. COORDINATE INDEPENDENCE

Coordinate independence is essential in physics, and its role in the context of Lorentz violation is well established [8,9]. However, many threshold analyses lack coordinate invariance, and occasionally Lorentz breaking is identified with the loss of coordinate independence. It is therefore appropriate to review some arguments behind this fundamental principle before discussing its consequences for dispersion relations.

A certain labeling scheme for events in space and time corresponds to a choice of coordinate system. Such a labeling of events is a pure product of human thought and thus arbitrary to a large extent. Despite being one of the most common and important tools in physics, coordinate systems fail to possess physical reality in the sense that the physics must remain independent of the choice of coordinates. This principle, also called observer invariance,
is one of the most fundamental in science. Mathematically, observer invariance can be implemented by choosing a spacetime manifold as the arena for physical events and certain tensors or spinors for the representation of physical quantities.

We also mention that coordinate invariance is much more general than Lorentz symmetry. For example, nonrelativistic classical mechanics and Newton’s law of gravitation fail to be Lorentz invariant but can be formulated in the coordinate-free language of three-vectors. Only when the spacetime manifold is taken to be lorentzian, the Lorentz transformations acquire a significant role: They implement changes between local Minkowski frames.

Lorentz violation is associated with nontrivial vacua described at low energies by nondynamical tensorial backgrounds. A tensor background can lead, for instance, to direction-dependent particle propagation, a situation comparable to the one inside certain crystals. One then says that particle Lorentz invariance is broken [8]. However, observer Lorentz invariance remains fully intact, so that locally one can still work with the metric \( g^{\mu\nu} = \text{diag}(1, -1, -1, -1) \), particle four-momenta still transform in the usual way under coordinate changes, and the conventional tensors and spinors still represent physical quantities.

The important difference between observer and particle Lorentz symmetry can be illustrated in the conventional context of a classical point particle of mass \( m \) and charge \( q \) in an external electromagnetic field \( F^{\mu\nu} \) obeying

\[
m \frac{dv^\nu}{d\tau} = q F^{\mu\nu} v_\nu.
\]

Here, \( v^\mu \) is the four-velocity of the particle and \( \tau \) is its proper time. In general, invariance under rotations of the particle’s trajectory (and thus particle Lorentz symmetry) is broken by the external \( F^{\mu\nu} \) resulting in the nonconservation of the charge’s angular momentum, for instance. However, Eq. (1) is a tensor equation valid in all coordinate systems maintaining observer Lorentz symmetry. We remark that in the above example, the background \( F^{\mu\nu} \) is a local electromagnetic field generated by other four-currents that can in principle be controlled. On the other hand, a background in the Lorentz-violating context is a global property of the effective vacuum outside of experimental control.

**Observer-invariant dispersion-relation ansatz.**

In the published literature, Lorentz-violating dispersion relations are usually taken to be of the form [43]

\[
\lambda_0^2 - \vec{\lambda}^2 = m^2 + \delta f(\lambda_0, \vec{\lambda}),
\]

where \( m \) is the usual mass parameter and \( \lambda^\mu = (\lambda_0, \vec{\lambda}) \) is the plane-wave four-vector (before the reinterpretation of negative energies, see Eq. (10)). The function \( \delta f(\lambda_0, \vec{\lambda}) \) controls the extent of the Lorentz-breaking. We proceed under the assumption that the dynamics of the free particle is described by a linear partial differential equation with constant coefficients. In the absence of nonlocalities, we obtain the polynomial ansatz

\[
\delta f(\lambda_0, \vec{\lambda}) = \sum_{n \geq 1} T^{\alpha\beta\ldots}_{(n)} \lambda_\alpha \lambda_\beta \ldots \cdot
\]

Here, \( T^{\alpha\beta\ldots}_{(n)} \) denotes a constant tensor of rank \( n \) parametrizing the violation of particle Lorentz symmetry. All the tensor indices \( \alpha, \beta, \ldots \) are understood to be distinct but each one is contracted with a momentum factor, so that all terms in the sum are observer Lorentz invariant. Note that the \( T^{(n)} \) can be taken as totally symmetric. The above arguments are summarized in Result (i): Lorentz-violating dispersion relations must satisfy the the requirement of coordinate independence, and therefore they hold in any frame and contain only nonnegative integer powers of \( \lambda_0 \) and \( |\vec{\lambda}| \) [44].

**Energy degeneracy.**

For a given \( \vec{\lambda} \), Eq. (2) is a polynomial in \( \lambda_0 \), which has generally multiple roots lifting the conventional energy degeneracy between particle, antiparticle, and possible spin-type states. This is intuitively reasonable because degeneracies usually arise through symmetries, and in the present context the number of symmetries is normally reduced. This raises the question what degree of physical Lorentz violation is described by dispersion-relation modifications preserving the usual degeneracy. This issue assumes particular importance in light of the fact that the majority of threshold analyses employ dispersion relations maintaining the conventional degeneracy.

For example, it is known that “doubly special relativities” [45,46] maintain the conventional number of spacetime symmetries and exhibit the ordinary equality of all particle and antiparticle energies for a given three-momentum. However, such approaches to the loss of Lorentz symmetry appear to be physically indistinguishable from the conventional case [47]. In the present context, degeneracy is maintained, e.g., when \( \delta f(\lambda_0, \vec{\lambda}) \) contains only \( \lambda_0 \)-independent and \( \lambda_0^2 \) terms. In certain models, one can then find transformations removing the Lorentz breaking from the particle species in question at the cost of introducing the violations in a different sector of the theory [25]. Then, Lorentz-violating and Lorentz-symmetric sectors can have the same number of spacetime symmetries and associated conserved currents but the symmetry generators and the conserved quantities differ in the two sectors. This could potentially lead to additional conservation constraints on particle reactions.

The above arguments support the conjecture that Lorentz violation can be completely removed from a theory with identical, degeneracy-preserving dispersion-relation corrections in all sectors. In summary, we are lead to Result (ii): For generality and to avoid possible triviality, degeneracy-lifting dispersion relations must be included in kinematical investigations. We remark that
terms with more than four powers of $\lambda_0$ generally result in more than four particle states, which could lead to interpretational difficulty.

**Rotational symmetry.** Most threshold analyses make the simplifying assumption of rotation invariance in certain frames, and then $\delta f(\lambda_0, \bar{\lambda})$ is expanded in powers of $|\bar{\lambda}|$ with $\lambda_0$ contributions absent. We remark in passing that this procedure is associated with potential triviality problems because the particle energies remain degenerate. But there is also another unsatisfactory aspect of this approach. Many authors include odd powers of $|\bar{\lambda}|$ into the expansion of $\delta f(\bar{\lambda})$ violating coordinate independence: The implementation of rotation symmetry requires that the $T_{(n)}$ are constructed from $\eta^{\mu\nu}$ and constant tensors with vanishing spacelike and mixed components. Then, odd powers of $|\bar{\lambda}|$ are absent because the spacelike and mixed contributions to $T_{(n)}$ are of $\delta^{jk}$-type, so that $\bar{\lambda}$ always appears contracted with itself. For example, the respective dispersion-relation corrections for $n = 1, \ldots, 4$ (up to constants) are:

$$
\begin{align*}
\lambda_0, \\
\lambda_0^2, \quad \bar{\lambda}^2, \\
\lambda_0^3, \quad \lambda_0 \bar{\lambda}^2, \\
\lambda_0^4, \quad \lambda_0^2 \bar{\lambda}^2, \quad \bar{\lambda}^4. 
\end{align*}
$$

(4)

For completeness, we mention that certain special choices of the tensors $T_{(n)}$ could generate effective corrections with odd powers of $|\bar{\lambda}|$. For instance, consider a case with an *unsuppressed* correction $\delta f(\lambda_0, \bar{\lambda})$ leading to the observer-invariant dispersion relation $\lambda^2 - 2m^2\lambda^2 + m^4 = N^2|\bar{\lambda}|^2$, where $\lambda^2 = \lambda^\nu\nu$, $N$ is a parameter for Lorentz breaking, and $j$ is an integer. Taking the square root yields $\lambda_0^2 - \bar{\lambda}^2 = m^2 \pm N|\bar{\lambda}|^j$. For appropriate values of $j$, odd powers of $|\bar{\lambda}|$ are generated. Note, however, the partial nondegeneracy. This yields Result (iii): *Odd powers of $|\bar{\lambda}|$ are excluded in dispersion-relations modifications $\delta f(\lambda_0, \bar{\lambda})$ when rotational invariance is assumed* [48]. An explicit example, which also demonstrates that this result cannot be bypassed by the common substitution $\lambda_0 \to |\bar{\lambda}|$, is discussed in Sec. 5.

**III. THE STANDARD-MODEL EXTENSION**

Implementing general dynamical features significantly increases the scope of particle-reaction analyses. However, the use of dynamics in threshold investigations has recently been questioned on the grounds of framework dependence [49]. We disagree with this claim and begin with a few remarks about purely kinematical analyses.

Although kinematics imposes powerful constraints on particle reactions, it provides only an incomplete description of reaction processes: An expected high-energy reaction can be suppressed by modified dispersion relations but also by novel symmetries, for example. Similarly, the presence at high energies of a reaction kinematically forbidden at low energies could perhaps be explained by additional channels due to the loss of low-energy symmetries or new undetected particles. Moreover, models of both acceleration mechanisms for UHECRs and shower development in the atmosphere involve conventional dynamics. Thus, in studying threshold bounds on Lorentz violation, assumptions outside kinematics such as dynamical quantum-field aspects cannot be eliminated completely.

Possible dynamical features are constrained by the requirement that known physics must be recovered in certain limits, despite some freedom in introducing dynamics compatible with a given set of kinematical rules. Moreover, it appears difficult and may even be impossible to find an effective theory containing the Standard Model with dynamics significantly different from the SME. We also mention that kinematics investigations are limited to only a few potential Lorentz-violating signatures from fundamental physics. From this viewpoint, it is desirable to explicitly implement dynamics of sufficient generality into the search for Lorentz breaking.

**The generality of the SME.** To appreciate the generality of the SME, we review the philosophy behind its construction [8,9]. One adds to the usual Standard-Model lagrangian $L_{SM}$ Lorentz-violating terms $\delta L$:

$$
L_{SME} = L_{SM} + \delta L.
$$

(5)

Here, $L_{SME}$ denotes the lagrangian of the SME. The correction $\delta L$ is formed by contracting standard-model field operators of unrestricted dimensionality with Lorentz-violating tensorial parameters yielding observer Lorentz scalars. It is thus apparent that the complete set of possible contributions to $\delta L$ yields the most general effective dynamical description of Lorentz breaking at the level of observer Lorentz-invariant quantum field theory.

Note that instead of constructing corrections to the energy-momentum relation, one proceeds at the level of the lagrangian. This superior approach yields a much more powerful and dynamical framework, despite employing a philosophy that is at its base *de facto* identical to the dispersion-relation case considered in the previous section.

Potential Planck-scale features, such as a certain discreteness of spacetime or a possible non-pointlike nature of elementary particles, are unlikely to invalidate the above effective-field-theory approach at present energies. On the contrary, the extremely successful Standard Model is normally viewed as an effective field theory approximating more fundamental physics. If the underlying theory indeed incorporates minute Lorentz-violating effects, it would appear contrived to consider low-energy effective models outside the framework of quantum fields. We mention in passing that Lorentz-symmetric aspects of candidate fundamental theories, such as new symmetries, novel particles, or large extra dimensions, are also
unlikely to require a low-energy description beyond effective field theory and can therefore be implemented into the SME, if necessary [57]. The above discussion indicates that dispersion-relation phenomenology offers at best narrow advantages in generality when compared to the SME.

Advantages of the SME. The SME allows the identification and direct comparison of virtually all presently feasible Lorentz and CPT tests. Moreover, the SME contains classical kinematics test models of relativity (such as Robertson’s framework, its Mansouri-Sexl extension, or the $c^2$ model) as limiting cases [25]. Another advantage of the SME concerns the implementation of additional desirable conditions besides observer Lorentz invariance. For instance, one can choose to impose translation invariance, power-counting renormalizability, $SU(3)\times SU(2)\times U(1)$ gauge symmetry, hermiticity, and point-like interactions, which further restricts the parameter space for Lorentz breaking. As in the dispersion-relation case, one can adopt simplifying choices, such as rotational invariance in certain frames. This latter simplification of the SME has been assumed in Ref. [29].

Concerning threshold investigations, the quadratic, translationally invariant sector of the SME determines possible one-particle dispersion relations, so that the correction (3) is constrained. Further restrictions are obtained by implementing the general conditions discussed in the previous paragraph.

For example, the free renormalizable gauge-invariant contribution to the photon sector of $\delta\mathcal{L}$ is given by [9]

$$\delta\mathcal{L} \supset (k_{AF})_\mu A_\lambda F^{\mu\nu} - \frac{1}{4} (k_F)_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} ,$$

where $A_\mu$ is the usual four-potential, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ denote the field-strength tensor and its dual, respectively, and $(k_{AF})_\mu$ and $(k_F)_{\mu\nu\rho\sigma}$ are constant parameters controlling the Lorentz violation. If in addition rotation invariance is assumed for the photon, suitable transformations move the Lorentz violation due to the $(k_F)$ term into other sectors of the theory and map $(k_{AF})^0 \rightarrow (k_{AF})^0 [25]$, so that only one Lorentz-violating parameter $\xi = (k_{AF})^0$ remains. Then, the usual methods yield

$$\lambda^\mu \lambda^\mu = 4 \xi^2 \lambda^\mu \lambda^\mu - 4 (\xi \lambda_0)^2 = 0 .$$

Equation (7) is only one example of many other natural dispersion relations never considered in UHECR analyses.

As an additional advantage, the SME provides the basis for the calculation of reaction rates, a determining factor for observational relevance. An example is given by vacuum Čerenkov radiation [29,50]. The above discussion supports Result (iv): *Threshold analyses within the SME ensure compatibility with dynamics and permit a much broader scope, while maintaining greatest possible generality* [51]. The above result strongly suggests that particle-reaction investigations are best performed in the context of the SME.

IV. POSITIVITY AND CAUSALITY

Positivity and causality are fundamental principles in conventional physics and have been investigated in the Lorentz-violating context [11]. However, we are not aware of any threshold analysis discussing these principles carefully. Therefore, it seems appropriate to review some important aspects of positivity and causality before investigating the implementation of these properties into particle-reaction studies.

A postulate of Special Relativity states that velocities of material bodies and radiation are limited by the vacuum light speed. This postulate both contains a notion of causality and leads to the Lorentz transformations. This fact results in the common misconception that Lorentz breaking implies superluminal propagation, and hence causality violations. However, in many conventional situations involving a background, Lorentz symmetry is broken but causality is maintained. For instance, the anisotropic propagation of electromagnetic waves inside certain crystals is normally causal [52,9]. Moreover, in such a situation the total conserved energy is clearly positive definite for all observers. Thus, the principles of positivity and causality are *a priori* independent and distinct from the principle of Lorentz symmetry. On the one hand, positivity and causality lead, e.g., to the spin-statistics theorem [53,54], which is a cornerstone of quantum field theory. On the other hand, polynomial Lorentz-violating dispersion relations fail to satisfy these requirements above scales associated with the underlying theory [11]. It is therefore natural to ask whether positivity or causality violations become acceptable in the presence of Lorentz breaking.

Positivity. Lorentz violation introduces positivity problems through spacelike four-momenta for particles [11], which can then have negative energies in an arbitrary frame. In general, these negative energies can neither be eliminated by a shift of the energy zero nor by a coordinate-independent reinterpretation as corresponding to antiparticles. This loss of positivity can lead to unconventional instabilities for one-particle states [11], but it may leave unaffected the stability of the vacuum. Note also the existence of a mechanism that avoids positivity problems despite the presence of spacelike momenta [7], so that such particles could potentially become admissible from a purely phenomenological viewpoint.

In light of the previous section, it is also necessary to investigate whether spacelike states can be incorporated into quantum field theory. Although it may be feasible to construct a Fock space of spacelike particles [55], we are unaware of any internally consistent interacting quantum field theories involving such particles as asymptotic states. The usual assumptions in perturbation theory, for example, seem to exclude negative energies and spacelike on-shell momenta.
Causality. Causality breakdown in the present context occurs for particles with superluminal group velocities [11]. For such particles, the chronology of events, and in particular cause and effect, become observer-dependent concepts. Note also that in most frames, particle propagation forward and backward in time can become possible. It then appears difficult to make meaningful physical predictions. We mention, however, that the causality breakdown due to small Lorentz-violating effects appears mild enough to permit the existence of concordant frames, in which particle propagation is always forward in time, so that closed causal loops and therefore many causal paradoxes could be avoided.

Although superluminal particles may potentially be acceptable for concordant observers, it is unlikely that this type of causality breakdown can be accommodated within the framework of relativistic quantum field theory. Generally, a hermitian Hamiltonian for massive fermions fails to exist in the majority of frames [11]. In addition, the usual covariant perturbative expansion relies on time ordering, an operation no longer coordinate invariant, when microcausality is violated [56]. Again, it appears that privileged observers would have to be introduced to apply the conventional methods.

Implications for threshold analyses. From a conservative viewpoint, it is natural to ask whether reaction-to occur \[\cdot\] to be conventional requires spacelike pions for the decay spacelike four-momenta or superluminal group velocities, scales of the fundamental theory and present-day low-energy physics, respectively. Then the point \((\vec{p},E)\) is the case for dimension four operators can be as low as \[\times \times 10^{20}\text{ eV. HICR}\]s with a spectrum extending beyond \(10^{20}\text{ eV are often employed to bound Lorentz violation or to suggest evidence for Lorentz breaking. Thus, imposing positivity and causality should be maintained in threshold investigations.}

The above arguments lead to Result (v): Positivity and causality should be maintained in threshold investigations. To enforce positivity and causality one could perform all reaction analyses well below the breakdown scale for these properties. However, in many practical applications this approach may be unsatisfactory because of the limited accessible momentum range. It thus seems desirable to employ dispersion relations for Lorentz violation that preserve positivity and causality at all scales. Such dispersion relations can be motivated in string field theory and have been investigated in the context of the SME [11].

V. FURTHER RESULTS

In this section, we consider additional, important kinematics issues that are best discussed by example. We focus on the rotationally invariant \(n = 3\) case, which has received a lot of attention in the literature [34–37,39,42]. We also take this opportunity to illustrate some of our results from the previous sections.

The \(\lambda^3\) correction. The standard approach to \(n = 3\) dispersion relations assumes

\[
\lambda_0^2 - \lambda^2 = m^2 + \frac{\lambda^3}{M},
\]

Note that Result (iii) implies that (9) has problems with coordinate invariance. Note also the fourfold degeneracy of the energies, an undesirable feature in light of Result (ii). Moreover, it will become clear below that dispersion relations of the type (9) develop positivity or causality violations at energies that are phenomenologically accessible, when \(M\) is taken to be a Planck-scale mass.

We first consider a positive \(M\). Equation (9) has the roots \(\lambda_\pm = \pm (|\lambda|^3/M + \lambda^2 + m^2)^{1/2}\). We proceed by assuming that the usual reinterpretation of the negative-energy roots \(\lambda_-\) is still applicable, so that these solutions correspond to positive-energy reversed-momentum particle states:

\[
p_+ = (E_+,\vec{p}), \quad E_+ = \lambda_+ (\vec{p}), \quad p_- = (E_-,\vec{p}), \quad E_- = \lambda_- (\vec{p}).
\]

The respective particle and antiparticle energies \(E_+\) and \(E_-\) are then given by

\[
E_\pm (\vec{p}) = \sqrt{\frac{|\lambda|^3}{M} + \vec{p}^2 + m^2}.
\]

Note that spacelike particle momenta are absent. The corresponding group velocities \(\vec{v}_g^\pm\) obey

\[
\vec{v}_g^\pm (\vec{p}) = \frac{1 + 3|\vec{p}|/2M}{\sqrt{|\vec{p}|^4/M + \vec{p}^2 + m^2}} \vec{p}.
\]

It can be shown that above momentum scales \(p_c \approx \sqrt{m^2/M/2}\) both the particle and the antiparticle develop superluminal group velocities violating causality.
Suppose we had chosen the correction term $|\bar{\lambda}|^3/M$ to enter the dispersion relation (9) with a minus instead of a plus sign. The resulting particle energies and group velocities are then obtained by the replacement $M \rightarrow -M$ in Eqs. (11) and (12), respectively. In this case, superluminal group velocities are absent, but positivity problems due to spacelike momenta occur at scales $p_\ell = \sqrt{m^2 M}$. At three-momenta $|\vec{p}| > 2M/3$ the particle energy decreases with increasing $|\vec{p}|$, so that monotonicity of $E_\pm(|\vec{p}|)$ is lost. Moreover, at momenta above the scale $M$, dispersion relation (9) admits imaginary particle energies. Both of these features signal the breakdown of validity of (9). However, if $M$ is of the order of the Planck mass, the required particle momenta are phenomenologically uninteresting.

In the context of (9), we next consider the photon decay $\gamma \rightarrow e^+ + e^-$ into an electron-positron pair, which is kinematically forbidden in conventional physics. For all particles, we assume the same $M$ parameter, and we set $m = 0$ for the photon. Energy-momentum conservation requires

$$k^\mu = p_+^\mu + p_-^\mu, \quad (13)$$

where $k^\mu$ is the photon four-momentum, and $p_+^\mu$ and $p_-^\mu$ are the four-momenta of the positron and electron, respectively. In App. A, we show rigorously that the decay is kinematically allowed provided the photon three-momentum obeys $|\vec{k}| \geq \sqrt{4m^2 M}$. Then, causality problems occur because at least one decay product must be superluminal. If a dispersion relation of the type (9) but with the opposite sign of the correction term is used, the above photon-decay process is kinematically forbidden within the validity range $|\vec{p}| < 2M/3$. This fact is also demonstrated rigorously in App. A.

The $|\vec{\lambda}|^3$ correction. Consider now enforcing observer Lorentz invariance by allowing two simultaneous signs for the correction term in (9). The particle and antiparticle energies are then

$$E_\pm^{(a)}(\vec{p}) = \sqrt{(-1)^a \frac{|\vec{p}|^3}{M} + \vec{p}^2 + m^2}, \quad (14)$$

where $\alpha = 1, 2$ labels the two possible particle (antiparticle) energies, which perhaps correspond to different spin-type states. As a result, six kinematically distinct decays have to be considered. Note, however, that additional conservation laws associated with rotational invariance may preclude some of the six reactions. A proper investigation of this case therefore requires dynamical concepts. This is our Result (vi): The effects of assumed symmetries, such as rotational invariance, must be incorporated into threshold analyses.

The $\lambda_0 |\bar{\lambda}|^2$ correction. Another $n = 3$ coordinate-independent dispersion relation is

$$\lambda_0^2 - |\bar{\lambda}|^2 = m^2 + \frac{\lambda_0 |\bar{\lambda}|^2}{M}. \quad (15)$$

The usual reinterpretation (10) yields the respective particle and antiparticle energies $E_+$ and $E_-:

$$E_\pm(\vec{p}) = \sqrt{\frac{\vec{p}^4}{4M^2} + \vec{p}^2 + m^2 \pm \frac{\vec{p}^2}{2M}}. \quad (16)$$

Note that the particle-antiparticle degeneracy is lifted, and only the antiparticles develop spacelike momenta, and thus difficulty with positivity, above the scale $p_\ell = \sqrt{m^2 M}$. The particle momenta $(E_+, \vec{p})$ remain timelike for all $\vec{p} \neq \vec{0}$. The corresponding group velocities are

$$v_g^\pm(\vec{p}) = \left(\frac{\vec{p}^2 + 2M^2}{\sqrt{\vec{p}^4 + 4M^2\vec{p}^2 + 4M^2m^2}} \pm 1\right) \frac{\vec{p}}{M}. \quad (17)$$

Thus, at any given nonzero three-momentum the particle travels faster than the antiparticle. Moreover, one can verify that above the scale $p_\ell \sim \sqrt{m^2 M}$ the particle speed becomes superluminal leading to causality problems. The antiparticle always remains subluminal. For $|\vec{p}| \gg M$, the antiparticle’s speed goes to zero, a feature indicating the validity breakdown of (15). In any case, this momentum range appears phenomenologically uninteresting at the present time.

Consider again photon decay into an electron-positron pair. We now assume a dispersion relation of the type (15), take the lepton and photon corrections to be controlled by the same parameter $M$, and set $m = 0$ for the photon. For notational simplicity we define the positron as the particle and the electron as the antiparticle. Because of the two possible incoming photon states $\gamma_+ \Gamma$ and $\gamma_- \Gamma$, two kinematically distinct processes must be investigated. The subscripts + and − correspond to ones for the particle energy in (16). In App. B, it is demonstrated that the decay $\gamma_+ \rightarrow e^+ + e^-$ is allowed above a certain threshold. If the observed value $m = 0.511$ MeV for the electron and positron masses is used, and $M$ is taken to be the Planck mass, the numerically determined threshold value for the incoming photon three-momentum is $|\vec{k}_{\text{min}}| \approx 7.21$ TeV. Appendix B also contains a proof that the decay channel in question is kinematically forbidden for $\gamma_-$ photons.

Suppose we had chosen the Lorentz-violating correction to enter the dispersion relation with the opposite sign. In such a situation, the roles of particle and antiparticle are interchanged. However, apart from this trivial reinterpretation, the above discussion remains unaffected. In particular, the threshold for photon decay into an electron-positron pair is left unchanged.

The above example demonstrates that, contrary to claims in the literature, a $\lambda_0 |\bar{\lambda}|^2$ correction kinematically permits photon decay irrespective of the sign of the correction. In addition, the discussion in this section identifies the common assumption $\lambda_0 \simeq |\bar{\lambda}|$ as the source of this confusion. In a general context, we arrive at Result (vii): In threshold analyses, many approximations, such as $\lambda_0 \simeq |\bar{\lambda}|$, and others leading to additional degeneracies are typically invalid. We remark that Result (vii) applies also to the conventional case.
VI. CONCLUSION

This work has considered Lorentz-violating dispersion-relation modifications and some of their implications in the search for possible signatures for fundamental physics. More specifically, we have discussed the role of a dynamical framework and the requirements of coordinate independence, positivity, and causality in the subject.

The consequences of these principles are summarized in the various Results (i) to (vii) given in the text. Correct threshold investigations within the SME are automatically compatible with these requirements.

None of the particle-reaction analyses known to the author is consistent with all of the Results (i) to (vii). It would therefore be of great interest to revisit many threshold studies implementing the findings of the present work.

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APPENDIX A: THRESHOLDS IN THE $|\vec{x}|^3$ CASE

Implementing momentum conservation, we write $\vec{k}$, $\vec{p}$, and $\vec{k} - \vec{p}$ for the photon, positron, and electron three-momenta, respectively. Then, the dispersion relation (9) (for the photon with $m = 0$) yields the respective particle energies $k_0(\vec{k})$, $p_0(\vec{p})$, and $p_0(\vec{k} - \vec{p})$, which obey the conservation equation

$$k_0(\vec{k}) = p_0(\vec{p}) + p_0(\vec{k} - \vec{p}) . \quad (A1)$$

At $\vec{k} = \vec{0}$, the reaction is forbidden because the left-hand side of (A1) is smaller than the right-hand side for all $\vec{p}$. To find the threshold for the reaction, one can imagine to increase $|\vec{k}|$ until $k_0(\vec{k})$ becomes large enough to equal the minimum of the right-hand side of (A1) viewed as a function $\vec{p}$. The energy of the decay products is smallest, when the outgoing three-momenta are parallel [60]: Suppose adding transverse momenta to $\vec{p}$ and $\vec{k} - \vec{p}$ taken as parallel. This would increase the total outgoing energy because $p_0$ is strictly monotonic for positive arguments. A similar reasoning can be employed to exclude antiparallel final configurations.

Next, we show that for a given $\vec{k}$, the minimum of the outgoing energy is attained, when the three-momentum is shared equally by the decay products, i.e., $\vec{p} = \vec{k}/2$. Equation (A2) would then yield the threshold quoted in the text. Using the expression for the particle energies (11) and implementing the above considerations yields:

$$\sqrt{\frac{k^3}{M} + k^2} = \sqrt{\frac{p^3}{M} + p^2 + m^2 + (p \leftrightarrow p - k)} , \quad (A2)$$

where $k = |\vec{k}|$ and $p = |\vec{p}| \leq k$. The first derivative of the right-hand side of (A2) with respect to $p$ vanishes at $p = k/2$ consistent with the presence of an extremum. To complete the argument, we have to confirm that $p = k/2$ is the location of a minimum and that for $p \in [0, k]$ all values of the right-hand side of (A2) lie above the one at $p = k/2$. This will be the case if the second derivative of right-hand side of (A2) with respect to $p$ is nonnegative on $[0, k]$.

Consider the positron energy and take the second derivative:

$$\frac{\partial^2}{\partial p^2} p_0(p) = g(p) - h(p) . \quad (A3)$$

Here, the functions $g$ and $h$ are given by

$$g(p) = \frac{3p + M}{(M^2 + M^2 p^2 + M^2 m^2)^{3/2}}$$

$$h(p) = \frac{M(3p^2/2 + p)^2}{(M^2 + M^2 p^2 + M^2 m^2)^{3/2}} . \quad (A4)$$

To see that $g > h$ for $p \in [0, k]$, we begin with the trivial inequality $0 < 3p^3 + 4Mp^3 + 12Mm^2p + 4M^2m^2$. Addition of $(3p^2 + 2Mp)^2$ to both sides of this inequality yields $(3p^2 + 2Mp)^2 < 4(3p + M)(p^3 + Mp^2 + Mm^2)$. We finally multiply both sides with $M/4(M^2 + M^2 p^2 + M^2 m^2)^{3/2}$ to obtain $\partial^2 p_0(p)/\partial p^2 > 0$. Replacing $p \rightarrow k - p$ in the above argument shows that $\partial^2 p_0(k - p)/\partial p^2 > 0$. This establishes the positivity of the second derivative of the right-hand side of (A2) with respect to $p$, which concludes our proof.

We next consider the case with the opposite sign in (9) and show that the decay $\gamma \rightarrow e^+_L + e^-_R$ is kinematically forbidden within the validity range $0 < k, \vec{p} < 2M/3$. Since monotonicity still holds, similar considerations as above imply that we must demonstrate

$$\sqrt{\frac{k^3}{M} + k^2} < \sqrt{\frac{p^3}{M} + p^2 + m^2 + (p \leftrightarrow p - k)} . \quad (A5)$$

for any value of $k$ and $p$ less than $2M/3$. Let us denote the right-hand side of the above inequality (A5) by $R(k, p)$. We observe that in the validity range, $R$ can be decreased setting $m = 0$:

$$\sqrt{p^2 - \frac{p^3}{M} + \sqrt{(k - p)^2 - \frac{(k - p)^3}{M}}} < R(k, p) . \quad (A6)$$

Inequality (A6) will continue to hold, when its left-hand side $L(k, p)$ is minimized with respect to $p$. If we can demonstrate that the minimum is attained at either $p = 0$ or $p = k$, then (A6) reduces to (A5), and we are done.

We will show that $L(k, p)$ has a local maximum at $p/2$ and decreases monotonically in directions away from the maximum. Indeed, the derivative of $L$ with respect to $p$...
vanishes at $p/2$. Negativity of the second derivative of $L$ would establish the desired result. One can show that
\[ \frac{\partial^2}{\partial p^2} L(k,p) = G(p) - H(p) + (p \leftrightarrow k - p), \]  
(A7)
where the functions $G$ and $H$ are defined by
\[ G(p) = \frac{M - 3p}{(M^2 p^2 - M p^3)^{1/2}}, \]
\[ H(p) = \frac{M(Mp - 3p^2)^2}{(M^2 p^2 - M p^3)^{3/2}}. \]  
(A8)
We start from $0 > -p^3(4M - 3p)$, which holds within the validity range. Adding $(2M p - 3p^2)^2$ to both sides of this inequality yields $(2M p - 3p^2)^2 < 4(M^2 p^2 - p^3)(M - 3p)$. Multiplication with $M/(M^2 p^2 - M p^3)^{3/2}$ shows that $G(p) < H(p)$. Since $k - p$ is also within the range of validity, we can infer $G(k - p) < H(k - p)$. Together with (A7), these results establish the negativity claim.

APPENDIX B: THRESHOLDS IN THE $\lambda_0 \bar{\lambda}^2$ CASE

To see that the decay is permitted, consider the special case of the decay $\gamma^+ \rightarrow e^+ e^-$, in which the $e^+$ particle has zero three-momentum and the $\gamma^+$ and $e^-$ particles both have three-momentum $\vec{k}$. Energy conservation and Eq. (16) then imply
\[ \sqrt{\frac{k^4}{4M^2} + k^2} = m + \sqrt{\frac{k^4}{4M^2} + k^2 + m^2} - \frac{k^2}{M}, \]  
(B1)
where $k = |\vec{k}|$, as before. Note that at $k = 0$, the left-hand side is smaller than the right-hand side. For large $k$, the situation is vice-versa: The left-hand side grows quadratically, whereas the right-hand side decreases quadratically. By continuity, there must be some value of $k$, for which (B1) is satisfied, and the decay is allowed.

For completeness, we demonstrate that the decay of $\gamma^-$ is forbidden in the present context. Arguments similar to the ones in App. A imply that we have to show the validity of
\[ \sqrt{\frac{k^4}{4M^2} + k^2} < \sqrt{\frac{p^4}{4M^2} + p^2 + m^2} + \frac{(k - p)k}{M}, \]
\[ + \sqrt{\frac{(k - p)^4}{4M^2} + (k - p)^2 + m^2} \]  
(B2)
for $p \in [0, k]$, where $p = |\vec{p}|$ is the three-momentum magnitude of the $e^-$ particle. The right-hand side $r(k,p)$ of (B2) can be reduced by replacing in the second term the factor of $k$ by $p$ and setting $m = 0$ under the square roots:
\[ \sqrt{\frac{p^4}{4M} + p^2 + (p \leftrightarrow k - p) + \frac{(k - p)k}{M}} < r(k,p). \]  
(B3)
This inequality remains true, when the left-hand side is minimized with respect to $p$. If we can show that the minimum is attained at $p = 0$ or at $p = k$, the inequalities (B3) and (B2) become identical, and the claim follows.

We will establish that $r(k,p)$ has a local maximum at $p = k/2$ and decreases in directions away from this maximum. The necessary condition $\partial r/\partial p = 0$ at $p = k/2$ can be verified straightforwardly. We also show that $\partial^2 r/\partial p^2 < 0$ for $p \in [0, k]$. An explicit expression involving the second derivative is given by:
\[ M \frac{\partial^2}{\partial p^2} r(k,p) = F(p) + F(k - p) - 2, \]  
(B4)
where the function $F$ is defined by
\[ F(p) = \frac{p(6M^2 + p^2)}{(p^2 + 4M^2)^{3/2}}. \]  
(B5)
Thus, it suffices to prove $F < 1$. We observe that $0 < 64M^6 p^2 + 12M^4 p^4$ and $p^2(6M^2 + p^2)^2$ to both sides of this inequality. Thus, the right-hand side can be cast into the form $p^2(2^2 + 4M^2)^3$. Dividing by this expression yields $F^2 < 1$, which completes the proof.

[1] V.A. Kostelecky and S. Samuel, Phys. Rev. D 39, 683 (1989); 40, 1886 (1989); Phys. Rev. Lett. 63, 224 (1989); 66, 1811 (1991); V.A. Kostelecky and R. Potting, Nucl. Phys. B 359, 545 (1991); Phys. Lett. B 381, 89 (1996); Phys. Rev. D 63, 046007 (2001); V.A. Kostelecky, M. Perry, and R. Potting, Phys. Rev. Lett. 84, 4541 (2000).
[2] G. Amelino-Camelia et al., Nature 393, 763 (1998).
[3] D. Sudarsky, L. Urrutia, and H. Vucetich, gr-qc/0211101.
[4] F.R. Klinkhamer, Nucl. Phys. B 578, 277 (2000).
[5] J. Alfaro, H.A. Morales-Técotl, and L.F. Urrutia, Phys. Rev. Lett. 84, 2318 (2000); Phys. Rev. D 65, 103509 (2002).
[6] S.M. Carroll et al., Phys. Rev. Lett. 87, 141601 (2001); Z. Guralnik et al., Phys. Lett. B 517, 450 (2001); A. Anisimov et al., hep-ph/0106356; C.E. Carlson et al., Phys. Lett. B 518, 201 (2001).
[7] V.A. Kostelecky, R. Lehnert, and M.J. Perry, astro-ph/0212003.
[8] D. Colladay and V.A. Kostelecky, Phys. Rev. D 55, 6760 (1997).
[9] D. Colladay and V.A. Kostelecky, Phys. Rev. D 58, 116002 (1998).
[10] R. Jackiw and V.A. Kostelecky, Phys. Rev. Lett. 82, 3572 (1999).
[11] V.A. Kostelecky and R. Lehnert, Phys. Rev. D 63, 065008 (2001).
[12] V.A. Kostelecky, C.D. Lane, and A.G.M. Pickering, Phys. Rev. D 65, 056006 (2002).
L.R. Hunter et al., O. Bertolami, D. Colladay, V.A. Kostelecký, and R. Potkowski, in V.A. Kostelecký, ed., KTeV Collaboration, H. Nguyen, in V.A. Kostelecký, ed., CPT and Lorentz Symmetry II (World Scientific, Singapore, 2002); OPAL Collaboration, R. Ackerstaff et al., Z. Phys. C 76, 401 (1997); DELPHI Collaboration, M. Feindt et al., preprint DELPHI 97-98 CONF 80 (1997); BELLE Collaboration, K. Abe et al., Phys. Rev. Lett. 86, 3228 (2001); FOCUS Collaboration, J.M. Link et al., Phys. Lett. B 556, 7 (2003).

D. Colladay and V.A. Kostelecký, Phys. Lett. B 344, 259 (1995); Phys. Rev. D 52, 6224 (1995); Phys. Lett. B 511, 209 (2001); V.A. Kostelecký and R. Van Kooten, Phys. Rev. D 54, 5585 (1996); V.A. Kostelecký, Phys. Rev. Lett. 80, 1818 (1998); Phys. Rev. D 61, 016002 (2000); 64, 076001 (2001); N. Isgur et al., Phys. Rev. Lett. B 515, 333 (2001).

O. Bertolami, D. Colladay, V.A. Kostelecký, and R. Potting, Phys. Lett. B 395, 178 (1997).

L.R. Hunter et al., in V.A. Kostelecký, ed., CPT and Lorentz Symmetry (World Scientific, Singapore, 1999); D. Bear et al., Phys. Rev. Lett. 85, 5038 (2000); M.A. Humphrey et al., Phys. Rev. A 62, 063405 (2000); V.A. Kostelecký and C.D. Lane, Phys. Rev. D 60, 116010 (1999); J. Math. Phys. 40, 6245 (1999); R. Bluhm et al., Phys. Rev. Lett. 88, 090801 (2002); D. Sudarsky, L. Urrutia, H. Vucetich, Phys. Rev. Lett. 89, 231301 (2002).

D.F. Phillips et al., Phys. Rev. D 63, 111101 (2001).

R. Bluhm et al., Phys. Rev. Lett. 82, 2254 (1999).

H. Dehmelt et al., Phys. Rev. Lett. 83, 4694 (1999); R. Mittleman et al., Phys. Rev. Lett. 83, 2116 (1999); G. Gabrielse et al., Phys. Rev. Lett. 82, 3198 (1999).

R. Bluhm et al., Phys. Rev. Lett. 79, 1432 (1997); Phys. Rev. D 57, 3932 (1998).

R. Bluhm and V.A. Kostelecký, Phys. Rev. Lett. 84, 1381 (2000).

B. Heckel, in V.A. Kostelecký, ed., CPT and Lorentz Symmetry II (World Scientific, Singapore, 2002).

S. Carroll, G. Field, and R. Jackiw, Phys. Rev. D 41, 1231 (1990).

V.A. Kostelecký and M. Mewes, Phys. Rev. Lett. 87, 251304 (2001).

V.A. Kostelecký and M. Mewes, Phys. Rev. D 66, 056005 (2002).

H. Müller et al., Phys. Rev. D 67, 056006 (2003).

J.A. Lipa et al., Phys. Rev. Lett. 90, 060403 (2003).

V.W. Hughes et al., Phys. Rev. Lett. 87, 111804 (2001); R. Bluhm et al., Phys. Rev. Lett. 84, 1098 (2000).

S. Coleman and S.L. Glashow, Phys. Rev. D 59, 116008 (1999).

V. Barger, S. Pakvasa, T. Weiler, and K. Whisnant, Phys. Rev. Lett. 85, 5055 (2000).

W. Kluźniak, Astropart. Phys. 11, 117 (1999); astro-ph/9905308.

T. Kifune, Astrophys. J. 518, L21 (1999).

O. Bertolami and C.S. Carvalho, Phys. Rev. D 61, 103002 (2000).

R. Aloisio et al., Phys. Rev. D 62, 053010 (2000).

G. Amelino-Camelia and T. Piran, Phys. Lett. B 497, 265 (2001); Phys. Rev. D 64, 036005 (2001); G. Amelino-Camelia, Phys. Lett. B 528, 181 (2002).

T. Jacobson, S. Liberati, and D. Mattingly, Phys. Rev. D 66, 081302 (2002); hep-ph/0209264.

T.J. Konopka and S.A. Major, New J. Phys. 4, 57 (2002).

H. Vankov and T. Stanev, Phys. Lett. B 538, 251 (2002).

J. Alfaro and G. Palma, hep-th/0208193.

A.R. Frey, hep-th/0301189.

V. Gharibyan, hep-ex/0303010.

R.C. Myers and M. Pospelov, hep-ph/0301124.

In the context of the renormalizable sector of the SME, dispersion relations are typically fourth-order polynomials in $\lambda_0$, so that (2) together with ansatz (3) is inconvenient. However, if $\delta f$ is allowed to contain unsuppressed terms, SME dispersion relations can be generated.

Negative or fractional powers of $|\vec{\lambda}|$ could arise when the dispersion relation (2) with the ansatz (3) is rearranged. For example, if it is solved for $\lambda_0 = \lambda_0(|\vec{\lambda}|)$.

G. Amelino-Camelia, Phys. Lett. B 510, 255 (2001).

J. Magueijo and L. Smolin, Phys. Rev. Lett. 88, 190403 (2002).

J. Lukierski and A. Nowicki, hep-th/0203065; J. Rembieliński and K.A. Smolinski, hep-th/0207031; D.V. Ahluwalia, M. Kirchbach, and N. Dadhich, gr-qc/0212128; D. Grumiller, W. Kummer, and D.V. Vassilevich, hep-th/0301061.

Odd powers of $|\vec{\lambda}|$ could arise from equations of motion containing nonlocal operators of the type $\sqrt{\Delta}$, where $\Delta$ is the laplacian. However, such nonlocalities are presently unmotivated in candidate fundamental theories, so that we exclude such operators in the present work.

G. Amelino-Camelia, gr-qc/0212002.

R. Lehner, in preparation.

For explicit photon and fermion dispersion relations, see Refs. [9] and [11], respectively. Fermion-eigenenergy approximations can be found in R. Lehner, UMI-30-54449.

See, for example, G. Diener, Phys. Lett. A 223, 327 (1996); Phys. Lett. A 235, 118 (1997).

W. Pauli, Phys. Rev. 58, 716 (1940).

R.F. Streater and A.S. Wightman, PCT, Spin and Statistics, and All That (Benjamin Cummings, London, 1964).

G. Feinberg, Phys. Rev. 159, 1089 (1967).

O.W. Greenberg, Phys. Rev. Lett. 89, 231602 (2002).

See, for example, M.S. Berger and V.A. Kostelecký, Phys. Rev. D 65, 091701(R) (2002).

A mass term would yield a Lorentz-invariant (but gauge-symmetry violating) contribution, which is not of interest in the present context.

The allowed phase space for the decay products in case of a lightlike pion four momentum is a set of measure zero leading at best to a suppressed rate for the reaction.

See also D. Mattingly, T. Jacobson, and S. Liberati, hep-ph/0211466.