THE SIMPLEST MINIMAL FREE RESOLUTIONS IN $\mathbb{P}^1 \times \mathbb{P}^1$

NICOLÁS BOTBOL, ALICIA DICKENSTEIN, AND HAL SCHENCK

Abstract. We study the minimal bigraded free resolution of an ideal with three generators of the same bidegree, contained in the bihomogeneous maximal ideal $\langle s, t \rangle \cap \langle u, v \rangle$ of the bigraded ring $K[s, t; u, v]$. Our analysis involves tools from algebraic geometry (Segre-Veronese varieties), classical commutative algebra (Buchsbaum-Eisenbud criteria for exactness, Hilbert-Burch theorem), and homological algebra (Koszul homology, spectral sequences). We treat in detail the case in which the bidegree is $(1, n)$. We connect our work to a conjecture of Fröberg-Lundqvist on bigraded Hilbert functions, and close with a number of open problems.

1. Introduction

In this chapter, we consider bigraded minimal free resolutions in the first non-trivial case. Let $R = K[s, t; u, v]$ be the bigraded polynomial ring, where $\{s, t\}$ are of degree $(1, 0)$ and $\{u, v\}$ are of degree $(0, 1)$; $R$ is graded by $\mathbb{Z}^2$. For $d = (d_1, d_2)$, we consider a three dimensional subspace $W = \text{Span}\{f_0, f_1, f_2\} \subseteq R_d$, with the additional constraint that

\begin{equation}
I_W = \langle f_0, f_1, f_2 \rangle \text{ satisfies } \sqrt{I_W} = \langle s, t \rangle \cap \langle u, v \rangle.
\end{equation}

This generic condition arises from the natural geometric condition of being base-point free, defined in §1.2.2 below. We study the minimal free resolution of $I_W$ and we give precise results when $d = (1, n)$ and $n \geq 3$.

Example 1.1. For $d = (1, 1)$ the bigraded Betti numbers of $I_W$ are always

\[
\begin{array}{cccccc}
0 & \leftarrow I_W & \leftarrow & (-1, -1)^3 & \oplus & (-2, -3) \\
& & \oplus & (-2, -2)^3 & \oplus & (-3, -3) \\
& & & \oplus & (-3, -2)^2 & \leftarrow 0
\end{array}
\]

The degree $(2, 2)$ syzygies are Koszul. The first syzygies of degree $(1, 3)$ and $(3, 1)$ involve only one set of variables, and arise from the vanishing of a determinant (see Lemma 3.1). The reader is encouraged to work out the remaining differentials.

2020 Mathematics Subject Classification. Primary 14M25; Secondary 14F17.

Key words and phrases. Bihomogeneous ideal, Syzygy, Free resolution, Segre map.

Dickenstein is supported by ANPCyT PICT 2016-0398, UBACYT 20020170100048BA, and CONICET PIP 11220150100473, Argentina.

Schenck is supported by NSF 1818646, Fulbright FSP 5704.
The \( d = (1,2) \) case is more complex, and \([8]\) shows that there are two possible bigraded minimal free resolutions for \( I_W \). The resolution type is determined by how \( \mathbb{P}(W) \subseteq \mathbb{P}(R_{1,2}) = \mathbb{P}^5 \) meets the image \( \Sigma_{1,2} \) of the Segre map \( \mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^5 \) of factorizable polynomials.

1.1. Motivation from geometric modeling. In geometric modeling, it is often useful to approximate a surface in \( \mathbb{P}^3 \) with a rational surface of low degree. Most commonly the rational surfaces used are \( \mathbb{P}^1 \times \mathbb{P}^1 \) or \( \mathbb{P}^2 \), and the resulting objects are known as tensor product surfaces and triangular surfaces. See, for example, \([3],[6],[7],[9],[26]\). An efficient way to compute the implicit equation is via approximation complexes \([20],[21]\) which use syzygy data as input.

A tensor product surface is mapped to \( \mathbb{P}^3 \) by a four dimensional subspace \( V \subseteq \mathbb{R}^d \). For \( d = (1,2) \), Zube describes the singular locus in \([28],[29]\), and in \([14],[16]\), Elkadi-Galligo-Lê use the geometry of a dual scroll to analyze the image. When \( \sqrt{I_V} = \langle s,t \rangle \cap \langle u,v \rangle \), \([25]\) shows that there are exactly six types of free resolution possible, and analyzes the approximation complexes for the distinct resolutions. Degan \([10]\) examines the situation when the subspace has basepoints. For a three dimensional subspace \( W \subseteq V \), the ideals \( I_V \) and \( I_W \) are related via linkage in \([25]\).

1.2. Mathematical background. We start with a quick review of the cast of principal mathematical players, referring to \([12],[13]\) for additional details.

1.2.1. Bigraded Betti numbers.

**Definition 1.2.** For a bihomogeneous ideal \( I \subseteq R = \mathbb{K}[s,t;u,v] \) and \( d \in \mathbb{Z}^2 \), the bigraded Betti numbers are

\[
\beta_{i,a} = \dim \mathbb{K} \text{Tor}_i (R/I, \mathbb{K})_a.
\]

For all nonnegative integers \( i \) and all bidegrees \( a \), \( \beta_{i,a} \) is the number of copies of \( R(-a) \) appearing in the \( i \)th module of the minimal free resolution of \( I \). Since \( a \in \mathbb{Z}^2 \), the \( \beta_{i,a} \) cannot be displayed in the \( \mathbb{Z} \)-graded Betti table format \([13]\). Bigraded regularity is studied in \([1],[22],[24]\), and multigraded regularity was introduced in \([23]\).

**Example 1.3.** \([8], \text{Theorem 7.8}\) Let \( \Sigma_{1,2} \) be the Segre variety of \( \mathbb{P}^1 \times \mathbb{P}^2 \subseteq \mathbb{P}^5 = \mathbb{P}(U) \), where \( U \) has basis \( \{su^2, suv, sv^2, tu^2, tuv, tv^2\} \). Then \( W \cap \Sigma_{1,2} \) is a smooth conic iff \( I_W \) has the bigraded Betti numbers as below.

\[
\begin{array}{ccccccc}
0 & \leftarrow & I_W & \leftarrow & (-1,-6) & \oplus & (-2,-6)^2 \\
& & & \oplus & (-2,-4)^3 & \oplus & (3,4)^2 \\
& & & \oplus & (3,4) & \oplus & (3,6) & \leftarrow & 0 \\
(-3,-2) & \oplus & (3,6) & \leftarrow & (-3,-6) & \leftarrow & 0 \\
\end{array}
\]

For example, \( \beta_{1,(2,4)} = 3 \) and \( \beta_{2,(3,4)} = 2 \).
1.2.2. Bigraded algebra and line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$. As noted earlier, the constraint that $\sqrt{I_W}$ is the bihomogeneous maximal ideal in (1.1) arises as a natural geometric condition, and we give a quick synopsis; for additional details, see §V.I of [19].

A line bundle $\mathcal{L}$ on the abstract variety $\mathbb{P}^1 \times \mathbb{P}^1$ is characterized by a choice of $d \in \mathbb{Z}^2$ and we write $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d)$ for $\mathcal{L}$. Although the global sections $H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d)) = \mathbb{R}^d$ are not functions on $\mathbb{P}^1 \times \mathbb{P}^1$, their ratios give well defined functions on $\mathbb{P}^1 \times \mathbb{P}^1$ and so zero sets of sections are defined.

The upshot is that to realize $\mathbb{P}^1 \times \mathbb{P}^1$ as a subvariety of $\mathbb{P}^n$, we choose an $n + 1$ dimensional subspace $W \subseteq \mathbb{R}^d$ with $d \in \mathbb{Z}^2_{> (0,0)}$. As long as the $f_i \in W$ do not simultaneously vanish on $\mathbb{P}^1 \times \mathbb{P}^1$, this gives a regular map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^n$. The condition that the $f_i$ do not simultaneously vanish at a point of $\mathbb{P}^1 \times \mathbb{P}^1$ is exactly the condition (1.1); in this situation $W$ is said to be basepoint free. For example, if $W = \text{Span}\{su, sv, tu\}$, then $[(0 : 1), (0 : 1)] \in \mathbb{P}^1 \times \mathbb{P}^1$ is a basepoint of $W$.

1.2.3. Koszul homology and bigraded Hilbert series. Koszul homology is defined and discussed in §2. We will give in Definition 2.8 the precise conditions under which a basepoint free system of polynomials $f = \{f_0, f_1, f_2\} \subseteq R_d$ is generic. In this case, we make a conjecture about the first Koszul homology module for generic $W$, and connect it to a conjecture of Fröberg-Lundqvist [15] on the bigraded Hilbert series of $R/I_W$.

1.3. Roadmap of this chapter. Below is an overview of the sections which make up this chapter.

- In §2, we study the Koszul homology of $I$; the first homology encodes the non-Koszul first syzygies. The spectral sequence of the Čech-Koszul double complex has a single $d_3$ differential, and we explain the connection to local cohomology $H^*_B$. We make a conjecture about the first Koszul homology module for generic $W$, and connect it to a conjecture of Fröberg-Lundqvist [15] on the bigraded Hilbert series of generic bigraded ideals.

- In §3, we use tools from commutative algebra such as the Hilbert-Burch theorem to shed additional light on the first syzygies.

- In §4, we connect the minimal free resolution to the image of the Segre variety $\Sigma_{1,n}$, obtaining canonical syzygies in certain degrees, without any assumptions on genericity. The geometry of $W \cap \Sigma_{1,n}$ plays a key role.

- In §5 we prove results on higher Segre varieties, in particular about how the geometry of the intersection of $W$ with such varieties influences the free resolution. We close with a number of questions.

2. Koszul homology $H_1(K_*(f,R))$ and the generic case

We start this section with an overview of Koszul homology. We then prove in Theorem 2.2 a characterization of the first Koszul homology associated to our ideal $I_W$ under the assumption (1.1). Corollary 2.5 then gives a concrete representation of $H_1$, that we make explicit in Examples 2.6 and 2.7 for factorizable polynomials. Section 2.4 treats the generic case (see Definition 2.8). In this case, we specify some
values of the dimensions \((H_1)_a\) for \(a \in \mathbb{Z}_{\geq 0}^2\) and we state Conjecture \([2.10]\) about these dimensions. We prove in Proposition \([2.17]\) that our Conjecture is equivalent to Conjecture \([2.14]\) by Ralf Fröberg and Samuel Lundqvist (Conjecture 8 in [15]).

2.1. Koszul Homology. For notation, we write \(a = (s, t),\ b = (u, v),\ B = a \cap b\) and \(m = R_+ = (s, t, u, v)\)

Definition 2.1. For a sequence of polynomials \(f = \{f_0, \ldots, f_m\}\) the Koszul complex \(K_\bullet := K_\bullet(f, R)\) is the complex

\[
\cdots \rightarrow \Lambda^j(R^{m+1}) \overset{\delta_j}{\rightarrow} \Lambda^{j-1}(R^{m+1}) \rightarrow \cdots
\]

where

\[
\delta_j(e_{n_1} \wedge \cdots \wedge e_{n_j}) \mapsto \sum_{i=1}^{j} (-1)^i f_{n_i} \cdot (e_{n_1} \wedge \cdots \hat{e}_{n_i} \cdots \wedge e_{n_j})
\]

The \(i\)\(^{th}\) Koszul homology is \(H_i(K_\bullet)\); Koszul cohomology is \(H^1(Hom_R(K_\bullet, R))\).

The Koszul complex is exact iff \(f\) is a regular sequence. We will focus on the case where \(f = \{f_0, f_1, f_2\}\) is a basepoint free subset of \(R_4\). Hence

\[
K_\bullet(f, R) : 0 \rightarrow R(-3d) \overset{\delta_3}{\rightarrow} R(-2d)^3 \overset{\delta_2}{\rightarrow} R(-d)^3 \overset{\delta_1}{\rightarrow} R \rightarrow 0.
\]

Let \(Z_i\) and \(B_i\) be the modules of Koszul \(i\)\(^{-th}\) cycles and boundaries, graded so that the inclusion maps \(Z_i, B_i \subset K_i\) are of degree \((0, 0)\), and let \(H_i = Z_i/B_i\) denote the \(i\)\(^{-th}\) Koszul homology module. Since \(\delta_1(p_1, p_2, p_3) = \sum_{i=1}^{3} p_i f_i\),

\[
H_0 = \text{coker}(\delta_1) = R/I_W.
\]

Since \(\sqrt{I_W} = B\), the codimension of \(I_W\) is two, so since \(f\) has three generators, \(f\) is not a regular sequence, and thus \(H_1 \neq 0\). Our assumption \([1.1]\) that \(\text{rad}(I_W) = B\) means that \(\text{depth}(I_W) = 2\), and then \(H_2 = H_3 = 0\).

From the definition of Koszul homology, the syzygy module \(\text{Syz}(f) := \ker(\delta_1)\). Since \(H_1 \neq 0\), the map \(\delta_2\) in the Koszul complex \([2.1]\) factors through \(\text{Syz}(f)\) as \(R(-2d)^3 \overset{\delta_2}{\rightarrow} \text{Syz}(f)\) but is not surjective. The module \(\text{im}(\delta_2)\) is called the module of Koszul syzygies. Thus, the size of non-Koszul syzygies is measured by \(H_1\).

2.2. Determining \(H_1(K_\bullet(f, R))\). Since \(\text{rad}(I_W) = B\), the modules \(H_0\) and \(H_1\) are supported on \(B\). In particular, we have that \(H_i^B(H_1) = 0\) for \(i > 0\) and hence, \(H_i^B(H_1) = H_1\). This says that the Koszul complex \([2.1]\) is not acyclic globally, but it is acyclic off \(V(B)\), i.e. that for every prime \(p \not\in B\) the localization \((K_\bullet(f, R))_p\) of \([2.1]\) at \(p\) is acyclic.

Consider the extended Koszul complex of \([2.1]\)

\[
K_\bullet : 0 \rightarrow R(-3d) \overset{\delta_3}{\rightarrow} R(-2d)^3 \overset{\delta_2}{\rightarrow} R(-d)^3 \overset{\delta_1}{\rightarrow} R \rightarrow R/I_W \rightarrow 0.
\]

For the complex \([2.2]\) we have that \(H_i = 0\) if \(i \neq 1\). The following theorem characterizes \(H_1\).

Theorem 2.2. There is an isomorphism of bigraded \(R\)-modules

\[
H_1 \cong \ker \left( H^B_3(R(-3d)) \overset{\delta}{\rightarrow} (H^B_3(R(-2d))^3 \right).
\]
Proof. Consider the Čech-Koszul double complex \( \check{C}_B^\bullet(K_\bullet) \) that is obtained from \( \check{C}_B^\bullet(-) \) by applying the Čech functor \( \check{C}_B^\bullet(-) \).

Consider the two spectral sequences that arise from the double complex \( \check{C}_B^\bullet(K_\bullet) \). We will denote by \( \check{E} \) the spectral sequence that arises taking first homology horizontally, this is, computing first the Koszul homology, and by \( vE \) the spectral sequence that is obtained by computing first the Čech cohomology. The second page of the spectral sequence of the horizontal filtration is:

\[
\check{H}^i_j = H^i_j(H_j(K_\bullet)).
\]

Since \( H^i_j(1) = 0 \) for \( i > 0 \) and \( H^i_j(1) = H_1 \), we have

\[
\check{H}^i_j = \begin{cases} H_1 & \text{for } j = 1 \text{ and } i = 0 \\ 0 & \text{otherwise.} \end{cases}
\]

We conclude that

\[
H^\bullet_\bullet(H_\bullet) \Rightarrow H_1.
\]

The second spectral sequence has

\[
\check{E}^i_j = H^i_j(K_j),
\]

where \( K_j \) is the \( i \)-th module from the right in Equation (2.2). Precisely, we have

\[
\begin{align*}
\check{E}^1_{-1} &= H^1(R/I_W) \\
\check{E}^1_{0} &= H^1(R) \\
\check{E}^1_{1} &= H^1(R(-d))^3 \\
\check{E}^1_{2} &= H^1(R(-2d))^3 \\
\check{E}^1_{3} &= H^1(R(-3d))^3
\end{align*}
\]

Therefore, the first page of the vertical spectral sequence is

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \omega_R^*(3d) & \longrightarrow & (\omega_R^*(2d))^3 & \longrightarrow & (\omega_R^*(d))^3 & \longrightarrow & \omega_R^* & \longrightarrow & H^1_3(R/W) \\
0 & \longrightarrow & H^1_3(R(-3d)) & \longrightarrow & H^1_3(R(-2d))^3 & \longrightarrow & H^1_3(R(-d))^3 & \longrightarrow & H^1_3(R) & \longrightarrow & H^1_3(R/W) \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^1_3(R/I_W) & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^0_3(R/I_W) & \longrightarrow & 0
\end{array}
\]

By comparing both spectral sequences, we conclude that

\[
H_1 \cong \ker \left( H^1_3(R(-3d)) \xrightarrow{\delta} (H^1_3(R(-2d))^3) \right).
\]

Corollary 2.3. The sequence

\[
0 \rightarrow R(-3d) \xrightarrow{\delta_3} R(-2d)^3 \xrightarrow{\delta_2} \operatorname{Syz}(f) \rightarrow H^1_3(R(-3d)) \xrightarrow{\delta_1} (H^1_3(R(-2d))^3)
\]

is exact.

Proof. From equation (2.2), \( 0 \rightarrow R(-3d) \xrightarrow{\delta_3} R(-2d)^3 \xrightarrow{\delta_2} \operatorname{Syz}(f) \rightarrow H^1_3(R(-3d)) \xrightarrow{\delta_1} (H^1_3(R(-2d))^3) \) is exact. Theorem 2.2 gives \( H_1 \cong \ker \left( H^1_3(R(-3d)) \xrightarrow{\delta} (H^1_3(R(-2d))^3) \right) \). The result follows by connecting the two sequences. \( \square \)
2.3. Understanding \((H_1)_n\). We have the following consequence of Theorem 2.2.

**Corollary 2.4.**

\[ \text{Supp}_{\mathbb{Z}}(H_1) \subset -N \times N + (3d_1 - 2, 3d_2) \cup N \times -N + (3d_1, 3d_2 - 2). \]

**Proof.** A direct computation using the Mayer-Vietoris sequence yields

1. \( H^2_B(R) = H^2_m(R) \oplus H^2_m(R) = (\omega^*_R \otimes_R \mathbb{K}) \oplus (R_1 \otimes_K \omega^*_R) \),
2. \( H^2_B(R) = H^2_m(R) = \omega^*_R \),
3. \( H^2_B(R) = 0 \) for all \( i \neq 2, 3 \),

where \( \omega^*_S \) denotes the canonical dualizing module of \( S \). Since we have that \( \text{Supp}_{\mathbb{Z}}(H^2_B(R)) = -N \times N + (-2, 0) \cup N \times -N + (0, -2) \), by shifting we get that

\[ \text{Supp}_{\mathbb{Z}}(H^2_B(R)) \subset -N \times N + (3d_1 - 2, 3d_2) \cup N \times -N + (3d_1, 3d_2 - 2). \]

**Corollary 2.5.** Consider the map \( H^2_B(R(-3d)) \buildrel \delta \over \rightarrow (H^2_B(R(-2d)))^3 \). For every \( a = (a_1, a_2) \), we get

\[ (H^1)_{(a_1, a_2)} \cong \ker \left( \begin{array}{c}
\begin{pmatrix}
R_{(3d_1 - a_1, -3d_2 + a_2)} & \delta_0 \\
R_{(-3d_1 + a_1, 3d_2 - a_2)} & \delta_2
\end{pmatrix}
\end{array} \right)^3,
\]

is an isomorphism of \( \mathbb{K} \)-modules. Identifying the target with

\[ R^3_{(3d_1 - a_1, -3d_2 + a_2)} \oplus R^3_{(-3d_1 + a_1, 3d_2 - a_2)}, \]

we have

\[ \delta_a = \left( \begin{array}{cc}
\phi_1 & 0 \\
0 & \phi_2
\end{array} \right), \text{ with }
\]

\[ \phi_1 : R_{(-3d_1 + a_1, 3d_2 - a_2)} \rightarrow R^3_{(-3d_1 + a_1, 3d_2 - a_2)}, \]

\[ \phi_2 : R_{(3d_1 - a_1, -3d_2 + a_2)} \rightarrow R^3_{(3d_1 - a_1, -3d_2 + a_2)}. \]

For \( d_1, d_2 \geq 2 \), the previous result gives a description of the kernel \((H^1)_{(a_1, a_2)}\):

| \( \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_1} \) | \( \text{ker}(\phi_2) \) | \( R_{(3d_1 - a_1, -3d_2 + a_2)} \) | 0 | 0 |
|---|---|---|---|---|
| \( 3d_1 - 1 \) | 0 | 0 | 0 | 0 |
| \( 2d_2 - 2, 3d_2 - 2 \) | 0 | 0 | 0 | \( R_{(-3d_1 + a_1, 3d_2 - a_2 - 2)} \) |
| \( -\infty, 2d_2 - 2 \) | 0 | 0 | 0 | \( \text{ker}(\phi_1) \) |
| \( -\infty, 2d_1 - 2 \) | \( (3d_1, -\infty) \) | \( (2d_1 - 2, 3d_2 - 2) \) | \( 3d_1 - 1 \) | \( (3d_1, +\infty) \) |

The next examples illustrate the map \( \delta_a \) of Corollary 2.5 in a particular case in which the three polynomials \( f_i \) can be factored as two polynomials with bidegrees \((1, 0)\) and \((0, 0)\).

**Example 2.6.** Let \( d = (1, n) \), \( f_0 = su^n \), \( f_1 = tv^n \), \( f_2 = (s + t)(u^n + v^n) \), and \((a_1, a_2) = (3, n)\). Then

\[ (H^1)_{(3, n)} = \ker \left( \left( R_1 \otimes_K \omega^*_R \right)_{(0, -2n + 2)} \rightarrow \left( R_1 \otimes_K \omega^*_R \right)_{(1, -n + 2)} \right)^3 \]

and with the standard identification of the canonical dualizing modules, one has

\[ (H^1)_{(3, n)} \cong \ker \left( R_{(0, 2n - 2)} \rightarrow R_{(1, n - 2)} \right)^3 \]
The map $\delta_{(3,n)}$ in Equation [2.4] is given by multiplication by $f_i$. Precisely, given $a \geq 0$, $\delta_{(3,n)}$ is as follows

$$\frac{1}{uv} u^a v^{2n-2-a} \mapsto \frac{1}{uv} \left( \frac{1}{u^a v^{2n-2-a}} f_0, \frac{1}{u^a v^{2n-2-a}} f_1, \frac{1}{u^a v^{2n-2-a}} f_2 \right).$$

Thus, fixing a basis $B$ for $(R_1 \otimes_K \omega_R^*)_{(0,-2n+2)}$ and also fixing a basis $B'$ for $(R_1 \otimes_K \omega_R^*)_{(1,-n+2)}$, $|\delta_{(3,n)}|_{BB'}$ is a $(3 \cdot 2(n-1)) \times (2n-1)$-matrix given by the coefficients $\text{coef}_{B'}((f_0, f_1, f_2) \cdot B_i)$ of the $i$-th element of $B$ multiplied by one of the $f_j$ ($j$ depending on the row), written in the basis $B'$.

We now exhibit the matrices in Example 2.6 in bidegree $(1,6)$.

**Example 2.7.** Set for instance $n = 6$ (so $d = (1,6)$), $|\delta_{(3,n)}|_{BB'}$ is a $(3 \cdot 10) \times 11$-matrix. One can take $B = \left\{ \frac{1}{w v}, \frac{1}{w^2 v}, \ldots, \frac{1}{w^6 v} \right\}$ and $B' = \left\{ \frac{1}{w^3 v}, \frac{1}{w^4 v}, \frac{1}{w^5 v}, \frac{1}{w^6 v} \right\} \times \left\{ (1,0,0), (0,1,0), (0,0,1) \right\}$.

In this case, one has that the 10-tuple, corresponding to the 'upper third' of the first column of $|\delta_{(3,n)}|_{BB'}$, induced by multiplying by $f_0$ is

$$\text{coef}_{B'}((f_0 \cdot B_1) = \text{coef}_{B'} \left( f_0 \cdot \frac{1}{uw} \right) = \text{coef}_{B'} \left( \frac{1}{uw} s \right) = (1,0,0,\ldots,0).$$

And, because of the structure of multiplication on $\omega_R^*$, it is easy to see that in $(R_1 \otimes_K \omega_R^*)_{(1,-n+2)}$, $f_{j} \cdot u^{-6} v^{-6} = 0$. Thus, the 6th column of $|\delta_{(3,n)}|_{BB'}$ is zero, and the rest are not.

The matrix $|\delta_{(3,n)}|_{BB'}$ has the following shape

$$
\begin{pmatrix}
\text{Id}_{5 \times 5} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \text{Id}_{5 \times 5} & 0 \\
\text{Id}_{5 \times 5} & 0 & \text{Id}_{5 \times 5}
\end{pmatrix},
$$

where $\text{Id}_{5 \times 5}$ is the $5 \times 5$-identity matrix.

Finally, we conclude that $\text{corank}(|\delta_{(3,n)}|_{BB'}) = 1$. Since the matrix above induces a morphism from $k^{11} \to k^{30}$, we have that

$$H_{\Gamma} \left( (3,n) \right) = \dim(\ker(\delta_{(3,n)}|_{BB'})) = \text{corank}(|\delta_{(3,n)}|_{BB'}) = 1.$$

This in particular says that there is only one non-Koszul syzygy spanning every other non-Koszul syzygy.

Examples 2.6 and 2.7 are explained by Theorem 4.7.

2.4. The generic case. We give the definition of generic bihomogeneous polynomials $f_i$ of the same bidegree $d \in \mathbb{Z}_d^2$. Note that our assumption (1.1) implies that no $d_i$ could be equal to 0.

**Definition 2.8.** Given $d \in \mathbb{Z}_d^2$ and $f = \{f_0, f_1, f_2\}$ of bidegree $d$ satisfying (1.1), we say that $f$ is generic if the maps $\phi_1$ and $\phi_2$ in (2.3) in Corollary 2.5 have full rank, for any $a \in \mathbb{Z}_d^2$.

This condition of full rank is not true only under the assumption (1.1). For instance, the factorizable polynomials in Example 2.7 are not generic, but our
Conjecture 2.10 below states that the maps have full rank for polynomials with generic coefficients.

If we denote by $n_d(a)$ the difference of dimensions:

$$n_d(a) := \dim_k \left( \frac{R(3d_1-a_1-2, 3d_2+a_2)}{\oplus R(3d_1+a_1, 3d_2-a_2-2)} \right) - 3 \dim_k \left( \frac{R(2d_1-a_1-2, -2d_2+a_2)}{\oplus R(-2d_1+a_1, 2d_2-a_2-2)} \right) \in \mathbb{Z}[a,d],$$

we have by Corollary 2.5 that

$$\dim_k (H_1)_{(a_1,a_2)} \geq n_d(a).$$

For any real number $c$, denote

$$c_+ = \max(c,0), \quad c_- = \max(0,-c).$$

Note that for any $c \in \mathbb{R}$, $c = c_+ - c_-$ and only one of these two numbers can be positive.

Given $d, a \in \mathbb{Z}^{\geq 0}_2$, we set

$$\text{dom}_d(a) := (0, -3d_1+a_1+1)_+ + (0, 3d_2-a_2-1)_+ + (0, 3d_1-a_1-1)_+ + (0, -3d_2+a_2+1)_+$$

and

$$\text{cod}_d(a) := (0, -2d_1+a_1+1)_+ + (0, 2d_2-a_2-1)_+ + (0, 2d_1-a_1-1)_+ + (0, -2d_2+a_2+1)_+.$$

The following Lemma is straightforward taking into account that a linear map $V_1 \to V_2$ between two $K$-vector spaces of finite dimension is of maximal rank if and only if the dimension of its kernel of equals $(\dim_k(V_1) - \dim_k(V_2))_+.$

**Lemma 2.9.** Let $f = \{f_0, f_1, f_2\}$ of bidegree $d$ satisfying (1.1). Then, $f$ is generic if and only if

$$n_d(a) = (\text{dom}_d(a) - 3 \text{cod}_d(a))_+.$$  

In this case, for any $a \in \mathbb{Z}^{\geq 0}_2$, we have the equality:

$$\dim_k (H_1)_{(a_1,a_2)} = n_d(a).$$

In fact, we conjecture that this is indeed the generic behavior.

**Conjecture 2.10.** There exists a nonempty open set in the space of coefficients of the polynomials $f_i$ where $f$ is generic according to Definition 2.8 and hence $\dim_k (H_1)_a = n_d(a)$ for any $a \in \mathbb{Z}^{\geq 0}_2$ by Lemma 2.9.

**Remark 2.11.** Note that Corollary 2.5 proves that Conjecture 2.10 is always true for polynomials $f_i$ satisfying assumption (1.1) outside of the range where we have $(a_1 \geq 3d_1$ and $d_2 \leq a_2 \leq 2d_2 - 2)$ and $(a_2 \geq 3d_2$ and $d_1 \leq a_1 \leq 2d_1 - 2)$.

2.5. The Fröberg-Lindqvist conjecture on bigraded Hilbert series. For any bidegree $a$, we denote by $\chi_{x^a} (\ell)$ the Euler characteristic of the $\ell$-strand of the Koszul complex (2.1) and let $S(x,y) = \sum_{a \in \mathbb{Z}^{\geq 0}_2} \chi_{x^a} (\ell)x^{a_1}y^{a_2}$. Then,

$$S(x,y) = \frac{(1 - x^{d_1}y^{d_2})^3}{(1 - x)^2(1 - y)^2}.$$  

We denote by $S(x,y)_+$ the series supported in $\mathbb{Z}^{\geq 0}_2$ with coefficients $\chi_{x^a} (\ell)_+$. The following lemmas are straightforward.

**Lemma 2.12.** $S(x,y)_+$ and $S(x,y)_-$ are also rational functions of $x, y.$
Lemma 2.13. Define regions

\[ A_1 = \{ a_1 < d_1 \text{ or } a_2 < d_2 \} \]
\[ A_2 = \{ d_1 \leq a_1 < 2d_1 \text{ and } d_2 \leq a_2 \} \text{ or } (d_1 \leq a_1 \text{ and } d_2 \leq a_2 < 2d_2) \]
\[ A_3 = \{ 2d_1 \leq a_1 < 3d_1 \text{ and } 2d_2 \leq a_2 \} \text{ or } (a_1 < 3d_1 \text{ and } 2d_2 \leq a_2 < 3d_2) \]
\[ A_4 = \{ 3d_1 \leq a_1 \text{ and } 3d_2 \leq a_2 \} \]

For any \( a \in \mathbb{Z}^2_{\geq 0} \), the coefficient \( \chi_{\cdot a}(a) \) equals the following:

\[
\begin{align*}
(a_1 + 1)(a_2 + 1) & \quad \text{in } A_1 \\
(a_1 + 1)(a_2 + 1) - 3(a_1 - a_2 + 1)(a_2 - d_2 + 1) & \quad \text{in } A_2 \\
(a_1 + 1)(a_2 + 1) - 3(a_1 - d_1 + 1)(a_2 - d_2 + 1) + 3(a_1 - 2d_1 + 1)(a_2 - 2d_2 + 1) & \quad \text{in } A_3 \\
0 & \quad \text{in } A_4
\end{align*}
\]

The following table shows in which bidegrees the Euler characteristic \( \chi_{\cdot a}(a) \) is positive, negative or zero.

| 3d2, +∞ | 3d2 − 1 | (2d2 − 3d2 − 2) | (d2 − 1, 2d2 − 2) | (0, d2 − 1) |
|---|---|---|---|---|
| + | + | + | + | + |
| +/hiperb/− | + | + | + | + |
| − | 0 | 0 | 0 | 0 |
| 0 | − | − | − | − |

Here the notation \+)/hiperb/−\ means that there is a hyperbola where \( \chi_{\cdot a} \) vanishes, separating the positive from the negative values. Here is an equivalent version of the Fröberg and Lindqvist Conjecture 8 from [15].

Conjecture 2.14. There exists a nonempty open set in the space of coefficients of the polynomials \( f_i \) for which \( \dim(R/I_W)_a = \chi_{\cdot a} \) for any \( a \in \mathbb{Z}^2_{\geq 0} \).

We now prove that Conjecture 2.10 and 2.14 are indeed equivalent.

Lemma 2.15. Let \( f = \{ f_0, f_1, f_2 \} \) be bihomogeneous polynomials of bidegree \( d \in \mathbb{Z}^2_{>0} \) satisfying \( [1.1] \). We have the equality:

\[
\chi_{\cdot a}(a) = \dim(R/I_W)_a - \dim(H_1)_a.
\]

As we remarked in Section 2.1, our assumption \( [1.1] \) implies that \( H_2 = H_3 = 0 \)
and so the proof of Lemma 2.15 is immediate.

We compare the conjectural dimension \( n_d(a) \) of \( H_{1a} \) with the coefficients of \( S \).

Lemma 2.16. For any \( a \in \mathbb{Z}^2_{\geq 0} \) we have the equality:

\[
\dim n_d(a) = \chi_{\cdot a}(a).
\]

Proof. By Lemma 2.13 are sixteen domains of polynomiality of \( \chi_{\cdot a} \). Consider for instance the case \( a_1 > 3d_1 - 1, 2d_2 - 1 < a_2 \leq 3d_2 - 1 \). Then, \( n_d(a) = (3d_1 - a_1 - 1)(3d_2 - a_2 - 1) \) while \( \chi_{\cdot a}(a) = (a_1 + 1)(a_2 + 1) - 3(-d_1 + a_1 + 1)(-d_2 + a_2 + 1) + 3(-2d_1 + a_1 + 1)(-2d_2 + a_2 + 1) \) and it is a simple computation to check that they coincide. The other cases are similar. \( \square \)

By Lemma 2.12 the generating series \( T(x, y) = \sum_a \dim(H_1)_a x^{a_1} y^{a_2} \) is a rational function. Hence

Proposition 2.17. Conjectures 2.10 and 2.14 are equivalent.
Proof. Assume \( \dim (H_1)_a = n_d(a) \). Using Lemma 2.16 we substitute this value in the statement of Lemma 2.15:

\[
\chi_{\bullet}(a) = \chi_{\bullet}(a)_+ - \chi_{\bullet}(a)_- = \dim(R/I_W)_a - \chi_{\bullet}(a)_-,
\]

which says that \( \dim(R/I_W)_a = \chi_{\bullet}(a)_+ \). The converse is similar. \( \Box \)

Remark 2.18. By Remark 2.11, we have that \( \dim (H_1)_a = n_d(a) \) is true for polynomials \( f_i \) satisfying assumption (1.1) except in the ranges \( (a_1 \geq 3d_1 \text{ and } d_2 \leq a_2 \leq 2d_2 - 2) \) and \( (a_2 \geq 3d_2 \text{ and } d_1 \leq a_1 \leq 2d_1 - 2) \). So, we deduce from the proof of Proposition 2.17 that Conjecture 2.14 is true outside these ranges.

We end this section with an easy corollary.

Corollary 2.19. If Conjectures 2.10 and 2.14 hold, then for any \( f \) regular either \( R/I_W = 0 \) or \( H_1 = 0 \) for any \( a \in \mathbb{Z}_{\geq 0} \).

Proof. If the conjectures are valid for any \( f \) regular, we have for any bidegree \( a \) that \( \dim(R/I_W)_a = \chi_{\bullet}(a)_+ \) and \( \dim(H_1)_a = \chi_{\bullet}(a)_- \), and as we remarked after (2.5), at most one of these numbers can be nonzero. \( \Box \)

3. Koszul homology \( H_1(K_{\bullet}(f, R)) \) for \( d = (1, n) \)

From now on, we specialize our study to bidegrees of the form \( d = (1, n) \), always assuming that \( W \) is basepoint free. This case is both the natural sequel to the study of the \((1, 2)\) case studied in [8], as well as a key ingredient for better understanding the general case. It splits the analysis into separate parts, in a way that we make precise below. Theorem 3.7 relates the Betti numbers \( \beta_{1,a} \) with the Koszul homology of \( H_1 \) with respect to the sequence \( \{s, t, u, v\} \), for any bidegree \( d \).

For degree \((1, n)\), Corollary 2.5 yields the following description of \( (H_1)_{(a_1,a_2)} \):

\[
(H_1)_{(a_1,a_2)} \cong \ker \left( \begin{array}{c} R_{(1-a_1,-3n+a_2)} \oplus R_{(3+a_1,3n-a_2-2)} \\ R_{(1-a_1,2n-a_2)} \oplus R_{(2+a_1,2n-a_2-2)} \end{array} \right)^3. 
\]

The table given in Section 2.3 reduces to

| \( [3n, +\infty) \) | \( R_{(1-a_1,-3n+a_2)} \) | 0 | 0 | 0 |
| --- | --- | --- | --- | --- |
| \( 3n - 1 \) | 0 | 0 | 0 | 0 |
| \( (2n-2, 3n-2) \) | 0 | 0 | 0 | \( R_{(-3+a_1,3n-a_2-2)} \) |
| \( (-\infty, 2n-2) \) | 0 | 0 | \( \ker(\phi_1) \) | 3, +\infty |

with

\[
\phi_1 : R_{(-3+a_1,3n-a_2-2)} \to R_{(-2+a_1,2n-a_2-2)}.
\]

So the region where interesting behavior occurs is in multidegree \((a_1, a_2)\), with

\[
a_2 \geq 3n \quad \text{and} \quad a_1 = 1
\]

or

\[
a_1 \geq 3 \quad \text{and} \quad a_2 \leq 2n - 2
\]

Since we need \( I_W \) to be nonzero, we have the constraint that \( a_1 \geq d_1, a_2 \geq d_2 \), so for \( d = (1, n) \), the only region of interest is bidegree \((a_1, a_2)\), with

\[
a_1 \geq 3 \quad \text{and} \quad 2n - 2 \geq a_2 \geq n,
\]

corresponding to \( \ker(\phi_1) \) defined in 2.3. We study \( n \geq 3; \ n = 2 \) is analyzed in [8].
3.1. **Tautological first syzygies: degrees (1, *) and (2, *).**

**Lemma 3.1.** There is a unique minimal first syzygy on $I_W$ in bidegree (1, 3n).

**Proof.** Because $\phi_2 : \mathbb{K} \to 0$, $\ker(\phi_2) \simeq \mathbb{K}$, and we can describe the syzygy explicitly as follows (it is of the type appearing in Lemma 6.1 of [8].) Write

\[
\begin{align*}
  f_0 &= s \cdot p_0 + t \cdot q_0 \\
  f_1 &= s \cdot p_1 + t \cdot q_1 \\
  f_2 &= s \cdot p_2 + t \cdot q_2,
\end{align*}
\]

with the $p_i, q_i \in k[u, v]_n$. Then

\[
\det \begin{bmatrix} f_0 & p_0 & q_0 \\ f_1 & p_1 & q_1 \\ f_2 & p_2 & q_2 \end{bmatrix} = 0,
\]

so the $2 \times 2$ minors in $q_i$ and $p_i$ give a syzygy with entries of bidegree $(0, 2n)$, hence of bidegree $(1, 3n)$ on $I_W$. It is minimal since any syzygy of lower degree would be of the form $(0, d)$ with $d < 2n$. This would force $W$ to have basepoints; to see this note that a syzygy $(s_0, s_1, s_2)$ of bidegree $(0, d)$ must be in the kernel of the map

\[
\theta = \begin{bmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{bmatrix},
\]

by splitting out the $s$ and $t$ components. But $\theta$ gives a map $\mathcal{O}_{\mathbb{P}^1}^2(-n) \to \mathcal{O}_{\mathbb{P}^1}^2$ with zero cokernel, because the rank of $\theta$ drops on the locus of the $2 \times 2$ minors of $\theta$; such a point would be a basepoint of $W$. A Chern class computation shows $\ker(\theta) \simeq \mathcal{O}_{\mathbb{P}^1}(-3n)$; in fact it consists of the $2 \times 2$ minors of $\theta$. \hfill \Box

**Proposition 3.2.** The syzygy of Lemma 3.1 and the three Koszul syzygies generate a pair of minimal second syzygies of bidegree (2, 3n). Furthermore, there is a minimal third syzygy of bidegree (3, 3n).

**Proof.** Let

\[
\begin{align*}
  f_0 &= s \cdot p_0 + t \cdot q_0 \\
  f_1 &= s \cdot p_1 + t \cdot q_1 \\
  f_2 &= s \cdot p_2 + t \cdot q_2,
\end{align*}
\]

with the $a_i, b_i \in k[u, v]_n$, and consider the submatrix $A$ of $\partial_1$ generated by the syzygy of Lemma 3.1 and the three Koszul syzygies:

\[
\begin{bmatrix}
  q_1 p_2 - p_1 q_2 & t \cdot q_1 + s \cdot p_1 & t \cdot q_2 + s \cdot p_2 & 0 \\
  p_0 q_2 - q_0 p_2 & -t \cdot q_0 + s \cdot p_0 & 0 & t \cdot q_2 + s \cdot p_2 \\
  q_0 p_1 - p_0 q_1 & 0 & -(t \cdot q_0 + s \cdot p_0) & -(t \cdot q_1 + s \cdot p_1)
\end{bmatrix}
\]

The columns of the matrix $A'$

\[
\begin{bmatrix}
  s & t & 0 \\
  q_2 & -p_2 & f_2 \\
  -q_1 & p_1 & -f_1 \\
  q_0 & -p_0 & f_0
\end{bmatrix}
\]

are in the kernel of $A$; the rightmost column is the second Koszul syzygy on $I_W$. As $[t, -s, 1]$ is in the kernel of $A'$, we see that there is a third syzygy of bidegree $(3, 3n)$. Note that by Theorem 3.4 the second Koszul syzygy is not minimal, but can be represented in terms of the syzygies appearing in Theorem 3.4. \hfill \Box
The results of Section 2.3 show that there are no minimal first syzygies in bidegree $(2, *)$ except for the Koszul syzygies in degree $(2, 2n)$. This can be seen explicitly, as follows. First, a syzygy with entries of bidegree $(1, m)$ satisfies

$$(sg_0 + th_0)f_0 + (sg_1 + th_1)f_1 + (sg_2 + th_2)f_2 = 0,$$

with the $f_i$ as in the previous lemma. Note that $\langle p_0, p_1, p_2 \rangle$ and $\langle q_0, q_1, q_2 \rangle$ are both basepoint free on $\mathbb{P}^1$; for otherwise vanishing of $\{p_0, p_1, p_2, t\}$ would give a basepoint on $\mathbb{P}^1 \times \mathbb{P}^1$ and also for for $\{q_0, q_1, q_2, s\}$. If $(u_0 : v_0) \in \mathbb{P}^1$ is a point where the rank

$$[p_0 \ p_1 \ p_2 \ p_0 \ p_1 \ p_2 \ q_0 \ q_1 \ q_2]$$

is one, then $s = u_0, t = v_0$ is a basepoint of $I_W$. Using that $f_i = sp_i + tq_i$, multiplying out and collecting the coefficients of the $\{s^2, st, t^2\}$ terms shows that

$$[g_0 \ g_1 \ g_2 \ h_0 \ h_1 \ h_2]$$

is in the kernel of the matrix

$$M = \begin{bmatrix}
  p_0 & p_1 & p_2 & 0 & 0 & 0 \\
  q_0 & q_1 & q_2 & p_0 & p_1 & p_2 \\
  0 & 0 & 0 & q_0 & q_1 & q_2
\end{bmatrix}.$$

The remarks above show that the kernel is free of rank three, with first Chern class $6n$. The matrix $K$ below satisfies these properties and clearly $MK = 0$:

$$K = \begin{bmatrix}
  -p_1 & 0 & p_2 \\
  p_0 & -p_2 & 0 \\
  0 & p_1 & p_0 \\
  -q_1 & 0 & -q_2 \\
  q_0 & -q_2 & 0 \\
  0 & q_1 & q_0
\end{bmatrix}.$$

By the Buchsbaum-Eisenbud criterion, $K = \ker(M)$. But $K$ consists of exactly the Koszul syzygies. From this it follows that the lowest possible nonzero multidegree in the degree $(1, 0)$ variables for a non-tautological first syzygy is $(3, m + n) = (2, m) + (1, n)$, with $m \geq 0$. We tackle this next.

3.2. First syzygies of degree $(3, *)$. The next theorem gives a complete description of the first syzygies with entries of degree $(2, m)$, hence which are of total degree $(3, m + n)$.

**Theorem 3.3.** For the first Betti numbers,

$$\beta_{1,(3,*)} \in \{1, \ldots, 5\}$$

and all possible values between one and five occur.

**Proof.** Theorems [4.4, 4.7, 4.8] treat the case where $W$ meets the Segre variety $\Sigma_{1,n}$ in a smooth conic $C$ or 3 noncollinear points $Z$. In these situations we may choose a basis so $W = \text{Span}\{g_0h_0, g_1h_1, g_2h_2\}$ with $g_i$ degree $(1, 0)$ and $h_i$ degree $(0, n)$. Theorems [4.7] and [4.8] give explicit resolutions for $I_W$ in these cases.

- When $W \cap \Sigma_{1,n} = C$ there is a single syzygy of bidegree $(3, n)$.
- $W \cap \Sigma_{1,n} = Z$ there are two syzygies of bidegrees $(3, n + \mu), (3, 2n - \mu)$.
- The remaining cases are covered in Theorem 3.4 below.

$\square$
Theorem 3.4. Suppose \( \{f_0, f_1, f_2\} = (s q_0 + t q_3, s q_1 + t q_4, s q_2 + t q_5) \), with the \( q_i \) linearly independent (so \( n \geq 5 \)). Then there are exactly five minimal first syzygies whose entries are quadratic in \( \{s, t\} \), obtained from the Hilbert-Burch matrix \( [13] \) \( N \) for the ideal \( Q = \langle q_0, \ldots, q_5 \rangle \). If the columns of \( N \) have degrees \( \{b_1, \ldots, b_5\} \), then the syzygies on \( I_W \) are of degree \( \{(3, 2n - b_1), \ldots, (3, 2n - b_5)\} \).

Proof. A syzygy with entries of bidegree \((2, m)\) satisfies

\[
(s^2 a_0 + s t a_1 + t^2 a_2) f_0 + (s^2 b_0 + s t b_1 + t^2 b_2) f_1 + (s^2 c_0 + s t c_1 + t^2 c_2) f_2 = 0,
\]

so using that \( f_i = s q_i + t q_{i+3} \), multiplying out and collecting the coefficients of the \( \{s^3, s^2 t, s t^2, t^3\} \) terms shows that \( \{a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2\} \) is in the kernel of the matrix

\[
M = \begin{bmatrix}
q_0 & q_1 & q_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
q_3 & q_4 & q_5 & q_0 & q_1 & q_2 & 0 & 0 & 0 \\
0 & 0 & 0 & q_3 & q_4 & q_5 & q_0 & q_1 & q_2 \\
0 & 0 & 0 & 0 & 0 & q_3 & q_4 & q_5 & 0
\end{bmatrix}.
\]

Since \( W \) is basepoint free, it follows that as a sheaf, the cokernel of

\[
\mathcal{O}^9_{\mathbb{P}^1}(-n) \xrightarrow{M} \mathcal{O}^4_{\mathbb{P}^1}
\]

is zero, hence the kernel of \( M \) is a rank five free module with first Chern class \( 9n \). If \( N \) denotes the Hilbert-Burch matrix of \( Q \) then \( N \) is a \( 6 \times 5 \) matrix whose maximal minors are \( Q \). Write \( n^i_j \) for the determinant of the submatrix of \( N \) obtained by omitting rows \( i, j \) and column \( k \) (convention-indexing starts with \( 0 \)), and consider the matrix \( K \)

\[
\begin{bmatrix}
-n_{12} & -n_{02} & -n_{01} & n_{15} - n_{04} & -n_{23} & n_{05} - n_{02} & n_{04} - n_{01} & -n_{13} & -n_{45} & -n_{45} & n_{04} \\
-n_{12} & -n_{02} & -n_{01} & n_{15} - n_{04} & -n_{23} & n_{05} - n_{02} & n_{04} - n_{01} & -n_{13} & -n_{45} & -n_{45} & n_{04} \\
-n_{12} & -n_{02} & -n_{01} & n_{15} - n_{04} & -n_{23} & n_{05} - n_{02} & n_{04} - n_{01} & -n_{13} & -n_{45} & -n_{45} & n_{04} \\
-n_{12} & -n_{02} & -n_{01} & n_{15} - n_{04} & -n_{23} & n_{05} - n_{02} & n_{04} - n_{01} & -n_{13} & -n_{45} & -n_{45} & n_{04} \\
-n_{12} & -n_{02} & -n_{01} & n_{15} - n_{04} & -n_{23} & n_{05} - n_{02} & n_{04} - n_{01} & -n_{13} & -n_{45} & -n_{45} & n_{04} \\
-n_{12} & -n_{02} & -n_{01} & n_{15} - n_{04} & -n_{23} & n_{05} - n_{02} & n_{04} - n_{01} & -n_{13} & -n_{45} & -n_{45} & n_{04}
\end{bmatrix}.
\]

Entries of the \( i^{th} \) row of \( K \) correspond to combinations of certain \( 4 \times 4 \) minors of the submatrix \( N_i \) obtained by deleting the \( i^{th} \) column of \( N \). A computation shows that \( M \cdot K^t = 0 \) and therefore

\[
0 \rightarrow \bigoplus_{i=1}^5 \mathcal{O}_{\mathbb{P}^1}(-2n + b_i) \xrightarrow{K^t} \mathcal{O}^9_{\mathbb{P}^1}(-n) \xrightarrow{M} \mathcal{O}^4_{\mathbb{P}^1} \rightarrow 0
\]

is exact, by the Buchsbaum-Eisenbud criterion.

\[ \square \]

Remark 3.5. If the \( q_i \) are not linearly independent, then the basepoint free assumption means they span a space of dimension 5 or 4, or fall under Theorems \[4.7, 4.8\]. When \( \dim \text{Span}\{q_0, \ldots, q_5\} \in \{4, 5\} \), the matrix \( N \) is \( 5 \times 4 \) or \( 4 \times 3 \) and the argument of Theorem \[3.4\] works with appropriate modifications, which we leave to the interested reader.
Corollary 3.6. The tautological syzygies constructed in §3.1 and the syzygies of Theorem 3.4 are independent.

Proof. The syzygies constructed in Theorem 3.4 cannot be in the span of the tautological syzygies of §3.1 because their degree in the \{u, v\} variables is lower than that of the tautological syzygies. On the other hand, the tautological syzygies cannot be in the span of the syzygies of Theorem 3.4 as the tautological syzygies have lower degree in the \{s, t\} variables. □

3.3. Computing first Betti numbers, the general setting. The Koszul homology of the module \(H_1\) is computed from the complex \(\mathcal{M}_* := \mathcal{M}_*((s, t, u, v), H_1)\):

\[(3.1) \quad \mathcal{M}_*: 0 \rightarrow H_1(-4) \xrightarrow{\phi_1} H_1(-3)^3 \xrightarrow{\phi_2} H_1(-2)^6 \xrightarrow{\phi_3} H_1^4 \xrightarrow{\phi_4} H_1 \rightarrow 0.\]

The bigraded complex \(\mathcal{M}_*\) has the following shape:

\[
\begin{array}{ccccccc}
0 & \rightarrow & H_1(-2, -2) & \rightarrow & H_1(-2, -1)^2 & \oplus & H_1(-2, 0) \\
& & \oplus & \rightarrow & H_1(-1, -1)^4 & \oplus & H_1(-1, 0)^2 \\
& & & \oplus & \rightarrow & H_1(0, 0) & \rightarrow \end{array}
\]

We denote by \(H(\mathcal{M})_i\) the \(i\)-th homology module.

Theorem 3.7. For any \(a \in \mathbb{Z}^2\), we have the equality

\[(3.2) \quad \beta_{1,a} = \dim_{\mathbb{K}}(H(\mathcal{M})_{1,a}) - \dim_{\mathbb{K}}(H(\mathcal{M})_{2,a}).\]

Proof. First, observe that \(\beta_{1,a} \neq 0\) iff \((H_1)_a\) is not spanned by the images of \((H_1)_{a-e_1}\) and \((H_1)_{a-e_2}\). Hence \(\beta_{1,a} = \dim_{\mathbb{K}}(H(\mathcal{M})_{0,a})\). As \(\dim_{\mathbb{K}}(H(\mathcal{M})_{0,a})\) is the alternating sum of \(\dim_{\mathbb{K}}(H(\mathcal{M})_{i,a})\) for \(i > 0\), it suffices to show that

\(H(\mathcal{M})_3 = 0 = H(\mathcal{M})_4.\)

Let \(K_{i,m} = K_i((s, t, u, v); R)\) and \(K_{i,f} = K_i(f; R)\) denote the Koszul complex of \((s, t, u, v)\) and the Koszul complex of \(f\) on \(R\) respectively. We consider the two spectral sequences coming from the double complex \(C_{i,j} = K_{i,m} \otimes_R K_{i,f}\).

\[h E_{i,j}^1 = H_i(K_{*,m}) \otimes_R K_{i,f}\]

\[v E_{i,j}^1 = K_{i,m} \otimes_R H_j(K_{*,f})\]

Since \(K_{*,m}\) is acyclic, \(H_i(K_{*,m}) = 0\) iff \(i \neq 0\), and \(H_0(K_{*,m}) = \mathbb{K}\). Thus,

\[h E_{i,j}^1 = \begin{cases} k^0 & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}\]

On the vertical spectral sequence, one has:

\[v E_{i,j}^1 = \begin{cases} K_{i,m} \otimes_R H_0(K_{*,f}) & \text{if } j = 0 \\ K_{i,m} \otimes_R H_1(K_{*,f}) = \mathcal{M} & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}\]

Comparing the abutment of both spectral sequences, we have \(v E_{2,1}^2 = H(\mathcal{M})_4 = 0\), and \(v E_{3,1}^2 = H(\mathcal{M})_3 = 0\) and \(v E_{4,0}^2 = H_4(K_{*,m} \otimes_R H_0(K_{*,f})) = 0.\) □
Example 3.8. The homologies $H(M)_i, i = 1, 2$ in (3.2) might be both nonzero. For instance, let $d = (1, 6)$. Below we list all nonzero, non-Koszul degree first Betti numbers for generic $f$:

$$
(3.3) \quad \beta_{1,(3,10)} = 1, \beta_{1,(3,11)} = 4, \beta_{1,(4,10)} = 3, \beta_{1,(6,9)} = 2, \beta_{1,(1,18)} = 1.
$$

The sum of all these numbers equals 11. On the other side, the sum of the dimensions of all $\dim_K(H(M)_{1,a})$ equals 18 and the sum of the dimensions of all $\dim_K(H(M)_{2,a})$ equals 7. Indeed, $11 = 18 - 7$. We plot in Figure 1 below the bidegrees $a$ with nonzero $\beta_{1,a}$ in (3.3), together with the curve $n_{(1,6)}(a) = 1$.

![Figure 1. The generic case with $d = (1, 6)$](image)

The values of $n_{(1,6)}(a)$ are given below, where the column on the left represents $a_1 = 0$, and the row on the bottom $a_2 = 0$.

| 0 3 0 0 0 0 0 0 0 0 0 | 0 3 0 0 0 0 0 0 0 0 0 |
| 0 2 0 0 0 0 0 0 0 0 0 | 0 2 0 0 0 0 0 0 0 0 0 |
| 0 1 0 0 0 0 0 0 0 0 0 | 0 1 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 1 2 3 4 5 6 7 8 | 0 0 0 1 2 3 4 5 6 7 8 |
| 0 0 0 2 4 6 8 10 12 14 16 | 0 0 0 2 4 6 8 10 12 14 16 |
| 0 0 0 3 6 9 12 15 18 21 24 | 0 0 0 3 6 9 12 15 18 21 24 |
| 0 0 0 4 8 12 16 20 24 28 32 | 0 0 0 4 8 12 16 20 24 28 32 |
| 0 0 0 5 10 15 20 25 30 35 40 | 0 0 0 5 10 15 20 25 30 35 40 |
| 0 0 0 6 12 18 24 30 36 42 48 | 0 0 0 6 12 18 24 30 36 42 48 |
| 0 0 0 1 5 9 13 17 21 25 29 | 0 0 0 1 5 9 13 17 21 25 29 |
| 0 0 0 0 0 2 4 6 8 10 12 | 0 0 0 0 0 2 4 6 8 10 12 |
| 0 0 0 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 |
The values of the Betti numbers in (3.3) can be deduced from Theorem 3.7 and Lemma 2.9. A necessary condition is that $n(1,6)(a) \geq 1$. For instance, we have that

$$n(1,6)(3,10) = 1, n(1,6)(3,11) = 6, \text{ and } n(1,6)(4,10) = 5,$$

so

$$\beta_{1,(3,11)} = n(1,6)(3,11) - 2n(1,6)(3,10) = 6 - 2 = 4.$$  

Similarly,

$$\beta_{1,(4,10)} = n(1,6)(4,10) - 2n(1,6)(3,10) = 5 - 2 = 3.$$  

On the other side,

$$\beta_{1,(6,9)} = n(2,(6,9)) = 2, \text{ and } \beta_{1,(1,18)} = n(2,(1,18)) = 1.$$  

For a final example, $\beta_{1,(5,10)} = 0$ because

$$n(2,(5,10)) = 9 = 3n(2,(3,10)) + 2(n(2,(4,10)) - n(2,(3,10))) = 2n(2,(4,10)) - n(2,(3,10)),$$

and $\beta_{1,(1,19)} = 0$ because $n(1,6)(1,19) < 2n(1,6)(1,18)$.

**Example 3.9.** Consider the bidegree $d = (1,42)$ and $f$ generic. We list all bidegrees $a$ with nonzero, non-Koszul first Betti number:

$$(7,67), (6,68), (9,66), (8,67), (7,68), (6,69), (5,70), (10,66), (4,72),$$

$$(12,65), (3,75), (3,76), (17,64), (18,64), (33,63), (1,126).$$

This is the ordered list of the corresponding Betti numbers: 2, 3, 5, 8, 5, 8, 9, 3, 7, 6, 2, 3, 3, 1, 2, 1. In this example, there are minimal generators of the syzygy module in degrees $a$, $a - (1,0)$ and $a - (0,1)$, for $a = (7,68)$. We focus on the bidegrees $(7,68), (6,68), (7,67)$, marked with solid diamonds in Figure 2. Note that we also show a few other bidegrees but we do not display all bidegrees with nonzero, non-Koszul first Betti number in the list above. All these bidegrees must satisfy that $n(1,42)(a) > 0$. We also plot the curve $n(1,42)(a) = 1$. Again, the values of the Betti numbers can be deduced from Theorem 3.7 and Lemma 2.9.

**Figure 2.** The generic case with $d = (1,42)$
4. Factorization of sections of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, n)$ and the Segre variety $\Sigma_{1,n}$

Recall the Segre variety $\Sigma_{r,s}$ is the image of the regular map

$$\mathbb{P}^r \times \mathbb{P}^s \xrightarrow{\sigma_{r,s}} \mathbb{P}^{r+s}$$

given by multiplication

$$(x_0 : \ldots : x_r), (y_0 : \ldots : y_s) \mapsto (x_0y_0 : \ldots : x_0y_s : x_1y_0 : \ldots : x_ry_s).$$

In general one has the following diagram

$$\xymatrix{ \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,i))) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(n-i))) \ar[r] \ar[dr]_{\psi_i} & \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,n))) \ar[d]_{\pi} \\ & \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,n))) }$$

The composition of the Segre map

$$\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1))) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, n-1))) = \mathbb{P}^3 \times \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{4n-1},$$

with the projection $\pi$ onto $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,n)))$ is given with respect to the basis $\{su^n, \ldots, sv^n, t u^n, \ldots, t u^n\}$ for $H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,n))$ by

$$(a_0 : \ldots : a_3) \times (b_0 : \ldots : b_{n-1}) \mapsto (a_0b_0 : a_0 b_1 + a_1 b_0 : a_0 b_2 + a_1 b_1 : \ldots : a_0 b_{n-1} + a_1 b_{n-2} : a_1 b_{n-1} : a_2 b_0 : a_2 b_1 : a_2 b_2 + a_3 b_1 : \ldots : a_2 b_{n-1} + a_3 b_{n-2} : a_3 b_{n-1}).$$

For example, when $n = 2$, the image of $\psi_1$ is a quartic hypersurface

$$Q = V(x_2^2 x_3^2 - x_1 x_2 x_3 x_4 + x_0 x_2 x_4^2 + x_1^2 x_3 x_5 - 2x_0 x_2 x_3 x_5 - x_0 x_1 x_4 x_5 + x_0^2 x_5^2).$$

**Definition 4.1.** The image of the composite map $\psi_i$ is $\Sigma'_{2i+1,n-i}$.

Therefore for $i = 0$ we have $\Sigma_{1,n} = \Sigma'_{1,n}$, but for $i \geq 1$ the variety $\Sigma'_{2i+1,n-i}$ is a linear projection of $\Sigma_{2i+1,n-i}$, with $\text{codim}(\Sigma'_{2i+1,n-i}) = n-i$. In particular $\Sigma'_{2i+1,n-i}$ is a hypersurface in $\Sigma_{2i+1,n-i}$, and

$$\Sigma_{1,n} = \Sigma'_{1,n} \subseteq \Sigma'_{3,n-1} \subseteq \Sigma'_{5,n-2} \subseteq \ldots \subseteq \Sigma'_{2n-3,2} \subseteq \Sigma'_{2n-1,1} \subseteq \mathbb{P}^{2n+1}.$$
4.1. Intersection with $\Sigma_{1,n}$. In the case where $n = 2$, [8] shows that the only way in which $W$ can meet $\Sigma_{1,2}$ in a curve is if the curve is a smooth conic. This phenomenon persists, but we need a bit more machinery.

**Theorem** (Burau-Zeuge [5]) If $L$ is a linear space cutting each $n$-dimensional ruling of $\Sigma_{1,n}$ in at most a single point, then $L \cap \Sigma_{1,n}$ is a rational normal curve.

**Lemma 4.3.** If $W$ meets $\Sigma_{1,n}$ in a curve, then it must be a smooth conic.

**Proof.** First, suppose $W$ contains a $\mathbb{P}^1$ fiber of $\Sigma_{1,n}$, so that $W$ has basis $\{l_1s, l_2s, q\}$ with $l_i$ corresponding to points on the $\mathbb{P}^1$. Then $\mathcal{V}(s, q) \neq \emptyset$ and $W$ is not basepoint free. Next, since $W$ is linear, it cannot meet a $\mathbb{P}^n$ fiber $F$ of $\Sigma_{1,n}$ in more than two noncollinear points, for then it would be contained in $F$ and hence violate Lemma 4.2. If $W$ meets $F$ in two points, since $W$ and $F$ are both linear, $W \cap F$ is a line $L$, and if $F$ is the fiber over the point $l$ of $L$, then $W = \{al, bl, c\}$ and since $\mathcal{V}(l, c)$ is nonempty on $\mathbb{P}^1 \times \mathbb{P}^1$, $W$ would have basepoints. In particular, $W$ can meet each $\mathbb{P}^n$ fiber in at most a point, so by the result of Burau-Zeuge, $W \cap \Sigma_{1,n}$ is a rational normal curve. As $W \simeq \mathbb{P}^2$, the curve must be a smooth conic. \qed

**Theorem 4.4.** $W \cap \Sigma_{1,n}$ is a smooth conic iff $I_W$ has a bidegree $(3, n)$ first syzygy.

**Proof.** Suppose $I_W$ has a minimal first syzygy of bidegree $(3, n)$,

$$a(s, t) \cdot f_0 + b(s, t) \cdot f_1 + c(s, t) \cdot f_2 = 0.$$ 

If $\langle a(s, t), b(s, t), c(s, t) \rangle \neq \langle s^2, st, t^2 \rangle$, then it must be generated by two bidegree $(2, 0)$ quadrics $\{a(s, t), b(s, t)\}$ with no common factor. Changing basis for $I_W$, the syzygy involves only $f_0, f_1$, which implies $f_0, f_1 = -b(s, t)g, a(s, t)g$ for some $g$. This is impossible by degree considerations, so after a change of basis for $W$, we may assume the $f_i$ satisfy

$$s^2 \cdot f_0 + st \cdot f_1 + t^2 \cdot f_2 = 0.$$ 

Now we switch perspective, and consider $[f_0, f_1, f_2]$ as a syzygy on $[s^2, st, t^2]$. Since the syzygies on the latter space are generated by the columns of

$$\begin{bmatrix} t & 0 \\ -s & -t \\ 0 & s \end{bmatrix},$$

we have

$$f_0 = ta_0$$

$$f_1 = sa_0 + ta_1$$

$$f_2 = sa_1,$$

with the $a_i \in k[u, v]$. In particular, $\{a_0, a_1\}$ are a basepoint free pencil of $H^0(\mathcal{O}_{\mathbb{P}^1}(n))$, and we may parameterize as in the proof of Theorem 4.1 of [8] so that $W \cap \Sigma_{1,n}$ is a smooth conic. On the other hand, if $W \cap \Sigma_{1,n}$ is a smooth conic, the proof follows as in Theorem 4.1 of [8]. \qed

**Example 4.5.** If $W = \{su^n, tv^n, sv^n + tu^n\}$, then since $\Sigma_{1,n}$ is given by the $2 \times 2$ minors of the matrix

$$\begin{bmatrix} x_0 & \cdots & x_n \\ x_{n+1} & \cdots & x_{2n+1} \end{bmatrix},$$
with \(x_i = su^{i-1}v^i\) for \(i \in \{0, \ldots, n\}\) and \(tu^{i-1}v^i\) for \(i \in \{n+1, \ldots, 2n+2\}\), so (dualizing) \(W = V(x_1, \ldots, x_n, x_{n+2}, \ldots, x_{2n+1}, x_n - x_{n+1}) \subseteq \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, n)))^\vee\), and using coordinates \(\{x_0, x_n, x_{2n+1}\}\) for \(W\), \(W \cap \Sigma_{1,n} = W \cap V(x_0x_{2n+1} - x_n^2)\).

**Remark 4.6.** [Effective criterion] Notice that given \(I_W\), Theorem 4.4 gives an effective way to understand the geometry of \(W\). Combined with the following result we obtain a complete description of the minimal free resolution of \(I_W\) just by computing whether or not \(I_W\) has a bidegree \((3, n)\) first syzygy.

### 4.2. Minimal free resolutions determined by the geometry of \(W \cap \Sigma_{1,n}\).

We now examine situations where \(W \cap \Sigma_{1,n}\) has special geometry.

**Theorem 4.7.** \(W \cap \Sigma_{1,n}\) is a smooth conic iff \(I_W\) has bigraded Betti numbers:

\[
\begin{align*}
0 \leftarrow I_W \leftarrow (-1, -n)^3 & \overset{\partial_1}{\leftarrow} (-2, -2n)^3 & \overset{\partial_2}{\leftarrow} (-3, -2n)^2 & \overset{\partial_3}{\leftarrow} (-3, -3n) \leftarrow 0 \\
& \oplus & \oplus & \oplus \\
& (\cdot, -3n) & (\cdot, -2n) & (\cdot, -n)
\end{align*}
\]

**Proof.** In fact, we will show more, exhibiting the differentials in the minimal free resolution. By Theorem 4.4 we may choose the \(f_i\) so that

\[
\begin{align*}
f_0 &= ta_0 \\
f_1 &= sa_0 + ta_1 \\
f_2 &= sa_1,
\end{align*}
\]

Then the syzygy described by Lemma 3.1 is \((a_1^2, -a_0a_1, a_0^2)\) and we also have the bidegree \((2, 0)\) syzygy \((s^2, -st, t^2)\). Consider the \(\partial_i\) below.

\[
\partial_1 = \begin{bmatrix}
a_1^2 & f_1 & f_2 & 0 & s^2 \\
-a_0a_1 & -f_0 & 0 & f_2 & -st \\
a_0^2 & 0 & -f_0 & -f_1 & t^2
\end{bmatrix}
\]

\[
\partial_2 = \begin{bmatrix}
t & s & 0 & 0 \\
-a_1 & 0 & 0 & -s \\
a_0 & -a_1 & -s & t \\
0 & a_0 & t & 0 \\
0 & 0 & a_1 & a_0
\end{bmatrix}
\]

\[
\partial_3 = \begin{bmatrix}
s \\
-t \\
a_0 \\
-a_1
\end{bmatrix}
\]

A check shows \(\partial_i \partial_{i+1} = 0\), exactness follows by Buchsbaum-Eisenbud [E].

**Example 1.1** is a consequence of Theorem 4.7, because for \(d = (1, 1)\), \(W\) is basepoint free iff it meets \(\Sigma_{1,1}\) in a smooth conic. When \(W \cap \Sigma_{1,n}\) contains three distinct noncollinear points, the resolution of \(I_W\) is also completely determined.
**Theorem 4.8.** If $|W \cap \Sigma_{1,n}|$ is finite and contains three noncollinear points, then the bigraded Betti numbers of $I_W$ are

\[
\begin{array}{cccc}
(-1, -3n) & \oplus & (-2, -2n)^3 & \oplus \\
(-3, -n - \mu) & \oplus & (-3, -2n)^3 & \oplus \\
\end{array}
\]

with $0 < \mu \leq \lfloor n/2 \rfloor$.

**Proof.** As in the previous theorem, we will describe the differentials in the minimal free resolution. Since $W \cap \Sigma_{1,n}$ contains three noncollinear points, we may choose a basis so that $W = \{l_0g_0,l_1g_1,l_2g_2\}$ with the $l_i$ of bidegree $(1,0)$ and the $g_i$ of bidegree $(0,n)$. If the $g_i$ are not linearly independent, then changing basis we see that there are constants $a,b,c,d$ with

\[
W = (sg_0, tg_1, (as + bt) \cdot (c_0g_0 + bg_1)) = (sg_0, tg_1, acsg_0 + bdtg_1),
\]

so $(bdt^2, acs^2, -st)$ is a bidegree $(2,0)$ syzygy on $I_W$ and Theorem 4.4 applies.

So we may assume $\{g_0, g_1, g_2\}$ are linearly independent; suppose the Hilbert-Burch matrix for $\{g_1, g_2, g_3\}$ has columns of degree $a$ and $n - \mu$. In this case, in addition to the three Koszul syzygies and the syzygy of Lemma 3.1, the Hilbert-Burch syzygies can be lifted: if $(b_0, b_1, b_2)$ is a syzygy of degree $a$ on the $g_i$, then $(l_1l_2b_0, l_0l_2b_1, l_0l_1b_2)$ is a syzygy of bidegree $(3, n + \mu)$ on $I_W$, and similarly for the syzygy of degree $n - \mu$. Note that $g_0 = b_1c_2 - c_1b_2$, $g_1 = c_0b_2 - b_0c_2$, $g_2 = b_0c_1 - c_0b_1$. A priori, these need not be minimal, but by constructing the remaining differentials and applying the Buchsbaum-Eisenbud criterion, we will see that they are. Changing basis, we may assume $W$ has basis $\{sg_0, tg_1, (as + bt)g_2\}$, hence the syzygy of Lemma 3.1 takes the form $(-ag_1g_2, -bg_0g_2, sg_0g_1)$, and

\[
\partial_1 = \begin{bmatrix}
-ag_1g_2 & tg_1 & (as + bt)g_2 & 0 & 0 & t(as + bt)b_0 & t(as + bt)c_0 \\
-bg_0g_2 & -sg_0 & 0 & (as + bt)g_2 & s(as + bt)b_1 & s(as + bt)c_1 \\
g_0g_1 & 0 & -sg_0 & -tg_1 & sbt_2 & stc_2
\end{bmatrix}
\]

A check shows that the two matrices below satisfy $\partial_2\partial_1 = 0$ and $\partial_3\partial_2 = 0$:

\[
\partial_2 = \begin{bmatrix}
t & s & 0 & 0 & 0 \\
ag_2 & -bg_2 & as + bt & 0 & 0 \\
g_0 & g_1 & 0 & t & 0 \\
g_0 & 0 & 0 & c_2 & c_1 \\
g_0 & 0 & 0 & -b_2 & -b_1 & -b_0
\end{bmatrix}
\]

\[
\partial_3 = \begin{bmatrix}
s & 0 & 0 & 0 & 0 \\
t & -s & 0 & 0 & 0 \\
g_2 & 0 & 0 & c_1 & c_0 \\
g_1 & 0 & 0 & -b_2 & -b_1 & -b_0
\end{bmatrix}
\]

Applying the Buchsbaum-Eisenbud criterion shows the complex is indeed exact. Since $n + \mu \leq 2n - \mu$, then it follows that $\mu \leq \lfloor n/2 \rfloor$.

**Remark 4.9.** If $n = 2$ then $|W \cap \Sigma_{1,n}|$ is generically finite and generically contains three noncollinear points. For $n \geq 3$ this is not the case, so this closed condition is very restrictive. Moreover, for $n = 2, 3$, from Theorem 4.8 one has that $\mu = 1$. 

\[\square\]
5. Higher Segre Varieties

Since $\Sigma_{2i+1,n-i}'$ has codimension $n-i$, unless $i = n-1$ or $n-2$, the intersection $W \cap \Sigma_{2i+1,n-i}'$ is generically empty. Theorems 4.7 and 4.8 illustrate the principle that when $W \cap \Sigma_{2i+1,n-i}' \neq \emptyset$ and $i \leq n-3$, special behavior can occur. The next theorem makes this explicit when $|W \cap \Sigma_{2i+1,n-i}'|$ is finite and contains at least three noncollinear points.

**Theorem 5.1.** Suppose $W$ has basis $\{g_0h_0, g_1h_1, g_2h_2\}$ with $g_j \in H^0(\mathcal{O}_{P^1 \times P^1}(1,i))$ and $h_j \in H^0(\mathcal{O}_{P^1}(n-i))$, with $0 \leq i \leq n-1, 3 \leq n$. Then

1. $\{h_0, h_1, h_2\}$ and $\{g_0, g_1, g_2\}$ are basepoint free.
2. A syzygy $\{a_0, a_1, a_2\}$ on the $h_i$ lifts to a syzygy (possibly non-minimal)
   \[ \{g_1g_2a_0, g_0g_2a_1, g_0g_1a_2\} \]
   on $I_W$, and similarly for a syzygy on the $g_i$.
3. If $\{h_0, h_1, h_2\}$ is a pencil, then there is a bidegree $(3,2i+n)$ syzygy on $I_W$.
4. If $\{h_0, h_1, h_2\}$ is not a pencil, then it has a Hilbert-Burch matrix with columns of degrees $\{n-i-\mu, \mu\}$ in the $\{s,t\}$ variables. These give rise to syzygies of type (2) above of bidegree $(3, n+2i+\mu)$ and $(3, 2n+i-\mu)$.

**Proof.** For (1), if the $g_i$ or $h_i$ are not basepoint free, then neither is $W$, and for (2), the result is immediate. For (3), if the $h_i$ are a pencil, then $I_W = \langle g_0h_0, g_1h_1, g_2(a_0h_0 + b_0h_1) \rangle$ for constants $a, b$, and so $(ag_1g_2, bg_0g_2, -g_0g_1)$ is the desired syzygy, and (4) follows by applying (2) to the Hilbert-Burch syzygies.

The previous theorem deals with the situation where there is a basis for $W$ where all three elements factor in the same way. Even if only one or two elements factor into components of degree $(1,1)$ and $(0,n-1)$. The next example shows that the converse to Conjecture 5.2 need not hold.

**Example 5.3.** Suppose $a, b, c$ are basepoint free elements of bidegree $(1, 1)$, and $d, e, f$ are generic elements of bidegree $(0,4)$, with $I_W = \langle ad, be, cf \rangle$. A computation shows that the Betti numbers of $I_W$ are as in the diagram below.
Lemma 3.1 explains the $(1, 15)$ syzygy, and there are three Koszul syzygies. By construction, $W \cap \Sigma_{3,4}$ consists of three points, and the bigraded Betti numbers for this example agree with the generic case; Theorem 3.4 explains the five $(3, 9)$ syzygies, whereas Theorem 5.1 only accounts for two of them.

**Theorem 5.4.** Suppose $W = \text{Span}\{g_0h_0, g_1h_1, g_2h_2\}$, with $g_i$ of degree $(1, 1)$ and the $h_i$ a pencil of degree $(0, n - 1)$, so that

$$I_W = \langle g_0h_0, g_1h_1, g_2(ah_0 + bh_1) \rangle.$$ 

If $W \cap \Sigma_{1,n}$ is empty and $W \cap \Sigma'_{3,n-1}$ contains three noncollinear points, then there are minimal first syzygies of degrees

$$\{(−1, −3n), (−2, −2n)^3, (−3, −n − 2), (−3, −2n + 1)^2, (−6, −2n + 2)\}$$

**Proof.** The syzygies of degree $(1, 3n)$ and $(2, 2n)$ are tautological. The syzygy of degree $(3, n + 2)$ can be explained by Theorem 5.1 (3), but it is more enlightening to treat the syzygies of degree $(3, *)$ as a group. Write $f$ as in Theorem 3.4

$$\{sq_0 + tq_1, sq_2 + tq_3, sq_4 + tq_5\}.$$ 

Using that the $h_i$ are a pencil and expanding, we find that

$$\text{Span}\{q_0, \ldots, q_5\} = \text{Span}\{uh_0, vh_0, uh_1, vh_1\}.$$ 

Since $\{h_0, h_1\}$ are basepoint free, they are a complete intersection, so the Hilbert-Burch matrix for $\{uh_0, vh_0, uh_1, vh_1\}$ is

$$\begin{pmatrix} v & 0 & p_0 \\ -u & 0 & p_1 \\ 0 & v & p_2 \\ 0 & -u & p_3 \end{pmatrix},$$

with the $p_i$ of degree $n - 2$. In particular, the columns have degrees $\{b_1, b_2, b_3\} = \{1, 1, n − 2\}$, which by Theorem 5.4 yields syzygies in degrees

$$\{(3, 2n − 1), (3, 2n − 1), (3, n + 2)\}.$$ 

By Corollary 3.6, the seven syzygies constructed so far are independent. We next prove there exists a unique first syzygy of degree $(6, 2n − 2)$. Let $(s_0, s_1, s_2)$ be a syzygy on $I_W$, and rewrite it as below

$$(5.1) \quad s_0g_0h_0 + s_1g_1h_1 + s_2g_2(ah_0 + bh_1) = 0 = (s_0g_0 + as_2g_2)h_0 + (s_1g_1 + bs_2g_2)h_1.$$ 

So $(s_0g_0 + as_2g_2, s_1g_1 + bs_2g_2)$ is a syzygy on the complete intersection $(h_0, h_1)$, which implies that

$$S = \begin{pmatrix} s_0g_0 + as_2g_2 \\ s_1g_1 + bs_2g_2 \end{pmatrix},$$

is in the image of the Koszul syzygy $K$ on $\{h_0, h_1\}$:

$$K = \begin{pmatrix} h_1 \\ -h_0 \end{pmatrix}. $$

One possibility is $S = 0$, which leads to a syzygy of Theorem 5.1 type 3, of bidegree $(3, n + 2)$, which we have accounted for, so we may suppose $S$ is nonzero. This means $S = p \cdot K$ for some polynomial $p$. Since the $h_i$ are degree $(0, n − 1)$, the lowest possible degree for the $s_i$ in the $(0, 1)$ variables is $n − 2$. We show that in degree $(a, 2n − 2)$, there is a unique minimal syzygy of degree $(6, 2n − 2)$ which is not a multiple of the syzygy of degree $(3, n + 2)$.
To see this, we write out Equation 5.1 collecting the coefficients of
\[ \{u^{2n-2}, vu^{2n-1}, \ldots, v^{2n-2}\} \].

Each \( s_i \) is of degree \( n-2 \) in the \((0, 1)\) variables, so there are \( 3(n-1) \) columns, and we obtain a \( 2n-1 \times 3n-3 \) matrix
\[ O^{3n-3}_{P^1}(-1) \xrightarrow{\psi} O^{2n-1}_{P^1}. \]
The coefficients in the \((1, 0)\) variables of the \( s_i \) (written as polynomials in the \((0, 1)\) variables) correspond to elements of the kernel of this matrix. The nonzero entries of \( \psi \) come from the six linear forms in the \((1, 0)\) variables, obtained by writing the \((1, 1)\) components \( g_i \) as
\[ g_i = a_i u + b_i v, \text{ with } \{a_i, b_i\} \in \mathbb{K}[s, t]. \]

Since \( W \) is basepoint free and \( W \cap \Sigma_{1,n} = \emptyset \), the cokernel of \( \psi \) is zero, so \( \ker(\psi) \) has rank \( n-2 \), with first Chern class \( 3 - 3n \). The key is that the unique syzygy of degree \((3, n+2)\) generates \( n-3 \) independent syzygies of degree \((3, 2n-2)\), which follows from the computation
\[ h^0(O_{P^1}((2n-2) - (n+2))) = h^0(O_{P^1}(-n - 4)) = n - 3. \]
Since \( \ker(\psi) \) has rank \( n-2 \), this means there is a single additional element in the kernel, which has first Chern class
\[ 3(n-3) - (3n - 3) = -6, \]
yielding a unique first syzygy of degree \((6, 2n-2)\). \( \square \)

**Remark 5.5.** If \( W \cap \Sigma_{1,n} \neq \emptyset \), some of the \( g_i \) will factor. If all three factor, we are in the situation of Theorem 4.8. If only one or two factor, then the first syzygy of degree \((6, 2n-2)\) changes to a syzygy of degree \((4, 2n-2)\) if \( |W \cap \Sigma_{1,n}| = 2 \), and to a syzygy of degree \((5, 2n-2)\) if \( |W \cap \Sigma_{1,n}| = 1 \).

**Example 5.6.** With the hypotheses of Theorem 5.4, computations suggest that the bigraded Betti numbers are

\[
\begin{array}{c}
\begin{array}{cccccccc}
0 & \leftarrow I_W & \leftarrow (-1,-n)^3 & \oplus & (-2,-2n)^3 & \oplus & (-2,-3n)^2 & \oplus & (-3,-2n-2) & \oplus & (-6,-2n+1)^2 & \oplus & (-6,-2n+2) & \leftarrow 0 \\
& & \delta_1 & \oplus & \delta_2 & \oplus & \delta_3 & \oplus & \delta_4 & \oplus & \delta_5 & \oplus & \delta_6 & \\
& & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \\
\end{array}
\end{array}
\]

Note that the numbers
\[ \beta_{2,\{*, -3n\}} \text{ and } \beta_{3,\{*, -3n\}} \]
are explained by Proposition 3.2; one way to prove the diagram above is the correct betti table would be to determine explicitly the \((6, 2n-2)\) syzygy, and then write down the differentials and apply the Buchsbaum-Eisenbud criterion for exactness.
5.2. Connections to the Hurwitz discriminant and Sylvester map. We consider the case \( d = (1, n) \) with \( n = 5 \) at the particular bidegree \((3, 8)\). The aim is to point out the tip of the iceberg of the relation between non generic rank behavior in the matrix of \( \phi_1 \) from Section 2.3 and the different intersections of \( W \) with the variety of elementary tensors associated to all the decompositions of \((1, n) = (1, i) + (0, n - i)\).

We set \( n = 5 \). According to Theorem 3.4 when the \( f_i \) are generic there are syzygies in degree \((3, m)\) for \( m \geq 2 \cdot 5 - 0 - 1 = 9 \) (\( \kappa = 0 \) for \( n = 5 \)). But in this bidegree, we can read in the table of Section 2.3 that the matrix of \( \phi_1 \) has always full rank in the basepoint free case.

We then move one step to bidegree \((3, 8) = (3, 2 \cdot 5 - 2)\) and we try to detect the existence of nontrivial syzygies there. In this bidegree \( \phi_1 : R_{(0, 5)} \to R^3_{(1, 0)} \) so we get a \( 6 \times 6 \) matrix \( M \), constructed as follows, according to Example 2.6. Write

\[
\ell = s \left( \sum_{j=0}^{5} p_{\ell j} u^j v^{5-j} \right) + t \left( \sum_{j=0}^{5} q_{\ell j} u^j v^{5-j} \right), \quad \text{for } \ell = 0, 1, 2.
\]

Then

\[
M = \begin{pmatrix}
p_{00} & \cdots & p_{05} \\
p_{10} & \cdots & p_{15} \\
p_{20} & \cdots & p_{25} \\
q_{00} & \cdots & q_{05} \\
q_{10} & \cdots & q_{15} \\
q_{20} & \cdots & q_{25}
\end{pmatrix}.
\]

Let \( S_{(3,8)} \) denote the hypersurface \( V(\det(M)) \). Consider the 9-dimensional variety \( \Sigma'_{7,2} \subset \mathbb{P}^{11} \) of bihomogeneous polynomials factoring as the product of a degree \((1,3)\) polynomial and a degree \((0,2)\) polynomial. We expect \( \dim(\Sigma'_{7,2} \cap W) = 0 \). It would be interesting to relate the geometry of \( S_{(3,8)} \) to special features of this intersection (for example, finitely many points but with special properties, or a curve). Note that \( W \) lives in the Grassmanian of planes in \( \mathbb{P}^{11} \) and there is a polynomial, called the Hurwitz discriminant by Sturmfels [27], which vanishes whenever the intersection \( \Sigma'_{7,2} \cap W \) does not consist of \( \deg(\Sigma'_{7,2}) \) many points.

5.3. Concluding remarks. We close with a number of questions:

1. What happens when there are many “low degree” first syzygies? As shown in [11] and [25], linear first syzygies impose strong constraints.
2. \( W \) is a point of \( G(2, 2n + 1) \). How does the Schubert cell structure impact the free resolution of \( I_W \)?
3. What happens in other bidegrees? For other toric surfaces?
4. Are there special cases such as in Theorem 4.7, Theorem 4.8, Theorem 5.4 which are of interest to the geometric modeling community?
5. For computation of toric cohomology, it is sufficient to have a complex with homology supported in \( B \), rather than an exact sequence. This is studied by Berkesch-Erman-Smith in [2], and is a very active area of research.

Acknowledgments. Most of this paper was written while the third author was visiting Universidad de Buenos Aires on a Fulbright grant, and he thanks the Fulbright foundation for support and his hosts for providing a wonderful visit. All computations were done using Macaulay2 [17].
References

1. A. Aramova, K. Crona, and E. De Negri, Bigeneric initial ideals, diagonal subalgebras and bigraded Hilbert functions, *J. Pure Appl. Algebra* **150** (2000), 215–235.
2. C. Berkesch, D. Erman, G.G. Smith, Virtual resolutions for a product of projective spaces, *Algebraic Geometry*, **7** (2020), 460–481.
3. N. Botbol, A. Dickenstein, M. Dohm, Matrix representations for toric parametrizations, *Comput. Aided Geom. D.* **26** (2009), 757–771.
4. D. Buchsbaum, D. Eisenbud, What makes a complex exact?, *J. Algebra* **25** (1973), 259–268.
5. W. Burau, J. Zeuge, Über den Zusammenhang zwischen den Partitionen einer natürlichen Zahl und den linearen Schnitten der einfachsten Segremannigfaltigkeiten. *J. Reine Angew. Math* **274-75** (1975), 104-111.
6. L. Busé, M. Chardin, Implicitizing rational hypersurfaces using approximation complexes, *J. Symb. Comput.* **40** (2005), 1150–1168.
7. D. Cox, Curves, surfaces and syzygies, in “Topics in algebraic geometry and geometric modeling”, *Contemp. Math.* **334** (2003) 131–150.
8. D. Cox, A. Dickenstein, H. Schenck, A case study in bigraded commutative algebra, in “Syzygies and Hilbert Functions”, edited by Irena Peeva, Lecture notes in Pure and Applied Mathematics 254, (2007), 67–112.
9. D. Cox, R. Goldman, M. Zhang, On the validity of implicitization by moving quadrics for rational surfaces with no basepoints, *J. Symb. Comput.* **29** (2000), 419–440.
10. W.L.F. Degen, The types of rational (2,1)-Bézier surfaces. *Comput. Aided Geom. D.* **16** (1999), 639–648.
11. E. Duarte, H. Schenck, Tensor product surfaces and linear syzygies, *P. Am. Math. Soc.*, **144** (2016), 65–72.
12. D. Eisenbud, *Commutative Algebra with a view towards Algebraic Geometry*, Springer-Verlag, Berlin-Heidelberg-New York, 1995.
13. D. Eisenbud, *Geometry of Syzygies*, Springer-Verlag, Berlin-Heidelberg-New York, 2005.
14. M. Elkadi, A. Galligo and T. H. Lê, Parametrized surfaces in $P^3$ of bidegree (1,2), *Proceedings of the 2004 International Symposium on Symbolic and Algebraic Computation*, ACM, New York, 2004, 141–148.
15. R. Fröberg, S. Lundqvist, Questions and conjectures on extremal Hilbert series. *Rev. Union Mat. Arqnt.* **59** (2018), 415–429.
16. A. Galligo, T. H. Lê, General classification of (1,2) parametric surfaces in $P^3$, in “Geometric modeling and algebraic geometry”, Springer, Berlin, (2008) 93–113.
17. D. Grayson, M. Stillman, Macaulay2, a software system for research in algebraic geometry, Available at http://www.math.uiuc.edu/Macaulay2/.
18. J. Harris, *Algebraic Geometry, A First Course*, Springer-Verlag, Berlin-Heidelberg-New York, 1992.
19. R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, Berlin-Heidelberg-New York, 1977.
20. J. Herzog, A. Simis, W. Vasconcelos Approximation complexes of blowing-up rings, *J. Algebra* **74** (1982), 466–493.
21. J. Herzog, A. Simis, W. Vasconcelos Approximation complexes of blowing-up rings II, *J. Algebra* **82** (1983), 53–83.
22. J. W. Hoffman and H. H. Wang, Castelnuovo-Mumford regularity in biprojective spaces, *Adv. Geom.* **4** (2004), 513–536.
23. D. Maclagan, G.G. Smith, Multigraded Castelnuovo-Mumford regularity, *J. Reine Angew. Math.* **57** (2004), 179-212.
24. T. Römer, Homological properties of bigraded algebras, *Illinois J. Math.*, **45** (2001), 1361–1376.
25. H. Schenck, A. Seceleanu, J. Validashti, Syzygies and singularities of tensor product surfaces of bidegree $(2,1)$, *Math. Comp.* **83** (2014), 1337–1372.
26. T. W. Sederberg, F. Chen, Implicitization using moving curves and surfaces, in *Proceedings of SIGGRAPH*, 1995, 301–308.
27. B. Sturmfels, The Hurwitz form of a projective variety. *J. Symb. Comput.*, **79** (2017), 186–196.
28. S. Zube, Correspondence and (2,1)-Bézier surfaces, *Lith. Math. J.* **43** (2003), 83–102.
29. S. Zube, Bidegree (2,1) parametrizable surfaces in $P^3$, *Lith. Math. J.* **38** (1998), 291–308.
