Shadows and convexity of surfaces

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Abstract

We study the geometry and topology of immersed surfaces in Euclidean 3-space whose Gauss map satisfies a certain two-piece-property, and solve the “shadow problem” formulated by H. Wente.

1. Introduction

Let $M$ be a closed oriented 2-dimensional manifold, $f: M \to \mathbb{R}^3$ be a smooth immersion into Euclidean 3-space, and $n: M \to \mathbb{S}^2$ be a unit normal vectorfield, or the Gauss map, induced by $f$. Then for every unit vector $u \in \mathbb{S}^2$ (corresponding to the direction of light) the shadow, $S_u$, is defined by

$$S_u := \{ p \in M : \langle n(p), u \rangle > 0 \},$$

where $\langle \cdot, \cdot \rangle$ is the standard innerproduct. If $f$ is a convex embedding, i.e., $f$ maps $M$ homeomorphically to the boundary of a convex body, then it is intuitively clear that $S_u$ is a connected subset of $M$ for each $u$. In 1978, motivated by problems concerning the stability of constant mean curvature surfaces, H. Wente [17] appears to have been the first person to study the converse of this phenomenon, which has since become known as the “shadow problem” [13]: Does connectedness of the shadows $S_u$ imply that $f$ is a convex embedding? In this paper we prove:

**Theorem 1.1.** $f$ is a convex embedding if and only if, for every $u \in \mathbb{S}^2$, $S_u$ is simply connected.

Furthermore we show that the additional condition implied by the word “simply” in the above theorem is necessary:

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Theorem 1.2. There exists a smooth embedding of the torus, \( f: \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3 \), such that for all \( u \in \mathbb{S}^2 \), \( S_u \) is connected.

Thus, connectedness of the shadows in general is not strong enough to ensure convexity or even determine the topology; however, we can show:

Theorem 1.3. If \( M \) is topologically a sphere, and, for every \( u \in \mathbb{S}^2 \), \( S_u \) is connected, then \( f \) must be a convex embedding.

In short, the answer to the above question is yes, provided that either the shadows are simply connected, or \( M \) is a sphere; otherwise, the answer is no. This settles Wente’s shadow problem in 3-space. See [7] and [5] for motivations behind this problem and relations to constant mean curvature surfaces.

Note 1.4. The immersion \( f: M \to \mathbb{S}^2 \) has connected shadows if and only if for every great circle \( C \subset \mathbb{S}^2 \), \( n^{-1}(\mathbb{S}^2 - C) \) has exactly two components. That is, the Gauss map satisfies a two-piece-property [3] similar to that formulated by T. Banchoff [2], and further developed by N. Kuiper [12].

Note 1.5. For a great circle \( C \subset \mathbb{S}^2 \), the number of components of \( n^{-1}(\mathbb{S}^2 - C) \) has been called the vision number with respect to a direction perpendicular to \( C \). This terminology is due to J. Choe, who conjectured [5, p. 210] that there always exists a direction with respect to which the vision number of \( f: M \to \mathbb{R}^3 \) is greater than or equal to \( 4 - \chi(M) \) where \( \chi \) is the Euler characteristic. Theorem 1.2 gives a counterexample to this conjecture.

2. Regularity of horizons and shadow boundaries

First we need to establish some basic regularity results regarding the generic behavior of shadows. For each \( u \in \mathbb{S}^2 \), define the shadow function \( \sigma_u: M \to \mathbb{R} \) by

\[
\sigma_u(p) := \langle n(p), u \rangle.
\]

\( H_u := \sigma_u^{-1}(0) \) is called the horizon [5] in the direction \( u \). It is easy to see that in general \( \partial S_u \neq H_u \neq \partial S_{-u} \), where \( \partial \) denotes the boundary; however, using Sard’s theorem, we can show

Proposition 2.1. For almost all \( u \in \mathbb{S}^2 \) (in the sense of Lebesgue measure) \( H_u \) is a regular curve. Thus for these \( u \), both \( \partial S_u \) and \( \partial S_{-u} \) are regular curves as well. Further, if \( H_u \) is connected, then \( \partial S_u = H_u = \partial S_{-u} \).

We say that \( \Gamma \subset M \) is a regular curve if for each \( p \in \Gamma \) there is an open neighborhood \( U \) of \( p \) in \( M \) and a homeomorphism \( \varphi: U \to \mathbb{R}^2 \) such that \( \varphi(U \cap \Gamma) = \mathbb{R} \). In particular, unless stated otherwise, a regular curve needs not be differentiable.
Proof. Let $T_pM$ be the tangent plane of $M$ at $p$ which we identify with a subspace of $\mathbb{R}^3$ (by identifying $T_pM$ with $f_*(T_pM)$, and parallel translating the elements of $f_*(T_pM)$ to the origin in $\mathbb{R}^3$; $f_*$ denotes the differential of $f$). Let $UTM := \{(p,v) : p \in M, v \in T_pM, \|v\| = 1\}$ denote the unit tangent bundle of $M$, and $\tau$ be the mapping given by

$$UTM \ni (p,v) \mapsto \tau(p,v) = (p, \langle v, u \rangle) \in \mathbb{S}^2.$$ 

By Sard’s theorem almost every $u \in \mathbb{S}^2$ is a regular value of $\tau$; consequently, for such $u$, $\tau^{-1}(u)$ is a regular curve in $UTM$.

Now let $\pi$ be the mapping defined by

$$UTM \ni (p,v) \mapsto \pi(p,v) = \frac{\pi(p)}{\|\pi(p)\|} \in \mathbb{S}^2,$$

and let $u$ be a regular value of $\tau$. Note that $\pi$ is injective on $\tau^{-1}(u)$. As $\tau^{-1}(u)$ is compact, this implies that $\pi: \tau^{-1}(u) \to M$ is an embedding. Further note that

$$\pi(\tau^{-1}(u)) = \{p \in M : u \in T_pM\} = \{p \in M : \langle n(p), u \rangle = 0\} = H_u.$$

Thus $H_u$ is a regular curve. But then, it follows that $\partial S_u$ and $\partial S_{-u}$ are each open in $H_u$, which yields that $\partial S_u$ and $\partial S_{-u}$ are both regular curves as well. Finally, since these shadow boundaries are also closed in $H_u$, it follows that whenever $H_u$ is connected we have $\partial S_u = H_u = \partial S_{-u}$. \hfill $\square$

Note 2.2. Suppose that there is an open set $U \subset \mathbb{S}^2$, such that, for all $u \in U$, both $S_u$ and $S_{-u}$ are simply connected. Then $M$ is homeomorphic to $\mathbb{S}^2$; because, by the above proposition, there exists a $u_0 \in U$ such that $H_{u_0}$ is a regular curve. Consequently the closures $\overline{S}_u$ and $\overline{S}_{-u}$ are homeomorphic to disks. Further, since by assumption $M - H_{u_0}$ is made up of a pair of simply connected components, $H_{u_0}$ is connected. Thus by the above proposition $\partial S_{-u_0} = \partial S_{u_0}$. So $M$ is homeomorphic to a pair of disks glued together along their boundaries.

By smooth we mean differentiable of class $C^\infty$, and for convenience we always assume that the immersion $f: M \to \mathbb{R}^3$ is smooth, though in this paper it is enough that $f$ be $C^3$.

Note 2.3. The embedding $\pi: \tau^{-1}(u) \to M$ in the above proposition is smooth, when $u$ is a regular value of $\tau$. In particular, $H_u$ is smooth for almost all $u \in \mathbb{S}^2$. To see this let $(p,v) \in \tau^{-1}(u)$. Then $u \in T_pM$. Let $v \in T_pM$ with $\langle u, v \rangle = 0$. Then $c(t) := (p, \cos(t)u + \sin(t)v)$ parametrizes the fiber $UTM$ of the unit tangent bundle. Note that

$$\tau_{(p,v)}(c'(0)) = \frac{d}{dt}\bigg|_{t=0} c(t) = (p, \cos(t)u + \sin(t)v) \bigg|_{t=0} = v \neq 0.$$
On the other hand,
\[ T_{(p,u)}(\tau^{-1}(u)) = \{ X \in T_{(p,u)}(UTM) : \tau_{(p,u)}(X) = 0 \}. \]

Thus \( c'(0) \notin T_{(p,u)}(\tau^{-1}(u)) \), which implies that \( \tau^{-1}(u) \) is never tangent to any of the fibers \( UT_pM \) of the unit tangent bundle. So \( \pi|_{\tau^{-1}(u)} \) is a smooth immersion.

Next we need a local regularity result for the horizons and shadow boundaries. The Gaussian curvature \( K : M \to \mathbb{R} \) is defined by \( K(p) := \det(n_*(p)) \).

**Proposition 2.4.** If \( K(p) \neq 0 \) for some \( p \in M \), then there exists a neighborhood \( U \) of \( p \) such that for all \( u \in T_pM, H_u \cap U \) is a smooth regular curve and \( \partial S_u \cap U = H_u \cap U = \partial S_{-u} \cap U \).

**Proof.** Since \( \det(n_*) = S(p) \neq 0 \), then, by the inverse function theorem, \( n \) is a diffeomorphism between small neighborhoods \( U \) of \( p \) in \( M \) and \( V \) of \( n(p) \) in \( \mathbb{S}^2 \). Let \( S_u := \{ x \in \mathbb{S}^2 : \langle x, u \rangle > 0 \} \). Then \( \partial S_u = \partial S_{-u} \) is a regular curve. Thus, since \( S_u = n^{-1}(S_u) \) and \( S_{-u} = n^{-1}(S_{-u}) \), the proof follows. \( \square \)

**Note 2.5.** If \( K(p) = 0 \), then \( H_u \) may not be regular for all \( u \in T_pM \); however, typically \( H_u \) will be regular for most \( u \in T_pM \); because, for \( u \in T_pM \), the differential of \( \sigma_u \) at \( p \) is given by
\[ (d\sigma_u)_p(\cdot) = (\cdot, n_*(u)). \]

So if \( n_*(u) \neq 0 \), e.g., \( u \) is not an asymptotic direction, then \( d\sigma_u \) is nonzero at \( p \). Consequently, by the implicit function theorem, \( \sigma_u^{-1}(\sigma_u(p)) = \sigma_u^{-1}(0) = H_u \) is a smooth regular curve near \( p \).

### 3. Critical points of height functions

The next set of preliminary results we need involves some basic applications of Morse theory [14]. For every \( u \in \mathbb{S}^2 \), let the height function \( h_u : M \to \mathbb{R} \), associated to the immersion \( f : M \to \mathbb{R}^3 \), be defined by
\[ h_u(p) := \langle f(p), u \rangle. \]

Recall that \( p \) is a critical point of \( h_u \) if the differential map \( (dh_u)_p : T_pM \to \mathbb{R} \) is zero. Since \( (dh_u)_p(\cdot) = \langle \cdot, u \rangle \), it follows that \( p \) is a critical point of \( h_u \) if and only if \( u = \pm n(p) \). If all of its critical points are nondegenerate, \( h_u \) is a Morse function.

**Lemma 3.1.** (i) \( h_u \) is a Morse function if and only if \( K \neq 0 \) at all critical points of \( h_u \). (ii) \( h_u \) is a Morse function for almost all \( u \in \mathbb{S}^2 \). (iii) The set \( U \subset \mathbb{S}^2 \) such that for all \( u \in U \) \( h_u \) is a Morse function is open.
Though the above is fairly well-known (e.g., see [3, pp. 11–12]), we include a brief proof for completeness.

Proof. If \( p \) is a critical point of \( h_u \), then, as a standard computation shows, the Hessian of \( h_u \) is given by

\[
\text{Hess} h_u(\cdot, \cdot) = \pm \langle \cdot, n_p(\cdot) \rangle.
\]

Thus \( h_u \) is a Morse function if and only if at each critical point \( p \), \( K(p) = \det(n_p) \neq 0 \). This is equivalent to requiring that both \( u \) and \( -u \) be regular values of \( n \), because \( p \) is a critical point of \( h_u \) if and only if \( u = \pm n(p) \). Let \( U \subset S^2 \) be the set of all such values. Then, by Sard’s theorem, \( S^2 - U \) has measure zero. Further, since \( M \) is compact, and the set of critical points of \( n \) is closed, it follows that the set of critical values of \( n \) is closed as well, so \( U \) is open.

The following is implicit in a paper of Chern and Lashof [4].

**Lemma 3.2.** If \( f \) is not a convex embedding, then there exists a Morse height function \( h_u \) with at least three critical points.

Proof. Let \( \#C(h_u) \) denote the number of critical points of \( h_u \). Since \( p \) is a critical point of \( h_u \) if and only if \( n(p) = \pm u \), we have:

\[
\int_{S^2} \#C(h_u) \, du = \int_{S^2} \#n^{-1}(\pm u) \, du = 2 \int_M |\det(n_*)| \, dV = 2 \int_M |K| \, dV.
\]

The second equality above is just an application of the area formula [6, Thm. 3.2.3], where \( dV \) denotes the volume element on \( M \). Suppose that \( f \) is not a convex embedding. Then, by a well-known theorem of Chern and Lashof [4],

\[
\int_M |K| \, dV > 4\pi.
\]

Combining the above expressions yields a lower bound for the average number of critical points:

\[
\frac{1}{4\pi} \int_{S^2} \#C(h_u) \, du > 2.
\]

So since, by Lemma 3.1, \( h_u \) is a Morse function for almost every \( u \in S^2 \), it follows that there exists a Morse function such that \( \#C(h_u) > 2 \).

4. Triplets on the boundaries of simply connected domains

Here we develop some elementary topological methods whose motivation will become more clear in the next section.
Definition 4.1. By a domain we mean a connected open subset $\Omega \subset M$. We say $\Omega$ is adjacent to a triplet of points $\{p_1, p_2, p_3\} \subset M$ if $p_i \in \partial \Omega$. $\Omega$ is regular near $p_i$ if there are open neighborhoods $U_i$ of $p_i$ and homeomorphisms $\varphi_i: U_i \to \mathbb{R}^2$ which map $U_i \cap \Omega$ into the upper half-plane. A simple closed curve $T \subset \overline{\Omega}$ is a triangle of $\Omega$ (with vertices at $\{p_1, p_2, p_3\}$) if $p_i \in T$, and $T - \{p_1, p_2, p_3\} \subset \Omega$.

The following lemma, though quite elementary, is more subtle than it might at first appear (see Note 4.3).

Lemma 4.2. Every domain $\Omega$ adjacent to $\{p_1, p_2, p_3\}$ admits a triangle. Further if $\Omega$ is simply connected and regular near $p_i$, then any pair of such triangles may be homotoped to each other through a family of triangles of $\Omega$.

Proof. Since $\Omega$ is open and connected, there exists a regular arc $A_{12} \subset \Omega$ whose end points are $p_1$ and $p_2$. Since $A_{12}$ is regular, there exists a component $(\Omega - A_{12})^+$ of $\Omega - A_{12}$ which contains $p_3$ in its closure. Let $A_{23} \subset (\Omega - A_{12})^+$ be a regular arc with end points on $p_2$ and $p_3$. Then, similarly, there exists a component $((\Omega - A_{12})^+ - A_{23})^+$ of $(\Omega - A_{12})^+ - A_{23}$ which contains $p_1$ in its closure. Finally, let $A_{31} \subset ((\Omega - A_{12})^+ - A_{23})^+$ be a regular arc with end points at $p_3$ and $p_1$. The union of these three arcs, and their endpoints, gives the desired triangle.

Now suppose that $\Omega$ is simply connected and regular near $p_i$. Let $T$ and $T'$ be a pair of triangles of $\Omega$, and let $A_{12}$ and $A'_{12}$ be arcs of $T$ and $T'$ respectively which connect $p_1$ and $p_2$. Since $\Omega$ is regular near $p_i$, we may homotope $A_{12}$ (while keeping its end points fixed) by a small perturbation near $p_1$ so that $A_{12}$ and $A'_{12}$ coincide along a segment near $p_1$. Similarly, we may assume that they coincide near $p_2$ as well. Then it remains to homotope proper subarcs of $A_{12}$ and $A'_{12}$ which coincide at a pair of end points in $\Omega$. Since $\Omega$ is simply connected, these subarcs may be homotoped to each other while keeping the end points fixed. Thus $A_{12}$ and $A'_{12}$ are homotopic through a family of arcs of $\Omega$ with end points at $p_1$ and $p_2$. Other arcs of $T$ may be similarly homotoped to their counterparts in $T'$, which completes the proof.

Note 4.3. Without the regularity assumption near $p_i$, the second claim in the above lemma is not true in general: Suppose for instance that $\Omega \subset \mathbb{R}^2$ is an open disk of radius 1 centered at the origin, and with segment $[0, 1)$ removed. Set $p_1 = (0, 0)$, $p_2 = (1/2, 0)$, and $p_3 = (1, 0)$. Then a triangle of $\Omega$ which lies above the $x$-axis may not be homotoped to one lying below the $x$-axis.
Proposition 4.4. For a fixed orientation of $M$, every simply connected domain $\Omega$ which is adjacent to and regular near a triple of (distinct) points $\{p_1, p_2, p_3\} \subset M$ uniquely determines a permutation $\alpha_\Omega$ of $\{p_1, p_2, p_3\}$ such that (i) if $\Omega$ and $\Omega'$ have a triangle in common, then $\alpha_\Omega = \alpha_{\Omega'}$; and (ii) if $\partial \Omega = \partial \Omega'$ is a regular curve, and $\Omega$ and $\Omega'$ are distinct, then $\alpha_\Omega \neq \alpha_{\Omega'}$.

Proof. By Lemma 4.2 there exists a triangle $T$ of $\Omega$. $T$ bounds a simply connected subdomain $U$ of $\Omega$. Since $M$ is oriented, $U$ inherits a preferred sense of orientation, which in turn induces an orientation, or a sense of direction, on $T$. This direction induces a permutation of $\{p_1, p_2, p_3\}$ in the obvious way: If as we move along $T$ and pass $p_1$ we reach $p_2$ before reaching $p_3$, then we set the induced permutation $\alpha_\Omega$ to be the cycle $(p_1, p_2, p_3)$; otherwise, the induced permutation is the cycle $(p_1, p_3, p_2)$. It is clear that these permutations depend continuously on $T$. Thus, since by Lemma 4.2, all triangles of $\Omega$ are homotopic, it follows that $\alpha_\Omega$ does not depend on the choice of $T$ and is therefore well defined; and furthermore, if $\Omega$ and $\Omega'$ have a triangle in common then $\alpha_\Omega = \alpha_{\Omega'}$.

Now suppose that $\partial \Omega = \partial \Omega'$ is a regular curve, and $\Omega$ and $\Omega'$ are distinct. Then $\Omega$ and $\Omega'$ induce opposite orientations on $\partial \Omega$ which in turn gives rise to distinct permutations of $\{p_1, p_2, p_3\}$ (since $\Omega$ is simply connected, $\partial \Omega$ is connected). But by small perturbations, $\partial \Omega$ may be homotoped to a triangle of $\Omega$, just as well as it may be homotoped to a triangle of $\Omega'$. Thus the orientations which $\Omega$ and $\Omega'$ induce on $\partial \Omega$ are consistent with the orientations which $\Omega$ and $\Omega'$ induce on their own triangles respectively. So $\alpha_\Omega \neq \alpha_{\Omega'}$. □

5. Proof of Theorem 1.1

First we show that if $f$ is a convex embedding, then $S_u$ is simply connected for all $u \in S^2$. To see this let $\Pi$ be a plane perpendicular to $u$ and let $\pi: \mathbb{R}^3 \to \Pi$ be the orthogonal projection. Then $D := \pi(f(M))$ is a convex subset of $\Pi$ with interior points. In particular, $\text{int}(D)$ is homeomorphic to an open disk. Since $f(M)$ is convex and by definition $\langle n(p), u \rangle > 0$ for all $p \in S_u$, it is not hard to verify that $f(S_u)$ is a graph over $\text{int}(D)$. Thus $\pi \circ f: S_u \to \text{int}(D)$ is a homeomorphism.

Now we prove the other direction: Assume that for every $u \in S^2$, $S_u$ is simply connected; we have to show that $f$ is a convex embedding. The proof is by contradiction:

Lemma 5.1. If $f$ is not a convex embedding, then there exists a pair of orthogonal vectors $u_0, v_0 \in S^2$ such that (i) $h_{u_0}$ is a Morse function with at least three critical points, and (ii) $\partial S_{v_0} = H_{v_0} = \partial S_{-v_0}$ is a regular curve.
Proof. By Lemma 3.2, there exists a unit vector $u \in S^2$ such that the corresponding height function $h_u$ is a Morse function and has at least three critical points. Further, it follows from Lemma 3.1, that this $u$ may be chosen from an open set $U \subset S^2$.

Let $u^\perp := \{v \in S^2 : \langle u, v \rangle = 0\}$. Then $U^\perp := \cup_{u \in U} u^\perp$ is open. Consequently, by Proposition 2.1, there exits a $v_0 \in U^\perp \subset U^\perp$ such that $H_{v_0}$ is a regular curve. Further, since the complement of $H_{v_0}$ consists of a pair of simply connected domains, $H_{v_0}$ is connected. Thus, again by Proposition 2.1, $\partial S_{v_0} = H_{v_0} = \partial S_{-v_0}$ is a regular curve.

Let $\tilde{v}_0 \in S^2$ be a vector orthogonal to both $u_0$ and $v_0$, and set

$$(1) \quad v(\theta) := \cos(\theta) v_0 + \sin(\theta) \tilde{v}_0.$$  

Let $p_i$, $i = 1, 2, 3$, be a fixed triple of (distinct) critical points of $h_{u_0}$.

**Lemma 5.2.** For all $\theta \in \mathbb{R}$, $S_{v(\theta)}$ is a domain adjacent to and regular near $p_i$.

Proof. If $p_i$ is a critical point of $h_{u_0}$, then $n(p_i) = \pm u_0$. So $\sigma_{v(\theta)}(p_i) = \langle v(\theta), \pm u_0 \rangle = 0$, which yields that $p_i \in H_{v(\theta)}$. Since $h_{u_0}$ is a Morse function, then, by Lemma 3.1, $K(p_i) \neq 0$. So by Proposition 2.4, there exists a neighborhood $U_i$ of $p_i$ such that $\partial S_{v(\theta)} \cap U_i = H_{v(\theta)} \cap U_i = \partial S_{-v(\theta)} \cap U_i$, which completes the proof.

It now follows from Proposition 4.4 that each $S_{v(\theta)}$ induces a permutation of $\{p_1, p_2, p_3\}$ which we denote by $\alpha_{\theta} := \alpha_{(S_{v(\theta)})}$. Further, by the same proposition and since $\partial S_{v_0} = \partial S_{-v_0}$ is a regular curve, it follows that $\alpha_0 \neq \alpha_\pi$. On the other hand, letting $\text{Sym}$ denote the symmetric group, we claim that the mapping

$$\mathbb{R} \ni \theta \mapsto \alpha_{\theta} \in \text{Sym}\{\{p_1, p_2, p_3\}\}$$

is locally constant, which, since $[0, \pi]$ is connected, would imply that $\alpha_0 = \alpha_\pi$. This contradiction, which would complete the proof, follows from Proposition 4.4 and the following:

**Lemma 5.3.** For each $\theta_0 \in \mathbb{R}$ there exists an $\varepsilon > 0$ such that if $|\theta - \theta_0| < \varepsilon$ then $S_{v(\theta)}$ and $S_{v(\theta_0)}$ have a common triangle (with vertices at $\{p_1, p_2, p_3\}$).

Proof. Recall that, since $h_{u_0}$ is a Morse function, then, by Lemma 3.1, $K(p_i) \neq 0$ which yields that $n$ is a local diffeomorphism at $p_i$. Therefore, by Proposition 2.4, in a neighborhood $W$ of $\{p_1, p_2, p_3\}$, $\partial S_{v(\theta)} = H_{v(\theta)} = n^{-1}(v^\perp(\theta))$ where $v^\perp(\theta)$ denotes the great circle in $S^2$ orthogonal to $v(\theta)$. So, since $v^\perp(\theta)$ depends continuously on $\theta$, it follows that, in $W$, $\partial S_{v(\theta)}$ depends continuously on $\theta$ as well.
Let $T$ be a triangle of $S_{v(\theta_0)}$. Since $S_{v(\theta_0)}$ is open, after a perturbation of $T$ we may assume that the arcs of $T$ are smooth and meet $\partial S_{v(\theta_0)}$ transversely (recall that, by Proposition 2.4, $\partial S_{v(\theta_0)}$ is smooth near $p_i$). Thus, by the above paragraph, it follows that if $|\theta - \theta_0| < \varepsilon_1$, for some sufficiently small $\varepsilon_1 > 0$, then $T$ meets $\partial S_{v(\theta)}$ transversely as well. Then it follows that for some neighborhood $W$ of $\{p_1, p_2, p_3\}$, $(T - \{p_1, p_2, p_3\}) \cap W \subset S_{v(\theta)}$ for all $\theta$ such that $|\theta - \theta_0| < \varepsilon_1$.

Next note that $T - W$ is compact, and the mapping $\theta \mapsto \sigma_{v(\theta)}$ is continuous; therefore, since by assumption $\sigma_{v(\theta_0)} > 0$ on $T - W$, it follows that there exists an $\varepsilon_2 > 0$ such that $\sigma_{v(\theta)} > 0$ on $T - W$ for all $\theta$ such that $|\theta - \theta_0| < \varepsilon_2$. This yields that $T - W \subset S_{v(\theta)}$ for all $\theta$ such that $|\theta - \theta_0| < \varepsilon_2$.

From the previous two paragraphs it follows that, setting $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$, we have $(T - \{p_1, p_2, p_3\}) \subset S_{v(\theta)}$ for all $\theta$ such that $|\theta - \theta_0| < \varepsilon$, which completes the proof.

Note 5.4. Theorem 1.1 does not remain valid if the shadows are defined as the sets where $\langle n(p), u \rangle \geq 0$. For instance, the standard torus of revolution would be a counterexample.

Note 5.5. Theorem 1.1 does not remain valid without the compactness assumption; the hyperbolic paraboloid given by the graph of $z = xy$ would be a counterexample. This follows because here the unit normal vectorfield $n$ is a homeomorphism into a hemisphere. Thus the preimage of any open hemisphere under $n$ is simply connected.

6. Proof of Theorem 1.2

Definition 6.1. We say an immersion $\gamma: S^1 \simeq \mathbb{R}/2\pi \to \mathbb{R}^3$ is a skew loop if it has no pair of distinct parallel tangent lines; i.e,

$$\gamma'(t) \times \gamma'(s) \neq 0$$

for all $t, s \in [0, 2\pi)$, $t \neq s$.

A specific example of a skew loop, formulated by Ralph Howard [11], is as follows:

Example 6.2. Let $\gamma(t) := (x(t), y(t), z(t))$, where

$$x(t) := -\cos(t) - \frac{1}{20} \cos(4t) + \frac{1}{10} \cos(2t),$$

$$y(t) := +\sin(t) + \frac{1}{10} \sin(2t) + \frac{1}{20} \sin(4t),$$

$$z(t) := \frac{46}{75} \sin(3t) - \frac{2}{15} \cos(3t) \sin(3t),$$
and \( t \in [0, 2\pi] \). A computation of the tangential indicatrix \( T(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|} \) shows that \( T(t) \neq \pm T(s) \) for all \( t, s \in [0, 2\pi], \ t \neq s \). Thus \( \gamma \) is a skew loop.

Figure 1 shows the pictures of a tube built around \( \gamma(S^1) \).

If \( \gamma: S^1 \to \mathbb{R}^3 \) is an immersion, then the unit normal bundle of \( \gamma \) consists of all pairs \((p, \nu) \in S^1 \times S^2\) such that \( \langle \gamma'(p), \nu \rangle = 0 \). Since this bundle is homeomorphic to a torus, the following proposition yields Theorem 1.2.

**Proposition 6.3.** Let \( \gamma: S^1 \to \mathbb{R}^3 \) be a skew loop and \( M \) be the unit normal bundle of \( \gamma \). For \( \varepsilon > 0 \), define \( f_\varepsilon: M \to \mathbb{R}^3 \) by

\[
 f_\varepsilon(p, \nu) := \gamma(p) + \varepsilon \nu.
\]

Then, for \( \varepsilon \) sufficiently small, \( f_\varepsilon \) is a smooth immersion, and for all \( u \in S^2 \), \( S_u \) is connected. If \( \gamma \) is an embedding, then \( f_\varepsilon \) is an embedding as well.

**Proof.** That \( f_\varepsilon \) is a smooth immersion and is an embedding when \( \gamma \) is embedded follows from the tubular neighborhood theorem. Let \( n: M \to S^2 \) be the unit normal vector field given by \( n(p, \nu) = \nu \), and \( \pi: M \to S^1 \) be given by \( \pi(p, \nu) = p \). For every \( p \in S^1 \), let \( F_p := \pi^{-1}(p) \) be the corresponding fiber. Note that \( n \) embeds \( F_p \) into the great circle in \( S^2 \) which lies in the plane perpendicular to \( T(p) \). Further recall that \( S_u = n^{-1}(S^2_u) \) where \( S^2_u \) is the open hemisphere determined by \( u \). Thus there are only two possibilities for each \( p \in S^1 \): either \( F_p \) intersects \( S_u \) in an open half-circle, or \( F_p \) is disjoint from \( S_u \). The latter occurs if and only if \( T(p) \) is parallel to \( u \), which, since \( \gamma \) is skew, can occur at most once. Hence, it follows that \( S_u \) is either homeomorphic to a disk or an annulus. In particular, \( S_u \) is connected for every \( u \in S^2 \). \( \Box \)

**Question 6.4.** Let \( M \) be a closed oriented 2-dimensional manifold with topological genus \( g(M) \geq 2 \). Does there exist an embedding, or an immersion, \( f: M \to \mathbb{R}^3 \) such that \( S_u \) is connected for all \( u \in S^2 \)?
Note 6.5. Skew loops were first discovered by B. Segre [16] to disprove a conjecture of H. Steinhaus (see also [15]). More recently, it has been shown that there exists a skew loop in each knot class [18], and every pair of knots may be realized with the same tangential indicatrix [1].

Note 6.6. A general procedure for constructing skew loops is as follows. Let $T \subset S^2$ be a smooth simple closed curve such that (i) the origin is contained in the interior of the convex hull of $T$, $(0, 0, 0) \in \text{int conv } T$, and (ii) $T$ does not contain any pair of antipodal points, $T \cap -T = \emptyset$. Figure 2 shows an example.

![Figure 2](image)

Let $T(s), s \in \mathbb{R}$, denote a periodic parametrization of $T$ by arclength. So, assuming $T$ has total length $L$, we have $T(s + L) = T(s)$. Since $(0, 0, 0) \in \text{int conv } T$, there exists a function $\rho(s)$ with period $L$ such that $\int_0^L \rho(s) T(s) \, ds = 0$ [10, p. 168]. Set

$$
\gamma(t) := \int_0^t \rho(s) T(s) \, ds.
$$

Then $\gamma(t + L) = \gamma(t)$. Further, $\gamma'(t)/\|\gamma'(t)\| = T(t)$. Thus $\gamma$ is a closed curve whose tangential spherical image coincides with $T$. Hence $\gamma$ is a skew loop.

Note 6.7. With the sole exception of ellipsoids, every closed surface immersed in $\mathbb{R}^3$ admits a skew loop [8].

7. Proof of Theorem 1.3

We follow a modified outline of the proof of Theorem 1.1, which again proceeds by contradiction. Suppose that $M$ is homeomorphic to $S^2$ and $S_u$ is connected for all $u \in S^2$. If $f$ is not a convex embedding, let $u_0$ and $v_0$ be as in Lemma 5.1, and $v(\theta)$ be as defined by (1).

Definition 7.1. The augmented shadow $\tilde{S}_{v(\theta)}$ is the union of $S_{v(\theta)}$ with all components $X$ of $H_{v(\theta)}$ such that $U - X \subset S_{v(\theta)}$ for an open neighborhood $U$ of $X$. 
Then $\tilde{S}_v(\theta)$ satisfies the conditions of the following lemma:

**Lemma 7.2.** If $U \subset \mathbb{S}^2$ is a connected open set, and $\mathbb{S}^2 - U$ is also connected and has an interior point, then $U$ is simply connected.

*Proof.* Let $p$ be an interior point of $\mathbb{S}^2 - U$. Then the stereographic projection maps $U$ into a connected open set with connected complement. Thus, by [9, Thm. 11.4.1], $U$ is simply connected. □

So $\tilde{S}_v(\theta)$ is simply connected. Further:

**Lemma 7.3.** For all $\theta \in \mathbb{R}$, $\tilde{S}_v(\theta)$ is a domain adjacent to and regular near $p_i$.

*Proof.* This follows just as in the proof of Lemma 5.2, once we observe that whenever $\partial S_v(\theta) = H_v(\theta) = \partial S_{-v}(\theta)$ is regular in some open neighborhood, then $\partial \tilde{S}_v(\theta)$, and $\partial S_v(\theta)$ coincide within that neighborhood. □

Thus each $\theta$ induces a permutation $\tilde{\alpha}_\theta := \alpha \sim_{(S_v(\theta))} \{p_1, p_2, p_3\}$ which satisfies the enumerated properties in Proposition 4.4. In particular $\tilde{\alpha}_0 \neq \tilde{\alpha}_\pi$, because since $\partial S_v(0) = \partial S_{-v}(0)$ is by Lemma 5.1 a regular curve, it follows that $\partial \tilde{S}_v(0) = \partial \tilde{S}_{-v}(0)$ is a regular curve as well. So it remains to verify the following lemma which shows that $\theta \mapsto \tilde{\alpha}_\theta$ is locally constant. This would yield that $\tilde{\alpha}_0 = \tilde{\alpha}_\pi$ which is the desired contradiction.

**Lemma 7.4.** For each $\theta_0 \in \mathbb{R}$ there exists an $\varepsilon > 0$ such that if $|\theta - \theta_0| < \varepsilon$ then $\tilde{S}_v(\theta)$ and $\tilde{S}_v(\theta_0)$ have a common triangle (with vertices at $\{p_1, p_2, p_3\}$).

*Proof.* This is an immediate consequence of Lemma 5.3 where it was proved that $S_v(\theta)$ and $S_v(\theta_0)$ have a triangle in common (the proof of Lemma 5.3 makes no use of the simply connectedness assumption on $S_v(\theta)$). □

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References

[1] C. Adams, C. Lefever, J. Othmer, S. Paik, A. Stier, and J. Tripp, An introduction to the supercrossing index of knots and the crossing map, preprint.

[2] T. F. Banchoff, The two-piece property and tight n-manifolds-with-boundary in $E^n$, Trans. Amer. Math. Soc. 161 (1971), 259–267.

[3] T. E. Cecil and P. J. Ryan, Tight and Taut Immersions of Manifolds, Pitman (Advanced Publishing Program), Boston, Mass., 1985.

[4] S.-s. Chern and R. K. Lashof, On the total curvature of immersed manifolds, Amer. J. Math. 79 (1957), 306–318.

[5] J. Choe, Index, vision number and stability of complete minimal surfaces, Arch. Rational Mech. Anal. 109 (1990), 195–212.

[6] H. Federer, Geometric Measure Theory, Die Grundlehren der mathematischen Wissenschaften 153, Springer-Verlag, New York, 1969.

[7] M. Ghomi, Solution to the shadow problem in 3-space, in Minimal Surfaces, Geometric Analysis and Symplectic Geometry, 2000, Adv. Stud. Pure Math. (To appear); preprint available at www.math.sc.edu/~ghomi.

[8] M. Ghomi and B. Solomon, Skew loops and quadric surfaces, preprint available at www.math.sc.edu/~ghomi.

[9] R. E. Greene and S. G. Krantz, Function Theory of One Complex Variable, John Wiley & Sons Inc., New York, 1997.

[10] M. Gromov, Partial Differential Relations, Springer-Verlag, New York, 1986.

[11] R. Howard, Mohammad Ghomi’s solution to the shadow problem, Lecture notes, available at www.sc.edu/~howard.

[12] N. H. Kuiper, Geometry in curvature theory, in Tight and Taut Submanifolds (Berkeley, CA, 1994), pp. 1–50, Cambridge Univ. Press, Cambridge, 1997.

[13] J. McCuan, Personal e-mail, June 23, 1998.

[14] J. Milnor, Morse Theory, Based on lecture notes by M. Spivak and R. Wells, Ann. of Math. Studies 51, Princeton Univ. Press, Princeton, NJ, 1963.

[15] J. R. Porter, A note on regular closed curves in $E^3$, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 18 (1970), 209–212.

[16] B. Segre, Global differential properties of closed twisted curves, Rend. Sem. Mat. Fis. Milano 38 (1968), 256–263.

[17] H. C. Wente, Personal e-mail, January 9, 1999.

[18] Y.-Q. Wu, Knots and links without parallel tangents. Bull. London Math. Soc., to appear.

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